AdS backgrounds and induced gravity

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Abstract

In this paper we look for AdS solutions to generalised gravity theories in the bulk in various spacetime dimensions. The bulk gravity action includes the action of a non-minimally coupled scalar field with gravity, and a higher-derivative action of gravity. The usual Einstein-Hilbert gravity is induced when the scalar acquires a non-zero vacuum expectation value. The equation of motion in the bulk shows scenarios where AdS geometry emerges on-shell. We further obtain the action of the fluctuation fields on the background at quadratic and cubic orders.
I. INTRODUCTION

When well known framework of quantum field theory (QFT) on flat spacetime was applied to Einstein-Hilbert gravity, it was seen that the resulting quantum theory had no well defined high-energy behavior. In the sense that the theory was plagued with ultraviolet divergences arising at each loop order which cannot be absorbed in previous existing terms, thereby making the theory non-renormalizable \[1-3\]. Moreover the gravitational coupling strength was seen to grow without a bound. To overcome these hurdles, the idea of asymptotic safety scenario was suggested which generalizes the notion of renormalizability \[4, 5\] (see also \[6\] for a review). Around the same time another proposal came into existence: considering gravity theory at high energies by incorporating higher-derivative terms \[7\], which was shown to be perturbatively renormalizable to all loops. Although the theory was seen to have problems with tree level unitarity, these problems were conjectured to vanish in quantum theory \[8, 9\]. Indeed, it was seen that under quantum corrections this modified theory evades the problems of higher-derivative ghosts \[10, 11\]. These theories have been thoroughly investigated with or without matter, at one-loop \[12-19\].

An interesting proposal born of induced gravity, where the quantum fluctuation of matter generates dynamics of gravity \[20-25\] with an Einstein-Hilbert term, was very actively investigated. The idea actually was proposed earlier \[26-28\] but later works of Adler and Zee made it more formal. Furthermore, it was realized that Einstein-Hilbert gravity could be generated from a Weyl-invariant theory of matter and gravity \[29, 30\]. Recently this proposal of induced gravity coupled with higher-derivative gravity has gained momentum \[31-36\], where low energy Einstein-Hilbert gravity is generated via symmetry breaking in UV well defined quantum theory of scale-invariant system. This new proposal has wide applications in cosmology too. In fact a lot of work has already been done in this direction \[37, 38\].

In this paper we consider AdS solutions in the generalised gravity theory, in particular the induced gravity. In the bulk of the gravity, there is a non-minimal coupling of the scalar field to the Ricci curvature. The action we consider is from induced gravity action, where one obtains the usual Einstein-Hilbert gravity, when the scalar field acquires a non-zero vacuum expectation value. For a review of induced gravity, see also \[25\]. We also add a curvature-squared term. The quadratic curvature-squared terms can appear in low energy effective action of string compactifications, see for example \[14\]. The curvature-squared terms can also appear in brane world-volume actions, and are relevant in brane-world models. Hence it is important to study the behavior of such actions.

We find AdS solutions from this action. AdS solutions are particularly interesting for the AdS/CFT correspondence. This correspondence \[39-41\] has provided a remarkable way to study quantum gravity by quantum field theory on the boundary of the spacetime. It reveals that the bulk spacetime dynamically emerges from the degrees of freedom on the boundary \[42-45\]. We further obtain the action of the fluctuation fields on the AdS background at quadratic and cubic orders.

The organization of the rest of this paper is as follows. In subsection II A we analyze the gravity action coupled non-minimally with a scalar field, with higher-derivative terms, and find AdS solutions. In subsection II B we expand the fluctuation fields around the AdS background that we obtained. In subsection II C we take care of the gauge fixing conditions and compute the corresponding ghost action. In subsections II D and II E we obtain the action of the fluctuation fields on the AdS background at quadratic and cubic orders.
respectively, which will be relevant to the correlation functions in the study of AdS/CFT. In subsection II F, we briefly discuss some relations to AdS/CFT for these AdS solutions. Finally, we draw some conclusions in section III.

II. ADS BACKGROUNDS AND BULK GRAVITY

A. Bulk Gravity and AdS

The bulk gravity action is

\[ S = \int d^d x \sqrt{-g} \left[ \frac{1}{2} \xi R \phi^2 + a R^2 - \frac{1}{2} (\partial \mu \phi)^2 - \frac{1}{2} m^2 \phi^2 - V(\phi) \right]. \] (1)

The total spacetime dimension is \( d \). The \( \phi \) is the scalar field in the bulk, \( R \) is the Ricci scalar for the corresponding bulk metric \( g_{\mu \nu} \), \( m^2 \) is the mass-squared of the scalar, while the potential \( V(\phi) \) dictates the self-interactions of the scalar. The coupling \( \xi \) is dimensionless and positive in all spacetime dimensions, i.e. \( \xi \geq 0 \). The signature taken is \( \{-,+,+,+,+\cdots\} \), while the spacetime dimension \( d \) is kept arbitrary. The term \( \frac{1}{2} \xi R \phi^2 \) in the action encodes the explicit non-minimal coupling of the scalar field \( \phi \) to the Ricci curvature \( R \).

For the Euclidean path integral to be bounded from below, we demand that \( a \geq 0 \).

We will study the solutions to the equation of motion of the above action to look for cases where AdS geometry can be realized. The equation of motion for the above system can be obtained by varying the action with respect to fields \( g_{\mu \nu} \) and \( \phi \). The equation of motion for the action given in Eq. (1) is given by

\[ a \left[ \frac{1}{2} g_{\mu \nu} R^2 - 2 RR_{\mu \nu} + 2 \nabla_{\mu} \nabla_{\nu} R - 2 g_{\mu \nu} \Box R \right] + \frac{1}{2} \xi \phi^2 \left( \frac{1}{2} g_{\mu \nu} R - R_{\mu \nu} \right) + \frac{1}{2} \xi (\nabla_{\mu} \nabla_{\nu} - g_{\mu \nu} \Box) \phi^2 - \frac{1}{2} \left( \frac{1}{2} g_{\mu \nu} (\partial \phi)^2 - \partial_{\mu} \phi \partial_{\nu} \phi \right) - \frac{1}{2} g_{\mu \nu} \left( \frac{1}{2} m^2 \phi^2 + V(\phi) \right) = 0, \] (2)

and

\[ \Box \phi + \xi R \phi - m^2 \phi - V'(\phi) = 0. \] (3)

The first and second equations come by varying the action with respect to fields \( g_{\mu \nu} \) and \( \phi \) respectively.

To search for maximally symmetric solutions such as the AdS geometry we take

\[ g_{\mu \nu} = \bar{g}_{\mu \nu}, \quad R = \bar{R}, \quad \phi = \bar{\phi}, \] (4)

where a bar on quantities denotes their background values on the solution. For these geometries

\[ \bar{R}_{\mu \nu \rho} = \frac{\bar{R}}{d(d-1)} (\bar{g}_{\mu \rho} \bar{g}_{\nu \sigma} - \bar{g}_{\mu \sigma} \bar{g}_{\nu \rho}), \quad \bar{R}_{\mu \nu} = \frac{\bar{R}}{d} \bar{g}_{\mu \nu}, \] (5)

and \( \bar{R} \) doesn’t depend on spacetime. On AdS spaces \( \bar{R} < 0 \), while \( \bar{R} > 0 \) for dS, and \( \bar{R} = 0 \) for flat spaces. The equation of motion simplifies very much if one chooses to look for maximally symmetric solutions. This gives the background geometry.
For maximally symmetric solutions, the simplified equation of motion for these is given by,

\[
\frac{a(d - 4)}{d} \ddot{R}^2 + \frac{\xi(d - 2)}{2d} \ddot{\phi}^2 \ddot{R} - V(\ddot{\phi}) - \frac{1}{2} m^2 \ddot{\phi}^2 = 0, \tag{6}
\]

\[
\xi \ddot{\phi} - m^2 \dot{\phi} - V'(\phi) = 0. \tag{7}
\]

Then we get

\[
\ddot{R} = \left( m^2 + \frac{V'(\phi)}{\phi} \right) \frac{1}{\xi}. \tag{8}
\]

The value \( \ddot{\phi} \) is the solution to the equation

\[
\frac{a(d - 4)}{\xi^2d} \left( m^2 + \frac{V'(\phi)}{\phi} \right)^2 - \frac{1}{d} m^2 \ddot{\phi}^2 - V(\phi) + \frac{d - 2}{2d} V'(\phi) \ddot{\phi} = 0. \tag{9}
\]

For AdS solution,

\[
m^2 + \frac{V'(\phi)}{\phi} < 0, \tag{10}
\]

hence \( \ddot{R} < 0 \) for the AdS geometry.

The gravitational dynamics is dictated by the non-minimal piece \( (R \phi^2) \). In the case when \( \phi \) acquires a nonzero vacuum expectation value (vev), the action acquires a simple form whose resemblance is that of Einstein-Hilbert gravity with cosmological constant term. The effective Newton’s constant \( G_N \) is extracted from the coefficient of the kinetic term of the metric fluctuation

\[
\left( \frac{1}{2} a \ddot{R} + \frac{\xi \ddot{\phi}^2}{8} \right) \sqrt{-\bar{g}} h_{\mu\nu} \Box h^{\mu\nu} = \frac{1}{16 \pi G_N} \sqrt{-\bar{g}} \frac{1}{4} h_{\mu\nu} \Box h^{\mu\nu}. \tag{11}
\]

(See Eq. (50) for detailed derivation.) The part of the residual action including the Einstein-Hilbert and the cosmological constant pieces, can be written as,

\[
S = \frac{1}{16 \pi G_N} \int d^d x \sqrt{-g} [R - \Lambda]. \tag{12}
\]

So from here one has \( \Lambda < 0 \) when one has an AdS geometry. The Newton’s constant is obtained after the scalar field receives the vev. The corresponding Newton’s constant is

\[
G_N = \frac{1}{8 \pi (4a \ddot{R} + \xi \ddot{\phi}^2)}. \tag{13}
\]

The positivity of the Newton’s constant requires that

\[
\frac{\ddot{\phi}^2}{|\ddot{R}|} > \frac{4a}{\xi}. \tag{14}
\]

We have that \( G_N = l_p^{d-2} \), and the mass dimension is \( \left[ G_N^{-1} \right] = [M^{d-2}] \). The \( d \)-dimensional Planck constant is

\[
l_p = [8 \pi (4a \ddot{R} + \xi \ddot{\phi}^2)]^{-1/(d-2)}. \tag{15}
\]

The radius of curvature of the AdS background is

\[
R_{AdS} = \sqrt{-\frac{\xi d(d - 1)}{m^2 + V'(\phi)/\phi}}. \tag{16}
\]
There is implicit dependence of $R_{AdS}$ on $a$. In the Planck unit,

$$\frac{R_{AdS}}{l_p} = \sqrt{-\xi d(d-1)} \frac{\left[8\pi (4aR + \xi \phi^2)\right]^2}{m^2 + V'(\phi)/\phi}. \quad (17)$$

Our solutions for the metric of the AdS background in Poincare coordinates is

$$ds^2 = \frac{R_{AdS}^2}{z^2} [dz^2 + \eta_{ab} dx^a dx^b], \quad (18)$$

where $z$ and $x^a$ with $a = 0, 1, ..., d - 2$ are $d$ dimensional coordiantes, $\eta_{ab}$ is a metric of $d - 1$ dimensional flat spacetime with $a, b = 0, 1, ..., d - 2$, the radius $R_{AdS}$ is (17), and $\bar{R}$ and $\bar{\phi}$ are (8).

We then consider examples of the potential $V(\phi)$ in various dimensions. In general dimensions including $d = 4$, we consider an example

$$V(\phi) = \frac{\lambda}{4} \phi^4 + \frac{\mu}{3} \phi^3, \quad \lambda > 0. \quad (19)$$

Then using equation (18), we get, for $d = 4$,

$$\bar{\phi} = -\frac{3m^2}{\mu}, \quad (20)$$

$$\bar{R} = \frac{m^2}{\xi} \left(\frac{9\lambda m^2}{\mu^2} - 2\right). \quad (21)$$

For the above AdS solutions with $\bar{R} < 0$, the condition on the parameters is $0 < m^2 < \frac{2\mu^2}{9\lambda}$ for $d = 4$. While for $d \neq 4$, the solutions are more complicated looking, and for $d \neq 4, a = 0$, we have a simple expression,

$$\bar{\phi} = -\frac{\mu}{3(d-4)\lambda} \left(d - 6 + \sqrt{(d-6)^2 + 36(d-4)\frac{\lambda m^2}{\mu^2}}\right), \quad (22)$$

$$\bar{R} = \frac{d}{\xi(d-4)} \left(m^2 + \frac{1}{3} \mu \bar{\phi}\right). \quad (23)$$

For the above AdS solutions with $\bar{R} < 0$, the condition on the parameters is: $0 < m^2 < \frac{2\mu^2}{9\lambda}$ for $4 < d \leq 6$, while $-\frac{(d-6)^2 \mu^2}{36(d-4)\lambda} \leq m^2 < \frac{2\mu^2}{9\lambda}$ for $d > 6$. These AdS solutions are obtained when the scalar field acquires a vacuum expectation value. Here we have shown explicitly the examples with the potential (19). One could also have more complicated form of $V(\phi)$.

The variation of fields on the background are $\phi = \bar{\phi} + \epsilon \varphi$ and $g_{\mu\nu} = \bar{g}_{\mu\nu} + \epsilon h_{\mu\nu}$. The potential can be expanded near $\bar{\phi}$ as

$$V(\phi) = V(\bar{\phi}) + \epsilon V'(\bar{\phi}) \varphi + \frac{1}{2} \epsilon^2 V''(\bar{\phi}) \varphi^2 + \frac{1}{6} \epsilon^3 V'''(\bar{\phi}) \varphi^3 + \frac{1}{24} \epsilon^4 V''''(\bar{\phi}) \varphi^4 + \cdots. \quad (24)$$

The derivatives of $V(\phi)$ near $\bar{\phi}$ will be used in the next subsections.

We also mention that more general solutions with non-constant $\bar{\phi}$ from Eq. (3) are more complicated, and one may use the useful techniques developed in [46, 47].
B. Expansions

In the last subsection we have found solutions that has an AdS geometry. For this we first compute the equation of motion by doing a generic variation of action with respect to fields. This will result in equation of motion. Moreover, if we expand the action Eq. (1) around the background dictated by the last subsection, in powers of fluctuation fields, then it is seen that the linear order terms in the fluctuations will correspond to equation of motion, the second order terms will give the kinetic behavior (propagator, masses), while the higher order terms will give the interactions. Here we will be only concerned with computation up to third order expansion of action given in Eq. (1). In the following we will give the expansions of various geometrical quantities, followed by second and third order expansions of the action. These expansions will be relevant for the two-point and three-point correlators in AdS/CFT.

Here we give details of the various series expansions that are used in this paper. The fields are decomposed around a fixed background, where the background carries the spacetime dependence which is the outcome of equation of motions of the theory. We have that

\[ g_{\mu\nu} = \bar{g}_{\mu\nu} + \epsilon h_{\mu\nu}, \quad \phi = \bar{\phi} + \epsilon \varphi. \]  

(25)

Here \( \epsilon \) is a small parameter which is an expansion parameter. It is incorporated in order to keep track of the expansion-order. Under this decomposition, the expansion of the inverse metric and the determinant of metric is given by,

\[ g^{\mu\nu} = \bar{g}^{\mu\nu} - \epsilon h^{\mu\nu} + \epsilon^2 h^\mu_\alpha h^{\alpha\nu} - \epsilon^3 h^\mu_\alpha h^\beta_\beta h^{\beta\nu} + \cdots, \]

\[ \sqrt{-g} = \sqrt{-\bar{g}} \left[ 1 + \frac{\epsilon}{2} h + \epsilon^2 \left( \frac{h^2}{8} - \frac{1}{4} h_{\mu\nu} h^{\mu\nu} \right) + \epsilon^3 \left( \frac{h^3}{48} - \frac{1}{8} h_{\mu\nu} h^{\mu\nu} + \frac{1}{6} h^\mu_\alpha h^\beta_\beta h^\alpha_\mu \right) + \cdots \right], \]  

(26)

where \( h = \bar{g}^{\mu\nu} h_{\mu\nu} \). The expansion of Christoffel connection can be obtained by plugging the expansion of metric and its inverse in the definition of the Christoffel connection, and collecting pieces at various orders of \( \epsilon \). This gives

\[ \Gamma^\alpha_{\mu\beta} = \bar{\Gamma}^\alpha_{\mu\beta} + \epsilon \left( \bar{\nabla}_\mu h^\beta_\beta + \bar{\nabla}_\beta h^\mu_\alpha - \bar{\nabla}_\alpha h^\mu_\beta \right) - \epsilon^2 h^{\mu\rho} \left( \bar{\nabla}_\alpha h^\rho_\beta + \bar{\nabla}_\beta h^\rho_\alpha - \bar{\nabla}_\rho h^\alpha_\beta \right) + \epsilon^3 h^\mu_\alpha h^\rho_\rho \left( \bar{\nabla}_\alpha h^{\rho\beta} + \bar{\nabla}_\beta h^{\rho\alpha} - \bar{\nabla}_\rho h^{\alpha\beta} \right) + \cdots. \]

(27)

From here the variation of Christoffel connection at various orders can be read easily. Using this one can perform a series expansion of the various curvature tensors: Riemann tensor, Ricci tensor and Ricci scalar. We define the Riemann tensor in the following manner,

\[ [\nabla_\mu, \nabla_\nu] V^\rho = R^\rho_{\mu\nu} V^\sigma, \quad R^\rho_{\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma} - \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma}, \]  

(28)

where \( V^\mu \) is an arbitrary vector field. The expansion of the Riemann tensor is given by,

\[ R^\rho_{\mu\nu} = \bar{R}^\rho_{\mu\nu} + \epsilon \left( \bar{\nabla}_\mu \bar{\Gamma}^{(1)\rho}_{\nu} - \bar{\nabla}_\nu \bar{\Gamma}^{(1)\rho}_{\mu} \right) + \epsilon^2 \left( \bar{\nabla}_\mu \bar{\Gamma}^{(2)\rho}_{\nu} - \bar{\nabla}_\nu \bar{\Gamma}^{(2)\rho}_{\mu} + \bar{\Gamma}^{(1)\rho}_{\nu} \bar{\Gamma}^{(1)\lambda}_{\mu} - \bar{\Gamma}^{(1)\rho}_{\mu} \bar{\Gamma}^{(1)\lambda}_{\nu} \right) + \epsilon^3 \left( \bar{\nabla}_\mu \bar{\Gamma}^{(3)\rho}_{\nu} - \bar{\nabla}_\nu \bar{\Gamma}^{(3)\rho}_{\mu} + \bar{\Gamma}^{(1)\rho}_{\nu} \bar{\Gamma}^{(2)\lambda}_{\mu} + \bar{\Gamma}^{(2)\rho}_{\nu} \bar{\Gamma}^{(1)\lambda}_{\mu} - \bar{\Gamma}^{(1)\rho}_{\mu} \bar{\Gamma}^{(2)\lambda}_{\nu} - \bar{\Gamma}^{(2)\rho}_{\mu} \bar{\Gamma}^{(1)\lambda}_{\nu} \right) + \cdots. \]

(29)
The Ricci tensor is then obtained by contracting the first and third index of the above expression of the Riemann tensor. Its expansion is given by,

\[
R_{\nu\sigma} = R_{\rho\nu}^{\rho\sigma} = \bar{R}_{\nu\sigma} + \epsilon \left( \bar{\nabla}_{\rho} \Gamma_{\nu}^{(1)\rho}_{\sigma} - \bar{\nabla}_{\nu} \Gamma_{\rho}^{(1)\rho}_{\sigma} \right) \\
+ \epsilon^2 \left( \bar{\nabla}_{\rho} \Gamma_{\nu}^{(2)\rho}_{\sigma} - \bar{\nabla}_{\nu} \Gamma_{\rho}^{(2)\rho}_{\sigma} + \Gamma_{\rho}^{(1)\lambda}_{\nu} \Gamma_{\rho}^{(1)\lambda}_{\sigma} - \Gamma_{\nu}^{(1)\rho}_{\rho} \Gamma_{\rho}^{(1)\lambda}_{\sigma} \right) \\
+ \epsilon^3 \left( \bar{\nabla}_{\rho} \Gamma_{\nu}^{(3)\rho}_{\sigma} - \bar{\nabla}_{\nu} \Gamma_{\rho}^{(3)\rho}_{\sigma} + \Gamma_{\rho}^{(1)\lambda}_{\nu} \Gamma_{\rho}^{(2)\lambda}_{\sigma} + \Gamma_{\rho}^{(2)\lambda}_{\nu} \Gamma_{\rho}^{(1)\lambda}_{\sigma} - \Gamma_{\nu}^{(1)\rho}_{\rho} \Gamma_{\rho}^{(2)\lambda}_{\sigma} - \Gamma_{\nu}^{(2)\rho}_{\rho} \Gamma_{\rho}^{(1)\lambda}_{\sigma} \right) + \cdots .
\]

(30)

From this we note that a useful handy expression that often enters these expansion. These expression in terms of fluctuation \( h_{\mu\nu} \) can be written in simple forms as,

\[
\Gamma_{\rho}^{(1)\rho}_{\sigma} = \frac{1}{2} \bar{\nabla}_{\rho} h_{\sigma} , \quad \Gamma_{\rho}^{(2)\rho}_{\sigma} = \frac{1}{2} h^{\alpha\rho} \bar{\nabla}_{\sigma} h_{\rho\alpha} , \quad \Gamma_{\rho}^{(3)\rho}_{\sigma} = \frac{1}{2} h^{\rho\theta} h^{\alpha\rho} \bar{\nabla}_{\sigma} h_{\rho\alpha} .
\]

(31)

One can use these to obtain the expansion of the Ricci tensor. At each order one can perform commutation of covariant derivatives to obtain expressions which involve divergences of \( h_{\mu\nu} \). At first order it is given by,

\[
R_{\nu\sigma}^{(1)} = \frac{1}{2} \left[ \bar{\nabla}_{\nu} \bar{\nabla}_{\rho} h^{\rho}_{\sigma} + \bar{\nabla}_{\sigma} \bar{\nabla}_{\rho} h^{\rho}_{\nu} + \bar{\nabla}_{\nu} \bar{\nabla}_{\rho} h^{\rho}_{\lambda} + \bar{\nabla}_{\lambda} h^{\lambda}_{\rho} - 2 \bar{\nabla}_{\rho}^{\lambda} h^{\rho}_{\lambda} - \Box h_{\nu\sigma} - \bar{\nabla}_{\nu} \bar{\nabla}_{\sigma} h \right] .
\]

(32)

At second order the expansion is given by,

\[
R_{\mu\nu}^{(2)} = \bar{\nabla}_{\mu} \Gamma_{\nu}^{(2)\nu}_{\rho} - \bar{\nabla}_{\nu} \Gamma_{\rho}^{(2)\nu}_{\nu} + \frac{1}{4} \left[ \bar{\nabla}^{\lambda} h \left( \bar{\nabla}_{\lambda} h_{\rho\nu} + \bar{\nabla}_{\nu} h_{\lambda\rho} - \bar{\nabla}_{\lambda} h_{\rho\nu} \right) - \bar{\nabla}_{\rho} h^{\rho\lambda} \bar{\nabla}_{\nu} h_{\rho\lambda} \right]
\]

\[
- \bar{\nabla}^{\lambda} h^{\rho}_{\mu} \bar{\nabla}_{\rho} h_{\lambda\nu} + \bar{\nabla}^{\lambda} h^{\rho}_{\mu} \bar{\nabla}_{\nu} h_{\rho\lambda} + \bar{\nabla}^{\rho} h^{\lambda}_{\mu} \bar{\nabla}_{\rho} h_{\lambda\nu} - \bar{\nabla}^{\rho} h^{\lambda}_{\mu} \bar{\nabla}_{\lambda} h_{\rho\nu} \right] .
\]

(33)

At third order the expansion is given by,

\[
R_{\mu\nu}^{(3)} = \bar{\nabla}_{\mu} \Gamma_{\nu}^{(3)\nu}_{\rho} - \bar{\nabla}_{\nu} \Gamma_{\rho}^{(3)\nu}_{\nu} - \frac{1}{4} \left( \bar{\nabla}_{\nu} h \right) h^{\alpha\lambda} \left( \bar{\nabla}_{\alpha} h_{\lambda\nu} + \bar{\nabla}_{\nu} h_{\lambda\alpha} - \bar{\nabla}_{\lambda} h_{\nu\alpha} \right)
\]

\[
- \frac{1}{4} \left( h^{\alpha\rho} \bar{\nabla}_{\rho} h_{\alpha\nu} \right) \left( \bar{\nabla}_{\nu} h_{\lambda\rho} + \bar{\nabla}_{\rho} h_{\nu\lambda} - \bar{\nabla}_{\nu} h_{\lambda\rho} \right) - \frac{1}{2} h^{\lambda\alpha} \left[ \bar{\nabla}_{\nu} h^{\rho}_{\nu} \bar{\nabla}_{\alpha} h_{\rho\nu} + \bar{\nabla}_{\nu} h^{\rho}_{\nu} \bar{\nabla}_{\nu} h_{\rho\nu} \right]
\]

\[
- \bar{\nabla}^{\rho} h^{\lambda}_{\nu} \bar{\nabla}_{\rho} h_{\alpha\nu} + \bar{\nabla}^{\rho} h^{\lambda}_{\nu} \bar{\nabla}_{\alpha} h_{\rho\nu} \right] .
\]

(34)

In order to obtain the expansion of the Ricci scalar one has to make use of both the expansion of the inverse metric and Ricci tensor. At first order this expansion is given by,

\[
R^{(1)} = \bar{\nabla}_{\mu} \bar{\nabla}_{\nu} h_{\nu\mu} - \Box h - \bar{R}_{\mu\nu} h^{\mu\nu} .
\]

(35)

At second order we have the following,

\[
R^{(2)} = \bar{R}_{\rho\nu\lambda\sigma} h^{\rho\lambda} h^{\nu\sigma} - h^{\sigma\rho} \bar{\nabla}_{\nu} \bar{\nabla}_{\rho} h^{\rho}_{\sigma} + \frac{1}{2} h^{\nu\sigma} \Box h_{\nu\sigma} + \frac{1}{2} h^{\nu\sigma} \bar{\nabla}_{\nu} \bar{\nabla}_{\rho} h + \frac{1}{2} \bar{\nabla}^{\lambda} h \bar{\nabla}^{\sigma} h_{\lambda\sigma}
\]

\[
- \frac{1}{4} \bar{\nabla}^{\lambda} h \bar{\nabla}_{\lambda} h^{\nu}_{\nu} \bar{\nabla}^{\rho} h^{\rho}_{\nu} - \frac{1}{2} \bar{\nabla}^{\lambda} h \bar{\nabla}^{\mu} \bar{\nabla}^{\nu\lambda} h_{\mu\nu} + \bar{g}^{\rho\sigma} \left( \bar{\nabla}_{\rho} \Gamma_{\nu}^{(2)\rho}_{\sigma} - \bar{\nabla}_{\nu} \Gamma_{\rho}^{(2)\rho}_{\sigma} \right) .
\]

(36)
At third order the expansion is more elaborate as there are more number of terms. We have,

\[
R^{(3)} = \tilde{g}^{\mu \nu} \left( \nabla^2 \Gamma^{(3)}_{\rho} \right) - \frac{1}{4} h^\lambda_\alpha (\nabla_\lambda h) \left( 2\nabla^\nu h_{\nu \alpha} - \nabla_\alpha h \right) + \frac{1}{4} (h_{\rho \lambda} \nabla^\alpha h^\rho_\lambda) \left( 2\nabla^\nu h_{\nu \alpha} - \nabla_\alpha h \right) - \frac{1}{2} h^\lambda_\alpha \left( \nabla_\nu h^\rho_\lambda \nabla^\nu_\alpha + \nabla_\lambda h^\rho_\nu \nabla_\alpha h^\nu_\rho + \nabla^\nu_\alpha \nabla_\nu h^\rho_\lambda \nabla_\rho h^\nu_\alpha \right) \\
+ \nabla^\nu h_{\nu \lambda} \nabla_\alpha h^\rho_\nu - h^\nu_\nu h^\theta_\alpha h^\alpha_\sigma \bar{R}_{\nu \sigma} + \frac{1}{2} h^\nu_\nu h^\alpha_\sigma \left( \nabla_\nu \nabla_\rho h^\rho_\nu + \nabla_\sigma \nabla_\rho h^\rho_\nu + \bar{R}_{\nu \lambda} h^\lambda_\sigma + \bar{R}_{\sigma \lambda} h^\lambda_\nu \right) - 2 \bar{R}_{\nu \mu \lambda} h^\rho_\lambda h^\nu_\sigma - \Box h_{\nu \sigma} - \nabla_\nu \nabla_\sigma h) - h^{\mu \nu} (\nabla^2 \Gamma^{(2)}_{\rho} + \nabla_\nu \nabla^2 \Gamma^{(2)}_{\rho} \right) + \frac{1}{4} h^{\mu \nu} \left[ \nabla^\lambda h \left( \nabla_\mu h_{\nu \lambda} + \nabla_\nu h_{\lambda \mu} \right) \right.
\]
\[- \delta \lambda h_{\mu \nu} - \nabla_\lambda h^{\rho \lambda} \nabla_\mu h_{\rho \lambda} - \nabla^\lambda h^\rho_\mu \nabla_\rho h_{\lambda \mu} + \nabla^\lambda h^\rho_\mu \nabla_\lambda h_{\rho \mu} + \nabla^\rho h_{\lambda \mu} \nabla_\rho h^\lambda_\mu - \nabla^\rho h^\lambda_\mu \nabla_\lambda h_{\rho \mu} \right].
\]

(37)

C. Gauge-fixing and Faddeev-Popov ghosts

Here in this subsection we will take into account the gauge fixing condition and compute the corresponding ghost action.

We analyze the diffeomorphism invariant action of the coupled system using background field method \[48, 49\]. It is advantageous, as by construction it preserves background gauge invariance. The field is decomposed into background and fluctuation. Keeping the background fixed the path-integral is then reduced to an integral over the fluctuations. The gravitational metric field is decomposed into background and fluctuation. To prevent over-counting of gauge-orbits in the path-integral measure, a constraint is applied on this fluctuation field, which results in appearance of auxiliary fields called ghosts. This procedure of systematically applying the constraint leading to ghost can be elegantly taken care of by the Faddeev-Popov prescription \[50, 51\]. The effective action generated after integrating over the fluctuation and auxiliary fields still enjoys invariance over the background fields.

The diffeomorphism invariance of the full action in Eq. (1) implies that for arbitrary vector field \( V^\rho \), the action should be invariant under the following transformation of the metric field variable,

\[
\delta_D \gamma_{\mu \nu} = \mathcal{L}_V \gamma_{\mu \nu} = V^\rho \partial_\rho \gamma_{\mu \nu} + \gamma_{\mu \rho} \partial_\nu V^\rho + \gamma_{\nu \rho} \partial_\mu V^\rho,
\]

(38)

where \( \mathcal{L}_V \gamma_{\mu \nu} \) is the Lie derivative of the quantum metric \( \gamma_{\mu \nu} \) along the vector field \( V^\rho \). Decomposing the quantum metric \( \gamma_{\mu \nu} \) into background (\( \tilde{g}_{\mu \nu} \)) and fluctuation (\( \epsilon h_{\mu \nu} \)) allows one to figure out the transformation of the fluctuation field while keeping the background fixed. This will imply the following transformation of \( h_{\mu \nu} \),

\[
\delta_D h_{\mu \nu} = \frac{1}{\epsilon} \left( \nabla_\mu V_\nu + \nabla_\nu V_\mu \right) + V^\rho \nabla_\rho h_{\mu \nu} + \epsilon h_{\mu \rho} \nabla_\nu V^\rho + \epsilon h_{\nu \rho} \nabla_\mu V^\rho,
\]

(39)

where \( \nabla \) is the covariant derivative whose connection is constructed using the background metric. This is the full transformation of the metric fluctuation field. For small \( \epsilon \) only the leading term is relevant. In the quantum theory it is seen that the leading term leads to one-loop effects while higher-loop comes from \( \mathcal{O}(\epsilon^0) \) terms. The invariance of the action is broken by choosing an appropriate gauge-fixing condition implemented via Faddeev-Popov procedure \[50\].
The gauge fixing action chosen for fixing the invariance under the transformation of the metric fluctuation field is given by,

\[ S_{gf} = \frac{1}{2\alpha} \int d^d x \sqrt{-\bar{g}} (\bar{\nabla}^\rho h_{\rho\mu}) \bar{g}^{\mu\nu} (\bar{\nabla}^\sigma h_{\sigma\nu}). \]  

This gauge-fixing action introduces the gauge-fixing parameter \( \alpha \). When \( \alpha = 0 \) the gauge condition is imposed sharply, resulting in Landau gauge \( F_\nu = \bar{\nabla}_\mu h_{\mu\nu} = 0 \). The breaking of gauge-invariance leads to Faddeev-Popov ghosts which is obtained following the prescription given in \([50]\). We introduce gauge-condition in the path-integral by multiplying the later with unity in the following form,

\[ 1 = \int \mathcal{D} F^V_\mu \exp \left[ \frac{i}{2\alpha} \int d^d x \sqrt{-\bar{g}} F^V_\mu \bar{g}^{\mu\nu} F^V_\nu \right], \]  

where \( F^V_\mu \) is the gauge transformed \( F_\mu \). The original path-integral (without gauge-fixing) is invariant under transformation Eq. (39) of the field \( h_{\mu\nu} \), and this implies that a change of integration variable from \( h_{\mu\nu} \) to \( h^V_{\mu\nu} \) doesn’t give rise to any Jacobian in the path-integral measure. However replacing the measure over \( F^V_\mu \) with the measure over \( V_\rho \) introduces a non-trivial Jacobian in the path-integral. This is obtained as follows,

\[ \mathcal{D} F^V_\mu = \det \left( \frac{\partial F^V_\mu}{\partial V_\rho} \right) \mathcal{D} V_\rho. \]  

In the background field formalism this Jacobian consists of background covariant derivative, background and fluctuation fields, and is independent of the transformation parameter \( V_\rho \). This implies that it can be taken out of the functional integral over \( V_\rho \). Changing the integration variable from \( h^V_{\mu\nu} \) to \( h_{\mu\nu} \), and ignoring the infinite constant generated by integrating over \( V_\rho \), gives us the gauge fixed path integral.

The Faddeev-Popov determinant in Eq. (42) is then exponentiated by making use of anti-commuting auxiliary fields. These auxiliary fields are known as Faddeev-Popov ghosts. The path integral of the full ghost sector is given by,

\[ \int \mathcal{D} \bar{C}_\mu \mathcal{D} C_\nu \exp \left[ -i \int d^d x \sqrt{-\bar{g}} \left\{ \bar{C}_\mu \left( \frac{\partial F_\nu}{\partial V_\rho} \right) C^\rho \right\} \right], \]  

where \( \bar{C}_\mu \) and \( C_\nu \) are Faddeev-Popov ghost fields arising from the gauge fixing in the gravitational sector.

In the case when \( F_\mu \) is given as in Eq. (40), the Faddeev-Popov ghost action is given by,

\[ S_{gh}^{FP} = - \int d^d x \sqrt{-\bar{g}} \bar{C}_\mu X^\mu_\rho C^\rho, \]  

where,

\[ X^\mu_\rho = \bar{g}^{\mu\nu} \left[ \frac{1}{\epsilon} (\bar{\nabla}_\rho \bar{\nabla}_\nu + \bar{g}_{\rho\nu} \Box) + \bar{\nabla}_\rho h_{\sigma\nu} \bar{\nabla}^\sigma + \bar{\nabla}^\sigma \bar{\nabla}_\rho h_{\sigma\nu} + \bar{\nabla}_\rho h_{\sigma\nu} \bar{\nabla}^\sigma + h_{\sigma\nu} \bar{\nabla}_\rho \right] + \bar{\nabla}_\rho h_{\sigma\nu} \bar{\nabla}^\sigma + h_{\sigma\rho} \bar{\nabla}^\sigma \bar{\nabla}_\nu. \]
Here the last several terms contain terms linear in $h_{\mu\nu}$. These are not relevant in doing one-loop computations, but at higher-loops they are important. The full action of the theory (which includes the coupled gravity and matter action, gauge fixing and ghost action) possesses BRS invariance. Furthermore, in flat spacetime there is a residual symmetry coming from dilatation invariance \[52\]. One can still make the following coordinate transformation

$$\omega^\mu(x) = \omega x^\mu,$$

where $\omega$ is a constant parameter. In flat spacetime this is a dilatation. This symmetry invariance is broken by imposing $h = 0$, where $h = \text{tr} h^\mu_\mu$. This doesn’t lead to generation of ghost action \[53\]. Furthermore, on AdS background with fixed AdS radius, dilatation invariance is broken. In curved space following \[53\] we use $h = 0$ as the trace-free condition. In fact this can be incorporated in the action of the theory by using a Lagrange multiplier \[52\]. Variation with respect to Lagrange multiplier leads to constraint $h = 0$. This can be viewed as a constraint imposed to break dilatation invariance. In the following we will impose the Landau gauge ($\nabla^\mu h_{\mu\nu} = 0$) and dilatation breaking ($h = 0$) to obtain the second and third order terms on the maximally symmetric background.

### D. Second order

In this subsection we compute the second order terms for the action given in Eq. (11). The full second variation on a general background without the usage of gauge conditions is a complicated expression consisting of several terms. This acquires a bit simpler form once integration by parts is done. To compute second order variation of the action one needs the expansion up to second order in $\epsilon$ of $\sqrt{-g}$, $g^{\mu\nu}$, $g_{\mu\nu}$, $\Gamma^\rho_\mu_\nu$, and $R$. These expansions are computed in subsection II.B. Here we make use of these expansions to compute the variations of the action, on the background in subsection II.A. The second variation of the $R\phi^2$ term, without integration by parts, is given by,

$$S_{R\phi^2} = \frac{1}{2} \varepsilon^{2} \xi \int d^d x \sqrt{-g} \left[ \left( \frac{1}{2} h^2 - \frac{1}{4} h_{\mu\nu} h^{\mu\nu} \right) \bar{R} \phi^2 + \phi^2 \left\{ \bar{R} \phi + 2\phi \left( \nabla_\mu \nabla_\nu h^{\mu\nu} - \Box h - \bar{R} h^{\mu\nu} \right) \right\} + \bar{R} \phi^2 \right],$$

where the expansions of Christoffel appear in form of total derivative and won’t contribute in the bulk studies. However they generate a surface term. The second variation of $R^2$ term of the action is given by,

$$S_{R^2} = a \varepsilon^2 \int d^d x \sqrt{-g} \left[ \left( \frac{1}{2} h^2 - \frac{1}{4} h_{\mu\nu} h^{\mu\nu} \right) \bar{R}^2 + \bar{R} (\nabla_\mu \nabla_\nu h^{\mu\nu} - \Box h - \bar{R} h^{\mu\nu}) \right.$$

$$+ \nabla_\mu \nabla_\nu h^{\mu\nu} \nabla_\alpha \nabla_\beta h^{\alpha\beta} + \Box h \Box h + \bar{R} \mu\nu h^{\mu\nu} \bar{R} \alpha\beta h^{\alpha\beta} - 2
$$

$$\nabla_\mu \nabla_\nu h^{\mu\nu} \Box h - 2 \nabla_\mu \nabla_\nu h^{\mu\nu} \bar{R} \alpha\beta h^{\alpha\beta} + 2 \Box h \bar{R} \alpha\beta h^{\alpha\beta} + 2 \bar{R} \left\{ \bar{R} \phi + 2\phi \left( \nabla_\mu \nabla_\nu h^{\mu\nu} - \Box h - \bar{R} h^{\mu\nu} \right) \right\} + \bar{R} \phi^2 \right],$$

10
\[ -\frac{1}{4} \nabla^\lambda h \nabla_\lambda h + \frac{1}{4} \nabla^\rho h^{\nu\lambda} \nabla_\rho h_{\nu\lambda} - \frac{1}{2} \nabla^\lambda h^{\mu\nu} \nabla_\rho h_{\mu\lambda} + \frac{1}{2} h(\nabla_\mu \nabla_\nu h^{\mu\nu} - \Box h - R_{\mu\nu} h^{\mu\nu}) + g^{\nu\sigma}(\nabla_\rho \Gamma_\nu^{(2)\rho} - \nabla_\nu \Gamma_\rho^{(2)\rho}) \bigg] . \] (48)

The second variation for the matter part is a bit simpler as the gravitational couplings are simpler. These are given by,

\[ S_{\text{matter}} = \varepsilon^2 \int d^d x \sqrt{-g} \left[ -\left( \frac{1}{8} h^2 - \frac{1}{4} h_{\mu\nu} h^{\mu\nu} \right) \left( \frac{1}{2} m^2 \bar{\phi}^2 + V(\bar{\phi}) \right) - \frac{1}{2} h(m^2 \bar{\phi} + V'(\bar{\phi})) \varphi + \frac{1}{2} \Box \varphi - \frac{1}{2} (m^2 + V''(\bar{\phi})) \varphi^2 \right] . \] (49)

These second order expansion terms are over generic background (with arbitrary background curvature but with constant \( \bar{\phi} \)) and in arbitrary dimensions. However this gets simplified once integration by parts is done and Landau gauge (\( \nabla^\mu h_{\mu
u} = 0 \)) and trace free condition (\( h = 0 \)) is employed. Under these constraints the second variation gets very simplified as many terms go away. Then the residual second variation is given by,

\[ S^{(2)} = \varepsilon^2 \int d^d x \sqrt{-g} \left[h_{\mu\nu} \left\{ \frac{1}{2} \xi \bar{\phi}^2 \left( \frac{1}{4} \Box - \frac{(d^2 - 3d + 4)\bar{R}}{4d(d-1)} \right) + a\bar{R} \left( \frac{1}{2} \Box - \frac{(d^2 - 5d + 8)\bar{R}}{4d(d-1)} \right) \right. \right. \\
+ \left. \left. \frac{1}{4} \left( \frac{1}{2} m^2 \bar{\phi}^2 + V(\bar{\phi}) \right) \right\} h^{\mu\nu} + \frac{1}{2} \varphi \left( \Box + \xi \bar{R} - m^2 - V''(\bar{\phi}) \right) \varphi \right] . \] (50)

The surface terms produced during this are included in appendix A.

E. Third order

The third order terms are needed to compute the cubic couplings. These will give the three-point correlators on the boundary of AdS spacetime, which will be relevant for AdS/CFT. At the third order the number of terms are quite many for each term of the action when expanded. We would make use of gauge conditions and requirement of maximally symmetric spacetime background. The third order terms coming from \( R\phi^2 \) piece of the action are

\[ S_{R\phi^2}^{(3)} = \frac{\varepsilon^3}{2} \xi \int d^d x \sqrt{-g} \left[ \bar{\phi}^2 \left( \frac{1}{6} h^{\mu\alpha} h^{\beta\gamma} h^{\rho\mu} R + R^{(3)} \right) + 2\bar{\phi} \varphi \left( -\frac{1}{4} h_{\mu\nu} h^{\mu\nu} \bar{R} + R^{(2)} \right) \right] , \] (51)

where we used the simplification that on a maximally symmetric background one has that the first variation of \( R^{(1)} \) is zero. The expressions for \( R^{(2)} \) and \( R^{(3)} \) are given in subsection \[. \]

For the case of \( R^2 \) piece of action the third variation is given by,

\[ S_{R^2}^{(3)} = a\varepsilon^3 \int d^d x \sqrt{-g} \left[ \frac{1}{6} h^{\mu\alpha} h^{\beta\gamma} h^{\rho\mu} \bar{R} + 2\bar{R} R^{(3)} \right] . \] (52)

We notice that this expansion has some similar structure with the expansion of \( R\phi^2 \) piece. The third order expansion for the matter sector is

\[ S_{\text{matter}}^{(3)} = \varepsilon^3 \int d^d x \sqrt{-g} \left[ \frac{1}{2} h^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - \frac{1}{2} V''(\bar{\phi}) \varphi^3 + \frac{1}{4} h^{\mu\nu} h_{\mu\nu} (m^2 + V'(\bar{\phi})) \varphi \right. \\
+ \frac{1}{6} h^{\mu\alpha} h^{\beta\gamma} h^{\rho\mu} \left( \frac{1}{2} m^2 \bar{\phi}^2 + V(\bar{\phi}) \right) \bigg] . \] (53)
We can combine all the pieces of the third order expansion of the action and write it in a
unified form. This is given by,

\[
S^{(3)} = \epsilon^3 \int d^d x \sqrt{-g} \left\{ \frac{(d + 2)\xi R\bar{\phi}^2}{12(d - 1)} + \frac{(d + 5)a R^2}{6(d - 1)} - \frac{1}{6} \left( \frac{1}{2} m^2 \bar{\phi}^2 + V(\bar{\phi}) \right) \right\} h^{\alpha \beta} h_{\alpha \beta} + \left( \frac{1}{2} \xi \bar{\phi}^2 + 2a \bar{R} \right) \left\{ -\left( \nabla_{\nu} h_{\lambda \rho} \right) h_{\mu \nu} + \frac{1}{2} \left( \nabla^\rho h_{\lambda \mu} \right) h_{\lambda \nu} + \frac{1}{2} h_{\mu \nu} \nabla_{\lambda} \nabla_{\mu} h_{\lambda \nu} \right\} + \frac{1}{4} \nabla_{\mu} \nabla_{\nu} h_{\lambda \rho} + \nabla_{\rho} h_{\mu \nu} \nabla_{\lambda} h_{\nu \lambda} \] 

\[
- \frac{1}{6} V''(\bar{\phi}) \bar{\phi}^3 + \frac{1}{4} h_{\mu \nu} \left( m^2 + V'(\bar{\phi}) \right) \bar{\phi} \] .
\] (54)

To obtain these one has to perform several integrations by parts which allowed us to make
use of the Landau gauge and trace-free condition. The surface terms generated in this
computation are included in appendix A.

F. AdS/CFT discussion

After the scalar field receives the vev, the gravity action has an Einstein-Hilbert term, as well as terms of matter fields and higher derivative terms. The Einstein-Hilbert term is
induced when the scalar acquires a non-zero vacuum expectation value. At the same time,
we obtained the maximally symmetric AdS background and the dynamics of its fluctuation
fields, and we can use the AdS/CFT correspondence \cite{39,41}. The dual field theory is in
its conformally invariant vacuum \cite{55}. The fluctuation fields on the AdS background are
obtained in the above subsections.

The fluctuation of the scalar field is \( \phi \). The quadratic part of the Lagrangian of \( \phi \) is

\[
L(\phi) = -\frac{1}{2} (\partial^2 \phi)^2 - \frac{1}{2} m^2_{\phi} \phi^2,
\] (55)

where

\[
m^2_{\phi} = m^2 - \xi R + V''(\bar{\phi}) = V''(\bar{\phi}) - V'(\bar{\phi})/\bar{\phi},
\] (56)

and we have used Eq. (5).

The operator dual to \( \phi \) is a scalar operator \( O \) with scaling dimension \( \Delta = \Delta \pm \) \cite{41,40},

\[
\Delta_{\pm} = \frac{d - 1}{2} \pm \sqrt{\left( \frac{d - 1}{2} \right)^2 + m^2_{\phi} R_{AdS}^2}
\]  

\[
= \frac{d - 1}{2} \pm \sqrt{\left( \frac{d - 1}{2} \right)^2 + \frac{V'(\bar{\phi}) - V''(\bar{\phi})/\bar{\phi}}{V'(\bar{\phi}) + m^2_{\phi}} \xi d(d - 1)},
\] (57)

where we used Eq. (56). The Breitenlohner-Freedman (BF) bound \cite{54} demands that

\[
\frac{1}{4} (d - 1) + \frac{V'(\bar{\phi}) - V''(\bar{\phi})/\bar{\phi}}{V'(\bar{\phi}) + m^2_{\phi}} \xi d \geq 0.
\] (58)

The corresponding two-point function of the scalar operator is

\[
\langle O(x)O(x') \rangle = \frac{dO}{|x - x'|^{2\Delta}},
\] (59)
where $d_O$ is the normalisation factor. The three-point functions are related to the cubic couplings in the third order variation of the action.

The AdS radius in the Planck unit is related to the number of degrees of freedom of the fields in the field theory side. We can derive that the central charge is

$$c_T \propto \left( \frac{R_{\text{AdS}}}{l_p} \right)^{d-2} = \left( \frac{-\xi d(d-1)}{m^2 + V'(\phi)/\phi} \right)^{(d-2)/2} \left[ 8\pi (4a\bar{R} + \xi \bar{\phi}^2) \right]. \quad (60)$$

The AdS solution is perturbatively stable when the fluctuation modes have mass-squared above the Breitenlohner-Freedman (BF) bound \[54\]. We find that the solutions are perturbatively stable, with the condition above, in which the AdS background is stable against small fluctuations.

III. CONCLUSIONS AND DISCUSSIONS

In this paper, we analyzed a higher derivative gravity model coupled non-minimally with scalar field with a general potential. The aim of the paper is to see whether such an action can have AdS geometry as solution to equation of motion. It has been seen that such an action can have an AdS geometry for a wide class of potential for field $\phi$ and certain choice of parameters. These actions have been widely studied in the context of induced gravity and higher-derivative gravity in four or more general spacetime dimensions, where the known low energy Einstein-Hilbert gravity emerges from a UV well defined model of gravity. Then it is natural to consider that such actions anticipate AdS geometry as solutions and also their relation to the extensively studied AdS/CFT.

Here, we start by looking into equation of motions to look for AdS geometry. We found the AdS solutions in various circumstances. We then compute the second and third variation of the action. For this we compute first the expansion of various geometrical quantities up to third order in the fluctuation fields. These expansions are then used to compute the expansion of the action of the full theory. We choose to work in Landau gauge where the propagation of longitudinal modes is suppressed. Moreover, in flat spacetime the usual harmonic gauge fixing condition doesn’t fix the invariance in the field $h_{\mu\nu}$ entirely. There is a residual freedom left which is caused by dilatation invariance. This can be fixed by using the trace-free condition. In AdS geometry with fixed AdS radius such dilatation invariance is broken, which means one lacks the freedom to have conformal transformation with arbitrary conformal factor. The trace-free condition has also been used in the context of asymptotic safety.

The expansion of the geometrical quantities is used to obtain the expansion of the action of the theory up to the third order. In holographic language the second order terms are used to obtain mass-parameter for various field modes, while the third order terms lead to various correlators on the boundary. More details of the study of these relations deserve further investigation and we leave them for future exploration.

We obtain a new scenario for AdS solutions, in particular in the context of induced gravity. This scenario can have various generalizations. We can couple other fields with the system. For example, we can add vector fields or fermions into the system. This scenario potentially has interesting applications for holographic correspondence. On the other hand, the modified gravity theory and the Einstein gravity have slightly different physical observables. For example, the wave form of the gravitational waves predicted from
scalar-tensor theory and the Einstein gravity are slightly different. Because the physical observables are slightly variant, hence understanding the detailed dictionary of holography for modified gravity theory is interesting and deserves future investigations.

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Appendix A: Boundary terms

Here in this appendix we write the boundary terms that are generated while performing integration by parts during the course of evaluation at various orders. These may play some role in CFT on the boundary.

At second order the boundary terms are given by,

\[
(\partial \mathcal{M})^{(2)} = \frac{1}{2} \left( \frac{1}{2} \xi \phi^2 + 2 a \dot{R} \right) \int \sqrt{-g} \, d^4 x \left[ -\frac{1}{2} \nabla_\nu (h^{\nu\sigma} \nabla_\rho h_\sigma) + \frac{1}{2} \nabla_\nu (h^{\nu\sigma} \nabla_\sigma h) - \frac{1}{4} \nabla_\lambda (h \nabla^\lambda h) + \frac{1}{4} \nabla_\rho (h^{\nu\lambda} \nabla_\rho h_{\nu\lambda}) - \frac{1}{2} \nabla_\lambda (h^{\nu\rho} \nabla_\rho h_{\nu\lambda}) - \frac{1}{2} \nabla_\lambda (\bar{g}^{\nu\sigma} \Gamma^{(2)^{\rho}}_{\nu\sigma}) - \nabla_\nu (\bar{g}^{\nu\sigma} \Gamma^{(2)}_{\rho \sigma}) - \frac{1}{2} \nabla_\nu (h^{\lambda\rho} h_\rho \nabla^{\nu} h_{\alpha\rho}) \right].
\]

At the third order the boundary terms are more complicated, as one has to perform more number of integrations by parts. And we have

\[
(\partial \mathcal{M})^{(3)} = \frac{1}{2} \left( \frac{1}{2} \xi \phi^2 + 2 a \dot{R} \right) \int \sqrt{-g} \, d^4 x \left[ \nabla_\rho (\bar{g}^{\nu\sigma} \Gamma^{(3)\rho}_{\nu\sigma}) - \nabla_\nu (\bar{g}^{\nu\sigma} \Gamma^{(3)}_{\rho \sigma}) - \frac{1}{2} \nabla_\nu (h^{\lambda\rho} h_\rho \nabla^{\nu} h_{\alpha\rho}) \right]
\]

\[+ \frac{1}{2} \nabla_\nu (h^{\lambda\rho} h_\rho \nabla_\lambda h_{\alpha\rho}) - \frac{1}{2} \nabla_\lambda (h^{\lambda\rho} h_\rho \nabla^{\nu} h_{\alpha\rho}) + \frac{1}{2} \nabla_\rho (h^{\lambda\rho} h_\rho \nabla_\lambda h_{\alpha\rho}) - \frac{1}{2} \nabla_\rho (h^{\lambda\rho} h_\rho \nabla_\lambda h_{\alpha\rho}) - \frac{1}{4} \nabla_\lambda (h^{\mu\nu} h_{\rho\lambda} \nabla^{\lambda} h_{\mu\nu}) - \frac{1}{4} \nabla_\lambda (h^{\mu\nu} h_{\rho\lambda} \nabla^{\lambda} h_{\mu\nu})
\]

\[+ \frac{\xi}{2} \int \sqrt{-g} \, d^4 x \left[ \nabla_\rho (\varphi \bar{g}^{\nu\sigma} \Gamma^{(2)^{\rho}}_{\nu\sigma}) - \nabla_\nu (\varphi \bar{g}^{\nu\sigma} \Gamma^{(2)}_{\rho \sigma}) + \frac{1}{2} \nabla_\lambda (\varphi h^{\mu\nu} \nabla^{\lambda} h_{\mu\nu}) + \frac{1}{4} \nabla_\lambda (\varphi h^{\mu\nu} \nabla^{\lambda} h_{\mu\nu}) \right].
\]

\[\hspace{1cm} (A1)
\]

\[\hspace{1cm} (A2)
\]

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