Wigner’s form of the Leggett-Garg inequality, No-Signalling in Time and Unsharp Measurement

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Wigner’s form of the local realist inequality is used to derive its temporal version for a single two-level system involving two-time joint probabilities. Such an equality may be regarded as a novel form of the Leggett-Garg inequality (LGI) constituting a necessary condition for macrorealism. The robustness of its quantum mechanical (QM) violation against unsharpness of measurement is investigated. It is found that there exists a range of values of the unsharpness parameter (characterizing imprecision of the relevant measurements) for which the usual LGI is satisfied, but Wigner’s form of LGI (WLGI) is violated, thereby implying that the QM violation of macrorealism cannot be tested using the usual LGI, but can be tested using WLGI. In showing this, we take into account the general form of usual LGI involving an arbitrary number of pairs of two-time correlation functions. A recently proposed another necessary condition for macrorealism, called ‘no-signalling in time’, is also probed, showing that its QM violation persists for arbitrarily unsharp measurements.

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I. INTRODUCTION

Complementing the long-pursued exploration of the nonclassical features of the microphysical world, investigation of the fundamental aspects and validity of QM at the macroscopic level has been attracting an increasing attention over the past decade [1–9]. Relevant to this line of study, a key ingredient is provided by the Leggett-Garg form of macrorealist inequality (LGI) [10, 11] which is a temporal analogue of Bell’s inequality involving testable temporal correlation functions, and its validity can be considered a necessary condition for what is regarded as macrorealism. LGI is derived from the notion of macrorealism which is characterized by the following assumptions - Realism: At any given instant, irrespective of any measurement, a macroscopic object is definitely in one of the available states having predetermined and definite values for all its observable properties. Non-invasive measurability: It is possible, at least in principle, to determine which of the states a system is in, without affecting the state itself or the system’s subsequent behaviour.

A wide range of studies have probed different aspects of LGI; see, for example, a recent comprehensive review [12]. Against this backdrop, an earlier unexplored variant of LGI is formulated in the present paper by developing an analogy to the argument invoked in deriving Wigner’s form of local realist inequality [13]. Here we use, instead of local realism, the assumptions of realism and noninvasive measurability in the context of a single two-level system. This is parallel to the way the standard form of LGI [10, 11] involving two-time correlation functions is obtained analogous to the Bell-CHSH local realist inequality [14, 15]. The quantum mechanical (QM) violation of thus obtained, what may be called, Wigner’s form of LGI (WLGI) is studied by considering a general two-state oscillation pertaining to a system oscillating between the degenerate eigenstates of a Hamiltonian where the oscillation is induced by an external field. In this context, the robustness of the QM violation of WLGI with respect to the imprecision of relevant measurements is investigated by using the formalism of what is known as unsharp measurement [16–20]. Interestingly, it is found that there exists a range of values of the unsharpness parameter characterizing the imprecision of relevant measurements for which QM satisfies the usual form of LGI, but violates WLGI, thereby signifying the non-equivalence between LGI and WLGI. In particular, the QM violation of WLGI persists for smaller values of the unsharpness parameter (that correspond to greater imprecision of measurements involved) compared to that for the usual form of LGI. Hence, for such unsharp measurements, the QM violation of macrorealism can be tested using WLGI, but not in terms of LGI. This result is, thus, of interest in the context of the question as regards the extent of imprecision or unsharpness of the relevant measurements for which the QM violation of macrorealism can be tested. While making such a comparison between WLGI and LGI, we take into account the general form of usual LGI involving n pairs of two-time correlation functions. It is found that, with respect to unsharpness of measurements, the QM violation of WLGI is, in general, more robust than that of LGI.

Furthermore, we consider the other proposed necessary condition for macro-realism known as ‘No-Signalling in Time’ (NSIT) suggested by Kofler and Brukner [21]
which assumes that the statistics of the outcomes of a measurement at any instant would not show any dependence on whether a prior measurement has been performed; in other words, NSIT is based on applying the condition of non-invasive measurability (NIM) at the statistical level whose violation would thus imply violation of NIM at the individual level. Here we may note that Leggett [10, 11] has argued that NIM can be regarded as a ‘natural’ corollary of the condition of realism by considering what is known as an ideal negative result measurement that can be invoked for testing NSIT, as has been noted by Kofler and Brukner [21]. Thus, NSIT can be regarded as providing an alternative necessary condition for macrorealism. By investigating the robustness of the QM violation of NSIT against unsharpness of measurement, strikingly, a strong result is obtained that, no matter whatever be the value of the unsharpness parameter characterizing the imprecision of measurements involved, QM violates NSIT. In other words, the QM violation of NSIT turns out to be most robust against unsharp measurement.

II. WIGNER’S FORM OF LGI

We begin by recapitulating Wigner’s original argument [13] that derived a local realist testable inequality for a pair of spatially separated spin-1/2 particles in the singlet state. This was based on assuming as a consequence of local realism, the existence of overall joint probabilities of the predetermined definite outcomes of measuring the relevant dichotomic observables of the two particles that would yield the pair-wise marginal joint probabilities which are actually measurable. In the scenario studied by Wigner, the spin components of each of the two spatially separated particles are taken to be measured along three respective directions, say, \( \hat{a}, \hat{b} \) and \( \hat{c} \). Then consider, for example, the observable joint probability of obtaining both the outcomes +1 if, say, \( \overrightarrow{a}, \hat{a} \) and \( \overrightarrow{b} \) are measured on the first and the second particle respectively, denoted as \( P(\hat{a}+, \hat{b}+) \). Using the perfect anticorrelation property of the singlet state in question, \( P(\hat{a}+, \hat{b}+) \) can be written as marginal in the form

\[
P(\hat{a}+, \hat{b}^+) = \rho(+, +; +, +) + \rho(+, +; +, -) + \rho(+, +; -, +) + \rho(+, +; -, -)
\]

with \( \rho(v_1(\hat{a}), v_1(\hat{b}), v_2(\hat{a}), v_2(\hat{b}), v_2(\hat{c})) \) to be the overall joint probability of the predetermined definite outcomes of measurements pertaining to all the relevant observables, where \( v_1(\hat{a}) \) represents the outcome \((\pm 1)\) of measurement of the observable \( \hat{a} \) for the first particle, and so on. Similarly, considering the expressions for the observable joint probabilities \( P(\hat{c}+, \hat{b}+) \) and \( P(\hat{a}+, \hat{c}+) \) as marginals, and assuming non-negativity of the overall joint probability distributions, it follows that

\[
P(\hat{a}+, \hat{b}^+) = P(\hat{a}+, \hat{c}^+) = P(\hat{c}+, \hat{b}^+) \leq 0 \quad (1)
\]

which is one of the possible forms of Wigner’s version of the local realist inequality.

Next, in order to obtain WLGI by developing an appropriate analogy with the preceding argument, we proceed as follows. Let us focus our attention on an ensemble of systems undergoing temporal evolution involving oscillation between the two states, say, 1 and 2, and let \( Q(t) \) be an observable quantity such that, whenever measured, it is found to take a value +1(-1) depending on whether the system is in the state 1(2). Now, consider a collection of sets of experimental runs, each set of runs starting from the same initial state. On the first set of runs, let \( Q \) be measured at times \( t_1 \) and \( t_2 \); on the second \( Q \) be measured at \( t_2 \) and \( t_3 \), and on the third at \( t_1 \) and \( t_3 \) (here \( t_1 < t_2 < t_3 \)). From such measurements one can then determine the pair-wise joint probabilities like \( P(\hat{Q}_1, \hat{Q}_2), P(\hat{Q}_2, \hat{Q}_3), P(\hat{Q}_1, \hat{Q}_3) \) where \( \hat{Q}_i \) is the outcome \((\pm 1)\) of measuring \( Q \) at \( t_i \), \( i = 1, 2, 3 \). In this context, it is possible to suitably adapt the argument leading to Wigner’s inequality (1) with the times \( t_i \) of measurement playing the role of apparatus settings and by assuming, as a consequence of the assumptions of realism and NIM, the existence of overall joint probabilities \( \rho(\hat{Q}_1, \hat{Q}_2, \hat{Q}_3) \) from which by appropriate marginalization the pair-wise joint probabilities can be obtained. For example, the observable joint probability \( P(\hat{Q}_2+, \hat{Q}_3-, \hat{Q}_1+) \) of obtaining the outcomes +1 and −1 for the sequential measurements of \( Q \) at the instants \( t_2 \) and \( t_3 \) respectively can be written as

\[
P(\hat{Q}_2+, \hat{Q}_3-, \hat{Q}_1+) = \sum_{Q_1=\pm} \rho(Q_1, +, -) \quad (2)
\]

Writing similar expressions for the other measurable marginal joint probabilities \( P(\hat{Q}_1-, \hat{Q}_3- \hat{Q}_2+) \) and \( P(\hat{Q}_1+, \hat{Q}_2+) \), we get

\[
P(\hat{Q}_1+, \hat{Q}_2+) + P(\hat{Q}_1-, \hat{Q}_3-) = P(\hat{Q}_2+, \hat{Q}_3-) \quad (3)
\]

Then, invoking non-negativity of the joint probabilities occurring on the RHS of Eq(3), the following form of WLGI is obtained in terms of three pairs of two-time joint probabilities

\[
P(\hat{Q}_2+, \hat{Q}_3-) - P(\hat{Q}_1+, \hat{Q}_2+) - P(\hat{Q}_1-, \hat{Q}_3-) \leq 0 \quad (4)
\]

Similarly, other forms of WLGI involving three pairs of two-time joint probabilities can be derived by using various combinations of the observable joint probabilities, which are as follows

\[
P(\hat{Q}_2+, \hat{Q}_3+) - P(\hat{Q}_1-, \hat{Q}_2+) - P(\hat{Q}_1+, \hat{Q}_3+) \leq 0 \quad (5a)
\]

\[
P(\hat{Q}_2+, \hat{Q}_3-) - P(\hat{Q}_1-, \hat{Q}_2+) - P(\hat{Q}_1+, \hat{Q}_3-) \leq 0 \quad (5b)
\]

\[
P(\hat{Q}_2+, \hat{Q}_3+) - P(\hat{Q}_1+, \hat{Q}_2+) - P(\hat{Q}_1-, \hat{Q}_3+) \leq 0 \quad (5c)
\]
\[ P(Q_1+, Q_3-) - P(Q_1+, Q_2-) - P(Q_2+, Q_3-) \leq 0 \quad (5d) \]
\[ P(Q_1+, Q_3-) - P(Q_1+, Q_2+) - P(Q_2-, Q_3-) \leq 0 \quad (5e) \]
\[ P(Q_1+, Q_3+) - P(Q_1+, Q_2-) - P(Q_2+, Q_3+) \leq 0 \quad (5f) \]
\[ P(Q_1+, Q_2-) - P(Q_1+, Q_3-) - P(Q_2-, Q_3-) \leq 0 \quad (5g) \]
\[ P(Q_1+, Q_2-) - P(Q_1+, Q_3+) - P(Q_2-, Q_3-) \leq 0 \quad (5h) \]
\[ P(Q_1+, Q_2+) - P(Q_1+, Q_3+) - P(Q_2+, Q_3-) \leq 0 \quad (5i) \]
\[ P(Q_1+, Q_2+) - P(Q_1+, Q_3-) - P(Q_2+, Q_3+) \leq 0 \quad (5j) \]
\[ P(Q_1+, Q_2-) + P(Q_2+, Q_3-) + \ldots + P(Q_{n-1}+, Q_n-) = P(Q_1+, Q_n-) + (n - 2)2^{n-2} \text{ non-negative terms} \]

From the above expression, the form of WLGI in terms of \( n \) pairs of two-time joint probabilities can be obtained as follows

\[ P(Q_1+, Q_n-) - \sum_{i=1}^{n-1} P(Q_{i+}, Q_{i+1}-) \leq 0 \quad (6) \]

Other various forms of the \( n \)-term WLGI can be similarly obtained by using different combinations of the joint probabilities for the outcomes (±1) corresponding to \( Q_i \)'s. However, for illustrating the basic relevant features concerning the efficacy of WLGI, it suffices for our subsequent treatment to confine our attention to essentially 3-term WLGI involving three pairs of two-time joint probabilities.

Next, considering a typical two-state oscillation, let us focus on a system oscillating between the two states \(| A \rangle\) and \(| B \rangle\) which are degenerate eigenstates of the Hamiltonian \( H_0 \) corresponding to energy \( E_0 \), with a perturbing Hamiltonian \( H' \) inducing oscillatory transition between these two states. Here \( \langle A | H' | B \rangle = \langle B | H' | A \rangle = \Delta E \), and \( \langle A | H' | A \rangle = \langle B | H' | B \rangle = E' \). At any instant, such a system is found to be either in the state \(| A \rangle\) or in the state \(| B \rangle\) corresponding to the measurement of the dichotonic observable \( Q = | A \rangle \langle A | - | B \rangle \langle B | \). Let the initial state at \( t_1 \) be of the general form \( \rho_0(t_1) = \langle \psi_0 | \psi_0 \rangle \) where

\[ \langle \psi_0 | = \cos(\theta)| A \rangle + \exp(i\phi)\sin(\theta)| B \rangle \quad (7) \]

For the above state, the probability of obtaining the measurement outcome, say, +1 at the instant \( t_1 \) is given by \( tr(\rho_0(t_1)P_+) \), and after this measurement, the pre-measurement state changes to the state given by \( \rho_+(t_1) = P_+\rho_0(t_1)P_+^\dagger/tr(\rho_0(t_1)P_+) \) where \( P_+ = | A \rangle \langle A | = P_+^\dagger \). Subsequently, the post-measurement state evolves under the Hamiltonian \( H = H_0 + H' \) to the state \( \rho'_+(t_2) = U_{\Delta t}\rho_+(t_1)U_{\Delta t}^\dagger \) at a later instant \( t_2 \) where \( U_{\Delta t} = \exp(-i\Delta t) \) taking \( h = 1 \) and \( \Delta t = t_2 - t_1 \). Then, considering the subsequent measurement of \( Q \) at the instant \( t_2 \), the QM value of, say, the joint probability of obtaining both the outcomes +1 at the instants \( t_1 \) and \( t_2 \) is given by

\[ P(Q_1+, Q_2+) = tr(\rho_0(t_1)P_+)tr(\rho'_+(t_2)P_+) = tr(U_{\Delta t}(P_+\rho_0(t_1)P_+)U_{\Delta t}^\dagger P_+) = \cos^2(\theta)\cos^2(\tau) \quad (8) \]

where \( \tau = \Delta E\Delta t \) (in the units of \( h \) = 1), and the expression for the unitary matrix \( U_{\Delta t} = \exp(-iH\Delta t) \) is as follows

\[ U_{\Delta t} = e^{-i(E_0+E')\Delta t}[\cos(\tau)I - i\sin(\tau)(|A\rangle \langle B | + | B \rangle \langle A |)] \quad (9) \]

Similarly, one can obtain the QM values of the other relevant joint probabilities occurring on the LHS of the 3-term WLGI's given by Eqs. (4)-(5) (taking \( t_2 - t_1 = t_3 - t_2 = \Delta t \). The QM violations of the inequalities (4) and (5) can thus be studied by maximizing the QM values of their respective left hand sides with respect to the quantity \( \tau = \Delta E\Delta t \).

It has been found that the QM violation of the inequalities (4)-(5) depend on the the initial state and, among all the 3-term WLGI's (4)-(5) and the set of other 3-term inequalities obtained from them, the maximum QM violation is obtained of the inequalities (4) and (5a) when the LHS = 0.5043. Considering, for example, the inequality (4), the maximum QM violation (≈ 0.5043) occurs for the initial state given by Eq. (7) when \( \theta \approx 1.07 \) rad, \( \phi \approx \pi/2 \) or \( 3\pi/2 \) corresponding to the choice of \( \tau = 1.009 \) or 2.135 (in the units of \( h = 1 \)) respectively.

In the discussions of the following two sections, we will be specifically using this form of 3-term WLGI given by the inequality (4).

### III. WIGNER’S FORM OF LGI AND UNSHARP MEASUREMENT

In the preceding discussions, we have taken the relevant measurements of the observable \( Q \) to be essentially ‘ideal’. Now, if the ‘non-idealness’ of actual measurements has to be taken into account, a natural question
arises as to what effect this would have on the QM violation of WLGI as compared to that for LGI. In order to address this question, we take recourse to the formalism of what is known as unsharp measurement [16–20] which can be regarded as a particular case of POVM. Note that for an ideal measurement of the dichotomic observable under consideration given by \( Q = |A⟩⟨A| − |B⟩⟨B| = P_+ − P_− \), the respective probabilities of the outcomes \( \pm 1 \) and the way a measurement affects the observed state are determined by the projection operators that can be written as \( P_\pm = (1/2)(I \pm Q) \) where \( I = |A⟩⟨A| + |B⟩⟨B| \). Now, in order to capture the effect of imprecision involved in a non-ideal measurement, using the formalism of unsharp measurement [16–20], a parameter \( \lambda \) known as the unsharpness parameter is introduced to characterise the non-idealness or unsharpness of a measurement by defining what are referred to as the effect operators given by

\[
F_\pm = (1/2)(I \pm \lambda Q) = \lambda P_\pm + (1 − \lambda)I/2
\]

where \((1 − \lambda)\) denotes the amount of white noise present in any unsharp measurement \((0 < \lambda \leq 1)\), and \( F_\pm \) are mutually commuting Hermitian operators with non-negative eigenvalues; \( F_+ + F_- = I \), while for \( \lambda = 1 \), \( F_\pm \) reduce to projection operators \( P_\pm \). Note that Eq. (10) can be rewritten as a linear combination of projection operators \( P_\pm \) in the following way

\[
F_\pm = \left( \frac{1 \pm \lambda}{2} \right) P_+ + \left( \frac{1 \mp \lambda}{2} \right) P_-
\]

Here an important point is that, instead of the projection operators used in the case of an ideal measurement, in an unsharp measurement, the operators \( F_\pm \) determine the respective probabilities of the outcomes and the way a premeasurement state changes due to measurement. This means that for an unsharp measurement pertaining to a given state \( \rho \), the probability of an outcome, say, +1 is given by \( tr(\rho F_+) \) for which the post-measurement state is given by \( (\sqrt{F_+} \rho \sqrt{F_+})/tr(\rho F_+) \). Thus, in a given experiment, by estimating the difference between the actually observed probability of an outcome and the corresponding predicted value for an ideal experiment, the unsharpness parameter \( \lambda \) pertaining to the experiment in question can be determined. This gives an operational significance to the parameter \( \lambda \).

Using Eq. (10) or (11) and by following the prescription outlined above, the QM values of the joint probabilities occurring in WLGI given by (4) are now calculated as follows for unsharp measurements pertaining to the two-state oscillation by taking \( \rho_0(t_1) = |ψ_0⟩⟨ψ_0| \) where \( |ψ_0⟩ \) is given by Eq.(7)

\[
P(Q_{1+}, Q_{2+}) = tr(U_{\Delta t}(\sqrt{F_+} \rho_0(t_1) \sqrt{F_+}) U_{\Delta t}^\dagger F_+ )
= \frac{1}{4}[\lambda^2 \cos(2\theta) + 2\lambda \cos(2\theta) \cos^2(\tau) + \lambda \sin(2\theta) \sin(2\tau) \sin(\phi) \sqrt{1 - \lambda^2} + 1]
\]

\[
P(Q_{2+}, Q_{3-}) = tr(U_{\Delta t}(\sqrt{F_+} U_{\Delta t}(\rho_0(t_1) U_{\Delta t}^\dagger) \sqrt{F_+}) U_{\Delta t}^\dagger F_-)
= \frac{1}{4}[2\lambda \sin^2(\tau)(\cos(2\theta \cos(2\tau) + \sin(2\theta \sin(2\tau) \sin(\phi)) - \lambda^2 \cos(2\tau)
+ \lambda \sqrt{1 - \lambda^2} \sin(2\tau) \sin(2\theta \cos(2\tau) - \cos(2\tau) \sin(2\tau) \sin(\phi)) + 1]
\]

\[
P(Q_{1-}, Q_{3-}) = tr(U_{\Delta t}(\sqrt{F_-} \rho_0(t_1) \sqrt{F_-}) U_{\Delta t}^\dagger F_-)
= \frac{1}{4}[\lambda^2 \cos(4\tau) - 2\lambda \cos(2\theta) \cos^2(2\tau) - \lambda \sin(2\theta) \sin(4\tau) \sin(\phi) \sqrt{1 - \lambda^2} + 1]
\]

In the above expressions (12a)-(12c), for the parameters characterizing the initial state, we now put \( \theta = 1.07 \) rad and \( \phi = 3\pi/2 \) or \( \pi/2 \). Recall that for these specific choices, as mentioned towards the end of the preceding section, the QM violation of the 3-term WLGI given by (4) is maximum using the joint probabilities calculated for ideal measurements. Next, for the QM joint probabilities pertaining to unsharp measurements, the LHS of WLGI (4) which is now a function of \( \tau \) and \( \lambda \) can be numerically maximized with respect to \( \tau \) for any value of \( \lambda \in (0, 1) \). It is then found that as \( \lambda \) increases, the maximum QM value of the LHS of (4) increases from negative values to positive values, crossing zero for \( \lambda = 0.69 \) (approx). Thus, within the range \( \lambda \leq 0.69 \), the maximum QM value of the LHS of (4) remains non-positive, implying that within this bound of \( \lambda \) for unsharp mea-
measurements, the QM predictions always satisfy the 3-term WLGI (4). In other words, 0.69 is the critical value of $\lambda$ above which, as measurements become more precise, WLGI (4) can be violated by QM. We shall now compare this critical value of $\lambda$ with that for LGI.

IV. COMPARISON BETWEEN WLGI AND LGI WITH RESPECT TO UNSHARP MEASUREMENT

The general form of usual LGI involving $n$ pairs of two-time correlation functions can be expressed in the following way [12]

$$
-n \leq K_n \leq n - 2 \quad \text{for odd } n \geq 3
$$

$$
-(n - 2) \leq K_n \leq n - 2 \quad \text{for even } n \geq 4
$$

(13)

where $K_n = C_{21} + C_{32} + C_{43} + \ldots + C_{n(n-1)} - C_{n1}$, and the correlation function $C_{ij} = \langle Q_i Q_j \rangle$. Considering unsharp measurements, the correlation function for any initial state is obtained in the following form

$$
\langle Q_i Q_j \rangle_{\text{unsharp}} = P(Q_i+, Q_j+) + P(Q_i-, Q_j-) - P(Q_i-, Q_j+) - P(Q_i+, Q_j-)
$$

$$
= \text{tr}(U_{\Delta t}(\sqrt{F_+^\dagger \rho(t_i) \sqrt{F_+}})U_{\Delta t}^\dagger F_+^\dagger)
+ \text{tr}(U_{\Delta t}(\sqrt{F_-^\dagger \rho(t_i) \sqrt{F_-}})U_{\Delta t}^\dagger F_-^\dagger)
- \text{tr}(U_{\Delta t}(\sqrt{F_-^\dagger \rho(t_i) \sqrt{F_+}})U_{\Delta t}^\dagger F_+^\dagger)
- \text{tr}(U_{\Delta t}(\sqrt{F_+^\dagger \rho(t_i) \sqrt{F_-}})U_{\Delta t}^\dagger F_-^\dagger)
$$

(14)

$$
= \lambda^2 \cos(2\tau) = \lambda^2 \langle Q_i Q_j \rangle_{\text{sharp}}
$$

where $\langle Q_i Q_j \rangle_{\text{sharp}}$ is the correlation function for sharp measurements corresponding to $\lambda = 1$. Using Eq. (14) and the result that for any given $n$, the maximum QM value of $K_n$ for sharp measurements has been found to be $n \cos(\pi/n)$ [12], it follows that for unsharp measurements, if the QM predictions are to satisfy the general form of LGI given by (13), the following inequality needs to hold good

$$
\lambda^2 n \cos(\pi/n) \leq n - 2
$$

(15a)

for any $n$, which implies

$$
\lambda \leq \sqrt{\frac{n - 2}{n \cos(\pi/n)}}
$$

(15b)

Note that as $n$ increases, the RHS of (15b) also increases, thereby implying an increase of the critical value of $\lambda$ (denoted by, say, $\lambda_c$) above which, as measurements become more precise, the QM results can violate the general form of LGI given by (13). The minimum value of $\lambda_c (= \sqrt{2/3} = 0.81)$ occurs for $n = 3$. For $n = 4$, $\lambda_c$ is given by $(1/2)^{1/4} = 0.84$ which is the same as the corresponding $\lambda_c$ [22] obtained for the Bell-CHSH inequality. Here it may also be noted that for an arbitrary spin system, LGI involving unsharp measurement is being separately studied [23].

Now, comparing the above mentioned minimum value of $\lambda_c (= 0.81)$ for LGI with the corresponding critical value of $\lambda_c (= 0.69)$ for the 3-term WLGI given by (4), it is seen that for the values of $\lambda$ lying within the range given by $0.69 < \lambda \leq 0.81$, the QM predictions can violate WLGI, but will satisfy LGI. In other words, for the range of values of $\lambda \in (0.69, 0.81)$ corresponding to unsharp or imprecise measurements, the QM violation of macrorealism can be tested using the 3-term WLGI, but not in terms of LGI. This, therefore, underscores the efficacy of WLGI and its non-equivalence with LGI. Such a comparison can be extended for WLGIs involving more than three pairs of two-time joint probabilities. Intuitively, though, it can be argued that, as the number of pairs of measurement increases, the robustness of the QM violation of WLGIs against unsharp measurement is expected to decrease. In a future work, we propose to study a comprehensive way of constructing the WLGIs and investigating their general features, analogous to the way higher order LGIs can be constructed from the 3-term case by analysing the relationship between LGI and the geometry of the ‘cut polytope’ [24].

V. NO-SIGNALLING IN TIME AND UNSHARP MEASUREMENT

As already mentioned in the introductory section, an alternative necessary condition for the validity of macrorealism has recently been proposed [21] by assuming that the outcome statistics of a measurement would remain unaffected by any prior measurement. This condition, referred to as ‘No-Signalling in Time’ (NSIT), is the statistical version of NIM used in deriving LGI and can be viewed as an analogue of the no-signalling condition for the spacelike separated measurements used in the EPR-Bohm scenario, with the difference that while any violation of the latter would violate special relativity, violation of NSIT is not inconsistent with special relativity. Now, in order to express NSIT in a mathematical form, let us again consider a system oscillating in time between two possible states, as discussed earlier. The probability of obtaining a particular outcome, say, +1 for the measurement of a dichotomic observable $Q$ at an instant, say, $t_2$, without any earlier measurement being performed, be denoted by $P(Q_2 = +1)$. NSIT requires that $P(Q_2 = +1)$ should be the same even when an earlier measurement of, say, $Q$, is made at an instant, say, $t_1$. In other words, if we denote by $P(Q_2 = +1|Q_1 = \pm 1)$ the probability of obtaining an outcome +1 for the measurement of $Q$ at the instant $t_2$ when an earlier measurement of $Q$ has been performed at $t_1$ having an outcome $\pm 1$,
NSIT can be expressed as the equality condition given by $P(Q_2 = +1) = P(Q_2 = +1 | Q_1 = \pm 1)$ which implies that

$$P(Q_2 = +1) = P(Q_1 + , Q_2 +) + P(Q_1 -, Q_2 +)$$  \hspace{1cm} (16)

where the terms on the RHS of Eq. (16) are the relevant joint probabilities.

Now, pertaining to the two-state oscillation between the states $|A\rangle$ and $|B\rangle$ with the state $\rho_0(t_1) = |\psi_0\rangle \langle \psi_0|$ at the instant $t_1$ where $|\psi_0\rangle = \cos \theta |A\rangle + \exp(i \phi) \sin \theta |B\rangle$, the QM violation of the condition given by Eq. (16) for ideal measurements can be obtained as follows, based on calculations similar to that involving Eq. (8) as discussed earlier

$$P(Q_2 = +1) - [P(Q_1 + , Q_2 +) + P(Q_1 -, Q_2 +)]$$

$$= \text{tr}(U_{\Delta t} \rho_0(t_1) U_{\Delta t}^\dagger P_+) - \text{tr}(U_{\Delta t} (P_+ \rho_0(t_1) P_+) U_{\Delta t}^\dagger P_+)$$

$$- \text{tr}(U_{\Delta t} (P_- \rho_0(t_1) P_-) U_{\Delta t}^\dagger P_+)$$

$$= \frac{1}{2} \sin(2\tau) \sin(2\theta) \sin(\phi)$$  \hspace{1cm} (17)

It can be seen from Eq. (17) that, for sharp or ideal measurements, the maximum QM violation of the NSIT condition as given by Eq. (16) is 1/2 corresponding to the choices $\theta = \pi/4, \phi = \pi/2$ and $\tau = \Delta E \Delta t = \pi/4$ (in the units of $\hbar = 1$). Next, taking into account the unsharpness of measurements involved, the QM violation of the NSIT condition of the form (16) is obtained as follows on the basis of calculations similar to that leading to Eqs. (12a)-(12c)

$$P(Q_2 = +1) - [P(Q_1 + , Q_2 +) + P(Q_1 -, Q_2 +)]$$

$$= \text{tr}(U_{\Delta t} \rho_0(t_1) U_{\Delta t}^\dagger F_+) - \text{tr}(U_{\Delta t} (\sqrt{F_+} \rho_0(t_1) \sqrt{F_+}) U_{\Delta t}^\dagger F_+)$$

$$- \text{tr}(U_{\Delta t} (\sqrt{F_-} \rho_0(t_1) \sqrt{F_-}) U_{\Delta t}^\dagger F_+)$$

$$= \frac{1}{2} \lambda \sin(2\tau) \sin(2\theta) \sin(\phi)(1 - \sqrt{1 - \lambda^2})$$  \hspace{1cm} (18)

It is then seen from Eq. (18) that, while the magnitude of the QM violation of NSIT depends on the value of the unsharpness parameter $\lambda$ (this violation is maximum for $\lambda = 1$ corresponding to sharp measurement), a particularly noteworthy feature is that unless the state at the instant $t_1$ is such that either $\sin(2\theta)$ or $\sin(\phi)$ vanishes, the QM violation of NSIT persists for any non-zero value of $\lambda$, i.e., for any arbitrarily unsharp measurement. This shows remarkable robustness of the QM violation of NSIT with respect to unsharp or non-ideal measurements.

**VI. CONCLUDING DISCUSSION**

Three different necessary conditions for the validity of macrorealism are considered, including the two earlier proposed conditions namely LGI, NSIT, and the alternative condition WLGI proposed in this paper. Comparison between these three conditions in terms of the robustness of their respective QM violations against unsharpness of the measurements involved is the central theme of the present paper. Our investigation reveals that the emergence of classicality in terms of satisfying the macrorealist inequality WLGI corresponds to greater imprecision or unsharpness of measurements than that for LGI. Next, coming to NSIT, Interestingly, we find that its QM violation occurs whatever be the unsharpness of the relevant measurements. Thus, corresponding to this quantum feature, classicality does not emerge, irrespective of how unsharp the relevant measurements are. Implication of this curious finding calls for further reflection. Here we may also note that Kofler and Brukner [25] have made an incisive analysis of how and to what extent classicality emerges within quantum theory for any spin system under an appropriate coarse-graining of measurements. Taking into account such works it should, therefore, be interesting to make a comprehensive comparison of the results of our present paper with that using different characterizations of coarse-grained measurements that are involved while probing the emergence of classicality within quantum theory under imprecise measurements.

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