Asymmetric shape and dynamic stability
of exciton-phonon solitons moving in a periodic medium

Dan Roubtsov† and Yves Lépine
Groupe de Recherche en Physique et Technologie des Couches Minces,
Département de Physique, Université de Montréal,
C.P. 6128, succ. Centre-ville, Montréal, QC, H3C 3J7, Canada

E. Nihan Önder
Institut de génie nucléaire, École Polytechnique de Montréal,
C.P. 6079, succ. Centre-ville, Montréal, QC, H3C 3A7, Canada
†e-mail: roubtsod@magellan.umontreal.ca

Abstract

Solitons are known to move ballistically through a medium without changement of their shape. In practice, the shape of moving inhomogeneous states changes, and a long lasting tail appears behind the soliton moving in a periodic medium. Such a behavior can be described within the model of exciton-phonon coherent states as a dynamic effect in soliton transport. We argue that the coupling between bosonic excitations of the medium, such as excitons, and elastic modes of it, such as phonons, can be responsible for these effects. We derive nonlinear dynamic equations for the excited medium in the long wavelength approximation (a generalized Zakharov system) and apply a kind of ballistic ansatz for the coherent state of bosons and displacement field. Like solitons, the quasistationary solution we obtain on this way can move through the medium ballistically. Unlike solitons and kinks, there are inhomogeneous corrections to the ballistic velocity and the coherent phase of the condensate that control the changement of the packet shape. In the limit of $T \to 0$, laws of conservation prescribe formation of the cloud of collective excitations around the quasistationary Bose-core of the packet, and the density of a growing boson-phonon tail behind the moving coherent part can be estimated. The total packet can be associated with an exciton-phonon comet with the quasistable coherent Bose-core and incoherent tail moving in the medium.

PACS numbers: 71.35.Lk, 05.30.Jp, 63.20.Ls, 64.60.Ht

submitted to Phys. Rev. B, preprint at arXiv:cond-mat/0008284v2, revised October 2000
1 Introduction

Some localized solutions of a special (and, actually, quite a narrow) class of nonlinear evolution equations are known to be able to move conserving their shape and interacting with each other like particles. The typical examples are the solitons of Nonlinear Schrödinger equation (NLS) and Korteweg-de-Vriese equation (KdV), the kinks of sine-Gordon equation (SG) to name a few [1], [2]. It is worthy of mentioning that these particular solutions conserve all the intervals of motions prescribed by dynamic equations. However, as the evolution equations mentioned above are the result of approximation of more complicated dynamic equations, the solitonic properties of their solutions are the approximation as well. The question up to which extent they survive in ‘reality’ leads to interesting physical problems that urge people to go beyond the theory of exactly solvable models [3], [4], [5].

Fortunately, the exact results obtained within refine mathematical models can be compared with the real physical experiments. In addition, numerical simulations within more realistic models are performed for many equations supporting solitonic solutions [6]. As a result, one can realize under which conditions the concept of soliton is a good approximation to interpret physical data. For example, the NLS equation is used to describe the phenomenon of Bose-Einstein condensation of dilute trapped gases [7] and sharp pulses of light in optical fibers [8], and the KdV equation is used in hydrodynamics to model the moving packets of surface waves in channels [9]. The SG equation models surface grow and reconstruction in material science [10].

In this article, we discuss the transport properties of excited states (generally of electron transition nature) in a medium (a periodic structure, such as a crystal or semiconductor structure, some periodic biological tissues, etc.). Under certain conditions, a coherent state, which involves both the electronic (or spin) excitations and the medium elastic excitations, can appear in such structures. This state turns out to be localized inside the medium and can travel ballistically through it. In Biological Physics, such a state is known as Davydov soliton [11], whereas in Condensed Matter Physics, one can mention, for example, Spin-Peierls Systems [12], Charge Density Waves [13], and anomalous transport of excitonic packets in semiconducting crystals and heterostructures [14]. In particular, we are motivated by the ballistic transport of excitons in pure 3D crystals, such as Cu$_2$0 [15]. However, the ballistic transport of localized coherent structures and a set of nonlinear equations being used to describe it can be found in modeling of different physical phenomena [4], [16], [17]. For example, modeling of the coherent excited states of electronic origin ends up with NLS equation, whereas the coherent states of the elastic medium are described by the wave equation with nonlinear anharmonic terms (NLW equation) [18], [19], [20]. In plasma physics, this is the case of Zakharov system of equations [21], [22], [23], while the Davey-Stewartson system can be mentioned in the context of fluid mechanics [24]. Note that the two equations comprising such a system, NLS and NLW, are quite different from the mathematical point of view, but they are coupled with each other [24], [25], [26] and this is the origin of the wealth of physical effects they can describe.

Not surprisingly for the moving soliton-like states in periodic media, a shape of experimentally registered signals is far from being look like a “true” soliton (e.g., α
1/cosh(\(L_0^{-1}(x - vt)\)) or kink. One can assume that nonzero temperature, scattering on impurities and just a noise factor can be responsible for this. In some cases, however, the character of changement hints on some regular dynamic reason for these effects. For example, an asymmetric form of the signal with the pronounced sharp front and a long lasting tail behind the single soliton are observed on experiment [15],[27],[28]. Note that the subsonic ballistic transport is under consideration, i.e., \(\langle v \rangle < c_s\), where \(\langle v \rangle\) is the speed of the packet and \(c_s\) is the (longitudinal) sound speed in the periodic medium. The similar effects, such as delocalized solitons, were also observed in different physical settings. For example, in a kind of pump and probe experiments with sharp pulses of light in fibers, the solitonic shape of a probe pulse was significantly changed by Stimulated Raman Scattering [29]; see also [30]. In fluid mechanics, one can mention the problem of wave breaking and appearance of the so-called peakons and compactons in the soliton-like solutions of generalized KdV equations [31]; see also [32]. In essence, the non-steady solitons are found to be able to move with an acceleration and generate “tails” behind themselves.

In contrast, the transition to diffusion regime is possible for the ballistic exciton-phonon droplets, and, in the case of amplification of the soliton-like state formed inside the exciton-phonon packet, its ballistic velocity is found to be almost unchanged during such a non-stationary process [15]. These facts hint at different physics controlling the moving coherent state formed by excitons and phonons.

In this article, we assume that the coupling between collective excitations of the periodic medium can be responsible for such effects in Condensed Matter Physics, and a proper dynamic model can reproduce them even at \(T \to 0\). To support this hypothesis, we consider a field model that admits a soliton-like solution and can be easily generalized without a loss of physical clarity.

This is a two field model, in which the excitations of the medium are modeled by a nonideal Bose-gas coupled with long wavelength modes of the displacement field. For example, two (or even several) interacting boson fields can model different phenomena in Condensed Matter Physics [18]. In the case of semiconductors, one can often disregard the influence of free fermions or fermionic complexes on the processes under consideration. It turns out that many processes involving photons, excitons, and phonons can be described by use of the language of interacting Bose-fields [33]. Moreover, if there is a branch of optically inactive excitons in a crystal, one can even exclude the photons from simple models dealing with such excitons. In addition, the lifetime of a moving exciton with \(\hbar k_0 \approx m_x c_s\) can be large enough in semiconductor structures, so that the transport properties of a packet of moving excitons can be observed experimentally [15],[34].

### 2 Hamiltonian of the model

The Hamiltonian of the medium can be taken in the following form:

\[
\hat{H} = H_{\text{gas}}(\hat{\psi}, \hat{\psi}^\dagger) + H_{\text{ph}}(\hat{u}, \hat{\pi}) + H_{\text{int}}(\hat{u}, \hat{\psi}^\dagger \hat{\psi}).
\]

Here, \(\hat{\psi}, \hat{\psi}^\dagger\) are the Bose-gas field operators; they stands in the non-relativistic Hamiltonian \(H_{\text{gas}}\), whereas \(\hat{u}\) is the displacement field operator and \(\hat{\pi}\) is the momentum density.
operator conjugate to $\hat{u}$. This pair stands in a Phonon Hamiltonian $H_{ph}$. We consider a nonideal gas of phonons in the long-wavelength approximation and take into account the acoustic branch only, e.g., the medium is a 3D crystal of the volume $V = L S_\perp$.

The Bose-gas Hamiltonian has the following form:

$$H_{gas} = \int d\mathbf{x} \hat{E}_g \hat{\psi}^\dagger(\mathbf{x}) + \frac{\hbar^2}{2m} \nabla \hat{\psi}^\dagger \nabla \hat{\psi}(\mathbf{x}) + \frac{\nu_0}{2} \left( \hat{\psi}^\dagger(\mathbf{x}) \right)^2 \left( \hat{\psi}(\mathbf{x}) \right)^2 + \frac{\nu_1}{3} \left( \hat{\psi}^\dagger(\mathbf{x}) \right)^3 \left( \hat{\psi}(\mathbf{x}) \right)^3,$$

where $\nu_0 > 0$ is the strength of two particle repulsive interaction, and $\nu_1 > 0$ is the strength of three particle one. As the two particle interaction ($\sim \nu_0 (\hat{\psi}^\dagger)^2 \hat{\psi}^2$) can be strongly renormalized because of interaction with other fields, we include the hard core interaction term (modeled by repulsion $\sim \nu_1 (\hat{\psi}^\dagger)^3 \hat{\psi}^3$ in $H_{gas}$). We assume that the ‘bare’ characteristic energies of the particle-particle interactions satisfy the following inequality:

$$0 < \nu_1/a_x^6 < (\ll) \nu_0/a_x^3 \simeq \text{const } Ry_x, \quad \text{const} \approx 10,$$

see [33] for discussion. Here, $a_x$ and $Ry_x$ are the exciton Bohr radius and characteristic Rydberg energy, respectively. We count the energy of a free particle from $\hat{E}_g = E_g - Ry_x > 0$, so that $E_k = \hat{E}_g + \hbar k^2/2m$. (For semiconducting materials, $E_g$ is the fundamental gap.)

As only the longitudinal phonons $\mathbf{u}_l$ interact with the Bose-field $\hat{\psi}, \hat{\psi}^\dagger$ in our model, one can exclude the transversal phonons ($\nabla \mathbf{u}_t = 0$) from $H_{ph}$. Then, it can be reduced to the following simple form:

$$H_{ph} = \int \frac{\tilde{\kappa}^2(\mathbf{x})}{2\rho} + \frac{\rho c_l^2}{2} \partial_j \hat{u}_s \partial_j \hat{u}_s(\mathbf{x}) + \frac{\rho c_l^4}{3} \kappa_3 \partial_j \hat{u}_s \partial_j \hat{u}_s \partial_j \hat{u}_s(\mathbf{x}) + \frac{\rho c_l^7}{4} \kappa_4 (\partial_j \hat{u}_s)^4 \, d\mathbf{x},$$

where $c_l$ is the longitudinal sound speed of the crystal, and the dimensionless parameters $\kappa_3$ and $\kappa_4$ account for cubic and quartic nonlinearities. (In this article, we will not take into account a quartic term, so $\kappa_4 \approx 0$.)

We take the gas-phonon interaction in the form of Deformation Potential:

$$H_{int} = \int \sigma_0 \partial_j \hat{u}_j(\mathbf{x}) \hat{\psi}^\dagger \hat{\psi}(\mathbf{x}) + \vartheta_0 \partial_j \hat{u}_j(\mathbf{x}) \left( -\hat{\psi}^\dagger \Delta \hat{\psi}(\mathbf{x}) \right) \, d\mathbf{x}, \quad (2)$$

where $\sigma_0$ and $\vartheta_0$ are the coupling constants. Note that this Hamiltonian is equivalent to

$$\tilde{H}_{int} = \int \sigma_0 \partial_j \hat{u}_j(\mathbf{x}) \hat{\psi}^\dagger \hat{\psi}(\mathbf{x}) + \vartheta_0 \partial_j \hat{u}_j(\mathbf{x}) \nabla \hat{\psi}^\dagger \nabla \hat{\psi}(\mathbf{x}) \, d\mathbf{x}. \quad (3)$$

Here, we choose $\sigma_0 > 0$ and do not fix the sign of $\vartheta_0$. Developing a theory, we have a freedom with the sign of $\vartheta_0$; its value, however, can be roughly estimated as $|\vartheta_0| \approx \hbar^2/2m$. (Recall that $\sigma_0 \sim E_g$.)

If the exciton Bohr radius $a_x$ is no more than several times larger than the lattice constant $a_t$, the second term in (2) can be important too. (In fact, this is an intermediate case between Frenkel exciton and Wannier one with $a_x \gg a_t$.) Then, the following coupling terms appear in Heisenberg equations

$$\tilde{E}_g \hat{\psi} \rightarrow \tilde{E}_g \hat{\psi} + \sigma_0 \partial_j \hat{u}_j \hat{\psi}, \quad (4)$$
\[-(\hbar^2/2m) \Delta \hat{\psi} \rightarrow -(\hbar^2/2m) \Delta \hat{\psi} - \vartheta_0 \partial_j \hat{u}_j \Delta \hat{\psi}. \]  

(5)

In terms of the lattice analog of the medium Hamiltonian, we made the hopping matrix element \( t (t_{ij} \text{ in } \hat{H}_{gas} \sim t_{ij} \hat{\psi}_i^\dagger \hat{\psi}_j) \) of the ‘lattice’ boson \( \hat{\psi}_j \) to be dependent on the ‘lattice’ deformation field \( \hat{u}_j \),

\[ t_{ij} \rightarrow t_{ij}(u_i, u_j) \approx t_{ij} + \tilde{\vartheta}_0 (u_i - u_j). \]

Meanwhile, the energy on a cite, \( \varepsilon_{0,i} = \varepsilon_0 \) (in \( \hat{H}_{gas} \sim \varepsilon_{0,i} \hat{\psi}_i^\dagger \hat{\psi}_i \)) depends on the lattice displacements too, f.ex.,

\[ \varepsilon_{0,i} \rightarrow \varepsilon_0 + \tilde{\sigma}_0 (u_{i+1} - u_{i-1}), \]

see [36] for discussion.

3 Dynamic equations

Our aim is to investigate a special class of solutions of Hamiltonian (1), namely, we will search for the localized moving excitations (or packets). In the case of \( \vartheta_0 = 0 \), such a solution exists and the packet can move through the crystal with the constant velocity \( v \) saving its shape in a manner as the solitons and kinks do in nonlinear media [11],[19]. To simplify the equations of motion, we choose the quasi-1D approximation. This means \( \hat{\psi}(x,t) \rightarrow \hat{\psi}(x,t), \hat{u}_x(x,t) \rightarrow \hat{u}_x(x,t) \). In fact, we assume that the packet is inhomogeneous along the \( Ox \) axis only, and \( v \parallel Ox \). Thus, it occupies all the area of \( S_\perp \). (For a discussion on the validity of 1D approximation, see [37].) Then, we can write the Heisenberg equations of motion as

\[ i\hbar \partial_t \hat{\psi}(x,t) = \]

\[ = \left( \tilde{E}_g - \frac{\hbar^2}{2m} \partial_x^2 - \vartheta_0 \partial_x \hat{u}_x(x,t) \partial_x^2 + \nu_0 \hat{\psi}_i^\dagger \hat{\psi}_i(x,t) + \nu_1 \hat{\psi}_i^\dagger \hat{\psi}_i^\dagger \hat{\psi}_i^\dagger \hat{\psi}_i(x,t) \right) \hat{\psi}(x,t) + \sigma_0 \partial_j \hat{u}_j(x,t) \hat{\psi}(x,t), \]

\[ \left( \partial_t^2 - c_i^2 \partial_x^2 \right) \hat{u}_x(x,t) - \]

\[ - c_i^2 2\kappa_3 \partial_x^2 \partial_x \hat{u}_x(x,t) = \rho^{-1} \sigma_0 \partial_x (\hat{\psi}_i^\dagger \hat{\psi}_i(x,t)) + \rho^{-1} \vartheta_0 \partial_x \left( \partial_x \hat{\psi}_i^\dagger \partial_x \hat{\psi}_i \right). \]  

(6)

(7)

After some energy (and momentum) was pumped into the medium, an excited state of it appears near the boundary where the external energy (and momentum) was absorbed, see Fig. 1. Due to initial conditions, the excited state is a localized one, and it can be modeled as a droplet consisting of excitons (Bose-particles) and phonons. Moreover, it can acquire an average momentum directed along the \( Ox \) axis because of unidirectional phonon production during the thermalization stage. Thus, the exciton-phonon droplet starts to move [39]. The important assumption, however, is the appearance of a moving coherent field, or a condensate, from the localized excited state of a medium. For example, if the medium is a 3D crystal or an array of channels embedded into a matrix, it has to be coupled with a thermostat at low temperature. Moreover, the medium has to posses an inner mechanism of fast thermalization of the electronic excited states, for example, by emission of phonons [38].
Figure 1: A medium, in which the boson-phonon soliton can propagate, is presented on Figures 1(a) and (b) in the form of the channel ‘abcd’. It has the dimensions |ab| = L, |bc| ~ √S⊥, and L ≫ Lch, where Lch is the characteristic width of the soliton. After some amount of energy has been pumped into the medium during a short time interval δt and absorbed near the boundary, a localized excited state is formed near the face ‘ad’. It is schematically shown on Figure 1 (a) as a mixture of excitons and phonons. If there is a mechanism of the momentum transfer to the excited state, the droplet begins to move toward the opposite face ‘bc’ with the velocity ⟨v⟩, see Fig. 1 (b). Such conditions can favor the appearance of a coherent boson-phonon state (an analog of Davydov soliton) moving ballistically along the axis Ox at T < Tc. In other words, a sort of Bose-condensate can appear because of the effective attraction among the bosons (excitons) at T < Tc, see Fig. 1 (b). The coherent state, however, is only a core of the total moving packet. The profile of the excitonic Bose-core, n_o(x, t) ~ |Ψ_o(x, t)|^2, is shown by the dashed line and the intensity of the elastic (phonon) part of the Bose-core, ∂xu_o, x(x, t), is represented by changements of the intensity of the background color on Fig. 1 (b).
It is known [10] that nonlinear elastic lattices support several types of localized excitations; some of them turn out to have the so-called nonzero direct current, \( u_x \approx u_0(x - v t) \). For example, a moving packet can consist of two parts, such as

\[
\hat{u}_x \approx u_0(x - v t) + \sum_k C(x - v t) e^{i(kx - \omega_k t)} \hat{b}_k + \text{h.c.,}
\]

and \( \bar{v} \) is the group velocity of the packet. On the other hand, the Bose-gas has a property of multiple occupancy, so that the coherent mode can be introduced through the substitute \( \hat{\psi}(x, t) \to \hat{\Psi}_o(x, t) \) and \( \int dx |\hat{\Psi}_o(x, t)|^2 \gg 1 \). Thus, in the limit of \( T \to 0 \) and within the long wavelength approximation, one can explore how Eqs. (6), (7) support the localized coherent solutions. In this article, we use the language and technique of Bose-Einstein condensation [11] that are suitable to take into account both the effect of many-particle coherency and the effect of many-particle nonlinearity. Note that the two parts of the coherent field, \( \hat{\Psi}_o(x, t) \) and \( u_0(x, t) \), are taken into account selfconsistently and we do not integrate out the phonons.

Substituting the coherent fields (c-functions) into operator equations (3) and (7), we obtain the following system of dynamic equations (\( \vartheta_0 = (h^2/2m) \bar{\vartheta}_0 \) and \( \bar{\vartheta}_0 \) is dimensionless):

\[
\begin{align*}
\imath \hbar \partial_t \hat{\Psi}_o(x, t) &= -\left( \hbar^2/2m \right) \bar{\vartheta}_0 \partial_x u_o(x, t) \partial_x^2 \hat{\Psi}_o(x, t) + \\
&\quad + \left( \bar{E}_g - \frac{\hbar^2}{2m} \partial_x^2 + \nu_0 |\Psi_o|^2(x, t) + \nu_1 |\Psi_o|^4(x, t) \right) \Psi_o(x, t) + \sigma_0 \partial_x u_o(x, t) \Psi_o(x, t),
\end{align*}
\]

\( \partial_t^2 u_o(x, t) - c_l^2 \partial_x^2 u_o(x, t) - 2c_l^2 \kappa_3 \partial_x^2 u_o \partial_x u_o(x, t) = \\
= \rho^{-1} \sigma_0 \partial_x (\Psi_o^* \Psi_o(x, t)) + \rho^{-1} \partial_0 \partial_x \left( \partial_x \Psi_o^* \partial_x \Psi_o(x, t) \right). \quad (9)
\]

This system can be considered as a generalization of the Zakharov system which appeared in Plasma Physics [24]. The main difference is that, in the case of collective phenomena in Condensed Matter Physics, Eqs. (8) and (9) define the main part of the moving packet. (It is called the coherent Bose-core in this article.) In fact, in the case of \( T \neq 0 \) and, generally, in all the nonstationary cases, these two equations have to be coupled with another system of equations on the so-called out-of-condensate excitations [11]. The dynamics of these excitation states can strongly influence the dynamics of the “parent” condensate.

In this article, we rely on a kind of adiabatic hypothesis to simplify the solution of the problem. If the occupancy of the coherent mode, \( N_o(t) = \int dx |\Psi_o(x, t)|^2 \), and the occupancy of the out-of-condensate cloud \( \delta N(t) = \int dx \langle \delta \psi^\dagger \delta \psi(x, t) \rangle \), change in time slowly and \( N_o(t) \gg \delta N(t) \), one can try to find some quasi-stationary solution for the coherent core \( \Psi_o(x, t) \) and balance its energy and momentum by a kind of “leakage” from the Bose-core into the incoherent out-of-condensate cloud and tail, \( N_o \to N_o - \delta N(t) \). This approach can be called the propagation of an exciton-phonon “comet” in a periodic medium, and it is valid within a finite time interval which is estimated below.

Recall that the standard ballistic ansatz with \( v = \text{const} \),

\[
u_o(x, t) = u_o(x - t v),
\]

\[\]
\[ \Psi_o(x, t) = \exp(i(\varphi_c + k_0 x)) \exp(-i\Omega(k_0) t) \psi_o(x - t v), \]

where \[ \hbar \Omega(k_0) = \tilde{E}_g + (\hbar^2 k_0^2/2m) + \mu, \quad \hbar k_0 = mv, \]
does not work properly in the case of \( \vartheta_0 \neq 0 \). Moreover, with such a ballistic ansatz, we always have \( v = v_s \), where \( v_s, x \propto \partial_x \varphi_c(x, t) = \text{const} \) is the superfluid velocity of the Bose-core. As the anomalous ballistic transport of exciton-phonon packets in semiconductor and quasi-1D structures resembles, to some extent, the well-known effect of superfluidity \[ 12 \], we apply the methods of Many-Particle Physics to Eqs. \( 8 \) and \( 9 \). We assume that the following substitutes for the ballistic velocity and the chemical potential,

\[ v \rightarrow v(x, t) \approx v \left( 1 \pm \zeta(v) \psi_o^2(x, t) + \ldots \right), \] \( (10) \)

\[ \mu \rightarrow \mu(x, t) \approx \mu \left( 1 \pm \tilde{\zeta}(v) \psi_o^2(x, t) + \ldots \right), \] \( (11) \)

are more appropriate to model the nonlinear dynamics of the ballistic boson-phonon packets in the case of \( \vartheta_0 \neq 0 \). However, within the model with Hamiltonian \( 8 \), one can start, for example, from

\[ v \rightarrow v(x, t) \approx v \mathcal{F}_1(\partial_x u_o (x, t)) \]

and derive expansions \( (10) \) and \( (11) \).

To simplify the dynamic equations, we choose the following ansatz:

\[ u_o(x, t) = u_o \left( x - t v \mathcal{F}_1(\partial_x u_o (x, t)) \right), \] \( (12) \)

\[ \Psi_o(x, t) = \exp \left( i\varphi_c + i k_0 \mathcal{F}_2(\partial_x u_o) x - i \Omega(k_0, \partial_x u_o) t \right) \psi_o \left( x - t v \mathcal{F}_1(\partial_x u_o(x, t)) \right), \] \( (13) \)

where

\[ \hbar\Omega(k_0, \partial_x u_o) \approx \tilde{E}_g + (\hbar^2 k_0^2/2m) \mathcal{F}_3(\partial_x u_o(x, t)) + \mu \mathcal{F}_4(\partial_x u_o(x, t)). \]

For \( \mathcal{F}_j \), we use the following expansion:

\[ \mathcal{F}_j \approx 1 + c_j \vartheta_0 \partial_x u_o(x, t) + \bar{c}_j \left( \vartheta_0 \partial_x u_o(x, t) \right)^2, \quad j = 1, \ldots, 4, \]

and \( \hbar k_0 = mv \). Note that, in some sense, ansatz \( (13) \) is the well-known representation

\[ \Psi_o(x, t) = \sqrt{n_o(x, t)} \exp(i\varphi_c(x, t)), \]

with the inhomogeneous fields \( n_o(x, t), v_s, x(x, t) \propto \partial_x \varphi_c(x, t) \), and \( \mu(x, t) \propto \partial_t \varphi_c(x, t) \). In this article, we will not write out and solve a set of nonlinear hydrodynamic equations \[ 17 \] on these variables. Instead, we are interested in a microscopic approach aimed to clarify the role of the coupling parameters, \( \vartheta_0 \) and \( \sigma_0 \). Obviously, for a packet moving nonuniformly, one might expect some deviation from the simple low

\[ \varphi_c(x, t) = \varphi_c + p_s x/\hbar - (\tilde{E}_g + p_s^2/2m + \mu) t/\hbar. \]

If a good initial approximation for such a deviation can be found, one can finish with a kind of relatively simple NLS equation on the envelope functions of the Bose-core,
\( \exp(i \varphi_c) \psi_o(x) \) and \( \partial_x u_o(x) \). Such a mathematical structure do appears from Eqs. (8) and (10) within the validity of the adiabatic hypothesis if one starts from quasistationary ansatz (12) and (13).

Recall that the following relation between \( \psi_o(x) \) and \( \partial_x u_o(x) \) is a good approximation for the selfconsistent boson-phonon coherent state at \( \vartheta_0 = 0 \),

\[
\partial_x u_o(x, t) = f(\psi_o^2(x, t)) \approx A_2 \psi_o^2(x, t) + A_4 \psi_o^4(x, t).
\]

In fact, this approximation is valid in a more general case (although it is a part of a more complicated relation, such as \( \partial_x u_o = f(\psi_o, \partial_x \psi_o) \)). Then, for the coherent phase \( \varphi_c(x, t) \), we can write out the following symmetric representation:

\[
\varphi_c(x, t) \approx \varphi_c + k_0 \{1 + \zeta_2(v) \psi_o^2(x, t) + \cdots\} x - \left( \frac{\hbar}{2m} \{1 + \zeta_3(v) \psi_o^2(x, t) + \cdots\} - \mu \{1 + \zeta_4(v) \psi_o^2(x, t) + \cdots\} \right) t/\hbar. \quad (14)
\]

The ballistic velocity of the condensate, \( x \to x - v(x, t)t \), is modified as follows:

\[
x \to x - v(1 + \zeta_1(v) \psi_o^2(x, t) + \cdots) t. \quad (15)
\]

Here, we introduce the parameters \( \zeta_j(v) \approx c_j \zeta_j(v) \propto \vartheta_0 \) and expect \( \zeta_1(v) \approx \zeta_2(v) \approx \zeta_3(v) \approx \zeta_4(v) \). If \( c_2 \approx 1 \), we get immediately \( c_1 = 1 + c_2 \approx 2 \) and \( c_3 = 1 + 2c_2 \approx 3 \) within the quasistationary approximation for the dynamic equations. The next field terms (\( \propto \bar{c}_j \partial_o^2 \psi_o^4(x, t) \)) in expansions (14), (13) can be defined through \( c_2 \) and \( \bar{c}_2 \) as well. However, they are small in a weakly nonlinear case and we will disregard them. The constant \( c_4 \) will be defined to simplify the quasistationary equation on \( \psi_o^2(x) \).

To clarify the dynamic properties of the generalized ballistic ansatz and define the unknown \( c_j \), we will calculate the energy and momentum of a moving soliton-like solution with the factors \( F_j(x, t) \). Meanwhile, the envelope functions, \( \exp(i \varphi_c) \psi_o(x) \) and \( \partial_x u_o(x) \), are found from the quasistationary equations one has to derive. These equations read

\[
\mu \{1 + c_4 \bar{\vartheta}_0 \partial_x u_o(x, t) + \cdots\} \psi_o(x, t) \approx -\left( \frac{\hbar^2}{2m} \bar{\vartheta}_0 \partial_x u_o(x, t) \partial_x^2 \psi_o(x, t) + v^2 \partial_x^2 \psi_o(x, t) \right) \psi_o(x, t) + \sigma_0 \partial_x u_o(x, t) \psi_o(x, t),
\]

\[
+ \left( -\frac{\hbar^2}{2m} \partial_x^2 + \nu_0 \psi_o^2(x, t) + \nu_1 \psi_o^4(x, t) \right) \psi_o(x, t) + \sigma_0 \partial_x u_o(x, t) \psi_o(x, t),
\]

\[
\quad + \left( \frac{\hbar^2}{2m} \bar{\vartheta}_0 \partial_x u_o(x, t) + v^2 \partial_x^2 \psi_o(x, t) \right) \psi_o(x, t) - c_t^2 \partial_x^2 \psi_o(x, t) - c_t^2 \left( 2\kappa_3 \partial_x^2 \psi_o + \partial_x \psi_o \partial_x \psi_o \right)
\]

\[
\approx \rho^{-1} \partial_x \psi_o(x, t) + \rho^{-1} \partial_x \partial_x \psi_o(x, t).
\]

Note that we did not include the terms depending on time explicitly into these equations. The consequent divergencies, however, can be cured by taking into account the incoherent parts of the packet. In this article, we propose the effect of slow dynamic redistribution of the occupation numbers between the condensate and out-of-condensate cloud, while both of these states can be easily calculated in the quasistationary approximation. Recall that dynamic mean field interaction between the condensate and noncondensed part of
the packet leads to the damping of both of them \[18\]. On the other hand, the time interval, during which the exciton-phonon packet with the coherent Bose-core moves in the periodic medium, is always finite, especially for the “clean” samples without lattice imperfections. Therefore, the adiabatic hypothesis can be used in our case.

To find a soliton-like solution of Eqs. \((16)\) and \((17)\), we reduce Eq. \((17)\) to the following form:

\[
-(c_l^2 - v^2) \partial_x^2 u_o(x, t) - c_l^2 \kappa_3 \partial_x^2 u_o \partial_x u_o(x, t) = \\
= \rho^{-1} \sigma_0 \partial_x(\psi_o^2(x, t)).
\]

Note that the interaction vertices \(\kappa_3\) and \(\sigma_0\) are slightly renormalized in this equation, \(\bar{\kappa}_3 = \kappa_3(\bar{\vartheta}_0), \bar{\sigma}_0 = \sigma_0(\bar{\vartheta}_0)\). For example, we can write \(\bar{\kappa}_3\) as follows \((c_2 \simeq 1, c_1 \simeq 2)\)

\[
\bar{\kappa}_3 = -|\kappa_3| + 2(v^2/c_l^2) |\bar{\vartheta}_0| < 0,
\]

and \(|\bar{\kappa}_3| < |\kappa_3|\) for \(\bar{\vartheta}_0 < 0\). Here, we choose \(\kappa_3 < 0\) \((\kappa_4 > 0)\) and assume the inequality

\[
|\kappa_3| > |\bar{\vartheta}_0|
\]

to be valid. Then, the sign and the order of value of \(\kappa_3\) remains the same after the renormalization. For example, we take \(|\kappa_3| \simeq 5 - 10\), whereas \(|\bar{\vartheta}_0| \simeq 1 - 2\), and, for \(v < c_l\), we can estimate \(v^2/c_l^2 \simeq 0.5 - 0.9\).

One can represent the solution of Eq. \((18)\) in the following form:

\[
\partial_x u_o(x, t) = -A_2 \psi_o^2(x, t) + A_4 \psi_o^4(x, t),
\]

where

\[
A_2 \approx \bar{\sigma}_0 / \rho (c_l^2 - v^2) = \gamma(v) \left(\sigma_0/Mc_l^2\right) a_l^3 > 0,
\]

\[
A_4 \approx \gamma(v) |\bar{\kappa}_3| A_2^2 > 0.
\]

and \(\rho \approx M/a_l^3, \gamma(v) = v^2/(c_l^2 - v^2) > 1\). Immediately, we can rewrite the factor \(\mathcal{F}\) for the ballistic velocity \((\bar{\vartheta}_0 < 0)\) as follows:

\[
\mathcal{F}_1(\partial_x u_o(x, t)) \to \mathcal{F}_1(|\Psi_o(x, t)|^2) \approx \\
\approx 1 + |\zeta_1(v)| \psi_o^2(x, t) - \bar{\zeta}_1(v) \psi_o^4(x, t).
\]

For \(\bar{\vartheta}_0 > 0\), we estimate

\[
\mathcal{F}_1(\partial_x u_o(x, t)) \to \mathcal{F}_1(|\Psi_o(x, t)|^2) \approx \\
\approx 1 - |\zeta_1(v)| \psi_o^2(x, t) + \bar{\zeta}_1(v) \psi_o^4(x, t).
\]

We use the following representation for \(|\Psi_o(x, t)|^2\):

\[
|\Psi_o(x, t)|^2 = \psi_o^2(x - v(x, t)t) \to \Phi_o^2 f^2(\bar{x}/L_0),
\]

where \(f(\bar{x}/L_0)\) is the dimensionless ‘shape’ function, \(|f(\bar{x}/L_0)| \leq 1\), \(\Phi_o\) is the amplitude of the coherent state, and \(x - v(x, t)t \to \bar{x}\). The characteristic width of the condensate is
estimated as \((3-6) L_0\). Although the terms \(\propto \psi_o^4\) in Eqs. (20) and (21) could be important in strong nonlinear regimes (as well as other omitted contributions, e.g., \(\propto (\partial_x \psi_o)^2\)), we assume \(\zeta_1(v) \Phi_o^2 < 1\) and \(\zeta_1(v) \Phi_o^2 \gg \bar{\zeta}_1(v) \Phi_o^4\) in this article. Then, we can roughly estimate the dynamic factors \(F_j\) as

\[
F_j(|\Psi_o(x,t)|^2) \approx 1 \pm c_j \zeta(v) \Phi^2_o f^2(x,t), \quad c_j \sim 1,
\]

and the question is how strong the factor \(F\) can deviate from 1. If \(|\bar{\psi}_0| \approx 1\) and \(\gamma(v) \approx 5-10 (v < c_i)\), we can estimate the dimensionless factor \(c_j \zeta_j(v)/a_i^3\) that enters Eq. (22) as follows

\[
c_j \zeta_j(v)/a_i^3 \simeq c_j |\bar{\psi}_0| \gamma(v) (\bar{\sigma}_0/Mc_i^2) \simeq 1.
\]

Then, the most important parameter in Eq. (22) remains to be \(a_i^3 \Phi_o^2 < 1\).

Now, we can qualitatively describe the dynamics of bosonic core of the packet, see Fig. 2. The top of the soliton \(|\Psi_o(x-v(x,t)t)|^2 \rightarrow \Phi_o^2 f^2(x/L_0)\) moves with

\[
v \rightarrow v \pm (\zeta_1(v) \Phi_o^2) v, \quad f_{top}(x/L_0) \sim 1,
\]

whereas the slopes of \(|\Psi_o(x,t)|^2\) move with

\[
v \rightarrow v \pm 10^{-1}(\zeta_1(v) \Phi_o^2) v, \quad f_{slope}(x/L_0) \sim 10^{-1}.
\]

The ballistic velocity of the tails of \(|\Psi_o(x,t)|^2\) remains to be almost unchanged, i.e. \(\bar{v} = v\). One can introduce the important parameter \(\delta v_{top}\) as

\[
\delta v_{top} \approx \bar{c}_1 |\zeta(v)| \Phi_o^2 v
\]

and the dimensionless ratio

\[
\frac{\delta v_{top} \ell}{L_0} \approx \bar{c}_1 |\zeta(v)| \Phi_o^2 \ell v / L_0.
\]

Then, the time scale \(\Delta t\), during which our quasistationary ansatz can be used to describe the dynamics of the core of the total solitonic packets, can be roughly estimated from \(\delta v_{top} \Delta t / L_0 \sim 1\). In fact, parameter (23) depends on time because the dynamic leakage from the Bose-core into the tail and coma of the “comet” leads to \(\Phi_o(t)\) and \(L_0(t)\). To estimate the values of \(|\mu|, a_i^3 \Phi_o^2\), and \(L_0\), we have to solve the stationary equation on \(\psi_o(x)\).

4 Solution of stationary equation

As a result of all the simplifications we made, we obtain this equation in the following form (we choose \(c_i = 1\)):

\[
\mu \left\{1 + \zeta(v) \psi_o^2(x) - \bar{\zeta}(v) \psi_o^4(x) \right\} \psi_o(x) =
\]

\[
= \left( -\frac{\hbar^2}{2m} \left\{1 + \zeta(v) \psi_o^2(x) - \bar{\zeta}(v) \psi_o^4(x) \right\} \partial_x^2 - |\bar{\nu}_0(v)| \psi_o^2(x) + \bar{\nu}_1(v) \psi_o^4(x) \right) \psi_o(x). \tag{24}
\]
This equation resembles combined Ablowitz-Ladik and Nonlinear Schrödinger Equations taken in the continuous limit \[37\]. Note that the boson interaction vertices are strongly renormalized because of interaction with the coherent phonon field \(\partial_x u_0\). We assume the following conditions to be valid

\[
\nu_0 > 0 \to \tilde{\nu}_0(v/c_l, \sigma_0) < 0, \quad \nu_1 > 0 \to \tilde{\nu}_1(v/c_l, \sigma_0, \kappa_3) > 0.
\]

(25)

If the dimensionless parameter \(\zeta(v) \Phi_o^2 \propto a_i^2 \Phi_o^2\) is small enough, one can reduce Eq. (24) to Nonlinear Schrödinger Equation. We use the representation

\[
\psi_o(x) = \Phi_o f(\beta(\Phi_o)x, \eta(\Phi_o)), \quad \mu = -|\mu| < 0,
\]

with unknown functions \(\beta(\Phi_o) \equiv L_0^{-1}\) and \(\eta(\Phi_o)\). Then, the following simple equation appears as an approximation of Eq. (24):

\[
|\mu| f(x) - |\tilde{\nu}_0(v)| \Phi_o^2 f^3(x) + \tilde{\nu}_1(v) \Phi_o^4 f^5(x) \approx \frac{\hbar^2}{2m} \beta^2(\Phi_o) \partial_x^2 f(x).
\]

(26)

This equation is the so-called “subcritical” NLS equation. In Condensed Matter Physics, it is used in the theory of superfluidity and Bose-Einstein condensation \[13\]. It is also applied in nonlinear optics for light pulses in the medium with a cubic-quintic nonlinearity \[15\], and it is known as the Lienard equation in the theory of exactly solvable nonlinear equations \[16\].

Here, we apply it to describe the coherent state of excitons and phonons with macroscopic occupancy. The following approximation is used for the renormalized vortices: \(\tilde{\nu}_0 \approx \nu_0\) and \(\tilde{\nu}_1(v) \approx \nu_1(v) + |\tilde{\nu}_0(v)| \zeta(v)\). To estimate their strength, we use the following formulas:

\[
|\tilde{\nu}_0(v)| = \gamma(v) \left(\sigma_0/Mc_l^2\right) (\sigma_0 a_i^2) - \nu_0 > 0,
\]

(27)

which as valid at \(\gamma(v) > \gamma_o\), or, equivalently, \(\nu_0 < v < c_l\) (for discussion, see \[19\], \[37\]),

\[
\tilde{\nu}_1(v) = (\gamma(v) |\kappa_3|) \left\{ \gamma(v) \left(\frac{\sigma_0}{Mc_l^2}\right) \right\}^2 (\sigma_0 a_i^6) + \nu_1.
\]

(28)

and

\[
\tilde{\nu}_1 \Phi_o^4 \approx \tilde{\nu}_1 \Phi_o^4 + |\tilde{\nu}_0| \Phi_o^2 \left(\zeta(v) \Phi_o^2\right).
\]

(29)

Note that in the case of \(\tilde{\nu}_0 < 0\) we obtain the enhanced three particle vertex, and the nonlinear corrections are important to estimate its value, \(\nu_1 \to \tilde{\nu}_1 \to \tilde{\nu}_1 > 0\).

The localized solution of Eq. (26) exists if the following inequalities are valid \[16\]:

\[
|\mu| < \mu^* = \frac{3}{16} \frac{|\tilde{\nu}_0|^2}{\tilde{\nu}_1}, \quad \Phi_o^2 < \Phi_o^{*2} = \frac{3}{4} \frac{|\tilde{\nu}_0|}{\tilde{\nu}_1}.
\]

(30)

Then, \(|\mu| = |\mu|(\Phi_o)| and we have

\[
|\mu|/\mu^* = 2 \Phi_o^2/\Phi_o^{*2} - \left(\Phi_o^2/\Phi_o^{*2}\right)^2 < 1.
\]
As a rough estimate, we can use the inverse formula $\Phi_o^2(\mu) \approx 0.5 (|\mu|/\mu^*) \Phi_o^{*2}$ valid at $|\mu|/\mu^* \ll 1$.

The representation $|\mu| = (\hbar^2/2m) L_0^{-2}$ leads to an easy estimate of the characteristic width of the soliton. Indeed, we can introduce the length $L_*$ through $\mu^* = (\hbar^2/2m) L_*^{-2}$ and obtain the following representation

$$\beta(\Phi_o) \to \beta(\mu) = \sqrt{\frac{2m}{\hbar^2}} \mu^*/|\mu|, \quad \beta(|\mu|) = L_*^{-1} \sqrt{|\mu|/\mu^*}. \quad (31)$$

Then, $L_0 = L_* \sqrt{\mu^*/|\mu|}$ and, always, $L_0 > L_*$. To estimate the important microscopic parameters $\mu^*$ and $L_*$, we write the renormalized vertices in the following form:

$$|\tilde{v}_0(v)| = \tilde{\alpha}_0(v) \text{Ry}_x \alpha_x^3 \text{ and } \tilde{v}_1(v) = \tilde{\alpha}_1(v) \text{Ry}_x \alpha_x^6,$$  

and estimate $\tilde{\alpha}_0 \simeq \tilde{\alpha}_1 \simeq 10^{-1}$. (In theory, $\tilde{\alpha}_0$ can vary from $\sim 10^{-2}$ to $\sim 1$ with change of $v$ within $v_0 < v < c_t \ [19,37]$.) Then, we have

$$\mu^*(v) = \frac{3}{16} \frac{\tilde{\alpha}_0^2}{\tilde{\alpha}_1} \text{Ry}_x \simeq 10^{-2} \text{Ry}_x, \quad L_*^2(v) \simeq \frac{\tilde{\alpha}_1}{\tilde{\alpha}_0^2} a_x^2 \sim 10 a_x^2. \quad (33)$$

As the following inequality is valid

$$\zeta_j(v) \Phi_o^2 < \zeta_j(v) \Phi_o^{*2},$$

we can define the upper limit of $\delta v_{\text{top}}/v \simeq \zeta_1(v) \Phi_o^2$ in our model. We have

$$\zeta_1(v) \Phi_o^{*2}(v) \approx |\tilde{\partial}_0| c_1 \left\{ \frac{\gamma(v) \sigma_0}{M c_t^2} \right\} \left( a_t^3 \Phi_o^{*2} \right) \simeq 0.1 - 1, \quad (34)$$

where

$$a_t^3 \Phi_o^{*2}(v) \simeq (\tilde{\alpha}_0(v)/\tilde{\alpha}_1(v)) (a_t^3/a_x^2) \simeq 0.1. \quad (35)$$

As a result, one can define the meaning of the weakly nonlinear case in our model. For example, if the parameter $\Phi_o^2/\Phi_o^{*2} < 0.1 - 0.5$, the assumption

$$\zeta(v) \Phi_o^4 \ll \zeta(v) \Phi_o^2$$

we made to simplify the factors $\mathcal{F}_j(\partial_x u_o(x,t))$ to $\mathcal{F}_j(\psi_o^2(x,t))$ is correct.

We write the exact solution of Eq. (28) in the following form:

$$\psi_o^2(x) = \frac{\Phi_o^{*2}(|\mu|/2\mu^*)}{\sqrt{1 - |\mu|/\mu^* \cosh^2(\beta(|\mu|) x) + (1/2)(1 - \sqrt{1 - |\mu|/\mu^*})}} \quad (36)$$

Note that

$$\psi_0^2(x = 0) = \Phi_o^2(|\mu|) = \Phi_o^{*2} (1 - \sqrt{1 - |\mu|/\mu^*}).$$

If $|\mu| \ll \mu^*$, we can use the following asymptotics ($\eta(|\mu|/\mu^*) \to 0$):

$$\psi_o(x) \simeq 2 \Phi_o \exp(-\beta(\Phi_o) |x|) \quad \text{at} \quad |x| > 2 \beta(\Phi_o)^{-1}. \quad (37)$$
In addition, the coherent phonon part $u_o(x,t)$ can be rewritten as follows
\[
\partial_x u_o(x) \approx -\gamma(v) \left( \sigma_{o}/M c_t^2 \right) (a_o^2 \Phi_o^*^2) \left( \Phi_o^2/\Phi_o^*^2 \right) f^2(\beta(|\mu|) x) + \\
+ \gamma(v) |\bar{\kappa}_3| \left\{ \gamma(v) \left( \sigma_{o}/M c_t^2 \right) (a_o^2 \Phi_o^*^2) \right\} \left( \Phi_o^2/\Phi_o^*^2 \right)^2 f^4(\beta(|\mu|) x).
\]

We apply the 3D normalization condition on the macroscopic wave function $\Psi_0(x,t)$ as follows
\[
\int |\Psi_0|^2(x,t) \, dx = N_o(t) \gg 1.
\]
Substituting the quasistationary ansatz into the integrand, we have
\[
S \int \psi_o^2(x) \left( x - v(1 + \zeta_1(v))\psi_o^2(x,t) + ... \right) dx = N_o(t) \rightarrow \\
\rightarrow S \int \psi_o^2(x') \, dx' = N_o.
\]
Here, $N_o \gg 1$ has a meaning of the number of particles inside the quasistationary condensate, (it is a macroscopic number), and $S \approx L_1^2 \approx S_1$, where $S_1$ is the cross section area of the crystal. Note that, due to the symmetry of $f(x/L_0)$ against $x \rightarrow -x$, the value of $N_o$ is conserved, but as a first approximation.

To understand how the macroscopic wave function is formed from the microscopic parameters of the theory (they are $\mu^*$, $\Phi_o^*$, $L_*$, and the renormalized vertices $\tilde{\nu}(v)$), it is useful to introduce the parameter $N^* = S L_* \Phi_o^*^2$ as well. We estimate $N^*$ as follows:
\[
N^*(v) \approx \left( 1/\sqrt{2 \bar{\alpha}_1(v)} \right) (S/a_o^2) \gg 1.
\]
Then, $N^*(v) \sim (S/a_o^2)$ is always the macroscopic number in 3D case ($\bar{\alpha}_1(v) \approx 10^{-1} - 10^{-2}$). In fact, Eq. (40) leads to an algebraic equation that relates the macroscopic parameters with the microscopic ones. As a result, $\sqrt{|\mu|/\mu^*}$ can be defined as a function of $\exp(2 N_o/N^*)$. To avoid cumbersome formulas, we use simple estimates:
\[
\sqrt{|\mu|/\mu^*} = f(N_o/N^*) \simeq N_o/N^* \quad \text{and} \quad L_0/L_* \simeq (N_o/N^*)^{-1}.
\]
They are valid up to $N_o/N^* \simeq 0.1 - 0.5$.

Thus, we can estimate the important dimensionless parameter that characterizes the deviation of the ballistic velocity $v(x,t)$ from $v = \text{const}$ and the chemical potential $\mu(x,t)$ from $|\mu| = \text{const}$. We can write
\[
\Phi_o^2/\Phi_o^*^2 = f(|\mu|/\mu^*) \approx (1/2)(|\mu|/\mu^*) \simeq (1/2)(N_o/N^*)^2.
\]
This means
\[
\Phi_o^2/\Phi_o^*^2 \simeq 10^{-3} \rightarrow 10^{-2} \quad \text{while} \quad N_o/N^*(v) \simeq 10^{-2} \rightarrow 10^{-1},
\]
and
\[
\Phi_o^2/\Phi_o^*^2 \simeq 0.2 \rightarrow 0.4 \quad \text{while} \quad N_o/N^*(v) \simeq 0.6 - 0.7 \rightarrow 1.
\]
As a result, the reasonable values for the macroscopic parameter \( \zeta(v) \Phi_o^2 \simeq \delta v_{\text{top}} / v \) can be estimated as

\[
\{ \zeta(v) \Phi_o^2 \} \left( \Phi_o^2 / \Phi_o^{*2} \right) \simeq 10^{-3} - 10^{-2},
\]

(44)

(the microscopic \( \zeta(v) \Phi_o^{*2} \sim 0.1 - 1 \)). Now, we can rewrite parameter (23) to emphasize the interplay between the microscopic and macroscopic parameters of the theory:

\[
\frac{\delta v_{\text{top}}}{L_o} \simeq \pm 0.5 c_1 \left\{ |\zeta(v)| \Phi_o^{*2} \right\} \frac{vt}{L_s} (N_o/N_s)^3; \quad \zeta(v) \propto \vartheta_0 \sigma_0.
\]

(45)

On Fig. 2, we show how the coherent part of the packet moves in the medium if one switch on the interaction \( \vartheta_0 \neq 0 \). The initial state is taken to be symmetric (the exact solution for \( \sigma_0 \neq 0, \vartheta_0 = 0 \) is used), and its evolution is obtained by use of quasistationary ansatz (12), (13) which conserves the amplitude and the characteristic width of the moving soliton. Although it is a rough estimate of the real dynamics of the Bose-core, it gives some understanding of how the microscopic exciton-phonon interaction controls the dynamics. In addition, it can be easily adjusted to a more realistic case of the moving core and out-of-condensate cloud.
5 Energy of the moving soliton

Recall that the standard ballistic boson-phonon soliton, \( \Psi_o(x,t) = \exp(i\varphi_c(x,t))\phi_o(x-vt) \) and \( \partial_xu_o(x-vt) \) with \( \varphi_c(x,t) = \varphi_o + k_o x - \omega_0 t, \) is a stationary state (\( \vartheta_0 = 0 \)), that is

\[
E_o = \int dx \mathcal{T}_0^0(x,t) = \text{const} \quad \text{and} \quad P_o \rightarrow P_{o,x} = \int dx \mathcal{P}_x(x,t) = \text{const}.
\]

At \( \vartheta_0 \neq 0 \), we obtained an analog of such a solution, starting from Eqs. (12), (13) with \( v \rightarrow v(x,t), k_o \rightarrow k_o(x,t), \) and \( \mu \rightarrow \mu(x,t) \). In this article, the quasistationary approximation is the most important one we used to simplify the dynamic equations for the core of the packet. Therefore, the exact calculation of \( E_o \) leads to \( \partial_t E_o \neq 0 \) for \( \Psi_o(x,t) = \exp(i\varphi_c(x,t))\psi_o(x-vt)t \) with \( N_o = \text{const} \) and \( L_0 = \text{const} \).

Within the validity of the adiabatic hypothesis (slow exchange between \( N_o(t) \) and \( \delta N(t) \)), one has, first, to calculate \( \partial_t E_o \) of the quasistationary Bose-core and, second, to cure the obtained divergency by \( \partial_t E_o = \partial_t E_o + \partial_t \delta E = 0 \) (\( T \rightarrow 0 \)).

To calculate the energy of the moving condensate, we have to integrate the zero component of the energy-momentum tensor \( \mathcal{T}_0^0 \) over the spatial coordinates. We have the following formula (written in the laboratory frame):

\[
\mathcal{T}_0^0(x,t) = \tilde{E}_g \Psi_o^* \Psi_o + \frac{\hbar^2}{2m} \nabla \Psi_o^* \nabla \Psi_o + \frac{\nu_0}{2} (\Psi_o^*)^2 \Psi_o^2 + \frac{\nu_1}{3} (\Psi_o^*)^3 \Psi_o^3 + \\
\frac{\rho}{2} (\partial_t u_o)^2 + \frac{\rho_c^2}{2} (\partial_x u_o)^2 + \frac{\rho_c^2}{3} \kappa_3 (\partial_x u_o)^3 + \sigma_0 \partial_x u_o \Psi_o^* \Psi_o + \dot{\vartheta}_0 \frac{\hbar^2}{2m} \partial_x u_o \nabla \Psi_o^* \nabla \Psi_o.
\]

After integration of \( \mathcal{T}_0^0 \), we conclude that there are no terms \( \propto t^1 \) in \( E_o(t) \),

\[
E = E_o + (\delta E)(vt/L_0)^2 + \cdots \approx e_0 N_o + \delta e_0 N_o (\delta v_{\text{top}} t / L_0)^2.
\]

In fact, the dependence on \( N_o \) is a nonlinear one,

\[
E \approx e_0 (N_o/N_*) N_o + \delta e_0 (N_o/N_*) N_o \{ \delta v_{\text{top}}(N_o) t / L_0(N_o) \}^2.
\]

We found \( \delta e_0 > 0 \) for both \( \vartheta_0 > 0 \) (\( \delta v_{\text{top}} < 0 \)) and \( \vartheta_0 < 0 \) (\( \delta v_{\text{top}} > 0 \)). Indeed, we can write out the expansion

\[
\delta e_0 \approx \delta e_0^{(0)} + \delta e_0^{(1)} (\zeta(v) \Phi_o^2 + \cdots) \approx \delta e_0^{(0)} > 0,
\]

and \( \delta e_0^{(1)} \sim \delta e_0^{(0)} \). Recall the structure of the stationary part, \( E_o = e_0 N_o = E_x + E_{\text{int}} + E_{\text{ph}} \).

We have

\[
e_0 \approx (\hbar^2 k_o^2/2m) (1 + \text{const} \zeta(v) \Phi_o^2 + \cdots) - \text{const}' |\mu| + e_{\text{ph}},
\]

where \text{const}, \text{const}' \sim 1 \) and \( e_{\text{ph}} \approx e_{\text{ph}}^{(2)} - e_{\text{ph}}^{(3)} \). Here, the harmonic contribution into \( E_{\text{ph}} \) is \( e_{\text{ph}}^{(2)} N_o > 0 \), and

\[
e_{\text{ph}}^{(2)} \approx \frac{M (v^2 + c_i^2)}{2} \left( \gamma(v) \frac{\sigma_0}{M c_i^2} \right)^2 (a_i^2 \Phi_o^2)^2 \{ 0.5 |\mu|/\mu^* \}.
\]
Figure 2: To model transport properties of the boson-phonon soliton in a periodic medium, we started from the symmetric soliton as an initial condition at $t = 0$. Dynamics of the boson (exciton) part of the packet is presented on Figs. 2(a) and 2(c) in form of the moving $|\Psi_o(x - v(x, t)t)|^2$ without any “leakage”. The phonon part (a moving kink of the displacement field of the medium) is depicted on Fig. 2(b). (As a rough estimate, $\partial_x u_o(x-v(x,t)t) \simeq -A_2 \psi_0^2(x-v(x,t)t)$. ) The interaction parameter $\zeta_1(v) \Phi_0^2 \simeq \delta v_{top}/v$ controls the changement in the packet shape. It is taken to be $+0.1$ (Figs. 2(a) and (b)) and $-0.1$ (only the Boson part of the packet is presented on Fig. 2(c)). Then, the visible changements occur after the packet has traveled the distance of $n L_0$ corresponding to $\delta v_{top} \Delta t/L_0 \simeq 0.5 - 1$. For $\delta v_{top}/v = \pm 0.1$, we estimate $n \sim 10$. Note that the presented result is a crude estimate of the dynamics of the Bose-core of the total moving packet. To proceed, one has to “dress” such a core with the out-of-condensate cloud and tail.
We can compare formula (47) with our estimate of \( \delta e^{(0)}_0 \):

\[
\delta e_0 \simeq \left( \frac{\hbar^2 k_0^2}{2m} \right) \left( c_2^2 / 4 c_1^2 \right) \text{const}_1 + |\mu| \text{const}_2 + e^{(2)}_{\text{ph}} \text{const}_3 > 0,
\]

where all the constants are small (as a rough estimate, \( \text{const}_j \sim 10^{-1} \)). Then, \( \delta e_0 \ll e_0 \).

The result (46) means that changement of the shape of the Bose-core, which was included into the generalized ballistic ansatz to satisfy the dynamic equations with \( \theta_0 \neq 0 \), costs energy for both \( \delta v_{\text{top}} > 0 \) and \( \delta v_{\text{top}} < 0 \) during the observation time. Note that the width of the soliton, \((3 - 6) L_0(N_o), L_0^{-2} \propto |\mu|(N_o)\), and the number of Bose-particles forming the soliton, \(N_o\), are conserved as it was presented on Fig. 2, but this is only the first approximation within the adiabatic hypothesis. It is possible, however, to balance the energy of the total boson-phonon packet. (Here, the image of an exciton-phonon “comet” helps: the Bose-core with \( N_o \gg 1 \) is only a part of the moving delocalized packet.)

First, we can calculate the value of \( \{ \partial_{N_o} E_o \} \delta N \). We take it as

\[
\{ \partial_{N_o} E_o \} \delta N \simeq \tilde{e}_0 \delta N,
\]

where we have the following formula for \( \tilde{e}_0 \), \( \tilde{e}_0 \neq e_0 \) in Eq.(47):

\[
\tilde{e}_0 \simeq \left( \frac{\hbar^2 k_0^2}{2m} \right) (1 + \text{const}_n \zeta(v) \Phi_0^2 + ...) - \text{const}_n' |\mu| + \tilde{e}_{\text{ph}},
\]

and \( \tilde{e}_{\text{ph}} \simeq 3 e^{(2)}_{\text{ph}} - 5 e^{(3)}_{\text{ph}} \). Second, we assume that there is a “leakage” from the moving localized state, and such a leakage occurs dynamically,

\[
\delta N(t) \simeq -\delta N_o(t) \simeq - \text{const}_o (\delta v_{\text{top}} t / L_0)^2 < 0.
\]

Then, the tail behind the condensate grows as

\[
\langle \delta N_{x,\text{tail}} \rangle \simeq \delta N_o(t) = \text{const}' N_o (\delta v_{\text{top}} t / L_0)^2 > 0.
\]

To simplify the model, we disregard the creation of inside excitations, or the out-of-condensate cloud around the Bose-core, in this article. Then, \( \text{const} \simeq \text{const}' \). This means the following terms,

\[
- \tilde{e}_0 \text{const}_o N_o (\delta v_{\text{top}} t / L_0)^2 + \delta e_0 N_o (\delta v_{\text{top}} t / L_0)^2
\]

and

\[
E_{\text{tail}}(t) \simeq \langle \hbar \omega_x \rangle \delta N_{x,\text{tail}}(t) + \langle \hbar \omega_{\text{ph}} \rangle \delta N_{\text{ph,\text{tail}}}(t),
\]

have to be included into the (total) energy of the moving packet. The time interval \( \Delta t \), during which our quasistationary solution for the core+tail can be used to describe the transport of the total packet, can be roughly estimated from the condition \( (\delta v_{\text{top}} \Delta t / L_0) \simeq 1 \), (see Eq. (55) and Fig. 3). To estimate the excitation energies \( \langle \hbar \omega_x \rangle \) and \( \langle \hbar \omega_{\text{ph}} \rangle \) in Eq. (53), we have to discuss the excitation spectrum of the moving condensate.
6 Excitations and the tail of soliton

Within the quasistationary approximation, the following decomposition of the field operators is used [7],[11]:

\[ \hat{\psi}(x, t) = \Psi_o(x, t) + \delta \hat{\psi}(x, t), \]
\[ \hat{u}(x, t) = u_o(x, t) + \delta \hat{u}(x, t), \]

where the fields \( \delta \hat{\psi} \) and \( \delta \hat{u}(x, t) \) describe the out-of-condensate particles. To consider the case in which the moving condensate emits excitations during the observation time, one has to introduce the fluctuation of the condensate, \( \delta \Psi_o(x, t) \) and \( \delta u_o(x, t) \), and the fluctuation of the quasistationary out-of-condensate cloud described by \( \delta \hat{\psi}' \) and \( \delta \hat{u}' \). If the number of the “lost” particles, \( \delta N \), is small during the observation time (but \( \delta N(t) \) is continuously growing), one can consider the moving coherent packet as a quasistable one. Therefore, we can write the field decomposition as follows:

\[ \hat{\psi}(x, t) = \exp(i \varphi_c(x, t, \psi_o(x, t))) \{ \psi_o(x - v(x, t)t, \mathbf{x}_\perp, t) \} + \delta \hat{\psi}_0(x, t) \]
\[ \hat{u}_j(x, t) = u_o(x - v(x, t)t) \delta_{ij} + \delta \hat{u}_0_j(x - v(x, t)t, \mathbf{x}_\perp, t) + \delta u_o(x, t) \delta_{ij} + \delta \hat{u}'_j(x, t). \]

In this article, we do not take into account the fluctuational parts in this decomposition and, strictly speaking, we can only estimate the kinetic effects, such as the proposed \( \partial_\lambda N_o(t) \neq 0 \) and \( \partial_\lambda \delta n(t) = \partial_\lambda \langle \delta \hat{\psi}^\dagger \delta \hat{\psi} \rangle(t) \neq 0 \). However, the linear quasistationary equations on \( \delta \hat{\psi}_0 \) and \( \delta \hat{u}_0_j \) can be used to find the excitation spectrum of the outside excitations that is required for Eq. (53). Then, assuming the quasistability of the Bose-core during the observation time \( \Delta t \), one can switch on the mechanism of occupancy redistribution between the macroscopically occupied mode \( \Psi_o(x, t, N_o) \) and the out-of-condensate excitations, e.g., \( \delta n_k = \langle \delta \hat{\psi}_k^\dagger \delta \hat{\psi}_k \rangle \).

Thus, we “dress” the quasistationary parts of the out-of-condensate fields by Eqs. (54) and (55) and obtain the following set of equations \( (x - v(x, t)t \rightarrow x) \):

\[ i \hbar \partial_t \delta \hat{\psi}_0(x, \mathbf{x}_\perp, t) = |\mu| \delta \hat{\psi}_0(x, t) - \frac{\hbar^2}{2m} \Delta \delta \hat{\psi}_0(x, t) - \frac{\hbar^2}{2m} \zeta(v) \psi_o^2(x) \Delta \delta \hat{\psi}_0(x, t) + \]
\[ + \left\{ (\nu_o + \nu_0 + |\mu| \zeta(v)) \psi_o^2(x) + (2 \nu_1 + \nu_1) \psi_o^4(x) \right\} \delta \hat{\psi}_0(x, \mathbf{x}_\perp, t) + \]
\[ + \left\{ \nu_o \psi_o^2(x) + 2 \nu_1 \psi_o^4(x) \right\} \delta \hat{\psi}_0(x, \mathbf{x}_\perp, t) + \sigma_{\text{eff}} \psi_o(x) \nabla \delta \hat{u}_0(x, \mathbf{x}_\perp, t), \]
\[ \left( \{ \partial_t - v(x, t) \partial_x \}^2 - c_l^2 \Delta \right) \delta \hat{u}_0(x, \mathbf{x}_\perp, t) - \]
\[ - c_l^2 2 \kappa_3 \partial_x u_o(x) \partial_x \delta \hat{u}_0(x, \mathbf{x}_\perp, t) - c_l^2 2 \kappa_3 \partial_x^2 u_o(x) \partial_x \delta \hat{u}_0(x, \mathbf{x}_\perp, t) = \]
\[ = \rho^{-1} \tilde{\sigma}_{\text{eff}} \partial_x \left( \psi_o(x) \delta \hat{\psi}_0(x, \mathbf{x}_\perp, t) + \text{h.c.} \right). \]
Here, we simplified the boson-phonon coupling terms and, roughly, $\sigma_{\text{eff}} \approx \tilde{\sigma}_{\text{eff}} \approx \sigma_0$. Note that the interaction between $\delta \psi_0$ and $\delta u_{0, x}$ is mediated through the condensate $\psi_0(x, t) \cdot \partial_x u_0(x, t)$ in the linear theory at $T \to 0$.

For the outside excitations, the first approximation of Eqs. (26) and (27) consists of taking uncoupled excitons and phonons. However, they live in a half of the medium ($x < 0$) and form a tail behind the condensate. Then, we have the excitonic excitations with

$$\hbar \omega_\mathcal{E}(k_x) = |\mu| + |\mu| (k_x L_0)^2, \quad |k_x| L_0 < 1,$$

$$\delta \psi(x, t) \sim (\sqrt{V/2})^{-1} \exp(i \varphi_k + i k_x x - i \omega_{\mathcal{E}} t) \hat{a}_k, \quad x < 0.$$

This is exactly a free exciton with the energy $\tilde{E}_\mathcal{E} + \hbar^2 (k_0 - |k_x|)^2 / 2m$ in the laboratory frame. It moves behind the condensate with $p_x = \hbar (k_0 - |k_x|) > 0$. For the acoustic phonons, we have $(k_x' > 0)$:

$$\hbar \omega_\text{ph}(k_x') = \hbar \omega_\mathcal{E}(k_x') - v \hbar k_x',$$

$$\delta \hat{u}_x(x, x', t) \sim \text{const}(k') \exp(i \varphi_{k'} + i k_x' x - i \omega_{\mathcal{E}}(k_x') - k_x' v t) \hat{b}_{k'} + \text{h.c.},$$

where $\text{const}(k') \propto (\sqrt{\rho/V/2} \omega_{\mathcal{E}}(k'))^{-1}$. (After $x \to x - vt$, we have free phonons in the tail, and $k_x'$ is the same in the laboratory frame.) The ‘resonance’ equation for the excitations written in the comoving frame has the following form:

$$|\mu| + |\mu| (k_x L_0)^2 = \hbar (c_l \mp v) k_x',$$

and it can be solved easily. Here, we use the following estimate for the ‘relevant’ phonons:

$$(k_x' L_0) \approx 0.5 \frac{\sqrt{|\mu| / mc_l^2 / 2}}{(1 \mp v / c_l)}.$$  

For $k_x' > 0$, we have $(k_x' L_0) \approx (1 - 10) \sqrt{|\mu| / mc_l^2 / 2}$. In this article, we do not go into detail of how the coupling between the outside excitons and phonons is formed within Eqs. (59), (57). Our aim is to show that it is possible to balance the energy of the total packet by taking into account the growing out-of-condensate tail (51) behind the coherent state (12), (13) with leakage (50).

Returning to Eq. (56), we estimate the characteristic energies of excitations as follows:

$$\langle \hbar \omega_\mathcal{E} \rangle \sim \hbar^2 k_0^2 / 2m \quad \text{and} \quad \langle \hbar \omega_\text{ph} \rangle \sim \text{const}_k |\mu|, \quad \text{const}_k \simeq 2 - 8.$$  

(The value of $\langle \hbar \omega_\text{ph} \rangle$ comes from the resonance condition at $k_{\text{ph}, x} > 0$.) We assume that the tail consists of excitons and phonons, and the selfconsistency condition between $\psi_x^2(x/L_0) \propto \Phi_0^2$ and, roughly, $\partial_x u_0(x/L_0) \propto \Phi_0^2$ leads to the coherent emission of the outside excitons and phonons, and $\Phi_0^2 \to \Phi_0^2 - \delta \Phi_0^2(t)$. Then, we have

$$\langle \delta \hat{N}_{x, \text{tail}} \rangle \simeq \langle \delta \hat{N}_{\text{ph}, \text{tail}} \rangle.$$
As a result, the energy of the total packet

\[ E(t) \approx e_o N_o(t) + \delta e_o N_o(t) (\delta v_{\text{top}}(t) t / L_0(t))^2 - \tilde{e}_0 \text{ const} N_o(t) (\delta v_{\text{top}}(t) t / L_0(t))^2 + \]

\[ + \langle \hbar \omega_{\text{s}} \rangle \text{ const}' N_o(t) (\delta v_{\text{top}}(t) t / L_0(t))^2 + \langle \hbar \omega_{\text{ph}} \rangle \text{ const}' N_o(t) (\delta v_{\text{top}}(t) t / L_0(t))^2 \]

(60)
is conserved if one takes \( \tilde{e}_0 \text{ const} \approx \delta e_o \) and \( \text{ const } \approx \text{ const}' \approx 0.1 \) in Eqs. (52), (53), and (60) (the last value is probably overestimated).

To address the problem of the dynamic stability, one has to estimate how many particles and phonons are emitted out from the moving coherent packet. We start from the same initial conditions as in Fig. 2. However, the occupations of the tail states start growing immediately as \( t > 0 \) while the occupancy of the coherent state starts decreasing. Using the conditions \( E(t) = E_o(t = 0) \) and \( N_o(t) + \delta N_{\text{tail}}(t) = N_o(t = 0) \), we obtain the dynamics presented on Fig. 3 by combination of numerics and analytic estimates. For example, after the transport has been observed during the time interval of \( \Delta t \) corresponding to \( \delta v_{\text{top}} \Delta t / L_0 \approx (1/4 - 1/2) \), the soliton develops the leakage of \( \delta N \sim (1 - 3) 10^{-2} N_o \) into the tail excitations, see Fig. 3. However, the value of \( \delta v_{\text{top}}(N_o) \Delta t / L_0(N_o) \) starts to decrease because of this leakage too. For instance, when the adjusted value of \( \delta v_{\text{top}}(N_o(t)) \Delta t' / L_0(N_o(t)) \) \( \approx 1 \), the leakage is estimated as \( \delta N \approx (1 - 2) 10^{-1} N_o \), see Fig. 3. Therefore, the condensate can be considered as a quasistable one during the time interval of \( \approx \Delta t' \). The following formulas can be used for estimates

\[ \Phi_o^2(t) S \perp 2L_0(t) \approx N_o(t), \]

(61)

\[ \delta n_{\text{tail}}(t) S \perp \tilde{L} \approx q(t) N_o, \]

(62)

where \( \delta n_{\text{tail}}(t) \approx q' \Phi_o^2(t), \tilde{L} = \tilde{q} L_0 \). If \( q(t) \ll 1 \) (e.g., \( q \leq 10^{-1} \) within the time interval \( \Delta t' \) on Fig. 3) and \( \tilde{q} \gg 1 \) (e.g., \( \tilde{q} \approx 10 - 50 \)), one can roughly estimate the dimensionless \( q' \) as \( \sim 10^{-1} q \). The dynamics presented on Fig. 3 is in a qualitative agreement with experimental data [15]. (It is interesting to mention that the similar problems of dynamic stability appear in the theory of the so-called Embedded Solitons in nonlinear optics [18].)
Figure 3: To model transport properties of the boson-phonon soliton in the case of effective dissipation (the “leakage”), we started from the symmetric soliton without a tail as an initial condition at $t = 0$. Dynamics of the boson (exciton) part of the packet is presented on Fig. 3 (a) in the form of moving \[ |\Psi_0(x - v(x, t)t)|^2 + \delta n(x, t), \] where $\delta n(x, t) \approx \delta n_{\text{tail}}$ at $x < -(2 - 3) L_0$. The coherent phonon part (a moving kink of the displacement field) is depicted on Fig. 3 (b); the phonon part of the tail $\langle (\partial_x \delta \hat{u}_x)^2 \rangle(t) \neq 0$ is not presented on this figure. The initial value of the interaction parameter $\zeta(v) \Phi_0^2 \approx \delta v_{\text{top}}/v$ is taken to be $+0.05$. Then, the visible changes occur after the packet has traveled the distance of $(20 - 30) L_0$, which corresponds to the effective value of $\delta v_{\text{top}}(t) \Delta t/L_0(t) \sim 1$. Note that the energy of the total moving packet (coherent part + tail) is conserved at $T \to 0$. 
7 Conclusion

When a quasistationary exciton-phonon soliton moves in a periodic medium, an asymmetric form of the ballistic signal can be developed as a dynamic effect due to exciton-phonon interaction. This effect has the tendency to be accumulated with time, and it can be clearly seen after the packet has traveled the distance of \( nL_0 \), \( n \gg 1 \). Here, the width of the packet is \( \approx (3 - 6)L_0 \), and the value of \( n \) depends on the strength of relevant interaction parameters, such as \( \vartheta_0 \) and \( \sigma_0 \). It can be estimated by the following formula (see Eq. (44)):

\[
\begin{align*}
    n \approx \left( \frac{|\delta v_{\text{top}}|}{v} \right)^{-1} &\sim 10^3 - 10^2.
\end{align*}
\]

To a first approximation, we introduced the coherent part of the packet and described its dynamics by use of the selfconsistent generalization of \( v = \text{const} \rightarrow v(x,t) \) and \( \varphi_c(x,t) = k_0x - \omega_0t \rightarrow \tilde{\varphi}_c(x,t) \) in such a way:

\[
\begin{align*}
    v &\xrightarrow{\vartheta_0 \neq 0} v(\partial_x u_o(x,t)) \rightarrow v(|\Psi_o(x,t)|^2) \\
    \varphi_c(x,t) &\xrightarrow{\vartheta_0 \neq 0} \tilde{\varphi}_c(x,t, \partial_x u_o(x,t)) \rightarrow \varphi_c(x,t, |\Psi_o(x,t)|^2).
\end{align*}
\]

This substitute was applied within the quasistationary approximation to simplify and solve the corresponding dynamic equations on the coherent Bose-core of the total packet. The incoherent part of it turned out to be equally important in dynamics of the total packet. Indeed, the quasistationary solution for the Bose-core cannot move in a periodic medium as a single soliton, but with the leakage into the out-of-condensate excitation states. Thus, the question on whether a moving conserving solution (i.e., a soliton conserving at least \( E_o, P_{o,x} \), and \( N_o \)) exists in the model with \( \vartheta_0 \neq 0 \) and \( \sigma_0 \neq 0 \) remains open. In this article, we showed that an exciton-phonon “comet” is a better image for the moving ballistic packet, and the core of it (a kind of the “nucleus” of such a comet) can be modeled by a coherent state, or a condensate.

The ansatz we used for the soliton-like solution can be applied within the validity of the adiabatic hypothesis (i.e., it is not the exact one of the dynamic equations). As a result, we could obtain an explicit dependence on time \( \propto (\delta v_{\text{top}} t / L_0)^2 \) for the exact energy and momentum of the moving soliton if it were the only one component of the moving packet. However, the technique of Bose-Einstein condensation allows to proceed with the approximate solution obtained in this article. It is known that the interaction between Bose-condensate and non-condensed particles leads to an effective dissipation for the condensate wave function at \( T \neq 0 \) [50]. We assume that an effective dissipation appears naturally in description of the transport properties of the coherent part, \( \Psi_o(x,t) \) and \( \partial_x u_o(x,t) \), at \( T \rightarrow 0 \). If the initial state of the packet was taken as a pure coherent state with a nonzero momentum, the moving condensate emits excitations, i.e., \( N_o \rightarrow N_o - \delta N(t) \) during the observation time. Then, the out-of-condensate cloud of collective excitations and, in particular, the tail consisting of the out-of-condensate excitons and phonons grow around and behind the moving coherent state, e.g., \( \langle \delta \hat{n}(x, t) \rangle \neq 0 \), \( x < -3L_0 \).

We argue that the emission effect can make the dynamics of the total moving packet conservative. Therefore, the coherent part of such a packet has to be considered as
a quasistable core (a “nucleus” of the “comet”) of the moving exciton-phonon droplet provided $\delta N_{\text{tail}}(t) \ll N_0(t)$ during the observation time. In addition, the coherent phase $\varphi_c(x, t)$ prescribed to the macroscopic wave function $\Psi_0(x, t)$ cannot be taken as a regular field under such conditions because of dephasing effects [51]. Its fluctuations has to be taken into account to clarify the coherent properties of the Bose-core. On the other hand, the rigorous approach to the dynamics of solitons with emission lies beyond the quasistationary approximation we used in this articles. A set of kinetic equations has to be applied to treat this problem in detail.

8 Acknowledgements

One of the authors (D.R.) thanks I. Loutsenko for critical reading of the manuscript.

References

[1] R. K. Dodd, J. C. Eilbeck, J. D. Gibbon, H. C. Morris, *Solitons and Nonlinear Wave Equations*, (Academic Press, 1982).

[2] E. Infeld, G. Rowlands, *Nonlinear Waves, Solitons, and Chaos*, (Cambridge University Press, Cambridge, 1990).

[3] A. R. Bishop, M. G. Forest, D. W. McLaughlin, and E. A. Overman II, Physica D 23, 293 (1986).

[4] D. E. Edmundson and R. H. Enns, Phys. Rev. A 151, 2491 (1995).

[5] Y. S. Kivshar, D. E. Pelinovsky, Phys. Rep. 331, 200 (2000).

[6] B. Fornberg, G. B. Whitham, Phil. Trans. Royal. Soc. London 289, 373 (1978); S. B. Wineberg, J. McGrath, E. Gabl, L. R. Scott, C. Southwell, Comp. Phys. 97, 311 (1991); C. Cercignani, D. H. Sattinger, *Scaling Limits and Models in Physical Processes*, (Birkhäuser, Boston-Basel-Berlin, 1998).

[7] *Bose-Einstein Condensation*, edited by A. Griffin, D. W. Snoke and S. Stringari (Cambridge University Press, Cambridge, 1995); F. Dalfovo, S. Giorgini, L. P. Pitaevskii, and S. Stringari Rev. Mod. Phys. 71, 463 (1999).

[8] A. Hasegawa, *Optical Solitons in Fibers*, (Springer-Verlag, Berlin, 1989); A. C. Newell, J.V. Moloney. *Nonlinear Optics*, (Addison-Wesley, Redwood City, 1992).

[9] T. B. Benjamin, J. Bona, J.J. Mahoney, Phil. Trans. Royal. Soc. London A. 272, 47 (1972).
[10] O. M. Braun, Y. S. Kivshar, Phys. Rep. 306, 1 (1998).

[11] A. S. Davydov, Solitons in Molecular Systems, (Reidel, Dordrecht, 1984); Davydov’s Soliton Revisited, edited by P. L. Christiansen, A. C. Scott, NATO ASI Series B: Physics 243, (Plenum Press, 1990).

[12] A. J. Heeger, S. Kivelson, J. R. Schrieffer, and W. P. Su, Rev. Mod. Phys 60, 781 (1988); D. Augier, D. Poilblanc, E. Sorensen, I. Affleck, Phys. Rev. B 58, 9110 (1998).

[13] Charge Density Waves in Solids, L. Gor’kov and G. Grüner eds., (Elsevier Sci. Publ., Amsterdam, 1989).

[14] J. L. Lin, J. P. Wolfe, Phys. Rev. Lett. 71, 122 (1993); E. Fortin, S. Fafard, A. Mysyrowicz, Phys. Rev. Lett. 70, 3951 (1993); H. Kondo, H. Mino, I. Akai, and T. Karasawa, Phys. Rev. B 58, 13835 (1998); L. V. Butov, A. I. Filin, Phys. Rev. B 58, 1980 (1998); V. Negoita, D. W. Snoke, K. Eberl, Phys. Rev. B 60, 2661 (1999).

[15] A. Mysyrowicz, E. Benson, and E. Fortin, Phys. Rev. Lett. 77, 896 (1996); E. Benson, E. Fortin, A. Mysyrowicz, Sol. Stat. Comm. 101, 313 (1997); E. Benson, E. Fortin, B. Prade and A. Mysyrowicz, Europhys. Lett. 40, 311 (1997); E. Fortin, E. Benson, and A. Mysyrowicz, Electrochem. Soc. Proceedings 98-25, 1 (1998).

[16] S. F. Mingaleev, P. L. Christiansen, Y. B. Gaididei, M. Johansson, K. O. Rasmussen, J. of Biolog. Phys. 25, 41 (1999); A. V. Zolotaryuk, K. H. Spatschek, A. V. Savin, Phys. Rev. B 54, 266 (1996); P. L. Christiansen, Y. B. Gaididei, S. F. Mingaleev, cond-mat/0003146.

[17] E. A. Bartnik, J. A. Tuszyński, Phys. Rev. E 48, 1516 (1993);

[18] Microscopic Aspects of Nonlinearity in Condensed Matter, ed. by A. R. Bishop, V. L. Pokrovsky, and A. Tognetti, (Plenum, New York, 1991).

[19] I. Loutsenko, D. Roubtsov, Phys. Rev. Lett. 78, 3011 (1997); ibid. 84, 3503 (2000).

[20] A. R. Vasconcellos, M. V. Mesquita, and R. Luzzi, Europhys. Lett. 49, 637 (2000).

[21] V. E. Zakharov, Sov. Phys. JETP 35, 908 (1972).

[22] C. Sulem, P.-L. Sulem, The nonlinear Schrödinger equation. Self-Focusing and Wave Collapse., (Springer-Verlag, New York Inc., 1999).

[23] L. Bergé, Phys. Rep. 303, 259 (1998).

[24] S. Novikov, S. V. Manakov, L. P. Pitaevskii, V. E. Zakharov, Theory of Solitons. The Inverse Scattering Method, (Consultants Bureau, New York and London, 1984).
[25] M. J. Ablowitz, P. A. Clarkson, *Solitons, Nonlinear Evolution Equations and Inverse Scattering*, (Cambridge University Press, New York, 1991).

[26] V. K. Mel’nikov, Comm. Math. Phys. **112**, 639 (1987); J. Math. Phys. **28**, 2603 (1987).

[27] G. B. Whitham, *Linear and Nonlinear waves*, (John Willey&Sons, New York, 1973).

[28] S. Georgihoiu, T. D. Bradrick, A. Philippetis and J. M. Beechem, Biophysical J. **70**, 1909 (1996);
S. O. Kelley, J. K. Barton, Science **283**, 375 (1999);
J. Schiller, G. Major, H. J. Koester, Y. Schiller, Nature **404**, 285 (2000).
A. Xie, L. van der Meer, W. Hoff, R. H. Austin, Phys. Rev. Lett. **84**, 5435 (2000).

[29] G. P. Agrawal, *Nonlinear Fiber Optics*, (Second Edition, Academic Press, New York 1995), and references therein.

[30] D. J. Kaup, T. I. Lakoba, B. A. Malomed, J. Opt. Soc. Am. B **14**, 1199 (1997).

[31] R. Camassa, D. D. Holm, Phys. Rev. Lett. **71**, 1661 (1993);
P. Rosenau, Phys. Rev. Lett. **73**, 1737 (1994).

[32] J. P. Boyd, *Weakly Nonlocal Solitary Waves and Beyond – All-Orders Asymptotics* (Kluwer, Dodrecht, Boston, London, 1998).

[33] J. Inoue, T. Brandes, and A. Shimizu, Phys. Rev. B **61**, 2863 (2000).

[34] C. Rocke, S. Zimmermann, A. Wixforth, J. P. Kotthaus, Phys. Rev. Lett. **78**, 4099 (1997).

[35] J. Shumway and D. M. Ceperley, [cond-mat/9907309](https://arxiv.org/abs/cond-mat/9907309).

[36] D. Hennig, G. P. Tsironis, Phys. Rep. **307**, 333 (1999).

[37] D. Roubtsov, Y. Lépine, Phys. Stat. Sol. B **210**, 127 (1998); Phys. Rev. B **61**, 5237 (2000).

[38] A. L. Ivanov, C. Ell, and H. Haug, Phys. Rev. E **55**, 6363 (1997); Phys. Rev. B **57**, 9663 (1998).

[39] A. E. Bulatov, S. G. Tikhodeev, Phys. Rev. B **46**, 15058 (1992);
S. G. Tikhodeev, G. A. Kopelevich, N. A. Gippius, Phys. Stat. Sol. B **206**, 45 (1998).

[40] A. J. Sievers and S. Takeno, Phys. Rev. Lett. **61**, 970 (1988);
S. Flach, C. R. Wills, Phys. Rep. **295**, 181 (1998);
G. Huang and B. Hu, Phys. Rev. B **58**, 9194 (1998).
[41] A. Griffin, Phys. Rev. B 53, 9341 (1996); in Bose-Einstein Condensation in Atomic Gases, edited by M. Inguscio, S. Stringari and C. Wieman (Italian Physical Society, 1999).
   E. Zaremba, A. Griffin, T. Nikuni, Phys. Rev. A 57, 4695 (1998).

[42] D. R. Tilley, J. Tilley, Superfluidity and Superconductivity (A. Hilger, Bristol, 1990).

[43] M. J. Ablowitz, J. F. Ladik, J. Math Phys. 17, 1011 (1976);
   M. Salerno, Phys. Rev. A 46, 6856 (1992).

[44] C. Josserand, Y. Pomeau, S. Rica, Phys. Rev. Lett. 75, 3150 (1995).

[45] D. Mihalache, D. Mazilu, L.-C. Crasovan, B. A. Malomed, F. Lederer, Phys. Rev. E 61, 7142 (2000).

[46] D. Kong, Phys. Lett. A 196, 301 (1995);
   B. Dey, A. Khare, C. N. Kumar, hep-th/9510054.

[47] G. Baym, B. Link, Phys. Rev. Lett. 69, 2959 (1992);
   E. Zaremba, T. Nikuni, A. Griffin, J. Low Temp. Phys. 116, 277 (1999);

[48] L. P. Pitaevskii, S. Stringari, Phys. Lett. A 235, 398 (1997);
   T. Nikuni, A. Griffin, cond-mat/0009333; J. E. Williams, A. Griffin, cond-mat/0003481.

[49] J. Yang, B. A. Malomed, D. J. Kaup, Phys. Rev. Lett. 83, 1959 (1999).

[50] H. T. C. Stoof, J. Low Temp. Phys. 114, 11 (1999);
   M. J. Bijlsma, H. T. C. Stoof, cond-mat/0007026.

[51] A. J. Leggett, F. Sols, Foundations of Phys. 21, 353 (1991); Phys. Rev. Lett. 81, 1344 (1998).