RAMSEY TRANSFER TO REDUCTS

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Abstract. We introduce a notion weaker than an infinitary interpretation which we call a semi-retraction (after [1]). We say a countable structure has the Ramsey property if its age does. For a countable structure $B$ with the Ramsey property we show a countable semi-retraction $A$ of $B$ must also have the Ramsey property. We introduce the notion of a color-homogenizing map that transfers the Ramsey property from one structure to another. We also introduce notation for what we call semi-direct product structures, after the group construction known to preserve the Ramsey property. We use color-homogenizing maps to give a finitary argument for why semi-direct product structures of structures with the Ramsey property must also have the Ramsey property. The last result is a characterization of NIP theories using a semi-direct product structure.

1. Introduction

Structural Ramsey theory is the study of partition properties of classes of first-order structures. The usual Ramsey theorem for finite sequences is a special case of a structural Ramsey theory result where the class of finite linear orders is the class of structures under consideration. To read a survey of some recent work in structural Ramsey theory please see [10]. In this paper we study a partition property for classes of ordered structures called the Ramsey property (RP) (see Definition 2.1 below for a formal definition.) In a slight departure from the usual we say a countable structure has RP if its age does.

In this paper we consider mechanisms by which RP may be transferred from one class of finite structures to another. We introduce the notion of semi-retraction (see Definition 3.3 below.) A semi-retraction has some elements in common with an infinitary interpretation of one structure in another (this latter definition is repeated in Definition 3.5 below.) There has already been some study of how interpretability of one structure in another may transfer RP: see “simply bi-definable” expansions in [7, Prop 9.1] and Ramsey expansions of a structure interpretable in a Ramsey structure in [2, Prop 3.8] (see [2, Def 2.19] for the definition of a “Ramsey structure”). It is known that countable structures $A$ and $B$ have homeomorphic automorphism groups if and only if $A$ and $B$ are infinitarily bi-interpretable (see [6, Cor 7.7 in Models and Groups] for a proof). By a well-known result from [7], a closed subgroup $G \leq S_\infty$ is extremely amenable if and only if $G$ is the automorphism group of a Fraïssé limit with RP. (This result is also surveyed in [10].) By the combination of these results, given linearly ordered Fraïssé limits $F_1, F_2$, if $F_1$ and $F_2$ are infinitarily bi-interpretable, then $F_1$ has RP if and only if $F_2$ has RP.

One interesting question that arises is: what are other mechanisms that transfer RP from one structure to another? In Theorem 3.8 we show that if $B$ has RP and $A$ is a semi-retraction of $B$, then $A$ also has RP. From this result we might wonder if it is enough for $A$ to be a reduct of $B$. In fact, we will see in Example 3.13 below that being a reduct is not sufficient. Besides studying semi-retractions we will look at color-homogenizing maps (see...
Definition 4.1 below) and semi-direct product structures (see Definition 5.7 below) building on a notion of product structures from [15].

In Section 2 we give standard model-theoretic and combinatorial notation as well as some background on the modeling property and generalized indiscernible sequences. The latter two notions are closely related to RP and we introduce them so we may use them in proofs. In Section 3 we define semi-retractions in Definition 3.3 and give the RP transfer result in Theorem 3.8. In Section 4 we define color-homogenizing maps in Definition 4.1 and prove the corresponding RP transfer result in Theorem 4.2. In Section 5 we define semi-direct product structures in Definition 5.7 and apply Theorem 4.2 to obtain a finitary argument for why the semi-direct product structure obtained from two classes each with RP itself has RP (Theorem 5.14). In Section 6, we deduce previously known examples of structures with RP as special cases of Theorem 5.14. We also prove a characterization of NIP theories using a generalized indiscernible sequence indexed by a semi-direct product structure in Corollary 6.5 (see [4] for more characterization-by-indiscernible-sequence results).

2. Preliminaries

2.1. Notation and Conventions.

(1) Structures

- **Structures** are first-order structures. Briefly, a first-order structure $A$ consists of an underlying set with interpretations of basic relation and function symbols from some pre-defined signature/language $L$. (See [5] as a reference for common model-theoretic terms.)
- The language of a structure $M$ is denoted $L(M)$.
- Given $L$-structures $A_1, A_2$, by $A_1 \cong A_2$ we mean that there is an $L$-isomorphism from $A_1$ to $A_2$. We may write $A_1 \cong_L A_2$ for clarity.
- For a structure $A$, $|A|$ denotes the underlying set of $A$.
- A substructure $A_0 \subseteq A$ is a structure such that $|A_0| \subseteq |A|$ and the relations on $A_0$ are interpreted as the restrictions of the relations on $A$.
- For two $L$-structures $A, B$, $A \subseteq B$ will mean that $A$ is a substructure of $B$.
- A copy of $A$ in $B$ will mean a substructure $A' \subseteq B$ where $A' \cong A$. We denote the set of all copies of $A$ in $B$ as $[B]_A$. By an increasing copy of $A$ we mean a copy of $A$ that is enumerated as a tuple $\sigma$ in increasing order, i.e. $a_i < a_j \iff i < j$.
- For structures $A, B$ with the same underlying set we say that $A$ is a reduct of $B$ if all basic relations in the language of $A$ are definable by quantifier-free formulas without parameters in the language of $B$.
- The age $\mathcal{K}$ of a structure $I$, $\mathcal{K} = \text{age}(I)$, is the collection of all finitely-generated substructures of $I$ up to isomorphism. In the case that the language of $I$ is relational, $\text{age}(I)$ is the collection of all finite substructures of $I$ up to isomorphism.
- Let $\mathcal{K}_{og}$ be the class of all finite graphs with binary edge relation $R$ whose vertices are ordered by relation $\prec$. By a random ordered graph we mean an isomorphic copy of the countable Fraïssé limit of the class $\mathcal{K}_{og}$. (See [F] for a reference on Fraïssé theory.)
- We will use “Flim” to abbreviate Fraïssé limit.

(2) Size and order.

- For a structure $A$, $||A||$ denotes the cardinality of $|A|$.
• By an ordered structure $\mathcal{I}$, we mean one that is linearly ordered by a binary relation in the language of the structure. The symbol $<$ is reserved for this linear order (unless $<_{\text{lex}}$ is used). By the underlying order of $\mathcal{I}$ we mean the reduct $I := (\{I\}, <)$.
• Tuples $\bar{a}$ are finite sequences $(a_i)_{i \in \omega}^n$ for some $n \in \omega$. We will often simply write $a$ for $\bar{a}$. By ran $\bar{a}$ we mean $\{a_i \mid i \in n\}$.
• Given two finite sequences $\bar{a}, \bar{b}$ from $\omega$ we say that $\bar{a} <_{\text{lex}} \bar{b}$ if either $\bar{a}$ is an initial segment of $\bar{b}$ or else at the least $j$ where $a_j \neq b_j$, we have $a_j < b_j$. We call $<_{\text{lex}}$ the lexicographic order.

(3) Conventions.
In this paper, we consider only ordered structures and only countable ages. Infinite structures will be locally finite. For simplicity, we work with only countable structures, as these can be used effectively to describe countable ages. All structures except the various reducts of the tree $I_{\text{stree}}$ (see Definition 3.1) are structures in some relational language.

(4) Types
• We say $\eta$ is a quantifier-free type in $A$ if it is a set of quantifier-free formulas in some finite list of variables $(v_0, \ldots, v_{n-1})$ in the language of $A$ that is satisfied by some tuple from $A$.
• The complete quantifier-free type of $\bar{a}$ in $A$, $\text{qftp}(\bar{a})$, or $\text{qftp}^A(\bar{a})$ for clarity, is just the maximal quantifier-free type in $A$ satisfied by $\bar{a}$.
• Given a structure $A$, we write $\bar{a} \equiv_A \bar{b}$ to denote: $A \models \varphi(\bar{a}) \iff A \models \varphi(\bar{b})$ for all $\varphi$ in the language of $A$.
• We write $\bar{a} \sim_A \bar{b}$ to denote that $A \models \theta[\bar{a}] \iff A \models \theta[\bar{b}]$ for all quantifier-free formulas $\theta$ in the language of $A$. In other words, $\text{qftp}^A(\bar{a}) = \text{qftp}^A(\bar{b})$. Equivalently, the map $a_i \mapsto b_i$ extends to an isomorphism of the structures generated by $\bar{a}$ and $\bar{b}$.

(5) Ramsey notions.
• For a finite integer $k \geq 1$, $[k] := \{1, 2, \ldots, k\}$.
• For a finite integer $k \geq 1$ a $k$-coloring of a set $X$ is any function $c : X \to [k]$. If $[k]$ is replaced by any other set of cardinality $k$, $c$ may still be referred to as a $k$-coloring, or even simply a finite coloring if the size of $k$ is unimportant.
• We repeat the following definition from [11, Intro part (D)]

Definition 2.1. We say that an age of finite structures has the Ramsey property (RP) if for all $A, B \in \mathcal{K}$ and finite integers $k \geq 1$ there exists $C \in \mathcal{K}$ such that for any $k$-coloring of $c$ of $\binom{C}{A}$, there is $B' \in \binom{C}{B}$ such that for any $A', A'' \in \binom{B'}{A}$, $c(A') = c(A'')$.

We say that $B'$ is a copy of $B$ homogeneous for $c$ (on copies of $A$).

• In this paper, we will refer to a structure $A$ as having RP if age($A$) has RP.

2.2. The modeling property. In the study of classification theory in model theory there has been significant use of generalized indiscernible sequences, called “$I$-indexed indiscernible sets” in [13]. An $I$-indexed indiscernible set is an $I$-indexed subset of a structure $\mathcal{U}$ that is homogeneous according to the $I$-indexing: the language of $\mathcal{U}$ does not make more distinctions on the subset than the relations in the language of $I$. 
**Definition 2.2.** Fix a structure $I$ and same-length tuples $a_i$ from some structure $U$, for all $i \in I$. We say that the set $(a_i : i \in I)$ is $I$-indexed indiscernible if for all finite $n \geq 1$, for all length-$n$ sequences $\vec{t}, \vec{j}$ from $I$

$$\vec{t} \sim_I \vec{j} \Rightarrow \forall \vec{a} \equiv_U U \forall \vec{a}$$

It is often desirable to find an $I$-indexed indiscernible set that witnesses specific definable configurations in $U$. For this, we define the notion of EM-type:

**Definition 2.3** ([12]). Given an $L'$-structure $I$, an $L$-structure $U$ and an $I$-indexed set of same-length tuples from $U$, $X = (a_i : i \in I)$, we define the Ehrenfeucht-Mostowski type (EM-type) of $X$ to be a syntactic type in variables $(x_i : i \in I)$ such that for $\psi$ from $L$, $\psi(a_i) \in \text{EMtp}(X)$ if and only if for all $(j_1, \ldots, j_n)$ from $I$ such that $(j_1, \ldots, j_n) \sim_I (i_1, \ldots, i_n)$, $\psi(x_{j_1}, \ldots, x_{j_n}) \in \text{EMtp}(X)$.

**Remark 2.4.** The EM-type of $X$ may be encoded by “rules” $\{(\eta_s, \psi_s)\}_s$ where $\eta_s$ is a quantifier-free type in the language of $I$, $\psi_s$ is a formula in the language of $U$, and for all $\vec{t}$ satisfying $\eta_s$ in $I$, $a_{\vec{t}}$ satisfies $\psi_s$ in $U$. This is captured in the Proposition below.

**Proposition 2.5.** Fix sets of length-$n$ tuples from $U$ indexed by $I$

$$X = (a_i : i \in I)$$

$$Y = (b_i : i \in I)$$

$Y \models \text{EMtp}(X)$ if for all quantifier-free types $\eta$ in $I$, if

$$(\forall \vec{t})(I \models \eta(\vec{t}) \Rightarrow U \models \varphi(\vec{a}))$$

then

$$(\forall \vec{t})(I \models \eta(\vec{t}) \Rightarrow U \models \varphi(\vec{b}))$$

The following property guarantees that the “rules” encoding an EM-type can always be witnessed in an $I$-indexed indiscernible set.

**Definition 2.6.** For an infinite structure $I$, $I$-indexed indiscernible sets have the modeling property if for any set of same-length tuples from a sufficiently-saturated structure $U$

$$X = (a_i : i \in I)$$

there exists an $I$-indexed indiscernible set

$$Y = (b_i : i \in I)$$

also in $U$ such that $Y \models \text{EMtp}(X)$.

We may say that $Y$ is locally-based on $X$.

The following result shows a connection between the modeling property and RP. The condition qfi stands for “quantifier-free types are isolated by quantifier-free formulas”.

**Theorem 2.7.** ([12, Thm 3.12]) Suppose that $I$ is a qfi, locally finite structure in a language $L'$ with a relation $<$ linearly ordering $I$. Then $I$-indexed indiscernible sets have the modeling property if and only if age($I$) has RP.

This is a generalization of the similar result in [11] that was for relational languages $L'$. 
Remark 2.8. In fact, it was later pointed out to the author that the qfi assumption is not needed (see Acknowledgements). To see this, in the argument for [12, Claim 3.13] we replace $L'$ with expansion $L''$ that contains predicates $p_A(\tau)$ for all complete quantifier-free types of finite substructures $A$ of $I$. Then we apply compactness to the type $S$ where we replace $T_\forall \cup \text{Diag}(I)$ with the diagram of $I$ in $L''$. It was noted in the proof for [12, Thm 3.12] that the qfi hypothesis was used only in the argument for Claim 3.13, and in this Remark we point out why it is not even needed there. In terms of the compactness argument presented, the more important assumption is that structures $A, B$ being described in the type $S$ are finite, and can be listed in finitely many variables.

3. Transfer by semi-retractions

The inspiration for the theorem below is the following example. We give the definitions of the Shelah tree, $I_{\text{stree}}$, the strong tree $I_{\text{strtree}}$ and the convexly ordered equivalence relation $I_{\text{eq}}$. An exposition of the proof for $I_{\text{stree}}$ and $I_{\text{strtree}}$ having RP is given in [8]. $I_{\text{eq}}$ is proven to have RP in [7, Thm 6.6] and this fact can be used to show that a witness to $k$-TP2 may be assumed to be “array indiscernible” (see [8, Lem 5.6] for an alternate proof.)

Definition 3.1. Let $A$ be a finite, countably infinite structure in a possibly different language. We say that an injection $h : A \rightarrow B$ is quantifier-free type-preserving (qftp-preserving) if for all $\bar{a}, \bar{b}$ from $A$.

Definition 3.2. We say that an injection $h : A \rightarrow B$ is quantifier-free type-preserving (qftp-preserving) if

For all $\bar{a}, \bar{b}$ from $A$.

Definition 3.3 (semi-retractions). Let $A, B$ be countably infinite structures in possibly different languages. We say that $A$ is a semi-retraction of $B$ if there exist quantifier-free type-preserving injections $g : A \rightarrow B$ and $f : B \rightarrow A$ such that for any quantifier-free type $\eta$ in $A$

(i) $A \models \eta(\bar{a}) \Rightarrow A \models \eta((f \circ g)(\bar{a}))$

Observation 3.4. For $f, g, A$ in the definition of semi-retraction, $f \circ g$ maps onto a copy of $A$ in $A$.

We repeat a related definition.

Definition 3.5. Fix structures $A$ and $B$.

- A subset $U$ of a finite power of $A$ is quasidefinable in $A$ if it is the union of Aut($A$)-orbits.
• \( \mathbf{g} \) is an infinitary interpretation of \( B \) in \( A \) if \( \mathbf{g} \) is a countable set of functions \( g_i : U_i \to B \) where \( U_i \) are quasidefinable subsets of some finite power of \( A \) such that
  
  1. \( \bigcup_i \text{Im} g_i = B \);
  2. \( \{(u, v) \in U_i \times U_j | g_i(u) = g_j(v)\} \) is always quasidefinable in \( A \); and
  3. for all quasidefinable relations \( R \) on \( B \), \( g_i^{-1}(R) \) is quasidefinable in \( A \).

• \( A \) and \( B \) are infinitarily bi-interpretable if there exist interpretations \( \mathbf{f} \) of \( A \) in \( B \) and \( \mathbf{g} \) of \( B \) in \( A \) whose compositions are homotopic to the identity maps, meaning \( \{(u, v) \in U_i \times U_i | (\mathbf{g} \circ \mathbf{f})(u) = v\} \) is always quasidefinable in \( A \), and the similar condition for \( \mathbf{f} \circ \mathbf{g} \).

**Definition 3.6** ([1]). \( A \) is a **retraction** of \( B \) if there exist interpretations \( \mathbf{f} \) of \( A \) in \( B \) and \( \mathbf{g} \) of \( B \) in \( A \) such that \( \mathbf{f} \circ \mathbf{g} \) is homotopic to the identity on \( A \).

**Remark 3.7.** We might ask to what extent a semi-retraction is related to a retraction? Recall that if \( A \) is ultrahomogeneous, then any \( \text{Aut}(A) \)-orbit on a finite power of \( A \) is defined by a quantifier-free type. Fix \( g : A \to B \) and \( f : B \to A \) witnessing that \( A \) is a semi-retraction of \( B \). If \( A, B \) are ultrahomogeneous, then the pullbacks of quasidefinable relations under \( f, g \) are quasidefinable. If we let \( \mathbf{f} = \{f\}, \mathbf{g} = \{g\} \), then condition (2) in Definition 3.5 comes for free. Since \( \mathbf{f} \circ \mathbf{g} \) is surjective onto a copy \( A' \) of \( A \) in \( A \), we might say that \( f : g(A) \to A' \) is the desired surjective map in condition (1) of Definition 3.5 except that there is no requirement in Definition 3.5 that either \( g \) be surjective or \( g(A) \) be quasidefinable. For this reason, we cannot conclude that \( \mathbf{f} \circ \mathbf{g} \) always give interpretations.

In fact, in Corollary 3.12, we have an example of a semi-retraction where \( g \) is not surjective and \( g(A) \) is not quasidefinable. Of further interest, \( g \circ f \) does not happen to be an embedding in this example, which shows that a semi-retraction can have a unidirectional nature unlike a bi-interpretation.

The following result is used to show that semi-retractions transfer RP.

**Theorem 3.8.** Let \( A, B \) be countably infinite structures in possibly different languages. Suppose \( A \) is a semi-retraction of \( B \). Furthermore, suppose \( B \)-indexed indiscernible sets have the modeling property. then \( A \)-indexed indiscernible sets have the modeling property.

**Corollary 3.9.** Suppose \( A \) is a semi-retraction of \( B \) and \( B \) has RP. Then \( A \) has RP.

**Proof.** By Theorem 3.8 and Theorem 2.7. \(\square\)

**Proof.** (proof of Theorem 3.8) Fix a set of same-length \( A \)-indexed tuples in some sufficiently-saturated structure \( U \)

\[
X = (c_i \mid i \in A)
\]

We want to find an \( A \)-indexed indiscernible set

\[
Y = (e_i \mid i \in A)
\]

in \( U \) such that \( Y \models EMtp(X) \).

Define, using \( f : B \to A \)

\[
X' = (c_{f(j)} \mid j \in B)
\]

By assumption there is a \( B \)-indexed indiscernible set

\[
Y' = (d_j \mid j \in B)
\]

in \( U \) such that \( Y' \models EMtp(X') \).

**Claim 3.10.** If we set \( e_i := d_{g(i)} \), then \( Y = (e_i \mid i \in A) \) is the desired set.
Proof. To see that $Y$ is an $A$-indexed indiscernible set: fix $\bar{\tau}_1 \sim_A \bar{\tau}_2$. Since $g$ is qftp-preserving, $g(\bar{\tau}_1) \sim_B g(\bar{\tau}_2)$. By $B$-indexed indiscernibility of $Y'$:

$$\overline{\eta(\bar{\tau}_1)} \equiv_U \overline{\eta(\bar{\tau}_2)}$$

i.e.

$$\overline{\tau}_1 \equiv_U \overline{\tau}_2$$

To see that $Y \models \text{EMtp}(X)$: Fix $\eta, \varphi$ such that

(1) \quad (\forall \bar{\tau})(A \models \eta(\bar{\tau}) \Rightarrow U \models \varphi(\overline{\eta}))

Also fix $\overline{\tau}$ so that

(2) \quad A \models \eta(\overline{\tau})

We wish to show that $\models \varphi(\overline{\eta})$.

Since $f$ is qftp-preserving, there are quantifier-free types $\delta_k$ in $B$ such that for any $\overline{\eta}$ from $B$:

(3) \quad A \models \eta(f(\overline{\eta})) \Leftrightarrow B \models \bigvee \delta_k(\overline{\eta})

We could say

$$f^{-1}(\eta(A)) = \bigcup \delta_k(B)$$

Thus via assumptions (1) and (3) we get:

(4) \quad (\forall k) : (\forall \overline{\eta})(B \models \delta_k(\overline{\eta}) \Rightarrow U \models \varphi(\overline{\eta}))

Since $Y' \models \text{EMtp}(X')$, this transfers to $Y'$:

(5) \quad (\forall k) : (\forall \overline{\eta})(B \models \delta_k(\overline{\eta}) \Rightarrow U \models \varphi(\overline{\eta}))

By condition (i) of Definition 3.3

(6) \quad A \models \eta(\overline{\tau}) \Rightarrow A \models \eta((f \circ g)(\overline{\tau}))

Observe that by (3) and letting $\overline{\eta} := g(\overline{\tau})$

(7) \quad A \models \eta(f \circ g)(\overline{\tau}) \Rightarrow B \models \bigvee \delta_k(g(\overline{\tau}))

So we conclude by (2), (6) and (7):

(8) \quad B \models \bigvee \delta_k(g(\overline{\tau}))

Apply this fact to (5) with $\overline{\eta} := g(\overline{\tau})$ to get

(9) \quad U \models \varphi(\overline{\eta})$

i.e.

(10) \quad U \models \varphi(\overline{\tau})$

as desired.
Remark 3.11. It is clear that the \((d_{g(i)} \mid i \in A)\) must be an A-indexed indiscernible set just because the map \(g\) is type-preserving and the \(d\)'s form a B-indexed indiscernible set. This does not require anything special about the type \(g(\eta)\) that \(g(\tau)\) has in \(B\) as a function of the type \(\eta\) of \(\tau\) in \(A\).

By Observation 3.14 every quantifier-free type \(\eta\) in \(A\) pulls back by \(f\) to a union of refining types \(\delta_k\) in \(B\). The \((d_{g(i)} \mid i \in A)\) witness the modeling property for \(A\)-indexed indiscernible sets precisely because the map \(g\) selects a type \(g(\eta)\) for \(\eta\) that is one of these \(\delta_k\), so that \((f \circ g)(\eta)\) computes correctly as \(\eta\).

Corollary 3.12. If \(I_{\text{strtree}}\) has the Ramsey property, then so does \(I_{\text{eq}}\).

Proof. We let \(B = I_{\text{strtree}}, A = I_{\text{eq}}\). Our referee for [8] kindly suggested that we deduce RP for \(A\) from \(B\) by constructing a special embedding \(g : A \to B\) that is qftp-preserving (see [8, Thm 5.5] for details). Define \(\eta_i = (0, \ldots, 0)\). Now let \(g\) take the \(i\)th equivalence class of \(I_{\text{eq}}\) in the \(\prec\)-order to \(\{\eta_i \upharpoonright (j + 1) : j < \omega\}\). In the [8] proof, \(f : B \to A\) is taken to be the identity map, and thus \(A\) is a semi-retraction of \(B\).

By Corollary 3.9 since \(B = I_{\text{strtree}}\) has RP, \(A = I_{\text{eq}}\) must have RP. 

Observe that \(I_{\text{eq}}\) is a reduct of \(I_{\text{strtree}}\) by defining \(E(a_1, a_2) \iff \neg(a_1 \ll a_2 \lor a_2 \ll a_1)\). One might wonder if this is enough to guarantee transfer of the Ramsey property, but that is not the case as the following example shows.

Example 3.13. Not every reduct of a structure with RP has RP. For example, define \(I_0 := I_{\text{strtree}} \upharpoonright \{\ll, \land, \ll_{\text{lex}}\}\) and \(I_t := I_0 \upharpoonright \{\ll, \ll_{\text{lex}}\}\). \(I_0\) was originally shown to have RP in [9] and \(I_t\) was shown not to have RP in [17] (see [16] or [12, Cor 3.19] for a discussion). This is even a case where \(I_t\) is a linearly ordered reduct of a countable structure with RP but still fails to have RP.

4. Transfer by Homogenized Coloring

We start with a technical definition.

Definition 4.1 (color-homogenizing maps). Fix countable structures \(A, B\), and finite integers \(m, k \geq 1\). Given a finite substructure \(B_0 \subseteq A\), a finite \(k\)-coloring \(c\) on \(m\)-tuples from \(B\) and an injection \(g : B_0 \to B\) we say that \(g\) is color-homogenizing for \(c\) and \(B_0\) if for all \(\tau, \overline{\tau}\) of length \(m\) from \(B_0\)

\[\tau \sim_A \tau \Rightarrow c(g(\tau)) = c(g(\overline{\tau}))\]

Theorem 4.2. Let \(A\) be an \(L\)-structure and \(B\) an \(L'\)-structure, both countably infinite. Suppose there is an injection \(f : B \to A\) such that for any \(m, k \geq 1\), any finite substructure \(B_0 \subseteq A\) and any \(k\)-coloring \(c\) on \(m\)-tuples from \(B\), there is a color-homogenizing map \(g\) for \(B_0\) and \(c\) such that \(f \circ g : B_0 \to A\) is an \(L\)-embedding.

Then, \(A\) has RP.

Proof. Fix finite substructures \(A_0, B_0\) of \(A\) and suppose \(||A_0|| = m\). Let \(c'\) be a finite \(k\)-coloring of \(\binom{A}{A_0}\). We may extend \(c'\) to a finite \(k\)-coloring on all finite increasing \(m\)-tuples from \(A\). It suffices to find a copy of \(B_0\) in \(A\) homogeneous for this coloring on copies of \(A_0\).

Define a pull-back coloring \(c\) on \(m\)-tuples \(\tau\) from \(B\) by: \(c(\tau) = c'(f(\tau))\).

By assumption, there is \(g : B_0 \to B\) color-homogenizing for \(B_0\) and \(c\).

Claim 4.3. \((f \circ g)(B_0) =: B'_0\) is homogeneous for \(c\) on copies of \(A_0\).
Proof. First of all, $B_0 \cong B_0'$ by assumption that $f \circ g$ is an embedding. To see that $B_0'$ is homogeneous, let $d_1, d_2$ be increasing copies of $A_0$ in $B_0'$. Take their pre-images under $f \circ g$: $\overline{f} := (f \circ g)^{-1}(d_1)$ and $\overline{g} := (f \circ g)^{-1}(d_2)$ in $B_0$. Since $d_1 \sim_A d_2$ and $f \circ g$ is an embedding, we have that $\overline{f} \sim_A \overline{g}$. By assumption on $g$, $c(g(\overline{f})) = c(g(\overline{g}))$. By definition of $c$, $c'(f(g(\overline{f}))) = c'(f(g(\overline{g})))$, i.e., $c'(d_1) = c'(d_2)$.

$\square$

$\square$

5. SEMI-DIRECT PRODUCT STRUCTURES

Definition 5.1. Given a linear order $O$ and countable relational structures $M_i$ for $i \in O$ all ordered by a symbol $<$ define the structure $\mathcal{U} := \mathcal{U}_{i \in O}(M_i)$ to have

- domain the disjoint union of the domains of the $M_i$, i.e. $|\mathcal{U}_{i \in O}(M_i)| = \bigcup_i |M_i|$, and
- language $L(\mathcal{U}) = \bigcup_i L(M_i) \cup \{P_i\}_{i \in O}$

for new unary predicates $P_i$. All relations from the $L(M_i)$ except $<$ are interpreted in the same way on $|\mathcal{U}_{i \in O}(M_i)|$: $<$ and the $P_i$ are interpreted as follows:

(i) we define $a <^\mathcal{U} b$ if and only if there exist $i, j \in O$ such that $a \in M_i, b \in M_j$ and either $i < j$ or else $i = j$ and $a <^M_i b$, and
(ii) $P_i^\mathcal{U} = |M_i|$.

Observation 5.2. $\mathcal{U}_{i \in O}(M_i)$ preserves certain properties of the $M_i$:

- $\mathcal{U}_{i \in O}(M_i)$ is linearly ordered by $<$.
- If each $M_i$ for $i \in O$ has properties HP, JEP and AP then age($\mathcal{U}_{i \in O}(M_i)$) can be easily seen to have these properties: for example, to solve an amalgamation problem it suffices to solve the problem as restricted to $I$ and each $M_i$ and then take the union.

Here we restate the product ramsey theorem for classes. The notation $(B_i)_{i=1}^s$ denotes all length-$s$ sequences $(A'_i)_{i=1}^s$ such that $A'_i \subseteq B_i$ and $A'_i \cong A_i$ for every $i \leq s$.

Theorem 5.3 (13, Thm 2]). Fix $r, s \geq 1$ and let $(K_i)_{i=1}^s$ be a sequence of classes with RP. Fix $(B_i)_{i=1}^s, (A_i)_{i=1}^s$ such that $B_i, A_i \in K_i$. There exist $C_i \in K_i$ such that for any coloring $p : (C_i)_{i=1}^s \rightarrow \{1, \ldots, r\}$, there exists a sequence $(B'_i)$, with $B'_i \cong B_i$ and some $\ell \in \{1, \ldots, r\}$ such that $p$ restricted to $(B'_i)_{i=1}^s$ is the constant function $\ell$.

Corollary 5.4. If each $M_i$ for $i \in O$ in Definition 5.1 has RP then $\mathcal{U}_{i \in O}(M_i)$ also has RP.

Proof. This follows by Theorem 5.3. This is because for any substructure $A \subseteq \mathcal{U}_{i \in O}(M_i)$ there is a finite sequence $t_1 < \ldots < t_s$ from $O$ such that there exist substructures $A_i \subseteq M_{t_i}$ where $A$ is the increasing union of the sequence $(A_i)_{i=1}^s$.

Definition 5.5. Given a finite substructure $B_0$ of some countable structure $A$

- Define Sub$(B_0)$ to be the set of all substructures of $B_0$ up to isomorphism.
- Define a $k$-coloring of $(A_{\text{Sub}(B_0)})$ to be a $k$-coloring of $\bigcup S \in \text{Sub}(B_0)$ $(A_S)$

Here is a slight restatement of RP that we will need.

Proposition 5.6. If $A$ has RP, then for any finite substructure $B_0$ of $A$, for any finite $k$-coloring $c$ of $(A_{\text{Sub}(B_0)})$, there is a copy $B'_0 \cong B_0$ in $A$ such that $B'_0$ is homogeneous for $c$ on copies of $S$ for all $S \in \text{Sub}(B_0)$. In an extension of our usual convention we say $B'_0$ is homogeneous for $c$. 

Proof. This is well-known (e.g. See Claim 4.16 in [11]) and can also be argued for using $A$-indexed indiscernible sets. We repeat the argument here. Let $\text{Sub}(B_0) = \{D_1, \ldots, D_m\}$. Let $c$ be a $k$-coloring of $\text{Sub}(B_0)$ in $A$. We can define $Z_n$ so that $Z_1 := B_0$ and $Z_n \to (Z_{n-1})^D_{n-1}$ for $2 \leq n \leq m + 1$. Now define $Y_1 := Z_{m+1}$. Given $Y_{n-1}$, obtain $Y_n$ as a copy of $Z_{m-(n-2)}$ in $Y_{n-1}$ homogeneous for $c$ in copies of $D_{m-(n-2)}$, for all $2 \leq n \leq m + 1$. Thus $Y_1 \supseteq Y_2 \supseteq \cdots \supseteq Y_{m+1}$ and ultimately $Y_{m+1}$ is a copy of $Z_1 = B_0$ homogeneous for $c$ on copies of $D_m, D_{m-1}, \ldots, D_1$. □

We generalize on Definition 5.1.

Definition 5.7 (semi-direct product structures). Let $\mathcal{I}$ be a countable relational $L_2$-structure ordered by $<$. Let each $I_i$ be a countable relational $L_1$-structure also ordered by $<$. Assume there is an age $\mathcal{K}$ such that $\text{age}(M_i) = \mathcal{K}$ for all $i \in I$. Define the structure $\mathcal{I} := \text{Sub}(\mathcal{I}_i)_{i \in I}$ to have

- domain the disjoint union of the domains of the $M_i$, i.e. $|\mathcal{I}_i| = \bigcup_i |M_i|$, and
- language $L(\mathcal{I}) = L_2 \cup L_1 \cup \{E\}$

for a new binary relation $E$. All relations from $L_1 \setminus \{<\}$ are interpreted in the same way on $|\mathcal{I}_i|$: $E$ and relations from $L_2$ (including $<$) are interpreted as follows:

(i) for any $n$-ary relation $R_{\ell} \in L_2 \setminus \{<\}$, we define

\[ R_{\ell}^c(a_0, \ldots, a_{n-1}) \iff \text{for any } a_i \in M_i \text{ for all } i < n \text{ and } \mathcal{I} \models R_{\ell}(t_0, \ldots, t_{n-1}) \]

\[ a <^I b \iff \text{for any } i, j \in I \text{ such that } a \in M_i, b \in M_j \text{ and either } i <^I j \text{ or else } i = j \text{ and } a <^{M_i} b. \]

Remark 5.8. In Definition 5.7, $\mathcal{I}$ may be identified with $\mathcal{I}/E$ by the map $t \mapsto [a]/E$ for any $a \in M_i$. Preservation of order and equality are easily verified and for any relation $R_{\ell} \in L_2 \setminus \{<\}$:

\[ R_{\ell}^c(t_0, \ldots, t_{n-1}) \Rightarrow \text{for any } a_i \in M_i \text{ for all } i < n, R_{\ell}^c(a_0, \ldots, a_{n-1}) \]

\[ \neg R_{\ell}^c(t_0, \ldots, t_{n-1}) \Rightarrow \text{for any } a_i \in M_i \text{ for all } i < n, \neg R_{\ell}^c(a_0, \ldots, a_{n-1}) \]

by definition.

Observation 5.9. $\mathcal{I}_i$ is linearly ordered by $<$.  

Definition 5.10. Let $\mathcal{I}, I, L_1, L_2, \{M_i\}_{i \in I}$ be as in Definition 5.7.

1. for a substructure $S \subseteq \mathcal{I}_i \subseteq \mathcal{I}$, define $\text{gr}(S)$ to be the $L_2$-substructure of $\mathcal{I}$ identified with $\{ [s]/E \mid s \in S \}$ as in Remark 5.8 above.

   We call $\text{gr}(S)$ the "underlying graph" of $S$.

2. for a tuple $\bar{v}$ from $\mathcal{I}_i \subseteq \mathcal{I}$, by $\text{gr}(\bar{v})$ we mean $\text{gr}(\text{ran} \bar{v})$.

3. for a substructure $C \subseteq \mathcal{I}_i \subseteq \mathcal{I}$, by $C^{\text{red}}$ we mean the $(L_1 \cup \{E\})$-reduct of $C$.

   For $C \subseteq \mathcal{I}_i \subseteq \mathcal{I}$, by $C^{\text{red}}$, we mean the $(L_1 \cup \{E\})$-reduct of $C$ as it is naturally interpreted.

Observation 5.11. $|\text{gr}(S)| = \{ t \in |\mathcal{I}| \mid |S| \cap |M_i| \neq \emptyset \}$

Proposition 5.12. If $\text{age}(\mathcal{I})$ and $\mathcal{K}$ have properties $HP$, $JEP$ and $AP$ then $\text{age}(\mathcal{I}_i \subseteq \mathcal{I})$ also has these properties.
Proof. We verify AP and the rest are argued similarly. Consider an amalgamation problem given by \( A \subseteq B_1, B_2 \) where these are all finite substructures of \( \mathcal{I}_{i \in I}(M_i) \). Then, \( \text{gr}(A) \subseteq \text{gr}(B_1), \text{gr}(B_2) \) is an amalgamation problem in \( \mathcal{I} \). Since \( \text{age}(\mathcal{I}) \) is assumed to have AP and \( L_2 \) is relational, there is a solution \( C' \) with underlying order \( C \) whose domain \( |C'| = |\text{gr}(B_1)| \cup |\text{gr}(B_2)| \). For any \( t \in |C'| \), let \( A_t \) be the \( L_1 \)-structure on domain \( |A| \cap |M_t| \), similarly define \( B_{1,t}, B_{2,t} \) for \( i = 1, 2 \). If \( A_t \neq \emptyset \), the amalgamation problem \( A_t \subseteq B_{1,t}, B_{2,t} \) has a solution \( C_t \) in \( M_t \), by assumption that \( \mathcal{K} \) has AP. If \( A_t = \emptyset \), the joint embedding problem \( B_{1,t}, B_{2,t} \) has a solution \( C_t \) in \( M_t \). The resulting structure \( C'_{t \in C'}(C_t) \) solves the original amalgamation problem.

**Proposition 5.13.** Let \( \mathcal{I}, I, L_1, L_2, \{ M_i \}_{i \in I}, \mathbb{I} \) be as in Definition 5.7

Suppose \( \overline{a}, \overline{b} \in |\mathbb{I}|^m \) such that

\[
qftp^\text{red}(\overline{a}) = qftp^\text{red}(\overline{b})
\]

Then,

(i) \( \text{gr}(\overline{a}) = \text{gr}(\overline{b}) \iff qftp^{\mathcal{I}_{i \in I}(M_i)}(\overline{a}) = qftp^{\mathcal{I}_{i \in I}(M_i)}(\overline{b}), \) and

(ii) \( \text{gr}(\overline{a}) \cong \text{gr}(\overline{b}) \iff qftp^\text{red}(\overline{a}) = qftp^\text{red}(\overline{b}) \)

Proof. To see the \( \Rightarrow \) direction of (i) recall that \( P_i^t = |M_i| \). By assumption (11), there exists \( q < \omega \) and sequences \( (t_i)_{i < q}, (s_i)_{i < q} \) such that for all \( j < m, i < q: a_j \in |M_{t_i}| \Leftrightarrow b_j \in |M_{s_i}| \).

In other words, \( a_j \in P_i^t \Leftrightarrow b_j \in P_i^s \). The assumption in (i) that \( \text{gr}(\overline{a}) = \text{gr}(\overline{b}) \) together with the linear order guarantee that \( t_i = s_i \) for all \( i < q \).

The following Theorem could be obtained by the technology in [7]. Set \( \mathcal{F} = \text{Flim}(\mathcal{K}), \mathcal{G} = \text{Flim}(\text{age}(\mathcal{I})) \). Let \( G = \text{Aut}(\mathcal{G}) \times \prod \text{Aut}(\mathcal{F}) \). \( G \) is the automorphism group of the Fraïssé limit of \( \mathcal{I}_{i \in I}(M_i) \) and can be seen to be extremely amenable by [7, Lem. 6.7]. We give an alternate finitary proof using Theorem 4.2.

**Theorem 5.14.** Let \( \mathcal{I}, I, L_1, L_2, \{ M_i \}_{i \in I}, \mathbb{I}, \mathcal{K} \) be as in Definition 5.7. Assume additionally that \( \mathcal{I} \) has RP and \( \mathcal{K} \) has RP.

Proof. Towards applying Theorem 4.2. Define \( B := U_{i \in I}(M_i), A := L_{i \in I}(M_i) \). Structures \( A, B \) share their underlying set which we will call \( X \). All instances of \( M \subseteq N \) in this proof should be read as “\( M \) is a substructure of \( N \).”

Define \( f : B \to A \) to be the identity on underlying sets. Fix an integer \( m \geq 1 \), a finite substructure \( B_0 \subseteq A \) (which we may assume to be of size at least \( m \)), and a \( k \)-coloring \( c \) on increasing \( m \)-tuples from \( B \). List representatives of all isomorphism types of substructures of \( B_0 \) of size \( m \) as:

\[
E_0, \ldots, E_{t-1}
\]

Let \( H := \text{gr}(B_0) \). Let \( \ell = k^l \). By assumption that \( \mathcal{I} \) has RP and Proposition 5.6 there is some finite substructure \( N \subseteq \mathcal{I} \) such that

\[
(12) \quad N \rightarrow (H)_\ell^{\text{Sub}(H)}
\]

We may assume \( H \subseteq N \). We construct a new \( L(B) \)-structure \( B_\text{big} \) such that \( \text{gr}(B_\text{big}) = N \) and such that for any \( H_1 \subseteq N \) such that \( H_1 \cong H \), there is some \( B_1 \subseteq B_\text{big} \) such that

\[
(13) \quad B_1^\text{red} \cong B_0^\text{red} \quad \text{and} \quad \text{gr}(B_1) = H_1
\]
The structures $B_1$ above clearly exist in age($B$), so one way to complete the construction would be to use AP in age($B$) to glue them together with additional structures to guarantee that $\text{gr}(B_{\text{big}}) = N$. By properties of an age, we may assume that $B_{\text{big}} \subseteq B$.

By Proposition 5.14, $U_{c \in \ell}(M_i)$ has RP, so by Proposition 5.6, there is a copy of $B_{\text{big}}^* \cong B_{\text{big}}$, $B_{\text{big}}^* \subseteq B$, that is homogeneous for $c$. By the predicates $P_i$ in $L(B)$, $\text{gr}(B_{\text{big}}^*) = N = \text{gr}(B_{\text{big}})$.

Define an $\ell$-coloring $c'$ of $(\text{Sub}(H))^N$ on structures $K \subseteq H \subseteq N$ as follows:

$$K \mapsto (k_0, \ldots, k_{l-1}) \in k^l = \ell$$

where we define

$$k_i := \begin{cases} 0 & \text{if } \text{gr}(E_i) \not\sim K \\ c(F_i) & \text{if } \text{gr}(E_i) \equiv K, \text{and there exists } F_i \subseteq B_{\text{big}}^*, F_i^{\text{red}} \cong E_i^{\text{red}}, \text{gr}(F_i) = K \end{cases}$$

$c'$ is well-defined by construction of $B_{\text{big}}$, Proposition 5.13(i) and homogeneity of $B_{\text{big}}^*$. $c'$ is an $\ell$-coloring of $\text{Sub}(H)$, so by choice of $N$ there is a copy $H^*$ of $H$ in $N$ homogeneous for $c'$. By construction of $B_{\text{big}}$, there exists $B_0^*$ in $B_{\text{big}}$ such that

$$B_0^{\text{red}} \cong B_0^{\text{red}} \text{ and } \text{gr}(B_0^*) = H^*$$

Let $g : B_0 \rightarrow B_0^*$ witness the isomorphism $B_0^{\text{red}} \cong B_0^{\text{red}}$. By Proposition 5.13(ii), since $H^* \cong H$, we know that $f \circ g$ is also an $L_2$-embedding, given that $f$ is the identity map. In particular, $g$ acts as an $L_2$-isomorphism on the underlying graphs.

This $g$ works for Theorem 4.2. Fix any $m$-tuples $\overline{a}_1 \sim_A \overline{a}_2$ from $B_0$. Since the tuples are isomorphic in $L(A) \supseteq L_1 \cup \{E\}$ and $g$ is an $L_1 \cup \{E\}$-embedding:

$$g(\overline{a}_1)^{\text{red}} \cong g(\overline{a}_2)^{\text{red}}$$

Because $g$ is an $L_2$-isomorphism on the underlying graphs, we get the first and third isomorphisms below:

$$\text{gr}(g(\overline{a}_1)) \cong K \text{ gr}(\overline{a}_1) \cong K \text{ gr}(g(\overline{a}_2)) \cong K$$

By choice of $B_0^*$ (with underlying graph $H^*$ homogeneous for $c'$), this is enough to get $c(g(\overline{a}_1)) = c(g(\overline{a}_2))$.

\[ \square \]

6. Applications

We introduce special notation to highlight the case when $\mathcal{I}$ is a linear order or a random ordered graph.

**Definition 6.1.**

1. Given a linear order $\mathcal{I}$ and ordered relational $L$-structures $(M_i \mid i \in \mathcal{I})$ each with age $K$, define $\mathcal{O}(M_i \mid i \in \mathcal{I}) := \mathcal{I}_{i \in \mathcal{I}}(M_i)$.

2. Given a random ordered graph $\mathcal{I}$ with underlying order $I$ and ordered relational $L$-structures $(M_i \mid i \in I)$ each with age $K$, define $\mathcal{R}(M_i \mid i \in \mathcal{I}) := \mathcal{I}_{i \in \mathcal{I}}(M_i)$.

**Example 6.2.** Let $L = \{<\}$ and $M_i = (\omega, <)$ for all $i \in \mathcal{I} := (\omega, <)$. Then $\mathcal{O}(M_i \mid i \in \mathcal{I})$ is isomorphic to the structure $I_{\text{eq}}$ defined in Definition 5.1

**Corollary 6.3.** If $K$ in Definition 6.1 has RP then
(1) \( O(M_i \mid i \in \mathcal{I}) \) has RP
(2) \( R(M_i \mid i \in \mathcal{I}) \) has RP

Remark 6.4. The result on \( O(M_i \mid i \in \mathcal{I}) \) was obtained by Leeb using the notation \( \text{Ord}(C) \) ([9], see [3] for a discussion). There are similarities also with the “cross-construction” argument in [14, Prop 1].

We end with a characterization of NIP theories using a semi-direct product structure obtained from the random ordered graph.

Corollary 6.5. Fix a random ordered graph \( \mathcal{I} \) in the language \( L_2 := \{R_2, <\} \) and random ordered graphs \( M_i \) for \( i \in \mathcal{I} \) in the language \( L_1 := \{R_1, <\} \). Let \( \mathcal{R} := \mathcal{R}(M_i \mid i \in \mathcal{I}) \). Let \( O \) be the reduct of \( \mathcal{R} \) to the language \( L_{eq} = \{E, <\} \). A theory \( T \) is NIP if and only if any \( \mathcal{R} \)-indexed indiscernible set in a sufficiently-saturated model \( U \) of \( T \) is an \( O \)-indexed-indiscernible set.

Proof. First observe that, by definition, \( \mathcal{R} \) is in the language \( L_{eq} = \{E, R_1, R_2, <\} \). Let \( I \) be the underlying order of \( \mathcal{I} \) and, for all \( i \in I \), let \( N_i \) be the underlying order of \( M_i \). We also observe that \( O = O(N_i \mid i \in I) \).

We sketch an argument that follows the characterization of NIP theories by ordered graph-indiscernible sets in [11, Thm 5.11]. The right-to-left direction is easy to show and relies on Corollary 6.3 combined with Theorem 2.7. Since \( \mathcal{R} \) has the RP, \( \mathcal{R} \)-indexed indiscernible sets have the modeling property. If \( T \) has IP, then there are parameters in a model of \( T \) witnessing this such that an \( \mathcal{R} \)-indexed indiscernible set in a sufficiently-saturated extension \( U \models T \) locally-based on these parameters cannot be indiscernible as indexed by the \( L_{eq} \)-reduct.

For the left-to-right direction, we can follow the argument in [11, Lem 5.4]. Assume there is an \( \mathcal{R} \)-indexed indiscernible set \( \langle a_i : i \in \mathcal{R} \rangle \subseteq U \models T \) that is not \( O \)-indexed indiscernible (we will show \( T \) has IP). By compactness, we may assume the set is indexed by \( \mathcal{R}^* \), the Fraïssé limit of age(\( \mathcal{R} \)) (exists by Proposition 5.12). This is convenient because the theory of \( \mathcal{R}^* \) eliminates quantifiers (see [3]).

Since \( \langle a_i : i \in \mathcal{R}^* \rangle \) is not \( O \)-indexed indiscernible, there exist \( n \)-tuples \( \bar{\tau}, \bar{j} \) from \( \mathcal{R}^* \) with the same quantifier-free type in \( O \) but such that

\[ \bar{\tau} \not\equiv_U \bar{\tau} \]

In this case there is some formula \( \theta \) in the language of \( U \) such that

\[ U \models \theta(\bar{\tau}), U \models \neg \theta(\bar{\tau}) \]

and so by indiscernibility,

\[ \bar{\tau} \not\equiv_{\mathcal{R}^*} \bar{j} \]

Since \( \mathcal{R}^* \) eliminates quantifiers, \( n \)-types are of the form

\[ q(x) := p(x_0, \ldots, x_{n-1}) \cup \{ R_1(x_i, x_j) : (i, j) \in C \} \cup \{ \neg R_1(x_i, x_j) : (i, j) \in (n \times n) \setminus C \} \]
\[ \cup \{ R_2(x_i, x_j) : (i, j) \in D \} \cup \{ \neg R_2(x_i, x_j) : (i, j) \in (n \times n) \setminus D \} \]

where \( p \) is a complete quantifier-free \( L_{eq} \)-type and \( C, D \subseteq n \times n \). Assume \( q_1 \) is the complete quantifier-free type of \( \bar{\tau} \) in \( \mathcal{R}^* \) and \( q_2 \) is the complete quantifier-free type of \( \bar{j} \). Since \( \bar{\tau} \not\equiv_{O} \bar{j} \), \( q_1, q_2 \) agree on their restriction \( p(x_0, \ldots, x_{n-1}) \) to the language \( L_{eq} \). Since \( \bar{\tau} \not\equiv_{\mathcal{R}^*} \bar{j} \), \( q_1, q_2 \) must differ in the \( R_1 \)- or \( R_2 \)-edges.

By \( \mathcal{R}^* \)-indexed indiscernibility, all tuples \( \bar{\tau} \) satisfying \( q_1 \) in \( \mathcal{R}^* \) yield image \( \bar{\tau} \) satisfying \( \theta \) in \( U \); similarly for \( q_2 \) and \( \neg \theta \). Thus transpositions of \( R_1 \) with \( \neg R_1 \) in \( q_1 \) on the pairs from
\( n \times n \) (similarly, \( R_2 \) with \( \neg R_2 \) on pairs mod \( E \) from \( n \times n \)) take the images \( \overline{a} \) of tuples \( \overline{a} \) from \( \mathcal{R}^* \) from \( \theta(U) \) to \( \neg \theta(U) \). Thus there is some one transposition that flips the value of \( \theta \) in \( U \). So we may assume one of two cases.

**Case 1:** the quantifier free types of \( \tau, \eta \) differ only on some pair of indices \( s, t < n \) such that \( E(i_s, i_t) \) (equivalently, \( E(j_s, j_t) \)), without loss of generality
\[
R_1(i_s, i_t), \neg R_1(j_s, j_t)
\]

**Case 2:** the quantifier free types of \( \tau, \eta \) differ only on some pair of indices \( s, t < n \) such that \( \neg E(i_s, i_t) \) (equivalently, \( \neg E(j_s, j_t) \)), without loss of generality
\[
R_2(i_m, i_n), \neg R_2(j_m, j_n)
\]
for all \( i_m \in [i_s]/E \) and \( i_n \in [i_t]/E \).

Assume Case 1. Let \( q^*(x_s, x_t, \overline{a}) := q_1(\overline{x}) \setminus \{ R_1(x_s, x_t) \} \). By assumption, \( q^* \) also equals \( q_2(\overline{a}) \setminus \{ \neg R_1(x_s, x_t) \} \) and thus is the common quantifier-free type of \( \tau, \eta \). By properties of Fraïssé limits, we can realize arbitrary finite bipartite graphs \( (A, B) \) as \( R_1 \)-subgraphs of the class \( [i_s]/E \) such that for some \( \overline{a} \) of length \( n-2 \) from \( \mathcal{R}^* \), for all \( a \in A, b \in B, \mathcal{R}^* \models q^*(a, b, \overline{a}) \). This allows the images of the tuples \( (a, b, \overline{c}) \) in \( U \) to satisfy IP using \( \theta \).

The argument for Case 2 is similar but in this case we first find the bipartite graph on pairs \( ([a]/E, [b]/E) \) in \( I \) with appropriate \( R_2 \)-relations to \( [\overline{c}]/E \) as dictated by the common quantifier-free type. Then all \( R_1 \)-configurations are easily found within the classes \( [a]/E, [b]/E, [c_i]/E \) to match the common quantifier-free type of \( \tau, \eta \).

\[\Box\]

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