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Estimates in the Generalized Morrey Space for Linear Parabolic Systems

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Estimates in the Generalized Morrey Spaces

for Linear Parabolic Systems

A thesis submitted for the partial fulfillment of the requirements for the degree of Master of Science

By

Matthew Scott McBride

B.S., Purdue University, 2005

2007

Wright State University
I HEREBY RECOMMEND THAT THE THESIS PREPARED UNDER MY SUPERVISION BY
Matthew Scott McBride ENTITLED Estimates in the Generalized Morrey Spaces for Linear
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Abstract

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The purpose of this paper is to study the parabolic system $u_t - D_\alpha(a_{ij}^{\alpha\beta}D_\beta u^j) = -\text{div} f^i$ in the generalized Morrey Space $L^{2,\lambda}_\varphi$. We would like to understand the regularity of the solutions of this system. It will be shown that 1: if $a_{ij}^{\alpha\beta} \in C(Q_T)$ then $Du \in L^{2,\lambda}_\varphi$, and 2: if $a_{ij}^{\alpha\beta} \in VMO(Q_T)$ then $Du \in L^{2,\lambda}_\varphi$. Moreover we will be able to obtain estimates on the gradient of the solutions to the system, which will tell us about the regularity of the solutions.
## Contents

1 Introduction .................................................. 1

2 Preliminaries .................................................. 2

3 Main Results ................................................... 6

4 References ..................................................... 9
1 Introduction

In this paper we will be investigating the following linear parabolic systems of the form

\[(1-1)\quad u_i^t - D_\alpha \left( a_{\alpha \beta}^{ij}(x,t) D_\beta u^j \right) = -\text{div} f^i(x,t) \quad i = 1, \ldots, N\]

where \(i, j = 1, \ldots, N; \alpha, \beta = 1, \ldots, n\) and the repeated indices denote summation such as
\[\sum_{i=1}^n \sum_{j=1}^n a_{ij} \xi^i \xi^j = a_{ij} \xi^i \xi^j\]

Throughout the paper we assume an uniform ellipticity condition, namely:

\[(1-2)\quad \Lambda^{-1} |\xi|^2 \leq a_{\alpha \beta}^{ij}(x,t) \xi_\alpha^i \xi_\beta^j \leq \Lambda |\xi|^2 \quad \text{where} \Lambda > 0, \xi \in \mathbb{R}^{(n+1)N}, (x,t) \in Q_T, Q_T = \Omega \times [0,T], \Omega \subset \mathbb{R}^n\]

The main purpose of this paper is to demonstrate that one can obtain the gradient estimates in generalized Morrey spaces \(L^{2,\lambda}_e\) for weak solutions of \(1-1\).
2 Preliminaries

Notations:

\[ B_R(x_0) = \{ x \in \mathbb{R}^n | |x - x_0| < R \} \text{-ball in } \mathbb{R}^n \text{ centered at } x_0 \text{ with radius } R \]
\[ z_0 = (x_0, t_0) \in \mathbb{R}^{n+1} \text{ and } z = (x, t) \in \mathbb{R}^{n+1} \text{ for } x \in \mathbb{R}^n \text{ and } t \in (0, T) \]
\[ Q_R(z_0) = B_R(x_0) \times (t_0 - R^2, t_0) \text{- parabolic cylinder in } \mathbb{R}^{n+1} \text{ vertexed at } z_0 \]

Boundary Terms of the Parabolic Cylinder:

The boundary of the parabolic cylinder consists of the lateral walls, the lower boundary, and the lower corners, however we will use \( \partial_p Q_R \) to denote the parabolic boundary of the parabolic cylinder.

Morrey Space for Parabolic Setting:

\[(2-1) \quad L_p^{\varphi, \lambda}(Q_T) = \left\{ f \in L^p(Q_T) \mid \sup_{z_0 \in Q_T, 0 \leq \rho \leq d} \left( \frac{1}{\varphi(\rho)} \int_{Q_T \cap Q_\rho(z_0)} |f|^p \, dz \right)^{\frac{1}{p}} < \infty \right\} \text{ with } 1 \leq p < \infty, 0 \leq \lambda \leq n + 2 \varphi \text{ is a continuous function on } [0, d], \varphi > 0 \text{ on } (0, d], d \text{ is the diameter of } Q_T = \Omega \times (0, T] \subset \mathbb{R}^n. \]

Lemma 2.1:

\( L_p^{\varphi, \lambda} \) is a Banach Space under the norm

\[ \|f\|_{L_p^{\varphi, \lambda}} = \sup_{z_0 \in Q_T, 0 \leq \rho \leq d} \left( \frac{1}{\varphi(\rho)} \int_{Q_T \cap Q_\rho(z_0)} |f|^p \, dz \right)^{\frac{1}{p}} \]

Proof:

So the “norm” must satisfy the three properties to classify as a norm, then it must be shown that the space is complete. \( ||f||_{L_p^{\varphi, \lambda}} \geq 0 \) is trivial similarly \( ||\alpha f||_{L_p^{\varphi, \lambda}} = ||\alpha|| \|f\|_{L_p^{\varphi, \lambda}} \) is quite obvious, though the triangle inequality must be shown since it is not very simple.

Consider

\[ \frac{1}{\varphi(\rho)} \left( \rho^{-\lambda} \int_{Q_T \cap Q_\rho(z_0)} |f + g|^p \, dz \right)^{\frac{1}{p}} \]

\[ \leq \frac{1}{\varphi(\rho)} \left( \rho^{-\lambda} \int_{Q_T \cap Q_\rho(z_0)} |f|^p \, dz \right)^{\frac{1}{p}} + \left( \rho^{-\lambda} \int_{Q_T \cap Q_\rho(z_0)} |g|^p \, dz \right)^{\frac{1}{p}} \]

via Minkowski’s inequality. The one applies the sup function to both sides which yields,

\[ \sup_{z_0 \in Q_T, 0 \leq \rho \leq d} \left( \rho^{-\lambda} \int_{Q_T \cap Q_\rho(z_0)} |f + g|^p \, dz \right)^{\frac{1}{p}} \leq \]

\[ \sup_{z_0 \in Q_T, 0 \leq \rho \leq d} \left( \rho^{-\lambda} \int_{Q_T \cap Q_\rho(z_0)} |f|^p \, dz \right)^{\frac{1}{p}} + \left( \rho^{-\lambda} \int_{Q_T \cap Q_\rho(z_0)} |g|^p \, dz \right)^{\frac{1}{p}} \]
\[ \leq \sup_{\varphi(\rho)} \frac{1}{\varphi(\rho)} \left( \rho^{-\lambda} \int_{Q_{T} \cap Q_{z}(z_0)} |f|^p \, dz \right)^{\frac{1}{p}} + \sup_{\varphi(\rho)} \frac{1}{\varphi(\rho)} \left( \rho^{-\lambda} \int_{Q_{T} \cap Q_{z}(z_0)} |g|^p \, dz \right)^{\frac{1}{p}}. \]

This implies that \( \|f + g\|_{L_p^{\lambda}} \leq \|f\|_{L_p^{\lambda}} + \|g\|_{L_p^{\lambda}} \). Therefore the triangle inequality is satisfied, and \( \|\cdot\|_{L_p^{\lambda}} \) is a norm on \( L_p^{\lambda} \).

Next it must be shown that \( L_p^{\lambda} \) is complete under the norm \( \|\cdot\|_{L_p^{\lambda}} \). Let \( (f_k)_{k=1}^\infty \) be a Cauchy sequence in \( L_p^{\lambda} \). Tchebyshev's inequality implies that \( \{\{ z \in Q_T \mid |f_k(z) - f_m(z)| > \varepsilon \} \} \leq \varepsilon^{-p} \int_{Q_T \cap Q_{z}(z_0)} |f_k - f_m|^p \, dz \). Therefore, there exists a subsequence \( (f_{k_j}) \) and \( f \) such that \( f_{k_j} \to f \) a.e. in \( Q_T \). For every \( \varepsilon > 0 \) there exists \( K \) such that \( \|f_{k_j} - f\|_{L_p^{\lambda}} < \varepsilon \) if \( k_j, k > K \). Let \( k_j \to \infty \) then by Fatou's lemma, one obtains, \( \|f - f_k\|_{L_p^{\lambda}} < \varepsilon \) for \( k > K \). Thus \( f \in L_p^{\lambda} \) by \( \|f\|_{L_p^{\lambda}} \leq \|f - f_k\|_{L_p^{\lambda}} + \|f_k\|_{L_p^{\lambda}} < \infty \) and \( \|f - f_k\|_{L_p^{\lambda}} \to 0 \) as \( k \to \infty \). Therefore \( L_p^{\lambda} \) is complete and hence it is a Banach space.

Morrey Space for \( p=2 \):

We consider the case of \( L_p^{\lambda}(Q) \) for \( p = 2 \). Define the Morrey space for \( p = 2 \) by:

\[
(2-2) \quad L_p^{2,\lambda}(Q_T) = \left\{ f \in L^2(Q_T) \mid \sup_{z_0 \in \tilde{Q}} \frac{1}{\varphi(\rho)} \left( \rho^{-\lambda} \int_{Q_{T} \cap Q_{z}(z_0)} |f|^2 \, dz \right)^{\frac{1}{2}} < \infty \right\}
\]

Definition 2.1:

A function \( h : [0, d] \to [0, \infty) \) is said to be almost increasing if there exists \( K_h \geq 1 \) such that \( h(s) \leq K_h h(t) \) for \( 0 \leq s \leq t \leq d \).

The next proposition is due to [Hu] which will be useful for the main results of this paper.

Proposition 2.1:

Let \( H \) be a non-negative almost increasing function in \([0, R_0]\) and \( F(f) > 0 \) on \((0, R_0]\). Suppose that

(a) There exists \( A, B, \varepsilon, \beta > 0 \) such that \( H(\rho) \leq \left( A \left( \frac{\rho}{R} \right)^\beta + \varepsilon \right) H(R) + BF(R) \) for \( 0 \leq \rho \leq R \leq R_0 \)

(b) There exists \( \gamma \in (0, \beta) \) such that \( \frac{\rho^2}{F(\rho)} \) is almost increasing in \((0, R_0]\)

Then there exists \( \varepsilon_0 = \varepsilon_0(A, \beta, \gamma) \) and \( C = C(A, \beta, \gamma, K_H, K) \) such that if \( \varepsilon < \varepsilon_0 \) then \( H(\rho) \leq C \frac{F(\rho)}{F(R_0)} H(R) + CBF(\rho) \).

BMO and VMO Spaces:

Definition 2.2:
Let $\psi \in C[0,d]$ and $\psi > 0$ on $[0,d]$ then $BMO_\psi(Q)$ is defined by:

\begin{align*}
\text{(2-4) } BMO_\psi(Q_T) = \\
\left\{ f \in L^2(Q_T) \mid \sup_{z_0 \in Q_T, \rho \leq d} \frac{1}{\psi(\rho)} \left( \iiint_{Q_T \cap Q_\rho(z_0)} |f(z) - f_{Q_T \cap Q_\rho(z_0)}(z_0)|^2 \, dz \right)^{\frac{1}{2}} < \infty \right\}
\end{align*}

where $f_A = \iiint_A f(z) \, dz$ and $A \subset \mathbb{R}^{n+1}$

**Definition 2.3:**

Letting $\psi = 1$ one defines $VMO(Q_T)$ by:

\begin{align*}
\text{(2-5) } VMO(Q_T) = \\
\left\{ f \in BMO(Q_T) \mid [f]_{BMO(Q_T, \sigma)} = \sup_{z_0 \in Q_T, \rho \leq \sigma} \left( \iiint_{Q_T \cap Q_\rho(z_0)} |f(z) - f_{Q_T \cap Q_\rho(z_0)}(z_0)|^2 \, dz \right)^{\frac{1}{2}} \to 0 \text{ as } \sigma \to 0 \right\}
\end{align*}

**Weak Solutions:**

We would like to discuss the energy estimates for the system (2-3) $u^i_t - D_\alpha \left( a^\alpha_{ij} D_\beta u^j \right) = 0$ in $Q_T$ and $a^\alpha_{ij}$ is constant. For $Q_R(z_0) \subset Q_T$, let $u^i \xi^2(x)(\eta(t))$ be a test function with $\xi \in C_0^\infty(B_R(x_0))$, $0 \leq \xi \leq 1$ and $|D\xi| \leq \frac{C}{R-\rho}$ with $B_\rho(x_0) \subset B_R(x_0) \subset \Omega$ and $\eta(t) = \begin{cases} t-(t_0-R^2) \frac{1}{R^2-\rho^2} & t \in (t_0 - R^2, t_0 - \rho^2) \\ 1 & t \in [t_0 - \rho^2, t_0) \end{cases}$.

Next we multiply the test function by (2-3) and use integration by parts.

\[ 0 = \int \int_{B_R(x_0) \times (t_0-R^2,t)} u^i_t \xi^2 \eta + a^\alpha_{ij} D_\beta u^j D_\alpha (u^i \xi^2 \eta) \]

The boundary term is zero by definition of $\eta$ and $\xi$.

\[ = \int \int_{B_R(x_0) \times (t_0-R^2,t)} \left( \frac{1}{2} |u_t|^2 \right)_t \xi^2 \eta + \int \int_{B_R(x_0) \times (t_0-R^2,t)} a^\alpha_{ij} D_\beta_\alpha D_\beta u^i \left( \xi^2 \eta_t + 2 \xi u^i D_\alpha \xi \right) \eta \]

Therefore by uniform ellipticity and Cauchy-Schwarz inequality:

\[ \int \int_{B_R(x_0)} |u(x,t)|^2 \xi^2(x) + C \int_{t_0-R^2}^t \int_{B_R(x_0)} |D\xi|^2 |Du|^2 \]

\[ \leq \frac{1}{2} \int_{t_0-R^2}^t \int_{B_R(x_0)} |u|^2 \xi^2 \eta_t + C \int_{t_0-R^2}^t \int_{B_R(x_0)} |D\xi|^2 |u|^2 \eta \leq C \int_{t_0-R^2}^t \int_{B_R(x_0)} |u|^2 \left( |D\xi|^2 \eta + \frac{1}{2} \xi^2 \eta_t \right) \]

Then since $|D\xi| \leq \frac{C}{R-\rho}$ and $\eta_t \leq \frac{C}{R-\rho}$ this implies

\[ \int \int_{B_R(x_0)} \frac{1}{2} |u(x,t)|^2 \xi^2 + \int_{t_0-R^2}^t \int_{B_R(x_0)} |Du|^2 \xi^2 \eta \leq C \int_{t_0-R^2}^t \int_{B_R(x_0)} |u|^2 \left( \frac{1}{(R-\rho)^2} + \frac{1}{R^2-\rho^2} \right) \]

Then this implies

\[ \sup_{t_0-R^2 \leq t \leq t_0} \int_{B_R(x_0)} |u(t)|^2 + \int \int_{Q_R(x_0)} |Du|^2 \leq \frac{C}{(R-\rho)^2} \int \int_{Q_R(x_0)} |u|^2 \]

We call this last line the energy estimate for (2-3). Then define $V_2(Q_T) = \left\{ u \mid u \in L^\infty(0,T;L^2(Q_T), Du \in L^2(Q_T)) \right\}$ This
space is the Sobolov space counterpart for parabolic equations.

Using energy estimates and Sobolev embedding theorem, one can get the Morrey estimate for (2-3) with constant coefficients.

Lemma 2.2:
Let \( u \in V_2(Q_T) \) solve the following,

\[
(2-2) \quad u_t^i - D_\alpha \left( a_{ij}^{\alpha \beta} D_\beta u^j \right) = 0 \quad \text{in} \quad Q_T = \Omega \times (0, T]
\]

Then for \( Q_{R(z_0)} \subset Q_T \) and \( 0 \leq \rho \leq R \)

\[
\int \int_{Q_{\rho(z_0)}} |D u|^2 \leq C \left( \frac{\rho}{R} \right)^{n+2} \int \int_{Q_{R(z_0)}} |D u|^2
\]

Proof:
By [Sc] one has \( \int \int_{Q_{\rho(z_0)}} |u|^2 \leq C \left( \frac{\rho}{R} \right)^2 \int \int_{Q_{R(z_0)}} |u|^2 \). Since \( a_{ij}^{\alpha \beta} \) is constant, by differentiating (2-3) we obtain that \( D_\alpha u \) is still a solution. Hence \( \int \int_{Q_{\rho(z_0)}} |D u|^2 \leq C \left( \frac{\rho}{R} \right)^{n+2} \int \int_{Q_{R(z_0)}} |D u|^2 \).

We will now turn our attention to the main results of this paper.
### 3 Main Results

In the preliminaries section the Morrey estimate we are interested in was shown for \( a_{ij}^{\alpha\beta} = \text{constant} \) by [Sc]. In this section we will extend this result to system (1-1). We establish the Morrey estimate first for the case \( a_{ij}^{\alpha\beta} \in C(\overline{Q_T}) \) and second for \( a_{ij}^{\alpha\beta} \in L^\infty(Q_T) \cap VMO(Q_T) \).

**Theorem 3.1:**

Let \( u \in V_2(Q_T) \) be a weak solution to \( u_i^j - D_\alpha \left( a^{\alpha\beta}_{ij}(z)D_\beta u^j \right) = -\text{div} f \ i = 1, \ldots N \text{ in } Q_T. \) Let \( a_{ij}^{\alpha\beta} \in C(\overline{Q_T}) \) and assume the uniform ellipticity condition and \( f^i \in L^{2,\lambda}_x(Q_T) \) also assume there exists \( \lambda, \gamma \) such that \( \lambda < \gamma < n + 2 \) and \( \frac{\gamma - \lambda}{\phi^2(r)} \) is almost increasing, then \( Du \in L^{2,\lambda}_x(Q_T) \) for any \( Q_T \subset Q_T \) for \( Q_R(z_0) \subset Q_T \) and \( r \leq R. \)

Moreover

\[
\int_{Q_R(z_0)} |Du|^2 \, dz \leq C \frac{\rho^{n+2}}{R^2} \int_{Q_R(z_0)} |Du|^2 + \int_{Q_R(z_0)} |Dv|^2
\]

Therefore one obtains

\[
\int_{Q_R(z_0)} |Du|^2 \leq 2 \int_{Q_R(z_0)} (|Du|^2 + |Dv|^2)
\]

### Proof:

Let \( w \) satisfy (3-1)

\[
w^i - D_\alpha \left( a^{\alpha\beta}_{ij}(z_0)D_\beta w^j \right) = 0 \quad \text{in } Q_R(z_0)
\]

then \( v = u - w \) will satisfy

\[
\begin{align*}
  v^i - D_\alpha \left( a^{\alpha\beta}_{ij}(z_0)D_\beta (v^j) \right) &= D_\alpha \left( \left( a^{\alpha\beta}_{ij}(z) - a^{\alpha\beta}_{ij}(z_0) \right)D_\beta w^j \right) - \text{div} f^i \quad \text{in } Q_R(z_0) \\
  v &= 0 \quad \text{on } \partial_R Q_R(z_0)
\end{align*}
\]

Obviously by lemma 2.2 one obtains

\[
\int_{Q_R(z_0)} |Du|^2 \leq 2 \int_{Q_R(z_0)} (|Du|^2 + |Dv|^2)
\]

\[
\leq C \left( \frac{\rho}{R} \right)^{n+2} \int_{Q_R(z_0)} |Du|^2 + \int_{Q_R(z_0)} |Dv|^2
\]

Multiplying (3-2) by \( v \), integrating and performing integration by parts, one obtains the following:

\[
\int_{Q_R(z_0)} v^i v^j + \int_{Q_R(z_0)} a^{\alpha\beta}_{ij}(z_0)D_\beta v^j D_\alpha v^i - \int_{Q_R(z_0)} \left| a^{\alpha\beta}_{ij}(z) - a^{\alpha\beta}_{ij}(z_0) \right| |Du||Dv| + |f||Dv|
\]

Since \( a^{\alpha\beta}_{ij} \in C(\overline{Q}) \) if \( R \) is small enough, then

\[
\int_{Q_R(z_0)} |Du||Dv| \leq C \int_{Q_R(z_0)} (|Du|^2 + |Dv|^2) + \int_{Q_R(z_0)} |f|^2 \quad \text{via Schwartz inequality.}
\]

This yields

\[
\int_{Q_R(z_0)} |Du|^2 \leq C \left( \frac{\rho}{R} \right)^{n+2} \int_{Q_R(z_0)} |Du|^2 + \int_{Q_R(z_0)} |Dv|^2
\]

Therefore one obtains

\[
\int_{Q_R(z_0)} |Du|^2 \leq C \left( \frac{\rho}{R} \right)^{n+2} \int_{Q_R(z_0)} |Du|^2 + \int_{Q_R(z_0)} |Dv|^2
\]

The desired result follows immediately from proposition 2.1 □
The following Reverse Holder Inequality for parabolic equations can be found in [St-Gi] and is used in the proof of lemma 3.1.

Proposition 3.1:

Let \( u \in V_2(Q_T) \) be a weak solution to \( u_i^j - D_a \left(a^\alpha_{ij}(z)D_\beta u^j \right) = 0 \) in \( Q_T \) for \( i = 1, \ldots, N \) and \( a^\alpha_{ij} \) satisfying the uniform ellipticity condition (1-2), then there exists some \( s > 2 \) such that \( Du \in L_{loc}^s(Q_T) \) and for every \( Q_R \subset Q_4R \subset Q_T \) one has

\[
\left( \iint_{Q_R} |Du|^s \, dz \right)^{\frac{1}{s}} \leq C \left( \iint_{Q_4R} |Du|^2 \, dz \right)^{\frac{1}{2}}
\]

Lemma 3.1:

Let \( u \in V_2(Q_T) \) be a weak solution to \( u_i^j - D_a \left(a^\alpha_{ij}(z)D_\beta u^j \right) = 0 \) in \( Q_T \) for \( i = 1, \ldots, N \). Assume that \( a^\alpha_{ij} \in L^\infty(Q_T) \cap VMO(Q_T) \) and the uniform ellipticity condition (1-2) holds. Then for any \( 0 < \mu < n+2 \) there exist \( R_0 \) and \( C \) depending only on \( n+2, N, \mu, \Lambda, \) and \( \left[a^\alpha_{ij}\right]_{VMO(Q_T)} \) such that for \( \rho \leq R \leq \min(R_0, \text{dist}(z_0, \partial_Q T)) \)

\[
\iint_{Q_\rho(z_0)} |Du|^2 \leq C \left( \frac{\rho}{R} \right)^\mu \iint_{Q_R(z_0)} |Du|^2 \, dz.
\]

Proof:

Let \( \left(a^\alpha_{ij}\right)_{z_R} = \iint_{Q_R(z_0)} a^\alpha_{ij} \, dx \, dt \). As in Theorem 3.1 let \( w \) and \( v = u - w \) satisfy respectively

\[
\begin{cases}
  w_i^j - D_a \left(a^\alpha_{ij}(z_0)D_\beta w^j \right) = 0 & \text{in } Q_R(z_0) \\
  w = u & \text{on } \partial_p Q_R(z_0)
\end{cases}
\]

and

\[
\begin{cases}
  v_i^j - D_a \left( \left(a^\alpha_{ij}\right)_{z_R} D_\beta v^j \right) = D_a \left( \left(a^\alpha_{ij}(z) - \left(a^\alpha_{ij}\right)_{z_R} \right) D_\beta u^j \right) & \text{in } Q_R(z_0) \\
  v = 0 & \text{on } \partial_p Q_R(z_0)
\end{cases}
\]

(3-3)

Similar to the proof of Theorem 3.1 one obtains

\[
\iint_{Q_R(z_0)} |Du|^2 \leq C \left( \frac{\rho}{R} \right)^{n+2} \iint_{Q_R(z_0)} |Du|^2 + C \iint_{Q_R(z_0)} |Dv|^2.
\]

We multiply (3-4) by \( v \) and perform an integration by parts to obtain

\[
\iint_{Q_R(z_0)} |Dv|^2 \leq C \iint_{Q_R(z_0)} \left| a^\alpha_{ij}(z) - \left(a^\alpha_{ij}\right)_{z_R} \right|^2 |Du|^2.
\]

Then by Holder’s inequality, along with \( a^\alpha_{ij} \in VMO \) and by proposition 3.1 one has

\[
\leq C \left( \iint_{Q_R(z_0)} \left| a^\alpha_{ij}(z) - \left(a^\alpha_{ij}\right)_{z_R} \right|^2 \right)^{\frac{1}{2}} \left( \iint_{Q_R(z_0)} |Du|^{2q} \right)^{\frac{1}{q}}
\]

\[
= C |Q_R(z_0)| \left( \iint_{Q_R(z_0)} \left| a^\alpha_{ij}(z) - \left(a^\alpha_{ij}\right)_{z_R} \right|^2 \right)^{\frac{1}{2}} \left( \iint_{Q_R(z_0)} |Du|^s \right)^{\frac{1}{s}}
\]
\[ \leq C |Q_R(z_0)| \varepsilon \left( \iint_{Q_R(z_0)} |Du|^2 \right)^{\frac{2}{\gamma}} \leq C \varepsilon \int_{Q_R} |Du|^2 \] and therefore one gets
\[
\int_{Q_R} |Du|^2 \leq \left( C \left( \frac{\rho}{R} \right)^{n+2} + \varepsilon \right) \int_{Q_R} |Du|^2 .
\]
It follows that
\[
\int_{Q_R} |Du|^2 \leq \left( C \left( \frac{\rho}{R} \right)^{n+2} + \varepsilon \right) \int_{Q_R} |Du|^2 .
\]
Then by using Proposition 2.1 one can achieve the desired result.

Our final theorem will establish the generalized Morrey estimate for system (1-1).

**Theorem 3.2:**

Let \( u \in V_{2}(Q_T) \) be a weak solution to \( u_i - D_a \left( a_{ij}^\alpha(z) D_\beta u^j \right) = -\text{div} f^i \) for \( i = 1, \ldots, N \) in \( Q_T \) with the uniform ellipticity condition. Suppose there exist \( \lambda, \gamma \) such that \( \lambda < \gamma < n + 2 \) and \( \frac{\gamma-\lambda}{\varphi(r)} \) is almost increasing. Assume \( a_{ij}^\alpha \in L^\infty(Q_T) \cap VMO(Q_T) \) and \( f^i \in L^{2,\lambda}(Q_T) \), then \( Du \in L^{2,\lambda}(Q) \) for any \( Q' \subset Q_T \) and for \( Q_R \subset Q_T \) and \( \rho \leq R \). Moreover one obtains the following interior estimate
\[
\int_{Q_R} |Du|^2 \leq C \frac{\rho^\lambda \varphi^2(\rho)}{R^\lambda \varphi^2(R)} \int_{Q_R} |Du|^2 + C \varphi^2(\rho) \| f \|^2_{L^{2,\lambda}} .
\]

**Proof:**

Again like in Theorem 3.1, let \( w \) and \( v = u - w \) satisfy respectively
\[
\begin{align*}
(3-5) \quad w_i - D_a \left( a_{ij}^\alpha(z) D_\beta w^j \right) &= 0 \quad \text{in } Q_R(z_0) \\
&\text{and} \\
(3-6) \quad v_i - D_a \left( a_{ij}^\alpha(z) D_\beta v^j \right) &= -\text{div} f^i \quad \text{in } Q_R(z_0) \\
v &= 0 \quad \text{on } \partial_p Q_R(z_0)
\end{align*}
\]
the following
\[
(3-7) \int_{Q_R(z_0)} |Du|^2 \leq C \left( \frac{\rho}{R} \right)^{\mu} \int_{Q_R(z_0)} |Du|^2 + C \int_{Q_R(z_0)} |Dv|^2 .
\]

Also multiplying (3-6) to \( v \) and performing integration by parts, just as in the proof of Theorem 3.1 one has
\[
\int_{Q_R(z_0)} |Dv|^2 \leq C \int_{Q_R(z_0)} |f| |Dv| .
\]
Applying Cauchy-Schwartz’s inequality one has
\[
(3-8) \int_{Q_R(z_0)} |Dv|^2 \leq C \int_{Q_R(z_0)} |f|^2 .
\]
By combining (3-7), (3-8) and the fact that \( f^i \in L^{2,\lambda}(Q_T) \) one obtains
\[
\int_{Q_R(z_0)} |Du|^2 \leq C \left( \frac{\rho}{R} \right)^{\mu} \int_{Q_R(z_0)} |Du|^2 + C \rho^\lambda \varphi^2(\rho) \| f \|^2_{L^{2,\lambda}} .
\]
Therefore the result will follow applying Proposition 2.1 \( \Box \)
4 References

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