ON THE INVARIANCE OF ESSENTIAL NORMALITY OF HILBERT MODULES

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Abstract. In this paper, we consider the biholomorphic invariance of essential normality of Hilbert modules on pseudoconvex domains, especially bounded symmetric domains. We obtain an invariance result related to essential normality of Hilbert modules and quotient Hilbert modules on pseudoconvex domains under Taylor functional calculus. Furthermore, we prove that the essential normality of Hilbert modules and quotient Hilbert modules determined by an analytic subset in the context of weighted Bergman spaces on irreducible bounded symmetric domains in finite dimensions are invariant under biholomorphic automorphism, and the case of general Hilbert modules corresponding to the Wallach set is also considered.

1. Introduction

A complex Hilbert space $H$ is called a Hilbert module over a complex unit algebra $A$, if there exists a representation of $A$ on $H$, and the module action is induced by the representation. The main case we are interested in is that $A = \mathbb{C}[z]$ is the polynomial algebra and $H$ equipped a bounded commuting operator tube $T = (T_1, \cdots, T_m)$, whose module actions are given by

$$q \cdot h := q(T_1, \cdots, T_m)h$$

for every $h \in H$ and $q \in \mathbb{C}[z]$, which is denoted by $(H, T)$. A closed subset $M \subset H$ is called a submodule denoted by $(M, T)$, if it is invariant under the module action, i.e. $A \cdot M \subset M$, and in this case its orthogonal complement $M^\perp$ is also a submodule which called a quotient module, whose module actions are given by

$$q \cdot h := P_{M^\perp}q(T_1, \cdots, T_m)h = q(P_{M^\perp}T_1, \cdots, P_{M^\perp}T_m)h,$$

since $P_{M^\perp}T_iP_{M^\perp}T_j = P_{M^\perp}T_iT_j$ for $i, j = 1, \cdots, m$, where $P_{M^\perp} : H \to M^\perp$ is the orthogonal projection.

The study of essentially normal Hilbert modules begins with Arveson [2, 3]. A Hilbert (sub)module is said to be $p$-essentially normal if all cross-commutators

$$[T_i, T_j^*] = T_iT_j^* - T_j^*T_i$$

belong to the Schatten class operator ideal $L^p$, where $p \in [1, \infty]$ and $L^\infty$ is the compact operator ideal, we refer the reader to [6, 9, 22] and references therein for more details about Schatten class. However, in the case of quotient module, each operator in (1.2) should be replaced by its compression. The essentially normal is also called essentially...
reductive by Douglas. The essentially normality of Hilbert modules of various holomorphic function spaces equipped with coordinate multipliers on the unit ball, the unit polydisc and the bounded strongly pseudoconvex domain have continuously attracted more attention [11, 13, 27, 28, 29], and the results reveal the beautiful relationship among algebraic geometry, complex geometry, operator theory and index theory. The following two conjectures play a dominant role in the development of the essentially normal essentially theory for Hilbert modules of holomorphic function spaces. The Arveson Conjecture predicts that every graded submodule of the Drury-Arveson module on the unit ball \( \mathbb{B}^n \) is \( p \)-essentially normal for \( p > n \). Douglas [10, 11] observes that in the case of the quotient module it should be \( p > d \), where \( d \) is the complex dimension of the affine algebraic variety involved, this is now called the Geometric Arveson-Douglas Conjecture. There are abundant works on the above conjectures on more general spaces such as the Bergman space and Hardy space, we refer the reader to the survey [15] for a thorough introduction.

This paper mainly concerns the biholomorphic invariance of essential normality of Hilbert modules on pseudoconvex domains which are determined by an analytic subset, especially on bounded symmetric domains. Let us now explain our main motivation. Recently, Douglas, Wang and Xia [12, 28] consider the essential normality for quotient Bergman modules on the unit ball which are determined by analytic subsets rather than merely affine algebraic varieties, and obtain some affirmative answers of the Geometric Arveson-Douglas Conjecture in mild assumptions. On the other hand, Davidson, Ramsey, Kennedy and Shalit [7, 19] prove that the essential normality of Hilbert modules on the unit ball which are determined by an algebraic variety is invariant under the unitary change of variables. It is natural to consider invariance of essential normality under the more general biholomorphism, however the image of an algebraic variety under biholomorphism is not an algebraic set in general but an analytic subset. That is the first reason why we consider the biholomorphic invariance of essential normality of Hilbert modules determined by an analytic subset. In [13], Engliš and Eschmeier prove by the blow-up of the origin that the quotient Bergman module on the unit ball determined by homogeneous algebraic variety which is smooth away the origin is unitary to a quotient Bergman module determined by an smooth submanifold in a strongly pseudoconvex domain of a complex analytic manifold, and then prove the Geometric Arveson-Douglas Conjecture in this case. This seems to indicate that one can develop the essentially normal theory of Hilbert modules on some complex analytic manifolds. Once we attempt to consider the essentially normal theory of Hilbert modules on complex analytic manifolds (say Stein manifolds), we have to handle the problem that the essential normality of Hilbert modules is independent on the choice of holomorphic coordinates. It is the second reason why we consider the biholomorphic invariance of essential normality of Hilbert modules.

To express our results in the clearest way, we are led to introduce the following terminologies and notations. Let \( \Omega \) be a pseudoconvex domain in the complex number space \( \mathbb{C}^n \), and \( (H, T) \) be a Hilbert module of holomorphic function spaces on \( \Omega \). Let \( (M, T) \) be a submodule of \( (H, T) \), its complement \( M^\perp \) is a quotient module, whose module action is given by (1.1). We denote \( S = (P_{M^\perp}T_1, \cdots, P_{M^\perp}T_n) \) by the compression of \( T \). In these notations, \( (M, T) \) is said to be \( p \)-essentially normal if the tube
The invariance of essential normality

Given $T$ satisfying $(1.2)$, and the quotient module $(M^\perp, S)$ is said to be $p$-essentially normal if the tube $S$ satisfying $(1.2)$. Our first main result involves in the Taylor functional calculus for the commuting operator tube.

**Theorem 1.** Suppose $\phi = (\phi_1, \ldots, \phi_n)$ is an arbitrary biholomorphic automorphism on a pseudoconvex domain $\Omega$, then the following are equivalent:

1. $(M^\perp, S)$ is $p$-essentially normal.
2. $(M^\perp, \widehat{S})$ is $p$-essentially normal.
3. $(M^\perp, \phi(\widehat{S}))$ is $p$-essentially normal.

Where $\widehat{S}$ is an arbitrary permissive linear transformation for the operator tube $S$, see the definition in Section 3, and $\phi(\widehat{S})$ is the Taylor functional calculus for $\widehat{S}$.

Our second result deals with the biholomorphic invariance of essential normality of Hilbert modules on irreducible bounded symmetric domains in $\mathbb{C}^n$, in what follows we use $(\Omega, z)$ to indicate that $\Omega$ is an irreducible bounded symmetric domain assigned a given coordinate $z$. It is well known that $\Omega$ can be realized as the open unit ball with respect to the so-called spectral norm in the Jordan triple system, thus we can identify $\Omega$ with the spectral unit ball, which is also called symmetric ball in [24]. Let $(M, T)$ is a submodule of a weighted Bergman module determined by an analytic subset and equipped with $T = (M_{z_1}, \ldots, M_{z_n})$, which is usual coordinate function multipliers, its quotient Bergman module is denoted by $(M^\perp, S)$ where $S$ is the compression of $T$. We will replace $T, S$ with $T_z, S_z$ respectively, if it is necessary to emphasize the given coordinate $z$. Let $\phi: \Omega \to \Omega$ be a biholomorphic automorphism, we denote $\phi^*(M^\perp)$ by the pull back of $M^\perp$ under $\phi$, which is a quotient submodule of the pull back of the weighted Bergman module, see Section 4 below for more details.

**Theorem 2.** Let $(\Omega, z)$ be an irreducible bounded symmetric domain in $\mathbb{C}^n$ and $M$ be a weighted Bergman submodule determined by an analytic subset. Suppose that $\phi: (\Omega, w) \to (\Omega, z)$ is an arbitrary biholomorphic automorphism with $z = \phi(w)$, then $(\phi^*(M^\perp), S_w)$ is $p$-essentially normal if and only if $(M^\perp, S_z)$ is $p$-essentially normal.

As a direct consequence of Theorem 2, we obtain a result that the essential normality of a quotient weighted Bergman submodule with the compression of coordinate multipliers on an irreducible bounded symmetric domain is invariant if the coordinate multipliers is replaced by arbitrary biholomorphic automorphism multipliers, see Corollary 4.9 below. After additional efforts, the similar results on the Hilbert modules corresponding to the continuous part of Wallach set are also obtained, see Theorem 5.5 below. Although the above two theorems only deal with the quotient Hilbert module, the same is also true for Hilbert submodules, see Corollary 3.9 and Corollary 4.8 below, there is no essential difference between proofs for quotient modules and submodules. As its application, one can apply the results to establish the Geometric Arveson-Douglas conjecture for new classes of examples from the known results as [19] dose. These results not only generalize some results on the unit ball [7] and cover all irreducible bounded symmetric domains, but also establish a connection between operator theory and geometry of bounded symmetric domains. This work can be also viewed as a first step toward developing the essentially normal theory of Hilbert modules on Stein manifolds.
We now explain briefly the main ideas of the proofs. Our proof of Theorem 1 is inspired by a result due to Connes [6] on the Dunford-Riesz functional calculus of the Schatten class commutator. Actually this idea goes back at least as far as Helton and Howe [16]. The seed of the main idea in the proof of Theorem 2 is the following two observations. The first is that a close subset of a weighted Bergman space on an irreducible bounded symmetric domain is a submodule over the polynomial algebra if and only if it is a submodule over a holomorphic function algebra on the irreducible bounded symmetric domain, which in fact indicates that the Bergman module category over the polynomial algebra and the Bergman module category over the holomorphic function algebra are the same on the bounded symmetric domain, see Proposition 4.3 and Remark 4.4 below. We remark that the construction of holomorphic function algebra is inspired by the notation of germ in the sheaf theory, see the beginning of the Section 4 below. The second is that the major advantage of Bergman module over the holomorphic function algebra is that there exist abundant module actions and the essential normality problem we concerned is independent on the choice of the two above module categories. Thus we consider the essential normality problem on the Bergman module category over the holomorphic function algebra rather than the polynomial algebra.

The paper is organized as follows. In Section 2, we review some of the standard facts on pseudoconvex domains and bounded symmetric domains in term of the Jordan triple Systems. In section 3, we give the proof of the Theorem 1. Section 4 is devoted to prove the Theorem 2. Section 5 discusses the case of Hilbert modules corresponding to the continuous part of Wallach set.

2. PRELIMINARIES

In this section, we will recall some basic results that will be used in this paper. We first present facts on pseudoconvex domains and analytic subsets, and the interested readers can consult [8, 14, 20] for details.

A domain \( \Omega \subset \mathbb{C}^n \) is called pseudoconvex if there exists a continuous plurisubharmonic function \( \rho \) on \( \Omega \) such that the set \( \{ z \in \Omega : \rho(z) < r \} \) is a relatively compact in \( \Omega \) for every real number \( r \in \mathbb{R} \), and such a function \( \rho \) is called an exhaustion function. It is well known that a domain in \( \mathbb{C}^n \) is pseudoconvex if and only if it is holomorphically convex. A domain \( \Omega \subset \mathbb{C}^n \) is called holomorphically convex if for every compact subset \( K \subset \Omega \) its holomorphically convex hull

\[
\hat{K}_\Omega := \{ z \in \Omega : |f(z)| \leq \sup_K |f| \}
\]

for every holomorphic function \( f \) is again compact. A basic fact is that every convex domain \( \Omega \) in \( \mathbb{C}^n \) is pseudoconvex. For a domain with \( C^2 \)-smooth boundary in \( \mathbb{C}^n \), it is pseudoconvex as above if and only it is Levi pseudoconvex. A domain with \( C^2 \)-smooth boundary is called Levi (strong) pseudoconvex if its Levi form of defining function is (strictly) positive on its boundary. Let \( \Omega \subset \mathbb{C}^n \) be a domain with \( C^2 \)-smooth boundary and a defining function \( \rho \), namely \( \rho : \mathbb{C}^n \rightarrow \mathbb{R} \) is a twice continuously differentiable function such that \( \text{grad} \rho \neq 0 \) for \( z \in \partial \Omega = \{ \rho = 0 \} \) and \( \Omega = \{ \rho < 0 \} \) (such a function always exists). The Levi form \( L_{\partial \Omega, z} (w) \) on holomorphic tangent space
\[ T_z^{1,0}(\partial \Omega) = \{ w = (w^1, \cdots, w^n) \in \mathbb{C}^n : w^i \frac{\partial \rho}{\partial z_i} = 0 \} \text{ at every point } z \in \partial \Omega \]

is defined by

\[
L_{\partial \Omega, z}(w) = \frac{1}{|\text{grad } \rho|} \frac{\partial^2 \rho}{\partial z_i \partial \bar{z}_j} w^i \bar{w}^j,
\]

where the Einstein summation convention is adopted. A classical example of strongly pseudoconvex domain is the unit ball, and every bounded strongly pseudoconvex domain with noncompact automorphism group is biholomorphic to the unit ball \([30]\). The notation of pseudoconvex and holomorphically convex apply to the complex analytic manifold, a holomorphically convex manifold must be (weak) pseudoconvex, but the converse is not true in general.

Let \( \Omega \) be a domain in \( \mathbb{C}^n \), in general \( \Omega \) can be a complex analytic manifold. A closed subset \( A \subset \Omega \) is said to be an analytic subset, if for every \( z_0 \in A \) there exists a neighborhood \( U \) of \( z_0 \) and holomorphic functions \( f_1, \cdots, f_m \in \mathcal{O}(U) \) such that

\[
A \cap U = \{ z \in U : f_1(z) = \cdots = f_m(z) \},
\]

where \( \mathcal{O}(U) \) is the collections of holomorphic functions on \( U \). Then \( f_1, \cdots, f_m \) are said to be local equations of \( A \) in \( U \). Note that the definition of the analytic subset is local. A nontrivial fact is that every analytic subset can be globally defined by finitely many holomorphic functions on Stein manifolds which in fact means that every analytic subset in a Stein manifold has a global equations \([14]\) Theorem 5.14], especially on pseudoconvex domains in \( \mathbb{C}^n \). Roughly speaking, a complex analytic manifold is called a Stein manifold, if it is holomorphically convex and has abundant holomorphic functions \([8, 14]\). It is easy to see that, if \( f : \Omega_1 \to \Omega_2 \) is a holomorphic map between two domains and \( A \subset \Omega_2 \) is an analytic subset, so is its preimage \( f^{-1}(A) \subset \Omega_1 \).

We now briefly recall some known facts on bounded symmetric domains and their Jordan theoretic description without proofs, we refer the reader to \([9, 21, 24, 25]\) and references therein for general background. A bounded domain \( \Omega \subset \mathbb{C}^n \) is called symmetric, if for every \( z \in \Omega \) there exists a \( s_z \in \text{Aut}(\Omega) \) such that \( s_z \circ s_z = \text{Id} \) and \( z \) is an isolate fixed point of \( s_z \), where \( \text{Aut}(\Omega) \) denoted by the biholomorphic automorphisms group of \( \Omega \). A bounded symmetric domain is said to be irreducible if it \( \Omega \) can not be written the Cartesian product of bounded symmetric domains of lower dimensions (up to biholomorphism). Let \( G \) be the identity component of the group of all biholomorphic automorphisms of \( \Omega \). Then \( \Omega = G/K \) is a realization of the non-compact Hermitian symmetric space \( G/K \), where \( K = \{ g \in G : g(0) = 0 \} \) is the isotropy subgroup of \( 0 \in \Omega \). It is well known \([21, 24]\) that \( Z \) can be equipped with the structure of a hermitian Jordan triple and \( \Omega \) can then be realized as the spectral unit ball in \( Z \), thus we identify \( \Omega \) with the unit ball in the spectral norm, the spectral unit ball is also called symmetric ball \([24]\). Note that any two norms of the finite dimensional complex vector space \( \mathbb{C}^n \) are equivalent, thus the topologies induced by the spectral norm and the usual Euclidian norm coincide on \( \mathbb{C}^n \). Since \( \Omega \) is circular one can show that \( K = \{ g \in GL(\mathbb{C}^n) : g(\Omega) = \Omega \} \). We denote the Jordan triple product by

\[
Z \times Z \times Z \to Z, \quad (u, v, w) \mapsto \{uw^*w\}
\]
which is complex linear in $u, w$ and complex conjugate linear in $v$. For each pair $(u, v) \in Z \times Z$ the linear endomorphism

$$B(u, v)w := w - 2\{uw^*w\} + \{vw^*v\}^*u$$

is called the Bergman endomorphism associated with $(u, v)$. One can show that $B(z, \xi) \in GL(Z)$ is invertible whenever $z, \xi \in \Omega$. A pair $(z, \xi) \in Z \times Z$ is said to be quasi-invertible if the Bergman endomorphism $B(z, \xi)$ is invertible, in this case

$$z^\xi := B(z, \xi)^{-1}(z - \{z^*z\})$$

is called the quasi-inverse of the pair $(z, \xi)$. Now fix $z_0 \in \Omega$. One can show [21] that $g_{z_0}(w) = z_0 + B(z_0, z_0)^{1/2}w - z_0$ defines a biholomorphic automorphism $g_{z_0} \in Aut(\Omega)$ called the Moebius transformation associated with $z_0$ which satisfies

$$g_{z_0}(0) = z_0, g_{z_0}(-z_0) = 0,$$

and its inverse is $g_{-z_0}$.

Due to Cartan and Harish-Chandra [21], there only exist six type irreducible bounded symmetric domains up to biholomorphism, and the others are the Cartesian product of irreducible bounded symmetric domains up to biholomorphism. We give two classical examples of bounded symmetric domains in the Jordan theoretic description.

**Example 2.1.** (1) Type $I_{r \times n}$. Let $Z = \mathbb{C}^{r \times n}$ be the complex $(r \times n)$-matrix space with $r \leq n$. Then

$$I_{r \times n} = \{u \in Z : \text{Id}_r - uu^* > 0\}$$

is an irreducible bounded symmetric domain, where $\text{Id}_r$ is the $(r \times r)$-unit matrix and $u^*$ is the conjugate matrix of $u$. The associated Jordan triple product on $Z$ is the generalized anti-commutator product

$$\{uv^*w\} := \frac{1}{2}(uv^*w + wv^*u)$$

for $u, v, w \in Z$, where the multiplication in the right side is the usual matrix multiplication. In this case, the Bergman endomorphism associated with $(u, v)$ is given by

$$B(u, v)w = (\text{Id}_r - uv^*)w(\text{Id}_n - vu^*)$$

and the quasi-inverse of the pair $(z, \xi)$ is

$$z^\xi = (\text{Id}_r - z\xi^*)^{-1}z.$$  

It implies that the corresponding generic polynomial $\Delta(u, v)$ is

$$\Delta(u, v) = \text{Det}(\text{Id}_r - uv^*).$$

The case of $r = 1$ is the unit ball $I_{1 \times n} = \mathbb{B}^n$ in the usual Euclidian norm.

(2) Polydisc $\mathbb{D}^n$. The unit polydisc $\mathbb{D}^n$ is the $n$-Cartesian product of the unit disk $\mathbb{D} = \mathbb{B}^1$ in $Z = \mathbb{C}^{1 \times n}$, i.e.

$$\mathbb{D}^n = \{u = (u_1, \cdots, u_n) \in Z : 1 - u_i\bar{u}_i > 0, i = 1, \cdots, n\}.$$  

The associated Jordan triple product on $Z$ is the generalized anti-commutator product

$$\{uv^*w\} := (u_1\bar{v}_1w_1, \cdots, u_n\bar{v}_nw_n)$$
for \( u, v, w \in \mathbb{Z} \). The Bergman endomorphism associated with \((u, v)\) is given by
\[
B(u, v)w = ((1 - u_1 \overline{v}_1)^2w_1, \cdots, (1 - u_n \overline{v}_n)^2w_n)
\]
and the quasi-inverse of the pair \((z, \xi)\) is
\[
z^\xi = \left( \frac{z_1}{1 - z_1 \xi_1}, \cdots, \frac{z_n}{1 - z_n \xi_n} \right).
\]
Then the corresponding generic polynomial \( \Delta(u, v) \) is
\[
\Delta(u, v) = \prod_{i=1}^{n} (1 - u_i \overline{v}_i).
\]

In the following of this paper, the bounded symmetric domain \( \Omega \subset \mathbb{Z} = \mathbb{C}^n \) we considered is always assumed to be irreducible. It is well known that there exists a unique generic polynomial \( \Delta(z, w) \) in \( z, \bar{w} \) and an analytic numerical invariant \( N \) satisfying the Bergman kernel of \( \Omega \) is given by
\[
K(z, w) = \Delta(z, w)^{-N}, \tag{2.1}
\]
where the invariant \( N \) is also called the genus of domain \( \Omega \), the generic polynomial \( \Delta(z, w) \) is also called Jordan triple determinant or a denominator of the quasi-inverse such that \( \Delta(0, 0) = 1 \). Denote the \( K \)-invariant normalized measure \( dv_\gamma \) on \( \Omega \) by
\[
dv_\gamma(w) = c_\gamma \Delta(w, w)^\gamma dv(w)
\]
for \( \gamma > -1 \), where \( c_\gamma \) is the normalized constant and \( K \) is the isotropic subgroup. Denote \( A^2(dv_\gamma) \) by the weighted Bergman space consisted of square integrable holomorphic functions with respect to the measure \( dv_\gamma \) on \( \Omega \), which is also denoted by \( A^2(\Omega, dv_\gamma) \). The Bergman kernel \( K_\gamma(z, w) \) of the spectral unit ball is given by
\[
K_\gamma(z, w) = \Delta(z, w)^{-(N+\gamma)},
\]
which is degenerated to the formula (2.1) when \( \gamma = 0 \). The weighted Bergman space \( A^2(dv_\gamma) \) is a reproducing kernel Hilbert function space, since
\[
f(z) = \langle f, K_{\gamma,z} \rangle_\gamma = \int_\Omega f(w) \overline{K_{\gamma,z}(w)} dv_\gamma(w), \quad z \in \Omega \tag{2.2}
\]
for every \( f \in A^2(dv_\gamma) \), where
\[
K_{\gamma,w}(z) = K_\gamma(z, w) = \Delta(z, w)^{-(N+\gamma)}, \quad z, w \in \Omega,
\]
is also called the reproducing kernel of \( A^2(dv_\gamma) \).

By [24] the natural action of \( K \) on \( \mathcal{P}(Z) = \mathbb{C}[z] \) induces the Peter-Schmid-Weyl decomposition
\[
\mathcal{P}(Z) = \sum_{m \geq 0} \mathcal{P}_m(Z),
\]
where \( m = (m_1, \cdots, m_r) \geq 0 \) runs over all integer partitions, namely
\[
m_1 \geq \cdots \geq m_r \geq 0.
\]
The decomposition is irreducible under the action of $K$ and is orthogonal under the Fischer-Fock (or Segal-Bargmann) inner product $\langle \cdot, \cdot \rangle_F$, see [24, Section 2.7]. For a $r$-tuple $s = (s_1, \cdots, s_r) \in \mathbb{C}^r$, the Gindikin Gamma function [9, 24] is given by

$$\Gamma_\Omega(s) = (2\pi)^{\frac{ar(r-1)}{4}} \prod_{j=1}^{r} \Gamma(s_j - \frac{a}{2}(j-1)),$$

of the usual Gamma function $\Gamma$ whenever the right side is well defined, where $r$ is the rank of $\Omega$ and $a, b$ are two numerical invariants associated the joint Peirce decomposition for a chosen a frame $e_1, \cdots, e_r$ of minimal tripotents such that the dimension count

$$n = r + \frac{a}{2}r(r-1) + br$$

holds and the genus $N$ is given by

$$N := 2 + a(r-1) + b.$$

The multi-variable Pochhammer symbol is

$$\binom{\lambda}{s} := \frac{\Gamma_\Omega(\lambda + s)}{\Gamma_\Omega(s)},$$

where $\lambda + s := (\lambda + s_1, \cdots, \lambda + s_r)$. It can be verified that

$$\binom{\lambda}{s} = \prod_{j=1}^{r} (\lambda - \frac{a}{2}(j-1))s_j$$

of the usual Pochhammer symbols $(\mu)_m = \prod_{j=1}^{m} (\mu + j - 1)$. It is known from [1, 24] that

$$\Delta(z, w)^{-\lambda} = \sum_{m \geq 0} \binom{\lambda}{m}K_m(z, w)$$

converges compactly and absolutely on $\Omega \times \Omega$, where $K_m$ is the reproducing kernel of $\mathcal{P}_m(Z)$ in the Fischer-Fock inner product, for $\lambda \in \mathbb{C}$. The Wallach set $W_\Omega$ with respect to $\Omega$ is defined to be the set consists of all $\lambda \in \mathbb{C}$ satisfying

$$\binom{\lambda}{m} \geq 0$$

for all integer partitions $m \geq 0$, which is exactly the set of value $\lambda$ such that $\Delta(z, w)^{-\lambda}$ is a positive kernel. The Wallach set $W_\Omega$ admits the following decomposition

$$W_\Omega = W_{\Omega,d} \cup W_{\Omega,c}$$

where $W_{\Omega,d} = \{\lambda = (j-1)\frac{a}{2}, j = 1, \cdots, r\}$ and $W_{\Omega,c} = \{\lambda > (r-1)\frac{a}{2}\}$.

The set $W_{\Omega,c}$ is called the continuous part, in this case the function $\Delta(z, w)^{-\lambda}$ is the reproducing kernel of the Hilbert holomorphic function space

$$H^2_\lambda(\Omega) = \sum_{m \geq 0} \mathcal{P}_m(Z),$$

whose inner product is introduced by

$$(p, q)_\lambda = \frac{1}{\binom{\lambda}{m}} \langle p, q \rangle_F$$

(2.3)
for all \( p, q \in \mathcal{P}_m(Z), m \geq 0 \). The weighted Bergman space \( A^2(dv_\gamma) \) is coincided with \( H^2_{N+\gamma}(\Omega), \gamma > -1 \) and the classical Hardy space defined in [24, Definition 2.8.4] is coincided with \( H^2_\Omega \). The set \( W_{\Omega,d} \) is called the discrete part, in the case of \( \lambda = (j-1)^2 \), the function \( \Delta(z,w)^{-\lambda} \) is the reproducing kernel of the Hilbert holomorphic function space

\[
\mathcal{H}^2_\lambda(\Omega) = \sum_{m \geq 0, m_j = 0} \mathcal{P}_m(Z),
\]

whose inner product is introduced by

\[
(p, q)_\lambda = \frac{1}{\langle \lambda \rangle_m} \langle p, q \rangle_F
\]

for all \( p, q \in \mathcal{P}_m(Z) \) where \( m \geq 0, m_j = 0 \).

### 3. The Proof of Theorem 1

In this section, we will prove Theorem 1, which gives a criterion of the essential normality of Hilbert modules by Taylor functional calculus on pseudoconvex domains. We first introduce some notations involved.

Let \( \Omega \subset \mathbb{C}^n \) be a pseudoconvex domain, and \((H, T)\) be a Hilbert module over the polynomial algebra \( \mathbb{C}[z] \) whose module actions are given by

\[
q \cdot h := q(T_1, \cdots, T_n)h
\]

for every \( h \in H \) and \( q \in \mathbb{C}[z] \), where \( H \) is a Hilbert space of holomorphic functions on \( \Omega \). Let \((M, T)\) be a submodule of \((H, T)\), its orthogonal complement \( M^\perp \) is called a quotient module, whose module action is given by

\[
q \cdot h := q(S_1, \cdots, S_n)h,
\]

where \( S_i = P_{M^\perp} T_i, i = 1 \cdots, n \) and \( P_{M^\perp} : H \to M^\perp \) is the orthogonal projection. For a holomorphic polynomial \( q \in \mathbb{C}[z] \), denote \( S_q \) by

\[
S_q := q(S_1, \cdots, S_n) = P_{M^\perp} q(T_1, \cdots, T_n).
\]

The following two operator identities are elementary, which will be immediately verified by direct calculations.

**Lemma 3.1.** Let \( U, V, W \) are bounded operators on a Banach space, then the following hold.

1. \([UV, W] = U[V, W] + [U, W]V\).
2. If \( U \) is invertible, then \( [U^{-m}, V] = U^{-m}[V, U^m]U^{-m} = U^{-(m+1)}[U, V] - U^{-(m+1)}[U^{m+1}, V]U^{-m} \), for every integer \( m \geq 0 \).

**Proposition 3.2.** The following are equivalent.

1. \([S_i, S_j^*] \in \mathcal{L}^p \) for all \( i, j = 1, \cdots, n \).
2. \([S_h, S_q^*] \in \mathcal{L}^p \) for all \( h, q \in \mathbb{C}[z] \).

**Proof.** It comes from (1) of Lemma 3.1 and mathematical induction. \( \square \)
For a commuting bounded operator tube $T = (T_1, \ldots, T_n)$ on a Hilbert space $H$, denote $Sp(T)$ by its Taylor spectrum, which is a nonempty compact subset of the closed polydisc $\Delta_T = \{z \in \mathbb{C}^n : |z_i| \leq r(T_i), i = 1, \ldots, n\}$, where $r(T_i) = \lim_m \|T_i^m\|^\frac{1}{m}$ is the spectral radius of $T_i$. Let $B(H)$ be the set bounded linear operators and $O(Sp(T))$ be the set of holomorphic functions on a neighborhood of the compact set $Sp(T)$, then there exists a unique continuous algebraic homomorphism

$$O(Sp(T)) \to (T)^\prime \subset B(H),$$

which satisfies

$$1(T) = 1, z_i(T) = T_i, i = 1, \ldots, n,$$

and the following spectrum theorem, where $(T)^\prime$ is the bicommutant of the algebra generated by the tube $T$. It is called the Taylor functional calculus \[23\]. Suppose that $f = (f_1, \ldots, f_n) : \Omega \to \mathbb{C}^n$ is a holomorphic map and denote $f(T) = (f_1(T), \ldots, f_n(T))$ which is also a commuting bounded operator tube, then the spectrum theorem holds:

$$f(Sp(T)) = Sp(f(T)).$$

If $f$ is a holomorphic function on a Levi pseudoconvex domain that contains $Sp(T)$, then the Martinelli type formula of Taylor functional calculus to $f$ holds. We formulate it in the following lemma.

**Lemma 3.3.** \[17, 26\] (Martinelli type formula) Let $\Omega \subset \mathbb{C}^n$ be a bounded Levi pseudoconvex domain with defining function $\eta$. Assume that the Taylor spectrum $Sp(T) \subset \Omega$. Then for every $f \in O(\Omega)$, the following integral formula holds,

$$f(T) = \frac{1}{(2\pi)^n} \int_\Omega f(z)M(z, T)^{-n}\omega(z). \quad (3.2)$$

Where $\omega$ is the $(n - 1, n)$-type Henkin differential form and

$$M(z, T) = \sum_{i=1}^n (z_i - T_i) \frac{\partial \eta}{\partial z_i}(z)$$

is invertible in $B(H)$ for $z \in \mathbb{C}^n \setminus Sp(T)$.

**Remark 3.4.** If the compact set $\Delta_T \subset \Omega$, then the Dunford-Riesz functional calculus for a single operator is enough for our purposes. We now give a brief explanation. Since $Sp(T) \subset \Delta_T \subset \Omega$, there exist polydiscs $\Delta_r = \{z \in \mathbb{C}^n : |z_i| < r_i, i = 1, \ldots, n\}$ and $\Delta_{r'} = \{z \in \mathbb{C}^n : |z_i| < r'_i, i = 1, \ldots, n\}$ such that $\Delta_T \subset \Delta_r \subset \Delta_{r'} \subset \bar{\Delta}_{r'} \subset \Omega$.

Note that a holomorphic function on a circular domain containing the origin has a homogeneous holomorphic polynomial expansion which converges compactly, combining with the continuity and uniqueness of Taylor functional calculus, it implies that

$$f(T) = \frac{1}{(2\pi)^n} \int_{|\xi_1| = r_1} \cdots \int_{|\xi_n| = r_n} \frac{d\xi_1}{\xi_1 - T_1} \cdots \frac{d\xi_n}{\xi_n - T_n} f(\xi)$$

for $f \in O(\Omega)$, where the right side repeatedly uses the Cauchy integral formula.

The following composition law for Taylor functional calculus holds.
**Lemma 3.5.** Suppose that $O(Sp(T)) \subset \Omega_1$ and $g : \Omega_1 \rightarrow \Omega_2$ is a holomorphic map between two pseudoconvex domains. If $f : \Omega_2 \rightarrow \Omega_3$ is a holomorphic map, then

$$(f \circ g)(T) = f(g(T)).$$

*Proof.* It suffices to prove the scalar case that $f$ is a holomorphic function. From the spectrum theorem and the following Lemma 3.7, it follows that there exist relatively compact Levi pseudoconvex domains $D_1 = \{\eta_1 < 0\}, D_2 = \{\eta_2 < 0\}$ such that

$$Sp(T) \subset D_1 \subset \Omega_1$$

and

$$Sp(g(T)) = g(Sp(T)) \subset D_2 \subset \Omega_2,$$

where $\eta_1, \eta_2$ are defining functions. Then it infers from Martinelli type formulas of Taylor functional calculus that

$$f(g(T)) = \frac{1}{(2\pi \sqrt{-1})^d} \int_{\partial D_2} f(z) M_{\eta_2}(z, g(T))^{-d} w(z)$$

$$= \frac{1}{(2\pi \sqrt{-1})^d} \int_{\partial D_2} f(z) \left( \sum_{i=1}^{d} (z_i - g_i(T)) \frac{\partial \eta_2}{\partial z_i} \right)^{-d} w(z)$$

$$= \frac{1}{(2\pi \sqrt{-1})^d} \int_{\partial D_2} f(z) \left[ \int_{\partial D_1} \sum_{i=1}^{d} (z_i - g_i(u)) \frac{\partial \eta_2}{\partial z_i} M_{\eta_1}(u, T)^{-n} w(u) \right]^{-d} w(z)$$

$$= \frac{1}{(2\pi \sqrt{-1})^{d+n}} \int_{\partial D_2} f(z) \int_{\partial D_1} \sum_{i=1}^{d} (z_i - g_i(u)) \frac{\partial \eta_2}{\partial z_i} M_{\eta_1}(u, T)^{-n} w(u) w(z)$$

$$= \frac{1}{(2\pi \sqrt{-1})^{d+n}} \int_{\partial D_2} f(z) \left( \sum_{i=1}^{d} (z_i - g_i(u)) \frac{\partial \eta_2}{\partial z_i} \right) w(z) M_{\eta_1}(u, T)^{-n} w(u)$$

$$= \frac{1}{(2\pi \sqrt{-1})^n} \int_{\partial D_2} (f \circ g)(u) M_{\eta_1}(u, T)^{-n} w(u).$$

where $d$ is the complex dimension of the domain $\Omega_2$. The fourth equality comes from the fact that if the Levi pseudoconvex domains $D \supset Sp(T)$ and $f \in O(D)$ is invertible then $f(T)$ is invertible and its inverse is $f(T)^{-1} = \frac{1}{(2\pi \sqrt{-1})^n} \int_{\partial D} f(z)^{-1} M(z, T)^{-n} w(z)$. That finishes the proof.

**Remark 3.6.** The composition law for Taylor functional calculus should holds in more general case, however the current form is enough for our application.

**Lemma 3.7.** Suppose that $K$ is a compact subset in a pseudoconvex domain $\Omega$, then there exists a relatively compact Levi pseudoconvex domain $D$ such that $K \subset D \subset \Omega$.

*Proof.* Since $\Omega$ is pseudoconvex, without loss of generality, we can assume that $\rho$ is its exhaustion function which is continuous plurisubharmonic. The compactness of the set $K \subset \Omega$ implies that there exists a real number $c$ such that

$$K \subset E = \{\rho < c\},$$
since a continuous function can achieve its maximum on a compact set. By the standard smoothing process via convolution for a plurisubharmonic function, there exists a sequence of smooth plurisubharmonic functions \( \{ \rho_m \} \) such that \( \rho_m \) decreasingly converge to \( \rho \) on the domain \( \Omega \). By the decreasing monotone of the sequence \( \{ \rho_m \} \), it follows that for every \( x \in K \subset E \), there exists a plurisubharmonic function \( \rho_m \) satisfying \( \rho(x) \leq \rho_m(x) < c \), which means

\[
x \in \{ \rho_m < c \} \subset \{ \rho < c \} = E.
\]

Thus \( \cup_m \{ \rho_m - c < 0 \} \) is a covering of the set \( K \). Combining with the compactness of \( K \) and the decreasing monotone of the sequence \( \{ \rho_m \} \), it implies that there exists a plurisubharmonic function \( \rho_m_0 \) such that

\[
K \subset \{ \rho_{m_0} - c < 0 \} \subset E \subset \bar{E} \subset \Omega.
\]

Denote \( D = \{ \rho_{m_0} - c < 0 \} \), the smoothness of \( \rho_{m_0} \) implies that \( D \) is a pseudoconvex domain with smooth boundary, and \( \rho_{m_0} - c \) is an exhaustion function. Thus \( D \) is Levi pseudoconvex, by the equivalence of pseudoconvex and Levi pseudoconvex when the boundary is smooth. This completes the proof. \( \Box \)

**Remark 3.8.** We point out that the existence of Levi pseudoconvex domain in Lemma \( \ref{3.7} \) can be also derived directly from the fact that every pseudoconvex domain can be exhausted by Levi pseudoconvex domains, see \[20\]. Nevertheless, we give a direct proof here.

For every \( a > 0 \) and \( b \in \mathbb{C}^n \), we see that they determine a linear transformation of the operator tube \( T = (T_1, \ldots, T_n) \) in the form of \( aT + b \), such a linear transformation is called permissive for the pair \( (\Omega, T) \), if the compact set

\[
Sp(aT + b) = a \cdot Sp(T) + b \subset \Omega.
\]

For a given pair \( (\Omega, T) \), there always exist infinitely many permissive linear transformations. In the sequel, we will use the notation \( \hat{T} = (\hat{T}_1, \ldots, \hat{T}_n) \) to stand for an arbitrary permissive linear transformation for a given pair \( (\Omega, T) \).

**Theorem 1.** Suppose \( \phi = (\phi_1, \ldots, \phi_n) \) is an arbitrary biholomorphic automorphism on a pseudoconvex domain \( \Omega \), then the following are equivalent.

1. \( (M^\perp, S) \) is \( p \)-essentially normal.
2. \( (M^\perp, \hat{S}) \) is \( p \)-essentially normal.
3. \( (M^\perp, \phi(\hat{S})) \) is \( p \)-essentially normal.

**Proof.** Suppose \( \hat{S} = aS + b \), where \( a > 0 \) and \( b = (b_1, \ldots, b_n) \in \mathbb{C}^n \). Then

\[
[\hat{S}_i, \hat{S}_j^*] = [aS_i + b_i, aS_j^* + \bar{b}_j] = a^2[S_i, S_j^*],
\]

for all \( i, j = 1, \ldots, n \). It implies the equivalence between (1) and (2).

It remains to prove the equivalence of (2) and (3). We first prove that (2) implies (3). Since \( \Omega \) is pseudoconvex, without loss of generality, we assume that \( \rho \) is its exhaustion function which is continuous plurisubharmonic. The compactness of the set \( Sp(aT + b) = a \cdot Sp(T) + b \subset \Omega \) and Lemma \( \ref{3.7} \) show that there exists a relatively compact Levi pseudoconvex domain \( D \) with defining function \( \eta \) such that

\[
Sp(aT + b) \subset D = \{ \eta < 0 \} \subset \Omega.
\]
Applying Martinelli type formulas of Taylor functional calculus to holomorphic functions \( \phi_i \in \mathcal{O}(D) \), it implies that
\[
\phi_i(\hat{S}) = \frac{1}{(2\pi)^n} \int_{\partial D} \phi_i(z)M(z, \hat{S})^{-n} \omega(z),
\]
where \( \omega \) is the \((n - 1, n)\)-type Henkin differential form and
\[
M(z, \hat{S}) = \sum_{i=1}^{n} (z_i - \hat{S}_i) \frac{\partial \eta}{\partial z_i}(z)
\]
is invertible in \( B(M) \) for \( z \in \mathbb{C}^n \setminus Sp(\hat{S}) \). Note that the operator equality (2) of Lemma 3.1 it follows that
\[
[M(z, \hat{S})^{-n}, \hat{S}_j^*] = M(z, \hat{S})^{-n} [\hat{S}_j^*, M(z, \hat{S})^n] M(z, \hat{S})^{-n}
\]
\[
= M(z, \hat{S})^{-n} [\hat{S}_j^*, \left( \sum_{i=1}^{n} (z_i - \hat{S}_i) \frac{\partial \rho}{\partial z_i}(z) \right)^n] M(z, \hat{S})^{-n}
\]
Then \( z \mapsto \phi(z)[M(z, \hat{S})^{-n}, \hat{S}_j^*] \in \mathcal{L}^p \) is continuous on \( \partial D \) by Proposition 3.2. Thus \( [\phi_i(\hat{S}), \hat{S}_j^*] \in \mathcal{L}^p \) by the formula (3.3), for all \( i, j = 1, \cdots, n \). Observe that an operator belongs to Schatten class \( \mathcal{L}^p \) if and only if its adjoint belongs to Schatten class \( \mathcal{L}^p \). It follows that
\[
[\hat{S}_j, \phi_i(\hat{S})^*] = [\phi_i(\hat{S}), \hat{S}_j^*]^* \in \mathcal{L}^p.
\]
By using of Martinelli type formulas of Taylor functional calculus again, we conclude that
\[
[\phi_i(\hat{S}), \phi_j(\hat{S})]^* = [\phi_j(\hat{S}), \phi_i(\hat{S})^*]^* \in \mathcal{L}^p,
\]
for all \( i, j = 1, \cdots, n \). Thus (2) implies (3).

Similarly, the same argument for \( \phi^{-1} \) and the commuting operator tube \( \phi(\hat{S}) = (\phi_1(\hat{S}), \cdots, \phi_n(\hat{S})) \) implies that \( [\phi^{-1}_i(\phi(\hat{S})), \phi^{-1}_j(\phi(\hat{S}))]^*] \in \mathcal{L}^p \) if \( [\phi_i(\hat{S}), \phi_j(\hat{S})^*] \in \mathcal{L}^p \). Then Lemma 3.5 shows that
\[
[\hat{S}_i, \hat{S}_j^*] = [\phi^{-1}_i(\phi(\hat{S})), \phi^{-1}_j(\phi(\hat{S}))]^*] \in \mathcal{L}^p,
\]
for all \( i, j = 1, \cdots, n \). This proves that (3) implies (2), and the proof is complete. \( \square \)

Similarly, the same is also true for the submodule \((M, T)\).

**Corollary 3.9.** Suppose \( \phi = (\phi_1, \ldots, \phi_n) \) is an arbitrary biholomorphic automorphism on a pseudoconvex domain \( \Omega \), then the following are equivalent.

1. \((M, T)\) is p-essentially normal.
2. \((M, \hat{T})\) is p-essentially normal.
3. \((M, \phi(\hat{T}))\) is p-essentially normal.

### 4. The proof of Theorem 2

This section is mainly devoted to prove Theorem 2, which shows the essential normality of Hilbert modules and quotient Hilbert modules determined by an analytic subset in the setting of Bergman spaces are invariant under biholomorphic automorphism on irreducible bounded symmetric domains.
Let $\Omega$ be an irreducible bounded symmetric domain in $\mathbb{C}^n$ and $\mathcal{O}(\check{\Omega})$ be the set of holomorphic functions on a neighborhood of $\check{\Omega}$. We now briefly explain why $\mathcal{O}(\check{\Omega})$ is called an algebra. Two functions $f_1, f_2 \in \mathcal{O}(\check{\Omega})$ are said to be equivalent if there exists an open neighborhood $D \supset \Omega$ such that the restrictions on $D$ are equal, i.e. $f_1|_D = f_2|_D$, which denoted by $f_1 \sim f_2$ and whose equivalence classes denoted by $[f_1]$. Suppose $f_1$ is holomorphic on $D_1 \supset \check{\Omega}$ and $f_2$ is holomorphic on $D_2 \supset \check{\Omega}$ and $D_1, D_2$ are open. We define three binary operations for equivalence classes as follows:

$$
[f_1] + [f_2] := [f_1|_{D_1 \cap D_2} + f_2|_{D_1 \cap D_2}],
$$

$$
[f_1][f_2] := [f_1|_{D_1 \cap D_2}f_2|_{D_1 \cap D_2}],
$$

$$
c[f_1] := [cf_1], \quad \forall c \in \mathbb{C}.
$$

(4.1)

It clearly that the definitions in (4.1) are well defined. Thus $\mathcal{O}(\check{\Omega})/\sim$ becomes a complex unit algebra under the operations in (4.1). Note that $[1]$ is its unit where 1 is the constant function valued 1 in a neighborhood of $\check{\Omega}$. We see that $[f]$ is invertible in $\mathcal{O}(\check{\Omega})/\sim$ if and only if $[f]$ has a representative which has no zeros in $\check{\Omega}$. By abuse of notation, we continue to write $f, \mathcal{O}(\check{\Omega})$ instead of $[f], \mathcal{O}(\check{\Omega})/\sim$ respectively. In fact, $\mathcal{O}(\check{\Omega})$ is the inductive limit of the sets of sections of the holomorphic function sheaf on $\mathbb{C}^n$ over the sets contain $\check{\Omega}$. Recall that $A^2(dv_\gamma)$ is the weighted Bergman space with respect to the measure $dv_\gamma, \gamma > -1$ on $\Omega$. We can equip $A^2(dv_\gamma)$ with a natural $\mathcal{O}(\check{\Omega})$-module structure by the multiplication of functions, namely

$$
\begin{align*}
\tilde{f} \cdot h &:= \tilde{f}|_{\check{\Omega}} h,
\end{align*}
$$

for every $f \in \mathcal{O}(\check{\Omega})$ and $h \in A^2(dv_\gamma)$, which is well defined. Let $M$ be an $\mathcal{O}(\check{\Omega})$-submodule and $M^\perp$ be its orthogonal complement, we denote the operator $S_f : M^\perp \to M^\perp$ with symbol $f \in \mathcal{A}(\check{\Omega})$ by

$$
S_f h := P_{M^\perp}(fh),
$$

for every $h \in M^\perp$, where $\mathcal{A}(\check{\Omega})$ is the set of bounded holomorphic functions on $\check{\Omega}$ and $P_{M^\perp} : A^2(dv_\gamma) \to P_{M^\perp}$ is the projection. This coincides with the definition (3.1) when $T = (M_{z_1}, \cdots, M_{z_n})$ is the coordinate multiplier and $f$ is a polynomial, and in this case $S_i = S_{z_i}, i = 1, \cdots, n$. The following lemma implies that $M^\perp$ is an $\mathcal{O}(\check{\Omega})$-submodule, whose module actions are given by

$$
\begin{align*}
\tilde{f} \cdot h &:= S_f h,
\end{align*}
$$

for every $h \in M^\perp$ and $f \in \mathcal{O}(\check{\Omega})$.

**Lemma 4.1.** Suppose $f, g \in \mathcal{O}(\check{\Omega})$, then the following operator identity holds on $M^\perp$,

$$
S_{fg} = S_f S_g.
$$

**Proof.** Suppose $h \in M^\perp$, by definition we obtain

$$
S_f S_g h = P_{M^\perp}(f P_{M^\perp}(gh))
= P_{M^\perp}(f(Id - P_M)(gh))
= P_{M^\perp}(gh) - P_{M^\perp}(f(P_M(gh)))
= S_fgh,
$$

for all $h \in M^\perp$. The proof is complete.
where \( P_M : A^2(dv_\gamma) \to P_M \) is the projection. The third identity holds because that \( M \) is an \( O(\bar{\Omega}) \)-module.

**Corollary 4.2.** If \( f \in O(\bar{\Omega}) \) and has no zeros in \( \bar{\Omega} \), then \( S_f \) is invertible on \( M^\perp \) and its inverse is

\[
S_f^{-1} = S_{f^{-1}}.
\]

**Proof.** Without loss of generality, we can suppose that \( f \) is holomorphic on an open set \( D \supset \bar{\Omega} \). Note that the set \( D_f = \{ z \in D : f(z) \neq 0 \} \) is open, and the assumption means that \( \bar{\Omega} \subset D_f \). Thus \( f \) is invertible on \( D_f \) and its inverse \( f^{-1} \in O(\bar{\Omega}) \). Combining with Lemma 4.1, it follows that the desired operator identity. □

It is clear that a close set of \( A^2(dv_\gamma) \) is a \( C[z] \)-module if it is an \( O(\bar{\Omega}) \)-module, the Corollary 4.2 indicates that the \( O(\bar{\Omega}) \)-module action is more abundant than \( C[z] \)-module action, since a nonconstant polynomial is not invertible in the polynomial algebra \( C[z] \). Even so, the following lemma shows that the two kinds of Bergman modules are in fact equivalent on bounded symmetric domains.

**Proposition 4.3.** Suppose that \( \Omega \) is a bounded symmetric domain in \( \mathbb{C}^n \) and \( M \) is a close set of \( A^2(dv_\gamma) \), then \( M \) is a \( C[z] \)-module if and only if \( M \) is an \( O(\bar{\Omega}) \)-module.

**Proof.** The sufficiency part is clear, it is enough to prove the necessity part. Suppose \( M \) is a \( C[z] \)-module of \( A^2(dv_\gamma) \) and \( f \in O(\bar{\Omega}) \), we have to prove that \( f \cdot g \in M \) for every \( g \in M \). Since \( f \in O(\bar{\Omega}) \), there exists an open set \( D \) which contains the compact set \( \bar{\Omega} \) such that \( f \) is holomorphic on \( D \). Observe that the open set \( \Omega \) is the unit ball in the spectral norm and the topology induced by the spectral norm are coincide with the topology induced by the usual Euclidian norm. Combining with the Lebesgue number lemma, it follows that there exists a \( \delta > 0 \) satisfying

\[
\Omega \subset (1 + \delta)\Omega \subset (1 + \delta)\bar{\Omega} \subset D.
\]

Note that \( (1 + \delta)\Omega \) is a circular domain containing the origin, it implies that \( f \in O(D) \subset O((1 + \delta)\Omega) \) has a homogeneous expansion

\[
f(z) = \sum_{i=0}^{\infty} f_i(z)
\]

on the domain \((1 + \delta)\Omega\) where each \( f_i \) is \( i \)-homogeneous holomorphic polynomial, which converges compactly on \((1 + \delta)\Omega\). Hence the expansion \((4.2)\) converges uniformly on \( \Omega \) since \( \Omega \subset \bar{\Omega} \subset (1 + \delta)\Omega \). Thus

\[
f \cdot g = \lim_{i} f_i \cdot g \in M,
\]

as \( M \) is closed in \( A^2(dv_\gamma) \), which completes the proof. □

**Remark 4.4.** (1) In the proof of Proposition 4.3, we prove and use the fact that every bounded symmetric domain has a neighborhood basis of bounded symmetric domains and especially a pseudoconvex (or Stein) neighborhood basis. The same is true on strongly pseudoconvex domains, however it is false for pseudoconvex domains in general, see [5] and references therein.

(2) For a moment, we will reinterpret Proposition 4.3 from the point of view of category. Denote \( C[z]-\text{Bmod} \) by the category of Bergman modules on \( \Omega \), whose objects
are closed subsets with $\mathbb{C}[z]$-module actions of weighted Bergman space $A^2(dv_\gamma)$ for some $\gamma > -1$ and morphisms are continuous $\mathbb{C}[z]$-module homomorphisms. Similarly, $\mathcal{O}(\bar{\Omega})$-Bmod is denoted by the category of Bergman modules on $\Omega$, whose objects are closed subsets with $\mathcal{O}(\bar{\Omega})$-module actions of weighted Bergman space $A^2(dv_\gamma)$ for some $\gamma > -1$ and morphisms are continuous $\mathcal{O}(\bar{\Omega})$-module homomorphisms. Let $M_1 \xrightarrow{\varphi} M_2$ be a morphism in $\mathbb{C}[z]$-Bmod. Then Proposition 4.3 implies that $M_1, M_2$ are two objects in $\mathcal{O}(\bar{\Omega})$-Bmod. Moreover, since $\varphi$ is continuous, combining with (4.3) follows that $M_1 \xrightarrow{\varphi} M_2$ is actually a morphism in $\mathcal{O}(\bar{\Omega})$-Bmod. That introduces a functor $\iota$ satisfying

$$\iota : \mathbb{C}[z] \text{-Bmod} \rightarrow \mathcal{O}(\bar{\Omega}) \text{-Bmod},$$

$$M_1 \xrightarrow{\iota} M_2 \xrightarrow{\iota} M_1 \xrightarrow{\iota} M_2.$$  

Clearly $\mathcal{O}(\bar{\Omega})$-Bmod is a subcategory of $\mathbb{C}[z]$-Bmod in general, whose injection functor is denoted by $\tau$. Then by direct check shows that $\iota \tau = \text{Id}$ and $\tau \iota = \text{Id}$. The above shows that if $\Omega$ is a bounded symmetric domain then

$$\mathbb{C}[z] \text{-Bmod} = \mathcal{O}(\bar{\Omega}) \text{-Bmod}$$

and $\iota$ is the identity functor (or an isomorphism). As an application, we deduce that $(M_i, d_i)$ is a chain complex in the category $\mathbb{C}[z]$-Bmod if and only if $(M_i, d_i) = (\iota(M_i), \iota(d_i))$ is a chain complex in the category $\mathcal{O}(\bar{\Omega})$-Bmod. And in this case $\iota$ introduces the identity (or isomorphism) of homology groups. Especially, $(M_i, d_i)$ is a resolution of finite length of the module $M$ in $\mathbb{C}[z]$-Bmod if and only if $(M_i, d_i)$ is a resolution of finite length of the module $M$ in $\mathcal{O}(\bar{\Omega})$-Bmod, see [10] for the definition of resolution.

Note that the operator problems we concerned is independent on the choice of the two above module categories, thus in the sequel, we will consider the essential normality problems in the context of $\mathcal{O}(\bar{\Omega})$-modules rather than $\mathbb{C}[z]$-modules. Suppose $\bar{V}$ is an analytic subset on the closure $\bar{\Omega}$, namely there exists a neighborhood $D$ of $\bar{\Omega}$ such that $\bar{V}$ is an analytic set in $D$. We denote $M_V$ by

$$M_V := \{ f \in A^2(dv_\gamma) : f(z) = 0, \forall z \in V \},$$

where $V = \bar{V} \cap \Omega$. Since evaluation functionals are continuous on $A^2(dv_\gamma)$, it implies that $M_V$ is an $\mathcal{O}(\bar{\Omega})$-submodule of $A^2(dv_\gamma)$. By the Remark 4.3 $\Omega$ has a Stein neighborhood $D'$ satisfying $\Omega \subset D' \subset D$. Then there exist finitely many holomorphic functions $f_1, \cdots, f_m$ such that

$$V \subset \bar{V} \cap D' = \{ z \in D' : f_1(z) = \cdots = f_m(z) = 0 \},$$

by the fact mentioned in Section 2. Thus $V = \{ z \in \Omega : f_1(z) = \cdots = f_m(z) = 0 \}$. Note that $f_1, \cdots, f_m \in \mathcal{O}(\bar{\Omega})$, then $M_V$ is nontrivial whenever $V$ is proper in $\Omega$. In this paper, a submodule of $A^2(dv_\gamma)$ which is determined by an analytic subset is in the sense of (4.4).

Let $\phi : \Omega_1 \rightarrow \Omega_2$ be a holomorphic map between two bounded symmetric domains. The pull back of functions by the holomorphic map $\phi$ is given by

$$\phi^* : \mathcal{O}(\Omega_2) \rightarrow \mathcal{O}(\Omega_1), \quad f \mapsto \phi^*(f) = f \circ \phi,$$
which is a complex algebraic (module) homomorphism and especially a linear operator. The measure $dv$, can also be pulled back by the holomorphic map $\phi$, which is defined by

$$\phi^*(dv_{\gamma})(w) = c_\gamma \Delta(\phi(w), \phi(w)) dv(\phi(w)).$$

When $\phi : \Omega_1 \to \Omega_2$ is biholomorphic, then $\phi^* : A^2(\Omega_2, dv_{\gamma}) \to A^2(\Omega_1, \phi^*(dv_{\gamma}))$ is a unitary operator. Not to be confused with the concept of adjoint of an operator. Let $M$ be an $O(\Omega_1)$-submodule of $A^2(\Omega_1, dv_{\gamma})$ which is determined by an analytic subset. In this paper, we say $\phi : \Omega_1 \to \Omega_2$ is biholomorphic if $\phi$ is the restriction of a biholomorphism between neighborhoods of $\Omega_1, \Omega_2$. In this case, the pull back $\phi^*(M) = \{\phi^*(f) : f \in M\}$ is an $O(\Omega_2)$-submodule of $A^2(\Omega_2, \phi^*(dv_{\gamma}))$, which is determined by the analytic subset $\phi^{-1}(V) = \{\phi^{-1}(z) : z \in V\}$ in the sense of (4.4). Moreover, $\phi^*(M)$ is unitarily equivalent to $M$. Then the pull back $\phi^*(M^\perp)$ is a quotient module and unitarily equivalent to $M^\perp$, which implies that

$$\phi^* P_{M^\perp} = P_{\phi^*(M^\perp)} \phi^*. \quad (4.5)$$

**Proposition 4.5.** Suppose that $\phi : (\Omega_1, w) \to (\Omega_2, z)$ is a biholomorphism, let $M$ be a submodule of $A^2(\Omega_2, dv_{\gamma})$ determined by an analytic subset, then the following hold.

1. $M_{z_i}^M = (\phi^*)^{-1} M_{\phi_i(w)}^\phi \phi^*$.
2. $S_{z_i}^{M^\perp} = (\phi^*)^{-1} S_{\phi_i(w)}^{\phi^*(M^\perp)} \phi^*$.

**Proof.** (1) Let $f \in M$, since $\phi^*$ is an algebraic homomorphism and $\phi^*(z_i) = w_i$, it follows that

$$\phi^*(z_i f) = \phi^*(z_i \phi^*(f)) = \phi_i(w) \phi^*(f) \in \phi^*(M),$$

which implies

$$\phi^* M_{z_i} = M_{\phi_i(w)} \phi^*,$$

i.e. $M_{z_i} = (\phi^*)^{-1} M_{\phi_i(w)} \phi^*$, for $i = 1, \cdots, n$.

(2) Similarly, combining with (4.5), it follows that

$$\phi^* S_{z_i} f = \phi^* P_{M^\perp} M_{z_i} f$$

$$= \phi^* P_{M^\perp} (\phi^*)^{-1} \phi^* M_{z_i} f$$

$$= \phi^* P_{M^\perp} (\phi^*)^{-1} \phi^*(z_i f)$$

$$= \phi^* P_{M^\perp} (\phi^*)^{-1} \phi_i(w) \phi^* f$$

$$= P_{\phi^*(M^\perp)} \phi_i(w) \phi^* f$$

for every $f \in M^\perp$ and $i = 1, \cdots, n$. Thus $S_{z_i} = (\phi^*)^{-1} S_{\phi_i(w)} \phi^*$, $i = 1, \cdots, n$. $\Box$

We are now in a position to prove Theorem 2, we first prove it in a special case.

**Lemma 4.6.** Let $\Omega$ be an irreducible bounded symmetric domain in $\mathbb{C}^n$ and $M$ be a submodule. Suppose that $\phi : (\Omega, w) \to (\Omega, z)$ is a linear automorphism that is $\phi \in K$, then $(\phi^*(M^\perp), w)$ is $p$-essentially normal if and only if $(M^\perp, z)$ is $p$-essentially normal.

**Proof.** Since $\phi : (\Omega, w) \to (\Omega, z)$ is a linear automorphism, without loss of generality, we can assume $z_i = \phi_i(w) = a_{ij}^2 w_j$, where $(a_i^j)$ is an invertible matrix and the Einstein
summation convention is adopted. It follows from Proposition 4.5 that
\[
\phi^*[S_{z_i}^{M^\perp}, S_{z_j}^{M^\perp*}](\phi^*)^{-1} = [S_{\phi_i(w)}^{\phi^*(M^\perp)}, S_{\phi_j(w)}^{\phi^*(M^\perp)*}]
\]
\[
= [S_{a_{i,j}^w}^{\phi^*(M^\perp)}, S_{a_{j,i}^w}^{\phi^*(M^\perp)*}]
\]
\[
= a_{i,j}^w [S_{w_i}^{\phi^*(M^\perp)}, S_{w_j}^{\phi^*(M^\perp)*}],
\]
where \(i, j = 1, \cdots, n\). It follows that \([S_{z_i}^{M^\perp}, S_{z_j}^{M^\perp*}] \in \mathcal{L}^p\) if \([S_{w_i}^{\phi^*(M^\perp)}, S_{w_j}^{\phi^*(M^\perp)*}] \in \mathcal{L}^p\).
Thus \((M^\perp, z)\) is \(p\)-essentially normal if \((\phi^*(M^\perp), w)\) is \(p\)-essentially normal. Since the matrix \((a_{i,j}^w)\) is invertible, we denote its inverse by \((b_{i,j}^w)\), it implies that \(w_i = b_{i,j}^w z_j\).
The same argument shows that \((\phi^*(M^\perp), w)\) is \(p\)-essentially normal if \((M^\perp, z)\) is \(p\)-essentially normal.

**Theorem 2.** Let \((\Omega, z)\) be an irreducible bounded symmetric domain in \(\mathbb{C}^n\) and \(M\) be a weighted Bergman submodule determined by an analytic subset. Suppose that \(\phi : (\Omega, w) \rightarrow (\Omega, z)\) is an arbitrary biholomorphic automorphism with \(z = \phi(w)\), then \((\phi^*(M^\perp), S_w)\) is \(p\)-essentially normal if and only if \((M^\perp, S_z)\) is \(p\)-essentially normal.

**Proof.** Without loss of generality, we suppose \(\phi(0) = z_0\). Note that \(g_{z_0}(w) = z_0 + B(z_0, z_0)w - z_0\) satisfying \(g_{z_0}(0) = z_0\), where \(z^* = B(z, \xi)^{-1}(z - \{z\xi^*z\})\) is the quasi-inverse of the pair \((z, \xi)\) \(\in \Omega \times \Omega\). Thus \(g_{z_0}^{-1} \circ \phi \in \text{Aut}(\Omega)\) and \(g_{z_0}^{-1} \circ \phi(0) = 0\), it means that \(g_{z_0}^{-1} \circ \phi \in K\). It indeed shows that the biholomorphism \(\phi\) admits the factorization \(\phi = g_{z_0} \circ k\) where \(k \in K\), which means the biholomorphism \(\phi\) satisfies the following commutative diagram

\[
\begin{array}{ccc}
\Omega & \xrightarrow{k} & \Omega \\
\downarrow{\phi} & & \downarrow{g_{z_0}} \\
\Omega & & \Omega
\end{array}
\]

By Lemma 4.6, the essential normality of Hilbert modules and quotient Hilbert modules is invariant under the linear automorphism \(k\), so it suffices to prove the conclusion in the case of \(\phi = g_{z_0}\). On the other hand, by [24, Section 1.5], if the quasi-inverse of the pair \((z, \xi) \in \mathbb{C}^n \times \mathbb{C}^n\) exists, then its quasi-inverse can be written in the rational form of
\[
z^* = \frac{p(z, \xi)}{\Delta(z, \xi)},
\]
where \(p(z, \xi)\) is a \(\mathbb{C}^n\)-valued polynomial in \(z, \xi\), and \(\Delta(z, \xi)\) is the Jordan triple determinant and has no common factors with \(p(z, \xi)\). Moreover, by [21, Section 7.5], the quasi-inverse of the pair \((z, \xi)\) exists whenever the spectrum norm satisfying \(\|z\| \cdot \|\xi\| < 1\), and \(\Delta(z, \xi) \neq 0\) whenever the quasi-inverse of the pair \((z, \xi)\) exists. Therefore, for every \(z_0 \in \Omega = \{\|z\| < 1\}\), the biholomorphism \(g_{z_0} \in \text{Aut}(\Omega)\) is in fact holomorphic on the irreducible bounded symmetric domain \(\Omega_{z_0} = \{\|z\| < \frac{1}{\|z_0\|}\} \supset \Omega\) and \(g_{z_0} : \Omega \rightarrow \Omega\) is biholomorphic by the following Lemma 4.7. Thus each component function of \(g_{z_0}\) can be uniquely extended to a holomorphic function in \(\mathcal{O}(\Omega_{z_0})\), it implies that each component function of \(g_{z_0}\) is a rational function and belongs to \(\mathcal{O}(\Omega)\). So, we can write
\[
g_{z_0} = \left(\frac{p_1}{q_1}, \cdots, \frac{p_n}{q_n}\right),
\]
Lemma 4.7. For every \( \square \), and \( q_i \) have no zeros in \( \bar{\Omega} \). Applying Proposition 4.5 to the case of \( \phi = g_{z_0} \), and combining with Lemma 3.1 and Corollary 4.2, it implies that
\[
\phi^*[S_{z_i}^{M^*}, S_{z_j}^{M^*}](\phi^*)^{-1} = [S_{\phi_i}^{\phi^*(M^*)}, S_{\phi_j}^{\phi^*(M^*)}] = [S_{\phi_i}^{\phi^*(M^*)}S_{q_i}^{\phi^*(M^*)}, S_{\phi_j}^{\phi^*(M^*)}] = S_{\phi_i}^{\phi^*(M^*)}[S_{q_i}^{\phi^*(M^*)}, S_{\phi_j}^{\phi^*(M^*)}]S_{\phi_j}^{\phi^*(M^*)} - 1 \]
where
\[
U_1 = S_{\phi_i}^{\phi^*(M^*)}S_{\phi_j}^{\phi^*(M^*)}S_{q_i}^{\phi^*(M^*)} - 1 \quad S_{q_j}^{\phi^*(M^*)} - 1 [S_{\phi_i}^{\phi^*(M^*)}, S_{\phi_j}^{\phi^*(M^*)}S_{q_j}^{\phi^*(M^*)} - 1 S_{q_i}^{\phi^*(M^*)} - 1 ,
U_2 = -S_{\phi_i}^{\phi^*(M^*)}S_{q_i}^{\phi^*(M^*)} - 1 [S_{q_i}^{\phi^*(M^*)}, S_{\phi_j}^{\phi^*(M^*)}]S_{q_j}^{\phi^*(M^*)} - 1 S_{q_j}^{\phi^*(M^*)} - 1 ,
U_3 = -S_{\phi_j}^{\phi^*(M^*)}S_{q_j}^{\phi^*(M^*)} - 1 [S_{\phi_j}^{\phi^*(M^*)}, S_{\phi_i}^{\phi^*(M^*)}]S_{q_i}^{\phi^*(M^*)} - 1 S_{q_i}^{\phi^*(M^*)} - 1 ,
U_4 = [S_{\phi_i}^{\phi^*(M^*)}, S_{\phi_j}^{\phi^*(M^*)}]S_{q_j}^{\phi^*(M^*)} - 1 S_{q_i}^{\phi^*(M^*)} - 1 ,
\]
for all \( i, j = 1, \ldots, n \). It follows that \( [S_{z_i}^{M^*}, S_{z_j}^{M^*}] \in \mathcal{L}^p \) if \( [S_{w_i}^{\phi^*(M^*)}, S_{w_j}^{\phi^*(M^*)}] \in \mathcal{L}^p \) by Proposition 3.2 and the fact that \( \mathcal{L}^p \) is an operator ideal. It implies that \((M, z)\) is \( p \)-essentially normal if \((\phi^*(M), w)\) is \( p \)-essentially normal. Similarly, the same argument for the automorphism \( \phi^{-1} : (\Omega, z) \to (\Omega, w) \) implies that \((\phi^*(M), w)\) is \( p \)-essentially normal if \((M, z)\) is \( p \)-essentially normal. This completes the proof. \( \square \)

**Lemma 4.7.** For every \( z_0 \in \Omega \), then \( g_{z_0} : \Omega \to \bar{\Omega} \) is biholomorphic.

**Proof.** It suffices to show that \( g_{z_0} \) is biholomorphic on \( \Omega_{z_0} = \{ \| z \| < \frac{1}{\| z_0 \|} \} \supset \bar{\Omega} \) by the holomorphic version of the inverse function theorem. To show that \( g_{z_0} \) is biholomorphic on \( \Omega_{z_0} \), it is enough to show that the Jacobi matrix \( g_{z_0}' \) of \( g_{z_0}(z) = z_0 + B(z_0, z_0) \frac{1}{2} z^{-z_0} \) is nondegenerate on \( \Omega_{z_0} \). By [18 (2.18)], we conclude that
\[
g_{z_0}'(z) = B(z_0, z_0) \frac{1}{2} B(z, -z_0)
\]
for \( z \in \Omega_{z_0} \). Then the quasi-invertibility of \( z^{-z_0} \) on \( \Omega_{z_0} \) implies that \( g_{z_0}' \) is nondegenerate on \( \Omega_{z_0} \). \( \square \)

Similarly, the same is also true for the subdomain \((M, T_z)\).

**Corollary 4.8.** Let \((\Omega, z)\) be an irreducible bounded symmetric domain in \( \mathbb{C}^n \) and \( M \) be a Bergman submodule determined by an analytic subset. Suppose that \( \phi : (\Omega, w) \to (\Omega, z) \) is an arbitrary biholomorphic automorphism with \( z = \phi(w) \), then \((\phi^*(M), T_w)\) is \( p \)-essentially normal if and only if \((M, T_z)\) is \( p \)-essentially normal.

As a consequence of Theorem 2, it implies that the essential normality of a quotient Bergman submodule with the compression of coordinate multipliers determined
by an analytic subset on a bounded symmetric domain is invariant if the coordinate multipliers is replaced by biholomorphic automorphism multipliers.

**Corollary 4.9.** Let $(\Omega, z)$ be an irreducible bounded symmetric domain in $\mathbb{C}^n$ and $(M^\perp, S)$ be a quotient Bergman submodule determined by an analytic subset, where $S$ is the compression of coordinate multipliers. Suppose that $\phi : \Omega \to \Omega$ is an arbitrary biholomorphic automorphism, then the following are equivalent.

1. $[S_{z_i}, S^*_{z_j}] \in L^p$ for all $i, j = 1, \ldots, n$.
2. $[S_{\phi_i}, S^*_{\phi_j}] \in L^p$ for all $i, j = 1, \ldots, n$.

**Proof.** It comes from Theorem 2 and Proposition 4.5. □

5. The case on the Wallach set

In the previous section we consider the essential normality of Hilbert modules in the context of weighted Bergman space on irreducible bounded symmetric domains. However, as mentioned in the end of Section 2, on an irreducible bounded symmetric domain, the weighted Bergman spaces is a fraction of the reproducing kernel Hilbert holomorphic function spaces corresponding to the continuous part of the Wallach set. In this section, we will consider that the biholomorphic invariance of essential normality of Hilbert modules on the more general holomorphic function space determined by the Wallach set.

The main ideal of the proof is the similar to the case of the weighted Bergman modules, we consider the operator problems on the $O(\bar{\Omega})$-module category. However, the situation is a few different in general. When $\lambda > N - 1$, it is trivial to equip the weighted Bergman spaces $H^2_\lambda(\Omega)$ with module actions by the multiplication of functions over algebras $P(Z)$ and $O(\bar{\Omega})$ since those norms or inner products are induced by the integration with suitable probability measures. However, in general we requires additional efforts to equip $H^2_\lambda(\Omega)$ with an $O(\bar{\Omega})$-module action by the the multiplication of functions when it is a Hilbert module over the algebra $P(Z)$.

We first observe that the holomorphic function space $H^2_\lambda(\Omega)$ on an irreducible bounded symmetric domain $\Omega$ determined by the discrete part $W_{\Omega,d}$ of the Wallach set $W_\Omega$ is not a Hilbert module over the polynomial algebra $P(Z) = \mathbb{C}[z]$. However, for every continuous point $\lambda \in W_{\Omega,c}$, the function space $H^2_\lambda(\Omega)$ is always a Hilbert module over $P(Z)$ by the coordinate function multipliers, which is in fact a restatement of [1, Theorem 4.1] and [4, Theorem 1.1]. We formulate it in the following lemma.

**Lemma 5.1.** Let $\lambda \in W_\Omega$, then $(H^2_\lambda(\Omega), T)$ is a Hilbert module over $P(Z)$ with the coordinate function multipliers if and only if $\lambda \in W_{\Omega,c}$. Moreover, the Taylor spectrum $Sp(T) = \bar{\Omega}$ whenever $\lambda \in W_{\Omega,c}$.

**Proof.** By [1, Theorem 4.1] and [4, Theorem 1.1], it implies that the function space $(H^2_\lambda(\Omega), T)$ with the coordinate function multipliers is always a Hilbert module over $P(Z)$ and the Taylor spectrum $Sp(T) = \bar{\Omega}$ whenever $\lambda \in W_{\Omega,c}$. Thus it is sufficient to prove that $H^2_\lambda(\Omega)$ is not a Hilbert module over $P(Z)$ by the coordinate function multipliers if $\lambda \in W_{\Omega,d}$. Suppose it were false. Then we could find $\lambda_0 = \frac{3}{2}(j_0 - 1)$ for some $1 \leq j_0 \leq r$ such that $H^2_{\lambda_0}(\Omega)$ is a Hilbert module over $P(Z)$ by the coordinate
function multipliers. By the construction of $H^2_{\lambda_0}(\Omega)$, we know that

$$H^2_{\lambda_0}(\Omega) = \sum_{m \geq 0, m_{\lambda_0} = 0} P_m(Z).$$

Note that the constant function $1 \in H^2_{\lambda_0}(\Omega)$, hence

$$P(Z) \cdot 1 = P(Z) \subset H^2_{\lambda_0}(\Omega),$$

which contradicts $P(Z) = \sum_{m \geq 0} P_m(Z)$, and the proof is finished. \(\square\)

In the Section 4, we define an $O(\overline{\Omega})$-module action by the function multipliers on the weighted Bergman spaces $H^2_{N+\gamma}(\Omega) = A^2(dv_\gamma), \gamma > -1$. Notice that $N + \gamma > N - 1 > (r - 1)\frac{a}{2}$.

Actually, we can define an $O(\overline{\Omega})$-module action by the function multipliers on $H^2_{\lambda}(\Omega)$ for all $\lambda \in W_{\Omega,c} = \{\lambda > (r - 1)\frac{a}{2}\}$. To cover the gap $\{(r - 1)\frac{a}{2} < \lambda \leq N - 1\}$, we need the following lemma. To make its proof more precise, we introduce the following notations. Let $I_{\Omega}$ be the set of all integer partitions with length $r$. Define

$$I_{\Omega}(0) = \{0\},$$
$$I_{\Omega}(j) = \{(m_1, \cdots, m_j, 0, \cdots, 0) : m_1 \geq \cdots \geq m_j > 0\}, 1 \leq j \leq r - 1,$$
$$I_{\Omega}(r) = \{(m_1, \cdots, m_r) : m_1 \geq \cdots \geq m_r > 0\}.$$

It obvious that

$$I_{\Omega} = \bigcup_{j=0}^{r} I_{\Omega}(j) \quad \text{and} \quad I_{\Omega}(i) \cap I_{\Omega}(j) = \emptyset$$

whenever $i \neq j$.

**Lemma 5.2.** Suppose $(r - 1)\frac{a}{2} < \lambda_1 < \lambda_2$, then the imbedding

$$i : H^2_{\lambda_1}(\Omega) \rightarrow H^2_{\lambda_2}(\Omega), \quad f \mapsto i(f) = f$$

is continuous.

**Proof.** Let $f \in H^2_{\lambda}(\Omega)$ for $\lambda \in W_{\Omega,c}$ and $f = \sum_{m \geq 0} f_m$ be its Peter-Schmid-Weyl decomposition. Then

$$\|f\|_{\lambda} = \sum_{m \geq 0} \|f_m\|_{\lambda}.$$ 

Hence it suffices to prove that there exists an uniform positive constant $C$ satisfying

$$\|p_m\|_{\lambda_2} \leq C\|p_m\|_{\lambda_1}$$

for all $p_m \in P_m$ and $m \geq 0$. Combining with the formula (2.3), it suffices to prove that there exists an uniform positive constant $C$ satisfying

$$\frac{(\lambda_1)^m}{(\lambda_2)^m} \leq C \quad (5.1)$$
for every \( m \geq 0 \). By the definition of the multi-variable Pochhammer symbol, we obtain

\[
\frac{(\lambda_1)^m}{(\lambda_2)^m} = \frac{\Gamma(\lambda_1 + m)}{\Gamma(\lambda_1)} \frac{\Gamma(N + \gamma)}{\Gamma(\lambda_2 + m)}
\]

\[
= \frac{\Gamma(\lambda_2)}{\Gamma(\lambda_1)} \prod_{j=1}^{r} \frac{\Gamma(\lambda_1 + m_j - (j - 1)\frac{2}{2})}{\Gamma(\lambda_2 + m_j - (j - 1)\frac{2}{2})}.
\]

(5.2)

It is easy to see that

\[
\frac{(\lambda_1)^0}{(\lambda_2)^0} = \frac{\Gamma(\lambda_1)}{\Gamma(\lambda_1)} \prod_{j=1}^{r} \frac{\Gamma(\lambda_1 - (j - 1)\frac{2}{2})}{\Gamma(\lambda_2 - (j - 1)\frac{2}{2})}.
\]

(5.3)

Note that Stirling’s formula implies that there exists an uniform positive constant \( C_j \) satisfying

\[
\frac{\Gamma(\lambda_1 + m_j - (j - 1)\frac{2}{2})}{\Gamma(\lambda_2 + m_j - (j - 1)\frac{2}{2})} \leq C_j \frac{1}{m_j^{\lambda_2 - \lambda_1}}.
\]

for \( j = 1, \ldots, r \). Then (5.2) implies that

\[
\frac{(\lambda_1)^m}{(\lambda_2)^m} \leq \frac{\Gamma(\lambda_1)}{\Gamma(\lambda_1)} \prod_{j=1}^{r} \frac{\Gamma(\lambda_1 - (j - 1)\frac{2}{2})}{\Gamma(\lambda_2 - (j - 1)\frac{2}{2})} \prod_{j=1}^{k} \frac{1}{m_j^{\lambda_2 - \lambda_1}}
\]

(5.4)

for every \( m \in I_\Omega(j), 1 \leq j \leq r - 1 \). Similarly,

\[
\frac{(\lambda_1)^m}{(\lambda_2)^m} \leq \frac{\Gamma(\lambda_1)}{\Gamma(\lambda_1)} \prod_{j=1}^{r} \frac{\Gamma(\lambda_1 - (j - 1)\frac{2}{2})}{\Gamma(\lambda_2 - (j - 1)\frac{2}{2})}
\]

(5.5)

for every \( m \in I_\Omega(r) \). Thus (5.1) follows by (5.3), (5.4) and (5.5). This completes the proof.

\[\square\]

**Lemma 5.3.** Let \( \lambda \in W_{\Omega,c} \), then \( H^2_{\lambda}(\Omega) \) is an \( \mathcal{O}(\bar{\Omega}) \)-module whose module actions are the multiplication of functions. In particular, \( \mathcal{O}(\Omega) \subset H^2_{\lambda}(\Omega) \) is dense.

**Proof.** The case of \( \lambda > N - 1 \) is clearly, since in this case \( H^2_{\lambda}(\Omega) \) is weighted Bergman spaces whose inner products is introduced by a finite volume measures on \( \Omega \). Thus it suffices to consider the case of \( \lambda \in \{(r - 1)\frac{2}{2} < \lambda \leq N - 1\} \). Since the Taylor spectrum \( Sp(T) = \bar{\Omega} \) by Lemma 5.1, it implies that we can define an \( \mathcal{O}(\bar{\Omega}) \)-module actions on \( H^2_{\lambda}(\Omega) \) by Taylor functional calculus, namely for every \( f \in \mathcal{O}(\Omega) \)

\[ f \cdot h := f(T)h \]

(5.6)

for every \( h \in H^2_{\lambda}(\Omega) \). Since \( f \in \mathcal{O}(\bar{\Omega}) \), there exists an open set \( D \) which contains the compact set \( \bar{\Omega} \) such that \( f \) is holomorphic on \( D \). It follows from [4.2] that there exists a \( \delta > 0 \) satisfying

\[ \Omega \subset (1 + \delta)\Omega \subset (1 + \delta)\bar{\Omega} \subset D \]
and \( f \in \mathcal{O}(D) \subset \mathcal{O}((1 + \delta)\Omega) \) has a homogeneous polynomial expansion

\[
f(z) = \sum_{i=0}^{\infty} f_i(z)
\]
on the domain \((1 + \delta)\Omega\) where each \( f_i \) is \( i \)-homogeneous holomorphic polynomial, which converges uniformly on \((1 + \frac{\delta}{2})\Omega\). Thus \( f(T) = \lim_i f_i(T) \) on the strongly operator topology, which yields

\[
f \cdot h = f(T)h = \lim_i f_i(T)h = \lim_i f_ih
\]
for every \( h \in H^2_{\lambda}(\Omega) \). On the other hand, it follows from Lemma 5.2 that the imbedding \( H^2_{\lambda}(\Omega) \subset H^2_N(\Omega) = A^2(dv) \) is continuous because \((r - 1)\frac{a}{2} < \lambda < N\). Then \( f_ih \) is convergent in \( H^2_N(\Omega) \). Observe that \( f = \sum_{i=0}^{\infty} f_i \) converges uniformly on \((1 + \frac{\delta}{2})\Omega\). Thus \( f_ih \) converges to \( fh \) in \( H^2_N(\Omega) \). Hence

\[
f \cdot h = \lim_i f_ih = fh
\]
by the continuity of the evaluation functional. This implies that we can define an \( \mathcal{O}(\bar{\Omega}) \)-module action by the multiplication of functions, which is coincided with the \( \mathcal{O}(\bar{\Omega}) \)-module action (5.6) defined by the Taylor function calculus.

□

As a consequence, we can generalize Proposition 4.3 to the following form.

**Corollary 5.4.** Suppose that \( \Omega \) is an irreducible bounded symmetric domain in \( \mathbb{C}^n \) and \( M \) is a close set of \( H^2_{\lambda}(\Omega) \) for \( \lambda \in W_{\Omega,c} \), then \( M \) is a \( \mathbb{C}[z] \)-module if and only if \( M \) is an \( \mathcal{O}(\bar{\Omega}) \)-module.

Thus all notations we defined in Section 4 for the weighted Bergman modules can be generalized to the current situation of the Hilbert modules \( H^2_{\lambda}(\Omega) \) for \((r - 1)\frac{a}{2} < \lambda \leq N - 1\), except the pull back of measures. It is an interesting question to construct integral formulas with suitable finite measures \( dv_{\lambda} \) for the \( K \)-invariant inner products associated with the Wallach points, we refer the reader to [1, 4, 24]. It is known that the \( K \)-invariant inner products on \( H^2_{\lambda}(\Omega) \) for \( \lambda \in W_{\Omega,c} \) can be realized as the integration with a unique probability measures \( dv_{\lambda} \) supported inside \( \bar{\Omega} \) if and only if

\[
\lambda \in \{ \lambda_j = N - 1 - (j - 1)\frac{a}{2}, j = 1, \cdots, r \} \cup \{ \lambda > N - 1 \}.
\]

Moreover, the measure \( dv_{\lambda} \) is quasi-invariant if \( \lambda = \lambda_j, j = 1, \cdots, r \), namely

\[
dv_{\lambda_j}(g(z)) = |\det(g'(z))|^\\frac{2\lambda}{a} dv_{\lambda_j}(z)
\]
for all \( g \in G \) and \( z \in \bar{\Omega} \). In fact \( dv_{\lambda_j} \) supported on the \( G \)-orbit \( G \cdot (e_1 + \cdots + e_j) \) which is a part of the topological boundary \( \partial \Omega \). Hence in the spacial case of \( \lambda = \lambda_j \) we can still pull back the measures \( dv_{\lambda_j}, j = 1, \cdots, r \). However, in general we will directly pull back the inner product rather than the measure as follows. Let \( \phi : \Omega \to \Omega \) be a biholomorphic automorphism. The pull back of functions is given by

\[
\phi^* : \mathcal{O}(\Omega) \to \mathcal{O}(\Omega), \quad f \mapsto \phi^*(f) = f \circ \phi,
\]
which is a complex algebraic (module) homomorphism and especially a linear operator. For $H^2_\lambda(\Omega), \lambda \in W_{\Omega,c}$, its pull back under $\phi$ is denoted by $\phi^*(H^2_\lambda(\Omega))$, we equip it an inner product as
\[
\langle \phi^*(f), \phi^*(g) \rangle_{\lambda, \phi^*} := \langle f, g \rangle_{\lambda}
\]
whenever $f, g \in H^2_\lambda(\Omega)$, which is called the pull back of the inner product. By direct check we see that the inner product (5.7) is $K$-invariant and $\phi^*(H^2_\lambda(\Omega))$ is completed in this inner product, which coincides with the integration with respect to the pull backed measure whenever the inner product of $H^2_\lambda(\Omega)$ can be realized as the integration with respect to a probability measure. Thus $\phi^* : H^2_\lambda(\Omega) \to \phi^*(H^2_\lambda(\Omega))$ is a unitary operator.

Furthermore, if $M$ is an $O(\bar{\Omega})$-submodule in $H^2_\lambda(\Omega)$ determined by an analytic subset, then its pull back $\phi^*(M)$ is also an $O(\bar{\Omega})$-submodule in $\phi^*(H^2_\lambda(\Omega))$ determined by the pull back of the analytic subset. The similar is also true for the quotient submodule.

Now we list the main results of the biholomorphic invariance of essential normality of quotient Hilbert modules in $H^2_\lambda(\Omega)$ for $\lambda \in W_{\Omega,c}$; the case of weight Bergman modules of $\lambda > N - 1$ is our main Theorem 2 above, moreover the proof of the case of $(r-1)\frac{2}{2} < \lambda \leq N - 1$ is similar to the case of weighted Bergman modules and omitted.

**Theorem 5.5.** Let $(\Omega, z)$ be an irreducible bounded symmetric domain in $\mathbb{C}^n$ and $M$ be a submodule in $H^2_\lambda(\Omega), \lambda \in W_{\Omega,c}$ determined by an analytic subset. Suppose that $\phi : (\Omega, w) \to (\Omega, z)$ is an arbitrary biholomorphic automorphism with $z = \phi(w)$, then $(\phi^*(M^\perp), S_w)$ is $p$-essentially normal if and only if $(M^\perp, S_z)$ is $p$-essentially normal.

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