A KAC MODEL WITH EXCLUSION

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ABSTRACT. We consider a one dimensional Kac model with conservation of energy and an exclusion rule: Fix a number of particles $n$, and an energy $E > 0$. Let each of the particles have an energy $x_j \geq 0$, with $\sum_{j=1}^n x_j = E$. For some $\epsilon$, the allowed configurations $(x_1, \ldots, x_n)$ are those that satisfy $|x_i - x_j| \geq \epsilon$ for all $i \neq j$. At each step of the process, a pair $(i,j)$ of particles is selected uniformly at random, and then they “collide”, and there is a repartition of their total energy $x_i + x_j$ between them producing new energies $x^*_i$ and $x^*_j$ with $x^*_i + x^*_j = x_i + x_j$, but with the restriction that exclusion rule is still observed for the new pair of energies. This process bears some resemblance to Kac models for Fermions in which the exclusion represents the effects of the Pauli exclusion principle. However, the “non-quantized” exclusion rule here, with only a lower bound on the gaps, introduces interesting novel features, and a detailed notion of Kac’s chaos is required to derive an evolution equation for the evolution of rescaled empirical measures for the process, as we show here.

1. INTRODUCTION

The first attempts to formulate kinetic equations for colliding particles that satisfy Boson or Fermion statistics go back at least to the works of Nordheim [9] and Uehling and Uhlenbeck [15]. To make a rigorous derivation of these equations starting from the Schrödinger equation for a large system of particles has proven very difficult. See [2] for a review. To understand the classical spatially homogeneous Boltzmann equation Mark Kac introduced a Markov jump process to mimic a real $n$-particle system and from this model he could rigorously derive a simplified (one dimensional) Boltzmann equation [8]. A similar kind of jump process has been studied by Colangeli et al. [5], who derive a kinetic equation from a particle system with discretized phase space with exclusion.

We investigate a Kac model on the simplex with exclusion, but without dividing the simplex into cells, paying close attention to questions concerning Kac’s notion of chaos for the model. Before we introduce our model with exclusion, it will be helpful to recall Kac’s notion of chaos in the context of the corresponding model, in which states are characterized by their energy only, without exclusion.

Consider a system of $n$ (indistinguishable) particles with a total energy $E_n$, and assume that the state of a particle is determined by its energy $x_j \geq 0$. The phase space of this system is then the simplex

$$S_{E_n} := \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n_+ : \sum_{j=1}^n x_j = E_n \right\}.$$ (1)

Let $\sigma_n$ denote the uniform probability measure in $S_{E_n}$. In its simplest form, the Kac walk on the simplex $S_{E_n}$ is the process in which binary collisions occur in a Poisson stream of
jump times, with the expected waiting time between jumps being $1/n$, and when a jump occurs, a pair $(i, j)$, $1 \leq i < j \leq n$ is selected uniformly at random, and then the energy of the pair is redistributed by the “collision”, a new energy $x_j^*$ for the $i$-th particle is selected uniformly at random from $[0, x_i + x_j]$, and then $x_j^*$ is fixed by $x_i^* + x_j^* = x_i + x_j$. It is easy to see that the uniform probability measure is the unique invariant measure for this process, and the single particle marginals in equilibrium are certain beta distributions. The rate of approach to equilibrium has been studied by Giroux and Ferland [7]. The original Kac process [8] takes place on the $n - 1$-dimensional sphere consisting of vectors $(v_1, \ldots, v_n)$ such that $\sum_{j=1}^{n} v_j^2 = E_n$. The process described above is the image of the process on the sphere under the change of variables $x_j = v_j^2$.

As Kac discovered, the Kac process on the sphere propagates chaos, and it follows readily that the process on the simplex does as well. This means the following: For any probability density $F_n$ with respect to the uniform probability measure on $S_{E_n}$, consider the empirical distribution

$$\mu_n = \frac{1}{n} \sum_{j=1}^{n} \delta(x - \bar{x}_j) \quad \text{where} \quad \bar{x}_j = \frac{n}{E_n} x_j,$$

where $(x_1, \ldots, x_n)$ is distributed according to $F_n$. Note that

$$\int_{0}^{\infty} x \, d\mu_n = 1$$

for every $(\bar{x}_1, \ldots, \bar{x}_n) \in S_{E_n}$. The notion of chaos concerns sequences, indexed by $n$ of probability measures or densities on $S_{E_n}$. We use upper case letters, e.g. $F_n$, to denote such densities. While in some contexts it is natural to reserve upper-case lets for the distribution functions of probability densities on the line, here such distribution functions do not come into play, and it is more natural to use upper and lower case to distinguish between probability densities on the high dimensional space $S_{E_n}$ and probability densities on $\mathbb{R}_+$.

Now let $g(x)$ be any probability density on $\mathbb{R}_+$ with $\int_{0}^{\infty} x g(x) \, dx = 1$. A sequence $\{F_n\}$ of probability densities on $S_{E_n}$ is called $g$-chaotic in the sense of Kac in case the sequence $\{\mu_n\}$ of empirical distributions as specified above converges in probability to $g(x) \, dx$. The notion of chaos is often presented directly in terms of the probability measures: Consider a sequence of probability measures $\{m_n\}_{n=1}^{\infty}$ where the $m_n$ are symmetric probability distributions on $E^n$, the $n$-fold product of a metric space $E$. Then $\{m_n\}_{n=1}^{\infty}$ is said to be $m$-chaotic for some probability measure $m$ on $E$, if for every $k \geq 1$ and functions $\phi_j \in C(E)$, $j = 1, \ldots, k$ the following limit holds:

$$\lim_{n \to \infty} \int_{E} \cdots \int_{E} \phi_1(x_1) \cdots \phi_k(x_k) m_n(dx_1, \ldots, dx_n) - \prod_{j=1}^{k} \int_{E} \phi_j(x) m(dx) = 0.$$

In fact, this definition is equivalent to the definition given in terms of the empirical measures, as proven e.g. in [14].

Kac’s main result in [8] (for the spherical case) is that if one starts with a chaotic sequence $\{F_n\}$ of initial data that is $g$-chaotic, and if for each $t > 0$ one lets $\{F_{n,t}\}$ denote the sequence of densities resulting from running the evolution for a time $t$, then this sequence is $g_t$-chaotic for some density $g_t$, and moreover, $g_t$ is the unique solution of a certain non-linear Boltzmann-like equations starting from the initial data $g$. Thus, this Boltzmann-like equation gives a complete description, in the large $n$ limit of the evolution of the scaled empirical distribution under the Kac process provided one starts with chaotic initial data.

Since $S_{E_n}$ is very close to being a product space, it is possible to construct $g$-chaotic initial data for any probability density $g$ satisfying $\int_{0}^{\infty} x g(x) \, dx = 1$, $\int_{0}^{\infty} x^p g(x) \, dx < \infty$ and $g \in L^p(\mathbb{R}_+)$ for some $p > 1$: One takes $\prod_{j=1}^{n} g(\bar{x}_j)$, and restricts it to simplex $S_{E_n}$, and normalizes [3]. By the Central Limit Theorem, under $\prod_{j=1}^{n} g(\bar{x}_j)$, $\sum_{j=1}^{n} \bar{x}_j$ is with high probability very close to $n$, and so the mass is tightly concentrated on $S_{E_n}$. As long
as one does not look at too many coordinates at once, one cannot see the effects of the restriction. In the physics literature, this is known as the equivalence of ensembles. A related result can be found in [14], where $g$ is a density on $\mathbb{R}^k$, and the simplex is replaced by a set $x_1 + \cdots + x_n = na \in \mathbb{R}^k$.

The restrictions that $\int_0^\infty x^2g(x)dx < \infty$ and $g \in L^p(\mathbb{R}_+)$ for some $p > 1$ may then be removed in a limiting process [3], and thus one has a construction of chaotic initial data for every meaningful initial density $g$. The corresponding nonlinear Boltzmann-like equation that governs the evolution of the large $n$ empirical measure may then be studied in terms of the linear Kolmogorov equation associated to the Kac process on $S_{E_n}$.

That is, Kac had found an interesting way to study, by probabilistic means, a class of non-linear equations of a type that arise in kinetic theory. The method relies on the introduction of a family of stochastic processes indexed by $n$, the number of particles. Because of constraints such as $\sum_{j=1}^n x_j = E_n$ that correspond to conservation laws in the particle system, the $x_j$ are not independent, but their dependence is weak enough, for a wide class of sequences of probability measures including $\{\sigma_n\}$, that the empirical measure in (2) becomes non-random as $n \to \infty$.

In the model introduced next, we consider another type of kinematic constraint. In addition to the conservation of energy, we impose an exclusion condition. This brings dependencies of a new type into consideration, and we show that Kac’s notion of chaos is not enough to identify the evolution of a limiting density. Therefore a new approach is required, and a stronger notion of chaos, and one such approach is developed here.

1.1. The incorporation of exclusion. For Fermions, the Pauli exclusion principle asserts that a state (here characterized by its energy) only can be occupied by at most one particle.

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The process that we consider is the following: Again, the collision times arrive in a Poisson stream with expected waiting time equal to $1/n$, and again, when a jump time occurs, a pair $(i, j)$, $1 \leq i < j \leq n$ is selected uniformly at random. The energy of the two particles is then reapportioned as before, with $x_i^*$ chosen uniformly from $[0, x_i + x_j]$ and then $x_j^* = x_i^* + x_j - x_i^*$, except the jump only occurs if the new configuration $(x_1, \ldots, x_i^*, \ldots, x_j^*, \ldots, x_n)$ of energy levels satisfies the exclusion condition; i.e., only if it belongs to $S_{E_n, \epsilon}$. It is easy to see that $\sigma_{n, \epsilon}$ is the invariant measure for this process, and since the process is reversible, it is natural to refer to it as the equilibrium measure.

While $S_{E_n, \epsilon}$ is non-empty whenever $E_n > cn(n - 1)/2$, if $E_n$ is not too much larger than this value, the spacing between most levels will be very close to $\epsilon$. Think of a long line of parked cars with no marked spaces. For a new pair of cars to park, they must both find gaps of sufficient width. If there is a constraint on the sum of their distances from the start of the line, there may be no way for them to park. In terms of our model, if two cars pull out and look for different spaces, it may be that their only option is to return to the spaces they had (or to swap).

We shall find interesting large $n$ limits only if the energies $E_n$ grow with $n$ in a certain way. Define

$$\alpha_n := \frac{cn(n - 1)}{E_n}.$$  

Then

$$E_n - \frac{cn(n - 1)}{2} = \left(1 - \frac{\alpha_n}{2}\right)E_n$$
is the excess energy, the difference between the minimum energy for a configuration of \( n \) particles satisfying the exclusion constraint and the available energy. Clearly we must have \( 0 \leq \alpha_n \leq 2 \). We shall be studying sequences of probability measures \( \{ F_{n,\alpha_n} \} \) on \( S_{E_n,\epsilon} \) with \( E_n \) and \( n \) related by

\[
\lim_{n \to \infty} \alpha_n = \alpha \in [0,2[ .
\]

As before, we rescale the variables with the average energy,

\[
\bar{x}_j = \frac{n}{E_n} x_j ,
\]

and define the empirical distribution

\[
\mu_n := \frac{1}{n} \sum_{j=1}^{n} \delta(x - \bar{x}_j) .
\]

We also need to rescale \( \epsilon \), and set

\[
\tilde{\epsilon}_n = \frac{\epsilon n}{E_n} = \frac{\alpha_n}{n - 1} .
\]

Because, \( \sum_{j=1}^{n} \bar{x}_j = n \) for every \( (x_1, \ldots, x_n) \in S_{E_n,\epsilon} \), one always has that

\[
\int_{0}^{\infty} x \text{d}\mu_n = 1 .
\]

The exclusion limits the amount of mass the \( \mu_n \) can assign to any half open interval \([a,b]\) in \( \mathbb{R}_+ \): There can be at most \((b - a)/\tilde{\epsilon}_n\) particles in this interval, and hence

\[
\int_{[a,b]} d\mu_n \leq \frac{1}{n\tilde{\epsilon}_n} (b - a) = \frac{n - 1}{n\alpha_n} (b - a) .
\]

It follows from (13) that if \( \mu_n \) almost surely converges vaguely to \( g(x)dx \) along a sequence with \( \alpha_n \to \alpha \), then

\[
g(x) \leq \frac{1}{\alpha} ,
\]

and provided no mass escapes,

\[
\int_{0}^{\infty} g(x)dx = 1 .
\]

In what follows we will only use the rescaled variables \( \bar{x}_j \) and \( \tilde{\epsilon}_n \), but suppress the tildes from the notation. Moreover, in this scaling \( E_n = n \), and therefore \( S_{E_n,\epsilon} \) becomes \( S_{n,\epsilon} \).

At this point we can define a notion of chaos for our class of models:

**DEFINITION 1.1.** Let \( \alpha > 0 \) and let \( f(x) \) be a probability density on \( \mathbb{R}_+ \). We define a sequence \( \{ F_n \}_{n \geq 2} \) of probability measures on \( S_{n,\epsilon} \) to be \((\alpha,f)\)-chaotic if \( (x_1, \ldots, x_n) \) is random with distribution \( F_n \), and the empirical measures \( \mu_n = \frac{1}{n} \sum_{j=1}^{n} \delta(x - x_j) \) converge in probability to \( f(x)dx \) as \( n \to \infty \) and \( \alpha_n := c\epsilon(n - 1)/E_n \to \alpha \).

Let \( \mathcal{P}_t \) be the semigroup associated to a Markov process on \( S_{n,\epsilon} \), that is, \( F_n(x,t) = \mathcal{P}_t F_n(x,0) \). Following Kac, we say that the semigroup \( \mathcal{P}_t \) propagates chaos with parameter \( \alpha \) in case whenever \( \{ F_n(x,0) \} \) \((\alpha,f_0)\)-chaotic, then \( \{ F_n(x,t) \} \) is \((\alpha,f_t)\)-chaotic for some probability density \( f_t \) on \( \mathbb{R}_+ \).

In the Kac process that we study here, pairs of particle will interact by redistributing their energies \( x_i \) and \( x_j \) to a new pair \( x_i^* \) and \( x_j^* \) with \( x_i + x_j = x_i^* + x_j^* \) provided the gaps around \( x_i^* \) and \( x_j^* \) are large enough for the exclusion constraint to be satisfied. Let \( x \in \mathbb{R}_+ \). Then for all sufficiently large \( n \), and all \( (x_1, \ldots, x_n) \in S_{n,\epsilon} \), \( x < \max_{1 \leq k \leq n} \{ x_k \} \). Let \( x_{(j)} \) and \( x_{(j+1)} \) be the pair of consecutive energies such that \( x \in [x_{(j)},x_{(j+1)}] \). Define the gap at energy \( x \) to be

\[
\zeta(x) := x_{(j+1)} - x_{(j)} = \frac{\alpha}{n - 1} .
\]
Only when \( \zeta \geq \frac{\alpha}{n-1} \) is it possible for an interaction to result in either \( x^+_n \in [x(j), x(j+1)] \) or \( x^-_n \in [x(j), x(j+1)] \), since only in this case is the minimum spacing \( \frac{\alpha}{n-1} \) (in the scaled variable) available above and below some energy in this interval.

It is probably intuitively clear, and will be shown later on, that the evolution of the empirical density depends strongly on distribution of the energy gaps: For a given probability density \( f(x) \) as in Definition 1.1, and any \( 0 < \alpha < 2 \), there are different \((\alpha, f)\)-chaotic sequences \( \{F_n\}_{n \geq 2} \) that have very different gap distributions, and this will result in different sorts of interactions being favored in the process, and thus to different results for \( f_t \) under the time evolution. Thus, this definition as it stands will not lead to a well-defined evolution equation for the limiting density \( f_t \). We must bring in information on the gaps.

**DEFINITION 1.2.** Let a sequence \( \{F_n\}_{n \geq 2} \) be \((\alpha, f)\)-chaotic according to Definition 1.1. We say that \( \{F_n\}_{n \geq 2} \) is \((\alpha, f)\)-chaotic in detail if for any \( x \in \mathbb{R}_+ \), the random interval \( [x(j), x(j+1)] \) that contains \( x \), the gap length \( \zeta_{x,n} = x(j+1) - x(j) - \alpha/(n-1) \) satisfies

\[
\lim_{n \to \infty} \mathbb{P}[(n-1)\zeta_{x,n}/\alpha > r] = e^{-\frac{\alpha f(x)}{\alpha - \alpha f(x)}} r.
\]

We say that the semigroup \( \mathcal{P}_t \) propagates detailed chaos with parameter \( \alpha \) in case whenever \( \{F_n(x,0)\} \) is \((\alpha, f_0)\) chaotic in detail, then the same holds for \( \{F_n(x,t)\} \) for some probability density \( f_t \) on \( \mathbb{R}_+ \).

As we shall show below, this particular gap distribution specified in (16) is the only one that is possible: If the gap lengths are asymptotically exponential, and the empirical distribution is asymptotically deterministic with density \( f \), then the exponential rates must be related to \( f \) as specified in (16). Thus one could formulate the definition less specifically, only requiring that the gap lengths are asymptotically exponential with some rate.

This is probably the simplest generalization of the notion of chaos to our class of models with the exclusion constraint. We consider four questions concerning the Kac model on the simplex with exclusion:

(1) Does the \( \{\sigma_{n,c}\} \) of equilibrium measures satisfy the detailed chaos condition when \( \alpha_n \to \alpha \)? If so, what is the limiting density \( f_\alpha \) for which this sequence is \((\alpha, f_\alpha)\)-chaotic, and how does \( f_\alpha \) compare with the Fermi-Dirac distribution, which one might expect in a "quantized" model; i.e., one in which parking spaces are marked with lines?

(2) For which probability densities \( g \) on \( \mathbb{R}_+ \) that satisfy (14) and (15) do there exist sequences that satisfy \((\alpha, g)\)-chaos and detailed \((\alpha, g)\)-chaos conditions?

(3) Is detailed chaos propagated, and if so, what is the equation that governs the evolution of the limiting marginal densities?

(4) At which energy levels in equilibrium do collisions occur a rate bounded away from zero, and at which energy levels are the collisions "frozen out"?

Theorem 2.1 gives a positive answer to the first question, explicitly identifying \( f_\alpha \), which is not the Fermi-Dirac distribution; see Figure 1. Theorem 2.1 provides quantitative bounds on the rate at which \( W_1(\mu_n, f_\alpha)dx \to 0 \) in probability, where \( W_1 \) is the Kantorovich-Rubinstein transport metric. Mass transport methods are the basis of a number of our proofs.

Theorem 3.10 answers the second question—such chaotic sequences exist for all densities satisfying the two necessary requirements (14) and (15). Such sequences can be constructed in qualitatively different ways, and we provide two examples of constructions, the second one given in Theorem 3.13. Other results in this section provide quantitative chaos estimates, again in the \( W_1 \) metric for a broad class of densities \( g \) satisfying mild regularity hypotheses.

In section 4 we derive under the assumption of propagation of detailed chaos, the Boltzmann-like equation that governs the evolution of the limiting empirical measure. The equation resembles the Uehling-Uhlenbeck equation of quantum kinetic theory, but with a
different “exclusion factor” corresponding to our different exclusion model. But this exclusion factor turns out to depend on the chaotic sequence: Definition 1.1 is not restrictive enough to uniquely determine the evolution of the limiting empirical measure, though the additional information on the gaps provided in Definition 1.2 is enough.

We prove that the limiting densities $f_\alpha$ obtained from equilibrium measures $\{\sigma_{n,\epsilon}\}$ are stationary solutions to the Boltzmann-Kac equation. We do not prove that propagation of chaos according to either Definition 1.1 or Definition 1.2 holds, but we do provide numerical evidence that detailed $(\alpha, f)$-chaoticity is propagated, and also that if initial data are only $(\alpha, f_0)$-chaotic, without the correct exponential gap distribution (16) for $\alpha$ and $f_0$, this is actually improved by the evolution: The gap distribution converges rapidly to the correct exponential distribution, so that in this sense it appear that not only is chaos propagated, but it strengthens. The numerical evidence for this is presented in Section 5, and further results are available as supplementary material.

2. The empirical distribution with exclusion

Equip the rescaled state space, still denoted $S_{n,\epsilon}$, and defined in (5), with the uniform probability measure $\sigma_{n,\epsilon}$. Let $E$ denote expectation with respect to this probability measure. Then $x_1, \ldots, x_n$ become random variables.

For two probability measures $\mu$ and $\nu$ on $\mathbb{R}_+$, let $W_1(\mu, \nu)$ denote the Kantorovich-Rubinstein distance between $\mu$ and $\nu$. Recall that

\begin{equation}
W_1(\nu, \nu) = \sup \left\{ \left| \int_0^\infty \chi d\mu - \int_0^\infty \chi d\nu \right| : \chi \in \text{Lip}_1 \right\},
\end{equation}

\text{Lip}_1 denotes the class of 1-Lipschitz functions; i.e., functions $\chi$ such that $|\chi(x) - \chi(y)| \leq |x - y|$ for all $x, y$. Note that we may restrict to $\chi \in \text{Lip}_1$ and $\chi(0) = 0$ without changing anything.

**THEOREM 2.1.** For $\alpha_n = \epsilon n(n - 1)/E_n \to \alpha$, the sequence of uniform probability measures on $S_{n,\epsilon}$ is $(\alpha, f_\alpha)$-chaotic in detail, where

\begin{equation}
f_\alpha(x) = \frac{d}{dx} \phi^{-1}(x) = \frac{1}{\phi'(\phi^{-1}(x))}.
\end{equation}

and

\begin{equation}
\phi(\xi) := (1 - \alpha/2) \log \left( \frac{1}{1 - \xi} \right) + \alpha \xi.
\end{equation}

Moreover, the sequence of empirical measures $\{\mu_n\}$, defined as in (10), is such that there is a constant $C$ such that for any $\delta > 0$ and all sufficiently large $n$,

\begin{equation}
\mathbb{P}\{W_1(\mu_n, f_\alpha dx) > \delta\} \leq \frac{1}{\delta} \left( \frac{C}{\sqrt{n}} + \frac{3}{2} |\alpha_n - \alpha| \right).
\end{equation}

The theorem is a corollary of Theorem 3.5, except for the statement of detailed chaoticity, which is proven in Section 3.2. The function $\phi$ in equation (19) is known as the quantile function for the distribution with density $f_\alpha$ and it is derived as a limit of explicitly constructed quantile functions for each $n$. We refer to [4] for similar functions related to Young diagrams and shape functions associated with random permutations.

Theorem 2.1 shows that although the exclusion introduces new dependencies between the random variables $x_1, \ldots, x_n$ that are far more complicated than those induced by $\sum_{j=1}^n x_j = n$ which would be the only constraint in the absence of exclusion, these new dependencies are not an obstacle to chaos in the sense of Kac: If $\sigma_{n,\epsilon}$ denote the law of $(x_1, \ldots, x_n)$ and $\alpha_n \to \alpha$, then $\{\sigma_{n,\epsilon}\}$ is $(\alpha, f_\alpha)$-chaotic.

While the form of $\phi(\xi)$ is simple, it seems difficult to express the function $f$ in closed form, but it clearly differs from the Fermi-Dirac density that is the relevant expression in a quantized setting, although it does resemble it for large values of $\alpha$. The function $f$ is plotted for some different values of $\alpha$ in Figure 1.
Figure 1. The function $\phi(\xi)$ for $\alpha = 0.0, 1.0, 1.5, 1.9$ (left), and the density $f(x)$ for the same values of $\alpha$. Curiously, for $\xi = \xi_0 \approx 0.797$ (the solution of $1 - \xi - e^{-2\xi} = 0$), $\phi(\xi)$ is independent of $\alpha$, and hence the fraction of the mass of $f(x)$ in $0 \leq x \leq 2\xi_0$ (indicated by the dashed line) is $\xi_0$ for all values of $\alpha$.

2.1. Parameterization of $S_{n,\epsilon}$ by the standard simplex. We shall make use of a parameterization of the state space $S_{n,\epsilon}$ in terms of the standard simplex

$$S_1 := \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n_+ : \sum_{j=1}^n x_j = 1 \right\}.$$ (21)

We first define $S^*_{n,\epsilon}$ to be the subset consisting of all $(x_1, \ldots, x_n)$ with $x_1 < x_2 < \cdots < x_n$. Up to a set of measure zero, one recovers $S_{n,\epsilon}$ by taking the union over all permutations

$$\bigcup_{\pi \in S_n} \left\{ (x_{\pi(1)}, \ldots, x_{\pi(n)}) : (x_1, \ldots, x_n) \in S^*_{n,\epsilon} \right\}.$$ (22)

The measures we study are all symmetric under interchange of particles, and hence it suffices to parameterize $S^*_{n,\epsilon}$.

Lemma 2.2. For $\tilde{(z_1, \ldots, z_n)} \in S_1$, define $T_n(\tilde{z}_1, \ldots, \tilde{z}_n)$ to be the vector in $\mathbb{R}^n_+$ whose $j$th component $x_j$ is given by

$$x_j = n \left(1 - \frac{\alpha_n}{2}\right) \frac{\tilde{z}_1}{n} + \frac{\tilde{z}_2}{(n-1)} + \cdots + \frac{\tilde{z}_j}{n+1-j} + \frac{j - 1}{n-1} \alpha_n.$$ (23)

Then $T_n$ provides a one-to-one parameterization of $S^*_{n,\epsilon}$ by $S_1$, and moreover $\sigma_{n,\epsilon}$ is the push-forward of the uniform probability measure on $S_1$ under $T_n$, averaged over permutations.

Proof. First note that

$$\sum_{j=1}^n \left( \frac{\tilde{z}_1}{n} + \frac{\tilde{z}_2}{(n-1)} + \cdots + \frac{\tilde{z}_j}{n+1-j} \right) = 1$$

and

$$\sum_{j=1}^n \frac{j - 1}{n-1} \alpha_n = \frac{n\alpha_n}{2},$$

so that $\sum_{j=1}^n x_j = n$, and for $j > 1$,

$$x_j - x_{j-1} = n \left(1 - \frac{\alpha_n}{2}\right) \frac{\tilde{z}_j}{n+1-j} + \epsilon_n \geq \epsilon_n.$$ (24)

Thus the image of $T_n$ lies in $S_{n,\epsilon}$, Moreover, (23) shows that $T_n$ is invertible, and gives an explicit formula for the inverse from which one sees, by the same computations that $T^{-1}_n(S_{n,\epsilon}) \subset S_1$. This proves the statements about the parameterization. The proof of the description of $\sigma_{n,\epsilon}$ in terms of $T_n$ is somewhat more involved.
We begin by considering the case with no exclusion ($\epsilon = 0$): The uniform density is also the equilibrium distribution of a set of particles at equilibrium, so that for $\phi \in C(\mathbb{R}^n)$,
\[
\mathbb{E}(\phi(x_1, \ldots, x_n)) = \frac{1}{Z} \int_{0 < x_1 + \cdots + x_{n-1} < n} \phi(x_1, \ldots, x_n) \, dx_1 \, dx_2 \cdots dx_{n-1}
\]
(24)
\[
= \frac{1}{Z} \int_{0 < x_1 + \cdots + x_{n-1} < n} \sum_{\pi} \phi_{\pi}(x_1, \ldots, x_n) \, dx_1 \, dx_2 \cdots dx_{n-1},
\]
where $x_n = n - x_1 - \cdots - x_{n-1}$, and, in the second row, $\phi_{\pi}$ denotes the composition of $\phi$ with the permutation operator $\pi : (x_1, \ldots, x_n) \mapsto (x_{\pi_1}, x_{\pi_2}, \ldots, x_{\pi_n})$, and the sum is taken over all permutations. The normalizing factor $Z$ is given by
\[
(25)
\int_{0 < x_1 < \cdots < x_n} dx_1 \, dx_2 \cdots dx_{n-1}.
\]
Here we have parameterized $S_n$ with its projection on $\{(x_1, \ldots, x_{n-1}) \mid x_j > 0, \, x_1 + \cdots + x_{n-1} < n\}$, and set $d\sigma(x_1, \ldots, x_{n-1}) = dx_1 \cdots dx_{n-1}$ without the factor $\sqrt{n}$ which may anyway be absorbed into $Z$.

Now consider the case $\epsilon > 0$: The expectation in eq. (24) can then be computed with the same integrals, but adding the restriction that $x_j - x_{j-1} > \epsilon_n$ for all $j > 1$. Therefore we set $z_j = x_j - x_{j-1} - \epsilon_n > 0$ for $1 < j < n$ and set $z_1 = x_1$. This yields the following change of variables:
\[
x_1 = z_1
\]
\[
x_2 = x_1 + \epsilon_n + z_2 = z_1 + z_2 + \epsilon_n
\]
\[
\cdots
\]
\[
x_{n-1} = z_1 + \cdots + z_{n-1} + (n - 2)\epsilon_n
\]
\[
x_n = n - (n - 1)z_1 - (n - 2)z_2 - \cdots - z_{n-1} - \epsilon_n(n - 1)(n - 2)/2.
\]
The Jacobian of $(z_1, \ldots, z_{n-1}) \mapsto (x_1, \ldots, x_{n-1})$ has determinant one, and hence to compute the integrals in eq. (24), it is enough to find the domain of $(z_1, \ldots, z_{n-1})$. For each $j < k \leq n$, and each $1 \leq m \leq n - j$, we have $x_{j+m} \geq x_j + m\epsilon_n$, and evidently this is the smallest value $x_{j+m}$ can take, given $x_j$. Therefore,
\[
n - j \sum_{k=1}^{n-j} x_k = \sum_{m=1}^{n-j} x_{j+m} \geq (n - j)x_j + \frac{(n + 1 - j)(n - j)\epsilon_n}{2}.
\]
Rearranging terms,
\[
x_j \leq \frac{1}{n + 1 - j} \left( n - x_1 - x_2 - \cdots - x_{j-1} - \epsilon_n \frac{(n + 1 - j)(n - j)}{2} \right).
\]
Since $x_k = z_1 + \cdots + z_k + (k - 1)\epsilon_n$,
\[
\sum_{k=1}^{j-1} x_k = \sum_{k=1}^{j-1} (j - k)z_k + \epsilon_n \frac{(j - 1)(j - 2)}{2},
\]
and since
\[
(j - 1)(j - 2) + (n + 1 - j)(n - 1) + 2(j - 1)(n + 1 - j) = n(n - 1),
\]
\[
\begin{align*}
z_j &< \frac{1}{n+1-j} \left( n - \sum_{k=1}^{j-1} (j-k)z_k - \epsilon_n \frac{(j-1)(j-2)}{2} - \epsilon_n \frac{(n+1-j)(n-j)}{2} \right) \\
&\quad - z_1 - \cdots - z_{j-1} - (j-1)\epsilon_n \\
(28) &\Rightarrow \frac{1}{n+1-j} \left( n \left( 1 - \frac{\alpha_n}{2} \right) - \sum_{k=1}^{j-1} (n+1-k)z_k \right).
\end{align*}
\]

Define
\[
\tilde{z}_j := \frac{n+1-j}{n} \frac{1}{1-\alpha_n/2} z_j.
\]

Then (28) becomes \(\tilde{z}_j \leq 1 - \sum_{k=1}^{j-1} \tilde{z}_k\). Using this notation, the version of Equation (24) with exclusion can be written
\[
E(\phi(x_1,\ldots,x_n)) = \frac{\mathcal{J}_{n,\epsilon}}{Z_{n,\epsilon}} \int_0^1 d\tilde{z}_1 \int_0^{1-\tilde{z}_1} d\tilde{z}_2 \cdots \int_0^{1-\tilde{z}_1-\cdots-\tilde{z}_{n-2}} d\tilde{z}_n \cdots \\
\cdots \int_0^{1-\tilde{z}_1-\cdots-\tilde{z}_{n-2}} d\tilde{z}_{n-1} \sum_n \phi_n(x_1,x_2,\ldots,x_n),
\]

where \(\mathcal{J}_{n,\epsilon}\) is the Jacobian corresponding to the change of variables given in (29). Taking \(\varphi\) to be the constant function 1, it is evident that
\[
Z_{n,\epsilon} = \frac{\mathcal{J}_{n,\epsilon}}{1} \left( n - \frac{n\alpha_n}{2} \right)^{n-1},
\]

which gives the value of \(Z_{n,\epsilon}\). However, we only need to know that \(\mathcal{J}_{n,\epsilon}/Z_{n,\epsilon} = 1\), and then observe that the substitution (29) transforms \(x_j = z_1 + \cdots + z_{n-1} + (j-1)\epsilon_n\) into (22) for all \(j < n\).

**Remark 2.3.** Lemma 2.2 provides a convenient method for sampling \((x_1,\ldots,x_n)\): simply take \((\tilde{z}_1,\ldots,\tilde{z}_n)\) uniformly from the standard \(n\)-simplex, i.e. \(\tilde{z}_1 + \cdots + \tilde{z}_n = 1\), and compute the \(x_j\) according to the formula (22).

**Remark 2.4.** Because \(T_n\) is continuous and invertible with a continuous inverse, it sets up a one-to-one correspondence between symmetric Borel probability measures on \(S_{n,\epsilon}\) and Borel probability measures on \(S_1\). This correspondence provides a useful way to think about symmetric Borel probability measures on \(S_{n,\epsilon}\) in terms of partitions of the excess energy. In the rescaled variables, the excess energy is
\[
n \left( 1 - \frac{\alpha_n}{2} \right) = \sum_{j=1}^{n} \left( 1 - \frac{\alpha_n}{2} \right) \tilde{z}_j.
\]

Thus one may think of \(\{\tilde{z}_1,\ldots,\tilde{z}_n\}\) as specifying a partition of the excess energy into \(n\) components
\[
\{ n \left( 1 - \frac{\alpha_n}{2} \right) \tilde{z}_j \}_{j=1}^{n}.
\]

The first term in the partition may be understood as making an equal contribution of
\[
n \left( 1 - \frac{\alpha_n}{2} \right) \tilde{z}_1 \]

to the energy of each particle, and in the same way the second component makes equal contributions of
\[
n \left( 1 - \frac{\alpha_n}{2} \right) \tilde{z}_2.
\]
to the energy of each of the last \( n - 1 \) particle and so forth. Adding these up, together with the total excluded energies, one arrives at (22).

3. PRE-CHAOTIC SEQUENCES OF PROBABILITY MEASURES ON \( S_1 \)

We now identify a class of sequences of measures on \( S_1 \) whose push-forwards under \( T_n \) will be shown to be \((\alpha, g)\)-chaotic on \( S_n, w_n \) in the sense of Definition 1.1.

For each \( n \), let \( \tau_n \) be a Borel probability measure on \( S_1 \). Also define the function

\[
\begin{align*}
    w_n(\xi) &= n\mathbb{E}_{\tau_n}[\tilde{z}_k] & \text{for } \frac{k - 1}{n} < \xi \leq \frac{k}{n} & 1 \leq k \leq n, \\
    w(0) &= n\mathbb{E}_{\tau_n}[\tilde{z}_1].
\end{align*}
\]

and \( w(0) = n\mathbb{E}_{\tau_n}[\tilde{z}_1] \). Here \( \mathbb{E}_{\tau_n} \) is the expectation with respect to the measures \( \tau_n \). That is, for \( \xi > 0, w_n(\xi) = n\mathbb{E}[\tilde{z}_{n\xi}] \) where \( \lceil n\xi \rceil \) is the least integer \( k \) such that \( n\xi \leq k \). Note that for each \( n \),

\[
\int_0^1 w_n(\xi)d\xi = \sum_{k=1}^{\lfloor n\xi \rfloor} \mathbb{E}_{\tau_n}[\tilde{z}_k] = 1,
\]

so that \( w_n \) is a probability density.

**DEFINITION 3.1.** Let \( w : [0, 1] \rightarrow \mathbb{R}_+ \) be a continuous probability density. A sequence \( \{\tau_n\} \) of probability measures on \( S_1 \) is \( w \)-pre-chaotic in case

\[
\begin{align*}
    n\mathbb{E}_{\tau_n}[\tilde{z}_j] &= w(j/n) + r(j/n) & \text{for a continuous function } r(s) \text{ decreasing to zero when } s \rightarrow 0, \text{ where for each } 0 < \xi_+ < 1, \\
    &\text{and each } \epsilon > 0, \text{ there is an } n_\epsilon \text{ so that } \\
    \lvert r(j/n) \rvert &< \epsilon & \text{for all } n > n_\epsilon, j < n\xi_+.
\end{align*}
\]

and moreover, for some constant \( C < \infty \) depending only on \( \xi_+ \),

\[
\begin{align*}
    \text{Var}[\tilde{z}_i] &\leq \frac{C}{n} \epsilon & \text{and } \text{Cov}(\tilde{z}_j, \tilde{z}_k) \leq \frac{C}{n^2} \epsilon & \text{for all } n > n_\epsilon, j, k < n\xi_+, \\
    \text{for all } n, j \text{ and } k.
\end{align*}
\]

**Remark 3.2.** By Lemma 2.2, the equilibrium distribution \( \sigma_{n,\epsilon} \) arises when the random partition in (31) is determined by choosing \( (\tilde{z}_1, \ldots, \tilde{z}_n) \) from a flat Dirichlet distribution; i.e., the uniform density on \( S_1 \), and then the random variables \( \tilde{z}_j \) satisfy

\[
\begin{align*}
    \mathbb{E}[\tilde{z}_i] &= \frac{1}{n}, \\
    \text{Var}[\tilde{z}_i] &= \frac{(n - 1)}{n^2(n + 1)}, \\
    \text{Cov}[\tilde{z}_i, \tilde{z}_j] &= \frac{-1}{n^2(n + 1)}.
\end{align*}
\]

Moreover, it is clear that for each \( n, w_n(\xi) = 1 \) for all \( n \) and \( \xi \). In this case, \( w \) is continuous on the closed interval \( [0, 1] \) and hence is bounded at 1 also, though the definition allows for \( w(t) \) to diverge as \( t \uparrow 1 \). Later, we shall see that we need this generality.

The next lemma will be used several times in what follows.

**LEMMA 3.3.** Let \( f \) and \( g \) be two non-negative integrable functions on \( [0, 1] \) such that

\[
\left| \int_0^1 (f(\xi) - g(\xi))d\xi \right| \leq a.
\]

Then for all \( 0 < \xi_+ < 1, \)

\[
\int_0^1 |f(\xi) - g(\xi)|d\xi \leq 2\int_0^{\xi_*} |f(\xi) - g(\xi)|d\xi + 2\int_{\xi_*}^{1} g(\xi)d\xi + a.
\]
Proof. We have
\[
\int_0^1 |f(\xi) - g(\xi)|\,d\xi = \int_0^\xi |f(\xi) - g(\xi)|\,d\xi + \int_{\xi}^1 |f(\xi) - g(\xi)|\,d\xi \\
\leq \int_0^\xi |f(\xi) - g(\xi)|\,d\xi + \int_{\xi}^1 f(\xi)\,d\xi + \int_{\xi}^1 g(\xi)\,d\xi
\]
Next,
\[
\int_{\xi}^1 f(\xi)\,d\xi = \int_{\xi}^1 g(\xi)\,d\xi + \int_{\xi}^1 (f(\xi) - g(\xi))\,d\xi \\
\leq \int_{\xi}^1 g(\xi)\,d\xi + a - \int_{\xi}^\xi (f(\xi) - g(\xi))\,d\xi \\
\leq \int_{\xi}^1 g(\xi)\,d\xi + a + \int_{\xi}^\xi |f(\xi) - g(\xi)|\,d\xi .
\]
\[\square\]

Our first application is the following:

**Lemma 3.4.** Let \( \{\tau_n\} \) be a \( w \) pre-chaotic sequence, and let \( w_n \) be defined in terms of \( \tau_n \) as in (32). Then
\[
\lim_{n \to \infty} \int_0^1 |w_n(\xi) - w(\xi)|\,d\xi = 0 .
\]

**Proof.** Pick \( \epsilon > 0 \), and choose \( 0 < \xi_* < 1 \) such that \( \int_{\xi_*}^1 w(t)\,dt < \epsilon \). By hypothesis \( w \) is continuous on \([0, \xi_*]\,\) and for all \( n > n_\epsilon \),
\[
\int_{\xi_0}^{\xi_*} |w_n(\xi) - w(\xi)|\,d\xi = \sum_{k < n\xi_*} \int_{(k-1)/n}^{k/n} |w(k/n) + r(k,n) - w(\xi)|\,d\xi \\
\leq \sum_{k < n\xi_*} \int_{(k-1)/n}^{k/n} |w(k/n) - w(\xi)|\,d\xi + \epsilon
\]
If \( \omega \) denotes the modulus of continuity of \( w \) on \([0, \xi_*]\),
\[
|w(k/n) - w(\xi)| \leq \omega(1/n) \quad \text{on} \quad \left[\frac{k-1}{n}, \frac{k}{n}\right].
\]
Thus
\[
\int_{\xi_0}^{\xi_*} |w_n(\xi) - w(\xi)|\,d\xi \leq \omega(1/n) + \epsilon
\]
for all sufficiently large \( n \). By Lemma 3.3, for all sufficiently large \( n \),
\[
\int_{\xi_0}^1 |w_n(\xi) - w(\xi)|\,d\xi \leq \omega(1/n) + 4\epsilon.
\]
Since \( \epsilon > 0 \) is arbitrary, the lemma is proved. \[\square\]

3.1. **Chaotic sequences of probability measures on \( S_{n,\alpha_n} \).** In this section we prove the following:

**Theorem 3.5.** Let \( w \) be a probability density on \([0, 1]\) that is continuous on \([0, 1]\), and let \( \{\tau_n\} \) be a \( w \) pre-chaotic sequence of probability densities on \( S_1 \). Fix a sequence of energies \( \{E_n\} \) with \( \alpha_n = \varepsilon(n-1)/E_n \to \alpha \), and define the maps \( T_n \) in terms of \( \alpha_n \). Let \( \tau_n \) denote the push forward of \( \tau_n \) onto \( S_{n,\alpha_n} \), averaged over permutations. Let \( \{\mu_n\} \) be the sequence of empirical measures associated to \( \{\tau_n\} \). Then
\[
\lim_{n \to \infty} W_1(\mu_n, g(x)\,dx) = 0,
\]
where

\[
W_1(\mu_n, g(x)\,dx) = \int_{\Omega} \max \{|f(x) - g(x)|, 0\}\,dx
\]
where \( g \) is a probability density on \( \mathbb{R}_+ \) related to \( w \) as follows: Define the increasing function \( \phi \) on \([0,1]\) by

\[
\phi(\xi) = \left(1 - \alpha/2\right) \int_0^\xi \frac{w(t)}{1-t} \, dt + \alpha \xi
\]

and then

\[
g(x) = \frac{1}{\phi'(\phi^{-1}(x))}.
\]

Theorem 3.5 gives conditions for \( \{\hat{\tau}_n\} \) to be \((\alpha, g)\) chaotic for a probability density \( g \) on \( \mathbb{R}_+ \) that is determined by \( \alpha \) and \( w \). Notice that as long as \( w(1) \neq 0 \) we have \( \lim_{\xi \to 1^-} \phi(\xi) = \infty \), and if in addition \( \alpha > 0 \) or if \( w \) does not vanish on any interval, then \( \phi \) is strictly increasing, so that \( \phi \) is invertible from \([0,1]\) to \([0,\infty]\), and evidently it is differentiable. It is also possible to invert the relation between \( g \) and \( w \), so that given an appropriate density \( g \), one can find the \( w \) for which (39) and (40) yield \( g \):  

**THEOREM 3.6.** Let \( \alpha \in [0,2] \). Let \( g(x) \) be a probability density on \( \mathbb{R}_+ \) such that

\[
g(x) < \frac{1}{\alpha} \text{ a.s. and } \int_0^\infty xg(x) \, dx = 1.
\]

Let \( G(x) = \int_0^x g(t) \, dt \) denote the distribution function of \( g \), and for \( \xi \in [0,1] \) define

\[
w(\xi) := \frac{1}{1 - \alpha/2} \left( \frac{1}{g(G^{-1}(\xi))} - \alpha \right) (1 - \xi).
\]

Then \( w \) is a probability density on \([0,1]\), and with \( \phi \) defined as in (39)

\[
g(x) = \frac{1}{\phi'(\phi^{-1}(x))},
\]

and

\[
\frac{\alpha g(x)}{1 - \alpha g(x)} = \frac{2\alpha - 1 - \phi^{-1}(x)}{2 - \alpha} \frac{1}{w(\phi^{-1}(x))}.
\]

**Proof.** We compute, using the change of variables \( x := G^{-1}(\xi) \),

\[
(1 - \alpha/2) \int_0^1 w(\xi) \, d\xi = \int_0^\infty (1 - \alpha g(x))(1 - G(x)) \, dx
\]

\[
= \int_0^\infty (1 - G(x)) \, dx - \alpha \int_0^\infty \alpha g(x)(1 - G(x)) \, dx
\]

\[
= \int_0^\infty xg(x) \, dx - \frac{1}{2} \alpha.
\]

Thus, whenever, \( g(x) < 1/\alpha \) almost everywhere and \( \int_0^\infty xg(x) \, dx = 1 \), \( w(x) \) is a probability density on \([0,1]\).

With this choice of \( w(\xi) \) in (39), we find

\[
\phi(\xi) = \int_0^\xi \left( \frac{1}{g(G^{-1}(t))} - \alpha \right) \, dt + \alpha \xi = \int_0^\xi \frac{1}{g(G^{-1}(t))} \, dt.
\]

It follows that \( \phi'(\xi) = 1/g(G^{-1}(\xi)) = (G^{-1}(\xi))' \) and then since \( \phi(0) = G^{-1}(0) = 0 \), \( \phi(\xi) = G^{-1}(\xi) \). Thus, \( G(x) = \phi^{-1}(x) \), and (43) is valid.

Finally, by (43),

\[
\frac{\alpha g(x)}{1 - \alpha g(x)} = \frac{\alpha}{\phi'(\phi^{-1}(x)) - \alpha},
\]

and then since \( \phi'(\xi) = (1 - \alpha/2) \frac{w(\xi)}{1 - \xi} + \alpha \), (44) follows. \( \square \)
As an example, consider \( g(x) = e^{-x} \), which satisfies (41) as long as \( \alpha \leq 1 \). Then \( G(x) = 1 - e^{-x} \), and then \( G^{-1}(\xi) = -\log(1 - \xi) \). Therefore,

\[
w(\xi) = \frac{1}{1 - \alpha/2} (1 - \alpha(1 - \xi)),
\]
which is bounded on all of \([0, 1]\). By Theorem 3.5 and Theorem 3.6, for all \( \alpha \leq 1 \), there exists a \((\alpha, g)\)-chaotic sequence.

We now prepare to prove Theorem 3.5. The first step is to encode the empirical distribution into a random function as follows: Define a random function \( \psi_n : [0, 1] \to \mathbb{R}^+ \) by setting \( x_0 = 0 \), and then

\[
\psi_n(\xi) := x_{k-1} \quad \text{for} \quad \frac{k-1}{n} \leq \xi < \frac{k}{n}, \quad 1 \leq k \leq n.
\]

Explicitly,

\[
(47) \quad \psi_n(\xi) = x_{\lfloor n\xi \rfloor} = \left(1 - \frac{\alpha n}{2}\right) \sum_{j=1}^{\lfloor n\xi \rfloor} \frac{\tilde{z}_j}{n} + \frac{\alpha_n([n\xi] - 1)\downarrow}{n-1},
\]

where \([n\xi]\) is the largest integer \( k \) such that \( k \leq n\xi \). The point of the definition is this: Let \( \chi \) be any \( 1 \)-Lipschitz function on \( \mathbb{R}^+ \) with \( \chi(0) = 0 \). Then on account of (12), \( \chi \) is, with probability \( 1 \), integrable with respect to the empirical distribution \( \mu_n \), and one has

\[
(48) \quad \int_0^\infty \chi d\mu_n = \int_0^1 \chi(\psi_n(\xi)) d\xi + \frac{1}{n} \chi(x_n).
\]

Define \( \rho_n \) to be the push-forward under \( \psi_n \) of the uniform measure on \([0, 1]\), so that we can rewrite (48) as

\[
\mu_n = \rho_n + \frac{1}{n} \delta(x-x_n) - \frac{1}{n} \delta(x).
\]

Had we used the ceiling function \( \lceil \cdot \rceil \) in place of the floor function \( \lfloor \cdot \rfloor \), we would have had \( \mu_n = \rho_n \), and then we would have

\[
\int_0^1 \psi_n(\xi) d\xi = \frac{1}{n} \sum_{j=1}^n x_j = 1,
\]
so that \( \psi_n \) would be a random probability distribution. This would be convenient, but then some estimates that follow would be more complicated. It is easy to estimate the small difference:

**LEMMA 3.7.** We have

\[
(49) \quad \lim_{n \to \infty} \frac{1}{n} \mathbb{E}[x_n] = 0,
\]
and for all \( \delta > 0 \),

\[
(50) \quad \lim_{n \to \infty} \mathbb{P}\{W_1(\mu_n, \rho_n) > \delta\} = 0.
\]

**Proof.** Since \( \chi \) is 1-Lipschitz with \( \chi(0) = 0 \),

\[
\left| \int_0^\infty \chi d\mu_n - \int_0^\infty \chi d\rho_n \right| = \frac{1}{n} |\chi(x_n)| \leq \frac{1}{n} x_n,
\]
and hence \( W_1(\mu_n, \rho_n) \leq \frac{1}{n} x_n \). Thus, once we have proved (49), (50) follows by Markov’s inequality.

Now (36) yields

\[
\mathbb{E}[x_n] = \mathbb{E} \left( n \left(1 - \frac{\alpha_n}{2}\right) \left(\frac{\tilde{z}_1}{n} + \frac{\tilde{z}_2}{n-1} + \cdots + \frac{\tilde{z}_n}{1}\right) + \alpha_n \right).
\]
Pick \(0 < \xi_* < 1\), and split the sum into two pieces
\[
\sum_{k=1}^{n} \frac{\mathbb{E}[\tilde{z}_k]}{n-k+1} = \sum_{k \leq \lfloor n \xi_* \rfloor} \frac{\mathbb{E}[\tilde{z}_k]}{n-k+1} + \sum_{k > \lfloor n \xi_* \rfloor} \frac{\mathbb{E}[\tilde{z}_k]}{n-k+1}.
\]

The last term satisfies
\[
\sum_{k > \lfloor n \xi_* \rfloor} \frac{\mathbb{E}[\tilde{z}_k]}{n-k+1} \leq \sum_{k > \lfloor n \xi_* \rfloor} \mathbb{E}[\tilde{z}_k] \leq \int_{\xi_* - 1/n}^{1} w_n(\xi) d\xi,
\]
and by Lemma 3.4, for and \(\epsilon > 0\) sufficiently large \(n\), this is bounded above by
\[
\int_{\xi_* - 1/n}^{1} w(\xi) d\xi + \epsilon
\]
uniformly in \(\xi_*\). Because
\[
\lim_{\xi_* \to 1} \int_{\xi_*}^{1} w(\xi) d\xi = 0.
\]
we can choose \(\xi_* < 1\) so that
\[
\sum_{k > \lfloor n \xi_* \rfloor} \frac{\mathbb{E}[\tilde{z}_k]}{n-k+1} < \epsilon
\]
for all sufficiently large \(n\). Next,
\[
n \sum_{k \leq \lfloor n \xi_* \rfloor} \frac{\mathbb{E}[\tilde{z}_k]}{n-k+1} = \sum_{k \leq \lfloor n \xi_* \rfloor} \frac{n \mathbb{E}[\tilde{z}_k]}{1 - (k - 1)/n} = \sum_{k \leq \lfloor n \xi_* \rfloor} \frac{(w_n(k - 1)/n) 1}{1 - (k - 1)/n}
\]
By (33), for all \(\epsilon > 0\) this is bounded by
\[
\int_{0}^{\xi_*} \frac{w(t)}{1-t} dt + 2\epsilon
\]
for all sufficiently large \(n\). Altogether, for all \(\epsilon > 0\) and all sufficiently large \(n\),
\[
\frac{1}{n} \mathbb{E}[x_n] \leq (1 - \alpha_n/2) \left( \frac{1}{n} \int_{0}^{\xi_*} \frac{w(t)}{1-t} dt + 2\epsilon + \alpha_n \right) + \epsilon
\]
for all sufficiently large \(n\). Since \(\epsilon > 0\) is arbitrary, this proves (49). \(\Box\)

Now let \(n \to \infty\) with \(\alpha_n \to \alpha\). We shall show below that if \(\{\tilde{\tau}_n\}\) is the push forward of a \(w\)-pre-chaotic sequence \(\{\tau_n\}\) of probability densities on \(S_1\), and \(\mu_n\) is the corresponding sequence of empirical measures, then along this limit, the variance of \(\psi_n(\xi)\) converges to zero, and moreover, its expectation \(\phi_n(\xi) := \mathbb{E}[\psi_n(\xi)]\) converges to a limiting function \(\phi := \lim_{n \to \infty} \phi_n\). In this case
\[
\lim_{n \to \infty} \int_{0}^{\infty} \chi d\mu_n = \int_{0}^{1} \chi(\phi(\xi)) d\xi = \int_{0}^{\infty} \chi(x) f_n(x) dx
\]
with convergence in probability, where \(f_n(x) := 1/\phi'(\phi^{-1}(x))\), as in Theorem 2.1.

Computing the expectation of \(\psi_n(\xi)\), we see that
\[
\phi_n(\xi) = \left(1 - \frac{\alpha_n}{2}\right) \sum_{j=1}^{\lfloor n \xi \rfloor} \frac{\mathbb{E}[\tilde{z}_j]}{1 - \frac{j-1}{n}} + \frac{\alpha_n (\lfloor n \xi \rfloor - 1) \xi}{n - 1}
\]
(51)
\[
= \left(1 - \frac{\alpha_n}{2}\right) \frac{1}{n} \sum_{j=1}^{\lfloor n \xi \rfloor} \frac{w_n(j/n)}{1 - \frac{j-1}{n}} + \frac{(\lfloor n \xi \rfloor - 1) \xi}{(n - 1)\xi} \alpha_n \xi.
\]
For \(\frac{j-1}{n} \leq t \leq \frac{j}{n}\)
\[
\frac{1}{1-t} - \frac{1}{n (1-t) t^2} \leq \frac{1}{1-\frac{j-1}{n}} \leq \frac{1}{1-t},
\]
(52)
and therefore
\[
\int_0^\xi \frac{w_n(t)}{1-t} \, dt - \frac{1}{n} \int_0^\xi \frac{w_n(t)}{(1-t)^2} \, dt \leq \frac{1}{n} \sum_{j=1}^{\lfloor \xi n \rfloor} w_n(j/n) \leq \int_0^\xi \frac{w_n(t)}{1-t} \, dt.
\]

Setting
\[
\tilde{\phi}_n(\xi) := (1 - \alpha_n/2) \int_0^\xi \frac{w_n(t)}{1-t} \, dt + \frac{(\lfloor \xi n \rfloor - 1)_+}{(n-1)^2} \alpha_n \xi \geq \phi_n(\xi),
\]
we have
\[
|\phi_n(\xi) - \tilde{\phi}_n(\xi)| \leq (1 - \alpha_n/2) \frac{1}{n} \int_0^\xi \frac{w_n(t)}{(1-t)^2} \, dt \leq (1 - \alpha_n/2) \frac{C}{n} \frac{1}{1 - \xi}.
\]

Note also that
\[
1 - 2 \frac{1}{\xi n - 2} \leq \frac{(\lfloor \xi n \rfloor - 1)_+}{(n-1)^2} \leq 1,
\]
and therefore, if we assume that \(\alpha_n \to \alpha\),
\[
\phi_n(\xi) \to (1 - \alpha/2) \int_0^\xi \frac{w(t)}{1-t} \, dt + \alpha \xi =: \phi(\xi)
\]
when \(n \to \infty\).

We then have from (53), (55) and (56) for all \(\xi\),
\[
|\phi(\xi) - \phi_n(\xi)| \leq |\phi(\xi) - \tilde{\phi}_n(\xi)| + |\tilde{\phi}_n(\xi) - \phi_n(\xi)|
\]
\[
\leq (1 - \alpha/2) \int_0^\xi \frac{|w(t) - w_n(t)|}{1-t} \, dt + \left(1 - \frac{(\lfloor \xi n \rfloor - 1)_+}{(n-1)^2}\right) \alpha
\]
\[
+ \tilde{\phi}_n(\xi) - \phi_n(\xi).
\]

Now define \(\nu_n\) to be the probability measure on \(\mathbb{R}_+\) that is the push-forward of the uniform probability measure on \([0,1]\) under \(\phi_n\), and let \(\nu\) be determined by \(\phi\) in the same way.

**Lemma 3.8.** We have
\[
\lim_{n \to \infty} \int_0^1 |\phi_n(\xi) - \phi(\xi)| \, d\xi = 0.
\]

And
\[
\lim_{n \to \infty} W_1(\nu_n, \nu) = 0.
\]

**Proof of Lemma 3.8.** Let \(\chi \in \text{Lip}_1\). Then
\[
\left| \int_0^1 \chi(\phi_n(\xi)) \, d\xi - \int_0^1 \chi(\phi(\xi)) \, d\xi \right| \leq \int_0^1 |\phi_n(\xi) - \phi(\xi)| \, d\xi.
\]

It remains to show (59). We estimate each of the terms coming from (58).

Suppose first that \(\alpha_n = \alpha\) for all \(n\). To bound the integral of the first term on the right in (58), change the order of integration:
\[
\int_0^1 \left( \int_0^\xi \frac{|w(t) - w_n(t)|}{1-t} \, dt \right) \, d\xi = \int_0^1 \left( \int_0^1 \frac{|w(t) - w_n(t)|}{1-t} \, d\xi \right) \, dt
\]
\[
= \int_0^1 |w(t) - w_n(t)| \, dt,
\]
and by Lemma 3.4, the right side vanishes in the limit \(n \to \infty\). Making the obvious addition and subtraction argument, we see that the same conclusion holds under the assumption that \(\lim_{n \to \infty} \alpha_n = \alpha\).
Next, to estimate $\int_0^1 \left(1 - \frac{[(\xi n)-1]}{(n-1)\xi^n}ight)\,d\xi$ we break the integral up into two pieces, and use (56) away from $\xi = 0$:

$$
\int_0^1 \left(1 - \frac{[(\xi n)-1]}{(n-1)\xi^n}\right)\,d\xi \leq \int_0^{1/n} 1\,d\xi + \int_{1/n}^1 \frac{1}{\xi} - \frac{2}{n-2}\,d\xi = \frac{1}{n} + \frac{2\log n}{n-2},
$$

and this too vanishes in the limit $n \to \infty$.

Finally, to estimate $\int_0^1 (\tilde{\phi}_n(\xi) - \phi_n(\xi))\,d\xi$, we break the integral up into two pieces, but at the other end, and use (55):

$$
\int_0^{1-1/n} (\tilde{\phi}_n(\xi) - \phi_n(\xi))\,d\xi \leq (1 - \alpha/2) C n \int_0^{1-1/n} \frac{1}{1 - \xi} - \frac{1}{\xi} \,d\xi = (1 - \alpha/2) C \frac{\log n}{n}
$$

while

$$
\int_{1-1/n}^1 (\tilde{\phi}_n(\xi) - \phi_n(\xi))\,d\xi \leq \int_{1-1/n}^1 \tilde{\phi}_n(\xi)\,d\xi \leq (1 - \alpha/2) \int_{1-1/n}^1 \log(1 - \xi)\,d\xi + \frac{1}{n} \alpha
$$

$$
= (1 - \alpha/2) \frac{\alpha + 1 + \log n}{n}.
$$

To pass to the general case, let $\tilde{\phi}$ denote the function $\phi$ with $\alpha$ replaced by some $\alpha_n \in [0, 2]$. Then it is easy to see that $\int_0^{1} |\phi - \tilde{\phi}|\,d\xi \leq \frac{2}{\alpha} |\alpha - \alpha_n|$. Now one more application of the triangle inequality yields (59) in general.

**Lemma 3.9.** $\lim_{n \to \infty} P\{W_1(\rho_n, \nu_n) > \delta\} = 0$. Recall here that $\rho_n$ is the pushforward of the uniform measure on $[0, 1]$ by the map $\psi_n$ from eq. (47).

**Proof.** Take $\chi \in \text{Lip}_1$ and estimate

$$
\left| \int_0^1 \chi(\psi_n(\xi))\,d\xi - \int_0^1 \chi(\phi_n(\xi))\,d\xi \right| \leq \int_0^1 |\chi(\psi_n(\xi)) - \chi(\phi_n(\xi))|\,d\xi,
$$

uniformly in $\chi$, and hence $W_1(\rho_n, \nu_n) \leq \int_0^1 |\psi_n(\xi) - \phi_n(\xi)|\,d\xi$. By Markov’s inequality, for any $\delta > 0$,

$$
P\{W_1(\rho_n, \nu_n) > \delta\} \leq \frac{1}{\delta} E \left( \int_0^1 |\psi_n(\xi) - \phi_n(\xi)|\,d\xi \right).
$$

Now note that $\int_0^1 \psi_n(\xi)\,d\xi = \frac{1}{n} \sum_{j=0}^{n-1} x_j = 1 - \frac{1}{n} x_n$, and likewise

$$
\int_0^1 \phi_n(\xi)\,d\xi = \frac{1}{n} \sum_{j=0}^{n-1} E[x_j] = 1 - \frac{1}{n} E[x_n].
$$

Therefore,

$$
\left| \int_0^1 \psi_n(\xi)\,d\xi - \int_0^1 \phi_n(\xi)\,d\xi \right| \leq \frac{1}{n} |x_n - E[x_n]|.
$$

Then by Lemma 3.3,

$$
\int_0^1 |\psi_n(\xi) - \phi_n(\xi)|\,d\xi \leq 2 \int_0^\xi |\psi_n(\xi) - \phi_n(\xi)|\,d\xi + 2 \int_{\xi}^1 \phi_n(\xi)\,d\xi + \frac{1}{n} |x_n - E[x_n]|.
$$

Next, by Lemma 3.8

$$
\limsup_{n \to \infty} \int_0^1 \phi_n(\xi)\,d\xi \leq \int_0^\xi \phi(\xi)\,d\xi + \lim_{n \to \infty} \int_0^1 |\phi_n(\xi) - \phi(\xi)|\,d\xi = \int_0^1 \phi(\xi)\,d\xi,
$$

and the proof is complete.
and \( \frac{1}{n} \mathbb{E}[x_n - \mathbb{E}[x_n]] \leq \frac{2}{n} \mathbb{E}[x_n] \) which tends to zero by Lemma 3.7. Therefore,

\[
\limsup_{n \to \infty} \mathbb{E} \left[ \int_0^1 |\psi_n(\xi) - \phi_n(\xi)| \, d\xi \right] \leq 2 \limsup_{n \to \infty} \mathbb{E} \left[ \int_{\xi_*}^\xi |\psi_n(\xi) - \phi_n(\xi)| \, d\xi \right] + 2 \int_{\xi_*}^\xi \phi(\xi) \, d\xi.
\]

We next show that the first term on the right is zero. Pick \( \epsilon > 0 \). Then by (35), there is a constant \( C \) depending only on \( \xi_* \) such that for some \( n_\epsilon \)

\[
\mathbb{E}[\tilde{z}_i] \leq \frac{C}{n} \quad \text{and} \quad \mathbb{E}[(\tilde{z}_j, \tilde{z}_k)] \leq \frac{C}{n^2} \epsilon \quad \text{for all} \quad n > n_\epsilon, \ j, k < n\xi_* .
\]

We then have from Eq. (47 and (51) that for all \( \xi < \xi_* \)

\[
\mathbb{E}[\tilde{z}_j] = \mathbb{E}[(\psi_n(\xi) - \phi_n(\xi))]^2 \leq \mathbb{E} \left[ \left( 1 - \frac{\alpha_n}{2} \right)^2 \sum_{j=1}^{[n\xi]} \tilde{z}_j - \mathbb{E} \left[ \tilde{z}_j \right] \right] ^2 + \sum_{j, k=1}^{[n\xi]} \frac{C\epsilon}{n} \left( 1 - \frac{j-1}{n} \right) \left( 1 - \frac{k-1}{n} \right).
\]

Using the bounds on \( \mathbb{E}[\tilde{z}_j] \) and \( \mathbb{E}[(\tilde{z}_j, \tilde{z}_k)] \) from Eq. (35), for all sufficiently large \( n \),

\[
\mathbb{E}[\tilde{z}_j] < \left( 1 - \frac{\alpha_n}{2} \right)^2 \sum_{j=1}^{[n\xi]} \frac{C\epsilon}{n} \left( 1 - \frac{j-1}{n} \right) ^2 \left( \sum_{j, k=1}^{[n\xi]} \frac{C\epsilon}{n} \left( 1 - \frac{j-1}{n} \right) \left( 1 - \frac{k-1}{n} \right) \right).
\]

The first of the terms in the parentheses is smaller than

\[
\epsilon C \int_0^\xi \frac{1}{(1 - \xi) ^2} \, d\xi = \epsilon C \frac{1}{1 - \xi},
\]

and the second is smaller than

\[
\epsilon C \left( \frac{1}{n} \sum_{k=1}^{[n\xi]} \frac{1}{1 - \frac{k-1}{n}} \right) ^2 \leq \epsilon C \left( \log(1 - \xi) ^2 \right),
\]

where, as above, we have used (53) and its analog for \( (1 - \xi) ^{-2} \). It follows that for all \( \xi < \xi_* \),

\[
\mathbb{E}[\tilde{z}_j] = \epsilon C ((1 - \xi) ^{-1} + (\log(1 - \xi) ^2) \right).
\]

Therefore, for all \( n > n_\epsilon \),

\[
\limsup_{n \to \infty} \mathbb{E} \left[ \int_0^1 |\psi_n(\xi) - \phi_n(\xi)| \, d\xi \right] \leq \epsilon C ((1 - \xi_* ^{-1} + (\log(1 - \xi_*) ^2))^{1/2}.
\]
Since $\epsilon > 0$ is arbitrary, this proves that
\[
\limsup_{n \to \infty} \mathbb{E} \left[ \int_0^1 |\psi_n(\xi) - \phi_n(\xi)| \, d\xi \right] \leq \int_0^1 \phi(\xi) \, d\xi
\]
for all $0 < \xi < 1$. However, $\phi$ is integrable, we can choose $\xi_*$ to make this arbitrarily small. Thus,
\[
\lim_{n \to \infty} \mathbb{E} \left[ \int_0^1 |\psi_n(\xi) - \phi_n(\xi)| \, d\xi \right] = 0.
\]

The main assertion now follows from (61).

**Proof of Theorem 3.5.** By the triangle inequality,
\[
W_1(\nu, \rho_n) \leq W_1(\nu, \nu_n) + W_1(\nu_n, \rho_n) + W_1(\rho_n, \nu_n).
\]

Now applying Lemma 3.7, Lemma 3.8 and Lemma 3.9 yields the result yields (38). Then
\[
\lim_{n \to \infty} \mu_n(\{0, x\}) = \lim_{n \to \infty} \int_0^1 1_{\{0, x\}}(\psi_n(\xi)) \, d\xi = \lim_{n \to \infty} \int_0^1 1_{\{0, x\}}(\phi(\xi)) \, d\xi = \phi^{-1}(x),
\]
the cumulative distribution function of the limiting empirical measure is $\phi^{-1}(x)$ and hence limiting empirical measure has the density
\[
g(x) = \frac{d}{dx} \phi^{-1}(x) = \frac{1}{\phi'(\phi^{-1}(x))}.
\]

3.2. **Detailed chaoticity of the equilibrium sequence.** As our first application of Theorem 3.5, we identify the limiting equilibrium density $f_\alpha$, and prove the detailed $(\alpha, f_\alpha)$-chaoticity of the equilibrium sequence:

**Proof of Theorem 2.1.** By Lemma 2.2, the sequence of uniform probability measures on $S_{n, T}$ are obtained by averaging over permutations the push-forwards under the map $T_n$ described there of the flat Dirichlet measure on the standard simplices of the same dimension. In this case we have at fixed $n$ that $w_j = 1/n$ for all $j$, and $w(t) = 1$. By Remark 3.2, the sequence of flat Dirichlet measures on the standard simplices is $w$-chaotic in the sense of Definition 3.1 for $w = 1$. For $w(t) = 1$ for all $t$, $\phi(\xi)$ is given by (19), and this identifies the limiting density $f_\alpha$.

We next show that detailed $(\alpha, f_\alpha)$ chaoticity holds for the sequence. The gap length $\zeta_{x,n} = x_{(j+1),n} - x_{j,n} = \alpha/(n-1)$ satisfies
\[
\frac{n - 1}{\alpha} \zeta_{x,n} = \frac{2 - \alpha (n-1) \tilde{z}_j}{2^a - 1 - \frac{z}{n}} \quad \text{where} \quad j = \lfloor n \phi^{-1}(x) \rfloor.
\]
Each $\tilde{z}_j$ has a Beta$(1, n-1)$ distribution, and hence the probability density for $(n-1) \tilde{z}_j$ is $\frac{1}{n-1} (1 - z/(n-1))^{n-2}$ which converges to $e^{-z}$ as $n \to \infty$. Therefore
\[
\lim_{n \to \infty} \mathbb{P}[(n-1)\zeta_{x,n}/\alpha > r] \to e^{-\frac{2n(1-\phi^{-1}(x))}{2^n}}.
\]
and by (44), this is equivalent to (16). with $g = f_\alpha$.

The rate information is easily extracted from the Lemmas since all but Lemma 3.8 give rates. However, one can extract a rate from the proof of Lemma 3.8 by considering the last two displayed inequalities in the proof. The details are left to the reader. □
3.3. Chaotic sequences by non-flat Dirichlet measures. The construction just provided leads to other chaotic sequences of probability measures on $S_{n,\epsilon_n}$. Instead of pushing forward the flat Dirichlet measure on $S_1$, we can push forward more general Dirichlet distributions, and as we show in this section this leads to the construction of $(\alpha, g)$-chaotic sequences for all probability densities $g$ on $\mathbb{R}_+$ that satisfy (14) and (15). However, except in the case of the flat Dirichlet measures, these sequences will not satisfy the detailed chaoticity.

Let $w = (w_1, \ldots, w_n)$ be a probability measure on $\{1, \ldots, n\}$. That is, $w_j \geq 0$ for all $j$ and $\sum_{j=1}^n w_j = 1$. Then

$$h_w(z_1, \ldots, z_n) = \left(\prod_{j=1}^n \Gamma(nw_j)\right)^{-1} \frac{1}{\Gamma(n)} \prod_{j=1}^n z_j^{nw_j - 1}. \tag{69}$$

is the density for a Dirichlet distribution on $S_1$ with concentration parameters $w$. Random variables $(z_1, \ldots, z_n)$ with this distribution satisfy

$$\mathbb{E}[z_j] = w_j, \quad \text{Var}[z_j] = \frac{w_j(1 - w_j)}{n + 1},$$

$$\text{Cov}[z_i, z_j] = -\frac{w_i w_j}{n + 1}. \tag{70}$$

Now fix $\alpha \in [0, 2]$, and a sequence $\alpha_n \subset [0, 2]$ with $\alpha_n \to \alpha$. Let $\tau_n$ denote the Dirichlet distribution on the standard simplex in $n$ dimensions with the distribution given by (69) and this choice of $(w_1, \ldots, w_n)$. Then (37) holds on account of the continuity of $w$, and (35) holds with $C = \max_{\xi \in [0,1]} \{w(\xi)\}$ which is finite by the continuity of $w$. Thus $\{\tau_n\}$ is a $w$-pre-chaotic sequence.

Now fix $\alpha \in [0, 2]$, and a sequence $\alpha_n \subset [0, 2]$ with $\alpha_n \to \alpha$. Let $\tau_n$ denote the probability measure on $S_{n,\epsilon_n}$ obtained by pushing forward $\tau_n$ under the map specified in Lemma 2.2 at the value $\alpha_n$, and then averaging over permutations. By Theorem 3.5, $\{\tilde{\tau}_n\}$ is $(\alpha, g)$ chaotic where

$$g(x) = \frac{1}{\phi'(\phi^{-1}(x))} \quad \text{and} \quad \phi(\xi) = \left(1 - \alpha/2\right) \int_0^\xi \frac{w(t)}{1 - t} dt + \alpha \xi. \tag{71}$$

Provided $\int_0^1 (1 - t)^{-1} w(t) dt = \infty$, $\phi$ increases strictly from 0 at $\xi = 0$ to $\infty$ at $\xi = 1$, and in fact, $\phi'(\xi) \geq \alpha$ for all $\xi$.

As before, let $\nu_n$ denote the push-forward of the uniform probability measure on $[0, 1]$ under $\phi_n$, and let $\nu$ be defined in the same way in terms of $\phi$, so that $\nu = g(\xi) d\xi$ where $g(\xi) = 1/\phi'(\phi^{-1}(\xi))$. Thus we have:

**THEOREM 3.10.** Let $w$ be a continuous probability density on $[0, 1]$. For each $n$ and each $1 \leq j \leq n$, define $w_j = \int_{\alpha_j/n}^{\alpha_{j-n}/n} w(\xi) d\xi$. Let the energies $E_n$ be chosen so that $\alpha_n \to \alpha$. Equip $S_{n,\epsilon_n}$ with the probability measure that is the push-forward under $\tau_n$ (See Lemma 2.2) of the Dirichlet measure specified in (69) using these weights, averaged under permutations. Then this sequence is $(\alpha, g)$-chaotic where $g$ is given by (71).

However, the chaotic sequences obtained in this manner do not satisfy detailed chaos. In this construction, for each $n$ and $x_j(z)$, each $\tilde{x}_j(z)$ has a Beta$(nw_j, n - nw_j)$ distribution, and hence the probability density for $(n - 1)\tilde{x}_j(z)$ converges to a non-exponential Gamma distribution unless $w_j \to 1/n$. To obtain sequence that satisfies detailed chaos, we must push forward a different class of pre-chaotic measures on the standard simplices. In the next subsection, we describe one way of doing this.
Remark 3.11. If the concentration parameters in the Dirichlet distribution are multiplied by a common factor $K$, and hence eq. (69) is replaced by

$$h^K_w(\tilde{z}_1, \ldots, \tilde{z}_n) = \left( \prod_{j=1}^n \Gamma(Kn w_j) \right)^{-1} \Gamma(Kn) \prod_{j=1}^n \tilde{z}_j^{Kn w_j - 1},$$

we still have a Dirichlet distribution with the same expected values but with the variance covariance multiplied by the factor $1/K$. Equation (70) becomes

$$E[\tilde{z}_j] = w_j,$$

$$\text{Var}[\tilde{z}_i] = \frac{w_j(1 - w_j)}{K(n + 1)},$$

$$\text{Cov}[\tilde{z}_i, \tilde{z}_j] = -\frac{w_i w_j}{K(n + 1)}.$$

Therefore all estimates leading to the proof of Theorem 3.10 are still valid, and therefore these measures are $(\alpha, g)$-chaotic as well. But increasing $K$ implies that the random variables $\tilde{z}_j$ become more concentrated around their mean $w_j$. And while this does not change the limiting density $g$, it changes the gap distribution for all finite $n$, and we will see in Section 4 that this is a fundamental difference for the limiting dynamics of the particle system.

3.4. Detailed chaoticity via order statistics. The construction that we now give uses another probability density $h(\eta)$ on $[0, 1]$ that has to do with the excess energy distribution.

The distribution of the random points $x_j \in \mathbb{R}_+$ is determined by the distribution of empty intervals $[0, x_1], [x_1 + \epsilon, x_2]$, or equivalently, as we have seen in Lemma 2.2, by the random variables $z_j$ in (26). These specify a random partition

$$\{[0, a_1], (a_1, a_2], \ldots, (a_{n-1}, 1]\}$$

of $[0, 1]$ into $n$ parts with $a_j - a_{j-1} = \tilde{z}_j$ and $a_0 = 0$. This random partition is closely related to a partition of the excess energy. Recall that the fraction of the total energy that is excess energy is $(1 - \alpha/2)$. Given a probability density $g(x)$ on $\mathbb{R}_+$ that satisfies (41), $(1 - \alpha g(x))$ represents probability that the interval $[x, x + dx]$ is unoccupied. Opening up a gap in $[x, x + dx]$ would raise the energy of all the particles with energy higher than $x$ by $dx$. Thus this would make a contribution of

$$(1 - \alpha g(x))(1 - G(x))$$

to the total excess energy, where $G$ is defined as in Theorem 3.6. Therefore, the fraction of the excess energy that can be ascribed to gaps in $[x, x + dx]$ is

$$h(x)dx = \frac{1}{1 - \alpha/2} (1 - \alpha g(x))(1 - G(x))dx.$$

One readily checks that $h(x)$ is indeed a probability density. Let $H(x)$ denote its cumulative distribution function. Out of $G$ and $H$ we define two maps from $[0, 1]$ to $[0, 1]$, namely $G \circ H^{-1}$ and $H \circ G^{-1}$. We may use these two maps to push forward the uniform distribution on $[0, 1]$ onto $[0, 1]$ itself, producing two new probability measures on $[0, 1]$.

Define

$$\psi(\eta) = \frac{g(H^{-1}(\eta))}{h(H^{-1}(\eta))}.$$  

and note that the cumulative distribution function of $\psi$ is $\Psi(\eta) = G(H^{-1}(\eta))$. Likewise define

$$w(\xi) = \frac{h(G^{-1}(\xi))}{g(G^{-1}(\xi))}.$$
and note that the cumulative distribution function of \( w \) is \( H(G^{-1}(\eta)) \). Also, note that

\[
(77) \quad w(\xi) = \frac{1}{\psi(\Psi^{-1}(\eta))}.
\]

From (75), and the definition of \( h \) in terms of \( g \),

\[
(78) \quad \psi(\eta) = \frac{1}{1 - \alpha/2} \left( \frac{1}{g(G^{-1}(\Psi(\eta)))} - \alpha \right) (1 - \Psi(\eta)) .
\]

When there is an \( \epsilon > 0 \) such that \( g(x) \leq (\alpha + \epsilon)^{-1} \),

\[
(79) \quad \psi(\eta) \geq \frac{\epsilon}{1 - \alpha/2} (1 - \Psi(\eta)) ,
\]

and this provides a lower bound on \( \psi \) on any interval \([0, \eta_*]\), for any \( \eta_* < 1 \).

Likewise, we may recover \( w(\xi) \) as in equation (42) from the formula for \( h \). When there is an \( \epsilon > 0 \) so that \( 1/g(x) \geq \alpha + \epsilon \) for all \( x \),

\[
w(x) \geq \epsilon(1 - \xi),
\]

which provides a uniform lower bound on \( w(\xi) \) on \([0, \xi_*] \) for any \( \xi_* < 1 \). We see that \( w \) is the probability density on \([0, 1]\) associated to \( g \) through Theorem 3.6.

Let \( \Phi_j, j = 1, 2 \) be the cumulative distribution functions of two strictly positive probability densities \( \phi_j, j = 1, 2 \) respectively, on intervals \([a_j, b_j]\), \( j = 1, 2 \). Then \( \Phi_1^{-1} \circ \Phi_2 : [a_2, b_2] \to [a_1, b_1] \), and as is readily checked, \( \Phi_1^{-1} \circ \Phi_2 \) pushes \( \phi_2(x)dx \) onto \( \phi_1(x)dx \). In particular, if \( \phi_2(x)dx \) is the uniform distribution on \([0, 1]\); i.e.; \( \Phi_2(x) = x \) for all \( x \in [0, 1]\), \( \Phi_1^{-1} \) pushes forward the uniform distribution on \([0, 1]\) onto \( \phi_1(x)dx \) on \([a_1, b_1]\). That is, for all continuous functions \( \chi \) on \([a_1, b_1]\),

\[
\int_{a_1}^{b_1} \chi(x)\phi_1(x)dx = \int_{0}^{1} \chi(\Phi_1^{-1}(y))dy.
\]

In particular if \( \xi \) is a random variable that is uniformly distributed on \([0, 1]\), \( \Phi_1^{-1}(\xi) \) is a random variable with the law \( \phi_1(x)dx \). Therefore, if \( \xi_1, \ldots, \xi_{n-1} \) are the order statistics of \( n - 1 \) i.i.d uniformly distributed random variables, \( \Phi_1^{-1}(\xi_1), \ldots, \Phi_1^{-1}(\xi_{n-1}) \) are the order statistics of \( n - 1 \) independent samples from the law \( \phi_1(1)dx \).

**Lemma 3.12.** Let \( F(\eta) \) be the cumulative distribution function of a continuous probability density \( f(\eta) \) on \([0, 1]\) that is uniformly positive on \([0, \eta_*]\) for all \( 0 < \eta_* < 1 \); i.e., for some \( \alpha > 0 \), \( f(\eta) \geq \alpha \) for \( 0 \leq \eta \leq \eta_* \). Let \( \xi_1, \ldots, \xi_{n-1} \) be the order statistics of \( n - 1 \) i.i.d uniformly distributed random variables, \( \Phi_1^{-1}(\xi_1), \ldots, \Phi_1^{-1}(\xi_{n-1}) \) are the order statistics of \( n - 1 \) independent samples from the law \( \phi_1(x)dx \).

Moreover let \( \eta_j = F^{-1}(\xi_j) \) and set \( \tilde{\eta}_j = \eta_j - \eta_{j-1} \). Then for \( \eta_j, \eta_k \leq \eta_* \),

\[
(80) \quad \mathbb{E}[\tilde{\eta}_j] = \mathbb{E}\left[ \frac{\lambda_j}{f(\eta_j)} \right] + \frac{1}{n} r_1 ,
\]

\[
(81) \quad \text{Var}[\tilde{\eta}_j] = \mathbb{E}\left[ \left( \frac{\lambda_j}{f(\eta_j)} \right)^2 \right] + \frac{1}{n^2} r_2 ,
\]

\[
(82) \quad \text{Cov}[\tilde{\eta}_j, \tilde{\eta}_k] = \frac{1}{n^2} r_3 ,
\]

where \( r_1, r_2, \) and \( r_3 \) converge to zero as \( n \to \infty \) with a rate depending on the modulus of continuity of \( f \) and the lower bound on \( f \) on \([0, \eta_*]\). If \( f \) is Lipschitz continuous and uniformly bounded below on all of \([0, 1]\), then \( |r_i| < C/n \), with \( C \) depending on \( f \).

**Proof.** Related results for order statistics can be found e.g. in [1]. The present result is precisely adapted for our situation. We know that \( \lambda_j = \xi_j - \xi_{j-1} \) are distributed as a flat Dirichlet distribution, and hence that \( \mathbb{E}[\lambda_i] = 1/n \), and that \( \text{Var}[\lambda_j] = 1/n^2 + O(1/n^3) \) and \( \text{Cov}[\lambda_j, \lambda_k] = -1/n^3 + O(1/n^4) \).
First, since \( (F^{-1})'(t) = \frac{1}{f(F^{-1}(t))} \),

\[
\tilde{z}_j = \int_{\xi_{j-1}}^{\xi_j} \frac{1}{f(F^{-1}(s))} ds = \lambda_j \frac{1}{f(\eta_j)} + u_j ,
\]
where

\[
u_j := \int_{\xi_{j-1}}^{\xi_j} \left( \frac{1}{f(F^{-1}(s))} - \frac{1}{f(F^{-1}(\xi_j))} \right) ds .
\]

If \( f(s) \) is Lipschitz continuous and bounded below by \( a > 0 \) for \( F^{-1}(\xi_j) \leq \eta_j \), then

\[
\left| \frac{1}{f(F^{-1}(s))} - \frac{1}{f(F^{-1}(\xi_j))} \right| \leq \frac{\lambda_j^2}{a^2} .
\]

Therefore \( E[\nu_j] \leq C E[\lambda_j^2] \leq \frac{1}{a^2 \eta_j^2} \). Otherwise, if \( f \) is only continuous, there is a function \( \omega(\delta) > 0 \), the modulus of continuity, with \( \lim_{\delta \to 0} \omega(\delta) = 0 \) such that

\[
\sup_{|t_1 - t_2| < \delta} |f(t_1) - f(t_2)| \leq \omega(\delta) .
\]

Then, for \( \delta \) small enough, and \( \eta_j \leq \eta_* \) and \( f \geq a > 0 \) on \([0, \eta_*] \),

\[
E[|\nu_j|] = E[|u_j|1_{\lambda_j < \delta}] + E[|u_j|1_{\lambda_j \geq \delta}] \leq E[\lambda_j \omega(\delta)1_{\lambda_j < \delta}] + \frac{1}{a}P[\lambda_j > \delta] \leq \frac{1}{n} \omega(\delta) + \frac{1}{a} \frac{E[\lambda_j^2]}{\delta^2} \leq \frac{1}{n} \omega(\delta) + \frac{1}{a} \frac{1}{n^2 \delta^2} .
\]

Choosing \( \delta = n^{-1/3} \), we find,

\[
r_1 \leq \frac{1}{n} \left( \omega(n^{-1/3}) + \frac{1}{a} n^{-1/3} \right) .
\]

The proof of (81), which we omit, is very similar. To estimate the covariance we write

\[
\text{Cov}[\tilde{z}_j, \tilde{z}_k] = E[\tilde{z}_j \tilde{z}_k] - E[\tilde{z}_j]E[\tilde{z}_k] ,
\]
and

\[
\tilde{z}_j \tilde{z}_k = \int_{\xi_{j-1}}^{\xi_j} \int_{\xi_{k-1}}^{\xi_k} \frac{1}{f(F^{-1}(s))} \frac{1}{f(F^{-1}(t))} ds dt = \lambda_j \lambda_k \frac{1}{f(\eta_j)f(\eta_k)} \int_{\xi_{j-1}}^{\xi_j} \left( \frac{1}{f(F^{-1}(s))} - \frac{1}{f(\eta_j)} \right) ds \int_{\xi_{k-1}}^{\xi_k} \frac{1}{f(F^{-1}(t))} dt + \lambda_j \frac{1}{f(\eta_j)} \int_{\xi_{j-1}}^{\xi_j} \left( \frac{1}{f(F^{-1}(s))} - \frac{1}{f(\eta_j)} \right) ds \int_{\xi_{k-1}}^{\xi_k} \frac{1}{f(F^{-1}(t))} dt + \lambda_j \frac{1}{f(\eta_j)} \int_{\xi_{k-1}}^{\xi_k} \left( \frac{1}{f(F^{-1}(t))} - \frac{1}{f(\eta_k)} \right) dt .
\]

For the first term we note that \( E[\lambda_j \lambda_k] = n^{-2} + \text{Cov}[\lambda_j, \lambda_k] \), and by estimates like the ones used to estimate \( r_1 \), we find that the remaining terms are \( o(n^{-2}) \), or even \( O(n^{-2}) \) if \( f(s) \) is Lipschitz. Hence computing the covariance in (85) by taking the expectation of (86) and using (80) yields (82).

For the following theorem, recall the constraints for \( g \) as stated in equation (41), and the definition of \( h \) in (74), and that \( G \) and \( H \) are their cumulative distribution functions.
THEOREM 3.13. Let $\psi(\eta)$ be defined by (75), and let $\eta_j$, $j = 1, \ldots, n - 1$ be the order statistics of $n - 1$ i.i.d. random variables with distribution $\psi(\eta)d\eta$, and let $\eta_0 = 0, \eta_n = 1$. With $\tilde{z}_i = \eta_i - \eta_{i-1}$, $i = 1, \ldots, n$ this induces a measure on the standard $n - 1$ dimensional simplex $S_\alpha$, whose push-forward to $S_{\alpha,g}$ is $(\alpha, g)$-chaotic. Let $x > 0$, and let $|x_{(j)}, x_{(j+1)}|$ be the random interval that contains $x$. If the density $\psi(\eta)$ is continuous, then the gap length $z_{x,n} = x_{(j+1)} - x_{(j)} - \alpha/(n - 1)$ satisfies

\begin{equation}
\lim_{n \to \infty} P[|n - 1)z_{x,n}/\alpha > r] \to e^{-\alpha w(x)/r}.
\end{equation}

Proof. By Lemma 3.12 applied with $f = \psi$, together with (77), which implies that

\begin{equation}
\frac{1}{\psi(\eta_j)} = w(\xi_j).
\end{equation}

This permits us to rewrite (80), (81) and (82) in terms of $w$, we see that the sequence of laws of $(\tilde{z}_1, \ldots, \tilde{z}_n)$, averaged over permutations, are $w$ pre-chaotic family on $S_1$. Here $w$ is given as in (42). Then by Theorem 3.5, if we fix a sequence of energies $\{E_n\}$ with $\alpha_n = \alpha(n - 1)/E_n \to \alpha$, and define the maps $T_n$ in terms of $\alpha_n$ as in Lemma 22, the sequence of their push forwards onto $S_{\alpha,g}$, averaged over permutations, is $(\alpha, g)$-chaotic.

The statement about the gap distributions then follows from (83) with $f(\eta) = \frac{g(\eta_g^{-1}(\xi_j))}{h(\eta_g^{-1}(\xi_j))}$ so that

\begin{equation}
f(\eta_j) = \frac{g(\eta_g^{-1}(\xi_j))}{h(\eta_g^{-1}(\xi_j))}.
\end{equation}

It follows from (76) that $w(\xi)d\xi$ results from pushing the excess energy distribution forward onto $[0, 1]$ using the distribution function $G$. That is $h(x) = w(G(x))g(x)$. In equilibrium, this excess energy density is uniform; i.e., $w(\xi) = 1$, and the approach to equilibrium for our process can be thought of as the approach of the excess energy distribution to uniform. This is illustrated in Figure 2, which shows the cumulative excess energy for a couple of different densities $g(x)$, and the excess energy per particle as a function of the position $x$ of a particle, $(1 - \alpha g(x))G(x)/g(x)$.

**Figure 2.** The graphs show the particle density $g(x)$ (red), the corresponding excess energy distribution $w(x)$ corresponding to $\alpha = 1$ (blue), and the excess energy per particle, $w(x)/g(x)$ (green). To the left, the density is the equilibrium density $f_\alpha(x)$ as derived in Section 2, and to the right $g(x) = c_1/(1 + (c_2x - 1)^2)(1 + (c_2x - 4)^2)$ with the constants $c_1$ and $c_2$ chosen to make $g(x)$ a probability density with mean 1. We see that the density $f_\alpha(x)$ is equivalent to distributing the excess energy uniformly among the particles.

Lemma 3.12 and Theorem 3.13 provide a means of sampling the empirical distributions $\mu_n$: Let $\xi_{(1)}$, $\ldots, \xi_{(n-1)}$ be the order statistics of $n - 1$ independent samples from the uniform distribution on $[0, 1]$ and then form $\eta_1, \ldots, \eta_{n-1}$ through $\eta_j = \Psi^{-1}(\xi_{(j)})$, which then gives
us the order statistics of \( n - 1 \) independent samples from \( \psi(\eta) \). Then with \( \eta_0 = 0 \) and \( \eta_n = 1 \), we define \( \tilde{z}_j = \eta_j - \eta_{j-1} \), from which we recover a sample of \( (x_1, \ldots, x_n) \). We illustrate this way of sampling the empirical distribution in Figure 3, where \( g \) is as in Figure 2, but with \( \alpha = 1.5 \). Here the density is close to the maximal density \( \frac{2}{3} \), which leads to slow convergence of the empirical measures.

**Remark 3.14.** The construction of the uniform measure on the simplex \( S_1 \) by independent i.i.d uniform random variables on the unit interval is not new here, and can be found for example in [16], where the authors study the asymptotics of partitions of a number \( n \) into a sum of \( m \) integers.

**Remark 3.15.** This construction also illustrates the difference between \((\alpha, g)\)-chaoticity and detailed \((\alpha, g)\)-chaoticity, and why detailed \((\alpha, g)\) chaos is difficult to express in terms of marginal distributions in the way Kac defined chaos (see eq. (4)). Assuming that \( g \) is continuous as before, we have a one to one map between a point \((x_1, \ldots, x_n) \in S_{n,\epsilon}^E\) and points \( \xi_j = \Psi(\eta_j) \) where \( \eta_j = \sum_{i=1}^j \tilde{z}_i \in [0,1] \), with \( \eta_0 = 1 \). After symmetrization the push forward of a measure \( \sigma_{n,\epsilon} \) on \( S_{n,\epsilon}^E \) by this map gives rise to a symmetric measure \( \lambda_{n,\epsilon} \). One could compute the marginal distributions of \( \sigma_{n,\epsilon} \) and \( \lambda_{n,\epsilon} \) and see that if the sequence \{\sigma_{n,\epsilon}\} is \((\alpha, g)\) chaotic, then \{\lambda_{n,\epsilon}\} is chaotic with respect to the uniform measure on \([0,1]\), and the other way around, and the fact that \((\alpha, g)\)-chaoticity of the sequence \{\sigma_{n,\epsilon}\} corresponds to the usual notion of chaos \{\lambda_{n,\epsilon}\} is encoded in the maps \( T_n \). Detailed chaos can be expressed as saying that at the scale \( 1/n \), the points \( x_j \) after symmetrization behave as a Poisson point process: take any point \( \bar{\xi} \in [0,1] \), and \( a, b > 0 \). Then number of points \( \xi_j \in [\bar{\xi} - a/n, \bar{\xi} + b/n] \) will converge to a Poisson distribution with parameter \( b - a \) when \( n \to \infty \). Hence detailed chaos says in a sense that the set of points \( x_1, \ldots, x_n \) in the limit are as random as possible, given the constraint that their laws \( \sigma_{n,\epsilon} \) are \((\alpha, g)\)-chaotic.

**Figure 3.** The graphs show the particle density \( g(x) \) (thin, red), and empirical histograms of samples (blue, thick) as defined in Theorem 3.13. To the left \( n = 50 \), and to the right \( n = 1000 \). The number of independent samples is 5000 and the bin width in the histograms is 0.05.

### 4. The Kac Process

#### 4.1. Specification of the Master Equation.

Kac’s result [8] for the original Kac model is that propagation of chaos as described in the introduction is sufficient to identify an evolution equation for the limiting densities, the Kac-Boltzmann equation. When the jumps are constrained by the exclusion principle the situation is more subtle, and propagation of chaos according to the definition 1.1 is not enough to identify a limiting equation. In this section we will present the Kac-process, and derive a limiting Kac-Boltzmann equation that is valid under the assumption of chaos according to Definition 1.2, with an exponential gap distribution as in Theorem 3.13.

The jump process is then as follows: With \( x = (x_1, \ldots, x_n) \in S_{n,\epsilon} \),
(1) pick a random waiting time \( t \), exponentially distributed with rate \( \alpha \)
(2) pick \( 1 \leq j < k \leq n \) uniformly among possible pairs, and let \( \tilde{x}_{j,k} = \frac{x_j + x_k}{2} \).
(3) Let \( (x_j^*, x_k^*) = (\tilde{x}_{j,k}(1 - \xi), \tilde{x}_{j,k}(1 + \xi)) \), where \( \xi \) is chosen uniformly in the \([-1,1] \).
(4) If \((x_1, \ldots, x_j, \ldots, x_k, \ldots, x_n) \in S_{n,\epsilon,\alpha}\), then let \( x^* = (x_1, \ldots, x_j^*, \ldots, x_k^*, \ldots, x_n) \), else do nothing, i.e. let \( x^* = x \)

Note that the distribution of two particle energies after a collision would be exactly the same if the step (3) were replaced by

(3b) Pick \( \xi \) randomly from \([-1,1]\), and let

\[
\begin{align*}
x_j^* &= x_j + \xi \tilde{x}_{j,k} \pmod{x_j + x_k} \\
x_k^* &= x_k - \xi \tilde{x}_{j,k} \pmod{x_j + x_k}.
\end{align*}
\]

where \( \pmod{x_j + x_k} \) here simply means that if \( x_j + \xi \tilde{x}_{j,k} > x_j + x_k \), then \( (x_j + x_k) \) is added or subtracted to map \( x^*_j \) back into the interval \( 0 \leq x^*_j < (x_j + x_k) \).

This collision process is reversible for any fixed \( \xi \), and can also be naturally generalized to collision models which favor small energy exchanges in the collision, or for “grazing collision limits”, which are interesting in the classical setting.

However, for the purpose of writing down the generator of the process, the version as originally described is simplest. Let \( \mathcal{L} \) denote the generator. Then for any continuous function \( F \) on \( S_{n,\epsilon,\alpha} \)

\[
\mathcal{L} F(x) = \frac{1}{n - 1} \sum_{j<k} \int_{-1}^{1} 1_{S_{n,\epsilon,\alpha}}(x^*_{j,k,\xi})[F(x^*_{j,k,\xi}) - F(x)]d\xi
\]

where \( x = (x_1, \ldots, x_n) \), and \( x^*_{j,k,\xi} = (x_1, \ldots, x_{j-1}, (1 - \xi), \ldots, x_{j+k}(1 + \xi), \ldots, x_n) \). Recall that the process satisfies detailed balance, and is reversible, so that the operator \( \mathcal{L} \) on the \( L^2 \) space given by the invariant measure is self-adjoint. The Kolmogorov forward equation, or what is the same thing, the Master Equation, of the process is then

\[
\frac{\partial}{\partial t} F(x, t) = \mathcal{L} F(x, t).
\]

Let \( \mathcal{P}_t \) denote the semigroup associated to (89), so that if \( F(x, t) \) denotes the solutions with initial data \( F(x, 0) \), \( F(x, t) = \mathcal{P}_t F(x, 0) \).

### 4.2. The exclusion factor.

To compute exactly the probability that the outcome from step (3) results in a jump as defined in (4) is difficult, but it is possible to derive formula for the limit as \( n \to \infty \) under the assumption that the limiting gap distribution is known and that the events that \( x_j^* \) and \( x_k^* \) are admissible positions for particles are independent. We also assume here that the density \( g(x) \) is continuous.

First, to see why propagation of chaos in the sense of Kac is not enough to identify a limiting equation we compare two different chaotic sequences that are \((n, f_n)\)-chaotic, where \( f_n \) is the equilibrium density as found in Theorem 2.1. We take the empirical measures with the \( x_j \) defined as in (22), and \( \alpha_n = \alpha \) for simplicity. On the other hand taking \( \tilde{x}_j = 1/n \), for \( j = 1, \ldots, n \) provides another chaotic sequence. In this latter sequence the gaps between particles are deterministic, \( x_{j+1} - x_j = (1 - \alpha/2)/(n - j) \), and this means that to fit a new particle of size \( \alpha/(n - 1) \) into an interval we must have \( \frac{j}{n} > 1 - \frac{2 - \alpha}{2n} \), which is positive when \( \alpha > 2/3 \), and therefore for all \( x \) smaller than

\[
x_j \geq \left(1 - \frac{\alpha}{2}\right) \sum_{k=1}^{[1 - \frac{j}{n}]} \frac{1}{n + 1 - k} + \frac{3 - \alpha}{2} - \frac{\alpha}{n}.
\]
which converges to
\[ \tilde{x}_\alpha = \log \left( \frac{2\alpha}{2 - \alpha} \right) + \frac{3\alpha - 2}{2}. \]
when \( n \to \infty \). So if \( \alpha > 2/3 \) this \((\alpha, f_\alpha)\)-chaotic sequence does not allow any jump into an interval \([0, \tilde{x}_\alpha] \). On the other hand, for the sequence constructed in Section 2, where the \((\tilde{z}_1, \ldots, \tilde{z}_n)\) taken from the flat Dirichlet distribution, the \( \tilde{z}_j \) are close to being exponentially distributed with mean \( 1/n \). Hence for all \( j \) there is a positive probability that the \( j \)-th gap is bigger than \( \alpha/(n - 1) \), and therefore jumps are possible to any point in the interval \([0, \infty] \), although the probability will be very small in intervals near the origin if \( \alpha \) is large.

In the following calculation we neglect the probability that \( x_j^* \) belongs to one of the gaps created when \( x_j \) and \( x_k \) are lifted out, i.e. when the particles fall back into nearly the same point as where they started. The probability that this happens converges to zero at the order \( 1/n \), and can be neglected unless the excess energy is very small. Also in this case, the effect of such a jump on the density will be very small, and therefore to see an effect of this one would need to consider the process over a very long time scale. It could be interesting to study this situation in a diffusive scaling, and to analyze as certain models for competing particle systems and rank based interacting diffusions \([10, 13, 12]\).

For any \( x \), consider an interval \([x - \delta/2, x + \delta/2]\), where \( \delta \) is assumed to be small and eventually converging to \( 0 \). We will call a point \( x_\ast \) in this interval admissible if it satisfies the exclusion constraint, given the particles that are already present in the interval. The expected number of points \( n \) when \( x \) belongs to this interval will be \( m + 1 = n \int_{x - \delta/2}^{x + \delta/2} g(y) \, dy \sim n \delta \, g(x) \) due to the assumption that \( g \) is continuous. Now let \( x - \delta/2 < x_{(0)} < x_{(2)} < \ldots < x_{(m)} < x + \delta/2 \) be the positions of the \( m \) particles belonging to this interval, renumbered for convenience here, and let \( \zeta_{j+1} = x_{(i)} - x_{(j)} - \alpha/(n - 1) \) be the gaps between particles. For fixed \( \delta \), we have that \( x_{(0)} \to x - \delta/2 \) and \( x_{(m)} \to x + \delta/2 \) in probability, and therefore the error in considering only the interval \([x_{(0)}, x_{(m)}]\) will vanish in the limit as \( n \to \infty \). For a given gap \( \zeta_i \), the interval available for putting a new particle \( x_\ast \) is \( (\zeta_i - \alpha/(n - 1)) ] \). In a jump, \( x_\ast \) is chosen uniformly over any interval \( \zeta_1 \), and therefore

\[ \mathbb{P}[x_\ast \text{ is admissible } \mid x_\ast \in [x_{(0)}, x_{(m)}]] = \frac{\frac{1}{m} \sum_{i=1}^{m} (\zeta_i - \frac{\alpha}{n - 1}) \mathbb{I}_{\zeta_i > \alpha/(n - 1)}}{\frac{1}{m}(x_{(m)} - x_{(0)})} \]

which holds for any particle configuration, if the interval \([x_{(0)}, x_{(m)}]\) does not contain the particles that are selected for collision. The probability that \( x_\ast \) is admissible can now be computed by taking the expectation of the right hand side of equation (90) with respect to the other particles. To continue we make the following assumption:

**Assumption 4.1.** For \( n \to \infty \), one may take \( \delta = \delta_n \to 0 \) such that \( m \to \infty \) in probability, and such that \((n - 1)\zeta_i/\alpha \) are asymptotically i.i.d with a density \( \rho_x \).

This holds for the two constructions of chaotic sequences given in Theorem 3.10 and Theorem 3.13 if the density \( g \) is continuous. By the law of large numbers, the denominator of the righthand side of equation (90) is asymptotically \( E[\zeta_i] + \frac{\alpha}{n - 1} \sim \frac{1}{(n - 1)g(x)} \), and the enumerator is asymptotic to

\[ \frac{\alpha}{n - 1} \mathbb{E} \left[ \frac{n - 1}{\alpha} \zeta_i - 1 \right] = \frac{\alpha}{n - 1} \int_1^\infty (s - 1) \rho_x(s) \, ds, \]

and therefore, for any interval not containing \( x_j \) or \( x_k \) we have

\[ \lim_{\delta \to 0} \lim_{n \to \infty} \mathbb{P}[x_\ast \text{ is admissible } \mid x_\ast \in [x - \delta/2, x + \delta/2]] = \alpha g(x) \int_1^\infty (s - 1) \rho_x(s) \, ds. \]
For example, with the chaotic sequence from Theorem 3.13, \( \rho_x(s) = \frac{\alpha g(x)}{1 - \alpha g(x)} \exp \left( -\frac{\alpha g(x)}{1 - \alpha g(x)} s \right) \), and we find in the limit that
\[
P[x_+ \text{ is admissible}] = \frac{\alpha^2 g(x_+)^2}{1 - \alpha g(x_+)} \int_1^\infty (s - 1) e^{-\frac{\alpha g(x_+)}{1 - \alpha g(x_+)} s} \, ds
\]
\[
= (1 - \alpha g(x_+)) \exp \left( -\frac{\alpha g(x_+)}{1 - \alpha g(x_+)} \right).
\]
Here we recognize the first factor \( 1 - \alpha g(x_+) \) as the exclusion factor in the Uehling-Uhlenbeck equation for discrete energy levels, and the second exponential factor reflects the fact that the continuous spacing of gaps is a less efficient use of the available excess energy.

With the chaotic sequences constructed through the Dirichlet distribution as in Theorem 3.10 the distribution of the gaps are Beta-distributions, as shown in equation (73), which gives the density for the distribution of a gap in the partition of excess energy as
\[
\frac{\Gamma(Kn)}{\Gamma(Kn w_j) \Gamma(Kn(1 - w_j))} z^{Knw_j - 1} (1 - z)^{Kn(1 - w_j) - 1},
\]
and hence, because \( \zeta_j = (1 - \alpha/2) \frac{z_j}{1 - z} \) that \( s = (n - 1)\zeta_j/\alpha \) has density
\[
\rho_{x,n}(s) = cs^{Knw_j} \left( 1 - \frac{s}{\lambda_j} \right)^{Kn(1 - w_j) - 1},
\]
where \( c \) is a normalizing constant and \( \lambda_j = \frac{(n - 1)\alpha}{2 - \alpha} \). Because \( \frac{1}{n} \sim G(x) \) we have asymptotically
\[
nw_j = n \int_{(j-1)/n}^{j/n} w(x) \, dx \sim w(G(x)) = \frac{2\alpha}{2 - \alpha} \frac{1 - \alpha g(x)}{\alpha g(x)} (1 - G(x)) \equiv w_{g(x), G(x)},
\]
and \( Kn/\lambda_n \sim K \frac{2 - \alpha}{2\alpha} (1 - G(x)) \). It follows that
\[
\rho_{x,n}(s) \rightarrow \rho_{g(x), G(x)}(s) = s^{Knw_G(x)G(x) - 1} \exp \left( -\frac{K}{2\alpha} \frac{2 - \alpha}{(1 - G(x))} s \right).
\]
One can now obtain a formula similar to equation (93) corresponding to the density \( \rho_{g(x), G(x)} \).

The notable difference is that with this density the probability that a point \( x_+ \) is admissible asymptotically does not only depend on the limiting density \( g(x) \) but also on the cumulative distribution function \( G(x) \).

Hence, when analyzing the limiting behavior of the Kac process for this \( n \)-particle system, it is important to take the gap distribution into account. We formulate this asymptotic result for the exponential gap distribution as a proposition:

**Proposition 4.2.** Let \( g(x) \) be a continuous probability density on \( [0, \infty) \), and let \( ((x_1, ..., x_\infty))_{n=2}^\infty \) be a chaotic sequence constructed as in Theorem 3.13. There is a sequence \( \delta_n \rightarrow 0 \) such that if \( x_+ \) is chosen uniformly in an interval \([x, x + \delta_n]\), then
\[
\lim_{n \rightarrow \infty} \mathbb{P}[x_+ \text{ is admissible}] = (1 - g(x)) \exp \left( -\frac{\alpha g(x)}{1 - \alpha g(x)} \right)
\]

4.3. **The Boltzmann equation.** For the original Kac process, it is enough to prove propagation of chaos to identify an equation that describes the evolution of a density in the limit of infinitely many particles. Here the situation is more complicated, because the asymptotic gap distribution is important. We conjecture that the process defined here propagates chaos with exponential gap distribution, but we do not have a proof. The conjecture is supported by numerical simulations that are presented in Section 5. Under the assumptions that detailed chaos is propagated, it is then possible to write down the corresponding
kinetic equation, and compare this with the corresponding Kac and Uehling-Uhlenbeck
equations; a formal proof would follow along the same lines as Kac’s original derivation.

**THEOREM 4.3.** Suppose that the evolution specified by (89) propagates chaos with paramater \( \alpha \), and that the asymptotic gap distribution is exponential as in Theorem 3.13. Then the limiting empirical distribution \( g_t \) evolves according

\[
\frac{\partial}{\partial t} g(x, t) = Q[g](x, t),
\]

where

\[
Q[g](x) = \frac{1}{2} \int_0^\infty \int_{-1}^1 \left( g(x')g(y')\Pi(\alpha g(x))\Pi(\alpha g(y)) - g(x)g(y)\Pi(\alpha g(x'))\Pi(\alpha g(y')) \right) d\xi dy,
\]

\[
x' = (1 - \xi)(x + y)/2,
\]

\[
y' = (1 + \xi)(x + y)/2,
\]

and

\[
\Pi(u) = (1 - u) \exp \left( -\frac{u}{1 - u} \right).
\]

The function \( \Pi(u) \) specifies the effects of the exclusion constraint which slows down the
evolution. The function is plotted here in figure 4, together with the function \( \Pi(u) = 1 - u \),
which is the corresponding factor in the Uehling-Uhlenbeck equation. With the car parking
analogy from the introduction, this factor quantifies how much less efficient it is to let cars
park at will along a road compared to using fixed parking slots. In Kac’s original paper
there is no exclusion factor, and in that case \( \Pi(u) \) is constant, equal to 1.

\[\text{FIGURE 4. The exclusion factor as a function of } u \text{ (blue), compared with the fermionic factor } 1 - u\]

Note that \( \alpha^{-1} \) is the maximum density possible, and hence that \( \alpha f(x) = 1 \) implies
that the particles are densely packed near \( x \). The exclusion factor reduces the effective
jump rate much more strongly than the usual factor \( 1 - u \) from the Boltzmann equation
for Fermions, and is a significant difference between the continuous setting that we study
here, and the discrete, quantized models. In fig. 5 we plot the function \( \Pi(\alpha f(x)) \), i.e. the
exclusion factor evaluated at the equilibrium density as a function of the energy \( x \), which
indicates that particles will very seldom get a new energy close to \( x = 0 \), and therefore
that the rate of convergence to equilibrium could be very low.
4.4. Properties of the collision operator. The collision operator $Q[g]$ as defined in equation (100) is amenable to very much the same manipulations as the ordinary collision operator for the Boltzmann equation, except that, in addition to the mass, there is only one conserved quantity, the energy.

THEOREM 4.4. Let $Q[g]$ be defined as in equation (100). Then the following holds:

For any $a, b \in \mathbb{R}$, and any $g(x)$ satisfying $\int_{-\infty}^{\infty} xg(x) \, dx = 1$

\begin{equation}
\int_{0}^{\infty} (a + bx)Q[g](x) \, dx = 0,
\end{equation}

Let $f_\alpha(x)$ defined by equation (18) and (19). Then

\begin{equation}
Q[f_\alpha](x) = 0.
\end{equation}

If $g(x, t)$ is a solution to equation (99), then

\begin{equation}
\frac{d}{dt} \int_{0}^{\infty} g(x, t) \log \left( \frac{\alpha g(x, t)}{1 - \alpha g(x, t)} \right) \, dx \leq 0.
\end{equation}

Proof. Let

\begin{equation}
R(x', y', x, y) = (g(x')g(y')\Pi(\alpha g(x))\Pi(\alpha g(y)) - g(x)g(y)\Pi(\alpha g(x'))\Pi(\alpha g(y'))).
\end{equation}

Here $x'$ and $y'$ depend on a parameter $\xi$ as defined in equation (101). Formally, for any $h(x)$, a change of variables gives

\begin{equation}
\frac{1}{2} \int_{0}^{\infty} \int_{0}^{\infty} \int_{-1}^{1} R(x', y', x, y) h(x) \, d\xi \, dx \, dy =
\end{equation}

\begin{equation}
\int_{0}^{\infty} \frac{1}{u} \int_{0}^{u} \int_{0}^{u} R(z, u - z, v, u - v) h(v) \, dv \, dz \, du.
\end{equation}

We see, just as for the usual Boltzmann equation, that $R(x', y', x, y)$ is symmetric with respect to the changes $(x, y) \rightarrow (y, x)$ and anti symmetric with respect to changing $(x', y', x, y) \rightarrow (x, y, x', y')$, and therefore the righthand side of equation (107) is

\begin{equation}
\frac{1}{4} \int_{0}^{\infty} \frac{1}{u} \int_{0}^{u} \int_{0}^{u} R(z, u - z, v, u - v) \, (h(v) + h(u - v) - h(z) - h(u - z)) \, dv \, dz \, du =
\end{equation}

\begin{equation}
\frac{1}{8} \int_{0}^{\infty} \int_{0}^{\infty} \int_{-1}^{1} R(x', y', x, y) (h(x) + h(y) - h(x') - h(y')) \, d\xi \, dx \, dy,
\end{equation}

Figure 5. The exclusion factor $\Pi(\alpha f_\alpha(x))$ for $\alpha = 1.8$, as a function of $x$. The thin red curve shows the logarithm of the same function, and the equilibrium distribution $f_\alpha(x)$ scaled by a factor $\alpha$ is given in blue.
Hence (114). To prove (104), we write $R(x', y', x, y)$ as
\begin{equation}
\alpha^2 g(x')g(y')g(x)g(y)
- \frac{\Pi(\alpha g(x)) \Pi(\alpha g(y))}{\alpha g(x)}
- \frac{\Pi(\alpha g(x')) \Pi(\alpha g(y'))}{\alpha g(y')}
= \frac{\Pi(\alpha g(x)) \Pi(\alpha g(y))}{\alpha g(x) g(y)}
- \frac{\Pi(\alpha g(x')) \Pi(\alpha g(y'))}{\alpha g(y')}
\end{equation}
Next we take $g(x) = f_\alpha(x)$ and define
\begin{equation}
r(x) = \log \frac{\Pi(\alpha f_\alpha(x))}{\alpha f_\alpha(x)}
= - \log (\alpha f_\alpha(x)) + \log (1 - \alpha f_\alpha(x)) - \frac{\alpha f_\alpha(x)}{1 - \alpha f_\alpha(x)}
\end{equation}
Then
\begin{equation}
r'(x) = -f'_\alpha(x) \left( \frac{1}{f_\alpha(x)} + \frac{\alpha}{1 - \alpha f_\alpha(x)} + \frac{\alpha}{(1 - \alpha f_\alpha(x))^2} \right)
\end{equation}
\begin{equation}
= -f'_\alpha(x) \frac{1}{f_\alpha(x)(1 - \alpha f_\alpha(x))^2}
\end{equation}
On the other hand $f_\alpha(x)$ satisfies
\begin{equation}
f_\alpha(x) = \frac{1}{\phi'(F(x))}
\end{equation}
where $F(x) = \int_0^x f(y) \, dy$ and
\begin{equation}
\phi(\xi) = \left( 1 - \frac{\alpha}{2} \right) \log \frac{1}{1 - \xi} + \alpha \xi.
\end{equation}
Therefore
\begin{equation}
\frac{1}{f_\alpha(x)} = \phi'(F(x)) = \left( 1 - \frac{\alpha}{2} \right) \frac{1}{1 - F(x)} + \alpha
\end{equation}
and
\begin{equation}
f'_\alpha(x) = - \frac{\phi''(F(x))}{\phi'(F(x))^2} f(x) = - \phi''(F(x)) f(x)^3,
\end{equation}
which when inserted into (111) gives
\begin{equation}
r'(x) = \left( 1 - \frac{\alpha}{2} \right) \frac{1}{(1 - F(x))^2} \left( \frac{1}{f_\alpha(x)} - \alpha \right)^2
= \left( 1 - \frac{\alpha}{2} \right)^{-1}
\end{equation}
Hence $r(x)$ is a linear function, and because the parenthesis in equation (109) is
\begin{equation}
\exp(r(x) + r(y)) - \exp(r(x') + r(y'))
\end{equation}
and $x + y = x' + y'$ we see that $R(x', y', x, y)$ vanishes when $g(x) = f_\alpha(x)$. Therefore not only does $Q[f_\alpha(x)]$ vanish, but the whole integrand, which is to say that the collision process satisfies a detailed balance condition also after passing to the limit.

Finally, to prove (105) we write
\begin{equation}
\frac{\partial}{\partial t} \left( g \log \left( \frac{\alpha g}{1 - \alpha g} \right) \right)
= \frac{\partial g}{\partial t} \left( \log \left( \frac{\alpha g}{1 - \alpha g} \right) + \frac{\alpha g}{1 - \alpha g} \right)
= -Q[g(x)]r_g(x),
\end{equation}
where $r_g(x)$ is the expression in (110) with $f_\alpha$ replaced by $g$. Using the expression in (108), we then find
\begin{equation}
\frac{d}{dt} \int_0^\infty g(x, t) \log \left( \frac{\alpha g(x, t)}{1 - \alpha g(x, t)} \right) \, dx
= - \frac{1}{8} \int_0^\infty \int_0^\infty \int_{-1}^1 R(x', y', x, y) \, (r_g(x) + r_g(y) - r_g(y') - r_g(x')) \, d\xi \, dx \, dy
= - \frac{1}{8} \int_0^\infty \int_0^\infty \int_{-1}^1 g(x) g(y) g(x') g(y') \left( e^{r_g(x) + r_g(y)} - e^{r_g(x') + r_g(y')} \right) \times
\left( r_g(x) + r_g(y) - r_g(x') - r_g(y') \right) \, d\xi \, dx \, dy \leq 0,
\end{equation}
which proves (105). Therefore
\[
\int_{0}^{\infty} g(x) \log \left( \frac{\alpha g(x)}{1 - \alpha g(x)} \right) dx
\]
is an entropy for the Boltzmann equation (100).

5. Simulation Results

We present here simulations to illustrate the results presented in the previous sections, and to provide support for the conjecture that the Kac process on $S_{n, \epsilon}$ propagates detailed chaos according to Definition 1.2, and moreover to investigate the long time behavior of solutions of different types.

The sampling of initial data has been done as described by Theorem 3.10 and Theorem 3.13. A very large number of random numbers have been used, and in particular for simulations with a large number of particles, it is necessary to use random numbers with high precision. We have generated random numbers with 64 bits precision, using routines from the GNU Scientific Library [6].

In order to avoid having to compute the distance between the new energy of a particle, $x_j^*$ with the energies of all other particles, which would imply a computational cost of $O(n)$ for each jump, the $x_j$ are kept in an ordered list, which is implemented as minor modification of the AVL-tree as described by Ben Pfaff [11]. In this way the computational cost of one collision grows as $O(\log(n))$.

In the first example the initial distributions are $(\alpha, f_\alpha)$-chaotic, i.e. chosen to converge to the equilibrium distribution with $\alpha = 1$. We compare sampling initial data that are equidistributed (which corresponds to taking samples as in Remark 3.11 with $K \to \infty$ in Equation (72), and a Dirichlet distributed initial data with $\alpha = 0$. These are shown at $t = 0$ (Fig. 6), $t = 0.1$ (Fig. 7), and $t = 10.0$ (Fig. 8), showing that although the initial distributions are equilibrium-chaotic, they are not at equilibrium for this jump process. Fig. 9 shows the gap distribution at $t = 0$ and $t = 10$, and includes also the result when the initial distribution is a true equilibrium for this process, with asymptotically exponentially distributed energy gaps. For these cases, the gap distribution is very close to exponential at $t = 10$, which supports our conjecture that this property is propagated in time. More simulation results can be found in the supplementary material.

FIGURE 6. The graphs show the equilibrium distribution (green) and the result from 5000 independent samples of the empirical distribution with $n = 1000$, counted in bins of width 0.2. The black step function shows the mean outcome, and the blue dots illustrate the distribution of counts in the bins, with the area of the dots proportional to the number of samples with the same count. The exclusion parameter $\alpha = 1.0$. 
Figure 7. The graphs show the outcome of the same simulation as in Figure 6 at time $t = 0.1$. This shows that although the initial data in both cases are equilibrium chaotic, the non-equilibrium state gives quite different behavior of the evolution.

Figure 8. These graphs represent the solutions of the same simulation as in Figure 6 at time $t = 10.0$. Here the two simulations give the same result, a convergence to the true equilibrium.

Figure 9. The graphs show the gap distribution at time $t = 0$ (left) and $t = 2$ (right) for initial data with equal spacing of the excess energy (red) and the Dirichlet 0.01- distribution (blue). At $t = 2$ this is presented in logarithmic scale to show that the distribution becomes exponential as conjectured. The red and blue dots almost overlap here. At $t = 0$ the red curve represents a Dirac measure, and the blue curve shows that with the Dirichlet 0.01- distribution, most gaps are very close to zero, and the excess energy is essentially distributed to a few very large gaps.

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In this supplementary section we present numerical simulations intended to illustrate the results of the paper. The first examples show the stationary distribution for several values of $n$ and the parameter $\alpha$. These were obtained by sampling from the invariant density as described in Section 2. The influence of the boundary $x = 0$ is clearly visible, and extends to an interval $x \leq C\varepsilon$, where $C$ depends on $\alpha$. Not surprisingly, the influence of the boundary is much stronger when $\alpha$ is close to 2, and the density is very high. In this case the distance between two particles is not much larger than the exclusion distance $\varepsilon$. When $\alpha$ is close to 2 the distribution at low energies is essentially discrete, as shown by the very oscillatory behavior of the distribution. This is also illustrated showing the distribution of $x_j$, for $j = 1, 2, 3, 4$ and for some larger numbers. When $\alpha = 1.8$, for example, it is only at $j \sim 50$, that the distribution of $x_j$ and $x_{j+1}$ begin to overlap. But as the width of the interval is proportional to $\varepsilon$ the oscillations of the density $f_n(x)$ for $x$ larger than any fixed value $x_0 > 0$ disappears when $n \to \infty$. Some results illustrating this are presented in Fig. 10 to Fig. 15.

**Figure 10.** The empirical distribution from $4 \times 10^8$ samples from $S_{n,\epsilon n}$ with $n = 10, \alpha = 1.0$ (left). In all these images the red curve shows the limiting distribution when $n \to \infty$. Right: the distribution of the four particles with lowest energy, and with the highest energy taken from $2 \times 10^6$ samples. The unit in the $x$-axis is $\epsilon$, the minimal energy gap between particles.
Figure 11. The empirical distribution from $4 \times 10^8$ samples from $S_{n,\epsilon_n}$ with $n = 10, \alpha = 1.8$ (left). The green curve shows averages of the empirical distribution over intervals of size 0.2. Right: the distribution of the four particles with lowest energy, and with the highest energy taken from $2 \times 10^6$ samples.

Figure 12. The empirical distribution from $10^8$ samples from $S_{n,\epsilon_n}$ with $n = 100, \alpha = 1.0$ (left). Right: the distribution of the four particles with lowest energy, and with the highest energy taken from $2 \times 10^6$ samples.
FIGURE 13. The empirical distribution from $10^8$ samples from $S_{n,\epsilon_n}$ with $n = 100, \alpha = 1.8$ (left). The green curve shows averages of the empirical distribution over intervals of size 0.03. Right: the distribution of the four particles with lowest energy, and with the highest energy taken from $2 \times 10^6$ samples.

FIGURE 14. Left: The empirical distribution from $10^8$ samples from $S_{n,\epsilon_n}$ with $n = 1000, \alpha = 1.0$. Right: the distribution of the four particles with lowest energy, and with the highest energy taken from $2 \times 10^6$ samples.
The following simulation results concern the dynamical process close to equilibrium. These simulations are motivated partially by the fourth question presented in the introduction:

(4) At which energy levels in equilibrium do collisions occur at a rate bounded away from zero, and at which energy levels are the collisions "frozen out".

The first series of figures compare simulations around the stationary state with $\alpha = 1.8$ and with $\alpha = 1.0$. The simulations were carried out with the number of particles $n = 1000$, a number which is sufficiently large to give a good agreement with the limiting distribution, and a reasonable computational time. The initial configuration was sampled from the uniform distribution on $S_{n,\epsilon n}$. The simulation was then run up to time $T_{\text{end}} = 10^6$, and which with $n = 10^3$ implies that the number of (attempted) jumps during the simulation was $10^9$. With $\alpha = 1.8$, the particle distribution is very dense, and therefore the number of successful jumps was only $5.6 \times 10^5$, while with $\alpha = 1.0$ the number of successful jumps was $6.6 \times 10^7$.

To begin, Figure 16 shows a time series of the four particles with lowest energy (left: $\alpha = 1.8$, right: $\alpha = 1.0$), and Figure 17 shows histograms of the particle positions at these times, to be compared with Fig 15 and Fig 14.
Figure 17. Histograms of the 10000 samples of each particle energy, as shown in Fig. 16. These graphs should be compared with Fig. 15 and Fig. 14.

If instead of keeping track of the particles ordered from lowest energy, we follow some tagged particles along the flow, a very different picture is seen. Fig. 18 shows the paths of the particles that initially had the lowest and highest energy, and the corresponding histograms are given in Fig. 19. While in the picture to the right, with $\alpha = 1$, the long time behavior of the two tagged particles are the same, and generate histograms just like the equilibrium density, the pictures to the left, with $\alpha = 1.8$ the particle with lowest initial energy always stays at the bottom, and the particle that initially had the highest energy, while moving around quite a lot certainly does not regenerate the equilibrium density.

Figure 18. Sample paths of the tagged particles with lowest (blue) and highest energy (orange) in the initial configuration, for $\alpha = 1.8$ (left) and $\alpha = 1.0$ (right). To the left, also the path of the second highest energy is plotted (green), and the blue path is multiplied by a factor 1000 for clarity.

Figure 19. Histograms of the sample paths shown in Fig. 18. For $\alpha = 1.0$ (right), the two path histograms correspond very well with the theoretically computed equilibrium density for $n = \infty$ (red curve), whereas for $\alpha = 1.8$, neither of the tagged particles has a histogram which is similar to the equilibrium density.
How this comes about is partially illustrated in Fig. 20 which shows two scatterplots of the points \((x_k, x_k^*)\), i.e., the energy before and after a collision for one of the particles involved in a collision. The graphs contain 200000 points, and clearly show that with \(\alpha = 1.8\) there is not enough space at low energies for other outcomes of a collision than to fall back in the same energy gap as where the particle was before the collision.

Figure 20. A plot of one of the pairs \((x_j, x_j^*)\), the energy before and after collision for one of the particles involved in the first 100000 collisions in a simulation. In the left, \(\alpha = 1.8\), and to the right, \(\alpha = 1.0\). The orange curve shows \(f(x) \Pi(\alpha f(x))\). The plots appear to be symmetric around \(x = x^*\), as one should expect if detailed balance holds.

The following results come from simulations starting from initial distributions far from equilibrium. The dynamical process depends strongly on the value of \(\alpha\), because in areas with high particle densities, very few collisions attempts will actually result in a change. The figures 21, 22, and 23 are the results of simulations where the initial data are taken as the function \(g(x)\) as shown to the right of Figure 2 of the main article. The initial function is plotted in red and the equilibrium density in green. The black step functions shows the empirical histogram from the simulation, and the blue dots illustrate the variation around the histogram means. With a higher density (\(\alpha\) large) the rate of convergence to equilibrium is slower.

Figure 21. The initial data is the function \(g\) as in Fig. 2, with \(\alpha = 1.0\) to the left and \(\alpha = 1.5\) to the right. The blue points show the distribution of simulation results for each histogram bin. The number of particles is 1000 and the number independent samples is 5000.
Figure 22. At time $t = 0.1$, the result is already very close to equilibrium when $\alpha = 1.0$ (left), but not when $\alpha = 1.5$ (right).

Figure 23. At time $t = 10$, the solution has still not converged to equilibrium when $\alpha = 1.5$ (right), because here, for small $x$ the particle density is close to its maximum, and there is not much space for collisions to take place.

An even more extreme situation is shown in the final two examples. In Figure 24, the initial density is a single step function, where all the particles are equally spaced, and pushed as far towards the lower energy as possible. As discussed in Section 3 of the paper, such sequences are chaotic but not in the detailed sense. The simulations were carried out with $n = 10000$ particles. And we see that with $\alpha = 1.8$ (left of Fig. 24), the single step function is nearly a frozen state, there is hardly any change over the simulation period, which in this case is $T = 1000$. Initially the only possible jumps are those where the two involved particles fall back to essentially the same energy level they had before the collision. This is a slow diffusive motion, that may be compared with, and perhaps possible to analyze in the same way as models for competing particle systems and rank based interacting diffusions [10, 13, 12]. On the other hand, with $\alpha = 1.0$, there is enough space between the particles to allow for long jumps, and the convergence towards equilibrium is much faster so that already after time $T = 10$ the distribution is close to equilibrium (the right side of the same figure).

In the final example, shown in Figure 25 the initial configuration is one where the first $n - 1$ are pushed even closer together to lower energies, and the $n^{th}$ particle is given sufficiently high energy to give the same total energy. Here we see a convergence towards equilibrium also when $\alpha = 1.8$ (left), and faster for $\alpha = 1.0$. What happens is that when the particle with highest energy (outside the range of the graph) interacts with one of the other particles, the group of particles will be torn apart, leaving space for long jumps, and then the convergence to equilibrium can be seen also there.

Of course one can construct a sequence of interactions that transforms the initial data of Fig. 24 into the initial data of Fig. 25, and therefore one would expect convergence to equilibrium also from this initial configuration, but it may take a very long time to happen.
Figure 24. The black step function shows the empirical histogram after time $T = 1000$ for $\alpha = 1.8$ (left), and after time $T = 1.0$ for $\alpha = 1.0$ (right). In the initial configuration the $x_i$ are put at equal distance, corresponding to the step function represented in red.

Figure 25. The graphs show simulations results as in Fig. 24, except that the initial data consists of an initial step function, constructed as above, but with 99% of the mass compressed to a tighter configuration, and the remaining particles put at a higher energy (near $x = 11$) to keep the same initial energy. Here we observe a convergence towards equilibrium also when $\alpha = 1.8$, although this happens very slowly.