Quantizations and Global Hypoellipticity for Pseudodifferential Operators of Infinite Order in Classes of Ultradifferentiable Functions

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Abstract. We study the change of quantization for a class of global pseudodifferential operators of infinite order in the setting of ultradifferentiable functions of Beurling type. The composition of different quantizations as well as the transpose of a quantization are also analysed, with applications to the Weyl calculus. We also compare global $\omega$-hypoellipticity and global $\omega$-regularity of these classes of pseudodifferential operators.

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1. Introduction

In the present paper, we deal with the change of quantization in the class of global pseudodifferential operators introduced by Jornet and the author in [2]. The symbols are of infinite order with exponential growth in all the variables, in contrast to the approach of Zanghirati [27] and Fernández, Galbis, and Jornet [17], who treat pseudodifferential operators of infinite order in the local sense and infinite order only in the last variable, for Gevrey classes and for classes of ultradifferentiable functions of the Beurling type in the sense of Braun, Meise, and Taylor [10]. In [2,17], the composition of two operators is given in terms of a suitable symbolic calculus. On the other hand, Prangoski [25] studies pseudodifferential operators of global type and infinite order for ultradifferentiable classes of Beurling and Roumieu type in the sense of Komatsu. We refer also to [11,12,14,23] and the references therein to find other papers discussing pseudodifferential operators defined in global classes (especially Gelfand–Shilov classes).
The appropriate setting in the present paper and in [2] is the space of (non-quasianalytic) global ultradifferentiable functions defined by Björck [3], characterized as those $f \in S(\mathbb{R}^d)$, i.e., in the Schwartz class, such that for all $\lambda > 0$ and all $\alpha \in \mathbb{N}_0^d$ both
\[
\sup_{x \in \mathbb{R}^d} e^{\lambda \omega(x)} |\partial^\alpha f(x)| \quad \text{and} \quad \sup_{\xi \in \mathbb{R}^d} e^{\lambda \omega(\xi)} |\partial^\alpha \hat{f}(\xi)|
\]
are finite, $\omega$ denoting a (non-quasianalytic) weight function in the sense of [10]. These spaces are always contained in the Schwartz class, and they equal the Schwartz class for the case $\omega(t) = \log(1 + t)$, $t > 0$, not considered in our setting.

The notion of hypoellipticity comes from the problem of determining whether a distribution solution to the partial differential equation $Pu = f$ is a classical solution or not. The authors in [17] provide adequate conditions for the construction of a (left) parametrix for their symbols, which guarantee the hypoellipticity in the desired class in [16]. For the operators defined in [25], the corresponding construction of parametrices is done in Cappiello, Pilipović, and Prangoski [13]. Here, we develop the method of the parametrix in Sect. 5 for the class of operators introduced in [2], but also for every quantization of the pseudodifferential operator. In particular, we obtain a sufficient condition for any quantization of a pseudodifferential operator to be $\omega$-regular in the sense of Shubin [26] (see the definition of $\omega$-regularity at the beginning of Sect. 5). Recently, in [1] to Boiti, Jornet, Oliaro, and the author use the global parametrix method presented here to define a suitable Weyl wave front set for $S'_{\omega}(\mathbb{R}^d)$ and complete the characterization of global wave front sets given in [6].

As we mention at the beginning, one of the goals of the present paper is to extend the results in [2] by adapting them for a valid change of quantization for these symbols (see Sects. 3 and 4). Namely, we follow the ideas for the change of quantization set within the framework of global symbol classes of Shubin [26, §23]. In [25], it is considered the change of quantization and its corresponding symbolic calculus for classes in the sense of Komatsu [20], also in the Roumieu setting. Nonetheless, as pointed out in [2], whenever the weight $\omega$ is under the mild condition
\[
\exists H > 1 : 2\omega(t) \leq \omega(Ht) + H, \quad t > 0,
\]
the classes of ultradifferentiable functions are equally defined either by weights as in [10] or by sequences as in [20] (see Bonet, Meise, and Melikhov [8]). Thus, if the weight sequence $(M_p)_p$ satisfies only stability under ultradifferential operators, as assumed in [25], our classes of symbols (and amplitudes) might not coincide with the ones defined in [25]. It turns out that, even only in the Beurling setting, we are discussing different cases.

Finally, in Sect. 6, inspired by Boggiatto, Buzano, and Rodino [4], we show that some $\omega$-hypoelliptic symbols are stable under change of quantization and we compare the notions of $\omega$-regularity and $\omega$-hypoellipticity following the ideas of [5].
2. Preliminaries

We begin with some notation on multi-indices. Throughout the text, we will denote by $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$ a multi-index of dimension $d$. The length of $\alpha$ is $|\alpha| = \alpha_1 + \cdots + \alpha_d$. For two multi-indices $\alpha$ and $\beta$, we write $\beta \leq \alpha$ for $\beta_j \leq \alpha_j$, when $j = 1, \ldots, d$. Moreover, $\alpha! = \alpha_1! \cdots \alpha_d!$, and if $\beta \leq \alpha$, then $(\alpha \beta) := \frac{\alpha!}{\beta!(\alpha - \beta)!}$. For $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, we have $x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$. We write $\partial^\alpha = \left( \frac{\partial}{\partial x_1} \right)^{\alpha_1} \cdots \left( \frac{\partial}{\partial x_d} \right)^{\alpha_d}$ and we set

$$D^\alpha = D_{x_1}^{\alpha_1} \cdots D_{x_d}^{\alpha_d},$$

where $D_{x_j}^{\alpha_j} = (-i)^{\alpha_j} \left( \frac{\partial}{\partial x_j} \right)^{\alpha_j}$, $j = 1, \ldots, d$.

In our setting, we work with weight functions as the ones defined by Braun, Meise, and Taylor [10].

**Definition 2.1.** A non-quasianalytic weight function $\omega : [0, +\infty[ \to [0, +\infty]$ is a continuous and increasing function which satisfies:

1. $\exists L \geq 1 : \omega(2t) \leq L(\omega(t) + 1), \ t \geq 0$,
2. $\int_1^{+\infty} \frac{\omega(t)}{t^2} dt < +\infty$,
3. $\log(t) = o(\omega(t))$ as $t \to \infty$,
4. $\varphi_\omega : t \mapsto \omega(e^t)$ is convex.

We extend the weight function $\omega$ to $\mathbb{C}^d$ in a radial way: $\omega(z) = \omega(|z|)$, $z \in \mathbb{C}^d$, where $|z|$ denotes the Euclidean norm.

From Definition 2.1(1), we immediately have:

$$\omega(x + y) \leq L(\omega(x) + \omega(y) + 1), \quad x, y \in \mathbb{R}^d.$$  \hfill (2.1)

For $z \in \mathbb{C}^d$, we denote $(z) := \sqrt{1 + |z|^2}$. From (2.1), we have

$$\omega((z)) \leq \omega(1 + |z|) \leq L \omega(z) + L(1 + \omega(1)), \quad z \in \mathbb{C}^d.$$  \hfill (2.2)

**Definition 2.2.** Given a weight function $\omega$, the Young conjugate $\varphi_\omega^* : [0, \infty[ \to [0, \infty]$ of $\varphi_\omega$ is defined as

$$\varphi_\omega^*(t) := \sup_{s \geq 0} \{st - \varphi_\omega(s)\}.$$ 

When the weight function $\omega$ is clear or irrelevant in the context, we simply denote $\varphi_\omega$ and $\varphi_\omega^*$ by $\varphi$ and $\varphi^*$. From now on, we assume that $\omega|_{[0,1]} \equiv 0$, which implies that $\varphi^*(0) = 0$ (in particular, this gives that $\omega(1) = 0$ in formula (2.2)). Moreover, it is known that $\varphi^*$ is convex, and the function $\varphi^*(x)/x$ is increasing for $x > 0$ and $\varphi^{**} := (\varphi^*)^* = \varphi$ (see [10]). From [19, Remark 2.8(c)] it is not difficult to see (cf. [7, Lemma A.1]):

**Proposition 2.3.** If a weight function $\omega$ satisfies $\omega(t) = o(t^a)$ as $t \to +\infty$ for some $0 < a \leq 1$, then for every $B > 0$ and $\lambda > 0$, there exists $C > 0$, such that

$$B^n n! \leq C e^{a\lambda \varphi^*(\frac{2}{B})}, \quad n \in \mathbb{N}_0.$$ 

The following result can be found in [10].
Lemma 2.4. (1) Let $L > 0$ be such that $\omega(et) \leq L(\omega(t) + 1)$. Then

$$\lambda L^n \varphi^\ast\left(\frac{y}{L^n}\right) + ny \leq \lambda \varphi^\ast\left(\frac{y}{\lambda}\right) + \lambda \sum_{j=1}^{n} L^j$$

for every $y \geq 0$, $\lambda > 0$, $n \in \mathbb{N}$.

(2) For all $s, t, \lambda > 0$, we have

$$2\lambda \varphi^\ast\left(\frac{s + t}{2\lambda}\right) \leq \lambda \varphi^\ast\left(\frac{s}{\lambda}\right) + \lambda \varphi^\ast\left(\frac{t}{\lambda}\right) \leq \lambda \varphi^\ast\left(\frac{s + t}{\lambda}\right).$$

We will consider without losing generality with no explicit mention that the constant $L \geq 1$ that comes from Definition 2.1($\alpha$) fulfils the condition of Lemma 2.4. For more results involving $\varphi^\ast$, see, for instance, [2,10,17] and [7, Lemma A.1].

We deal with a class of global ultradifferentiable functions, which extends the classical Schwartz class with the use of weight functions. It was introduced by Björck [3], but only considering a subadditive weight function $\omega$ (so the following definition is slightly more general than the given by Björck).

Definition 2.5. For a weight $\omega$ as in Definition 2.1, we define $S_{\omega}(\mathbb{R}^d)$ as the set of all $u \in L^1(\mathbb{R}^d)$, such that ($u$ and its Fourier transform $\hat{u}$ belong to $C^\infty(\mathbb{R}^d)$ and)

(i) for each $\lambda > 0$ and $\alpha \in \mathbb{N}_0^d$, $\sup_{x \in \mathbb{R}^d} e^{\lambda \omega(x)} |D^\alpha u(x)| < +\infty$;

(ii) for each $\lambda > 0$ and $\alpha \in \mathbb{N}_0^d$, $\sup_{\xi \in \mathbb{R}^d} e^{\lambda \omega(\xi)} |D^\alpha \hat{u}(\xi)| < +\infty$.

The corresponding strong dual is denoted by $S'_{\omega}(\mathbb{R}^d)$ and is the set of all the linear and continuous functionals $u : S_{\omega}(\mathbb{R}^d) \to \mathbb{C}$. We say that an element of $S'_{\omega}(\mathbb{R}^d)$ is an $\omega$-temperate ultradistribution.

The space $S_{\omega}(\mathbb{R}^d)$ has been studied for different purposes by many authors. We refer, for instance, to [5,7,18] for some examples of publications that treat different problems in the setting of the class $S_{\omega}(\mathbb{R}^d)$. We recall here [2, Lemma 2.11], which will be useful below.

Lemma 2.6. If $f \in S(\mathbb{R}^d)$, then $f \in S_{\omega}(\mathbb{R}^d)$ if and only if for every $\lambda, \mu > 0$ there is $C_{\lambda,\mu} > 0$ such that for all $\alpha \in \mathbb{N}_0^d$ and $x \in \mathbb{R}^d$, we have

$$|D^\alpha f(x)| \leq C_{\lambda,\mu} e^{\lambda \varphi^\ast\left(\frac{|\alpha|}{\lambda}\right)} e^{-\mu \omega(x)}.$$

From now on, $m$ denotes a real number and $0 < \rho \leq 1$. In the following, we consider global symbols and global amplitudes of infinite order defined very similarly to the ones in [2, Definitions 3.1 and 3.2]. The unique difference is the factor $e^{m\omega(x,\xi)}$ in the case of symbols and $e^{m\omega(x,y,\xi)}$ in the case of amplitudes, which are more suitable for our purposes. We observe that these definitions are equivalent to those in [2]. In fact, when considering symbols for example, it is enough to use that there exist $A, B > 0$, such that $A(\omega(x) + \omega(\xi)) \leq \omega(x, \xi) \leq B(\omega(x) + \omega(\xi) + 1)$ for every $x, \xi \in \mathbb{R}^d$. 
Definition 2.7. A global symbol (of order $m$) in $\text{GS}_p^{m,\omega}$ is a function $p(x, \xi) \in C^\infty(\mathbb{R}^{2d})$, such that for all $n \in \mathbb{N}$, there exists $C_n > 0$ with

$$|D^\alpha_x D^\beta_\xi p(x, \xi)| \leq C_n \langle (x, \xi) \rangle^{-\rho|\alpha+\beta|} e^{n\rho\varphi^*\left(\frac{|\alpha+\beta|}{n}\right)} e^{m\omega(x, \xi)},$$

for all $\alpha, \beta \in \mathbb{N}_0^d$ and $x, \xi \in \mathbb{R}^d$.

Definition 2.8. A global amplitude (of order $m$) in $\text{GA}_p^{m,\omega}$ is a function $a(x, y, \xi) \in C^\infty(\mathbb{R}^{3d})$, such that for all $n \in \mathbb{N}$, there exists $C_n > 0$ with

$$|D^\alpha_x D^\beta_y D^\gamma_\xi a(x, y, \xi)| \leq C_n \langle x-y \rangle^\rho|\alpha+\beta+\gamma| \langle (x, y, \xi) \rangle^{\rho|\alpha+\beta+\gamma|} e^{n\rho\varphi^*\left(\frac{|\alpha+\beta+\gamma|}{n}\right)} e^{m\omega(x, y, \xi)},$$

for all $\alpha, \beta, \gamma \in \mathbb{N}_0^d$ and $x, y, \xi \in \mathbb{R}^d$.

In [2], we introduce global pseudodifferential operators on $\mathcal{S}_\omega(\mathbb{R}^d)$ by means of oscillatory integrals for global amplitudes as in Definition 2.8 (see [2, Proposition 3.3]). It turns out that the action of a pseudodifferential operator on a function in $\mathcal{S}_\omega(\mathbb{R}^d)$ can be written as an iterated integral [2, Theorem 3.7], and it is continuous and linear from $\mathcal{S}_\omega(\mathbb{R}^d)$ into itself. In fact, we use these properties to state the following definition:

Definition 2.9. Given a global amplitude $a(x, y, \xi) \in \text{GA}_p^{m,\omega}$ (as in Definition 2.8), we define the associated global pseudodifferential operator $A : \mathcal{S}_\omega(\mathbb{R}^d) \to \mathcal{S}_\omega(\mathbb{R}^d)$ by

$$A(f)(x) := \int \left( \int e^{i(x-y) \cdot \xi} a(x, y, \xi) f(y) dy \right) d\xi, \quad f \in \mathcal{S}_\omega(\mathbb{R}^d).$$

Moreover, this operator can be extended linearly and continuously to an operator $\tilde{A}$ from $\mathcal{S}_\omega'(\mathbb{R}^d)$ into $\mathcal{S}_\omega'(\mathbb{R}^d)$ [2, Proposition 3.10].
ultradistributions of Beurling type in $\Omega$. The following continuous embeddings hold:

$$\mathcal{E}_\omega' (\mathbb{R}^d) \subseteq \mathcal{S}_\omega' (\mathbb{R}^d) \subseteq \mathcal{D}'_\omega (\mathbb{R}^d).$$

We recall that the space $\mathcal{S}_\omega (\mathbb{R}^d)$, as well as its strong dual $\mathcal{S}_\omega' (\mathbb{R}^d)$, are stable under Fourier transform (see for instance [3]).

Since the global amplitudes have exponential growth in all the variables, it becomes useful a particular kind of integration by parts to understand the behaviour of a pseudodifferential operator in this setting. Following [25], but with a different point of view, we use in [2] entire functions with prescribed exponential growth in terms of a weight function $\omega$. The existence of this type of entire functions was proven by Braun [9] and Langenbruch [21]. In several variables, we have a similar result:

**Theorem 2.10.** ([2], Theorem 2.16) Let $\omega : [0, \infty] \to [0, \infty]$ be a continuous and increasing function satisfying the conditions (a), (γ), and (δ) of Definition 2.1. Then, there are a function $G \in \mathcal{H} (\mathbb{C}^d)$ and some constants $C_1, C_2, C_3, C_4 > 0$, such that

i') $\log |G(z)| \leq \omega(z) + C_1$, $z \in \mathbb{C}^d$;

ii') $\log |G(z)| \geq C_2 \omega(z) - C_4$, $z \in \tilde{U} := \{ z \in \mathbb{C}^d : |\text{Im}(z)| \leq C_3 (|\text{Re}(z)| + 1) \}$.

We also need the notion of $\omega$-ultradifferential operator with constant coefficients. Let $G$ be an entire function in $\mathbb{C}^d$ with $\log |G| = O(\omega)$. For $\varphi \in \mathcal{E}_\omega (\mathbb{R}^d)$, the map $T_G : \mathcal{E}_\omega (\mathbb{R}^d) \to \mathbb{C}$ given by

$$T_G (\varphi) := \sum_{\alpha \in \mathbb{N}^d_0} \frac{D^\alpha G(0)}{\alpha!} D^\alpha \varphi(0)$$

defines an ultradistribution $T_G \in \mathcal{E}_\omega' (\mathbb{R}^d)$ with support equal to $\{0\}$. The convolution operator $G(D) : \mathcal{D}'_\omega (\mathbb{R}^d) \to \mathcal{D}'_\omega (\mathbb{R}^d)$ defined by $G(D)(\mu) = T_G * \mu$ is said to be an ultradifferential operator of $\omega$-class.

**Proposition 2.11.** Let $G$ be the entire function given in Theorem 2.10 and $n \in \mathbb{N}$. If

$$G^n(z) = \sum_{\alpha \in \mathbb{N}^d_0} b_\alpha z^\alpha, \quad z \in \mathbb{C}^d$$

denotes the $n$th power of $G$, then there exist $C, K > 0$, such that

$$|b_\alpha| \leq e^{nC} e^{-nC \varphi^*(\frac{||\alpha||}{n\pi})}, \quad \alpha \in \mathbb{N}^d_0;$$

$$|G^n(\xi)| \geq C^{-n} e^{nK \omega(\xi)}, \quad \xi \in \mathbb{R}^d.$$

The following result characterizes those operators whose kernel is a function in $\mathcal{S}_\omega (\mathbb{R}^d)$. These operators are fundamental to understand the symbolic calculus. The proof is standard.

**Proposition 2.12.** Let $T : \mathcal{S}_\omega (\mathbb{R}^d) \to \mathcal{S}_\omega (\mathbb{R}^d)$ be a pseudodifferential operator. The following assertions are equivalent:

1. $T$ has a linear and continuous extension $\tilde{T} : \mathcal{S}_\omega' (\mathbb{R}^d) \to \mathcal{S}_\omega (\mathbb{R}^d)$. 
(2) There exists \( K(x, y) \in S_\omega(\mathbb{R}^d) \), such that
\[
(T \varphi)(x) = \int K(x, y) \varphi(y) \, dy, \quad \varphi \in S_\omega(\mathbb{R}^d).
\]

Any operator \( T : S_\omega(\mathbb{R}^d) \to S_\omega(\mathbb{R}^d) \) which satisfies (1) or (2) of Proposition 2.12 is called \( \omega \)-regularizing.

3. Symbolic Calculus for Quantizations

We generalize the symbolic calculus developed in [2] for quantizations.

**Definition 3.1.** We define \( \text{FGS}_\rho^{m, \omega} \) to be the set of all formal sums \( \sum_{j \in \mathbb{N}_0} a_j (x, \xi) \), such that \( a_j (x, \xi) \in C_\infty(\mathbb{R}^{2d}) \) and there is \( R \geq 1 \), such that for every \( n \in \mathbb{N} \), there exists \( C_n > 0 \) with
\[
|D_x^\alpha D_\xi^\beta a_j (x, \xi)| \leq C_n \langle (x, \xi) \rangle^{-\rho(|\alpha+\beta|+j)} e^{n\rho \varphi^* \left( \frac{|\alpha+\beta|+j}{n} \right)} e^{m \omega(x, \xi)}
\]
for each \( j \in \mathbb{N}_0, \alpha, \beta \in \mathbb{N}_0^d \), and \( \log \left( \frac{\langle (x, \xi) \rangle}{R} \right) \geq \frac{n}{j} \varphi^* \left( \frac{j}{n} \right) \).

**Definition 3.2.** Two formal sums \( \sum a_j \) and \( \sum b_j \) in \( \text{FGS}_\rho^{m, \omega} \) are said to be equivalent, denoted by \( \sum a_j \sim \sum b_j \), if there is \( R \geq 1 \), such that for each \( n \in \mathbb{N} \), there exist \( C_n > 0 \) and \( N_n \in \mathbb{N} \) with
\[
|D_x^\alpha D_\xi^\beta \sum_{j < N} (a_j - b_j)| \leq C_n \langle (x, \xi) \rangle^{-\rho(|\alpha+\beta|+N)} e^{n\rho \varphi^* \left( \frac{|\alpha+\beta|+N}{n} \right)} e^{m \omega(x, \xi)},
\]
for every \( N \geq N_n, \alpha, \beta \in \mathbb{N}_0^d \), and \( \log \left( \frac{\langle (x, \xi) \rangle}{R} \right) \geq \frac{n}{N} \varphi^* \left( \frac{N}{n} \right) \).

The following construction has been carried out in [2] following the lines of [17, Theorem 3.7]. Let \( \Phi \in \mathcal{D}_\sigma(\mathbb{R}^d) \), where \( \sigma \) is a weight function which satisfies \( \omega(t^{1/\rho}) = O(\sigma(t)) \), as \( t \to +\infty \), and
\[
|\Phi(t)| \leq 1, \quad \Phi(t) = 1 \text{ if } |t| \leq 2, \quad \Phi(t) = 0 \text{ if } |t| \geq 3.
\]
Let \( (j_n) \) be a sequence of natural numbers, such that \( j_n/n \to \infty \) as \( n \) tends to infinity. For each \( j_n \leq j < j_{n+1} \), we set
\[
\varphi_j(x, \xi) := 1 - \Phi \left( \frac{\langle x, \xi \rangle}{A_{n,j}} \right), \quad A_{n,j} = R e^{\pi \varphi^* \left( \frac{\pi}{A_{n,j}} \right)},
\]
where \( R \geq 1 \) is the constant which appears in Definition 3.1. It is understood that \( \varphi_0 = 1 \). We have shown in [2] that \( \varphi_j \in \text{GS}_\rho^{0, \omega} \). Moreover, if \( \sum_j a_j \in \text{FGS}_\rho^{m, \omega} \), then, by [2, Theorem 4.6],
\[
a(x, \xi) := \sum_{j=0}^\infty \varphi_j(x, \xi) a_j(x, \xi)
\]
is a global symbol in \( \text{GS}_\rho^{m, \omega} \), equivalent to \( \sum_j a_j \) in \( \text{FGS}_\rho^{m, \omega} \).

Now, we extend some results in [2] for quantizations. In what follows, \( \tau \) stands for a real number. Let \( k \in \mathbb{N}_0 \) denote the minimum natural number satisfying
\[
|\tau| + |1 - \tau| \leq 2^k.
\]
Furthermore, for any $m \in \mathbb{R}$, we denote
\[ m' = mL^k, \]
where $L \geq 1$ is the constant of Lemma 2.4. We observe that $m' = m$ if and only if $0 \leq \tau \leq 1$.

**Lemma 3.3.** If $b(x, \xi) \in G^{m, \omega}_p$ and $\tau \in \mathbb{R}$, then
\[ a(x, y, \xi) := b((1 - \tau)x + \tau y, \xi) \]
is a global amplitude in $G^{\max\{0, m'\}, \omega}_p$.

**Proof.** The following inequality is easy to check:
\[ \langle x, y, \xi \rangle \leq \sqrt{\bar{b}(\tau)}\langle x - y \rangle \langle ((1 - \tau)x + \tau y, \xi) \rangle, \quad x, y, \xi \in \mathbb{R}^d, \tau \in \mathbb{R}. \]
We take $\tilde{p} \in \mathbb{N}$, such that $\max\{|1 - \tau|, |\tau|, (\sqrt{\bar{b}(\tau)})^p\} \leq e^{\tilde{p}}$. By assumption, for all $\lambda > 0$, there exists $C_\lambda > 0$, such that ($L \geq 1$ is the constant of Lemma 2.4)
\[ |D_x^\alpha D_y^\beta D_\xi^\gamma a(x, y, \xi)| \leq |1 - \tau|^{\alpha|\alpha|} |\tau|^{\beta|\beta|} C_\lambda \langle ((1 - \tau)x + \tau y, \xi) \rangle^{-\rho|\alpha + \beta + \gamma|} \times e^{\lambda L^2 \tilde{p}^\rho \varphi^* \left( \frac{|\alpha + \beta + \gamma|}{\lambda L^4 p} \right)} e^{m\omega((1 - \tau)x + \tau y, \xi)}. \]
The choice of $\tilde{p}$ gives $|1 - \tau|^{\alpha|\alpha|} |\tau|^{\beta|\beta|} (\sqrt{\bar{b}(\tau)})^\rho|\alpha + \beta + \gamma| \leq e^{2\tilde{p} \rho|\alpha + \beta + \gamma|}$. Then, by (2.3), we get
\[ e^{\lambda L^2 \tilde{p}^\rho \varphi^* \left( \frac{|\alpha + \beta + \gamma|}{\lambda L^4 p} \right)} \leq e^{\lambda \rho \varphi^* \left( \frac{|\alpha + \beta + \gamma|}{\lambda} \right) \rho} \varphi^{\sum_{j=1}^{2\tilde{p}} L^j}. \]
Finally, since $\omega$ is radial and increasing, applying $k$ times property (a) of the weight function $\omega$, we get for $m \geq 0$
\[ e^{m\omega((1 - \tau)x + \tau y, \xi)} \leq e^{m\omega(2^k(x, y, \xi))} \leq e^{m'\omega(x, y, \xi)} e^{mL^k + mL^{k-1} + \cdots + mL}. \]
\[ (3.5) \]

**Corollary 3.4.** Let $\varphi_j$ be the function in (3.2). For all $\lambda > 0$, there exists $C_\lambda > 0$, such that
\[ |D_x^\alpha D_y^\beta D_\xi^\gamma \varphi_j((1 - \tau)x + \tau y, \xi)| \leq C_\lambda \langle ((1 - \tau)x + \tau y, \xi) \rangle^{-\rho|\alpha + \beta + \gamma|} e^{\lambda \rho \varphi^* \left( \frac{|\alpha + \beta + \gamma|}{\lambda} \right)}, \]
for every $\alpha, \beta, \gamma \in \mathbb{N}_0^d$ and $x, y, \xi \in \mathbb{R}^d$. Hence, $\varphi_j((1 - \tau)x + \tau y, \xi) \in G^{0, \omega}_p$ for all $\tau \in \mathbb{R}$.

Here, we generalize [2, Lemma 4.7] to readapt it to our context.

**Lemma 3.5.** Let $a(x, y, \xi)$ be an amplitude in $G^{m, \omega}_p$ and let $A$ be the corresponding pseudodifferential operator. For each $u \in S_\omega(\mathbb{R}^d)$
\[ A(u) = S_\omega(\mathbb{R}^d) - \sum_{j=0}^{\infty} A_j(u), \]
where $A_j$ is the pseudodifferential operator defined by the amplitude
\[ (\varphi_j - \varphi_{j+1})((1 - \tau)x + \tau y, \xi)a(x, y, \xi), \quad j \in \mathbb{N}_0. \]
Proof. By Corollary 3.4, \((\varphi_j - \varphi_{j+1})((1 - \tau)x + \tau y, \xi)a(x, y, \xi)\in GA^m_{\rho_\omega}\). Since \(A_{n,N+1} \to \infty\) as \(N \to \infty\), proceeding as in [2, Proposition 3.3], one can show that, for each \(\lambda > 0\) there exists \(s\) for some power \(O\) and \(\Phi(0) = 1\), being \(\Phi\)

We recall that

Since \(A\)

We show that this limit is, for all \(\tau \in \mathbb{R}\), equal to \(A\) in \(L(S_\omega(\mathbb{R}^d), S'_\omega(\mathbb{R}^d))\).

We recall that

\[
(1 - \varphi_{N+1})((1 - \tau)x + \tau y, \xi) = \Phi \left( \frac{(1 - \tau)x + \tau y, \xi}{A_{n,N+1}} \right)
\]

and \(\Phi(0) = 1\), being \(\Phi \in D_\sigma(\mathbb{R}^{2d})\) the function in (3.1) with \(\omega(t^{1/\rho}) = O(\sigma(t))\), \(t \to \infty\). We claim that for each \(f, g \in S_\omega(\mathbb{R}^d)\),

\[
\int \int \int e^{i(x-y)\cdot \xi} \left( \Phi \left( \frac{(1 - \tau)x + \tau y, \xi}{k} \right) - 1 \right) a(x, y, \xi) f(y) g(x) dy dx \to 0 \quad (3.6)
\]

as \(k \to \infty\). We use the following identity to integrate by parts with the ultradifferential operator \(G(D)\) associated with the entire function in Proposition 2.11:

\[
e^{i(x-y)\cdot \xi} = \frac{1}{G^s(\xi)} G^s(-D_y) e^{i(x-y)\cdot \xi}, \quad (3.7)
\]

for some power \(s \in \mathbb{N}\) that we will determine later. Then, the integrand in the left-hand side of (3.6) equals

\[
e^{i(x-y)\cdot \xi} = \frac{1}{G^s(\xi)} G^s(D_y) \left( \Phi \left( \frac{(1 - \tau)x + \tau y, \xi}{k} \right) - 1 \right) a(x, y, \xi) f(y) g(x)
\]

\[
e^{i(x-y)\cdot \xi} = \frac{1}{G^s(\xi)} \sum_{\eta \in \mathbb{N}^d} b_\eta \sum_{\eta_1 + \eta_2 + \eta_3 = \eta} \frac{\eta!}{\eta_1!\eta_2!\eta_3!} \left( \frac{\tau}{k} \right)^{|\eta_1|} D_{\eta_1} f(\Phi \left( \frac{(1 - \tau)x + \tau y, \xi}{k} \right) - 1)
\]

\[
\times D_{\eta_2} a(x, y, \xi) D_{\eta_3} f(y) g(x).
\]

Therefore, the integral in (3.6) is equal to

\[
\sum_{\eta \in \mathbb{N}^d} b_\eta \sum_{\eta_1 + \eta_2 + \eta_3 = \eta} \frac{\eta!}{\eta_1!\eta_2!\eta_3!} \left( \frac{\tau}{k} \right)^{|\eta_1|} \int \int \int e^{i(x-y)\cdot \xi} \frac{1}{G^s(\xi)}
\]

\[
\times D_{\eta_1} a(x, y, \xi) D_{\eta_2} f(y) g(x) dy dx.
\]

From Proposition 2.11, there are \(C_1, C_2, C_3 > 0\) (depending only on \(G\)), such that for all \(\eta \in \mathbb{N}_0^d\) and \(\xi \in \mathbb{R}^d\), we have

\[
|b_\eta| \leq e^{sC_1} e^{-sC_1 \varphi^*(\frac{|\eta|}{\pi \rho_1})}, \quad \frac{1}{G^s(\xi)} \leq C_s e^{-sC_2 \omega(\xi)}, \quad (3.8)
\]

It follows from Definition 2.8 (see for example [2, Lemma 2.6]) that for all \(\lambda > 0\) there exists \(C_\lambda > 0\), such that \((L \geq 1\) is the constant of Lemma 2.4)

\[
|D_{\eta_2} a(x, y, \xi)| \leq C_\lambda e^{\lambda L^3 \varphi^*(\frac{|\eta_2|}{\lambda L^3})} e^{m\omega(x, y, \xi)}.
\]
Since \( f, g \in S_\omega(\mathbb{R}^d) \), there exist \( C'_{\lambda,m}, C_m > 0 \), such that
\[
|D_y^{\eta_1} f(y)| \leq C'_{\lambda,m} e^{\lambda L^3 \varphi^*\left(\frac{|\eta_1|}{x^2}\right)} e^{-(mL+1)\omega(y)}, \quad |g(x)| \leq C_m e^{-(mL+1)\omega(x)}.
\]
For \( \eta_1 = 0 \), we have \( \Phi \equiv 1 \) if \( |((1-\tau)x + \tau y, \xi)| \leq 2k \), and for \( |\eta_1| > 0 \), it follows that \( D_y^{\eta_1} \left( \Phi \left( \frac{(1-\tau)x + \tau y, \xi}{k} \right) - 1 \right) = D_y^{\eta_1} \Phi \left( \frac{(1-\tau)x + \tau y, \xi}{k} \right) \) is zero for \( |((1-\tau)x + \tau y, \xi)| \leq 2k \); therefore, we can assume that \( |((1-\tau)x + \tau y, \xi)| > 2k \). In particular, we have
\[
1 \leq \frac{1}{2k} |((1-\tau)x + \tau y, \xi)| \leq \frac{1}{k} |(1-\tau) + |\tau||(|x| + 1)(|y| + 1)(|\xi| + 1).
\]
As \( \Phi \in D_\sigma(\mathbb{R}^{2d}) \subseteq D_\omega(\mathbb{R}^{2d}) \), there exists \( C''_{\lambda} > 0 \), such that
\[
|\tau|^{\eta_1} |D_y^{\eta_1} \left( \Phi \left( \frac{(1-\tau)x + \tau y, \xi}{k} \right) - 1 \right)| \leq C''_{\lambda} e^{\lambda L^3 \varphi^*\left(\frac{|\eta_1|}{x^2}\right)}, \quad \eta_1 \in \mathbb{N}^d.
\]
For \( m \geq 0 \) (if \( m < 0 \), then \( m \omega(x, y, \xi) < 0 \)), since
\[
m \omega(x, y, \xi) \leq m L \omega(x) + m L \omega(y) + m L \omega(\xi) + m L,
\]
it is enough to take \( s \in \mathbb{N} \) satisfying \( sC_2 \geq mL + 1 \) to get \( e^{-sC_2 + mL} \omega(\xi) \leq e^{-\omega(\xi)} \), and therefore, the integrals are convergent by condition (\( \gamma \)) of the weight \( \omega \). On the other hand, since \( \sum \frac{\eta!}{\eta_1! \eta_2! \eta_3!} = 3^{|\eta|} \leq e^{2|\eta|} \), by Lemma 2.4, we have
\[
\sum_{\eta_1 + \eta_2 + \eta_3 = \eta} \frac{\eta!}{\eta_1! \eta_2! \eta_3!} e^{\lambda L^3 \varphi^*\left(\frac{|\eta|}{x^2}\right)} e^{\lambda L^3 \varphi^*\left(\frac{|\eta_1|}{x^2}\right)} e^{\lambda L^3 \varphi^*\left(\frac{|\eta_2|}{x^2}\right)} \leq e^{\lambda L \varphi^*\left(\frac{|\eta|}{x^2}\right)} e^{\lambda L^2 + \lambda L^3}.
\]
Now, the series
\[
\sum_{\eta \in \mathbb{N}^d} e^{-sC_1 \varphi^*\left(\frac{|\eta|}{x^2}\right)} e^{\lambda L \varphi^*\left(\frac{|\eta|}{x^2}\right)}
\]
converges provided \( \lambda > sC_1 \) (see [2, (3.5), (3.6)]). Thus, there exists \( C > 0 \), such that
\[
\left| \int \int \int e^{i(x-y)\cdot \xi} \left( \Phi \left( \frac{(1-\tau)x + \tau y, \xi}{k} \right) - 1 \right) a(x, y, \xi) f(y) g(x) d y d \xi d x \right| \leq C \frac{1}{k} \to 0,
\]
and hence, (3.6) is satisfied. \( \square \)

The next result is the corresponding extension of [2, Proposition 4.8].

**Lemma 3.6.** Let \( \sum p_j \in FGS_{\rho,\omega}^n \) and let \( (C_n)_n, (C'_n)_n \) be the sequences of constants that appear in Definition 3.1 and in the estimate of the derivatives of \( \varphi_j \) in Corollary 3.4. We denote \( D_n := C_{2nL_{\rho,\omega} + 1} \) and \( D'_n := C'_{nL_{\rho,\omega} + 1} \), where \( L \geq 1 \) is the constant of Lemma 2.4 and \( p \in \mathbb{N} \) is so that \( 3\langle \tau \rangle \leq e^p \), for a fixed \( \tau \in \mathbb{R} \). Consider \( (j_n)_n, j_n \in \mathbb{N}, \) such that \( j_1 = 1, j_n < j_{n+1}, j_n/n \to \infty \) and
\[
D_{n+1}D'_{n+1} \sum_{j=j_n+1}^{\infty} (2R)^{-\rho_j} \leq \frac{D_nD'_n}{2} \sum_{j=j_n}^{j_{n+1}-1} (2R)^{-\rho_j}, \quad n \in \mathbb{N},
\]
and moreover,
\[
\frac{n}{j} \varphi^a \left( \frac{j}{n} \right) \geq \max \{ n, \log D_n, \log D'_n \}, \quad \text{for } j \geq j_n.
\]

If
\[
a(x, \xi) := \sum_{j=0}^{\infty} \varphi_j(x, \xi) p_j(x, \xi),
\]
then the associated pseudodifferential operator \( A \) is the limit in \( L(S_\omega(\mathbb{R}^d), S'_\omega(\mathbb{R}^d)) \) of the sequence of operators \( S_{N, \tau} : S_\omega(\mathbb{R}^d) \to S_\omega(\mathbb{R}^d) \), where each \( S_{N, \tau} \) is a pseudodifferential operator with amplitude
\[
\sum_{j=0}^{N} (\varphi_j - \varphi_{j+1}) \left( (1 - \tau)x + \tau y, \xi \right) \sum_{l=0}^{j} p_l((1 - \tau)x + \tau y, \xi).
\]

**Proof.** For each \( j \in \mathbb{N}_0 \), one can show that
\[
(\varphi_j - \varphi_{j+1})((1 - \tau)x + \tau y, \xi) \sum_{l=0}^{j} p_l((1 - \tau)x + \tau y, \xi)
\]
is a global amplitude in \( GA^\rho_{\max\{0,m'\}, \omega} \), \( m' \) being set in (3.4). Hence, the function
\[
\sum_{j=0}^{N} (\varphi_j - \varphi_{j+1}) \left( \sum_{l=0}^{j} p_l \right) = \sum_{j=0}^{N} \varphi_j p_j - \varphi_{N+1} \sum_{l=0}^{N} p_l
\]
is a global amplitude in \( GA^\rho_{\max\{0,m'\}, \omega} \).

Now, we prove that \( S_{N, \tau} \to A \) in \( L(S_\omega(\mathbb{R}^d), S'_\omega(\mathbb{R}^d)) \) as \( N \to \infty \). As in the proof of [2, Proposition 4.8], it is enough to show that, for any \( f, g \in S_\omega(\mathbb{R}^d) \), \( \langle (S_{N, \tau} - A)f, g \rangle \to 0 \) as \( N \to \infty \). Note that \( A \) and \( S_{N, \tau}, N = 1, 2, \ldots \) act continuously on \( S_\omega(\mathbb{R}^d) \). Thus
\[
\langle (S_{N, \tau} - A)f, g \rangle = \int (S_{N, \tau} - A)f(x)g(x)dx
\]
for every \( f, g \in S_\omega(\mathbb{R}^d) \), where \( \varphi_j, \varphi_N, p_j, p_l, \) and \( a \) are evaluated at \( ((1 - \tau)x + \tau y, \xi) \).

We show that, for each \( f, g \in S_\omega(\mathbb{R}^d) \)
\[\text{a)} \int \left( \int \int e^{i(x-y) \cdot \xi} \left( \sum_{j=N+1}^{\infty} \varphi_j((1 - \tau)x + \tau y, \xi)p_j((1 - \tau)x + \tau y, \xi) \right)f(y)dyd\xi \right)g(x)dx \]
and
\[\text{b)} \int \left( \int \int e^{i(x-y) \cdot \xi} \left( \varphi_{N+1}((1 - \tau)x + \tau y, \xi)p_0((1 - \tau)x + \tau y, \xi) \right)f(y)dyd\xi \right)g(x)dx \]
tend to zero when $N \to \infty$.

Let us show that the integral in a) goes to zero. We integrate by parts with formula (3.7) for some $s \in \mathbb{N}$ to be determined later. Then

$$
e^{i(x-y)\cdot \xi} \frac{1}{G^s(\xi)} G^s(D_y) \left( \sum_{j=N+1}^{\infty} \varphi_j \cdot p_j \cdot f(y) \right)
$$

$$= e^{i(x-y)\cdot \xi} \frac{1}{G^s(\xi)} \sum_{\eta \in [N]^d} b_\eta \sum_{\eta_1+\eta_2+\eta_3=\eta} \frac{\eta!}{\eta_1!\eta_2!\eta_3!} \times \sum_{j=N+1}^{\infty} \tau^{j_1+j_2} D_y^{\eta_1} \varphi_j \cdot D_y^{\eta_2} p_j \cdot D_y^{\eta_3} f(y).$$

Hence, we can reformulate the integral in a) as

$$\int \left( \int \frac{1}{G^s(\xi)} \sum_{\eta \in [N]^d} b_\eta \sum_{\eta_1+\eta_2+\eta_3=\eta} \frac{\eta!}{\eta_1!\eta_2!\eta_3!} \tau^{j_1+j_2} \right) \times \int e^{i(x-y)\cdot \xi} \sum_{j=N+1}^{\infty} D_y^{\eta_1} \varphi_j \cdot D_y^{\eta_2} p_j \cdot D_y^{\eta_3} f(y) dy d\xi g(x) dx. \quad (3.9)$$

When $\varphi_j \neq 0$, and $j_1 \leq j < j_{n+1}$, we have $\log(\frac{((1-\tau)x+\tau y, \xi)}{2R}) \geq \frac{n}{j} \varphi^*(\frac{j}{n})$ (see (3.2)). By Corollary 3.4, for each $n \in \mathbb{N}$, the following estimate holds (as in the hypotheses of this lemma, we denote $D'_n = C'_n L^{p+1} > 0$)

$$|D_y^{\eta_1} \varphi_j ((1-\tau)x+\tau y, \xi)| \leq D'_n e^{nL^{p+1} \varphi^*}(\frac{\eta_1}{nL^{p+1}}).$$

Moreover, for that $n \in \mathbb{N}$ (as in the hypotheses of this lemma, we denote $D_n = C_{2nL^{p+1}} > 0$); by (2.4), we have

$$|D_y^{\eta_2} p_j ((1-\tau)x+\tau y, \xi)|
\leq D_n e^{2nL^{p+1} \varphi^*}(\frac{\eta_2}{2nL^{p+1}}) \left( ((1-\tau)x+\tau y, \xi) \right) - \rho(\eta_2+j) e^{\omega((1-\tau)x+\tau y, \xi)}
\leq D_n e^{nL^{p+1} \varphi^*}(\frac{\eta_2}{nL^{p+1}}) \left( ((1-\tau)x+\tau y, \xi) \right) - \rho j e^{\omega((1-\tau)x+\tau y, \xi)}
\leq D_n e^{nL^{p+1} \varphi^*}(\frac{\eta_2}{nL^{p+1}}) (2R)^{-\rho j} e^{\omega((1-\tau)x+\tau y, \xi)}.$$

Property (γ) of Definition 2.1 yields that there exists $C > 0$, such that $\langle x \rangle \leq C e^\omega(x)$, $x \in \mathbb{R}^d$. Then, using (2.2)

$$e^{m\omega((1-\tau)x+\tau y, \xi)} \leq e^{(m+3)\omega((1-\tau)x+\tau y, \xi)} e^{-3\omega((1-\tau)x+\tau y, \xi))}
\leq e^{(m+3)L\omega((1-\tau)x+\tau y, \xi)} e^{(m+3)L} C^3 \langle ((1-\tau)x+\tau y, \xi) \rangle^{-3}
\leq C e^{(m+3)L\omega((1-\tau)x+\tau y, \xi)} e^{(m+3)L} e^{-3\frac{2}{\eta} \varphi^*}(\frac{\eta}{\rho}).$$

By (3.5) [k being as in (3.3)], we obtain

$$e^{(m+3)L\omega((1-\tau)x+\tau y, \xi)} \leq e^{(m+3)L^{k+1} \omega(x,y,\xi)} e^{(m+3)L^{k+1} \omega(x,y,\xi)} e^{\omega(y) + \omega(\xi)} e^{(m+3)L^{k+2} \omega(y) + \omega(\xi)} e^{(m+3)L^{k+2} \omega(y) + \omega(\xi)}.$$
Take $0 < \ell < n$. Later, an additional condition will be imposed on $\ell$. Since $f, g \in S(\mathbb{R}^d)$, there are $C''_\ell > 0$, which depends on $\ell, m$, and on $\tau$, and $D > 0$ that depends on $m$ and on $\tau$, such that
\[
|D_{yj} f(y)| \leq C''_\ell e^{\ell L^{k+1} \varphi^* \left( \frac{|y|}{\ell L^{k+1}} \right)} e^{-(m+3)L^{k+2}+1|\varphi(y)|};
\]
\[
|g(x)| \leq De^{-(m+3)L^{k+2}+1|\varphi(x)|}.
\]

Lemma 2.4, the fact that $\sum_{\eta_1+\eta_2+\eta_3=\eta} \frac{\eta!}{\eta_1!\eta_2!\eta_3!} |\tau|^{|\eta_1+\eta_2}|e^{nL^{k+1} \varphi^* \left( \frac{|\eta_1|}{nL^{k+1}} \right)} e^{nL^{k+1} \varphi^* \left( \frac{|\eta_2|}{nL^{k+1}} \right)} e^{\ell L^{k+1} \varphi^* \left( \frac{|\eta_3|}{\ell L^{k+1}} \right)} \leq (\tau)^{|\eta|} e^{\ell L \varphi^* \left( \frac{|\eta|}{\ell L} \right)} e^{\ell L \sum_{n=1} n L^r} L^r.$

Thus, from (3.8), we estimate (3.9) by
\[
\int \left( \int C_2s e^{-sC_2 \varphi(\xi)} \sum_{\eta \in \mathbb{N}_0^d} e^{-sC_1 \varphi^* \left( \frac{|\eta|}{sC_1} \right)} \left( \int \sum_{j=1}^{\infty} D_n D'_n e^{\ell L \varphi^* \left( \frac{|\eta|}{\ell L} \right)} e^{\ell L \sum_{n=1} n L^r} L^r \right. \right. \times (2R)^{-\rho j} e^{(m+3)L^{k+2} \omega(x)+\omega(y)+\omega(\xi))} \times e^{-3\omega(\xi)} \frac{1}{2} C''_\ell e^{-(m+3)L^{k+2}+1\omega(x)} dy, dx.
\]

Take $s \in \mathbb{N}_0$, such that $sC_2 \geq (m+3)L^{k+2}+1$. Choosing $\ell \geq sC_1$, we obtain that the series depending on $\eta \in \mathbb{N}_0^d$
\[
\sum_{\eta \in \mathbb{N}_0^d} e^{-sC_2 \varphi^* \left( \frac{|\eta|}{sC_2} \right)} e^{\ell L \varphi^* \left( \frac{|\eta|}{\ell} \right)}
\]
converges (see [2, (3.6)]). The constant depending on $n$ is $D_n D'_n$. We get for $j_1 \leq n + 1 < j_1+1$, the following estimate for the integral in $a)$:
\[
E_\ell \left( \int e^{-\varphi^* \left( \frac{|\eta|}{sC_1} \right)} \left( \int e^{-\varphi(\eta)} dy \right) \left( \int e^{-\varphi(\xi)} dx \right) \right) \sum_{n=1}^{\infty} \sum_{j=j_n}^{j_{n+1}-1} \frac{D_n D'_n}{(2R)^{\rho j} e^{3\omega(\xi)}} \sum_{\eta \in \mathbb{N}_0^d} e^{-sC_2 \varphi^* \left( \frac{|\eta|}{sC_2} \right)} e^{\ell L \varphi^* \left( \frac{|\eta|}{\ell} \right)},
\]
where $E_\ell$ is a constant depending on $\ell$. The last three integrals converge by property $(\gamma)$ of the weight function. By assumption, we have $3\omega(\xi) \geq \log D_n + \log D'_n + n$. This finally proves that the integral in $a)$ converges to zero as $N$ tends to infinity.

For the limit in $b)$, we can proceed as in [2, Proposition 4.8] with the above techniques. □

The next example recovers [2, Example 4.9] for $\tau = 0$. The proof is straightforward and is left to the reader.

Example 3.7. Let $a(x, y, \xi)$ be an amplitude in $GA^{m, \omega}_\rho$ and let
\[
p_j(x, \xi) := \sum_{|\beta+\gamma|=j} \frac{1}{|\beta|!|\gamma|!} |(1-\tau)|^{|\gamma|} \partial_x^{\beta+\gamma} (-D_x)^{\beta} D_y^\gamma a(x, y, \xi)|_{y=x}.
\]
Then, the series $\sum_{j=0}^{\infty} p_j(x, \xi)$ is a formal sum in $\text{FGS}_\rho^{\max\{m, mL\}, \omega}$ for all $\tau \in \mathbb{R}$.

The following lemma is taken from [17, Lemma 3.11].

**Lemma 3.8.** Let $m \geq n$ and $\frac{1}{e} e^{\frac{n}{m} \varphi^*(\frac{t}{m})} \leq t \leq e^{\frac{n}{t} \varphi^*(\frac{t}{n})}$ for $t > 0$. Then

$$e^{n \varphi^*(\frac{t}{n})} \geq e^{(n-1)\omega(t)} e^{2n \varphi^*(\frac{t}{2n})},$$

for $j$ large enough.

These two lemmas are easy to prove.

**Lemma 3.9.** Let $\tau \in \mathbb{R}$ and let $k \in \mathbb{N}_0$ as in (3.3). Then, we have

$$\omega(x, y) \leq L^2 \omega((1 - \tau)x + \tau y) + L^{k+2} \omega(y - x) + \sum_{j=1}^{k+2} L^j, \quad x, y \in \mathbb{R}^d.$$

**Lemma 3.10.** For all $\tau \in \mathbb{R}$, the inequality

$$|v|^2 \leq C(|v + t\tau w|^2 + |v - t(1 - \tau)w|^2)$$

holds for all $v, w \in \mathbb{R}^d$, $0 \leq t \leq 1$, where $C = 2 \max\{(1 - \tau)^2, \tau^2\}$.

The following result shows that any pseudodifferential operator can be written as a quantization modulo an $\omega$-regularizing operator and is needed to understand the composition of two different quantizations in the next section. For the proof, it is fundamental the fact that the kernel $K$ of a pseudodifferential operator behaves like a function in $\mathcal{S}_\omega(\mathbb{R}^d)$ in the complement of a strip $\Delta_r = \{(x, y) \in \mathbb{R}^{2d} : |x - y| < r\}$ around the diagonal of $\mathbb{R}^{2d}$, for some $r > 0$. In other words, if $\chi$ is as in [2, Lemma 5.1], then $\chi K \in \mathcal{S}_\omega(\mathbb{R}^{2d})$ [2, Theorem 5.2].

**Theorem 3.11.** Let $a(x, y, \xi)$ be an amplitude in $\text{GA}_{m, \omega}^{m, \omega}$ with associated pseudodifferential operator $A$. Then, for any $\tau \in \mathbb{R}$, we can write $A$ uniquely as

$$A = P_\tau + R,$$

where $R$ is an $\omega$-regularizing operator and $P_\tau$ is the pseudodifferential operator given by

$$P_\tau u(x) = \iint e^{i(x-y) \cdot \xi} p_\tau((1 - \tau)x + \tau y, \xi) u(y) dy d\xi, \quad u \in \mathcal{S}_\omega(\mathbb{R}^d),$$

being $p_\tau \in \text{FGS}_\rho^{\max\{m, mL\}, \omega}$. Moreover, we have

$$p_\tau(x, \xi) \sim \sum_{j=0}^{\infty} p_j(x, \xi)$$

$$= \sum_{j=0}^{\infty} \sum_{|\beta| + |\gamma| = j} \frac{1}{\beta! \gamma!} \tau^{|\beta|} (1 - \tau)^{|\gamma|} \partial_\xi^{\beta+\gamma} (-D_x)^{\beta} D_y^\gamma a(x, y, \xi)|_{y=x}.$$

The symbol $p_\tau(x, \xi)$ is called $\tau$-symbol of the pseudodifferential operator $A$. When $\tau = 0, 1, 1/2$, these symbols are called the left, right, and Weyl symbols of $A$. 

Proof. We consider the sequence \((j_n)\) as in the statement of Lemma 3.6, with \(\frac{n}{j} \varphi^* \left( \frac{j}{n} \right) \geq \max \{n, \log (C_{4nL^{v+3}}), \log (D_{4nL^{v+3}})\}\), where \((C_n)\) and \((D_n)\) denote the sequences of constants that come from Definition 2.8 and Corollary 3.4, and \(\tilde{p} \in \mathbb{N}_0\) is so that \(\max \{|1-\tau|, 2|\tau|\} \leq e^{\tilde{p}}.\) Put

\[ p_j(x, \xi) := \sum_{|\beta|+|\gamma|=j} \frac{1}{\beta!\gamma!} (1-\tau)^{|\gamma|} \partial_\xi^{\beta+\gamma} (-D_x)^{\beta} D_y^{\gamma} a(x, y, \xi) |_{y=x}. \]

By Example 3.7, \(\sum_j p_j \in \text{FGS}^{\max\{m,mL\}}_{\rho}.\) Now, we write

\[ p_\tau(x, \xi) := \sum_{j=0}^{\infty} \varphi_j(x, \xi) p_j(x, \xi), \]

where \((\varphi_j)\) is the sequence described in (3.2). By [2, Theorem 4.6], we obtain that \(p_\tau(x, \xi) \in \text{GS}^{\max\{m,mL\}}_{\rho}\) and \(p_\tau \sim \sum p_j.\) We set, for \(u \in S_\omega(\mathbb{R}^d),\)

\[ P_\tau u(x) := \int \int e^{i(x-y)\xi} p_\tau((1-\tau)x+y, \xi) u(y) dy d\xi. \]

By Lemma 3.6, \(P_\tau\) is the limit of \(S_{N,\tau}\) in \(L(S_\omega(\mathbb{R}^d), S'_\omega(\mathbb{R}^d)),\) where \(S_{N,\tau}\) is the pseudodifferential operator with amplitude \(\sum_{j=0}^{N} (\varphi_j - \varphi_{j+1})((1-\tau)x+y, \xi)(\sum_{i=0}^{j} p_i((1-\tau)x+y, \xi))\) in \(\text{GA}^{\max\{0,m'L\}}_{\rho}, m'\) as in (3.4). On the other hand, from Lemma 3.5, \(A = \sum_{N=0}^{\infty} A_N,\) where \(A_N\) is the pseudodifferential operator with amplitude \(a(x, y, \xi)(\varphi_N - \varphi_{N+1})((1-\tau)x+y, \xi)\) in \(\text{GA}^{m,\omega}_{\rho} \subseteq \text{GA}^{\max\{0,m'L\}}_{\rho}.\) Thus, for \(u \in S_\omega(\mathbb{R}^d),\)

\[ Au(x) = \sum_{N=0}^{\infty} \int \int e^{i(x-y)\xi} ((\varphi_N - \varphi_{N+1})((1-\tau)x+y, \xi)) a(x, y, \xi) u(y) dy d\xi \]

and

\[ P_\tau u(x) = \lim_{N \to \infty} \int \int e^{i(x-y)\xi} \left[ \sum_{j=0}^{N} (\varphi_j - \varphi_{j+1})((1-\tau)x+y, \xi)(\sum_{i=0}^{j} p_i((1-\tau)x+y, \xi)) \right] u(y) dy d\xi. \]

Hence, we can write \(A - P_\tau\) as the series \(\sum_{N=0}^{\infty} P_{N,\tau},\) where each \(P_{N,\tau}\) corresponds to the pseudodifferential operator associated with the amplitude in \(\text{GA}^{\max\{0,m'L\}}_{\rho}.\)

\[ (\varphi_N - \varphi_{N+1})((1-\tau)x+y, \xi) a(x, y, \xi) - \sum_{j=0}^{N} p_j((1-\tau)x+y, \xi). \]

Our purpose is to show that the kernel \(K\) of \(A - P_\tau\) belongs to \(S_\omega(\mathbb{R}^{2d}).\) To that purpose, we write
\[
K(x, y) = \sum_{N=0}^{\infty} K_N(x, y)
\]
\[
= \sum_{N=0}^{\infty} \int_{\mathbb{R}^{2d}} e^{i(x-y) \cdot \xi} (\varphi_N - \varphi_{N+1})((1 - \tau)x + \tau y, \xi) \left( a(x, y, \xi) - \frac{\partial^\beta}{\partial^\beta x} \right) + \mathcal{O}_{\varphi_x}(1) \right) d\xi.
\]

Now, we take \( r > 0 \) and \( \chi \in \mathcal{E}_\omega(\mathbb{R}^{2d}) \), such that \( \chi \equiv 1 \) in \( \mathbb{R}^{2d} \setminus \Delta_2 r \), and \( \chi \equiv 0 \) in \( \Delta_r \) (see [2, Lemma 5.1]). Then, we can write
\[
K = \chi K + (1 - \chi) \lim_{N \to \infty} \sum_{j=0}^{N} K_j.
\]

We follow the lines of [26, Theorem 23.2], and also the scheme of the proof of [17, Theorem 3.13], as well as [2, Theorem 5.4]. We make the following change of variables:
\[
\begin{align*}
\omega &= (1 - \tau)x + \tau y; \\
w &= x - y.
\end{align*}
\]

Similarly as in [17, Theorem 3.13], we write the partial sums of \( K \) as
\[
\sum_{j=0}^{N} K_j = K_0 + \sum_{j=1}^{N} I_j + \sum_{j=1}^{N} Q_j - W_N,
\]
where
\[
I_j(x, y) := \sum_{\beta+\gamma=j} \sum_{\alpha \leq \beta+\gamma} \frac{(\beta + \gamma)!}{\beta! \gamma!} \frac{1}{\alpha!(\beta + \gamma - \alpha)!} \times \int e^{i(x-y) \cdot \xi} |(1 - \tau)^{\gamma}|(1 - \tau)^{\gamma} D_\xi^\alpha \varphi_j(v, \xi)(\partial_x^\beta \partial_y^\gamma D_\xi^\beta + \alpha a)(v, v, \xi) d\xi;
\]
\[
Q_j(x, y) := \sum_{\beta+\gamma=j+1} \sum_{\alpha \leq \beta+\gamma} \frac{(\beta + \gamma)!}{\beta! \gamma!} \frac{1}{\alpha!(\beta + \gamma - \alpha)!} \times \int e^{i(x-y) \cdot \xi} D_\xi^\alpha (\varphi_j - \varphi_{j+1})(v, \xi) D_\xi^{\gamma + \gamma - \alpha} \omega_{\beta, \gamma}(x, y, \xi) d\xi;
\]
\[
\omega_{\beta, \gamma}(x, y, \xi) := (j + 1) \int_0^1 (\partial_x^\beta \partial_y^\gamma a)(v + t \tau w, v - (1 - \tau) t w, (1 - t)^2 d t;
\]
\[
W_N(x, y) := \sum_{\beta+\gamma=1 \leq \beta+\gamma} \sum_{\alpha \leq \beta+\gamma} \frac{(\beta + \gamma)!}{\beta! \gamma!} \frac{1}{\alpha!(\beta + \gamma - \alpha)!} \times \int e^{i(x-y) \cdot \xi} |(1 - \tau)^{\gamma}|(1 - \tau)^{\gamma} D_\xi^\alpha \varphi_{N+1}(v, \xi)
\]
\[
(\partial_x^\beta \partial_y^\gamma D_\xi^{\beta + \gamma - \alpha} a)(v, v, \xi) d\xi.
\]

We have \( \chi K \in \mathcal{S}_\omega(\mathbb{R}^{2d}) \) [2, Lemma 5.1, Theorem 5.2]. Moreover, it is easy to see that \( K_0 \in \mathcal{S}_\omega(\mathbb{R}^{2d}) \). Indeed, we have
\[
K_0(x, y) = \int e^{i(x-y) \cdot \xi} (1 - \varphi_1)((1 - \tau)x + \tau y, \xi)(a(x, y, \xi) - a(x, x, \xi)) d\xi.
\]

Since \( 1 - \varphi_1 \in \mathcal{S}_\omega(\mathbb{R}^{2d}) \), following [2, Lemma 3.5(a)], one obtains the desired property for \( K_0 \) by Lemma 3.9.
First step. First of all, we compute $D_x^\theta D_y^\epsilon I_j(x, y)$ for $\theta, \epsilon \in \mathbb{N}_0^d$. We use integration by parts with the formula

$$e^{i(x-y)\cdot \xi} = \frac{1}{G(y-x)}G(-D\xi)e^{i(x-y)\cdot \xi},$$

for a suitable power $G^s(D)$ of $G(D)$, being $G(\xi)$ the entire function considered in Proposition 2.11. We obtain as in [2, Theorem 5.4]

$$D_x^\theta D_y^\epsilon I_j(x, y) = \sum_{|\beta+\gamma|=j \atop 0 \leq \beta, \gamma \leq j} \frac{(\beta + \gamma)!}{\beta!\gamma!} \frac{1}{\alpha!(\beta + \gamma - \alpha)!} \frac{1}{G^s(y-x)} \sum_{\eta \in \mathbb{N}_0^d} b_\eta \times$$

$$\times \sum_{\eta_1 + \eta_2 + \eta_3 = \eta} \frac{\theta!}{\theta_1!\theta_2!\theta_3!} \frac{\epsilon!}{\epsilon_1!\epsilon_2!\epsilon_3!} \frac{(\theta_1 + \epsilon_1)!}{(\theta_1 + \epsilon_1 - \eta_1)!} \frac{(\theta_2 + \epsilon_2)!}{(\theta_2 + \epsilon_2 - \eta_2)!} \frac{(\theta_3 + \epsilon_3)!}{(\theta_3 + \epsilon_3 - \eta_3)!} \tau^{\beta+\gamma+\alpha+\eta_3}

\int e^{i(x-y)\cdot \xi} \varphi_j(v, \xi) D_x^{\beta_2} D_y^{\epsilon_2} \varphi_j(v, \xi) D_x^{\beta_1} D_y^{\epsilon_1} (\varphi_j - \varphi_j+1)(v, \xi)\,d\xi.$$

Fix $\lambda > 0$ and set $n \geq \lambda$ large enough that may depend on $\tau, m, \rho, L, \text{and } R$. According to Lemma 3.9, it is enough to take $s \in \mathbb{N}$, such that $sC_2 \geq \lambda L^{k+2}$, where $C_2 > 0$ comes from (3.8) and $k \in \mathbb{N}_0$ as in (3.3). For the convergence of the series depending on $n \in \mathbb{N}_0^d$, let $n$ satisfy in addition that $n \geq sC_1$, where $C_1 > 0$ comes from (3.8). Now, proceeding as in [17, Theorem 3.13] (and using Proposition 2.3 and Lemma 3.8), we can show that $\sum_{j=1}^\infty I_j \in S_\omega(\mathbb{R}^{2d}).$

Second step. Since $1-\chi$ is supported in $\Delta_{2r}$, we estimate $|D_x^\theta D_y^\epsilon Q_j(x, y)|$ for $\theta, \epsilon \in \mathbb{N}_0^d, (x, y) \in \Delta_{2r}$. By the formula of integration by parts given in (3.11) for a suitable power of $G(D)$, $G^s(D)$, we have

$$D_x^\theta D_y^\epsilon Q_j(x, y) = \sum_{|\beta+\gamma|=j+1 \atop 0 \leq \beta, \gamma \leq j+1} \frac{(\beta + \gamma)!}{\beta!\gamma!} \frac{1}{\alpha!(\beta + \gamma - \alpha)!} \frac{1}{G^s(y-x)} \sum_{\eta \in \mathbb{N}_0^d} b_\eta \times$$

$$\times \sum_{\eta_1 + \eta_2 + \eta_3 = \eta} \frac{\theta!}{\theta_1!\theta_2!\theta_3!} \frac{\epsilon!}{\epsilon_1!\epsilon_2!\epsilon_3!} \frac{(\theta_1 + \epsilon_1)!}{(\theta_1 + \epsilon_1 - \eta_1)!} \frac{(\theta_2 + \epsilon_2)!}{(\theta_2 + \epsilon_2 - \eta_2)!} \frac{(\theta_3 + \epsilon_3)!}{(\theta_3 + \epsilon_3 - \eta_3)!} \tau^{\beta+\gamma+\alpha+\eta_3}

\int e^{i(x-y)\cdot \xi} \varphi_j(v, \xi) D_x^{\beta_2} D_y^{\epsilon_2} \varphi_j(v, \xi) D_x^{\beta_1} D_y^{\epsilon_1} (\varphi_j - \varphi_j+1)(v, \xi)\,d\xi.$
where \( \omega_{\beta \gamma}(x, y, \xi) \) is defined in (3.10). Fix \( \lambda > 0 \) and take \( n \geq \lambda \) to be determined later. We consider in this step \( \tilde{\nu} \in \mathbb{N} \), such that
\[
\max \{2(1 + |\tau|), (1 + 2r)^p\} \leq e^{\tilde{\nu}p}.
\]
We put \( \tilde{n} \in \mathbb{N}_0 \), \( \tilde{n} \geq n \), such that (where \( q \in \mathbb{N}_0 \) satisfies \( 2q \geq 3R \))
\[
\tilde{n} \geq \frac{L^{q+1}}{\rho}(\lambda L^{\tilde{\nu}+2} + mL^3 + 1) + 1.
\]
By Lemma 3.10 and the properties of \( \varphi^* \), proceeding as in the proof of the second step of [2, Theorem 5.4] we obtain, for some \( \tilde{C}_n > 0 \)
\[
|D_x^{\beta}D_y^{\gamma}(D_\xi^{\beta+\gamma-\alpha+\eta_3}\omega_{\beta \gamma})(x, y, \xi)| \\
\leq \tilde{C}_n e^{16nL^{p+1}} \rho \sum_{p=1}^{3p+1} L^p e^{ml^j\xi_{j+1}} \int (|v(\xi)| - \rho)^{|2\beta+2\gamma-\alpha|} \\
x e^{16\tilde{n}L^{\tilde{\nu}+1}} \rho \varphi^*\left(\frac{|2\beta+2\gamma-\alpha|}{16L^{p+1}}\right) e^{ml^j\omega(v)} e^{ml^{j+1}\omega(w)} e^{ml^j\omega(\xi)} \int_0^1 |1-t|^j dt.
\]
For the estimate of the derivatives of \( Q_j(x, \xi) \), we can proceed similarly as in the first step to show finally that \( (1 - \chi) \sum_{j=1}^{N} Q_j \in S_\omega(\mathbb{R}^{2d}) \).

**Third step.** Let \( T_N : S_\omega(\mathbb{R}^d) \to S_\omega(\mathbb{R}^d) \) be the operator with kernel \((1 - \chi)W_N\). As in the proof of [2, Theorem 5.4], it follows that \( (T_N) \) converges to an operator \( T : S_\omega(\mathbb{R}^d) \to S_\omega(\mathbb{R}^d) \) in \( L(S_\omega(\mathbb{R}^d), S_\omega(\mathbb{R}^d)) \). We show that \( T = 0 \).

To this aim, fix \( N \in \mathbb{N} \), \( j_n \leq N + 1 < j_{n+1} \) and set \( a_N := \Re e^{\frac{n+1}{N+1} \varphi^*(\frac{N+1}{n})} \).

For the support of the derivatives of \( \varphi_{N+1} \), we may assume that
\[
2a_N \leq ((1 - \tau)x + \tau y, \xi) \leq 3a_N.
\]
For \( f, g \in S_\omega(\mathbb{R}^d) \), we have
\[
\langle T_N f, g \rangle = \int T_N f(x)g(x)dx = \int \left( \int (1 - \chi)(x, y)W_N(x, y)f(y)dy \right)g(x)dx.
\]
Fixed \( N \in \mathbb{N} \), we can use Fubini’s theorem (since \( f, g \in S_\omega(\mathbb{R}^d) \) and \( |\xi| \leq 3a_N \) and we obtain
\[
\langle T_N f, g \rangle = \int \left( \int \sum_{|\beta + \gamma| = 1} \sum_{0 \leq \alpha \leq |\beta + \gamma|} \frac{(\beta + \gamma)!}{\beta!\gamma!} \\
\times \frac{1}{\alpha!(\beta + \gamma - \alpha)!} \left\{ \int e^{i(x-y) \cdot \xi} e^{(1 - \tau)|\gamma|} f(y)(1 - \chi)(x, y)dy \right\}g(x)dx.
\]
An integration by parts with (3.11) for a suitable power \( s \in \mathbb{N} \), to be determined, gives
\[
e^{i(x-y) \cdot \xi} \frac{1}{G^s(\xi)} \sum_{\eta \in \mathbb{N}_0} b_{\eta} \sum_{\eta_1 + \eta_2 + \eta_3 + \eta_4 = \eta} \frac{\eta!}{\eta_1!\eta_2!\eta_3!\eta_4!} \tau^{\gamma_1} \chi^{\gamma_2} D^\gamma_y D^\gamma_x \varphi_{N+1}(x, \xi) \\
\times D^\gamma_y (\partial^\alpha_x \partial^\gamma_y D^{\beta+\gamma-\alpha}_\xi f)(y)(1 - \chi)(x, y)g(x).
\]
Thus, we obtain
\[
\langle T_N f, g \rangle = \sum_{|\beta + \gamma|=1}^{N} \sum_{0 \neq \alpha \leq \beta + \gamma} \frac{(\beta + \gamma)!}{\alpha!} \sum_{\eta \in \mathbb{N}_d^4} b_\eta \prod_{\eta_{1} + \eta_{2} + \eta_{3} + \eta_{4} = \eta} \eta_i! \eta_j! \eta_k! \eta_l! \\
\times \tau^{\eta_1 + \beta} (1 - \tau)^{\eta_2} \eta_3 \eta_4 (\eta_5 + \eta_6) \int e^{i(x-y) \xi} \frac{1}{G^s(\xi)} \int \gamma^2 \varphi_{N+1}(v, \xi) \\
\times \gamma^2 \eta \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma 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Set \( s \in \mathbb{N} \), such that \( sC_2 \geq (mL + L)L^{k+1} + 1 \), and take \( \ell \geq sC_1 \) to get that the series is convergent. It is easy to see for such \( s \in \mathbb{N} \) that there exists \( C_k > 0 \) such that

\[
E' e^{m\omega(v,v,\xi)} e^{-(mL+L)L^{k+1}+1}\omega(y)+\omega(x)) \leq C_k e^{\omega((v,\xi)))} e^{-\omega(x)} e^{-\omega(y)} e^{-\omega(\xi)}.
\]

Therefore, we have

\[
\iint_{2aN \leq (v,\xi) \leq 3aN} e^{-\omega((v,\xi)))} e^{-\omega(x)} e^{-\omega(y)} e^{-\omega(\xi)} dyd\xi dx \leq e^{-\omega(2aN)} \iint_{\mathbb{R}^d} e^{-\omega(x)} e^{-\omega(y)} e^{-\omega(\xi)} dyd\xi dx,
\]

By property (\( \gamma \)) of Definition 2.1, there exists \( C > 0 \), such that \( 3\log(t) \leq \omega(t) + C_t, t \geq 0 \). Thus

\[
e^{-\omega(2aN)} \leq (2aN)^{-3} e^C.
\]

We recall that \( C_n D_n \) is the only constant that depends on \( n \). By the choice of the sequence \( (j_n) \), we have

\[
e^n C_n D_n \leq a_N^3.
\]

Hence, there exists \( C' > 0 \), such that

\[
|\langle T_N f, g \rangle| \leq C' \sum_{1 \leq l \leq N} \frac{1}{(2R)^{2\ell}} \int_{|\beta+\gamma| = 0} \left( \frac{e^{\tilde{\rho}+1}}{(2R)^{2\rho}} \right)^{1} \frac{C_n D_n}{a_N^3} 
\]

Since the series converges for \( R \geq 1 \) large enough (which may depend on \( \tau \)), and since \( n \to \infty \) when \( N \to \infty \), we show that \( |\langle T_N f, g \rangle| \) tends to zero when \( N \to \infty \).

It only remains to prove the uniqueness of the pseudodifferential operator modulo an \( \omega \)-regularizing operator. We notice that every global amplitude as in Definition 2.8 defines an \( \omega \)-ultradistribution. Then, as in [23,26], the identities in \( S'_{\omega}(\mathbb{R}^{2d}) \) for the Fourier transform

\[
K_{\tau}(x,y) = (2\pi)^d \mathcal{F}_{\xi \to -x-y}^{-1}(a_{\tau}((1-\tau)x + \tau y, \xi))
\]

and

\[
a_{\tau}(v,\xi) = (2\pi)^{-d} \mathcal{F}_{w \to \xi}(K_{\tau}(v + \tau w, v - (1-\tau)w))
\]

yield the uniqueness of the \( \tau \)-symbol, since the kernel \( K_{\tau} \) is also unique. \( \square \)

As a consequence of Theorem 3.11, we can describe the precise relation between different quantizations for a given global symbol in terms of equivalence of formal sums as the following result shows.
**Theorem 3.12.** If $a_{\tau_1}(x, \xi)$ and $a_{\tau_2}(x, \xi)$ are the $\tau_1$ and $\tau_2$-symbol of the same pseudodifferential operator $A$, then

$$a_{\tau_2}(x, \xi) \sim \sum_{j=0}^{\infty} \sum_{|\alpha|=j} \frac{1}{\alpha!} (\tau_1 - \tau_2)^{|\alpha|} \partial_\xi^\alpha D_x^\alpha a_{\tau_1}(x, \xi).$$

**Proof.** By Theorem 3.11, the pseudodifferential operator $A$ is determined via the $\tau_1$-symbol $a_{\tau_1}((1 - \tau_1)x + \tau_1 y, \xi)$ modulo an $\omega$-regularizing operator. Again, by Theorem 3.11, its $\tau_2$-symbol has the following asymptotic expansion:

$$a_{\tau_2}(x, \xi) \sim \sum_{j=0}^{\infty} \sum_{|\alpha|=j} \frac{(-1)^{|\beta|}}{\beta! \gamma!} \tau_2^{\beta}(1 - \tau_2)^{|\gamma|} \partial_\xi^{\beta + \gamma} D_x^\beta D_y^\gamma (a_{\tau_1}((1 - \tau_1)x + \tau_1 y, \xi)|_{y=x})$$

$$= \sum_{j=0}^{\infty} \sum_{|\alpha|=j} \left( \sum_{\beta + \gamma = \alpha} \frac{1}{\beta! \gamma!} ((1 - \tau_2)\tau_1 - \tau_2(1 - \tau_1))^{|\gamma|} \partial_\xi^{\beta + \gamma} D_x^\beta D_y^\gamma a_{\tau_1}(x, \xi) \right)$$

$$= \sum_{j=0}^{\infty} \sum_{|\alpha|=j} \frac{1}{\alpha!} ((1 - \tau_2)\tau_1 - \tau_2(1 - \tau_1))^{|\alpha|} \partial_\xi^\alpha D_x^\alpha a_{\tau_1}(x, \xi).$$

\[\square\]

### 4. Transposition and Composition of Operators

By [2, Proposition 3.10], we deduce that if $A$ has as amplitude $a((1 - \tau)x + \tau y, \xi)$, then its transpose $^tA$ has the amplitude $a((1 - \tau)y + \tau x, -\xi)$. Hence, if $a_{\tau}(x, \xi)$ is the $\tau$-symbol of $A$, then $^t a_{1-\tau}(x, \xi)$ is the $(1 - \tau)$-symbol of $^tA$ given by

$$^t a_{1-\tau}((1 - \tau)x + \tau y, \xi) := a_{\tau}((1 - \tau)y + \tau x, -\xi). \quad (4.1)$$

In particular, we have $^t a_{\tau}(x, \xi) = a_{1-\tau}(x, -\xi)$. On the other hand, for $\tau = 0$, $^t a_1(y, -\xi)$ coincides with $a_0(x, \xi)$. Now, we show the corresponding generalization of [2, Proposition 5.5].

**Theorem 4.1.** Let $A$ be the pseudodifferential operator with $\tau$-symbol $a_{\tau}(x, \xi)$. Then, its transpose restricted to $S_{\omega}(\mathbb{R}^d)$ can be decomposed as $^tA = Q + R$, where $R$ is an $\omega$-regularizing operator and $Q$ is the pseudodifferential operator associated to the $\tau$-symbol given by

$$q(x, \xi) \sim \sum_{j=0}^{\infty} \sum_{|\alpha|=j} \frac{1}{\alpha!} (1 - 2\tau)^{|\alpha|} \partial_\xi^\alpha D_x^\alpha a_{\tau}(x, -\xi).$$

**Proof.** By assumption, we deduce that $^tA$ has the $(1 - \tau)$-symbol $^t a_{1-\tau}(x, \xi)$ given by formula (4.1) restricted to $y = x$. Moreover, from Theorem 3.12, the $\tau$-symbol of $^tA$ satisfies
\[
\imath a_\tau(x, \xi) \sim \sum_{j=0}^{\infty} \sum_{|\alpha|=j} \frac{1}{\alpha!} (1 - 2\tau)^{|\alpha|} \partial_\xi^\alpha D_x a_{\tau}(x, \xi) \\
= \sum_{j=0}^{\infty} \sum_{|\alpha|=j} \frac{1}{\alpha!} (1 - 2\tau)^{|\alpha|} \partial_\xi^\alpha D_x a_\tau(x, -\xi).
\]

Let us deal with the composition of two pseudodifferential operators given by their corresponding quantizations of symbols.

**Theorem 4.2.** Let \(a_{\tau_1}(x, \xi) \in GS^{m_1, \omega}_\rho\) be the \(\tau_1\)-symbol of \(A_1\) and \(b_{\tau_2}(x, \xi) \in GS^{m_2, \omega}_\rho\) be the \(\tau_2\)-symbol of \(A_2\), being \(A_1\) and \(A_2\) their corresponding pseudodifferential operators. The \(\tau\)-symbol \(c_\tau(x, \xi) \in GS^{m_1+m_2, \omega}_\rho\) of \(A_1 \circ A_2\) has the asymptotic expansion

\[
\sum_{j=0}^{\infty} \sum_{\substack{|\alpha+\beta-\alpha_1-\alpha_2|=j \\ \alpha+\beta=\gamma+\delta}} c_{\alpha\beta\gamma\delta\alpha_1\alpha_2} \partial_\xi^\gamma D_x^\alpha a_{\tau_1}(x, \xi) \cdot \partial_\xi^\delta D_x^\beta b_{\tau_2}(x, \xi),
\]  

(4.2)

where the coefficients \(c_{\alpha\beta\gamma\delta\alpha_1\alpha_2}\) are

\[
\frac{(2\pi)^d}{\gamma!\delta!} \sum_{k,l=0}^{\infty} \sum_{|\alpha_1|=k \atop |\alpha_2|=l} (-1)^{|\alpha_1+\alpha_2|} \\
\times \left( \frac{\alpha + \beta - \alpha_1 - \alpha_2}{\alpha - \alpha_1} \right) \left( \frac{\gamma}{\alpha_1} \right) \left( \frac{\delta}{\alpha_2} \right) \tau^{|\alpha_1|} (1 - \tau)^{|\beta_1|} \tau_1^{|\alpha_1|} (1 - \tau_2)^{|\alpha_2|}.
\]

**Proof.** We first assume \(\tau_1=0\) and \(\tau_2=1\). In this case, \(a_{\tau_1}((1-\tau_1)x + \tau_1 y, \xi)\) and \(b_{\tau_2}((1-\tau_2)x + \tau_2 y, \xi)\) coincide with \(a_0(x, \xi)\) and \(b_1(y, \xi)\). Then

\[
(A_1 \circ A_2)u(x) = \int e^{ix \cdot \xi} a_0(x, \xi) A_2 u(\xi) d\xi, \quad x \in \mathbb{R}^d.
\]

It is not difficult to see that \(A_2 u(x) = \tilde{I}(-x)\), where \(I(\xi) = \int e^{-iy \cdot \xi} b_1(y, \xi) u(y) dy\). Hence, \(\tilde{A}_2 u(\xi) = (2\pi)^d I(\xi)\) and

\[
(A_1 \circ A_2)u(x) = \int \int e^{i(x-y) \cdot \xi} c(x, y, \xi) u(y) d\xi dy, \quad x \in \mathbb{R}^d,
\]

where \(c(x, y, \xi) = (2\pi)^d a_0(x, \xi) b_1(y, \xi)\) is an amplitude in \(GA^{m_1+m_2, \omega}_\rho\). Therefore, by Theorem 3.11, the \(\tau\)-symbol \(c_\tau(x, \xi)\) has the asymptotic expansion

\[
c_\tau(x, \xi) \sim (2\pi)^d \sum_{j=0}^{\infty} \sum_{|\beta+\gamma|=j} \frac{(-1)^{|\beta|}}{\beta!\gamma!} \tau^{|\beta|} (1 - \tau)^{|\gamma|} \partial_\xi^\gamma D_x^\beta D_y^\gamma (a_0(x, \xi) b_1(y, \xi)) \big|_{y=x}
\]

(4.3)

\[
= (2\pi)^d \sum_{j=0}^{\infty} \sum_{|\beta+\gamma|=j} \frac{(-1)^{|\beta|}(\beta + \gamma)!}{\delta!\epsilon!\gamma!} \tau^{|\beta|} (1 - \tau)^{|\gamma|} \partial_\xi^\gamma D_x^\beta a_0(x, \xi) \cdot \partial_\xi^\epsilon D_x^\gamma b_1(x, \xi).
\]

(4.4)
For the general case, by Theorem 3.12, we have
\[ a_0(x, \xi) \sim \sum_{|\alpha_1|=j_1}^{\infty} \frac{1}{\alpha_1!} \tau_1^{[\alpha_1]} \partial_{\xi \alpha_1} D_x^{\alpha_1} a_\tau_1(x, \xi); \]
\[ b_1(x, \xi) \sim \sum_{|\alpha_2|=j_2}^{\infty} \frac{(-1)^{|\alpha_2|}}{\alpha_2!} (1 - \tau_2)^{|\alpha_2|} \partial_{\xi \alpha_2} D_x^{\alpha_2} b_{\tau_2}(x, \xi). \]
Thus, from (4.4), we get
\[ c_\tau(x, \xi) \sim (2\pi)^d \sum_{j=0}^{\infty} \sum_{|\beta+\gamma|=j}^{\infty} \frac{(-1)^{|\beta|}|\beta + \gamma|!}{\delta!|\beta|!|\gamma|!} \tau_\beta (1 - \tau)^{|\gamma|} \]
\[ \times \partial_\xi^\beta D_x^\beta \left( \sum_{|\alpha_1|=j_1}^{\infty} \sum_{|\alpha_2|=j_2}^{\infty} \frac{1}{\alpha_1!} \tau_1^{[\alpha_1]} \partial_{\xi \alpha_1} D_x^{\alpha_1} a_\tau_1(x, \xi) \right) \]
\[ \times \partial_\xi^\gamma D_x^\gamma \left( \sum_{|\alpha_2|=j_2}^{\infty} \sum_{|\alpha_2|=j_2}^{\infty} \frac{(-1)^{|\alpha_2|}}{\alpha_2!} (1 - \tau_2)^{|\alpha_2|} \partial_{\xi \alpha_2} D_x^{\alpha_2} b_{\tau_2}(x, \xi) \right). \]
We make the change of variables \( \gamma' = \alpha_1 + \delta, \alpha' = \alpha_1 + \beta, \delta' = \alpha_2 + \epsilon, \beta' = \alpha_2 + \gamma \). Then
\[ c_\tau(x, \xi) \sim (2\pi)^d \sum_{j=0}^{\infty} \sum_{\alpha'_1+\beta'_1=\alpha'_2+\alpha'_2=\tau_\beta} \frac{1}{\gamma'!\delta'!} \partial_{\xi}^{\beta'} D_x^{\beta'} a_{\tau_1}(x, \xi) \partial_{\xi}^{\gamma'} D_x^{\gamma'} b_{\tau_2}(x, \xi) \]
\[ \times \sum_{k,l=0}^{\infty} \sum_{|\alpha_k|=k}^{\infty} \frac{(-1)^{|\alpha_k'|-\alpha_k+\alpha_k'}}{|\alpha_k'|-\alpha_k+\alpha_k'|} \]
\[ \times \frac{(\alpha'_1+\beta'_1-\alpha_1-\alpha_2)!}{(\alpha'-\alpha_1)!((\beta'-\alpha_2)! \alpha_1!(\gamma'-\alpha_1)! \alpha_2!(\delta'-\alpha_2)!)} \]
\[ \times \tau_1^{[\alpha'_1]} (1 - \tau)^{|\beta'-\alpha_2|} \tau_1^{[\alpha_2]} (1 - \tau_2)^{|\alpha_2|}. \]
The proof follows, since
\[ \frac{(\alpha'_1+\beta'_1-\alpha_1-\alpha_2)!}{(\alpha'-\alpha_1)!((\beta'-\alpha_2)! \alpha_1!(\gamma'-\alpha_1)! \alpha_2!(\delta'-\alpha_2)!)} \]
\[ = \left( \alpha'_1+\beta'_1-\alpha_1-\alpha_2 \right) \left( \gamma' \right) \left( \delta' \right) \left( \alpha_1 \right) \left( \alpha_2 \right). \]
\[ \square \]
The coefficients appearing in formula (4.2) are sometimes simplified for some particular \( \tau \in \mathbb{R} \). For example, if \( \tau = 0 \), by formula (4.3), we obtain
\[ c(x, \xi) = c_0(x, \xi) \sim (2\pi)^d \sum_{j=0}^{\infty} \sum_{|\gamma|=j} \frac{1}{\gamma!} \partial_{\xi}^{\gamma} D_y^{\gamma} (a_0(x, \xi) b_1(y, \xi)) \big|_{y=x}. \]
On the other hand, from formula (4.1), \( b_1(x, \xi) = t b_0(x, -\xi) \). Hence, by [2, Lemma 5.6], we have

\[
c_0(x, \xi) \sim (2\pi)^d (a_0(x, \xi) \circ b_0(x, \xi)) = (2\pi)^d \sum_{j=0}^{\infty} \sum_{|\gamma|=j} \frac{1}{\gamma!} \partial_\xi^\gamma a_0(x, \xi) D_x^\gamma b_0(x, \xi),
\]

which in particular gives [2, Theorem 5.7] (cf. [26, Theorem 23.7]).

Another interesting case is when dealing with \( \tau = 1/2 \). We will obtain it as a consequence of a more general result (cf. [26, Problem 23.2]). First, we need a lemma, taken from [4, Theorem 5.5]:

**Lemma 4.3.** The formula

\[
\frac{(\beta + \gamma)!}{(\beta + \gamma - \epsilon)! \beta! \gamma!} = \sum_{0 \leq \delta \leq \beta} \frac{1}{(\beta - \delta)!(\beta - \epsilon + \gamma - \delta)! \delta!(\delta - \beta + \epsilon)!}
\]

holds for all \( \beta, \gamma, \epsilon \in \mathbb{N}_0^d \) with \( \epsilon \leq \beta + \gamma \).

**Example 4.4.** Given two pseudodifferential operators \( A \) and \( B \), the \( \tau \)-symbol of the composition operator \( C = A \circ B \) is given by

\[
c_\tau(x, \xi) \sim (2\pi)^d \sum_{j=0}^{\infty} \sum_{\beta + \gamma = j} (-1)^{|\beta|} \beta! \gamma! \tau^{(|\beta| \beta) (1 - \tau)^{|\gamma|}} (\partial_\xi^\beta D_x^\beta a_\tau(x, \xi)) (\partial_\xi^\gamma D_x^\gamma b_\tau(x, \xi)).
\]

**Proof.** Formula (4.4) states that \( c_\tau(x, \xi) \) is equivalent to (since \( \delta = \beta + \gamma - \epsilon \))

\[
(2\pi)^d \sum_{j=0}^{\infty} \sum_{\beta + \gamma = j} (-1)^{|\beta|} \tau^{(|\beta| \beta) (1 - \tau)^{|\gamma|}} \times \sum_{\epsilon \leq \beta + \gamma} \frac{(\beta + \gamma)!}{(\beta + \gamma - \epsilon)! \epsilon! \beta! \gamma!} \partial_\xi^{\beta + \gamma - \epsilon} D_x^{\beta} a_0(x, \xi) \cdot \partial_\xi^{\gamma} D_x^{\gamma} b_1(x, \xi).
\]

Moreover, by Lemma 4.3, it is equal to

\[
(2\pi)^d \sum_{j=0}^{\infty} \sum_{\beta + \gamma = j} (-1)^{|\beta|} \tau^{(|\beta| \beta) (1 - \tau)^{|\gamma|}} \times \sum_{\epsilon \leq \beta + \gamma} \sum_{0 \leq \delta \leq \beta} \sum_{\beta - \epsilon \leq \delta \leq \beta - \epsilon + \gamma} \frac{1}{(\beta - \delta)!(\beta - \epsilon + \gamma - \delta)! \delta!(\delta - \beta + \epsilon)!} \partial_\xi^{\beta + \gamma - \epsilon} D_x^{\beta} a_0(x, \xi) \cdot \partial_\xi^{\gamma} D_x^{\gamma} b_1(x, \xi).
\]

We put \( \mu = \beta - \delta, \nu = \beta - \epsilon + \gamma - \delta, \) and \( \theta = \delta - \beta + \epsilon \). Therefore

\[
c_\tau(x, \xi) \sim (2\pi)^d \sum_{j=0}^{\infty} \sum_{\nu + \theta + \mu + \delta = j} (-1)^{|\mu + \delta|} \mu! \nu! \delta! \theta! \tau^{(|\mu + \delta| \mu + \delta) (1 - \tau)^{|\nu + \theta|}} \times \partial_\xi^{\nu + \theta} D_x^{\mu + \delta} a_0(x, \xi) \cdot \partial_\xi^{\beta + \mu} D_x^{\nu + \theta} b_1(x, \xi),
\]
and taking \( j = j_1 + j_2 + j_3, j_1, j_2, j_3 \in \mathbb{N}_0 \), we have

\[
c_w(x, \xi) \sim (2\pi)^d \sum_{j_1=0}^{\infty} \sum_{|\nu + \mu| = j_1} (-1)^{|\mu|} \frac{|\tau|^{|\mu|}}{\mu! \nu!} (1 - \tau)^{|\nu|} \\
\times \partial^\nu_x D_x^\mu \left( \sum_{j_2=0}^{\infty} \sum_{|\delta| = j_2} (-1)^{|\delta|} \frac{|\tau|^{|\delta|}}{\delta!} \partial^\delta_x D_x^\delta a_0(x, \xi) \right) \\
\times \partial^\nu_x D_x^\mu \left( \sum_{j_3=0}^{\infty} \sum_{|\theta| = j_3} \frac{1}{\theta!} (1 - \tau)^{|\theta|} \partial^\theta_x D_x^\theta b_1(x, \xi) \right).
\]

We get the result, since Theorem 3.12 gives

\[
a_{\tau}(x, \xi) \sim \sum_{k=0}^{\infty} \sum_{|\delta| = k} (-1)^{|\delta|} \frac{|\tau|^{|\delta|}}{\delta!} (1 - \tau)^{|\delta|} \partial^\delta_x D_x^\delta a_0(x, \xi),
\]

\[
b_{\tau}(x, \xi) \sim \sum_{k=0}^{\infty} \sum_{|\theta| = k} \frac{1}{\theta!} (1 - \tau)^{|\theta|} \partial^\theta_x D_x^\theta b_1(x, \xi).
\]

\[\square\]

**Corollary 4.5.** Given two pseudodifferential operators \( A \) and \( B \), the Weyl symbol of the composition operator \( C = A \circ B \) is given by

\[
c_w(x, \xi) \sim (2\pi)^d \sum_{j=0}^{\infty} \sum_{|\beta + \gamma| = j} (-1)^{\beta} \frac{|\beta|!|\gamma|!}{\beta! \gamma!} 2^{-|\beta + \gamma|} (\partial^\gamma_x D_x^\beta a_w(x, \xi))(\partial^\beta_x D_x^\gamma b_w(x, \xi)).
\]

### 5. Parametrices and \( \omega \)-regularity

In this section, we give a sufficient condition for \( \omega \)-regularity of a global pseudodifferential operator. We say that a pseudodifferential operator \( P : \mathcal{S}_\omega'(\mathbb{R}^d) \to \mathcal{S}_\omega'(\mathbb{R}^d) \) is \( \omega \)-regular if given \( u \in \mathcal{S}_\omega'(\mathbb{R}^d) \), such that \( Pu \in \mathcal{S}_\omega'(\mathbb{R}^d) \), we have \( u \in \mathcal{S}_\omega'(\mathbb{R}^d) \). See [5] for a study of \( \omega \)-regularity of linear partial differential operators with polynomial coefficients using quadratic transformations (cf. [22] for the non-isotropic case).

We use the well-known method of the construction of a parametrix for the symbol of the operator, using symbolic calculus. We follow the lines of [16,27]. From [24], we know that a weight function \( \sigma \) is equivalent to a subadditive weight function if and only if it satisfies

\[
(\alpha_0) \ \exists C > 0, \ \exists t_0 > 0 \ \forall \lambda \geq 1 : \ \sigma(\lambda t) \leq \lambda C \sigma(t), \quad t \geq t_0.
\]

We refer to [15,24] for applications and characterizations of property \((\alpha_0)\) on the weight function. The following result is taken from [16, Lemma 3.3].

**Lemma 5.1.** Let \( \omega \) be a subadditive weight function. For all \( \lambda > 0 \) and \( j, k \in \mathbb{N} \), we have

\[
\frac{e^{\lambda \varphi_\omega^*(\frac{j}{k})}}{j!} \frac{e^{\lambda \varphi_\omega^*(\frac{k}{j})}}{k!} \leq \frac{e^{\lambda \varphi_\omega^*(\frac{j+k}{k})}}{(j+k)!}.
\]

\[\square\]
The following lemma states Vandermonde’s identity.

**Lemma 5.2.** For any $m, n, r \in \mathbb{N}_0$, we have
\[
\sum_{k=0}^{r} \binom{m}{k} \binom{n}{r-k} = \binom{m+n}{r}.
\]

**Lemma 5.3.** If $\sum_j a_j \in \text{FGS}_p^{m_1,\omega}$ and $b(x, \xi) \in \text{GS}_p^{m_2,\omega}$, then $\sum_j a_j(x, \xi)b(x, \xi) \in \text{FGS}_p^{m_1+m_2,\omega}$.

The following result is in the spirit of Zanghirati [27] and Fernández, Galbis, and Jornet [16] (see also Cappiello, Pilipović, and Prangoski [13]).

**Theorem 5.4.** Let $\omega$ be a weight function and let $\sigma$ be a subadditive weight function with $\omega(t^{1/p}) = o(\sigma(t))$ as $t \to \infty$. Let $p(x, \xi) \in \text{GS}_p^{[m],\omega}$ be such that, for some $R \geq 1$:

(i) $|p(x, \xi)| \geq \frac{1}{R} e^{-|m|\omega(x, \xi)}$ for $\langle (x, \xi) \rangle \geq R$;

(ii) There exist $C > 0$ and $n \in \mathbb{N}$, such that
\[
|D_x^\alpha D_\xi^\beta p(x, \xi)| \leq C^{\alpha+\beta} |\langle (x, \xi) \rangle |^{-\rho|\alpha+\beta|} e^{\frac{1}{n} \omega^*(n|\alpha|)} e^{\frac{1}{n} \omega^*(n|\beta|)} |p(x, \xi)|,
\]
for $\alpha, \beta \in \mathbb{N}_0^d$, $\langle (x, \xi) \rangle \geq R$.

Then, there exists $q(x, \xi) \in \text{GS}_p^{[m],\omega}$, such that $q \circ p \sim 1$ in $\text{FGS}_p^{[m],\omega}$.

**Proof.** We set
\[
q_0(x, \xi) = \frac{1}{p(x, \xi)}, \quad \langle (x, \xi) \rangle \geq R.
\]
We show by induction on $|\alpha + \beta|$ that there exists $C_1 > 0$, such that
\[
|D_x^\alpha D_\xi^\beta q_0(x, \xi)| \leq C_1^{\alpha+\beta} |\langle (x, \xi) \rangle |^{-\rho|\alpha+\beta|} e^{\frac{1}{n} \omega^*(n|\alpha|)} e^{\frac{1}{n} \omega^*(n|\beta|)} |q_0(x, \xi)|
\]
(5.1) for all $\alpha, \beta \in \mathbb{N}_0^d$, $\langle (x, \xi) \rangle \geq R$. Indeed, the inequality is true for $\alpha = \beta = 0$. Now, differentiating the formula $p(x, \xi)q_0(x, \xi) = 1$, we obtain
\[
p(x, \xi)D_x^\alpha D_\xi^\beta q_0(x, \xi) = - \sum_{0 \neq (\alpha, \beta) \leq (\alpha, \beta)} \frac{\alpha!}{\alpha!(\alpha - \hat{\alpha})!} \frac{\beta!}{\beta!(\beta - \hat{\beta})!} D_x^\alpha D_\xi^\beta p(x, \xi) D_x^{\alpha - \hat{\alpha}} D_\xi^{\beta - \hat{\beta}} q_0(x, \xi).
\]

Now, we assume that the inequality (5.1) is true for $(\hat{\alpha}, \hat{\beta}) < (\alpha, \beta)$. Using condition (ii), we obtain
\[
|p(x, \xi)D_x^\alpha D_\xi^\beta q_0(x, \xi)|
\]
\[
\leq \sum_{0 \neq (\hat{\alpha}, \hat{\beta}) \leq (\alpha, \beta)} \frac{\alpha!}{\alpha!(\alpha - \hat{\alpha})!} \frac{\beta!}{\beta!(\beta - \hat{\beta})!} C_1^{\hat{\alpha}+\hat{\beta}} |\langle (x, \xi) \rangle |^{-\rho|\hat{\alpha}+\hat{\beta}|} e^{\frac{1}{n} \omega^*(n|\hat{\alpha}|)} e^{\frac{1}{n} \omega^*(n|\hat{\beta}|)} |p(x, \xi)|
\]
\[
\times C_1^{\alpha-\hat{\alpha}+\beta-\hat{\beta}} |\langle (x, \xi) \rangle |^{-\rho|\alpha-\hat{\alpha}+\beta-\hat{\beta}|} e^{\frac{1}{n} \omega^*(n|\alpha-\hat{\alpha}|)} e^{\frac{1}{n} \omega^*(n|\beta-\hat{\beta}|)} |q_0(x, \xi)|.
\]
Since \( |\alpha|! \frac{1}{n!} \frac{\varphi_\alpha^*(n|\alpha|)}{\alpha!} \frac{\beta!}{\beta!} \frac{\varphi_\beta^*(n|\beta|)}{\beta!} \leq \frac{|\alpha|!}{|\alpha|!} \frac{\beta!}{|\beta|!} \frac{|\beta|!}{|\beta|!} \), we obtain, by Lemma 5.1
\[
|\alpha|! \frac{1}{\alpha!} \frac{\varphi_\alpha^* (n|\alpha|)}{\alpha!} \frac{\beta!}{\beta!} \frac{\varphi_\beta^* (n|\beta|)}{\beta!} \leq e^{\frac{1}{n} \varphi_\alpha^* (n|\alpha|)} e^{\frac{1}{n} \varphi_\beta^* (n|\beta|)}.
\]
Thus
\[
|D_x^a D_\xi^\beta q_0 (x, \xi)| \leq C_1^{[\alpha + \beta]} \langle (x, \xi) \rangle^{-\rho|\alpha + \beta|} e^{\frac{1}{n} \varphi_\alpha^* (n|\alpha|)} e^{\frac{1}{n} \varphi_\beta^* (n|\beta|)} |q_0 (x, \xi)|
\times \sum_{0 \neq (\alpha, \beta) \leq (\alpha, \beta)} \left( \frac{C}{C_1} \right)^{[\alpha + \beta]}.
\]
Finally, the fact that
\[
\sum_{0 \neq (\alpha, \beta) \leq (\alpha, \beta)} \left( \frac{C}{C_1} \right)^{[\alpha + \beta]} \leq \sum_{k=1}^{\infty} \left( \frac{C}{C_1} \right)^{[\alpha + \beta]} \leq \sum_{k=1}^{\infty} \left( \frac{dC}{C_1} \right)^k
\]
completes the proof of (5.1) if we take \( C_1 > 0 \), such that
\[
\sum_{k=1}^{\infty} \left( \frac{dC}{C_1} \right)^k < 1.
\]
For \( j \in \mathbb{N} \), we define recursively
\[
q_j (x, \xi) := -q_0 (x, \xi) \sum_{0 < |\alpha + \gamma| \leq j} \frac{(-1)^{|\xi|}}{e^{|\gamma|}} (1 - \tau)^{|\gamma|} \langle \partial_\xi^\gamma D_x^\alpha q_j - |\epsilon + \gamma| (x, \xi) \rangle (\partial_\xi^\gamma D_x^\alpha p (x, \xi)).
\]
We show that there exist constants \( C_2, C_3 > 0 \) with \( C_1 < C_2 < C_3 \), such that
\[
|D_x^a D_\xi^\beta q_j (x, \xi)| \leq C_2^{[\alpha + \beta]} C_3^{[\alpha + \beta]} \langle (x, \xi) \rangle^{-\rho (|\alpha + \beta| + 2j)} e^{\frac{1}{n} \varphi_\alpha^* (n|\alpha + \beta| + 2j)} e^{m|\omega(x, \xi)},
\]
for all \( \alpha, \beta \in \mathbb{N}_0^d \), \( \langle (x, \xi) \rangle \geq R \). We proceed by induction on \( j \in \mathbb{N}_0 \). First, observe that formula (5.1) implies formula (5.2) for \( j = 0 \), since \( |q_0 (x, \xi)| \leq Re^{m|\omega(x, \xi)} \langle (x, \xi) \rangle \geq R \). Now, suppose that (5.2) holds for all \( 0 \leq l < j \) (where \( C_3 > C_2 > C_1 \), and \( C_2, C_3 > 0 \) are large enough). Then, by the definition of \( q_j (x, \xi) \), we have
\[
|D_x^a D_\xi^\beta q_j (x, \xi)| \leq \sum_{0 < |\alpha + \gamma| \leq j} \frac{(-1)^{|\xi|}}{e^{|\gamma|}} (1 - \tau)^{|\gamma|} |D_x^a D_\xi^\alpha q_j - |\epsilon + \gamma| (x, \xi) \rangle |D_x^a D_\xi^\alpha p (x, \xi)|.
\]
We use formula (5.1) for the derivatives of \( q_0 (x, \xi) \), the inductive hypothesis (5.2) for the ones of \( q_j - |\mu| (x, \xi) \), and condition (ii) for the derivatives of \( p(x, \xi) \). All this implies
\[
|D_x^a D_\xi^\beta q_j (x, \xi)| \leq \sum_{0 < |\alpha + \gamma| \leq j} \frac{(-1)^{|\xi|}}{e^{|\gamma|}} (1 - \tau)^{|\gamma|} |D_x^a D_\xi^\alpha q_j - |\epsilon + \gamma| (x, \xi) \rangle |D_x^a D_\xi^\alpha p (x, \xi)|.
\]
Then, as 
\[ \sum_{0 < |x| + |\gamma| \leq j} \frac{1}{e!} |\tau|^{\epsilon} \leq |\tau|^{1 - \gamma} |C_2|^{\alpha_2 + \epsilon + \beta_2 + \gamma} |C_3|^{-1 - \gamma} \]
\[ \times \left( (x, \xi) - \rho((\alpha + \beta) + 2j) \right) e^{m|\omega(x, \xi)} \]
\[ \times C_1^{\alpha_1 + \gamma + \beta_3 + \epsilon} |(x, \xi) - \rho|^{\alpha_1 + \gamma + \beta_3 + \epsilon} e^{\frac{1}{\alpha} \varphi_3^* (n|\beta_3 + \epsilon)} |p(x, \xi)| \]
\[ = \sum_{\alpha_1 + \alpha_2 + \alpha_3 = \alpha \beta_1 + \beta_2 = \beta} \frac{\alpha!}{\alpha_1! \alpha_2! \alpha_3!} \frac{\beta!}{\beta_1! \beta_2! \beta_3!} C_1^{\alpha_1 + \beta_1} \]
\[ \times e^{\frac{1}{\beta} \varphi_3^* (n|\alpha_1)} e^{\frac{1}{2} \varphi_3^* (n|\beta_1)} \sum_{0 < |x| + |\gamma| \leq j} \frac{1}{e!} |\tau|^{\epsilon} \leq |\tau|^{1 - \gamma} \]
\[ \times e^{\frac{1}{\alpha} \varphi_3^* (n|\alpha_2 + \beta_2 + 2j - |\epsilon + \gamma|)} C_2^{\alpha_1 + \gamma + \beta_3 + \epsilon} e^{\frac{1}{\alpha} \varphi_3^* (n|\alpha_3 + \gamma)} |e^{\frac{1}{\alpha} \varphi_3^* (n|\beta_3 + \epsilon)} . \] (5.3)

To estimate the right-hand side of (5.3), we multiply and divide by
\[ (|\alpha_2 + \beta_2| + 2j - |\epsilon + \gamma|)! |\alpha_3 + \gamma||\beta_3 + \epsilon|! \]

Then, as
\[ \frac{\alpha!}{\alpha_1! \alpha_2! \alpha_3!} \frac{\beta!}{\beta_1! \beta_2! \beta_3!} \leq \frac{|\alpha|!}{|\alpha_1|! |\alpha_2|! |\alpha_3|!} \frac{|\beta|!}{|\beta_1|! |\beta_2|! |\beta_3|!} \]

we have, by Lemma 5.1
\[ e^{\frac{1}{\beta} \varphi_3^* (n|\alpha_1)} e^{\frac{1}{2} \varphi_3^* (n|\beta_1)} e^{\frac{1}{\alpha} \varphi_3^* (n|\alpha_2 + \beta_2 + 2j - |\epsilon + \gamma|)} \frac{1}{|\alpha_1|!} \frac{|\beta_1|!}{(|\alpha_2 + \beta_2| + 2j - |\epsilon + \gamma|)!} \frac{1}{|\alpha_3 + \gamma|!} \frac{|\beta_3 + \epsilon|!}{(\beta_1 + \beta_3)!} \]
\[ \leq e^{\frac{1}{\beta} \varphi_3^* (n|\alpha + \beta + 2j|)} \]

Now, we see that
\[ \frac{|\alpha|!}{|\alpha_2|! |\alpha_3|!} \frac{|\beta|!}{|\beta_3|!} \frac{|\alpha_3 + \gamma||\beta_3 + \epsilon|!}{|\alpha_2 + \beta_2| + 2j - |\epsilon + \gamma|)!} \]
\[ \leq 2^{(|\alpha_1 + \alpha_3| + 2|\beta_1 + \beta_3| + |\epsilon + \gamma|)!} . \] (5.4)

Indeed, we multiply and divide by \(|\alpha_1 + \alpha_3| + |\beta_1 + \beta_3| + |\epsilon + \gamma|)!\) to get, by the properties of the multinomial coefficients
\[ \frac{|\alpha|!}{|\alpha_2|! |\alpha_3|!} \frac{|\beta|!}{|\beta_3|!} \frac{|\alpha_3 + \gamma||\beta_3 + \epsilon|!}{(|\alpha_2 + \beta_2| + 2j - |\epsilon + \gamma|)!} \]
\[ \leq 2^{1} \frac{|\alpha|!}{|\alpha_2|! |\alpha_3|!} \frac{|\beta|!}{|\beta_3|!} \frac{|\alpha_3 + \gamma||\beta_3 + \epsilon|!}{(|\alpha_2 + \beta_2| + 2j - |\epsilon + \gamma|)!} \]

As we have, for \(\alpha = \alpha_1 + \alpha_2 + \alpha_3\),
\[ \frac{|\alpha|!}{|\alpha_1|! |\alpha_2|! |\alpha_3|!} = \frac{|\alpha_1 + \alpha_3|!}{|\alpha_1|! |\alpha_3|!} \left( \frac{|\alpha|}{|\alpha_2|} \right) \leq 2^{(|\alpha_1 + \alpha_3|)}, \]

(and in the same way for \(\beta = \beta_1 + \beta_2 + \beta_3\), we deduce formula (5.4) by Lemma 5.2. We then have from (5.3)
\[ |D_x^\alpha D_\xi^\beta q_3(x, \xi)| \leq \sum_{\alpha_1 + \alpha_2 + \alpha_3 = \alpha \beta_1 + \beta_2 + \beta_3 = \beta} \frac{2^{(|\alpha_1 + \alpha_3| + 2|\beta_1 + \beta_3|)C_1^{\alpha_1 + \beta_1}C_2^{\alpha_2 + \beta_2}C_3^{\alpha_3 + \beta_3}|}}{} \]
\[
\frac{1}{\epsilon^{|\gamma|}} |\tau| |\ell^{|\ell^{|\gamma|}} C_2^{\ell^{|\gamma|}} C_3^{\ell^{|\gamma|}} \cdot C_3^{|\ell^{|\gamma|}} C_2^{|\ell^{|\gamma|}}. \]
\]

Since
\[
C_2^{\alpha + \beta} C_3^{\beta} \sum_{\alpha_1 + \alpha_2 + \alpha_3 = \alpha} \left( \frac{2C_1}{C_2} \right)^{\alpha_1 + \beta_1} \left( \frac{2C_1}{C_2} \right)^{\alpha_3 + \beta_3} \]
\[
\leq C_2^{\alpha + \beta} C_3^{\beta} \sum_{\alpha_1 + \alpha_2 + \alpha_3 = \alpha} \left( \frac{2C_1}{C_2} \right)^{\alpha_1 + \alpha_2 + \alpha_3 = \alpha} \]
\[
\leq C_2^{\alpha + \beta} C_3^{\beta} \sum_{k=0}^{\infty} \sum_{|\eta| = k} \left( \frac{2C_1}{C_2} \right)^{k}, \]
we take \( C_2 > 0 \) large enough, so that
\[
\sum_{k=0}^{\infty} \left( \frac{2dCC_1}{C_2} \right)^{k} < 2.
\]
In addition, we put \( C_3 > 0 \) large enough satisfying
\[
\sum_{0 < |\ell| \leq j} \frac{1}{\ell!} \left( \frac{CC_2}{C_3} \right)^{|\ell|} \sum_{0 < |\gamma| \leq j} \frac{1}{\gamma!} \left( \frac{CC_2}{C_3} \right)^{|\gamma|} \]
\[
\leq \left( \sum_{0 < k \leq j} \frac{1}{k!} \left( \frac{d^2CC_2}{C_3} \max \{ |\tau|, 1 - |\tau| \} \right)^{k} \right)^{2} \]
\[
\leq \left( \sum_{\ell=1}^{\infty} \frac{1}{k!} \left( \frac{d^2CC_2}{C_3} \max \{ |\tau|, 1 - |\tau| \} \right)^{k} \right)^{2} < 1/2.
\]
This proves (5.2). Furthermore, by [2, Lemma 2.9(1)], we have that for all \( \ell \in \mathbb{N} \), there exists \( C_\ell > 0 \), such that, for each \( j \),
\[
|D_x^\alpha D_\xi^\beta q_j(x, \xi)| \leq C_\ell C_2^{\alpha + \beta} C_3^{\beta} \langle (x, \xi) \rangle^{-\rho(|\alpha + \beta| + |2j|)} e^{d^2CC_2 \max \{ |\tau|, 1 - |\tau| \}} e^{\gamma|x|},
\]
for all \( \alpha, \beta \in \mathbb{N}^d \) and \( \langle (x, \xi) \rangle \geq R \) and, in particular, the estimate of Definition 3.1 follows.

Now, we extend \( q_j(x, \xi) \) to \( C^\infty (\mathbb{R}^{2d}) \) for each \( j \in \mathbb{N}_0 \). To this aim, we take \( \phi \in \mathcal{D}_\sigma (\mathbb{R}^{2d}) \), supported in \( \langle (x, \xi) \rangle \in \mathbb{R}^{2d} : \langle (x, \xi) \rangle \leq 2R \) and equal to 1 when \( \langle (x, \xi) \rangle \leq R \). Then, we set \( \tilde{q}_j(x, \xi) := q_j(x, \xi)(1 - \phi)(x, \xi) \), which satisfies \( \tilde{q}_j(x, \xi) = q_j(x, \xi) \) if \( \langle (x, \xi) \rangle > 2R \) and vanishes if \( \langle (x, \xi) \rangle \leq R \). It is easy to see that \( 1 - \phi \in \mathcal{G}^{\infty}_p \). Hence, by Lemma 5.3, \( \tilde{q}_j(x, \xi) \in \mathcal{G}S^{\infty}_p \).

We identify \( \tilde{q}_j = q_j \) and we show that \( \sum q_j \circ p \sim 1 \). For \( j > 0 \), by the definition of \( q_j(x, \xi) \), we have
\[
q_j(x, \xi)p(x, \xi) = - \sum_{0 < |\ell| + |\gamma| \leq j} \frac{(-1)^{|\gamma|}}{\ell!|\gamma|!} (1 - \tau)^{|\ell|} (1 - |\tau|)^{|\gamma|} (\partial_\xi^\alpha D_\xi^\beta q_{j-|\ell|+|\gamma|}(x, \xi))(\partial_\xi^\alpha D_\xi^\beta p(x, \xi))
\]
\[
= -r_j(x, \xi) + q_j(x, \xi)p(x, \xi),
\]
where \( \sum r_j := \sum q_j \circ p \) (cf. [2, Proposition 4.13]). Thus, \( r_j(x, \xi) = 0 \) for \( j > 0 \). Also, by the definition of composition, \( r_0(x, \xi) = q_0(x, \xi)p(x, \xi) = 1 \).
if $\langle (x, \xi) \rangle > 2R$, which shows that $\sum q_j \circ p \sim 1$. Since $\sum q_j$ is a formal sum in $\text{FGS}_\rho^{m, \omega}$, by [2, Theorem 4.6], there exists $q(x, \xi) \in \text{GS}_\rho^{m, \omega}$, such that $q \sim \sum q_j$. Finally, [2, Proposition 4.14] yields $q \circ p \sim 1$, and the proof is complete. □

Corollary 5.5. Let $\omega$ be a weight function and let $\sigma$ be a weight function that satisfies $(a_0)$ with $\omega(t^{1/\rho}) = o(\sigma(t))$ as $t \to \infty$. If $p(x, \xi) \in \text{GS}_\rho^{m, \omega}$ satisfies the hypotheses of Theorem 5.4, any quantization of the corresponding pseudodifferential operator $P$ is $\omega$-regular.

Proof. By Theorem 5.4, there is a pseudodifferential operator $Q$, such that $Q \circ P = I + R$, being $I$ the identity operator and $R$ an $\omega$-regularizing operator (as a direct consequence of Theorems 4.2 and 3.11 for $\tau = 0$). Then, $u = Q(Pu) - Ru \in S_\omega(\mathbb{R}^d)$ for any $u \in S'_\omega(\mathbb{R}^d)$ with $Pu \in S_\omega(\mathbb{R}^d)$. The same argument is valid for an arbitrary quantization. □

6. Global $\omega$-hypoellipticity for Mixed Classes

In what follows, $m, m_0 \in \mathbb{R}$, $m_0 \leq m$, $0 < \rho \leq 1$, and for any given weight function $\omega$, $\sigma$ denotes a Gevrey weight function, i.e., $\sigma(t) = t^a$, for some $0 < a < 1$, such that

$\omega(t^{1/\rho}) = o(\sigma(t))$, $t \to \infty$. 

(6.1)

Definition 6.1. Let $a \in \text{GS}_\rho^{m, \omega}$. We say that $a$ is an $\omega$-hypoelliptic symbol in the class $\text{HGS}_\rho^{m, m_0; \omega}$, and we write $a \in \text{HGS}_\rho^{m, m_0; \omega}$, if there exist a Gevrey weight function $\sigma$ satisfying (6.1) and $R \geq 1$, such that

(i) There exist $C_1, C_2 > 0$, such that

$C_1 e^{m_0 \omega(x, \xi)} \leq |a(x, \xi)| \leq C_2 e^{m \omega(x, \xi)}$, $\langle (x, \xi) \rangle \geq R$.

(ii) There exist $C > 0$, $n \in \mathbb{N}$, such that

$|D^\alpha_x D^\beta_\xi a(x, \xi)| \leq C |(x, \xi)|^{\rho |\alpha + \beta|} e^{\frac{1}{n} \varphi_\sigma(n |\alpha|)} e^{\frac{1}{n} \varphi_\sigma(n |\beta|)} |a(x, \xi)|$,

for $\langle (x, \xi) \rangle \geq R$, $\alpha, \beta \in \mathbb{N}_0^d$.

We show in Theorem 6.8 that Definition 6.1 is independent on the quantization $\tau$ for the case $m_0 = m$. Hence, we extend [4, Proposition 8.4], showing that $\omega$-hypoelliptic symbol classes are not perturbed by a change of quantization. We observe that any pseudodifferential operator defined by an $\omega$-hypoelliptic symbol is also $\omega$-regular by Theorem 5.4, but the converse is not true. For instance, the twisted Laplacian in $\mathbb{R}^2$,

$L = \left(D_x - \frac{1}{2} y \right)^2 + \left(D_y - \frac{1}{2} x \right)^2$

is $\omega$-regular for every weight function $\omega$ as it is shown in [5, Example 5.4], but its corresponding symbol is not $\omega$-hypoelliptic for any given weight function $\omega$ by [5, Remark 5.5].
For technical reasons, the class of global symbols for which Theorem 6.8 holds needs to be smaller than the one introduced in Sect. 2. Namely, we need to introduce some kind of mixed conditions. The following is the corresponding definition for symbols:

**Definition 6.2.** We say that \( a \in \widetilde{\text{GS}}^{m,\omega}_\rho \) if \( a \in C^\infty(\mathbb{R}^{2d}) \) and there exists a Gevrey weight function \( \sigma \) satisfying (6.1), such that for all \( \lambda > 0 \), there is \( C_\lambda > 0 \) with

\[
|D_x^\alpha D_y^\beta a(x,\xi)| \leq C_\lambda \langle (x,\xi) \rangle^{-\rho|\alpha+\beta|} e^{\lambda \varphi_*^\beta\rho(\lambda)} e^{m\omega(x,\xi)}, \quad \alpha,\beta \in \mathbb{N}^d, \ x,\xi \in \mathbb{R}^d.
\]

Definitions 6.1 and 6.2 are independent of the weight function \( \sigma \), since given two Gevrey weight functions \( \sigma_1 \) and \( \sigma_2 \) with (6.1), the Gevrey weight function \( \sigma(t) := \min\{\sigma_1(t),\sigma_2(t)\} \), \( t > 1 \), satisfies (6.1) too.

According to condition (6.1), we have, by [2, Lemma 2.9(1)], that for all \( \lambda,\mu > 0 \), there exists \( C > 0 \), such that

\[
\lambda \varphi_*^\sigma\rho\left(\frac{j}{\lambda}\right) \leq C + \mu \rho \varphi_\omega^\rho\left(\frac{j}{\mu}\right), \quad j \in \mathbb{N}_0.
\]  

(6.2)

As an immediate consequence, we have \( \widetilde{\text{GS}}^{m,\omega}_\rho \subseteq \text{GS}^{m,\omega}_\rho \).

**Lemma 6.3.** Let \( a \in \widetilde{\text{GS}}^{m,\omega}_\rho \). Then, \( a \in H\text{GS}^{m,\omega}_\rho \) if and only if there exist \( R \geq 1 \) and \( C' > 0 \), such that \( |a(x,\xi)| \geq C'_1 e^{m\omega(x,\xi)} \) for \( \langle (x,\xi) \rangle \geq R \).

**Proof.** The necessity is obvious. For the sufficiency, since \( a \in \widetilde{\text{GS}}^{m,\omega}_\rho \), for \( \sigma \) as in (6.1), there exists \( C > 0 \) with

\[
|D_x^\alpha D_y^\beta a(x,\xi)| \leq C \langle (x,\xi) \rangle^{-\rho|\alpha+\beta|} e^{\lambda \varphi_*^\beta\rho(\lambda)} e^{m\omega(x,\xi)}, \quad \alpha,\beta \in \mathbb{N}^d, \ x,\xi \in \mathbb{R}^d,
\]

(6.3)

which in particular yields

\[
C'_1 e^{m\omega(x,\xi)} \leq |a(x,\xi)| \leq C e^{m\omega(x,\xi)}, \ \quad \langle (x,\xi) \rangle \geq R.
\]  

(6.4)

This shows Definition 6.1(i). For condition (ii), by (2.4), \( e^{\lambda \varphi_*^\beta\rho(\lambda)} \leq e^{\frac{1}{2} \varphi_*^\beta(2|\beta|)} \). Thus, by (6.3) and (6.4), we have (since \( C_1' \leq C' \))

\[
|D_x^\alpha D_y^\beta a(x,\xi)| \leq \left(\frac{C}{C_1'}\right)^{|\alpha+\beta|} \langle (x,\xi) \rangle^{-\rho|\alpha+\beta|} e^{\frac{1}{2} \varphi_*^\beta(2|\beta|)} |a(x,\xi)|,
\]

for \( \langle (x,\xi) \rangle \geq R, \alpha,\beta \in \mathbb{N}^d \). Since \( a \in \widetilde{\text{GS}}^{m,\omega}_\rho \subseteq \text{GS}^{m,\omega}_\rho \), the result follows.

\( \square \)

Similar mixed conditions are imposed to amplitudes and formal sums.

**Definition 6.4.** An amplitude \( a(x,y,\xi) \in C^\infty(\mathbb{R}^{3d}) \) belongs to \( \widetilde{\text{GA}}^{m,\omega}_\rho \) if there exists a Gevrey weight function \( \sigma \) satisfying (6.1), such that for all \( \lambda > 0 \), there is \( C_\lambda > 0 \) with

\[
|D_x^\alpha D_y^\beta D_y^\gamma a(x,y,\xi)| \leq C_\lambda \langle (x,y,\xi) \rangle^{-\rho|\alpha+\beta+\gamma|} e^{\lambda \varphi_*^\beta\rho(\lambda)} e^{m\omega(x,\xi)},
\]

for all \( \alpha,\beta,\gamma \in \mathbb{N}^d, \ x,y,\xi \in \mathbb{R}^d \).
**Definition 6.5.** A formal sum \( \sum p_j \) is in \( \widetilde{\text{FGS}}_{\rho}^{m,\omega} \) if \( p_j \in C^\infty(\mathbb{R}^{2d}) \) and there exists a Gevrey weight function \( \sigma \) satisfying (6.1) and \( R \geq 1 \), such that for all \( n \in \mathbb{N} \), there exists \( C_n > 0 \), such that

\[
|D_x^a D_\xi^\beta p_j(x, \xi)| \leq C_n \langle (x, \xi) \rangle^{-\rho(|\alpha + \beta| + j)} e^{n\varphi^*_\sigma(\frac{1}{n})} e^{m\omega(x, \xi)},
\]

for each \( j \in \mathbb{N}_0 \), \( \alpha, \beta \in \mathbb{N}_0^d \), \( \log \left( \frac{\langle (x, \xi) \rangle}{R} \right) \geq \frac{n}{2} \varphi^*_\omega(\frac{1}{n}) \).

**Definition 6.6.** We say that \( \sum a_j \sim \sum b_j \) in \( \text{FGS}_{\rho}^{m,\omega} \) if there exist a Gevrey weight function \( \sigma \) satisfying (6.1) and \( R \geq 1 \), such that for all \( n \in \mathbb{N} \), there exist \( C_n > 0 \), \( N_n \in \mathbb{N} \), such that

\[
|D_x^a D_\xi^\beta \sum_{j < N} (a_j - b_j)| \leq C_n \langle (x, \xi) \rangle^{-\rho(|\alpha + \beta| + N)} e^{n\varphi^*_\sigma(\frac{|\alpha + \beta| + N}{n})} e^{m\omega(x, \xi)},
\]

for all \( N \geq N_n \), \( \alpha, \beta \in \mathbb{N}_0^d \), \( \log \left( \frac{\langle (x, \xi) \rangle}{R} \right) \geq \frac{n}{N} \varphi^*_\omega(\frac{N}{n}) \).

Again by (6.2), it is also clear that \( \widetilde{\text{GA}}_\rho^{m,\omega} \subseteq \text{GA}_{\rho}^{m,\omega} \) and \( \widetilde{\text{FGS}}_{\rho}^{m,\omega} \subseteq \text{FGS}_{\rho}^{m,\omega} \).

The amplitudes introduced in Definition 6.4 do not have exponential growth in the variable \( y \) to avoid the increasing in the order \( m \in \mathbb{R} \) in some results in Sect. 3. For instance, if \( a \in \text{GA}_{\rho}^{m,\omega} \), then following Example 3.7

\[
p_j(x, \xi) := \sum_{j=0}^{\infty} \sum_{|\beta + \gamma| = j} \frac{1}{\beta! \gamma!} \tau^{|\beta|} (1 - \tau)^{|\gamma|} \partial_\xi^{\beta + \gamma} (-D_x)^\beta D_\xi^\gamma a(x, y, \xi) \bigg|_{y=x} 
\in \text{FGS}_{\rho}^{m,\omega}.
\]

(6.5)

It is easy to check that \( \varphi_j \) (defined in (3.2)) belongs to \( \text{GS}_{\rho}^{0,\omega} \). Hence, the corresponding symbolic calculus is developed in the same manner as for the global symbol class \( \text{GS}_{\rho}^{m,\omega} \). In particular, by [2, Theorem 4.6], we have from (6.5)

\[
p_{\tau}(x, \xi) := \sum_{j=0}^{\infty} \varphi_j(x, \xi) p_j(x, \xi) \in \text{GS}_{\rho}^{m,\omega}
\]

for all \( \tau \in \mathbb{R} \). Such symbol is called is the \( \tau \)-symbol of the pseudodifferential operator associated with the amplitude \( a(x, y, \xi) \in \text{GA}_{\rho}^{m,\omega} \). In addition, as a consequence of Theorem 3.11, we obtain Theorem 3.12 for mixed classes.

**Theorem 6.7.** Let \( \tau_1, \tau_2 \in \mathbb{R} \). If \( a_{\tau_1}(x, \xi), a_{\tau_2}(x, \xi) \in \text{GS}_{\rho}^{m,\omega} \) are the \( \tau_1 \)-symbol and the \( \tau_2 \)-symbol of the pseudodifferential operator \( A \), then

\[
a_{\tau_2}(x, \xi) \sim \sum_{j=0}^{\infty} \sum_{|\alpha| = j} \frac{1}{\alpha!} (\tau_1 - \tau_2)^{|\alpha|} \partial_\xi^\alpha D_x^\alpha a_{\tau_1}(x, \xi)
\]

in \( \text{FGS}_{\rho}^{m,\omega} \).

Now, we are ready to prove the main theorem of this section.

**Theorem 6.8.** Let \( \tau_1, \tau_2 \in \mathbb{R} \) and let \( a_{\tau_1} \in \text{GS}_{\rho}^{m,\omega} \). If \( a_{\tau_1} \in \text{HGS}_{\rho}^{m,m;\omega} \), then \( a_{\tau_2} \in \text{HGS}_{\rho}^{m,m;\omega} \).
Proof. By \((6.6)\), we have \(a_{\tau_2} \in \widetilde{GS}^{m,\omega}_\rho\). Therefore, by Lemma \(6.3\), it is enough to show that there exist \(R \geq 1, D > 0\), such that
\[
|a_{\tau_2}(x, \xi)| \geq De^{m\omega(x, \xi)} \tag{6.7}
\]
for \(\langle (x, \xi) \rangle \geq R\). In fact, by assumption, by the same result, there are \(R_1 \geq 1, D_1 > 0\), such that
\[
|a_{\tau_1}(x, \xi)| \geq D_1 e^{m\omega(x, \xi)} \tag{6.8}
\]
for \(\langle (x, \xi) \rangle \geq R_1\). By Theorem \(6.7\) and Definition \(6.6\), there exist a Gevrey weight function \(\sigma_1\) satisfying \((6.1)\) and \(R_2 \geq 1\), such that there exist \(C_1 > 0, N_1 \in \mathbb{N}\)
\[
|a_{\tau_2}(x, \xi) - \sum_{j < N} \sum_{|\alpha| = j} \frac{1}{\alpha!}(\tau_1 - \tau_2)^{\alpha} \partial_\xi^\alpha D_x^\alpha a_{\tau_1}(x, \xi)| \leq C_1 (\langle (x, \xi) \rangle)^{-\rho N} e^{\sigma_1(N)} e^{m\omega(x, \xi)}
\]
for \(N \geq N_1\) and \(\log \left(\frac{\langle (x, \xi) \rangle}{R_2}\right) \geq \frac{1}{N} \varphi_\omega^*(N)\). By \((6.2)\), there exists \(A_1 > 0\), such that \(\varphi_\omega^*(N) \leq A_1 + \rho \varphi_\omega^*(N)\) for all \(N \in \mathbb{N}\). Then
\[
|a_{\tau_2}(x, \xi) - \sum_{j < N} \sum_{|\alpha| = j} \frac{1}{\alpha!}(\tau_1 - \tau_2)^{\alpha} \partial_\xi^\alpha D_x^\alpha a_{\tau_1}(x, \xi)| \leq C_1 e^{A_1} R_3^{-\rho N} e^{m\omega(x, \xi)},
\]
\[
\tag{6.9}
\]
for all \(N \geq N_1\) and \(\langle (x, \xi) \rangle \geq R_3 e^{\frac{1}{N} \varphi_\omega^*(N)}\), where \(R_3 \geq R_2\) will be determined later.

We fix \(N = N_1 \in \mathbb{N}\) and we claim that
\[
\left| \sum_{j=0}^{N-1} \sum_{|\alpha| = j} \frac{1}{\alpha!}(\tau_1 - \tau_2)^{\alpha} \partial_\xi^\alpha D_x^\alpha a_{\tau_1}(x, \xi) \right| \geq \frac{D_1}{2} e^{m\omega(x, \xi)}, \tag{6.10}
\]
if \(\langle (x, \xi) \rangle\) is large enough. The inequality is immediate for \(N = 1\) by \((6.8)\) for \(\langle (x, \xi) \rangle \geq R_1\), so we shall assume that \(N > 1\). First, we estimate
\[
\left| \sum_{j=1}^{N-1} \sum_{|\alpha| = j} \frac{1}{\alpha!}(\tau_1 - \tau_2)^{\alpha} \partial_\xi^\alpha D_x^\alpha a_{\tau_1}(x, \xi) \right|.
\]
Since \(a_{\tau_1}(x, \xi) \in \widetilde{GS}^{m,\omega}_\rho\), there exists a Gevrey weight function \(\sigma_2\) satisfying \((6.1)\), such that there is \(C_2 > 0\) with
\[
|D_x^\alpha D_\xi^\alpha a_{\tau_1}(x, \xi)| \leq C_2 \langle (x, \xi) \rangle^{-2\rho} e^{2\varphi_\omega^*(N-1)} e^{m\omega(x, \xi)},
\]
for all \(x, \xi \in \mathbb{R}^d\) and \(1 \leq |\alpha| \leq N - 1\). Again, by \((6.2)\), there exists \(A_2 > 0\), such that \(\varphi_\omega^*(N-1) \leq A_2 + \rho \varphi_\omega^*(N-1)\). Consider \(\langle (x, \xi) \rangle\) large enough, so that
\[
\langle (x, \xi) \rangle \geq R_4 e^{\varphi_\omega^*(N-1)},
\]
with \(R_4 \geq 1\) to be determined. Then
\[
|D_x^\alpha D_\xi^\alpha a_{\tau_1}(x, \xi)| \leq C_2 e^{2A_2} \langle (x, \xi) \rangle^{-2\rho} e^{2\varphi_\omega^*(N-1)} e^{m\omega(x, \xi)} \leq C_2 e^{2A_2} (R_4)^{-2\rho} e^{m\omega(x, \xi)},
\]
for $\langle (x, \xi) \rangle \geq R_4 e^{\varphi_\omega(N-1)}$, $1 \leq |\alpha| \leq N - 1$. On the other hand, by formula [23, (0.3.1)], we obtain

$$\sum_{j=1}^{N-1} \sum_{|\alpha|=j} \frac{|\tau_1 - \tau_2|^{|\alpha|}}{\alpha!} \leq \sum_{j=1}^{N-1} \frac{(d|\tau_1 - \tau_2|)^j}{j!} \leq e^{d|\tau_1 - \tau_2|}.$$

Therefore, we deduce

$$\left| \sum_{j=1}^{N-1} \sum_{|\alpha|=j} \frac{1}{\alpha!} (\tau_1 - \tau_2)^{|\alpha|} \partial^\alpha_x D^\alpha_x a_{\tau_1}(x, \xi) \right| \leq C_2 e^{2A_2(R_4)^{-2}} e^{d|\tau_1 - \tau_2|} e^{m\omega(x, \xi)},$$

(6.11)

for $\langle (x, \xi) \rangle \geq R_4 e^{\varphi_\omega(N-1)}$. Hence, by the triangular inequality, from formulas (6.11) and (6.8), we have

$$\left| \sum_{j=0}^{N-1} \sum_{|\alpha|=j} \frac{1}{\alpha!} (\tau_1 - \tau_2)^{|\alpha|} \partial^\alpha_x D^\alpha_x a_{\tau_1}(x, \xi) \right| \geq D_1 e^{m\omega(x, \xi)} - C_2 e^{2A_2(R_4)^{-2}} e^{d|\tau_1 - \tau_2|} e^{m\omega(x, \xi)}$$

$$\geq \frac{D_1}{2} e^{m\omega(x, \xi)},$$

which shows (6.10) provided $R_4$ be so that

$$(R_4)^{2\rho} \geq \frac{2}{D_1} C_2 e^{2A_2} e^{d|\tau_1 - \tau_2|},$$

and $\langle (x, \xi) \rangle \geq \max\{R_1, R_4 e^{\varphi_\omega(N-1)}\}$. Finally, we obtain by (6.10) and (6.9)

$$|a_{\tau_2}(x, \xi)| \geq \frac{D_1}{2} e^{m\omega(x, \xi)} - C_1 e^{A_1 R_3^{-\rho N}} e^{m\omega(x, \xi)} \geq \frac{D_1}{4} e^{m\omega(x, \xi)}$$

if $R_3^{\rho N} \geq \frac{4}{D_1} C_1 e^{A_1}$ and $\langle (x, \xi) \rangle \geq R := \max\{R_1, R_4 e^{\varphi_\omega(N-1)}, R_3 e^{\frac{1}{\lambda} \varphi_\omega(N)}\}$. Then, (6.7) is satisfied for $D = \frac{D_1}{4} > 0$ and $R \geq 1$, and the proof is complete. \hfill \square

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