VOLUMES OF DOUBLE TWIST KNOT CONE-MANIFOLDS

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Abstract. We give explicit formulae for the volumes of hyperbolic cone-manifolds of double twist knots, a class of two-bridge knots which includes twist knots and two-bridge knots with Conway notation $C(2n, 3)$. We also study the Riley polynomial of a class of one-relator groups which includes two-bridge knot groups.

1. Introduction

Let $K$ be a hyperbolic knot and $X_K$ the complement of $K$ in $S^3$. Let $\rho_{\text{hol}}$ be a holonomy representation of $\pi_1(X_K)$ into $\text{PSL}_2(\mathbb{C})$. Thurston [Th] showed that $\rho_{\text{hol}}$ can be deformed into a one-parameter family $\{\rho_\alpha\}$ of representations to give a corresponding one-parameter family $\{C_\alpha\}$ of singular complete hyperbolic manifolds. These $\alpha$’s and $C_\alpha$’s are called the cone-angles and hyperbolic cone-manifolds of $K$, respectively.

We consider the complete hyperbolic structure on a knot complement as the cone-manifold structure of cone-angle zero. It is known that for a two-bridge knot $K$ there exists an angle $\alpha_K \in [\frac{2\pi}{3}, \pi)$ such that $C_\alpha$ is hyperbolic for $\alpha \in (0, \alpha_K)$, Euclidean for $\alpha = \alpha_K$, and spherical for $\alpha \in (\alpha_K, \pi)$ [HLM, Ko1, Po, PW]. In [HLM] a method for calculating the volumes of two-bridge knot cone-manifolds were introduced but without explicit formulae. Explicit volume formulae for hyperbolic cone-manifolds of knots are known for the following knots: $4_1$ [HLM, Ko1, Ko2, MR], $5_2$ [Me], twist knots [HMP] and two-bridge knots with Conway notation $C(2n, 3)$ [HL]. In this paper we will calculate the volumes of cone-manifolds of double twist knots, a class of two-bridge knots which includes twist knots and two-bridge knots with Conway notation $C(2n, 3)$.

Let $J(k, l)$ be the knot/link as in Figure 1, where $k, l$ denote the numbers of half twists in the boxes. Positive (resp. negative) numbers correspond to right-handed (resp. left-handed) twists. Note that $J(k, l)$ is a knot if and only if $kl$ is even, and is the trivial knot if $kl = 0$. Furthermore, $J(k, l) \cong J(l, k)$ and $J(-k, -l)$ is the mirror image of $J(k, l)$. Hence, without loss of generality, we will only consider $J(k, 2n)$ for $k > 0$ and $|n| > 0$. The knot $J(2, 2n)$ is known as a twist knot, and $J(3, 2n)$ is the two-bridge knot with Conway notation $C(-2n, 3)$. In general, the knot $J(k, 2n)$ is called a double twist knot. It is a hyperbolic knot if and only if $|k|, |2n| \geq 2$ and $J(k, 2n)$ is not the trefoil knot. We will now exclusively consider the hyperbolic $J(k, 2n)$ knots.

To state our main results we introduce the Chebychev polynomials of the second kind $S_j(\omega)$. They are recursively defined by $S_0(\omega) = 1, S_1(\omega) = \omega$ and $S_j(\omega) = \omega S_{j-1}(\omega) - S_{j-2}(\omega)$ for all integers $j$.

Let $X_K(\alpha)$ be the hyperbolic cone-manifold with underlying space $S^3$ and with singular set $K$ of cone-angle $\alpha$. The volume of $X_{J(k, 2n)}(\alpha)$ is given as follows.
Theorem 1. We have

\[ \text{Vol}(X_{J(2m+1,2n)}(\alpha)) = \int_{\alpha}^{\pi} \log \left| \frac{S_m(z) - M^2 S_{m-1}(z)}{M^2 S_m(z) - S_{m-1}(z)} \right| d\omega. \]

Here \( M = e^{i\phi} \) and \( z, \) with \( \text{Im} \left( \frac{S_m(z) S_{m-1}(z)}{S_m(z) - S_{m-1}(z)} \right) \leq 0, \) is a zero of the Riley polynomial

\[ \Phi_{J(2m+1,2n)}(M, z) = S_n(t_m) - d_m S_{n-1}(t_m), \]

where

\[ t_m = M^2 + M^{-2} + 2 - z - (z - 2)(z - M^2 - M^{-2}) S_m(z) S_{m-1}(z), \]
\[ d_m = 1 - (z - M^2 - M^{-2}) S_m(z) (S_m(z) - S_{m-1}(z)). \]

Theorem 2. We have

\[ \text{Vol}(X_{J(2m,2n)}(\alpha)) = \int_{\alpha}^{\pi} \log \left| \frac{(S_m(z) - S_{m-1}(z)) - M^2 (S_{m-1}(z) - S_{m-2}(z))}{M^2 (S_m(z) - S_{m-1}(z)) - (S_{m-1}(z) - S_{m-2}(z))} \right| d\omega. \]

Here \( M = e^{i\phi} \) and \( z, \) with \( \text{Im} \left( \frac{(S_m(z) - S_{m-1}(z)) S_{m-1}(z) - S_{m-2}(z)}{S_m(z) - S_{m-1}(z)} \right) \leq 0, \) is a zero of the Riley polynomial

\[ \Phi_{J(2m,2n)}(M, z) = S_n(\tilde{t}_m) - \tilde{d}_m S_{n-1}(\tilde{t}_m) = 0, \]

where

\[ \tilde{t}_m = 2 + (z - 2)(z - M^2 - M^{-2}) S_{m-1}^2(z), \]
\[ \tilde{d}_m = 1 + (z - M^2 - M^{-2}) S_{m-1}(z) (S_m(z) - S_{m-1}(z)). \]

Remark 1.1. For a fixed integer \( m, \) the Riley polynomial \( P_n := \Phi_{J(2m+1,2n)}(M, z) \) can be defined recursively by

\[ P_n = \left( M^2 + M^{-2} + 2 - z - (z - 2)(z - M^2 - M^{-2}) S_m(z) S_{m-1}(z) \right) P_{n-1} - P_{n-2} \]

for all integers \( n, \) with initial conditions \( P_0 = 1 \) and

\[ P_1 = 1 + (z - M^2 - M^{-2}) S_{m-1}(z) (S_m(z) - S_{m-1}(z)). \]

Similarly, the Riley polynomial \( Q_n := \Phi_{J(2m,2n)}(M, z) \) can be defined recursively by

\[ Q_n = \left( 2 + (z - 2)(z - M^2 - M^{-2}) S_{m-1}^2(z) \right) Q_{n-1} - Q_{n-2} \]

for all integers \( n, \) with initial conditions \( Q_0 = 1 \) and

\[ Q_1 = 1 - (z - M^2 - M^{-2}) S_{m-1}(z) (S_{m-1}(z) - S_{m-2}(z)). \]
With appropriate changes of variables, we obtain the volume formulae for cone-manifolds of two-bridge knots with Conway notation $C(2n, 3)$ in [HL] and for cone-manifolds of twist knots in [HMP] by setting $m = 1$ in Theorems 1 and 2 respectively.

The paper is organized as follows. In Section 2 we study the Riley polynomial of a class of one-relator groups which includes two-bridge knot groups. In Section 3 we apply the result in Section 2 to calculate the Riley polynomial of the double twist knot $J(k, 2n)$ and then we prove Theorems 1 and 2.

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2. Nonabelian representations

In this section we will study nonabelian $SL_2(\mathbb{C})$-representations of a class of one-relator groups which includes two-bridge knot groups. Like the case of two-bridge knot groups, the set of nonabelian $SL_2(\mathbb{C})$-representations of each group in this class is described by a single polynomial in two variables which we call the Riley polynomial of the group. We will present three different approaches to the Riley polynomial.

Let $F_{a,b} = \langle a, b \rangle$ be the free group on two letters $a$ and $b$. For a word $u \in F_{a,b}$ let $\tilde{u}$ be the word obtained from $u$ by replacing $a$ and $b$ by $b^{-1}$ and $a^{-1}$ respectively.

We consider the group

$$G = \langle a, b \mid wa = bw \rangle,$$

where $w$ is a word in $F_{a,b}$ with $w \neq 1$ and $\tilde{w} = w^{-1}$.

The knot group of a two-bridge knot always has a presentation of the form (2.1). Indeed, two-bridge knots are those knots admitting a projection with only two maxima and two minima. The double branched cover of $S^3$ along a two-bridge knot is a lens space $L(p, q)$, which is obtained by doing a $p/q$ surgery on the unknot. Such a two-bridge knot is denoted by $b(p, q)$. Here $p$ and $q$ are relatively prime odd integers, and we can always assume that $p > |q| \geq 1$. It is known that $b(p', q')$ is ambient isotopic to $b(p, q)$ if and only if $p' = p$ and $q' \equiv q \pm 1 \pmod{p}$, see e.g. [BZ]. The knot group of the two-bridge knot $b(p, q)$ has a presentation of the form $\langle a, b \mid wa = bw \rangle$ where $a, b$ are meridians, $w = a^{\varepsilon_1}b^{\varepsilon_2}c^{\varepsilon_3} \cdots a^{\varepsilon_{2p-2}}b^{\varepsilon_{2p-1}}$ and $\varepsilon_i = (-1)^{(i+1)p/q}$ for $1 \leq i \leq p - 1$. Since $\varepsilon_i = \varepsilon_{p-i}$, we have

$$\tilde{w} = b^{-\varepsilon_1}a^{-\varepsilon_2}b^{-\varepsilon_3}a^{-\varepsilon_4} \cdots b^{-\varepsilon_{p-2}}a^{-\varepsilon_{p-1}} = b^{-\varepsilon_{p-1}}a^{-\varepsilon_{p-2}} \cdots b^{-\varepsilon_2}a^{-\varepsilon_1} = w^{-1}.$$

We now consider representations of $G$ into $SL_2(\mathbb{C})$. Two representations $\rho, \rho' : G \to SL_2(\mathbb{C})$ are called conjugate if there exists a matrix $C \in SL_2(\mathbb{C})$ such that $\rho'(g) = C \rho(g) C^{-1}$ for all $g \in G$. In this paper we study nonabelian representations up to conjugation. A representation $\rho : G \to SL_2(\mathbb{C})$ is called nonabelian if the image $\rho(G)$ is a nonabelian subgroup of $SL_2(\mathbb{C})$. Suppose $\rho$ is nonabelian. Since $a$ and $b = waw^{-1}$ are conjugate, up to conjugation we can assume that

$$\rho(a) = \begin{bmatrix} M & 1 \\ 0 & M^{-1} \end{bmatrix} \quad \text{and} \quad \rho(b) = \begin{bmatrix} M & 0 \\ r & M^{-1} \end{bmatrix},$$

where $(M, r) \in \mathbb{C}^* \times \mathbb{C}^*$ satisfies the matrix equation $\rho(w)\rho(a) = \rho(b)\rho(w)$.

We will show that this matrix equation is equivalent to a single equation in $M$ and $y$, which we call the Riley polynomial of the group $G$.

For a word $u$ in $F_{a,b}$, we write $\rho(u) = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}$ where $u_{ij} \in \mathbb{C}$. 
2.1. The Riley polynomial. To solve the matrix equation $\rho(w)\rho(a) = \rho(b)\rho(w)$, we follow Riley’s approach in [Ri].

**Lemma 2.1.** Suppose $u$ is a word in $F_{a,b}$ with $u \neq 1$ and $\tilde{u} = u^{-1}$. Then we have

\begin{equation}
    u_{21} = ru_{12}.
\end{equation}

**Proof.** We use induction on the length $\ell(u)$ of $u$. Note that $\ell(u) \geq 2$. If $\ell(u) = 2$ then it is easy to see that $u = g_1g_2$, where $\{g_1, g_2\} = \{a, b\}$ and $\varepsilon = \pm 1$. Equality (2.3) holds true by a direct calculation.

Suppose $\ell(u) > 2$. Write $u = gvh$, where $g, h \in \{a^\pm 1, b^\pm 1\}$ and $v \in F_{a,b}$ with $\ell(v) = \ell(r) - 2$. Since $\tilde{u} = u^{-1} = h^{-1}v^{-1}g^{-1}$, we have $h^{-1} = \tilde{g}$, $g^{-1} = \tilde{h}$ and $v^{-1} = \tilde{v}$. It follows that $\{g, h\} = \{a, b\}$ or $\{g, h\} = \{a^{-1}, b^{-1}\}$.

We consider the case $(g, h) = (a, b)$. Then $u = avb$. By induction hypothesis we have

$$
    \rho(v) = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}
$$

with $v_{21} = rv_{12}$. Hence

$$
    \rho(u) = \rho(a)\rho(v)\rho(b) = \begin{bmatrix} M & 1 \\ 0 & M^{-1} \end{bmatrix} \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} \begin{bmatrix} M & 0 \\ r & M^{-1} \end{bmatrix} = \begin{bmatrix} M^2v_{11} + Mrv_{12} + Mv_{21} + rv_{22} & u_{12} + M^{-1}v_{22} \\ v_{21} + M^{-1}rv_{22} & M^{-2}v_{22} \end{bmatrix}.
$$

It follows that $u_{21} = v_{21} + rM^{-1}v_{22} = r(u_{12} + M^{-1}v_{22}) = ru_{12}$.

The cases $(g, h) = (b, a)$, $(a^{-1}, b^{-1})$ and $(b^{-1}, a^{-1})$ can be proved similarly. \qed

By Lemma 2.1 we have $\rho(w) = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix}$ with $w_{21} = rw_{12}$. Then

$$
    \rho(w)\rho(a) - \rho(b)\rho(w) = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix} \begin{bmatrix} M & 1 \\ 0 & M^{-1} \end{bmatrix} - \begin{bmatrix} M & 0 \\ r & M^{-1} \end{bmatrix} \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix} = \begin{bmatrix} 0 & w_{11} - (M - M^{-1})w_{12} \\ -rw_{11} + (M - M^{-1})w_{21} & w_{21} - rw_{12} + w_{21} \\ 0 & w_{11} - (M - M^{-1})w_{12} \end{bmatrix}.
$$

Hence $\rho(w)\rho(a) = \rho(b)\rho(w)$ if and only if

$$
    w_{11} - (M - M^{-1})w_{12} = 0.
$$

We call $w_{11} - (M - M^{-1})w_{12}$ the Riley polynomial of the group $G$. It describes the set of nonabelian representations of $G$ into $SL_2(\mathbb{C})$.

2.2. Lê’s approach. In this subsection we determine the universal $SL_2(\mathbb{C})$-character ring of the group $G$ defined in (2.1). As a consequence, we obtain another description of the set of nonabelian $SL_2(\mathbb{C})$-representations of $G$. We will follow Lê’s approach in [Le].

We first recall the definitions of the character variety and universal character ring of a group. The set of representations of a finitely generated group $H$ into $SL_2(\mathbb{C})$ is an algebraic set defined over $\mathbb{C}$, on which $SL_2(\mathbb{C})$ acts by conjugation. The set-theoretic quotient of the representation space by that action does not have good topological properties, because two representations with the same character may belong to different orbits
of that action. A better quotient, the algebro-geometric quotient denoted by $\chi(H)$ (see [CS, LM]), has the structure of an algebraic set. There is a bijection between $\chi(H)$ and the set of all characters of representations of $H$ into $SL_2(\mathbb{C})$, hence $\chi(H)$ is usually called the character variety of $H$.

The character variety of $H$ is determined by the traces of some fixed elements $h_1, \ldots, h_k$ in $H$. More precisely, we can find $h_1, \ldots, h_k$ in $H$ such that for every element $h$ in $H$ there exists a polynomial $P_h$ in $k$ variables such that for any representation $\rho : H \to SL_2(\mathbb{C})$ we have $\text{tr} \rho(h) = P_h(x_1, \ldots, x_k)$ where $x_i := \text{tr} \rho(h_i)$. The universal character ring of $H$ is defined to be the quotient of the polynomial ring $\mathbb{C}[x_1, \ldots, x_k]$ by the ideal generated by all expressions of the form $\text{tr} \rho(u) - \text{tr} \rho(v)$, where $u$ and $v$ are any two words in the letters $g_1, \ldots, g_k$ which are equal in $H$, see e.g. [BH]. The universal character ring of $H$ is actually independent of the choice of $h_1, \ldots, h_k$. The quotient of the universal character ring of $H$ by its nilradical is equal to the character ring of $H$, which is the ring of regular functions on the character variety $X(H)$.

Recall that $F_{a,b}$ is the free group on two letters $a$ and $b$. The character variety of $F_{a,b}$ is isomorphic to $\mathbb{C}^3$ by the Fricke-Klein-Vogt theorem, see [Fr, Vo]. It follows that for every word $u \in F_{a,b}$ there is a unique polynomial $P_u$ in 3 variables such that for any representation $\rho : F_{a,b} \to SL_2(\mathbb{C})$ we have $\text{tr} \rho(u) = P_u(x, x', y)$ where $x := \text{tr} \rho(a)$, $x' := \text{tr} \rho(b)$ and $y := \text{tr} \rho(ab)$.

**Lemma 2.2.** For $u \in F_{a,b}$ we have that $x - x'$ divides $\text{tr} \rho(u) - \text{tr} \rho(\tilde{u})$ in $\mathbb{C}[x, x', y]$.

**Proof.** Since $\tilde{u}$ is the word obtained from $u$ by replacing $a$ by $b^{-1}$ and $b$ by $a^{-1}$, we have

$$\text{tr} \rho(\tilde{u}) = P_u(tr \rho(b^{-1}), tr \rho(a^{-1}), tr \rho(b^{-1}a^{-1})) = P_u(x', x, y).$$

Hence $\text{tr} \rho(u) - \text{tr} \rho(\tilde{u}) = P_u(x, x', y) - P_u(x', x, y)$ is divisible by $x - x'$ in $\mathbb{C}[x, x', y]$. □

**Proposition 2.3.** [TR] Prop.2.1 Let $H = \langle a, b | u = v \rangle$, where $u, v \in F_{a,b}$. Then the universal character ring of $H$ is the quotient of the polynomial ring $\mathbb{C}[x, y, z]$ by the ideal generated by the four polynomials $\text{tr} \rho(u) - \text{tr} \rho(v)$, $\text{tr} \rho(ua^{-1}) - \text{tr} \rho(va^{-1})$, $\text{tr} \rho(ub^{-1}) - \text{tr} \rho(vb^{-1})$ and $\text{tr} \rho(uab^{-1}) - \text{tr} \rho(va^{-1}b^{-1})$.

Recall that $G = \langle a, b | wa = bw \rangle$ where $\tilde{w} = w^{-1}$. We now describe the universal character ring of $G$. In this case, since $a$ and $b = waw^{-4}$ are conjugate, we have $\text{tr} \rho(a) = \text{tr} \rho(b)$. This means that $x = x'$.

**Theorem 3.** The universal character ring of $G$ is the quotient of the polynomial ring $\mathbb{C}[x, y]$ by the principal ideal generated by the polynomial $\text{tr} \rho(bwa^{-1}) - \text{tr} \rho(w)$.

**Proof.** By Proposition 2.3 the universal character ring of $G$ is the quotient of the polynomial ring $\mathbb{C}[x, z]$ by the ideal generated by the four polynomials

$$\text{tr} \rho(wa) - \text{tr} \rho(bw),$$
$$\text{tr} \rho((wa)a^{-1}) - \text{tr} \rho((bw)a^{-1}) = \text{tr} \rho(w) - \text{tr} \rho(bwa^{-1}),$$
$$\text{tr} \rho((wa)b^{-1}) - \text{tr} \rho((bw)b^{-1}) = \text{tr} \rho(wab^{-1}) - \text{tr} \rho(w),$$
$$\text{tr} \rho((wa)a^{-1}b^{-1}) - \text{tr} \rho((bw)a^{-1}b^{-1}) = \text{tr} \rho(wb^{-1}) - \text{tr} \rho(wa^{-1}).$$
Since $x = x'$, by Lemma 2.2 we have $\text{tr } \rho(u) = \text{tr } \rho(\tilde{u})$ for all $u \in F_{a,b}$. Hence

$$\text{tr } \rho(wa) - \text{tr } \rho(bw) = \text{tr } \rho(\tilde{w}\tilde{a}) - \text{tr } \rho(bw) = \text{tr } \rho(\tilde{w}\tilde{a}) - \text{tr } \rho(bw) = \text{tr } (w^{-1}b^{-1}) - \text{tr } \rho(bw) = 0,$$

$$\text{tr } \rho(wab^{-1}) - \text{tr } \rho(w) = \text{tr } \rho(\tilde{w}ab^{-1}) - \text{tr } \rho(w) = \text{tr } \rho(\tilde{w}ab^{-1}) - \text{tr } \rho(w) = \text{tr } (w^{-1}b^{-1}a) - \text{tr } \rho(w) = \text{tr } (w^{-1}b^{-1}a) - \text{tr } \rho(w),$$

$$\text{tr } \rho(wb^{-1}) - \text{tr } \rho(wa^{-1}) = \text{tr } \rho(\tilde{w}b^{-1}) - \text{tr } \rho(\tilde{w}a) - \text{tr } \rho(wa^{-1}) = \text{tr } (\tilde{w}b^{-1}) - \text{tr } \rho(\tilde{w}a) - \text{tr } \rho(wa^{-1}) = \text{tr } (w^{-1}a) - \text{tr } \rho(wa^{-1}) = 0.$$ 

The theorem then follows. \hfill \Box

Now suppose $\rho : G \to \text{SL}_2(\mathbb{C})$ is a nonabelian representation of the form (2.2). Then

$$\text{tr } \rho(bwa^{-1}) - \text{tr } \rho(w) = (w_{11} + w_{22} - rw_{11} - M^{-1}w_{21} + rMw_{12}) - (w_{11} + w_{22}) = -r(w_{11} - Mw_{12} + M^{-1}r^{-1}w_{21}).$$

Hence, by Theorem 3 we have $\rho(wa) = \rho(bw)$ if and only if

$$w_{11} - (Mw_{12} - M^{-1}r^{-1}w_{21}) = 0.$$ 

We call $w_{11} - (Mw_{12} - M^{-1}r^{-1}w_{21})$ the Lê polynomial of $G$.

**Remark 2.4.** Since $w_{21} = rw_{12}$ (by Lemma 2.1), it is easy to see that the Lê equation is equal to the Riley equation.

### 2.3. Mednykh’s approach.

In this subsection we study the set of nonabelian $\text{SL}_2(\mathbb{C})$-representations of the group $G$ defined in (2.1), following Mednykh’s approach in [HMP].

**Lemma 2.5.** Suppose $C \in \text{SL}_2(\mathbb{C})$. Then $C^2 = -I$ if and only if $\text{tr } C = 0$.

**Proof.** By the Cayley-Hamilton theorem we have $C^2 - (\text{tr } C)C + I = 0$. It follows that $C^2 + I = 0$ if and only if $\text{tr } C = 0$. \hfill \Box

**Proposition 2.6.** Suppose $C \in \text{SL}_2(\mathbb{C})$ such that $C\rho(b) = \rho(a^{-1})C$ and $C^2 = -I$. Then for any word $u \in F_{a,b}$ we have

$$(2.4) \quad C\rho(u) = \rho(\tilde{u})C.$$

**Proof.** By taking the inverse of $C\rho(b) = \rho(a^{-1})C$ we get $C\rho(b^{-1}) = \rho(a)C$.

Since $C = -C^{-1}$ we have

$$C\rho(a) = -C^{-1}\rho(a) = -\rho(a^{-1}C)^{-1} = -(C\rho(b))^{-1} = -\rho(b)^{-1}C^{-1} = \rho(b^{-1})C.$$ 

By taking the inverse of $C\rho(a) = \rho(b^{-1})C$ we get $C\rho(a^{-1}) = \rho(b)C$. Hence

$$C\rho(u) = \rho(\tilde{u})C.$$ 

for $u \in \{a^\pm 1, b^\pm 1\}$.

We now prove (2.4) by induction on the length $\ell(u)$ of $u \in F_{a,b}$. If $\ell(u) = 1$, then $u \in \{a^\pm 1, b^\pm 1\}$. By the above arguments we have $C\rho(u) = \rho(\tilde{u})C$.

Suppose $\ell(u) > 1$. Then we can write $u = gv$ where $g \in \{a^\pm 1, b^\pm 1\}$ and $v \in F_{a,b}$ with $\ell(v) = \ell(u) - 1$. By induction hypothesis $C\rho(v) = \rho(\tilde{v})C$. We have

$$\rho(\tilde{u})C = \rho(\tilde{g})\rho(\tilde{v})C = \rho(\tilde{g})C\rho(v) = C\rho(g)\rho(v) = C\rho(u).$$ 

The proposition follows. \hfill \Box
Now consider \( G = \langle a, b \mid wa = bw \rangle \) where \( w \in F_{a,b} \) with \( \tilde{w} = w^{-1} \). Suppose \( \rho : F_{a,b} \to SL_2(\mathbb{C}) \) is a nonabelian representation of the form (2.2).

Let \( C = \begin{bmatrix} 0 & -1/\sqrt{r} \\ \sqrt{r} & 0 \end{bmatrix} \). Then it is easy to check that \( C \rho(b) = \rho(a^{-1})C \) and \( C^2 = -I \).

By Proposition 2.6 we have

\[
0 = C \rho(w) = \rho(w) C.
\]

Hence \( \rho(bwa^{-1}w^{-1}) = \rho(bwa^{-1}\tilde{w})C = -\rho(bwa^{-1})C \rho(w)C = -\rho(bw)C \rho(bw)C \).

We call \( w_{11} = (Mw_{12} - M^{-1}r^{-1}w_{21}) \) the Mednykh polynomial of \( G \). Note that it is exactly the Lê equation, which is equal to the Riley polynomial (by Lemma 2.1).

### 3. Proofs of Theorems 1 and 2

In this section we apply the result in Section 2 to calculate the Riley polynomial of the double twist knot \( J(k, 2n) \) and then we prove Theorems 1 and 2.

#### 3.1. The Riley polynomial

By [HS] the knot group of \( K = J(k, 2n) \) has a presentation \( \pi_1(X_K) = \langle a, b \mid w^n a = bw^n \rangle \), where \( a, b \) are meridians and

\[
w = \begin{cases} 
(ba^{-1})^m ba(b^{-1}a)^m, & \text{if } k = 2m + 1, \\
(ba^{-1})^m (b^{-1}a)^m, & \text{if } k = 2m.
\end{cases}
\]

Suppose \( \rho : \pi_1(X_K) \to SL_2(\mathbb{C}) \) is a nonabelian representation. Taking conjugation if necessary, we can assume that \( \rho \) has the form

\[
(3.1) \quad \rho(a) = \begin{bmatrix} M & 1 \\ 0 & M^{-1} \end{bmatrix} \quad \text{and} \quad \rho(b) = \begin{bmatrix} M & 0 \\ 2 - z & M^{-1} \end{bmatrix}
\]

where \((M, z) \in \mathbb{C}^* \times \mathbb{C}\) satisfies the matrix equation \( \rho(w^n a) = \rho(bw^n) \). Note that \( z = \text{tr } \rho(ab^{-1}) = M^2 + M^{-2} + 2 - \text{tr } \rho(ab) \).

The following lemmas are elementary, see e.g. [Tr2].

**Lemma 3.1.** We have

\[
S_j^2(\omega) - \omega S_j(\omega) S_{j-1}(\omega) + S_{j-1}^2(\omega) = 1.
\]

**Lemma 3.2.** Suppose \( V = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} \in SL_2(\mathbb{C}). \) Then

\[
V^j = \begin{bmatrix} S_j(v) - v_{22} S_{j-1}(v) & v_{12} S_{j-1}(v) \\ v_{21} S_{j-1}(v) & S_j(v) - v_{11} S_{j-1}(v) \end{bmatrix},
\]

where \( v := \text{tr } V = v_{11} + v_{22} \).
3.1.1. The case $k = 2m + 1$. In this case we have $w = (ba^{-1})^mba(b^{-1}a)^m$.

**Proposition 3.3.** We have $\rho(w) = \begin{bmatrix} w_{11} & w_{12} \\ (2-z)w_{12} & w_{22} \end{bmatrix}$ where
\[
w_{11} = M^2 S_m^2(z) - 2M^2 S_m(z)S_{m-1}(z) + (2 + M^2 - z)S_{m-1}^2(z),
\]
\[
w_{12} = (S_m(z) - S_{m-1}(z))(MS_m(z) - M^{-1}S_{m-1}(z)),
\]
\[
w_{22} = (M^{-2} + 2 - z)S_m^2(z) - 2M^{-2}S_m(z)S_{m-1}(z) + M^{-2}S_{m-1}^2(z).
\]

**Proof.** Since $\rho(ba^{-1}) = \begin{bmatrix} 1 \\ M^{-1}(2-z) \end{bmatrix}$, by Lemma 3.2 we have
\[
\rho((ba^{-1})^m) = \begin{bmatrix} S_m(z) - (z-1)S_{m-1}(z) & -MS_{m-1}(z) \\ M^{-1}(2-z)S_{m-1}(z) & S_m(z) - S_{m-1}(z) \end{bmatrix}.
\]
Similarly,
\[
\rho((b^{-1}a)^m) = \begin{bmatrix} S_m(z) - (z-1)S_{m-1}(z) & M^{-1}S_{m-1}(z) \\ M(z-2)S_{m-1}(z) & S_m(z) - S_{m-1}(z) \end{bmatrix}.
\]
Since $\rho(w) = \rho((ba^{-1})^mba(b^{-1}a)^m)$, the lemma follows by a direct calculation.

**Proposition 3.4.** The Riley polynomial of $J(2m + 1, 2n)$ is
\[
\Phi_{J(2m+1,2n)}(M, z) = S_n(t_m) - (1 - (z - M^2 - M^{-2})S_m(z)(S_m(z) - S_{m-1}(z)))S_{n-1}(t_m)
\]
where $t_m := \text{tr} \rho(w) = (M^2 + M^{-2} + 2 - z) - (z - 2)(z - M^2 - M^{-2})S_m(z)S_{m-1}(z)$.

**Proof.** By Proposition 3.3 we have
\[
t_m = w_{11} + w_{22} = (M^2 + M^{-2} + 2 - z)(S_m^2(z) + S_{m-1}^2(z)) - 2(M^2 + M^{-2})S_m(z)S_{m-1}(z).
\]
Since $S_m^2(z) + S_{m-1}^2(z) = 1 + zS_m(z)S_{m-1}(z)$ (by Lemma 3.1), we have
\[
t_m = (M^2 + M^{-2} + 2 - z)(1 + zS_m(z)S_{m-1}(z)) - 2(M^2 + M^{-2})S_m(z)S_{m-1}(z)
\]
\[
= (M^2 + M^{-2} + 2 - z) - (z - 2)(z - M^2 - M^{-2})S_m(z)S_{m-1}(z).
\]
Since $\rho(w) = \begin{bmatrix} w_{11} & w_{12} \\ (2-z)w_{12} & w_{22} \end{bmatrix}$, by Lemma 3.2 we have
\[
\rho(w^n) = \begin{bmatrix} S_n(t_m) - w_{22} S_{n-1}(t_m) & w_{12} S_{n-1}(t_m) \\ (2-z)w_{12} S_{n-1}(t_m) & S_n(t_m) - w_{11} S_{n-1}(t_m) \end{bmatrix}.
\]
Hence the Riley polynomial is
\[
\Phi_K(M, z) = S_n(t_m) - w_{22} S_{n-1}(t_m) - (M - M^{-1})w_{12} S_{n-1}(t_m)
\]
\[
= S_n(t_m) - (w_{22} - (M - M^{-1})w_{12}) S_{n-1}(t_m).
\]
Since $S_m^2(z) + S_{m-1}^2(z) - zS_m(z)S_{m-1}(z) = 1$ we have
\[
w_{22} + (M - M^{-1})w_{12}
\]
\[
= (1 + M^2 + M^{-2} - z)S_m^2(z) - (M^2 + M^{-2})S_m(z)S_{m-1}(z) + S_{m-1}^2(z)
\]
\[
= 1 - (z - M^2 - M^{-2})S_m(z)(S_m(z) - S_{m-1}(z)).
\]
The formula for $\Phi_K(M, z)$ then follows. \qed
3.1.2. The case $k = 2m$. In this case we have $w = (ba^{-1})^m(b^{-1}a)^m$.

**Proposition 3.5.** We have $\rho(w) = \begin{bmatrix} w_{11} & w_{12} \\ (2 - z)w_{12} & w_{22} \end{bmatrix}$ where

$$
\begin{align*}
w_{11} &= S_m^2(z) + (2 - 2z)S_m(z)S_{m-1}(z) + (1 + 2M^2 - 2z - M^2z + z^2)S_{m-1}^2(z), \\
w_{12} &= (M^{-1} - M)S_m(z)S_{m-1}(z) + (M^{-1} + M - M^{-1}z)S_{m-1}^2(z), \\
w_{22} &= S_m^2(z) - 2S_m(z)S_{m-1}(z) + (1 + 2M^{-2} - M^{-2}z)S_{m-1}^2(z).
\end{align*}
$$

**Proposition 3.6.** The Riley polynomial of $J(2m, 2n)$ is

$$
\Phi_{J(2m, 2n)}(M, z) = S_n(\bar{t}_m) - \left(1 + (z - M^2 - M^{-2})S_{m-1}(z)\left(S_m(z) - S_{m-1}(z)\right)\right)S_{n-1}(\bar{t}_m)
$$

where $\bar{t}_m := \text{tr} \rho(w) = 2 + (z - 2)(z - M^2 - M^{-2})S_{m-1}^2(z)$.

**Remark 3.7.** Similar formulae for the Riley polynomial of $J(2m + 1, 2n)$ and $J(2m, 2n)$ have already been obtained in [MPL, MT].

3.2. The canonical longitude. Recall that $X_K$ is the complement of the knot $K$. The boundary of $X_K$ is a torus $\mathbb{T}^2$. There is a standard choice of a meridian $\mu$ and a longitude $\lambda$ on $\mathbb{T}^2$ such that the linking number between the longitude and the knot is zero. We call $\lambda$ the canonical longitude of $K$ corresponding to the meridian $\mu$.

Let $\mu = a$ be the meridian of $K = J(k, 2n)$ and $\lambda$ the canonical longitude corresponding to $\mu$. Then we have

$$
\lambda = \begin{cases} 
\bar{w}^nw^n\mu^{-4n}, & \text{if } k = 2m + 1, \\
\bar{w}^nw^n, & \text{if } k = 2m,
\end{cases}
$$

where $\bar{w}$ is the word in the letters $a, b$ obtained by writing $w$ in the reversed order. With the representation $\rho$ in (3.1), by [HS] we have $\rho(\lambda) = \begin{bmatrix} L & * \\ 0 & L^{-1} \end{bmatrix}$, where $L = -\bar{w}_{12}/w_{12}$.

Here $\bar{w}_{ij}$ is obtained from $w_{ij}$ by replacing $M$ by $M^{-1}$.

From Propositions 3.3 and 3.5 we have the following.

**Proposition 3.8.** (i) If $k = 2m + 1$ then

$$
L = -M^{-4n} \frac{M^{-1}S_m(z) - MS_{m-1}(z)}{MS_m(z) - M^{-1}S_{m-1}(z)}.
$$

(ii) If $k = 2m$ then

$$
L = -\frac{M^{-1}(S_m(z) - S_{m-1}(z)) - M(S_m(z) - S_{m-2}(z))}{M(S_m(z) - S_{m-1}(z)) - M^{-1}(S_{m-1}(z) - S_{m-2}(z))}.
$$

3.3. Proof of Theorems 1 and 2. We begin with a simple lemma.

**Lemma 3.9.** Suppose $z_1, z_2 \in \mathbb{C}$ and $\omega \in \mathbb{R}$. Then

$$
|z_1 - e^{i\omega}z_2|^2 = |z_1|^2 + |z_2|^2 - 2 \text{Re}(z_1\overline{z_2}) \cos \omega - 2 \text{Re}(z_1\overline{z_2}) \sin \omega.
$$

Hence $|z_1 - e^{i\omega}z_2| \geq |z_1 - e^{-i\omega}z_2|$ if and only if $\text{Im}(z_1\overline{z_2}) \sin \omega \leq 0$. Moreover, $|z_1 - e^{i\omega}z_2| = |z_1 - e^{-i\omega}z_2|$ if and only if $\text{Im}(z_1\overline{z_2}) \sin \omega = 0$. 
For a hyperbolic two-bridge knot $K$, by [HLM] [Kol] [Po] [PW] there exists an angle $\alpha_K \in \left(\frac{2\pi}{3}, \pi\right)$ such that $X_K(\alpha)$ is hyperbolic for $\alpha \in (0, \alpha_K)$, Euclidean for $\alpha = \alpha_K$, and spherical for $\alpha \in (\alpha_K, \pi)$.

We first consider the case $K = J(2m+1, 2n)$. For $\alpha \in (0, \alpha_K)$, by the Schlafli formula the volume of a hyperbolic cone-manifold $X_K(\alpha)$ is given by

$$\text{Vol}(X_K(\alpha)) = \int_{\alpha}^{\alpha_K} \frac{\ell_\omega}{2} d\omega,$$

where $\ell_\omega$ is the real length of the longitude of the cone-manifold $X_K(\omega)$. See e.g. [HMP].

For each $\alpha \leq \omega \leq \alpha_K$, $\ell_\omega$ is calculated as follows. Suppose $\rho : \pi_1(X_K) \to SL_2(\mathbb{C})$ is a nonabelian representation of the form (3.1), where $M = e^{i\omega/2}$ and $z$ is a zero of the Riley polynomial $\Phi_K(M, z)$. The complex length of a canonical longitude $\lambda$ of $K$ is the complex number $\gamma_\lambda$ module $2\pi \mathbb{Z}$ satisfying

$$\text{tr} \rho(\lambda) = 2 \cosh \frac{\gamma_\lambda}{2}.$$

Then $\ell_\omega = |\text{Re}(\gamma_\lambda)|$. By Proposition 3.8 we have $\rho(\lambda) = \begin{bmatrix} L & * \\ 0 & L^{-1} \end{bmatrix}$ where

$$L = -M^{-4n} \frac{M^{-1}S_m(z) - MS_{m-1}(z)}{MS_m(z) - M^{-1}S_{m-1}(z)}.$$

For the volume, we can either choose $|L| \geq 1$ or $|L| \leq 1$. We choose $L$ with $|L| \geq 1$. Since $M = e^{i\omega/2}$, by Lemma 3.9 we have $\text{Im}(S_m(z)S_{m-1}(z)) \leq 0$. Moreover $|L| = 1$ if and only if $\text{Im}(S_m(z)S_{m-1}(z)) = 0$.

Since $\ell_\omega = |\text{Re}(\gamma_\lambda)| = 2 \log |L|$ we have

$$\text{Vol}(X_{J(2m+1, 2n)}(\alpha)) = \int_{\alpha}^{\alpha_K} \frac{\ell_\omega}{2} d\omega = \int_{\alpha}^{\alpha_K} \log |L| d\omega.$$

For $\alpha_K < \alpha \leq \pi$, by [PW] Prop.6.4 all the characters are real. In particular $z = \text{tr} (ab^{-1}) \in \mathbb{R}$, and hence $|L| = 1$, for $\alpha_K < \alpha \leq \pi$. Hence

$$\text{Vol}(X_{J(2m+1, 2n)}(\alpha)) = \int_{\alpha}^{\pi} \log |L| d\omega = \int_{\alpha}^{\pi} \log \left| \frac{S_m(z) - M^2S_{m-1}(z)}{M^2S_m(z) - S_{m-1}(z)} \right| d\omega.$$

This completes the proof of Theorem 1. The proof of Theorem 2 is similar. In that case we apply Propositions 3.5, 3.6 and the fact that

$$|L| = \left| \frac{(S_m(z) - S_{m-1}(z)) - M^2(S_{m-1}(z) - S_{m-2}(z))}{M^2(S_m(z) - S_{m-1}(z)) - (S_{m-1}(z) - S_{m-2}(z))} \right| \geq 1$$

if and only if $\text{Im} \left( \frac{S_m(z) - S_{m-1}(z)}{S_{m-1}(z) - S_{m-2}(z)} \right) \leq 0$.

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