THE AFFINE INVARIANT OF GENERALIZED SEMITORIC SYSTEMS

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ABSTRACT. A generalized semitoric system $F := (J, H): M \to \mathbb{R}^2$ on a symplectic 4-manifold is an integrable system whose essential properties are that $F$ is a proper map, its set of regular values is connected, $J$ generates an $S^1$-action and is not necessarily proper. These systems can exhibit focus-focus singularities, which correspond to fibers of $F$ which are topologically multi-pinched tori. The image $F(M)$ is a singular affine manifold which contains a distinguished set of isolated points in its interior: the focus-focus values $\{(x_i, y_i)\}$ of $F$. By performing a vertical cutting procedure along the lines $\{x := x_i\}$, we construct a homeomorphism $f: F(M) \to f(F(M))$, which restricts to an affine diffeomorphism away from these vertical lines, and generalizes a construction of Vũ Ngọc. The set $\Delta := f(F(M)) \subset \mathbb{R}^2$ is a symplectic invariant of $(M, \omega, F)$, which encodes the affine structure of $F$. Moreover, $\Delta$ may be described as a countable union of planar regions of four distinct types, where each type is defined as the region bounded between the graphs of two functions with various properties (piecewise linear, continuous, convex, etc). If $F$ is a toric system, $\Delta$ is a convex polygon (as proven by Atiyah and Guillemin-Sternberg) and $f$ is the identity.

1. Introduction

Let $(M, \omega)$ be a symplectic $2n$-manifold, that is, $M$ is a smooth $2n$-dimensional manifold and $\omega$ is a non-degenerate closed $2$-form. Throughout this paper we assume that $M$ is connected. However, we do not assume that $M$ is compact.

1.1. Definitions. Motivated by [At82, GS82, PRV12, PV09, PV11, Vu07], we introduce in this paper a particular class of classical Liouville integrable systems of the so-called “generalized semitoric type”.

Definition 1.1 An integrable system on $(M, \omega)$ is given by a map $F: M \to \mathbb{R}^n$ whose components $f_1, \ldots, f_n: M \to \mathbb{R}$ are Poisson commuting smooth functions which generate vector fields $\mathcal{X}_{f_1}, \ldots, \mathcal{X}_{f_n}$ (via pairing with $\omega$) that are linearly independent at almost every point. A singularity of $F$ is a point in $M$ where this linear independence fails to hold. A singular fiber of $F$ is a level set of $F$ which contains at least one singularity of $F$.

In this article we assume that $n = 2$, and use the index free notation $f_1 = J$ and $f_2 = H$.

Definition 1.2 An $S^1$-action on $(M, \omega)$ is Hamiltonian if there exists a smooth map $J: M \to \mathbb{R}$, the momentum map, such that

$$\omega(\mathcal{X}_M, \cdot) = -dJ,$$

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Figure 1.1. The singular Lagrangian fibration \( F: M \to \mathbb{R}^2 \) of a generalized semitoric system with three isolated singular values \( c_1, c_2, c_3 \). The generic fiber is a 2-dimensional torus, the singular fibers are lower dimensional tori, points, or multipinched tori. For each \( \vec{\epsilon} \in \{-1, 1\}^2 \) we construct, in Theorems B and C, a homeomorphism \( f_{\vec{\epsilon}}: F(M) \to \mathbb{R}^2 \) such that \( (f_{\vec{\epsilon}} \circ F)(M) \) is a “nice region” of \( \mathbb{R}^2 \), which is a symplectic invariant. The notion of “nice region” is made precise in Definition 4.2.

where \( \mathcal{X}_M \) is the infinitesimal generator of the action.

In this article we construct a symplectic invariant when \( F \) is of generalized semitoric type. We refer to Section 8.2 for a quick review of the notions concerning singularities used in the following definition.

**Definition 1.3** An integrable system \( F := (J, H): M \to \mathbb{R}^2 \) on \((M, \omega)\) is generalized semitoric if:

(H.i) \( J \) is the momentum map of an effective Hamiltonian circle action.
(H.ii) The singularities of \( F \) are non-degenerate with no hyperbolic blocks.
(H.iii) \( F \) is a proper map (i.e., the preimages of compact sets are compact).
(H.iv) \( J \) has connected fibers, and the bifurcation set of \( J \) is discrete (here discrete includes multiplicity: that is, for any critical value \( x \) of \( J \), there exists a small neighborhood \( V \ni x \) such that the critical set of \( J \) in the preimage \( J^{-1}(V) \) only contains a finite number of connected components.)

**Remark 1.4** (H.iv) implies that the fibers of \( F \) are also connected by [PRV12]. (H.iii), (H.iv) are implied by (H.i),(H.ii) when \( J \) is proper. In some simple physical models like the spherical pendulum (Example 6.1), \( J \) is not proper but (H.iii), (H.iv) still hold.

A typical generalized semitoric system is depicted in Figure 1.1.

For background material on integrable systems and group actions, see [PV11a].
1.2. **Singularities.** The class of systems in Definition 1.3 may have the so-called *focus-focus singularities*, and give rise to fibers of $F$ which are multi-pinched tori. Focus-focus singularities appear in algebraic geometry [GS06] and symplectic topology, e.g., [LS10, Sy01, Vi2013] (in the context of Lefschetz fibrations they are sometimes called *nodes*), and include simple physical models from mechanics such as the spherical pendulum ([AM78]).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.jpg}
\caption{In general $F(M) \subseteq \mathbb{R}^2$ is not convex. The interior of $F(M)$ contains two isolated singular values $c_1 = (x_1, y_1)$ and $c_2 = (x_2, y_2)$. By performing a vertical cutting procedure along the lines $\ell_i := \{x := x_i\}$, we construct a homeomorphism $f: F(M) \to f(F(M))$, which restricts to an affine diffeomorphism away from these vertical lines. The right hand side figure displays the associated polygon with the distinguished lines.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.jpg}
\caption{A cartographic projection of $F$. It is a symplectic invariant of $F$, see Theorem C.}
\end{figure}

1.3. **Toric systems: Atiyah and Guillemin–Sternberg Theory.** If $M$ is compact and $F := (J, H)$ is the momentum map of an effective Hamiltonian 2-torus action, all assumptions above hold by the Atiyah and Guillemin-Sternberg...
Theorem ([At82], [GS82]) and $F$ does not possess any focus-focus singularity. In this case $(M, \omega, F)$ is called a toric system or a symplectic toric manifold.

Toric systems have been thoroughly studied in the past thirty years (in any dimension) and, at least from the point of view of symplectic geometry, a complete picture emerged in the compact case due to the aforementioned results of Atiyah [At82], Guillemin-Sternberg [GS82], and a classification result due to Delzant [De88]. The first two papers showed that the image $\mu(M)$ of the momentum map $\mu: M \to \mathbb{R}^k$ of a Hamiltonian $\mathbb{T}^k$-action on a $2n$-dimensional compact connected symplectic manifold $(M, \omega)$ is a convex polytope in $\mathbb{R}^k$, which is a symplectic invariant.

1.4. **Goal of this article.** In the present article we will extend to generalized semitoric systems the results in Section 1.3, inspired by an extension of the Atiyah-Guillemin-Sternberg result to (non-generalized) semitoric systems recently achieved by Vù Ngòc [Vu07]. Using Morse theory and the Duistermaat-Heckman Theorem for proper momentum maps, he dealt with integrable systems $F: M \to \mathbb{R}^2$ of semitoric type for which, in addition to assumptions (H.i)-(H.iv), $J: M \to \mathbb{R}$ is proper. Then he performed a cutting procedure along the vertical lines going through the isolated singularities of the image $F(M)$ of the system and constructed a convex polygon from it, which is an invariant of $F$; see Figure 1.2.

The difficulty of the generalized situation considered in this article is due to the fact that the Duistermaat-Heckman theorem does not hold for nonproper $J$ (Remark 4.4), and neither does standard Morse theory. This has striking consequences for the statement of our extension: while the invariant in [Vu07] is a class of convex polygons as in Figure 1.2, ours is a union of planar regions of various types (to be precisely defined later), which looks, in general, like Figure 1.3. This invariant encodes the singular affine structure induced by the (singular) Lagrangian fibration $F: M \to \mathbb{R}^2$ on the base $F(M)$. Its construction and properties appear in Theorems B, C, D. This affine structure also plays a role in parts of symplectic topology, mirror symmetry, and algebraic geometry, see for instance Auroux [Au09], Borman-Li-Wu [BLW13], Kontsevich-Soibelman [KS06]. Integrable systems exhibiting semitoric features appear in the theory of symplectic quasi-states, see Eliashberg-Polterovich [EP10].

Theorem E shows that there are many simple examples in which the invariant, which is the most natural planar representation of the singular affine structure of the system, has a non-polygonal, non-convex, form.

2. **Toric and semitoric systems**

Although this section is not original, we put previous results in a general framework which is better suited for expressing our new results in the following section.

2.1. **The set of semitoric images.** Let $\mathcal{P}(\mathbb{R}^2)$ be the set of subsets of $\mathbb{R}^2$ and

$$\mathfrak{S} := \left\{ \mathbb{Z}, \mathbb{Z}^+, \mathbb{Z}^-, \{1, \ldots, N\}_{N>0}, \emptyset \right\},$$

where $\mathbb{Z}$ denotes the set of integer numbers.
where \( \mathbb{Z}^+ = \{ n \in \mathbb{Z} \mid n \geq 0 \} \) and \( \mathbb{Z}^- = \{ n \in \mathbb{Z} \mid n \leq 0 \} \).

Let
\[
T := \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix},
\]
and consider the group \( T \) whose elements are the matrices \( T^k, k \in \mathbb{Z} \), composed with vertical translations. This gives rise to the quotient space \( \mathcal{P}^T(\mathbb{R}^2) := \mathcal{P}(\mathbb{R}^2)/T \).

2.2. Action on \( \mathcal{P}^T(\mathbb{R}^2) \times \mathbb{R}^z \times \mathbb{N}^z \). A vertical line \( L \subset \mathbb{R}^2 \) splits \( \mathbb{R}^2 \) into two half-spaces. Let \( u \in \mathbb{Z} \). We define a map \( t^u \) acting on \( \mathbb{R}^2 \) as follows. On the left half space defined by \( L \), we let the map \( t^u \) act as the identity. On the right half space, with an origin placed arbitrarily on \( L \), \( t^u \) acts as the matrix \( T^u \).

**Definition 2.1** Let \( Z \in \mathfrak{S} \) and let \( \tilde{x} \in \mathbb{R}^z \). Let \( n \in \mathbb{Z} \) and denote by \( \mathcal{L}^x_n \) the vertical line through \( (\tilde{x}(i), 0) \). We define the action of \( u \in \mathbb{Z}^z \) on \( \mathcal{P}^T(\mathbb{R}^2) \times \mathbb{R}^z \) by
\[
\tilde{u} \cdot (X, \tilde{x}) = \left( \prod_{\tilde{u}(i) \neq 0, \ n \in \mathbb{Z}} \mathcal{L}^x_n \right)(X, \tilde{x}).
\]

Let \( \tilde{k} \in \mathbb{N}^z \). We finally define the action of \( \tilde{e} \in \{-1, 1\}^z \) on \( \mathcal{P}^T(\mathbb{R}^2) \times \mathbb{R}^z \times \mathbb{N}^z \) by the formula
\[
\tilde{e} \cdot (X, \tilde{x}, \tilde{k}) = \left( \tilde{e} \cdot \tilde{k} \right) \cdot (X, \tilde{x}), \tilde{k} \right),
\]
where \( \tilde{e} \cdot \tilde{k} := i \mapsto \frac{1 - \tilde{e}(i) k(i)}{2} \). We denote the \( \{-1, 1\}^z \)-orbit space by \( \mathcal{B}_{GST}(Z) := \mathcal{P}^T(\mathbb{R}^2) \times \mathbb{R}^z \times \mathbb{N}^z / \{-1, 1\}^z \).

2.3. Affine invariant for semitoric systems. Let \( F = (J, H): M \to \mathbb{R}^2 \) be a semitoric system, i.e., in addition to assumptions (H.i)-(H.iv), the map \( J: M \to \mathbb{R} \) is proper. There exists a unique \( Z \in \mathfrak{S} \) such that \( \tilde{x} \in \mathbb{R}^z \) is the tuple of images by \( J \) of focus-focus values \( c_i = (x_i, y_i) \) of \( F \) ordered by non-decreasing values, and \( \tilde{k} \in \mathbb{N}^z \) such that \( \tilde{k}(i) \) is the number of focus-focus critical points in the fiber \( F^{-1}(c_i) \).

Let \( \mathcal{L}^x := (\mathcal{L}_{x_i})_{i \in Z} \) where \( \mathcal{L}_{x_i} \) is the unique vertical line in \( \mathbb{R}^2 \) through \( (x_i, 0) \). For each fixed \( \tilde{e} \in \{-1, 1\}^z \), Vu Ngoc constructed \([Vu07, \text{Theorem 3.8 and Proposition 4.1}]\) an equivalence class of convex polygons in \( \mathbb{R}^2 \)
\[
(\Delta_x \mod T) \in \mathcal{P}^T(\mathbb{R}^2).
\]
by performing a cutting procedure along the vertical lines \( \mathcal{L}_{x_i} \). The “choice” of cuts is given by \( \tilde{e} \), where a positive sign corresponds to an upward cut, and a negative sign corresponds to a downward cut.

**Definition 2.2** Let \( (M, \omega, F) \) be a semitoric system. Define:
\[
(\Delta(M, \omega, F) := (\Delta_x \mod T, \tilde{x}, \tilde{k}) \mod \{-1, 1\}^z \in \mathcal{B}_{GST},
\]
where \( \tilde{e}(i) = 1 \) for all \( i \in Z \) and the action of \( \{-1, 1\}^z \) is defined above.

**Definition 2.3** Let \( \mathcal{M}_T \) be the set of toric systems. Let \( Z \in \mathfrak{S} \) and let \( \mathcal{M}_{ST}(Z) \) and \( \mathcal{M}_{GST}(Z) \) be the sets of semitoric and generalized semitoric systems \( F = \)}.
\((J, H)\), with the images of the focus-focus values of \(F\) by \(J\) indexed by the set \(Z\). Let
\[
\mathcal{M}_{\text{ST}} := \bigsqcup_{Z \in \mathfrak{F}} \mathcal{M}_{\text{ST}}(Z); \quad \mathcal{M}_{\text{GST}} := \bigsqcup_{Z \in \mathfrak{F}} \mathcal{M}_{\text{GST}}(Z); \quad \mathfrak{B}_{\text{GST}} := \bigsqcup_{Z \in \mathfrak{F}} \mathfrak{B}_{\text{GST}}(Z).
\]

We now recall the notion of isomorphism for generalized semitoric systems, which coincide with the notion introduced in [Vu07] for proper semitoric systems.

**Definition 2.4** The generalized semitoric systems \((M_1, \omega_1, F_1 := (J_1, H_1))\) and \((M_2, \omega_2, F_2 := (J_2, H_2))\) are isomorphic if there exists a symplectomorphism \(\varphi : M_1 \to M_2\) such that \(\varphi^*(J_2, H_2) = (J_1, h(J_1, H_1))\) for a smooth \(h\) such that \(\frac{\partial h}{\partial H_1} > 0\).

Notice that the set \(\mathcal{M}_{\text{T}}\) is not invariant under these isomorphisms. Hence we introduce the following definition.

**Definition 2.5** A generalized semitoric system is said to be of toric type if it is isomorphic to a toric system. We denote by \(\mathcal{M}_{\text{TT}}\) the set of semitoric systems of toric type.

**Remark 2.6** Clearly \(\mathcal{M}_{\text{T}} \subset \mathcal{M}_{\text{ST}} \subset \mathcal{M}_{\text{GST}}\).

**Theorem 2.7** ([Vu07]). The class of convex polygons (2.2) is an invariant of the isomorphism type of \(F\).

For a system satisfying properties (Hi)–(Hiv), in this article we will construct a more general symplectic invariant by unwinding the (singular) affine structure induced by \(F\) on \(F(M)\), which extends (2.1). The fact that \(J\) may not be proper complicates the situation a lot because the Duistermaat-Heckman theorem does not hold for nonproper momentum maps (Remark 4.4), and standard Morse theory essentially breaks down for nonproper maps. This is why systems with non-proper \(J\) were excluded in [PV09, PV11].

### 3. Summary result: Theorem A

As a consequence of Theorems B, C, and D (stated and proved in the next sections), we obtain the following statement, which is less explicit (less useful for computations) but provides a summary of the paper.

**Definition 3.1** Recall that if \((M, \omega, F) \in \mathcal{M}_{\text{T}}\) then \(F(M)\) does not contain focus-focus singular values, and \(F(M)\) is a convex polygon. If \((M, \omega, F) \in \mathcal{M}_{\text{ST}}\), let \(\Delta(M, \omega, F)\) be as in (2.2). Consider the maps
\[
(3.1) \quad \mathcal{C}_{\text{ST}} : \mathcal{M}_{\text{ST}} \ni (M, \omega, F) \mapsto \Delta(M, \omega, F) \in \mathfrak{B}_{\text{GST}}
\]
\[
(3.2) \quad \mathcal{C}_{\text{T}} : \mathcal{M}_{\text{T}} \ni (M, \omega, F) \mapsto (F(M) \mod \mathcal{T}, \emptyset, \emptyset) \mod \{-1, 1\}^Z \in \mathfrak{B}_{\text{GST}}
\]

\(^1\)while \(F(M)\) is neither generally convex, nor an invariant.
and
\[(3.3)\]
\[\mathcal{C}_{TT} : \mathcal{M}_{TT} \ni (M, \omega, F) \rightarrow (F'(M) \mod \mathcal{T}, \emptyset, \emptyset) \mod \{-1, 1\}^2 \in \mathcal{B}_{GST},\]
where \(F'\) is any toric momentum map isomorphic to \(F\) as a semitoric system.

**Definition 3.2** If \(\mathcal{F}\) is a family of integrable systems containing \(\mathcal{M}_T\), a cartographic invariant is any map \(\mathcal{C} : \mathcal{F} \rightarrow \mathcal{B}_{GST}\) extending \(\mathcal{C}_T\) in (3.3) and invariant under isomorphism.

It follows from the Atiyah-Guillemin-Sternberg theory and [Vu07] that the maps \(\mathcal{C}_T, \mathcal{C}_{TT}\) and \(\mathcal{C}_{ST}\) are cartographic invariants. Notice that it is straightforward to check from Definition 2.4 that \(\mathcal{C}_{TT}\) is indeed well defined.

**Theorem A.** Let \(\mathcal{C}_{ST}, \mathcal{C}_{TT}\) and \(\mathcal{C}_T\) be the cartographic invariants defined in (3.1) and (3.3). Then there exists a cartographic invariant \(\mathcal{C}_{GST} : \mathcal{M}_{GST} \rightarrow \mathcal{B}_{GST}\) such that the diagram
\[(3.4)\]
\[\xymatrix{ \mathcal{M}_T \ar[r]_{\mathcal{C}_T} & \mathcal{M}_{TT} \ar[r]_{\mathcal{C}_{TT}} & \mathcal{M}_{ST} \ar[r]_{\mathcal{C}_{ST}} & \mathcal{M}_{GST} \ar[r]_{\mathcal{C}_{GST}} & \mathcal{B}_{GST} }\]
is commutative.

We will prove several theorems which together imply Theorem A and which are more informative because the cartographic invariant is explicitly constructed. It would be interesting to prove Theorem A (in particular, defining the maps involved) for integrable systems on origami manifolds (see [DGP11]) and on orbifolds (see [LT97]), where, as far as we know, integrable systems have not been studied.

4. Main results: Theorems B, C, D, E

For simplicity, from now on, we use the term “semitoric” to refer to integrable systems satisfying (Hi)–(Hiiv), that is, we drop the word “generalized”.

Let \((M, \omega)\) be a connected symplectic 4-manifold and \(F := (J, H) : M \rightarrow \mathbb{R}^2\) a semitoric system. Next we prepare the grounds for the main theorems of the paper. Let \(B_r \subset B\) is the set of regular values of \(F\). Since \(F\) is proper we know that the set of focus-focus critical values of \(F\) is discrete. We denote by \(c_i := (x_i, y_i), i \in Z\), the focus-focus critical values of \(F\), ordered so that \(x_i \leq x_{i+1}\), and \(k_i\) is the number of critical points in \(F^{-1}(c_i)\). Given \(\mathcal{E} = (\epsilon_i)_{i \in Z} \in \{-1, +1\}^Z\), we define the vertical closed half line originating at \(c_i = (x_i, y_i)\) by
\[\mathcal{L}^{\epsilon_i}_i := \{(x, y) \in \mathbb{R}^2 \mid \epsilon_i y \geq \epsilon_i y_i\}\]
for each \(i \in Z\), which is pointing up from \(c_i\) if \(\epsilon_i = 1\) and down if \(\epsilon_i = -1\). Define \(\ell^{\epsilon_i}_i := B \cap \mathcal{L}^{\epsilon_i}_i \subset \mathbb{R}^2\).

For any \(c \in B\), define \(I_c := \{i \in Z \mid c \in \ell^{\epsilon_i}_i\}\) and the map \(k : \mathbb{R}^2 \rightarrow \mathbb{Z}\) by
\[(4.1)\]
\[k(c) := \sum_{i \in I_c} \epsilon_i k_i,\]
with the convention that if $I_c = \emptyset$ then $k(c) = 0$. The sum is finite thanks to (H.iv). Let $\ell^\epsilon := k^{-1}(\mathbb{Z} \setminus \{0\})$.

For the necessary background on affine manifolds in the discussion which follows, readers may consult the appendix (Section 8.3). We write $\mathbb{A}_\mathbb{Z}^2$ for $\mathbb{R}^2$ equipped with its standard integral affine structure with automorphism group $\text{Aff}(2, \mathbb{Z}) := \text{GL}(2, \mathbb{Z}) \rtimes \mathbb{R}^2$.

The integral affine structure on $B_r$, which in general is not the affine structure induced by $\mathbb{A}_\mathbb{Z}^2$, is defined for instance in [Vu07, Section 3] or [HZ1994, Appendix A2]; see also Section 8.3: affine charts near regular values are given by action variables $f : U \subset F(M) \rightarrow \mathbb{R}^2$ on open subsets $U$ of $F(M)$ with the induced subspace topology and any two such charts differ by the action of $\text{Aff}(2, \mathbb{Z})$.

Let $X$ and $Y$ be smooth manifolds and $A \subset X$. A map $f : A \rightarrow Y$ is said to be smooth if every point in $A$ admits an open neighborhood in $X$ on which $f$ can be smoothly extended. The map $f$ is called a diffeomorphism onto its image if $f$ is injective, smooth, and its inverse $f^{-1} : f(A) \rightarrow A$ is a smooth map, in the sense above.

The following theorem is a generalization of [Vu07, Theorem 3.8].

**Theorem B.** Let $F : M \rightarrow \mathbb{R}^2$ be a semitoric system in $\mathcal{M}_{GST}(\mathbb{Z})$, for some $Z \in \mathfrak{F}$. For every $\ell^\epsilon \in \{-1, +1\}^\mathbb{Z}$ there exists a homeomorphism $f_{\ell^\epsilon} : B \rightarrow f_{\ell^\epsilon}(B) \subseteq \mathbb{R}^2$ of the form $f_{\ell^\epsilon}(x, y) = (x, f_{\ell^\epsilon}^{(2)}(x, y))$ such that:

1. (P.i) the restriction $f_{\ell^\epsilon}|_{(B\setminus{\ell^\epsilon})}$ is a diffeomorphism onto its image, with positive Jacobian determinant;
2. (P.ii) the restriction $f_{\ell^\epsilon}|_{(B_r\setminus{\ell^\epsilon})}$ sends the integral affine structure of $B_r$ to the standard integral affine structure of $\mathbb{A}_\mathbb{Z}^2$;
3. (P.iii) the restriction $f_{\ell^\epsilon}|_{(B_r\setminus{\ell^\epsilon})}$ extends to a smooth multi-valued map $B_r \rightarrow \mathbb{R}^2$

and for any $i \in \mathbb{Z}$ and $c \in \ell^\epsilon_i \setminus \{c_i\}$, we have

\[
\lim_{(x,y) \to c, x < x_i} df_{\ell^\epsilon}(x,y) = T_{k(c)} \lim_{x \to c, x > x_i} df_{\ell^\epsilon}(x,y),
\]

where $k(c)$ is defined in (4.1).

Such an $f_{\ell^\epsilon}$ is unique modulo a left composition by a transformation in $T$.

In toric case, $f_{\ell^\epsilon}(x, y) = (x, y)$, as was mentioned in Section 3.

**Definition 4.1** The map $f_{\ell^\epsilon}$ in Theorem B is a cartographic map\(^2\) for $F$ and its image $f_{\ell^\epsilon}(B)$ is a cartographic projection of $F$.

**Definition 4.2** Let $R$ be a subset of $\mathbb{R}^2$. We say that $R$ has type I if there is a convex polygon $\Delta \subset \mathbb{R}^2$ and an interval $I \subseteq \mathbb{R}$ such that

\[
R = \Delta \cap \{ (x,y) \in \mathbb{R}^2 \mid x \in I \}.
\]

\(^2\)since they lay out the affine structure of $F$ on two dimensions.
We say that $R$ has type II if there is an interval $I \subseteq \mathbb{R}$ and $f: I \to \mathbb{R}$, $g: I \to \mathbb{R}$ such that $f$ is piecewise linear, continuous, and convex, $g$ is lower semicontinuous, and
\[
\mathcal{R} = \left\{ (x,y) \in \mathbb{R}^2 \mid x \in I \text{ and } f(x) \leq y < g(x) \right\}.
\]
We say that $\mathcal{R}$ has type III if there is an interval $I \subseteq \mathbb{R}$ and $f: I \to \mathbb{R}$, $g: I \to \mathbb{R}$ such that $f$ is upper semicontinuous, $g$ is piecewise linear continuous and concave, and
\[
\mathcal{R} = \left\{ (x,y) \in \mathbb{R}^2 \mid x \in I \text{ and } f(x) < y \leq g(x) \right\}.
\]
We say that $R$ has type IV if there is an interval $I \subseteq \mathbb{R}$ and $f, g: I \to \mathbb{R}$ such that $f$ is upper semicontinuous, $g$ is lower semicontinuous, and
\[
\mathcal{R} = \left\{ (x,y) \in \mathbb{R}^2 \mid x \in I \text{ and } f(x) < y < g(x) \right\}.
\]

In the following statement we call a discrete sequence a sequence such that for every value $c$, there is a neighborhood of $c$ which contains only the image of a finite number of indices.

**Theorem C.** Let $F = (J, H): M \to \mathbb{R}^2$ be a semitoric system and let $f_\epsilon$ be a cartographic map for $F$. Let
\[
K^+ := \left\{ x \in J(M) \mid J^{-1}(x) \cap H^{-1}([0, +\infty)) \text{ is compact} \right\},
\]
and
\[
K^- := \left\{ x \in J(M) \mid J^{-1}(x) \cap H^{-1}((-\infty, 0]) \text{ is compact} \right\}.
\]
Suppose that the topological boundaries $\partial K^+$ and $\partial K^-$ in $J(M)$ are discrete. Then there exists an increasing sequence $(x_j)_{j \in \mathbb{Z}}$ in $\mathbb{R}$, and sets $\mathcal{C}_j \subseteq \mathbb{R}^2$, $j \in \mathbb{Z}$, such that:

(P.1) for each $j \in \mathbb{Z}$, the set $\mathcal{C}_j$ has type I, II, III, or IV associated to $(x_j, x_{j+1})$;
(P.2) $f_\epsilon(B) = \bigcup_{j \in \mathbb{Z}} \mathcal{C}_j$;
(P.3) for every $j \in \mathbb{Z}$, and every regular value $x$ of $J$, the volume $V(x) \leq +\infty$ of $J^{-1}(x)$ is equal to the Euclidean length of the vertical line segment $(\{x\} \times \mathbb{R}) \cap \mathcal{C}_j$.

In some cases (for instance if $M$ is compact), only a finite number of the $x_j$’s are relevant.

Suppose that $F: M \to \mathbb{R}^2$ is the momentum map of a Hamiltonian $\mathbb{T}^2$-action on a compact connected symplectic 4-manifold. Then the cartographic projection of $F$ is a compact convex polygon in $\mathbb{R}^2$; see [At82] and [GS82]. If $F: M \to \mathbb{R}^2$ is a semitoric system for which $J$ is proper, then any cartographic projection of $F$ is a convex polygon in $\mathbb{R}^2$, which may be bounded or unbounded, and which is always a closed subset of $\mathbb{R}^2$; see [Vu07, Theorem 3.8].

**Example 4.3** Figure 1.1 shows the regular and singular focus-focus fibers of singular Lagrangian fibration $f \circ F: M \to \bigcup_{j \in \mathbb{Z}} \mathcal{C}_j$ in Theorem C. There are two focus-focus singular fibers, $F^{-1}(c_i)$, $i = 1, 2$. The value $c_1$ has multiplicity $k_1 = 2$ and $c_2$ has multiplicity $k_2 = 3$. ☜
Remark 4.4 Concerning Theorem C(P.3), note that the Duistermaat-Heckman theorem does not hold for nonproper momentum maps. Indeed, let $M = S^2 \times S^2$ with $F = (z_1, z_2)$ (toric momentum map). Let $f: [-1, 1] \to (-1, 1]$ be continuous. Let $M' = F^{-1}(\{(x,y) \mid x \in [-1,1], y < f(x)\})$. The set $M'$ is an open subset of $M$, and $\mu = z_1$ is a momentum map for a Hamiltonian $S^1$-action on $M'$. Furthermore, $\mu$ is not proper because $\mu^{-1}(x) = F^{-1}(\{(x,y) \mid y < f(x)\})$ is not closed. Now let $V(x)$ be the symplectic volume of $M'_x$ where $M'_x = M' \cap \mu^{-1}(x)/S^1 = S^2 \cap \{z_2 < f(x)\}$. (See Figure 4.1.) Then $V(x) = \text{vol}(S^2) [(1 + f(x))/2] = 2\pi(1 + f(x))$. So $V(x)$ is not piecewise linear in general, in contrast with Duistermaat-Heckman [DH82]. This shows that the Duistermaat-Heckman theorem may not hold when the $S^1$-momentum map is not proper. Notice that the full map $F|_M$ is not proper, but we can easily modify it as follows. Let $g(x,y) = (x,1/(f(x) - y))$. Then $F' = g \circ F|_{M'}$ is proper, and the $S^1$-momentum map is not modified. Thus $F'$ is a generalized semitoric system.

\[
\text{Figure 4.1. The reduced manifold } M'_x.
\]

In the following definition, we use the terminology of sections 2 and 3.

Definition 4.5 Let $C_{\text{GST}}(F)$ be defined as follows. Let $\bar{c} = (\epsilon_i)_{i \in Z}$ with $\epsilon_i = 1$ for all $i \in Z$. Then

\begin{equation}
(4.3) \quad C_{\text{GST}}(F) := (f_{\bar{c}}(B) \mod T) \mod \{-1, 1\}^Z \in \mathcal{B}_{\text{GST}}.
\end{equation}

Theorem D. The map $C_{\text{GST}}: M_{\text{GST}} \to \mathcal{B}_{\text{GST}}$ is a cartographic invariant.

Proof. Let $F_1: M_1 \to \mathbb{R}^2$ and $F_2: M_2 \to \mathbb{R}^2$ be semitoric systems, and let $f_{\bar{c}_1}$, $f_{\bar{c}_2}$ be the corresponding cartographic maps defined by Theorem B. If $F_1$ and $F_2$ are isomorphic, they have the same leaf space, with identical induced integral affine structures. Thus, from Theorem B, (P.ii), there must be a transformation $t \in T$ such that $f_{\bar{c}_1} = t \cdot f_{\bar{c}_2}$. Then the result follows from (4.3). \qed

We conclude with a result which shows that there are semitoric systems with a cartographic projection which may not occur as the cartographic projection of a toric or semitoric system $(J, H): M \to \mathbb{R}^2$ with proper $J$. 
**Theorem E.** There exists an uncountable family of semitoric integrable systems \( \{ F_\lambda : M \to \mathbb{R}^2 \}_\lambda \), with cartographic map \( f_{\lambda, \epsilon} \), such that the following properties hold:

- **(E.1)** \( F_\lambda : M \to \mathbb{R}^2 \) is proper;
- **(E.2)** \( B_\lambda := F_\lambda(M) \) is unbounded in \( \mathbb{R}^2 \);
- **(E.3)** \( f_{\lambda, \epsilon}(B_\lambda) \) is a bounded in \( \mathbb{R}^2 \);
- **(E.4)** \( B_\lambda \) is not a convex region;
- **(E.5)** \( B_\lambda \) is not open and is not closed in \( \mathbb{R}^2 \);
- **(E.6)** \( F_\lambda \) is isomorphic to \( F_{\lambda'} \) if and only of \( \lambda = \lambda' \);
- **(E.7)** for every \( i \in \{ I, II, III, IV \} \) there exists \( \lambda \) such that \( f_{\lambda, \epsilon}(B) \) as in Theorem C(P.ii), is a union of regions in \( \mathbb{R}^2 \) of types I, II, III, and IV, in which at least one of them has type \( i \).

Motivated by [PV11], the following inverse type question is natural. Let \( C := \bigcup_{j \in \mathbb{N}} C_j \) be a connected set, where \( C_j \subset \mathbb{R}^2 \) is a region of type I, II, III, or IV. Does there exist a semitoric system \( F : M \to \mathbb{R}^2 \) with \( B := F(M) \) such that \( f_{\epsilon}(B) = C \), where \( f_{\epsilon} \) is a cartographic map for \( F \) ?

The classifications of Delzant [De88] and [PV11] give partial answers to this question. Note that here we are not claiming uniqueness; in fact, it follows from [PV11] that there are many semitoric systems which realize the same \( C \).

## 5. Proof of Theorem B

The proof is close to [Vu07], but our construction is more transparent thanks to the use of a recent result in [PRV12]. Let \( \Sigma_J \) be the bifurcation set of \( J \). We fix a point \( q_0 = (x_0, y_0) \in B_r, \) such that \( x_0 \notin \Sigma_J \). Since the fibers of \( J \) are connected by (H.iv), we know from [PRV12, Theorem 4.7] that the fibers of \( F \) are also connected. By the Liouville-Mineur-Arnold Theorem (see, [HZ1994, Appendix A2]), there exists a diffeomorphism \( g : U \subset B_r \to g(U) \subset \mathbb{R}^2 \), with positive Jacobian determinant, defined on an open neighborhood \( U \) of \( q_0 \) which, without loss of generality, we may assume to be simply connected, such that \( A = (A_1, A_2) = g \circ F \) are local action variables. Since \( J \) is the momentum map of an effective Hamiltonian \( S^1 \)-action, it has to be free on the regular fibers (see for instance [DK00, Theorem 2.8.5]). Hence, we may assume \( A_1 = J \) (see [Vu07, point 2 of the proof of Theorem 3.8]). Repeating this argument with an open cover of \( B_r \), we may fix an affine atlas of \( B_r \) such that all transition functions belong to the group \( T \) (see Section 2.1).

We divide the proof into four steps: the first four treat the generic case in which the lines in \( \ell^c \) are pairwise distinct, whereas the last step deals with the non-generic case. We warn the reader that statements (P.i)-(P.iii) are proven in the first three steps, but the claim that \( f_{\epsilon} \) is a homeomorphism onto its open image is proven in Step 4.

The homeomorphism \( f_{\epsilon} \) with the required properties is constructed from the developing map of the universal cover \( p_r : \tilde{B}_r \to B_r \), chosen with \( q_0 \) as base point (see Section 8.3). Let \( \tilde{G}_c : \tilde{B}_r \to \mathbb{R}^2 \) be the unique developing map such
that $\tilde{G}_\gamma(\gamma) = g(\gamma(1))$ for paths $\gamma$ contained in $U$, such that $\gamma(0) = q_0$. The goal is to use $\tilde{G}_\epsilon$ in order to extend $g$ to the whole image $B = F(M)$.

We begin the proof by assuming that the half lines in $\ell^c$ do not overlap.

**Step 1.** ($B_r \setminus \ell^c$ is simply connected). By [PRV12, Theorem 4.7], $B$ is a region of $\mathbb{R}^2$ which is between the graphs of two continuous functions defined on the same interval. These graphs cannot intersect above an interior point of the interval, because this would imply that the interior of $B_r$ is not connected, which is known to be false because $F$ is proper (see [PRV12, Theorem 3.6]). This proves that $B_r \setminus \ell^c$ is simply connected.

**Step 2.** (Proof of (P.i) and (P.ii) on $B_r \setminus \ell^c$). Hence, the developing map $\tilde{G}_\gamma : B_r \rightarrow \mathbb{R}^2$ induces a unique affine map $G_\gamma : B_r \setminus \ell^c \rightarrow \mathbb{R}^2$ by the relation

$$G_\gamma \circ p_r := \tilde{G}_\gamma,$$

i.e., if $c \in B_r \setminus \ell^c$ and $\gamma$ is a smooth path in $B_r \setminus \ell^c$ connecting $q_0$ to $c$, then $G_\gamma(c) := \tilde{G}_\gamma(\gamma)$. Note that $G_\gamma|U = g$.

The definition implies that $G_\gamma$ is a local diffeomorphism. We show now that $G_\gamma$ is injective. Since $A_1 = J$, $G_\gamma|U$ is of the form $G_\gamma(x, y) = (x, h^U_\gamma(x, y))$ for some smooth function $h^U_\gamma : U \rightarrow \mathbb{R}$. Because we have an affine atlas of $B_r$ with transition functions in $T$, the affine map $G_\gamma$ must preserve the first component $x$, i.e. there exists a smooth function $h_\gamma : B_r \setminus \ell^c \rightarrow \mathbb{R}$, extending $h^U_\gamma$ such that

$$G_\gamma(x, y) = (x, h_\gamma(x, y))$$

for all $(x, y) \in B_r \setminus \ell^c$. Since $G_\gamma$ is a local diffeomorphism, $\frac{\partial h_\gamma}{\partial y}$ never vanishes, which implies that for each fixed $x$, all the maps $y \mapsto h_\gamma(x, y)$ are injective. Hence $G_\gamma$ is injective and thus a global diffeomorphism $B_r \setminus \ell^c \rightarrow G_\gamma(B_r \setminus \ell^c) \subset \mathbb{R}^2$.

This proves (P.i) on $B_r \setminus \ell^c$ by choosing $f_\gamma := G_\gamma$ and (P.ii) because $G_\gamma$ is an affine map.

**Step 3.** (Extension of the developing map to $B \setminus \ell^c$ and proof of (P.i) and (P.iii)). By the description of the image of $F$ in [PRV12, Theorem 5], we simply need to extend $G_\gamma$ at elliptic critical values. But the behavior of the affine structure at an elliptic critical value $\epsilon$ is well known (see [MZ04]): there exist a smooth map $a : V \rightarrow \mathbb{R}^2$, where $V$ is an open neighborhood of $c \in \mathbb{R}^2$, and a symplectomorphism $\varphi : F^{-1}(V) \rightarrow M_Q$ onto its image such that

$$(5.1) \quad a \circ F|_{F^{-1}(V)} = Q \circ \varphi : F^{-1}(V) \rightarrow \mathbb{R}^2,$$

where $Q$ is the “normal form” of the same singularity type as $F$, given by $Q = (x_1^2 + \xi_1^2, \xi_2)$ (rank 1 case) or $Q = (x_1^2 + \xi_2^2, x_2^2 + \xi_2^2)$ (rank 0 case). Here $M_Q = \mathbb{R}^2 \times T^*T^1 = \mathbb{R}^2 \times T^1 \times \mathbb{R}$ (rank 1) or $M_Q = \mathbb{R}^4$ (rank 0). It follows from the formula for $Q$ that $Q$ is generated by a Hamiltonian $T^2$-action, and therefore $a$ is an affine map. On the other hand, since $F$ and $Q$ have the same singularity type, the ranks of $dF$ and $dQ$ must be equal, and the dimensions of the spaces spanned by the Hessians must be the same as well. Computing the Taylor expansion of (5.1) shows that $da(c)$ has to be invertible. Thus, $a$
is a diffeomorphism onto its image. Therefore $a|_{B_r \cap V}$ is a chart for the affine structure of $B_r$.

Thus there exists a unique affine map $A \in \text{Aff}(2, \mathbb{Z})$ such that
\[(G_{\tilde{c}})|_{Br \cap V} = A \circ a|_{Br \cap V}\]
and we may simply extend $G_{\tilde{c}}$ to $B_r \cup V$ by letting
\[(G_{\tilde{c}})|_V = A \circ a.\]
Because $a$ is a diffeomorphism into its image, we see that $G_{\tilde{c}}$ remains a local diffeomorphism. This proves (P.i) with $f_{\tilde{c}}|_{B \setminus \ell^c} := G_{\tilde{c}}$.

The fact that $G_{\tilde{c}}$ extends to a smooth multi-valued map $B_r \to \mathbb{R}^2$ follows from the smoothness of the universal cover as in [Vu07, Section 3]. Formula (4.2) follows from the calculation of the monodromy around focus-focus singularities, which is carried out exactly as in [Vu07, pages 921-922] since it relies only on the properness of $F$ (and not on the properness of $J$). This proves (P.iii).

**Step 4.** *(Extension to a homeomorphism $B \to \mathbb{R}^2$).* Finally we show that $G_{\tilde{c}}$ may be extended to a homeomorphism $f_{\tilde{c}} : B \to f_{\tilde{c}}(B) \subset \mathbb{R}^2$, which will prove the theorem if no half lines in $\ell^c$ overlap.

Because of (P.iii), if $c_0 \in \ell^c$, but $c_0$ is not a focus-focus value, it follows that $G_{\tilde{c}}$ has a unique continuation to $c_0$, from the left, and a unique continuation from the right. As in [Vu07, Proof of Theorem 3.8], the fact that these continuations coincide follows from the fact that the affine monodromy around a focus-focus singularity leaves the vertical line through $c_0$ pointwise invariant. That $G_{\tilde{c}}(c)$ has a limit as $c$ approaches the focus-focus value follows from the $z \log z$ behavior of $G_{\tilde{c}}$, see [Vu03, Section 3].

Let $f_{\tilde{c}} : B \setminus \{c_i \mid i \in \mathbb{Z}\} \to \mathbb{R}^2$ be this continuous extension of $G_{\tilde{c}}$. Because of (P.iii), the extensions of the vertical derivative $\partial_y f_{\tilde{c}}$ from the left or from the right coincide on $\ell^c$. Since any extension of $G_{\tilde{c}}(x, y) = (x, h_{\tilde{c}}(x, y))$ is a local diffeomorphism, $\partial_y h_{\tilde{c}}$ cannot vanish on $\ell^c$. Thus, $f_{\tilde{c}}|_{\ell^c}$ is injective.

This implies that $f_{\tilde{c}}$ is injective on $B \setminus \{c_i \mid i \in \mathbb{Z}\}$.

Extend by continuity the map $f_{\tilde{c}}$ to $\{c_i \mid i \in \mathbb{Z}\}$. So far, we have shown that $f_{\tilde{c}} : B \to \mathbb{R}^2$ is a continuous injective map which is an affine diffeomorphism off $\ell^c$. It remains to be shown that $(f_{\tilde{c}})^{-1}$ is continuous on $f_{\tilde{c}}(B)$. Since $f_{\tilde{c}}$ is a diffeomorphism off $\ell^c$, we only have to show that $(f_{\tilde{c}})^{-1}$ is continuous at points of $f_{\tilde{c}}(\ell^c)$.

Let $c_0 = (x_0, y_0) \in \ell^c$ and $\tilde{G}_{\tilde{c}} : U \to \tilde{G}_{\tilde{c}}(U)$ be an affine chart which coincides with $f_{\tilde{c}}$ on the left hand-side of $c_0$ in $U$, that is, on
\[U_{\text{left}} := \{(x, y) \in U \mid x \leq x_0\}.\]
Then,
\[(f_{\tilde{c}})^{-1}|_{f_{\tilde{c}}(U_{\text{left}})} = \tilde{G}_{\tilde{c}}^{-1}|_{f_{\tilde{c}}(U_{\text{left}})}\]
and hence it is continuous on $f_{\tilde{c}}(U_{\text{left}})$. Similarly, it is proved that $(f_{\tilde{c}})^{-1}|_{f_{\tilde{c}}(U_{\text{right}})}$ is continuous on $U_{\text{right}}$, which shows that $(f_{\tilde{c}})^{-1}$ is continuous at $f_{\tilde{c}}(c_0)$ for any $c_0 \in \ell^c$. 

Finally, we need to prove the continuity of \((f_{\ell})^{-1}\) at all points \(f_{\ell}(c_i)\), where \(c_i = (x_i, y_i)\), \(i \in \mathbb{Z}\), are the focus-focus values in \(B\). Let \(\ell_i\) be the vertical line containing \(c_i\). Let us use the following local description of the behavior of \(f_{\ell}\) at \(c_i\), \([\text{Vu03}], \text{[Vu07], Proof of Theorem 3.8}]: for all \((x, y) \in U \setminus \ell_i,\)

\[
f_{\ell}(x, y) = (x, \Re(z \log z + g(x, y)),
\]

where \(z = \hat{y}(x, y) + ix \in \mathbb{C}\), \(g\) and \(\hat{y}\) are smooth functions and \(\hat{y}(0, 0) = y_0\). It follows that \(\frac{\partial f_{\ell}}{\partial y}\) is continuous near \(c_i\) (which is in agreement with (4.2)) and is equivalent, as \(z \to 0\), to \(K \ln(x^2 + y^2)\) for some constant \(K > 0\). Hence we get the lower bound

\[
\left| \frac{\partial f_{\ell}}{\partial y}\right| \geq C > 0
\]

for some constant \(C\), if \((x, y)\) is in a small neighborhood \(V = [x_i - \eta, x_i + \eta] \times [y_i - \eta, y_i + \eta]\) of \(c_i\), for some \(\eta > 0\). For simplicity of notation, let us assume for instance that \(\epsilon_i = 1\); the case \(\epsilon_i = -1\) is treated similarly. Hence, for any fixed \(x \in [x_i - \eta, x_i + \eta]\), the function \(y \mapsto f_{\ell}(x, y)\) is invertible on \([y_i, y_i + \eta]\) and has bounded derivative, uniformly for \(x \in [x_i - \eta, x_i + \eta]\). Hence, the inverse \((f_{\ell})^{-1}\) extends by continuity at \(f(c_i) = f(x_i, y_i)\). The limit of the inverse at this point must equal \(y_i\) since \(f_{\ell}\) is injective. This shows that \((f_{\ell})^{-1}\) is continuous at the point \(f_{\ell}(c_i)\).

This concludes the proof of Theorem B in case there is no overlap of vertical lines in \(\ell^c\).

**Step 5. (Proof in the case of overlapping lines in \(\ell^c\)).** If on the other hand there are overlaps of vertical lines in \(\ell^c\), then \(B_r \setminus \ell^c\) may not be simply connected. In this case, for each \(c \in B_r \setminus \ell^c\), we need to choose a path \(\gamma_c\) joining \(q_0\) to \(c\) inside \(B_r \setminus \{c_i \mid i \in \mathbb{Z}\}\), which we do as follows. We replace the focus-focus critical values \(c_i\) which lie in the same vertical line by nearby points \(\tilde{c}_i\), in such a way that their \(x\)-coordinates are all pairwise distinct. This turns the corresponding set \(B_r \setminus \ell^c\) into a simply connected set; thus, up to homotopy, there is a unique path \(\gamma_c\) joining \(q_0\) to \(c\) inside \(B_r \setminus \ell^c\), and we can always assume that this path avoids the true focus-focus values \(c_i\).

The homotopy class of \(\gamma_c\) depends on the choice of ordering of the \(x\)-coordinates of the points \(\tilde{c}_i\). However, we claim that the value

\[
G_{\ell}(c) := \tilde{G}_{\ell}([\gamma_c])
\]

is well defined. Indeed, decomposing a permutation as a product of transpositions of the form \((i, i + 1)\) or \((i + 1, i)\), it suffices to consider only the case where we permute two points, \(\tilde{c}_i\) and \(\tilde{c}_{i+1}\), which lie in adjacent vertical lines. In this case, one can check that the homotopy class \([\gamma_c]\) is modified by a commutator \(g_i g_{i+1} g_i^{-1} g_{i+1}^{-1}\), where \(g_i, i \in \mathbb{Z}\), is a set of generators of the fundamental group of \(B_r \setminus \{c_i \mid i \in \mathbb{Z}\}\). But the monodromy representation is Abelian, due to the global \(S^1\) action (see [CVN02]). It follows that, as required, the value \(\tilde{G}_{\ell}([\gamma_c])\) is invariant under this transposition.

Now that \(G_{\ell}\) is defined, the previous proof for (P.i) and (P.ii) remains valid. The formula in (P.iii) follows from the fact that the monodromy representation is Abelian.
6. Proof of Theorem C and the spherical pendulum example

Proof of Theorem C. The proof is divided into three steps.

Step 1. Let \( f_\varepsilon : B \to f_\varepsilon(B) \subset \mathbb{R}^2 \) be the homeomorphism in Theorem B. Let \( H^+, H^- : J(M) \to \mathbb{R} \) be the functions defined by \( H^+(x) := \sup_{J^{-1}(x)} H \) and \( H^-(x) := \inf_{J^{-1}(x)} H \). Since \( J \) is Morse-Bott with connected fibers (see, e.g., [PR11, Theorem 3]) we may apply [PRV12, Theorem 5.2] which states that \( H^+, H^- \) are continuous and \( F(M) = (\text{hypograph of } H^+) \cap (\text{epigraph of } H^-) \).

Since \( H^+, H^- \) are continuous and \( F \) is proper, one can check that the sets \( K^+, K^- \) defined in the theorem are open in \( J(M) \). Hence we have the following equality of sets, where the four sets on the right hand side are open and disjoint: \( J(M) = (K^+ \cap K^-) \cup (K^+ \setminus K^-) \cup (K^- \setminus K^+) \cup (J(M) \setminus (K^+ \cup K^-)) \). By assumption, \( \partial K^+ + \partial K^- \) are discrete, and therefore there exists a countable collection of intervals \( \{I_j\}_{j \in \mathbb{Z}} \), whose interiors are pairwise disjoint, such that each \( I_j \) is contained in one of the above four sets \((K^+ \cap K^-), (K^+ \setminus K^-), (K^- \setminus K^+) \text{ or } (J(M) \setminus (K^+ \cup K^-))\), and such that \( J(M) = \bigcup_{j \in \mathbb{Z}} I_j \).

By letting for every \( j \in \mathbb{Z} \), \( C^\varepsilon_j := f_\varepsilon(I_j \times \mathbb{R}) \cap F(M) \subset I_j \times \mathbb{R} \), we obtain \( f_\varepsilon(F(M)) = \bigcup_{j \in \mathbb{Z}} C^\varepsilon_j \).

Step 2. (Proof of (P.1) and (P.2)). We consider the four cases.

1. If \( I_j \subset (K^+ \cap K^-) \), then the fibers of \( J \) are compact, and hence the analysis carried out in [Vu07, Theorem 3.8, (v)] applies. This implies that \( C^\varepsilon_j \) is of type I.

2. Consider now \( I_j \subset (K^- \setminus K^+) \). Let \( x \in I_j \). Since \( J^{-1}(x) \cap H^{-1}((-\infty, 0]) \) is compact, \( H_-(x) \) is finite. On the other hand, \( H_+(x) \) must be \(+\infty\); otherwise, \( F^{-1}(\{x\} \times [0, H_+(x)]) \) would be compact, by the properness of \( F \). This would imply that \( J^{-1}(x) \) is compact, a contradiction.

Let \( y \in H(J^{-1}(x)) \). Recall that \( f_\varepsilon(x, y) = (x, f_\varepsilon^{(2)}(x, y)) \) and that \( \frac{\partial f_\varepsilon^{(2)}}{\partial y} \) is continuous on \( F(M) \) (see (4.2)). Since \( \frac{\partial f_\varepsilon^{(2)}}{\partial y} > 0 \), the image \( f_\varepsilon(I_j \times \mathbb{R}) \cap F(M) = C^\varepsilon_j \) has the form

\[
\left\{ (x, z) \mid x \in I_j, \ h^-_\varepsilon(x) \leq z < h^+_\varepsilon(x) \right\},
\]

where

\[
h^-_\varepsilon(x) := \min_{y \in J^{-1}(x)} f^{(2)}_\varepsilon(x, y) = f^{(2)}_\varepsilon(x, H_-(x)) \in \mathbb{R}
\]

\[
h^+_\varepsilon(x) := \sup_{y \in J^{-1}(x)} f^{(2)}_\varepsilon(x, y) = \lim_{y \to +\infty} f^{(2)}_\varepsilon(x, y) \in \mathbb{R}.
\]

We have used the fact that \( f_\varepsilon \) is a homeomorphism, so that the point \((x, h^+_\varepsilon(x))\) cannot belong to \( C^\varepsilon_j \). The function \( h^+_\varepsilon \) is a pointwise limit of continuous functions, so it is continuous on a dense set. However, we need to show that it is lower semicontinuous.

The new map

\[
\left( J, f^{(2)}_\varepsilon(J, H) \right) = f_\varepsilon \circ F
\]
satisfies the hypothesis of the following slight variation of [PRV12, Theorem 5.2] for continuous maps (the proof of which is identical by line): Let \( \hat{M} \) be a connected smooth four-manifold. Let \( \hat{F} = (\hat{J}, \hat{H}) : \hat{M} \to \mathbb{R}^2 \) be a continuous map. Suppose that the component \( \hat{J} \) is a smooth non-constant Morse-Bott function with connected fibers. Let \( \hat{H}^+, \hat{H}^- : \hat{J}(\hat{M}) \to \mathbb{R} \) be defined by \( \hat{H}^+(x) := \sup_{\hat{J}^{-1}(x)} \hat{H} \) and \( \hat{H}^-(x) := \inf_{\hat{J}^{-1}(x)} \hat{H} \). Then the functions \( \hat{H}^+ \) and \( -\hat{H}^- \) are lower semicontinuous. This statement gives the required semicontinuity in the statement of Theorem C.

The analysis of the graph of \( h_\varepsilon \), which corresponds to the elliptic critical values and possible cuts due to focus-focus singularities, was carried out in [Vu07, Theorem 3.8]: it is continuous, piecewise linear, and convex. Thus, \( C_\varepsilon^J \) is of type II.

3. The fact that \( I_j \subset (K^+ \setminus K^-) \) implies that \( C_\varepsilon^J \) is of type III can be proved in a similar way to (2).

4. Finally, let \( I_j \subset J(M) \setminus (K^+ \cup K^-) \). In this case, we must have, for any \( x \in I_j \), \( \hat{H}_+(x) = +\infty \) and \( \hat{H}_-(x) = -\infty \). Therefore, \( f_\varepsilon((I_j \times \mathbb{R}) \cap F(M)) = C_\varepsilon^J \) has the form

\[
\left\{ (x, z) \mid x \in I_j, \lim_{y \to -\infty} f_\varepsilon^{(2)}(x, y) < z < \lim_{y \to +\infty} f_\varepsilon^{(2)}(x, y) \right\},
\]

where the limits are understood in \( \mathbb{R} \). Thus, \( C_\varepsilon^J \) is of type IV.

This proves (P.1).

**Step 3.** *(Proof of (P.3)).* By the action-angle theorem, \( (A_1, A_2) := f_\varepsilon \circ F \) is a set of action variables near \( F^{-1}(x, y) \) with

\[
A_1 = J, \quad A_2 = A_2(J, H).
\]

We have a symplectomorphism \( U \to T^2_\theta \times \mathbb{R}^2_A \), where \( U \) is a saturated neighborhood of the fiber \( F^{-1}(x, y) \), and the symplectic form on \( T^2_\theta \times \mathbb{R}^2_A \) is given by \( dA_1 \wedge d\theta_1 + dA_2 \wedge d\theta_2 \). We have

\[
U \cap J^{-1}(x) = A_1^{-1}(x) = \left\{ (\theta, A) \mid \theta \in T^2, \ A_1 = x \right\}.
\]

Since the normalized Liouville volume form is \( (2\pi)^{-2}dA_1 \wedge dA_2 \wedge d\theta_1 \wedge d\theta_2 \), the induced volume form on \( U \cap J^{-1}(x) \) is \( (2\pi)^{-2}dA_2 \wedge d\theta_1 \wedge d\theta_2 \). In other words, the push-forward by \( A_2 \) of the Liouville measure on \( J^{-1}(x) \) has a constant density 1 against the Lebesgue measure \( dA_2 \). This gives the result because the set of critical points of \( H \) in \( J^{-1}(x) \) has zero-measure in \( J^{-1}(x) \). This concludes the proof of Theorem C.

\( \square \)

**Example 6.1** *(Spherical Pendulum)* Semitoric systems with proper \( F = (J, H) \) but non-proper \( J \) include many simple integrable systems from classical mechanics, such as the spherical pendulum, which we now recall. The phase space of the spherical pendulum is \( M = T^*S^2 \) with its natural exact symplectic form. Let the circle \( S^1 \) act on the sphere \( S^2 \subset \mathbb{R}^3 \) by rotations about the vertical axis. Identify \( T^*S^2 \) with \( TS^2 \), using the standard Riemannian metric on
$S^2$, and denote its points by $(q, p) = (q^1, q^2, q^3, p_1, p_2, p_3) ∈ T^*S^2 = TS^2$, $\|q\|^2 = 1$, $q \cdot p = 0$. Working in units in which the mass of the pendulum and the gravitational acceleration are equal to one, the integrable system $F := (J,H) : TS^2 → \mathbb{R}^2$ is given by the momentum map of the (co)tangent lifted $S^1$-action on $TS^2$,

\begin{equation}
J(q^1, q^2, q^3, p_1, p_2, p_3) = q^1 p_2 - q^2 p_1,
\end{equation}

and the classical Hamiltonian

\begin{equation}
H(q^1, q^2, q^3, p_1, p_2, p_3) = \frac{(p_1)^2 + (p_2)^2 + (p_3)^2}{2} + q^3,
\end{equation}

the sum of the kinetic and potential energy. The momentum map $J$ is not proper because the sequence $\{(0,0,1,n,n,0)\}_{n \in \mathbb{N}} \subset J^{-1}(0) \subset TS^2$ does not contain any convergent subsequence. The Hamiltonian $H$ is proper since $H^{-1}([a,b])$ is a closed subset of the compact subset of $TS^2$ for which $2(a-1) ≤ \|p\|^2 ≤ 2(b+1)$. Therefore, $F$ is also proper. In this case, $F(M)$ is depicted in Figure 6.1 and the cartographic invariant of $(M,F)$ is represented in Figure 6.3; we call it $\Delta(F)$.

There is precisely one elliptic-elliptic singularity at $((0,0,-1),(0,0,0))$, one focus-focus singularity at $((0,0,1),(0,0,0))$, and uncountably many transversally-elliptic type singularities. The range $F(M)$ and the set of critical values of $F$, which equals its bifurcation set, are given in Figure 6.1. The image under $F$ of the focus-singularity is the point $(0,1)$. The image under $F$ of the elliptic-elliptic singularity is the point $(0,-1)$. We know that the image by $J$ of critical points of $F$ of rank zero is the singleton \{0\}. Hence (one of the two representatives of) $\Delta(F)$ has no vertex in both regions $J < 0$ and $J > 0$. In each of these regions, there is only one connected family of transversally elliptic singular values. This means that $\Delta(F)$ in these region consist of a single (semi-infinite)

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{image.png}
\caption{Image of of $F := (J,H)$ given by (6.1) and (6.2). The edges are the image of the transversally-elliptic singularities (rank 1), the vertex is the image of the elliptic-elliptic singularity (rank 1), and the dark dot in the interior is the image of the focus-focus singularity (rank 0). All other points are regular (rank 2).}
\end{figure}
Figure 6.2. Fiber of $F := (J, H)$ given by (6.1) and (6.2) over the focus-focus critical value $(0, 1)$.

edge. We can arbitrarily assume that, in the region $J < 0$, the edge in question is the negative real axis $\{(y, 0) \mid y < 0\}$. Then we have a vertex at the origin $(x = 0, y = 0)$.

Figure 6.3. One of the two cartographic projections of the spherical pendulum.

We still need to compute the slope of the edge corresponding to the region where $J > 0$. For this, we apply [Vu07, Theorem 5.3], which states that the change of slope can be deduced from the isotropy weights of the $S^1$ momentum map $J$ and the monodromy index of the focus-focus point. (We need to include the focus-focus point because its $J$-value is the same as the $J$-value of the elliptic-elliptic point.) So we compute these weights now. The vertex of the polygon corresponds to the stable equilibrium at the South Pole of the sphere. We use the variables $(q_1, q_2, p_1, p_2)$ as canonical coordinates on the tangent plane to the South Pole. In these coordinates, the quadratic approximation of $J$ is in fact exact, and equal to $J^{(2)} = q_1^2 p_2 - q_2^2 p_1$. Now consider the following change of coordinates:

\begin{equation}
(6.3) \quad x_1 := \frac{q_2 - q_1}{\sqrt{2}}, \quad x_2 := \frac{p_1 + p_2}{\sqrt{2}}, \quad \xi_1 := \frac{p_1 - p_2}{\sqrt{2}}, \quad \xi_2 := \frac{q_1 + q_2}{\sqrt{2}}.
\end{equation}

This is a canonical transformation and the expression of $J^{(2)}$ in these variables is $J^{(2)} = \frac{1}{2}(x_2^2 + \xi_2^2) - \frac{1}{2}(x_1^2 + \xi_1^2)$. Since the Hamiltonian flows of $\frac{1}{2}(x_2^2 + \xi_2^2)$ and $\frac{1}{2}(x_1^2 + \xi_1^2)$ are $2\pi$-periodic, this formula implies that the isotropy weights of $J$ at this critical point are $-1$ and $1$. From [Vu07], we know that the difference between the slope of the edge in $J > 0$ and the slope of the edge in $J < 0$ must be equal to $\frac{1}{ab} + k$, where $a$ and $b$ are the isotropy weights, and $k$ is the
monodromy index. For the spherical pendulum, \( k = 1 \) because there is only one simple focus-focus point. Thus the new slope is \( \frac{1}{\alpha} + k = 1 + 1 = 2 \). This leads to the polygonal set depicted in Figure 6.3.

7. Proof of Theorem E

We give here the outline of the construction of a family of integrable systems defined on an open subset of \( S^2 \times S^2 \), leading to the proof of Theorem E.

**Step 1. (Construction of suitable smooth functions.)** Let

\[ \Omega := [-1, 1] \times [-1, 1] \setminus \{0\} \times [0, 1]. \]

Let \( \chi : [-1, 1] \to \mathbb{R} \) be any \( C^\infty \)-smooth function such that \( \chi(z_2) \equiv 1 \) if \( z_2 \leq 0 \) and \( 0 < \chi(z_2) \neq 1 \) if \( z_2 > 0 \). Define \( f : \Omega \to \mathbb{R} \) by

\[
(7.1) \quad f(z_1, z_2) = \begin{cases} 
1 & \text{if } z_1 \leq 0; \\
\chi(z_2) & \text{if } z_1 > 0.
\end{cases}
\]

and note that it is smooth on \( \Omega \).

**Step 2. (Definition of a connected smooth 4-manifold \( M \).)** Let \( S^2 \) be the unit sphere in \( \mathbb{R}^3 \) and \( M := S^2 \times S^2 \setminus \{(x_1, y_1, z_1), (x_2, y_2, z_2) \in S^2 \times S^2 \mid z_1 = 0, z_2 \geq 0\} \), where a point in the first sphere has coordinates \((x_1, y_1, z_1)\) and a point in the second sphere has coordinates \((x_2, y_2, z_2)\). Since \( M \subset S^2 \times S^2 \) is an open subset, it is a smooth manifold. Moreover, \( M \) is connected.

**Step 3. (Definition of a smooth 2-form \( \omega \in \Omega^2(M) \).)** Let \( \pi_i : S^2 \times S^2 \to S^2 \) be the projection on the \( i \)-th copy of \( S^2 \), \( i = 1, 2 \). Let \( \omega_i := \pi^* \omega_{S^2} \) where \( \omega_{S^2} \) is the standard area form on \( S^2 \). Define the 2-form \( \omega \) on \( M \) by

\[
(7.2) \quad \omega_{(m_1, m_2)} := (\omega_1)_{m_2} + f(z_1, z_2) (\omega_2)_{m_2}
\]

for every \( (m_1, m_2) \in M \). Since \( f \) is smooth by Step 1, \( \omega \) is also smooth, i.e., \( \omega \in \Omega^2(M) \).

**Step 4. (The 2-form \( \omega \) is symplectic.)** One can check that \( \omega \) is closed because \( \frac{\partial f}{\partial z_1} = 0 \), and that \( \omega \) is non-degenerate because \( f \neq 0 \).

**Step 5. (\( (M, \omega) \) with \( J := z_1, H := z_2 \) satisfies \( \{J, H\} = 0 \) and \( J \) is a momentum map for a Hamiltonian \( S^1 \)-action.)** We let \( S^1 \) act on \( M \) by rotation about the (vertical) \( z_1 \)-axis of the first sphere and trivially on the second sphere. The infinitesimal generator of this action equals the vector field \( \mathcal{X}_{(x_1, y_1, z_1), (x_2, y_2, z_2)} := ((-y_1, x_1, 0), (0, 0, 0)) \). This immediately shows that \( J = z_1 \) is a moment map for this action.

**Step 6. (\( (M, \omega) \) with \( J := z_1, H := z_2 \) is a generalized semitoric system with only elliptic singularities.)** A direct verification shows that the rank zero critical points are precisely \((N_1, N_2), (N_1, S_2), (S_1, N_2), \) and \((S_1, S_2)\), where \( N_i, S_i \) are the North and South Poles on the first and second spheres, respectively. One can verify that these critical points are non-degenerate, in the sense that a generic combination \((ia, -ia, ibf(1), -ibf(1))\) of the linearizations of the vector fields \( \mathcal{X}_J \) and \( \mathcal{X}_H \) at each of these points \((0, 0, if(1), -if(1))\) and \((0, 0, i, -i)\) has four distinct eigenvalues. Thus these singularities are of elliptic-elliptic type. The rank one critical points are \((N_1, (x_2, y_2, z_2)), (S_1, (x_2, y_2, z_2)), ((x_1, y_1, z_1), N_2)\) with \( z_1 \neq 0 \), and \(((x_1, y_1, z_1), S_2)\). Another simple computation shows that all
of them are non-degenerate and of transversally elliptic type. It follows that
\( J := z_1, H := z_2 \) is an integrable system with only non-degenerate singularities,
of either elliptic-elliptic or transversally elliptic type. Hence \( (J := z_1, H := z_2) \)
is a generalized semitoric system.

Since the range of \( F \) is
\[
(7.3) \quad F(M) = [-1, 1] \times [-1, 1] \setminus \{ z_1 = 0, z_2 \geq 0 \},
\]
is not a closed set (see also Figure 7.1), it follows that \( F \) is not a proper map.

\[
\text{Figure 7.1. The image } F(M).
\]

**Step 7.** *(Modify } F \text{ suitably to turn it into a proper map which still defines a semitoric system.)* Consider the smooth function \( g : \Omega \to \mathbb{R}^2 \) defined by
\[
g(z_1, z_2) = \left( z_1, \frac{z_2 + 2}{z_1 + h(z_2)} \right),
\]
where \( h(z_2) \geq 0 \), \( h(z_2) = 0 \) if and only if \( z_2 \geq 0 \), and \( h'(z_2) < 0 \) for \( z_2 < 0 \). Define \( \widetilde{F} := F \circ g = \left( J, \frac{H + 2}{J + h(D)} \right) : M \to \mathbb{R}^2 \). Since the Jacobian of \( \widetilde{F} \) is
\[
\frac{1}{(z_1^2 + h(z_2))^2} \left( z_1^2 + h(z_2) - h'(z_2)(z_2 + 2) \right) > 0
\]
(recall that \( h'(z_2) \leq 0 \) and \( z_1^2 + h(z_2) > 0 \) for \( (z_1, z_2) \in \Omega \)), it follows that \( \widetilde{F} \) is a local diffeomorphism. In order to show that \( \widetilde{F} \) is proper, it suffices to prove
that \( \widetilde{F}^{-1}(K_1 \times K_2) \) is compact if \( K_1 \) and \( K_2 \) are closed intervals of \( \mathbb{R} \); since
the second component of \( g \) is always positive, we can assume, without loss of
generality, that \( K_2 = [a, b) \) with \( a > 0 \). To show that \( \widetilde{F} \) is proper, we begin
by analyzing \( g^{-1}(K_1 \times K_2) \). We have \((z_1, z_2) \in g^{-1}(K_1 \times K_2)\) if and only if
\( z_1 \in K_1 \) and \( 0 < a \leq \frac{z_2 + 2}{z_1 + h(z_2)} \leq b \), which is implies that
\[
\frac{1}{b} \leq \frac{z_2 + 2}{b} \leq z_1^2 + h(z_2).
\]
Hence either \( z_1^2 \geq 1/2b \) or \( h(z_2) \geq 1/2b \) Thus the set \( g^{-1}(K_1 \times K_2) \) lies inside
the set \( \Omega_b \) in Figure 7.2. Since \( g^{-1}(K_1 \times K_2) \) is closed and obviously bounded,
as a subset of the compact set \( \Omega_b \), it follows that \( g^{-1}(K_1 \times K_2) \) is compact in
\( \mathbb{R}^2 \). Therefore,
\[
\widetilde{F}^{-1}(K_1 \times K_2) = F^{-1} \left( g^{-1}(K_1 \times K_2) \right)
\]
is compact in \( S^2 \times S^2 \) and is obviously contained in \( M \), by construction. We conclude that \( \tilde{F}^{-1}(K_1 \times K_2) \) is compact in \( M \), endowed with the subspace topology.

Note that \( J \) is not proper because \( J^{-1}(0) \) is not compact. However, \( \tilde{F} \) is a general semitoric system and \( \tilde{F} \) is proper.

**Step 8. (Finding the image \( \tilde{F}(M) \).)** Let

\[
X := \left( [-1,0] \times [-1,1] \right) \cup \left( (0,1] \times [-1,1] \right) \cup \left( \{0\} \times [-1,0] \right).
\]

It follows from (7.3) (see also Figure 7.1) that

\[
\tilde{F}(M) = g(F(M)) = \left\{ \left( z_1, -\frac{z_2 + 2}{z_1^2 + h(z_2)} \right) \mid (z_1, z_2) \in X \right\}.
\]

Note that the second component of \( g \) is an even function of \( z_1 \) and hence the range \( \tilde{F}(M) \) is symmetric about the vertical axis in \( \mathbb{R}^2 \). A straightforward analysis shows that \( \tilde{F}(M) \) is the following region in \( \mathbb{R}^2 \):

\[
\left\{ (x, y) \in \mathbb{R}^2 \mid 0 < |x| \leq 1, \quad \frac{1}{x^2 + h(-1)} \leq y \leq \frac{3}{x^2} \right\} \bigcup \left( \{0\} \times \left[ \frac{1}{h(-1)}, \infty \right) \right);
\]

see Figure 7.3.

Note that the closed segment \([-1,1] \times \{-1\} \subset F(M)\) is mapped by \( g \) to the lower curve in Figure 7.3, the two half-open segments \([-1,1] \setminus \{0\} \times \{1\}\) to the two upper curves, the two closed vertical segments to the two closed vertical segments, and the half-open interval \( \{0\} \times [-1,0) \) to the infinite half-open interval \( \{0\} \times [1/h(-1), \infty) \).

**Step 9. (Construction of the cartographic representation.)** We shall construct the cartographic invariant in Theorem C from \( \tilde{F}(M) \) by flattening out the horizontal curves and setting the height between them at the value given by the volume of the corresponding reduced phase space. For each \(|x| \leq 1\), let \( \ell(x) \) denote the volume of the reduced manifold \( J^{-1}(x)/S^1 \). Then, by Theorem C, the cartographic invariant associated to the general semitoric system \((M, \tilde{F})\) is given by the formula

\[
\Delta = \left\{ (x, y) \in \mathbb{R}^2 \mid 0 < |x| \leq 1, \quad 0 \leq y \leq \ell(x) \right\} \cup \{0\} \times [0,2\pi).
\]
Using the definition (7.1) of $f$, a direct computation shows that if $x < 0$ then $J^{-1}(x)/S^1 = \{x\} \times S^2$, and hence
\[
\ell(x) = \int_{S^2} f(x, Z_2) \mathrm{d}\theta \wedge \mathrm{d}Z_2 = 2\pi \int_{-1}^{1} f(x, Z_2) \mathrm{d}Z_2 = 4\pi,
\]
because for $x < 0$, we have $f(x, Z_2) = 1$ for any $Z_2 \in [-1, 1]$. Similarly, if $x > 0$ then, as before, the reduced space is $J^{-1}(x)/S^1 = \{x\} \times S^2$, and hence
\[
\ell(x) = 2\pi \int_{1}^{1} \chi(Z_2) \mathrm{d}Z_2.
\]
If $x = 0$, then the reduced space $J^{-1}(0)/S^1$ is the southern hemisphere of the second factor and hence $\ell(0) = 2\pi$. Therefore, the cartographic invariant is given in Figure 7.4.

![Figure 7.3](image.png)

**Figure 7.3.** The set $\bar{F}(M)$ with the choice $h(-1) = 1$.

![Figure 7.4](image.png)

**Figure 7.4.** A representative of $\Delta(M, \bar{F})$.

We have so far shown (E.1)-(E.5). Theorem D implies (E.6). We have left to show (E.7).

To conclude the proof, we modify the construction above in order to illustrate the existence of unbounded cartographic invariants with fibers of infinite length. As we shall see, most of the computations of the previous example remain valid.
Let
\[ N := S^2 \times S^2 \setminus \left\{ ((x_1, y_1, z_1), (x_2, y_2, z_2)) \in S^2 \times S^2 \mid z_1 = 0, z_2 \geq 0 \right\} \]
(7.4)
\[ \cup \left\{ ((x_1, y_1, z_1), (x_2, y_2, z_2)) \in S^2 \times S^2 \mid z_1 \geq 0, z_2 = 1 \right\}. \]

As in the previous example, \( N \) is open and connected. Moreover, because it is a subset of \( M \), the restriction of the form \( \Omega \) given by (7.2), is a symplectic form. Similarly, \( J = z_1, H = z_2 \) defines an integrable system on \( N \) and \( J \) is the momentum map of a Hamiltonian \( S^1 \)-action. The computations in the previous example show that we have the same singularities, all of them non-degenerate. If \( F = (J, H) \), its image is
\[ F(N) = [-1, 1] \times [-1, 1] \setminus \left( \{ z_1 = 0, z_2 \geq 0 \} \cup \{ 0 \leq z_1 \leq 1, z_2 = 1 \} \right) \]
(7.5)
(see Figure 7.5) which is not a closed set, and hence \( F \) is not a proper map.

Define
\[
\begin{align*}
g(z_1, z_2) := & \left( z_1, \frac{z_2 + 2}{((z_1 - 1)^2 + h(z_1))(z_1^2 + h(z_2))} \right) \\
\end{align*}
\]
and \( \tilde{F} := g \circ F \), where \( F := (z_1, z_2); h \) is as in the previous example. To see that \( g \) is a local diffeomorphism, it suffices to note that the Jacobian determinant of \( g \) has the expression \( (\Delta - (z_2 + 2) \frac{\partial \Delta}{\partial z_2})/\Delta^2 \), where \( \Delta := ((z_1 - 1)^2 + h(z_1))(z_1^2 + h(z_2)) \). Since \( \Delta > 0 \) and
\[ \frac{\partial \Delta}{\partial z_2} = 2(z_2 - 1)(z_1^2 + h(z_2)) + ((z_1 - 1)^2 + h(z_1))h'(z_2) < 0, \]
it follows that the Jacobian determinant of \( g \) is strictly positive. As in the previous example, one can check that \( g^{-1}(K_1 \times K_2) \) is a compact subset of \( \mathbb{R}^2 \), where \( K_i, i = 1, 2 \), are closed bounded intervals in \( \mathbb{R} \). The argument given in the previous example shows then that \( \tilde{F} \) is a proper map. Therefore, \( (N, \tilde{F}) \) is a proper general semitoric system. The image \( \tilde{F} \) is given in Figure 7.6.

Finally, to determine the possible affine invariants associated to this system, we need to compute \( \ell(x) \), the volume of the reduced manifold \( J^{-1}(x)/S^1 \). As
Figure 7.6. The image $\tilde{F}(N)$ with the choice $h(-1) = 1$.

Before, we compute

$$\ell(x) = \begin{cases} 
4\pi, & \text{if } x < 0 \\
2\pi, & \text{if } x = 0 \\
2\pi(1 + \alpha), & \text{if } x > 0 
\end{cases}$$

where $\alpha := \int_0^1 \chi(z_2)dz_2 \geq 0$. The possible cartographic invariants are given in Figure 7.7. This proves (E.7).

Figure 7.7. A representative of the cartographic invariants depending on $\alpha$. 
Remark 7.1 When $M$ is compact (so $J,H,F$ are all proper), the cartographic invariant of $F$ is a polygon which is related to the classification of Hamiltonian $S^1$-spaces by Karshon [Ka99], as explained in [HSS13].

8. Appendix

8.1. Bifurcation set. Let $M$ and $N$ be smooth manifolds. A smooth map $f : M \to N$ is said to be locally trivial at $n_0 \in f(M)$, if there is an open neighborhood $U \subset N$ of $n_0$ such that $f^{-1}(n)$ is a smooth submanifold of $M$ for each $n \in U$ and there is a smooth map $h : f^{-1}(U) \to f^{-1}(n_0)$ such that $f \times h : f^{-1}(U) \to U \times f^{-1}(n_0)$ is a diffeomorphism. The bifurcation set $\Sigma_f$ consists of all the points of $N$ where $f$ is not locally trivial.

It is known that the set of critical values of $f$ is included in the bifurcation set and that if $f$ is proper this inclusion is an equality (see [AM78, Proposition 4.5.1] and the comments following it).

8.2. Linearization of singularities. Let $(M,\omega)$ be a connected symplectic 4-manifold, $F = (f_1, f_2)$ an integrable system on $(M,\omega)$, and $m \in M$ a critical point of $F$, i.e., the rank of the derivative (tangent map) $d_m F : T_m M \to \mathbb{R}^2$ of $F$ is either 0 or 1. If $d_m F = 0$, $m$ is said to be non-degenerate if the Hessians $\text{Hess } f_1(m)$, $\text{Hess } f_2(m)$ span a Cartan subalgebra of the symplectic Lie algebra of quadratic forms on the symplectic vector space $(T_m M, \omega_m)$. If rank$(d_m F) = 1$, we may assume that $d_m f_1 \neq 0$. Let $\iota : S \to M$ be an embedded local 2-dimensional symplectic submanifold through $m$ such that $T_m S \subset \ker(d_m f_1)$ and $T_m S$ is transversal to the Hamiltonian vector field $\mathcal{X}_{f_1}$ defined by the function $f_1$. This is possible by the classical Hamiltonian Flow Box Theorem ([AM78, Theorem 5.2.19]), also known as the Darboux-Carathéodory Theorem ([PV11a, Theorem 4.1]). It is easily seen that the definition does not depend on the choice of $S$. The point $m$ is called transversally non-degenerate if $\text{Hess}(\iota^*f_2)(m)$ is a non-degenerate symmetric bilinear form on $T_m S$.

For the notion of non-degeneracy of a critical point in arbitrary dimensions see [Ve78] and [Vu06, Section 3]). In this paper, we need the following property of non-degenerate critical points ([El84, El90], [VW10]) in terms of the Williamson normal form ([Wi36]), which we state in any dimension but will only use in dimension 4.

Theorem 8.1 (Eliasson). Let $F = (f_1, \ldots, f_n) : M \to \mathbb{R}^n$ be an integrable system and $m \in M$ a non-degenerate critical point of $F$. Then there are local symplectic coordinates $(x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$ about $m$, in which $m$ is represented as $(0, \ldots, 0)$, such that $\{f_i, q_j\} = 0$, for all $i, j$, where the $q_1, \ldots, q_n$ are defined on an neighborhood of $(0, \ldots, 0)$ in $\mathbb{R}^n$ and have one of the following expressions:

(a) Elliptic component: $q_j = (x_j^2 + \xi_j^2)/2$, where $1 \leq j \leq n$.
(b) Hyperbolic component: $q_j = x_j \xi_j$, where $1 \leq j \leq n$.
(c) Focus-focus component: $q_j-1 = x_j-1 \xi_j - x_j \xi_{j-1}$ and $q_j = x_j-1 \xi_{j-1} + x_j \xi_j$ where $2 \leq j \leq n - 1$. 

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(d) Non-singular component: \( q_j = \xi_j \), where \( 1 \leq j \leq n \).

If \( m \) does not have hyperbolic components, then the system of equations \( \{ f_i, q_j \} = 0 \), for all \( i, j \), may be replaced by \((F - F(m)) \circ \varphi = g \circ (q_1, \ldots, q_n)\), where \( \varphi = (x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)^{-1} \) and \( g \) is a diffeomorphism from a neighborhood of \((0, \ldots, 0)\) in \( \mathbb{R}^n \) onto another such neighborhood, with \( g(0, \ldots, 0) = (0, \ldots, 0) \).

If \( M \) is 4-dimensional and \( F \) has no hyperbolic singularities, \((q_1, q_2)\) is:

(T.1) if \( m \) is a critical point of \( F \) of rank zero, then \( q_j \) is one of

(i) \( q_1 = (x_1^2 + \xi_1^2)/2 \) and \( q_2 = (x_2^2 + \xi_2^2)/2 \).

(ii) \( q_1 = x_1\xi_2 - x_2\xi_1 \) and \( q_2 = x_1\xi_1 + x_2\xi_2 \);

on the other hand,

(T.2) if \( m \) is a critical point of \( F \) of rank one, then

(iii) \( q_1 = (x_1^2 + \xi_1^2)/2 \) and \( q_2 = \xi_2 \).

A non-degenerate critical point is called elliptic-elliptic, focus-focus, or transversally-elliptic if both components \( q_1, q_2 \) are of elliptic type, \( q_1, q_2 \) together correspond to a focus-focus component, or one component is of elliptic type and the other component is \( \xi_1 \) or \( \xi_2 \), respectively.

For the spherical pendulum, see Figure 6.1 where the critical points of \( F \) lie in \( F(TS^2) \).

8.3. Affine manifolds. An affine \( n \)-dimensional manifold is a smooth manifold admitting an atlas whose change of chart maps are in the affine group of \( \mathbb{R}^n \), i.e., in

\[
\text{Aff}(n, \mathbb{R}) := \text{GL}(n, \mathbb{R}) \ltimes \mathbb{R}^n
\]

\[
:= \left\{ \begin{bmatrix} U & u \\ 0 & 1 \end{bmatrix} \mid U \in \text{GL}(n, \mathbb{R}), \ u \in \mathbb{R}^2 \right\} \subset \text{GL}(n+1, \mathbb{R}).
\]

An integral affine \( n \)-dimensional manifold is an affine manifold admitting an atlas whose change of chart maps are in \( \text{Aff}(n, \mathbb{Z}) := \text{GL}(n, \mathbb{Z}) \ltimes \mathbb{Z}^n \), i.e., \( U \in \text{GL}(n, \mathbb{Z}) \) in the definition above.

Let \( M \) be a connected \( n \)-dimensional manifold, \( m_0 \in M \), and \( p : \widetilde{M} \to M \) its universal covering manifold, i.e., the set of homotopy classes of smooth paths \( \lambda : [0, 1] \to M \) starting at \( \lambda(0) = m_0 \) and keeping the endpoints fixed; \( p([\lambda]) := \lambda(1) \). Recall that \( \widetilde{M} \) is a smooth simply connected \( n \)-dimensional manifold and that \( p \) is a covering map. The group of deck transformations of \( p \), i.e., all diffeomorphisms \( \chi : \widetilde{M} \to \widetilde{M} \) such that \( p \circ \chi = p \), is isomorphic to the first fundamental group \( \pi_1(M) \) (based at \( m_0 \)).

If \( M \) is, in addition, an affine manifold (see, e.g., [GH84, Section 2.3] for more information), then \( p \) induces an affine manifold structure on \( \widetilde{M} \) by requiring \( p \) to be an affine map, i.e., its local representative is affine in any pair of local charts.

A developing map for \( M \) is an affine immersion \( \zeta : \widetilde{M} \to \mathbb{R}^n \). It is well-known (see, e.g., [GH84, page 641]) that each connected affine manifold has at least one developing map and that if \( \zeta' : \widetilde{M} \to \mathbb{R}^n \) is another developing map then there is a unique \( A \in \text{Aff}(n, \mathbb{R}) \) such that \( \zeta' = A\zeta \). In addition, for any developing map \( \zeta : \widetilde{M} \to \mathbb{R}^n \), there is a unique equivariant monodromy homomorphism
\[
\mu : \pi_1(M) \to \text{Aff}(n, \mathbb{R}), \ i.e., \zeta([\lambda \ast \gamma]) = \mu([\lambda])\zeta([\gamma]) \text{ for any } [\lambda] \in \pi_1(M) \text{ and } [\gamma] \in \tilde{M}, \text{ where } \ast \text{ denotes composition of paths by concatenation.}
\]

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