DENOMINATORS OF BOUNDARY SLOPES FOR (1,1)-KNOTS

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Abstract. We show that for every odd integer \( n > 1 \) the \((2, -3, n)\)-pretzel knot is a hyperbolic \((1,1)\)-knot whose exterior in \( S^3 \) contains an essential surface with boundary slope \( 2(n - 1)^2/n \) and Euler characteristic \( -n \).

1. Introduction

Embedded essential (i.e., incompressible and \( \partial \)-incompressible) surfaces have long been crucial to the study of 3-manifolds. The most general strategy for constructing essential surfaces in a 3-manifold \( M \) is established in [4], which shows how to associate essential surfaces to ideal points of irreducible curves in the character variety equal to the complex algebraic set of characters of \( \text{SL}_2 \mathbb{C} \)-representations of \( \pi_1(M) \). This technique is especially effective in identifying the boundary slopes of \( M \), i.e., the elements of \( \pi_1(\partial M) \) represented by boundary components of essential surfaces. These in turn are helpful in understanding the Dehn fillings on \( M \), which is important because every closed 3-manifold can be obtained by Dehn filling on the exterior of a knot or link in \( S^3 \). Results in this vein include [3] for fillings that yield manifolds with cyclic fundamental group and [1] for fillings that yield manifolds with finite fundamental group. Investigations of boundary slopes for knot exteriors in \( S^3 \) include [11] for two-bridge knots, [9] for two-bridge links, and [10] for Montesinos knots \( K(p_1/q_1, \ldots, p_n/q_n) \) obtained by connecting \( n \geq 3 \) rational tangles of non-integral slopes \( p_1/q_1, \ldots, p_n/q_n \) with each pair \( (p_i, q_i) \) relatively prime (these conditions will be assumed henceforth) in a simple cyclic pattern that yields a knot if just one \( q_i \) is even or if all \( q_i \) are odd and the number of odd \( p_i \) is odd; these are classified by the following (Classification Theorem 1.2 of [17]).

Theorem 1.1. Montesinos knots \( K(p_1/q_1, \ldots, p_n/q_n) \) with \( n \geq 3 \) are classified by the sum \( \sum p_i/q_i \) and the vector \( (p_1/q_1, \ldots, p_n/q_n) \mod 1 \) up to cyclic permutation and reversal of order.

Two-bridge knots are the Montesinos knots with \( n < 3 \), and \((q_1, \ldots, q_n)\)-pretzel knots are the Montesinos knots \( K(1/q_1, \ldots, 1/q_n) \); we also recall the following modifications of Theorem III of [13] and Corollary 5 of [15] respectively.

Theorem 1.2. A \((q_1, \ldots, q_n)\)-pretzel knot with \( n \geq 3 \) is a torus knot if and only if \( n = 3 \) and \((q_1, q_2, q_3) \) is a cyclic permutation of either \((-2\epsilon, 3\epsilon, 3\epsilon)\) or \((-2\epsilon, 3\epsilon, 5\epsilon)\) where \( \epsilon = \pm 1 \).

Proposition 1.3. A Montesinos knot is hyperbolic if it is not a torus knot.

Following [14], the tunnel number \( t(K) \) of a knot \( K \) in \( S^3 \) is the minimum number of mutually disjoint arcs \( \{\tau_i\} \) properly embedded in \( S^3 \setminus K \) such that the exterior of \( K \cup (\cup \tau_i) \) is a handlebody, and \( K \) has a \((g,b)\)-decomposition if there is a genus \( g \) Heegaard splitting \( \{W_1, W_2\} \) of \( S^3 \) such that \( K \) intersects each \( W_i \)
in a $b$-string trivial arc system; then $t(K) \leq g + b - 1$, so $K$ has tunnel number one if it admits a (1,1)-decomposition, i.e., is a (1,1)-knot, a type of knot that has attracted much attention recently (e.g., [7] and [8] investigate closed and meridional essential surfaces in (1,1)-knot exteriors). A related result for Montesinos knots is the following taken from Theorem 2.2 and a closing remark of [14].

**Theorem 1.4.** If $K = K(p_1/q_1,p_2/q_2,p_3/q_3)$ is a Montesinos knot with $q_1 = 2$ and $q_2 \equiv q_3 \equiv 1 \mod 2$ up to cyclic permutation of the indices, then $K$ has tunnel number one and admits a (1,1)-decomposition.

The denominators of boundary slopes are particularly important; e.g., [6] conjectures that there are no essential surfaces in two-bridge link exteriors with boundary in only one component and slope $1/n$ if $n \geq 6$. This in turn implies that all tunnel number one knot exteriors containing a closed essential surface of genus at least two are hyperbolic. Another result concerning the denominators of boundary slopes is the following upper bound taken from Theorem 1.1 of [12].

**Theorem 1.5.** If $K$ is a Montesinos knot with $n \geq 3$ rational tangles and is not a $(-2,3,t)$-pretzel knot for odd $t \geq 3$, $S$ an essential surface properly embedded in $S^3 \setminus K$ with boundary slope $p/q$ where $q > 0$ and $p$ and $q$ are relatively prime, and $\chi(S)$ and $\#b(S)$ the Euler characteristic and number of boundary components of $S$, then $q \leq -\chi(S)/\#b(S)$.

In this paper, we use an algorithm of [10] also presented in [2], [12], and [16] to show that the upper bound in Theorem 1.5 is achieved for an infinite class of hyperbolic (1,1)-knots by proving the following.

**Theorem 1.6.** For any odd integer $n > 1$, the $(2, -3, n)$-pretzel knot, which is the Montesinos knot

$$K_n = K\left(\frac{1}{n}, \frac{2}{3}, -\frac{1}{2}\right),$$

is a hyperbolic (1,1)-knot whose exterior in $S^3$ contains an essential surface with boundary slope $2(n - 1)^2/n$ and Euler characteristic $-n$.

![Figure 1](image_url)
The $(2, −3, n)$-pretzel knot in Figure 1 is the Montesinos knot $K(1/2, −1/3, 1/n)$ and hence $K_n$ by Theorem 1.1. This is not a torus knot by Theorem 1.2 and thus is hyperbolic by Proposition 1.3. Theorem 1.4 establishes that $K_n$ has tunnel number one and admits a $(1,1)$-decomposition, i.e., is a $(1,1)$-knot. That $S^3 \setminus K_n$ contains an essential surface with boundary slope $2(n − 1)^2/n$ and Euler characteristic $−n$ will be shown in Section 3 after reviewing the algorithm of [10] in Section 2.

2. Preliminaries

Let $K = K(p_1/q_1, \ldots, p_n/q_n)$ be a Montesinos knot with $n \geq 3$. Our computation of boundary slopes for $K$ follows the algorithm of [10] as also presented in [2], [12], and [16]. See those for details, but briefly [10] shows how to associate candidate surfaces to admissible edgopath systems in a graph $D$ in the $uv$-plane whose vertices $(u, v)$ correspond to projective curve systems $[a, b, c]$ on the 4-punctured sphere carried by the train track in Figure 2(a) via $u = b/(a + b)$ and $v = c/(a + b)$. Specifically, the vertices of $D$ are:

- the $\infty$-tangle $(\infty)$ in Figure 2(b) with $uv$-coordinates $(-1, 0)$,
- the $p/q$-circles $⟨p/q⟩$ whose $uv$-coordinates $(1, p/q)$ correspond to the projective curve system $[0, q, p]$, and
- the $p/q$-tangles $⟨p/q⟩$ whose $uv$-coordinates $((q − 1)/q, p/q)$ correspond to the projective curve system $[1, q − 1, p]$.

![Figure 2. Curve systems and tangles.](image)

If $|ps − qr| = 1$, then $[(p/q), (r/s)]$ is a non-horizontal edge in $D$ connecting $⟨r/s⟩$ to $⟨p/q⟩$; the remaining edges in $D$ are:

- the horizontal edges $[⟨p/q⟩, (p/q)∧]$ connecting $⟨p/q⟩∧$ to $⟨p/q⟩$,
- the vertical edges $[⟨m⟩, ⟨m + 1⟩]$ connecting $⟨m + 1⟩$ to $⟨m⟩$, and
- the infinity edges $[⟨\infty⟩, (m)]$ connecting $⟨m⟩$ to $⟨\infty⟩$

for any integer $m$; Figure 3 shows part of the graph $D$.

Rational points $(p/q, r/s) ∈ D ∩ \mathbb{Q}^2$ need not be vertices of $D$ and correspond to the projective curve systems $[s(q − p), sp, rq]$. If $[(p/q), (r/s)]$ is a non-horizontal edge in $D$, then $\frac{k}{m} (p/q) + \frac{km}{m} (r/s)$ is a rational point on this edge with coordinates

\[
\begin{align*}
\frac{k(q − s) + m(s − 1)}{k(q − s) + ms}, & \quad \frac{k(p − r) + mr}{k(q − s) + ms},
\end{align*}
\]

for rational $k$ and integer $m$. The algorithm of [10] constructs these edges by finding a path from a vertically oriented edge $(1, p/q)$ to an edge $(p/q, r/s)$ via an expression of the form $[k/p, k/q]$ with $k/p + k/q = 1$. The steps in the algorithm of [10] are:

1. Start at $(1, p/q)$ and move vertically down to $(p/q, r/s)$.
2. If the vertical distance from $(p/q, r/s)$ to $(1, p/q)$ is $m$, then $m = r/s − p/q$.
3. Choose $k$ such that $k/p + k/q = 1$ and $k/q = r/s − p/q$.
4. The rational point on the edge is $\frac{k}{m} (p/q) + \frac{km}{m} (r/s)$.

This algorithm is used to find the essential surface with boundary slope $2(n − 1)^2/n$ and Euler characteristic $−n$. The essential surface is constructed by starting at $⟨\infty⟩$ and following the algorithm to find a path to $⟨p/q⟩$ for each $p/q$ in the set $[1/n, 2/n, 3/n, \ldots, n/n]$, and then connecting these paths to form an essential surface with boundary slope $2(n − 1)^2/n$ and Euler characteristic $−n$.
An edgepath in $\mathcal{D}$ is a piecewise linear path $[0, 1] \to \mathcal{D}$ that begins and ends at rational points (not necessarily vertices) of $\mathcal{D}$. An admissible edgpath system $\gamma = (\gamma_1, \ldots, \gamma_n)$ is an $n$-tuple of edgpaths in $\mathcal{D}$ such that:

(E1) Each starting point $\gamma_i(0)$ lies on the horizontal edge $[\langle p_i/q_i \rangle, \langle p_i/q_i \rangle^\circ]$, and $\gamma_i$ is constant if $\gamma_i(0) \neq \langle p_i/q_i \rangle$.

(E2) Each $\gamma_i$ is minimal, i.e., it never stops and retraces itself, and it never travels along two sides of a triangle in $\mathcal{D}$ in succession.

(E3) The ending points $\gamma_1(1), \ldots, \gamma_n(1)$ all lie on a vertical line (i.e., have the same $u$-coordinates), and their $v$-coordinates sum to zero.

(E4) Each $\gamma_i$ proceeds monotonically from right to left where traversing vertical edges is permitted, i.e., if $0 \leq t_1 < t_2 \leq 1$, then the $u$-coordinate of $\gamma_i(t_1)$ is at least as great as the $u$-coordinate of $\gamma_i(t_2)$.

The aforementioned curve systems on the 4-punctured sphere describe how the boundaries of 3-balls that decompose $S^3$ and each contain a tangle of $K$ intersect properly embedded surfaces in $S^3 \setminus K$, and [10] shows how to associate a finite number of candidate surfaces to each admissible edgpath system; their importance is the following (Proposition 1.1 in [10]).

**Proposition 2.1.** Every incompressible, $\partial$-incompressible surface in $S^3 \setminus K$ with non-empty boundary of finite slope is isotopic to one of the candidate surfaces.

Given an admissible edgpath system $\gamma = (\gamma_1, \ldots, \gamma_n)$ in $\mathcal{D}$, the final $r$-value of each edgpath $\gamma_i$ is the denominator of the $v$-coordinate at the point where the rightward extension of the final edge of $\gamma_i$ intersects the vertical line $u = 1$. The sign of the final $r$-value is negative if this final edge travels downward from right to left. The cycle of final $r$-values of $\gamma$ is the $n$-tuple of final $r$-values of the edgpaths $\gamma_1, \ldots, \gamma_n$; its importance is the following (Corollary 2.4 of [10]).

![Figure 3. Part of the graph $\mathcal{D}$.](image_url)
Proposition 2.2. A candidate surface is incompressible unless the cycle of final \( r \)-values of its associated admissible edgepath system has one of the following forms: \( (0, r_2, \ldots, r_n), (1, \ldots, 1, r_n) \), or \( (1, \ldots, 1, 2, r_n) \).

To compute the boundary slope of a candidate surface, \([10]\) establishes the following algorithm. The twist number of a candidate surface \( S \) associated to an admissible edgepath system \( \gamma \) is \( \tau(S) = 2(e_- - e_+) \), where \( e_+ \) (\( e_- \)) is the number of edges of \( \gamma \) that travel upward (downward) from right to left (infinity edges are not counted). Fractional values of \( e_\pm \) correspond to edges of \( \gamma \) that only traverse a fraction of an edge in \( D \), i.e., the segment from \( \langle r/s \rangle \) to \( \frac{p}{q} \langle r/s \rangle + \frac{m}{m} \langle r/s \rangle \) counts as the fraction \( k/m \) of an edge. The boundary slope of \( S \) is \( \tau(S) - \tau(\Sigma) \), where \( \Sigma \) is a Seifert surface for \( K \) that is a candidate surface found in the following manner described on pages 460-461 of \([10]\).

Remark 2.3. A candidate surface associated to an admissible edgepath system \( \gamma = (\gamma_1, \ldots, \gamma_n) \) is a Seifert surface for \( K(p_1/q_1, \ldots, p_n/q_n) \) if one \( q_i \) is even and each \( \gamma_i \) is a minimal edgepath from \( \langle p_i/q_i \rangle \) to \( \langle \infty \rangle \) whose mod 2 reduction uses only one edge of the triangle in \( D \) with vertices \( \langle \infty \rangle \), \( \langle 0 \rangle \), and \( \langle 1 \rangle \) such that the number of odd penultimate vertices of the \( \gamma_i \) is even.

To compute the Euler characteristic of a candidate surface \( S \) associated to an admissible edgepath system \( \gamma = (\gamma_1, \gamma_2, \gamma_3) \) for \( K(p_1/q_1, p_2/q_2, p_3/q_3) \), we observe the following algorithm from Lemma 2.2 and the proof of Theorem 2.1 in \([10]\). Define the length \( |\gamma_i| \) of each \( \gamma_i \) by counting the length of a full edge as 1 and the length of a partial edge from \( \langle r/s \rangle \) to \( \frac{p}{q} \langle r/s \rangle + \frac{m}{m} \langle r/s \rangle \) as \( k/m \). If \( \gamma_i \) is not constant, let \( m_i \) be the least positive integer such that \( m_i |\gamma_i| \in \mathbb{Z} \), \( m = \text{lcm}(m_1, m_2, m_3) \), and \( \chi(\gamma_i) = m(2 - |\gamma_i|) \). Since \( \gamma \) is an admissible edgepath system, the ending points \( \gamma_i(1) \) all have the same \( u \)-coordinate \( b/(a+b) \); the Euler characteristic of \( S \) is

\[
\chi(S) = \sum_{i=1}^{3} \chi(\gamma_i) - 4a - b.
\]

3. Proof of Theorem 1.6

We now prove our result restated here for convenience.

Theorem 1.6. For any odd integer \( n > 1 \), the \((2, -3, n)\)-pretzel knot, which is the Montesinos knot

\[
K_n = K\left(\frac{1}{n}, \frac{2}{3}, -\frac{1}{2}\right),
\]

is a hyperbolic \((1,1)\)-knot whose exterior in \( S^3 \) contains an essential surface with boundary slope \( 2(n-1)^2/n \) and Euler characteristic \(-n\).

Proof. Again, the \((2, -3, n)\)-pretzel knot is the Montesinos knot \( K(1/2, -1/3, 1/n) \) and hence \( K_n \) by Theorem 1.1, see Figure 1. This is not a torus knot by Theorem 1.2 and thus is hyperbolic by Proposition 1.3. Theorem 1.4 establishes that \( K_n \) has tunnel number one and admits a \((1,1)\)-decomposition, i.e., is a \((1,1)\)-knot. We now use the algorithm of \([10]\) described in Section 2 and Formula (2) to show that \( S^3 \setminus K_n \) contains an essential surface with boundary slope \( 2(n-1)^2/n \) and Euler characteristic \(-n\).
Let $\gamma$ be the edgepath system in Figure 4 given by

\[
\begin{align*}
\gamma_1 &= \left[ \frac{n-1}{n} (0) + \frac{1}{n} \left\langle \frac{1}{n} \right\rangle, \left\langle \frac{1}{n} \right\rangle \right] \\
\gamma_2 &= \left[ \frac{1}{n} (0) + \frac{n-1}{n} \left\langle \frac{1}{2} \right\rangle, \left\langle \frac{2}{3} \right\rangle \right] \\
\gamma_3 &= \left[ \frac{1}{n} (-1) + \frac{n-1}{n} \left\langle -\frac{1}{2} \right\rangle, \left\langle -\frac{1}{2} \right\rangle \right].
\end{align*}
\]

To obtain an associated candidate surface, we verify that $\gamma$ satisfies conditions (E1-4):

(E1) $\gamma_1(0) = (1/n)$ lies on the horizontal edge $[(1/n), (1/n)]$, $\gamma_2(0) = (2/3)$ lies on the horizontal edge $[(2/3), (2/3)]$, and $\gamma_3(0) = (-1/2)$ lies on the horizontal edge $[(-1/2), (-1/2)]$; none of the $\gamma_i$ are constant.

(E2) No $\gamma_i$ stops and retraces itself or travels along two sides of a triangle in $D$ in succession.

(E3) Using Formula 1,

\[
\begin{align*}
\gamma_1(1) &= \left[ \frac{n-1}{n} (0) + \frac{1}{n} \left\langle \frac{1}{n} \right\rangle \right] = \left( \frac{n-1}{2n-1}, \frac{1}{2n-1} \right) \\
\gamma_2(1) &= \left[ \frac{1}{n} (0) + \frac{n-1}{n} \left\langle \frac{1}{2} \right\rangle \right] = \left( \frac{n-1}{2n-1}, \frac{n-1}{2n-1} \right) \\
\gamma_3(1) &= \left[ \frac{1}{n} (-1) + \frac{n-1}{n} \left\langle -\frac{1}{2} \right\rangle \right] = \left( \frac{n-1}{2n-1}, \frac{-n}{2n-1} \right),
\end{align*}
\]

which all lie on a vertical line (i.e., have the same $u$-coordinates), and their $v$-coordinates sum to zero.

(E4) Each $\gamma_i$ proceeds monotonically from right to left.
We now find a Seifert surface for $K_n$ reductions that use only the edges $⟨∞⟩$ edgepath system. Let $e$ be the edgepath system in Figure 5 given by $⟨−n⟩$ is a full edge and another $1/n$ by Formula (2).

Since this final edge travels downward from right to left, the final $r$-value of $γ_1$ is $1−n$, the negative of the denominator of the $v$-coordinate of this point of intersection.

Similarly, the final edge of $γ_2$ connects $(1/2) = (1/2, 1/2)$ to $γ_2(1)$, so its slope is 1, and its rightward extension intersects the vertical line $u = 1$ at the point $(1, 1)$. Since this final edge travels downward from right to left, the final $r$-value of $γ_2$ is $−1$, the negative of the denominator of the $v$-coordinate of this point of intersection.

Lastly, the final edge of $γ_3$ connects $⟨−1/2⟩ = (1/2, −1/2)$ to $γ_3(1)$, so its slope is 1, and its rightward extension intersects the vertical line $u = 1$ at the point $(1, 0)$. Since this final edge travels downward from right to left, the final $r$-value of $γ_1$ is $−1$, the negative of the denominator of the $v$-coordinate of this point of intersection regarding 0 as 0/1.

Thus, the cycle of final $r$-values of $γ$ is $(1−n, −1, −1)$, so $S_n$ is incompressible by Proposition 2.2. To compute its Euler characteristic, we note $|γ_1| = (n−1)/n$, $|γ_2| = (n+1)/n$, and $|γ_3| = 1/n$, so $n$ is the least positive integer such that $n|γ_i| ∈ \mathbb{Z}$ for all $i$. Thus, $χ(γ_1) = n+1$, $χ(γ_2) = n−1$, and $χ(γ_3) = 2n−1$. The $u$-coordinate of the ending points $γ_i(1)$ is $(n−1)/(2n−1)$, so $a = n$, $b = n−1$, and $χ(S) = −n$ by Formula 3.

To compute the boundary slope of $S_n$, we note that $γ_1$ is $(n−1)/n$ of an edge, $γ_2$ is a full edge and another $1/n$ of an edge, and $γ_3$ is $1/n$ of an edge, all of which travel downward from right to left, so $e_+ = 0$, $e_− = (2n+1)/n$, and $τ(S_n) = (4n+2)/n$. We now find a Seifert surface for $K_n$ that is a candidate surface for an admissible edgepath system. Let $δ$ be the edgepath system in Figure 5 given by

\[
δ_1 = \left[ ⟨∞⟩, ⟨1⟩, \frac{1}{2}, \ldots, \frac{1}{n} \right]
\]

\[
δ_1 = \left[ ⟨∞⟩, ⟨0⟩, \frac{1}{2}, \frac{2}{3} \right]
\]

\[
δ_2 = \left[ ⟨∞⟩, ⟨−1⟩, −\frac{1}{2} \right].
\]

We first verify that $δ$ satisfies conditions (E1-4):

(E1) $δ_1(0) = ⟨1/n⟩$ lies on the horizontal edge $⟨(1/n), (1/n)^0⟩$, $δ_2(0) = ⟨2/3⟩$ lies on the horizontal edge $⟨(2/3), (2/3)^0⟩$, and $δ_3(0) = ⟨−1/2⟩$ lies on the horizontal edge $⟨(−1/2), (−1/2)^0⟩$; none of the $δ_i$ are constant.

(E2) No $δ_i$ stops and retraces itself or travels along two sides of a triangle in $D$ in succession.

(E3) Each $δ_i(1) = ⟨∞⟩ = (−1, 0)$, so they all lie on a vertical line (i.e., have the same $u$-coordinates), and their $v$-coordinates sum to zero.

(E4) Each $δ_i$ proceeds monotonically from right to left.

Hence, $δ$ is an admissible edgepath system with one $q_i$ even and each $δ_i$ a minimal edgepath from $(p_i/q_i)$ to $⟨∞⟩$ with penultimate vertices $(1)$, $(0)$, and $−(1)$ and mod 2 reductions that use only the edges $[⟨∞⟩, ⟨1⟩], [⟨∞⟩, ⟨0⟩], and [⟨∞⟩, ⟨1⟩]$ respectively,
so an associated candidate surface $\Sigma_n$ is a Seifert surface for $K_n$ by Remark 2.3.

Ignoring the infinity edges, $\delta_1$ consists of $n - 1$ edges traveling upward from right to left, $\delta_2$ consists of two edges traveling downward from right to left, and $\delta_3$ consists of one edge traveling downward from right to left, so $e_+ = n - 1$, $e_- = 3$, and $\tau(\Sigma_n) = 8 - 2n$. Therefore, the boundary slope of $S_n$ is $\tau(F_n) - \tau(\Sigma_n) = 2(n - 1)^2/n$. ■

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References

[1] S. Boyer and X. Zhang. A proof of the finite filling conjecture. J. Differential Geom., 59(1):87–176, 2001.
[2] E. Chesebro and S. Tillmann. Not all boundary slopes are strongly detected by the character variety. Comm. Anal. Geom., 15(4):695–723, 2007.
[3] M. Culler, C. Gordon, J. Luecke, and P. Shalen. Dehn surgery on knots. Ann. of Math. (2), 125(2):237–300, 1987.
[4] M. Culler and P. Shalen. Varieties of group representations and splittings of 3-manifolds. Ann. of Math. (2), 117(1):109–146, 1983.
[5] N. Dunfield. A table of boundary slopes of Montesinos knots. Topology, 40(2):309–315, 2001.
[6] M. Eudave-Muñoz. Incompressible surfaces in tunnel number one knot complements. Topology Appl., 98(1-3):167–189, 1999. II Iberoamerican Conference on Topology and its Applications (Morelia, 1997).
[7] M. Eudave-Muñoz. Incompressible surfaces and (1, 1)-knots. J. Knot Theory Ramifications, 15(7):935–948, 2006.
[8] M. Eudave-Muñoz and E. Ramírez-Losada. Meridional surfaces and (1, 1)-knots. Trans. Amer. Math. Soc., 361(2):671–696, 2009.
[9] W. Floyd and A. Hatcher. The space of incompressible surfaces in a 2-bridge link complement. Trans. Amer. Math. Soc., 305(2):575–599, 1988.
[10] A. Hatcher and U. Oertel. Boundary slopes for Montesinos knots. *Topology*, 28(4):453–480, 1989.
[11] A. Hatcher and W. Thurston. Incompressible surfaces in 2-bridge knot complements. *Invent. Math.*, 79(2):225–246, 1985.
[12] K. Ichihara and S. Mizushima. Bounds on numerical boundary slopes for Montesinos knots. *Hiroshima Math. J.*, 37(2):211–252, 2007.
[13] A. Kawauchi. Classification of pretzel knots. *Kobe J. Math.*, 2(1):11–22, 1985.
[14] K. Morimoto, M. Sakuma, and Y. Yokota. Identifying tunnel number one knots. *J. Math. Soc. Japan*, 48(4):667–688, 1996.
[15] U. Oertel. Closed incompressible surfaces in complements of star links. *Pacific J. Math.*, 111(1):209–230, 1984.
[16] Y-Q. Wu. The classification of toroidal Dehn surgeries on Montesinos knots. *Comm. Anal. Geom.*, 19(2):305–345, 2011.
[17] H. Zieschang. Classification of Montesinos knots. In *Topology (Leningrad, 1982)*, volume 1060 of *Lecture Notes in Math.*, pages 378–389. Springer, Berlin, 1984.

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