On Galois–Gauss sums
and the square root of the inverse different

by

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In memory of Professor Andrzej Schinzel

1. Introduction. About thirty years ago, Erez initiated the study of the Hermitian–Galois module structure of the square root of the inverse different of finite Galois extensions of number fields.

To discuss this, we fix a finite Galois extension $L/K$ of number fields, with $G = \text{Gal}(L/K)$, and write $\mathcal{O}_L$ for the ring of algebraic integers of $L$ and $\mathcal{D}_{L/K}$ for the different of $L/K$.

We note that there can exist at most one fractional ideal $\mathcal{A}_{L/K}$ of $L$ for which there is an equality of fractional ideals

$$ (\mathcal{A}_{L/K})^2 = (\mathcal{D}_{L/K})^{-1}. $$

If such an ideal $\mathcal{A}_{L/K}$ exists, then (unsurprisingly) it is referred to as the ‘square root of the inverse different’ of $L/K$. More importantly, it will necessarily be both self-dual with respect to the canonical trace pairing $L \times L \to K$ of $L/K$ and also stable under the action of $G$ so that it can be considered as a module over the integral group ring $\mathbb{Z}[G]$.

We henceforth assume that $L/K$ is ‘weakly ramified’ in the sense of Erez [16]. We recall that this means that the second ramification subgroup in $G$ (in the lower numbering) of every place of $K$ is trivial and that $L/K$ being tamely ramified is a sufficient, but not necessary, condition for this to be satisfied.

Suppose $G$ has odd order. Hilbert’s classical formula for the valuation of $\mathcal{D}_{L/K}$ at each prime ideal implies that the fractional ideal $\mathcal{A}_{L/K}$ satisfying (1) exists. Erez [16] has shown that $\mathcal{A}_{L/K}$ is a projective $\mathbb{Z}[G]$-module and,

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if $L/K$ is tamely ramified, the class $[A_{L/K}]$ in the reduced projective class group $\text{Cl}(\mathbb{Z}[G])$ defined by $A_{L/K}$ is trivial (that is, equivalently, $A_{L/K}$ is a free $\mathbb{Z}[G]$-module).

Motivated by this result, and several other results (including extensive numerical computations in [32] and [33]), Vinatier has made the following conjecture (cf. [32, Conj.] and [10, §1.2]).

**Conjecture 1.1 (Vinatier).** If $L/K$ is a weakly ramified Galois extension of number fields of odd degree, then $A_{L/K}$ is a free $\mathbb{Z}[G]$-module.

### 1.1. The conjecture of Erez.

We now assume that $L/K$ is any weakly ramified extension for which a fractional ideal $A_{L/K}$ satisfying (1) exists.

The first investigation of $A_{L/K}$ in this case was undertaken by Caputo and Vinatier in the article [10]. By making a detailed study of certain torsion modules first considered by Chase [12], they were able to prove that if $L/K$ is tamely ramified (so that the $\mathbb{Z}[G]$-modules $A_{L/K}$ and $\mathcal{O}_L$ are both projective) and locally abelian (i.e. the decomposition group in $G$ of every ramified place of $L/K$ is abelian), then in $\text{Cl}(\mathbb{Z}[G])$,

$$[A_{L/K}] = [\mathcal{O}_L].$$

In this tamely ramified case one also knows, by Taylor’s celebrated proof of Fröhlich’s conjecture (in [30]), that

$$[\mathcal{O}_L] = W_{L/K},$$

where $W_{L/K}$ denotes the Cassou-Noguès–Fröhlich root number class, which is defined in terms of Artin root numbers attached to non-trivial irreducible symplectic characters of $G$.

By comparing the equalities (2) and (3), Caputo and Vinatier were, in particular, able to deduce the existence of a tamely ramified extension $L/K$ in which the $\mathbb{Z}[G]$-module $A_{L/K}$ is not free, thereby showing (in view of Conjecture 1.1) that the general theory of $A_{L/K}$ can have features that do not seem apparent in the case of extensions of odd degree.

Next we recall that, independently of any ramification hypotheses, Chinburg [13] has defined a canonical ‘$\Omega(2)$-invariant’ $\Omega(L/K, 2)$ in $\text{Cl}(\mathbb{Z}[G])$, has shown that $\Omega(L/K, 2) = [\mathcal{O}_L]$ whenever $L/K$ is tamely ramified, and has conjectured, as a natural generalisation of the equality (3) proved by Taylor, that in all cases one should have

$$\Omega(L/K, 2) \overset{?}{=} W_{L/K}.$$  

This conjectural equality has by now been extensively studied in the literature and is also known to have a natural interpretation as a consequence of the compatibility with respect to the functional equation of Artin $L$-functions of an important special case of the equivariant Tamagawa number conjecture.
More concretely, following the results in [1] and a comparison of the equalities (2)–(4), we consider a version of the conjecture made by Erez (cf. [10, Quest. 2] and [1, Th. 1.5]).

**Conjecture 1.2.** Let $L/K$ be a weakly ramified Galois extension of number fields for which $A_{L/K}$ exists, and set $G = \text{Gal}(L/K)$. Then, in $\text{Cl}(\mathbb{Z}[G]),$

$$\Omega(L/K, 2) = [A_{L/K}] + J_{2, S, L/K}.$$ 

Here $J_{2, S, L/K}$ is an element depending on the sign of the second Adams-operator twisted modified Galois–Jacobi sums attached to irreducible symplectic characters. In particular, the class in $\text{Cl}(\mathbb{Z}[G])$ defined by $J_{2, S, L/K}$ has order at most 2 and is not always trivial in the class group (see §5.1).

If one assumes the validity of (4), then Conjecture 1.2 includes Conjecture 1.1 as a special case (if $G$ is of odd order, and so has no non-trivial irreducible symplectic characters, then $W_{L/K} = J_{2, S, L/K} = 0$), and is perhaps the central question in the theory of $A_{L/K}$ for weakly ramified extensions of arbitrary degree.

**1.2. The conjecture of Bley, Burns and Hahn.** Taking motivation from a somewhat different direction, Bley, Burns and Hahn (in [4], and also in the related PhD thesis [19] by Hahn) have recently introduced techniques of relative algebraic $K$-theory to formulate a precise conjectural link between $A_{L/K}$ and twisted Galois–Gauss sums, and have thereby explained how much of the theory developed by Erez, by Erez and Taylor and by Vinatier can be refined.

Let $K_0(\mathbb{Z}[G], \mathbb{Q}^c[G])$ denote the relative $K_0$-group of the ring inclusion $\mathbb{Z}[G] \to \mathbb{Q}^c[G]$. Suppose that $L/K$ is of odd degree. Bley, Burns and Hahn define a canonical relative element $a_{L/K}$ of $K_0(\mathbb{Z}[G], \mathbb{Q}^c[G])$ that is sent to $[A_{L/K}]$ by the canonical projection map $\partial_G : K_0(\mathbb{Z}[G], \mathbb{Q}^c[G]) \to \text{Cl}(\mathbb{Z}[G])$. They were able to prove in [4, Th. 5.2] that the element $a_{L/K}$ belongs to the torsion subgroup $K_0(\mathbb{Z}[G], \mathbb{Q}[G])_{\text{tor}}$ of the subgroup $K_0(\mathbb{Z}[G], \mathbb{Q}[G]) \subset K_0(\mathbb{Z}[G], \mathbb{Q}^c[G])$, and that $a_{L/K}$ has good functorial properties under change of extension $L/K$.

Motivated by these facts and extensive numerical computation, the authors were led to formulate the following conjecture.

**Conjecture 1.3 ([4 Conj. 10.7]).** If $L/K$ is any weakly ramified Galois extension of number fields of odd degree, then $a_{L/K} = c_{L/K}$ in $K_0(\mathbb{Z}[G], \mathbb{Q}[G])$.

Here $c_{L/K}$ is a canonical ‘idelic twisted unramified characteristic’ element such that it also belongs to $K_0(\mathbb{Z}[G], \mathbb{Q}[G])_{\text{tor}}$ and enjoys the same functoriality properties under change of extension as does $a_{L/K}$. Recently the present author has provided new evidence [24] for this conjecture in the setting of extensions of odd prime power degree.
1.3. Main results. We find it tempting, therefore, to wonder whether there are any useful links between Conjecture \[1\] and the general approach of Bley, Burns and Hahn.

The first thing to note in this regard is that all of the constructions in [4] are made under the assumption that \(G\) has odd order and so, to explore possible links to Conjecture \[1\], it is necessary to define analogues of the relative elements \(a_{L/K}\) and \(c_{L/K}\) under the weaker hypothesis that \(A_{L/K}\) exists (rather than that \(G\) has odd order).

Fortunately, this is not very difficult; for the details see §3.2 and §3.3.1. In addition, we are also able to use the methods developed in [4] to show that, in this more general setting, the elements \(a_{L/K}\) and \(c_{L/K}\) have many of the key properties established (for extensions of odd degree) in [4].

However, a much more serious issue that arises in this setting is that the (generalised) element \(a_{L/K}\) involves Galois–Gauss sums attached to virtual characters that are obtained by taking the image of Artin characters under the second Adams operator \(\psi_2\).

The point here is that, whilst the approach of [4] is rooted in the methods of Fröhlich and Taylor and hence makes systematic use of functorial behaviour under passage to subgroups (or equivalently, via the appropriate ‘Hom-description’, under induction of characters from subgroups), if the order of \(G\) is even, then the operator \(\psi_2\) does not always commute with induction of characters from subgroups.

For this reason, computing the Galois–Gauss sums attached to \(\psi_2\)-twisted irreducible symplectic characters can be considerably more difficult in extensions of even degree compared to extensions of odd degree.

In some cases, we are able to overcome these challenges by using a detailed analysis of Galois–Gauss sums of \(\psi_2\)-twisted characters that has recently been made in joint work [1] of ours with Agboola, Burns and Caputo. This result relies principally both on a purely representation-theoretic analysis of the extent to which \(\psi_2\) commutes with induction functors and on the explicit computation of the Artin root number of dihedral characters that is given by Fröhlich and Queyrut [18].

In particular, by combining these results with the general approach of Erez [16] and with the results of Bley and Cobbe [5] (which themselves rely heavily on those of [26]), we are able to prove the following results.

We remark that claim (i) generalises the result [4, Cor. 8.4] of Bley, Burns and Hahn (which is precisely the same result in the case of odd degree extensions), and claim (ii) generalises the result [2] of Caputo and Vinatier (see Lemma 5.5(i)).

**Theorem 1.4.** Let \(L/K\) be a finite weakly ramified Galois extension of number fields for which \(A_{L/K}\) exists and set \(G := \text{Gal}(L/K)\).
(i) $a_{L/K} = c_{L/K}$ in $K_0(\mathbb{Z}[G], \mathbb{Q}^c[G])$ provided that every place $v$ of $K$ that is wildly ramified in $L$ has the following three properties and no such $v$ is 2-adic:

(a) the decomposition subgroup in $G$ of any place of $L$ above $v$ is abelian;
(b) the inertia subgroup in $G$ of any place of $L$ above $v$ is cyclic;
(c) the completion of $K$ at $v$ is absolutely unramified.

(ii) Conjecture 1.2 is valid if the conditions of claim (i) are satisfied and, in addition, the inertia degree of each wildly ramified place $v$ in $L/K$ is prime to the absolute degree of the completion $K_v$.

Moreover, it follows that both Conjectures 1.2 and 1.3 hold in cases where the extensions are tamely ramified.

We note that, whilst claim (ii) for the tame case was first proved in the jointly authored article [1], the argument that we present here closely follows the classical methods of Erez [16] (see §5.2 below) rather than the methods of Agboola and McCulloh [2] and so it is different in key respects from that in [1] §§4–6.

Finally, we remark that Theorem 1.4(i) makes it natural to wonder whether, with our generalised definitions of $a_{L/K}$ and $c_{L/K}$, the equality $a_{L/K} = c_{L/K}$ conjectured by Bley, Burns and Hahn should be valid for all weakly ramified Galois extensions $L/K$ for which the fractional ideal $\mathcal{A}_{L/K}$ exists.

This is a question that we feel deserves further careful attention.

However, since we have no evidence for an affirmative answer to it that goes beyond that in Theorem 1.4 and in particular we lack the sort of numerical evidence that is provided in [4], we shall end this introduction by simply posing the following question.

**Question 1.5.** Let $L/K$ be a finite weakly ramified Galois extension of number fields for which $\mathcal{A}_{L/K}$ exists and set $G := \text{Gal}(L/K)$. Then, in $K_0(\mathbb{Z}[G], \mathbb{Q}^c[G])$, does one always have

$$a_{L/K} = c_{L/K}^?$$

We emphasise that Conjecture 1.2 and Question 1.5 may not be compatible (further details can be found in Remark 5.9).

2. Preliminaries

2.1. Notations. For a unital ring $A$, we write $A^\times$ for the multiplicative group of invertible elements of $A$, and $\zeta(A)$ for the centre of $A$. By an $A$-module, we shall always mean a left $A$-module. For each pair of $A$-modules $M$ and $M'$ we write $\text{Is}_A(M, M')$ for the set of $A$-module isomorphisms from $M$
to $M'$, and $\text{Aut}_A(M)$ for the group of $A$-module automorphisms of $M$. We also let $\mathcal{P}(A)$ denote the category of finitely generated projective $A$-modules.

Fix a finite group $\Gamma$, and a Dedekind domain $R$ of characteristic 0, with field of fractions $F$, and let $E$ be an extension field of $F$. We let $\mathcal{O}_E$ denote the ring of integers of the field $E$. For any $R[\Gamma]$-module $M$, we write $M_E := E \otimes_R M$, and $M_v := R_v \otimes_R M$ with $R_v$ the completion of $R$ at $v$, where $v$ is a (non-zero) prime ideal of $R$.

Let $E/F$ be a finite Galois extension of fields. We let $\text{Gal}(E/F)$ denote the Galois group of $E/F$, and we write the action of $\text{Gal}(E/F)$ on $E$ as $x \mapsto g(x)$ for $x \in E$ and $g \in \text{Gal}(E/F)$. We fix a separable closure $F^c$ of $F$, and write $\Omega_F$ for the absolute Galois group $\text{Gal}(F^c/F)$ of $F$. For convenience, we take $\mathbb{Q}^c$ to be the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$.

Throughout this article, we say $F$ is a number field if it is a finite extension of $\mathbb{Q}$ contained in $\mathbb{Q}^c$, and that $F$ is a local field if it is a finite extension of $\mathbb{Q}_v$ for some finite place $v$.

For a finite $\Gamma$, we write $\hat{\Gamma}$ (resp. $\hat{\Gamma}_\ell$) for the set of $\mathbb{Q}^c$-valued (resp. $\mathbb{Q}^c_\ell$-valued) irreducible characters of $\Gamma$, and $R_\Gamma$ (resp. $R_{\Gamma,\ell}$) for the additive group generated by $\hat{\Gamma}$ (resp. $\hat{\Gamma}_\ell$).

### 2.2. Algebraic $K$-theory.

We fix a finite group $\Gamma$ and a Dedekind domain $R$ of characteristic 0 with $F := \text{Frac}(R)$, and $E$ is any field extension of $F$.

#### 2.2.1. The long exact sequence of relative $K$-theory.

We denote by $K_0(R[\Gamma], E[\Gamma])$ the algebraic $K_0$-group of the ring inclusion $R[\Gamma] \xrightarrow{\subseteq} E[\Gamma]$ described in [29] p. 215 and we recall that it is generated by the symbols $[P, \phi, Q]$ where $P, Q \in \mathcal{P}(R[\Gamma])$ and $\phi \in \text{Is}_{E[\Gamma]}(P_E, Q_E)$. We also recall that there is a canonical decomposition

$$K_0(R[\Gamma], F[\Gamma]) \cong \bigoplus_v K_0(R_v[\Gamma], F_v[\Gamma]),$$

where $v$ runs over all non-Archimedean places of $F$. This isomorphism is induced by the diagonal localisation homomorphism $(\pi_{\Gamma,v})_v$ (cf. [15] discussion below (49.12))), where for each non-zero prime ideal $v$ of $R$, we write

$$\pi_{\Gamma,v} : K_0(R[\Gamma], F[\Gamma]) \to K_0(R_v[\Gamma], F_v[\Gamma])$$

for the homomorphism that sends the class of $[P, \phi, Q]$ to the class of $[P_v, F_v \otimes_F \phi, Q_v]$.

We also write $K_1(R[\Gamma])$ (resp. $K_1(E[\Gamma])$) for the Whitehead group of the ring $R[\Gamma]$ (resp. $E[\Gamma]$), and $\text{Cl}(\mathbb{Z}[\Gamma])$ for the reduced projective class group of $R[\Gamma]$ (as in [15] §49A)]. In particular, we note that the reduced projective class group of $R[\Gamma]$ is isomorphic to the locally free class group of $R[\Gamma]$ (see [15] Remark (49.11)(iv)); we therefore do not distinguish the notation for the two groups.
of these two groups for the case of $R[\Gamma]$, and the latter group plays an important part in the ‘classical’ Galois module theory (see [17, Chap. I, §2] for more details).

Then there exists a commutative diagram (taken from [29, Th. 15.5])

\[
\begin{array}{cccccc}
K_1(R[\Gamma]) & \longrightarrow & K_1(E[\Gamma]) & \xrightarrow{\partial_1^{R,E,\Gamma}} & K_0(R[\Gamma], E[\Gamma]) & \xrightarrow{\partial_0^{R,E,\Gamma}} & \text{Cl}(R[\Gamma]) \\
\vert & & \uparrow{\iota_1} & & \uparrow{\iota_2} & & \\
K_1(R[\Gamma]) & \longrightarrow & K_1(F[\Gamma]) & \xrightarrow{\partial_1^{R,F,\Gamma}} & K_0(R[\Gamma], F[\Gamma]) & \xrightarrow{\partial_0^{R,F,\Gamma}} & \text{Cl}(R[\Gamma])
\end{array}
\]

in which the rows are the respective long exact sequences of relative $K$-theory (with the morphisms in the second row being completely analogous to those in the first): the homomorphism $\partial_1^{R,E,\Gamma}$ sends each pair $[E[\Gamma]^n, \phi]$ with $\phi$ in $\text{Aut}_{E[\Gamma]}(E[\Gamma]^n)$ to the class of $[R[\Gamma]^n, f, R[\Gamma]^n]$, where $f$ is given by $$(R[\Gamma]^n)_E \xrightarrow{\cong} E[\Gamma]^n \xrightarrow{\phi} E[\Gamma]^n \cong (R[\Gamma]^n)_E;$$ for each pair $P$ and $Q$ in $\mathcal{P}(R[\Gamma])$ and each isomorphism $\phi : P_E \cong Q_E$ of $E[\Gamma]$-modules, the homomorphism $\partial_0^{R,E,\Gamma}$ sends the class of $[P, \phi, Q]$ to the difference $[P] - [Q]$. The vertical maps $\iota_1$ and $\iota_2$ are the natural scalar extension morphisms (these maps are injective and will usually be regarded as inclusions).

2.2.2. The reduced norm map. Suppose $E$ is either a number field or a $p$-adic field (for some prime $p$). The ‘reduced norm’ map discussed by Curtis and Reiner [15, §45A] induces an injective homomorphism of abelian groups $\text{Nrd}_{E[\Gamma]} : K_1(E[\Gamma]) \rightarrow \zeta(E[\Gamma])^\times$, where $\zeta(E[\Gamma])$ denotes the centre of the ring $E[\Gamma]$. We then define a subgroup of $\zeta(E[\Gamma])^\times$ by setting $\zeta(E[\Gamma])^\times^+ := \text{im}(\text{Nrd}_{E[\Gamma]})$, and we make much use of the following facts about this group (cf. [14, Ths. (7.45) & (7.48)]).

**Lemma 2.1 (The Hasse–Schilling–Maass norm theorem).**

(i) $\zeta(E[\Gamma])^\times^+$ contains $(\zeta(E[\Gamma])^\times)^2$.

(ii) $\zeta(E[\Gamma])^\times^+ = \zeta(E[\Gamma])^\times$ in each of the following cases:

(a) $E$ is algebraically closed;

(b) $E$ is a number field and $\Gamma$ has no irreducible symplectic characters;

(c) $E$ is $p$-adic (for any prime number $p$).

(iii) If $E = \mathbb{Q}$, then $\zeta(E[\Gamma])^\times^+ = \zeta(E[\Gamma])^\times \cap \zeta(\mathbb{R}[\Gamma])^\times^+$.

2.2.3. The extended boundary map of Burns and Flach. The following construction (introduced by Burns and Flach in [9, §4.2]) is key to the formulation of arithmetic conjectures in relative $K$-groups.
Lemma 2.2. There exists a canonical homomorphism
\[ \delta \Gamma : \zeta(Q[\Gamma])^\times \to K_0(Z[\Gamma], Q[\Gamma]) \]
of abelian groups that has the following properties:

(i) The connecting homomorphism \( \partial_{Z, Q, \Gamma} \) in (7) is equal to \( \delta \Gamma \circ \text{Nrd}_{Q[\Gamma]} \).

(ii) If \( x \in \zeta(Q[\Gamma])^\times \), then \( \delta \Gamma(x) = \partial_{Z, Q, \Gamma}((\text{Nrd}_{Q[\Gamma]}(x))^{-1}) \). In particular, if \( \text{Nrd}_{Q[\Gamma]} \) is bijective, then \( \delta \Gamma = \partial_{Z, Q, \Gamma} \circ (\text{Nrd}_{Q[\Gamma]})^{-1} \).

(iii) Fix a prime \( \ell \) and write \( j_{\ell,*} \) for the natural projection map \( K_0(Z[\Gamma], Q[\Gamma]) \to K_0(Z_{\ell}[\Gamma], Q_{\ell}[\Gamma]) \) (as in (6)). Then, regarding \( \zeta(Q[\Gamma])^\times \) as a subgroup of \( \zeta(Q_{\ell}[\Gamma])^\times \), one has
\[ j_{\ell,*} \circ \delta \Gamma = \partial_{Z_{\ell}, Q_{\ell}, \Gamma} \circ (\text{Nrd}_{Q_{\ell}[\Gamma]})^{-1}. \]

Remark 2.3. The homomorphism \( \delta \Gamma \) in Lemma 2.2 extends to a group homomorphism \( \zeta(R[\Gamma])^\times \to K_0(Z[\Gamma], R[\Gamma]) \) (this was the original construction of Burns and Flach) but not, in general, to a group homomorphism \( \zeta(C[\Gamma])^\times \to K_0(Z[\Gamma], C[\Gamma]) \) (see, for example, Breuning [7, Prop. 2.11]).

Remark 2.4. Following the explicit recipe given in [9, §4.2, Lem. 9], for each \( x \) in \( \zeta(Q[\Gamma])^\times \) one has
\[ \delta \Gamma(x) = \partial_{Z, Q, \Gamma}((\text{Nrd}_{Q[\Gamma]}(x^2))^{-1}) - \sum_{\ell} \partial_{Z_{\ell}, Q_{\ell}, \Gamma}((\text{Nrd}_{Q_{\ell}[\Gamma]}(x_{\ell}))) \]
where the sum is over all primes \( \ell \), and \( x_{\ell} \) denotes \( x \) regarded as an element of \( \zeta(Q_{\ell}[\Gamma])^\times = \zeta(Q_{\ell}[\Gamma])^{\times +} \). Here we use the fact that \( x^2 \in \zeta(Q[\Gamma])^{\times +} \) (by Lemma 2.2(i)), and regard each group \( K_0(Z_{\ell}[\Gamma], Q_{\ell}[\Gamma]) \) as a subgroup of \( K_0(Z[\Gamma], Q[\Gamma]) \) by the canonical decomposition (5).

2.2.4. Parametrising central elements. Let \( E \) be an algebraically closed field of characteristic 0. We fix a finite group \( \Gamma \), and write \( \Gamma := \hat{\Gamma}(E) \) for the set of \( E \)-valued irreducible characters of \( \Gamma \), and \( R_{\Gamma} \) for the additive group generated by \( \hat{\Gamma} \). For each \( \chi \) in \( \hat{\Gamma} \), we obtain a primitive idempotent of the centre \( \zeta(E[\Gamma]) \) of \( E[\Gamma] \) by setting \( e_{\chi} = \frac{\chi(1)}{|\Gamma|} \sum_{g \in \Gamma} \chi(g^{-1})g \). In addition, the standard orthogonality relations for irreducible characters imply that the set \( \{ e_{\chi} \}_{\chi \in \hat{\Gamma}} \) is an \( E \)-basis of \( \zeta(E[\Gamma]) \) (see, for example, [27, §6.3, Exer. 6.4]).

It follows that each element of \( \zeta(E[\Gamma]) \), respectively \( \zeta(E[\Gamma])^\times \), can be written uniquely in the form
\[ x = \sum_{\chi \in \hat{\Gamma}} e_{\chi} \cdot x_{\chi}, \quad \text{with } x_{\chi} \in E, \text{ respectively } x_{\chi} \in E^\times, \text{ for all } \chi. \]

For what follows, it is helpful to extend the assignment \( \lambda \mapsto x_{\lambda} \) to all \( \lambda \in R_{\Gamma} \) by multiplicativity (as in [4, §4A]), i.e.
\[ x_{\lambda} = \prod_{\chi \in \hat{\Gamma}} x_{\lambda}^{(\lambda \cdot \chi)_{\Gamma}} \quad \text{for all } \lambda \in R_{\Gamma}. \]
In later arguments, we will use the following consequence of this decomposition. Lemma 2.1 implies that \( \text{im}(\text{Nrd}_{R[\Gamma]}) = \zeta(R[\Gamma])^\times \) is equal to the subgroup of \( \zeta(C[\Gamma])^\times \) that is defined by the following explicit conditions on the individual coefficients \( x_\chi \):

\[
\left\{ \sum_{\chi \in \hat{\Gamma}} e_\chi \cdot x_\chi \ \middle| \ x_\chi = \overline{x}_\chi \text{ for all } \chi \in \hat{\Gamma}(\mathbb{C}), \right. \\
\text{and } x_\chi > 0 \text{ for all } \chi \in \text{Symp}(\Gamma) \}
\]

where \( \overline{x}_\chi \) denotes the complex conjugation of \( x_\chi \), and \( \text{Symp}(\Gamma) \) denotes the set of irreducible complex symplectic characters of \( \Gamma \).

2.2.5. Induction functors. We let \( R \) denote either \( \mathbb{Z} \) or \( \mathbb{Z}_\ell \) for a prime number \( \ell \), and \( F \) denotes the corresponding fields \( \mathbb{Q} \) or \( \mathbb{Q}_\ell \). Suppose \( E \) is a finite field extension of \( F \) contained in \( F^c \). Let \( G \) be a finite group, and let \( J \) be a subgroup of \( G \); we let \( \text{res}_G^J \) denote the restriction homomorphism \( R_G \rightarrow R_J \).

We write \( i^G_J \) for the induction functor \( P(R[\Gamma]) \rightarrow P(R[\Gamma]) \) obtained by applying \( R[\Gamma] \otimes_{R[J]} \), and similarly for the \( E[G]\)-module, noting that it preserves isomorphisms and short exact sequences. This functor induces a homomorphism of relative \( K \)-groups that we denote by

\[
i^G_J : K_0(R[\Gamma], E[\Gamma]) \rightarrow K_0(R[G], E[G]),
\]

\[
[P, \phi, Q] \mapsto [i^G_J P, i^G_J \phi, i^G_J Q].
\]

We also define a map \( \tilde{i}^G_J : \zeta(F^c[\Gamma])^\times \rightarrow \zeta(F^c[G])^\times \) by setting, for each \( x \in \zeta(F^c[\Gamma])^\times \) and \( \chi \in \hat{G} \) (in terms of the decomposition in \( \text{(8)} \)),

\[
\tilde{i}^G_J(x)_{\chi} = \prod_{\phi \in \hat{J}} x_{\phi}^{\langle \text{res}_G^J \chi, \phi \rangle_J},
\]

where \( \langle -, - \rangle_J \) denotes the standard inner product on \( R_J \). We will frequently use the following results (taken from \( \text{[3, p. 581]} \)).

Proposition 2.5.

(i) If \( x \in \zeta(\mathbb{Q}[\Gamma])^\times \), then \( \tilde{i}^G_J(x) \in \zeta(\mathbb{Q}[G])^\times \).

(ii) Set \( \delta_{R,E,\Gamma} := \partial_{R,E,\Gamma}(\text{Nrd}_{E[\Gamma]})^{-1} : \zeta(E[\Gamma])^\times \rightarrow K_0(R[\Gamma], E[\Gamma]) \). Then

\[
i^G_J \circ \delta_{R,E,\Gamma} = \delta_{R,E,G} \circ \tilde{i}^G_J.
\]

2.2.6. Taylor’s Fixed Point Theorem. To finish this section, we shall recall an important result of Taylor. Fix a finite group \( \Gamma \) and a prime number \( \ell \) and write \( \mathbb{Q}_\ell \) for the maximal tamely ramified extension of \( \mathbb{Q}_\ell \) in \( \mathbb{Q}_\ell^c \). We also write \( \mathcal{O}_\ell \) for the valuation ring of \( \mathbb{Q}_\ell \) and consider the associated
homomorphism of relative \( K \)-groups,

\[
j_{\ell,*}^\dagger : K_0(\mathbb{Z}_\ell[\Gamma], \mathbb{Q}^c_\ell[\Gamma]) \to K_0(\mathcal{O}^t_\ell[\Gamma], \mathbb{Q}^c_\ell[\Gamma]),
\]

\[
[P, \phi, Q] \mapsto [\mathcal{O}^t_\ell \otimes \mathbb{Z}_\ell P, \phi, \mathcal{O}^t_\ell \otimes \mathbb{Z}_\ell Q].
\]

Then, by Taylor’s Fixed Point Theorem [31 Chap. 8, §1], the composite homomorphism \( K_0(\mathbb{Z}_\ell[\Gamma], \mathbb{Q}^c_\ell[\Gamma]) \to K_0(\mathbb{Z}_\ell[\Gamma], \mathbb{Q}^c_\ell[\Gamma]) \) is injective, where the first map is the inclusion induced by the relevant case of diagram (7).

For each \( u \in \text{GL}_n(\mathbb{Q}^c_\ell[\Gamma]) \) and each character \( \chi \in \hat{\Gamma}_\ell \) with corresponding representation \( T_\chi \) over \( \mathbb{Q}^c_\ell \), we recall from [17 Chap. I, §2, and Chap. II, §2] the generalised determinant map \( \text{Det} : \text{GL}_n(\mathbb{Q}^c_\ell[\Gamma]) \to \text{Hom}(\mathbb{R}_{\Gamma,\ell}, \mathbb{Q}^{\times c}_\ell) \) given by \( (\text{Det}(u))(\chi) := \text{det}(T_\chi(u)) \), where \( \text{det} \) is the determinant map. This definition is independent of the choice of \( T_\chi \) and depends only on \( \chi \).

We also recall from [3] (48)\( \) the canonical isomorphism

\[
\text{Hom}(\mathbb{R}_{\Gamma,\ell}, \mathbb{Q}^{\times c}_\ell) \simeq \zeta(\mathbb{Q}^c_\ell[\Gamma])^{\times}.
\]

In this way, for every \( u \in \text{GL}_n(\mathbb{Q}^c_\ell[\Gamma]) \) and every \( \chi \in \mathbb{R}_{\Gamma,\ell} \), there is a canonical identification (see also [17 Chap. II, §1, Lem. 1.6, and §2, Lem. 2.1])

\[
(\text{Det}(u))(\chi) = \text{Nrd}_{\mathbb{Q}^c_\ell[\Gamma]}(u)\chi \in \mathbb{Q}^{\times c}_\ell.
\]

We will often use the fact (taken from [3] (49)) that for the composite homomorphism

\[
\delta_{\mathcal{O}^t_\ell, \mathbb{Q}^c_\ell, \Gamma, \cdot} : (\mathbb{Q}^c_\ell[\Gamma])^{\times} \xrightarrow{\text{Nrd}_{\mathbb{Q}^c_\ell[\Gamma]}^{-1}} K_1(\mathbb{Q}^c_\ell[\Gamma]) \xrightarrow{\det_{\mathcal{O}^t_\ell, \mathbb{Q}^c_\ell, \Gamma, \cdot}^1} K_0(\mathcal{O}^t_\ell[\Gamma], \mathbb{Q}^c_\ell[\Gamma]),
\]

one has

\[
\ker(\delta_{\mathcal{O}^t_\ell, \mathbb{Q}^c_\ell, \Gamma, \cdot}) = \text{Det}(\mathcal{O}^t_\ell[\Gamma]^{\times}).
\]

### 2.3. Adams operations and Galois–Jacobi sums.

In this final subsection, we recall the definitions of Adams operations on the ring of virtual characters, modified Galois–Gauss sums and the Galois–Jacobi sums considered in [3] §4A.

#### 2.3.1. Adams operations.

Let \( E \) denote either \( \mathbb{Q}^c \) or \( \mathbb{Q}^c_\ell \). We fix a finite group \( \Gamma \) and write \( \hat{\Gamma} := \hat{\Gamma}(E) \) with \( R_\Gamma \) the additive group generated by \( \hat{\Gamma} \).

For each natural number \( k \) and each \( \chi \in R_\Gamma \), we define a class function \( \psi_k(\chi) \) on \( \Gamma \) by setting \( \psi_k(\chi)(\gamma) = \chi(\gamma^k) \) for all \( \gamma \in \Gamma \). The function \( \psi_k \) from \( R_\Gamma \) to the set of class functions on \( \Gamma \) is called the \( k \)-th Adams operator for \( \Gamma \) (see [14 Prop. (12.8)]). The following general results about Adams operators will be very useful in what follows (in the case of \( k = 2 \)).

For a character \( \chi \) of \( \Gamma \) with representation \( T_\chi : \Gamma \to \text{GL}_n(E) \), we write \( \text{det}_\chi \) for the map induced by sending each \( g \in \Gamma \) to \( \text{det}(T_\chi(g)) \). We note that if \( \chi \) is a linear character, then it can be identified with \( \text{det}_\chi \).
Proposition 2.6.

(i) $\psi_k$ is an endomorphism of $R_\Gamma$. In particular, $\psi_k(\chi) \in R_\Gamma$ for each $\chi \in R_\Gamma$.

(ii) $\psi_k$ commutes with the restriction and the inflation functors, and with the action of $\Omega_{Q}$ on $R_\Gamma$.

(iii) $\det_\psi_k(\chi) = (\det_\chi)^k$ for each $\chi \in R_\Gamma$.

(iv) If $k$ is prime to the order of $\Gamma$, then $\psi_k$ commutes with the induction functors, and a character $\chi$ is irreducible if and only if $\psi_k(\chi)$ is irreducible.

Proof. Claim (i) is stated in [14, Cor. (12.10)] and claims (ii) and (iii) are taken from [16, Prop.-Def. 3.5]. The assertions in claim (iv) follow from [21, p. 15] and [27, §9.1, Exer. 9.4] respectively.

Remark 2.7. If $k$ is not prime to $|\Gamma|$, then $\psi_k$ does not in general commute with the induction functors or preserve irreducibility. It is also important to note that, if $|\Gamma|$ is even, then $\psi_2$ does not in general preserve the set of symplectic characters (for an explicit example, see [1, Lem. 7.3(d)]).

Remark 2.8. For each pair of integers $m$ and $n$, and each natural number $k$, the canonical decomposition (8) and the generalised notation (9) allow us to define an endomorphism $m + n \cdot \psi_{k,*}$ of the multiplicative group $\zeta(E[\Gamma])^\times$ in the following way: for each element $x$ of $\zeta(E[\Gamma])^\times$, the element $(m + n \cdot \psi_{k,*})(x)$ is uniquely specified by the condition that

$$(m + n \cdot \psi_{k,*})(x) \chi := x_{m\chi + n \cdot \psi_k(\chi)} = (x_{\chi})^m \cdot (x_{\psi_k(\chi)})^n$$

for every irreducible character $\chi$ in $\hat{\Gamma}$.

We also write $m + n \cdot \psi_{k,*}$ for the homomorphism

$$\text{Hom}(R_\Gamma, E^\times) \to \text{Hom}(R_\Gamma, E^\times)$$

defined by setting $(m + n \cdot \psi_{k,*})(f)(\chi) = f(m\chi + n \cdot \psi_k(\chi))$ for all $f \in \text{Hom}(R_\Gamma, E^\times)$ and $\chi \in R_\Gamma$.

In this way, for each $u \in \text{GL}_n(Q_{\ell}[\Gamma])$ and $\chi \in \hat{\Gamma}_{\ell}$, we can identify

$$((m + n \cdot \psi_{k,*})(\text{Det}(u)))(\chi) = (\text{Det}(u))(m\chi + n \cdot \psi_k(\chi)) = N_{\text{Det}(u)}(m\chi + n \cdot \psi_k(\chi)) = (m + n \cdot \psi_{k,*})(N_{\text{Det}(u)}(m\chi + n \cdot \psi_k(\chi))))$$

where the second equality follows from (15).

2.3.2. The unramified characteristic. We recall the definition of ‘unramified characters’ from [17, Chap. I, §5, (5.6)]. Fix a finite Galois extension $E/F$ of non-Archimedean local fields and set $\Gamma := \text{Gal}(E/F)$; we write $\Gamma_0$ for the inertia subgroup of $\Gamma$. 


Definition 2.9.

(i) An irreducible character $\chi \in \hat{\Gamma}$ is said to be unramified if $\Gamma_0$ is contained in the kernel of $\chi$ (or equivalently if $\chi(\gamma) = \chi(1)$ for all $\gamma \in \Gamma_0$). If $\chi$ is not unramified, then it is ramified.

(ii) Each $\chi \in R_\Gamma$ can be written uniquely as a finite sum $\chi = \sum_{i \in I} m_i \cdot \mu_i$, where each $m_i$ is an integer and each $\mu_i$ is in $\hat{\Gamma}$. The unramified part of $\chi$ is the element of $R_\Gamma$ obtained by setting $n(\chi) := \sum_{i \in I} m_i \cdot n(\mu_i)$ with

$$n(\mu_i) := \begin{cases} 
\mu_i & \text{if } \mu_i \text{ is unramified}, \\
0 & \text{if } \mu_i \text{ is ramified}.
\end{cases}$$

We next recall the ‘unramified characteristic’ defined in [17, Chap. IV, §1, (1.1)].

Definition 2.10.

(i) For each $\phi \in \hat{\Gamma}$, the unramified characteristic of $\phi$ is defined by setting $y(F, \phi) = \begin{cases} 
1 & \text{if } \phi \text{ is ramified}, \\
(-1)^{\phi(1)} \det(\phi(\sigma)) & \text{if } \phi \text{ is unramified},
\end{cases}$

where $\sigma$ is the Frobenius element in $\Gamma/\Gamma_0$ lifted to $\Gamma$.

(ii) The equivariant unramified characteristic of $E/F$ is defined by $y_{E/F} := \sum_{\chi \in \hat{\Gamma}} e_{\chi} \cdot y(F, \chi)$.

In particular, [17] Chap. IV, §1, Th. 29(i)] implies that, for all prime $\ell$, (19)

$$y_{E/F} \in \zeta(Q[\Gamma])^\times \subseteq \zeta(Q[\Gamma_\ell])^\times.$$  

2.3.3. Modified global Galois–Gauss sums. Let $L/K$ be a finite Galois extension of number fields and set $G := \Gal(L/K)$. In terms of (8), we define the following ‘equivariant’ elements in $\zeta(Q[G])^\times$.

If $v$ is a place of $K$, we fix a place $w$ of $L$ above $v$ and write $G_w$ for the decomposition subgroup of $w$ in $G$, and for each $\chi \in R_G$ we let $\chi_v$ denote the restriction of $\chi$ to $G_w$, and therefore regard $\chi_v$ as characters of the (local) Galois group $\Gal(L_w/K_v)$ via identifying $G_w = \Gal(L_w/K_v)$.

Definition 2.11.

(i) The equivariant Galois–Gauss sum of $L/K$ is $\tau_{L/K} := \sum_{\chi \in \hat{G}} e_{\chi} \cdot \tau(K, \chi)$, where $\tau(K, \chi)$ is the global Galois–Gauss sum (cf. [17] Chap. I, §5, (5.22)).

(ii) The global unramified characteristic of a virtual character $\chi \in R_G$ is defined by setting $y(K, \chi) = \prod_{v|d_L} y(K_v, \chi_v)$, where $d_L$ denotes the (absolute) discriminant of $O_L$. 
(iii) The equivariant unramified characteristic of $L/K$ is
\[ y_{L/K} := \sum_{\chi \in \hat{G}} e_{\chi} \cdot y(K, \chi) \in \zeta(\mathbb{Q}[G])^\times. \]

(iv) The modified equivariant Galois–Gauss sum of $L/K$ is $\tau'_{L/K} := \tau_{L/K} \cdot y_{L/K}^{-1}$.

(v) The absolute Galois–Gauss sum of $L/K$ is
\[ \tau_{L/K}^\dagger := \sum_{\chi \in \hat{G}} e_{\chi} \cdot \tau(\mathbb{Q}, \text{ind}_K^Q \chi). \]

(vi) The induced discriminant of $L/K$ is $\tau_G^K := \sum_{\chi \in \hat{G}} e_{\chi} \cdot \tau^{(1)}_K$ with $\tau_K := \tau(\mathbb{Q}, \text{ind}_K^Q 1_K)$.

2.3.4. Galois–Jacobi sums. Let $L/K$ be a finite Galois extension of number fields and set $G := \text{Gal}(L/K)$. In the following definition, we use a particular case of the endomorphisms defined in Remark 2.8.

**Definition 2.12.** The second Galois–Jacobi sum for $L/K$ is the element of $\zeta(\mathbb{Q}^c[G])^\times$ obtained by setting $J_{2,L/K} := (\psi_2, 2)(\tau_{L/K})$.

For the next result, we let $\text{Ver}_{K/Q} : \Omega_Q^{ab} \to \Omega_K^{ab}$ denote the transfer map, and $\text{v}_{K/Q}$ denote the cotransfer map from the linear characters of $\Omega_K$ to linear characters of $\Omega_Q$ (that is, the dual of the transfer $\text{Ver}_{K/Q}$). For each $\chi \in R_G$, we observe that the function $\text{det}_\chi$ is a linear character of $G$, which we can consider as lifted to a linear character of $\Omega_K$ via the surjection $\Omega_K \to G$ of Galois groups. And so we can obtain a linear character $\text{v}_{K/Q} \text{det}_\chi$ of $\Omega_Q$.

The proof of the following result is the same as in [4, Lem. 4.4]. We include a complete argument below for clarity.

**Proposition 2.13.** $J_{2,L/K}$ belongs to $\zeta(\mathbb{Q}[G])^\times$.

**Proof.** Taking account of Remark 2.8 and Definition 2.12 it suffices to show that the assignment $\chi \mapsto \tau(K, \psi_2(\chi) - 2\chi)$ is $\Omega_Q$-equivariant for each $\chi \in \hat{G}$.

First, we recall from [17, Chap. III, §3, Th. 20B] that, for all $\chi \in R_G$ and $\omega \in \Omega_Q$, the Galois action on (global) Galois–Gauss sums is given by
\[ \tau(K, \chi^{\omega^{-1}}) = \tau(K, \chi) \cdot (\text{v}_{K/Q} \text{det}_\chi)(\omega). \]

We also recall from Proposition 2.6(i) that $\psi_2(\chi)$ is again a virtual character in $R_G$ and so
\[ \tau(K, \psi_2(\chi))^{\omega^{-1}} = \tau(K, \psi_2(\chi)) \cdot (\text{v}_{K/Q} \text{det}_{\psi_2(\chi)})(\omega). \]

Thus, it suffices to show that $((\text{v}_{K/Q} \text{det}_\chi)(\omega))^2 = (\text{v}_{K/Q} \text{det}_{\psi_2(\chi)})(\omega)$, and this follows since $\text{det}_{\psi_2(\chi)} = (\text{det}_\chi)^2$ by Proposition 2.6(iii). □
Remark 2.14. If $E/F$ is a finite Galois extension of local fields and if $\Gamma = \text{Gal}(E/F)$, then one obtains a local second Galois–Jacobi sum by setting $J_{2,E/F} := (\psi_{2,*} - 2)(\tau_{E/F})$, where $\tau_{E/F}$ is defined to be the local analogue of the equivariant Galois–Gauss sum from Section 2.3.3.

For a number field $K$, we write $S_f(K)$ for the set of all non-Archimedean places of $K$. The following properties of these elements are important.

Proposition 2.15. For each finite place $v$ of $K$, fix some $w \in L$ above $v$ and identify $\text{Gal}(L_w/K_v)$ with the decomposition subgroup $G_w$ of $w$ in $G$.

(i) $J_{2,L/K} = \prod_{v \in S_f(K)} i_{G_w}(J_{2,L_w/K_v})$.

(ii) $(1 - \psi_{2,*})(y_{L/K}) = \prod_{v|d_L} i_{G_w}((1 - \psi_{2,*})(y_{L_w/K_v}))$.

(iii) $J_{2,L/K} \cdot (1 - \psi_{2,*})(y_{L/K}) = \tau_K^G \cdot (\psi_{2,*} - 1)(\tau_{L/K}^\dagger(\tau_{L/K})^{-1}$.

Proof. To prove (i), by the decomposition results of global Galois–Gauss sums (see, for example, [17, Chap. III, §2, Cor. to Th. 18]), one has

$$\tau(K, \psi_2(\chi) - 2\chi) = \prod_{v \in S_f(K)} \tau(K_v, \psi_2(\chi_v) - 2\chi_v)$$

where the second equality follows from Proposition 2.6(ii). Then the result follows from the definition of $i_{G_w}$ given in (12).

Claim (ii) can be obtained using the same argument, but replacing the decomposition result with the definition of the global unramified characteristic in Definition 2.11(ii).

Claim (iii) can be established by explicitly comparing the definitions (see [4, (4.5)]), noting that the assumption that $G$ has odd order is not necessary in this context.

Remark 2.16. Using the notation from Remark 2.14 analogous arguments to those for Propositions 2.13 and 2.15(ii) show that local second Galois–Jacobi sums have the following properties:

(i) $J_{2,E/F} \in \zeta(\mathbb{Q}[\Gamma])^\times$. The proof is exactly the same as in the global case, with the Galois action of local Galois–Gauss sums (cf. [25, p. 42, Th. 5.1]).

(ii) $J_{2,E/F} \cdot (1 - \psi_{2,*})(y_{E/F}) = \tau^\Gamma_F \cdot (\psi_{2,*} - 1)(\tau_{E/F}'(\tau_{E/F}')^{-1}$, where $\tau_{E/F}'$ and $\tau_{E/F}'$ are defined to be the local analogues of the elements described in Definition 2.11.

3. The square root of the inverse different and relative $K$-theory.

In this section, we first recall the basic properties of the square root $A_{L/K}$
of the inverse different of a Galois extension $L/K$ that is weakly ramified (in the sense of Erez [16]). We then generalise the construction by Bley, Burns and Hahn of a canonical element in relative $K$-theory that compares $A_{L/K}$ with Galois–Gauss sums.

3.1. Weakly ramified extensions. Throughout this section, we assume $L/K$ is a finite Galois extension of number fields and $G := \text{Gal}(L/K)$. The importance of weak ramification [16, Def. 2.1] for our theory is explained by the following result.

**Proposition 3.1.** Suppose that the square root of the inverse different of $L/K$ exists. Then the following conditions are equivalent:

(i) $A_{L/K}$ is a projective $O_K[G]$-module.
(ii) For every finite place $v$ of $K$, the $O_{K_v}[G_w]$-module $A_{L_w/K_v}$ is free.
(iii) $L/K$ is weakly ramified.

**Remark 3.2.** In [10, p. 109, footnote 1], Caputo and Vinatier first pointed out (without proof) that the original argument [16, Th. 1] of Erez for extensions of odd degree can be extended to show that if $L/K$ is weakly ramified and $A_{L/K}$ exists, then it must be a projective $O_K[G]$-module.

The equivalence of the above conditions (i) and (ii) follows from a general result of integral representation theory (see, for example, [14, Th. (32.11)]).

To prove the equivalence of (ii) and (iii), we fix a finite Galois extension $E/F$ of $p$-adic fields for which $A_{E/F}$ exists, set $\Gamma = \text{Gal}(E/F)$ and write $p_E$ for the maximal ideal of $O_E$. It is sufficient to prove that $A_{E/F}$ is a free $O_E[\Gamma]$-module if and only if $E/F$ is weakly ramified. Our argument relies on the following result of Köck.

**Lemma 3.3 (Köck, [22, Th. 1.1]).** Fix an integer $n$. Then the fractional $O_E$-ideal $p_E^n$ is free over $O_F[\Gamma]$ if and only if both $E/F$ is weakly ramified and $n \equiv 1 \pmod{|\Gamma_1|}$.

This result immediately implies that (ii) implies (iii), and also shows that (iii) implies (ii) provided that, defining $n$ by the equality $A_{E/F} = p_E^n$, one has $n \equiv 1 \pmod{|\Gamma_1|}$.

To prove this, we recall (from [28, Chap. IV, §2, Prop. 4]) Hilbert’s formula for the valuation of the different $D_{E/F}$ of $E/F$:

$$
\text{ord}_E(D_{E/F}) = \sum_{i=0}^{\infty} (|\Gamma_i| - 1).
$$
We set \( I := \Gamma_0 \), \( W := \Gamma_1 \) and \( C := I/W \), and we recall from [28, Chap. IV, §2, Cor. 4 to Prop. 7] that \( |C| = |I|/|W| \) is prime to \( |W| \). Then, since \( \Gamma_2 \) is assumed to be trivial, Hilbert’s formula implies that

\[
\begin{align*}
n &= \text{ord}_E(\mathcal{A}_{E/F}) = -\frac{\text{ord}_E(\mathcal{D}_{E/F})}{2} = -\frac{1}{2}(|I| - 1 + |W| - 1) \\
&= -\frac{1}{2}(|C| \cdot |W| - 1 + |W| - 1) = -\frac{|C| + 1}{2} \cdot |W| + 1.
\end{align*}
\]

If \( p = 2 \), then \( |W| \) is even and so \( |C| \) is odd and hence \((|C| + 1)/2\) is an integer. So, this formula implies the required congruence \( n \equiv 1 \pmod{|W|} \).

If \( p > 2 \), then \( |W| \) is odd and hence, since \( n \) is an integer, this formula implies that \( |C| + 1 \) is even, and therefore \( n \equiv 1 \pmod{|W|} \), as required.

This completes the proof of Proposition 3.1.

**Remark 3.4.** If \( E/F \) is a weakly ramified Galois extension of \( p \)-adic fields, with \( \Gamma = \text{Gal}(E/F) \), then the order of the inertia subgroup \( I \) is odd if either

(i) \( E/F \) is tamely ramified and \( \mathcal{A}_{E/F} \) exists, or

(ii) \( p \) is odd and \( \mathcal{A}_{E/F} \) exists.

The claim in case (i) follows directly from Hilbert’s formula (20) and the fact that, in this case, \( \Gamma_i \) is trivial for all \( i > 0 \). Case (ii) is valid since \( |W| \) is odd in this case and so the computation (21) implies that \( |C| = |I|/|W| \), and hence also \( |I| \), are odd.

### 3.2. The canonical relative elements of Bley, Burns, and Hahn.

In this section, we extend the key definition of Bley, Burns, and Hahn [4] to all weakly ramified Galois extensions \( L/K \) of number fields for which the inverse different is a square. We also show that this element decomposes as a sum of canonical elements arising from the extensions obtained by completing \( L/K \) at the places that ramify in \( L/K \).

**3.2.1. The global element.** Let \( L/K \) be a weakly ramified finite Galois extension of number fields with Galois group \( G := \text{Gal}(L/K) \). In this section, we define the canonical relative elements of \( K_0(\mathbb{Z}[G], \mathbb{Q}^c[G]) \) using the construction in [4, §2A3 and §5].

To start, we identify the set \( \Sigma(L) \) of field embeddings \( L \to \mathbb{Q}^c \) with the set of field embeddings \( L \to \mathbb{C} \) and consider the free \( \mathbb{Z}[G] \)-module \( H_L := \prod_{\sigma \in \Sigma(L)} \mathbb{Z} \) (upon which \( G \) acts via its natural composition action on \( \Sigma(L) \)).

We then consider the isomorphism \( \kappa_L : \mathbb{Q}^c \otimes_{\mathbb{Q}} L \to \prod_{\sigma \in \Sigma(L)} \mathbb{Q}^c = \mathbb{Q}^c \otimes_{\mathbb{Z}} H_L \) of \( \mathbb{Q}^c[G] \)-modules that sends \( z \otimes l \) for each \( z \in \mathbb{Q}^c \) and \( l \in L \) to the vector \( (\sigma(l)z)_{\sigma \in \Sigma(L)} \). Then, for any full projective \( \mathbb{Z}[G] \)-sublattice \( \mathcal{L} \) of \( L \), we obtain an element of \( K_0(\mathbb{Z}[G], \mathbb{Q}^c[G]) \) by setting

\[ \Delta(\mathcal{L}) := [\mathcal{L}, \kappa_L, H_L] \in K_0(\mathbb{Z}[G], \mathbb{Q}^c[G]). \]
In particular, if $A_{L/K}$ exists, then (by Proposition 3.1) it gives rise to a well-defined element $\Delta(A_{L/K})$.

We shall investigate relations between this element and the element of $\zeta(\mathbb{Q}[G])^\times$ obtained by setting

$$T^{(2)}_{L/K} := (\tau^G_K \cdot (\psi_2, - 1)(\tau'_{L/K}))^{-1},$$

where $\tau^G_K$ and $\tau'_{L/K}$ are as defined in $\S 2.3.3$.

As a first step, we prove the following result.

**Lemma 3.5.** Suppose that $A_{L/K}$ exists and fix $x \in K_1(\mathbb{Q}[G])$ such that $\partial^1_{z,\mathbb{Q}^c,G}(x) - \Delta(A_{L/K})$ is in the subgroup $K_0(\mathbb{Z}[G], \mathbb{Q}[G])$ of $K_0(\mathbb{Z}[G], \mathbb{Q}[G])$. Then

$$\text{Nrd}_{\mathbb{Q}^c[G]}(x) \cdot T^{(2)}_{L/K} \in \zeta(\mathbb{Q}[G])^\times.$$  

**Proof.** We recall from Proposition 2.13 that $J_{2,L/K} \in \zeta(\mathbb{Q}[G])^\times$. It is also clear that (by the explicit definition of $y_{L/K}$) the element $(1 - \psi_2)(y_{L/K})$ is in $\zeta(\mathbb{Q}[G])^\times$. Upon recalling the equality of Proposition 2.15 (iii), we are therefore reduced to showing that, for the specified element $x$, the element $\text{Nrd}_{\mathbb{Q}^c[G]}(x) \cdot (\tau^\dagger_{L/K})^{-1}$ is in $\zeta(\mathbb{Q}[G])^\times$.

To do this, we fix an isomorphism $\lambda : \mathbb{Q} \otimes_{\mathbb{Z}} H_L \cong L$ of $\mathbb{Q}[G]$-modules. Then the automorphism $\kappa_L \circ (\mathbb{Q}^c \otimes_{\mathbb{Q}} \lambda)$ of $\mathbb{Q}^c \otimes_{\mathbb{Z}} H_L$ defines an element $\langle \lambda \rangle$ of $K_1(\mathbb{Q}[G])$ for which

$$\partial^1_{z,\mathbb{Q}^c,G}(\langle \lambda \rangle) = [H_L, \kappa_L \circ (\mathbb{Q}^c \otimes_{\mathbb{Q}} \lambda), H_L] = [H_L, (\mathbb{Q}^c \otimes_{\mathbb{Q}} \lambda), A_{L/K}] + \Delta(A_{L/K}).$$

In particular, since $[H_L, (\mathbb{Q}^c \otimes_{\mathbb{Q}} \lambda), A_{L/K}] = [H_L, \lambda, A_{L/K}] \in K_0(\mathbb{Z}[G], \mathbb{Q}[G])$, the above two equalities can be combined with the assumption $\partial^1_{z,\mathbb{Q}^c,G}(x) - \Delta(A_{L/K}) \in K_0(\mathbb{Z}[G], \mathbb{Q}[G])$ to show $\partial^1_{z,\mathbb{Q}^c,G}(\langle \lambda \rangle \cdot x^{-1})$ is in $K_0(\mathbb{Z}[G], \mathbb{Q}[G])$.

We also note that the commutativity of diagram (7) implies that if there is an $\alpha \in K_1(\mathbb{Q}[G])$ such that $\partial^1_{z,\mathbb{Q}^c,G}(\alpha)$ is in $K_0(\mathbb{Z}[G], \mathbb{Q}[G])$, then $\alpha$ must be in the image of $K_1(\mathbb{Q}[G])$ in $K_1(\mathbb{Q}[G])$. Given this argument, we can therefore deduce that $\langle \lambda \rangle$ differs from $x$ by an element of $K_1(\mathbb{Q}[G])$ and so it is enough to show that

$$\text{Nrd}_{\mathbb{Q}^c[G]}(\langle \lambda \rangle) \cdot (\tau^\dagger_{L/K})^{-1} \in \zeta(\mathbb{Q}[G])^\times.$$

This is proved by Burns and Bley [3] Prop. 3.4 and Rem. 3.5. □

The result of Lemma 3.5 allows us to make the following definition, in which we use the extended boundary homomorphism $\delta_G : \zeta(\mathbb{Q}[G])^\times \to K_0(\mathbb{Z}[G], \mathbb{Q}[G])$ from Lemma 2.2.

**Definition 3.6.** Suppose $A_{L/K}$ exists and fix $x \in K_1(\mathbb{Q}^c[G])$ such that $\partial^1_{z,\mathbb{Q}^c,G}(x) - \Delta(A_{L/K}) \in K_0(\mathbb{Z}[G], \mathbb{Q}[G])$. Then the canonical relative element of Bley, Burns and Hahn is the element of $K_0(\mathbb{Z}[G], \mathbb{Q}[G])$ that is obtained...
by setting

$$a_{L/K} := \Delta(A_{L/K}) = \partial_{Z,Q,c,G}^1(x) + \delta_G(Nrd_{Q^c[G]}(x) \cdot T_{L/K}^{(2)}).$$

The basic properties of this element are recorded in the following result. We note that claim (ii) below shows that $a_{L/K}$ does generalise (for odd degree extensions) the element defined by Bley, Burns and Hahn [4].

In claim (iii), we use the following notation: for each prime $\ell$ and embedding $j^c_\ell : \mathbb{Q}^c \to \mathbb{Q}^c_\ell$ of fields, we also write $j^c_\ell$ for the induced ring homomorphism $\mathbb{Q}^c[G] \to \mathbb{Q}^c_\ell[G]$, and we consider the homomorphism of abelian groups

$$(22) \quad j^c_{\ell,*} : K_0(\mathbb{Z}[\Gamma], \mathbb{Q}^c[\Gamma]) \to K_0(\mathbb{Z}_\ell[\Gamma], \mathbb{Q}^c_\ell[\Gamma]),$$

$$[P, \phi, Q] \mapsto [P_\ell, Q^c_\ell \otimes_{\mathbb{Q}^c,j^c_\ell \phi} \phi, \ell],$$

where $Q^c_\ell \otimes_{\mathbb{Q}^c,j^c_\ell} \phi$ denotes the composite isomorphism

$$\mathbb{Q}^c_\ell \otimes_{\mathbb{Z}_\ell} (\mathbb{Z}_\ell \otimes \mathbb{Q}) \xrightarrow{\sim} \mathbb{Q}^c_\ell \otimes_{\mathbb{Q}^c} (\mathbb{Q}^c \otimes_{\mathbb{Z}} \mathbb{Q})$$

$$\xrightarrow{\sim} \mathbb{Q}^c_\ell \otimes_{\mathbb{Z}_\ell} (\mathbb{Z}_\ell \otimes \mathbb{Q})$$

with $Q^c_\ell \otimes_{\mathbb{Q}^c,j^c_\ell} \phi$ — the tensor product obtained by using $j^c_\ell$ to regard $Q^c_\ell$ as a $Q^c$-module.

**Proposition 3.7.** Suppose $A_{L/K}$ exists.

(i) $a_{L/K}$ is independent of the choice of $x$.

(ii) If $|G|$ is odd, then $a_{L/K}$ is equal to the element defined by Bley, Burns and Hahn in [4 §5].

(iii) Fix a prime $\ell$ and an embedding $j^c_\ell : \mathbb{Q}^c \to \mathbb{Q}^c_\ell$. Then, in $K_0(\mathbb{Z}_\ell[G], \mathbb{Q}_\ell[G])$,

$$j^c_{\ell,*}(a_{L/K}) = j^c_{\ell,*}(\Delta(A_{L/K})) + \delta_{\mathbb{Z}_\ell,Q^c_\ell,G}(j^c_\ell(T_{L/K}^{(2)})},$$

where $j_{\ell,*}$ denotes the natural projection homomorphism $K_0(\mathbb{Z}[\Gamma], \mathbb{Q}[\Gamma]) \to K_0(\mathbb{Z}_\ell[\Gamma], \mathbb{Q}_\ell[\Gamma])$ in [3], and $\delta_{\mathbb{Z}_\ell,Q^c_\ell,G}$ denotes the composite homomorphism

$$(23) \quad \delta_{\mathbb{Z}_\ell,Q^c_\ell,G} = \partial_{\mathbb{Z}_\ell,Q^c_\ell,G}^1 \circ (Nrd_{Q^c_\ell[G]})^{-1} : \mathbb{Z}(Q^c_\ell[G])^\times \to K_0(\mathbb{Z}_\ell[G], \mathbb{Q}_\ell[G]).$$

(iii) For any $x$ as in the definition of $a_{L/K}$, we define an element of $\text{Cl}(\mathbb{Z}[G])$ by setting

$$W_{L/K}^{(2)} := \partial_{\mathbb{Z}_\ell,Q,G}(\delta_G(Nrd_{Q^c[G]}(x) \cdot T_{L/K}^{(2)})),$$

This element is independent of $x$, has order dividing 2 and vanishes if $G$ has no irreducible symplectic characters. In addition,

$$\partial_{\mathbb{Z}_\ell,Q,G}(a_{L/K}) = [A_{L/K}] + W_{L/K}^{(2)}.$$  

**Proof.** Throughout this argument, we fix $x \in K_1(\mathbb{Q}^c[G])$ as in the definition of $a_{L/K}$. We also set $y := T_{L/K}^{(2)}$. 


To prove claim (i), we choose another \( x' \in K_1(\mathbb{Q}^c[G]) \) that has the same property as \( x \). Then
\[
\partial_{\mathbb{Z},\mathbb{Q}^c,G}(x^{-1}x') = \partial_{\mathbb{Z},\mathbb{Q}^c,G}(x') - \partial_{\mathbb{Z},\mathbb{Q}^c,G}(x) \in K_0(\mathbb{Z}[G], \mathbb{Q}[G]),
\]
and so diagram (7) implies that \( x^{-1}x' \in K_1(\mathbb{Q}[G]) \). It follows that
\[
\delta_G(\text{Nrd}_{\mathbb{Q}^c[G]}(x')y) = \delta_G(\text{Nrd}_{\mathbb{Q}^c[G]}(x'x^{-1})\text{Nrd}_{\mathbb{Q}^c[G]}(x)y)
\]
\[
= \delta_G(\text{Nrd}_{\mathbb{Q}[G]}(x'x^{-1})\text{Nrd}_{\mathbb{Q}^c[G]}(x)y)
\]
\[
= \delta_G(\text{Nrd}_{\mathbb{Q}[G]}(x')) + \delta_G(\text{Nrd}_{\mathbb{Q}^c[G]}(x)y)
\]
\[
= \partial_{\mathbb{Z},\mathbb{Q}^c,G}(x') - \partial_{\mathbb{Z},\mathbb{Q}^c,G}(x) + \delta_G(\text{Nrd}_{\mathbb{Q}^c[G]}(x)y),
\]
where the fourth equality is given by Lemma 2.2(i), and the last one again follows from the commutativity of (7).

For claim (ii), we first note that if \( G \) has odd order, then
\[
\zeta(\mathbb{Q}[G])^{\times +} = \zeta(\mathbb{Q}[G])^{\times}
\]
(see Lemma 2.1(ii)). Therefore, in this case, Lemma 2.2(ii) implies
\[
\delta_G(\text{Nrd}_{\mathbb{Q}^c[G]}(x)y) = \partial_{\mathbb{Z},\mathbb{Q}^c,G}((\text{Nrd}_{\mathbb{Q}[G]})^{-1}(\text{Nrd}_{\mathbb{Q}^c[G]}(x)y))
\]
\[
= \partial_{\mathbb{Z},\mathbb{Q}^c,G}(x \cdot (\text{Nrd}_{\mathbb{Q}^c[G]})^{-1}(y))
\]
\[
= \partial_{\mathbb{Z},\mathbb{Q}^c,G}(x) + \partial_{\mathbb{Z},\mathbb{Q}^c,G}((\text{Nrd}_{\mathbb{Q}^c[G]})^{-1}(y))
\]
and hence \( a_{L/K} \) is equal to the element \( \Delta(A_{L/K}) + \partial_{\mathbb{Z},\mathbb{Q}^c,G}((\text{Nrd}_{\mathbb{Q}^c[G]})^{-1}(y)) \) considered by Bley, Burns and Hahn in [1].

Turning to claim (iii), we write \( j^c_{\ell, *} \) for the homomorphism \( K_1(\mathbb{Q}^c[G]) \to K_1(\mathbb{Q}^c[G]) \) induced by the embedding \( j^c_\ell \). Then Lemma 2.2(iii) implies
\[
j_{\ell, *}(\delta_G(\text{Nrd}_{\mathbb{Q}^c[G]}(x)y)) = \partial_{\mathbb{Z},\mathbb{Q}^c,G}(j^c_{\ell, *}(\text{Nrd}_{\mathbb{Q}^c[G]}(x)y))
\]
\[
= \partial_{\mathbb{Z},\mathbb{Q}^c,G}(j^c_{\ell, *}(x) \cdot (\text{Nrd}_{\mathbb{Q}^c[G]})^{-1}(j^c_\ell(y)))
\]
\[
= \partial_{\mathbb{Z},\mathbb{Q}^c,G}(j^c_{\ell, *}(x)) + \delta_{\mathbb{Z},\mathbb{Q}^c,G}(j^c_\ell(y))
\]
\[
= j^c_{\ell, *}(\partial_{\mathbb{Z},\mathbb{Q}^c,G}(x)) + \delta_{\mathbb{Z},\mathbb{Q}^c,G}(j^c_\ell(y)).
\]

For the last equality, one can check directly from the explicit definition in (22) and the connecting homomorphism in (7) that
\[
\partial_{\mathbb{Z},\mathbb{Q}^c,G} \circ j^c_{\ell, *} = j^c_{\ell, *} \circ \partial_{\mathbb{Z},\mathbb{Q}^c,G}.
\]
Thus, the equality in (iii) follows directly from the definition of \( a_{L/K} \) as an explicit sum.
Finally, the second displayed equality in (iv) is true since the explicit definition of \(a_{L/K}\) implies that
\[
\partial_{\mathbb{Z}, \mathbb{Q}, G}^0(a_{L/K}) \\
= \partial_{\mathbb{Z}, \mathbb{Q}^c, I}^0(\Delta(A_{L/K})) - \partial_{\mathbb{Z}, \mathbb{Q}^c, G}(\partial_{\mathbb{Z}, \mathbb{Q}^c, G}(x)) + \partial_{\mathbb{Z}, \mathbb{Q}, G}(\delta_G(\text{Nrd}_{\mathbb{Q}^c[G]}(x)y)) \\
= \partial_{\mathbb{Z}, \mathbb{Q}, G}^0([A_{L/K}, \kappa_L, H_L]) + W_{L/K}^{(2)} \\
= [A_{L/K}] - [H_L] + W_{L/K}^{(2)} = [A_{L/K}] + W_{L/K}^{(2)}.
\]

Here, the second equality follows from the explicit definition of \(W_{L/K}^{(2)}\) and the fact that the exactness of (7) implies that \(\partial_{\mathbb{Z}, \mathbb{Q}, G}^0 \circ \partial_{\mathbb{Z}, \mathbb{Q}^c, G}\) is zero. The third equality follows from the explicit definition of the connecting homomorphism \(\partial_{\mathbb{Z}, \mathbb{Q}, G}^0\), and the last equality is given by the fact that \(H_L\) is a free \(\mathbb{Z}[G]\)-module.

Since both \(a_{L/K}\) and \(A_{L/K}\) are independent of \(x\) (the first by claim (i) and the second obviously), the above equality implies that \(W_{L/K}^{(2)}\) is independent of \(x\).

To prove that \(2 \cdot W_{L/K}^{(2)} = 0\), we first note that, by Lemma 2.1(i), \((\text{Nrd}_{\mathbb{Q}^c[G]}(x)y)^2\) is in \(\zeta(\mathbb{Q}[G])^\times\) and hence
\[
2 \cdot W_{L/K}^{(2)} = \partial_{\mathbb{Z}, \mathbb{Q}, G}^0(\delta_G((\text{Nrd}_{\mathbb{Q}^c[G]}(x)y)^2)) \\
= \partial_{\mathbb{Z}, \mathbb{Q}, G}^0(\partial_{\mathbb{Z}, \mathbb{Q}, G}^1((\text{Nrd}_{\mathbb{Q}[G]})^{-1}((\text{Nrd}_{\mathbb{Q}^c[G]}(x)y)^2))) = 0.
\]

Here, the second equality follows from Lemma 2.2(ii), and the last one follows from the fact that \(\partial_{\mathbb{Z}, \mathbb{Q}, G}^0 \circ \partial_{\mathbb{Z}, \mathbb{Q}, G}^1\) is the zero map.

In a similar way, one can deduce from Lemmas 2.2(ii) and 2.1(ii)(b) that \(W_{L/K}^{(2)}\) vanishes whenever \(G\) has no irreducible symplectic characters.

**Remark 3.8.** A natural problem is to explicitly describe the difference between the class \(W_{L/K}^{(2)}\) defined in Proposition 3.7(iv) and the Cassou-Noguës–Fröhlich root number class \(W_{L/K}\), which plays a key role in classical Galois module theory. We shall consider the problem again in §5.2.

**3.2.2. The local element.** Fix a finite Galois extension \(E/F\) of local fields of residue characteristic \(\ell\) and set \(\Gamma := \text{Gal}(E/F)\). We use exactly the same recipe as Bley, Burn and Hahn [4 §7A] to define the canonical local relative element of \(K_0(\mathbb{Z}_\ell[\Gamma], \mathbb{Q}_\ell^c[\Gamma])\).

**Definition 3.9.** Fix an embedding \(j_{\ell}^c : \mathbb{Q}^c \to \mathbb{Q}_\ell^c\) of fields. We define an element of \(K_0(\mathbb{Z}_\ell[\Gamma], \mathbb{Q}_\ell^c[\Gamma])\) by setting
\[
a_{E/F} := \Delta(A_{E/F}) + \delta_{\mathbb{Z}_\ell, \mathbb{Q}_\ell^c, I}(j_{\ell}^c(T_{E/F}^{(2)})) - U_{E/F},
\]
Galois–Gauss sums

where \(\delta_{\mathbb{Z}_\ell, \mathbb{Q}_\ell, r}\) is defined as in [23] (with \(G\) replaced by \(\Gamma\)) and \(U_{E/F}\) is the canonical ‘unramified’ element of \(K_0(\mathbb{Z}_\ell[\Gamma], \mathbb{Q}_\ell[\Gamma])\) defined by Breuning [8].

Here, the elements \(\Delta(\mathcal{A}_{E/F})\) and \(T_{E/F}^{(2)}\) are defined as local analogues of the elements defined in the previous section.

Next, for each \(a\) in \(E\) generating a normal basis of \(E/F\), and each character \(\chi\) of a representation \(T_{\chi} : \Gamma \rightarrow \text{GL}_n(\mathbb{Q}_\ell^c)\), we recall from [17, Chap. I, §4, and Chap. III, §3, (3.1)] the definitions of the resolvent element and the (local) norm resolvent:

\[
(a|\chi) := \det \left( \sum_{g \in \Gamma} g(a) T_{\chi}(g^{-1}) \right), \quad \mathcal{N}_{F/\mathbb{Q}_\ell}(a|\chi) = \prod_{\omega}(a|\chi^{-1})^{\omega},
\]

where \(\omega\) runs through a transversal of \(\Omega_F\) in \(\Omega_{\mathbb{Q}_\ell}\).

We now follow Breuning in giving an explicit description of \(a_{E/F}\) in terms of norm resolvents. To do this, we fix a \(\mathbb{Z}_\ell\)-basis \(\{a_\sigma\}_{\Sigma(F)}\) of \(\mathcal{O}_F\) and set \(\delta_F := \det(\tau(a_\sigma))_{\tau, \sigma \in \Sigma(F)} \in \mathbb{Q}_\ell^c\).

**Proposition 3.10.**

(i) The element \(a_{E/F}\) is independent of the choice of embedding \(j^c\) and is in \(K_0(\mathbb{Z}_\ell[\Gamma], \mathbb{Q}_\ell[\Gamma])\).

(ii) (Breuning) Fix \(a \in E\) such that \(\mathcal{A}_{E/F} = \mathcal{O}_F[\Gamma] \cdot a\). Then

\[
\sum_{\chi \in \hat{\Gamma}} (\delta_F^{\chi(1)} \cdot \mathcal{N}_{F/\mathbb{Q}_\ell}(a|\chi)) e_\chi \in \zeta(\mathbb{Q}_\ell^c[\Gamma])^\times
\]

and, in \(K_0(\mathbb{Z}_\ell[\Gamma], \mathbb{Q}_\ell^c[\Gamma])\), one has

\[
\Delta(\mathcal{A}_{E/F}) = \delta_{\mathbb{Z}_\ell, \mathbb{Q}_\ell, r} \left( \sum_{\chi \in \hat{\Gamma}} (\delta_F^{\chi(1)} \cdot \mathcal{N}_{F/\mathbb{Q}_\ell}(a|\chi)) e_\chi \right).
\]

(iii) (Breuning) For each embedding \(k : \mathbb{Q}_c \rightarrow \mathbb{Q}_\ell^c\), the element \(\delta_F/k(\tau_F)\) is in \((\mathcal{O}_\ell^c)^\times\).

(iv) (Breuning) Let \(j^t_{\ell,*}\) be the homomorphism defined in [13]. Then \(U_{E/F} \in \ker(j^t_{\ell,*})\).

**Proof.** Since the terms \(\Delta(\mathcal{A}_{E/F})\) and \(U_{E/F}\) obviously do not depend on \(j^c\), the element \(a_{E/F}\) is independent of \(j^c\) if \(j^c(T_{E/F}^{(2)})\) is independent of the choice.

In addition, from the equality in Remark 2.16(ii), one has \((T_{E/F}^{(2)})^{-1} = \tau_{E/F}^t \cdot (J_{2,E/F} \cdot (1 - \psi_{2,*})(y_{E/F}))\). We also note that Remark 2.16(i) and the explicit definition of \(y_{E/F}\) combine to imply that

\[
(J_{2,E/F} \cdot (1 - \psi_{2,*})(y_{E/F})) \in \zeta(\mathbb{Q}[\Gamma])^\times.
\]
It is therefore sufficient to show that $\tau_{E/F}^\dagger$ is independent of the choice of $j^c_\ell$, and this is proved by Breuning [8, Lem. 2.2].

Thus, to prove (i), it is enough to show that $a_{E/F} \in K_0(\mathbb{Z}_\ell[\Gamma], \mathbb{Q}_\ell[\Gamma])$, and this follows directly from (24) and the containment

$$\Delta(A_{E/F}) - \delta_{\mathbb{Z}_\ell, \mathbb{Q}_\ell, \Gamma}(j^c_\ell(\tau_{E/F}^\dagger)) - U_{E/F} \in K_0(\mathbb{Z}_\ell[\Gamma], \mathbb{Q}_\ell[\Gamma]),$$

proved by Bley, Burns and Hahn [4, Prop. 7.1].

Claims (ii), (iii) and (iv) are proved by Breuning [7, Lemmas 4.16, 4.29 and 4.4].

**Remark 3.11.** Let $\mathbb{Q}_\ell^t$ denote the maximal tamely ramified extension of $\mathbb{Q}_\ell$, and $O_\ell^t$ denote the valuation ring of $\mathbb{Q}_\ell^t$. As a consequence of Taylor’s Fixed Point Theorem (which is discussed in §2.2.6), we will often consider the image of $a_{E/F}$ in $K_0(O_\ell^t[\Gamma], \mathbb{Q}_\ell[\Gamma])$. To address the problem, we set

$$a'_{E/F} := \Delta(A_{E/F}) + \delta_{\mathbb{Z}_\ell, \mathbb{Q}_\ell, \Gamma}(j^c_\ell(T_{E/F}^{(2)})).$$

Then Definition 3.9 and Proposition 3.10(iv) combine to imply that the images in $K_0(O_\ell^t[\Gamma], \mathbb{Q}_\ell[\Gamma])$ of $a_{E/F}$ and $a'_{E/F}$ coincide. We will use this fact frequently.

**3.2.3. From global to local.** By adapting an argument of Bley, Burns and Hahn, we shall now describe the precise connection between the elements defined in §3.2.1 for number fields and in §3.2.2 for local fields.

To do this, we fix a weakly ramified Galois extension $L/K$ of number fields for which $A_{L/K}$ exists, and we set $G := \text{Gal}(L/K)$. For each place $v$ of $K$, we fix a place $w$ of $L$ lying above $v$ and identify the Galois group of $L_w/K_v$ with the decomposition subgroup $G_w$ of $w$ in $G$.

We recall from (11) the induction homomorphisms on relative $K$-groups. For each prime $\ell$, we identify the group $K_0(O_\ell^t[\Gamma], \mathbb{Q}_\ell[\Gamma])$ with a subgroup of $K_0(\mathbb{Z}[G], \mathbb{Q}[G])$ by means of the decomposition (5).

**Proposition 3.12.** Let $L/K$ be a weakly ramified finite Galois extension of number fields, with $G := \text{Gal}(L/K)$, for which $A_{L/K}$ exists. Then, in $K_0(\mathbb{Z}[G], \mathbb{Q}[G])$,

$$a_{L/K} = \sum_\ell \sum_{v|\ell} i_{G_w, \mathbb{Q}_\ell}(a_{L_w/K_v}),$$

where the first sum is over all rational primes $\ell$ and the second is over all places $v$ of $K$ of residue characteristic $\ell$.

**Proof.** The proof is exactly the same as for [4, Th. 7.6] using Proposition 3.7(iii), with the last line of the latter proof replaced by appealing to the result of Lemma 3.13 below. ■
For the next result, we fix an arbitrary prime number \( \ell \) with a field embedding \( j_{\ell}^* : \mathbb{Q}^c \to \mathbb{Q}_{\ell}^c \). We also let \( \mathcal{O}_{\ell}^c \) denote the valuation ring of the maximal tamely ramified extension of \( \mathbb{Q}_{\ell} \) and recall the composite homomorphisms \( j_{\ell,*}^\tau \) from \([13]\) and \( \delta_{\mathcal{O}_{\ell}^c, \mathcal{Q}_{\ell}^c, \Gamma} \) from \([16]\) (for which the finite group \( \Gamma \) will be clear from the context). Set \( \delta_{G,\ell} := \delta_{\mathbb{Z}_{\ell}, \mathcal{Q}_{\ell}^c, G} \) (as given in \([23]\)).

**Lemma 3.13.** If \( v \) is a non-Archimedean place of \( K \) that does not divide \( \ell \), then

\[
\begin{align*}
(i) \quad j_{\ell}^c((\psi_{2,*} - 2)(\tau_{L_w/K_v} y_{L_w/K_v}^{-1})) & \in \ker(\delta_{\mathcal{O}_{\ell}^c, \mathcal{Q}_{\ell}^c, \Gamma_{\ell}^c)). \\
(ii) \quad \prod_{v|d_{\ell}, v \not\mid \ell} \tilde{i}_{G_v}^c (j_{\ell}^c((\psi_{2,*} - 2)(\tau_{L_w/K_v} y_{L_w/K_v}^{-1}))) & \text{vanishes, where } d_{\ell} \text{ denotes the discriminant of } \mathcal{O}_L.
\end{align*}
\]

In particular, the image under \( j_{\ell,*}^\tau \circ \delta_{G,\ell} \) of the product

\[
\prod_{v|d_{\ell}, v \not\mid \ell} \tilde{i}_{G_v}^c (j_{\ell}^c((\psi_{2,*} - 2)(\tau_{L_w/K_v} y_{L_w/K_v}^{-1})))
\]

vanishes, where \( d_{\ell} \) denotes the discriminant of \( \mathcal{O}_L \).

**Proof.** For the first claim we shall adapt an argument of Agboola, Burns, Caputo and the present author [1 Prop. 6.6(a)]. To do this, we write \( p \) for the residue characteristic of \( v \) (so that \( p \not= \ell \)) and \( \mathbb{Q}(p^\infty) \) for the subextension of \( \mathbb{Q}^c \) that is generated by all \( p \)-power order roots of unity. We also fix an \( \ell \)-adic place of \( \mathbb{Q}^c \), and for each finite extension \( M \) of \( \mathbb{Q} \) in \( \mathbb{Q}(p^\infty) \), we write \( M_{\ell} \) for its completion at the fixed \( \ell \)-adic place, and \( \mathcal{O}_{M_{\ell}} \) for the valuation ring of \( M_{\ell} \).

Now, since \( p \not= \ell \), the result [20, p. 10, (3.3b')]) of Holland and Wilson and the identification [14] combine to imply that, for a large enough subfield \( M \) of \( \mathbb{Q}(p^\infty) \), one has \( j_{\ell}^c(\tau_{L_w/K_v} y_{L_w/K_v}^{-1}) \in \ker(\mathcal{O}_{M_{\ell}}[\mathcal{G}_w]^\times) \). It follows that

\[
j_{\ell}^c((\psi_{2,*} - 2)(\tau_{L_w/K_v} y_{L_w/K_v}^{-1})) = (\psi_{2,*} - 2)(j_{\ell}^c(\tau_{L_w/K_v} y_{L_w/K_v}^{-1}))
\]

\[
\in (\psi_{2,*} - 2)(\ker(\mathcal{O}_{M_{\ell}}[\mathcal{G}_w]^\times)) \subseteq \ker(\mathcal{O}_{M_{\ell}}[\mathcal{G}_w]^\times),
\]

where the equality is given by Proposition [2.6(ii)], the containment follows from the identification \([18]\), and the inclusion is a consequence of the fact that, since \( M_{\ell} \) is an unramified extension of \( \mathbb{Q}_{\ell} \), the result [11] Th. 1 of Cassou-Noguès and Taylor proves \( \psi_{2,*}(\ker(\mathcal{O}_{M_{\ell}}[\mathcal{G}_w]^\times)) \subseteq \ker(\mathcal{O}_{M_{\ell}}[\mathcal{G}_w]^\times) \).

We also note that \( \ker(\mathcal{O}_{M_{\ell}}[\mathcal{G}_w]^\times) \subseteq \ker(\mathcal{O}_{\ell}^c[G]^\times) \) as \( M_{\ell} \subseteq \mathbb{Q}_{\ell}^c \) and so claim (i) follows from \([17]\).

Claim (ii) follows from the results above, the result for induction homomorphisms in \([17]\) Chap. II, §3, Th. 12(ii)], the fact (from [3] p. 580]) that \( j_{\ell,*}^\tau \circ \delta_{G,\ell} \) and \( \delta_{\mathcal{O}_{\ell}^c, \mathcal{Q}_{\ell}^c, \Gamma} \) coincide, and the fact that the kernel of each homomorphism \( \delta_{\mathcal{O}_{\ell}^c, \mathcal{Q}_{\ell}^c, \Gamma} \) is equal to \( \ker(\mathcal{O}_{\ell}^c[G]^\times) \) (see \([17]\)).

**3.3. The conjecture of Bley, Burns and Hahn.** In this section, we recall the central conjecture formulated by Bley, Burns and Hahn [4]. To
do this, we first recall a variant of the classical ‘unramified characteristic’ introduced in [4].

**3.3.1. The twisted idelic unramified characteristic.** To start, we consider the case of local fields.

**Definition 3.14.** Let $E/F$ be a finite Galois extension of $\ell$-adic fields and set $\Gamma := \text{Gal}(E/F)$. Then the twisted unramified characteristic of $E/F$ is the element of $K_0(\mathbb{Z}_\ell[\Gamma], \mathbb{Q}_\ell[\Gamma])$ obtained by setting

$$\mathfrak{c}_{E/F} := \delta_{\mathbb{Z}_\ell, \mathbb{Q}_\ell, \Gamma}((1 - \psi_{2,*})(y_{E/F})),$$

where $y_{E/F}$ is the equivariant unramified characteristic, and $\delta_{\mathbb{Z}_\ell, \mathbb{Q}_\ell, \Gamma}$ denotes the composite homomorphism

$$\partial_{\mathbb{Z}_\ell, \mathbb{Q}_\ell, \Gamma} \circ (\text{Nrd}_{\mathbb{Q}_\ell[\Gamma]})^{-1} : \zeta(\mathbb{Q}_\ell[\Gamma])^\times \to K_0(\mathbb{Z}_\ell[G], \mathbb{Q}_\ell[\Gamma]).$$

The following result extends the observations made in [4, Rem. 7.5].

**Lemma 3.15.** Let $E/F$ and $\Gamma$ be as above. Suppose that $\Gamma$ has odd order or $\Gamma$ is abelian and its inertia subgroup $I := \Gamma_0$ has odd order if $|\Gamma|$ is even.

(i) In $\zeta(\mathbb{Q}[\Gamma])^\times$, one has $(1 - \psi_{2,*})(y_{E/F}) = (1 - e_I) + \sigma^{-1}e_I$, where $\sigma$ is a lift to $\Gamma$ of the Frobenius element in $\Gamma/I$, and $e_I$ is the idempotent $|I|^{-1} \sum_{g \in I} g$.

(ii) The element $\mathfrak{c}_{E/F}$ vanishes if either $E/F$ is tamely ramified or both totally ramified and of odd degree.

**Proof.** Assume $\Gamma$ is either abelian or of odd order. Then, if $\phi$ is an irreducible character of $\Gamma$, so is $\psi_2(\phi)$ (see Proposition 2.6(iv) for $\Gamma$ of odd order, and if $\Gamma$ is abelian, one has $\phi(1) = \psi_2(\phi)(1) = 1$).

In particular, if $\phi$ is both irreducible and unramified (that is, trivial on $I$), then $\phi$ is linear, and so $\psi_2(\phi)$ is also both linear and unramified. Thus,

$$\partial_{\mathbb{Z}_\ell, \mathbb{Q}_\ell, \Gamma} \circ (\text{Nrd}_{\mathbb{Q}_\ell[\Gamma]})^{-1} : \zeta(\mathbb{Q}_\ell[\Gamma])^\times \to K_0(\mathbb{Z}_\ell[G], \mathbb{Q}_\ell[\Gamma]).$$

The fourth equality here follows from Proposition 2.6(iii) and the fact that $\phi(1) = \psi_2(\phi)(1) = 1$.

Now we also assume that $I$ has odd order. Then $I = \{g^2 : g \in I\}$. In this case, if an irreducible character $\phi$ is ramified (that is, non-trivial on $I$), then $\psi_2(\phi)$ is also ramified, and so the unramified parts (see Definition 2.9(ii)) $n(\phi)$ and $n(\psi_2(\phi))$ both vanish. Hence,

$$\partial_{\mathbb{Z}_\ell, \mathbb{Q}_\ell, \Gamma} \circ (\text{Nrd}_{\mathbb{Q}_\ell[\Gamma]})^{-1} = (1 - \psi_{2,*})(y_{E/F}) = y(F, \phi - \psi_2(\phi)) = y(F, \phi)y(F, \psi_2(\phi))^{-1} = (1 - e_I) + \sigma^{-1}e_I.$$

The fourth equality here follows from Proposition 2.6(iii) and the fact that $\phi(1) = \psi_2(\phi)(1) = 1$.

Write $\widehat{\Gamma}_R$ and $\widehat{\Gamma}_U$ for the sets of ramified and unramified irreducible characters of $\Gamma$. Then, by putting together (26) and (27), we get
Lemma 3.15(ii) implies that these definitions do, in fact, agree.

Involves sums of 

\[ K_{10} \]

and set 

\[ W_{12} \]

where 

\[ L/K \]

and set 

\[ \Gamma_{14} \]

The last equality is true since \( \sum_{\phi \in \hat{I}_U} e_{\phi} = e_{I} \) and \( \phi(\sigma)e_{\phi} = \sigma e_{\phi} \) for all \( \phi \in \hat{I}_U \). This proves claim (i).

Turning to claim (ii), we first note that if \( E/F \) is tamely ramified, then \( |I| \) is prime to \( \ell \). In this case, \( \mathbb{Z}_\ell[I] \) is the maximal \( \mathbb{Z}_\ell \)-order in \( \mathbb{Q}_\ell[I] \), and so \( e_{I} \in \mathbb{Z}_\ell[I] \) and \( \sigma^{-1}e_{I} \in \mathbb{Z}_\ell[I] \).

Addition, since all irreducible characters \( \phi \in \hat{I}_U \) are linear, \([26]\) and \([27]\) combine to imply that

\[ (1 - e_{I}) + \sigma^{-1}e_{I} = \text{Nrd}_{\mathbb{Q}_\ell[I]}((1 - e_{I}) + \sigma^{-1}e_{I}). \]

Now, since \( (1 - e_{I}) + \sigma^{-1}e_{I} \) is a unit of \( \mathbb{Z}_\ell[I] \), the formula from (i) can therefore be combined with the exactness of \((7)\) to deduce that \( c_{E/F} = \delta_{\mathbb{Z}_\ell,\mathbb{Q}_\ell,I}((1 - e_{I}) + \sigma^{-1}e_{I}) \) vanishes in \( K_{0}(\mathbb{Z}_\ell[I], \mathbb{Q}_\ell[I]) \).

Finally, if \( \Gamma \) is equal to \( I \) and has odd order, then \( \sigma^{-1}e_{I} = e_{I} \) and so the formula in (i) implies that

\[ (1 - \psi_{2,\ast})(y_{E/F}) = (1 - e_{I}) + \sigma^{-1}e_{I} = (1 - e_{I}) + e_{I} = 1. \]

Therefore, in this case \( c_{E/F} = 0. \)

We now define a global analogue of the above twisted unramified characteristic.

**Definition 3.16.** Let \( L/K \) be a finite Galois extension of number fields and set \( G := \text{Gal}(L/K) \). Then the **idelic twisted unramified characteristic** of \( L/K \) is the following element of \( K_{0}(\mathbb{Z}[G], \mathbb{Q}[G]) \):

\[ c_{L/K} := \sum_{v \in \mathcal{W}_{L/K}} i_{G_{w},\mathbb{Q}_{\ell}}^{G_{w},\mathbb{Q}_{\ell}}(c_{L_w/K_v}), \]

where \( \mathcal{W}_{L/K} \) denotes the (finite) set of non-Archimedean places \( v \) of \( K \) that ramify wildly in \( L/K \), and \( \ell \) denotes the residue characteristic of \( v \). Here \( i_{G_{w},\mathbb{Q}_{\ell}}^{G_{w},\mathbb{Q}_{\ell}}(c_{L_w/K_v}) \) is considered as an element of \( K_{0}(\mathbb{Z}[G], \mathbb{Q}[G]) \) via the inclusion \( K_{0}(\mathbb{Z}[G], \mathbb{Q}_{\ell}[G]) \subset K_{0}(\mathbb{Z}[G], \mathbb{Q}[G]) \) induced by \((5)\) for each prime \( \ell \).

**Remark 3.17.** (i) The formal definition of \( c_{L/K} \) given above differs from that given (for odd degree extensions) in [4 (8.3)] since the latter involves sums of \( i_{G_{w},\mathbb{Q}_{\ell}}^{G_{w},\mathbb{Q}_{\ell}}(c_{L_w/K_v}) \) over all non-Archimedean places of \( v \).

However, Lemma 3.15(ii) implies that these definitions do, in fact, agree.

(ii) For each rational prime \( \ell \), one has

\[ j_{\ell,\ast}(c_{L/K}) = \begin{cases} 0 & \text{if } \ell \notin \mathcal{W}_{L/K}^Q, \\
\sum_{v|\ell} i_{G_{w},\mathbb{Q}_{\ell}}^{G_{w},\mathbb{Q}_{\ell}} c_{L_w/K_v} & \text{if } \ell \in \mathcal{W}_{L/K}^Q, \end{cases} \]
where $\mathcal{W}_{L/K}^{Q}$ denotes the set of residue characteristics of places of $K$ that ramify wildly in $L/K$.

**3.3.2. Interpretation of Conjecture 1.3.** We recall from Proposition 3.7(ii) that, since Conjecture 1.3 only considers the case of $L/K$ of odd degree, the element $a_{L/K}$ introduced in Definition 3.6 agrees with the corresponding element defined in [4, §5].

**Remark 3.18.** It is known that this conjecture provides the first concrete link between the theory of the square root of the inverse different and the general framework of Tamagawa number conjectures that originated with Bloch and Kato [6] (see [4, Introduction]). This link, however, will play no significant role in our results.

The strongest theoretical evidence that Bley, Burns and Hahn offer in support of Conjecture 1.3 is recalled in the following result.

**Proposition 3.19 ([4, Cor. 8.4]).** Let $L/K$ be a weakly ramified finite Galois extension of number fields of odd degree, and let $DT(\mathbb{Z}[G])$ be the torsion subgroup of $K_{0}(\mathbb{Z}[G], \mathbb{Q}[G])$.

(i) Conjecture 1.3 is valid modulo $DT(\mathbb{Z}[G])$.

(ii) Conjecture 1.3 is valid provided that every place $v$ of $K$ that is wildly ramified in $L$ has the following three properties:

(a) the decomposition subgroup in $G$ of any place of $L$ above $v$ is abelian;

(b) the inertia subgroup in $G$ of any place of $L$ above $v$ is cyclic;

(c) the completion of $K$ at $v$ is absolutely unramified.

**Remark 3.20.** (i) Since $\mathcal{c}_{L/K}$ is in $K_{0}(\mathbb{Z}[G], \mathbb{Q}[G])$ and has finite order, the meaning of claim (i) of Proposition 3.19 is that $a_{L/K} \in DT(\mathbb{Z}[G])$. The result is proved by reduction to tamely ramified extensions (see [4] §8B1 for the proof) that depends on the functorial properties of $a_{L/K}$ and $\mathcal{c}_{L/K}$ (see [4, Th. 6.1 and Rem. 8.9]).

(ii) Claim (ii) of Proposition 3.19 is proved by adapting methods developed by Pickett and Vinatier [26] (which we will follow for the proof of Proposition 4.4 in §4.2).

(iii) Aside from Proposition 3.19, the only other pieces of evidence in support of Conjecture 1.3 are (1) the numerical computations described in [4] §10 that verify the conjecture for all Galois extensions of $\mathbb{Q}$ of degree 27 (in Theorem 10.2) and for a certain family of Galois extensions of $\mathbb{Q}$ of degree 63 (in Theorem 10.5); (2) the computation by the present author [24] of explicit upper bounds on the order of $a_{L/K}$ and $\mathcal{c}_{L/K}$ when $G$ is a $p$-group and $p$ is unramified in $K$, where $p$ is an odd prime.
4. Results in the relative $K$-group. In this section, we prove Theorem 1.4(i), which extends Proposition 3.19 to extensions of arbitrary degree where the square root of the inverse different exists and where no wildly ramified place is 2-adic.

In view of the decomposition result for $a_{L/K}$ in Proposition 3.12 and of Definition 3.16 of $c_{L/K}$, it is enough to show that the stated equality holds for each place $v$ of $K$ that ramifies in $L$ (and satisfies the stated hypotheses).

4.1. Tamely ramified local extensions. In this section, we will show that, for a tamely ramified extension of local fields, the canonical relative element defined in §3.2.2 vanishes in the relative $K$-group whenever it exists.

In §4.1.1 we state Theorem 4.1, which serves as a local variant of Theorem 1.4(i) for tamely ramified extensions. The proof of this theorem is provided immediately after a key result, Lemma 4.2. In §4.1.2 we introduce some necessary preparations and reductions for Lemma 4.2. The proof of the lemma is then presented in §4.1.3.

4.1.1. Statement of the results

**Theorem 4.1.** Let $E/F$ be a finite Galois extension of $p$-adic fields and set $\Gamma := \text{Gal}(E/F)$. If $E/F$ is tamely ramified and $A_{E/F}$ exists, then $a_{E/F} = 0$ in $K_0(Z_p[\Gamma], Q_p[\Gamma])$.

For the next result, we fix an embedding $j^c : Q^c \to Q_p$. We also write $j^c_p$ for the induced embedding $\zeta(Q^c[G])^\times \to \zeta(Q^c_p[G])^\times$. We let $Q^c_p$ denote the maximal tamely ramified extension of $Q_p$, and let $O^c_p$ denote the valuation ring of $Q^c_p$.

Recall the notations $\hat{\Gamma}, R_\Gamma$ and $\hat{\Gamma}_p, R_{\Gamma,p}$ from §2.1. We note that $j^c_p$ also gives rise to a bijection $R_\Gamma \to R_{\Gamma,p}$ by sending $\chi$ to the function $\chi^j$ that is defined by setting $\chi^j(g) := j^c_p(\chi(g))$ for all $g \in \Gamma$. In this way, for each $\phi \in R_{\Gamma,p}$ with $\phi = \chi^j$ for some $\chi \in R_\Gamma$, the virtual character $\chi$ is equal to the function $\phi^{j^{-1}}$ defined by setting $\phi^{j^{-1}}(g) := j^{-1}_\ell(\phi(g))$, and hence we write

$$j^c_p(\tau'_{E/F}) = \sum_{\phi \in \hat{\Gamma}_p} e_\phi \cdot j^c_p(\tau'(F, \phi^{j^{-1}})) \in \zeta(Q^c_p[\Gamma])^\times.$$  

Here $\tau'_{E/F}$ is the local analogy to the element described in Definition 2.11.

The following extends the result [16. Lem. 8.4] of Erez for tamely ramified extensions of odd degree, which uses results of Fröhlich and Taylor (see [17 Chap. III & IV]) pertaining to Fröhlich’s conjecture. We shall follow their steps in proving it.
Lemma 4.2. Fix an element \( a \in E \) such that \( \mathcal{A}_{E/F} = \mathcal{O}_F[I] \cdot a \). Then there exists \( u \in \mathcal{O}_p^t[I] \times \) such that (in terms of (15)), for all \( \chi \in \hat{I}_p \),
\[
\frac{N_{F/Q_p}(a|\chi)}{\mathcal{J}_p^c(\tau'(F, \psi_2(\chi^{j-1}) - \chi^{j-1}))} = (\text{Det}(u))(\chi).
\]

Before we prove this lemma, we shall first show that it implies Theorem 4.1.

By Taylor’s Fixed Point Theorem (see §2.2.6), in order to prove Theorem 4.1 it is enough to show that the image of \( a_{E/F} \) in \( K_0(\mathcal{O}_p^t[I], \mathbb{Q}_p^c[I]) \) vanishes. Taking account of Remark 3.11, it is therefore sufficient to show that the image of \( a'_{E/F} \) in \( K_0(\mathcal{O}_p^t[I], \mathbb{Q}_p^c[I]) \) vanishes.

To do this, we set \( \delta_{I,p} := \delta_{Z_p, \mathbb{Q}_p^c, I} \) and \( v := \delta_F/j_p^c(\tau_F) \). By the explicit formula in Proposition 3.10(ii), one has
\[
a'_{E/F} = \delta_{I,p} \left( (j_p^c(T_{E/F}^{(2)})) \cdot \sum_{\chi \in \hat{I}_p} (\delta_F(\tau_F))^{(1)} \cdot N_{F/Q_p}(a|\chi) e_\chi \right)
\]
\[
= \delta_{I,p} \left( \sum_{\chi \in \hat{I}_p} \left( (\delta_F/j_p^c(\tau_F))^{(1)} \cdot \frac{N_{F/Q_p}(a|\chi)}{\mathcal{J}_p^c(\tau'(F, \psi_2(\chi^{j-1}) - \chi^{j-1}))} \right) e_\chi \right)
\]
\[
= \delta_{I,p}(x_1) + \delta_{I,p} \left( \sum_{\chi \in \hat{I}_p} \frac{N_{F/Q_p}(a|\chi)}{\mathcal{J}_p^c(\tau'(F, \psi_2(\chi^{j-1}) - \chi^{j-1}))} e_\chi \right)
\]

with \( x_1 := \sum_{\chi \in \hat{I}_p} v^{(1)} e_\chi = \text{Nrd}_{\mathbb{Q}_p^c[I]}(v) \). Since \( v \) is a unit of \( \mathcal{O}_p^t \) (see Proposition 3.10(iii)), the image of \( \delta_{I,p}(x_1) \) in \( K_0(\mathcal{O}_p^t[I], \mathbb{Q}_p^c[I]) \) vanishes (by the exactness of (7)).

Now, assuming Lemma 4.2, we have \( x_2 = \sum_{\chi \in \hat{I}_p} x_{2,\chi} e_\chi \) such that (in terms of (15)), for all \( \chi \in \hat{I}_p \),
\[
x_{2,\chi} := \frac{N_{F/Q_p}(a|\chi)}{\mathcal{J}_p^c(\tau'(F, \psi_2(\chi^{j-1}) - \chi^{j-1}))} = (\text{Det}(u))(\chi),
\]

where \( u \in \mathcal{O}_p^t[I] \times \). Therefore the image in \( K_0(\mathcal{O}_p^t[I], \mathbb{Q}_p^c[I]) \) of \( \delta_{I,p}(x_2) \) vanishes as a consequence of (17) and the fact that \( j_{p^*} \circ \delta_{I,p} = \delta_{\mathcal{O}_p^t, \mathbb{Q}_p^c, I} \). This completes the proof of Theorem 4.1.

4.1.2. Reduction to inertia subgroups. In this section, we recall the fractional ideals in \( \mathbb{Q}_p^c \) generated by norm resolvents and Galois–Gauss sums from [17] Chap. III] and the functorial behaviour of both elements. We let \( U(\mathbb{Q}_p^c) \) denote the group of units of the ring \( \mathbb{Z}_p^c \) of integers of \( \mathbb{Q}_p^c \).

Fix \( a \in E \) that generates a normal basis of \( E/F \). Then, for each \( \chi \in R_{I,p} \), we write \( P(E/F, \chi) \) for the class of \((a|\chi)\) modulo \( U(\mathbb{Q}_p^c) \) and we recall from [17] p. 106, the last paragraph] that \( P(E/F, \chi) \) is independent of the choice
of \( a \). We also define \( N_{F/Q_p} P(E/F, \chi) = \prod_{\omega} P(E/F, \chi^{\omega^{-1}})^{\omega} \), where \( \{\omega\} \) is a transversal of \( \Omega_F \) in \( \Omega_{Q_p} \). One can see (directly from the definition) that this is the fractional ideal in \( Q_p^c \) generated by \( N_{F/Q_p} (a|\chi) \) (with \( a \) as above), and it is independent of the choice of \( \omega \) (cf. \cite{[17]} Chap. I, §4, Prop. 4.4).

We shall abbreviate \( N_{F/Q_p} P(E/F, \chi) \) to \( N_{F/Q_p} P(\chi) \). For \( \phi \in R_F \), we let \( (f_p^c(\tau(F, \phi))) \) denote the class of \( f_p^c(\tau(F, \phi)) \) modulo \( U(Q_p^c) \). We again write \( I \) for the inertia subgroup \( I_0 \) of \( \Gamma \) and \( B := E^I \) (so that \( E/B \) is of odd degree by Remark 3.4(i) and \( B/F \) is unramified). We set \( \text{res} := \text{res}_I^\Gamma \).

The following results rely heavily on the methods and result of \cite{[16]} Th. 5.2 and arguments in \cite{[17]} Chap. III, §§5–7.

**Proposition 4.3.** Suppose that \( E/F \) is tamely ramified.

(i) For all \( \chi \in R_{\Gamma} \), one has \( (\tau(B, \text{res} \chi)) = (\tau(F, \chi))^{[B:F]} \) as ideals.

(ii) For all \( \chi \in R_{\Gamma,p} \), the following are valid:

(a) Fix \( a \in E \) such that \( A_{E/F} = \mathcal{O}_F[I] \cdot a \), and \( b \in E \) such that \( A_{E/F} = \mathcal{O}_B[I] \cdot b \). Then \( (b|\text{res} \chi)_{E/B} = (a|\chi)_{E/F} \cdot (\text{Det}(\lambda))(\chi) \), where \( \lambda \in \mathcal{O}_B[I] \cdot x \).

(b) \( N_{B/Q_p} P(\text{res} \chi) = N_{F/Q_p} P(\chi)^{[B:F]} \).

(c) \( N_{F/Q_p} P(\chi) = (j_p^c(\tau(F, \psi_2(\chi^j - 1)) - \chi^j - 1)) \).

**Proof.** Claim (i) is given in \cite{[17]} Cor. 1 to Th. 25.

Now we move to claim (ii). In the rest of this proof, we shall consider \( \chi \in R_{\Gamma,p} \).

For part (a), we recall the set \( \text{Map}(\Gamma, E)^I \) of maps \( \Gamma \to E \) of \( I \)-sets (more precisely, the set of maps \( f : \Gamma \to E \) such that \( h(f(g)) = f(hg) \) for all \( h \in I \) and \( g \in \Gamma \)). It is a \( B \)-algebra via \( (xf)(g) = xf(g) \) for all \( x \in B, f \in \text{Map}(\Gamma, E)^I, g \in \Gamma \), and it is a \( \Gamma \)-module via \( (gf)(g') = f(g'g) \) for all \( f \in \text{Map}(\Gamma, E)^I, g, g' \in \Gamma \). Then there is an isomorphism of \( B[I] \)-modules (taken from \cite{[17]} Chap. III, §6, Lem. 6.2])

\[
j : B \otimes_F E \to \text{Map}(\Gamma, E)^I
\]

defined by \( (j(x \otimes y))(g) = xg(y) \) for \( x \in B, y \in E, \) and \( g \in \Gamma \), and it induces an isomorphism of \( \mathcal{O}_B[I] \)-modules

\[
\mathcal{O}_B \otimes_{\mathcal{O}_F} A_{E/F} \to \text{Map}(\Gamma, A_{E/F})^I.
\]

(See, for example, the argument of \cite{[10]} Lem. 2.4 and the isomorphism given in \cite{[10]} (8)) with the notations \( E = K, B = F, F = k, \) and \( A_{E/F} \) represented by \( \mathcal{P}_K^{-n} \).

We also construct a map \( h_b : \Gamma \to E \) associated to \( b \) by setting (as in \cite{[17]} Chap. III, §6, (6.7))

\[
h_b(g) = \begin{cases} 
g(b) & \text{if } g \in I, \\
0 & \text{if } g \in \Gamma \setminus I. 
\end{cases}
\]
We note that $h_b$ is in $\text{Map}(\Gamma, \mathcal{A}_{E/F})^I$ and it generates $\text{Map}(\Gamma, \mathcal{A}_{E/F})^I$ freely over $\mathcal{O}_B[\Gamma]$ (this can be shown by the argument given in [17] with $\mathcal{O}_L$ replaced by $\mathcal{A}_{E/F}$).

In this way, $j^{-1}(h_b)$ is a free generator of $\mathcal{O}_B \otimes \mathcal{O}_F \mathcal{A}_{E/F}$ over $\mathcal{O}_B[\Gamma]$. Then there exists $\lambda \in \mathcal{O}_B[\Gamma]^\times$ such that $j^{-1}(h_b) = \lambda (1 \otimes a)$. The rest of the proof of (a) is just copying down [17, Chap. III, §6, (6.9)–(6.12)], with $j^{-1}g$ replaced by $j^{-1}(h_b)$.

With the result in part (a), the proof for part (b) is exactly as in [17, bottom of p. 131] (see [17, Chap. I, §4, Prop. 4.4] for the Galois action formula on resolvents).

Now, given the results in claim (i) and (ii)(b), we are reduced to showing that the equality stated in (c) is valid for $E/B$ (which is of odd degree), and this is proved by Erez [16, Th. 5.2 and §7.1]. Indeed,

\[
(j_p^c(\tau(F, \psi_2(\chi^j-1) - \chi^{j-1})))^{[B:F]} = (j_p^c(\tau(B, \text{res}(\psi_2(\chi^j-1) - \chi^{j-1}))))
\]
\[
= (j_p^c(\tau(B, \psi_2(\text{res} \chi^j-1)) - \text{res} \chi^{j-1}))
\]
\[
= N_{B/Q_p} P(\text{res} \chi) = N_{F/Q_p} P(\chi)^{[B:F]}.
\]

Here, the second equality follows from Proposition 2.6(ii), and the third is given by the result of Erez. 

**4.1.3. Proof of Lemma 4.2.** To prove Lemma 4.2 as Erez pointed out in [16, Lem. 8.4] for extensions of odd degree, one only needs to copy the argument for [17, Chap. IV, §1, Th. 31], replacing Theorems 23 and 25 by our Propositions 4.3(ii)(c) and (ii)(a) respectively. For a complete proof, we refer the reader to the author’s PhD thesis [23, §6.1.4].

**4.2. Special cases of weakly ramified extensions, and proof of Theorem 1.4(i).** The proof of the following result is exactly the same as for [4, Th. 8.1].

**Proposition 4.4.** Fix an odd prime number $p$. Let $E/F$ be an abelian and weakly and wildly ramified Galois extension of $\mathbb{Q}_p$ such that $F/\mathbb{Q}_p$ is unramified, and write $\Gamma := \text{Gal}(E/F)$. Suppose $\mathcal{A}_{E/F}$ exists and the inertia subgroup of $\Gamma$ is cyclic. Then $\mathfrak{a}_{E/F} = \mathfrak{c}_{E/F}$ in $K_0(\mathbb{Z}_p[\Gamma], \mathbb{Q}_p[\Gamma])$.

We note that, under these hypotheses, the computations by Bley and Cobbe [5, §5] and those of Pickett and Vinatier [26, §3.2] do not rely on the assumption that $\Gamma$ is of odd degree, but rather on the assumptions that $\Gamma$ is abelian and that the inertia subgroup $I_0$ of $\Gamma$ has odd order $p$ (see [5, Rem. 1 and §3.1] and [26, Cor. 3.4]). A complete proof of Proposition 4.4 can be found in [23, §7.1].
Taking account of Proposition 3.12 and Definition 3.16, Theorem 1.4(i) from the introduction now follows immediately from Theorem 4.1 and Proposition 4.4.

5. Elements in the class group. In view of the ‘classical’ Galois module theory, in this section, we consider the stable-isomorphism class in the class group defined by the square root of the inverse different.

5.1. Equivariant symplectic Galois–Jacobi sums. In this section, we define the element (in Conjecture 1.2) that is associated to symplectic Galois–Jacobi sums, by using the technique of relative algebraic $K$-theory. (For the approach concerning the classical ‘Hom-description’, we refer the reader to [II §8] and the proof of Lemma 5.5(ii) below).

For a real number $x$, we write $\text{sign}(x) \in \{\pm 1\}$ for the sign of $x$.

**Definition 5.1.**

(i) Let $E/F$ be a Galois extension of non-Archimedean local fields and set $\Gamma := \text{Gal}(E/F)$. For each $\chi \in \hat{\Gamma}$, we set

\begin{equation}
J'_{2,S}(E/F, \chi) := \begin{cases} 
\text{sign}(\tau'(F, \psi_2(\chi) - 2\chi)) & \text{if } \chi \in \text{Symp}(\Gamma), \\
1 & \text{otherwise}.
\end{cases}
\end{equation}

In terms of (8), we then define the *equivariant modified symplectic Galois–Jacobi sum* of $E/F$ by setting

\begin{equation}
J'_{2,S,E/F} := \sum_{\chi \in \hat{\Gamma}} e_{\chi} \cdot J'_{2,S}(E/F, \chi).
\end{equation}

(ii) Let $L/K$ be a finite Galois extension of number fields with $G := \text{Gal}(L/K)$. We define the global *equivariant symplectic Galois–Jacobi sum* of $L/K$ by setting

\begin{equation}
J'_{2,S,L/K} := \prod_{v \in S_f(K)} \tilde{t}_{iG_w}^G(J'_{2,S,L_w/K_v}),
\end{equation}

where, for each place $v \in S_f(K)$, $G_w$ denotes the decomposition subgroup in $G$ of some fixed place $w$ of $L$ above $v$.

**Remark 5.2.** We recall from [II Prop. 8.2] that, for a local extension $E/F$ with $\Gamma := \text{Gal}(E/F)$, the element $\psi_{2,*}(y_{E/F})$ does not always lie in $\zeta(Q[\Gamma])^\times$, and therefore it behaves differently compared to the case when $|\Gamma|$ is odd (where $\zeta(Q[\Gamma])^\times = \zeta(Q[\Gamma])^\times$), and to the element $y_{E/F}$ (see [19]). This property combines with the fact [II Th. 7.1] that, if $E/F$ is tamely ramified, then $\tau(F, \psi_2(\chi) - 2\chi) > 0$ for all $\chi \in \text{Symp}(\Gamma)$, to motivate the definitions given in this section.
We note that $J_{2,S,L/K}'$ has order at most 2 and is in $\zeta(\mathbb{Q}[G])^\times$. Additionally, we will make use of the following properties of this element in the subsequent section.

In what follows, we write $J_{2,L/K}' := (\psi_2,*) - 2)\tau_{L/K}' \in \zeta(\mathbb{Q}[G])^\times$ for the modified second Galois–Jacobi sum for $L/K$.

**Proposition 5.3.** The product $J_{2,L/K} \cdot (1 - \psi_2,*)$ $y_{L/K} \cdot (J_{2,S,L/K}')^{-1}$ is in $\zeta(\mathbb{Q}[G])^\times$.

**Proof.** In light of Propositions 2.15 and 2.5(i), we deduce from (10) and Definition 5.1 that $J_{2,L/K}'(J_{2,S,L/K}')^{-1} \in \zeta(\mathbb{Q}[G])^\times$. Next, the argument for Proposition 2.15 implies that $y_{L/K} = \prod_{v|d} i_{G_v}(y_{L_v/K_v})$ and so the claimed result follows from (19) and Proposition 2.5(i).

**Definition 5.4.** By using the extended boundary map (see §2.2.3) and the projection homomorphism given in (7), we define an element of $\text{Cl}(\mathbb{Z}[G])$ by setting

$J_{2,S,L/K} := \partial_{\mathbb{Z},\mathbb{Q},G}^0 \circ \delta_G(J_{2,S,L/K}')$.

The following results show that the class $J_{2,S,L/K}$ is often, but not always, trivial. Part (ii) is a consequence of [11, Th. 9.3 and §10], and the proof is analogous to the argument for [3, Prop. 3.1].

We also remark that Conjecture 1.2 holds for the extensions considered in (ii), as it holds for any tamely ramified extension where the square root of the inverse different exists (which will be shown in the next subsection).

For any integer $m \geq 1$, we write $H_{4m}$ for the generalised quaternion group of order $4m$.

**Lemma 5.5.** Let $L/K$ be a finite Galois extension of number fields and write $G := \text{Gal}(L/K)$. The following claims are valid in the class group:

(i) If $L/K$ is locally abelian or of odd degree, then $J_{2,S,L/K} = 0$.
(ii) Let $K$ be an imaginary quadratic field such that $\text{Cl}(\mathcal{O}_K)$ contains an element of order 4. For any sufficiently large prime $\ell$ with $\ell \equiv 3$ (mod 4), there exist infinitely many tamely ramified extensions $L/K$ with $G := \text{Gal}(L/K) \cong H_{4\ell}$ such that $A_{L/K}$ exists and $J_{2,S,L/K} \neq 0$.

**Proof.** For claim (i), one needs only note that in this case the decomposition subgroup $G_w$ in $G$ of every place $w$ of $L/K$ is either abelian or of odd order, and so there is no irreducible symplectic character of $G_w$.

Turning to part (ii), for ease of notation, we write $x := J_{2,S,L/K}'$ and so (in terms of [8]) we let $x_\chi$ denote the individual coefficient that corresponds to each $\chi \in \hat{\Gamma}$. Recalling from Remark 2.4 that

$$\delta_G(x) = \partial_{\mathbb{Z},\mathbb{Q},G}^1((\text{Nrd}_{\mathbb{Q}[G]})^{-1}(x^2)) - \sum_p \partial_{\mathbb{Z}_p,\mathbb{Q}_p,G}^1((\text{Nrd}_{\mathbb{Q}_p[G]})^{-1}(x_p)),$$
where the sum runs over all rational primes $p$, we deduce that
\[-J_{2,S,L/K} = J_{2,S,L/K} = \partial_{Z,Q,G}^0 \left( \sum_p \partial_{Z_p,Q_p,G}^1 ((\text{Nrd}_{Q_p[G]}^{-1}(x_p))) \right),\]
as $\partial_{Z,Q,G}^0 \circ \partial_{Z,Q,G}^1$ is the zero homomorphism (due to the exactness of (7)).

To show that this term does not vanish, we first recall from [17, Chap. I, §2, Th. 1] the isomorphism
\[c_G : \frac{\text{Hom}(R_G, J(Q^c))^{\Omega_q}}{\text{Det}(U(Z[G])) \cdot \text{Hom}(R_G, Q^{c\times})^{\Omega_q}} \approx \text{Cl}(Z[G]),\]
where $J(Q^c)$ denotes the group of ideles of the field $Q^c$, and $U(Z[G])$ the ideles of $Z[G]$. We also view $Q^{c\times}$ as a subgroup of $J(Q^c)$ via the natural diagonal embedding.

In the rest of this proof, for each prime $\ell$ (here we allow $\ell = \infty$ by setting $Q^c_{\ell} = \mathbb{C}$ and $Q_{\ell} = \mathbb{R}$) we fix a field embedding $j_\ell^G : Q^c \to Q^c_{\ell}$ and for each $a \in Q^c$ we write $a_\ell := j_\ell^G(a)$.

We then define an element $f_\chi$ of $\text{Hom}(R_G, J(Q^c))^{\Omega_q}$ by setting,
\[f_\chi(\chi)_{\ell} := \begin{cases} 1 & \text{if } \ell = \infty, \\ x_{\chi,\ell} & \text{otherwise,} \end{cases} \quad \text{for each } \chi \in \hat{G}.\]

We note that the ‘Hom-description’ isomorphism and the surjective map $\partial_{Z,Q,G}$ combine to imply that $J_{2,S,L/K} = c_G(f_\chi)$ (cf. [14, p. 88 and Rem. 3.2]).

Next we define $g_\chi \in \text{Hom}(R_G, J(Q^c))^{\Omega_q}$ by setting, for each $\chi \in \hat{G},$
\[g_\chi(\chi)_{\ell} := \begin{cases} x_{\chi,\ell}^2 & \text{if } \ell = \infty \text{ and } \chi \in \text{Symp}(G), \\ x_{\chi,\ell} & \text{if } \ell = \infty \text{ and } \chi \notin \text{Symp}(G), \\ 1 & \text{otherwise}. \end{cases}\]

Since $x^2 \in \zeta(Z[G])^{\times^+}$, one has $g_\chi \in \text{Det}(U(Z[G]))$. We also write $h_\chi$ for the element in $\text{Hom}(R_G, Q^{c\times})^{\Omega_q}$ defined by $h_\chi(\chi) = x_\chi$ for each $\chi \in \hat{G}$. We therefore have $c_G(g_\chi h_\chi^{-1}) = 0$, and hence $J_{2,S,L/K} = c_G(f_\chi) = c_G(f_\chi \cdot g_\chi h_\chi^{-1})$.

Finally, by an explicit comparison of the definitions above with [1, Def. 8.3 and 8.5], one can conclude that $J_{2,S,L/K}$ is equal to the class $J_\infty^*(L/K)$ of [1], and the claimed result now follows from [1, Ths. 1.7 & 9.3].

### 5.2. New evidence for Conjecture 1.2

In this section, we show that the results presented in the previous sections can be combined with the central result of Bley and Cobbe [5] to prove Conjecture 1.2 for a special class of weakly ramified extensions and thereby extend [1] Th. 1.5.

First, we state a series of properties the global extensions should satisfy.

**Hypothesis 5.6.** Let $L/K$ be a finite Galois extension of number fields that is at most weakly ramified and set $G := \text{Gal}(L/K)$. We assume the following:
(i) The square root of the inverse different $A_{L/K}$ exists.

For every place $v$ of $K$ that is wildly ramified in $L$, we also assume:

(ii) $v$ is lying above an odd prime number.
(iii) The decomposition subgroup in $G$ of any place of $L$ above $v$ is abelian.
(iv) The inertia subgroup in $G$ of any place of $L$ above $v$ is cyclic.
(v) The completion of $K$ at $v$ is absolutely unramified.
(vi) The inertia degree of $v$ in $L/K$ is prime to the absolute degree of the completion $K_v$.

Now, we restate Theorem 1.4(ii) involving the Cassou-Noguès–Fröhlich root number class, $W_{L/K}$, and present its proof.

**Theorem 5.7.** Let $L/K$ be a weakly ramified finite Galois extension of number fields and set $G := \text{Gal}(L/K)$.

(i) Suppose $L/K$ satisfies conditions (i)–(v) in Hypothesis 5.6. Then, in $\text{Cl}(\mathbb{Z}[G])$,
$$[A_{L/K}] = W_{L/K} + J_{2,S,L/K}.$$  

(ii) Suppose $L/K$ satisfies all of the conditions in Hypothesis 5.6. Then Conjecture 1.2 is valid for $L/K$.

We note that claim (ii) implies that Conjecture 1.2 holds for the cases where $L/K$ is tamely ramified.

**Proof of Theorem 5.7.** To prove part (i), we note that Theorem 1.4(i) implies that
$$\partial_{\mathbb{Z},\mathbb{Q},G}^0(a_{L/K} - c_{L/K}) = 0$$
in the class group. In addition, [4, Lem. 8.7 and Rem. 8.8] imply that under the stated hypothesis, the image of $c_{L/K}$ in the class group vanishes.

Then, by comparing the claimed equality with Proposition 3.7(iv), the result follows directly from Proposition 5.8 below.

Claim (ii) follows immediately from (i) and [5, Cor. 2].

The following result relies on [3, Prop. 3.1] concerning the ‘equivariant global epsilon constant’.

**Proposition 5.8.** Suppose that $L/K$ satisfies Hypothesis 5.6(i). Then
$$W_{L/K}^{(2)} = W_{L/K} + J_{2,S,L/K}.$$  

**Proof.** We recall (from Proposition 3.7(iv)) the explicit expression of $W_{L/K}^{(2)}$:
where the second equality follows from Proposition 2.15(iii).

We first claim that the second term of (29) is equal to $\mathcal{J}_{2,S,L/K} = -\mathcal{J}_{2,S,L/K}$. Indeed,

$$\partial_{Z,Q,G}^0 \left( \delta_G(J_{2,L/K} \cdot y_{L/K}^{(2)}) \right) = \mathcal{J}_{2,S,L/K}$$

Next, to prove that the first term of (29) is equal to $W_{L/K} = -W_{L/K}$, we recall the (global) $\zeta(\mathbb{R}[G])$-valued ‘epsilon factor’ function $\epsilon(s)$ of a complex variable $s$ given by Bley and Burns [3, §3] with $\epsilon_{L/K} := \epsilon(0) \in \zeta(\mathbb{R}[G])^\times$. In particular, by [3, Proposition 3.4 and Remark 3.5], one has

$$\text{Nrd}_{\mathbb{R}[G]}(y) \cdot (\epsilon_{L/K})^{-1} = \text{Nrd}_{\mathbb{Q}[G]}(x) \cdot (\tau_{L/K}^\dagger)^{-1} \in \zeta(\mathbb{Q}[G])^\times$$

with a suitable choice of $y$ (and the choice of $x$ as in the proof of Lemma 3.5). Upon recalling the explicit definition of the extended boundary homomorphism on $\zeta(\mathbb{R}[G])^\times$ from [9, (44)], one sees that

$$\partial_{Z,Q,G}^0 \left( \delta_G(\text{Nrd}_{\mathbb{Q}[G]}(x) \cdot (\tau_{L/K}^\dagger)^{-1}) \right) = \partial_{Z,Q,G}^0 \left( \delta_G(\text{Nrd}_{\mathbb{R}[G]}(y) \cdot (\epsilon_{L/K})^{-1}) \right)$$

$$= \partial_{Z,Q,G}^0(\partial_{Z,R,G}^0(y)) = -\partial_{Z,\mathbb{R},G}^0(\delta_G(\epsilon_{L/K})) = -W_{L/K}.$$

Here, the second equality follows from Lemma 2.2(ii) (extended to $\zeta(\mathbb{R}[G])^\times$, see Remark 2.3) and the right two vertical maps of (7) (with $R, F, E$ and $\Gamma$ replaced by $\mathbb{Z}, Q, \mathbb{R}$ and $G$ respectively), and the last equality is given by [3, Prop. 3.1] (we note that the equivariant global epsilon constant $\mathcal{E}_{L/K}$ defined in [3, p. 551] is equal to $\delta_G(\epsilon_{L/K})$). \hfill \blacksquare

**Remark 5.9.** Even if we assume Chinburg’s $\Omega(2)$-conjecture (see [4]), it is possible that Conjecture 1.2 and Question 1.5 are not compatible.

To see this, we note that Propositions 3.7(iv) and 5.8 combine to imply that the image in $\text{Cl}(\mathbb{Z}[G])$ of the equality in Question 1.5 can be written as $[\mathcal{A}_{L/K}] + W_{L/K} + \mathcal{J}_{2,S,L/K} = \partial_{Z,Q,G}^0(\epsilon)$. Suppose that both Conjecture 1.2 and Question 1.5 are compatible. Then, the equality in Question 1.5 can be written as $[\mathcal{A}_{L/K}] + W_{L/K} + \mathcal{J}_{2,S,L/K} = \partial_{Z,Q,G}^0(\epsilon)$, and hence

$$\partial_{Z,Q,G}^0(\delta_G(\text{Nrd}_{\mathbb{Q}[G]}(x) \cdot (\tau_{L/K}^\dagger)^{-1})) = \partial_{Z,Q,G}^0 \left( \delta_G(\text{Nrd}_{\mathbb{R}[G]}(y) \cdot (\epsilon_{L/K})^{-1}) \right)$$

$$= \partial_{Z,Q,G}^0(\partial_{Z,R,G}^0(y)) = -\partial_{Z,\mathbb{R},G}^0(\delta_G(\epsilon_{L/K})) = -W_{L/K}.$$

This contradicts our assumption that Conjecture 1.2 and Question 1.5 are compatible. Therefore, it is possible that Conjecture 1.2 and Question 1.5 are not compatible.
and (4) hold; then the affirmative answer to Question 1.5 is consistent with Conjecture 1.2 provided that $\partial_{\mathbb{Z}_L,\mathbb{Q},\mathcal{G}}(c) = 0$.

However, despite being known to vanish for a range of classes of extensions, such as those in Theorem 5.7(i) (taken from [4, Lem. 8.7 and Rem. 8.8]) and a certain family of weakly and wildly ramified non-abelian extensions (see [4, §9B3]), it remains an open question whether the projection of $c_{L/K}$ to $\text{Cl}(\mathbb{Z}[G])$ vanishes in general.

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