Local Correlations in the Super Tonks–Girardeau Gas

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We study the local correlations in the super Tonks–Girardeau gas, a highly excited, strongly correlated state obtained in quasi one-dimensional Bose gases by tuning the scattering length to large negative values using a confinement-induced resonance. Exploiting a connection with a relativistic field theory, we obtain results for the two-body and three-body local correlators at zero and finite temperature. At zero temperature our result for the three-body correlator agrees with the extension of the results of Cheianov et al. [Phys. Rev. A 73, 051604(R) (2006)], obtained for the ground–state of the repulsive Lieb–Liniger gas, to the super Tonks–Girardeau state. At finite temperature we obtain that the three-body correlator has a weak dependence on the temperature up to the degeneracy temperature $T_D$. We also find that for temperatures larger than $T_D$ the values of the three-body correlator for the super Tonks–Girardeau gas and the corresponding repulsive Lieb–Liniger gas are rather similar even for relatively small couplings.

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I. INTRODUCTION

The experimental capability of tailoring tightly confined trapping potentials opened the way to realizing one-dimensional (1D) interacting systems with ultracold atoms [1, 2]: when the transverse motion of atoms is confined to zero-point oscillations, the effective Hamiltonian describing their equilibrium properties and their dynamics is 1D. It is then possible to experimentally simulate paradigmatic 1D many-body models: 1D interacting Fermi gases provide an experimental realization of the Gaudin–Yang model [3], while 1D Bose gases are very well described by the Lieb–Liniger (LL) model (see the reviews [4, 5]). These models have been theoretically studied for decades since they display a rich variety of non mean-field features: the study of their equilibrium properties motivated the developments of analytical and numerical techniques and, at the same time, they were used as a benchmark for testing non-perturbative techniques.

The experimental realization of the LL model with ultracold bosons not only renewed the interest in the equilibrium and dynamical properties of the model, but also called for the study of its excited, strongly correlated states. Indeed, tuning the effective 1D scattering length it is possible to vary the coupling constant $\gamma$ of the LL model: $\gamma$ can be made very large close to a confinement-induced resonance [6]. For large positive $\gamma$ one approaches the Tonks–Girardeau (TG) limit [7], while suddenly switching to the other side of the resonance one can prepare a highly excited many-body state, the super Tonks–Girardeau (STG) state [8]. Exploiting a confinement-induced resonance and using a gas of Cesium atoms, the STG state has been recently realized [9]: the crossover from the TG to the STG regime has been studied by determining the collective mode frequencies and the dynamics through the crossover has been characterized by measuring the particle loss and the expansion [9].

Several properties of the STG gas have been discussed recently in the literature [8–12]: in [10] it was shown that the STG gas corresponds to a highly excited state in the Bethe ansatz solution of the LL model with attractive interactions (i.e., negative $\gamma$), characterized by real Bethe roots. In [11] the connection between the STG gas and a 1D hard sphere Bose gas (with hard sphere diameter almost equal to the 1D scattering length) was discussed. The realization of effective STG gases in strongly attractive 1D Fermi gases has also been proposed and investigated [13, 14]. The dynamics in the crossover between the TG and the STG gases when the coupling constant $\gamma$ is suddenly quenched from a positive value (in which the system is in the ground state) to a negative value was considered in [12, 15]: after the quench the system is in a metastable state, however, solving the exact dynamics it is possible to see that such a state can be stable for rather long times ($\sim 100\text{ms}$). The metastability of the STG gas can be understood in terms of the very small overlap between the ground state of the TG gas and the collapsed cluster states.

In this paper we focus on the computation of local two- and three-body correlations of the STG gas, as these important quantities determine the rates of inelastic processes, such as photoassociation in pair collisions and three-body recombination. We use a recently introduced method [16] which allows for the determination of equilibrium expectation values in the repulsive LL gas: extending this approach to the STG state, we can compute two- and three-body correlations in such a highly excited, strongly correlated state both at zero and finite temperature. At $T = 0$ we find that our results for the three-body
correlator \( g_3 \) are in agreement with the extension to the STG state of the results of [17], obtained for the ground-state of the repulsive Lieb–Liniger gas. At finite temperature, \( g_3 \) displays for intermediate values of the coupling constant a weak dependence on the temperature up to the degeneracy temperature \( T_D \); furthermore for \( T \gtrsim T_D \) the ratio between the three-body correlation of the STG gas for a coupling constant \( \gamma \) (with \( \gamma < 0 \)) and the corresponding value computed at equilibrium for the LL gas with coupling constant \( |\gamma| \) is rather close to one even for relatively small values of \( |\gamma| \) (i.e. for \( |\gamma| \gtrsim 20 \)).

The plan of the paper is the following: in Section II we review basic facts of the LL model and introduce the STG state, while Section III is devoted to the computation of local expectation values in the STG gas. In Section IV we show our results for zero temperature, and the findings for finite temperature are discussed in Section V. In Section VI we present our conclusions.

II. THE SUPER TONKS–GIRARDEAU GAS

The LL Hamiltonian describes \( N \) non-relativistic bosons of mass \( m \) in one dimension, interacting via a two-body \( \delta \)-potential [18]:

\[
H = -\frac{\hbar^2}{2m} \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + 2\lambda \sum_{i<j} \delta(x_i - x_j) \tag{1}
\]

The coupling \( \lambda \) of the 1D \( \delta \)-function contact potential in the Hamiltonian (1) can be expressed in terms of the parameters of the three-dimensional Bose gas [6] as

\[
\lambda = \frac{\hbar^2 a_{3D}}{ma_\perp^2} \frac{1}{1 - Ca_{3D}/a_\perp}, \tag{2}
\]

where \( a_{3D} \) is the three-dimensional scattering length of the Bose gas [1, 2], \( a_\perp = \sqrt{\hbar/m\omega_\perp} \) is the harmonic oscillator length of the transverse confinement with trap frequency \( \omega_\perp \) and \( C \approx 1.0326 \) is a constant. The effective coupling constant of the LL model is given by the dimensionless quantity

\[
\gamma = \frac{2m\lambda}{\hbar^2 n}, \tag{3}
\]

where \( n = N/L \) is the density of the gas (\( L \) is the length of the system).

The limit \( \gamma \ll 1 \) is the weak coupling limit where the Bogoliubov approximation gives a good estimate of the ground–state energy of the system [18]. For large positive \( \gamma \) one approaches the TG limit [7], where the combined effect of the reduced dimensionality and the strong repulsion leads to an effective Pauli exclusion and the ground–state wave function can be mapped to the wave function of free fermions. A value of \( \gamma \approx 5 \) was reached in [19], while in [20], using an additional shallow optical lattice along the longitudinal direction, effective values \( \gamma \approx 200 \) for \( \gamma \) were achieved. As one can see from (2), by tuning \( a_{3D} \sim a_\perp \) one can have large \( \gamma \) and pass from positive to negative values of \( \gamma \) [6, 9].

In the LL model temperatures are usually expressed in units of the quantum degeneracy temperature

\[
k_B T_D = \frac{\hbar^2 n^2}{2m}, \tag{4}
\]

in the following we use the scaled temperature

\[
\tau = \frac{T}{T_D}. \tag{5}
\]

Notice that defining the thermal De Broglie wavelength \( \lambda_T = \sqrt{2\pi\hbar^2/mk_BT} \) [1, 2], one has \( n\lambda_T = 2\sqrt{\tau} \), showing that for \( T \sim T_D \) degeneracy effects arise. For 1D tubes having \( N \sim 100 \) atoms and size \( L \sim 10\mu m \) (corresponding to longitudinal frequencies \( \omega_z \sim 2\pi \times (1 - 5) \) Hz), the degeneracy temperature \( T_D \) is \( \sim 300nK \). For a transverse trapping confinement \( \omega_\perp \sim 2\pi \times 5kHz \), one has \( \hbar\omega_\perp/2 \sim k_B \times 100nK \), and then scaled temperatures as low as \( \tau \sim 0.3 \) are realistically reachable.

As shown by Lieb and Liniger in their original paper [18], the eigenvalue problem of the Hamiltonian (1) can be solved in terms of a coordinate Bethe ansatz. The equations of motion are just free Schrödinger equations in the domain where the coordinates of the particles are all distinct. If we denote by \( R_1 \) the
subset of the configuration space where \( x_1 < x_2 < \cdots < x_N \), the solution of the equations in \( R_1 \) is given by the Bethe wave function
\[
\chi_N(x_1, x_2, \ldots, x_N) = \sum_P a(P) e^{i \sum_{j=1}^N P(k_j)x_j},
\]
where \( \sum_P \) denotes a sum over permutations of the wavevectors \( \{k_1, \ldots, k_n\} \) characterizing the state. For configurations outside \( R_1 \) the solution is easily obtained using the symmetry of \( \chi_N \) with respect to the \( x_i \). For the permutations \( P : (k, l, k_{\alpha_3}, \ldots, k_{\alpha_N}) \) and \( Q : (l, k, k_{\alpha_3}, \ldots, k_{\alpha_N}) \) the relation between the coefficients in the sum appearing in (6) is given by
\[
a(Q) = \frac{k - l - ik}{k - l + ik} a(P),
\]
where \( \kappa = \frac{2m}{\gamma} = n\gamma \). Hence the wave function gets multiplied by the factor \( a(Q)/a(P) \) whenever two particles with momenta \( p_1 = \hbar k \) and \( p_2 = \hbar l \) are exchanged, therefore the two-body \( S \)-matrix of the Lieb–Liniger model is
\[
S_{LL}(k, \lambda) = \frac{k - i\gamma n}{k + i\gamma n},
\]
where \( \hbar k = \hbar k_1 - \hbar k_2 \) is the momentum difference.

With periodic boundary conditions the momenta in (6) are constrained by the Bethe equations
\[
e^{ik_i L} = - \prod_{i=1}^N \frac{k_j - k_i + i\gamma n}{k_j - k_i - i\gamma n}, \quad j = 1, \ldots, N,
\]
and the ground–state energy is given by \( E = \frac{k^2}{2m} \sum_{j=1}^N k_j^2 \). The roots of these equations for the ground state are all real for \( \gamma > 0 \) [21] and the energy for large coupling \( \gamma \gg 1 \) is given by
\[
\frac{E}{N} \approx \frac{\pi^2 \hbar^2 n^2}{6m} \left( 1 + \frac{2}{\gamma} \right)^{-2}.
\]
The thermodynamical properties of the LL model for \( \gamma > 0 \) can be obtained by the thermodynamic Bethe ansatz (TBA), as shown originally by Yang and Yang [22]: in Appendix A we summarize the TBA equations for the repulsive LL gas.

For negative coupling, \( \gamma < 0 \), the ground–state is a cluster-like state having complex Bethe roots with energy [23, 24]
\[
E = -\frac{\pi^2 n^2}{12} N(N^2 - 1).
\]
Due to the attraction, the particles tend to collapse in the same region of the space and the system does not have a well-defined thermodynamic limit: the ground–state energy per particle \( E/N \to -\infty \) for \( N \to \infty \).

However, the existence of a stable gas-like state has been proposed [8]: the STG state. This state is an eigenstate of the attractive \( (\gamma < 0) \) LL model characterized by Bethe roots which are all real. It was shown [10] that this is a highly excited state in the attractive regime which is, however, stable in a wide range of coupling strength. As it was recently confirmed in the experiment [9], the STG state can be created from a TG gas by an abrupt change of the sign of the interaction. The large kinetic energy inherited from the TG acts like a Fermi pressure and it cannot be quenched instantly. Moreover, for large coupling strengths the two wave functions are almost identical and their overlap is very large. A detailed analysis of the root distribution can be found in [10, 12]: it is found that the configuration of the real roots corresponds to the roots of the repulsive LL gas but with the sign of the coupling changed.

These results suggest a simple way to calculate the energy and other quantities for the STG gas: one can use the equations and formulae for the repulsive \( (\gamma > 0) \) LL gas for \( \gamma < 0 \). In this way in [8, 10] the following asymptotic expression for the energy of the STG gas with coupling constant \( -|\gamma| \) (with \( |\gamma| \gg 1 \)) was obtained:
\[
e(\gamma < 0) \approx \frac{\pi^2}{3} \left( 1 - \frac{2}{|\gamma|} \right)^{-2},
\]
where \( \pi^2 \approx 10 \).
where \( e(\gamma < 0) = E(\gamma < 0)/N k_B T_D \) [see eqn (A7)]. The corresponding asymptotic expression for the ground-state energy of a repulsive LL gas with coupling constant \( \gamma > 0 \) is (for \( \gamma \gg 1 \))
\[
e(\gamma) \approx \frac{\pi^2}{3} \left(1 + \frac{2}{\gamma} \right),
\]
as can be seen directly from eqn (10).

The energy of the STG gas at finite values of \(-|\gamma|\) can be obtained at \( T = 0 \) using the integral equations reported in Appendix A: in these equations, valid for the ground-state of the LL model with \( \gamma > 0 \), one has to replace \( \gamma \) with \(-|\gamma|\). The result is plotted in Fig. 1, where we show for comparison the exact energies calculated from the LL integral equations for the STG and LL gases.

To simplify the notations in this figure as well as in the following figures and sections, for the STG results the coupling \( \gamma \) should be understood as the positive parameter \(-\gamma = |\gamma|\), while the corresponding LL results refer to equilibrium values for the LL gas with \( \gamma > 0 \).

### III. LOCAL CORRELATORS IN THE SUPER TONKS–GIRARDEAU GAS

In this section we present a discussion of the way to compute local correlators in the STG gas both at zero and finite temperatures. The quantities we are interested in are defined as
\[
g_k = \left\langle \psi^{\dagger} k \psi^k \right\rangle_{nk}, \tag{13}
\]
where \( k = 1, 2, \ldots \). Equilibrium local correlators for the repulsive LL gas were discussed in several papers: exact results based on the Yang–Yang equations and the Hellmann–Feynman theorem are available for two-body correlations [25, 26], while the local three-body correlation \( g_3 \) was determined at zero temperature in [17]. Asymptotic expressions for \( g_k \) for large and small \( \gamma \) were presented in [25]. Results for \( g_3 \) at finite temperature were obtained in [16].

To compute the correlators \( g_k \) in the STG we use the method recently introduced in [16]: this approach is based upon the fact that in \((1+1)\) dimensions the repulsive LL model can be obtained as a suitable non-relativistic limit of an integrable relativistic quantum field theory, the sinh–Gordon (sh-G) model. This model is defined by the Lagrangian density
\[
\mathcal{L}_{sh-G} = \frac{1}{2} \left[ \left( \frac{\partial \phi}{\partial t} \right)^2 - (\nabla \phi)^2 \right] - \frac{\mu^2}{g^2} \cosh(g \phi), \tag{14}
\]
where \( \phi = \phi(x,t) \) is a real scalar field, \( c \) is the speed of light and the parameter \( \mu \) is related to the physical mass \( m \) by \( \mu^2 = \pi a m^2 c^2 / \hbar^2 \sin(\pi \alpha) \), where \( \alpha = hc g^2 / (8\pi + hc g^2) \) [27]. The energy \( E \) and the momentum \( P \) of a particle can be written as \( E = mc^2 \cosh \theta, P = mc \sinh \theta \), where \( \theta \) is the rapidity. Since the sh-G dynamics is ruled by an infinite number of conservation laws, all its scattering processes are purely elastic and can be factorized in terms of the two-body S-matrix [27]
\[
S_{sh-G}(\theta, \alpha) = \frac{\sinh \theta - i \sin(\alpha \pi)}{\sinh \theta + i \sin(\alpha \pi)}, \tag{15}
\]
integer. means taking its finite part, i.e. omitting all the terms of the form $\langle \leftarrow \theta \rangle = 1 \theta_{conn} = \theta_{conn}$. This formula contains both the pseudo-energy $\epsilon(\theta)$ satisfying the TBA equations of the sh-G model and the connected diagonal form factor of the operator $\mathcal{O}$ at temperature $T$ and at finite density is given by
\begin{equation}
\langle \mathcal{O} \rangle = \frac{\text{Tr} \left( e^{-(H-\mu N)/(kB T)} \mathcal{O} \right)}{\text{Tr} \left( e^{-(H-\mu N)/(kB T)} \right)} .
\end{equation}
In a relativistic integrable model the above quantity can be expressed as [29]
\begin{equation}
\langle \mathcal{O} \rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{-\infty}^{\infty} \left( \prod_{i=1}^{n} \frac{d\theta_i}{2\pi} f(\theta_i) \right) \langle \mathcal{O}(0) | \tilde{\theta} \text{conn} \rangle ,
\end{equation}
where $f(\theta_i) = 1/(1 + e^{\epsilon(\theta_i)})$ and $\tilde{\theta} \equiv \theta_1, \ldots, \theta_n$ (\tilde{\theta} \equiv \theta_{n+1}, \ldots, \theta_1) denote the asymptotic states entering the traces in (18). This formula contains both the pseudo-energy $\epsilon(\theta)$ satisfying the TBA equations of the sh-G model and the connected diagonal form factor of the operator $\mathcal{O}$. The latter is defined as $\langle \mathcal{O}(0) | \tilde{\theta} \text{conn} \rangle = \mathcal{F}(\lim_{n_1 \to 0} \langle 0 | \mathcal{O} | \tilde{\theta} \rangle, \tilde{\theta} = \theta_{n+1}, \ldots, \theta_1)$ where $\tilde{\eta} \equiv \eta_{n+1}, \ldots, \eta_1$ and $\mathcal{F}$ in front of the expression means taking its finite part, i.e. omitting all the terms of the form $\eta_i/\eta_j$ and $1/n^p$ where $p$ is a positive integer.

Since the form factors in the sh-G model are exactly known [30], one can compute local correlators in the LL model [16, 28]:
\begin{equation}
\langle \psi^k | \psi^k \rangle = \left( \frac{2k}{k} \right)^{-1} \left( \frac{\hbar^2}{2m} \right)^{-k} \sum_{n=1}^{\infty} \frac{1}{n!} \int_{-\infty}^{\infty} \left( \prod_{i=1}^{n} \frac{d\theta}{2\pi} f(\theta_i) \right) \tilde{F}_{2n,conn}^k(p_1, \ldots, p_n) .
\end{equation}
Here $f(p) = 1/(1 + e^{\epsilon(p)})$ where $\epsilon(p)$ is the solution of the non-relativistic TBA equations and
\begin{equation}
\tilde{F}_{2n,conn}^k(p_i) = \lim_{c \to 0, \gamma \to 0} \left( \frac{1}{mc} \right)^n \tilde{F}_{2n,conn}^k(\{ \theta_i = \frac{p_i}{mc} \})
\end{equation}

FIG. 2: Difference between our evaluation of $g_\perp(\gamma)$ and the exact value $g_\perp = 1$ as a function of $\gamma$ at $T = 0$ for the STG gas. The solid lines correspond (from top to bottom) to the evaluation of $g_\perp$ using the first one, two and three terms in the series eqn (20). Inset: the same quantity as a function of the scaled temperature $\tau$ for $\gamma = 10$. where $\theta$ is the rapidity difference of the two particles. It is then easy to see that taking simultaneously the non-relativistic and weak-coupling limits of the sh-G model such that
\begin{equation}
g \to 0, \; c \to \infty, \; gc = 4\sqrt{\lambda}/\hbar = \text{fixed} ,
\end{equation}
its $S$-matrix (15) becomes identical to the $S$-matrix (8) of the LL model [16, 28]. Notice that the coupling $\lambda$ does not need to be small, i.e. with this mapping one can study the LL model at arbitrarily large values of the dimensionless coupling $\gamma$. The mapping between the two models goes beyond the identity of their $S$-matrix: it extends both to their Lagrangians and TBA equations (details can be found in [28]).

To perform the non-relativistic limit of the sh-G model, one has to express the real scalar field in the form
\begin{equation}
\phi(x, t) = \sqrt{\frac{\hbar^2}{2m}} \left( \psi(x, t) e^{-i\frac{mc^2}{\hbar^2} t} + \psi^\dagger(x, t) e^{i\frac{mc^2}{\hbar^2} t} \right) ,
\end{equation}
and, when the limit $c \to \infty$ of the Lagrangian (or other expressions of $\phi$) is taken, of omitting all the oscillating terms [16, 28]. At equilibrium the expectation value of an operator $\mathcal{O} = \mathcal{O}(x)$ at temperature $T$ and at finite density is given by
\begin{equation}
\langle \mathcal{O} \rangle = \text{Tr} \left( e^{-(H-\mu N)/(kB T)} \mathcal{O} \right) = \frac{\text{Tr} \left( e^{-(H-\mu N)/(kB T)} \right)}{\text{Tr} \left( e^{-(H-\mu N)/(kB T)} \right)} .
\end{equation}
FIG. 3: Two-body correlator \( g_2 \) as a function of \( \gamma \) at \( T = 0 \) for the STG gas. The solid line corresponds to the result obtained using the Hellmann–Feynman theorem (22), while the dashed lines (a), (b) and (c) are the results taking the first one, two and three terms in the series (20). The dotted lines (from the right) correspond to the asymptotic expansions (26) and (28).

The double limit (16) of the connected form factors. Note that this is a completely non-relativistic formula where only the form factors (21) have their origin in the sh-G model. However, following the arguments of [31] about the connection between the non-relativistic and relativistic form factors, the quantities (21) should also be derived solely from the LL model.

Eqn (20) allows for the computation of the local correlators at equilibrium for the repulsive LL model, once the TBA equations with positive \( \gamma \) are solved for the pseudo-energy \( \varepsilon \) and the solution is inserted in (20) together with the form factor defined by (21). The way the method needs to be modified for negative \( \gamma \) depends on the state for which we want to compute expectation values. If one is interested in computing the local correlators in the ground-state of the attractive LL model, one should use as a starting relativistic field theory the sine–Gordon model [32, 33] instead of the sh-G model. In the cold atom setup [9] this would correspond to a very slow, adiabatic switch from the positive to the negative side of the confinement-induced resonance such that the ground-state is reached. For a sudden switch when the meta-stable STG state is obtained, the Bethe roots remain real and their positions can be obtained from the Bethe ansatz equations of the LL model with negative \( \gamma \). Since all the expectation values finally depend only on the Bethe roots, they can consistently be obtained by changing the sign of the coupling constant. Correspondingly our formula eqn (20) has to be used with negative \( \gamma \). Of course, to be consistent one has to use the solutions of the TBA equations and the form factors with negative \( \gamma \).

Eqn (20) provides a series expansion in terms of the form factors. For small values of \( |\gamma| \) we would have to sum many terms of the series. However, in the case of the STG gas we are interested in large values of \( |\gamma| \) and as we discuss it in the next section, the series is rapidly converging and already the first few terms give very accurate results.

In the next two sections we present the results for the STG local correlators and we compare them with available results from the literature. The two-body correlator \( g_2 \) can be exactly determined [25, 26] via the Hellmann–Feynman theorem:

\[
\langle \psi^\dagger \psi^\dagger \psi \psi \rangle = \frac{1}{L} \frac{dH}{d\lambda} = \frac{d}{d\lambda} \left( \frac{E}{L} \right) \implies g_2 = \frac{de}{d\gamma}.
\]  

(22)

For \( T > 0 \) the theorem gives

\[
\langle \psi^\dagger \psi^\dagger \psi \psi \rangle = \frac{d}{d\lambda} \left( \frac{F}{L} \right),
\]  

(23)

where the free energy \( F \) can be calculated from the TBA approach [see eqn (A4b)]. In dimensionless variables

\[
g_2(\gamma, \tau) = \tau \frac{d}{d\gamma} \left( \alpha - \int_{-\infty}^{\infty} \frac{dq}{2\pi} \log(1 + e^{-\varepsilon(q)}) \right),
\]  

(24)

where \( \alpha = \mu/k_B T \) with \( \mu \) being the chemical potential. In [28] a compact form without derivatives was
FIG. 4: Three-body correlator $g_3$ as a function of $\gamma$ at $T = 0$ for the STG gas (we also plot for comparison the corresponding LL result). The STG solid line corresponds to the extension of $[17]$ to negative values of $\gamma$, while the LL solid line is the plot of $[17]$ for $\gamma > 0$. The two dashed lines are the results taking the first one and two terms in the series $[20]$. The dotted line corresponds to the large $\gamma$ asymptotic result $[27]$.

derived

$$g_2 = \frac{2}{\gamma} \int_{-\infty}^{\infty} dq \frac{g(q)}{1 + e^{\varepsilon(q)}} q^2 - \frac{\tau}{2\pi} \int_{-\infty}^{\infty} dq \log(1 + e^{-\varepsilon(q)}) .$$

(25)

As previously discussed, the above equations are also valid for the STG gas provided that $\gamma$ (and $\lambda$ and $\beta$ in Appendix A) are chosen to be negative.

For large $\gamma$ there are asymptotic expansions in the literature $[16, 25]$; e.g., for $g_2(\gamma)$ at $T = 0$ we have from (12) and the Hellmann–Feynman theorem

$$g_2(\gamma) \approx \frac{4\pi^2}{3\gamma^2} \left( 1 + \frac{2}{\gamma} \right)^{-3},$$

(26)

while for $g_3(\gamma)$ we will use the leading order result of $[25]$ which is invariant under the sign change $\gamma \rightarrow -\gamma$:

$$g_3(\gamma) \approx \frac{16\pi^6}{15\gamma^6}.$$

(27)

Let us recall again our convention described at the end of Section II that for the STG results $\gamma$ denotes the absolute value of the effective coupling constant.

IV. RESULTS FOR $T = 0$

In this section we present our results for the two- and three-body correlators at zero temperature obtained by the method discussed in the previous section. To test the convergence of our series in Fig. 2 we show the difference between our evaluation of $g_1(\gamma)$ and the exact value $g_1 = \langle \psi^\dagger \psi \rangle / n = 1$. Although the convergence is slower than for the repulsive LL gas, for $\gamma = 10$ the error using the first three terms in the series expansion is around 1%.

In Fig. 3 we plot our results for $g_2(\gamma)$ at $T = 0$. Our findings are compared with the exact result obtained using the Hellmann–Feynman theorem (22) and with the asymptotic expansions (26). We also plot the more accurate third order strong-coupling expansion obtained in [16]:

$$g_2 = \frac{4\pi^2}{3\gamma^2} \left( 1 + \frac{6}{\gamma} + \left( 24 - \frac{8}{9}\pi^2 \right) \frac{1}{\gamma^2} \right).$$

(28)

For intermediate values of $\gamma$ (in the region $5 \lesssim \gamma \lesssim 25$) our results are much better than the asymptotic results. A comparison with the repulsive LL results shows also that $g_2$ for the STG gas in the intermediate region is significantly larger than the corresponding quantity for the LL gas.

We show the results for $g_3(\gamma)$ at $T = 0$ in Fig. 4. Our results are plotted together with the result of $[17]$ for the repulsive case. We also show the results for $g_3$ obtained by extending the formula for $g_3$ of $[17]$...
to negative values of $\gamma$, corresponding to the STG gas. One can see that the first two terms of the series 
20 are in excellent agreement with the findings obtained from [17] already for $\gamma \gtrsim 10$. The comparison 
between the STG and the LL gas shows also that the former is much more subjected to three-body 
recombination: at $\gamma \sim 30$, for example, $g_{3}^{\text{(STG)}}/g_{3}^{\text{(LL)}} \sim 3$.

V. RESULTS FOR $T > 0$

In this section we show our finite temperature results for $g_2$ and $g_3$. To test the reliability of the series 
expansion (20) at finite temperature, we plot in the inset of Fig. 2 the deviations $1 - g_1(\gamma)$ from the exact 
result $g_1 = 1$ for a fixed value of $\gamma$. It is interesting to observe that for large coupling the convergence is 
faster than the LL result (not shown there). From the figure we can also see that the convergence slightly 
improves with increasing temperature.

In Fig. 5 $g_2$ is plotted as a function of the scaled temperature for three different values of $\gamma$, together 
with the results obtained from the Hellmann–Feynman theorem (23) which agree very well with our 
expansion. It can be seen in Fig. 5 that $g_2$ is approximately constant for temperatures $T \lesssim T_D$; e.g. for 
$\gamma = 100$ we find $g_2(\tau = 1)/g_2(\tau = 0) \approx 1.1$. For higher values of $\gamma$ the temperature effects get relevant at 
slightly lower temperatures. Comparing $g_2(\tau)$ for the STG and LL gases it is also possible to see that for 
the STG gas temperature effects become important at slightly higher temperatures than for the LL gas.

In Figs. 6-7 we finally plot our results for $g_3$ for the STG gas at finite temperature. In Fig. 6 we show 
g_3 for the STG gas as a function of $\gamma$ for two different scaled temperatures $\tau = 1$ and $\tau = 10$. In the 
inset we plot $g_3$ for the STG and LL gases as a function of $\tau$ for $\gamma = 10$ and $\gamma = 30$: it is clear that for 
the STG gas $g_3$ is larger than the corresponding LL value and that in both cases temperature effects are 
present at $T \gtrsim T_D$ (for the same $\gamma$ significant deviations from the $T = 0$ results start at slightly higher 
temperatures for the STG gas).

In Fig. 7 we plot the ratio $g_{3}^{\text{(STG)}}/g_{3}^{\text{(LL)}}$ between the three-body correlators of the STG and LL gases 
as functions of $\gamma$ for three different scaled temperatures. In the inset of the same figure $g_{3}^{\text{(STG)}}/g_{3}^{\text{(LL)}}$ is 
plotted for two values of $\gamma$ as a function of $\tau$. From Fig. 7 one can see that for $T = T_D$ ($\tau = 1$) this ratio 
is very large for intermediate values of the coupling and it is not very different from the $T = 0$ result. For 
large values of $\gamma$ the ratio becomes smaller: at $\gamma = 30$ the ratio is $\sim 3$ for $\tau = 1$ and $\sim 1.5$ for $\tau = 10$.

VI. CONCLUSIONS

We studied the local correlations in the super Tonks–Girardeau gas, a highly excited, strongly correlated 
state obtained in quasi one-dimensional Bose gases when the scattering length is tuned to large negative 
values using a confinement-induced resonance. After introducing the Lieb–Liniger model we discussed 
the main properties of the super Tonks–Girardeau gas which was recently realized in the experiment 
reported in [9]. We focused on the computation of the local correlators: using a relation with a relativistic 
field theory, we obtained results for the two-body and three-body local correlators at zero and finite 
temperature. At $T = 0$ we showed that the three-body correlator agrees with the extension of the results
FIG. 6: Three-body correlator $g_3$ for the STG gas as a function of $\gamma$ for two different scaled temperatures $\tau = 1, 10$. The two dashed lines for each value of $\tau$ are the results taking the first one (top) and two (bottom) terms in the series (20). Inset: $g_3$ for the STG and LL gases as functions of the scaled temperature $\tau$ for two different values of $\gamma = 10, 30$.

FIG. 7: Ratio between the three-body correlator $g_3$ of the STG gas and the corresponding quantity for the LL gas at three different scaled temperatures $\tau = 0, 1, 10$ (from top to bottom). The $\tau = 0$ solid line is obtained using the results of [17]. Inset: the same quantity as a function of $\tau$ for two different couplings $\gamma = 10, 30$. In both cases the dashed lines are obtained using two terms of the series (20).

of Cheianov et al. [17] obtained for the ground–state of the repulsive Lieb–Liniger gas, to the super Tonks–Girardeau state. At finite temperature, the three-body correlator for intermediate values of the coupling constant has a very weak dependence on the temperature up to the degeneracy temperature $T_D$. We also showed that the value of $g_3$ for larger temperatures at even relatively small coupling constants $\gamma$ is rather similar to the corresponding value of the repulsive Lieb–Liniger gas.

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Appendix A: Thermodynamical Bethe ansatz equations for the Lieb–Liniger model

In this Appendix we summarize the TBA equations describing the equilibrium properties of a repulsive LL gas. In the limit $N \to \infty$, $L \to \infty$ with the density $n$ fixed, the discrete energy levels of the system get encoded in an energy level density function $\rho(p)$ and in the density $\rho^{(r)}(p)$ of the occupied levels. The ratio between the two densities $\rho$ and $\rho^{(r)}$ defines the pseudo-energy $\varepsilon(p)$ through the relation

$$\frac{\rho(p)}{\rho^{(r)}(p)} = 1 + e^{\varepsilon(p)},$$

(A1)

and this quantity, together with the densities, satisfies a coupled set of integral equations. Using the rescaled quantities

$$q \equiv \frac{p}{n\hbar}, \quad \alpha \equiv \frac{\mu}{k_B T}, \quad g(q) \equiv \hbar n \rho(n\hbar q),$$

(A2)
where $\mu$ is the chemical potential, $T$ is the temperature and $k_B$ is the Boltzmann constant, the equations read

$$
\varepsilon(q) = -\alpha + \frac{q^2}{\tau} - \int_{-\infty}^{\infty} dq' \frac{2\gamma}{(q-q')^2 + \gamma^2} \log \left(1 + e^{-\varepsilon(q')}\right),
$$

(A3a)

$$
g(q) = \frac{1}{2\pi} + \int_{-\infty}^{\infty} dq' \frac{2\gamma}{(q-q')^2 + \gamma^2} \frac{g(q')}{1 + e^{\varepsilon(q')}},
$$

(A3b)

and

$$
1 = \int_{-\infty}^{\infty} \frac{g(q)}{1 + e^{\varepsilon(q)}} dq.
$$

(A3c)

The physical parameters of the problem are $\lambda$, $T$ and $n$, but only the dimensionless combinations $\gamma$ and $\tau$ enter the results. The chemical potential (or the dimensionless fugacity-like parameter $\alpha$) gets fixed by the constraint given by the last equation.

Once the TBA integral equations (A3) are solved, the ground–state energy $E$ and the free energy $F$ of the system are expressed as

$$
\frac{E}{L} = \int_{-\infty}^{\infty} dp \frac{p^2}{2m} \rho^{(e)}(p) = \frac{\hbar^2}{2m} n^3 \int_{-\infty}^{\infty} dq \frac{g(q)}{1 + e^{\varepsilon(q)}},
$$

(A4a)

$$
\frac{F}{L} = nk_B T \left(\alpha - \frac{1}{2\pi} \int_{-\infty}^{\infty} dq \log \left(1 + e^{-\varepsilon(q)}\right)\right).
$$

(A4b)

At zero temperature the energy level density gets a compact support, i.e. it is different from zero only on an interval (which we denote by $[-B, B]$) and, correspondingly, the TBA equations simplify. Applying a different rescaling,

$$
k \equiv \frac{p}{B}, \quad \nu(k) \equiv h \rho^{(e)}(Bk), \quad \beta \equiv \frac{2m}{\hbar} \lambda \equiv \frac{\hbar n \gamma}{B},
$$

(A5)

we arrive at the LL integral equations

$$
1 = \frac{\gamma}{\beta} \int_{-1}^{1} \nu(k) dk,
$$

(A6a)

$$
\nu(k) = \frac{1}{2\pi} + \frac{1}{2\pi} \int_{-1}^{1} dk' \frac{2\beta}{(k-k')^2 + \beta^2} \nu(k'),
$$

(A6b)

while the ground–state energy is given by

$$
\frac{E}{L} = \int_{-B}^{B} dp \rho^{(e)}(p) \frac{p^2}{2m} = \frac{\hbar^2}{2m} n^3 \left(\frac{\gamma}{\beta}\right)^3 \int_{-1}^{1} dk \nu(k) k^2 \equiv \frac{\hbar^2}{2m} n^3 e(\gamma).
$$

(A7)

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