Out of Equilibrium Dynamics of the Inflaton Re-examined

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Abstract

In a series of papers Boyanovsky et al. have studied the evolution of an inflaton with a negative mass squared and a quartic self coupling using the Closed Time Path (CTP) formalism relevant for out-of-equilibrium dynamics. In this paper we comment on various aspects of these works. We first compare their approach to alternate approaches to study inflaton dynamics and point out that the use of the CTP formalism gives the same results as standard field theory in the Hartree and leading order large N approximations. We then redervie using the WKB approximation the large momentum mode functions of the inflaton needed for renormalisation and point out some differences with the previously obtained results. We also argue that the WKB approximation is valid only for large $k/a$ and not for large $k$ as apparently assumed in the above mentioned works. We comment on the renormalisation prescription adopted in these works and finally discuss how it differs from another more commonly used prescription.

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I. INTRODUCTION

In the new inflationary scenario the expansion of the universe is driven by the nearly constant potential energy of a slowly rolling scalar field. In Refs. [1,2], Boyanovsky et al. have discussed in great detail the evolution of an inflaton with a negative mass squared and a quartic self coupling using the Schwinger-Keldysh Closed Time Path (CTP) formalism [3,4] which is suited for non-equilibrium or dynamical situations. The advantage of the CTP formalism is that it allows one to obtain \( \langle in|in \rangle \) matrix elements of field operators (as opposed to \( \langle out|in \rangle \) matrix elements of standard field theory) which are relevant in non-equilibrium or dynamical situations [5]. \(^1\) In an earlier work [7] it had been shown (in the context of Minkowski spacetime) that if unstable modes grow large then a perturbative treatment within the CTP formalism will be insufficient and one needs to invoke a non-perturbative treatment. Therefore in Ref. [1,2] both the Hartree and the large \( N \) non-perturbative approximations are used to obtain the equation of motion for the inflaton.

Renormalisation was achieved in Refs. [1,2] by subtracting off the divergent ultraviolet behaviour based on the renormalisation scheme employed in Ref. [6]. The ultraviolet behaviour was obtained by analytically solving the equations of motion for the large momentum fluctuations of the inflaton field after applying the Hartree or large \( N \) approximations. These approximations render the equations of motion linear and thus amenable to techniques such as the WKB method.

In this work we first restate the equations of motion and the expression for the inflaton fluctuations obtained in Refs. [1,2]. We then compare this with earlier approaches to studying inflaton dynamics. We next rederive the expressions for the large \( k \) mode functions of the inflaton.

\(^1\)Alternatively, in Ref. [6] the authors use a Gaussian ansatz for the density matrix, which is evolved in time with a time dependent Hamiltonian using the von Neumann equation, and this is used instead of the CTP formalism to obtain the necessary expectation values for the dynamical system.
inflaton field needed for renormalisation, after introducing more generalised initial conditions with finite temperature corrections to the initial mass of the inflaton. Since some of our results differ from those in Refs. [1,2] and related works, and since the calculations are involved, we have provided all the details and pointed out where we differ from previous works. We have also commented upon certain aspects of the renormalisation procedure. We then compare the renormalisation scheme adopted in Refs. [1,2] with schemes adopted by other authors. Lastly we provide some numerical results.

To study the evolution of the inflaton in its potential we choose initial conditions for the field modes consistent with those specified in Ref. [2]. We work in the context of the large $N$ approximation and introduce an $N$-plet $\vec{\Phi}$ to represent the inflaton. As in Ref. [1] we work in the context of a quenched approximation in which the initial state of the universe at the onset of inflation at $t = 0$ corresponds to a state of positive mass $\Phi$ particles in thermodynamic equilibrium at a temperature $T_i > T_c$, the critical temperature, while the subsequent evolution is for a universe at $T = 0$ in which the mass term for $\Phi$ becomes negative. We assume that the distribution of $\Phi$ particles freezes out soon after the onset of inflation at a temperature close to $T_i$.

Our major conclusions are the following:

- The equations of motion obtained in Refs. [1,2] after invoking the CTP formalism are the same as those obtained using standard field theory. This is because of the Hartree and lowest order large $N$ approximations adopted in Refs. [1,2].

- The WKB solutions obtained in Refs. [1,2] for large $k$ (i.e. $k \gg m_R$, where $m_R$ is the renormalised inflaton mass parameter) are actually valid for large $k/a$ (i.e. $k/a \gg m_R$). Since the renormalisation scheme of Refs. [1,2] involves subtracting the contribution of modes with $k > m_R$ the counterterms used in Refs. [1,2] do not match the bare quantities for $m_Ra > k > m_R$.

- The dominant modes relevant for inflaton evolution in new inflation where the inflaton has a negative mass squared are the low momentum modes. The counterterms used
in Refs. [1,2] do not have any significant effect on the contribution of these modes and hence, despite our above criticism of the counterterms, one gets the same results as in the more standard prescription for removing ultraviolet divergences for the inflaton.

- In the more standard prescription to deal with ultraviolet divergences for the inflaton one subtracts the contribution of modes with $k/a > H$. In Refs. [1,2] the intent of the renormalisation scheme appears to be to subtract the contribution of modes with $k > m_R$. While for the case of an inflaton with a negative mass squared it turns out that both schemes give the same results, for an arbitrary field the renormalisation scheme of Refs. [1,2] should be used with care. We illustrate this with an example of a massless field in de Sitter space.

II. EQUATIONS OF MOTION AND INITIAL CONDITIONS FOR INFLATON MODES, AND TOTAL FLUCTUATIONS

The Lagrangian density that we consider in the context of a spatially flat Robertson-Walker universe is given by,

$$\mathcal{L} = a^3(t) \left[ \frac{1}{2} \ddot{\Phi}^2(x) - \frac{1}{2} \frac{(\nabla \Phi(x))^2}{a^2(t)} - V(\Phi(x)) \right],$$  \hspace{1cm} (1)

$$V(\Phi) = \frac{Nm^4}{2\lambda} + \frac{1}{2} M_g^2 \Phi^2 + \frac{\lambda}{8N} \Phi^4$$  \hspace{1cm} (2)

$\Phi$ is an $N$-plet and $M_g^2 = -m^2 + \xi \mathcal{R}$. $\mathcal{R}$ is the Ricci scalar and is given by

$$\mathcal{R} = 6 \left( \frac{\dot{a}(t)}{a(t)} + \frac{\dot{a}^2(t)}{a^2(t)} \right).$$  \hspace{1cm} (3)

The scale factor can be obtained dynamically using the Einstein equation. (In the numerical solutions in Ref. [2] $H$ was obtained dynamically while it was taken as a constant in the simulations of Ref. [1].)

Below we sketch the large $N$ approximation as discussed in greater detail in Refs. [8,1].

We then present the equations of motion and discuss the initial conditions. In this section we also discuss other approaches to study inflaton dynamics.
Non-perturbative treatment of non-equilibrium dynamics can be studied by approximations such as the Hartree approximation and the large $N$ expansion [1]. These approximations are similar at leading order in $1/N$. Here we shall work with the large $N$ approximation. In the large $N$ approximation the Lagrangian can be written in terms of $\lambda/N$ as above and in the spirit of Ref. [1] it is assumed below that $\lambda$ is small enough that the large $N$ approximation is valid even when $N = O(1)$. (We shall choose $\lambda = 10^{-12}$ later.) Following Ref. [8,1], only leading order terms in the large $N$ approximation are retained below.

To facilitate the use of the large $N$ approximation to study the inflationary phase transition the inflaton is written as

$$\Phi(x,t) = (\sigma(x,t), \pi(x,t)),$$

where $\pi$ is an $N-1$-plet, and we let

$$\sigma(x,t) = \sigma_0(x,t) + \rho(x,t) ; \quad \langle \sigma(x,t) \rangle = \sigma_0(x,t) ; \quad \langle \rho(x,t) \rangle = 0. \quad (4)$$

To apply the large $N$ approximation $\sigma_0$ and $\pi$ are defined by

$$\sigma_0(x,t) = \sqrt{N}\phi(t), \quad \pi(x,t) = \psi(x,t) \left(1, 1, \cdots, 1\right). \quad (5)$$

$\phi$ mimics the mean of the inflaton field while $\psi$ and $\rho$ correspond to fluctuations. It is implicitly assumed here that the mean field is a function of time only.

The mode functions $U_k(t)$ of the inflaton are defined as

$$\psi(x) = \int \frac{d^3k}{(2\pi)^3\sqrt{2}} [U_k(t)a_k e^{ikx} + h.c.] \quad (6)$$

with $[a_k, a_k^\dagger] = (2\pi)^3\delta(k - k')$. The leading order in $1/N$ equations of motion for $\phi$ and $U_k(t)$ then are

$$\ddot{\phi}(t) + 3H\dot{\phi}(t) + M_g^2\phi(t) + \frac{\lambda}{2}\phi^3(t) + \frac{\lambda}{2}\dot{\phi}(t)\langle \psi^2(t) \rangle = 0, \quad (7)$$

$$\left[ \frac{d^2}{dt^2} + 3H \frac{d}{dt} + \omega_k^2(t) \right] U_k(t) = 0, \quad (8)$$
and the effective frequency $\omega_k(t)$ is given by

$$\omega^2_k(t) = \frac{k^2}{a^2(t)} + M^2(t),$$

(9)

where

$$M^2(t) = M_g^2 + \frac{\lambda}{2} \phi^2(t) + \frac{\lambda}{2} \langle \psi^2(t) \rangle,$$

(10)

and

$$\langle \psi^2(t) \rangle = \int \frac{d^3k}{(2\pi)^3} \frac{|U_k(t)|^2}{2} \coth\left[\frac{W_{k0}}{2T_i}\right].$$

(11)

By $\langle \psi^2(t) \rangle$ we mean $\langle \psi^2(\vec{x}, t) \rangle(t)$. By translational invariance $\langle \psi^2(\vec{x}, t) \rangle(t)$ depends only on time.

The coth function reflects the initial thermal particle distribution and it is assumed that $\psi$ goes out of equilibrium at a temperature close to $T_i$. We obtain $T_i$ by specifying that inflation starts when the energy density in radiation falls to one-tenth of the constant term in the potential energy. Assuming the number of relativistic degrees of freedom to be 100, we get $T_i \sim 200 m_R$. $W_{k0}$ is related to the initial frequencies of the modes and is defined later.

At leading order in $1/N$, $\rho$ does not enter in the equations of motion for $\phi$ and $U_k$. Furthermore the contribution of $\rho$ to the energy density is smaller by a power of $N$ compared to that of the other fields. Hence it is not included in the analysis.

The above equations of motion are similar to those in earlier works that do not invoke the CTP formalism, such as Ref. [9]. Modifying Eqs. (2.13) and (2.14) of Ref. [9] for our potential (and our definition of the Fourier modes) gives

$$\ddot{U}_k + 3H\dot{U}_k + \left(\frac{k^2}{a^2} + M_g^2 + 3\frac{\lambda}{2} \langle \psi^2(t) \rangle' \right)U_k = 0$$

(12)

where

$$\langle \psi^2(t) \rangle' \equiv \text{out} \langle \psi^2(\vec{x}, t) \rangle_{\text{in}} = \int \frac{d^3k}{(2\pi)^3} \frac{|U_k(t)|^2}{2}.$$

(13)
The similarity in our equations and those of Ref. [9] \(^2\), though the latter are obtained without invoking the CTP formalism, is because we are working to lowest order in \(1/N\), and at this order both the \(\langle \text{out} | \text{in} \rangle\) and the \(\langle \text{in} | \text{in} \rangle\) formalisms give the same equations of motion. This may be verified by looking at the field equations obtained using standard field theory and the CTP formalism in Ref. [5]. Eqs. (4.27)-(4.34) of Ref. [5] give the equations of motion in the CTP formalism including terms of order \(1/N\) for the mean field and the momentum modes and these reduce to equations obtained using standard field theory, namely, Eqs. (2.20) and (2.21) of Ref. [5] for the mean field and Eq. (12) above for the momentum modes, if we ignore terms of order \(1/N\). \(^3\) Note that at lowest order in \(1/N\) the equal time Wightman functions \(i\langle \psi^2(t) \rangle\) obtained using the CTP formalism (also referred to as \(G_>, G_<\)) and the equal time Feynman propagator obtained in standard field theory that appear in the equations of motion are the same (see Sec. II of Ref. [10]).

Another approach to studying inflaton dynamics is provided in Ref. [11] in which the mean field \(\langle \Phi \rangle\) is taken to be 0 and the dynamics is studied by evolving \(\langle \Phi^2 \rangle\) by defining an effective action which is a function of \(\langle \Phi^2 \rangle\) rather than of \(\langle \Phi \rangle\), similar to the Cornwall-Jackiw-Tomboulis method \([12]\) in which one studies the two particle irreducible effective action. In Ref. [11] the authors do not invoke the CTP formalism. In Ref. [13] the CJT method is combined with the CTP formalism to study the evolution of a \(\lambda \phi^4\) scalar field with a tachyonic mass which is suddenly brought into contact with a heat bath at zero temperature (but in Minkowski spacetime). The CJT method directly gives an equation of motion for the two-point function. In contrast, above, as in Refs. [1,2,5], we obtain the time-

\(^2\)The factor of 3 with \(\lambda\) above is because the cactus approximation in Ref. [9] is analogous to the Hartree approximation, and the leading order large \(N\) and Hartree approximations differ by this factor of 3 \([1]\). \(\phi(t)\) is absent in the equation of motion for \(U_k\) as the mean field is absent in Ref. [9], and temperature effects are ignored.

\(^3\)Curved spacetime terms are absent in Ref. [5].
dependent two-point function by solving for the mode functions $U_k$ and then substitute them in the expression for the two-point function Eq. (11). Note also when comparing the above approaches that when one uses the large $N$ approximation one is expanding in powers of $\lambda/N$ while in the CJT method one is expanding in powers of $\bar{h}$. The reader is also referred to Ref. [14] in which the time-dependent two-point function is obtained for a system undergoing a phase transition with a quench albeit in Minkowski space time and without a time-dependent effective mass after the quench.

**Initial conditions:**

As discussed in Ref. [2], for $\phi(0) \ll H$ and $\dot{\phi}(0) \approx 0$ the growth of $\phi$ does not contribute much to the evolution of the inflaton. We assume $\phi(0)=0$ and $\dot{\phi}(0) = 0$ which implies that the mean field does not evolve with time. This is consistent with the argument presented in Ref. [11] that for a symmetric potential in de Sitter space the expectation values of the fields will be 0 in the absence of sources because the finiteness of the effective spatial volume due to the presence of an event horizon allows for tunneling between the states with expectation value $\phi$ and $-\phi$. Note that if $\phi(t) = 0$ then the field $\Phi$ can be represented as an $N$-plet of the form $\psi''_N(1,1,...)$. However, in the large $N$ limit, working with an $N-1$-plet as we do gives the same results.

We assume that the initial size of the inflating region is $O(H^{-1})$ and only include modes with $k/a_0 \geq 2H(0)$ in our expressions for $\langle \psi^2 \rangle$, energy $\epsilon$, pressure $p$, etc. [16]. (In Refs. [1,2] all modes are included. Our numerical results indicate that there is no significant effect of our lower momentum cutoff.) Then, as discussed further below, the initial conditions for $U_k$ for modes with $k/a_0 \geq 2H(0)$ are given by

$^4$Timescales for this tunneling and thermal initial states are not discussed in Ref. [11]. The non-zero overlap between the vacua associated with the two minima of our potential when the volume is finite may also be deduced from the discussion in Ref. [15]. It is because the vacua are related by a non-zero transformation when the volume is finite.
\[ U_k(0) = \frac{1}{\sqrt{W_{k0}}} ; \dot{U}_k(0) = [-iW_{k0} - H(0)]U_k(0), \tag{14} \]

where

\[ W_{k0}^2 = k^2 + a_0^2[(-1 + r^2)m^2 + (\xi - \frac{1}{6})R(0) + \frac{\lambda}{2}\phi^2(0) + \frac{\lambda}{2}\langle\psi^2(0)\rangle]. \tag{15} \]

The term \( r^2m^2 \) represents the corrections to the initial inflaton mass, predominantly thermal corrections, due to the interactions of the inflaton with other fields. We assume \( \frac{1}{2}\langle\psi^2(0)\rangle \) is small compared to other terms in \( W_{k0}^2 \) and so ignore it while entering the initial conditions in our numerical programme. Our choice of the factor 2 in the lower momentum cutoff ensures that the initial frequencies defined above are real with our choice later of \( H(0) = 2m \) for all relevant values of \( k \) irrespective of the value of \( r \), and that the contributions related to the initial fluctuations in \( \psi \) can indeed be ignored in the initial frequencies. (The existence of the low momentum cutoff allows us to avoid the issue of imaginary frequencies at \( t = 0 \) for lower momentum modes, which has to be dealt with in Ref. [2].)

Refs. [1,2] differ in the initial conditions for the mode functions, as discussed in the section on the WKB approximation. Refs. [1] and [2] also differ in the initial temperature taken in the initial conditions for the inflaton. In Ref. [1], the authors studied a quenched approximation from an initial temperature \( T_i > T_c \) to \( T \sim 0 \). The initial conditions for the modes of the inflaton field are specified at temperature \( T_i \sim 10^7 m_R \). In Ref. [2], the initial temperature is taken to be 0. As was noted in Sec. IVA of Ref. [1], this gives a much longer duration of inflation. \( T_i \) enters in both the \( \coth \) function and the choice of \( r \). Below we show that it is the influence via the former that causes a change in the duration of inflation. This may also be deduced from the arguments of Ref. [1].

III. THE LARGE \( K \) SOLUTION AND RENORMALISATION

WKB:

We now discuss the WKB approximation used to obtain the mode functions and their derivatives for large values of \( k \). The renormalised \( \langle\psi^2\rangle, \epsilon \) and \( \epsilon + p \) are then obtained by
subtracting the contribution of the large momentum modes. Our calculation is similar to that of Ref. [17]. We provide a complete derivation so as to include some details not provided in the literature. Furthermore our results differ from those in the literature and we point out the differences.

The large $k$ mode functions for a scalar field $\Phi$ with a quartic potential were obtained in Ref. [7] for evolution in Minkowski space and later in Refs. [6] and [1] for evolution in FRW and de Sitter spacetimes respectively. These were reobtained in Refs. [17,2] with a different set of initial conditions than in Refs. [6,1]. The new initial conditions for the mode functions were obtained by first defining them for the field modes in conformal spacetime. With the new initial conditions, Ref. [17] focussed on radiation and matter dominated backgrounds while the inflationary era was studied in Ref. [2]. Below we obtain the large $k$ mode functions for $\Phi$ with initial conditions similar to those in Refs. [2] and further generalise the initial conditions to include high temperature corrections to the initial effective mass, i.e., the $r^2m^2$ term in $W_{k0}$. The latter was included in Ref. [1] but not in Ref. [2]. In Refs. [17,2] it was stated that the large $k$ behaviour of the mode functions, which is used to define the counterterms for renormalisation, is independent of initial conditions if one chooses the new initial conditions. As we see below, this is true only if temperature corrections to the inflaton mass at the onset of inflation are 0.

To obtain the large $k$ mode functions, we work in conformal spacetime. Defining the conformal time $\tau$ from $d\tau = dt/a(t)$ the mode functions in conformal time are defined via $U_k(t) = f_k(\tau)/C(\tau)$ where $C(\tau) = a(t)$ when $\tau$ and $t$ are related as above. We have chosen the normalization of the scale factor such that $a(0) = C(\tau_0) = 1$.

The equation of motion for $f_k$ is (see the Appendix of Ref. [17] for more details)

$$\left[ \frac{d^2}{d\tau^2} + k^2 + \mathcal{M}(\tau)^2 \right] f_k(\tau) = 0$$

with

$$\mathcal{M}^2 = C^2[-m^2 + (\xi - \frac{1}{6})\mathcal{R}] + \frac{\lambda}{2}\bar{\phi}^2 + \frac{\lambda}{2}\langle\bar{\psi}^2\rangle$$

(17)
where the tilde refers to the conformally rescaled field, i.e., \( \phi(x,t) = \tilde{\phi}(x,\tau)/C(\tau) \), etc. \( R(\tau) = 6C''(\tau)/C^3(\tau) \) where ' refers to differentiation with respect to \( \tau \). The above expression for the effective mass differs from that in Eq. (A7) in Ref. [17].

The initial conditions for the conformal time mode functions \( f_k \) are chosen to correspond to an initial density matrix that represents a system in local thermodynamic equilibrium and which commutes with the Hamiltonian in conformal spacetime at \( \tau = \tau_0 \) [17]. Therefore, for modes with \( k/a_0 \geq 2H(0) \),

\[
f_k(\tau_0) = \frac{1}{\sqrt{W_{k0}}} ; \quad \dot{f}_k(\tau_0) = -iW_{k0}f_k(\tau_0) ; \quad W_{k0} = \sqrt{k^2 + M_0^2}.
\]

\( M_0 \) is the initial effective mass in terms of conformal time.

\[
M_0^2 = C^2(\tau_0)[(1 + r^2)m^2 + (\xi - \frac{1}{6})R(\tau_0)] + \frac{\lambda}{2}\bar{\phi}^2(\tau_0) + \frac{\lambda}{2}\langle\bar{\psi}^2(\tau_0)\rangle
\]

Note that the temperature corrections \( r^2m^2 \) above is not included in Eq. (17) which is valid at times after the quench. The presumption here is that the other fields to which the inflaton couples go out of equilibrium during the quench at a temperature much below \( T_i \) at which stage their contribution to the effective mass of the inflaton is small and can be ignored, and that any subsequent growth in their fluctuations during inflation does not lead to a significant contribution to the effective inflaton mass.

These initial conditions for modes with \( k/a_0 \geq 2H(0) \) translate to

\[
U_k(0) = \frac{1}{\sqrt{W_{k0}}} ; \quad \dot{U}_k(0) = [-iW_{k0} - H(0)]U_k(0),
\]

where \( W_{k0} \) is \( \mathcal{W}_{k0} \) expressed in comoving coordinates.

Dividing the equation of motion for \( f_k \) Eq. (16) by \( k^2/m^2 \), we treat \( m/k \equiv \delta \) as a small parameter and apply WKB theory to solve for \( f_k \). We shall seek solutions for \( k \geq 10m \). Using a WKB solution for \( f_k \) of the form [18]

\[
D_k(\tau) \sim \exp\left[\frac{1}{\delta}S(\tau)\right] = \exp\left[\frac{1}{\delta}\sum_{n=0}^{\infty} \delta^n S_n \right] \equiv e^{\int_{\tau_0}^{\tau} R_k(\tau')d\tau'},
\]

we obtain two solutions, \( D_{k1}(\tau) \) and its complex conjugate, which agree with the expressions in Eqs. (A14) and (A15) of Ref. [17], namely,
\[ R_k(\tau) = -i k + R_{0,k}(\tau) - i \frac{R_{1,k}(\tau)}{k} + \frac{R_{2,k}(\tau)}{k^2} - i \frac{R_{3,k}(\tau)}{k^3} + \frac{R_{4,k}(\tau)}{k^4} + \cdots \]  \hspace{1cm} (22)

and its complex conjugate. The coefficients are given by

\[
\begin{align*}
R_{0,k} &= 0 ; \\
R_{1,k} &= \frac{1}{2} M^2(\tau) ; \\
R_{2,k} &= -\frac{1}{2} R'_{1,k} = -\frac{1}{4} M^2(\tau) \\
R_{3,k} &= \frac{1}{2} \left( R'_{2,k} - R^2_{1,k} \right) = -\frac{1}{8} M^{2''}(\tau) - \frac{1}{8} M^4(\tau) \\
R_{4,k} &= -\frac{1}{2} \left( R'_{3,k} + 2 R_{1,k} R_{2,k} \right) = \frac{1}{16} M^{2''}(\tau) + \frac{1}{8} M^4(\tau).
\end{align*}
\]  \hspace{1cm} (23)

Imposing the initial condition for \( f_k(\tau_0) \) implies

\[
f_k(\tau) = \frac{1}{2\sqrt{W_{k0}}} [(1 + \gamma) D_{k1}(\tau) + (1 - \gamma) D'_{k1}(\tau)].
\]  \hspace{1cm} (24)

\( \gamma \) is then obtained by imposing the initial condition on \( f'_k(\tau_0) \). We find that the real and imaginary parts of \( \gamma \) are given by

\[
\begin{align*}
\gamma_R &= 1 + \frac{g_1}{k^2} + \frac{g_2}{k^4} \\
\gamma_I &= \frac{g_3}{k^3} + \frac{g_4}{k^5}
\end{align*}
\]  \hspace{1cm} (25)

where

\[
\begin{align*}
g_1 &= \frac{1}{7} (M^2_0 - M^2_{0+}) \\
g_2 &= -\frac{1}{8} M^{2''}_{0+} - \frac{1}{8} (M^4_0 - M^4_{0+}) - \frac{1}{4} M^2_{0+} (M^2_0 - M^2_{0+}) \\
g_3 &= M^2_{0+} \\
g_4 &= -\frac{1}{16} M^{2''}_{0+} - \frac{1}{2} M^2_{0+} M^2_{0+} + \frac{1}{8} M^2_0 M^2_{0+}
\end{align*}
\]  \hspace{1cm} (26)

and \( M^2_{0+} \equiv M^2(\tau_0) \). Note that \( M^2_0 - M^2_{0+} = r^2 m^2 \) since \( M^2_0 \) and \( M^2_{0+} \) are the effective masses in conformal spacetime just before and after the onset of inflation.

Expressing \( D_{k1}(\tau) \) as \( e^{X+iY} \) and using the above relations we get

\[
|f_k|^2 = \frac{1}{W_{k0}} e^{2X} [\cos^2 Y + \gamma_R^2 \sin^2 Y - 2\gamma_I \cos Y \sin Y + \gamma_I^2 \sin^2 Y]
\]  \hspace{1cm} (27)

Ultimately the above expression will be integrated over all \( k \). Expressing trigonometric functions as exponentials there are terms in the integrand that for large \( k \) go as
\[ k^{2k-1-l} \exp[\pm 2i k(\tau - \tau_0)], \ l \geq 0. \] For large \( k(\tau - \tau_0) \) we set their integrals to 0. So we replace \( \cos^2 Y \) and \( \sin^2 Y \) by \( \frac{1}{2} \) and \( \cos Y \sin Y \) by 0 and write the effective \(|f_k|^2\) as

\[
|f_k|^2 = \frac{1}{2W_{k0}} e^{2X}[1 + \gamma_R^2]
= \frac{1}{k} \left[ 1 - \frac{1}{k^2} \frac{\mathcal{M}^2(\tau)}{2} + \frac{1}{k^4} \left\{ \frac{\mathcal{M}'(\tau)}{8} + \frac{3\mathcal{M}^4(\tau)}{8} + \frac{\mathcal{M}_0^2 - \mathcal{M}_{0+}^2}{8} \right\} \right],
\]

(28)

where we have also ignored the term proportional to \( \gamma_I^2 \) since it is of order \( 1/k^6 \) and can be ignored in the large \( k \) limit. \(|f'_k|^2\) is given by

\[
|f'_k|^2 = \frac{1}{W_{k0}} e^{2X} \left\{ \left[ R_{kR} \cos Y - R_{kI} \sin Y - \gamma_I(R_{kR} \sin Y + R_{kI} \cos Y) \right]^2 \\
+ \gamma_R^2 (R_{kR} \sin Y + R_{kI} \cos Y)^2 \right\},
\]

(29)

where \( R_{kR} \) and \( R_{kI} \) are the real and imaginary parts of \( R_k \) defined in Eq. (21). Again, the effective \(|f'_k|^2\) relevant for subsequent integration over all \( k \) and excluding terms of higher order than \( 1/k^3 \) is

\[
|f'_k|^2 = \frac{1}{2W_{k0}} e^{2X}[1 + \gamma^2_R R_{kI}^2]
= k \left[ 1 + \frac{1}{k^2} \frac{\mathcal{M}^2(\tau)}{2} \\
+ \frac{1}{k^4} \left\{ -\frac{\mathcal{M}'(\tau)}{8} - \frac{\mathcal{M}^4(\tau)}{8} + \frac{\mathcal{M}_0^2 - \mathcal{M}_{0+}^2}{4} \right\} \right],
\]

(30)

(We obtain \(|f_k|^2\) to \( O(1/k^5) \) and \(|f'_k|^2\) to \( O(1/k^3) \) since the former is multiplied by a factor of \( k^2 \) in the expression for energy and pressure. However in the expression for \( \langle \psi^2 \rangle \) we only keep terms to \( O(1/k^3) \) in \(|f_k|^2\).)

We reiterate that the terms in Eqs. (27) and (29) set to 1/2 and 0 attain these values only after integration over all \( k \). There is no mention of this in the literature. Furthermore, during inflation \( \tau = -1/(aH) \). Treating \( H \) as a constant equal to \( 2m \) we can easily check that for \( k \geq 3m \), \( k(\tau - \tau_0) >> 1 \) is valid within two e-foldings or so of inflation. Once inflation ends the universe expands as \( t^n \) \((n < 1)\) and \( \tau = \frac{n}{1-n}(aH)^{-1} \). Then \( \tau \) is positive and increases with time. Hence \( k(\tau - \tau_0) >> 1 \) will continue to be valid. However for \( k \leq 2m \), \( k(\tau - \tau_0) >> 1 \) will never hold during the inflationary era. Therefore later we shall use the above large \( k \) expressions only for \( k \geq 10m_R \), where \( m_R \) is the renormalised mass.
parameter, rather than for \( k \geq m_R \) as in Refs. [1,2], while subtracting off the divergences during renormalisation.

As mentioned earlier, one can see that it is only for \( r = 0 \), i.e. when \( \mathcal{M}_0^2 = \mathcal{M}_{0+}^2 \), that the large \( k \) subtractions needed to renormalise the divergences are independent of initial conditions. For \( r = 0 \), our result for \( |f_k(\tau)|^2 \) agrees with Eq. (A24) of Ref. [17]. Our result for \( |f'_k(\tau)|^2 \) disagrees with Eq. (A25) of Ref. [17]. However it agrees with the corresponding expression in Eq. (5.24) of the review article [19] indicating a possible typographical error in Ref. [17]. (Note that the expression for \( |f_k(\tau)|^2 \) \((r = 0)\) in Eq. (5.24) of Ref. [19] disagrees with both our expression and that of Ref. [17] indicating a possible typographical error in Ref. [19].)

The corresponding expressions for the comoving time mode functions are

\[
|U_k(t)|^2 = \frac{|f_k(\tau)|^2}{C^2(\tau)}
\]

\[
= \frac{1}{ka^2(t)} - \frac{1}{2ka^2(t)} B(t) + \frac{1}{8k^5 a^2(t)} \left[ 3B(t)^2 + a(t)\dot{a}(t)\dot{B}(t) + a(t)^2\ddot{B}(t) + (B_0 - B_{0+})^2 \right] + \mathcal{O}(1/k^7)
\]

\[
= S^{(2)} + \mathcal{O}(1/k^7),
\]

\[
|\dot{U}_k(t)|^2 = \frac{1}{C^2(\tau)} \left[ \frac{|f_k(\tau)|^2}{C^2(\tau)} + \left( H^2 - \frac{H}{C(\tau)} \frac{d}{d\tau} \right) |f_k(\tau)|^2 \right]
\]

\[
= \frac{k}{a^4(t)} + \frac{1}{2ka^4(t)} \left[ B(t) + 2\dot{a}(t)^2 \right]
\]

\[
= \frac{1}{8k^3 a^4(t)} \left[ -B(t)^2 - a(t)^2\ddot{B}(t) + 3a(t)\dot{a}(t)\dot{B}(t) - 4\dot{a}^2(t)B(t) + (B_0 - B_{0+})^2 \right]
\]

\[
+ 2B(t)\dot{B}(t) + \mathcal{O}(1/k^5)
\]

\[
= S^{(1)} + \mathcal{O}(1/k^5),
\]

where \( B(t) \) is \( \mathcal{M}^2(\tau) \) expressed in comoving coordinates and \( B_0 \) and \( B_{0+} \) are similarly defined.

**Renormalisation:**

As in Ref. [1,2] we use the above expressions to write the renormalised \( \langle \psi^2 \rangle \), \( \langle \psi^2 \rangle_R \), as (hereafter the subscript \( R \) refers to renormalised quantities)
\[ \langle \psi^2(t) \rangle_R = \int \frac{d^3k}{(2\pi)^3} \coth \left( \frac{W_{ka}}{2T_i} \right) \left[ |U_k(t)|^2 \Theta(k - 2H(0)) - \Theta(k - \kappa) \left( \frac{1}{ka^2} - \frac{B}{2k^3a^2} \right) \right]. \tag{35} \]

\( \kappa \) represents a renormalisation scale. As mentioned above, the large \( k \) expressions used in the subtraction are valid for \( k \geq 10m_R \) (after two e-foldings). Hence we take \( \kappa \) to be \( 10m_R \) rather than \( m_R \) as in Ref. [2]. We also choose to not rescale \( \langle \psi^2 \rangle_R \) to be 0 at \( t = 0 \). As in Ref. [1] we shall take \( H(0) = 2m_R \).

\( B(t) \) is now expressed in terms of renormalised parameters making use of the condition, \( M_R^2(t) = M^2(t) \) discussed in Sec. IV of Ref. [6]. (Note that \( M^2(t) \) is written in terms of bare parameters.) Therefore

\[ B(t) = a^2(t) \left[ -m_R^2 + (\xi_R - \frac{1}{6})\mathcal{R}(t) + \frac{\lambda_R}{2} \phi^2(t) + \frac{\lambda}{2} \langle \psi^2(t) \rangle_R \right]. \tag{36} \]

Since \( B \) contains \( \langle \psi^2 \rangle_R \) one has to solve Eq. (35) to obtain an explicit expression for \( \langle \psi^2 \rangle_R \).

Besides the presence of the first \( \Theta \) function, our approach above also differs from that of Ref. [1,2] in that the second \( \Theta \) function applies to all terms in the subtraction and not just to the last term that suffers from an infrared singularity. The difference between the subtractions under the prescription above and that of Refs. [1,2] is finite and so both are valid. However, as we discuss below, a prescription that subtracts all terms in an expansion for \( |U_k|^2 \) for large \( k \) modes makes it simpler to obtain the expressions for the renormalised energy and energy plus pressure that satisfy the energy conservation equation.

Similar to the approach in Refs. [1,2], the renormalised energy and energy plus pressure

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5The expression for \( \langle \psi^2(t) \rangle_R \) in Eq. (3.18) of Ref. [1] is missing one subtraction term (and the coupling). We thank D. Cormier for confirming that this is a typographical error. There are similar errors in Eq. (3.20) of Ref. [1] and in the expression for the renormalised fluctuations in Sec. V of Ref. [2].
are defined as \(^6\) \(^7\)

\[
\frac{\epsilon_R}{N} = \frac{m_R^4}{2\lambda} + \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} M_{pl}^2 \phi^2 + \frac{\lambda_R}{8} \phi^4 \\
+ \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \coth[\frac{W_{k0}/(2\theta)}{2(\pi^2)}] \left[ \dot{U}_k^2 \Theta(k - 2H(0)) - S^{(1)} \Theta(k - \kappa) \\
+ \omega_{kR}^2 (|U_k|^2 \Theta(k - 2H(0)) - S^{(2)} \Theta(k - \kappa)) \right] - \frac{\lambda_R}{8} \langle \psi^2 \rangle_R
\]

\[(37)\]

\[
\frac{(p + \varepsilon)_R}{N} = \dot{\phi}^2 + \int \frac{d^3k}{2(2\pi)^3} \coth[\frac{W_{k0}/(2\theta)}{2(\pi^2)}] \left[ \dot{U}_k^2 \Theta(k - 2H(0)) - S^{(1)} \Theta(k - \kappa) \\
+ \frac{k^2}{3a^2} (|U_k|^2 \Theta(k - 2H(0)) - S^{(2)} \Theta(k - \kappa)) \right]
\]

\[(38)\]

The renormalised energy and pressure should satisfy the energy conservation equation

\[
\dot{\epsilon}_R(t) + 3 h(t)(p + \varepsilon)_R(t) = 0.
\]

For the energy conservation equation to hold the \(\Theta\) function for the subtracted terms should be chosen compatible with the subtraction in \(\langle \psi^2 \rangle_R\). This has been done in Ref. [1] and above. Refs. [1,2] differ from each other in their subtractions for the energy and energy plus pressure and we find that the expressions for \(\epsilon_R\) and \((p + \varepsilon)_R\) in Ref. [2] do not satisfy the energy conservation equation because the lower momentum cutoff for the different subtracted terms are not chosen appropriately. We find that a simple prescription

\(^6\)There is an error in the last term in the expression for the renormalised energy density in Eq. (3.21) of Ref. [1].

\(^7\)An alternate prescription of obtaining the renormalised energy-momentum tensor using adiabatic subtraction is provided in Ref. [20], albeit for chaotic inflation. In this prescription one expands bare quantities in powers of the derivatives of \(\ln a\) which is equivalent to expanding about solutions for Minkowski spacetime.
is to adopt the same lower momentum cutoff for all the subtracted terms in \( \langle \psi^2 \rangle_R, \epsilon_R \) and \( (p + \epsilon)_R \), and not to rescale \( \langle \psi^2 \rangle_R \) at \( t = 0 \) to be 0.

An issue that appears to have been overlooked in Refs. [17,2] is that the WKB solutions obtained above are also valid only for \( k^2 \gg \mathcal{M}^2 \). In fact, one can see that if one guesses a solution for Eq. (16) of the form \( \exp[i \int (k^2 + \mathcal{M}^2)^{1/2} \, d\tau] \) then the exponent will approximate to a series of the form Eq. (22) (in the limit of slowly varying \( \mathcal{M}^2 \)) only if \( \mathcal{M}^2 \ll k^2 \). Because of the factor \( C(\tau) \) in \( \mathcal{M} \) this condition breaks down very quickly for the lower momentum modes in the integrals over \( k \). Alternatively, the validity condition for the WKB solutions is written as \( \delta^n S_{n+1} \ll \delta^{n-1} S_n \) as \( \delta \to 0 \) [18], i.e. succeeding terms in the series in the exponent of the WKB solution should be smaller than previous ones. Applying this condition to the non-zero terms in the series for \( R_k \) we find that it holds only for \( k^2 \gg \mathcal{M}^2 \). Similarly, in the expansion for \( |U_k|^2 \) in powers of \( 1/k \) in Eq. (3.2) of Ref. [1], the second term is smaller than the first only if \( k^2/a^2 \gg M^2(t) - \mathcal{R}/6 \), and similarly for the second and third terms. Thus the subtractions being carried out do not match the true solution for increasingly larger numbers of modes as inflation progresses. (Though the WKB series will not approximate the true solution for \( k < \mathcal{M} \) this may not interfere with the energy conservation equation as the individual terms in the series are obtained consistently order by order from the equation of motion.)

Now, for the inflaton field the dominant contribution to \( \langle \psi^2 \rangle_R, \epsilon_R \) and \( p_R \) come from low momentum modes. When a mode becomes unstable, i.e., when its effective frequency \( \omega^2_{kR} \) becomes negative, it grows exponentially fast. Obviously low momentum modes become unstable earlier and hence start growing earlier. Furthermore, after a few e-foldings of inflation the contribution of additional modes that become unstable does not add much to the contribution of the modes that left in the first few e-foldings. As discussed above, for the dominant low momentum modes the WKB solutions are not valid approximations of their behaviour. However, as seen in Fig. 2, the subtractions carried out in Refs. [1,2] based on the WKB solutions do not affect the contribution of these modes much. This may explain why the numerical solutions of Ref. [2] did not show any conflict with the energy
conservation equation even though the counterterms were subtracted improperly. Note that for a field with a positive mass squared, as also considered in Ref. [17], this explanation will not hold.

**A comparison with the standard prescription**

We now compare the renormalisation schemes adopted in Refs. [1,2] based on Ref. [6] with the approach outlined in Ref. [9]. (See also Ref. [16].) The basic difference is that in Ref. [9] all modes that have left the horizon during inflation are included in the integral over $k$ in $\langle \psi^2 \rangle$. This provides an ultraviolet cutoff of $Ha$ for the integral. (In this sub-section we assume $H$ is constant during inflation for easy comparison with Refs. [9,16] where this is also assumed.) In the scheme adopted in Refs. [1,2] all divergent terms in the WKB solution obtained above are subtracted for modes $k \geq \kappa$ from $|U_k|^2$ to obtain $\langle \psi^2 \rangle_R$. In the former scheme more and more modes contribute to the integral in $\langle \psi^2 \rangle_R$ as they cross the horizon but this does not seem to be the goal in the latter scheme. But it is precisely the contribution of the newer modes leaving that gives the standard time dependence of $\langle \psi^2 \rangle_R$ for a massless or massive field, as in Ref. [21].

For example, in the case of a massless field for which we know the exact solution for $U_k$, the prescription of Ref. [9] gives 8

\[
\langle \psi^2 \rangle_R = \frac{1}{(2\pi)^3} \int_H^{Ha} 4\pi k^2 dk \frac{H^2}{2k^3} \left(1 + \frac{k^2}{H^2a^2}\right)
\approx \frac{H^3t}{4\pi^2}.
\]

(40)

(The second term is ignored as it is suppressed by $a^2$.) Presuming that the intent in the scheme of Ref. [6] was to subtract the contribution of modes greater than some renormalisation scale $\kappa$, and making use of the solution obtained in Ref. [9] to correctly subtract the contribution of all modes $k > \kappa$, $\langle \psi^2 \rangle_R$ will be

---

8As stated in Ref. [9], this approach provides a simple but intuitive way of obtaining $\langle \psi^2 \rangle_R$. A more rigourous derivation with proper regularisation and renormalisation is given in Ref. [22].
\[ \langle \psi^2 \rangle_R = \frac{1}{(2\pi)^3} \int_0^\kappa 4\pi k^2 dk \frac{H^2}{2k^3} \left( 1 + \frac{k^2}{H^2 a^2} \right) \approx \frac{H^2}{4\pi^2} \ln(\kappa/H). \]  

(41)

(We have chosen the lower momentum cutoff to be \( H \) in keeping with Ref. [9].) One sees that the two prescriptions give very different results which must be kept in mind while calculating the effect of fluctuations.

For our field \( \psi \) with a negative mass squared the renormalisation prescription based on Ref. [6] implies

\[ \langle \psi^2 \rangle_R = \int_{2H}^{\infty} \frac{4\pi k^2 dk}{(2\pi)^3} \frac{1}{2} \left| U_k(t) \right|^2 \coth\left[ W_{k_0}/(2T_i) \right] - \int_0^\kappa \frac{4\pi k^2 dk}{(2\pi)^3} \frac{1}{2} \left[ \left| U_K(t) \right|^2 \coth\left[ W_{k_0}/(2T_i) \right] \right] \]  

\[ = \int_{2H}^{\infty} \frac{4\pi k^2 dk}{(2\pi)^3} \frac{1}{2} \left| U_k(t) \right|^2 \coth\left[ W_{k_0}/(2T_i) \right] \]  

\[ + \int_0^{\left| M \right|} \frac{4\pi k^2 dk}{(2\pi)^3} \frac{1}{2} \left( \left| U_k(t) \right|^2 - \left| U_K(t) \right|^2 \right) \coth\left[ W_{k_0}/(2T_i) \right], \]  

(42)

(43)

where \( U_K(t) \) are the WKB solutions and \( \left| U_K(t) \right|^2 \) above contains terms of order \( 1/k \) and \( 1/k^3 \). As discussed earlier, the WKB solutions \( U_K(t) \) in the second integrals are not an approximation for the corresponding \( U_k(t) \) for \( k \leq \left| M \right| \). (Eq. (43) is written for times when \( \kappa \leq \left| M \right| \), which is all times if \( \kappa = m_R \) and after 2 e-foldings if \( \kappa = 10m_R \).) On the other hand, the renormalisation scheme of Ref. [9] would imply

\[ \langle \psi^2 \rangle_R = \int_{2H}^{H a} \frac{4\pi k^2 dk}{(2\pi)^3} \frac{1}{2} \left| U_k(t) \right|^2 \coth\left[ W_{k_0}/(2T_i) \right] \]  

(44)

(for times when \( H a > 2H \), i.e., after 1 e-folding).

Both expressions for \( \langle \psi^2 \rangle_R \) appear different. (Since they differ by a finite amount, in principle, both are valid.) However, in Fig. 1 we have plotted \( \langle \psi^2 \rangle_R \) obtained under both renormalisation schemes and they are similar. For the case of the inflaton with a negative mass squared lower momentum modes are growing modes and the contribution of the lowest few modes \( k \leq 20m_R \) dominates the integrals in \( \langle \psi^2 \rangle_R \). For these modes the WKB subtractions do not affect their contributions, as seen in Fig. 2. Therefore once \( |M| \) and \( H a \)
become larger than $20m_R$, which happens within 3 e-foldings, $\langle \psi^2 \rangle_R$ obtained under both renormalisation schemes are similar.

**Some additional comments:**

In Ref. [1] the authors cite the possibility using the growth of fluctuations of the inflaton as a new means of evolving the inflaton in its potential and providing a graceful exit from the inflationary phase. However, following the evolution of the fluctuations rather than the mean field to study the inflationary phase transition has been studied earlier in Refs. [21,9,11].

In Sec. V(D) of Ref. [2] the authors introduce the notion of a zero mode assembly consisting of super-horizon sized low momentum modes, which acts as an effective zero mode after a few e-foldings of inflation even in the absence of $\phi(t)$. The authors have also discussed the classicalisation of these low momentum fluctuations. We point out that this is similar to ideas discussed in Sec. IIIA of Ref. [23] and in Sec. 8.3 of Ref. [16] in which the early evolution of the inflaton is due to the growth of quantum fluctuations $\langle \psi^2 \rangle_R$ but after some time $t_{cl}$ one can study inflation by solving the classical equation of a motion for a purely time dependent field with the initial value at $t_{cl}$ taken (in Ref. [16]) to be $\langle \psi^2(t_{cl}) \rangle_R^{1/2}$.

**IV. NUMERICAL RESULTS**

In this section we present our results after numerically solving the large set of differential equations representing a continuum of $k$-values. The numerical code is based on fourth order Runge-Kutta for evolution of the mode functions and IMSL algorithms for interpolation of discrete data and integration to obtain $\langle \psi^2 \rangle_R$. We have chosen $\lambda = 10^{-12}$ consistent with the calculation of density perturbations in Ref. [2] and we set $\xi_R = 0$. For numerical purposes we follow Ref. [1] in introducing the following dimensionless quantities

\[
\bar{t} = m_R t ; \quad h = \frac{H}{m_R} ; \quad q = \frac{k}{m_R} \quad (45)
\]

\[
g = \frac{\lambda_R}{8\pi^2} ; \quad g\Sigma(\bar{t}) = \frac{\lambda_R}{2m_R^2} \langle \psi^2(t) \rangle_R ; \quad T_{ii} = T_i/m_R . \quad (46)
\]
We treat $H$ as a constant. Assuming $m_R \sim 10^{13}$ GeV we set $h = 2$.

In Fig. 1 we present the growth of $g\Sigma$ under the renormalisation schemes of Ref. [6] and Ref. [9]. The plot following the renormalisation scheme of Ref. [6] contains the modifications suggested above, namely, only modes $k/a_0 > 2H(0)$ are included in the integrals over $k$, the theta function for the subtracted quantities applies to all terms being subtracted and the renormalisation scale $\kappa$ is taken to be $10m_R$.\(^9\) It is not possible to distinguish between the two curves and the quantum fluctuations grow dramatically between $\tilde{t} = 60$ and 100 and approach an asymptotic value of 1.

In Fig. 2 we plot $g\Sigma$ with and without the subtraction of Eq. (42). The two curves lie on top of each other. This implies that the effect of the subtraction is negligible.

In Fig. 3 we plot $g\Sigma$ for different values of $T_{ii}$. The initial temperature $T_i$ enters in two places in the calculation of the inflaton dynamics in Ref. [1]. Firstly, it appears in the temperature dependent correction to the effective mass for $\psi$ in the initial conditions ($r$ is taken to be $T_i/T_c$ in Ref. [1]) and secondly, it appears in the expressions for $\langle \psi^2 \rangle$, energy, etc. in the factor $\coth[W_{k_0}/(2T_i)]$. To distinguish between these two effects we choose $T_{ii} = 0, 200$ and $10^7$ and take $r = 0$ for the first case and $r = 2$ for the other cases. For $T_{ii} = 0$, $g\Sigma$ begins to grow appreciably at $\tilde{t} \approx 70$ and achieves an asymptotic value of 1. This agrees well with the results in Ref. [2]. For $T_{ii} = 10^7$, $g\Sigma$ begins to grow appreciably at $\tilde{t} \approx 30$. This again agrees well with the results in Ref. [1]. The corresponding timescale for $T_{ii} = 200$ is 60. Comparing the plots for $T_{ii} = 200$ and $10^7$ with the same value of $r$ we see that it is the presence of the coth term representing initial thermal fluctuations of the $\psi$ field that plays a substantial role in modifying the duration of inflation. As a further test we have also varied $r$ in $W_{k_0}$ till $r = 10$ and seen that there is no significant dependence on $r$.

The inflationary phase transition is completed when the effective mass for the inflaton modes $-1 + g\Sigma$ achieves a value of 0. On the other hand, the inflationary era ends when

\(^9\)Subsequent figures follow the renormalisation scheme of Ref. [6] with the same modifications.
\[ \ddot{a} < 0. \] Now \[ \ddot{a} = -\frac{4\pi}{3}(\epsilon_R + 3p_R) \] and hence inflation ends when \( p_R \) becomes greater than \(-\frac{1}{3}\epsilon_R\). Since \( \dot{H}/H^2 = -\frac{3}{2}(1 + p_R/\epsilon_R) \), inflation ends when \( \dot{H}/H^2 \) becomes less than -1. Since we do not treat the Hubble constant dynamically we take the breakdown of the constant \( H \) assumption, when the fluctuations begin to grow dramatically, to be an indicator of the end of inflation. As the fluctuations grow the universe will transit from an exponentially inflating universe to a power law inflating universe and will then enter the reheating era. Therefore, \( h\ell_e \), where \( \ell_e \) is the time when the fluctuations begin to grow dramatically, gives us a lower bound on the number of e-foldings of inflation.

Understandably, the existence of frozen-in initial thermal fluctuations in \( \psi \) decreases the time required for the total fluctuations \( \langle \psi^2 \rangle_R \) to reach its final value. If one assumes that the inflaton was in thermal equilibrium prior to the onset of inflation then it is likely it will go out of equilibrium soon after the onset of inflation and the above shows that it is important to include the thermal fluctuations of the initial state while estimating the duration of inflation. If the inflaton goes out of thermal equilibrium earlier than the onset of inflation then the contribution of the thermal fluctuations frozen in at the higher temperature can be even larger.

**V. CONCLUSION**

In this paper we have firstly compared the approach of Refs. [1,2] with that of earlier works to study the dynamics of the inflaton. We point out that for the Hartree and the lowest order large \( N \) approximations employed in Refs. [1,2] with the CTP formalism the equations of motion are akin to those obtained in standard field theory. We have reobtained the large momentum mode solutions for the inflaton field using the WKB method and pointed out some differences with earlier results. Because of these differences and the very involved nature of these calculations we have included many explicit details of the calculations. We have also included temperature corrections in the initial effective mass of the inflaton field and pointed out that the assertion in Refs. [17,2] that their choice of initial conditions
in conformal spacetime gives counterterms that are initial conditions independent is only true if there are no thermal corrections to the initial inflaton mass. We have discussed different aspects of the renormalisation prescription adopted in Refs. [1,2]. In particular, we have pointed out that the asymptotic expressions used in Refs. [1,2] for the large momentum modes during renormalisation are valid for large $k/a$ and not large $k$. We have also compared the renormalisation prescription of Refs. [1,2] with that of earlier works and pointed out their differences. Despite these differences the results of these different approaches are the same because for an inflaton with a negative mass squared it is the low momentum modes, and not the high momentum modes regularised by renormalisation, that give the dominant contribution.

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Figure Captions:

**Fig. 1:** Renormalised fluctuations of the inflaton field, $g\Sigma$, are shown as a function of normalised time $\bar{t}$ for the two renormalisation schemes discussed in the text (with constant $H$). $r = 2$, $h = H/m_R = 2$ and the normalised initial temperature $T_{ii} = T_i/m_R = 200$. The solid line corresponds to the (modified) renormalisation scheme of Ref. [6] while the points correspond to the renormalisation scheme of Ref. [9].

**Fig. 2:** Fluctuations of the inflaton field, with and without the WKB subtractions (solid line and circles respectively), are shown as a function of normalised time $\bar{t}$. All parameters are as in Fig. 1.

**Fig. 3:** $g\Sigma$ vs. $\bar{t}$ for $T_{ii} = 0, 200$ and $10^7$. $r = 2$ for $T_{ii} = 200$ and $10^7$ and $r = 0$ for $T_{ii} = 0$. $h = 2$ for all plots.
Figure 2

\[ T_{ii} = 200 \]
Figure 3

\[ \frac{\theta \Sigma}{T} \]

- \( T_{ii} = 10^7 \)
- \( T_{ii} = 200 \)
- \( T_{ii} = 0 \)