Quantum probabilistic teleportation via entangled coherent states

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Abstract

We study the entanglement of entangled coherent states in vacuum environment by employing the entanglement of formation and propose a scheme to probabilistically teleport a coherent superposition state via entangled coherent states, in which the amount of classical information sent by Alice is restricted to one bit. The influence of decoherence due to photon absorption is considered. It is shown that decoherence can improve the mean fidelity of probabilistic teleportation in some situations.

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I. INTRODUCTION

Quantum entanglement plays an important role in various fields of quantum information, such as quantum computation [1], quantum teleportation [2,3], dense coding [4] and quantum communication [5], etc. Entanglement can exhibit the nature of a nonlocal correlation between quantum systems that have no classical interpretation. It has been noted [2,3] that quantum teleportation can be viewed as an achievable experimental technique to quantitatively investigate quantum entanglement. Several quantum teleportation schemes of both discrete and continuous variables have been proposed [2,3,6,7]. Continuous variable quantum teleportation of an unknown coherent states has been realized experimentally by employing a two-mode squeezed vacuum state as an entanglement resource [3]. Recently, the teleportation schemes via the entangled coherent states have been studied [8].

In this Letter, we study the entanglement of entangled coherent states in vacuum environment by employing the entanglement of formation [9] and find that the entanglement of formation of the entangled coherent states is sensitive with the relative phase $\phi$ when the amplitude $|\alpha|$ is very small. Then, we propose a scheme of probabilistic teleportation via entangled coherent states, in which the amount of classical information sent by Alice is restricted to one bit. In this probabilistic teleportation scheme, a coherent superposition state can be probabilistically perfectly teleported via a properly chosen entangled coherent state. When the interaction with the vacuum environment is addressed, the mean fidelity of the scheme is studied. It is shown that the decoherence due to interaction with the environment can improve the mean fidelity in some specific situations.

II. ENTANGLEMENT OF ENTANGLED COHERENT STATES

In this section, we investigate the entanglement properties of entangled coherent states by analyzing the entanglement of formation. The pure entangled coherent states is defined by,

$$|C(\alpha_1, \alpha_2, \phi)\rangle = N_\phi(|\alpha_1\rangle_1|\alpha_2\rangle_2 + e^{i\phi}|\alpha_1\rangle_1 - \alpha_2\rangle_2),$$

where the normalization constant $N_\phi = [2 + 2 \cos \phi \exp(-2|\alpha_1|^2 - 2|\alpha_2|^2)]^{-1/2}$ and $|\pm \alpha_1, \alpha_2\rangle$ are coherent states.

Many measures of entanglement have been introduced and analyzed [9,10,11]. Here, we adopt the entanglement of formation to analyze the entanglement properties of pure or mixed entangled coherent states. The entanglement of formation of pure entangled coherent states $|C(\alpha_1, \alpha_2, \phi)\rangle$ reduces to the von Neumann entropy of the reduced density matrix $\rho_2 \equiv \text{Tr}_1(|C(\alpha_1, \alpha_2, \phi)\rangle\langle C(\alpha_1, \alpha_2, \phi)|)$, i.e., $E(|C(\alpha_1, \alpha_2, \phi)\rangle) = -\text{Tr}\rho_2 \log_2 \rho_2$. The reduced density matrix $\rho_2$ can be expressed as

$$\rho_2 \equiv \text{Tr}_1(|C(\alpha_1, \alpha_2, \phi)\rangle\langle C(\alpha_1, \alpha_2, \phi)|)$$

$$= N_\phi^2[1 + \exp(-2|\alpha_1|^2)][1 + \cos \phi \exp(-2|\alpha_2|^2)]|\chi_+\rangle\langle \chi_+|$$
\[ + N_\phi^2[1 - \exp(-2|\alpha_1|^2)][1 - \cos \phi \exp(-2|\alpha_2|^2)]|\chi_-\rangle\langle \chi_-|, \] (2)

where
\[ |\chi_+\rangle = [2 + 2 \cos \phi \exp(-2|\alpha_2|^2)]^{-1/2}([\alpha_2] + e^{i\phi}|\alpha_2\rangle), \]
\[ |\chi_-\rangle = [2 - 2 \cos \phi \exp(-2|\alpha_2|^2)]^{-1/2}([\alpha_2] - e^{i\phi}|\alpha_2\rangle). \] (3)

It is easy to obtain the two non-negative eigenvalues \( \lambda_\pm \) of \( \rho_2 \) as follows,
\[
\lambda_\pm = \frac{1}{2} \pm \sqrt{\frac{1}{4} - N_\phi^4[1 - \exp(-4|\alpha_1|^2)][1 - \exp((-4|\alpha_2|^2])}. \] (4)

Then, the entanglement of formation of entangled coherent states \(|C(\alpha_1, \alpha_2, \phi)\rangle\) is given by
\[ E(|C(\alpha_1, \alpha_2, \phi)\rangle) = -\lambda_+ \log_2 \lambda_+ - \lambda_- \log_2 \lambda_- . \] (5)

The entanglement of formation \( E \) of the entangled coherent states with the relative phase \( \phi = \pi \) is shown in Fig.1. In Fig.2, we plot the entanglement of formation of the entangled coherent states \(|C(\alpha_1, \alpha_2, \phi)\rangle\) as the function of \(|\alpha_1| = |\alpha_2| = |\alpha| \) and the relative phase \( \phi \). It is shown that \(|C(\alpha, \alpha, \pi)\rangle\) is a maximally entangled pure state in \( 2 \times 2 \) Hilbert space. When, the relative phase \( \phi \) varies from 0 to \( 2\pi \), the degree of entanglement firstly increases, and achieve a maximal value, then decreases to a minimal value.

Next, we discuss the influence of decoherence on the entanglement of \(|C(\alpha_1, \alpha_2, \phi)\rangle\). When the entangled coherent states \(|C(\alpha_1, \alpha_2, \phi)\rangle\) are disposed to a vacuum environment, the states decohere and become mixed state \( \rho(t) \), where \( t \) stands for the decoherence time. The time evolution of \( \rho(t) \) satisfies the following master equation [12],
\[
\frac{\partial \rho(t)}{\partial t} = \sum_{i=1}^{2} \gamma \hat{a}_i \rho(t) \hat{a}_i^\dagger - \frac{\gamma}{2}(\hat{a}_i^\dagger \hat{a}_i \rho(t) + \rho(t) \hat{a}_i^\dagger \hat{a}_i), \] (6)

where \( \gamma \) is the decay rate, and \( \hat{a}_i, \hat{a}_i^\dagger (i = 1, 2) \) are the \( i \)th mode annihilation and creation operators. The solution of the master equation can be obtained
\[
\rho(t) = \sum_{n_1,n_2=0}^{\infty} \frac{(1 - e^{-\gamma t})^{n_1 + n_2}}{n_1! n_2!} \tilde{L}(t) \hat{a}_1^{n_1} \hat{a}_2^{n_2} \rho(0) \hat{a}_1^{\dagger n_1} \hat{a}_2^{\dagger n_2} \tilde{L}(t), \] (7)

where \( \tilde{L}(t) = \exp[-\frac{\gamma}{2}(\hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2)t] \). Substituting \( \rho(0) = |C(\alpha_1, \alpha_2, \phi)\rangle \langle C(\alpha_1, \alpha_2, \phi)| \) into the Eq.(7), we can express the mixed state \( \rho(t) \) as,
\[
\rho(t) = N_\phi^2(|\alpha_1(t)|\alpha_2(t)\rangle \langle \alpha_1(t)| \langle \alpha_2(t)| + | - \alpha_1(t)| - \alpha_2(t)\rangle \langle -\alpha_1(t)| - \alpha_2(t)| \]
\[ + \beta_{12}|\alpha_1(t)|\alpha_2(t)\rangle \langle -\alpha_1(t)|(-\alpha_2(t) + \beta_{12} - \alpha_1(t))\rangle - \alpha_2(t)\rangle \langle \alpha_2(t)|, \] (8)

where \( \beta_{12} = \exp[-i \phi - 2d^2(|\alpha_1|^2 + |\alpha_2|^2)], \) \( d = \sqrt{1 - e^{-\gamma t}}, \) and \(| \pm \alpha_i(t) \rangle = | \pm \alpha_i e^{-\frac{\gamma t}{2}} \rangle \) \( (i = 1, 2) \). In order to calculate the entanglement of formation of \( \rho(t) \), we introduce two orthogonal vector \(|1\rangle_i \) and \(|0\rangle_i (i = 1, 2) \) as follows,
\[
|1\rangle_i = |\alpha_i(t)\rangle_i, \quad |0\rangle_i = \frac{1}{\sqrt{1 - \eta_i^2}}(| - \alpha_i(t)\rangle_i - \eta_i |\alpha_i(t)\rangle_i), \] (9)
where \( \eta_i = \exp(-2e^{-\gamma t}|\alpha_i|^2) \). By making use of the orthogonal vectors \(|1\rangle_i\) and \(|0\rangle_i\), we reexpress \(|\alpha_i(t)\rangle_i\) and \(|-\alpha_i(t)\rangle_i\) as,

\[
|\alpha_i(t)\rangle_i = |1\rangle_i, \quad |-\alpha_i(t)\rangle_i = \eta_i|1\rangle_i + \sqrt{1 - \eta_i^2}|0\rangle_i.
\] (10)

Being encoded as above, \( \rho(t) \) in Eq.(8) becomes a mixed two-qubit state,

\[
\rho(t) = N_\phi^2 \left\{ \frac{1}{2} \left[ (1 + (\beta_{12} + \beta_{12}^*)\eta_1\eta_2 + \eta_1^2\eta_2^2)|11\rangle\langle 11| + \eta_1^2(1 - \eta_2^2)|10\rangle\langle 10| \\
+ \eta_2^2(1 - \eta_1^2)|01\rangle\langle 01| + (1 - \eta_1^2)(1 - \eta_2^2)|00\rangle\langle 00| \\
+ (\beta_{12} + \eta_1\eta_2)\eta_1\sqrt{1 - \eta_2^2}|11\rangle\langle 10| + (\beta_{12} + \eta_1\eta_2)\eta_2\sqrt{1 - \eta_1^2}|11\rangle\langle 01| \\
+ (\beta_{12} + \eta_1\eta_2)\sqrt{(1 - \eta_1^2)(1 - \eta_2^2)}|11\rangle\langle 00| + \eta_1\eta_2\sqrt{(1 - \eta_1^2)(1 - \eta_2^2)}|10\rangle\langle 01| \\
+ \eta_1(1 - \eta_2^2)\sqrt{1 - \eta_1^2}|10\rangle\langle 00| + \eta_2(1 - \eta_1^2)\sqrt{1 - \eta_2^2}|01\rangle\langle 01| \right\} + h.c.
\] (11)

For a mixed two-qubit state, \( E \) is equal to \( h(\frac{1}{2} + \frac{1}{2}\sqrt{1 - C^2}) \), where \( h \) is the binary entropy function \( h(x) = -x \log_2 x - (1-x) \log_2 (1-x) \) and \( C \) is the concurrence defined by[10]

\[ C = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}, \] (12)

where the \( \lambda_i \) (i = 1, 2, 3, 4) are the square roots of the eigenvalues in decreasing order of magnitude of the “spin-flipped” density matrix operator \( \hat{R} = \rho(\sigma^y \otimes \sigma^y)\rho^*(\sigma^y \otimes \sigma^y) \).

In Fig.3, we plot the entanglement of formation \( E \) as a function of the degree of decay \( d = \sqrt{1 - e^{-\gamma t}} \) and the relative phase \( \phi \) for (a)\(|\alpha| = 0.5 \) and (b)\(|\alpha| = 0.2 \). Here we have set \(|\alpha_1| = |\alpha_2| = |\alpha| \). The numerical results show that the entanglement of formation is sensitive with the relative phase \( \phi \) when the amplitude \(|\alpha| \) is very small. In Fig.4, the entanglement of formation \( E \) is displayed as a function of \(|\alpha_1| \) and \(|\alpha_2| \) for (a)\(\phi = \frac{\pi}{2} \) and (b)\(\phi = \pi \) with \( d = 0.5 \). It is shown that the entanglement of entangled coherent states with small amplitude \(|\alpha| \) is more robust against the decoherence than the one with large amplitude.

### III. THE PROBABILISTIC TELEPORTATION SCHEME WITH RESTRICTED CLASSICAL INFORMATION

In this section, we present the scheme of probabilistic teleportation via entangled coherent states. In this scheme, the amount of classical information is restricted to only one bit.

We assume that Alice wants to teleport a coherent superposition state

\[
|\psi\rangle_a = A_+|\alpha\rangle + A_-|-\alpha\rangle
\] (13)

via the pure entangled coherent channel \(|C(\alpha, \alpha, \pi)\rangle\rangle\), where \( A_\pm \) are unknown parameters. The state \(|\psi\rangle_a \) can be represented as

\[
|\psi\rangle_a = A'_+|\psi_+\rangle + A'_-|\psi_-\rangle,
\] (14)
where $|\psi_\pm\rangle$ are usual even and odd coherent states, defined by

$$
|\psi_\pm\rangle = N_\pm(|\alpha'\rangle \pm |-\alpha'\rangle), \quad N_\pm = [2 \pm 2 \exp(-2|\alpha'|^2)]^{-1/2}.
$$

For teleporting the state $|\psi\rangle_a$, Alice and Bob shares the quantum channel $|C(\alpha, \alpha, \pi)\rangle$. If $|C(\alpha, \alpha, \pi)\rangle$ is disposed into the vacuum, it becomes a quantum noise channel, described by

$$
\rho(t) = N_\alpha^2\{|\alpha(t)\rangle\langle\alpha(t)| + |-\alpha(t)\rangle\langle-\alpha(t)| - \alpha(t)\langle\alpha(t)|(-\alpha(t)|
\quad + \beta|\alpha(t)\rangle\langle\alpha(t)|(-\alpha(t)| + \beta^*|\alpha(t)| - \alpha(t)\rangle\langle\alpha(t)|\}
$$

where $N_\alpha = (2 - 2e^{-4|\alpha'|^2})^{-1/2}$, $\beta = -\exp(-4d^2|\alpha|^2)$ and $|\pm \alpha(t)\rangle = |\pm \alpha e^{-2d^2t}\rangle$.

In Ref.

\cite{[8]}, the standard teleportation schemes via the entangled coherent states have been studied. The amplitude of the teleported state in the scheme of Ref.

\cite{[8]} is assumed to decrease with the decoherence if the entangled coherent state is disposed into vacuum. Here, we proposed a probabilistic teleportation scheme as follows. For a fixed $|\psi\rangle_a$, which Alice want to teleport to Bob, Alice performs a Bell state measurement on the state $|\psi\rangle_a$ and her part of the quantum channel. If the state projects the whole system into the states $|\Phi_+\rangle \langle\Phi_+| \otimes \{\langle\Psi_+| |\psi\rangle_a\langle\psi| \otimes \rho(t)\}|\Phi_+\rangle\}$, where $|\Phi_+\rangle$ is the one of the following and four Bell states defined by

$$
|\Phi_\pm\rangle = \frac{\sqrt{2}}{2}(|\psi_+\rangle \langle\psi_-| \pm |\psi_-\rangle \langle\psi_+|)
$$

then the teleportation is successful, which occurs in the probabilistic manner. The mean probability of successful teleportation is

$$
P_s = N_+^2 N_a^2[1 + \exp(2e^{-\gamma t}|\alpha|^2 - 4|\alpha|^2)] \exp(-|\alpha'|^2 - e^{-\gamma t}|\alpha|^2)[\cosh(2e^{-\gamma t}|\alpha\alpha|) - 1]
\quad + N_-^2 N_a^2[1 - \exp(2e^{-\gamma t}|\alpha|^2 - 4|\alpha|^2)] \exp(-|\alpha'|^2 - e^{-\gamma t}|\alpha|^2)[\cosh(2e^{-\gamma t}|\alpha\alpha|) + 1],
$$

here, we have assumed that $\alpha'$ and $\alpha$ have the same phase. In this probabilistic teleportation scheme, Bob need not know the details of the channel. The classical information sent from Alice to Bob is just only S (Success) or F (Failure), which determines whether Bob accept the state or not. In the following, we calculate the mean fidelity $F$ defined as follows\cite{[13]},

$$
F = \frac{\langle\psi|\rho|\psi\rangle_a}{a}\n\quad = 2 \exp(-|\alpha'|^2 - e^{-\gamma t}|\alpha|^2)\{2(1 - \beta) \ln[\frac{\nu(1 - \beta\eta)}{\mu(1 + \beta\eta)}] + \frac{\mu\nu}{\mu(1 + \beta\eta) - \mu(1 + \beta\eta)}
\quad + (1 + \beta)(\mu^2 + \nu^2)\nu^2(1 - \beta\eta)^2 - \mu^2(1 + \beta\eta)^2 - 2\mu\nu(1 - \beta^2\eta^2) \ln[\nu(1 - \beta\eta) - \mu(1 + \beta\eta)]\}\},
$$

where, $\eta = \exp(-2e^{-\gamma t}|\alpha|^2)$, $\mu = N_+^2 \cosh^2(|e^{-\frac{d^2}{4}}\alpha\alpha|)$ and $\nu = N_-^2 \sinh^2(|e^{-\frac{d^2}{4}}\alpha\alpha|)$. In Fig.5, we plot the mean fidelity $F$ as a function of the degree of decay $d$ and the
initial amplitude $|\alpha|$ of the entangled coherent states with $|\alpha'| = 1$. It is seen that the mean fidelity revivals during the decoherence process in some situations. This means that the decoherence due to interaction with the vacuum environment can improve the mean fidelity in some specific situations. In Fig.6., we show that the decoherence can enhance the mean fidelity above the classical limit $2/3$ (The plot range is from $2/3$ to 0.68).

**IV. CONCLUSION**

In this Letter, we study the entanglement of entangled coherent states in the vacuum environment by employing the entanglement of formation. The results indicate that the entanglement of formation of the entangled coherent states is sensitive with the relative phase $\phi$ when the amplitude $|\alpha|$ is very small. Furthermore, we present a probabilistic teleportation scheme, in which the amount of classical information sent by Alice is restricted to one bit. In this probabilistic teleportation scheme, the decoherence due to interaction with the environment can enhance the mean fidelity in some specific situations. It is interesting to verify the probabilistic teleportation scheme experimentally.

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Figure 1: The entanglement of formation $E$ as a function of the amplitude $|\alpha_1|$ and $|\alpha_2|$ with $\phi = \pi$.

**Figure Caption**

**FIG.1.** The entanglement of formation $E$ as a function of the amplitude $|\alpha_1|$ and $|\alpha_2|$ with $\phi = \pi$.

**FIG.2.** The entanglement of formation $E$ as a function of the amplitude $|\alpha|$ and the relative phase $\phi$.

**FIG.3.** The entanglement of formation $E$ as a function of the degree of decay $d$ and the relative phase $\phi$ for (a)$|\alpha| = 0.5$ and (b)$|\alpha| = 0.2$.

**FIG.4.** The entanglement of formation $E$ as a function of the amplitude $|\alpha_1|$ and $|\alpha_2|$ for (a)$\phi = \frac{\pi}{2}$ and (b)$\phi = \pi$ with the degree of decay $d = 0.5$.

**FIG.5.** The mean fidelity $F$ as a function of the degree of decay $d$ and the initial amplitude $|\alpha|$ with $|\alpha'| = 1$.

**FIG.6.** The same as Fig.5.
Figure 2: The entanglement of formation $E$ as a function of the amplitude $|\alpha|$ and the relative phase $\phi$. 
Figure 3: The entanglement of formation $E$ as a function of the degree of decay $d$ and the relative phase $\phi$ for (a) $|\alpha| = 0.5$ and (b) $|\alpha| = 0.2$. 
Figure 4: The entanglement of formation $E$ as a function of the amplitude $|\alpha_1|$ and $|\alpha_2|$ for (a) $\phi = \frac{\pi}{2}$ and (b) $\phi = \pi$ with the degree of decay $d = 0.5$. 
Figure 5: The mean fidelity $F$ as a function of the degree of decay $d$ and the initial amplitude $|\alpha|$ with $|\alpha'| = 1$.

Figure 6: The same as Fig.5.