Elements of information-theoretic derivation of the formalism of quantum theory.

A. GRINBAUM
CREA, Ecole Polytechnique,
1 rue Descartes 75005 Paris, France
E-mail: grinbaum@poly.polytechnique.fr

Information-theoretic derivations of the formalism of quantum theory have recently attracted much attention. We analyze the axioms underlying a few such derivations and propose a conceptual framework in which, by combining several approaches, one can retrieve more of the conventional quantum formalism.

1 Strategy

Initially formulated by John Wheeler 1,2, the program of deriving quantum formalism from information-theoretic principles has been receiving lately much attention. Thus, Jozsa 3 promotes a viewpoint which "attempts to place a notion of information at a primary fundamental level in the formulation of quantum physics". Fuchs 4 presents his program as follows: "The task is not to make sense of the quantum axioms by heaping more structure, more definitions... on top of them, but to throw them away wholesale and start afresh. We should be relentless in asking ourselves: From what deep physical principles might we derive this exquisite mathematical structure?.. I myself see no alternative but to contemplate deep and hard the tasks, the techniques, and the implications of quantum information theory."

In a similar fashion, Rovelli 5 distinguishes a philosophical problem of interpretation from a mathematical problem of derivation of quantum mechanical formalism from the first principles. He writes, "... quantum mechanics will cease to look puzzling only when we will be able to derive the formalism of the theory from a set of simple physical assertions ('postulates', 'principles') about the world. Therefore, we should not try to append a reasonable interpretation to the quantum mechanics formalism, but rather to derive the formalism from a set of experimentally motivated postulates". Rovelli refers to his own work as a point of view and not as interpretation: "From the point of view discussed here, Bohr's interpretation, consistent histories interpretations, as well as many worlds interpretation, are all correct". Rovelli's point of view, i.e. informational treatment of quantum mechanics, thus serves a formal criterion or a filter that permeates certain interpretations and not others. In other words,
treatment of quantum mechanics on information-theoretic grounds entails that some interpretations of quantum theory will be with certainty inapplicable but a number of other interpretations will all remain possible. Such a result can be naturally expected from any novel formal development of quantum theory that remains in the area of science as opposed to philosophy.

2 Axioms

2.1 The choice of axioms

Any formal derivation of quantum mechanics, in particular those using Bayesian methods and quite promising for someone who believes in information-theoretic foundations of physics, requires a definite conceptual background on which such a derivation will further operate. As it is often the case, to give a rigorous axiomatic system that could provide the necessary background, is a difficult task. Below we analyze some three proposed solutions, by Rovelli, by Fuchs, and by Brukner and Zeilinger.

Elementary act of measurement is understood by Rovelli as yes-no question. Brukner and Zeilinger use the term "proposition" which generalizes the notion of binary question. Still, if one looks into where from the term "proposition" appears, one finds in two formulations of Zeilinger’s fundamental principle for quantum mechanics:

FP1 An elementary system represents the truth value of one proposition.

FP2 An elementary system carries one bit of information.

It seems that Zeilinger’s choice of these two principles strongly suggests that the following phrase in BZ reflects the view of the authors on the fundamental issue and thus puts them very close to Rovelli’s position: "Yes-no alternatives are representatives of basic fundamental units of all systems."

Fuchs starts directly with the Hilbert space and the full structure of quantum mechanics. He describes measurements not by projectors but by positive operator-valued measures. This allows one to think that he will not agree with a definition of primitive measurements as consisting of exclusive yes-no alternatives, where the word "exclusive" leads to mathematically representing yes-no questions as orthogonal projectors. Still, Fuchs mentions some of the basic assumptions that he makes in his derivation.

Rovelli and BZ each then pose two axioms.

Axiom 1:
• Rovelli: "There is a maximum amount of relevant information that can be extracted from a system."
• Fuchs: Doesn’t follow the axiomatic approach; states that "There is maximal information about a system."
• Brukner and Zeilinger: "The information content of a quantum system is finite."

Axiom 2 :

• Rovelli: "It is always possible to acquire new information about a system."
• Fuchs: Doesn’t follow the axiomatic approach; states that "There will always be questions that we can ask of a system for which we cannot predict the outcomes."
• Brukner and Zeilinger: Introduce the notion of total information content of the system; state that there exist mutually complementary propositions; state that total information content of the system is invariant under a change of the set of mutually complementary propositions.

In spite of a quite striking analogy between the axioms chosen by different authors, as for the following derivation of quantum mechanics, they do not proceed in the same manner. We shall now have a closer look at the axioms and derivation techniques.

2.2 Discussion of the axioms

Axiom 1 marks a crucial point of departure from classical physics. Newtonian physics employs mathematics of continuum to represent the world and, therefore, any calculation of complete information about, say, a particle position would require an infinitely long computation. This fact has profoundly disturbed many physicists, with most prominently Feynman saying, "It always bothers me that, according to the laws as we understand them today, it takes a computing machine an infinite number of logical operations to figure out what goes on in no matter how tiny a region of space and no matter how tiny a region of time,... why should it take an infinite amount of logic to figure out what one tiny piece of space-time is going to do?" Axiom 1 also goes in line with Wheeler’s "no continuum" principle.

While it seems intuitively plausible to accept Axiom 1, its interpretation is not straightforward. Each author imposes his own interpretation by choosing a
suitable translation into his language; for Rovelli it means that there exist complete sets of yes-no questions that could provide one, abstractly, with complete information about a system; for Brukner and Zeilinger it means that everything that is there to a system is represented by a complete set of mutually complementary propositions.

However, it appears that Axiom 1 can raise yet a different issue. Our intuition is that essential finiteness applies, not to the system to which we address yes-no questions but to the system plus the observer who asks these questions. Formal development of this idea will appear in section 3. Philosophical argument goes as follows: it is not true, that in order to know a Newtonian coordinate we need the knowledge of infinitely many decimal digits. The latter should not make us worry, for we are endowed with an ability to create a special code (a new concept), which will substitute in the thinking the undesired infinity. The same works for computation: Feynman’s argument from the infinity of logical operations must include the possibility of “hiding” Newtonian infinities under the ”and so forth” concept, for which one may specify operational rules. Consequently, the requirement of finiteness applies to the observer-observed system. In a similar manner, in BZ terms, essential finiteness applies to system plus the one who chooses the propositions to be tested on the system, i.e. the observer. We do not have the intuition that ”one cannot know the infinity” but, rather, that one cannot have infinite knowledge.

How is this understanding of the finiteness axiom related to the one adopted by Rovelli and other authors? With the assumption of universality of quantum theory, one can deduce the old point of view. Indeed, universality allows us to treat the border line between theory and meta-theory in quantum mechanics as flexible. Any given observer (meta-theoretical entity) can be included in the theory proper by taking the point of view of an observer external to the one in question. If the amount of information remains finite in spite of the arbitrary choice of the frontier between the system and the observer, we can eliminate from consideration any previously given observer at the price of redefining the question-asking party. Imagine, for example, a computer solving in Maple software some field theory renormalization problem. Renormalization is about removing infinities, so if a computer were let to solve this task without conceptualizing infinities by means of a previously learned renormalization technique, it would have never arrived at any result. Abstract amount of information in the system is infinite in this example, but the amount of relevant information is finite. What is relevant, decides the observer who translates relevancy into concepts that he employs for the computation on a system, or equivalently into a specific manner to ask some yes-no questions and not others (see section 3.5).
Going to the extreme of definitions, what is information and what is not decides the observer, and it is because of this that the amount of information is finite. Had we had the liberty to call information anything we want, there would be no intuitively clear argument showing why this "anything" must be finite. Finiteness thus has to include the observer, and thanks to the universality of quantum mechanics can be in the limit reduced to finiteness in the sense of Rovelli.

Axiom 2 beautifully corresponds to Wheeler’s dictum (adopted after Philip Anderson) "More is different". If one wants to get more information, this will be different information; or one can always get more information, and it will be different information. Though in the original "More is different" was used in the context of complexity theory, it can as well, as a basic principle, apply to information-based quantum theory. At it will be seen below, for Rovelli Axiom 2 allows to introduce probabilities and to deduce an analogue of the Born rule; for BZ Axiom 2 leads to imposing a certain structure on the information space. This axiom is responsible for the departure from classicality, which is not yet fully accommodated by Axiom 1 alone.

2.3 Possible development

Fuchs uses as a priori given the structure of Hilbert space; his task is to deduce some of the operational structure of quantum mechanics, namely, density matrices. Rovelli, on the contrary, is interested in deducing quantum mechanics from the axioms and does not show a way to deduce most of further structure, to start with the superposition principle (apart from introducing it as an axiom). Fuchs uses a decision-theoretic (Bayesian) approach to derive the superposition principle. He refers to Rovelli’s paper in his own, and one is left free to suggest that many of his axiomatic assumptions, on which he doesn’t clearly comment, might be similar to Rovelli’s ones, apart from the key issue of how to define measurement. Indeed, Fuchs insists on the fundamental character of positive operator-valued measures. This may not seem intuitively evident. But because Fuchs leaves the axiomatic foundations of the Bayesian approach open, even if we dismiss the necessity to define measurement as POVM, there still remains an opportunity to introduce the latter in the theory. POVM have a natural description as conventional von Neumann measurements on an ancilla system and thus to Rovelli’s axiomatic derivation of the Hilbert space structure one may try to add an account of inevitability of ancilla systems and naturally obtain from this the needed POVM description. This will be attempted in section

Brukner and Zeilinger proceed differently. If information is primary, they
argue, then any formalism should be a formalism dealing with information and not with some other notions. Therefore BZ construct an information space where they apply the axioms and use the formalism to deduce testable predictions. BZ do not refer to physical space or to Hilbert space in their construction. Thus they do not have access to algebra allowing a reconstruction of the state space out of the operator Hilbert space. Therefore, because of this change of scenery, they are bound to postulate more properties of mutually complementary propositions than Rovelli or Fuchs. Namely, they postulate the homogeneity of parameter space. BZ’s self-imposed terminological limitation to abstract information space does not seem viable for philosophical, i.e. extra-scientific, reasons: it renders the formalism less transparent in use, while introduction of supplementary axioms does not make it conceptually clearer than traditional formalisms.

To continue, the question is how to extract a useful approach from the juxtaposition of Rovelli’s and Fuchs’s proposals. Rovelli, as said before, shows a way to construct the Hilbert space structure from two axioms. Unlike Brukner and Zeilinger, we do not call this Hilbert space information space but simply physical space, for there is no other space in the whole construction that would be the physical space. BZ’s information space is what the physical space is\footnote{Our use of the verb to be does not imply that we hold any form of realism. We merely refer here to common usage of the term physical space.}.

Next, with Hilbert space in hand, we use Fuchs’s derivation based on Gleason’s theorem to deduce density matrices and their properties. This requires essentially one more step: we need to introduce POVM as measurements, as discussed above. Once we’ve introduced ancilla systems, we can operationally redefine measurement as described by POVM.

To move further in combining Rovelli’s and Fuchs’s proposals, after the axiomatic stage, we either need a sort of decision-theoretic approach to derive the formal consequences of the necessary intersubjective accord of measurement results (for Fuchs, for example, via a version of the de Finetti theorem), or we need to use an algebraic approach so that the constructed Hilbert space be treated as space of operators corresponding to observables. This latter option will be investigated below.

3 Reconstruction of the quantum formalism

3.1 Key metaphor

We are guided by the computer metaphor. Indeed, the strategic task is to give a reformulation of quantum theory in information-theoretic terms. A theory that operates with the notion of information can be compared to software
as opposed to a theory that operates with the notion of energy which can be compared to hardware. Ideally one would wish to see all "hardware" or energetic language disappear from the formulation of the theory, so that only "software" or informational language remain.

3.2 I-observer and P-observer

We are usually interested in information about (knowledge of) the chosen system and we disregard particular ways in which we have obtained this information. All that counts is knowledge that can be useful in future or, in other words, relevant knowledge or relevant information. This is why one usually does not pay attention to the very process of interaction between the system being measured and the measuring system, and one treats measuring system as a meta-theoretic, i.e. non-physical, apparatus. To give an example, for some experiment a physicist may need to know the proton mass but he will not at all be interested in how this quantity was measured, unless he is a narrow specialist whose interest is in measuring particle masses. Particular ways to gain knowledge are irrelevant, while knowledge itself is highly relevant and useful. Some of the experiments where one is interested in the measurement as a physical process are discussed in 14. From now on we assume that measurement details are irrelevant, perhaps at the price of redefining what is measurement.

In a practical setting, though, information is always physical. This is to say that there always is some physical support of information, some hardware. The necessity of the physical support requires that we proceed in the following manner: first, treat the measurement interaction as physical; then, disregard the fact that it was physical and reformulate the theory in terms of measurement results only.

To start, make a distinction between two parts of the world: quantum system \( S \), which is the system of interest, and the observer. The observer, in the spirit of the software-hardware metaphor, consists of an informational agent ("I-observer") and of the physical realization of the observer ("P-observer"). There is no I-observer without P-observer. Reciprocally, there is no sense in calling P-observer an observer unless there is I-observer (otherwise P-observer is just a physical system as any). Hence, the two components of the "larger observer" are not in any way separate or orthogonal to each other; on the contrary, these are merely two viewpoints, and the difference is but descriptive.

3.3 Hilbert space

P-observer interacts with the quantum system and thus provides for the physical basis of measurement. I-observer is only interested in the measurement
result, i.e. information per se, and he gets information by reading it from P-
observer. The act of reading or getting information is here a common linguistic
expression and not a physical process since I-observer and P-observer are not
physically distinct. In fact, the concept of "being physical" only applies to P-
observer, and by definition the physical content of the "larger observer" is all
contained in P-observer. I-observer as informational agent is meta-theoretic,
and hence the fact that its interaction with P-observer, or the act of "read-
ing information", is unphysical. To give a mathematical meaning to this act,
we assume that getting information is described as yes-no questions asked by
I-observer to P-observer.

To follow Rovelli’s construction [4], the set of questions will be denoted
$W(P) = \{Q_i, i \in I\}$. According to Axiom 1, there is a finite number $N$ that
characterizes P-observer’s informational capacity. The number of questions in
$I$, though, can be much larger than $N$, as some of these questions are not
independent. In particular, they may be related by implication ($Q_1 \Rightarrow Q_2$),
union ($Q_3 = Q_1 \lor Q_2$), and intersection ($Q_3 = Q_1 \land Q_2$). One can define an
always false ($Q_0$) and an always true question ($Q_\infty$), negation of a question
($\neg Q$), and a notion of orthogonality as follows: if $Q_1 \Rightarrow \neg Q_2$, then $Q_1$ and
$Q_2$ are orthogonal ($Q_1 \perp Q_2$). Equipped with these structures, and under the
non-trivial assumption that union and intersection are defined for every pair
of questions, $W(P)$ is an orthomodular lattice.

One needs to make a few more steps to obtain the Hilbert space structure.
As follows from Axiom 1, one can select in $W(P)$ a set $c$ of $N$ questions that are
independent from each other. In the general case, there exist many such sets
$c, d, \text{etc.}$ If I-observer asks the $N$ questions in the family $c$ then the obtained
answers form a string
\[
\mathbf{s}_c = [e_1, \ldots, e_N]_c.
\]
This string represents the "raw" information that I-observer got from P-observer
as a result of asking the questions in $c$. Note that this is not yet information
about the quantum system $S$ that the I-observer ultimately wants to have, but
only a process due to functional separation within the "larger observer".

The string $s_c$ can take $2^N$ values and, since these outcomes are by con-
struction mutually exclusive, we can define new questions $Q_c^{(1)} \ldots Q_c^{(2^N)}$ such
that the yes answer to $Q_c^{(i)}$ corresponds to the string of answers $s_c^{(i)}$:
\[
Q_c^{(i)} = \neg Q_1 \land \neg Q_2 \land \ldots \land \neg Q_{N-i+1} \land Q_{N-i+2} \land \ldots \land Q_N.
\]
To these questions we refer as to "complete questions". By taking all possible
unions of sets of complete questions $Q_c^{(i)}$ of the same family $c$ one constructs
a Boolean algebra that has $Q_c^{(i)}$ as atoms.
Alternatively, one can consider a different family $d$ of $N$ independent yes-no questions and obtain another Boolean algebra with different complete questions as atoms. It follows, then, from Axiom 1 that the set of questions $W(P)$ that can be asked to P-observer is algebraically an orthomodular lattice containing subsets that form Boolean algebras. This is precisely the algebraic structure formed by the family of linear subsets of Hilbert space.

It is interesting to note that in approaches that start with an abstract $C^*$-algebra of operators one needs to use the Gelfand-Naimark-Segal construction to obtain a representation of this algebra as algebra of operators on a Hilbert space. In the present approach, information-theoretic axioms are evoked to obtain a similar result, namely, to show that operators form a Hilbert space.

3.4 Born rule

From the second Rovelli’s axiom it follows immediately that there are questions such as answers to these questions are not determined by $s_c$. Define, in general, as $p(Q, Q_c^{(i)})$ the probability that a yes answer to $Q$ will follow from the string $s_c^{(i)}$. Given two complete strings of answers $s_c$ and $s_b$, we can then consider the probabilities

$$p^{ij} = p(Q_b^{(i)}, Q_c^{(j)}).$$

From the way it is defined, the $2^N \times 2^N$ matrix $p^{ij}$ cannot be completely arbitrary. First, we must have

$$0 \leq p^{ij} \leq 1.$$

Then, if information $s_c^{(j)}$ is available about the system, one and only one of the outcomes $s_b^{(i)}$ may result. Therefore

$$\sum_i p^{ij} = 1.$$

If we assume that $p(Q_b^{(i)}, Q_c^{(j)}) = p(Q_c^{(j)}, Q_b^{(i)})$ then we also get

$$\sum_j p^{ij} = 1.$$

If pursued further in an attempt to deduce probability amplitudes, this derivation, however, encounters some difficulties. To get the result, Rovelli

\[\text{bThis introduction of probabilities does not yet commit one to any particular view on what probabilities are. Personally, the author believes in the transcendental deduction of the structure of probabilities}\]
postulates explicitly the superposition principle. We, too, introduce a new assumption to obtain more of the structure of quantum theory. Namely, we postulate non-contextuality and use Gleason’s theorem to deduce density matrices. It remains an open question if non-contextuality as an intuitively made assumption is welcome or must be rejected as too strong. In mathematical terms, it states that probabilities can be defined for a projector independently of the family of projectors of which it is a member, or that in \( p(Q_b^{(i)}, Q_c^{(j)}) \) with fixed \( Q_b^{(i)} \) probability will be the same had the fixed question belonged not to the family \( b \) but to some other family \( d \). One can then prove a theorem due to Gleason:

**Theorem (Gleason)** Let \( f \) be any function from 1-dimensional projections on a Hilbert space of dimension \( d > 2 \) to the unit interval, such that for each resolution of the identity \( \{\Pi_k\}, k = 1 \ldots d, \sum_{k=1}^{d} \Pi_k = I, \sum_{k=1}^{d} f(\Pi_k) = 1 \). Then there exists a unique density matrix \( \rho \) such that \( f(\Pi_k) = \text{Tr}(\rho \Pi_k) \).

### 3.5 Unitary dynamics

One last step before we move to quantum theory of the system \( S \) is to obtain unitary dynamics. Following Rovelli, any question can be labelled by the time variable \( t \) indicating the time at which it is asked. Denote as \( t \to Q(t) \) the one-parameter family of questions defined by the same procedure performed at different times. Assume that time evolution is a symmetry in the theory. In the context of our approach the latter word ”theory” includes theory of P-observer and of the quantum system \( S \). Then recall that the set \( W(P) \) has the structure of a set of linear subspaces in the Hilbert space, and the set of all questions at time \( t_2 \) to the P-observer part of the physically interacting conjunction of two systems, must be isomorphic to the set of all questions at time \( t_1 \). Therefore, the corresponding family of linear subspaces must have the same structure; it follows that there must be a unitary transformation \( U(t_2 - t_1) \) such that

\[
Q(t_2) = U(t_2 - t_1)Q(t_1)U^{-1}(t_2 - t_1).
\]

It is straightforward to see that these unitary matrices form an abelian group and \( U(t_2 - t_1) = \exp[-i(t_2 - t_1)H] \), where \( H \) is a self-adjoint operator in the Hilbert space, the Hamiltonian.

In a practical setting, it is from the past or the future of a given experiment, in particular from the intentions of the experimenter, that one can learn which information about the experiment is relevant and which is not. What is relevant can either be encoded in the preparation of the experiment or selected by the experimenter a posteriori. In all cases, the notion of relevance does not enter into the formalism which solely describes the measurement within the
context of the experiment. All that is "allowed to be known" inside the formal framework is that there (a) is (b) some relevant information. What is relevant is reflected in the choice of questions that are asked by I-observer.

Interaction between P-observer and the quantum system should be viewed as physical interaction between just any two physical systems. Still, because I-observer then reads information from P-observer and because we aren’t interested in the posteriority of relations between P-observer and the quantum system, we can treat P-observer as an ancillary system in course of its interaction with $S$. Such an ancillary system would have interacted with $S$ and then would be subject to a standard measurement described mathematically on its Hilbert space via a set of orthogonal yes-no projection operators.

So far, for P-observer we have the Hilbert space and the standard Born rule. The fact that P-observer is treated as ancillary system allows to transfer some of this structure on the quantum system $S$. A new non-trivial assumption has to be made, that the time dynamics that has previously arisen in the context of P-observer alone, also applies to the I-observer and to $S$. In other words, there is only one time in the system. Time of I-observer is the one in which one can grasp the meaning of the words "past" and "future" as used above in relation with the experimental setting and the notion of relevant information: it is in this time that there is a "before the experiment" and an "after the experiment". Times of physical systems, such as $S$ or P-observer, are times in which their dynamics takes place.

Now, both the physical interaction of P-observer with $S$ and the process of asking questions by I-observer to P-observer take place in one and the same time. Since (a) until I-observer asks the question that he chooses to ask, sets of questions at different times are isomorphic and evolution is unitary, and (b) time at which I-observer asks the question only depends on I-observer and considerations of relevance that must not enter into the formalism, then one concludes that the interaction between the quantum system and P-observer must respect the unitary character all until the decoupling of the ancilla. Now write,

$$\rho_{SP} \rightarrow U \rho_{SP} U^\dagger.$$ 

After asking a question corresponding to a projector $\Pi_b$, probability of the yes answer will be given by

$$P(b) = Tr \left( U(\rho_S \otimes \rho_P) U^\dagger(I \otimes \Pi_b) \right).$$

Because the systems decouple, trace can be decomposed into

$$P(b) = Tr_S(\rho_S E_b),$$
where all presence of the ancilla is hidden in the operator

$$E_b = \text{Tr} \left( (I \otimes \rho_p) U (I \otimes \Pi_b) U^\dagger \right),$$

which acts on the quantum system $S$ alone. This operator is positive-semi-

4 Conclusion

We have shown how to obtain a description of quantum measurement via
POVM at the condition of disregarding completely the physical interaction
during measurement and the existence of P-observer. If one is only interested in
a formal description of how I-observer acquires information about the quantum
system $S$, this is done via POVM and the Born rule following from Gleason’s
theorem. To be mentioned here, Gleason’s theorem also admits a generalization
from von Neumann’s orthogonal projector measures to POVM. One gets
therefore a description of measurement as used in quantum information theory,
and one can now continue the development of the theory in the conventional way. In agreement with the intuition expressed in the key metaphor, all
"hardware" language is eliminated and the theory can be formulated in the
"software" language alone.

Formal deduction of the results concerning the Hilbert space, however,
was not completely rigorous. Rovelli acknowledges it in his disclaimer, "I
do not claim any mathematical nor philosophical rigor". Indeed, the fact that
yes-no questions form an orthomodular lattice containing subsets that form
Boolean algebras only commits one to the structure of union of Hilbert spaces
and not of a single Hilbert space. Thus, this can happen to be the union
of primitive Hilbert spaces, which allow for a classical and not a quantum
interpretation. Generally speaking, the structure will be the one of the Hilbert
space with superselection rules. One needs then to use Axiom 2 to show that
the possibility to ask in every situation some new informative question excludes
classicality. Completion of this program remains an open problem.

Introduction of space and time "by hand" in any algebraic approach to
quantum mechanics is certainly quite unsatisfactory. One would wish to see
how space and time arise naturally from the formalism. This may require
a fully rigorous algebraic approach involving von Neumann algebras and the
GNS construction for the Hilbert space. However, the author is only aware of
one way to introduce time in this framework. This leaves open the question
of link between time and space, and of the possibility to use the two notions together to obtain the evolution equation.

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