A NOTE ON THE CENTERS OF A CLOSED CHAIN OF CIRCLES

ÁKOS G.HORVÁTH

ABSTRACT. In this note we prove that the centers of a closed chain of circles for which every two consecutive members meet in the points of two given circles form a tangent polygon of a conic.

1. Introduction

An interesting recent elementary statement on circles is the so-called Dao’s Theorem on six circles (see in [1], [3], [2] and [8]). It states that if we have a cyclic hexagon and consider the six triangles defined by the lines of its three consecutive sides, respectively, then the centers of the circumscribed circles of these triangles form a Brianchon hexagon. (Brianchon hexagon holds the property that its main diagonals are concurrent, they are joining at the so-called Brianchon point of the hexagon. See e.g. in [6]) Two consecutive circles of the chain intersect each other in at most two points, one of them is a vertex of the original hexagon and they can join in another common point, too. This latter is called the "second point of intersection" of the above two circles. The following question can be arisen immediately: What is the original hexagon and they can join in another common point, too. This latter is called the "second point of intersection" of the above two circles. The following question can be arisen immediately: What is the speciality of the hexagon defined by these "second points of intersection"?

On the other hand, Miquel’s Six-Circles Theorem (see in [7]) can be formulated in the following way: If we have two cyclic quadrangle \(P_1P_2P_3P_4\) and \(Q_1Q_2Q_3Q_4\) for which the quadruples \(P_1Q_1Q_2P_2\), \(P_2Q_2Q_3P_3\), \(P_3Q_3Q_4P_4\) are cyclic then the last quadruple of this type \(P_4Q_4Q_1P_1\) is also cyclic. The circum-circles of the last four quadruples form a closing chain of intersecting circles with the property that the points of intersections belong to two other circles transversal to each of the circle of the chain, respectively. G. Gévy proved in the following extension of this theorem (see in [4]):

**Theorem 1** ([4]). Let \(\alpha\) and \(\beta\) be two circles. Let \(n > 2\) be an even number, and take the points \(P_1, \ldots, P_n\) on \(\alpha\) and \(Q_1, \ldots, Q_n\) on \(\beta\), such that each quadruple \(P_1Q_1Q_2P_2\), \(\ldots, P_{n-1}Q_{n-1}Q_nP_n\) is cyclic. Then the quadruple \(P_nQ_nQ_1P_1\) is also cyclic.

In the case of \(n = 6\) the getting configuration of circles very similar to the configuration of Dao’s theorem, so it is not to surprising that L. Szilassi observed that the centers of the circles forms a Brianchon hexagon, but he has given no proof. G. Gévy gave a proof of this statement in [4] using projective geometry of the three-space. Our short paper contains another (simpler and shorter) proof of this statement, using only the Euclidean geometry of the plane. This proof also leads to a generalization of the statement of \(n = 6\) to any even value of \(n\).

We note that the condition of Dao’s theorem and the condition of Theorem 2 of our paper is independent to each other. An easy example for a Dao’s construction when the second points of intersection are not cocyclic can be seen in Fig. 1.

In Duong’s paper we can find the refine of this problem: Prove that if the original cyclic hexagon is also a Brianchon hexagon then the hexagon of the second points of intersection is also cyclic. (See Problem 6 in [8] or Problem 2 in [9].)

2. A theorem on the chain of intersecting circles

We prove the following theorem:

**Theorem 2.** Let \(c(K)\) and \(c(L)\) be two circles with respective centers \(K\) and \(L\). Let \(n > 2\) be an even number, and take the points \(P_1, \ldots, P_n\) on \(c(K)\) and \(Q_1, \ldots, Q_n\) on \(c(L)\), such that each quadruple \(P_1Q_1Q_2P_2\), \(\ldots, P_nQ_nQ_1P_1\) is cyclic. Denote by \(O_i\) the center of the circle \(c(O_i)\) circumscribed the quadrangle \(P_iQ_iQ_{i+1}P_{i+1}\). Then for each \(i\) the line \(O_iO_{i+1}\) are tangent to a fized conic with foci \(K\) and \(L\), respectively.

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**Proof.** We prove that if we reflect the point $K$ to the successive sides of the polygon $O_1 \ldots O_n$ then we get points on a circle with center $L$. To this we have to consider only one cyclic quadrangle, e.g. $P_1Q_1Q_2P_2$ (see Fig. 2).

![Figure 2. The inscribed conic in the case of ellipse.](image)

If $Q_1P_1P_2\angle$ is denoted by $\beta$ and $P_1P_2Q_2\angle$ is denoted by $\gamma$ then $P_1Q_1Q_2\angle = \pi - \gamma$ and $P_2Q_2Q_1\angle = \pi - \beta$. Reflect the point $K$ to the lines $O_1O_2$ and $O_nO_1$, respectively. Denote by $U$ and $T$ the getting points and observe that the quadrangles $KP_1Q_1T$ and $KP_2Q_2U$ are symmetric trapezoids, respectively. Hence we have $|Q_1T| = |KP_1| = |KP_2| = |Q_2U|$. Since $KP_1Q_1\angle = P_1Q_1T\angle$ then $TQ_1Q_2\angle = \alpha + \beta + \gamma - \pi$. Similarly we also have that $KP_2Q_2\angle = P_2Q_2U\angle$ implying that $UQ_2Q_1\angle = \alpha + \beta + \gamma - \pi$, too. Hence the quadrangle $TQ_1Q_2U$ is also a symmetric trapezoid implying that the perpendicular bisector of the segments $TU$ and $Q_1Q_2$ agree. Consequently, the distance of the points $T$ and $U$ from the center $L$ is equal to each other. Hence the conic defined by the foci $K$ and $L$ and the tangent line $O_nO_1$ agree with the conic defined by the same pair of foci and the tangent line $O_1O_2$. The similar reasoning for the next quadrangle $P_3Q_3P_4$ implies that this conic is the same as the conic defined by the foci $K$, $L$ and the tangent line $O_2O_3$ and so one...This proves the theorem. □

**Corollary 1** (Theorem 2 in [5]). If $n = 6$ the polygon defined by the centers $O_i$ is a Brianchon hexagon by the Brianchon’s theorem on conics.

Let $n \geq 6$ be an even number. We can state now the following:
Theorem 3. We take modulo $n$ that natural numbers which we use as an index. Denote the point of intersection of the lines $O_iO_{i+1}$ and $O_kO_{k+1}$ by $O_{i,k}$, where $O_{i,i+1} = O_{i+1}$ for all values of $i$. Then for $i < j < k < l < m < n$ we have that the points $O_{i,j}$, $O_{j,k}$, $O_{k,l}$, $O_{l,m}$, $O_{m,n}$ and $O_{n,i}$ form a Brianchon hexagon.

Figure 3. The inscribed conic in the case of parabola.

Remark 1. Observe that in the case when $K$ is an inner point of the circle $c(L)$ containing the points $Q_i$ then the conic is an ellipse and if the point $K$ is an outer point of $c(L)$ then the conic is an hyperbola. The case of the parabola can be got from that situation when the point $L$ is at infinity meanings that $c(L)$ is a line. In fact, in this case the segment $UT$ is parallel to $c(L)$ and the examined lines tangent to a parabola, respectively (see Fig.)

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Department of Geometry, Budapest University of Technology and Economics, H-1521 Budapest, Hungary
E-mail address: ghorvath@math.bme.hu