Elementary branching: waves, rays, decoherence

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Abstract
The ‘elementary particle’ of branched flow is the splitting of waves and rays into localised branches at an individual cusped caustic. Localisation is the result of noise in the incident beam: decoherence for waves, here modelled analytically by random phases, or irregularity in the rays, here modelled by transverse smoothing. The two models coincide as the localisation emerges with increasing noise.

Keywords: Pearcey, caustic, cusp, diffraction

(Some figures may appear in colour only in the online journal)

1. Introduction
The remarkable phenomenon of branched flow, in which waves split repeatedly while travelling through a smoothly and gently disordered refracting medium, has been studied in different contexts and from different perspectives. Examples where branching occurs (mostly two-dimensional) are: electrons issuing from a quantum point contact [1–3], tsunamis passing over an ocean whose depth varies [4], rogue waves [5], microwaves [6], and recently light, in beautiful experiments and simulations [7] with incident beams and plane waves confined within soap films whose thickness varies. Investigations have centred on several aspects, including: statistics [8–12], ray and wave descriptions [10], controlling flow along the branches [13], and persisting localisation of the branches [7]. The subject has been reviewed [4].

Because the inhomogeneities vary slowly on the wavelength scale, explanations of branching should involve the rays of geometrical optics (or classical trajectories in the quantum case). Individual rays cannot split, because ray dynamics is deterministic. But any wave field, even a narrow beam, corresponds to a family of rays, and a family has the holistic property that no individual ray possesses: caustics, i.e. envelopes of the ray family, on which focusing occurs. And caustics can possess branches; in two dimensions, these are smooth curves that originate in pairs at cusp point singularities (see figure 2(b) later). This association of branching with cusped caustics is recognised [4] (see the sketched figure 2 of [14] for an early example).

My purpose here is to concentrate on what might be regarded as the ‘elementary particle’ of branched flow: details of the intensity near an cusp. It is necessary to investigate this, because although the intensity of a cusped caustic is strong along the branches issuing from it, refraction fills the region between the branches with rays, and diffraction decorates the geometrical rays with interference detail: specifically, the Pearcey pattern [15]. The suppression of this structure, leading to intensity localised on separate branches, is the result of noise in the incident beam—not to be confused with the smoothly varying fixed disorder in the refractive-index distribution. Its effect will be calculated using two models, one for waves and one for geometrical rays. The aim is not to simulate any particular observation, but to provide analytical ‘minimal models’ [16] exhibiting the splitting as simply as possible.

Section 2 describes the diffraction pattern and corresponding geometrical ray intensity for waves or rays propagating from an initial phase screen. This is standard material, included to illustrate how an individual cusp occurs without the complication of repeated branching. In section 3, noise is
modelled for waves by parameter-dependent random phases in the initial wavefront; in quantum terminology, this corresponds to a mixed state rather than a pure state, and interference is suppressed by decoherence, resulting in the emergence of distinct branches, with the intensity described by integrals of Airy functions (technical details are presented in the appendix). This suppression has recently been observed [17]. Section 4 describes the corresponding ray theory; the geometrical counterpart of phase noise and a mixed state is an incident ray family that is no longer a normal congruence [18]: it comprises beams travelling in different directions. This is modelled by smoothing the ray intensity. Section 5 shows that in the strong-noise or short-wave asymptotic limits the cusped ray and wave branched patterns are the same.

2. Pearcey and ray caustic

This reprise of standard material is included to illustrate with the simplest model how the Pearcey wave and the cusped ray caustic arise.

The model is the free-space evolution of a monochromatic wave $\psi(x,z)$ from a gently varying initial wavefront with profile $f(x)$, modulating a locally plane wave. This is given by the familiar paraxial diffraction integral for wavelength $\lambda = 2\pi/k$, expressing the superposition of elementary waves (figure 1):

$$ \psi(x,z) = \sqrt{\frac{ik}{2\pi z}} \exp(ikz) \int_{-\infty}^{\infty} dx' \exp \left( ik \left( -f(x') + \frac{(x-x')^2}{2z} \right) \right). $$  

(1)

As discussed in section 7 of [19], the global geometro-optics caustics consist of smooth curves (fold catastrophes), beginning and ending in the far field ($z \to \infty$), each containing a cluster of an odd number of alternately back- and forward-pointing cusp points, corresponding to the zeros of the derivative $f_{xx}$, between successive zeros of $f_x$.

A perfect point focus would be generated by a quadratic wavefront (because spheres are paraxially modelled by parabolas). To get the simplest profile that generates a single cusp, aberration must be included. Locally, a quartic modulation suffices:

$$ f(x) = \frac{x^2}{2F} - \frac{x^4}{4L^2}. $$  

(2)

The focus is at $z = F, x = 0$, and the parameter $L$ determines the separation of the curved focal caustic lines emanating from the cusp point. Convenient scalings to represent the wave (1) in standard dimensionless form are

$$ x' = \sqrt{\frac{3}{2k}} \left( \frac{L^3}{k} \right)^{1/4} \xi, \quad x = -\frac{FX}{\sqrt{2(kL)^{3/4},}} \quad z = F + \frac{F^2Z}{\sqrt{kL}}, $$  

(3)

after which expansion of the phase in (1) near the focus, neglect of a phase factor depending only on $x$, and with a convenient prefactor, gives

$$ \psi(X,Z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\xi \exp \left( i \left( \xi^4 - \xi^2Z + \xi X \right) \right). $$  

(4)

This is the Pearcey function [15, 20], with $Z$ reversed compared to the usual representation. Its intensity is illustrated in figure 2(a). Note the $k$ dependences of the scalings (3); they indicate that in physical units the fringe spacing near the cusp is $O(\lambda^{3/d})$ transversely, and $O(\lambda^{1/2})$ longitudinally. And the physical amplitude prefactor, ignored in (4), is proportional to $k^{1/4}$, showing that the wave intensity at the cusp point diverges as $1/\lambda$ in the geometrical optics limit. These scaling indices are special cases of the ‘twinkling exponents’ [21] associated with the wave patterns (‘diffraction catastrophes’) describing the interference structures decorating more general caustics locally described by catastrophe theory [19, 22–25]. Although the particular model (1) and (2) generates a cusp explicitly in a simple way, the Pearcey integral (4), and the $k$ scalings (3), are universal: catastrophe theory guarantees that they are stable under perturbation [22], and so give the local description of waves near any cusped caustic, including those being studied here, where branching occurs.
The family of geometrical rays is determined by the real variables $\xi = \xi_j$ for which the phase in (4) is stationary (vanishing first derivative), and the caustic $X_c(Z)$, which is the focal curve enveloped by the rays, is determined by the additional condition that the second derivative vanishes. Thus
\[
\text{rays: } 4\xi^3 - 2\xi Z + X = 0 \Rightarrow \xi_j(X,Z) \\
\text{caustic: } 6\xi^2 - Z = 0 \Rightarrow X_c^2 = \frac{8}{27} Z^2.
\]
(5)

Inside the cusp, i.e. $|XL < X_c$, there are three real rays $\xi_j$; outside, there is one. The rays and caustic are illustrated in figure 2(b). The geometrical-optics intensity associated with this family of rays, diverging on the caustic, is given by the Jacobian of the transformation from $\xi$ on the initial wavefront to $X$ at height $Z$:
\[
I_{\text{geom}}(X,Z) = \int_{\infty}^{-\infty} d\xi \delta \left( 4\xi^3 - 2\xi Z + X \right) \\
= \frac{1}{2} \sum_{\xi_j(X,Z) \text{ real}} \frac{1}{1 - 6\xi_j^2(X,Z)}.
\]
(6)

As explained in appendix 2 of [19], cusps occur in aberration theory, as constituents of more complicated caustics, in particular the hyperbolic umbilic, whose singular section consists of two branches meeting at a finite angle, i.e. pure coma.

3. Wave decoherence

Of several possible ways to suppress interference by incorporating noise, we choose to introduce a random phase modulation $\Phi(\xi)$ into $P(X,Z)$:
\[
P_r(X,Z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\xi \exp \left( i \left( \xi^4 - \xi^2 X + \xi X \right) \right) \exp(i\Phi(\xi)).
\]
(7)

The modulation-averaged intensity is a double integral. With $\Phi$ chosen as a smooth stationary Gaussian random function, the integrand can be averaged explicitly, leading to
\[
I_{\text{wave,sm}}(X,Z,H) = \left\langle \left| P_r(X,Z) \right|^2 \right\rangle \\
= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta \exp \left( E(\xi,\eta;X,Z,H) \right),
\]

\[
E(\xi,\eta;X,Z,H) = i \left( \xi^4 - \eta^4 \right) - iZ \left( \xi^2 - \eta^2 \right) + iX(\xi - \eta) - \frac{1}{2} H^2 (\xi - \eta)^2.
\]
(8)

The evaluation involves the decoherence factor, for which we choose the simplest local model:
\[
(\exp(\Phi(\xi) - \Phi(\eta))) = \exp \left( -\frac{1}{2} \left( \Phi(\xi) - \Phi(\eta) \right)^2 \right) \\
= \exp \left( -\frac{1}{2} H^2 (\xi - \eta)^2 \right).
\]
(9)

The first equality can be derived by expanding the exponential and averaging term by term. In the second equality, the parameter $H$ is a convenient dimensionless descriptor of the slope of the random-phase difference between nearby points. In physical units, the scalings (3) imply that if the slope of the random contribution to the initial wavefront profile is $\theta(x^c)$, then
\[
\left\langle \theta^2 \right\rangle = \frac{H^2}{2(kL)^{3/2}}.
\]
(10)

The model (9) applies strictly for small $\xi - \eta$, but is accurate for all $\xi - \eta$ if $\left\langle \Phi^2 \right\rangle \gg 1$, i.e. for strong noise.

In (8), the two amplitudes are coupled by the term $H^2 \xi \eta$ in the exponent. Either of the $\xi$ or the $\eta$ integrals can be evaluated, leaving the other integral which involves the Pearcey function or its conjugate. However, a simpler form is obtained by some transformations, explained in the appendix, that enable (10) to be reduced to the sum of two single integrals involving the Airy function:
\[
I_{\text{wave,sm}}(X,Z,H) = \int_{0}^{\infty} \frac{du}{(3u)^{1/3}} \\
\times \sum_{\pm} \left[ \arctan \left( \frac{H^2}{4(3u)^{2/3}} \left( 1 + \frac{12u}{H^2} (4u^3 - 2uZ \pm X) \right) \right) \right] \\
\times \exp \left( -\frac{H^2}{4(3u)^{2/3}} (4u^3 - 2uZ \pm X) \right).
\]
(11)

This is illustrated in figure 3 for several values of $H$. It is clear how interference detail is suppressed as $H$ increases, leading to separation of the intensity into two localised branches.
**4. Ray smoothing**

A simple way to incorporate noise in the ray intensity (6) is by Gauss-smoothing laterally, with $X$ width $H$. This could be produced by a smooth random perturbation of the initial wavefront, analogous to $\Phi(\xi)$ that generates decoherence in the wave description, because the varying slope of the perturbation changes the slopes of the rays, so they are randomly shifted when they reach the height $Z$, corresponding to incident rays that do not form a normal congruence. The resulting $H$-smoothed ray intensity is

\[
I_{\text{geom},\text{sm}}(X,Z,H) = \frac{1}{\sqrt{2\pi H}} \int_{-\infty}^{\infty} dX' \exp \left( \frac{-(X-X')^2}{2H^2} \right) I_{\text{geom}}(X,Z)
\]

\[
= \frac{1}{\sqrt{2\pi H}} \int_{-\infty}^{\infty} d\xi \exp \left( \frac{-(4\xi^3 - 2Z\xi^2 + X)^2}{2H^2} \right).
\]

This is illustrated in figure 4 for the same values of $H$ as in figure 3.

For $H = 0.2$, the intensity is concentrated close to the caustics, where it reaches large values, approaching the singular geometrical-optics intensity (6). As $H$ increases, the geometrical caustic is increasingly smoothed, and for $H = 2$ it closely resembles the corresponding smoothed wave intensity in figure 3.

**5. Asymptotic connection between smoothings**

The large $H$ resemblance between the smoothed wave and ray patterns (see $H = 2$ in figures 3 and 4) is not accidental. When $H$ is large—that is, for strong noise or short wavelength (cf (9))—the argument of the Airy functions in the wave intensity (11) is large and positive, so $\text{Ai}$ can be approximated by its leading-order asymptotics

\[
\text{Ai}(x) \approx \frac{1}{2\sqrt{\pi}x^{1/4}} \exp \left( -\frac{2}{3}x^{3/2} \right), \quad (x > 1).
\]

Then the leading orders in the large $H$ expansions of the exponent in the approximated $\text{Ai}$ and the exponential factors cancel, leading to precisely the smoothed ray intensity (12). Below the cusp, i.e. $Z < 0$, the ray and wave intensities are similar even for smaller $H$, because for waves there is no interference to suppress, and the geometrical ray pattern is already smooth (in this region there is only one ray and no caustic). Figure 5 illustrates the similarity of the wave and ray patterns for large $H$ and also $Z < 0$.

A special case of the asymptotic connection is the wave intensity at the cusp point $X = Z = 0$. From (11), in dimensionless scaled units,

\[
I_{\text{wave,sm}}(0,0,H) = \frac{2}{3\sqrt{\pi}} \int_{0}^{\infty} du \frac{u}{11/3} \text{Ai} \left( \frac{H^2 + 48u^4}{4(3u)^{11/3}} \right)
\]

\[
\times \exp \left( \frac{H^2 + 72Hu^4}{108u^2} \right).
\]

In physical units, this intensity is multiplied by a factor proportional to $\kappa^{12}$. The small and large $H$ limiting forms are

\[
I_{\text{wave,sm}}(0,0,H) \approx \frac{2}{3\sqrt{\pi}} \int_{0}^{\infty} du \frac{u}{11/3} \text{Ai} \left( \frac{4u^4}{3\sqrt{\pi}} \right) = |P(0,0)|^2
\]

\[
= \frac{\Gamma(\frac{1}{3})^2}{8\pi} = 0.523025
\]

\[
\approx \frac{1}{H} \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} du \exp \left( -\frac{8u^6}{H^2} \right) = \frac{\Gamma(\frac{1}{3})}{6\sqrt{\pi}H^{11/3}}.
\]

\[
= 0.52341 \frac{H^{11/3}}{H^{11/3}},
\]

(15)
in which the Ai asymptotics (13) has been used for $H \gg 1$; the geometrical-optics formula (12) gives the same result. Thus, the fractional suppression of the wave intensity at the cusp point is, in physical units, is the ratio

$$r(H) = \frac{I_{\text{wave,sm}}(0,0,0)}{I_{\text{wave,sm}}(0,0,H)} \rightarrow \frac{0.999265}{0.793117} \sqrt{\frac{H^2}{3 L}} \langle \theta^2 \rangle^{1/3}.$$ (16)

The factor $1/\sqrt{\kappa}$ cancels the $\sqrt{\kappa}$ wave intensity at the Percey maximum, reflecting the $\kappa$-independent geometrical optics limit of large $H$. Figure 6 shows $r(H)$, illustrating the suppression and the limits (15).

6. Concluding remarks

This calculation has modelled how the wave and ray patterns near a cusp take the form of localised branches, as the result of noise blurring both the wave interference structure and the geometrical caustic singularities. Since only an individual cusp is involved, the simple phase screen model suffices.

Phase screens can generate quite complicated caustics, including clusters of cusps (e.g. figure 7), and for random screens the statistics can be calculated explicitly (section 7 of [19]). But the phase screen model cannot generate repeated branching along caustic curves exhibited by branched flow: this requires the refractive inhomogeneity to persist as the wave propagates.

The ray bending underlying the formation of repeated cusps depends, paraxially, on the transverse variation of refractive index (or potential in the case of quantum waves). Beyond the phase screen, the simplest model is of transverse variations that are propagation-independent, for example the refractive index corresponding to the ‘volume grating’

$$n(x) = n_0 + n_1 \cos qx$$ (17)

for propagation along $z$. This model originated in the 1930s as a description of the diffraction of light by ultrasound [26, 27]. But as figure 8 shows, it still does not generate repeated branching. Instead, there is a proliferation of individual cusps, leading to increasing number of contributing rays that get denser and fill the $x$ axis ergodically as $z \rightarrow \infty$ [28]; in this limit, the ray intensity and the corresponding wave intensity coincide when smoothed, but this is different from the ‘elementary particle’ smoothing considered here, because it acts on the superposition of many caustics.

Repeated branching requires the refractive inhomogeneities to vary longitudinally as well as transversely—as in all physical examples and simulations of branched flow. An interesting question, not addressed here, is whether the further focusing associated with this longitudinal variation, combined with wave or ray noise, is required to explain the observed persistent localisation of the branches [7] beyond their births at cusps.

Another smoothing effect not considered here is chromaticity. Most media are dispersive, so with incident white light the

\[ Z \]
\[ F(x) \]
\[ x \]

Figure 7. Caustics (red) from phase screen (black) with profile $f(x) = \cos(x) + \sin(\sqrt{2}x) + 0.05 \cos(6x)$.

Figure 8. Proliferation of cusps in the propagation-independent refractive index. (17)
differently coloured rays will separate, albeit only slightly in paraxial branched flow: the branches will be weakly coloured. Dispersion is also the basis of Newton’s explanation of the colours of rainbows, which are much stronger because rainbow optics is not paraxial. An additional and in a sense more fundamental source of colour is diffraction. This contributes to the colours of natural rainbows [29, 30]; in gravitational lensing, which is refractively achromatic, diffraction is the only potentially observable source of colour, and the corresponding spectral distortion of the Airy patterns decorating caustic curves has been described analytically (section 4 of [31]). The analogous effect for the cusps in branched flow would be worth exploring.

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Appendix. Calculation of wave integral (11)

In the intensity (10), the integration variables $\xi, \eta$ appear quartically in the exponent $E$. The transformation

$$
\xi = u + \frac{i}{6\omega} \left( \frac{t}{|u|^{1/3}} - \frac{iH^2}{6\omega} \right), \quad \eta = u - \frac{i}{6\omega} \left( \frac{t}{|u|^{1/3}} + \frac{iH^2}{6\omega} \right),
$$

(A1)

to new variables $u, t$, results in an exponent where the $t$ dependence is cubic:

$$
E(\xi, \eta; X, Z, H) = \frac{1}{4} t^3 \text{sgn} u + \frac{iH^3}{4 \cdot 3^{1/3} |u|^{1/3}} \left( 4u^3 - 2uZ + X \right) \left( 1 + \frac{12u}{H^2} \left( 4u^3 - 2uZ + X \right) \right),
$$

(A2)

If the $u$ integral is separated into contributions with $u \geq 0$ and $u < 0$, the resulting two $t$ integrals can be evaluated in terms of standard Airy functions, leading directly to (11). The integrands in (11) decrease rapidly for large $u$, so the integrals converge. But in the range represented in figure 3 the factors in the integrands vary over hundreds of orders of magnitude, so numerical integration is not straightforward. To generate figure 3, I used a judicious combination of the factors, and incorporated Airy asymptotics.

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