\-DOLD-KAN CORRESPONDENCE VIA REPRESENTATION THEORY

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Abstract. In this paper we give a purely derivator theoretical reformulation and proof of a classic result of Happel and Ladkani, showing that it occurs uniformly across stable derivators and it is then independent of coefficients. The resulting equivalence provides a bridge between homotopy theory and representation theory: indeed, we explain how our result is a derivator-theoretic version of the \-Dold-Kan correspondence for bounded cochain complexes. Moreover, our equivalence can also be realised as an action of a spectral bimodule in the setting of universal tilting theory developed by Groth and Šťovíček.

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1. Introduction

Over algebraically closed fields, one of the best understood examples of finite dimensional algebras are path algebras over Dynkin quivers \[7\] and their quotients by admissible ideals \[2\]. The most relevant theory in this setting is due to Auslander and Reiten \[3, 4\]. This theory, in particular, motivates why, in order to understand algebras, we study not only the category of finitely generated modules but also the associated bounded derived category. At this level, equivalences are usually found by applying the derived version of the Morita theory due to Rickard \[35\]. This theory, which is meant as a generalization of the tilting theory developed by Happel and Ringel \[20\], is based on the study of the so called tilting complex (see also \[19\]). Then, one obtains the desired equivalences as derived tensor products by tilting complexes.

Let \(A_n\) be the Dynkin quiver of type \(A\) with \(n\) vertices. For a commutative ring \(k\), Ladkani \[28\] studied the construction of new tilting complexes realising derived equivalences between tensor products of \(k\)-algebras over Dynkin quivers of type \(A\) and \(k\)-algebras over the \(A_n\) quiver with relations

\[
\begin{align*}
A_n: & \quad 0 \xrightarrow{\alpha_0} 1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-2}} n-1.
\end{align*}
\]

A trivial case of these equivalences, which is also a consequence of the work by Happel \[18\], can be stated as follows.

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Theorem 1.1. [28, Corollary 1.2] Let $I$ be the ideal of the path algebra $kA_n$ generated by the relations $\alpha_i+1\alpha_i = 0$ for $i = 0, \cdots, n-3$. Then, there is an equivalence of derived categories of modules

$$(1.1) \quad D(kA_n/I) \cong D(kA_n).$$

We aim to enhance the equivalence (1.1) and for this purpose the language of derivators (see [11]) turns out to be the most convenient one. Derivators, in fact, are meant to be a minimal extension of a derived category with a well behaved calculus of homotopy limits and colimits. The easiest example of a derivator consists of diagrams in a bicomplete category where, in fact, the left and right Kan extensions can always be computed pointwise (see [34]). The definition of derivators axiomatises this property which, for derived categories in Theorem 1.1, is guaranteed if we consider the derived category of diagram categories (coherent diagrams) instead of diagrams in the derived category (incoherent diagrams). The passage from incoherent diagrams to coherent ones can be studied via homotopical epimorphisms which we present in Section 4 as they are a fundamental tool to enhance and prove our main result (Theorem 6.1).

Specifically, in this article we work with stable derivators which, by definition, admit zero objects and whose homotopy pushout squares and homotopy pullback squares coincide. Stable derivators, introduced by Heller [21, 22] and Grothendieck [17], were then studied further by Franke [8], Keller [26] and Maltsiniotis [31, 32]. They are of general interest because, with the additional hypothesis of being strong (cf. [11, Definition 1.8]), the underlying category of a strong and stable derivator is always a triangulated category. They are interesting for us because, given a Grothendieck category $\mathcal{G}$, the derived category of diagrams in $\mathcal{G}$ forms a stable derivator.

In this article, we give a purely derivator-theoretic reformulation and proof of Theorem 1.1; this shows that the phenomenon occurs uniformly across stable derivators and is independent of the ring of coefficients $k$. Namely, given a stable derivator $\mathcal{D}$, we can refine the derived category $D(kA_n)$ with the stable derivator shifted by the free category of $A_n$ (Proposition 2.6), which we denote by $\mathcal{D}^A_n$. To generalize the derived category $D(kA_n/I)$, we consider the shifted derivator $\mathcal{D}^{\tilde{A}(n,2)}$ where $\tilde{A}(n,2)$ is a suitable poset (see picture 5.1). Then, to express the relations generating the ideal $I$, we introduce the new notion of strict full subderivator (Section 3) and we define $\mathcal{D}^{A(n,2)}$ as a particular strict full subderivator of $\mathcal{D}^{\tilde{A}(n,2)}$. We show how $\mathcal{D}^{A(n,2)}$ is an enhancement of $D(kA_n/I)$ and explain the problem of enhancing the derived category of a quiver with relations via strict full subderivators in Section 5.

Section 6 is dedicated to our main result:

Theorem 1.2. (Theorem 6.1) There is an equivalence of stable derivators

$$(1.2) \quad \mathcal{D}^{A(n,2)} \quad i^n \quad \mathcal{D}^A_n$$

where $i^n$ and $G^n$ are suitable compositions of left and right Kan extensions.

We observe that this result is related to [5, Corollary 9.15] where the different approach, involving hypercubes, leads to a different but equivalent definition of $\mathcal{D}^{A(n,2)}$.

While the first part of this article is dedicated to introduce and prove the main Theorem 6.1 which enhances a result in representation theory, Section 7 and Section 8 aim to explain how this result is closely related to homotopy theory so that, for this reason, it provides a bridge between these two areas. Indeed, looking at the equivalence (1.2), we observe that we are working with coherent chain complexes on the left hand side and filtered objects on the right hand side. This interpretation suggests a link with the $\infty$-Dold-Kan correspondence [30, Theorem 1.2.4.1] (Theorem 7.9). In fact, in this context, one can ask what is the relation between filtered objects and the $\infty$-category of coherent chain complexes [24, Definition 35.1] arising from the generalization of the classical Dold-Kan correspondence [30, Remark 1.2.4.3].
This question was already answered by Ariotta in [1, Theorem 4.7] and, since we know how to canonically associate a stable derivator to a stable $\infty$-category (Example 2.5), we observe that

**Proposition 1.3.** *(Proposition 7.12)* If we restrict to bounded chain complexes and bounded filtrations then Theorem 1.2 (Theorem 6.1) is the derivator-theoretical version of [1, Theorem 4.7] (Theorem 7.10).

In particular, we are able to see the relation between these results through the refinement of the mesh category described by Groth and Šťovíček in [14, Theorem 4.6] (Theorem 7.4). Finally, in Section 8 we see how (1.2) is an equivalence given by an universal tilting bimodule ([14, Section 10]). Namely, since every stable derivator is a closed module over the derivator of spectra [6] and thanks to the tilting theory for derivators [14, Theorem 8.5] (Theorem 8.3), we can write the functor $G^n$ in (1.2) as a canceling tensor product:

**Proposition 1.4.** *(Proposition 8.4)* The following equivalence of functors holds

$$G^n \cong T_n \otimes_{[A_n]} -$$

Here $T_n$ is a spectral bimodule which we also explain how to compute.

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## 2. Preliminaries on derivators

We recall some basics about the theory of derivators. More details can be found in [11, 10]. Let $\text{Cat}$ be the 2-category of small categories and $\text{CAT}$ the 2-category of large categories – we disregard logic and “size” issues here.

**Definition 2.1.** A 2-functor $\mathcal{D} : \text{Cat}^{\text{op}} \to \text{CAT}$ is called prederivator.

Heuristically, a prederivator can be viewed as a collection of “homotopy categories of diagrams”. Indeed, a typical example is the represented prederivator which is associated to a bicomplete category $C \in \text{CAT}$

$$\mathcal{D}_C : A \mapsto C^A$$

where $C^A$ is the category of functors from $A$ to $C$. Prederivators form a 2-category $\mathcal{PD}ER$ whose morphisms are pseudo-natural transformations and transformations of prederivators are modifications.

Given a prederivator $\mathcal{D}$ and a functor $u : A \to B$ in $\text{Cat}$, the application of $\mathcal{D}$ to $u$ gives two categories $\mathcal{D}(A), \mathcal{D}(B)$ and a functor

$$\mathcal{D}(u) = u^* : \mathcal{D}(A) \to \mathcal{D}(B)$$

which is called restriction. Similarly, given two functors $u, v : A \to B$ and a natural transformation $\alpha : u \to v$, by applying $\mathcal{D}$, we get an induced natural transformation

$$\alpha^* : u^* \to v^*$$

Let now $e \in \text{Cat}$ be the terminal category, i.e. the category with one object and identity morphism only. For an object $a \in A$, we denote by $a : e \to A$ the unique functor sending the object of $e$ to $a$. The restriction of such functor

$$a^* : \mathcal{D}(A) \to \mathcal{D}(e)$$
is called evaluation and it takes values in the underlying category \( \mathcal{D}(e) \). For two objects \( X, Y \) and a morphism \( f : X \to Y \) in \( \mathcal{D}(A) \) we denote by
\[
f_a : X_a \to Y_a
\]
their images under \( a^* \). Moreover, we call \( \mathcal{D}(A) \) the category of coherent \( A \)-shaped diagrams in \( \mathcal{D} \). Given an object \( X \in \mathcal{D}(A) \), we can define a map
\[
dia_A(X) : A \to \mathcal{D}(e)
\]
which assigns, to any element of \( A \), its evaluation functor. We call this map underlying incoherent diagram. Underlying coherent diagrams yield themselves a functor
\[
dia_A : \mathcal{D}(A) \to \mathcal{D}(e) \quad X \mapsto dia_A(X).
\]
The point of constructing derivators is that, in general, \( dia_A \) is not an equivalence: coherent diagrams cannot be determined by their underlying diagrams, even not up to isomorphism. The defining axioms of derivators aim to solve this problem by requiring (homotopy) completeness properties. In order to state them we need the following definition:

**Definition 2.2.** Let \( u : A \to B \in \text{Cat} \) and consider the restriction functor \( u^* : \mathcal{D}(B) \to \mathcal{D}(A) \). When they exist, we call the left adjoint of the restriction \( u_! : \mathcal{D}(A) \to \mathcal{D}(B) \) left Kan extension and the right adjoint \( u_* : \mathcal{D}(A) \to \mathcal{D}(B) \) right Kan extension.

We can observe that, when \( B = e \), we have a unique functor \( \pi = \pi_A : A \to e \) and the left Kan extension \( \pi_! = \text{colim}_A \) is a colimit functor and the right Kan extension is \( \pi_* = \text{lim}_A \) a limit functor. For the represented derivator (2.1) Kan extensions exist because they exist for bicomplete categories and, in particular, they can be calculated pointwise (see [34, X.3.1]). Having a formula to compute Kan extensions will be another important axiom of derivators. To give such formulas it is useful to represent canonical transformations through squares: let \( \mathcal{D} \) be a derivator, assume the Kan extensions always exist and consider natural transformation \( \alpha : up \to vq \in \text{Cat} \) then we can write \( \alpha \) as follows:

\[
\begin{array}{ccc}
D & \xrightarrow{p} & A \\
\downarrow{q} & & \downarrow{u} \\
B & \xrightarrow{v} & C
\end{array}
\]

Thanks to the adjunction unit \( \eta \) and counit \( \epsilon \), we get the so called canonical mate transformations

\[
\begin{align*}
q \cdot p^* & \xrightarrow{q \cdot p^*} q \cdot p^* \cdot u_! \xrightarrow{\alpha} q \cdot q^* \cdot v_! \xrightarrow{\epsilon} v^* \cdot u_! & \quad \text{and} \\
nu \cdot v_! & \xrightarrow{\nu \cdot v_!} p \cdot p^* \cdot v_! \xrightarrow{\alpha} p \cdot q^* \cdot v_! \xrightarrow{\epsilon} p \cdot q^*.
\end{align*}
\]

We say that the square (2.2) is homotopy exact if, for every prederivator \( \mathcal{D} \), the canonical mates (2.3) and (2.4) are isomorphisms. In particular, it is possible to show that (2.3) is an isomorphism if and only if this is the case for (2.4). Homotopy exact squares are compatible with pasting i.e. the passage to canonical mates (2.3) and (2.4) is functorial with respect to horizontal and vertical pasting. Consequently, horizontal and vertical pastings of homotopy exact squares are homotopy exact ([11, Lemma 1.14]).

**Definition 2.3.** Let \( u : A \to B \) be a functor in \( \text{Cat} \) and \( b \) an object in \( B \). The slice category \( (u/b) \) consists of pairs \( (a,f) \) where \( a \) is an object in \( A \) and \( f : u(a) \to b \) a morphism in \( B \).
A morphisms between two objects \((a, f)\) and \((a', f')\) is a morphism \(a \to a'\) in \(B\) making the following triangle commute:

\[
\begin{array}{ccc}
u(a) & \rightarrow & u(a') \\
\downarrow f & & \downarrow f' \\
\downarrow b & \nearrow & \\
& \end{array}
\]

The slice category \((b/u)\) is defined dually.

Relevant squares for us are **slice squares**:

\[
\begin{array}{ccc}
(p/A) & \rightarrow & A \\
\downarrow p & & \downarrow u \\
\downarrow e & \nearrow & \\
(b, u) & \rightarrow & B
\end{array}
\]

which come with canonical transformations \(u \circ p \to b \circ \pi\) and \(b \circ \pi \to u \circ q\). Here the functor \(p: (u/b) \to A\) is the projection onto the first component and \(q\) is defined dually. Other examples of homotopy exact squares can be found in [33] and [11, 10, 15].

We are now ready to give the definition of a derivator

**Definition 2.4.** A prederivator \(\mathcal{D}: \text{Cat}^{\text{op}} \to \text{CAT}\) is called **derivator** if it satisfies the following axioms:

(Der1) \(\mathcal{D}: \text{Cat}^{\text{op}} \to \text{CAT}\) takes coproducts to products, i.e., the canonical map \(\mathcal{D}(\coprod A_i) \to \prod \mathcal{D}(A_i)\) is an equivalence. In particular, \(\mathcal{D}(\emptyset)\) is equivalent to the terminal category.

(Der2) For any \(A \in \text{Cat}\), a morphism \(f: X \to Y\) in \(\mathcal{D}(A)\) is an isomorphism if and only if the morphisms \(f_a: X_a \to Y_a\), for any \(a \in A\), are isomorphisms in \(\mathcal{D}(e)\).

(Der3) For every functor \(u: A \to B\), his restriction has both the left \(u_l\) and right \(u_r\) Kan extension.

(Der4) For any functor \(u: A \to B\) and any \(b \in B\) the slice squares (2.5) are homotopy exact.

By axioms (Der1) and (Der3) we can see that \(\mathcal{D}(A)\) has small categorical coproducts and products, and, in particular, initial and terminal objects.

Derivators form a category a \(\text{DER}\) which is a full sub-2-category of \(\mathcal{PDER}\): morphisms of derivators are simply morphisms of underlying prederivators and, similarly, natural transformations are modifications.

Let us now look at some relevant examples of derivators.

**Example 2.5.**

1. The represented prederivator \(\mathcal{D}_{\text{C}}\) (2.1) is itself a derivator. Indeed, the Kan extensions \(u_l, u_r\) are then the ordinary Kan extension functors and the underlying category is isomorphic to \(\text{C}\) itself.

2. Let \(\text{C}\) be a Quillen model category (see e.g. [36, 23]) with weak equivalences \(\text{W}\). We can define the underlying **homotopy derivator** \(\mathcal{Ho}(\text{C})\) by formally inverting the pointwise weak equivalences

\[\mathcal{Ho}(\text{C}) : A \mapsto (\text{C}^A)((\text{W}^A)^{-1}).\]

The functors \(u_l, u_r\) are derived versions of the functors of \(y(\text{C})\) and the underlying category of \(\mathcal{Ho}(\text{C})\) is the homotopy category \(\text{Ho}(\text{C}) = \text{C}[\text{W}^{-1}]\) of \(\text{C}\).

3. Let \(\text{C}\) be a bicomplete ∞-category \(\text{C}\) in the sense of Joyal [25] and Lurie [29] (see [13] for an introduction). We can define the prederivator \(\mathcal{Ho}(\text{C})\) by

\[\mathcal{Ho}(\text{C}) : A \mapsto \text{Ho}(\text{C}^N(A))\]

where \(N(A)\) is the nerve of \(A\). A sketch proof that for bicomplete ∞-categories this yields the **homotopy derivator** of \(\text{C}\) can be found in [10].
The axioms of derivators allows us to define new derivators out of given ones.

**Proposition 2.6.** Let \( \mathcal{D}, \mathcal{E} \) be derivators, \( A, B \in \text{Cat} \). The following properties holds:

1. \( \mathcal{D}^B : A \mapsto \mathcal{D}(B \times A) \)
   is a derivator which we call **shifted derivator** (see [11, Theorem 1.25]).

2. \( \mathcal{D}^{\text{op}} : A \mapsto \mathcal{D}(A^{\text{op}})^{\text{op}} \)
   is a derivator which we call **opposite derivator**

3. the **product** of two derivators
   \( (\mathcal{D} \times \mathcal{E}) : A \mapsto \mathcal{D}(A) \times \mathcal{E}(A) \)
   is a derivator.

4. These derivators are suitably compatible with each other.

The shifted derivator is the derivator we will mostly work with because it allows to study the homotopy theory of coherent diagrams of shape \( B \) in \( \mathcal{D} \).

Here are some fundamental properties which we will need in the next sections.

**Proposition 2.7.**

1. Kan extensions along fully faithful functors are fully faithful. Namely, let \( u : A \rightarrow B \) be a fully faithful functor, then the unit \( \eta : \text{id} \rightarrow u^*u_! \) and the counit \( \epsilon : u^*u_* \rightarrow \text{id} \) are isomorphisms ([11, Proposition 1.20]).

2. Kan extensions and restrictions in unrelated variables commute. Namely, given two functors \( u : A \rightarrow B \) and \( v : C \rightarrow D \) then the commutative square

   \[
   \begin{array}{ccc}
   A \times C & \overset{u \times \text{id}}{\longrightarrow} & B \times C \\
   \downarrow{\text{id} \times v} & & \downarrow{\text{id} \times v} \\
   A \times D & \overset{u \times \text{id}}{\longrightarrow} & B \times D \\
   \end{array}
   \]

   is homotopy exact ([11, Proposition 2.5]).

3. Right adjoint functors are homotopy final. If \( u : A \rightarrow B \) is a right adjoint, then the square

   \[
   \begin{array}{ccc}
   A & \overset{u}{\longrightarrow} & B \\
   \downarrow{\pi_A} & & \downarrow{\pi_B} \\
   e & \overset{u^*}{\longrightarrow} & e \\
   \end{array}
   \]

   is homotopy exact i.e., the canonical mate \( \text{colim}_A u^* \rightarrow \text{colim}_B \) is an isomorphism ([11, Proposition 1.18]). In particular, if \( b \in B \) is a terminal object, then there is a canonical isomorphism \( b^* \cong \text{colim}_B \).

Since Kan extensions and restrictions in unrelated variables commute, we can have parametrized versions of restriction and Kan extension functors. Namely, for a derivator \( \mathcal{D} \) and a functor \( u : A \rightarrow B \), there are adjunctions of derivators

\[
(u_!, u^*): \mathcal{D}^B \rightleftarrows \mathcal{D}^A \\
(u^*, u_*): \mathcal{D}^B \rightleftarrows \mathcal{D}^A.
\]

which are defined internally to the 2-category \( \text{DER} [14, 11] \). Moreover, if \( u \) is fully faithful then \( u_!, u^* \) are. This observation is important because, as a consequence, they induce equivalences of derivators onto their essential images.

We have another important consequence of this proposition: given a map \( u : A \rightarrow B \) and a derivator \( \mathcal{D} \), thanks to the structure of a shifted derivator, the restriction functor \( u^* : \mathcal{D}(B) \rightarrow \mathcal{D}(A) \) is equal to the restriction \( u^* : \mathcal{D}(B \times e) \rightarrow \mathcal{D}(A \times e) \) and hence to \( u^* : \mathcal{D}^B(e) \rightarrow \mathcal{D}^A(e) \). This also holds for the left \( u_! \) and right \( u_* \) Kan extensions. We can then state the following proposition.
Proposition 2.8 ([11, Cor. 2.6]). Let $\mathcal{D}$ be a derivator, $C$ a small category and let $u : A \to B$ be a fully faithful functor. An object $X \in \mathcal{D}^C(B)$ lies in the essential image of $u^! : \mathcal{D}^C(A) \to \mathcal{D}^C(B)$ if and only if $X_c$ lies in the essential image of $u^! : \mathcal{D}(A) \to \mathcal{D}(B)$ for all $c \in C$.

Finally, given a (pre)derivator $\mathcal{D}$, we write $X \in \mathcal{D}$ to indicate that there is a small category $A$ such that $X \in \mathcal{D}(A)$.

2.9. Stable derivators. In this article we aim to enhance an equivalence of triangulated categories. Higher categorical enhancements of triangulated categories, have the additional properties of being stable, which means that they in some sense behave like abelian categories. In this subsection we then recall some basics about stable derivators.

Definition 2.10. A derivator $\mathcal{D}$ is pointed if $\mathcal{D}(e)$ has a zero object.

Proposition 2.11. If $\mathcal{D}$ is pointed then so are the shifted derivators $\mathcal{D}^B$ and its opposite $\mathcal{D}^{\text{op}}$.

In particular, for any $A \in \text{Cat}$, $\mathcal{D}(A)$ have zero objects which are preserved by restriction and Kan extension functors. If $\mathcal{D}$ is a pointed derivator, some inclusion functors of small categories are especially interesting because we can describe the image of their Kan extensions.

Definition 2.12. A functor $u : A \to B$ is a sieve if it is fully faithful and if for any morphism $b \to u(a)$ in $B$ there exists an $a' \in A$ with $u(a') = b$. A cosieve is defined dually.

Note that, by Proposition 2.7, if $u : A \to B$ is a (co)sieve then the Kan extensions $u_!, u_*$ are fully faithful.

Sieves and cosieves are what we need to extend diagrams by zero objects.

Proposition 2.13 ([11, Prop. 1.23]). Let $\mathcal{D}$ be a pointed derivator.

1. Let $u : A \to B$ be a cosieve. Then the homotopy left Kan extension $u_!$ is fully faithful and $X \in \mathcal{D}(B)$ lies in the essential image of $u_!$ if and only if $X_b$ is zero for all $b \in B - u(A)$.

2. Let $u : A \to B$ be a sieve. Then the homotopy right Kan extension $u_*$ is fully faithful and $X \in \mathcal{D}(B)$ lies in the essential image of $u_*$ if and only if $X_b$ is zero for all $b \in B - u(A)$.

In particular, when $u$ is cosieve we call the functor $u_!$ left extension by zero and, when $u$ is a sieve, we call $u_*$ right extension by zero.

Stable derivators are pointed derivators with an additional property. Let $[n]$ be the poset $(0 < \cdots < n)$ considered as a category. The commutative square $\square = [1] \times [1]$, 

\[
\begin{array}{ccc}
(0, 1) & \longrightarrow & (1, 1) \\
\uparrow & & \uparrow \\
(0, 0) & \longrightarrow & (1, 0)
\end{array}
\]

(2.8)

comes with full subcategories $i_* : \downarrow \to \square$ and $i^* : \triangledown \to \square$ obtained by removing the terminal object and the initial object, respectively. Since both inclusions are fully faithful, so are $(i_*)_! : \mathcal{D} \to \mathcal{D}^\square$ and $(i^*)_* : \mathcal{D}^\triangledown \to \mathcal{D}^\square$.

Definition 2.14. A square $X \in \mathcal{D}^\square$ is cocartesian if it lies in the essential image of $(i_*)_!$ and it is cartesian if it lies in the essential image of $(i^*)_*$. A square which is both cartesian and cocartesian is called bicartesian.

In the proof of the Theorem 6.1 we will often need to check whether a square contained in a larger diagram is (co)cartesian. This will be possible thanks to the following lemma.

Proposition 2.15 ([11, Prop. 3.10]). Let $i : \square \to B$ be a square in $B$ and let $u : A \to B$ be a functor.
(1) Assume that the induced functor \( \tilde{i} : (B - i(1,1))_{i(1,1)} \) has a left adjoint and that \( i(1,1) \) does not lie in the image of \( u \). Then for all \( X = u(Y) \in \mathcal{D}(B), Y \in \mathcal{D}(A) \), the induced square \( i^*(X) \) is cocartesian.

(2) Assume that the induced functor \( \tilde{i} : (B - i(0,0))_{i(0,0)} \) has a right adjoint and that \( i(0,0) \) does not lie in the image of \( u \). Then for all \( X = u_*(Y) \in \mathcal{D}(B), Y \in \mathcal{D}(A) \), the induced square \( i^*(X) \) is cartesian.

Remark 2.16. By Proposition 2.7, we have that if \( u : A \to B \) is a fully faithful functor, then the unit \( \eta : id \to u^*u_! \) and the counit \( \epsilon : u_*u^* \to id \) are isomorphisms. Then an object \( X \in \mathcal{D}(B) \) belongs to the essential image of \( u_* \) if and only if the counit is an isomorphism. Dually we can state the same property for \( u^* \). As a consequence, by Proposition 2.15, we have that the essential image of \( u_* \) (resp. \( u^* \)) consists of all the objects \( X \) such that \( i^*(X) \) is cocartesian (resp. cartesian).

Definition 2.17. A pointed derivator is stable if the classes of cartesian squares and co-cartesian squares coincide. These squares are then called bicartesian.

Different characterizations of stable derivators are given in [10, Theorem 7.1] and [15, Corollary 8.13].

Proposition 2.18. (1) If \( \mathcal{D} \) is stable derivator then so are the shifted and opposite derivators \( \mathcal{D}^B \) and \( \mathcal{D}^{op} \) ([11, Proposition 4.3]).

(2) If \( \mathcal{D} \) and \( \mathcal{E} \) are stable derivators, then so is the product \( \mathcal{D} \times \mathcal{E} \).

Example 2.19. (1) Let \( \mathcal{G} \) be a Grothendieck category. We have a stable combinatorial model category for complexes over a Grothendieck category and quasi-iso’s as weak equivalences [37, Example 3.11]. Recall that the derived category \( D(\mathcal{G}) \) is the localization of the category of chain complexes at the class of quasi-isomorphisms. Then, by Example 2.5 and [11, Proposition 1.30] the derivator associated to a Grothendieck category is

\[ \mathcal{D} : A \mapsto D(\mathcal{G}^A) \]

and since the model category is stable the derivator is. In particular, we can choose \( \mathcal{G} \) the module category of an algebra over a quiver and we can use this example to enhance Theorem 1.1.

(2) There are many Quillen equivalent stable model categories of spectra such that the homotopy category is the stable homotopy category \( \text{SHC} \). The homotopy derivator \( \mathcal{S}p \) associated to any of these model categories is stable. We will refer to it as the derivator of spectra, it will play an essential role in Section 8.

(3) Homotopy derivators of stable \( \infty \)-categories and stable model categories are stable [10].

More examples can be found in [15].

The reason why we are interested in stable derivators is the following proposition.

Theorem 2.20 ([11, Theorem 4.16]). If \( \mathcal{D} \) is a strong and stable derivator then its underlying category \( \mathcal{D}(c) \) is triangulated.

The additional hypothesis of being strong requires that, for any \( A \in \text{Cat} \), the partial underlying diagram functor

\[ \mathcal{D}(A \times [1]) \to D(A)^{[1]} \]

is full and essentially surjective. The example of derivators we refer to in this paper, such as homotopy derivators of model categories and \( \infty \)-categories, are strong as they are represented derivators. This concept is then not essential for the purpose of this paper.
We can observe that we need to work with stable derivator in order to get a triangulated underlying category by looking at a bicartesian square $X \in \mathcal{D}_{\square}$. Indeed, if $X_{(0,1)}$ is the zero object then

$X_{(0,0)} \to X_{(1,0)} \to X_{(1,1)}$

is a triangle in $\mathcal{D}(e)$ and, in particular, $X_{(1,1)}$ is the cone of the morphism $X_{(0,0)} \to X_{(1,0)}$.

We conclude this subsection with the following definition which we will use to state Theorem 6.1.

**Definition 2.21.** A morphism of derivators is **right exact** if it preserves initial objects and pushouts. Dually we can define **left exact** morphism. A morphism which is both right and left exact is called **exact**.

For stable derivators these three notions clearly coincide.

2.22. **Total cofiber construction.** We now want to recall the total cofiber construction because this is the key idea behind the proof of our main result Theorem 6.1. More details can be found in [12].

**Construction 2.23.** Consider $\downarrow = \square - \{(1,1)\}$ the full subcategory of the square obtained by removing the final object and consider the category $\overline{K}_{1,2}^3$ in the diagram below. This is the cocone on the square obtained by adjoining a new terminal object $(2,1)$,

\[
\begin{array}{ccc}
(2,1) \\
| \downarrow \\
(0,1) & \longrightarrow & (1,1) \\
| \\
(0,0) & \longrightarrow & (1,0)
\end{array}
\]

Associated to this category there are the fully faithful inclusions of the source and target square

\[
(2.9) \quad s = s_\downarrow : \square \to \overline{K}_{1,2}^3 \quad \text{and} \quad t = t_\downarrow : \square \to \overline{K}_{1,2}^3.
\]

where the image of $s$ is given by all objects except $(2,1)$ and the image of $t$ is given by all objects except $(1,1)$.

**Proposition 2.24 ([12, Prop. 2.2]).** Let $\mathcal{D}$ be a derivator and let $s, t : \square \to \overline{K}_{1,2}^3$ be the inclusions of the source and target squares.

1. The morphism $t_! : \mathcal{D}_\square \to \mathcal{D}_{\overline{K}_{1,2}^3}$ is fully faithful and $Y \in \mathcal{D}_{\overline{K}_{1,2}^3}$ lies in the essential image of $t_!$ if and only if the source square $s^*Y$ is cocartesian.

2. A square $X \in \mathcal{D}_{\square}$ is cocartesian if and only if the following canonical comparison map is an isomorphism,

\[
(2.10) \quad \text{can} = \text{can}(X) : t_!(X)_{(1,1)} \to t_!(X)_{(2,1)}.
\]

**Definition 2.25. ([12, Def. 2.4])** Let $\mathcal{D}$ be a pointed derivator. The **total cofiber** of $X \in \mathcal{D}_\square$ is the cone of the comparison map (2.10). In formulas we set

\[
t_{\text{cof}}(X) = C(\text{can}(X)) \in \mathcal{D}
\]

where $C : \mathcal{D}^{[1]} \to \mathcal{D}$ is the cone morphism (see [11, Subsection 3.3]). The definition of the **total fiber** $t_{\text{fib}}(X) \in \mathcal{D}$ is dual.
3. STRICT FULL SUBDERIVATORS

We dedicate a section to strict full subderivators as they are among the fundamental tools through which we prove the main Theorem and, in particular, they allow us to enhance the derived category $D(kA_n/I)$ (cf. 1.1) at level of derivators (Example 2.19).

First we want to recall the already existing definition of full subderivator.

**Definition 3.1.** Let $\mathcal{D}$ be a derivator, a **full subprederivator** $\mathcal{D}'$ of $\mathcal{D}$ is a full sub-2-functor of $\mathcal{D}$ i.e. it is a 2-functor $\mathcal{D}' : \text{Cat}^{\text{op}} \to \text{CAT}$ such that, for any $I \in \text{Cat}$, $\mathcal{D}'(I) \subseteq \mathcal{D}(I)$ is a full subcategory. In particular, if $u : A \to B$ is a functor, then the restriction $(u)^* : \mathcal{D}'(B) \to \mathcal{D}'(A)$ is given by $(u^*)_{|\mathcal{D}'(A)}$ that implies the following diagram to be commutative

$$
\begin{array}{ccc}
\mathcal{D}'(A) & \xrightarrow{i_A} & \mathcal{D}(A) \\
(u)^* \downarrow & & u^* \downarrow \\
\mathcal{D}'(B) & \xrightarrow{i_B} & \mathcal{D}(B)
\end{array}
$$

where $i$ is the inclusion functor.

From this definition naturally follows the definition of full subderivator, namely a full subprederivator which is also a derivator. Given $\mathcal{D}'$ full subderivator of $\mathcal{D}$, by (Der3), for every restriction functor $(u)^*$, there exist both the left $(u)^!$ and the right $(u)^*$ Kan extension. What cannot be guaranteed is that these Kan extensions are compatible with the Kan extensions of $\mathcal{D}$. In order to obtain this compatibility condition we then introduce the new notion of a **strict full subderivator** i.e. a full subderivator which preserves Kan extensions. This notion must not be confused with the notion of strictly full subderivator.

**Definition 3.2. A strict full subderivator** $\mathcal{D}'$ of a derivator $\mathcal{D}$ is a full subprederivator such that it satisfies (Der1), (Der3) and moreover we require that the left and right Kan extensions of $\mathcal{D}$ restricted to $\mathcal{D}'$ i.e. if $u : A \to B$ is a functor, then $(u)^!$, $(u)^* : \mathcal{D}'(A) \to \mathcal{D}'(B)$ are given by $u^!_{|\mathcal{D}'(A)}$, $u^*_{|\mathcal{D}'(A)}$ respectively, up to the canonical equivalences arising from the adjunctions. Then, we have the following commutative diagram

$$
\begin{array}{ccc}
\mathcal{D}'(A) & \xrightarrow{i_A} & \mathcal{D}(A) \\
(u)^! \downarrow & & u^! \downarrow \\
\mathcal{D}'(B) & \xrightarrow{i_B} & \mathcal{D}(B)
\end{array}
$$

where $i$ is the inclusion functor, the restrictions strictly commutes and the Kan extensions commutes up to the canonical equivalences arising from the adjunctions.

**Proposition 3.3.** Any strict full subderivator $\mathcal{D}' \subseteq \mathcal{D}$ is a derivator.

**Proof.** (Der2) Consider a small category $A \in \text{Cat}$, and let $f : X \to Y$ be a morphism in $\mathcal{D}'(A)$. If $f$ is an isomorphism then it is preserved by the evaluation functor $(a^*)^! : \mathcal{D}'(A) \to \mathcal{D}'(e)$. Hence $(a^*)^!(f) = f_a : X_a \to Y_a$ is an isomorphism. On the other hand, for any $a \in A$, by definition we have the following commutative diagram

$$
\begin{array}{ccc}
\mathcal{D}'(A) & \xrightarrow{i_A} & \mathcal{D}(A) \\
(a^*)^! \downarrow & & a^! \downarrow \\
\mathcal{D}'(e) & \xrightarrow{i_e} & \mathcal{D}(e)
\end{array}
$$

If $(a^*)^!(f)$ is an isomorphism then also $i_a(a^*)^!(f) = a^!(i_A(f))$ is an isomorphism. Being a derivator, $\mathcal{D}$ satisfies (Der2) then $i_A(f)$ and hence $f$ are isomorphisms.

(Der4) By the property (Der4) of the derivator $\mathcal{D}$ we have the isomorphism $\pi_{fp^*} \sim b^*u_!$. Then, composing with the inclusion functor, we get the map $\pi_{fp^*i_A} \sim b^*u_!i_A$ which
is again an isomorphism. We have the following diagram

\[
\begin{array}{ccc}
\pi p^* i_A & \sim & b^* u_I i_A \\
\downarrow & & \downarrow \\
\pi i_{u/b}(p^*)' & \sim & b^*_I (u)' \\
\downarrow & & \downarrow \\
i_e(\pi)'(p^*)' & \sim & i_e(b^*)'(u)'
\end{array}
\]

where all the vertical arrows are equivalences. The last isomorphism implies the desired one \((\pi)'(p^*)' \to (b^*)'(u)\). \hfill \Box

In this paper we will work with particular strict full subderivators that we can describe as follows.

**Definition 3.4.** Given a stable derivator \( \mathcal{D} \) and a small category \( A \in \text{Cat} \), we define what will be shown in Proposition 3.6 to be strict full subderivators of the shifted derivator \( \mathcal{D}^A \).

1. The strict full subderivator spanned by all the coherent diagrams where we require an arrow to be an isomorphism. Namely, consider the functor \( k : e \to [1] \) that chooses the initial object and the inclusion functor \( j : [1] \to A \) that chooses a morphism in \( A \). For any \( I \in \text{Cat} \) we define
   \[ \mathcal{E}^\text{is}_A(I) := \{ X \in \mathcal{D}^A(I) = \mathcal{D}(A \times I) : (j \times \text{id})^*(X) \in \text{essIm}(k \times \text{id})! \} \].
2. The strict full subderivator spanned by all the coherent diagrams which vanish in one position. Namely, consider the canonical functor \( k : \emptyset \to e \) and the choice of an object \( j : e \to A \). For any \( I \in \text{Cat} \) we define
   \[ \mathcal{E}^\text{va}_A(I) := \{ X \in \mathcal{D}^A(I) = \mathcal{D}(A \times I) : (j \times \text{id})^*(X) \in \text{essIm}(k \times \text{id})! \} \].
3. The strict full subderivator spanned by all the coherent diagrams where we require a square to be bicartesian. Namely, consider the incusion functor \( k : \square \to \square \) and the choice of a commutative square \( j : \square \to A \). For any \( I \in \text{Cat} \) we define
   \[ \mathcal{E}^\text{bi}_A(I) := \{ X \in \mathcal{D}^A(I) = \mathcal{D}(A \times I) : (j \times \text{id})^*(X) \in \text{essIm}(k \times \text{id})! \} \].

**Definition 3.5.** A stable strict full subderivator is a strict full subderivator that is stable as a derivator.

**Proposition 3.6.** Let \( \mathcal{D} \) be a stable derivator, \( A \in \text{Cat} \) then \( \mathcal{E}^\text{is}_A \), \( \mathcal{E}^\text{va}_A \) and \( \mathcal{E}^\text{bi}_A \) are stable strict full subderivators of \( \mathcal{D}^A \).

**Proof.** We prove the statement only for \( \mathcal{E}^\text{is}_A \) since the other cases are analogous. For simplicity we will denote \( \mathcal{E}^\text{is}_A \) by \( \mathcal{E} \), \( (\text{id}_A \times u) \) by \( u_A \) for any \( A \in \text{Cat} \) and \( (j \times \text{id}_J) \), \( (k \times \text{id}_I) \) respectively by \( j_I \), \( k_I \). First we want to prove that \( \mathcal{E} \) is a subprederivator i.e. given a functor \( u : J \to I \), we want to verify that if \( X \in \mathcal{E}(I) \) then \( u^*_A(X) \in \mathcal{E}(J) \). We have the following commutative diagram.

\[
\begin{array}{ccc}
\mathcal{D}^I(e) & \overset{u^*_I}{\longrightarrow} & \mathcal{D}^J(e) \\
\downarrow^{k_I} & & \downarrow^{k_J} \\
\mathcal{D}^I([1]) & \overset{u^*_I{[1]}}{\longrightarrow} & \mathcal{D}^J([1]) \\
\downarrow^{j_I} & & \downarrow^{j_J} \\
\mathcal{D}^I(A) & \overset{u^*_I}{\longrightarrow} & \mathcal{D}^J(A)
\end{array}
\]
where we identify \( \mathcal{D}^B(A) \) with \( \mathcal{D}A^B(A) \) via the natural isomorphism \( B \times A \cong A \times B \) for any \( A, B \in \text{Cat} \). By definition we can restate the thesis saying that if \( X \in \mathcal{D}^I(e) \) and \( j^*_I(X) \cong k^I_l(Y) \) then \( u^*_A(X) \in \mathcal{D}^I(A) \) is such that \( j^*_J(u^*_A(X)) \in \text{ess Im} k^J_l \).

This is guaranteed by the following equivalences

\[
j^*_J(u^*_A(X)) = u^*_A[j^*_I(X) \cong k^I_l(Y) \cong k^J_l u^*_A(Y)].
\]

To prove (Der1), it is enough to consider the following commutative diagram

\[
\begin{array}{ccc}
\mathcal{D}^I(e) & \xrightarrow{F^e} & \prod_i \mathcal{D}^I_i(e) \\
\downarrow{k^I_l} & & \downarrow{k^I_l} \\
\mathcal{D}^I([1]) & \xrightarrow{F^I} & \prod_i \mathcal{D}^I_i([1]) \\
\uparrow{j^*_I} & & \uparrow{j^*_I} \\
\mathcal{D}^I(A) & \xrightarrow{F^A} & \prod_i \mathcal{D}^I_i(A)
\end{array}
\]

where \( F^e, F^I, \) and \( F^A \) are equivalences coming from the property (Der1) for the derivator \( \mathcal{D} \). Then, by diagram chasing as before, we can conclude. (Der3) follows by proving that the Kan extensions are the ones in \( \mathcal{D} \) restricted to \( \mathcal{E} \). It can be verified again by diagram chasing, thank to Proposition 2.7.

Finally, since the Kan extensions are the restriction of the ones in \( \mathcal{D}^A \), it is straightforward that \( \mathcal{E} \) is a stable derivator.

In order to write the derivator whose underlying category is \( D(kA_n/I) \) (see (1.1)) and to prove Theorem 6.1 we will need to define strict full subderivators with more than one vanishing position, isomorphism or bicartesian square. The following proposition suggests us that it will be enough to define them as intersections.

**Proposition 3.7.** Let \( \mathcal{D} \) be a stable derivator, \( A \in \text{Cat} \). Intersections of strict full subderivators \( \mathcal{E}^\text{iso}_A, \mathcal{E}^\text{bi}_A \) and \( \mathcal{E}^\text{iso}_A \) of \( \mathcal{D}^A \) are still strict full subderivators of \( \mathcal{D}^A \).

**Proof.** The strategy is the same as in the proof of Proposition 3.6. 

### 4. Homotopical epimorphisms

Let \( \mathcal{D} \) be a derivator and, as in the previous section, consider the functor

\[
k : e \to [1] \in \text{Cat}
\]

which chooses the initial object in \([1]\). By definition of the left Kan extension, an **isomorphism in the underlying category** is an element \( Y \) in the essential image of \( k_l : \mathcal{D}(e) \to \mathcal{D}([1]) \).

Consider a small category \( A \), a coherent diagram \( X \in \mathcal{D}(A) \), and the functor

\[
j : [1] \to A
\]
which chooses a morphism in \( A \). Analogously to the previous case, the map we choosed through \( j : [1] \to A \) is an isomorphism in the coherent diagram if \( j^*(X) \) is in the essential image of the left Kan extension \( k_! : \mathcal{D}(e) \to \mathcal{D}([1]) \).

Unfortunately, being an isomorphism in the coherent diagram does not imply that the morphism we choosed through \( j : [1] \to A \) is an isomorphism in the (incoherent) diagram \( A \).

In order to detect the cases where we can lift the isomorphisms in coherent diagrams to the incoherent ones, we need homotopical epimorphisms.

**Definition 4.1.** ([16, Def. 3.8]) A functor \( u : A \to B \) is a homotopical epimorphism if for every derivator \( \mathcal{D} \) the restriction functor \( u^* : \mathcal{D}(B) \to \mathcal{D}(A) \) is fully faithful.

**Remark 4.2.** Note that, by Proposition 2.7, \( u : A \to B \) is a homotopical epimorphism if and only if the square

\[
\begin{array}{ccc}
A & \xrightarrow{u} & B \\
\downarrow & & \downarrow \text{id} \\
B & \xrightarrow{\text{id}} & B \\
\end{array}
\]

is homotopy exact.

To understand better the problem of lifting the isomorphisms we look at an example. Consider the following small category

\[
\begin{array}{ccc}
a & \xrightarrow{c} & b \\
\downarrow & & \downarrow \\
c & \xrightarrow{d} & d
\end{array}
\]

we call it \( \tilde{B} \). Let \( \mathcal{D}^B \) be the strict full subderivator of \( \mathcal{D}^\tilde{B} \) where we require the arrow \( a \to c \) to be an isomorphism. This isomorphism is then an isomorphism in the coherent diagram. Being able to lift this isomorphism to the incoherent diagram amounts to prove that the map

\[
\begin{array}{ccc}
a & \xrightarrow{\eta} & b \\
\downarrow & & \downarrow \\
c & \xrightarrow{d} & d
\end{array}
\]

is an homotopical epimorphism. We denote the target category \( B_{a \sim c} \).

To understand whether a given functor is or not a homotopical epimorphism there are some criteria which we now illustrate. Homotopical epimorphisms and the following criteria are among the fundamental tools which will allow us to prove Theorem 6.1.

**Proposition 4.3** ([16, Prop. 8.2]). Let \( u : A \to B \) be essentially surjective, let \( \mathcal{D} \) be a derivator, and let \( u^* : \mathcal{D}(B) \to \mathcal{D}(A) \) be the restriction morphism. Let us assume further that \( \mathcal{E} \subseteq \mathcal{D}^A \) is a full subderivator such that

1. the essential image \( \text{essIm}(u^*) \) lies in \( \mathcal{E} \), i.e., \( \text{essIm}(u^*) \subseteq \mathcal{E} \subseteq \mathcal{D}^A \), and
2. the unit \( \eta : X \to u^* u_! X \) is an isomorphism for all \( X \in \mathcal{E} \).

Then \( u^* : \mathcal{D}(B) \to \mathcal{D}(A) \) is fully faithful and \( \text{essIm}(u^*) = \mathcal{E} \). In particular, \( \mathcal{E} \) is a derivator.

**Construction 4.4.** ([16, Construction. 8.4]) Let \( \mathcal{D} \) be a derivator, \( A \in \text{Cat} \) and let \( a \in A \). Associated to the square

\[
\begin{array}{ccc}
e & \xrightarrow{a} & A \\
\downarrow & & \downarrow \pi_A \\
e & \xrightarrow{\pi_A} & e
\end{array}
\]
there is the canonical mate
\[(4.1)\quad a^* \to \text{colim}_A.\]

As a special case relevant in later applications, given a functor \( u: A \to B \) and \( a \in A \) we consider the functor \( p: (u/ua) \to A \). Whiskering the mate \((4.1)\) in the case of \((a, \text{id}: ua \to ua) \in (u/ua)\) with \( p^* \) we obtain a canonical map
\[(4.2)\quad a^* = (a, \text{id}_{ua})^* p^* \to \text{colim}_{(u/ua)} p^*.\]

**Lemma 4.5** ([16, Lemma. 8.7]). Let \( D \) be a derivator, \( u: A \to B \), and \( a \in A \). The component of the unit \( a^* \eta: a^* \to a^* u^* u! \) is isomorphic to \( a^* \to \text{colim}_A u^* \) (4.2). In particular, \( \eta_a \) is an isomorphism if and only if this is the case for (4.2).

We will later apply the previous lemma in situations in which \( u: A \to B \) is an homotopy final functor and the slice category admits a terminal object. For this purpose we collect the following result.

**Lemma 4.6** ([16, Lemma. 8.8]). Let \( u: A \to B \) be a homotopy final functor and let \( a \in A \).

1. The map \( u(a)^* \to \text{colim}_B (4.1) \) is naturally isomorphic to \( a^* u^* \to \text{colim}_A u^* \), the whiskering of an instance of (4.1) with \( u^* \).
2. If \( A \) admits a terminal object \( \infty \), then the map \( a^* \to \text{colim}_A (4.1) \) is naturally isomorphic to \( a^* \to \infty^* \).

We are now ready to study the functor \( v \)

\[
\begin{array}{ccc}
 a & \xrightarrow{\sim} & b \\
\downarrow & & \downarrow \\
 c & \xrightarrow{\sim} & d
\end{array} \quad \xrightarrow{v} \quad \begin{array}{ccc}
 a & \xrightarrow{\sim} & b \\
\downarrow & & \downarrow \\
 c & \xrightarrow{\sim} & d
\end{array}
\]

This is a surjective functor so, in particular, it is essentially surjective on objects and \( \text{ess\,Im}(v^*) \subseteq D^B \).

By Proposition 4.3, we then only need to check that the unit \( \eta: X \to v^* v_! X \) is an isomorphism for all \( X \in D^B \).

Thanks to (Der2), it suffices to check that the map
\[
i^* \eta: i^* X \to i^* v^* v_! X
\]
is an isomorphism for every object \( i \in \widetilde{B} \). By Lemma 4.5 we equivalently show that the map
\[(4.3)\quad i^* X \to \text{colim}_{(v/v(i))} p^* X
\]
is an isomorphism for every \( i \in \widetilde{B} \). Due to the shape of \( B_{w=c} \) the interesting case we should look at is when \( i = d \). Indeed, the slice category \( v/v(d) \) looks as \( \widetilde{B} \) and it is straightforward to see that the colimit of such a category is not, in general, isomorphic to the evaluation in \( d \). Consequently, \( v \) is not an homotopical epimorphism and it shows then a case where, even if there is an isomorphism in the coherent diagram, it is not possible to lift it to the incoherent one.

5. Enhancement of a quiver with relations

The main goal of this paper is to enhance Theorem 1.1. While we can easily get \( D(kA_n) \) as underlying category of the shifted derivator \( D^{An} \), the derivator whose underlying category is \( D(kA_n/I) \) is not straightforwardly defined. For this reason we introduced strict full subderivators (Section 3). In particular, they allow to enhance in a direct and natural way the relations we have on the quiver.
Definition 5.1. Let $\mathcal{D}$ be a stable derivator and fix $n \in \mathbb{N}$, we consider the following subposet of $[n-2] \times [n-2]$:

\[
\begin{array}{c}
(0,0) \rightarrow (1,0) \\
(0,1) \rightarrow (1,2) \\
(2,1) \rightarrow (3,2) \\
(2,3) \rightarrow (n-2, n-3) \\
(n-3, n-2) \rightarrow (n-2, n-2)
\end{array}
\]

In particular, we have that every square commutes. We call this shape $\bar{A}(n, 2)$ and we denote by $\mathcal{D}^{\bar{A}(n, 2)}$ the strict full subderivator of $\mathcal{D}^{\bar{A}(n, 2)}$ spanned by all coherent diagrams of the shape $\bar{A}(n, 2)$ which vanish at $(i, i+1)$, for $i = 0, \cdots, n - 3$ (cf. 3.4).

Proposition 5.2. Let $\mathcal{D}$ be a stable derivator, $\mathcal{D}^{\bar{A}(n, 2)}$ is the derivator enhancement of $D(kA_n/I)$

Proof. We recall that, by Example 2.19, the derivator of the Grothendieck category $\text{Mod}(kA_n/I)$ is the homotopy derivator associated to the combinatorial model category for complexes over $\text{Mod}(kA_n/I)$. Its underlying category is $D(kA_n/I)$. In particular, we want to show that $D(kA_n/I)$ is equivalent to the underlying category of $\mathcal{D}^{\bar{A}(n, 2)}$ which is the derived category $D(k^{A(n, 2)})$,

here $k^{A(n, 2)}$ is the category of functors from $\bar{A}(n, 2)$ to $\text{Mod} k$ satisfying the conditions coming from the structure of the strict full subderivator. Namely, the vanishing conditions on $\mathcal{D}^{\bar{A}(n, 2)}$ imply that, in correspondence of the objects $(i, i+1)$ for $i = 0, \cdots, n - 3$ we have acyclic complexes in the underlying category. These acyclic complexes will be denoted by the letters $A_i$ for $i = 1, \cdots, n - 2$. Let us now define the quasi-isomorphisms which give the desired derived equivalence.

Given an object

\[
\begin{array}{c}
A_{n-2} \rightarrow Z \\
A_{n-3} \rightarrow Y \\
A_2 \rightarrow X \\
A_1 \rightarrow W \\
U \rightarrow V
\end{array}
\]

in $D(k^{A(n, 2)})$, via the obvious projections it is trivially quasi isomorphic to
where $M_1 = \begin{pmatrix} l & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, $M_2 = \begin{pmatrix} p \\ u \end{pmatrix}$, $M_3 = \begin{pmatrix} q & 0 \end{pmatrix}$, $M_4 = \begin{pmatrix} g & 0 \\ q & 0 \end{pmatrix}$, $M_5 = \begin{pmatrix} r \\ v \end{pmatrix}$, $M_6 = \begin{pmatrix} s \\ 0 \end{pmatrix}$, $M_7 = \begin{pmatrix} h \\ s & 0 \end{pmatrix}$, $M_8 = \begin{pmatrix} z \\ 1 \end{pmatrix}$, $M_9 = \begin{pmatrix} l & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. We can observe that (5.3) is also quasi isomorphic to

where $N_1 = \begin{pmatrix} g & -p \\ q & -u \end{pmatrix}$, $N_2 = \begin{pmatrix} h & -r \\ s & -v \end{pmatrix}$, $N_3 = \begin{pmatrix} l & -z \end{pmatrix}$. The quasi isomorphism is given by identities where possible, zero maps to the zero objects and the following morphisms:

It is now easy to check that if we consider the non zero objects in (5.4) and the maps between them, we get an object in $D(kA_n/I)$. □

The structure of strict full subderivators was the key to construct the enhancement for the derived category of $A_n$ with relations given by the ideal $I$. However, whether this structure can enhance a generic quiver with relations is still an open question. Indeed, the most intuitive
poset we can think about as enhancement of $D(kA_n/I)$ is without the arrows between the acyclic objects, however, they play a fundamental role from the homotopy theory point of view. We now explain better why through an example on $A(n, 4)$. Suppose that those arrows are not necessary, then the restriction of the following inclusion

\[ \xymatrix{ (0, 1) \ar[r]^{u} \ar[d] & (2, 1) \ar[d] \cr (0, 0) \ar[r] & (1, 0) } \]

\[ \tilde{A}(1, 2, -) \quad \tilde{A}(4, 2) \]

should give an equivalence of derivators between $D^{A(4, 2, -)}$ and $D^{A(4, 2)}$, where $D^{A(4, 2, -)}$ is a strict full subderivator of $D^{A(4, 2, -)}$ with same vanishing condition as $D^{A(4, 2)}$. But this does not happen: given an object $X \in D^{A(4, 2, -)}$, we can compute $(u_{1}(X))_{(1, 2)}$ and see that it is different from the zero object. In fact, by (Der4), computing $(u_{1}(X))_{(1, 2)}$ amounts to computing the colimit of the following category.

\[ O_2 \]
\[ O_1 \]
\[ U \quad V \]

If $U$ is not the zero object then it is easy to check that this colimit is also different from zero. However, it should be zero by definition of the strict full subderivator $D^{A(4, 2)}$.

In conclusion, the absence of the arrows between the acyclic objects modifies the homotopy (co)limits of the poset in a way which does not allow to enhance $D(kA_n/I)$ anymore.

6. Main theorem and proof

In this section we state and prove the main result of this article.

Observe that the free category of $A_n$ is isomorphic to $[n - 1]$.

**Theorem 6.1.** Let $\mathcal{D}$ be a stable derivator. Then, for any integer $n \geq 3$, there exists an equivalence of stable derivators

\[ \xymatrix{ D^{A(n, 2)} \ar[r]^{i^n} & D^{A_n} \ar[l]_{G^n} } \]

which is natural with respect to exact morphisms.

**Proof.** The proof is divided in 7 steps and is based only on Kan extensions and restrictions of inclusion functors between posets which, for the case $n = 4$, are all illustrated in Example 6.2. We prove this result by induction: we start with the case $n = 3$.

**First step: Case $n = 3$.**

If $n = 3$, we can consider $\tilde{A}(3, 2)$ as a poset of the shape $\square = [1] \times [1]$. We want to construct a chain of equivalences whose composition will give us the desired one. Since all the functors we will consider in this step are Kan extensions of fully faithful functors, thanks to Proposition 2.7, it will suffice to check their essential images. Following the Construction
2.23, we define the inclusion of the target square functor

\[
\begin{array}{c}
(0,1) \rightarrow (2,1) \\
| \quad | \\
(0,0) \rightarrow (1,0)
\end{array}
\quad \quad \quad
\begin{array}{c}
(0,1) \rightarrow (1,1) \rightarrow (2,1) \\
| \quad | \\
(0,0) \rightarrow (1,0)
\end{array}
\]

\[\tilde{A}(3,2) \quad \quad \quad \tilde{K}^3_{1,2}\]

Recalling the conditions on \(\mathcal{D}(n,2)\), by Proposition 2.24 and since \(\mathcal{D}\) is in particular a pointed derivator, an element \(X \in \mathcal{D} \tilde{K}^3_{1,2}\) belongs to the essential image of

\[(i^3_{1,1})!: \mathcal{D}(n,2) \rightarrow \mathcal{D} \tilde{K}^3_{1,2}\]

if and only if the square

\[(6.2)\]

\[
\begin{array}{c}
X_{(0,1)} \rightarrow X_{(1,1)} \\
| \quad | \\
X_{(0,0)} \rightarrow X_{(1,0)}
\end{array}
\]

is cocartesian and \(X_{(0,1)} = 0\). Since \(\mathcal{D}\) is stable, the square (6.2) is also cartesian. We denote by \(\mathcal{D} \tilde{K}^3_{1,2}\) this essential image. By the characterization we have just given, \(\mathcal{D} \tilde{K}^3_{1,2}\) is a strict full subderivator of \(\mathcal{D} \tilde{K}^3_{1,2}\). Since \(X_{(0,1)} = 0\), we can observe that \(X_{(1,1)}\) is the cone of the map \(X_{(0,0)} \rightarrow X_{(1,0)}\).

Consider now a new inclusion of posets

\[
\begin{array}{c}
(1,1) \rightarrow (2,1) \\
| \quad | \\
(1,0)
\end{array}
\quad \quad \quad
\begin{array}{c}
(0,1) \rightarrow (1,1) \rightarrow (2,1) \\
| \quad | \\
(0,0) \rightarrow (1,0)
\end{array}
\]

\[A_3 \quad \quad \quad \tilde{K}^3_{1,3}\]

It is a cosieve, hence, by Proposition 2.13,

\[(i^3_{1,3})!: \mathcal{D} A_3 \rightarrow \mathcal{D} \tilde{K}^3_{1,3}\]

is the left extension by zero. Thus \((i^3_{1,3})!\) induces an equivalence onto the strict full subderivator \(\mathcal{D} \tilde{K}^3_{1,3} \subset \mathcal{D} \tilde{K}^3_{1,3}\) spanned by all diagrams which vanish at \((0,1)\). Finally, we can include \(\tilde{K}^3_{1,3}\) in \(\tilde{K}^3_{1,2}\) through the map

\[
\begin{array}{c}
(0,1) \rightarrow (1,1) \rightarrow (2,1) \\
| \quad | \\
(1,0)
\end{array}
\quad \quad \quad
\begin{array}{c}
(0,1) \rightarrow (1,1) \rightarrow (2,1) \\
| \quad | \\
(0,0) \rightarrow (1,0)
\end{array}
\]

\[\tilde{K}^3_{1,3} \quad \quad \quad \tilde{K}^3_{1,2}\]
Then, by Proposition 2.15 and since $\mathcal{D}$ is a pointed derivator, the right Kan extension
\[(i_{1,2}^3)_* : \mathcal{D}K_{1,3} \to \mathcal{D}K_{1,2} \]
induces an equivalence onto the essential image that consists of the objects $X \in \mathcal{D}K_{1,2}$ such that the square (6.2) is cartesian and vanishes in the position (0,1) i.e. it induces an equivalence onto $\mathcal{D}K_{1,2}$. Since both the Kan extensions $(i_{1,2}^3)_*$ and $(i_{1,3}^3)_*$ are fully faithful, the composition $(i_{1,2}^3)_*(i_{1,3}^3)_*$ is fully faithful and then it induces an equivalence onto the strict full subderivator $\mathcal{D}K_{1,2}$ so that we can consider the inverse equivalence
\[((i_{1,2}^3)_*(i_{1,3}^3)_*)^{-1} = (i_{1,3}^3)_!^{-1}(i_{1,2}^3)_*^{-1} : \mathcal{D}K_{1,2} \to \mathcal{D}A_3.\]

Since we also have the equivalence given by $(i_{1,1}^3)_!$, we get the desired one by considering the following composition:
\[\mathcal{D}A^{(2,3)} (i_{1,1}^3)_! \xrightarrow{(i_{1,2}^3)_*} \mathcal{D}K_{1,2} \xrightarrow{(i_{1,2}^3)_*^{-1}} \mathcal{D}K_{1,3} \xrightarrow{(i_{1,3}^3)_!^{-1}} \mathcal{D}A_3.\]

In the next steps we will denote by $\tilde{K}_{1,4}$ the coherent diagram of shape $A_3$.

**Second step:** $\mathcal{D}A^{(n,2)} \cong \mathcal{D}K_{1,2} \cong \mathcal{D}K_{1,3} \cong \mathcal{D}K_{1,4}$ for $n \geq 4$.

In this step we consider the generic case $\mathcal{D}A^{(n,2)}$ and we explain the first part of the inductive passage to reduce the problem to the case $\mathcal{D}A^{(n-1,2)}$. We apply the same strategy as in the **First Step** in order to "delete" the first object subject to the vanishing condition in $\mathcal{D}A^{(n,2)}$. After this procedure we get a poset with only $n-3$ objects subject to the vanishing conditions as in $\mathcal{D}A^{(n-1,2)}$.

We consider the following posets and inclusions, for $n \geq 4$.

Our aim is again to construct a chain of fully faithful functors whose composition will hence induce an equivalence onto the essential image. By Proposition 2.7, all the functors we will consider in this step are fully faithful then we will only check their essential images.
Recalling Proposition 3.7, let us define $\mathcal{D}_{\tilde{K}_{l,m}^n}$ as the strict full subderivator of $\mathcal{D}_{\tilde{K}_{l,m}^n}$, for $m = 2, 3, 4$, spanned by all the coherent diagrams which vanish at $(i, i + 1)$ for
\[
\begin{cases}
i = 0, \ldots, n - 3 & \text{if } m = 2, 3 \\
i = 1, \ldots, n - 3 & \text{if } m = 4.
\end{cases}
\]

Consider the inclusion $i^n_{1,1}$, by Proposition 2.15 and since $\mathcal{D}$ is a pointed derivator, the essential image of the left Kan extension
\[
(i^n_{1,1})! : \mathcal{D}^A(n,2) \rightarrow \mathcal{D}_{\tilde{K}_{l,2}^n}
\]
of the objects $X \in \mathcal{D}_{\tilde{K}_{l,2}^n}$ such that the square (6.2) is cocartesian. We denote by $\mathcal{D}_{\tilde{K}_{l,3}^n}$ this essential image. As in the previous step, we can observe that $\mathcal{D}_{\tilde{K}_{l,2}^n}$ is a strict full subderivator of $\mathcal{D}_{\tilde{K}_{l,3}^n}$ and $X(1,1)$ is the cone of the map
\[
X_{(0,0)} \rightarrow X_{(1,0)}.
\]
Consider now the map $i^n_{1,3}$, it is the inclusion of a cosieve. Hence it follows from Proposition 2.13 that
\[
(i^n_{1,3})! : \mathcal{D}_{\tilde{K}_{l,4}^n} \rightarrow \mathcal{D}_{\tilde{K}_{l,3}^n}
\]
is the left extension by zero. We denote by $\mathcal{D}_{\tilde{K}_{l,3}^n}$ the essential image of this functor. $\mathcal{D}_{\tilde{K}_{l,3}^n}$ is a strict full subderivator of $\mathcal{D}_{\tilde{K}_{l,3}^n}$ and, since $\mathcal{D}$ is in particular a pointed derivator, it coincides with $\mathcal{D}_{\tilde{K}_{l,3}^n}$.

We can observe that $\mathcal{D}_{\tilde{K}_{l,2}^n}$ is also the essential image of the right Kan extension of $i^n_{1,2}$
\[
(i^n_{1,2})_* : \mathcal{D}_{\tilde{K}_{l,3}^n} \rightarrow \mathcal{D}_{\tilde{K}_{l,2}^n}
\]
since, by Proposition 2.15 it consists of the objects $X \in \mathcal{D}_{\tilde{K}_{l,2}^n}$ such that the square (6.2) is cartesian. Then, as in the previous step, since we are working with stable derivators, we can conclude that the essential image of the functor $(i^n_{1,2})_*(i^n_{1,3})!$ coincides with the essential image of $(i^n_{1,1})!$; and we get the desired equivalence given by the following composition of functors
\[
\mathcal{D}^A(n,2) \xrightarrow{(i^n_{1,1})!} \mathcal{D}_{\tilde{K}_{l,n}^2} \xrightarrow{(i^n_{1,2})_*} \mathcal{D}_{\tilde{K}_{l,3}^n} \xrightarrow{(i^n_{1,3})!} \mathcal{D}_{\tilde{K}_{l,4}^n} = \mathcal{D}_{\tilde{K}_{l,n}^4}.
\]

Here the last equality holds since $(i^n_{1,3})!$ is an equivalence and then the essential image of the inverse equivalence, denoted by $\mathcal{D}_{\tilde{K}_{l,4}^n}$, coincides with $\mathcal{D}_{\tilde{K}_{l,n}^4}$.

**Third step: Definition of $\mathcal{D}_{\tilde{K}_{l,m}^n}$ for $l = 2, \ldots, n - 2$, $m = 1, \ldots, 4$, $n \geq 4$.**

In the previous step we got the derivator $\mathcal{D}_{\tilde{K}_{l,4}^n}$ with only $n - 3$ objects subject to the vanishing conditions. Still, $\mathcal{D}_{\tilde{K}_{l,4}^n} \neq \mathcal{D}^A(n-1,2)$ and, in particular, the difference between them is the arrow $(1, 0) \rightarrow (1, 1)$. The idea is then to manage this arrow by "bending" it and construct a 3-dimensional poset where, by fixing the third coordinate, we find the poset $\tilde{A}(n - 1, 2)$ (see Fourth Step). The notation for these new posets will be as follows:

$\tilde{K}_{l,m}^n$ where $n$ comes from $A(n,2)$, $m = 1, \ldots, 4$ indicates the passages of the procedure in the Second Step and $l = 2, \ldots, n - 2$ is the index for the new third dimension: in particular, $l - 1$ equals the number of arrows we "bent". We then define the strict full subderivators of $\mathcal{D}_{\tilde{K}_{l,m}^n}$ we need.

Consider the poset of the shape

$\tilde{K}_{l,m}^{n-l+1} \times [l - 1]$ for $l = 2, \ldots, n - 2$, $m = 1, \ldots, 4$ where $\tilde{K}_{l,m}^{n-l+1}$ denotes the poset $\tilde{A}(n - l + 1, 2)$. 
Thanks to Proposition 3.7, we define $\mathcal{D}^{R_{n,m}}$ to be the strict full subderivator of $\mathcal{D}^{K_{1,m}^{-l+1}\times[l-1]}$ spanned by all coherent diagrams which vanish at

$$(x, x + 1, y) \text{ for } x = 0, \ldots, n - l - 2, \quad y = 0, \ldots, l - 1$$

and in addition to this condition we require the arrows

$$(x, x - 1, y) \mapsto (x, x - 1, y + 1)$$

$$(x - 1, x, y) \mapsto (x - 1, x, y + 1)$$

$$(n - l - 1, n - l - 1, y) \mapsto (n - l - 1, n - l - 1, y + 1)$$

to be isomorphisms for $x = 1, \ldots, n - l - 1, y = 0, \ldots, l - 2$. Namely, we want $\mathcal{D}^{R_{n,m}}$ to be the intersection between the strict full subderivator satisfying the vanishing conditions described above and the one satisfying the isomorphism conditions described above.

To use a shorter notation we will denote $K_{1,m}^{-l+1}\times[l-1]$ by $\tilde{K}_{l,m}^n$.

**Fourth step:** $\mathcal{D}^{K_{1,4}^n} \cong \mathcal{D}^{\tilde{K}_{2,1}^n}$ for $n \geq 4$.

In this step we explain how we formally pass from the 2-dimensional poset $K_{1,4}^n$ to the 3-dimensional poset $\tilde{K}_{2,1}^n$ and we prove the equivalence between the strict full subderivators $\mathcal{D}^{K_{1,4}^n}$ and $\mathcal{D}^{\tilde{K}_{2,1}^n}$ by defining an homotopical epimorphism (Section 4).

We define an epimorphism

$$i_{1,4}^n: K_{2,1}^n \to K_{1,4}^n$$

$$(x, y, z) \mapsto (x + 1, y + 1)$$

$$(0, 0, 0) \mapsto (1, 0).$$

We want to prove that it is an homotopical epimorphism and in particular that

$$(i_{1,4}^n)^*: \mathcal{D}^{K_{1,4}^n} \to \mathcal{D}^{\tilde{K}_{2,1}^n}$$

is an equivalence. In order to apply Proposition 4.3 we have to show that $\text{essIm}(i_{1,4}^n)^* \subseteq \mathcal{D}^{\tilde{K}_{2,1}^n}$ and that the unit

$$\eta: Y \to (i_{1,4}^n)^*(i_{1,4}^n)^*Y$$

is an isomorphism for every $Y \in \mathcal{D}^{\tilde{K}_{2,1}^n}$. Clearly $i_{1,4}^n$ is surjective then it is essentially surjective on objects and $\text{essIm}(i_{1,4}^n)^* \subseteq \mathcal{D}^{\tilde{K}_{2,1}^n}$. To show that the unit is an isomorphism, by (Der2), it suffices to check the invertibility of $\eta$ at every object $(x, y, z) \in \tilde{K}_{2,1}^n$ i.e. to check that the map

$$(x, y, z)^*\eta: (x, y, z)^*Y \to (x, y, z)^*(i_{1,4}^n)^*(i_{1,4}^n)^*Y$$

is an isomorphism for every $(x, y, z) \in \tilde{K}_{2,1}^n$. By Lemma 4.5 we equivalently show that the map

$$(x, y, z), \text{id}_{i_{1,4}^n((x,y,z))}^*p^*Y \to \text{colim}_{i_{1,4}^n/(i_{1,4}^n, (x,y,z))}^*p^*Y$$

is an isomorphism. We can observe that $((x, y, 1), \text{id}_{i_{1,4}^n((x,y,1))})$ is the the terminal object of the slice category $i_{1,4}^n/\tilde{i}_{1,4}^n((x, y, 1))$, then we have two cases:

1. For $z = 1$, by Lemma 4.6, (6.3) is an isomorphism if and only if

$$(x, y, z), \text{id}_{i_{1,4}^n((x,y,z))}^*p^*Y \to ((x, y, z), \text{id}_{i_{1,4}^n((x,y,z))}^*p^*Y$$

is, then we can conclude since this map is the restriction of the identity.

2. For $z = 0$, by Lemma 4.6, (6.3) is an isomorphism if and only if

$$(x, y, z), \text{id}_{i_{1,4}^n((x,y,z))}^*p^*Y \to ((x, y, 1), \text{id}_{i_{1,4}^n((x,y,1))}^*p^*Y$$
is, but in this case we get one of the maps we required to be isomorphisms in the definition of the strict full subderivator $\mathcal{D}^{K_{2,1}}$ in the previous step of the proof.

**Fifth step:** $\mathcal{D}^{K_{2,1}} \cong \mathcal{D}^{K_{n,2}} \cong \mathcal{D}^{K_{n,3}} \cong \mathcal{D}^{K_{n,4}}$ for $l = 2, \ldots, n - 2, n \geq 4$.

In this step we explain how we can apply the procedure in the **Second step** to the cases where $l > 1$.

By the first and the second step of the proof, for any stable derivator we have the following equivalence for any $n \geq 3$.

$$\mathcal{D}(A(n, 2)) = \mathcal{D}K_{1, 1}^{n} \xrightarrow{(i_{1, 1}^n)} \mathcal{D}K_{1, 2}^{n} \xrightarrow{(i_{1, 2}^n)^{-1}} \mathcal{D}K_{1, 3}^{n} \xrightarrow{(i_{1, 3}^n)^{-1}} \mathcal{D}K_{1, 4}^{n}.$$ 

If, as stable derivator, we now consider $\mathcal{C} = \mathcal{D}[l-1]$ then we have the following situation

$$\mathcal{D}K_{1, 1}^{n-4} \times [l-1] \xrightarrow{(i_{1, 1}^{n-4} \times \text{id})} \mathcal{D}K_{1, 2}^{n-4} \times [l-1] \xrightarrow{(i_{1, 2}^{n-4})^{-1}} \mathcal{D}K_{1, 3}^{n-4} \times [l-1] \xrightarrow{(i_{1, 3}^{n-4})^{-1}} \mathcal{D}K_{1, 4}^{n-4}$$

where all the horizontal top arrows are then equivalences and we define $\mathcal{D}K_{n, m}$ for any $m = 1, \ldots, 4, l = 2, \ldots, n - 2$, as the strict full subderivator given by the intersection

$$\mathcal{D}K_{n, m}^{n-4} \times [l-1] \cap \hat{\mathcal{R}}_{i, n,m}.$$ 

We can observe that $\mathcal{D}^{K_{2,1}}$ coincides with $\hat{\mathcal{R}}_{2,1}$ by definition. The bottom maps are the restrictions of the top ones to $\mathcal{D}^{K_{l,m}}$: they are well defined by Proposition 3.6. We want to show that the bottom maps are still equivalences. These functors are clearly faithfully then we only have to check that the essential images coincides with $\mathcal{D}K_{n, m}$ for $m = 2, 3, 4$, $l = 2, \ldots, n - 2$. Consider the functors $f_i: e \to [l-1]$ for $i = 0, \ldots, l-1$, which choose the object $i \in [l-1]$ and consider an object $X^m \in \mathcal{D}K_{n, m}$ for $m = 2, 3, 4$. By Proposition 2.8

$$X^m \in \text{essIm}(i_{n, m-1}^n) \iff (\text{id} \times f_i)^*X^m \in \text{essIm}(i_{n, m-1}^n)$$

for $\quad \star \in \{\emptyset, -1\}, \heartsuit \in \{!, \ast\}$. This is guaranteed, thanks to Proposition 2.7, by the commutativity of following diagram

$$\begin{array}{ccc}
\mathcal{D}K_{n, m-1} & \xrightarrow{(i_{n, m-1}^n)} & \mathcal{D}K_{n, m} \\
(\text{id} \times f)^* & \uparrow & \uparrow (\text{id} \times f)^* \\
\mathcal{D}K_{n, m} & \xrightarrow{(i_{n, m}^n)} & \mathcal{D}K_{n, m} \\
\end{array}$$

and by the top arrows in the diagram above which are equivalences.

**Sixth step:** $\mathcal{D}^{K_{l,1}} \cong \mathcal{D}^{K_{l+1,1}}$ for $l = 2, \ldots, n - 3, n \geq 4$.

In this step we explain the homotopical epimorphism in the **Fourth step** to the cases where $l > 1$. In particular, we need to define a composition of two homotopical epimorphisms.

We first define the new poset $\hat{K}_{l,5}$ to be the full subposet of $\hat{K}_{l,4}$ spanned by all the objects different from $(1, 0, 1), \ldots, (1, 0, l - 1)$. As in the third step we then define $\mathcal{D}^{K_{l,5}}$ to be the strict full subderivator of $\mathcal{D}^{K_{l,5}}$ spanned by all coherent diagrams which vanish at

$$(x, x + 1, y) \text{ for } x = 1, \ldots, n - l - 2, \quad y = 0, \ldots, l - 1$$

and in addition to this condition we require the arrows

$$(x, x - 1, y) \to (x, x - 1, y + 1)$$

$$(n - l - 1, n - l - 1, y) \to (n - l - 1, n - l - 1, y + 1)$$
to be isomorphisms for \( x = 1, \cdots, n - l - 1, y = 0, \cdots, l - 2 \) and the arrows
\[
(x-1,x,y) \rightarrow (x-1,x,y+1)
\]
to be isomorphisms for \( x = 2, \cdots, n - l - 1, y = 0, \cdots, l - 2 \).

We now define an epimorphism
\[
i_{l,4}^n : \overline{K}_{l,4}^n \rightarrow \overline{K}_{l,5}^n
\]
\[
(x,y,z) \mapsto (x,y,z), \quad \text{if} \quad (x,y) \neq (1,0)
\]
\[
(0,0,0) \mapsto (1,0,0).
\]

With techniques similar to those used in the fourth step of the proof, it is possible to verify that \( i_{l,4}^n \) is an homotopical isomorphism if we consider the restriction
\[
(i_{l,4}^n)^* : \mathcal{D}K_{l,5}^n \rightarrow \mathcal{D}K_{l,4}^n.
\]

The second map we want to consider is
\[
i_{l,5}^n : \overline{K}_{l+1,1}^n \rightarrow \overline{K}_{l,5}^n
\]
\[
(x,y,z) \mapsto (x+1,y+1,z-1), \quad \text{if} \quad z \neq 0
\]
\[
(x,y,0) \mapsto (x+1,y+1,0), \quad \text{if} \quad (x,y) \neq (0,0)
\]
\[
(0,0,0) \mapsto (1,0,0).
\]

Again with techniques similar to those used in fourth step of the proof, it is possible to verify that \( i_{l,5}^n \) is an homotopical isomorphism if we consider the restriction
\[
(i_{l,5}^n)^* : \mathcal{D}K_{l,5}^n \rightarrow \mathcal{D}K_{l+1,1}^n.
\]

Then we have the following desired equivalences
\[
\mathcal{D}K_{l,4}^n \overset{((i_{l,5}^n)^*)^{-1}}{\rightarrow} \mathcal{D}K_{l,5}^n \overset{(i_{l,5}^n)^*}{\rightarrow} \mathcal{D}K_{l+1,1}^n.
\]

**Seventh step:** \( \mathcal{D}K_{n-2,4}^n \cong \mathcal{D}A_n \) for \( n \geq 4 \).

In this step we explain the last passages of the proof and, in particular, we define the homotopical epimorphisms which give the equivalence to \( \mathcal{D}A_n \).

We can observe that, by definition,
\[
\mathcal{D}K_{n-2,4}^n \subset \mathcal{D}K_{n-3}^{n-2,4 \times [n-2-1]} = \mathcal{D}K_{l,4}^{l \times [n-3]} = \mathcal{D}A_3 \times [n-3].
\]

Similarly as in the previous step we will now conclude the proof defining maps between posets that turn out to be equivalences given by homotopical epimorphisms. As before we can construct the equivalence
\[
\mathcal{D}K_{n-2,4}^n \overset{((i_{n-2,4}^n)^*)^{-1}}{\rightarrow} \mathcal{D}K_{n-2,5}^n.
\]

We now define the following new map
\[
i_{n-2,5}^n : \overline{K}_{n-2,5}^n \rightarrow A_n
\]
\[
(1,1,z) \mapsto z + 2
\]
\[
(1,0,z) \mapsto 1
\]
\[
(2,1,z) \mapsto n.
\]

With techniques similar to those used in the fourth step of the proof, it is possible to show that
\[
(i_{n-2,5}^n)^* : \mathcal{D}A_n \rightarrow \mathcal{D}K_{n-2,5}^n
\]
is an equivalence given by an homotopical epimorphism. Then, we get the last equivalence by taking the inverse of \( (i_{n-2,5}^n)^* \).
Conclusion.
We can conclude the proof considering the equivalence given by the composition of the ones we built in each step

\[
(i_{n-2,3}^n)^{-1}(i_{n-2,4}^n)^{-1}(i_{n-2,5}^n)^{-1}(i_{n-2,3}^n)^{-1}(i_{n-2,2}^n)^{-1}(i_{n-2,1}^n)^{-1}(i_{n-3,4}^n)^{-1}(i_{n-3,3}^n)^{-1} \cdots
\]

We call this composition \( i^n \). Since we only extended by zeroes, added cocartesian squares and restricted through homotopical epimorphisms, it is possible to check that this equivalence is natural with respect to exact morphisms.

\[\square\]

Example 6.2. We draw here the posets involved in the proof for the case \( n = 4 \). The red objects are the ones we required to be zero objects and the double line arrows are the isomorphisms.

\[(0.1) \quad (0.0) \quad (1.0) \quad (1.1) \quad (2.1) \quad (2.2)\]

\[\tilde{K}_{1,1}^4 \quad \tilde{K}_{1,2}^4 \quad \tilde{K}_{1,3}^4 \quad \tilde{K}_{1,4}^4\]

Remark 6.3. A similar result can be deduced from Falk Beckert’s work [5, Corollary 9.15]. There, Beckert uses hypercubes to enhance quiver with relations: it is possible to show that his enhancement is equivalent to the one in this paper. For the case \( n = 3 \) the considered hypercube shape coincides with the poset \( A_1(3,2) \). For greater \( n \), Beckert gets a \( (n-2) \)-dimensional hypercube. Here we propose a more convenient and natural enhancement of quiver with relations which deals only with 2-dimensional posets. Moreover, we give a very elementary proof of Theorem 6.1 which involves only simple Kan extensions and restrictions.

7. ∞-Dold-Kan correspondence via representation theory

In this section we aim to explain a connection between representation theory and homotopy theory. Indeed, Theorem 6.1, which arises as an enhancement of an equivalence in representation theory, is actually also the enhancement of a fundamental result in homotopy theory.
theory such as the Dold-Kan correspondence. In particular, we will see how the equivalence in Theorem 6.1 is, at the level of derivators, the bounded version of the equivalence obtained by Ariotta in [1, Theorem 4.7], which is a reformulation of the \( \infty \)-Dold-Kan correspondence. To be able to compare Theorem 6.1 and [1, Theorem 4.7] we will need the construction of the coherent Auslander-Reiten quiver developed by Groth and Šťovíček in [14] which we present in Subsection 7.1 together with his connection to Theorem 6.1.

7.1. Coherent Auslander-Reiten quiver. We briefly recall the definitions we need. A quiver \( Q \) consists of a set of vertices \( Q_0 \) and a set of arrows \( Q_1 \). We can associate to \( Q \) the repetitive quiver \( \hat{Q} \) whose vertices are pairs \( (k, q) \) with \( k \in \mathbb{Z} \) and \( q \in Q \) and for every arrow \( \alpha : q_1 \to q_2 \) in \( Q_1 \) there are arrows \( \alpha : (k, q_1) \to (k, q_2) \) and \( \alpha^* : (k, q_2) \to (k + 1, q_1) \) in \( \hat{Q}_1 \). We denote by \( M_n \) the category obtained from the repetitive quiver of \( A_{n+2} \) by forcing all squares to commute.

Example 7.2. The category \( M_4 \) looks like:

\[
\begin{array}{cccccc}
(-3,5) & \rightarrow & (-2,5) & \rightarrow & (-1,5) & \rightarrow (0,5) \\
(-3,4) & \rightarrow & (-2,4) & \rightarrow & (-1,4) & \rightarrow (0,4) \\
(-3,3) & \rightarrow & (-2,3) & \rightarrow & (-1,3) & \rightarrow (0,3) \\
(-3,2) & \rightarrow & (-2,2) & \rightarrow & (-1,2) & \rightarrow (0,2) \\
(-3,1) & \rightarrow & (-2,1) & \rightarrow & (-1,1) & \rightarrow (0,1) \\
(-3,0) & \rightarrow & (-2,0) & \rightarrow & (-1,0) & \rightarrow (0,0) \\
\end{array}
\]

Observe that \( M_n \) can be also considered as a poset.

Construction 7.3. Let now \( \mathcal{D} \) be a stable derivator. Following [14, Section. 4] we construct a coherent diagram of shape \( M_n \) satisfying certain exactness and vanishing conditions. Note that there is a fully faithful functor

\[
i_n : A_n \to M_n \\
l \mapsto (0, l)
\]

which we consider as an inclusion. This embedding factors as a composition of inclusions of full subcategories

\[
i_n : A_n \xrightarrow{s_1} K_1 \xrightarrow{s_2} K_2 \xrightarrow{s_3} K_3 \xrightarrow{s_4} M_n
\]

where

1. \( K_1 \) is obtained from \( A_n \) by adding the objects \( (k, n+1) \) for \( k \geq 0 \) and \( (k, 0) \) for \( k > 0 \),
2. \( K_2 \) contains all objects from \( K_1 \) and the objects \( (k, l) \), \( k > 0 \), and
3. \( K_3 \) is obtained from \( K_2 \) by adding the objects \( (k, n+1) \) for \( k < 0 \) and \( (k, 0) \) for \( k \leq 0 \).

The inclusion \( s_4 \) thus adds the remaining objects in the negative \( k \)-direction. By Proposition 2.7, associated to these fully faithful functors there are fully faithful Kan extension functors

\[
\begin{array}{cccccc}
\mathcal{D}A_n & \xrightarrow{(s_1)_*} & \mathcal{D}K_1 & \xrightarrow{(s_2)_*} & \mathcal{D}K_2 & \xrightarrow{(s_3)_*} & \mathcal{D}K_3 & \xrightarrow{(s_4)_*} & \mathcal{D}M_n \\
\end{array}
\]

Let us denote by \( \mathcal{D}^{M_n,ex} \subseteq \mathcal{D}M_n \) the full subderivator spanned by all coherent diagrams which vanish at \( (k, 0), (k, n+1) \) for all \( k \in \mathbb{Z} \) and which make all squares bicartesian. The objects with the vanishing condition are depicted in red in Picture (7.1). Observe that, by
Proposition 3.6, $\mathcal{D}^{\text{M.n}, \text{ex}}$ is actually a strict full subderivator and then, in particular, it is a derivator.

This construction brings us to the following theorem.

**Theorem 7.4 ([14, Theorem. 4.6]).** Let $\mathcal{D}$ be a stable derivator. Then (7.3) induces an equivalence of stable derivators

$$
\mathcal{D}^A_n \xleftrightarrow{F_n} \mathcal{D}^{\text{M.n}, \text{ex}}
$$

which is natural with respect to exact morphisms. Moreover, the inclusion $\mathcal{D}^{\text{M.n}, \text{ex}} \to \mathcal{D}^{\text{M.n}}$ is exact.

Our main Theorem 6.1 can be described through the above Theorem 7.4.

In Theorem 6.1, we found the equivalence $\mathcal{D}^{A(n,2)} \rightleftharpoons \mathcal{D}^A_n$ and equivalences between $\mathcal{D}^A_n$ and $\mathcal{D}^{K^m_l}$ for every $l = 1, \cdots, n - 2$, $m = 1, \cdots, 5$. Then it is straightforward that we can also find the following equivalences

$$
\mathcal{D}^{A(n,2)} \rightleftharpoons \mathcal{D}^{\text{M.n}, \text{ex}}_n, \quad \mathcal{D}^{K^m_l} \rightleftharpoons \mathcal{D}^{\text{M.n}, \text{ex}}_n
$$

for every $l = 1, \cdots, n - 2$, $m = 1, \cdots, 5$.

**Example 7.5.** We now explicitly describe these equivalences for $n = 4$, the general case will then be easy to deduce from this particular one. Consider the following inclusions:

- $j^4 : A(4,2) = \widetilde{K}^4_{1,4} \to M_4$,
  
  $$(0,0) \mapsto (-3,1), \quad (1,0) \mapsto (-3,4), \quad (0,1) \mapsto (-2,0),$$
  
  $$(1,2) \mapsto (-2,5), \quad (2,1) \mapsto (0,1), \quad (2,2) \mapsto (0,4).$$
  
  Except for the objects we required to be zeroes, we underlined this map with the green color in Pictures (6.5) and (7.1).

- $j^4_{1,2} : \widetilde{K}^4_{1,2} \to M_4$,
  
  $$(1,1) \mapsto (-2,3)$$
  
  and on the remaining objects it acts as the inclusion $j^4$ described above.

- $j^4_{1,4} : \widetilde{K}^4_{1,4} \to M_4$, it consists of the inclusion $j^4_{1,2}$ restricted to the full subposet $\widetilde{K}^4_{1,4} \subset \widetilde{K}^4_{1,2}$.

- $j^4_{2,1} : \widetilde{K}^4_{2,1} \to M_4$,
  
  $$(0,0,0) \mapsto (-3,4), \quad (0,0,1) \mapsto (-2,3), \quad (0,1,0) \mapsto (-3,5), \quad (0,1,1) \mapsto (-2,5),$$
  
  $$(1,0,0),(1,0,1) \mapsto (0,1), \quad (1,1,0),(1,1,1) \mapsto (0,4)$$

- $j^4_{2,2} : \widetilde{K}^4_{2,2} \to M_4$,
  
  $$(0,0,0) \mapsto (-3,4), \quad (0,0,1) \mapsto (-2,3), \quad (0,1,0) \mapsto (-3,5), \quad (0,1,1) \mapsto (-2,5),$$
  
  $$(1,1,0) \mapsto (0,2), \quad (1,0,0),(1,0,1) \mapsto (0,1), \quad (1,1,1) \mapsto (0,3), \quad (2,1,0),(2,1,1) \mapsto (0,4).$$
We can represent this map as follows

\[
\begin{array}{cccc}
(0,0) & (0,1) & (0,1.1) & (0.5) \\
(-2.1) & (-2.4) & (-1.4) & (2,1.2) \\
(-3.1) & (-3.3) & (-1.3) & (1,1.1) \\
(-3,0) & (-2,0) & (-1,0) & (0,0) \\
\end{array}
\]

for \( z = 0, 1 \).

- \( j_{2,4}^4 : \widetilde{K}_{2,4}^4 \to M_4 \), \( j_{2,5}^4 : \widetilde{K}_{2,5}^4 \to M_4 \), \( i_4 : A_4 \to M_4 \). They are the restrictions of \( j_{2,2}^4 \) to the full subposets \( \widetilde{K}_{2,4}^4 \), \( \widetilde{K}_{2,5}^4 \) and \( A_4 \) respectively. Note that \( i_4 \) is the map we have already defined (7.2).

The fact that in \( M_4 \) every square commutes makes all these inclusions fully faithful. If we now consider the restrictions of the inclusions defined above, they are then fully faithful and it is possible to show that they give us the following equivalences between strict full subderivators

\[
\begin{array}{cccc}
\mathcal{D}^{A(4,2)} & \sim & \mathcal{D}^{K_{1,2}^4} & \sim & \mathcal{D}^{K_{1,4}^4} & \sim & \mathcal{D}^{K_{2,4}^4} & \sim & \mathcal{D}^{K_{2,5}^4} & \sim & \mathcal{D}^{A_4} \\
\sim & \sim & \sim & \sim & \sim & \sim & \sim & \sim & \sim & \sim & \sim \\
(j_2^4)^*_n & (j_{1,2}^4)_n^* & (j_{1,4}^4)_n^* & \cdots & (j_{2,4}^4)_n^* & (j_{2,5}^4)_n^* & \cdots & \cdots & \cdots & \cdots & \cdots \\
\mathcal{D}^{M_4,ex} \\
\end{array}
\]

To generalize the Example 7.5 above, we consider the following inclusion for \( n \in \mathbb{N} \):

- If \( n \) even, \( j^n : \tilde{A}(n,2) \to M_n \)

\[
\begin{array}{c}
(0,0) \mapsto \left( -\left( (n - 2)/2 \right) (n - 1), 1 \right), \quad (n,n) \mapsto (0,n), \\
(p,p-1) \mapsto \begin{cases} \left( \left( (n-p+1)/2 \right)(n-1), n \right) & \text{if } p \text{ odd} \\
\left( \left( (n-p+2)/2 \right)(n-1), 1 \right) & \text{if } p > 0, \text{ even}, 
\end{cases} \\
(p-1,p) \mapsto \begin{cases} \left( \left( (n-p+1)/2 \right)(n-1) + 1, n+1 \right) & \text{if } p \text{ odd} \\
\left( \left( (n-p)/2 \right)(n-1), 1 + n \right) & \text{if } p > 0, \text{ even}, 
\end{cases}
\end{array}
\]

- If \( n \) odd, \( j^n : A(n,2) \to M_n \)

\[
\begin{array}{c}
(0,0) \mapsto \left( \left( (n+1)/2 \right)(n-1), n \right), \quad (n,n) \mapsto (0,n), \\
(p,p-1) \mapsto \begin{cases} \left( \left( (n+p+2)/2 \right)(n-1), 1 \right) & \text{if } p \text{ odd} \\
\left( \left( (n+p+1)/2 \right)(n-1), n \right) & \text{if } p > 0, \text{ even}, 
\end{cases} \\
(p-1,p) \mapsto \begin{cases} \left( \left( (n+p)/2 \right)(n-1), 1 + n \right) & \text{if } p \text{ odd} \\
\left( \left( (n+p+1)/2 \right)(n-1) + 2, 0 \right) & \text{if } p > 0, \text{ even}.
\end{cases}
\end{array}
\]

As before, it is then possible to show the following result.
Theorem 7.6. Let $D$ be a stable derivator. We can always construct the following commutative diagram whose arrows are equivalences.

\[
\begin{array}{ccc}
D^{A(n,2)} & \xrightarrow{i^n} & D^{A_n} \\
(j^n)^* & \xrightarrow{(t_n)^*} & (g_{n,ex})^*
\end{array}
\]

(7.4)

Here, $(j^n)^*$ being an equivalence follows from the Theorem 6.1 where we show that $i^n$ is an equivalence.

Remark 7.7. One can define different symmetries in the stable derivator $D^{M,ex}$; they are described in [14, Sections 4, 5 and 12]. The most relevant for us are the shift functor and the Auslander-Reiten translation. These two functors are, respectively, the restrictions of the following two maps [14, (4.10),(5.9)]:

\[
f_n: M_n \rightarrow M_n, \quad (k, l) \mapsto (k + l, n + 1 - l)
\]

\[
t_n: M_n \rightarrow M_n, \quad (k, l) \mapsto (k - 1, l)
\]

We call $g_n$ any finite compositions of $f_n$, $t_n$, $f_n^{-1}$, $t_n^{-1}$. Then we have that $g_n j^n$ gives us another embedding of $A(n,2)$ in $M_n$ and $(g_n j^n)^*$ is an equivalence.

7.8. ∞-Dold-Kan correspondence. Let $A$ be an abelian category, the classical Dold-Kan correspondence [30, Theorem 1.2.3.7] asserts that the category $\text{Fun}(\Delta^{op}, A)$ of simplicial objects of $A$ is equivalent to the category $\text{Ch}_{\geq 0}(A)$ of (homologically) nonnegatively graded chain complexes. If we replace $A$ with a bicomplete stable ∞-category $C$, we can get an analog of the classical Dold-Kan correspondence at level of ∞-categories.

Theorem 7.9 ([30, Theorem 1.2.4.1]). The ∞-categories

\[
\text{Fun}(N(\Delta^{op}), C) \text{ and } \text{Fun}(N(\mathbb{Z}_{\geq 0}), C)
\]

are equivalent to one another.

Here, $N$ is the nerve functor and $\text{Fun}(N(\mathbb{Z}_{\geq 0}), C)$ can be thought of the bounded ∞-category of filtered objects. $\text{Fun}(N(\mathbb{Z}_{\geq 0}), C)$ is a full subcategory of the ∞-category of complete filtered objects $\widehat{\text{Fun}}(N(\mathbb{Z}), C)$ i.e. of those filtrations whose limit vanishes. It turns out that $\widehat{\text{Fun}}(N(\mathbb{Z}), C)$ is equivalent to a suitable ∞-category of coherent chain complexes that is defined as follows. We recall that $\text{Ch}$ [24, Definition 35.1] is the pointed category whose objects are $\mathbb{Z} \cup \{pt\}$ and whose arrows are given by

\[
\text{Ch}(m, n) = \begin{cases} 
\{\partial_n, 0\} & \text{if } m = n - 1 \\
\{\text{id}, 0\} & \text{if } m = n \\
\{0\} & \text{otherwise}
\end{cases}
\]

where, by definition, $\partial_{n-1} \partial_n = 0$ and $\{pt\}$ is a zero object. Then we can define the ∞-category $\text{Ch}(C)$ as the category of functors $\text{Ch} \rightarrow C$ preserving the zero object. More formally we have the following result by Ariotta.

Theorem 7.10 ([1, Theorem 4.7]). There exists an equivalence of stable ∞-categories

\[
\text{Fun}(N(\mathbb{Z}), C) \xrightarrow{\sim} \text{Ch}(C).
\]

It happens that the equivalence we proved in the main Theorem 6.1 is the same as the equivalence in Theorem 7.10 if we consider it at level of derivators and we restrict to bounded
coherent chain complexes. In the following subsection we will explain how these results are related.

7.11. ∞-Dold-Kan correspondence via Theorem 6.1. We will now describe, objectwise, the modification which gives the equivalence between Theorem 6.1 and Theorem 7.10.

Let

\[ \text{Fun}^{[0,n]}(N(\mathbb{Z}), \mathcal{C}) \]

be the full subcategory of \( \text{Fun}(N(\mathbb{Z}), \mathcal{C}) \) such that the images of the arrows \( i - 1 \to i \), for \( i > n - 1 \), are isomorphisms and the images of the objects \( i \), for \( i < 0 \), are zero objects. We consider also the full subcategory of bounded chain complexes

\[ \text{Ch}_{[n,0]}(\mathcal{C}) \]

in \( \text{Ch}(\mathcal{C}) \). By [1, Remark 3.24], the essential image of the equivalence (7.5) restricted to \( \text{Fun}^{[0,n-1]}(N(\mathbb{Z}), \mathcal{C}) \), is \( \text{Ch}_{[n-1,0]}(\mathcal{C}) \). Then (7.5) induces an equivalence

\[
(7.6) \quad \text{Fun}^{[0,n-1]}(N(\mathbb{Z}), \mathcal{C}) \xrightarrow{\mathcal{A}} \text{Ch}_{[n-1,0]}(\mathcal{C}).
\]

between the full subcategories.

By Example 2.5, the homotopy category \( \text{Ho}(\text{Fun}(N(\mathbb{Z}), \mathcal{C})) \) is the homotopy derivator of \( \mathcal{C} \) evaluated in \( \mathbb{Z} \). If we consider the full inclusion

\[ k_n : [n - 1] \to \mathbb{Z}, \]

then we have that the restriction \( k_n^* \) gives an equivalence

\[ \text{Ho}(\text{Fun}^{[0,n-1]}(N(\mathbb{Z}), \mathcal{C})) \xrightarrow{k_n^*} \text{Ho}_{\mathcal{C}}(A_n). \]

Indeed, \( \text{Ho}(\text{Fun}^{[0,n-1]}(N(\mathbb{Z}), \mathcal{C})) \) is the essential image of \( (k_n)_! \). Moreover, we have a map

\[
(7.7) \quad \text{Fun}^{[0,n-1]}(N(\mathbb{Z}), \mathcal{C}) \xrightarrow{\mathcal{A}} \text{Ch}_{[n-1,0]}(\mathcal{C}) \]

\[ k_n \]

whose restriction gives a functor

\[
(7.7) \quad \text{Ho}(\text{Fun}^{[0,n-1]}(N(\mathbb{Z}), \mathcal{C})) \xrightarrow{k_n^*} \text{Ho}_{\mathcal{C}}(A_n). \]

Proposition 7.12. If we restrict to bounded chain complexes and bounded filtrations then Theorem 6.1 is the derivator-theoretical version of Theorem 7.10.

Proof. It suffices to show that we have the following commutative diagram where all the maps are equivalences:

\[
(7.7) \quad \text{Ho}(\text{Fun}^{[0,n-1]}(N(\mathbb{Z}), \mathcal{C})) \xrightarrow{\mathcal{A}} \text{Ho}(\text{Ch}_{[n-1,0]}(\mathcal{C})) \]

\[ k_n \]

\[
(7.7) \quad \text{Ho}_{\mathcal{C}}(A_n) \xrightarrow{u_n^*} \text{Ho}_{\mathcal{C}}(A(n, 2)) \]

We already discussed why \( k_n^* \) is an equivalence and, since the passage to the homotopy category preserves equivalences of \( \infty \)-categories, also \( \mathcal{A} \) is an equivalence which is induced by Theorem 7.10. Even if we would expect the equivalence in Theorem 6.1 as bottom arrow
which makes the diagram commutative, this is not sufficient: we need a slightly more complicated composition of functors which factors through the higher Auslander-Reiten quiver (Section 7.1). Indeed, in [1, Remark 3.24] Ariotta draws a diagram which describes the equivalence in his Theorem (look also at [1, Remark 4.9]); we can observe that such diagram has the shape $M_n$ and that there are the same conditions of zero objects, commutativity and bicartesian squares which characterize $\mathcal{G}^{M_n,ex}$ (Construction 7.3). Indeed, in the context of $\infty$-categories, thanks to [29, Proposition 4.3.2.15], we can define bicartesian squares through Kan extensions. Moreover, when we consider the functor category

$$\text{Ho}(\text{Fun}(N(M_n), C)),$$

the presence of the nerve implies the commutativity of the diagram we get in the underlying category. Then, the coherent diagram we get by looking at $\text{Ho}_C(M_n)^{ex}$, is exactly Ariotta’s diagram in [1, Remark 3.24].

In Section 7.1, we saw that the equivalence of Theorem 6.1 factors through $\mathcal{G}^{M_n,ex}$ (Theorem 7.6). In particular, if we take $\mathcal{D}$ to be the homotopy derivator $\text{Ho}_C$, we can also have the following commutative diagram

$$\begin{array}{ccc}
\text{Ho}_C(A_n) & \xrightarrow{G^n} & \text{Ho}_C(A(n, 2)) \\
\downarrow & & \downarrow \\
\text{Ho}_C(M_n)^{ex} & \xrightarrow{(j^n)^*} & \text{Ho}_C(M_n)^{ex}
\end{array} \quad (7.8)$$

The embedding $i_n$ describes precisely the same objects as in [1, Remark 3.24]. But if we substitute $G^n$ with $u_n^*\mathcal{D}$ in (7.8), we don’t get a commutative diagram anymore. This is because $u_n^*\mathcal{D}$ differs from $G^n$ by an autoequivalence of $\text{Ho}_C(M_n)^{ex}$. To explain better how to find this autoequivalence, we look at the particular case $n = 3$.

**Example 7.13.** We draw the diagram $M_3$ to compare Theorem 6.1 and Theorem 7.10 through Theorem 7.4.

$$\begin{array}{c}
\cdots \\
(-2.1) & \cdots \\
(-2.2) & (-2.3) & \cdots \\
(-2.0) & (-1.0) & (0.0) & (1.0) & (2.0) & (3.0) & (4.0) & \cdots \\
\cdots & \cdots \\
(-1.1) & (-1.2) & \cdots \\
(0.1) & (0.2) & \cdots \\
(1.1) & (1.2) & \cdots \\
(2.1) & (2.2) & \cdots \\
(3.1) & (3.2) & \cdots \\
(4.1) & (4.2) & \cdots \\
\cdots & \cdots \\
\text{This diagram comes from Construction 7.3 behind Theorem 7.4 where the functor}
\end{array}$$

$$i_3 : A_3 \to M_3$$

embeds $A_3$ in the red coordinates $(0, 1), (0, 2), (0, 3)$. The green coordinates are the ones describing the embedding

$$j^3 : A(3, 2) \to M_3$$

in Theorem 7.6. The blue coordinates describe the link with [1, Remark 3.24] and so with the equivalence (7.5). The autoequivalence we are searching for is the one which allows to pass from the complex given by the green objects $(-2, 3), (0, 1), (0, 3)$ to the one given by the blue objects $c^0, c^1, c^2$. Namely, let $C$ be a complex in $\text{Ch}_{[2,0]}(\mathcal{C})$

$$C : \cdots \to 0 \to C^2 \to C^1 \to C^0 \to 0 \to \cdots$$

As we mentioned before, the image of $C$ under functor $\mathcal{F}$ (7.5) coincides with the objects in the coordinates $(0, 1), (0, 2), (0, 3)$, where $A_3$ was embedded through $i_3$. Let

$$b_3 : \square \to M_3$$
be the embedding whose image is the square with vertices \( c^0, c^1, c^2, (4, 0) \). We can observe that \( b_3^* \) has the following form

\[
\phi_{M_3, \text{ex}} \xrightarrow{f_3^*(t_3)^{-1}} \phi_{M_3, \text{ex}} \xrightarrow{(j^3)^*} \phi_{A(3, 2)},
\]

where \( f_3, t_3 \) were defined in Remark 7.7. In particular, since all the maps in (7.10) are equivalences, \( b_3^* \) is an equivalence. This means that if we consider an object \( X \in \phi_{M_3, \text{ex}} \), then we have the following isomorphisms

\[
f_3^* (t_3)^{-1}(X)_{(-2, 3)} \cong X_{c^2}, \quad f_3^* (t_3)^{-1}(X)_{(0, 1)} \cong X_{c^1}, \quad f_3^* (t_3)^{-1}(X)_{(0, 3)} \cong X_{c^0}, \quad f_3^* (t_3)^{-1}(X)_{(-2, 4)} \cong X_{(4, 0)} \cong 0.
\]

Moreover, considering the map

\[
b_3^*(i_3)^{-1} : \phi_{A^3} \rightarrow \phi_{A(3, 2)},
\]

we get the same equivalence as Ariotta in [1, Theorem 4.7].

From the above example we can describe the general case. The maps \( j^n \) defined at the end of Section 7.1 give the position of the green objects in the diagram \( M_n \). From this embedding we can get Ariotta’s complex by composing with an autoequivalence \( b_n^* \) which, as in Example 7.13, is a composition of shift functors \( f_n^* \) and Auslander-Reiten translations \( t_n^* \). In conclusion, the equivalence between Theorem 6.1 and Theorem 7.10 is given by the following commutative diagram:

\[
\begin{array}{ccc}
\text{Ho(Fun}^{[0,n-1]}(N(\mathbb{Z}, C)) & \xrightarrow{\Phi} & \text{Ho(Ch}_{[n-1,0]}(\mathbb{C})) \\
k_n^* & & u_n^* \\
\downarrow & & \downarrow \\
\text{Ho}_C(A_n) & \xrightarrow{(i_n^*)^{-1}} & \text{Ho}_C(M_{n})^{ex} \xrightarrow{b_n^*} \text{Ho}_C(A(n, 2))
\end{array}
\]

where also \( u_n^* \) is an equivalence because all the other functors are and the diagram commutes.

\[\square\]

8. Universal tilting theory

In this section we want to show one more link that our main Theorem 6.1 has with homotopy theory. In particular, we aim to prove that the functors which give the equivalence (6.1) can be realized as tensor products with spectral bimodules. This is a universal version of the (derived) Morita Theory developed by J. Rickard [35]. In particular, we have the following well known result.

**Theorem 8.1** ([35]). Let \( k \) be a commutative ring and \( A, B \) \( k \)-algebras which are flat as modules over \( k \). The following are equivalent.

1. There is a \( k \)-linear triangle equivalence \( \Phi : \text{D(Mod} A) \rightarrow \text{D(Mod} B) \).
2. There is a complex of \( A-B \)-modules \( X \) such that the total left derived functor

\[
- \otimes^L_A X : \text{D(Mod} A) \rightarrow \text{D(Mod} B)
\]

is an equivalence.

As we saw in Example 2.19, derived categories are only the underlying categories of a specific stable derivator. It makes then sense to think about generalizing Rickard’s result and
writing it at level of derivators. We need to recall that every stable derivator is canonically a closed module over the derivator of spectra $\mathcal{S}p$ [6, Appendix A.3]. Thus, if $\mathcal{D}$ is a stable derivator, then there is a canonical action
\[ \otimes: \mathcal{S}p \times \mathcal{D} \to \mathcal{D} \]
which, for every $A, B, C \in \text{Cat}$, allows us to define the so called canceling tensor product [9, Section 5]
\[ \otimes[A]: \mathcal{S}p(B \times A^{\text{op}}) \times \mathcal{D}(A \times C) \to \mathcal{D}(B \times C) \]
\[ (X, Y) \mapsto X \otimes_{[A]} Y. \]
It is worth it to recall also that we refer to an object of $\mathcal{S}p(B \times A^{\text{op}})$ as spectral bimodule.

Our aim in this section, is to apply a derivator enhancement of Rickard’s result to the functor $G^n: \mathcal{D}^A \to \mathcal{D}^{A(n,2)}$ in Theorem 6.1. In particular, we want to show that $G^n$ is equivalent to a functor whose components are canceling tensor products i.e. for every $B \in \text{Cat}$, we want an equivalence between $\mathcal{D}(A_n \times B)$ and $\mathcal{D}(A(n,2) \times B)$ whose form is the canceling tensor product by an object. We recall that an object in $\mathcal{D}(A(n,2) \times B)$ is an object in $\mathcal{D}(A(n,2) \times B)$ subject to some vanishing conditions (cf. Definition 5.1). Thus, it comes naturally to search for a spectral bimodule in $\mathcal{S}p(A(n,2) \times A^{\text{op}})$. As it is proved and defined in [9, Theorem 5.9], the unit of the canceling tensor product is given by the identity profunctor
\[ I_{A_n} \in \mathcal{S}p(A_n \times A^{\text{op}}) \cong \mathcal{S}p^A_n(A_n^{\text{op}}). \]
Applying the main Theorem 6.1 to $\mathcal{S}p$, we obtain the following equivalence
\[ (8.1) \quad \mathcal{S}p^A_n \cong \mathcal{S}p^{A(n,2)}. \]

We can then define a particular spectral bimodule $T_n \in \mathcal{S}p(A(n,2) \times A^{\text{op}})$ to be the image under the equivalence (8.1) of the identity profunctor $I_{A_n}$:
\[ \mathcal{S}p^A_n(A_n^{\text{op}}) \cong \mathcal{S}p^{A(n,2)}(A_n^{\text{op}}) \cong \mathcal{S}p(A(n,2) \times A^{\text{op}}) \]
\[ I_{A_n} \mapsto T_n. \]
Then, for every small category $B$, it is possible to define an action of the bimodule $T_n$ on $\mathcal{D}(A_n \times B)$ via the canceling tensor product
\[ \otimes_{[A_n]}: \mathcal{S}p(A(n,2) \times A_n^{\text{op}}) \times \mathcal{D}(A_n \times B) \to \mathcal{D}(A(n,2) \times B). \]
Namely, we can define the functor
\[ T_n \otimes_{[A_n]} -: \mathcal{D}(A_n \times B) \to \mathcal{D}(A(n,2) \times B) \]
\[ X \mapsto T_n \otimes_{[A_n]} X. \]

What is left to discuss is why the functor $G^n$ in (6.1) is isomorphic to $T_n \otimes_{[A_n]} -$. For this purpose we recall the following definition:

**Definition 8.2 ([14, Definition 8.1]).** Let $\mathcal{D}$ be a stable derivator and let $A, B \in \text{Cat}$. A morphism $\mathcal{D}^A \to \mathcal{D}^B$ is **left admissible** if it can be written as a composition of

- (LA1) restriction morphisms $u^*: \mathcal{D}^B \to \mathcal{D}^A$,
- (LA2) left Kan extensions $u_i: \mathcal{D}^A \to \mathcal{D}^B$,
- (LA3) right Kan extensions $u_*: \mathcal{D}^A \to \mathcal{D}^B$ along fully faithful functors which amount precisely to adding a cartesian square or right Kan extensions along countable compositions of such functors, and
- (LA4) right extensions by zero $u_*: \mathcal{D}^A \to \mathcal{D}^B$ for sieves $u: A' \to B'$.

Dually, we can define a **right admissible** morphism.
We observe that, by the construction in the proof of Theorem 6.1, the functor $G^n$ is left admissible. We can then apply the following Theorem stating that every left admissible morphism is a canceling tensor product.

**Theorem 8.3** ([14, Theorem 8.5]). Let $\mathcal{D}$ be a stable derivator and let $F: \mathcal{D}^A \to \mathcal{D}^B$ be a morphism. If $F$ is left admissible then there is a bimodule $M \in \mathcal{S}_p(B \times A^{op})$ and a natural isomorphism

$$F \cong M \otimes_{[A]} - : \mathcal{D}^A \to \mathcal{D}^B.$$ 

The proof of this Theorem show why in our case the module $M$ is given by $T_n$ and then we can conclude with the following proposition

**Proposition 8.4.** The following equivalence of functors holds

$$G^n \cong T_n \otimes_{[A_n]} -.$$ 

Similarly, also the functor $i^n$ in (6.1) is isomorphic to a canceling tensor product.

**Declarations**

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