Torsional surface waves in an inhomogeneous layer over a gravitating anisotropic porous half-space

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Abstract. The present work aims to deal with the propagation of torsional surface wave in an inhomogeneous layer over a gravitating anisotropic porous half space. The inhomogeneous layer exhibits the inhomogeneity of quadratic type. In order to show the effect of gravity the equation for the velocity of torsional wave has been obtained. It is also observed that for a layer over a homogeneous half space without gravity, the torsional surface wave does not propagate. An attempt is also made to assess the possible propagation of torsional surface waves in that medium in the absence of the upper layer. The effects of inhomogeneity factors and porosity on the phase velocity are depicted by means of graphs.

1. Introduction
The study of surface waves propagating over the surface of homogeneous and inhomogeneous half space is important to seismologists and in understanding the causes, estimation of damage due to earthquakes. A quite good amount of valuable information about the propagation of elastic waves is the monograph written by Ewing et al. [1]. Numerous papers on the subject have been published in various journals. A lot of information on the effect of heterogeneity in the study of surface waves due to line-load was given by Vrettos [2, 3]. Unfortunately the literature available on torsional waves in an inhomogeneous layer over an inhomogeneous half space are very much less than that of Rayleigh Waves, Love waves and Stoneley Waves. Lord Rayleigh [4], in his remarkable paper, showed that the isotropic homogenous elastic half-space does not allow a Torsional surface wave to propagate. Bhattacharya [5] has investigated the torsional wave propagation in a two layered circular cylinder with imperfect bond. Later on, Meissner [6] pointed out that in an inhomogeneous elastic half space with quadratic variation of shear modulus and density varying linearly with depth, torsional surface waves do exist. Torsional surface waves in an initially stressed cylinder have been studied by Dey and Dutta [7], and the existence and propagation of torsional surface waves in an elastic half space with void pores has been discussed by Dey et al. [8]. The propagation of torsional surface waves in non-homogeneous and anisotropic medium with polynomial and exponential variation in rigidity and constant density has been discussed by Dey et al. [9]. Whether torsional surface waves can propagate in the presence of a gravity field if the medium is elastic or dry sandy has been studied by Dey et al. [10]. The same authors [11] studied torsional surface waves will always propagate if the medium is taken as sandy or elastic in gravitating earth under initial stress. The presence of initial stress effects on the propagation of torsional surface waves in a non-homogeneous
anisotropic medium which has been studied by Dey et al. [12].

The present paper attempts to study the possibility of propagation of torsional surface waves in an inhomogeneous layer over a gravitating anisotropic porous half space. It is observed that the presence of a gravity field always allow the torsional surface wave to propagate. The anisotropy also has a great effect in enhancing the velocity of torsional waves. The effect of inhomogeneity, porosity and gravity on the velocity of torsional wave was studied graphically.

2. Formulation of the problem

Let us consider a model which consists of an inhomogeneous elastic layer of thickness \( H \), over anisotropic porous half space under gravity as shown in Figure 1. The inhomogeneity has been considered both in rigidity and density. The surface of contact is the plane \( z = 0 \) and the \( z \)-axis is taken to be positive downwards. The wave is assumed to propagate along radial direction.

![Figure 1. Geometry of the problem.](image)

3. Solution of the problem

3.1. Solution for non-homogeneous elastic layer

The upper medium has been considered as a non-homogeneous elastic layer. The dynamical equation of motion for the torsional surface waves propagating in the radial direction can be written as

\[
\frac{\partial \sigma_{\theta \theta}}{\partial r} + \frac{\partial \sigma_{z \theta}}{\partial z} + \frac{2\sigma_{\theta \theta}}{r} = \rho(z) \frac{\partial^2 v}{\partial t^2},
\]

where \( v(r, z, t) \) is the displacement along the \( \theta \) (azimuthal) direction. The stresses are related to the displacement component by

\[
\sigma_{r \theta} = \mu(z) \left( \frac{\partial v}{\partial r} - \frac{v}{r} \right), \quad \sigma_{z \theta} = \mu(z) \left( \frac{\partial v}{\partial z} \right).
\]

Using equation (2), equation (1) takes the form

\[
\mu(z) \left( \frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} - \frac{1}{r^2} \right) + \frac{\partial}{\partial z} \left( \mu(z) \frac{\partial v}{\partial z} \right) = \rho(z) \frac{\partial^2 v}{\partial t^2}.
\]

where \( \mu \) and \( \rho \) are the rigidity and the mass density of the media respectively, assumed to be function of depth \( z \). Harmonic wave solution of Eq. (3) is of the form

\[
v = V(z) J_1(kr) \exp(i\omega t),
\]
where $V(z)$ is the solution of the following equation:

$$ \frac{d^2 V(z)}{dz^2} + \frac{\mu'(z)}{\mu(z)} \frac{dV(z)}{dz} - k^2 \left( 1 - \frac{c^2}{c_s^2} \right) V(z) = 0. \quad (5) $$

In the above equation, $c = \omega/k$ is the torsional wave velocity, $c_s = \sqrt{\mu(z)/\rho(z)}$, $\omega$ is the angular frequency, $k$ is the angular wave number, and $J_1(Kr)$ is the first order Bessel function of first kind. For the layer we have considered

$$ \mu = \mu_0 \left( 1 + \frac{z}{b} \right)^2, \quad \rho = \rho_0 \left( 1 + \frac{z}{b} \right)^2. \quad (6) $$

Assume

$$ V = \frac{V_1}{\sqrt{\mu}}. \quad (7) $$

Now using (6) and (7), the equation for the upper layer, equation (5) takes the form

$$ \frac{d^2 V_1}{dz^2} - \lambda^2 V_1 = 0, \quad (8) $$

where

$$ \lambda^2 = k^2 \left( 1 - \frac{c^2}{c_s^2} \right), \quad c_s = \sqrt{\mu_0/\rho_0}. \quad (9) $$

So, the solution of equation (8) is obtained as

$$ V_1 = A_1 e^{\lambda z} + A_2 e^{-\lambda z}. \quad (10) $$

Therefore

$$ v = v_0 \text{(say)} = \frac{A_1 e^{\lambda z} + A_2 e^{-\lambda z}}{\sqrt{\mu_0 \left( 1 + \frac{z}{b} \right)}} J_1(kr) e^{i\omega t}. \quad (11) $$

### 3.2. Solution for the porous half space

The half space is under the action of gravity field. The origin of the co-ordinate system is located at the considered point source on the free surface as shown in figure 1. Let $r$ and $\theta$ be the radial and circumferential co-ordinates, respectively. It is assumed that the torsional wave travels in the radial direction and that all mechanical properties associated with it are independent of $\theta$. For the torsional wave $u = w = 0$ and $v = v(r, z, t)$, and the equation of motion in anisotropic porous medium under gravity may be written as Biot [13]:

$$ \frac{\partial \sigma_{r\theta}}{\partial r} + \frac{\partial \sigma_{z\theta}}{\partial z} + 2 \sigma_{r\theta} + \frac{\partial}{\partial r} \left( d' g z e_{z\theta} \right) - d' g z \frac{\partial}{\partial z} \left( \frac{1}{2} \frac{\partial v}{\partial r} + \frac{v}{r} \right) = d' \frac{\partial^2 v}{\partial t^2}, \quad (12) $$

where $v(r, z, t)$ is displacement along the $r$ direction and $g$ is acceleration due to gravity. For a gravitating anisotropic porous medium the stress is related to strain by

$$ \sigma_{r\theta} = 2N e_{r\theta}, \quad \sigma_{z\theta} = 2L e_{z\theta}, \quad e_{r\theta} = \frac{1}{2} \left( \frac{\partial v}{\partial r} - \frac{v}{r} \right), \quad e_{z\theta} = \frac{1}{2} \left( \frac{\partial v}{\partial z} \right), \quad (13) $$

where $N$ and $L$ are the shear modulus of elasticity along the $r$ and $z$ directions, respectively. Equation (12) using the above relation takes the form

$$ \left( N - \frac{1}{2} d' g z \right) \left( \frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} - \frac{1}{r^2} \right) v + \left( L - \frac{1}{2} d' g z \right) \frac{\partial^2 v}{\partial z^2} - \frac{d' g}{2} \frac{\partial v}{\partial z} = d' \frac{\partial^2 v}{\partial t^2}, \quad (14) $$
where \( d' = \rho_{11} - \frac{c^2}{2G} \) and the dynamic co-efficients \( \rho_{11}, \rho_{12}, \rho_{22} \) take into account the inertia effects of the moving fluid and are related to the mass densities of the solid \( \rho_s \) and fluid \( \rho_w \) as written by Biot [14, 15] in the equations

\[
\rho_{11} + \rho_{12} = (1 - f) \rho_s, \quad \rho_{12} + \rho_{22} = f \rho_w,
\]

so that the mass densities of the bulk material are

\[
\rho' = \rho_{11} + 2\rho_{12} + \rho_{22} = \rho_s + f (\rho_w - \rho_s).
\]

The dynamic co-efficients, moreover, obey the inequalities, as written by Biot [14]

\[
\rho_{11} > 0, \quad \rho_{22} > 0, \quad \rho_{12} < 0, \quad \rho_{11}\rho_{22} - \rho_{12}^2 > 0,
\]

where \( f \) is the porosity of the layer. The parameters may be non-dimensionalized as

\[
\gamma_{11} = \frac{\rho_{11}}{\rho'}, \quad \gamma_{12} = \frac{\rho_{12}}{\rho'}, \quad \gamma_{22} = \frac{\rho_{22}}{\rho'}, \quad \text{and} \quad d = \frac{d'}{\rho'} = \frac{d''}{\rho'} = \frac{\gamma_{11} - \gamma_{12}^2}{\gamma_{22}}.
\]

Thus one obtains the following:

(i) \( d \rightarrow 1 \), when the layer is non-porous solid;
(ii) \( d \rightarrow 0 \), when the layer is fluid;
(iii) \( 0 < d < 1 \) when the layer is poro-elastic.

For the wave propagating along the \( r \) direction one may assume the solution of (14) as

\[
v = V(z) J_1(kr) \exp (i\omega t).
\]

Here \( V \) is the solution of

\[
V'' - \frac{Gkd}{2(1 - \frac{Gkd}{2})} V' - k^2 \left[ \frac{N}{L} \left( 1 - \frac{GkdLz}{2N} \right) - \frac{c^2d}{c_1^2 \left( 1 - \frac{Gkd}{2} \right)} \right] V = 0,
\]

where \( c = \frac{c_2}{N} \) is the velocity of the torsional wave, \( c_1 = \left( \frac{L}{p} \right)^\frac{1}{2} \) is the velocity of the shear wave in an anisotropic elastic medium along the \( r \) direction, \( G = \frac{Gd}{L} \) is Biot's gravity parameter, \( k \) is the wave number, and \( J_1(kr) \) is the Bessel function of the first kind and of order one. Substituting \( V = \frac{\phi(z)}{(1 - \frac{GkdLz}{2})^\frac{1}{2}} \) in equation (18), we have

\[
\phi''(z) + \left[ \frac{k^2G^2d^2}{(1 - \frac{Gkd}{2})^2} - k^2 \left\{ \frac{N}{L} \left( 1 - \frac{GkdLz}{2N} \right) - \frac{c^2d}{c_1^2 \left( 1 - \frac{Gkd}{2} \right)} \right\} \right] \phi(z) = 0.
\]

Again substituting \( \delta = \frac{4}{c'^2d} \left( 1 - \frac{Gkd}{2} \right) \) in Equation (19), it may be reduced to

\[
\phi''(\delta) + \left[ -\frac{1}{4} + \frac{m}{\delta} + \frac{1}{4\delta^2} \right] \phi(\delta) = 0,
\]

where \( m = \frac{c^2}{c'^2d} + \frac{1}{c'^2d} \left( 1 - \frac{N}{L} \right) \). Equation (20) is known as the Whittaker equation [16], whose solution is

\[
\phi(\delta) = A_3 W_{-m,0}(-\delta) + A_4 W_{m,0}(\delta).
\]
As the solution should vanish at \( z \to \infty \), that is, for \( \delta \to -\infty \) we may take the solution as

\[
\phi (\delta) = A_3 W_{-m,0} (-\delta).
\]

Hence, the solution of Equation (14) may be written as

\[
v = v_1 \text{ (say)} = A_3 W_{-m,0} \left[ -\frac{4Gd}{Gd^2 \left( 1 - \frac{Gdkz}{2} \right)} \right] \left( 1 - \frac{Gdkz}{2} \right)^{1/2} J_1 (kr) e^{i\omega t}.
\] 

(22)

4. Boundary conditions

The appropriate boundary conditions are as follows:

- Upper surface is stress free i.e., at the upper boundary \( z = -H \),

\[
\mu_0 \frac{\partial v_0}{\partial z} = 0.
\] 

(23)

- At the interface \( z = 0 \)

  Continuity of the displacement components,

\[
v_0 = v_1.
\] 

(24)

  Continuity of stress components,

\[
\mu_0 \left( \frac{\partial v_0}{\partial z} \right) = L \left( \frac{\partial v_1}{\partial z} \right).
\] 

(25)

5. Results

Expanding the Whittaker function up to the quadratic term (as in the expansion, depth \( z \) occurs in the denominator of each term with higher powers and surface waves vanish with increase in depth), and substituting into boundary conditions (23), (24) and (25), we get

\[
A_1 e^{-\lambda H} \left( \lambda \left( 1 - \frac{H}{b} \right) - \frac{1}{b} \right) - A_2 e^{\lambda H} \left( \lambda \left( 1 - \frac{H}{b} \right) + \frac{1}{b} \right) = 0,
\] 

(26)

\[
\frac{A_1 + A_2}{\sqrt{\mu_0}} = A_3 e^{2\delta} \left( -\frac{4}{Gd} \right)^{-m} Y_1,
\] 

(27)

\[
\sqrt{\mu_0} \left[ A_1 \left( -\lambda + \frac{1}{b} \right) + A_2 \left( \lambda + \frac{1}{b} \right) \right] = 2A_3 Lke^{2\delta} \left( -\frac{4}{Gd} \right)^{1/2} Y_2.
\] 

(28)

Now eliminating \( A_i \) \((i = 1, 2, 3)\) form (26), (27) and (28), we get

\[
\frac{X_2 \left( \lambda - \frac{1}{b} \right) e^{\lambda H} - X_1 \left( \lambda + \frac{1}{b} \right) e^{-\lambda H}}{X_2 e^{\lambda H} + X_1 e^{-\lambda H}} = 2k \left( \frac{L}{\mu_0} \right) \left( \frac{Y_2}{Y_1} \right),
\] 

(29)

where \( X_1, X_2, Y_1, Y_2 \) are defined in Appendix.
6. Particular cases

6.1. Case I
When \( N = L = \mu_1 \) (say), \( b \to \infty \) and \( G \to 0 \), that is, when the both of layer and half-space become homogeneous and isotropic and no gravity be present, then equation (29) takes the form

\[
\tan \left[ kH \sqrt{ \frac{c^2}{c_0^2} - 1 } \right] = \left( \frac{\mu_1}{\mu_0} \right) \sqrt{ \frac{1 - \frac{c^2}{c_1^2}}{\frac{c^2}{c_0^2} - 1} },
\]

which is well-known dispersion equation of Love type wave in a homogeneous crustal layer over a homogeneous half-space.

6.2. Case II
In the absence of the upper layer \( (H \to 0) \), then the velocity equation of (29) takes the form

\[
\frac{c}{c_1} = \left[ \frac{G}{(2 + Gd)} + \frac{1}{d} \left( \frac{N}{L} - 1 \right) \right]^{\frac{1}{2}}.
\]

This result coincides with Gupta et al. [17]. The above equation gives the following informations:

(i) In the absence of \( G \), the wave is still available whose velocity is \( \frac{c}{c_1} = \left[ \frac{1}{d} \left( \frac{N}{L} - 1 \right) \right]^{\frac{1}{2}} \).
(ii) In case of isotropic medium, in the absence of layer and gravity \( (G) \) wave does not propagate.
(iii) For poro-elastic medium \( 0 < d < 1 \), it is expected that the velocity will be more than that of elastic medium (i.e., \( d = 1 \)).
(iv) When the medium is perfectly liquid \( (d \to 0) \), the velocity of torsional wave tends to zero, because we have, \( d = d' / \rho' \) as \( d \to 0, \rho' \to \infty \).

So \( c_1 = \sqrt{\frac{L}{\rho'}} \to 0 \), i.e., velocity of the shear wave in anisotropic elastic medium is zero.

7. Numerical computation
Based on the dispersion equation (29), numerical results of phase velocity are provided to show the effect of various inhomogeneity on the propagation of torsional surface waves for the following values of elastic constants.

For the upper layer [18]:

\[ \mu_0 = 93 \times 10^9 N/m^2, \quad \rho_0 = 7450 Kg/m^3. \]

For the half-space [19]:

\[ L = 7.5 \times 10^9 N/m^2, \quad N = 6 \times 10^9 N/m^2, \quad \rho' = 3364 Kg/m^3. \]
Figure 2. Effect of inhomogeneity ($H/b$) in the propagation of torsional wave.

Figure 2 is representing the different phase velocity of torsional waves for the different values of the inhomogeneity parameter ($H/b$) where the Biots Gravity parameter and the porosity of the half-space is taken to be fixed as 0.5 and 0.1, respectively, and it has been found that as the value of the inhomogeneity parameter ($H/b$) increases, the phase velocity of the torsional wave increases for a particular dimensionless wave number ($kH$).

Figure 3. Effect of porosity ($d$) in the propagation of torsional wave.

In Figure 3, attempt has been made to show the effect of porosity ($d$) on the propagation of torsional waves. The graph has been plotted for different values of porosity where the inhomogeneity parameter and Biots Gravity parameter is kept fixed at 0.1 and 0.5, respectively. This shows that as the value of the porosity increases, the phase velocity of the torsional wave decreases for a particular dimensionless wave number ($kH$).

The graph in Figure 4 has been drawn to show the effect of Biots Gravity parameter on torsional wave propagation. The value of the inhomogeneity parameter and the porosity are 0.1 and 0.3, respectively. Here we see that as the value of the Biots Gravity parameter increases, the phase velocity of the torsional wave decreases for a particular dimensionless wave number ($kH$).
8. Conclusion
In this problem the propagation of torsional surface waves in an inhomogeneous layer of finite thickness over a gravitating porous half-space has been studied. The displacements in the layer and in the half space have been derived separately. Then using the asymptotic expansion of the Whittakers function the dispersion relation has been computed. Effects of inhomogeneity parameter, porosity and gravity factor on the phase velocity for fixed wave number are depicted by means of Graphs. And these show that the dimensionless phase velocity decreases as the non-dimensional wave number increases. As the gravity and porosity increases the phase velocity decreases and as the anisotropy increases the phase velocity increases.

Appendix
\[ X_1 = \lambda \left( 1 - \frac{H}{b} \right) - \frac{1}{b}, \]
\[ X_2 = \lambda \left( 1 - \frac{H}{b} \right) + \frac{1}{b}, \]
\[ Y_1 = 1 + \frac{(m+0.5)^2}{\pi^2} + \frac{(m+0.5)^2(m+1.5)^2}{2(\frac{\pi}{b})^2}, \]
\[ Y_2 = \left( \frac{1}{2} - \frac{m}{\pi^2} \right) \left\{ 1 + \frac{(m+0.5)^2}{\pi^2} + \frac{(m+0.5)^2(m+1.5)^2}{2(\frac{\pi}{b})^2} \right\} - \left\{ \frac{(m+0.5)^2}{\pi^2} + \frac{(m+0.5)^2(m+1.5)^2}{4(\pi^2)} \right\}. \]

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