On a $c$-number quantum $\tau$-function

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ABSTRACT

We first review the properties of the conventional $\tau$-functions of the KP and Toda-lattice hierarchies. A straightforward generalization is then discussed. It corresponds to passing from differential to finite-difference equations; it does not involve however the concept of operator-valued $\tau$-function nor the one associated with non-Cartanian (level $k \neq 1$) algebras. The present study could be useful to understand better $q$-free fields and their relation to ordinary free fields.

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1 Contribution to the Proceedings of III International Conference on Mathematical Physics, String Theory and Quantum Gravity, Alushta, June, 1993

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1 Introduction

The $\tau$-functions originally associated with integrable Toda-lattice hierarchies, have been identified as sections of determinant bundles over the universal moduli space \cite{1, 2}, and also seen to admit a group-theoretical interpretation \cite{3}. They are now attracting attention again because of their appearance as non-perturbative partition functions in quantum field/string theory \cite{4}. This is giving a new stimulus to the search of a broader interpretation of these $\tau$-function which would go beyond free-fermion models and commuting differential flows \cite{5}. The two most obvious directions of generalization consist in moving from free fermions to generic conformal models on one hand and from continuous 2-dimensional spectral surfaces to their quantum and/or discrete counterparts on the other hand. In the group-theoretical approach this corresponds to going beyond the level $k = 1$ Kac-Moody algebras and from “classical” to quantum groups respectively.

The purpose of this note is to discuss briefly some preliminary material, concerning the “quantum $\tau$-functions” that follow from the second direction of generalization. We concentrate here on a very particular aspect of the corresponding theory, i.e. on a possible relation of these “quantum $\tau$-function” to the conventional “classical” Toda-lattice $\tau$-function interpreting the former one as a $c$-numbers. We proceed by rephrasing results obtained in refs. \cite{6}, where solutions to finite-difference analogues of Hirota equations are given. These solutions are found for the “first” time-variable. We present here one particular way to introduce the other “times”. In this construction the $c$-number “quantum $\tau$-function” is obtained from the “classical” one, merely through a change of variable which appears to be a kind (albeit special) of Miwa transformation. Within this framework there is no need to go beyond the standard Segal-Wilson infinite Grassmannian and the standard free fermions (no $q$-fermions are required). Relations to generic operator-valued objects as well as to other aspects of the theory of quantum $\tau$-functions \cite{7} will be discussed elsewhere.

2 Conventional $\tau$-function

We begin by reminding the reader of the definition of the conventional $\tau$-function for the Toda-lattice and KP hierarchies. There are actualy several more or less equivalent definitions. Any
generalization should prove natural irrespective of which definition is adopted. For the sake of completeness and future use we give below the whole list. In this note however, we will deal mostly with only one of these definitions.

The conventional $\tau$-function can be characterized by any of the following properties:

1) It satisfies a bilinear Hirota equation, which for the KP hierarchy looks like

$$
\oint \frac{dz}{z} V^+ \{z|t\} \otimes V^- \{z|t'\} \tau\{t\} \otimes \tau\{t'\} = 0,
$$

$$
V^\pm \{z|t\} = \exp \left( \pm \sum_{n>0} \frac{1}{nz^n} \frac{\partial}{\partial t_n} \right).
$$

2) It can be written as a correlator of free fermions,

$$
\tau_N \{t, \bar{t}\} = \langle N| e^{H\{t\}} G e^{\bar{H}\{\bar{t}\}} |N\rangle,
$$

where

$$
H\{t\} = \sum_{n>0} t_n J_{+n}, \quad \bar{H}\{\bar{t}\} = \sum_{n>0} \bar{t}_n J_{-n},
$$

$$
J(z) = \sum_{n=-\infty}^{\infty} J_n z^{n-1} = \tilde{\psi}(z) \psi(z) = \partial \phi(z), \quad G = \exp \left( \sum_{m,n} A_{mn} \tilde{\psi}_n \psi_n \right),
$$

$$
\psi(z) = \sum_{i \in \mathbb{Z}} \psi_i z^i, \quad \tilde{\psi}(z) = \sum_{i \in \mathbb{Z}} \tilde{\psi}_i z^{-i},
$$

$$\tilde{\psi}_n |N\rangle = 0, \text{ for } n \geq N, \quad \psi_n |N\rangle = 0 \text{ for } n < N .$$

3) In Miwa coordinates, the KP $\tau$-function can be represented in the determinant form:

$$
\tau\{t\} = \frac{\det \Psi_a (\lambda_b)}{\Delta(\lambda)},
$$

where

$$
t_n = t_n^{(0)} + \frac{1}{n} \sum_\alpha \lambda_\alpha^{-n}, \quad \bar{t}_n = \bar{t}_n^{(0)} + \frac{1}{n} \sum_\alpha \bar{\lambda}_\alpha^n,
$$

and $\Delta(\lambda) = \prod_{\alpha > \beta} (\lambda_\alpha - \lambda_\beta)$. The analogous representation of the generic Toda-lattice $\tau$-function is a little more sophisticated, see refs. [3, 5, 8] for details. Sometimes it is convenient to interpret the $\lambda_\alpha$’s as eigenvalues of some matrix $\Lambda$ [4].

The Miwa coordinates can be used to introduce a sort of Fourier transform with respect to the variables $\{t, \bar{t}\}$,

$$
\tau_N \{t, \bar{t}\} = \hat{\tau}\{p(\lambda), \bar{p}(\bar{\lambda})\},
$$
with
\[ t_n = t_n^{(0)} + \frac{1}{n} \int_{d\lambda} p(\lambda) \lambda^{-n}, \quad \bar{t}_n = \bar{t}_n^{(0)} + \frac{1}{n} \int_{d\bar{\lambda}} \bar{p}(\bar{\lambda}) \bar{\lambda}^n. \] (7)

If \( p(\lambda) = \sum_a p_a \delta(\lambda - \lambda_a) \) and all non-vanishing \( p_a = 1 \), we recover the parameterization (8).

In terms of \( \hat{\tau}(p_a, \bar{p}_a) \) the Hirota equations become finite-difference equations [10]:
\[ (\lambda_a - \lambda_b) \hat{\tau}_N(p_a + 1, p_b + 1; p_c) \hat{\tau}_N(p_a, p_b; p_c + 1) + \]
\[ + (\text{cyclic permutations of } a, b, c) = 0 \text{ for any three } \lambda_a, \lambda_b, \lambda_c; \]
\[ \hat{\tau}_N(p + 1, \bar{p}) \hat{\tau}_N(p, \bar{p} + 1) - \hat{\tau}_N(p + 1, \bar{p} + 1) \hat{\tau}_N(p, \bar{p}) = \]
\[ = \frac{\lambda_b}{\lambda_a} \hat{\tau}_{N+1}(p + 1, \bar{p} + 1) \hat{\tau}_{N-1}(p, \bar{p}) \text{ for any two } \lambda_a, \lambda_b; \] (9)

These equations are an immediate consequence of (1). Actually, there is an infinite set of equations, but it is sufficient to consider only (8) in order to fix the KP flows associated with positive times, the conjugate equation is necessary to fix the KP flows w.r.t. negative times. Moreover, it is enough to consider only (9) in order to obtain the complete Toda-lattice hierarchy.

4) Most relevant to our discussion is another determinant formula for the Toda-lattice \( \tau \)-function [3, 8]:
\[ \tau_N\{t, \bar{t}\} = \text{det}_{-\infty < i, j \leq N} H_{ij}\{t, \bar{t}\}, \] (10)

where the matrix \( H_{ij} \) is characterized by the property,
\[ \frac{\partial}{\partial t_k} H_{ij} = H_{i+k,j} = \left( \frac{\partial}{\partial t_1} \right)^k H_{ij}; \]
\[ \frac{\partial}{\partial \bar{t}_k} H_{ij} = H_{i,j+k} = \left( \frac{\partial}{\partial \bar{t}_1} \right)^k H_{ij}. \] (11)

These equations can be solved by a sort of Fourier transform, and their general solution is provided by an arbitrary measure function \( \mu(z, \bar{z}) \):
\[ H = \int_{dzd\bar{z}} \left( \prod_{k=1}^{\infty} e^{t_k z_k^k} \right) \mu(z, \bar{z}) \left( \prod_{k=0}^{\infty} e^{\bar{t}_k \bar{z}_k^k} \right), \quad T_{lm} = \int_{dzd\bar{z}} z^l \mu(z, \bar{z}) \bar{z}^m, \] (12)

which implies that
\[ H_{ij}\{t, \bar{t}\} = \sum_{l,m} P_{l-i}\{t\} T_{lm} P_{m-j}\{\bar{t}\}. \] (13)
$T_{lm}$ is already independent of $t$ and $\bar{t}$ (in fact, $T_{lm}$ describes the adjoint action of the element of the Grassmannian $G$ on fermionic modes). The Schur polynomials $P_{m}\{t\}$ are defined by

$$
\prod_{k=0}^{\infty} e^{t_k z^k} = \sum_{m=0}^{\infty} P_{m}\{t\} z^m, \quad (14)
$$

so that

$$
\frac{\partial}{\partial t_k} P_{m}\{t\} = P_{m-k}\{t\} = \left( \frac{\partial}{\partial t_1} \right)^k P_{m}\{t\}. \quad (15)
$$

The object, defined by the r.h.s. of eq. (10), satisfies the set of Hirota equations and thus is a Toda-lattice $\tau$-function as a corollary of (11). However, the origin of the Hirota equations from this point of view is somewhat more general than eq. (11).

### 3 The origin of the bilinear equation

The Hirota equations or bilinear identities have different interpretations according to the different constructions which are used to interpret the $\tau$-function:

1) From the point of view of integrable hierarchies these just appear as a compact way to write the entire hierarchy in terms of a single identity (1).

2) In the formalism of free fermions the simplest origin of the bilinear identities is the formula for the quadratic Casimir operator of the corresponding version of $GL(\infty) \otimes GL(\infty)$, $\sum_{n=-\infty}^{\infty} \bar{\psi}_n \otimes \psi_{-n}$, which, together with the embedding of $GL(\infty)$ into the universal enveloping of $\hat{U}(1)_{k=1}$, straightforwardly implies (1).

3) Eqs. (8), (9) for vanishing multiplicities $p_a, p_b, p_c, p, \bar{p}$ are immediate consequences of the determinant formula (4). Eqs. (8), (9) for arbitrary integer $p$’s can also be obtained in this way, if $p_a > 1$ is interpreted as the number (multiplicity) of coincident ”eigenvalues” $\lambda_a$. If $\lambda_b$ is smoothly moved along the spectral surface and reaches the position of $\lambda_a$, then $p_a$ changes abruptly: $p_a \rightarrow p_a + p_b$. Within this context all vanishing $p_a$ can be considered as associated with $\lambda_a = \infty$, and the change of $p_a$ by one $p_a \rightarrow p_a + 1$ is interpreted as ”bringing” one extra ”eigenvalue” $\lambda_a$ from infinity.

4) The most general bilinear identities can be deduced by the following procedure. Introduce two operators, $\hat{M}$ and $\hat{\bar{M}}$, acting on the infinite-dimensional matrix $H_{ij}$:

$$
\hat{M} : H_{ij} \rightarrow H_{i+1,j}; \quad \hat{\bar{M}} : H_{ij} \rightarrow H_{i,j+1}. \quad (16)
$$
Select some $n \times n$ submatrix of $H_{ij}$ and consider its determinant,
\[ \tau_{(n)}[\alpha, \bar{\alpha}|\beta, \bar{\beta}] = \det_{\alpha \leq i \leq \beta} H_{ij}, \quad n = \beta - \alpha = \bar{\beta} - \bar{\alpha}. \quad (17) \]

Then $\hat{M}$ and $\hat{M}'$ act on $\tau_{(n)}$ by shifting the $\alpha$’s and $\beta$’s:
\[ \hat{M} : \alpha \to \alpha + 1, \quad \beta \to \beta + 1, \quad \bar{\alpha} \to \bar{\alpha}, \quad \bar{\beta} \to \bar{\beta}; \]
\[ \hat{M}' : \alpha \to \alpha, \quad \beta \to \beta, \quad \bar{\alpha} \to \bar{\alpha} + 1, \quad \bar{\beta} \to \bar{\beta} + 1, \quad (18) \]
and the following bilinear relation is true:
\[ (I \otimes \hat{M} \hat{M}' - \hat{M} \otimes \hat{M}') \tau_{(n)}[\alpha, \bar{\alpha}|\beta, \bar{\beta}] \tau_{(n)}[\alpha, \bar{\alpha}|\beta, \bar{\beta}] = \tau_{(n+1)}[\alpha, \bar{\alpha}|\beta + 1, \bar{\beta} + 1] \tau_{(n-1)}[\alpha + 1, \bar{\alpha} + 1|\beta, \bar{\beta}]. \quad (21) \]

This formula is an identity, provided the action of $\hat{M}$ and $\hat{M}'$ is defined according to (14). With the same definition any two elements of the matrix $H_{ij}$ can be related through
\[ H_{ij} = \hat{M}^i \hat{M}'^j H, \quad (22) \]
(where $H = H_{00}$). Thus
\[ \tau_{(n)}[\alpha, \bar{\alpha}|\beta, \bar{\beta}] = \det_{\alpha \leq i < \beta} \left( \hat{M}^i \hat{M}'^j H \right). \quad (23) \]

One can now reformulate the statement and say that (23) is always a solution to (21), without any references to the matrix $H_{ij}$ and the rules (14). This is the general statement about the determinant solution to the bilinear equations which we will need in this paper.

\[ ^1 \text{It is a particular ($p = 2$) case of the general identity for the minors of any matrix,} \]
\[ \sum_{i_p} H_{r_{i_p}} \hat{H}_{i_1 \ldots i_p|j_1 \ldots j_p} = \frac{1}{p!} \sum_{p'} (-)^p \hat{H}_{i_1 \ldots i_{p-1}|j_{p(1)} \ldots j_{p(p-1)}} \delta_{r_{p(p)}} \delta_{j_{p(p)}}, \quad (19) \]
where the sum on the r.h.s. is over all permutations of the $p$ indices and $\hat{H}_{i_1 \ldots i_p|j_1 \ldots j_p}$ denotes the determinant (minor) of the matrix, which is obtained from $H_{ij}$ by removing the rows $i_1 \ldots i_p$ and the columns $j_1 \ldots j_p$. Using the fact that $(H^{-1})_{ij} = \hat{H}_{i|j}/\hat{H}$, this identity can be rewritten as
\[ \hat{H} \hat{H}_{i_1 \ldots i_p|j_1 \ldots j_p} = \left( \frac{1}{p!} \right)^2 \sum_{p', p''} (-)^{p} (-)^{p'} \hat{H}_{i_{p(1)} \ldots i_{p(p-1)}|j_{p(1)} \ldots j_{p(p-1)}} \delta_{i_{p(p)}} \delta_{j_{p(p)}} \hat{H}_{i|j}. \quad (20) \]

Taking now $p = 2$ and $i_1 = \alpha, \quad i_2 = \beta, \quad j_1 = \bar{\alpha}, \quad j_2 = \bar{\beta}$ one obtains (21).
Let us note, however, that this solution has the form of the determinant of a finite matrix. Such objects are usually interpreted as \( \tau \)-functions of forced hierarchies \([1]\). Generic \( \tau \)-functions can be obtained from these through a limiting procedure \( \alpha, \bar{\alpha} \to \infty \):

\[
\tau_N = \lim_{n \to \infty} \tau(n)[N - n, N - n|N, N].
\] (24)

We will use in the following both determinant representations since the infinite matrix one is natural from the viewpoint of the fermionic realization, while solution (23) is convenient to discuss algebraic properties.

It might be good at this point to explain, how the conventional differential Hirota equation can be deduced from this general construction. It arises from a particular choice of operators \( \hat{M} \) and \( \hat{\bar{M}} \). Namely, consider a function \( F(X, \bar{X}) \) and define:

\[
\hat{M}F(X, \bar{X}) = F(qX, \bar{X}), \quad \hat{\bar{M}}F(X, \bar{X}) = F(X, q\bar{X}) \quad \text{(multiplicative action)}
\] (25)

or

\[
\hat{M}f(x, \bar{x}) = f(x + \epsilon, \bar{x}), \quad \hat{\bar{M}}f(x, \bar{x}) = f(x, \bar{x} + \bar{\epsilon}) \quad \text{(additive action)}.
\] (26)

One can also introduce an algebraic version of the group action, specified by the operator \( \hat{M} \). It is defined in terms of the “derivative” operator:

\[
\hat{D} = \hat{\sigma}(\hat{M} - I),
\] (27)

and similarly for \( \hat{\bar{D}} \). In the additive case \( \hat{\sigma} \) simply acts as multiplication by a \( c \)-number, \( \hat{\sigma} = \epsilon^{-1} \). In the multiplicative case \( \hat{\sigma} \) is usually defined to act as multiplication by \( \frac{1}{(q-1)X} \), (analogously, \( \hat{\bar{\sigma}} \) is defined to act as multiplication by the factor \( \frac{1}{(\bar{q}-1)\bar{X}} \)) and has a non-trivial commutation relation with \( \hat{M} \):

\[
\hat{\sigma}\hat{M} = q\hat{M}\hat{\sigma}.
\] (28)

As a corollary of the above definitions, we have:

\[
\hat{D}^n = [\hat{\sigma}(\hat{M} - I)]^n = \hat{\sigma}^n \prod_{k=0}^{n-1} \left( \hat{M}q^{-k} - I \right) = \hat{\sigma}^n \hat{M}^n q^{-n(n-1)/2} \left( \hat{M}^{-1}; q \right)_n.
\] (29)

\footnote{Often a more symmetric definition \( \hat{D} = \hat{\sigma}(\hat{M}^{1/2} - \hat{M}^{-1/2}) \) is used. It gives expressions in a more symmetric form like the analogue of \([2]\), \( \Delta(\hat{D}) = \hat{D} \otimes \hat{M}^{1/2} + \hat{M}^{-1/2} \otimes \hat{D} \big|_{\text{diag}} \), but makes formulas somewhat more lengthy. Since this is not important for our presentation, we stick here to the simpler definitions.}
When acting on a product of two functions \( F(X)G(X) \) (i.e. on the “diagonal” of the tensor product) \( \hat{M} \) is characterized by the following comultiplication property:

\[
\Delta(\hat{M}) = \hat{M} \otimes \hat{M} \mid_{\text{diag}}, \quad \text{i.e.}
\]

\[
\hat{M}[F(X)G(X)] = F(qX)G(qX) = [\hat{M} F(X)][\hat{M} G(X)].
\]

Since \( \hat{\sigma} \) acts on the diagonal as multiplication,

\[
\Delta(\hat{\sigma}) \mid_{\text{diag}} = \hat{\sigma},
\]

we conclude that

\[
\Delta(\hat{D}) \mid_{\text{diag}} = \Delta \left( \hat{\sigma}(\hat{M} - I) \right) \mid_{\text{diag}} = \hat{\sigma} \Delta(\hat{M} - I) \mid_{\text{diag}} =
\]

\[
= \hat{\sigma} \left( \hat{M} \otimes \hat{M} - I \otimes I \right) \mid_{\text{diag}} = \left( \hat{D} \otimes I + I \otimes \hat{D} + \hat{\sigma}^{-1} \hat{D} \otimes \hat{D} \right) \mid_{\text{diag}} =
\]

\[
= \left( \hat{D} \otimes \hat{M} + I \otimes \hat{D} \right) \mid_{\text{diag}}.
\]

Since \( \hat{\sigma}^{-1} = (q-1)X \) vanishes at \( q = 1 \), we obtain in this “classical” case the ordinary Leibnitz rule \( \Delta(\hat{D}) = \hat{D} \otimes I + I \otimes \hat{D} \mid_{\text{diag}} \).

The bilinear identity (21) can be, of course, rewritten in terms of the operators \( \hat{D} \) and \( \hat{\bar{D}} \) instead of \( \hat{M} \) and \( \hat{\bar{M}} \). Indeed, using the action of the operators \( \hat{M} \) and \( \hat{\bar{M}} \) (25), (26), one can write:

\[
\tau_{(n)}(qX, \bar{q} \bar{X})\tau_{(n)}(X, \bar{X}) = \tau_{(n)}(qX, \bar{X})\tau_{(n)}(X, \bar{q} \bar{X}) = \tau_{(n-1)}(X, \bar{X})\tau_{(n+1)}(qX, \bar{q} \bar{X})
\]

in the multiplicative variables, or

\[
\tau_{(n)}(x + \epsilon, \bar{x} + \bar{\epsilon})\tau_{(n)}(x, \bar{x}) - \tau_{(n)}(x + \epsilon, \bar{X})\tau_{(n)}(x, \bar{X} + \bar{\epsilon}) =
\]

\[
= \tau_{(n-1)}(x, \bar{x})\tau_{(n+1)}(x + \epsilon, \bar{x} + \bar{\epsilon})
\]

in the additive variables. It can be equally rewritten in a form similar to (21):

\[
\left( I \otimes \hat{M} \hat{\bar{M}} \right) \tau_{(n)} = I \otimes \hat{M} \hat{\bar{M}} \tau_{(n-1)} \tau_{(n+1)},
\]

or, using (27):

\[
\left( I \otimes \hat{\bar{D}} \hat{D} \right) \tau_{(n)} = I \otimes \hat{\bar{D}} \hat{D} \tau_{(n-1)} \tau_{(n+1)}.
\]
More important, the same is true for $\tau(n)$ as given in (23) (if it is considered as the $\tau$-function of a forced hierarchy – see above). Indeed,

$$\tau^{(q)} = \det_{0 \leq i,j < N} \hat{M}^i \hat{M}^j H = \det(I + \hat{\sigma}^{-1} \hat{D})^i (I + \hat{\sigma}^{-1} \hat{D})^j H = (q\bar{q})^{(N-1)(N-2)/2} \hat{\sigma}^{-N(N-1)/2} \hat{\sigma}^{N(N-1)/2} \det\hat{D}^i \hat{D}^j H.$$  \hfill (37)

One is actually free to choose an arbitrary normalization of the $\tau$-function. In particular, a natural choice, which cancels the $\sigma$-factors in the r.h.s. of (36) (compare with (8), (9)), is

$$\hat{\tau}^{(q)}_N = \det \hat{D}^i \hat{D}^j H.$$ \hfill (38)

These normalized $\tau$-functions satisfy the equation

$$\left( I \otimes \hat{D} \hat{D} - \hat{D} \otimes \hat{D} \right) \hat{\tau}_N \hat{\tau}_N = \left( I \otimes \hat{M} \hat{M} \right) \hat{\tau}_{N-1} \hat{\tau}_{N+1} \hfill (39)$$

which should be compared with (35). In the $q = 1$ case, formula (38) is exactly the classical expression of the theory of the Toda-lattice $\tau$-functions, included as p.4) in the list of sect. 2. For $q \neq 1$ this result was first derived in refs. [6] and interpreted there as the determinant formula for solutions of a finite-difference Hirota equation.

4 Introduction of higher time-variables

There is still one ingredient missing in our discussion for generic $q$ (we consider $|q| < 1$ whenever it might lead to ambiguity). The thing is that the operators $\hat{M}$ and $\hat{D}$ are defined as acting on functions of a single variable $X$. Comparison with the $q = 1$ case implies that $X$ plays the role of the first time-variable $X = T_1$ (and $\bar{X} = \bar{T}_1$ while $N$ can be considered as one of the zero-times). According to this, (36) is only the subset of Hirota equations, associated with the first time-variables. A way to introduce higher times is suggested by the relation (13) for the $q = 1$ case. A natural generalization is

$$\hat{D}_k H = (\hat{D}_1)^k H, \quad \hat{D}_k H = (\hat{D}_1)^k H.$$ \hfill (40)

where $\hat{D}_k$ is a finite-difference operator w.r.t. the $k$-th time $T_k$,

$$\hat{D}_k F(T_k) = F(q_k T_k) - F(T_k) / (q_k - 1) T_k.$$ \hfill (41)
We reserve the possibility to take \( q_k \neq q \equiv q_1 \). For example, the choice

\[ q_k = q^k \tag{42} \]

seems to be a reasonable alternative.

It was a corollary of (11) that \( H_{ij} = \partial^i \bar{\partial}^j H = \partial_i \bar{\partial}_j H \) acquired the form (13), which is important for establishing the relation to free fermions. Let us now derive the analogue of (13) for \( q \neq 1 \). The generic solution to eqs. (40) is given by an integral formula involving \( q \)-exponentials

\[ H = \int dzd\bar{z} \left( \prod_{k=1}^{\infty} \hat{e}_{q_k}(T_k z^k) \right) \mu(z, \bar{z}) \left( \prod_{k=0}^{\infty} \hat{e}_{q_k}(\bar{T}_k \bar{z}^k) \right). \tag{43} \]

The measure \( \mu(z, \bar{z}) \) is not specified by (40).

Introduce now the \( q \)-Schur polynomials \( P_n^{(q)} \{ T \} \) through

\[ \prod_{k=0}^{\infty} \hat{e}_{q_k}(T_k z^k) = \sum_{n=0}^{\infty} P_n^{(q)} \{ T \} z^n. \tag{44} \]

It follows that \( \hat{D}_k P_n^{(q)} \{ T \} = P_{n-k}^{(q)} \{ T \} \) and that

\[ H_{ij}^{(q)} = \hat{D}^i \hat{D}^j H = \sum_{l=1}^{\infty} T_{lm} P_n^{(q)} \{ T \} \] \( T_{lm} = \int dzd\bar{z} \left( \frac{\mu(z, \bar{z})}{z^m} \right). \tag{45} \]

The next question is that of the free-fermion representation of \( \tau^{(q)} = \det H_{ij}^{(q)} \). It could at first seem natural to look for a representation in terms of \( q \)-fermions, \( q \)-oscillators etc. However it turns out that the most straightforward option is to stay with the ordinary free fermions, the same as for \( q = 1 \), and thus, with the ordinary Segal-Wilson Grassmannian.

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3 Let us remind the reader that the \( q \)-exponential \( e_q(x) = 1/E_q(-x) \) is characterized by the following set of properties:

1. \( D^+ \hat{e}_q(x) = \hat{e}_q(x) \); \( \hat{e}_q(x) \equiv e_q((1 - q)x) \) and \( \lim_{q \to 1} \hat{e}_q(x) = e^x \);
2. \( e_q(x) = \sum_{k \geq 0} \frac{x^k}{(q;q)_k} \), \( E_q(x) = \sum_{k \geq 0} \frac{k^{(k-1)/2} x^k}{(q,q)_k} \), where \( (a,q)_k \equiv \prod_{i=0}^{k-1} (1 - aq^i) = \frac{(a,q)_\infty}{(aq^k,q)_\infty} \);
3. \( e_q(x) = \frac{1}{(x,q)_\infty} \), thus \( E_q(x) = -(x,q)_\infty \) and \( \theta_{00}(x) \equiv \sum_{k=-\infty}^{\infty} q^{k^2/2} x^k = (q,q)_\infty E_q(q^{1/2}x)E_q(q^{1/2}x^{-1}) \);
4. \( E_q(x)E_q(y) = E_q(x+y) \) and \( e_q(y)e_q(x) = e_q(x+y) \), provided \( xy = qyx \).

The first three properties explain the origin of the \( q \)-special functions as solutions to finite difference equations (i.e. to various periodicity constraints), the last property indicates why these same functions occur in the study of non-commutative algebras and problems of quantum mechanics and quantum field theory.
(This phenomenon has already been observed in ref. [12].) The origin of this possibility lies in the fact that \( q \)-Schur polynomials and other relevant objects can be obtained from their \( q = 1 \) counterparts by a simple and universal change of time-variables. Indeed,

\[
\prod_{k=1}^{\infty} \hat{e}_{q_k}(T_k z^k) = \prod_{k=1}^{\infty} e^{t_k z^k},
\]

provided the \( t \)'s are expressed in terms of the \( T \)'s according to

\[
\sum_{k=1}^{\infty} t_k z^k = \sum_{n,k=1}^{\infty} \frac{T_k^n(1-q_k)^n}{n(1-q_k^n)} z^{nk}.
\]

Thus

\[
P_{k}^{(q)} \{ T \} = P_{k} \{ t \}
\]

and

\[
H_{ij}^{(q)} \{ T, \bar{T} \} = H_{ij} \{ t, \bar{t} \}, \quad \tau_{N}^{(q)} \{ T, \bar{T} \} = \tau_{N} \{ t, \bar{t} \},
\]

where \( H_{ij} \) satisfies eqs. (11), the entire set of equivalences given in sect. 2 remaining valid.

Because of this, \( \tau_{N}^{(q)} \{ T, \bar{T} \} \) can be represented as

\[
\tau_{N}^{(q)} \{ T, \bar{T} \} = \tau_{N} \{ t, \bar{t} \} \equiv \langle N| e^{H(t)} G e^{R(\bar{t})}|N \rangle
\]

with some \( G = \exp \left( \sum_{m,n} A_{mn} \hat{\psi}_{n} \hat{\bar{\psi}}_{m} \right) \) and

\[
H \{ t \} = \sum_{n>0} \hat{t}_n J_{+n} \overset{(47)}{=} \sum_{n,k=0}^{\infty} \frac{T_k^n(1-q_k)^n}{n(1-q_k^n)} J_{+nk},
\]

\[
\bar{H} \{ \bar{t} \} = \sum_{n>0} \hat{\bar{t}}_n J_{-n} = \sum_{n,k=0}^{\infty} \frac{\bar{T}_k^n(1-q_k)^n}{n(1-q_k^n)} J_{-nk}.
\]

5 Miwa transformation and discrete Hirota equations

The bilinear identity for \( q \neq 1 \), when written in the form (34), is spectacularly similar to “the discrete Hirota equation” [3]. That arises when the ordinary \( (q = 1) \) Hirota equation is represented in terms of the Miwa coordinates. This is perhaps a little less surprising once we know the relation (47), which, for the variable \( T_1 \), is nothing else than the Miwa transformation (3). One difference is that

\[
t_k = \frac{1}{k} \frac{((1-q)T_1)^k}{1-q^k} = \frac{1}{k} \sum_{l \geq 0} ((1-q)q^l T_1)^k
\]
is essentially a simultaneous change of multiplicities \( p(q^{-l}\lambda_1) \) at all the points \( q^{-l}\lambda_1, \ l \geq 0 \) on the spectral curve, rather than at a single point \( \lambda_1 = ((q - 1)T_1)^{-1} \). Another difference is that eq. (34) is written in terms of the additive variables \( x, \bar{x} \), while \( T_1, \bar{T}_1 \) are associated with the multiplicative ones \( T_1 = X, \bar{T}_1 = \bar{X} \). We shall not go into a detailed discussion of these differences here but will instead address another question.

What do the higher time-variables \( T_k \) correspond to in this context? Note first of all that there are generalizations of the discrete Hirota equation (9), corresponding to simultaneous variation of the multiplicities \( p(\lambda) \) at several arbitrary points \( \lambda \) on the spectral curve, not necessarily related by a “semi-periodicity” condition like (52). It is among these equations that we should look for counterparts of the \( T_k \) variations. In fact, the relevant set of points \( \lambda \), associated with the variable \( T_k \) is

\[
\left\{ e^{2\pi ia/k} \lambda_k q_k^{-l/k} \right| a = 0, \ldots, k - 1; \ l \geq 0 \right\}, \quad \lambda_k = ((1 - q_k)T_k)^{-1/k}.
\]

(Here we find a first reason to prefer the choice \( q_k = q^k \) instead of \( q_k = q \) since in the former case we have \( q_k^{1/k} = q^l \).) The action of \( D/DT_k \) on \( \tau_{(q)}^N \) can be now described as the insertion of the \( k \)-fermion non-local operator

\[
\Psi \left( \lambda_k e^{2\pi ia/k} \right) = : \prod_{a=0}^{k-1} \psi(\lambda_k e^{2\pi ia/k}) : \quad (54)
\]

(note that only points with \( l = 0 \) contribute):

\[
\tilde{M}_{T_k} \tau_{(q)}^N \sim \left\langle N + k \left| \Psi \left( \lambda_k e^{2\pi ia/k} \right) e^{H(t)} G e^{\bar{H}(\bar{t})} \right| N \right\rangle,
\]

\[
\tilde{M}_{\bar{T}_k} \tau_{(q)}^N \sim \left\langle N \left| e^{H(t)} G e^{\bar{H}(\bar{t})} \bar{\Psi} \left( \lambda_k e^{2\pi ia/k} \right) \right| N + k \right\rangle. \quad (55)
\]

Thus, we conclude that the time-variables \( T_k \) are associated with certain types of Miwa transformations, distinguished by a particular selection of points, where the multiplicities \( p(\lambda) \) are simultaneously changed. One peculiarity is the “semi-periodicity” requirement, forcing to introduce all the points \( q^{-l}\lambda, \ 0 \leq l < \infty \) together with any \( \lambda \). This is a usual thing in \( q \)-analysis and \( q \)-free field theory (see, e.g. [12]), which can be easily formalized in terms of Jackson integrals. Another peculiarity is the association of the multiples of \( k \)-th roots of unity with every \( T_k \)-variable. This feature is, indeed, closely related to the notion of \( k \)-reduction of the KP hierarchy, being, in fact, somewhat “orthogonal” to that of reduction (\( k \)-reduction implies that the dependence on the variable \( t_{kn} \) is completely eliminated, while
Miwa transformation, associated with $T_k$ does instead introduce exactly this dependence and nothing else).

### 6 Towards $q$-Matrix models

The last promising direction for further research that will be mentioned in this note concerns the search for “$q$-matrix models”. We restrict ourselves here to several preliminary remarks concerning the relevant eigenvalue models and the associated conformal theories [3,4], leaving the issue of matrices and quantum groups for a more detailed presentation.

The partition functions of the eigenvalue models are usually represented in multiple integral form

$$Z = \left[ \prod_i \int_{dz_i d\bar{z}_i} \mu(z_i, \bar{z}_i) \right] \Delta(z) \Delta(\bar{z}) \prod_{i,k} e^{t_k z^i_k} \prod_{i,k} e^{\bar{t}_k \bar{z}^i_k}. \quad (56)$$

The measure $\mu(z, \bar{z})$ in this expression depends on the concrete theory and for the case of two-matrix model is equal to $e^{-c(z-\bar{z})^2}$. This integral can be easily rewritten [3,4] as $\det H_{ij}$ with

$$H_{ij} = \int_{dzd\bar{z}} z^{i-1} \bar{z}^{j-1} \mu(z, \bar{z}) \prod_k e^{t_k z^i} \prod_k e^{\bar{t}_k \bar{z}^i} \quad (57)$$

which evidently satisfies (11) and, therefore, $Z$ is a Toda-lattice $\tau$-function. The simplest model arises when $c \to \infty$, i.e. $\mu(z, \bar{z}) = \delta(z, \bar{z})$. It is called the one-matrix model and corresponds to the reduction of the Toda-lattice hierarchy to the Toda-chain one. The $\tau$-function of this hierarchy depends only on the difference of times $t_k - \bar{t}_k$ and is generally described by a particular matrix $H_{ij} = H_{i+j}$. The relevant matrix model integral is

$$Z^{(1mm)} = \left[ \prod_i \int_{dz_i} \right] \Delta^2(z) \prod_{i,k} e^{t_k z^i_k}. \quad (57)$$

it gives rise to

$$H^{(1mm)}_{ij} = H_{i+j} = \int_{dz} z^{i+j-2} \prod_k e^{T_k z^i_k}. \quad (58)$$

There is an obvious way to deform these eigenvalue integrals. One can just change all exponentials for $q$-exponentials. This is not enough, however, to clarify the connection with quantum groups. The following remarks makes this relation a little clearer. The thing is that the appearance of Van-der-Mondians in (56) is crucial for the representation of such integrals in the form $\det H_{ij}$, characteristic of $\tau$-functions associated to (forced) Toda-lattice hierarchies. However, Van-der-Mondians are not very natural objects in the theory of quantum groups. Their two natural analogues are either:
(i) the quantum dimension which looks essentially like $\prod_{i>j}(q^{z_i-z_j} - q^{-z_i-z_j})$ \[3\] and is unsuitable if the deformed integral still needs to be a Toda $\tau$-function, or

(ii) an object like

$$\prod_{i>j}(qz_i - z_j)(z_i - z_j)$$

which arises in the theory of $q$-free fields ($q$-affine algebras) [14]. Remarkably, in the course of integration, it is possible to change variables to replace (59) by $\Delta^2(z)$ and to obtain the appropriate determinant formula. Indeed,

$$\left[ \prod_i \int_{dz_i} \right] \prod_{i>j} (z_i - z_j)(qz_i - z_j) \prod_{i,k} e^{t_k z_i} = \frac{\prod_{a=1}^N (1 - q^a)}{(1 - q)^N} \det H_{ij} =$$

$$= \frac{\prod_{a=1}^N (1 - q^a)}{(1 - q)^N} \frac{1}{N!} \left[ \prod_i \int_{dz_i} \right] \prod_{i>j} (z_i - z_j)^2 \prod_{i,k} e^{t_k z_i}$$

with $H_{ij}$ defined in \[58\].

The partition functions of the eigenvalue models can be usually represented as correlators in two-dimensional conformal field theory \[4, 8\]. This is especially useful for the study of Ward identities. In particular, for the one-matrix model we have:

$$Z^{(1mm)} = \left[ \prod_i \int_{dz_i} \right] \left\langle e^{\frac{1}{\sqrt{2}} \sum_{k>0} t_k J_k} \prod_i e^{\sqrt{2} \phi(z_i)} \right\rangle_N,$$

(61)

where $\phi(z)$ is the free scalar field and $J(z) \equiv \partial \phi(z)$.

To deform this model in the way described above, one should perform the substitution of times \[31\]

$$\sum_{k>0} t_k J_k \rightarrow \sum_{n,k>0} T^n_k \frac{(1 - q)^n}{n(1 - q^n)} J_{nk}.$$  

(62)

Note now that the square of the Van-der-Mondian in \[57\] arises from the operator product expansion of the $\hat{sl}(2)_{k=1}$-currents in \[31\]:

$$\prod_i e^{\sqrt{2}\phi(z_i)} = \prod_i J^+(z_i) \sim \prod_{i>j} (z_i - z_j)^2.$$  

(63)

If these currents are replaced by those of the $\hat{sl}_q(2)_{k=1}$ algebra [14], we get

$$\prod_i J^+_q(z_i) \sim \prod_{i>j} (z_i - z_j)(qz_i - z_j).$$

(64)

We thus obtain an argument in favor of the integrable system \[30\]. The l.h.s. of \[60\] is reproduced in this fashion only for a particular choice of $q$-free fields. (It depends on the
precise choice of factors in front of the negative and positive harmonics of the field $\phi(z)$; this can be done in different ways, without changing rules like (64).

A very important ingredient in the theory of eigenvalue models is the fact that one has explicit expressions for the Ward identities which usually take the simple and recognizable form of Virasoro and $W$-constraints. In our $q$-matrix models, the Ward identities should presumably obey some kind of $q$-Virasoro and $q$-$W$ constraints. We shall not go into any details here, we shall merely note one important identity, that arises if the choice

$$q_k = q^k$$

(65)

is made. This choice respects the fact that the argument of the $q$-exponential, while linear in $T_k$, involves the $k$-th power of $z$. Because of this

$$\frac{D}{Dz} \hat{e}_{q^k}(T_k z^k) = \frac{\hat{e}_{q^k}(T_k z^k q^k) - \hat{e}_{q^k}(T_k z^k)}{(q - 1)} = \frac{q^k - 1}{q - 1} T_k \frac{D}{D T_k} \hat{e}_{q^k}(T_k z^k).$$

(66)

This does not immediately lead to some simple formulas for the constraints, because of the twisted nature of the $q$-Leibnitz rule: in actual calculations, for models like (57), every $D/Dz_a$ will be acting on the product of $\hat{e}_{q^k}(T_k z^k)$ with different $k$’s, while every $D/D T_k$ will be acting on the product of $\hat{e}_{q^k}(T_k z^k)$ with different $a$’s. Integration by parts is also somewhat more complicated than in the $q = 1$ case. To make this integration possible, one should certainly use the adequate notion of integration, inverse to that of finite-differentiation. It is essentially provided by the Jackson integral,

$$\int_q F(X) = (1 - q) \sum_l F(q^l) q^l.$$ 

(67)

Usually the sum on the r.h.s. is defined over all integers. Then, the integral of a total derivative is given by the boundary terms, $\int_q \hat{D} F(X) = F(\infty) - F(0)$. In order to get $q$-Virasoro constraints, it would be useful to have the relation $\int_q \hat{D} F(X) = 0$. For this purpose it will not be enough to simply use the integral (67), the reason being that it tends, in

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4 The fact that the full set of Ward-identites (constraints) forms this kind of algebras is very general, at least, as general as the existence of some free-field representation of partition functions, in turn equivalent to their identification as $\tau$-functions. It is remarkable that in the simplest matrix models these algebras arise naturally in some simple and well known representations.
the classical limit, to an integral over the semi-axis \((0, \infty)\) and that there is no need for a reasonable function \(F(X)\) to vanish at zero.

There is however another integral which in the classical limit goes to an integral over the entire real axis, namely,

\[
\oint q F(X) = (1 - q) \sum_{l=-\infty}^{+\infty} \left\{ F(q^l) + F(-q^l) \right\} q^l
\]

and

\[
\oint \hat{D}F(X) = F(\infty) - F(-\infty).
\]

A simple example of a function, which decreases rapidly at both ends of the integration domain (i.e. as \(X \to \pm \infty\)) is \(e_q(-X^2) = \prod_{k=0}^{\infty} (1 + q^k X^2)^{-1}\).

7 Acknowledgements

We are indebted to A. Gerasimov, S. Kharchev and A. Zabrodin for numerous and stimulating discussions. A. Mironov and A. Morozov are also grateful to Centre de Recherches Mathématiques, Université de Montréal for the kind hospitality. The work of A. Mironov is partially supported by grant 93-02-14365 of the Russian Foundation of Fundamental Research, that of L. Vinet through funds provided by NSERC (Canada) and FCAR (Quebec).

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