Dual formulation of causal gain-scheduled output feedback controller design using parameter-dependent Lyapunov functions

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ABSTRACT
This paper is concerned with the design problem of causal gain-scheduled output feedback controllers for linear parameter-varying systems using parameter-dependent Lyapunov functions. This research topic has already been addressed by several researchers, and several effective methods have been correspondingly proposed in terms of parametrically dependent matrix inequalities. This paper addresses the dual formulation of one of the methods, and successfully derives the counterpart result in continuous-time case but points out the intractability of deriving the counterpart result in discrete-time case. It is also shown, in discrete-time case, that the intractability disappears and a design method similar to an existing method is successfully derived if the causality with respect to scheduling parameters is abandoned, namely, if one-step ahead scheduling parameters are available. A toy example is introduced to confirm that, in continuous-time case, the derived dual formulation gives the same performance as the previously proposed method for the dual state-space representation of the example.

1. Introduction
It is well known that gain-scheduled (GS) control is a powerful method for controlling systems modelled as linear parameter-varying (LPV) systems. In general, it is impossible to measure all states in practical systems. Many researchers have been therefore encouraged to work on the design problem of GS output feedback (GSOF) controllers for LPV systems since 1990s, as the design technique adopted in H∞ control problem for Linear Time-Invariant (LTI) systems is also applicable to GSOF controller design for LPV systems. The first proposition, i.e. [1], uses parametrically independent Lyapunov functions, i.e. constant Lyapunov functions, then parameter-dependent Lyapunov functions (PDLFs) are used to incorporate parameter variation speed in continuous-time (CT) case, e.g. [2–4], and bounded parameter variations in discrete-time (DT) case, e.g. [5]. However, in those papers, the use of PDLFs breaks the causality of GSOF controllers with respect to (w.r.t.) scheduling parameters; that is, GSOF controllers depend on the current scheduling parameters as well as their derivatives in the CT case or one-step ahead scheduling parameters in the DT case. Due to this property, the designed GSOF controllers in the references above are not practical, i.e. not implementable to practical systems.

On this issue, three major remedies have been proposed in the CT case. The first one is to design GSOF controllers using structured PDLFs in order to escape from the causality issue [4], the second one is to incorporate filters with GSOF controllers to use the scheduling parameters as the derivatives of the filtered scheduling parameters [6], and the third one is to design GSOF controllers with slightly complicated design conditions to escape from the causality issue [7,8]. On the last remedy, in [9] (the conference version is [10]), a new-bounding approach using Elimination lemma [11,12] has been used to derive a new design condition with reduced conservatism, and it is proved that the new method recovers the methods in [4,8] as special cases.

In the DT case, another remedy to use extended Linear Matrix Inequality (LMI) with constant auxiliary matrices has been proposed in [13]. However, it has been illustrated by a practical example in [9] that constant auxiliary matrices inevitably introduce conservatism compared to the counterpart method in the DT case of the third remedy for the CT case. Along with the extended LMI approach, a design method of GSOF controllers depending on the current and one-step behind scheduling parameters has been recently proposed in [14]. The online calculation of the GSOF controllers is slightly complicated; however, it has been proved that it is possible to design GSOF controllers with further reduced conservatism by using further past scheduling parameters.

In this paper, we address the dual formulation of the methods in [9] for designing causal GSOF controllers for LPV systems using the same approach in [7–9].
and successfully show the counterpart result for the CT case; however, we must point out the intractability of deriving the counterpart result for the DT case. Though, if the GSOF controllers can depend on one-step ahead scheduling parameters, namely, the causality w.r.t. scheduling parameters are abandoned, then it is shown that a similar method to the one in [5] is derived.

The remainder of this paper is as follows: In Section 2, we show the problem setup; in Section 3, we first show the dual formulation of the method in [9] in the CT case, then point out the intractability and show a design method of non-causal GSOF controllers (similarly to [5]) in the DT case; in Section 4, design results using a toy example, which is used in [8,10,15,16], for the CT case are first shown, then design results for a practical DT example in [9] with non-causal GSOF controllers designed using the method in this paper and causal GSOF controllers designed using the method in [9] are shown, and finally conclusions are given.

Notations: 0 denotes an appropriately dimensional zero matrix, I denotes an appropriately dimensional identity matrix (if necessary, the dimension is denoted by the subscript), \( \mathbb{R}_+, \mathbb{R}^n, \mathbb{R}^{n \times m} \) and \( \mathcal{S}^1_+ \), respectively, denote the sets of positive real scalars, \( n \times m \)-dimensional real vectors, \( n \times m \)-dimensional real matrices and \( n \times n \)-dimensional positive definite real matrices, the symbol \( * \) in a matrix represents an abbreviated asymmetric term, \( \text{diag}(X_1, \ldots, X_m) \) denotes a block diagonal matrix composed of \( X_1, \ldots, X_m \), and \( \text{He}[X] \) for a square matrix \( X \) denotes \( X + X^T \). For a time-varying vector \( x \), \( \delta[x] \) denotes \( \dot{x} := \frac{dx}{dt} \) and \( x^+ := x(k+1) \) in the CT case and the DT case, respectively.

2. Problem setup

2.1. LPV plant

Let us consider the following continuous-/discrete-time LPV plant.

\[
G(\theta) = \begin{bmatrix} \delta[x] \\ z \\ y \end{bmatrix} = \begin{bmatrix} A(\theta) & B_1(\theta) & B_2(\theta) \\ C_1(\theta) & D_{11}(\theta) & D_{12}(\theta) \\ C_2(\theta) & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ w \\ u \end{bmatrix},
\]

where \( x \in \mathbb{R}^n, w \in \mathbb{R}^n, u \in \mathbb{R}^n, z \in \mathbb{R}^n \) and \( y \in \mathbb{R}^n \) respectively denote the state with a zero vector at the initial time, the external input, the control input, the performance output and the measurement output, and the matrices \( A(\theta) \), etc. are supposed to have compatible dimensions. The vector \( \theta = [\theta_1 \ \theta_2 \ \ldots \ \theta_q] \) represents the mutually independent scheduling parameters whose values are available in real time. The scheduling parameters \( \theta_i \) and their derivatives \( \dot{\theta}_i \) (CT case) or the deviations per a single sampling period \( \theta_i^+ - \theta_i \) (DT case) are supposed to lie in bounded intervals.

Thus, the following holds with \textit{a priori} defined convex polytope \( \Lambda_\theta \).

\[
(\theta, \delta[\theta]) \in \Lambda_\theta
\]

2.2. GSOF controller

We now define a full-order GSOF controller to be designed.

\[
K(\theta) = \begin{bmatrix} \delta[x_K] \\ u \end{bmatrix} = \begin{bmatrix} A_K(\theta) & B_K(\theta) \\ C_K(\theta) & D_K(\theta) \end{bmatrix} \begin{bmatrix} x_K \\ y \end{bmatrix},
\]

where \( x_K \in \mathbb{R}^n \) denotes the state with a zero vector at the initial time, and the matrices \( A_K(\theta), \text{ etc.}, \text{ all of which should be designed, are supposed to have compatible dimensions.} \)

In a sharp contrast to some papers in the literature, e.g. [4] in the CT case and [5] in the DT case, the GSOF controller (3) should be independent of the derivative of the scheduling parameters \( \theta \) (CT case) or the one-step ahead scheduling parameters \( \theta^+ \); that is, the controller \( K(\theta) \) is required to be causal w.r.t. scheduling parameters.

2.3. Problem definition

The closed-loop system comprising \( G(\theta) \) and \( K(\theta) \) is given as \( G_{cl}(\theta) \):

\[
G_{cl}(\theta) = \begin{bmatrix} \delta[x_{cl}] \\ z \end{bmatrix} = \begin{bmatrix} A_{cl}(\theta) & B_{cl}(\theta) \\ C_{cl}(\theta) & D_{cl}(\theta) \end{bmatrix} \begin{bmatrix} x_{cl} \\ w \end{bmatrix},
\]

where \( x_{cl} = [x^T_1 x^T_K]^T \) denotes the state, and the matrices \( A_{cl}(\theta), \text{ etc.} \) are straightforwardly calculated as follows:

\[
\begin{bmatrix} A_{cl}(\theta) & B_{cl}(\theta) \\ C_{cl}(\theta) & D_{cl}(\theta) \end{bmatrix} = \begin{bmatrix} A(\theta) & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} B_1(\theta) \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} D_{11}(\theta) \\ D_{12}(\theta) \end{bmatrix}.
\]

Our problem is defined below.

\textbf{Problem 2.1:} For given scalar \( \gamma \in \mathbb{R}_+ \), design a GSOF controller \( K(\theta) \) defined in (3) such that the closed-loop system \( G_{cl}(\theta) \) is stabilized and (5) is satisfied for all admissible trajectories \( (\theta, \delta[\theta]) \in \Lambda_\theta \).

\[
||z||_2^2 < \gamma^2 ||w||_2^2
\]

2.4. Brief review of methods in [9]

In this subsection, a brief review of our previous result in [9] is given with a comparison of the methods in the literature.
The key technique used in [9] is Elimination lemma [11,12], which is given below.

Lemma 2.1 ([9]): Suppose that matrices \( Q_0 \in \mathbb{R}_{++}^n \), \( Q_1 \in \mathbb{R}^{n \times l} \) and \( Q_2 \in \mathbb{R}^{l \times n} \) are given. Then, the following three conditions, i.e. (6), (7) and (8), are all equivalent.

\[
Q_0 - H_0 \{ Q_2 \} Q_2 > 0
\]  
(6)

\[
\exists \bar{\mathbf{R}} \in \mathbb{R}^{l \times l} \text{ s.t. } \begin{bmatrix} Q_0 & Q_1 \\ 0 & 0 \end{bmatrix} + H_0 \begin{bmatrix} 0 \\ 1 \end{bmatrix} > 0
\]
(7)

\[
\exists \tilde{\mathbf{R}} \in \mathbb{R}^{l \times l} \text{ s.t. } \begin{bmatrix} Q_0 & Q_2 \\ 0 & 0 \end{bmatrix} + H_0 \begin{bmatrix} 0 \\ 1 \end{bmatrix} > 0
\]
(8)

2.4.1. Continuous-time case

The following condition to guarantee the induced \( L_2 \) norm (5) with \( X_{cl}(\theta) \in \mathbb{S}^n_+ \) is well known. (A formulation using \( X_{cl}(\theta)^{-1} \) is given in [3].)

\[
\begin{bmatrix} H_0 \{ A_{cl}(\theta) X_{cl}(\theta) \} - \dot{X}_{cl}(\theta) & * \\
C_{cl}(\theta) X_{cl}(\theta) & \gamma I \\
* & * - \gamma I \end{bmatrix} > 0
\]
(9)

Based on (9) with \( X_{cl}(\theta) = \begin{bmatrix} \mathcal{X}(\theta) & \mathcal{Y}(\theta) \\ \mathcal{Y}(\theta) & \mathcal{Y}(\theta) \end{bmatrix} \), whose structure can be set without loss of generality (see [16]) as long as the same parametrization in [4,17] is adopted, the following lemma, in which \( \mathcal{Y}(\theta) \) is obtained as \( \mathcal{X}(\theta) - \mathcal{Z}(\theta)^{-1} \), has been proposed with help of Lemma 2.1 to maintain the causality of \( K(\theta) \).

Lemma 2.2 ([9]): For given \( \gamma \in \mathbb{R}_+ \), if there exist matrices \( \mathcal{R}(\theta, \dot{\theta}) \in \mathbb{R}^{n \times n} \) and \( \mathcal{K}(\theta) \in \mathbb{R}^{(n+n_n) \times (n+n_n)} \), and continuously differentiable matrices \( \mathcal{X}(\theta), \mathcal{Z}(\theta) \in \mathbb{S}^n_+ \) such that (10) and (11) hold for all possible pairs \((\theta, \dot{\theta}) \in \Lambda_\theta \), then the controller \( K(\theta) \), whose state-space matrices are given in (13), stabilizes the closed-loop system \( G_{cl}(\theta) \) and satisfies (5) for all admissible trajectories \((\theta, \dot{\theta}) \in \Lambda_\theta \).

The following condition to guarantee the induced \( L_2 \) norm (5) with \( X_{cl}(\theta) \in \mathbb{S}^n_+ \) is well known. (A formulation using \( X_{cl}(\theta)^{-1} \) is given in [3].)

\[
\begin{bmatrix} \mathcal{X}(\theta) & 1 \\
1 & \mathcal{Z}(\theta) \end{bmatrix} > 0
\]
(10)

\[
-\Upsilon^{c}(\theta) - \text{diag} \left( \begin{bmatrix} -\dot{\mathcal{X}}(\theta) & 0 & 0 \\ \mathcal{Z}(\theta) & 0 & 0 \end{bmatrix} \right) > 0
\]

\[
\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + H_0 \begin{bmatrix} [ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} I]N \end{bmatrix} \mathcal{R}(\theta, \dot{\theta}) \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \mathcal{Z}(\theta) > 0,
\]

where \( \Upsilon^{c}(\theta) \) is

\[
\begin{bmatrix} H_0 \{ \Upsilon_A(\theta) \} & * & \Upsilon_B(\theta) \\
\Upsilon_C(\theta) & -\gamma I & \Upsilon_D(\theta) \\
* & -\gamma I & -\gamma I \end{bmatrix}
\]

Applying Lemma 2.1 to (11) after the factorization \( \mathcal{R}(\theta, \dot{\theta}) = R(\theta, \dot{\theta}) \mathcal{Z}(\theta)^{-1} \) with some \( R(\theta, \dot{\theta}) \in \mathbb{R}^{n \times n} \) leads to the following inequality directly:

\[
-\Upsilon^{c}(\theta) - \text{diag} \left( \begin{bmatrix} -\dot{\mathcal{X}}(\theta) & 0 & 0 \\ \mathcal{Z}(\theta) & 0 & 0 \end{bmatrix} \right) > 0.
\]

Then, the same technique in [4] is used to prove the assertion. Due to the created term \( \mathcal{Z}(\theta) \mathcal{Z}(\theta)^{-1} \), the state-space matrices of GSOF controller \( K(\theta) \) are independent of the derivatives of scheduling parameters, as shown in (13); that is, the causality of GSOF controllers w.r.t. scheduling parameters is kept.

However, as the condition (11) is a Bilinear Matrix Inequality (BMI), the following lemma with the use of a line search parameter \( r \in \mathbb{R}_+ \) instead of a full matrix \( \mathcal{R}(\theta, \dot{\theta}) \) has been proposed with some conservatism admitted.

Lemma 2.3 ([9]): For given \( \gamma \in \mathbb{R}_+ \), if there exist a scalar \( r \in \mathbb{R}_+ \), a matrix \( \mathcal{K}(\theta) \in \mathbb{R}^{(n+n_n) \times (n+n_n)} \), and continuously differentiable matrices \( \mathcal{X}(\theta), \mathcal{Z}(\theta) \in \mathbb{S}^n_+ \) such that (10) and (11), in which \( \mathcal{R}(\theta, \dot{\theta}) \) is replaced by \( rI \), hold for all possible pairs \((\theta, \dot{\theta}) \in \Lambda_\theta \), then the controller \( K(\theta) \), whose state-space matrices are given in (13), stabilizes the closed-loop system \( G_{cl}(\theta) \) and satisfies (5) for all admissible trajectories \((\theta, \dot{\theta}) \in \Lambda_\theta \).

Remark 2.1: Some conservatism due to the structural constraint of \( \mathcal{R}(\theta, \dot{\theta}) = rI \) is introduced in Lemma 2.3; however, it has been proved in [9] that Lemma 2.3 is no more conservative than the method in [8], which is also no more conservative than the method using partially constant Lyapunov matrix (i.e. \( \mathcal{Z}(\theta) = \mathcal{Z} \)) in [4].
2.4.2. Discrete-time case

The following condition to guarantee the induced \( l_2 \) norm (5) with \( X_{cl}(\theta) \in S^+_n \) is well-known [18].

\[
\begin{bmatrix}
X_{cl}(\theta) & * & * & 0 \\
A_{cl}(\theta)X_{cl}(\theta) & X_{cl}(\theta^+) & 0 & B_{cl}(\theta) \\
C_{cl}(\theta)X_{cl}(\theta) & 0 & \gamma I & D_{cl}(\theta)
\end{bmatrix} > 0
\]  \hspace{1cm} (14)

Based on (14) with the same parametrization of \( X_{cl}(\theta) \) in the CT case, the following has been proposed with help of Lemma 2.1 to escape from the multiplicatons of matrices depending on the current scheduling parameters \( \theta \) and one-step ahead scheduling parameters \( \theta^+ \).

**Lemma 2.4 ([9]):** For given \( \gamma \in \mathbb{R}_+ \), if there exist matrices \( R(\theta, \theta^+) \in \mathbb{R}^{n \times n} \), \( X(\theta), Z(\theta) \in S^+_n \), and \( K(\theta) \in \mathbb{R}^{(n+n_u) \times (n+n_y)} \) such that (15) holds for all possible pairs \( (\theta, \theta^+) \in \Lambda_\theta \), then the controller \( K(\theta) \), whose state-space matrices are given in (13), stabilizes the closed-loop system \( G_{cl}(\theta) \) and satisfies (5) for all admissible trajectories \( (\theta, \theta^+) \in \Lambda_\theta \).

\[
\begin{bmatrix}
\Upsilon^d_{rev}(\theta, \theta^+) & * & 0 \\
0 & Z(\theta^+) - Z(\theta) & 0 & 0 \\
\text{He} & \left[ \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix} \right] & R(\theta, \theta^+) & \times \\
\left[ \begin{bmatrix} 0 & 1 \end{bmatrix} + \frac{1}{2}Z(\theta) \right] & \left[ \begin{bmatrix} 0 & 0 \end{bmatrix} \right] & Z(\theta)^+ & \left[ \begin{bmatrix} 0 & 0 \end{bmatrix} \right]
\end{bmatrix} > 0, \hspace{1cm} (15)
\]

where \( \Upsilon^d_{rev}(\theta, \theta^+) \) is

\[
\Upsilon^d(\theta, \theta^+) + \text{diag} \left( \begin{bmatrix} 0 & 1 \end{bmatrix}, Z(\theta) \right), 0, 0
\]

with \( \Upsilon^d(\theta, \theta^+) \) defined as

\[
\begin{bmatrix}
X(\theta) & 1 & Z(\theta) \\
1 & 0 & \gamma I
\end{bmatrix}
\]

\[
\begin{bmatrix}
\Upsilon_A(\theta) & * & * & 0 \\
\text{diag} \left( X(\theta^+), 0 \right) & \gamma I & \Upsilon_B(\theta) \\
\Upsilon_C(\theta) & 0 & \gamma I & \Upsilon_D(\theta)
\end{bmatrix}
\]

using the same definitions of \( \Upsilon_A(\theta) \), etc. in Lemma 2.2.

Remark 2.2: Similar to the CT case, some conservatism due to the structural constraint of \( R(\theta, \theta^+) = rI \) is introduced in Lemma 2.5; however, it has been proved in [9] that Lemma 2.5 is no more conservatism than the design method using partially constant Lyapunov matrix, i.e. \( Z(\theta) = Z \).

### 3. Dual formulation

In this section, we try to show the dual formulations of Lemma 2.2 and Lemma 2.4 for Problem 2.1. To this end, the following analysis conditions are adopted here with \( P_{cl}(\theta) = X_{cl}(\theta)^{-1} \):

\[
\begin{bmatrix}
\text{He} & \left[ \begin{bmatrix} P_{cl}(\theta) & A_{cl}(\theta) \end{bmatrix} + \hat{P}_{cl}(\theta) \right] & * & P_{cl}(\theta)B_{cl}(\theta) \\
\text{C}_{cl}(\theta) & * & -\gamma I & D_{cl}(\theta)
\end{bmatrix} > 0
\]  \hspace{1cm} (17)

for the CT case, and

\[
\begin{bmatrix}
P_{cl}(\theta) & * & * & 0 \\
P_{cl}(\theta)^+ & 0 & P_{cl}(\theta)^+B_{cl}(\theta) & P_{cl}(\theta^+)A_{cl}(\theta) \\
\text{C}_{cl}(\theta) & 0 & \gamma I & D_{cl}(\theta) \\
0 & * & \gamma I & \gamma I
\end{bmatrix} > 0
\]  \hspace{1cm} (18)

for the DT case.
We first parametrize matrix $P_d(\theta)$ as follows:

$$P_d(\theta) = \begin{bmatrix} \mathcal{P}(\theta) & \mathcal{Q}(\theta) \\ \mathcal{Q}(\theta) & \mathcal{Q}(\theta) \end{bmatrix}.$$  \hfill (19)

This can be set without loss of generality as long as the reason is detailed below.

In [4], by following [17], matrix $X_d(\theta)$ is parametrized as

$$X_d(\theta) = \begin{bmatrix} \mathcal{A}(\theta) & \mathcal{I}(\theta) \\ \mathcal{M}(\theta)^T & \mathcal{I}(\theta) \end{bmatrix}^{-1}$$

with $\mathcal{A}(\theta)$ and $\mathcal{I}(\theta)$ being set as decision matrices, and $\mathcal{M}(\theta)$ and $\mathcal{N}(\theta)$ being set as arbitrary non-singular matrices satisfying the equation $\mathcal{M}(\theta)^T \mathcal{N}(\theta)^T = \mathbf{I} - \mathcal{A}(\theta) \mathcal{I}(\theta)$.

In Lemma 2.2 and Lemma 2.4, $\mathcal{X}(\theta)$ and $\mathcal{Z}(\theta)$ are set as follows:

$$\mathcal{X}(\theta) = \mathcal{A}(\theta), \quad \mathcal{Z}(\theta) = \mathcal{I}(\theta)$$

with a special factorization of $\mathcal{M}(\theta)$ and $\mathcal{N}(\theta)$ being set as $\mathcal{M}(\theta) = \mathcal{I}(\theta)$ and $-\mathcal{I}(\theta)$, respectively.

Now, as $P_d(\theta)$ is defined as $X_d(\theta)^{-1}$, $P_d(\theta)$ is straightforwardly calculated as

$$P_d(\theta) = \begin{bmatrix} \mathcal{I}(\theta) & \mathcal{N}(\theta) \\ \mathcal{N}(\theta)^T & -\mathcal{N}(\theta)^T \mathcal{A}(\theta) \mathcal{M}(\theta)^T \end{bmatrix}$$

using the matrices defined in [4]. If $\mathcal{M}(\theta)$ and $\mathcal{N}(\theta)$ are set as

$$\mathcal{M}(\theta) = -\mathcal{A}(\theta), \quad \mathcal{N}(\theta) = \mathcal{I}(\theta) - \mathcal{A}(\theta)^{-1},$$

then $P_d(\theta)$ is straightforwardly calculated as

$$P_d(\theta) = \begin{bmatrix} \mathcal{I}(\theta) & \mathcal{I}(\theta) - \mathcal{A}(\theta)^{-1} \\ \mathcal{I}(\theta) - \mathcal{A}(\theta)^{-1} & \mathcal{I}(\theta) - \mathcal{A}(\theta)^{-1} \end{bmatrix}.$$  \hfill (20)

Thus, without loss of generality, $P_d(\theta)$ is parametrized as in (19) with $\mathcal{P}(\theta)$ and $\mathcal{Q}(\theta)$ being set as $\mathcal{I}(\theta)$ and $\mathcal{I}(\theta) - \mathcal{A}(\theta)^{-1}$, respectively.

### 3.1. Continuous-time case

Using the parametrization of $P_d(\theta)$, the following is successfully derived for Problem 2.1.

**Theorem 3.1:** For given $\gamma \in \mathbb{R}_+$, if there exist matrices $\bar{R}(\theta, \dot{\theta}) \in \mathbb{R}^{n \times n}$ and $\bar{K}(\theta) \in \mathbb{R}^{(n+n_a) \times (n+n_y)}$, and continuously differentiable matrices $\mathcal{P}(\theta), \mathcal{S}(\theta) \in \mathbb{S}^n_+$ such that (20) and (21) hold for all admissible trajectories $(\theta, \dot{\theta}) \in \Lambda_\theta$, then the controller $K(\theta)$, whose state-space matrices are given in (23), stabilizes the closed-loop system $G_d(\theta)$ and satisfies (5) for all admissible trajectories $(\theta, \dot{\theta}) \in \Lambda_\theta$. Theorem 3.1:

$$\begin{bmatrix} \mathcal{P}(\theta) & \mathcal{I} \\ \mathcal{S}(\theta) \end{bmatrix} > 0$$  \hfill (20)

$$\begin{bmatrix} \mathcal{P}(\theta) \\ \mathcal{S}(\theta) \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$  \hfill (21)

where $\tilde{K}(\theta)$ is

$$\begin{bmatrix} \mathcal{H}(\bar{A}(\theta)) & -\gamma I \\ -\gamma I & -\gamma I \end{bmatrix} \begin{bmatrix} \mathcal{P}(\theta) \mathcal{A}(\theta) & 0 \\ \mathcal{A}(\theta) \mathcal{S}(\theta) & \mathcal{P}(\theta) \mathcal{B}_1(\theta) \end{bmatrix} \begin{bmatrix} \mathcal{P}(\theta)^{-1} & 0 \\ 0 & \mathcal{D}_1(\theta) \end{bmatrix}.$$  \hfill (22)

Proof: It is confirmed from (20) that matrix $\mathcal{S}(\theta)$ is non-singular, thus $\bar{R}(\theta, \dot{\theta})$ can be factorized as $\bar{R}(\theta, \dot{\theta}) \mathcal{S}(\theta)^{-1}$ with some matrix $\bar{R}(\theta, \dot{\theta}) \in \mathbb{R}^{n \times n}$. Then, inequality (21) is equivalently expressed as (8) with the following matrices.

$$\begin{bmatrix} 0 \\ \mathcal{P}(\theta) \mathcal{A}(\theta) \mathcal{S}(\theta) \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$  \hfill (23)

Proof: It is confirmed from (20) that matrix $\mathcal{S}(\theta)$ is non-singular, thus $\bar{R}(\theta, \dot{\theta})$ can be factorized as $\bar{R}(\theta, \dot{\theta}) \mathcal{S}(\theta)^{-1}$ with some matrix $\bar{R}(\theta, \dot{\theta}) \in \mathbb{R}^{n \times n}$. Then, inequality (21) is equivalently expressed as (8) with the following matrices.

$$Q_0 = -\tilde{R}(\theta) - \mathcal{S}(\theta)^{-1} \mathcal{S}(\theta) \geq 0$$

$$Q_2 = \begin{bmatrix} \mathcal{S}(\theta)^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathcal{S}(\theta)^{-1} & 0 \\ 0 & 0 \end{bmatrix}^T$$

Therefore, the following inequality is straightforwardly derived:

$$-\tilde{R}(\theta) - \mathcal{S}(\theta)^{-1} \mathcal{S}(\theta) \geq 0.$$
This inequality is confirmed to be equivalent to the following:

\[
\begin{bmatrix}
T(\theta) & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} (\text{L.H.S. of (17)}) \begin{bmatrix}
T(\theta) & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}^T > 0,
\]

where \( P_d(\theta) \) is in (19), and \( T(\theta) = \left[ \frac{I}{S(\theta)} - S(\theta) \right] \) with \( S(\theta) = (P(\theta) - Q(\theta))^{-1} \), after the change-of-variables in (23) is applied. Similarly, the following is also confirmed:

\[
\hat{P}_d(\theta) := T(\theta) P_d(\theta) T(\theta)^T = \begin{bmatrix}
P(\theta) & I \\
I & S(\theta)
\end{bmatrix}.
\]

Since \( S(\theta) \) is non-singular, the positivity of \( P_d(\theta) \) and the satisfaction of (17) are proved.

Since the condition (21) is a BMI, the following lemma with use of a line search parameter \( \bar{r} \in \mathbb{R}_+ \) instead of a full matrix \( \bar{R}(\theta, \hat{\theta}) \) is proposed with some conservatism admitted.

**Lemma 3.2:** For given \( \bar{\gamma} \in \mathbb{R}_+ \), if there exist a scalar \( \bar{r} \in \mathbb{R}_+ \), a matrix \( \bar{K}(\theta) \in \mathbb{R}^{(n+n_0) \times (n+n_0)} \), and continuously differentiable matrices \( \bar{P}(\theta), S(\theta) \in \mathbb{S}_+^n \) such that (20) and (21), in which \( \bar{R}(\theta, \hat{\theta}) \) is replaced by \( \bar{r} \) hold for all possible pairs \( (\theta, \hat{\theta}) \in \Lambda_{\theta} \), then the controller \( K(\theta) \), whose state-space matrices are given in (23), stabilizes the closed-loop system \( G_d(\theta) \) and satisfies (5) for all admissible trajectories \( (\theta, \hat{\theta}) \in \Lambda_{\theta} \).

Then, similarly to [9], the following holds.

**Theorem 3.3:** The following holds:

\[
\bar{\gamma}^c \geq \bar{\gamma}^P \geq \bar{\gamma}^{EL} \geq \bar{\gamma}^{non},
\]

where

\[
\begin{align*}
\bar{\gamma}^c &:= \min_{\bar{r}, \bar{K}(\theta), \text{constant } \bar{P} \text{ and } S} \bar{\gamma} \text{ s.t. (20) and } \bar{T}^c(\theta) < 0, \\
\bar{\gamma}^P &:= \min_{\bar{r}, \bar{K}(\theta), \bar{P}(\theta), \text{constant } S} \bar{\gamma} \text{ s.t. (20) and } \bar{T}^i(\theta) + \text{diag} \left( \text{diag} \left( \bar{P}(\theta), 0 \right), 0, 0 \right) < 0, \\
\bar{\gamma}^{EL} &:= \min_{\bar{r}, \bar{K}(\theta), \bar{P}(\theta), S(\theta), \bar{R}(\theta, \hat{\theta})=\bar{r} \in \mathbb{R}_+} \bar{\gamma} \text{ s.t. (20) and } (21), \\
\bar{\gamma}^{non} &:= \min_{\bar{r}, \bar{K}(\theta), \bar{P}(\theta), S(\theta), \bar{R}(\theta, \hat{\theta})} \bar{\gamma} \text{ s.t. (20) and } \bar{T}^i(\theta) + \text{diag} \left( \text{diag} \left( \bar{P}(\theta), -\bar{S}(\theta) \right), 0, 0 \right) < 0.
\end{align*}
\]

**Proof:** The relation \( \bar{\gamma}^c \geq \bar{\gamma}^P \) is obvious due to the use of parameter-dependent \( \bar{P}(\theta) \) in \( \bar{\gamma}^P \). The relation \( \bar{\gamma}^{EL} \geq \bar{\gamma}^{non} \) is also obvious due to the structural constraint for \( \bar{R}(\theta, \hat{\theta}) \), and \( \bar{\gamma}^{EL} \geq \bar{\gamma}^{non} \) is further obvious because the inequality \( \bar{T}^i(\theta) + \text{diag} \left( \text{diag} \left( \bar{P}(\theta), -\bar{S}(\theta) \right), 0, 0 \right) < 0 \) is included in (21). Thus, we only prove \( \bar{\gamma}^P \geq \bar{\gamma}^{EL} \). In pursuing \( \bar{\gamma}^P \), if \( \bar{T}^i(\theta) + \text{diag} \left( \text{diag} \left( \bar{P}(\theta), 0 \right), 0, 0 \right) < 0 \) holds, then there exists a sufficiently small \( \bar{r} \) satisfying \( \bar{T}^c(\theta) + \text{diag} \left( \text{diag} \left( \bar{P}(\theta), 0 \right), 0, 0 \right) < 0 \). This inequality is equivalent to (21) with \( \bar{R}(\theta, \hat{\theta}) = \bar{I} \) and constant \( S \). Thus, \( \bar{\gamma}^P \geq \bar{\gamma}^{EL} \) holds due to the freedom of \( \bar{r} \) in pursuing \( \bar{\gamma}^{EL} \).

**Remark 3.1:** Note that only GSOF controllers designed by the optimization of \( \bar{\gamma}^{non} \) depend on \( \bar{\theta} \) and \( \hat{\theta} \) as in [4], and all the other GSOF controllers depend only on \( \theta \), i.e. causal GSOF controllers.

### 3.2. Discrete-time case

Similarly to the CT case, \( P_d(\theta) \) defined in (19) is used for (18). Multiplying matrix \( \text{diag}(T(\theta), T(\theta^+), I, I) \) and its transpose from the left and the right, respectively, to (18) gives the following:

\[
\begin{bmatrix}
\bar{P}(\theta) \\
T(\theta^+) P_d(\theta^+) A_d(\theta) T(\theta)^T \\
C_d(\theta) T(\theta)^T \\
0 \\
0 \\
T(\theta^+) P_d(\theta^+) B_d(\theta) \\
D_d(\theta) \\
\gamma I
\end{bmatrix}
> 0,
\]

(25)

where the detailed expressions of \( T(\theta^+) P_d(\theta^+) A_d(\theta) T(\theta)^T, T(\theta^+) P_d(\theta^+) B_d(\theta), C_d(\theta) T(\theta)^T \) and \( D_d(\theta) \) are shown below. (The meaning of underline and doubly underlined parts will be described later.)

\[
T(\theta^+) P_d(\theta^+) A_d(\theta) T(\theta)^T =
\begin{bmatrix}
P(\theta^+) B_2(\theta) D_k(\theta) + Q(\theta^+) B_k(\theta) C_2(\theta) + P(\theta^+) A(\theta) \\
A(\theta) + B_1(\theta) D_k(\theta) C_2(\theta) \\
\bar{P}(\theta^+) A(\theta) S(\theta) \\
\bar{P}(\theta^+) B_2(\theta) D_k(\theta) + Q(\theta^+) B_k(\theta) C_2(\theta) S(\theta)
\end{bmatrix}
\]

(26)

\[
T(\theta^+) P_d(\theta^+) B_d(\theta) =
\begin{bmatrix}
P(\theta^+) B_1(\theta) \\
+ P(\theta^+) B_2(\theta) D_k(\theta) + Q(\theta^+) B_k(\theta) D_{21}(\theta)
\end{bmatrix}
\]

(27)

\[
C_d(\theta) T(\theta)^T =
\begin{bmatrix}
D_{12}(\theta) D_k(\theta) C_2(\theta) + D_{12}(\theta) C_1(\theta) S(\theta) \\
D_{12}(\theta) [D_k(\theta) C_2(\theta) - C_k(\theta)] S(\theta)
\end{bmatrix}
\]

(28)
If we apply change-of-variables similar to (23), \( C_d(\theta) \) \( T(\theta)^T \) and \( D_d(\theta) \) become affine with respect to new decision matrices in the same way as the CT case, namely, the following expressions, which are affine w.r.t. decision matrices \( (C_k(\theta), D_k(\theta) \) and \( S(\theta)) \), are derived using \( C(\theta) := [D_k(\theta)C_2(\theta) - C_k(\theta)]S(\theta) \) and \( D_k(\theta) := D_k(\theta) \).

\[
C_d(\theta)T(\theta)^T = \begin{bmatrix} D_{12}(\theta)D_k(\theta)C_2(\theta) + C_1(\theta) \\
D_{12}(\theta)C_k(\theta) + C_1(\theta)S(\theta) \end{bmatrix} \\
D_d(\theta) = D_{11}(\theta) + D_{12}(\theta)D_k(\theta)D_{21}(\theta)
\]

However, an intractability arises in \( T(\theta)^+P_d(\theta)^+B_d(\theta) \), since the underlined part i.e. \( P(\theta)^+B_2(\theta)D_k(\theta) + Q(\theta)^+B_k(\theta) \), which is set as a new decision matrix in [5], contains the multiplications of matrices depending on the current and one-step ahead scheduling parameters.

The term \( T(\theta)^+P_d(\theta)^+A_d(\theta)T(\theta)^T \) contains \( P(\theta)^+B_2(\theta)D_k(\theta) + Q(\theta)^+B_k(\theta) \) as well as further complicated terms (i.e. the doubly underlined part).

Due to these intractable terms, the dual formulation of [9] cannot be derived in the DT case at the current moment.

However, as mentioned above, if causality w.r.t. scheduling parameters is abandoned then non-causal GSOF controller, whose state-space matrices are given in (31), can be designed similarly to [5]. The method is given below.

**Theorem 3.4:** For given \( \bar{\gamma} \in \mathbb{R}_+ \), if there exist matrices \( P(\theta), S(\theta) \in \mathbb{S}_+^n \), and \( \bar{K}(\theta) \in \mathbb{R}^{(n+m)\times(n+m)} \) such that (30) holds for all possible pairs \( (\theta, \theta^+) \in \Lambda_\theta \), then the controller \( K(\theta) \), whose state-space matrices are given in (31), stabilizes the closed-loop system \( G_d(\theta) \) and satisfies (5) for all admissible trajectories \( (\theta, \theta^+) \in \Lambda_\theta \).

\[
\begin{bmatrix}
\bar{P}_d(\theta) & * & * & 0 \\
\bar{Y}_A(\theta, \theta^+) & \bar{P}_d(\theta^+) & 0 & \bar{Y}_B(\theta, \theta^+) \\
\bar{Y}_C(\theta) & 0 & \gamma I & \bar{Y}_D(\theta) \\
0 & * & * & \gamma I
\end{bmatrix} > 0, \quad (30)
\]

where

\[
\bar{P}_d(\theta) = \begin{bmatrix} P(\theta) & I \\
I & S(\theta) \end{bmatrix}
\]

and

\[
\begin{bmatrix}
\bar{Y}_A(\theta, \theta^+) & \bar{Y}_B(\theta, \theta^+) \\
\bar{Y}_C(\theta) & \bar{Y}_D(\theta)
\end{bmatrix}
\]

are defined as

\[
\begin{bmatrix} \mathcal{P}(\theta^+)A(\theta) & 0 \\
A(\theta)S(\theta) & \mathcal{P}(\theta^+)B_1(\theta) \end{bmatrix}
\]

\[
\begin{bmatrix} 1 & 0 \\
0 & B_2(\theta) \end{bmatrix} \bar{K}(\theta) \begin{bmatrix} 0 & I \\
C_2(\theta) & 0 \end{bmatrix} \begin{bmatrix} 0 & I \\
0 & D_{21}(\theta) \end{bmatrix}.
\]

\[
\begin{bmatrix} A_k(\theta, \theta^+) & B_k(\theta, \theta^+) \\
C_k(\theta) & D_k(\theta) \end{bmatrix} = \begin{bmatrix} Q(\theta^+)^{-1} & -Q(\theta^+)^{-1}P(\theta^+)B_2(\theta) \\
0 & 1 \end{bmatrix}
\]

\[
\times \left( \bar{K}(\theta) - \begin{bmatrix} \mathcal{P}(\theta^+)A(\theta)S(\theta) & 0 \\
0 & 0 \end{bmatrix} \right), \quad (31)
\]

where \( Q(\theta) = \mathcal{P}(\theta) - S(\theta)^{-1} \).

**4. Numerical examples**

We show two CT toy examples and one DT practical example below. All results are obtained by using SDPT3 [19].

**4.1. Continuous-time toy example No. 1**

We consider the same toy example as in [8,10,16], which has been originally used in [15]. The state-space matrices of LPV system \( G(\theta) \) in (1) with a single scheduling parameter \( \Lambda_\theta = [-3, 3] \times [-1, 1] \) are given below.

\[
\begin{bmatrix}
A(\theta) & B_1(\theta) \\
C_1(\theta) & D_{11}(\theta) \\
C_2(\theta) & D_{21}(\theta) \\
0 & D_{22}(\theta)
\end{bmatrix}
\]

\[
= \begin{bmatrix}
-4 & 3 & 5 & 1 & 0 \\
0 & 7 & -5 & -2 & 16 \\
0.1 & -2 & -3 & 1 & -10 \\
1 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 2 & 0
\end{bmatrix}
\]

\[
+ \theta \begin{bmatrix}
1 & 0 & 1 & 0 & 1 \\
2 & 0 & -5 & 0 & -5 \\
2 & 5 & 1.5 & 0 & 3.5 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}, \quad (32)
\]

We first design GSOF controllers using Lemma 3.2 for the LPV system with dual state-space representation of (32) with the line search parameter \( r \) set as 400 points linearly gridded over a logarithmic scale in \([10^{-10}, 10^0]\). The decision matrices are all set to be parametrically affine w.r.t. the scheduling parameter \( \theta \). Even in this setup, the inequality (21) is not parametrically affine. Therefore, we use slack variable
approach [20] to solve (21), similarly to [8,10,16]. Other than slack variable approach, it is possible to apply Sum-Of-Squares (SOS) relaxation [21,22], Polya’s theorem relaxation [23], etc. However, our focus is not the relaxation method for solving parametrically dependent LMIs, thus we use the same method as in [10] for fair comparison.

The result is shown as red line in Figure 1. The minimum is given as 21.73 at $\tilde{r} = 0.2652$. The result of Lemma 2.3 for (32) with the same setup for line search parameter and decision matrices is also shown as broken black line in the same figure. The minimum is given as 21.73 at $r = 0.2652$. As expected, these confirm that the dual formulation, i.e. Lemma 3.2, for the LPV system with the dual state-space representation gives the same performance as Lemma 2.3 for the LPV system with the state-space representation in (32).

We next design GSOF controllers using Lemma 3.2 for the LPV system with the state-space representation of (32) with the same setup for line search and decision matrices. The result is shown as blue line in Figure 1. The minimum is given as 22.48 at $\tilde{r} = 0.9439$. In this example, Lemma 2.3 is less conservative than Lemma 3.2.

4.2. Continuous-time toy example No. 2

Let us consider a slightly modified problem, namely, only $C_2(\theta)$ is revised as $C_2(\theta) = [0\;1\;0] + 0.2[1\;-1\;-1]$. The design results using Lemma 3.2 (blue line) and Lemma 2.3 (broken black line) with the same setup for line search and decision matrices are shown in Figure 2. The minimum of Lemma 3.2 is given as 70.97 at $\tilde{r} = 0.8410$, and the minimum of Lemma 2.3 is given as 73.04 at $r = 1.3345$. In this case, Lemma 3.2 is less conservative than Lemma 2.3.

Thus, in general, note that conservatism between Lemmas 2.3 and 3.2 depends on problems.

| $T$   | $\gamma$ by Lemma 2.5 | $\tilde{\gamma}$ by Theorem 3.4 |
|-------|-----------------------|-------------------------------|
| 7.0   | 2.752                 | 2.752                         |
| 7.5   | 2.738                 | 2.738                         |
| 8.0   | 2.726                 | 2.726                         |
| 8.5   | 2.716                 | 2.716                         |
| 9.0   | 2.707                 | 2.707                         |
| 9.5   | 2.699                 | 2.699                         |
| 10.0  | 2.691                 | 2.691                         |
4.3. Discrete-time practical example

The same DT practical example in [9], i.e. discrete-time GSOF flight controller design problem for the lateral-directional motion around wing level flight of MuPAL-α [24], is next addressed. The problem setup is completely the same as in [9], the details are omitted here.

We design causal GSOF controllers by Lemma 2.5, and non-causal GSOF controllers by Theorem 3.4. The results are shown in Table 1, where “T” in the table denotes the design parameter defining control performance. (The results using Lemma 2.5 are the same as in [9].) In this example, the achievable control performance with causal GSOF controllers designed by Lemma 2.5 is the same as that with non-causal GSOF controllers designed by Theorem 3.4. The numerical complexity of these two methods is slightly different. The numbers of decision matrices in Lemma 2.5 and Theorem 3.4 are the same, i.e. 1337, since the line search parameter \( r \) in Lemma 2.5 is given a priori; however, the row numbers of LMI in Lemma 2.5 and Theorem 3.4 are different, namely, the former has 36, 984 rows and the latter has 30, 552 rows for 402 gridding points of the admissible region of \( \alpha \). This is due to that Lemma 2.5 has additional rows and columns to alleviate the problematic term which introduces non-causality to GSOF controllers.

For our addressed practical example, the minimized induced \( L_2 \) norm by Lemma 2.5 and Theorem 3.4 is coincidentally the same under our problem setup; however, in general, Theorem 3.4 is less conservative than Lemma 2.5 due to the permission of non-causal GSOF controllers.

5. Conclusions

On the GSOF controller design for LPV systems using PDLFs, we successfully show the dual formulation of our previous result for continuous-time case; however, the counterpart result for discrete-time case cannot be straightforwardly derived due to the complicatedly multiplied terms of the matrices depending on the current scheduling parameters and one-step ahead scheduling parameters.

Disclosure statement

No potential conflict of interest was reported by the author(s).

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