HUNEKE’S DEGREE-COMPUTING PROBLEM

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ABSTRACT. We deal with a problem posted by Huneke on the degree of generators of symbolic powers.

1. INTRODUCTION

Let \( A_m := K[x_1, \ldots, x_m] \) be the polynomial ring of \( m \) variables over a field \( K \). We drop the subscript \( m \), when there is no doubt of confusion. Let \( I \) be an ideal of \( A \). Denote the \( n \)-th symbolic power of \( I \) by \( I^{(n)} := \bigcap_{p \in \text{Ass}(I)} (I^n A_p \cap A) \). Huneke [7] posted the following problem:

Problem 1.1. (Understanding symbolic powers). Let \( p \triangleright A \) be a homogeneous and prime ideal generated in degrees \( \leq D \). Is \( p^{(n)} \) generated in degrees \( \leq Dn \)?

The problem is clear when \( m < 3 \). We present three observations in support of Huneke’s problem. The first one deals with rings of dimension 3:

Observation A. Let \( I \triangleright A_3 \) be a radical ideal and generated in degrees \( \leq D \). Then \( I^{(n)} \) generated in degrees \( < (D + 1)n \) for all \( n \gg 0 \).

The point of this is to connect the Problem 1.1 to the fruitful land \( H^0_m(\cdot) \). This unifies our interest on symbolic powers as well as on the (LC) property. The later has a role on tight closure theory. It was introduced by Hochster and Huneke.

Corollary 1.2. Let \( I \triangleleft A_3 \) be any radical ideal. There is \( D \in \mathbb{N} \) such that \( I^{(n)} \) is generated in degrees \( \leq Dn \) for all \( n > 0 \).

Observation B. Let \( I \triangleleft A_4 \) be an ideal of linear type generated in degrees \( \leq D \). Then \( I^{(n)} \) is generated in degrees \( \leq Dn \).

I am grateful to Hop D. Nguyen for suggesting the following example to me:

Example 1.3. There is a radical ideal \( I \triangleleft A_6 \) generated in degrees \( \leq 4 \) such that \( I^{(2)} \) does not generated in degrees \( \leq 2.4 = 8 \).

There is a simpler example over \( A_7 \). Via the flat extension \( A_6 \to A_n \) we may produce examples over \( A_n \) for all \( n > 5 \). One has the following linear growth formula for symbolic powers:

Observation C. Let \( I \) be any ideal such that the corresponding symbolic Rees algebra is finitely generated. There is \( E \in \mathbb{N} \) such that \( I^{(n)} \) is generated in degrees \( \leq En \) for all \( n > 0 \).

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Observation C for monomial ideals follows from [8, Thorem 2.10] up to some well-known facts. In this special case, we can determine $E$:

**Corollary 1.4.** Let $I$ be a monomial ideal. Let $f$ be the least common multiple of the generating monomials of $I$. Then $I^{(n)}$ is generated in degrees $\leq \deg(f)n$ for all $n > 0$.

We drive this sharp bound when $I$ is monomial and radical not only by the above corollary, but also by an elementary method. There are many examples of ideals such as $I$ such that the corresponding symbolic Rees algebra is not finitely generated but there is $D \in \mathbb{N}$ such that $I^{(n)}$ is generated in degrees $\leq Dn$ for all $n > 0$. Indeed, Roberts constructed a prime ideal $p$ over $A_3$ such that the corresponding symbolic Rees algebra were not be finitely generated. However, we showed in Corollary 1.2 that there is $D$ such that $p^{(n)}$ is generated in degrees $\leq Dn$ for all $n > 0$. These suggest the following:

**Problem 1.5.** Let $I \triangleleft A$ be any ideal. There is $f \in \mathbb{Q}[X]$ such that $I^{(n)}$ is generated in degrees $\leq f(n)$ for all $n > 0$. Is $f$ linear?

Section 2 deals with preliminaries. The reader may skip it, and come back to it as needed later. In Section 3 we present the proof of the observations. Section 4 is devoted to the proof of Example 1.3. We refer the reader to [1] for all unexplained definitions in the sequel.

### 2. PRELIMINARIES

We give a quick review of the material that we need. Let $R$ be any commutative ring with an ideal $a$ with a generating set $a := a_1, \ldots, a_r$. By $H^i_a(M)$, we mean the $i$-th cohomology of the Čech complex of a module $M$ with respect to $a$. This is independent of the choose of the generating set. For simplicity, we denote it by $H^i_a(M)$. We equip the polynomial ring $A$ with the standard graded structure. Then, we can use the machinery of graded Čech cohomology modules.

**Notation 2.1.** Let $L = \bigoplus_{n \in \mathbb{Z}} L_n$ be a graded $A$-module and let $d \in \mathbb{Z}$.

i) The notation $L(d)$ is referred to the $d$-th twist of $L$, i.e., shifting the grading $d$ steps.

ii) The notation $\text{end}(L)$ stands for $\sup\{n : L_n \neq 0\}$.

iii) The notation $\text{beg}(L)$ stands for $\inf\{n : L_n \neq 0\}$.

**Discussion 2.2.** Denote the irrelevant ideal $\bigoplus_{n \geq 0} A_n$ of $A$ by $m$.

i) We use the principals that $\sup\{\emptyset\} = -\infty$, $\inf\{\emptyset\} = +\infty$ and that $-\infty < +\infty$.

ii) Let $M$ be a graded $A$-module. Then $H^i_m(M)$ equipped with a $\mathbb{Z}$-graded structure and $\text{end}(H^i_m(M)) < +\infty$.

**Definition 2.3.** The Castelnuovo-Mumford regularity of $M$ is

$$\text{reg}(M) := \sup\{\text{end}(H^i_m(M)) + i : 0 \leq i \leq \dim M\}.$$  

The $\text{reg}(M)$ computes the degrees of generators in the following sense.
Fact 2.4. A graded module $M$ can be generated by homogeneous elements of degrees not exceeding $\text{reg}(M)$.

The following easy fact translates a problem from symbolic powers to a problem on Čech cohomology modules.

Fact 2.5. Let $I \triangleleft A$ be a radical ideal of dimension one. Then $I^{(n)}/I^n = H^0_m(A/I^n)$.

We will use the following results:

Lemma 2.6. (See [2]) Let $I \triangleleft A$ be a homogeneous ideal such that $\dim A/I \leq 1$. Then $\text{reg}(I^n) \leq n \text{reg}(I)$ for all $n$.

Also, see [5]. Let us recall the following result from [4] and [10]. The regularity of $\text{reg} I^{(n)}$ is equal to $dn + e$ for all large enough $n$. Here $d$ is the smallest integer $n$ such that

$$(x : x \in I, \text{ and } x \text{ is homogeneous of degree at most } n)$$

is a reduction of $I$, and $e$ depends only on $I$. In particular, $e$ is independent of $n$.

Lemma 2.7. (See [2] Corollary 7]) Let $I \triangleleft A$ be a homogeneous ideal with $\dim A/I = 2$. Then $\text{reg} I^{(n)} \leq n \text{reg}(I)$.

The above result of Chandler generalized in the following sense:

Lemma 2.8. (See [8] Corollary 2.4]) Let $I \triangleleft A$ be a homogeneous ideal with $\dim A/I \leq 2$. Denotes the maximum degree of the generators of $I$ by $d(1)$. There is a constant $e$ such that for all $n > 0$ we have $\text{reg} I^{(n)} \leq nd(1) + e$.

3. PROOF OF THE OBSERVATIONS

If symbolic powers and the ordinary powers are the same, then Huneke’s bound is tight. We start by presenting some non-trivial examples to show that the desired bound is very tight. Historically, these examples are important.

Example 3.1. (This has a role in [8] Page 1801]) Let $R = \mathbb{Q}[x,y,z,t]$ and let $I := (xz, xt^2, y^2z)$. Then $I^{(2)}$ is generated in degrees $\leq 6 = 2 \times 3$. Also, $\text{beg}(I^{(2)}) = 2 \text{beg}(I)$.

Proof. The primary decomposition of $I$ is given by

$$I = (x, y^2) \cap (z, t^2) \cap (x, z).$$

By definition

$$I^{(2)} = (x, y^2)^2 \cap (z, t^2)^2 \cap (x, z)^2 = (x^2z^2, x^2zt^2, xy^2z^2, x^2t^4, xy^2z^2, y^4z^2).$$

Thus, $I^{(2)}$ is generated in degrees $\leq 6$. Clearly, $\text{beg}(I^{(2)}) = 4 = 2 \times \text{beg}(I)$. □

Example 3.2. Let $R = \mathbb{Q}[a, b, c, d, e, f]$ and let $I := (abc, abf, ace, ade, adf, bcd, bde, bcf, cdf, cef)$. Then $I^{(2)}$ is generated in degrees $\leq 6 = 2 \times 3$. Also, $\text{beg}(I^{(2)}) = 5 < 6 = 2 \times \text{beg}(I)$. 
Sturmfels showed that $\operatorname{reg}_1(I^2) = 7 > 6 = 2 \operatorname{reg}_1(I)$. Also, see Discussion 3.3.

**Proof.** This deduces from Corollary 1.4. Let us prove it by hand. The method is similar to Example 3.1. We left to reader to check that $I^{(2)}$ is generated by the following degree 5 elements

$$ \{bcdef, acdef, abdef, abcef, abcde, abcdf, abcdg\}, $$

plus to the following degree 6 elements

$$ \{c^2e^2f^2, bc^2e^2f^2, b^2c^2e^2f^2, c^2def^2, ab^2ef^2, c^2d^2f^2, \}

$$

\begin{align*}
acd^2f^2, a^2d^2ef, a^2bdf^2, a^2b^2f, b^2de^2f, a^2c^2ef, \\
- a^2d^2ef, b^2c^2f, a^2b^2cf, b^2d^2e^2, abd^2e^2, a^2d^2e^2, \\
a^2cde^2, a^2c^2e^2, b^2de^2e, a^2bc^2e, b^2c^2d^2, ab^2c^2d, a^2b^2c^2e^2. \\
- \end{align*}$

Thus $I^{(2)}$ is generated in degrees $\leq 6 = 2 \times 3$. Clearly, $\operatorname{deg}(I^{(2)}) = 5 < 6 = 2 \times \operatorname{deg}(I)$.

**Proposition 3.3.** Let $I \subset A$ be a homogeneous ideal such that $\dim A/I \leq 1$. Then $I^{(n)}$ generated in degrees $\leq \max \{n \operatorname{reg}(I) - 1, nD\}$ for all $n$.

**Proof.** Let $D$ be such that $I$ is generated in degree $\leq D$. Recall that $D \leq \operatorname{reg}(I)$. We note that $I^n$ generated in degree $\leq Dn$. As $\dim(A/I) = 1$ and in view of Fact 2.4, one has $I^{(n)}/I^n = H^0_m(A/I^n)$. Now look at the exact sequence

$$ 0 \longrightarrow I^n \rightarrow I^{(n)} \xrightarrow{\pi} I^{(n)}/I^n \longrightarrow 0. $$

Suppose $\{f_1, \ldots, f_s\}$ is a homogeneous system of generators for $I^n$. Also, suppose $\{\overline{g}_1, \ldots, \overline{g}_s\}$ is a homogeneous system of generators for $I^{(n)}/I^n$, where $g_j \in I^{(n)}$ defined by $\pi(g_j) = \overline{g}_j$. Hence, $\operatorname{deg}(g_j) = \operatorname{deg}(\overline{g}_j)$.

Let us search for a generating set for $I^{(n)}$. To this end, let $x \in I^{(n)}$. Hence $\pi(x) = \sum j s_j \overline{g}_j$ for some $s_j \in A$. Thus $x - \sum j s_j g_j \in \ker(\pi) = \operatorname{im}(\rho) = I^n$. This says that $x - \sum j s_j g_j = \sum i r_i f_i$ for some $r_i \in A$. Therefore, $\{f_1, \ldots, f_s\}$ is a homogeneous generating set for $I^{(n)}$.

Recall that $\operatorname{deg}(f_i) \leq nD$. Fixed $n$, and let $1 \leq j \leq s$. Its enough to show that

$$ \operatorname{deg}(g_j) \leq n \operatorname{reg}(I) - 1. $$

Keep in mind that $m \geq 3$. One has $\operatorname{depth}(A_m) = m > 2$. By [1] Proposition 1.5.15(e)], $\operatorname{grade}(m, A) \geq 2$. Look at

$$ 0 \longrightarrow I^n \longrightarrow A \longrightarrow A/I^n \longrightarrow 0. $$

This induces the following exact sequence:

$$ 0 \simeq H^0_m(A) \longrightarrow H^0_m(A/I^n) \longrightarrow H^1_m(I^n) \longrightarrow H^1_m(A) \simeq 0. $$

Thus, $H^0_m(A/I^n) \simeq H^1_m(I^n)$. In view of Lemma 2.6

$$ \operatorname{end}(H^0_m(I^n)) + 1 \leq \operatorname{reg}(I^n) \leq \operatorname{reg}(I)n. $$

Therefore, $\operatorname{end}(I^{(n)}/I^n) < \operatorname{reg}(I)n$. By Fact 2.4

$$ \operatorname{deg}(g_j) = \operatorname{deg}(\overline{g}_j) \leq \operatorname{end}(I^{(n)}/I^n) < \operatorname{reg}(I)n.$$
as claimed.

**Definition 3.4.** The ideal $I$ has a linear resolution if its minimal generators all have the same degree and the nonzero entries of the matrices of the minimal free resolution of $I$ all have degree one.

**Discussion 3.5.** In general powers of ideals with linear resolution need not to have linear resolutions. The first example of such an ideal was given by Terai, see [3]. It may be worth to note that his example were used in Example 3.2 for a different propose.

However, we have:

**Corollary 3.6.** Let $I \subset A$ be an ideal with a linear resolution, generated in degrees $\leq D$ such that $\dim A/I \leq 1$. Then $I^{(n)}$ generated in degrees $\leq Dn$ for all $n$.

**Proof.** It follows from Definition 3.4 that $\text{reg}(I) = D$. Now, Proposition 3.3 yields the claim. □

**Theorem 3.7.** Let $I \subset A_4$ be an (radical) ideal generated in degrees $\leq D$. There is an integer $E$ such that $I^{(n)}$ is generated in degrees $\leq En$. Suppose in addition that $I$ is of linear type. Then $I^{(n)}$ is generated in degrees $\leq Dn$.

**Proof.** We note that $I^n$ generated in degree $\leq Dn$. Suppose first that $\text{ht}(I) = 1$. Then $I = (x)$ is principal, because height-one radical ideals over unique factorization domains are principal. In this case $I^{(n)} = (x^n)$, because it is a complete intersection. In particular, $I^{(n)}$ generated in degrees $\leq Dn$. The case $\text{ht}(I) = 3$ follows by Corollary 3.6. Then without loss of the generality we may assume that $\text{ht}(I) = 2$. Suppose $I^{(n)}$ generated in degrees $\leq D_n$. Then

$$D_n \leq \text{reg}(I^{(n)}) \leq n \text{reg}(I^{3.4}) nD.$$ 

The proof in the linear-type case is complete. □

**Theorem 3.8.** Let $I \subset A_3$ be a homogeneous radical ideal, generated in degrees $\leq D$ and of dimension 1. Then $I^{(n)}$ generated in degrees $<(D+1)n$ for all $n \gg 0$.

**Proof.** Keep the proof of Theorem 3.7 in mind. Then, we may assume $\dim(A_3/I) = 1$. By the proof of Proposition 3.3 we need to show $H^1_m(I^n)$ generated in degrees $<(D+1)n$ for all $n \gg 0$. By [10], $\text{reg}(I^n) = a(I)n + b(I)$ for all $n \gg 0$. This is well-known that $a(I) \leq D$, see [10]. For all $n > b(I)$ sufficiently large,

$$\text{end}(H^1_m(I^n)) + 1 \leq \text{reg}(I^n) = a(I)n + b(I) \leq Dn + b(I) \leq (D+1)n,$$

as claimed. □

**Corollary 3.9.** Let $I \subset A_3$ be any radical ideal. There is $D \in \mathbb{N}$ such that $I^{(n)}$ is generated in degrees $\leq Dn$ for all $n > 0$.

*In a paper by Rees [11], there is a height-one prime ideal (over a normal domain) such that non of its symbolic powers is principal.*
Proof. Suppose \( I \) is generated in degrees \( \leq E \) for some \( E \). Let \( n_0 \) be such that \( I^{(n)} \) generated in degrees \( < (E+1)n \) for all \( n > n_0 \), see the above theorem. Let \( \ell_i \) be such that \( I^{(n)} \) generated in degrees \( < \ell_i \). Now, set \( e_i := \floor{\frac{\ell_i}{n}} + 1 \). Then \( I^{(n)} \) is generated in degrees \( < e_i n \) for all \( n \). Let 
\[
D := \sup \{ E, e_i : 1 \leq i \leq n_0 \}.
\]
Clearly, \( D \) is finite and that \( I^{(n)} \) is generated in degrees \( \leq Dn \) for all \( n > 0 \). \( \square \)

Corollary 3.10. Let \( I \triangleleft A_3 \) be a homogeneous radical ideal. Then \( I^{(n)} \) and \( I^n \) have the same reflexive-hull.

Proof. Set \( A := A_3 \). Without loss of the generality, we may assume that \( \text{ht}(I) = 2 \). Set \( (\cdot)^* := \text{Hom}_A(\cdot, A) \). We need to show \( (I^{(n)})^* \simeq (I^n)^* \). As, \( I^{(n)}/I^n = H^0_{m}(A/I^n) \) is of finite length, \( \text{Ext}^i_A(I^{(n)}/I^n, A) = 0 \) for all \( i < 3 \), because depth\( (A) = 3 \). Now, we apply \( (\cdot)^* \) to the following exact sequence
\[
0 \rightarrow I^n \rightarrow I^{(n)} \rightarrow I^{(n)}/I^n \rightarrow 0,
\]
to observe \( (I^{(n)})^* \simeq (I^n)^* \). From this we get the claim. \( \square \)

To prove Observation C we need:

Fact 3.11. (See [9, Theorem 3.2]) Let \( I \) be a monomial ideal in a polynomial ring over a field. Then the corresponding symbolic Rees algebra is finitely generated.

Here, we present the proof of Observation C:

Proposition 3.12. Let \( I \) be any ideal such that the corresponding symbolic Rees algebra is finitely generated (e.g. \( I \) is monomial). There is \( D \in \mathbb{N} \) such that \( I^{(n)} \) is generated in degrees \( \leq Dn \) for all \( n > 0 \).

Proof. (Suppose \( I \) is monomial. Then \( \mathcal{R} := \bigoplus_{i \geq 0} I^{(i)} \) is finitely generated, see Fact 3.11.) The finiteness of \( \mathcal{R} \) gives an integer \( \ell \) such that \( \mathcal{R} = R[I^{(i)} : i \leq \ell] \). Let \( e_i \) be such that \( I^{(i)} \) is generated in degree less or equal than \( e_i \) for all \( i \in \mathbb{N} \). Set \( d_i := \floor{\frac{\ell_i}{n}} + 1 \) for all \( i \leq \ell \). The notation \( D \) stands for \( \max\{d_i : \text{for all } i \leq \ell\} \). We note that \( D \) is finite. Let \( n \) be any integer. Then
\[
I^{(n)} = \sum j_1^{(j_1)} \ldots j_n^{(j_n)}, \quad \text{where } j_1 + \ldots + j_n = n \text{ and } 1 \leq j_i \leq \ell \quad \text{for all } 1 \leq i \leq n.
\]
This implies that
\[
e_n \leq e_{j_1} + \ldots + e_{j_n} \\
\leq j_1 d_{j_1} + \ldots + j_n d_{j_n} \\
\leq j_1 D + \ldots + j_n D \\
= D(j_1 + \ldots + j_n) \\
= Dn,
\]
as claimed. \( \square \)

Corollary 3.13. Let \( I = (f_1, \ldots, f_t) \) be a monomial ideal. Set \( E := \deg f_1 + \ldots + \deg f_t \). Then \( I^{(n)} \) is generated in degrees \( \leq En \) for all \( n > 0 \).
Proof. We may assume $I \neq 0$. Thus, $\text{ht}(I) \geq 1$. Let $f$ be the least common multiple of the generating monomials of $I$. In view of [8 Thorem 2.9] and for all $n > 0$, 

$$\text{reg}(I^{(n)}) \leq (\deg f)n - \text{ht}(I) + 1 \quad (\ast)$$

The notation $e_n$ stands for the maximal degree of the number of generators of $I^{(n)}$. Due to Fact 2.4 we have $e_n \leq \text{reg}(I^{(n)})$. Putting this along with $(\ast)$ we observe that

$$e_n \leq \text{reg}(I^{(n)}) \leq (\deg f)n - \text{ht}(I) + 1 \leq (\deg f)n$$

for all $n > 0$. It is clear that $\deg f \leq \deg(\prod f_i) = \sum_i \deg f_i = D$. \hfill $\Box$

One may like to deal with the following sharper bound:

Corollary 3.14. Let $I$ be a monomial ideal and let $f$ be the least common multiple of the generating monomials of $I$. Then $I^{(n)}$ is generated in degrees $\leq \deg f n$ for all $n > 0$.

The following is an immediate corollary of Corollary 3.14. Let us prove it without any use of advanced technics such as the Castelnuovo-Mumford regularity.

Remark 3.15. Let $I$ be a monomial radical ideal generated in degrees $\leq D$. Then $I^{(n)}$ is generated in degrees $\leq Dn$. Indeed, first we recall a routine fact. By $[u, v]$ we mean the least common multiple of the monomials $u$ and $v$. Denote the generating set of a monomial ideal $K$ by $G(K)$. Also, if $K = (u : u \in G(K))$ and $L = (v : v \in G(L))$ are monomial, then

$$K \cap L = \langle [u, v] : u \in G(K), \text{ and } v \in G(L) \rangle \quad (\ast)$$

Now we prove the desired claim. Let $I$ be a radical monomial ideal generated in degrees $\leq D$. The primary decomposition of $I$ is of the form

$$I = (X_{i_1}, \ldots, X_{i_k}) \cap \ldots \cap (X_{j_1}, \ldots, X_{j_l})$$

Set

$$\sum := \{X_i \in p_i \setminus \bigcup_{j \neq i} p_j \text{ for some } i\}.$$ 

In view of $(\ast)$, we see that $|\sum| = D$. By definition,

$$I^{(n)} = (X_{i_1}, \ldots, X_{i_k})^n \cap \ldots \cap (X_{j_1}, \ldots, X_{j_l})^n \quad (\ast)$$

Recall that

$$(X_{i_1}, \ldots, X_{i_k})^n = (X_{i_1}^{m_1} \cdots X_{i_k}^{m_k} : \text{where } m_1 + \cdots + m_k = n) \quad (\ast, \ast)$$

Combining $(\ast)$ along with $(\ast, \ast)$ and $(\ast)$ we observe that any monomial generator of $I^{(n)}$ is of degree less or equal than $Dn$. 
4. PROOF OF EXAMPLE 4.3

We start by a computation from Macaulay2.

**Lemma 4.1.** Let $A := \mathbb{Q}[x, y, z, t, a, b]$ and let $M := (x(x - y)ya, (x - y)ztb, yz(xa - tb))$. Set $f := xy(x - y)ztab(ya - tb)$. The following holds:

i) $(M^2 :_A f) = (x, y, z)$,

ii) $M$ is a radical ideal and all the associated primes of $M$ have height 2. In fact $\text{Ass}(M) = \{(b, a), (b, x), (y, x), (z, x), (t, x), (b, y), (z, y), (t, y), (a, t), (a, z), (z, x - y), (x - y, ya - tb)\}$

**Proof.**

i1 : R=QQ[x,y,z,t,a,b]
o1 = R
i2 : M=ideal(x*(x-y)*y*a,(x-y)*z*t*b,y*z*(x*a-t*b))
o2 = ideal(x(x-y)ya,(x-y)ztb,yz(xa-tb))
o3 : Ideal of R
i3 : Q = quotient(I, f[x,y*(x-y)*z*t*a*b*(y*a-t*b)])
o3 = ideal(z, y, x)
o4 : Ideal of R
i4 : associatedPrimes M
o4 = {ideal(b,a), ideal(b,x), ideal(y,x), ideal(z,x), ideal(t,x), ideal(b,y), ideal(z,y), ideal(t,y), ideal(a,t), ideal(a,z), ideal(z,x-y), ideal(x-y, y*a-t*b)}
o4 : List

It is easy to see that $M$ is radical. These prove the items i) and ii). □

**Lemma 4.2.** Adopt the above notation. Then $M^{(2)} = M^2 + (f)$.

**Sketch of Proof.** Denote the set of all associated prime ideals of $M$ by $\{p_i : 1 \leq i \leq 12\}$ as listed in Lemma 4.1. Revisiting Lemma 4.1 we see that $M$ is radical. Thus $M = \bigcap_{1 \leq i \leq 12} p_i$. By definition,

$$M^{(2)} = \bigcap_{1 \leq i \leq 12} p_i^2$$

For Simplicity, we relabel $A := p_2^2, B := p_3^2$ and so on. Finally, we relabel $L := p_{12}^2$. In order to compute this intersection we use Macaulay2.

i5 : Y = intersect(A,B,C,D,E,F,G,H,I,X,K,L)
o5 : Ideal of R
i6 : N = ideal(x*y*(x-y)*z*t*a*b*(y*a-t*b))
o7 : Ideal of R
i8 : M*M+N == Y
o8 = true

The output term “true” means that the claim “$M^{(2)} = M^2 + (f)$” is true. □
Now, we are ready to present:

**Example 4.3.** Let $A := \mathbb{Q}[x, y, z, t, a, b]$ and $J := (x(x - y)ya, (x - y)ztb, yz(xa - tb))$. Then $J$ generated by degree-four elements and $J^{(2)}$ has a minimal generator of degree 9.

**Proof.** Let $f$ be as of Lemma 4.1. In the light of Lemma 4.1 $(J^2 :_A f) = (x, y, z)$. Thus, $f \notin J^2$. This means that $f$ is a minimal generator of $J^{(2)}$. Since $\deg(f) = 9$ we get the claim. □

A somewhat simpler example (in dimension 7) is:

**Example 4.4.** Let $A := \mathbb{Q}[x, y, z, a, b, c, d]$ and $I := (xyab, xzcd, yz(ab - cd))$. Then $I$ is radical, binomial, Cohen-Macaulay of height 2. Clearly, $I$ is generated in degree 4. But $I^{(2)}$ has a minimal generator of degree 9.

**Proof.** Let $f = xyzabcd(ac - bd)$. By using Macaulay2, we have $(I^2 :_A f) = (x, y, z)$. The same computation shows that $I$ is radical, binomial, Cohen-Macaulay of height 2. Clearly, $I$ is generated in degree 4. But $I^{(2)} = I^2 + (f)$, and $f$ is a minimal generator of $I^{(2)}$ of degree 9. □

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