I. INTRODUCTION

In a series of recent papers ([2]-[3]) we have studied and compared how the "constants" $G$ and $\Lambda$ may vary in different theoretical frameworks for several metrics. These different theoretical frameworks are: general relativity with time varying constants (TVC), scalar cosmological models with non-interacting scalar and matter fields and TVC, and the last of the studied models is the usual scalar-tensor theory with a dynamical cosmological constant which seems to be the most natural theoretical model to study the possible variation of the gravitational and the cosmological constants. In those recent works we have been able to state and prove general results valid for all the geometries (all the Bianchi types as well as for the FRW geometries) within the context of self-similar solutions (SSS). We have focused our attention on this class of solutions since, as has been pointed out by Coley ([7]), the self-similar models play an important role in describing the asymptotic dynamics of the Bianchi models. A large class of orthogonal spatially homogeneous models (including all class B models) are asymptotically self-similar at the initial singularity and are approximated by exact perfect fluid or vacuum self-similar power-law models. In the same way, exact self-similar power-law models can also approximate general Bianchi models at intermediate stages of their evolution and which is also important, self-similar solutions can describe the behaviour of Bianchi models at late times. Working under the hypothesis of self-similarity allows us to find exact power-law solutions in such a way that we may compare the obtained solution for each studied model.

In this paper we extend our program by considering the important case of the interacting scalar and matter fields within the framework of scalar cosmological models, and of course, we formulate the corresponding model with TVC. Therefore the aim of the present work consists in studying and comparing, by calculating the exact solution, how the constants $G$ and $\Lambda$ may vary in different theoretical models and with different geometries under the self-similar hypothesis. We apply the outlined program to study three different geometries which generalize the FRW ones, which are Bianchi V, VII$_{0}$ and IX, under the self-similarity hypothesis. We put special emphasis on calculating exact power-law solutions which allow us to compare the different models. In all the studied cases we arrive to the conclusion that the solutions are isotropic and noninflationary while the cosmological constant behaves as a positive decreasing time function (in agreement with the current observations) and the gravitational constant behaves as a growing time function.

The non-interacting case of a scalar and matter fields and its TVC generalization.

As a part of our enlarging program, we now study the interesting case of an interacting scalar and matter fields. We compare three “a priori” (in principle) different approaches. The first one proposed by Maia et al ([16]) where the authors do not split the resulting conservation equation following a thermodynamical approach. The second one has been proposed by Wetterich ([21]). In this case the author consider the splitting conservation equations and takes as the possible coupling between the scalar and matter field the function $q^0 = \delta \rho_m$, where

$$q^0 = \delta \rho_m, \quad \delta \rho_m = \frac{1}{2} \left( \dot{\rho}_m + 3 \rho_m \dot{a}^2 \right).$$
where the constant $\delta$ must be negative. The third of the studied cases is the one proposed by Billard et al \( [1] \) where in analogy with the model proposed by Wetterich but following a different point of view the authors consider as a coupling function $Q = \delta H \rho_m$, where, in this case the constant $\delta$ must be positive in order to satisfy the second law of thermodynamics.

- We conclude the study of the scalar cosmological models by considering the case of an interacting scalar and matter field with TVC. For this purpose we split the generalized conservation equation and consider as a possible coupling function $Q = \delta H \rho_m$, with $\delta > 0$.

In section IID we consider a general scalar-tensor theory with a dynamical cosmological constant, $\Lambda$.

Once we have presented the program, then we apply it, step by step, to three different geometries which are Bianchi V, Bianchi VII\(_0\) and Bianchi IX, since they may be considered as an anisotropic generalization of the FRW geometries. In section III we study the Bianchi V model, while in sections IV and V we studied the models Bianchi VII\(_0\) and IX, respectively. We begin each section by deducing the metric through their Killing vector fields (KVF) and outlining the Einstein tensor. Then we go on to study the classical, scalar and scalar-tensor models as we have described previously. In each case we calculate the standard solution, i.e., with the constants acting as true constants and its TVC solution in order to show how such hypotheses modify the standard solution. We end in section VI with a brief conclusions.

In the appendix A we have included the proofs of the results stated in section II concerning the interacting scalar and matter models. We give a rigorous proofs of the results by using the Lie group method. We also explore a generalization which allows us to consider different coupling functions between the scalar and the matter fields. In appendix B we prove the results for the case of an interacting scalar and matter fields with TVC.

### II. THE MODELS

#### A. Self-similar solutions

Throughout the paper, $\mathcal{M}$ will denote the usual smooth (connected, Hausdorff, 4-dimensional) spacetime manifold with smooth Lorentz metric $g$ of signature $(-, +, +, +)$ (see for example \([21]\)). Thus $\mathcal{M}$ is paracompact. A semi-colon and the symbol $\mathcal{L}$ denote the covariant and Lie derivative, respectively. We shall use a system of units where $c = 1$. For a metric, $g$, and for a vector field $\mathcal{H} \in \mathfrak{X}(\mathcal{M})$, \(((\mathcal{H} = \partial_{t_i} (t, x, y, z) \delta_{x_i})_{i=1}^4\)) the homothetic equation reads (\([23]\) and \([3]\)):

$$\mathcal{L}_H g = 2g.$$  \hspace{1cm} (1)

In the case of the Bianchi models, this equation brings us to obtain that the scale factors behave as follows

$$a = a_0 (t + t_0)^{a_1}, \quad b = b_0 (t + t_0)^{a_2}, \quad d = d_0 (t + t_0)^{a_3},$$  \hspace{1cm} (2)

where $(a_i)_{i=1}^3 \in \mathbb{R}^+$. In that follows we define, $H = h (t + t_0)^{-1}$, with $h = a_1 + a_2 + a_3$. In each of the studied cases we will get restrictions on the constants $(a_i)_{i=1}^3$.

We define the deceleration parameter as

$$q = \frac{3}{h} - 1,$$  \hspace{1cm} (3)

and the anisotropic quantities, $A (\vec{a} - \vec{b})$, and $W^2$ (\([23]\)):

$$A = \frac{\sigma^2}{H^2}, \quad W^2 = I_4,$$  \hspace{1cm} (4)

that give us a measure of the anisotropy. $\sigma^2$ is the shear scalar and $I_4 = C^{abcd}C_{abcd}$ is the Weyl scalar.

#### B. The classical models

The field equations (FE) read

$$R_{ij} - \frac{1}{2} R g_{ij} = 8\pi G T_{ij} - \Lambda g_{ij}, \quad T_{ij}^1 = 0,$$  \hspace{1cm} (5)

where we consider vacuum solutions by making $T_{ij} = \Lambda = 0$, and the classical perfect fluid solutions if $\Lambda = 0$. For a perfect fluid (PF) the energy-momentum tensor is defined by:

$$T_{ij}^m = (\rho + p) u_i u_j + p g_{ij},$$  \hspace{1cm} (6)

where $\rho$ is the energy density of the fluid, $p$ the pressure and they are related by the equation of state $p = \omega \rho$, $(\omega \in (-1, 1))$, $u_i$ is the 4–velocity. From the conservation equation, $T_{ij}^1 = 0$, and taking into account that the scale factors are given by Eq. \((7)\), then it is easy to arrive to the conclusion that the energy density behaves as

$$\rho = \rho_0 (t + t_0)^{-2}.$$  \hspace{1cm} (7)

In order to take into account the variations of $G$ and $\Lambda$ we use the Bianchi identities (see for example \([12, 14]\))

$$\left( R_{ij} - \frac{1}{2} R g_{ij} \right)^{ij} = (8\pi G T_{ij} - \Lambda g_{ij})^{ij},$$  \hspace{1cm} (8)

which read:

$$8\pi G [\rho' + \rho (1 + \omega) H] = -\Lambda' - 8\pi G' \rho,$$  \hspace{1cm} (9)

($'$ means time derivative) in our case we obtain (assuming the additional condition, $T_{i,j}^1 = 0$):

$$\rho' + \rho (1 + \omega) H = 0, \quad \Lambda' = -8\pi G' \rho.$$  \hspace{1cm} (10)

It could be proven that the solution for this model (and for every Bianchi model) has the following form

$$\rho = \rho_0 (t + t_0)^{-\alpha}, \quad G = G_0 (t + t_0)^{\alpha - 2},$$

$$\Lambda = \Lambda_0 (t + t_0)^{-2},$$  \hspace{1cm} (11)

where $\alpha = h (\omega + 1) \in \mathbb{R}$, and $\rho_0, G_0, \Lambda_0 \in \mathbb{R}$.
C. Scalar models

We consider the following cases:

1.- For a scalar field $\phi$, the stress-energy tensor may be written in the following form (8):

$$T_{ij}^\phi = (\rho_\phi + \rho_\phi') u_i u_j + p_\phi g_{ij},$$

(12)

where the energy density and the pressure of the fluid due a scalar field are given by

$$\rho_\phi = \frac{1}{2} \dot{\phi}^2 + V(\phi), \quad p_\phi = \frac{1}{2} \dot{\phi}^2 - V(\phi),$$

(13)

and conservation equation now reads (the Klein-Gordon equation (KG))

$$\ddot{\phi} + H \dot{\phi} + \frac{d}{d\phi} V = 0.$$  

(14)

In a previous work (2) we have proven that the unique form for the scalar field, $\phi$, and the potential, $V(\phi)$, compatible with the self-similar solution (SSS) are given by

$$\phi = \pm \sqrt{\alpha} \ln (t + t_0), \quad V = \beta \exp \left( \frac{2}{\sqrt{\alpha}} \phi \right),$$  

(15)

where $\alpha \in \mathbb{R}^+$, while $\beta \in \mathbb{R}$.

2.- We would like to study how the gravitational constant varies when we are considering a scalar field. For this purpose, by using the Bianchi identity, we may define by, $G(t)T^\phi_i = 0$, we get

$$\phi' \left( \ddot{\phi} + H \dot{\phi} + \frac{d}{d\phi} V \right) = -\frac{G'}{G} \rho_\phi,$$

(16)

which is the modified KG equation. In the G-var framework, in (2) we have proven that the main quantities behave as follows

$$\phi = \phi_0 (t + t_0)^{-\alpha}, \quad V = \beta (t + t_0)^{-2(\alpha + 1)},$$  

$$G = G_0 (t + t_0)^{\gamma},$$

(17)

where $\alpha, \phi_0, G_0, \beta \in \mathbb{R}$, and $g = 2\alpha$. Note that they must verify $G\dot{\phi}^2 \approx t^{-2}$, $GV \approx t^{-2}$. For a similar approach in the context of a FRW model with holographic dark energy with varying gravitational constant see for example (12).

Scalar and matter fields models. In this case the stress-energy tensor may be defined by, $T = T^m + T^\phi$, where $T^m$ is defined by Eq. (6) and $T^\phi$ by Eq. (12). We may consider two cases: The non-interacting and the interacting one (with their TVC versions).

3.- In the non-interacting case we already know how each quantity must behave since we consider the conservation equations ($\text{div} T^m = 0 = \text{div} T^\phi$) so they behave as

$$\phi = \pm \sqrt{\alpha} \ln (t + t_0), \quad V = \beta \exp \left( \frac{2}{\sqrt{\alpha}} \phi \right),$$  

$$\rho = \rho_0 (t + t_0)^{-2}.$$  

(18)

This class of solutions are known in the literature as scaling cosmological solutions (4 and the references therein).

4.- For the non-interacting case with $G$-var we consider, by taking into account the Bianchi identity, the following modified KG equation (see 2 for details)

$$G \ddot{\phi} + G (\ddot{\rho} + \ddot{\phi}) H = -G' \dot{\rho},$$

(19)

where $\ddot{\rho} = \rho_0 + \rho_\phi$, and $\ddot{\phi} = \rho_0 + \rho_\phi$, which we may split into

$$\dot{\rho}_m + (\rho_0 + \rho_\phi) \theta = -\frac{G'}{G} \rho, \quad \phi' \left( \Box \phi + \frac{dV}{d\phi} \right) = -\frac{G'}{G} \rho,$$

(20)

(21)

where we already know how each quantity behaves, namely

$$\phi = \phi_0 (t + t_0)^{-\alpha}, \quad V = \beta (t + t_0)^{-2(\alpha + 1)},$$

$$G = G_0 (t + t_0)^{\gamma}, \quad \rho = \rho_0 (t + t_0)^{-2(\alpha + 1)}.$$  

(22)

5.- In the case of an interacting perfect fluid with a scalar field, interacting quintessence, we may consider several possibilities. In these models, it is considered that the scalar field decays into the perfect fluid and therefore it may be interpreted as a model with dark matter (DM) coupled to dark energy (DE). In order to analyze this case we consider three different approaches:

1. **Approach 1.** (16) In this case the conservation equation reads

$$\dot{\rho}_m + (\omega + 1) \rho_0 H = -\phi' \left( \Box \phi + \frac{dV}{d\phi} \right).$$

(23)

In the appendix we will prove that the behaviour of the quantities are given by Eq. (18). In (16) the authors interpret the scheme from a thermodynamical point of view, a la Prigogine (20) i.e. a model with matter creation.

2. **Approach 2.** (23) In this case the conservation equation is split as follows

$$\dot{\rho}_m + (\omega + 1) \rho_0 H = -\phi' q^\phi,$$

(24)

$$\Box \phi + \frac{dV}{d\phi} = q^\phi,$$

(25)

where $q^\phi = \delta \rho_m$. Therefore we have obtained, see appendix, that the quantities behave as Eq. (18). Setting $\delta = 0$, we regain the non-interacting case. The only constrain for the model is $\delta < 0$. The author considers the possible coupling of the scalar field to matter from previous works (27).

3. **Approach 3.** (4) In this case the conservation equations ($\left( T^m \right)^{ij}_{\alpha \beta} = \delta H \rho_m$, and $\left( T^\phi \right)^{ij}_{\alpha \beta} = \delta H \rho_\phi$).
\(-\delta H \rho_m\) read
\[
\rho_m' + (\omega + 1) \rho_m H = Q, \quad (26)
\]
\[
\phi' \left( \frac{\Box \phi + dV}{d\phi} \right) = -Q, \quad (27)
\]
where \(Q = \delta H \rho_m\). Again, in the appendix we will show that the behaviour of the quantities are given by Eq. \([19]\). As above, the energy of the scalar field is transferred to the matter field. The only restriction is to assume \(\delta > 0\), in such a way that the second law of thermodynamics is verified \([18]\). Note that if \(\delta = 0\), then the non-interacting case is regained. The authors obtain the interacting term, \(\delta H \rho_m\), by taking into account the conformal transformation that relates the scalar-tensor theory to the scalar one.

In the appendix we will discuss the most general form of the function \(Q\), in the framework of the self-similar solutions.

6.- To end, we consider the case of interacting fluids within the framework of \(G\)--varying. For this model the modified KG equation reads
\[
\rho_m' + \phi'' \phi' + \frac{dV}{d\phi} \phi' + H \left( \phi'^2 + (\rho_m + p_m) \right) = -G' \frac{G}{G} \left( \rho_m + \frac{1}{2} \phi'^2 + V \right), \quad (28)
\]
where we may split (as above) it in order to get
\[
\rho_m' + (\rho_m + p_m) H + G' \frac{G}{G} \rho_m = Q, \quad (29)
\]
\[
\phi' \left( \phi'' + \phi'H + \frac{dV}{d\phi} \right) + G' \frac{G}{G} \rho_m = -Q, \quad (30)
\]
with for example
\[
Q = \delta H \rho_m. \quad (31)
\]
In this case, the solution takes the form (see the appendix for details):
\[
\phi = \phi_0 (t + t_0)^{-\alpha}, \quad V(t) = \beta (t + t_0)^{-2(\alpha + 1)},
\]
\[
G = G_0 (t + t_0)^{2\alpha}, \quad \rho_m = \rho_0 (t + t_0)^{-2(\alpha + 1)}. \quad (32)
\]
For an alternative point of view see for example \([1]\).

D. Cosmological models with dynamical \(\Lambda\) in scalar-tensor theories

We start with the action for the a general scalar-tensor theory of gravitation with \(\Lambda (\phi)\),
\[
S = \frac{1}{16\pi G_*} \int d^4x \sqrt{-g} \left[ \phi R - \frac{\omega (\phi) g^{ij} \phi, i \phi, j}{\phi} + 2\phi \Lambda (\phi) \right] + S_{NG}, \quad (33)
\]
where \(g = \det(g_{ij})\), \(G_*\) is Newton’s constant, \(S_{NG}\) is the action for the nongravitational matter. The arbitrary functions \(\omega (\phi)\) and \(\Lambda (\phi)\) distinguish the different scalar-tensor theories of gravitation, \(\Lambda (\phi)\) is a potential function and plays the role of a cosmological constant, and \(\omega (\phi)\) is the coupling function of the particular theory \([28]\).

The explicit field equations are
\[
R_{ij} - \frac{1}{2} g_{ij} R = 8\pi G T_{ij} + \frac{\Lambda (\phi)}{\phi^2} \left( \phi, i \phi, j - \frac{1}{2} g_{ij} \phi, \phi \right) + \frac{1}{\phi} (\phi, i - g_{ij} \phi, \phi), \quad (34)
\]
\[
(3 + 2\omega (\phi)) \Box \phi = 8\pi T - \frac{d\omega}{d\phi} \phi, i \phi, \phi - 2\phi \left( \phi, \phi \phi, - \Lambda (\phi) \right), \quad (35)
\]
where, \(T = T^i_i\), is the trace of the stress-energy tensor, defined by Eq. \([10]\). The gravitational coupling \(G_{\text{eff}}(t)\) is given by
\[
G_{\text{eff}}(t) = \frac{2\omega + \Lambda (\phi)}{2\omega + 3} \frac{G_*}{\phi(t)}. \quad (36)
\]

In a recent work (see \([3]\)) we have proven that the self-similar solution admitted for the FE \([31, 35]\) have the following form
\[
\phi = \phi_0 (t + t_0)^{\tilde{n}}, \quad \Lambda (\phi) = \Lambda_0 \phi^{-\frac{(\tilde{n} + \alpha)}{2}}, \quad (37)
\]
with \(\tilde{n} + \alpha = 2\), therefore \(\Lambda (t) = \Lambda_0 (t + t_0)^{-2}\). The Brans-Dicke parameter is constant, \(\omega (\phi) = \text{const}\.), and \(\rho = \rho_0 (t + t_0)^{-\alpha}\), \(\alpha = (1 + \gamma) h\). Note that in this case we have changed the notation and now the parameter of the EoS is \(\gamma\), so \(p = \gamma \rho\).

III. BIANCHI V MODELS

A. The metric

We consider (see for example \([21]\)) the following Killing vectors fields (KVF)
\[
\xi_1 = \partial_y, \quad \xi_2 = \partial_z, \quad \xi_3 = \partial_x + my \partial_y + mz \partial_z, \quad (38)
\]
where \(m \in \mathbb{R}\ \{0\}\), such that, \([\xi_i, \xi_j] = C_{ij}^k \xi_k\), i.e. \([\xi_1, \xi_2] = 0\), \([\xi_2, \xi_3] = m \xi_1\), and \([\xi_1, \xi_3] = m \xi_1\). With these KVF we obtain the following metric
\[
ds^2 = -dt^2 + a(t)^2 dx^2 + b(t)^2 e^{-2mx} dy^2 + d(t)^2 e^{-2mx} dz^2. \quad (39)
\]
The Einstein tensor reads
\[ \frac{b'}{ba} + \frac{dd'}{a} + \frac{dd'}{b} - \frac{3m^2}{a^2} = G_0^0, \quad (40) \]
\[ -2\frac{a'}{a} \frac{d'}{b} + \frac{d'}{d} = G_1^1, \quad (41) \]
\[ \frac{b''}{b} + \frac{a''}{a} \frac{d'}{b} + \frac{md^2}{a^2} = G_2^1, \quad (42) \]
\[ \frac{d''}{d} + \frac{a''}{a} \frac{d'}{b} - \frac{md^2}{a^2} = G_2^2, \quad (43) \]
\[ \frac{b''}{b} + \frac{a''}{a} + \frac{2m^2}{ba} - \frac{md^2}{a^2} = G_3^2. \quad (44) \]

From Eq. (1) we obtain the following homothetic vector field
\[ \mathcal{H} = t\partial_t + (1 - a_2) y\partial_y + (1 - a_3) z\partial_z, \quad (45) \]
with the following constraints:
\[ a = a_0 (t + t_0), \quad b = b_0 (t + t_0)^{a_2}, \quad d = d_0 (t + t_0)^{a_3}, \quad (46) \]
where \( a_1 = 1 \) and \((a_i)_{i=2}^3 \in \mathbb{R}\), and therefore, \( H = h(t + t_0)^{-1} \), with \( h = 1 + a_2 + a_3 \). Note that we have obtained a non-singular solution.

### B. Classical solutions

#### 1. Vacuum solution

The only possible solution is, \( a_2 = a_3 = 1 \), and \( m = \pm 1 \), hence the solution is
\[ a = a_0 (t + t_0), \quad b = b_0 (t + t_0), \quad d = d_0 (t + t_0), \quad (47) \]
with \( m = \pm 1 \). Therefore \( q = 0 \). With these results the metric yields
\[ ds^2 = -dt^2 + (t + t_0)^2 (dx^2 + e^{\pm 2m} (dy^2 + dz^2)), \quad (48) \]
and therefore it admits more KVF, \( \xi_4 = -m x \partial_x + m y \partial_y, \)
\[ \xi_5 = -2m x \partial_x - 2m x y \partial_y + (m^2 y^2 - m^2 z^2 + e^{2mx}) \partial_z, \]
and \( \xi_6 = 2m y \partial_x + (m^2 y^2 - m^2 z^2 - e^{2mx}) \partial_y + 2m x y \partial_z \). This metric is the Milne form of a flat space-time (see Chapter 9, Eq. (9.8) of [24]).

#### 2. Perfect fluid solution

As we already know, the behaviour of the solution is given by Eq. (17), so it only remains to know the value of the constants \((a_i)_{i=2}^3 \) and \( \omega \), the parameter of the equation of state. We have found the following solution for the FE (40-44) with the conservation equation, \( \text{div} T = 0 \), where the stress-energy tensor is defined by Eq. (4),
\[ a_1 = a_2 = a_3 = 1, \quad q = 0, \quad \omega = \frac{1}{3}. \quad (49) \]
and \( \forall m \in (-1,1) \setminus \{0\} \). With these results the metric collapses to Eq. (48), it does not inflate, \( q = 0 \), and it is only valid for the EoS, \( \omega = -\frac{1}{3} \), which is no strange since for example the MSS for the Bianchi I model is only valid for \( \omega = 1 \). A simple calculation shows us that the solution is isotropic since the anisotropic parameters \( A \) and \( \mathcal{W}^2 \) vanish. From the DS point of view, the solution is stable (44) since it is the FRW model with negative curvature.

#### 3. Time varying constants model

In this case the behaviour of the solution is given by Eq. (11). The FE for this model are described by Eqs. (40-44) with the conservation equations (10), where \( \alpha = (\omega + 1) \) and \( h = (1 + a_2 + a_3) \), and the stress-energy tensor is defined by Eq. (4). Therefore we have found the next solution
\[ a_1 = a_2 = a_3 = 1, \quad q = 0, \quad \alpha = 3 (\omega + 1), \quad (50) \]
\[ G_0 = \frac{(1 - m^2)}{4\pi \rho_0 (\omega + 1)}, \quad \Lambda_0 = 3 (1 - m^2) - \frac{(1 - m^2)}{\omega + 1}, \quad (51) \]
with the following behaviour for \( G \) and \( \Lambda \)
\[ G \approx \begin{cases} \text{decreasing} & \forall \omega \in (-1,-1/3) \\ \text{constant} & \omega = -1/3 \\ \text{growing} & \forall \omega \in (-1/3,1] \end{cases}, \]
\[ \Lambda_0 \approx \begin{cases} \text{negative} & \forall \omega \in (-1,-1/3) \\ \text{vanish} & \omega = -1/3 \\ \text{positive} & \forall \omega \in (-1/3,1] \end{cases}, \]
this solution is valid \( \forall \omega \in (-1,1), \) \( m \in (-1,1) \setminus \{0\} \). As above, with these results the metric collapses to Eq. (48) and therefore it does not inflate. It is isotropic and valid \( \forall \omega \in (-1,1) \). As it is observed, if we set \( \omega = -1/3 \), then the solution collapses to the above one with \( G = \text{const.} \) and \( \Lambda = 0 \). With regard to the behaviour of \( G \) and \( \Lambda \), we only may say that "if" we take into account the current observations (13) which suggest us that \( \Lambda_0 > 0 \), then the solution is only valid \( \forall \omega \in (-1/3,1] \) which means that \( G \) is a growing time function, \( G = G_0 (t + t_0)^{\alpha - 2} \), \( \alpha > 2, \forall \omega \in (-1/3,1] \).

### C. Scalar models

#### 1. Scalar model

The FE for this model are described by Eqs. (40-44) with the stress-energy tensor is defined by Eqs. (12), and the conservation equation given by Eq. (14). Hence taking into account Eq. (15) we find the following result
\[ a_1 = a_2 = a_3 = 1, \quad q = 0, \quad \forall m \in (-1,1) \setminus \{0\}, \quad \alpha = \beta = 2 (1 - m^2). \quad (51) \]
With these results the metric collapses to Eq. (48), so it is not inflationary and isotropic. As it has been proven in (see for example [13] and for a extensive review of results and all the references therein) we cannot say anything about its dynamical behaviour since the obtained value of $\alpha$ escape to this study. Nevertheless the study of the solution through perturbations shows us that it is stable. The potential behaves like the dynamical cosmological constant $V \sim \Lambda \sim t^{-2}$.

2. Scalar model with $G$-var

In this case, the model is described by the FE (40-44) with the stress-energy tensor defined by Eq. (12-13) while the conservation equations are defined by Eq. (10). The solution behaves as (17). We find the following solution

$$a_1 = a_2 = a_3 = 1, \quad q = 0,$$

$\forall m \in (-1, 1) \setminus \{0\}$, therefore the metric collapses to Eq. (48), so the solution is isotropic and not inflationary, while

$$\beta = \alpha^2, \quad G_0 = \frac{2 (1 - m^2)}{\alpha^2} > 0.$$  (53)

With these results we do not know how $G \sim (t + t_0)^{2\alpha}$ may behave. Note that if $\alpha \in (-1, \infty) \setminus \{0\}$, the potential, $V(t) \sim (t + t_0)^{-2(\alpha + 1)}$, behaves as a decreasing time function, as a positive dynamical cosmological constant, but $G$ is decreasing if $\alpha \in (-1, 0)$, and it behaves as a growing time function iff $\alpha > 0$.

3. Non-interacting scalar and matter model

The geometric part of the FE for this model are given by Eqs. (40-44) with the stress-energy tensor defined by Eqs. (6 and 12-13) while the conservation equations are defined by Eqs. (div $T^m = 0 = \text{div} T^\phi$). The solution behaves as (18). We find the following solution

$$a_1 = a_2 = a_3 = 1, \quad q = 0,$$

$$\alpha = \beta, \quad \rho_0 = 3 \left(1 - m^2 - \beta \right),$$  (54)

as it is observed the metric collapses to Eq. (48), so the solution is not inflationary but it is isotropic and only valid for $\omega = -\frac{1}{3}$. We have found the restriction, $\beta < 2 (1 - m^2), \forall m \in (-1, 1) \setminus \{0\}$. Therefore $\alpha = \beta \in (0, 2)$.

4. Non-interacting scalar and matter fields with $G$-var

The FE for the model are given by Eqs. (40-44). The stress-energy tensor is defined by Eqs. (6 and 12-13) while the conservation equations are defined by Eqs (20-21). The solution behaves as (22), therefore we have obtained the following solution

$$a_1 = a_2 = a_3 = 1, \quad q = 0,$$

$\forall m \in (-1, 1) \setminus \{0\}$, so the metric collapses to Eq. (48), i.e. the solution is not inflationary and isotropic and it is only valid for $\omega = -\frac{1}{3}$. The rest of the parameters are

$$\beta = \alpha^2, \quad \rho_0 = \frac{3}{G_0} (1 - m^2) - 3 \alpha^2,$$  (55)

so, in order to get $\rho_0 > 0$, we have the following restriction, $\alpha^2 < \frac{(1 - m^2)}{G_0}$. Unfortunately with these results we do not know how $G \sim (t + t_0)^{2\alpha}$ may behave. Only by “assuming” that the potential, $V(t) \sim (t + t_0)^{-2(\alpha + 1)}$, mimics the dynamic of $\Lambda$ we may bound the value of $\alpha$, that is, $\alpha \in (-1, \infty) \setminus \{0\}$, and therefore, as above, we may say that $G$, is decreasing if $\alpha \in (-1, 0)$, and it behaves as a growing time function iff $\alpha > 0$.

5. Interacting scalar and matter model

a. Approach 1. The FE are given by Eq. (40-44) and the stress-energy tensor defined by Eqs. (6 and 12-13). The conservation equation by Eq. (23). Since the behaviour of the quantities are given by Eq. (18), then we have found the following solution

$$a_1 = a_2 = a_3 = 1, \quad q = 0,$$

$$\alpha = 2 \left(1 - m^2\right) - \rho_0 (\omega + 1),$$

$$\beta = 2 \left(1 - m^2\right) + \frac{\rho_0}{2} (\omega - 1),$$  (56)

$\forall m \in (-1, 1) \setminus \{0\}$, and $\forall \omega \in (-1, 1]$, with the restriction $\rho_0 < \frac{2(1 - m^2)}{(\omega + 1)^2}$. With these results the metric collapses to Eq. (48), therefore the solution does not inflate and it is isotropic. For example, if $\omega = 0$, (which may stands for a mixture of dark matter and dark energy) then

$$\beta = 1 - m^2 + \frac{\alpha}{2}, \quad \rho_0 = 2 \left(1 - m^2\right) - \alpha,$$

with $\alpha < 2 \left(1 - m^2\right)$ in $(0, 2)$.

b. Approach 2. We have found the following solution for the FE (40-44) with the stress-energy tensor is defined by Eqs. (6 and 12-13) and the conservation equations given by Eqs. (24-25), The behaviour of the quantities is given by Eq. (18), so

$$a_1 = a_2 = a_3 = 1, \quad q = 0,$$

$$\beta = 2 \left(1 - m^2\right) - \frac{(1 - \omega)}{(\omega + 1)} \left(2 \left(1 - m^2\right) - \alpha\right),$$

$$\rho_0 = \frac{2 \left(1 - m^2\right) - \alpha}{\omega + 1}, \quad \alpha < 2 \left(1 - m^2\right),$$

$$\delta = -\frac{1}{\sqrt{\alpha}} (3\omega + 1),$$  (57)
where $\omega > -1/3$, and $\forall m \in (-1, 1) \setminus \{0\}$, otherwise $\delta \geq 0$ (remember that in this case $\delta$ must be negative). With these results the metric collapses to Eq. (32) so it is isotropic and non-inflationary. For example if we set $\omega = 0$, then the solution reduces to:

$$\beta = 1-m^2 + \frac{\alpha}{2}, \quad \rho_0 = 2\left(1-m^2\right) - \alpha, \quad \delta = -\frac{1}{\sqrt{\alpha}}.$$ 

with $\alpha < 2\left(1-m^2\right)$.

c. Approach 3. The FE (10-14) with the stress-energy tensor is defined by Eqs. (6) and (12-13) while the conservation equations are given by Eqs. (26-27). As in the above studied cases the behaviour of the quantities is given by Eq. (18), so we get the following solution

$$a_1 = a_2 = a_3 = 1, \quad q = 0,$$

$$\rho_0 = \frac{2\left(1-m^2\right) - \alpha}{\omega + 1}, \quad \alpha < 2\left(1-m^2\right),$$

$$\beta = 2\left(1-m^2\right) - \frac{(1-\omega)}{2(\omega + 1)} \left(2\left(1-m^2\right) - \alpha\right),$$

$$\delta = \frac{1}{3} + \omega, \quad \rho = \rho_0 (t + t_0)^{-2}, \quad (58)$$

and therefore the solution is isotropic and non-inflationary since the metric collapses to Eq. (18). As it is observed, we have obtained the same solution as in the approach 2. In this model, there is a critical value for $\omega$, $\omega = -1/3$, thus $\omega > -1/3$, and $\forall m \in (-1, 1) \setminus \{0\}$, otherwise $\delta \leq 0$ (remember that in this case $\delta$ must be positive). As in the above studies cases if we set $\omega = 0$ then we get

$$\beta = 2\left(1-m^2\right) + \frac{\alpha}{2}, \quad \rho_0 = 2\left(1-m^2\right) - \alpha, \quad \delta = \frac{1}{3},$$

with $\alpha < 2\left(1-m^2\right)$.

As it is observed for $\omega = 0$ we have obtained the same results in the three approaches. The main difference with regard to the non-interacting case is that in this case the solution is valid for $\forall \omega\in(-1/3,1)$, except in the approach 1 where $\omega \in (-1, 1]$.

6. Interacting scalar and matter fields with G-var

The FE for this model are given by Eqs. (10-14). The stress-energy tensor is defined by Eqs. (6) and (12-13) while the conservation equations are given by Eqs. (24-27). In this case the behaviour of the quantities are given by Eq. (32), so we get the following solution with $Q = \delta H \rho_m$,

$$a_1 = a_2 = a_3 = 1, \quad q = 0,$$

with $\forall m \in (-1, 1) \setminus \{0\}$, so the solution is non-inflationary and isotropic. The rest of parameters are

$$\omega = \frac{1}{G_0\rho_0} \left[2\left(1-m^2\right) - G_0 \left(\alpha^2 + \rho_0\right)\right],$$

$$\beta = \frac{1}{2G_0} \left[6\left(1-m^2\right) - G_0 \left(\alpha^2 + 2\rho_0\right)\right],$$

$$\delta = \frac{1}{3G_0\rho_0} \left[6\left(1-m^2\right) - G_0 \left(3\alpha^2 + 2\rho_0\right)\right], \quad (59)$$

therefore the solution is valid for $\forall \omega\in(-1,1]$.

If we set $\omega = 0$, then we obtain:

$$\rho_0 = \frac{1}{G_0} \left[2\left(1-m^2\right) - G_0 \alpha^2\right],$$

$$\beta = \frac{1}{2G_0} \left[2\left(1-m^2\right) + G_0 \alpha^2\right], \quad \delta = \frac{1}{3},$$

so we have the following restriction, $\alpha^2 < 2\left(1-m^2\right)$, since $\rho_0 > 0$. With these results we do not know how $G \sim (t + t_0)^{2\alpha}$ may behave. Note that if $\alpha \in (-1, \infty) \setminus \{0\}$, the potential, $V(\phi) \sim (t + t_0)^{-2(\alpha+1)}$, behaves as a decreasing time function, as a positive dynamical cosmological constant, but $G$ is decreasing if $\alpha \in (-1, 0)$, and it behaves as a growing time function iff $\alpha > 0$.

D. Scalar tensor model

In this case the effective stress-energy tensor takes the following form

$$T_0^0 = \frac{8\pi}{\phi} \rho - H \frac{\phi'}{\phi} + \frac{\omega}{2} \left(\frac{\phi'}{\phi}\right)^2 + \Lambda (\phi), \quad (60)$$

$$T_1^1 = -\frac{8\pi}{\phi} \rho - \frac{\phi'}{\phi} \left(\frac{d' d + b'}{b}\right) - \frac{\omega}{2} \left(\frac{\phi'}{\phi}\right)^2 - \frac{\phi''}{\phi} + \Lambda (\phi) \quad (61)$$

$$T_2^2 = -\frac{8\pi}{\phi} \rho - \frac{\phi'}{\phi} \left(\frac{a' d + a'}{a}\right) - \frac{\omega}{2} \left(\frac{\phi'}{\phi}\right)^2 - \frac{\phi''}{\phi} + \Lambda (\phi) \quad (62)$$

$$T_3^3 = -\frac{8\pi}{\phi} \rho - \frac{\phi'}{\phi} \left(\frac{a' a + b'}{b}\right) - \frac{\omega}{2} \left(\frac{\phi'}{\phi}\right)^2 - \frac{\phi''}{\phi} + \Lambda (\phi) \quad (63)$$

and conservation equations

$$\left(3 + 2\omega (\phi)\right) \left(\frac{\phi''}{\phi} + H \frac{\phi'}{\phi}\right) - 2 \left(\Lambda - \phi \frac{d\Lambda}{d\phi}\right) = \frac{8\pi}{\phi} \left(\rho - 3p\right), \quad (64)$$

$$\rho' + (\rho + p) H = 0. \quad (65)$$

For the FE (34-35) and taking into account Eq. (37)
we find the next solution.

\[ a_1 = a_2 = a_3 = 1, \quad \rho_0 = 0, \quad m = m, \quad \dot{n} = -1 - 3\gamma, \]
\[ \rho_0 = -\frac{1}{8\pi(1 + \gamma)} \left( \omega + 1 \right) \left( 3\gamma + 1 \right)^2 + 6\gamma + 2m^2, \]
\[ \Lambda_0 = -\frac{\left( \left( \gamma - 1 \right) \left( 3\gamma + 1 \right)^2 \omega + 2 \left( 3\gamma + 1 \right) m^2 - 1 \right)}{2 \left( 1 + \gamma \right)}, \]

(66)

therefore, the metric collapses to Eq. (68), so the solution is isotropic and non-inflationary. \( \rho_0 = 0 \), iff

\[ \gamma = -\frac{1}{3}m^2, \quad \omega = -1, \]

while, \( \dot{n} = 0 \iff \gamma_c = -\frac{1}{3}. \) If \( \gamma < -1/3 \) then \( \dot{n} > 0 \) and \( \dot{n} < 0 \forall \gamma \in (-1/3, 1]. \)

For example, if we set \( \omega = 3300 \) and \( m = \pm 1/2, \) then \( \rho_0 > 0 \iff \gamma \in I_1, \) where \( I_1 = (-0.34054, -0.32633), \) but if \( m = \pm 50, \) then \( \rho_0 < 0, \forall \gamma \in (-1, 1), \) and if \( m = \pm0.001, \rho_0 > 0 \iff \gamma \in I_0, \) with \( I_0 = (-0.34164, -0.32523). \) While \( \Lambda_0 > 0, \forall \gamma \in (-1, 1) \setminus J_2, \) where \( J_2 = (-0.333446, -0.333333). \) As it is observed \( \Lambda_0 < 0, \) if

\[ \gamma = -\frac{1}{3}, \quad \gamma = \frac{1}{3}\omega \left( \omega \pm 2\sqrt{2\omega (2\omega - 3m^2 + 3)} \right), \]

note that \( \Lambda_0 < 0, \forall \gamma \in I_2. \)

Therefore this solution has only physical meaning \( \forall \gamma \in I_1, \) i.e. a small neighborhood of \( \gamma_c, \) \( E(\gamma_c = -\frac{1}{3}) = I_1, \) note that \( I_2 \subset I_1. \) For example for \( \gamma_c, \) then \( \rho_0 > 0 \) but \( \Lambda_0 = 0 \) and \( \dot{n} = 0, \) therefore \( \phi = \phi_0 \) and this means that \( G = G_0 \) (i.e. constant), i.e. for this specific value, the solution is the same than the one obtained in the classical model for a perfect fluid. In the same way we may compare the JBD solution with the one obtained in the classical situation where \( G = G_0 \) and \( \Lambda \) vary. If \( \gamma > \gamma_c, \) \( \forall \gamma \in I_1 \) then \( \dot{n} < 0 \) and therefore \( G_{\text{eff}} \approx \phi^{-1}, \) is a growing time function, and \( \Lambda \) behaves as a positive decreasing time function. Note that \( \dot{n} > 0, \forall \gamma \in (-0.34164, \gamma_c), \)

\( \dot{n} = 0 \) if \( \gamma = \gamma_c, \) and \( \dot{n} < 0 \forall \gamma \in (\gamma_c, -0.32633). \)

IV. BIANCHI VII₀

A. The metric

We have the following Killing vector fields (21):

\[ \xi_1 = \partial_y, \quad \xi_2 = \partial_z, \quad \xi_3 = \partial_x - z\partial_y + y\partial_z, \]

(67)

with \( [\xi_1, \xi_2] = 0, [\xi_2, \xi_3] = -\xi_1, [\xi_1, \xi_3] = \xi_2, \) so we may consider that \( C_{13}^2 = 1, C_{23}^1 = -1, \) and therefore we have the following metric

\[ ds^2 = -dt^2 + a^2 dx^2 + (b^2 \cos^2 x + d^2 \sin^2 x) dy^2 + 2 \cos x \sin x (b^2 - d^2) dy dz + (b^2 \sin^2 x + d^2 \cos^2 x) dz^2. \]

(68)

The homothetic vector field is obtained from Eq. (I)

\[ \mathcal{H} = (t + t_0) \partial_t + (1 - a_2) y \partial_y + (1 - a_2) z \partial_z, \]

(69)

so the scale factors must behave as

\[ a = a_0 (t + t_0), \quad b = b_0 (t + t_0)^{a_2}, \quad d = d_0 (t + t_0)^{a_2}, \]

(70)

with \( a_1 = 1, a_2 = a_3 \in \mathbb{R}^+. \) We emphasize the fact that we have been able to obtain non-singular scale factors. So the restrictions are:

\[ a_1 = 1, \quad a_2 = a_3, \]

(71)

hence the metric collapses to the following one

\[ ds^2 = -dt^2 + (t + t_0)^2 dx^2 + (t + t_0)^{2a_2} (dy^2 + dz^2), \]

(72)

thus it belongs to a locally rotationally symmetric (LRS) BVII₀ which looks like a LRS BI. In this way we find a new KVF, \( \xi_4 = \partial_x. \)

The FE read:

\[ a'' \frac{b'}{a} + a' b' d' + b' d' + 2 \left( d^2 - \frac{b^2}{d^2} \right) = 8\pi G T_1, \]

(73)

\[ b'' \frac{b'}{d} + d'' \frac{b'}{d} + b' d' + 1 \left( -2 + \frac{d^2}{b^2} + \frac{b^2}{d^2} \right) = -8\pi G T_2, \]

(74)

\[ \frac{b''}{d} - d'' \frac{b'}{d} + a' \frac{b'}{a} - a' d' + 1 \left( \frac{d^2}{b^2} - \frac{d^2}{b^2} \right) = 0, \]

(75)

\[ \frac{a''}{a} + b'' \frac{a'}{a} + a' \frac{b'}{a} + 1 \left( \frac{b^2}{d^2} - \frac{3d^2}{b^2} \right) = -8\pi G T_3, \]

(76)

\[ \frac{a''}{a} + d'' \frac{a'}{a} + a' \frac{d'}{a} + 1 \left( d^2 - \frac{3b^2}{d^2} \right) = -8\pi G T_4, \]

(77)

and the conservation equation(s) for the different studied cases.

B. Classical solutions

1. Vacuum solution

There is a solution with \( a_2 = 0. \) Therefore the metric collapses to this one

\[ ds^2 = -dt^2 + (t + t_0)^2 dx^2 + dy^2 + dz^2. \]

(78)

This metric is the Taub form of a flat space-time (see chapter 9 Eq. (9.6) of [24] and [10]).
2. Perfect fluid solution

The FE are given by Eqs. (73-77) with the conservation equations, div $T = 0$. The stress-energy tensor is defined by Eq. (6), and the general behaviour for the solution is given by Eq. (77). We find only two solutions

1. This solution is only valid if $\omega = 1$,

$$ds^2 = -dt^2 + (t + t_0)^2 \, dx^2 + dy^2 + dz^2,$$

which is the same metric as the obtained one in the vacuum solution.

2.

$$ds^2 = -dt^2 + (t + t_0)^2 \left( dx^2 + dy^2 + dz^2 \right),$$

i.e. $a_1 = a_2 = a_3 = 1$, $q = 0$, a flat FRW and only valid for $\omega = -\frac{1}{3}$. A simple calculation shows us the obtained solution is non-inflationary, $q = 0$, and isotropic since the anisotropic parameters $(A, W)$ vanish.

3. Time-varying constants scene

In this case the behaviour of the solution is given by Eq. (11). The FE (73-77) with the conservation equations (15), where $\alpha = (\omega + 1) \, h$ and $h = (1 + 2a_2)$. and the stress-energy tensor is defined by Eq. (8). Therefore we have found the next solution by taking into account the obtained SS restrictions for the scale factors given by eq. (71). $G$ behaves as follows:

$$G = G_0 \, (t + t_0)^{\alpha_0 - 2}, \ G_0 = \frac{A}{4\pi \rho_0 (\omega + 1) \alpha},$$

where $A = 2a_2 + a_3^2$, while the cosmological “constant” behaves as:

$$\Lambda = A \left( 1 - \frac{2}{\alpha} \right) (t + t_0)^{-2} = \Lambda_0 (t + t_0)^{-2}.$$

We find the following solution

$$a_1 = a_2 = a_3 = 1, \ q = 0, \ \rho = \rho_0 (t + t_0)^{-(\omega + 1)},$$

$$G = G_0 (t + t_0)^{3(\omega + 1) - 2}, \ \Lambda = \Lambda_0 (t + t_0)^{-2},$$

valid $\forall \omega \in (-1, 1]$, and therefore the metric collapses to Eq. (80). We find the following behaviour for $G$ and $\Lambda$

$$G \approx \begin{cases} \text{decreasing} & \forall \omega \in [-1, -1/3) \\ \text{constant if} & \omega = -1/3 \\ \text{growing} & \forall \omega \in (-1/3, 1] \end{cases}$$

$$\Lambda_0 \approx \begin{cases} \text{vanish if} & \omega = -1/3 \\ \text{positive} & \forall \omega \in (-1/3, 1] \end{cases}.$$

As it is observed the solution is valid $\forall \omega \in (-1, 1]$, instead of only for $\omega = -1/3$. For this value of EoS the solution collapses to the PF solution, i.e. $G = \text{const.}$ and $\Lambda = 0$. The metric collapses to the given one by Eq. (85), so it is non-inflationary and isotropic. As in the case of the BV solution, only under the assumption of $\Lambda > 0$ (19), we may say that $G$ behaves as a growing time function.

C. Scalar models

1. Scalar model

The FE are given by Eqs. (73-77) with the stress-energy tensor is defined by Eqs. (12-13). For this model the conservation equation is given by Eq. (15). Hence, by taking into account Eq. (15) we find the following results

1. Vacuum solution

$$a_1 = 1, \ a_2 = a_3 = 0, \ \alpha = \beta = 0,$$

therefore we have obtained the same solution as in the vacuum case.

2. A non-trivial solution

$$a_1 = a_2 = a_3 = 1, \ q = 0, \ \alpha = \beta = 2,$$

for this value of EoS the metric collapses to Eq. (84), i.e. the solution is non-inflationary and isotropic. From the DS point of view the solution is stable (75).

2. Scalar model with $G$-var

The FE of this model are given by Eqs. (73-77) while the stress-energy tensor is defined by Eqs. (12-13). The solution behaves as

$$\phi = \phi_0 \, (t + t_0)^{-\alpha}, \ V(t) = \beta \, (t + t_0)^{-(\alpha + 1)},$$

$$G = G_0 \, (t + t_0)^{2\alpha}.$$

We find the following solution

$$a_1 = a_2 = a_3 = 1, \ q = 0, \ \alpha = 1, \ \beta = 1, \ G_0 = 2.$$

As in the previous studied models, the metric collapses to Eq. (80), so it is non-inflationary and isotropic. As it is observed, since $\alpha = 1$, then the potential $V$ is positive and behaves as $V \sim t^{-4}$, while $G \sim t^2$, i.e. $G$ is a growing time function. Note that $GV \sim t^{-2}$, while in the standard approach $\Lambda \sim t^{-2}$. Note that in the classical approach $\Lambda$ is not multiplied by $G$ while in the quintessence approach the potential $V$, is multiplied by $G$. 
3. Non-interacting scalar and matter model

The model is described by the following FE (73-77) with the stress-energy tensor defined by Eqs. (8) and (12) [13] while the following conservation equations is defined by Eqs. (div $T^\mu = 0 = div T^\rho$). The solution behaves as (22). We find the following solution

$$a_1 = a_2 = a_3 = 1, \quad q = 0,$$

$$\alpha = \beta = 2 \left(1 - \frac{\rho_0}{3}\right), \quad \rho_0 \in (0, 3), \quad \omega = -\frac{1}{3}. \quad (86)$$

With these results the metric collapses to Eq. (80) and therefore it is non-inflationary and isotropic. The solution is only valid for $\omega = -\frac{1}{3}$, with the restriction $\rho_0 \in (0, 3)$ so if $\rho_0 \ll 1$ then the solution collapses to the one obtained in the scalar model with $\alpha = \beta = 2$.

4. Non-interacting scalar and matter fields with G-var

The FE are given by Eqs. (73-77) with the stress-energy tensor defined by Eqs. (8) and (12) [13] and the following conservation equations given by Eqs. (20-21). The solution behaves as (22), therefore we have obtained the following solution

$$a_1 = a_2 = a_3 = 1, \quad q = 0,$$

$$\alpha = 1 = \beta, \quad \rho_0 = \frac{3}{G_0} - \frac{3}{2}, \quad \omega = -\frac{1}{3}. \quad (87)$$

In order to get, $\rho_0 > 0$, we have the following restriction: $G_0 < 2$. The metric collapses to Eq. (80) and therefore it is non-inflationary and isotropic and only valid for $\omega = -\frac{1}{3}$. As in the scalar solution with $G$-var, since $\alpha = 1$, then the potential $V$ is positive and behaves as $V \sim t^4$, while $G \sim t^2$, i.e. $G$ is growing time function.

5. Interacting scalar and matter model

a. Approach 1 The model is described by Eqs. (73-77) where the stress-energy tensor is defined by Eqs. (8) and (12) [13] and the following conservation equation (23). Since the behaviour of the quantities are given by Eq. (18), then we have found the following solution

$$a_1 = a_2 = a_3 = 1, \quad q = 0,$$

$$\alpha = 2 - \rho_0 (\omega + 1), \quad \beta = 2 + \frac{1}{2} \rho_0 (\omega - 1), \quad \delta = -\frac{1 + 3 \omega}{\sqrt{-2 - \rho_0 (\omega + 1)}}, \quad \forall \omega \in (-1/3, 1], \quad (88)$$

this solution is valid $\forall \omega \in (-1, 1)$, instead of a unique value of $\omega$ as in the case of non-interacting fluids. We find only a restriction, $\rho_0 < (\omega + 1)$. The metric collapses to Eq. (80) and therefore it is isotropic and non-inflationary. For example if we set $\omega = 0$, then we get

$$\alpha = 2 (\beta - 1), \quad \rho_0 = 2 (1 - \beta), \quad \forall \beta \in (1, 2).$$

b. Approach 2 We obtain the following solution for the FE (73-77) with stress-energy tensor is defined by Eqs. (8) and (12) [13] and the following conservation equations given by Eqs. (24-25). The behaviour of the quantities are given by Eq. (18),

$$a_1 = a_2 = a_3 = 1, q = 0,$$

$$\alpha = 2 - \rho_0 (\omega + 1), \quad \beta = 2 + \frac{1}{2} \rho_0 (\omega - 1), \quad \gamma = -\frac{1 + 3 \omega}{\sqrt{2 - \rho_0 (\omega + 1)}}, \quad \forall \omega \in (-1/3, 1], \quad \delta \leq 0, \quad (89)$$

valid $\forall \omega \in (-1/3, 1], \quad \delta \leq 0$, with the same restriction, $\rho_0 < (\omega + 1)^2$. The metric collapses to Eq. (80) so it is isotropic and non-inflationary. For example if we set $\omega = 0$, then we get

$$\delta = -\frac{1}{\sqrt{\alpha}}, \quad \beta = 1 + \frac{\alpha}{2}, \quad \rho_0 = 2 - \alpha,$$

$\forall \alpha \in (0, 2)$. Observe that we get the same result as in the previous case.

c. Approach 3 In this case the FE are given by Eqs. (73-77) while the stress-energy tensor is defined by Eqs. (8) and (12) [13] and the conservation equations are given by Eqs. (26-27). As in the above studied cases the behaviour of the quantities are given by Eq. (18), so we get the following solution

$$a_1 = a_2 = a_3 = 1, q = 0,$$

$$\alpha = 2 - \rho_0 (\omega + 1), \quad \beta = 2 + \frac{1}{2} \rho_0 (\omega - 1), \quad \delta = \frac{1}{3} + \omega, \quad \forall \omega \in (-1/3, 1], \quad \delta \geq 0 \text{ with the restriction, } \rho_0 < (\omega + 1)^2. \quad (90)$$

valid $\forall \omega \in (-1/3, 1], \quad \delta \geq 0 \text{ with the restriction, } \rho_0 < (\omega + 1)^2$. The metric collapses to Eq. (80) and therefore it is non-inflationary and isotropic. For example if we set $\omega = 0$, then we get

$$\delta = \frac{1}{3}, \quad \beta = 1 + \frac{\alpha}{2}, \quad \rho_0 = 2 - \alpha,$$

$\forall \alpha \in (0, 2)$. As it is observed we have obtained the same solution in the three cases. The main difference with the non-interacting case is that for these models the solution is valid for all values of $\omega$, i.e. $\forall \omega \in (-1, 1)$, which is more realistic. In approach 1 we have obtained $\omega \in (-1, 1]$.

6. Interacting scalar and matter fields with G-var

The FE (73-77) with the stress-energy tensor is defined by Eqs. (8) and (12) [13] while the conservation equations are given by Eqs. (29-30). In this case the behaviour of the quantities are given by Eq. (32), so we get the following
solution with $Q = \delta H \rho_m$, \( a_1 = a_2 = a_3 = 1, \ q = 0, \)
\[
\omega = \frac{1}{G_0 \rho_0} [2 - G_0 (1 + \rho_0)],
\]
\[
\beta = \frac{1}{2G_0} [6 - G_0 (1 + 2\rho_0)], \ \alpha = 1,
\]
\[
\delta = \frac{1}{3G_0 \rho_0} [6 - G_0 (3 + 2\rho_0)].
\]
(91)

The metric collapses to Eq. (80) so it is isotropic and non-inflationary. As in the previous cases, since $\alpha = 1$, then the potential $V$ is positive and behaves as $V \sim t^{-4}$, while $G \sim t^2$, i.e. $G$ is growing time function. If we set $\omega = 0$, then we obtain:
\[
\alpha = 1, \ \rho_0 = \frac{1}{G_0} [2 - G_0],
\]
\[
\beta = \frac{1}{2G_0} [2 + G_0], \ \delta = \frac{1}{3},
\]
in order to get, $\rho_0 > 0$, we have the following restriction $G_0 < 2$.

D. Scalar tensor model

The effective stress-energy tensor takes the following form
\[
T^0_0 = \frac{8\pi}{\phi} \rho - H \frac{\phi'}{\phi} + \frac{\omega}{2} \left( \frac{\phi'}{\phi} \right)^2 + \Lambda (\phi),
\]
(92)
\[
T^0_0 = \left( \frac{d'}{d} - \frac{b'}{b} \right) \frac{\phi'}{\phi},
\]
(93)
\[
T^1_1 = -\frac{8\pi}{\phi} \rho - \frac{\phi'}{\phi} \left( \frac{d'}{d} + \frac{\phi'}{\phi} \right) - \frac{\omega}{2} \left( \frac{\phi'}{\phi} \right)^2 - \frac{\phi''}{\phi} + \Lambda (\phi),
\]
(94)
\[
T^2_2 = -\frac{8\pi}{\phi} \rho - \frac{\phi'}{\phi} \left( \frac{a'}{a} + \frac{\phi'}{\phi} \right) - \frac{\omega}{2} \left( \frac{\phi'}{\phi} \right)^2 - \frac{\phi''}{\phi} + \Lambda (\phi),
\]
(95)
\[
T^3_3 = -\frac{8\pi}{\phi} \rho - \frac{\phi'}{\phi} \left( \frac{d'}{d} + \frac{a'}{a} \right) - \frac{\omega}{2} \left( \frac{\phi'}{\phi} \right)^2 - \frac{\phi''}{\phi} + \Lambda (\phi).
\]
(96)

while the conservation equations are:
\[
(3 + 2\omega) \left( \frac{\phi''}{\phi} + \frac{H \phi'}{\phi} \right) - 2 \left( \Lambda - \phi \frac{dM}{d\phi} \right) = \frac{8\pi}{\phi} (\rho - 3p),
\]
(97)
\[
\rho' + (\rho + p) H = 0.
\]
(98)

For the FE (34-35) and taking into account Eq. (37) we find the next solution
\[
a_1 = a_2 = a_3 = 1, \ q = 0,
\]
\[
\phi_0 = 1, \ 
\dot{n} = -1 - 3\gamma,
\]
\[
\rho_0 = -\omega (3\gamma + 1)^2 + 9\gamma^2 + 12\gamma + 1
\]
\[
8\pi (1 + \gamma)
\]
\[
\Lambda_0 = \frac{\omega (1 - \gamma) (3\gamma + 1)^2 + 2 (3\gamma + 1)}{2 (1 + \gamma)}.
\]
(99)

With these results, the metric collapses to Eq. (80) and therefore it is isotropic and non-inflationary. Therefore, $\rho_0 = 0$, iff
\[
\gamma = -\frac{\omega \pm \sqrt{2\omega + 3} + 2}{3\omega}, \ \omega \neq -1, \ \omega \neq \frac{1}{2},
\]
while, $\Lambda_0 = 0$, iff
\[
\gamma = -\frac{1}{3}, \ \gamma = \omega \pm \sqrt{2\omega + 3}, \ \omega 
eq 0, \ \omega \neq \frac{1}{2},
\]
If we fix $\omega = 3300, (22)$ then we get
\[
\rho_0 > 0 \iff \gamma \in I_1, \ I_1 = (-0.3416, -0.3252),
\]
\[
\Lambda_0 < 0 \iff \gamma \in I_2, \ I_2 = (-0.3334, -0.3333),
\]
in fact $\Lambda_0 \geq 0$ if $\gamma \notin I_2$. Therefore this solution is only valid $\forall \gamma \in I_1$, a small neighborhood of $\gamma_c$, $\mathcal{C}(\gamma_c = -\frac{1}{3}) = I_1$, note that $I_2 \subset I_1$, but if we take into account the current observations ($19$) which suggest us that the cosmological constant must be positive then we arrive to the conclusion that the solution is valid $\forall \gamma \in I_1 \setminus I_2$. If $\gamma > \gamma_c$, $\forall \gamma \in I_1$ then $\dot{n} < 0$ and therefore $G_{\text{eff}} \approx \phi^{-1}$, is a growing time function, and $\Lambda$ behaves as a positive decreasing time function. For example if $\gamma = -1/3$, which means that $\dot{n} = 0$, then
\[
\rho_0 = \frac{3}{8\pi}, \ \Lambda_0 = 0, \ G_{\text{eff}} = G_0,
\]
compare with the perfect fluid solution and with classical model with TVC. Note that $\dot{n} > 0, \forall \gamma \in (-0.3416, \gamma_c)$, $\dot{n} = 0$ if $\gamma = \gamma_c$, and $\dot{n} < 0, \forall \gamma \in (\gamma_c, -0.3252)$.

V. Bianchi IX

A. The metric

The Killings are (21):
\[
\xi_1 = \partial_y,
\]
\[
\xi_2 = \cos y \partial_x - \cot x \sin y \partial_y + \frac{\sin y}{\sin x} \partial_z,
\]
\[
\xi_3 = -\sin y \partial_x - \cot x \cos y \partial_y + \frac{\cos y}{\sin x} \partial_z,
\]
therefore, $[\xi_1, \xi_2] = \xi_3$, $[\xi_1, \xi_3] = -\xi_2$, $[\xi_2, \xi_3] = \xi_1$. Algebra brings us to get

\[ ds^2 = -dt^2 + \left( a^2(t) \sin^2 z + b^2(t) \cos^2 z \right) dx^2 + 2 \left( b^2(t) - a^2(t) \right) \sin z \sin x \cos x dy dx + \left( a^2(t) \sin^2 x \cos^2 z + b^2(t) \sin^2 x \sin^2 z + d^2(t) \cos^2 x \right) dy^2 + 2d^2(t) \cos x dy dz + d^2(t) dz^2. \]

(100)

The metric admits the following HVF

\[ \mathcal{H} = (t + t_0) \partial_t + (1 - a_2) y \partial_y + (1 - a_3) z \partial_z, \]

(101)

so the scale factors must behave as

\[ a = a_0 (t + t_0), \quad b = b_0 (t + t_0)^{a_2}, \quad d = d_0 (t + t_0)^{a_3}, \]

and the restrictions on the constants $(a_i)_{i=1}^3 \in \mathbb{R}$, are

\[ a_1 = 1, \quad a_2, a_3 \in \mathbb{R}. \]

(102)

For this model the FE read:

\[ \frac{a'}{a} = \frac{b'}{b} + \frac{d'}{d} + \frac{1}{2} \left( \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{d^2} \right) \]

\[ - \frac{1}{4} \left( \frac{b'^2}{a^2d^2} + \frac{b^2}{a^2d^2} + \frac{d'^2}{b^2a^2} \right) = 8\pi GT_1, \]

(103)

\[ \frac{a''}{a} - \frac{b''}{b} + \frac{d''}{d} + \frac{1}{2} \left( \frac{1}{a^2} - \frac{1}{b^2} + \frac{2}{a^2d^2} \right) = 0, \]

(104)

\[ \frac{b''}{b} + \frac{d''}{d} + \frac{a''}{a} + \frac{1}{2} \left( \frac{1}{b^2} + \frac{1}{d^2} - \frac{1}{a^2} \right) \]

\[ + \frac{1}{4} \left( \frac{a'^2}{b^2d^2} + \frac{a'^2}{a^2b^2} - \frac{3a'^2}{a^2b^2} \right) = -8\pi GT_2, \]

(105)

\[ \frac{a''}{a} + \frac{d''}{d} + \frac{a''}{a} + \frac{1}{2} \left( \frac{1}{a^2} + \frac{1}{d^2} - \frac{1}{b^2} \right) \]

\[ + \frac{1}{4} \left( \frac{a'^2}{b^2d^2} + \frac{a'^2}{b^2a^2} - \frac{3a'^2}{a^2b^2} \right) = -8\pi GT_3, \]

(106)

\[ \frac{b''}{b} - \frac{a'^2}{a} \frac{b'}{b} - \frac{b'^2}{a^2} - \frac{d'}{b^2a^2} \]

\[ + \frac{1}{4} \left( \frac{1}{b^2} - \frac{1}{d^2} + \frac{b'^2}{a^2} - \frac{d'^2}{a^2d^2} - \frac{d'^2}{a^2d^2} \right) = 0, \]

(107)

\[ \frac{a''}{a} + \frac{b''}{b} + \frac{a''}{a} + \frac{1}{2} \left( \frac{1}{a^2} + \frac{1}{b^2} - \frac{1}{d^2} \right) \]

\[ + \frac{1}{4} \left( \frac{a'^2}{b^2d^2} + \frac{b'^2}{a^2d^2} - \frac{3a'^2}{a^2b^2} \right) = -8\pi GT_4, \]

(108)

and the conservation equation(s).

B. Classical solutions

1. Vacuum solution

There is not SS solution for the vacuum model.

2. Perfect fluid solution

The FE are given by Eqs. (103-108) with the conservation equation, $T_{ij}^t = 0$. The stress-energy tensor is defined by Eq. (10), and the general behaviour for the solution is given by Eq. (7). We find the following solution

\[ a_1 = a_2 = a_3 = 1, \quad q = 0, \quad \omega = -1/3. \]

(109)

In this way the metric collapses to this one:

\[ ds^2 = -dt^2 + (t + t_0)^2 (dx^2 + dy^2 + dz^2) + 2(t + t_0)^2 \cos x dy dz \]

(110)

and therefore there is another KVF, $\xi_4 = \partial_z$. As in the previous models this solution is isotropic, $(A, W^2)$ vanish and it is only valid for the EoS $\omega = -1/3$. The obtained solution is stable from the dynamical point of view (7).

3. Time varying constants framework

In this framework the FE are given by Eqs. (103-108) with the conservation equations (10), where $\alpha = (\omega + 1) h$ and $h = (1 + a_2 + a_3)$, and the stress-energy tensor is defined by Eq. (3). In this case the behaviour of the solution is given by Eq. (11). Therefore we have found the next solution by taking into account the obtained SS restrictions for the scale factors given by Eq. (12).

\[ a_1 = a_2 = a_3 = 1, \quad q = 0, \]

\[ G_0 = \frac{15}{48\pi \rho_0 (\omega + 1)} \quad \forall \omega \in (-1, 1), \]

\[ \Lambda_0 = \frac{15}{4c^2} \left( 1 - \frac{2}{3(\omega + 1)} \right) (t + t_0)^{-2}, \]

(111)

as above the metric collapses to Eq. (110), so it is isotropic and non-inflationary. The behaviour of the “constants” is the following one:

\[ G \approx \begin{cases} \text{decreasing} & \forall \omega \in [-1, -1/3) \\ \text{constant} \text{ if } \omega = -1/3 \\ \text{growing} \text{ if } \omega \in (-1/3, 1) \end{cases} \]

\[ \Lambda_0 \approx \begin{cases} \text{negative} \forall \omega \in [-1, -1/3) \\ \text{vanish} \text{ if } \omega = -1/3 \\ \text{positive} \forall \omega \in (-1/3, 1) \end{cases} \]

i.e. we have found the same behaviour as in the previous studied models.
C. Scalar models

1. Scalar model

The FE are given by Eqs. (103-108) while the stress-energy tensor is defined by Eqs. (12-13). For this model the conservation equation given by Eq. (16). Hence, by taking into account Eq. (15) we find the following non-trivial solution

\[ a_1 = a_2 = a_3 = 1, \quad \alpha = \beta = \frac{5}{2} \]  

(112)

therefore the metric collapses to Eq. (110) and the solution is isotropic and non-inflationary. As it is showed in (7) the solution is stable from the DS point of view.

2. Scalar model with G-var

For this model the FE are given by Eqs. (103-108). The stress-energy tensor is defined by Eq. (12-13) while the conservation equation is defined by Eq. (10). The solution behaves as (7), we find the following solution

\[ a_1 = a_2 = a_3 = 1, \quad q = 0, \]
\[ \alpha = 1, \quad \beta = 1, \quad G_0 = \frac{5}{2} \]  

(113)

where, as it is observed, the metric collapses to Eq. (110) and therefore the solution is isotropic and non-inflationary. Since, \( \alpha = 1 \), then we may say that \( G \sim t^2 \), i.e. it is a growing time function while the potential is positive and decreasing, it behaves as, \( V \sim t^{-\gamma} \).

3. Non-interacting scalar and matter model

In this case, the FE for the model are described by Eqs. (103-108), and the stress-energy tensor is defined by Eqs. (6) and (12-13) while the conservation equations are defined by Eqs. (24-25). The solution behaves as (18). We find the following solution

\[ a_1 = a_2 = a_3 = 1, \quad q = 0, \]
\[ \alpha = \beta, \quad \rho_0 = \frac{3}{2} (\frac{5}{2} - \beta), \]
\[ \beta \in (0, \frac{5}{2}), \quad \omega = -\frac{1}{3} \]  

(114)

therefore the metric collapses to Eq. (110) and the solution is non-inflationary and isotropic. As in the previous models, the solution is only valid for \( \omega = -\frac{1}{3} \), while the parameters \( (\alpha, \beta) \) are only bounded.

4. Non-interacting scalar and matter fields with G-var

The model is described by the FE (103-108), with the stress-energy tensor defined by Eqs. (6) and (12-13) and the corresponding conservation equations given by Eqs. (20-21). The solution behaves as (22), therefore we have obtained the following solution

\[ a_1 = a_2 = a_3 = 1, \quad q = 0, \]
\[ \alpha = 1 = \beta, \]
\[ \rho_0 = \frac{3}{4G_0} (5 - 2G_0), \quad \omega = -\frac{1}{3}. \]  

(115)

so we have the following restriction, \( G_0 < \frac{5}{2} \), in order to get \( \rho_0 > 0 \). The metric collapses to Eq. (110) so the solution is isotropic and non-inflationary and only valid for \( \omega = -\frac{1}{3} \). Note that \( \alpha = 1 \), so \( G \) is a growing time function, \( G \sim t^2 \), while the potential is positive and behaves as a decreasing time function.

5. Interacting scalar and matter model

a. Approach 1

The model is described by the FE (103-108) with the stress-energy tensor is defined by Eqs. (6) and (12-13) and conservation equation (23). Since the behaviour of the quantities are given by Eq. (18), then we have found the following solution

\[ a_1 = a_2 = a_3 = 1, \quad q = 0, \]
\[ \alpha = \frac{5}{2} - \rho_0 (\omega + 1), \quad \beta = \frac{5}{2} + \frac{1}{2} \rho_0 (\omega - 1), \]  

(116)

finding that the solution is valid \( \forall \omega \in (-1, 1] \), with the restriction, \( \rho_0 < \frac{5}{2(\omega + 1)} \). The metric collapses to Eq. (110) which means that the solution is non-inflationary and isotropic. For example if we set \( \omega = 0 \), then we get

\[ \alpha = 2\beta - \frac{5}{2}, \quad \rho_0 = 5 - 2\beta, \quad \forall \beta \in \left( \frac{5}{4}, \frac{5}{2} \right). \]

b. Approach 2

The FE for this model are described by Eqs. (103-108) with the stress-energy tensor is defined by Eqs. (6) and (12-13) and the conservation equations are given by Eqs. (24-25). The behaviour of the quantities is given by Eq. (18), so we get the following solution

\[ a_1 = a_2 = a_3 = 1, \quad q = 0, \]
\[ \alpha = \frac{5}{2} - \rho_0 (\omega + 1), \quad \beta = \frac{5}{2} + \frac{1}{2} \rho_0 (\omega - 1), \]
\[ \rho_0 = \frac{2(1 + 3\omega)}{\sqrt{10 - 4G_0 (\omega + 1)}}, \]  

(117)

finding the restriction, \( \rho_0 < \frac{5}{2(\omega + 1)}, \, \forall \omega \in (-1/3, 1], (\delta \leq 0) \). With these results, the metric collapses to Eq. (110) and therefore the solution is isotropic and non-inflationary. For example if we set \( \omega = 0 \), then we get

\[ \delta = -\frac{1}{\sqrt{\alpha}}, \quad \beta = \frac{5}{4} + \frac{\alpha}{2}, \]
\[ \rho_0 = \frac{5}{2} - \alpha, \quad \forall \alpha \in \left( 0, \frac{5}{2} \right), \]

and therefore we have obtained the same result as in the approach 1, as it is expected.
c. Approach 3 The FE with the stress-energy tensor is defined by Eqs. (6 and [12-13]) and conservation equations given by Eqs. (20-27) admit the following solution, note that the behaviour of the quantities is given by Eq. (18),

\[
\begin{align*}
\alpha &= a_1 = a_2 = a_3 = 1, \\
q &= 0, \\
\omega &= 2 - \frac{5}{2} (\omega + 1), \\
\delta &= \frac{1}{3} + \omega, \\
\beta &= \frac{5}{2} + \frac{1}{2} \rho_0 (\omega - 1), \quad (118)
\end{align*}
\]

with \( \rho_0 = \rho_0 \), finding the restriction, \( \rho_0 < \frac{5}{3} \), \( \omega < -1/3 \). Therefore the metric collapses to Eq. (110), which means that the solution is non-inflationary and isotropic. If we set \( \omega = 0 \), then it is obtained the following result:

\[
\begin{align*}
\delta &= \frac{1}{3}, \\
\beta &= \frac{5}{2} + \alpha, \\
\rho_0 &= \frac{5}{2} \alpha, \quad \forall \alpha \in (0, \frac{5}{2}),
\end{align*}
\]

and therefore, once again we have shown that the three approaches are identical.

6. Interacting scalar and matter fields with G-var

The model is described by the FE with the stress-energy tensor defined by Eqs. (6 and [12-13]) while the conservation equations are given by Eqs. (20-27). In this case the behaviour of the quantities are given by Eq. (112), so we get the following solution with, \( Q = \delta H \rho_m \),

\[
\begin{align*}
a_1 &= a_2 = a_3 = 1, \\
q &= 0, \quad \alpha = 1, \\
\omega &= \frac{1}{2G_0 \rho_0} [5 - 2G_0 (1 + \rho_0)], \\
\beta &= \frac{1}{4G_0} [15 - 2G_0 (1 + 2 \rho_0)], \\
\delta &= \frac{1}{6G_0 \rho_0} [15 - 2G_0 (3 + 2 \rho_0)].
\end{align*}
\]

The metric collapses to Eq. (110), so the solution is isotropic and non-inflationary. Since \( \alpha = 1 \), then \( G \) is growing, it behaves as \( G \sim t^2 \), while the potential behaves as a positive decreasing time function. Setting \( \omega = 0 \), which may be interpreted as a model with interacting DM and DE, then we obtain:

\[
\begin{align*}
\alpha &= 1, \quad \rho_0 = \frac{1}{2G_0} [5 - 2G_0], \\
\beta &= \frac{1}{4G_0} [5 + 2G_0], \quad \delta = \frac{1}{3},
\end{align*}
\]

so we have the following restriction, \( G_0 < \frac{5}{2} \), in order to get \( \rho_0 > 0 \).

D. Scalar tensor model

The effective stress-energy tensor takes the following form

\[
\begin{align*}
T_1^\prime &= \frac{8\pi}{\phi} \rho - H \frac{\phi'}{\phi} + \frac{\omega}{2} \left( \frac{\psi'}{\phi} \right)^2 + \Lambda (\phi), \\
T_2^\prime &= \frac{8\pi}{\phi} \rho - \frac{\phi'}{\phi} \left( \frac{d' + b'}{a} \right) - \frac{\omega}{2} \left( \frac{\psi'}{\phi} \right)^2 - \frac{\psi''}{\phi} + \Lambda (\phi), \\
T_3^\prime &= \frac{8\pi}{\phi} \rho - \frac{\phi'}{\phi} \left( \frac{a + b'}{a} \right) - \frac{\omega}{2} \left( \frac{\psi'}{\phi} \right)^2 - \frac{\psi''}{\phi} + \Lambda (\phi), \\
T_4^\prime &= \frac{8\pi}{\phi} \rho - \frac{\phi'}{\phi} \left( \frac{a'}{a} + \frac{b'}{b} \right) - \frac{\omega}{2} \left( \frac{\psi'}{\phi} \right)^2 - \frac{\psi''}{\phi} + \Lambda (\phi),
\end{align*}
\]

with the conservation equations

\[
(3 + 2\omega (\phi)) \left( \frac{\phi''}{\phi} + H \frac{\phi'}{\phi} \right) - 2 \left( \Lambda = \phi \frac{d\Lambda}{d\phi} \right) = \frac{8\pi}{\phi} (\rho - 3p),
\]

\[
\rho' + (\rho + p) H = 0.
\]

For the FE and taking into account Eq. (87) we find the next solution

\[
\begin{align*}
a_1 &= a_2 = a_3 = 1, \\
q &= 0, \quad \phi_0 = 1, \\
\rho_0 &= - \left[ 2\omega (3\gamma + 1)^2 + 18\gamma^2 + 24\gamma + 1 \right], \\
\Lambda_0 &= \left[ 2\omega \left( (1 - \gamma) (3\gamma + 1)^2 \right) + 5 (3\gamma + 1) \right],
\end{align*}
\]

therefore, the metric collapses to Eq. (110), so the solution is isotropic and non-inflationary. \( \rho_0 = 0 \), iff

\[
\gamma = - \frac{2\omega \sqrt{2} (5\omega + 7) + 4}{6 (\omega + 1)}, \quad \omega \neq -1,
\]

and the cosmological constant vanish when, \( \Lambda_0 = 0 \), iff

\[
\gamma = - \frac{1}{3}, \quad \gamma = \frac{2\omega \sqrt{2} (8\omega + 15)}{6\omega}, \quad \omega \neq 0.
\]

If we fix \( \omega = 3300 \), (22) then we get

\[
\rho_0 > 0 \iff \gamma \in I_1, \quad I_1 = (-0.3426, -0.3242), \quad \Lambda_0 < 0 \iff \gamma \in I_2, \quad I_2 = (-0.3335, -0.3333),
\]

in fact \( \Lambda_0 \geq 0 \) if \( \gamma \notin I_2 \). Note that \( I_2 \subset I_1 \). Therefore this solution is only valid if \( \gamma \in I_1 \), i.e. for a small neighborhood of \( \gamma_c, \quad E(\gamma_c = -\frac{1}{3}) = I_1 \), note that \( I_2 \subset I_1 \). As in the above cases, if we consider that \( \Lambda_0 \) must be positave then we arrive to the conclusion that the solution is valid.
\( \forall \gamma \in I_1 \backslash I_2 \). If \( \gamma > \gamma_c \), \( \forall \gamma \in I_1 \) then \( \tilde{n} < 0 \) and therefore \( G_{\text{eff}} \approx \phi^{-1} \), is a growing time function, and \( \Lambda \) behaves as a positive decreasing function. Setting \( \gamma = -1/3 \), then \( \tilde{n} = 0 \), and therefore

\[
\rho_0 = \frac{15}{32\pi}, \quad \Lambda_0 = 0, \quad G_{\text{eff}} = G_0,
\]

i.e. we obtain the classical perfect fluid solution.

VI. CONCLUSIONS

In this paper we have studied how the constants \( G \) and \( \Lambda \) may vary in different theoretical models as general relativity with a perfect fluid, scalar cosmological models ("quintessence" models) with and without interacting scalar and matter fields and a scalar-tensor model with a dynamical \( \Lambda \). We have applied the outlined program to study three different geometries which generalize the FRW ones, which are Bianchi V, VII_0 and IX, under the self-similarity hypothesis. These geometries are in principle homogeneous but anisotropic. We have put special emphasis in calculating exact power-law solutions which allow us to compare the different models.

As we have shown, in all the studied cases, we arrive to very similar conclusions. The first of them is that the solutions are isotropic (\( A, W^2 \) vanish) in spite of considering anisotropic geometries, and noninflationary (\( \dot{\gamma} = 0 \)).

With regard to the classical solutions i.e. solutions obtained within the general relativity (with a perfect fluid as matter model) and its TVC model, we arrive to the conclusion that the obtained solutions are only valid for a unique value of the EoS, \( \omega = -1/3 \). In the Bianchi V model the metric collapses to a metric with 6 KVF, therefore, as we already know, the FE generalize the case FRW with negative curvature. In the Bianchi VII_0 model the metric collapses to the flat FRW and for the Bianchi IX the obtained metric has only 4 KVF, nevertheless the FE generalize the FRW with positive curvature. In the case of TVC models, we have arrived to the conclusion that in all the cases the solutions are valid \( \forall \omega \in (-1, 1) \). For \( \omega = -1/3 \) the solutions collapse to the standard solution with \( G = \text{const.} \) and \( \Lambda = 0 \). If we consider the current observations which suggest \( \Lambda_0 > 0 \), then we arrive to the conclusion that the solution is valid only for \( \forall \omega \in (-1/3, 1) \) and that the gravitational "constant", \( G(t) \sim t^{4\omega + 1} \), behaves as a growing time function while \( \Lambda \sim t^{-2} \).

With regard to the scalar cosmological models, we conclude that in all the cases, the metrics collapse to the ones obtained for the case of a perfect fluid model. All the obtained solutions are stable from the dynamical system point of view and therefore relevant from the physical point of view. In the case of the scalar models with \( \Lambda \)-var, we arrive to the conclusion that in the cases BVII_0 and BIX, \( G \) behaves as a growing time function, \( G(t) \sim t^2 \), and the potential, which mimics a dynamical cosmological constant behaves as a positive decreasing function, \( V \sim t^{-4} \). In the BV model, since it depends on more parameters like \( m \), then the obtained solution is not so precise in such a way that we only may say that if \( \alpha \in (-1, \infty) \backslash \{0\} \), the potential, \( V(t) \sim (t + t_0)^{-2(\alpha + 1)} \), behaves as a decreasing time function, as a positive dynamical cosmological constant, but \( G \) is decreasing if \( \alpha \in (-1, 0) \), and it behaves as a growing time function iff \( \alpha > 0 \). In the non-interacting case, we arrive to the conclusion that the obtained solutions are only valid for a unique value of the EoS, \( \omega = -1/3 \). In the non-interacting case with \( G \)-var, the solutions are only valid for \( \omega = -1/3 \), while \( G \) and \( \Lambda \) behave like in the case of the scalar model with \( G \)-var, so we have no new information about their behavior. For the interesting cases of interacting scalar and matter fields we arrive to the conclusion that for all the cases the solutions are very similar. For the approaches 2 and 3 the solutions are the same and only difference with respect to approach 1 is the range of validity for \( \omega \), the parameter of the EoS. The main difference with respect to the non-interacting case is that the solutions are valid \( \forall \omega \in (-1/3, 1] \) in approaches 2 and 3 while in approach 1 \( \forall \omega \in (-1, 1] \). This situation is more realistic from the physical point of view. For the critical value \( \omega = -1/3 \) the solutions collapse to the non-interacting case in approaches 2 and 3. For the case of interacting scalar and matter models with \( G \)-var we conclude that the solutions are valid \( \forall \omega \in (-1/3, 1] \), while \( G \) behaves as a growing time function \( G(t) \sim t^2 \), and the potential behaves as a positive decreasing function, \( V \sim t^{-4} \), as in the above studied models.

As we have commented along the paper, the SSS may be considered as asymptotic solutions for general solutions. In the same way we have pointed out that the PF and the scalar solutions are stable from the DS point of view \( (22, 31) \). Since in the TVC scheme we have obtained the same behaviour for the scale factors than in these solutions, we may "conjecture" that the solutions obtained in this framework are also stable, note that always we obtain relationships like that, \( G_{\text{pm}} \sim t^{-2} \sim G_{\rho_0} \). A simple perturbation analysis may be carried out in order to prove this conjecture.

For the scalar-tensor theories we arrive in all the cases that the solution is only valid for a small a small neighborhood of \( \gamma_c , \mathcal{E}(\gamma_c = -\frac{1}{4}) = I_1 \). For \( \gamma_c , \) then \( \rho_0 > 0 \) but \( \Lambda_0 = 0 \) and \( \tilde{n} = 0 \), therefore \( \phi = \phi_0 \) and this means that \( G = G_0 \) (i.e. constant), i.e. for this specific value, the solution is the same than the one obtained in the classical model for a perfect fluid. In the same way we may compare the JBD solution with the one obtained in the classical situation when \( G \) and \( \Lambda \) vary. If \( \gamma > \gamma_c \), \( \forall \gamma \in I_1 \) then \( \tilde{n} < 0 \) and therefore \( G_{\text{eff}} \approx \phi^{-1} \), is a growing time function, and \( \Lambda \) behaves as a positive decreasing time function. Note that \( \tilde{n} > 0 \), \( \forall \gamma \in \mathcal{E}, \gamma < \gamma_c, \tilde{n} = 0 \) if \( \gamma = \gamma_c \), and \( \tilde{n} < 0 \forall \gamma \in \mathcal{E}, \gamma > \gamma_c \). Once again, if we take into account the current observations, then we arrive to the conclusion that \( G \) behaves as a growing time function \( G(t) \sim t^{3\gamma - 1} \), and the cosmological constant behaves as a positive decreasing function, \( \Lambda \sim t^{-2} \), as in the above.
studied models. We have performed the same calculation considering the scalar-tensor theory defined by

\[ S = \frac{1}{8\pi} \int d^4x \sqrt{-g} \left\{ \frac{1}{2} \left[ \phi R - \frac{\omega}{\phi} \phi_\alpha \phi_\alpha - 2U(\phi) \right] + \mathcal{L}_M \right\} \]

arriving to the same solutions that the obtained ones where, which is not obvious as we have pointed out in (2).

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Appendix A: Interacting scalar and matter models

In this appendix we will prove the results stated in section II. We shall study the admitted form for the potential, as well as, for the energy density, in the SS framework. For this purpose we study the different DE through the Lie group method (11).

1. Approach 1

We may rewrite Eq. (A7) in an alternative way

\[ \rho_m' + r \rho_m t^{-1} = -\phi'' \left( \phi' + h_0 t^{-1} + \frac{dV}{d\phi} \right), \] (A1)

where \( r = (\omega + 1) h, H = ht^{-1} \). Lie’s method applied to Eq. (A1) brings us to get

\[ t^2 \xi_{\phi\phi} = 0, \] (A2)
\[ -2t^2 \xi_{t\phi} + 2ht \xi_\phi + t^2 \eta_{\phi\phi} = 0, \] (A3)
\[ 2t^2 \eta_{tt} - t^2 \xi_{tt} + 3t^2 \xi_\phi V_\phi + th \xi_t - h \xi = 0, \] (A4)

\[ t^2 \eta V_{\phi\phi} + t^2 \eta_{tt} + 4t^2 \xi_\phi (tp' + rp) + t^2 V_\phi (2\xi_t - \eta_\phi) + h t \eta_t = 0, \] (A5)
\[ tp' (3t \xi_t - 2t \phi_t + r \xi) + \]
\[ r \rho (3t \xi_t - 2t \eta_\phi - \xi) + t^2 \xi \rho'' = 0, \] (A6)
\[ \eta_t (r \rho + tp') = 0. \] (A7)

Then, we may impose some symmetries in such a way that condition on the integrability of the functions \( V \) and \( \rho \) are obtained. For example, the symmetry \( (\xi = t, \eta = 1) \), which brings us to get the invariant solution \( \phi = \ln t \), we get, from Eq. (A3)

\[ V_{\phi\phi} + 2V_\phi = 0, \iff V = \exp (-2\phi), \] and from Eq. (A13)

\[ \rho'' = -(3 + r) \frac{p'}{t} - 2r \frac{\rho}{t^2} \iff \rho = C_1 t^{-2} + C_2 t^{-r}, \] (A8)

setting \( C_2 = 0 \), we get

\[ \phi = \ln t, \quad V = \exp (-2\phi), \quad \rho = C_1 t^{-2}. \] (A9)

2. Approach 2

In this model the conservation equations read:

\[ \rho_m' + (\omega + 1) \rho_m H = -\phi' q^\phi, \] (A10)
\[ \Box \phi + \frac{dV}{d\phi} = q^\phi, \] (A11)

If we assume such functional relationship, \( q^\phi = \beta \rho_m \), then we obtain for Eq. (A11) the following system of PDE

\[ t^2 \xi_{\phi\phi} = 0, \] (A12)
\[ -2t^2 \xi_{t\phi} + 2ht \xi_\phi + t^2 \eta_{\phi\phi} = 0, \] (A13)
\[ 3t^2 \xi_\phi (V_\phi - \beta \rho) + th \xi_t - h \xi + 2t^2 \eta_{tt} - t^2 \xi_{tt} = 0, \] (A14)

\[ t^2 \eta V_{\phi\phi} + t^2 \eta_{tt} + 2t^2 \xi_t (V_\phi - \beta \rho) - \]
\[ t^2 \eta_\phi (V_\phi - \beta \rho) + h t \eta_t - \beta^2 t^2 \xi \rho' = 0. \] (A15)

The symmetry \( (\xi = t, \eta = 1) \implies \phi = \ln t \), brings us to obtain the following restriction, from Eq. (A15) we get

\[ V_{\phi\phi} + 2V_\phi = \beta (tp' + 2\rho), \]
\[ V = \exp (-2\phi), \quad \beta = t^{-2}, \]

While Eq. (A10) admits the following solutions

\[ \phi = -\frac{1}{\beta} (r \ln t - \ln \rho) + C_1, \]

in an alternative way

\[ \rho = \exp \left[ -\beta \phi - r \ln t + \beta C_1 \right] = \rho_0 \frac{1}{r^r + \beta}, \]

\[ r + \beta = 2. \]

Therefore we have obtained:

\[ \phi = \ln t, \quad V = \exp (-2\phi), \quad \rho = \rho_0 t^{-2}. \] (A16)
3. Approach 3

The conservation equations read

\[ \rho'_m + (\omega + 1) \rho_m H = \delta H \rho_m, \quad (A17) \]
\[ \phi' \left( \nabla \phi + \frac{dV}{d\phi} \right) = -\delta H \rho_m, \quad (A18) \]

From Eq. (A17) we get
\[ \rho_m = \rho_0 t^{b(\delta-(\omega+1))}, \]
and therefore Eq. (A18) yields
\[ \phi' \left( \phi'' + h\phi' t^{-1} + \frac{dV}{d\phi} \right) = -At^a, \quad (A19) \]

with \( a = h(\delta-(\omega+1))-1 \), \( A = \delta h \rho_0 \). We have the next system of PDE for Eq. (A19)

\[ \xi_{\phi\phi} t^2 = 0, \quad (A20) \]
\[ -2t^2 \xi_{t\phi} + 2ht \xi_{\phi} + t^2 \eta_{\phi\phi} = 0, \quad (A21) \]
\[ 3t^2 \xi_{\phi\phi} V_{\phi} + th \xi_t - h \xi_t + 2t^2 \eta_{t\phi} - t^2 \xi_{tt} = 0, \quad (A22) \]
\[ 3At^{2+a} \xi_t - 2At^{2+a} \eta_{\phi} + aAt^{1+a} \xi = 0, \quad (A23) \]
\[ At^{2+a} \eta_t = 0. \quad (A24) \]

\[ t^2 \eta_{\phi\phi} V_{\phi} + t^2 \eta_{tt} + 4At^{2+a} \xi_{\phi} V_{\phi} - \]
\[ t^2 \eta_{t\phi} V_{\phi} + h \eta_t + 2t^2 \xi_{t} V_{\phi} = 0, \quad (A25) \]

The symmetry \( (\xi = t, \eta = 1) \), bring us to get from Eq. (A29)
\[ t^2 \eta_{\phi\phi} V_{\phi} + 2t^2 \xi_{t} V_{\phi} = 0, \]
\[ 3At^{2+a} \xi_t - aAt^{1+a} \xi = 0, \]
simplifying
\[ V_{\phi\phi} + 2aV_{\phi} = 0 \quad V = V_0 \exp(-2a\phi), \]

where \( a = -3 \). Therefore we get:
\[ \rho_m = \rho_0 t^{-2}, \quad \phi = \ln t, \quad V = \exp(-2\phi). \quad (A26) \]

4. A generalization

In this model we try to generalize the previous approaches. We consider a function \( Q = \delta H f(t) \), \( f(t) \) is an unknown function. Therefore applying the same procedure we would like to determine the admitted form for \( f \) in order to obtain a SSS. For this purpose we consider the following system of DE

\[ \rho'_m + (\omega + 1) \rho_m H = \delta H f(t), \quad (A27) \]
\[ \phi' \left( \nabla \phi + \frac{dV}{d\phi} \right) = -\delta H f(t), \quad (A28) \]

From Eq. (A27) we get
\[ \rho = \left( \delta h \int f(t) t^{-1} dt + C_1 \right) t^{-r}. \quad (A29) \]

where \( r = h(\omega + 1) \), and \( H = ht^{-1} \). In the same way we may analyze Eq. (A28) through the Lie group method
\[ \phi' \left( \phi'' + h\phi' t^{-1} + \frac{dV}{d\phi} \right) = -\delta h t^{-1} f(t), \quad (A30) \]

getting the following system of PDE:

\[ \xi_{\phi\phi} t^2 = 0, \quad (A31) \]
\[ -2t^2 \xi_{t\phi} + 2ht \xi_{\phi} + t^2 \eta_{\phi\phi} = 0, \quad (A32) \]
\[ 3t^2 \xi_{\phi\phi} V_{\phi} + th \xi_t - h \xi_t + 2t^2 \eta_{t\phi} - t^2 \xi_{tt} = 0, \quad (A33) \]
\[ 3\delta h t f(t) \xi_t - 2\delta h f(t) \eta_t + \delta h (t f'(t) - f) \xi = 0, \quad (A34) \]
\[ \delta h f(t) \eta_t = 0, \quad (A35) \]
\[ t^2 \eta_{\phi\phi} + t^2 \eta_{tt} + 4\delta h f(t) \xi_{\phi} V_{\phi} - \]
\[ t^2 \eta_{t\phi} V_{\phi} + h \eta_t + 2t^2 \xi_{t} V_{\phi} = 0. \quad (A36) \]

The symmetry \((\xi = t, \eta = 1)\), bring us to get from Eq. (A36)
\[ V_{\phi\phi} + 2V_{\phi} = 0 \quad V = V_0 \exp(-2\phi), \]
\[ t f'(t) + 2f = 0 \quad f = f_0 t^{-2}, \]

then Eq. (A29) gives
\[ \rho = \left( \delta h \int t^{-3} dt + C_1 \right) t^{-r}, \quad (A37) \]

and therefore
\[ \rho = C_1 t^{-2} + C_2 t^{-r} \]

Therefore, setting \( C_2 = 0 \), we have obtained the following results:
\[ \rho_m = \rho_0 t^{-2}, \quad \phi = \ln t, \quad V = \exp(-2\phi), \quad f = f_0 t^{-2}. \quad (A38) \]

As we can see we can consider any function \( f \) that behaves as \( f = f_0 t^{-2} \). This fact justifies the employ of \( f = \rho_m \), as in the previous approaches, but we can also consider other forms for \( f \) as for example, \( f = \rho_\phi \approx t^{-2} \), or \( f = \rho_m + \rho_\phi \approx t^{-2} \), etc.
Appendix B: Matter and scalar fields with $G$–varying

In this appendix we will study the equation

$$
\rho'_{m} + \phi''_{\phi} \phi' + \frac{dV}{d\phi} \phi' + H \left( \phi'^{2} + (\rho_{m} + p_{m}) \right)
= -G' \left( \rho_{m} + \frac{1}{2} \phi'^{2} + V \right),
$$

that we may study as follows. In the first case and in analogy with the previous appendix we will study the whole equation i.e. Eq. (B1) without splitting it while in the second case we will split it for a suitable coupling function.

1. Case 1

Eq. (B1) may be written in the following form

$$
\left( \phi'' + \left( ht^{-1} + \frac{G'}{2G} \right) \phi' + \frac{dV}{d\phi} \phi \right) \phi
+ \frac{G'}{G} V + \rho'_{m} + \left( rt^{-1} + \frac{G'}{G} \right) \rho_{m} = 0,
$$

where $r = (1 + \gamma) h$ and $H = ht^{-1}$. The standard procedure brings us to get the following system of PDE

$$
t^{2}G^{2} \xi_{\phi \phi} = 0,  \tag{B2}
$$

$$
-2t^{2}G^{2} \eta_{\phi \phi} - 4t^{2}G^{2} \xi_{t \phi} + 2tG \xi_{\phi} (2hG + tG') = 0,  \tag{B3}
$$

$$
6tG^{2} \xi_{\phi} (\rho + tV_{\phi}) + 4t^{2}G^{2} \eta_{\phi \phi} - 2t^{2}G^{2} \xi_{tt} +
$$

$$
(2hG^{2} - t^{2}G'^{2} + t^{2}GG'') \xi + tG (2hG + tG') \xi_{t} = 0,  \tag{B4}
$$

$$
4t^{2}G^{2}V_{\phi} \xi_{t} - 2t^{2}G^{2}V_{\phi} \eta_{\phi} +
8tG \left( tG' (V + \rho) + tG' \rho \right) \xi_{\phi} +
$$

$$
2t^{2}G^{2} \eta_{tt} + 2t^{2}G^{2} V_{\phi \phi} \eta + tG (2hG + tG') \xi_{t} = 0,  \tag{B5}
$$

$$
2t^{2} G' G' V_{\phi} \eta + 6tG (tG'V + rG + tG' \rho + tG' \rho') \xi_{t} -
4tG (tG'V + rG + tG' \rho + tG' \rho) \xi_{\phi} +
$$

$$
2t^{2} G \left( G''V + G' \rho' + t^{-1} G \rho' + G'' \rho \right) \xi
+ 2t^{2}G (G'' - rGt^{-2} - G^{-1} G^{2} (V + \rho)) \xi = 0,  \tag{B6}
$$

$$
-2t^{2}G^{2} \left( \rho + G^{-1} G' V + t^{-1} r \rho + G' \rho \right) \eta_{t} = 0.  \tag{B7}
$$

The symmetry ($\xi = t$, $\eta = \alpha \phi$) then we get.

From Eq. (B4) we obtain

$$
G'' = \frac{G^{2}}{G} - \frac{G'}{t}, \quad \Rightarrow \quad G = G_{0} t^{\beta}.
$$

Now from Eq. (B3) we get

$$
V_{\phi} = \left( \frac{\alpha - 2}{\alpha} \right) \frac{V_{\phi}}{\phi} \Rightarrow V (\phi) = V_{0} \phi^{\frac{2}{\beta} (\alpha - 1)}.
$$

From Eq. (B6) we obtain since $\phi = t^{\alpha}$ and $V (\phi) = V_{0} \phi^{\frac{2}{\beta} (\alpha - 1)} = V_{0} t^{2 (\alpha - 1)}$, and $G = G_{0} t^{\beta}$ then we get

$$
\rho'' = -c_{1} \frac{\rho'}{t} - c_{2} \frac{\rho}{t^{2}} - c_{3} t^{2 (\alpha - 2)},
$$

finding that a solution is:

$$
\rho_{m} = \rho_{0} (t + t_{0})^{2 (\alpha - 1)}.  \tag{B8}
$$

Therefore we get (after redefining the constants)

$$
\phi = \phi_{0} (t + t_{0})^{-\alpha}, \quad V (t) = \beta (t + t_{0})^{-2 (\alpha + 1)},
$$

$$
G = G_{0} (t + t_{0})^{\beta}, \quad \rho_{m} = \rho_{0} (t + t_{0})^{-2 (\alpha + 1)},  \tag{B9}
$$

with $g = 2 \alpha$, since $G \rho \approx t^{-2}$, and $p_{m} = \omega \rho_{m}$.

2. Case 2

In this case we may split Eq. (B1) as follows

$$
\rho'_{m} + (\rho_{m} + p_{m}) H + \frac{G'}{G} \rho_{m} = \delta H \rho_{m},  \tag{B10}
$$

$$
\phi' \left( \phi'' + \phi' H + \frac{dV}{d\phi} \phi \right) + \frac{G'}{G} \phi_{\phi} = -\delta H \rho_{m},  \tag{B11}
$$

in such a way that if we study both equations through the LG method then we get.

For Eq. (B11) we get:

$$
t^{2}G^{2} \xi_{\phi \phi} = 0,  \tag{B12}
$$

$$
2t^{2}G^{2} \eta_{\phi \phi} - 4t^{2}G^{2} \xi_{t \phi} + 2tG \xi_{\phi} (2hG + tG') = 0,  \tag{B13}
$$

$$
4t^{2}G^{2} V_{\phi} \xi_{t} - 2t^{2}G^{2} V_{\phi} \eta_{\phi} +
8tG \left( tG' (V + \rho) + tG' \rho \right) \xi_{\phi} +
$$

$$
2t^{2}G^{2} \eta_{tt} + 2t^{2}G^{2} V_{\phi \phi} \eta + tG (2hG + tG') \xi_{t} = 0,  \tag{B15}
$$

$$
2t^{2} G' G' V_{\phi} \eta + 6tG (tG'V + rG + tG' \rho + tG' \rho') \xi_{t} -
4tG (tG'V + rG + tG' \rho + tG' \rho) \xi_{\phi} +
$$

$$
2t^{2} G \left( G''V + G' \rho' + t^{-1} G \rho' + G'' \rho \right) \xi
+ 2t^{2}G (G'' - rGt^{-2} - G^{-1} G^{2} (V + \rho)) \xi = 0,  \tag{B6}
$$

$$
-2t^{2}G^{2} \left( \rho + G^{-1} G' V + t^{-1} r \rho + G' \rho \right) \eta_{t} = 0.  \tag{B7}
$$

The symmetry ($\xi = t$, $\eta = \alpha \phi$) then we get.

From Eq. (B4) we obtain

$$
G'' = \frac{G^{2}}{G} - \frac{G'}{t}, \quad \Rightarrow \quad G = G_{0} t^{\beta}.
$$

Now from Eq. (B3) we get

$$
V_{\phi} = \left( \frac{\alpha - 2}{\alpha} \right) \frac{V_{\phi}}{\phi} \Rightarrow V (\phi) = V_{0} \phi^{\frac{2}{\beta} (\alpha - 1)}.
$$

From Eq. (B6) we obtain since $\phi = t^{\alpha}$ and $V (\phi) = V_{0} \phi^{\frac{2}{\beta} (\alpha - 1)} = V_{0} t^{2 (\alpha - 1)}$, and $G = G_{0} t^{\beta}$ then we get

$$
\rho'' = -c_{1} \frac{\rho'}{t} - c_{2} \frac{\rho}{t^{2}} - c_{3} t^{2 (\alpha - 2)},
$$

finding that a solution is:

$$
\rho_{m} = \rho_{0} (t + t_{0})^{2 (\alpha - 1)}.  \tag{B8}
$$

Therefore we get (after redefining the constants)

$$
\phi = \phi_{0} (t + t_{0})^{-\alpha}, \quad V (t) = \beta (t + t_{0})^{-2 (\alpha + 1)},
$$

$$
G = G_{0} (t + t_{0})^{\beta}, \quad \rho_{m} = \rho_{0} (t + t_{0})^{-2 (\alpha + 1)},  \tag{B9}
$$

with $g = 2 \alpha$, since $G \rho \approx t^{-2}$, and $p_{m} = \omega \rho_{m}$.
\[ 2t^2GG'V_{\phi\phi}\eta + 6tG(tG^2V_{\phi} + G\delta \rho ) \xi_t - 4tG(tG^2V_{\phi} + G\delta \rho ) \eta_\phi + \]

\[ 2 (\delta hG^2 (t^2 \rho - \rho ) + 2t^2V_{\phi} (GG''-G'^2)) \xi = 0, \quad (B16) \]

\[ 2tG (tG^2V_{\phi} + G\delta \rho ) \eta = 0. \quad (B17) \]

The symmetry \((\xi = t, \eta = \alpha \phi) \implies \phi = t^\alpha\), bring us to get from Eq. \((B14)\)

\[ G'' = \frac{G'^2}{G} - \frac{G'}{t} \implies G = G_0 t^\beta. \]

From Eq. \((B15)\)

\[ V_{\phi\phi} = \frac{(\alpha - 2)}{\alpha} \frac{V_{\phi}}{\phi} \implies V = V_0 \phi^{\frac{2(\alpha-1)}{\alpha}}. \]

Now, from Eq. \((B16)\) we obtain an ODE for the energy density

\[ 2 (1 - \alpha) \rho + \rho' t = 0, \quad \implies \rho = \rho_0 t^{2(\alpha-1)}. \quad (B18) \]

Note that for Eq. \((B10)\) we get a direct integration since

\[ \rho_m' + (1 + \gamma - \delta) H \rho_m + \frac{G'}{G} \rho_m = 0, \]

then \(\rho_m G = K t^{-r}\), where \(r = (1 + \gamma - \delta)\).

Therefore we have arrived to the same conclusion as in the above case i.e.:

\[ \phi = \phi_0 (t + t_0)^{-\alpha}, \quad V(t) = \beta (t + t_0)^{-2(\alpha+1)}, \]

\[ G = G_0 (t + t_0)^{\beta}, \quad \rho_m = \rho_0 (t + t_0)^{-2(\alpha+1)}, \quad (B19) \]

with \(g = 2\alpha\), since \(G \rho \approx t^{-2}\).