Universal critical exponent in class D superconductors

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We study a physical system consisting of non-interacting quasiparticles in disordered superconductors that have neither time-reversal nor spin-rotation invariance. This system belongs to class D within the recent classification scheme of random matrix ensembles (RME) and its phase diagram contains three different phases: metallic and two distinct localized phases with different quantized thermal Hall conductances. We find that critical exponents describing different transitions (insulator-to-insulator and insulator-to-metal) are identical within the error of numerical calculations and also find that critical disorder of the insulator-to-metal transition is energy independent.

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The properties of quasiparticles in disordered superconductors have been intensively investigated. Those systems are representatives of new symmetry classes different from the familiar three classes in normal disordered conductors and in the Wigner-Dyson random matrix ensembles. A list of additional random matrix ensembles, determined by these new symmetry classes has been established about 10 years ago. These additional random matrix ensembles describe zero-dimensional problems, and are appropriate to model a small grain of a superconductor in the ergodic highly conducting limit. In the corresponding higher-dimensional systems of the same symmetry classes there can be transitions between metallic, localized, or quantized Hall phases for quasiparticles.

The symmetry class D (which we address in this paper) may be realized in superconductors with broken time-reversal invariance, and either broken spin-rotation invariance (as in d-wave superconductors with spin-orbit scattering) or spinless or spin-polarized fermions (as in certain p-wave states). The associated changes in quasiparticle dynamics must be probed by energy transport, since neither charge density nor spin are conserved. In this paper we present extensive numerical results on a model first introduced by Cho and Fisher (CF) (see a detailed description below), which has a particularly rich phase diagram in two dimensions. We use advanced numerical calculations, proposed in Ref. 1 and described in detail in 2, to overcome round-off errors in calculations of renormalized localization lengths. We also apply, for the first time, an optimization algorithm to determine both critical exponent and critical disorder for the insulator-to-metal transition and critical exponent for the insulator-to-insulator transition. Our results suggest the existence of the universal critical exponent \( \nu = 1.4 \pm 0.2 \) for the thermodynamical localization length \( \xi \) for both types of transitions mentioned above, an energy dependent value of the critical disorder \( W_{cr} \approx 0.19 \pm 0.02 \) and show collapse of all data on one curve with thermodynamic localization length \( \xi \sim \frac{\epsilon(W_{cr} - W)}{M} \).

The original network model was proposed to describe transitions between plateaux in the quantum Hall effect (QHE). QHE is realized in a two-dimensional electron gas subjected to a strong perpendicular magnetic field and a random potential. When random potential varies smoothly (its correlation length is much larger than the magnetic length) a semiclassical description becomes relevant: electrons move along the lines of constant potential. When two equipotential lines come close to each other (near a saddle point), tunneling is feasible. In the network model, electrons move along unidirectional links forming closed loops in analogy with semiclassical motion on contours of constant potential. Scattering between links is allowed at nodes in order to map tunneling through saddle point potentials. Propagation along links yields a random phase, thus links are presented by diagonal matrices with elements in the form exp(\( i \phi \)). Transfer matrix for one node relates a pair of incoming and outgoing amplitudes on the left to a corresponding pair on the right; it has the form

\[
T = \begin{pmatrix} \cosh \theta & \sinh \theta \\ \sinh \theta & \cosh \theta \end{pmatrix}.
\]

The node parameter \( \theta \) is related to the electron energy in the following way

\[
\epsilon = -\frac{2}{\pi} \ln(\sinh \theta),
\]

where \( \epsilon \) is a relative distance between the electron energy and the barrier height. It is easy to see that the most "quantum" case (equal probabilities to scatter to the left and to the right) is at \( \epsilon = 0 \) (\( \theta = 0.8814 \)), in fact, numerical calculations show that there is an extended state at that energy.

Numerical simulations on the network model are performed on a system with fixed width \( M \) and periodic boundary conditions in the transverse direction. Multiplying transfer matrices for \( N \) slices and then diagonalizing the resulting total transfer matrix, it is possible to extract the smallest Lyapunov exponent \( \lambda \) (the eigenvalues of the transfer matrix are exp(\( \lambda N \))). The localization length \( \xi_M \) is proportional to 1/\( \lambda \). Repeating calculations for different system widths and different energies it is possible to show that the localization length \( \xi_M \) satisfies a scaling relation

\[
\frac{\xi_M}{M} = f \left( \frac{M}{\xi(\epsilon)} \right).
\]
When renormalized localization length $\xi_M/M$ becomes $M$-independent it strongly suggests that the system is in the critical state. Indeed, it means that when you double the system size $M$, the localization length $\xi_M$ also doubles, or, from the mathematical standpoint, the only way to make Eq. (3) $M$-independent and consistent is to put the thermodynamic localization length $\xi$ equal to infinity (extended state).

For a D class symmetry a Bogoliubov - de Gennes Hamiltonian may be written in terms of a Hermitian matrix. The corresponding time evolution operator is real, restricting the generalized phase factors to be $O(N)$ matrices for a model in which $N$-component fermions propagate on links, and to the values $\pm 1$ for $N = 1$, the case that was studied. There are three models: random bond Ising model\textsuperscript{11,12}, supporting two different localized phases, uncorrelated $O(1)$ model\textsuperscript{13}, where phases on the links are independent random variables and all states are extended, and the model first introduced by Cho and Fisher (CF)\textsuperscript{5} where scattering phases with the value $\pi$ appear in correlated pairs. Each model has two parameters: the first one is a disorder concentration $W$, such that there is a probability $W (1 - W)$ to have a phase 0 ($\pi$) on a given link. The second parameter is an energy $\epsilon$ describing scattering at the nodes. For the CF model, the phase diagram (updated version of which is presented in Fig. 3) in the $\epsilon$-$W$ plane has three distinctive phases: metallic, and two insulating phases characterized by different Hall conductances. The sensitivity to the disorder is a distinctive feature of class D.

The existence of region of extended states means that the smallest Lyapunov exponent at each particular energy is zero or extremely small. This point was discussed in detail in Ref. 3. The advanced algorithm using the structure of the transfer matrix imposed by current conservation and the symmetry of class D suggested there allowed to increase the accuracy of the data significantly. However, in the vicinity of the critical points the noise increases and in order to identify the values of critical disorder and critical energy is zero or extremely small. As one approaches the critical point, the values of all renormalized lengths increase significantly (tending to infinity). In Fig. 1b it is shown that all the data fit onto one curve when we scale the horizontal axis as $\epsilon M^{1/\nu}$ with the critical exponent of the thermodynamic localization length $\nu = 1.4$. Indeed, substituting into the right-hand side of Eq. (3) $\xi \sim \epsilon^{-\nu}$ one can get the variable of the horizontal axis in Fig. 1b. We have performed such calculations for different fixed values of the disorder $W$ in the range $[0.05; 0.15]$. The optimization program have found very close values of $\nu$ for all the data.

Next, we address the most interesting problem - definition of the critical disorder and corresponding critical exponent. We have fixed the value of energy $\epsilon$ and vary $M$ and $W$. The renormalized localization lengths are shown in Fig. 2a for a fixed $\epsilon = 0.1$. We then use the optimization program to find optimal values for both the critical disorder $W_{cr}$ and critical exponent. The results of the scaling are shown in Fig. 2b. Surprisingly, the value of the critical exponent found in this way was identical (within the error of numerical calculations) as for a fixed disorder. The critical disorder found for a wide range of different fixed values of energy $\epsilon (\epsilon = 0.1, 0.2, \ldots 1)$ has the same value $W_{cr}$ independent on $\epsilon$. This way of definition of $W_{cr}$ by the optimization procedure looks more promising, especially taking into account that when one
looks for $\xi_M/M$ to become $M$-independent in the standard procedure, the errors increase significantly and one needs to take $M$ as big as possible and to trust only to the largest values of $M$. We, therefore, suggest that our method leads to the phase diagram shown in Fig. 3 with the vertical critical line separating insulator and metal (for energies $\epsilon$ not very close to zero). We, however, cannot determine the exact position of the multicritical point at $\epsilon = 0$ (studied previously5,14), which is underlined by the dots in Fig. 3.

![FIG. 3: Updated phase diagram for a CF model with metallic, insulating and quantized Hall phases.](image)

The very fact that we have obtained almost identical values of the critical exponent $\nu = 1.4 \pm 0.2$ for both types of transition, insulator-to-insulator transition with $\xi \sim \epsilon^{-\nu}$ and insulator-to-metal transition with $\xi \sim (W_{cr} - W)^{-\nu}$ (which is rather different from anything obtained before, e.g. two different critical exponents for a class C5,14) has led us to the last fit. We have suggested that the thermodynamic localization length should depend on the product of both distances of energy and disorder from the critical ones with the same critical exponent in the following way $\xi \sim [\epsilon(W_{cr} - W)]^{-\nu}$. Indeed, all the data for the renormalized localization lengths obtained for various $\epsilon$, $W$ and $M$ collapse on the same curve after suggested fit as shown in Fig. 4.

![FIG. 4: Collapse of all of the data on a single curve with universal critical exponent $\nu = 1.4$ and universal critical disorder $W_{cr} = 0.19$.](image)

Detailed analytical explanations for the divergence of the localization length in various types of class D systems for a one-dimensional model ($M = 2$) were presented earlier. Here we wish to add a qualitative argument for a two-dimensional system. First, consider a system without disorder ($W = 0$). Then, there are no phases on the links, tunneling alone defines a quasiparticle behaviour, and for any non-zero energy makes system an insulator. For any $\epsilon \neq 0$ at any node quasiparticle prefers to turn into the same direction circling closed trajectories. Furthermore, because the probability of a tunneling event is proportional to $\exp(-\pi \epsilon)$, it is trivial to derive that after $N$ tunneling events the probability becomes $\exp(-\pi \epsilon N)$, and, considering a number of tunneling events $N$ as a distance, one immediately obtains the localization length as $\xi \sim \epsilon^{-1}$. We have found that those statements, mentioned above, are correct numerically as well. For $W = 0$ and $\epsilon \neq 0$ all states are ideally localized ($\xi_M$ is $M$-independent) and fit onto one curve depending on $\xi/M$ with $\xi \sim \epsilon^{-1}$. Now let us introduce disorder. It causes the appearance of phase $\pi$ on some links. Those phases can cause a constructive interference for different trajectories leading from one point of the network model to another, making therefore an extended state. For a $O(1)$ model phases on the different links are uncorrelated, and therefore even a small disorder $W$ introduces that constructive interference immedi-
ately. In the Cho-Fisher model that we consider in this paper, phases on both sides of the node are correlated (the same), therefore small disorder does not mean an immediate delocalization, but it is clear that the stronger the disorder is the larger is the opportunity for the state to be extended.

To summarize, we have studied critical exponents describing the divergence of thermodynamic localization lengths for two both types of transition (insulator-to-insulator and insulator-to-metal) of the CF model and found that within the error of numerical calculations both exponents are equal to \( \nu = 1.4 \pm 0.2 \). We have used optimization procedure in order to determine not only the best fit of the data leading to the critical exponent, but also to look for a critical disorder if the energy is fixed.

We have found that the value of the critical disorder is energy independent (for energies \( \epsilon \) not very close to zero). Obviously, we cannot rule out completely some possible weak dependence of the critical disorder on the energy, especially when \( \epsilon \to \infty \), but for a wide range of energies studied, the critical disorder is \( W_{\text{cr}} = 0.19 \pm 0.02 \).

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