ON THE LIFTS OF $F_a(5, 1)$—STRUCTURE ON TANGENT AND COTANGENT BUNDLE

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Abstract. This paper consists of three main sections. In the first part, we obtain the complete lifts of the $F_a(5, 1)$—structure on tangent bundle. We have also obtained the integrability conditions by calculating the Nijenhuis tensors of the complete lifts of $F_a(5, 1)$—structure. Later we get the conditions of to be the almost holomorphic vector field with respect to the complete lifts of $F_a(5, 1)$—structure. Finally, we obtained the results of the Tachibana operator applied to the vector fields with respect to the complete lifts of $F_a(5, 1)$—structure on tangent bundle. In the second part, all results obtained in the first section investigated according to the horizontal lifts of $F_a(5, 1)$—structure in tangent bundle $T(M^n)$. In finally section, all results obtained in the first and second section were investigated according to the horizontal lifts of the $F_a(5, 1)$—structure in cotangent bundle $T^*(M^n)$.

1. Introduction

The investigation for the integrability of tensorial structures on manifolds and extension to the tangent or cotangent bundle, whereas the defining tensor field satisfies a polynomial identity has been an actively discussed research topic in the last 50 years, initiated by the fundamental works of Kentaro Yano and his collaborators, see for example [17]. Also, the idea of $F$—structure manifold on a differentiable manifold developed by Yano [14], Ishihara and Yano [7], Goldberg [6] and among others. Moreover, Yano and Patterson [15, 16] studied on the horizontal and complete lifts from a differentiable manifold $M^n$ of class $C^\infty$ to its cotangent bundles. Andree has studied the structure defined by a tensor field $F \neq 0$ of type $(1, 1)$ satisfying $F^5 + F = 0$ [1]. Later Ram Nivas and C.S. Prasad [11] studied on more form $F_a(5, 1)$—structure. This paper consist of three main sections. In the first part,
we obtain the complete lifts of the $F_a(5,1)$--structure on tangent bundle. We have also obtained the integrability conditions by calculating the Nijenhuis tensors of the complete lifts of $F_a(5,1)$--structure. Later we get the conditions of to be the almost holomorphic vector field with respect to the complete lifts of $F_a(5,1)$--structure. Finally, we obtained the results of the Tachibana operator applied to the vector fields with respect to the complete lifts of $F_a(5,1)$--structure on tangent bundle. In the second part, all results obtained in the first section investigated according to the horizontal lifts of $F_a(5,1)$--structure in tangent bundle $T(M^n)$. In finally section, all results obtained in the first and second section were investigated according to the horizontal lifts of the $F_a(5,1)$--structure in cotangent bundle $T^*(M^n)$.

Let $M^n$ be an $n$--dimensional differentiable manifold of class $C^\infty$. Suppose there exist on $M^n$, a $(1,1)$ tensor field $F(\neq 0)$ satisfying [11]

$$F^5 - a^2 F = 0,$$

where $a$ is a complex number not equal to zero. If $a = i$ where $i = \sqrt{-1}$, our structure takes the form $F^5 + F = 0$ studied by Andreou [1].

Let us define on $M^n$, the operators $l$ and $m$ as follows:

$$l = (F^4/a^2) \text{ and } m = I - (F^4/a^2).$$

$I$ being unit tensor field.

In view of equations (1) and (2), we have

$$l^2 = l, \quad m^2 = m \quad \text{and} \quad l + m = I.$$  \hspace{1cm} (3)

For a tensor field $F(\neq 0)$ of type $(1,1)$ satisfying (1) the operators $l$ and $m$ defined by (2), when applied to the tangent space of $M^n$ at a point, are complementary projection operators.

Thus there exist complementary distributions $L$ and $M$ corresponding to the projection operators $l$ and $m$ respectively. If the rank of $F$ is constant every where or equal to $r$, the dimensions of $L$ and $M$ are $r$ and $n - r$ respectively [10]. Us call such a structure as $F_a(5,1)$--structure of rank $r$ [11].

For a tensor field $F(\neq 0)$ of type $(1,1)$ admitting $F_a(5,1)$--structure and for the projection operators $l$ and $m$ given by (2) we have

$$F l = l F = F, \quad F m = m F = 0.$$  \hspace{1cm} (4)

and

$$F^2 l = l F^2 = F^2, \quad F^2 m = m F^2 = 0.$$  \hspace{1cm} (5)

In the manifold $M^n$ endowed with $F_a(5,1)$--structure, the $(1,1)$ tensor field $\tilde{F}$ given by $\tilde{F} = l - m = (2F^4/a^2) - I$ gives an almost product structure [9].

1.1. Complete Lift of $F_a(5,1)$--Structure on Tangent Bundle. Let $M^n$ be an $n$--dimensional differentiable manifold of class $C^\infty$ and $T_p(M^n)$ the tangent space at a point $p$ of $M^n$ and

$$T(M^n) = \bigcup_{p \in M^n} T_p(M^n)$$
is the tangent bundle over the manifold \( M^n \).

Let us denote by \( T^r_s(M^n) \), the set of all tensor fields of class \( C^\infty \) and of type \((r, s)\) in \( M^n \) and \( T(M^n) \) be the tangent bundle over \( M^n \). The complete lift of \( F^C \) of an element of \( T^1_1(M^n) \) with local components \( F^h_i \) has components of the form \( F^C = \begin{bmatrix} F^h_i & 0 \\ \delta^h_i & F^h_i \end{bmatrix} \).

Now we obtain the following results on the complete lift of \( F \) satisfying \( F^5 - a^2 F = 0 \).

Let \( F, G \in T^1_1(M^n) \). Then we have

\[
(FG)^C = F^C G^C. \tag{7}
\]

Replacing \( G \) by \( F \) in (7) we obtain

\[
(FF)^C = F^C F^C \text{ or } (F^2)^C = (F^C)^2. \tag{8}
\]

Now putting \( G = F^4 \) in (7) since \( G \) is \((1, 1)\) tensor field therefore \( F^4 \) is also \((1, 1)\) so we obtain \((FF)^C = F^C (F^4)^C \) which in view of (8) becomes

\[
(F^5)^C = (F^C)^5. \tag{9}
\]

Taking complete lift on both sides of equation \( F^5 - a^2 F = 0 \) we get

\[
(F^5)^C - (a^2 F)^C = 0
\]

which in consequence of equation (9) gives

\[
(F^C)^5 - a^2 F^C = 0. \tag{10}
\]

Let \( F \) satisfying \((1, 1)\) be an \( F^- \)structure of rank \( r \) in \( M^n \). Then the complete lifts \( l^C = (F^4)^C \) of \( l \) and \( m^C = I - (F^4)^C \) of \( m \) are complementary projection tensors in \( T(M^n) \). Thus there exist in \( T(M^n) \) two complementary distributions \( L^C \) and \( M^C \) determined by \( l^C \) and \( m^C \), respectively.

1.2. **Horizontal Lift of \( F_{(5, 1)}^- \)Structure on Tangent Bundle.** Let \( F^h_i \) be the component of \( F \) at \( A \) in the coordinate neighbourhood \( U \) of \( M^n \). Then the horizontal lift \( F^H \) of \( F \) is also a tensor field of type \((1, 1)\) in \( T(M^n) \) whose components \( \tilde{F}^A_B \) in \( \pi^{-1}(U) \) are given by

\[
F^H = F^C - \gamma(\nabla F) = \begin{pmatrix} F^h_i & 0 \\ -\Gamma^h_i F^l_j + \Gamma^l_i F^h_j & F^h_i \end{pmatrix}.
\]

Let \( F, G \) be two tensor fields of type \((1, 1)\) on the manifold \( M \). If \( F^H \) denotes the horizontal lift of \( F \), we have

\[
(FG)^H = F^H G^H. \tag{11}
\]

Taking \( F \) and \( G \) identical, we get

\[
(F^H)^2 = (F^2)^H. \tag{12}
\]
Multiplying both sides by $F^H$ and making use of the same (12), we get

$$(F^H)^3 = (F^3)^H$$

and so on. Thus it follows that

$$(F^H)^4 = (F^4)^H, \quad (F^H)^5 = (F^5)^H.$$  \hspace{1cm} (13)

Taking horizontal lift on both sides of equation $F^5 - a^2 F = 0$ we get

$$(F^5)^H - (a^2 F)^H = 0$$

view of (13), we can write

$$(F^5)^H - a^2 F^H = 0. \hspace{1cm} (14)$$

### 2. Main Results

#### 2.1. The Nijenhuis Tensor

$N_{(F^5)^C} (F^C)^C (X^C, Y^C)$ of the Complete Lift $F^5$ on Tangent Bundle $T(M^n)$.

**Definition 1.** Let $F$ be a tensor field of type $(1,1)$ admitting $F_5(5,1)$–structure in $M^n$. The Nijenhuis tensor of a $(1,1)$ tensor field $F$ of $M^n$ is given by

$$N_F = [FX, FY] - F[X, FY] - F[FX, Y] + F^2 [X, Y]$$  \hspace{1cm} (15)

for any $X, Y \in \mathfrak{X}(M^n)$. The condition of $N_F(X, Y) = N(X, Y) = 0$ is essential to integrability condition in these structures.

The Nijenhuis tensor $N_F$ is defined local coordinates by

$$N_{ij}^k \partial_k = (F_i^s \partial_k F_j^s - F_j^s \partial_k F_i^s - \partial_i F_j^s F_k^s + \partial_j F_i^s F_k^s) \partial_k,$$

where $X = \partial_i$, $Y = \partial_j$, $F \in \mathfrak{X}(M^n)$.

**Definition 2.** Let $X$ and $Y$ be any vector fields on a Riemannian manifold $(M^n, g)$, we have

$$[X^H, Y^H] = [X, Y]^H - (R(X, Y) \nu)^V,$$

$$[X^H, Y^V] = (\nabla X Y)^V,$$

$$[X^V, Y^V] = 0,$$

where $R$ is the Riemannian curvature tensor of $g$ defined by

$$R(X, Y) = [\nabla X, \nabla Y] - \nabla_{[X, Y]}.$$  \hspace{1cm} (17)

In particular, we have the vertical spray $u^V$ and the horizontal spray $u^H$ on $T(M^n)$ defined by

$$u^V = u^i (\partial_i)^V = u^i \partial_i, \quad u^H = u^i (\partial_i)^H = u^i \delta_i,$$

where $\delta_i = \partial_i - u^j \Gamma_{ji}^s \partial_s$. $u^V$ is also called the canonical or Liouville vector field on $T(M^n)$. 
Theorem 3. The Nijenhuis tensor $N_{(F^5)C(F^5)C} (X^C, Y^C)$ of the complete lift of $F^5$ vanishes if the Nijenhuis tensor of the $F$ is zero.

Proof. In consequence of Definition 1 the Nijenhuis tensor of $(F^5)^C$ is given by

$$N_{(F^5)C(F^5)C} (X^C, Y^C) = [(F^5)^C X^C, (F^5)^C Y^C] - (F^5)^C [(F^5)^C X^C, Y^C]$$

$$- (F^5)^C [X^C, (F^5)^C Y^C] + (F^5)^C (F^5)^C [X^C, Y^C]$$

$$= a^4 \{(FX)^C, (FY)^C \} - (F)^C ((FX)^C, Y^C)$$

$$- (F)^C [X^C, (FY)^C] + (F)^C (F)^C [X^C, Y^C] \}$$

$$= a^4 \{(FX, FY) - F[FX, Y]$$

$$- F[X, FY] + F^2 [X, Y])^C$$

$$= a^4 N(X, Y)^C$$

\[ \square \]

Theorem 4. The Nijenhuis tensor $N_{(F^5)C(F^5)C} (X^C, Y^V)$ of the complete lift of $F^5$ vanishes if the Nijenhuis tensor $F$ is zero.

Proof.

$$N_{(F^5)C(F^5)C} (X^C, Y^V) = [(F^5)^C X^C, (F^5)^C Y^V] - (F^5)^C [(F^5)^C X^C, Y^V]$$

$$- (F^5)^C [X^C, (F^5)^C Y^V] + (F^5)^C (F^5)^C [X^C, Y^V]$$

$$= a^4 \{(FX)^C, (FY)^V \} - (F)^C ((FX)^C, Y^V)$$

$$- (F)^C [X^C, (FY)^V] + (F)^C (F)^C [X, Y]^V \}$$

$$= a^4 \{(FX, FY)^V - (F[FX, Y])^V$$

$$- (F[X, FY])^V - (F^2 [X, Y])^V \}$$

$$= a^4 N(X, Y)^V$$

\[ \square \]

Theorem 5. The Nijenhuis tensor $N_{(F^5)C(F^5)C} (X^V, Y^V)$ of the complete lift of $F^5$ vanishes.

Proof. Thus $[X^V, Y^V] = 0$ for all $X, Y \in \mathfrak{g}_0 (M^n)$, easily we get

$$N_{(F^5)C(F^5)C} (X^V, Y^V) = 0.$$
2.2. The Purity Conditions of Sasakian Metric with Respect to $(F^5)^C$ on $T(M^n)$.

**Definition 6.** The Sasaki metric $Sg$ is a (positive definite) Riemannian metric on the tangent bundle $T(M^n)$ which is derived from the given Riemannian metric on $M$ as follows:

$Sg(X^H, Y^H) = g(X, Y)$,  \hspace{1cm} (19)

$Sg(X^H, Y^V) = Sg(X^V, Y^H) = 0$,  \hspace{1cm}

$Sg(X^V, Y^V) = g(X, Y)$

for all $X, Y \in \mathfrak{X}_0(M^n)$.

**Theorem 7.** The Sasaki metric $Sg$ is pure with respect to $(F^5)^C$ if $\nabla F = 0$ and $F = a^2 I$, where $I$ = identity tensor field of type $(1, 1)$.

Proof. $S(\tilde{X}, \tilde{Y}) = Sg((F^5)^C \tilde{X}, \tilde{Y}) - Sg(\tilde{X}, (F^5)^C \tilde{Y})$ if $S(\tilde{X}, \tilde{Y}) = 0$ for all vector fields $\tilde{X}$ and $\tilde{Y}$ which are of the form $X^V, Y^V$ or $X^H, Y^H$ then $S = 0$.

i)  \hspace{1cm} $S(X^V, Y^V) = Sg((F^5)^C X^V, Y^V) - Sg(X^V, (F^5)^C Y^V)$

\hspace{1cm} $= a^2 \{ Sg((FX)^V, Y^V) - Sg(X^V, (FY)^V) \}$

\hspace{1cm} $= a^2 \{ (g(FX, Y))^V - (g(X, FY))^V \}$

ii)  \hspace{1cm} $S(X^V, Y^H) = Sg((F^5)^C X^V, Y^H) - Sg(X^V, (F^5)^C Y^H)$

\hspace{1cm} $= -a^2 \ Sg(X^V, (FY)^H + (\nabla_y F) Y^H)$

\hspace{1cm} $= -a^2 \ Sg(X^V, ((\nabla F) u) Y^V)$

\hspace{1cm} $= -a^2 \ (g(X, ((\nabla F) u) Y)^V)$

iii)  \hspace{1cm} $S(X^H, Y^H) = Sg((F^5)^C X^H, Y^H) - Sg(X^H, (F^5)^C Y^H)$

\hspace{1cm} $= a^2 \ Sg((FX)^H + (\nabla_y F) X^H, Y^H)$

\hspace{1cm} $= a^2 \ Sg(X^H, (FY)^H + (\nabla_y F) Y^H)$

\hspace{1cm} $= a^2 \ { g((FX), Y)^V - g(X, (FY))^V }$
Theorem 11. Let \( \varphi \in \mathfrak{g}(M^n) \), and \( \mathfrak{g}(M^n) = \sum_{s=0}^{\infty} \mathfrak{g}_s(M^n) \) be a tensor algebra over \( R \). A map \( \phi_{\varphi} \mid_{r+s}^0 : \mathfrak{g}(M^n) \to \mathfrak{g}(M^n) \) is called as Tachibana operator or \( \phi_{\varphi} \) operator on \( M^n \) if

a) \( \phi_{\varphi} \) is linear with respect to constant coefficient,

b) \( \phi_{\varphi} : \mathfrak{g}(M^n) \to \mathfrak{g}_{r+s+1}(M^n) \) for all \( r \) and \( s \),

c) \( \phi_{\varphi}(K \otimes L) = (\phi_{\varphi}K) \otimes L + K \otimes \phi_{\varphi}L \) for all \( K, L \in \mathfrak{g}(M^n) \),

d) \( \phi_{\varphi,X}Y = -(L_Y \varphi)X \) for all \( X, Y \in \mathfrak{g}_0(M^n) \), where \( L_Y \) is the Lie derivation with respect to \( Y \) (see [3, 5–8]),

e) \[ (\phi_{\varphi,X}Y)Y = (d(\iota_Y \eta)) \varphi X - (d(\iota_Y (\eta \varphi)))X + \eta (L_Y \varphi)X \]

for all \( \eta \in \mathfrak{g}^0(M^n) \) and \( X, Y \in \mathfrak{g}_0(M^n) \), where \( \iota_Y \eta = \eta Y = \eta \otimes Y, \mathfrak{g}_r(M^n) \) the module of all pure tensor fields of type \( (r, s) \) on \( M^n \) with respect to the affinor field, \( \otimes \) is a tensor product with a contraction \( C \) (see 13 for applied to pure tensor field).

Remark 9. If \( r = s = 0 \), then from e), d) and c) of Definition 8 we have \( \phi_{\varphi,X}(\iota_Y \eta) = \phi X(\iota_Y \eta) - X(\iota_{\varphi,Y} \eta) \) for \( \iota_Y \eta \in \mathfrak{g}^0_0(M^n) \), which is not well-defined \( \phi_{\varphi} \)-operator. Different choices of \( Y \) and \( \eta \) leading to same function \( f = \iota_Y \eta \) do get the same values. Consider \( M^n = R^2 \) with standard coordinates \( x, y \). Let \( \varphi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \).

Consider the function \( f = 1 \). This may be written in many different ways as \( \iota_Y \eta \). Indeed taking \( \eta = dx \), we may choose \( Y = \frac{\partial}{\partial x} \) or \( Y = \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} \). Now the right-hand side of \( \phi_{\varphi,X}(\iota_Y \eta) = \phi X(\iota_Y \eta) - X(\iota_{\varphi,Y} \eta) \) is \( \phi X \) in the first case, and \( \phi X \) in the second case. For \( X = \frac{\partial}{\partial x} \), the latter expression is \(-1 \neq 0 \). Therefore, we put \( r + s > 0 \).

Remark 10. From d) of Definition 8 we have

\[ \phi_{\varphi,X}Y = [\varphi X, Y] - \varphi[X, Y]. \]

By virtue of

\[ [fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X \]

for any \( f, g \in \mathfrak{g}^0_0(M^n) \), we see that \( \phi_{\varphi,X}Y \) is linear in \( X \), but not \( Y \).

Theorem 11. Let \( \phi_{\varphi} \) be the Tachibana operator and the structure \( (F^5)^C - a^2 F^C = 0 \) defined by Definition 8 and 14, respectively. If \( L_Y F = 0 \), then all results with respect to \( (F^5)^C \) is zero, where \( X, Y \in \mathfrak{g}_0(M) \), the complete lifts \( X^C, Y^C \in \mathfrak{g}_0^C(T(M)) \) and the vertical lift \( X^V, Y^V \in \mathfrak{g}_0^C(T(M)). \)

i) \( \phi_{(F^5)^C,X}Y^C = -a^2 ((L_Y F)X)^C \)
ii) $\phi_{(F^5)C}c_X c Y^V = -a^2 ((L_Y F) X)^V$

iii) $\phi_{(F^5)C}X^V Y^C = -a^2 ((L_Y F) X)^V$

iv) $\phi_{(F^5)C}X^V Y^V = 0$

Proof. i)

$$\phi_{(F^5)C}c_X c Y^C = -(L_Y c (F^5)^C)X^C$$

$$= a^2 \{-L_Y c (FX)^C + (F)^C L_Y c X^C\}$$

$$= -a^2 ((L_Y F) X)^C$$

ii)

$$\phi_{(F^5)C}c_X c Y^V = -(L_Y v (F^5)^C)X^C$$

$$= -L_Y v (F^5)^C X^C + (F^5)^C L_Y v X^C$$

$$= a^2 \{-L_Y v (FX)^V + (F)^C L_Y v X^C\}$$

$$= -a^2 ((L_Y F) X)^V$$

iii)

$$\phi_{(F^5)C}X^V Y^C = -(L_Y c (F^5)^C)X^V$$

$$= -L_Y c (F^5)^C X^V + (F^5)^C L_Y c X^V$$

$$= a^2 \{-L_Y c (FX)^V + (F)^C L_Y c X^V\}$$

$$= -a^2 ((L_Y F) X)^V$$

iv)

$$\phi_{(F^5)C}X^V Y^V = -(L_Y v (F^5)^C)X^V$$

$$= -L_Y v (F^5)^C X^V + (F^5)^C L_Y v X^V$$

$$= 0$$

$\square$

**Theorem 12.** If $L_Y F = 0$ for $Y \in M$, then its complete lift $Y^C$ to the tangent bundle is an almost holomorphic vector field with respect to the structure $(F^5)^C - a^2 F^C = 0$.

Proof. i)

$$(L_Y c (F^5)^C)X^C = L_Y c (F^5)^C X^C - (F^5)^C L_Y c X^C$$

$$= a^2 \{L_Y c (FX)^C - (F)^C L_Y c X^C\}$$

$$= a^2 ((L_Y F) X)^C$$
The Structure \((F^5)^H - a^2 F^H = 0\) on Tangent Bundle \(T(M^n)\).

**Theorem 13.** The Nijenhuis tensor \(N_{(F^5)^H} (X^H, Y^H)\) of the horizontal lift of \(F^5\) vanishes if the Nijenhuis tensor of the \(F\) is zero and \((-\hat{R}(FX,Y)u) + (F(\hat{R}(FX,Y)u)) - ((F)^2(\hat{R}(X,Y)u))\)^\(V\) = 0.

**Proof.**

\[
N_{(F^5)^H} (X^H, Y^H) = \left[ (F^5)^H X^H, (F^5)^H Y^H \right] - (F^5)^H \left[ (F^5)^H X^H, Y^H \right] - (F^5)^H \left[ X^H, (F^5)^H Y^H \right] + (F^5)^H \left[ (F^5)^H X^H, Y^H \right] = a^4 \{(FX, FY) - (F) [FX, Y] - (F) [X, FY] - (F) [X, Y] \} + (F(\hat{R}(FX,Y)u)) - ((F)^2(\hat{R}(X,Y)u)) \}
\]

If \(N_{\hat{F}F} (X, Y) = 0\) and \((-\hat{R}(FX,Y)u) + (F(\hat{R}(FX,Y)u)) + (F(\hat{R}(X,Y)u)) - ((F)^2(\hat{R}(X,Y)u))\)^\(V\) = 0, then we get \(N_{(F^5)^H} (X^H, Y^H) = 0\). The theorem is proved.

Where \(\hat{R}\) denotes the curvature tensor of the affine connection \(\hat{\nabla}\) defined by \(\hat{\nabla}_X Y = \nabla_Y X + [X, Y]\) (see \[17\] p.88-89).

**Theorem 14.** The Nijenhuis tensor \(N_{(F^5)^H} (X^H, Y^V)\) of the horizontal lift of \(F^5\) vanishes if the Nijenhuis tensor of the \(F\) is zero and \(\nabla F = 0\).

**Proof.**

\[
N_{(F^5)^H} (X^H, Y^V) = \left[ (F^5)^H X^H, (F^5)^H Y^V \right] - (F^5)^H \left[ (F^5)^H X^H, Y^V \right] - (F^5)^H \left[ X^H, (F^5)^H Y^V \right] + (F^5)^H \left[ (F^5)^H X^H, Y^V \right] = a^4 \{(FX, FY)^V - (F) [FX, Y]^V - (F) [X, FY]^V - ((F)^2 [X, Y])^V + (\nabla_F FX)^V - (F(\nabla_Y FX))^V \}
\]
Theorem 15. The Nijenhuis tensor $N_{(F^5)\mathcal{H}}(X^V, Y^V)$ of the horizontal lift of $F^5$ vanishes.

Proof. Because of $[X^V, Y^V] = 0$ for $X, Y \in M$, easily we get

$$N_{(F^5)\mathcal{H}}(X^V, Y^V) = 0.$$ 

\[ \square \]

Theorem 16. The Sasakian metric $Sg$ is pure with respect to $(F^5)^\mathcal{H}$ if $F = a^2I$, where $I$ = identity tensor field of type $(1,1)$.

Proof. $S(\bar{X}, \bar{Y}) = Sg((F^5)^\mathcal{H} \bar{X}, \bar{Y}) - Sg(\bar{X}, (F^5)^\mathcal{H} \bar{Y})$ if $S(\bar{X}, \bar{Y}) = 0$ for all vector fields $\bar{X}$ and $\bar{Y}$ which are of the form $X^V, Y^V$ or $X^H, Y^H$ then $S = 0$.

i) $$S(X^V, Y^V) = Sg((F^5)^\mathcal{H} X^V, Y^V) - Sg(X^V, (F^5)^\mathcal{H} Y^V) = a^2\{Sg((FX)^V, Y^V) - Sg(X^V, (FY)^V)\}$$

ii) $$S(X^V, Y^H) = Sg((F^5)^\mathcal{H} X^V, Y^H) - Sg(X^V, (F^5)^\mathcal{H} Y^H) = -a^2 Sg(X^V, (FY)^H) = 0$$

iii) $$S(X^H, Y^H) = Sg((F^5)^\mathcal{H} X^H, Y^H) - Sg(X^H, (F^5)^\mathcal{H} Y^H) = a^2\{(Sg((FX)^H, Y^H) - Sg(X^H, (FY)^H)\}$$

\[ \square \]

Theorem 17. Let $\phi_{\rho}$ be the Tachibana operator and the structure $(F^5)^\mathcal{H} - a^2F^H = 0$ defined by Definition 3 and 4, respectively. If $L_Y F = 0$ and $F = a^2I$, then all results with respect to $(F^5)^\mathcal{H}$ is zero, where $X, Y \in \mathfrak{X}_0(M)$, the horizontal lifts $X^H, Y^H \in \mathfrak{X}_0(T(M^n))$ and the vertical lift $X^V, Y^V \in \mathfrak{X}_0(T(M^n))$

i) $\phi_{(F^5)^\mathcal{H}}X^H Y^H = -a^2\{-(L_Y F)X^H + (\hat{R}(Y, F)X^H)U - (F(\hat{R}(Y, X)U)\}\$,

ii) $\phi_{(F^5)^\mathcal{H}}X^V Y^V = a^2\{-(L_Y F)Y^V + ((\nabla_Y F)X^V)\}$. 

\[ \square \]
iii) $\phi_{(F^5)^H X^V Y^H} = a^2 \{-((L_Y F) X)^V - (\nabla_{FX} Y)^V + (F (\nabla_X Y))^V\}$,

iv) $\phi_{(F^5)^H X^V Y^V} = 0$.

Proof. i)

$\phi_{(F^5)^H X^H Y^H} = -(L_{YH} (F^5)^H X^H)
= -L_{YH} (F^5)^H X^H + (F^5)^H L_{YH} X^H
= -a^2 [Y,FX]^H + a^2 \gamma \tilde{R} [Y,FX] + a^2 (F [Y,X])^H - a^2 (F \nabla (Y,X) \cdot V)^V
= -a^2 \{-((L_Y F) X)^H + (\tilde{R} (Y,FX) \cdot V
- (F (\nabla_X Y))^V\}$

ii)

$\phi_{(F^5)^H X^H Y^V} = -(L_{YV} (F^5)^H X^H)
= -L_{YV} (F^5) X^H + (F^5)^H L_{YV} X^H
= -a^2 [F,Y]^V + a^2 (\nabla_{YF} X)^V
+ a^2 (F [Y,X])^V - a^2 (F (\nabla_Y X))^V
= a^2 \{-((L_Y F) X)^V + ((\nabla_Y F) X)^V\}$

iii)

$\phi_{(F^5)^H X^V Y^H} = -(L_{YH} (F^5)^H X^V)
= -L_{YH} (F^5) X^V + (F^5)^H L_{YH} X^V
= a^2 [Y,FX]^V - a^2 (\nabla_{FX} Y)^V + a^2 (F [Y,X])^V + a^2 (F (\nabla_X Y))^V
= a^2 \{-((L_Y F) X)^V - (\nabla_{FX} Y)^V + (F (\nabla_X Y))^V\}$

iv)

$\phi_{(F^5)^H X^V Y^V} = -(L_{YV} (F^5)^H X^V)
= -a^2 L_{YV} (FX)^V + a^2 (F)^H L_{YV} X^V
= 0$

\[\square\]

2.4. The Structure $(F^5)^H - a^2 F^H = 0$ on Cotangent Bundle. In this section, we find the integrability conditions by calculating Nijenhuis tensors of the horizontal lifts of $F_n(5,1)$—structure. Later, we get the results of Tachibana operators applied to vector and covector fields according to the horizontal lifts of $F_n(5,1)$—structure.
in cotangent bundle $T^*(M^n)$. Finally, we have studied the purity conditions of Sasakian metric with respect to the lifts of the structure.

Let $F, G$ be two tensor fields of type $(1, 1)$ on the manifold $M$. If $F^H$ denotes the horizontal lift of $F$, we have

$$F^H G^H + G^H F^H = (FG + GF)^H$$

Taking $F$ and $G$ identical, we get

$$(F^H)^2 = (F^2)^H \quad (20)$$

Multiplying both sides by $F^H$ and making use of the same (20), we get

$$(F^H)^3 = (F^3)^H \quad (21)$$

and so on. Thus it follows that

$$(F^H)^4 = (F^4)^H \quad (22)$$

Since $F$ gives on $M$ the $F_a(5; 1)$ structure, we have

$$F^5 - a^2 F = 0. \quad (23)$$

Taking horizontal lift, we obtain

$$(F^5)^H - a^2 F^H = 0. \quad (24)$$

In view of (23), we can write

$$(F^5)^H - a^2 F^H = 0. \quad (25)$$

**Theorem 18.** The Nijenhuis tensor $N_{(F^5)^H, (F^5)^H}(X^H, Y^H)$ of the horizontal lift $F^5$ vanishes if $F = a^2 I$ on $M$.

**Proof.** The Nijenhuis tensor $N(X^H, Y^H)$ for the horizontal lift of $F^5$ is given by

$$N_{(F^5)^H, (F^5)^H}(X^H, Y^H) = \frac{\left([F^5]^H X^H, [F^5]^H Y^H\right)}{[F^5]^H [F^5]^H [X^H, Y^H]} - (F^5)^H [(F^5)^H X^H, Y^H]$$

$$\quad - (F^5)^H [X^H, (F^5)^H Y^H] + (F^5)^H [(F^5)^H X^H, Y^H]$$

$$= a^4 \left\{ \left([F^H X^H, (F^H)^H Y^H] - (F^H)^H [F^H X^H, Y^H] \right) 
\quad - (F^H)^H [X^H, (F^H)^H Y^H] + (F^H)^H [(F^H)^H X^H, Y^H] \right\}$$

$$= a^4 \left\{ \left\{ [FX, FY] - F[(FX), Y] - F[X, FY] \right\}^H + \gamma \left\{ R(FX, FY) - R((FX), Y) F \right\} 
\quad - R(X, FY) F^2 + R(X, Y) F^2 \right\}$$

Let us suppose that $F = a^2 I$ on $M$. Thus, the equation becomes

$$N_{(F^5)^H, (F^5)^H}(X^H, Y^H) = a^4 \left\{ \left\{ [X, Y] - [X, Y] - [X, Y] + [X, Y] \right\}^H 
\quad + \gamma \left\{ R(X, Y) - R(X, Y) - R(X, Y) + R(X, Y) \right\} \right\}.$$
Therefore, it follows
\[ N_{\{F^5\}^\mu\{F^5\}^\mu}(X^H, Y^H) = 0 \]
\[ \square \]

**Theorem 19.** The Nijenhuis tensor \( N_{\{F^5\}^\mu\{F^5\}^\mu}(X^H, \omega^V) \) of the horizontal lift \( F^5 \) vanishes if \( \nabla F = 0 \).

**Proof.**
\[
N_{\{F^5\}^\mu\{F^5\}^\mu}(X^H, \omega^V) = \left[(F^5)^H X^H, (F^5)^H \omega^V\right] - (F^5)^H [(F^5)^H X^H, \omega^V]
\]
\[ - (F^5)^H [X^H, (F^5)^H \omega^V] + (F^5)^H (F^5)^H [X^H, \omega^V] \]
\[ = a^4 \left\{ (\nabla_{FX}(\omega \circ F))^V - ((\nabla_{FX}) \circ F)^V \right\}
\]
\[ - (\omega \circ F)^{V^2} + \left(\nabla_{X \omega}^V \circ F \right)^2 \}
\]
\[ = a^4 \left\{ (\omega \circ (\nabla_{FX} F) - (\omega \circ (\nabla_{FX} F))^V \right\}
\]
where \( F \in \mathfrak{X}^1(M), X \in \mathfrak{X}^0_1(M), \omega \in \mathfrak{X}^0_0(M) \). The theorem is proved. \[ \square \]

**Theorem 20.** The Nijenhuis tensor \( N_{\{F^5\}^\mu\{F^5\}^\mu}(\omega^V, \theta^V) \) of the horizontal lift \( F^5 \) vanishes.

**Proof.** Because of \( [\omega^V, \theta^V] = 0 \) and \( \omega \circ F \in \mathfrak{X}^0_0(M) \) on \( T^*(M^n) \), the equation becomes
\[ N_{\{F^5\}^\mu\{F^5\}^\mu}(\omega^V, \theta^V) = 0. \]
\[ \square \]

**Theorem 21.** Let \( (F^5)^H \) be a tensor field of type \((1,1)\) on \( T^*(M^n) \). If the Tachibana operator \( \phi_{\circ} \) applied to vector and covector fields according to horizontal lifts of \( F^5 \) defined by \[ 23 \] on \( T^*(M^n) \), then we get the following results.

i) \( \phi_{\{F^5\}^\mu X^H Y^H} = a^2 \left\{ -((L_Y F)X)^H - (pR(Y,F)X)^V \right\}
\]
\[ + ((pR(Y,F)X)^F)^V \}, \]

ii) \( \phi_{\{F^5\}^\mu X^H \omega^V} = a^2 \left\{ (\nabla_{FX} \omega)^V - ((\nabla_{X \omega}) \circ F)^V \right\} \}

iii) \( \phi_{\{F^5\}^\mu \omega^V X^H} = -a^2 ((\omega \circ (\nabla_{X} F))^V \}

iv) \( \phi_{\{F^5\}^\mu \omega^V \theta^V} = 0, \)

where horizontal lifts \( X^H, Y^H \in \mathfrak{X}^0_1(T^*(M^n)) \) of \( X, Y \in \mathfrak{X}^0_1(M^n) \) and the vertical lift \( \omega^V, \theta^V \in \mathfrak{X}^0_0(T^*(M^n)) \) of \( \omega, \theta \in \mathfrak{X}^0_1(M^n) \) are given, respectively.

**Proof.** i)
\[
\phi_{\{F^5\}^\mu X^H Y^H} = -((L_Y (F^5)^H)X)^H
\]
\[ \begin{align*}
- L_{Y H} (F^5)^H X^H + (F^5)^H L_{Y H} X^H \\
= a^2 \{-((L_{Y F})X)^H - (pR(Y, F X))^V \\
+ (pR(Y, X))F^V \} \\
\end{align*} \]

\[ \phi_{(F^5)^H} \omega^V = -(L_{\omega V} (F^5)^H)X^H \]

\[ = -L_{\omega V} (F^5)^H X^H + (F^5)^H L_{\omega V} X^H \]

\[ = -a^2 L_{\omega V} (F X)^H - a^2 (F^H (\nabla_X \omega))^V \]

\[ = a^2 \{ (\nabla_{F X} \omega)^V - ((\nabla_X \omega) \circ F)^V \}, \]

\[ \phi_{(F^5)^H} \omega^V X^H = -(L_{\omega V} (F^5)^H) \omega^V \]

\[ = -a^2 (\nabla_X (\omega \circ F))^V + a^2 ((\nabla_X \omega) \circ F)^V \]

\[ = -a^2 (\omega \circ (\nabla_X F))^V \]

\[ \phi_{(F^5)^H} \omega^V \theta^V = -(L_{\theta V} (F^5)^H) \omega^V \]

\[ = -L_{\theta V} (F^5)^H \omega^V + (F^5)^H L_{\theta V} \omega^V \]

\[ = 0 \]

**Definition 22.** A Sasakian metric \( S g \) is defined on \( T^*(M^n) \) by the three equations

\[ S g(\omega^V, \theta^V) = (g^{-1}(\omega, \theta))^V = g^{-1}(\omega, \theta) o \pi, \] \( (26) \)

\[ S g(\omega^V, Y^H) = 0, \] \( (27) \)

\[ S g(X^H, Y^H) = (g(X, Y))^V = g(X, Y) \circ \pi. \] \( (28) \)

For each \( x \in M^n \) the scalar product \( g^{-1} = (g^{ij}) \) is defined on the cotangent space \( \pi^{-1}(x) = T^*_x(M^n) \) by

\[ g^{-1}(\omega, \theta) = g^{ij} \omega_i \theta_j, \] \( (29) \)

where \( X, Y \in \mathfrak{X}_0(M^n) \) and \( \omega, \theta \in \Omega^1(M^n) \). Since any tensor field of type \((0, 2)\) on \( T^*(M^n) \) is completely determined by its action on vector fields of type \( X^H \) and \( \omega^V \) (see [17], p.280), it follows that \( S g \) is completely determined by equations \((26), (27)\) and \((28)\).

**Theorem 23.** Let \((T^*(M^n), S g)\) be the cotangent bundle equipped with Sasakian metric \( S g \) and a tensor field \((F^5)^H\) of type \((1, 1)\) defined by \((25)\). Sasakian metric \( S g \) is pure with respect to \((F^5)^H\) if \( F = a^2 I \) (\( I = \text{identity tensor field of type } (1, 1)\)).
Proof. We put

\[ S(\tilde{X}, \tilde{Y}) = S g((F^5)^H \tilde{X}, \tilde{Y}) - S g(\tilde{X}, (F^5)^H \tilde{Y}). \]

If \( S(\tilde{X}, \tilde{Y}) = 0 \), for all vector fields \( \tilde{X} \) and \( \tilde{Y} \) which are of the form \( \omega^V, \theta^V \) or \( X^H, Y^H \), then \( S = 0 \). By virtue of \((F^5)^H = \alpha^2 F^H \) and \([26], [27], [28]\), we get

i)

\[ S(\omega^V, \theta^V) = S g((F^5)^H \omega^V, \theta^V) - S g(\omega^V, (F^5)^H \theta^V) = S g((\alpha^2 F)^H \omega^V, \theta^V) - S g(\omega^V, (\alpha^2 F)^H \theta^V) = \alpha^2 (S g((\omega \circ F)^V, \theta^V) - S g(\omega^V, (\theta \circ F)^V)). \]

ii)

\[ S(X^H, \theta^V) = S g((F^5)^H X^H, \theta^V) - S g(X^H, (F^5)^H \theta^V) = S g((\alpha^2 F)^H X^H, \theta^V) - S g(X^H, (\alpha^2 F)^H \theta^V) = \alpha^2 (S g((F X)^H, \theta^V) - S g(X^H, (\omega \circ F)^V)) = 0. \]

iii)

\[ S(X^H, Y^H) = S g((F^5)^H X^H, Y^H) - S g(X^H, (F^5)^H Y^H) = S g((\alpha^2 F)^H X^H, Y^H) - S g(X^H, (\alpha^2 F)^H Y^H) = \alpha^2 (S g((F X)^H, Y^H) - S g(X^H, (FY)^H)). \]

Thus, \( F = \alpha^2 I \), then \( S g \) is pure with respect to \((F^5)^H \).

\[ \square \]

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