MAXIMAL DIVISORIAL IDEALS AND $t$-MAXIMAL IDEALS

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ABSTRACT. We give conditions for a maximal divisorial ideal to be $t$-maximal and show with examples that, even in a completely integrally closed domain, maximal divisorial ideals need not be $t$-maximal.

INTRODUCTION

The $v$-operation and the $t$-operation are the the best known and most useful star operations; mainly because the structure of certain semigroups of $t$-ideals reflects the multiplicative properties of an integral domain. In this context an important role is played by the prime and the maximal $v$- and $t$-ideals.

Since the $t$-operation is a star operation of finite type, a domain $R$ has always $t$-maximal ideals. On the other hand, the set of $v$-maximal ideals may be empty.

In this paper we deal with the following question:

Assume that $M$ is a $v$-maximal ideal of $R$, is $M$ necessarily a $t$-maximal ideal?

We show that although the answer is positive in a large class of domains, namely in the class of $v$-coherent domains, it is negative in general. In fact we give two examples of a $v$-maximal ideal $P$ that is not a $t$-maximal ideal. In the first example $P$ is an upper to zero of a completely integrally closed polynomial ring, thus $P$ is $v$-invertible. In the second example $P$ is a strongly divisorial ideal of an integrally closed semigroup ring.

1. Preliminaries and notations

Throughout this paper $R$ will denote an integral domain with quotient field $K$. We will refer to a fractional ideal as an ideal and will call a fractional ideal contained in $R$ an integral ideal.

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We recall that a star operation is an application $I \rightarrow I^*$ from the set $F(R)$ of nonzero ideals of $R$ to itself such that:

1. $R^* = R$ and $(aI)^* = aI^*$, for all $a \in K \setminus \{0\}$;
2. $I \subseteq I^*$ and $I \subseteq J \Rightarrow I^* \subseteq J^*$;
3. $I^{**} = I^*$.

General references for systems of ideals and star operations are [13, 16, 17, 22]. We denote by $f(R)$ the set of nonzero finitely generated ideals of $R$. A star operation $*$ is of finite type if, for each $I \in F(R)$, $I^* = \cup\{J^* | J \subseteq I$ and $J \in f(R)\}$. To any star operation $*$, we can associate a star operation $*_{f}$ of finite type by defining $I^{*_{f}} = \cup\{J^* | J \subseteq I$ and $J \in f(R)\}$. Clearly $I^{*_{f}} \subseteq I^*$.

The $v$- and the $t$-operations are particular star operations, defined in the following way.

For a pair of nonzero ideals $I$ and $J$ of a domain $R$ we let $(J : I)$ denote the set $\{x \in K | xI \subseteq J\}$ and $(J : R I)$ denote the set $\{x \in R | xI \subseteq J\}$. We set $I_v = (R : (R : I))$ and $I_t = \bigcup J_v$ with the union taken over all finitely generated ideals $J$ contained in $I$.

The $t$-operation is the finite type star operation associated to the $v$-operation.

A nonzero ideal $I$ is called a $*$-ideal if $I = I^*$. Thus a nonzero ideal $I$ is a $v$-ideal, or is divisorial, if $I = I_v$, and it is a $t$-ideal if $I = I_t$. Note that $I$ is a $t$-ideal if and only if $J_v \subseteq I$ whenever $J$ is finitely generated and $J \subseteq I$.

The set $F_*(R)$ of $*$-ideals of $R$ is a semigroup with respect to the $*$-multiplication, defined by $(I, J) \rightarrow (IJ)^*$, with unity $R$.

We say that an ideal $I \in F(R)$ is $*$-invertible if $I^*$ is a unit in the semigroup $F_*(R)$. In this case the $*$-inverse of $I$ is $(R : I)$. Thus $I$ is $*$-invertible if and only if $(I(R : I))^* = R$. Invertible ideals are ($*$-invertible) $*$-ideals.

We have $I \subseteq I^* \subseteq I_v$, so that any divisorial ideal is a $*$-ideal and any $*$-invertible ideal is $v$-invertible. In particular a divisorial ideal is a $t$-ideal and a $t$-invertible ideal is $v$-invertible.

A nonzero ideal $I$ is $*$-finite if $I^* = J^*$ for some finitely generated ideal $J$. Since the $v$- and the $t$-operation coincide on finitely generated ideals and since $I_t = J_v$ implies $I_v = J_v$, an ideal $I$ is $t$-finite if and only if $I_v = J_v$ (equivalently $(R : I) = (R : J)$) for some finitely generated ideal $J \subseteq I$. It follows that the set $f_v(R)$ of the $v$-finite divisorial ideals coincides with the set of the $t$-finite $t$-ideals. $f_v(R)$ is a sub-semigroup of $F_v(R)$. 
An ideal $I$ is $t$-invertible if and only if it is $v$-invertible and both $I$ and $(R : I)$ are $t$-finite. Hence the set of the $t$-invertible $t$-ideals of $R$, here denoted by $T(R)$, is the largest subgroup of $f_v(R)$. The importance of the notion of $t$-invertibility is well illustrated in [28].

Denoting by $Inv(R)$ the group of the invertible ideals of $R$, we have

$Inv(R) \subseteq T(R) \subseteq f_v(R) \subseteq F_v(R) \subseteq F_t(R) \subseteq F(R)$

and

$Inv(R) \subseteq f(R)$. 

Several important classes of domains may be characterized by the fact that some of these inclusions are equalities. For example $R$ is a Prüfer domain if and only if $Inv(R) = f(R)$ [13], it is a Krull domain if and only if $T(R) = F_t(R)$ [21] and it is a Prüfer $v$-multiplication domain, for short a PeMD, if and only if $T(R) = f_v(R)$ [16]. A Mori domain is a domain satisfying the ascending chain condition on integral divisorial ideals and has the property that $f_v(R) = F_t(R)$. Noetherian and Krull domains are Mori. A recent reference for Mori domains is [1]. The class of domains with the property that $F_v(R) = F_t(R)$ have been studied by several authors [3, 18, 26, 4, 5]. A domain such that $F_t(R) = F(R)$ is called in [20] a TV-domain. Examples of TV-domains are given in [20, 23, 7]. Mori and pseudovaluations domains which are not valuation domains are TV-domains.

2. When a maximal divisorial ideal is $t$-maximal

A prime $*$-ideal is called a $*$-prime. A $*$-maximal ideal is an ideal that is maximal in the set of the proper integral $*$-ideals. A $v$-maximal ideal is also called a maximal divisorial ideal. A $*$-maximal ideal is a prime ideal (if it exists).

If $*$ is a star operation of finite type, an easy application of the Zorn Lemma shows that the set $*\text{Max}(R)$ of the $*$-maximal ideals of $R$ is not empty. Moreover, for each $I \in F(R)$, $I^* = \cap_{M \in \text{Max}(R)} I^* R_M$ [16]. In particular the set of the $t$-maximal ideals is not empty and $I_t = \cap_{M \in \text{Max}(R)} I_t R_M$. On the contrary, the set of maximal divisorial ideals may be empty, like for example when $R$ is a rank-one nondiscrete valuation domain.

If $M$ is a $*$-maximal ideal that is not $*$-invertible, then $M = (M(R : M))^*$ and so $(M : M) = (R : M)$. An ideal $I$ with the property that $(R : I) = (I : I)$ is called strong. A strong ideal is never $*$-invertible and we have just seen that a $*$-maximal ideal is either $*$-invertible or strong.

An ideal which is strong and divisorial is called strongly divisorial.
Proposition 2.1.

(1) If $M$ is a maximal divisorial ideal of $R$, then $M = x^{-1}R \cap R$, for some element $x \in K$. Hence $(R : M) = (R + xR)_v$.

(2) If $P$ is a prime divisorial ideal of $R$ such that $(R : P) = R + xR$, for some element $x \in K$, then $P$ is maximal divisorial.

Proof. (1) If $x \in (R : M) \setminus R$, then $M \subseteq x^{-1}R$ and $R \not\subseteq x^{-1}R$. Since an intersection of divisorial ideals is divisorial and $M$ is $v$-maximal, we have $M = x^{-1}R \cap R = (R : R + xR)$.

(2) Let $Q$ be a proper divisorial ideal containing $P$. Since $Q$ is divisorial, $(R : Q) \not\subseteq R$. Since $(R : Q) \subseteq (R : P) = R + xR$, we see that there exists an element $y \in R$ such that $xy \in (R : Q) \setminus R$. Thus $y \not\in P$, and $xyQ \subseteq R$. Since $P = (R : R + xR)$, we obtain that $yQ \subseteq P$. Since $P$ is a prime ideal, we conclude that $Q \subseteq P$. Hence $P$ is maximal divisorial. □

In a Mori domain $R$, all the prime divisorial ideals are of the form $x^{-1}R \cap R = (R : R + xR)$ [19, Corollary 2.5].

A domain has the property that each $t$-maximal ideal is divisorial if and only if every ideal $I$ such that $(R : I) = R$ is $t$-finite [20, Proposition 2.4]. A domain of this type is called an $H$-domain in [15]. A TV-domain is clearly an $H$-domain, but the converse is not true [20, 2].

The following proposition gives conditions for a divisorial prime ideal to be a $t$-maximal ideal. A proof can be found in [10].

Proposition 2.2.

(1) A $v$-invertible divisorial prime is maximal divisorial;
(2) A $v$-finite maximal divisorial ideal is $t$-maximal;
(3) A $v$-finite $v$-invertible divisorial prime is $t$-invertible;
(4) A $t$-invertible $t$-prime is $t$-maximal.

We remark that in general a $*$-invertible $*$-prime need not be $*$-maximal (for example a principal prime ideal is not necessarily a maximal ideal) and that a (non-prime) $v$-finite $v$-invertible divisorial ideal need not be $t$-invertible [6].

Corollary 2.3. Assume that each maximal divisorial ideal of $R$ is a $t$-maximal ideal. Then each $v$-invertible divisorial prime is a $t$-invertible $t$-maximal ideal.

Proof. Let $P$ be a $v$-invertible divisorial prime. By Proposition 2.2, $P$ is maximal divisorial and so $t$-maximal. Since $P$ is not strong, then it is $t$-invertible. □

In general, if each $v$-invertible divisorial prime of $R$ is $t$-invertible, it is not true that each $v$-invertible ideal is $t$-invertible. This last property
is in fact equivalent to $R$ being an $H$-domain \[25\] Proposition 4.2]. The ring of entire functions is not an $H$-domain, but all its divisorial primes are $t$-invertible (see for example \[10\] Section 2). A $v$-coherent domain is a domain $R$ with the property that, for each finitely generated ideal $J$, the ideal $(R : J)$ is $v$-finite. This class of domains was first studied (under a different name) in \[25\] and is very large, properly including $PvMD$’s, Mori domains and coherent domains \[25\] \[14\]. (A domain is coherent if each finitely generated ideal is finitely presented, or, equivalently, if the intersection of each pair of finitely generated ideals is finitely generated.)

**Proposition 2.4.** If $R$ is $v$-coherent, then each maximal divisorial ideal is $t$-maximal.

*Proof.* Let $M$ be a maximal divisorial ideal of $R$. Then $M = x^{-1}R \cap R = (R : R + xR)$ for some $x \in K$ (Proposition \[2.1\]). Since $R$ is $v$-coherent, then $M$ is $v$-finite and so $t$-maximal by Proposition \[2.2\].

A domain $R$ is completely integrally closed if and only if $F_v(R)$ is a group under $v$-multiplication \[13\]. If $F_v(R) = T(R)$, then $R$ is a completely integrally closed $H$-domain, equivalently a Krull domain \[10\] \[15\].

A divisorial prime of a completely integrally closed domain, being $v$-invertible, is always maximal divisorial by Proposition \[2.2\]. We will see in the next section that it need not be $t$-maximal. As a matter of fact, a divisorial prime $P$ of a completely integrally closed domain has height one and $P$ is $t$-maximal if and only if it is $v$-finite, if and only if it is $t$-invertible \[10\] Theorem 2.3].

A completely integrally closed $v$-coherent domain is a (completely integrally closed) $PvMD$. In this case each divisorial prime is $t$-maximal by Corollary \[2.3\].

We now turn to the case of polynomial rings.

We denote by $X$ a set of independent indeterminates over $R$ and by $R[X]$ the polynomial ring in this set of indeterminates. It is well known that the correspondence $I \mapsto I[X]$ induces inclusion preserving injective maps $t(R) \hookrightarrow t(R[X])$ and $D(R) \longrightarrow D(R[X])$. Moreover, $M$ is a $t$-maximal ideal, respectively a maximal divisorial ideal, of $R[X]$ such that $M \cap R \neq (0)$, if and only if $M = (M \cap R)[X]$ and $M \cap R$ is a $t$-maximal ideal, respectively a maximal divisorial ideal, of $R$ (see for example \[8\] Lemma 2.1 and \[27\] Theorem 3.6]).

Thus, if each maximal divisorial ideal of $R[X]$ is $t$-maximal, $R$ has the same property.
On the other hand, Example 3.1 in the next section shows that if $M \cap R = (0)$, then $M$ may be maximal divisorial but not $t$-maximal.

A prime ideal $Q$ of $R[X]$ such that $Q \cap R = (0)$ is called an upper to zero. $Q$ is an upper to zero of height one if and only if $Q = fK[X] \cap R[X]$ for some polynomial $f \in R[X]$, irreducible in $K[X]$ [12, Lemma 2.1]. In one indeterminate, all the uppers to zero are of this form.

Recall that if $R$ is integrally closed and $f$ is a nonzero polynomial of $R[X]$, then $fK[X] \cap R[X] = f(R : c(f))[X]$ [13, Corollary 34.9]. (Here $c(f)$ denotes the content of $f$, that is the fractional ideal of $R$ generated by the coefficients of $f$.) Hence if $R$ is integrally closed, an upper to zero of height one is always divisorial and if $R$ is completely integrally closed, an upper to zero of height one, being $v$-invertible, is always maximal divisorial.

In general, an upper to zero is $t$-maximal if and only if it is $t$-invertible; in this case it has height one [12, Section 3]. We now show that a similar result holds for the $v$-operation.

**Proposition 2.5.** A divisorial upper to zero is a maximal divisorial ideal if and only if it is $v$-invertible. In this case it has height one.

**Proof.** A divisorial $v$-invertible prime is always maximal divisorial (Proposition 2.2 (1)).

Conversely, let $P \subseteq R[X]$ be an upper to zero that is maximal divisorial. Then $P = \frac{f}{g}R[X] \cap R[X] \subseteq fK[X] \cap R[X]$, for some $f, g \in R[X]$, $g \neq 0$ (Proposition 2.1(1)). Since $P \cap R = (0)$ and $f = \frac{f}{g}g \in P$, then $f \notin R$. We may also assume that $f$ and $g$ are coprime in $K[X]$.

Let $h = f\alpha \in fK[X] \cap R[X]$, with $\alpha \in K[X]$. There is a nonzero $c \in R$ such that $c\alpha \in R[X]$. Hence $ch = (c\alpha)f = (c\alpha)\frac{f}{g}g \in P$. Since $c \notin P$, then $h \in P$.

We conclude that $P = fK[X] \cap R[X]$ has height one. In addition, $\frac{f}{g} \in (R[X] : P)$, but $\frac{f}{g} \notin (P : P)$. Otherwise $g = \frac{f}{g}f \in P$ and so $g = \frac{f}{g}t$ for some $t \in R[X]$. Then $f$ divides $g^2$ in $K[X]$, which is impossible, because $f$ and $g$ are coprime and $f \notin K$.

It follows that $P$ is not strong and, being maximal divisorial, is $v$-invertible. □

The following result was proved in [15] for one indeterminate.

**Proposition 2.6.** $R$ is an $H$-domain if and only if $R[X]$ is an $H$-domain.

**Proof.** An extended prime $P[X]$ is a $t$-maximal ideal, respectively a maximal divisorial ideal, if and only if so is $P$, [8, Lemma 2.1] and
A $t$-maximal upper to zero is $t$-invertible by [12, Theorem 2.3]. Hence it is divisorial.

The domain $R$ is said to be a $UMT$-domain if every upper to zero of $R[X]$ is a $t$-maximal ideal [21]. This property is stable under polynomial extensions, in fact $R$ is a $UMT$-domain if and only if $R[X]$ is a $UMT$-domain [8, Theorem 2.4]. The integrally closed $UMT$-domains are exactly the PvMDs [21, Proposition 3.2].

The following proposition is immediate.

**Proposition 2.7.** Assume that $R$ is an $UMT$-domain. Then each maximal divisorial ideal of $R$ is $t$-maximal if and only if $R[X]$ has the same property.

We conclude this section recalling that it is not known whether $R$ $v$-coherent implies that $R[X]$ is $v$-coherent. This is true under the additional hypothesis that $R$ is integrally closed [25]. In this case, each prime of $R[X]$ upper to zero is divisorial $v$-finite. When $R$ is $v$-coherent and completely integrally closed (thus a completely integrally closed PvMD), each upper to zero of $R[X]$ is $t$-maximal (and $t$-invertible).

3. MAXIMAL DIVISORIAL IDEALS THAT ARE NOT $t$-MAXIMAL

In this section we give two examples of a maximal divisorial ideal $P$ of an integral domain $R$ that is not a $t$-maximal ideal. In the first example $R$ is a completely integrally closed polynomial ring in one indeterminate and $P$ is an upper to zero, thus $P$ is $v$-invertible. In the second example $R$ is an integrally closed semigroup ring and $P$ is strongly divisorial.

**Example 3.1.** An upper to zero $P$ of a completely integrally closed polynomial ring $R[X]$ that is maximal divisorial but not $t$-maximal. $P$ is necessarily $v$-invertible.

Let $y, z$ and $t = \{t_n(n \geq 1)\}$ be independent indeterminates over a field $k$. Let $S$ be the semigroup of monomials $f$ of $k[y, z, t]$ satisfying the conditions $\deg_{y, z} f \geq \deg_{t_n} f$ for all $n \geq 1$, and let $R = k[S]$ the semigroup ring over $k$ generated by $S$.

Set

$$P = (y + zX)K[X] \cap R[X],$$

where $K$ is the field of fractions of $R$ and $X$ is an indeterminate over $R$. Then $R$ (and so also $R[X]$) is completely integrally closed, and $P$ is a maximal divisorial ideal of $R[X]$ that is not $t$-maximal.

**Proof.**
(1) \( R[X] \) is completely integrally closed.

It is enough to show that \( R \) is completely integrally closed. Since \( R = k[S] \) is a semigroup ring over the field \( k \), by [13, Corollary 12.7 (2)] to this end it suffices to show that the semigroup \( S \) is completely integrally closed.

Let \( u, v, w \in S \) so that \( u(\frac{w}{w})^m \in S \) for all \( m \geq 1 \). Fix \( n \geq 1 \). Then \( \deg_{y,z}(u(\frac{w}{w})^m) \geq \deg_{t^n}(u(\frac{w}{w})^m) \) for all \( m \). Hence

\[
\deg_{y,z} u + m \deg_{y,z}(\frac{v}{w}) \geq \deg_{t^n} u + m \deg_{t^n}(\frac{v}{w}).
\]

Divide by \( m \) and let \( m \) go to \( \infty \) to obtain that \( \deg_{y,z}(\frac{v}{w}) \geq \deg_{t^n}(\frac{v}{w}) \). The same argument shows that \( \frac{w}{w} \) is a monomial, that is has a nonnegative degree in each indeterminate. It follows that \( \frac{w}{w} \in S \); thus \( S \) is completely integrally closed.

(2) \( P \) is an upper to zero of \( R[X] \) that is a \( v \)-invertible maximal divisorial ideal.

\( P \) is clearly an upper to zero. Since \( R \) is integrally closed, then \( P = (y + zX)(R : (y, z))|X] \) by [13, Corollary 34.9], hence \( P \) is divisorial. But \( R[X] \) is completely integrally closed; thus \( P \) is \( v \)-invertible and so maximal divisorial (Proposition 2.2).

(3) \( P \) is not t-maximal.

Let \( Q = (y, z)k[y, z, t] \cap R \). Then \( QR[X] \) is a proper t-ideal of \( R[X] \) properly containing \( P \).

To verify this, let \( F \) be a finite subset of \( QR[X] \). Let \( t_n \) be an indeterminate that does not occur in the polynomials in the set \( F \). Then \( t_n \cdot f \in R[X] \) for all \( f \in F \), so \( t_n \in (R[X] : F) \). If \( g \in (F)_v \), then \( gt_n \in R[X] \). Hence \( \deg_{y,z} gt_n \geq 1 \) and \( g \in Q \). It follows that \( (F)_v \subseteq Q \), so \( Q \) is a t-ideal.

\( \square \)

Example 3.2. An example of a strong maximal divisorial ideal of an integrally closed domain \( R \) that is not t-maximal.

Let \( k \) be a field and let \( Y, Z, X = \{X_n : n \geq 1\}, T = \{T_n : n \geq 1\} \) be independent indeterminates over \( k \). Let \( S \) be the set of monomials \( f \) in \( k[Y, Z, X, T] \) satisfying the following two conditions:

(a) If \( Z \) occurs in \( f \), then some \( X_n \) occurs in \( f \).
(b) For all \( n \), if \( T_n \) occurs in \( f \), then either \( Y \) or \( X_i \) occurs in \( f \) for some \( i \leq n \).

Clearly, \( S \) is a semigroup containing \( X \) and \( Y \). Let \( R = k[S] \) be the semigroup ring over \( S \) and set

\[
P = (X)k[Y, Z, X, T] \cap R.
\]
Then $R$ is integrally closed and $P$ is a strong maximal divisorial ideal of $R$ that is not $t$-maximal.

**Proof.** We will use repeatedly that $P$ is a monomial ideal of $R$.

1. $R$ is integrally closed.
   
   By [14, Corollary 12.11 (2)], it is enough to show that the monoid $S$ is integrally closed. If $f$ is an element in the quotient group of $S$ such that $f^n \in S$ for some $n \geq 1$, then $f$ is a monomial. Since $f^n$ satisfies conditions (a)-(b), it is clear that $f$ also satisfies them, thus $f \in S$. We conclude that $R$ is integrally closed.

2. $P = RZ^{-1} \cap R$. Hence $P$ is a divisorial ideal.
   
   Clearly, any monomial in $ZP$ satisfies conditions (a)-(b), hence $ZP \subseteq R$. Thus $P \subseteq RZ^{-1} \cap R$.

   For the reverse inclusion, it is enough to show that any monomial $f \in RZ^{-1}$ belongs to $P$. Since $Zf \in R$, we see that $Zf$ satisfies conditions (a)-(b) and so does $f$, thus $f \in R$. Using again that $Zf \in R$, we see that some $X_n$ occurs in $f$, hence $f \in P$.

3. $(R : P) = R[Z]$.
   
   Using conditions (a)-(b), we see that $R[Z] \subseteq (R : P)$.

   For the reverse inclusion, let $u$ be a quotient of monomials in $(R : P)$. Since $uX_1, uX_2 \in R$, we see that $uX_1$ and $uX_2$ are monomials, hence, by factoriality, $u$ also is a monomial. Let $u = Z^k u_0$, where $k \geq 0$, $u_0$ is a monomial and $Z$ does not occur in $u_0$. Choose a positive integer $N$ such that $N > i$ for all $T_i$’s occurring in $u$. Since $Z^k u_0 X_N \in R$, we see that $u_0$ satisfies condition (b); hence $u_0 \in R$, so $u \in R[Z]$.

4. $P$ is a strong maximal divisorial ideal.
   
   We have $(R : P) = R[Z] \subseteq (P : P)$, thus $(R : P) = (P : P)$, that is, $P$ is strong.

   Assume that $P$ is not maximal divisorial, so there is a divisorial ideal $Q$ properly containing $P$. Let $f \in Q \setminus P$. We may assume that no $X_n$ occurs in $f$, thus $Z$ does not occur in $f$ either by condition (a) above. Let $g \in (R : Q) \setminus R$, thus $g \in (R : P) = R[Z], g = \sum_{i=0}^{n} a_i Z^i$, where $a_0, \ldots, a_n \in R$. We may assume that $a_n Z^n \notin R$, thus $n \geq 1$. We also may assume that no $X_i$ occurs in $a_n$. Thus no $X_i$ occurs in $fa_n$, which implies that $fa_n Z^n \notin R$. Since $R = k[S]$, we obtain that $fg = fa_n Z^n + \cdots \notin R$, a contradiction.

5. The ideal $M = (S)R$ is a maximal ideal of $R$ properly containing $P$ and is a $t$-ideal.
Clearly $M$ is a maximal ideal containing $P$. Since $Y \in M \setminus P$, we have $P \subseteq M$.

To show that $M$ is a $t$-ideal, let $F$ be a finite subset of $M$ and let $N$ be a positive integer such that $N > i$ for each $T_i$ occurring in some element of $F$. From conditions (a)-(b) it follows that $M \subseteq (X, Y)k[Y, Z, X, T]$. Hence $T_N F \subseteq R$. Thus $(F)_v \subseteq (R : T_N) \cap R$. Since $T_N \notin R$ and since $(R : T_N) \cap R$ is a monomial ideal, we obtain that $(R : T_N) \cap R \subseteq (S)R = M$. It follows that $(F)_v \subseteq M$ and that $M$ is a $t$-ideal.

\[\square\]

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