Superradiant stability of dyonic black holes in string theory

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When a scalar wave perturbation is properly scattering off a charged or rotating black hole, the energy of the reflected scalar wave may be amplified. This is a superradiant process. If this amplification process can occur back and forth through certain confining mechanism, it will lead to strong instability of the black hole.

In this paper, the superradiant stability is investigated for a special kind of dyonic black holes in string theory. Although the dyonic black hole has a similar spacetime metric with a electrically charged RN black hole, it is found that the dyonic black hole is more unstable than a RN black hole due to the coupling between magnetic charge of the dyonic black hole and the impinging electrically charged scalar wave. We find two superradiantly stable regions in the parameter space for the dyonic black holes and charged massive scalar perturbation.

I. INTRODUCTION

The (in)stability problem of black holes is an interesting topic in black hole physics. Regge and Wheeler[1] proved that the spherically symmetric Schwarzschild black hole is stable under perturbations. The stability problems of rotating or charged black holes are more complicated due to the significant effect of superradiance. Superradiance effect can occur in both classical and quantum scattering processes [2–4, 6, 15]. When a charged bosonic wave is impinging upon a charged rotating black hole, the wave reflected by the event horizon is amplified if the wave frequency $\omega$ lies in the following superradiant regime

$$\omega < m\Omega + e\Phi,$$

(1)
where $e$ is the charge of the bosonic wave, $m$ is the azimuthal number, $\Omega$ is the angular velocity of black hole horizon and $\Phi$ is the electromagnetic potential of the black hole [7–13]. This means that when the incoming wave is scattered, the wave extracts rotational energy from rotating black holes and electromagnetic energy from charged black holes. According to the black hole bomb mechanism proposed by Press and Teukolsky [14], if there is a mirror between the black hole horizon and space infinity, the amplified wave can be reflected back and forth between the mirror and the black hole and grows exponentially. This leads to the superradiant instability of the black hole.

The superradiant properties of different kinds of black holes have been extensively studied in the literature (for a review, see [15]). A rotating Kerr black hole can be superradiantly unstable caused by a massive scalar field when the parameters of the black hole and the scalar field are in certain parameter spaces. In this case, the mass of the scalar field acts as a natural mirror. A lot of work has been done to identify these parameter spaces, e.g. [16–25]. A rapidly spinning black hole that is impinged upon by a complex and massive vector field is discussed in [26, 27]. The superradiant instability of rotating black holes in curved space is also reported [28, 36].

It is known that superradiant instability can also occur for charged black holes. When there is a mirror or a cavity outside a Reissner-Nordstrom (RN) black hole horizon, this black hole is superradiantly unstable in certain parameter spaces [37–42]. In flat background, RN black holes have been proved to be superradiantly stable against charged massive scalar perturbation [46–49]. It is pointed that when the parameters of a RN black hole and a charged massive scalar field satisfy superradiant conditions, there is no effective trapping potential outside the black hole horizon, which acts as a mirror to reflect the superradiant modes [47]. When charged black holes are in curved backgrounds, such as (anti-)de Sitter space, these backgrounds provide natural reflecting boundary conditions and support the existence of superradiant instability [50–54]. The discussion of black hole superradiant property has also been extended to the analogue of RN black hole in string theory and it has been shown that stringy RN black hole is superradiantly stable against charged massive scalar perturbation [55, 56]. But when a mirror is introduced, superradiant modes exist and the stringy RN black hole becomes unstable [57–59].

Magnetic field can trigger superradiant instability for a black hole. In [60, 61], the authors use an approximation method to show that when a scalar field is propagating on the Ernst background, the magnetic field will induce an effective mass $\mu_{\text{eff}} \propto B$ (where $B$ is the magnetic field strength) for the scalar field, leading to the superradiant instability. In a further work [62], the authors considered Kerr-Newman black holes immersed in a uniform magnetic field and showed that the magnetic field can confine scalar perturbations leading to long-lived modes, which trigger superradiant instabilities.

Besides electrically charged stringy RN black hole, another kind of interesting black holes in string theory are dyonic black holes [63–65], which have both electric and magnetic charges. In our previous work [47], we showed that electrically charged RN
black holes are superradiantly stable under massive charged scalar perturbation. In this paper, we will consider the superradiant behavior of the dyonic black holes under a charged massive scalar perturbation. For our case, the metric of the dyonic black hole is similar to a electrically charged RN black hole, however, the magnetic field also leads to the superradiant instability of the dyonic black holes and scalar perturbation system. In Section II, we describe the dyonic black hole and scalar perturbation system. In Section III, we analyse the radial equation of motion for the scalar and find two superradiant stable parameter regions for the system. Section IV is devoted to a summary.

II. DYONIC BLACK HOLE UNDER A SCALAR PERTURBATION

In low energy effective theory of string theory in four dimension, there are dyonic black hole solutions which have both electric charge $Q_e$ and magnetic charge $Q_m$. It is convenient to use natural units in which $G = c = 1$. The metric of the dyonic black hole in string theory is

$$ds^2 = - \frac{(r - r_+)(r - r_-)}{r^2 - r_0^2} dt^2 + \frac{r^2 - r_0^2}{(r - r_+)(r - r_-)} dr^2 + (r^2 - r_0^2)(d\theta^2 + \sin^2 \theta d\phi^2),$$

where

$$r_0 = \frac{Q_m^2 - Q_e^2}{2M}, \quad r_{\pm} = M \pm (M^2 + r_0^2 - Q_m^2 - Q_e^2)^{1/2}. \tag{3}$$

$M$ is the mass of the black hole. The dilaton field $\phi$ is $e^{2\phi} = \frac{r^2}{r^2 - r_0^2}$. This black hole has an event horizon at $r_+$, inner horizon at $r_-$ and a singular sphere at $r_0$.

In this paper, we consider a special dyonic black hole with equal electric and magnetic charge $Q_e = Q_m = Q$. Then, the above metric becomes to a RN-like metric,

$$ds^2 = - \frac{(r - r_+)(r - r_-)}{r^2} dt^2 + \frac{r^2}{(r - r_+)(r - r_-)} dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2),$$

where $r_{\pm} = M \pm (M^2 - 2Q^2)^{1/2}$. In this case, the dilaton filed $\phi = 0$. This metric has event horizon at $r_+$ and inner horizon at $r_-$ and there is no singular sphere. *Although this metric is similar to RN black hole, we will see that its superradiance behavior is different from RN black hole due to the magnetic charge.*

When a charged massive scalar perturbation $\Psi$ is impinging on the dyonic black hole, the dynamics of the charged massive scalar field is described by the Klein-Gordon (KG) equation

$$[(\nabla^\nu - iqA^\nu)(\nabla_\nu - iqA_\nu) - \mu^2]\Psi = 0,$$

where $q$ and $\mu$ are the charge and mass of the scalar field. The nonzero vector field components are $A_t = -Q/r$ and $A_\phi = -Q \cos \theta$ which describe electric and magnetic fields of the black hole.
The solution of the KG equation can be decomposed as the following form

$$\Psi(t, r, \theta, \phi) = R(r)Y(\theta)e^{im\phi}e^{-i\omega t}. $$  \hspace*{1cm} (6)

$m$ is azimuthal harmonic index and $\omega$ is the frequency of the scalar perturbation. Then, the angular part of the KG equation is

$$\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \frac{\partial}{\partial \theta} Y) + (\lambda - \frac{m^2 + q^2 Q^2 + 2mqQ \cos \theta}{\sin^2 \theta}) Y = 0. $$  \hspace*{1cm} (7)

Substituting $\eta = \cos \theta$ into the angular equation, one obtains the Fuchs equation

$$\frac{d^2 Y}{d\eta^2} + \left( \frac{1}{\eta - 1} + \frac{1}{\eta + 1} \right) \frac{dY}{d\eta} + \left[ -\frac{(m + qQ)^2}{2(\eta - 1)} + \frac{(m - qQ)^2}{2(\eta + 1)} - \lambda \right] \frac{Y}{(\eta - 1)(\eta + 1)} = 0, $$  \hspace*{1cm} (8)

which has three singularities ($-1, 1, \infty$). The solutions of Fuchs equation (8) can be expressed by hypergeometric functions as

$$Y(\eta) = C(1 - \eta)^{\frac{1}{2}(m+qQ)}(1 + \eta)^{\frac{1}{2}(m-qQ)}F_1(\alpha, \beta; \gamma; \frac{1+\eta}{2}), $$  \hspace*{1cm} (9)

where $C$ is a normalization constant. The parameters $\alpha, \beta$ and $\gamma$ are given by

$$\begin{align*}
\alpha &= m + \frac{1}{2}(1 + \sqrt{1 + 4\lambda}) \\
\beta &= m + \frac{1}{2}(1 - \sqrt{1 + 4\lambda}) \\
\gamma &= 1 + m - qQ \\
\end{align*} $$  \hspace*{1cm} (10)

Since $Y(\eta)$ should be finite when $-1 \leq \eta \leq 1$, we have the following conclusions,

- $\beta$ is a negative integer, the separation constant $\lambda = l(l + 1)$ and $l > m$.
- Finiteness of factor $(1 - \eta)^{\frac{1}{2}(m-qQ)}$ in (8) implies constraints between the angular quantum numbers $(l, m)$ of the scalar field and the charges of the scalar field and black hole,

$$m > qQ, \ l(l + 1) > q^2 Q^2. $$  \hspace*{1cm} (11)

Different from the cases of RN black hole and the electrically charge black hole in string theory, due to the interaction between the charged scalar and magnetic filed of the black hole, the solution of angular equation $Y(\eta)$ will be no longer hypergeometric function. Lower bounds on the angular quantum numbers of charged scalar modes appear.

The radial part of the Klein-Gordon equation is given by

$$\Delta \frac{d^2}{dr^2}(\Delta \frac{dR}{dr}) + UR = 0, $$  \hspace*{1cm} (12)

where $\Delta = (r - r_+)(r - r_-)$, and

$$U = (\omega r^2 - qQr)^2 - (r - r_+)(r - r_-)(-q^2 Q^2 + \mu^2 r^2 + l(l + 1)). $$  \hspace*{1cm} (13)
In order to study the superradiance (in)stability of the black hole against the massive charged perturbation, the asymptotic solutions of the radial wave equation near the horizon and at infinity will be considered with proper boundary conditions.

Defining the tortoise coordinate \( \gamma \) by the equation

\[
\frac{d\gamma}{dr} = \frac{r^2}{\Delta},
\]

and a new radial function as \( \tilde{R} = rR \), the radial wave equation (12) can be written as

\[
\frac{d^2\tilde{R}}{d\gamma^2} + \tilde{U}\tilde{R} = 0,
\]

where

\[
\tilde{U} = \frac{U}{r^4} + \frac{2\Delta^2}{r^6} - \frac{\Delta d\Delta}{r^5}.
\]

The asymptotic forms of the new potential \( \tilde{U} \) at the two boundaries are

\[
\lim_{r \to r_+} \tilde{U} = (\omega - \frac{qQ}{r_+})^2, \quad \lim_{r \to +\infty} \tilde{U} = \omega^2 - \mu^2.
\]

For the existence of superradiance modes, the chosen boundary conditions are ingoing wave at the horizon (\( y \to -\infty \)) and bound states (exponentially decaying modes) at spatial infinity (\( y \to +\infty \)). Then the radial wave equation has the following asymptotic solutions

\[
\tilde{R} \sim \begin{cases} 
  e^{-i(\omega - \frac{qQ}{r_+})y}, & y \to -\infty (r \to r_+) \\
  e^{-\sqrt{\mu^2 - \omega^2}y}, & y \to +\infty (r \to +\infty).
\end{cases}
\]

It is obvious that \( \tilde{R} \) describes exponentially decaying bound state modes as \( r \to +\infty \) when \( \omega^2 < \mu^2 \).

### III. STABILITY ANALYSIS OF DYONIC BLACK HOLE

Now we analyze whether there is a trapping well outside the black hole event horizon when the parameters of the scalar field and the black hole satisfy the bound state condition \( \omega^2 < \mu^2 \) and the superradiance condition \( 0 < \omega < qQ/r_+ \). It is convenient to define a new radial function \( \psi \) by

\[
\psi = \Delta^{1/2} R.
\]

Using this new function, the radial equation (5) can be written in the form of a Shrodinger-like wave equation

\[
\frac{d^2\psi}{dr^2} + (\omega^2 - V)\psi = 0,
\]
where the effective potential is

\[
V = \omega^2 + \frac{1}{2\Delta^2} (\Delta - U - (r - M)^2).
\]

(22)

In order to see if there exists a trapping well outside the event horizon, we should analyze the shape of the effective potential \(V\). It is obvious that

\[
V \to -\infty, \quad \text{as} \quad r \to r_-; \quad V \to -\infty, \quad \text{as} \quad r \to r_+;
\]

\[
V \to \mu^2 + \frac{-4M\omega^2 + 2q\phi + 2M\mu^2}{r} + O\left(\frac{1}{r^2}\right), \quad \text{as} \quad r \to +\infty.
\]

(23)

When \(\omega\) satisfies superradiant condition, \(\omega < \frac{\beta_0}{\mu^3}\), and bound state condition, \(\omega < \mu\), we can prove

\[
-4M\omega^2 + 2q\phi + 2M\mu^2 > -4M\omega^2 + 2q\phi + 2M\omega = 2M\omega\left(\frac{\phi}{\mu} - \omega\right) > 2M\omega\left(\frac{\phi}{\mu} - \omega\right) > 0.
\]

(24)

This means the effective potential has at least one maximum in the region \(r_- < r < r_+\) and one maximum outside the event horizon \((r > r_+)\). If there was only one maximum outside the event horizon for the effective potential, there will be no trapping well and the black hole will be stable. Defining a new radial variable \(z = r - r_-\), we conclude that if the derivative of the effective potential, \(V'(z)\), has only two positive roots, the black hole will be superradiantly stable.

The derivative of the effective potential is

\[
V'(z) = -\frac{2}{\Delta^2}((\Delta - U - (z + r_- - M)^2)\frac{d\Delta}{dz} + \frac{1}{2}\Delta U').
\]

(25)

We just care about the roots of \(V'(z)\), so that only the numerator of \(V'(z)\) is needed for our discussion. The numerator of \(V'(z)\) can be written as a polynomial of \(z\),

\[
f(z) = az^4 + bz^3 + cz^2 + dz + e,
\]

(26)

where

\[
a = -2M\mu^2 + 4M\omega^2 - 2q\phi, \tag{27}
\]

\[
b = 4(8M^2 - 6Mr_+ + r_+^2)\omega^2 - 4q(5M - 2r_+)\rho - 2(l + f - 2q^2 \phi + (6M^2 - 6Mr_+ + r_+^2)\mu^2), \tag{28}
\]

\[
c = 12(2M - r_+)^2 \omega^2 - 18q(2M - r_+^2)\omega - 6l(M - r_+)(l + 1) - 3Mq^2 \phi^2 + 2r_+q^2 \phi^2 + \mu^2(M - r_+)(2M - r_+^2), \tag{29}
\]

\[
d = 4(4M - 3r_+)(2M - r_+)^2 \omega^2 - 4q(7M - 5r_+)(2M - r_+^2)\omega - 4(2M^2 - 3Mr_+ + r_+^2)\mu^2
\]

\[+4(7M^2 - 9Mr_+ + 3r_+^2)q^2 \phi^2 - 4(l(l + 1) - 1)(M - r_+)^2, \tag{30}
\]

\[
e = 2(r_+ - r_-)^2(q\phi - q\rho) + \frac{1}{2}(r_+ - r_-)^3. \tag{31}
\]
The four roots of \( f(z) = 0 \) are denoted by \( z_1, z_2, z_3, z_4 \). They satisfy the following relations

\[
\begin{align*}
z_1 * z_2 * z_3 * z_4 &= e/a, \\
z_1 * z_2 + z_1 * z_3 + z_1 * z_4 + z_2 * z_3 + z_2 * z_4 + z_3 * z_4 &= c/a.
\end{align*}
\] (33) (34)

Some comment on the roots is needed. In principle, these four roots are in complex plane. According to discussion below equation (24), we already have two real roots. If there is a complex root, then its complex conjugate is also a root. In this case, the dyonic black hole is superradiantly stable. In the following discussion, we suppose a worse case that all four roots are real. From the superradiant conditions \( \varphi < qQ/r_+ \) and bound state condition \( \varphi < \mu \), we can prove that \( a < 0 \) and \( e < 0 \). From the asymptotic behaviors of the effective potential analyzed before, we also know that there are at least two maxima for the effective potential \( V(z) \) when \( z > 0 \) and they correspond to two positive roots \( z_1, z_2 \) of \( V'(z) = 0 \). The equation (33) means the another two roots \( z_3, z_4 \) are both positive or negative. If they are both negative, then there is no trapping well outside the event horizon and the black hole is superradiantly stable. From equation (34), we can find that \( c/a < 0 \) (i.e. \( c > 0 \)) is a sufficient condition for negative \( z_3, z_4 \) and therefore a sufficient condition for superradiantly stable dyonic black holes.

Now, let’s consider the parameter regions of dyonic black hole and scalar field where \( c > 0 \). Coefficient \( c \) can be treated as a quadratic polynomial of \( \varphi, c = c(\varphi) \) and the coefficient of the quadratic term is positive. The intercept of \( c(\varphi) \) is

\[
c(0) = -6\{(M - r_+)l(l + 1) - 3Mq^2Q^2 + 2r_+q^2Q^2 + \mu^2(M - r_+)(2M - r_+)^2\}. \] (35)

The discriminant of the quadratic equation, \( c(\varphi) = 0 \), is

\[
\Delta_c = \left(-18qQr_+^2\right)^2 - 4\left(12r_+^3\right)\left(3\left(-r_+ \left(l^2 + l - 3q^2Q^2 - 1\right) + r_+ \left(l^2 + l - q^2Q^2 + 1\right) - 2M - \mu^2r_+ + \mu^2r_+^2r_+\right)\right) = 36(2M - r_+)^3 \left(8l^2(M - r_+) + 8l(M - r_+) + 32\mu^2M^3 - 64\mu^2M^2r_+ - 6Mq^2Q^2 + 40\mu^2Mr_+^2 + 7q^2Q^2r_+ - 8\mu^2r_+^3\right). \] (36)

In the following, we will discuss two cases for \( c > 0 \) and find out the relevant parameter regions.

1. **Case I:** \( \Delta_c < 0 \Rightarrow c > 0 \)

By the definition, we have \( 2M > r_+ > M, \) and \( \Delta_c < 0 \) is equivalent to

\[
8l^2(M - r_+) + 8l(M - r_+) + 32\mu^2M^3 - 64\mu^2M^2r_+ - 6Mq^2Q^2 + 40\mu^2Mr_+^2 + 7q^2Q^2r_+ - 8\mu^2r_+^3 < 0. \] (37)

The above inequality can be rewritten as

\[
\mu^2 > \frac{q^2Q^2}{r_+^2} \frac{x^2(7x - 6)}{8(x - 2)^2(x - 1)} - \frac{l(l + 1)}{r_+^2}. \] (38)
where \( x = r_*/M \), and from the definition of \( r_* \), we have \( 1 < x < 2 \). When the parameters of black hole and dyonic black hole satisfy Eq.(38), the dyonic black hole is superradiantly stable against the scalar perturbation. Two examples of the effective potential in this case are shown in Fig1.

![Effective Potential in Case I](image)

**FIG. 1:** Two examples of the effective potential in case I. The black hole mass are chosen as \( M = 10500 \) and \( M = 10000 \) for the blue curve and orange curve respectively. The other parameters are chosen as \( l = \omega = 5, \mu = 20, q = 15, Q = 7000 \).

2. **Case II:** when \( \Delta_c > 0 \), the two roots of \( \varsigma(\omega) = 0 \) are \( \omega_+ \), then \( 0 < \omega < \omega_- \Rightarrow c > 0 \)

The explicit forms of \( \omega_\pm \) are

\[
\omega_\pm = \frac{18qQr^2 \pm \sqrt{\Delta_c}}{24r_-^3}.
\]

(39)

The condition \( \Delta_c > 0 \) requires that \( \mu \) must satisfy

\[
\mu^2 < \frac{q^2Q^2}{r_-^2} - \frac{x^2(7x - 6)}{8(x - 2)^2(x - 1)} \frac{l(l + 1)}{r_-^2}.
\]

(40)

We know that \( \phi \) satisfies superradiant condition, \( \phi < qQ/r_+ \). If \( qQ/r_+ < \omega_- \), then we have \( 0 < \omega < \omega_- \Rightarrow c > 0 \). So we consider the constraint \( qQ/r_+ < \omega_- \),

\[
qQ/r_+ < \frac{18qQr^2 - \sqrt{\Delta_c}}{24r_-^3} = \frac{3qQ}{4r_-} - \frac{\sqrt{\Delta_c}}{24r_-^3},
\]

\[
\Rightarrow \frac{\sqrt{\Delta_c}}{24r_-^3} < \frac{3qQ}{4r_-} - qQ/r_+. \]

(41)

Then the above inequality is equivalent to

\[
\frac{3qQ}{4r_-} - qQ/r_+ > 0 \quad \text{(} r_+/r_- > 4/3\text{)}, \quad 18qQr_-^2 - 24r_-^3qQ > r_+ \sqrt{\Delta_c}.
\]

(42)
It is convenient to square the second inequality of the above so that we have

\[ r_+^2 (2M - r_+)^2 \mu^2 - 16M^2 q^2 Q^2 + 20Mq^2 Q^2 r_+ + (l(l + 1) - 7q^2 Q^2) r_+^2 > 0, \]

which can be simplified as

\[ \mu^2 > \frac{q^2 Q^2}{r_+^2} \frac{7r_+^2 - 20Mr_+ + 16M^2}{r_+^2} - \frac{l(l + 1)}{r_+^2} = \frac{q^2 Q^2}{r_+^2} \frac{7x^2 - 20x + 16}{(x - 2)^2} - \frac{l(l + 1)}{r_+^2}. \]  

(43)

One can check that the above condition is consistent with Eq. (40),

\[ \frac{q^2 Q^2}{r_+^2} \frac{7x^2 - 20x + 16}{(x - 2)^2} - \frac{l(l + 1)}{r_+^2} < \mu^2 < \frac{q^2 Q^2}{r_+^2} \frac{x^2(7x - 6)}{8(x - 2)^2(x - 1)} - \frac{l(l + 1)}{r_+^2}. \]  

(44)

So, together with the constraint on mass and charge ratio of dyonic black hole, a superradiantly stable parameter region of the scalar and dyonic black hole system in case II is

\[ \frac{q^2 Q^2}{r_+^2} \frac{7x^2 - 20x + 16}{(x - 2)^2} - \frac{l(l + 1)}{r_+^2} < \mu^2 < \frac{q^2 Q^2}{r_+^2} \frac{x^2(7x - 6)}{8(x - 2)^2(x - 1)} - \frac{l(l + 1)}{r_+^2}, \]  

(45)

\[ \frac{r_+}{r_-} > \frac{4}{3} \left( \frac{Q}{M} < \frac{2 \sqrt{6}}{7} \right). \]  

(46)

There are two examples of the effective potential in this case are shown in Fig 2.

FIG. 2: Two examples of the effective potential in case II. The black hole mass are chosen as \( M = 10000 \sqrt{2} + 1000 \) and \( M = 10000 \sqrt{2} + 1010 \) for the blue curve and orange curve respectively. The other parameters are chosen as \( l = 5, \omega = 7, \mu = 15, q = 14, Q = 10000 \).
3. Case III: when $\Delta > 0$, the two roots of $c(\omega) = 0$ are $\omega_{\pm}$, then $\omega > \omega_{\pm} \Rightarrow c > 0$

We know that $\omega$ satisfies superradiant condition, $\omega < qQ/r_+$. In this case, we need $\omega_{\pm} < \omega < qQ/r_+ \Rightarrow c > 0$. Then we consider the constraint $qQ/r_+ > \omega_{\pm}$,

$$
\begin{align*}
qQ/r_+ &> \frac{18qQr_+^2 + \sqrt{\Delta_c}}{24r_+^3} = \frac{3qQ}{4r_-} + \frac{\sqrt{\Delta_c}}{24r_+^3}, \\
\Leftrightarrow \frac{\sqrt{\Delta_c}}{24r_+^3} &< \frac{-3qQ}{4r_-} + qQ/r_+.
\end{align*}
$$

(47)

Then the above inequality is equivalent to

$$
\frac{3qQ}{4r_-} - qQ/r_+ < 0 \ (r_+/r_- < 4/3), \quad -(18qQr_+^2 - 24r_+^3qQ) > r_+ \sqrt{\Delta_c}.
$$

(48)

It is convenient to square the second inequality of the above so that we have

$$
(2M - r_+)^2 \mu^2 - 16M^2q^2Q^2 + 20Mq^2Q^2r_+ + (l(l + 1) - 7q^2Q^2)r_+^3 > 0,
$$

which can be simplified as

$$
\mu^2 > \frac{q^2Q^2 7r_+^2 - 20Mr_+ + 16M^2}{r_+^2} - \frac{l(l + 1) + 16}{r_+^3} = \frac{q^2Q^2 7x^2 - 20x + 16}{r_+^2} - \frac{l(l + 1)}{r_+^3}.
$$

(49)

One can check that the above condition is consistent with Eq. (40),

$$
\frac{q^2Q^2 7x^2 - 20x + 16}{r_+^2} < \frac{l(l + 1) + 16}{r_+^3} < \mu^2 < \frac{x^2(-6 + 7x)}{8(-2 + x)^2(-1 + x)} - \frac{l(l + 1)}{r_+^3}.
$$

(50)

In case III, the superradiantly stable parameter space of the scalar and dyonic black hole system is

$$
\begin{align*}
\frac{q^2Q^2 7x^2 - 20x + 16}{r_+^2} - \frac{l(l + 1)}{r_+^3} &< \mu^2 < \frac{q^2Q^2 7x^2 - 20x + 16}{r_+^2} - \frac{l(l + 1)}{r_+^3}, \\
r_+ &< \frac{4}{5} \left( \frac{Q}{M} > \frac{2\sqrt{6}}{7} \right),
\end{align*}
$$

(51)

$$\omega_{\pm} < \omega < qQ/r_+.
$$

(52)

(53)

Similarly, we draw two examples of the effective potential in this case are shown in Fig3.
FIG. 3: Two examples of the effective potential in case III. The black hole mass and frequency are chosen as $M = 10000 \sqrt{2} + 10, \omega = 6$ and $M = 10000 \sqrt{2} + 10, \omega = 6.55$ for the blue curve and orange curve respectively. The other parameters are chosen as $l = 5, \mu = 15, q = 10, Q = 10000$.

IV. SUMMARY

In this paper, we study the superradiant stability of the dyonic black holes in string theory against a charged massive bosonic perturbation. Although the metric of a dyonic black hole is similar to that of a RN black hole, the magnetic field of the dyonic black hole makes the scalar and black hole system more unstable than RN black hole. The system is not superradiantly stale in whole parameter space and we discuss three cases for the superradiantly stable region. For a general dyonic black hole (with the equal electric and magnetic charges) the superradiantly stable parameter region of the system is

$$\mu^2 > \frac{\mu^2 Q^2}{r_c^2} \frac{x^2(7x^2 - 6)}{8(x^2 - 1)^2} - \frac{h(x+1)}{r_c^2} (x = r_+/M).$$

If the ratio between the black hole charge and the black hole mass satisfies $\frac{Q}{M} < \frac{2 \sqrt{6}}{7}$, the superradiantly stable parameter region of the system becomes larger, which is $\mu^2 > \frac{\mu^2 Q^2}{r_c^2} \frac{x^2(x^2 - 20x + 16)}{(x^2 - 2)^2} - \frac{h(x+1)}{r_c^2}$. For the case $\frac{Q}{M} > \frac{2 \sqrt{6}}{7}$, the superradiantly stable parameter is

$$\mu^2 > \frac{\mu^2 Q^2}{r_c^2} \frac{x^2(x^2 - 20x + 16)}{(x^2 - 2)^2} - \frac{h(x+1)}{r_c^2} \text{ and } \omega_s < \omega < qQ/r_+.$$ 

For each case, we present a picture to show the superradiantly stable effective potential.

Recently, it has been pointed out that the magnetically charged black holes have some interesting features\cite{67}. They can have long-lived life even their masses are not large. They may have strong magnetic fields, which results in restoring the electroweak symmetry in some regions around them. It is worth noting that a relevant interesting feature is the Hawking radiation effects are enhanced by the magnetic fields for near extremal black holes. In the near extremal limit, the Hawking emission modes are in the superradiant region. In this sense, the magnetic fields enhanced the emission of superradiant modes. This is consistent with our result that magnetic fields make black hole system more unstable. It will be interesting to study further about magnetically charged black hole systems.
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