Dynamic Aspect of Two Dimensional Single Server Markovian Queueing Model With Multiple Vacations and Reneging

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Abstract:

This work deals with “the dynamic analysis of Two-dimensional M/M/1 queuing model with reneging and multiple vacations”. Customers renege according to negative exponential distribution. Dynamic aspect is more appropriate in understanding the behaviour of the system. Two dimensions represent respectively the number of arrivals at, and departure from, the system at a given time. The system starts with “i units at the time “t=0”. Allowing server to take vacation makes queuing model more feasible in studying the waiting time system appropriately. For example in ticket booking counter, messages to be delivered, patients form queue to have appointments before the clinic open or arrival of doctor. The solution for this model is obtained recursively with the help of Laplace transformation and results are achieved without involving complex functions.

Keywords: Two-dimensional state model; Markovian Queueing system; Transient analysis, multiple vacations; Laplace transform, Reneging.

1. Introduction

Queuing system is one of the most active research topics in operations research over the past few decades due to its wide practical application in many areas such as production, supermarkets, telecommunication and many more. Different situations generate different type of queues e.g. vacations queues, cyclic queues, tandem queues and so on.

Classical M/M/1 queue; obtained on the basis of one dimensional state model represents the number of customers in the system at a given time and it does not provide any information about the number that joined the system or is served upto that time. Thus, to study the queuing system more explicitly for arrivals and departures there is another technique known as Two Dimensional model. Pegden was the one who introduced the concept of two dimensional for classical M/M/1 queue [1]. Later on many authors have worked on this idea for obtaining transient solution.

Vacation queues make system more pragmatic and flexible in study of waiting line system e.g. call centers with multitasking employees, border crossing situation etc. Single vacation, exhaustive vacation, multiple vacations are some sort of vacation policies found in literature. Cooper was the first to study vacation models and defined vacation as “when the server finishes serving a unit and finds the system empty, however, it goes away for a length of time called a
vacation” [2]. Levy and Yechalli studied server idle time and its utilization for some supplementary jobs [3]. Surveys on vacation queuing systems are available in Doshi [4], Ke et al. [5]. In multiple vacations policy, server keeps on taking vacations until it finds at least one customer waiting in the system at the instant of vacation completion. Chowdhury obtained steady state behavior of queuing system under multiple vacations policy [6]. Baba derived steady state results for G/M/1 queue with vacations [7].

Queuing system incorporating reneging has wide application in communication, modelling and so on. “Reneging can be defined as reluctance of customer to remain in the line after joining it and leaving the system without being served”. Haight discussed the Markovian queue with reneging [8]. Cheng analyzed markovian queue with unreliable server and impatient customers [9]. Anker and Graffian studied Markovian queue with reneging [10,11].

Queues comprising multiple vacations and reneging have fascinated a lot of researchers. Padmavaty et al. derived the steady state probabilities of vacation queues with impatient customers [12]. Vijaya Laxmi and Jyothsna used supplementary variable and recursive techniques to derived steady state probabilities for model with reneging, heterogeneous servers and multiple vacations[13].

But the time independent solution does not reveal the real picture of the system under consideration, thus transient solution helps us to provide better understanding of the system. Ammar obtained transient solutions of two non-homogeneous servers with impatient customers using PGF [15].

All the above mentioned authors have worked on the concept of queueing models with the assumption “there is no customer present in the system initially”. However, in most of the practical situations, there are customers present even when the server is not available to queue e.g. arrivals of patients before opening of a clinic or customer waiting for reservation before ticket counter. Our main interest in the present work is to develop the model with initially “i” customers with multiple vacations and reneging.

2. Assumptions and Notations:

1. Arrivals are Poissonian with mean rate λ.
2. Service times are “exponentially distributed with parameter μ”.
3. “The vacation time follows an exponential distribution with parameter w”.
4. Inter-arrival times, service times and others involved in the system are statistically not dependent.
5. Each customer upon joining the queue will wait a certain length of time for his service to begin. If it does not begin by then, he will get impatient (reneged) and may leave the queue without getting service. The reneging times follow exponential distribution with parameter ξ.
6. At time t=0 we have considered ‘i’ customers in the system.
7. “The system state is given by \((n,k)\), where \(n\) is the number of arrival and \(k\) is the number of departure up to time \(t\)”, i.e.

\[
P(i, 0) = \sum_{k=0}^{\infty} P_{i+k,k,Y}(i, 0) = 1
\]  

(1)

\[
\delta_{n,k} = \begin{cases} 
1; & \text{when } n = k \\
0; & \text{when } n \neq k 
\end{cases}
\]

The Laplace transform of \(F(t)\) is given as:

\[
\tilde{F}(s) = \int_{0}^{\infty} e^{-st} F(t) dt \quad \text{Re}(s) > 0
\]  

(2)

The Laplace inverse of \(G(p)\) is

\[
\tilde{H}_{m_1,m_2,m_3}(t) = \sum_{m=1}^{\infty} \sum_{l=0}^{m} \frac{t^{m-1} e^{-at}}{(m-l)! (l-1)!} \times \text{d}^{l-1} \text{d}^{l-1} \frac{G(p)}{P(p)} (p-a_i)^{m_x} |a_i \neq a_x \text{ for } i \neq x
\]

where \(P(p) = (p-a_1)^{m_1}(p-a_2)^{m_2} \cdots (p-a_y)^{m_y} \) and

\[
G(p) \text{ is the polynomial of degree } < m_1 + m_2 + m_3 + \cdots + m_y - 1
\]

3. TWO-DIMENSIONAL STATE MODEL
Define

\[ P_{n,k,B}(r,t) = \text{The probability of exactly } n \text{ arrivals, } k \text{ departures and } r- \text{ customers remain in the system by time } t \text{ and the server is busy corresponding to the queue}. \]

\[ P_{n,k,V}(r,t) = \text{The probability of exactly } n \text{ arrivals, } k \text{ departures and } r- \text{ customers remain in the system by time } t \text{ and the server is on vacation}. \]

\[ P_n(r,t) = \text{The probability that there are exactly } n \text{ arrivals and } k \text{ departures and } r- \text{ customers remain in the system by time } t. \]

The difference-differential equations directing the system are:

\[
\frac{d}{dt} P_{n,k,V}(r,t) = -(\lambda + w) P_{n,k,V}(r,t) + \lambda P_{n-1,k,V}(r,t) \quad ; 0 \leq k < n, r \geq 1
\]

\[
\frac{d}{dt} P_{n,n,V}(0,t) = -\lambda P_{n,n,V}(0,t) + \mu P_{n,n-1,B}(1,t) \left(1 - \delta_{n,0}\right) \quad ; n \geq 0
\]

\[
\frac{d}{dt} P_{n,k,B}(r,t) = - (\lambda + \mu + (r-1)\xi) P_{n,k,B}(r,t) + \lambda P_{n-1,k,B}(r+1,t) \left(1 - \delta_{n,n-1}\right) + w P_{n,k,V}(r,t) + (\mu + r\xi) P_{n,k-1,B}(r,t) \quad ; 0 \leq k < n, r \geq 1
\]

Clearly,

\[ P_{n,k}(r,t) = P_{n,k,V}(r,t) + P_{n,k,B}(r,t)(1 - \delta_{n,k}) \quad ; n \geq k \geq 0
\]

Finding of the Problem

We solve the above equations (4) to (7) by first having there Laplace transforms and then recursively solving then

\[ P_{n,k,V}(0,s) = \frac{1}{(s+\lambda)} \delta_{(i,0)} P_{0,0,V}(0,0) \quad n=0=k
\]

\[ P_{n,0,V}(n,s) = \lambda^n P_{n,0,0}(s) \delta_{(i,0)} P_{0,0,V}(0,0) + \sum_{x=1}^{\infty} \lambda^{n-x} P_{n-x+1,0,0}(s) \delta_{(i,x)} P_{x,0,V}(x,0) \quad ; n > 0,
\]

\[ P_{n,0,B}(n,s) = w \sum_{y=1}^{\infty} \frac{\lambda^{n-y}}{\prod_{x=0}^{\infty} (s+\lambda+\mu+(n-x-1)\xi)} \delta_{(i,y)} P_{y,0,V}(y,s) \quad ; n > 0
\]

\[ P_{n,k,V}(r,s) = \lambda^{n-k} \mu \frac{\lambda+w+\lambda_0}{\prod_{x=k}^{\infty} (s+\lambda+\mu+(n-x-1)\xi)} \delta_{(i,y)} P_{k,0,V}(0,0) + \sum_{x=k+1}^{\infty} \lambda^{n-x} \prod_{x=k}^{\infty} (s) \delta_{(i,x-k)} P_{x,k,V}(x-k,0) \quad ; n > k \geq 0
\]
\[ P_{n,k,B}(r,s) = \sum_{y=k+1}^{\infty} \frac{\lambda^{y-y} \{(\mu + (y-k) \cdot \xi)\}_{y}}{\prod_{x=0}^{y} (s + \lambda + x + \mu + (n-x-k-1) \cdot \xi)} \cdot P_{y,k-1,B}(y-k+1,s) \]
\[ + S_{n,k,B}(r,s) \sum_{y=k+1}^{\infty} \frac{(\lambda^{y-y} \cdot \xi)}{\prod_{x=0}^{y} (s + \lambda + x + \mu + (n-x-k-1) \cdot \xi)} \cdot P_{y,k,V}(y-k,s) \quad ; n > k \geq 0 \quad (12) \]

\[ \bar{P}_{n,n,V}(0,s) = \frac{\mu}{(s+\lambda)} \cdot \bar{P}_{n,n-1,B}(1,s) + \frac{1}{(s+\lambda)} \cdot \delta_{(i,0)} \cdot \bar{P}_{n,n,V}(0,0) \quad ; n>0 \quad (13) \]

We solve the above equations (8) to (13) by taking their Inverse Laplace transforms and we get

\[ P_{n,k,V}(0,t) = e^{-\lambda t} \cdot (\delta_{(i,0)} \cdot P_{0,0,V}(0,0)) \quad ; n=k=0 \quad (14) \]

\[ P_{n,0,V}(n,t) = \lambda^{n} \cdot H_{n,1,0}^{\lambda + w_{x,0}}(t) \cdot (\delta_{(i,0)} \cdot P_{0,0,V}(0,0)) + \sum_{x=1}^{\infty} \lambda^{n-x} \cdot H_{n-x+1,0,0}(t) \cdot (\delta_{(i,x)} \cdot P_{0,0,V}(0,0)) \]
\[ P_{n,0,B}(n,t) = w \cdot \sum_{y=1}^{\infty} \frac{\lambda^{y-y}}{\prod_{x=0}^{y} (s + \lambda + x + (n-x-k-1) \cdot \xi)} \cdot (\delta_{(i,y)} \cdot P_{y,0,V}(y,s)) \quad ; n>0 \quad (15) \]

\[ P_{n,k,V}(r,t) = \lambda^{n-k} \cdot \mu \cdot H_{n-k,1,0}^{\lambda + w_{x,0}}(t) \cdot P_{k,k-1,B}(1,s) + \lambda^{n-k} \cdot H_{n-k,1,0}^{\lambda + w_{x,0}}(t) \cdot (\delta_{(i,0)} \cdot P_{k,k,V}(0,0)) + \sum_{x=k+1}^{\infty} \lambda^{n-x} \cdot H_{n-x+1,0,0}(t) \cdot (\delta_{(i,x-k)} \cdot P_{k,k,V}(x-k,0)) \quad ; n > k \geq 0 \quad (17) \]

\[ P_{n,k,B}(r,t) = \sum_{y=k+1}^{\infty} \frac{\lambda^{y-y} \cdot \{(\mu + (y-k) \cdot \xi)\}_{y}}{\prod_{x=0}^{y} (s + \lambda + x + \mu + (n-x-k-1) \cdot \xi)} \cdot P_{y,k-1,B}(y-k+1,t) \]
\[ + S_{n,k,B}(r,t) \sum_{y=k+1}^{\infty} \frac{(\lambda^{y-y} \cdot \xi)}{\prod_{x=0}^{y} (s + \lambda + x + \mu + (n-x-k-1) \cdot \xi)} \cdot P_{y,k,V}(y-k,t) \quad ; n > k \geq 0 \quad (18) \]

\[ P_{n,n,V}(0,t) = \mu e^{-\lambda t} \cdot P_{n,n-1,B}(1,t) + e^{-\lambda t} \cdot (\delta_{i,0} \cdot P_{n,n,V}(0,0)) \quad ; n>0 \quad (19) \]

4. Substantiation of Model

The Laplace Transform \( \bar{P}_{n_i}(r,s) \) of the probability \( P_{n_i}(r,t) \) that exactly \( n \) unit arrives by the time \( t \):
\[(a) \quad \bar{P}_n(s) = \sum_{k=0}^{n} \left( \bar{P}_{n,k,v}(r, s) + \bar{P}_{n,k,b}(r, s)(1 - \delta_{n,k}) \right) = \sum_{k=0}^{n} \bar{P}_{n,k}(r, s) = \frac{\lambda^n}{(s+\lambda)^{n+1}}\]

And its Inverse Laplace transform is
\[P_n(r, t) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}\]

(b) “The Laplace transform of mean number of arrivals is”:
\[\sum_{n=0}^{\infty} n \cdot \bar{P}_n(r, s) = \frac{\lambda}{s^2}\]

And its Inverse Laplace transform is
\[\sum_{n=0}^{\infty} n \cdot P_n(r, t) = \lambda, t\]

\[\sum_{n=0}^{\infty} \sum_{k=0}^{n} \left( \bar{P}_{n,k,v}(r, s) + \bar{P}_{n,k,b}(r, s)(1 - \delta_{n,k}) \right) = \frac{1}{s}\]

\[\sum_{n=0}^{\infty} \sum_{k=0}^{n} \left( \bar{P}_{n,k,v}(r, t) + \bar{P}_{n,k,b}(r, t)(1 - \delta_{n,k}) \right) = 1\]

5. Numerical Results

A. The numerical results for the following:

(a) \[\sum_{k=0}^{n} P_{n,k}(r, t)\]
(b) \[\sum_{k=0}^{n} P_{n,k,b}(r, t)\]
(c) \[\sum_{k=0}^{n} P_{n,k,v}(r, t)\]

are computed and summarized for different sets of parameters presented in Table – (I) which is built on the following associations.

\[\Pr \{n \text{ arrivals in } (0,t)\} = \frac{e^{-\lambda t}(\lambda t)^n}{n!} = \sum_{k=0}^{n} P_{n,k}(r, t) = P_n(r, t)\]

The results derived from this model coincide with the last column of “Pegden & Rosenshine (1982)”

| \(\lambda\) | \(\mu\) | \(w\) | \(\xi\) | \(T\) | \(n\) | \(e^{-\lambda t} * (\lambda t)^n\) | \(\sum_{k=0}^{n} P_{n,k,v}(r, t)\) | \(\sum_{k=0}^{n} P_{n,k,b}(r, t)\) | \(\sum_{k=0}^{n} P_{n,k}(r, t)\) |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 1 | 1 | 3 | 1 | 0.149361 | 0.126885 | 0.022476 | 0.149361 |
| 1 | 2 | 1 | 1 | 3 | 3 | 0.224042 | 0.137283 | 0.086759 | 0.224042 |
| 1 | 2 | 1 | 1 | 3 | 5 | 0.100819 | 0.043788 | 0.057031 | 0.100819 |
| 2 | 2 | 1 | 1 | 3 | 1 | 0.014873 | 0.012634 | 0.002238 | 0.014873 |
B. “Server’s utilization time $U(t)$” is:

$$U(t) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} P_{n,k,B}(r, t)$$

C. “Server’s vacation time $V(t)$”:

$$V(t) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} P_{n,k,V}(r, t)$$

Here, the $U(t)$ and $V(t)$ are evaluated for $\lambda=1$, $w=1$, $\xi=0.1$, $i=1$, $\mu=2$.

| t  | U(t)    | V(t)    | Server's Total time |
|----|---------|---------|---------------------|
| 1  | 0.337408| 0.642287| 0.9796952           |
| 2  | 0.300916| 0.522748| 0.8236645           |
| 3  | 0.215984| 0.429081| 0.6450656           |
| 4  | 0.12208 | 0.311370| 0.4334507           |
| 5  | 0.062952| 0.202297| 0.2652503           |

6. Graphical Representation
Graphical representation of Utilization time and Vacation time has been shown in the Fig.1, Fig.2 for $\lambda=1$, $w=1$, $\xi=0.1$, $i=1$, $\mu=2$ respectively and for different values of time (t)

**Conclusion:**

In this paper, we have obtained transient probabilities for Two-dimensional M/M/1 queuing model with reneging and multiple vacations. The initially ‘i’ concept makes the model more realistic than those studied in early literature. Finally numerical analysis and performance measures have been evaluated with the help of maple software.

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