NOTES ABOUT THE CARATHÉODORY NUMBER

IMRE BÁRÁNY AND ROMAN KARASEV

ABSTRACT. In this paper we give sufficient conditions for a compactum in \( \mathbb{R}^n \) to have Carathéodory number less than \( n+1 \), generalizing an old result of Fenchel. Then we prove the corresponding versions of the colorful Carathéodory theorem and give a Tverberg type theorem for families of convex compacta.

1. Introduction

The Carathéodory theorem \cite{6} asserts that every point \( x \) in the convex hull of a set \( X \subset \mathbb{R}^n \) is in the convex hull of one of its subsets of cardinality at most \( n+1 \). In this note we give sufficient conditions for the Carathéodory number to be less than \( n+1 \) and prove some related results. In order to simplify the reasoning we always consider compact subsets of \( \mathbb{R}^n \).

There are results about lowering the Carathéodory constant: A theorem of Fenchel \cite{10,11} asserts that a compactum \( X \subset \mathbb{R}^n \) either has the Carathéodory number \( \leq n \) or can be separated by a hyperplane into two non-empty parts. By separated we mean “divided by a hyperplane disjoint from \( X \) into two non-empty parts”. In order to state more results we need formal definitions:

Definition 1.1. For a compactum \( X \subset \mathbb{R}^n \) denote by \( \text{conv}_{k+1} X \) the sets of points \( p \in \mathbb{R}^n \) that can be expressed as a convex combination of at most \( k+1 \) points in \( X \). We denote by \( \text{conv} X \) (without subscript) the standard convex hull of \( X \).

Definition 1.2. The Carathéodory number of \( X \) is the smallest \( k \) such that \( \text{conv} X = \text{conv}_{k} X \).

Remark 1.3. So Carathéodory’s theorem \cite{6} is equivalent to the equality \( \text{conv} X = \text{conv}_{n+1} X \) when \( X \subset \mathbb{R}^n \). We will give an alternative definition for \( \text{conv}_{k} X \) in Section 4 as the \( k \)-fold join of \( X \).

Definition 1.4. A compactum \( X \subset \mathbb{R}^n \) is \( k \)-convex if every linear image of \( X \) to \( \mathbb{R}^k \) is convex.

We give some examples of \( k \)-convex sets. What is needed in Fenchel’s theorem is 1-convexity and every connected set is 1-convex. The \( k \)-skeleton of a convex polytope is \( k \)-convex (though for such \( k \)-convex sets most results of this paper are trivial). In \cite{5} (see also \cite{11} Chapter II, § 14) it is shown that the image of the sphere under the Veronese map \( v_2 : S^{n-1} \rightarrow \mathbb{R}^{n(n+1)/2} \) (with all degree 2 monomials as coordinates) is 2-convex.

In \cite{11} Corollary 1] the following remarkable result is proved:

2010 Mathematics Subject Classification. 52A35, 52A20.

Key words and phrases. Carathéodory’s theorem, Helly’s theorem, Tverberg’s theorem.

The work of both authors was supported by ERC Advanced Research Grant No 267195 (DISCONV).

The first author acknowledges support from Hungarian National Research Grant No 78439. The second author is supported by the Dynasty Foundation, the President’s of Russian Federation grant MK-113.2010.1, the Russian Foundation for Basic Research grants 10-01-00096 and 10-01-00139, the Federal Program “Scientific and scientific-pedagogical staff of innovative Russia” 2009-2013, and the Russian government project 11.G34.31.0053.
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Theorem 1.5 (Hanner–Rådström, 1951). If $X$ is a union of at most $n$ compacta $X_1, \ldots, X_n$ in $\mathbb{R}^n$ and each $X_i$ is $1$-convex then $\text{conv}_n X = \text{conv} X$.

It is also known [13, 4] that a convex curve in $\mathbb{R}^n$ (that is a curve with no $n + 1$ points in a single affine hyperplane) has Carathéodory number at most $\lfloor \frac{n+2}{2} \rfloor$. It would be interesting to obtain some nontrivial bounds for the Carathéodory number of the orbit $Gx$ of a point $x$ in a representation $V$ of a compact Lie group $G$ in terms of $\dim V$, $\dim G$ (or the rank of $G$). The latter question is mentioned in [17, Question 3] and would be useful in results like those in [16].

In Section 2 of this paper we show that the Carathéodory number is $\leq k + 1$ for $(n-k)$-convex sets. In Section 4 we prove the corresponding analogue of the colorful Carathéodory theorem, and in Section 6 we give a related Tverberg-type result.

2. THE CARATHÉODORY NUMBER AND k-CONVEXITY

We are going to give a natural generalization of the reasoning in [11]:

Theorem 2.1. Suppose $X_1, \ldots, X_{n-k}$ are compacta in $\mathbb{R}^n$ and $p$ does not belong to $\text{conv}_{k+1} X_i$ for any $i$. Then there exists an affine $k$-plane $L \ni p$ that has empty intersection with any $X_i$.

Remark 2.2. If we replace $\text{conv}_{k+1} X_i$ by the honest convex hull $\text{conv} X_i$ then the result is simply deduced by induction from the Hahn–Banach theorem.

Remark 2.3. In [15] a somewhat related result was proved: For a compactum $X \subset \mathbb{R}^n$ and a point $p \notin X$ there exists an affine $k$-plane $L$ (for a prescribed $k < n$) such that the intersection $L \cap K$ is not acyclic modulo 2. Here acyclic means having the Čech cohomology of a point.

The proof of Theorem 2.1 is given in Section 3. Now we deduce the following generalization of Fenchel’s theorem [10]:

Corollary 2.4. If a compactum $X \subset \mathbb{R}^n$ is $(n-k)$-convex then $\bigcup_{i=1}^{n-k} \text{conv}_{k+1} X_i = \bigcup_{i=1}^{n-k} \text{conv} X_i$.

Proof. Assume the contrary and let $p \in \text{conv} X \setminus \bigcup_{i=1}^{n-k} \text{conv}_{k+1} X_i$. Applying Theorem 2.1 to the family $X, \ldots, X$ we find a $k$-dimensional $L \ni p$ disjoint from $X$. Now project $X$ along $L$ with $\pi : \mathbb{R}^n \to \mathbb{R}^{n-k}$. Since $X$ is $(n-k)$-convex $\pi(L)$ must be separated from $X$ by a hyperplane. Hence $L$ is separated from $X$ by a hyperplane and therefore $p$ cannot be in $\text{conv} X$.

Remark 2.5. In the above lemma and its proof we could consider $n-k$ different $(n-k)$-convex compacta $X_1, \ldots, X_{n-k}$ and by the same reasoning obtain the following conclusion:

$$\bigcup_{i=1}^{n-k} \text{conv}_{k+1} X_i = \bigcup_{i=1}^{n-k} \text{conv} X_i.$$

But this result trivially follows from Corollary 2.4 by taking the union.

Remark 2.6. For the image $v_2(S^{n-1})$ of the Veronese map the Carathéodory constant is roughly of order $n$, see [11, Chapter II, § 14, Theorem 14.3]. Hence Corollary 2.4 is not optimal for this set.
Let us replace $X_i$ by a smooth nonnegative function $\rho_i$ such that $\rho_i > 0$ on $X_i$ and $\rho_i = 0$ outside some $\varepsilon$-neighborhood of $X_i$. Let $p$ be the origin.

Assume the contrary: for any $k$-dimensional linear subspace $L \subset \mathbb{R}^n$ some intersection $L \cap X_i$ is nonempty. The space of all possible $L$ is the Grassmann manifold $G^k_n$. Denote by $D_i$ the open subset of $G^k_n$ consisting of $L \in G^k_n$ such that $\int_L \rho_i > 0$. Note that $0$ cannot lie in the convex hull $\text{conv}(L \cap X_i)$ because in this case by the ordinary Carathéodory theorem $0$ would be in $\text{conv}_{k+1}(L \cap X_i) \subseteq \text{conv}_{k+1}X_i$, contradicting the hypothesis. Hence (if we choose small enough $\varepsilon > 0$) the “momentum” integral

$$m_i(L) = \int_L \rho_i x \, dx$$

never coincides with $0$ over $D_i$. Obviously $m_i(L)$ is a continuous section of the canonical vector bundle $\gamma : E(\gamma) \to G^k_n$, which is nonzero over $D_i$. Now we apply:

**Lemma 3.1.** Any $n - k$ sections of $\gamma : E(\gamma) \to G^k_n$ have a common zero because of the nonzero Euler class $e(\gamma)^{n-k}$.

This lemma is a folklore fact, see \cite{8, 21} for example. Applying this lemma to the sections $m_i$ we obtain that the sets $D_i$ do not cover the entire $G^k_n$. Hence some $L \in G^k_n$ has an empty intersection with every $X_i$. $\square$

**Remark 3.2.** In the proof of Theorem 1.5 in \cite{11} instead of finding a zero of a section of a vector bundle over $\mathbb{R}P^{n-1}$ some analogue of the Brouwer fixed point theorem is used for a convex subset of the sphere $S^{n-1}$.

### 4. The Colorful Carathéodory Number

Let us introduce some notation and restate the colorful Carathéodory theorem \cite{2}.

**Definition 4.1.** Denote $A \ast B$ the geometric join of two sets $A, B \subset \mathbb{R}^n$, which is

$$\{ta + (1-t)b : a \in A, b \in B, \text{ and } t \in [0,1]\}.$$ 

This is actually the alternative definition of $\text{conv}_k X$ as $X \ast \cdots \ast X$.

**Theorem 4.2** (Bárány, 1982). If $X_1, \ldots, X_{n+1} \subset \mathbb{R}^n$ are compacta and $0 \in \text{conv} X_i$ for every $i$ then $0 \in X_1 \ast X_2 \ast \cdots \ast X_{n+1}$.

It is possible to reduce the Carathéodory number $n + 1$ assuming the $(n - k)$-convexity of $X_i$, thus generalizing Corollary \cite{2.4}

**Theorem 4.3.** Let $0 \leq k \leq n$. If $X_1, \ldots, X_{k+1} \subset \mathbb{R}^n$ are $(n - k)$-convex compacta and $0 \in \text{conv} X_i$ for every $i$ then $0 \in X_1 \ast X_2 \ast \cdots \ast X_{k+1}$.

*Proof.* We use the classical scheme \cite{2} along with the degree reasoning used in \cite{6, 3, 18, 1} in the proof of different generalizations of the colorful Carathéodory theorem.

Consider the case $k = n - 1$ first. In this case we have $n$ sets and 1-convexity. Let $x_1, \ldots, x_n$ be the system of representatives of $X_1, \ldots, X_n$ such that the distance from $S = \text{conv}\{x_1, \ldots, x_n\}$ to the origin is minimal. If this distance is zero then we are done. Otherwise assume that $z \in S$ minimizes the distance.

Let $z = t_1 x_1 + \cdots + t_n x_n$, a convex combination of the $x_i$s. If $t_i = 0$ then we observe that $0 \in \text{conv} X_i$ and we can replace $x_i$ by another $x'_i$ so that new simplex $S' = \text{conv}\{x_1, \ldots, x_{i-1}, x'_i, x_{i+1}, \ldots, x_n\}$ is closer to the origin than $S$. So we may assume
that all the coefficients $t_i$ are positive and $z$ is in the relative interior of $S$. This also implies that $S$ is $(n - 1)$-dimensional, i.e., there is a unique hyperplane containing $S$.

Consider the hyperplane $h \ni 0$ parallel to $S$. Applying the definition of 1-convexity to the projection along $h$ we obtain that there exists a system of representatives $y_i \in X_i \cap h$. The set

$$f(B) = \{x_1, y_1\} \ast \{x_2, y_2\} \ast \ldots \ast \{x_n, y_n\}$$

is a piece-wise linear image of the boundary of a crosspolytope, which we denote by $B$. Note that for every facet $F$ of $B$, the vertices of the simplex $f(F)$ form a system of representatives for $\{X_1, \ldots, X_n\}$. In particular, $S = f(F)$ for some facet $F$ of $B$. The line $\ell$ through the origin and $z$ intersects the simplex $S = f(F)$ transversally and so it must intersect some other $f(F')$ (where $F' \neq F$ is a facet of $B$) because of the parity of the intersection index. The intersection $\ell \cap f(F')$ is on the segment $[0, z]$ and cannot coincide with $z$. Therefore $f(F')$ is closer to the origin than $S$. This is a contradiction with the choice of $S$. Thus the case $k = n - 1$ is done.

The case $k = 0$ of this theorem is trivial by definition, the case $k = n$ corresponds to the colorful Carathéodory theorem. Now let $0 < k < n - 1$. Consider again a system of representatives $x_1, \ldots, x_{k+1}$ minimizing the distance $\operatorname{dist}(0, \operatorname{conv}\{x_1, \ldots, x_{k+1}\})$. Put $S = \operatorname{conv}\{x_1, \ldots, x_{k+1}\}$. As above the closest to the origin point $z \in S$ must lie in the relative interior of $S$ if $z \neq 0$.

Let $L \subset \mathbb{R}^n$ be the $k$-dimensional linear subspace parallel to $S$. As in the first proof using $(n-k)$-convexity we select $y_i \in L \cap X_i$. Then we map naturally the boundary $B$ of a $(k+1)$-dimensional crosspolytope to the geometric join

$$f(B) = \{x_1, y_1\} \ast \{x_2, y_2\} \ast \ldots \ast \{x_{k+1}, y_{k+1}\}.$$ 

Note that $f(B)$ is contained in the $(k + 1)$-dimensional linear span of $S$ and $L$, so by the parity argument as above the image under $f$ of some face of $B$ must be closer to the origin than $S$.

\hspace{1cm} \hfill \Box

Remark 4.4. In this proof in the case $k < n - 1$ we can choose some $(k + 1)$-dimensional subspace $M \subset \mathbb{R}^n$ and a system of representatives $\{x_1, \ldots, x_{k+1}\}$ for $M \cap X_1, \ldots, M \cap X_{k+1}$. Then we can make the steps reducing $\operatorname{dist}(0, \operatorname{conv}\{x_1, \ldots, x_{k+1}\})$ so that the system of representatives always remains in $M$.

5. A Topological Approach to Theorem 4.3

Theorem 4.3 can also be deduced from the following lemma:

Lemma 5.1. Let $\xi : E(\xi) \to X$ be a $k$-dimensional vector bundle over a compact metric space $X$. Let $Y_1, \ldots, Y_{k+1}$ be closed subspaces of $E(\xi)$ such that for every $i$ the projection $\xi|_{Y_i} : Y_i \to X$ is surjective. If $\xi(\xi) \neq 0$ then for some fiber $V = \xi^{-1}(x)$ the geometric join

$$(Y_1 \cap V) \ast \ldots \ast (Y_{k+1} \cap V)$$

contains $0 \in V$.

Remark 5.2. The Euler class here may be considered in integral cohomology or in the cohomology mod 2. The proof passes in both cases so we omit the coefficients from the notation.

Reduction of Theorem 4.3 to Lemma 5.1 for $k < n$. Take a linear subspace $M \subset \mathbb{R}^n$ of dimension $k + 1$. For every $k$-dimensional linear subspace $L \subset M$ all the intersections $L \cap X_i$ are nonempty. All such $L$ constitute the canonical bundle $\gamma$ over $G_{k+1}^k = \mathbb{R}P^k$ with
nonzero Euler class by Lemma \[3.1\]. For any fixed \(i\) the union of sets \(L \cap X_i\) constitute a closed subset of \(E(\gamma)\) that we denote by \(Y_i\). By Lemma \[5.1\] for some \(L\) the join
\[(Y_1 \cap L) \ast \cdots \ast (Y_{k+1} \cap L) = (X_1 \cap L) \ast \cdots \ast (X_{k+1} \cap L)\]
must contain the origin. \(\square\)

Now we prove Lemma \[5.1\]. The proof has much in common with the results of \[15\]. The main idea is that fiberwise acyclic (up to some dimension) subsets of the total space of a vector bundle behave like sections of that vector bundle.

Let \(Y = Y_1 \ast_X \cdots \ast_X Y_{k+1}\) be the abstract fiberwise join over \(X\), that is the set of all formal convex combinations
\[t_1 y_1 + t_2 y_2 + \cdots + t_{k+1} y_{k+1},\]
where \(t_i\) are nonnegative reals with unit sum and \(y_i \in Y_i\) are points such that
\[\xi(y_1) = \cdots = \xi(y_{k+1}).\]

Denote the natural projection \(\eta : Y \to X\). Any formal convex combination \(y \in Y\) defines a corresponding "geometric" convex combination \(f(y)\) in the fiber \(\xi^{-1}(\eta(y))\) depending continuously on \(y\). It is easy to check that \(f(y)\) can be considered as a section of the pullback vector bundle \(\eta^*(\xi)\) over \(Y\).

For any point \(x \in X\) its preimage under \(\eta\) is a join of \((k + 1)\) nonempty sets
\[(Y_1 \cap \xi^{-1}(x)) \ast \cdots \ast (Y_{k+1} \cap \xi^{-1}(x))\]
and therefore \(\xi^{-1}(x)\) is \((k - 1)\)-connected. Hence the Leray spectral sequence for the Čech cohomology \(H^*(Y)\) with \(E_2^{p,q} = H^p(X; H^q(\eta^{-1}(x)))\) (the coefficient sheaf is the direct image of the homology of the total space) has empty rows number 1, \ldots, \(k - 1\) and its differentials cannot kill the image of \(e(\xi)\) in \(E_r^{k,0}\). Hence \(\eta^*(e(\xi)) = e(\eta^*(\xi))\) remains nonzero over \(Y\) and by the standard property of the Euler class for some \(y \in Y\) the section \(f(y)\) must be zero. \(\square\)

Remark 5.3. In this proof we essentially use the inequality \(k < n\). So the colorful Carathéodory theorem is not a consequence of Lemma \[5.1\] at least in our present state of knowledge.

The subsets \(Y_i\) in Lemma \[5.1\] can be considered as set-valued sections. The same technique proves the following:

Theorem 5.4. Let \(B\) be an \(n\)-dimensional ball and \(f_i : B \to 2^B \setminus \emptyset\) for \(i = 1, \ldots, n+1\) be set-valued maps with closed graphs (in \(B \times B\)). Then for some \(x \in B\) the inclusion holds:
\[x \in f_1(x) \ast \cdots \ast f_{n+1}(x)\]

Proof. We may assume that all sets \(f_i(x)\) are in the interior of \(B\), because the general case is reduced to this one by composing \(f_i\) with a homothety with scale \(1 - \varepsilon\) and going to the limit \(\varepsilon \to +0\).

It is known \[12\] that for a single-valued map \(f : B \to \text{int} B\) (considered as a section of the trivial bundle \(B \times \mathbb{R}^n \to B\)) a fixed point \((x = f(x))\) is guaranteed by the relative Euler class \(e(f(x) - x) \in H^n(B, \partial B)\). Then the proof proceeds as in Lemma \[5.1\] by lifting \(e(f(x) - x)\) to the abstract fiberwise join of graphs of \(f_i\) over the pair \((B, \partial B)\) and using the properties of the relative Euler class of a section. \(\square\)

Corollary 5.5. Suppose \(X_1, \ldots, X_{n+1}\) are compacta in \(\mathbb{R}^n\) and \(\rho\) is a continuous metric on \(\mathbb{R}^n\). For any \(x \in \mathbb{R}^n\) denote by \(f_i(x)\) the set of farthest point from \(x\) in \(X_i\) (in the metric \(\rho\)). Then for some \(x \in \mathbb{R}^n\) we have
\[x \in f_1(x) \ast \cdots \ast f_{n+1}(x)\].
Remark 5.6. If we denote by $f_i(x)$ the closest points in $X_i$ then this assertion becomes almost trivial without using any topology.

6. The Carathéodory number and the Tverberg property

Tverberg’s classical theorem [19] says the following:

**Theorem 6.1** (Tverberg, 1966). Every set of $(n+1)(r-1)+1$ points in $\mathbb{R}^n$ can be partitioned into $r$ parts $X_1, \ldots, X_r$ so that the convex hulls $\text{conv } X_i$ have a common point.

From the general position considerations it is clear that the number $(n+1)(r-1)+1$ cannot be decreased. But we are going to decrease it after replacing a finite point set by a family of convex compacta. Let us define the Carathéodory number for such families:

**Definition 6.2.** Suppose $\mathcal{F}$ is a family of convex compacta in $\mathbb{R}^n$. The Carathéodory number of $\mathcal{F}$ is the least $\kappa$ such that for any subfamily $\mathcal{G} \subseteq \mathcal{F}$

$$\text{conv } \bigcup \mathcal{G} = \bigcup_{\mathcal{H} \subseteq \mathcal{G}, |\mathcal{H}| \leq \kappa} \text{conv } \bigcup \mathcal{H}. $$

We denote the Carathéodory number of $\mathcal{F}$ by $\kappa(\mathcal{F})$.

Again, from the Carathéodory theorem [6] it follows that $\kappa(\mathcal{F}) \leq n+1$. Another observation is that Corollary 2.4 guarantees that $\kappa(\mathcal{F}) \leq k+1$ if the union of every subfamily $\mathcal{G} \subseteq \mathcal{F}$ is $(n-k)$-convex.

Now we state the analogue of Tverberg’s theorem:

**Theorem 6.3.** Suppose $\mathcal{F}$ is a family of convex compacta in $\mathbb{R}^n$, $r$ is a positive integer, and

$$|\mathcal{F}| \geq r\kappa(\mathcal{F}) + 1.$$  

Then $\mathcal{F}$ can be partitioned into $r$ subfamilies $\mathcal{F}_1, \ldots, \mathcal{F}_r$ so that

$$\bigcap_{i=1}^r \text{conv } \bigcup \mathcal{F}_i \neq \emptyset.$$

**Remark 6.4.** Note the following: If $\kappa(\mathcal{F}) = n+1$ then taking a system of representatives for $\mathcal{F}$ and applying the Tverberg theorem we obtain a weaker condition: $|\mathcal{F}| \geq (r-1)(n+1) + 1$.

**Remark 6.5.** This theorem originated in discussions with Andreas Holmsen, who established the same result in the special case $n = 2$, $\kappa(\mathcal{F}) = 2$, and with $|\mathcal{F}| \geq 2r$ (not $2r+1$).

**Proof of Theorem 6.3.** We again use a minimization argument, combined with Sarkaria’s tensor trick [20]. Let $|\mathcal{F}| = m$, $\kappa = \kappa(\mathcal{F})$, and

$$\mathcal{F} = \{C_1, C_2, \ldots, C_m\}.$$

Put the space $\mathbb{R}^n$ to $A = \mathbb{R}^{n+1}$ as a hyperplane given by the equation $x_{n+1} = 1$. Consider a set $S$ of vertices of a regular simplex in some $(r-1)$-dimensional space $V$ and assume that $S$ is centered at the origin.

Now define the subsets of $V \otimes A$ by

$$X_i = S \otimes C_i,$$

and consider a system of representatives $(x_1, x_2, \ldots, x_m)$ for the family of sets $\mathcal{G} = \{X_1, X_2, \ldots, X_m\}$. Such a system gives rise to a partition $\{P_s : s \in S\}$ of $\{1, \ldots, m\}$ the following way. For $s \in S$ define

$$P_s = \{i \in \{1, \ldots, m\} : x_i = s \otimes c_i, \text{ for some } c_i \in C_i\}.$$
One form of Sarkaria’s trick, Lemma 2 in [1] says that $0 \in \text{conv}\{x_1, \ldots, x_m\}$ if and only if $\bigcap_{s \in S} \text{conv}\{c_i : i \in P_s\} \neq \emptyset$. Based on this we choose a system of representatives $(x_1, \ldots, x_m)$ of $\mathcal{G}$ so that the distance between 0 and $\text{conv}\{x_1, x_2, \ldots, x_m\}$ is minimal. If this distance is zero then the required partition of $\mathcal{F}$ is given by the sets $\{C_i \in \mathcal{F} : i \in P_s\}$, $s \in S$.

Assume that the minimal distance is not zero. Then it is attained on some convex combination

$$x_0 = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_m x_m.$$  

We claim that $\alpha_i > 0$ for all $i \in \{1, \ldots, m\}$. Assume, for instance that $\alpha_1 = 0$, and $x_1 = s \otimes c_1$ for some $c_1 \in C_1$ and $s \in S$. Now $x_1$ can be replaced by $t \otimes c_1$ for any $t \in S$ as such a change does not influence $x_0$. The distance minimality condition implies that all the points $t \otimes c_1$ are separated from the origin by a hyperplane in $V \otimes A$, which is the support hyperplane for the ball, centered at the origin and touching $\text{conv}\{x_1, \ldots, x_m\}$. Obviously

$$\sum_{t \in S} t \otimes c_1 = 0,$$

so the points $t \otimes c_1, t \in S$ are not separated from the origin. This contradiction completes the proof of the claim.

The above convex combination representing $x_0$ can be written as

$$x_0 = \sum_{s \in S} s \otimes \left( \sum_{i \in P_s} \alpha_i c_i \right).$$

Assume first that no $P_s$ is the emptyset. Define $c(s) = \sum_{i \in P_s} \alpha_i c_i$ and $\alpha(s) = \sum_{i \in P_s} \alpha_i > 0$. Then $c(s)/\alpha(s)$ is a convex combination of elements $c_i \in C_i, i \in P_s$. Thus $c(s)/\alpha(s) \in \bigcup_{i \in P_s} C_i$. According to the definition of the Carathéodory number, there is a subset $P'_s \subset P_s$, of size at most $\kappa$, such that $c(s)/\alpha(s) \in \bigcup_{i \in P'_s} C_i$ for every $s \in S$. This means that there are $c'_i \in C_i$ for all $i \in P'_s$ such that $c(s)/\alpha(s) \in \text{conv}\{c'_i : i \in P'_s\}$, in other words, $c(s) = \sum_{i \in P'_s} \alpha'_i c'_i$ with positive $\alpha'_i$ satisfying $\sum_{i \in P'_s} \alpha'_i = \alpha(s)$. Thus

$$x_0 = \sum_{s \in S} s \otimes \left( \sum_{i \in P'_s} \alpha'_i c_i \right).$$

In this case the minimum distance is attained on the convex hull of no more that $r \kappa$ elements as each $|P'_s| \leq \kappa$. But $m > r \kappa$ contradicting the claim.

Finally we have deal with the (easy) case when some $P_s = \emptyset$. The above argument works, with no change at all, for the non-empty $P_s$ implying that $x_0$ can be written as the convex combination of at most $(r - 1) \kappa$ elements. Again $m > (r - 1) \kappa$ and the same contradiction finishes the proof.

**Acknowledgment.** We thank Peter Landweber who has drawn our attention to those old results by Fenchel, Hanner, and Rådström and Alexander Barvinok for discussions and examples of $k$-convexity.

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NOTES ABOUT THE CARATHÉODORY NUMBER

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E-mail address: barany@renyi.hu

IMRE BÁRÁNY, RÉNYI INSTITUTE OF MATHEMATICS, HUNGARIAN ACADEMY OF SCIENCES, PO Box 127, 1364 Budapest, Hungary; and DEPARTMENT OF MATHEMATICS, UNIVERSITY COLLEGE LONDON, GOWER STREET, LONDON WC1E 6BT, ENGLAND

E-mail address: r.n_karasev@mail.ru

ROMAN KARASEV, DEPT. OF MATHEMATICS, MOSCOW INSTITUTE OF PHYSICS AND TECHNOLOGY, INSTITUTSKY PER. 9, DOLGOPRUDNY, RUSSIA 141700; and LABORATORY OF DISCRETE AND COMPUTATIONAL GEOMETRY, YAROSLAVL STATE UNIVERSITY, SOVETSKAYA ST. 14, YAROSLAVL’, RUSSIA 150000