ERGODIC BOUNDARY AND POINT CONTROL FOR LINEAR STOCHASTIC PDES DRIVEN BY A CYLINDRICAL LÉVY PROCESS

KAREL KADLEC AND BOHDAN MASŁOWSKI

Charles University, Prague
Faculty of Mathematics and Physics
Czech Republic

Abstract. An ergodic control problem is studied for controlled linear stochastic equations driven by cylindrical Lévy noise with unbounded control operator in a Hilbert space. A family of optimal controls is shown to consist of those asymptotically achieving the feedback form that employs the corresponding Riccati equation. The formula for optimal cost is given. The general results are applied to stochastic heat equation with boundary control and to stochastic structurally damped plate equations with point control.

1. Introduction. For controlled deterministic and stochastic linear, distributed parameter systems, an important family of controls are those acting on the boundary or at specific points of the domain. The deterministic theory has been developed by numerous authors, notably by I. Lasiecka and R. Triggiani (cf. the monograph [21] and the references therein). The stochastic counterpart has been studied especially in the case when the random perturbations take the form of space-dependent Gaussian noise that is white in time and, possibly, correlated in the space variable. The first results of this kind have appeared in [5] and [7] employing also some ideas from [13] where an analogous problem for distributed control and quadratic cost has been studied. In these papers also adaptive ergodic control for parameter-dependent systems was considered and the usual self-tuning and self-optimizing properties of the adaptive control law have been shown. Some of these results were extended to semilinear controlled stochastic evolution equations (typically of reaction-diffusion type), cf. [13], [8], [14] and [15]. There seems to exist only few works on controlled stochastic infinite dimensional evolution systems for the driving noise being not Gaussian and white in time. In [9] and [10], the finite and infinite time horizon control problems for linear SPDEs driven by fractional Brownian motion are considered, in [12] the LQ problem for SPDEs with bilinear Gauss-Volterra noise is investigated. Finally, Lévy-driven controlled equations have been studied in [19], where pathwise optimization for linear-quadratic ergodic control is shown in the case of exponentially stable uncontrolled systems and square-integrable noise processes.

In the present paper, the controlled evolution equation in a Hilbert space $H$,

$$
dX^U(t) = (AX^U(t) + BU(t))dt + dL(t), \quad X^U(0) = x, \quad (1)
$$

2020 Mathematics Subject Classification. 60H15, 93E20.
Key words and phrases. Lévy-driven SPDEs, Ergodic control, Controlled Lévy driven equations.

This research was supported by GAČR grant no. 19-07140S.
is studied where \( A \) is a linear, densely defined, operator on \( H \) generating an analytic semigroup, \( U \) is an integrable and progressively measurable control taking values in (possibly, different) Hilbert space. The operator \( B \) may be unbounded, which makes it possible to apply the general result to parabolic-like systems with boundary or point control in the setting analogous to [21] or [5]. The driving process \( L \) is a cylindrical Lévy process on \( H \) which was studied, for example, in [28] and [27], and the cost functional is quadratic, defined as a average mean value cost in long run. Under appropriate stability and detectability conditions the optimal family of controls has been found, which are controls asymptotically achieving the feedback form that employs the corresponding Riccati equation with unbounded coefficients. The general results are applied to the stochastic heat equation with control on the boundary and to point control of some stochastic plate equations.

The paper is divided into five Sections. Section 2 is preliminary, the control problem is rigorously described, the general hypotheses are formulated and some basic facts from theory of cylindrical Lévy processes are recalled. In Section 3 an Itô type formula for quadratic functionals is proved that is applicable in our case. The main difficulty comes from the fact that the driving process is just cylindrical and both the drift term and the control operator are unbounded hence there is no strong solution in such case and the standard Itô formula cannot be used directly. Section 4 contains the main result of the paper. Optimal controls are found in a standard class of stabilizing controls and the formula for optimal cost is given. Section 5 contains three examples: The stochastic heat equation and two examples of stochastic plate equations with structural damping and stochastic Kelvin-Voigt equation.

2. Preliminaries. Let \( \mathbb{H} = (\mathbb{H}, | \cdot |_\mathbb{H}) \) and \( \mathbb{Y} = (\mathbb{Y}, | \cdot |_\mathbb{Y}) \) be real separable Hilbert spaces. We define cylindrical subsets of \( \mathbb{H} \) as sets of the form

\[
C(h_1, \ldots, h_n) := \{ h \in \mathbb{H} : \langle h, h_1 \rangle_\mathbb{H}, \ldots, \langle h, h_n \rangle_\mathbb{H} \in B \},
\]

where \( \{ h_1, \ldots, h_n \} \subset \mathbb{H}, B \in \mathcal{B}(\mathbb{R}^n) \), \( n \in \mathbb{N} \). For arbitrary \( V \subset \mathbb{H} \),

\[
C(V) = \{ C(V, B) : B \in \mathcal{B}(\mathbb{R}^n) \}
\]

and we denote \( C(V) = \sigma(C(V)) \). A function \( \nu : C(\mathbb{H}) \to \mathbb{R} \) is a cylindrical measure on \( C(\mathbb{H}) \) if it is a measure on \( C(K) \) for each \( K \) finite subset of \( \mathbb{H} \). Let \( K \) be a finite subset of \( \mathbb{H} \) and let \( f : (\mathbb{H}, C(K)) \to (\mathbb{C}, \mathcal{B}(\mathbb{C})) \). The integral

\[
\int_{\mathbb{H}} f(z) d\nu(z)
\]

is well defined as a complex valued Lebesgue integral if it exists (\( \nu \) is a measure on \( C(K) \)). For more details cf. [28].

Let \((\Omega, \mathcal{A}, \mathcal{F}, P)\) be a complete filtered probability space. We define a cylindrical random variable \( Z \) on \((\Omega, \mathcal{A}, P)\) in \( \mathbb{H} \) as a linear and continuous map \( \mathbb{H} \to \mathbb{L}^0(\Omega, \mathcal{A}, P) \) and set \( Z(u) = (Z, u) \). A cylindrical process \( L \) on \((\Omega, \mathcal{A}, P)\) in \( \mathbb{H} \) is a family \((L(t), t \geq 0)\) of cylindrical random variables on \((\Omega, \mathcal{A}, P)\) in \( \mathbb{H} \). The characteristic function of cylindrical random variable \( Z \) is defined as

\[
\phi_Z(h) = \mathbb{E}e^{i\langle Z, h \rangle_{\mathbb{H}}}, \quad h \in \mathbb{H}.
\]

A cylindrical process \( L \) on \((\Omega, \mathcal{A}, \mathcal{F}, P)\) in \( \mathbb{H} \) is a cylindrical Lévy process if the stochastic process

\[
((\langle L(t), h_1 \rangle, \ldots, \langle L(t), h_n \rangle), \quad t \geq 0)
\]
is a Lévy process on $(\Omega, A, \mathcal{F}, P)$ in $\mathbb{R}^n$ for each $\{h_1, \ldots, h_n\} \subset \mathbb{H}$, $n \in \mathbb{N}$. We have

$$\phi_L(t) = e^{(ip(h)-\frac{1}{2}q(h))t + \int_{S\mathbb{F}} e^{t(z,h)\mathbb{F}} - 1 - i(z,h)\mathbb{I}_{B_\mathbb{F}}(z,h)\mathbb{F} dz},$$

where $p : \mathbb{H} \to \mathbb{R}$ is a continuous mapping, $q : \mathbb{H} \to \mathbb{R}$ is a quadratic form and $\nu$ is a cylindrical measure on $\mathcal{C}(\mathbb{H})$ such that $\nu \circ \langle h, \cdot \rangle_{\mathbb{H}}^{-1}$ is Lévy measure on $\mathcal{B}(\mathbb{R})$ of the scalar Lévy process $\langle L, h \rangle = (\langle L(t), h \rangle, t \geq 0)$ for each $h \in \mathbb{H}$. The cylindrical measure $\nu$ is called cylindrical Lévy measure of $L$ and $(p, q, \nu)$ are called cylindrical characteristics of $L$.

The cylindrical process $L$ has weak second moments if $\langle L, h \rangle = (\langle L(t), h \rangle, t \geq 0)$ has finite second moments for all $h \in \mathbb{H}$. The cylindrical process $L$ is a cylindrical martingale if $\langle L, h \rangle$ is a martingale for all $h \in \mathbb{H}$.

We are concerned with the controlled stochastic evolution equation

$$dX^U(t) = (AX^U(t) + BU(t)) dt + \Phi dL(t), \quad X^U(0) = x,$$

(2)

where $x \in \mathbb{H}$, $A$ is the infinitesimal generator of an analytic semigroup $S$ on $\mathbb{H}$. For $\beta > 0$ large enough the operator $-A + \beta \mathbb{I}$ is strictly positive (in the sequel, $\beta > 0$ is fixed). For $\alpha > 0$ we denote by $\mathbb{D}^\alpha_A$ the domain of the fractional power $(-A + \beta \mathbb{I})^\alpha$ equipped with the graph norm $|y|_{\mathbb{D}^\alpha_A} = |(-A + \beta \mathbb{I})^\alpha x|_{\mathbb{H}}$ and by $\mathbb{D}^\alpha_{\mathbb{H}}$, the domain of fractional power $(-A^* + \beta \mathbb{I})^\alpha$ equipped with the graph norm $|y|_{\mathbb{D}^\alpha_{\mathbb{H}}} = |(-A^* + \beta \mathbb{I})^\alpha x|_{\mathbb{H}}$.

We assume:

(A1) $\Phi \in \mathcal{L}(\mathbb{H})$ and there exists $\delta \in (0, \frac{1}{2})$ such that $\Phi^*(-A^* + \beta \mathbb{I})^{-\frac{1}{2} + \delta}$ is Hilbert-Schmidt.

(A2) $B : \mathbb{D}(B) \subset \mathcal{Y} \to \mathbb{D}(A^*)'$, the dual of $\mathbb{D}(A^*)$ with respect to the topology of $\mathcal{Y}$, and $(\beta \mathbb{I} - A)^{\epsilon - 1} B \in \mathcal{L}(\mathcal{Y}, \mathbb{H})$, for a given $\epsilon \in (0,1]$.

(A3) We have that

$$U \in L^{p,loc}_{\mathcal{F}}(\mathbb{R}^+, \mathcal{Y}),$$

where $L^{p,loc}_{\mathcal{F}}(\mathbb{R}^+, \mathcal{Y})$ denotes the space of all $\mathcal{F}$-progressively measurable processes from $L^{p,loc}(\mathbb{R}^+, \mathcal{Y})$, where

$$L^{p,loc}(\mathbb{R}^+, \mathcal{Y}) = \left\{ Y : \mathbb{R}^+ \times \mathcal{X} \to \mathcal{Y} \text{ measurable; } \forall t > 0 : E \int_0^t |Y(s)|^p ds < \infty \right\}$$

and $p > \max\{2, \frac{1}{\alpha} \}$ is given. $U$ has the meaning of control process and $\mathcal{U} = L^{p,loc}_{\mathcal{F}}(\mathbb{R}^+, \mathcal{Y})$ is the space of admissible controls.

(A4) $L$ is a cylindrical Lévy process on $(\Omega, A, \mathcal{F}, P)$ in $\mathbb{H}$ with weak second moments and cylindrical characteristics $(p, 0, \nu)$ where

$$p(h) = -\int_{\mathbb{H}} \langle z, h \rangle_{\mathbb{H}} \mathbb{I}_{B_\mathbb{H}}(\langle z, h \rangle_{\mathbb{H}}) \nu(dz),$$

and $B_\mathbb{H}$ denotes the complement of the set $B_{\mathbb{R}}$ in the set $\mathbb{R}$. Note that the integral exists as $L$ has weak second moments (and therefore weak first moments). The characteristic function of $L$ is

$$E e^{i\langle L(t), h \rangle} = e^{\frac{1}{2} \int_0^t E (e^{i\langle z, h \rangle_{\mathbb{H}} - 1 - i\langle z, h \rangle_{\mathbb{H}}} \nu(dz)), \quad h \in \mathbb{H}.}$$

It means that $L$ has zero Gaussian part and its Lévy measure $\nu$ is cylindrical. $L$ is a cylindrical martingale as $\langle L, h \rangle$ is centered for all $h \in \mathbb{H}$ and we can
write for all $h \in \mathbb{H}$

$$\langle L(t), h \rangle = \int_0^t \int_{\mathbb{R} \setminus \{0\}} u\tilde{N}_h(ds, du), \ t \in \mathbb{R}_+,$$

where $\tilde{N}_h$ is compensated Poisson measure of $\langle L(t), h \rangle$ with Lévy measure $\nu \circ \langle u, \cdot \rangle^{-1}$ (cf. [28]).

For each Hilbert-Schmidt operator $\phi$ on $\mathbb{H}$ and each $t \geq 0$ there exists a square integrable random variable $L_{\phi}$ such that for all $h \in \mathbb{H}$

$$\langle L(t), \phi^* h \rangle = \langle L_{\phi}(t), h \rangle_{\mathbb{H}}.$$

The process $L_{\phi} = (L_{\phi}(t), \ t \geq 0)$ is a càdlàg square integrable martingale ([28], Corollary 4.4). For simplicity we put $\phi L = L_{\phi}$. We are using characteristic function of $\phi L$ for each $h \in \mathbb{H}$ and $t \geq 0$:

$$E e^{i\langle \phi L(t), h \rangle_{\mathbb{H}}} = E e^{i\langle \phi^* L(t), h \rangle} = e^{\int_0^t (e^{i\langle z, \phi^* h \rangle_{\mathbb{H}} - 1 - i\langle z, \phi^* h \rangle_{\mathbb{H}}})d\nu(z)}$$

$$= e^{\int_0^t (e^{i\langle z, h \rangle_{\mathbb{H}} - 1 - i\langle z, h \rangle_{\mathbb{H}}})d(\nu \circ \phi^{-1})(z)}$$

(the integral of cylindrical measurable functions with respect to $\nu$ is the standard Lebesgue integral). It means that $\phi L$ is Lévy process in $\mathbb{H}$ without Gaussian part and with Lévy measure $\nu_{\phi} = \nu \circ \phi^{-1}$ ($\nu \circ \phi^{-1}$ is indeed a Radon measure with strong second moments, cf. [28], p. 15). We can write

$$L_{\phi}(t) = \int_0^t \int_{\mathbb{H}} z\tilde{N}_{\phi}(ds, dz), \ t \in \mathbb{R}_+,$$

where $\tilde{N}_{\phi}$ is a compensated Poisson measure of $L_{\phi}$. For each $T : \mathbb{R}_+ \to L(\mathbb{H})$ and $t \geq 0$ we have that

$$\int_0^t T(s)\phi dL(s) = \int_0^t T(s)d\phi L(s),$$

where the first integral is defined as in [28] and the second integral is defined as in [27]. This can be shown by the standard approximation approach as both integrals are defined as limits in $L^2$ of the same approximating random sequence.

The mild solution $X \in L^{p,loc}_{\mathcal{F}}(\mathbb{R}_+, \mathbb{H})$ of the stochastic evolution equation (2) is defined by the formula

$$X^U(t) = S(t)x + \int_0^t S(t-s)BU(s)ds + \int_0^t S(t-s)\Phi dL(s), \ t \geq 0, \quad (3)$$

where the stochastic integral $\int_0^t S(t-s)\Phi dL(s), \ t \geq 0$, is defined as in [28]. By (A2) we may write

$$\int_0^t S(t-s)\Phi dL(s) = \int_0^t S(t-s)(-A^* + \beta I)^{\frac{1}{2}-\delta}d(-A^* + \beta I)^{-\frac{1}{2}+\delta}\Phi L(s), \ t \geq 0,$$

cf. [27]. We can easily see that both integrals in (3) exist. In fact, we have the following.

**Lemma 2.1.** Let

$$Z(t) = \int_0^t S(t-r)\Phi dL(r),$$

$$\tilde{Z}(t) = \int_0^t S(t-r)BU(r)dr,$$
for some
for each
t that

\[ t \geq 0. \] Then \( Z, \hat{Z} \in L^{p, \text{loc}}_{\mathbb{F}}(\mathbb{R}_+, \mathbb{H}) \) and, consequently,

\[ X^U \in L^{p, \text{loc}}_{\mathbb{F}}(\mathbb{R}_+, \mathbb{H}). \]

**Proof.** The fact that \( \hat{Z}(\cdot) = \int_0^t S(\cdot - r)BU(r)dr \in L^{p, \text{loc}}_{\mathbb{F}}(\mathbb{R}_+, \mathbb{H}) \) has been proved in [5], Lemma 2.1.

Let \( t \in \mathbb{R}_+ \) and define

\[ Y(s) = \int_0^s S(t - r)\Phi dL(r), \quad s \in [0, t]. \]

The process \( (Y(s), s \in [0, t]) \) is an \( \mathbb{H} \)-valued martingale, hence

\[ \mathbf{E}[|Z(t)|^p] = \mathbf{E}[|Y(t)|^p] \leq c_1 \mathbf{E}(Y)^{\frac{p}{2}} \]

for some \( c_1 > 0 \) by the Burkholder inequality (cf. [27] Theorem 6.13). It follows that

\[ \mathbf{E}[|Z(t)|^p] \leq c_1 \left( \int_0^t \left| S(t - r)(-A + \beta I)^{\frac{1}{2} - \delta}(-A + \beta I)^{-\frac{1}{2} + \delta \phi}_H \right|^2 dr \right)^{\frac{p}{2}} \]

\[ \leq c_2 \left( \int_0^t \left| S(t - r)(-A^* + \beta I)^{\frac{1}{2} - \delta}I_{\mathbb{L}(\mathbb{H})} \right|^2 dr \right)^{\frac{p}{2}} \left| (-A^* + \beta I)^{-\frac{1}{2} + \delta \phi}_H \right|^2 \]

\[ \leq c_3 \left( \int_0^t \frac{1}{(t - r)^{1-2\delta}} dr \right)^{\frac{p}{2}} \]

for each \( t \in [0, T] \) and therefore we have \( c_5 > 0 \) such that

\[ \int_0^T \mathbf{E}[|Z(t)|^p] dt \leq c_3 \int_0^T \left( \int_0^t \frac{1}{(t - r)^{1-2\delta}} dr \right)^{\frac{p}{2}} dt \leq c_5 < \infty. \]

\[ \square \]

The weak solution of (2) is defined by the equation

\[ \langle a, X^U(t) \rangle = \langle a, x \rangle_{\mathbb{H}} + \int_0^t \langle A^* a, X^U(s) \rangle_{\mathbb{H}} ds + \int_0^t \langle B^* a, U(r) \rangle_{\mathbb{Y}} dr + \langle L(t), \Phi^* a \rangle, \]

\( a \in \mathcal{D}(A^*), \ t \in \mathbb{R}_+ \). Using standard arguments we obtain the following statement.

**Proposition 1.** \( X^U \) is the mild solution of (2) if and only if \( X^U \) is the weak solution of (2).

The proof is based on the Yosida approximation of the equation (2) (cf. (12) below) and Theorem 9.15 in [27] where the equivalence is proved for standard \( \mathbb{H} \)-valued Lévy processes.

Now we introduce the control problem studied in the paper. Set

\[ J(U, t) = \int_0^T \left( \langle QX^U(s), X^U(s) \rangle_{\mathbb{H}} + \langle RU(s), U(s) \rangle_{\mathbb{Y}} \right) ds, \quad (4) \]

where \( T > 0, \ X^U \) is the solution of (2), \( Q \in \mathcal{L}(\mathbb{H}) \) is a symmetric positive semi-definite operator and \( R \in \mathcal{L}(\mathbb{Y}) \) is a symmetric positive definite operator, i.e.

\[ \langle Ry, y \rangle_{\mathbb{Y}} \geq r|y|^2, \quad y \in \mathbb{Y}, \]

holds for a constant \( r > 0 \).
Ergodic cost functional is defined as the “mean average cost per time unit in long run”, that is
\[ \tilde{J}(U) = \lim_{t \to \infty} \inf \frac{E J(U,t)}{t}, \quad U \in \mathcal{U}. \] (6)

To solve the ergodic control problem in a class of controls \( U_0 \subset \mathcal{U} \) means to find \( V_0 \in \mathbb{R} \) and \( U_0 \in U_0 \) such that
\[ \tilde{J}(U) \geq V_0, \quad U \in U_0, \] (7)
and
\[ \lim_{t \to \infty} \frac{E J(U_0,t)}{t} = V_0. \] (8)

Then \( U_0 \) and \( V_0 \) are called the optimal control and the optimal cost, respectively.

Consider the stationary Riccati equation
\[ \langle a, A^* V b \rangle_{\mathbb{H}} + \langle A^* V a, b \rangle_{\mathbb{H}} + \langle Qa, b \rangle_{\mathbb{H}} - \langle R^{-1} B^* V a, B^* V b \rangle_{\mathbb{H}} = 0, \] (9)
\( a, b \in \mathcal{D}(A^*) \).

As usual, we impose the standard stabilizability and detectability conditions on the triple of operators \((A,B,Q)\). Assume there exist \( H \in L(\mathbb{H}, \mathbb{Y}) \) and \( \bar{H} \in L(\mathbb{Y}, \mathbb{H}) \) such that the semigroups generated by the operators
\( A + BH \) (for stabilizability) and
\( A + \bar{H} \sqrt{Q} \) (for detectability) are exponentially stable. Then, as we can see in [30]:
1. The equation (9) has a unique solution in the class of non-negative and self-adjoint linear operators on \( \mathbb{H} \) and, moreover, \( V \in \mathcal{L}(\mathbb{H}, \mathcal{D}_{A}^{1-\epsilon}) \).
2. The semigroup \( S_V \) generated by \( AV = A - BR^{-1}B^* V \) is exponentially stable, more specifically, there exist constants \( M_0 > 0, \omega > 0 \), such that
\[ |S_V(t)|_{\mathcal{L}(\mathbb{H})} \leq M_0 e^{-\omega t}, \] (10)
for each \( t \geq 0 \).

By (9) there exists \( h(\cdot) \), a continuous extension of \( \langle A^* V \cdot, \cdot \rangle_{\mathbb{H}} \) on \( \mathbb{H} \) such that for some \( c \in \mathbb{R}_+ \)
\[ |h(y)| \leq c|y|_{\mathbb{H}}^2 \] (11)
holds for all \( y \in \mathbb{H} \).

3. The Itô formula. The standard Itô formula (see e.g. [16], [24], Theorem D.2 in [27]) may be used for the processes which are strong solutions of Lévy-driven SEE. In the paper, however, the noise in SEE is only cylindrical Lévy process and the operators \( A \) and \( B \) are not bounded on the state space. Therefore the strong solutions do not exist. In this section, we prove a modification of Itô formula which is satisfactory to establish the main result of this paper, more specifically, is applicable to weak/mild solutions and functionals of the form \( \langle \cdot, V \cdot \rangle_{\mathbb{H}} \) where \( V \) is a self-adjoint, nonnegative operator on \( \mathbb{H} \), taking values in the space \( \mathcal{D}_{A}^{1-\epsilon} \). The method of Yosida approximations which we adopt here may be applied to much wider class of functionals (cf. [5] in Gaussian case) under appropriate set of conditions. For brevity, we do not consider the general case here.

The proof is based on approximation of SEE (2) by SEEs which have strong solutions and which fulfill the assumptions of the standard Itô formula. To this end we define \( R(\lambda) = 1 + \lambda R(\lambda, A) = \lambda (\lambda I - A)^{-1} \) for \( \lambda > \beta \). Due to our assumptions, \( R(\lambda)^{\frac{1}{2}}(\lambda - \Phi) \) is Hilbert-Schmidt, therefore \( R(\lambda) \Phi \) is Hilbert-Schmidt and
$L_\lambda(\cdot) = R(\lambda)\Phi L(\cdot)$ has the nuclear incremental covariance $R(\lambda)\Phi \Phi^*(R(\lambda))^*$. Our approximate SEE takes the form:

$$dX_\lambda^U(t) = (AX_\lambda^U(t) + R(\lambda)BU(t))dt + R(\lambda)dL_\lambda(t), \quad X(0) = x_\lambda,$$

(12)

where $x_\lambda = R(\lambda)x$ and $\lambda > \beta$. By Theorem 9.24 in [27] there exist càdlàg versions of $X_\lambda, \lambda > \beta$, which we consider in the sequel.

We aim at verifying the formula:

$$E\langle X_U(t), VX_U(t) \rangle_H - \langle x, Vx \rangle_H$$

$$= 2E \int_0^t h(X_U(s))ds + E \int_0^t 2\langle B^*VX_U(s), U(s) \rangle_\gamma ds + t\Pi$$

for all $t \in [0, T]$, where we assume:

$$\lim_{\lambda \to \infty} Tr(VR^2(\lambda)\Phi \Phi^*(R^2(\lambda))^*) = \Pi < \infty.$$  

(13)

Here $Tr$ denotes the operator trace and $h : H \to \mathbb{R}$ satisfies (11) and extends the function $\langle A^*V, \cdot \rangle_H$.

The proof has four steps:

1. Lemma 3.1 allows us to approximate the solution of the equation (2) by solutions of the equations (12).
2. Lemma 3.2 states the existence of the strong solution of the equation (12).
3. In Lemma 3.3, we apply the standard Itô formula to the equation (12).
4. In Lemma 3.4, we apply limit theorems to the formula proved in the Lemma 3.3.

**Lemma 3.1.** 1. There exists a sequence $\{\lambda_n\}_{n \in \mathbb{N}}$, such that $\lambda_n \to \infty$, $n \to \infty$, and for all $t \in [0, T]$ $\lim_{n \to \infty} X_{\lambda_n}^U(t) = X_U(t)$ a.s.,

2. for each $T > 0$ there exists a constant $C > 0$ such that

$$\sup_{n \in \mathbb{N}} \sup_{t \in [0, T]} E|X_{\lambda_n}(t)|^p \leq C,$$

(14)

where $p$ was introduced in (A3).

**Proof.** 1. Since $R(\lambda) \to I$ strongly as $\lambda \to \infty$, we have that $x_\lambda \to x$ and there exists a $c > 0$ such that $|R(\lambda)|_{\mathcal{L}(H)} \leq c$, $\lambda > 0$. As in Lemma 3.2. in [5] we obtain

$$\lim_{\lambda \to \infty} \left| \int_0^t S(t-s)R(\lambda)BU(s)ds - \int_0^t S(t-s)BU(s)ds \right|_H = 0, \quad t \in [0, T],$$

and

$$\sup_{\lambda > \beta, t \in [0, T]} \left| \int_0^t S(t-s)R(\lambda)BU(s)ds \right|_H < \infty \quad a.s.$$  

By Theorem 3.50 in [27] we have for fixed $t > 0$:

$$E \left| \int_0^t S(t-r)\Phi dl(r) - \int_0^t S(t-r)R(\lambda)dL_\lambda(r) \right|_H^2$$

$$= E \left| \int_0^t S(t-r) (I - R^2(\lambda)) \Phi dl(r) \right|_H^2$$
\[ \leq c_1 \int_0^t |S(t - r) (I - R^2(\lambda)) \Phi|_{HS}^2 dr \]
\[ = c_1 \int_0^t |S(t - r)(-A + \beta I)^{\frac{1}{2}}(-A + \beta I)^{\frac{1}{2}} (I - R^2(\lambda)) \Phi|_{HS}^2 dr \]
\[ \leq c_2 \int_0^t |S(t - r)(-A + \beta I)^{\frac{1}{2}} (I - R^2(\lambda)) \Phi|_{LS}^2 dr \]
\[ \leq c_3 \int_0^t \frac{1}{(t - r)^{1-\sigma}} dr \left| (-A + \beta I)^{\frac{1}{2}} (I - R^2(\lambda)) \Phi \right|_{HS}^2 \]
\[ = c_4 \text{Tr} \left( (-A + \beta I)^{\frac{1}{2}} (I - R^2(\lambda)) \Phi \Phi^* (I - R^2(\lambda))^* \right), \]
for some constants \( c_1, c_2, c_3, c_4 > 0 \). This converges to 0 for \( \lambda \to \infty \) (cf. Lemma 3.2. in [5]). Therefore we have a sequence \( \{\lambda_n\}_{n \in \mathbb{N}} \) such that \( \lambda_n \to \infty, n \to \infty \), and
\[ \lim_{n \to \infty} \int_0^t S(t - r) R(\lambda_n) dL_{\lambda_n}(r) = \int_0^t S(t - r) dL(r), \ a.s. \]

2. For \( t \in [0, T] \) we have that
\[ \mathbf{E} \left( \left| \int_0^t S(t - s) R(\lambda_n) B U(s) ds \right|^p \right) \leq c \mathbf{E} \left( \left| \int_0^t S(t - s)(\beta I - A)^{1-\sigma} R(\lambda_n) (\beta I - A)^{\frac{1}{2}} B |\mathcal{L}(\mathbb{H})| U(s) |\mathcal{L}(\mathbb{H})| ds \right|^p \right) \]
\[ \leq \mathbf{E} \left( \int_0^t \frac{1}{(t - s)^{1-\sigma}} |U(s)|_{\mathcal{L}(\mathbb{H})} |\mathcal{L}(\mathbb{H})| (\beta I - A)^{\frac{1}{2}} B |\mathcal{L}(\mathbb{H})| U(s) |\mathcal{L}(\mathbb{H})| ds \right)^p \leq c \left( \int_0^T \frac{1}{s^{1-\sigma}} ds \right)^{p-1} \int_0^T \mathbf{E} |U(s)|_{\mathcal{L}(\mathbb{H})}^p ds \leq c \]
for a universal constant \( c > 0 \) (which may be different from line to line). Here we used the well-known fact that the operators \( R(\lambda_n) \) are uniformly bounded in \( \mathcal{L}(\mathbb{H}) \). Also, similarly as in the proof of lemma 2.1, we obtain
\[ \mathbf{E} \left( \left| \int_0^t S(t - s) R(\lambda_n) dL_{\lambda_n}(s) \right|^p \right)_{\mathcal{H}} \]
\[ \leq c \left( \int_0^t \left| S(t - s)(\beta I - A)^{\frac{1}{2}} R^2(\lambda_n)(\beta I - A)^{\frac{1}{2}} \Phi \right|_{HS}^2 ds \right)^\frac{p}{2} \]
\[ \leq c \left| (\beta I - A)^{\frac{1}{2}} \Phi \right|_{HS}^p |R^2(\lambda_n)|_{\mathcal{L}(\mathbb{H})}^\frac{p}{2} \left( \frac{1}{(t - s)^{1-2\sigma}} \right)^\frac{p}{2} \leq c \]
for a universal constant \( c > 0 \) independent of \( n \in \mathbb{N} \) and \( t \in [0, T] \) which completes the proof of (14).

\[ \square \]

The steps of the proof of the following lemma are the same as the steps of the proof in Lemma 3.1 in [5]. Therefore we skip the proof in this paper.

**Lemma 3.2.** For \( \lambda > \beta \) (12) has the unique strong solution.

Now we can apply the standard Itô formula to the strong solutions of (12). We use this fact in the following lemma.
Lemma 3.3. Assume (A1)-(A4), \( V \in L(\mathbb{H}, \mathbb{D}^{1,-}_A) \) is non-negative and self-adjoint operator on \( \mathbb{H} \). Then

\[
E \langle X^U_\lambda(t), VX^U_\lambda(t) \rangle_{\mathbb{H}} - \langle x_\lambda, Vx_\lambda \rangle_{\mathbb{H}} = E \int_0^t 2 \langle VX^U_\lambda(s), AX^U_\lambda(s) \rangle_{\mathbb{H}} ds \\
+ E \int_0^t 2 \langle B^* R^*(\lambda) VX^U_\lambda(s), U(s) \rangle_{\mathbb{H}} ds + t Tr(V R^2(\lambda) \Phi^* (R^2(\lambda))^*)
\]

Proof. \( X_\lambda \) is the strong solution of (12), therefore using the Theorem D.2 in [27] we obtain

\[
\langle X^U_\lambda(t), VX^U_\lambda(t) \rangle_{\mathbb{H}} - \langle x_\lambda, Vx_\lambda \rangle_{\mathbb{H}} = \int_0^t 2 \langle VX^U_\lambda(s), dX^U_\lambda(s) \rangle_{\mathbb{H}} \\
+ \sum_{s \leq t} \left( \langle X^U_\lambda(s), VX^U_\lambda(s) \rangle_{\mathbb{H}} - \langle X^U_\lambda(s-), VX^U_\lambda(s-) \rangle_{\mathbb{H}} - 2 \langle VX^U_\lambda(s-), \Delta X^U_\lambda(s) \rangle_{\mathbb{H}} \right)
\]

\[
= \int_0^t 2 \langle VX^U_\lambda(s), AX^U_\lambda(s) \rangle_{\mathbb{H}} ds + \int_0^t 2 \langle VX^U_\lambda(s-), R^2(\lambda) \Phi dL(s) \rangle_{\mathbb{H}} \\
+ \int_0^t 2 \langle VX^U_\lambda(s-), R(\lambda) B U(s) \rangle_{\mathbb{H}} ds + \sum_{s \leq t} \langle \Delta X^U_\lambda(s), V \Delta X^U_\lambda(s) \rangle_{\mathbb{H}}
\]

\[
= \int_0^t 2 \langle VX^U_\lambda(s), AX^U_\lambda(s) \rangle_{\mathbb{H}} ds + \int_0^t 2 \langle \Phi^* (R^2(\lambda))^* VX^U_\lambda(s-), dL(s) \rangle_{\mathbb{H}} \\
+ \int_0^t 2 \langle B^* R^*(\lambda) VX^U_\lambda(s-), U(s) \rangle_{\mathbb{H}} ds + \sum_{s \leq t} \left| \Delta V^{\frac{1}{2}} R^2(\lambda) \Phi L(s) \right|^2_{\mathbb{H}}.
\]

where \( \Delta X^U_\lambda(s) = X^U_\lambda(s) - X^U_\lambda(s-) \) (note that the symbols \( \sum_{s \leq t} \) in the formula make sense because the summands are different from zero only in a countable number of points) and

\[
\Delta V^{\frac{1}{2}} R^2(\lambda) \Phi L(s) = V^{\frac{1}{2}} R^2(\lambda) \Phi L(s) - V^{\frac{1}{2}} R^2(\lambda) \Phi L(s-)
\]

\[
= V^{\frac{1}{2}} (R^2(\lambda) \Phi L(s) - R^2(\lambda) \Phi L(s-)) = V^{\frac{1}{2}} \Delta R^2(\lambda) \Phi L(s).
\]

Note that \( V^{\frac{1}{2}} R^2(\lambda) \Phi L(s) \) is a square integrable Lévy processes. As

\[
E \left| \int_0^t 2 \langle \Phi^* (R^2(\lambda))^* VX^U_\lambda(s-), dL(s) \rangle_{\mathbb{H}} \right| < \infty,
\]

we obtain

\[
E \int_0^t 2 \langle \Phi^* (R^2(\lambda))^* VX^U_\lambda(s-), dL(s) \rangle_{\mathbb{H}} = 0.
\]

We can see that

\[
E \sum_{s \leq t} \left| \Delta V^{\frac{1}{2}} R^2(\lambda) \Phi L(s) \right|_{\mathbb{H}}^2 = t Tr(V R^2(\lambda) \Phi^* (R^2(\lambda))^*)
\]

and the result follows. \( \Box \)

In the following Lemma, we apply this result to the weak/mild solution of (2). Here, we use Lemma 3.1.
Lemma 3.4. Assume (A1)-(A4), \( V \in \mathcal{L}(\mathbb{H}, \mathbb{D}^{1,-}_{A^-}) \) is non-negative and self-adjoint on \( \mathbb{H} \) and let (13) hold. Assume further that there exists a continuous function \( h : \mathbb{H} \rightarrow \mathbb{R} \) satisfying (11) extending the function \( \langle A^* V, \cdot \rangle_{\mathbb{H}} \). Then for all \( t \in [0,T] \) we have

\[
E\langle X^U(t), VX^U(t) \rangle_{\mathbb{H}} - \langle x, Vx \rangle_{\mathbb{H}} = 2E \int_0^t h(X^U(s)) ds + E \int_0^t 2 \langle B^* VX^U(s), U(s) \rangle_{\mathbb{H}} ds + tH.
\]

Proof. By Lemma 3.3, we have:

\[
E\langle X^U(t), VX^U(t) \rangle_{\mathbb{H}} - \langle x, Vx \rangle_{\mathbb{H}} = E \int_0^t 2 \langle VX^U(s), AX^U(s) \rangle_{\mathbb{H}} ds
\]

\[+ E \int_0^t 2 \langle B^* R^*(\lambda)VX^U(s), U(s) \rangle_{\mathbb{H}} ds + tTr(V R^2(\lambda) \Phi \Phi^*(R^2(\lambda))^*) \quad (15)
\]

Lemma 3.1 yields a sequence \( \{\lambda_n\}_{n \in \mathbb{N}} \), such that \( \lambda_n \rightarrow \infty, n \rightarrow \infty \), and for all \( t \in [0,T] \) a.s.

\[
\lim_{n \rightarrow \infty} \langle X^U(t), VX^U(t) \rangle_{\mathbb{H}} = \langle X^U(t), VX^U(t) \rangle_{\mathbb{H}}
\]

and

\[
\lim_{n \rightarrow \infty} h(X^U(t)) = h(X^U(t)).
\]

Also,

\[
\left|\langle B^* R^*(\lambda)VX^U(s), U(s) \rangle_{\mathbb{H}} - \langle B^* VX^U(s), U(s) \rangle_{\mathbb{H}}\right|
\]

\[
\leq \left|\langle B^* R^*(\lambda)VX^U(s), U(s) \rangle_{\mathbb{H}} - \langle B^* VX^U(s), U(s) \rangle_{\mathbb{H}}\right|
\]

\[+ \left|\langle B^* R^*(\lambda)VX^U(s), U(s) \rangle_{\mathbb{H}} - \langle B^* R^*(\lambda)VX^U(s), U(s) \rangle_{\mathbb{H}}\right|
\]

\[= \left|\langle B^* (R^*(\lambda) - I)VX^U(s), U(s) \rangle_{\mathbb{H}} + \langle B^* R^*(\lambda)(VX^U(s) - U^U(s)), U(s) \rangle_{\mathbb{H}}\right|
\]

for \( s \in [0,T] \). As

\[
\lim_{\lambda \rightarrow \infty} |(R^*(\lambda) - I)VX^U(s)|_{\mathbb{D}^{1,-}_{A^-}} = 0,
\]

and \( B^* \in \mathcal{L}(\mathbb{D}^{1,-}_{A^-}, \mathbb{H}) \), we have

\[
\lim_{\lambda \rightarrow \infty} \left|\langle B^* (R^*(\lambda) - I)VX^U(s), U(s) \rangle_{\mathbb{H}}\right|
\]

\[\leq \lim_{\lambda \rightarrow \infty} |B^* (R^*(\lambda) - I)VX(s)|_{\mathbb{H}}|U(s)|_{\mathbb{H}} = 0.
\]

It is easy to see that for each \( y \in \mathbb{H} \)

\[
\lim_{z \rightarrow y} |\langle B^* R^*(\lambda)V(z - y), U(s) \rangle_{\mathbb{H}}| = 0
\]

uniformly with respect to \( \lambda > \beta \). Therefore we obtain

\[
\lim_{n \rightarrow \infty} |\langle B^* R^*(\lambda_n)V(X^U_{\lambda_n}(s) - U^U(s)), U(s) \rangle_{\mathbb{H}}| = 0, \quad \text{a.s.}
\]

Furthermore, taking into account (11) and since \( p > 0 \) we may use Lemma 3.1 (2) to conclude that the sequences of functions

\[
t \mapsto \langle X^U_{\lambda_n}(t), VX^U_{\lambda_n}(t) \rangle_{\mathbb{H}}, \quad s \mapsto \langle X^U_{\lambda_n}(s), AX^U_{\lambda_n}(s) \rangle_{\mathbb{H}}
\]

are uniformly integrable on \( \Omega \) and \((0,t) \times \Omega, \) respectively. Let us show that the family

\[
s \mapsto 2 \langle B^* R^*(\lambda_n)VX^U_{\lambda_n}(s), U(s) \rangle_{\mathbb{H}}
\]

is uniformly integrable on \((0,t) \times \Omega\) as well. We have that

\[
|\langle B^* R^*(\lambda_n)VX^U_{\lambda_n}(s), U(s) \rangle_{\mathbb{H}}|
\]
Let $\text{Proposition 2.}$ of the preceding Lemma is satisfied.

4. **Main results.**

The optimal cost is $p$ with $\Pi = \mathcal{P}(\mathbb{H}, \mathbb{D}_{A^*}^{-\delta})$. Then (13) is fulfilled, where $\xi q$ is the feedback form $U_t$. The detectability conditions, imposed in Section 2, are satisfied. Consider the control in the strong operator topology and converges to $\xi q$. The main result is stated below.

**Proposition 2.** Let $V \in \mathcal{L}(\mathbb{H}, \mathbb{D}_{A^*}^{-\delta})$ be non-negative and self-adjoint on $\mathbb{H}$ and let one of the following conditions be satisfied.

1. $\Phi$ is Hilbert-Schmidt,
2. $V$ is nuclear,
3. $V \in \mathcal{L}(\mathbb{D}_{A^*}^{\delta - \frac{1}{2}}, \mathbb{D}_{A^*}^{-\delta})$.

Then (13) is fulfilled, where $\Pi = \text{Tr}(V \Phi \Phi^*)$ in case of (1) and (2) and $\Pi = \text{Tr}((R^*(\beta))^{\frac{1}{4} - \frac{1}{2}} V \Phi \Phi^*(R^*(\beta))^{\frac{1}{4} - \delta})$ in case of (3).

**Proof.** This Proposition is a simple consequence of Proposition 3.4 in [5].

4. **Main results.** In this section, we give the main result of this paper. Throughout this section we assume that (A1)-(A4), (13), and the stabilizability and detectability conditions, imposed in Section 2, are satisfied. Consider the control in the feedback form $U_0(t) = K(t)X(t)$, $t > 0$, where $K : \mathbb{R} \rightarrow \mathcal{L}(\mathbb{H}, \mathbb{Y})$ is continuous in the strong operator topology and converges to $K_0 = - R^{-1} B^* V$ in $\mathcal{L}(\mathbb{H}, \mathbb{Y})$ as $t \rightarrow \infty$. The main result is stated below.

**Theorem 4.1.** Assume (A1)-(A4), (13), and let $(A, B, Q)$ be stabilizable and detectable. Then the feedback control $U_0(t)$, $t \geq 0$, is an optimal control for the ergodic problem (7) - (8) in the class $\mathcal{U}$ of all controls from the space $U$ satisfying

$$
\mathbb{E} \langle VX^U(t), X^U(t) \rangle_{\mathbb{H}} \rightarrow 0, \quad t \rightarrow \infty.
$$

The optimal cost is $V_0 = \Pi$.

The proof of Theorem 4.1 consists of three steps:

1. We have that

$$
\lim_{t \rightarrow \infty} \inf_{U} \frac{\mathbb{E}J(U, t)}{t} \geq \Pi
$$

for all controls $U$ from the space $\mathcal{U}$ satisfying (16).

2. The feedback control $U_0(t), t \geq 0$, satisfies (16).

3. The feedback control $U_0(t), t \geq 0$, is an optimal control for the ergodic problem (7) - (8), that is

$$
\lim_{t \rightarrow \infty} \frac{\mathbb{E}J(U_0, t)}{t} = \Pi.
$$
We prove these three facts in three separate lemmas.

**Lemma 4.2.** Let $V$ be the solution of the Riccati equation (9). Then (17) holds true.

**Proof.** Using Lemma 3.4 we obtain

\[
E \left\langle X U(t), V X U(t) \right\rangle_H - \langle x, V x \rangle_H = 2E \int_0^t h(X U(s))ds + E \int_0^t 2 \langle B^* V X U(s), U(s) \rangle_Y ds + t \Pi.
\]

We have, using (9),

\[
E \left\langle X U(s), V X U(s) \right\rangle_H - \langle x, V x \rangle_H = -E \int_0^t \left( \langle X U(s), Q X U(s) \rangle_H + \langle U(s), RU(s) \rangle_Y \right) ds + \Pi
\]

\[
+ E \int_0^t \langle U(s), RU(s) \rangle_Y ds
\]

\[
\frac{d}{dt} \left( \langle X U(s), V U(s) \rangle_H - \langle x, V x \rangle_H \right)
\]

\[
= -E \frac{d}{dt} \left( \langle X U(s), Q X U(s) \rangle_H + \langle U(s), RU(s) \rangle_Y \right) ds + \Pi
\]

\[
+ E \int_0^t \langle R^{-1} B^* V X U(s), U(s) \rangle_Y ds.
\]

As

\[
E \int_0^t \langle R^{-1} B^* V X U(s), U(s) \rangle_Y ds \geq 0, \ t \geq 0,
\]

we obtain (17) by (16). \qed

As the next step, we prove in the following lemma that the feedback control $U_0(t)$ satisfies (16), that is $U_0$ is in the set of controls where we want to find the optimal one.

**Lemma 4.3.** We have that

\[
E \left\langle X^{U_0}(t), V X^{U_0}(t) \right\rangle_H \to 0, \ t \to \infty.
\]

**Proof.** Recall (10) and let $\epsilon^* > 0$ be such that

\[
\theta = \theta^* = (3\epsilon^* \Gamma(\epsilon))^{\frac{1}{2}} < \omega,
\]

where $\Gamma$ stands for the Euler Gamma-function. As $K(t)$ converges to $K_0$ in the uniform operator topology as $t \to \infty$, we can find $r > 0$ such that for all $s \geq r$:

\[
|K(s) - K_0|^2_{L(H, Y)} < \epsilon^*.
\]

Using Proposition 1, we can see that $X^{U_0}$ is the mild solution of

\[
dX^{U_0} = (AX^{U_0} + BU_0) dt + \Phi dL(t), \ X^{U_0}(0) = x,
\]

iff $X^{U_0}$ satisfies the mild formula

\[
X^{U_0}(t) = SV(t - r)X^{U_0}(r) + \int_r^t SV(t - s)B(K(t) - K_0)X^{U_0}(s)ds
\]

\[
+ \int_r^t SV(t - s)\Phi dL(s), \ t \geq r.
\]
Hence we obtain

\[ E \left\langle X^{U_0}(t), VX^{U_0}(t) \right\rangle_{\mathbb{H}} \leq E|X^{U_0}(t)|^2_{\mathbb{H}} \]

\[ \leq 3|V|^2_{\mathbb{H}}|S_V(t-r)|^2_{\mathbb{H}} E|X^{U_0}(r)|^2_{\mathbb{H}} \]

\[ + 3|V|^2_{\mathbb{H}} \left( \int_r^t |S_V(t-s)B|_{\mathcal{L}(\mathbb{H},\mathbb{H})} |K(s) - K_0|_{\mathcal{L}(\mathbb{H},\mathbb{H})} |X^{U_0}(s)|_{\mathbb{H}} ds \right)^2 \]

\[ + 3E \left| \int_r^t S_V(t-s)\Phi dL(s) \right|^2_{\mathbb{H}} \]

\[ \leq 3M_0 e^{-\omega(t-r)} E|X^{U_0}(r)|^2_{\mathbb{H}} + 3 \int_r^t \frac{e^{-\omega(t-s)}}{(t-s)^{1-\epsilon}} |K(s) - K_0|^2_{\mathcal{L}(\mathbb{H})} E|X(s)|^2_{\mathbb{H}} ds + c_1 \]

\[ \leq c_2 + 3e^* \int_r^t \frac{e^{-\omega(t-s)}}{(t-s)^{1-\epsilon}} E|X^{U_0}(s)|^2_{\mathbb{H}} ds + c_1 \]

for some constants \(c_1\) and \(c_2\) dependent only on \(r\).

Following the same steps of as in the proof of Lemma 7.1.1. in [18] (the general-ized Gronwall Lemma), we obtain:

\[ E|X^{U_0}(t)|^2_{\mathbb{H}} \leq c_2 + c_3 \int_r^t E'\left(\theta(t-s)\right)e^{-\omega(t-s)} ds + c_1, \]

for a constant \(c_3\), where \(E'(z)\) is asymptotically

\[ \frac{\epsilon^{-1}}{\Gamma(\epsilon)}, \quad z \to 0_+, \]

\[ \frac{e^z}{\epsilon}, \quad z \to \infty. \]

Therefore we can find \(t_0 > r\) such that for all \(t > t_0\):

\[ E|X^{U_0}(t)|^2_{\mathbb{H}} \leq c_4 \int_r^t \frac{e^{(\theta-\omega)(t-s)}}{(t-s)^{1-\epsilon}} ds + c_5 < c_6 \]

for some constants \(c_4, c_5\) and \(c_6\) dependent only on \(r\). It follows that

\[ \frac{E \left\langle X^{U_0}(t), VX^{U_0}(t) \right\rangle_{\mathbb{H}}}{t} \to 0, \quad t \to \infty, \]

which concludes the proof. \(\square\)

Thus we know that \(U_0\) is admissible control for the ergodic problem (7) (8). It remains to show that \(U_0\) is the optimal one.

**Lemma 4.4.** We have that

\[ \lim_{t \to \infty} \frac{EJ(U_0, t)}{t} = \Pi. \]  

\(23\)

*Proof.* Recall that \(U_0(t) = K(t)X(t)\) where \(K(t) \to K_0\) in \(\mathcal{L}(\mathbb{H}, \mathbb{Y})\). Following the steps of the proof of Lemma 4.2, we obtain

\[ E \left\langle X^{U_0}(s), VX^{U_0}(s) \right\rangle_{\mathbb{H}} \leq \langle x, Vx \rangle_{\mathbb{H}} \]

\[ = -E \left\langle \frac{J(U_0, t)}{t} + \Pi \right\rangle + E \int_0^t \left\langle -K_0X^{U_0}(s) + K(s)X^{U_0}(s), R(-K_0X^{U_0}(s) + K(s)X^{U_0}(s)) \right\rangle_{\mathbb{H}} ds \].
It is easy to see that
\[
\lim_{t \to \infty} \frac{\mathbb{E} \int_0^t \langle -K_0X^{U_0}(s) + K(s)X^{U_0}(s), R(-K_0X^{U_0}(s) + K(s)X^{U_0}(s)) \rangle \, ds}{t} = \lim_{t \to \infty} c_1 \frac{\mathbb{E} \int_0^t |K_0 - K(s)|_{L(\mathbb{H}, \mathbb{Y})} |X^{U_0}(s)|^2 \, ds}{t} = 0.
\]
Hence in virtue of Lemma 4.3 we obtain (23). \qed

5. Examples.

5.1. Example. (Boundary control for stochastic heat equation). The stochastic heat equation formally written as
\[
v_t(t, x) = \Delta v(t, x) + l(t, x), \quad (t, x) \in \mathbb{R}_+ \times G, \tag{24}
v_{\nu}(t, x) + h(x)v(t, x) = u(t, x), \quad (t, x) \in \mathbb{R}_+ \times \partial G,
v(0, x) = v_0(x), \quad x \in G,
\]
is considered where \(v_{\nu}(t, x)\) is normal derivative of \(v\) in \((t, x) \in \mathbb{R}_+ \times \partial G\) in direction \(\nu\), the outward normal to \(\partial G\), \(h \in \mathcal{C}^\infty(\partial G)\), \(h \geq 0\) and \(l\) formally stands for Lévy noise on \(L^2(\partial G)\).

The above system is rewritten in the form (2) with \(\mathbb{H} = L^2(G), \mathbb{Y} = L^2(\partial G), A = \Delta\) on the domain \(D(A) = \{ f \in H^2(G) : f_{\nu}(x) + h(x)f(x) = 0, \ x \in \partial G \}\).

The operator \(A\) generates an analytic semigroup and there exists \(\beta \geq 0\) such that \((A - \beta I)\) is strictly negative. To define the operator \(B\) we follow the lines of the standard approach developed for deterministic equations in [2], [21] (see also [5], [9] for various modifications in stochastic cases). Consider the elliptic problem
\[
\Delta z - \beta z = 0
\]
on \(G\) and
\[
z_{\nu} + hz = -g
\]
on \(\partial G\). The solution map \(M : g \mapsto -z\) is an element of \(\mathcal{L}(L^2(\partial G), \mathcal{D}_A^-)\) for \(\epsilon < \frac{3}{4}\).

The operator \(B\) is then defined as the composition \(B = AM\), where \(\hat{A}\) is the isomorphic extension of the operator \(A\) into \(\mathcal{D}_A^-\). Therefore, \(B \in \mathcal{L}^\epsilon(\mathcal{Y}, \mathcal{D}_A^-)\), which verifies the condition (A2). Finally, to verify (A1) with \(\delta = \frac{1}{2}\), the diffusion term \(\Phi^\epsilon\) may be arbitrary Hilbert-Schmidt operator on \(\mathbb{H}\). On the other hand, if the space dimension \(n = 1\), (A1) holds true for any \(\Phi \in \mathcal{L}(\mathbb{H})\) with \(\delta < \frac{1}{4}\).

5.2. Example. (Point control of stochastic plate equation with structural damping). Consider the equation
\[
p_{tt}(t, x) - \Delta p_t(t, x) + \Delta^2 p(t, x) = \mathbb{I}_{x = x_0}u(t) + l(t, x), \quad (t, x) \in \mathbb{R}_+ \times G, \tag{25}
p(0, x) = p_0, \quad p_t(0, x) = p_1, \quad x \in G,
p(t, x) = p_l(t, x) = 0, \quad (t, x) \in \mathbb{R}_+ \times \partial G,
\]
where \(G \subset \mathbb{R}^n\) is an open and bounded domain with a sufficiently smooth boundary \(\partial G\), \(n \leq 3\), \(x_0 \in G\), \(l\) formally stands for a (space-dependent) Lévy noise. Define the cost functional
\[
J(u, T) = \int_0^T \left( |p(t)|_{H^2(G)}^2 + |p_t(t)|_{H^1(G)}^2 + |u(t)|^2 \right) \, dt,
\]
where \(H^2\) denotes the Sobolev space \(\{ y \in L^2(G) : D^\alpha y \in L^2(G), |\alpha| \leq 2 \\}\).
The deterministic case has been studied, in [3] and [30]. Let $\mathbb{A} = \Delta^2$ on the domain
\[ \mathcal{D}(\mathbb{A}) = \{ h \in H^4(G) : h(x) = \Delta h(x) = 0, \ x \in \partial G \} . \]

The equation (25) is rewritten in the form (2),
\[ dX^U(t) = (AX^U(t) + BU(t)) dt + \Phi dL(t), \ X^U(0) = x. \]

We set
\[ H = \mathcal{D}(\mathbb{A}^{1/2}) \times L^2(G) = (H^2(G) \cap H_0^1(G)) \times L^2(G), \ \ Y = \mathbb{R}, \]
\[ A = \begin{pmatrix} 0 & I \\ -\mathbb{A} & \mathbb{A}^{1/2} \end{pmatrix}, \]
\[ Bu = \begin{pmatrix} 0 \\ I_x = x_0 u \end{pmatrix}, \]
\[ \Phi = \begin{pmatrix} 0 & 0 \\ 0 & \Phi_1 \end{pmatrix}, \]
\[ R = Q = I, \]
where $\Phi_1 \in \mathcal{L}(L^2(G))$ is Hilbert-Schmidt, $X_0 = (v_0, v_1)^T \in H$ and $L$ is a cylindrical Lévy process on $\mathbb{H}$.

The operator $A$ generates an exponentially stable analytic semigroup (cf. [3]). In [30] it is shown that $B \in \mathcal{L}(Y, \mathbb{D}_{\mathbb{A}}^{-1})$ for $\epsilon \in (\frac{1}{2} - \frac{n}{4})$. In this case all conditions imposed in the paper are satisfied and the optimal ergodic control may be obtained by means of Theorem 4.1.

5.3. Example. (Point control of the stochastic Kelvin-Voigt plate equation). Consider the problem
\[ w_{tt}(t, x) + \Delta^2 w(t, x) + \rho \Delta^2 w(t, x) = I_{x=x_0} u(t) + l(t, x), \quad (t, x) \in \mathbb{R}_+ \times G, \]
\[ w(0, x) = w_0, \ w_t(0, x) = w_1, \ x \in G, \]
\[ \Delta w(t, x) + (1 - \mu)B_1 w(t, x) = \frac{\partial \Delta w(t, x)}{\partial \nu} + (1 - \mu)B_1 w(t, x) = 0, \quad (t, x) \in \mathbb{R}_+ \times \partial G, \]
where $\mu \in (0, \frac{1}{2})$, $\rho > 0$, $x_0 \in G \subset \mathbb{R}^n$, $n \leq 2$. Furthermore, the boundary operators $B_1$ and $B_2$ are given by
1. For $n = 1$: $B_1 = B_2 = 0$,
2. For $n = 2$:
\[ B_1 w = 2\nu_1 \nu_2 w_{x,y} - \nu_1^2 w_{y,y} - \nu_2^2 w_{x,x}, \]
\[ B_2 w = \frac{\partial}{\partial \nu}(\nu_1^2 - \nu_2^2) w_{x,y} + \nu_1 \nu_2 (w_{y,y} - w_{x,x}), \]
where $\frac{\partial}{\partial \nu}$ is tangential derivative.

As in the previous example, $l$ formally represents a (space-dependent) Lévy noise. Define the cost functional
\[ J(u, T) = \int_0^T (|w(t)|_{H^2(G)}^2 + |w_t(t)|_{L^2(G)}^2 + |u(t)|^2) dt. \]

The deterministic case ($l \equiv 0$) is analyzed in [30]. Let $\mathbb{A} = \Delta^2$ on the domain
\[ \mathcal{D}(\mathbb{A}) = \{ h \in H^4(G) : h(x) + (1 - \mu)B_1 h(x) \}
\[ = \frac{\partial \Delta h(x)}{\partial \nu} + (1 - \mu)B_1 h(x) = 0, \ x \in \partial G \} . \]
We rewrite the equation (26) in the form (2),

\[ dX^U(t) = \left( AX^U(t) + BU(t) \right) dt + \Phi dL(t), \quad X^U(0) = x. \]

Here

\[ H = D(\mathcal{A}^{1/2}) \times L^2(G) = (H^2(G) \cap H^1_0(G)) \times L^2(G), \quad Y = \mathbb{R}, \]

\[ A = \begin{pmatrix} 0 & 1 \\ -\mathcal{A} & -\nu \mathcal{A} \end{pmatrix}, \]

the operators \( B, \Phi, R \) and \( Q \) take the same form as in the previous example, \( X_0 = (w_0, w_1)^T \in \mathbb{H} \) and \( L \) is a cylindrical Lévy process on \( \mathbb{H} \).

In [30] it is shown that \( A \) generates an exponentially stable analytic semigroup and \( B \in L(Y, D_{\mathcal{A}^{-1}}) \) for \( \epsilon = \frac{1}{2} \). Hence all conditions imposed in the paper are satisfied.

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Received July 2019; revised December 2019.

E-mail address: maslov@karlin.mff.cuni.cz
E-mail address: kadlec@karlin.mff.cuni.cz