Pseudo Algebraically Closed Extensions

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by

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Chapter 1

Introduction

1.1 Background and Motivation

1.1.1 Hilbert’s Tenth Problem

Hilbert’s tenth problem asks for a general algorithm to decide whether a given polynomial Diophantine equation with integer coefficients has a solution in integers. In 1972 Matijasevich gave a negative solution to that problem relying on the works of Davis, Putnam, and J. Robinson since the 1930’s.

These developments led Robinson to ask Hilbert’s tenth problem w.r.t. the ring of all algebraic integers. In 1987 Rumely indeed found an algorithm for solving Diophantine problems over the ring of all algebraic integers using the following local-global principle. Let $\mathbb{Q}$ be the field of all algebraic numbers and $\mathbb{Z}$ the ring of all algebraic integers. Then a variety $V$ over $\mathbb{Q}$ has an integral point, i.e. $V(\mathbb{Z}) \neq \emptyset$, if and only if $V$ has an integral point in each completion of $\mathbb{Q}$.

Jarden and Razon obtained Rumely’s local-global principle for an abundance of algebraic integer rings [18, 19] and then derived an affirmative solution of Hilbert’s tenth problem for those rings. The main innovation made by Jarden-Razon is the following key definition [17].

Definition 1.1.1. Let $M$ be a field and $K$ a subset. Then $M/K$ is called pseudo algebraically closed (PAC) (or, alternatively, we say that $M$ is a PAC extension of $K$) if for every absolutely irreducible polynomial $f(T, X) \in M[T, X]$ which is separable in $X$ there are infinitely many $(a, b) \in K \times M$ for which $f(a, b) = 0$.

Note that this definition generalizes the classical notion of PAC fields (taking $K = M$), which is central in the subject Field Arithmetics (some books on this subject are [9, 22, 34]).
Let us describe Jarden-Razon result on Rumely’s local-global principle more precisely. Let $K$ be a field. Consider an $e$-tuple of Galois automorphisms $\sigma = (\sigma_1, \ldots, \sigma_e) \in \text{Gal}(K)^e$. Then $K_s(\sigma) = \{x \in K_s \mid \sigma_i(x) = x, \forall i\}$ denotes the fixed field of $\sigma$ in the separable closure $K_s$ and $\langle \sigma \rangle$ denote the closed subgroup generated by $\sigma_1, \ldots, \sigma_e$.

**Theorem A.** Let $R$ be a countable Hilbertian domain, $K$ its quotient ring, and $e \geq 1$ an integer. Then $K_s(\sigma)/R$ is PAC and $\langle \sigma \rangle$ is free of rank $e$ for almost all $\sigma \in \text{Gal}(K)^e$ (with respect to the Haar measure).

Recall that a Hilbertian domain is an integral domain $R$ with the following feature. For every irreducible $f \in K[X, Y]$ which is separable in $Y$ there exists $a \in R$ s.t. $f(a, Y)$ is irreducible over $K$. For example, the ring of integers of an arbitrary global field is a Hilbertian domain.

A classic result of Jarden [9, Theorems 18.5.6 and 18.6.1] asserts that $K_s(\sigma)$ is PAC and $\langle \sigma \rangle$ is free of rank $e$. The rest of the theorem, i.e. the PACness of $K_s(\sigma)/R$, is the main result in [17].

Now we are ready to present the extension of Rumely’s local-global principle.

**Theorem B.** Let $K$ be a global field. For almost all $\sigma \in \text{Gal}(K)^e$ the following holds: Let $V$ be a variety defined over $M = K_s(\sigma)$. Then $V$ has an integral point in $M$ if and only if $V$ has an integral point in each completion (w.r.t. primes of the ring of integers).

This theorem is a special case of [19, Theorem 2.5].

1.1.2 Connections to Other Areas

Above we described the original motivation for the definition of PAC extensions. Later on Jarden and Razon sharpened Theorem [3] in a series of papers (see [19] for the latest version). However some new applications to PAC extensions having no relation to Hilbert’s tenth problem started to come out.

**Abundance of Hilbertian Fields**

A Galois extension $N/K$ is said to be unbounded if the set $\{\text{ord}(\sigma) \mid \sigma \in \text{Gal}(N/K)\}$ is unbounded. In [26] Razon applies Theorem [A] to find an abundance of Hilbertian fields:

**Theorem C.** Let $K$ be a countable Hilbertian field and $N/K$ an unbounded abelian extension. Then for almost all $\sigma \in \text{Gal}(K)^e$ every subextension $M$ of $K_s(\sigma)N/N$ is Hilbertian.
The following two results show some nice properties of a field $K$ having a “non-degenerate” PAC extension. The main idea lying behind these results is that such fields satisfy some weak Hilbert’s irreducibility theorem, as described further below in the introduction. Let us briefly describe the results.

**Dirichlet’s Theorem for Polynomial Rings**

The Artin-Kornblum analog of Dirichlet’s theorem on primes in arithmetical progressions asserts the following for a finite field $F$.

**Theorem D.** For any coprime $a, b \in F[X]$ and sufficiently large positive integer $n$ there exists $c \in F[X]$ for which $a + bc$ is irreducible of degree $n$.

A natural continuation is to generalize Dirichlet’s theorem to infinite fields. Note that, obviously, Dirichlet’s theorem does not hold for $\mathbb{C}$ or $\mathbb{R}$. A more subtle but still easy observation which the author learnt from Jarden is that over a Hilbertian field $K$ Dirichlet’s theorem does hold (see Chapter 7 for details). In this work we prove that Dirichlet’s theorem holds for a field $K$ provided it has a PAC extension.

**Theorem I.** Let $K$ be a field, $a, b \in K[X]$ coprime, and $n$ sufficiently large. Assume that there exists a PAC extension $M/K$ and a separable extension $N/M$ of degree $n$. Then there exist infinitely many $c \in K[X]$ such that $a + bc$ is irreducible of degree $n$.

This result appears in [1].

**Scaled Trace Forms**

Let $L/K$ be a separable extension of degree $n$. Then $L$ is equipped with the non-degenerate quadratic trace form. Namely, $x \mapsto \text{Tr}_{L/K}(x^2)$, for $x \in L$. A generalization of the trace form is the following scaled trace form. Fix $\lambda \in L^\times$. Then the scaled (by $\lambda$) trace from is the non-degenerate quadratic form on $L$ defined by

$$x \mapsto \text{Tr}_{L/K}(\lambda x^2),$$

for all $x \in L$.

Scharlau [31] and Waterhouse [35] (independently) prove that over a Hilbertian field $K$ every non-degenerated quadratic form is isomorphic to a scaled trace form. In a joint work with Kelmer [2], we extend Scharlau-Waterhouse proof to fields having a PAC extension.
**Theorem E.** Let $K$ be a field. Assume that there exists a PAC extension $M/K$ and a separable extension $N/M$ of degree $n$. Then every non-degenerate quadratic form of degree $n$ over $K$ is isomorphic to a scaled trace form.

**Remark 1.1.2.** The latter two theorems make the property that a field $K$ has a PAC extension $M/K$ and a separable extension $N/M$ of degree $n$ interesting. Below we find some families of fields having this property for many $n$’s, e.g., pro-solvable extensions of $\mathbb{Q}$.

### 1.2 The Galois Structure of PAC Extensions

We develop an elementary machinery to study field extensions. The essential ingredient is a generalization of embedding problems to field extensions. Then we apply it to PAC extensions. In particular we get a key property – the lifting property (Proposition 3.4.6). Let us describe some of the consequences of this developments.

#### 1.2.1 Restrictions on PAC Extensions

In [17] (where PAC extensions firstly appear) Jarden and Razon find some Galois extensions $M$ of $\mathbb{Q}$ such that $M$ is PAC as a field but $M$ is a PAC extension of no number field. (For this a heavy tool is used, namely Faltings’ theorem.) This led them to ask whether this is a coincidence or a general phenomenon.

In [16] Jarden answers this question by showing that the only Galois PAC extension of an arbitrary number field is its algebraic closure. Jarden does not use Faltings’ theorem, but different kind of results. Namely, Frobenius’ density theorem, Neukirch’s characterization of $p$-adically closed fields among all algebraic extensions of $\mathbb{Q}$, and the special property of $\mathbb{Q}$ that it has no proper subfields (!). For that reason Jarden’s method is restricted to number fields.

The next step is to generalize Jarden’s results to a finitely generated infinite field $K$. Elaborating Jarden-Razon original method, i.e. using Faltings’ theorem (and Grauert-Manin theorem in positive characteristic), Jarden and the author [2] generalize Jarden’s result to $K$.

**Theorem F.** Let $K$ be a finitely generated infinite field. The only Galois PAC extension of $K$ is its separable closure.
Using the lifting property mentioned above and wreath products, we can elementarily and easily reprove Theorem F. Moreover, our proof uses no special features of finitely generated fields, so it generalizes Theorem F to arbitrary fields.

**Theorem II.** Let $M/K$ be a proper separable algebraic PAC extension. Then the Galois closure of $M/K$ is the separable closure of $K$.

In particular, if $M/K$ is a Galois PAC extension, the either $M = K$ or $M = K_s$.

The Galois structure of a PAC extension $M/K$ has more restrictions. For example in Corollary 4.2.4 we classify all finite PAC extensions.

**Theorem III.** Let $M/K$ be a finite extension. Then either $M/K$ is PAC if and only if one of the following holds.

(a) $K_0$ is real closed and $K$ is the algebraic closure of $K_0$.

(b) $K_0$ is PAC and $K/K_0$ is purely inseparable.

Interestingly, this theorem implies a positive answer to a seemingly unrelated problem, namely Problem 18.7.8 of [9] in case of finitely generated fields: For a given Hilbertian field $K$, this problem asks whether the following ‘bottom theorem’ holds. For almost all $\sigma \in \text{Gal}(K)^e$ the field $M = K_s(\sigma)$ has no subfield $L$ such that $1 < [M : L] < \infty$.

In [11] Haran proves an earlier version of the bottom theorem, namely with the additional condition that $K \subseteq L$. Note that if, e.g., $K = \mathbb{F}_p(x)$, then $[M : M^p] = p$ for any algebraic separable extension $M/K$ (see discussion before Theorem 8.2.2 for more details). Therefore the bottom theorem fails for this $K$ for trivial reasons. Hence the condition ‘$M/L$ is separable’ should be added to the formulation of the bottom theorem.

Now the bottom theorem for $K$ a finitely generated infinite field is valid. The main ingredient in the proof is, as mentioned above, Theorem III.

**Theorem IV.** The bottom theorem holds for all finitely generated infinite fields.

1.2.2 Descent of Galois Groups

In [27] Razon proves for a PAC extension $M/K$ that every separable extension $N/M$ descends to a separable extension $L/K$.

**Theorem G.** Let $M/K$ be a PAC extension and $N/M$ a separable algebraic extension. Then there exists a separable algebraic extension $L/K$ that is linearly disjoint from $M$ over $K$ such that $N = ML$. 

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We generalize Razon’s result and get a stronger descent result:

**Theorem V.** Let $M/K$ be a PAC extension and $N/M$ a finite Galois extension. Assume $\text{Gal}(N/M) \leq H$, where $H$ is regularly realizable over $K$. Then there exists a Galois extension $L/K$ such that $\text{Gal}(L/K) \leq H$ and $N = LM$.

Indeed, applying Theorem $V$ to the symmetric group $H = S_n$ yields a proof of Theorem $G$ (see proof of Corollary 4.1.4). It is interesting to note that the original proof of Theorem $G$ is similar but very specific: One only considers the regular realization of $H = S_n$ generated by the generic polynomial $f(T_1,\ldots,T_n, X) = X^n + T_1X^{n-1} + \cdots + T_n$.

Let me explain the name ‘descent’ attached to Theorem $V$. If a finite Galois group $G = \text{Gal}(N/M)$ over $M$ regularly occurs over $K$, then, by taking $H = G$ in Theorem $V$ we get that $G$ occurs over $K$ (since $G = \text{Gal}(L/K)$ in that case). Thus $G$ descends to a Galois group over $K$.

As a consequence of this and of the fact that abelian groups are regularly realizable over any field, we get, for example, that

$$M_{\text{ab}} = MK_{\text{ab}}.$$ 

### 1.3 Projective Pairs

There is a strong connection between PAC fields and projective profinite groups.

**Theorem H.** The absolute Galois group of an arbitrary PAC field is projective, and vice-versa, every projective profinite group can be realized as the absolute Galois group of a PAC field.

Ax proved the first assertion [9, Theorem 11.6.2] and Lubotzky and v.d. Dries proved the second one [9, Corollary 23.1.2].

We introduce a similar characterization for PAC extensions of PAC fields and **projective pairs**. A projective pair $(\Gamma, \Lambda)$ is composed of a profinite group $\Lambda$ and a closed subgroup $\Gamma$ for which any double finite embedding problem is weakly solvable (see Definition 5.1.1).

**Theorem VI.**  (a) Let $M$ be a PAC extension of a PAC field $K$. Then $(\text{Gal}(M), \text{Gal}(K))$ is a projective pair.

(b) Let $M$ be an algebraic extension of a PAC field $K$. Then $M/K$ is PAC if and only if $(\text{Gal}(M), \text{Gal}(K))$ is projective.
(c) Let \((\Gamma, \Lambda)\) be a projective pair. Then there exists a PAC extension \(M\) of a PAC field \(K\) such that \(\Gamma \cong \text{Gal}(M), \Lambda \cong \text{Gal}(K)\).

Thus any result about projective pairs immediately yields an analog for PAC extensions of PAC fields. This connection motivates the study of projective pairs.

Interestingly, although we cannot directly apply the results about projective pairs to analogous results on PAC extensions (of non PAC field) we can apply the ideas and technics. And indeed the elementary machinery mentioned in Section 1.2 comes from this analogy.

The study of projective pairs in this work go along two perspectives. First we prove the analogs of results about PAC extensions. For example, the analog of Theorem II asserts that the only normal projective pairs (i.e. \(\Gamma \triangleleft \Lambda\)) are when \(\Gamma = 1\) or \(\Gamma = \Lambda\). Sometimes in the group theoretic setting the results become stronger, e.g., the analog of Theorem V implies the following

**Theorem VII.** Let \((\Gamma, \Lambda)\) be a projective pair. Then \(\Lambda = N \rtimes \Gamma\), for some \(N \triangleleft \Lambda\).

In the same time we find families of examples of projective pairs. By the transitivity of PAC extensions (Proposition 3.4.8) we get new families of PAC extensions. For example, the following is a special case of Corollary 5.4.2.

**Theorem VIII.** Every projective group \(P\) can be realized as the absolute Galois group of a PAC extension of some Hilbertian field \(K\). Moreover if the rank of \(P\) is at most countable, then we can take \(K = \mathbb{Q}_{ab}\).

Unfortunately, this does not lead us to discover new PAC extensions of \(\mathbb{Q}\).

### 1.4 Weak Hilbert’s Irreducibility Theorem

There is a deep connection between embedding problems, irreducible specializations of polynomials, and rational points on varieties. The following theorem is a good evidence for this connection.

**Theorem I.** Let \(K\) be a PAC field. Then \(K\) is Hilbertian if and only if \(K\) is \(\omega\)-free.

(Recall that the PACness means that all varieties have rational points, Hilbertianity implies that all polynomials have the irreducible specialization property, and \(\omega\)-freeness asserts that all finite embedding problems are solvable.) In their work on Frobenius fields [8], Haran, Fried, and Jarden refined the above connection.
We extract from the previous result the exact statement that connects embedding problems, irreducible specializations of polynomials, and rational points on varieties (see Lemma 6.1.2). Next we apply that general criterion to a field \( K \) which has PAC extensions. This gives a weak Hilbert’s irreducibility theorem for \( K \) (Proposition 6.2.1). One nice corollary is in the case where the polynomial is the “most irreducible.”

**Theorem IX.** Let \( K \) be a field and \( f(T,X) \in K(T)[X] \) an irreducible polynomial of degree \( n \) in \( X \) whose Galois group over \( \bar{K}(T) \) is \( S_n \). Further assume that there exists a PAC extension \( M/K \) and a separable extension \( N/M \) of degree \( n \). Then there exist infinitely many \( a \in K \) for which \( f(a,X) \) is irreducible over \( M \), and hence over \( K \).

This result is essential in the proofs of Theorems 1 and E.

### 1.5 Fields Having PAC Extensions

In light of the above results, it is a significant feature for a field \( K \) to have non-trivial PAC extensions.

We focus on two directions. First we generalize Theorem A in case \( K \) is a finitely generated field. Recall that a finitely generated infinite field is always Hilbertian.

**Theorem X.** Let \( K \) be a finitely generated infinite field and \( e \geq 1 \) a positive integer. Then for almost all \( \sigma \in \text{Gal}(K)^e \) and for every field \( F \subseteq K_s(\sigma) \) which is not algebraic over a finite field the extension \( K_s(\sigma)/F \) is PAC.

We do not know whether this result is true in the more general case where \( K \) is an arbitrary countable Hilbertian field (e.g. \( K = \mathbb{Q}_{ab} \)). A positive answer would imply the bottom theorem for countable Hilbertian fields.

Next we go in the opposite direction, that is, we consider a fixed field \( F \) and try to find PAC extensions of it. More precisely we ask the following

**Question 1.5.1.** For what positive integers \( n \) there exist a PAC extension \( M/F \) and a separable extension \( N/M \) of degree \( n \)?

In several cases we can give a satisfactory answer as presented in the following result.

**Theorem XI.** Let \( K \) be a countable Hilbertian field and \( F/K \) a separable algebraic extension. Then there exist a PAC extension \( M/F \) and a separable extension \( N/M \) of degree \( n \) in the following cases.
(a) $F/K$ is pro-solvable and $n \geq 5$.

(b) There exists a prime $p$ such that $p \nmid [\hat{F} : K]$ (as supernatural numbers), where $\hat{F}$ is the Galois closure of $F/K$.

This result has several applications. For example one can extend Theorem E to the following

**Theorem J.** Let $K$ be a Hilbertian field (of any cardinality) and let $F/K$ be a pro-solvable extension (resp. an extension whose Galois closure is prime to $p$). Then for every $n \geq 5$ (resp. $n \geq 1$) any non-degenerate quadratic form of dimension $n$ is isomorphic to a scaled trace form over $F$.

This result appear in [3].
Chapter 2

Preliminaries in Galois Theory

This chapter sets up the necessary background in Galois theory and finite group theory needed for this work. It mainly fixes notation and gives some of the basic properties which will be used later. Also some of the technical details of the upcoming chapters are hidden in this chapter. The expert reader may wish to skip this expository part, and return to it when needed.

2.1 Permutational Galois Groups

Let $F/K$ be a Galois extension. The Galois group $G = \text{Gal}(F/K)$ is equipped with several natural actions coming from polynomials and subgroups. We describe two of these actions below.

Firstly let $f(X) \in K[X]$ be a separable monic polynomial which splits over $F$. Then $f(X) = \prod_{i=1}^{n}(X - \alpha_i)$, where all $\alpha_i \in F$ and are distinct. Since $G$ fixes the coefficients of $f(X)$, it permutes the roots of $f$. Properties of $f$ are encoded in this action (this idea goes back to Galois himself), e.g., the following

Lemma 2.1.1. Let $R$ be the set of roots of $f$.

(a) $R$ generates $F$ over $K$ if and only if the action of $G$ on $R$ is faithful.

(b) There is a correspondence between the factorization $f(X) = \prod_{i=1}^{m} f_i(X)$ of $f(X)$ into irreducible polynomials over $K$ and the decomposition $R = \bigcup_{i=1}^{m} R_i$ of $R$ into $G$-orbits $R_i$, given by $f_i(X) = \prod_{\alpha \in R_i} (X - \alpha)$ (under a suitable rearrangement of the factors $f_i$).

(c) $f(X)$ is irreducible over $K$ if and only if $(G, R)$ is transitive.
Proof. Let \( E \subseteq F \) be the field generated by \( R \). By the Galois correspondence \( E = F \) if and only if no \( \sigma \in G \) fixes \( E \). This implies (a).

For (b) notice that the coefficients of each \( f_i(X) = \prod_{\alpha \in R_i} (X - \alpha) \) are \( G \)-invariant, hence \( f_i \in K[X] \). Furthermore, if \( f_i \) were reducible, by Galois correspondence, there would been a proper nonempty subset of \( R_i \) which is \( G \)-invariant, a contradiction. (c) is a special case of (b).

Let us describe the second type of actions. Let \( H \leq G \) be a subgroup. Then \( G \) acts naturally on the cosets \( G/H \) of \( H \) in \( G \) by left multiplication. This action is (non-canonically) isomorphic to the previous action. Namely, let \( E = F^H \) be the fixed field of \( H \) in \( F \), let \( \alpha_1, \ldots, \alpha_n \in E \) be a generating set of \( E/K \), i.e. \( E = K(\alpha_1, \ldots, \alpha_n) \), and let \( f(X) \in K[X] \) be the minimal monic polynomial with \( \alpha_1, \ldots, \alpha_n \) as roots. Then \( G \) permutes the roots of \( f(X) \), and, by the Galois correspondence, \( \text{Gal}(F/E) = H \). This implies that the stabilizer of \( \alpha_1, \ldots, \alpha_n \) is \( H \).

### 2.2 Embedding Problems

A profinite group is defined as an inverse limit of finite groups, or equivalently as a totally disconnected compact Hausdorff group \([9, \text{Lemma 1.1.7}]\). The former description makes it plausible to characterize a profinite group \( \Gamma \) via finite objects. If \( \Gamma \) is finitely generated, then it is determined (up-to-isomorphism) by the set
\[
\{ \Gamma/N \mid N \text{ is open and normal in } \Gamma \}/\text{isom}
\]
of all finite quotients \([9, \text{Proposition 16.10.7}]\).

However in general this set of finite quotient does not determine \( \Gamma \). For example, the direct product \( \Gamma_1 = \prod_G G \) of all finite groups \( G \) and the free profinite group of countable rank \( \Gamma_2 = \hat{F}_\omega \), both have all finite groups as quotients, but \( \Gamma_1 \not\cong \Gamma_2 \) (since, e.g., \( \Gamma_2 \) has no torsion). Therefore a more refined finite object – finite embedding problems – is needed.

Let us present two classical results before going into details.

- A profinite group \( \Gamma \) is projective if and only if every finite embedding problem is weakly solvable \([28, \text{Lemma 7.6.1}]\).

- Let \( \Gamma \) be a profinite group of infinite rank \( \kappa \). Then \( \Gamma \) is free if and only if every non-trivial finite embedding problem has \( \kappa \) solutions \([9, \text{Theorem 25.1.7}]\).

\(^1\)If \( G \) is an infinite group, then we assume that it is profinite and that \( H \) is open in \( G \).

\(^2\)in the topological sense.
2.2.1 Embedding Problems for Profinite Groups

Let $\Gamma$ be a profinite group. An embedding problem for $\Gamma$ is a diagram

$$
\begin{array}{ccc}
\Gamma & \xrightarrow{\exists \theta} & G \\
\downarrow & & \downarrow \\
A & \xrightarrow{\mu} & A,
\end{array}
$$

where $G$ and $A$ are profinite groups and $\mu$ and $\alpha$ are (continuous) epimorphisms. In short we write $(\mu, \alpha)$.

A solution of $(\mu, \alpha)$ is an epimorphism $\theta: \Gamma \to G$ such that $\alpha \theta = \mu$. If $\theta$ satisfies $\alpha \theta = \mu$ but is not necessarily surjective, we say that $\theta$ is weak solution. In particular, a profinite group $G$ is a quotient of $\Gamma$ if and only if the embedding problem $(\Gamma \to 1, G \to 1)$ is solvable.

If $G$ is finite (resp. $\alpha$ splits), we say that the embedding problem is finite (resp. split).

Two embedding problems $(\mu: \Gamma \to A, \alpha: G \to A)$ and $(\nu: \Gamma \to B, \beta: H \to B)$ are said to be equivalent if there exist isomorphisms $i: G \to H$ and $j: A \to B$ for which the following diagram commutes.

$$
\begin{array}{ccc}
G & \xrightarrow{\alpha} & A \\
| & | & | \\
H & \xrightarrow{\beta} & B \\
| & | & | \\
\Gamma & \xleftarrow{\mu} & \Gamma
\end{array}
$$

It is evident that any (weak) solution of $(\mu, \alpha)$ corresponds to a (weak) solution of $(\nu, \beta)$ and vice versa.

The following lemma gives an obvious, but useful, criterion for a weak solution to be a solution (i.e. surjective).

**Lemma 2.2.1.** A weak solution $\theta: \Gamma \to G$ of an embedding problem $(\mu: \Gamma \to A, \alpha: G \to A)$ is surjective if and only if $\ker(\alpha) \leq \theta(\Gamma)$.

**Proof.** Suppose $\ker(\alpha) \leq \theta(\Gamma)$. Let $g \in G$, put $a = \alpha(g)$, and let $f \in \mu^{-1}(a)$. Then $\theta(f)^{-1}g \in \ker(\alpha) \leq \theta(\Gamma)$, and hence $g \in \theta(\Gamma)$. The converse is obvious. \qed

2.2.2 Embedding Problems for Fields

Let $K$ be a field with a separable closure $K_s$. The absolute Galois group $\text{Gal}(K) = \text{Gal}(K_s/K)$ is profinite. This defines the notion of an embedding problem for $K$. 
More precisely, an embedding problem for a field $K$ is an embedding problem

$$(\mu: \text{Gal}(K) \to A, \alpha: G \to A)$$

for $\text{Gal}(K)$. If we denote by $L$ the fixed field of $\ker(\mu)$, then $\mu$ factors as $\mu = \bar{\mu}\mu_0$, where $\mu_0: \text{Gal}(K) \to \text{Gal}(L/K)$ is the restriction map and $\bar{\mu}: \text{Gal}(L/K) \to A$ is an isomorphism. Then the embedding problems $(\mu, \alpha)$ and $(\mu_0, \bar{\mu}^{-1}\alpha)$ are equivalent. So, from now on, we shall assume that $A = \text{Gal}(L/K)$ and $\mu$ is the restriction map (unless we explicitly specify differently).

(2.1)

$$\begin{array}{ccc}
\text{Gal}(K) & \xymatrix{
\exists \theta \ar[d]_\mu \\
G & \ar[l]_\alpha \text{Gal}(L/K)
}
\end{array}$$

Another piece of notation is that of solution fields: Let $\theta: \text{Gal}(K) \to G$ be a weak solution of $(\mu, \alpha)$. The fixed field of $\ker(\theta)$ is called the solution field.

In terms of fields, a weak solution of an embedding problem corresponds to a $K$-embedding of the field $L$ inside the solution field $F$ in such a way that the restriction map $\text{Gal}(F/K) \to \text{Gal}(L/K)$ coincides with $\alpha$. In particular, the embedding problems $(\mu, \alpha)$ and the embedding problem defined by the restriction map, i.e. $(\mu, \text{res}: \text{Gal}(F/K) \to \text{Gal}(L/K))$, are equivalent.

### 2.2.3 Rational and Geometric Embedding Problems

We define embedding problems coming from geometric objects.

**Definition 2.2.2.** Let $E$ be a finitely generated regular extension of $K$, let $F/E$ be a Galois extension, and let $L = F \cap K_s$, where $K_s$ is a separable closure of $K$). Then the restriction map $\alpha: \text{Gal}(F/E) \to \text{Gal}(L/K)$ is surjective, since $E \cap K_s = K$. Therefore

(2.2)

$$\begin{array}{ccc}
\text{Gal}(K) & \xymatrix{
\text{Gal}(F/E) & \ar[l]_\alpha \text{Gal}(L/K) \\
\text{Gal}(K) & \ar[d]_\mu \\
\text{Gal}(L/K)
}
\end{array}$$

is an embedding problem for $K$. We call such an embedding problem geometric embedding problem.
If \( E = K(t_1, \ldots, t_e) \) is a field of rational functions over \( K \) then we call (2.2) which gets the form

\[
\text{(2.3) } \quad \xymatrix{ \text{Gal}(K) \ar[d]^\mu \ar[r]^\alpha \ar@{_{(}->}^\cong & \text{Gal}(L/K) } \quad \text{rational embedding problem.}
\]

It seems that the rationality of an embedding problem depends on \( e \geq 1 \). However Lemma 2.4.5 shows that if the condition is satisfied for some \( e \geq 1 \), then it also holds for \( e = 1 \).

**Lemma 2.2.3.** Every finite embedding problem is equivalent to a geometric embedding problem.

See [9, Lemma 11.6.1] for the proof of this result. Now we can restrict and discuss only geometric embedding problems.

We say that a finite group is **regularly realizable** if there exists a Galois extension \( F/K(t) \) such that \( F \) is regular over \( K \) and \( G \cong \text{Gal}(F/K(t)) \). In other words, in (2.3) we have \( L = K \). The following classical result asserts that certain families of finite groups are regularly realizable over any field:

**Theorem 2.2.4.** All abelian groups, all symmetric groups, and all alternating groups are regularly realizable over any field.

A proof can be found e.g. in [9]. (This result is by no mean the state of art.)

**2.3 Fiber Products**

Let \( \alpha_1 : G_1 \to A \) and \( \alpha_2 : G_2 \to A \) be epimorphisms of (profinite) groups. Then the fiber product is defined by

\[
G_1 \times_A G_2 = \{ (g_1, g_2) \in G_1 \times G_2 \mid \alpha_1(g_1) = \alpha_2(g_2) \} \leq G_1 \times G_2.
\]

It is equipped with two natural projection maps \( \pi_i : G_1 \times_A G_2 \to G_i \) defined by \( \pi_i(g_1, g_2) = g_i, \ i = 1, 2 \).

Fiber products rise up ‘naturally’ in many situations. We shall use them mainly to find a convenient embedding problem that dominates a given embedding problem.
**Definition 2.3.1.** An embedding problem \((\mu': \Gamma \to A', \alpha': G' \to A')\) for a profinite group \(\Gamma\) dominates an embedding problem \((\mu: \Gamma \to A, \alpha: G \to A)\) if every (weak) solution \(\theta'\) of \((\mu', \alpha')\) induces a (weak) solution \(\theta\) of \((\mu, \alpha)\) via a commutative diagram

\[
\begin{array}{ccc}
G' & \xrightarrow{\alpha'} & A' \\
\downarrow{\pi} & & \downarrow{\nu} \\
G & \xrightarrow{\alpha} & A.
\end{array}
\]

Here \(\pi: G' \to G\) is surjective.

The following lemma gives a dominating embedding problem via fiber products.

**Lemma 2.3.2.** Let \((\mu: \Gamma \to A, \alpha: G \to A)\) be an embedding problem for a profinite group \(\Gamma\). Let \(\mu': \Gamma \to A'\) be an epimorphism such that \(\ker(\mu') \leq \ker(\mu)\). Let \(G' = G \times_A A'\) and let \(\alpha': G' \to A'\) be the corresponding projection map. Then the embedding problem \((\mu', \alpha')\) dominates \((\mu, \alpha)\). Moreover

(a) Assume that \(\theta: \Gamma \to G\) is a weak solution and \(\ker(\mu') \leq \ker(\theta)\). Then \((\mu', \alpha')\) splits.

(In particular, if \((\mu, \alpha)\) splits, then so does \((\mu', \alpha')\).)

(b) A weak solution \(\theta\) of \((\mu, \alpha)\) induces a weak solution \(\theta'\) of \((\mu', \alpha')\) defined by \(\theta'(x) = (\theta(x), \mu'(x))\).

**Proof.** The commutative diagram (2.4), where \(\pi\) is the projection map and \(\nu\) is induced by \(\mu'\) and \(\mu\) implies that \((\mu', \alpha')\) dominates \((\mu, \alpha)\), as needed.

(b) is trivial.

Finally we prove (a). The map \(\theta \times \mu': \Gamma \to G'\) sends \(x \in \Gamma\) to the pair \((\theta(x), \mu'(x))\). Let \(K = \ker(\theta \times \mu')\). Clearly \(K = \ker(\theta) \cap \ker(\mu')\) and hence by assumption \(K = \ker(\mu')\). Therefore \(\theta \times \mu'\) induces a map \(\beta': A' \to G'\). Then \(\alpha' \beta' \mu' = \alpha'(\theta \times \mu') = \mu'\); hence \(\alpha' \beta' = \text{id}\).

A fiber product also describes the Galois group of the compositum of two Galois extensions. The restriction maps are then realized as the projection maps:

**Lemma 2.3.3.** Let \(M_1\) and \(M_2\) be Galois extensions of a field \(K\) and let \(L = M_1 \cap M_2\). Then \(\text{Gal}(M_1M_2/K)\) is canonically isomorphic to \(\text{Gal}(M_1/K) \times_{\text{Gal}(L/K)} \text{Gal}(M_2/K)\). Under this isomorphism the restriction maps coincide with the projection maps.
Proof. The canonical isomorphism
\[
\text{Gal}(M_1 M_2 / K) \to \text{Gal}(M_1 / K) \times_{\text{Gal}(L / K)} \text{Gal}(M_2 / K)
\]
is given by \( \sigma \mapsto (\sigma |_{M_1}, \sigma |_{M_2}) \). (See [21, VI §1 Theorem 1.14].)

There is a connection between fiber products and independent solutions of embedding problems. Let \( \Psi = \{ \theta_i \mid i \in I \} \) be a family of solutions of a finite embedding problem \((\mu : \Gamma \to A, \alpha : G \to A)\) for a profinite group \( \Gamma \). We call \( \Psi \) independent if \( \{ \ker(\theta_i) \mid i \in I \} \) is an independent family w.r.t. the Haar measure of \( \ker(\mu) \). That is to say, \( \Psi \) is independent if and only if
\[
(\ker(\mu) : \bigcap_{i \in I_0} \ker(\theta_i)) = \prod_{i \in I_0} (\ker(\mu) : \ker(\theta_i))
\]
for every finite subset \( I_0 \subseteq I \).

We denote
\[
G^I_A = \{(g_i) \in G^I \mid \alpha(g_i) = \alpha(g_j), \forall i, j \in I\}
\]
for the fiber product of \( I \) copies of \( G \). For each \( i \in I \) let \( \pi_i : G^I_A \to G \) be the \( i \)th projection map. We have an epimorphism \( \alpha_I : G^I_A \to A \) defined by \( \alpha_I = \alpha \pi_i \), for some \( i \in I \). Clearly this definition is independent of the choice of \( i \in I \).

**Lemma 2.3.4.** If \( \theta : \Gamma \to G^I_A \) is a solution of \((\mu, \alpha_I)\), then \( \{ \theta_i = \pi_i \theta \mid i \in I \} \) is an independent family of solutions of \((\mu, \alpha)\).

**Proof.** For finite \( I_0 \subseteq I \) set \( \pi_{I_0} : G^I_A \to G^{I_0}_A \) and denote \( \theta_{I_0} = \pi_{I_0} \theta \).

The above diagram with \( I_0 = \{ i \} \) implies that \( \theta_i \) is a solution, for any \( i \in I \).

Since
\[
1 = \theta_{I_0}(x) = \pi_{I_0}(\theta(x)) \iff \forall i \in I_0, 1 = \pi_i(\theta(x)) = \theta_i(x),
\]
we get that \( \bigcap_{i \in I_0} \ker(\theta_i) = \ker(\theta_{I_0}) \). Thus
\[
(\ker(\mu) : \bigcap_{i \in I_0} \ker(\theta_i)) = (\ker(\mu) : \ker(\theta_{I_0})) = \frac{|G^{I_0}_A|}{|G|} = \left( \frac{|G|}{|A|} \right)^{|I_0|}
\]
\[
= \prod_{i \in I_0} (\ker(\mu) : \ker(\theta_i)).
\]
This shows that \( \{ \theta_i \mid i \in I \} \) is independent family of solutions, as needed.

\[ \]  

2.4 Places

In this section we recall some of the definitions and basic properties of places, for reference see [9].

A place \( \varphi \) of a field \( F \) is a map of \( \varphi : F \to M \cup \{ \infty \} \) such that \( M \) is a field and the following properties hold.

(a) \( x + \infty = \infty + x = \infty, \forall x \in M \).

(b) \( x \cdot \infty = \infty \cdot x = \infty, \forall x \in M^* := M \setminus \{ 0 \} \).

(c) \( \varphi(x + y) = \varphi(x) + \varphi(y) \) (whenever the right hand side is defined).

(d) \( \varphi(xy) = \varphi(x)\varphi(y) \) (whenever the right hand side is defined).

(e) \( \exists x, y \in F \) such that \( \varphi(x) = \infty \) and \( \varphi(y) \neq 0, \infty \).

If \( F \) and \( M \) are extensions of a common field \( K \), and \( \varphi(x) = x \) for every \( x \in K \), we say that \( \varphi \) is a \( K \)-place.

Every place \( \varphi \) of \( F \) comes with a local ring \( \mathcal{O}_\varphi = \{ x \in F \mid \varphi(x) \neq \infty \} \) whose maximal ideal is \( m_\varphi = \{ x \in \mathcal{O}_\varphi \mid \varphi(x) = 0 \} \). The quotient field \( \mathcal{O}_\varphi/m_\varphi \) is canonically isomorphic to the residue field \( \bar{F} = \{ \varphi(x) \mid x \in \mathcal{O}_\varphi \} \). Places are said to be equivalent if they have the same local ring. In particular, equivalent places have isomorphic residue fields.

Let \( E/K \) be a regular extension and let \( F/E \) be a finite Galois extension. Assume that \( \varphi \) is a \( K \)-place of \( E \). Then, by Chevalley’s lemma, the place \( \varphi \) extends to a place \( \Phi : F \to M \cup \{ \infty \} \), where \( M \) is the algebraic closure of \( \bar{E} \) [9, Proposition 2.3.1]. Assume from now on that \( F/E \) is separable. Then \( F/E \) is, in fact, Galois and the Galois group \( \text{Gal}(\bar{F}/\bar{E}) \) is then called the residue group.

Let \( L = F \cap K_s \), then \( L \subseteq \bar{F} \). (Indeed, since \( K \subseteq \mathcal{O}_\varphi \), \( L \) is integral over \( K \), and \( \mathcal{O}_\varphi \) is integrally closed, it follows that \( L \subseteq \mathcal{O}_\varphi \).) By composing \( \Phi \) with an appropriate Galois automorphism of \( \bar{F}/K \), we may assume that \( \Phi \) is an \( L \)-place.

We call the subgroup

\[
D_{\Phi/\varphi} = \{ \sigma \in \text{Gal}(F/E) \mid \sigma \mathcal{O}_\varphi = \mathcal{O}_\varphi \} = \{ \sigma \in \text{Gal}(F/E) \mid \sigma m_\varphi = m_\varphi \}
\]
of Gal(F/E) the **decomposition group** of Φ/ϕ. The fixed field of the decomposition group in F is (as expected) the **decomposition field** of Φ/ϕ. The decomposition field is the maximal subextension of F/E having the same residue field as of E.

There is a natural epimorphism from the decomposition group to the residue group, namely σ ∈ DΦ/ϕ ↦→ ̄σ ∈ Gal( ¯F/¯E), where ̄σ is defined by ̄σ(x + mΦ) = σx + mΦ. The kernel of this epimorphism is called the **inertia group**, and is denoted by IΦ/ϕ, viz.

IΦ/ϕ = {σ ∈ Gal(F/E) | x − σ(x) ∈ mΦ, ∀x ∈ OΦ}.

If the inertia group is trivial, the place Φ is said to be **unramified** over ϕ (or ϕ is unramified in F). In other words, Φ is unramified over ϕ if and only if the decomposition group is isomorphic to the residue group.

From now on we assume the F, E are function fields. That is, we assume that F and E are finite separable extensions of K(t), where t = (t1, . . . , te) is a finite e-tuple of variables for some e ≥ 1.

**Lemma 2.4.1.** Let x ∈ F be such that F = E(x). Then there exists a nonzero g(t) ∈ K[t] such that for every K-place ϕ of F if a = ϕ(t) is finite and g(a) ̸= 0, then ̄x = ϕ(x) is finite, ϕ is unramified over E, and F = ̄E(̄x).

**Proof.** Let f(t, X) ∈ K[t, X] be the irreducible polynomial of x over K(t). Let g(t) ∈ K[t] be the product of the leading coefficient of f (as a polynomial in X) and its discriminant. Since f is separable, g(t) ̸= 0.

Let R = K[t, g(t)^−1] and let S, U be the integral closures of R in F, E, respectively. Then S = R[x] ([9 Definition 6.1.3]), in particular S = U[x]. Conclude that S/U is a ring cover. Now [9 Lemma 6.1.4] finishes the proof.

Let x ∈ F be a primitive element of F/E, i.e. F = E(x) and let g(t) ∈ K[t] from the above lemma. Assume that a = ϕ(t) is finite and g(a) ̸= 0. Let f be the irreducible polynomial of x over E.

Then ̄f = ϕ(f) is a separable polynomial (since the Φ is unramified over E). Therefore Φ induces a bijective map between the set X of all roots of f and the set ̄X of all roots of ̄f. We thus have an isomorphism ρ: S_X → S_̄X. Recall that DΦ/ϕ embeds into S_X (since Gal(F/E) does) and that Gal(F/E) embeds into S_̄X. Then the following result holds (cf. [9 Lemma 6.1.4]).

**Lemma 2.4.2.** The restriction of ρ: S_X → S_̄X to the decomposition group DΦ/ϕ coincides with the natural map DΦ/ϕ → Gal(̄F/̄E).
The above natural map relates to embedding problems in the following way. An unramified place $\Phi/\varphi$ for which $\overline{E} = K$ induces a map $\Phi^*: \text{Gal}(K) \to \text{Gal}(F/E)$ which factors through the residue group and whose image $\Phi^*(\text{Gal}(K)) = D_{\varphi}/\varphi$ is the decomposition group. We have

$$\Phi(\Phi^*(\sigma)x) = \sigma\Phi(x),$$

for all $\sigma \in \text{Gal}(K)$ and $x \in \mathcal{O}_\Phi$. Thus if $\Phi$ is an $L$-place, then $\text{res}_{F,L} \circ \Phi^* = \text{res}_{K_s,L}$. In other words, $\Phi^*$ is a weak solution of the embedding problem

$$\text{(res: Gal}(K) \to \text{Gal}(L/K), \text{res: Gal}(F/E) \to \text{Gal}(L/K)).}$$

Such weak solutions are called geometric and will be discussed below.

Let $\Psi$ be another place of $F$ lying over $\varphi$, then $\Phi = \Psi\sigma$ for some Galois automorphism $\sigma \in \text{Gal}(F/E)$, the decomposition groups are conjugate (by the same $\sigma$) and $\Psi^*$ and $\Phi^*$ differ by this conjugation. Therefore, to make notation simpler, and when there is no risk of confusing, we omit $\Phi$ from the notation. For example, $D_\varphi$ stands for $D_{\Phi^*}$ for some extension $\Phi$ of $\varphi$ and $\varphi^*$ is actually $\Phi^*$ for some $\Phi$ lying above $\varphi$; we shall say that $\varphi$ is unramified at $E$ if $\Phi/\varphi$ is unramified, etc.

Let $t = (t_1, \ldots, t_e)$ be an $e$-tuple of variables over $K$. Any $a_1, \ldots, a_e \in K_s$ defines a specialization $K[t] \to K[a]$. This specialization can be extended to a $K$-place $\varphi$ of $K(t)$ into $K_s \cup \{\infty\}$. If $e = 1$, the place $\varphi$ is uniquely determined by $a = a_1 \in K_s$, and, in particular, the residue field is $K(a)$.

However, if $e > 1$, then $\varphi$ is not uniquely determined by $a$ and even the residue field need not be $K(a)$. (The reason for this is that an element such as $\varphi(\frac{t_i-a_i}{t_j-a_j})$, $i \neq j$, can be defined to be $\infty$ or any other element of $K_s$.) Nevertheless, we can extend the specialization $t \to a$ to a $K$-place $\varphi$ of $K(t)$ such that its residue field is $K(a)$, and moreover, for every finite extension $L/K$ the residue field of $L(t)$ is $L(a)$ (under any extension of $\varphi$ to an $L$-place of $L(t)$), see [9, Lemma 2.2.7].

2.4.1 Points on Varieties and Places

Let $V$ be an algebraic variety defined over $K$. In this work varieties always stay irreducible over the algebraic closure (that is $V \otimes_K \bar{K}$ is irreducible). Let $E$ denote the function field of $V$. Let $a \in V(K)$ and let $U = \text{Spec}R$ be an affine neighborhood of $a$. Then $a$ defines a $K$-homomorphism $s_a: R \to K$.

The homomorphism $s_a$ can be extended to a $K$-place of $E$. Similarly to the case of extending specializations to places, there are several ways to extends $s_a$ to $E$ when $V$ is
not a curve and it can be extended to a $K$-place of $E$ with residue field $\bar{E} = K(a)$.

Let $\nu: V \to \mathbb{A}^{\dim(V)}$ be a finite separable morphism over $K$. Then the corresponding field extension $E/K(t)$ is finite and separable. Let $a \in K^s_{\dim(V)}(= \mathbb{A}^{\dim(V)}(K_s))$ and take $b \in \nu^{-1}(a) \subseteq V(K_s)$. By Lemma 2.4.1, there exists $0 \neq g(t) \in K[t]$ such that if $g(a) \neq 0$, then the specialization $t \mapsto a$ can be extended to a $K$-place of $E$ such that $\bar{K(t)} = K(a)$ and $\bar{E} = K(b)$.

### 2.4.2 Geometric Solutions of Embedding Problems

Consider a geometric embedding problem

$$(\mu: \text{Gal}(K) \to \text{Gal}(L/K), \alpha: \text{Gal}(F/E) \to \text{Gal}(L/K))$$

for a field $K$. As we saw before, a place $\varphi$ of $E$ that is unramified in $F$ and with residue field $\bar{E} = K$ induces a weak solution $\varphi^*$ whose image is the decomposition group.

#### Definition 2.4.3.

A weak solution $\theta: \text{Gal}(K) \to \text{Gal}(F/E)$ of a geometric embedding problem $(\mu, \alpha)$ is called geometric if there exists a $K$-place $\varphi$ of $E$ unramified in $F$ such that $\theta = \varphi^*$.

Geometric solutions are compatible with scalar extensions.

#### Lemma 2.4.4.

Let $(\mu, \alpha)$ be a geometric embedding problem. Let $M/K$ be a Galois extension with $L \subseteq M$. Then the geometric embedding problem

$$(\mu': \text{Gal}(K) \to \text{Gal}(M/K), \alpha': \text{Gal}(F/M) \to \text{Gal}(M/K)),$$

where $\alpha'$ and $\mu'$ are the corresponding restriction maps dominates the embedding problem $(\mu, \alpha)$ with respect to the restriction maps. Furthermore, if $\psi^*$ is a geometric (weak) solution of $(\mu', \alpha')$, then $(\psi|_F)^*$ is a geometric (weak) solution of $(\mu, \alpha)$.

**Proof.** As $E/K$ is regular and by Lemma 2.3.3, we have

$$\text{Gal}(F(N/E) = \text{Gal}(F/E) \times_{\text{Gal}(L/E)} \text{Gal}(M/E) \cong \text{Gal}(F/E) \times_{\text{Gal}(L/K)} \text{Gal}(M/K)$$

and the projection maps coincide with the restriction maps. Thus $(\mu', \alpha')$ dominates $(\mu, \alpha)$ (Lemma 2.3.2). Now let $\psi^*$ be a geometric (weak) solution of $(\mu', \alpha')$. For $\varphi = \psi|_F$, we have that $\varphi$ is unramified in $F$ and $\text{res}_{FM,F} \circ \psi^* = \varphi^*$, as needed.

Combination of Matsusaka-Zariski Theorem and Bertini-Noether Lemma reduces the transcendence degree in (2.3) to 1.
Lemma 2.4.5. Let $K$ be an infinite field, $L/K$ a finite Galois extension, and $(u, t)$ an $e + 1$-tuple of algebraically independent elements over $K$. Consider a rational embedding problem (2.3). Then there exists a solution of

$$(\mu_u : \text{Gal}(K(u)) \to \text{Gal}(L/K), \alpha : \text{Gal}(F/K(t)) \to \text{Gal}(L/K))$$

whose solution field is regular over $L$.

Proof. Let $x \in F$ be integral over $L[t]$ such that $F = K(t, x)$. Let $f(T, X) \in K[T, X]$ be the absolutely irreducible polynomial that is separable and monic in $X$ for which $f(t, x) = 0$.

Take two variables $U, V$. Matsusaka-Zariski Theorem [9, Proposition 10.5.4] implies that $f(T_1, \ldots, T_{e-1}, U + VT_1, X)$ is irreducible in the ring $\tilde{L}[T_1, \ldots, T_{e-1}, X]$, where $\tilde{L}$ is an algebraic closure of $L(U, V)$. Let $h(T_1, \ldots, T_r) \in K[T]$ be the non-zero polynomial given in Lemma 2.4.1 with respect to the extension $F/K(t)$.

By Bertini-Noether Lemma [9, Proposition 10.4.2] there exists a nonzero $c(U, V) \in L[U, V]$ such that for any $\alpha_e, \beta_e \in K$ satisfying $c(\alpha_e, \beta_e) \neq 0$ the monic polynomial $f(T_1, \ldots, T_{e-1}, \alpha_e + \beta_e T_1, X)$ remains absolutely irreducible over $K$. Since $K$ is infinite, there exist $\alpha_e, \beta_e \in K$ with $\beta_e \neq 0$ such that $c(\alpha_e, \beta_e) \neq 0$ and $h(T_1, \ldots, \alpha_e + \beta_e T_1) \neq 0$.

Induction on $e$ yields $\alpha_i, \beta_i \in K$, $\beta_i \neq 0$, $i = 2, \ldots, e$ such that $g(T, X) = f(T, \alpha_2 + \beta_2 T, \ldots, \alpha_e + \beta_e T, X)$ is an absolutely irreducible polynomial and $h(T, \alpha_2 + \beta_2 T, \ldots, \alpha_e + \beta_e T, X) \neq 0$.

Extend the specialization $t \mapsto (u, \alpha_2 + \beta_2 u, \ldots, \alpha_e + \beta_e u)$ to an $L$-place $\varphi$ of $F/K(t)$ and denote by $F_0/K(u)$ the residue field extension [9, Lemma 2.2.7]. By Lemma 2.4.1, $F_0 = L(u, x_0)$, where $x_0 = \varphi(x)$ is a root of $g(u, X)$. Hence $F_0$ is regular over $L$. The place $\varphi$ also induces a geometric weak solution $\varphi^* : \text{Gal}(K(u)) \to \text{Gal}(F/K(t))$ of $(\mu_u, \alpha)$ whose residue field is $F_0$. But

$$[F_0 : K(t)] = [F_0 : L(t)][L : K] = \deg_X g(t, X)[L : K] = \deg_X f(t, X)[L : K] = [F : L(t)][L : K] = [F : K(t)],$$

so $\varphi^*$ is surjective, and thus the assertion follows.

2.5 Wreath Products

In this section we introduce wreath products, and the more general notion of twisted wreath products.
2.5.1 Definition

Let $A, G_0 \leq G$ be finite groups. Assume that $G_0$ acts on $A$ (from the right). Let

$$\text{Ind}_{G_0}^G(A) = \{ f: G \to A \mid f(\sigma \rho) = f(\sigma)^\rho, \forall \sigma \in G, \rho \in G_0 \} \cong A^{(G:G_0)}.$$ 

Here multiplication is component-wise, i.e. $(fg)(\sigma) = f(\sigma)g(\sigma)$. Then $G$ acts on $\text{Ind}_{G_0}^G(A)$ by $f^\sigma(\tau) = f(\sigma \tau)$. We define the **twisted wreath product** to be the semidirect product

$$A \wr_{G_0} G = \text{Ind}_{G_0}^G(A) \rtimes G,$$

i.e., an element in $A \wr_{G_0} G$ can be written uniquely as $f \sigma$, where $f \in \text{Ind}_{G_0}^G(A)$ and $\sigma \in G$.

The multiplication is then given by $(f \sigma)(g \tau) = fg^{\sigma^{-1}}\tau$. The twisted wreath product is equipped with the quotient map $\alpha: A \wr_{G_0} G \to G$ defined by $\alpha(f \sigma) = \sigma$.

If $G_0 = 1$, the twisted wreath product is simply called **wreath product**. We abbreviate $A \wr_1 G$ and write $A \wr G$ for the wreath product.

The next result states that $A \times G_0$ embeds in $A \wr_{G_0} G$.

**Lemma 2.5.1.** For each $a \in A$ let $f_a \in \text{Ind}_{G_0}^G(A)$ be defined by

$$f_a(\sigma) = \begin{cases} a^\sigma & \text{if } \sigma \in G_0 \\ 1 & \text{otherwise.} \end{cases}$$

Then the map $\rho: A \times G_0 \to A \wr_{G_0} G$ defined by $\rho(a \sigma) = f_a \sigma$ is a monomorphism.

**Proof.** Clearly $\rho$ is injective. Since $(f_a)^\tau = f_{a^\tau}$ for any $a \in A$ and $\tau \in G_0$, the map $\rho$ is a homomorphism. \qed

2.5.2 Twisted Wreath Products and Embedding Problems

The following result strengthens Lemma 2.2.1 in the case of embedding problems with twisted wreath products.

**Lemma 2.5.2.** Let $(\varphi: \Gamma \to G, \alpha: A \wr_{G_0} G \to G)$ be a finite embedding problem for a profinite group $\Gamma$ and let $\theta: \Gamma \to A \wr_{G_0} G$ be a weak solution. Assume that $A = \{ f_a \mid a \in A \} \leq \theta(\Gamma)$. Then $\theta$ is surjective.

**Proof.** The image of $\theta$ is $G$-invariant, hence $A^\sigma \leq \theta(\Gamma)$ for every $\sigma \in G$. Therefore

$$\text{Ind}_{G_0}^G(A) = \prod_{\sigma \in G} A^\sigma \leq \theta(\Gamma).$$

But $\text{Ind}_{G_0}^G(A) = \ker(\alpha)$; hence Lemma 2.2.1 implies that $\theta$ is surjective. \qed

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There is a close relation between solutions of embedding problem and twisted wreath products. An application of this connection is Haran’s diamond theorem \cite{12}. We bring below a simple result in that spirit, but first we need a simple preparation.

We claim that the map \( \pi : \text{Ind}^G_{G_0}(A) \times G_0 \rightarrow A \times G_0 \) defined by \( \pi(f\sigma) = f(1)\sigma \) is an epimorphism: It is obvious that \( \pi \) is surjective. For any \( f \in \text{Ind}^G_{G_0}(A) \) and \( \sigma \in G_0 \) we have

\[
\pi(f^\sigma) = f(\sigma) = f(1)^\sigma = \pi(f)^\sigma,
\]

and hence \( \pi \) is also a homomorphism. We call \( \pi \) the **Shapiro map**. Note that the map \( \rho \) defined in Lemma 2.5.1 is a section of \( \pi \), i.e. \( \pi \rho \) is the identity map on \( A \times G_0 \).

Lemma 2.5.3. Let \( \Gamma \leq \Lambda \) be profinite groups. Let \( \nu : \Lambda \rightarrow G \) be an epimorphism onto a finite group, let \( G_0 = \nu(\Gamma) \), and set \( \mu = \nu|\Gamma \). Assume \( G_0 \) acts on a finite group \( A \), and let \( \theta \) be a weak solution of \( (\nu : \Lambda \rightarrow G, \alpha : A \wr G_0 G \rightarrow G) \). Then \( \eta = \pi \theta|_\Gamma \) is defined and is a weak solution of \( (\mu : \Gamma \rightarrow G_0, \beta : A \times G_0 \rightarrow G_0) \).

Proof. Note that since \( \alpha \theta(\Gamma) = \nu(\Gamma) = G_0 \), we have \( \theta(\Gamma) \leq \text{Ind}^G_{G_0}(A) \times G_0 \). So \( \eta = \pi \theta|_\Gamma \) is well defined. It is evident that \( \eta \) is a weak solution of \( (\mu, \beta) \).

We shall need the following technical result.

Lemma 2.5.4. Let \( A \wr G_0 G \) be a twisted wreath product of finite groups and denote by \( \pi : \text{Ind}^G_{G_0}(A) \times G_0 \rightarrow A \times G_0 \) the corresponding Shapiro map. Let \( i : G_0 \rightarrow A \times G_0 \) be some splitting of the quotient map \( \alpha_0 : A \times G_0 \rightarrow G_0 \). Then there exists a splitting \( j : G \rightarrow A \wr G_0 G \) of the quotient map \( \alpha : A \wr G_0 G \rightarrow G \) such that \( j(\sigma) \in \text{Ind}^G_{G_0}(A) \times G_0 \) and \( \pi(j(\sigma)) = i(\sigma) \) for every \( \sigma \in G_0 \).

Proof. For each \( \sigma \in G_0 \) let \( a_\sigma = i(\sigma)^{-1} \), i.e. \( i(\sigma) = a_\sigma \sigma \). Since \( i \) is a homomorphism, for every \( \sigma, \tau \in G_0 \) we have

\[
(2.5) \quad a_{\sigma\tau} = a_\sigma a_\tau^{-1}.
\]

(Note that substituting \( \sigma = \tau = 1 \) we get that \( a_1 = 1 \).) Choose a set of representatives \( R \) of left cosets of \( G_0 \) in \( G \), i.e. any \( \sigma \in G \) uniquely factorizes as \( \sigma = \sigma' \rho \), \( \sigma' \in G_0 \) and \( \rho \in R \). Assume that \( 1 \in R \). Then define \( a_\sigma = a_{\sigma'} \).

We note that \((2.5)\) holds, under the extended definition, for every \( \sigma \in G_0 \) and \( \tau \in G \). Indeed, assume \( \tau = \tau' \rho \), \( \tau' \in G_0 \) and \( \rho \in R \). Then \( a_\tau = a_{\tau'} \) and \( a_{\sigma\tau} = a_{\sigma\tau'} \). Now using \((2.5)\) with \( \sigma, \tau' \in G_0 \) we get

\[
a_{\sigma\tau} = a_{\sigma\tau'} = a_\sigma a_{\tau'}^{-1} = a_\sigma a_\tau^{-1}.
\]
It suffices to construct $f_\sigma \in \text{Ind}^G_{G_0}(A)$ for each $\sigma \in G$ such that

\begin{align}
(2.6) & \quad f_\sigma(1) = a_\sigma, \\
(2.7) & \quad f_{\sigma\tau}(\rho) = f_\sigma(\rho)f_\tau(\sigma^{-1}\rho)
\end{align}

for all $\sigma, \tau \in G$. Indeed, assume we done that. Then we set $j(\sigma) = f_\sigma \sigma$ and we have $j(\sigma) \in \text{Ind}^G_{G_0}(A) \rtimes G_0$ for $\sigma \in G_0$. Now (2.7) implies that $j$ is a homomorphism and (2.6) implies that $\pi(j(\sigma)) = i(\sigma)$ for all $\sigma \in G_0$. (Note that substituting $\tau = \rho = 1$ in (2.7) one gets $f_1(\sigma^{-1}) = f_\sigma(1)^{-1}f_\sigma(1) = 1$, i.e. $f_1 = 1$.)

Let $f_\tau(\sigma) = a_{\sigma^{-1}\sigma\tau}^{-1}$ (this definition comes from (2.7) with $\rho = 1$ and (2.6)). Then, clearly, (2.6) holds. For $\sigma, \tau, \rho \in G$ we have

\[ f_\sigma(\rho)f_\tau(\sigma^{-1}\rho) = a_{\rho^{-1}}^{-1}a_{\rho^{-1}\sigma}^{-1}a_{\rho^{-1}\sigma\tau}^{-1} = a_{\rho^{-1}}^{-1}a_{\rho^{-1}\sigma\tau}^{-1} = f_{\sigma\tau}(\rho), \]

hence (2.7) holds. Next let $\sigma, \tau \in G$ and $\rho \in G_0$. By (2.5), we get that

\[ f_\sigma(\tau\rho) = a_{\rho^{-1}\tau^{-1}}^{-1}a_{\rho^{-1}\tau^{-1}\sigma}^{-1}a_{\rho^{-1}\tau^{-1}\sigma}\tau = (a_{\rho^{-1}\tau^{-1}}^{-1}a_{\rho^{-1}\tau^{-1}\sigma})\rho = (f_{\sigma\tau})\rho, \]

that is to say, $f_\sigma \in \text{Ind}^G_{G_0}(A)$, as needed.

\begin{proposition}
Let $A$ and $H$ be finite groups, let $H_0, G$ be subgroups of $H$, and let $G_0 = G \cap H_0$. Assume that $H_0$, and hence also $G_0$, acts on $A$ and that there exists a splitting $i: G \to A \times G_0 \leq A \times H_0$ of the projection map $A \times G_0 \to G_0$. Then there exists an embedding $j: G \to A \wr_{H_0} H$ such that the diagram

\[ \begin{array}{ccc}
G_0 & \xrightarrow{j|_{G_0}} & \text{Ind}^H_{H_0}(A) \times H_0 \\
& \searrow & \downarrow \pi \\
& & A \times H_0
\end{array} \]

commutes. (Here $\pi$ is the Shapiro map.)
\end{proposition}

\begin{proof}
Let $f \in \text{Ind}^G_{G_0}(A)$. Note that $G/G_0$ naturally embeds into $H/H_0$ by mapping $\sigma G_0$ to $\sigma H_0$ and its image is the set $(G \cdot H_0)/H_0$. Extend $f$ to $\tilde{f}: H \to A$ by setting $\tilde{f}(\sigma h) = f(\sigma)^h$ if $\sigma \in G$, $h \in H_0$ and $f(\sigma) = 1$ if $\sigma \in H \setminus (G \cdot H_0)$. Then $\tilde{f} \in \text{Ind}^H_{H_0}(A)$. Moreover the map $\varphi: A \wr_{G_0} G \to A \wr_{H_0} H$ defined by $\varphi(f \sigma) = \tilde{f} \sigma$, $f \in \text{Ind}^G_{G_0}(A)$, $\sigma \in G$ is an embedding.
\end{proof}
Now, by the previous lemma, for \( i: A \times G_0 \) there exists \( j': G \rightarrow A_{\text{wr}} G_0 \) such that \( \pi_0(j'(\sigma)) = i(\sigma) \) for all \( \sigma \in G_0 \). (Here \( \pi_0 \) is the Shapiro map w.r.t. \( A_{\text{wr}} G_0 \).) Let \( j = \varphi j' \) we show below that \( \pi(j(\sigma)) = i(\sigma) \) for all \( \sigma \in G_0 \).

Indeed, let \( \sigma \in G_0 \). Denote \( j'(\sigma) = f\sigma, f \in \text{Ind}_G^G(A) \); then \( j(\sigma) = \varphi(f\sigma) = \tilde{f}\sigma \). Thus

\[ \pi(j(\sigma)) = \tilde{f}(1)\sigma = f(1)\sigma = \pi_0(f\sigma) = \pi_0(j'(\sigma)) = i(\sigma), \]
as needed.

### 2.5.3 Twisted Wreath Product in Fields

**Definition 2.5.6.** Let \( \hat{F}/K \) be a Galois extension whose Galois group is \( A_{\text{wr}} G_0 \). Set \( I = \text{Ind}_G^G(A) \). To the chain of subgroups

\[ 1 \leq \{ f \in I \mid f(1) = 1 \} \leq I \leq I \rtimes G_0 \leq A_{\text{wr}} G_0 \]

there corresponds a tower of fields (in the inverse order)

\[ (2.8) \quad K \subseteq L_0 \subseteq L \subseteq F \subseteq \hat{F}. \]

In particular, \( G_0 = \text{Gal}(L/L_0) \) and \( G = \text{Gal}(L/K) \). Then we say that \( (2.8) \) realizes \( A_{\text{wr}} G_0 \).

**Remark 2.5.7.** Consider an embedding problem \( (\mu: \text{Gal}(K) \rightarrow G, \alpha: A_{\text{wr}} G_0 \rightarrow G) \), where \( G = \text{Gal}(L/K) \) and \( \mu \) is the restriction map. Then \( (2.8) \) realizes \( A_{\text{wr}} G_0 \) implies that \( \theta: \text{Gal}(K) \rightarrow \text{Gal}(\hat{F}/K) \) is a solution.

The following lemma, due to Haran [13], enables us to descend split embedding problems in terms of twisted wreath products.

But first let us recall several facts. Let \( M \) be a field, \( N/M \) a Galois extension, and \( F/M(t) \) a Galois extension such that, \( N \subseteq F \), \( F/N \) is regular, and \( t \) is a transcendental
element over $M$. Now if the restriction map $\beta: \text{Gal}(F/M(t)) \to \text{Gal}(N/M)$ splits, then $F = EN$, where $E$ is the fixed field in $F$ of the image of $\text{Gal}(N/M)$ under some splitting of $\beta$. Then $E/M$ is regular and $M(t) \subseteq E$.

Let $x \in E$ be an element for which $E = M(t,x)$ and let $f(t,X) \in M[t,X]$ be its irreducible polynomial over $M(t)$. Then $f$ is absolutely irreducible (since $E/M$ is regular) and $f$ is Galois over $N(t)$ (since $F = EN = N(t,x)$).

**Lemma 2.5.8.** Let $M/K$ be a separable algebraic extension, $t$ a transcendental element over $M$, and consider a rational finite split embedding problem

$$(\mu: \text{Gal}(M) \to \text{Gal}(N_1/M), \beta: \text{Gal}(F/M(u)) \to \text{Gal}(N_1/M))$$

for $M$. In particular, $F/N_1$ is regular. Let $f(u,X) \in M[u,X]$ be as above, i.e., $f$ is absolutely irreducible, Galois over $N_1(u)$, and a root of which generates $F/N_1(u)$. Assume that there exists a finite Galois extension $L/K$ satisfying

(a) $f(u,X) \in L_0[u,X]$, where $L_0 = L \cap M$,

(b) $f(u,X)$ is Galois over $L(u)$, and

(c) $N_1 \subseteq N$, where $N = ML$.

Set $G = \text{Gal}(L/K)$, $G_0 = \text{Gal}(L/L_0) \cong \text{Gal}(N/M)$, let $\varphi: \text{Gal}(K) \to G$ be the restriction map, and let $A = \ker(\beta) = \text{Gal}(F/N_1(u)) \cong \text{Gal}(FL/N(u))$. Then

(2.9) \[
(\varphi: \text{Gal}(K) \to G, \alpha: A \wr_{G_0} G \to G)
\]

is rational. (Here $\text{Gal}(N_1/M)$ acts on $A$ via a splitting of $\beta$ and $G_0$ acts on $A$ via the restriction map $G_0 \to \text{Gal}(N_1/M)$.)
Proof. Let \( c_1, \ldots, c_n \) be a basis of \( L_0/K \) and let \( t = (t_1, \ldots, t_n) \) be an \( n \)-tuple of algebraically independent elements over \( L_0 \). Then by [13, Lemma 3.1], there exist fields \( F_0, \hat{F}_0 \) such that

(a) \( K(t) \subseteq L_0(t) \subseteq F_0 \subseteq \hat{F}_0 \) realizes \( A \wr_{G_0} G \),

(b) \( \hat{F}_0/L \) is regular, and

(c) \( F_0 = L(t)(z) \), where \( \text{irr}(z, L(t)) = f(\sum_1^n c_i t_i, Z) \in L_0[t, X] \).

In particular, \( \theta : \text{Gal}(K(t)) \rightarrow \text{Gal}(\hat{F}_0/K(t)) \) is a solution of (2.9) whose solution field is regular over \( L \). This means that the embedding problem is rational. 

Remark 2.5.9. The connection between solutions of the two embedding problems in the above lemma is much deeper. In [13], Haran establishes this connection and applies it to prove his diamond theorem. This theorem states a general sufficient condition for a separable extension of a Hilbertian field to be Hilbertian.

Corollary 2.5.10. Let \( K \subseteq L \subseteq M \) be a tower of separable algebraic extensions such that \( L/K \) is a finite Galois extension with a Galois group \( G = \text{Gal}(L/K) \). Let \( A \) be a finite group which is regularly realizable over \( K \). Then the embedding problem

\[(\text{res} : \text{Gal}(K) \rightarrow G, \alpha : A \wr G \rightarrow G)\]

is rational.

Proof. By assumption there exists a regular extension \( F_0 \) of \( K \) and \( t \in F_0 \) such that \( F_0/K(t) \) is a Galois extension with a Galois group \( A = \text{Gal}(F_0/K(t)) \). Let \( x \in F_0 \) be a primitive element and \( f(t, X) = \text{irr}(x, K(t)) \). We use Lemma 2.5.8 with \( F_1 = F_0M, N_1 = M, \) and \( L \) to get the assertion. 

2.5.4 Permutational Wreath Product

Often wreath products occur in nature as permutation groups. We do not present here the most general setting, for a more general definition see e.g. [23].

Let \( A, G \) be finite groups. Assume that \( A \) acts on some set \( X \). Then \( A \wr G \) acts on \( X \times G \) by the following rule.

\[(f \sigma)(x, \tau) = (f(\sigma \tau)x, \sigma \tau), \quad \forall f \sigma \in A \wr G, \ x \in X, \ \text{and} \ \tau \in G.\]
This action is well defined since
\[ f\sigma f'(x,\tau) = f\sigma(f'(\sigma\tau)x,\sigma\tau) = (f(\sigma\tau)f'(\sigma\tau)x,\sigma\sigma\tau) = (f^{\sigma^{-1}}(\sigma\tau)x,\sigma\sigma\tau) = f^{\sigma^{-1}}\sigma'(x,\tau). \]

**Proposition 2.5.11.** If the action of \( A \) on \( X \) is faithful (resp. transitive), then so is the action of \( A \wr G \) on \( X \times G \).

**Proof.** Clear. \qed

A group \( A \) is of **degree** \( n \) if \( A \) acts faithfully on a set \( X \) with \( n \) elements. The symmetry group \( S_n \) is maximal with respect to the property that any group of degree \( n \) can be embedded inside \( S_n \) as a permutation group. In what follows we show that \( S_n \wr G \) is the analogous ‘maximal’ permutation group if we consider permutation groups with an epimorphism onto \( G \) whose kernel has degree \( n \).

Let us be more precise. Let \( \alpha \): \( H \rightarrow G \) be an epimorphism of finite groups. Let \( X = \{1, \ldots, n\} \), and let \( \beta \): \( H \rightarrow G \) be an epimorphism of finite groups. Assume that \( H \) acts finely on \( X \times G \). Then there exists an embedding \( \nu \): \( H \rightarrow S_n \wr G \) that respects the actions on \( X \times G \) such that \( \alpha \nu = \beta \).
Proof. Define \( \nu : H \to S_n \wr G \) by setting \( \nu(h) = f_hg \), where \( g = \beta(h) \) and \( f_h(\sigma)(x) = h_{g^{-1}\sigma}(x) \), for \( x \in X \). It is obvious that \( \alpha\nu = \beta \). Also \( \nu \) respects the action on \( X \times G \). Indeed,
\[
\nu(h)(x, \tau) = (f_h(\beta(h)\tau)x, \beta(h)\tau) = (h_\tau(x), \beta(h)\tau) = h(x, \tau).
\]

As the actions of \( H \) and \( S_n \wr G \) on \( X \times G \) are faithful, we get that \( \nu \) is injective. It remains to show that \( \nu \) is a homomorphism.

Let \( h_1, h_2 \in H \) and \( g_1 = \alpha(h_1), g_2 = \alpha(h_2) \). Since \( \nu(h_1h_2) = f_{h_1h_2}g_1g_2 \) and \( \nu(h_1)\nu(h_2) = f_{h_1}f_{h_2}^{g_1^{-1}}g_1g_2 \), it suffices to verify that \( f_{h_1h_2} = f_{h_1}f_{h_2}^{g_1^{-1}} \). By (2.10) we have
\[

f_{h_1h_2}(\sigma)(x) = (h_1h_2)_{g_2^{-1}g_1^{-1}\sigma}(x) = (h_1)_{g_1^{-1}\sigma}(h_2)_{g_2^{-1}g_1^{-1}\sigma}(x) = f_{h_1}(\sigma)f_{h_2}(g_1^{-1}\sigma)(x) = (f_{h_1}f_{h_2}^{g_1^{-1}})(\sigma)(x),
\]
as needed. \( \square \)

2.5.5 The Embedding Theorem

The wreath product has the following interesting property that any extension of \( A \) and \( G \) can be embedded in \( A \wr G \). We will not use this result here.

**Theorem 2.5.13.** Let \( \xymatrix{1 \ar@{->}[r] & A \ar@{->}[r] & H \ar@{->}[r]^\pi & G \ar@{->}[r] & 1} \) be an exact sequence of groups. Then there exists an embedding \( i : H \to A \wr G \) such that \( \alpha i = \pi \), where \( \alpha : A \wr G \to G \) is the projection map.

For a proof see [23, Corollary 2.10].
Chapter 3

Double Embedding Problems and PAC Extensions

This chapter constitutes the technical backbone of the thesis. The study of the Galois structure of a field $K$ can be carried out by finite embedding problems. We introduce the notion of double embedding problems for a field extension $K/K_0$ which consists on two compatible embedding problems – one for $K$ and one for $K_0$.

It is known (although in different terminology) that over a PAC field every solution of a geometric finite embedding problem is geometric (see Subsection 2.4.2 for definitions). We go in a parallel way, and characterize the PACness of an extension in terms of geometric solutions of certain double embedding problems.

Surprisingly, a stronger property is valid – every solution of an embedding problem for $K$ (under some regularity condition) can be lifted to a geometric solution of the whole double embedding problem. This key property is called the lifting property and it is the main result of the chapter. We also give a stronger, but a bit more technical, lifting property for PAC extensions of finitely generated fields.

This group theoretic approach proves to be extremely efficient. In the following chapters we apply it to the study of the Galois structure PAC extensions and other applications.

3.1 Basic Properties of PAC Extensions

Recall the definition of a PAC field.

Definition 3.1.1. A field $K$ is called PAC if every nonempty absolutely irreducible variety defined over $K$ has a $K$-rational point.

In [17], Jarden and Razon introduce the more general notion of PAC extensions:
**Definition 3.1.2.** A field extension $K/K_0$ is said to be **PAC** if for every absolutely irreducible variety $V$ of dimension $e \geq 1$ defined over $K$ and for every separable dominating rational map $\nu: V \to \mathbb{A}^e$ there exists $b \in V(K)$ such that $a = \nu(b) \in K_0^e$.

**Remark 3.1.3.** In fact, [17] considers a more general setting, that is to say, it allows $K_0$ to be a subring or even a subset of $K$, see Definition 1.1.1.

**Remark 3.1.4.** Note that $K_0$ must be infinite. Indeed, if $K_0$ were finite, then $\nu^{-1}(K_0^e)$ would be also finite, and thus $\tilde{V} = V \setminus \nu^{-1}(K_0^e)$ would have no point $b \in \tilde{V}(K)$ satisfying $\nu(a) = b$.

**Remark 3.1.5.** If an extension $K/K_0$ is PAC, then the field $K$ is obviously PAC. In particular, $K$ is PAC if and only if $K/K$ is.

If $K$ is a $\mathbb{Z}_l$ extension of a finite prime field $\mathbb{F}_p$, then $K$ is PAC [9]. However, any proper subfield of $K$ is finite. Hence $K$ is not a PAC extension of any proper subfield.

In zero characteristic there are also examples of PAC fields which are PAC extensions of no proper subfields (Corollary 4.2.2).

The following proposition gives several equivalent definitions of PAC extensions in terms of polynomials and places, including a reduction to plane curves. A proof of that proposition essentially appears in [17]. Nevertheless, for the sake of completeness, we give here a formal proof.

**Proposition 3.1.6.** The following conditions are equivalent for a field extension $K/K_0$.

1. (6a) $K/K_0$ is PAC.

2. (6b) For every absolutely irreducible polynomial $f(T, X) \in K[T_1, \ldots, T_e, X]$ that is separable in $X$, and nonzero $r(T) \in K[T]$ there exists $(a, b) \in K_0^e \times K$ for which $r(a) \neq 0$ and $f(a, b) = 0$.

3. (6c) For every absolutely irreducible polynomial $f(T, X) \in K[T, X]$ that is separable in $X$, and nonzero $r(T) \in K(T)$ there exists $(a, b) \in K_0 \times K$ for which $r(a) \neq 0$ and $f(a, b) = 0$.

4. (6d) For every finitely generated regular extension $E/K$ with a separating transcendence basis $t = (t_1, \ldots, t_e)$ and every nonzero $r(t) \in K(t)$, there exists a $K$-place $\varphi$ of $E$ unramified over $K(t)$ such that $E = K$, $K_0(t) = K_0$, $a = \varphi(t)$ is finite, and $r(a) \neq 0, \infty$. 

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(1.6e) For every finitely generated regular extension $E/K$ of transcendence degree $1$ with a separating transcendence basis $t$ and every nonzero $r(t) \in K[t]$ there exists a $K$-place $\varphi$ of $E$ unramified over $K(t)$ such that $\bar{E} = K$, $\bar{K}_0(t) = K_0$, $a = \varphi(t) \neq \infty$, and $r(a) \neq 0, \infty$.

Proof. The proof of [17, Lemma 1.3] gives the equivalence between (1.6a), (1.6b), and (1.6c). Obviously (1.6d) implies (1.6e), so it suffices to prove that (1.6b) implies (1.6d) and that (1.6e) implies (1.6c).

(1.6b) $\Rightarrow$ (1.6d): Let $x \in E/K(t)$ be integral over $K[t]$ such that $E = K(t, x)$. Let $f(T, X) \in K[T, X]$ be the absolutely irreducible polynomial which is monic and separable in $X$ and for which $f(t, x) = 0$. Let $g(t) \in K[t]$ be the polynomial given in Lemma 2.4.1 for the extension $E/K(t)$. We have $(a, b) \in K_0 \times K$ such that $f(a, b) = 0$ and $g(a)r(a) \neq 0, \infty$. Extend the specialization $t \mapsto a$ to a $K$-place $\varphi$ of $E$ with the following properties to conclude the implication: (1) $\varphi(x) = b \neq \infty$ (this is possible since $x$ is integral over $K[t]$); (2) $\bar{K}_0(t) = K_0$ and $\bar{E} = K(b) = K$ (Lemma 2.4.1); (3) $\varphi$ is unramified over $K(t)$ (Lemma 2.4.1).

(1.6c) $\Rightarrow$ (1.6c): Let $f(T, X) = \sum_{k=0}^n a_k(T)X^k$ and $r(T)$ be as in (1.6c). Set $r'(T) = r(T)a_n(T)$. Let $t$ be a transcendental element and let $x \in \bar{K}(t)$ be such that $f(t, x) = 0$. Let $E = K(t, x)$. Then $E$ is regular over $K$ and separable over $K(t)$. Applying (1.6c) to $E$ and $r'(t)$ we get a $K$-place $\varphi$ of $E$ satisfying the following properties. (1) $a = \varphi(t) \in K_0$ which implies that $b = \varphi(x)$ is finite, since $\varphi(f(t, x)) = 0$ and $f(a, X)$ has a nonzero leading coefficient; (2) $\bar{E} = K$, which concludes the proof since $b \in \bar{E} = K$.

3.2 Geometric Solutions and PAC Fields

The following result characterizes when a solution is geometric in terms of a rational place of some regular extension.

Proposition 3.2.1. Let $K$ be a field and consider a geometric embedding problem

(3.1) $\mu: \text{Gal}(K) \to \text{Gal}(L/K), \alpha: \text{Gal}(F/E) \to \text{Gal}(L/K))$

for $K$. Let $\theta: \text{Gal}(K) \to \text{Gal}(F/E)$ be a weak solution. Then there exists a finite separable extension $\hat{E}/E$ such that $\hat{E}/K$ is regular and for every place $\varphi$ of $E/K$ that is unramified in $F$ the following two conditions are equivalent.

(a) $\varphi$ extends to a place $\Phi$ of $F$ such that $\bar{E} = K$ and $\Phi^* = \theta$. 

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(b) \( \varphi \) extends to a \( K \)-rational place of \( \hat{E} \).

**Proof.** First we consider the special case when \( \text{Gal}(F/E) \cong \text{Gal}(L/K) \). Then there is a unique solution of \((\mu, \alpha)\), namely \( \theta = \alpha^{-1}\mu \). Let \( \Phi \) be an extension of \( \varphi \) to \( F \) and take \( \hat{E} = E \). It is trivial that (a) implies (b). Assume (b). Then \( \Phi^* \) is defined, and from the uniqueness, \( \Phi^* = \theta \).

Next we prove the general case. Let \( M/K \) be a Galois extension such that \( \text{Gal}(M) = \ker(\theta) \) (in particular, \( L \subseteq M \)) and let \( \hat{F} = FM \).

\[
\begin{array}{c}
F \\
\vdots \\
\hat{E} \\
E \\
\vdots \\
EL \\
\vdots \\
EM \\
K \\
\vdots \\
L \\
\vdots \\
M
\end{array}
\]

As \( F \) and \( M \) are linearly disjoint over \( L \), the fields \( F \) and \( EM \) are linearly disjoint over \( EL \). We have

\[
\text{Gal}(\hat{F}/E) = \text{Gal}(F/E) \times_{\text{Gal}(L/K)} \text{Gal}(M/K).
\]

Define \( \hat{\theta}: \text{Gal}(K) \to \text{Gal}(\hat{F}/E) \) by \( \hat{\theta}(\sigma) = (\theta(\sigma), \sigma|_M) \). Let \( \hat{E} \) denote the fixed field of \( \hat{\theta}(\text{Gal}(K)) \) in \( \hat{F} \). Then \( \hat{\theta} \) is a solution in

\[
\begin{array}{c}
\text{Gal}(K) \\
\hat{\theta} \\
\text{Gal}(\hat{F}/\hat{E}) \xrightarrow{\hat{\alpha}} \text{Gal}(L/K).
\end{array}
\]

Here \( \hat{\alpha} \) is the restriction map. In particular, \( \hat{E}/K \) is regular. Also \( \ker(\hat{\theta}) = \ker(\theta) \cap \text{Gal}(M) = \text{Gal}(M) \), so \( \text{Gal}(\hat{F}/\hat{E}) \cong \text{Gal}(M/K) \).

Assume \( \varphi \) extends to a \( K \)-rational place \( \hat{\varphi} \) of \( \hat{E} \). Extend \( \varphi \) \( L \)-linearly to a place \( \hat{\Phi} \) of \( \hat{F} \). Then \( \hat{\Phi}/\hat{\varphi} \) is unramified. Let \( \Phi = \hat{\Phi}|_F \). Then by the first part \( \Phi^* = \hat{\theta} \). Lemma 2.4.4 then asserts that \( \Phi^* = \theta \).

On the other hand, assume that \( \varphi \) extends to a place \( \Phi \) of \( F \) such that \( \hat{E} = K \) and \( \Phi^* = \theta \). Extend \( \Phi \) \( M \)-linearly to a place \( \hat{\Phi} \) of \( \hat{F} \). Then, since \( \text{res}_{\hat{F},F}(\Phi^*) = \Phi^* \) and \( \text{res}_{\hat{F},M}(\hat{\Phi}^*) = \text{res}_{K,M}, \) we have

\[
\hat{\Phi}^*(\sigma) = (\Phi^*(\sigma), \sigma|_M) = \hat{\theta}.
\]

Therefore the residue field of \( \hat{E} \) is also \( K \). \( \square \)
Remark 3.2.2. In the proof it was shown that $\hat{E} \subseteq FM$, where $M$ is the solution field of $\theta$.

Remark 3.2.3. In the proof we showed that $\ker \hat{\theta} = \text{Gal}(M)$, that is $\hat{\alpha}$ is an isomorphism. Thus $\hat{E}M = \hat{F}$. This implies that our proof is a group theoretic formulation of the field crossing argument.

Remark 3.2.4. Proposition 3.2.1 sharpens earlier works of Roquette on PAC Hilbertian fields [9, Corollary 27.3.3], Fried-Haran-Jarden on Frobenius fields [9, Proposition 24.1.4], and of Dèbes on Beckmann-Black problem [5].

Although in each of the earlier works the authors prove a slightly weaker version of the proposition, e.g., Fried-Haran-Jarden consider PAC fields and Dèbes considers the case where $L = K$, the proofs are essentially the same as the proof given here. In other words one can easily extend the proof of [9, Lemma 24.1.1] or alternatively the proof of [5, Proposition 2.2] to get a proof of this proposition. In fact, the author of this work chose the former.

The innovation of our result is the focus on the solution $\theta$, rather on its image which is the decomposition group. This observation is the basis of the result on this dissertation.

Proposition 3.2.1 is extremely useful. For example, the following result which is in fact a reformulation of [9, Lemma 24.1.1] (excluding the part on $p$-independent elements) is a straightforward conclusion of the proposition.

**Corollary 3.2.5.** Every weak solution of a finite embedding problem (2.2) for a PAC field $K$ is geometric.

**Proof.** By Proposition 3.2.1 for each weak solution there exists a finitely generated regular extension $\hat{E}$ of $K$ such that the solution is geometric if and only if there exists a $K$-rational place of $\hat{E}$. Thus, as $K$ is PAC, there is always such a $K$-rational place.

**Proof of Lemma 24.1.1 of Fried-Jarden.** Let $K$ be a PAC field, $S/R$ a regular finitely generated Galois ring cover over $K$, and $F/E$ the corresponding fraction fields extension. Let $L$ denote the algebraic closure of $K$ in $F$. Let $E'$ be a subextension of $F/E$ such that the restriction map $\alpha' : \text{Gal}(F/E') \to \text{Gal}(L/K)$ is surjective. Assume the existence of an epimorphism $\gamma : \text{Gal}(K) \to \text{Gal}(F/E')$ such that $\alpha' \gamma = \nu$, where $\nu : \text{Gal}(K) \to \text{Gal}(L/K)$ is the restriction map. Let $M$ be the fixed field of $\ker \gamma$ in $K$.

We have to prove the existence of an epimorphism $\varphi : S \to M$ such that $\varphi(R) = K$ and $D_{\varphi} = \text{Gal}(F/E')$. In the notation of this work, $\gamma$ is a solution of $(\nu, \alpha')$. The previous corollary implies that $\gamma = \varphi^*$, for a place $\varphi$ of $E$ unramified in $F$ and such $E = K$. Thus

$$D_{\varphi} = \varphi^*(\text{Gal}(K)) = \gamma(\text{Gal}(K)).$$
In fact we can choose $\varphi$ such that $S \subseteq \mathcal{O}_\varphi$, since the only requirement is that the residue field of $\hat{E}$ is $K$. Then $\varphi(S) = \hat{F} = M$ and $\varphi(R) = \hat{E} = K$.

## 3.3 Double Embedding Problems

In this section we generalize the notion of embedding problems to field extensions. The reader is advised to recall the definitions and notation concerning embedding problems, if he needs to.

### 3.3.1 The Definition of Double Embedding Problems

Let $K/K_0$ be a field extension. A double embedding problem (DEP) for $K/K_0$ consists on two embedding problems, one $(\mu: \text{Gal}(K) \to G, \alpha: H \to G)$ for $K$ and one $(\mu_0: \text{Gal}(K_0) \to G_0, \alpha_0: H_0 \to G_0)$ for $K_0$ which are compatible in the following sense. $H \leq H_0, G \leq G_0$, and if we write $i: H \to H_0$ and $j: G \to G_0$ for the inclusion maps, then the following diagram commutes.

\[
\begin{array}{ccc}
\text{Gal}(K_0) & \xrightarrow{\mu} & G_0 \\
\xleftarrow{\exists \theta_0?} & & \\
H_0 & \xrightarrow{\alpha_0} & G_0 \\
\xleftarrow{i} & & \\
\text{Gal}(K) & \xrightarrow{\exists \theta?} & G \\
\xleftarrow{\exists \theta_0?} & & \\
H & \xrightarrow{\alpha} & G \\
\end{array}
\]

Given a DEP for $K/K_0$, we refer to the corresponding embedding problem for $K$ (resp. $K_0$) as the lower (resp. the higher) embedding problem. We call a DEP finite if the higher (and hence also the lower) embedding problem is finite.

A weak solution of a DEP (3.2) is a weak solution $\theta_0$ of the higher embedding problem which restricts to a solution $\theta$ of the lower embedding problem via the restriction map $\text{Gal}(K) \to \text{Gal}(K_0)$. In case $K/K_0$ is a separable algebraic extension, the restriction map is the inclusion map, and hence the condition on $\theta_0$ reduces to $\theta_0(\text{Gal}(K)) \leq H$. To emphasize the existence of $\theta$, we usually regard a weak solution of a DEP as a pair $(\theta, \theta_0)$ (where $\theta$ is the restriction of $\theta_0$ to $\text{Gal}(K)$).
3.3.2 Regularly Solvable Double Embedding Problems

Consider a double embedding problem (3.2) and let $L_0$ and $L$ be the fixed fields of the kernels of $\mu_0$ and $\mu$, respectively. Then we have isomorphisms $\bar{\mu}_0: \text{Gal}(L_0/K_0) \to G_0$ and $\bar{\mu}: \text{Gal}(L/K) \to G$. Hence (as in the case of embedding problems) replacing $G_0$ and $G$ with $\text{Gal}(L_0/K_0)$ and $\text{Gal}(L/K)$ (and replacing correspondingly all the maps) gives us an equivalent DEP. The compatibility condition is realized as $L = L_0K$.

**Lemma 3.3.1.** Every DEP for which the upper embedding problem is rational is equivalent to the following DEP:

\[
\begin{array}{ccccccc}
\text{Gal}(K_0) & \xrightarrow{\mu_0} & \text{Gal}(L_0/K_0) \\
\text{Gal}(F_0/K(t)) & \xrightarrow{\alpha_0} & \text{Gal}(L_0/K_0) \\
\text{Gal}(K) & \xrightarrow{\mu} & \text{Gal}(L/K) \\
\text{Gal}(F/E) & \xrightarrow{\alpha} & \text{Gal}(L/K).
\end{array}
\]

Moreover, the results even holds if we take $t = t$ a single variable.

**Proof.** Let (3.2) be a rational double embedding problem. By definition, it means that the higher embedding problem is rational, so we may replace the higher embedding problem of (3.2) with an equivalent embedding problem as in (3.3). By Lemma 2.4.5 we may assume that $t = t$.

Let $F = F_0K$ and $L = L_0K$. The compatibility condition implies that $H$ embeds into $\text{Gal}(F_0/K_0(t))$ (via $i$) as a subgroup of $\text{Gal}(F_0/(L_0 \cap K)(t)) \cong \text{Gal}(F/K(t))$. Let $E \subseteq F$ be the fixed field of $H$, i.e., $H = \text{Gal}(F/E)$. Under this embedding, $\alpha: \text{Gal}(F/K(t)) \to \text{Gal}(L/K)$ is the restriction map. Therefore $\alpha(H) = \text{Gal}(L/K)$ implies that $E \cap L = K$, and hence $E$ is regular over $K$.

\[\square\]
Definition 3.3.2. A double embedding problem as in the above lemma is called rational double embedding problem.

Remark 3.3.3. The converse of the above lemma is also valid, that is to say, assume we have a rational double embedding problem as in (3.3), i.e. a finitely generated regular extension $E/K$, a separating transcendence basis $t$ for $E/K$, and a finite Galois extension $F_0/K_0(t)$ such that $E \subseteq F$, where $F = F_0K$. Then all the restriction maps in (3.3) are surjective, and hence (3.3) defines a finite double embedding problem.

3.3.3 Geometric Solutions of Double Embedding Problems

First recall that a weak solution $\theta$ of a geometric embedding problem

$$(\mu: \text{Gal}(K) \rightarrow \text{Gal}(L/K), \alpha: \text{Gal}(F/E) \rightarrow \text{Gal}(L/K))$$

is geometric if $\theta = \varphi^*$, for some $K$-place $\varphi$ of $E$ that is unramified in $F$ and under which the residue field of $E$ is $K$. Then we call a weak solution $(\theta, \theta_0)$ of (3.3) geometric solution if $(\theta, \theta_0) = (\varphi^*, \varphi_0^*)$, where $\varphi^*$ is a geometric solution of the lower embedding problem and $\varphi_0 = \varphi|_{K_0}$.

Note that since $\varphi_0^*$ is a solution of the higher embedding problem, the residue field of $K_0(t)$ is $K_0$. In particular, if $\varphi(t)$ is finite, then $\varphi(t) \in K_0^c$. Also note that for a $K$-place of $E$ that is unramified at $F$ and such that $\bar{E} = K$ and $\bar{K_0(t)} = K_0$, the pair $(\varphi^*, \varphi_0^*)$ is indeed a weak solution of (3.3), since $\varphi_0^* = \text{res}_{K, K_0}\varphi^*$.

3.4 The Lifting Property

In this section we formulate and prove the lifting property. First we reduce the discussion to separable algebraic extensions by showing that if $K/K_0$ is PAC, then $K \cap K_{0s}/K_0$ is PAC and $\text{Gal}(K) \cong \text{Gal}(K \cap K_{0s})$ via the restriction map. Then we characterize separable algebraic PAC extensions in terms of geometric solutions of double embedding problems. From this characterization we establish the lifting property. Finally we prove a strong (but complicated) version of the lifting problem to PAC extensions of finitely generated fields.

3.4.1 Reduction to Separable Algebraic Extensions

In [17, Corollary 1.5] Jarden and Razon show

Lemma 3.4.1 (Jarden-Razon). If $K/K_0$ is PAC, then so is $K \cap K_{0s}/K_0$. 

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Moreover, we have:

**Theorem 3.4.2.** Let $K/K_0$ be a PAC extension. Then $K \cap K_0s/K_0$ is PAC and the restriction map $\text{Gal}(K) \to \text{Gal}(K \cap K_0s)$ is an isomorphism.

**Proof.** It suffices to show that $K_s = K_0sK$. Let $L/K$ be a finite Galois extension with Galois group $G$ of order $n$. Embed $G$ into the symmetric group $S_n$. Let $F_0/K_0(t)$ be a regular realization of $S_n$ with $F_0$ algebraically independent from $K$ over $K_0$ (Theorem 2.2.4). Then $F = F_0K$ is regular over $K$ and $\text{Gal}(F/K(t)) \cong S_n$. Furthermore

$$\text{Gal}(FL/K(t)) = \text{Gal}(FL/L(t)) \times \text{Gal}(FL/F) \cong \text{Gal}(F/K(t)) \times \text{Gal}(L/K) \cong S_n \times G.$$

Let $E$ be the fixed field of the subgroup $\Delta = \{(g, g) \mid g \in G\}$ in $FL$, i.e., $\text{Gal}(FL/E) \cong \Delta$. By Galois correspondence, $S_n\Delta = S_n \times G$ implies that $E \cap L = K$ and $1 = \Delta \cap S_n = G \cap \Delta$ implies that $FL = EL = FE$. In particular, $E/K$ is regular.

Let $x_F, x_E, x_{FL}$ be primitive elements of $F/K(t)$, $E/K(t)$, and $FL/K(t)$, respectively. Let $h(t) \in K[t]$ be the products of the corresponding polynomials given in Lemma 2.4.1.

As $K/K_0$ is PAC, there is a $K$-place $\varphi$ of $E$ such that $\bar{E} = K$, $\bar{K}_0(t) = K_0$, $a = \varphi(t)$ is finite, and $h(a) \neq 0$. Extend $\varphi$ to an $L$-place $\Phi$ of $FL$.

Then, $FL = EL$ implies that $\bar{FL} = \bar{EL} = L(\Phi(x_E)) = L$. However, as $F = F_0K$, it follows that $\bar{F} = \bar{F}_0K$, hence, $L = \bar{FL} = \bar{FE} = K(\Phi(x_E), \Phi(x_F)) = \bar{F} = \bar{F}_0K \subseteq K_0sK$, as needed.

**Remark 3.4.3.** The above theorem also follows from the main result of [27]. However we shall prove that result using the theorem.

PAC fields have a nice elementary theory. Since $K$ and $K \cap K_0s$ are PAC fields and since they have isomorphic absolute Galois groups, they are elementary equivalent under some necessary condition:
Corollary 3.4.4. Let \( K/K_0 \) be a separable PAC extension. Assume that \( K \) and \( K \cap K_0 \) have the same imperfect degree. Then \( K \) is an elementary extension of \( K \cap K_0 \).

Proof. The assertion follows from \[9\] Corollary 20.3.4 and Theorem 3.4.2.

3.4.2 Characterization of Separable Algebraic PAC Extensions

Proposition 3.4.5. Let \( K/K_0 \) be a separable algebraic field extension. The following conditions are equivalent:

(4.5a) \( K/K_0 \) is PAC.

(4.5b) For every finite rational double embedding problem (3.3) for \( K/K_0 \) and every nonzero rational function \( r(t) \in K(t) \), there exists a geometric weak solution \( (\varphi^*, \varphi_0^*) \) such that \( a = \varphi(t) \) is finite and \( r(a) \neq 0, \infty \).

(4.5c) For every finite rational double embedding problem (3.3) for \( K/K_0 \) with \( t = t \) a transcendental element there exist infinitely many geometric weak solution \( (\varphi^*, \varphi_0^*) \).

Proof. The implication (4.5a) \( \Rightarrow \) (4.5b) follows from Proposition 3.1.6 (part 1.6d) and the definition of geometric weak solutions.

The implication (4.5b) \( \Rightarrow \) (4.5c) is immediate.

(4.5c) \( \Rightarrow \) (4.5a): We apply Proposition 3.1.6 and show that (1.6e) holds. Let \( E/K \) be a regular extension with a separating transcendence basis \( t \) and let \( r(t) \in K[t] \) be nonzero. Choose \( F_0 \) to be a finite Galois extension of \( K_0(t) \) such that \( E \subseteq F_0K \) (such \( F_0 \) exists since \( K/K_0 \) is separable and algebraic) and let \( F = F_0K, L = F \cap K_0, \) and \( L_0 = F_0 \cap K_0 \). By assumption there are infinitely many geometric weak solution \( (\varphi^*, \varphi_0^*) \) of the DEP

Since for only finitely many solutions \( \varphi(t) \) is infinite or \( r(\varphi(t)) = 0, \infty \) we can find a solution such that \( \varphi(t) \neq \infty \) and \( r(\varphi(t)) \neq 0, \infty \). In particular, \( \bar{E} = K \) and \( \bar{K}_0(t) = K_0 \), as required in (1.6e).
Let $K/K_0$ be a PAC extension and consider a rational DEP for $K/K_0$. The following key property – the lifting property – asserts that any weak solution of the lower embedding problem can be lifted to a geometric weak solution of the DEP.

**Proposition 3.4.6** (The lifting property). Let $K/K_0$ be a PAC extension, let $\theta: \text{Gal}(K) \to \text{Gal}(F/E)$ be a weak solution of the lower embedding problem in (3.3). Then there exists a geometric weak solution $(\varphi^*, \varphi_0^*)$ of (3.3) such that $\theta = \varphi^*$.

Moreover, if $r(t) \in K(t)$ is nonzero, we can choose $\varphi$ such that $a = \varphi(t) \in K_0^e$ and $r(a) \neq 0, \infty$.

**Proof.** By Proposition 3.2.1 there exists a finite separable extension $\hat{E}/E$ that is regular over $K$ with the following property. If $\Phi$ is a $K$-rational place of $\hat{E}$ and if $\varphi = \Phi|_E$ is unramified in $F$, then $\varphi^* = \theta$.

By the PACness of $K/K_0$ there exists a $K$-rational place $\Phi$ of $\hat{E}$ such that $\varphi = \Phi|_E$ is unramified in $F$, the residue field of $K_0(t)$ is $K_0$, $a = \Phi(t)$ is finite, and $r(a) \neq 0, \infty$. If we let $\varphi_0 = \varphi|_{K_0(t)}$, then we get that $(\varphi^*, \varphi_0^*)$ is a geometric weak solution and that $\varphi^* = \theta$. \hfill \Box

A first and an easy consequence is the transitivity of PAC extensions. To the best of our knowledge there is no other way to prove this property in the literature.

**Lemma 3.4.7.** Let $K_0 \subseteq K_1 \subseteq K_2$ be a tower of separable algebraic extensions. Assume that $K_2/K_1$ and $K_1/K_0$ are PAC extensions. Then $K_2/K_0$ is PAC.

**Proof.** Let

\[
((\mu_0: \text{Gal}(K_0) \to \text{Gal}(L_0/K_0), \alpha_0: \text{Gal}(F_0/K_0(t)) \to \text{Gal}(L_0/K_0)),

(\mu_2: \text{Gal}(K_2) \to \text{Gal}(L_2/K_2), \alpha_2: \text{Gal}(F_2/E) \to \text{Gal}(L_2/K_2))
\]

be a rational finite DEP for $K_2/K_0$. By Lemma 3.4.5 it suffices to find a geometric weak solution to $((\mu_0, \alpha_0), (\mu_2, \alpha_2))$. Set $F_1 = F_0 K_1$, $L_1 = L_0 K_1$. Then, since $K_2/K_1$ is PAC there exists a weak solution $(\varphi_2^*, \varphi_1^*)$ of the double embedding problem defined by the lower
part of the following commutative diagram.

(3.4)

![Diagram](image)

Now we lift \( \varphi^*_1 \) to a geometric weak solution \((\varphi^*_1, \varphi^*_0)\) of the DEP for \(K_1/K_0\) defined by the higher part of the diagram. This is possible by the lifting property applied to the PAC extension \(K_1/K_0\).

Since \( \varphi^*_0 \mid_{\text{Gal}(K_2)} = \varphi^*_1 \mid_{\text{Gal}(K_2)} = \varphi^*_2 \) we get that \((\varphi^*_2, \varphi^*_0)\) is a geometric weak solution of the DEP we started from. \(\square\)

**Proposition 3.4.8.** Let \( \kappa \) be an ordinal number and let

\[ K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots \subseteq K_\kappa \]

be a tower of separable algebraic extensions. Assume that \(K_{\alpha+1}/K_\alpha\) is PAC for every \( \alpha < \kappa \) and that \( K_\alpha = \bigcup_{\beta < \alpha} K_\beta \) for every limit \( \alpha \leq \kappa \). Then \( K_\kappa/K_0 \) is PAC.

**Proof.** We apply transfinite induction. Let \( \alpha \leq \kappa \). If \( \alpha \) is a successor ordinal, then the assertion follows from the previous lemma.
Let $\alpha$ be a limit ordinal. Let

\begin{equation}
\text{Gal}(K_\alpha) \to \text{Gal}(F_\alpha/E_\alpha) \to \text{Gal}(L_\alpha/K_\alpha)
\end{equation}

be a rational DEP for $K_\alpha/K_0$. Here $F_\alpha = F_0K_\alpha$ and $L_\alpha = L_0K_\alpha$. Now since all the extensions are finite, there exists $\beta < \alpha$ such that $\text{Gal}(F_\alpha/K_\alpha) \cong \text{Gal}(F_\beta/K_\beta)$ and $\text{Gal}(L_\alpha/K_\alpha) \cong \text{Gal}(L_\beta/K_\beta)$ (via the corresponding restriction maps), where $F_\beta = F_0K_\beta$ and $L_\beta = L_0K_\beta$.

Induction gives a weak solution of the double embedding problem (3.5) with $\beta$ replacing $\alpha$. This weak solution induces a weak solution of (3.5) via the above isomorphisms.

\section{3.5 Strong Lifting Property for PAC Extensions of Finitely Generated Fields}

Let $K_0$ be a finitely generated field (over its prime field). In this section we prove a strong lifting property for PAC extensions $K/K_0$. The additional ingredient is the Mordell conjecture for finitely generated fields (now a theorem due to Faltings in characteristic 0 \cite{Faltings} and to Grauert-Manin in positive characteristic \cite[page 107]{Manin}).

The following lemma is based on the Mordell conjecture.

\begin{lemma} [\cite[Proposition 5.4]{Iz}]
Let $K_0$ be a finitely generated infinite field, $f \in K_0[T,X]$ an absolutely irreducible polynomial which is separable in $X$, $g \in K_0[T,X]$ an irreducible polynomial which is separable in $X$, and $0 \neq r \in K_0[T]$. Then there exist a finite purely inseparable extension $K'_0$ of $K_0$, a nonconstant rational function $q \in K'_0(T)$, and a finite subset $B$ of $K'_0$ such that $f(q(T),X)$ is absolutely irreducible, $g(q(a),X)$ is irreducible in $K'_0[X]$, and $r(q(a)) \neq 0$ for any $a \in K'_0 \setminus B$.

Let $K/K_0$ be an extension. Consider a rational double embedding problem (3.3) for $K/K_0$ (with $t = t$). For any subextension $K_1$ of $K/K_0$ we have a corresponding rational
double embedding problem. Namely

\[
\begin{align*}
\text{Gal}(K_1) & \xrightarrow{\mu_1} \text{Gal}(L_1/K_1) \\
\text{Gal}(F_1/K_1(t)) & \xrightarrow{\alpha_1} \text{Gal}(L_1/K_1) \\
\text{Gal}(F/E) & \xrightarrow{\alpha} \text{Gal}(L/K),
\end{align*}
\]

where \( F_1 = F_0K_1 \), \( L_1 = L_0K_1 \), and \( \mu_1 \) and \( \alpha_1 \) are the restriction maps.

Assume that \( K'_1/K_1 \) is a purely inseparable extension. Then the double embedding problem (3.6) remains the same if we replace all fields by their compositum with \( K'_1 \).

**Proposition 3.5.2** (Strong Lifting Property). Let \( K \) be a PAC extension of a finitely generated field \( K_0 \). Let

\[
\mathcal{E}(K) = (\mu: \text{Gal}(K) \to \text{Gal}(L/K), \alpha: \text{Gal}(F/E) \to \text{Gal}(L/K))
\]
be an embedding problem as in (2.2), and let \( \theta: \text{Gal}(K) \to \text{Gal}(F/E) \) be a weak solution of \( \mathcal{E}(K) \). Then there exist a finite subextension \( K_1/K_0 \) and a finite purely inseparable extension \( K'_1/K_1 \) satisfying the following properties.

(a) For any rational double embedding problem (3.3) (whose lower embedding problem is \( \mathcal{E}(K) \)), we can lift \( \theta \) to a weak solution \( (\theta, \theta_1) \) of the double embedding problem (3.6) in such a way that \( \theta_1 \) is surjective.

(b) The solution \( (\theta, \theta_1) \) is a geometric solution of the double embedding problem that we get from (3.6) by replacing all fields with their compositum with \( K'_1 \).

**Proof.** By Proposition 3.2.1 there exists a finite separable \( \hat{E}/E \) that is regular over \( K \) such that a \( K \)-place \( \varphi \) of \( E \) that is unramified in \( F \) can be extended to a place \( \Phi \) of \( F \) such that \( \Phi^e = \theta \) if and only if \( \varphi \) extends to a \( K \)-rational place of \( \hat{E} \). Let \( f(t, X) \in K[t, X] \) be an absolutely irreducible polynomial whose root \( x \) generates \( \hat{E}/K(t) \), i.e. \( \hat{E} = K(t, x) \). Let \( M \) be the fixed field of \( \ker(\theta) \) in \( K_s \). Then \( M/K \) is a finite Galois extension. Let \( h(X) \in K[X] \) be a Galois irreducible polynomial whose root generates \( M/K \).

\[\text{That is to say, } E \text{ is regular (of transcendence degree 1) over } K, \text{ } F/E \text{ is finite and Galois, } L = F \cap K_s, \text{ and all the maps are the restriction maps.}\]
Let $K_1$ be a finite subextension of $K/K_0$ that contains the coefficients of $f$ and $h$ and such that $h$ is Galois over it. Let $M_1$ be the splitting field of $h$ over $K_1$ and let $L_1, F_1$ be as in the corresponding rational double embedding problem (3.6). Then $\text{Gal}(M/K) \cong \text{Gal}(M_1/K_1)$, and thus also $\text{Gal}(L/K) \cong \text{Gal}(L_1/K_1)$.

\[
\begin{array}{c}
E & \rightarrow & EM \\
\downarrow & & \downarrow \\
K(t) & \rightarrow & M(t) \\
\downarrow & & \downarrow \\
K_0(t) & \rightarrow & K_1(t) \rightarrow & L_1(t) \rightarrow & M_1(t) \\
\downarrow & \rightarrow & \downarrow \\
F_1 & \rightarrow & \\
\end{array}
\]

Let $g(t, X) \in K_1[T, X]$ be an irreducible polynomial whose root generates $F_1/K_1(t)$. Choose $r(t) \in K_1(t)$ such that $r(a) \neq 0$ implies that the prime $(t - a)$ is unramified in $F_1$ and that the leading coefficients of $f(t, X)$ and $g(t, X)$ do not vanish at $a$. Let $K'_1/K_1$ be the purely inseparable extension, $B \subseteq K'_1$ the finite subset, and $q \in K'_1(T)$ the nonconstant rational function that Lemma 3.5.1 gives for $K_1, g, f, r$. Let $K' = KK'_1$.

Since $K'/K'_1$ is PAC ([17, Corollary 2.5]) there exist $a \in K'_1 \setminus B$ and $b \in K'$ for which $f(q(a), b) = 0$ (Proposition 3.1.6). Extend $t \mapsto q(a)$ to a $K'$-rational place $\Phi$ of $\hat{E}K'_1$. Then $\varphi = \Phi|_{EK'_1}$ is unramified in $FK'_1$ (since $r(q(a)) \neq 0$). It follows that $(\varphi^*, \varphi^*_1)$ is a geometric weak solution of the DEP $((\mu, \alpha), (\mu_1, \alpha_1))$ that we get from (3.6) by replacing all fields with their compositum with $K'_1$.

Moreover, since $F_1K'_1/K'_1(t)$ is generated by $g(t, X)$ and $g(q(a), X)$ is irreducible, we get that $\varphi^*_1$ is surjective. This proves (b). Now assertion (a) follows since $(\varphi^*, \varphi^*_1)$ is a (not necessarily geometric) solution of $((\mu, \alpha), (\mu_1, \alpha_1))$. □
Chapter 4

On the Galois Structure of PAC Extensions

In [17] Jarden and Razon use a deep theorem of Faltings to prove that some specific Galois extensions of \( \mathbb{Q} \) are not PAC extensions of any number field. Then they ask whether this is a general phenomenon or a coincidence.

In [16] Jarden settles the above question by showing that the only Galois PAC extension of a number field is its algebraic closure. This result is based on Frobenius’ density theorem, on Neukirch’s characterization of \( p \)-adically closed fields among all algebraic extensions of \( \mathbb{Q} \), and on the special property of \( \mathbb{Q} \) that it has no proper subfields (!).

In [2] Jarden and the author elaborate the Jarden-Razon method and extend Jarden’s answer to the larger family of finitely generated fields \( K \) using the Mordell conjecture (proved by Faltings in characteristic 0 and Grauert-Manin in positive characteristic).

Using the lifting property, we can elementarily reprove all the above results. This new proof does not use any of the properties of finitely generated fields, and hence is valid for an arbitrary field \( K \) (Theorem 4.2.3). This result and other results appearing in this chapter (e.g. descent results) manifest that the right approach to PAC extension is via double embedding problems.

4.1 Descent Features

The following result allows us to descend Galois groups in a PAC extension.

**Theorem 4.1.1.** Let \( K/K_0 \) be a PAC extension, let \( \varphi: \text{Gal}(K) \to \text{Gal}(K_0) \) be the restriction map, and let \( \theta: \text{Gal}(K) \to G \) be an epimorphism onto a finite group \( G \). Furthermore, let \( i: G \hookrightarrow G_0 \) be an embedding of \( G \) into a finite group \( G_0 \) which is regularly realizable.
over $K_0$. Then there exists a homomorphism $\theta_0 : \text{Gal}(K_0) \to G_0$ such that $\theta_0 \varphi = i\theta$:

$$
\begin{array}{ccc}
\text{Gal}(K) & \xrightarrow{\varphi} & \text{Gal}(K_0) \\
\downarrow & & \downarrow \\
G & \xrightarrow{i} & G_0.
\end{array}
$$

**Proof.** The map $\theta : \text{Gal}(K) \to G$ is a solution of the lower embedding problem of the double embedding problem

$$(4.1)$$

Since $G_0$ is regularly realizable, the DEP is in fact rational. Thus $\theta$ extends to a geometric weak solution $(\theta, \theta_0)$ of (4.1) by the lifting property (Proposition 3.4.6).

In case $G = G_0$, Theorem 4.1.1 immediately yields:

**Corollary 4.1.2.** Let $K/K_0$ be a PAC extension. Let $G$ be a finite Galois group over $K$ that is regularly realizable over $K_0$. Then $G$ also occurs as a Galois group over $K_0$.

Since every abelian group is regularly realizable over any field (Theorem 2.2.4), we get the following

**Corollary 4.1.3.** Let $K/K_0$ be a PAC extension. Then $K^{ab} = K_0^{ab} K$.

We reprove [27, Theorem 5] using Theorem 4.1.1 and the fact that the symmetric group is regularly realizable over any field. This proof provides an insight into Razon’s original technical proof.

**Corollary 4.1.4 (Razon).** Let $K/K_0$ be a PAC extension. Let $L/K$ be a separable algebraic extension. Then there exists a separable algebraic extension $L_0/K_0$ that is linearly disjoint from $K$ over $K_0$ for which $L = L_0 K$. 

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Proof. First assume that $[L : K]$ is finite. Let $M$ be the Galois closure of $L/K$, $G = \text{Gal}(M/K)$, $G' = \text{Gal}(M/L)$, and $\theta: \text{Gal}(K) \to G$ the quotient map. The action of $G$ on the cosets $\Sigma = G/G'$ admits an embedding $i: G \to S_\Sigma$.

As $S_\Sigma$ is regularly realizable (Theorem 2.2.4), Theorem 4.1.1 gives a homomorphism $\theta_0: \text{Gal}(K_0) \to S_\Sigma$ for which $\theta_0|_{\text{Gal}(K)} = i\theta$ (note that $\theta_0|_{\text{Gal}(K_0)} = \varphi\theta_0$ in the diagram above). In particular, the image $H$ of $\theta_0$ contains $i(G)$, and hence $H$ is transitive. Thus $(H : H') = |\Sigma| = [L : K]$, where $H'$ is the stabilizer in $H$ of the coset $G' \in \Sigma$. Also, as the subgroup $G' \leq G$ is the stabilizer in $G$ of the coset $G' \in \Sigma$, it follows that $H' \cap i(G) = i(G')$. Hence $i^{-1}(H') = G'$.

Let $M_0$ be the fixed field of $\ker(\theta_0)$ (i.e. $\text{Gal}(M_0) = \ker(\theta_0)$) and let $L_0 \subseteq M_0$ be the corresponding fixed field of $H'$ (i.e. $\text{Gal}(L_0) = \theta_0^{-1}(H')$). Since $i$ is an embedding and $\theta_0|_{\text{Gal}(K)} = i\theta$, we have

$$\text{Gal}(L) = \theta^{-1}(G') = \theta^{-1}i^{-1}(H') = \varphi^{-1}\theta_0^{-1}(H') = \varphi^{-1}(\text{Gal}(L_0)) = \text{Gal}(L_0K).$$

Hence $L = L_0K$. In addition, $L_0$ is linearly disjoint from $K$, since $[L_0 : K_0] = (H : H') = [L : K]$, as needed.

The case where $L/K$ is an infinite extension follows from Zorn’s Lemma. The main point is that for a tower of algebraic extensions $L_1 \subseteq L_2 \subseteq L_3$, $L_3/L_1$ is separable if and only if both $L_2/L_1$ and $L_3/L_2$ are. The details can be found in [27].

Remark 4.1.5. In the last proof $H'$ was the stabilizer of a point of a subgroup of $S_n$. This stabilizer is, in general, not normal even if $L/K$ is Galois. That is to say, $L_0/K_0$ need not be Galois, even if $L/K$ is.

Example 4.1.6. In this example we build a field tower $K_0 \subseteq K_1 \subseteq K$ such that $K/K_1$ is PAC, $K/K_0$ is not PAC, and $K_1/K_0$ is finite.

Let $K_0 = \mathbb{Q}_{\text{sol}}$ the maximal pro-solvable extension of $\mathbb{Q}$. Then $K_0$ has no extensions of degree 2. This imply that if $K$ is an extension of $K$ and there exists $L/K$ of degree 2, then $K/K_0$ is not PAC. Indeed, had $K/K_0$ PAC, we would have got $L_0/K_0$ of degree 2 such that $L = L_0K$ (Corollary 4.1.4), a contradiction.

Let $K_1/K_0$ be a finite proper extension and $e \geq 1$ an integer. Weissauer’s theorem [9] Theorem 13.9.1] asserts that $K_1$ is Hilbertian. Hence for almost all $\sigma \in \text{Gal}(K_1)^e$ the extension $K/K_1$ is PAC and the absolute Galois group of $K$ is free on $e$ generators, where $K = \mathbb{Q}(\sigma)$. In particular, $K$ has an extension of degree 2, so by the previous paragraph, $K/K_0$ is not PAC.
4.2 Restrictions on the Galois Structure of $K/K_0$

Not every extension can be a PAC extension. This section reveals some of the restrictions.

Warning: The reader is advised to recall the definitions and basic properties of wreath products appearing in Section 2.5 before continuing any further.

Lemma 4.2.1. Let $K/K_0$ be a proper Galois PAC extension. Then $K$ is separably closed.

Proof. Let $A$ be a Galois group over $K$ and let $\theta: \text{Gal}(K) \to A$ be a corresponding epimorphism. Let $G$ be a nontrivial finite quotient of $\text{Gal}(K/K_0)$ and $\varphi: \text{Gal}(K_0) \to G$ the restriction map. We may assume that $A \leq S_n$ for some large $n$. Define an embedding $i: A \to S_n \wr G$ by setting $i(a) = (f_a, 1)$, where $f_a(1) = a$ and $f_a(\sigma) = 1$ for any nontrivial $\sigma \in G$. As $S_n$ is regularly realizable over any field, $(\varphi, S_n \wr G \to G)$ is rational over $K_0$ (Corollary 2.5.10).

By the lifting property we can extend the solution $\theta$ of the lower embedding problem to a weak solution $(\theta, \theta_0)$ of the DEP

As $\text{Gal}(K)$ is normal in $\text{Gal}(K_0)$, we have that $i(A)$ is invariant under the image of $\text{Gal}(K_0)$. Let $1 \neq \sigma \in G$, extend it to $\hat{\sigma} \in \text{Gal}(K_0)$, and let $f\sigma = \theta_0(\hat{\sigma})$. Then $i(A) \cap i(A)^f = i(A) \cap i(A)^\sigma = 1$. But $i(A)$ is invariant under $f\sigma$; we thus get that $i(A) = 1$. Therefore $K$ is separably closed, as desired. 

Some Galois extensions of $\mathbb{Q}$ are known to be PAC as fields. So we get below PAC fields which are not PAC over any proper subfield.

Corollary 4.2.2. Let $K$ be a Galois extension of $\mathbb{Q}$ which is not algebraically closed. Then $K$ is a PAC extension of no proper subfield. This result holds even if $K$ is PAC. In particular it holds in the following cases.
(a) the Galois hull $K = \mathbb{Q}[\sigma]$ of $\mathbb{Q}$ in $\mathbb{Q}(\sigma)$, for almost all $\sigma \in \text{Gal}(\mathbb{Q})^e$ [9, Theorem 18.10.2].

(b) $K = \mathbb{Q}_{\text{tot}, R}(i)$, where $\mathbb{Q}_{\text{tot}, R}$ is the maximal real Galois extension of $\mathbb{Q}$ and $i^2 = -1$ [25].

(c) The compositum $K = \mathbb{Q}_{\text{sym}}$ of all Galois extensions of $\mathbb{Q}$ with a symmetric Galois group [9, Theorem 18.10.3].

Proof. If $K/\mathbb{Q}$ is Galois, then, since any subfield $K_0$ of $K$ contains $\mathbb{Q}$, $K/K_0$ is Galois and hence not PAC (as an extension).

A somewhat stronger result is the following.

**Theorem 4.2.3.** The Galois closure of an algebraic separable proper PAC extension is the separable closure.

Proof. Let $K/K_0$ be a proper PAC extension and let $N$ be its Galois closure. By Corollary 4.1.4, there exists a separable extension $N_0/K_0$, linearly disjoint from $K$ over $K_0$, such that $N = N_0K$. In particular, $N/N_0$ is a proper Galois PAC extension, which implies, by Lemma 4.2.1, that $N$ is separably closed.

Theorem 4.2.3 is a generalization of Chatzidakis’ result from 1986. Chatzidakis prove that if $K$ is a countable Hilbertian and $e \geq 1$ an integer, then for almost all $\sigma \in \text{Gal}(K)^e$ the field $K_s(\sigma)$ is a Galois extension of no proper subextension $K \subseteq K' \subsetneq K_s(\sigma)$. (To see this, recall that $K_s(\sigma)/K$ is PAC for almost all $\sigma$.)

Let us discuss finite PAC extensions. It is clear that every PAC field is a PAC extension of itself. Another trivial example for a finite PAC extension is $C/R$, where $C$ is the algebraic closure of a real closed field $R$. In this case $[C : R] = 2$ [21, VI §Corollary 9.3].

A family of finite PAC extensions are the purely inseparable ones: Let $K$ be a purely inseparable extension of a PAC field $K_0$. Then $K/K_0$ is PAC [17, Corollary 2.3]. Hence if $K/K_0$ is finite purely inseparable extension and $K_0$ is PAC, then $K/K_0$ is also a finite PAC extension.

The following result asserts that, in fact, these are all the finite PAC extensions.

**Corollary 4.2.4.** Let $K/K_0$ be a finite extension. Then $K/K_0$ is PAC if and only if either

(a) $K_0$ is real closed and $K$ is the algebraic closure of $K_0$, or

(b) $K_0$ is PAC and $K/K_0$ is purely inseparable.
Proof. Let $K_1$ be the maximal separable extension of $K_0$ contained in $K$. Then $K/K_1$ is purely inseparable [21 V§6 Proposition 6.6]. By [17 Corollary 2.3] $K_1/K_0$ is PAC, and in particular $K_1$ is a PAC field. If $K_1 = K_0$, we are done.

Assume $K_1 \neq K_0$. By Theorem 4.2.3, the Galois closure $N$ of $K_1/K_0$ is the separable closure. Hence, by Artin-Schreier Theorem 21 Corollary 9.3 $N$ is, in fact, algebraically closed and $K_0$ is real closed (recall that $1 < [N : K_0] < \infty$). In particular, the characteristic of $K$ is 0, and hence $K_1 = K$. \qed
Chapter 5
Projective Pairs

In this chapter we define the group theoretic analog of PAC extensions – projective pairs. This analogy motivates the study of these pairs.

This study influences the understanding of PAC extension in two ways. First we have a characterization of PAC extensions of PAC fields in term of projective pairs.

**Proposition 5.0.5.** (a) Let $M$ be a PAC extension of a PAC field $K$. Then the pair $(\text{Gal}(M), \text{Gal}(K))$ is projective.

(b) Let $M$ be an algebraic extension of a PAC field $K$. Then $M/K$ is PAC if and only if the restriction map $(\text{Gal}(M), \text{Gal}(K))$ is projective.

(c) Let $(\Gamma, \Lambda)$ be a projective pair. Then there exists a separable algebraic PAC extension $M$ of a PAC field $K$ such that $\Gamma \cong \text{Gal}(M)$, $\Lambda \cong \text{Gal}(K)$.

(See the definition of a projective pair and the proof of this result in the body of the chapter.)

Although we cannot directly translate the results on projective pairs to results on PAC extensions (of a non-PAC field), the ideas from the study of the former do apply to the latter. This explains the extensive use of group theoretic methods which appears in the previous chapters. In particular, the key property, the lifting property, comes from group theoretic considerations.

Using Proposition 5.0.5 we can transfer result about PAC extensions to result about projective pairs. Nevertheless in this chapter we give direct proofs without going via Proposition 5.0.5 for two reasons.

First the group theoretic group are usually simpler and sometimes give a stronger results. The second reason is that we work in a slightly more general setting. Namely we
work in the category of pro-\(C\) groups, where \(C\) is a Melnikov formation of finite groups (see below).

5.1 The Basic Properties of Projective Pairs

5.1.1 Definitions

Throughout this chapter we fix a Melnikov formation \(C\) of finite groups. That means that \(C\) is closed under fiber products and given a short exact sequence

\[
1 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 1
\]

we have that \(A, C \in C\) if and only if \(B \in C\). In particular, \(C\) is closed under direct products. The exact sequence

\[
1 \longrightarrow A^{(G:G_0)} \longrightarrow A \text{wr}_{G_0} G \longrightarrow G \longrightarrow 1
\]

implies that \(A, G \in C\) if and only if \(A \text{wr}_{G_0} G \in C\).

The following three families are examples of Melnikov formations. The family of all finite groups; the family of all \(p\)-groups, for a prime \(p\); the family of all solvable groups.

Let \(\Gamma \leq \Lambda\) be pro-\(C\) groups. A \(C\)-DEP for the pair \((\Gamma, \Lambda)\) is a commutative diagram

(5.1)

where \(G, H, A, B \in C\), \(A \leq B, G \leq H\), \(i, j, \varphi\) are the inclusion maps, and \(\alpha, \mu, \beta, \nu\) are surjective. Therefore a \(C\)-DEP consists of two compatible \(C\) embedding problems: the lower embedding problem for \(\Gamma\) and the higher embedding problem for \(\Lambda\).

In case \(C\) is the family of all finite groups, we omit the \(C\) notation and say that (5.1) is a double embedding problem (as in (3.2)). Sometimes we abbreviate (5.1) and write \(((\mu, \alpha), (\nu, \beta))\).

A \(C\)-DEP is said to be split if the higher embedding problem splits, i.e., in (5.1) there exists \(\beta' : B \rightarrow H\) for which \(\beta \beta' = \text{id}_B\). If, in addition, the lower embedding problem splits, then we call the DEP doubly split.
If we allow the groups $G, H, A, B$ to be pro-$C$, then (5.1) is a pro-$C$-DEP.

Giving a weak solution $\eta: \Gamma \to G$ of the lower embedding problem and a weak solution $\theta: \Lambda \to H$ of the higher embedding problem, we say that $(\eta, \theta)$ is a weak solution of (5.1) if $\eta$ and $\theta$ are compatible, i.e. $j\eta = \theta\varphi$.

Note that a weak solution of (5.1) is completely determined by $\theta$. Indeed, $\theta$ induces a solution $\eta$ of the C-DEP if and only if $\theta(\Gamma) \leq G$, and then $\eta = \theta|_{\Gamma}$.

**Definition 5.1.1.** A pair $(\Gamma, \Lambda)$ of pro-$C$ groups is called $C$-projective if any $C$-DEP is weakly solvable.

Let us begin with some basic properties of projective pairs. The first is trivial.

**Proposition 5.1.2.** A pro-$C$ group $\Lambda$ is $C$-projective if and only if the pair $(1, \Lambda)$ is $C$-projective.

**Lemma 5.1.3.** Consider a $C$-DEP (5.1) for a pair $(\Gamma, \Lambda)$ of pro-$C$ groups. Assume that both the higher and lower embedding problems are weakly solvable. Then (5.1) is dominated by a doubly split $C$-DEP.

**Proof.** Let $\theta: \Lambda \to H$ be a weak solution of the higher embedding problem and $\eta: \Gamma \to G$ a weak solution of the lower embedding problem. Choose an open normal subgroup $N \leq \Lambda$ such that $N \leq \ker(\theta)$ and $\Gamma \cap N \leq \ker(\eta)$.

Let $\hat{B} = \Lambda/N$, $\hat{A} = \Gamma/\Gamma \cap N$ and let $\hat{H} = H \times_B \hat{B}$, $\hat{G} = G \times_A \hat{A}$. Then the following commutative diagram defines a dominating $C$-DEP.

\[
\begin{array}{ccc}
\Gamma & \rightarrow & \Lambda \\
\downarrow & & \downarrow \\
\hat{G} & \rightarrow & \hat{A} \\
\downarrow & & \downarrow \\
G & \overset{\alpha}{\rightarrow} & A \\
\hat{\Lambda} & \rightarrow & \hat{B} \\
\downarrow & & \downarrow \\
\hat{H} & \overset{\beta}{\rightarrow} & B \\
\end{array}
\]

(Here all the maps are canonically defined.) Lemma 2.3.2 implies that this $C$-DEP doubly splits.

**Corollary 5.1.4.** Let $(\Gamma, \Lambda)$ be a pair of pro-$C$ groups and suppose that $\Lambda$ is $C$-projective. Then $(\Gamma, \Lambda)$ is $C$-projective if and only if every doubly split $C$-DEP is weakly solvable.

**Proof.** Since $\Lambda$ is $C$-projective, $\Gamma$ is also $C$-projective. In other words, every finite embedding problem for $\Lambda$ (resp. $\Gamma$) is weakly solvable. Lemma 5.1.3 implies that every $C$-DEP for $(\Gamma, \Lambda)$ is dominated by a doubly split $C$-DEP.

The converse is trivial.

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Proposition 5.1.5. Let $(\Gamma, \Lambda)$ be a $C$-projective pair. Then any pro-$C$-DEP for $\varphi$ is weakly solvable.

Proof. We follow the proof of [9, Lemma 22.3.2]. In order to solve pro-$C$-DEPs for $(\Gamma, \Lambda)$ we need to solve more general pro-$C$-DEPs, in which the maps of the higher embedding problem are not necessarily surjective.

In case of $C$-DEPs, we can solve such $C$-DEPs. Indeed, assume that in (5.1) $\nu, \mu$ are not surjective. First $\ker(\alpha), \ker(\beta) \in C$ since $C$ is a Melnikov formation. Next $\nu(\Gamma), \mu(\Lambda) \in C$ since $\Gamma, \Lambda$ are pro-$C$. Finally $\alpha^{-1}(\nu(\Gamma)), \beta^{-1}(\mu(\Lambda)) \in C$, again since $C$ is a Melnikov formation.

Replace $A, B$ with $\nu(\Gamma), \mu(\Lambda)$ and $G, H$ with $\alpha^{-1}(\nu(\Gamma)), \beta^{-1}(\mu(\Lambda))$. In this new $C$-DEP all the maps are surjective. So by assumption there is a weak solution.

Let us move to the more general case of pro-$C$-DEP: Consider a pro-$C$-DEP (5.1) and write $K = \ker(\beta)$. We prove the assertion in two steps.

**Step A: Finite Kernel.** Assume $K$ is finite. Then $G$ is open in $KG$ since $(KG : G) \leq |K|$. Choose an open normal subgroup $U \leq H$ for which $U \cap KG \leq G$ and $K \cap U = 1$ (note that $K$ is finite and $H$ is Hausdorff). Then $U \cap KG = U \cap G$. By the second isomorphism theorem (in the group $UG$) we have that $(KG \cap UG : G) = (U \cap (KG \cap UG) \cap G) = (U \cap KG : U \cap G) = 1$, i.e.

\[(5.2) \quad (KG) \cap (UG) = G.\]

Write $\bar{H} = H/U$, let $\pi: H \to \bar{H}$ be the quotient map, $\bar{G} = \pi(G)$, $\bar{B} = B/\beta(U)$, $\bar{A} = A/A \cap \beta(U)$ and $\bar{\beta}: H \to \bar{B}$, $\bar{\alpha}: G \to \bar{A}$ the epimorphisms induced from $\beta, \alpha$, respectively.

Since $\bar{H} \in C$, there is a homomorphism $\bar{\varphi}: \Lambda \to \bar{H}$ with $\bar{\varphi}(\Gamma) \leq \bar{G}$ (let $\bar{\eta} = \bar{\varphi}|_\Gamma$) for which

\[
\begin{array}{ccc}
G & \xto{\alpha} & A \\
\downarrow \eta & & \downarrow \alpha \\
G & \xto{\bar{\alpha}} & \bar{A}
\end{array}
\quad \quad \begin{array}{ccc}
1 & \xto{\bar{\alpha}} & K \\
\downarrow \bar{\alpha} & & \downarrow \beta \\
1 & \xto{\bar{\alpha}} & \bar{A}
\end{array}
\quad \quad \begin{array}{ccc}
\Lambda & \xto{\mu} & B \\
\downarrow \mu & & \downarrow \bar{\beta} \\
\Lambda & \xto{\mu} & B
\end{array}
\]

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are commutative diagrams. The right square in the right embedding problem is a cartesian square, since \( K \cap U = 1 \) ([9, Example 22.2.7(c)]). Hence we can lift \( \bar{\theta} \) to \( \theta: \Lambda \to H \) such that \( \beta \theta = \mu \) (see [9, Lemma 22.2.1]). We claim that \( \theta(\Gamma) \leq G \). Indeed,

\[
A \geq \mu(\Gamma) = \beta(\theta(\Gamma)),
\]
hence \( \theta(\Gamma) \leq K\beta^{-1}(A) = K\alpha^{-1}(A) = KG \). Also,

\[
\bar{G} \geq \bar{\theta}(\Gamma) = \pi(\theta(\Gamma)),
\]
hence \( \theta(\Gamma) \leq UG \). Then, from (5.2) we have \( \theta(\Gamma) \leq (KG) \cap (UG) = G \), as claimed.

**Step B: The General Case.** We use Zorn’s Lemma. Consider the family of pairs \( (L, \theta) \) where \( L \subseteq K \) is normal in \( H \), \( \theta \) is a weak solution of the following embedding problem, and \( \theta(\Gamma) \subseteq GL/L \).

\[
\Lambda \quad \quad \quad \quad \quad \theta
\]
\[
1 \longrightarrow K/L \longrightarrow H/L \quad \beta \quad \quad \quad \quad \quad B \quad \longrightarrow 1.
\]
We say that \( (L, \theta) \leq (L', \theta') \) if \( L \subseteq L' \) and

\[
\Lambda
\]
\[
H/L \quad \theta
\]
\[
H/L' \longrightarrow B
\]

is commutative. For a chain \( \{(L_i, \theta_i)\} \) we define a lower bound \( (L, \theta) \) by \( L = \bigcap_i L_i \) and \( \theta = \lim \theta_i \) (note that \( \theta(\Gamma) \subseteq GL/L \) by [9, Lemma 1.2.2(b)]). By Zorn’s Lemma there exists a minimal element \( (L, \theta) \) in the family. We claim that \( L = 1 \). Otherwise, there is an open normal subgroup \( U \) of \( H \) with \( L \not\subseteq U \). Part A gives (since \( L/U \cap L \) is finite) a weak solution \( \theta' \) of the following embedding problem such that \( \theta'(\Gamma) \subseteq G(U \cap L)/(U \cap L) \).

\[
1 \longrightarrow L/U \cap L \longrightarrow H/U \cap L \longrightarrow H/L.
\]

Hence \( (L, \theta) \) is not minimal. This contradiction implies that \( L = 1 \), as claimed.
Recall that a pro-$\mathcal{C}$ group $\Lambda$ is projective if and only if any short exact sequence of pro-$\mathcal{C}$ groups
\[
1 \longrightarrow K \longrightarrow \Delta \longrightarrow \Lambda \longrightarrow 1
\]
splits. Similar characterization is given in the next result for a pair $(\Gamma, \Lambda)$ of pro-$\mathcal{C}$ groups.

**Corollary 5.1.6.** Let $(\Gamma, \Lambda)$ be a pair of pro-$\mathcal{C}$ groups. Then $(\Gamma, \Lambda)$ is $\mathcal{C}$-projective if and only if the rows of any exact commutative diagram of pro-$\mathcal{C}$ groups

\[
\begin{array}{ccc}
\Delta & \xrightarrow{\beta} & \Lambda \\
\alpha \downarrow & & \beta' \downarrow \\
E & \xrightarrow{\alpha'} & \Gamma \\
\eta \downarrow & & \downarrow \\
1 & & 1
\end{array}
\]
compatibly split. That is to say, there exists a splitting $\beta': \Lambda \to \Delta$ of $\beta$ (i.e. $\beta\beta' = \text{id}$) such that $\beta'\varphi(\Gamma) \leq \psi(E)$ and $\alpha'$ defined by $\psi\alpha' = \beta'\varphi$ is a splitting of $\alpha$.

**Proof.** Since the splittings of an epimorphism $\gamma: M \to N$ correspond bijectively to solutions of the embedding problem $(\text{id}: N \to N, \gamma: M \to N)$, the assertion follows immediately from Proposition 5.1.5.

**5.1.2 Basic Properties and Characterizations**

**Proposition 5.1.7 (The lifting property).** Let $(\Gamma, \Lambda)$ be $\mathcal{C}$-projective and consider a pro-$\mathcal{C}$-DEP for $(\Gamma, \Lambda)$. Then any weak solution $\eta$ of the lower embedding problem can be lifted to a weak solution $(\hat{\eta}, \hat{\theta})$ of the DEP.

**Proof.** Let (5.1) be a pro-$\mathcal{C}$ DEP and let $\eta: \Gamma \to G$ be a weak solution of the lower embedding problem. Define $\hat{H} = H \times_B \Lambda$ and let $\hat{G} = \{(\eta(\gamma), \gamma) \mid \gamma \in \Gamma\} \leq G \times_A \Gamma$. Since $\hat{G} \cong \Gamma$, there is a unique weak solution of the lower embedding problem of

\[
\begin{array}{ccc}
\hat{G} & \xrightarrow{\hat{\alpha}} & \Gamma \\
\downarrow \text{id} & & \downarrow \text{id} \\
\hat{H} & \xrightarrow{\hat{\beta}} & \Lambda
\end{array}
\]

Now by Proposition 5.1.5 there exists a weak solution $(\hat{\eta}, \hat{\theta})$. Hence $\hat{\eta}(\gamma) = \hat{\alpha}^{-1}(\gamma) = (\eta(\gamma), \gamma)$. This implies that $(\eta, \theta)$ is a solution of the DEP, where $\theta = \pi\hat{\theta}$ and $\pi: \hat{H} \to H$ is the quotient map. 

\[\square\]
The next result follows from the lifting property using the same argument that implied Corollary 5.1.6 from Proposition 5.1.5.

**Corollary 5.1.8.** Let \((\Gamma, \Lambda)\) be a \(\mathcal{C}\)-projective pair and consider a diagram as in Corollary 5.1.6. Then any splitting \(\alpha'\) of \(\alpha\) can be lifted to a splitting \(\beta'\) of \(\beta\).

**Proposition 5.1.9 (Transitivity).** Let \(\Lambda_3 \leq \Lambda_2 \leq \Lambda_1\) be pro-\(\mathcal{C}\) groups. Then

(a) If \((\Lambda_3, \Lambda_1)\) is \(\mathcal{C}\)-projective, then \((\Lambda_3, \Lambda_2)\) is \(\mathcal{C}\)-projective.

(b) If \((\Lambda_3, \Lambda_2)\) and \((\Lambda_2, \Lambda_1)\) are \(\mathcal{C}\)-projective, then so is \((\Lambda_3, \Lambda_1)\).

**Proof.** To show (a) it suffices to solve a doubly split \(\mathcal{C}\)-DEP

\[
\begin{array}{ccc}
\Lambda_3 & \xrightarrow{\varphi_3} & \Lambda_2 \\
\downarrow \alpha_3 & & \downarrow \varphi_2 \\
G_3 & \xrightarrow{} & K_2 \\
\end{array}
\]

\[
\begin{array}{ccc}
1 & \xrightarrow{} & G_2 \\
\downarrow \alpha_2 & & \downarrow \varphi_2 \\
A_2 & \xrightarrow{} & 1 \\
\end{array}
\]

for \((\Lambda_3, \Lambda_2)\). Fix a splitting of \(\alpha_2\) to identify \(G_2\) and \(K_2 \rtimes A_2\). We can assume that \(\ker(\varphi_2) = \Lambda_2 \cap L\), where \(L\) is open normal subgroup of \(\Lambda_1\). (Otherwise, we can choose an open normal subgroup \(L\) of \(\Lambda_1\) such that \(\ker(\varphi_2) \geq \Lambda_2 \cap L\) and replace \(A_i\) with \(\Lambda_i / (\Lambda_i \cap L)\) and \(G_i\) with the corresponding fiber product, \(i = 2, 3\).)

Let \(A_1 = \Lambda_1 / L\), let \(\varphi_1: \Lambda_1 \to A_1\) be the quotient map. Embed \(G_3\) inside \(K_2 \wr A_2 A_1\) as in Proposition 2.5.5. In particular, under the Shapiro map

\[
\pi: \text{Ind}^{A_1}_{A_2}(K_2) \rtimes A_2 \to K_2 \rtimes A_2
\]

we have \(\pi(G_3) = G_3\). The \(\mathcal{C}\)-DEP

\[
\begin{array}{ccc}
\Lambda_3 & \xrightarrow{\varphi_3} & \Lambda_1 \\
\downarrow \alpha_3 & & \downarrow \varphi_1 \\
G_3 & \xrightarrow{} & K_2 \wr A_2 A_1 \\
\end{array}
\]

has a weak solution \((\eta_3, \eta_1')\) since \((\Lambda_3, \Lambda_1)\) is \(\mathcal{C}\)-projective. Let \(\eta_i = \pi \eta_i'|_{\Lambda_i}, i = 2, 3\). Then \((\eta_3, \eta_2)\) is a weak solution of (5.3) and the proof of (a) is concluded.

The proof of (b) is analogous to the proof of Lemma 3.4.7, one merely need to disregard the word ‘geometric’ everywhere it appears.
For the sake of completeness we give a proof based on exact sequences. Consider the following commutative diagram with injective rows.

\[
\begin{array}{ccc}
\Delta_1 & \xrightarrow{\alpha_1} & \Lambda_1 & \xrightarrow{} & 1 \\
\Delta_2 & \xrightarrow{\alpha_2} & \Lambda_2 & \xrightarrow{} & 1 \\
\Delta_3 & \xrightarrow{\alpha_3} & \Lambda_3 & \xrightarrow{} & 1
\end{array}
\]

By Corollary 5.1.6 it suffices to show that the higher and lower rows compatibly split. As \((\Lambda_3, \Lambda_2)\) is \(C\)-projective, the two bottom rows compatibly split, let \(\alpha'_3, \alpha'_2\) be the corresponding sections. Now by Corollary 5.1.8, \(\alpha'_2\) extends to a section \(\alpha'_1\) of \(\alpha_1\). Thus \(\alpha'_3, \alpha'_1\) are compatible sections, as needed.

Remark 5.1.10. In the above proposition it might happen that \((\Lambda_3, \Lambda_1)\) is \(C\)-projective but \((\Lambda_2, \Lambda_1)\) is not. For example, take \(\Lambda_1\) to be \(C\)-projective, take \(1 \neq \Lambda_2 \triangleleft \Lambda_1\) and \(\Lambda_3 = 1\). Then \((1, \Lambda_1)\) is \(C\)-projective (Proposition 5.1.2) while if \(\Lambda_2 \neq \Lambda_1\), then \((\Lambda_2 \leq \Lambda_1)\) is not (Proposition 5.3.2).

Projective pairs behave well under taking subgroups.

**Proposition 5.1.11.** Let \((\Gamma, \Lambda)\) be a \(C\)-projective pair, let \(\Lambda_0 \leq \Lambda\) be a subgroup, and write \(\Gamma_0 = \Lambda_0 \cap \Gamma\). Then \((\Gamma_0, \Lambda_0)\) is \(C\)-projective.

**Proof.** By Lemma 5.1.3 it suffices to show that any doubly split \(C\)-DEP for \((\Gamma_0, \Lambda_0)\) is weakly solvable. Let

\[((\mu_1: \Gamma_0 \to A_1, \alpha_1: G_1 \to A_1), (\nu_1: \Lambda_0 \to B_1, \beta_1: H_1 \to B_1))\]

be a doubly split \(C\)-DEP for \((\Gamma_0, \Lambda_0)\).

The case where \(\Lambda_0\) is open in \(\Lambda\). Choose an open normal subgroup \(N\) of \(\Lambda\) such that \(N \leq \ker \nu_1\). Then \(N \cap \Gamma_0 \leq \ker(\mu_1)\). Let \(\nu: \Lambda \to B = \Lambda/N\) be the quotient map. Let \(B_0\), \(A\), and \(A_0\) be the respective images of \(\Lambda_0\), \(\Gamma\), and \(\Gamma_0\) under \(\nu\) and let \(\nu_0\), \(\mu\), and \(\mu_0\) be the restrictions of \(\nu\) to the respective subgroups. Let \(G_0 = G_1 \times_{A_1} A_0\), \(H_0 = H_1 \times_{B_1} B_0\) and let \(\alpha_0: G_0 \to A_0\) and \(\beta_0: H_0 \to B_0\) be the projection maps. Then the doubly split \(C\)-DEP \(((\mu_0, \alpha_0), (\nu_0, \beta_0))\) dominates \(((\mu_1, \alpha_1), (\nu_1, \beta_1))\) (Lemma 2.3.2). Hence it suffices to weakly solve \(((\mu_0, \alpha_0), (\nu_0, \beta_0))\).
Clearly $A_0 \leq B_0 \cap A$. Let $\bar{x} \in B_0 \cap A$. Then $\bar{x} = \nu(x)$, where $x = yn$, $x \in \Lambda_0$, $y \in \Gamma$, and $n \in N$. Since $N \leq \Lambda_0$ we get that $y = xn^{-1} \in \Lambda_0$, hence $y \in \Gamma_0$. Since $\nu(y) = \nu(x) = \bar{x}$ we have $\nu(\Gamma_0) \geq \nu(\Lambda_0) \cap \nu(\Gamma)$. Consequently $A_0 = B_0 \cap A$.

If $\alpha'_0: A_0 \to G_0$ is a splitting of $\alpha_0$, then it suffices to find a weak solution $(\eta_0, \theta_0)$ such that $\eta_0(\Gamma_0) = \alpha'_0(A_0)$. Therefore, we can replace $G_0$ by $\alpha'_0(A_0)$ to assume that $G_0 \cong A_0$.

Let $\beta'_0: B_0 \to H_0$ be a splitting of $\beta_0$. Identify $H_0$ with $K \times B_0$ via $\beta'_0$, where $K = \ker \beta_0$. This identification induces an embedding $i: A_0 \to K_0 \times B_0$ satisfying $\beta_0i = \alpha_0$.

Let $\beta: K \wr_{B_0} B \to B$ be the quotient map and $\pi: \text{Ind}^B_{B_0}(K) \times B_0 \to K \times B_0$ the corresponding Shapiro map. Proposition 2.5.5 gives an embedding $j: A \to K \wr_{B_0} B$ such that $\pi j|A_0 = i$, and in particular $\beta j(a) = a$, for all $a \in A$. Let $\alpha = \beta|j(A): j(A) \to A$.

Since $(\Gamma, \Lambda)$ is $C$-projective, there exists a weak solution $(\eta, \theta)$ of $((\mu, \alpha), (\nu, \beta))$. Set $\eta_0 = \pi \eta|\Gamma_0$ and $\theta_0 = \pi \theta|A_0$. Then $(\eta_0, \theta_0)$ is a weak solution of $((\mu_0, \alpha_0), (\nu_0, \beta_0))$.

The general case. Fix an open normal subgroup $N$ of $\Lambda$ such that $N \cap \Lambda_0 \leq \ker \nu_1$. Then $N \cap \Gamma_0 \leq \ker (\mu_1)$. We can extend $\nu_1$ to $\Lambda_0N$ by setting $\nu_1(\lambda n) = \nu_1(\lambda)$ for each
\[ \lambda \in \Lambda_0 \text{ and } n \in N. \] Similarly, \( \Gamma_0(N \cap \Gamma) \) is an open subgroup of \( \Gamma \) and \( \mu_1 \) extends to it. Note that the restriction of \( \nu_1 \) to \( \Gamma_0(N \cap \Gamma) \) equals \( \mu_1 \).

By [9, Lemma 1.2.2(b)] we have an open subgroup \( \Lambda' \) of \( \Lambda \) that contains \( \Lambda_0N \) and such that \( \Lambda' \cap \Gamma \leq \Gamma_0(N \cap \Gamma) \). Let \( \Gamma' = \Lambda' \cap \Gamma \). Now we have the \( C \)-DEP

\[
(\mu_1 : \Gamma' \to A_1, \alpha_1 : G_1 \to A_1), (\nu_1 : \Lambda' \to B_1, \beta_1 : H_1 \to B_1)
\]

for \((\Gamma', \Lambda')\). By the first part \((\Gamma', \Lambda')\) is \( C \)-projective, hence there exists a weak solution \((\eta, \theta)\) of this DEP. Now if we set \( \eta_0 = \eta|_{\Gamma_0} \) and \( \theta_0 = \theta|_{\Lambda_0} \), we get the weak solution \((\eta_0, \theta_0)\) of the original DEP.

**Remark 5.1.12.** The analogous result for PAC extensions – if \( M/K \) is a PAC extension and \( L/K \) algebraic, then \( ML/L \) is PAC – is proved with a descent argument (see [17, Lemma 2.1]). This has suggested us that wreath product should be the tool for this group theoretic version.

**Corollary 5.1.13.** Let \((\Gamma, \Lambda)\) be a \( C \)-projective pair and let \( N \leq \Gamma \). Then there exists \( M \leq \Lambda \) such that \( N = \Gamma \cap M \) and \( \Gamma \cdot M = \Lambda \). Moreover, if \( N \triangleleft \Gamma \), then \( M \triangleleft \Lambda \).

**Proof.** Let \( \tilde{N} = \bigcap_\sigma N^\sigma \) and let \( \eta : \Gamma \to \Gamma/\tilde{N} \) be the natural quotient map. Lift \( \eta \) to a solution \((\eta, \theta)\) of the DEP

\[
((\Gamma \to 1, \Gamma/\tilde{N} \to 1), (\Lambda \to 1, \Gamma/\tilde{N} \to 1)).
\]

Let \( \hat{M} = \ker(\theta) \) and \( M = \theta^{-1}(N/\tilde{N}) \). Then (since \( \eta = \theta|_{\Gamma} \))

\[
N = \eta^{-1}(N/\tilde{N}) = \Gamma \cap \theta^{-1}(N/\tilde{N}) = \Gamma \cap M.
\]

Since \( \theta(\Gamma) = \Gamma/\tilde{N} \), it follows that \( \Gamma \hat{M} = \Lambda \), and in particular, \( \Gamma \cdot M = \Lambda \).

To conclude the proof, note that if \( N \triangleleft \Gamma \), then \( N = \tilde{N} \), and hence \( M = \hat{M} \). So \( M \triangleleft \Lambda \), as needed.

Taking \( N = 1 \) in the above result we get the following splitting corollary.

**Corollary 5.1.14.** If \((\Gamma, \Lambda)\) is \( C \)-projective, then \( \Lambda \cong M \rtimes \Gamma \) for some \( M \triangleleft \Lambda \).

### 5.2 Families of Projective Pairs

In this section, for simplicity, we assume that \( C \) is the family of all finite groups.
5.2.1 Free Products

Let $\Gamma_1, \Gamma_2$ be profinite groups. The (profinite) free product $\Gamma = \Gamma_1 * \Gamma_2$ is defined by the following universal property. Let $(\mu: \Gamma \to A, \alpha: H \to A)$ be an arbitrary embedding problem for $\Lambda$. Let $A_1 = \mu(\Gamma_1), A_2 = \mu(\Gamma_2), H_1 = \alpha^{-1}(A_1), \text{and } H_2 = \alpha^{-1}(A_2)$. Also let $\mu_1, \mu_2$ the respective restriction of $\mu$ to $\Gamma_1, \Gamma_2$ and $\alpha_1, \alpha_2$ the respective restriction of $\alpha$ to $H_1, H_2$. Then for every weak solutions $\theta_1, \theta_2$ of $(\mu_1, \alpha_1)$ and $(\mu_2, \alpha_2)$ there exists a unique lifting $\theta$ to a solution of $(\mu, \alpha)$.

**Proposition 5.2.1.** Let $\Gamma$ be a free factor of a projective group $\Lambda$. Then $(\Gamma, \Lambda)$ is projective.

**Proof.** By assumption $\Lambda = \Gamma * N$. The subgroups $\Gamma, N$ are projective as closed subgroups of a projective group. Consider a finite DEP (5.1). Let $A_1 = \nu(N)$ and $G_1 = \beta^{-1}(A_1)$. Let $\eta_1, \eta$ be respective weak solutions of $(\mu_1, \alpha_1)$ and $(\mu, \alpha)$, where $\mu_1 = \nu|_N$ and $\alpha_1 = \beta|_{G_1}$.

By the definition of free products, we can lift $\eta_1, \eta$ to a weak solution $\theta$ of $(\nu, \beta)$. Hence $(\eta, \theta)$ is a weak solution of (5.1). \hfill \Box

The following result appears in [15].

**Lemma 5.2.2** (Haran and Lubotzky). Let $\kappa$ be a cardinal and let $P$ be a projective profinite group of rank $\leq \kappa$. Then $\hat{F}_\kappa \cong P * \hat{F}_\kappa$.

As a consequence we get a family of examples of projective pairs inside a free group.

**Corollary 5.2.3.** Let $\Lambda$ be a free profinite group of infinite rank $\kappa$. Then for any projective group $\Gamma$ of rank $\leq \kappa$ there exists an embedding of $\Gamma$ in $\Lambda$ such that $(\Gamma, \Lambda)$ is projective.

5.2.2 Random Subgroups

Let us start by fixing some notation. We write $e$-tuples in bold face letters, e.g., $b = (b_1, \ldots, b_e)$. For a homomorphism of profinite groups $\beta: H \to B$, we write that $\beta(h) = b$ if $\beta(h_i) = b_i$ for all $i = 1, \ldots, e$. Let $C$ be a coset of a subgroup $N^e$ in $H^e$, where $N \leq \ker(\beta)$. By abuse of notation we write $\beta(C)$ for the unique element $b \in B^e$ such that $\beta(c) = b$ for every $c \in C$.

**Proposition 5.2.4.** Let $\Lambda = \hat{F}_\omega$. Then for almost all $\sigma \in \Lambda^e$, $(\langle \sigma \rangle, \Lambda)$ is projective.

**Proof.** As a profinite group, $\Lambda$ is equipped with a normalized Haar measure $m$. Let

\[
\begin{array}{ccc}
\Lambda & \xrightarrow{\mu} & H \\
\downarrow \rho & \searrow \beta & \downarrow \text{in}
\end{array}
\]

\[
\begin{array}{ccc}
H & \xrightarrow{\beta} & B \\
\end{array}
\]
be a finite embedding problem for $\Lambda$, let $b \in B^e$, let $A = \langle b \rangle$ be the subgroup of $B$ generated by $b$, and let $h \in H^e$ be such that $\beta(h) = b$. Define $\Sigma = \Sigma(b, h, \mu, \beta) \subseteq \Lambda^e$ to be the following set.

$$\Sigma = \{ \sigma \in \Lambda^e \mid (\mu(\sigma) = b) \Rightarrow (\exists \theta : \Lambda \to H, (\beta \theta = \mu) \land (\theta(\sigma) = h)) \},$$

that is to say, all $\sigma \in \Lambda^e$ such that there exists a weak solution $\theta$ of \[5.4\] with $\theta(\sigma) = h$, provided $\mu(\sigma) = b$. Note that $\Sigma = (\Sigma \cap C) \cup (\Lambda^e \setminus C)$, where $C$ is the coset of $\ker(\mu)^e$ in $\Lambda^e$ for which $\mu(C) = b$.

We break the proof into three parts. In the first two we show that $m(\Sigma \cap C) = m(C)$, and hence $m(\Sigma) = 1$.

**Part A: Construction of solutions.** Let

$$\Delta = \{ (b_i) \in H^N \mid \beta(b_i) = \beta(b_j) \ \forall i, j \in N \}.$$

It is equipped with canonical projections $\pi_i : \Delta \to H_i$, $i \in N$. Set $\hat{\beta} : \Delta \to B$ by $\hat{\beta}(x) = \beta \pi_i(x)$, $x \in \Delta$. Note that this definition is independent of $i$ and $\hat{\beta}$ is an epimorphism.

Let $\theta : \Lambda \to \Delta$ be a solution of $(\mu : \Lambda \to B, \hat{\beta} : \Delta \to B)$ (for the existence of $\theta$ see [28, Theorem 3.5.9]). Then for every $i \in N$ the map $\theta_i = \pi_i \theta$ is a solution of \[5.4\]. Moreover, by Lemma 2.3.4 the set $\{ \ker(\theta_i) \}$ is an independent set of subgroups of $\ker(\mu)$.

**Part B: Calculating $m(\Sigma)$.** For each $i \in N$ take the coset $X_i$ of $\ker(\theta_i)^e$ with $\theta_i(X_i) = h$. Then, since

$$\mu(X_i) = \beta \theta_i(X_i) = \beta(h) = b,$$

it follows that $X_i \subseteq C$. Moreover, from it follows Part A that $\{ X_i \mid i \in N \}$ is an independent set in $C$.

By Borel-Cantelli Lemma [3, Lemma 18.3.5], $\sum_i m_C(X_i) = \sum_i \frac{|H|^e}{|H|^e} = \infty$ implies that $m_C(X) = 1$, where $m_C$ is the normalized Haar measure on $C$ and $X = \bigcap_{i=1}^{\infty} \cup_{j=i}^{\infty} X_i$. So it suffices to show that $X \subseteq \Sigma$.

Indeed, let $\sigma \in X$. Then $\sigma \in X_i$ for some $i$. It implies that $\theta_i$ is a solution of \[5.4\] and that $\theta_i(\sigma) = h$. Hence $\sigma \in \Sigma$ and $X \subseteq \Sigma$, as desired.

**Part C: Conclusion.** Let $\Upsilon$ be the intersection of all $\Sigma(b, h, \mu, \beta)$. Since there are only countably many of them and each is of measure 1, we have $m(\Upsilon) = 1$. Let $\sigma \in \Upsilon$ and let $\Gamma = \langle \sigma \rangle$.

Then $\Gamma \leq \Lambda$ is projective. Indeed, consider a double embedding problem as in \[5.1\] and choose $h \in G$ such that $\beta(h) = \mu(\sigma)$. Then, since $\sigma \in \Sigma(\mu(\sigma), h, \mu, \beta)$, there exists an homomorphism $\theta : \Lambda \to H$ such that $\theta(\Gamma) = \langle \theta(\sigma) \rangle = \langle h \rangle \leq G$. \qed
Remark 5.2.5. In the above theorem we actually prove that for almost all $\sigma \in \Lambda^e$ the pair $((\sigma), \Lambda)$ has the following strong lifting property. For any embedding problem (5.1) and for any $h \in G^e$ that satisfies $\alpha(h) = \mu(\sigma)$ there exists a weak solution $\theta: \Lambda \rightarrow B$ with $\theta(\sigma) = h$.

5.3 Restrictions on Projective Pairs

We prove the analogs for projective pairs of PAC extensions properties.

Lemma 5.3.1. Let $(\Gamma, \Lambda)$ be a $C$-projective pair and assume that $\Gamma \lhd \Lambda$. Then either $\Gamma = 1$ or $\Gamma = \Lambda$.

Proof. Let $N \lhd \Gamma$ be an open normal subgroup of $\Gamma$ and let $D \lhd \Lambda$ be an open normal subgroup of $\Lambda$ such that $\Gamma \leq D$. Write $A = \Gamma/N$ and $G = \Lambda/D$ and let $\eta: \Gamma \rightarrow A$ and $\mu: \Lambda \rightarrow G$ be the natural quotient maps. Identify $A$ with the subgroup $\{f_a \mid a \in A\}$ of $A \wr G$. (Recall that $f_a(1) = a$ and $f_a(\sigma) = 1$ for any $\sigma \neq 1$.)

Extend $\eta$ to a weak solution $\theta$ of $\Gamma \rightarrow \Lambda$.

Lemma 2.5.2 implies that $\theta$ is surjective (since $A = \theta(\Gamma) \leq \theta(\Lambda)$). Since $\Gamma \lhd \Lambda$ we have $A \lhd A \wr G$. This evidently implies that either $A = 1$ or $G = 1$, and hence the assertion. $\square$

Proposition 5.3.2. Let $(\Gamma, \Lambda)$ be a $C$-projective pair such that $\Gamma \neq \Lambda$. Then $\bigcap_{x \in \Lambda} \Gamma^x = 1$.

Proof. Let $\Gamma_0 = \bigcap_{x \in \Lambda} \Gamma^x$. By Corollary 5.1.13 there exists $\Lambda_0$ such that $\Gamma_0 = \Lambda_0 \cap \Gamma$ and $\Gamma\Lambda_0 = \Lambda$. In particular $\Gamma_0 \neq \Lambda_0$. Moreover, by Proposition 5.1.11 $(\Gamma_0, \Lambda_0)$ is a $C$-projective pair. But since $\Gamma_0 \lhd \Lambda_0$ and $\Gamma_0 \neq \Lambda_0$, Lemma 5.3.1 implies $\Gamma_0 = 1$. $\square$

Corollary 5.3.3. Let $(\Gamma, \Lambda)$ be a $C$-projective pair. Assume that $\Gamma$ is open in $\Lambda$. Then $\Gamma = \Lambda$.

Proof. Assume that $\Gamma \neq \Lambda$ (and in particular, $\Lambda \neq 1$). Since $\Gamma$ is open, the normal core $\bigcap_{x \in \Lambda} \Gamma^x$ is also open. By Proposition 5.3.2, $\bigcap_{x \in \Lambda} \Gamma^x = 1$. Consequently $\Lambda/\bigcap_{x \in \Lambda} \Gamma^x = \Lambda$ is finite. This contradicts the fact that $\Lambda$ is $C$-projective. $\square$
Proposition 5.3.4. Let $\Lambda$ be a $C$-profinite group and $\Gamma$ a $p$-Sylow subgroup. Assume that $\Lambda$ has a non-abelian simple quotient that is divisible by $p$. Then the pair $(\Gamma, \Lambda)$ is not $C$-projective.

Proof. Assume the contrary, i.e. $(\Gamma, \Lambda)$ is $C$-projective. Hence, by Corollary 5.1.14, we have that $\Lambda = M \rtimes \Gamma$. Note that $p \nmid (\Lambda : \Gamma) = |M|$ since $\Gamma$ is $p$-Sylow.

Let $\psi : \Lambda \to S$ be an epimorphism onto a non-abelian simple group of order divisible by $p$. Then $\psi(M) \neq S$. On the one hand, $\psi(M) = 1$, since $\psi(M) \triangleleft S$. On the other hand, $\psi(\Gamma)$ is a proper subgroup of $S$. (Otherwise $S$ would be a $p$-group, which is solvable.) The assertion now follows from the contradiction $S = \psi(\Lambda) = \psi(M)\psi(\Gamma) = \psi(\Gamma) < S$. 

5.4 Applications to PAC Extensions

5.4.1 Proof of Proposition 5.0.5

Let $M$ be a PAC extension of a PAC field $K$. Since $\text{Gal}(M)$ and $\text{Gal}(M \cap K_s)$ are isomorphic and $M \cap K_s/K$ is PAC (Theorem 3.4.2), we can assume that $M/K$ is separable and algebraic. Let $\Gamma = \text{Gal}(M)$ and $\Lambda = \text{Gal}(K)$.

Since $K$ is PAC, $\Lambda$ is projective [9, Theorem 11.6.2]. By Corollary 5.1.4, to show that $\varphi$ is projective it suffices to solve a doubly split double embedding problem (5.1). Over PAC fields any finite split embedding problem is rational [25, 14], and hence (5.1) is rational by definition. Hence there exists a weak solution (Proposition 3.4.5).

Next assume that $M/K$ is algebraic and separable and that $(\text{Gal}(M), \text{Gal}(K))$ is projective. By Proposition 3.4.5, to show that $M/K$ is PAC it suffices to geometrically solve (in the weak sense) all finite rational double embedding problems. Since $(\text{Gal}(M), \text{Gal}(K))$ is projective, the double embedding problem is weakly solvable. Moreover, all weak solutions are geometric (Corollary 3.2.5), as needed.

Let $(\Gamma, \Lambda)$ be a projective pair. We would like to find a PAC extension $M/K$ such that $\Gamma = \text{Gal}(M)$, $\Lambda = \text{Gal}(K)$.

By [9, Corollary 23.1.2], there exists a PAC field $K$ such that $\text{Gal}(K) \cong \Lambda$ (since $\Lambda$ is projective). Let $M$ be the fixed field of $\Gamma$, i.e., $\text{Gal}(M) = \Gamma$. Since $(\Gamma, \Lambda)$ is projective, by the third paragraph, $M/K$ is PAC.

5.4.2 New Examples of PAC Extensions

Proposition 5.4.1. Let $K_0$ be a field which has a PAC extension $K/K_0$. Assume that $\text{Gal}(K)$ is free of infinite rank $\kappa$. Then for any projective group $P$ of rank $\leq \kappa$ there exists
a PAC extension $M/K_0$ such that $P \cong \text{Gal}(M)$.

Proof. By Theorem 3.4.2, we can assume that $K/K_0$ is a separable algebraic extension. By Corollary 5.2.3 and Proposition 5.0.5 it follows that there exists a (separable algebraic) PAC extension $M/K$ with $\text{Gal}(M) \cong P$. Now transitivity of PAC extensions (Proposition 3.4.8) implies that $M/K_0$ is PAC.

Recall that a Galois extension $N/K$ is unbounded if the set $\{ \text{ord}(\sigma) \mid \sigma \in \text{Gal}(N/K) \}$ is unbounded.

**Corollary 5.4.2.** Let $P$ be a projective group of at most countable rank, let $E$ be a countable Hilbertian field, and let $K_0/E$ be an unbounded abelian extension. Then $K_0$ has a PAC extension $M$ such that $P \cong \text{Gal}(M)$.

Proof. In the proof of [26, Proposition 3.8] it is shown that there exists a PAC extension $K/K_0$ such that $\text{Gal}(K) \cong \hat{F}_\omega$. Hence the assertion follows from the previous proposition.

**Corollary 5.4.3.** Let $P$ be a projective profinite group. Then there exists a Hilbertian field $K$ and a PAC extension $M/K$ such that $\text{Gal}(M) \cong P$. 

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Chapter 6

Weak Hilbert’s Irreducibility Theorem

Recall that a field $K$ is **Hilbertian** if it satisfies the following condition.

- Every irreducible polynomial $f(T, X) \in K(T)[X]$ that is separable in $X$ admits a specialization $T \mapsto a \in K$ under which $f(a, X)$ remains irreducible.

Although it seems that Hilbertianity and PACness are contradictory, there is a surprising link between them.

**Theorem 6.0.4.** A PAC field $K$ is Hilbertian if and only if it is $\omega$-free.

Here a field $K$ is **$\omega$-free** if

- Every finite embedding problem for $K$ is solvable.

Roquette proved the right-to-left direction of this theorem (see [9, Corollary 27.3.3]). The opposite direction was later proved by Fried-Völklein (in characteristic zero) [10] and Pop (in arbitrary characteristic) [25]. It is important to mention that Haran and Jarden reprove this result using the simpler method of algebraic patching [14].

In this chapter we extend Theorem 6.0.4 and establish a quantitative form – the “weak Hilbert’s irreducibility theorem.” Note that Theorem 6.0.4 assumes that all curves have rational points (i.e. $K$ is PAC), and under this assumption it connects the property $\Delta$ of all finite embedding problems for $K$ with the property $\Delta$ of all irreducible polynomials. In the proof of Theorem 6.0.4, in order to show that a specialization keeps the polynomial irreducible, a stronger object is preserved – the Galois group of the polynomial.

Our quantitative form, on the other hand, deals with one (arbitrary) polynomial. We find a specific finite embedding problem with the property that if it is ‘transitively solvable,’
then there is an irreducible specialization, provided that a certain curve has a point. The main ingredient is Proposition 3.2.1. In the case that $K$ is PAC, this condition, i.e. the existence of a transitive solution, becomes necessary and sufficient for the existence of an irreducible specialization. It is somewhat surprising that, in contrast to the above, there might exist an irreducible specialization even if the Galois group of the polynomial does not occur as Galois group over $K$.

Next we apply weak Hilbert’s irreducibility theorem to a field $K$ which has a PAC extension. We get some sufficient conditions to have irreducible specializations of a polynomial in terms of solutions of some double embedding problem. This result has some interesting applications which will be discuss in the following chapters.

### 6.1 Embedding Problems and Polynomials

Let $E$ be a finitely generated regular extension of a field $K$. For a separable polynomial $f(X) \in E[X]$, let $F/E$ be its splitting field, and let $L = F \cap K_s$. Recall that $\text{Gal}(F/E)$ acts faithfully on the roots of $f(X)$. Set $A = \text{Gal}(L/K)$ and $G = \text{Gal}(F/E)$. Let $G_0 \leq G$ be the stabilizer of some fixed root $x \in F$ of $f$.

Now $f$ admits the **induced embedding problem**

\[(6.1)\quad \mathcal{E}_f = (\nu: \text{Gal}(K) \to A, \alpha: G \to A)\]

with a **distinguished** subgroup $G_0$. Here $\nu$ and $\alpha$ are the restriction maps.

Let $\theta: \text{Gal}(K) \to G$ be a weak solution of the induced embedding problem, and denote its image by $D = \theta(\text{Gal}(K))$. If $D$ acts transitively on the roots of $f$, we say that $\theta$ is **transitive**. Since $G_0$ is the stabilizer of the action, a solution $\theta$ is transitive if and only if $(G : G_0) = (D : D \cap G_0)$, that is, transitivity is determined by the distinguished subgroup. The following lemma trivially holds.

**Lemma 6.1.1.** If $\theta$ is a solution (i.e. surjective), then $\theta$ is transitive.

Next we give a criterion for a polynomial $f(X) \in E[X]$ to remain irreducible after applying a place on $E$.

**Lemma 6.1.2.** Let $E$ and $f$ be as above and let $\varphi$ be a $K$-rational place of $E/K$. Assume that $\varphi(f)$ is well defined, separable and of the same degree of $f$. Then $\varphi(f)$ is irreducible over $K$ if and only if the geometric solution $\varphi^*$ of the induced embedding problem is transitive.
Proof. Let $X \subseteq F$ be the set of all roots of $f$ and $\bar{X} \subseteq K$, the set of all roots of $\phi(f)$. Then $\phi(X) = \bar{X}$. By assumptions, $|X| = \deg f = \deg \phi(f) = |\bar{X}|$, and thus $\phi$ bijectively maps $X$ onto $\bar{X}$.

Now $\phi(f)$ is irreducible if and only if $\text{Gal}(K)$ acts transitively on $\bar{X}$. Recall that $\phi(\phi^*(\sigma)(x)) = \sigma(\phi(x))$ for all $\sigma \in \text{Gal}(K)$ and $x \in X$ (see Lemma 2.4.2). Let $x_1, x_2 \in X$ and $\phi(x_1), \phi(x_2) \in \bar{X}$. Then

$$\exists \sigma \in \text{Gal}(K), \sigma(\phi(x_1)) = \phi(x_2) \iff \exists \sigma \in \text{Gal}(K), \phi(\phi^*(\sigma)(x_1)) = \phi(x_2) \iff \exists \sigma \in \text{Gal}(K), \phi^*(\sigma)(x_1) = x_2.$$ 

In other words, $\text{Gal}(K)$ acts transitively on $\bar{X}$ if and only if $\phi^*$ acts transitively on $X$, that is $\phi^*$ is transitive.

The previous result in particular asserts that for an irreducible polynomial to have a place under which it remains irreducible it is necessary that the induced embedding problem have a transitive solution. This condition also suffices, provided that certain regular extension has a $K$-rational place:

**Lemma 6.1.3.** Let $E/K$ be a finitely generated regular extension. Let $f \in E[X]$ be irreducible and separable. Assume that the induced embedding problem is transitively solvable. Then there exists a finite separable extension $\hat{E}/E$ that is regular over $K$ satisfying the following property. For any $K$-rational place $\phi$ of $\hat{E}$, $\phi(f)$ is irreducible, provided that $\phi(f)$ is well defined, separable, and of the same degree as $f$.

**Proof.** Let $\theta$ be a transitive solution of the induced embedding problem. Let $\hat{E}$ be as in Proposition 3.2.1. Then the residue field of $\hat{E}$ is $K$ implies that $\phi^* = \theta$. Thus Lemma 6.1.2 gives the irreducibility of $\phi(f)$.

As every finitely generated regular extension of a PAC field has a rational place we get the following result.

**Proposition 6.1.4.** Let $K$ be a PAC field, let $E/K$ be a finitely generated regular extension, and let $f(X) \in E[X]$ be an irreducible polynomial. Then there exists a place $\phi$ of $E$ under which $\phi(f)$ is well defined, irreducible, and of the same degree as $f$ if and only if the induced embedding problem $\mathcal{E}_f$ is transitively solvable.

**Remark 6.1.5.** To the best of our knowledge in previous works, to find an irreducible specialization for a polynomial one always preserves the Galois group of the polynomial.
The innovation of this work is that we do not insist that the Galois group be preserved in order that \( f \) remains irreducible. This approach proves to be useful in applications. For example in Chapter \( \square \) an irreducible specialization of a polynomial whose Galois group is the symmetric group \( S_n \) is needed. The results of those chapters are valid e.g. for any pro-solvable extension \( K \) of \( \mathbb{Q} \). Over such \( K \) it may happen that \( S_n \) does not occur. Hence our observation is indeed important to those applications.

Indeed, assume that \( \mathbb{Q}_{\text{sol}} \) and assume that \( S_n \) occurs as a Galois group over it. If \( n \geq 2 \), then as \( \mathbb{Z}/2\mathbb{Z} \) is a quotient of \( S_n \), we get that \( \mathbb{Z}/2\mathbb{Z} \) occurs as a Galois group over \( \mathbb{Q}_{\text{sol}} \). This contradiction implies that \( n = 1 \).

### 6.2 Weak Hilbert’s Irreducibility Theorem for PAC Extensions

In this section we consider a more classical setting. Instead of an arbitrary regular extension \( E/K \), we take \( E = K(\mathbf{t}) \), where \( \mathbf{t} \) is an \( e \)-tuple of algebraically independent transcendental elements. In other words, we consider a polynomial \( f(T,X) \) defined over \( K \) that is irreducible in the ring \( K[T,X] \) and asks whether there is an irreducible specialization for \( f \). That is to say, a map \( T \mapsto a \in K^e \) for which \( f(a,X) \in K[X] \) is irreducible, separable, and \( \deg(f(a,X)) = \deg_X(f(T,X)) \).

Let \( K \) be a field, \( f \in K[T,X] \) an irreducible polynomial that is separable in \( X \), and \( \mathbf{t} \) a tuple of algebraically independent transcendental elements. Let \( F \) be the splitting field of \( f(\mathbf{t},X) \) over \( E = K(\mathbf{t}) \), \( L = F \cap K_s \), and

\[
(6.1) \quad \mathcal{E}_f = (\mu: \text{Gal}(K) \to A, \alpha: G \to A)
\]

the induced embedding problem (see (6.1)). Here \( G = \text{Gal}(F/K(\mathbf{t})) \) and \( A = \text{Gal}(L/K) \).

For a field extension \( M/K \) that is algebraically independent of \( K(\mathbf{t}) \) we define a corresponding embedding problem

\[
(6.2) \quad \mathcal{E}_f(M) = (\mu: \text{Gal}(M) \to B, \beta: H \to B).
\]

Here \( N = LM, \tilde{F} = FM, B = \text{Gal}(N/M) \cong \text{Gal}(L/M \cap L) \), and \( H = \text{Gal}(\tilde{F}/M(\mathbf{t})) \cong \text{Gal}(F/M \cap L(\mathbf{t})) \). Then \( B \leq A \) and \( H \leq G \), so \( \beta = \alpha|_H \).

\[1\] If \( f(T,X) \) is irreducible in \( K[T,X] \), then \( f(\mathbf{t},X) \) is irreducible over \( K(\mathbf{t}) \) (by Gauss’ lemma). On the other hand, assume that \( f(\mathbf{t},X) \in K(\mathbf{t})[X] \) is irreducible over \( K(\mathbf{t}) \). Then there exists \( c(\mathbf{t}) \in K(\mathbf{t}) \) such that \( f^*(T,X) = c(\mathbf{t})f(T,X) \in K[T,X] \) and is primitive. Hence \( f^* \) is irreducible in the ring \( K[T,X] \).
Note that the pair $(\mathcal{E}_f(M), \mathcal{E}_f)$ is a double embedding problem for $M/K$.

**Proposition 6.2.1.** Let $K$ be a field, $f \in K[T, X]$ an irreducible polynomial that is separable in $X$, and $\mathcal{E}_f = (\mu : \text{Gal}(K) \to A, \alpha : G \to A)$ the induced embedding problem as in (6.1) with distinguished subgroup $G_0$. Further assume that there exists a PAC extension $M/K$ (which is algebraically independent of $K(t)$) and let $\mathcal{E}_f(M)$ be the corresponding embedding problem as in (6.2).

Each of the following conditions suffices for the existence of an irreducible specialization $T \mapsto a \in K^e$ for $f$.

(a) $\mathcal{E}_f(M)$ is solvable.

(b) $(G : G_0) = (H : H_0)$, where $H_0 = H \cap G_0$ and $\mathcal{E}_f(M)$ is transitively solvable w.r.t. the distinguished subgroup $H_0$.

**Proof.** Let $r(t)$ be the product of the leading coefficient of $f$ and its discriminant. Assume there exists a weak solution $\eta : \text{Gal}(M) \to H$ of $\mathcal{E}_f(M)$.

By the lifting property (Proposition 3.4.6) we can lift $\eta$ to a geometric weak solution $\varphi^*$ of $\mathcal{E}_f$ satisfying $a = \varphi(t)$ is finite (and hence in $K^e$) and $r(a) \neq 0$. In particular, $\varphi$ is unramified in $F$.

For (a) assume that $\eta$ is surjective. Then its image $\eta(\text{Gal}(M))$ contains $\ker(\beta)$. But $\ker(\beta) = \ker(\alpha)$, hence $\ker(\alpha) \leq \eta(\text{Gal}(M)) \leq \varphi^*(\text{Gal}(K))$. By Lemma 2.2.1 $\varphi^*$ is surjective. Consequently Lemma 6.1.2 implies the assertion.

For (b) assume that $(G : G_0) = (H : H_0)$ and that $\eta$ is transitive. Let $C = \eta(\text{Gal}(M))$ be the image of $\eta$. Since $\eta$ is transitive we have $(C : C \cap H_0) = (H : H_0)$ (see discussion after the definition of transitive solutions). Let $D = \varphi^*(\text{Gal}(K))$. Then $C \leq H \cap D$, hence $C \cap H_0 = C \cap H \cap G_0 = C \cap G_0$. It implies that

$$(G : G_0) \geq (D : D \cap G_0) \geq (C : C \cap H_0) = (H : H_0) = (G : G_0).$$
Hence \((G : G_0) = (D : D \cap G_0)\), i.e. \(\varphi^*\) is transitive. Then Lemma 6.1.2 implies the assertion.

Remark 6.2.2. In the proof we actually showed that \(f(a, X)\) is irreducible over \(M\).

Remark 6.2.3 (On the condition \([b]\)). In the notation of the above proposition, recall that \(G_0\) is the stabilizer of a root \(x \in F\) of \(f(t, X)\), so \((G : G_0) = \deg_X f\). Now let \(g(t, X)\) be the irreducible polynomial of \(x\) over \(M(t)\). Then \(g\) divides \(f\) in the ring \(M(t)[X]\) and \((H : H_0) = \deg_X g\). Thus \((H : H_0) = (G : G_0)\) if and only if \(f\) is irreducible over \(M(t)\).

In particular, if \(f\) is absolutely irreducible, the condition \((G : G_0) = (H : H_0)\) always holds.

Another example in which the condition \((G : G_0) = (H : H_0)\) holds is when \(A \cong B\). Indeed, since \(\ker(\alpha) = \ker(\beta)\) we get

\[|G| = |A| |\ker(\alpha)| = |B| |\ker(\beta)| = |H|,\]

and thus \(G \cong H\). Consequently \((G : G_0) = (H : H_0)\).

In what follows we apply Proposition 6.2.1 to polynomials which are in some sense the most irreducible. These cases are important to applications, since by transcendental constructions, it is usually easier to get extreme irreducibilities.

Call a polynomial \(f(T, X) \in K[T, X]\) of degree \(n\) in \(X\) a **stable symmetric polynomial** if the Galois group of \(f(t, X) \in \tilde{K}(t)\) is the symmetric group \(S_n\). Regard stable symmetric polynomial as the most irreducible polynomials. A stable symmetric polynomial is absolutely irreducible and its splitting field over \(K(t)\) is regular over \(K\). Thus the induced embedding problem is actually the following realization problem.

\[\mathcal{E}_f = (\text{Gal}(K) \to 1, S_n \to 1)\]

Here the distinguished subgroup is \(S_{n-1}\) (all permutations that fix 1).

Remark 6.2.4. Let \(E/K\) be a regular extension. One can asks whether there exists a separating transcendence basis \(t\) of \(E/K\) such that a polynomial \(f(t, X) \in K(t)[X]\) whose root generates \(E/K(t)\) is a stable symmetric polynomial of degree \(n\).

The stability theorem (see [9, Theorem 18.9.3]) asserts that indeed for infinitely many positive integers \(n\) there exists such a separating transcendence basis.

We have the following nice sufficient condition to have irreducible specialization of stable symmetric polynomials.
Corollary 6.2.5. Let $K$ be a field, let $r(T) \in K[T]$ non-constant, and let $f(T, X) \in K[T, X]$ be a stable symmetric polynomial of degree $n$. Assume that there exists a PAC extension $M/K$ and a separable extension $N/M$ of degree $n$. Then there exists $a \in K^e$ such that $r(a) \neq 0$ and $f(a, X)$ is separable irreducible polynomial of degree $n$.

Proof. Since $F$ (the splitting field of $f(t, X)$ over $K(t)$) is regular over $K$, we have $\text{Gal}(FM/M(t)) \cong \text{Gal}(F/K(t)) \cong S_n$. Hence

$$\mathcal{E}_f(M) = (\text{Gal}(M) \to 1, S_n \to 1)$$

is the corresponding embedding problem.

Let $\hat{N}$ be the Galois closure of $N/M$ and let $\eta: \text{Gal}(M) \to \text{Gal}(\hat{N}/M)$ be the restriction map. The action of $\text{Gal}(\hat{N}/M)$ on the cosets of $\text{Gal}(\hat{N}/N)$ induces an embedding $\text{Gal}(\hat{N}/M) \to S_n$ with a transitive image. Identify $\text{Gal}(\hat{N}/M)$ with its image in $S_n$. Hence $\eta$ is a transitive solution of the corresponding embedding problem. By Proposition 6.2.1 the assertion follows.

6.3 Some Corollaries

Theorem 6.3.1 from the introduction asserts that a countable Hilbertian field has a PAC extension with free absolute Galois group of rank $e$, for an arbitrary positive integer $e \geq 1$. The following result implies that having such PAC extension is also sufficient for Hilbertianity.

Corollary 6.3.1. Let $K$ be a field. Assume that there exists a PAC extension $M/K$ whose absolute Galois group $\text{Gal}(M)$ is free of rank $e$, for infinitely many positive integers $e$. Then $K$ is Hilbertian.

Proof. Let $f \in K[T, X]$ be an irreducible polynomial that is separable in $X$ and let $\mathcal{E}_f$ be its induced embedding problem as in (6.1). Take a PAC extension $M/K$ whose absolute Galois group $\text{Gal}(M)$ is free of rank $\geq |G|$.

Let $\mathcal{E}_f(M)$ be the corresponding embedding problem as in (6.2). We have

$$\text{rank} \text{Gal}(M) \geq |G| \geq |H| \geq \text{rank } H.$$

Hence, by the embedding property [9, Proposition 17.7.3], $\mathcal{E}_f(M)$ is solvable. It follows that $f$ has an irreducible specialization (Proposition 6.2.1).

If $K$ has a PAC extension $M$ whose absolute Galois group is $\omega$-free (i.e. any finite embedding problem for $M$ is solvable) then we get the following result of Razon [26, Lemma 2.2].
Corollary 6.3.2 (Razon). Let $K$ be a field and $M/K$ a PAC extension whose absolute Galois group is $\omega$-free. Then $M$ is Hilbertian over $K$. In particular, $K$ is Hilbertian.

Proof. Let $f(T, X) \in K[T, X]$ be an irreducible polynomial that is separable in $X$. Since over $M$ every finite embedding problem is solvable, by Proposition 6.2.1 there exists an irreducible specialization $T \mapsto a \in K^e$ for $f$. Moreover, $f(a, X)$ is irreducible over $M$ (Remark 6.2.2).

We pose a natural question.

Question 6.3.3. Let $K$ be a finitely generated infinite field. Does there exist a PAC extension $M/K$ whose absolute Galois group is $\omega$-free?

We even do not know the answer to the following simpler problem in the case when $K = \mathbb{Q}$.

Question 6.3.4. Let $K$ be a finitely generated infinite field and let $M/K$ be a PAC extension. Is $\text{Gal}(M)$ finitely generated?
Chapter 7

Dirichlet’s Theorem for Polynomial Rings

7.1 Introduction and the Main Result

Dirichlet’s classical theorem about primes in arithmetic progressions states that if $a, b$ are relatively prime positive integers, then there are infinitely many $c \in \mathbb{N}$ such that $a + bc$ is a prime number. Following a suggestion of E. Landau, Kornblum [20] proved an analog of Dirichlet’s theorem for the ring of polynomials $F[X]$ over a finite field $F$. Later, Artin refined Kornblum’s result and proved that if $a(X), b(X) \in F[X]$ are relatively prime, then for every sufficiently large integer $n$ there exists $c(X) \in F[X]$ such that $a(X) + b(X)c(X)$ is irreducible of degree $n$ in $F[X]$ [29, Theorem 4.8].

To avoid repetition, we shall say that Dirichlet’s theorem holds for a polynomial ring $F[X]$ and for a set of positive integers $\mathcal{N}$, if for any relatively prime polynomials $a, b \in F[X]$ there exist infinitely many $c \in F[X]$ such that $a + bc$ is irreducible of degree $n$, provided that $n \in \mathcal{N}$ is sufficiently large.

Jarden raised the question of whether the Artin-Kornblum result generalizes to other fields. Of course, if $F$ is algebraically closed, then the polynomial $a(X) + b(X)c(X)$ is reducible unless it is of degree 1. On the other hand, if $F$ is Hilbertian, then Dirichlet’s theorem trivially holds. Indeed, $a(X) + b(X)Y$ is irreducible, and hence there are infinitely many $c \in F$ such that $a(X) + b(X)c$ is irreducible in $F[X]$. To get irreducible polynomials of higher degree in this case, just choose $c_0(X) \in F[X]$ relatively prime to $a(X)$ and of high degree, and then repeat the above argument for $a(X) + b(X)c_0(X)Y$.

By the weak Hilbert’s irreducibility theorem, this simple argument can be extended to fields having PAC extensions. We start by introducing a piece of notation: For a field $F$...
we let $\mathcal{N}(F)$ be the set of all positive integers for which there exists a PAC extension $M/F$ and a separable extension $N/M$ such that $n = [N : M]$.

**Theorem 7.1.1.** Let $F$ be a field and $\mathcal{N} = \mathcal{N}(F)$. Then Dirichlet’s theorem holds for $F$ and $\mathcal{N}$.

In Section 8.3 we calculate $\mathcal{N}(F)$ for certain families of fields $F$. For example, if $F$ is a pro-solvable extension of a countable Hilbertian fields, then $\mathcal{N}(F)$ contains every $n \geq 5$.

Let $n \in \mathcal{N}$ be sufficiently large. By Corollary 6.2.5 and the above argument, it suffices to find $c \in F[Y]$ of degree $n$ such that $f(Y, X) = a(X) + b(X)c(X)Y$ is of degree $n$ and its Galois group over $\bar{F}(Y)$ is $S_n$. So the proof of Theorem 7.1.1 reduces to the following

**Proposition 7.1.2.** Let $F$ be an infinite field with an algebraic closure $\bar{F}$ and let $a, b \in F[X]$ be relatively prime polynomials. Then for any $n \geq 2 \max(\deg a, \deg b) + \log n(1 + o(1))$ there exists $c(X) \in F[X]$ such that $f(X, Y) = a(X) + b(X)c(X)Y$ is irreducible over $F(Y)$ of degree $n$ in $X$ and $\text{Gal}(f(X, Y), \bar{F}(Y)) \cong S_n$.

Note that each infinite algebraic extension $F$ of a finite field $K$ is PAC [9, Corollary 11.2.4]. By Theorem 7.1.1, $F[X]$ satisfies Dirichlet’s theorem for $\mathcal{N} = \{n \mid \exists N/M, n = [N : M]\}$. This result already follows from a quantitative form of the result of Artin-Kornblum. Nevertheless, our proof has the advantage that the constructions are essentially explicit: The polynomial $c(X)$ in Theorem 7.1.1 equals to the polynomial $c(X)$ appearing in Proposition 7.1.2 times some factor, say $\alpha$, coming from the PAC property. The construction in Proposition 7.1.2 is explicit as it uses nothing but the Euclidean algorithm.

## 7.2 Polynomials over Infinite Fields

### 7.2.1 Calculations with Polynomials

The following result is a special case of Gauss’ Lemma.

**Lemma 7.2.1.** A polynomial $f(X, Y) = a(X) + b(X)Y \in F[X, Y]$ is irreducible if and only if $a(X)$ and $b(X)$ are relatively prime.

**Lemma 7.2.2.** Let $a, b, c \in F[X]$ such that $\gcd(a, b) = 1$ and $c \neq 0$. Then there exists a finite subset $S \subseteq F$ such that for each $\alpha \in F \setminus S$ the polynomials $a + \alpha b$ and $c$ are relatively prime. Moreover, if $b' \neq 0$, we may choose $S$ such that $a + \alpha b$ is separable.
Proof. Let \( S = \{ -\frac{a(\gamma)}{b(\gamma)} \mid \gamma \in \tilde{F}, b(\gamma) \neq 0 \text{ and } c(\gamma) = 0 \} \cap F \). Then \( a + ab \) has no common zero in \( \tilde{F} \) with \( c \) for each \( \alpha \notin S \), hence these polynomials are relatively prime.

Let \( d(Y) \in F[Y] \) be the discriminant of \( a(X) + Yb(X) \) over \( F(Y) \), then \( b'(X) \neq 0 \) implies that \( \frac{\partial}{\partial X}(a(X) + b(X)Y) \neq 0 \), and hence \( d(Y) \neq 0 \). In this case add all the roots of \( d(Y) \) to \( S \).

**Lemma 7.2.3.** Let \( a, b, p_1, p_2 \in F[X] \) be pairwise relatively prime polynomials and let \( \alpha_1, \alpha_2 \in F \) be distinct nonzero elements. Then for each \( n > \deg p_1 + \deg p_2 \) there exists \( c \in F[X] \) of degree \( n \) and separable \( h_1, h_2 \in F[X] \) such that \( a = p_i h_i + b c \alpha_i \) and \( \gcd(h_i, a p_i) = 1 \) for \( i = 1, 2 \).

Proof. Write \( b_i = b c \alpha_i \). Since \( \gcd(b_i p_{3-i}, p_i) = 1 \) for \( i = 1, 2 \), there are \( c_1, c_2, h_{1,0}, h_{2,0} \in F[X] \) such that

\[
(7.1) \quad a = p_i h_{i,0} + b_i c_i p_{3-i} \quad i = 1, 2,
\]

with \( \deg c_i < \deg p_i \). For \( \bar{c} = c_1 p_2 + c_2 p_1 \) and \( h_{i,1} = h_{i,0} - c_{3-i} b_i \), we have

\[
(7.2) \quad a = p_i h_{i,1} + b_i \bar{c} \quad i = 1, 2.
\]

Here \( h_{i,1} \) is relatively prime to \( b_i \), since \( a \) is. Taking \((7.1)\) with \( i = 2 \) and \((7.2)\) with \( i = 1 \), we get

\[
p_1 h_{1,1} \equiv a - b_1 \bar{c} \equiv a - b_1 c_2 p_1 \equiv b_2 c_2 p_1 - b_1 c_2 p_1 \equiv p_1 b c_2 (\alpha_2 - \alpha_1) \pmod{p_2}.
\]

Therefore \( h_{1,1} \) is relatively prime to \( p_2 \), since \( b c_2 p_1 \) is (by \((7.1)\) with \( i = 2 \)). Similarly, \( h_{2,1} \) is relatively prime to \( p_1 \).

Take \( c = \bar{c} + p_1 p_2 s \) for some \( s \in F[X] \). Then, for \( h_i = h_{i,1} - b_i p_{3-i} s \), we have

\[
a = p_i h_i + b_i c \quad i = 1, 2.
\]

To conclude the proof it suffices to find \( s \in F[X] \) such that \( h_1 \) and \( h_2 \) are separable, \( \gcd(h_i, a p_i) = 1 \), and \( \deg c = n \). Choose \( s \in F[X] \) for which \( \deg s = n - (\deg p_1 + \deg p_2) \geq 1 \), \( b p_i s' \neq 0 \), and \( \gcd(s, h_{i,1}) = 1 \) for \( i = 1, 2 \) (e.g., \( s(X) = (X - \beta)^{n-1} (X - \gamma) \), where \( \beta, \gamma \in F \) are not roots of \( h_{1,1} h_{2,1} b p_1 p_2 \)).

By Lemma 7.2.2 with \( h_{i,1} \), \( b p_{3-i} s \), \( a p_i \) (for \( i = 1, 2 \)) we get a finite set \( S \subseteq F \) such that for each \( \alpha \in F \setminus S \) the polynomial \( h_{i,1} - \alpha b p_{3-i} s \) is separable and relatively prime to \( a p_i \).

Replace \( s \) with \( \alpha s \), for some \( \alpha \neq 0 \) for which \( \alpha \beta, \alpha \gamma \in F \setminus S \), if necessary, to assume that \( \alpha_1, \alpha_2 \in F \setminus S \). This \( s \) has all the required properties. \( \square \)
7.2.2 Other Lemmas

The next lemma gives a sufficient criterion for a transitive group to be primitive and to be the symmetric group (cf. [33, Lemma 4.4.3]).

**Lemma 7.2.4.** Let $A \leq S_n$ be a transitive group and let $e$ be a positive integer in the segment $\frac{n}{2} < e < n$ such that $\gcd(e, n) = 1$. Then, if $A$ contains an $e$-cycle, it is primitive. Moreover, if $A$ contains also a transposition, then $A = S_n$.

**Proof.** Let $\Delta \neq \{1, \ldots, n\}$ be a block of $A$. We have $|\Delta| \leq \frac{n}{2}$, since $|\Delta| \mid n$. For the first assertion, it suffices to show that $|\Delta| = 1$, and since $\gcd(e, n) = 1$, it even suffices to prove that $|\Delta| \mid e$. Without loss of generality assume that $\sigma = (1 \, 2 \, \cdots \, e) \in A$ and $1 \in \Delta$. Then $\{1, \ldots, e\} \not\subseteq \Delta$, since $e > \frac{n}{2} \geq |\Delta|$. Hence $\Delta \neq \sigma \Delta$, and hence $\Delta \cap \sigma \Delta = \emptyset$. As $\sigma(x) = x$ for any $n \geq x > e$, we have $\Delta \subseteq \{1, \ldots, e\}$. Consequently, $\Delta$ is a block of $\langle \sigma \rangle$, so $|\Delta| \mid e$.

The second assertion follows since a primitive group containing a transposition is the symmetric group [6, Theorem 3.3A].

The following number-theoretic lemma will be needed later.

**Lemma 7.2.5.** For any positive integers $n, m$ and prime $p$ satisfying $n \geq 2m + \log n(1 + o(1))$, there exists an integer $e$ in the segment $n - m > e > n/2$ such that $\gcd(e, np) = 1$.

**Proof.** Let $e$ be

\[
\begin{align*}
\frac{n}{2} + 2, & \quad \text{if } n \text{ is even but not divisible by } 4, \\
\frac{n}{2} + 1, & \quad \text{if } n \text{ is divisible by } 4, \text{ or} \\
\frac{n+1}{2}, & \quad \text{if } n \text{ is odd.}
\end{align*}
\]

Then $e$ is the first integer greater than $\frac{n}{2}$ for which $\gcd(e, n) = 1$. If $p \nmid e$, we are done (and we only need $n > 2m + 4$). Next assume that $p \mid e$ (and hence $p \mid n$). Firstly, if $n$ is even but not divisible by 4, then the next candidate $e' = e + 2$ works, since $\gcd(e', n) = 1$ and $p \nmid e'$ (otherwise, $p$ divides $e' - e = 2$, hence $e$ is even, a contradiction). Next, if $n$ is divisible by 4, then the first relatively prime to $n$ integer greater than $e$ is $e' = \frac{n}{2} + q$, where $q$ is the smallest prime not dividing $n$.

If $p \mid e'$, then $p \mid (e' - e) = q - 1$. In particular, $p < q$, and hence $p \mid n$ by minimality of $q$, a contradiction. Therefore $p \nmid e'$. Finally, if $n$ is odd, the same argument will show that $e' = \frac{n+1}{2}$ is relatively prime to $np$, where now $q$ is the smallest odd prime not dividing $n$.

It remains to evaluate $q$ – a standard exercise in number theory: Let $\omega(n)$ be the number of distinct prime divisors of $n$. Then $q$ is no more than the $(\omega(n) + 2)$-th prime.
number. Since the \( k \)-th prime equals to \( k \log k(1 + o(1)) \) and
\[
\omega(n) \leq \frac{\log n}{\log \log n} (1 + o(1))
\]
[24, Theorem 2.10], we have
\[
q \leq \omega(n) \log(\omega(n))(1 + o(1)) = \log n(1 + o(1)).
\]
Note that for \( n = 4 \prod_{2 < l < q} l \) (i.e., 4 times the product of all the odd prime numbers less than \( q \)) the inequality is in fact equality. Thus the estimation \( n > 2m + \log(n)(1 + o(1)) \) is the best possible.

The following result is very well known, however, for the sake of completeness, we give a proof.

**Proposition 7.2.6.** Let \( F \) be an algebraically closed field of characteristic \( l \geq 0 \). Let \( E/K \) be a separable extension of degree \( n \) of algebraic function fields of one variable over \( F \). Assume that a prime divisor \( p \) of \( K \) factors as
\[
p = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_r^{e_r}
\]
in \( E \). Assume either

(a) \( l = 0 \), or

(b) \( l > 0 \) and \( \gcd(e_i, l) = 1 \), for \( i = 1, \ldots, r \), or

(c) \( l = e_r = 2 \) and \( e_1, \ldots, e_{r-1} \) are odd.

Then the Galois group of the Galois closure of \( E/K \) (as a subgroup of \( S_n \)) contains an element of cyclic type \( (e_1, \ldots, e_r) \).

**Proof.** The completion \( \hat{K} \) of \( K \) at \( p \) is a field of Laurent series over \( F \) [32, Theorem II.2], say \( \hat{K} = F((Y)) \). Let \( x \) be a primitive element of \( E/K \), integral at \( p \) and let \( f \) be its irreducible polynomial over \( K \).

Then \( E = K[X]/(f) \). Let \( f = f_1 \cdots f_r \) be the factorization of \( f \) over \( \hat{K} \) into a product of distinct separable irreducible monic polynomials. Then by [32, Theorem II.1], the completion of \( E \) at \( \mathfrak{p}_i \) is \( \hat{K}[X]/(f_i) \), and hence \( \deg f_i = e_i \), for \( i = 1, \ldots, r \).

If either \( l = 0 \), or \( l > 0 \) and \( \gcd(e_i, l) = 1 \), then \( F((Y)) \) has a unique extension of degree \( e_i \), namely \( F((Y^{1/e_i})) \) [31, III§6] which is cyclic. We thus get that the splitting field
of \( f \) over \( F((Y)) \) is \( F((Y^{1/e})) \), where \( e = \text{lcm}(e_1, \ldots, e_r) \), unless \( l = e_r = 2 \) and then the splitting field of \( f \) is the compositum of \( F((Y^{1/e'})) \) with an extension of degree 2, where \( e' = \text{lcm}(e_1, \ldots, e_{r-1}) \) is odd. In both cases the Galois group of \( f \) over \( F((Y)) \) is cyclic of order \( e \). Its generator \( \sigma \) acts cyclicly on the roots of each of the \( f_i \)'s. Consequently, the cyclic type of \( \sigma \) is \((e_1, \ldots, e_r)\), as required. \( \square \)

### 7.2.3 Proof of Proposition 7.1.2

Let \( f(X,Y) = a(X) + b(X)Y \in F[X,Y] \) be an irreducible polynomial. For a large integer \( n \) we need to find \( c(X) \in F[X] \) such that \( f(X, c(X)Y) = a(X) + b(X)c(X)Y \) is irreducible of degree \( n \) and the Galois group of \( f(X, c(X)Y) \) over \( \tilde{F}(Y) \) is \( S_n \).

Lemma 7.2.5 with \( m = \max\{\deg a(X), 2 + \deg b(X)\} \) and \( p = \text{char}(F) \) gives (for \( n \geq 2m + \log n(1 + o(1)) \)) a positive integer \( e > \frac{n}{2} \) such that

\[
(7.3) \quad n > \max\{\deg a(X), e + 2 + \deg b(X)\},
\]

\[
(7.4) \quad \gcd(e, np) = 1 \text{ (or } \gcd(e, n) = 1, \text{ if } p = 0).\]

Let \( \alpha_1 \neq \alpha_2 \) and \( \gamma_1 \neq \gamma_2 \) be elements of \( F \) such that \( \alpha_i \) is nonzero and \( \gamma_i \) is not a root of \( a(X)b(X), i = 1, 2 \). In Lemma 7.2.3 we have constructed (for \( a, b, p_1 = (X - \gamma_1)^e, p_2 = (X - \gamma_2)^2, \alpha_1, \) and \( \alpha_2 \)) a polynomial \( c(X) \in F[X] \) of degree \( \deg c = n - \deg b(X) \) which is relatively prime to \( a(X) \) such that

\[
(7.5) \quad f(X, c(X)\alpha_1) = a(X) + \alpha_1 b(X)c(X) = (X - \gamma_1)^e h_1(X),
\]

\[
(7.6) \quad f(X, c(X)\alpha_2) = a(X) + \alpha_2 b(X)c(X) = (X - \gamma_2)^2 h_2(X).
\]

Here \( h_1(X), h_2(X) \in F[X] \) are separable polynomials which are relatively prime to \( (X - \gamma_1)a(X), (X - \gamma_2)a(X) \) respectively. In particular \( \gcd(a, c) = 1 \), and hence \( f(X, c(X)Y) \) is irreducible (Lemma 7.2.1). By (7.3), \( \deg_X f(X, c(X)Y) = \deg b(X) + \deg c(X) = n \).

Taking (7.4), (7.5), and (7.6) in mind, Proposition 7.2.6 with \( p = (Y - \alpha_1) \) gives us an \( e \)-cycle in \( \text{Gal}(f(X, c(X)Y), \tilde{F}(Y)) \) and with \( p = (Y - \alpha_2) \) gives a transposition. Thus \( \text{Gal}(f(X, c(X)Y), \tilde{F}(Y)) = S_n \) (Lemma 7.2.4) as needed. \( \square \)

**Question 7.2.7.** Let \( f(X,Y) \in F[X,Y] \) be an absolutely irreducible polynomial. For large \( n \), is there a polynomial \( c(X) \in F[X] \) for which (a) \( f(X, c(X)Y) \) is an \( X \)-stable polynomial of degree \( n \)? (b) \( \text{Gal}(f(X, c(X)Y), \tilde{F}(Y)) \cong S_n \)?
Chapter 8

Families of PAC Extensions

The aim of this chapter is to produce several new families of PAC extensions. First we find that some fields (with high probability in some sense) are PAC extensions over any subfield which is not algebraic over a finite field. As an application we solve Problem 18.7.8 in [9] for finitely generated fields.

In the second section we fix a field and find PAC extensions of it. Specifically we get that an extension of a countable Hilbertian field having certain group theoretic features (e.g. pro-solvable extensions) has many PAC extensions. This implies the results of Chapter 7 and Theorem 7 for those fields.

8.1 PAC Extensions over Subfields

Recall that Jarden and Razon prove that if $K$ is a countable Hilbertian field, then $K_s(\sigma)/K$ is PAC for almost all $\sigma \in \text{Gal}(K)^e$. In this section we generalize this result.

Let us start by introducing some notation. Let

$$f_1(T_1, \ldots, T_e, X_1, \ldots, X_n), \ldots, f_m(T, X) \in K[T, X]$$

be irreducible polynomials and $g(T) \in K[T]$ nonzero. The corresponding Hilbert set is the set of of all irreducible specializations $T \mapsto a$ for $f_1, \ldots, f_m$ under which $g$ does not vanish, i.e.

$$H_K(f_1, \ldots, f_m; g) = \{a \in K^r \mid \forall i \ f_i(a, X) \text{ is irreducible in } K[X] \text{ and } g(a) \neq 0\}.$$ 

Now $K$ is Hilbertian if any Hilbert set is nonempty provided that $f_i = f_i(T, X)$ is separable in $X$ for each $i$. (Some authors use the terminology ‘$K$ is separable Hilbertian’.)
A stronger property is that any Hilbert set for \( K \) is nonempty. We call such a field s-Hilbertian.

In case the characteristic of \( K \) is zero, these two properties coincide. If the characteristic of \( K \) is positive, there is a simple criterion for a Hilbert field to be s-Hilbertian.

But first let us recall the following notion. Let \( K \) be a field of positive characteristic \( p > 0 \). Then the subset of all \( p \)-th-powers \( K_p \) is a subfield of \( K \). If \( [K : K_p] = 1 \), we call \( K \) perfect, otherwise we say it is imperfect.

Theorem 8.1.1 (Uchida). Let \( K \) be a Hilbertian field of positive characteristic. Then \( K \) is s-Hilbertian if and only if \( K \) is imperfect.

Definition 8.1.2. A field \( E \) is said to be Hilbertian over a subset \( K \) if \( H_E(f_1, \ldots, f_m; g) \cap K^r \neq \emptyset \) for any irreducible \( f_1, \ldots, f_m \in E[T, X] \) that are separable in \( X \) and any nonzero \( g(T) \in E[T] \). If furthermore \( H_E(f_1, \ldots, f_m; g) \cap K^r \neq \emptyset \) for any irreducible \( f_1, \ldots, f_m \in E[T, X] \), then we say that \( E \) is s-Hilbertian over \( K \).

Note that a field \( K \) is Hilbertian (resp. s-Hilbertian) if and only if it is Hilbertian (resp. s-Hilbertian) over itself.

We begin with the observation that the proof of [17, Proposition 3.1] gives the following stronger statement:

Theorem 8.1.3 (Jarden-Razon). Let \( E \) be a countable field that is Hilbertian over a subset \( K \). Then for almost all \( \sigma \in \text{Gal}(E)^e \) the fields \( E_s(\sigma) \) and \( \tilde{E}(\sigma) \) are PAC over \( K \).

(Recall that a field \( E \) is a PAC extension of a subset \( K \) if for any \( f(T, X) \in K[T, X] \) that is absolutely irreducible and separable in \( X \) there exists \( (a, b) \in K \times E \) for which \( f(a, b) = 0 \).) Next we wish to find new PAC extensions, and we start by finding Hilbertian fields over other fields.

Lemma 8.1.4. Let \( K \) be an s-Hilbertian field over a subset \( S \) and \( E/K \) a purely transcendental extension. Then \( E \) is s-Hilbertian over \( S \).

Proof. Let \( f_1(T, X), \ldots, f_r(T, X) \in E[T, X] \) be irreducible polynomials and \( 0 \neq g(T) \in E[T] \). Since \( E = K(u_\alpha | \alpha \in A) \), where \( \{u_\alpha | \alpha \in A\} \) is a set of variables, we can assume that \( f_i(T, X) = g_i(u, T, X) \), where

\[
\begin{align*}
g_1(u, T, X), \ldots, g_r(u, T, X) \in K[u, T, X]
\end{align*}
\]

for some finite tuple of variables \( u \).
Since $K$ is s-Hilbertian over $S$, there exists a tuple $a$ of elements in $S$ such that all $f_i(a, X) = g_i(u, a, X)$ are irreducible in $K[u, X]$ and $g(a) \neq 0$. But the elements in $\{u_\alpha \mid \alpha \in A\}$ are algebraically independent, so all $f_i(a, X) = g_i(u, a, X)$ are irreducible in the larger ring $E[X]$. 

**Proposition 8.1.5.** Let $K$ be an s-Hilbertian field over a subset $S$ and let $E/K$ be a finitely generated extension. Then $E$ is Hilbertian over $S$. Moreover, if $E/K$ is also separable, then $E$ is even s-Hilbertian over $S$.

**Proof.** Choose a transcendence basis $t$ for $E/K$, i.e., $K(t)/K$ is purely transcendental and $E/K(t)$ is finite. Let $H \subseteq E^r$ be a separable Hilbert set for $E$. By [9, Proposition 12.3.3], there exists a separable Hilbert set $H_1 \subseteq K(t)^r$ such that $H_1 \subseteq H$. By Lemma 8.1.4, we get that $H_1 \cap S^r \neq \emptyset$, and hence the assertion.

If $E/K$ is also separable, then we can choose $t$ to be a separating transcendence basis, that is, we can assume that $E/K(t)$ is separable. Now the same argument as above work for any Hilbert set $H \subseteq E^r$ (using [9, Corollary 12.2.3] instead of [9, Proposition 12.3.3]).

Combining the results that we attained so far we enlarge the family of PAC extensions:

**Theorem 8.1.6.** Let $e \geq 1$ be an integer, let $K$ be a countable field which is s-Hilbertian over some subset $S$, and let $E/K$ be a finitely generated extension. Then for almost all $\sigma \in \text{Gal}(E)^e$ the fields $E_s(\sigma)$ and $\tilde{E}(\sigma)$ are PAC over $S$.

In particular, the result is valid when $K$ is a countable s-Hilbertian field (and $S = K$).

**Corollary 8.1.7.** Let $e \geq 1$ be an integer and let $K$ be a finitely generated infinite field (over its prime field). Then for almost all $\sigma \in \text{Gal}(K)^e$ the field $K_s(\sigma)$ is a PAC extension of any subfield which is not algebraic over a finite field. Moreover, if $K$ is of characteristic 0, then $K_s(\sigma)$ is also PAC over any subring.

**Proof.** First assume that $K$ is of characteristic 0. Then any ring contains $\mathbb{Z}$, so it suffices to show that $K_s(\sigma)/\mathbb{Z}$ is PAC for almost all $\sigma \in \text{Gal}(K)^e$. And indeed, since $\mathbb{Q}$ is Hilbertian over $\mathbb{Z}$, Theorem 8.1.6 implies that $K_s(\sigma)/\mathbb{Z}$ is PAC for almost all $\sigma$.

Next assume that the characteristic of $K$ is $p > 0$. Since any field $F$ which is not algebraic over $\mathbb{F}_p$ contains a rational function field $\mathbb{F}_p(t)$, it suffices to show that $K_s(\sigma)/\mathbb{F}_p(t)$ is PAC for almost all $\sigma \in \text{Gal}(K)^e$ and any $t \in K_s(\sigma) \setminus \tilde{\mathbb{F}}_p$.

Set $G = \text{Gal}(K)^e$ and let $\mu$ be its normalized Haar measure. For any $t \in K_s \setminus \tilde{\mathbb{F}}_p$, we define a subset $\Sigma_t \subseteq G$ as follows.

\[
(8.1) \quad \Sigma_t = \{\sigma \in G \mid \text{if } t \in K_s(\sigma), \text{ then } K_s(\sigma)/\mathbb{F}_p(t) \text{ is PAC}\}.
\]
We claim that \( \mu(\Sigma_t) = 1 \). Indeed, let \( E = K(t) \). Then \( E/K \) is a finite separable extension. Let \( H = \text{Gal}(E)^e \) be the corresponding open subgroup of \( G \).

Note that \( t \in K_s(\sigma) \) if and only if \( \sigma \in H \). Then the definition of \( \Sigma_t \) implies that

\[
\Sigma_t = (H \cap \Sigma_t) \cup (G \setminus H).
\]

Hence it suffices to show that \( \mu(H \cap \Sigma_t) = \mu(H) \), or equivalently, \( \nu(H \cap \Sigma_t) = 1 \), where \( \nu \) denotes the normalized Haar measure on \( H \).

Since \( \mathbb{F}_p(t) \) is Hilbertian ([9, Theorem 13.3.5]) and imperfect ([9, Lemma 2.7.2]), Uchida’s theorem implies that \( \mathbb{F}_p(t) \) is s-Hilbertian. Also \( E/\mathbb{F}_p(t) \) is finitely generated because \( K \) is.

Finally, since \( H \cap \Sigma_t \) is the set of all \( \sigma \in \text{Gal}(E)^e \) for which \( E_s(\sigma)/\mathbb{F}_p(t) \) is PAC, and since \( E_s = K_s \), Theorem 8.1.6 implies that \( \nu(H \cap \Sigma_t) = 1 \), as desired.

8.2 An Application

In this section we address Problem 18.7.8 of [9], the so called ‘bottom theorem’. Let \( K \) be a Hilbertian field and \( e \geq 1 \) an integer. The problem asks whether for almost all \( \sigma \) the field \( M = K_s(\sigma) \) has no cofinite subfield (that is, \( N \not\subseteq M \) implies \( [M : N] = \infty \)).

Note that the Hilbertian field \( K = \mathbb{F}_p(t) \) has imperfect degree \( p \), i.e., \( [K : \mathbb{F}_p] = p \). Moreover, the imperfect degree is preserved under separable extensions (see [9, Lemma 2.7.3]), and hence every separable extension \( M/K \) satisfies \( [M : M^p] = p \). In other words, the problem requires a small modification.

Conjecture 8.2.1. Let \( K \) be a Hilbertian field and \( e \geq 1 \) an integer. Then for almost all \( \sigma \in \text{Gal}(K)^e \) and for every proper subfield \( N \not\subseteq K_s(\sigma) \), if the extension \( K_s(\sigma)/N \) is separable, then it is infinite.

In [11] Haran proves an earlier version of this conjecture, namely with the additional assumption that \( K \subseteq N \). The following result settles the conjecture for finitely generated fields.

Theorem 8.2.2. Let \( K \) be a finitely generated infinite field and \( e \geq 1 \) an integer. Then for almost all \( \sigma \in \text{Gal}(K)^e \) and for every proper subfield \( N \not\subseteq K_s(\sigma) \), if the extension \( K_s(\sigma)/N \) is separable, then it is infinite.

Proof. Clearly \( N \) contains a finitely generated infinite field \( F \). By Corollary 8.1.7 \( K_s(\sigma)/F \) is PAC. Hence \( K_s(\sigma)/N \) is PAC. The assertion now follows from Corollary 4.2.4. \( \square \)
8.3 Algebraic Extensions of Hilbertian Fields

Throughout this section let $K$ be a countable Hilbertian field and $F/K$ an extension. Since $K$ has an abundance of algebraic PAC extensions (Theorem 8.1.6) we can look on the following family of PAC extensions of $F$.

$$\{MF \mid M/K \text{ is an algebraic PAC extension}\}.$$

In this section we discuss when this family contains non separably-closed fields. For applications it is more important to understand the following set of positive integers.

$$\mathcal{N}(F) = \{n \geq 1 \mid \exists M/F \text{ PAC and a separable extension } N/M \text{ such that } n = [N : M]\}.$$

Note that always $1 \in \mathcal{N}(F)$. Also if there exists a non separably-closed PAC extension $M/F$, then $\mathcal{N}(F)$ is infinite. Indeed, $M$ is not a formally real field since $X^2 + Y^2 + 1 = 0$ has a solution. Therefore, by Artin-Schreier Theorem [21, Corollary 9.3], $M$ has infinitely many separable extensions.

We start by giving a general result about the infinitude of $\mathcal{N}(F)$ in terms of the absence of free subgroups in $\text{Gal}(F/K)$.

**Theorem 8.3.1.** Assume $F/K$ is Galois and for some $e \geq 1$ no closed subgroup of $\text{Gal}(F/K)$ is isomorphic to $\hat{F}_e$. Then $\mathcal{N}(F)$ is infinite.

**Proof.** Let $\sigma \in \text{Gal}(K)^e$ be an $e$-tuple such that $M = K_s(\sigma)$ is a PAC extension of $K$ and $\langle \sigma \rangle \cong \hat{F}_e$ (Theorem A).

It suffices to show that $MF \neq K_s$. If $MF = K_s$, then

$$\text{Gal}(F/F \cap M) \cong \text{Gal}(K_s/M) = \langle \sigma \rangle \cong \hat{F}_e,$$

a contradiction. $\Box$

In what follows we shall describe $\mathcal{N}(F)$ more explicitly for more specific extensions. We shall use the following simple group theoretic lemma.

**Lemma 8.3.2.** Let $N \leq N_0 \leq G$ be profinite groups such that $N$ is normal in $G$. Let $H$ be a quotient of $G$ such that $H$ and $G/N$ have no common nontrivial quotients. Then, $H$ is a quotient of $N_0$. In particular, if $H$ has an open subgroup of index $n$, so does $N_0$.

**Proof.** Let $U \vartriangleleft G$ such that $G/U = H$. Since $G/NU$ is a common quotient of $G/U$ and $G/N$, we get that $G/NU = 1$, so $G = NU$. Therefore also $N_0U = G$, and hence $N_0/N_0 \cap U \cong N_0U/U = G/U = H$. $\Box$
8.3.1 Pro-solvable Extensions

Recall that a finite separable field extension is called **solvable** if the Galois group of its Galois closure is solvable. Then an algebraic separable field extension is called **pro-solvable** if it is the compositum of solvable extensions.

**Theorem 8.3.3.** Let \( F \) be a pro-solvable extension of a countable Hilbertian field \( K \) and \( e \geq 2 \). Then for almost all \( \sigma \in \text{Gal}(K)^e \) the field \( M = FK_s(\sigma) \) is a PAC extension of \( F \) and it has a separable extension of every degree \( > 4 \). In particular

\[
\{5, 6, 7, \ldots\} \subseteq N(F).
\]

**Proof.** Let \( \hat{F} \) be the Galois closure of \( F/K \), so \( \text{Gal}(\hat{F}/K) \) is pro-solvable. For almost all \( \sigma \in \text{Gal}(K)^e \) the field \( K_s(\sigma) \) is a PAC extension of \( K \) and its absolute Galois group \( G = \langle \sigma \rangle \) is a free profinite group of rank \( e \) (Theorem A). Fix such \( \sigma \) and write \( M = FK_s(\sigma) \). Then \( M/F \) is PAC. Let \( N_0 = \text{Gal}(M) \) be the absolute Galois group of \( M \) and let \( N = \text{Gal}(\hat{FK}_s(\sigma)) \). Then \( N \leq N_0 \leq G \), \( N \) is normal in \( G \), and \( G/N = \text{Gal}(\hat{FK}_s(\sigma)/K_s(\sigma)) \). The restriction map \( \text{Gal}(\hat{FK}_s(\sigma)/K_s(\sigma)) \rightarrow \text{Gal}(\hat{F}/K) \) is an embedding, so \( G/N \) is pro-solvable.

Let \( n > 4 \), we show that \( M \) has a separable extension of degree \( n \). By the Galois correspondence, it suffices to show that \( N_0 \) has an open subgroup of index \( n \). As \( G \) is free of rank \( \geq 2 \) it has \( A_n \) (the alternating group) as a quotient. Now \( A_n \) and \( G/N \) have no nontrivial common quotients (as \( G/N \) is pro-solvable and \( A_n \) is simple). Lemma 8.3.2 with \( H = A_n \) implies that \( N_0 \) has an open subgroup of index \( n \) (since \( (A_n : A_{n-1}) = n \)). \( \square \)

8.3.2 Prime-to-\( p \) Extensions

Let \( p, m, k \) be positive integers such that \( p \) is prime, \( p \nmid m \), and \( p \mid \varphi(m) \). Here \( \varphi \) is Euler’s totient function. Fix \( \alpha \in (\mathbb{Z}/m\mathbb{Z})^* \) of (multiplicative) order \( p \). Then \( \mathbb{Z}/p^k\mathbb{Z} \) acts on \( \mathbb{Z}/m\mathbb{Z} \) (from the left) by \( xb := \alpha^x b, x \in \mathbb{Z}/p^k\mathbb{Z}, b \in \mathbb{Z}/m\mathbb{Z} \).

Consider the semidirect product \( H = \mathbb{Z}/m\mathbb{Z} \rtimes \mathbb{Z}/p^k\mathbb{Z} \) of all pairs \((a, x), a \in \mathbb{Z}/m\mathbb{Z} \) and \( x \in \mathbb{Z}/p^k\mathbb{Z} \) with multiplication given by

\[
(a, x)(b, y) = (a + \alpha^x b, x + y).
\]

In particular

\[
(a, 1)^n = (a(1 + \alpha + \alpha^2 + \cdots + \alpha^{n-1}), n) = \left( \frac{a(1 - \alpha^n)}{1 - \alpha}, n \right).
\]

We embed \( \mathbb{Z}/m\mathbb{Z} \) and \( \mathbb{Z}/p^k\mathbb{Z} \) in \( H \) in the natural way.
Lemma 8.3.4. Let $p, m, k$, and $H = \mathbb{Z}/m\mathbb{Z} \rtimes \mathbb{Z}/p^k\mathbb{Z}$ be as above. Then

a. $H$ is quasi-$p$ (i.e. it is generated by its $p$-Sylow subgroups).

b. The only prime-to-$p$ quotient of $H$ is the trivial quotient.

c. For $n \mid |H| = p^km$, there exists a subgroup of $H_0$ of order $n$.

Proof. a. The elements $(0, 1)$ and $(1, 1) = (1, 0)(0, 1)$ generate $H$, so it suffices to show that their order divides $p^k$ (and in fact equals to $p^k$). We have $(0, 1)^{p^k} = (0, p^k) = (0, 0)$ and $(1, 1)^{p^k} = \left(\frac{1 - \alpha^{p^k}}{1 - \alpha}, 0\right) = (0, 0)$.

b. follows from a.: Indeed, let $\bar{H} = H/N$ be a quotient of $H$ with order prime-to-$p$. Thus $p^k$ divides the order of $N$, and hence $N$ contains a $p$-Sylow subgroup of $H$. As $N \triangleleft H$, it contains all the $p$-Sylow subgroups. By a., $N = H$ and $\bar{H} = 1$, as desired.

To show c., assume $n \mid p^km$. Then $n = n_0q$, where $n_0 \mid m$ and $q \mid p^k$. Let $A, B$ be the unique subgroups of $\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/p^k\mathbb{Z}$ of order $n_0, q$, respectively. Because of the uniqueness, $A$ is $B$-invariant. Hence $A \rtimes B \leq H$. Obviously $H_0 = A \rtimes B$ is of order $n$. □

Recall that a finite extension $F/K$ is prime-to-$p$ if $p$ does not divide $[F : K]$. An infinite algebraic extension is prime-to-$p$ if any finite subextension is.

Theorem 8.3.5. Let $p$ be a prime, let $F$ be a separable extension of a countable Hilbertian field $K$ whose Galois closure is prime-to-$p$ over $K$. Let $e \geq 2$. Then for almost all $\sigma \in \text{Gal}(K)^e$ the field $M = FK_\sigma(\sigma)$ is a PAC extension of $F$ and it has a separable extension of every degree. In particular, $N(F) = \mathbb{Z}_+^e$.

Proof. Let $\hat{F}$ be the Galois closure of $F/K$, so every finite quotient of $\text{Gal}(\hat{F}/K)$ has order prime to $p$. By Theorem A, for almost all $\sigma \in \text{Gal}(K)^e$ the field $M_0 = K_\sigma(\sigma)$ has a free absolute Galois group of rank $e$, namely $G = \langle \sigma \rangle$, and $M = M_0F$ is a PAC extension of $F$. Let $N_0 = \text{Gal}(M)$ be the absolute Galois group of $M$ and let $N = \text{Gal}(\hat{F}K_\sigma(\sigma))$. Then $N \leq N_0 \leq G$, $N$ is normal in $G$, and $G/N = \text{Gal}(\hat{F}K_\sigma(\sigma)/K_\sigma(\sigma))$. The restriction map $G/N \to \text{Gal}(\hat{F}/K)$ is an embedding, so every finite quotient of $G/N$ has order prime to $p$ (because it is a subgroup of a finite quotient of $\text{Gal}(\hat{F}/K)$).

By the Galois correspondence, it suffices to show that $N_0$ has open subgroups of any index. Let $n$ be a positive integer prime to $p$ and $k \geq 1$. Let $l$ be a prime number such that $l \nmid n$ and $p \mid l - 1$ (such a prime $l$ exists since there are infinitely many primes in the
arithmetic progression \( l \equiv 1 \pmod{p} \)) and let \( m = nl \). Then \( p \nmid m \) and \( p \mid \varphi(m) \). Let \( H = \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/p^k\mathbb{Z} \) as in Lemma 8.3.4. Then \( H \) and \( G/N \) have no nontrivial common quotients (by Lemma 8.3.4b.). By Lemma 8.3.2 and Lemma 8.3.4c., \( N_0 \) has open subgroups of index \( n \) and \( np^k \), i.e., of any index. \( \square \)
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