NON-COMMUTATIVE $q$-EXPANSIONS

MAHESH KAKDE

King’s College London

Abstract. In this short note we partially answer a question of Fukaya and Kato by
constructing a $q$-expansion with coefficients in a non-commutative Iwasawa algebra
whose constant term is a non-commutative $p$-adic zeta function.

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Notations and Set up

We use the following notation and set up throughout the paper. Fix an odd prime $p$.
For a pro-finite group $G$ we define the Iwasawa algebra $Λ(G) := \lim_{\leftarrow} Ζ_p[G/U]$, where
$U$ runs through open normal subgroups of $G$. If $G$ is a compact $p$-adic Lie group with
a closed normal subgroup $H$ such that $G/H ∼= Ζ_p$, the additive group of $p$-adic integers
then we have the canonical Ore set of $[3]$ defined as

$$S := \{ f ∈ Λ(G) : Λ(G)/Λ(G)f \text{ is a f.g. } Λ(H) - \text{module} \}.$$ 

Put $\widehat{Λ(G)}_S$ for the $p$-adic completion of the localisation $Λ(G)_S$.

The extension $Q(μ_p\infty)$ of $Q$ obtained by adjoining all $p$-power roots of 1 contains a
unique extension of $Q$ with Galois group isomorphic to $Ζ_p$. We denote this extension by
$Q_{cycl}$, the cyclotomic $Ζ_p$-extension of $Q$. If $L$ is any number field, then the cyclotomic
$Ζ_p$-extension of $L$ is defined as $L_{cycl} := LQ_{cycl}$. For any number field $L$, the ring of
integers of $L$ is denoted by $O_L$. 

E-mail address: mahesh.kakde@kcl.ac.uk.
Date: version 2.
Throughout $F$ will denote a totally real number field of degree $r := r_F := [F : \mathbb{Q}]$. Let $\Sigma := \Sigma_F$ denote a finite set of finite places of $F$. If $L$ is an extension of $F$, then we put $\Sigma_L$ for the set of places of $L$ above $\Sigma$. If there is no confusion we will often write $\Sigma$ for $\Sigma_L$. For any subset $O$ of $F$, we write $O^+$ for the set of totally positive elements of $O$. Throughout $F_\infty$ will denote a totally real Galois extension of $F$ such that

1. $F_{\text{cyc}} \subset F_\infty$.
2. $F_\infty$ is unramified outside $\Sigma$.
3. $G := \text{Gal}(F_\infty/F)$ is a $p$-adic Lie group.

We put $A_F(G)$ (often written simply as $A(G)$, where $F$ is clear from the context) for the ring $\hat{\Lambda}(G)\lbrack \lbrack q \rbrack \rbrack$ of all formal power series

$$a_0 + \sum_{\mu \in O_F^+} a_\mu q^\mu.$$ 

1. Introduction

The theory of $p$-adic modular forms essentially began with the paper of Serre [15]. It was generalised by Katz [11] and Deligne-Ribet [4] and used to construct $p$-adic $L$-functions for CM and totally real number fields respectively. The theory of $\Lambda$-adic modular forms was systematically developed by Hida. Since then they have formed a central tool in number theory and have most notably been used to prove main conjectures of commutative Iwasawa theory (Wiles [18], Skinner-Urban [17] etc.). The main conjecture of non-commutative Iwasawa theory was formulated by Coates-Fukaya-Kato-Sujatha-Venkakob [3] for elliptic curves without complex multiplication and more generally in Fukaya-Kato [6]. In an unpublished manuscript Kato [10] proved a case of non-commutative main conjecture for totally real fields by computing $K_1(\Lambda(G))$ and $K_1(\Lambda(G)_S)$ for a certain group $G$ and then proving congruences between certain between abelian $p$-adic zeta functions by proving the congruences first between $\Lambda$-adic Hilbert Eisenstein series. Abelian $p$-adic zeta functions appear in constant terms of these Eisenstein series (see theorem 1). At the end of the paper Kato mentions the following question of Fukaya - Is there a $\Lambda$-adic modular form, with non-commutative ring $\Lambda$, whose constant term is the non-commutative $p$-adic $L$-function. We cannot answer this question completely but we do construct a $q$-expansion (in certain cases; see theorem 13 for a precise statement) whose constant term is a non-commutative $p$-adic zeta function. The evaluation of this $q$-expansion at Artin characters is closely related to Hilbert Eisenstein series (see corollary 14).

The content of the article are as follows: in section 2 we recall the result of Deligne and Ribet on Hilbert Eisenstein series. In section 3 we prove the M"{o}bius-Wall congruences for the Eisenstein series from section 2. As well as giving a slight generalisation of the congruences proven by Ritter-Weiss [14] this section simplifies the exposition. As usual the congruences are actually proven directly for non-constant coefficients of the standard $q$-expansion of the Eisenstein series in theorem 1. The congruence for the constant terms, i.e. $p$-adic zeta functions, can then be deduced from the $q$-expansion principal for Hilbert modular forms. These congruences are used in [14], [9] (generalising [10]) to construct non-commutative $p$-adic zeta function and prove the main conjecture for totally real number fields. In any case, we get the M"{o}bius-Wall congruences for the
Λ-adic Eisenstein series in theorem [11]. In section [12] we give a description of $K_1(A_\mathbb{Q}(G))$ for certain $G$ (see [13] for details). For simplicity we work only over $\mathbb{Q}$ but the result should hold over other totally real number fields. In section [14] we use this description along with the Möbius-Wall congruences for Λ-adic Eisenstein series to construct an element in $K_1(A_\mathbb{Q}(G))$ whose constant term equals the non-commutative $p$-adic zeta function.

2. Λ-adic modular Eisenstein series

In this section we assume that $G$ is commutative i.e. $F_\infty/F$ is an abelian extension. Recall the following result of Deligne and Ribet.

**Theorem 1** (Deligne-Ribet [4], theorem 6.1). There exists a $\Lambda(G)$-adic Hilbert modular Eisenstein series $E(F_\infty/F)$ with standard $q$-expansion given by

$$2^{-r} \zeta(F_\infty/F) + \sum_{\mu \in O_F^+} \left( \sum_{\sigma_a \in G} \frac{\sigma_a}{N_F a} \right) q^\mu,$$

where $\zeta(F_\infty/F)$ is the $p$-adic zeta function, $a$ runs through all ideals of $O_F$ coprime to $\Sigma$, $\sigma_a \in G$ is the Artin symbol of $a$, $N_F a \in \mathbb{Z}_p$ is the norm of $a$ and $q^\mu = e^{2\pi i r_F/ \mathbb{Q}(\mu)}$.

In particular, for any finite order character $\chi$ of $G$ and any positive integer $k$ divisible by $p - 1$, the evaluation of $E(F_\infty/F)$ at $\chi \kappa^k$ (here $\kappa$ is the cyclotomic character of $F$) has standard $q$-expansion

$$2^{-r} L_{\Sigma}(\chi, 1 - k) + \sum_{\mu \in O_F^+} \left( \sum_{\sigma_a \in G} \chi(\sigma_a) N_F a^{k-1} \right) q^\mu.$$

**Proposition 2.** If $\beta \in O_F^+$ divisible only by primes in $\Sigma$, then there exists a Hecke operator $U_\beta$ such that the action of $U_\beta$ on the standard $q$-expansion of $\Lambda(G)$-adic forms is as follows: if the standard $q$-expansion of $f$ is

$$c_0 + \sum_{\mu \in O_F^+} c(\mu) q^\mu,$$

then the standard $q$-expansion of $f|_{U_\beta}$ is

$$c_0 + \sum_{\mu \in O_F^+} c(\beta \mu) q^\mu.$$

**Proof.** See [13] lemma 6].

Let $K$ be a subfield of $F$. Then the Hilbert modular variety of $K$ can be diagonally embedded in that of $F$. Restricting Hilbert modular forms on $F$ along this diagonal gives Hilbert modular forms over $K$. We denote this map by $\text{Res}_{F/K}$.

**Proposition 3.** If the standard $q$-expansion of $f$ is $c_0 + \sum_{\mu \in O_F^+} c(\mu) q^\mu$, then the standard $q$-expansion of $\text{Res}_{F/K}(f)$ is

$$c_0 + \sum_{\eta \in O_K^+} \left( \sum_{\mu : \tau_{F/K}(\mu) = \eta} c(\mu) \right) q^\eta,$$
3. The Möbius-Wall congruences for the Eisenstein series

In this section we assume that $G$ is a $p$-adic Lie group. Let

$$S^0(G) := \{U : U \text{ is an open subgroup of } G\}$$

Put $F_U := F^{U\infty}_\infty$, the field fixed by $U$ and put $K_U := F^{U\infty}_\infty$, the field fixed by the commutator subgroup of $U$. Therefore $\text{Gal}(K_U/F_U) = U^{ab}$, the abelianisation of $U$. For $V, U \in S^0(G)$, with $V \subset U$, the transfer homomorphism $\text{ver} : U^{ab} \to V^{ab}$ induces a ring homomorphism

$$\text{ver} : A(U^{ab}) \to A(V^{ab}),$$

which is identity on the coefficients and $q$. If $V$ is a normal subgroup of $U$, then we can define a map

$$\sigma^U_V : A(U^{ab}) \to A(V^{ab}),$$

given by

$$x \mapsto \sum_{g \in U/V} gxg^{-1}.$$

Recall the definition of Möbius function on finite groups. It takes value 1 on the trivial group and then defined recursively as

$$\sum_{P' \subset P} \mu(P') = 0.$$

**Theorem 4.** For every $V, U \in S^0(G)$, with $V$ a normal subgroup of $U$ we put

$$\mathcal{G} := \sum_{V \subset W \subset U} \mu(W/V) \text{ver} \left( \text{Res}_{F_W/F_U} \left( E(K_W/F_W) \right) \right) |u_{[W:V]}|.$$ 

Then the standard $q$-expansion of $\mathcal{G}$ lies in $\text{Im}(\sigma^U_V)$.

**Proof.** We follow the proof in [14, lemma 6]. Let $\mu \in O^+_F$. Then the $\mu$th coefficient of the standard $q$-expansion of $\mathcal{G}$ is

$$\sum_{V \subset W \subset U} \mu(W/V) \text{ver} \left( \sum_{(\alpha, a)} \frac{\sigma_a}{N_{F_W} a} \right),$$

where $\alpha$ in the second summation runs through all element of $O^+_F$ such that $tr_{F_W/F_U}(\alpha) = [U : W] \mu$ and $a$ runs through integral ideals of $F_W$ coprime to $\Sigma$ and containing $\alpha$. Take $M$ to be the set of all pairs $(\alpha, a)$ with $\alpha \in O^+_F$, such that $tr_{F_U/F_V}(\alpha) = [U : V] \mu$ and $a$ an integral ideal of $F_V$ coprime to $\Sigma$ and containing $\alpha$. Then $U$ acts on $M$ and the above sum can be written as

$$\sum_{V \subset W \subset U} \mu(W/V) \left( \sum_{(\alpha, a) \in M | W} \frac{\sigma_a}{N_{F_W} a^{1/[W:V]}} \right).$$

Now fix $(\alpha, a) \in M$ and let $W_0$ be the stabiliser of $(\alpha, a)$. Then the coefficient of $\sigma_a$ in the above sum is

$$\sum_{V \subset W \subset W_0} \mu(W/V) \frac{1}{(N_{F_V} a)^{1/[W:V]}} = \sum_{V \subset W \subset W_0} \mu(W/V) (N_{F_W} a)^{-[W_0:W]}.$$
We get the coefficient of $\sigma_{\varpi(a)}$ for every $g \in U/W_0$. Hence to show the congruence it suffices to show that for any finite group $P$ and any unit $r$ in $\mathbb{Z}_p$ we have

$$\sum_{P' \subseteq P} \mu(P') r^{[P:P']} \equiv 0 \pmod{|P|\mathbb{Z}_p}.$$  

We use [7, corollary 3.9]. Let $|P| = p^k \cdot t$ with $t$ an integer co-prime to $p$. Let $t'$ be a divisor of $t$. By taking $n = p^k \cdot t'$, the subgroup $H$ to be the identity we deduce from loc. cit. that $\sum_{|P'|} \mu(P')$ is divisible by $p^k$, where $P'$ runs through all subgroups of $P$ whose order divides $p^k \cdot t'$. Since this holds for arbitrary $t'$ we deduce that the sum $\sum_{P'} \mu(P')$ is divisible by $p^k$, where $P'$ runs through all subgroups of $P$ whose order is divisible by $t'$ and divides $p^k \cdot t'$. Now by [7, corollary 4.9] we have that $p^k$ divides $p \cdot \mu(P')$, where $p^k$ is the largest power of $p$ dividing $|P'|$. Therefore $\mu(P') r^{[P:P']} \equiv \mu(P') z'^t \pmod{|P|\mathbb{Z}_p}$ for any subgroup $P'$ of $P$ or order $p^k \cdot t'$ and where $z$ is the $(p-1)$st root of 1 in $\mathbb{Z}_p$ congruence to $r$ modulo $p$. Therefore

$$\sum_{P'} \mu(P') r^{[P:P']} \equiv z'^t \left(\sum_{P'} \mu(P')\right) \equiv 0 \pmod{|P|\mathbb{Z}_p},$$

where the $P'$ runs through all subgroups $P'$ of $P$ whose is divisible by $t'$ and divides $p^k \cdot t'$. This proves congruence in equation (11) and hence the theorem.

\[\square\]

**Remark 5.** We may replace $G$ by $\text{Res}_{F_1/F}(G)$ and the conclusion of the theorem still clearly holds. Though this is not important here, in cases of Eisenstein series over other groups (i.e. other than $\text{GL}_2$) this may be useful because there are cases when $q$-expansion principal may be known to hold over $F$ but not over extensions of $F$.

### 4. $K_1$ of some Iwasawa algebras

Detailed proofs of results in this section will appear in [2]. From now on we also assume that $F = \mathbb{Q}$. Let $G$ be a compact $p$-adic Lie group of the form $H \times \Gamma$, where $H \cong \mathbb{Z}_p^d$ and $\Gamma$ is an open subgroup of $\mathbb{Z}_p^\times$ and containing $1 + p\mathbb{Z}_p$. Furthermore, we assume that the action of $\Gamma$ on $H$ is diagonal. Put $\Gamma_0 := \Gamma$ and $\Gamma_i := 1 + p^i\mathbb{Z}_p \subset \mathbb{Z}_p^\times$ for $i \geq 1$. We put $\delta := [\Gamma : \Gamma_1]$. Put $G_i := H \times \Gamma_i$ for $i \geq 0$. Let $A(G)$ be the free abelian group generated by absolutely irreducible finite order (Artin) representations of $G$. Then we have a natural map

$$\text{Det} : K_1(A(G)) \to \text{Hom}(A(G), A(\Gamma)^\times).$$

Define $SK_1(A(G)) := \text{Ker}(\text{Det})$. Put $K'_1(A(G)) := K_1(A(G))/SK_1(A(G))$.

**Remark 6.** We expect $SK_1(A(G))$ to be trivial in this case but make no attempt to prove it here. This would be in analogy with the fact that $SK_1(A(G))$ is trivial ([12 proposition 12.7]).

**Definition 7.** Define a map

$$\theta := \prod_{i \geq 0} \theta_i : K'_1(A(G)) \to \prod_{i \geq 0} A(G_i^\times)^\times,$$
where each $\theta_i$ is the composition $K'_1(A(G)) \to K'_1(A(G_i)) \to A(G_i^ab)^\times$ of the norm map and the natural surjection.

**Some more maps:** Let $0 \leq j \leq i$. We have two natural maps

$$N := N_{i,j} : A(G_j^{ab})^\times \to A(G_i/[G_j,G_j])^\times$$

and the natural projection

$$\pi := \pi_{i,j} : A(G_i^{ab}) \to A(G_i/[G_j,G_j]).$$

We have the transfer homomorphism $\text{ver} : G_j^{ab} \to G_i^{ab}$ which induces a ring homomorphism, again denoted by $\text{ver}$

$$\text{ver} : A(G_j^{ab}) \to A(G_i^{ab}),$$

which acts as identity on $q$. We have a $\mathbb{Z}_p$-linear map from section \[ \sigma_i : A(G_i^ab) \to A(G_i^ab), \]

The image of the map $\sigma_i$ lies in the subring $A(G_i^{ab})^G$, the part fixed by $G$. In fact, the image $\sigma_i$ is an ideal in this ring (but not in the ring $A(G_i^{ab})$).

**Definition 8.** Let $\hat{\Phi} \subset \prod_{i \geq 0} (A(G_i^{ab})^\times)^G$ consisting of all tuples $(x_i)_i$ satisfying the congruence

$$(C) \quad \text{ver}(x_{i-1}) \equiv x_i (\text{mod } \text{Im}(\sigma_i))$$

**Definition 9.** Let $\Phi \subset \hat{\Phi}$ consisting of all tuples $(x_i)_i$ satisfying the functoriality

$$(F) \quad \text{For all } 0 \leq j \leq i \quad N_{i,j}(x_j) = \pi_{i,j}(x_i).$$

We define one more map before stating the main theorem of this section. We define $\eta_0 : A(G_0^{ab}) \to A(G_0^{ab})$ to be

$$\eta_0(x) = \frac{x^\delta}{\prod_{k=0}^{p-1} \bar{\omega}^k(x)},$$

where $\omega$ is a character of $G_0^{ab}$ inflated from an order $\delta$ character of $\Gamma$ and $\bar{\omega}$ is the map induced by $g \mapsto \omega(g)g$. For every $i \geq 1$, we define

$$\eta_i : A(G_i^{ab}) \to A(G_i^{ab})$$

by

$$x \mapsto \frac{x^p}{\prod_{k=0}^{p-1} \bar{\omega}_i^k(x)},$$

where $\omega_i$ is a character of $G_i^{ab}$ inflated from an order $p$ character of $\Gamma_i$ and $\bar{\omega}_i$ is the map on $A(G_i^{ab})$ induced by $g \mapsto \omega_i(g)g$. We put

$$\eta := \prod_{i \geq 0} \eta_i : \prod_{i \geq 0} A(G_i^{ab}) \to A(G_i^{ab})$$

**Theorem 10.** (1) $\theta$ induces an isomorphism between $K'_1(A(G))$ and $\Phi$.

(2) The inclusion $\Phi \hookrightarrow \hat{\Phi}$ has a natural section.
Proof. We give a rough idea of the proof with details and more general results appearing in [2].

(1) By definition of $K'_i$ and the fact that every irreducible Artin representation of $G$ is induced from a one dimensional Artin representation of $G_i$ for some $i$ ([16, proposition 25]) it is clear that the map $\theta$ is injective. It is also easy to show that the image of $\theta$ lies in $\Phi$. Let $(x_i)_i \in \Phi$. Then the inverse image of $(x_i)_i$ in $K'_i(A(G))$ is constructed as follows: firstly we may and do assume that $x_0 = 1$ since the map $K_1(A(G)) \to A(G_{ab})^\times$ is surjective. For every $i \geq 1$ put $y_i := \frac{x_i}{\text{ver}(x_{i-1})}$. Define

$$X := \prod_{i \geq 1} \eta_i(y_i)^{(i-1)p^i} \in K'_i(A(G)).$$

There are several points that need explaining - firstly, $\eta_i(y_i)$ lies in $A(G_{ab})^\times$. However, it can be show that the map $K'_1(A(G_i)) \to A(G_{ab})^\times \xrightarrow{\text{Res}} \eta_i(A(G_{ab})^\times)$ splits and hence $\eta_i(y_i)$ makes sense as an element of $K'_1(A(G))$. It can be shown using conditions (F) and (C) that $\eta_i(y_i)$ is a $(p-1)p^i = p[G_i : G]$th power in $K'_1(A(G))$. Using the fact that $x_0 = 1$, we can show that $\eta_i(y_i)$ actually lies in the image, denoted by $K'_1(A(G), J)$ of $K_1(A(G), J)$ in $K_1(A(G))$ and that it has a unique $(p-1)p^i$th root in $K_1'(A(G), J)$. Here $J$ is the kernel of $A(G) \to A(C_{ab})$. Lastly, one needs to show that the infinite product converges. Each of this step is non-trivial and crucially uses integral logarithm map (or rather it generalisation to rings like $A(G)$) of R. Oliver and M. Taylor.

(2) By the above we may define the natural section as follows: Let $(x_i)_i \in \Phi$ and we may again assume that $x_0 = 1$. Define $z_0 = 1$ and for $i \geq 1$ define

$$z_i = \prod_{j \geq i} \eta_j(y_j)^{p^{j-i}} \in A(G_{ab})^\times,$$

with $y_i$ defined as above. Then one can check that $(z_i)_i \in \Phi$ and gives a natural section of the inclusion.

Definition 11. We define the natural section of the inclusion $\Phi \subset \tilde{\Phi}$ given by the above theorem by $s$.

Corollary 12. If $(x_i)_i \in \tilde{\Phi}$, then there is a unique element $x \in K'_1(A(G))$ such that $\theta(x) = s((x_i)_i)$.

5. NON-COMMUTATIVE $q$-EXPANSIONS

We continue with the notation of the previous section. Therefore $F = \mathbb{Q}$. Let $F_1 := F_{G_i}^\infty$ and $K_1 := F_{F_{G_i}^\infty}^{G_i,G_i}$. We put $\mathcal{E}_i$ for the standard $q$-expansion of the $\Lambda(G_{ab})$-adic Hilbert modular form $\text{Res}_{F_1/F}(\mathcal{E}(K_i/F_i))|_{U_{q_p}}$ from section 2. As $F_1$ is an abelian extension of $\mathbb{Q}$ and $G_1$ is pro-$p$ we know by the theorem of Ferrero-Washington [5] that $\mathcal{E}_i \in A(G_{ab})^\times$ (it is enough to show that the constant term, i.e. the $p$-adic zeta functions $\zeta(K_i/F_i)$, of $\mathcal{E}_i$ are units in $\Lambda(G_{ab})_S$. This is well-known, for example see [3] lemma 1.7 and 1.14).

Theorem 13. The tuple $(\mathcal{E}_i)_i$ lies in $\tilde{\Phi}$. Hence there exists $\mathcal{E} \in K'_1(A(G))$ such that $\theta(\mathcal{E}) = s((\mathcal{E}_i)_i)$. The “constant term” of $\mathcal{E}$, i.e. its image under the map $K'_1(A(G)) \to K'_1(A(G) |_S)$ mapping $q$ to 0, takes $\mathcal{E}$ to the $p$-adic zeta function for $F_{\infty}/F$. 

Proof. It is clear from the explicit expression that each $E_i$ is fixed under the conjugation action of $G$. The congruence condition (C) follows from the Möbius-Wall congruences as follows: taking $V = G_i$ and $U = G_1$ and noting that $\mu(\mathbb{Z}/p^n\mathbb{Z})$ is zero unless $0 \leq n \leq 1$, we get that the standard $q$-expansion of

\begin{equation}
\text{ver}(\text{Res}_{F_i/F_1}(\mathcal{E}(K_i/F_1))/U_{p_i}) - \text{Res}_{F_i/F_1}(\mathcal{E}(K_i/F_1))/U_{p_i}
\end{equation}

lies in $\text{Im}(\sigma_{G_1}^{G_i})$ by theorem 3. Now $\text{ver}(\mathcal{E}_{i-1}) - \mathcal{E}_i$ is obtained by applying $\text{Res}_{F_1/Q}$ and $U_\delta$ to (2) and taking its standard $q$-expansion. Hence $\text{ver}(\mathcal{E}_{i-1}) - \mathcal{E}_i \in \text{Im}(\sigma_1)$. The second assertion follows from corollary 12.

The last assertion follows from the commutative diagram

$K_i'(A(G)) \xrightarrow{\theta} K_i'(\Lambda(G))$

\[\prod_{i \geq 0} A(G_i^{ab}) \xrightarrow{\theta} \prod_{i \geq 0} \Lambda(G_i^{ab}),\]

where the horizontal maps are $q \mapsto 0$.

The following corollary tells us something about evaluation of $\mathcal{E}$ at elements of $A(G)$.

**Corollary 14.** Let $\rho \in A(G)$. Then there exist $i$ and a one dimensional character $\chi$ of $G_i$ such that $\rho = \text{Ind}_{G_1}^{G_i} \rho$. Then

$$\rho(\mathcal{E}) = \prod_{j \geq i} \left( \chi(\eta_j(\mathcal{E}_j))^{[\Gamma_j:\Gamma_i]} \right).$$

**Remark 15.** There are many examples of the totally real extensions of $\mathbb{Q}$ with Galois group $G$ whose form is as in the previous section. For example if $p$ is an irregular prime, then the maximal abelian pro-$p$ extension of $\mathbb{Q}(\mu_p\infty)$ is isomorphic to $\mathbb{Z}_p^d$ for some positive integer $d$. If Vandiver’s conjecture is true for $p$ then the action of $\text{Gal}(\mathbb{Q}(\mu_p\infty)^+/\mathbb{Q})$ on $\mathbb{Z}_p^d$ is diagonal.

**Remark 16.** Even though the modifications we have to make to the $\Lambda$-adic Eisenstein series to get a lift to the non-commutative case are somewhat complicated, the result is rather formal i.e. if a “$q$-expansion” satisfies a congruence, then its modification given above can be lifted. The congruences seem to hold for Eisenstein series over other groups (see for example [1]). Hence their modifications should also have a lift. Is there a more conceptual description of these lifts?

**Remark 17.** We also remark that $(\mathcal{E}_i)_i$ does not lie in $\Phi$ because they do not satisfy the functoriality condition (F).

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