The Effective Dirac Algebra by Interaction Gauge Field

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Abstract

Conventional quantum field theory (QFT) is set on flat Minkowski spacetime, where all computable quantities are calculated from the flat metric $\eta_{\mu\nu}$. We can redefine the metric of spacetime from the Dirac algebra. In this paper, we study how an quantum electrodynamic interaction can alter the normal gamma matrix to an effective one and result in a shift in the metric perturbatively. We also study how the spin operator is changed under the interaction that contribute to an effective spin operator.

1 Introduction

The Dirac equation is the foundation of quantum electrodynamics (QED) which explains the origin of anti-particles. The solution of the Dirac equation represents the relativistic spin-$\frac{1}{2}$ fermions and anti-fermions [1, 2]. The Dirac equation can be obtained by minimizing the free QED action (in natural units) [3, 4, 2],

$$S = \int d^4x \left( -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + i \bar{\psi} \gamma^\mu \partial_\mu \psi - m \bar{\psi} \psi \right),$$

(1)

where $F_{\mu\nu}$ is the gauge field strength, $\psi$ is the Dirac spinor, $\gamma^\mu$ is the Dirac gamma matrix and $m$ is the mass of the fermion. The equation of motion for fermion would be the Dirac equation $i \gamma^\mu \partial_\mu \psi - m \bar{\psi} \psi = 0$ [3]. When the interaction is turned on, the action is modified by a minimum substitution of covariant derivative $\partial_\mu \rightarrow D_\mu = \partial_\mu + ie A_\mu$, where $A_\mu$ is the interaction spin-1 photon gauge field [3, 4, 2],

$$S = \int d^4x \left( -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + i \bar{\psi} \gamma^\mu D_\mu \psi - m \bar{\psi} \psi \right).$$

(2)

By minimizing the action the Dirac equation is modified to

$$i \gamma^\mu \partial_\mu \psi - e \gamma^\mu A_\mu \psi - m \bar{\psi} \psi = 0.$$

(3)

Quantum field theory is set on flat Minkowski spacetime and the gamma matrices satisfy the Dirac algebra is [3]

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}1,$$

(4)

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where $\mathbf{1}$ is the $4 \times 4$ identity matrix. Therefore the anti-commutation relation of two gamma matrices just gives the natural Minkowski metric tensor. We can look into the opposite perspective by defining the metric of the field theory from its form of gamma matrices.

In this paper we would like to study on how gauged interaction can modify the current flat spacetime by means of absorbing the interaction in the $\gamma^\mu$ vertex in such a way that we redefine the metric using the new effective $\gamma^\mu$ vertex $\Gamma^\mu$. The metric is then defined by

$$\{\Gamma^\mu, \Gamma^\nu\} = 2g^\mu\nu\mathbf{1},$$
\hspace{1cm} (5)

Perturbatively,

$$\{\Gamma^\mu, \Gamma^\nu\} = 2(\eta^\mu\nu - h^\mu\nu)\mathbf{1},$$
\hspace{1cm} (6)

In such way the spacetime is curved by the existence of interaction. Although such curvature might be small or even negligible, it is worth studying how the new metric behaves and how the spacetime geometry is changed. Next we can will find out the effective $\gamma^5$ matrix and effective spin operator. The spin of the fermion will be expected to change and deviated from $\frac{1}{2}$, very slightly, due to the effect of gauge interaction.

2 Effective Dirac Algebra

We would revisit the Dirac equation and investigate it in a new perspective. For our purpose we will keep the constants $\hbar$ and $c$ in the equations. We rewrite the free Dirac equation in the form of

$$\psi(x) = \frac{i\hbar}{mc}\gamma^\mu\partial_\mu\psi(x) = \frac{i\lambda}{2\pi}\gamma^\mu\partial_\mu\psi(x),$$
\hspace{1cm} (7)

where $\lambda = \frac{\hbar}{mc}$ is the Compton wavelength. For the Dirac equation with electromagnetic interaction, we rewrite it as

$$\psi(x) = \frac{i\hbar}{mc} + \frac{1}{\eta^\rho A_\rho(x)}\gamma^\mu\partial_\mu\psi(x) = \frac{i\lambda}{2\pi} 1 + \frac{1}{2\pi} \frac{\lambda e A_\rho(x)}{\gamma^\rho} \gamma^\mu\partial_\mu\psi(x).$$
\hspace{1cm} (8)

Or in tensor notation we have

$$\psi_c = \frac{i\lambda}{2\pi} \left(1 + \frac{\lambda e}{2\pi} A_\rho(x) \gamma^\rho\right) \gamma^{\mu a} \partial_\mu \psi_b.$$  
\hspace{1cm} (9)

Therefore with interaction, we have an extra term contributed by the photon gauge field $A_\mu$. Using the Feynman notation $\hat{A} = \gamma^\mu A_\mu$, define the functional

$$f[\hat{A}(x)] = \frac{1}{1 + \frac{\lambda e}{2\pi} \hat{A}(x)}.$$  
\hspace{1cm} (10)

The functional can be expanded perturbatively,

$$f[\hat{A}(x)] = \frac{1 - \frac{\lambda e}{2\pi} \hat{A}}{1 - \frac{\lambda^2 e^2}{4\pi^2} \hat{A}^2} = \left(1 + \frac{\lambda^2 e^2}{4\pi^2} \hat{A}^2 - \frac{\lambda^4 e^4}{16\pi^4} \hat{A}^4 + O(e^6)\right) \left(1 - \frac{\lambda e}{2\pi} \hat{A}\right)$$
\hspace{1cm} (11)

$$= 1 - \frac{\lambda e}{2\pi} \hat{A} + \frac{\lambda^2 e^2}{4\pi^2} \hat{A}^2 - \frac{\lambda^3 e^3}{8\pi^2} \hat{A}^2 + \frac{\lambda^4 e^4}{16\pi^4} \hat{A}^4 + O(e^6).$$
Thus the series can be decomposed into the sum of odd and even powers of $A$, where for odd-power series it is $\gamma^\nu$ dependent, while of even-power series it is independent of $\gamma^\rho$. Define

$$ f_{\text{even}}[A_\rho] = \sum_{k=0,2,4,...}^\infty \left( \frac{\lambda e}{2\pi} \right)^k A^k = \frac{1}{1 - \frac{\lambda^2 e^2}{4\pi^2} A^2}, $$

(12)

$$ f_{\text{odd}}[\gamma^\rho, A_\rho] = \sum_{k=1,3,5,...}^\infty \left( \frac{\lambda e}{2\pi} \right)^k \gamma^\rho A^\rho A^{k-1} = \frac{\lambda e}{2\pi} f_{\text{even}}, $$

(13)

and we have

$$ f[A] = f_{\text{even}}[A_\rho] - f_{\text{odd}}[\gamma^\rho, A_\rho]. $$

(14)

Using $c = 1$ in natural unit, we can define the effective mass $\mathcal{M}(x)$ in terms of the free mass $m$,

$$ \mathcal{M}(x) = m \left( 1 + \frac{\lambda e}{2\pi} \gamma^\mu A_\mu(x) \right). $$

(15)

Alternatively we can define the effective gamma matrix as

$$ \Gamma^\mu_L = f[A] \gamma^\mu. $$

(16)

The subscript $L$ indicates that $f[A]$ is on the left-hand side. Yet we can also define the effective gamma matrix by having $f[A]$ at the right hand side, but we need to introduce a commutator for compensating such change

$$ \psi = \frac{i\lambda}{2\pi} \gamma^\mu f[A] \partial_\mu \psi - \frac{i\lambda}{2\pi} [\gamma^\mu, f[A]] \partial_\mu \psi. $$

(17)

Then we remain to evaluate the commutator. Using the identity $\{ \gamma^\mu, A \} = 2A^\mu$, one can show that

$$ f[A] \gamma^\mu = \left( \gamma^\mu - \frac{\lambda e}{\pi} A^\mu \right) f_{\text{even}} + \gamma^\mu f_{\text{odd}}. $$

(18)

It follows that

$$ \{ \gamma^\mu, f[A] \} = \frac{\lambda e}{\pi} A^\mu f_{\text{even}} - 2\gamma^\mu f_{\text{odd}} = \frac{\lambda e}{\pi} (A^\mu - \gamma^\mu A) f_{\text{even}}. $$

(19)

Using equation (12), then we finally obtain

$$ \{ \gamma^\mu, f[A] \} = \frac{\lambda e}{\pi} \left( \frac{A^\mu - \gamma^\mu A}{1 - \frac{\lambda^2 e^2}{4\pi^2} A^2} \right). $$

(20)

Therefore the Dirac equation with electromagnetic interaction can be written as order of $e$ as

$$ \psi = \frac{i\lambda}{2\pi} \Gamma^\mu_R \partial_\mu \psi - \frac{i\lambda^2 e}{2\pi^2} \left( \frac{A^\mu - \gamma^\mu A}{1 - \frac{\lambda^2 e^2}{4\pi^2} A^2} \right) \partial_\mu \psi $$

$$ = \frac{i\lambda}{2\pi} \Gamma^\mu_R \partial_\mu \psi + \left( \frac{\lambda^2 e}{2\pi^2} + \frac{\lambda^4 e^3}{8\pi^4} + O(e^5) \right) (A^\mu - \gamma^\mu A) i\partial_\mu \psi. $$

(21)

Finally, we would like to study how can the interaction change the geometry of flat spacetime. First we know that QFT is defined in flat space time by the flat metric $\eta_{\mu\nu}$, which can be generated from the Dirac gamma matrices,

$$ \{ \gamma^\mu, \gamma^\nu \} = 2\eta^{\mu\nu} \mathbf{1}, $$

(22)
The full derivation is given as follow. First consider and substituting equation (25) into equation (24), we get
\[ \gamma \text{ where } \Gamma \]
To further simply terms of \( \gamma, f[A] \), then using \( \mu, \Gamma \)
Substituting this result back into equation(26), we finally obtain the desired equation
Therefore we have the full form of curved (inverse) metric matrix as follow,
\[ g^{\mu\nu} \text{ where } \Gamma \]

where \( \mathbf{1} \) is the \( 4 \times 4 \) identity matrix. Now with the interaction, the effective gamma matrices \( \Gamma \) follows the new Dirac algebra,
\[ \{ \Gamma_L^\mu, \Gamma_L^\nu \} = 2g^{\mu\nu}\mathbf{1}, \quad (23) \]

where \( g^{\mu\nu} \) is the curved metric, and it depends on \( \lambda, e, \gamma^\rho \) and the gauge field \( A^\rho(x) \). The full derivation is given as follow. First consider
\[ \{ \Gamma_L^\mu, \Gamma_L^\nu \} = \{ f[A] \gamma^\mu, f[A] \gamma^\nu \} = f[A] \gamma^\mu f[A] \gamma^\nu + f[A] \gamma^\nu f[A] \gamma^\mu. \quad (24) \]

Then using
\[ \gamma^\mu, f[A] = \gamma^\mu f[A] - f[A] \gamma^\mu = \gamma^\mu f[A] [\gamma^\mu, f[A]] + f[A] \gamma^\mu \quad (25) \]

and substituting equation (25) into equation (24), we get
\[ \{ \Gamma_L^\mu, \Gamma_L^\nu \} \]
\[ = f[A] \left[ \{ \gamma^\mu, f[A] \} + f[A] \gamma^\mu \right] \gamma^\nu + f[A] \left[ \{ \gamma^\nu, f[A] \} + f[A] \gamma^\nu \right] \gamma^\mu \]
\[ = f[A] \left[ \gamma^\mu, f[A] \right] \gamma^\nu + f[A] \left[ \gamma^\nu, f[A] \right] \gamma^\mu + f^2[A] (\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\nu) \]
\[ = 2f^2[A] \eta^{\mu\nu} + f[A] \frac{\lambda e}{\pi} \left( \frac{A^\mu - \gamma^\mu A}{1 - \frac{\lambda e^2}{4\pi^2}A^2} \right) \gamma^\nu + f[A] \frac{\lambda e}{\pi} \left( \frac{A^\nu - \gamma^\nu A}{1 - \frac{\lambda e^2}{4\pi^2}A^2} \right) \gamma^\mu \]
\[ = 2f^2[A] \eta^{\mu\nu} + \frac{\lambda e}{\pi} f[A] \left( \frac{1}{1 - \frac{\lambda e^2}{4\pi^2}A^2} \right) (A^\mu \gamma^\nu + A^\nu \gamma^\mu - \gamma^\mu A^\nu - \gamma^\nu A^\mu) \]

To further simply terms of \( \gamma^\mu A^\nu, \gamma^\nu A^\mu \), we use the identity
\[ \{ \gamma^\mu, A \} = 2A^\mu \implies A^\mu = 2A^\mu - \gamma^\mu A. \quad (27) \]

Then
\[ - (\gamma^\mu A^\nu + \gamma^\nu A^\mu) \]
\[ = \left( \gamma^\mu (2A^\nu - \gamma^\nu A) + \gamma^\nu (2A^\mu - \gamma^\mu A) \right) \]
\[ = - \left( 2\gamma^\mu A^\nu + 2\gamma^\nu A^\mu - (\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu) A \right) \]
\[ = -(2\gamma^\mu A^\nu + 2\gamma^\nu A^\mu - 2\eta^{\mu\nu} A). \quad (28) \]

Substituting this result back into equation(26), we finally obtain the desired equation for the anti-commutator for the effective Dirac algebra,
\[ \{ \Gamma_L^\nu, \Gamma_L^\mu \} = 2f^2[A] \eta^{\mu\nu} + \frac{\lambda e}{\pi} f[A] \left( \frac{1}{1 - \frac{\lambda e^2}{4\pi^2}A^2} \right) (2\eta^{\mu\nu} A - \gamma^\mu A^\nu - \gamma^\nu A^\mu). \quad (29) \]

Therefore we have the full form of curved (inverse) metric matrix as follow,
\[ g^{\mu\nu} \text{ where } \Gamma \]
\[ g^{\mu\nu} \mathbf{1} = f^2[A] \eta^{\mu\nu} + \frac{\lambda e}{2\pi} f[A] \left( \frac{1}{1 - \frac{\lambda e^2}{4\pi^2}A^2} \right) (2\eta^{\mu\nu} A - \gamma^\mu A^\nu - \gamma^\nu A^\mu). \quad (30) \]

The metric can be obtained by taking the trace both sides,
\[ g_{\mu\nu} = \frac{1}{4} \text{Tr} \left\{ f^2[A] \eta_{\mu\nu} + \frac{\lambda e}{2\pi} f[A] \left( \frac{1}{1 - \frac{\lambda e^2}{4\pi^2}A^2} \right) (2\eta_{\mu\nu} A - \gamma_\mu A_\nu - \gamma_\nu A_\mu) \right\}. \quad (31) \]
Next we are interested in how this can be resulted into a weak perturbation theory for the metric. Since \( f[A] \) can be worked out perturbatively, \( g_{\mu\nu} \) can be expected to be an expansion series of the coupling \( e, \lambda \) and gauge field \( A^\mu \),

\[
g^{\mu\nu}(\lambda, e, A^\rho) = \eta^{\mu\nu} - h^{\mu\nu}(\lambda, e, A^\rho). \tag{32}
\]

Therefore, the effective Dirac algebra can be regarded as the original Dirac algebra with some perturbative corrections from the interaction,

\[
\{\Gamma^\mu_L, \Gamma^\nu_R\} = \{\gamma^\mu, \gamma^\nu\} + \text{perturbative corrections}. \tag{33}
\]

Here we would like to carry out perturbation theory so we expand in the powers of the coupling \( O(e^n) \), and do it up to the second order. Consider

\[
f[A] = \frac{1}{1 + \frac{\lambda e}{2\pi A}} = 1 - \frac{\lambda e}{2\pi A} + \frac{\lambda^2 e^2}{4\pi^2 A^2} + O(e^3) \tag{34}
\]

Thus

\[
f^2[A] = \left( 1 - \frac{\lambda e}{2\pi A} + \frac{\lambda^2 e^2}{4\pi^2 A^2} + O(e^3) \right) \left( 1 - \frac{\lambda e}{2\pi A} + \frac{\lambda^2 e^2}{4\pi^2 A^2} + O(e^3) \right)
= 1 - \frac{\lambda e}{2\pi A} + \frac{\lambda^2 e^2}{4\pi^2 A^2} - \frac{\lambda e}{2\pi A} + \frac{\lambda^2 e^2}{4\pi^2 A^2} + O(e^3)
= 1 - \frac{\lambda e}{\pi} A + \frac{3\lambda^2 e^2}{4\pi^2} A^2 + O(e^3) \tag{35}
\]

Therefore the first term in equation \((29)\) reads

\[
2f^2[A] = 2\eta^{\mu\nu} + \left( \frac{2\lambda e}{\pi A} + \frac{3\lambda^2 e^2}{2\pi^2} A^2 + O(e^3) \right) \eta^{\mu\nu} \tag{36}
\]

Next we consider the expansion for the second terms in equation \((29)\). Notice that

\[
\frac{1}{1 - \frac{\lambda^2 e^2}{4\pi^2 A^2}} = 1 + \frac{\lambda^2 e^2}{4\pi^2 A^2} + O(e^4), \tag{37}
\]

then all together we have

\[
\frac{\lambda e}{\pi} \left( 1 - \frac{\lambda e}{2\pi A} + \frac{\lambda^2 e^2}{4\pi^2 A^2} + O(e^3) \right) \left( 1 + \frac{\lambda^2 e^2}{4\pi^2} A^2 + O(e^4) \right) (2\eta^{\mu\nu} A - \gamma^\mu A^\nu - \gamma^\nu A^\mu)
= \frac{\lambda e}{\pi} \left( 1 - \frac{\lambda e}{2\pi A} + \frac{3\lambda^2 e^2}{4\pi^2} A^2 + O(e^3) \right) (2\eta^{\mu\nu} A - \gamma^\mu A^\nu - \gamma^\nu A^\mu)
= \left( \frac{2\lambda e}{\pi} A - \frac{\lambda^2 e^2}{\pi^2} A^2 \right) \eta^{\mu\nu} - \left( \frac{\lambda e}{\pi} - \frac{\lambda^2 e^2}{2\pi^2 A} \right) (\gamma^\mu A^\nu + \gamma^\nu A^\mu) + O(e^3). \tag{38}
\]

Combining the results in equation \((36)\) and equation \((38)\), we obtain

\[
\{\Gamma^\mu_L, \Gamma^\nu_R\} = 2g^{\mu\nu} \mathbf{1} = \left( 2 + \frac{\lambda^2 e^2}{2\pi^2 A^2} \right) \eta^{\mu\nu} \mathbf{1} - \left( \frac{\lambda e}{\pi} - \frac{\lambda^2 e^2}{2\pi^2 A} \right) B^{\mu\nu} + O(e^3), \tag{39}
\]

where we define the symmetric tensor \( B^{\mu\nu} = \gamma^\mu A^\nu + \gamma^\nu A^\mu \).
The inverse perturbation metric matrix is thus
\[ h^{\mu\nu} = \frac{\lambda^2 e^2}{4\pi^2} A^2 \eta^{\mu\nu} + \left( \frac{\lambda e}{2\pi} - \frac{\lambda^2 e^2}{4\pi^2} A \right) B^{\mu\nu} + O(e^3). \] (40)

To obtain the metric tensor, we take the trace both sides, thus the full metric is just
\[ ds^2 = \frac{1}{8} \text{Tr} \{ \Gamma_{\mu\nu} \} dx^\mu dx^\nu. \] (41)

Notice that \( \text{Tr} \gamma^\mu = 0 \) and \( \text{Tr} \gamma^\mu \gamma^\nu = 4\eta^{\mu\nu} \), then
\[
4g_{\mu\nu} = \left( 1 + \frac{\lambda^2 e^2}{4\pi^2} A^2 \right) \eta^{\mu\nu} \text{Tr} 1 + \frac{\lambda^2 e^2}{4\pi^2} A^2 \left( A_{\nu} A^\rho \text{Tr} (\gamma_{\rho} \gamma_{\mu}) + A_{\mu} A_{\nu} \text{Tr} (\gamma_{\mu} \gamma_{\nu}) \right) = 4 \left( 1 + \frac{\lambda^2 e^2}{4\pi^2} A^2 \right) \eta_{\mu\nu} + \frac{2\lambda^2 e^2}{\pi^2} A_{\mu} A_{\nu}. \] (42)

Hence, we obtain
\[
g_{\mu\nu} = \left( 1 + \frac{\lambda^2 e^2}{4\pi^2} A^2 \right) \eta_{\mu\nu} + \frac{\lambda^2 e^2}{2\pi^2} A_{\mu} A_{\nu}. \] (43)

Therefore, the perturbation metric tensor up to second order of coupling is
\[
h_{\mu\nu} = \frac{\lambda^2 e^2}{4\pi^2} (A^2 \eta_{\mu\nu} + 2A_{\mu} A_{\nu}). \] (44)

Therefore, the spacetime geometry is changed by the tiny amount given by equation (43). We would like to calculate how the geometry changes. Using the linearized perturbation theory of general relativity, the Riemannian tensor is given by [3]
\[
R_{\mu\nu\rho\sigma} = \frac{1}{2} (\partial_{\rho} \partial_{\nu} h_{\mu\sigma} + \partial_{\sigma} \partial_{\mu} h_{\nu\rho} - \partial_{\sigma} \partial_{\nu} h_{\mu\rho} - \partial_{\rho} \partial_{\mu} h_{\nu\sigma}). \] (45)

Explicitly,
\[
R_{\mu\nu\rho\sigma} = \frac{\lambda^2 e^2}{4\pi^2} \left( \partial_{\rho} A_{\beta} (\eta_{\mu\sigma} \partial_{\nu} - \eta_{\mu\nu} \partial_{\sigma}) A^\beta + \partial_{\nu} A_{\beta} (\eta_{\rho\sigma} \partial_{\mu} - \eta_{\rho\mu} \partial_{\sigma}) A^\beta \\
+ A_{\alpha} (\eta_{\mu\nu} \partial_{\rho} - \eta_{\mu\rho} \partial_{\nu}) \partial_{\sigma} A^\alpha + A_{\sigma} (\eta_{\nu\rho} \partial_{\mu} - \eta_{\rho\mu} \partial_{\nu}) \partial_{\nu} A_{\rho} \\
+ \frac{1}{2} \left( (A_{\nu} \partial_{\rho} - A_{\rho} \partial_{\nu}) F_{\nu\mu} + (A_{\rho} \partial_{\nu} - A_{\nu} \partial_{\rho}) F_{\rho\mu} + F_{\mu\nu} F_{\rho\sigma} \\
+ \partial_{\rho} A_{\mu} \partial_{\nu} A_{\sigma} - \partial_{\sigma} A_{\mu} \partial_{\rho} A_{\nu} + \partial_{\sigma} A_{\nu} \partial_{\rho} A_{\mu} - \partial_{\rho} A_{\mu} \partial_{\nu} A_{\sigma} \right) \right),
\] (46)

where \( F_{\mu\nu} = \partial_{\nu} A_{\mu} - \partial_{\mu} A_{\nu} \) is the field strength tensor. The Ricci tensor is given by [6]
\[
R_{\mu\nu} = \frac{1}{2} \left( \partial_{\sigma} \partial_{\mu} h^\sigma_{\nu} + \partial_{\sigma} \partial_{\nu} h^\sigma_{\mu} - \partial_{\mu} \partial_{\nu} h - \square h_{\mu\nu} \right). \] (47)

For simplicity we will impose the Lorentz gauge condition that \( \partial_{\mu} A^{\mu} = 0 \). The first term of the Ricci tensor is
\[
\partial_{\sigma} \partial_{\nu} h^\sigma_{\mu} = \frac{\lambda^2 e^2}{4\pi^2} \left( \partial_{\rho} \partial_{\nu} A_{\rho}^2 + 2 \partial_{\nu} A_{\sigma} \partial_{\sigma} A_{\mu} + 2 A_{\sigma} \partial_{\sigma} \partial_{\nu} A_{\mu} \right). \] (48)
The second term of the Ricci tensor is
\[
\partial_\sigma \partial_\mu h^\sigma_\nu = \frac{\lambda^2 e^2}{4\pi^2} (\partial_\sigma \partial_\mu A^2 + 2 \partial_\mu A^\sigma \partial_\sigma A_\nu + 2 A^\sigma \partial_\sigma \partial_\mu A_\nu). \tag{49}
\]
Since \( h = h^\rho_\rho = \frac{\lambda^2 e^2}{4\pi^2} (6A^2) \), the third term is just simply \(-\partial_\mu \partial_\nu = \frac{-\lambda^2 e^2}{2\pi^2} (\partial_\mu \partial_\nu A^2)\).

Lastly we have the forth term as
\[
-\Box h_{\mu\nu} = \frac{\lambda^2 e^2}{4\pi^2} \left( - \eta_{\mu\nu} \Box A^2 - 2 (A_\nu \Box A_\mu + A_\mu \Box A_\nu + 2 \partial_\rho A_\mu \partial_\rho A_\nu) \right). \tag{50}
\]

The Ricci scalar is given by
\[
R = \partial_\mu \partial_\nu h_{\mu\nu} - \Box h = \frac{\lambda^2 e^2}{4\pi^2} (-5 \Box A^2 + 2 \partial_\alpha A^\beta \partial_\beta A^\alpha). \tag{51}
\]
Finally the Einstein tensor is defined by
\[
G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} R = 8\pi G T_{\mu\nu}. \tag{52}
\]
Thus we obtain
\[
G_{\mu\nu} = \frac{\lambda^2 e^2}{4\pi^2} \left( 2 (\Box \eta_{\mu\nu} - \partial_\mu \partial_\nu) A^2 + \partial_\mu A^\sigma \partial_\sigma A_\nu + \partial_\nu A^\sigma \partial_\sigma A_\mu + A^\sigma \partial_\sigma (\partial_\mu A_\nu + \partial_\nu A_\mu) \right. \\
- A_\nu \Box A_\mu - A_\mu \Box A_\nu - 2 \partial_\rho A_\mu \partial_\rho A_\nu - \eta_{\mu\nu} A^\alpha \partial_\alpha A^\alpha \left.), \tag{53}
\]
which will be a non-linear equation.

Finally we would study how the action changes under such change in the geometry. Any generic quantum field theory in curved spacetime is written as
\[
G = \int d^4x \sqrt{-g} \mathcal{L}, \tag{54}
\]
where \( g \) is the determinant of the curved metric and \( \mathcal{L} \) is any Lagrangian. Since we know the metric \( g_{\mu\nu}(x) \) as the perturbation of the flat Minkowski metric by \( g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x) \) and here we simply write \( g(x) = \eta + h(x) \) as in the matrix notation. We would like to find out \( \sqrt{-g} \) in terms of the perturbation metric \( h \). Then we would have the correction from the gauge field by
\[
\sqrt{-g} = 1 + f(e, \lambda, A^2) + O(A^4). \tag{55}
\]
Note that the correction term must be in the power of \( A^2 \) because \( \sqrt{-g} \) is a scalar quantity.

We will begin with the identity of the determinant. Let \( X \) be a matrix, then we have the following identity as an invariant polynomial of \( X \) (an equation which is invariant under similarity transformation of \( X \to U^{-1} X U \))
\[
\det(1 + X) = 1 + \operatorname{Tr} X + \frac{1}{2!} \left( \operatorname{Tr}(X^2) - (\operatorname{Tr} X)^2 \right) + \cdots + \det X. \tag{56}
\]
Note that the matrix variable of the determinant is \( 1 + X \), but we have \( \eta + h \). Therefore we need to make some slight changes. First consider
\[
r^{\rho\mu} g_{\mu\nu} = \eta^{\rho\mu} \eta_{\mu\nu} + \eta^{\rho\mu} h_{\mu\nu}(x) \\
= \delta^{\rho}_{\nu} + \eta^{\rho\mu} h_{\mu\nu}(x) \\
= 1 + \eta^{-1} h(x). \tag{57}
\]
Therefore we have
\[
\det(\eta^{-1}g) = \det(1 + \eta^{-1}h)
\]
\[
\implies (\det \eta^{-1})(\det g) = 1 + \text{Tr}(\eta^{-1}h) + \frac{1}{2!} \left(\text{Tr}(\eta^{-1}h)^2 - (\text{Tr} \eta^{-1}h)^2\right) + \cdots \quad (58)
\]
As \(\det \eta^{-1} = -1\),
\[
-\det g = -\det(1 + \eta^{-1}h) = 1 + \text{Tr}(\eta^{-1}h) + \frac{1}{2!} \left(\text{Tr}(\eta^{-1}h)^2 - (\text{Tr} \eta^{-1}h)^2\right) + \cdots \quad (59)
\]
therefore
\[
-\det g = 1 + \eta^{\mu\nu}h_{\mu\nu} + \frac{1}{2!} \left(\eta^{\mu\nu}\eta^{\rho\sigma}h_{\mu\rho}h_{\nu\sigma} - \eta^{\mu\nu}\eta^{\rho\sigma}h_{\mu\nu}h_{\rho\sigma}\right) + O(h^3). \quad (60)
\]
Up to first order, it follows that the differential of the determinant is
\[
d \det g = -\eta^{\mu\nu} dh_{\mu\nu} = -d(\eta^{\mu\nu} h_{\mu\nu}) = -d\text{Tr} h = -\text{Tr} dh. \quad (61)
\]
Or we can write
\[
\frac{d \det g}{dh_{\mu\nu}} = -\eta^{\mu\nu}. \quad (62)
\]
This can also be understood as from the taylor expansion of \(-\det g\),
\[
-\det g = 1 + \frac{d \det g}{dg_{\mu\nu}} dg_{\mu\nu} = 1 + \frac{d \det g}{dh_{\mu\nu}} dh_{\mu\nu} = 1 + \eta^{\mu\nu} dh_{\mu\nu} \quad (63)
\]
because \(dg_{\mu\nu} = d(\eta_{\mu\nu} + h_{\mu\nu}) = dh_{\mu\nu}\). Then we have
\[
\sqrt{-\det g} = \sqrt{1 + \eta^{\mu\nu} h_{\mu\nu} + O(h^2)} \approx 1 + \frac{1}{2} \eta^{\mu\nu} h_{\mu\nu} = 1 + \frac{1}{2} \text{Tr} h. \quad (64)
\]
Now since
\[
\eta^{\mu\nu} h_{\mu\nu} = \frac{3\lambda^2 e^2 A^2}{2\pi^2}, \quad (65)
\]
therefore the action in such curved spacetime is
\[
S = \int d^4x \left(1 + \frac{3\lambda^2 e^2 A^2}{2\pi^2} + O(A^4)\right) \mathcal{L}. \quad (66)
\]
Hence we have the extra correction as \(\left(\frac{3\lambda^2 e^2 A^2}{2\pi^2} + O(A^4)\right)\mathcal{L}\) due to the correction of gauge field.

### 3 Effective \(\gamma^5\) and Spin operator

The \(\gamma^5\) is a parity operator defined by \(\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3\), which is the only operator that commutes with all other \(\gamma^\mu\) matrices \([2]\). The spin operator is constructed by the \(\gamma^5\) matrix by, in the Dirac representation \([2]\),
\[
S^i = \frac{1}{2} \gamma^5 \gamma^0 \gamma^i \quad (67)
\]
for \( i = 1, 2, 3 \), where \( S^1, S^2 \) and \( S^3 \) correspond to the \( S_x, S_y, S_z \) spin components respectively. We will see how the interaction with gauge field effectively changes the spin of the Dirac fermion. The effective \( \gamma^5 \) operator becomes

\[
\Gamma_5^L = i \Gamma_0^L \Gamma_1^L \Gamma_2^L \Gamma_3^L = i f[A] \gamma^0 f[A] \gamma^1 f[A] \gamma^2 f[A] \gamma^3 .
\]

(68)

Up to second order, after some algebras one finds,

\[
\Gamma_5^L = \prod_{k=0}^{3} \left( 1 - \frac{\lambda e^2}{2\pi^2} \lambda + \frac{\lambda^2 e^2}{4\pi^2} A^2 + O(e^3) \right) \gamma^k \\
= i \left( \gamma^0 \gamma^1 - \frac{\lambda^2 e^2}{4\pi^2} \left( - \gamma^0 A \gamma^1 + 2 A^2 \gamma^0 \gamma^1 - A \gamma^0 \gamma^1 \right) + O(e^3) \right) \\
\times \left( \gamma^2 \gamma^3 - \frac{\lambda^2 e^2}{4\pi^2} \left( - \gamma^2 A \gamma^3 + 2 A^2 \gamma^2 \gamma^3 - A \gamma^2 \gamma^3 \right) + O(e^3) \right).
\]

(69)

Then it follows that,

\[
\Gamma_5^L = \gamma^5 + i \frac{\lambda^2 e^2}{4\pi^2} \left( 4 A^2 \gamma^0 \gamma^1 \gamma^2 \gamma^3 - \gamma^0 \gamma^1 \gamma^2 A \gamma^3 - \gamma^0 \gamma^1 \gamma^2 \gamma^3 \\
- \gamma^0 \gamma^1 A \gamma^2 \gamma^3 - A \gamma^0 \gamma^1 \gamma^2 \gamma^3 \right) + O(e^3) .
\]

(70)

With some manipulation finally we can write \( \Gamma_5^L \) as some correction of the original \( \gamma^5 \) matrix with some extra correction terms

\[
\Gamma_5^L = \left( 1 + \frac{3 \lambda^2 e^2}{2\pi^2} A^2 \right) \gamma^5 + F(A^\mu, \gamma^\nu) + O(e^3) ,
\]

(71)

where

\[
F(A^\mu, \gamma^\nu) = - \frac{i \lambda^2 e^2}{2\pi^2} \left( \gamma^0 \gamma^1 \gamma^2 A^{(3)} + A^{(0)} \gamma^1 \gamma^2 \gamma^3 + 2 \gamma^0 \gamma^1 A^{(2)} A^{(3)} - \left( \gamma^0 \gamma^1 A^{(2)} + \gamma^1 \gamma^2 A^{(0)} \right) \gamma^3 \right).
\]

(72)

We can see that only if \( A^\mu \) vanishes we get back \( \gamma^5 \).

Next we compute the effective spin operator. We have

\[
\Sigma^i = \frac{1}{2} \Gamma_5^L \Gamma_0^L \Gamma_i^L .
\]

(73)

and we want to express the effective spin operator \( \Sigma^i \) in terms of the original \( S^i \) with
some extra corrections. We can use the result in equation (70),

\[
    \Sigma^i = \frac{1}{2} \left( \gamma^5 + i \frac{\lambda^2 e^2}{4\pi^2} \left( 4A^2 \gamma^0 \gamma^1 \gamma^2 \gamma^3 - \gamma^0 \gamma^1 \gamma^2 \gamma^3 - \gamma^0 A \gamma^1 \gamma^2 \gamma^3 \right) - \gamma^0 A^2 \gamma^0 A \gamma^0 + \frac{\lambda^2 e^2}{4\pi^2} A^2 \gamma^0 \right) \times \left( \gamma^0 \frac{\lambda e^2}{4\pi^2} A \gamma^0 + \frac{\lambda^2 e^2}{4\pi^2} A^2 \gamma^0 \right)
\]

\[
    = S^i + \frac{\lambda^2 e^2}{2\pi^2} \left( A^3 \gamma^5 \gamma^0 \gamma^j - \frac{i}{4} \gamma^0 \gamma^1 \gamma^2 A \gamma^3 \gamma^0 \gamma^j - \frac{i}{4} \gamma^0 \gamma^1 A \gamma^2 A \gamma^3 \gamma^0 \gamma^j - \frac{i}{4} \gamma^0 \gamma^1 \gamma^2 \gamma^3 \gamma^0 \gamma^j \right)
\]

Employing the identity of \( \{ \gamma^\mu, A \} = 2A^\mu \) and the fact that \( A A = A^2 \) to the fifth term and the sixth term in the last line of equation (73), we get respectively,

\[
    -\frac{i}{4} A \gamma^0 A \gamma^1 \gamma^2 \gamma^3 \gamma^0 \gamma^j = -\frac{i}{2} A^0 \gamma^1 \gamma^2 \gamma^3 \gamma^0 \gamma^j + \frac{1}{4} A^2 \gamma^5 \gamma^0 \gamma^j
\]

and

\[
    -\frac{i}{4} \gamma^0 \gamma^1 \gamma^2 \gamma^3 \gamma^0 A \gamma^j = -\frac{i}{2} \gamma^0 \gamma^1 \gamma^2 \gamma^3 \gamma^0 A^j + \frac{1}{4} \gamma^5 \gamma^0 \gamma^j.
\]

After some simplification, we finally obtain the desired formula,

\[
    \Sigma^i = \left( 1 + \frac{3\lambda^2 e^2}{2\pi^2} A^2 \right) S^i + \frac{\lambda^2 e^2}{4\pi^2} \{ S^i, A \} + F(A^\mu, \gamma^\nu, \gamma^5),
\]

where

\[
    F(A^\mu, A^\nu, \gamma^5) = \frac{\lambda^2 e^2}{\pi^2} \left( \gamma^5 A^0 \gamma^j + \frac{i}{2} A^i \gamma^1 \gamma^2 \gamma^3 - \frac{i}{4} \gamma^0 \gamma^1 \gamma^2 A \gamma^3 \gamma^0 \gamma^j \right)
\]

Thus we can see that how origin spin operator is corrected by the gauge field, coupling and the anti-commutator \( \{ S^i, A \} \). We also clearly see that when the gauge field is turned off, we get back the original spin operator \( S^i \).

4 Conclusion

In this paper, we have derived an explicit formula for the curved metric that is defined by the effective gamma matrix due to gauge field interaction. We also worked out the perturbation metric \( h_{\mu\nu} \) by series expansion from the effective gamma matrix and studied how the action will be corrected. Finally, we demonstrated how the parity operator and spin operator are corrected under the effective gamma matrix in the presence of gauge field interaction.
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