Extended hydrodynamics from Enskog’s equation
The bidimensional case

Hideaki Ugawa
Departamento de Física, FCFM, Universidad de Chile, Santiago, Chile

October 26, 2018

Abstract

A heat conduction problem is studied using extended hydrodynamic equations obtained from Enskog’s equation for a simple case of two planar systems in contact through a porous wall. One of the systems is in equilibrium and the other one in a steady conductive state. The example is used to put to test the predictions which has been made with a new thermodynamic formalism.

PACS: 51.10.+y 05.20.Jj 44.10.+i 05.70.Ln
kinetic theory; Enskog dense gases; Heat conduction; Nonequilibrium thermodynamics

1 Introduction

Today Enskog’s original kinetic theory is known as the standard Enskog theory (SET) [1, 2, 3] because after the pioneer work of van Beijeren and Ernst [4] there are several new versions of Enskog’s theory collectively called revised Enskog’s theory (RET) [5]. Among the latter there are versions that have been extended to describe condensed matter [6]. To Navier-Stokes level both SET and RET lead to the same results [4, 7], whether or not an external force is present.

In the present article we make use of extended hydrodynamic equations for the bidimensional case [9]. They are more complete than a linear approximation but still they are the result of an approximation scheme that we explain elsewhere. Using a strategy as in Ref. [8] and approximations defined in Ref. [9] we obtain in Sec. 5 the same hydrodynamic equations for SET and RET.

In this article we apply our extended hydrodynamics to a one dimensional steady heat conductive state. There is much work on this as, for example, the experimental results in [10] or the theoretical ones in [11, 12, 13]. Recently Kim and Hayakawa [14] studied this problem for hard core and Maxwellian particles using Boltzmann’s equation combined with Chapman-Enskog’s method. They tried a test and criticized the analysis of the nonequilibrium steady state thermodynamics (SST) proposed by Sasa and Tasaki [15]. In the last reference the authors state that if there is gas in a one dimensional heat conductive configuration in contact, through a porous wall, with an equilibrium gas state, then a pressure difference must appear in the direction of the heat flow. We analyze this double system making use of the extended hydrodynamic equations derived from Enskog’s equation using Grad’s moment expansion method [16]. Our conclusions differ from those in [15].

The organization of the present article is as follows. In Sec. 2 the configuration of these systems is drawn schematically, in Sec. 3 the condition for the two systems to be in contact via the central porous plate is introduced: the upper and lower plates are normal plates; the central plate has many small pores through which the gas can pass. In Sec. 4 we give the basic equations used in this paper. Comments are in Sec. 5. Finally, our discussion and conclusions are written in Sec. 6 and 7 respectively.

[published in Physica A, 354, 77-87, 2005]
2 Definition of the system

Sasa and Tasaki [15] proposed an interesting system consisting of a nonequilibrium steady state subsystem in contact with a subsystem in equilibrium as explained below. This system has three plates as shown in Fig. 1 and there is gas between them. The upper and lower plates (plates 1 and 3) are normal plates. The central plate (plate 2) has pores through which gas can pass.

Following Sasa and Tasaki, we consider the system consisting of three infinite parallel plates 1, 2 and 3 separated by a distance $L$. The $Y$ axis is defined perpendicular to them while an $X$ axis is placed on plate 2. The pores in plate 2 are distributed homogeneously. Plates 1 and 2 have fixed temperature $T_1$ while plate 3 has a different (fixed) temperature $T_2$.

After a sufficiently long time, by effusion, some of the gas passes through plate 2 and the gas between plates 1 and 2 reaches an equilibrium state. The system between the plates 2 and 3 reaches a nonequilibrium steady state with translation symmetry along the $X$ axis.

We assume that the typical distances between pores is very small and that the diameter of the pores is also very small, so that the ratio between such lengths and the mean free path is much smaller than unity. Having no external force and no hydrodynamic velocity there is no heat flux parallel to the plates. The system is in a static configuration.

3 The contact condition

In general, there is a difference between the temperatures of the plates and the temperatures of the gas in contact with them, this is a well-known effect called thermal slip. However, for simplicity sake, we assume that the temperature of plate 2 and the gas in contact on both sides of it are equal, namely we are neglecting the Knudsen layer.

The velocity and the peculiar velocity of the gas will be denoted by $c$ and $C$, respectively. The condition that there is no mass flux through plate 2 is

$$
\int_{-\infty}^{\infty} dc_x \int_{0}^{\infty} dc_y c_y f_{\text{equil.}} + \int_{-\infty}^{\infty} dc_x \int_{0}^{0} dc_y c_y f_{y=0} = 0,
$$

where $f_{\text{equil.}} = n_{\text{eq}} \left[ \frac{1}{2\pi T_1} \right] \exp \left[ \frac{-c^2}{2T_1} \right]$ is the equilibrium distribution function associated to the gas between plates 1 and 2 and $n_{\text{eq}}$ is the uniform number density in this same region while $f_{y=0}$ is the nonequilibrium distribution function between plates 2 and 3 evaluated at $y = 0$. Next, it is necessary to see how the two distributions satisfy condition (1).

Imposing condition (1) yields

$$
n_{\text{eq}} = \frac{1}{2} \left[ n(0) + P_{yy} \right] - \delta \frac{\chi}{2Kn} \left[ n(0) + P_{yy} \right] n(0) + \delta^2 \frac{\chi^2}{2Kn^2} \left[ n(0) + P_{yy} \right] n(0)^2.
$$

Figure 1: The plates of the system as described in the text.
In addition, the total mass conservation law for the system is
\[ n_{eq} + \int_{0}^{1} n(y)dy = 2. \] (3)

Above we are using dimensionless fields and dimensionless variables in general. The fields \( n \) (number density), \( P_{ij} \) (pressure tensor), \( \mathbf{Q} \) (net heat flux vector), and \( T \) (temperature) generally depend on the coordinate \( y \), where \( p_{ij} \) and \( \mathbf{q}^k \) are the symmetric and traceless part of the pressure tensor and the kinetic part of the heat flux vector respectively. These hydrodynamic fields are defined according to the following sum rules:

\[ \int f dx = n(y), \] (4)

\[ \int C(y)f dx = 0, \] (5)

\[ \int \frac{1}{2} C(y)^2 f dx = n(y)T(y), \] (6)

\[ \int C_i(y)C_j(y) f dx = n(y)T(y)\delta_{ij} + p_{ij}(y), \] (7)

\[ \int \frac{1}{2} C(y)C^2(y) f dx = q^k(y). \] (8)

We also use the following dimensionless numbers

\[ Kn = \frac{8\sqrt{2} L}{\pi \ell}, \quad \text{Knudsen number}, \quad \delta = \frac{q}{L} = Kn \rho_0 \]

where \( \sigma \) is the particle’s diameter, \( \ell \) their mean free at equilibrium and \( \rho_0 \) is the mean area density.

### 4 Balance equations

The basic concrete equations solved here are the following.

- In the case of the linearized Boltzmann-Grad method (LBG): \( P_{yy}(y) \equiv P_{yy} = \text{constant}, \quad P_{xy}(y) \equiv P_{xy} = \text{constant}, \quad Q_y(y) \equiv Q_y = \text{constant}, \quad n(y)T(y) = P_{xx} = P_{yy}, \quad \frac{dT(y)}{dy} + \frac{2Q_y}{Kn\sqrt{\pi} T(y)} = 0. \) (9)

- In the case of the Enskog-Grad method (EG): \( P_{yy}(y) \equiv P_{yy} = \text{constant}, \quad Q_y(y) \equiv Q_y = \text{constant}, \quad P_{xy}(y) = p_{xy}(y) = 0, \quad P_{yy} = -\left[1 + \frac{\delta}{Kn} \chi n(y)\right] p_{xx}(y) + \left[1 + 2\frac{\delta}{Kn} \chi n(y)\right] n(y)T(y), \) (10)

\[ Q_y = \left[1 + 3\frac{\delta}{Kn} \chi n(y)\right] q^k_y(y) - \delta^2 \frac{2}{\sqrt{\pi}Kn} \chi n(y)^2 \sqrt{T(y)} \frac{dT(y)}{dy}, \] (11)

\[ -\frac{1}{2} \frac{dq^k_y(y)}{dy} = \frac{8}{\sqrt{\pi} Kn} \chi \sqrt{T(y)} \left[n(y)p_{xx}(y) - \frac{q^k_y(y)^2}{128T(y)^4}\right] \]

\[ + \frac{\delta}{4Kn} \chi \left[5q^k_y(y) \frac{dn(y)}{dy} + 3n(y) \frac{dq^k_y(y)}{dy}\right] \]

\[ - \frac{\delta^2}{\sqrt{\pi} Kn} \chi n(y)^2 \sqrt{T(y)} \left[2 \frac{dn(y)}{dy} \frac{dT(y)}{dy} + \frac{1}{2} \frac{n(y)}{T(y)} \left(\frac{dn(y)}{dy}\right)^2 + n(y) \frac{d^2 T(y)}{dy}\right], \] (12)
and

\[ -T(y) \frac{dp_{xx}(y)}{dy} - p_{xx}(y) \frac{dT(y)}{dy} - n(y)T(y) \frac{dT(y)}{dy} + T(y) \frac{n(y)p_{xx}(y)}{dy} \frac{dn(y)}{dy} \]

\[ - \frac{p_{xx}(y)}{n(y)} \frac{dp_{xx}(y)}{dy} \]

\[ = - \frac{4}{\sqrt{\pi}T(y)Kn} \left[ n(y)T(y) - p_{xx}(y) \right] \sigma_y^k(y) \]

\[ + \frac{\delta}{Kn} \chi \left[ \frac{7}{2} n(y)T(y) \frac{dp_{xx}(y)}{dy} + \frac{7}{2} T(y) p_{xx}(y) \frac{dn(y)}{dy} + 2p_{xx}(y) \frac{dT(y)}{dy} \right] \]

\[ - 8n(y)T(y)^2 \frac{dn(y)}{dy} - 7n(y)^2 T(y) \frac{dT(y)}{dy} \]

(13)

The substitution of \( \delta = 0 \) and \( \chi = 1 \) in the EG equations yields the equations corresponding to the nonlinearized Boltzmann-Grad method (NLBG).

### 5 The pressure difference between the equilibrium and nonequilibrium sides

All the results we describe in what follows were obtained using perturbation methods choosing \( T_2 > T_1 \) and using \( \epsilon = (T_2 - T_1)/T_1 \) as the perturbation parameter. We solve the system of equations and their boundary (contact) conditions up to \( \epsilon^6 \). We choose \( \delta = 0.001 \). In such case \( Kn \) is in inverse proportion to \( \rho_0 \). We choose Henderson’s expression \[17\] as the concrete expression for \( \chi \):

\[ \chi = 1 - \frac{7}{16} \frac{n}{(1 - \rho_0)^2} \]

(14)

We calculate the pressure in both sides of plate 2. Using \( P_{yy} \) for the nonequilibrium steady state side and the pressure \( P_{eq} \) which is estimated by the state equation for the equilibrium side

\[ P_{eq} \equiv \left[ 1 + \delta \frac{2}{Kn} \chi(n)\right] n_{eq}, \]

(15)

the pressure difference \( \Delta P \) is defined by

\[ \Delta P = P_{yy} - P_{eq}. \]

(16)

Note that \( n(0) = n_{eq} \) and \( \Delta P = 0 \) to first order in \( \epsilon \). Hence, we rewrite our results in the following way

\[ n(0) = n_{eq} \left[ 1 + \lambda_n \frac{Q_y^2}{n_{eq}^2} \right], \]

(17)

\[ \Delta P = \lambda_{\Delta P} \frac{Q_y^2}{n_{eq}} \]

(18)

where \( \lambda_n \) and \( \lambda_{\Delta P} \) are constants. Furthermore, it is possible to rewrite \( P_{yy} \):

\[ P_{yy} = n(0) \left[ 1 + \frac{\delta}{Kn} \chi(n(0)) \right] \left[ 1 + \lambda_{yy}^n \frac{Q_y^2}{n(0)^2} \right]. \]

(19)

Tables 1 and 2 give the values of these constants for \( \epsilon = 0.05 \) and 0.1, respectively. The value and sign of \( \Delta P \) depend on \( \epsilon \) and \( Kn \). Table 3 gives the value of \( \lambda_{\Delta P} \) obtained by first order EG, that is, up to \( \delta \) for
| Kn  | $\lambda_n \times 10^{-2}$ | $\lambda_{\Delta P} \times 10^{-2}$ | $\lambda_{yy}^{\nu}/\lambda_n$ |
|-----|--------------------------|--------------------------|--------------------------|
| 0.005 | -0.360727 | 1.43059 | -3.45019 |
| 0.01  | 0.6175405 | -0.37938 | -1.17319 |
| 0.02  | 0.5891347 | -0.51577 | -1.66426 |
| 0.05  | 0.4851223 | -0.46437 | -1.87508 |
| 0.1   | 0.4309258 | -0.44057 | -1.93856 |
| 0.2   | 0.4158376 | -0.41159 | -1.96948 |

Table 1: The values of $\lambda_n$, $\lambda_{\Delta P}$ and $\lambda_{yy}^{\nu}/\lambda_n$ for $\epsilon = 0.05$ in the case of EG

| Kn  | $\lambda_n \times 10^{-2}$ | $\lambda_{\Delta P} \times 10^{-2}$ | $\lambda_{yy}^{\nu}/\lambda_n$ |
|-----|--------------------------|--------------------------|--------------------------|
| 0.005 | -0.405391 | 1.49646 | -3.19368 |
| 0.01  | 0.614064 | -0.37163 | -1.16022 |
| 0.02  | 0.590175 | -0.51564 | -1.66031 |
| 0.05  | 0.486046 | -0.40465 | -1.87365 |
| 0.1   | 0.431343 | -0.43093 | -1.93785 |
| 0.2   | 0.416123 | -0.41182 | -1.96124 |

Table 2: The values of $\lambda_n$, $\lambda_{\Delta P}$ and $\lambda_{yy}^{\nu}/\lambda_n$ for $\epsilon = 0.1$ in the case of EG

$\epsilon = 0.05$ and 0.1, respectively. In this case, the pressure difference also exists and its value and sign depend on $\epsilon$ and $Kn$, too.

In the case of LBG, since $p_{xx} = 0$ then $\lambda_n = \lambda_{\Delta P} = \lambda_{yy}^{\nu} = 0$. There is no pressure difference in this case.

On the other hand, for the case of NLBG, the substitution of $\delta = 0$ in Eqs. (2), (10), (11), (12), (15), (16), (17) and (19) leads to $\lambda_n = 1/256$, $\lambda_{\Delta P} = -1/256 < 0$ and $\lambda_{yy}^{\nu} = -1/128$ where for $\lambda_n$ and $\lambda_{\Delta P}$ it is correct to consider only up to second order in $Q_y$. Hence the osmotic pressure difference does exist and its value is constant and negative.

Furthermore, for the case of EG, $\lambda_{yy}^{\nu}/\lambda_n = -2 - \delta \frac{\sqrt{\pi}K_n}{Kn} m_n \neq -2$.

We analyze the pressure difference $\Delta P$ from another point of view. It is sufficient to calculate $\Delta P$ up to $\epsilon^2$. It is given by

$$\Delta P = \epsilon^2 \frac{\pi Kn^2}{4096} \left[ 215\rho_0^2 - \frac{52}{\chi} \rho_0 - \frac{1}{\chi^2} \right]$$

$$= \epsilon^2 \frac{\pi Kn^2}{4096(\rho_0 - 16)^2} [15335 \rho_0^4 - 69024 \rho_0^3 + 81344 \rho_0^2 - 9216 \rho_0 - 1024].$$

It is seen that the sign of $\Delta P$ changes from negative to positive approximately at $\rho_0 = 0.2$, whereas it is always negative in the NLBG and to first order in the EG’s case.

Besides the system far from equilibrium, we are also interested in a region extremely close to the equilibrium condition. Therefore, we analyze the case without the strong nonlinear term, namely, we eliminate the terms involving $q_y^k(y)^2$ and $p_{xx}(y)q_y^k(y)$ in Eqs. (12) and (13). In this case, up to $\delta$,

$$P_{yy} = n(0) - \delta \sqrt{\pi Q_y} \left( \frac{dn(y)}{dy} \right)_{y=0},$$

$$\Delta P = -\delta \frac{\sqrt{\pi}Q_y}{32n(0)} \left( \frac{dn(y)}{dy} \right)_{y=0}. \tag{21}$$

Assuming that the derivative $dn(y)/dy$ of the density at plate 2 has the same sign as $Q_y$ (this is normaly
\[ \Delta P = \epsilon^2 \pi \frac{Kn}{128 \chi} [9 \chi \rho_0 - 2] = \epsilon^2 \pi \frac{Kn}{128 \chi} \left[ \frac{9 \rho_0 (16 - 7 \rho_0)}{16 (1 - \rho_0)^2} - 2 \right], \]

\[ \lambda_n = \frac{\chi}{16} \left[ 1 - \frac{9}{2} \chi \rho_0 \right], \quad \frac{\lambda_{yy}}{\lambda_n} = -2 + 2 \chi \rho_0 + 9 (\chi \rho_0)^2. \]  

Table 3: The values of \( \lambda_{\Delta P} \) for \( \epsilon = 0.05 \) and 0.1 in the case of the first order EG

Table 4: The values of \( \lambda_n, \lambda_{\Delta P} \) and \( \frac{\lambda_{yy}}{\lambda_n} \) for \( \epsilon = 0.05 \) in the case of EG without a strong nonlinear term.
\begin{tabular}{|c|c|c|c|}
\hline
Kn & \(\lambda_n \times 10^{-2}\) & \(\lambda_{\Delta P} \times 10^{-2}\) & \(\lambda_{yy}/\lambda_n\) \\
\hline
0.005 & -0.598881 & 1.66316 & -2.60733 \\
0.01 & 0.340322 & -0.15760 & -0.98090 \\
0.02 & 0.259368 & -0.22086 & -1.62996 \\
0.05 & 0.119439 & -0.11385 & -1.86873 \\
0.1 & 0.061951 & -0.06598 & -1.93656 \\
0.2 & 0.031501 & -0.03117 & -1.96879 \\
\hline
\end{tabular}

Table 5: The values of \(\lambda_n\), \(\lambda_{\Delta P}\) and \(\lambda_{yy}/\lambda_n\) for \(\epsilon = 0.1\) in the case of EG without a strong nonlinear term.

6 Discussion

In Ref. [15] the authors argue that there is a pressure difference at plate 2, namely the pressure in one side of the plate is different to that on the other side. They call this new pressure which acts on the central plate the “flux induced osmosis” (FIO). We consider the existence of FIO proposed by [15] identifying \(\Delta P\) as the pressure difference defined in Sec. 5.

In Ref. [15], the following criteria are stated:
1. \(\Delta P > 0\) regardless of the sign of \(Q_y\).
2. \(P_{yy}\) is a function of the nonequilibrium quantities: \(T_1\), the nonequilibrium steady heat flow \(Q_y\), and it is related to the equilibrium quantity \(P_{eq}\) as long as the nonequilibrium and equilibrium temperature at both sides of plate 2 coincide,

\[
\frac{n(0)}{n_{eq}} = \left(\frac{\partial P_{yy}}{\partial P_{eq}}\right)_{T_1,Q_y}
\]  

(23)

where \(T_1\) is the thermodynamic temperature of plate 2.

In Ref. [14], for a system of hard spheres and of maxwellian particles—which obey Boltzmann’s equation and which obey the BGK equation [18]—using the Chapman-Enskog method, it is shown that criterion 1 in Ref. [15] is valid but criterion 2 is not valid.

On the other hand, for the hard disk’s system, from our present scheme based on Enskog’s equation we obtain that
1. Criterion 1 in Ref. [15] is not obeyed in the case of NLBG: \(\Delta P\) is always negative independent of the sign of \(Q_y\).
2. In the case of LBG, (23) is valid. However, substitution of (17) and (19) into (23) leads to \(\lambda_{yy}/\lambda_n = -2\). This is correct only in the case of NLBG. Hence, criterion 2 in Ref. [15] is not satisfied either.

In the formulation of SST it is assumed that the number of particles and the size of the system is infinite but that the number density of the system is finite [14]. This condition implies that the terms \(O(\delta^2)\) in the collision terms can be neglected. Table 2 still indicates that under such condition the osmotic pressure difference is not always positive. Especially when the system is extremely close to equilibrium, Eq. (21) implies that \(\Delta P\) is always negative.

However when the system is extremely close to equilibrium, Eq. (22) implies that \(\Delta P\) is always positive. This result coincides with the hard sphere case in Ref. [14].

Furthermore, the condition under which there is no heat flux at the porous wall is

\[
\int_{-\infty}^{\infty} dc_x \int_{0}^{\infty} dc_y \frac{c_y}{2} C^2 f_{\text{equil.}} + \int_{-\infty}^{\infty} dc_x \int_{-\infty}^{0} dc_y \frac{c_y}{2} C^2 f_{y=0} = 0.
\]  

(24)

Using Eq. (1) yields

\[
q_y^k(0) = \sqrt{\frac{2}{\pi}} [n_{eq} - n(0)].
\]  

(25)
The above condition is not satisfied without introducing a difference between the temperature of the gas and the porous wall except in the LBG case. Therefore, except in the LBG case, it must be difficult to maintain the equilibrium state between plates 1 and 2 even if the heat conductivity of plate 2 is extremely high. Here, as a rough simplification, let us introduce the temperature of the gas in contact with plate 2, \( T_g \neq T_1 \). Substituting \( \delta = 0 \) in Eqs. (2), (10), (11), (12), (15), (16), (17), (18) and (19), using the no-mass-flux condition given by an equation similar to (1), yields,

\[
\Delta P = n(0) \left[ 1 - \frac{\sqrt{T_g}}{2} - \frac{1}{2 \sqrt{T_g}} \right] + \left[ \frac{1}{\sqrt{T_g}} - 2 \right] \frac{Q_g^2}{256n(0)}. \tag{26}
\]

As \( T_1 < T_g < T_2 \) it follows that \( 1 < T_g < 1 + \epsilon \), and putting \( \delta^2 = 0 \) in the right side of Eq. (26), not only for NLBG but also for LBG, \( \Delta P < 0 \). The behavior of the pressure difference changes qualitatively even if we restrict the analysis to Boltzmann’s regime. This implies that the estimation of \( \Delta P \) is a very delicate problem. Even if one can prepare the walls which satisfy Eq. (23) and estimate the pressure difference in Enskog’s regime, it is difficult to know what physical meaning lies behind such case.

7 Conclusions

We have analyzed a simple nonequilibrium steady state system inspired by [15]. Our study refers only to a hard disk system and analyze in great detail its behavior using our extended hydrodynamic equations [9] using various approximations. Since we obtain that the osmotic pressure difference is negative in many cases for which Eq. (23) is not satisfied we cannot agree with [15].

We have assumed that the pores in plate 2 are small enough and we have not considered the problem about reflections on the wall at all. As we point out in the last part of Sec. 6, the boundary (contact) condition is very delicate. We recognize that a more sophisticated analysis is necessary.

However, in cases when strong nonlinearities can be neglected and the system is quite close to equilibrium—so that higher order terms in \( \epsilon \) do not contribute—then the osmotic pressure is positive. This implies the possibility of the existence of FIO.

In addition, in the full-paper [15], the authors point out that Eq. (28) is directly related to the condition at the wall and this condition is essential to construct the formalism of SST in a complete form which gets a new nonequilibrium extensive quantity which determines the degree of nonequilibrium. Therefore, to clarify the problem, the measure of the pressure difference is done only for the case of a wall obeying Eq. (28). Within this context, they still recognize the results of [14] as implying the existence FIO. Hence we also think that it is worth estimating the pressure difference starting from Eq. (28). If one were to analyze the problem in such a way then the boundary (contact) condition would have to be reconsidered to solve the kinetic equation. In other words, one would have to evaluate \( \Delta P \) under the rather complex conditions required by kinetic theory that would lead to satisfy Eq. (28). This has not been done.

The SST formalism is quite interesting and the present study has only put to test the possible existence of FIO.

Finally, we briefly comment about EIT: Extended irreversible thermodynamics [19, 20, 21]. For an ideal gas, Refs. [20, 21] studied a problem quite similar to the one in the present article. In Ref. [20] the authors estimated the pressure difference without considering a special wall. They assumed that the direction of the heat flow is parallel to the interface and their results are very interesting. Furthermore, in [21] the authors estimated the difference between the pressures which are parallel and perpendicular to the heat flow.

For a hard disk, applying LBG to the systems of [20, 21], it is easily possible to get similar equations to [9] and to see that the pressure difference predicted by [20] is positive and that the difference predicted by [21] is zero. In both cases, of course, it is totally unnecessary to use conditions [1] and [4].

Acknowledgements
The author wishes to express his sincere gratitude to Prof. P. Cordero who has guided him into the investigation using kinetic theory, given him the basic theme related to the present investigation and conducted many important discussions. The author also wishes to thank Prof. Dino Risso and Prof. Rodrigo Soto for many helpful hints. The scholarship from Mecesup UCh 0008 project is gratefully acknowledged.

References

[1] S. Chapman and T. G. Cowling, *Mathematical Theory of Non-Uniform Gases 3\textsuperscript{rd} ed.*, Cambridge, 1970.
[2] J. H. Ferziger and H. G. Kaper, *Mathematical Theory of Transport Processes*, North-Holland, 1972.
[3] P. Réssibois and M. De Leener, *Classical Kinetic Theory of Fluids*, McGraw-Hill, 1965.
[4] H. van Beijeren and M. H. Ernst, Physica 68, 437 (1973).
[5] N. Bellomo, *The Enskog Equation*, World Scientific, 1991.
[6] T. R. Kirkpatrick, S. P. Das and M. H. Ernst, J. Piasecki, J. Chem. Phys. 92, 3768 (1990).
[7] M. López de Haro and V. Garzó, Physica A 197, 98 (1993).
[8] D. Risso and P. Cordero, Phys. Rev. E 56 489 (1997); *ibid* Phys. Rev. 57 7365 (1998).
[9] H. Ugawa, in preparation
[10] W. P. Teagan and G. S. Springer, Phys. Fluids 11, 497 (1968).
[11] E. P. Gross and S. Ziering, Phys. Fluids 2, 701 (1959).
[12] T. Ohwada, K. Aoki and Y. Sone, *Rarefield Gas Dynamics: Theoretical and Computational Techniques*, Progress in Astronautics and Aeronautics Matin Summerfield Series Editor-in-Chief, 118, 70 (1989).
[13] T. Ohwada, Phys. Fluids 8, 2153 (1996).
[14] Kim. Hyeon-Deuk and H. Hayakawa, J. Phys. Soc. Jpn. 72, 1904 (2003).
[15] S. Sasa and H. Tasaki, cond-mat/0108365 (2001); *ibid*. cond-mat/0411052 (2004). The former is the letter-paper and the latter is the full-paper. The word “µ-wall” is named in this full-paper for the first time.
[16] H. Grad, Comm. Pure Appl. Math. 2, 331 (1949)

H. Grad, *Principle of the Kinetic Theory of Gases* in Handbuch der Physik, Vol. XII, edited by S. Flunge, Springer, 1958.

[17] D. Henderson, Mol. Phys. 30, 971 (1975).
[18] P. L. Bhatnagar, E. P. Gross and M. Krook, Phys. Rev. 94, 511 (1954).
[19] D. Jou, Casa-Vázquez and G. Lebon, *Extended Irreversible Thermodynamics*, Springer, 1996 (2nd edition).
[20] R. Domínguez and D. Jou, Phys. Rev. E 51, 158 (1995).
[21] J. Camacho and D. Jou, Phys. Rev. E 52, 3940 (1995).