THE INVERSE AND THE COMPOSITION IN THE SET OF FORMAL LAURENT SERIES

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Abstract. The aim of this article is to investigate the issues of multiplicative inverses and composition in the set of formal Laurent series. We show the lack of uniqueness of inverse series to formal Laurent series. Some necessary and sufficient conditions for the existence of inverse series to a certain type formal Laurent series are provided. Moreover, we define a general composition of formal Laurent series and investigate the Right Distributive Law and the Chain Rule for formal Laurent series. Finally, we provide a sufficient condition for the boundary convergence of formal Laurent series.

1. Introduction

Let $S$ be a ring, let $\mathbb{N}$ denote the set of all positive integers and let $l \in \mathbb{N}$. A formal power series on $S$ is defined as a mapping $f : (\mathbb{N} \cup \{0\})^l \mapsto S$. If $l = 1$, we denote $f(z) = \sum_{n=0}^{\infty} a_n z^n$, where $a_n \in S$ for every $n \in \mathbb{N} \cup \{0\}$. This definition was introduced in [19]. Denote by $\mathcal{X}(S)$ the set of all formal power series over $S$. The main operations defined on $\mathcal{X}(S)$ seem to be the multiplication and the composition. The necessary and sufficient condition for the existence of the general composition of formal power series was provided in [15]. Let us add that the proof of the main result of [15] is quite long and it is split into a few Lemmas. The reader can find a much simpler proof of that result in the paper [2].

Let us emphasize that formal power series have found applications in many areas such as algebra (see e.g. [1,3,5]), differential equations (see e.g. [6,20]) or combinatorics (see e.g. [19]).

Formal Laurent series are a natural extension of formal power series. However, one should point out that their properties are quite different from the analogous properties of formal power series (see e.g. [14]). For example, in general, the product of two given formal Laurent series does not have to exist and the inverse in the set of formal Laurent series does not have to be unique, unlike to formal power series (see [12], Thm. 1.1.9.). Let us add that the composition of formal Laurent series with formal power series was considered e.g. in the paper [14].

In this paper we are going mainly to focus on the inverse and the composition in the set of formal
Laurent series. First, we establish that every formal Laurent series has either no inverse series, a unique inverse series or uncountably many inverse series. We show the connection between this fact and the lack of associativity of multiplication in the set of formal Laurent series (see Prop. 3.5). Next, we provide some necessary and sufficient conditions for a certain type formal Laurent series to have no inverse or to have one, or to have infinitely many inverses. For that purpose we use the theory of infinite systems of linear algebraic equations, investigated for example in \cite{7,8,17}. We also define a general composition of formal Laurent series and use it to examine the Right Distributive Law and the Chain Rule, which hold if one considers formal power series \cite{10}. In the last section of this paper we examine the boundary convergence of formal Laurent series. We extend some results concerning this problem proved in \cite{13}.

2. Preliminaries

In this section we are going to collect some basic definitions and results which will be needed in the sequel. Let us begin with the following

Remark 2.1 (\cite{18}). For a sequence \((a_n)_{n \in \mathbb{Z}}\) of complex numbers, we define the infinite sum \(\sum_{n = -\infty}^{\infty} a_n\) (or, equivalently, \(\sum_{n = -\infty}^{\infty} a_n\)) as

\[
\sum_{n = -\infty}^{\infty} a_n := \sum_{n = 1}^{\infty} a_{-n} + \sum_{n = 0}^{\infty} a_n
\]

provided both series on the right side are convergent. In other cases we say that the sum on the left side is divergent.

Now, we collect some basic notions related to formal Laurent series.

Definition 2.2 (\cite{14}). A formal Laurent series on \(\mathbb{C}\) is defined as a mapping \(g : \mathbb{Z} \mapsto \mathbb{C}\). We denote \(g(z) = \sum_{n \in \mathbb{Z}} b_n z^n\), where \(b_n \in \mathbb{C}\) for every \(n \in \mathbb{Z}\). We denote by \(\mathbb{L}(\mathbb{C})\), or simply by \(\mathbb{L}\), the set of formal Laurent series over \(\mathbb{C}\). The zero formal Laurent series is defined as \(S_0 := \sum_{n \in \mathbb{Z}} a_n z^n\), where \(a_n = 0\) for every \(n \in \mathbb{Z}\). The unit formal Laurent series is defined as \(S_1 := \sum_{n \in \mathbb{Z}} a_n z^n\), where \(a_0 = 1\), \(a_n = 0\) for \(n \neq 0\). The series \(g^+ := \sum_{n=0}^{\infty} b_n z^n\) and \(g^- := \sum_{n=1}^{\infty} b_{-n} z^{-n}\) are called the regular and the principal part of \(g\), respectively. We also denote \(\hat{g} = \sum_{n \in \mathbb{Z}} b_{-n} z^n\) for \(g = \sum_{n \in \mathbb{Z}} b_n z^n\)

We define the addition and the scalar multiplication of formal Laurent series by formulas

\[
\begin{align*}
(1) \quad (f + g)(z) &= \sum_{n \in \mathbb{Z}} (b_n + a_n) z^n, \\
(2) \quad (cg)(z) &= \sum_{n \in \mathbb{Z}} c b_n z^n,
\end{align*}
\]

where \(g(z) = \sum_{n \in \mathbb{Z}} b_n z^n\), \(f(z) = \sum_{n \in \mathbb{Z}} a_n z^n \in \mathbb{L}\) and \(c \in \mathbb{C}\).
Theorem 2.6 \([15]\). Let \( g(z) = \sum_{n \in \mathbb{Z}} b_n z^n \), \( f(z) = \sum_{n \in \mathbb{Z}} a_n z^n \in \mathbb{L} \). We define the product of \( f \) and \( g \) as
\[
(fg)(z) := \sum_{k \in \mathbb{Z}} d_k z^k,
\]
where \( d_k = \sum_{m \in \mathbb{Z}} b_m a_{k-m} \) for every \( k \in \mathbb{Z} \), provided \( \sum_{m \in \mathbb{Z}} b_m a_{k-m} \) is convergent for every \( k \in \mathbb{Z} \). Otherwise we say that the product of \( f \) and \( g \) does not exist. We define \( \mathbb{L}(g) := \{ f \in \mathbb{L} : fg \text{ exists} \} \) for every \( g \in \mathbb{L} \).

Some basic properties of formal Laurent series are collected in

Proposition 2.4 \([14]\). Let \( f, g, h \in \mathbb{L} \). Then
\begin{enumerate}
\item \( f \in \mathbb{L}(g) \) if and only if \( g \in \mathbb{L}(f) \),
\item \( \mathbb{L}(f) \neq \emptyset \),
\item \( f \in \mathbb{L}(g) \Rightarrow \alpha f \in \mathbb{L}(g) \) for every \( \alpha \in \mathbb{C} \),
\item \( f, h \in \mathbb{L}(g) \Rightarrow f + h \in \mathbb{L}(g) \).
\end{enumerate}

It is also easy to check that the product of formal Laurent series is commutative if it exists.

Finally, we recall basic definitions and facts concerning the composition of formal series.

Definition 2.5 \([15]\). Let \( S \) be a ring with a metric and let \( \mathbb{X} \) be the set of all formal power series over \( S \). Fix \( g(z) = \sum_{k=0}^{\infty} b_k z^k \). We define a subset \( \mathbb{X}_g \subset \mathbb{X} \) as
\[\mathbb{X}_g = \{ f \in \mathbb{X} : f(z) = \sum_{k=0}^{\infty} a_k z^k, \sum_{n=0}^{\infty} b_n a_k^{(n)} \in S, k \in \{0, 1, 2, \ldots \} \},\]
where \( f^n(z) = \sum_{k=0}^{\infty} a_k^{(n)} z^k \) for \( n \in \{1, 2, \ldots \} \) and \( f^0(z) = 1 \). The mapping \( T_g : \mathbb{X}_g \mapsto \mathbb{X} \) such that
\[T_g(f)(z) = \sum_{k=0}^{\infty} c_k z^k, \text{ where } c_k = \sum_{n=0}^{\infty} b_n a_k^{(n)}\]
is called the composition of \( g \) and \( f \) and we denote it as \( g \circ f \).

An important role in the theory of formal power series plays the following

Theorem 2.6 \([15]\). Let \( S \) be a ring with a metric and let \( \mathbb{X} \) be the set of all formal power series over \( S \). Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \), \( g(z) = \sum_{n=0}^{\infty} a_n z^n \) be formal power series. Denote \( \deg(f) = \sup\{n \in \mathbb{N} \cup \{0\} : a_n \neq 0 \} \) and assume \( \deg(f) \neq 0 \). Then the composition \( g \circ f \) exists if and only if
\[\sum_{n=k}^{\infty} b_n a_0^{n-k} \in S \text{ for all } k \in \mathbb{N} \cup \{0\} \tag{2.1}\]

Definition 2.7 \([14]\). Let \( g = \sum_{n \in \mathbb{Z}} b_n z^n \in \mathbb{L} \). Then the formal derivative of \( g \) is the formal Laurent series defined by
\[g'(z) = \sum_{n \in \mathbb{Z}} (n+1)b_{n+1} z^n.\]
Remark 2.8. Let \( g = \sum_{n=0}^{\infty} b_n z^n \), \( f = \sum_{n=0}^{\infty} a_n z^n \in \mathbb{X}(\mathbb{C}) \). Then \( g \circ f \in \mathbb{X}(\mathbb{C}) \) if and only if \( g^{(k)}(a_0) \in \mathbb{C} \) for every \( k \in \mathbb{N} \) where \( g^{(k)} \) denotes the \( k \)th formal derivative of \( g \).

Definition 2.9. Let \( g(z) = \sum_{k \in \mathbb{Z}} a_k z^k \in \mathbb{L}(\mathbb{C}) \). Define

\[
\mathbb{X}_g = \{ f \in \mathbb{X}(\mathbb{C}) : f(z) = \sum_{k=0}^{\infty} a_k z^k, a_0 \neq 0, \sum_{n \in \mathbb{Z}} b_n a_k^{(n)} \in S, k \in \{0, 1, 2, \ldots\}, \}
\]

where \( f^n(z) = \sum_{k=0}^{\infty} a_k^{(n)} z^k \), \( f^{-n}(z) = [f^{-1}(z)]^n = \sum_{k=0}^{\infty} a_k^{(-n)} z^k \) for \( n \in \{1, 2, \ldots\} \) and \( f^0(z) = 1 \). If \( \mathbb{X}_g \neq \emptyset \), the mapping \( T_g : \mathbb{X}_g \hookrightarrow \mathbb{X} \) such that

\[
T_g(f)(z) = \sum_{k=0}^{\infty} c_k z^k, \quad \text{where} \quad c_k = \sum_{n \in \mathbb{Z}} b_n a_k^{(n)}
\]

is called the composition of \( g \) and \( f \) and we denote it as \( g \circ f \).

Theorem 2.10. Let \( g(z) = \sum_{k \in \mathbb{Z}} a_k z^k \in \mathbb{L}(\mathbb{C}) \) and \( f(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathbb{X}(\mathbb{C}) \), \( a_0 \neq 0 \). Then \( g \circ f(z) \in \mathbb{X}(\mathbb{C}) \) if and only if

\[
\sum_{n=k}^{\infty} b_n a_0^{n-k} \in \mathbb{C} \quad \text{and} \quad \sum_{n=k}^{\infty} b_n^{-1} a_0^{n-k} \in \mathbb{C}
\]

for all \( k \in \mathbb{N} \cup \{0\} \).

3. Definition and basic properties of inverses of formal Laurent series

Definition 3.1. Let \( f \in \mathbb{L} \). We say that \( g \in \mathbb{L} \) is an inverse of \( f \), if and only if \( fg = S_1 \) (which is equivalent to \( gf = S_1 \)). We denote the set of all inverse series of a given formal Laurent series \( f \) as \( R(f) \).

If one considers the set \( \mathbb{X}(\mathbb{C}) \) of formal power series, then the inverse of a given series is unique if it exists (see [12]). Moreover, we have the following

Theorem 3.2. Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathbb{X}(\mathbb{C}) \) be a formal power series over \( \mathbb{C} \). Then there exists such a formal power series \( g \in \mathbb{X}(\mathbb{C}) \) that \( fg = S_1 \), if and only if \( a_0 \neq 0 \).

However, this case is much different in the set of formal Laurent series. First, we are going to show that if \( f \) is a formal Laurent series, then one of the following situations occurs:

1. \( f \) has no inverse series,
2. \( f \) has exactly one inverse series,
3. \( f \) has uncountably many inverse series.

Let us consider the following examples.
Example 3.3. Let \( f = \sum_{n \in \mathbb{Z}} c_n z^n \), where \( c_n = c_0 q^n \) for all \( n \in \mathbb{Z} \), \( c_0, q \in \mathbb{C} \), \( q \neq 0 \). Assume that there exists at least one formal Laurent series \( g = \sum_{n \in \mathbb{Z}} d_n z^n \) such that \( fg = S_1 \). Then we have

\[
0 = \sum_{m=-\infty}^{\infty} d_m c_{1-m} = q \sum_{m=-\infty}^{\infty} d_m c_{-m} = q,
\]

which is a contradiction since \( q \neq 0 \). That proves \( f \) does not possess any inverse series.

Example 3.4. Consider \( f = \sum_{n \in \mathbb{Z}} a_n z^n \), where \( a_n = 1 \) for \( n \geq 0 \) and \( a_n = 0 \) for \( n < 0 \). Assume that a formal Laurent series \( g = \sum_{n \in \mathbb{Z}} b_n z^n \) is an inverse of \( f \). Then \( \sum_{m=0}^{\infty} b_{-m} = 1 \) and \( \sum_{m=0}^{\infty} b_{n-m} = 0 \) for \( n \neq 0 \). Therefore

\[
b_n = \sum_{m=0}^{\infty} b_{n-m} - \sum_{m=0}^{\infty} b_{n-1-m} = \begin{cases} 0, & n \in \mathbb{Z} \setminus \{0,1\}, \\ -1, & n = 1, \\ 1, & n = 0. \end{cases}
\]

(3.1)

It is easy to check that if \( g = \sum_{n \in \mathbb{Z}} b_n z^n \), where \( b_n \) are given by (3.1), then \( fg = S_1 \). This proves that \( f \) possesses exactly one inverse series \( g(z) = 1 - z \).

Example 3.5. Consider \( f = \sum_{n \in \mathbb{Z}} a_n z^n \), where \( a_0 = 1 \), \( a_1 = -1 \) and \( a_n = 0 \) for \( n \notin \{0,1\} \). A formal Laurent series \( \sum_{n \in \mathbb{Z}} b_n z^n \) is an inverse of \( f \), if and only if

\[
\sum_{m=-\infty}^{\infty} a_m b_{n-m} = b_n - b_{n-1} = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0. \end{cases}
\]

Therefore any formal Laurent series \( \sum_{n \in \mathbb{Z}} b_n z^n \), where

\[
b_n = \begin{cases} c, & n \geq 0 \\ c-1, & n < 0, \end{cases}
\]

for some \( c \in \mathbb{C} \), is an inverse of \( f \), so \( f \) has infinitely many inverse series.

Examples 3.4 and 3.5 show that the fact that \( g \) is the only inverse of \( f \) does not imply that \( f \) is the only inverse of \( g \).

It has yet to be proved that the formal Laurent series cannot have a finite, but greater than 1, or a countable number of inverse series.

Proposition 3.6. Assume that a formal Laurent series \( f \) has two different inverse series \( g_1, g_2 \). Then \( f \) has uncountably many different inverses.

Proof. Let \( k_1, k_2 \in \mathbb{C} \), \( k_1 + k_2 \neq 0 \). Then \( f(k_1 g_1 + k_2 g_2) = (k_1 + k_2)S_1 \), so \( f \frac{k_1 g_1 + k_2 g_2}{k_1 + k_2} = S_1 \). Therefore for any \( k_1, k_2 \in \mathbb{C} \), \( k_1 + k_2 \neq 0 \), the formal Laurent series \( \frac{k_1 g_1 + k_2 g_2}{k_1 + k_2} \) is an inverse of \( f \), which completes the proof. \( \square \)
The proof of the next proposition shows the connection between the fact that the inverse of a 
formal Laurent series is not always unique and the lack of associativity of multiplication of formal 
Laurent series.

**Proposition 3.7.** The multiplication of formal Laurent series is not associate.

**Proof.** Assume that the multiplication in $\mathbb{L}$ is associate, that is for any $f, g, h \in \mathbb{L}$, if $gh \in \mathbb{L}(f)$, then $fg \in \mathbb{L}(h)$ and $f(gh) = (fg)h$. We will now prove two properties being consequences of this 
assumption.

1. Let $f \in \mathbb{L}$. If there exists such $g \in \mathbb{L}$, $g \neq S_0$, that $gf = S_0$, then $f$ has no inverse series. 
   Assume that an inverse $f^{-1}$ of $f$ exists. Then $g = g(ff^{-1}) = (gf)f^{-1} = S_0f^{-1} = S_0$, which 
is a contradiction. Therefore $f$ has no inverses.

2. For all $f \in \mathbb{L}$, if $f^{-1}$ exists, then it is unique.
   Assume that there exist such $g_1, g_2 \in \mathbb{L}$ that $g_1 \neq g_2$ and $fg_1 = fg_2 = S_1$. Then $f(g_1 - g_2) = S_0$. We have $g_1 - g_2 \neq S_0$, so by (1), we infer that $f$ has no inverse series, which is a 
contradiction.

However, the inverse series of formal Laurent series is not unique in general (see Example 3.5), so 
the property (2) led us to a contradiction. Therefore the multiplication of formal Laurent series is 
not associate in general. □

**Remark 3.8.** Let $f, g, h \in \mathbb{L}$ and assume $fg, gh, f(gh), (fg)h \in \mathbb{L}$. Denote the coefficients of 
$f, g, h, (fg)h, f(gh)$ by $f_n, g_n, h_n, a_n, b_n$, respectively. We have

$$a_n = \sum_{t \in \mathbb{Z}} \sum_{s \in \mathbb{Z}} f_s g_{t-s} h_{n-t},$$

$$b_n = \sum_{m \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} f_{n-m} g_k h_{m-k} = \sum_{s \in \mathbb{Z}} \sum_{t \in \mathbb{Z}} f_s g_{t-s} h_{n-t}$$

(where we substituted $s = n - m$, $t = k + n - m$). The only one difference between $a_n$ and $b_n$ is the 
arangement of indexes with respect to which we add up the coefficients above. However, using the 
Rearrangement Theorems such as [21], Thm. 2.50, 7.50 (which proofs can be easily performed 
alogously for triple series) and Remark 2.1, one can obtain some sufficient conditions for $a_n$ and 
$b_n$ to be equal, for instance:

**Proposition 3.9.** Let $f, g, h \in \mathbb{L}$ and denote the coefficients of $f, g, h$ as $f_n, g_n, h_n$, respectively. 
Assume $f_n, g_n, h_n \geq 0$ for all $n \in \mathbb{Z}$. Then if $fg \in \mathbb{L}(h)$, then $gh \in \mathbb{L}(f)$ and $(fg)h = f(gh)$.

For a more detailed study of the existence and equality of the coefficients $a_n$ and $b_n$, the reader 
is referred to [21].

The lack of general associativity of the multiplication of formal Laurent series and the lack of 
general uniqueness of inverses of formal Laurent series make the issue of composition of formal 
Laurent series more complex (according to our best knowledge, one can find in the literature the
definition of composition of formal Laurent series and formal power series- see e.g. \[14\]). That’s because it requires considering expressions of the form \(f^{-n} := (f^{-1})^n := f^{-1} \cdot \ldots \cdot f^{-1}\) (\(n\) factors), and it is not possible to unambiguously assign to a given series \(f\) its inverse series \(f^{-1}\).

4. The existence and computation of inverses of formal Laurent series

In this section we are going to consider the problem of the existence and computation of inverse series to formal Laurent series. To solve this problem, we will use some elements of the theory of infinite systems of linear equations (see e.g. \[7\], \[17\]).

4.1. Algorithm I. In this subsection we are going to compute inverses of formal Laurent series that satisfy some assumptions. First, we will state an obvious lemma, which will be needed in the sequel.

Lemma 4.1. Let \((a_n)_{n \in \mathbb{Z}}\) and \((d_n)_{n \in \mathbb{Z}}\) be two sequences of complex numbers. Define \(\mu_{n,s} = a_{n-(s-1)\lfloor s/2 \rfloor} \cdot (-1)^s\lfloor s/2 \rfloor\) and \(y_s = d_{(s-1)\lfloor s/2 \rfloor} \cdot (-1)^s\lfloor s/2 \rfloor\); \(n \in \mathbb{Z}, s \in \mathbb{N}\). Now fix \(n \in \mathbb{Z}\). If the series \(\sum_{m=-\infty}^{\infty} a_{n-m}d_m\) is convergent, then the series \(\sum_{s=1}^{\infty} \mu_{n,s}y_s\) is convergent and

\[
\sum_{m=-\infty}^{\infty} a_{n-m}d_m = \sum_{s=1}^{\infty} \mu_{n,s}y_s.
\]

Let \(f = \sum_{n \in \mathbb{Z}} a_n z^n\). A series \(g = \sum_{n \in \mathbb{Z}} d_n z^n\) is an inverse of \(f\), if and only if the following infinite system of linear equations is satisfied:

\[
\sum_{m \in \mathbb{Z}} d_m a_{n-m} = \begin{cases} 1, & n = 0, \\ 0, & n \neq 0. \end{cases} \quad (4.1)
\]

The index \(n\) occurring above will be called the index of the equation. There are some known methods for solving such infinite systems, but they assume that the equations and unknowns are indexed by positive integers only. Therefore, we will now transform the system (4.1), so that it meets this requirement.

Proposition 4.2. If (4.1) holds for some sequences \((d_m)_{m \in \mathbb{Z}}\) and \((a_m)_{m \in \mathbb{Z}}\), then

\[
\sum_{s=1}^{\infty} u_{j,s}y_s = b_j \quad (j \in \mathbb{N}), \quad (4.2)
\]

where

\[
u_{j,s} = a_{(-1)^j \lfloor s/2 \rfloor \cdot (-1)^{s\lfloor s/2 \rfloor}}, \quad y_s = d_{(-1)^s \lfloor s/2 \rfloor}, \quad b_j = \begin{cases} 1, & j = 1, \\ 0, & j \in \mathbb{Z} \setminus \{1\}. \end{cases} \quad (4.3)
\]

What is more, if (4.2) holds and the series \(\sum_{m \in \mathbb{Z}} a_{n-m}d_m\), where \(a_n, d_n\) come from (4.3), converges for all \(n \in \mathbb{Z}\), then (4.1) holds.
We will denote the determinant of the matrix $W$. We have proved that if (4.1) holds, then (4.2) holds. Now, see that if we can also change the way of numbering the equations in the above system from all integers $(n)$ to positive integers only $(j)$, obtaining (4.2).

We have proved that if (4.1) holds, then (4.2) holds. Now, see that if $\sum a_m d_{n-m}$ converges for all $n \in \mathbb{Z}$, where $a_m$, $d_m$ are calculated from (4.3) for some $y_s$, $u_j$, satisfying (4.2), then, by Lemma 4.1 we can analogously prove the other way around, which completes the proof.

Remark 4.3. Under the above notation, we have

$$d_m = \begin{cases} \ y_{2m}, & m > 0 \\ \ y_{1-2m}, & m \leq 0. \end{cases} \quad (4.5)$$

Definition 4.4. The system (4.2), that is

$$W[f]y = b, \quad (4.6)$$

where

$$b = [b_1 \ b_2 \ ...]^T, \ y = [y_1 \ y_2 \ ...]^T$$

and

$$W[f] = \begin{bmatrix} u_{1,1} \ u_{1,2} \ & \ u_{1,n} \ & \ \vdots \ & \ \vdots \ & \ \vdots \\ u_{2,1} \ u_{2,2} \ & \ u_{2,n} \ & \ \vdots \ & \ \vdots \ & \ \vdots \\ : \ & : \ & \ : \ & \ : \ & \ : \\ u_{n,1} \ u_{n,2} \ & \ u_{n,n} \ & \ \vdots \ & \ \vdots \ & \ \vdots \end{bmatrix} = \begin{bmatrix} a_0 \ a_{-1} \ a_1 \ a_{-2} \ a_2 \ \vdots \ & \ & \ & \ & \vdots \\ a_1 \ a_0 \ a_2 \ a_{-1} \ a_3 \ \vdots \ & \ & \ & \ & \vdots \\ a_{-1} \ a_{-2} \ a_0 \ a_{-3} \ a_1 \ \vdots \ & \ & \ & \ & \vdots \\ a_2 \ a_1 \ a_3 \ a_0 \ a_4 \ \vdots \ & \ & \ & \ & \vdots \\ a_{-2} \ a_{-3} \ a_{-1} \ a_{-4} \ a_0 \ \vdots \ & \ & \ & \ & \vdots \end{bmatrix}, \quad (4.7)$$

will be called the inverse system of the formal Laurent series $f$.

We will denote the determinant of the matrix $W[f]$ as $|W[f]|$. Let us note that, similarly to [9], we define the determinant of an infinite matrix $A = [a_{i,j}]_{i,j \in \mathbb{Z}^+}$ by the formula $|A| = \lim_{n \to \infty} |A_n|$ (where $|A_n|$ is the determinant of the matrix $A_n = [a_{i,j}]_{i,j \in \{1, \ldots, n\}}$ if this limit exists; otherwise we say that the determinant of $A$ does not exist. We also call $|A_n|$ the $n$th principal minor of $A$. For more information about infinite matrices, their determinants and their applications to the theory of infinite linear systems, the reader is referred to [7], [16].

Remark 4.5. In the following considerations, we will use the concept of the strictly particular solution to an infinite system of equations. It is one of the particular solutions to the system, satisfying an infinite version of the Cramer rule (to see the full definition, the reader is referred to [7]). It is known that an infinite system is consistent if and only if its strictly particular solution exists.
We will now consider formal Laurent series $f$ satisfying the following conditions:

1. $|W[f]|$ exists,
2. $|W[f]| \neq 0$,
3. all principal minors $|W[f]|_k$, $k \in \mathbb{N}$, of the matrix $W[f]$ are different from 0.

Let $f = \sum_{n \in \mathbb{Z}} a_n z^n$ satisfy (1)-(3). Therefore, by [7, Thm. 2.1.], the matrix $W[f]$ can be written in the form

$$W[f] = \begin{bmatrix}
d_{1,1} & 0 & 0 & \cdots & 0 \\
d_{2,1} & d_{2,2} & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
d_{n,1} & d_{n,2} & \cdots & d_{n,n} & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots
\end{bmatrix} = \begin{bmatrix}
c_{1,1} & c_{1,2} & \cdots & c_{1,n} & \cdots \\
0 & c_{2,2} & \cdots & c_{2,n} & \cdots \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & c_{n,n} & \cdots \\
\vdots & \vdots & \ddots & \ddots & \ddots
\end{bmatrix}, \quad (4.8)$$

where the coefficients $d_{i,j}$, $c_{i,i}$ satisfy the recursive relations

$$d_{i,k} = \frac{u_{i,k} - \sum_{j=1}^{k-1} d_{i,j}c_{j,k}}{c_{k,k}} \quad (i \geq k), \quad c_{i,k} = \frac{u_{i,k} - \sum_{j=1}^{i-1} d_{i,j}c_{j,k}}{d_{i,i}} \quad (i < k), \quad (4.9)$$

and the diagonal coefficients $c_{i,i}$, $i \in \mathbb{N}$ are arbitrary - let us then define $c_{i,i} = 1$ for all $i \in \mathbb{N}$; we denote $\sum_{j=1}^{0} d_{i,j}c_{j,1} := 0$. Let us emphasize that we multiply infinite matrices analogously to finite matrices, that is, using the notations above, $u_{i,j} = \sum_{m=1}^{\infty} d_{i,m}c_{m,j}$ for all $i, j \in \mathbb{N}$. Also, see that for all $i \in \mathbb{N}$, $c_{i,i}, d_{i,i} \neq 0$, because $|W[f]| \neq 0$.

Now, by [7, Theorem 2.2, Corollary 2.2, Note 1], a both side inverse matrix of $D$ exists (that is, there exists such a matrix $D^{-1} = (d_{i,j}^{(-1)})_{i,j \in \mathbb{Z}_+}$ that $DD^{-1} = D^{-1}D = I$, where $I = (I_{i,j})_{i,j \in \mathbb{Z}_+}$, $I_{i,i} = 1$ and $I_{i,j} = 0$ for $i \neq j$). The system (4.10) is then equivalent to the system $Cy = D^{-1}b$, that is

$$\sum_{p=0}^{\infty} c_{j,j+p}y_{j+p} = \beta_j, \quad j \in \mathbb{N}, \quad (4.10)$$

where $\beta_n = d_{n,1}^{(-1)}$, $n \in \mathbb{N}$. By the results from [7], the matrix $D^{-1}$ is triangular, so it is easy to check that, using the definition of matrix multiplication, the coefficients $\beta_n$ satisfy the following recursion:

$$\beta_1 = \frac{1}{d_{1,1}}, \quad \beta_n = -\frac{1}{d_{n,n}} \sum_{i=1}^{n-1} d_{n,i} \beta_i. \quad (4.11)$$
By [9, Thm. 8, 9], the system (4.10) is consistent (possesses a solution), if and only if its strictly particular solution exists, that is, denoting

\[
A_p(j) = \begin{cases} 
  \sum_{p=0}^{k-1} (-1)^{p-1-k} c_{j+k,j+p} A_k(j), & p > 0, \\
  1, & p = 0,
\end{cases}
\]

the series

\[
B(j) = \sum_{p=0}^{\infty} (-1)^p A_p(j) B_{j+p}
\]  \hspace{1cm} (4.12)

is convergent for all \( j \in \mathbb{N} \) and satisfies (4.10), that is

\[
\sum_{p=0}^{\infty} c_{j,j+p} B(j + p) = \beta_j, \quad j \in \mathbb{N}.
\]  \hspace{1cm} (4.13)

Then \( y_j = B(j) \) is a solution to (4.10). It is worth mentioning that if (4.10) is consistent, then (see [7, Thm. 3.8])

\[
B(j) = \frac{|C(j)|}{|C|} = |C^{(j)}|, \quad j \in \mathbb{N},
\]  \hspace{1cm} (4.14)

where \( |C| = \prod_{i=1}^{\infty} c_{i,i} = 1 \) is the determinant of the matrix \( C \) and \( C^{(j)} \) is a matrix constructed by replacing the \( j \)-th column of \( C \) with the sequence \( (\beta_i) \). See that we index the equations in (4.10) by positive integers, unlike [7], where indexes begin with 0, hence \( j \) instead of \( j + 1 \) in the above formula.

**Corollary 4.6.** Let \( f \in \mathbb{L} \) satisfy (1)-(3), that is \( |W[f]| \) exists, \( |W[f]| \neq 0 \) and \( |W[f]_k| \neq 0 \) for all \( k \in \mathbb{N} \). Then \( f \) possesses an inverse series, if and only if (4.13) holds (where \( B(j) \) are calculated from (4.12) or, equivalently, (4.12)) and the series \( \sum_{m \in \mathbb{Z}} d_m a_{n-m}, n \in \mathbb{Z} \), where \( d_n \) are calculated from (4.5) for \( y_j = B(j) \), are convergent. Then \( \sum_{n \in \mathbb{Z}} d_n z^n \) is an inverse of \( f \).

By Corollary 4.6, we obtain an inverse of a formal Laurent series satisfying (1)-(3) (if it has an inverse). However, we know a formal Laurent series may have more than one inverse. The following result gives the method of finding all inverses of a formal Laurent series \( f \) satisfying (1)-(3):

**Theorem 4.7.** Let \( f \) be a formal Laurent series satisfying (1)-(3) that has at least one inverse and let \( c_{i,j}, i, j \in \mathbb{N} \) denote the coefficients from (4.8).

Put \( S_1(n,j) = c_{n-1,n} \) and

\[
S_{n-j}(n,j) = c_{j,j+1} + \sum_{p=2}^{n-j} (-1)^{p+1} c_{j,j+p} \prod_{k=1}^{p-1} (S_{n-j-k}(n,j))^{-1}
\]

for every \( n \in \mathbb{N} \) and \( j \in \{1, ..., n - 2\} \).

Then \( f \) has infinitely many inverses, if and only if
\( (1) \lim_{n \to \infty} S_{n-j}(n,j) = S(j) \text{ exists for all } j \in \mathbb{N}, \)

\( (2) \text{for every } j \in \mathbb{N}, \sum_{p=1}^{\infty} (-1)^p c_{j,p} \prod_{k=0}^{p-1} (S(j+k))^{-1} = 1, \)

\( (3) \text{for all } n \in \mathbb{Z}, \text{the series } \sum_{n \in \mathbb{Z}} a_m d'_n, \) where

\[
d'_n = \begin{cases} 
v_{2n}, & n > 0, \\
v_{1-2n}, & n \leq 0, \end{cases} \quad v_i = (-1)^i \prod_{k=0}^{i-1} (S(k))^{-1}, \quad i \in \mathbb{N},
\]

is convergent.

Then all inverses of \( f \) are given by the general formula:

\[
f^{-1} = \sum_{n \in \mathbb{Z}} (d_n + cd'_n)z^n, \quad (4.15)
\]

where \( c \) is an arbitrary constant and \( d_n \) are coefficients from Corollary 4.6.

**Proof.** Let \( y^{(s)} = [y_1^{(s)} \ y_2^{(s)} \ldots]^T \) be the strictly particular solution to \( (4.10) \). See that \( [y_1 \ y_2 \ldots]^T := y \neq y^{(s)} \) is also a solution to \( (4.10) \), if and only if, denoting \( y^{(r)} = y - y^{(s)} \),

\[
\sum_{p=0}^{\infty} c_{j,j+p} y^{(r)}_{j+p} = 0, \quad j \in \mathbb{N}.
\]

The claim now results directly from \([7], \text{Thm. 5.2}\) and Remark 4.3. \( \square \)

**Example 4.8.** Consider the formal Laurent series \( f = \sum_{n \in \mathbb{Z}} a_n z^n \), where

\[
a_n = \begin{cases} 
sin n, & n > 0, \\
cos n, & n \leq 0. \end{cases}
\]

It can be checked numerically that \( |W[f]| = \frac{1}{2} \) and \( |W[f]| k \) \( \neq 0 \) for all \( k \in \mathbb{N} \), so we can find inverses of \( f \) using the method presented above.

The Gaussian system \( (4.10) \) was truncated to the first 500 rows and columns for the algorithm to be able to perform numerical calculations. It turns out \( f \) possesses exactly one inverse (not all the limits \( \lim_{n \to \infty} S_{n-j}(j) \) exist - mainly because there are some \( k \in \mathbb{N} \), for which \( d_{k,k+1} = 0 \)). The coefficients \( d_n \) of this inverse, rounded to four decimals points, are presented in Table 1 for \( n \in \{-10, ..., 10\} \).

4.2. **Algorithm II.** In this section, we give another method for calculating the inverses of formal Laurent series satisfying some conditions (other than the ones considered in Section 4.1).
Therefore to find all solutions to (4.1), we can find all solutions to (4.16) corresponding to all possible sequences \((u)\). Let us first rewrite (4.16), so that the equations are indexed with positive integers only. We get

\[
\begin{align*}
\sum_{m=0}^{\infty} a_{n-m}d_m &= u_n \quad (n \in \mathbb{Z}) \\
\sum_{m=1}^{\infty} a_{n+m}d_{-m} &= \begin{cases}
1 - u_n, & n = 0 \\
-u_n, & n \neq 0
\end{cases} \quad (n \in \mathbb{Z}),
\end{align*}
\]

(4.16)

where \((u_n)\) is a sequence of some complex numbers. Let us notice that if a sequence \((d_m)\) is a solution of (4.1), then it is a solution of (4.16) for some sequence \((u_n)\) of complex numbers. What is more, if a sequence \((d_m)\) is a solution of (4.16) for a sequence \((u_n)\), then it is a solution of (4.1). Therefore to find all solutions to (4.1), we can find all solutions to (4.16) corresponding to all possible sequences \((u_n)\).

Let us first rewrite (4.16), so that the equations are indexed with positive integers only. We get

\[
\begin{align*}
\sum_{m=0}^{\infty} a_{(-1)^m j}d_{-m} &= s_j \quad (j \in \mathbb{N}) \\
\sum_{m=1}^{\infty} a_{(-1)^m j}d_{m} &= \begin{cases}
1 - s_j, & j = 1 \\
-s_j, & j \neq 1
\end{cases} \quad (j \in \mathbb{N}),
\end{align*}
\]

(4.16)

where \((s_j)_{j \in \mathbb{N}}\) is a sequence of some complex numbers. Let \(x_i = d_{i-1}\) and \(y_i = d_{-i}\) for all \(i \in \mathbb{N}\). We can rewrite the above systems as

\[
\begin{align*}
\sum_{i=1}^{\infty} u_{i,j}x_i &= s_j \quad (j \in \mathbb{N}) \\
\sum_{i=1}^{\infty} u_{i,j}y_i &= \begin{cases}
1 - s_j, & j = 1 \\
-s_j, & j \in \mathbb{N} \setminus \{1\}
\end{cases},
\end{align*}
\]

(4.16)
or, denoting \( s = [s_1 \ s_2 \ ...]^T \), \( t = [t_1 \ t_2 \ ...]^T := [1-s_1 \ -s_2 \ -s_3 \ ...]^T \), \( x = [x_1 \ x_2 \ ...]^T \), \( y = [y_1 \ y_2 \ ...]^T \),

\[
A_1 x = s, \quad A_2 y = t.
\]  

(4.17)

Now fix a sequence of complex numbers \((s_j)_{j \in \mathbb{N}}\). By [7], Thm. 2.1, we calculate the Gaussian elimination for matrices \(A_1, A_2\):

\[
A_1 = D_1 C_1 = \begin{bmatrix}
  d_{1,1} & 0 & 0 & \ldots & 0 & \ldots \\
  d_{2,1} & d_{2,2} & 0 & \ldots & 0 & \ldots \\
  \vdots & \vdots & \ddots & \ddots & \vdots & \ddots \\
  d_{n,1} & d_{n,2} & \ldots & d_{n,n} & 0 & \ldots \\
  \vdots & \vdots & \ddots & \ddots & \vdots & \ddots \\
  \end{bmatrix}
\]  

\[
A_2 = D_2 C_2 = \begin{bmatrix}
  \alpha_{1,1} & 0 & 0 & \ldots & 0 & \ldots \\
  \alpha_{2,1} & \alpha_{2,2} & 0 & \ldots & 0 & \ldots \\
  \vdots & \vdots & \ddots & \ddots & \vdots & \ddots \\
  \alpha_{n,1} & \alpha_{n,2} & \ldots & \alpha_{n,n} & 0 & \ldots \\
  \vdots & \vdots & \ddots & \ddots & \vdots & \ddots \\
  \end{bmatrix}
\]  

where the following recursions hold:

\[
d_{i,k} = \frac{v_{i,k} - \sum_{j=1}^{k-1} d_{i,j}c_{j,k}}{c_{k,k}}, \quad \alpha_{i,k} = \frac{u_{i,k} - \sum_{j=1}^{k-1} \alpha_{i,j} \beta_{j,k}}{\beta_{k,k}} \quad (i \geq k),
\]  

(4.18)

\[
c_{i,k} = \frac{v_{i,k} - \sum_{j=1}^{i-1} d_{i,j}c_{j,k}}{d_{i,i}}, \quad \beta_{i,k} = \frac{u_{i,k} - \sum_{j=1}^{i-1} \alpha_{i,j} \beta_{j,k}}{\alpha_{i,i}} \quad (i \leq k),
\]  

(4.19)

and the diagonal coefficients \(c_{i,i}, \beta_{i,i}, i \in \mathbb{N}\) are arbitrary - let us then define \(c_{i,i} = \beta_{i,i} = 1\) for all \(i \in \mathbb{N}\); we denote \(\sum_{j=1}^{0} d_{i,j}c_{j,1} = \sum_{j=1}^{0} \alpha_{i,j} \beta_{j,1} := 0\). Also, see that for all \(i \in \mathbb{N}\), \(c_{i,i}, d_{i,i}, \alpha_{i,i}, \beta_{i,i} \neq 0\), because we assumed \(|A_1|, |A_2| \neq 0\).

Now, by [7], Thm. 2.2, Corollary 2.2, Note 1], both side inverse matrices of \(D_1, D_2\) exist and are triangular. It follows that (4.17) are equivalent to

\[
C_1 x = D_1^{-1} s, \quad C_2 y = D_2^{-1} t,
\]  

(4.20)

that is,

\[
\sum_{p=0}^{\infty} c_{j,j+p} x_{j+p} = \sum_{k=1}^{j} d_{j,k}^{(-1)} s_k \quad (j \in \mathbb{N}) \quad \text{and} \quad \sum_{p=0}^{\infty} \beta_{j,j+p} y_{j+p} = \sum_{k=1}^{j} \alpha_{j,k}^{(-1)} t_k \quad (j \in \mathbb{N}),
\]  

(4.21)
where $d_{i,j}^{(-1)}$, $\alpha_{i,j}^{(-1)}$ are the elements of matrices $D_1^{-1}$, $D_2^{-1}$, respectively; one can easily check that the following recursions hold:

\[
\begin{cases}
  d_{m,m}^{(-1)} = \frac{1}{d_{m,m}}, \\
  d_{n,m}^{(-1)} = -\frac{1}{d_{n,m}} \sum_{i=m}^{n-1} d_{n,i}d_{i,m}^{(-1)}, \quad n > m,
\end{cases}
\]

consistent (possess a solution), if and only if their strictly particular solutions

\[
\begin{align*}
  S & = \phi^{(-1)}(s), \\
  T & = \psi^{(-1)}(s),
\end{align*}
\]

Denote $\phi_j(s) = \sum_{k=1}^{j} d_{j,k}^{(-1)} s_k$, $\psi_j(s) = \sum_{k=1}^{j} \alpha_{j,k}^{(-1)} t_k$. By [9], Thm. 8, 9, both systems (4.21) are consistent (possess a solution), if and only if their strictly particular solutions exist, that is, denoting

\[
\begin{align*}
  A_0^{(1)}(j) & = 1, \\
  A_p^{(1)}(j) & = \sum_{p=0}^{k-1} (-1)^{p-1-k} c_{j+k,j+p} A_k^{(1)}(j), \quad p > 0, \\
  A_0^{(2)}(j) & = 0, \\
  A_p^{(2)}(j) & = \sum_{p=0}^{k-1} (-1)^{p-1-k} \beta_{j+k,j+p} A_k^{(2)}(j), \quad p > 0,
\end{align*}
\]

the series

\[
B^{(1)}(j, s) = \sum_{p=0}^{\infty} (-1)^p A_p^{(1)}(j) \phi_{j+p}(s), \\
B^{(2)}(j, s) = \sum_{p=0}^{\infty} (-1)^p A_p^{(2)}(j) \psi_{j+p}(s)
\]

are convergent for all $j \in \mathbb{N}$ and satisfy (4.21), that is

\[
\sum_{p=0}^{\infty} c_{j,j+p} B^{(1)}(j + p, s) = \phi_j(s) \quad \text{and} \quad \sum_{p=0}^{\infty} \beta_{j,j+p} B^{(2)}(j + p, s) = \psi_j(s). \quad (4.22)
\]

Then $x_j(s) = B^{(1)}(j, s)$ is a solution to the first system in (4.21), and $y_j(s) = B^{(2)}(j, s)$ is a solution to the second system in (4.21). It is worth mentioning that if the systems (4.21) are consistent, then (see [7], Thm. 3.8)

\[
B^{(1)}(j, s) = \frac{|C_1^{(j)}|}{|C_1|} = |C_1^{(j)}|, \\
B^{(2)}(j, s) = \frac{|C_2^{(j)}|}{|C_2|} = |C_2^{(j)}|, \quad j \in \mathbb{N}, \quad (4.23)
\]

where the notations are analogous to those in (4.14).

**Definition 4.10.** Let $f$ be a formal Laurent series satisfying conditions $(1^*)$, $(2^*)$. We denote the set of all sequences $s$, for which both systems (4.20) are consistent as $SC(f)$.

Now, let $s \in SC(f)$. We have already found one solution to (4.20) corresponding to $s$ - namely, $x_j(s) = B^{(1)}(j, s)$, $y_j(s) = B^{(2)}(j, s)$. Now we will find all solutions to (4.20) corresponding to $s$.

Denote $S_1(n, j) = c_{n-1,n}$, $T_1(n, j) = \beta_{n-1,n}$ and

\[
S_{n-j}(n, j) = c_{j,j+1} + \sum_{p=2}^{n-j} (-1)^{p+1} c_{j,j+p} \prod_{k=1}^{p-1} (S_{n-j-k}(n, j))^{-1},
\]

\[
T_{n-j}(n, j) = \beta_{j,j+1} + \sum_{p=2}^{n-j} (-1)^{p+1} \beta_{j,j+p} \prod_{k=1}^{p-1} (T_{n-j-k}(n, j))^{-1};
\]
see that the values of $S_{n-j}(n,j)$, $T_{n-j}(n,j)$ do not depend on the choice of $s$. By [7], Thm. 5.2, the homogenous system $\sum_{p=0}^{\infty} c_{j,j+p}x_{j+p} = 0$ ($j \in \mathbb{N}$) possesses a nontrivial solution, if and only if the limit $S(j) := \lim_{n \to \infty} S_{n-j}(n,j)$ exists for all $j \in \mathbb{N}$ and

$$
\sum_{p=1}^{\infty}(-1)^{p}c_{j,j+p}\prod_{k=0}^{p-1}(S(j+k))^{-1} = 1. \quad (4.24)
$$

If the above condition holds, then the general solution to the first system in (4.20) has the form

$$
X_{j}(s,c_{1}) = x_{j}(s) + c_{1}(-1)^{j}\prod_{k=0}^{j-1}(S(k))^{-1}, \text{ where } x_{j}(s) = B^{(1)}(j,s) \text{ and } c_{1} \in \mathbb{C} \text{ is arbitrary}. \quad (4.25)
$$

Analogously, the homogenous system $\sum_{p=0}^{\infty} \beta_{j,j+p}y_{j+p} = 0$ ($j \in \mathbb{N}$) possesses a nontrivial solution, if and only if the limit $T(j) := \lim_{n \to \infty} T_{n-j}(n,j)$ exists for all $j \in \mathbb{N}$ and

$$
\sum_{p=1}^{\infty}(-1)^{p}\beta_{j,j+p}\prod_{k=0}^{p-1}(T(j+k))^{-1} = 1. \quad (4.26)
$$

If the above condition holds, then the general solution to the second system in (4.20) has the form

$$
Y_{j}(s,c_{2}) = y_{j}(s) + c_{2}(-1)^{j}\prod_{k=0}^{j-1}(T(k))^{-1}, \text{ where } y_{j}(s) = B^{(1)}(j,s) \text{ and } c_{2} \in \mathbb{C} \text{ is arbitrary}. \quad (4.27)
$$

Let us recall that $R(f)$ denotes the set of all inverses of a formal Laurent series $f \in L$.

**Corollary 4.11.** Let $f$ be a formal Laurent series satisfying (1*), (2*). Then, using the notations from (4.25), (4.27),

- if none of the conditions (4.24), (4.26) is satisfied, then

$$
R(f) = \left\{ \sum_{n \in \mathbb{Z}} d_{n}(s)z^{n}, \quad d_{n}(s) = \begin{cases} x_{n+1}(s), & n \geq 0 \\ y_{-n}(s), & n < 0 \end{cases} : s \in SC(f) \right\};
$$

- if only the condition (4.24) is satisfied, then

$$
R(f) = \left\{ \sum_{n \in \mathbb{Z}} d_{n}(s,c_{1})z^{n}, \quad d_{n}(s,c_{1}) = \begin{cases} X_{n+1}(s,c_{1}), & n \geq 0 \\ y_{-n}(s), & n < 0 \end{cases} : c_{1} \in \mathbb{C}, s \in SC(f) \right\};
$$

- if only the condition (4.26) is satisfied, then

$$
R(f) = \left\{ \sum_{n \in \mathbb{Z}} d_{n}(s,c_{2})z^{n}, \quad d_{n}(s,c_{2}) = \begin{cases} x_{n+1}(s), & n \geq 0 \\ Y_{-n}(s,c_{2}), & n < 0 \end{cases} : c_{2} \in \mathbb{C}, s \in SC(f) \right\};
$$

- if both conditions (4.24), (4.26) are satisfied, then

$$
R(f) = \left\{ \sum_{n \in \mathbb{Z}} d_{n}(s,c_{1},c_{2})z^{n}, \quad d_{n}(s,c_{1},c_{2}) = \begin{cases} X_{n+1}(s,c_{1}), & n \geq 0 \\ Y_{-n}(s,c_{2}), & n < 0 \end{cases} : c_{1},c_{2} \in \mathbb{C}, s \in SC(f) \right\}.\]
Notice that \( f \) has no inverses, if and only if \( SC(f) = \emptyset \), and \( f \) has exactly one inverse, if and only if \( SC(f) \) has exactly one element and the conditions (4.24), (4.26) are not satisfied.

**Remark 4.12.** Denote the columns of \( A_1(f) \) as \( l_1, l_2, \ldots \) and the columns of \( A_2(f) \) as \( m_1, m_2, \ldots \). Let us notice that, using the above notation, \( W[f] = [l_1 \ l_2 \ m_1 \ m_2 \ l_4 \ldots] \).

Now, consider the formal Laurent series \( f = S_1 + \sum_{n=1}^{\infty} a_n z^n \), \( a_1, a_2, \ldots \in \mathbb{C} \). It is easy to check that the determinant and all principal minors of \( W[f] \) are equal to 1, so one can use the method from Section 4.1 to calculate the inverses of \( f \). However, it is also easy to check that the determinant of \( A_1(f) \) is equal to 0, so one cannot determine the inverses of \( f \) using the method from Section 4.2.

The question whether there exists a formal Laurent series \( f \) such that one can obtain \( f^{-1} \) using the method from Section 4.2, but cannot obtain \( f \)'s inverses using the method from Section 4.1 remains open.

5. Composition of formal Laurent series

**Definition 5.1.** We define the \( n \)th power of a formal Laurent series \( f \) (if it exists) by the recursion \( f^n = ff^{n-1} \) \((n \in \mathbb{N})\). We also set \( f^0 := S_1 \) and, if there exists an inverse \( f^{-1} \) of \( f \), we define \( f^{-n} := (f^{-1})^n \) (however, this expression is not unique in general).

If \( f = \sum_{n \in \mathbb{Z}} a_n z^n \), then we denote \( f^k = \sum_{n \in \mathbb{Z}} a_n^{(k)} z^n \). We also denote \( P_k(\mathbb{L}) = \{ f \in \mathbb{L} : f^k \text{ exists} \} \).

See that \( \mathbb{L} = P_0(\mathbb{L}) = P_1(\mathbb{L}) \supset P_2(\mathbb{L}) \supset P_3(\mathbb{L}) \supset \ldots \). We denote \( P_\infty(\mathbb{L}) = \bigcap_{k \in \mathbb{N}} P_k(\mathbb{L}) \).

**Definition 5.2.** Let \( g = \sum_{n \in \mathbb{Z}} b_n z^n \in \mathbb{L} \). We define \( \deg_+(g) = \sup\{ n \in \mathbb{N} \cup \{0\} : b_n \neq 0 \} \) and \( \deg_-(g) = \sup\{ n \in \mathbb{N} : b_n \neq 0 \} \) (sup\( \emptyset := 0 \)). Notice that \( \deg_+(g) \), \( \deg_-(g) \) may be equal to \( +\infty \).

Notice that, using the definition of the product of formal Laurent series, one can obtain

\[
a_n^{(k)} = \sum_{m_1 \in \mathbb{Z}} \sum_{m_2 \in \mathbb{Z}} \ldots \sum_{m_k \in \mathbb{Z}} a_{n-m_1} a_{m_1-m_2} \ldots a_{m_{k-2}-m_{k-1}} a_{m_{k-1}-m_{k-2}} \ldots a_{m_2-m_1} a_{m_1} \text{ if } \left( \sum_{n \in \mathbb{Z}} a_n z^n \right)^k \in \mathbb{L}. \tag{5.1}
\]

We will now define the composition of two formal Laurent series.

**Definition 5.3.** Let \( g = \sum_{n \in \mathbb{Z}} b_n z^n \in \mathbb{L} \). We set

\[
\mathbb{X}_g = \{ f = \sum_{n \in \mathbb{Z}} a_n z^n \in P_{\deg_+(g)}(\mathbb{L}) : \exists f^{-1} \in R(f) \cap P_{\deg_-(g)}(\mathbb{L}) \sum_{m=-\deg_-(g)}^{\deg_+(g)} b_m a_n^{(m)} \in \mathbb{C} \text{ for all } n \in \mathbb{Z} \} \tag{5.2}
\]

if \( g \) is not a formal power series (that is \( \deg_-(g) > 0 \)) and

\[
\mathbb{X}_g = \{ f = \sum_{n \in \mathbb{Z}} a_n z^n \in P_{\deg_+(g)}(\mathbb{L}) : \sum_{m=0}^{\deg_+(g)} b_m a_n^{(m)} \in \mathbb{C} \text{ for all } n \in \mathbb{Z} \} \tag{5.3}
\]
if \( g \) is a formal power series (that is \( \text{deg}_{-}(g) = 0 \)).

If \( \mathbb{X}_g \) is nonempty, we define a mapping \( T_g : \mathbb{X}_g \to \mathbb{L} \) as

\[
T_g(f) = g \circ f := \sum_{n \in \mathbb{Z}} c_n z^n, \quad \text{where} \quad c_n = \sum_{m=-\text{deg}_{-}(g)}^{\text{deg}_{+}(g)} b_m a_n^{(m)} \quad \text{for every} \quad n \in \mathbb{Z}.
\] (5.4)

If \( \text{deg}_{-}(g) > 0 \), we call \( g \circ f \) the composition of \( g \) and \( f \) with respect to the inverse \( f^{-1} \in R(f) \) (see that \( g \circ f \) does not have to be unique in general when \( \text{deg}_{-}(g) > 0 \)).

**Remark 5.4.** If it does not cause any misunderstandings or if a given property holds for the composition \( g \circ f \) with respect to any \( f^{-1} \in R(f) \), then we simply call \( g \circ f \) the composition of \( f \) and \( g \) instead of the composition of \( f \) and \( g \) with respect to \( f^{-1} \).

**Remark 5.5.** It is obvious that if \( f \in \mathbb{X}_g \) and \( f \in \mathbb{X}_{g+h} \), then \( f \in \mathbb{X}_{g+h} \) and \( (g+h) \circ f = g \circ f + h \circ f \).

It is also obvious, that if \( f \in \mathbb{X}_g \), then \( f \in \mathbb{X}_{cg} \) and \( (cg) \circ f = cg \circ f \) for all \( c \in \mathbb{C} \).

What is more, \( g \circ f \) with respect to an inverse \( f^{-1} \) of \( f \) is equal to \( \tilde{g} \circ f^{-1} \) with respect to the inverse \( f^{-1} \) if \( g \circ f \) with respect to the inverse \( f^{-1} \) exists.

Let us now investigate whether the composition of formal Laurent series satisfies the Chain Rule and the Right Distributive Law as in the case of formal power series (see [12], [14], [15], [10] for more details).

**5.2. The Chain Rule.** The Chain Rule for formal power series can be formulated in the following way:

**Theorem 5.6 (11, Thm. 5.5.3.).** Let \( g(z) = \sum_{n=0}^{\infty} b_n z^n \) and \( f(z) = \sum_{n=0}^{\infty} a_n z^n \), \( \text{deg}(f) \neq 0 \), be two formal power series over \( \mathbb{C} \). Then \( g' \circ f \) exists, if and only if \( g \circ f \) exists. Moreover,

\[
(g \circ f)'(z) = g'(f(z)) f'(z)
\]

if \( g \circ f \) exists.

**Remark 5.7.** Similarly to [4], Remark 18, the assumption \( \text{deg}(f) \neq 0 \) must be included in the above theorem. Indeed, the proof of Theorem 5.5.3. in [11] is based on Lemma 5.5.2, whose proof uses Theorem 5.4.1 (the general composition theorem). However, one of the assumptions of that theorem is that \( \text{deg}(f) \neq 0 \). This assumption is, unfortunately, omitted in Theorem 5.5.3 in [11].

We will now investigate whether an analogous theorem is true for the composition of formal Laurent series. First, let us consider the following examples:

**Example 5.8.** (1) Let \( f(z) = 1 - z \) and \( g \in \mathbb{L} \). We have \( f' = -S_1 \), so \( f' \circ g = -S_1 \in \mathbb{L} \) and \( (f' \circ g)g' = -g' \in \mathbb{L} \). Moreover, \( (f \circ g)' = (S_1 - g)' = -g' \in \mathbb{L} \), so \( (f \circ g)' = (f' \circ g)g' \).
(2) Let $f = \sum_{n \in \mathbb{Z}} \frac{1}{n^2} z^n$, $g = S_1$. It is easy to check that $f \circ g$ exists (so $(f \circ g)'$ exists), since $\sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{1}{n^2} \in \mathbb{C}$. However, $f' \circ g = (\sum_{n \in \mathbb{Z} \setminus \{-1\}} \frac{1}{n+1})S_1 \notin \mathbb{L}$, so $f' \notin X_g$ and therefore $(f' \circ g)'$ does not exist (although $g' = S_0$).

(3) Let $f = z^2$, $g = \sum_{n \in \mathbb{Z}} \frac{1}{n^2} z^n$. It is easy to check that $g \in P_2(\mathbb{L})$, since $\sum_{m \in \mathbb{Z} \setminus \{0,n\}} \frac{1}{m-m} \in \mathbb{C}$ for all $n \in \mathbb{Z}$. Therefore $f \circ g$ exists. What is more, $f' \circ g = (2z) \circ g = 2g \in \mathbb{L}$. However, see that $g'g'$ does not exist, since the series $\sum_{m \in \mathbb{Z} \setminus \{0,n+1\}} \frac{1}{m}(n-m+1)\frac{1}{n-m+1}$ is divergent for all $n \in \mathbb{Z}$. Therefore $(f' \circ g)g'$ does not exist.

The above examples show that the Chain Rule holds for some but not all formal Laurent series; it does not even always hold even if $f = z^k$, $k \in \{2,3,\ldots\}$ - indeed, one can have $g \in P_k(\mathbb{L})$ but $g^{k-1} \notin \mathbb{L}(g')$.

It can be proven, however, that the Chain Rule holds for formal Laurent series satisfying some additional assumptions. Indeed, we have the following

**Proposition 5.9.** Let $f = \sum_{n \in \mathbb{Z}} a_n z^n \in \mathbb{L}$ and let $g = \sum_{n \in \mathbb{Z}} b_n z^n$ be a formal Laurent series such that

1. $g \in P_{\deg_+(f)}(\mathbb{L})$ and, if $\deg_-(f) < 0$, then $g^{-1} \in P_{\deg_-(f)+1}(\mathbb{L})$ for some $g^{-1} \in R(g);$
2. $(g^m)' = mg^{m-1}g'$ for all $m \in \{-\deg_-(f),\ldots,\deg_+(f)\} \setminus \{0\}$.

Then

1. if $f' \circ g \in \mathbb{L}$, then $(z^{-1}f) \circ g \in \mathbb{L};$
2. if $f \circ g \in \mathbb{L}$, then, denoting $(f \circ g)' = \sum_{n \in \mathbb{Z}} d_n z^n,$

$$d_n = \sum_{m \in \{-\deg_-(f),\ldots,\deg_+(f)\} \setminus \{0\}} \sum_{k \in \mathbb{Z}} ma_mb_k^{(m-1)}(n-k+1)b_{n-k+1} \text{ for all } n \in \mathbb{Z}. \quad (5.5)$$

What is more, $f \circ g \in \mathbb{L}$, if and only if the above double series is convergent for all $n \in \mathbb{Z}$.
3. if $(f' \circ g)g' \in \mathbb{L}$, then, denoting $(f' \circ g)g' = \sum_{n \in \mathbb{Z}} e_n z^n,$

$$e_n = \sum_{k \in \mathbb{Z}} \sum_{m \in \{-\deg_-(f),\ldots,\deg_+(f)\} \setminus \{0\}} ma_mb_k^{(m-1)}(n-k+1)b_{n-k+1} \text{ for all } n \in \mathbb{Z}. \quad (5.6)$$

What is more, $(f' \circ g)g' \in \mathbb{L}$, if and only if $f' \circ g \in \mathbb{L}$ and the above double series is convergent for all $n \in \mathbb{Z}$.
4. if $f' \circ g \in \mathbb{L}$ and one of the series $(5.5)$, $(5.6)$ is absolutely convergent for all $n \in \mathbb{Z}$, then $f \circ g,(f' \circ g)g' \in \mathbb{L}$ and $(f \circ g)' = (f' \circ g)g'$.

**Proof.** (1) since $f' \circ g$ exists, the series

$$\sum_{m \in \{-\deg_-(f)-1,\ldots,\deg_+(f)-1\}\setminus\{-1\}} (m+1)a_{m+1}b_n^{(m)}$$
is convergent for all \( n \in \mathbb{Z} \). By \([14]\), Lemma 4.1, the series
\[
\sum_{m \in \{-\deg(f)-1,...,\deg(f)-1\} \setminus \{0\}} a_{m+1} b_n^{(m)}
\]
is also convergent for all \( n \in \mathbb{Z} \), which completes this part of the proof.

(2) since \( g \) satisfies (1\(^*\), (2\(^*\)), we have \((n+1)b_{n+1}^{(m)} = m \sum_{k \in \mathbb{Z}} b_k^{(m-1)}(n+k+1) b_{n-k+1} \) for all
\( m \in \{-\deg(f),...,\deg(f)\} \setminus \{0\} \) and, of course, \((n+1)b_{n+1}^{(0)} = 0 \) for all \( n \in \mathbb{Z} \). Therefore, for all \( n \in \mathbb{Z} \),
\[
d_n = (n+1) \sum_{m \in \{-\deg(f),...,\deg(f)\} \setminus \{0\}} a_m b_{n+1}^{(m)}
\]
which completes this part of the proof.

(3) we have
\[
e_n = \sum_{k \in \mathbb{Z}} \sum_{s \in \{-\deg(f)-1,...,\deg(f)-1\} \setminus \{0\}} (s+1)a_{s+1} b_k^{(s)}(n-k+1) b_{n-k+1}
\]
\[
= \sum_{k \in \mathbb{Z}} \sum_{m \in \{-\deg(f),...,\deg(f)\} \setminus \{0\}} ma_m b_k^{(m-1)}(n-k+1) b_{n-k+1}
\]
for all \( n \in \mathbb{Z} \), which completes the proof.

(4) the last statement of the proposition is a direct result of (2), (3) and \([21]\), Thm. 7.50.

\[\square\]

5.3. The Right Distributive Law. The Right Distributive Law for formal power series can be formulated in the following way:

**Theorem 5.10** \([11]\), Thm. 5.6.2., [4], Remark 18). Let \( A(x) = \sum_{n=0}^{\infty} a_n x^n \), \( B(x) = \sum_{n=0}^{\infty} b_n x^n \) and
\( P(x) = \sum_{n=0}^{\infty} p_n x^n \), \( \deg(P) \neq 0 \) be three formal power series over \( \mathbb{C} \). The right distributive law, that is,
\[
(A \circ P)(B \circ P) = (AB) \circ P
\]
holds if both \( A \circ P \) and \( B \circ P \) exist.

We will now investigate whether an analogous theorem is true for the composition of formal Laurent series. First, let us consider the following examples:

**Example 5.11.**

(1) Let \( f(z) = 1 - z , \ g(z) = 2z, \ h \in P_{2}(\mathbb{L}) \). It follows directly from \([5,4]\) and the definition of multiplication of formal Laurent series that \( f \circ h = S_1 - h \in \mathbb{L} \),
\( g \circ h = 2h \in \mathbb{L} \), \( fg = 2z - z^2 \in \mathbb{L} \), \( (fg) \circ h = 2h - h^2 \in \mathbb{L} \), \( (f \circ h)(g \circ h) = 2h - h^2 \in \mathbb{L} \), so
\( (f \circ h)(g \circ h) = (fg) \circ h \).
(2) Let \( f = \sum_{n \in \mathbb{Z}} a_n z^n \), where \( a_n = \frac{(-1)^n}{\sqrt{|n|}} \) for \( n \neq 0 \) and \( a_0 = 0 \). Let also \( h = S_1 \) and \( g = 2f \). It is obvious that \( \sum_{n \in \mathbb{Z}} a_n \in \mathbb{C} \). Denote \( \sum_{n \in \mathbb{Z}} a_n = A \). By (5.4), \( f \circ h = AS_1 \in \mathbb{L}, \ g \circ h = 2AS_1 \in \mathbb{L}, \)
so \( (f \circ h)(g \circ h) = 2A^2S_1 \in \mathbb{L} \). Now, let us notice that
\[
\sum_{m \in \mathbb{Z}} 2a_m a_{-m} = \sum_{m \in \mathbb{Z}} \frac{2}{|m|},
\]
which is of course divergent.
Therefore \( f \notin \mathbb{L}(g) \), so \( (fg) \circ h \) does not exist.

(3) Let \( f(z) = z^l, g(z) = z^m, l, m \in \mathbb{Z}_+ \) and let \( h \in P_{max(m,l)}(\mathbb{L}) \setminus P_{l+m}(\mathbb{L}) \). Then \( f \circ h, g \circ h \in \mathbb{L} \)
and \( fg \in \mathbb{L} \). However, \( (fg) \circ h \notin \mathbb{L} \). Now let \( f(z) = z^2 \) and \( g(z) = z^{-1} \). Assume \( h \in \mathbb{L} \setminus P_2(\mathbb{L}) \)
possesses an inverse \( h^{-1} \). Then \( g \circ h, (fg) \circ h \in \mathbb{L} \), but \( f \circ h \notin \mathbb{L} \).

Finally, let \( f(z) = g(z) = z^2, h = \sum_{n \in \mathbb{Z}} h_n z^n \in P_4(\mathbb{L}) \). We have \( f \circ h = g \circ h = \sum_{n \in \mathbb{Z}} h_n(2) z^n \),
where \( h_n(2) = \sum_{m_1 \in \mathbb{Z}} h_{n-m_1} h_{m_1} \) and \( (fg) \circ h = \sum_{n \in \mathbb{Z}} h_n(4) z^n \), where
\[
h_n(4) = \sum_{m_3 \in \mathbb{Z}} \sum_{m_2 \in \mathbb{Z}} \sum_{m_1 \in \mathbb{Z}} h_{n-m_3} h_{m_3-m_2} h_{m_2-m_1} h_{m_1}, \quad (5.7)
\]
Moreover, \( (f \circ h)(g \circ h) \in \mathbb{L} \), if and only if the series
\[
\sum_{m_3 \in \mathbb{Z}} \sum_{m_2 \in \mathbb{Z}} \sum_{m_1 \in \mathbb{Z}} h_{m_1} h_{m_3-m_1} h_{m_2} h_{n-m_3-m_2} = \sum_{m_3 \in \mathbb{Z}} \sum_{m_2 \in \mathbb{Z}} \sum_{m_1 \in \mathbb{Z}} h_{n-M_3} h_{M_3-M_2} h_{M_2-m_1} h_{m_1}, \quad (5.8)
\]
(where we made the substitution \( M_2 = m_1 + m_2, M_3 = m_2 + m_3 \) is convergent for all \( n \in \mathbb{Z} \).
However, see that the series \( (5.7), (5.8) \) have the same terms but different order
of summation, so one cannot state in general that \( (5.7) \) is convergent for all \( n \in \mathbb{Z} \), if and
only if \( (5.8) \) is convergent for all \( n \in \mathbb{Z} - \) or that these series have equal sums if they both
converge.

The above examples show that the Right Distributive Law holds for some but not all formal
Laurent series. It can be proven, however, that the Right Distributive Law holds for some formal
Laurent series satisfying additional assumptions. Indeed, we have the following

**Proposition 5.12.** Let \( f = \sum_{n=0}^{\infty} a_n z^n, g = \sum_{n=0}^{\infty} b_n z^n \) be two formal power series whose coefficients
belong to \( \mathbb{R}_+ \cup \{0\} \) and let \( h = \sum_{n \in \mathbb{Z}} c_n z^n \) be a formal Laurent series such that \( c_n \geq 0 \) for all \( n \in \mathbb{Z} \),
h \in \mathbb{X}_f \cap \mathbb{X}_g \) (so \( h \in P_{max(\deg(f),\deg(g))}(\mathbb{L}) \)) and \( h^k h^{l-k} = h^l \) for all \( 0 \leq k \leq l \leq \max(\deg(f), \deg(g)) \).
Then \( (f \circ h)(g \circ h) \in \mathbb{L} \), if and only if \( (fg) \circ h \in \mathbb{L} \). Moreover, \( (f \circ h)(g \circ h) = (fg) \circ h \) if the series
on both sides exist.

**Proof.** Assume that \( (f \circ h)(g \circ h) \in \mathbb{L} \) and denote its coefficients by \( d_n \). We have, by the Mertens
Theorem,
\[
d_n = \sum_{m \in \mathbb{Z}} \left( \sum_{k=0}^{\infty} a_k c_m^{(k)} \right) \left( \sum_{k=0}^{\infty} b_k c_{n-m}^{(k)} \right) = \sum_{m \in \mathbb{Z}} \sum_{k=0}^{\infty} a_k b_k c_{m}^{(k)} c_{n-m}^{(k)}, \quad \text{for } n \in \mathbb{Z}.
\]
What is more, \((f \circ h)(g \circ h) \in \mathbb{L}\), if and only if \(f \circ h, g \circ h \in \mathbb{L}\) and the above triple series is convergent. Now, assume that \((fg) \circ h \in \mathbb{L}\) and denote its coefficients by \(e_n\). We have

\[
e_n = \sum_{k=0}^{\infty} \sum_{s=0}^{k} a_s b_{k-s} c_n^{(k)} = \sum_{k=0}^{\infty} \sum_{s=0}^{k} \sum_{m \in \mathbb{Z}} a_s b_{k-s} c_m^{(s)} c_n^{(k-s)}
\]

What is more, \((fg) \circ h \in \mathbb{L}\), if and only if the above triple series is convergent. The claim is now a direct result of [21, Thm. 2.50].

The next proposition can be proven analogously to the previous one (using Thm. 7.50 from [21], instead of Thm. 2.50).

**Proposition 5.13.** Denote \(f^* = \sum_{n \in \mathbb{Z}} |f_n| z^n\) for every \(f = \sum_{n \in \mathbb{Z}} f_n z^n \in \mathbb{L}\). Let \(f = \sum_{n=0}^{\infty} a_n z^n\), \(g = \sum_{n=0}^{\infty} b_n z^n\) be two formal power series over \(\mathbb{C}\) and let \(h = \sum_{n \in \mathbb{Z}} c_n z^n\) be a formal Laurent series such that

- \(h \in \mathbb{X}_f \cap \mathbb{X}_g\) and \(h^k h^{l-k} = h^l\) for all \(0 \leq k \leq l \leq \max(\deg(f), \deg(g))\),
- \(h^* \in \mathbb{X}_f^* \cap \mathbb{X}_g^*\) and \((h^*)^k (h^*)^{l-k} = (h^*)^l\) for all \(0 \leq k \leq l \leq \max(\deg(f), \deg(g))\).

Then if either \((f^* \circ h^*)(g^* \circ h^*) \in \mathbb{L}\) or \((f^* g^*) \circ h^* \in \mathbb{L}\), then \((f \circ h)(g \circ h), (fg) \circ h \in \mathbb{L}\) and \((f \circ h)(g \circ h) = (fg) \circ h\).

**Remark 5.14.** Let \(f, g \in \mathbb{L}\) and let \(g^{-1}\) be an inverse of \(g\). Assume \(g^{-1} \circ f, g \circ f\) exist. See that \(g^{-1} \circ f = (g \circ f)^{-1}\), if and only if \(g, g^{-1}, f\) satisfy the Right Distributive Law, that is \((g \circ f)(g^{-1} \circ f) = (gg^{-1}) \circ f = S_1 \circ f = S_1\).

Finally, let us emphasize that the problem of finding general conditions for the existence of composition of any two given formal Laurent series in a general case still remains open.

### 6. Boundary convergence of formal Laurent series

The problem of boundary convergence of power series was considered e.g. in [13]. Now we will extend the results from this paper to Laurent series, using the methods of formal analysis. First, let us recall the well-known

**Theorem 6.1 ([18]).** A Laurent series \(f = \sum_{n \in \mathbb{Z}} a_n z^n\), \(z \in \mathbb{C}\) is

- convergent if \(|x| \in (r, R)\),
- divergent if \(|x| \notin [r, R]\),

where \(r := \limsup_{n \to \infty} |a_{-n}|^{1/n}\), \(1/R := \limsup_{n \to \infty} |a_n|^{1/n}\).

Moreover, we will also use an obvious
Lemma 6.2. Let $r_f, R_f$ be the radii of convergence of a Laurent series $f$. Let $f'$ be the formal derivative of $f$ and denote its radii of convergence as $r'_f, R'_f$, respectively. Then $r_f = r'_f$ and $R_f = R'_f$.

The main result of this section states as follows.

Theorem 6.3. Let $g(z)$ be a Laurent series and let $r, R$ ($0 < r < R < +\infty$) be its radii of convergence. Then

1. if $(g^+)^{(k)}(a) \in \mathbb{C}$ ($(g^+)^{(k)}$ denotes the $k$th order formal derivative of the regular part of $g$) for every $k \in \mathbb{N} \cup \{0\}$ for some $a \in \mathbb{C}$, $|a| = R$, then $g^{(k)}(z)$ converges absolutely for every $k \in \mathbb{N} \cup \{0\}$ for every $z \in \mathbb{C}$ such that $r < |z| \leq R$;

2. if $(g^-)^{(k)}(b)$ converges absolutely ($(g^-)^{(k)}$ denotes the $k$th order formal derivative of the principal part of $g$) for every $k \in \mathbb{N} \cup \{0\}$ for some $b \in \mathbb{C}$, $|b| = r$, then $g^{(k)}(z)$ converges absolutely for every $k \in \mathbb{N} \cup \{0\}$ for every $z \in \mathbb{C}$ such that $r \leq |z| < R$.

Proof. We know by [13], Lemma 2.4, that if $(g^+)^{(k)}(a) \in \mathbb{C}$ for every $k \in \mathbb{N} \cup \{0\}$ for some $a \in \mathbb{C}$, $|a| = R$, then $(g^+)^{(k)}$ converges absolutely on the closed disk $\{z \in \mathbb{C} : |z| \leq R\}$ for every $k \in \mathbb{N} \cup \{0\}$. We also know that $g^-$ converges absolutely for $|z| > r$, so using Lemma 6.2, $g^{(k)} = (g^+)^{(k)} + (g^-)^{(k)}$ converges absolutely for every $z \in \mathbb{C}$ such that $r < |z| \leq R$ for every $k \in \mathbb{N} \cup \{0\}$, which ends the first part of the proof.

Now, let us notice that $h(z) = \sum_{n=1}^{\infty} a_n z^n$ is a power series with the radius of convergence $r_h = \frac{1}{r}$.

By [13], Lemma 2.4, if $h^{(k)}(c) \in \mathbb{C}$ for every $k \in \mathbb{N} \cup \{0\}$ for some $c \in \mathbb{C}$, $|c| = r_h = \frac{1}{r}$, then $h^{(k)}(z)$ converges absolutely for every $k \in \mathbb{N} \cup \{0\}$ for every $z \in \mathbb{C}$ such that $|z| \leq \frac{1}{r}$. For every $k \in \mathbb{Z}$, the $k$th order formal derivatives of $h$ and $g^-$ are equal to

$$h^{(k)}(z) = \sum_{n=0}^{\infty} (n+1)...(n+k)a_{-n-k}z^n,$$

$$g^{(-k)}(z) = z^{-2k}(-1)^k \sum_{n=-k+1}^{\infty} (n+2k-1)...(n+k)a_{-n-k}z^{-n}.$$

However, we have, for every $z \in \mathbb{C}$,

$$\lim_{n \to \infty} \frac{(n+k)...(n+2k-1)a_{-n-k}z^n}{(n+1)...(n+k)a_{-n-k}z^n} = 1,$$

so $g^{(-k)}(\frac{1}{r})$ converges absolutely for some $z \in \mathbb{C}$, if and only if $h^{(k)}(z)$ converges absolutely. It follows that if $(g^-)^{(k)}(b)$ converges absolutely for every $k \in \mathbb{N} \cup \{0\}$ for some $b \in \mathbb{C}$, $|b| = r$, then $(g^-)^{(k)}(z)$ converges absolutely for every $k \in \mathbb{N} \cup \{0\}$ for every $z \in \mathbb{C}$ such that $|z| \geq r$. Therefore, using Lemma 6.2, $g^{(k)}$ converges absolutely for every $k \in \mathbb{N} \cup \{0\}$ for every $z \in \mathbb{C}$, such that $r \leq |z| < R$, which ends the proof.

\[\square\]

Corollary 6.4. Let $g = \sum_{n \in \mathbb{Z}} b_n z^n$ be a Laurent series and let $r, R$ ($0 < r < R < +\infty$) be its radii of convergence. If $(g^+)^{(k)}(a) \in \mathbb{C}$ for every $k \in \mathbb{N} \cup \{0\}$ for some $a \in \mathbb{C}$, $|a| = R$ and $(g^-)^{(k)}(b)$
converges absolutely for every \( k \in \mathbb{N} \cup \{0\} \) for some \( b \in \mathbb{C} \), \(|b| = r\), then \( g^{(k)} \) converges absolutely for every \( k \in \mathbb{N} \cup \{0\} \) for every \( z \in \mathbb{C} \) such that \( r \leq |z| \leq R \).

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