THE GENERALIZED BUSEMANN-PETTY PROBLEM WITH WEIGHTS

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Abstract. The generalized Busemann-Petty problem asks whether origin-symmetric convex bodies in $\mathbb{R}^n$ with smaller $i$-dimensional sections necessarily have smaller volume. We study the weighted version of this problem corresponding to the physical situation when bodies are endowed with mass distribution and the relevant sections are measured with attenuation.

1. Introduction

Let $G_{n,i}$ be the Grassmann manifold of $i$-dimensional linear subspaces of $\mathbb{R}^n$, and let $\text{vol}_i(\cdot)$ denote the $i$-dimensional volume function, $1 \leq i \leq n$. Is it true that for origin-symmetric convex bodies $K$ and $L$ in $\mathbb{R}^n$, the inequality

\begin{equation}
\text{vol}_i(K \cap \xi) \leq \text{vol}_i(L \cap \xi) \quad \forall \xi \in G_{n,i}
\end{equation}

implies

\begin{equation}
\text{vol}_n(K) \leq \text{vol}_n(L)
\end{equation}

This question is known as the generalized Busemann-Petty problem. For $i = n - 1$, the problem was posed by Busemann and Petty [2] in 1956. It has a long history, and the answer is affirmative if and only if $n \leq 4$; see [3], [8], [11]. For the generalized Busemann-Petty problem the following statements are known. If $i = 2, n = 4$, an affirmative answer follows from that in the case $i = n - 1$. If $3 < i \leq n - 1$, the negative answer was given by Bourgain and Zhang [1]; see also [8], [12]. For the special case, when $K$ is a body of revolution, the answer for $i = 2$ and 3 is affirmative [5], [14], [12]. The case, when $K$ is an arbitrary origin-symmetric convex body and $i = 2$ and 3, is still open.

In a recent paper [16], Zvavitch considered the Busemann-Petty problem ($i = n - 1$) in a more general setting, when volumes under consideration are evaluated with respect to general measures satisfying

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certain conditions; see also [15] where the case of the Gaussian measure was considered. Motivated by these papers, we extend the results from [16] to sections of arbitrary dimension $1 \leq i \leq n - 1$ and study a weighted version of the generalized Busemann-Petty problem. Our approach is new in the sense that it relies on elementary properties of Radon transforms on the sphere and does not invoke the Fourier transform techniques as in [16]. Main results are presented by Theorems 3.1 and 3.3. Diverse geometric inequalities that follow from those theorems are exhibited in Section 4.

The generalized Busemann-Petty problem with weights can be given a physical meaning, when bodies under consideration are endowed with mass distribution and the relevant sections are measured with inevitable attenuation.

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2. Preliminaries

We use the following notation: $S^{n-1}$ is the unit sphere in $\mathbb{R}^n$; $\sigma_{n-1} = 2\pi^{n/2}/\Gamma(n/2)$ is the area of $S^{n-1}$; $e_1, e_2, \ldots, e_n$ denote the coordinate unit vectors. In the following $SO(n)$ is the special orthogonal group of $\mathbb{R}^n$; $SO(n-1)$ stands for the subgroup of $SO(n)$ preserving $e_n$. If $i$ is an integer, $1 \leq i \leq n - 1$, then $G_{n,i}$ denotes the Grassmann manifold of $i$-dimensional linear subspaces of $\mathbb{R}^n$. For $\gamma \in SO(n)$, and $\xi \in G_{n,i}$, we denote by $d\gamma$ and $d\xi$ the corresponding $SO(n)$-invariant measures with total mass 1.

For continuous functions $f(\theta)$ on $S^{n-1}$ and $\varphi(\xi)$ on $G_{n,i}$, the totally geodesic Radon transform $R_i f$ and its dual $R^*_i \varphi$ are defined by

\[
(R_i f)(\xi) = \int_{S^{n-1} \cap \xi} f(\theta) \, d\xi \theta, \quad (R^*_i \varphi)(\theta) = \int_{\xi \ni \theta} \varphi(\xi) \, d\theta \xi,
\]

where $d\xi \theta$ and $d\theta \xi$ denote the induced measures on the corresponding manifolds $S^{n-1} \cap \xi$ and \{ $\xi \in G_{n,i} : \xi \ni \theta$ \}; see [6], [10]. The precise meaning of the second integral is

\[
(R^*_i \varphi)(\theta) = \int_{SO(n-1)} \varphi(r_{\theta} \gamma p_0) \, d\gamma, \quad \theta \in S^{n-1},
\]

where $p_0 = \mathbb{R}e_{n-i+1} + \ldots + \mathbb{R}e_n$ is the coordinate $i$-dimensional plane and $r_{\theta} \in SO(n)$ is a rotation satisfying $r_{\theta} e_n = \theta$. The corresponding duality relation reads

\[
\frac{1}{\sigma_{i-1}} \int_{G_{n,i}} (R_i f)(\xi) \varphi(\xi) d\xi = \frac{1}{\sigma_{n-1}} \int_{S^{n-1}} f(\theta)(R^*_i \varphi)(\theta) d\theta
\]
and is applicable provided the integral in either side is finite for \( f \) and \( \varphi \) replaced by \(|f|\) and \(|\varphi|\), respectively.

The Radon transform \( R_i \) and its dual extend as linear bounded operators from \( L^1(S^{n-1}) \) to \( L^1(G_{n,i}) \) and from \( L^1(G_{n,i}) \) to \( L^1(S^{n-1}) \), respectively. Moreover, they can be defined for finite Borel measures. Specifically, if \( \mu \) is such a measure on \( G_{n,i} \), then, according to (2.3), \( R_i^* \mu \) is a finite Borel measure on \( S^{n-1} \) (i.e., a linear continuous functional on \( C(S^{n-1}) \)) defined by

\[
(R_i^* \mu, f) = \frac{\sigma_{n-1}}{\sigma_{i-1}} \int_{G_{n,i}} (R_i f)(\xi) \, d\mu(\xi), \quad f \in C(S^{n-1}).
\]

For instance, if \( \mu \) is the unit mass on the circle \( S^{n-1} \cap \mathbb{R}^i \), then \( R_i^* \mu \) assigns to \( f \) the integral of \( f \) over this circle multiplied by \( \sigma_{n-1}/\sigma_{i-1} \).

Let \( K \) be an origin-symmetric star body in \( \mathbb{R}^n \). The radial function of \( K \) is defined by

\[
\rho_K(\theta) = \sup\{\lambda \geq 0 : \lambda \theta \in K\}, \quad \theta \in S^{n-1},
\]

and represents the Euclidean distance from the origin to the boundary of \( K \) in the direction of \( \theta \). If \( \xi \) is an \( i \)-dimensional subspace of \( \mathbb{R}^n \), \( 1 \leq i \leq n \), then

\[
\text{vol}_i(K \cap \xi) = i^{-1} \int_{S^{n-1} \cap \xi} \rho_K^i(\theta) \, d\xi \theta.
\]

If \( 1 \leq i < n \) this is just \( i^{-1}(R_i \rho_K^i)(\xi) \). The body \( K \) is called infinitely smooth if \( \rho_K(\theta) \in C^\infty_{\text{even}}(S^{n-1}) \).

We will need the following elementary inequality which is a slight generalization of Lemma 1 from [16].

**Lemma 2.1.** Let \( a, b > 0 \) and suppose that \( \alpha(r) \) and \( \beta(r) \) are positive continuous functions on \((0, \max\{a, b\})\) such that \( r^{n-i} \alpha(r)/\beta(r) \) is nondecreasing on \((0, \max\{a, b\})\). Then

\[
\int_0^a r^{n-1} \alpha(r) \, dr - a^{n-i} \frac{\alpha(a)}{\beta(a)} \int_0^a r^{i-1} \beta(r) \, dr \\
\leq \int_0^b r^{n-1} \alpha(r) \, dr - a^{n-i} \frac{\alpha(a)}{\beta(a)} \int_0^b r^{i-1} \beta(r) \, dr.
\]
Proof. This inequality is equivalent to

\[ a^{n-i} \frac{\alpha(a)}{\beta(a)} \int_a^b r^{i-1} \beta(r) \, dr \leq \int_a^b r^{n-1} \alpha(r) \, dr. \]

The latter is obvious by taking into account that \( r^{n-i} \alpha(r)/\beta(r) \) is non-decreasing, no matter \( a < b \) or \( a > b \).

\[ \square \]

3. Main theorems

Let \( K \) be an origin-symmetric star body in \( \mathbb{R}^n \) with the radial function \( \rho_K(\theta) \). Given nonnegative measurable functions \( u \) and \( v \) on \( \mathbb{R}^n \), we denote

\[ V_u(K \cap \xi) = \int_{K \cap \xi} u(x) \, dx, \quad V_v(K) = \int_K v(x) \, dx, \]

provided these integrals are well defined. The functions \( u \) and \( v \) can be given a physical meaning to be the attenuated mass distribution and the true mass distribution, respectively. In polar coordinates we have

\[ V_u(K \cap \xi) = \int_{S^{n-1} \cap \xi} d\theta \int_0^{\rho_K(\theta)} r^{i-1} u(r\theta) \, dr, \quad V_v(K) = \int_{S^{n-1}} d\theta \int_0^\infty r^{n-1} v(r\theta) \, dr. \]

The second integral is finite for any locally integrable function \( v \). The first one is represented as the Radon transform

\[ (3.1) \quad V_u(K \cap \xi) = (R_i b_K)(\xi), \quad b_K(\theta) = \int_0^{\rho_K(\theta)} r^{i-1} u(r\theta) \, dr. \]

It is finite (at least for almost all \( \xi \in G_{n,i} \)) if \( |x|^{i-n} u(x) \) is locally integrable. This follows from duality \( \odot \odot \), according to which (set \( \varphi \equiv 1 \))

\[ \frac{\sigma_{n-1}}{\sigma_{i-1}} \int_{G_{n,i}} V_u(K \cap \xi) \, d\xi = \frac{\sigma_{n-1}}{\sigma_{i-1}} \int_{G_{n,i}} (R_i b_K)(\xi) = \int_{S^{n-1}} b_K(\theta) \, d\theta = \int_{S^{n-1}} d\theta \int_0^{\rho_K(\theta)} r^{i-1} u(r\theta) \, dr = \int_K |x|^{i-n} u(x) \, dx. \]

For technical reasons we impose some more restrictions on \( u \) and \( v \) and consider a class of weights satisfying the following conditions:
(a) $u(x)$ is an even function which is positive and continuous for $x \in \mathbb{R}^n \setminus \{0\}$ and such that $|x|^{1-n}u(x)$ is locally integrable;

(b) $v(x)$ is a nonnegative, even, locally integrable function, and the function $v_\theta(r) = v(r\theta)$ is continuous in $r > 0$ for almost all $\theta \in S^{n-1}$;

(c) (the comparison condition) The function $a_\theta(r) = r^{n-i} \frac{v(r\theta)}{u(r\theta)}$ is nondecreasing for almost all $\theta \in S^{n-1}$.

The conditions (a)-(c) look pretty sophisticated but they allow us to consider weights $v$ which are discontinuous on the unit sphere; see Example 4.4. The comparison condition (c) restricts our class of admissible weights, and the case when (c) fails remains open. However, this condition has a certain physical meaning: if attenuation is too strong, we cannot retrieve desired information from measurements.

Given a symmetric star body $K$ in $\mathbb{R}^n$, we introduce a comparison function

\begin{equation}
(3.2) a_K(\theta) \equiv a_\theta(\rho_K(\theta)) = \rho_K^{n-i}(\theta) \frac{v(\rho_K(\theta)\theta)}{u(\rho_K(\theta)\theta)}.
\end{equation}

**Theorem 3.1.** Let $2 \leq i \leq n-1$ and suppose that $u$ and $v$ satisfy the conditions (a)-(c) above. If the comparison function $a_K(\theta)$ is represented by the dual Radon transform of a positive measure $\mu$ on $G_{n,i}$, i.e., $a_K = R_i^* \mu$, then for any symmetric star body $L$ in $\mathbb{R}^n$, satisfying

\begin{equation}
(3.3) \int_{K \cap \xi} u(x) \, dx \leq \int_{L \cap \xi} u(x) \, dx, \quad \forall \xi \in G_{n,i},
\end{equation}

we have

\begin{equation}
(3.4) \int_K v(x) \, dx \leq \int_L v(x) \, dx.
\end{equation}

A few words are in order on how one should interpret the key equality $a_K = R_i^* \mu$. Note that by (a) and (b), the functions $b_K$ and $b_L$ are continuous, and $a_K \in L^1(S^{n-1})$. On the other hand, $R_i^* \mu$ is a measure; see definition (2.4). The equality $a_K = R_i^* \mu$ means that $\int_{S^{n-1}} a_K(\theta) f(\theta) \, d\theta = (R_i^* \mu, f)$ for any $f \in C(S^{n-1})$ or $R_i^* \mu$ is an absolutely continuous measure (with respect to the Lebesgue measure on $S^{n-1}$) with density $a_K$.

**Proof of Theorem 3.1.** The result is an immediate consequence of the following inequalities:

\begin{equation}
(3.5) \int_{S^{n-1}} a_K(\theta) b_K(\theta) \, d\theta \leq \int_{S^{n-1}} a_K(\theta) b_L(\theta) \, d\theta.
\end{equation}
(3.6) \[ V_v(K) - \int_{S^{n-1}} a_K(\theta)b_K(\theta) \, d\theta \leq V_v(L) - \int_{S^{n-1}} a_K(\theta)b_L(\theta) \, d\theta, \]

in which \( a_K(\theta), b_K(\theta) \) and \( b_L(\theta) \) are defined by (3.2) and (3.1). The inequality (3.5) can be easily obtained if we write (3.3) as (3.3) as \( (R_i \cdot b_K)(\xi) \leq (R_i \cdot b_L)(\xi) \) and make use of the definition (2.4):

\[ \int_{S^{n-1}} a_K(\theta)b_K(\theta) \, d\theta = \left( R_i^* \cdot b_K \right)(\xi) \leq \left( R_i^* \cdot b_L \right)(\xi) = \int_{S^{n-1}} a_K(\theta)b_L(\theta) \, d\theta. \]

The inequality (3.6) can be derived from (2.6) if we set \( a = \rho_K(\theta), b = \rho_L(\theta), \alpha(r) = v(r\theta), \beta(r) = u(r\theta). \) This gives

\[ \int_0^{\rho_K(\theta)} r^{n-1}v(r\theta) \, dr - \rho_K^{-i}(\theta) \frac{v(\rho_K(\theta)\theta)}{u(\rho_K(\theta)\theta)} \int_0^{\rho_K(\theta)} r^{i-1}u(r\theta) \, dr \]

\[ \leq \int_0^{\rho_L(\theta)} r^{n-1}v(r\theta) \, dr - \rho_K^{-i}(\theta) \frac{v(\rho_K(\theta)\theta)}{u(\rho_K(\theta)\theta)} \int_0^{\rho_L(\theta)} r^{i-1}u(r\theta) \, dr \]

or

\[ \int_0^{\rho_K(\theta)} r^{n-1}v(r\theta) \, dr - a_K(\theta)b_K(\theta) \leq \int_0^{\rho_L(\theta)} r^{n-1}v(r\theta) \, dr - a_K(\theta)b_L(\theta). \]

Integrating the latter over \( S^{n-1} \), we obtain (3.6). \( \square \)

**Remark 3.2.** 1. We did not include the case \( i = 1 \) in Theorem 3.1 because in this case the implication (3.3) \( \Rightarrow \) (3.4) is true for any non-negative \( u \) and \( v \) satisfying the condition (a) and (b) above.

The next theorem shows that the assumption \( a_K = R_i^* \mu, \mu > 0, \) in Theorem 3.1 is crucial. Namely, if it fails, then there exist origin-symmetric convex bodies \( K \) and \( L \) such that \( V_u(K \cap \xi) \leq V_u(L \cap \xi) \) for all \( \xi \in G_{n,i} \), but \( V_v(K) > V_v(L) \). More precisely, the following statement holds.

**Theorem 3.3.** Let \( u \) and \( v \) satisfy the conditions (a)-(c) above. Suppose also that \( v \) is positive and both functions are infinitely differentiable
away from the origin. Given an infinitely smooth origin-symmetric convex body $L \subset \mathbb{R}^n$ with positive curvature, let

$$a_L(\theta) \equiv a_{\theta}(\rho_L(\theta)) = \rho_L^{n-i}(\theta) \frac{v(\rho_L(\theta) \theta)}{u(\rho_L(\theta) \theta)}$$

be represented by the dual Radon transform $R^*_i \varphi$ of a function $\varphi \in C^\infty(G_{n,i})$ which is negative for some $\xi \in G_{n,i}$. Then there is a convex symmetric body $K$ in $\mathbb{R}^n$ such that

$$\int_{K \cap \xi} u(x) \, dx \leq \int_{L \cap \xi} u(x) \, dx, \quad \forall \xi \in G_{n,i},$$

but

$$\int_{K} v(x) \, dx > \int_{L} v(x) \, dx.$$

Proof. We start with some comments that might be useful for understanding the essence of the matter. Since the mapping $R^*_i : C^\infty(G_{n,i}) \to C^\infty(S^{n-1})$ is "onto", the function $a_L(\theta)$ is represented as $R^*_i \varphi$ for some $\varphi \in C^\infty(G_{n,i})$ automatically. Such a function $\varphi$ is not unique for $1 < i < n - 1$, because $R^*_i$ is non-injective in this case. The theorem actually assumes that there as at least one representative of the class \{\varphi + \ker(R^*_i)\} which is negative somewhere on $G_{n,i}$.

As in the previous theorem, the result will follow if define $K$ satisfying the following inequalities:

$$\int_{S^{n-1}} a_L(\theta) b_K(\theta) \, d\theta > \int_{S^{n-1}} a_L(\theta) b_L(\theta) \, d\theta,$$

$$V_i(K) - \int_{S^{n-1}} a_L(\theta) b_K(\theta) \, d\theta \geq V_i(L) - \int_{S^{n-1}} a_L(\theta) b_L(\theta) \, d\theta.$$

The body $K$ can be defined as follows. Since $\varphi$ is smooth, then there exist $\delta > 0$ and $\theta_0 \in S^{n-1}$ such that $\varphi(\xi)$ is negative for all $\xi$ in the open domain $\Omega_\delta = \{\xi \in G_{n,i} : d(S^{n-1} \cap \xi, \theta_0) < \delta\}$, $d(\cdot, \cdot)$ being the geodesic distance on $S^{n-1}$. Consider the spherical cap $B = \{\theta : d(\theta, \theta_0) < \delta\}$, and let $B'$ denote the symmetric cap centered at $-\theta_0$. Choose a non-negative function $g \in C^\infty_{\text{even}}(S^{n-1})$, $g \not\equiv 0$, supported by $B \cup B'$ Then $g_1 = R_0 g$ is a $C^\infty$ negative function supported by $\Omega_\delta$, and by duality [2.3] we have

$$\int_{S^{n-1}} a_L g = \int_{S^{n-1}} g R^*_i \varphi = \frac{\sigma_{n-1}}{\sigma_{i-1}} \int_{G_{n,i}} g_1 \varphi > 0.$$
Now we define an origin-symmetric convex body $K$ so that

$$b_K(\theta) = b_L(\theta) + \varepsilon g(\theta),$$

assuming $\varepsilon > 0$ sufficiently small (the proof of validity of this definition almost coincides with that of Proposition 2 in [16]). Multiplying (3.13) by $a_L$ and integrating over $S^{n-1}$, we get

$$\int_{S^{n-1}} a_L b_K = \int_{S^{n-1}} a_L b_L + \varepsilon \int_{S^{n-1}} a_L g.$$ 

Owing to (3.12), this gives (3.10). The proof of (3.11) is similar to that of (3.6) in Theorem 3.1 and relies on the inequality (2.6) in which one should set $a = \rho_L(\theta)$, $b = \rho_K(\theta)$, $\alpha(r) = v(r\theta)$, $\beta(r) = u(r\theta)$. □

4. Corollaries and partial results

Theorems 3.1 and 3.3 give rise to a series of statements. Some of them are new and others were obtained before in a more complicated way. Below we present a few examples.

4.1. The case of equal weights. Let $u$ be a positive even functions on $\mathbb{R}^n$ which is continuous away from the origin and $|x|^{1-n}u(x)$ is locally integrable. Suppose the weights in Theorems 3.1 and 3.3 are equal, i.e., $v \equiv u$. Then $a_K(\theta) = \rho_{n-i}^K(\theta)$ and we have the following statement.

Corollary 4.1.

(i) If $\rho_{n-i}^K = R_{i}^\ast \mu$ where $\mu$ is a positive measure on the Grassmannian $G_{n,i}$, then for any symmetric star body $L$ in $\mathbb{R}^n$, satisfying

$$(4.1) \quad V_u(K \cap \xi) \leq V_u(L \cap \xi) \quad \forall \xi \in G_{n,i}$$

we have $V_u(K) \leq V_u(L)$.

(ii) Let $L$ be an infinitely smooth origin-symmetric convex body in $\mathbb{R}^n$ so that $\rho_{n-i}^L = R_{i}^\ast \varphi$ for some $\varphi \in C^\infty(G_{n,i})$. If $\varphi(\xi) < 0$ for some $\xi \in G_{n,i}$, then there is a convex symmetric body $K$ in $\mathbb{R}^n$ which obeys (4.1) and $V_u(K) > V_u(L)$.

For $i = n - 1$ this statement was proved by A. Zvavitch [16] who used the Fourier transform approach. The key question is what can one say about validity of the representation

$$\rho_{n-i}^K = R_{i}^\ast \mu, \quad \mu \geq 0.$$ 

(4.2)

It is known [1], [7], [12], that if $i > 3$, then there is an infinitely smooth origin-symmetric strictly convex body for which (4.2) fails, and we are

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1 Origin-symmetric star bodies with this property were called in [13] $i$-intersection bodies. See [9] and [12] for $i = n - 1$. 

in the situation of the statement (ii) above. In the special case \( i = n - 1 \) corresponding to the Busemann-Petty problem with equal weights, this gives a negative answer to this problem for all \( n > 4 \). If \( n = 3, 4 \), the validity of (4.2) for \( i = n - 1 \) was proved by different methods in a series of publications; see, e.g., [3], [8], [14], [11], and references therein.

The cases \( i = 2 \) and \( i = 3 \) when \( n > 4 \) are the most difficult. In these cases the validity of (4.2) is known only for bodies of revolution [5], [12]. For arbitrary convex bodies the problem is still open.

4.2. The case of power weights. Let \( u(x) = |x|^\alpha \), \( v(x) = |x|^\beta \).

Then the conditions (a)-(c) have the form
\[
0 < \alpha + i \leq \beta + n. \tag{4.3}
\]
The function \( a_K(\theta) \) is \( \rho_K(\theta)^{\beta + n - \alpha - i} \). Representation of this function by the dual Radon transform of a positive measure and the relevant generalization of the Busemann-Petty problem was studied in [12]. By making use of Erdelyi-Kober fractional integrals, it was proved, that for every \( i > 3 \), there exist an infinitely smooth origin-symmetric strictly convex body \( L \) of revolution for which the representation \( \rho_L(\theta)^{\beta + n - \alpha - i} = R_\mu^i \mu \) fails to be true with \( \mu > 0 \). By Theorem 3.3 it follows that if \( i > 3 \) and \( 0 < \alpha + i \leq \beta + n \), then there exists a convex symmetric body \( K \) such that
\[
\int_{K \cap \xi} |x|^\alpha \, dx \leq \int_{L \cap \xi} |x|^\alpha \, dx \quad \forall \xi \in G_{n,i}, \quad \int_K |x|^\beta \, dx > \int_L |x|^\beta \, dx. \tag{4.4}
\]
For \( i = 2 \) and 3, the representation \( \rho_K(\theta)^{\beta + n - \alpha - i} = R_\mu^i \mu \), \( \mu > 0 \), corresponding to Theorem 3.1 is known to be true in the case \( \alpha + i + 1 = \beta + n \) [12]. We observe that it is also true if \( \alpha + i = \beta + n \) because in this case the equality \( 1 = R_\mu^i \mu \) trivially holds with \( \mu \equiv 1 \). More subtle results in the cases \( i = 2 \) and 3, covering the whole domain (4.3), were obtained for bodies of revolution; see [12] for details. For arbitrary symmetric convex bodies, the case \( \alpha + i \neq \beta + n \) \((i = 2, 3)\) remains open. The case \( \alpha + i > \beta + n \) contradicts (4.3) and is also open because it does not fall into the scope of Theorems 3.1 and 3.3 (in this case the condition (c) is not satisfied).

It is worth exhibiting the particular case \( \beta = 0; \ i = 2, 3 \), when the implication
\[
\int_{K \cap \xi} |x|^\alpha \, dx \leq \int_{L \cap \xi} |x|^\alpha \, dx \quad \forall \xi \in G_{n,i} \implies \text{vol}_n(K) \leq \text{vol}_n(L)
\]
holds provided \( \alpha = n - i - 1 \) and \( \alpha = n - i \). It may fail if \( \alpha < 0 \) and the question is open for \( 0 \leq \alpha < n - i \) \((\alpha \neq n - i - 1)\) and \( \alpha > n - i \).
4.3. More general homogeneous weights. The case $\alpha - \beta = n - i$ in the previous subsection when $a_K(\theta) \equiv 1$ deserves special mentioning. In this case, owing to Theorem 3.1, the implication

$$
\int_{K \cap \xi} |x|^\alpha \, dx \leq \int_{L \cap \xi} |x|^\alpha \, dx \forall \xi \in G_{n,i} \implies \int_{K} |x|^\beta \, dx \leq \int_{L} |x|^\beta \, dx
$$

is valid for all symmetric star bodies $K$ and $L$ and all $0 < i < n$. This observation can be essentially generalized. One can ask the following question: For which more general homogeneous weights the implication

$$
V_u(K \cap \xi) \leq V_u(L \cap \xi) \forall \xi \in G_{n,i} \implies V_v(K) \leq V_v(L)
$$

is independent of the choice of symmetric star bodies $K$ and $L$, i.e., $a_K(\theta)$ is independent of $K$? The following theorem answers this question.

**Theorem 4.2.** Let $u$ and $v$ be homogeneous functions of degree $\alpha$ and $\beta$, respectively, which satisfy the conditions (a)-(c) above. Suppose that $\alpha - \beta = n - i$ and there is a function $\varphi \in L^1(G_{n,i})$ such that

$$
v(\theta) = u(\theta) (R_i^* \varphi)(\theta)
$$

for almost all $\theta \in S^{n-1}$. Then the implication (4.7) holds for any symmetric star bodies $K$ and $L$ in $\mathbb{R}^n$.

**Proof.** The statement is a consequence of Theorem 3.1 because for any symmetric star bodies $K$,

$$
a_K(\theta) = \rho_K^{n-i}(\theta) \frac{u(\rho_K(\theta) \theta)}{u(\rho_K(\theta) \theta)} = \rho_K^{n-i}(\theta) \frac{\rho_K^\beta(\theta) u(\theta)}{\rho_K^\alpha(\theta) u(\theta)} = v(\theta) \frac{u(\theta)}{u(\theta)} = (R_i^* \varphi)(\theta).
$$

\qed

**Example 4.3.** Let us consider the weight functions

$$
u(x) = |x|^\gamma w_\gamma(x),
$$

where

$$
w_\gamma(x) = (1 - x_n^2/|x|^2)^{(\gamma+i-n)/2} = (|x'|/|x|)^{\gamma+i-n}, \quad x' = (x_1, \ldots, x_{n-1}).
$$

Suppose that $\alpha > -i$, $\alpha - \beta = n - i$, and $\gamma > 0$. It is known (see Example 2.5 in [10]) that $w_\gamma(\theta) = (R_i^* m_\gamma)(\theta)$ with

$$
m_\gamma(\xi) = \frac{\sigma_{n-2} \Gamma((i - 1 + \gamma)/2)}{\pi^{(i-1)/2} \sigma_{n-i-1} \Gamma(\gamma/2)} \sin^{\gamma+i-n} [d(e_n, S^{n-1} \cap \xi)],
$$
being the geodesic distance on $S^{n-1}$. By Theorem 4.2 for any symmetric star bodies $K$ and $L$ in $\mathbb{R}^n$, the inequality

$$\int_{K \cap \xi} |x|^\alpha \, dx \leq \int_{L \cap \xi} |x|^\alpha \, dx \quad \forall \xi \in G_{n,i}$$

implies

$$\int_K |x|^\beta w_\gamma(x) \, dx \leq \int_L |x|^\beta w_\gamma(x) \, dx.$$

In particular (set $\alpha = 0$) for any $\gamma > 0$,

$$\int_K |x'|^{\gamma+i-n} |x|^{-\gamma} \, dx \leq \int_L |x'|^{\gamma+i-n} |x|^{-\gamma} \, dx$$

provided $\int_{K \cap \xi} dx \leq \int_{L \cap \xi} dx \quad \forall \xi \in G_{n,i}$.

We conclude this article by laying stress on the question that is of major importance in Theorems 3.1 and 3.3: Is it possible to represent the comparison function $a_K(\theta)$ by the dual Radon transform of a positive measure? This question is difficult even in the case $i = n - 1$ when the corresponding Radon transform (it is known as the Minkowski-Funk transform) is actually self-adjoint and injective. The case $1 < i < n - 1$ is much more difficult because the dual Radon transform is non-injective for such $i$ (it has a nontrivial kernel). These difficulties have been overcome so far only in some particular cases using the tools fractional calculus, the Fourier analysis, and known facts from the theory of Radon transforms.

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