Affine structures on filiform Lie algebras

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The aim of this note is to prove that every non characteristically nilpotent filiform algebra is provided with an affine structure. We generalize this result to the class of nilpotent algebras whose derived algebra admits non singular derivation.

1 Affine structures on Lie algebras

Definition 1.1 An affine connexion on a manifold $M$ is a law $\nabla$ which gives for every vectorfield $X$ an endomorphism $\nabla_X$ of $\mathcal{D}^1(M)$ the space of vectorfields on $M$ satisfying the two conditions

(1) $\nabla_{fX+gY} = f\nabla_X + g\nabla_Y$;
(2) $\nabla_X(fY) = f\nabla_X(Y) + (Xf)Y$

for $f, g \in C^\infty(M)$, $X, Y \in \mathcal{D}^1(M)$.

If $M$ is an Lie group $G$, then the affine connexion $\nabla$ is called left invariant if the connexion $\nabla'$ given by

$$\nabla'_X(Y) = (\nabla_X(\Phi Y))\Phi^{-1}$$

$X, Y \in \mathcal{D}^1(M)$

satisfies

$$\nabla' = \nabla$$

for every left translation $\Phi$. It is equivalent to say that the left translations are affine mapping on the affine Lie group $(G, \nabla)$.

Definition 1.2 The torsion of the affine connexion $\nabla$ is the tensor $T$ defined by

$$T(X,Y) = \nabla_X(Y) - \nabla_Y(X) - [X,Y]$$

The curvature of $\nabla$ is given by the tensor $C$ defined by

$$C(X,Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}$$
In this work we consider the left invariant connection satisfying $T = 0, C = 0$.

Let $\mathfrak{g}$ be the Lie algebra of the affine group $G$. As the operator $\nabla$ is left invariant, it induces a mapping, always noted $\nabla$:

$$\nabla : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}.$$ 

If we put $X.Y = \nabla X Y$ for every $X, Y \in (\mathfrak{g})$, then the condition on the curvature and torsion imply that this product satisfies:

1) $X. (Y.Z) - Y. (X.Z) = (X.Y) . Z - (Y.X) . Z$

2) $X.Y - Y.X = [X, Y]$ for every $X, Y, Z \in \mathfrak{g}$.

**Definition 1.3** A such operator $\nabla$ on the Lie algebra $\mathfrak{g}$ is called an affine structure on $\mathfrak{g}$.

2 Affine structure on nilpotent Lie algebras

The problem of existence of affine structures on nilpotent Lie algebras has been put by John Milnor. Recently, Benoist has proposed examples of 11-dimensional nilpotent Lie algebras which are not endowed with such structure. These Lie algebras have the following structure

$$\begin{align*}
[X_1, X_i] &= X_{i+1} \quad i = 2, \ldots, 10 \\
[X_2, X_4] &= X_6 \\
[X_2, X_6] &= -5X_8 + 2X_9 + 2tX_{10} \\
[X_2, X_8] &= \frac{26}{25}X_{10} + \frac{29}{25}X_{11} \\
[X_3, X_4] &= 3X_7 - X_8 - tX_9 \\
[X_3, X_6] &= -\frac{421t}{80}X_9 - \frac{62}{25}X_{10} + \frac{448 + 15352t}{2000}X_{11} \\
[X_3, X_8] &= \frac{321}{80}X_{11} \\
[X_4, X_6] &= \frac{472}{25}X_{10} - \frac{24}{25}X_{11} \\
[X_5, X_6] &= \frac{1377}{80}X_{11} \\
[X_2, X_3] &= X_5 \\
[X_2, X_7] &= -2X_7 + X_8 + tX_9 \\
[X_2, X_9] &= \frac{13}{40}X_9 + \frac{47}{40}X_{10} + \frac{448 + 24754t}{2000}X_{11} \\
[X_3, X_5] &= 3X_8 - X_9 - tX_{10} \\
[X_3, X_7] &= -\frac{39}{5}X_{10} + \frac{24}{25}X_{11} \\
[X_4, X_5] &= \frac{24}{5}X_9 - \frac{24}{25}X_{10} + \frac{448 - 35252t}{2000}X_{11} \\
[X_4, X_7] &= -\frac{189}{16}X_{11} \quad t \in \mathbb{R}
\end{align*}$$

**Classical examples**
1. Dimensions less than 7

Every Lie nilpotent Lie algebras of dimension less or equal to 7 admits an affine structure.
2. Symplectic Lie algebras

Let \( g \) a \( 2p \)-dimensional Lie algebra endowed to a symplectic form \( \theta \in \Lambda^2 g^* \).
It satisfies \( d\theta = 0 \) where

\[
d\theta(X,Y,Z) = \theta(X,[Y,Z]) + \theta(Y,[Z,X]) + \theta(Z,[X,Y]).
\]

For every \( X \in g \) let \( f(X) \) be defined by

\[
\theta(\text{ad}X(Y),Z) = \theta(Y,f(X)(Z)).
\]

Then \( \nabla_X Y = f(X)(Y) \) is an affine structure \( g \).

3. Lie algebras admitting a regular derivation

Such an algebra is necessary nilpotent. More there exists a diagonalizable
regular derivation. Let \( f \) be such derivation. For every \( X \in g \) we put

\[
\nabla_X = f^{-1} \circ \text{ad}X \circ f.
\]

Then

\[
\nabla_X Y - \nabla_Y X = f^{-1}([X,f(Y)] - [Y,f(X)])
= f^{-1}(f([X,Y]))
= [X,Y].
\]

We have also

\[
\nabla_X \nabla_Y - \nabla_Y \nabla_X = f^{-1} \circ \text{ad}X \circ f \circ f^{-1} \circ \text{ad}Y \circ f
- f^{-1} \circ \text{ad}Y \circ f \circ f^{-1} \circ \text{ad}X \circ f
= f^{-1} \circ \text{ad}[X,Y] \circ f
= \nabla_{[X,Y]}.
\]

This operator \( \nabla \) defines an affine structure on \( g \).

3 Non characteristically filiform algebra

Recall that a Lie algebra is called characteristically nilpotent is every derivation
is nilpotent. Examining the counter examples of Benoist and Burde we the
following conjecture becomes natural:

**Conjecture** [Kh] Every nilpotent Lie algebra which does not admit affine
structure is characteristically nilpotent.

Of course, the converse is false. There exist seven dimensional characteristically
nilpotent Lie algebras and these algebras have affine structures.

The aim of this section is to prove the following result :

**Theorem 3.1** Every filiform non characteristically nilpotent Lie algebra admits
an affine structure
Proof. A $n$-dimensional nilpotent Lie algebra is called filiform if the central descending sequence satisfies:

$$\mathfrak{g} \supset C^1 \mathfrak{g} \supset \cdots \supset C^{n-2} \mathfrak{g} \supset \{0\} = C^{n-1} \mathfrak{g}$$

and we have

$$\begin{align*}
\dim C^1 \mathfrak{g} &= n - 2, \\
\dim C^i \mathfrak{g} &= n - i - 1, & i = 1, \ldots, n - 1.
\end{align*}$$

For a non characteristically nilpotent Lie algebra $\mathfrak{g}$, let us call rank of $\mathfrak{g}$ the dimension of a maximal exterior torus of derivations (a maximal abelian subalgebra of $\text{Der}(\mathfrak{g})$ of which elements are semi-simple derivations). We have the following results [G.K]:

1. If the Lie algebra $\mathfrak{g}$ is filiform, its rank $r(\mathfrak{g})$ satisfies

$$r(\mathfrak{g}) \leq 2.$$ 

2. Every filiform Lie algebras of rank 2 is isomorphic to $L_n$ or $Q_n$, where $L_n$ and $Q_n$ are the $n$-dimensional filiform Lie algebras defined by

$$L_n : \{ [Y_1, Y_j] = Y_{1+j}, \quad j = 2, \ldots, n - 1 \}$$

$$Q_n = \left\{ \begin{array}{ll}
[Y_1, Y_j] = Y_{1+j}, & j = 2, \ldots, n - 1 \\
[Y_i, Y_{n-i+1}] = (-1)^{i+1} Y_n, & i = 2, \ldots, n/2 \end{array} \right.$$

For each Lie algebra, a maximal exterior torus is precisely determined.

If $\mathfrak{g} = L_n$, there exists a torus generated by the diagonal derivations:

$$f_1(Y_i) = 0, \quad f_1(Y_i) = Y_i, \quad 2 \leq i \leq n$$

$$f_2(Y_1) = Y_1, \quad f_2(Y_i) = iY_i, \quad 2 \leq i \leq n$$

the basis $\{Y_i\}$ being as above.

If $\mathfrak{g} = Q_n$, the basis $\{Y_i\}$ is not a basis of eigenvectors for a diagonalizable derivation. We can consider the new basis given by

$$Z_1 = Y_1 - Y_2, Z_2 = Y_2, \ldots, Z_n = Y_n$$

This basis satisfies

$$[Z_1, Z_j] = Z_{1+j}, \quad j = 2, \ldots, n - 2; [Z_i, Z_{n-i+1}] = (-1)^{i+1} Z_n, \quad i = 2, \ldots, n/2$$

Then the diagonal derivations

$$f_1(Z_1) = 0, \quad f_1(Z_i) = Z_i, \quad 2 \leq i \leq n - 1; \quad f_1(Z_n) = 2Z_n$$

$$f_2(Z_1) = Z_1, \quad f_2(Z_i) = (i - 2)Z_i, \quad 2 \leq i \leq n - 1; \quad f_2(Z_n) = (n - 3)Z_n.$$ 

generates a maximal exterior torus of derivations.

3. Every filiform Lie algebra of rank 1 and dimension $n$ is isomorphic to one of the following Lie algebras
i) $A_n^k (\lambda_1, ..., \lambda_{t-1}) , \ t = \left\lfloor \frac{n+k}{2} \right\rfloor , \ 2 \leq k \leq n-3$

\[
\begin{cases}
   [Y_i, Y_i] = Y_{i+1}, & i = 2, ..., n-1 \\
   [Y_i, Y_{i+1}] = \lambda_{i-1}Y_{2i+k-1} , & 2 \leq i \leq t \\
   [Y_i, Y_j] = a_{ij}Y_{i+j+k-2} , & 2 \leq i \leq j \ \ i+j+k-2 \leq n
\end{cases}
\]

ii) $B_n^k (\lambda_1, ..., \lambda_{t-1}) , \ n = 2m , \ t = \left\lfloor \frac{n+k}{2} \right\rfloor , \ 2 \leq k \leq n-3$

\[
\begin{cases}
   [Y_i, Y_i] = Y_{i+1} & i = 2, ..., n-2 \\
   [Y_i, Y_{n-i+1}] = (-1)^{i+1} Y , & i = 2, ..., n-1 \\
   [Y_i, Y_{i+1}] = \lambda_{i-1}Y_{2i+k-1} , & i = 2, ..., t \\
   [Y_i, Y_j] = a_{ij}Y_{i+j-k-2} , & 2 \leq i, j \leq n-2, i+j+k-2 \leq n-2, \ j \neq i+1
\end{cases}
\]

iii) $C_n (\lambda_1, ..., \lambda_t) , \ n = 2m+2 , \ t = m-1$

\[
\begin{cases}
   [Y_i, Y_i] = Y_{i+1} & i = 2, ..., n-2 \\
   [Y_i, Y_{n-i+1}] = (-1)^{i-1} Y_n , & i = 2, ..., m+1 \\
   [Y_i, Y_{n-i-2k+1}] = (-1)^{i+1} \lambda_k Y_n
\end{cases}
\]

The non defined brackets are equal to zero. In this theorem, $[x]$ denotes the integer part of $x$ and $(\lambda_1, ..., \lambda_t)$ are non simultaneously vanishing parameters satisfying polynomial equations associated to the Jacobi conditions. Moreover, the constants $a_{ij}$ satisfy

\[ a_{ij} = a_{ij+1} + a_{i+1,j} \]

and $a_{ii+1} = \lambda_{i-1}$.

We can easily see that the filiform algebra $L_n, Q_n$ or of type $A^n$ or $B^n$ admit regular derivations. Then they admits affine structure. Let us consider the case $C^n$. This algebra is of rank 1. The exterior torus of derivation is generated by

\[ f(Y_i) = 0, f(Y_i) = Y_i, \ \ i = 2, ..., n-1, \ \ f(Y_n) = 2Y_n. \]

Thus every derivation is singular.

**Lemma 3.2** The restriction of the derivation $f$ to the derived subalgebra $D(g)$ is a regular derivation of $D(g)$.

Let us consider a vectorial endomorphism $g$ of $g$ which leaves invariant $D(g)$, and such that the restriction to $D(g)$ satisfies $f \circ g = Id$. Then the bilinear mapping given by

\[ \nabla_X = g \circ adX \circ f. \]

defines an affine structure on $C^n$. In fact

\[ \nabla_X(Y) - \nabla_Y(X) = g \circ adX \circ f(Y) - g \circ adY \circ f(X) = g(f[X, Y]) \]

because $f$ is a derivation. As $g = f^{-1}$ on the derived subalgebra, we can deduce

\[ \nabla_X(Y) - \nabla_Y(X) = [X, Y]. \]
In the same way
\[ \nabla_X \nabla_Y(Z) - \nabla_Y \nabla_X(Z) = g[X, [Y, f(Z)]] - g[Y, [X, f(Z)]] = -g[f(Z), [X, Y]] \]

Then
\[ \nabla_X \nabla_Y(Z) - \nabla_Y \nabla_X(Z) = \nabla_{[X,Y]}(Z) \]

This proves the theorem.

4 A theorem of existence of affine structure

The previous proof gives the following result

**Theorem 4.1** Let \( g \) be a nilpotent Lie algebra admitting a derivation of which restriction to the derived subalgebra is regular. Then this algebra admits an affine structure.

**BIBLIOGRAPHY**

[Be] Besnoit Y. Une nilvariété non affine. *J.Diff.Geom.* 41 (1995), 21-52.
[Bu1] Burde D. Left invariant affine structure on reductive Lie groups. *J. Algebra.* 181, (1996), 884-902.
[Bu2] Burde D. Affine structures on nilmanifolds. *Int. J. of Math*, 7 (1996), 599-616.
[G.K] Goze M., Khakimdjanov Y., *Nilpotent Lie algebras*. Kluwer editor. 1995.
[G.R] Goze M., Remm E., Affine structures on abelian Lie algebras. Linear Algebra and its Applications. 360, 2003. 215-230.
[Kh] Khakimdjanov Y., Characteristically nilpotent and affine Lie algebra. *Actes Colloque Vigo 2000* To appear.
[R] Remm E., Structures affines sur les algèbres de Lie et opérades Lie-admissibles. Thèse, Mulhouse (2001).
[R2] Remm E., Structures affines on contact Lie algebra. xxx: math.RA/0109077