Upscaling of Forchheimer flows

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Abstract

In this work we propose upscaling method for nonlinear Forchheimer flow in heterogeneous porous media. The generalized Forchheimer law is considered for incompressible and slightly-compressible single-phase flows. We use recently developed analytical results (Aulisa et al., 2009) [1] and formulate the resulting system in terms of a degenerate nonlinear flow equation for the pressure with the nonlinearity depending on the pressure gradient. The coarse scale parameters for the steady state problem are determined so that the volumetric average of velocity of the flow in the domain on fine scale and on coarse scale are close. A flow-based coarsening approach is used, where the equivalent permeability tensor is first evaluated following streamline methods for linear cases, and modified in order to take into account the nonlinear effects. Compared to previous works (Garibotti and Peszynska, 2009) [2], (Durlofsky and Karimi-Fard) [3], this approach can be combined with rigorous mathematical upscaling theory for monotone operators, (Efendiev et al., 2004) [4], using our recent theoretical results (Aulisa et al., 2009) [1]. The developed upscaling algorithm for nonlinear steady state problems is effectively used for variety of heterogeneities in the domain of computation. Direct numerical computations for average velocity and productivity index justify the usage of the coarse scale parameters obtained for the special steady state case in the fully transient problem. For nonlinear case analytical upscaling formulas in stratified domain are obtained. Numerical results were compared to these analytical formulas and proved to be highly accurate.

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1. Introduction

In recent years, using near well data, such as core data, engineers have been able to create increasingly complex and detailed geocellular models. This compels taking into account highly heterogeneous geological parameters of reservoirs. Such descriptions typically require a high number of computational cells which is difficult to simulate, e.g., in well optimization problems and history matching. To reduce the computational complexity, some type of coarsening and upscaling procedures are needed. The geological parameters, such as permeability or transmissibility and porosity, should be upscaled for each coarse-grid block.

The variety of approaches for upscaling and multiscale methods of fine scaled geological parameters have been proposed for the linear Darcy case (see [5–10]). These approaches include upscaling methods, see [5,8–10] and multiscale methods [4–7]. In both approaches, a goal is to represent the solution on a coarse grid where each coarse-grid block consists of a union of connected fine-grid blocks. In upscaling methods, the upscaled permeability is calculated in each coarse-grid block by solving local problems with specified boundary conditions and calculating the average of the flux. Local problems can be solved in extended domains for computing the effective properties. In multiscale methods, the local multiscale basis functions are computed instead of local effective properties and these basis functions are coupled via a global formulation. The extensions of these methods to nonlinear flows, such as Forchheimer flow, are carried out in several papers, see [2,3] which are closely related to our work. In [3], the authors consider the use of iterative upscaling techniques where at each iteration, local-global upscaling technique is used. The work [2] closely relates to our work. In [2], the authors start with a full nonlinear upscaling where the upscaled conductivity is a nonlinear function of the pressure gradient. The starting point of our approach follows [2]. In [2], the authors further use special nonlinear forms for upscaled Forchheimer flows that simplify the upscaling calculations. In the current paper, our goal is to carry out a nonlinear upscaling using
new formulations of Forchheimer flows. We emphasize that the monotonicity of the fine-scale operator (as discussed in [2]) is important for formulating full nonlinear homogenization.

In current paper, we utilize recent finding [1], where Forchheimer equation is written in an equivalent form using monotone nonlinear permeability function depending on gradient of pressure. This equivalent formulation reduces the original system of equations for pressure and velocity to one nonlinear parabolic or elliptic equation for pressure only. The ellipticity constant of this equation degenerates as the pressure gradient converges to infinity. The rate of the degeneration is effectively controlled by the order of Forchheimer polynomial and the structure of the coefficients has the important monotonicity properties (see Proposition III.6 and Lemma III.10 from [1]). It allows to prove results on the well-posedness of the initial boundary value problem and apply numerical homogenization theory.

In this paper we present the upscaling algorithm for fluid flow in incompressible media for two types of fluids, incompressible and slightly compressible. Steady state problem for incompressible flow reduces to the degenerate elliptic equation, however the corresponding problem for compressible fluid reduces to time dependent degenerate parabolic equation.

In this paper we first introduce and investigate the upscaling procedure for the time independent problem in case of incompressible fluid. In case of time dependent problem the question one should address is that while the solution is time dependent, the upscaled parameters are time independent for incompressible media. We use the upscaled parameters obtained for steady state case in the time dependent problem. This procedure is justified by the results obtained in our papers [11,12] and the numerical experiment presented in this article. Namely, we will relate the fine scale fully transient solution to the special pseudo steady state (PSS) solution. This solution has a form \( At + W(x) \), where \( A \) is a constant and \( W(x) \) is a solution of auxiliary steady state boundary value problem for the equation with non zero RHS. According to our results in [11,12] under some assumptions the pseudo steady state pressure and velocity serve as pseudo attractors for fully transient pressure and velocity. To upscale the steady state equation we determine the coarse scale porosity and nonlinear permeability, so that the average volumetric velocity of the flow is preserved.

To evaluate the efficiency of the described method in the time dependent case we compare the productivity index (PI) of the well on fine and coarse grids. The PI is inversely proportional to the difference between the average of pressure in the reservoir and on the well. We select the PI as a criteria for the evaluation of the upscaling method as it is widely used by the engineers, see [13–16] and references therein. In the numerical examples we calculate the difference between the values of the PIs on fine and coarse grids. Our numerical results show that the proposed algorithm provides accurate results for different heterogeneities and nonlinearities in steady state case. Resulting transient velocity and PI on coarse scale also provide accurate approximation of corresponding transient parameters on fine scale for heterogeneous fields considered in the paper. Clearly the accuracy of the proposed method depends on heterogeneities as in a single-phase upscaling, [17], i.e., for highly heterogeneous fields, the accuracy of the method will decrease. The main goal of this paper is to propose a method to handle the nonlinearities and, thus, we do not consider highly heterogeneous fields [17].

The paper is structured as follows. In Section 2.1 we introduce \( g \)-Forchheimer equations, review their properties and formulate the problem. In Section 2.2 we obtain the form of the coarse scale equation for generalized Forchheimer flow. In Section 2.3 we introduce the special steady state equation which will be used for upscaling for flow of slightly compressible fluid. We then discuss the theoretical results justifying the use of upscaled parameters from steady state problem in time dependent problem. Section 3 is devoted to description of upscaling algorithm. In Section 4 we obtain the explicit analytical upscaling formulas in case of incompressible fluid for nonlinear Forchheimer flow in stratified region. In contrast with the linear case, the formulas for nonlinear case may depend on boundary data, see (42) and (44). In Sections 5.1 and 5.2 we present the numerical results for the incompressible and slightly compressible flows correspondingly. In Section 5.3 we test the usage of the parameters obtained by upscaling steady state equation in transient case. We present the theoretical and numerical result on convergence of transient velocity and PI to corresponding steady state values for upscaled problem.

2. Problem statement and preliminary results

2.1. Generalized Forchheimer equation

Let \( \Omega \subset \mathbb{R}^n \) be the flow domain. Darcy equation describes the linear dependence of velocity \( u \) on the pressure gradient \( \nabla p \)

\[
\nabla \cdot (\mu \nabla p) = f
\]

where \( \mu \) is the viscosity of the fluid. Forchheimer equation, [18], is known to generalize Darcy’s equation to take into account inertial terms and has been introduced in the literature in several forms. E.g., see [19, p. 182]

\[
\begin{align*}
\text{Two term law: } & u + \beta(x)||u||u = -\frac{1}{\mu}k(x)\nabla p, \\
\text{Three term law: } & u + a_1(x)||u||u + a_2(x)||u||^2u = -\frac{1}{\mu}k(x)\nabla p, \\
\text{Power law: } & u + b_1(x)||u||^{m-1}u = -\frac{1}{\mu}k(x)\nabla p. \\
\end{align*}
\]

From here on \( ||\cdot|| \) is the \( L_2 \) vector norm. Coefficients \( \beta(x), a_1(x), a_2(x), b_1(x) \) and power \( m \) are empirical.

All these relations can be written in a compact form as

\[
g(||u||, x)u = -\frac{1}{\mu}k(x)\nabla p, \tag{3}
\]

for some function \( g(s, x) \geq 0 \) for \( s \geq 0 \). We will refer to (3) as \( g \)-Forchheimer (momentum) equation. For simplicity from now on we assume the viscosity \( \mu = 1 \).

To develop rigorous numerical homogenization concepts for Forchheimer flow, we use the results in [1] which allows writing (3) as a monotone relation for \( \nabla p \) (see Eq. (6) and the discussion after it). Moreover, this allows obtaining the well-posedness results of the corresponding initial boundary value problem and allows estimating the residual error in numerical homogenization because of monotonicity. It was shown in [1] that the monotone relation between velocity and gradient of pressure exists for general functions \( g(s, x) \) in the form

\[
g(s, x) = 1 + \sum_{l=1}^{l} a_l(x)s^{l} = 1 + a_1(x)s^{10} + a_2(x)s^{20} + \ldots + a_l(x)s^{10}, \tag{4}
\]

where \( l \geq 0, \) the exponents satisfy \( 0 < \alpha \leq \alpha_{l+1} \), and the coefficients \( a_l(x) \geq 0, j = 1, \ldots, l \). Thus defined function \( g \) in (3) includes all the known cases of Forchheimer flow (1) and (2).

In order to make further constructions we rewrite (3) in the equivalent form solving \( u \) implicitly in terms of \( \nabla p \). To that purpose using the notation \( h(s) = sg(s, x) = ||u|| \) and \( s = h^{-1}(||u||) \), where \( s, ||u|| \geq 0 \), we introduce the function
Using (5) we rewrite (3) in a form
\[ u = -G(\|k(x)\|; x) k(x) \nabla p. \] (6)
which we call generalized (nonlinear) Darcy equation.

In [1] it is shown that the function \( F(\zeta) = G(\|\zeta\|; x) \zeta \) is strictly monotone on bounded sets. Namely, \( (F(\zeta_1) - F(\zeta_2)) \cdot (\zeta_1 - \zeta_2) \geq C\|\zeta_1 - \zeta_2\|^2 \). We also obtained an asymptotic relation for \( G \) in the homogeneous case when \( G(\|\zeta\|; x) = G(\|\zeta\|) \):
\[ G(\|\zeta\|) \sim \frac{1}{1 + \|\zeta\|^2}, \quad a = \frac{\zeta_1}{\zeta_1 + 1}. \] (7)

Remark 2.1. In the particular case of two-term Forchheimer law, the nonlinear permeability coefficient \( G \) can explicitly be written
\[ G(\|\zeta\|; x) = \frac{2}{1 + \sqrt{1 + 4\|\zeta\|^2}}. \] (8)

The \( g \)-Forchheimer equation written in the form (6) allows reducing the dynamical system to single nonlinear equation of pressure. Namely, we consider the continuity equation
\[ \phi(x) \frac{\partial p}{\partial t} = -\nabla \cdot (\rho u), \] (9)
where \( \rho \) is the density of the fluid, and \( \phi \) is the rock porosity. For incompressible fluid (\( \rho = \text{const} \)), (9) reduces to \( \nabla \cdot u = 0 \) and combined with the flow equation (6) results in the degenerate elliptic equation of pressure only for steady-state flow
\[ \nabla \cdot (G(\|k(x)\|; x) k(x) \nabla p) = 0. \] (10)
From (7) it follows that the degeneracy in Eq. (10) is the same as for \( \nu \)-Laplace type equation for big values of gradient of pressure.

For slightly compressible fluid (such as the compressible liquid) the equation of state takes the form, see [13],
\[ \rho(p) = \rho_0 e^{p}, \] (11)
where \( \gamma \) is the inverse of the compressibility constant.

Substituting (11) in (9) we get
\[ \phi \frac{\partial p}{\partial t} = -\frac{1}{\gamma} \nabla \cdot u - u \cdot \nabla p. \] (12)
For slightly compressible fluids \( \gamma \) is of order \( 10^{-3} \). From experimental data it follows that the impact of the second term in RHS of Eq. (12) is negligible, see [13–15,20]. Thus the engineers drop the term \( u \cdot \nabla p \) in their practice, arriving to equation
\[ \phi \frac{\partial p}{\partial t} = -\frac{1}{\gamma} \nabla \cdot u. \] (13)
Combining (13) with (6) we obtain the degenerate parabolic equation for pressure
\[ \gamma \phi(x) \frac{\partial p}{\partial t} = \nabla \cdot (G(\|k(x)\|; x) k(x) \nabla p). \] (14)

In what follows we consider two orthogonal grids in domain \( \Omega \): fine \( x = (x_1, x_2) \)-scale and coarse \( X = (X_1, X_2) \)-scale (see Fig. 1). Eqs. (10) and (14) describe the fluid flow on the fine grid. Our aim is to devise an upscaling algorithm for the parameters \( k(x), \phi(x) \) and \( G(\|k(x)\|; x) \) in Eqs. (10) and (14) and obtain the corresponding coarse scale equations.

2.2. Coarse scale equation in case of incompressible fluid
To obtain the coarse scale equation for incompressible case, we first rewrite Eq. (10) in each coarse block \( \Omega_c \) in a form:
\[ \nabla \cdot (K(\zeta; x) \nabla p; x) = 0 \quad \text{in} \quad \Omega_c, \] (15)
where \( K(\zeta; x) = G(\|\zeta\|; x) \zeta \). We assume \( p = \eta \cdot x \) on \( \partial \Omega_c \), where vector \( \eta = (\eta_1, \eta_2) \) (and \( \eta = (\eta_1, \eta_2, \eta_3) \) in 3D). Our aim is to find the form of Eq. (15) on coarse scale depending on coarse scale parameters \( k' \) and \( G' \). We then solve Eq. (15) in each coarse block. In each coarse block \( \Omega_c \) we define
\[ K'(\eta) = \langle K(\eta; x) \rangle = \langle G(\|\eta\|; x) \eta \rangle. \]
Here
\[ \langle f \rangle = \frac{1}{|\Omega_c|} \int_{\Omega_c} f \, d\Omega_c \]
is the volumetric average of the function over \( \Omega_c \).

We would like to find the upscaled tensor \( k' \) and a scalar \( G' \), depending on \( k' \), so that
\[ K'(\eta) = G'(\eta) k' \eta. \]
Then, the upscaled equation takes the form
\[ \nabla \cdot (G'(\nabla p') k' \nabla p') = 0. \] (16)
It follows that the coarse scale function \( G' \) depends on the vector \( \nabla p' \), while the fine scale function \( G \) depends on the scalar \( \|k(x)\| \nabla p\| \).

2.3. Coarse scale equation in case of slightly compressible fluid
Unlike the steady state equation (10) for incompressible fluid, Eq. (14) for slightly compressible flow is transient in time. Parameters \( k, G \) and \( \phi \) on the fine scale are, however, time independent.
We want to find the upscaled parameters \( k', G' \) and \( \Phi' \) on the coarse scale which are time independent as well. It is difficult to use the original Eq. (14) for the upscaling procedure directly. Instead we will relate the fine scale transient pressure and velocity to the special pseudo steady state solution of Eq. (14) which will be defined below.

Let \( \Omega \) be the domain with the boundary \( \Gamma \) consisting of two parts \( \Gamma = \Gamma_x \cup \Gamma_i \). The no-flux condition is imposed on \( \Gamma_x \)

\[
u \cdot v_{i} = 0
\]

(17)

and prescribed total flux condition is imposed on \( \Gamma_i \)

\[
\int_{\Gamma_i} u \cdot v d\Sigma = Q(t),
\]

(18)

where \( u \) is the velocity as in (6) and \( v \) is the outer normal to the boundary \( \Gamma \).

In [1] it was proved that there exists a special solution \( p_i(x,t) \) of Eq. (14) with boundary condition (17) on \( \Gamma_x \) such that

\[
\frac{\partial p_i}{\partial t} = \text{const} = -A \ \text{for all } t.
\]

(19)

Such solution is called Pseudo Steady State (PSS). From definition (15) of PSS solution it follows that the corresponding production rate is constant \( Q(t) = Q = A \Omega / \gamma = \text{const} \) and \( p_i(x,t) \) can be written as

\[
p_i(x,t) = -\gamma \frac{Q}{\Omega} t + W(x) + C,
\]

(20)

where \( W(x) \) is called a basic profile and is a solution of the steady state BVP

\[
\nabla \cdot (G(|k(x)|\nabla W)|x)k(x)\nabla W) = -\gamma \frac{Q}{\Omega} \Phi(x),
\]

(21)

\[
u_i(x) \cdot v|_{\Gamma_i} = 0,
\]

(22)

\[W|_{\Gamma_i} = \phi_0(x),\]

(23)

with given function \( \phi_0(x) \) and constant \( C \), see [1]. Notice that \( \nabla p_i = \nabla W \) and the corresponding PSS velocity \( u_i(x) = -G(|k|\nabla p)|x|k \nabla p_i = -G(|k|\nabla W)|x|k \nabla W \)

(24)

is time independent.

The steady state BVP (21)-(23) will be used to find the upsampling parameters for fully transient Eq. (14). On coarse scale the steady state equation (21) will take the form

\[
\nabla \cdot (G'k'\nabla p') = -A\Phi'.
\]

(25)

Following the standard approach (cf. [8]), we aim to formulate the upscaling algorithm to find \( k', G' \) and \( \Phi' \) so that

\[
||\{u_i\}|_{\Omega} - \{u_i\}|_{\Omega}|| \text{ is sufficiently small.}
\]

(26)

Here

\[
\{u_i\}_{\Omega} = \frac{1}{|\Omega|} \int_{\Omega_i} u_i d\Omega, \quad \{u_i\}_{\Omega} = \frac{1}{|\Omega|} \int_{\Omega} u_i dU,
\]

(27)

where \( u_i \) and \( u' \) are the steady state velocities on fine and coarse scales, correspondingly.

If initial data is not of basic profile then solution \( p(x,t) \) of the original Eq. (14) and the corresponding velocity \( u(x,t) \) are time dependent. Thus in order to justify the usage of upscaling criteria (26) for general case one should prove convergence of the corresponding time dependent quantity to the time independent one. This property was obtained in [11,12] under certain conditions on the boundary data. Namely, let

\[
\psi(x,t) = p(x,t)|_{\Gamma_x} - \frac{1}{|\Gamma_x|} \int_{\Gamma_x} p(x,t)d\Sigma
\]

and \( \phi(x) = \phi_0(x) - \frac{1}{|\Gamma|} \int_{\Gamma} \phi_0(x) d\Sigma \)

be the deviations from the average on the boundary \( \Gamma_x \) of the trace of transient solution \( p(x,t) \) and basic profile \( W(x) \) correspondingly.

We proved that if the differences \( Q(t) - \phi(x,t) - \phi(x) \) converge in certain sense to zero at time infinity (see [12], Section 3.2), then the PSS velocity \( u_i(x) \) serves as the steady-state attractor for the fully transient velocity \( u(x,t) \) with any initial data:

\[
\int_{\Omega} ||u(x,t) - u_i(x)||^2 dx \to 0 \quad \text{as } t \to \infty.
\]

This justifies the usage of criteria (26) for the upscaling of coefficients \( k, G, \Phi \) in fully transient problem.

3. Numerical upscaling algorithm

In this section we present the numerical upscaling algorithm for the steady state equations (21) and (10). We consider 2D rectangular region \( \Omega \), with horizontal size \( L_1 \) and vertical size \( L_2 \).

The porous media on the fine scale is considered to be isotropic, and permeability tensor \( k(x) \) is a scalar function \( k(x_1, x_2) \). The fine scaled equation (21) or (10) with parameters \( k(x_1, x_2), \phi(x_1, x_2) \) and \( G(|k|\nabla p)|x_1, x_2) \) is upscaled to the coarse scale equation (25) or (16) with parameters \( k'(x_1, x_2), \Phi'(x_1, x_2) \) and \( G'(|k'|\nabla p)|x_1, x_2) \) so that condition (26) is satisfied. The \( u_i \) and \( u'_i \) in (26) are the velocities on the fine and coarse scale correspondingly.

Our approach is purely local, so the algorithm is described for a single coarse block \( \Omega \), with boundary \( \partial \Omega \). For simplicity we take \( \Omega \) to be the rectangle \([0, L_1] \times [0, L_2] \) with the area \( |\Omega| = L_1 \cdot L_2 \).

For each coarse block the following two-step procedure is performed:

Step 1: the equivalent permeability tensor \( k' \) is obtained using linear upscaling methods;

Step 2: the equivalent nonlinear coefficient \( G' \) is obtained using \( k' \).

Step 1. Procedure to obtain \( k' \). In order to obtain full permeability tensor \( k' \) we use the standard local procedure via volume averages of velocity and pressure gradients, see for example [10]. We solve two flow problems in each block with periodic boundary conditions. Namely, let \( p_1 \) and \( p_2 \) be the solutions of the fine scale equation in coarse block \( \Omega \)

\[
\nabla \cdot (k(x_1, x_2) \nabla p) = 0,
\]

(28)

with boundary conditions:

\[
\begin{align*}
p_1(x_1, 0) & = p_1(x_1, L_2) \quad \text{for } x_1 \in [0, L_1]; \\
p_2(0, x_2) & = p_2(L_1, x_2) \quad \text{for } x_2 \in [0, L_2]; \\
u_1(x_1, 0) & = -u_1(x_1, L_2) = v_2; \\
u_2(0, x_2) & = -u_2(L_1, x_2) = v_1; \\
p_1(0, x_2) & = 0; \quad p_1(L_1, x_2) = 1; \\
p_2(x_1, 0) & = 0; \quad p_2(x_1, L_2) = 1.
\end{align*}
\]

(29)

Here \( u_i \) is the velocity vector corresponding to the pressure distribution \( p_i, i = 1, 2 \).

The four elements of the upscaled permeability \( k' \) are then calculated from two vector equations:

\[
u_i = -k' \nabla p_i, \quad i = 1, 2.
\]

(30)

The upscaled porosity \( \Phi' \) is computed via integral averaging on the coarse block following classical approach, e.g. [10]:

\[
\Phi' = \langle \phi(x_1, x_2) \rangle = \frac{1}{|\Omega|} \int_{\Omega} \phi(x_1, x_2) d\Sigma.
\]

(31)

Step 2. Procedure to obtain \( G' \). We use the upscaled permeability \( k' \) to determine the nonlinear coefficient \( G' \) via pure local averaging. As it was mentioned in Section 2.2, unlike the fine-scale...
function $G$ depending on $\|k\nabla p\|$ the upscaled $G$ depends on the vector $\nabla p^c$ itself. Let $\eta = (\eta_1, \eta_2)$ be the gradient of pressure in coarse block $\Omega$. For fixed $\eta$, the $G$ is a constant. If $p^c$ is the solution of coarse scale equation (16) with boundary condition

$$p^c|_{\partial \Omega} = \eta_1 x_1 + \eta_2 x_2,$$

then it is also the solution of equation $\nabla \cdot (k^c \nabla p^c) = 0$ with the same boundary condition.

We determine $G$ so that

$$\|\langle u^c \rangle\| = \|\langle u \rangle\|,$$

where $u = -G^c k^c \nabla p^c$ is the velocity on coarse scale and $u = -G \nabla p$ is the velocity on fine scale corresponding to the solution $p(x,t)$ of (10) with boundary condition $p|_{\partial \Omega} = \eta_1 x_1 + \eta_2 x_2$. Then for fixed $\eta_1, \eta_2$ we have:

$$G((\eta_1, \eta_2)) = \frac{\|\langle u \rangle\|}{\|\langle k^c \nabla p^c \rangle\|}.$$ (34)

Using formula (34) we numerically construct the table of values of $G^c$ for $\eta_1, \eta_2 \in (-\infty, \infty)$. It follows that $G^c(0,0) = 1$, $G^c(\eta_1, \eta_2) \to 1$ if $\|\eta\| \to 0$, $G^c(\eta_1, \eta_2) \to 0$ if $\|\eta\| \to \infty$ and $G^c$ possesses certain symmetry: $G^c(\eta_1, \eta_2) = G^c(-\eta_1, -\eta_2)$.

It is thus sufficient to consider $\eta_1 \in (-\infty, \infty)$ and $\eta_2 \geq 0$ only. It is worth mentioning that the special attention should be paid to the way the domain for the $\eta$ is discretized. Taking the grid to be too fine makes the calculations overly expensive, however the sparse grid does not allow to capture the features of nonlinearity of the process. We will use the nonuniform grid, where the subsequent point is calculated on the basis of the deviation between the preceding values of the function. Namely, $\eta_1$ and $\eta_2$ are taken from the set $s_n, n = 0, 1, 2, \ldots$, where the first three values are taken a priori: $s_0 = 0$, and $s_1, s_2$ to be small enough. Next value $s_{n+1}$ is chosen so that

$$G_{n+1} = G_{n} - \varepsilon_n, \quad G_{s} = G_{s}(s_{n}, s_{n}), \quad m = n, n + 1$$

for some fixed value $\varepsilon_n$. The stopping criteria for the computation is $G_{n} \leq \varepsilon$ and $|G_{n+1}| \leq \varepsilon_\delta$ so that the value of $G$ as well as its variation are sufficiently small.

The shape of function $G$ is presented on Fig. 2.

### 4. Analytical upscaling for the layered porous media

Here we will present the analytical upscaling formula for the nonlinear Forchheimer flow of incompressible fluid in layered porous media. Review of existing analytical methods can be found for example in [21]. All the results discussed in [21] are for the linear case only, and to our best knowledge there are no analytical upscaling formulas in the nonlinear case.

Consider a rectangular region $R$ of horizontal size $L$ and vertical size $H$. The region has a horizontal multilayer structure, and is composed by $n$ layers, see Fig. 3. Each $i$th layer, $i = 1, \ldots, n$, has vertical size $h_i$ and is characterized by constant isotropic permeability $k_i$ and $g$-Forchheimer polynomial $g(s,x) = g_i(s)$ with constant coefficients or, equivalently, by the nonlinear function $G_i = G_i(k_i \nabla p_i)$. We assume that the type of nonlinearity is the same for each layer, while the coefficients of $g$-polynomials can be different.

Under these assumptions, in each $i$th layer, $i = 1, \ldots, n$, Eqs. (3), (6) and (10) yield

$$g_i(\|u_i\|) u_i = -k_i \nabla p_i,$$ (35)
$$u_i = -G_i k_i \nabla p_i,$$ (36)
$$\nabla \cdot (G_i k_i \nabla p_i) = 0.$$ (37)

Here $u_i$, $p_i$ and $Q_i$ are, correspondingly, velocity, pressure and the total boundary flux in $i$th layer.

We assume that flow within the whole block $R$ is subject to the equation with the same type of nonlinearity as in each layer. We aim to find the equivalent homogeneous block permeability $k^c$ and nonlinear coefficient $G^c = G^c(k^c \nabla p^c)$ for two types of flow: flow parallel to the layers (Section 4.1) and flows perpendicular to the layers (Section 4.2). The upscaled parameters are determined so that the total flux of the system stays the same as with nonhomogeneous parameters. The comparison between the obtained analytical results and numerical computations using the method in Section 3 is presented in Section 5.1. Note, that the case when $g_i = \text{const}$ is the same as Darcy case and the upscaling formulas for $k^c$ are the same as in [19].

#### 4.1. Flow parallel to the layers

We impose the following boundary conditions on boundaries of $R$

- $p_i = p^c = p_0$, on the left boundary, $i = 1, \ldots, n$,
- $p_i = p^c = p_n$ on the right boundary, $i = 1, \ldots, n$,
- $u \cdot v = 0$ on the bottom and top boundaries,

where $p_1 > p_n$. Under these conditions the flow is parallel to the layers and the solution of Eq. (37) is linear in $x$. The pressure gradient is constant and is equal to $\nabla p = (\eta_1, 0)$, where $\eta_1 = (p_1 - p_n)/L$. In each layer the vertical velocity component $u_i$ is identically zero, while the horizontal component $u_i$ is constant in each layer and, according to (36), is equal

$$u_i = u_i = -G_i(k_i \eta_1) k_i \eta_1.$$ (38)

On the other hand the outgoing flux is equal to incoming flux and is equal to the sum of fluxes in the $i$th layer:

$$Q = \sum_{i=1}^{n} Q_i = -\sum_{i=1}^{n} u_i h_i = \eta_1 \sum_{i=1}^{n} G_i k_i h_i.$$ (39)

The flux is zero on the top and bottom boundaries.

We now consider the analogous block with the same boundary conditions and permeability $k^c$ and nonlinear function $G^c$ resulting in the same flux $Q$. In this case the flux is

$$\begin{array}{c|c|c|c}
\hline
k_1, G_1 & h_1 \\
\vdots & \vdots \\
k_n, G_n & h_n \\
\hline
\end{array}$$

![](image)
\( Q = -u'H = G'k'H \eta_1, \)

where \( u' \) is constant horizontal component of upscaled velocity. Expression for \( Q \) above and (39) yield

\[
\eta_1 \sum_{i=1}^{n} G_k h_i = G'k'H \eta_1, \quad \text{or} \quad u' = \frac{1}{H} \sum_{i=1}^{n} u_i h_i. \tag{40}
\]

First we consider the limiting linear Darcy case \( G_i = 1. \) In this case \( G' = 1. \) We then find an expression for \( k' \)

\[
k' = \frac{1}{H} \sum_{i=1}^{n} k_i h_i. \tag{41}
\]

In view of (41) the general expression for \( G' \) follows from (40)

\[
G' = \frac{1}{H} \frac{1}{k} \sum_{i=1}^{n} G_k h_i = \frac{\sum_{i=1}^{n} G_k h_i}{\sum_{i=1}^{n} k_i h_i}. \tag{42}
\]

Formulas (41) and (42) can be generalized in case when instead of layers we have a continuous variation of the parameters \( k = k(x_2), \) \( G = G(k_\eta, x_2); \)

\[
k' = \frac{1}{H} \int_{X_1}^{X_2} k(x_2) dx_2, \quad G' = \frac{1}{H} \frac{1}{k} \int_{X_1}^{X_2} G(k_\eta, x_2) k(x_2) dx_2. \tag{43}
\]

Alternatively, using g-Forchheimer equation (35) with \( \| u_i \| = u, \) the upscaled formula for the g-polynomial can be obtained:

\[
\frac{1}{G'(u')} = \frac{\sum_{i=1}^{n} \frac{1}{\beta_i} h_i}{\sum_{i=1}^{n} k_i h_i}. \]

From here we can obtain the upscaled coefficients \( a_j, j = 1, \ldots l \) corresponding to the power \( s^2, \) for the g-polynomial in domain R. In particular in case of two-terms law as in Remark 2.1 the upscaled Forchheimer coefficient can be obtained explicitly in the form

\[
\beta'' = \frac{\sum_{i=1}^{n} \beta_i k_i^2 h_i}{\left( \sum_{i=1}^{n} h_i k_i \right)^2} = \frac{\sum_{i=1}^{n} \beta_i k_i^2 h_i}{k'^2}, \tag{44}
\]

where \( \beta_i \) is coefficient corresponding to ith layer.

The coefficient \( \beta'' \) depends explicitly on \( \eta_1. \) The two limiting cases are

\[
\lim_{\eta_1 \to 0} \beta'' = \frac{\sum_{i=1}^{n} \beta_i k_i^2 h_i}{\left( \sum_{i=1}^{n} h_i k_i \right)^2} = \frac{\sum_{i=1}^{n} \beta_i k_i^2 h_i}{k'^2},
\]

and

\[
\lim_{\eta_1 \to \infty} \beta'' = \frac{\sum_{i=1}^{n} \beta_i h_i}{\left( \sum_{i=1}^{n} h_i \frac{k_i}{\beta_i} \right)^2} = \left( \frac{k'}{\sum_{i=1}^{n} \beta_i h_i} \right)^2.
\]

In case when the parameters \( k = k(x_2), \) \( \beta = \beta(x_2) \) are continuous functions, the expression (44) for \( \beta'' \) yields

\[
\beta'' = \frac{1}{H} \int_{X_1}^{X_2} \beta(x_2) \left( \frac{2 k(x_2)}{1 + \sqrt{\alpha k(x_2)}} \right)^2 dx_2 = \frac{1}{H} \int_{X_1}^{X_2} \left( \frac{2 k(x_2)}{1 + \sqrt{\alpha k(x_2)}} \right)^2 dx_2.
\]

4.2. Flow perpendicular to the layers

Let consider the same geometry and let impose the following boundary conditions:

- \( p_i |_{x_2 = 0} = p'_i |_{x_2 = 0} = p_o, \) on the bottom boundary,
- \( \rho |_{x_2 = 0} = 0, \) on the top boundary,
- \( u \cdot v = 0 \) on the left and right boundaries.

In this case the flow is perpendicular to the layers and the horizontal velocity component \( u_i \) is identically zero, while the vertical component of velocity \( u_i = u \) is constant in each layer. The pressure gradient in ith layer is equal to

\[
\nabla p_i = \left( 0, \eta_{2i} \right), \quad \text{where} \quad \eta_{2i} = \frac{p_i - p_{i-1}}{h_i},
\]

where \( p_i \) is the pressure measured at the top of ith layer for \( i = 1, \ldots, n. \) Then, according to (36), vertical component of velocity is equal to \( u = -G_k h \eta_{2i}. \)

It thus follows that the flux is constant and in each layer is equal to \( Q = -uL = G_k h \eta_{2i} L. \) Then the pressure gradient in ith layer is

\[
\eta_{2i} = \frac{Q}{G_k h L}, \quad i = 1, \ldots, n. \tag{45}
\]

We again want to identify the equivalent homogeneous parameters \( k' \) and \( G' \) in the region \( R \) resulting in the same flux \( Q' = Q. \) The pressure gradient in the domain \( R \) is

\[
\eta_{2i} = \frac{p_i - p_{i-1}}{H} = \frac{1}{H} \sum_{i=1}^{n} \eta_{2i} h_i. \tag{46}
\]

We get the expression for the flux

\[
Q = -u' L = G' k' \eta_{2i}. \tag{47}
\]

Plugging (46) in (47) and using (45) we get

\[
\frac{1}{G' k'} = \frac{1}{H} \sum_{i=1}^{n} \frac{h_i}{G_k h L} \tag{48}
\]

and it follows:

\[
\frac{1}{k'} = \frac{1}{H} \sum_{i=1}^{n} \frac{h_i}{k_i}, \quad \frac{1}{G'} = \frac{1}{H} \sum_{i=1}^{n} \frac{h_i}{G_k h L}. \tag{49}
\]

Alternatively, using g-Forchheimer equation (35) \( g_i(u) = -k\eta_{2i} \), the upscaled formula for the g-polynomial can be obtained:

\[
g' (u) = k' \frac{1}{H} \sum_{i=1}^{n} g_i (u) \frac{h_i}{k_i} = \frac{\sum_{i=1}^{n} g_i (u) h_i}{\sum_{i=1}^{n} k_i}, \tag{50}
\]

and thus

\[
a_j' = k' \frac{1}{H} \sum_{i=1}^{n} a_i h_i \frac{1}{k_i} = \frac{\sum_{i=1}^{n} a_i h_i}{\sum_{i=1}^{n} k_i}, \quad j = 1, \ldots, l. \tag{51}
\]

It is important to mention that the obtained analytical formula (44) for \( \beta'' \) in case of the flow parallel to the layers depends on the boundary data \( \eta_1 = (p_1 - p_0)/L \) at the same time analytical formulas (51) for the flow perpendicular to the layers do not depend on
boundary data. This fact once more highlights the essential difference between the linear Darcy case in comparison to nonlinear Forchheimer flow.

5. Numerical results

In this section we numerically illustrate the upscaling algorithm described in Section 3 for the incompressible and slightly compressible fluids. All the simulations were performed using COMSOL Multiphysics. The considered cases of the permeability distribution on fine scale are presented in Fig. 4. Note, that the permeability presented in Fig. 4(c) is generated using log-normal distribution. The obtained upsampling errors are relatively small, since we did not consider large heterogeneities, but instead focused on the upsampling method for the nonlinear flow.

5.1. Numerical results for incompressible fluid

In this section we present the numerical results for incompressible fluid. Several approaches are compared: the upscaling algorithm Section 3 and the analytical formulas obtained in Section 4.

On the fine scale the pressure is subject to Eq. (10) in the region \( \Omega = [0, L_1] \times [0, L_2] \) with the boundary conditions

\[
p(0, x_2) = 0; \quad p(L_1, x_2) = 1; \quad \frac{\partial p}{\partial x_2}(x_1, 0) = \frac{\partial p}{\partial x_2}(x_1, L_2) = 0.
\]

We report the relative error in the averaged velocities originated from the upscaling of Eqs. (10)–(16):

\[
E = \frac{\|\langle u \rangle - \langle u' \rangle\|}{\|\langle u \rangle\|}.
\]

The errors for the layered system are reported in Table 1 (flow perpendicular to the layers, permeability as in Fig. 4) and Table 2 (flow parallel to the layers, permeability as in Fig. 4(b)). The results for the case of log-normal realization of \( k \) (see Fig. 4(c)) are reported in Table 3.

For each case we compare results obtained in three different approaches of averaging \( G \): 1) analytical formulas (42) for flow parallel to the system (denoted by “Av p’’); 2) numerical approach described in Section 3 formula (34) (denoted by “Num”).

It should be noticed that the tensor \( k \) is evaluated in all cases using the numerical procedure discussed in Section 3 formula (30), and never using the analytical formulas. By using (30) in the linear case (\( \beta = 0 \)), the error for stratified reservoirs (either perpendicular to the flow or parallel to the flow) is always zero, because the solution in both cases is one-dimensional and the \( k \) is taken to match the total flux in each cell exactly. Thus, the errors reported in the Tables 1, 2 arise only by the use of the upscaled parameter \( G \).

We only used the numerical evaluated \( k \) because for the linear case it gives the smallest error (zero in Tables 1, 2 and less than 2% in Table 3), and because we are mostly interested in isolating and analyzing the error arising by using three different \( G \) evaluations: parallel to the flow averaging \((G_{Av})\), perpendicular to the flow averaging \((G_{av})\), and numerical one \((G_{num})\).

It should be noticed that in Tables 1–3 each triplet in the first column (\( \beta = 0 \)) corresponds to single simulation (thus it is reported once) since \( k \) is evaluated using formula (30) and the \( G \) value is identically zero.

Our aim is to estimate the efficiency of upscaling formulas for the nonlinear parameters. The calculations are performed for two-term Forchheimer law, with nonlinear function \( G \) as in (8). In this case the coefficient \( \beta(x) \) in (8) is taken with its relative magnitude \( \Delta \beta/\beta_{min} = 1.10.100 \), where \( \Delta \beta \) is the difference between the maximum value of \( \beta(x) \) and the minimum value \( \beta_{min} \).

The coarse grid is considered to contain 20 \( \times \) 20 blocks where each of the coarse-grid block contains 20 \( \times \) 20 fine blocks, relative magnitude of the permeability is \( \|\Delta k\|/k_{min} = 10 \), where \( \Delta k \) is the difference between the maximum value of \( k(x) \) and the minimum value \( k_{min} \). For the layered system we consider both fine and coarse grids to be square. For the log-normal distribution of \( k \) three different cases are considered: \( H_1/H_2 = 0.1 \). Here \( H_1, H_2 \) and \( h_1, h_2 \) are the size of coarse and fine cells correspondingly.

For the layered system, the corresponding averaging formula gives the exact result. In general however they show different performances. Formulas derived for the parallel flow are consistently
better than those derived for perpendicular flow (Table 1, line Av =), and Table 2, line Av ↑, and Table 3).

From Tables 2 and 3 it can be seen that the accuracy of both averaging formulas and numerical approach decreases as the relative magnitude of nonlinear coefficient \( \beta \) increases. In Table 1 the relative error obtained with the average formula \( E(G_{Av}) \) does not show monotonicity with respect to \( \beta \), however it remains very small for all cases (less than 0.1%).

### 5.2. Numerical results for slightly compressible fluid

Numerical results for upscaling in case of slightly-compressible flow are presented in Tables 4–7. The coarse grid is considered to contain 4 × 4 blocks where each of the coarse-grid block contains 64 × 64 fine blocks.

On the fine scale the pressure is subject to Eq. (21) in the region \( \Omega = [0, L_1] \times [0, L_2] \) with the boundary conditions

\[
\begin{align*}
\rho (L_1, x_2) = 0; & \quad \frac{\partial p}{\partial x_1}(0, x_2) = \frac{\partial p}{\partial x_2}(x_1,0) = \frac{\partial p}{\partial x_2}(x_1, L_2) = 0.
\end{align*}
\]

(53)

We report the relative errors in the average velocity and the PI between exact and upcaled solution, given by (52) and \( |PI – PI^*|/|PI| \).

The calculations are performed for linear Darcy case and two-term Forchheimer law, with \( G(x, x_2) \) as in (8). Four distributions of the fine permeability field \( k \) are considered, see Fig. 4.

While there is no fundamental law relating porosity \( \phi \), permeability \( k \) and Forchheimer coefficient \( \beta \), there is a plethora of empirical approximations, see for example [22] and references therein. One of the commonly used relations between \( \phi \) and \( k \) is of the form

\[
\phi \sim k^{\alpha_0},
\]

where \( \alpha_0 = 0.2222, 0.2272, 0.3333, 0.1961 \), (see [23–26]). In our calculations we take \( \phi = 0.1 \cdot k^{2} \) where \( \alpha = 0.33, 0.25, 0.2 \). The review of the formulas for the Forchheimer coefficient \( \beta \) available in the literature can be found in [27]. One of the recent studies of dependence of Forchheimer coefficients on porosity can be found in [22,28]. Following [29] we take

\[
\beta = \frac{\phi}{k^{1/2}}.
\]

Numerical results show that the proposed upscaling algorithm provides small errors for the upscaled average velocity and productivity index. In particular, the relative errors are less than 5% in all cases. Though, we use a different expression for the case of compressible flow compared to incompressible flow, we observe that the velocity errors become larger for nonlinear flows.

### 5.3. Upscaling of transient problem for the slightly compressible flow

In general if the solution of (14) is not of the form (20) where the basic profile \( W(x) \) is the solution of BVP (21)–(23), the pressure \( p(x, t) \) and velocity \( u(x, t) \) are time dependent. At the end of Section 2.3 we discussed the usage of criteria (26) for the upscaling of the fully transient problem.

To evaluate if it is justified to use the upscaled parameters from steady state problem in time dependent problem we made a comparison between the productivity index on coarse and fine scale using the coefficients \( k', G' \), and \( \Phi' \) on coarse scale. Productivity

### Table 1

Upscaling errors for numerical and analytical methods, permeability of Fig. 4(a).

| \( \beta = 0 \) | \( \frac{\Delta}{\Delta_{av}} - 1 \) | \( \frac{\Delta}{\Delta_{av}} - 10 \) | \( \frac{\Delta}{\Delta_{av}} - 100 \) |
|----------------|-------------------------------|-------------------------------|-------------------------------|
| \( E(G_{Av}) \) | 8.5e−3                        | 7.9e−3                        | 8.63e−3                      |
| \( E(G_{Av}) \) | 1.87e−2                       | 1.44e−2                       | 1.08e−1                      |
| \( E(G_{Av}) \) | 6.52e−3                       | 9.65e−3                       | 1.70e−2                      |
| \( \frac{\Delta}{\Delta_{av}} - 1 \) | 1.38e−2                       | 1.52e−2                       | 2.76e−2                      |
| \( E(G_{Av}) \) | 1.35e−2                       | 6.55e−2                       | 1.20e−1                      |
| \( E(G_{Av}) \) | 9.53e−3                       | 1.25e−2                       | 1.65e−2                      |
| \( \frac{\Delta}{\Delta_{av}} - 0.1 \) | 1.86e−2                       | 2.21e−2                       | 3.87e−2                      |
| \( E(G_{Av}) \) | 1.11e−2                       | 5.59e−2                       | 9.33e−2                      |
| \( E(G_{Av}) \) | 1.09e−3                       | 1.54e−2                       | 1.99e−2                      |

### Table 2

Upscaling errors for numerical and analytical methods, permeability of Fig. 4(b).

| \( \beta = 0 \) | \( \frac{\Delta}{\Delta_{av}} - 1 \) | \( \frac{\Delta}{\Delta_{av}} - 10 \) | \( \frac{\Delta}{\Delta_{av}} - 100 \) |
|----------------|-------------------------------|-------------------------------|-------------------------------|
| \( E(G_{Av}) \) | 0                             | 0                             | 0                             |
| \( E(G_{Av}) \) | 5.36e−4                       | 2.36e−3                       | 1.7e−2                        |
| \( E(G_{Av}) \) | 0                             | 0                             | 0                             |

### Table 3

Upscaling errors for numerical and analytical methods, log-normal distribution of permeability.

| \( \frac{\Delta}{\Delta_{av}} - 1 \) | \( \frac{\Delta}{\Delta_{av}} - 10 \) | \( \frac{\Delta}{\Delta_{av}} - 100 \) |
|-------------------------------|-------------------------------|-------------------------------|
| \( E(G_{Av}) \) | 8.5e−3                        | 7.9e−3                        | 8.63e−3                      |
| \( E(G_{Av}) \) | 1.87e−2                       | 1.44e−2                       | 1.08e−1                      |
| \( E(G_{Av}) \) | 6.52e−3                       | 9.65e−3                       | 1.70e−2                      |
| \( \frac{\Delta}{\Delta_{av}} - 1 \) | 1.38e−2                       | 1.52e−2                       | 2.76e−2                      |
| \( E(G_{Av}) \) | 1.35e−2                       | 6.55e−2                       | 1.20e−1                      |
| \( E(G_{Av}) \) | 9.53e−3                       | 1.25e−2                       | 1.65e−2                      |
| \( \frac{\Delta}{\Delta_{av}} - 0.1 \) | 1.86e−2                       | 2.21e−2                       | 3.87e−2                      |
| \( E(G_{Av}) \) | 1.11e−2                       | 5.59e−2                       | 9.33e−2                      |
| \( E(G_{Av}) \) | 1.09e−3                       | 1.54e−2                       | 1.99e−2                      |
index is routinely used by engineers in estimation of available reserves and optimizing well recovery efficiency (see [13–15]). It is defined as follows. Let $p(x, t)$ be the solution of BVP in region $U$ for Eq. (14) with boundary conditions (17) and (18). The Productivity Index/Diffusive Capacity (PI) is defined as the ratio

$$J(t) = \frac{Q(t)}{\overline{pU}(t) - \overline{pR}(t)},$$

where $\overline{pU}(t) - \overline{pR}(t)$ is a pressure drawdown; and

$$\overline{pU}(t) = \frac{1}{|U|} \int_U p(x, t) \, dx, \quad \overline{pR}(t) = \frac{1}{|R|} \int_R p(x, t) \, ds.$$

Thus the coarse coefficients $k$, $G$, and $\Phi$, obtained for the steady state equation (21), can be used to calculate fully transient productivity index on coarse scale.

Numerical experiment confirms the discussion above. We solve time dependent equation (14) with boundary conditions (53) on fine and coarse scales. The fine scale permeability $k$ is taken to be as in Fig. 4(c). The coarse scale permeability is taken to be $k'$ obtained by upsampling the PSS equation (see Section 5.2). Figs. 5 and 6 present the time dependence of velocity and the PI on coarse and fine scales. The time dependent values are also compared to the PSS values, which are constant in time. As it can be seen from the graphs, in the long term the coarse scale time dependent velocity and PI calculated using the upsampled parameters from the steady state problem provide good approximation of the corresponding fine scale values.

Fig. 5. Time dependence of the average velocity on fine and coarse scales: I is PSS ($u_i$) on the fine scale; II is PSS ($u_i$) on the coarse scale; III is ($u(t)$) on the fine scale; IV is $u(t)$ on the coarse scale.

6. Conclusions

- The developed upscaling algorithm for nonlinear steady state problems can be effectively used for $\gamma$-Laplacian type equations of the form (10) and (21) and for variety of heterogeneities in the domain of computation.
- The coarse scale parameters $k$, $G$, and $\Phi$ are determined so that the volumetric average of velocity of the flow in the domain on fine scale and on coarse scale are close enough.
- The numerical results show that the proposed method can be used to approximate the productivity index (PI) of the well in the bounded domain on the coarse scale.
- Analytical upscaling formulas in stratified domain are obtained for the nonlinear case.
- In our results for the nonlinear problems, the upscaled parameters depend on the range of boundary data.
- Our results on asymptotic behavior of fully transient velocity and PI and actual numerical computations justify the usage of the coarse scale parameters $k$, $G$, and $\Phi$ obtained for the steady state case in the fully transient problem [14].

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Appendix A

The results in the paper use constant upscaled permeability fields which are more relevant in practical setups. These results can be improved by adding extra degrees of freedom such as dividing the coarse grid block into smaller coarse regions or using higher order interpolation. In this appendix, we show a use of higher order interpolation which gives an improved results (see Tables 8–11).

Here we consider the upscaled $k'$ and $\Phi'$ to be of a form

$$k' = \frac{K_{11}}{K_{21}} \frac{K_{12}}{K_{22}},$$

where $K = K_0 + K_1(x_1 - c_1) + K_2(x_2 - c_2);$$$

$$\Phi' = \Phi_0 + \Phi_1(x_1 - c_1) + \Phi_2(x_2 - c_2) + \Phi_0[(x_1 - c_1)^2 - (x_2 - c_2)^2].$$
Here $c = (c_1, c_2)$ is the central point of the coarse cell $\Omega_c$ and $K_1, K_{12}, K_{21}, K_0, K_1, K_2, \Phi_0, \Phi_1, \Phi_2$ and $\Phi_c$ are constants to be determined.

First, the permeability tensor $k'$ is determined, and then it is used to upscale the porosity $\phi$.

I. Permeability $k'$. We first obtain the polynomial $K$ in (A.2) as the least square approximation of permeability $k(x_1, x_2)$:

$$K_0 = \frac{1}{|\Omega_c|} \int_{\Omega_c} k(x_1, x_2) \, d\Omega_c;$$

$$K_i = \frac{\int_{\Omega_c} k(x_1, x_2)(x_1 - c_i) \, d\Omega_c}{\int_{\Omega_c} (x_1 - c_i)^2 \, d\Omega_c}, \quad i = 1, 2. \quad (A.4)$$

We will now use the constant elements $K_{11}, K_{12}, K_{21}$ and $K_{22}$ of matrix $k'$ to “correct” $K(x_1, x_2)$ so that the average velocities on
the fine and coarse scales are the same. For this purpose we will modify our approach presented in Section 3. We consider two linear fine-scale equations with zero RHS, the exact Eq. (28) and the averaged equation
\[
\nabla \cdot (\langle R(x_1, x_2) \rangle \nabla p) = 0.
\]
(A.5)

Each equation is solved twice. Namely, let \( p_1 \) and \( p_2 \) be the solutions of (28) and (A.5), correspondingly, subject to boundary conditions (29a) and let \( p_3 \) and \( p_4 \) be the solutions of (28) and (A.5), correspondingly, subject to boundary conditions (29b).

We now equate the velocity averages on fine and coarse scale, with the coarse scale velocity \(-\mathbf{K} \nabla \langle p \rangle, i = 1, 2\), “corrected” with the elements of the matrix \( \mathbf{k}' \):
\[
\frac{1}{|\Omega|} \int_{\Omega} -k \nabla p_1 \, d\Omega = \frac{1}{|\Omega|} \int_{\Omega} -k \nabla p_c \, d\Omega, \quad i = 1, 2.
\]
(A.6)

Solving the linear system of four Eqs. (A.6) gives the values of \( K_{11}, K_{12}, K_{21}, K_{22} \).

II. Porosity \( \Phi \). To find the coefficients \( \Phi_A, \Phi_B, \Phi_C \) and \( \Phi_D \) determined so that the boundary flux \( k \nabla p \) through the faces of the coarse cell corresponding to the up-scaled solution \( p \) and fine scale solution \( p_c \) are equal. Due to the linearity of Eq. (A.7) solution \( p \) is the linear combination
\[
P = \Phi_A \cdot P_A + \Phi_B \cdot P_B + \Phi_C \cdot P_C + \Phi_D \cdot P_D,
\]
(A.9)

where \( P_a = A, B, C, D \), are the solutions of BVP
\[
-k \nabla \mathbf{f}(x_1, x_2) = \Phi_i(x_1, x_2),
\]
(A.10)

with the RHS
\[
f_i(x_1, x_2) = \begin{cases} 
1 & \text{for } \lambda = A, \\
-x_i - c_i & \text{for } \lambda = B, \\
x_2 - c_2 & \text{for } \lambda = C, \\
(x_1 - c_1)^2 - (x_2 - c_2)^2 & \text{for } \lambda = D.
\end{cases}
\]

The boundary fluxes through four faces of the coarse cell are then related by the same expression as (A.9). Solving the resulting system of four equations for \( \Phi_i, \lambda = A, B, C, D \), we obtain the expression for \( \Phi' \).

The corresponding numerical results are presented in Tables 8–11. For the reader’s convenience the results presented in Tables 4–7 are included here once again for the comparison. We compare errors for the upscaled algorithm with \( k \) and \( \Phi' \) calculated using different approaches described above. Four cases are considered for each of linear and nonlinear case.

(i) \( k' \) and \( \Phi' \) are calculated as in Section 3 (denoted as \( k' - C, \Phi' - C \));
(ii) \( k' \) is as in (41), while \( \Phi' \) is a constant (31) (denoted as \( k' - P, \Phi' - C \));
(iii) \( k' \) is as in Section 3, while \( \Phi' \) is as in (A.3) (denoted as \( k' - C, \Phi' - P \));
(iv) \( k' \) is as in (41) and \( \Phi' \) is as in (A.3) (denoted as \( k' - P, \Phi' - P \)).

Approach (iv) provides consistently better results for both error in PI and velocity. Though it is computationally more expensive, the increase is negligible in nonlinear case where the main computational expense comes from obtaining the function \( \Phi' \). Approach (ii) is routinely comparable to (iv), and for some permeability distributions (\( k \) as in Fig. 4(a), \( \tau = 1/4, 1/5 \)) is even better. With that it only amounts to computation of coefficients (A.4). As expected, approach (iii) is routinely the worst, as there is not sufficient information for correct estimation of coefficients of polynomial (A.3).

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