Can the Couplings in the Fermion-Higgs Sector of the Standard Model be Strong?

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Abstract

We present results for the renormalized quartic self-coupling $\lambda_R$ and the Yukawa coupling $y_R$ in a lattice fermion-Higgs model with two SU(2)$_L$ doublets, mostly for large values of the bare couplings. One-component ('reduced') staggered fermions are used in a numerical simulation with the Hybrid Monte Carlo algorithm. The fermion and Higgs masses and the renormalized scalar field expectation value are computed on $L^324$ lattices where $L$ ranges from 6 to 16. In the scaling region these quantities are found to have a $1/L^2$ dependence, which is used to determine their values in the infinite volume limit. We then calculate the $y_R$ and $\lambda_R$ from their tree level definitions in terms of the masses and renormalized scalar field expectation value, extrapolated to infinite volume. The scalar field propagators can be described for momenta up to the cut-off by one fermion loop renormalized perturbation theory and the results for $\lambda_R$ and $y_R$ come out to be close to the tree level unitarity bounds. There are no signs that are in contradiction with the triviality of the Yukawa and quartic self-coupling.

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1 Introduction

The experiments in high energy physics are so far consistent with the predictions of the perturbative Standard Model. Two particles still await experimental confirmation, the Higgs boson and the top quark. The top quark seems to be on the verge of discovery. Radiative corrections and high precision experiments restrict the mass of the top quark to an interval of about 120-180 GeV [1], whereas the experiments of the CDF collaboration give already a lower bound of 91 GeV with 95% confidence level [2]. This indicates that the Yukawa coupling of the top quark will be relatively small and can still be treated within perturbation theory. Similar restrictions on the mass of the Higgs particle are rather weak [3]. A lower bound on the Higgs boson mass was obtained in ref. [4], where it was shown that it cannot be lighter than approximately 50-100 GeV, provided the mass of the top quark lies in the range 100-200 GeV. The determination of upper bounds on the Higgs mass is a non-perturbative problem and lies outside the scope of weak coupling expansions [5]. For recent analytical studies going beyond ordinary perturbation theory using large $N$ techniques see ref. [6].

It is not excluded by the present status of the experiments that there exists also a fourth fermion generation with a neutrino heavier than half the $Z$ boson mass. Since the systematics of the masses of the known fermions are unclear until now there is no reliable extrapolation to the masses of a fourth generation. Calculations of the radiative corrections to the $\rho$ parameter indicate that the mass split within the isospin doublets of a possible fourth generation must be extremely small. However, the constraints obtained by a comparison of experimental data with perturbative calculations may be doubtful if the relevant couplings are large.

Experimentally it is important to know the largest possible values of the Higgs boson and heavy fermion masses $m_H$ and $m_F$ in the Standard Model. Theoretically this is an issue of self-consistency of the model. Large ratios $m_H/v_R$ and $m_F/v_R$ (where $v_R = 246$ GeV is the electroweak scale) imply a large renormalized scalar self-coupling $\lambda_R$ and Yukawa coupling $y_R$. But it is well known that the Standard Model is suspected to have the property called ‘triviality’ and that $\lambda_R$ and $y_R$ can only increase if the regularization scale $\Lambda$ decreases. For energies larger than $\Lambda$ the theory is not consistent anymore and ‘new physics’ has to take over. If $m_H$ and/or $m_F$ get so large that $\Lambda$ gets comparable in magnitude, the theory looses self-consistency and new physics is noticeable at that mass scale. This scale is somewhat fuzzy because the definition of $\Lambda$ depends on the method of regularization and because it is not clear how large the ratio $m_{H,F}/\Lambda$ can be before new physics takes over.

One can get an impression of the sensitivity to $\Lambda$ by assuming that the one-loop perturbative $\beta$-functions are valid for momenta up to $\Lambda$ and identifying the bare couplings with the running couplings evaluated at $\Lambda$. Integration of the renormalization group equations leads to a relation between the bare couplings at the regulator scale and renormalized couplings at the electroweak scale. The renormalized couplings vanish logarithmically when $\Lambda$ decreases to infinity at fixed bare couplings. Varying the bare couplings at fixed $\Lambda$ the renormalized couplings go through an allowed region in the coupling constant space. Examples for such allowed regions in a fermion-Higgs model with two isospin doublets and gauge couplings switched off are shown in fig. [1], where we plotted the ratio $m_\sigma/v_R = \sqrt{2\lambda_R}$ as a function of $y_R = m_F/v_R$ (we shall use $m_H$ and $m_\sigma$ interchangeably). The solid and dotted curves represent the boundaries of such allowed regions for two different cut-off values. The cut-off for the dotted curve is by a factor two larger than for the solid curve. Such plots were
first presented in ref. [7] were Λ was chosen at the GUT scale, and discussed further in ref. [8].

The upper bound curve for \( \lambda_R \) does not depend very much on \( y_R \) and the lower bound curve for \( \lambda_R \) increases rapidly when the value of \( y_R \) is raised. The upper (lower) bound curve corresponds to infinite (zero) bare quartic coupling and a bare Yukawa coupling ranging from zero to infinity. Both curves join in a point where \( y_R \) and \( \lambda_R \) have their largest values. When increasing Λ to infinity, the allowed region shrinks to the origin and its boundary depends only weakly on Λ. For smaller regularization scales the sensitivity to Λ increases.

The above exploration ignores of course the fact that infinite bare couplings are outside the region of validity of the one-loop \( \beta \)-functions. To overcome this we perform numerical simulations using the lattice regularization with lattice distance \( a \). The regularization scale may be identified with the largest possible momentum, which is \( \pi/a \) for bosons and \( \pi/2a \) for fermions, or simply with the inverse lattice distance \( 1/a \). The three values reflect the unavoidable arbitrariness in the definition of Λ. We shall use Λ = \( \pi/2a \) as the scale where the model breaks down and ‘new physics’ takes over. In order to retain some scaling we shall limit the ratios \( m_{F,H}/\Lambda \lesssim 0.4 \), which corresponds to \( a m_{F,H}/\Lambda \lesssim 0.7 \). More precise definitions of upper bounds have been studied in terms of limits to scaling violations [9, 10, 11, 12, 13].

It is reasonable first to investigate the question of upper bounds within a simplified fermion-Higgs model and to ignore all couplings in the electroweak theory which are small and can be treated well within perturbation theory. For an overview about the recent progress in this field using the lattice regularization we refer to ref. [14].

Using the lattice regularization for a quantum field theory that involves fermionic field variables one is confronted with the phenomenon of species doubling: On a hypercubic lattice each fermion is accompanied by 15 doubler fermions with degenerate mass. This fermion doubling gives rise to the following two problems: a) Eight of these species doublers couple with an opposite chiral charge [15] and spoil the chiral couplings to the SU(2)_L \( \otimes \) U(1)_Y gauge fields. Several proposals have been developed to overcome this problem [14], but so far none of them has been shown to succeed in formulating a chiral gauge theory on the lattice. Recently, a thorough investigation of one of the proposals [14] has raised strong doubts on its success [17], although this is not generally accepted [18]. b) A second and more serious problem is caused by the large number of mass degenerate fermion species. A straightforward, non-chiral transcription of a lattice fermion-Higgs model with one SU(2) doublet on the lattice would lead to a theory with 16 mass degenerate doublets, which is not observed in nature. It is therefore important to develop methods which allow to reduce this number of fermions. At the moment there exist two different approaches, the mirror fermion method [19] and the staggered fermion approach [20, 21, 22], which enable us to reduce the number of isospin doublets to one. In a numerical simulation with the Hybrid Monte Carlo algorithm (HMCA) this number has to be doubled in both cases.

In the mirror fermion model [19] each fermion field is paired up with a mirror fermion field. This ‘doubling’ of the fermionic degrees of freedom allows for chirally invariant Wilson mass terms to remove species doublers of the original and the mirror fermions from the spectrum [23]. The physical mirror fermion can also be decoupled from the spectrum (in the fermion-Higgs model) by a proper choice of the bare coupling parameters and the model then describes one mass degenerate doublet in the scaling region [24, 25].
Figure 1: The $m_\sigma$-$m_F$ plot for the fermion-Higgs model with two mass degenerate doublets. The masses are given in units of $v_R$. The various points in this figure represent our non-perturbative numerical results for the ratios $m_\sigma/v_R$ and $m_F/v_R$ in the infinite volume limit. The values for $av_R = \pi v_R/2\Lambda$, given beside the frame indicate the dependence of the cut-off $\Lambda$. The symbols correspond to those in the phase diagram, fig. 2. The vertical and horizontal dashed lines give the tree level unitarity bounds for the Yukawa and quartic self-coupling. The solid and dotted lines were obtained by integrating the one-loop $\beta$-functions. The cut-off parameter for the dotted line is twice as large as for the solid line.
In this paper we will follow a different proposal [21, 22] which is based on the ‘reduced’ or ‘real’ staggered fermion formalism [26, 27]. The fermion-Higgs model we will investigate in this paper has already been explored in ref. [28], where we presented preliminary results. The usual euclidean staggered fermions on a four-dimensional hypercubic lattice describe four flavors in the scaling region. By using the ‘reduced’ staggered formalism the number of staggered flavors can be reduced to two. These two staggered flavors are coupled to the Higgs field, leading to a model with only one doublet in the scaling region. Since the staggered flavor symmetry group is discrete, the full O(4) symmetry is broken by the Yukawa coupling to a discrete subgroup and it is necessary to add two scalar field counterterms to recover the full symmetry in the scaling region [28]. We will not add these counterterms, but present several analytical and numerical results which show that the effects of the symmetry breaking are small in the parameter region we are interested in.

The mirror fermion model and our model with reduced staggered fermions are expected to reproduce in the scaling region the target model, which is described by the continuum action

$$S_0 = S_H + S_F$$

with

$$S_H = -\int d^4x \left\{ \frac{1}{2} \frac{1}{2} \text{Tr} \left\{ (\partial_\mu \phi)^\dagger (\partial^\mu \phi) \right\} + \frac{m_0^2}{2} \frac{1}{2} \text{Tr} \left\{ (\phi^\dagger \phi) \right\} + \frac{\lambda_0}{4} \frac{1}{2} \text{Tr} \left\{ (\phi^\dagger \phi)^2 \right\} \right\}$$

$$S_F = -\int d^4x \sum_{j=1}^{N_D} \overline{\psi}_j \gamma^\mu \partial_\mu \psi_j + y_0 \frac{1}{2} \left( \sum_{j=1}^{N_D} (\overline{\psi}_{L,j} \phi \psi_{R,j} + \overline{\psi}_{R,j} \phi^\dagger \psi_{L,j}) \right)$$.

Here \( \phi(x) \) is a 4-component scalar field in the \( 2 \times 2 \) matrix notation, \( \psi_j(x) \) is an SU(2) doublet and \( N_D \) denotes the number of SU(2) doublets. The bare coupling parameters \( m_0^2 \), \( \lambda_0 \) and \( y_0 \) are respectively the mass parameter, the quartic self-coupling of the scalar field and the Yukawa coupling.

We can regularize this model by introducing a four-dimensional hypercubic lattice with lattice spacing \( a \) and replacing the derivatives \( \partial_\mu \phi(x) \) by \( (\phi_{x+\hat{\mu}} - \phi_x)/a \). It is convenient to rescale the fields

$$\phi(x) \rightarrow \sqrt{2\kappa} \phi_x/a$$,

$$\psi(x) \rightarrow \psi_x/a^{3/2}$$

and reparametrize the coupling parameters

$$\left( a m_0 \right)^2 = \frac{1 - 2\lambda}{\kappa} - 8$$,

$$\lambda_0 = \frac{\lambda}{\kappa^2}$$,

$$y_0 = \frac{y}{\sqrt{2\kappa}}$$.

As usual we shall mostly use lattice units, i.e. \( a = 1 \). The actions (1.2) and (1.3) then get replaced by

$$S_H = \sum_x \left\{ \kappa \sum_\mu \frac{1}{2} \text{Tr} \left( \phi^\dagger_x \phi_{x+\hat{\mu}} + \phi^\dagger_{x+\hat{\mu}} \phi_x \right) - \frac{1}{2} \text{Tr} \left[ (\phi^\dagger_x \phi_x + \lambda (\phi^\dagger_x \phi_x - I)^2) \right] \right\}$$

$$S_F = -\sum_x \sum_{j=1}^{N_D} \left\{ \frac{1}{2} \left( \overline{\psi}_{x,j} \gamma_\mu \psi_{x+\hat{\mu},j} - \overline{\psi}_{x+\hat{\mu},j} \gamma_\mu \psi_{x,j} \right) + y \left( \overline{\psi}_{x,L,j} \phi_x \psi_{R,j} + \overline{\psi}_{R,j} \phi^\dagger_x \psi_{L,j} \right) \right\}$$.

This model was extensively investigated in ref. [29, 30]. Because of species doubling it describes \( 16 \times N_D \) doubllets in the continuum limit (naive fermions).
Figure 2: Phase diagram of the reduced staggered fermion model with $N_D = 2$. The solid lines indicate the position of the phase transitions between the FM, PM, AM and FI phases. At the points marked by the various symbols, we have performed numerical simulations on a sequence of different lattices ranging in size from $6^324$ to $16^324$. These symbols will be used throughout most of the figures of this paper. The dots represent some points at $\kappa = 0$ where we carried out calculations only on a $12^324$ lattice.

In this paper we will use the reduced staggered fermion method which reduces this large number of SU(2) doublets to $N_D$. The reduced staggered fermion formalism will be recalled in sect. 2. For the study of the largest possible quartic coupling we consider only $\lambda = \infty$ which corresponds to radially frozen Higgs fields, i.e. $\phi_x^1 \phi_x = 1$.

In the last part of this introduction we shall summarize the main results. The phase diagram for the model with $N_D = 2$ obtained in ref. [28] is shown in fig. 2. There are four different phases in the $\kappa$-$y$ plane: A broken or ferromagnetic (FM), a symmetric or paramagnetic (PM) phase, an antiferromagnetic (AM) phase and a ferrimagnetic (FI) phase. We will restrict ourselves in this paper to the broken or ferromagnetic (FM) phase. The AM and FI phases have to be regarded most probably as lattice artefacts since their existence seems to be related strongly to the hypercubic lattice geometry. An interesting aspect of the phase diagram is that the FM phase extends into the negative $\kappa$ region which is not
accessible in the continuum parametrization (1.5). We note that $y_0 \to \infty$ for $\kappa \searrow 0$ and according to eq. (1.5) $y_0$ would become imaginary for $\kappa < 0$. Our numerical results indicate that the whole PM-FM phase transition line falls into the same universality class and there is no reason to ignore the negative $\kappa$ region. On the other hand our quantitative results for the renormalized Yukawa coupling suggest that the region with $\kappa < 0$ may not be relevant since $y_R$ saturates when $\kappa \searrow 0$ while keeping $v_R$ constant and does not increase any more when $\kappa$ is lowered beyond $\kappa = 0$. The various symbols in fig. 2 mark the points in the phase diagram where we have carried out the numerical simulation. The points are in three different regions which are labeled by the Roman numerals (I), (II) and (III).

In fig. 1 we have displayed the Higgs mass $m_\sigma$ as a function of the fermion mass $m_F$. The masses on the axis are given in units of the electroweak scale $v_R \approx 246$GeV. The plot contains only our infinite volume results for the ratios $m_\sigma/v_R = \sqrt{2}\lambda_R$ and $m_F/v_R = y_R$. The symbols in this figure label the position of the various points in the phase diagram and correspond to those in fig. 2. The values of $awR$ listed beside the figure indicate the cut-off values of the various points. The arrows mark the tree level unitarity bounds for the Yukawa and quartic self-coupling. The graph shows that the numerical results are very close to these bounds which implies that the renormalized couplings are not very strong. Also our other results are in accordance with the triviality of the Yukawa and quartic self-coupling. It is remarkable that the solid curve, which was obtained by integrating the one-loop $\beta$-function is not very far from the numerical results.

The outline of the paper is as follows: In sect. 2 we introduce the model. Sect. 3 contains a perturbative one-loop calculation for the Goldstone and Higgs particle propagators and a discussion of the effects of the O(4) symmetry breaking. In sect. 4 we describe the numerical methods which we have used for the determination of the Higgs mass, the wave-function renormalization constant of the Goldstone propagator and the fermion mass. Sect. 5 deals with the extrapolation to infinite volume and discusses the infinite volume results for the renormalized Yukawa and quartic self-coupling. A summary of our results is given in sect. 6.

2 The model

We start this section by recalling the reduced staggered fermion method which allows a reduction of the number of 16 fermion species by a factor eight. In the second part we will explain how the two staggered flavors can be coupled to the Higgs field.

Let us start from the naive euclidean action

$$S_K = -\sum_{x,\mu} \frac{1}{2} (\bar{\psi}_x \gamma_\mu \psi_{x+\hat{\mu}} - \bar{\psi}_{x+\hat{\mu}} \gamma_\mu \psi_x) ,$$

(2.1)

where the field $\psi_x$ is a usual four-component Dirac spinor on the lattice. Because of the fermion doubling phenomenon the action (2.1) leads to a model which can be represented in the continuum by the following action

$$S_K = -\int d^4x \sum_{j=1}^{N_F} \bar{\psi}_j(x) \phi_j(x) , \quad N_F = 16 ,$$

(2.2)

and it describes 16 massless fermions rather than one. Using the usual staggered fermions this number can be reduced by a factor four. This is achieved by performing a spin-
diagonalization transformation \([11]\) on the fields \(\psi_x\) and \(\chi_x\) in eq. \((2.1)\)

\[
\psi_x^\alpha = \sum_{\beta=1}^{4} (\gamma^x)^{\alpha\beta} \chi_x^\beta, \quad \chi_x^\alpha = \sum_{\beta=1}^{4} \chi_x^{\beta} ((\gamma^x)_{\beta\alpha}) ,
\]

where \(\gamma^x \equiv \gamma_1^x \gamma_2^x \gamma_3^x \gamma_4^x\). After this transformation the action \((2.1)\) is a sum of four identical terms

\[
S_K = - \sum_{\alpha=1}^{4} \sum_{\mu=0,1}^{4} \frac{1}{2} \eta_{\mu x} (\chi_x^\alpha \chi_{x+\hat{\mu}}^\alpha - \chi_{x+\hat{\mu}}^\alpha \chi_x^\alpha) .
\]  

(2.4)

The sign factor \(\eta_{\mu x}\) in \((2.4)\) is given by \(\eta_{\mu x} = (-1)^{x_1+\ldots+x_{\mu-1}}\) and the staggered fields \(\chi_x\) and \(\chi_{\bar{x}}\) are four-component complex Grassmann variables. Using only one component and dropping the suffix \(\alpha\), it can be shown that the staggered fermion field \(\chi_x\) describes four Dirac fermions in the continuum limit. The flavor and spin indices of the four staggered fermions are spread out over the lattice and do not appear in an explicit form. To have control over the spin-flavor structure it is convenient to introduce the \(4 \times 4\) matrix fields \([20]\)

\[
\Psi_x = \frac{1}{8} \sum_b \gamma^{x+b} \chi_{x+b}, \quad \chi_x = \frac{1}{8} \sum_b (\gamma^{x+b})^\dagger \chi_{x+b} .
\]  

(2.5)

where in contrast to eq. \((2.3)\) we sum over the 16 corners of a unit lattice hypercube, \(b_{\mu} = 0,1\) and the fields \(\chi_x\) are one-component complex Grassmann variables. The row (column) matrix index of the \(\Psi_x\) field represents the spin (flavor) label and vice versa for \(\chi_x\). Since the \(\Psi\) field contains 16 times as many degrees of freedom as the \(\chi\) field, not all components \(\Psi_{\alpha \kappa}\) are independent. Their Fourier modes \(\tilde{\Psi}(p)\) defined in the restricted momentum interval \(-\pi/2 < p_{\mu} \leq \pi/2\), however, are independent \([22]\). The staggered fermion action may now be written in the form

\[
S_K = - \sum_{x \mu} \frac{1}{2} \text{Tr}(\Psi_x \gamma \mu \Psi_{x+\hat{\mu}} - \Psi_{x+\hat{\mu}} \gamma \mu \Psi_x) .
\]  

(2.6)

This form reduces in the classical continuum limit to the action \((2.2)\) with \(N_F = 4\).

This number of staggered flavors can be reduced once more by a factor two by defining the \(\chi_x\) fields on the odd sites and the \(\chi_{\bar{x}}\) on the even sites of the hypercubic lattice, i. e.

\[
\chi_x \rightarrow \frac{1}{2} (1 - \varepsilon_x) \chi_x , \quad \chi_{\bar{x}} \rightarrow \frac{1}{2} (1 + \varepsilon_x) \chi_{\bar{x}}
\]  

(2.7)

where \(\varepsilon_x = (-1)^{x_1+x_2+x_3+x_4}\). The insertion of these relations into eq. \((2.3)\) gives

\[
\Psi_x = \frac{1}{8} \sum_b \gamma^{x+b} \frac{1}{2} (1 - \varepsilon_{x+b}) \chi_{x+b} , \quad \chi_x = \frac{1}{8} \sum_b (\gamma^{x+b})^\dagger \frac{1}{2} (1 + \varepsilon_{x+b}) \chi_{x+b} .
\]  

(2.8)

When inserting the relations \((2.8)\) into eq. \((2.6)\) one can reproduce the action for reduced (‘real’ or ‘Majorana-like’) staggered fermions \([27]\)

\[
S_K = - \frac{1}{2} \sum_{x \mu} \eta_{\mu x} \chi_x \chi_{x+\hat{\mu}} .
\]  

(2.9)

The restriction of the fields \(\chi\) and \(\chi\) to odd and even sites corresponds to the projections

\[
\Psi \rightarrow \frac{1}{2} (\Psi - \gamma_5 \Psi \gamma_5) , \quad \Psi \rightarrow \frac{1}{2} (\Psi + \gamma_5 \Psi \gamma_5) .
\]  

(2.10)
This implies the following structure for the matrix fields

\[
\Psi = \begin{pmatrix} \psi_L & 0 \\ 0 & \psi_R \end{pmatrix}, \quad \Psi = \begin{pmatrix} 0 & \psi_L \\ \psi_R & 0 \end{pmatrix},
\]

where \( \psi_L, \psi_R, \psi_L \) and \( \psi_R \) are \( 2 \times 2 \) matrices. The row and column indices of the \( \psi_L \) and \( \psi_R \) fields are respectively the Weyl-spinor and flavor labels, and vice versa for \( \psi_L \) and \( \psi_R \).

The model in eq. (2.9) is invariant under the staggered fermion (SF) symmetry group which includes:

a) Shifts by one lattice distance

\[
\chi_x \rightarrow \zeta_{\mu x} \chi_{x+\hat{\mu}},
\]

where \( \zeta_{\mu x} = (-1)^{x_{\mu+1} + \cdots + x_4} \).

b) \( 90^\circ \) rotation

\[
\chi_x \rightarrow S_R(R^{-1} x) \chi_{R^{-1} x},
\]

where \( R = R^\sigma \rho \) is the rotation \( x_\rho \rightarrow x_\sigma, x_\sigma \rightarrow -x_\rho, x_\tau \rightarrow x_\tau \) with \( \tau \neq \rho, \sigma \) and \( S_R(x) = \frac{1}{2} (1 + \eta_\rho \eta_\sigma - \zeta_\rho \zeta_\sigma + \eta_\rho \eta_\sigma \zeta_\rho \zeta_\sigma) \).

c) Lattice parity

\[
\chi_x \rightarrow (-1)^{x_1 + x_2 + x_3} I \chi_x,
\]

where \( I = I_s \) is the lattice parity transformation \( x_4 \rightarrow x_4, x_\tau \rightarrow -x_\tau \) for \( \tau = 1, 2, 3 \).

d) Global U(1) symmetry

\[
\chi_x \rightarrow e^{i \alpha \varepsilon_x} \chi_x,
\]

where \( \alpha \) is a real phase.

The transformation (2.12) can be interpreted as a discrete flavor transformation [27, 32]. The invariance of (2.9) under the symmetry (2.13) implies fermion number conservation. Next we shall couple the two reduced staggered flavors to the Higgs field such that we recover in the scaling region the target model of eq. (1.3) with one SU(2) doublet. We demand that the final form of the action is invariant under the SF symmetry group transformations since this ensures the staggered flavor interpretation in the scaling region.

In order to couple the reduced staggered fermion flavors to the Higgs field we first introduce the \( 4 \times 4 \) matrix

\[
\Phi = \begin{pmatrix} 0 & \phi^i \\ \phi_i & 0 \end{pmatrix} = - \sum \mu \varphi_\mu \gamma_\mu, \quad \text{(2.16)}
\]

where \( \varphi_\mu, \mu = 1, \ldots, 4 \) denote the usual O(4) components of the Higgs fields. When using the relations (2.11) and (2.16) one can show that the action

\[
S_F = -\sum_x \left[ \sum \mu \frac{1}{2} \text{Tr}(\bar{\Psi}_x \gamma_\mu \Psi_{x+\hat{\mu}} - \bar{\Psi}_{x+\hat{\mu}} \gamma_\mu \Psi_x) + y \text{Tr}(\bar{\Psi}_x \Psi_x \Phi_x^T) \right]
\]

reduces in the classical continuum limit to the action of the target model in eq. (1.3) with \( N_D = 1 \). After inserting the transformations (2.8) into eq. (2.17) the final form of the fermionic action in terms of the \( \chi \) fields reads

\[
S_F = -\frac{1}{2} \sum_{x,\mu} \chi_x \chi_{x+\hat{\mu}} (\eta_{\mu x} + y \varepsilon_\mu \zeta_{\mu x} \varphi_\mu) = -\frac{1}{2} \sum_{x,\mu} \chi_x M_{xy} \chi_y,
\]

where

\[
\varphi_{\mu x} = \frac{1}{16} \sum_b \varphi_{\mu, x-b}
\]

(2.19)
is the average of the scalar field over a lattice hypercube. The hypercubic Yukawa coupling arises naturally from the summation over the corners of a lattice hypercube which we introduced in the definition of the Ψ matrices in eq. (2.8). The fermion matrix $M$ in eq. (2.18) is antisymmetric and real. The final form of the lattice action at $\lambda = \infty$ is then given by

$$S = 2\kappa \sum_{x\mu} \varphi_{\mu x} \varphi_{\mu,x+\hat{\mu}} - \frac{1}{2} \sum_{x\mu} \chi_{x} \chi_{x+\hat{\mu}} (\eta_{\mu x} + y \bar{\varphi}_{\mu x} \varphi_{\mu x}),$$  \hspace{1cm} (2.20)

with $\sum_{\mu=1}^{4} \varphi_{\mu x}^{2} = 1$. The action (2.20) is invariant under the SF symmetry group if the Higgs field transforms in the following way under:

a) Shifts by one lattice distance:

$$\varphi_{\mu x} \to (1 - 2\delta_{\mu \rho}) \varphi_{\mu,x+\hat{\rho}} ,$$  \hspace{1cm} (2.21)

b) $90^\circ$ rotations:

$$\varphi_{\mu x} \to R_{\mu \nu} \varphi_{\nu ,R^{-1}(x+n) - n} ,$$  \hspace{1cm} (2.22)

where $n = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$,

c) and lattice parity:

$$\varphi_{\mu x} \to (2\delta_{\mu 4} - 1) \varphi_{\mu ,I x} .$$  \hspace{1cm} (2.23)

The additional shifts by the vector $n$ in eq. (2.22) are needed because the relation between the Ψ and χ fields in eq. (2.8) is not manifestly rotationally covariant. The invariance under rotation would have been more transparent if we would have associated the Ψ and ϕ fields from the beginning with the dual lattice which is shifted by the vector $n$ with respect to the lattice for the χ fields. The scalar field does not transform under the U(1) symmetry (2.15).

The action is not invariant, however, under the full O(4) flavor group, but one expects to be able to recover this invariance in the scaling region. In the scaling region operators with dimension larger than four become irrelevant. There are, however, two operators with dimension four which respect the discrete symmetries (2.21)-(2.23), but break O(4):

$$O^{(1)} = \sum_{x\mu} \varphi_{\mu x}^{4}, \quad O^{(2)} = \frac{1}{2} \sum_{x\mu} (\varphi_{\mu,x+\hat{\mu}} - \varphi_{\mu x})^{2} .$$  \hspace{1cm} (2.24)

We will show in the next section that these terms are indeed generated by the quantum fluctuations. In order to recover the full O(4) symmetry one has to add these operators as counterterms to the action (2.20)

$$S \to S + \varepsilon_{0} O^{(1)} + \delta_{0} O^{(2)},$$  \hspace{1cm} (2.25)

and tune the coefficients $\varepsilon_{0}$ and $\delta_{0}$ as a function of the bare parameters $\kappa$ and $y$ such that the O(4) invariance gets restored in the scaling region. In this paper we will not add these counterterms, but present several analytic and numerical results which show that the effect of the symmetry breaking is small in the parameter region of interest.

### 3 One-loop fermion effects on the scalar propagator

In the pure O(4) model the numerically measured scalar field propagator has a momentum dependence that is nearly of the free field form because the renormalized self-couplings are small. The Yukawa interaction with the fermions affects the scalar propagator in three ways: the masses and wave-functions are renormalized, the additional self-energy gives rise to a
more complicated momentum dependence than the almost pure pole found in the O(4) model, and with our staggered fermion method the fermions induce O(4) symmetry breaking effects. The last two effects can be studied in one fermion loop renormalized perturbation theory. This is useful even at large bare couplings because the renormalized couplings turn out to be relatively small. By taking into account the one-loop effects we can then more reliably extract the scalar masses and wave-function renormalization constants from the numerical data and estimate the effect of O(4) symmetry breaking on the renormalized couplings.

If the fermion effects are neglected the scalar sector of the model has the following approximate effective action,

\[
S_{\text{eff}} \approx - \sum_x \left[ \frac{1}{2} \partial_{\mu} \varphi_R \partial^{\mu} \varphi_R + \frac{m_R^2}{2} \varphi_R \varphi_R + \frac{\lambda_R}{4} (\varphi_R \varphi_R)^2 \right], \tag{3.1}
\]

where \( \partial_{\mu} \varphi = \varphi_{x+\hat{\mu}} - \varphi_x \) and the subscript \( R \) denotes renormalized quantities. In the broken phase we can decompose the scalar field in a Higgs mode, \( \sigma \), and three Goldstone modes, \( \pi^a, a = 1, 2, 3 \), according to

\[
\varphi_R = v_R e_4^\mu + \varphi'_R = (v_R + \sigma_R) e_4^\mu + \pi^a_R e^a_R. \tag{3.2}
\]

The \( \{ e_\mu \} \) form an orthogonal set of O(4) unit vectors and \( v_R \) is the scalar field expectation value. Before taking fermion effects into account, the choice of \( e_\mu \)'s is arbitrary and after substituting the decomposition (3.2) into the action (3.1) one finds the usual tree level relations for the masses of the Higgs and Goldstone modes,

\[
m^2 = 2\lambda_R v_R^2, \quad m^2 = 0, \quad v^2_R = -\frac{m^2_R}{\lambda_R}, \tag{3.3}
\]

as well as the three point interactions \( \lambda_R v_R \sigma_\alpha R + \sigma_R \pi^a_R \pi^a_R R \). The scalar propagator

\[
G_{\mu\nu}(k) = \left\langle \frac{1}{V} \sum_{x,y} \varphi'_\mu \varphi'_\nu \exp(ik(x-y)) \right\rangle \tag{3.4}
\]
is given by

\[
G^{-1}_{\mu\nu}(k) = l_{\mu\nu}(\hat{k}^2 + m^2_\sigma) + t_{\mu\nu}(\hat{k}^2 + m^2_\pi), \quad \hat{k}^2 = 2 \sum_\mu (1 - \cos k_\mu), \tag{3.5}
\]

\[
l_{\mu\nu} = e^4_{\mu} e^4_{\nu}, \quad t_{\mu\nu} = e^a_{\mu} e^a_{\nu}, \quad l_{\mu\nu} + t_{\mu\nu} = \delta_{\mu\nu}, \tag{3.6}
\]

where \( l_{\mu\nu} \) and \( t_{\mu\nu} \) are the longitudinal and transverse projectors onto the \( \sigma \) and \( \pi \) subspaces.

To find the effect of the staggered fermions on the scalar propagator we compute the one-loop diagrams shown in fig. 3. The Feynman rules for the fermions can be derived from the action

\[
S_F = -\frac{1}{2} \sum_{x,\mu} \chi_x \chi_{x+\hat{\mu}} (\eta_{\mu x} + m_F e^4_{\mu} e_{\mu x}) - \frac{1}{2} y_R \sum_{x,\mu} \chi_x \chi_{x+\hat{\mu}} \varphi'_R e_{\mu x} \zeta_{\mu x}, \tag{3.7}
\]

which results after inserting (3.2) into eq. (2.18). The fermion mass in the first term is given by the usual tree level relation

\[
m_F = y_R v_R. \tag{3.8}
\]
The bar on \( \varphi' \) indicates hypercubic averaging, as in (2.19).
Figure 3: Feynman diagrams for the one fermion loop contributions to the vacuum expectation value (a) and to the scalar field self-energy (b and c) in the FM phase.

We shall use in the following the staggered fermion formalism developed in refs. [27, 32]. The fermion propagator follows from the first term in eq. (3.7)

\[ S_{AB}(p) \delta(p - q) = \sum_{x,y} e^{-i(p + \pi_A)x} \langle \chi_x \chi_y \rangle e^{i(q + \pi_B)y} \]

The lattice momentum of the fermion is in the restricted interval \( p_\mu \in (-\pi/2, \pi/2] \), and \( \pi_A, A = 1, \ldots, 16 \), are the momentum four-vectors with components equal to 0 or \( \pi \) and \( V \) is the lattice volume. The 16-dimensional gamma and flavor matrices \( \Gamma_\mu \) and \( \Xi_\mu \) form a Clifford algebra with \( \{ \Gamma_\mu, \Gamma_\nu \} = 2 \delta_{\mu\nu}, \{ \Xi_\mu, \Xi_\nu \} = 2 \delta_{\mu\nu}, [\Gamma_\mu, \Xi_\nu] = 0 \).

The second term in (3.7) contains the interaction with the scalar field \( \phi'(\mu) \). The corresponding vertex function is given by

\[ \varphi(p, q, k) = -y_R h(k) e^{ik_\mu/2} \cos(q - \frac{1}{2}k_\mu) e^{i\pi_\mu} \delta(p - q + k + \pi_A + \pi_B + \pi_x + \pi_\zeta_\mu), \]

where \( n = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \) is the vector introduced earlier in (2.22). The quantities \( p + \pi_A \) and \( q + \pi_B \) are the outgoing and incoming wave vectors of the fermion field, with \( p \) and \( q \) in the restricted interval mentioned above, and \( k \) is the outgoing momentum of the scalar field \( \varphi'(\mu) \). The \( \pi_x \) and \( \pi_\zeta_\mu \) are defined such that \( \epsilon_x = \exp(i\pi_x) \) and \( \zeta_\mu x = \exp(i\pi_\zeta x) \). The factor \( h(k) \) is due to the hypercubic fermion-scalar coupling. In the classical continuum limit the momenta in lattice units approach zero and

\[ \Gamma_\mu(p, -q, k) \rightarrow -y_R \Xi_5 \delta(p - q + k). \]

Note that \( y_R \) is a bare Yukawa coupling as it does not contain at this stage the effects of the fermion interactions.

Let us first consider the one loop fermion effect on the vacuum expectation value of the scalar field (cf. fig. 3a):

\[ \sum_x e^{-i k x} \langle \varphi_{\mu x} \rangle = v_R \epsilon_4 \delta(k) - \frac{1}{2} N_D \sum_\nu \left[ G_{\mu\nu}(k) \sum_p \frac{1}{16} \text{Tr} \{ \Gamma_\nu(p, -p, k) S(p) \} \right] \]

(3.13)
\[ V_{\text{eff}}(\varphi) = \frac{N_D}{2} \sum_p \frac{1}{16} \text{Tr} \ln \sum_\mu \left[ -i \Gamma_\mu \sin p_\mu + y_R \varphi_R \Xi_\mu \Xi_5 \Gamma_5 \cos p_\mu \right] \]  

where \( N_D \) is the number of fermion doublets (which is two in our numerical work) and \( \sum_p \) is a normalized sum over the lattice momenta. The direction of spontaneous symmetry breaking \( e_\mu^4 \) is compatible with the fermion loop correction if the latter has no transverse component, i.e. \( \sum_\nu t_{\mu \nu} I_\nu = 0 \). This is the case for \( e_\mu^4 = (\pm 1, 0, 0, 0), \ldots, (0, 0, 0, \pm 1), (\pm 1, \pm 1, 0)/\sqrt{2}, \ldots, (\pm 1, \pm 1, \pm 1)/\sqrt{3}, \ldots, (\pm 1, \pm 1, \pm 1, \pm 1)/2 \). By studying the one fermion loop effective potential \( V_{\text{eff}}(\varphi) \) for small \( y_R \) we found in ref. [28] that only \( e_\mu^4 = (\pm 1, \pm 1, \pm 1, \pm 1)/2 \) are local minima which correspond to a ground state; the others are saddle points or local maxima. We checked this conclusion by a numerical study of the \( \langle \varphi_\mu \rangle \) probability distribution. In the following we shall understand \( e_\mu^4 \) to be one of the 16 unit vectors

\[ e_\mu^4 = \frac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1). \]  

Note that the hypercubic coupling prohibits a fermion induced staggered magnetization \( \langle \varphi_\mu \rangle \propto \varepsilon_x \). One way to see this is that the factor \( h(k) \) in \( \Gamma_\mu(p, -p, k) \) in (3.13) vanishes when \( k_\mu = \pi \).

We now turn to the fermion contribution to the scalar self-energy corresponding to the diagrams b and c shown in fig. 3. The tadpole contribution of fig. 3b is given by the following expression:

\[
\Sigma^{(b)}_{\mu \nu}(k) = \frac{1}{2} y_R^2 N_D (3 l_{\mu \nu} + t_{\mu \nu}) \sum_p \frac{\cos^2 p_\mu}{D(p)} ,
\]

\[
D(p) = \sum_p \left[ \sin^2 p_\rho + \frac{1}{4} m_F^2 \cos^2 p_\rho \right] ,
\]

which is independent of \( k \). Diagram c in fig. 3 leads to the contribution

\[
\Sigma^{(c)}_{\mu \nu}(k) = \frac{1}{2} y_R^2 N_D |h(k)|^2 e^{\frac{i}{2}(k_\mu - k_\nu)} \sum_p \cos(p + \frac{1}{2} k_\mu) \cos(p + \frac{1}{2} k_\nu) \nu
\]

\[
\frac{1}{16} \text{Tr} \{ \Xi_5 \Gamma_5 \Xi_\mu S(p) \Xi_5 \Gamma_5 \Xi_\nu S(p + k) \}
\]

\[
= \frac{1}{2} y_R^2 N_D |h(k)|^2 e^{\frac{i}{2}(k_\mu - k_\nu)} \sum_p \frac{\cos(p + \frac{1}{2} k_\mu) \cos(p + \frac{1}{2} k_\nu)}{D(p)D(p + k)} \left\{ -\delta_{\mu \nu} [\sin p \cdot \sin(p + k) \right.
\]

\[ + \frac{1}{4} m_F^2 \cos p \cdot \cos(p + k) \right] + l_{\mu \nu} m_F^2 [\cos p_\mu \cos(p + k)_\nu + \mu \leftrightarrow \nu] \right\} .
\]

For \( k = 0 \) this reduces to

\[
\Sigma^{(c)}_{\mu \nu}(0) = \frac{1}{2} y_R^2 N_D \left[ -\delta_{\mu \nu} \sum_p \frac{c_\rho^2}{D} + l_{\mu \nu} 2 m_F^2 \sum_p \frac{c_\rho^2 c_\sigma^2}{D^2} \right] ,
\]
where $D$ was defined in (3.18) and we use the notation

$$c_\mu = \cos p_\mu, \quad s_\mu = \sin p_\mu, \quad s^2 = \sum_\mu \sin^2 p_\mu.$$ (3.21)

The total fermion contribution to the scalar self energy equals $\Sigma_{\mu\nu}(k) = \Sigma_{\mu\nu}^{(b)}(k) + \Sigma_{\mu\nu}^{(c)}(k)$. In an $O(4)$ symmetric model the transverse parts of the zero momentum $\Sigma_{\mu\nu}^{(b)}$ and $\Sigma_{\mu\nu}^{(c)}$ in (3.17) and (3.20) would cancel such that $m_\pi$ remains zero. Using $\delta_{\mu\nu} = l_{\mu\nu} + t_{\mu\nu}$ we see such a cancellation in the first term of (3.20). However, despite its factor $l_{\mu\nu}$, the second term also contains a transverse part, as follows from

$$l_{\mu\nu} \sum_p \frac{c_\mu^2 c_\nu^2}{D^2} = l_{\mu\nu} \sum_p \frac{c_\mu^2 c_\nu^2}{D^2} + \delta_{\mu\nu} \frac{1}{4} \sum_p \delta_1 \frac{1}{D^2}.$$ (3.22)

So we find

$$\Sigma_{\mu\nu}(0) = l_{\mu\nu} \left[ \frac{1}{2} y_R^2 N_D \sum_p \left( \frac{2 c_\mu^2}{D} + 2m_F^2 \frac{c_\mu^2}{D^2} \right) + 2 \varepsilon_R v_\pi^2 \right] + t_{\mu\nu} 2 \varepsilon_R v_\pi^2,$$ (3.23)

$$\varepsilon_R = \frac{y_R^4 N_D}{8} \sum_p \frac{c_1^2 - c_1^2 s_2^2}{D^2} = \varepsilon_N D y_R^4,$$ (3.24)

leading to a pion mass $\propto \varepsilon_R$. The values of $\varepsilon_N$ range from 0.0054 to 0.0043 for $m_F$ ranging from 0 to 0.5.

In ref. [28] we followed a slightly different strategy and computed the coefficient $\varepsilon_R$ of the term $\varepsilon_R \Sigma_{\mu\nu}^{(c)}$ in the effective potential (3.15). There we found the same coefficient $\varepsilon_R$ as computed here from the two point function.

The continuum limit of $\Sigma_{\mu\nu}(k)$ can be calculated by separating the integration region ($\sum_p \rightarrow \int_{-\pi/2}^{\pi/2} d^4p/\pi^4$) into a small ball around the origin and the outer region (see e.g. [22]), which leads to the form

$$\Sigma_{\mu\nu}(k) = y_R^2 N_D \left\{ l_{\mu\nu}(c_2 + c_0 m_F^2 - \frac{3 m_F^2}{2 \pi^2} \ln m_F^2) + \delta_{\mu\nu} k^2(\tau - \frac{1}{4 \pi^2} \ln m_F^2) \right. \notag$$

$$- \frac{1}{2 \pi^2} \int_0^1 dx \left[ \delta_{\mu\nu}(m_F^2 + 3x(1-x)k^2) + l_{\mu\nu} 2m_F^2 \right] \ln \left[ 1 + x(1-x) \frac{k^2}{m_F^2} \right] \right\} \notag$$

$$+ 2 \varepsilon_R v_\pi^2 l_{\mu\nu} + \delta_{\mu\nu} \delta_{\mu\nu} k^2.$$ (3.25)

Here $c_2, c_0$ and $\tau$ are numerical coefficients which we have not calculated explicitly. The term $\delta_{\mu\nu} \delta_{\mu\nu} k^2$ corresponds to the second $O(4)$ symmetry breaking term of dimension four in eq. (2.24). The coefficient $\delta_{\mu\nu}$ can be expressed as

$$\delta_{\mu\nu} = \frac{1}{2} y_R^2 N_D \sum_p \left[ \frac{1}{2} \frac{c_1^2 - s_1^2}{D^2} - a \frac{3 c_1^2 s_1^2 - 12 c_1^2 s_2^2 - c_1^4 + c_1^2 s_1^2}{D^2} - 2a^2 \frac{c_1^4 s_1^2 - c_1^2 s_1^2 s_2^2}{D^2} \right. \notag$$

$$+ \frac{m_F^2}{4} \left\{ \frac{3}{2} \frac{c_1^2 s_1^2 - c_1^2 s_2^2}{D^2} - \frac{3}{2} \frac{c_1^2 (c_1^2 - c_2^2)}{D^2} + a \frac{2 c_1^4 (-c_1^2 + 3 c_2^2 + 3 s_1^2 - 2 s_2^2) - 4 c_1^2 c_2^2 (c_3^2 + 3 s_1^2 - s_2^2)}{D^3} \right. \notag$$

$$+ a^2 \frac{8 c_1^2 (c_1^2 s_1^2 - c_1^2 s_2^2) - 16 c_1^2 c_2^2 (c_1^2 s_1^2 - c_1^2 s_2^2)}{D^4} \right\} = f_8 N_D y_R^2.$$ (3.26)
with $a = 1 - m_F^2/4$ (The expression (3.26) differs from that in ref. [28] which was incomplete). The values of $f_\delta$ range from 0.0026 to 0.0017 for $m_F$ ranging from 0 to 0.5.

The quadratic and logarithmic divergencies of the continuum limit correspond to the $c_{-2}$ term limit. Here we defined $\lambda_R$ by its appearance in (3.27). This definition is based on the effective tree level action

$$S_{eff} \approx - \int d^4x \left[ \frac{1}{2} \partial_\mu \varphi_{\nu} \partial_\mu \varphi_{\nu} + \frac{m_R^2}{2} \varphi_{R\mu} \varphi_{R\nu} + \frac{\lambda_R}{4} (\varphi_{R\mu} \varphi_{R\nu})^2 + \sum_\mu \left[ \frac{\delta_R}{2} (\partial_\mu \varphi_{R\mu})^2 + \varepsilon_R \varphi_{R\mu}^4 \right] \right],$$

which leads to (3.27) with $\Sigma^{(sub)} \to 0$.

To determine the scalar propagator from (3.27) we use the $\sigma, \pi$ basis,

$$G_R^{\alpha\beta}(k) = e_\mu^\alpha e_\nu^\beta G_{R\mu\nu}(k), \quad \Sigma^{(sub)}_{\mu\nu}(k) = e_\mu^\alpha e_\nu^\beta \Sigma^{(sub)}_{\mu\nu}(k)$$

$$G_{\sigma R}(k) = l_{\mu\nu} G_{\mu\nu}(k) = \left[ 2(\lambda_R + \varepsilon_R) v_R^2 + (1 + \frac{\delta_R}{4}) k^2 + \Sigma^{(4)}_{\mu\nu}(k) + O(\delta_R^2) \right]^{-1},$$

$$G_{\pi R}(k) = \frac{1}{3} t_{\mu\nu} G_{\mu\nu} = \frac{1}{3} \sum_{a=1}^3 \left[ 2 \varepsilon_R v_R^2 + (1 + \frac{\delta_R}{4}) k^2 + \Sigma^{(4)}_{\mu\nu}(k) + O(\delta_R^2) \right]^{-1}. $$

In the $\sigma, \pi$ basis the transverse and longitudinal projectors are of course diagonal, but the $\delta_R$ term has an off-diagonal piece. To establish (3.31) and (3.32) for the scalar propagators it is convenient to use the basis vectors $e^1 = \frac{1}{2}(1, -1, -1, 1)$, $e^2 = \frac{1}{2}(-1, 1, -1, 1)$ and $e^3 = \frac{1}{2}(-1, -1, 1, 1)$ for the modes orthogonal to $e^4 = \frac{1}{2}(1, 1, 1, 1)$. With this choice the scalar part of the $\delta_R$ term takes an $O(4)$ invariant form, $\delta_R \delta^a_\mu \delta^b_\mu k^2/4$, and gives an additional wave-function renormalization by a factor $1 + \delta_R/4$. The off-diagonal terms $\delta_R \sum_\mu (\epsilon^a_\mu \epsilon^b_\mu - \delta^a_\mu / 4) k^2_\mu$ lead to corrections of order $\delta_R^2$ in $G_{\sigma,\pi}$ which may be neglected.

From eq. (3.31) and (3.32) we read off the following one fermion loop estimate for the
Higgs and Goldstone masses,

\[
m^2_\sigma = \frac{2(\lambda_R' + \varepsilon_R)v_R^2}{1 + \delta_R/4}, \quad m^2_\pi = \frac{2\varepsilon_Rv_R^2}{1 + \delta_R/4}. \tag{3.33}
\]

In ref. [28] we found that this result for the Goldstone mass was in reasonable agreement with the numerical data, which will be supported by our results in sect. 4. This motivates us to take also the above result for the Higgs mass seriously and use it for an improved definition of the quartic coupling which is corrected for the O(4) symmetry breaking,

\[
\lambda_R' = \frac{m^2_\sigma}{2v_R^2}(1 - \frac{m^2_\pi}{m^2_\sigma}(1 + \frac{\delta_R}{4})). \tag{3.34}
\]

We shall compare this with the usual definition

\[
\lambda_R = \frac{m^2_\sigma}{2v_R^2} \tag{3.35}
\]

in our numerical work to be discussed below. After checking that the corrections are small, we may neglect them.

To this one loop order the coupling \( y_R \) may be interpreted to include also the fermion feedback on the scalars. In the spirit of renormalized perturbation theory, \( (3.31) \) and \( (3.32) \) are expected to be valid whenever the renormalized couplings are sufficiently small, even when the bare couplings are large.

It can be seen from \( (3.25) \) and \( (3.28) \) that \( \Sigma^{(\text{sub})} \) becomes important for \( k^2 \gtrsim 4m^2_F \). We shall use expressions \( (3.31) \) and \( (3.32) \) in analyzing our scalar propagator data, using for \( \Sigma^{(\text{sub})} \) its full lattice form and \( k^2 \to \hat{k}^2 \rho \) as in \( (3.5) \).

4 Finite volume results for the fermion and scalar propagators

In this section we turn to the numerical simulations. After reporting on some technical details, we discuss our methods for analyzing the fermion and scalar propagators and we present the finite volume results.

We have measured the propagators in three different regions of the bare \((\kappa, y)\) parameter space, \( \kappa = 0.29-0.31, \ y = 0.7 \) (I), \( \kappa = 0, \ y = 3.6-4.8 \) (II) and \( \kappa = -0.25, \ y = 5.6-5.8 \) (III). These regions with the points at which we have performed the simulations have been indicated in the phase diagram of fig. 2. The region (I) was included to get a crude idea of the \( y_R \) dependence of \( \lambda_R \) for intermediate values of the Yukawa coupling. The regions (II) and (III) were chosen to study the model at large values of the bare Yukawa coupling. Region (II) would correspond to \( y_0 = \infty \) in the continuum parametrization of the action given in eq. (1.3).

We have restricted our calculations to the FM phase, in particular, we did not increase \( y \) and lower \( \kappa \) further beyond region (III), since from comparing the results in region (II) and (III) we observed that the renormalized couplings \( \lambda_R \) and \( y_R \) remain almost constant when lowering \( \kappa \) beyond \( \kappa = 0 \), while keeping \( a w_R \) roughly fixed. We expect this trend to continue also beyond the multicritical point A. The bosonic particle spectrum in the FI phase, however, differs from that in the FM phase. It contains in addition to the usual
Higgs and Goldstone particles so-called ‘staggered’ Higgs and Goldstone particles which are associated with the antiferromagnetic ordering in that phase \cite{30}.

We carried out the simulation on volumes of size $V = L^3 \times T$ where the spatial extent ranged from $L = 6$ to 16 and the time extent was always kept fixed at $T = 24$. The trajectory length $\tau$ in the HMCA was set equal to one and we have tuned the step size $\delta \tau$ such that the acceptance rate fell into a range between 70 and 80\%. In general $\delta \tau$ had to be reduced slightly when the value of $\kappa$ was lowered and the value of $y$ was raised. We used for the fermions periodic boundary conditions in spatial directions and antiperiodic boundary conditions in the time direction. The scalar fields had periodic boundary conditions in all directions. Depending on the autocorrelation times for the scalar field expectation value we have generated 5,000 to 20,000 trajectories, which resulted in reasonably small statistical errors for the various observables. We could afford large lattices and high statistics because the staggered fermion matrix in eq. (2.18) is relatively small and the conjugate gradient inversions were found to converge excellently.

4.1 Fermion propagator

The fermion propagator in the momentum space representation is defined by the expression

$$S_{AB}(p) = \left\langle \frac{1}{V} \sum_{x,y} e^{i(p+\pi A)x} M^{-1}_{xy} e^{-i(p+\pi B)y} \right\rangle,$$

where $M_{xy}$ is the fermion matrix defined in (2.18). Assuming that loop effects are small, we can use the free fermion result in eq. (3.3) to parametrize $S_{AB}$. This suggests to measure
the projections

\[ S^\Gamma_\mu = \frac{1}{16} \text{Tr} \{ \Gamma_\mu S \} , \quad S^\Xi_\mu = \frac{1}{16} \text{Tr} \{ \Gamma_5 \Xi_5 \mu S \} . \]  

(4.2)

On a finite volume, the scalar field will tunnel from one minimum of the effective potential to another, where the average field value \( \sum_x \varphi_{\mu x} / V \) is proportional to \((\pm 1, \pm 1, \pm 1, \pm 1)\) in these minima. The propagator component \( S^\Gamma_\mu \) is the same in all minima, but \( S^\Xi_\mu \) is proportional to \( \sum_x \varphi_{\mu x} / V \) and would vanish in a finite volume.

In practise we have measured \( S^\Gamma_4(p) \) by averaging over rotated scalar field configurations. These global O(4) rotations were chosen such that \( \sum_x \varphi_{\mu x} / V \propto \delta_{\mu 4} \) on all configurations that went into the averaging. The fermion momenta were chosen along the 4-direction:

\[ p_4 = \left( n - \frac{1}{2} \right) \frac{2\pi}{T} , \quad n = -\frac{T}{4} + 1, \ldots, \frac{T}{4} \]  

(4.3)

and \( p_j = 0 \). Typical results are shown in fig. 4, where we have plotted \( \sin p_4 / S^\Gamma_4(p_4) \) for several values of \( y \) and \( \kappa = 0 \) as a function of \( \sin^2 p_4 \). The linear behavior which is characteristic for weakly interacting fermions is seen to hold remarkably well over the full range of momenta.

In order to extract the fermion mass from these data we adopt the free fermion parametrization,

\[ \frac{\sin p_4}{S^\Gamma_4(p_4)} \approx \frac{(1 - \frac{m_F^2}{Z_F}) \sin^2 p_4 + \frac{m_F^2}{Z_F}}{Z_F} \],

(4.4)

and fitted the parameters \( m_F \) and \( Z_F \). This formula follows from (3.9) with the modification that the vector \( e_4^\mu \) in the mass part, which corresponds to the direction of symmetry breaking, now is the unit vector in the 4-direction.

The results of the fits are collected in table 1. The fermion mass found from \( S^\Xi_4 \) with the appropriate free fermion fit gave the same results within 10%. This small discrepancy decreases on larger lattices, presumably because \( \langle \varphi_\mu \rangle \) gets frozen in one of the minima.

### 4.2 Scalar propagator

We have measured the following momentum space Green functions,

\[ G_\sigma(p) = G_{44}(p), \quad G_\pi(p) = \frac{1}{3} \sum_{m=1}^3 G_{mm}(p) , \]  

(4.5)

with

\[ G_{\mu\nu}(p) = \left\langle \frac{1}{V} \sum_{xy} \varphi_{\mu x} \varphi_{\nu y} e^{ip(x-y)} \right\rangle , \]  

(4.6)

which is identical to (3.4) for non-zero momenta \( p \). The scalar fields in \( G_{\mu\nu}(p) \) have been rotated such that \( \sum_x \varphi_x / V \propto \delta_4 \), as done for the computation of the fermion propagator. For the definition of \( G_{\sigma,\pi} \) in (3.30)-(3.32) this is just a change of the basis such that \( e_4^\mu \rightarrow \delta_4 \), \( G_\sigma = l_\mu G_{\mu\nu} \rightarrow G_{44} \), \( G_\pi = \frac{1}{3} t_\mu G_{\mu\nu} \rightarrow \frac{1}{3} \sum_{m=1}^3 G_{mm} \), as in (1.5). Note, however, that our fitting formulas (3.31) and (3.32) should be unchanged, i.e. we still use in \( \Sigma_{(sub)} \) the \( e_4^\mu \) of eq. (3.16) (The dynamical fermions in the Monte Carlo process ‘experience’ of course the preferred local minima (3.16) and not \( e_4^\mu = \delta_4 \)).
Table 1: Results for the fermionic observables $m_F$, $Z_F$, $y_R$ and $\tilde{y}$ in the regions (I), (II) and (III) of the phase diagram (c.f. fig. 2) for several values of $L$ and $T = 24$. 

| $\kappa$ | $y$ | $L$ | $m_F$     | $Z_F$     | $y_R$     | $\tilde{y}$ |
|----------|-----|-----|-----------|-----------|-----------|------------|
| (I)      | 0.29| 0.7 | 6         | 0.1844(1) | 0.9768(6) | 0.69(2)   | 0.72(3)   |
|          |     |     | 8         | 0.1496(7) | 0.980(2)  | 0.79(4)   | 0.72(8)   |
|          |     |     | 10        | 0.129(1)  | 0.981(4)  | 0.90(7)   | 0.6(2)    |
|          | 0.30|     | 6         | 0.230(1)  | 0.976(2)  | 0.75(2)   | 0.77(5)   |
|          |     |     | 8         | 0.2074(4) | 0.9794(6) | 0.82(2)   | 0.75(9)   |
|          |     |     | 10        | 0.198(1)  | 0.9807(2) | 0.85(3)   | 0.6(2)    |
|          |     |     | 12        | 0.1940(4) | 0.9809(2) | 0.83(2)   | 0.9(2)    |
| (II)     | 0.00| 3.6 | 12        | 0.2004(8) | 0.847(1)  | 2.32(7)   | 1.89(3)   |
|          |     |     | 3.8       | 0.488(1)  | 0.800(4)  | 2.61(6)   | 2.37(2)   |
|          |     |     | 8         | 0.394(2)  | 0.821(7)  | 2.51(4)   | 2.25(4)   |
|          |     |     | 10        | 0.355(3)  | 0.8259(6) | 2.48(4)   | 2.33(4)   |
|          |     |     | 12        | 0.333(3)  | 0.826(3)  | 2.48(5)   | 2.34(4)   |
|          |     |     | 16        | 0.318(2)  | 0.825(1)  | 2.50(5)   | 2.28(8)   |
|          | 4.0 | 6   | 0.5804(3) | 0.777(8)  | 2.85(5)   | 2.50(9)   |
|          |     |     | 8         | 0.509(1)  | 0.799(3)  | 2.75(2)   | 2.50(4)   |
|          |     |     | 10        | 0.491(5)  | 0.805(6)  | 2.72(4)   | 2.52(7)   |
|          |     |     | 12        | 0.482(2)  | 0.807(2)  | 2.68(5)   | 2.58(9)   |
|          | 4.2 | 6   | 0.674(5)  | 0.75(1)   | 2.95(6)   | 2.93(9)   |
|          |     |     | 12        | 0.624(1)  | 0.790(2)  | 2.91(5)   | 2.7(1)    |
|          | 4.8 | 12  | 0.9839(5) | 0.7596(7) | 3.36(5)   | 3.3(2)    |
| (III)    | -0.25| 5.6 | 6         | 0.492(1)  | 0.775(5)  | 2.83(4)   | 2.42(3)   |
|          |     |     | 8         | 0.400(1)  | 0.794(2)  | 2.65(4)   | 2.41(4)   |
|          |     |     | 10        | 0.360(3)  | 0.799(2)  | 2.68(4)   | 2.30(5)   |
|          |     |     | 12        | 0.338(4)  | 0.799(2)  | 2.61(4)   | 2.42(2)   |
|          | 5.8 | 6   | 0.5480(3) | 0.758(4)  | 3.02(4)   | 2.51(5)   |
|          |     |     | 8         | 0.475(2)  | 0.778(3)  | 2.85(4)   | 2.58(4)   |
|          |     |     | 10        | 0.448(5)  | 0.785(3)  | 2.75(4)   | 2.63(5)   |
|          |     |     | 12        | 0.446(1)  | 0.781(3)  | 2.84(3)   | 2.66(5)   |
In fig. 5 we show a representative example of the measured inverse propagators $G_{\sigma,\pi}(p)^{-1}$, as a function of the bosonic lattice version of the momentum squared, $\hat{p}^2 = 2 \sum_\mu (1 - \cos p_\mu)$. To reduce statistical errors we have computed the propagator in the small momentum range for all possible lattice momenta, and averaged over results with the same value of $\hat{p}^2$.

For sufficiently small momenta, $\hat{p}^2 \ll m_F^2$, $m_{\sigma,\pi}$ one expects a linear $\hat{p}^2$ dependence, $G_{\sigma,\pi}^{-1} = (\hat{p}^2 + m_{\sigma,\pi}^2)/Z_{\sigma,\pi}$ and a naive linear fit was commonly applied to extract $m_{\sigma,\pi}$ and $Z_{\sigma,\pi}$, obtained from a linear fit. This bias can be large on small lattices, where even the smallest momenta are too large to neglect the curvature. Therefore we followed the same strategy as in ref. [30] and included the subtracted self-energy $\Sigma_{(\text{sub})}(p)$ (cf. eqs. (3.31) and (3.32)) to parametrize the non-linear $\hat{p}^2$ dependence. Furthermore it was found that the bending over of $G_{\sigma,\pi}^{-1}$ for very large momenta, $\hat{p}^2 \geq 4$ (see fig. 5), could be reproduced by including the pole contribution coming from the ‘staggered’ scalar modes. Apart from the usual Higgs and three Goldstone bosons the spectrum in the FM phase contains also four ‘staggered’ scalar modes which are associated with the antiferromagnetic ordering in the FI phase. These modes produce a pole in the diagonally opposite corner of the Brillouin zone, at $16 - \hat{p}^2 = (m_{\Phi}^T)^2$. The mass $m_{\Phi}^T$ of these particles is usually large, but it becomes small when approaching point A in the phase diagram. See ref. [30] for a detailed discussion of the staggered scalar modes.

These considerations lead us to use the following parametrization for the inverse scalar propagators in the FM phase,

$$G_{\sigma,\pi}(p) \approx Z_{\sigma,\pi}/\left(\hat{p}^2 + m_{\sigma,\pi}^2 + \frac{\hat{y}^2}{y_R^2} \Sigma_{(\text{sub})}(p)\right) + Z_{\Phi}^T/(16 - \hat{p}^2 + (m_{\Phi}^T)^2) . \quad (4.7)$$

This requires us to fit five parameters: the mass and wave-function renormalization factors for the scalar particles ($\sigma$ or $\pi$) and for the staggered bosons and the coefficient of the subtracted self-energy, $\hat{y}$. This last parameter measures the strength of the renormalized three point Yukawa interaction and should be close to $y_R$ obtained from the tree level relation with the fermion mass. Notice that $\Sigma_{(\text{sub})}/y_R^2$ contains no free parameters, since we use the measured value of the fermion mass in the loop sum. The subtracted self-energy $\Sigma_{(\text{sub})}(p)$ is computed numerically for the lattice sizes used in the simulations. The coefficient of the term linear in $\hat{p}^2$ is computed with the finite difference approximation $[\partial \Sigma(k)/\partial k^2]_{k=0} \approx (\Sigma(h_4) - \Sigma(0))/h_4^2$, where $h_4$ is the smallest non-zero lattice momentum in the time direction.

If the renormalized Yukawa coupling is weak, one expects that the one-loop expression (4.7) gives a good description of the non-linear $\hat{p}^2$ dependence of the inverse propagator. Conversely, an accurate description of the data over a wide momentum range with the one-loop form (4.7) indicates that the Yukawa coupling must be weak. As will be seen shortly, this is the case in our model and including the self-energy term leads to stable values for the mass and wave-function renormalization factor even on small lattices.

We applied this method of analysis to the scalar propagators in all three regions of the phase diagram. In region (I) we found larger autocorrelations which urged us to increase the statistics to 20,000 trajectories. Region (III) was not very different from region (II). Since the AM phase is closer here, we found a stronger effect of the staggered scalar modes on the scalar propagator for large momenta, but this could still very well be fitted with the Ansatz.
Figure 5: The inverse scalar propagators $G_{\sigma}^{-1}(p)$ (fig. a) and $G_{\pi}^{-1}(p)$ (fig. b) as a function of $p^2$ at the point $(\kappa, y) = (0, 3.8)$. The lattice size is $12^3 \times 24$. The circles were obtained by fitting the Ansatz (4.7) to the Monte Carlo data which are represented by the crosses.
Figure 6: The inverse scalar propagator $G_\pi(p)$ is plotted for the full momentum range as a function of $p^2$. The computation was performed at the point $(\kappa, y) = (-0.25, 5.6)$ (region (III)) on a lattice of size $12^324$. The points in the lower figure were obtained by fitting the Ansatz (4.7) to the Monte Carlo data which are displayed in the upper figure.
| $\kappa$ | $y$ | $L$ | $v_R$  | $m_\pi$ | $m_\sigma$ | $Z_\pi$ | $Z_\sigma$ | $m_\sigma/v_R$ |
|-------|-----|-----|--------|---------|-----------|--------|----------|-------------|
| (I)   | 0.29| 0.7 | 6.0    | 0.267(9)| 0.09(4)   | 0.58(2)| 0.76(5)  | 1.12(2)     | 2.2(1)      |
|       |     | 8   | 0.19(1)| 0.17(5) | 0.44(1)   | 1.0(1) | 1.13(1)  | 2.4(2)      |
|       |     | 10  | 0.14(1)| 0.18(6) | 0.38(2)   | 1.3(2) | 1.17(2)  | 2.6(2)      |
|       | 0.30| 6   | 0.31(1)| 0.13(3) | 0.72(3)   | 0.96(6)| 1.23(3)  | 2.4(1)      |
|       |     | 8   | 0.254(7)| 0.14(3) | 0.63(2)   | 1.19(6)| 1.25(2)  | 2.48(9)     |
|       |     | 10  | 0.235(8)| 0.17(4) | 0.65(2)   | 1.32(9)| 1.31(2)  | 2.8(1)      |
|       |     | 12  | 0.235(6)| 0.13(4) | 0.65(1)   | 1.28(7)| 1.33(2)  | 2.75(9)     |
|       | 0.31| 6   | 0.346(7)| 0.13(3) | 0.84(3)   | 1.04(4)| 1.20(4)  | 2.4(1)      |
|       |     | 8   | 0.313(8)| 0.10(4) | 0.80(3)   | 1.21(6)| 1.23(4)  | 2.6(1)      |
|       |     | 10  | 0.30(1) | 0.14(4) | 0.84(2)   | 1.34(9)| 1.28(2)  | 2.8(1)      |
|       |     | 12  | 0.290(7)| 0.14(4) | 0.81(2)   | 1.38(7)| 1.27(2)  | 2.79(9)     |
|       | 0.00| 3.6 | 12     | 0.086(3)| 0.15(2)   | 0.28(2)| 0.26(1)  | 0.289(3)    | 3.2(3)      |
|       | 3.8 | 6   | 0.187(4)| 0.15(1) | 0.74(4)   | 0.34(1)| 0.56(1)  | 4.0(2)      |
|       |     | 8   | 0.157(3)| 0.16(2) | 0.64(1)   | 0.35(1)| 0.500(4)| 4.1(1)      |
|       |     | 10  | 0.143(2)| 0.11(2) | 0.55(2)   | 0.35(1)| 0.464(5)| 3.9(1)      |
|       |     | 12  | 0.134(3)| 0.11(3) | 0.50(1)   | 0.35(1)| 0.441(3)| 3.69(8)     |
|       |     | 16  | 0.127(3)| 0.17(3) | 0.50(2)   | 0.36(1)| 0.410(7)| 3.9(2)      |
|       | 4.0 | 6   | 0.203(4)| 0.18(2) | 0.81(3)   | 0.40(1)| 0.58(1)  | 4.0(2)      |
|       |     | 8   | 0.185(2)| 0.15(1) | 0.72(1)   | 0.412(6)| 0.555(6)| 3.90(7)     |
|       |     | 10  | 0.181(2)| 0.12(2) | 0.68(2)   | 0.414(9)| 0.537(7)| 3.8(1)      |
|       |     | 12  | 0.180(3)| 0.11(4) | 0.65(2)   | 0.41(1)| 0.528(9)| 3.6(1)      |
|       | 4.2 | 6   | 0.229(4)| 0.14(2) | 0.78(4)   | 0.41(1)| 0.58(2)  | 3.4(2)      |
|       |     | 12  | 0.214(4)| 0.18(2) | 0.87(2)   | 0.47(2)| 0.64(1)  | 4.1(1)      |
|       | 4.8 | 12  | 0.293(4)| 0.29(2) | 1.34(3)   | 0.54(2)| 0.90(2)  | 4.6(1)      |
| (III) | -0.25| 5.6 | 6.0    | 0.174(2)| 0.17(1)   | 0.76(2)| 0.180(4)| 0.293(4)    | 4.4(1)      |
|       |     | 8   | 0.151(2)| 0.11(2) | 0.65(1)   | 0.175(4)| 0.268(3)| 4.34(9)     |
|       |     | 10  | 0.135(2)| 0.17(2) | 0.59(1)   | 0.192(4)| 0.251(2)| 4.4(1)      |
|       |     | 12  | 0.129(1)| 0.13(1) | 0.53(1)   | 0.185(2)| 0.237(1)| 4.13(6)     |
|       | 5.8 | 6   | 0.182(2)| 0.18(1) | 0.84(3)   | 0.209(4)| 0.321(9)| 4.6(2)      |
|       |     | 8   | 0.167(2)| 0.16(2) | 0.74(2)   | 0.207(4)| 0.302(5)| 4.4(1)      |
|       |     | 10  | 0.163(2)| 0.11(2) | 0.69(2)   | 0.205(4)| 0.288(6)| 4.2(2)      |
|       |     | 12  | 0.157(2)| 0.16(1) | 0.63(1)   | 0.213(3)| 0.271(2)| 4.00(7)     |

Table 2: Results for the bosonic observables $v_R$, $m_{\sigma,\pi}$, $Z_{\sigma,\pi}$ and the ratio $m_\sigma/v_R$ in the regions (I), (II) and (III) of the phase diagram at several values of $L$ and $T = 24$. 
We have fitted the inverse Goldstone propagators over the momentum range \(0 < \hat{p}^2 < 1.3\) and the \(\sigma\) particle propagator over the interval \(2(1 - \cos(2\pi/T)) < \hat{p}^2 < 1.7\). In these intervals the effect of the staggered pole in eq. (1.7) is negligible and could be dropped. In fig. 3 we show an example for such a fit at a point which falls into the region (II). The result of the fit is shown by the open circles and it is seen that the agreement with the measured data (represented by the crosses) is excellent. In fig. 5 we show the result of a fit over the full momentum range at a point which belongs to the region (III), this time including the staggered scalar pole. Again it is seen that the quality of the fit is very good, even for large momenta. The resulting values for \(m_{\pi,\sigma}, Z_{\pi,\sigma}\) and \(\bar{y}\) are consistent with those which were obtained from the fit over the restricted momentum interval. The fitted values for \(m_{\sigma,\pi}^2\) and \(Z_{\sigma,\pi}\) are collected in table 4. The fitted values for \(\bar{y}\) can be found in table 4. They are seen to be roughly equal to the \(y_R\) values, which gives further support to the consistency of our analysis and indicates that the renormalized Yukawa coupling is not very strong.

By using the one fermion loop fit method we achieved a considerable improvement of the scalar propagator analysis. In ref. [28] we have used linear fits, but this was possible only on the \(12^324\) lattice since on smaller lattices the systematic errors due to the curvature at small momenta were unacceptably large. This problem prohibited a study of the finite size effects and the extrapolation to infinite volume. The linear fits on the \(12^324\) lattice produced slightly larger values of \(m_\sigma\) and smaller values of \(Z_\pi\). We emphasize that with the fitting method used in this paper the systematic error for \(m_\sigma\) could be reduced by almost an order of magnitude.

The renormalized couplings \(\lambda_R\) and \(y_R\), listed in the tables 4 and 2, were obtained from the tree level relations (3.3) and (3.8). For the computation of these couplings we have to determine the renormalized scalar field expectation value

\[
v_R = \frac{v}{\sqrt{Z_\pi}},
\]

where \(v\) is the unrenormalized scalar field expectation value. As we mentioned in sect. 4.1, on a finite lattice the system is observed to tunnel from one minimum to another and after averaging over many configurations one would find zero for the scalar field expectation value. To compensate for this drift it has proven to be useful to rotate the magnetization vector for each configuration first into a certain direction, before taking the ensemble average [34]. On a finite lattice \(v\) can then be defined by the relation

\[
v = \langle |\frac{1}{V} \sum_x \varphi_{\mu x}| \rangle.
\]

As in the pure O(4) model we use \(Z_\pi\) for the renormalization of the scalar field expectation value in eq. (1.8), rather than \(Z_\sigma\), because the Higgs particle is unstable.

One should be aware that the fitted value of \(Z_\pi\) contains the factor \((1+\delta_R/4)^{-1}\), cf. (3.32), which implies that \(v'_R = v_R(1 + \delta_R/4)^{-1/2}\) should be used for the calculation of corrected couplings \(\lambda'_R\) and \(y'_R\) from eqs. (3.8) and (3.34),

\[
y'_R = y_R(1 + \frac{\delta_R}{4})^{1/2}
\]

\[
\lambda'_R = \frac{m^2_\sigma}{2v^2_R}(1 - \frac{m^2_\pi}{m^2_\sigma})(1 + \frac{\delta_R}{4})^2.
\]
The Goldstone mass as a function of $y$ for $\kappa = 0$. The lattice size is $12^3 24$. The measured values of $m_\pi$ are represented by the squares and the one-loop result for $m_\pi$ obtained from eq. (3.33) by the diamonds.

The ratios $R_y = (y_R - y_R')/y_R'$ and $R_\lambda = (\sqrt{2\lambda_R} - \sqrt{2\lambda_R'})/\sqrt{2\lambda_R'}$ give a measure for the O(4) symmetry breaking effects. These ratios were computed numerically from eq. (4.10) and (4.11), where the quantities $\varepsilon_R$ and $\delta_R$ were determined by means of eqs. (3.24) and (3.26), with the measured values of the fermion mass and $y_R$ supplied. For the data listed in tables 1 and 2 we find

$$|R_y| < 5\% , \quad |R_\lambda| < 6\% ,$$

(4.12)
in a parameter region with $m_F < 0.5$ and $m_\sigma < 0.7$. In general the numerical values of $|R_y|$ and $|R_\lambda|$ decrease with increasing lattice volume (see also fig. 10).

The accurate description of the momentum dependence of the inverse scalar propagator over a wide momentum range, using one fermion loop renormalized perturbation theory indicates that $y_R$ is in the perturbative range and it suggests that also our computation of the O(4) symmetry breaking effects should be reliable. This was found to be the case in ref. [28], where we compared the measured values of the Goldstone mass with the one-loop result $m_\pi^2 \propto \varepsilon_R$ of eq. (3.33). In fig. 7 we repeat this comparison using our high statistics data. Again it is seen that the one-loop result gives a reasonable description of the data. The increase of $m_\pi$ for $y$ values approaching the phase transition around $y \approx 3.6$ is due to finite size rounding which is not contained in the one-loop formula. Motivated by this result we shall assume that also the one-loop corrections appearing in (4.10) and (4.11) give a reasonable account of the remaining symmetry breaking effects.
5 Infinite volume results for the renormalized couplings

From the finite volume results for the fermion and Higgs mass we want to obtain estimates of the renormalized couplings \( y_R \) and \( \lambda_R \) in the infinite volume. Therefore we need to extrapolate the results of the previous section to the infinite volume. In subsect. 5.1 we shall discuss how these extrapolations have been done. The resulting infinite volume estimates for the renormalized couplings are discussed in subsect. 5.2.

5.1 \( 1/L^2 \) extrapolation to infinite volume

If the spectrum contains massless Goldstone bosons at infinite volume, this gives rise to finite size effects in finite volume quantities which vanish \( \propto 1/L^2 \). In ref. [30] this was found to hold for the scalar field expectation value, the Higgs mass and also for the fermion mass. In our model the Goldstone particles have a mass in an infinite volume due to the O(4) symmetry breaking. On a finite volume this is expected to give rise to deviations from the linear \( 1/L^2 \) dependence only when the lattice extent \( L \) increases beyond the Goldstone correlation length, \( L > O(1/m_{\pi}) \). Small deviations imply that the symmetry breaking effects are small. On the other hand, on the smaller lattices one expects additional non-leading finite-size effects. A pragmatic way to proceed is to apply the linear \( 1/L^2 \) extrapolation as long as no significant deviations are observed in the numerical data.

In fig. 8 we plotted the observables \( v_R \), \( m_\sigma \) and \( m_F \) obtained at the three different points of region (I) as a function of \( 1/L^2 \). Fig. 9 shows the \( 1/L^2 \) dependence of the same quantities for the points in the regions (II) and (III). The dashed lines are linear fits to all data points. It is seen that the linear \( 1/L^2 \) behavior holds satisfactory well within the statistical error bars in all three regions of the phase diagram. In region (I), where we expect the symmetry breaking effects to be even smaller than in the other regions, the agreement with a linear \( 1/L^2 \) dependence is indeed best. In regions (II) and (III), only the \( m_F \) values at the points which are most distant from the phase transition show a slight indication of the expected systematic deviation for large volumes. However, when comparing the results obtained in regions (II) and (III) to the ones of region (I), we conclude that this effect is rather small and we interpret this observation as further evidence that the effect of O(4) symmetry breaking in our model is small.

Also the couplings themselves show a \( 1/L^2 \) behavior as can be seen in fig. 10, where we have displayed as an example the ratio \( \sqrt{2\lambda_R} = m_\sigma/v_R \) (open symbols) as a function of \( 1/L^2 \) for two points in region (II) and (III) in the phase diagram. We showed in the previous paragraph that both \( m_\sigma \) and \( v_R \) are proportional to \( 1/L^2 \), \( v_R \simeq c_1 + c_2/L^2 \), \( m_\sigma \simeq d_1 + d_2/L^2 \), with relatively small coefficients \( c_2 \) and \( d_2 \). From an expansion in powers of \( 1/L^2 \) we expect also the ratio \( m_\sigma/v_R \) to obey an approximate \( 1/L^2 \) behavior. In addition we have also included in the graph the results for the corrected ratio \( \sqrt{2\lambda'_R} \) (filled symbols) which were obtained by means of eq. (4.11). Also here we expect, based on an expansion of the right-hand side of eq. (4.11) in powers of \( 1/L^2 \), an approximate linear \( 1/L^2 \) dependence which is also seen in the numerical results. Similar results hold also for the coupling \( y'_R \) which we defined in eq. (1.10). In general we find that the linear \( 1/L^2 \) dependence is better fulfilled for the corrected ratios than for the uncorrected ones, which gives some evidence that the symmetry breaking effects in the corrected ratios are small. It is amusing that in all the cases the infinite volume extrapolations of the corrected and uncorrected ratios, shown by the dotted and dashed lines in fig. 10, respectively, lead within the error bars to the same
Figure 8: The quantities $v_R$, $m_F$ and $m_\sigma$ as a function of $1/L^2$ for the three different coupling points of the region (I). The extent of the lattice in the time direction was kept fixed at $T = 24$. The dashed lines were obtained by a linear fit to all the data at different lattices and the symbols drawn at $1/L^2 = 0$ are the result of the infinite volume extrapolation.
Figure 9: The same as fig. 8, but now for the regions (II) and (III).
Figure 10: The ratios $\sqrt{2\lambda_R} = m_\sigma/v_R$ (open symbols) and $\sqrt{2\lambda'_R}$ (filled symbols) as a function of $1/L^2$ for $(\kappa, y) = (0, 4.0)$ (diamonds) and $(-0.25, 5.8)$ (squares). The extent in the time direction was kept fixed at $T = 24$. The dashed lines (for the open symbols) and the dotted lines (for the filled symbols) were obtained by linear fits to all data at different lattices and the symbols drawn at $1/L^2 = 0$ are the results of the infinite volume extrapolation.
5.2 Renormalized couplings

A comprehensive summary of our infinite volume results for the renormalized couplings is given in fig. 1 where we have plotted the ratio \( m_\sigma / v_R = \sqrt{2 \lambda_R} \) as a function of \( m_F / v_R = y_R \) for the various points of the regions (I), (II) and (III). The symbols were chosen such that they match with those in fig. 2 so that the reader can easily find out at which bare couplings in the phase diagram the various points have been obtained. All points have roughly the same value of the cut-off in units of the scalar field expectation value, \( a v_R \approx 0.15 - 0.25 \). The unsystematic scattering (as a function of the cut-off) of the points in the region around \( y_R = 2.5, \sqrt{2 \lambda_R} = 4 \) is still within one standard deviation and is presumably due to uncertainties in the infinite volume extrapolation. The figure shows that the data points obtained in the regions (II) and (III) fall almost on top of each other. The renormalized couplings appear to saturate when \( \kappa \rightarrow 0 \) and not to increase further when \( \kappa \) is lowered beyond zero, always keeping \( a v_R \) roughly fixed.

We furthermore compare in fig. 1 the numerical results for \( \sqrt{2 \lambda_R} \) and \( y_R \) with the tree level unitarity bounds marked by the horizontal and vertical arrows below and beside the axis. The tree level unitarity bound for \( \sqrt{2 \lambda_R} \) was taken from ref. 9. The bound on \( y_R \) for a fermion-Higgs model with \( N_D \) doublets is given by

\[
y_R \lesssim \sqrt{4\pi/N_D},
\]

i. e. \( y_R \lesssim 2.5 \) for \( N_D = 2 \). The points obtained in the regions (II) and (III) are very close to the tree level unitarity bounds which indicates that the couplings are not very strong.
This is also consistent with our observation that the numerical results for the scalar propagators could well be described in the full momentum range by the one fermion loop expressions.

For comparison we have displayed in fig. 1 also the \( y_R \) dependence of \( \sqrt{2 \lambda_R} \) using the assumption that the one-loop \( \beta \)-function is a good approximation to the full \( \beta \)-function. The one-loop \( \beta \)-functions for the Yukawa and quartic self-coupling in a fermion-Higgs model with \( N_D \) doublets are given by the expressions:

\[
\beta_y(\overline{y}, \overline{\lambda}) \equiv \frac{d\overline{y}(t)}{dt} = \frac{N_D}{4\pi^2} \overline{y}^3, \tag{5.2}
\]

\[
\beta_\lambda(\overline{y}, \overline{\lambda}) \equiv \frac{d\overline{\lambda}(t)}{dt} = \frac{3}{2\pi^2} \overline{\lambda}^2 + \frac{N_D}{\pi^2} \overline{\lambda} \overline{y}^2 - \frac{N_D}{\pi^2} \overline{y}^4, \tag{5.3}
\]

where \( \overline{\lambda} \) and \( \overline{y} \) are the running couplings and \( t = \ln(\mu/\Lambda_1) \). The quantity \( \mu \) is an energy scale and \( \Lambda_1 \) is the cut-off of the one-loop \( \beta \)-function model. In order to find the relations for the renormalized couplings we have integrated these \( \beta \)-functions numerically from the cut-off scale \( \mu = \Lambda_1 \) down to the physical scale \( \mu = v_R \), with the identifications \( \overline{y}(0) = y_0 \), \( \overline{y}(t_R) = y_R \), where \( t_R = \ln(v_R/\Lambda_1) \). Since the simulations were carried out at \( \lambda = \infty \), we have chosen \( \lambda_0 = 100 \approx \infty \) and varied the starting value for \( y_0 \) in the interval \( 0 \leq y < 50 \). The solid and dotted curves in fig. 1 correspond to two different values of the ratio \( v_R/\Lambda_1 \). For the solid curve this ratio was adjusted such that the agreement with the numerical results is best. It is remarkable that the shape of the curve is in reasonable agreement with our data. The ratio \( v_R/\Lambda_1 \) of the dotted curve is by a factor two smaller than that of the solid curve.

Let us now turn to the question of the determination of upper bounds on \( m_\sigma \) and \( m_F \). Reading off an upper bound at \( m_\sigma = 0.7/a \), we find \( m_\sigma/v_R \lesssim 3 \) in region (I) of the phase diagram and \( m_\sigma/v_R \lesssim 4 \) in regions (II) and (III). The fermion mass increases along the phase transition line (keeping \( a m_\sigma \) constant) until \( y \approx 4 \) where \( \kappa_c(y) \) becomes negative. From then on it appears to remain roughly constant. Also the field expectation value does not change much for \( y \gtrsim 4 \) and we find \( m_F/v_R = 0.9, 2.6 \) and 2.6 in regions (I), (II) and (III) respectively. The observed saturation of the renormalized Yukawa coupling indicates that we can read off the upper bound for \( y_R \) at \( y \approx 4 \).

It is interesting to compare our results for \( \lambda_R \) with those obtained in the mirror fermion model of ref. [14], which also describes two doublets of light fermions. Since the mirror fermion model has a quite different bare action with a different bare parameter space, a comparison of the \( y_R \) dependence of \( \lambda_R \) at the same values of \( a v_R \) would give some idea to what extent these results are model dependent. Such a comparison is shown in fig. 1 where we have plotted the ratio \( m_\sigma/v_R \) against \( m_F/v_R \) for various models. The three diamonds represent some infinite volume results in the reduced staggered fermion model which have roughly the same cut-off, \( a v_R = 0.15-0.21 \). In contrast to our results, the results in the mirror fermion model (full circles) [21] were not extrapolated to infinite volume. The cut-off values on the \( 6^3 12 \) lattice, \( a v_R = 0.31-0.39 \), are consistent with our results on the \( 6^3 24 \) lattice (see table 2). It can be seen from fig. 1 that the points obtained from the mirror model are within error bars on the same curve as our results.

We included in fig. 1 also some infinite volume results which were obtained in the pure O(4) model (\( N_D = 0 \), square, \( a v_R = 0.20(2) \)) [33] and in the naive fermion model (\( N_D = 32 \), open circle, \( a v_R = 0.20(3) \)) [30]. It is seen that in the naive fermion model, where \( m_F/v_R \) is relatively small, the fermion effect on \( \lambda_R \) is negligible, whereas the fermions give a moderate
Figure 11: Comparison of the results for the ratio $m_\sigma/v_R$ as a function of $m_F/v_R$ obtained in various models. The open symbols represent infinite volume results for the $O(4)$ model ($N_D = 0$, square) [35], the naive fermion model ($N_D = 32$, open circle) [30], and the reduced staggered fermion model ($N_D = 2$, diamond), whereas the full circles show finite volume ($6^3 12$) results obtained in the mirror fermion model ($N_D = 2$) [23].
increase of $\lambda_R$ in the model with two doublets. In both models the maximal values for $y_R$ appear to be close to the tree level unitarity bound given in eq. (5.1).

An interesting speculation has been that, even though the Yukawa and quartic self-couplings are trivial, they might still be sufficiently large as to give rise to interesting non-perturbative effects, like formation of a $\rho$ bound state [36]. We find, however, that the largest values of $\lambda_R$ and $y_R$ we can obtain are very close to their tree level unitarity bounds, which suggests that they are still on the edge of the perturbative regime. This indication is supported by the observation here and also in ref. [30] for the 32 doublet model, that the numerical results for the scalar propagator can be very accurately described using one-loop renormalized perturbation theory. It is therefore most likely that renormalized perturbation theory gives a complete and accurate description of the physics of Yukawa models.

6 Summary and discussion

In this paper we have used ‘reduced’ or ‘real’ staggered fermions in a high statistics investigation of a fermion-Higgs model. The two staggered flavors are coupled to the O(4) Higgs field, leading to a model with a single isospin doublet in the scaling region. In a simulation with the Hybrid Monte Carlo algorithm this number of doublets has to be doubled. Since the species doubler degrees of freedom are used as physical flavor-spin components, there is no redundancy in the fermion field in this approach and the model can be very efficiently simulated on large lattices.

The Yukawa coupling to the staggered flavors breaks the O(4) symmetry and it requires two scalar field counterterms to restore this symmetry in the scaling region. Even without these counterterms, this O(4) symmetry breaking has only a small effect on the values of the renormalized couplings, at least in the physically relevant scaling region of the phase diagram where we have done our simulations. This follows from our findings that a) the deviations from the $1/L^2$ finite size effects are very small, b) the Goldstone mass is much smaller than the Higgs mass and c) that the one fermion loop estimate of the symmetry breaking correction to the quartic coupling is found to be small. The one fermion loop estimate for the symmetry breaking effects is presumably reliable, because is gives a reasonable prediction for the Goldstone mass.

The renormalized Yukawa coupling $y_R$ and the renormalized quartic self-coupling $\lambda_R$ were determined from the tree level relations $y_R = m_F/v_R$ and $\lambda_R = m_\sigma^2/(2v_R^2)$, where $v_R = v/\sqrt{Z_\pi}$ is the renormalized field expectation value. On a finite lattice the masses $m_F$ and $m_\sigma$ and the Goldstone wave-function renormalization constant $Z_\pi$ were computed from the scalar and fermion propagators in momentum space. The momentum dependence of the scalar propagators differs from that of a free propagator and in particular on small lattices it is important to include the one fermion loop self-energy to get a reliable estimate for $m_\sigma$ and $Z_\pi$. In order to investigate the finite size dependence of our results and extrapolate them to the infinite volume, we have simulated on a sequence of lattices ranging in size from $6^324$ to $16^324$. The finite volume results for $m_\sigma$, $v_R$ and $m_F$ show a $1/L^2$ dependence, which was used to extrapolate these observables to the infinite volume.

A comprehensive summary of our infinite volume results is shown in fig. 1, where we have plotted the ratio $m_\sigma/v_R = \sqrt{2\lambda_R}$ as a function of $m_F/v_R = y_R$. The results were obtained
in three different regions of the phase diagram (c.f. fig. 2) and always at $\lambda = \infty$. The ratio $m_\sigma/v_R$ increases a little bit for increasing $y_R$ and becomes at large $y_R$ slightly larger than the tree level unitarity bound. The maximal value of the Yukawa coupling is roughly equal to its tree level unitarity bound. This indicates that the couplings are not very strong. Further evidence for the perturbative nature of the renormalized couplings comes from the fact that the numerical results for the scalar field propagators can be nicely described by taking into account the one fermion loop contribution to the self-energy. We find even for the points in region (III) an almost perfect agreement over the full range of momenta. Furthermore fig. 1 shows that the numerical values for $m_\sigma/v_R$ and $m_F/v_R$ are in qualitative agreement with a simple model based on extending the one-loop $\beta$-function to infinite couplings (solid curve). In conclusion we can say that all our results are consistent with the triviality scenario and there is no evidence for a strong coupling sector.

Our results are obtained in a particular regularization of the SU(2) fermion-Higgs model: using the lattice regularization with reduced staggered fermions and two isospin doublets in the scaling region. One would like to see to what extent the results are model dependent. Therefore we have compared our results with those of ref. [25] which uses mirror fermions with two doublets in the scaling region. Looking at the $y_R$ dependence of $\lambda_R$ at roughly the same value of the cut-off in all models, we find consistent results within error bars, c.f. fig. 1.

The main result of our investigation therefore is that the renormalized quartic and Yukawa couplings cannot be strong, unless the cut-off is unacceptably low. In fact we find that one-loop renormalized perturbation theory is applicable even for the maximally strong couplings. More quantitatively we find for the upper bounds on the masses of the Higgs particle and the heavy fermion in our model, $m_\sigma \lesssim 4v_R$ and $m_F \lesssim 2.6v_R$ for cut-off values $\Lambda \gtrsim 2.5 m_{\sigma,F}$. From experience in the O(4) model with modifications in the regularization (by including dimension six operators [10] or using improved actions [11]) we expect that these numbers for the upper bounds may be stretched by perhaps 20-30%. Larger changes have been recently found with the Pauli-Villars regularization scheme in the large $N$ limit [13].

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