Provably adaptive reinforcement learning in metric spaces

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Abstract

We study reinforcement learning in continuous state and action spaces endowed with a metric. We provide a refined analysis of a variant of the algorithm of Sinclair, Banerjee, and Yu (2019) and show that its regret scales with the zooming dimension of the instance. This parameter, which originates in the bandit literature, captures the size of the subsets of near optimal actions and is always smaller than the covering dimension used in previous analyses. As such, our results are the first provably adaptive guarantees for reinforcement learning in metric spaces.

1 Introduction

In reinforcement learning (RL), an agent learns to select actions to navigate a state space and accumulates reward. In terms of theoretical results, the majority of results address the tabular setting, where the number of states and actions are finite and comparatively small. However, tabular problems are rarely encountered in practical applications, as state and action spaces are often large and may even be continuous. To address these practically relevant settings, a growing body of work has developed algorithmic principles and guarantees for reinforcement learning in continuous spaces.

In this paper, we contribute to this line of work on reinforcement learning in continuous spaces. We consider episodic RL where the joint state-action space is endowed with a metric and we posit that the optimal \( Q^* \) function is Lipschitz continuous with respect to this metric. This setup has been studied in several recent works establishing worst case regret bounds that scale with the covering dimension of the metric space (Song and Sun, 2019; Sinclair et al., 2019; Touati et al., 2020). While these results are encouraging, the guarantees are overly pessimistic, and intuition from the special case of Lipschitz bandits suggests that much more adaptive guarantees are achievable. In particular, while the Lipschitz contextual bandits setting of Slivkins (2014) is a special case of this setup, no existing analysis recovers his adaptive guarantee that scales with the zooming dimension of the problem.

Our contribution. We give the first analysis for reinforcement learning in metric spaces that scales with the zooming dimension of the instance instead of the covering dimension of the metric space. The zooming dimension, originally defined by Kleinberg et al. (2019) in the context of Lipschitz bandits, measures the size of the set of near-optimal actions and can be much smaller than the covering dimension in favorable instances. For reinforcement learning, the natural generalization is to measure near-optimality relative to the \( Q^* \) function; this recovers the definition of Kleinberg et al. (2019) and Slivkins (2014) for bandits and contextual bandits, respectively as special cases. As a consequence, our guarantees also strictly generalize theirs to the multi-step reinforcement learning setting. In addition, our guarantee addresses an open problem of Sinclair et al. (2019) by characterizing problems where refined guarantees are possible.

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Our result is based on a refined analysis of a variant of the algorithm of Sinclair et al. (2019). This algorithm uses optimism to select actions and an adaptive discretization scheme to carefully refine a coarse partition of the state-action space to focus (“zoom in”) on promising regions. Adaptive discretization is essential for obtaining instance-dependent guarantees, but the bounds in Sinclair et al. (2019) do not reflect this favorable behavior.

At a technical level, the main challenge is that, unlike in bandits, we cannot upper bound the number of times a highly suboptimal arm will be selected by the optimistic strategy. Analysis for the bandit setting uses these upper bounds to prove that the adaptive discretization scheme will not zoom in on suboptimal regions, which is crucial for the instance-dependent bounds. However, in RL, the algorithm actually can zoom in on and select actions in suboptimal regions, but only when there is significant error at later time steps. Thus, in the analysis, we credit error incurred from a highly suboptimal region to the later time steps, so we can proceed as if we never zoomed in on this region at all. Formally, this analysis uses the clipped regret decomposition of Simchowitz and Jamieson (2019) as well as a careful bookkeeping argument to obtain the instance-dependent bound.

Changes from the initial version. The present version of the paper corrects an error in the version published in NeurIPS 2020. The differences are both in the algorithm, which is no longer identical to that of Sinclair et al. (2019), and in the analysis, which is somewhat more involved. The changes address an issue that arises when a child ball inherits updates from its parent, which results in each sample appearing many times with the same weight $\alpha_t^i$ in the recursive regret decomposition used in the tabular analysis of Jin et al. (2018), displayed in (3). This ultimately compromises the final regret bound, which crucially uses that these weights form a convergent series.

The fix is that child balls no longer inherit data from the parent so that every interaction tuple (of state, action, reward, next state) results in exactly one update. This ensures that the $\alpha_t^i$ weight sequences converge, but is also problematic, as child balls are initialized with large bonuses, so the confidence sum does not capture the zooming property we hope to exhibit. We resolve this latter issue with a buffering phase where a child ball is slowly updated but is never played. Specifically, once a parent ball has received enough samples, we split it and mark the children as buffering. While the children are buffering, we continue to use the parent for action selection and mostly continue to update the parent, but every $H + 1^{st}$ update is instead performed on the child. Once the child has enough updates that the bonus is small, we move it out of the buffering phase and can safely use it for decision making.

Unfortunately, the buffering approach means that the parent ball is periodically chosen but not updated, which again results in a weight sequence where some terms (specifically every $H^{th}$ term) appears twice. However this sequence is much more benign than the one that arises if we re-use samples. Indeed, we can show that this new sequence is convergent via a new amortizing argument that relates it to the original one in Jin et al. (2018).

The final challenge is that now the parent ball remains active for much longer. This results in a final regret bound that now scales polynomially with $\Lambda$, the maximum number of children that a parent can have (or the doubling constant of the metric space), and additionally is polynomially worse in its dependence on the horizon $H$ than the bound claimed in the NeurIPS 2020 version of the paper. On the other hand, the new bound still captures the adaptive and zooming nature of the algorithm.

2 Preliminaries

We consider a finite-horizon episodic reinforcement learning setting in which an agent interacts with an MDP, defined by a tuple $(S, A, H, P, r)$. Here $S$ the state space, $A$ is the action space, $H \in \mathbb{N}$ is the horizon, $P$ is
the transition operator and \( r \) is the reward function. Formally, \( \mathbb{P} : S \times A \to \Delta(S) \) and \( r : S \times A \to [0, 1] \)
where \( \Delta(\cdot) \) denotes the set of distributions over its argument.\(^1\)

A (nonstationary) policy \( \pi \) is a mapping from states to distributions over actions for each time. Every policy has non-stationary value and action-value functions, defined as

\[
V^\pi_h(x) := \mathbb{E}_\pi \left[ \sum_{h'=h}^H r_{h'}(x_{h'}, a_{h'}) \mid x_h = x \right], \quad Q^\pi_h(x, a) := r_h(x, a) + \mathbb{E} \left[ V^\pi_{h+1}(x') \mid x, a \right].
\]

Here \( \mathbb{E}_\pi [\cdot] \) denotes that all actions are chosen by policy \( \pi \) and transitions are given by \( \mathbb{P} \). The optimal policy \( \pi^* \) and optimal action-value function \( Q^* \) are defined recursively as

\[
Q^*_h(x, a) := r_h(x, a) + \mathbb{E} \left[ \max_{a'} Q^*(x', a') \mid x, a \right], \quad \pi^*_h(x) = \arg\max_a Q^*_h(x, a).
\]

The optimal value function \( V^*_h \) is defined analogously.

The agent interacts with the MDP for \( K \) episodes, where in episode \( k \) the agent picks a policy \( \pi_k \) and we generate the trajectory \( \gamma_k = (x^k_1, a^k_1, r^k_1, x^k_2, a^k_2, r^k_2, \ldots, x^k_H, a^k_H, r^k_H) \) where (1) \( x^k_1 \) is chosen adversarially, (2) \( a^k_h = \pi_k(x^k_h) \), (3) \( x^k_{h+1} \sim \mathbb{P}(\cdot \mid x^k_h, a^k_h) \), (4) \( r^k_h = r(x^k_h, a^k_h) \). We would like to choose actions to maximize the cumulative rewards \( \sum_{h=1}^H r^k_h \).

Equipped with these definitions, we can state our performance criterion. Over the course of \( K \) episodes, we would like to accumulate reward that is comparable to the optimal policy, formalized via the notion of regret:

\[
\text{Reg}(K) := \sum_{k=1}^K \left( V^*_1(x^k_1) - \sum_{h=1}^H r^k_h \right).
\]

In particular, we seek algorithms with regret rate that is sublinear in \( K \). Note that we have not assumed that \( |S| \) and \( |A| \) are finite, and we also allow for the starting state \( x^k_1 \) to be chosen adversarially in each episode.

### 2.1 Metric spaces.

Instead of assuming that \( |S| \) and \( |A| \) are finite, we will poset a metric structure on these spaces. We recall the key definitions for metric spaces. A space \( Y \) equipped with a function \( \mathcal{D} : Y \times Y \to \mathbb{R}_+ \) is a metric space if \( \mathcal{D} \) satisfies (a) \( \mathcal{D}(y, y') = 0 \) iff \( y = y' \) (b) \( \mathcal{D} \) is symmetric, and (c) \( \mathcal{D} \) satisfies the triangle inequality \( \mathcal{D}(x, y) \leq \mathcal{D}(x, z) + \mathcal{D}(z, y) \). If these properties hold then \( \mathcal{D} \) is called a metric. For a radius \( r > 0 \), we use the notation \( B(y, r) := \{ y' \in Y : \mathcal{D}(y, y') < r \} \) to denote the open ball centered at \( y \) with radius \( r \). For a subset \( Y' \subseteq Y \) the diameter is defined as \( \text{diam}(Y') := \sup_{y, y' \in Y'} \mathcal{D}(y, y') \). We also use the standard notions of covering and packing to measure the size of metric spaces.

**Definition 1** (Notions of size). We define the following notions of size for a metric space.

- A covering of \( Y \) at scale \( r \) (also called an \( r \)-covering) is a collection of subsets of \( Y \), each with diameter at most \( r \), whose union equals \( Y \). The minimum number of subsets that form an \( r \)-covering is the \( r \)-covering number, denoted \( N_r(Y) \).

- A packing of \( Y \) at scale \( r \) (also called an \( r \)-packing) is a collection of points \( Z \subset Y \) such that \( \min_{z 
eq z' \in Z} \mathcal{D}(z, z') \geq r \). The maximum number of points that form an \( r \)-packing is the \( r \)-packing number, denoted \( N^\text{pack}_r(Y) \).

\(^1\)Deterministic rewards simplifies the presentation but has no bearing on the final results. In particular, we can handle stochastic bounded rewards with minimal modification to the proofs.
• An \( r \)-net of \( Y \) is an \( r \)-packing \( S \subset Y \) for which \( \{ B(y, r) \}_{y \in S} \) covers \( Y \).

• Define the doubling constant \( \Lambda(Y) := \max_{r > 0, y \in Y} N_{r/2}(B(y, r)) \), which is the maximum number of balls of radius \( r/2 \) required to cover some ball of radius \( r \).

These definitions also apply to subsets of the metric space, which will be important for our development. Also note that \( N_{2r}^{\text{pack}}(Y) \leq N_r(Y) \leq N_1^{\text{pack}}(Y) \).

### 2.2 Main Assumptions

We now state the main assumptions that we adopt in our analysis. These or closely related assumptions are standard in the literature on bandits and reinforcement learning in metric spaces (Song and Sun, 2019; Sinclair et al., 2019; Touati et al., 2020).

**Assumption 1.** \((S \times A, \mathcal{D})\) is a metric space with finite diameter \( \text{diam}(S \times A) = d_{\text{max}} < \infty \).

**Assumption 2.** For every \( h \in [H] \), \( Q_h^* \) is \( L \)-Lipschitz continuous with respect to \( \mathcal{D} \):

\[
\forall (x, a), (x', a') : |Q_h^*(x, a) - Q_h^*(x', a')| \leq L \cdot \mathcal{D}((x, a), (x', a')).
\]

Additionally \( V_h^* \) is \( L \)-Lipschitz with respect to the metric \( \mathcal{D}_S : (x, x') \mapsto \min_{a, a'} \mathcal{D}((x, a), (x', a')) \):

\[
\forall x, x' : |V_h^*(x) - V_h^*(x')| \leq L \cdot \min_{a, a'} \mathcal{D}((x, a), (x', a')).
\]

Assumption 1 is a basic regularity condition, while the first part of Assumption 2 imposes continuity of the \( Q^* \) function. In particular, Lipschitz-continuity characterizes how the metric structure influences the reinforcement learning problem. These assumptions appear in prior work, and we note that (1) is strictly weaker than assuming that \( \mathbb{P} \) is Lipschitz continuous (Kakade et al., 2003; Ortner and Ryabko, 2012). The second part of Assumption 2 reflects an additional structural assumption on the problem, which is a departure from previous work. In detail, (2) posits that the optimal value function \( V_h^* \) is \( L \)-Lipschitz with respect to a metric defined only on the states that is derived from the original one. This metric is dominated by the original one since for each \( (x, x', a) \) we have \( \min_{a_1, a_2} \mathcal{D}((x, a_1), (x', a_2)) \leq \mathcal{D}((x, a), (x', a)) \), so this assumption is not directly implied by (1). However, whenever \( \mathcal{D} \) is sub-additive in the sense that \( \mathcal{D}((x, a), (x', a')) \leq \mathcal{D}_S(x, x') + \mathcal{D}_A(a, a') \), then the assumption holds trivially. Sub-additivity holds for most metrics of interest, including those induced by \( \ell_p \) norms for \( p \geq 1 \). As such, we do not view this assumption as particularly restrictive.

### 2.3 Related work

Reinforcement learning in the tabular setting, where the state and action spaces are finite, is relatively well-understood (Azar et al., 2017; Dann et al., 2017; Zanette and Brunskill, 2019). Of this line of work, the two most related papers are those of Jin et al. (2018) and Simchowitz and Jamieson (2019). Our results build on the model-free/martingale analysis of Jin et al. (2018), which has been used in recent work on RL in metric spaces (Song and Sun, 2019; Sinclair et al., 2019; Touati et al., 2020). We also employ techniques from the gap-dependent analysis of Simchowitz and Jamieson (2019). In particular, we use a version of their “clipping” argument, as we will explain in Section 5.

Moving beyond the tabular setting, several papers study reinforcement learning in metric spaces, originating with the results of Kakade et al. (2003) (c.f., Ortner and Ryabko (2012); Ortner (2013); Song and Sun (2019); Ni et al. (2019); Sinclair et al. (2019); Touati et al. (2020)). Of these, the most related result is that of Sinclair et al. (2019) who study the adaptive discretization algorithm and give a worst-case regret...
analysis, showing that the algorithm has a regret rate of $K^{d+1}$ where $d$ is the covering dimension of the metric space. Essentially the same results appear in Touati et al. (2020), although the algorithm is slightly different. However, none of these results give sharper instance-dependence guarantees that reflect benign problem structure, as we will obtain.

For the special case of (contextual) bandits, several instance-dependent guarantees that yield improved regret rates exist (Auer et al., 2007; Valko et al., 2013; Kleinberg et al., 2019; Bubeck et al., 2011; Slivkins, 2014; Krishnamurthy et al., 2019). For non-contextual bandits, the results and assumptions vary considerably, but most results quantify a benign instance in terms of the size of the set of near-optimal actions. The formulation that we adopt is the notion of zooming dimension, which measures the growth rate of the $r$-packing number of the set of $O(r)$-suboptimal arms. This notion has been used in several works on bandits and contextual bandits in metric spaces, and we will recover some of these results as a special case of our main theorem.

3 Main Results

Our main result is a regret bound that scales with the zooming dimension. We introduce this parameter with a sequence of definitions. First, we define the gap function, which describes the sub-optimality of an action $a$ for state $x$.

**Definition 2 (Gap).** For any $(x, a) \in S \times A$, for $h \in [H]$, the stage-dependent sub-optimality gap is

$$\text{gap}_h(x, a) := V^*_h(x) - Q^*_h(x, a).$$

We use the gaps to define the subset of the metric space that is near-optimal.

**Definition 3 (Near-optimal set).** We define near-optimal set as

$$\mathcal{P}_{Q^*}^r := \left\{ (x, a) \in S \times A : \text{gap}_h(x, a) \leq \left( \frac{2(H+1)}{d_{\max}} + 2L \right) r \right\}.$$

Intuitively, $\mathcal{P}_{Q^*}^r$ is the set of state-action pairs with gap that is $O(r)$ at stage $h$. The constant in the definition is a consequence of our analysis, but it is quite similar to the constant in the definition of Slivkins (2014) for contextual bandits. In particular, he considers $d_{\max} = 1, H = 1, L = 1$ and obtains a constant of 12, while we obtain a constant of 6 in this case.

Finally, we define the zooming number and the zooming dimension.

**Definition 4 (Zooming number and dimension).** The $r$-zooming number is the $r$-packing number of the near-optimal set $\mathcal{P}_{Q^*}^r$, that is $N^\text{pack}_r(\mathcal{P}_{Q^*}^r)$. The stage-dependent zooming dimension is defined as

$$z_{h,c} := \inf \left\{ d > 0 : N^\text{pack}_r(\mathcal{P}_{Q^*}^r) \leq cr^{-d}, \forall r \in (0, d_{\max}] \right\}.$$ 

The zooming dimension for the instance as the largest among all stages $z_c = \max_{h \in [H]} z_{h,c}$.

Intuitively, the zooming dimension measures how the near-optimal region grows as we change the sub-optimality level $r$. Importantly, we use $r$ both to parametrize the radius in the packing number and the sub-optimality. Thus, the zooming number captures how many $r$-separated points can be packed into the $O(r)$ sub-optimal region.
Adaptive Q-learning has the following regret

\[ \text{Reg}(K) \leq \tilde{O}\left( (H^{3/2} + \sqrt{H \Lambda}) \inf_{r_0 \in [0, d_{\max}]} \left( \sum_{h=1}^{H} \sum_{r=d_{\max}2^{-i}, r \geq r_0} N_r^\text{pack}(P_{h,r}^{Q^*}) \frac{d_{\max} \sqrt{H \Lambda}}{r} + \frac{K r_0}{d_{\max}} \right) \right), \]

where \( x \in S \times A \) is near-optimal actions for \( x \).

Figure 1: An example where the zooming dimension is 1 while the covering dimension is 2.

With these definitions, we can now state the main theorem.

**Theorem 1.** For any initial states \( \{x_k^k : k \in [K]\} \), and any \( \delta \in (0, 1) \), with probability at least, \( 1 - \delta \), Adaptive Q-learning has the following regret:

\[ \text{Reg}(K) \leq \tilde{O}\left( (H^{3/2} + \sqrt{H \Lambda}) \inf_{r_0 \in [0, d_{\max}]} \left( \sum_{h=1}^{H} \sum_{r=d_{\max}2^{-i}, r \geq r_0} N_r^\text{pack}(P_{h,r}^{Q^*}) \frac{d_{\max} \sqrt{H \Lambda}}{r} + \frac{K r_0}{d_{\max}} \right) \right), \]

Before turning to a discussion of the theorem, we state some corollaries. First, by optimizing \( r_0 \), we obtain a regret bound in terms of the zooming dimension.

**Corollary 2.** For any initial states \( \{x_k^k : k \in [K]\} \), and any \( \delta \in (0, 1) \), with probability at least, \( 1 - \delta \), Adaptive Q-learning has:

\[ \text{Reg}(K) \leq \tilde{O}\left( H^{3/2} + \frac{\sqrt{H \Lambda}}{d_{c+2}} \right), \]

for any constant \( c > 0 \).

Finally, we recover the regret rate of Slivkins (2014) in the special case of contextual bandits.

**Corollary 3 (Contextual bandits).** If \( H = 1 \), then Adaptive Q-learning has regret:

\[ \tilde{O}\left( \frac{K}{d_{c+2}} \right), \]

which recovers the regret rate of Slivkins (2014).

We now turn to the remarks:

- Theorem 1 gives a regret bound that depends on the packing numbers of the near-optimal set (Definition 3). This bound should be compared with the “metric-specific” regret guarantee of Sinclair et al. (2019) or the “refined regret bound” of Touati et al. (2020). Both of these results have the same form as ours, but with \( N_r^\text{pack}(S \times A) \) in the place of \( N_r^\text{pack}(P_{h,r}^{Q^*}) \). As \( P_{h,r} \subset S \times A \), our bound improves on theirs in this sense, at the cost of a \( \sqrt{H \Lambda} \) additional dependence.

- The more-interpretable bound is in terms of the zooming dimension (Definition 4), which highlights the dependence on the number of episodes \( K \). We obtain a regret rate of \( \frac{K}{d_{c+2}} \) for any constant \( c > 0 \), which should be compared with the non-adaptive rate \( K^{\frac{d_{c+2}}{d_{c+2}}} \) that scales with the covering dimension (Song and Sun, 2019; Sinclair et al., 2019; Touati et al., 2020).\(^3\) As the zooming dimension can be smaller than covering dimension (recall Figure 1), this bound demonstrates a polynomial improvement over non-adaptive approaches.

\(^2\) Throughout the paper \( \tilde{O}(\cdot) \) suppresses logarithmic dependence in its argument.

\(^3\) We always treat \( c \) as a universal constant, so its dependence in the regret bounds is suppressed.
• Corollary 3 shows that our bound recovers the guarantee from Slivkins (2014), although his bound does not require that (2) holds. We give a more detailed explanation on the necessity of (2) in Section 5. Nevertheless, the fact that we essentially recover his bound suggests that our results are the natural generalization to multi-step RL.

• Finally, we remark that we can instantiate the result in the tabular setting with finite $S, A$ by taking the metric to be $D((x, a), (x', a')) = 1\{(x, a) \neq (x', a')\}$. In this case we obtain a “partial” gap-dependent bound of the form:

$$\text{poly}(H) \cdot \left(\sqrt{|S|K} + \sum_{h=1}^{H} \sum_{x \in \mathcal{S}} \sum_{a : \text{gap}_h(x, a) > 0} \frac{\log(K)}{\text{gap}_h(x, a)}\right).$$

This is not a fully gap-dependent bound because of the $\sqrt{|S|K}$ term, but it does recover an intermediate result of Simchowitz and Jamieson (2019). In particular, this confirms that the model-free methods can achieve a partial gap-dependent guarantee for the tabular setting.

### 4 Algorithm

As we have mentioned, the algorithm is based on the Adaptive $Q$-learning algorithm of Sinclair et al. (2019). The pseudocode is presented in Algorithm 1. The algorithm adaptively partitions the state-action space to focus on the informative regions, and uses optimism to explore the space and drive the agent to regions with high reward. Compared to Sinclair et al. (2019), the main difference is that when a new partition is formed it does not inherit the value and sample count from its parent. Instead, it will go through an additional buffering phase before it is activated and used for action selection.

During the execution, the algorithm creates many balls $B \subset S \times A$ for each stage $h$. For stage $h$ and episode $k$, we use $\mathcal{P}_h^k$ to denote the set of active balls, and $\mathcal{F}_h^k$ to denote the set of buffering balls. When a set of balls are created, they are first moved to the buffering set, and balls in this set will not be used for decision making, but may occasionally be updated.

Each ball $B$ is associated with (1) a radius, denoted $r(B)$, (2) a domain, denoted $\text{dom}_h^k(B)$, (3) several counters and thresholds related to the amount of data it has seen, and (4) an optimistic estimate of $Q_h^k$. The radius of a ball is $r(B) := \text{diam}(B)$ and the domain $\text{dom}_h^k(B)$ is the set of points contained in this ball, but not in any other active ball with a smaller radius. Formally,

$$\text{dom}_h^k(B) := B \setminus \{\cup_{B' \in \mathcal{P}_h^k : r(B') < r(B)} B'\}.$$

For the counters, $n_h^k(B)$ denotes the number of times ball $B$ has been updated at stage $h$ and episode $k$, while $\bar{n}_h^k(B)$ denotes the number of times ball $B$ has been “played” or used for decision making. These two counters will not be equivalent in general. We also use two thresholds: $N_{\text{split}}(B)$ is the number of updates we must perform before we split $B$ into smaller balls, and $N_{\text{min}}(B)$ is the number of updates we must perform before moving $B$ from the buffering set $\mathcal{F}_h^k$ to the active set $\mathcal{P}_h^k$. These latter two are defined as:

$$N_{\text{split}}(B) := 4N_{\text{min}}(B) := \left(\frac{d_{\text{max}}}{r(B)}\right)^2.$$

When a ball is split in line 9 the resulting balls are called children and denoted $C(B)$. Finally, each ball maintains a scalar $Q_h^k(B)$ which serves as an upper bound on $\max_{(x, a) \in B} Q_h^k(x, a)$.

In stage $h$ of episode $k$, we select the action for state $x_h^k$ as follows: we consider all the smallest active balls that contains $x_h^k$, defined as “relevant” balls

$$\text{rel}_h^k(x) := \{B \in \mathcal{P}_h^k \mid \exists a, (x, a) \in \text{dom}_h^k(B)\}.$$
Algorithm 1 Adaptive $Q$-learning with zooming dimension

1: For $h \in [H]$, initialize $\mathcal{F}_{h}^1 = \emptyset$, $\mathcal{P}_{h}^1$ to be a $\frac{d_{\max}}{H}$-net of $S \times A$.
2: For $B \in \mathcal{P}_{h}^1$, define $Q_{h}(B) = H \cdot n_{h}^k(B) = n_{h}^k(B) = 0$.
3: for each episode $k = 1, 2, \ldots, K$ do
    4:     Receive $x_{h}^k$.
    5:     for stage $h = 1, 2, \ldots, H$ do
        6:         $B_{h}^k = \arg\max_{B \in \mathcal{P}_{h}^k} Q_{h}^k(B)$, $\tilde{n}_{h}^k(B_{h}^k) = n_{h}^k(B_{h}^k) + 1$.
        7:         Play action $a_{h}^k$ for some $(x_{h}^k, a_{h}^k) \in \text{dom}_{h}^k(B_{h}^k)$, receive $r_{h}^k, x_{h+1}^k$.
        8:         if $\tilde{n}_{h}^k(B_{h}^k) > \text{N}_{\text{split}}(B_{h}^k)$ then
            9:             if $B_{h}^k$ is not split then split $B_{h}^k$.
            10:            Create a set of children $C(B_{h}^k) = \{ r_{h}^k \}$-net of $\text{dom}_{h}^k(B_{h}^k)$.
            11:            Set $\mathcal{F}_{h}^{k+1} = \mathcal{F}_{h}^k \cup C(B_{h}^k)$.
        12:         else if $\tilde{n}_{h}^k(B_{h}^k) \mod (H + 1) = 0$ then
            13:             Find $B' \in C(B_{h}^k)$ such that $(x_{h}^k, a_{h}^k) \in B'$ and set $B_{h}^k = B'$.
        14:     end if
        15:     end if
    16:     Update $n_{h}^{k+1}(B_{h}^k) = n_{h}^k(B_{h}^k) + 1$ and set $t = n_{h}^{k+1}(B_{h}^k)$.
    17:     $V_{h+1}(x_{h+1}) = \min \left\{ H, \max_{B \in \mathcal{P}_{h+1}^k} Q_{h+1}^k(B) \right\}$.
    18:     $Q_{h+1}^k(B_{h}^k) = (1 - \alpha_t)Q_{h}^k(B_{h}^k) + \alpha_t(r_{h}^k + b_t + V_{h+1}(x_{h+1}))$.
    19:     if $B_{h}^k \in \mathcal{F}_{h}^k$ and $n_{h}^{k+1}(B_{h}^k) \geq \text{N}_{\text{min}}(B_{h}^k)$ then
        20:         Move $B_{h}^k$ from $\mathcal{F}_{h}^k$ to $\mathcal{P}_{h}^k$, i.e. $\mathcal{F}_{h}^{k+1} = \mathcal{F}_{h}^k \setminus \{ B_{h}^k \}, \mathcal{P}_{h}^{k+1} = \mathcal{P}_{h}^k \cup \{ B_{h}^k \}$.
        21:         Set $\tilde{n}_{h}^{k+1}(B_{h}^k) = n_{h}^k(B_{h}^k)$
    22:     end if
    23: end for
    24: Advance all other algorithm state (i.e., $\mathcal{P}_{h}^{k+1} \leftarrow \mathcal{P}_{h}^k$, etc., if not explicitly updated above)
25: end for

Among the relevant balls, the algorithm selects the ball $B_{h}^k$ with the highest $Q_{h}^k(B)$ value and plays an arbitrary action such that $(x_{h}^k, a) \in \text{dom}_{h}^k(B_{h}^k)$. We almost always update the ball that we play, except sometimes we invoke line 13 where we rebind $B_{h}^k$ to be one of the children. In this case, we play a certain ball but then update its child. At the end of the episode, we update the estimated $Q$ value $Q_{h}^k(B_{h}^k)$ and increment the sample count $t = n_{h}^k(B_{h}^k) + 1$. The update rule is a form of optimistic $Q$ learning

$$ Q_{h+1}^k(B_{h}^k) = (1 - \alpha_t)Q_{h}^k(B_{h}^k) + \alpha_t(r_{h}^k + b_t + V_{h+1}(x_{h+1})), $$

where the $\alpha_t$ is the learning rate and $b(t)$ is the bonus added to ensure that $Q_{h}^k$ is optimistic. Formally,

$$ \alpha_t := \frac{H + 1}{H + t}, \quad b_t := 2\sqrt{\frac{H^3 \log \left(4HK/\delta \right)}{t}} + \frac{4L\max_{k} \sqrt{HK + \lambda + 1}}{\sqrt{t}}. $$

For all other balls at stage $h$, we set $Q_{h+1}^k(B) \leftarrow Q_{h}^k(B)$, with no update.

We split a ball $B$ as soon as $n_{h}^k(B) \geq \text{N}_{\text{split}}(B)$. When splitting, we create a set of new “children” balls with radius $r(B)/2$ that forms an $r(B)/2$-net of $\text{dom}_{h}^k(B)$. These “children” are added to the buffering set.
$F^k_h$. Once a ball $B \in F^k_h$ receives $N_{\min}(B)$ updates, we move it to the active set $P^k_h$ and we can use it for action selection. This splitting rule leads to the following invariant:

**Lemma 4** (Lemma 5.3 in Sinclair et al. (2019)). For every $(h, k) \in [H] \times [K]$, we have

1. (Covering) The domains of balls in $P^k_h$ covers $S \times A$.
2. (Separation) For any two balls of radius $r$, their centers are at distance at least $r$.

The last component to describe is the warm-starting process for balls in the buffering phase, which is the main difference compared with the algorithm of Sinclair et al. (2019). This process works as follows. A ball $B$ with $n^k_h \geq N_{\text{split}}(B)$ updates may still be chosen for action selection if some of its children are still buffering (if no child is buffering, then, by the definition of dom$_h^k$, B cannot be selected). During this phase, every $H + 1$ times that we play ball $B$, we instead use the sample to update one of the children, specifically the one that contains $(x^k_h, a^k_h)$. In this way, balls in the buffering set $F^k_h$ slowly accumulate samples and eventually can be moved to the active set.

## 5 Proof sketch

In this section we describe the main steps of the proof, with details deferred to the appendix.

It is worth reviewing prior regret analyses for episodic RL (Jin et al., 2018). The arguments establish a regret decomposition that relates the estimate $V^k_h$ to $V^\pi_h$, the expected reward collected in episode $k$. The decomposition is recursive in nature, involving differences between $Q^k_h$ and $Q^\pi_h$. These are controlled by the update rule and the design of the learning rate. In particular, we can bound $Q^k_h - Q^\pi_h$ by an immediate “surplus” $\beta_t$ and the downstream value function error. Formally for any ball $B$ with $(x, a) \in \text{dom}_h^k(B)$

$$Q^k_h(B) - Q^\pi_h(x, a) \leq 1_{t=0}H + \sum_{i=1}^t \alpha^i_t(V^k_{h+1} - V^\pi_{h+1})(x^k_{h+1}) + \beta_t,$$

where $t = n^k_h(B)$, $\alpha^i_t = \alpha_i \prod_{j=i+1}^t (1 - \alpha_j)$ and $\beta_t = 2\sum_{i=1}^t \alpha^i_t b_i$. Here $k_i$ is the index of the episode where $B$ was updated for the $i$th time. Summing over all episodes and grouping terms appropriately (and ignoring the buffering process for now), we obtain

$$\sum_{k=1}^K (V^k_h - V^\pi_h)(x^k_h) \leq \sum_{k=1}^K (H1_{[n^k_h=0]} + \beta_n^k + \xi^k_h) + (1 + 1/H) \sum_{k=1}^K \left(V^k_{h+1} - V^\pi_{h+1}\right)(x^k_{h+1}),$$

where $\xi^k_{h+1}$ is a stochastic term that can be ignored for this discussion. Note that, as long as $V^k_h$ is optimistic (which we will verify), this also provides a bound on the regret.

For the tabular setting, Jin et al. (2018) use this regret decomposition to obtain a worst-case bound. The leading term arises from the “surplus” term $\beta_n^k$, which leads to a poly$(H)\sqrt{SAK}$ regret bound for the tabular setting. On the other hand for our setting, the splitting rule and the buffering scheme implies that, for any ball $B$, we must have $n^k_h \leq (H\Lambda + \Lambda + 1) (d_{\text{max}}/r(B))^2$, as we will show. We can use this to obtain a bound that depends on the number of active balls at each scale $r$ times $d_{\text{max}}/r$. If we could bound the number of active balls at scale $r$ in terms of the packing number $N_{r}^{\text{pack}}(P^k_{h,r})$, then we would obtain the instance-dependent bound.

Unfortunately, this is not possible. In general, the algorithm will activate balls outside of the near-optimal region, because we may have to select a highly suboptimal ball many times to reduce downstream overestimation error. So indeed the number of active balls at scale $r$ could be much larger than the packing of the near-optimal set.
where \( \xi \) hence the gaps, are Lipschitz. We recall that (2) is implied by (1) if the metric is sub-additive.

We address this with the following key observation. If the surplus \( \beta_{h}^{k} \) is small compared to gap, and we choose this ball, it must be the case that the downstream regret is quite large, otherwise we would not have chosen this ball. If this is true, we can account for the surplus by adding a small constant fraction of the future regret. In otherwords, we can “clip” the surplus to zero once it is proportional to the gap, and we only pay a constant factor in the recursive term. This is the clipping trick developed by Simchowitz and Jamieson (2019) to establish gap dependent bounds for tabular MDP. Formally instead of (3), we have the following lemma.

**Lemma 5** (Clipped upper bound). For any \( \delta \in (0, 1) \) with probability at least \( 1 - \delta/2 \), \( \forall h \in [H] \),

\[
Q^{k}_{h}(B_{h}) - Q^{\ast}_{h}(x^{k}_{h}, a^{k}_{h}) \leq (1 + 1/H) \left( H1_{|t=0|} + \sum_{i=1}^{t} \alpha_{i}^{k}(V^{k}_{h+1} - V^{\ast}_{h+1})(x^{k}_{h+1}) \right) + \text{clip} \left[ \beta_{t} \left| \frac{\text{gap}_{h}(x^{k}_{h}, a^{k}_{h})}{H + 1} \right| \right],
\]

where \( t = n_{h}^{k}(B) \), \( \alpha_{i}^{k} = \alpha_{i}^{k} \prod_{j=1}^{t} (1 - \alpha_{j}) \) and \( \beta = 2 \sum_{i=1}^{t} \alpha_{i}^{k} b_{i} \) and \( \text{clip}[\mu | \nu] := \mu \{ \mu \geq \nu \} \).

This bound should be compared with (3). On one hand the recursive term is multiplied by \( 1 + 1/H \), but, on the other, we are able to clip the surpluses \( \beta_{t} \). The former will exponentiate but will asymptote to \( e \), while the latter is crucial for our instance dependent bounds.

Using this lemma, we can bound the difference between \( V^{k}_{h} \) and \( V^{\pi}_{h} \).

**Lemma 6** (Clipped recursion, informal). For any \( \delta \in (0, 1) \), with probability at least \( 1 - \delta/2 \), \( \forall h \in [H] \),

\[
\sum_{k=1}^{K} (V^{k}_{h} - V^{\pi}_{h})(x^{k}_{h}) \leq \sum_{k=1}^{K} (1 + 1/H) \left( H1_{|n_{h}^{k}(B_{h}) = 0|} + \text{clip} \left[ \beta_{n_{h}^{k}(B_{h})} \left| \frac{\text{gap}_{h}(x^{k}_{h}, a^{k}_{h})}{(H + 1)} \right| \right] + \xi_{h+1}^{k} \right) + (1 + 2/H)^{3} \sum_{k=1}^{K} (V^{k}_{h+1} - V^{\pi}_{h+1})(x^{k}_{h+1}),
\]

where \( \xi_{h+1}^{k} \) is conditionally centered random variable with range \( H \).

We bound \( V^{k}_{1} - V^{\pi}_{1} \), and by optimism the regret, by applying Lemma 6 recursively.

The last step is to show that the sum of clipped surpluses can be related to the zooming dimension. First note that for any ball \( B \), the buffering process implies that it is updated at least \( 1/4 \left( d_{\max}/r(B) \right)^{2} \) times before it becomes activated. If it becomes activated but only contains points with large gap, we can always clip the surplus term. Thus all active balls \( B \) that have \( r(B) \leq \min_{x,a \in B} \text{gap}(x, a) \) do not contribute to the regret.

Next, if a ball with radius \( r \) contains a point where the gap is small, we cannot appeal to clipping. However, by Lipschitzness, all points in the ball must have small gaps, which means that this ball is contained in the near optimal set at scale \( r \). As above, the surplus for each of these balls contributes at most \( d_{\max}/r \) to the regret. Then, since all balls with radius \( r \) are at least \( r \) apart and we only incur regret for those entirely contained in the near-optimal region, we obtain the bound that depends on \( N_{r}^{\text{pack}}(\mathcal{P}_{h,r}^{g}) \).

**Remarks on Assumption 2.** We give some intuition on why our proof requires (2), which is slightly stronger than what is required for the zooming dimension analysis of Slivkins (2014) for contextual bandits. In Slivkins (2014), the optimistic selection rule ensures that the context-action pairs chosen by the algorithm have small gap, but this is not true in the multi-step setting. In the RL setting, we might select an action (in a ball) with a large gap because the downstream regret is large. In this case, we can clip the surplus, but we can only clip at the minimum gap among all \((x, a)\) pairs in the ball. To obtain a zooming dimension bound, we must argue that this ball is contained in the near-optimal set, but this requires that the value functions, and hence the gaps, are Lipschitz. We recall that (2) is implied by (1) if the metric is sub-additive.
6 Discussion

In this paper, we give a refined analysis of a variant of the Adaptive Q-learning algorithm of Sinclair, Banerjee and Yu (2019) for sample efficient reinforcement learning in metric spaces. We show that the algorithm has a regret bound that depends on the zooming dimension of the instance, with rate $K^{\frac{z+1}{z+2}}$ when the zooming dimension is $z$. This improves on the worst-case bound that depends on the covering dimension, and can be much better when the $Q^*$ function concentrates quickly onto a low-dimensional set of actions. The bound also recovers that of Slivkins (2014) for contextual bandits in metric spaces, under a slightly stronger assumption. The key technique is the clipped regret decomposition of Simchowitz and Jamieson (2019), which we complement with a novel buffering process and a corresponding book-keeping argument. Our results show that adaptivity to benign instances is possible in RL with metric spaces, and partially mitigate the curse of dimensionality in such settings.

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References

Peter Auer, Ronald Ortner, and Csaba Szepesvári. Improved rates for the stochastic continuum-armed bandit problem. In Conference on Learning Theory, 2007.

Mohammad Gheshlaghi Azar, Ian Osband, and Rémi Munos. Minimax regret bounds for reinforcement learning. In International Conference on Machine Learning, 2017.

Sébastien Bubeck, Rémi Munos, Gilles Stoltz, and Csaba Szepesvári. X-armed bandits. Journal of Machine Learning Research, 2011.

Christoph Dann, Tor Lattimore, and Emma Brunskill. Unifying pac and regret: Uniform PAC bounds for episodic reinforcement learning. In Advances in Neural Information Processing Systems, 2017.

Chi Jin, Zeyuan Allen-Zhu, Sebastien Bubeck, and Michael I Jordan. Is Q-learning provably efficient? In Advances in Neural Information Processing Systems, 2018.

Sham Kakade, Michael J Kearns, and John Langford. Exploration in metric state spaces. In International Conference on Machine Learning, 2003.

Robert Kleinberg, Aleksandrs Slivkins, and Eli Upfal. Bandits and experts in metric spaces. Journal of the ACM, 2019.

Akshay Krishnamurthy, John Langford, Aleksandrs Slivkins, and Chicheng Zhang. Contextual bandits with continuous actions: Smoothing, zooming, and adapting. In Conference on Learning Theory, 2019.

Chengzhuo Ni, Lin F Yang, and Mengdi Wang. Learning to control in metric space with optimal regret. In Allerton Conference on Communication, Control, and Computing, 2019.

Ronald Ortner. Adaptive aggregation for reinforcement learning in average reward markov decision processes. Annals of Operations Research, 2013.
Ronald Ortner and Daniil Ryabko. Online regret bounds for undiscounted continuous reinforcement learning. In *Advances in Neural Information Processing Systems*, 2012.

Max Simchowitz and Kevin G Jamieson. Non-asymptotic gap-dependent regret bounds for tabular MDPs. In *Advances in Neural Information Processing Systems*, 2019.

Sean R Sinclair, Siddhartha Banerjee, and Christina Lee Yu. Adaptive discretization for episodic reinforcement learning in metric spaces. *ACM Conference on Measurement and Analysis of Computing Systems*, 2019.

Aleksandrs Slivkins. Contextual bandits with similarity information. *Journal of Machine Learning Research*, 2014.

Zhao Song and Wen Sun. Efficient model-free reinforcement learning in metric spaces. *arXiv:1905.00475*, 2019.

Ahmed Touati, Adrien Ali Taiga, and Marc G Bellemare. Zooming for efficient model-free reinforcement learning in metric spaces. *arXiv:2003.04069*, 2020.

Michal Valko, Alexandra Carpentier, and Rémi Munos. Stochastic simultaneous optimistic optimization. In *International Conference on Machine Learning*, 2013.

Andrea Zanette and Emma Brunskill. Tighter problem-dependent regret bounds in reinforcement learning without domain knowledge using value function bounds. In *International Conference on Machine Learning*, 2019.
A Appendix

In this section we provide a detailed proof for the main theorem. First we state some facts about the learning rate and the algorithm. The first lemma regarding the learning rate sequence is directly from Jin et al. (2018).

**Lemma 7** (Lemma 4.1 from Jin et al. (2018)). Let \( \alpha_i^t := \alpha_i \prod_{j=i+1}^t (1 - \alpha_j) \). Then for every \( i \geq 1 \):

\[
\sum_{t=i}^{\infty} \alpha_i^t = 1 + \frac{1}{H}.
\]

The next lemma, also regarding the learning rate sequence, is new. This lemma shows how the skipped updates that arise due to our buffering scheme do not significantly compromise the regret bound. The proof is based on an amortizing argument, which is illustrated in Figure 2.

**Lemma 8.** Fix \( i \geq \frac{H}{2} \) and \( T_0 \geq 0 \). Consider the sequence \( \{\alpha_i^t\}_{t \geq i} \), and let \( \{\tilde{\alpha}_i^t\}_{t \geq i} \) be the sequence formed by repeating every \( H \)-th term in \( \{\alpha_i^t\}_{t \geq i} \) starting at \( t = T_0 \). Then

\[
\sum_{t=i}^{\infty} \tilde{\alpha}_i^t \leq \left(1 + \frac{2}{H}\right)^2.
\]

**Proof.** We can rewrite the sum of \( \tilde{\alpha}_i^t \) as

\[
\sum_{t=i}^{\infty} \tilde{\alpha}_i^t = \sum_{t=i}^{\infty} \alpha_i^t + \sum_{j=0}^{\infty} \alpha_{T_0+jH}^i.
\]

We will show how to absorb the second sum into the first, and we will use a crediting scheme as in Figure 2. The first observation is that

\[
\alpha_{T_0+jH}^i \leq \frac{1}{H} \sum_{k=0}^{H-1} \alpha_{T_0+jH-k}^i.
\]
since $\alpha_i^j$ is a decreasing sequence. This observation immediately addresses any term for which the $H$ previous terms appear in the first sequence. This is any term where $T_0 + jH - i \geq H$.

We just have to handle the terms where $j$ is such that $T_0 + jH - i < H$. In this case we must have $j = 0$. Using the fact that $i \geq H^2$ we obtain

$$\alpha_{T_0}^i \leq \alpha_i^i \leq \frac{H + 1}{H + H^2} \leq \frac{1}{H}.$$ 

Putting these two observations together, we have

$$\sum_{t=i}^{\infty} \alpha_i^t = \sum_{t=i}^{\infty} \alpha_i^t + \sum_{j=0}^{\infty} \alpha_{T_0+jH}^i = \sum_{t=i}^{\infty} \alpha_i^t + \sum_{j=1}^{\infty} \alpha_{T_0+jH}^i + \alpha_{T_0}^i \leq \sum_{t=i}^{\infty} \frac{1}{H} \sum_{t=i}^{\infty} \alpha_i^t + \frac{1}{H} \leq \left(1 + \frac{1}{H}\right)^2 + \frac{1}{H} \leq \left(1 + \frac{2}{H}\right)^2.$$ 

The last step uses Lemma 7.

The next lemma establishes some basic facts on the counters used in the algorithm.

**Lemma 9.** For any $k \in [K]$, $h \in [H]$ and ball $B \in F_h^k$, $B' \in P_h^k$ we have

$$n_h^k(B) \in [0, N_{\min}(B) - 1], \quad n_h^k(B) = 0$$

$$N_{\min}(B') \leq n_h^k(B') \leq n_h^k(B) \leq (H \Lambda + \Lambda + 1) \left(\frac{d_{\max}(r_{B^k})}{r(B)}\right)^2 =: N_{\max}(B)$$

*Proof.* For $B \in F_h^k$, the upper bound on the number of updates comes directly from the rule to move $B$ into the active set. Additionally, the ball in the buffering set will not be played according to the definition of $rel_h^k(x)$, so $n_h^k(B) = 0$. For $B'$, at the time it is added to $P_h^k$ we have $n_h^k(B') = n_h^k(B')$. It will only be updated if it is played, but if it is played it is not necessarily updated, so we have $n_h^k(B') \leq n_h^k(B')$.

The final bound is less obvious. A ball $B$ will no longer be played if all of its children are in the active set, by definition of $rel_h^k(x)$. Further, by the definition of $dom_h^k$, when a ball “passes down” its update to a child, that child must be in the buffering set (otherwise the state action pair is not in the domain of $B$). Before splitting, $B$ is played at most $\left(\frac{d_{\max}(r_{B^k})}{r(B)}\right)^2$ times. After splitting, each child will be updated at most $\frac{1}{4} \left(\frac{d_{\max}(r_{B^k})}{r(B)}\right)^2 = \left(\frac{d_{\max}(r_{B^k})}{r(B)}\right)^2$ before being moved to the active set. Since we have at most $\Lambda$ children (by definition of the doubling constant) and we play the parent ball $(H + 1)$ times for each update to a child, we obtain the bound $(H \Lambda + \Lambda + 1) \left(\frac{d_{\max}(r_{B^k})}{r(B)}\right)^2$.

Next we prove an elementary bound on the bias incurred by some ball.

**Lemma 10.** For any $(x, a, h, k) \in S \times A \times [H] \times [k]$ and ball $B \in P_h^k$ with $(x, a) \in dom_h^k(B)$, if $B$ is updated at step $h$ in episodes $k_1 < k_2 < \cdots < k_t < k$, where $t = n_h^k(B)$, then

$$\sum_{i=1}^{t} \alpha_i^t |Q_h^*(x_{h}^{k_i}, a_{h}^{k_i}) - Q_h^*(x, a)| \leq 4Ld_{\max} \sqrt{(H \Lambda + \Lambda + 1) \frac{1}{\sqrt{t}}}.$$ 

*Proof.* By Lemma 9, we have $n_h^k(B) \leq (H \Lambda + \Lambda + 1) \left(\frac{d_{\max}(r_{B^k})}{r(B)}\right)^2$. Re-arranging, we find that $r(B) \leq d_{\max} \sqrt{(H \Lambda + \Lambda + 1) \frac{1}{n_h^k(B)}}$. Of course we always have $\alpha_i^t \leq 1$, and so, by Lipschitzness we have

$$\sum_{i=1}^{t} \alpha_i^t |Q_h^*(x_{h}^{k_i}, a_{h}^{k_i}) - Q_h^*(x, a)| \leq 2Lr(B) \sum_{i=1}^{t} \alpha_i^t \leq 2Ld_{\max} \sqrt{(H \Lambda + \Lambda + 1) \frac{1}{\sqrt{t}}}.$$
To bound the regret, our starting point is an upper bound on the difference between the optimistic \(Q\)-function and the optimal \(Q^*\) function.

**Lemma 11.** For any \(\delta \in (0, 1)\) if \(\beta_t := 2 \sum_{i=1}^t \alpha_i b(i)\) then

\[
\beta_t \leq 8 \sqrt{\frac{H^3 \log(4HK/\delta)}{t}} + 16 \frac{Ld_{\max} \sqrt{(H\Lambda + \Lambda + 1)}}{\sqrt{t}}.
\]

Additionally, with probability at least \(1 - \delta/2\) the following holds simultaneously for all \((x, a, h, k) \in S \times A \times [H] \times [K]\) and ball \(B\) such that \((x, a) \in \text{dom}_h^k(B)\):

\[
0 \leq Q_h^k(B) - Q_h^*(x, a) \leq 1_{[t=0]} H + \beta_t + \sum_{i=1}^t \alpha_i (V_{h+1}^{k_i} - V_{h+1}^*)(x_{h+1}^{k_i}),
\]

where \(t = n_h^k(B)\), and \(k_1 < \cdots < k_t\) are the episodes where \(B\) was previously updated by the algorithm.

**Proof.** This is a modified version of Lemma E.7 from Sinclair et al. (2019). The proof is exactly the same as the original, except that we use a larger bonus term \(b(i)\) to account for larger upper bound in Lemma 10.

This bound contains three parts. The first is an upper bound for the first step when there is no data. The second term, \(\beta_t\), is the surplus that we add to ensure optimism. The third part is an “average” of the estimated future regret. The key observation is that when \(\beta_t\) is small, it can be absorbed into the future regret. In this way, we can clip \(\beta_t\) proportionally to the future regret which enables a form of gap dependent regret bound. This clipping feature is demonstrated in the next lemma. Recall the definition \(\text{clip}[\mu | \nu] := \mu \mathbb{1}\{\mu \geq \nu\}\).

**Lemma 12 (Clipped upper bound).** For any \(\delta \in (0, 1)\) if \(\beta_t := 2 \sum_{i=1}^t \alpha_i b(i)\). With probability at least \(1 - \delta/2\), \(\forall h \in [H], k \in [K]\),

\[
Q_h^k(B_h^k) - Q_h^*(x_h^k, a_h^k) \leq \left(1 + \frac{1}{H}\right) \left(1_{[t=0]} H + \sum_{i=1}^t \alpha_i (V_{h+1}^{k_i} - V_{h+1}^*)(x_{h+1}^{k_i}) + \text{clip}\left[\beta_t \mid \text{gap}_h(x_h^k, a_h^k)/(H + 1)\right]\right).
\]

**Proof.** We use \(a_h^* : \mathcal{X} \to A\) to denote a mapping from the state to the optimal action at stage \(h\). By the definition of the gap

\[
\text{gap}_h(x_h^k, a_h^k) = Q_h^*(x_h^k, a_h^*(x_h^k)) - Q_h^*(x_h^k, a_h^k) \leq Q_h^k(B_h^{k^*}) - Q_h^*(x_h^k, a_h^k)
\]

\[
\leq Q_h^k(B_h^k) - Q_h^*(x_h^k, a_h^k) \leq 1_{[t=0]} H + \beta_t + \sum_{i=1}^t \alpha_i (V_{h+1}^{k_i} - V_{h+1}^*)(x_{h+1}^{k_i}),
\]

where \(B_h^{k^*}\) is any ball in \(\text{rel}_h^k(x_h^k)\) such that \((x_h^k, a_h^*(x_h^k)) \in \text{dom}_h^k(B_h^{k^*})\) (note that such a ball must exist). The first inequality is by the lower bound of Lemma 11, namely the optimism of \(Q_h^k\). The second uses the selection rule of choosing the ball with the largest \(Q_h^k(B)\) among those in \(\text{rel}_h^k(x_h^k)\). The third inequality is by the upper bound of Lemma 11.

Now we consider two cases, if \(\beta_t > \text{gap}_h(x_h^k, a_h^k)/(H + 1)\), the bound is trivially implied by Lemma 11. If \(\beta_t \leq \text{gap}_h(x_h^k, a_h^k)/(H + 1)\), then

\[
\text{gap}_h(x_h^k, a_h^k) \leq 1_{[t=0]} H + \beta_t + \sum_{i=1}^t \alpha_i (V_{h+1}^{k_i} - V_{h+1}^*)(x_{h+1}^{k_i})
\]

\[
\leq 1_{[t=0]} H + \sum_{i=1}^t \alpha_i (V_{h+1}^{k_i} - V_{h+1}^*)(x_{h+1}^{k_i}) + \text{gap}_h(x_h^k, a_h^k)/(H + 1)
\]

15
By re-arranging to move all gap terms to the left hand side, we have
\[
gap_h(x^k_h, a^k_h) \leq \frac{H + 1}{H} \left( 1_{[t=0]} H + \sum_{i=1}^t \alpha_i(V_{h+1}^k - V_{h+1}^*)(x_{h+1}^k) \right)
\]

By Lemma 11 and our assumption
\[
Q_h^k(B^k_h) - Q_h^*(x^k_h, a^k_h) \leq 1_{[t=0]} H + \beta_t + \sum_{i=1}^t \alpha_i(V_{h+1}^k - V_{h+1}^*)(x_{h+1}^k)
\]
\[
< 1_{[t=0]} H + \text{gap}_h(x^k_h, a^k_h)/(H + 1) + \sum_{i=1}^t \alpha_i(V_{h+1}^k - V_{h+1}^*)(x_{h+1}^k)
\]
\[
\leq \left( 1 + \frac{1}{H} \right) \left( 1_{[t=0]} H + \sum_{i=1}^t \alpha_i(V_{h+1}^k - V_{h+1}^*)(x_{h+1}^k) \right) .
\]

The next step is to replace the future regret to $V^*$ with the future regret of $V^{\pi_k}$, so that we can solve for the $h = 1$ case recursively.

**Lemma 13 (Clipped recursion).** For any $\delta \in (0, 1)$ if $\beta_t := 2 \sum_{i=1}^t \alpha_i b(i)$. With probability at least $1 - \delta/2$, $\forall h \in [H], k \in [K],$
\[
\sum_{k=1}^K (V_h^k - V_h^{\pi_k})(x_h^k) \leq \sum_{k=1}^K \left( 1 + \frac{1}{H} \right) \left( H1_{[n_k(B^k_h)=0]} + \xi^k_{h+1} + \text{clip} \left[ \beta_{n_k(B^k_h)} \left\{ \frac{\text{gap}_h(x^k_h, a^k_h)}{H + 1} \right\} \right] \right)
\]
\[
+ \left( 1 + \frac{2}{H} \right) \sum_{k=1}^K (V_h+1^k - V_h^{\pi_k})(x_h^k),
\]
where $\xi^k_{h+1} = \mathbb{E} \left[ V_{h+1}^*(x) - V_{h+1}^{\pi_k}(x) \mid x_h^k, a_h^k \right] - (V_{h+1}^* - V_{h+1}^{\pi_k})(x_{h+1}^k).
\]

**Proof.** First, consider stage $h$ in episode $k$ and let $B^k_h$ be the ball that is chosen. Define $t = n_k^h(B^k_h)$. Then by applying the previous lemma, we have
\[
V_h^k(x_h^k) - V_h^{\pi_k}(x_h^k) = \max_{B \in \text{rel}_h^k(x_h^k)} Q_h^k(B) - Q_h^*(x_h^k, a_h^k) = Q_h^k(B_h^k) - Q_h^*(x_h^k, a_x^k)
\]
\[
= Q_h^k(B_h^k) - Q_h^*(x_h^k, a_h^k) + Q_h^*(x_h^k, a_h^k) - Q_h^*(x_h^k, a_x^k)
\]
\[
\leq \left( 1 + \frac{1}{H} \right) \left( 1_{[t=0]} H + \sum_{i=1}^t \alpha_i(V_{h+1}^k - V_{h+1}^*)(x_{h+1}^k) \right) + \text{clip} \left[ \beta_t \left\{ \frac{\text{gap}_h(x^k_h, a^k_h)}{H + 1} \right\} \right]
\]
\[
+ (V_{h+1}^* - V_{h+1}^{\pi_k})(x_{h+1}^k) + \xi^k_{h+1}.
\]

Summing over episodes, let $t^k = n_h^k(B^k_h)$ and let $k_i(B^k_h)$ be the episode where $t^k_h$ is incremented for the $i^{th}$ time.
\[
\sum_{k=1}^K V_h^k(x_h^k) - V_h^{\pi_k}(x_h^k) \leq \sum_{k=1}^K \left( 1 + \frac{1}{H} \right) \left( 1_{[n_h^k(B^k_h)=0]} H + \text{clip} \left[ \beta_{n_h^k(B^k_h)} \left\{ \frac{\text{gap}_h(x^k_h, a^k_h)}{H + 1} \right\} \right] \right)
\]
\[
+ \left( 1 + \frac{1}{H} \right) \sum_{k=1}^K \sum_{i=1}^{t^k_h} \alpha_{n_h^k(B^k_h)} (V_{h+1}^k(B^k_h) - V_{h+1}^*)(x_{h+1}^k) + \sum_{k=1}^K (V_{h+1}^* - V_{h+1}^{\pi_k})(x_{h+1}^k) + \xi^k_{h+1} .
\]
For any ball $B$, let $T_0(B)$ be the first time that it is played but not updated, i.e., the first time that $\tilde{n}_h^k(B) > n_h^k(B)$. In the terminology of lemma 8, for any ball $B$, we define the sequence $\tilde{\alpha}_t^i(T_0(B))$ with this value of $T_0$. Now, using the observation in Jin et al. (2018); Song and Sun (2019), we can rearrange the second term and use lemma 8:

$$\sum_{k=1}^K \sum_{i=1}^{n_h^k} \alpha_{n_h^k}^i(V^k_{h+1} - V^*_h)(x^k_{h+1}) \leq \sum_{k=1}^K (V^k_{h+1} - V^*_h)(x^k_{h+1}) \sum_{t=n_h^k}^\infty \alpha_{t}^n_k(T_0(B_h^k))$$

$$\leq \left(1 + \frac{2}{H}\right)^2 \sum_{k=1}^K (V^k_{h+1} - V^*_h)(x^k_{h+1}).$$

The first inequality is based on the following reasoning: The left hand side is “backward” looking, in the sense that for each episode $k$ the expression involves the previous updates to the ball played. On the other hand, the right hand side is “forward” looking, in that episode $k$ also results in an update to some ball (which may not be the one that is played), and so it appears every subsequent time the latter ball is played. Thus, rather than looking at the previous updates to the ball played in episode $k$, we can look at the future plays of the ball updated in episode $k$. This is why we switch the weight sequence from $\tilde{\alpha}_t^i$ to $\alpha_{t}^n$, where recall that the latter has every $H$th term repeated, possibly after some initial burn-in phase.

Since $V^\pi_h(x^k_{h+1}) \leq V^*_h(x^k_{h+1})$, we have

$$\left(1 + \frac{1}{H}\right) \left(1 + \frac{2}{H}\right)^2 \sum_{k=1}^K (V^k_{h+1} - V^*_h)(x^k_{h+1}) \sum_{k=1}^K (V^*_h - V^\pi_h)(x^k_{h+1})$$

$$\leq \left(1 + \frac{2}{H}\right)^3 \sum_{k=1}^K (V^k_{h+1} - V^*_h)(x^k_{h+1}) \sum_{k=1}^K (V^*_h - V^\pi_h)(x^k_{h+1})$$

$$= \left(1 + \frac{2}{H}\right)^3 \sum_{k=1}^K (V^k_{h+1} - V^\pi_h)(x^k_{h+1}).$$

So we have

$$\sum_{k=1}^K (V^k_{h+1} - V^\pi_h)(x^k_{h+1}) \leq \sum_{k=1}^K \left(1 + \frac{1}{H}\right) \left(H1_{[n_h^k(B^k_h)=0]} + \xi_{h+1}^k + \text{clip} \left[\beta_{n_h^k(B^k_h)} \cdot \frac{\text{gap}_h(x^k_{h+1}, a_h^k)}{H + 1}\right]\right)$$

$$+ \left(1 + \frac{2}{H}\right)^3 \sum_{k=1}^K (V^k_{h+1} - V^\pi_h)(x^k_{h+1}).$$

There are two terms that we need to bound. The $\xi_{h+1}^k$ term can be bounded by a concentration argument as shown in Sinclair et al. (2019).

**Lemma 14** (Azuma–Hoeffding bound, Lemma E.9 from Sinclair et al. (2019)). For any $\delta \in (0, 1)$, with probability at least $1 - \delta/2$

$$\sum_{h=1}^H \sum_{k=1}^k \xi_{h+1}^k \leq 2\sqrt{2H^3K \log(4HK/\delta)}.$$
where \( c_1 = \frac{2(H+1)}{d_{\text{max}}} + 2L \). Define the stage-dependent zooming number as

\[
z_{h,c} = \inf\{d > 0 : |p_h^{Q^*}| \leq cr^{-d}\}.
\]

The following is our key lemma that bounds surplus \( \beta_t \) using the zooming number.

**Lemma 15.**

\[
\sum_{h=1}^{H} \sum_{k=1}^{K} \text{clip}[\beta_{n_h^k}, \frac{\text{gap}_h(x_k^k, a_h^k)}{H + 1}] \leq \sum_{h=1}^{H} 32(\sqrt{H^3 \log(4HK/\delta)} + Ld_{\text{max}}\sqrt{2HA})
\]

\[
\inf_{r_0 \in (0,d_{\text{max}}]} \left( \sum_{r=d_{\text{max}}2^{-i}, r \geq r_0} N_{r}^{\text{pack}}(p_h^{Q^*}) \frac{2d_{\text{max}}\sqrt{2HA}}{r} + \frac{2Kr_0}{d_{\text{max}}} \right).
\]

**Proof.** Let \( c_2 = 16(\sqrt{H^3 \log(4HK/\delta)} + Ld_{\text{max}}\sqrt{2HA}) \). By Lemma 11 we have

\[
\beta_{n_h^k} \leq 16(\sqrt{H^3 \log(4HK/\delta)} + Ld_{\text{max}}\sqrt{HA} + \Lambda + 1) \frac{1}{\sqrt{n_h^k}} \leq \frac{c_2}{\sqrt{n_h^k}}.
\]

Let \( N_{\text{min}}(B) = \frac{1}{4} \left( \frac{d_{\text{max}}}{r(B)} \right)^2 \), and \( N_{\text{max}}(B) = \left( \frac{d_{\text{max}}}{r(B)} \right)^2 (HA + \Lambda + 1) \). Considering Lemma 9, we know that whenever \( \beta_{n_h^k} \) appears in our regret bound (which only happens when a ball is played), we have

\[
N_{\text{min}}(B) \leq n_h^k(B) \leq N_{\text{max}}(B).
\]

Letting \( \text{gap}_h(B) = \min_{(x,a) \in B} \text{gap}_h(x, a) \) be the minimum gap \( B \), we can rearrange the sum for each ball.

\[
\sum_{k=1}^{K} \text{clip} \left[ \beta_{n_h^k(B_h^k)} \frac{\text{gap}_h(x_k^k, a_h^k)}{H + 1} \right] \leq \sum_{B \in P_h^K} \sum_{n=N_{\text{min}}(B)}^{N_{\text{max}}(B)} \text{clip} \left[ \frac{c_2}{\sqrt{n}} \frac{\text{gap}_h(B)}{H + 1} \right]
\]

\[
\leq c_2 \sum_{B \in P_h^K} \sum_{n=N_{\text{min}}(B)}^{N_{\text{max}}(B)} \text{clip} \left[ \frac{1}{\sqrt{n}} \frac{\text{gap}_h(B)}{H + 1} \right]
\]

The last step is due to the fact that \( c_2 > 1 \) and if \( \frac{c_2}{\sqrt{n}} < \frac{\text{gap}_h(B)}{H+1} \) then \( \frac{1}{\sqrt{n}} < \frac{\text{gap}_h(B)}{H+1} \). Now, ignoring clipping, the inner sum can be bounded by

\[
\sum_{n=N_{\text{min}}(B)}^{N_{\text{max}}(B)} \frac{1}{\sqrt{n}} \leq \int_{i=0}^{N_{\text{max}}(B)} \frac{1}{\sqrt{i + N_{\text{min}}(B)}} \leq \frac{2d_{\text{max}}\sqrt{HA + \Lambda + 1}}{r(B)}.
\]

With clipping, we consider two cases.

**Case 1:** If \( \text{gap}_h(B) \geq \frac{2(H+1)r(B)}{d_{\text{max}}} \), then the regret on ball \( B \) will always be clipped:

\[
\frac{1}{\sqrt{n_h^k(B)}} \leq \frac{1}{\sqrt{N_{\text{min}}(B)}} = \frac{2r(B)}{d_{\text{max}}} \leq \frac{\text{gap}_h(B)}{H + 1}.
\]

**Case 2:** If \( \text{gap}_h(B) < \frac{2(H+1)r(B)}{d_{\text{max}}} \), then we will pay \( 2d_{\text{max}}\sqrt{2HA}/r(B) \) for this ball. However, we will show that this ball also belongs to the near optimal set, so that we do not incur this term too many times.
Let \((x_c, a_c)\) be the center of \(B\) and \((x_m, a_m)\) be the point that has the minimum gap, i.e. the point that achieves \(\text{gap}_h(B)\). Using the assumption that \(Q^*\) and \(V^*\) are Lipschitz:

\[
\text{gap}_h(x_c, a_c) - \text{gap}_h(B) = Q^*_h(x_c, a_c(x_c)) - Q^*_h(x_c, a_c) - (Q^*_h(x_m, a^*_h(x_m)) - Q^*_h(x_m, a_m)) \\
\leq 2Lr(B),
\]

so we know that all the points in \(B\) have small gaps relative to \(r\). In particular,

\[
\text{gap}_h(x_c, a_c) \leq \text{gap}_h(B) + 2Lr(B) \leq \frac{2(H+1)r(B)}{d_{\text{max}}} + 2Lr(B).
\]

Thus, we have \((x_c, a_c) \in \mathcal{P}_{h,r(B)}^Q\). Now we are ready bound the sum. Note that for a ball \(B \in \mathcal{P}_{h}^K\), either \(B\) gets clipped, or the center of \(B\) is in \(\mathcal{P}_{h,r(B)}^Q\). Since all the balls of radius \(r\) are at least \(r\) apart, we can have at most \(N_{\text{pack}}^{Q}(\mathcal{P}_{h,r})\) in the latter case.

\[
\sum_{k=1}^{K} \text{clip} \left[ \beta_{n_h} \left| \frac{\text{gap}_h(x_k^h, a_k^h)}{H+1} \right| \right] \\ \leq \sum_{B \in \mathcal{P}_{h}^K} \sum_{n=N_{\text{min}}(B)}^{N_{\text{max}}(B)} \text{clip} \left[ \frac{c_2}{\sqrt{n}} \left| \frac{\text{gap}_h(B)}{H+1} \right| \right] \\
\leq c_2 \inf_{r_0 \in [0,d_{\text{max}}]} \left( \sum_{r=d_{\text{max}}2^{-i}, r \geq r_0} N_{\text{pack}}^{Q}(\mathcal{P}_{h,\tau}) \frac{2d_{\text{max}} \sqrt{2H} \Lambda}{r} + \frac{2Kr_0}{d_{\text{max}}} \right).
\]

The second term uses the fact that for any ball \(B\) with \(r(B) \leq r_0\), we have \(N_{\text{min}}(B) \leq \frac{1}{4} \left( \frac{d_{\text{max}}}{r_0} \right)^2 \) \(\square\).

Now we are ready to prove that for any ball \(B\) with \(r(B) \leq r_0\), we have \(N_{\text{min}}(B) \leq \frac{1}{4} \left( \frac{d_{\text{max}}}{r_0} \right)^2 \).

**Proof of Theorem 1.** We apply Lemma 13 recursively.

\[
\sum_{k=1}^{K} (V^*_1 - V_{1}^\pi)(x^k_1) \\
\leq (H+1) + \sum_{k=1}^{K} \left( 1 + \frac{1}{H} \right) \left( \xi_k + \text{clip} \left[ \beta_{n_h}(B_k^h) \left| \frac{\text{gap}_h(x_k^h, a_k^h)}{H+1} \right| \right] \right) + \left( 1 + \frac{2}{H} \right) \sum_{k=1}^{K} (V^*_2 - V_{2}^\pi)(x^k_2) \\
\leq \sum_{h=1}^{H} (1 + \frac{2}{H} \frac{3(h-1)}{2H}) + \sum_{h=1}^{H} \left( 1 + \frac{2}{H} \right) \sum_{k=1}^{3h} \left( \xi_k + \text{clip} \left[ \beta_{n_h}(B_k^h) \left| \frac{\text{gap}_h(x_k^h, a_k^h)}{H+1} \right| \right] \right) \\
\leq 404H^2 + 404 \sum_{h=1}^{H} \sum_{k=1}^{K} \left( \text{clip} \left[ \beta_{n_h}(B_k^h) \left| \frac{\text{gap}_h(x_k^h, a_k^h)}{H+1} \right| \right] + \xi_k \right).
\]

Here we are using that \(1 + 2/H)^{3H} \leq \left( (1 + 2/H)^{H/2} \right)^6 \leq e^6 \leq 404\). Next, we use the Azuma-Hoeffding inequality above to obtain:

\[
404 \sum_{h=1}^{H} \sum_{k=1}^{K} \xi_{h+1} \leq 808 \sqrt{2H^3K \log(4HK/\delta)}.
\]
Finally, we use Lemma 15 to bound the clipped surplus term:

\[
404 \sum_{h=1}^{H} \sum_{k=1}^{K} \text{clip} \left[ \beta_{n_k(B_h^k)} \left( \frac{\text{gap}(x_h^k, a_h^k)}{H + 1} \right) \right] \\
\leq 12928 \sum_{h=1}^{H} \left( \sqrt{H^3 \log(4HK/\delta)} + Ld_{\text{max}} \sqrt{2HL} \right) \\
\times \inf_{r_0 \in (0, d_{\text{max}}]} \left( \sum_{r=d_{\text{max}}2^{-i}, r \geq r_0} N_{\text{pack}}^r(p^*_{h,r}) \frac{2d_{\text{max}} \sqrt{2HL}}{r} + \frac{2Kr_0}{d_{\text{max}}} \right).
\]

Combining the above bounds, we obtain the theorem. \(\square\)