Pulse encoding for ZTE imaging:
RF excitation without dead time penalty

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A. Signal models and image reconstruction in algebraic ZTE MRI

ZTE with hard pulse

The time-domain signal obtained for one radial readout in ZTE MRI with hard-pulse excitation can be described by

\[ s_0(t) = \int_{\text{FOV}} \rho(r) e^{j\gamma G r t} \, dr \]  

(1)

where encoding is performed with constant gradient strength \( G \), \( r \) is the spatial coordinate along the direction of the gradient, \( \rho(r) \) is the projection of the imaged object onto this direction, the integration limited to the FOV indicates finite support of the object (i.e. that all signal is assumed stemming from inside this range), and transverse relaxation is ignored \((1,2)\). Signal corresponding to negative times is obtained by inverting \( G \), which is omitted throughout the present description for convenience.

To enable using Fourier transform for writing Eq. 1, replacement with either \( \omega = \gamma G r \) or \( k = \gamma G t \) is commonly used, leading to either

\[ s_0(t) = \int_{\text{BW}} \rho(\omega) e^{j\omega t} \, d\omega = iFT[\rho(\omega)](t) \]  

(2)

or

\[ s_0(k) = \int_{\text{FOV}} \rho(r) e^{jkr} \, dr = iFT[\rho(r)](k) \]  

(3)

with the equivalent Fourier pairs in time-frequency domain or k-real space\(^1\).

ZTE with sweep pulse

In sweep ZTE \((2)\), the extended RF pulse of duration \( 2\Delta t_p \) can be modeled as a continuous series of hard sub-pulses with complex amplitudes \( p(\tau) \) at times \( \tau \in [-\Delta t_p, \Delta t_p] \). Each sub-pulse generates a signal coherence

\[ s(\tau, t) = p(\tau) s_0(t - \tau) = p(\tau) s_0(t_{\text{enc}}) \]  

(4)

corresponding to the primary signal of Eq. 2 weighted with \( p(\tau) \). The encoding time \( t_{\text{enc}} = t - \tau \) describes how long gradient encoding took effect after excitation. The total signal at time \( t \) is obtained by integration over the pulse duration, which is equivalent to the convolution of the primary signal \( s_0 \) with the pulse shape \( p \):

\(^1\) Note that strictly \( s_0(t) = s_0(k/\gamma G) \) and \( \rho(r) = \rho(\omega/\gamma G) \) but the scaling factors are dropped here for convenience.
Supporting Information Figure S1: Sweep ZTE signal model. The sweep pulse can be modeled as a continuous series of hard pulses with complex amplitudes $p(\tau)$, each generating a signal coherence $p(\tau)s_0(t-\tau)$ (of which only a few are represented here). In this view, the total signal $s(t)$ results from the integral of the signal coherences and can equivalently be represented by the convolution between the complex pulse amplitude $p(\tau)$ and the primary signal $s_0(t)$ (Equ. 5).

The convolution in time domain can alternatively be described via multiplication of pulse spectrum $P(\omega) = iFT[p(t)]$ and proton density $\rho(\omega)$ in frequency space

$$s(t) = (p \ast s_0)(t) = iFT[P(\omega)\rho(\omega)](t)$$

(6)
Discretization and matrix representation

To prepare algebraic image reconstruction, signal equation (6) is discretized and represented as matrix-vector multiplications

\[ s = E \rho = E_G P \rho \quad (7) \]

with the full encoding matrix \( E_{m,j} = \sum_i p_i \exp(i \omega_j (t_m - \tau_i)) \), the gradient encoding matrix \( E_{Gm,j} = \exp(i \omega_j t_m) \) and the pulse matrix \( P_{j,j'} = \delta_{j,j'} P(\omega_j) \) holding the pulse spectrum on the diagonal.

Equation 7 shows that the full encoding can be viewed as two independent sequential operations:

1) The proton density \( \rho \) is multiplied by the pulse spectrum \( P \) in frequency domain.
2) The result of step 1 is transferred to time domain by application of \( E_G \), which only contains information about gradient encoding and acquisition timing only.

Image reconstruction

1D projection images are obtained from \( \hat{\rho} = F s \), where \( F \) is the pseudoinverse of the encoding matrix

\[ F = E^\dagger = (E_G P)^\dagger \quad (8) \]

With \( E_G \) being of full column rank (given sufficient time-domain oversampling) and the diagonal matrix \( P \) of full rank (given a non-zero pulse spectrum), this can be rewritten as (3)

\[ F = (E_G P)^\dagger = P^\dagger E_G^\dagger = P^{-1} F_G \quad (9) \]

This shows that also image reconstruction can be viewed as two independent sequential operations:

1) Reconstruction of an intermediate image \( \hat{\rho}_G = F_G s \), assuming RF excitation with a hard pulse.
2) Division of the image by the pulse spectrum: \( \hat{\rho} = P^{-1} \hat{\rho}_G \).

According to this, pulse correction occurs in final image space and cannot support filling the dead-time gap, independently on sequence parameters such as oversampling.

ZTE with pulse profile encoding

Using pulse information to fill the dead-time gap

In the following section, it is demonstrated that, opposed to the findings above, pulse information can indeed be exploited to partly fill the dead-time gap if more than one pulse is considered in the signal model. In this case, pulse correction and inversion of gradient encoding cannot be separated into independent steps, thus pulse information and finite support assumption jointly serve for filling the gap. This greatly facilitates image reconstruction, especially when the dead-time is dominated by the pulse, i.e. T/R switching time and filter group delay are small compared to half the pulse duration.
Using two different RF-pulses

Independent pulse information may be introduced into the signal model by considering two ZTE acquisitions performed with different RF pulses. A graphical representation of Eq. 7 for this case is shown in Supporting Information Figure S2.

\[
\begin{align*}
S & = E_G \\
& = P \rho
\end{align*}
\]

Supporting Information Figure S2: Illustration of vector and matrices corresponding to a signal model in which k-space is acquired twice, each time with a different pulse. The gradient encoding matrix \(E_G\) is composed of two submatrices (grey) corresponding to the two acquisitions. The pulse matrix \(P\) is made of two diagonal matrices concatenated along the lines, in which diagonal elements consists of two different pulse spectrums \(P1\) and \(P2\) respectively.

Opposed to the situation of Eq. 8, the matrix \(P\) is only of full column rank but in general not anymore of full row rank. Therefore, the conversion of Eq. 9 does not hold and separation of the reconstruction into two independent parts corresponding to gradient and RF pulse is not possible. Hence, gradient and pulse information are combined during image reconstruction. In this way, pulse knowledge and finite support assumption are jointly used to enable to recovery of the data missed during the dead-time gap.

Boundary conditions in k-space

The boundary conditions at the borders of the k-space support may influence image reconstruction, especially when using only one, long sweep pulse. Their meaning and treatment are discussed in this section.
When targeting an image with spatial resolution $\Delta r$, encoding must be performed up to $k_{\text{max}} = \frac{\pi}{\Delta r} = \gamma G t_{\text{max}}$. Therefore, in sweep ZTE, data is collected until $t = t_{\text{max}} + \Delta t_p$ since only by that time the last coherence created at $\Delta t_p$ has experienced full encoding (Supporting Information Figure S 3). However, any sample acquired after $t = t_{\text{max}} - \Delta t_p$ (region highlighted in green) also contains primary signals $s_0(t_{\text{enc}})$ that are excessively modulated, i.e. with $t_{\text{enc}} > t_{\text{max}}$ (2) (grey region).

There is different approaches to treat this signal in the model underlying image reconstruction, two of which are considered here:

1) The object is assumed to be discrete with non-zero values separated by distance $\Delta r$. This implies periodicity in k-space with period $2k_{\text{max}}$, hence $s_0(t_{\text{enc}} > t_{\text{max}}) = s_0(k > k_{\text{max}}) = s_0(k - 2k_{\text{max}})$. This assumption is implemented in Eq. 7 by discretization of the target image, although for the usually continuous imaged objects it is actually violated. However, it can often be assumed that inherent spoiling and $T_2^*$ decay render the involved magnetization negligible, which has generally been confirmed by experimental results.
2) The latter assumption is used to neglect excessively modulated signal at all, i.e. \( s_0(t > t_{max}) = 0 \). This is equivalent to assuming that each signal point acquired after \( t = t_{max} - \Delta t_p \) was created by a different RF pulse, consisting of only part of the original pulse. Hence, in the green range, different pulses act in the encoding, and similarly to the two-pulse case above, this information may be used together with the finite support to recover data missed during the dead-time. However, because of the low correlation between data around \( t = 0 \) and \( t = t_{max} \), this will usually hardly play a role. Paradoxically, with the pulse duration increasing towards \( t_{max} \) and the green region approaching the border of the dead-time gap, image reconstruction will actually be supported by the present approach.

In both approaches, it is not useful to have a pulse duration longer than \( t_{max} \) since the signal portion created at the beginning of the pulse is already encoded to outside the usable k-range when data acquisition starts. To use this data, reconstruction could be initially performed with finer spatial discretization, followed by a reduction to the target resolution.

\( T_2^* \) relaxation

In the derivation provided above, \( T_2^* \) relaxation was omitted for sake of convenience. Although the related effect is generally known, it is derived here for the specific case considered in this work.

In the presence of transverse relaxation Eq. 2 becomes

\[
\begin{align*}
s_0(t) &= \int_{BW} p(\omega) e^{-\frac{|t|}{T_2}} e^{i\omega t} d\omega = i\mathcal{F}\{p(\omega)(t) e^{-\frac{|t|}{T_2}}\} \\
&= i\mathcal{F}\{p(\omega)(t) \} i\mathcal{F}\{L(\omega,T_2^*)\}(t)
\end{align*}
\]

with the Lorentzian function \( L(\omega,T_2^*) = \mathcal{F}\{ e^{-\frac{|t|}{T_2}}\} \). The equivalent representation in the frequency domain is

\[
\mathcal{F}\{s_0(t)\}(\omega) = p(\omega) * L(\omega,T_2^*)
\]

showing that for hard-pulse ZTE, \( T_2^* \) relaxation translates into an image blurring corresponding to a convolution of the image with a Lorentzian.

In analogy, with a sweep pulse, Eq. 6 becomes

\[
\begin{align*}
s(t) &= i\mathcal{F}\{P(\omega)(p(\omega) * L(\omega,T_2^*))\}(t)
\end{align*}
\]

with its frequency domain representation

\[
\mathcal{F}\{s(t)\}(\omega) = P(\omega)(p(\omega) * L(\omega,T_2^*))
\]

demonstrating that for both sweep-ZTE and PE-ZTE the same image blurring occurs as for hard-pulse ZTE, with the only difference that the blurred image is additionally multiplied by the pulse spectrum.
References

(1) Haacke EM, Ph D, Louis S, Brown RW, Thompson MR. Magnetic Resonance Imaging Physical Principles and Sequence Design. Wiley; 1999.

(2) Weiger M, Hennel F, Pruessmann KP. Sweep MRI with algebraic reconstruction. Magn Reson Med 2010;64:1685–95.

(3) Greville TNE. Note on the Generalized Inverse of a Matrix Product. SIAM Rev 1966;8:518–21.
B. Calculation of phase shift in frequency-swept RF pulses induced by off-resonance in center frequency

Sweep pulses are created by changing the frequency $f$ over the pulse duration $T_p$ in order to cover the whole imaging bandwidth $BW$ during excitation. In practice, the frequency is swept linearly over a period $T_{Lin} < T_p$. In the linear range, the frequency is described by

$$ f(t) = t \frac{BW}{T_{Lin}} + f_0 \quad \text{with} \quad t \in \left[ -\frac{T_{Lin}}{2}, \frac{T_{Lin}}{2} \right] $$

where the center frequency $f_0$ is set to the NMR frequency of the observed nucleus.

The pulse phase $\phi(t)$ induced by a frequency modulation $f(t)$ is

$$ \phi(t) = 2\pi \int_{-\frac{T_{Lin}}{2}}^{t} f(\tau) \, d\tau = 2\pi \left( \frac{BW}{2T_{Lin}} \, t^2 + f_0 \, t \right) + C $$

with $C = 2\pi \left( \frac{BW}{2T_{Lin}} \frac{T_{Lin}^2}{4} - f_0 \frac{T_{Lin}}{2} \right) = T_{Lin} \left( \frac{BW - 4f_0^2}{8} \right)$.

$\phi(t)$ is a second-order polynomial, which can be rewritten as

$$ \phi(t) = \frac{\pi BW}{T_{Lin}} \left( t + T_{Lin} \frac{f_0}{BW} \right)^2 + T_{Lin} \left( -\frac{\pi f_0^2}{BW} - \frac{f_0}{2} + \frac{BW}{8} \right) $$

By adding an off-resonance $\Delta f$ to the center frequency $f_0$ this expands to

$$ \phi(t, \Delta f) = \frac{\pi BW}{T_{Lin}} \left( t + T_{Lin} \frac{f_0 + \Delta f}{BW} \right)^2 + T_{Lin} \left( -\frac{\pi (f_0 + \Delta f)^2}{BW} - \frac{f_0 + \Delta f}{2} + \frac{BW}{8} \right) $$

From that, we see that the off-resonance will:

1) shift the quadratic phase along the time axis,
2) change the phase offset.

The time shift $\Delta t$ of the quadratic phase induced by the application of an off-resonance $\Delta f$ is

$$ \Delta t = \frac{T_{Lin} \Delta f}{BW} $$

which can also be expressed in units of Nyquist dwells $dt = 1/BW$ as

$$ \Delta t / dt = T_{Lin} \Delta f $$
C. SNR calculations

The SNR of head images was calculated with two different methods:

1) Local SNR: A region of interest with homogeneous intensity is selected in a magnitude image and the average signal intensity (sig) is divided by its standard deviation (std) (Price R, et al. Quality assurance methods and phantoms for magnetic resonance imaging: Report of AAPM nuclear magnetic resonance Task Group No. 1a. Med Phys 1990;2).

Supporting Information Figure S4: Local SNR. The region of interest is displayed in violet, chosen in a brain region with homogenous intensity. As noise standard deviation varies only by a few percent, the SNR difference is dominated by changes in the intensity.
2) Global SNR: The global SNR is calculated by dividing the average intensity (measured over a large 3D region in the head) by the standard deviation of a reconstructed noise image (real part).

**Supporting Information Figure S5: Global SNR.** A slice of the 3D mask in which intensity (sig) and noise standard deviation (std) are measured is displayed in violet. The mask was created by selecting intensity values $I > 0.3 \times \max(I)$, with $\max(I)$ the maximum intensity in the magnitude image. To create the noise image, a noise sample was acquired without RF excitation. Then, a complete dataset with same noise statistics was generated and reconstructed to an image. The noise values obtained with this technique are very close to those measured locally in the magnitude image. However, the global intensities differ more strongly than the local intensities, leading to even higher relative SNR for PE-ZTE.
D. Experimental limitation of PE-ZTE

This document illustrates the current experimental limitation of the pulse duration in PE-ZTE imaging. In the main manuscript it is shown that the conditioning of the encoding matrix remains low even with pulses lasting several tens of Nyquist dwells, indicating that the use of pulses of several hundreds of microseconds would be feasible for typical bandwidths on clinical scanners. However, the PE-ZTE concept is demonstrated with pulses of up to 6 Nyquist dwells and 35 µs. Their duration is currently limited by the emergence of an artifact as seen in Supporting Information Figure S6.

Supporting Information Figure S6: Current experimental limitation of pulse duration in PE-ZTE imaging. Sweep pulses of different durations were used to image a water bottle. Images were reconstructed from the same dataset but with different field of view (FOV) and transmit-receive switching times (Δt_T/R). To create a longer Δt_T/R, the first data points were discarded for reconstruction. The total dead time Δt (= Δt_p + Δt_T/R, with Δt_p = half the pulse duration) is given below each image in Nyquist dwells dt = 1/BW (with bandwidth BW). All images are normalized to the same signal value, allowing comparison of their relative intensities. Experimental parameters: pulse power = 63 W, resolution = 2.3 x 2.3 x 2.3 mm, T/R = 1 ms, pulse off-resonance = ± 6 kHz (pulse = 60 µs), ± 4 kHz (pulse = 80 µs), ± 2 kHz (pulse = 150 µs), BW = 250 kHz (FOV = 300 mm) and 350 kHz (FOV = 420 mm).
The first row in Supporting Information Figure S6 shows that an artifact emerges when long pulse durations are used. Its intensity increases with pulse length. As illustrated in the second row, cleaner images are obtained when the data probed during the first 5 μs following the dead time is skipped, suggesting that the data acquired early contains spurious signal. Indeed, any remaining model violation should rather lead to larger artifacts when $\Delta t_{T/R}$ increases. In the third row, bandwidth and in turn relative gap size are increased by reconstructing the images at larger FOV. This shows that the artifact does not depend directly on the number of missed Nyquist dwells $d t$ since image quality is preserved.

The nature of the spurious signal is still to be determined. At this stage, the most likely hypothesis is that it arises from RF hardware ringdown that accumulates with increasing pulse duration. Suppressing such ringdown is technically feasible (Brunner DO, Pruessmann KP. Coil Ringdown Suppression by Broadband Forward Compensation. Proc ISMRM 22 2014:951) and could potentially be used to remove this artifact.

To conclude, we demonstrate in this document that clean images can be reconstructed using pulses with durations of up to 80 μs, leading to gaps larger than 10 Nyquist dwells. Above this limit and in the current experimental conditions, the artifact deteriorates image quality considerably. However, it is also demonstrated that the PE-ZTE principle as such is not limited in this way, since images with otherwise good quality could be obtained with pulses of more than 100 μs and several tens of Nyquist dwells. In case the artifact source can be eliminated and with the current reconstruction approach, the useful pulse duration would then probably be limited to the maximum encoding time $t_{\text{max}}$ (c.f. Supporting information A).