MOMENTS CALCULATION FOR THE DOUBLE TRUNCATED MULTIVARIATE NORMAL DENSITY

Manjunath, B G and Stefan Wilhelm

Key Words: multivariate normal; double truncation; moment generating function; marginal density function.

ABSTRACT

In the present article we deal with the problem of computing the first and second moments for the rectangularly double truncated multivariate normal density. Our primary aim is to extend the derivation of Tallis (1961) to general \( \mu \), \( \Sigma \) and for double truncation. Indeed we also deduce a simple computer algorithm for computing the first two moments and the bivariate marginal density.

1. INTRODUCTION

In Tallis (1961) for the singly truncated multivariate normal distribution, moment generating function (m.g.f) and explicit expressions are derived under the correlation matrix \( R \). Amemiya (1974) and Lee (1979) extended the derivation to zero mean and deduced the relationship between the first two moments. Lee (1983) derived a very simple recursive relationship between moments of any orders, see also Gupta and Tracy (1976). Due to the specific structure of the recursive relationship, these recurrent conditions are not sufficient for the computation of high order moments. In this article, we will generalize the Tallis (1961) result.

In the following we will note few literatures, on different forms of truncation and their moments calculation. Tallis (1963) introduces elliptical and radial truncation for

\(^{1}\) Corresponding author.

E-mail addresses: bgmanjunath@gmail.com (Manjunath, B.G.), wilhelm@financial.com (Stefan Wilhelm).
multivariate normal population. In conjunction to moments calculation for elliptically truncated normal, see Kotz et al. (2000). In Tallis (1965) for plan truncated normal variates, moments calculation and their relation to rectangle truncation has been derived. For a brief introduction on different types of truncation, see Kotz et al. (2000).

The $d$–dimensional normal density with location parameter vector $\mu \in \mathbb{R}^d$ and non-singular covariance matrix $\Sigma$ is given by,

$$\phi_{\mu, \Sigma}(x) = \frac{1}{(2\pi)^{d/2}|\Sigma|^{1/2}} \exp \left\{ -\frac{1}{2} (x - \mu)' \Sigma^{-1} (x - \mu) \right\}, \quad x \in \mathbb{R}^d. \quad (1)$$

The pertaining distribution function is denoted by $\Phi_{\mu, \Sigma}(x)$. Correspondingly, the multivariate truncated normal density, truncated at $a$ and $b$, in $\mathbb{R}^d$, is defined as

$$\phi_{a, \mu, \Sigma}(x) = \begin{cases} \frac{\phi_{\mu, \Sigma}(x)}{P\{a \leq X \leq b\}}, & \text{for } a \leq x \leq b, \\ 0, & \text{otherwise}, \end{cases} \quad (2)$$

Denote, $\alpha = P\{a \leq X \leq b\}$

2. MOMENT GENERATING FUNCTION

By the definition, $d$–dimensional m.g.f is given by,

$$m(t) = \int_a^b e^{tx} f(x) dx,$$

Clearly, the above integral is an $d$–fold integral. The m.g.f for (2) is given by

$$m(t) = \frac{1}{\alpha(2\pi)^{d/2}|\Sigma|^{1/2}} \int_a^b \exp \left\{ -\frac{1}{2} \left[ (x - \mu)' \Sigma^{-1} (x - \mu) - 2t'x \right] \right\} dx \quad (3)$$

For simplification the assertion is derived for $\mu = 0$ without loss of generality, i.e. we translate all variables and truncation points by $-\mu$. At the end of this section, our result will be extended to general $\mu$. Now, consider only the exponent term in (3) with $\mu = 0$,

$$-\frac{1}{2} \left[ x' \Sigma^{-1} x - 2t'x \right].$$

By simplification, we will have

$$\frac{1}{2} t' \Sigma t - \frac{1}{2} \left[ (x - \xi)' \Sigma^{-1} (x - \xi) \right].$$
where, $\xi = \Sigma t$.

The m.g.f for the double truncated multivariate normal is,

$$m(t) = \frac{e^T}{\alpha (2\pi)^{d/2} |\Sigma|^{1/2}} \int_{a}^{b} \exp \left\{ -\frac{1}{2} \left[ (x - \xi)' \Sigma^{-1} (x - \xi) \right] \right\} dx$$  \hspace{1cm} (4)

where, $T = \frac{1}{2} t' \Sigma t$.

Further simplification it reduces to,

$$m(t) = \frac{e^T}{\alpha (2\pi)^{d/2} |\Sigma|^{1/2}} \int_{a-\xi}^{b-\xi} \exp \left\{ -\frac{1}{2} x' \Sigma^{-1} x \right\} dx$$  \hspace{1cm} (5)

For notational convenience, (5) can be written as,

$$m(t) = e^T \Phi_{a\Sigma}$$  \hspace{1cm} (6)

3. FIRST AND SECOND MOMENT CALCULATION

Differentiate equation (6) with respect to $t_i$, we get

$$\frac{\partial m(t)}{\partial t_i} = e^T \frac{\partial \Phi_{a\Sigma}}{\partial t_i} + \Phi_{a\Sigma} \frac{\partial e^T}{\partial t_i}.$$  \hspace{1cm} (7)

The evaluated terms in the above equation are,

$$\frac{\partial e^T}{\partial t_i} = e^T \sum_{k=1}^{d} \sigma_{i,k} t_k$$

and

$$\frac{\partial \Phi_{a\Sigma}}{\partial t_i} = \frac{\partial}{\partial t_i} \int_{a_1^*}^{b_1^*} \ldots \int_{a_i^*}^{b_i^*} \ldots \int_{a_d^*}^{b_d^*} \phi_{a\Sigma}(x) dx \ldots dx_1$$  \hspace{1cm} (8)

where $a_i^* = a_i - \sum_{k=1}^{d} \sigma_{i,k} t_k$ and $b_i^* = b_i - \sum_{k=1}^{d} \sigma_{i,k} t_k$.

So, (8) reduces to,

$$\frac{\partial \Phi_{a\Sigma}}{\partial t_i} = \sum_{k=1}^{d} \sigma_{i,k} (F_k(a_k^*) - F_k(b_k^*))$$  \hspace{1cm} (9)

where,

$$F_i(x) = \int_{a_1}^{b_1} \ldots \int_{a_{i-1}}^{b_{i-1}} \int_{a_{i+1}}^{b_{i+1}} \ldots \int_{a_d}^{b_d} \phi_{a\Sigma}(x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_d) dx_1 \ldots dx_{i-1} dx_{i+1} \ldots dx_d.$$  \hspace{1cm} (10)
Note that at \( t_k = 0 \), for \( k = 1, 2, \ldots, d \) then \( a_i^* = a_i \) and \( b_i^* = b_i \). So, \( F_i(x) \) will be the \( i \)-th marginal density. From (7) – (9) for all \( t_k = 0 \), for \( k = 1, 2, \ldots, d \). Hence, the first moment is given by,

\[
E(X_i) = \frac{\partial m(t)}{\partial t_i} \bigg|_{t=0} = \sum_{k=1}^{d} \sigma_{i,k} (F_k(a_k) - F_k(b_k)). 
\] (11)

Now, again differentiate (7) with respect to \( t_j \), we have

\[
\frac{\partial^2 m(t)}{\partial t_j \partial t_i} = e^T \frac{\partial^2 \Phi_{a\Sigma}}{\partial t_j \partial t_i} + \frac{\partial \Phi_{a\Sigma}}{\partial t_i} \frac{\partial e^T}{\partial t_j} + \Phi_{a\Sigma} \frac{\partial^2 e^T}{\partial t_i \partial t_j} + \frac{\partial e^T}{\partial t_i} \frac{\partial \Phi_{a\Sigma}}{\partial t_j} 
\] (12)

The required evaluated terms in the above equation are,

\[
\frac{\partial^2 e^T}{\partial t_j \partial t_i} = \sigma_{i,j}. 
\]

Clearly, for the first term in (12) we need to differentiate (9) with respect to \( t_j \). We have,

\[
\frac{\partial^2 \Phi_{a\Sigma}}{\partial t_j \partial t_i} = \sum_{k=1}^{d} \left( \sigma_{i,k} \frac{\partial F_k(a_k^*)}{\partial t_j} \right) - \sum_{k=1}^{d} \left( \sigma_{i,k} \frac{\partial F_k(b_k^*)}{\partial t_j} \right) 
\] (13)

Now, merely consider the partial differentiation of marginal density with respect to \( t_j \), by simplification we have,

\[
\frac{\partial F_k(a_k^*)}{\partial t_j} = \frac{\partial}{\partial t_j} \int_{a_1^*}^{b_1^*} \ldots \int_{a_{k-1}^*}^{b_{k-1}^*} \int_{a_{k+1}^*}^{b_{k+1}^*} \ldots \int_{a_d^*}^{b_d^*} \Phi_{a\Sigma}(x_1, \ldots, x_{k-1}, a_k^*, x_{k+1}, \ldots, x_d) dx_{-k} 
\]

\[
= \frac{\sigma_{i,k} a_k^* F_k(a_k^*)}{\sigma_{k,k}} + \sum_{q \neq k} \left( \sigma_{i,q} \frac{\partial \sigma_{j,k}}{\partial \sigma_{k,k}} \right) \left( F_{k,q}(a_k^*, a_q^*) - F_{k,q}(a_k^*, b_q^*) \right). 
\] (14)

where,

\[
F_{k,q}(x, y) = \int_{a_1}^{b_1} \ldots \int_{a_{k-1}}^{b_{k-1}} \int_{a_{k+1}}^{b_{k+1}} \ldots \int_{a_q}^{b_q} \int_{a_{q+1}}^{b_{q+1}} \ldots \int_{a_d}^{b_d} \Phi_{a\Sigma}(x, y, x_{-k-q}) dx_{-k-q} 
\] (15)

The above equation (14) is deduced from Lee (1979), pp. 167. Note that for all \( t_k = 0 \), then the term \( F_{k,q}(x, y) \) will be a bivariate marginal density.

Similarly, \( \frac{\partial F_k(b_k^*)}{\partial t_j} \) can be obtained by replaying \( a_k^* \) by \( b_k^* \). From (12) – (15) at all \( t_k = 0 \) and
hence the second moment is given by,

$$E(X_i, X_j) = \frac{\partial^2 m(t)}{\partial t_j \partial t_i} \bigg|_{t=0}$$

$$= \sum_{k=1}^{d} \frac{\sigma_{j,k} (a_k F_k(a_k) - b_k F_k(b_k))}{\sigma_{k,k}}$$

$$+ \sum_{k=1}^{d} \sigma_{i,k} \sum_{q \neq k} (\sigma_{j,q} - \frac{\sigma_{k,q} \sigma_{j,k}}{\sigma_{k,k}}) \left( (F_{k,q}(a_k, a_q) - F_{k,q}(a_k, b_q)) - (F_{k,q}(a_k, a_q) - F_{k,q}(b_k, b_q)) \right). \quad (16)$$

For general $\mu$, the first moment is,

$$E(X_i) = \sum_{k=1}^{d} \sigma_{i,k} (F_k(a_k) - F_k(b_k)) + \mu_i. \quad (17)$$

and the second moment is unchanged because covariance matrix is unaltered due to location transformation.

4. BIVARIATE MARGINAL DENSITY COMPUTATION

For computing the bivariate marginal density in this section we mainly follow Tallis (1961), pp. 223 and Leppard and Tallis (1989) who implicitly used the bivariate marginal density as part of the moment calculation, evaluated at the integration bounds. However, we extend it to the double truncated case and state the function for all points within the support region.

Without loss of generality we use a $z$-transformation for all variates $x = (x_1, \ldots, x_d)'$ as well as for all lower and upper truncation points $a = (a_1, \ldots, a_d)'$ and $b = (b_1, \ldots, b_d)'$, resulting in a $N(0, R)$ distribution with correlation matrix $R$ for the standardized untruncated variates. In this section we treat all variables as if they are $z$-transformed, leaving the notation unchanged.

For computing the bivariate marginal $F_{q,r}(x_q, x_r)$ with $a_q \leq x_q \leq b_q, a_r \leq x_r \leq b_r, q \neq r$, we use the fact that for truncated normal densities the conditional densities are truncated normal again. The following relationship holds for $x_s, z_s \in \mathbb{R}^{d-2}$ if we condition on
\( x_q = c_q \) and \( x_r = c_r \) (\( s \neq q \neq r \)):

\[
\alpha^{-1} \varphi_d(x_s, x_q = c_q, x_r = c_r; R) = \alpha^{-1} \varphi(c_q, c_r, \rho_{qr}) \varphi_{d-2}(z_s; R_{qr})
\]

(18)

where

\[
z_s = \frac{(x_s - \beta_{sq}c_q - \beta_{sr}c_r)}{\sqrt{(1 - \rho_{sq}^2)(1 - \rho_{sr,q}^2)}}
\]

(19)

and \( R_{qr} \) is the matrix of second-order partial correlation coefficients for \( s \neq q \neq r \). \( \beta_{sq,r} \) and \( \beta_{sr,q} \) are the partial regression coefficients of \( x_s \) on \( x_q \) and \( x_r \) respectively and \( \rho_{sr,q} \) is the partial correlation coefficient between \( x_s \) and \( x_r \) for fixed \( x_q \).

Integrating out \((d - 2)\) variables \( x_s \) leads to \( F_{qr}(x_q, x_r) \) as a product of a bivariate normal density \( \varphi(x_q, x_r) \) and a \((d - 2)\)-dimension normal integral \( \Phi_{d-2} \):

\[
F_{qr}(x_q = c_q, x_r = c_r) = \int_{a_1}^{b_1} \cdots \int_{a_{d-1}}^{b_{d-1}} \int_{a_{d-1}}^{b_{d-1}} \int_{a_{d-1}}^{b_{d-1}} \varphi_{d-2}(x_s, c_q, c_r) dx_s
\]

(20)

\[
= \alpha^{-1} \varphi(c_q, c_r, \rho_{qr}) \Phi_{d-2}(A^q_{rs}; B^q_{rs}; R_{qr})
\]

(21)

where \( A^q_{rs} \) and \( B^q_{rs} \) denote the lower and upper integration bounds of \( \Phi_{d-2} \) given \( x_q = c_q \) and \( x_r = c_r \):

\[
A^q_{rs} = \frac{(a_s - \beta_{sq}c_q - \beta_{sr}c_r)}{\sqrt{(1 - \rho_{sq}^2)(1 - \rho_{sr,q}^2)}}
\]

(22)

\[
B^q_{rs} = \frac{(b_s - \beta_{sq}c_q - \beta_{sr}c_r)}{\sqrt{(1 - \rho_{sq}^2)(1 - \rho_{sr,q}^2)}}
\]

(23)

The computation of \( F_{qr}(x_q, x_r) \) just needs the evaluation of the normal integral \( \Phi_{d-2} \) in \( d - 2 \) dimensions, which is readily available in most statistics software packages (for example the function \texttt{pmvnorm()} in the R package \texttt{mvtnorm}). We implemented the bivariate marginal density function \texttt{dtmvnorm.marginal2()} in the R package \texttt{tmvtnorm} where the code is publicly available.

5. NUMERICAL EXAMPLES

EXAMPLE 1

We will use the following bivariate example with \( \mu = (0.5, 0.5)^T \) and covariance matrix \( \Sigma \)

\[
\Sigma = \begin{pmatrix} 1 & 1.2 \\ 1.2 & 2 \end{pmatrix}
\]

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Figure 1: Contour plot for the bivariate truncated density function

as well as lower and upper truncation points $a = (-1, -\infty)^T, b = (0.5, 1)^T$, i.e. $x_1$ is doubly, while $x_2$ is singly truncated. The bivariate marginal density $F_{q,r}(x, y)$ is the density function itself and is shown in figure 1, the one-dimensional densities $F_k(x)$ ($k = 1, 2$) in figure 2.

The moments calculation for our example results in truncated mean $\mu^* = (-0.152, -0.388)^T$ and covariance matrix

$$
\Sigma^* = \begin{pmatrix}
0.163 & 0.161 \\
0.161 & 0.606
\end{pmatrix}
$$

Figures 3 and 4 show the evolution of a Monte-Carlo estimate for the elements of the mean vector and the covariance matrix respectively for growing sample sizes. Furthermore, the 95% confidence interval obtained from Monte Carlo using the full sample of 10000 items is shown. All confidence intervals contain the true theoretical value, but Monte Carlo estimates still show substantial variation even with a sample size of 10000. Simulation from a truncated multivariate distribution and calculating the sample mean or the sample covariance respectively also leads to consistent estimates of $\mu^*$ and $\Sigma^*$. Since the rate of convergence of the MC estimator is $O(1/\sqrt{n})$, one has to ensure sufficient Monte Carlo it-
Figure 2: Marginal densities $F_k(x)$ ($k = 1, 2$) for $x_1$ and $x_2$ obtained from Kernel density estimation and from direct calculation.

EXAMPLE 2
Let $\mu = (0, 0, 0)^T$, the covariance matrix

$$
\Sigma = \begin{pmatrix}
1.1 & 1.2 & 0 \\
1.2 & 2 & -0.8 \\
0 & -0.8 & 3
\end{pmatrix}
$$

and the lower and upper truncation points $a = (-1, -\infty, -\infty)^T, b = (0.5, \infty, \infty)^T$, then the only truncated variable is $x_1$, which is furthermore uncorrelated with $x_3$.

Our formula results in $\mu^* = c(-0.210, -0.229, 0)'$ and

$$
\Sigma^* = \begin{pmatrix}
0.174 & 0.190 & 0.0 \\
0.190 & 0.898 & -0.8 \\
0 & -0.8 & 3.0
\end{pmatrix}
$$
Figure 3: Paths of the Monte-Carlo estimator for $\mu^*$

Figure 4: Paths of the Monte-Carlo estimator for $\Sigma^*$
For this special case of only $k < d$ truncated variables $x_1, \ldots, x_k$, the remaining $d - k$ variables $x_{k+1}, \ldots, x_d$ can be regressed on the truncated variables, and a simple formula for the mean and covariance matrix can be given (see Kotz et al. (2000), p.70). Let the covariance matrix $\Sigma$ of $x_1, \ldots, x_d$ be partitioned as

$$
\Sigma = \begin{pmatrix}
V_{11} & V_{12} \\
V_{21} & V_{22}
\end{pmatrix}
$$

(24)

where $V_{11}$ denotes the $k \times k$ covariance matrix of $x_1, \ldots, x_k$. The mean vector and the covariance matrix of all $d$ variables can be computed as

$$
(\xi'_1, \xi'_1 V_{11}^{-1} V_{12})
$$

(25)

and

$$
\begin{pmatrix}
U_{11} & U_{11} V_{11}^{-1} V_{12} \\
V_{21} V_{11}^{-1} U_{11} & V_{22} - V_{21} (V_{11}^{-1} - V_{11}^{-1} U_{11} V_{11}^{-1}) V_{12}
\end{pmatrix}
$$

(26)

where $\xi'_1$ and $U_{11}$ are the mean and covariance of the $x_1, \ldots, x_k$ after truncation.

The mean vector and standard deviation for the univariate truncated $x_1$ are

$$
\xi_1 = \mu^*_1 = \sigma_{11} \frac{\phi_{\mu_1,\sigma_{11}}(a_1) - \phi_{\mu_1,\sigma_{11}}(b_1)}{\Phi_{\mu_1,\sigma_{11}}(b_1) - \Phi_{\mu_1,\sigma_{11}}(a_1)}
$$

$$
\sigma^*_1 = \sigma_{11} + \sigma_{11} \frac{a_1 \phi_{\mu_1,\sigma_{11}}(a_1) - b_1 \phi_{\mu_1,\sigma_{11}}(b_1)}{\Phi_{\mu_1,\sigma_{11}}(b_1) - \Phi_{\mu_1,\sigma_{11}}(a_1)}
$$

Letting $U_{11} = \sigma^*_1$ and inserting $\xi_1$ and $U_{11}$ into equations (25) and (26), one can verify our result. However, the crux in using the Johnson/Kotz formula is the need to first compute the moments of the truncated variables. But this is exactly the subject of our paper.

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