Impact of Interleaver Pruning on Properties of Underlying Permutations

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Abstract

In this paper we address the issue of pruning (i.e., shortening) a given interleaver via truncation of the transposition vector of the mother permutation and study its impact on the structural properties of the permutation. This method of pruning allows for continuous uninterrupted data flow regardless of the permutation length since the permutation engine is a buffer whose leading element is swapped by other elements in the queue. The principle goal of pruning is that of construction of variable length and hence delay interleavers with application to iterative soft information processing and concatenated codes, using the same structure (possibly in hardware) of the interleaver and deinterleaver units. We address the issue of how pruning impacts the spread of the permutation and also look at how pruning impacts algebraically constructed permutations. We note that pruning via truncation of the transposition vector of the permutation can have a catastrophic impact on the permutation spread of algebraically constructed permutations. To remedy this problem, we propose a novel lifting method whereby a subset of the points in the permutation map leading to low spread of the pruned permutation are identified and eliminated. Practical realization of this lifting is then proposed via dummy symbol insertion in the input queue of the Finite State Permuter (FSP), and subsequent removal of the dummy symbols at the FSP output.

Index Terms

Permutation, pruning, concatenated codes, variable length turbo codes, iterative soft decoding, interleaver.

I. INTRODUCTION

INTERLEAVERS appear in many modern communication link designs at the physical layer. Their primary function is to scramble and de-correlate the data at the interleaver output, permitting the generation of extrinsic information needed in many iterative signal processing paradigms that are by now a standard procedure to achieve performances at the physical layer near the theoretical limits. They have a long history but one of their first systematic uses were in connection with concatenated channel codes [1-26]. An example of this is in Parallel Concatenated Convolutional Codes (PCCC or Turbo codes) which are potential candidates for many applications due to their excellent performance near the capacity limit [1], [2], [3]. Upon proper termination, the PCCC can essentially be interpreted as a large block code which we denote as $C_p$.

The principle goal of pruning is that of construction of variable length and hence delay interleavers with application to iterative soft information processing and concatenated codes. In a series of recent papers [4], [5], [6] M. M. Mansour has developed a systematic and efficient technique for determination of permutation inliers that can be used for construction of variable length interleavers. Our approach is fundamentally different.

In [7] we presented a systematic iterative technique for construction of interleavers tailored to given constituent Recursive Systematic Convolutional (RSC) codes leading to significant performance gains for a given length interleaver. This improvement is manifested in the distance spectrum of $C_p$, and the resulting interleavers lead to PCCCs with better distance spectra compared to other interleaver design techniques proposed in the literature.

In [8] we focused on the design of variable length interleavers with application to turbo codes, starting from any generic interleaver of maximal length. We presented a pruning method suitable for practical implementation and applicable to any interleaver and demonstrated the average optimality of the pruning strategy. Variable length block coding may be of significant interest in practice since a major component of the Quality Of Service (QOS) is the delay. Since the encoding and decoding delay associated with turbo codes are essentially dominated by the block length (or the data frame length), it is of considerable practical interest to be able to construct variable length Turbo codes.

This paper explores the theoretical properties of the general pruning method presented in [8] and addresses the impact of pruning on permutation spread and on structured algebraically constructed permutations. Our focus on algebraic constructions will be on Quadratic Permutation Polynomials (QPP) [9], [10] and we shall demonstrate that general pruning as proposed in [8] can have catastrophic impact on permutation spread. We then present a novel technique of lifting permutation points leading to low spread via dummy symbol insertion in the sequence to be interleaved to remedy this problem.

There are several important features of the pruning method presented in [8] that we wish to recall before we proceed with the rest of the paper:

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1) the technique presented in [2] is an interleaver growth algorithm (i.e., opposite of pruning) with one major difference with the pruning strategy presented in [3]. During the construction of an interleaver of length \( K + 1 \) from one of length \( K \), one has a choice of \( K + 1 \) interleavers to select from in order to minimize some appropriate cost function. In pruning an interleaver of length \( K + 1 \) to one of length \( K \) using the method proposed in [3], one has only one choice. Hence, growing an interleaver via the method of [7], one ends up with an implicitly prune optimized (in a narrow sense) interleaver. Starting with an arbitrary interleaver, the pruning may not necessarily lead to optimal results in any strict sense. What we have shown is average optimality of the approach in the context of channel coding;

2) the technique presented in [8] and summarized below allows any interleaver performing a given permutation to be reduced in size with gradual performance degradation of the resulting PCCC consistent with what one would obtain via redesign of the interleaver each time, but without the need for changing almost completely the operation of the interleaver.

The rest of the paper is organized as follows. In section II we provide a review of the transposition realization of any given permutation and its use in realization of the interleaver as a Finite State Permuter (FSP) originally introduced in [7]. In section III we present the core results on interleaver pruning and growth and how this process impacts the underlying permutations. We also address the issue of how pruning impacts the spread of the permutation. Finally we look at how pruning impacts algebraically constructed permutations. We note that pruning via truncation of the transposition vector of the permutation can have a catastrophic impact on the permutation spread. To remedy this problem, we propose a novel lifting method whereby a subset of the points in the permutation map leading to low spread of the pruned permutation are identified and eliminated. Practical realization of this lifting is then proposed via dummy symbol insertion in the input queue of the FSP, and subsequent removal of the dummy symbols at the FSP output. Section IV is devoted to conclusions.

II. The Finite State Permuter and Transposition Vector of a Permutation

Design of concatenated signal processing schemes and in particular, Parallel or Serial concatenated codes, is tied to the realization of the interleaver and deinterleaver. One approach for implementation is that of Random Access Memory (RAM) units whereby the data are written in a particular order and read according to the permutation to be implemented by the interleaver, and its inverse to be implemented by the deinterleaver. While this approach is obvious, it can be very slow compared to a purely hardware based solution and it is by no means clear how the effective length of the permutation can be reduced, without significant changes in the read/write order of the elements and without significant degradation in the performance of the resulting concatenated codes (in practice we need to satisfy both requirements).

In [7], we have presented an alternative technique for the realization of the interleaver (deinterleaver) as a finite state permuter (i.e., a serial queue implementing transpositions) whose operation was described via a transposition vector that uniquely identifies the permutation. The representation of the permutation using its transposition vector preserves information about the prefix symbol substitution property of a permutation which directly correlates with how the interleaver (deinterleaver) operates on the error events and their time shifted versions which are of relevance in concatenated code design (see [7] for details).

We start this section by recalling some results from [7]. Consider an indexed set of elements \( x_1 x_2 x_3 \ldots x_N \). A given interleaver performs a particular permutation of this set of elements. The permutation \( \pi \) acts on the indices of the elements. Henceforth, the notation \( \pi(i) = j \) is used to mean that the \( j \)-th input symbol is carried to the \( i \)-th position at the output. It is a basic result in group theory [25] that any permutation \( \pi \) on a set of elements \( S \) can be written as a product of disjoint cycles, and \( S \) may be divided into disjoint subsets such that each cycle operates on a different subset. A cycle of length two is called a transposition. It is easy to verify that any finite cycle can be written as a product of transpositions. Hence, we conclude that transpositions represent the elementary constituents of any permutation.

The Finite State Permuter (FSP) introduced in [7], is a realization of an interleaver in the form of a sliding window transposition box of fixed length equal to the delay of the permutation it implements on its input sequence. An FSP has the property that a transposition performed at a given time slot is responsible for the generation of the output at the same time slot. Operation of the FSP can be understood by thinking of a sliding window as a queue. To generate any possible sequence of outputs, it is sufficient to exchange the head of the queue with the element that is to be ejected at that time slot.

Any permutation on a finite set of elements can be represented using a unique transposition vector associated with its FSP realization. As an example consider the permutation

\[
\pi = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 \\
4 & 3 & 1 & 2 & 5
\end{pmatrix}.
\]  

Consider the queue model of an FSP and assume that data enters from left to right as depicted in Fig. [1]

Let us label the transpositions to be performed sequentially with the head of the queue to generate the desired outputs, using positive integers. Let the integer 1 denote the case whereby no transposition is performed with the head of the queue and the element at the head of the queue is simply ejected. Then the transposition vector (to be read from left to right) that fully defines permutation \( \pi \) is \( T_\pi = (4, 2, 2, 1, 1) \). The delay of this permutation is 3 (i.e., one less than the largest element of the transposition vector), and the FSP can be implemented using three memory cells. Any permutation on \( N \) elements uniquely
defines a transposition vector of size \( N \). Conversely, any transposition vector of size \( N \), defines a unique permutation on \( N \) elements. Note that when synthesizing a permutation using the transposition vector \( T \), the \( k \)-th element of vector \( T \), when scanned from right to left, can only assume values in the set \( \{1, 2, \ldots, k\} \).

The description of a permutation using the transposition vector of its associated FSP realization turns out to be quite useful for iterative interleaver construction and pruning. This is because of a prefix symbol substitution property that is best described through an example. Consider the example transposition vector \( T_\pi = (4, 2, 2, 1, 1) \) associated with permutation \( \pi \) above. Take the binary sequence 10110 labeled from left to right. Permutation \( \pi \) maps this sequence to 00111. Consider a new transposition vector \( T_{\pi_2} = (3, T_\pi) = (3, 4, 2, 2, 1, 1) \) associated with the permutation

\[
\pi_2 = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 \\
3 & 5 & 4 & 2 & 1 & 6
\end{pmatrix}.
\]

Note that \( \pi_2 \) looks quite different from \( \pi \), yet its output corresponding to the shifted sequence 010110 is 001110 which is a shifted version of the output generated by \( \pi \). In essence, the descriptions of \( \pi \) and \( \pi_2 \) using the transposition vectors preserves information about the prefix symbol substitution property of these two permutations on error patterns whereby the first transposition exchanges a zero with a zero, or a one with a one.

By its very nature a FSP can operate in continuous un-interrupted mode. Data simply enters the FSP from right, there is an initial delay equal to the permutation delay after which a sequence of transpositions are applied continuously. As the permuted sequence exits the FSP, a new block of data to be permuted enters sequentially until the FPS buffer is full again and the sequence of transpositions is repeated. Such a continuous flow mode is generally un-achievable with any RAM structure which requires us to read the entire content of the RAM, wait until the RAM is re-loaded and the process is repeated. Since pruning boils down to the truncation of the transposition vector, this continuous flow property is maintained regardless of the pruning length. In other words, the block length can change continuously without incurring any delay in changing the interleaving length.

### III. Basic Results on Interleaver Pruning and Growth

We need to be able to go from one representation of a permutation to another easily. The first pseudo-code below generates the transposition vector corresponding to a given permutation of length \( N \). We start with a vector of labels \( v = [1, 2, 3, \ldots, N] \) and transform this via suitable transpositions to the permuted set of labels as specified by the permutation \( P \) (we call this routine, perm2trans):

```plaintext
for j=1 to N;
    for k=j to N;
        if v(k)=P(j), then
            T(j)=k-j+1;
            tmp=v(k);
            v(k)=v(j);
            v(j)=tmp;
            break
        end
end
```
The complexity of this routine is upper bounded by $0.5(N+1)N$, i.e., it is quadratic in $N$. The main reason for this is that at each step the contents of the buffer holding the data needs to be examined to find the next element that needs to be swapped with the head of the queue to get the desired output. The next pseudo code generates the permutation vector from the transposition vector to come up with the desired permutation (we call this routine, \textit{trans2perm}):

\begin{verbatim}
for \( j=1 \text{ to } N; \\
    s=P(j); \\
    P(j)=P(T(j)+j-1); \\
    P(T(j)+j-1)=s;
end
\end{verbatim}

Clearly the complexity of this routine is $O(N)$.

Next, suppose a permutation $π_N$ of length $N$ with transposition vector $T_N$ is \textit{grown} to a permutation of length $(N+1)$, $π_{N+1}$ via growing the transposition vector to $T_{N+1} = [j, T_N]$, whereby $j \in \{1, 2, ..., N+1\}$. the question is what is the relationship between $π_{N+1}$ and $π_N$? We have the following result in connection with this:

\textbf{Theorem-1}:

\[ π_{N+1}(1) = j \]

if $j \neq 1$, then

\[ π_{N+1}(π_N^{-1}(k) + 1) = k + 1 \quad k = 1, 2, ..., N \ & k \neq (j - 1); \]

\[ π_{N+1}(π_N^{-1}(j - 1) + 1) = 1 \quad \text{otherwise,} \]

\[ π_{N+1}(π_N^{-1}(k) + 1) = k + 1, \quad k = 1, 2, ..., N. \]

\textbf{Proof}:

The first transposition is $j$ and we swap the element at position 1 with $j$. The order of the elements in the \textit{queue} (see figure) becomes $v = [2, 3, ..., (j - 1), 1, (j + 1), ..., N, N+1]$ where element 1 is in the $(j - 1)$-th position from left in vector $v$ that is now of length $N$. The rest of the transposition vector $T_N$ executes $π_N$ on the elements of $v$ and it is easy to see that:

- element 2 appears at $(π_N^{-1}(1) + 1)$-st position in the output sequence;
- element 3 appears at $(π_N^{-1}(2) + 1)$-st position in the output sequence, and so on;
- element 1 appears at $(π_N^{-1}(j - 1) + 1)$-st position in the output sequence provided $j \neq 1$. If in fact $j = 1$, we simply eject element 1 and vector $v$ above is an ordered sequence from 2 to $N$ and $(π_N^{-1}(k) + 1) = k + 1 \forall k = 1, 2, 3, ..., N$. □

An example illustrates the process. Suppose $T_N = [4, 2, 2, 1, 1]$ and $T_{N+1} = [3, T_N]$. Using the \textit{trans2perm} routine we obtain the two permutations $π$ and $π_2$ in equations (12) above. To verify the result on $π_2$ using the result of Theorem-1 above, we need the inverse permutation $π^{-1}$ which is given by:

\[ π^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 2 & 1 & 5 \end{pmatrix} \quad (3) \]

We note that $j = 3 \neq 1$, hence we get:

\[ π_2(1) = 3 \]
\[ π_2(π^{-1}(1) + 1) = π_2(4) = 1 + 1 = 2 \]
\[ π_2(π^{-1}(2) + 1) = π_2(5) = 1 \quad \text{since } 2π_2j-1 \]
\[ π_2(π^{-1}(3) + 1) = π_2(3) = 3 + 1 = 4 \]
\[ π_2(π^{-1}(4) + 1) = π_2(2) = 4 + 1 = 5 \]
\[ π_2(π^{-1}(5) + 1) = π_2(6) = 5 + 1 = 6 \]

The pruning process is just the opposite of the growth whereby we prune the interleaver of length $(N+1)$ with transposition vector $T_{N+1} = [j, T_N]$ by dropping the first transposition $j$. From Theorem-1 the following result is immediate:

if $j \neq 1$, then:
\[
\pi_N^{-1}(j-1) = \pi_{N+1}^{-1}(1) - 1 \\
\pi_N^{-1}(k) = \pi_{N+1}^{-1}(k+1) - 1, \ k = 1, 2, \ldots, N & k \neq (j-1)
\]

if \( j = 1 \), then:

\[
\pi_N^{-1}(k) = \pi_{N+1}^{-1}(k+1) - 1, \ k = 1, 2, \ldots, N
\]

We can think of the transposition vector as an operator acting on a vector \( s \) and producing a vector with permuted elements \( \hat{s} \) (we could think of the permutation really the same way). Notationally, we can write \( \hat{s} = T_{N+1}(s) \) where, \( s \) is a vector of elements of length \( (N+1) \) and \( \hat{s} \) is the permuted vector. Clearly, if \( T_{N+1}^{-1} \) is the transposition vector of the inverse permutation, we must have \( T_{N+1}^{-1}T_{N+1}(s) = s \), hence, we can write \( T_{N+1}^{-1}T_{N+1} = T_{I_{N+1}} \) where \( T_{I_{N+1}} = [1, 1, \ldots] \) is the transposition vector of the identity permutation. Given how the transposition vector operates on the elements of vector \( s \) in the queue, it is possible to derive the following result:

\[
\text{if } T_{N+1} = [j, T_1] \text{ and } \\
T_{N+1}^{-1} = [k, T_2], \\
\text{then } T_1T_2 = T_{I_N} \\
\text{consequently, } T_1 = T_N \quad T_2 = T_{N}^{-1}
\]

in other words, \( T_N \) and \( T_N^{-1} \) are the transposition vectors of the pruned permutation and its inverse. The explanation for this results is as follows. The first transposition \( j \) swaps element 1 with \( j \). The content of the queue after this transposition is \( v = [2, 3, \ldots, (j-1), 1, (j+1), \ldots, N, N+1] \). To determine whether \( T_1 \) and \( T_2 \) are inverses of each other, we relabel these elements into \( \hat{v} = [1, 2, \ldots, (j-2), (j-1), j, \ldots, N] \) and see the effect after the application of \( T_1 \) and \( T_2 \). The position of the element 1 in the queue after the application of \( T_{N+1}^{-1} \) must be \( k \) since \( T_N^{-1} = [k, T_2] \). After the first swap the rest of the elements in the queue must be from 2 to \( (N+1) \) in a scrambled order. Relabeling these elements again by subtracting one from each element without changing their scrambled order we end up with the set of elements from 1 to \( N \). Now, the combined action of \( T_{N+1}^{-1} \) and \( T_{N+1}^{-1} \) should restore the sequence to its original order. This is only possible if the relabeled sequences are permuted and subsequently put back in successive order after the application of \( T_1 \) followed by \( T_2 \). This by definition implies that \( T_1 = T_N \), \( T_2 = T_N^{-1} \).

The parsing of the transposition vector can occur at any point desired and the result would continue to hold. In other words:

\[
\text{if } T_N = [j_1, j_2, \ldots j_n, T_1] \text{ and } \\
T_N^{-1} = [k_1, k_2, \ldots k_n, T_2], \\
\text{then } T_1T_2 = T_{I_{N-n}} \\
\text{consequently, } T_1 = T_{N-n} \quad T_2 = T_{N}^{-1}
\]

In light of this and given our result on impact of pruning, we can now give a result on pruning in terms of the permutation itself and not its inverse. In particular, if \( T_{N+1} = [j, T_N] \) and \( T_{N+1}^{-1} = [k, T_N^{-1}] \), we can prove that:

**Theorem-2:**

if \( k \neq 1 \), then:

\[
\pi_N(k-1) = \pi_{N+1}(1) - 1 \\
\pi_N(n) = \pi_{N+1}(n+1) - 1, \ n = 1, 2, \ldots, N & n \neq (k-1)
\]

if \( k = 1 \), then:

\[
\pi_N(n) = \pi_{N+1}(n+1) - 1, \ n = 1, 2, \ldots, N
\]

**Proof:** is obvious in light of Theorem-1 and the relations (7). □
A. Impact of Pruning on Permutation Spread

One important issue concerns the spread of the permutation which appears to play a key role for instance in the performance of Turbo codes. In this subsection, we wish to examine the impact of our pruning method on the spread properties of the underlying permutations. Given the recursive nature of the pruning and growth processes, we anticipate to develop recursive relationships for the impact of pruning and growth on permutation spread. First, let us give a generic expression for the spread of permutation \( \pi_{N+1} \), denoted \( \text{sprd}(\pi_{N+1}) \), as follows:

\[
\text{sprd}(\pi_{N+1}) = \min_{n_1, n_2} |\pi_{N+1}(n_2) - \pi_{N+1}(n_1)| + |n_2 - n_1|
\]  

(9)

Given our general result above on pruning, we have the following result on spread properties of pruned interleaver:

**Theorem-3:**

if \( k \neq 1 \), let:

\[
\alpha = \min_{m=1,2,...,N, m \neq (k-1)} [|\pi_{N+1}(m+1) - \pi_{N+1}(1)| + |m - k + 1|]
\]  

(10)

then,

\[
\text{sprd}(\pi_N) \geq \text{sprd}(\pi_{N+1}) \text{ if } \alpha \geq \text{sprd}(\pi_{N+1})
\]

\[
\text{sprd}(\pi_N) = \alpha \text{ if } \alpha < \text{sprd}(\pi_{N+1})
\]

if \( k = 1 \), then:

\[
\text{sprd}(\pi_N) \geq \text{sprd}(\pi_{N+1})
\]

(11)

**Proof:**

Note that if \( k \neq 1 \):

\[
|\pi_N(n_2) - \pi_N(n_1)| + |n_2 - n_1| = |\pi_{N+1}(n_2 + 1) - \pi_{N+1}(n_1 + 1)| + |(n_2 + 1) - (n_1 + 1)|
\]

\[
\forall n_2, n_1 = 1, 2, ..., N, n_2, n_1 \neq (k-1)
\]

minimization of \(|\pi_N(n_2) - \pi_N(n_1)| + |n_2 - n_1|\) over all \( n_1, n_2 \) except for \( n_1 \) or \( n_2 = (k-1) \) is the same as minimization of \(|\pi_{N+1}(n_2 + 1) - \pi_{N+1}(n_1 + 1)| + |(n_2 + 1) - (n_1 + 1)|\) over the same range of indices. Since the search space is reduced relative to the case where we search for minimum of \(|\pi_{N+1}(n_2) - \pi_{N+1}(n_1)| + |n_2 - n_1|\) over all possible values of \( n_1, n_2 = 1, 2, ..., (N+1) \), the result can only be a number greater than or equal to \( \text{sprd}(\pi_{N+1}) \). The only remaining case that needs to be examined, is when either \( n_1 = (k-1) \) or \( n_2 = (k-1) \) and that leads to our definition of \( \alpha \).

If \( k = 1 \), then from equation (3), the set of numbers used to find the spread is really the same for \( \pi_N \) and \( \pi_{N+1} \) except that the search space for finding the spread of \( \pi_{N+1} \) is larger, hence the computed minimum for \( \pi_{N+1} \) can only be equal or smaller and therefore the spread of \( \pi_N \) can only be equal or larger than that of \( \pi_{N+1} \). \( \square \)

B. Impact of Pruning on Algebraically Constructed Interleavers

A variety of algebraically constructed permutations have been proposed in the literature. The idea is to have a formula to construct permutations having certain desirable properties (e.g., high spread). The main requirement in any algebraic mapping to produce a permutation is closure and the requirement that no two integers are mapped via the permutation to the same number (i.e., the function is bijective). In connection with such permutations, the main result of interest is Theorem-2 and in particular, the relation \( \pi_N(n) = \pi_{N+1}(n + 1) - 1 \) which is a linear relationship suggesting that the core property that the mapping is obtained through an algebraic function, for the most part, remains intact after pruning.

An example class of algebraic permutations are the Quadratic Permutation Polynomials (QPPs) \( [9, 10] \) of length \( K \) defined through the relationship:

\[
\pi_K(j) = j \cdot h + j^2 \cdot b + c \pmod{K} , j = 0, 1, ..., (K-1).
\]

(12)

Note that in this paper, the permutation starts with index 1, while in above definition, the first index is 0. To be consistent with our indexing, we can easily write:

\[
\pi_K(j) = (j - 1) \cdot h + (j - 1)^2 \cdot b + c \pmod{K+1} , j = 1, ..., K.
\]

(13)

where, there are two cases to consider for the construction to be valid. In case-1, \( \text{gcd}(h, K) = 1 \) and let \( K = \prod_i p_i^{n_i} \) be the prime factorization of \( K \). If 2 is not a prime factor or a prime factor with power greater than or equal to 2, then the prime factors of \( b \) must include at least all prime factors of \( K \) other than 2 (possibly with different powers), \( b = m \prod_i p_i^{n_i}, m \mid p_i \& n_i \geq 1 \). In case-2, 2 is a prime factor with power of 1, \( \text{gcd}(h, K/2) = 1 \), and \( h+b \) must be odd, \( b = m \prod_i p_i^{n_i}, m \mid p_i p_i \neq 2, \& n_i \geq 1 \).
1 \& 2 \nmid m if \( h \) is even. A toy example is \( K = 3 \times 5, h = 2, \gcd(h, K) = 1, b = 3 \times 5 \) leading to \( \pi_{15}(j) = 2j + 15j^2 + c \pmod{15}, j = 0, 1, 2, \ldots, 14 \), where, \( c < K \) is an arbitrary shift parameter so that \( \pi_{15}(0) = c \).

The transposition vector of a generic QPP is easily seen to be of the form \( T_K = [c + 1, T_{K-1}] \). Let the transposition vector of the inverse permutation be \( T_{K}^{-1} = [n, T_{K-1}^{-1}] \), then:

\[
\begin{align*}
\pi_{K-1}(n - 1) &= \pi_K(1) - 1 = c \\
\pi_{K-1}(l) &= \pi_K(l + 1) - 1 = l \cdot h + l^2 \cdot b + c \pmod{K}, \quad l = 1, 2, \ldots, (K - 1), \ l \neq (n - 1).
\end{align*}
\]

if \( n \neq 1 \):

\[
\begin{align*}
\pi_{K-1}(l) &= \pi_K(l + 1) - 1 = l \cdot h + l^2 \cdot b + c \pmod{K}, \\
&= l = 1, 2, \ldots, (K - 1).
\end{align*}
\]

The algebraic structure of the permutation is indeed retained, albeit, with the same modulus operation as before. It is not difficult to show that the repeated iteration of the pruning via truncation of the transposition vector always leads to a permutation that is algebraically defined via the same equation but with shifted indexes over a smaller subset of overall indexes (i.e., that is the only way the permutation can remain to be a valid bijective map). Indeed, suppose a QPP permutation of length \( K \) is pruned by truncating its transposition vector by \( M \) positions. Repeated application of the above result suggests that there is a large fraction of the permutations (provided \( M/K \) is small) that must satisfy the following:

\[
\pi_{K-M}(l) = (l + M - 1) \cdot h + (l + M - 1)^2 \cdot b + c \pmod{K} + M - 1 \tag{14}
\]

In terms of the spread properties of the pruned interleaver, the key parameter to compute is \( \alpha \) as defined in equation (10):

\[
\begin{align*}
\alpha &= \min_S ||\pi_K(l + 1) - \pi_K(1)| + |l - n + 1|| \\
&= \min_S ||l \cdot h + l^2 \cdot b + c \pmod{K} - c| + |l - n + 1||
\end{align*}
\]

assuming \( c < (K - 1) \), where the set \( S = \{ l = 1, 2, \ldots, (K - 1), \ l \neq (n - 1) \} \). For a fixed set of parameters \( (h, b) \), we can easily obtain a plot of \( g(l) = |l \cdot h + l^2 \cdot b + c \pmod{K} - c| + |l - n + 1| \), then for any given \( n, \alpha \) is the minimum of \( g(l) \). As an example, take the QPP \( \pi(j) = 63j + 128j^2 + 347 \pmod{2048} \), then \( g(l) = |63l + 128l^2 + 347 \pmod{2048} - 347| + |l - 1180 + 1| \) is plotted in figure [2]. We note that repeated application of the pruning based on truncation of the transposition vector can have a catastrophic impact on the spread properties of the pruned interleaver, i.e., maintenance of the algebraic property of the permutation is no guarantee of its continued performance when it comes to having high spread. However, in light of equation (14), this problem can be remedied as follows:

1) Much of the spread properties of the QPP permutations (and indeed many other algebraically constructed permutations), comes from the good distribution of points in the XY plane when we plot points with coordinates \( (j, \pi(j)) \). Such properties are embodied in the original algebraic expression used in definition of the permutation itself;
2) Truncation of the transposition vector of the permutation as proposed here for construction of variable length interleavers may cause folding of certain number of points due to the equation $\pi_{K-1}(n-1) = \pi_{K}(1) - 1 = c$ when $n \neq 1$, in the same region containing the points satisfying the core algebraic relationship defining the mother permutation. The effect on spread can be catastrophic because of such folding. To illustrate, consider the QPP mother permutation of length 2048 defined via:

$$
\pi(i) = 63 \cdot (i - 1) + 128 \cdot (i - 1)^2 + c \pmod{2048} + 1 \\
i = 1, 2, \ldots, 2048.
$$

and suppose we truncate the transposition vector of this permutation by just 10 positions. The scatter diagram of the resulting permutation of length 2038 is shown in figure 3.

The folding of the points causes the spread of the resulting permutation to drop from 64 for the length 2048 mother permutation down to 2, even though almost all the other points are well distributed in the XY plane, hence, leading to high spreads.

3) The folding problem can be easily resolved in effect by lifting such points hence reducing the permutation size from its original target length, but restoring the spread properties of the pruned permutation. There are two issues to address, a) how do we identify such points?, and b) how do we actually implement a systematic variable length interleaver using the transposition vector paradigm? The identification of the valid points from those we would like to lift is actually quite straightforward, we simply check and see if the relation $\pi_{K-M}(l) = (l + M - 1) \cdot h + (l + M - 1)^2 \cdot b + c \pmod{K} + M - 1$ holds for index $l$, if it does not, we flag the index as invalid and lift the point. The question of how do we actually lift the point in practice in the context of sequential operation of the finite state permuter is trickier. To handle this problem we proceed as follows:

- in the sequential entry of data in the queue used to hold the data and implement the transpositions we insert dummy symbols at locations where we wish to lift the points from the permutation map;
- we truncate the transposition vector to an initial target length $(K - M)$ by eliminating the first $M$ transpositions from left, in the transposition vector of the mother interleaver;
- we apply the truncated transpositions to the data in queue containing the dummy as well as valid symbols to be permuted and run the FSP to generate the output. We then simply discard the dummy symbols from the output sequence and get the permuted data exhibiting the desirable spread properties.

An example illustrates the procedure for interleaver pruning with lifting that leads to maintenance of the desirable spread properties of the pruned interleaver. Take the QPP defined via:

$$
\pi(i) = 63 \cdot (i - 1) + 128 \cdot (i - 1)^2 + c \pmod{2048} + 1 \\
i = 1, 2, \ldots, 2048.
$$

and let the offset $c = 0$ for convenience (this does not change the spread properties of the pruned permutations). Suppose we set $M = 500$ and truncate the transposition vector of the above permutation by 500 positions. The scatter plot of the mother length 2048 QPP permutation and the lifted pruned permutation is shown in figure 4. While the target length of the pruned
interleaver is \((2048 - 500) = 1548\), the actual length of the pruned interleaver after lifting is 1169 and hence, 379 points in the permutation map had to be lifted. The original spread of the mother permutation is 64, while that of lifted pruned interleaver is 43. If lifting was not done, the spread of the length 1548 permutation would have been 2. The scatter plot of the length 1169 lifted pruned permutation showing regular and well disbursed spread of points in the XY plane is shown in figure 5.

**IV. CONCLUSIONS**

Systematic methods of constructing variable length interleavers with application to variable-rate, delay and decoding complexity channel coding and possibly other application areas, are rare. On the surface, the problem may be non-existent in that one could hypothetically design a set of interleavers with varying lengths and store the information and use a purely software approach by writing and reading into a RAM. There are several drawbacks to such an approach, one is in terms of implementation complexity that can be significantly higher than more hardware oriented approaches, the other is speed whereby a hardware based solution can generally run much faster than a RAM based approach and finally the fact that one must design the system a-priori and by its very nature the design is inflexible (i.e., one could not change the interleaving block length at will). The idea of using a good mother interleaver and constructing a variable length interleaver out of it via a clever strategy is clearly appealing and that is what this paper and the approach presented in [8], [4], [5], [6] is all about. In coding, a good low rate code is punctured to get higher rate codes. In this paper, the transposition vector of a good interleaver can be truncated to get lower block length and hence delay interleavers. Of course, one must define what the goodness criterion is. In this paper...
we have focused on permutation spread which broadly speaking plays a key role in concatenated coding applications.

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