A COLLATZ-WIELANDT CHARACTERIZATION OF THE SPECTRAL RADIUS OF ORDER-PRESERVING HOMOGENEOUS MAPS ON CONES

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Abstract. Several notions of spectral radius arise in the study of nonlinear order-preserving positively homogeneous self-maps of cones in Banach spaces. We give conditions that guarantee that all these notions give the same value. In particular, we give a Collatz-Wielandt type formula, which characterizes the growth rate of the orbits in terms of eigenvectors in the closed cone and super-eigenvectors in the interior of the cone. This characterization holds when the cone is normal and when a quasi-compactness condition, involving an essential spectral radius defined in terms of \( k \)-set contractions, is satisfied.

1. Introduction

Non-linear self-maps of a Banach space preserving the order associated to a (closed, convex, and pointed) cone arise in a number of fields, including population dynamics [Per07], entropy maximization and scaling problems [MS69, BLN94], renormalization operators and fractal diffusions [Sab97, Met05], mathematical economy [Mor64], optimal filtering and optimal control [Bou95], zero-sum games [Kol92, RS01, Ney03], the latter being related with tropical geometry [AGG11]. They turn out to share many of the features of nonnegative matrices, as shown by a number of works establishing non-linear analogues of classical results of Perron-Frobenius theory, including [KR48, Bir57, Bir67, Hop63, Kra64, Bir67, Pot77, Bus73, Bus86, Nus88, Nus89, Kra01, NVL99, GG04, ACLN06]. See [LN11] for a recent overview.

A classical problem, for a map \( f \) leaving invariant a subset \( D \) of a Banach space \( X \), is to characterize the (maximal) growth rate of the orbits of \( f \):

\[
r(f, D) := \sup_{x \in D} \limsup_{k \to \infty} \frac{\|f^k(x)\|^1}{k},
\]

where \( f^k := f \circ \cdots \circ f \) denotes the \( k \)th iterate of \( f \).

When \( f \) is positively homogeneous (meaning that \( f \) commutes with the product with a positive constant), and when \( f \) preserves the order associated to a cone \( C \), we may look for non-linear sub and super-eigenvectors, \( u, v \in C \setminus \{0\} \), satisfying respectively

\[
    f(u) \leq \lambda u, \quad f(v) \geq \mu v,
\]

(1.1)

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where $\lambda, \mu > 0$; see Sections 2 and 3 for more background. In particular, if the cone is normal, it is easily verified that if $x$ is such that

$$av \leq x \leq bu$$

for some positive constants $a$ and $b$, then

$$\mu \leq \limsup_{k \to \infty} \|f^k(x)\|^{1/k} \leq \lambda.$$ 

A fortiori, the same conclusion persists if we require $u$ or $v$ to be eigenvectors, i.e., to satisfy the equality in (1.1). It is natural to ask whether the bounds obtained in this way are tight. The main result of this paper shows that the answer is positive, under rather general circumstances.

**Theorem 1.1 (Generalized Collatz-Wielandt Theorem).** Let $C$ be a normal cone of non-empty interior in a Banach space, and let $f$ be a positively homogeneous self-map of $C$ that preserves the order associated to $C$. Suppose that $f$ is uniformly continuous on every bounded subset of $C$. Then, the equalities

$$r(f, C) = \max\{\mu \geq 0 \mid \exists v \in C \setminus \{0\}, f(v) = \mu v\}$$

$$= \inf\{\lambda > 0 \mid \exists u \in \text{int} C, f(u) \leq \lambda u\}$$

hold as soon as the cone essential spectral radius of $f$ is inferior to the latter infimum.

The notion of cone essential spectral radius appearing in (1.1) is a non-linear extension of the notion of essential spectral radius of linear maps introduced in [Nus70]. It is defined using generalized measures of non-compactness and $k$-set contractions (Section 3). Note that when some iterate of the map $f$ is compact, the cone essential spectral radius of $f$ is zero. Then, the assumption of the theorem is satisfied as soon as $r(f, C) > 0$.

Theorem 1.1 follows from Theorem 7.3 below. Note the lack of symmetry between (1.2) and (1.3): we use the notation max in (1.2) to indicate that the set has a maximum element, whereas the infimum in (1.3) is not attained in general. Thus, this theorem states in particular that there exists $v \in C \setminus \{0\}$ such that $f(v) = r(f, C)v$. In other words, the maximal growth rate of the orbits coincides with the maximal non-linear eigenvalue of $f$ associated to an eigenvector in the (closed) cone $C$.

The classical theorem of Collatz and Wielandt concerns the case in which $X = \mathbb{R}^n$, $C = \mathbb{R}^n_+$ and $f$ is linear. Nussbaum extended this result in [Nus86, Th. 3.1] to non-linear continuous self-maps of $\mathbb{R}^n_+$ preserving the order associated to the cone $\mathbb{R}^n_+$. Theorem 1.1 should also be compared with a result of Mallet-Paret and Nussbaum, who showed that under the assumption that the cone essential spectral radius of $f$ is strictly less than $r(f, C)$, there exists an integer $m$ and a vector $v$ such that $f^m(v) = (r(f, C))^mv$, see [MPN02, Th. 3.1]. An interest of a Collatz-Wielandt type characterization lie in their strong duality nature: it provides certificates (the eigenvector $v \in C$ and the super-eigenvector $u \in \text{int} C$) allowing one to bound the growth rate from above and from below.

In the course of proving Theorem 1.1, we establish several non-linear fixed-point results of independent interest. In particular, Theorem 5.1 which completes the previously mentioned result of [MPN02], shows that when the cone ordering of the Banach space $X$ induces a lattice, under the same compactness assumption, there
exists a vector \( v \in C \setminus \{0\} \) such that \( f(v) \geq r(f,C)v \). Corollary 5.4 gives further assumptions under which \( v \) can be chosen so that the equality holds.

In Section 6, we consider the situation in which the map \( f \) is defined over the whole Banach space and \( C \) is reproducing, meaning that \( X = C - C \). Then, we show that the growth rate of the orbit of a vector \( x \) is maximal when \( x \in C \) or \( x \in -C \), i.e.,

\[
r(f,X) = \max(r(f,C),r(f,-C)).
\]

(1.4)

We also show (Theorem 6.2) that, again under a compactness assumption, 0 is the unique fixed point of \( f \) in \( X \) if and only if \( r(f,X) < 1 \). In other words, the uniqueness of the fixed point implies every orbit converges to this fixed point with a geometric rate. This result can be used in combination with the ones of the companion article [AGN06], which gives sufficient conditions to check the uniqueness of the fixed point of a semidifferentiable nonexpansive mapping.

The main result (Theorem 1.1) is established in Section 7.

As an application, we finally show (Section 8) that the spectral radius commutes with suprema or infima of families of maps satisfying a selection properties. This is motivated by zero-sum games, in which Shapley operators are naturally defined as infima or suprema of more elementary operators.

We finally note that in a recent work [MPN10], carried out after the present one, Mallet-Paret and Nussbaum pointed out some imperfections of the notion of essential spectral radius of [MPN02] which is used here. The generalization of the present results to the setting of [MPN10] will be considered in a further work.

2. Preliminary results about Hilbert’s and Thompson’s metric

We next recall classical notions about cones and establish results which will be used in the following sections. See [Nus88, Chapter 1] and [Nus94, Section 1] for more background.

A subset \( C \) of a real vector space \( X \) is called a cone (with vertex 0) if \( tC := \{tx \mid x \in C\} \subset C \) for all \( t \geq 0 \). If \( f \) is a map from a cone \( C \) of a vector space \( X \) to a cone \( C' \) of a vector space \( Y \), we shall say that \( f \) is (positively) homogeneous (of degree 1) if \( f(ty) = tf(y) \), for all \( t > 0 \) and \( y \in C \). We say that the cone \( C \) is pointed if \( C \cap (-C) = \{0\} \). A convex pointed cone \( C \) of \( X \) induces on \( X \) a partial ordering \( \leq_C \), which is defined by \( x \leq_C y \) iff \( y-x \in C \). If the choice of \( C \) is obvious, we shall write \( \leq \) instead of \( \leq_C \). When \( X \) is a topological vector space, we say that \( C \) is proper if it is closed convex and pointed. Note that in [Nus88], a cone is by definition what we call here a proper cone. We next recall the definition of Hilbert’s and Thompson’s metrics associated to a proper cone \( C \) of a topological vector space \( X \).

Let \( x \in C \setminus \{0\} \) and \( y \in X \). We define \( M(y/x) \) by

\[
M(y/x) := \inf \{b \in \mathbb{R} \mid y \leq bx\},
\]

(2.1)

where the infimum of the empty set is by definition equal to \( +\infty \). Similarly, we define \( m(y/x) \) by

\[
m(y/x) := \sup \{a \in \mathbb{R} \mid ax \leq y\},
\]

(2.2)

where the supremum of the empty set is by definition equal to \( -\infty \). We have \( m(y/x) = -M(-y/x) \) and if in addition \( y \in C \setminus \{0\}, m(y/x) = 1/M(x/y) \) (with the convention \( 1/(+\infty) = 0 \)). Since \( C \) is pointed and closed, we have \( M(y/x) \in \mathbb{R} \) and \( m(y/x) \in \mathbb{R} \), and

\[
r(f,X) = \max(r(f,C),r(f,-C)).
\]

(1.4)
and \(D\) such that \(x \psi a \leq y \leq bx\), as soon as \(m(y/x) > -\infty\). Symmetrically \(m(y/x) \in \mathbb{R} \cup \{-\infty\}\) and \(m(y/x) x \leq y\), as soon as \(m(y/x) < +\infty\).

We shall say that two elements \(x\) and \(y\) in \(C\) are comparable and write \(x \sim_C y\) or \(x \sim y\) if there exist positive constants \(a > 0\) and \(b > 0\) such that \(ax \leq y \leq bx\). If \(x, y \in C \setminus \{0\}\) are comparable, we define
\[
d_H(x, y) = \log M(y/x) - \log m(y/x),
\]
\[
d_T(x, y) = \log M(y/x) \vee (-\log m(y/x)),
\]
where we use the notation \(a \vee b = \max(a, b)\). We adopt the convention that \(d_H(0, 0) = d_T(0, 0) = 0\). If \(u \in C\), the set of elements comparable with \(u\),
\[
C_u := \{x \in C \mid x \sim u\},
\]
is called a part of \(C\). If \(C\) has nonempty interior \(\text{int} C\), and \(u \in \text{int} C\), then \(C_u = \text{int} C\). In general, \(C_u \cup \{0\}\) is a pointed convex cone but \(C_u \cup \{0\}\) is not closed. The map \(d_T\) is a metric on \(C_u\), called Thompson’s metric. The map \(d_H\) is called the Hilbert projective metric on \(C_u\). The term “projective metric” is justified by the following properties: for all \(x, y, z \in C_u\), \(d_H(x, z) \leq d_H(x, y) + d_H(y, z)\), \(d_H(x, y) = d_H(y, x) \geq 0\) and \(d_H(x, y) = 0\) iff \(y = \lambda x\) for some \(\lambda > 0\).

From now on, we will assume that \(X = (X, \|\cdot\|)\) is a Banach space. We denote by \(X^*\) the space of continuous linear forms over \(X\), and by \(C^* := \{\psi \in X^* \mid \psi(x) \geq 0 \ \forall x \in C\}\) the dual cone of \(C\). If \(f\) is a map between two ordered sets \((D, \leq)\) and \((D', \leq')\), we shall say that \(f\) is order-preserving if \(f(x) \leq f(y)\) for all \(x, y \in D\) such that \(x \leq y\). Then, any element of \(C^*\) is a homogeneous and order-preserving map from \((X, \leq_C)\) to \([0, +\infty)\). If the cone \(C\) is proper, the Hahn-Banach theorem implies that for all \(u \in C \setminus \{0\}\), there exists \(\psi \in C^*\) such that \(\psi(u) > 0\). For such a \(\psi\), we have \(\psi(x) > 0\) for all \(x \in C_u\). More generally, if \(q : C_u \rightarrow (0, +\infty)\) is homogeneous and order-preserving, we shall write
\[
\Sigma_u = \{x \in C_u \mid q(x) = q(u)\}.
\]
Then, \(d_H\) and \(d_T\) are equivalent metrics on \(\Sigma_u\). Indeed, as shown in [Nus88, Remark 1.3, p. 15]:
\[
\frac{1}{2} d_H(x, y) \leq d_T(x, y) \leq d_H(x, y), \ \forall x, y \in \Sigma_u. \tag{2.3}
\]
More precisely:
\[
1 \leq M(y/x) \leq e^{d_H(x, y)} \ \forall x, y \in \Sigma_u. \tag{2.4}
\]
To see this, let us apply \(q\) to the inequality \(y \leq M(y/x) x\). Using that \(q\) is order-preserving and homogenous, and that \(q(x) = q(y)\), we get \(M(y/x) \geq 1\). By symmetry, \(M(x/y) \geq 1\), hence \(\log M(y/x) = d_H(x, y) - \log M(x/y) \leq d_H(x, y)\).

We say that a cone \(C\) is normal if \(C\) is proper and there exists a constant \(M\) such that \(\|x\| \leq M\|y\|\) whenever \(0 \leq x \leq y\). Every proper cone \(C\) in a finite dimensional Banach space \((X, \|\cdot\|)\) is necessarily normal. We shall need the following result of Thompson.

**Proposition 2.1** ([Tho63 Lemma 3]). Let \(C\) be a normal cone in a Banach space \((X, \|\cdot\|)\). For all \(u \in C \setminus \{0\}\), \((C_u, d_T)\) is a complete metric space.

The next proposition follows from a general result of Zabreiko, Krasnosel’skii and Pokornyi [ZKP71] (see [Nus88, Theorem 1.2 and Remarks 1.1 and 1.3] and a previous result of Birkhoff [Bir62]). When \(q \in C^*\), it follows from Proposition 2.1
together with Eqn (2.3) and the property that $\Sigma_u$ is closed in the topology of the Thompson’s metric $d_T$.

**Proposition 2.2.** Let $C$ be a normal cone in a Banach space $(X, \| \cdot \|)$. Let $u \in C \setminus \{0\}$ and let $q : C_u \rightarrow (0, +\infty)$, be homogenous and order-preserving with respect to $C$. Define $\Sigma_u = \{ x \in C_u \mid q(x) = 1 \}$. Then, $(\Sigma_u, d)$ and $(\Sigma_u, d_T)$ are complete metric spaces.

Given $u \in C \setminus \{0\}$, we define the linear space

$$X_u = \{ x \in X \mid \exists a > 0, -au \leq x \leq au \}.$$

Let $M$ and $m$ be defined as in (2.1) and (2.2). We equip $X_u$ with the norm:

$$\| x \|_u = M(x/u) \vee (-m(x/u)) = \inf \{ a > 0 \mid -au \leq x \leq au \}.$$  \hspace{1cm} (2.5)

**Proposition 2.3** ([Nus94, Proposition 1.1]). Let $C$ be a normal cone of nonempty interior in a Banach space $(X, \| \cdot \|)$. If $u \in \text{int } C$, then $X_u = X$ and $\| \cdot \|$ and $\| \cdot \|_u$ are equivalent norms on $X$.

We say that a cone $C$ of a Banach space $(X, \| \cdot \|)$ is *reproducing* if $X = C - C := \{ x - y \mid x, y \in C \}$. The following observation is standard. We include a proof for the convenience of the reader.

**Proposition 2.4.** A cone $C$ in a Banach space $(X, \| \cdot \|)$ with nonempty interior is reproducing.

**Proof.** If $C$ is a cone and has nonempty interior, take $u \in \text{int } C$. Then, $B_e(u) = \{ z \mid \| z - u \| \leq \epsilon \} \subset C$ for some $\epsilon > 0$. Consider $x \in X$. If $x = 0$, $x \in C$. If $x \neq 0$, $u \pm \frac{\epsilon}{\| x \|} x \in C$, which implies that $\pm \frac{1}{2} x + \frac{\| x \|}{2\epsilon} u \in C$, and so, $x = (\frac{1}{2} x + \frac{\| x \|}{2\epsilon} u) - (-\frac{1}{2} x + \frac{\| x \|}{2\epsilon} u) \in C - C$. \hspace{1cm} $\Box$

We shall finally need the following well known elementary property (see for instance [Nus88, Tho63, Bus73, Pot77]):

**Lemma 2.5.** Let $C$ be a proper cone and $u \in C \setminus \{0\}$. If $f : C_u \rightarrow C$ is order-preserving and homogeneous, then $f(C_u) \subset C_{f(u)}$ and $f$ is nonexpansive both in Hilbert’s projective metric $d_H$ and in Thompson’s metric $d_T$.

### 3. Spectral radius notions and $k$-set contractions

In this section, we recall some results of [MNP02] concerning spectral radii of non-linear maps and $k$-set contractions.

Let $C$ be a cone of a Banach space $(X, \| \cdot \|)$. If $h$ is a homogeneous map (of degree 1) from $C$ to a normed vector space $(Y, \| \cdot \|)$, we define:

$$\| h \|_C := \sup_{x \in C \setminus \{0\}} \frac{\| h(x) \|}{\| x \|}.$$  \hspace{1cm} (3.1)

If $h$ is continuous at point 0, then $\| h \|_C < +\infty$. Indeed, since $h(0) = 0$, by continuity of $h$, there exists $\delta > 0$ such that $\| h(x) \| \leq 1$ for all $x \in C$ such that $\| x \| \leq \delta$. Hence, by homogeneity of $h$, $\| h \|_C \leq 1/\delta$. 

Consider now a homogeneous map \( h \) from \( C \) to \( C \). Following [MPN02], we define:

\[
\tilde{r}(h, C) = \lim_{k \to \infty} \|h^k\|_{C}^{1/k} = \inf_{k \geq 1} \|h^k\|_{C}^{1/k},
\]

\[(3.2a)\]

\[
r(h, C) = \sup_{x \in C} \mu(x) \quad \text{where } \mu(x) = \limsup_{k \to \infty} \|h^k(x)\|^{1/k},
\]

\[(3.2b)\]

\[
\hat{r}(h, C) = \sup \{ \lambda \geq 0 \mid \exists x \in C \setminus \{0\}, h(x) = \lambda x \},
\]

\[(3.2c)\]

\[
\hat{r}(h, C) = \sup_{k \geq 1} (\hat{r}(h^k, C))^{1/k}.
\]

\[(3.2d)\]

When \( C = X \), \( C \) will be omitted in the previous notation. The equality of the limit and the infimum in \((3.2a)\) follows from \( \|h^{k+\ell}\| \leq \|h^k\| \|h^\ell\| \). The number \( \hat{r}(h, C) \) is called Bonsall’s cone spectral radius of \( h \), \( r(h, C) \) is called the cone spectral radius of \( h \), and \( \tilde{r}(h, C) \) is called the cone eigenvalue spectral radius of \( h \). We have the following elementary inequalities.

**Proposition 3.1** ([MPN02] Eqn (2.9) and Prop. 2.1). If \( h \) is a homogeneous self-map of a convex pointed cone in a Banach space, then

\[ 0 \leq \hat{r}(h, C) \leq \tilde{r}(h, C) \leq r(h, C) \leq \hat{r}(h, C). \]

The following result, taken from [MPN02], shows that the equality \( r(h, C) = \hat{r}(h, C) \) holds in several common situations.

**Theorem 3.2** ([MPN02] Th. 2.2 and its following remark, Th. 2.3]). Let \( C \) be a convex pointed cone of a Banach space \( (X, \| \cdot \|) \). Let \( h : C \to C \) be a continuous and homogeneous map. The equality

\[ r(h, C) = \hat{r}(h, C) \]

\[(3.3)\]

holds if either \( C \) is normal and \( h \) is order-preserving with respect to \( C \), or \( C \) is closed and convex and \( h \) is linear, or \( C \) is proper and there exists \( m \geq 1 \) such that \( h^m \) is compact.

If \( C \) is a proper cone and \( h : C \to C \) is continuous and homogeneous, an example in [MPN02] shows that it may happen that \( r(h, C) < \hat{r}(h, C) \). It is not known whether such an example is possible if \( h \) is also order-preserving.

We now introduce the notions of \( k \)-set contraction and essential spectral radius, which rely on measures of noncompactness.

A map \( \nu \) from the set of bounded subsets of \( X \) to the set of real nonnegative numbers is called a homogeneous generalized measure of noncompactness if for all bounded subsets \( A, B \) of \( X \) and for all nonnegative scalars \( \lambda \),

\[ \nu(A) = 0 \iff \text{cl } A \text{ is compact} \]

\[(3.4a)\]

\[ \nu(A + B) \leq \nu(A) + \nu(B) \]

\[(3.4b)\]

\[ \nu(\text{cl conv}(A)) = \nu(A) \]

\[(3.4c)\]

\[ \nu(AA) = \lambda \nu(A) \]

\[(3.4d)\]

\[ A \subset B \implies \nu(A) \leq \nu(B). \]

\[(3.4e)\]

We use the notation \( \text{cl } A \) for the closure of \( A \) and \( \text{conv } A \) for the convex hull of \( A \). Note that our definition is slightly more general than that in [MPN02], which requires that \( \nu(A \cup B) = \nu(A) \lor \nu(B) \) holds for all bounded sets \( A, B \). This difference is sometimes convenient but will play no role here. In fact, we shall only need the following special case of the latter property, which follows from the definition above.
Proposition 3.3. Let \( \nu \) denote a homogeneous generalized measure of noncompactness on \( X \). Then, for all bounded subsets \( A, B \) of \( X \),

\[ \nu(A \cup B) = \nu(A) \] if \( B \) is relatively compact. \hspace{1cm} (3.5)

Proof. Observe first that for all bounded subsets \( A \) of \( X \), and for all \( x \in X \),

\[ \nu(A + \{x\}) = \nu(A). \] Indeed, since \( \{x\} \) is compact, it follows from (3.4b) that

\[ \nu(A + \{x\}) \leq \nu(A) + \nu(\{x\}) = \nu(A). \] The same argument shows that \( \nu(A) = \nu(A + \{x\} - \{x\}) \leq \nu(A + \{x\}) \), and so \( \nu(A + \{x\}) = \nu(A) \).

Let us now fix \( a_0 \in A \). Then, \( A + (B \cup \{a_0\}) - \{a_0\} \supset A \cup B \). By (3.4c),

\[ \nu(A \cup B) \leq \nu(A + (B \cup \{a_0\}) - \{a_0\}) = \nu(A + (B \cup \{a_0\})), \]

and by (3.4a) and (3.4b),

\[ \nu(A + (B \cup \{a_0\})) \leq \nu(A) + \nu(B \cup \{a_0\}) = \nu(A). \] Hence, \( \nu(A \cup B) \leq \nu(A) \). The opposite inequality follows from (3.4c). \qed

For every bounded subset \( A \) of \( X \), let \( \alpha(A) \) denote the infimum of all \( \delta > 0 \) such that there exists an integer \( k \) and \( k \) subsets \( S_1, \ldots, S_k \subset A \) of diameter at most \( \delta \), such that \( A = S_1 \cup \cdots \cup S_k \). The map \( \alpha \), introduced by Kuratowski and further studied by Darbo (see [MPN02] for references), is a particular case of a homogeneous generalized measure of noncompactness. It satisfies in addition, for all bounded sets \( A, B \subset X \),

\[ \alpha(A \cup B) = \alpha(A) \lor \alpha(B) \] \hspace{1cm} (3.6)

As another example, consider the Banach space \( X = C(W) \) of continuous functions from a compact metric space \( (W, d) \) to \( \mathbb{R} \). For all bounded subsets \( A \) of \( X \) and \( \delta > 0 \), define:

\[ \omega_\delta(A) := \sup \{|x(t) - x(s)| \mid x \in A, \text{ and } t, s \in W \text{ satisfying } d(s, t) \leq \delta\} \] \hspace{1cm} (3.7a)

and the “modulus of equicontinuity”

\[ \omega(A) = \inf_{\delta > 0} \omega_\delta(A) . \] \hspace{1cm} (3.7b)

The map \( \omega \) is a homogeneous generalized measure of noncompactness on \( X \) (see [MPN02]).

If \( h : D \subset X \to X \) is a map, we define

\[ \nu(h, D) = \inf \{ \lambda > 0 \mid \nu(h(A)) \leq \lambda \nu(A) \}, \] \hspace{1cm} for all bounded sets \( A \subset D \).

If in addition \( h(D) \subset D \), we define:

\[ \rho(h, D) = \lim_{k \to \infty} (\nu(h^k, D))^{1/k} = \inf_{k \geq 1} \nu(h^k, D)^{1/k} . \]

If \( C \) is a cone and \( h : C \to C \) is homogeneous and Lipschitz continuous with constant \( \kappa \), then \( \alpha(h, C) \leq \kappa \). A general map \( h : D \subset X \to X \), such that \( \nu(h, D) \leq k < 1 \) is called a \( k \)-set contraction with respect to the homogeneous generalized measure of noncompactness \( \nu \). If \( C \) is a cone and \( h : C \to C \) is homogeneous, \( \rho(h, C) \) is called the cone essential spectral radius of \( h \) associated to the homogeneous generalized measure of noncompactness \( \nu \). In general \( \nu(\cdot, C) \) depends on the homogeneous generalized measure of noncompactness \( \nu \). However, usually \( \rho(\cdot, C) \) does not depend on \( \nu \). For example, when \( \nu \) and \( \nu' \) are equivalent in the sense that there exist positive constants \( m \) and \( M \), such that for all bounded subsets \( A \) of \( C \):

\[ m \nu(A) \leq \nu'(A) \leq M \nu(A) . \]
then the cone spectral radii $\rho(\cdot, C)$ associated to $\nu$ and $\nu'$ coincide. When $X = C(W)$ as above, this property applies in particular to the modulus of equicontinuity $\omega$ and to the Kuratowski-Darbo measure of non-compactness $\alpha$, which satisfy $\alpha(A) \leq \omega(A) \leq 2\alpha(A)$, for all bounded subsets of $C(W)$, see [Nus71] Theorem 1.

From now on, we fix a homogeneous generalized measure of noncompactness $\nu$.

**Theorem 3.4** ([MPN02] Th. 3.1). Let $C$ be a proper cone of a Banach space $(X, \| \cdot \|)$, and $h : C \to C$ be a map that is continuous, homogeneous, and order-preserving with respect to $C$. Suppose that for some $m \geq 1$,

$$\nu(h^m, C))^{1/m} < r(h, C).$$

Then, there exists a vector $x_m \in C \setminus \{0\}$ satisfying

$$h^m(x_m) = (r(h, C))^m x_m . \quad (3.8)$$

This result implies that, under the same assumptions

$$r(h, C) = (\tilde{r}(h^m, C))^{1/m} = \tilde{r}(h, C), \quad (3.9)$$

in particular,

$$\rho(h, C) < r(h, C) \implies r(h, C) = \tilde{r}(h, C),$$

see [MPN02] Cor. 3.3.

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### 4. A nonlinear Fredholm-type property

We now introduce a nonlinear Fredholm-type property. If $(X, \| \cdot \|)$ and $(Y, \| \cdot \|)$ are Banach spaces, $D$ is a subset of $X$, and $h : D \to Y$ is a map, we shall say that $h$ has Property (F) when

(F) any sequence $\langle x_j \in D \mid j \geq 1 \rangle$, bounded in $X$, and such that $h(x_j) \to_{j \to \infty} 0,$ has a converging subsequence in $X$.

In the point set topology literature, Property (F) corresponds to the property that the restriction of $h$ to any closed bounded set of $X$ is proper at 0. If $X$ is finite dimensional, any continuous homogeneous map $h : X \to Y$ has Property (F). When $h$ is a bounded linear map, $h$ has Property (F) if, and only if, $h$ is a semi-Fredholm linear operator with index in $\mathbb{Z} \cup \{-\infty\}$, which means that $h$ has a finite dimensional kernel and a closed range, see for instance [Hör94] Proposition 19.1.3 or [Kat93] Chapter IV, Theorems 5.10 and 5.11. In the sequel, $\text{Id}$ denotes the identity map over any set.

**Lemma 4.1.** If $D$ is a subset of a Banach space $(X, \| \cdot \|)$, and if $h : D \to X$ is such that $\nu(h, D) < 1$, then $\text{Id} - h$ has Property (F).

**Proof.** Let $\langle x_j \in D \mid j \geq 1 \rangle$ be a bounded sequence such that $(\text{Id} - h)(x_j) \to 0$ when $j \to \infty$, and let $S = \{x_j \mid j \geq 1 \}$, $T = \{(\text{Id} - h)(x_j) \mid j \geq 1 \}$. Since $(\text{Id} - h)(x_j)$ converges when $j \to \infty$, the set $T$ is relatively compact in $X$, and by (3.4a), $\nu(T) = 0$. Since $S$ is bounded, $\nu(S)$ is finite. Since $x_j = (\text{Id} - h)(x_j) + h(x_j)$, we get that $S \subset T + h(S)$. Applying $\nu$ and using (3.4a) and (3.4b), we get

$$\nu(S) \leq \nu(T + h(S)) \leq \nu(T) + \nu(h(S)) = \nu(h(S)) \leq \nu(h, D) \nu(S)$$

and since $\nu(h, D) < 1$ and $\nu(S)$ is a finite nonnegative number, it follows that $\nu(S) = 0$. Using Property (3.4a), we get that $S$ is relatively compact in $X$. Therefore $\langle x_j \mid j \geq 1 \rangle$ has a converging subsequence in $X$. \qed
Proposition 4.2. If $C$ is a cone of a Banach space $(X, \|\cdot\|)$, if $h : C \to C$ is homogeneous and uniformly continuous on bounded sets of $C$, and if either $\rho(h, C) < 1$ or $\tilde{r}(h, C) < 1$, then $\text{Id} - h$ has Property (F) on $C$. Moreover, when $\tilde{r}(h, C) < 1$, $0$ is the unique fixed point of $h$ in $C$.

Proof. Let $(x_j \in C \mid j \geq 1)$ be a bounded sequence such that $(\text{Id} - h)(x_j) \to 0$ when $j \to \infty$. For all $n \geq 1$, we can write:

$$\langle \text{Id} - h^n(x_j) = \sum_{m=0}^{n-1} (h^m(x_j) - h^{m+1}(x_j)) \rangle.$$ (4.1)

We claim that for all $m \geq 0$,

$$h^m(x_j) - h^{m+1}(x_j) \to 0 \text{ when } j \to \infty.$$ (4.2)

Since (4.2) holds by assumption when $m = 0$, let us assume by induction that (4.2) holds for some $m \geq 0$. Since $h$ is continuous and homogeneous, it follows from (3.1) that the sequences $(h^m(x_j) \mid j \geq 1)$ and $(h^{m+1}(x_j) \mid j \geq 1)$ are bounded. Since $h$ is uniformly continuous on bounded sets, we get from (4.2) that $h^{m+1}(x_j) - h^{m+2}(x_j) \to 0$ when $j \to \infty$, which shows by induction that (4.2) holds for all $m \geq 0$. Combining (4.2) and (4.1), we get that for all $n \geq 1$,

$$\langle \text{Id} - h^n(x_j) \rangle = 0 \text{ when } j \to \infty.$$ (4.3)

Assume first that $\rho(h, C) < 1$ so that $\nu(h^n, C) < 1$ for some $n \geq 1$. Then, by Lemma 4.1, $\text{Id} - h^n$ has Property (F), and we deduce from (4.3) that $(x_j \mid j \geq 1)$ has a converging subsequence in $X$, so that $\text{Id} - h$ has Property (F).

Assume now that $\tilde{r}(h, C) < 1$, so that $\|h^n\|_C < 1$ for some $n \geq 1$. Since

$$\|x_j\| \leq \|x_j - h^n(x_j)\| + \|h^n(x_j)\| \leq \|x_j - h^n(x_j)\| + \|h^n\|_C \|x_j\|,$$

we obtain, using (4.3)

$$(1 - \|h^n\|_C)\|x_j\| \leq \|x_j - h^n(x_j)\| \to 0 \text{ when } j \to \infty,$$

due to $x_j$ converges to 0. This shows that $\text{Id} - h$ has Property (F). Finally, if $x$ is a fixed point of $h$, we get that $h^n(x) = x$, hence $\|x\| = \|h^n(x)\| \leq \|h^n\|_C \|x\|$, and since $\|h^n\|_C < 1$, we deduce that $x = 0$. \hfill \Box

5. Existence of eigenvectors of order-preserving maps in sup-semilattices

Let $C$ be a proper cone in a Banach space $X$. We say that a subset $D$ of $X$ is a sup-semilattice (in the ordering from $C$) if, for all $x, y \in D$, there exists $z \in D$ such that $x \leq z, y \leq z$ and $z \leq w$ for every $w \in D$ for which $x \leq w$ and $y \leq w$. Then, the element $z$ as above is unique and we shall write it $w = x \vee y$. If $C$ is not normal, it may easily happen (see the remark after Lemma 3.6 in [MPN02]) that $X$ is a sup-semilattice but that the map $(x, y) \mapsto x \vee y$ is not continuous. The notion of inf-semilattice is defined dually (by reversing the order). A lattice is an ordered set which is both an inf-semilattice and a sup-semilattice.

Theorem 5.1. Let $C$ be a proper cone in a Banach space $(X, \|\cdot\|)$, and assume that $C$ is a sup-semilattice in the partial ordering induced by $C$. Assume that $h : C \to C$ is continuous, homogeneous and order-preserving in the partial ordering from $C$, and that $\rho(h, C) < r(h, C)$. Then there exists $z \in C \setminus \{0\}$ such that

$$h(z) \geq r(h, C)z.$$
If, in addition, we assume that there exists \( \zeta \in C \setminus \{0\} \) such that

\[(a) \quad h(\zeta) \geq r(h,C)\zeta, \quad \text{and}\]
\[(b) \quad \{ \frac{h^k(\zeta)}{r(h,C)^k} | k \geq 0 \} \text{ is bounded,}\]

then \( h(x) = r(h,C)x \) for some \( x \in C \setminus \{0\} \), and, in particular, \( r(h,C) = r(h,C) \).

Proof. By replacing \( h \) by \( r^{-1}h \), where \( r = r(h,C) \), we can assume that \( r = r(h,C) = 1 \). Since \( \rho(h,C) < r(h,C) = 1 \), there exists \( N \geq 1 \) such that \( \nu(h^m,C) < 1 \), for all \( m \geq N \). Select a fixed \( m \geq N \). Theorem 5.3 implies that there exists \( x_m \in C \setminus \{0\} \) such that \( h^m(x_m) = x_m \). Define \( z := x_m \vee h(x_m) \vee \cdots \vee h^{m-1}(x_m) \), so \( z \geq h^j(x_m) \) for all \( j \geq 0 \). It follows that \( h(z) \geq h^{j+1}(x_m) \) for all \( j \geq 0 \) (h is order-preserving), and since \( h^m(x_m) = x_m, h(z) \geq h^j(x_m) \) for all \( j \geq 0 \). By the properties of the sup-semilattice, we must have \( h(z) \geq z \).

Now assume that there exists \( \xi \in C \setminus \{0\} \) satisfying (a) and (b). This implies that the sequence \( (h^k(\xi))_{k=0} \) is nondecreasing. Let \( S := \{ h^k(\xi) | k \geq 0 \} \), and for \( 0 \leq j \leq m-1 \), let \( S_j := \{ h^{km+j}(\xi) | k \geq 0 \} \). Notice that we have \( S_j = \{ h^j(\xi) \} \cup h^m(S_j) \). Since \( S_j \) is assumed to be bounded, we obtain, using (3.5), that

\[ \nu(S_j) = \nu(h^m(S_j)) \leq \nu(h^m,C) \nu(S_j) ; \]

and since \( \nu(h^m,C) < 1 \), we conclude that \( \nu(S_j) = 0 \), and \( \overline{S_j} \) is compact. It follows that \( S = \bigcup_{j=0}^{m-1} S_j \) has compact closure. Because \( \overline{S} \) is compact, there exists a strictly increasing sequence of integers \( k_i \) with \( h^j(\xi) \to w \). We claim that \( h^{\ell_i}(\xi) \to w' \), \( w' \neq w \), as \( i \to \infty \). For any fixed \( i \), we have \( k_j \geq \ell_i \) for sufficiently large \( j \), so \( h^{k_j}(\xi) \geq h^{\ell_i}(\xi) \) for all \( j \) large and \( w \geq h^{\ell_i}(\xi) \). If we now let \( i \) approach infinity we see that \( w \geq w' \). By symmetry of this argument, we also obtain \( w' \geq w \); so \( w = w' \). This contradicts our original assumption, so we must have that \( h^k(\xi) \to w \) as \( k \to \infty \) and \( h^j(\xi) \to w \) as \( k \to \infty \). However, the continuity of \( h \) at \( w \) implies that \( h^{j+1}(\xi) \to h(w) \), so \( h(w) = w \). Note that \( w \geq \zeta \), so \( w \neq 0 \).

Remark 5.2. We conjecture that if hypotheses are as above, but we do not assume the existence of \( \zeta \in C \setminus \{0\} \) satisfying (a) and (b) in Theorem 5.1, it is still true that \( h(x) = r(h,C)x \) for some \( x \in C \setminus \{0\} \). In general, fixed point theorems which deduce the existence of fixed points of a given map \( f \) by making assumptions about the behaviour of (large) iterates \( f^m \) of \( f \) have been called “asymptotic fixed point theorems”; see [Nus72, Nus77, Nus85] and references to the literature there. Proving asymptotic fixed point theorems is sometimes surprisingly difficult, and a number of old fundamental conjectures (see [Nus72, Nus77, Nus84]) remain open. Our conjecture here fits into this framework, inasmuch as the assumption \( \rho(h,C) < r(h,C) = 1 \) is an assertion about the behaviour of iterates of \( h \) and not directly about \( h \).

The hypotheses of Theorem 5.1 may seem difficult to verify. However, we shall see that by exploiting the concept of parts of a cone (see Section 2 for the definition), one can give natural assumptions which imply these hypotheses. It is convenient to prove a lemma first.

**Lemma 5.3.** Let \( C \) be a proper cone of a Banach space \( X \). Assume that \( h : C \to C \) is continuous, homogeneous and order-preserving in the partial ordering from \( C \) and...
that $\rho(h, C) < r(h, C) = 1$. Assume that $\zeta$ and $w \in C \setminus \{0\}$ are such that $h^k(\zeta) \leq w$ for all $k \geq 0$. Then $\{h^k(\zeta) \mid k \geq 0\}$ is bounded.

**Proof.** Let $S = \{h^k(\zeta) \mid k \geq 0\}$ and assume, by way of contradiction, that $S$ is unbounded in norm. It follows that there exists a strictly increasing sequence of integers $(k_i \mid i \geq 0)$ such that $\|h^{k_i}(\zeta)\| \geq \|h^{j}(\zeta)\|$ for $0 \leq j \leq k_i$ and $\|h^{k_i}(\zeta)\| \to \infty$. Define

$$z_i = \frac{h^{k_i}(\zeta)}{\|h^{k_i}(\zeta)\|}, \quad T = \{z_i \mid i \geq 0\},$$

and suppose that we can prove that $\text{cl} T$ is compact. Then, by taking a further subsequence, we can assume that $z_i$ tends to $z$ as $i \to \infty$, where $\|z\| = 1$ and $z \in C$. However, we have, since $\|h^{k_i}(\zeta)\| \to \infty$, $0 \leq \frac{w}{\|h^{k_i}(\zeta)\|} - z_i \to -z$, so $-z \in C$. Since $C$ is proper, we have a contradiction.

Thus it suffices to prove that $\text{cl} T$ is compact. Notice that if $n \geq 1$ is a positive integer, we can write

$$T = \{z_i \mid i \leq n\} \cup h^n\left(\left\{\frac{h^{k_i-n}(\zeta)}{\|h^{k_i}(\zeta)\|} \mid i > n\right\}\right).$$

Setting $B := \{x \in X \mid \|x\| \leq 1\}$, our assumptions imply that $\frac{h^{k_i-n}(\zeta)}{\|h^{k_i}(\zeta)\|} \in B \cap C$ for $i > n$, so $T \subset \{z_i \mid i \leq n\} \cup h^n(B \cap C)$. The latter equation yields $\nu(T) \leq \nu(h^n, C)\nu(B \cap C)$. Because $\rho(h, C) < 1$, we know that $\nu(h^n, C) \to 0$ as $n \to \infty$, so we conclude that $\nu(T) = 0$ and $\text{cl} T$ is compact. \qed

In the following corollary, recall that a periodic point $x$ of a map $h$ is a point such that $h^n(x) = x$ for some positive integer $n$.

**Corollary 5.4.** Let $C$ be a proper cone in a Banach space $(X, \| \cdot \|)$ and assume that $C$ is a sup-semilattice in the partial ordering induced by $\|$ . Assume that $h : C \to C$ is continuous, homogeneous and order-preserving in the partial ordering from $C$ and that $\rho(h, C) < r(h, C)$. Assume that there exists an integer $m$ such that if $x \in C \setminus \{0\}$ is any periodic point of $h$, $\{h^j(x) \mid j \geq 0\}$ is contained in the union of at most $m$ parts of $C$, i.e. there exists $u_1, \ldots, u_m \in C$, $u_i$ dependent on $x$, with

$$\{h^j(x) \mid j \geq 0\} \subset \bigcup_{i=1}^m C_{u_i}.$$

Then there exists $y \in C \setminus \{0\}$ with $h(y) = ry$, $r = r(h, C)$.

**Proof.** By replacing $h$ by $r^{-1}h$, where $r = r(h, C)$, we can assume that $r = r(h, C) = 1$. Select a prime integer $p$ such that $\nu(h^p) < 1$ and $p > m$, where $m$ is as in the statement of the corollary. Theorem 3.3 implies that there exists $x \in C \setminus \{0\}$ such that $h^p(x) = x$. We claim that $h^j(x)$ is comparable to $x$ for all $j \geq 0$. Since $p > m$, there exists $j$ with $x \sim h^j(x)$ and $0 < j < p$. Since $h$ is order-preserving and homogeneous, it follows that $x \sim h^k(x)$ for all integers $k \geq 1$. Since $p$ is prime, there exists $k \geq 1$ such that $k \equiv 1 \mod p$, so $x \sim h(x)$, and applying $h$ repeatedly we see that $x \sim h^s(x)$ for all $s \geq 0$.

Since $x \sim h^j(x)$ for $0 \leq j \leq p - 1$, there exists $\beta > 0$ such that $h^j(x) \leq \beta h^j(x)$ for $0 \leq j \leq p - 1$ and $0 \leq k \leq p - 1$. If we write $\zeta = x \lor h(x) \lor \cdots \lor h^{p-1}(x)$, it follows that $\zeta \leq \beta h^j(x)$ for $0 \leq k \leq p - 1$. A simple induction on $s$ implies that $h^s(\zeta) \leq \beta h^s(x)$ for $0 \leq k \leq p - 1$ and all $s \geq 0$. The latter equation implies that $h^s(\zeta) \leq \beta \zeta$ for all $s \geq 0$. So Lemma 3.3 implies that $\{h^s(\zeta) \mid s \geq 0\}$ is bounded.
The same argument as in Theorem 5.1 shows that \( h(\zeta) \geq \zeta \); and \( \zeta \in C \setminus \{0\} \) because \( \zeta \geq x \). Theorem 5.1 now implies that there exists \( y \in C \setminus \{0\} \) with \( h(y) = y \). □

Remark 5.5. Let hypotheses be as in Corollary 5.4 but do not assume the existence of an integer \( m \) as in the statement of Corollary 5.4. Instead assume that there exists an integer \( \mu \geq 1 \) and an integer \( N \geq 1 \) and point \( v_1, \ldots, v_\mu \) in \( C \) such that

\[
\hat{h}^N(C) \subset \bigcup_{r=1}^\mu C_{v_r}.
\]

Then it is easy to see that the hypotheses of Corollary 5.4 are satisfied with \( \mu = m \).

A very special case, which is sometimes assumed in the literature is to assume that \( \text{int} C \), the interior of \( C \), is nonempty and \( \hat{h}^N(C) \subset \{0\} \cup \text{int} C \) for some integer \( N \).

6. THE SPECTRAL RADIUS OF ORDER-PRESERVING MAPS OVER \( X \)

We now consider the situation in which \( f \) is defined on the whole Banach space \( X \), and determine the spectral radius of \( f \) by considering the restriction of the map to the cones \( C \) and \( -C \). Recall our convention to write \( r(h) \) for \( r(h, X) \), etc.

Lemma 6.1. Let \( C \) be a reproducing normal cone of a Banach space \((X, \| \cdot \|)\), and let \( h : X \to X \) be a map which is continuous, homogeneous, and order-preserving with respect to \( C \). Then, \( h(C) \subset C, h(-C) \subset -C \) and

\[
\hat{r}(h) = \max(\hat{r}(h, C), \hat{r}(h, -C)) = \max(r(h, C), r(h, -C)) = r(h).
\]

Proof. Since \( h \) is continuous and homogeneous, \( h(0) = 0 \). Since \( h \) is order-preserving with respect to \( C \), we deduce that \( h(C) \subset C \) and \( h(-C) \subset -C \), so that \( \hat{r}(h, C), \hat{r}(h, -C), r(h, C) \) and \( r(h, -C) \) are well defined. Moreover, \( h \) is also order-preserving with respect to \( -C \). It follows from Theorem 5.1 that \( \hat{r}(h, C) = r(h, C) \) and \( \hat{r}(h, -C) = r(h, -C) \), which shows the central equality in (6.1). By definition, \( \hat{r}(h) \geq \max(\hat{r}(h, C), \hat{r}(h, -C)) \) and \( r(h) \geq \max(r(h, C), r(h, -C)) \). It remains to show the reverse inequalities.

Let \( x \in X \). Since \( C \) is reproducing, there exist \( x^+ \) and \( x^- \in C \) such that \( x = x^+ - x^- \). Then, \( -x^- \leq x \leq x^+ \). Since \( h \) is order-preserving, we get \( h^n(-x^-) \leq h^n(x) \leq h^n(x^+) \) for all \( n \in \mathbb{N} \), hence \( 0 \leq h^n(x) - h^n(-x^-) \leq h^n(x^+) - h^n(-x^-) \). Since \( C \) is normal, it follows that \( \| h^n(x) - h^n(-x^-) \| \leq M \| h^n(x^+) - h^n(-x^-) \| \) for some positive constant \( M \), hence

\[
\| h^n(x) \| \leq M \| h^n(x^+) \| + (M + 1) \| h^n(-x^-) \|.
\]

Taking the power \( 1/n \) of this inequality and passing to the limit when \( n \) goes to infinity, we obtain: \( \mu(x) \leq \max(\mu(x^+), \mu(x^-)) \leq \max(r(h, C), r(h, -C)), \) where \( \mu \) is defined as in (3.22). Therefore, \( r(h) \leq \max(r(h, C), r(h, -C)) \).

Using (6.2) again, we deduce that

\[
\| h^n(x) \| \leq M \| h^n \|_C \| x^+ \| + (M + 1) \| h^n \|_{-C} \| x^- \| \leq (M + 1) \max(\| h^n \|_C, \| h^n \|_{-C})(\| x^+ \| + \| x^- \|) .
\]

Since \( C \) is a reproducing normal cone, the quantity

\[
\| x \| := \inf\{ \| x^+ \| + \| x^- \| \mid x^+, x^- \in C, x = x^+ - x^- \}, \quad x \in X,
\]

defines a norm on \( X \) which is equivalent to the initial norm \( \| \cdot \| \) (see [Sch71]). Hence, there exists \( M_0 > 0 \) such that \( \| x \| \leq M_0 \| x \| \) for all \( x \in X \). Since (6.3) holds for all \( x^+, x^- \in C \) such that \( x = x^+ - x^- \), we deduce that \( \| h^n(x) \| \leq \frac{M_0^{n+1}}{M+1} \| x \| \).
\[(M+1)\max(\|h^n\|_C,\|h^n\|_{-C}) \|x\| \leq (M+1)M_0\max(\|h^n\|_C,\|h^n\|_{-C})\|x\|,\]

and since this holds for all \(x \in X\), we obtain \(\|h^n\| \leq (M+1)M_0\max(\|h^n\|_C,\|h^n\|_{-C})\). It follows that \(\hat{r}(h) \leq \max(\hat{r}(h,C),\hat{r}(h,-C))\).

The following proposition identifies a situation in which the uniqueness of the fixed point of a map implies that every orbit converges to this fixed point with a geometric rate.

**Theorem 6.2 (Uniqueness implies contraction).** Let \(C\) be a reproducing normal cone of a Banach space \((X,\|\cdot\|)\), and let \(h : X \to X\) be a map that is continuous, homogeneous, and order-preserving with respect to \(C\). Then, \(\hat{r}(h) = r(h)\) and

\[
\nu(h) \leq r(h) \implies \{ r(h) = \hat{r}(h) \text{ and there exists } x \in (C \cup -C) \setminus \{0\}, h(x) = r(h)x \} .
\]

(6.4)

In particular, if we assume that \(r(h) \leq 1\) and that \(\nu(h) < 1\), we have:

\[
h \text{ has a unique fixed point, i.e. } (h(x) = x \implies x = 0) \iff r(h) < 1 .
\]

(6.5)

**Proof.** It follows from Lemma 6.1 that \(\hat{r}(h) = r(h)\). Assume first that \(\nu(h) < r(h)\). By Lemma 6.1, we deduce that either \(r(h) = r(h,C)\) or \(r(h) = r(h,-C)\). Consider for instance the case in which \(r(h) = r(h,C)\). Using \(\nu(h,C) \leq \nu(h)\) and Theorem 6.1 for \(m = 1\), we deduce that \(r(h,C) = \hat{r}(h,C)\) and that there exists \(x \in C \setminus \{0\}\) such that \(h(x) = r(h,C)x\), which shows (6.4). The case where \(r(h) = r(h,-C)\) is similar. Since \(\hat{r}(h) \leq r(h)\) and there exists \(x \in X \setminus \{0\}\) such that \(h(x) = r(h)x\), we obtain the equality \(\hat{r}(h) = r(h)\).

Assume now that \(r(h) \leq 1\) and that \(\nu(h) < 1\). Then, if \(r(h) \neq 1\), we get that \(r(h) = 1\), hence, by (6.4), there exists \(x \in (C \cup -C) \setminus \{0\}\) such that \(h(x) = r(h)x = x\), that is a non zero fixed point of \(h\), which shows the \(\Rightarrow\) implication in (6.5). The converse implication follows from \(\hat{r}(h) \leq r(h)\).

**Proposition 6.3.** Let \(C\) be a reproducing normal cone of a Banach space \((X,\|\cdot\|)\), such that \(X\) is a lattice for the order defined by \(C\). Let \(h : X \to X\) be a map which is continuous, homogeneous, and order-preserving with respect to \(C\), and such that for all \(x \in X\), the orbit \(\{h^k(x)\}_{k \in \mathbb{N}}\) is bounded. Then, \(\hat{r}(h) = r(h) \leq 1\). Moreover, if \(\rho(h) < 1\) and \(r(h) = 1\), then there exists \(x \in (C \cup -C) \setminus \{0\}\) such that \(h(x) = x\), and in particular \(r(h) = \hat{r}(h)\). Finally, if \(\rho(h) < 1\), then (6.5) holds.

**Proof.** Since all the orbits of \(h\) are bounded, \(r(h) \leq 1\). By Lemma 6.1, \(h(C) \subseteq C\), \(h(-C) \subseteq -C\), and \(\hat{r}(h) = r(h) = \max(h(r(h,C)),r(h,-C))\). Assume that \(r(h) = 1\) and that \(\rho(h) < 1\). Then, \(r(h,C) = 1\) or \(r(h,-C) = 1\). Assume first that \(r(h,C) = 1\). Since \(X\) is a lattice, \(C\) is a sup-semilattice for the order defined by \(C\) and since \(\rho(h,C) \leq \rho(h) < 1\), Theorem 6.1 shows that there exists \(x \in C \setminus \{0\}\) such that \(h(x) = x\). When \(r(h,-C) = 1\), a symmetrical argument shows that there exists \(x \in -C \setminus \{0\}\) such that \(h(x) = x\). Since \(\hat{r}(h) \leq r(h)\), this implies \(\hat{r}(h) = r(h)\).

If now \(\rho(h) < 1\) only, the same arguments as in the proof of Theorem 6.2 show that (6.5) holds.

**Remark 6.4.** When \(C\) is a reproducing normal cone of a Banach space \((X,\|\cdot\|)\), the condition that \(C\) is a sup-semilattice for the order defined by \(C\) together with that \(-C\) is a sup-semilattice for the order defined by \(-C\) is equivalent to the condition assumed in Proposition 6.3 that \(X\) is a lattice for the order defined by \(C\). When \(X\) is an AM-space with unit, i.e., when \(X\) is the space on continuous functions on
a compact space, equipped with the sup-norm \( \| \cdot \| \), and when \( C = X^+ \), then \( C \) and \( X \) satisfy the assumptions of Proposition 6.3.

We also have:

**Lemma 6.5.** Let \( C \) be a normal cone of a Banach space \( (X, \| \cdot \|) \), and let \( g \) and \( h : C \to C \) be two maps which are continuous and homogeneous. Assume that \( g \) is order-preserving and that \( g \leq h \), then

\[
\tilde{r}(g, C) = r(g, C) \leq r(h, C) \leq \tilde{r}(h, C).
\] (6.6)

**Proof.** The equality in (6.6) follows from Theorem 5.2 and the last inequality holds in general. Let \( x \in C \). We show by induction on \( n \) that \( 0 \leq g^n(x) \leq h^n(x) \) holds for all \( n \geq 1 \). Since \( x \in C \), \( g(C) \subset C \), and \( g \leq h \), we get \( 0 \leq g(x) \leq h(x) \). Assume now that \( 0 \leq g^n(x) \leq h^n(x) \) for some \( n \geq 1 \). Then, applying \( g \), which is order-preserving, we get \( g(0) \leq g^{n+1}(x) \leq g(h^n(x)) \). Since \( g \leq h \), we get \( g^{n+1}(x) \leq h^{n+1}(x) \), and since \( 0 = g(0) \), we deduce that \( 0 \leq g^{n+1}(x) \), which concludes the induction. Since \( C \) is normal, we deduce from \( 0 \leq g^n(x) \leq h^n(x) \) that \( \| g^n(x) \| \leq M \| h^n(x) \| \) for some positive constant \( M \). Since this holds for all \( x \in C \), it follows that \( r(g, C) \leq r(h, C) \). \( \square \)

**Corollary 6.6.** Let \( C \) be a reproducing normal cone of a Banach space \( (X, \| \cdot \|) \), and let \( f : X \to X \) be a map which is continuous, homogeneous, order-preserving, and convex. Then, \( \tilde{r}(f) = \tilde{r}(f, C) = r(f) = r(f, C) \).

**Proof.** Consider the map \( f^- : X \to X \) defined by \( f^-(x) = -f(-x) \). We have \( (f^-)^n(x) = -f^n(-x) \) for all \( x \in X \), hence \( \| (f^-)^n \|_C = \| f^n \|_{-C} \) for all \( n \in \mathbb{N} \). It follows that \( r(f^-, C) = r(f, -C) \) and \( \tilde{r}(f^-, C) = \tilde{r}(f, -C) \). Since \( f \) is homogeneous and convex, we get that for all \( x \in X \), \( 0 = f(0) = f(\frac{1}{2}x - \frac{1}{2}x) \leq \frac{1}{2}f(x) + \frac{1}{2}f(-x) \), hence \( -f(-x) \leq f(x) \), which shows that \( f^- \leq f \). Since \( f^- \) is order-preserving (with respect to \( C \)), we deduce from Lemma 6.5 that \( r(f^-, C) \leq r(f, C) \) and \( \tilde{r}(f^-, C) \leq \tilde{r}(f, C) \). With Lemma 6.1, this yields the assertion of the corollary. \( \square \)

7. Spectral radius and Collatz-Wielandt number

We now introduce an additional spectral radius notion. Let \( C \) be a cone in a Banach space \( (X, \| \cdot \|) \), assume \( C \) has nonempty interior \( \text{int} C \), and let \( h : C \to C \) be a homogeneous map. We define the Collatz-Wielandt number:

\[
\text{cw}(h, C) = \inf \{ \lambda > 0 \mid \exists x \in \text{int} C, \ h(x) \leq \lambda x \}.
\] (7.1)

In the sequel, when the choice of the cone \( C \) will be obvious, and in particular, when \( C = X \), the cone \( C \) will be omitted in the previous notation.

By the Hahn-Banach theorem, when \( C \) is proper, \( C = (C^*)^* \), and so, the inequality \( h(x) \leq \lambda x \) is equivalent to \( \psi(h(x)) \leq \lambda \psi(x) \), for all \( \psi \in C^* \). Observe that if \( \psi \in C^* \) is non-zero, then \( \psi \) is non-zero at every point of the interior of \( C \). It follows that

\[
\text{cw}(h, C) = \inf_{x \in \text{int} C} \sup_{\psi \in C^* \setminus \{0\}} \frac{\psi(h(x))}{\psi(x)}.
\] (7.2)

This formula extends the expression of the classical Collatz-Wielandt function which arises in Wielandt’s approach of the finite dimensional Perron-Frobenius theorem. The latter function can be written in a way similar to (7.2), in the special case when \( C \) is the standard positive cone in \( \mathbb{R}^n \). The formula is usually written with
the sup in \( (7.2) \) taken only over the set of coordinates forms on \( \mathbb{R}^n \), which of course does not change the value of the function.

The following result is obvious.

**Lemma 7.1.** Let \( C \) be a proper cone of a Banach space \((X, \| \cdot \|)\), with nonempty interior, and let \( g \) and \( h : C \to C \) be two maps which are homogeneous. If \( g \leq h \), then

\[
\text{cw}(g, C) \leq \text{cw}(h, C) .
\]

(7.3)

We also have:

**Lemma 7.2.** Let \( C \) be a proper cone of a Banach space \((X, \| \cdot \|)\), with nonempty interior, and let \( h : C \to C \) be a continuous, homogeneous and order-preserving map. Then, for all \( n \geq 1 \), we have

\[
\hat{r}(h, C) \leq (\text{cw}(h^n, C))^{1/n} \leq \text{cw}(h, C) .
\]

(7.4)

If in addition \( C \) is normal, we also have, for all \( n \geq 1 \),

\[
r(h, C) = \hat{r}(h, C) \leq (\text{cw}(h^n, C))^{1/n} .
\]

(7.5)

**Proof.** Let \( \lambda > 0 \) and \( x \in \text{int} \, C \) be such that \( h(x) \leq \lambda x \). Since \( h \) is homogeneous and order-preserving, we deduce that \( h^n(x) \leq \lambda^n x \) for all \( n \geq 1 \). Hence, \( \lambda^n \geq \text{cw}(h^n, C) \), and since this holds for all \( \lambda > 0 \) and \( x \in \text{int} \, C \) such that \( h(x) \leq \lambda x \), we get that \( (\text{cw}(h^n, C))^{1/n} \leq \text{cw}(h, C) \) and the second inequality of \( (7.4) \) is proved.

We now assume that \( \hat{r}(h, C) > 0 \), since otherwise the first inequality of \( (7.4) \) is trivial. Let \( n, m \geq 1 \), \( \lambda, \mu > 0 \), \( x, y \in \text{int} \, C \) and \( y \in C \setminus \{0\} \) be such that \( h^n(x) \leq \lambda^n x \) and \( h^m(y) = \mu^m y \). Then, \( h^{nm}(y) = \mu^{nm} y \) and, since \( h \) is homogeneous and order-preserving, \( h^{nm}(x) \leq \lambda^{nm} x \). Since \( x \in \text{int} \, C \) and \( y \in C \setminus \{0\} \), we get that \( b := M(y/x) \in (0, +\infty) \). Since \( y \leq bx \), using again the fact that \( h \) is homogeneous and order-preserving, it follows that \( \mu^{nm} y = h^{nm}(y) \leq bh^{nm}(x) \leq b\lambda^{nm} x \). Hence, \( M(y/x) \leq (\mu^{-1} \lambda)^{nm} b = (\mu^{-1} \lambda)^{nm} M(y/x) \), and since \( M(y/x) \in (0, +\infty) \), we deduce that \( \mu \leq \lambda \). Since this holds for all \( \lambda > 0 \) and \( \mu > 0 \) as above, we get

\[
(\hat{r}(h^m, C))^{1/m} \leq (\text{cw}(h^n, C))^{1/n} .
\]

Since this inequality holds for all \( n, m \geq 1 \), we obtain the first inequality of \( (7.4) \).

Assume now that \( C \) is normal. By Theorem 9.2, \( \hat{r}(h, C) = r(h, C) \). Since for all \( m \geq 1 \), \( h^m \) is continuous, homogeneous and order-preserving, and \( (\hat{r}(h, C))^m = \hat{r}(h^m, C) \) it is sufficient to prove the inequality of \( (7.5) \) for \( m = 1 \). Let \( \lambda > 0 \), let \( x \in \text{int} \, C \) be such that \( h(x) \leq \lambda x \) and let \( y \in C \). Then, \( b := M(y/x) \in (0, +\infty) \) and \( y \leq bx \). As above, it follows that \( h^k(y) \leq bh^k(x) \leq b\lambda^k x \), for all \( k \geq 1 \). Since \( C \) is normal, there exists \( M > 0 \) (independent of \( y \in C \) and \( k \)) such that \( ||h^k(y)|| \leq M b \lambda^k \|x\| \), for all \( k \geq 1 \). Hence, \( \mu(y) \leq \lambda \) holds for all \( y \in C \), and so \( r(h, C) \leq \lambda \). Since this holds for all \( \lambda > 0 \) and \( x \in \text{int} \, C \) such that \( h(x) \leq \lambda x \), this concludes the proof of \( (7.5) \). \( \square \)

The following theorem extends the characterization of the Perron root as the value of the Collatz-Wielandt function, which arises in finite dimensional (linear) Perron-Frobenius theory.

**Theorem 7.3.** Let \( C \) be a normal cone of a Banach space \((X, \| \cdot \|)\), with nonempty interior. Let \( h : C \to C \) be a continuous, homogeneous, and order-preserving map. Consider the following conditions:
(i) \( h \) is uniformly continuous on bounded sets of \( C \),
(ii) \( \nu(h, C) < \text{cw}(h, C) \),
(iii) \( \rho(h, C) < \text{cw}(h, C) \).

If (i) or (ii) holds, then
\[
r(h, C) = \text{cw}(h, C) .
\]

If ((i) and (iii)) or (ii) holds, then
\[
\hat{r}(h, C) = r(h, C) .
\]

and there exists an element \( x \in C \setminus \{0\} \) such that
\[
h(x) = r(h, C)x .
\]

To prove this result, we need the following lemmas.

**Lemma 7.4.** Let \( C \) be a proper cone of a Banach space \( (X, \| \cdot \|) \). For all \( u, x, y \in C \setminus \{0\} \) such that \( M(x/u) < +\infty \) and \( M(y/x) < +\infty \), we have
\[
M(y + u/x + u) \leq \frac{(M(y/x) \lor 1)M(x/u) + 1}{M(x/u) + 1} .
\]

**Proof.** Let \( u, x, y \) be as in the lemma, and denote \( b = M(y/x) \lor 1 \). Then, \( b \geq 1 \) and \( y \leq bx \). We need to find \( b_1 \geq 0 \) such that \( y + u \leq b_1(x + u) \). Since \( y \leq bx \), the latter inequality is satisfied if \( bx + u \leq b_1(x + u) \), or equivalently if
\[
(b - b_1)x \leq (b_1 - 1)u .
\]

This inequality holds when \( 1 \leq b_1 \leq b \) and
\[
(b - b_1)M(x/u) \leq (b_1 - 1) .
\]

Since \( b \geq 1 \), we can take \( b_1 = (bM(x/u) + 1)/(M(x/u) + 1) \), which yields the inequality of the lemma. \( \square \)

**Lemma 7.5.** Let \( C \) be a proper cone of a Banach space \( (X, \| \cdot \|) \), with nonempty interior \( \text{int} C \) and let \( d_H \) denote the Hilbert’s projective metric. Let \( q : C \setminus \{0\} \to (0, +\infty) \) be a homogeneous map preserving the order of \( C \), and let \( \Sigma := \{x \in C \setminus \{0\} | q(x) = 1\} \). Given \( u \in \text{int} C \), define the maps \( \Phi_u : C \setminus \{0\} \to \text{int} C \) and \( \Psi_u : C \setminus \{0\} \to \text{int} C \cap \Sigma \) by
\[
\Phi_u(x) = x + q(x)u \quad \text{and} \quad \Psi_u(x) = \frac{\Phi_u(x)}{q(\Phi_u(x))} .
\]

Then, for all \( v \in C \setminus \{0\} \) and \( R > 0 \), there exists a constant \( c = c_{u,v,R} \) (depending on \( u, v \) and \( R \)) such that \( 0 \leq c < 1 \) and
\[
d_H(\Psi_u(x), \Psi_u(y)) = d_H(\Phi_u(x), \Phi_u(y)) \leq c \ d_H(x, y) \quad \forall x, y \in B_R(v) ,
\]  
where \( B_R(v) := \{x \in C_v | d_H(x, v) \leq R\} \). In particular, \( \Psi_u|_{C_v \cap \Sigma} \) is nonexpansive and it is a contraction mapping on all bounded sets of the metric space \( (C_v \cap \Sigma, d) \).

Moreover, if \( C \) is normal and there exists \( \gamma > 0 \) such that
\[
\|x\| \leq \gamma q(x) \quad \forall x \in C \setminus \{0\} ,
\]
the image of \( C \setminus \{0\} \) by \( \Psi_u \) is a bounded set of \( \text{int} C \cap \Sigma, d \).
Proof. Since \( q(x) > 0 \) for all \( x \in C \setminus \{0\} \), it is clear that \( \Phi_u \) sends \( C \setminus \{0\} \) into int \( C \) and \( \Psi_u \) sends \( C \setminus \{0\} \) into int \( C \cap \Sigma \). Moreover, if (7.6) is shown for all \( R > 0 \), then \( d_H(\Psi_u(x), \Psi_u(y)) \leq d_H(x, y) \) for all \( x, y \in C_v \), which implies that \( \Psi_u|_{C_v \cap \Sigma} \) is nonexpansive in \( d_H \).

Let \( v \in C \setminus \{0\} \) and \( R > 0 \), and let us show (7.6). Since the Hilbert’s projective metric only depends on the lines generated by two vectors, we have \( d_H(\Psi_u(x), \Psi_u(y)) = d_H(\Phi_u(x), \Phi_u(y)) \) for all \( x, y \in C \setminus \{0\} \), which shows the equality in (7.6). Moreover, by homogeneity of \( q \), we get that for all \( x, w \in C \setminus \{0\} \), \( \Psi_u(x) = \Psi_u(x') \) and \( d_H(x, w) = d_H(x', w) \) where \( x' = \frac{x}{q(x)} \in \Sigma \), hence it is sufficient to show the inequality in (7.6) when \( x, y \in B_R(v) \cap \Sigma \). We can also assume without loss of generality that \( v \in \Sigma \). In addition, since \( d_H(x, y) = \log M(y/x) + \log M(x/y) \) for all \( x, y \in C \setminus \{0\} \) such that \( x, y \in C_v \), it is sufficient to show that
\[
M(\Phi_u(y)/\Phi_u(x)) \leq [M(y/x)]^c \quad \forall x, y \in B_R(v) \cap \Sigma.
\] (7.8)

Let \( x, y \in B_R(v) \cap \Sigma \). Since \( u \in \text{int} C \) and \( v \in C \setminus \{0\} \), we deduce that \( M_0 := M(v/u) \in (0, +\infty) \). From (2.4), we deduce that \( M(x/v) \leq e^{d_H(x,v)} \leq e^R \) (since \( q(x) = q(v) = 1 \) and \( x \in C_v \)), hence \( M(x/v) \leq M(x/v)M(v/u) \leq e^R M_0 \). Using this inequality together with (2.4) and Lemma 7.4, we get that
\[
M(\Phi_u(y)/\Phi_u(x)) \leq \mu + (1 - \mu) M(y/x) \quad \text{where} \quad \mu = \frac{1}{e^R M_0 + 1}.
\]

Hence, it is sufficient to show that there exists \( 0 \leq c < 1 \), independent of \( x, y \in B_R(v) \cap \Sigma \), such that
\[
\mu + (1 - \mu) M(y/x) \leq [M(y/x)]^c \ .
\]

Since \( d_H(x, y) \leq d_H(x, v) + d_H(v, y) \leq 2R \), it follows from (2.4) that \( 1 \leq M(y/x) \leq e^{2R} \). Hence, it is sufficient to show that
\[
\mu + (1 - \mu) \beta \leq \beta^c \quad \forall \beta \in [1, e^{2R}] \ .
\] (7.9)

Let \( 0 \leq c < 1 \) and define \( \varphi(\beta) = \mu + (1 - \mu) \beta - \beta^c \) for \( \beta > 0 \). We get that \( \varphi(1) = 0 \) and since \( \varphi \) is convex, the inequality (7.9) holds if, and only if, \( \varphi(e^{2R}) \leq 0 \), which is satisfied when
\[
c = \frac{\log(\mu + (1 - \mu)e^{2R})}{2R}.
\]

Since \( 0 < \mu < 1 \) and \( R > 0 \), we get that \( 0 < c < 1 \) and (7.6) is proved.

Assume now that (7.7) holds. Let \( x \in C \setminus \{0\} \). We get that \( q(x) \leq \Phi_u(x) \), hence \( m(\Phi_u(x)/u) \geq q(x) \). Since \( C \) is normal and \( u \in \text{int} C \), the norms \( \|\cdot\| \) and \( \|\cdot\|_u \) are equivalent in \( X \) (see Proposition 2.3). In particular, there exists a constant \( \beta > 0 \) such that \( \|x\|_u \leq \beta \|x\| \) for all \( x \in X \). From (7.7), we deduce that \( \|x\|_u \leq \beta \|x\| \leq \beta \gamma q(x) \). It follows that \( \Phi_u(x) \leq (\beta \gamma + 1) q(x) \), or equivalently \( M(\Phi_u(x)/u) \leq (\beta \gamma + 1) q(x) \). With \( m(\Phi_u(x)/u) \geq q(x) \), this yields \( d_H(\Psi_u(x), u) = d_H(\Phi_u(x), u) \leq \log(\beta \gamma + 1) \), which shows that the image of \( \Psi_u \) is bounded in \( \text{int} C \cap \Sigma \). \( \square \)

Remark 7.6. The conclusion of Lemma 7.5 remains valid, with the same constant \( c \), if the map \( \Phi_u \) (or \( \Psi_u \)) is replaced by the map \( C \setminus \{0\} \rightarrow \text{int} C \), \( x \mapsto x + u \), \( \Sigma \) is replaced by \( C \setminus \{0\} \), and Hilbert’s projective metric \( d_H \) is replaced by Thompson’s metric \( d_T \).

Proof of Theorem 7.3. We shall assume that \( h \) is a non identically zero map on \( C \), since otherwise all the assertions of the theorem are trivially true. By Theorem 3.2,
$r(h, C) = \tilde{r}(h, C)$, so we shall prove that $\tilde{r}(h, C) = \text{cw}(h, C)$. To do so, we shall construct an approximation of $h$ which has an eigenvector in the interior of $C$.

Let $u \in \text{int} \ C$. Since $C$ is a normal cone, $\| \cdot \|_u$ is a norm equivalent to $\| \cdot \|$ on $X$ (see Proposition 2.3), and since $\| \cdot \|_u$ is order-preserving on $C$, the restriction $q$ of $\| \cdot \|_u$ to $C \setminus \{0\}$ satisfies all the conditions of Lemma 7.3. Since $h$ is homogenous and order-preserving, $h$ sends $\text{int} \ C = C_u$ into $C_{h(u)}$ and $h$ is nonexpansive in Hilbert’s projective metric $d_H$ (see Lemma 2.5). Moreover, since $h$ is nonzero and continuous, $h(\text{int} \ C)$ cannot be $\{0\}$, hence $h(u) \neq 0$. Let $s > 0$, let $\Sigma$, $\Phi_{su}$ and $\Psi_{su}$ be defined as in Lemma 7.3 denote $h_s = \Phi_{su} \circ h$, $g_s = \Psi_{su} \circ h|_{\text{int} \ C \cap \Sigma}$ and $v = h(u)$. By the properties of $h$ and Lemma 7.3, $g_s$ is a self-map of $\text{int} \ C \cap \Sigma$, it is nonexpansive in $d_H$ and its image is bounded for the metric $d_H$. Moreover, since $h$ is nonexpansive in $d_H$, the image by $h$ of any bounded set of $(\text{int} \ C \cap \Sigma, d)$ is included in some $B_R(v)$ with $R > 0$, hence $g_s$ is a contraction mapping on any bounded set of $(\text{int} \ C \cap \Sigma, d)$, and in particular on the closure (for $d_H$) of the image of $g_s$. Since, by Proposition 2.5 (int $\ C \cap \Sigma, d$) is a complete metric space, it follows that $g_s$ admits a unique fixed point $x_s \in \text{int} \ C \cap \Sigma$. This implies that $x_s \in \text{int} \ C$, $q(x_s) = 1$ and

$$h_s(x_s) = h(x_s) + sq(h(x_s))u = \lambda x_s$$

(7.10)

for some $\lambda_s > 0$.

For all $s > 0$, $h_s : C \rightarrow C$, $x \mapsto h(x) + sq(h(x))u$ ($q$ and $h_s$ are extended by 0 at 0) is continuous, order-preserving and homogeneous. Hence, by (7.10) and (7.5), we get that \(\text{cw}(h_s, C) \leq \lambda_s \leq \tilde{r}(h_s, C)\) $\leq r(h_s, C) \leq \text{cw}(h_s, C)$, hence $\text{cw}(h_s, C) = \lambda_s$. Since $h \leq h_s \leq h$, for all $0 < s \leq t$, Lemma 7.1 implies that $\text{cw}(h, C) \leq \lambda_s \leq \lambda_t$. Denote $y_k = x_{1/k}$ and $\mu_k = \lambda_{1/k}$. The sequence $(\mu_k)_{k \geq 1}$ is nonincreasing, thus it converges towards some real $\mu \geq \text{cw}(h, C)$. Since $q(y_k) = 1$ for all $k$, the sequence $y_k$ is bounded in $(X, \| \cdot \|)$. Then, $\mu y_k - h(y_k) = (\mu - \mu_k)y_k + \frac{1}{k}q(h(y_k))u$ tends to 0 when $k$ goes to $\infty$.

Suppose first that $\nu(h, C) < \text{cw}(h, C)$. Then,$$
\nu(\frac{1}{\mu} h, C) < \text{cw}(h, C)/\mu \leq 1
$$
and by Lemma 4.1, $\mu \text{Id} - h$ has Property (F). Hence, $y_k$ has a converging subsequence, and since $h$ and $q$ are continuous and $C$ is closed, the limit $y$ of this subsequence satisfies $y \in C$, $q(y) = 1$ and $h(y) = \mu y$. Hence, $\mu \leq \tilde{r}(h, C) \leq r(h, C) = \tilde{r}(h, C) \leq \text{cw}(h, C) \leq \mu$, which implies that $\tilde{r}(h, C) = \text{cw}(h, C)$ and also proves that $\tilde{r}(h, C) = r(h, C)$ and that there exists $y \in C \setminus \{0\}$ such that $h(y) = r(h, C)y$.

Suppose now that $h$ is uniformly continuous on bounded sets of $C$, and assume by contradiction that $\tilde{r}(h, C) < \text{cw}(h, C)$. Then, $\tilde{r}(\frac{1}{\mu} h, C) < \text{cw}(h, C)/\mu \leq 1$ and by Proposition 4.1, $\mu \text{Id} - h$ has Property (F), and 0 is the unique fixed point of $\frac{1}{\mu} h$. Using Property (F), we conclude as above that there exists an element $y \in C$ such that $q(y) = 1$ and $h(y) = \mu y$, which contradicts the fact that 0 is the unique fixed point of $\frac{1}{\mu} h$. This shows that $\tilde{r}(h, C) = \text{cw}(h, C)$.

Suppose in addition that $\rho(h, C) < \text{cw}(h, C)$. Then, $\rho(\frac{1}{\mu} h, C) < \text{cw}(h, C)/\mu \leq 1$ and, by Proposition 4.1, $\mu \text{Id} - h$ has Property (F). As above, we conclude that there is a vector $y \in C$ such that $q(y) = 1$ and $h(y) = \mu y$. Therefore, $\mu \leq \tilde{r}(h, C) \leq r(h, C) = \tilde{r}(h, C) \leq \text{cw}(h, C) \leq \mu$, showing that $h(y) = r(h, C)y$. \qed
Corollary 7.7. Let $C$ be a normal cone of a Banach space $(X, \| \cdot \|)$, with nonempty interior. Let $h : C \to C$ be a continuous, homogeneous, and order-preserving map. If, for some $m \geq 1$, we have

$$\nu(h^m, C) < \text{cw}(h^m, C)$$

(7.11)

then,

$$\hat{r}(h^m, C) = (r(h, C))^m = (\hat{r}(h, C))^m = \text{cw}(h^m, C)$$

(7.12)

and there exists an element $x_m \in C \setminus \{0\}$ such that

$$h^m(x_m) = (r(h, C))^m x_m .$$

(7.13)

Proof. Theorem [7.3] shows that $\hat{r}(h^m, C) = r(h^m, C) = \hat{r}(h, C) = \text{cw}(h^m, C)$, and there exists an element $x_m \in C \setminus \{0\}$ satisfying $h^m(x_m) = r(h^m, C)x_m$. It follows readily from the definition of $\hat{r}(h, C)$ that $\hat{r}(h^m, C) = (\hat{r}(h, C))^m$. Since, by Theorem [5.2], $\hat{r}(h, C) = r(h, C)$, the corollary is proved. $\square$

Remark 7.8. The Collatz-Wielandt theorem of [GV11] was recently generalized in [Nus86] in a different way, to the case of order preserving and positively homogeneous self-maps of the interior of a finite dimensional cone with a family of geodesics in Thompson’s metric satisfying Busemann’s nonpositive curvature condition. Some of the conditions of [GV11] (finite dimension, nonpositive curvature) are considerably more demanding than the ones of Theorem [1.1] however, the Collatz-Wielandt type result of [GV11] remains valid even if the map cannot be extended continuously to the boundary of the cone, a property which we require here.

8. Spectral radius of a supremum or of an infimum of functions

Let $C$ be a proper cone of a Banach space $(X, \| \cdot \|)$, and let $(f_a)_{a \in A}$ be a family of maps $X \to X$. We say that $(f_a)_{a \in A}$ admits a upper selection if for all $x \in X$, there exists $a_x \in A$ such that $f_{a_x}(x) \geq f_b(x)$ for all $b \in A$. We denote by $\sup_{a \in A} f_a$ the map $X \to X$ which associate to $x \in X$, the element $f_{a_x}(x) \in X$, and call it the supremum of the family $(f_a)_{a \in A}$. We define symmetrically the notion of lower selection of $(f_a)_{a \in A}$ and the associated infimum which we denote by $\inf_{a \in A} f_a$.

Proposition 8.1. Let $C$ be a proper cone of a Banach space $(X, \| \cdot \|)$, and let $(f_a)_{a \in A}$ be a family of maps $X \to X$ that are homogeneous. Assume that $(f_a)_{a \in A}$ admits a upper selection and denote by $f$ its supremum. Then, $f$ is homogeneous and we have:

$$\hat{r}(f, C) \leq \sup_{a \in A} \hat{r}(f_a, C) .$$

(8.1)

Assume further that the maps $f_a$ are continuous and order-preserving with respect to $C$, that $f$ is continuous and that $C$ is a normal cone. Then, $f$ is order-preserving, and we have:

$$\sup_{a \in A} r(f_a, C) \leq r(f, C) \leq \max(\nu(f, C), \sup_{a \in A} r(f_a, C)) .$$

(8.2)

In particular, when $\nu(f, C) < r(f, C)$, we have

$$\hat{r}(f, C) = \sup_{a \in A} \hat{r}(f_a, C) = \sup_{a \in A} r(f_a, C) = r(f, C) ,$$

(8.3)

and the suprema are attained in (8.3).
Proof. For all $\lambda > 0$ and $x \in X$, we get $f_a(\lambda x) = \lambda f_a(x)$ for all $a \in A$, hence $f(x) = f_a(x) = \lambda f_a(\lambda x) \leq \lambda f(x)$, which shows that $f$ is homogeneous. Let now $x \in X$ and $\lambda > 0$ be such that $f(x) = \lambda x$. Since $(f_a)_{a \in A}$ admits an upper selection, there exists $a_x \in A$ such that $f_{a_x}(x) = f(x) = \lambda x$. It follows that $\lambda \leq \hat{r}(f_{a_x}, C) \leq \sup_{a \in A} \hat{r}(f_a, C)$. Since this holds for all $\lambda$ such that $f(x) = \lambda x$ for some $x \in C$, we deduce (8.1).

Now assume that the maps $f_a$ are continuous and order-preserving with respect to $C$, that $f$ is continuous and that $C$ is a normal cone. Let $x, y \in X$ be such that $x \leq y$. We get $f_a(x) \leq f_a(y)$ for all $a \in A$, hence $f(x) = f_{a_x}(x) \leq f_{a_x}(y) \leq f(y)$, which shows that $f$ is order-preserving. Using Lemma 6.5, we obtain that $r(f, C) \geq r(f_a, C)$ for all $a \in A$, which shows the first inequality in (8.2). Combining the previous inequalities with the fact that $\hat{r}(f_a, C) \leq r(f_a, C)$ for all $a \in A$, we get:

$$\hat{r}(f, C) \leq \sup_{a \in A} \hat{r}(f_a, C) \leq \sup_{a \in A} r(f_a, C) \leq r(f, C).$$

Then, the equalities in (8.3) follow from Theorem 3.3. Moreover, by (8.3), there exists $x \in X \setminus \{0\}$ such that $f(x) = \hat{r}(f, C)x$. Then, $f_{a_x}(x) = \hat{r}(f, C)x$, hence $\hat{r}(f, C) \leq \hat{r}(f_{a_x}, C)$ and since $\hat{r}(f_{a_x}, C) \leq r(f_{a_x}, C) \leq r(f, C) = \hat{r}(f, C)$, we get that the suprema are attained in (8.3). Finally, the second inequality in (8.2) follows from (8.3).

**Proposition 8.2.** Let $C$ be a proper cone of a Banach space $(X, \|\cdot\|)$, with nonempty interior, and let $(f_a)_{a \in A}$ be a family of maps $X \to X$ that are homogeneous. Assume that $(f_a)_{a \in A}$ admits a lower selection and denote by $f$ its infimum. Then, $f$ is homogeneous and we have:

$$\inf_{a \in A} cw(f_a, C) = \inf_{a \in A} f_a(x) = cw(f, C).$$

Assume in addition that the maps $f_a$ are continuous and order-preserving with respect to $C$, that $f$ is continuous and that $C$ is a normal cone. Then, $f$ is order-preserving, and we have:

$$r(f, C) \leq \inf_{a \in A} r(f_a, C).$$

When, in addition, either $\nu(f, C) < cw(f, C)$ or $f$ is uniformly continuous on bounded sets of $C$, the following equalities hold:

$$r(f, C) = \inf_{a \in A} r(f_a, C) = \inf_{a \in A} cw(f_a, C) = cw(f, C).$$

**Proof.** The proof of the properties that $f$ is homogenous of degree 1, or order-preserving are identical to that for Lemma 3.1. Also, the proof of (8.5) is identical to that of the first inequality in (8.2). From Lemma 7.1, we get that $cw(f, C) \leq \inf_{a \in A} cw(f_a, C)$. Let $\lambda > 0$ and $x \in \text{int} C$ be such that $f(x) \leq \lambda x$. By assumption, there exists $a_x \in A$ such that $f_{a_x}(x) = f(x) \leq \lambda x$, hence $\lambda \geq cw(f_{a_x}, C) \geq \inf_{a \in A} cw(f_a, C)$. since this holds for all $\lambda > 0$ and $x \in \text{int} C$ such that $f(x) \leq \lambda x$, we deduce that $cw(f, C) \geq \inf_{a \in A} cw(f_a, C)$, which shows (8.4). Finally, the equalities in (8.6) follow from (8.1), (8.5), Lemma 7.2 and Theorem 7.3.

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