AUTOMORPHISMS AND DEFORMATIONS OF CONFORMALLY KÄHLER, EINSTEIN–MAXWELL METRICS

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Abstract. We obtain a structure theorem for the group of holomorphic automorphisms of a conformally Kähler, Einstein–Maxwell metric, extending the classical results of Matsushima [25], Lichnerowicz [20] and Calabi [6] in the Kähler–Einstein, cscK, and extremal Kähler cases. Combined with previous results of LeBrun [19], Apostolov–Maschler [4] and Futaki–Ono [12], this completes the classification of the conformally Kähler, Einstein–Maxwell metrics on \( \mathbb{CP}^1 \times \mathbb{CP}^1 \). We also use our result in order to introduce a (relative) Mabuchi energy in the more general context of \((K, q, a)\)-extremal Kähler metrics in a given Kähler class, and show that the existence of \((K, q, a)\)-extremal Kähler metrics is stable under small deformation of the Kähler class, the Killing vector field \(K\) and the normalization constant \(a\).

1. Introduction

Let \((M, J)\) be a compact Kähler manifold of complex dimension \(m\) and \(g\) a \(J\)-compatible Kähler metric. Following [4], the Hermitian metric \(\tilde{g} = \frac{1}{f^2}g\) is said to be conformally Kähler, Einstein–Maxwell if \(\tilde{g}\) has

(a) \(J\)-invariant Ricci tensor, i.e.

\[
\text{Ric}_{\tilde{g}}(\cdot, \cdot) = \text{Ric}_{\tilde{g}}(J\cdot, J\cdot),
\]

(b) constant scalar curvature, i.e.

\[
\text{Scal}_{\tilde{g}} = \text{const}.
\]

These conditions extend to higher dimensions a 4-dimensional riemannian signature analogue of the Einstein–Maxwell equations in General Relativity, see [18, 1].

In [4], Apostolov–Maschler initiated a study of conformally Kähler, Einstein–Maxwell Kähler metrics in a framework similar to the famous Calabi problem [6] of finding extremal Kähler metrics in a given Kähler class, and set the existence problem of the conformally Kähler, Einstein–Maxwell Kähler metrics in a formal GIT picture, extending the work of Donaldson and Fujiki [7, 8] characterizing the Calabi extremal metrics as critical points of the norm of the corresponding moment map. In particular, fixing a Kähler class \(\Omega\) on \((M, J)\), a quasi-periodic real holomorphic vector field \(K\) with zeroes, and a real positive constant \(a > 0\), it was shown in [4] that there is a natural obstruction to the existence of conformally Kähler, Einstein–Maxwell Kähler metrics associated to the above data, similar to the Futaki invariant [10, 11] in the Kähler–Einstein and the constant scalar curvature Kähler (cscK) cases. More recently, Futaki–Ono [12] have characterized the latter obstruction in terms of a volume-minimizing condition on \(K\), reminiscent to the constant scalar curvature Sasaki case [24, 14, 22].

The purpose of this paper is to extend two fundamental results in the theory of extremal Kähler metrics to a more general context relevant the conformally Kähler, Einstein–Maxwell metrics described above. The first result is a suitable extension of Calabi’s Theorem [6] on the structure of the group of holomorphic automorphisms of a compact extremal Kähler manifold. To state it, let \(g\) be a Kähler metric on \((M, J)\) endowed with a Killing vector field \(K\) with zeroes. Hodge theory implies (see e.g. [16]) that \(K\) is hamiltonian with respect to the Kähler form \(\omega = gJ\), i.e.

\[\iota_K \omega = -df\]

for a smooth function on \(M\), called a Killing potential...
of \( K \). We normalize \( f = f_{(K,\omega,a)} \) by requiring \( \int_M f_{(K,\omega,a)} dv_\gamma = a > 0 \), where the positive real constant \( a \) is such that \( f_{(K,\omega,a)} > 0 \) on \( M \). Then, for any fixed real number \( q \) we define the \((K,q,a)\)-scalar curvature of \( g \) to be

\[
S_{(K,q,a)}(g) := f_{(K,\omega,a)}^2 \text{Scal}_g + 2q f_{(K,\omega,a)} \Delta_g \left( f_{(K,\omega,a)} - q(q-1)|K|^2_g \right),
\]

where \( \text{Scal}_g \) denotes the usual scalar curvature, \(| \cdot |_g \) is the tensor norm induced by \( g \), and \( \Delta_g \) stands for the riemannian Laplacian on functions.

The point of this definition is that the condition (1) above yields that the conformal factor \( f \) is a positive Killing potential of a Killing vector field \( K \) for \( g \), whereas the scalar curvature \( \text{Scal}_\tilde{g} \) of \( \tilde{g} = \frac{1}{f^2} g \) is given by the formula (3) with \( q = -(2m-1) \). Other choices of the weight \( q \) lead to other interesting geometric problems, as it was observed in [2]. We also notice that the GIT framework of [4] makes sense for any choice of the weight \( q \) as above, see Section 2 below.

**Definition 1.** Let \( g \) be a Kähler metric on \((M,J)\) endowed with a Killing vector field \( K \) as above, and \( a > 0 \) a real constant such the corresponding Killing potential \( f_{(K,\omega,a)} > 0 \) on \( M \).

We say that \( g \) is \((K,q,a)\)-extremal if its \((K,q,a)\)-scalar curvature given by (3) is a Killing potential, i.e. \( \Xi = J \text{grad}_g(\text{Scal}_{(K,q,a)}(g)) \) is a Killing vector field for \( g \).

The definition above incorporates the case when \( S_{(K,q,a)}(g) \) is constant, which in turn links to the initial motivation of studying conformally Kähler, Einstein–Maxwell metrics. We denote by \( \mathfrak{k}^K \) (resp. \( \mathfrak{e}^K \)) the centralizer of \( K \) in the Lie algebra of holomorphic vector fields (resp. Killing vector fields) of \((M,J)\) (resp. \((M,g)\)) and \( \text{Aut}^K_0(M,J) \) (resp. \( \text{Isom}^K_0(M,g) \)) the corresponding closed connected Lie groups. We then have:

**Theorem 1.** Suppose \((M,g,J)\) is a compact \((K,q,a)\)-extremal Kähler manifold. Then the group \( \text{Isom}^K_0(M,g) \) is a maximal compact connected subgroup of \( \text{Aut}^K_0(M,J) \). Furthermore, if \( S_{(K,q,a)}(g) = \text{const} \), then \( \text{Aut}^K_0(M,J) \) is a reductive complex Lie group.

This basic result yields that each compact \((K,q,a)\)-extremal Kähler manifold \((M,J,g)\) is invariant under a maximal torus \( T \) in the connected component of the identity of the reduced automorphism group \( \text{Aut}_{\text{red}}(M,J) \), with \( K \in \text{Lie}(T) \), thus linking to the point of view of [4]. In particular, we can deduce from Theorem 1 and [4, Theorem 3] that if \((M,J)\) is toric, i.e. \( T \) is \( m \)-dimensional, then the \((K,q,a)\)-extremal Kähler metrics are unique up to isometries in their Kähler classes (see Corollary 4 below). Concerning the existence of conformally Kähler, Einstein–Maxwell metrics, Theorem 1 and [4, Theorem 5] yield together a complete classification of the latter on the toric complex surfaces \( \mathbb{CP}^1 \times \mathbb{CP}^1 \) and the Hirzebruch surfaces \( \mathbb{F}_n = \mathbb{P}(O \oplus O(n)) \rightarrow \mathbb{CP}^1 \) in terms of explicit constructions given by either the Calabi Ansatz [18, 19, 17] or by the hyperbolic ambitoric ansatz [1] (a riemannian analogue of the Plebanski-Damianski explicit solutions [26]). In practice, however, the algorithm of [4, Theorem 5] allowing one to decide whether or not for a given Kähler class, a quasi-periodic holomorphic vector field \( K \) and a constant \( a > 0 \) there exists a compatible conformally–Kähler, Einstein–Maxwell metric is of considerable complexity, see [12]. The case \( \mathbb{CP}^1 \times \mathbb{CP}^1 \) has been successfully resolved by [18, 4] (see also [12]):

**Corollary 1.** Any conformally-Kähler, Einstein–Maxwell metric on \( \mathbb{CP}^1 \times \mathbb{CP}^1 \), must be toric, and if it is not a product of Fubini-Study metrics on each factor, it must be homothetically isometric to one of the metrics constructed in [18].

We also notice that similarly to the Kähler–Einstein and cscK cases [25, 20], Theorem 1 places an obstruction in terms of \( \text{Aut}^K_0(M,J) \) for \((M,J)\) to admit a Kähler metric of constant \((K,q,a)\)-scalar curvature, in particular a conformally Kähler, Einstein–Maxwell metric.
Corollary 2. Let \((M, J) = \mathbb{F}(\mathcal{O} + \mathcal{O}(1)) \rightarrow \mathbb{F}_n\) where \(E = (\mathcal{O} + \mathcal{O}(n)) \rightarrow \mathbb{C}P^1\) and \(\mathbb{F}_n = \mathbb{F}(E)\) is the \(n\)-th Hirzebruch complex surface. Denote by \(K\) the generator of the \(S^1\)-action on \(M\), corresponding to diagonal multiplications on the \(\mathcal{O}_E(1)\)-factor. Then \((M, J)\) admits no Kähler metric of constant \((bK, q, a)\)-scalar curvature for any values of \(b\) and \(q\).

We now describe our second result, which is a suitable modification of the stability of the existence of extremal Kähler metrics under deformation of the Kähler class, proved by LeBrun–Simanca in [21], see also [9]. In our extended context, and without loss of generality by using Theorem 1 above, we fix a maximal real torus \(\mathbb{T} \subset \text{Aut}_{\text{red}}(M, J)\), a real weight \(q\), and study the existence of a \(\mathbb{T}\)-invariant \((K, q, a)\)-extremal Kähler metric as a function of the Kähler class \(\Omega \in H^2_{\text{dr}}(M, \mathbb{R})\), the vector field \(K \in \text{Lie}(\mathbb{T})\), and the real constant \(a > 0\). We prove the following:

**Theorem 2.** Suppose (without loss of generality by Theorem 1) that \(\mathbb{T} \subset \text{Aut}_{\text{red}}(M, J)\) is a maximal real torus and \((M, J)\) admits a \(\mathbb{T}\)-invariant \((K, q, a)\)-extremal Kähler metric \((g, \omega)\). Then, for any \(g\)-harmonic, \(\mathbb{T}\)-invariant, \((1, 1)\)-form \(\alpha\), and any \(H \in \text{Lie}(\mathbb{T})\), there exist \(\varepsilon > 0\), such that for any real numbers \(|s| < \varepsilon\), \(|t| < \varepsilon\) and \(|u| < \varepsilon\), there exists a \((K + uH, q, a + s)\)-extremal Kähler metric in the Kähler class \([\omega + t\alpha]\).

This result provides an efficient way to obtain many new examples from known ones.

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2. A family of variational problems in Kähler geometry

In this section we recall the Apostolov–Maschler [4] moment map interpretation of the \((K, q, a)\)-scalar curvature. In [4], the case \(q = -2m + 1\) is considered, but their argument works for any weight \(q\) (see [2]).

Let \((M, J, \omega)\) be a compact Kähler manifold of real dimension \(2m \geq 4\). We denote by \(\mathfrak{h}_{\text{red}}\) the Lie algebra of the reduced automorphism group \(\text{Aut}_{\text{red}}(M, J)\), given by the real holomorphic vector field with zeros (see [16]). Let \(K \in \mathfrak{h}_{\text{red}}\) be a quasi-periodic Killing vector field generating a torus \(G \subset \text{Aut}_{\text{red}}(M, J)\). It is well known that \(G\) acts in an isometric hamiltonian way on \((M, \omega)\). Let \(f_{(K, \omega, a)} \in C^\infty(M, \mathbb{R})\) be the normalized positive Killing potential of \(K\), defined by the condition \(\int_M f_{(K, \omega, a)} \omega = a\).

We denote by \(\mathcal{K}^G(M, \omega)\) the space of all \(\omega\)-compatible, \(G\)-invariant Kähler structures on \((M, \omega)\), and consider the natural action of the infinite dimensional group \(\text{Ham}^G(M, \omega)\), of \(G\)-equivariant Hamiltonian transformations of \((M, \omega)\). We have the identification

\[
\text{Lie} \left( \text{Ham}^G(M, \omega) \right) \cong C^\infty_0(M, \mathbb{R})^G
\]

where \(C^\infty_0(M, \mathbb{R})^G\) denote the space of smooth \(G\)-invariant functions with zero mean value with respect to \(f_{(K, \omega, a)}^{-2}\), \(v_\omega = \frac{\omega^m}{m!}\) being the Riemannian volume form) endowed with the Poisson bracket.

For any \(q \in \mathbb{R}\), the space \(\mathcal{K}^G(M, \omega)\) carries a \(q\)-weighted formal Kähler structure \((J, \Omega^{(K, q, a)})\) given by ([7, 8, 4])

\[
\Omega_j^{(K, q, a)}(J_1, J_2) = \frac{1}{2} \int_M \text{Tr}(J J_1 J_2) f_{(K, \omega, a)}^q v_\omega,
\]

\[
J_j(J) = J J_1.
\]
where the tangent space of $K^G(M,\omega)$ at $J$ is identified with the space of smooth $G$-invariant sections $J$ of $\text{End}(TM)$ satisfying
$$\tilde{J}J + J\tilde{J} = 0, \quad \omega(\tilde{J},.) + \omega(.,\tilde{J}) = 0.$$ 
In what follows we denote by $g_J := \omega(.,J)$ the Kähler metric corresponding to $J \in K^G(M,\omega)$, and index all objects calculated with respect to $J$ similarly. On $C^0_\infty(M,\mathbb{R})^G$, we consider the scalar product given by,
$$\langle \phi, \psi \rangle_{(K,q,a)} = \int_M \phi_j f_J^{-2} v_\omega.$$ 

**Theorem 3.** [4] The action of $\text{Ham}^G(M,\omega)$ on $(K^G(M,\omega), J, \Omega^{(K,q,a)})$ is Hamiltonian with a momentum map given by the $(.,.)_{(K,q,a)}$-dual of the $(K,q,a)$-scalar curvature given by (3).

**Remark 1.**

(i) The weight $q = -2m + 1$ corresponds to the conformally Kähler, Einstein–Maxwell case studied in [4], and $S(J,q,a)$ computes the scalar curvature of the hermitian metric $\tilde{g}_J := f_{(K,\omega,a)}^{-2} g_J$. 
(ii) If $q = 0$, $S(J,q,a)(J)$ computes the so-called conformal scalar curvature $\tilde{\kappa}_J$ of the hermitian metric $\tilde{g}_J$ given by (see e.g. [15]),
$$\tilde{\kappa}_J = (2m - 1) \left\langle \tilde{\Omega}, \tilde{\Phi}_{\tilde{J}} \right\rangle_{\tilde{g}_J},$$

where $\tilde{\Phi}_{\tilde{J}} = \tilde{g}_J(J,.)$ is its fundamental 2-form of $(\tilde{g}_J, J)$ and $\tilde{\Omega}$ is the corresponding Weil tensor.
(iii) The weight $q = m - 1$ appears in the study of Levi-Kähler quotients (see e.g. [2]). 
(iv) For a real number $p$, one can define,
$$S(J,p,q,a)(J) := f_{(K,\omega,a)}^{p-2} S(J,q,a)(g_J). \quad (4)$$

Then the $\langle . \rangle_{(K,q,a-p+2,1)}$-dual of (4) is a momentum map for the action of $\text{Ham}^G(M,\omega)$ on $(K^G(M,\omega), J, \Omega^{(K,q,a)})$. Taking $(p,q) = (2, q)$ we obtain Theorem 3, wheres the value $(p,q) = (\frac{2}{m} - 1)$ corresponds to the Lejmi–Upmieer moment map given by the hermitian scalar curvature of $\tilde{g}_J$ (see [23]).

3. THE EXTENDED CALABI PROBLEM

3.1. The $(K,q,a)$-constant scalar curvature Kähler metrics. Following [4] we now fix the complex manifold $(M,J)$ and vary the Kähler form $\omega$ within a fixed Kähler class $\Omega \in H^2(M,\mathbb{R}) \cap H^{1,1}(M,\mathbb{C})$. We also fix the compact torus $G \subset \text{Aut}_{\text{red}}(M,J)$ generated by a quasi periodic vector field $K \in \text{Lie}(G)$, and denote by $K^G_{\Omega}(M,J)$ the space of $G$-invariant Kähler forms $\omega \in \Omega$. Let $f_{(K,\omega,a)}$ be the normalized Killing potential of $K$ with respect to $\omega$, with normalization constant $a > 0$, such that $f_{(K,\omega,a)} > 0$. As shown in [4, Lemma 1] we have $f_{(K,\omega,a)} > 0$ on $M$ for all $\omega' \in K^G_{\Omega}(M,J)$.

The space $K^G_{\Omega}(M,J)$ is a Frechet manifold given near $\omega \in K^G_{\Omega}(M,J)$ by the open subset of elements $\phi \in C^\infty(M,\mathbb{R})^G/\mathbb{R}$ such that $\omega + dd^*\phi > 0$ is positive definite. The tangent space of $K^G_{\Omega}(M,J)$ at $\omega$ is identified with $C^0_{\infty}(M,\mathbb{R})^G$, the space of $G$-invariant smooth functions with mean value 0 with respect to $f_{(K,\omega,a)}^{q-2} v_\omega$.

We then consider the following generalized Calabi problem on $K^G_{\Omega}(M,J)$ (see [4]):

**Problem.** For a weight $q \in \mathbb{R}$, a quasi-periodic vector field $K$ generating a torus $G$ in $\text{Aut}_{\text{red}}(M,J)$, $\omega \in K^G_{\Omega}(M,J)$ and $a > 0$ such that $f_{(K,\omega,a)} > 0$, does there exist $\phi \in C^0_{\infty}(M,\mathbb{R})^G$ such that $\omega + dd^*\phi$ is $(K,q,a)$-extremal?
In what follows we calculate the first variation of the \((K_q,a)\)-scalar curvature along \(\omega \in K_G(M,J)\). We denote by \(D\) the Levi-Civita connection and by \(\delta = D^*\) the co-differential of \((M,\omega,g)\). For a 1-form \(\alpha\) on \(M\), let \(D^\pm\alpha\) be the \(J\)-invariant (resp. \(J\)-anti-invariant) part of \(D\alpha\), i.e.
\[
(D^\pm\alpha)_{X,Y} = \frac{1}{2} \left( (D\alpha)_{JX,JY} \pm (D\alpha)_{X,Y} \right).
\]
There is a natural action on \(p\)-forms \(\psi\) induced by \(J\) as follows,
\[
(J\psi)(X_1,\cdots,X_p) = (-1)^p \psi(JX_1,\cdots,JX_p).
\]
The twisted differential and the twisted codifferential on \(p\)-forms are defined by,
\[
d^c = JdJ^{-1},
\]
\[
\delta^c = J\delta J^{-1}.
\]
To simplify notation we omit below the index \((K,\omega,a)\) of \(f(K,\omega,a)\).

**Lemma 1.** For any \(G\)-invariant 1-form \(\alpha\) we have
\[
2f^{2-q} \delta \delta \left( f^q D^{-\alpha} \right) = 2f^2 \delta \delta (D^{-\alpha}) - 2q f (\Delta \alpha, df)
\]
\[
+ 2q f (\Delta df, \alpha) + 2q f (\delta \alpha, df)
\]
\[
- q(q-1)(\alpha, d(df, df)) + q(q-1)(df, d(df, df)),
\]
where \((\cdot,\cdot)\) stand for the inner product of tensors induced by \(g\).

**Proof.** Indeed,
\[
2f^{2-q} \delta \delta \left( f^q D^{-\alpha} \right) = 2f^2 \delta \delta (D^{-\alpha}) + 2q f (\delta D^{-\alpha})(JK)
\]
\[
+ 2q f \delta ((D^{-\alpha})(JK, \cdot)) - 2q(q-1)(D^{-\alpha})(K,K).
\]
We consider the decomposition of the tensor \(D^{-\alpha}\) in symmetric and skew-symmetric parts \(\Psi\) and \(\Phi\), respectively,
\[
D^{-\alpha} = \Psi + \Phi.
\]
For any vector field \(X\) on \(M\) we have
\[
\delta (\Psi(X,\cdot)) = - (\Psi, DX^\flat) + (\delta \Psi)(X),
\]
\[
\delta (\Phi(X,\cdot)) = (\Phi, DX^\flat) - (\delta \Phi)(X).
\]
Using (6) for \(X = JK\) we get
\[
\delta (\Psi(JK,\cdot)) = (\delta \Psi)(JK),
\]
\[
\delta (\Phi(JK,\cdot)) = -(\delta \Phi)(JK).
\]
Thus,
\[
2f^{2-q} \delta \delta \left( f^q D^{-\alpha} \right) = 2f^2 \delta \delta (D^{-\alpha}) + 4q f (\delta \Psi)(JK) - 2q(q-1)(D^{-\alpha})(K,K).
\]
Using [16, Lemma 1.23.4] and \(2\Phi = da - Jda\) we have
\[
(\delta \Psi)(JK) = -(\delta D^{-\alpha}, df) + (\delta \Phi)(df^2)
\]
\[
= -\frac{1}{2} (\Delta \alpha, df) + \text{Ric}(\text{grad}_g f, \alpha^\sharp) + \frac{1}{2} (\delta \alpha, df) - \frac{1}{2} (\delta Jda, df)
\]
\[
= -\frac{1}{2} (\Delta \alpha, df) + (\Delta df, \alpha) - (\delta D^+ df, \alpha) + \frac{1}{2} (\delta \alpha, df)
\]
\[
= -\frac{1}{2} (\Delta \alpha, df) + \frac{1}{2} (\Delta df, \alpha) + \frac{1}{2} (\delta \alpha, df)
\]
where we have used the identity \((\delta Jd\alpha, df) = -(\delta^c d\alpha)(K) = L_K \delta^c \alpha = 0\) which holds since \(K\) is Killing. Furthermore,
\[
2(D^- \alpha)(K, K) = (D_K \alpha)(K) - (D_{JK} \alpha)(JK)
\]
\[
= -(df, d(\alpha, df)) + (d^c f, d(\alpha, d^c f)) - 2\alpha(D_K K)
\]
\[
= -(df, d(\alpha, df)) + (d^c f, d(\alpha, d^c f)) + (\alpha, d(df, df))
\]
\[
= -(df, d(\alpha, df)) + (\alpha, d(df, df)),
\]
since \((d^c f, d(\alpha, d^c f)) = L_K (\alpha, d^c f) = 0\) by the \(G\)-invariance of \(\alpha\). The result follows by substituting (8) and (9) in (7). This completes the proof. \(\square\)

**Definition 2.** We define the \((K, q, a)\)-Lichnerowicz operator
\[
L^g_{(K,q,a)} : C^\infty(M, \mathbb{R})^G \to C^\infty(M, \mathbb{R})^G,
\]
with respect to a metric \(g\) in \(K^G_{\Omega}(M, J)\) by
\[
L^g_{(K,q,a)}(\phi) = f^2_{(K,\omega,a)} \delta \left( f^2_{(K,\omega,a)} D^- (d\phi) \right),
\]
for \(\phi \in C^\infty(M, \mathbb{R})^G\).

**Proposition 1.** For any variation \(\dot{\omega} = dd^c \dot{\phi}\) of \(\omega\) in \(K^G_{\Omega}(M, J)\), the first variation of the \((K, q, a)\)-scalar curvature is given by
\[
\delta S_{(K,q,a)}(\dot{\phi}) = -2L^g_{(K,q,a)}(\dot{\phi}) + \left( dS_{(K,q,a)}(\omega), d\dot{\phi} \right).
\]

**Proof.** For a variation \(\dot{\omega} = dd^c \dot{\phi}\) in \(K^G_{\Omega}(M, J)\), the corresponding variations of \(f_{(K,\omega,a)}, \Delta, \text{Scal}, \text{Scal}_{\omega}\) are given by (see e.g. [16, 5]):
\[
\begin{align*}
\dot{v}_\omega &= - (\Delta \omega \dot{\phi}) v_\omega \\
\dot{f} &= (df, d\dot{\phi}) \\
\dot{\Delta} &= (dd^c \dot{\phi}, dd^c \dot{\phi}) \\
\dot{\text{Scal}} &= -2L^g(\dot{\phi}) + (d\text{Scal}(\omega), d\dot{\phi}),
\end{align*}
\]
where \(L^g(\dot{\phi}) = \delta \delta (D^- d\dot{\phi})\) is the usual Lichnerowicz operator. Then the first variation of the \((K, q, a)\)-scalar curvature is given by:
\[
\delta S_{(K,q,a)}(\dot{\phi}) = -2f^2L^g(\dot{\phi}) + f^2(d\text{Scal}(\omega), d\dot{\phi}) + \text{Scal}(\omega)(df^2, d\dot{\phi}) + 2qf \Delta d(\omega, df, d\dot{\phi})
\]
\[
+ 2q(df, d\dot{\phi}) \Delta d \omega f + 2q f (dd^c \dot{\phi}, df) - q(q - 1)(df, d(df, d\dot{\phi})).
\]
By (6) and the \(G\)-invariance of \(\phi\) we have
\[
(dd^c \dot{\phi}, dd^c f) = -\Delta(df, d\dot{\phi}) + (d\Delta \dot{\phi}, df).
\]
Thus,
\[
\delta S_{(K,q,a)}(\dot{\phi}) = -2f^2L^g(\dot{\phi}) + f^2(d\text{Scal}(\omega), d\dot{\phi})
\]
\[
+ \text{Scal}(\omega)(df^2, d\dot{\phi}) + 2qf (df, d\Delta \dot{\phi})
\]
\[
+ 2q(df, d\dot{\phi}) \Delta \omega f - q(q - 1)(df, d(df, d\dot{\phi})).
\]

On the other hand we have
\[
(dS_{(K,q,a)}(\omega), d\dot{\phi}) = f^2(d\text{Scal}(\omega), d\dot{\phi}) + (df^2, d\dot{\phi}) \text{Scal}(\omega)
\]
\[
+ 2q(\Delta f)(df, d\dot{\phi}) + 2q f (d\Delta f, d\dot{\phi}) - q(q - 1)(d\dot{\phi}, d(df, d\dot{\phi})).
\]
By taking the difference (12)-(13) we get exactly (5) for \(\alpha = d\dot{\phi}\), which, in turn, is equal to
\[-2L^g_{(K,q,a)}(\dot{\phi}).\]
Let $h^K_{\text{red}}$ denote the centralizer of $K$ in $h_{\text{red}}$ (i.e. the space of vector fields $H \in h_{\text{red}}$ such that $[H, K] = 0$) and $\text{Aut}^K_{\text{red}}(M, J)$ the closed connected Lie subgroup of $\text{Aut}_{\text{red}}(M, J)$ with Lie algebra $h^K_{\text{red}}$.

Let $\omega \in K^G_\Omega(M, J)$. To each element $\phi \in C_0^\infty(M, \mathbb{R})^G$ we associate a vector field $\hat{\phi}$ on $K^G_\Omega(M, J)$, equal to $\phi$ at any point of $K^G_\Omega(M, J)$. We then have $[\hat{\phi}, \hat{\psi}] = 0$ for any $\phi, \psi \in C_0^\infty(M, \mathbb{R})^G$. We will consider the natural action of $\text{Aut}^K_{\text{red}}(M, J)$ on $K^G_\Omega(M, J)$ defined by:

$$\gamma \cdot \omega = \gamma^* \omega.$$ 

Consider the 1-form $\sigma$ on $K^G_\Omega(M, J)$ given by:

$$\sigma_\omega(\hat{\phi}) = \int_M S_{(K, q, a)}(\omega) \phi f^{q-2} f^{\psi} v_\omega$$

where $\omega \in K^G_\Omega(M, J)$ and $\phi \in C_0^\infty(M, \mathbb{R})^G$.

**Proposition 2.** The 1-form $\sigma$ is $\text{Aut}^K_{\text{red}}(M, J)$—invariant and we have the following expression for its first variation,

$$\delta \left( \sigma(\hat{\phi}) \right)_\omega(\hat{\psi}) = -2 \int_M (D^- d\phi, D^- d\psi) f^{q-2} f^{\psi} v_\omega$$

$$- \int_M S_{(K, q, a)}(\omega)(d\psi, d\phi) f^{q-2} f^{\psi} v_\omega.$$ 

In particular $\sigma$ is closed.

**Proof.** Since $\text{Aut}^K_{\text{red}}(M, J)$ preserves the complex structure $J$ and $K$, the invariance of $\sigma$ under the action of $\text{Aut}^K_{\text{red}}(M, J)$ follows. Now we will calculate the first variation of the functional $\omega \mapsto \sigma_\omega(\hat{\phi})$. By (11) we have for each $\psi \in C_0^\infty(M, \mathbb{R})^G$,

$$\delta \left( \sigma(\hat{\phi}) \right)_\omega(\hat{\psi}) = \int_M \delta S_{(K, q, a)}(\omega) \phi f^{q-2} v_\omega + \int_M S_{(K, q, a)}(\omega) \phi (df^{q-2}, d\psi) v_\omega$$

$$- \int_M S_{(K, q, a)}(\omega) \phi f^{q-2} (\Delta_\omega \psi) v_\omega$$

$$= \int_M \delta S_{(K, q, a)}(\omega) \phi f^{q-2} v_\omega - \int_M (dS_{(K, q, a)}(\omega), d\psi) \phi f^{q-2} v_\omega$$

$$- \int_M S_{(K, q, a)}(\omega)(d\psi, d\phi) f^{q-2} v_\omega.$$ 

From (10) and the above formula we readily get (14). Thus,

$$(d\sigma)_\omega(\hat{\phi}, \hat{\psi}) = \delta \left( \sigma(\hat{\psi}) \right)_\omega(\hat{\phi}) - \delta \left( \sigma(\hat{\phi}) \right)_\omega(\hat{\psi}) - \sigma_\omega([\hat{\phi}, \hat{\psi}]) = 0.$$ 

i.e. $\sigma$ is a closed 1–form. \qed

**Remark 2.** One can alternatively elaborate along the lines of [4]. For $\omega \in K^G_\Omega(M, J)$ and $J \in K^G(M, \omega)$ fixed, we consider the path of Kähler metrics $\omega_t = \omega + dt^c \phi_t$ with $\phi_t \in C_0^\infty(M, \mathbb{R})^G$, $\phi_0 = 0$ and $\phi_t = \phi$. Using the equivariant Moser Lemma (see [4, Lemma 1]) there exists a family of $G$-equivariant diffeomorphisms $\Phi_t \in \text{Diff}^G_0(M)$ such that $\Phi_0 = \text{id}_M$ and $\Phi_t \cdot \omega = \omega_t$. Then we have a path $J_t = \Phi_t \cdot J$ in $K^G(M, \omega)$. Note that if $g_t = \omega_t(\cdot, J_t)$
then \( f^*_{(K,\omega,a)} = f_*(K,\omega,a) \circ \Phi_t \). We have,
\[
\delta \left( \sigma(\dot{\psi}) \right)_{\omega} (\dot{\phi}) = \frac{d}{dt} \left|_{t=0} \int_M S(K,q,a)(\omega_t) \psi f_{(K,\omega,a)}^{q-2} \nu_\omega \right.
\]
\[
= \frac{d}{dt} \left|_{t=0} \int_M \Phi_t^* \left( S(K,q,a)(J_t) \left( \psi \circ \Phi_t^{-1} \right) f_{(K,\omega,a)}^{q-2} \nu_\omega \right) \right.
\]
\[
= \frac{d}{dt} \left|_{t=0} \int_M S(K,q,a)(J_t) \left( \psi \circ \Phi_t^{-1} \right) f_{(K,\omega,a)}^{q-2} \nu_\omega \right.
\]
\[
= \int_M \left( \dot{J}, DJ \dot{\psi} \right) f_{(K,\omega,a)}^q \nu_\omega + \int_M S(K,q,a)(\omega) d\psi(Z) f_{(K,\omega,a)}^{q-2} \nu_\omega,
\]
where we used [4, Eq(9)] and that \( Z, \dot{J} \) are given by:
\[
Z = \frac{d}{dt} \left|_{t=0} \Phi_t^{-1} \circ \Phi = -\text{grad}_g \phi, \right.
\]
\[
\dot{J} = \frac{d}{dt} \left|_{t=0} \Phi_t.J = -\mathcal{L}_Z J. \right.
\]
It follows that,
\[
d\psi(Z) = -(d\psi, d\phi),
\]
\[
\left( \dot{J}, DJ \dot{\psi} \right) = -2 \left( D^- d\phi, D^- d\psi \right).
\]
We thus get,
\[
\delta \left( \sigma(\dot{\psi}) \right)_{\omega} (\dot{\phi}) = -2 \int_M \left( D^- d\phi, D^- d\psi \right) f_{(K,\omega,a)}^q \nu_\omega - \int_M S(K,q,a)(\omega)(d\psi, d\phi) f_{(K,\omega,a)}^{q-2} \nu_\omega.
\]
As shown in [4], and as it easily follows from (14), the following expression:
\[
c(\Omega,K,q,a) := \frac{\int_M S(K,q,a)(\omega) f_{(K,\omega,a)}^{q-2} \nu_\omega}{\int_M f_{(K,\omega,a)}^q \nu_\omega}
\]
is a topological constant (i.e. independent of the choice of \( \omega \) in the Kähler class \( \Omega \)). We consider the following 1-form, on \( \mathcal{K}^G_{\Omega}(M, J) \) given by:
\[
\tilde{\sigma}_\omega(\dot{\phi}) := \int_M \left( S(K,q,a)(\omega) - c(\Omega,K,q,a) \right) \phi f_{(K,\omega,a)}^{q-2} \nu_\omega.
\]

**Lemma 2.** The 1-form \( \tilde{\sigma} \) is closed.

**Proof.** We consider the 1-form \( \theta \) defined on \( \mathcal{K}^G_{\Omega}(M, J) \) by
\[
\theta_\omega(\dot{\phi}) = \int_M \phi f_{(K,\omega,a)}^{q-2} \nu_\omega.
\]
We have using (11),
\[
\delta(\theta_\omega(\dot{\phi}))(\dot{\psi}) = \int_M \phi(df^{q-2}, d\psi) \nu_\omega - \int_M \phi f^{q-2}(\Delta_{\omega} \psi) \nu_\omega
\]
\[
= - \int_M (d\psi, d\phi) f^{q-2} \nu_\omega.
\]
Thus,
\[
(d\theta)_\omega (\dot{\phi}, \dot{\psi}) = \delta \left( \theta(\dot{\psi}) \right)_\omega (\dot{\phi}) - \delta \left( \dot{\theta}(\dot{\phi}) \right)_\omega (\dot{\psi}) - \theta_\omega([\dot{\phi}, \dot{\psi}]) = 0,
\]
i.e. \( \theta \) is closed. By Proposition 2, \( \tilde{\sigma} = \sigma - c(\Omega,K,q,a) \cdot \theta \) is closed.

Since \( \mathcal{K}^G_{\Omega}(M, J) \) is contractible, \( \tilde{\sigma} \) is an exact form, so it admits a primitive functional.
**Definition 3.** We define the \((\Omega, K, q, a)\)-Mabuchi energy 
\[
\mathcal{M}_{(\Omega, K, q, a)} : \mathcal{K}_G^G(M, J) \to \mathbb{R}
\]
as minus the primitive of the one form \(\tilde{\sigma}\), i.e.
\[
\tilde{\sigma} = -d\mathcal{M}_{(\Omega, K, q, a)},
\]
normalized by \(\mathcal{M}_{(\Omega, K, q, a)}(\omega_0) = 0\) for some base point \(\omega_0 \in \mathcal{K}_G^G(M, J)\).

**Remark 3.** By its very definition, the Kähler metrics in \(\mathcal{K}_G^G(M, J)\) of constant \((\Omega, K, q, a)\)-scalar curvature are critical points of the \((\Omega, K, q, a)\)-Mabuchi functional.

3.2. The \((\Omega, K, q, a)\)-Futaki invariant. For \(\omega \in \mathcal{K}_G^G(M, J)\) and \(H \in \mathfrak{h}_{\text{red}}^K\), we denote by \(h'(H, \omega) + \sqrt{-1}L_h'(H, \omega) \in C_0^\infty(M, \mathbb{C})\) the normalized holomorphy potential of \(H\), i.e. \(h'(H, \omega)\) and \(h(H, \omega)\) are the normalized smooth functions such that,
\[
H = \text{grad}_v(h'(H, \omega)) + J\text{grad}_g(h(H, \omega)).
\]

Using the identification \(T_\omega \mathcal{K}_G^G(M, J) \cong C_0^\infty(M, \mathbb{R})^G\), the vector field \(JH\) defines a vector field \(\widetilde{JH}\) on \(\mathcal{K}_G^G(M, J)\), given by:
\[
\omega \mapsto L_{\widetilde{JH}}\omega = dd^c h(H, \omega),
\]
so that \(\widetilde{JH}_\omega = h(H, \omega)\). By the invariance of \(\tilde{\sigma}\) under the \(\text{Aut}_{\text{red}}^K(M, J)\)-action and Cartan’s formula we get,
\[
\mathcal{L}_{\widetilde{JH}} \tilde{\sigma} = d\left(\tilde{\sigma}_{\widetilde{JH}}\right) = 0.
\]

Then \(\omega \mapsto \tilde{\sigma}(\widetilde{JH})\) is constant on \(\mathcal{K}_G^G(M, J)\). We will use the following notation,
\[
\mathcal{F}_{(\Omega, K, q, a)}(H) := \tilde{\sigma}(\widetilde{JH}) = \int_M \left( S(K, q, a)(\omega) - c(\Omega, K, q, a) \right) h(H, \omega) f_{(K,\omega,a)}^{q-2} v_\omega.
\]

**Definition 4.** The linear map \(\mathcal{F}_{(\Omega, K, q, a)} : \mathfrak{h}_{\text{red}}^K \to \mathbb{R}\) will be called the \((\Omega, K, q, a)\)-Futaki invariant associated to the data \((\Omega, K, q, a)\).

**Remark 4.**

(i) Definition 4 is consistent with the one given in [4] for \(q = -2m + 1\), but it has the advantage to show that the \((\Omega, K, q, a)\)-Futaki invariant extends to the whole of \(\mathfrak{h}_{\text{red}}^K\), not just \(\text{Lie}(G)\).

(ii) For a Kähler class \(\Omega\) which admits \((K, q, a)\)-extremal metric \(\omega\), \(\mathcal{F}_{(\Omega, K, q, a)} = 0\) if and only if \(\omega\) is a \((K, q, a)\)-constant scalar curvature Kähler metric. In fact, for \(\Xi = J\text{grad}_g(S_{(K, q, a)}(\omega)) \in \mathfrak{h}_{\text{red}}^K\) we have
\[
\mathcal{F}_{(\Omega, K, q, a)}(\Xi) = \int_M \left( S_{(K, q, a)}(\omega) - c(\Omega, K, q, a) \right)^2 f_{(K,\omega,a)}^{q-2} v_\omega = 0,
\]
thus \(S_{(K, q, a)}(\omega) = c(\Omega, K, q, a)\).

4. **Proof of Theorem 1**

In this section we shall prove Theorem 1 from the introduction. We thus assume that \((g, \omega)\) is an \((K, q, a)\)-extremal metric on a compact, connected Kähler manifold \((M, J)\), and \(G \subset \text{Aut}_{\text{red}}(M, J)\) the torus generated by the quasi-periodic Killing vector field \(K\).

**Lemma 3.** For any \(G\)-invariant function \(\phi \in C^\infty(M, \mathbb{R})^G\) we have
\[
\mathcal{L}_\Xi \phi = -2 f_{(K,\omega,a)}^{2-q} \delta \left( f_{(K,\omega,a)}^q D^{-}(d^c \phi) \right),
\]
where \(\mathcal{L}_\Xi\) denotes the Lie derivative along the vector field \(\Xi = J\text{grad}_g(S_{(K, q, a)}(\omega))\).
Proof. We have,
\[
\mathcal{L}_\omega \phi = -(dS_{(K,q,a)}(\omega), \delta^e \phi) \\
= -f^2(d\text{Scal}(\omega), \delta^e \phi) - 2qf(d\Delta f, \delta^e \phi) + q(q-1)(\delta^e \phi, d(df, df)).
\]
By taking \(\alpha = \delta^e \phi\) in (5) we get,
\[
2f^2 - q(D^- d)^* f^q(D^- d^*) \phi = 2f^2 (D^- d)^* (D^- d^*) \phi + 2qf (d\Delta f, \delta^e \phi) - q(q-1)(\delta^e \phi, d(df, df)) = f^2 (dS_{\omega}, \delta^e \phi) + 2qf (d\Delta f, \delta^e \phi) - q(q-1)(\delta^e \phi, d(df, df)),
\]
where we have used (see [16, p.63, Eq.1.23.15]),
\[
(D^- d)^* (D^- d^*) \phi = (dS_{\omega}, \delta^e \phi),
\]
and the identity,
\[
(\delta dd^e \phi, df) = -(\delta^e d\delta^e \phi)(K) = -\mathcal{L}_K \delta^e \delta^e \phi = 0.
\]
\(\square\)

For a 1-forme \(\alpha\) we denote by \(D^{2,0} \alpha\) (resp. \(D^{0,2} \alpha\)) the \((2,0)\)-part (resp. \((0,2)\)-part) of the tensor \(D \alpha\). We define the \((K, q, a)\)-Calabi’s operators \(\mathbb{L}^{g,\pm}_{(K,q,a)}\) on \(C^\infty(M, \mathbb{C})^G\) by
\[
\mathbb{L}^{g, +}_{(K,q,a)}(F) = 2f^2 (D^{2,0} \alpha)^* f^q(D^{0,2} \alpha) D^{0,2} \alpha dF \\
\mathbb{L}^{g, -}_{(K,q,a)}(F) = 2f^2 (D^{0,2} \alpha)^* f^q(D^{2,0} \alpha) D^{2,0} \alpha dF.
\]
Recall that the space of hamiltonian Killing vector fields is given by (see [16])
\[
\mathfrak{k}_{\text{ham}} = \mathfrak{h}_{\text{red}} \cap \mathfrak{k}.
\]
The following Proposition is straifhtforword (see [16, Chapter 2]).

**Proposition 3.**

(i) Let \(H = \text{grad}_P P + J\text{grad}_Q Q\), where \(P, Q\) are real valued functions with zero mean, such that \([H, K] = 0\). Then \(H \in \mathfrak{h}_{\text{red}}\) if and only if \(\mathbb{L}^{g, +}_{(K,q,a)}(P + \sqrt{-1}Q) = 0\) and \(P, Q \in C^\infty(M, \mathbb{R})^G\) i.e. we have
\[
\mathfrak{h}_{\text{red}}^{K} \cong \ker(\mathbb{L}^{g, +}_{(K,q,a)}) \cap C^\infty_0(M, \mathbb{C})^G.
\]

(ii) For any \(F \in C^\infty(M, \mathbb{C})^G\) we have,
\[
\mathbb{L}^{g, \pm}_{(K,q,a)}(F) = \mathbb{L}^{g}_{(K,q,a)}(F) \pm \frac{\sqrt{-1}}{2} \mathcal{L}_\omega F.
\]

(iii) Let \(X\) be real holomorphic vector field. Then \(X \in \mathfrak{k}_{\text{ham}}^K\) if and only if there exists \(h \in C^\infty(M, \mathbb{R})^G\) such that \(X = J\text{grad}_h h\) and \(\mathbb{L}^{g}_{(K,q,a)}(h) = 0\).

**Theorem 4.** Suppose \((M, g, J)\) is a compact \((K, q, a)\)-extremal Kähler manifold. Then \(\mathfrak{k}^K\) admits the following \((\cdot, \cdot)_{(g,q,a)}\)-orthogonal decomposition
\[
\mathfrak{k}^K = \mathfrak{k}^{K(0)}_{(0)} \oplus \bigoplus_{\lambda > 0} \mathfrak{k}^K_{(\lambda)},
\]
where \(\mathfrak{k}^{K(0)}_{(0)}\) is the centralizer of \(\Xi\) in \(\mathfrak{k}^K\) and for \(\lambda > 0\), \(\mathfrak{k}^K_{(\lambda)}\) denote the subspace of elements \(X \in \mathfrak{k}^K\) such that \(\mathcal{L}_\omega X = \lambda JX\).

The subspace \(\mathfrak{k}^{K(0)}_{(0)}\) is a reductive complex Lie subalgebra of \(\mathfrak{k}^K\); it contains \(\mathfrak{k}_{\text{ham}}^K\) and \(J\mathfrak{k}_{\text{ham}}^K\) and is given by the \((\cdot, \cdot)_{(g,q,a)}\)-orthogonal sum of these three spaces:
\[
\mathfrak{k}^{K(0)}_{(0)} = \mathfrak{a} \oplus \mathfrak{k}_{\text{ham}}^K \oplus J\mathfrak{k}_{\text{ham}}^K.
\]
Moreover, each $h^K_{\lambda}, \lambda > 0$, is contained in the ideal $h^K_{\text{red}}$, so that we also have the following $(\cdot, \cdot)_{(g,q,a)}$-orthogonal decompositions:

$$h^K_{\lambda} = a \oplus h^K_{\text{red}_{\lambda}},$$

$$t^K_{\lambda} = a \oplus t^K_{\text{ham}_{\lambda}}.$$  

Proof. Let $X = X_H + \text{grad}_g P + J \text{grad}_g Q \in h^K$, where $X_H$ is the dual of the harmonic part of $\xi = X^0$ denoted $\xi_H$, and $P, Q \in C^\infty(M, \mathbb{R})$ with zero mean value. Since $K = J \text{grad}_g f$ is Killing we have

$$\mathcal{L}_K P = \mathcal{L}_K Q = 0.$$  

By (5) in Lemma 1 we have

$$2f^{2-q}(D^2) + f^q D^2 - \xi_H = 2f^2(D^2) + D^2 - \xi_H$$

$$= f^2(d\text{Scal}(\omega), \xi_H) + 2q f(d\Delta f, \xi_H)$$

$$- q(q-1)(\xi_H, df, df)$$

$$= J\mathcal{L}_\xi \xi_H = 0,$$

where we have used $(\xi_H, df) = 0$ and the fact that $\Xi$ is a Killing vector field. It follows that

$$0 = f^{2-q}(D^2) + f^q D^2 - \xi = f^{2-q}(D^2) + f^q D^2 - (dP + d^* Q) = \text{Re}\left(\mathcal{L}_{(K,q,a)}(P + \sqrt{-1}Q)\right).$$

Starting from $JX$ instead of $X$ we similarly get

$$\text{Im}\left(\mathcal{L}_{(K,q,a)}^*(P + \sqrt{-1}Q)\right) = 0.$$  

It follows that $\mathcal{L}_{(K,q,a)}^+(P + \sqrt{-1}Q) = 0$, then by Proposition 3(i) we have that $X_H$ and $\text{grad}_g P + J \text{grad}_g Q$ are real holomorphic vector fields, which proves (17) (for the decomposition of $t^K_{\lambda}$ we use the fact that $t_{\text{ham}} := t \cap h_{\text{red}}$ and $t \cap a = a$).

Since $\Xi$ is Killing and commutes with $K$, the operators $L_{(K,q,a)}^\pm$ commute. Then $L_{(K,q,a)}^-$ acts on $h^K_{\text{red}}$ and by Proposition 3(ii) this action is given by $-\sqrt{-1}\mathcal{L}_\Xi$. Since $L_{(K,q,a)}^+$ is $(\cdot, \cdot)_{(g,q,a)}$-self-adjoint and semi-positive, $h^K_{\text{red}}$ splits as

$$h^K_{\text{red}} = h^K_{\text{red},(0)} \oplus \left(\bigoplus_{\lambda > 0} h^K_{(\lambda)}\right),$$

where $h^K_{\text{red},(0)}$ is the kernel of $\mathcal{L}_\Xi$ in $h^K_{\text{red}}$ whereas, for each $\lambda > 0$, $h^K_{(\lambda)}$ is the subspace of elements $X \in h^K$ such that $\mathcal{L}_\Xi X = \lambda JX$. Using (17) we get (15) (Notice that $h^K_{(\lambda)} = h^K_{\text{red},(\lambda)}$ since $\Xi$ is Killing and commutes with $K$).

We have $a \oplus t^K_{\text{ham}} \oplus Jh^K_{\text{ham}} \subset h^K_{(0)}$. By Proposition 3(ii) the restriction of $\mathcal{L}_\Xi$ to ker $(L_{(K,q,a)}^+) \cap C^\infty_0(M, C)^G$ coincides with the restriction of $L_{(K,q,a)}^-$ to the same space. Then, using Proposition 3(iii), we obtain the converse inclusion, which proves (16).

Now we are in position to give a proof for Theorem 1.

Proof of Theorem 1. This is done as in the case where $(G = \{1\}, q = 0, a = 0)$ (see [16, 6]). Let $\mathfrak{s}$ be the Lie algebra of a connected, compact Lie subgroup, $S \subset \text{Aut}_K^\circ(M, J)$ containing $\text{Isom}_K^\circ(M, g)$. Suppose, for a contradiction, that there exists $X \in \mathfrak{s}$ that doesn’t belong to $t^K$. By Theorem 4, (see (15), (17) and (16)) we have the splitting

$$h^K = t^K \oplus Jt^K_{\text{ham}} \oplus \left(\bigoplus_{\lambda > 0} h^K_{(\lambda)}\right),$$
then we can assume that $X \in J\mathfrak{h}_{\operatorname{ham}}^K \oplus \left( \bigoplus_{\lambda > 0} \mathfrak{h}^K_{(\lambda)} \right)$. Let $X = X_0 + \sum_{\lambda > 0} X_\lambda$ be the corresponding decomposition of $X$, then for any positive integer $r$ we have

\[(L\Xi)^{2r} X = - \sum_{\lambda > 0} \lambda^{2r} X_\lambda \in \mathfrak{s}.
\]

It follows that each component $X_\lambda$ of $X$ is in $\mathfrak{s}$. We can therefore assume that $X \in \mathfrak{s}_\lambda := \mathfrak{s} \cap \mathfrak{h}^K_{(\lambda)}$ or $X \in J\mathfrak{h}_{\operatorname{ham}}^K \subset \mathfrak{s}_0$. Suppose that $X \in \mathfrak{s}_\lambda$ for some $\lambda > 0$. Let $B$ denote the Killing form of $\mathfrak{s}$.

We have the following exact sequence (see [3, Proposition 1.3]):

\[0 \to h_B(M) \to \mathfrak{h}(M) \to \mathfrak{h}(B) \to 0\]

where $\mathfrak{h}_B(M)$ denote the Lie algebra of holomorphic vector fields on $M$ which are tangent to the fibers of $\pi$. The proof of [3, Proposition 1.3] also shows that,

\[0 \to h^K_B(M) \to h^K(M) \to \mathfrak{h}(B) \to 0\]

where $h^K_B(M) = \operatorname{span}_{\mathbb{C}} \{ K, JK \}$ is the abelian sub-algebra generated by the vector fields $K$, $JK$. If $M$ admits a Kähler metric of constant $(bK, q, a)$-scalar curvature, then $h^K(M)$ must be reductive by Theorem 1. As $h^K_B(M)$ is in the center of $h^K(M)$, it would follow that $h(B)$ is reductive, which is not the case for $B = \mathbb{F}_n$ (see e.g. [5]). It follows that $M$ admits no Kähler metric of constant $(bK, q, a)$-scalar curvature.

Proof of Corollary 1. This follows from Corollary 3 and [4, Proposition 6] \hfill \square

Proof of Corollary 2. We have the following exact sequence (see [3, Proposition 1.3]):

\[0 \to h_B(M) \to \mathfrak{h}(M) \to \mathfrak{h}(B) \to 0\]

where $B = \mathbb{F}_n$ and $h_B(M)$ denote the Lie algebra of holomorphic vector fields on $M$ which are tangent to the fibers of $\pi$. The proof of [3, Proposition 1.3] also shows that,

\[0 \to h^K_B(M) \to h^K(M) \to \mathfrak{h}(B) \to 0\]

where $h^K_B(M) = \operatorname{span}_{\mathbb{C}} \{ K, JK \}$ is the abelian sub-algebra generated by the vector fields $K$, $JK$. If $M$ admits a Kähler metric of constant $(bK, q, a)$-scalar curvature, then $h^K(M)$ must be reductive by Theorem 1. As $h^K_B(M)$ is in the center of $h^K(M)$, it would follow that $h(B)$ is reductive, which is not the case for $B = \mathbb{F}_n$ (see e.g. [5]). It follows that $M$ admits no Kähler metric of constant $(bK, q, a)$-scalar curvature. \hfill \square
5. The \((K,q,a)\)-extremal Kähler metrics relatively to a maximal torus \(T\), and 
\((\mathbb{T}, K,q,a)\)-extremal vector field

Using Corollaries 3 and 4, we assume from now on that \(T\) is a fixed maximal torus in \(\text{Aut}_{\text{red}}(M,J)\) and \(K \in \text{Lie}(\mathbb{T})\). We denote by \(\Pi_g^{\mathbb{T}}\) the orthogonal projection with respect to the \(L^2\)-scalar product
\[
(\phi, \psi)_{(g,q,a)} := \int_M \phi \psi \gamma^{q-2}_{(K,\omega,a)} v_\omega
\]
defined on the Hilbert space \(L^2_T(M, \mathbb{R})\) onto the space \(P^g_T(M, \mathbb{R})\) of Killing potentials of the elements of \(\text{Lie}(\mathbb{T})\) relatively to \(g\) which is isomorphic to \(\mathbb{R} \oplus \text{Lie}(\mathbb{T})\). Then we have the following decomposition of the \((K,q,a)\)-scalar curvature,
\[
S_{(K,q,a)}(\omega) = S^T_{(K,q,a)}(\omega) + \Pi_g^{\mathbb{T}}(S_{(K,q,a)}(\omega)),
\]

**Definition 5.** We call \(S^T_{(K,q,a)}(\omega)\) the reduced \((K,q,a)\)-scalar curvature with respect to \(T\).

We say that \(\omega \in \mathcal{K}^T_{\Omega}(M,J)\) is \((K,q,a)\)-extremal relatively \(T\) if \(S^T_{(K,q,a)}(\omega)\) is identically zero.

**Remark 5.** Notice that by Corollary 3, any \((K,q,a)\)-extremal metric is extremal relatively to the maximal torus of \(\text{Aut}_{\text{red}}(M,J)\) containing \(K\).

Following [16, Proposition 4.11.1] we have,

**Definition 6.** For \(X,Y \in \mathfrak{h}_{\text{red}}\) with normalized complex potentials \(F^X_\omega, F^Y_\omega\) we define the \((\Omega, K,q,a)\)-Futaki-Mabuchi bilinear by the following expression
\[
\mathcal{B}_{(\Omega,K,q,a)}(X,Y) := \int_M F^X_\omega F^Y_\omega \gamma^{q-2}_{(K,\omega,a)} v_\omega,
\]
which is independent from the choice of \(\omega \in \mathcal{K}^T_{\Omega}(M,J)\).

We denote by \(Z^T_{\omega}(K,q,a)\) the vector field given by
\[
Z^T_{\omega}(K,q,a) := J\nabla_g \left( \Pi_g^{\mathbb{T}}(S_{(K,q,a)}(\omega)) \right).
\]
Then for all \(H \in \text{Lie}(\mathbb{T})\), we have
\[
\mathcal{F}_{(\Omega,K,q,a)}(H) = -\mathcal{B}_{(\Omega,K,q,a)} \left( H, Z^T_{\omega}(K,q,a) \right).
\]
From its very definition, the restriction of \(\mathcal{B}_{(\Omega,K,q,a)}\) to \(\text{Lie}(\mathbb{T})\) is negative definite. Then \(Z^T_{\omega}(K,q,a)\) is well-defined by the above expression, so it is an element of \(\text{Lie}(\mathbb{T})\), independent of the choice of \(\omega \in \mathcal{K}^T_{\Omega}(M,J)\).

**Definition 7.** We call \(Z^T(\Omega,K,q,a) \in \text{Lie}(\mathbb{T})\) the \((\Omega,K,q,a)\)-extremal vector field.

Now we consider the 1-form \(\zeta^T\) defined on \(\mathcal{K}^T_{\Omega}(M,J)\) by,
\[
\zeta^T(\phi) = \int_M \Pi_g^{\mathbb{T}}(S_{(K,q,a)}(\omega)) \phi \gamma^{q-2}_{(K,\omega,a)} v_\omega.
\]

**Lemma 4.** The 1-form \(\zeta^T\) is closed.

**Proof.** To simplify notations we denote \(z(\omega) := \Pi_g^{\mathbb{T}}(S_{(K,q,a)}(\omega))\). Then \(Z^T(\Omega,K,q,a) = J\nabla_g z(\omega)\). For a variation \(\dot{\omega} = d\mathbb{F} \phi\) in \(\mathcal{K}^T_{\Omega}(M,J)\), using (11) we have
\[
\dot{z}(\phi) = (d\phi, dz(\omega))_\omega
\]
and therefore,
\[
\delta \left( \zeta^T(\psi) \right) \omega (\hat{\phi}, \hat{\psi}) = \int_M (d\phi, dz(\omega)) \omega f^{q-2} v_\omega + \int_M z(\omega) \psi (d\phi, df^{q-2}) v_\omega \\
- \int_M z(\omega) \psi \Delta_\omega \phi f^{q-2} v_\omega \\
= - \int_M z(\omega) (d\phi, d\psi) f^{q-2} v_\omega.
\]
It follows that,
\[
\left( d\xi^T \right) \omega (\hat{\phi}, \hat{\psi}) = \delta \left( \zeta^T(\psi) \right) \omega (\hat{\phi}) - \delta \left( \zeta^T(\phi) \right) \omega (\hat{\psi}) = 0.
\]

Now we consider the 1-form \(\sigma^T\) on \(K^T(M, J)\) given by
\[
\sigma^T := \sigma - \zeta^T
\]
which is a closed 1-form by virtue of Proposition 2 and Lemma 4.

**Definition 8.** The relative Mabuchi energy \(\mathcal{M}^T_{(\Omega, K, q, a)}\) is defined by
\[
\sigma^T = -d\mathcal{M}^T_{(\Omega, K, q, a)},
\]
where the primitive \(\mathcal{M}^T_{(\Omega, K, q, a)}\) is normalized by requiring \(\mathcal{M}^T_{(\Omega, K, q, a)}(\omega_0) = 0\) for some base point \(\omega_0 \in K^T_0(M, J)\).

**Remark 6.** By its very definition, the critical points of the relative Mabuchi energy are the \(T^\ast\)-invariant \((K, q, a)\)-extremal metrics.

### 6. Proof of Theorem 2

Let \((M, J)\) be a compact Kähler manifold. We fix \(K \in h_{\text{red}}\), and \(q \in \mathbb{R}\). Suppose that \((g, \omega)\) is a \((K, q, a)\)-extremal Kähler metric on \(M\) with \(\Omega = [\omega]\). Without loss of generality, by Corollary 3, we can assume that \((g, \omega)\) is invariant under the action of a maximal torus \(T \subseteq \text{Aut}_{\text{red}}(M, J)\). Let \(\alpha\) be \(T^\ast\)-invariant \(g\)-harmonic \((1, 1)\)-form. We take \((\omega, \alpha) = 0\) to avoid trivial deformations of the form \(\alpha = \lambda \omega\). We denote by
\[
\omega_{t, \phi} := \omega + t \alpha + dd^c \phi,
\]
a \(T^\ast\)-invariant deformations of \(\omega\) for \(t \in \mathbb{R}\) and \(\phi \in C^\infty(M, \mathbb{R})^T\). We consider the following map,
\[
S : \mathbb{R}^3 \times C^\infty(M, \mathbb{R})^T \to C^\infty(M, \mathbb{R})^T
\]
defined by,
\[
S(s, t, u, \phi) := S_{(K + uH, q, a + s)}(\omega_{t, \phi}),
\]
so that \(S(0) = S_{(K, q, a)}(\omega) := S\). We denote \(f_{(s, t, u, \phi)} = f_{(K + uH, \omega_{t, \phi}, a + s)} > 0\) the hamiltonian function of \(K + uH\) with respect to \(\omega_{t, \phi}\), with normalization constant \(a + s\), so that \(f_0 = f_{(K, \omega, a)} := f\). We take \(k > n\) such that the Sobolev space \(L^2_k(M, \mathbb{R})^T\) form an algebra for the usual multiplication of functions, embadded in \(C^4(M, \mathbb{R})^T\). Then \(S\) defines a map
\[
S : \mathbb{R}^3 \times L^2_{k+4}(M, \mathbb{R})^T \to L^2_k(M, \mathbb{R})^T,
\]
and we have:
Lemma 5. The map $S$ is $C^1$ with Fréchet derivative in 0 given by

$$D_0 S = (A \ B \ C \ D)$$

with,

$$A = 2f \text{Scal}(\omega) + 2q\Delta(f),$$
$$B = -2\left(\rho_{(K,q,a)}(\omega), \alpha\right) + 2\lambda f S + 2q\Delta(f) + 2qf\Delta\lambda - 2q(q-1)(d\lambda, df),$$
$$C = 2f_{(H,\omega)} f \text{Scal}(\omega) - 2q(q-1)(K, H) + 2q \left[f_{(H,\omega)} \Delta(f) + f \Delta(f_{(H,\omega)})\right],$$
$$D(\phi) = -2L^g_{(K,q,a)}(\phi) + (dS, d\phi),$$

where $(\cdot, \cdot)$, $\text{grad}$, $\Delta$, the green operator $G$ are calculated with respect to $\omega$, and $\lambda : = -G(\alpha, dd^c f)$.

Proof. The expressions of $A$, $C$ and $D$ are straightforward. For the partial derivative with respect to $t$ we have $S_{(K,q,a)}(\omega) = 2\Lambda_{\omega} \left(\rho_{(K,q,a)}(\omega)\right)$ where (see [2])

$$\rho_{(K,q,a)}(\omega) = f^2 \rho(\omega) - qf dd^c f - \frac{1}{2} q(q-1) df \wedge d^c f,$$

with $\rho(\omega)$ is the Ricci form of $(g, \omega)$. By taking $X = -JK$ in [16, Lemma 5.2.4] we get,

$$\frac{\partial}{\partial t} \bigg|_{t=0} f_{s,t,u,\phi} = -G(\delta(\alpha(K),)) = -G(\alpha, dd^c f_{(K,\omega,\alpha)}) = \lambda,$$

and we have $\frac{\partial}{\partial t} \bigg|_{t=0} \rho(\omega_{t,\phi}) = 0$ since we assumed $(\omega, \alpha) = 0$. Thus,

$$B = 2 \left(\frac{\partial}{\partial t} \bigg|_{t=0} \Lambda_{\omega_{t,\phi}}\right) \rho_{(K,q,a)}(\omega) + 2\Lambda_{\omega} \left(\frac{\partial}{\partial t} \bigg|_{t=0} \rho_{(K,q,a+s)}(\omega_{t,\phi})\right)$$
$$= -2 \left(\rho_{(K,q,a)}(\omega), \alpha\right) + 2\lambda f S + 2q\Delta f + 2qf\Delta\lambda - 2q(q-1)(d\lambda, df).$$

We consider the following maps,

$$\mathcal{F}(s, t, u) := \mathcal{F}_{(\omega + tu), K + uH, q, a + s},$$
$$\mathcal{B}(s, t, u) := \mathcal{B}_{(\omega + tu), K + uH, q, a + s},$$
$$Z(s, t, u) := Z^\omega(\omega + tx), K + uH, q, a + s).$$

Lemma 6. The $t$-derivative of the character $\mathcal{F}(s, t, u)$ and the bilinear for $\mathcal{B}(s, t, u)$ in the point $(s, t, u) = 0$ is given by

$$\left(\frac{\partial}{\partial t} \bigg|_{t=0} \mathcal{F}(s, t, u)\right)(X) = \left\langle h_{(X,\omega)}, B_{(g,q,a)} \right\rangle + \left\langle h_{(X,\omega)}, (q - 2)\lambda f S \right\rangle_{(g,q,a)}$$
$$- \left\langle h_{(X,\omega)}, f^{2-q} \left(\alpha, dd^c G \left(f^{q-2}S\right)\right) \right\rangle_{(g,q,a)},$$

$$\left(\frac{\partial}{\partial t} \bigg|_{t=0} \mathcal{B}(s, t, u)\right)(X, Y) = \left\langle \alpha, f^{2-q} h_{(X,\omega)} dd^c G \left(h_{(Y,\omega)} f^{q-2}\right) \right\rangle_{(g,q,a)}$$
$$+ \left\langle \alpha, f^{2-q} G \left(h_{(X,\omega)} f^{q-2}\right) dd^c h_{(Y,\omega)} \right\rangle_{(g,q,a)}$$
$$+ (q - 2) \left\langle \alpha, f^{2-q} G \left(h_{(X,\omega)} h_{(Y,\omega)} f^{q-3}\right) dd^c f \right\rangle_{(g,q,a)},$$

for any $X = J \text{grad}_y(h_{(X,\omega)})$ and $Y = J \text{grad}_y(h_{(Y,\omega)})$ in $\text{Lie}(\mathbb{T})$ with $h_{X,\omega}$, $h_{Y,\omega}$ are the normalized real potential of $-JX$, $-JY$ respectively.
Proof. For the derivative of $\mathcal{F}(s, t, u)$ we have,

$$\mathcal{F}(s, t, u)(X) = \int_M S(s, t, u)h(X,\omega + \alpha)f_{(s,t,u)}^{q-2}\nu_{\omega + \alpha},$$

then,

$$\frac{\partial}{\partial t} \bigg|_0 \mathcal{F}(s, t, u)(X) = \int_M Bh(X,\omega)f_{q-2}\nu_{\omega} - (q - 2) \int_M S\lambda f_{q-3}h(X,\omega)\nu_{\omega}$$

$$- \int_M Sf_{q-2}\mathcal{G}(\delta(\alpha(X,\cdot)))\nu_{\omega}.$$

On the other hand,

$$\int_M Sf_{q-2}\mathcal{G}(\delta(\alpha(X,\cdot)))\nu_{\omega} = \int_M (\alpha(X,\cdot), d\mathcal{G}(f_{q-2}\mathcal{S}))\nu_{\omega}$$

$$= \int_M (\alpha, X^h \wedge d\mathcal{G}(f_{q-2}\mathcal{S}))\nu_{\omega}$$

$$= \int_M h_{\cdot}(\alpha, d\mathcal{G}(f_{q-2}\mathcal{S}))\nu_{\omega}$$

$$= \langle h(X,\omega), f_{2-q}^{2-q}(\alpha, d\mathcal{G}(f_{q-2}\mathcal{S})) \rangle_{(g,q,\alpha)}.$$

which gives the expression (19) for the $t$-derivative of $\mathcal{F}(s, t, u)$.

Now we calculate the $t$-derivative of $\mathcal{B}(s, t, u)$. We have

$$\mathcal{B}(s, t, u)(X, Y) = - \int_M h(X,\omega + \alpha)f_{(s,t,u)}^{q-2}\nu_{\omega + \alpha}.$$ 

Then

$$\frac{\partial}{\partial t} \bigg|_0 \mathcal{B}(s, t, u)(X, Y) = \int_M \mathcal{G}(\delta(\alpha(X,\cdot))) h_Y f_{q-2}\nu_{\omega}$$

$$+ \int_M \mathcal{G}(\delta(\alpha(Y,\cdot))) h_X f_{q-2}\nu_{\omega}$$

$$+ (q - 2) \int_M \mathcal{G}(\delta(\alpha(K,\cdot))) h_X h_Y f_{q-3}\nu_{\omega}.$$ 

On the other hand,

$$\int_M \mathcal{G}(\delta(\alpha(X,\cdot))) h_Y f_{q-2}\nu_{\omega} = \int_M (\alpha(X,\cdot), d\mathcal{G}(h_Y f_{q-2}))\nu_{\omega}$$

$$= \int_M (\alpha, d^X h_Y \wedge d\mathcal{G}(h_Y f_{q-2}))\nu_{\omega}$$

$$= \langle \alpha, f_{2-q}^2 h_Y d\mathcal{G}(h_Y f_{q-2}) \rangle_{(g,q,\alpha)},$$

$$\int_M \mathcal{G}(\delta(\alpha(Y,\cdot))) h_X f_{q-2}\nu_{\omega} = \int_M (\alpha(Y,\cdot), d\mathcal{G}(h_X f_{q-2}))\nu_{\omega}$$

$$= \int_M (\alpha, d^Y h_X \wedge d\mathcal{G}(h_X f_{q-2}))\nu_{\omega}$$

$$= \int_M (\alpha, \mathcal{G}(h_X f_{q-2}) d\mathcal{G}(h_Y))\nu_{\omega}$$

$$= \langle \alpha, f_{2-q}^2 \mathcal{G}(h_X f_{q-2}) d\mathcal{G}(h_Y) \rangle_{(g,q,\alpha)}.$$
Lemma 7.

(i) The s-derivative of $F(s, t, u)$ is given by

\[ \frac{\partial}{\partial s} F(s, t, u)(X) = \langle qf^{-1}S_{q-1}, h(X, \omega) \rangle_{(g, q, a)} \]

where $S_{q-1} := S(K, q-1, a)(\omega)$.

(ii) The u-derivative of $F(s, t, u)$

\[ \frac{\partial}{\partial u} F(s, t, u)(X) = \langle C + (q - 2)f^{-1}f(H, \omega)S, h(X, \omega) \rangle_{(g, q, a)} \]

(iii) The s-derivative of $B(s, t, u)$ is given by

\[ \frac{\partial}{\partial s} B(s, t, u) = (q - 2)B((\Omega, K, q-1), a) \]

(iv) The u-derivative of $B(s, t, u)$ is given by

\[ \frac{\partial}{\partial u} B(s, t, u)(X, Y) = (q - 2) \int_M h(X, \omega)h(Y, \omega)f(H, \omega)f^{-1}v_{\omega} \]

for any $X = J\text{grad}_g(h(X, \omega))$ and $Y = J\text{grad}_g(h(Y, \omega))$ in Lie($T$).

Lemma 8. Let $\omega$ be a $(K, q, a)$-extremal metric, we have

(i) The t-derivative of $Z(s, t, u)$ is given by

\[ \frac{\partial}{\partial t} Z(s, t, u) = J\text{grad}_g \left( \Pi^T_g [B + G(\alpha, ddS)] \right) \]

(ii) The s-derivative of $Z(s, t, u)$ is given by

\[ \frac{\partial}{\partial s} Z(s, t, u) = J\text{grad}_g \left( \Pi^T_g [f^{-1} (qS_{q-1} + S)] \right) \]

(iii)

\[ \frac{\partial}{\partial u} Z(s, t, u) = J\text{grad}_g \left( \Pi^T_g [C + 2(q - 2)f^{-1}f(H, \omega)S] \right) \]

Proof.

(i) We have $\frac{\partial}{\partial t} Z(s, t, u) = J\text{grad}_g(P_g)$ for some function $P_g \in P^T(M, \mathbb{R})$, since $Z(s, t, u) \in \text{Lie}(T)$ for all $(s, t, u)$. By (18), for all $X \in \text{Lie}(T)$ we have,

\[ \mathcal{B}(s, t, u)(Z(s, t, u), X) = -F(s, t, u)(X) \]

then

\[ \mathcal{B}(0) \left( \left. \frac{\partial}{\partial t} \right|_0 Z(s, t, u), X \right) = -\langle P_g, h(X, \omega) \rangle_{(g, q, a)} \]

\[ = -\left. \frac{\partial}{\partial t} \right|_0 F(s, t, u)(X) - \left( \left. \frac{\partial}{\partial t} \right|_0 \mathcal{B}(s, t, u) \right) (Z(0), X) \]
Lemma 9. For a $(K, q, a)$-extremal metric $\omega$ we have,

\begin{align}
\frac{\partial}{\partial t} \Pi^{T}_{(s,t,u,\phi)} S(s, t, u, \phi) &= \Pi^{T}_{g} B + (\Pi^{T}_{g} - I) (G(\alpha, d\bar{f} S)). \\
\frac{\partial}{\partial s} \Pi^{T}_{(s,t,u,\phi)} S(s, t, u, \phi) &= \Pi^{T}_{g} [f^{-1}(qS_q-1 + S)] \\
\frac{\partial}{\partial u_0} \Pi^{T}_{(s,t,u,\phi)} S(s, t, u, \phi) &= \Pi^{T}_{g} [C + 2(q - 2)f^{-1}f_{(H, \omega)}S].
\end{align}

Proof. We have $Z(s, t, u) = J\text{grad}_{g} (\Pi^{T}_{(s,t,u,\phi)} S(s, t, u, \phi))$ then

\[ J\text{grad}_{g} \left( \frac{\partial}{\partial t} \Pi^{T}_{(s,t,u,\phi)} S(s, t, u, \phi) \right) = \frac{\partial}{\partial t} Z(s, t, u) - J \left( \frac{\partial}{\partial t} \text{grad}_{g,\phi} \right) (\Pi^{T}_{g} S). \]

On the other hand we have,

\[ \left( \frac{\partial}{\partial t} \right) \text{grad}_{g,\phi} (\Pi^{T}_{g} S) = (\alpha(Z(0), .))^2 = \text{grad}_{g} (G(\alpha, d\bar{f} S)). \]

By (25) it follows that

\[ \frac{\partial}{\partial t_0} \Pi^{T}_{(s,t,u,\phi)} S(s, t, u, \phi) = \Pi^{T}_{g} B + (\Pi^{T}_{g} - I) (G(\alpha, d\bar{f} S)) + c. \]

By differentiating in $t = 0$ the equality,

\[ \int_{M} (\Pi^{T}_{g} S(\omega_t)) f_{(K, \omega_t, a)}^{q-2} = \int_{M} S(\omega_t) f_{(K, \omega_t, a)}^{q-2} \]

we get $c = 0$, which proofs (29). Similarly we can show (30) and (31).

Following LeBrun-Simanca’s arguments [21] we give a proof of Theorem 2.
Proof. Let $L^2_k(M, \mathbb{R})^{T, \perp}$ be the orthogonal complement of $P^\perp_g(M, \mathbb{R})$ with respect to $\langle \cdot, \cdot \rangle_{(g,q,a)}$ in $L^2_k(M, \mathbb{R})^T$. For $t \in (-\epsilon, \epsilon)$ and $\phi \in U$ where $U$ is a small neighborhood of the origin in $L^2_k(M, \mathbb{R})^T$. As in [21] by taking a smaller open set $U$ and smaller $\epsilon$ we may assume that,

$$\ker \left( \Id - \Pi_g \right) \circ \left( \Id - \Pi^T_{(s,t,u,\phi)} \right) = \ker \left( \Id - \Pi^T_{(s,t,u,\phi)} \right).$$

Now we consider the Leban-Simanca map

$$\Psi : (-\epsilon, \epsilon)^3 \times U \to (-\epsilon, \epsilon)^3 \times L^2_k(M, \mathbb{R})^{T, \perp}$$

defined by

$$\Psi(s, t, u, \phi) := \left( s, t, \left( \Id - \Pi^T_g \right) \circ \left( \Id - \Pi^T_{(s,t,u,\phi)} \right) S(s, t, u, \phi) \right).$$

Note that $\Psi(0) = 0$ and if $\Psi(s, t, u, \phi) = (s, t, u, 0)$ then $\omega_{t,\phi}$ is $(K + uH, q, a + s)$-extremal.

The map $\Psi$ is $C^1$ and its Fréchet derivative at the origin is given by:

$$D_0 \Psi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \Id - \Pi^T_g \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \Id - \Pi^T_g \end{pmatrix}$$

where $A$, $B$, and $C$ are given in Lemma 5. Indeed, by Lemma 9 we have,

$$\frac{\partial}{\partial \phi} \bigg|_0 \left( \Id - \Pi^T_{(s,t,u,\phi)} \right) S(s, t, u, \phi) = D(\phi) - \frac{\partial}{\partial \phi} \bigg|_0 \Pi^T_{(s,t,u,\phi)} S(s, t, u, \phi) = D(\phi) - (dS, d\phi) = -2f^{2,q}(D-d)^* f^q(D-d)\phi.$$
the solution is of regularity at least $C^4$. We conclude using a similar bootstrapping argument as in the case of extremal metrics [21, Proposition 4].

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