Modification of Laplace Adomian decomposition method for solving nonlinear Volterra integral and integro-differential equations based on Newton Raphson formula

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Abstract. In this paper, we establish a modified Laplace transform Adomian decomposition method for solving nonlinear Volterra integral and integro-differential equations. This technique is different from the standard Laplace Adomian decomposition method because of the terms involved in Adomian polynomials. Here, we have used Newton Raphson formula in place of the term $u_i$ in Adomian polynomials. The proposed scheme is investigated with some illustrative examples and has given reliable results.

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1. Introduction

For solving nonlinear functional equations, Adomian decomposition method was introduced by George Adomian in 1980 [4, 19, 23]. Basically, the technique provides an infinite series solution of the equation and the nonlinear term is decomposed into an infinite series of Adomian polynomials [1, 2, 5, 6, 8, 10, 14–17, 20, 22–24, 26–29]. Several linear and nonlinear ordinary, partial, deterministic and stochastic differential equations are solved easily and adequately by Adomian decomposition method [4, 13, 14, 19, 23]. In this work, Laplace transform technique in combination with Adomian decomposition method is presented and modified, which was first studied by Khuri in [14] to solve nonlinear differential equations. In [13], the authors investigated the method for solving coupled nonlinear partial differential equations. Laplace decomposition method was

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employed to logistic differential equations to find the numerical solutions in [10]. Chanquing and Jianhua studied the Adomian decomposition method to solve the nonlinear fractional differential equations in [27]. In [7], the technique was applied on delay differential equations. A comparison was made between Adomian decomposition and tau methods in [4] for finding the solution of Volterra integro-differential equations. Magdy and Mohamed [20] practiced Laplace decomposition method and Padé approximation to get the numerical solution of nonlinear system of partial differential equations. Further, a modified Laplace decomposition method was adopted for Lane-Emden type differential equations in [28]. Hence, there are numerous applications where Laplace Adomian decomposition method is used by many researchers.

In the present paper, we focus to solve nonlinear Volterra integral and integro-differential equations. Nonlinear Volterra integral equations arise in many scientific fields such as the population dynamics, spread of epidemics and semi-conductor devices [25]. Volterra integro-differential equations also emanated in many physical applications such as biological species coexisting together with increasing and decreasing rates of generating and in engineering applications such as heat transfer, diffusion process in general [3, 4, 21, 24]. Recently, many researchers investigated the solution of these problems. Extant methods are presented to solve these kinds of equations. In [18], quasilinearization technique was employed to solve Volterra integral equations. Kamugad et al. proposed the discretisation and interpolation method for Volterra integral equations [12]. In [3], a comparison was made between Laplace decomposition method, homotopy perturbation method and wavelet-Galerkin method for solving nonlinear Volterra integro-differential equations. Laplace transform combined with Adomian decomposition method is pertained already to solve nonlinear Volterra integral and integro-differential equations [9, 19, 23]. Our work is inspired from these. In this paper, we have followed the combined Laplace transform and Adomian decomposition method but while decomposing the nonlinear term using Adomian polynomials, we have substituted the term $u_i$ with Newton Raphson formula. As we know that Newton Raphson formula is used for finding the better approximate solution of real valued function. By adapting this change, we have achieved the approximate solutions which are in good agreement with the exact one.

The paper is organized as follows: In Section 2, the modified Laplace Adomian decomposition method is presented and discussed for Volterra integral equations. Section 3 summarizes the application of technique to nonlinear Volterra integro-differential equations. In Section 2 and 3, some numerical results are also given to clarify the method. The conclusions are drawn in last section.

2. Nonlinear Volterra integral equations of the second kind

Consider the following nonlinear Volterra integral equation with difference kernel i.e. $k(x, t) = k(x - t)$ defined as

$$u(x) = f(x) + \int_0^x k(x - t)F(u(t))dt,$$  \hspace{1cm} (1)
where \( f(x) \) is known real valued function and \( F(u(x)) \) is the nonlinear function of \( u(x) \).

Apply Laplace transform on both sides of (1). After that using the linear property and convolution theorem of Laplace transform, we have

\[
L[u(x)] = L[f(x)] + L[k(x - t)]L[F(u(x))].
\]  

The methodology consists of approximating the solution of (1) as an infinite series given by

\[
u(x) = \sum_{n=0}^{\infty} u_n(x).
\]

However, the nonlinear term \( F(u(x)) \) is decomposed as

\[
F(u(x)) = \sum_{n=0}^{\infty} A_n(x),
\]

where \( A'_n, s \) are modified Adomian polynomials which are based on Newton Raphson formula given by

\[
A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[ f \left( \sum_{i=0}^{n} \lambda^i \left( u_i - \frac{F(u_i)}{F'(u_i)} \right) \right) \right]_{\lambda=0}, n \geq 0.
\]

Substituting (3) and (4) into (2), we get

\[
L \left[ \sum_{n=0}^{\infty} u_n(x) \right] = L[f(x)] + L[k(x - t)]L \left[ \sum_{n=0}^{\infty} A_n(x) \right].
\]

Using the linearity property of Laplace transform, we get

\[
\sum_{n=0}^{\infty} L[u_n(x)] = L[f(x)] + L[k(x - t)] \sum_{n=0}^{\infty} L[A_n(x)].
\]

To determine the terms \( u_0(x), u_1(x), u_2(x), u_3(x) \ldots \) of infinite series, comparing both sides of (6), we have the following iterative scheme

\[
L[u_0(x)] = L[f(x)],
\]

In general, the relation is given by

\[
L[u_{n+1}(x)] = L[k(x - t)]L[A_n(x)].
\]

Employing the inverse Laplace transform to (7) and (8), we get

\[
u_0(x) = L^{-1} [L[f(x)]],
\]
\[ u_{n+1}(x) = L^{-1} \left[ L[k(x-t)]L[A_n(x)] \right]. \] (10)

Adapting the value of \( u_0(x) \) into (5) gives the value of \( A_0 \) and then using the general iterative relation (10), we get the values of \( u_1(x) \), \( u_2(x) \), \( u_3(x) \) and so on, which finally gives the solution (3) to the given Volterra integral equation.

The effectiveness of modified technique for solving Volterra integral equations is shown by following numerical examples. Here, we have also found the maximum absolute error estimation to show the adequacy of technique given as:

\[ e_j = \text{Max}|u_{ex} - u_{app}| \]

where \( e_j \) denotes the maximum absolute error at some \( x_j \) in the given interval.

**Example 1.** Consider the following Volterra integral equation [11, 25]

\[ u(x) = x + \int_0^x u^2(t) dt, \] (11)

which has the exact solution as \( u(x) = \tan x \).

**Solution.** Taking Laplace transform on both sides of (11) and using the linearity property of Laplace transform, we have

\[ L[u(x)] = L[x] + L \left[ \int_0^x u^2(t) dt \right], \]

that is

\[ L[u(x)] = \frac{1}{s^2} + \frac{1}{s} L[u^2(x)], \]

Using above technique, we have

\[ L \left[ \sum_{n=0}^{\infty} u_n(x) \right] = \frac{1}{s^2} + \frac{1}{s} \left[ \sum_{n=0}^{\infty} A_n(x) \right], \] (12)

where the nonlinear term \( F(u(x)) = u^2(x) \) is decomposed using the formula given by (5). Certain terms of modified Adomian polynomials are as follows:

\[ A_0 = \left(\frac{1}{2}\right)^2 u_0^2, \]

\[ A_1 = \left(\frac{1}{2}\right)^2 (u_0 u_1), \]

\[ A_2 = \left(\frac{1}{2}\right)^2 (2u_0 u_2 + u_1^2), \]

\[ A_3 = \left(\frac{1}{2}\right)^2 (2u_0 u_3 + 2u_1 u_2). \]
Pairing both sides of (12), gives
\[ L[u_0(x)] = \frac{1}{s^2}, \]
(13)

In general
\[ L[u_{n+1}(x)] = \frac{1}{s} L[A_n(x)]. \]
(14)

Applying inverse Laplace transform on both sides of (13), gives
\[ u_0(x) = x, \]
(15)

Using general relation, we have
\[ u_1(x) = \frac{x^3}{12}, \]
Continuing in this manner, we get
\[ u_2(x) = \frac{x^5}{120}, \]
\[ u_3(x) = \frac{x^7}{20160}, \]
\[ u_4(x) = \frac{31x^9}{362880}, \]

Subsequently, the approximate solution becomes
\[ u(x) = x + \frac{x^3}{12} + \frac{x^5}{120} + \frac{x^7}{20160} + \frac{31x^9}{362880} \ldots \]

The exact solution and the one obtained by our technique corresponding to distinct values of \( x \) are presented in Table 1 and demonstrated through figure 1. The absolute error laid out in the table admit that the solutions are very much close to the exact solution and the maximum absolute error is 0.0002.

**Example 2.** Solve the following Volterra integral equation [12]
\[ u(x) = 2x - \frac{x^4}{12} + 0.25 \int_0^x (x - t)u^2(t)dt, \]
(16)

having exact solution \( u(x) = 2x \).

**Solution.** Applying the modified decomposition method, we have
\[ L[u(x)] = L \left[ 2x - \frac{x^4}{12} \right] + 0.25L[x]L[u^2(x)], \]

The method assumes the series solution of function \( u(x) \)
\[ L \left[ \sum_{n=0}^{\infty} u_n(x) \right] = L \left[ 2x - \frac{x^4}{12} \right] + \frac{1}{4s^2} L \left[ \sum_{n=0}^{\infty} A_n(x) \right], \]
(17)
Table 1: Numerical Results for Example 1.

| x | Exact Solution | Approximate Solution | Absolute Error |
|---|----------------|----------------------|----------------|
| 0 | 0              | 0                    | 0.0000E+00     |
| 0.01 | 0.010000333 | 0.010000083          | 2.5001E-07     |
| 0.02 | 0.020002667 | 0.020000667          | 2.0004E-06     |
| 0.03 | 0.030009003 | 0.030002250          | 6.7530E-06     |
| 0.04 | 0.040021347 | 0.040005334          | 1.6013E-05     |
| 0.05 | 0.050041708 | 0.050010419          | 3.1289E-05     |
| 0.06 | 0.060072104 | 0.060018006          | 5.4097E-05     |
| 0.07 | 0.070114558 | 0.070028597          | 8.5961E-05     |
| 0.08 | 0.080171105 | 0.080042694          | 1.2841E-04     |
| 0.09 | 0.090243790 | 0.090060799          | 1.8299E-04     |
| 0.1  | 0.100334672 | 0.100083417          | 2.5126E-04     |

Comparing both sides of (17), gives the continual algorithm

$$L[u_0(x)] = L\left[2x - \frac{x^4}{12}\right],$$

In general

$$L[u_{n+1}(x)] = \frac{1}{4s^2}L[A_n(x)].$$

Taking inverse Laplace transform on above iterative steps, implies

$$u_0(x) = 2x - \frac{x^4}{12},$$

$$u_1(x) = \frac{x^{10}}{207360} - \frac{x^7}{2016} + \frac{x^4}{48}.$$
and so on.

Thus, the solution takes the form

\[
u_2(x) = -\frac{x^{16}}{4777574400} + \frac{37x^{13}}{905748480} - \frac{11x^{10}}{2903040} + \frac{x^7}{8064},
\]

The numerical results shown in Table 2 and Figure 2 illustrate the performance of proposed

| x   | Exact Solution | Approximate Solution | Absolute Error |
|-----|----------------|----------------------|----------------|
| 0   | 0              | 0                    | 0.0000E+00     |
| 0.05| 1              | 0.999999609          | 3.9063E-07     |
| 0.1 | 2              | 1.9999375            | 6.2500E-06     |
| 0.15| 3              | 2.99968359           | 3.1641E-05     |
| 0.2 | 4              | 3.99899995           | 1.0000E-04     |
| 0.25| 5              | 4.99755837           | 2.4416E-04     |
| 0.3 | 6              | 5.99493669           | 5.0633E-04     |
| 0.35| 7              | 6.9906187            | 9.3813E-04     |
| 0.4 | 8              | 7.98399391           | 1.6006E-03     |
| 0.45| 9              | 8.9743572            | 2.5643E-03     |
| 0.5 | 1              | 9.96090845           | 3.9092E-03     |

Figure 2: Comparison of Presented Approximate Solution with Exact solution.

method and maximum absolute error is 0.003.
3. Nonlinear Volterra integro-differential equations of the second kind

The nonlinear Volterra integro-differential equation of the second kind with difference kernel \( k(x, t) = k(x - t) \) is defined as

\[
u^{(i)}(x) = f(x) + \int_{0}^{x} k(x - t)F(u(t))dt,
\]

where \( u^{(i)}(x) \) denotes the \( i \)th derivative of \( u(x) \) w.r.t. \( x \), \( f(x) \) is known as source term and \( F(u(x)) \) is the nonlinear function of \( u(x) \).

The derivative property of Laplace transform is defined by

\[
L[u^{(i)}(x)] = s^iL[u(x)] - s^{i-1}u(0) - s^{i-2}u'(0) - \ldots - u^{(i-1)}(0).
\]

Taking Laplace transform on both sides of (21) and using the properties of Laplace transform, we get

\[
s^iL[u(x)] - s^{i-1}u(0) - s^{i-2}u'(0) - \ldots - u^{(i-1)}(0) = L[f(x)] + L[k(x - t)]L[F(u(x))].
\]

which implies

\[
L[u(x)] = \frac{1}{s}u(0) + \frac{1}{s^2}u'(0) + \ldots + \frac{1}{s^i}u^{(i-1)}(0) + \frac{1}{s^i}L[f(x)] + \frac{1}{s^i}L[k(x - t)]L[F(u(x))].
\]

Adopting the same process as described in Section 2, we obtain

\[
L \left[ \sum_{n=0}^{\infty} u_n(x) \right] = \frac{1}{s}u(0) + \frac{1}{s^2}u'(0) + \ldots + \frac{1}{s^i}u^{(i-1)}(0) + \frac{1}{s^i}L[f(x)] + \frac{1}{s^i}L[k(x - t)]L \left[ \sum_{n=0}^{\infty} A_n(x) \right].
\]

The linearity property of Laplace transform gives

\[
\sum_{n=0}^{\infty} L[u_n(x)] = \frac{1}{s}u(0) + \frac{1}{s^2}u'(0) + \ldots + \frac{1}{s^i}u^{(i-1)}(0) + \frac{1}{s^i}L[f(x)] + \frac{1}{s^i}L[k(x - t)] \sum_{n=0}^{\infty} L[A_n(x)].
\]

Matching both sides, we have the following recurrence relation

\[
L[u_0(x)] = \frac{1}{s}u(0) + \frac{1}{s^2}u'(0) + \ldots + \frac{1}{s^i}u^{(i-1)}(0) + \frac{1}{s^i}L[f(x)],
\]

In general, the relation is given by

\[
L[u_{n+1}] = \frac{1}{s^i}L[k(x - t)]L[A_n(x)].
\]

Applying the inverse Laplace transform to (26), we get the value of \( u_0(x) \), that will define the value of \( A_0 \). Using the value of \( A_0(x) \), \( u_1(x) \) is obtained. Continuing in this manner we will find \( u_n(x) \) from the general relation given by (27). After finding the components of infinite series, the series solution (3) follows. The proposed method will
be illustrated by using the following example.

**Example 3.** Consider the nonlinear Volterra integro-differential equation [3, 19, 21]

\[
\frac{d}{dx}u(x) = -1 + \int_{0}^{x} u^2(t) dt, \quad u(0) = 0. \tag{28}
\]

**Solution.** Taking Laplace transform on both sides of (28) and employing the initial condition, we get

\[
L[u'(x)] = L \left[-1 + \int_{0}^{x} u^2(t) dt\right],
\]

\[
L[u(x)] = \frac{1}{s^2} + \frac{1}{s^2} L[u^2(x)].
\]

Substituting the series form of \(u(x)\) gives

\[
L \left[ \sum_{n=0}^{\infty} u_n(x) \right] = -\frac{1}{s^2} + \frac{1}{s^2} L \left[ \sum_{n=0}^{\infty} A_n(x) \right], \tag{29}
\]

Matching both sides of (29) gives the iterative algorithm

\[
L[u_0(x)] = -\frac{1}{s^2}, \tag{30}
\]

\[
L[u_{n+1}(x)] = \frac{1}{s^2} L[A_n(x)]. \tag{31}
\]

Taking inverse Laplace transform on both sides of (30) and using the recursive relation (31), we get

\[
u_0(x) = -x,
\]

\[
u_1(x) = \frac{x^4}{48},
\]

\[
u_2(x) = -\frac{x^7}{4032},
\]

\[
u_3(x) = \frac{x^{10}}{387072},
\]

\[
u_4(x) = -\frac{x^{13}}{40255488}.
\]

The series solution is therefore given by

\[
u(x) = -x + \frac{x^4}{48} - \frac{x^7}{4032} + \frac{x^{10}}{387072} - \frac{x^{13}}{40255488} \ldots
\]

Table 3 and Figure 3 show that the approximate numerical solution compared with exact [21] is very superior having maximum absolute error 0.003.
4. Concluding Remarks

Newton Raphson formula using as a term in Adomian polynomials exhibits the tenability of combining Laplace transform technique and Adomian decomposition method to solve the nonlinear Volterra integral and integro-differential equations. This is the first time that the Adomian polynomials are modified using Newton Raphson formula. The solution given in tables and demonstrated through figures reveals that the approximate solution using the modified technique is very close to exact solution. Thus, the proposed technique is easy to implement and manifest the accuracy of solution.

References

[1] J Ahmad, F Hussain, and M Naeem. Laplace decomposition method for solving singular initial value problems. *Aditi Journal of Mathematical Physics*, 5:1–15, 2014.
REFERENCES

[2] B Ghazanfari and A Sepahvandzadeh. Adomian decomposition method for solving Bratu’s type equation. *Journal of Mathematics and Computer Science*, 8:236–244, 2014.

[3] D Bahuguna, A Ujlayan, and D N Pandey. A comparative study of numerical methods for solving an integro-differential equation. *Computers and Mathematics with Applications*, 57:1485–1493, 2009.

[4] N Bildik and M Inc. A comparison between Adomian decomposition and Tau methods. *Abstract and Applied Analysis*, 2013:1–5, 2013.

[5] J A S Cano. Adomian decomposition method and Taylor series method in ordinary differential equations. *International Journal of Research and Reviews in Applied Sciences*, 16:168–175, 2013.

[6] N Doan. Solution of the system of ordinary differential equations by combined Laplace transform Adomian decomposition method. *Mathematical and Computational Applications*, 17:203–211, 2012.

[7] D J Evans and K R Raslan. The Adomian decomposition method for solving delay differential equation. *International Journal of Computer Mathematics*, 00:1–6, 2004.

[8] J Fadaei. Application of Laplace Adomian decomposition method on linear and nonlinear system of PDEs. *Applied Mathematical Sciences*, 5:1307–1315, 2011.

[9] F A Hendi. Laplace Adomian decomposition method for solving the nonlinear Volterra integral equation with weakly kernels. *Studies in Nonlinear Sciences*, 2:129–134, 2011.

[10] S Islam, Y Khan, N Faraz, and F Austin. Numerical solution of logistic differential equations by using Laplace decomposition method. *World Applied Sciences Journal*, 8:1100–1105, 2010.

[11] H Jafari, M Ghorbani, and S Ghasempour. A note on exact solutions for nonlinear integral equations by a modified homotopy perturbation method. *New Trends in Mathematical Sciences*, 2:22–26, 2013.

[12] A V Kamyad, M Mehrabinezhad, and J S Nadjafi. A numerical approach for solving linear and nonlinear Volterra integral equations with controlled error. *International Journal of Applied Mathematics*, 40:1–6, 2010.

[13] M Khan, M Hussain, H Jafari, and Y Khan. Application of Laplace decomposition method to solve nonlinear coupled partial differential equations. *World Applied Sciences Journal*, 9:13–19, 2010.

[14] S A Khuri. A Laplace decomposition algorithm applied to a class of nonlinear differential equation. *Journal of Applied Mathematics*, 1:141–155, 2001.
[15] M A Koroma, S Widatalla, A F Kamara, and C Zhang. Laplace Adomian decomposition method applied to a two-dimensional viscous flow with shrinking sheet. 7:525–529, 2013.

[16] A Kumar and R D Pankaj. Laplace decomposition method to study solitary wave solutions of coupled nonlinear partial differential equation. ISRN Computational Mathematics, 2012:1–5, 2012.

[17] Y Lin and C K Chen. Modified Adomian decomposition method for double singular boundary value problems. Romanian Journal of Physics, 59:443–453, 2014.

[18] K Maleknejad and E Najafi. Numerical solution of nonlinear Volterra integral equations using the idea of quasilinearization. Communication in Nonlinear Science and Numerical Simulation, 16:93–100, 2011.

[19] J Manafianheris. Solving the integro-differential equations using the modified Laplace Adomian decomposition method. Journal of Mathematical Extension, 6:41–55, 2012.

[20] M A Mohamed and M S Torky. Numerical solution of nonlinear system of partial differential equations by the Laplace decomposition method and the Padé approximation. American Journal of Computational Mathematics, 3:175–184, 2011.

[21] L Saeedi, A Tari, and S H Momeni Masuleh. Numerical solution of a class of the nonlinear Volterra integro-differential equations. Journal of Applied Mathematics and Informatics, 31:65–77, 2013.

[22] N Singh and M Kumar. Adomian decomposition method for solving higher order boundary value problems. Mathematical Theory and Modeling, 2:11–22, 2011.

[23] A H Waleed. Solving nth-order integro-differential equations using the combined Laplace transform-Adomian decomposition method. Applied Mathematics, 4:882–886, 2013.

[24] A M Wazwaz. The combined Laplace transform-Adomian decomposition method for handling nonlinear Volterra integro-differential equations. Applied Mathematics and Computation, 216:1304–1309, 2010.

[25] A M Wazwaz. Linear and Nonlinear Integral Equations: Methods and Applications. Springer, 2011.

[26] S P Yan, H Jafari, and H K Jassim. Local fractional Adomian decomposition and function decomposition methods for Laplace equation within local fractional operators. Advances in Mathematical Physics, 2014:1–7, 2014.

[27] C Yang and J Hou. An approximate solution of nonlinear fractional differential equation by Laplace transform and Adomian polynomials. Journal of Information and Computational Science, 10:213–222, 2013.
[28] F K Yin, W Y Han, and J Q Song. Modified Laplace decomposition method for Lane-Emden type differential equations. *International Journal of Applied Physics and Mathematics*, 3:98–102, 2013.

[29] J B Yindoula, P Youssouf, G Bissanga, F Bassono, and B Some. Application of the Adomian decomposition method and Laplace transform method to solving the convection diffusion-dissipation equation. *International Journal of Applied Mathematical Research*, 3:30–35, 2014.