A perturbative re-analysis of $\mathcal{N}=4$ supersymmetric Yang–Mills theory

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Abstract

The finiteness properties of the $\mathcal{N}=4$ supersymmetric Yang–Mills theory are reanalyzed both in the component formulation and using $\mathcal{N}=1$ superfields, in order to discuss some subtleties that emerge in the computation of gauge dependent quantities. The one-loop corrections to various Green functions of elementary fields are calculated. In the component formulation it is shown that the choice of the Wess–Zumino gauge, that is standard in supersymmetric gauge theories, introduces ultraviolet divergences in the propagators at the one-loop level. Such divergences are exactly cancelled when the contributions of the fields that are put to zero in the Wess–Zumino gauge are taken into account.

In the description in terms of $\mathcal{N}=1$ superfields infrared divergences are found for every choice of gauge different from the supersymmetric generalization of the Fermi–Feynman gauge. Two-, three- and four-point functions of $\mathcal{N}=1$ superfields are computed and some general features of the infrared problem are discussed.

We also examine the effect of the introduction of mass terms for the (anti) chiral superfields in the theory, which break supersymmetry from $\mathcal{N}=4$ to $\mathcal{N}=1$. It is shown that in the mass deformed model no ultraviolet divergences appear in two-point functions. It argued that this result can be generalized to $n$-point functions, supporting the proposal of a possible of use of this modified model as a supersymmetry-preserving regularization scheme for $\mathcal{N}=1$ theories.
1 Introduction

The action of $\mathcal{N}=4$ four dimensional supersymmetric Yang–Mills theory was given for the first time in [1, 2] in the context of toroidal compactifications of the type I superstring. The theory has the maximal amount of supersymmetry allowed for a rigid supersymmetric theory in four dimensions, namely the symmetry generated by sixteen real supercharges, and has been proved to be finite, possessing a vanishing $\beta$-function. For this reason it has long been considered to be a rather trivial theory. However recent developments in the study of the correspondence with type IIB superstring theory on anti de Sitter space have shown that it is actually a very interesting quantum field theory, displaying very peculiar properties.

The field content of the theory, which is unique apart from the choice of the gauge group $G$, consists of six real scalars, four Weyl spinors and one vector, which are all in the adjoint representation of the gauge group $G$. In the Abelian case the theory is free, whereas in the non Abelian case it has a moduli space of vacua parameterized by the vacuum expectation values (vev’s) of the six real scalars. In the Coulomb phase, reached giving non vanishing vev’s to the scalars in the Cartan subalgebra of $G$, the theory is again free, while the origin of the moduli space corresponds to a highly non-trivial superconformal field theory.

The model is classically invariant under the $\mathcal{N}=4$ superconformal group and this property is supposed to be preserved at the quantum level [3, 4]. Furthermore it has a global $\text{SU}(4)\sim \text{SO}(6)$ symmetry, which naturally emerges in the compactification from ten dimensions on a six-torus and is identified, in a four dimensional perspective, with the R-symmetry group of automorphisms of the $\mathcal{N}=4$ superconformal algebra [5].

The $\mathcal{N}=4$ super Yang-Mills theory is supposed to exactly realize a generalization of the electric-magnetic duality of Montonen and Olive [6] (S-duality), thanks to the presence in the spectrum of an infinite tower of stable BPS dyonic states [7, 8, 9].

At the perturbative level the theory is finite up to three loops. The $\beta$-function has been shown to be vanishing both in the component formulation [10, 11, 12] and using superspace techniques [13, 14]. In particular the formulation in terms of $\mathcal{N}=1$ superfields has proved to be an extremely powerful tool in the perturbative analysis. Strong arguments have been proposed in order to extend the proof of the finiteness to all orders in perturbation theory [3, 4, 15, 16] and moreover general considerations from instanton calculus and duality arguments suggest that the same should hold at the non-perturbative level [17].

Following the proposal of Maldacena [18] of a correspondence relating (super) conformal field theories in $d$ dimensions and type IIB superstring theory on $d+1$ anti de Sitter space there has been recently a renewal of interest in $\mathcal{N}=4$ supersymmetric Yang–Mills theory. According to this conjectured duality both perturbative [19] and non-perturbative [20] contributions to correlation functions of gauge invariant composite operators in $\mathcal{N}=4$ super Yang–Mills theory should be related to type IIB superstring amplitudes in $\text{AdS}_5\times S^5$. Work in this context is leading to very interesting results in the study of peculiar properties of $\mathcal{N}=4$ Yang–Mills theory as a superconformal field theory.

The aim of this paper is to present a careful re-analysis of perturbation theory, which allows to point out various problems related to the choice of gauge. Both in the component field formulation and using $\mathcal{N}=1$ superfields the gauge fixing procedure appears very
subtle. As we will discuss in detail later, in both cases one finds ultraviolet and/or infrared divergences in off-shell Green functions.

Throughout the paper calculations will be carried out in Euclidean space. Two different formulations of the $\mathcal{N}=4$ supersymmetric Yang–Mills theory will be employed, one in terms of $\mathcal{N}=1$ superfields and the other in component (‘physical’) fields. Notations and conventions that will be used are those of [22]. The $\mathcal{N}=4$ “on-shell” multiplet can be obtained by combining three $\mathcal{N}=1$ chiral superfields and one $\mathcal{N}=1$ vector superfield, so that the six real scalars of the model are assembled into three complex scalars which, together with three of the Weyl spinors, form the chiral multiplets, the fourth spinor and the vector gives rise to the vector multiplet. In this description a SU(3)×U(1) subgroup of the SU(4) R-symmetry group is manifest. The component-field formulation that will be used is directly related to that in $\mathcal{N}=1$ superspace, actually it is obtained from the former by integrating over the Grassmannian coordinates of $\mathcal{N}=1$ superspace.

In the component formulation the propagators of the elementary fields are ultraviolet divergent in the Wess–Zumino (WZ) gauge and these infinities are exactly cancelled when the contributions of the ‘gauge-dependent’ fields, that are put to zero in the WZ gauge, are taken into account. The choice of the WZ gauge, that is almost unavoidable in explicit computations, introduces divergences that require a wave function renormalization. Different problems emerge in the formulation of the theory in terms of $\mathcal{N}=1$ superfields. Almost all of the calculations showing the vanishing of the quantum corrections to two- and three-point functions, that are presented in the literature, were performed in the supersymmetric generalization of the Fermi–Feynman gauge. With a different choice of gauge two- and three-point functions develop infrared singularities, leading to the result that the choice of the Fermi–Feynman gauge is somehow privileged. This conclusion was proposed for the first time in [21], and then it was discussed in the case of the $\mathcal{N}=4$ theory in [23]; however no possible explanation was proposed. Unlike those of the elementary fields, correlation functions of gauge invariant composite operators, that play a crucial rôle in the correspondence with AdS type IIB supergravity/superstring theory, should not suffer from problems related to the gauge fixing.

We will also discuss, in the superfield formulation, the effect of introducing of a mass term for the (anti) chiral superfields, which breaks supersymmetry from $\mathcal{N}=4$ down to $\mathcal{N}=1$. It is shown that this deformation of the model does not modify the ultraviolet properties of the original theory. This result was first proposed in [24], where it was proved that the inclusion of mass terms for the (anti) chiral superfields does not generate divergent corrections to the effective action. We will argue that this statement can be reinforced, showing that, at least at one loop, no new divergences, not even corresponding to wave function renormalizations, appear as a consequence of the addition of the mass terms. This result supports the claim put forward in [25, 26], where the ‘mass deformed’ $\mathcal{N}=4$ theory was proposed as a supersymmetry-preserving regularization scheme for a class of $\mathcal{N}=1$ theories.

The paper is organized as follows. Section 2 deals with the problems introduced by the choice of the Wess–Zumino gauge when the component formalism is used. The following sections report calculations performed in the $\mathcal{N}=1$ superfield formalism of two-, three- and four-point functions. A discussion of the results is presented in the concluding section.
2 Perturbation theory in components: problems with the Wess–Zumino gauge

As already remarked the field content of $\mathcal{N}=4$ super Yang–Mills theory can be obtained by coupling in a gauge invariant way one $\mathcal{N}=1$ vector superfield $V^a(x, \theta, \overline{\theta})$, $I = 1, 2, 3$, all in the adjoint representation of the gauge group $G$, so that the colour index $a$ takes the values $a = 1, 2, \ldots, \dim G$. A SU(3)×U(1) subgroup of the SU(4) R-symmetry group is manifest and under this global symmetry the chiral superfields $\Phi^I$ transform in the $3$, while the vector $V$ is a singlet.

The complete expression for $V$ is

\begin{equation}
V(x, \theta, \overline{\theta}) = C(x) + i\theta \chi(x) - i\overline{\theta} \overline{\chi}(x) + \frac{i}{2} \theta \partial_\sigma S(x) - \frac{i}{2} \overline{\theta} \partial_\overline{\sigma} S(x) - \theta \sigma^\mu \overline{\theta} A_\mu(x) + \\
+ i\overline{\theta} \partial_\overline{\mu} \partial_\mu \chi(x) - i\overline{\theta} \partial_\overline{\mu} \partial_\mu \overline{\chi}(x) + \frac{1}{2} \overline{\theta} \overline{\partial}_\overline{\mu} \partial_\mu D(x) + \frac{1}{2} \Box C(x) .
\end{equation}

The gauge transformations of the superfields take the form

$$\Phi \rightarrow \Phi' = e^{-i \Lambda} \Phi , \quad \Phi^\dagger \rightarrow \Phi^\dagger' = \Phi^\dagger e^{i \Lambda^\dagger}$$

and

$$V \rightarrow V' \quad \text{where} \quad e^{V'} = e^{-i \Lambda^\dagger} e^V e^{i \Lambda} ,$$

where $\Lambda$ is a matrix-valued chiral superfield. For infinitesimal gauge transformations use of the Becker–Hausdorff’s formula allows to write

\begin{equation}
\delta V = V' - V = i \mathcal{L}_{V/2} \left[(\Lambda + \Lambda^\dagger + \coth(\mathcal{L}_{V/2})(\Lambda - \Lambda^\dagger))\right] ,
\end{equation}

where $\mathcal{L}_A(B) = [A, B]$ is the Lie derivative and (3) is a compact form to be understood in terms of the power expansion of $\coth(\mathcal{L})$. By explicitly writing the component expansion of these relations one can show that it is always possible to put to zero the lower components, $C$, $\chi$ and $S$, of $V$ by a suitable gauge transformation. As a result for the superfield $V$ one obtains

\begin{align*}
V(x, \theta, \overline{\theta}) &= -\theta \sigma^\mu \overline{\theta} A_\mu(x) + i\theta \overline{\theta} \partial_\mu \chi(x) - i\overline{\theta} \theta \partial_\mu A_\mu(x) + \frac{1}{2} \theta \theta \overline{\theta} \overline{\theta} D(x) \\
V^2(x, \theta, \overline{\theta}) &= -\frac{1}{2} \theta \theta \overline{\theta} A_\mu(x) A^\mu(x) \\
V^n(x, \theta, \overline{\theta}) &= 0 , \quad \forall n \geq 3 .
\end{align*}

This choice is known as the Wess–Zumino gauge. Fixing the Wess–Zumino gauge still leaves with the ordinary non Abelian gauge freedom on the remaining fields.

In this section we will discuss perturbation theory in components. The one-loop correction to the propagators of elementary fields will be computed first in the Wess-Zumino gauge and then taking into account the contribution of the fields $C$, $\chi$ and $S$ that are absent in this gauge. We will show that the choice of the WZ gauge introduces ultraviolet divergences in the propagators.
The standard formulation of the model is obtained by eliminating the auxiliary fields, \( F^I \) from the chiral multiplet and \( D \) from the vector multiplet. This procedure in the WZ gauge results in a polynomial action with a scalar potential containing a quadrilinear term for the scalars. However in order to correctly deal with the lower components of the vector superfield \( V \), that are put to zero in the Wess–Zumino gauge, it is more convenient not to eliminate the auxiliary fields through their equations of motion, but rather keep them in the action: in the computation of Green functions the corresponding \( x \)-space propagators are \( \delta \)-functions.

The action is written in terms of the non Abelian field strength superfield \( W_{\alpha} \)

\[
W_{\alpha} = -\frac{1}{4} \overline{D} D e^{-V} D_{\alpha} e^{V} = \sum_{k=1}^{\infty} W_{\alpha}^{(k)},
\]

where

\[
W_{\alpha}^{(1)} = -\frac{1}{4} \overline{D} D D_{\alpha} V
\]

\[
W_{\alpha}^{(2)} = \frac{1}{8} \overline{D} D [V, D_{\alpha} V]
\]

and the terms \( W_{\alpha}^{(k)} \), with \( k \geq 3 \), contain \( k \) factors of \( V \) and vanish in the WZ gauge. In Euclidean space the action in the \( N=1 \) superfield formalism takes the form

\[
S^{(E)} = \int d^4x d^2\theta d^2\overline{\theta} \left\{ \left[ \frac{1}{4} W^{(1)\alpha} W^{(1)\alpha} \delta(\overline{\theta}) + \frac{1}{4} \overline{W}_{\alpha}^{(1)\alpha} \overline{W}^{(1)\alpha} \delta(\theta) - \frac{1}{8\alpha} \overline{D} D V D^2 V \right] + \Phi^{I}_I V \Phi^I + \left[ \left( \frac{1}{2} W^{(1)\alpha} W^{(2)\alpha} + \frac{1}{4} W^{(2)\alpha} W^{(2)\alpha} \right) \delta(\theta) + \left( \frac{1}{2} \overline{W}_{\alpha}^{(1)} W^{(2)\alpha} + \frac{1}{4} \overline{W}_{\alpha}^{(2)} \overline{W}^{(2)\alpha} \right) \delta(\theta) \right] + \frac{1}{2} \Phi^{I}_I V^2 \Phi^I + \ldots \right\},
\]

where a standard gauge fixing term has been included, whereas no ghost term is displayed since it will not be relevant for the computations to be discussed in this section. In equation (5) dots denote terms of higher order in \( V \), which do not contribute to the Green functions that will be considered and thus will be suppressed from now on. In the following calculations the Fermi–Feynman gauge, \( \alpha=1 \), will be used. With this choice one obtains\[1\]

\[
S = \int d^4x d^2\theta d^2\overline{\theta} \left\{ \left[ \frac{1}{2} \overline{V} \overline{\square} V + \Phi^{I}_I \Phi^I \right] + \left[ \left( \frac{1}{2} W^{(1)\alpha} W^{(2)\alpha} + \frac{1}{4} W^{(2)\alpha} W^{(2)\alpha} \right) \delta(\theta) + \left( \frac{1}{2} \overline{W}_{\alpha}^{(1)} W^{(2)\alpha} + \frac{1}{4} \overline{W}_{\alpha}^{(2)} \overline{W}^{(2)\alpha} \right) \delta(\theta) \right] + \Phi^{I}_I V \Phi^I + \frac{1}{2} \Phi^{I}_I V^2 \Phi^I \right\},
\]

where from the definition (4) one gets

\[
W_{\alpha}^{(1)} = -i \lambda_{\alpha} + \left[ \delta_{\alpha}^{\beta} D - \frac{1}{2} \delta_{\alpha}^{\beta} \overline{\square} C - \frac{i}{2} (\sigma^\mu \sigma^\nu)^{\alpha}^{\beta} (\partial_\mu A_\nu - \partial_\nu A_\mu) \right] \theta^\beta + \theta \theta^{\alpha}_\mu \sigma^\mu_\alpha \partial_\mu \overline{\lambda}.
\]

\[1\] From now on the superscript E will be suppressed. Euclidean signature is to be understood unless otherwise stated.
The explicit form of \( W^{(2)}_α \) is not necessary for the moment (see however equation (21) at the end of this section). Expansion of the action (3) in components using the complete expression of \( V \) gives

\[
S = S_0 + S_{\text{int}}.
\]

The free action \( S_0 \) comes from the terms \( \frac{1}{2} V \Box V \) and \( \Phi_I^i \Phi^I \) and reads

\[
S_0 = \int d^4x \left\{ \left( (\partial_\mu \varphi_1^{a\dagger})(\partial^\mu \varphi_1^{a\dagger}) + \bar{\psi}_I \gamma^\mu (\partial_\mu \psi_I^a) + F_I^{a\dagger} F_I^a \right) + \frac{1}{2} S_{\text{int}} \right\}.
\]

Note that the scalar field \( C \) and the spinor \( \chi \) have “wrong” physical dimension, resulting in non standard free propagators, as will be shown below. The interaction part \( S_{\text{int}} \) contains an infinite number of terms. The propagators of the fermions \( \psi^I \) and of the scalars \( \varphi^I \) belonging to the \( \mathcal{N} = 1 \) chiral multiplets will now be computed at one loop. The terms that are relevant for these calculations come from the expansion of \( \Phi_I^i \Phi^IF^I \Phi_{\text{int}} \) in (3). The latter generates tadpole type diagrams in the propagator \( \langle \varphi^I \varphi \rangle \), but not in \( \langle \bar{\psi} \psi \rangle \). The interaction part of the action to be considered is

\[
S_{\text{int}} = \int d^4x \left\{ ig f_{abc} \left\{ -\frac{1}{4} \varphi_1^{a\dagger} C^b (\Box \varphi_1^{c\dagger}) - \frac{1}{4} \varphi_1^{a\dagger} (\Box C^b) \varphi_1^{c\dagger} - \frac{1}{4} (\Box \varphi_1^{a\dagger}) C^b \varphi_1^{c\dagger} + \frac{1}{2} (\partial_\mu \varphi_1^{a\dagger}) C^b (\partial^\mu \varphi_1^{c\dagger}) + \frac{i}{2} \varphi_1^{a\dagger} F_1^{b\dagger} F_1^c - \frac{1}{2} \varphi_1^{a\dagger} \bar{D}^{b\dagger} \varphi_1^{c\dagger} + \frac{i}{2} \varphi_1^{a\dagger} A^b \partial_\mu \varphi_1^{c\dagger} - \frac{1}{2} \varphi_1^{a\dagger} A^b \varphi_1^{c\dagger} - \frac{1}{2} (\partial_\mu \varphi_1^{a\dagger}) A^b \varphi_1^{c\dagger} - \frac{1}{2} (\partial^\mu \varphi_1^{a\dagger}) A^b \varphi_1^{c\dagger} + \frac{1}{2} \varphi_1^{a\dagger} \bar{D}^{b\dagger} \varphi_1^{c\dagger} + \frac{i}{2} \varphi_1^{a\dagger} \psi_1^I \psi_1^J F_1^{I\dagger} F_1^{J\dagger} + \frac{1}{2} (\partial_\mu \chi^b \varphi_1^{c\dagger}) + \frac{1}{2} (\partial_\mu \varphi_1^{a\dagger}) \chi^b - \frac{1}{2} (\partial_\mu \varphi_1^{a\dagger}) \chi^b + \frac{1}{2} (\partial_\mu \psi_1^I) \varphi_1^{a\dagger} \chi^b + \frac{1}{2} (\partial_\mu \chi^b \varphi_1^{a\dagger}) \chi^b + \frac{1}{2} (\partial_\mu \varphi_1^{a\dagger}) \chi^b \right\} \cdot \frac{g^2}{2} f_{abc} f_{cde} \left\{ - C^b C^d \sigma^{de} \right\}
\]

The quadratic part of the action, \( S_0 \), can be written in a more compact form introducing
the notation

\[ B^a(x) = \begin{pmatrix} C^a(x) \\ D^a(x) \end{pmatrix}, \quad \mathcal{F}^a(x) = \begin{pmatrix} \chi^a(x) \\ \overline{\lambda}^a(x) \end{pmatrix}, \]

so that

\[ S_0 = \int d^4x \left\{ \left[ (\partial_\mu \varphi^a_\alpha)(\partial_\mu \varphi^a_\alpha) + \overline{\psi}_I \sigma^\mu(\partial_\mu \psi^I_\alpha) + F^a_I F^a_I \right] + \frac{1}{2} S^I_a \Box S^a + \frac{1}{2} A^a_\mu A^a_\mu + B^a_T MB^a + F^a_T N F^a \right\}, \]

where

\[ M = \frac{1}{2} \left( \begin{array}{ccc} \Box & \Box & 0 \\ \Box & 0 & \Box \\ 0 & \Box & \Box \end{array} \right), \quad N = \frac{1}{2} \left( \begin{array}{ccc} 0 & \Box & \Box \\ \Box & 0 & 0 \\ -\Box & 0 & \Box \end{array} \right). \]

Inverting the kinetic matrices \( M \) and \( N \) one gets the free propagators. From (12) it follows

\[ M^{-1} = \left( \begin{array}{ccc} 0 & 1 & \Box \\ 1 & \Box & 0 \end{array} \right), \quad N^{-1} = \left( \begin{array}{ccc} 0 & -\Box & 0 \\ -\Box & 0 & 1 \\ \Box & 0 & \Box \end{array} \right). \]

so that the free propagators are

- \( J, b \longrightarrow I, a \quad \rightarrow \quad \langle \varphi^b_J(x) \varphi^I_a(y) \rangle_{\text{free}} = -\frac{\delta^b_J \delta^I_a}{\Box} \delta(x-y) = \Delta^b_J(x-y) \)

- \( b \longrightarrow a \quad \rightarrow \quad \langle S^b(x) S^a(y) \rangle_{\text{free}} = \frac{2\delta^b_a}{\Box} \delta(x-y) = 2\Delta^b_a(x-y) \)

- \( b \longrightarrow \times \longrightarrow a \quad \rightarrow \quad \langle C^b(x) D^a(y) \rangle_{\text{free}} = \frac{\delta^b_a}{\Box} \delta(x-y) = \Delta^b_a(x-y) \)

- \( b \longrightarrow a \quad \rightarrow \quad \langle D^b(x) D^a(y) \rangle_{\text{free}} = \delta^b_a \delta(x-y) \)

- \( J, b \longrightarrow I, a \quad \rightarrow \quad \langle F^b_J(x) F^I_a(y) \rangle_{\text{free}} = \delta^I_J \delta^b_a \delta(x-y) \)


\[ \mu, b \quad \longrightarrow \quad \langle A^b_\mu(x)A^\nu_\nu(y) \rangle_{\text{tree}} = -\frac{\delta^b_\mu\delta^\nu_\nu}{\Box} \delta(x-y) = \Delta^b_{\mu\nu}(x-y) \]

\[ \alpha, b \quad \longrightarrow \quad \langle \chi^{b\alpha}(x)\lambda^\beta_\alpha(y) \rangle_{\text{tree}} = -\frac{\varepsilon^{\alpha\beta}\delta^b_\alpha}{\Box} \delta(x-y) = R^{b\alpha\beta}_\alpha(x-y) \]

\[ \dot{\alpha}, b \quad \longrightarrow \quad \langle \chi^{\dot{b}\dot{\alpha}}(x)\dot{\lambda}^\dot{\beta}_\alpha(y) \rangle_{\text{tree}} = -\frac{\varepsilon^{\dot{\alpha}\dot{\beta}}\delta^b_\alpha}{\Box} \delta(x-y) = \bar{R}^{\dot{b}\dot{\alpha}\dot{\beta}}_\alpha(x-y) \]

\[ \dot{\alpha}, b \quad \longrightarrow \quad \langle \chi^{\dot{b}\dot{\alpha}}(x)\lambda^\alpha_\alpha(y) \rangle_{\text{tree}} = \frac{\delta^b_\alpha\sigma^{\mu\dot{\alpha}}_\alpha\partial_\mu}{\Box} \delta(x-y) = S^{b\dot{\alpha}\alpha}_\alpha(x-y) \]

\[ \dot{\alpha}, I, b \quad \longrightarrow \quad \langle \psi^{\dot{b}\dot{\alpha}}_I(x)\psi^{\alpha J}_a(y) \rangle_{\text{tree}} = \frac{\delta^b_\alpha\sigma^{\mu J}_\alpha\partial_\mu}{\Box} \delta(x-y) = S^{bJ}_{a\dot{\alpha}\alpha}(x-y) \]

To summarize beyond the ordinary propagators for the physical fields, \( \varphi, \psi, \lambda \) and \( A_\mu \), and those for the auxiliary fields, \( F \) and \( D \), one further obtains the propagators \( \langle S^I S^I \rangle \), \( \langle CD \rangle \), \( \langle \chi \lambda \rangle \) and \( \langle \chi \lambda \rangle \). The latter are absent in the Wess–Zumino gauge.

### 2.1 One loop corrections to the propagator of the fermions belonging to the chiral multiplet

The one-loop correction to the propagator of the fermions \( \psi^I \) in the chiral multiplet is the simplest to compute. In the WZ gauge there are three contributions at the one loop level, that will be shown to lead to a logarithmically divergent result.

From the action (8), (9), with \( C=\chi=S=0 \) one obtains the following three diagrams:

\[ A_\mu A_\nu \quad \longrightarrow \quad \psi^I_j(x) \quad \longrightarrow \quad \psi^I_b(y) \quad \longrightarrow \quad A^{aI}_{b,j}(x;y) \]

\[ A_\mu A_\nu \quad \longrightarrow \quad \psi^I_j(x) \quad \longrightarrow \quad \psi^I_b(y) \quad \longrightarrow \quad B^{aI}_{b,j}(x;y) \]
Notice that the insertion of tadpoles such as

\[ \psi_j(x) \longrightarrow \psi_b^I(y) \rightarrow C_{bJ}^{\alpha I}(x; y) \]

in a diagram gives a vanishing result because all the propagators are diagonal in colour space, so that the tadpole contains a factor \( \delta_{ab} \varepsilon_{abc} \equiv 0 \). The same is true also for diagrams in \( \mathcal{N}=1 \) superspace that will be discussed in subsequent sections.

The three contributions depicted above can be easily evaluated and give the results

\[ A_{bJ}^{\hat{\alpha} \hat{\alpha} I}(x; y) = -\frac{1}{4} g^2 f^d_{ef} f^l_{mn} \int d^4x_1 d^4x_2 \left\{ \Delta_{\nu\mu}^{me}(x_2 - x_1) \left[ \bar{S}_{I\lambda}^{\hat{\beta}bI}(x - x_2) \sigma_\nu^{\hat{\beta}} \cdot \bar{S}_{dK}^{\hat{\gamma}\gamma L}(x_2 - x_1) \sigma_\lambda^{\hat{\gamma}} S_{bJ}^{\hat{\beta}aI}(x_1 - y) \right] \right\}, \]

\[ B_{bJ}^{\hat{\alpha} \hat{\alpha} I}(x; y) = -\frac{1}{4} g^2 \varepsilon^{LMN} \varepsilon^{PQR} f^d_{ef} f^l_{mn} \int d^4x_1 d^4x_2 \left\{ \Delta_{PL}^{ld}(x_2 - x_1) \cdot \left[ \bar{S}_{JN}^{\hat{\lambda}bI}(x - x_2) S_{\beta\beta MQ}^{em}(x_2 - x_1) S_{bJ}^{\hat{\beta}aI R}(x_1 - y) \right] \right\}, \]

\[ C_{bJ}^{\hat{\alpha} \hat{\alpha} I}(x; y) = -\frac{1}{2} g^2 f^d_{ef} f^l_{mn} \int d^4x_1 d^4x_2 \left\{ \Delta_{JL}^{IK}(x_2 - x_1) \cdot \left[ \bar{S}_{nK}^{\hat{\lambda}bI}(x - x_2) S_{\beta\beta MQ}^{em}(x_2 - x_1) S_{bJ}^{\hat{\beta}aI D}(x_1 - y) \right] \right\}. \]

Taking the Fourier transform one obtains

\[ \tilde{A}_{bJ}^{\hat{\alpha} \hat{\alpha} I}(p) = i \frac{g^2}{4} \delta_\beta^\lambda \delta_\mu^\alpha \bar{\sigma}_\beta^\lambda p_\lambda \bar{S}^{\hat{\beta}aI}(p) \left[ \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k + p/2)^2 (k - p/2)^2} \right], \]

\[ \tilde{B}_{bJ}^{\hat{\alpha} \hat{\alpha} I}(p) = -\tilde{C}_{bJ}^{\hat{\alpha} \hat{\alpha} I}(p) = \tilde{A}_{bJ}^{\hat{\alpha} \hat{\alpha} I}(p), \]

where

\[ \bar{S}^{\hat{\lambda}aI}(p) = -i \frac{\bar{\sigma}_\mu^\alpha p^\mu}{p^2}. \]
As anticipated above, the one-loop correction to the fermion propagator in the Wess–Zumino gauge turns out to be logarithmically ultraviolet-divergent

\[
\langle (\bar{\psi}\psi)_{bJ} \rangle_{\text{1-loop,WZ}}^{a_{\alpha a} I} = \bar{A}_{bJ}^{a_{\alpha a} I}(p) + \bar{B}_{bJ}^{a_{\alpha a} I}(p) + \bar{C}_{bJ}^{a_{\alpha a} I}(p) = \frac{i}{4} g^2 \delta_\alpha^\beta \delta_{J}^{\dagger} \tilde{S}^{\dagger}(p) \sigma_\beta^{\lambda} P_{\lambda} \tilde{S}(p) \left[ \int \frac{d^4 k}{(2\pi)^4} \frac{1}{(k + \frac{x}{2})^2 (k - \frac{x}{2})^2} \right]. \tag{17}
\]

Equation (17) shows that a logarithmically divergent wave function renormalization is required in the WZ gauge.

This divergence will now be shown to be a gauge artifact due to the choice of the WZ gauge. In fact if the fields \( C, \chi \) and \( S \) are included two further contributions must be added, corresponding to the diagrams

\[
\begin{align*}
\bar{\psi}_J(x) & \quad \chi \quad \bar{\psi}_b(y) \quad \rightarrow \quad D_{bJ}^{aI}(x; y) \\
\bar{\psi}_J(x) & \quad \chi \quad \bar{\psi}_b(y) \quad \rightarrow \quad E_{bJ}^{aI}(x; y)
\end{align*}
\]

The contribution of these two diagrams exactly cancels the divergence in equation (17) giving a net vanishing one-loop result for the propagator. In fact the computation of \( D \) and \( E \) gives

\[
D_{bJ}^{a_{\alpha a} I}(x; y) = -\frac{i}{4} g^2 f_{de} f_{fm} \int d^4 x_1 d^4 x_2 \left\{ \Delta_{J}^{I}(x_2 - x_1). \right. \left[ S_{mK}^{\gamma 
abla I} (x - x_2) \sigma_\gamma, \mu (\partial_\mu (x_2 - x_1)) \bar{S}_{bJ}^{\beta \alpha I}(x_1 - y) \right\}, \tag{18}
\]

In momentum space one finds

\[
\bar{D}_{bJ}^{a_{\alpha a} I}(p) = -\frac{i}{8} g^2 \delta_\alpha^\beta \delta_{J}^{\dagger} \tilde{S}^{\dagger}(p) \sigma_\beta^{\lambda} P_{\lambda} \tilde{S}(p) \left[ \int \frac{d^4 k}{(2\pi)^4} \frac{1}{(k + \frac{x}{2})^2 (k - \frac{x}{2})^2} \right],
\]

\[
\bar{E}_{bJ}^{a_{\alpha a} I}(p) = \bar{D}_{bJ}^{a_{\alpha a} I}(p).
\]

In conclusion summing up all the terms gives

\[
\langle (\bar{\psi}\psi)_{bJ} \rangle^{a_{\alpha a} I}_{\text{1-loop,WZ}} = \bar{A}_{bJ}^{a_{\alpha a} I}(p) + \bar{B}_{bJ}^{a_{\alpha a} I}(p) + \bar{C}_{bJ}^{a_{\alpha a} I}(p) + \bar{D}_{bJ}^{a_{\alpha a} I}(p) + \bar{E}_{bJ}^{a_{\alpha a} I}(p) = 0, \tag{19}
\]

so that the one-loop correction to the \( \langle \bar{\psi}\psi \rangle \) propagator without fixing the Wess–Zumino gauge is zero, as expected in \( \mathcal{N}=4 \) super Yang–Mills theory.
2.2 One loop corrections to the propagator of the scalars belonging to the chiral multiplet

The calculation of the propagator $\langle \varphi^\dagger \varphi \rangle$ is more complicated because more diagrams are involved. In particular tadpole type graphs, coming from the term $\Phi^\dagger V^2 \Phi$ in the action, are present. However the result is analogous to what was found for the $\langle \bar{\psi} \psi \rangle$ propagator: in the WZ gauge the one-loop correction is logarithmically ultraviolet-divergent, requiring a wave function renormalization, but when the contributions neglected in the WZ gauge are taken into account the total one-loop result vanishes.

The diagrams contributing to $\langle \varphi^\dagger \varphi \rangle$ at the one-loop level in the Wess–Zumino gauge are the following:

- $A_{\mu}A_{\nu}$
- $\varphi^a \rightarrow A^a_{b,\mu}(x; y)$
- $\bar{\chi}\lambda$
- $\bar{\psi}\psi$
- $\varphi^a \rightarrow B^a_{b,\mu}(x; y)$
- $\psi \bar{\psi}$
- $\varphi^a \rightarrow C^a_{b,\mu}(x; y)$
- $A_{\mu}A_{\nu}$
- $\varphi^a \rightarrow D^a_{b,\mu}(x; y)$
- $DD$
- $\varphi^a \rightarrow E^a_{b,\mu}(x; y)$
The last three diagrams are all tadpole corrections since the free propagators for the auxiliary fields $F$ and $D$ are $\delta$-functions, so that the corresponding lines shrink to a point. Each one of these diagrams is quadratically divergent. There is also a quadratically divergent contribution coming from the two diagrams $B(x, y)$ and $C(x, y)$ and the sum of all these terms exactly vanishes. The total sum gives a final result that is logarithmically divergent. In momentum space it is schematically of the form

$$\langle \phi^a \phi^b \rangle^{1\text{-loop}, WZ} \sim g^2 \delta_{IJ} \delta_a^b \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k + \frac{p}{2})^2 (k - \frac{p}{2})^2}. \quad (20)$$

Just like in the case of $\langle \bar{\psi} \psi \rangle$ this divergence can be reabsorbed by a wave function renormalization for the field $\phi$.

Taking into account the contribution of the fields $C$, $\chi$ and $S$ there are eight more diagrams to be calculated.
Single diagrams contain quadratically divergent terms that cancel in the sum leaving a total contribution that is again logarithmically divergent. This correction is exactly what is needed to cancel the divergence in (20). As a result the complete one-loop correction to the $\langle \phi^\dagger \phi \rangle$ propagator is actually zero.

Notice that the situation induced by the choice of the WZ gauge is a feature common to every supersymmetric gauge theory, since the contribution of the gauge-dependent fields $C, \chi$ and $S$ is in general logarithmically divergent. However in theories with less supersymmetry a logarithmic wave function renormalization is in any case unavoidable, so that this effect is completely irrelevant. On the contrary it becomes important in the $\mathcal{N}=4$ Yang–Mills theory and in general in finite theories. Of course the divergences encountered here are gauge artifacts and disappear in gauge invariant correlation functions. Examples of computations of correlators of gauge invariant operators, in which the choice of the WZ gauge does not lead to this kind of problems, have been considered within the discussion of the correspondence with type IIB superstring theory on AdS space, [19, 20].

The calculations we described seem to suggest the possibility of constructing improved Feynman rules in which the effect of the gauge dependent fields, set to zero in the WZ gauge, is dealt with by a suitable redefinition of the free propagators. However this pro-
gram proves extremely complicated when the propagators of fields in the vector multiplet or three- and more-point functions are considered. The calculation of these correlation functions, already at the one-loop level, requires many other terms to be included in the action $S_{\text{int}}$. In particular terms coming from $W^{(1)}W^{(2)}$ and $W^{(2)}W^{(2)}$ must be considered. The calculation of $W^{(2)}$ without the simplifications introduced by the Wess–Zumino gauge is rather lengthy and gives

$$W^{a(2)}_\alpha = \frac{-i}{2} f^{abc} \left\{ -i C^b \lambda^c_\alpha - \frac{1}{2} \sigma^{\mu}_{\alpha \dot{a}} \chi^{\dot{a}} \partial_\mu C^c - i \frac{1}{2} \sigma^{\mu}_{\alpha \dot{a}} \chi^{\dot{a}} A_\mu^c + \frac{1}{2} S^{\dot{a} \dot{b}} \chi^{\dot{c}} \right\} + \left[ \frac{1}{2} S^{\dot{b} \dot{c}} \right]
$$

$$+ \left[ \frac{1}{2} (\sigma^\mu \sigma^{\nu})_{\alpha}^\beta (A^b_\mu C^c + A^b_\alpha \partial_\nu C^c) - \frac{1}{2} \delta_\alpha^\beta C^b \square C^c - \frac{1}{2} \delta_\alpha^\beta \chi^3 \chi^c + \right.

+ \frac{1}{2} (\sigma^\mu \sigma^{\nu})_{\alpha}^\beta \partial_\mu C^b \partial_\nu C^c - \chi^{b \beta} \lambda_\alpha^c - \chi^{b \beta} \chi_\alpha^c - i \sigma^{\mu}_{\alpha \dot{a}} \chi^{\dot{a}} \partial_\mu \chi^{c \beta} +

+ i \gamma^\beta \sigma^{\mu}_{\alpha \dot{a}} \partial_\mu \chi^{\dot{a}} \chi^c + \frac{1}{2} (\sigma^\mu \sigma^{\nu})_{\alpha}^\beta A^b_\mu A^c_\mu \theta_\beta + \left[ C^b \sigma^{\mu}_{\alpha \dot{a}} \partial_\mu \chi^{\dot{a}} + \right.

- i \chi^{b}_{\alpha} D^c - \frac{1}{4} (\sigma^\mu \sigma^{\nu})_{\alpha}^\beta \chi^{b}_{\alpha} (\partial_\mu A^c_\nu - \partial_\nu A^c_\mu) - \frac{1}{2} A^b_\mu (\sigma^\mu \sigma^{\nu})_{\alpha}^\beta \partial_\mu \chi^c_\beta +

+ \frac{1}{2} \partial_\mu A^b_\mu \chi^c_\alpha - \frac{1}{2} \partial_\mu C^b (\sigma^\mu \sigma^{\nu})_{\alpha}^\beta \partial_\mu \chi^c_\beta - \frac{i}{2} \sigma^{\mu}_{\alpha \dot{a}} \partial_\mu \chi^{\dot{a}} S^c + S^b \chi^c_\alpha +

- i A^b_\mu \sigma^{\mu}_{\alpha \dot{a}} \chi^{\dot{a}} \right\} \theta_\alpha \right\} \quad (21)

Substituting (21) into $S_{\text{int}}$ results in a number of relevant interaction terms of the order of 100, making this formulation totally impractical in explicit calculations. In conclusion perturbation theory in components almost unavoidably requires the WZ gauge, but the latter introduces divergences in gauge dependent quantities. In the following sections the superfield formalism, that allows to avoid these problems, will be employed. However we will show that new difficulties related to “ordinary” gauge fixing emerge.

3 Perturbation theory in $\mathcal{N}=1$ superspace: propagators

The difficulties encountered in the previous section in the calculation of gauge dependent quantities, because of the divergences present in the Wess–Zumino gauge, can be overcome using the superfield formulation.

Because of the lack of a completely consistent $\mathcal{N}=4$ formulation, we will employ the $\mathcal{N}=1$ superfield formalism which has proved to be a powerful tool in the proof of the finiteness of the theory up to three loops. In this approach there is no particular difficulty in working without fixing the WZ gauge. If one does not choose to work in the WZ gauge the action is non polynomial, however a finite number of terms is relevant at each order in perturbation theory, so that only at very high order the choice of the Wess–Zumino gauge introduces significant simplifications. The aim of this section is to show that there
are other subtleties related to further fixing the gauge for the vector superfield, even if one does not work in the WZ gauge.

To be more general and for the purpose of studying the possibility of finding a supersymmetric regularization of a class of $\mathcal{N}=1$ theories, following the proposal of \[23, 26\], we will consider a formulation in $\mathcal{N}=1$ superspace of $\mathcal{N}=4$ super Yang–Mills theory deformed with the addition of mass terms for the (anti) chiral superfields.

$$S_m = -\int d^4 x d^2 \theta d^2 \bar{\theta} \left[ \frac{1}{2} m \delta_{IJ} \Phi^I \Phi^J \delta(\bar{\theta}) + \frac{1}{2} m^* \delta_{IJ} \Phi^I \Phi^J \delta(\theta) \right].$$  \hspace{1cm} (22)

The inclusion of this terms to the action breaks $\mathcal{N}=4$ supersymmetry down to $\mathcal{N}=1$. In \[24\] it was argued, by means of dimensional arguments, that this term should not affect the ultraviolet properties of the $\mathcal{N}=4$ theory. More precisely the statement of \[24\] is that no divergences appear in gauge invariant quantities, so that no divergent contribution to the quantum effective action is generated perturbatively. As a result the model obtained in this way would be an example of a finite $\mathcal{N}=1$ theory. The inverse construction, in which one deforms a $\mathcal{N}=1$ model to $\mathcal{N}=4$ super Yang–Mills plus a mass term, has been proposed in \[25, 26\] as a regularization procedure preserving supersymmetry. The calculations presented in this section will show that in the presence of the term (22) no divergence is generated, at the one-loop level, in the two-, three- and four-point Green functions that are computed. The results presented suggest the possibility of reinforcing the conclusions of \[24\]: namely the $\mathcal{N}=4$ theory augmented with (22) appears to be finite in the sense that no divergences appear in the complete $n$-point irreducible Green functions, at least at one loop. In particular no wave function renormalization is required. Notice that this result is what is actually necessary for the consistency of the approach advocated in \[23, 26\].

The complete action we will be using for our perturbative calculations is thus$^2$

$$S = S_{\mathcal{N}=4} + S_m$$

and reads

$$S = \int d^4 x d^2 \theta d^2 \bar{\theta} \left\{ \frac{1}{2} V^a \left[ -\Box P_T - \xi (P_1 + P_2) \Box \right] V_a + \Phi_i^a \Phi_i^a - \frac{1}{2} m \delta_{IJ} \Phi^I \Phi^J \delta(\bar{\theta}) + \right.$$ \hspace{1cm} 

$$- \frac{1}{2} m \delta_{IJ} \Phi_i^a \Phi_i^b \delta(\theta) + \frac{i}{\sqrt{2}} g f_{abc} \Phi_i^a V^b \Phi_i^c - \frac{g^2}{2} f_{abc} \Phi_i^a V^b V^c \Phi_i^d +$$ \hspace{1cm} 

$$- \frac{i}{16 \sqrt{2}} g f_{abc} \left[ \bar{\Phi}^i (D^a V^b) \right] V^b (D_\alpha V^c) - \frac{1}{128} g^2 f_{abc} f_{edc} V^a (D^a V^b) \left[ (D^2 V^c) (D_\alpha V^d) \right] +$$ \hspace{1cm} 

$$+ \ldots - \frac{1}{3!} g f_{abc} \left[ \varepsilon_{IJK} \Phi^i_a \Phi^j_b \Phi^K_c \delta(\bar{\theta}) + \varepsilon_{IJK} \Phi^i_a \Phi^j_b \Phi^K_c \delta(\theta) \right] + \left( C^a_c - C^c_a \right) +$$ \hspace{1cm} 

$$+ \frac{i}{2 \sqrt{2}} g f_{abc} \left[ C^{a \alpha} + \bar{C}^{a \alpha} \right] V^b \left( C^c_c + \bar{C}^c_c \right) - \frac{1}{8} g^2 f_{abc} f_{edc} \left( C^{a \alpha} + \bar{C}^{a \alpha} \right) V^b V^c \left( C^d_d + \bar{C}^d_d \right) + \ldots \right\},$$  \hspace{1cm} (23)

where dots stand for terms that are not relevant for the considerations of this paper.

Notice that in the action (24) a gauge fixing term corresponding to a family of gauges parameterized by $\alpha = \frac{1}{3}$ has been introduced. It will now be shown, by explicitly computing the propagators of both the chiral and the vector superfields, that the supersymmetric

\footnote{In the following the mass parameter $m$ will be taken to be real for simplicity of notation.}
generalization of the Fermi–Feynman gauge, corresponding to $\alpha = 1$, is somehow privileged (see also [23, 27]), since any other choice of the parameter $\alpha$ leads to infrared divergences in Green functions.

3.1 Propagator of the chiral superfield

The propagator of the chiral superfield is the simplest Green function to compute. The calculation will be reported in detail in order to illustrate the superfield technique.

It follows from the form of the action (24) that there are three diagrams contributing to the propagator $\langle \Phi \Phi^\dagger \rangle$ at the one loop level. The one-particle irreducible parts of these diagrams will be directly evaluated in momentum space using the improved super Feynman rules of [28]. The convention employed is that all momenta are taken to be incoming. The diagrams are the following

where $z = (x, \theta, \overline{\theta})$. The notations for the internal propagators and the corresponding $x$-space expressions are

$$ I, a \quad \longrightarrow \quad J, b \quad \longrightarrow \quad \langle \Phi^a(z) \Phi^b_j(z') \rangle_{\text{free}} = \delta^a_j \delta^b_a \frac{1}{\Box + m^2} \delta_8(z - z') $$

$$ a \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad b \quad \longrightarrow \quad \langle V^a(z)V_b(z') \rangle_{\text{free}} = -\frac{\delta^a_b}{\Box} [1 + (\alpha - 1)(P_1 + P_2)] \delta_8(z - z') $$

$P_1$ and $P_2$ in the $\langle VV \rangle$ propagator are the projectors [22]

$$ P_1 = \frac{1}{16} \frac{D^2 \overline{D}^2}{\Box}, \quad P_1 \Phi^\dagger = \Phi^\dagger, \quad P_1 \Phi = 0 $$
\[ P_2 = \frac{1}{16} \frac{D^2 D^2}{\Box}, \quad P_2 \Phi = \Phi, \quad P_2 \Phi^\dagger = 0. \]

Moreover in the presence of mass terms for \( \Phi \) and \( \Phi^\dagger \) there are extra \( \langle \Phi \Phi \rangle \) and \( \langle \Phi^\dagger \Phi^\dagger \rangle \) propagators, that will enter the calculation of the vector superfield propagator at one loop

\[ I, a \quad \begin{array}{c} \times \end{array} \quad J, b \quad \rightarrow \quad \langle \Phi^{\dagger I}(z) \Phi^I_b(z') \rangle_{\text{free}} = -\delta^I_J \delta^a_b \frac{m}{4} \frac{D^2}{\Box} (\Box + m^2) \delta_8(z - z') \]

Using the above expressions for the free propagators and the rules of [28], with the vertices read from the action [24], the three contributions can be evaluated without too much effort. For the Fourier transform of \( A(z, z') \) one finds

\[
\begin{align*}
\tilde{A}(p) &= \int \frac{d^4 k}{(2\pi)^4} d^2 \theta_1 d^2 \theta_2 d^2 \bar{\theta}_2 \left\{ \frac{i}{\sqrt{2}} g f_{abc} \Phi^{\dagger I}_I(p, \theta_1, \bar{\theta}_1) \left[ \left( -\frac{1}{4} \bar{D}_2^2 \right) \delta^I_J \delta^c_f \delta(1, 2) \right] \frac{i}{\sqrt{2}} g f_{efb} \Phi^{\dagger J}_J(-p, \theta_2, \bar{\theta}_2) \right\},
\end{align*}
\]

where \( \gamma = (\alpha - 1) \) and the compact notation \( \delta(1, 2) = \delta_2(\theta_1 - \theta_2) \delta_2(\bar{\theta}_1 - \bar{\theta}_2) \) has been introduced. The computation uses the properties of the Grassmannian \( \delta \)-function, which imply for example [23, 28]

\[
D_{1a} \delta(1, 2) = -\delta(1, 2) \bar{D}_{2a}, \quad \bar{D}_{1a} \delta(1, 2) = -\delta(1, 2) \bar{D}_{2a},
\]

\[
\bar{D}_{1a} D_{1a} \delta(1, 2) = \delta(1, 2) \bar{D}_{2a} D_{2a}, \quad \bar{D}^2_1 D^2_1 \delta(1, 2) = \delta(1, 2) \bar{D}^2 D^2_2,
\]

and integrations by parts on the Grassmannian variables in order to remove the \( D \) and \( \bar{D} \) derivatives from one \( \delta \), so that one \( \theta \) integration can be performed immediately. In this way one obtains an expression that is local in \( \theta \) as is expected from the \( \mathcal{N}=1 \) non-renormalization theorem [29]. From (24) one obtains

\[
\begin{align*}
\tilde{A}(p) &= -g^2 \delta_6^a \delta_J^I \left( \frac{1}{4} \right)^2 \int \frac{d^4 k}{(2\pi)^4} d^2 \theta_1 d^2 \theta_2 d^2 \bar{\theta}_2 \frac{1}{k^2[(p - k)^2 + m^2]} \left\{ \Phi^I_{\alpha I}(p, 1) \cdot \Phi^{\dagger J}_J(-p, 2) \left[ \bar{D}^2_1 D^2_1 \delta(1, 2) \right] \left[ 1 + \gamma(D^2_1 \bar{D}^2_1 + D^2_1 \bar{D}^2_1) \delta(1, 2) \right] \right\} = \tilde{A}_1(p) + \tilde{A}_2(p).
\end{align*}
\]

The first term is trivially calculated using

\[
\int d^2 \theta d^2 \bar{\theta} \left[ \bar{D}^2_1 D^2 \delta(\theta - \theta') \right] \delta(\theta - \theta') = 16
\]

\[
\int d^2 \theta d^2 \bar{\theta} \left[ \bar{D}^m D^n \delta(\theta - \theta') \right] \delta(\theta - \theta') = 0 \quad \text{if} \quad (m, n) \neq (2, 2)
\]

and gives

\[
\tilde{A}_1(p) = -2g^2 \delta_6^a \delta_J^I \int \frac{d^4 k}{(2\pi)^4} d^2 \theta d^2 \bar{\theta} \frac{1}{k^2[(p - k)^2 + m^2]} \left\{ \Phi^I_{\alpha I}(p, \theta, \bar{\theta}) \Phi^{\dagger J}_J(-p, \theta, \bar{\theta}) \right\}.
\]
In the computation of the second term one must use the (anti) commutators of covariant
derivatives, which in particular imply [22, 28]
\[ D^2 D^2 D^2 = 16 \Box D^2 \quad \quad D^2 D^2 = 16 \Box D^2. \] (27)

Then integration by parts gives
\[ \tilde{A}_2(p) = 2 \gamma g^2 \delta_0^i \delta_j \int \frac{d^4k}{(2\pi)^4} \frac{d^2 \theta d^2 \bar{\theta}}{k^4[(p-k)^2 + m^2]} \left\{ \Phi_{ai}^j(p, \theta, \bar{\theta}) \Phi^{bJ}(p, \theta, \bar{\theta}) \right\}. \] (28)

In conclusion from the first diagram one obtains two contributions, the first proportional
to \( \gamma \) and the second independent of it. The second diagram gives one single \( \gamma \)-independent
contribution. The Feynman rules give
\[ \tilde{B}(p) = \int \frac{d^4k}{(2\pi)^4} \frac{d^2 \theta d^2 \bar{\theta}}{(k^2 + m^2)[(p-k)^2 + m^2]} \left\{ \Phi_{ai}^j(p, \theta, \bar{\theta}) \Phi^{bJ}(p, \theta, \bar{\theta}) \right\}. \] (29)

From the last diagram one gets
\[ \tilde{C}(p) = \int \frac{d^4k}{(2\pi)^4} \frac{d^2 \theta d^2 \bar{\theta}}{(k^2 + m^2)[(p-k)^2 + m^2]} \left\{ \Phi_{ai}^j(p, \theta, \bar{\theta}) \Phi^{bJ}(p, \theta, \bar{\theta}) \right\}, \] (30)

from which one sees that the \( \tilde{C} \) contribution is absent in the gauge \( \alpha = 1 \), i.e. \( \gamma = 0 \).

Putting the various corrections together gives the following result. The sum of the
terms \( \tilde{A}_1(p) \) and \( \tilde{B}(p) \), which does not depend on \( \gamma \), is
\[ \tilde{A}_1(p) + \tilde{B}(p) = 2 g^2 \delta_0^i \delta_j \int \frac{d^4k}{(2\pi)^4} \frac{d^2 \theta d^2 \bar{\theta}}{k^4} \left\{ \Phi_{ai}^j(p, \theta, \bar{\theta}) \Phi^{bJ}(p, \theta, \bar{\theta}) \right\}. \]

This is finite and exactly vanishes for \( m = 0 \), i.e. in the limit in which the \( \mathcal{N}=4 \) theory
is recovered.
The sum of the terms $\tilde{A}_2(p)$ and $\tilde{C}(p)$ is proportional to $\gamma$ and reads

$$\tilde{A}_2(p) + \tilde{C}(p) = 2\gamma g^2 \delta^b A^I \int \frac{d^4k}{(2\pi)^4} d^2 \theta d^2 \bar{\theta} \left[ \Phi^a_I (p, \theta, \bar{\theta}) \Phi^I_b (-p, \theta, \bar{\theta}) \right].$$

Both terms in the last integral are infrared divergent. The first one is zero in the limit $m \to 0$, while the second one gives an infrared divergence that survives in the $\mathcal{N}=4$ theory, i.e., in the limit $m \to 0$. More explicitly putting

$$\tilde{A}_2(p) + \tilde{C}(p) = 2\gamma g^2 \delta^b A^I \left[ \Phi^a_I (p, \theta, \bar{\theta}) \Phi^I_b (-p, \theta, \bar{\theta}) \right] [I_1(p) + I_2(p)], \quad (31)$$

one has

$$I_1(p) = \int \frac{d^4k}{(2\pi)^4} \left\{ \frac{(p-k)^2}{k^4 [(p-k)^2 + m^2]} - \frac{1}{k^4} \right\} =$$

$$= - \int \frac{d^4k}{(2\pi)^4} \left\{ \frac{m^2}{k^4 [(p-k)^2 + m^2]} \right\} =$$

$$= -2m^2 \int_0^1 d\zeta \zeta \int \frac{d^4k}{(2\pi)^4} \frac{1}{[k^2 + (p^2\zeta + m^2)(1 - \zeta)]^3} =$$

$$= -\frac{2m^2}{2(4\pi)^2} \int_0^1 d\zeta \zeta \frac{1}{(p^2\zeta + m^2)(1 - \zeta)} =$$

$$= \frac{m^2}{(4\pi)^2} \left[ \frac{1}{(p^2 + m^2)} \log \epsilon + \frac{m^2}{p^2(p^2 + m^2)} \log \left( \frac{p^2 + m^2}{m^2} \right) \right],$$

where a standard Feynman parameterization has been used. Analogously

$$I_2(p) = \int \frac{d^4k}{(2\pi)^4} \left\{ \frac{p^2}{k^4 [(p-k)^2 + m^2]} \right\} =$$

$$= 2p^2 \int_0^1 d\zeta \zeta \int \frac{d^4k}{(2\pi)^4} \frac{1}{[k^2 + (p^2\zeta + m^2)(1 - \zeta)]^3} =$$

$$= \frac{p^2}{(4\pi)^2} \int_0^1 d\zeta \zeta \frac{1}{(p^2\zeta + m^2)(1 - \zeta)} =$$

$$= -\frac{1}{(4\pi)^2} \left[ \frac{p^2}{(p^2 + m^2)} \log \epsilon + \frac{m^2}{p^2 + m^2} \log \left( \frac{p^2 + m^2}{m^2} \right) \right].$$

In the above expressions an infrared regulator $\epsilon$ has been introduced. Notice that the total correction (31) is exactly zero on-shell, i.e., for $p^2 = m^2$.

To summarize the results, the propagator of the chiral superfields of the $\mathcal{N}=4$ super Yang–Mills theory in $\mathcal{N}=1$ superspace is logarithmically infrared-divergent for any choice of the gauge parameter $\alpha \neq 1$. This divergence corresponds to a wave function renormalization for the superfields $\Phi^I$. In the Fermi–Feynman gauge $\alpha = 1$ the one-loop correction exactly vanishes.
3.2 Propagator of the vector superfield

The one-loop calculation of the propagator of the vector superfield is much more complicated, because many more diagrams are involved producing a large number of contributions. The final result is however completely analogous: in the Fermi–Feynman gauge the one-loop correction is zero, whereas off-shell infrared divergences arise for $\alpha \neq 1$. In the presence of a mass term for the (anti) chiral superfields no new divergences are generated.

From a calculational viewpoint the new feature with respect to the $\langle \Phi \Phi^\dagger \rangle$ case is that there are also diagrams involving the ghosts. There are two multiplets of ghosts, described by the chiral superfields $C'$ and $C''$. The free propagators for these superfields will be denoted by

\[
\begin{align*}
\langle C'^a(z)C'^b(z') \rangle_{\text{free}} &= \delta^a_b \frac{1}{\Box} \delta_8(z - z') \\
\langle C''^a(z)C''^b(z') \rangle_{\text{free}} &= \delta^a_b \frac{1}{\Box} \delta_8(z - z')
\end{align*}
\]

The ghosts are treated exactly like ordinary chiral superfields with the only difference that there is a minus sign associated with loops, because $C'$ and $C''$ are anticommuting fields [28].

The corrections to the $\langle V V \rangle$ propagator at the one-loop level are given by the following diagrams:

\[
\begin{align*}
V^a(z) & \leadsto V_b(z') & \rightarrow A(z; z') \\
V^a(z) & \leadsto V_b(z') & \rightarrow B(z; z') \\
V^a(z) & \leadsto V_b(z') & \rightarrow C(z; z')
\end{align*}
\]
The contributions $A$ to $E$ are rather straightforward to evaluate much in the same way as the diagrams entering the $\langle \Phi \Phi \rangle$ propagator. The last two graphs are more involved because the free propagator for the $V$ superfield is more complicated for generic values of the parameter $\alpha$. Moreover the cubic and quartic vertices

$$- \frac{i}{16\sqrt{2}} g f_{abc} \left[ \overline{D}^2 (D^a V^c) \right] V^b (D_\alpha V^c)$$

$$- \frac{1}{128} g^2 f_{ab} \, f_{ced} V^a \left[ (\overline{D}^2 V^c) (D_\alpha V^d) \right] ,$$

lead to several terms corresponding to the many ways in which the covariant derivatives can act on the $V$ lines. The $V^3$ vertex in particular

$$\begin{array}{c}
\begin{array}{c}
\text{(32)}
\end{array}
\end{array}$$

gives rise to six different terms.

Schematically the calculation goes as follows. $\tilde{A}(p)$ is a new contribution that appears because of the addition of the mass terms (22); it is not present in the $\mathcal{N}=4$ theory, i.e. when $m=0$. It is useful to discuss separately the corrections coming from diagrams $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ and those obtained from $\tilde{D}_i$, $\tilde{E}_i$, $\tilde{F}$ and $\tilde{G}$, since the latter correspond to the one-loop contribution to the vector superfield propagator in the $\mathcal{N}=1$ supersymmetric Yang–Mills theory.

The first diagram, $\tilde{A}$, is logarithmically divergent and reads

$$\tilde{A}(p) = \frac{3}{2} g^2 \delta_{ab} \int \frac{d^4 k}{(2\pi)^4} d^2 \theta d^2 \overline{\theta} \frac{m^2}{(k^2 + m^2)[(p - k)^2 + m^2]} \left\{ V^a(p, \theta, \overline{\theta}) V^b(-p, \theta, \overline{\theta}) \right\} . \quad (32)$$

For the diagram $\tilde{B}$ application of the Feynman rules gives rise to three different contributions, one quadratically divergent and two logarithmically divergent

$$\tilde{B}(p) = \frac{3}{2} g^2 \delta_{ab} \int \frac{d^4 k}{(2\pi)^4} d^2 \theta d^2 \overline{\theta} \frac{1}{(k^2 + m^2)[(p - k)^2 + m^2]} \left\{ k^2 V^a(p, \theta, \overline{\theta}) V^b(-p, \theta, \overline{\theta}) - \right.$$  

$$\left. - \frac{i}{4} p_{\mu} \sigma_{\alpha\dot{\alpha}}^{\mu} V^a(p, \theta, \overline{\theta}) \left[ (\overline{D}^2 D^a) V^b(-p, \theta, \overline{\theta}) \right] + \frac{1}{4} V^a(p, \theta, \overline{\theta}) \left[ (\overline{D}^2 D^a) V^b(-p, \theta, \overline{\theta}) \right] \right\} . \quad (33)$$

The tadpole diagram $\tilde{C}$ gives a quadratically divergent result of the form

$$\tilde{C} = - \frac{3}{2} g^2 \delta_{ab} \int \frac{d^4 k}{(2\pi)^4} d^2 \theta d^2 \overline{\theta} \frac{1}{(k^2 + m^2)} \left\{ V^a(p, \theta, \overline{\theta}) V^b(-p, \theta, \overline{\theta}) \right\} . \quad (34)$$

Putting the three corrections $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ together gives a net result that is only logarithmically divergent

$$\tilde{A}(p) + \tilde{B}(p) + \tilde{C}(p) = - \frac{3}{2} g^2 \delta_{ab} \int \frac{d^4 k}{(2\pi)^4} d^2 \theta d^2 \overline{\theta} \frac{1}{(k^2 + m^2)[(p - k)^2 + m^2]} .$$
\[ \cdot \left( \frac{i}{4} p_\mu \sigma^{\mu}_{a\dot{a}} V^a(p, \theta, \bar{\theta}) \left[ (\overline{D}^4 D^a) V^b(-p, \theta, \bar{\theta}) \right] + \frac{1}{16} V^a(p, \theta, \bar{\theta}) \left[ (\overline{D}^2 D^2) V^b(-p, \theta, \bar{\theta}) \right] \right) . \quad (35) \]

Notice that in particular the logarithmically divergent contribution proportional to \( m^2 \) exactly cancels out. This is crucial because this correction would correspond to a mass renormalization for the vector superfield that is known to be excluded in any gauge theory as well as in supersymmetric theories in general.

The diagrams \( \tilde{D}_i \) and \( \tilde{E}_i \) are completely analogous to the previous ones, with the only difference that the mass does not appear in the denominators and there is a minus sign associated with the loops. Their sum is logarithmically divergent and takes the form

\[ \tilde{D}_1(p) + \tilde{D}_2(p) + \tilde{D}_3(p) + \tilde{E}_1(p) + \tilde{E}_2(p) = \frac{1}{16} g^2 \delta_{ab} \int \frac{d^4k}{(2\pi)^4} d^2\theta d^2\bar{\theta} \frac{1}{k^2(p-k)^2} \cdot \cdot \cdot \left( i p_\mu \sigma^{\mu}_{a\dot{a}} V^a(p, \theta, \bar{\theta}) \left[ (\overline{D}^4 D^a) V^b(-p, \theta, \bar{\theta}) \right] + \frac{1}{8} V^a(p, \theta, \bar{\theta}) \left[ (\overline{D}^2 D^2) V^b(-p, \theta, \bar{\theta}) \right] \right) . \quad (36) \]

All of the above corrections are independent of the gauge parameter \( \gamma = \alpha - 1 \) and must be summed to those coming from the last two diagrams. \( \tilde{F} \) exactly vanishes for any \( \alpha \), so that only \( \tilde{E} \) needs to be considered. This diagram produces in principle 72 corrections because the Feynman rules give rise to 18 terms (distributing the covariant derivatives associated with the two vertices), each of which splits into 4, since the free propagator itself contains two terms. Many of these contributions can be easily shown to vanish using the properties of the covariant derivatives. In particular one uses

\[ \overline{D}^2 D^a \overline{D}^2 D^2 = 0 , \]

which follows from \( \overline{D}^3 = 0 \) and use of the (anti) commutation relations for the \( D \)'s. It is useful to separate in the non-vanishing part terms proportional to \( \gamma \) and \( \gamma^2 \), from the \( \gamma \)-independent terms. The latter combine to give a logarithmically divergent correction that together with that of equation (35) cancel the correction (35) coming from the sum \( \tilde{A} + \tilde{B} + \tilde{C} \). Actually if \( m \neq 0 \) this sum is finite and exactly vanishes at \( m=0 \). As a result the only non-vanishing corrections to the vector superfield propagator at the one loop level come from terms proportional to \( \gamma \) and to \( \gamma^2 \) in \( \tilde{E}(p) \). The former contain an infrared divergent part of the form

\[ J^{(1)}(p) = c_1 \gamma^2 g^2 \delta_b^a \int \frac{d^4k}{(2\pi)^4} d^2\theta d^2\bar{\theta} \frac{\sigma^{\mu}_{a\dot{a}} \sigma^{\nu}_{b\dot{b}} P_\mu P_\nu}{k^4(p-k)^2} \left( V_a(p, \theta, \bar{\theta}) \left[ D^a \overline{D}^4 D^3 D^\beta V^b(-p, \theta, \bar{\theta}) \right] \right) . \]

but is ultraviolet finite. Furthermore there is a correction, finite both in the ultraviolet and in the infrared regions, proportional to \( \gamma^2 \), that reads

\[ J^{(2)}(p) = c_2 \gamma^2 g^2 \delta_b^a \int \frac{d^4k}{(2\pi)^4} d^2\theta d^2\bar{\theta} \frac{(\sigma^{\alpha\dot{\alpha}} \sigma^{\beta\dot{\beta}})}{k^4(p-k)^4} \cdot \cdot \cdot \left( \left[ (\overline{D}^2 D^a) V_a(p, \theta, \bar{\theta}) \right] + \left[ (D^2 \overline{D}^4) V^b(-p, \theta, \bar{\theta}) \right] \right) . \]

In conclusion in the \( \mathcal{N}=4 \) theory, \textit{i.e.} when \( m=0 \), the one-loop correction to the vector superfield propagator, just like that of the chiral superfield, is ultraviolet finite, but infrared
singular unless the Fermi–Feynman gauge, $\alpha=1$, is chosen, in which case it vanishes. Like in the case of the chiral superfield propagator the non-vanishing $\gamma$-dependent corrections could be reabsorbed by a wave function renormalization of the superfield $V$ and are zero on-shell, i.e. for $p^2=0$.

The proof of the finiteness of the theory in the presence of the mass terms (22) given in [24] is based on naive power counting, which gives for the superficial degree of divergence, $d$, of a diagram in $\mathcal{N}=1$ superspace

$$d = 2 - E - C,$$

where $E$ is the number of external (anti) chiral lines and $C$ the number of $\Phi\Phi$ or $\Phi^\dagger\Phi^\dagger$ propagators. In [24] it is also argued that for corrections to the effective action involving only $V$ superfields the requirement of gauge invariance reduces the degree of divergence to

$$d = -C.$$

However for the purposes of [25, 26] it appears crucial that no divergences, not even corresponding to a wave function renormalization, be present in complete $n$-point functions for any $n$. The computation of the two-point functions in this section has shown that this actually the case at one loop. An argument for the generalization of this result to Green functions with an arbitrary number of external $V$ lines will now be briefly sketched. First note that diagrams involving only vector and ghost superfields are not modified by the inclusion of (22), so that one must only consider graphs containing internal chiral lines. The ultraviolet properties of diagrams that are only logarithmically divergent in the original $\mathcal{N}=4$ theory are not modified by the presence of the mass in the propagators. Thus the contributions that we need to analyze are the quadratically divergent ones, which can acquire subleading logarithmic singularities, plus eventually new diagrams involving $\Phi\Phi$ and $\Phi^\dagger\Phi^\dagger$ propagators. The relevant quadratic divergences come from tadpole diagrams

![Diagram](https://example.com/diagram.png)

The only new diagram containing $\Phi\Phi$ and $\Phi^\dagger\Phi^\dagger$ propagators that must be considered is
The only covariant derivatives that are associated with the vertices in these graphs come from the functional derivatives and must act on the internal (anti) chiral lines to give a non-vanishing result. As a consequence the loop integral in the above diagrams is exactly the same as for the $\tilde{C}$ and $\tilde{A}$ corrections to the $VV$ propagator. Hence summing to the previous diagrams the contribution of

leads to a net logarithmically divergent correction that is exactly the same as the one obtained in the original $\mathcal{N}=4$ theory. Like in the latter case this divergence will be cancelled by the contributions coming from the other one-loop diagrams involving $V$ and ghost internal lines. The argument given here reinforces the results of [24], at least at the one-loop level, and supports the proposal, put forward in [25, 26], according to which the mass deformed $\mathcal{N}=4$ model can be considered a consistent supersymmetry-preserving regularization for (a class of) $\mathcal{N}=1$ theories.

Notice that for the previous discussion it is not necessary to consider equal masses for all the (anti) chiral superfields. The same results can be proved giving different masses to the three superfields. This can be easily understood since in each diagram considered in this section only one chiral/anti-chiral pair is involved, because the propagators are diagonal in ‘flavor’ space and the vertices containing vector and (anti) chiral superfields couple $\Phi_I^{\dagger}$ and $\Phi^I$ with the same index $I$. Basically this means that the above discussed cancellations apply separately to the contributions of each (anti) chiral superfield. From the viewpoint of the dimensional analysis of [24] having different masses $m_I$ is irrelevant.

As a consequence one can in particular give mass to only two of the (anti) chiral multiplets. This suggests the possibility of generalizing the approach of [25, 26] to the case of $\mathcal{N}=2$ super Yang–Mills theories. A discussion of the effect of a $\mathcal{N}=2$ mass term in $\mathcal{N}=4$ supersymmetric Yang–Mills theory can be found in [31].
4 Three- and four-point functions of $\mathcal{N}=1$ superfields

The computation of Green functions with three and four external legs will now be considered. From now on the Fermi–Feynman gauge will be assumed. The three-point functions are expected to suffer from infrared problems of the kind encountered in the preceding section. This issue will not be addressed here since the calculation with $\alpha \neq 1$ is quite involved, even at the one-loop level, because of the huge number of contributions. Four-point functions on the contrary should be infrared finite, because they are directly related to physical scattering amplitudes. Notice, however, that infrared divergences in the Green function with four external $V$ lines were found in \[32\] with $\alpha \neq 1$. In that paper beyond the infrared singularity, it was shown that the adimensionality of the superfield $V$ implies that it requires a non-linear renormalization, in the sense that the renormalized field $V_R$ will be a non-linear function of the bare field $V$

$$V_R = f(V).$$

An example of three-point function will be briefly discussed here and then the more interesting case of a four-point function will be studied in greater detail.

4.1 Three-point functions

The simplest three point function that one can consider corresponds to the correction to the vertex $\varepsilon_{IJK} \text{tr} (\Phi^I \Phi^J \Phi^K)$ (or $\varepsilon_{IJ^K} \text{tr} (\Phi^I \Phi^J \Phi^K)$), which is determined by one single diagram at the one loop level. However, as a less trivial example of computation of three-point functions, the one-loop correction to the vertex

$$\Phi^I_b(z) \rightarrow \Phi^I_b(z) + \Phi^c_I V_c(z) \rightarrow \Phi^I_b(z) \Phi^I_c V_c(z) \rightarrow \Phi^I_b(z) \Phi^I_c V_c(z)$$

will be considered here. The correction at the one-loop level comes from the following diagrams (the notation for the propagators is the same as in the previous section)
In the presence of mass terms for the chiral and antichiral super fields there are two additional contributions

\[ \tilde{\Phi}^a_I(p) \rightarrow \tilde{G}(p; q) \]

\[ \tilde{\Phi}^{aI}(p) \rightarrow \tilde{H}(p; q) \]

The calculation of these diagrams is completely analogous to those that were presented in the previous section. The last two diagrams are finite and vanish in the $\mathcal{N}=4$ theory, \textit{i.e.} at $m=0$, as they are proportional to $m^2$; they will not be considered here. The other contributions will be briefly studied in the limit $m=0$.

The diagram $\tilde{D}(p, q)$ is zero as a consequence of the contraction among the colour indices, so that there are five contributions to be calculated. Dimensional analysis gives a vanishing superficial degree of divergence $d$, corresponding to a logarithmic divergence, for the Green function under consideration. Single diagrams actually contain a logarithmically divergent term plus finite terms. A straightforward but rather lengthy computation, based on elementary $D$-algebra, allows to prove that the divergent part of all the diagrams is of the form

\[ I_{\log} = c \delta^I_{\mu} f^{abc} \int \frac{d^4k}{(2\pi)^4} d^2\theta d^2\bar{\theta} \frac{1}{k^2(p-k)^2} \left\{ \Phi^1_{af}(p, \theta, \bar{\theta}) V_b(-p - q, \theta, \bar{\theta}) \Phi^I_c(q, \theta, \bar{\theta}) \right\} , \]

where $c$ is a constant. Moreover one finds a finite contribution of the form

\[ I_{\text{finite}} = \delta^I_{\mu} f^{abc} \int \frac{d^4k}{(2\pi)^4} q^2 d^2\theta d^2\bar{\theta} \frac{\Phi^1_{af}(p, \theta, \bar{\theta})}{k^2(q + k)^2(p - k)^2} \left\{ c_1 \left[D^\alpha \mathcal{D}^2 D_\alpha V_b(-p-q, \theta, \bar{\theta})\right] \Phi^I_c(q, \theta, \bar{\theta}) \right\} + c_2 \sigma^{\mu\nu}_{\alpha\alpha}(q + k) \left[D^\alpha \mathcal{D}^5 V_b(-p-q, \theta, \bar{\theta})\right] \Phi^I_c(q, \theta, \bar{\theta}) , \]

where $c_1$ and $c_2$ are numerical constants. The sum of all the logarithmically divergent terms contained in the diagrams $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{E}$ and $\tilde{F}$ vanishes. The residual finite part can be shown to be zero as well.

In conclusion the one-loop correction to the three-point function $\langle \Phi^I V \Phi \rangle$ exactly vanishes in the Fermi–Feynman gauge. The same result can be shown to hold for the other three-point functions.
The superfield formalism does not lead to significant simplifications in the calculation of three-point functions with respect to the same computation in components (in the Wess–Zumino gauge!). Actually for the Green function considered here the number of diagrams to be evaluated is approximately the same as in the component formulation. However the power of superspace techniques becomes clear in the computation of four-point functions that will be considered in next subsection.

4.2 Four-point functions

The computation of four-point functions in components is extremely complicated even at the one-loop level and in the Wess–Zumino gauge. In this case the choice of the WZ gauge should not lead to extra divergences because the complete four-point function must finally give the physical scattering amplitude. In the \( \mathcal{N}=1 \) superfield formulation the calculation of four-point functions, though rather lengthy, is much more simple.

In this section the computation of the one-particle irreducible one-loop correction to the Green function \( \langle \Phi^\dagger \Phi \Phi \Phi^\dagger \rangle \) in the Fermi–Feynman gauge will be presented. There are several diagrams to be considered: each of them is free of ultraviolet divergences, as immediately follows from dimensional analysis. Moreover in the Fermi–Feynman gauge each single diagram is infrared finite. In conclusion one finds a finite and non-vanishing result.

The first subset of contributions corresponds to the diagram

\[
\begin{aligned}
\Phi_b^\dagger & \quad \Phi^\dagger_K \\
\Phi_l^{\alpha\dagger} & \quad \Phi_d^L
\end{aligned}
\]

\[\longrightarrow \quad \tilde{A}(p; q; r)\]

\(\Phi_1^{(A)acJL}(p, q, r)\)

where

\[
\Phi_1^{(A)acJL}(p, q, r) = \int \frac{d^3k}{k^2} \frac{\alpha^4}{(2\pi)^4} \frac{1}{k^2(p-k)(q+k)(q+k-r)^2} \cdot (D_{\tilde{a}}D_{\tilde{a}}) \Phi^\dagger_{\tilde{a}}(p, q, r, \tilde{a}) \Phi_{\tilde{a}}^L(r, q, r, \tilde{a}) \Phi_{\tilde{a}}^L(q, r, \tilde{a}) + 2 (D_{\tilde{a}}D_{\tilde{a}}) \Phi^\dagger_{\tilde{a}}(p, q, r, \tilde{a}).
\]
\[ \cdot [D^i \Phi^c_{\alpha}(p + q - r, \theta, \bar{\theta}) \Phi^c_{\alpha}(r, \theta, \bar{\theta}) \Phi^c_{\alpha}(q, \theta, \bar{\theta}) + 2 [D_{\alpha} \Phi^c_{\alpha}(p, \theta, \bar{\theta})].
\]
\[ \cdot [(D^2 \theta^i) \Phi^c_{\alpha}(p + q - r, \theta, \bar{\theta}) \Phi^c_{\alpha}(r, \theta, \bar{\theta}) \Phi^c_{\alpha}(q, \theta, \bar{\theta}) + 4 [(D^\alpha D_{\alpha}) \Phi^c_{\alpha}(p, \theta, \bar{\theta})].
\]
\[ \cdot [D_{\alpha} \Phi^c_{\alpha}(p + q - r, \theta, \bar{\theta}) \Phi^c_{\alpha}(r, \theta, \bar{\theta}) \Phi^c_{\alpha}(q, \theta, \bar{\theta})] \equiv 
\]
\[ \equiv \int \frac{d^4k}{(2\pi)^4} d^2\theta d^2\bar{\theta} \frac{1}{k^2(p - k)^2(q + k)^2(q + k - r)^2} K_{bdIK}^{acJL}(p, q, r; \theta, \bar{\theta}) \]  
\[ \text{and the constant } \kappa \text{ is a group theory factor defined by} 
\]
\[ f_{aef} f_{bgf} f_{ehg} f_{dhe} = \kappa(\delta_{a}^{b} \delta_{c}^{d} + \delta_{a}^{d} \delta_{c}^{b}); \]

\[ \tilde{A}_2(p, q, r) = \left( \frac{1}{4} \right)^2 g^4 4 \kappa(\delta_{a}^{b} \delta_{c}^{d} + \delta_{a}^{d} \delta_{c}^{b}) \delta_{L} \delta_{K} I_{2bdIK}^{(A)acJL}(p, q, r), \]

where

\[ I_{2bdIK}^{(A)acJL}(p, q, r) = \int \frac{d^4k}{(2\pi)^4} d^2\theta d^2\bar{\theta} \frac{1}{k^2(p - k)^2(q + k)^2(q + k - r)^2} \cdot K_{bdIK}^{acJL}(p, q, r; \theta, \bar{\theta}); \]  
\[ \text{and} \]

\[ \tilde{A}_3(p, q, r) = \left( \frac{1}{4} \right)^2 g^4 4 \kappa(\delta_{a}^{b} \delta_{c}^{d} + \delta_{a}^{d} \delta_{c}^{b}) \delta_{L} \delta_{K} I_{2bdIK}^{(A)acJL}(p, q, r), \]

where

\[ I_{3bdIK}^{(A)acJL}(p, q, r) = \int \frac{d^4k}{(2\pi)^4} d^2\theta d^2\bar{\theta} \frac{1}{k^2(p - k)^2(q + k)^2(q + k - r)^2} \cdot \cdot \cdot \]

\[ + 8i \sigma_{a\alpha}(p - k - r)_{\mu} [D^3 \Phi^c_{\alpha}(p, \theta, \bar{\theta})] [D^\alpha \Phi^c_{\alpha}(q, \theta, \bar{\theta})] \Phi^c_{\alpha}(p + q - r, \theta, \bar{\theta})]; \]

\[ \Phi^c_{\alpha}(r, \theta, \bar{\theta}) - 16(p - k - r)^2 \Phi^c_{\alpha}(p, \theta, \bar{\theta}) \Phi^c_{\alpha}(q, \theta, \bar{\theta}) \Phi^c_{\alpha}(p + q - r, \theta, \bar{\theta}) \Phi^c_{\alpha}(r, \theta, \bar{\theta}) \} ; \]

\[ \tilde{A}_4(p, q, r) = \left( \frac{1}{4} \right)^2 g^4 4 \kappa(\delta_{a}^{b} \delta_{c}^{d} + \delta_{a}^{d} \delta_{c}^{b}) \delta_{L} \delta_{K} I_{4bdIK}^{(A)acJL}(p, q, r), \]

where

\[ I_{4bdIK}^{(A)acJL}(p, q, r) = \int \frac{d^4k}{(2\pi)^4} d^2\theta d^2\bar{\theta} \frac{1}{k^2(p - k)^2(q + k)^2(q + k - r)^2} \cdot \cdot \cdot \]

\[ + 8i \sigma_{a\alpha}(q + k)_{\mu} [D^3 \Phi^c_{\alpha}(p, \theta, \bar{\theta})] [D^\alpha \Phi^c_{\alpha}(q, \theta, \bar{\theta})] [D^\alpha \Phi^c_{\alpha}(p + q - r, \theta, \bar{\theta})] \]

\[ - 16(q + k)^2 \Phi^c_{\alpha}(p, \theta, \bar{\theta}) \Phi^c_{\alpha}(q, \theta, \bar{\theta}) \Phi^c_{\alpha}(p + q - r, \theta, \bar{\theta}) \Phi^c_{\alpha}(r, \theta, \bar{\theta}) \} . \]
The second kind of contributions correspond to the diagram

\[ \tilde{\Phi}_b' \rightarrow \tilde{B}(p; q; r) \]

\[ \tilde{\Phi}_c^\dagger \]

\[ \tilde{\Phi}_d^L \]

\[ \tilde{\Phi}_l'^\dagger \]

\[ \tilde{\Phi}_d^L \]

In this case the diagrams obtained by crossing are identical to the one depicted here, so that they are accounted for by giving the correct weight to $\tilde{B}(p, q, r)$. One finds

\[ \tilde{B}(p, q, r) = \frac{g^4}{24} \kappa (\delta_{a}^{b} \delta_{c}^{d} + \delta_{a}^{d} \delta_{c}^{b})(\delta_{f}^{I} \delta_{L}^{K} + \delta_{L}^{I} \delta_{f}^{K}) I^{(B)ac,IL}_{bdIK}(p, q, r), \]

where

\[ I^{(B)ac,IL}_{bdIK}(p, q, r) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2(p-k)^2(q+r)^2(q+k)^2} \left\{ \{D^2 \Phi_i \}(p, \theta, \bar{\theta}) \Phi_i^\dagger(q, \theta, \bar{\theta}) \Phi_K^\dagger(p + q - r, \theta, \bar{\theta}) [D^2 \Phi_d^I(r, \theta, \bar{\theta})] + 
\right. 

\left. + 8i\sigma_{\alpha\beta}(p - k)_{\mu} [D^k \Phi_i \}(p, \theta, \bar{\theta}) \Phi_i^\dagger(q, \theta, \bar{\theta}) \Phi_K^\dagger(p + q - r, \theta, \bar{\theta}) [D^\alpha \Phi_d^I(r, \theta, \bar{\theta})] - 
\right. 

\left. - 16(p - k)^2 \Phi_i \}(p, \theta, \bar{\theta}) \Phi_i^\dagger(q, \theta, \bar{\theta}) \Phi_K^\dagger(p + q - r, \theta, \bar{\theta}) \Phi_d^I(r, \theta, \bar{\theta}) \} \right. \].

The next one-loop correction is

\[ \tilde{\Phi}_b' \rightarrow \tilde{C}(p; q; r) \]

\[ \tilde{\Phi}_c^\dagger \]

\[ \tilde{\Phi}_d^L \]

This contribution is trivially zero because it contains the product

\[ \delta_4(\theta_1 - \theta_2) \delta_4(\theta_1 - \theta_2) \equiv 0. \]

This would not be true in a gauge different from the Fermi–Feynman gauge, i.e. with $\alpha \neq 1$, because in that case there would be projectors acting on the $\delta$’s. The vanishing of $\tilde{C}(p, q, r)$, which is completely trivial in the superfield formulation, corresponds, in the component formulation, to a complicated cancellation among various terms coming from graphs with the same topology.
Another subset of diagrams includes

\[ \Phi_K^\dagger \quad \rightarrow \quad \tilde{D}(p; q; r) \]

as well as the crossed ones. There are three inequivalent crossed diagrams. The result of the calculation consists of the following four terms (\(\tilde{D}_1, \tilde{D}_2, \tilde{D}_3, \tilde{D}_4\))

\[
\tilde{D}_1(p, q, r) = \left( \frac{1}{4} \right)^2 \frac{g^4}{2} c(\delta_{ac}\delta^{bd} + \delta^b_d \delta^{ac}_c)(\delta^I_{aI} \delta^K_{cI} - \delta^I_{bI} \delta^K_{dI}) I^{(D)acJL}_{1 bdIK}(p, q, r),
\]

where

\[
I^{(D)acJL}_{1 bdIK}(p, q, r) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{2\theta d^2\theta} k^2(p - k)^2(k + r - p - q)^2(p - k - r)^2 \cdot \\
\cdot \left\{ [D^2\Phi^\dagger_I(p, \theta, \bar{\theta})] [D^2\Phi^\dagger_J(q, \theta, \bar{\theta})] \Phi^\dagger_K(p + q - r, \theta, \bar{\theta}) \Phi^\dagger_d(r, \theta, \bar{\theta}) + \\
+ 8i\sigma^\mu_{\alpha\alpha}(k + r - p) \mu \left[ D^\alpha\Phi^\dagger_I(p, \theta, \bar{\theta}) \right] \left[ D^\alpha\Phi^\dagger_J(q, \theta, \bar{\theta}) \right] \Phi^\dagger_K(p + q - r, \theta, \bar{\theta}) \Phi^\dagger_d(r, \theta, \bar{\theta}) \right\};
\] (42)

\[
\tilde{D}_2(p, q, r) = \left( \frac{1}{4} \right)^2 \frac{g^4}{2} c(\delta_{ac}\delta^{bd} + \delta^b_d \delta^{ac}_c)(\delta^I_{aI} \delta^K_{cI} - \delta^I_{bI} \delta^K_{dI}) I^{(D)acJL}_{2 bdIK}(p, q, r),
\]

where

\[
I^{(D)acJL}_{2 bdIK}(p, q, r) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{2\theta d^2\theta} k^2(p - k)^2(k + r - p - q)^2(p + q - k)^2 \cdot \\
\cdot \left\{ [D^2\Phi^\dagger_I(p, \theta, \bar{\theta})] \Phi^\dagger_J(q, \theta, \bar{\theta}) \Phi^\dagger_K(p + q - r, \theta, \bar{\theta}) \Phi^\dagger_d(r, \theta, \bar{\theta}) \right\} + \\
+ 8i\sigma^\mu_{\alpha\alpha}(k - p - q) \mu \left[ D^\alpha\Phi^\dagger_I(p, \theta, \bar{\theta}) \right] \Phi^\dagger_J(q, \theta, \bar{\theta}) \Phi^\dagger_K(p + q - r, \theta, \bar{\theta}) \left[ D^\alpha\Phi^\dagger_d(r, \theta, \bar{\theta}) \right] - \\
- 16(k - p - q)^2 \Phi^\dagger_I(p, \theta, \bar{\theta}) \Phi^\dagger_J(q, \theta, \bar{\theta}) \Phi^\dagger_K(p + q - r, \theta, \bar{\theta}) \Phi^\dagger_d(r, \theta, \bar{\theta}) \right\};
\] (43)

\[
\tilde{D}_3(p, q, r) = \left( \frac{1}{4} \right)^2 \frac{g^4}{2} c(\delta^d_c \delta^b_d + \delta_{ac}\delta^{bd})(\delta^I_{aI} \delta^K_{cI} - \delta^I_{bI} \delta^K_{dI}) I^{(D)acJL}_{3 bdIK}(p, q, r),
\]
where

\[ I_{1 \text{bdIK}}^{(D)acJL}(p, q, r) = \int \frac{d^4 k}{(2\pi)^4} \frac{d^2 \theta d^2 \bar{\theta}}{k^2(p-k)^2(k+r-p-q)^2(p+q-k)^2} \cdot \left\{ \Phi_{I}^{\ast \dagger}(p, \theta, \bar{\theta}) \Phi_{b}^{J}(q, \theta, \bar{\theta}) \left[D^2 \Phi_{K}^{\dagger}(p+q-r, \theta, \bar{\theta}) \right] \left[D^2 \Phi_{K}^{\dagger}(p+q-r, \theta, \bar{\theta}) \right] - 
\]

\[ -8i \sigma_{\alpha\sigma}(p-k-r)_{\mu} \Phi_{I}^{\ast \dagger}(p, \theta, \bar{\theta}) \Phi_{b}^{J}(q, \theta, \bar{\theta}) \left[D^2 \Phi_{K}^{\dagger}(p+q-r, \theta, \bar{\theta}) \right] \left[D^2 \Phi_{K}^{\dagger}(p+q-r, \theta, \bar{\theta}) \right] - \]

\[ -16(p-k-r)^2 \Phi_{I}^{\ast \dagger}(p, \theta, \bar{\theta}) \Phi_{b}^{J}(q, \theta, \bar{\theta}) \Phi_{K}^{\dagger}(p+q-r, \theta, \bar{\theta}) \Phi_{K}^{\dagger}(p+q-r, \theta, \bar{\theta}) \right\} ; \quad (44) \]

\[ \bar{D}_{4}(p, q, r) = \left( \frac{1}{4} \right)^2 \frac{g^4}{2} \kappa(\delta_{a}^{b} \delta_{c}^{d} + \delta_{ac} \delta_{bd}) (\delta_{I}^{J} \delta_{K}^{K} - \delta_{I}^{J} \delta_{K}^{K}) I_{1 \text{bdIK}}^{(D)acJL}(p, q, r) , \]

where

\[ I_{1 \text{bdIK}}^{(D)acJL}(p, q, r) = \int \frac{d^4 k}{(2\pi)^4} \frac{d^2 \theta d^2 \bar{\theta}}{k^2(p-k)^2(k+r-p-q)^2(p+q-k)^2} \cdot \left\{ \Phi_{I}^{\ast \dagger}(p, \theta, \bar{\theta}) \left[D^2 \Phi_{b}^{J}(q, \theta, \bar{\theta}) \right] \left[D^2 \Phi_{K}^{\dagger}(p+q-r, \theta, \bar{\theta}) \right] \Phi_{I}^{\dagger}(r, \theta, \bar{\theta}) - 
\]

\[ -8i \sigma_{\alpha\sigma}(p+q-k)_{\mu} \Phi_{I}^{\ast \dagger}(p, \theta, \bar{\theta}) \left[D^2 \Phi_{b}^{J}(q, \theta, \bar{\theta}) \right] \left[D^2 \Phi_{K}^{\dagger}(p+q-r, \theta, \bar{\theta}) \right] - \]

\[ \Phi_{I}^{\dagger}(r, \theta, \bar{\theta}) - 16(p+q-k)^2 \Phi_{I}^{\ast \dagger}(p, \theta, \bar{\theta}) \Phi_{b}^{J}(q, \theta, \bar{\theta}) \Phi_{K}^{\dagger}(p+q-r, \theta, \bar{\theta}) \Phi_{K}^{\dagger}(p+q-r, \theta, \bar{\theta}) \right\} . \quad (45) \]

The last family of one-loop diagrams consists of the one below plus again those obtained by crossing.

\[ \Phi_{I}^{\ast \dagger} \quad \Phi_{K}^{\dagger} \]

\[ \Phi_{b}^{J} \quad \Phi_{d}^{L} \]

\[ \rightarrow \quad \bar{E}(p; q; r) \]

The resulting contributions to the four-point Green function are the following four terms \((\bar{E}_{1}, \bar{E}_{2}, \bar{E}_{3}, \bar{E}_{4})\)

\[ \bar{E}_{1}(p, q, r) = \left( \frac{1}{4} \right)^2 \frac{g^4}{4} \kappa(\delta_{a}^{b} \delta_{c}^{d} + \delta_{ac} \delta_{bd}) \delta_{I}^{J} \delta_{K}^{K} I_{1 \text{bdIK}}^{(E)acJL}(p, q, r) , \]

where

\[ I_{1 \text{bdIK}}^{(E)acJL}(p, q, r) = \int \frac{d^4 k}{(2\pi)^4} \frac{d^2 \theta d^2 \bar{\theta}}{k^2(p-k)^2(p+q-k)^2} \cdot \left\{ \Phi_{I}^{\ast \dagger}(p, \theta, \bar{\theta}) \Phi_{b}^{J}(q, \theta, \bar{\theta}) \Phi_{K}^{\dagger}(p+q-r, \theta, \bar{\theta}) \Phi_{K}^{\dagger}(p+q-r, \theta, \bar{\theta}) \right\} ; \quad (46) \]
\[ \tilde{E}_2(p, q, r) = \left( \frac{1}{4} \right)^2 \frac{g^4}{4} \kappa (\delta_a^b \delta_c^d + \delta_{ac} \delta_{bd}) \delta^I_L \delta^K_J I_{2bdIK}^{(E)acJL}(p, q, r), \]

where

\[ I_{2bdIK}^{(E)acJL}(p, q, r) = \int \frac{d^4k}{(2\pi)^4} d^2\theta d^2\bar{\theta} \frac{1}{k^2(p-k)(r-k)^2} \cdot \cdot \{ \Phi^+_I(p, \theta, \bar{\theta}) \Phi^I_b(q, \theta, \bar{\theta}) \Phi^+_K(r, \theta, \bar{\theta}) \}; \]

\[ \tilde{E}_3(p, q, r) = \left( \frac{1}{4} \right)^2 \frac{g^4}{4} \kappa (\delta_a^b \delta_c^d + \delta_{ac} \delta_{bd}) \delta^I_L \delta^K_J I_{3bdIK}^{(E)acJL}(p, q, r), \]

where

\[ I_{3bdIK}^{(E)acJL}(p, q, r) = \int \frac{d^4k}{(2\pi)^4} d^2\theta d^2\bar{\theta} \frac{1}{k^2(p+q-k)(k-r)^2} \cdot \cdot \{ \Phi^+_I(p, \theta, \bar{\theta}) \Phi^I_b(q, \theta, \bar{\theta}) \Phi^+_K(r, \theta, \bar{\theta}) \}; \]

\[ \tilde{E}_4(p, q, r) = \left( \frac{1}{4} \right)^2 \frac{g^4}{4} \kappa (\delta_a^b \delta_c^d + \delta_{ac} \delta_{bd}) \delta^I_L \delta^K_J I_{4bdIK}^{(E)acJL}(p, q, r), \]

where

\[ I_{4bdIK}^{(E)acJL}(p, q, r) = \int \frac{d^4k}{(2\pi)^4} d^2\theta d^2\bar{\theta} \frac{1}{k^2(q+k-r-p)^2} \cdot \cdot \{ \Phi^+_I(p, \theta, \bar{\theta}) \Phi^I_b(q, \theta, \bar{\theta}) \Phi^+_K(r, \theta, \bar{\theta}) \}. \]

The sum of all the preceding terms results in a finite and non-vanishing total one-loop correction to the four point function. The final expression contains terms with six different tensorial structures

\[ \langle \Phi^I \Phi^I \Phi \rangle = \sum_{i=1}^{6} G^{(i)}, \]

where

\[ G^{(1)} = \kappa \left( \frac{1}{4} \right)^2 g^4 \delta_a^b \delta_c^d \delta^I_L \delta^K_J \left( \frac{1}{4} I_{1bdIK}^{(A)acJL} + \frac{1}{4} I_{3bdIK}^{(A)acJL} + \frac{1}{6} I_{bdIK}^{(B)acJL} - \frac{1}{2} I_{2bdIK}^{(D)acJL} \right) \]

\[ - \frac{1}{2} I_{4bdIK}^{(D)acJL} + \frac{1}{2} I_{1bdIK}^{(E)acJL} + \frac{1}{2} I_{3bdIK}^{(E)acJL} \]

\[ G^{(2)} = \kappa \left( \frac{1}{4} \right)^2 g^4 \delta_a^b \delta_c^d \delta^I_L \delta^K_J \left( \frac{1}{4} I_{1bdIK}^{(A)acJL} + \frac{1}{2} I_{1bdIK}^{(D)acJL} - \frac{1}{2} I_{2bdIK}^{(D)acJL} + \frac{1}{2} I_{3bdIK}^{(D)acJL} \right) \]

\[ - \frac{1}{2} I_{4bdIK}^{(D)acJL} + \frac{1}{2} I_{1bdIK}^{(E)acJL} + \frac{1}{2} I_{3bdIK}^{(E)acJL} \]

\[ G^{(3)} = \kappa \left( \frac{1}{4} \right)^2 g^4 \delta_a^b \delta_c^d \delta^I_L \delta^K_J \left( \frac{1}{4} I_{2bdIK}^{(A)acJL} + \frac{1}{6} I_{bdIK}^{(B)acJL} + \frac{1}{2} I_{2bdIK}^{(D)acJL} + \frac{1}{2} I_{4bdIK}^{(D)acJL} \right) \]
\[ G^{(4)} = \kappa \left( \frac{1}{4} \right)^2 g^A \delta_\alpha^b \delta_\beta^c \delta_\gamma^d \delta_\delta^I \delta_\epsilon^J \left( \frac{1}{4} I_{2 \beta \delta \iota} + \frac{1}{4} I_{1 \alpha \beta \epsilon} + \frac{1}{6} I_{1 \alpha \epsilon \iota} - \frac{1}{2} I_{1 \beta \delta \iota} - \right. \\
\left. - \frac{1}{2} I_{1 \alpha \beta \iota} + \frac{1}{2} I_{1 \alpha \epsilon \iota} + \frac{1}{2} I_{1 \beta \delta \iota} \right) \\
G^{(5)} = \kappa \left( \frac{1}{4} \right)^2 g^A \delta_\alpha^b \delta_\beta^c \delta_\gamma^d \delta_\delta^I \delta_\epsilon^J \left( \frac{1}{4} I_{3 \beta \delta \iota} + \frac{1}{6} I_{1 \alpha \beta \epsilon} + \frac{1}{2} I_{1 \beta \delta \iota} + \frac{1}{2} I_{1 \alpha \epsilon \iota} \right) \\
G^{(6)} = \kappa \left( \frac{1}{4} \right)^2 g^A \delta_\alpha^b \delta_\beta^c \delta_\gamma^d \delta_\delta^I \delta_\epsilon^J \left( \frac{1}{4} I_{3 \beta \delta \iota} - \frac{1}{2} I_{1 \alpha \beta \epsilon} + \frac{1}{2} I_{1 \beta \delta \iota} - \frac{1}{2} I_{1 \alpha \epsilon \iota} + \right. \\
\left. + \frac{1}{2} I_{1 \beta \delta \iota} + \frac{1}{2} I_{1 \alpha \epsilon \iota} + \frac{1}{2} I_{1 \beta \delta \iota} \right) \\
\]

Notice that, since in the Fermi–Feynman gauge the one-loop corrections to the propagators and vertices are zero, the total cross section is completely determined by the sum of the above contributions, which is non vanishing for on-shell external momenta, i.e. \( p^2 = q^2 = r^2 = 0 \).

In the presence of a mass term for the (anti) chiral superfields the expressions given above are modified by the presence of the mass in the free propagators and furthermore there are two additional sets of contributions corresponding to the diagrams

\[ \Phi^c_I \rightarrow \tilde{F}(p; q; r) \]

\[ \Phi^c_I \rightarrow \tilde{G}(p; q; r) \]

Both of these graphs give corrections proportional to \( m^2 \), that can be calculated much in the same way as the previous ones.

## 5 Discussion

The infrared divergences found in the calculation of the propagators of the chiral and the vector superfields are due to the fact that the vector superfield is dimensionless, so that
it contains in particular, as its lowest component, the scalar $C$ that is itself dimensionless and hence has a propagator which behaves, in momentum space, like

$$\langle (CC)(k) \rangle \sim \frac{1}{k^4}.$$  

The contribution of the scalar $C$ to the $\langle VV \rangle$ propagator leads to an infrared divergence whenever a diagram contains a loop involving a $VV$ line. In the Fermi–Feynman gauge the problem is not present because the choice $\alpha = 1$ gives a kinetic matrix of the form of equation (12) in the component expansion. The corresponding inverse matrix $M^{-1}$ in (13) shows that no $\langle CC \rangle$ free propagator is present. On the contrary any choice $\alpha \neq 1$ produces such a propagator, i.e. gives a matrix $M^{-1}$ with a non-vanishing $(M^{-1})_{11}$ entry, leading to the previously discussed infrared problem. An explicit calculation in components with $\alpha \neq 1$ and without fixing the Wess–Zumino gauge should display problematic infrared divergences analogous to those encountered here. These problems with infrared divergences are not peculiar of the $\mathcal{N}=4$ super Yang–Mills theory, but appear in any supersymmetric gauge theory. Analogous infrared divergences in Green functions have been observed in [27, 32]. The conclusion proposed in these papers is that there exist no way to remove the infrared divergences, so that the choice $\alpha=1$ is somehow necessary, at least for the computation of gauge-dependent quantities.

There are two general theorems concerning infrared divergences in quantum field theory. The first is the Kinoshita–Lee–Nauenberg theorem [33], which deals with infrared divergences in cross sections. It states that no infrared problem is present in physical cross sections of a renormalizable quantum field theory, if the appropriate sums over degenerate initial and final states, associated with soft and collinear particles, are considered. The second theorem was proved by Kinoshita, Poggio and Quinn [34] and concerns Green functions. The statement of the theorem is that the proper (one-particle irreducible) Euclidean Green functions with non-exceptional external momenta are free of infrared divergences in a renormalizable quantum field theory. A set of momenta $p_i, i = 1, 2, \ldots, n$ is said to be exceptional if any of the partial sums

$$\sum_{i \in I} p_i,$$  

vanishes. The reason for the absence of divergences is that the external momenta, if non-exceptional, play the role of an infrared regulator in the Green functions.

The proof of the theorem is based on dimensional analysis which does not work in the presence of a $\frac{1}{k^4}$ propagator. In particular it fails in the case of supersymmetric gauge theories in general gauges, because the adimensionality of the $V$ superfield implies that a propagator $\frac{1}{k^4}$ can appear in loop integrals. The apparent violation of the Kinoshita–Poggio–Quinn theorem can be understood from the viewpoint of the component formulation: supersymmetric gauge theories, being non-polynomial and thus containing an infinite number of interaction terms, are not formally renormalizable. The choice of the Wess–Zumino gauge makes the theory polynomial. In fact in this case no infrared divergence is found in Green functions and the general theorems are satisfied. However in the case of the $\mathcal{N}=4$ theory the choice of the WZ gauge results in a change of the ultraviolet properties of the model, at least for what concerns gauge dependent quantities, e.g. the propagators.
In theories with less supersymmetry, in which ultraviolet divergences are present the problem might be less severe, since the subtraction of the ultraviolet infinities could also cure infrared singularities.

The infrared problems discussed here are however gauge artifacts and cannot affect gauge invariant quantities. The mechanism leading to the cancellation of infrared divergences in gauge independent Green functions has not been verified in detail yet, but can be understood starting from the super Feynman rules. When the propagator $\langle VV \rangle$ is inserted into a Feynman graph it is connected to vertices which carry covariant derivatives. These derivatives come directly from the form of the action for vertices involving only $V$ fields and from the definition of the functional derivatives for vertices involving (anti) chiral superfields. The situation in diagrams for gauge invariant quantities is such that at least one covariant derivative can always be brought to act on the $V$ propagator by integrations by parts. In this way an additional factor of the momentum $k$ is generated in the numerator. More precisely for the infrared problematic term $-\frac{1}{k^4} (D^2 D^2 + D^2 \bar{D}^2) \delta_4(\theta - \theta')$ one gets

$$D_\alpha \left[ -\frac{1}{k^4} (D^2 D^2 + D^2 \bar{D}^2) \delta_4(\theta - \theta') \right] = -4i \frac{k_\mu \sigma_\mu^{\alpha \dot{\alpha}}}{k^4} \bar{D}_{\dot{\alpha}} D^2$$

and

$$D_{\dot{\alpha}} \left[ -\frac{1}{k^4} (D^2 D^2 + D^2 \bar{D}^2) \delta_4(\theta - \theta') \right] = 4i \frac{k_\mu \sigma_\mu^{\alpha \dot{\alpha}}}{k^4} D_\alpha \bar{D}^2.$$ 

A rigorous and general check of this mechanism in gauge invariant Green functions has not been achieved yet.

With a gauge choice different from the Fermi–Feynman gauge the computation of four-point Green functions is more complicated. Single diagrams involving vector superfield propagators contain new contributions, some of which are infrared divergent. The correction $\tilde{C}$ is not zero anymore, because there are projection operators acting on the $\delta$-functions. Moreover further diagrams must be included in the calculation of scattering amplitudes at the same order as a consequence of the non-vanishing of the one-loop correction to the propagators and vertices. For example one must consider diagrams such as

![Diagram](image)

as well as diagrams obtained by the insertion of one-loop corrections to the propagators in tree graphs. Notice that the inclusion of these contributions, each one separately infrared divergent, is fundamental in order to get a finite total cross section.
The techniques illustrated in this paper for the calculation of Green functions in the \( \mathcal{N}=1 \) formalism can be applied to correlation functions of composite operators as well. Green functions of gauge invariant composite operators such as those that form the multiplet of currents (see [35] for the explicit form of the multiplet in the Abelian case) play a crucial rôle in the correspondence with type IIB superstring theory on AdS space. The application of \( \mathcal{N}=1 \) superspace to this problem has not been considered here, but is under active investigation [36]. It can be shown that the extension of the formalism to the case of composite operators is in principle rather straightforward, the fundamental difference being that the complete Green functions and not the proper parts must be considered.

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References

[1] L. Brink, J. Scherk and J.H. Schwarz, “Supersymmetric Yang-Mills theories”, Nucl. Phys. B121 (1977) 77.

[2] F. Gliozzi, D.I. Olive and J. Scherk, “Supersymmetry, supergravity and the dual spinor model”, Nucl. Phys. B122 (1977) 253.

[3] M.F. Sohnius and P.C. West, “Conformal invariance in \( N = 4 \) supersymmetric Yang–Mills theory”, Phys. Lett. 100B (1981) 245.

[4] P.L. White, “Analysis of the superconformal cohomology structure of \( N=4 \) super Yang–Mills”, Class. Quant. Grav. 9 (1992) 413.

[5] R. Haag, J.T. Lopuszański and M. Sohnius, “All possible generators of supersymmetries of the \( S \)-matrix”, Nucl. Phys. B88 (1975) 257.

[6] K. Montonen and D.I. Olive, “Magnetic monopoles as gauge particles”, Phys. Lett. 66B (1977) 61.

[7] H. Osborn, “Topological charges for \( N=4 \) supersymmetric gauge theories and monopoles of spin one”, Phys. Lett. 83B 321.

[8] J.P. Gauntlett, “Low energy dynamics of \( N=2 \) supersymmetric monopoles”, Nucl. Phys. B411 (1994) 443, hep-th/9305068. J.D. Blum, “Supersymmetric quantum mechanics of monopoles in \( N=4 \) Yang–Mills theory”, Phys. Lett. B333 (1994) 92, hep-th/9401133.

[9] A. Sen, “Dyon-monopole bound states, self dual harmonic forms on the multimonopole moduli space”, Phys. Lett. 329B (1994) 217, hep-th/9402032.
[10] S. Ferrara and B. Zumino, “Supergauge invariant Yang–Mills theories”, *Nucl. Phys. B*79 (1974) 413.

[11] D.T.R. Jones, “Charge renormalization in a supersymmetric Yang–Mills theory”, *Phys Lett.* 72B (1977) 199; E. Poggio and H. Pendleton, “Vanishing of charge renormalization and anomalies in a supersymmetric gauge theory”, *Phys Lett.* 72B (1977) 200.

[12] A.A. Vladimirov and O.V. Tarasov, “Vanishing of the three-loop charge renormalization function in a supersymmetric gauge theory”, *Phys. Lett.* 96B (1980) 94.

[13] M. Grisaru, M. Roček and W. Siegel, “Zero value of the three-loop $\beta$ function in $N = 4$ supersymmetric Yang–Mills theory”, *Phys. Rev. Lett.* 45 (1980) 1063.

[14] W.E. Caswell and D. Zanon, “Zero three-loop beta function in the $N = 4$ supersymmetric Yang–Mills theory”, *Nucl. Phys.* B182 (1981) 125.

[15] P.S. Howe, K.S. Stelle and P.K. Townsend, “The relaxed hypermultiplet: an unconstrained $N = 2$ superfield theory”, *Nucl. Phys.* B214 (1983) 519; P.S. Howe, K.S. Stelle and P.C. West, “A class of finite four-dimensional supersymmetric field theories”, *Phys. Lett.* 124B (1983) 55; P.S. Howe, K.S. Stelle and P.K. Townsend, “Miraculous ultraviolet cancellations in supersymmetry made manifest”, *Nucl. Phys.* B236 (1984) 125.

[16] S. Mandelstam, “Light-cone superspace and the ultraviolet finiteness of the $N = 4$ model”, *Nucl. Phys.* B213 (1983) 149.

[17] M. Dine and N. Seiberg, “Comments on higher derivative operators in some SUSY field theories”, *Phys. Lett.* B409 (1997) 239, hep-th/9705057.

[18] J. Maldacena, “The large $N$ limit of superconformal field theories and supergravity”, *Adv. Theor. Math. Phys.* 2 (1998) 231, hep-th/9711200.

[19] E. Witten, “Anti de Sitter space and holography”, *Adv. Theor. Math. Phys.* 2 (1998) 253, hep-th/9802150; W. Mück and K.S. Viswanathan, “Conformal field theory correlators from classical field theory on anti-de Sitter space, I and II”, hep-th/9804035 and hep-th/9805145; D.Z. Freedman, S.D. Mathur, A. Matusis and L. Rastelli, “Correlation functions in the CFT$_d$/AdS$_{d+1}$ correspondence”, hep-th/9804058; G. Chalmers, H. Nastase, K. Schalm and R. Siebelink, “R-current correlators in $N=4$ Super Yang–Mills theory from anti-de Sitter supergravity”, hep-th/9805105; S. Lee, S. Minwalla, M. Rangamani and N. Seiberg, “Three-point functions of chiral operators in $D=4$ $N=4$ SYM at large $N$”, hep-th/9806074; D.Z. Freedman, E. D’Hoker and W. Skiba, “Field theory tests for correlators in the AdS/CFT correspondence”, hep-th/9807093; H. Liu and A.A. Tseytlin, “On four point functions in the CFT/AdS correspondence”, hep-th/9807097; D.Z. Freedman, S.D. Mathur, A. Matusis and L. Rastelli, “Comments on 4 point functions in the CFT/AdS correspondence”, hep-th/9808009; G. Chalmers and K. Schalm, “The large $N_c$ limit of four point functions in $N=4$ super Yang–Mills theory from
anti-de Sitter supergravity”, [hep-th/9810051].

H. Liu, “Scattering in anti-de Sitter space and operator product expansion”, [hep-th/9811152].

B. Eden, P.S. Howe, C. Schubert, E. Sokatchev and P.C. West, “Four point functions in $N=4$ supersymmetric Yang–Mills theory at two loops”, [hep-th/9811172].

[20] M. Bianchi, M.B. Green, S. Kovacs and G.C. Rossi, “Instantons in supersymmetric Yang–Mills and D-instantons in IIB superstring theory”, JHEP 08 (1998) 013, [hep-th/9807033].

N. Dorey, V.V. Khoze, M.P. Mattis and S. Vandoren, “Yang–Mills Instantons in the large-$N$ limit and the AdS/CFT correspondence”, [hep-th/9808157].

J.H. Brodie and M. Gutperle, “String corrections to four point functions in the AdS/CFT correspondence”, [hep-th/9809067].

[21] O. Piguet and A. Rouet, “Supersymmetric BPHZ renormalization 2: supersymmetric extension of pure Yang–Mills model”, Nucl. Phys. B108 (1976) 265.

[22] J. Wess and J. Bagger, Supersymmetry and supergravity, 2nd Edition, Princeton University Press (1992).

[23] D. Storey, “General gauge calculations in $N=4$ super Yang–Mills theory”, Phys. Lett. 105B (1981) 171.

[24] A.J. Parkes and P.C. West, “$N=1$ supersymmetric mass terms in the $N=4$ supersymmetric Yang–Mills theory”, Phys. Lett. 122B (1983) 365.

[25] N. Arkani-Hamed and H. Murayama, “Holomorphy, Rescaling Anomalies and exact $\beta$ functions in supersymmetric gauge theories”, [hep-th/9707133].

[26] S. Arnone, C. Fusi and K. Yoshida, “Exact renormalization group equation in presence of rescaling anomaly”, [hep-th/9812022].

[27] J.W. Juer and D. Storey, “One-loop renormalization of superfield Yang–Mills theories”, Nucl. Phys. B216 (1983) 185.

[28] M.T. Grisaru, M. Roček and W. Siegel, “Improved methods for supergraphs”, Nucl. Phys. B159 (1979) 429.

[29] J. Wess and B. Zumino, “A Lagrangian model invariant under supergauge transformations”, Phys. Lett. 49B (1974) 52; J. Iliopoulos and B. Zumino, “Broken supergauge symmetry and renormalization”, Nucl. Phys. B76 (1974) 310; S. Ferrara, J. Iliopoulos and B. Zumino, “Supergauge invariance and the Gell-Mann Low eigenvalue”, Nucl. Phys. B77 (1974) 41.

[30] S. Ferrara and O. Piguet, “Perturbation theory and renormalization of supersymmetric Yang–Mills theories”, Nucl. Phys. B93 (1975) 261; D. Capper and G. Leibbrandt, “On the degree of divergence of Feynman diagrams in superfield theories”, Nucl. Phys. B85 (1975) 492; K. Fujiikawa and W. Lang, “Perturbation calculations for the scalar multiplet in a superfield formulation”, Nucl. Phys. B88 (1975) 61.
[31] M.A. Namazie, A. Salam and J. Strathdee, “Finiteness of broken $N=4$ super Yang–Mills theory”, *Phys.Rev.* **D28** (1983) 1481.

[32] J.W. Juer and D. Storey, “Nonlinear renormalization in superfield gauge theories”, *Phys. Lett.* **119B** (1982) 125.

[33] T. Kinoshita, “Mass singularities of Feynman amplitudes”, *J. Math. Phys.* **3** (1962) 650; T.D. Lee and M. Nauenberg, “Degenerate systems and mass singularities”, *Phys. Rev.* **133** (1964) B1562.

[34] T. Kinoshita and A. Ukawa, “New approach to the singularities of Feynman amplitudes in the zero-mass limit”, *Phys. Rev.* **D13** (1976) 1573; E.C. Poggio and H.R. Quinn, “The infrared behavior of zero-mass Green’s functions and the absence of quark confinement in perturbation theory”, *Phys. Rev.* **D14** (1976) 578; G. Sterman, “Kinoshita’s theorem in Yang–Mills theories”, *Phys. Rev.* **D14** (1976) 2123.

[35] E. Bergshoeff, M. de Roo and B. de Wit, “Extended conformal supergravity”, *Nucl. Phys.* **B182** (1981) 173.

[36] M. Bianchi, S. Kovacs, G.C. Rossi and Ya. Stanev, in preparation.