Microcanonical Action and the Entropy of a Rotating Black Hole

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The authors have recently proposed a “microcanonical functional integral” representation of the density of quantum states of the gravitational field. The phase of this real–time functional integral is determined by a “microcanonical” or Jacobi action, the extrema of which are classical solutions at fixed total energy, not at fixed total time interval as in Hamilton’s action. This approach is fully general but is especially well suited to gravitating systems because for them the total energy can be fixed simply as a boundary condition on the gravitational field. In this paper we describe how to obtain Jacobi’s action for general relativity. We evaluate it for a certain complex metric associated with a rotating black hole and discuss the relation of the result to the density of states and to the entropy of the black hole.

1 DEDICATION
We dedicate this paper to Yvonne Choquet–Bruhat in honor of her retirement following a brilliant career. J.W.Y. would like to thank her for friendship, support, and their happy collaboration in “analysis, manifolds, and physics” [1].

2 INTRODUCTION
We are concerned here with the description of a stationary and axisymmetric rotating black hole in a closed system in thermodynamic equilibrium with its environment. The equilibrating radiation in the closed system surrounding the hole is the “environment”. This “radiation fluid” rotates at a constant angular velocity [2][3]. Unlike the more familiar non–relativistic case, here one must take into account explicitly the spatial finiteness of the system in order to avoid super–luminal velocities resulting from the rotation. Even though we shall ignore the explicit effects of the equilibrating radiation in this work, its presence–in–principle must be kept in mind in order to give the problem a physically and mathematically reasonable formulation.
The key conserved quantities in closed systems like ours are the total energy and the total angular momentum. Their “Massieu” conjugates \([4]\) are, respectively, inverse temperature \(\beta\), and \(\beta\) multiplied by an appropriate angular velocity. An important question is which among these quantities to specify in advance, that is, which “ensemble” picture to employ. We choose here to fix energy and angular momentum as in the microcanonical ensemble. This choice leads us to employ the general relativistic version of Jacobi’s action.

Jacobi’s form of the action principle \([5][6]\) involves variations at fixed energy, rather than the variations at fixed time used in Hamilton’s principle. The fixed time interval in Hamilton’s action becomes fixed inverse temperature \(\beta\) in the “periodic imaginary time” formulation, thus transforming Hamilton’s action into the appropriate (imaginary) phase for a periodic path in computing the canonical partition function from a “Euclidean” Feynman functional integral \([7]\). (We are here and in the next paragraph speaking only of energy and inverse temperature in order to simplify the discussion. Similar remarks hold for angular momentum and its conjugate.) In contrast, fixed energy is suitable for the microcanonical ensemble and, correspondingly, Jacobi’s action is the phase in an expression for the density of states as a real–time “microcanonical functional integral” (MCFI) \([8]\).

Let us characterize briefly the canonical and microcanonical pictures. Neither picture can hold with perfect precision even in principle when gravity is taken into account because the “infinite” heat bath of the canonical picture and the “perfectly adiabatic” walls of the microcanonical picture are both at variance with the known physics of the gravitational field. However, each picture still provides a useful framework for discussion. Furthermore, the long–range, unscreened nature of the gravitational interaction means that simple notions of ordinary statistical thermodynamics like “extensive”, “intensive”, etc., no longer apply in general. With gravity, statistical mechanics is inherently \textit{global}.

In the canonical picture, with a fixed temperature shared by all constituents of a system, there are no constraints on the energy. This feature simplifies combinatorial (counting) problems and leads to factorization of the partition function for weakly coupled constituents. For gravitating systems in equilibrium, the temperature is not spatially uniform because of red–shift and blue–shift effects. In such cases, the relevant temperature is that determined at the spatial two–boundary \(B\) of the system \([9]\). It can be specified by a boundary condition on the metric \([10][11]\). It is then used in conjunction with Hamilton’s principle, which is the form of the gravity action.
in which the metric is fixed on the history of the spatial two–boundary [12]. (The metric determines the lapse of proper time along the history of the boundary.) The case of a rotating, charged, stationary, axisymmetric black hole has been treated with (grand) canonical boundary conditions in [13]. See also [10] and [11].

With its constraint on the energy, the microcanonical picture leads to stability properties more robust than in the canonical case. However, the energy constraint can complicate combinatorial problems because the constituents of the system must all share from a common fixed pool of energy. For field theories, with a continuous infinity of degrees of freedom, the energy constraint restricts the entire phase space of the system unless gravity is taken into account. For gravitating systems, as a consequence of the equivalence principle, the total energy in a given finite region, including that of matter fields, can be given as an integral of certain geometrically well–defined derivatives of the metric over a finite two–surface bounding the system. In other words, one can find and employ a suitable expression for “quasi–local” gravitational energy that does not require an appeal to asymptotic regions. This quasi–local energy has the value, per unit proper time, of the Hamiltonian of the spatially bounded region, as discussed in detail in [12]. Therefore, if we specify as a boundary condition the energy per unit two–surface area, we have constrained the total energy simply by a boundary condition [8][12]. Thus, through the mediation of the gravitational field, the canonical and microcanonical cases are placed on similar footing and differ only in which of the conjugate variables [14], inverse temperature or energy, is specified on the boundary. The corresponding functional integrals, for the partition function or density of states, differ in which action gives the correct phase, Hamilton’s or Jacobi’s. We regard the MCFI for the density of states as the more fundamental [8].

We shall now outline a recent application of the above reasoning to the case of an axisymmetric stationary black hole. The MCFI, in a steepest descents approximation, shows that the density of states is the exponential of one–fourth of the area of the event horizon, thus confirming in this approximation the Bekenstein–Hawking expression for the entropy of a black hole [8]. Full details of the following are given in [8] and [12]. (An application of the MCFI to a simple harmonic oscillator–without gravity–has been given recently by the authors [15]. No approximations were required in this case and the exact energy spectrum was obtained.)

3 JACOBI’S ACTION FOR GRAVITY
We here analyze the action for pure general relativity. The method, however, can be just as well applied when matter is included and/or for higher–derivative theories of
Consider a region of spacetime $M = \Sigma \times I$ where $\Sigma$ are spatial slices and $I$ is an interval of the real line. The two–boundary of space $\Sigma$ is denoted by $B$, whose history is $^3B = B \times I$. The orthogonal intersection of any $\Sigma$ with $^3B$ is the two–boundary $B$ at some time. Thus a generic $B$ can be regarded as being embedded in $\Sigma$ with a spacelike unit normal $n^\mu$ tangent to $\Sigma$ at $B$, and also as being embedded in $^3B$ with a timelike unit normal $u^\mu$ such that $u_\mu n^\mu = 0$. The subspaces of $M$ that correspond to the endpoints of $I$ are spacelike hypersurfaces $t = t'$ and $t = t''$. The notation $\int_{t'}^{t''} d^3x$ denotes an integral over $t''$ minus an integral over $t'$. The spacetime metric is $g_{\mu\nu}$ and its scalar curvature is $\Re$. Then “Hamilton’s” action for general relativity, in which the metric is fixed on the boundary, is given by a variant of the Hilbert action, namely [12][16]

$$S[g] = \frac{1}{2\kappa} \int_M d^4x \sqrt{-g(\Re - 2\Lambda)} + \frac{1}{\kappa} \int_{t'}^{t''} d^3x \sqrt{h}K - \frac{1}{\kappa} \int_{^3B} d^3x \sqrt{-\gamma}\Theta - S^0, \quad (1)$$

where the metric and extrinsic curvature of the slices $\Sigma$ are $h_{ij}$ and $K_{ij}$, while those for $^3B$ are $\gamma_{ij}$ and $\Theta_{ij}$. (Latin letters $i, j, \ldots$ are used as tensor indices for both $\Sigma$ and $^3B$. No cause for confusion arises from this convention.) In (1), $\kappa = 8\pi G$ and we set Newton’s constant $G = 1$ henceforth. The term $S^0$ in (1) is a functional of the metric $\gamma_{ij}$ of $^3B$. The purpose of this term is to determine the “zero” of energy and momentum. It will not turn out to affect the Jacobi action for general relativity. ($S^0$ is analyzed in [12] but was not included in [16].)

The canonical form of “Hamilton’s” action (1) is [12]

$$S = \int_M d^4x [P^{ij}h_{ij} - N\mathcal{H} - V^i\mathcal{H}_i] - \int_{^3B} d^3x \sqrt{\sigma}[N\varepsilon - V^i j_i], \quad (2)$$

where $P^{ij}$ is the Arnowitt–Deser–Misner momentum conjugate to $h_{ij}$, and $\mathcal{H}$ and $\mathcal{H}_i$ are the usual Hamiltonian and momentum constraints. The lapse function is denoted by $N$ and the shift vector by $V^i$. In the surface term of (2), $\sigma$ denotes the determinant of the two–metric $\sigma_{ij}$ induced on $B$ as a surface embedded in $\Sigma$; likewise, $n_i$ denotes $B$’s unit normal in $\Sigma$ and $k_{ij}$ denotes the corresponding extrinsic curvature. The energy surface density $\varepsilon$ and momentum surface density $j_i$ are given by [12]

$$\varepsilon = \frac{1}{\kappa} k + \frac{1}{\sqrt{\sigma}} \frac{\delta S^0}{\delta N},$$

$$j_i = -\frac{2}{\sqrt{h}} \sigma_{ij} n_k P^{jk} - \frac{1}{\sqrt{\sigma}} \frac{\delta S^0}{\delta V^i}. \quad (3)$$
The total quasi–local energy is the integral of $\varepsilon$ over $B$. In obtaining (2), we have assumed that $S^0$, if present, is a linear functional of $N$ and $V^i$ on $3\mathcal{B}$. In the total Hamiltonian extracted directly from (2), the shift vector must satisfy $n_i V^i|_B = 0$. Each of these points is discussed in [12].

In the action (2), one varies $h_{ij}$, $P^{ij}$, $N$, and $V^i$ to obtain

$$
\delta S = (\text{terms giving the equations of motion}) + \int_{t'}^{t''} d^3x P^{ij} \delta h_{ij}
$$

$$
- \int_{\mathcal{B}} d^3x \sqrt{\sigma} \left[ \varepsilon \delta N - j_a \delta V^a - \left( N/2 \right) s^{ab} \delta \sigma_{ab} \right],
$$

where indices $a$, $b$, ... are used to denote tensors on $B$, i.e., $\Sigma$–tensors that are orthogonal to $n^i$. The surface stress tensor $s^{ab}$ on $B$ is given in [12] and does not concern us here. We see from (4) that suitable boundary conditions for $S$ are obtained by fixing the metric induced on the boundary elements $t'$, $t''$, and $3\mathcal{B}$ of $M$. In particular, the lapse function $N$ is fixed on $3\mathcal{B}$, where it determines proper time elements $N dt$ on $3\mathcal{B}$ along the unit normal $u^\mu$ associated with the foliation of $3\mathcal{B}$ by $B$; hence, we refer to $S$ as “Hamilton’s action” in canonical form.

What we define as the microcanonical action $S_m$ is, in essence, Jacobi’s action for general relativity. It is obtained from $S$ by a canonical transformation that changes the appropriate boundary conditions on $3\mathcal{B}$ from fixed metric components $N$, $V^a$, and $\sigma_{ab}$ to fixed energy surface density $\varepsilon$, momentum surface density $j_a$, and boundary metric $\sigma_{ab}$. Thus, define [8]

$$
S_m = S + \int_{\mathcal{B}} d^3x \sqrt{\sigma} \left[ N \varepsilon - V^a j_a \right]
$$

$$
= \int_{\mathcal{M}} d^4x \left[ P^{ij} \delta h_{ij} - N \mathcal{H} - V^i \mathcal{H}_i \right].
$$

From (4), it follows that the variation of $S_m$ is

$$
\delta S_m = (\text{terms giving the equations of motion}) + \int_{t'}^{t''} d^3x P^{ij} \delta h_{ij}
$$

$$
+ \int_{\mathcal{B}} d^3x \left[ \varepsilon \delta \left( \sqrt{\sigma} \varepsilon \right) - V^a \delta \left( \sqrt{\sigma} j_a \right) + \left( N \sqrt{\sigma}/2 \right) s^{ab} \delta \sigma_{ab} \right].
$$

This result shows that solutions of the equations of motion extremize $S_m$ with respect to variations in which $\varepsilon$, $j_a$, and $\sigma_{ab}$ are held fixed on the boundary $B$. Observe that the unspecified subtraction term $S^0$ does not appear in $S_m$. Nevertheless, the
variation (6) of $S_m$ involves $\varepsilon$, $j_a$, and $s^{ab}$, which do depend on $S^0$ through their definitions. However, all dependences on $S^0$ in the boundary variation terms of $\delta S_m$ actually cancel because of the requirement that $S^0$ be a linear functional of the lapse and shift [8][12]. Thus, neither $S_m$ nor its variation depends on $S^0$. In other words, as long as energy and momentum are to be fixed, as in $S_m$, a quantity like $S^0$ whose only role is to determine their “zero points” is irrelevant.

4 MICROCANONICAL FUNCTIONAL INTEGRAL

In [8] and [15] we showed that for nonrelativistic mechanics the density of states is given by a sum over periodic, real time histories, where each history contributes a phase determined by Jacobi’s action. In the case of nonrelativistic mechanics, the energy that is fixed in Jacobi’s action is just the value of the Hamiltonian that generates unit time translations. For a self–gravitating system, the Hamiltonian has a “many–fingered” character: space can be pushed into the future in a variety of ways, governed by different choices of lapse function $N$ and shift vector $V^i$. The value of the Hamiltonian incorporated into (2) depends on this choice. More precisely, the value of the Hamiltonian is determined by the choice of lapse and shift on the boundary $B$, since the lapse and shift on the domain of $\Sigma$ interior to $B$ are Lagrange multipliers for the (vanishing) Hamiltonian and momentum constraints. Accordingly, the energy surface–density $\varepsilon$ and momentum surface–density $j_a$ for a self–gravitating system play a role that is analogous to energy for a nonrelativistic mechanical system.

The above considerations lead us to propose that the density of states for a spatially finite, self–gravitating system is a functional of the energy surface–density $\varepsilon$ and momentum surface–density $j_a$. In addition to these energy–like quantities, the density of states is also a functional of the metric $\sigma_{ab}$ on the boundary $B$, which specifies the size and shape of the system. In the absence of matter fields, these make up the complete set of variables and $\nu[\varepsilon, j_a, \sigma_{ab}]$ is interpreted as the density of quantum states of the gravitational field with energy density, momentum density, and boundary metric having the values $\varepsilon$, $j_a$, and $\sigma_{ab}$. The action to be used in the functional integral representation of $\nu$ is $S_m$, which describes the gravitational field with fixed $\varepsilon$, $j_a$, and $\sigma_{ab}$. Note that $\varepsilon$, $j_a$, and $\sigma_{ab}$ replace the traditional thermodynamical extensive variables. Our variables are all constructed from the dynamical phase space variables ($h_{ij}$, $P^{ij}$) for the system, where the phase space structure is defined using the foliation of $M$ into spacelike hypersurfaces. (We expect this to be a defining feature of extensive variables for general systems of gravitational and matter fields.)

We propose [8] that the density of states of the gravitational field is defined formally
by
\[ \nu[\varepsilon, j, \sigma] = \sum_M \int \mathcal{D}H \exp(iS_m) . \] (Planck’s constant has been set to unity.) The sum over \( M \) refers to a sum over manifolds of different topologies. The three–boundary for each \( M \) is required to have topology \( \partial M = B \times S^1 \). If \( B \) has two–sphere topology, then the sum over topologies includes \( M = (\text{ball}) \times S^1 \), with \( \partial M = \partial(\text{ball}) \times S^1 = S^2 \times S^1 \). Another example is \( M = (\text{disk}) \times S^2 \), with \( \partial M = \partial(\text{disk}) \times S^2 = S^1 \times S^2 \). The action \( S_m \) that appears in Eq. (7) is the microcanonical action (5) of the previous section applied to the manifolds \( M \) with a single boundary component \( \partial M = 3B = B \times S^1 \). The functional integral (7) for \( \nu \) is a sum over Lorentzian metrics \( g_{\mu \nu} \). Note that the microcanonical action may require the addition of terms that depend on the topology of \( M \), such as the Euler number.

In the boundary conditions on \( \partial M = B \times S^1 \), the two–metric \( \sigma_{ab} \) that is fixed on the hypersurfaces \( B \) is typically real and spacelike. Likewise, the energy density \( \varepsilon \) is real, which requires the unit normal to \( \partial M \) to be spacelike. Therefore, the Lorentzian metrics on \( M \) must induce a Lorentzian metric on \( \partial M \), where the timelike direction coincides with the periodically identified \( S^1 \). Note, however, that there are no nondegenerate Lorentzian metrics on a manifold with topology \( M = (\text{disk}) \times S^2 \) that also induce such a Lorentzian metric on \( \partial M \). This implies that the formal functional integral (7) for the density of states must include degenerate metrics. (For a discussion of the role of degenerate metrics in classical and quantum gravity, see [17].)

Now consider the evaluation of the functional integral (7) for fixed boundary data \( \varepsilon, j_a, \sigma_{ab} \) that correspond to a stationary, axisymmetric black hole. That is, start with a real Lorentzian, stationary, axisymmetric, black hole solution of the Einstein equations, and let \( T = \text{constant} \) be stationary time slices that contain the closed orbits of the axial Killing vector field. Next, choose a topologically spherical two–surface \( B \) that contains the orbits of the axial Killing vector field, and is contained in a \( T = \text{constant} \) hypersurface. From this surface \( B \) embedded in a \( T = \text{constant} \) slice, obtain the data \( \varepsilon, j_a, \) and \( \sigma_{ab} \). In the functional integral for \( \nu[\varepsilon, j, \sigma] \), fix this data on each \( t = \text{constant} \) slice of \( \partial M \). Observe that, to the extent that the physical system can be approximated by a single classical configuration, that configuration will be the real stationary black hole that is used to induce the boundary data.

The functional integral (7) can be evaluated semiclassically by searching for four–
metrics $g_{\mu\nu}$ that extremize $S_m$ and satisfy the specified boundary conditions. Observe that the Lorentzian black hole geometry that was used to motivate the choice of boundary conditions is not an extremum of $S_m$, because it has the topology [Wheeler (spatial) wormhole] $\times$ [time] and cannot be placed on a manifold $M$ with a single boundary $S^2 \times S^1$. However, there is a related complex four–metric that does extremize $S_m$, and is described as follows. Let the Lorentzian black hole be given by

$$ds^2 = -\tilde{N}^2dT^2 + \tilde{h}_{ij}(dx^i + \tilde{V}^i dT)(dx^j + \tilde{V}^j dT) ,$$

where $\tilde{N}$, $\tilde{V}^i$, and $\tilde{h}_{ij}$ are $T$–independent functions of the spatial coordinates $x^i$. The horizon coincides with $\tilde{N} = 0$. Choose spatial coordinates that are “co–rotating” with the horizon. Then the proper spatial velocity of the spatial coordinate system relative to observers at rest in the $T = constant$ slices vanishes on the horizon, $(\tilde{V}^i/\tilde{N}) = 0$, and the Killing vector field $\partial/\partial T$ coincides with the null generator of the horizon. As shown in [13], the complex metric

$$ds^2 = -(-i\tilde{N})^2dT^2 + \tilde{h}_{ij}(dx^i - i\tilde{V}^i dT)(dx^j - i\tilde{V}^j dT) ,$$

where the coordinate $T$ is real, satisfies the Einstein equations everywhere on a manifold with topology $M = (disk) \times S^2$, with the possible exception of the points $\tilde{N} = 0$ where the foliation $T = constant$ degenerates. The locus of those points $\tilde{N} = 0$ is a two–surface called the “bolt” [18]. Near the bolt, the metric becomes

$$ds^2 \approx \tilde{N}^2dT^2 + \tilde{h}_{ij}dx^i dx^j ,$$

and describes a Euclidean geometry. The sourceless Einstein equations are not satisfied at the bolt if this geometry has a conical singularity in the two–dimensional submanifold that contains the unit normals $\tilde{n}^i$ to the bolt for each of the $T = constant$ hypersurfaces. However, there is no conical singularity if the circumferences of circles surrounding the bolt initially increase as $2\pi$ times proper radius. The circumference of such circles is given by $P\tilde{N}$, where $P$ is the period in coordinate time $T$. Therefore the absence of conical singularities is insured if the condition $P(\tilde{n}^i\partial_i\tilde{N}) = 2\pi$ holds at each point on the bolt, where $\tilde{n}^i$ is the unit normal to the bolt in one of the $T = constant$ surfaces. Because the unit normal is proportional to $\partial_i\tilde{N}$ at the bolt, this condition restricts the period in coordinate time $T$ to be $P = 2\pi/\kappa_H$, where $\kappa_H = [\partial_i(\tilde{h}_{ij}(\partial_j\tilde{N}))^{1/2}]_H$ is the surface gravity at the horizon of the Lorentzian black hole (8).

The lapse function and shift vector for the complex metric (9) are $N = -i\tilde{N}$ and $V^i = -i\tilde{V}^i$. Thus, (9) and the Lorentzian metric (8) differ only by a factor of $-i$.
in their lapse functions and shift vectors. In particular, the three–metric \( \tilde{h}_{ij} \) and its conjugate momentum \( \tilde{P}_{ij} \) coincide for the stationary metrics (8) and (9) [13]. Since the boundary data \( \varepsilon, j_a \), and \( \sigma_{ab} \) are constructed from the canonical variables only, the complex metric (9) satisfies the boundary conditions imposed on the functional integral for \( \nu[\varepsilon, j, \sigma] \).

The complex metric (9) with the periodic identification given above extremizes the action \( S_m \) and satisfies the chosen boundary conditions for the density of states \( \nu[\varepsilon, j, \sigma] \). Although this metric is not included in the sum over Lorentzian geometries (7), it can be used for a steepest descents approximation to the functional integral by distorting the contours of integration for the lapse \( N \) and shift \( V^i \) in the complex plane. Then the density of states is approximated by

\[
\nu[\varepsilon, j, \sigma] \approx \exp(iS_m[-i\tilde{N}, -i\tilde{V}, \tilde{h}]) ,
\]

where \( S_m[-i\tilde{N}, -i\tilde{V}, \tilde{h}] \) is the microcanonical action evaluated at the complex extremum (9). The density of states can be expressed approximately as

\[
\nu[\varepsilon, j, \sigma] \approx \exp(S[\varepsilon, j, \sigma]) ,
\]

where \( S[\varepsilon, j, \sigma] \) is the entropy of the system. Then the result (11) shows that the entropy is

\[
S[\varepsilon, j, \sigma] \approx iS_m[-i\tilde{N}, -i\tilde{V}, \tilde{h}]
\]

for the gravitational field with microcanonical boundary conditions.

In order to evaluate \( S_m \) for the metric (9), we start with the microcanonical action written in spacetime covariant form [8] and perform a canonical decomposition under the assumption that the manifold \( M \) has the topology of a punctured disk \( \times S^2 \). That is, the spacelike hypersurfaces \( \Sigma \) have topology \( I \times S^2 \), and the timelike direction is periodically identified \( (S^1) \). The outer boundary of the disk corresponds to the three–boundary element \( ^3B \) of \( M \) (denoted \( \partial M \) previously) on which the boundary values of \( \varepsilon, j_a \), and \( \sigma_{ab} \) are imposed. The inner boundary of the disk, the boundary of the puncture, appears as another boundary element \( ^3H \) for \( M \). (No data are specified at \( ^3H \).) The canonical decomposition results in [8]

\[
S_m = \int_M d^4x P^{ij} \dot{h}_{ij} - N \mathcal{H} - V^i \mathcal{H}_i + \int_{^3H} d^3\sqrt{\sigma} [n^i(\partial_i N)/\kappa + 2n_i V_j P^{ij} / \sqrt{h}] ,
\]

where the expression \( a_i = (\partial_i N)/N \) for the acceleration of the timelike unit normal has been used. The boundary term at \( ^3H \) was first given in [13].
Now evaluate the action $S_m$ on the punctured disk $\times S^2$ for the complex metric (9), and take the limit as the puncture disappears to obtain a manifold topology $M = (\text{disk}) \times S^2$. In this limit, the smoothness of the complex geometry is assured by the regularity condition on the period of $T$. Since the metric satisfies the Einstein equations, the Hamiltonian and momentum constraints vanish, and the terms $P^{ij} \dot{h}_{ij}$ also vanish by stationarity. Moreover, the second boundary term at $^3H$ is zero because the shift vector vanishes at the horizon. Thus, only the first of the boundary terms at $^3H$ survives. Evaluating this term for the complex metric (9), that is, for the lapse function $N = -i \tilde{N}$, and using the regularity condition $P = 2\pi / \kappa_H$, we find for the microcanonical action

$$S_m[-i \tilde{N}, -i \tilde{V}, \tilde{h}] = -\frac{i}{\kappa} \int_0^\rho d\tau \int d^2x \sqrt{\tilde{\sigma}} \tilde{n}^i \partial_i \tilde{N} = -\frac{i}{4} A_H,$$

(15)

where $A_H$ is the area of the event horizon for the Lorentzian black hole (8).

The result (15) for the microcanonical action evaluated at the extremum (9) leads to the following approximation for the entropy (13):

$$S[\varepsilon, j, \sigma] \approx \frac{1}{4} A_H.$$

(16)

The generality of the result (16) should be emphasized: The boundary data $\varepsilon$, $j_a$, and $\sigma_{ab}$ were chosen from a general stationary, axisymmetric black hole that solves the vacuum Einstein equations within a spatial region with boundary $B$. Outside the boundary $B$, the black hole spacetime need not be free of matter or be asymptotically flat. Thus, for example, the black hole can be distorted relative to the standard Kerr family. Furthermore, recall that the quantum–statistical system with this boundary data is classically approximated by the physical black hole solution that matches that boundary data. The result (16) shows that the entropy of the system is approximately 1/4 the area of the event horizon of the physical black hole configuration that classically approximates the contents of the system. It also should be emphasized that the entropy is independent of the term $S^0$ in (1) as has been shown in the framework of the canonical partition function [9].

Expression (16) is the principal result we wished to demonstrate here. The physical and mathematical limitations of our analysis and possible ways to overcome them are described in [8], which also discusses canonical and grand canonical boundary conditions for the rotating black hole. A further elaboration of the physical underpinnings and implications of our analysis of relativistic rotating systems is given in [19].
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