Probing Pseudo Nambu Goldstone Boson Dark Energy Models with Dark Matter - Dark Energy Interaction

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Abstract

We consider a dark energy scenario driven by a scalar field \( \Phi \) with a pseudo Nambu Goldstone boson (pNGB) type potential \( V(\phi) = \mu^4 (1 + \cos(\phi/f)) \). The pNGB originates out of breaking of spontaneous symmetry at a scale \( f \) close to Planck mass \( M_{\text{pl}} \). We consider two cases namely the quintessence dark energy and the other, where the standard pNGB action is modified by the terms related to Slotheon cosmology. We demonstrate that for this pNGB potential, high-\( f \) problem is better addressed when interaction between dark matter and dark energy is taken into account and that Slotheon dark energy scenario works even better over quintessence in this respect. To this end, a mass limit for dark matter is also estimated.

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1 Introduction

The observational data [1] - [4] reveal that our Universe is not only expanding with time but is undergoing an accelerated expansion. One of the most challenging problems of modern cosmology is to explain such a late time cosmic acceleration. The general conjecture is that a mysterious component of the Universe with negative pressure, broadly known as dark energy (DE) is responsible for the recent acceleration that accounts about 69% of the total mass energy content of the present Universe. In theory the commonly used candidate for dark energy is the cosmological constant \( \Lambda \) [5] in the Einstein’s equations introduced by Einstein. The popular and widely used dark energy model namely \( \Lambda \)CDM model that includes cosmological constant \( \Lambda \) and the cold dark matter (CDM) has some unsolved theoretical problems like the fine tuning problem [5] and cosmic coincidence problem [6].

Another concept to address the dark energy is to assume the existence of a scalar field \( \phi \) with a slow rolling potential \( V(\phi) \) as the source of dark energy. This scalar field \( \phi \), named as quintessence field, provide the dynamical nature of the dark energy, in contrast to the cosmological constant explanation according to which the dark energy is constant throughout the evolution of the Universe. Extensive studies have been done to study the nature of quintessence dark energy model [7, 8]. A well motivated alternative description of dark energy is given by modified gravity models of dark energy, inspired by the theories of extra dimensions [9, 10]. It is observed that to obey the observational results the scalar fields \( \phi \) should possess a very flat potential \( V(\phi) \) and very light mass.

Another dark sector component of the Universe is dark matter (DM) which contains about 25% of the total energy content of the present Universe. The existence of the dark matter is confirmed by various observational results [11] - [13]. In literature there are various attempts to explore the unknown nature of the dark matter by extending the Standard Model of Particle physics. Different types of dark matter candidates such as singlet scalar [14], singlet fermion [15, 16], vector [17], pseudoscalar [18] etc. are introduced by several authors but in the present context we concentrate on the singlet scalar dark matter candidate which is a well established model for particle dark matter.

We consider in this work the dark matter dark energy interaction. Till date there are no theoretical considerations or experimental observations that seem to suggest that such a possibility can not exist. We consider here a nonminimal coupling between the two dark sector components namely dark matter and dark energy instead to treat them independently. In literature various authors discussed the interacting dark energy (IDE) models [19] to address different phenomenological problems [20] - [31].

As mentioned a popular dark energy model is scalar field dark energy model where one
considers a slowly varying potential (slow roll) for the scalar field which the latter tracks as the potential changes over time. Even though the idea of the scalar field dark energy model is well motivated but in Ref. \[32\] Kolda and Lyth pointed out a serious problem for any slow rolling scalar field dark energy model. The problem is to prevent any additional terms to the field potential \(V(\phi)\), which would spoil the flatness of the potential. In order to avoid this problem, the dark energy models are considered where the light mass of the scalar field \(\phi\) is protected by a symmetry. Such scenario can be arisen if a pseudo Nambu Goldston boson (pNGB) acts as a dynamical dark energy field. This concept is studied in Refs. \[33\] and \[34\].

The spontaneous symmetry breaking of a global \(U(1)\) symmetry results in producing a massless Goldstone boson. Such spontaneous symmetry breaking leads to two modes, one is the massive radial mode and the other is the massless angular mode (named as NGB) at the symmetry breaking energy scale. A pseudo Nambu Goldstone boson is produced when the NGB acquires mass at the soft-explicit symmetry breaking scale, which is lower than the spontaneous symmetry breaking scale. A popular example of pNGB is Axion \[35, 36\]. As mentioned, pNGB could play the role of the quintessence dark energy field with the following form of the potential \[33\]

\[
V(\phi) = \mu^4 (1 + \cos(\phi/f)) .
\] (1)

In the above, \(f\) is the spontaneous global symmetry breaking scale which controls the steepness of the potential and \(\mu\) is the explicit global symmetry breaking scale. The above potential is periodic with period \(2\pi f\) and the field value \(\phi\) ranges from 0 to \(2\pi f\). This periodic potential is special because it stabilises the mass from quantum corrections \[32\] and suppresses the fifth-force constraints \[37\]. In Refs. \[38\] - \[40\] the authors found that generally for the quintessence dark energy with pNGB potential, \(f \geq M_{pl}\), \(M_{pl}\) being the reduced Planck mass and indicated that the larger value of \(f\) corresponds to the flatter potential. But it is very difficult to interpret such high value of \(f\) since such values of \(f\) are not compatible with the valid domain of field theory. For this range, quantum gravity corrections can not be controlled \[40\]. This problem is termed as high-\(f\) problem of pNGB quintessence dark energy model. Few authors have attempted to solve this high-\(f\) issue earlier \[41, 42\]. For example in Ref. \[41\] this has been suggested that N number of pNGB fields drive the late time acceleration and in Ref. \[42\] the authors addressed the issue by adding extra phenomenological terms to the quintessence lagrangian. Here in this work we explore a new approach where we consider interacting dark energy (IDE) model to tackle this high-\(f\) problem of pNGB quintessence.

In this work we consider the nonminimally coupled dark matter-dark energy scenario to address the high-\(f\) value problem. Dark matter dark energy (DM-DE) interactions are sometime marked by considering both dark energy and dark matter to be fluids or both of them to be
scalar fields or even one of them is a scalar field and other is a fluid. In this work we take both the dark energy and dark matter as real scalar fields and treat the interactions in a way more fundamental in nature. Here, we consider two dark energy models, the standard quintessence dark energy model and the Slotheon field dark energy model [43] - [45]. The Slotheon dark energy model is a modified gravity model, inspired by extra dimensional theories. The Dvali Gabadadze Porrati (DGP) model [46], an extra dimensional model, in its decoupling limit can be described by a scalar field named as Galileon field in Minkowski spacetime and the field obeys Galileon shift symmetry [47, 48]. The generalisation of the Galileon shift symmetry to curved spacetime leads to the Slotheon field [45]. In literature there are references where it is shown that in some cases such dark energy models fit better with the observational or theoretical constraints than the standard quintessence model [43, 49, 50, 51]. Therefore in this work too we evaluate our results for each of these two cases and compare them. For both the cases we consider nonminimal coupling between dark matter and dark energy and furnish our results for both the scenarios with and without DM-DE interactions.

This paper is organised as follows. In section 2 we briefly describe the dark matter-dark energy interactions in the general ground. Section 3 is devoted to explain the coupled (coupled to dark matter) quintessence dark energy model with pNGB potential while in section 4, coupled Slotheon field dark energy model with pNGB potential is described. We present our calculations and results in section 5 and finally in section 6 we narrate the summary and conclusions.

2 Dark Energy - Dark Matter Interaction

In this section we briefly narrate the forms of DM-DE interactions, that has been adopted in this work.

In order to describe the DM-DE interactions an energy exchange term $Q$ is introduced in the energy-momentum conservation equations as follows,

$$\dot{\rho}_{DE} + 3H(\rho_{DE} + p_{DE}) = Q,$$  \hspace{1cm} (2)

$$\dot{\rho}_{DM} + 3H\rho_{DM} = -Q,$$  \hspace{1cm} (3)

where $\rho_{DE}$ and $\rho_{DM}$ are the energy densities of dark energy and dark matter respectively and $H$ denotes the Hubble parameter while $\dot{A}$ represents time derivative of the variable $A$. In the above $Q$ refers to the energy density transfer between dark matter and dark energy and defines the DM-DE interaction coupling. In literature there are several studies of this interaction where the dark sector is considered to be a two component fluid with some usual forms for coupling $Q$ are adopted [52, 53]. But in interacting quintessence model where dark energy is assumed to be
a scalar field $\phi$, the coupling $Q$ is given as \[Q = f(\phi)\dot{\rho}_b,\] (4)

where $f(\phi)$ is a function of field $\phi$ and $\rho_b$ is the energy density of the background fluid. Substituting $\rho_b$ by $\rho_{dm}$ in eq. (4) one may obtain the DM-DE interaction coupling $Q$ in a model where dark matter interacts with the quintessence dark energy field [56].

A more fundamental approach to describe the DM-DE interactions is to treat both the dark matter and dark energy as fields [57, 58] (for our case we consider both the fields are real scalar fields). In this case the action contains kinetic term for dark energy field $\phi$, kinetic term for dark matter field $\chi$ and a total potential $V(\phi, \chi)$, where $V(\phi, \chi)$ includes the potentials for dark energy and dark matter candidates. Such an action can be written as,

\[S = \int d^4x \sqrt{-g} \left[ -\frac{1}{2} g^{\mu\nu} (\partial_\mu \phi \partial_\nu \phi + \partial_\mu \chi \partial_\nu \chi) - V(\phi, \chi) \right].\] (5)

In the above $g^{\mu\nu}$ denotes the metric tensor and $g$ is the determinant of the metric tensor. These models also lead to the coupling of interacting quintessence case (eq. (4)) when the interaction is of the form of dark matter mass term. This approach being more elementary we follow in this work this formalism and treat the DM-DE interaction when both the dark matter and dark energy are considered to be real scalar fields. The formalism is given below.

From the action given in eq. (5) we obtain the equation of motion of field $\phi$ and $\chi$ as

\[\ddot{\phi} + 3H\dot{\phi} + \frac{\partial V}{\partial \phi} = 0,\] (6)

\[\ddot{\chi} + 3H\dot{\chi} + \frac{\partial V}{\partial \chi} = 0,\] (7)

where $\ddot{A}$ defines the double derivative of the variable $A$ with respect to time. We also derive the energy density $\rho_d$ and pressure density $p_d$ from eq. (5) and the expressions are

\[\rho_d = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\dot{\chi}^2 + V(\phi, \chi),\] (8)

\[p_d = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\dot{\chi}^2 - V(\phi, \chi).\] (9)

It may be mentioned here that non-relativistic baryonic matter density $\rho_m$ and radiation energy density $\rho_r$ are not having any interactions with dark energy and evolve with scale factor $a$ as $\rho_m \propto a^{-3}$ and $\rho_r \propto a^{-4}$. But since dark matter and dark energy are coupled to each other, the evolutions of these components do not have this usual and simple relations.

Writing $V(\phi, \chi)$ as the sum of two terms as

\[V(\phi, \chi) = V_{DE}(\phi) + V_{DM}(\phi, \chi),\] (10)
where $V_{DE}(\phi)$ represents the non interacting part and the interacting part $V_{DM}(\phi, \chi)$ is written as
\[ V_{DM}(\phi, \chi) = \frac{1}{2} M^2(\phi) \chi^2 , \] (11)
the mass function $M^2(\phi)$ follows a form
\[ M^2(\phi) = m^2 + F(\phi) . \] (12)

In the above $m$ refers to the “bare” mass of the dark matter and $F(\phi)$ denotes a polynomial of the field $\phi$. It may be noted that in eq. (12) the $m^2$ term leads to the dark matter potential $\frac{1}{2}m^2\chi^2$ and the nature of the DM-DE interactions are defined by the function $F(\phi)$.

Now, for the interaction potentials in eq. (10) and eq. (11) the dark matter energy density $\rho_{DM}$ and pressure density $p_{DM}$, are obtained as
\[ \rho_{DM} = \frac{1}{2} \dot{\chi}^2 + \frac{1}{2} M^2(\phi) \chi^2 , \] (13)
\[ p_{DM} = \frac{1}{2} \dot{\chi}^2 - \frac{1}{2} M^2(\phi) \chi^2 . \] (14)
As the oscillations of the $\chi$ field is much more higher than the expansion, the values of $\dot{\chi}^2$, $\chi^2$ can be replaced by their averaged values over an oscillation period [59, 60],
\[ \langle \rho_{DM} \rangle = \langle \chi^2 \rangle = M^2(\phi) \langle \chi^2 \rangle , \] (15)
\[ \langle p_{DM} \rangle = 0 . \] (16)

Hence, from eq. (7, 13 - 16) we obtain
\[ \dot{\rho}_{DM} + 3H\rho_{DM} = \frac{1}{2} \dot{\phi} \frac{1}{M^2(\phi)} \frac{\partial M^2(\phi)}{\partial \phi} \rho_{DM} , \] (17)
which is similar to that given in eq. (3). See Appendix for detail calculations. Comparing eq. (4) with eq. (17) it can be easily noted that
\[ f(\phi) = -\frac{1}{2} \frac{\partial \ln M^2(\phi)}{\partial \phi} . \] (18)
Moreover, the formal solution of eq. (17) is given by
\[ \rho_{DM}(\phi, a) = n_o a^{-3} M(\phi) , \] (19)
with $n_o$ is an integration constant which in this case is attributed to the number density of dark matter and $a$ is the scale factor of the Universe. It may be noted that the above equation is the evolution equation of $\rho_{DM}$ for the coupled case.

It can be noted that the dynamics of the system can now be evaluated only with the equations of field $\phi$. In doing so the potential $V_{DE}(\phi)$ is to be replaced with an effective potential $V_{eff}(\phi, a)$ (see the Appendix). The effective potential is given as [60],
\[ V_{eff}(\phi, a) = V_{DE}(\phi) + \rho_{DM}(\phi, a) . \] (20)
3 Coupled Quintessence Dark Energy Model

In this section also, we consider standard quintessence scalar field $\phi$ as dark energy candidate and a canonical real scalar field $\chi$ for the candidate of dark matter. On the basis of the discussions in the previous section we write down the following action for coupled quintessence scenario

$$S = \int d^4x \sqrt{-g} \left( \frac{1}{2} M_{pl}^2 R - \frac{1}{2} g^{\mu\nu} (\partial_\mu \phi \partial_\nu \phi + \partial_\mu \chi \partial_\nu \chi) - V_{\text{DE}}(\phi) - \frac{1}{2} \beta \phi^2 \chi^2 \right) + S_m + S_r ,$$

(21)

where $M_{pl}$ and $R$ are reduced planck mass and Ricci scalar respectively and $\beta$ denotes the coupling constant for DM-DE interaction. In the above $S_m$ and $S_r$ refer to the action of baryonic matter and action of radiation component respectively. We consider the quintessence field with pNGB potential for our dark energy model, therefore the expression for the potential $V_{\text{DE}}(\phi)$ is similar as in eq. (1)

$$V_{\text{DE}}(\phi) = \mu^4 (1 + \cos(\phi/f)) .$$

(22)

Hence in our case the effective potential is given by

$$V_{\text{eff}}(\phi, a) = \mu^4 ((1 + \cos(\phi/f)) + n_o a^{-3} M(\phi) ,$$

(23)

where symbols have similar meanings as in eq. (19) and eq. (20) and here $M^2(\phi)$ is given by

$$M^2(\phi) = m^2 + 2 \beta \phi^2 .$$

(24)

By varying the action of eq. (21) with respect to the metric, the Friedmann equations are obtained for flat Friedmann-Robertson-Walker (FRW) Universe and the equations are

$$3 M_{pl}^2 H^2 = \rho_m + \rho_r + \frac{\dot{\phi}^2}{2} + V_{\text{eff}}(\phi, a) ,$$

(25)

$$M_{pl}^2 (2 \dot{H} + 3 H^2) = -\frac{\rho_r}{3} - \frac{\dot{\phi}^2}{2} + V_{\text{eff}}(\phi, a) ,$$

(26)

$$0 = \ddot{\phi} + 3 H \dot{\phi} + \frac{\partial V_{\text{eff}}(\phi, a)}{\partial \phi} .$$

(27)
Now in order to obtain the evolutions of the system it is suitable to introduce the following dimensionless variables

\[ x = \frac{\dot{\phi}}{\sqrt{6HM_{pl}}} , \]  
(28)

\[ y = \frac{\sqrt{V_{DE}(\phi)}}{\sqrt{3HM_{pl}}} , \]  
(29)

\[ \lambda = -M_{pl}\frac{dV_{DE}(\phi)}{d\phi} , \]  
(30)

\[ \xi = \frac{\phi}{m} . \]  
(31)

Using eqs. (25 - 27) and the dimensionless variables (eqs. (28 - 31)), the autonomous set of equations can be written as

\[ \frac{dx}{dN} = \frac{P}{\sqrt{6}} - x\frac{\dot{H}}{H^2} , \]  
(32)

\[ \frac{dy}{dN} = -y \left( \sqrt{\frac{3}{2}} \lambda x + \frac{\dot{H}}{H^2} \right) , \]  
(33)

\[ \frac{d\lambda}{dN} = -\sqrt{6}x\lambda^2 \left( \frac{V_{DE} d^2 V_{DE}}{d\phi^2} \right)^2 - 1 , \]  
(34)

\[ \frac{d\xi}{dN} = \sqrt{6}\frac{x}{b} . \]  
(35)

In the above, \( N = \ln a \) is the number of e-foldings, \( b = \frac{m}{M_{pl}} \) and

\[ \frac{\dot{H}}{H^2} = \frac{1}{2} \left( \frac{3\Omega_{do}}{a^3} - 3x^2 + 3y^2 - \Omega_r - 3 \right) , \]  
(36)

\[ P = -\frac{6\beta\xi\Omega_{do}}{a^3(2\beta\xi^2 + b)} - 3\sqrt{6}x + 3\lambda y^2 , \]  
(37)

\[ \frac{V_{DE} d^2 V_{DE}}{(d\phi d\phi)^2} = \frac{1}{2} \left( 1 - \frac{1}{\lambda^2(\frac{m}{M_{pl}})^2} \right) , \]  
(38)

where \( \Omega_{do} \) is the dark matter density parameter at the present epoch (i.e., at scale factor \( a = 1 \) or redshift \( z = 0 \)) and \( \Omega_r \) is the density parameter for radiation. The density parameter \( \Omega \) is defined as \( \Omega = \frac{\rho}{\rho_c} \) while \( \rho_c \) is the critical density of the Universe.

The effective equations of state parameter \( \omega_{\text{eff}} \) and equation of state parameter for dark energy
\( \omega_{DE} \) are obtained from eqs. (25 - 27) as

\[
\omega_{\text{eff}} = \frac{p_{\text{total}}}{\rho_{\text{total}}} = \frac{p_m + p_{DM} + p_{DE} + p_r}{\rho_m + \rho_{DM} + \rho_{DE} + \rho_r} = -1 - \frac{2\dot{H}}{3H^2},
\]

(39)

\[
\omega_{DE} = \frac{\omega_{\text{eff}} - \Omega_r}{\Omega_{DE}}.
\]

(40)

In the above \( \Omega_{DE} \) is dark energy density parameter and \( p_x \) and \( \rho_x \) define the pressure density and energy density of the component \( x \) (\( x \equiv \text{baryonic matter (m)}, \text{dark matter (DM)}, \text{dark energy (DE)} \) and radiation (\( r \)) of the Universe respectively. It is needless to mention here that density parameters \( \Omega_x \) and equation of state parameter \( \omega_{DE} \) are now obtained in terms of the dimensionless variables of eqs. (28 - 31) and the evolutions of these cosmological parameters can easily be derived by solving the set of coupled equations (eqs. (32 - 35)) with proper initial conditions.

### 4 Coupled Slotheon Dark Energy Model

The Slotheon field model \cite{45} is a scalar field model which is classified as a modified gravity model of dark energy. The Slotheon field model is inspired from Dvali Gavadaze Poratti (DGP) model \cite{46} - an extra dimensional model with one extra dimension. The DGP model in its decoupling limit \( r_c \to \infty \) \cite{61, 62} \((r_c \text{ separates } 4-\text{D and } 5-\text{D regimes and defined as } r_c = \frac{M_{pl}^2}{2M_5^2}, \text{where } M_{pl} \text{ and } M_5 \text{ are bulk and brane Planck masses respectively}) \) can be described by a scalar field (say \( \pi \)), dubbed as Galileon field \cite{47, 48}. Here the field \( \pi \) respects a shift symmetry known as Galileon shift and is given by \( \pi \rightarrow \pi + a + b_\mu x^\mu \) \cite{63}. The symbols \( a \) and \( b_\mu \) represent a constant and a constant vector respectively. The Slotheon field arises when this Galileon transformation is generalised to curved spacetime \cite{45} and the Slotheon field obeys this curved Galileon transformation,

\[
\pi(x) \rightarrow \pi(x) + c + c_{a} \int_{\Gamma,x_0}^{x} \gamma^a,
\]

(41)

where \( \gamma^a \) is a set of Killing vectors, \( x_0 \) is a reference point connected to another point \( x \) through the curve \( \Gamma \) while \( c \) and \( c_{a} \) are respectively a constant and a constant vector. It is observed in Ref. \cite{43} and \cite{44} that if Slotheon field model of dark energy is considered, then the slow rolling criteria will be more favoured than the standard quintessence dark energy model. This is because the former induces an extra friction which favours the slow rolling nature of the field. Moreover in Ref. \cite{49} it is demonstrated that the Swampland criteria are better satisfied with the Slotheon field model of dark energy over the quintessence model.

In this section we consider the Slotheon field \( \pi \) as dark energy and explore the behaviours of different cosmological parameters when it is coupled to the dark matter. The action of the
coupled Slotheon dark energy field is given as
\[
S = \int d^4x \sqrt{-g} \left( \frac{1}{2} M_{pl}^2 R - \frac{1}{2} g^\mu\nu \partial_\mu \pi \partial_\nu \pi + \frac{G^\mu\nu}{M^2} \partial_\mu \pi \partial_\nu \pi + g^\mu\nu \partial_\mu \chi \partial_\nu \chi - V_{DE}(\pi) - \frac{1}{2} m^2 \chi^2 - \beta \pi^2 \chi^2 \right) + S_m + S_r .
\]  
(42)

In the above \( G^\mu\nu \) is the Einstein tensor and \( M \) represents an energy scale and all the symbols are same as in eq. (21). It can be noted here that without the term \( \frac{G^\mu\nu}{M^2} \partial_\mu \pi \partial_\nu \pi \) in the action of eq. (42), both the actions of eqs. (21) and (42) are identical. In the Slotheon case also, we consider the pNGB potential as the field potential of dark energy. Here \( V_{DE}(\pi) \) is given as
\[
V_{DE}(\pi) = \mu^4 (1 + \cos(\pi/f)) .
\]  
(43)

Similar to eq. (23), the effective potential in this case is given by
\[
V_{\text{eff}}(\pi, a) = \mu^4 ((1 + \cos(\pi/f)) + n_o a^{-3} M(\pi) ,
\]  
(44)

and \( M^2(\pi) \) is
\[
M^2(\pi) = m^2 + 2\beta \pi^2 .
\]  
(45)

The friedmann equations for the action in eq. (42) can now be derived as
\[
3 M_{pl}^2 H^2 = \rho_m + \rho_r + \frac{\dot{\pi}^2}{2} + \frac{9 H^2 \dot{\pi}^2}{2 M^2} + V_{\text{eff}}(\pi, a) ,
\]  
(46)

\[
M_{pl}^2 (2 \dot{H} + 3 H^2) = -\frac{\rho_r}{3} - \frac{\dot{\pi}^2}{2} + V_{\text{eff}}(\pi, a) + (2 \dot{H} + 3 H^2) \frac{\dot{\pi}^2}{2 M^2} + \frac{2 H \dot{\pi} \ddot{\pi}}{M^2} ,
\]  
(47)

\[
0 = \dddot{\pi} + 3 H \ddot{\pi} + \frac{3 H^2}{M^2} \left( \dddot{\pi} + 3 H \ddot{\pi} + \frac{2 H \dot{\pi} \ddot{\pi}}{H} \right) + \frac{\partial V_{\text{eff}}(\pi, a)}{\partial \pi} .
\]  
(48)

It is useful to define the following dimensionless quantities to obtain the evolutions of the system
\[
x = \frac{\dot{\pi}}{\sqrt{6 H M_{pl}}} ,
\]  
(49)

\[
y = \frac{\sqrt{V_{DE}(\pi)}}{\sqrt{3 H M_{pl}}} ,
\]  
(50)

\[
\lambda = -M_{pl} \frac{d V_{DE}(\pi)}{d \pi} ,
\]  
(51)

\[
\xi = \frac{\pi}{m} ,
\]  
(52)

\[
\epsilon = \frac{H^2}{2 M^2} .
\]  
(53)
It may noted that eqs. (49 - 52) are similar to eqs. (28 - 31) of previous section and in this case we have an additional dimensionless variable \( \epsilon \), defined in eq. (53). This arises due to the term \( \frac{G_{\mu\nu}}{M_T} \partial_\mu \pi \partial_\nu \pi \) in the action of eq. (42). Now from eqs. (46 - 48) and eqs. (49 - 53) we get the set of autonomous equations as given below,

\[
\begin{align*}
\frac{dx}{dN} &= P \sqrt{6} - x \frac{\dot{H}}{H^2}, \\
\frac{dy}{dN} &= -y \left( \sqrt{\frac{3}{2}} \lambda x + \frac{\dot{H}}{H^2} \right), \\
\frac{d\lambda}{dN} &= -\sqrt{6}x \lambda^2 \left( \frac{V_{\text{DE}}}{dV_{\text{DE}}} \left( \frac{dV_{\text{DE}}}{d\pi} \right)^2 - 1 \right), \\
\frac{d\xi}{dN} &= \sqrt{6}x b, \\
\frac{d\epsilon}{dN} &= 2 \epsilon \frac{\dot{H}}{H^2}.
\end{align*}
\]

In the above equations

\[
\frac{\dot{H}}{H^2} = \frac{-x^2(6\epsilon + 1)(18\epsilon + 1) + 4\sqrt{6}x \epsilon \left( \lambda y^2 - \frac{2\beta \xi \Omega_{\text{do}}}{a^2(2\beta x^2 + b)} \right)}{4\epsilon \left( x^2(1 - 18\epsilon) - 1 \right) - \frac{2}{3}} + \frac{(6\epsilon + 1)(3y^2 - \Omega_r - 3 + 3\Omega_{\text{do}})}{4\epsilon \left( x^2(1 - 18\epsilon) - 1 \right) - \frac{2}{3}}.
\]

\[
P = \frac{6\Omega_{\text{do}} \left( \beta \xi (6x^2 \epsilon - 1) - 3\sqrt{6}bx \epsilon (2\beta x^2 + 1) \right)}{a^2 b (2\beta x^2 + 1)(6\epsilon (x^2(18\epsilon - 1) + 1)) = \frac{1}{12}} + \frac{3a^3 b (2\beta x^2 + 1) 12 \sqrt{6}x^3 \epsilon - 6\lambda x^2 y^2 \epsilon + \sqrt{6}x \left( -6y^2 \epsilon + 2\Omega_r \epsilon - 1 \right) + \lambda y^2}{a^3 b (2\beta x^2 + 1)(6\epsilon (x^2(18\epsilon - 1) + 1)) = \frac{1}{12}},
\]

where all the symbols have their meaning as mentioned earlier. The effective equation of state parameter \( \omega_{\text{eff}} \) and the equation of state of dark energy \( \omega_{\text{DE}} \) for Slotheon field dark energy model can now be constructed using eqs. (46 - 48) and they will be of the same forms as in eq. (39) and eq. (40). The variations of cosmological parameters \( \Omega_x \) and \( \omega_{\text{DE}} \) are obtained from solving the eqs. (54 - 58) with properly chosen initial conditions.

5 Calculations and Results

In this section we furnish the results we obtain from solving the equations in section 3 and section 4.
In Fig. 1, the evolutions of the density parameters ($\Omega$) of different components of the Universe as the functions of redshift $z$ are shown. In this figure, dynamics of the density parameters are plotted by considering the quintessence dark energy model coupled to the dark matter field. From Fig. 1, it can be observed that the Universe was radiation dominated at early epoch, i.e., for very high redshift values. The matter domination (dark matter and baryonic matter) initiated when $\ln(1 + z) \simeq 7$ or at $z \simeq 10^3$. At a much later stage when $\ln(1 + z) \simeq 0.4$, or $z \simeq 0.49$ the dark energy start dominating and the Universe enters into the phase of late time acceleration. From Fig. 1 we can estimate the present day density parameter values as $\Omega_{DE} \simeq 0.7$, $\Omega_M (= \Omega_{DM} + \Omega_m) \simeq 0.3$, $\Omega_r \simeq 0$. It can also be noted from the figure that for the coupled dark energy model considered here, i.e., the dark energy is coupled only to dark matter (and no coupling with baryonic matter), we have an Universe which at early stage has a radiation era then followed by matter dominated era and more recently experiencing the onset of an era of dark energy domination. These types of scenarios are well studied in Ref. [64]-[66].

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Variations of density parameters with redshift}
\end{figure}

In Fig. 2, we show the variations of the dark energy equation of state parameter $\omega_{DE}$ with redshift $z$ for standard quintessence dark energy model with potential given in eq. (22). Fig. 2 also indicates the thawing nature [67] of the dark energy, i.e., $\omega_{DE}$ is equal to $-1$ at early epochs which gradually deviates from the $-1$ with time, is clear. In Fig. 2, we compare the variations for $\omega_{DE}$ with $z$ when the DM-DE interactions are considered (red line in the plot, $\beta \neq 0$) with the same with no such interactions are included (the green line in the plot, $\beta = 0$). It is interesting to note that quintessence dark energy is more akin to $\Lambda$CDM model (where $\omega_{DE} = -1$, throughout the time of evolution) when the dark energy field is coupled to the dark matter field. In this case and in the following cases we take $\beta = 0.01$, $f = M_{pl}$, $m = 1$ keV otherwise stated. But it may be mentioned here that our results are independent of the value of $\beta$ and only effected by the presence or absence of the DM-DE interactions in the framework. This can be understood
from equations of section 3 and section 4. One observes from these equations that the parameter $\beta$ cancels out from the numerator and the denominator for the choices of initial conditions and values of other parameters, appropriate for this work.

![Figure 2: Evolutions of dark energy equation of state parameter with redshift for quintessence dark energy model with DM-DE interaction (red line) and in the absence of DM-DE interaction (green line).](image)

In Fig. 3 we plot the evolutions of dark energy equation of state $\omega_{DE}$ as a function of redshift $z$ for different initial values of the field ($\phi_i$ or $\pi_i$). We consider in Fig. 3(a) the quintessence dark energy model and in Fig. 3(b) the Slotheon field dark energy model. For both the cases it can be noted that for smaller values of $\phi_i$ (or $\pi_i$) the evolutions of $\omega_{DE}$ appears to resemble more to that for $\Lambda$CDM. Thus when initially the field is nearer to the top of the potential (from eq. 22 and eq. 43 it can be clearly observed that the potentials have their maximum value for $\phi_i = 0$ or $\pi_i = 0$ respectively), it will feel the steepness of the potential less severely and heads toward a slower rolling. We observe that if $\phi_i \simeq 0$ (or $\pi_i \simeq 0$) then $\omega_{DE}$ is almost equal to $-1$ all through, i.e., the field may not experience the slope of the potential and the effective dynamics is independent of $f$. 

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Figure 3: (a) (left panel) Variations of dark energy equation of state parameter with redshift for general quintessence dark energy model, considering different initial values of $\phi_i/f$. (b) (right panel) Variations of dark energy equation of state parameter with redshift for Slotheon field dark energy model, considering different initial values of $\pi_i/f$.

In Fig. 4, we plot the variations of the dark energy equation of state parameter $\omega_{DE}$ with red shift for both the dark energy models namely quintessence (the solid lines) and the Slotheon (dashed lines). It can be easily observed from the graph that the behaviour of $\omega_{DE}$ for the Slotheon model behaviour is closer to the same for ΛCDM model than what is obtained for the quintessence model. This is expected as the Slotheon field favours the slow roll criteria more than the quintessence field, which we have discussed earlier. It is also interesting to note here that for both the cases when interactions between dark matter and dark energy are considered ($\beta \neq 0$), the behaviours of the fields resemble more to the behaviour of ΛCDM model. In other words the fields better satisfy the slow rolling criteria.
In order to understand how sensitive the dark energy equation of state \( \omega_{DE} \) and the dark energy density parameter \( \Omega_{DE} \) at the present epoch are, to the variation of \( f \), we calculate these variations using the formalism given in section 3 and section 4 and the results are plotted in Fig. 5 (a and b) and in Fig. 6 (a and b).

In Fig. 5(a) (left panel of Fig. 5) the variations of the present value of the equation of state parameters \( \omega_{DE}^0 \) (value of \( \omega_{DE} \) at redshift \( z = 0 \)) with the spontaneous symmetry breaking scale \( f \) for both the models quintessence (marked with solid lines) and the Slotheon model (marked with dashed lines) are shown. We not only compare the variations for both the dark energy models considered here but also the results we obtain when DM-DE interaction (\( \beta \neq 0 \)) is included in the calculations and the case when such interaction is not considered (\( \beta = 0 \)). We adopt the range of the present value of \( \omega_{DE} \) to be \(-1.038 \leq \omega_{DE}^0 \leq -0.884 \) as given by PLANCK 2018 [68]. It is to be noted that in all the four cases shown in Fig. 5(a) the calculated values of \( \omega_{DE}^0 \) lie well within the PLANCK range for higher values of \( f \) while the former deviates from this range for smaller values of the scale \( f \). This is due to the fact that since \( f \) is associated with a steepness of the pNGB potentials, pNGB potential tends to be flat as \( f \) increases. A discussion is in order. For the quintessence case with \( \beta = 0 \) the value of \( \omega_{DE}^0 \) goes beyond the PLANCK range at \( f < 0.4M_{pl} \) while for \( f \geq 0.4M_{pl} \), \( \omega_{DE}^0 \) remains barely within the PLANCK range. The situation is much improved when DM-DE interaction is switched on (\( \beta \neq 0 \)). In this situation \( \omega_{DE}^0 \) lies well within the PLANCK range even upto \( f \sim 0.3M_{pl} \). Similar situation is observed for the Slotheon case too. But as seen from Fig. 5(a), Slotheon results are always better than those obtained from quintessence since for both \( \beta = 0 \) and \( \beta \neq 0 \), the range of \( f \) for which the \( \omega_{DE}^0 \) agree with the PLANCK limit are always larger for the later case. Similar conclusionc
can be drawn from Fig. 5(b) (right panel of Fig. 5) too where present value of dark energy density parameter $\Omega^0_{DE}$ (value of $\Omega_{DE}$ at $z = 0$) is plotted for various $f$ values and compare with PLANCK range given by $0.678 \leq \Omega^0_{DE} \leq 0.692$. In addition one also note that in case of $\Omega^0_{DE}$ (Fig. 5(b)) the Slotheon field results for both $\beta = 0$ and $\beta \neq 0$ are in better agreement with the PLANCK limit than those for quintessence considerations. Therefore from Fig. 5(a) and Fig. 5(b) we may conclude that the Slotheon field dark energy model with DM-DE interactions address the higher-$f$ problem most effectively.

![Graph of $\Omega^0_{DE}$ vs $f/M_{pl}$](image)

**Figure 5:** (a) (left panel) Variations of the present value (at $z = 0$ or $a = 1$) of the dark energy equation of state parameters $\omega^0_{DE}$ with $\frac{f}{M_{pl}}$ for quintessence dark energy model (solid lines) and Slotheon field dark energy model (dashed lines). Effects of absence and presence of the DM-DE interaction are also shown for both the fields. (b) (right panel) Same as Fig. 5(a) but for present value of dark energy density parameter $\Omega^0_{DE}$. See text for detail discussion.

In order to study how the nature of variation of $\omega^0_{DE}$ with $f$ changes for different “bare” dark matter masses $m$ in presence of DM-DE interaction ($\beta \neq 0$) we repeat the analyses shown in Fig. 5 for different choices of $m$. We found that when $m < 1$ keV the values of $\omega^0_{DE}$ for the chosen range of $\frac{f}{M_{pl}}$ do not lie within the PLANCK limit for $\omega^0_{DE}$. Again for $m > 1$ TeV all $\omega^0_{DE}$ values for the same choice of $\frac{f}{M_{pl}}$ range lie beyond the PLANCK limit for $\omega^0_{DE}$. In Fig. 6(a and b) we plot $\omega^0_{DE}$ vs $\frac{f}{M_{pl}}$ (Fig. 6(a)) and $\Omega^0_{DE}$ vs $\frac{f}{M_{pl}}$ (Fig. 6(b)) for the cases of Slotheon field and quintessence field for two values of “bare” dark matter masses namely 1 TeV and 1 keV and compare the results.

It is obvious that similar trend as in Fig. 5 is reflected in Fig. 6 too. Although even for small $f$-values the PLANCK result is satisfied for $\omega^0_{DE}$ for both the quintessence and Slotheon dark energy case when $m = 1$ keV (Fig. 6(a)) but for Slotheon case the variation plot of $\omega^0_{DE}$ lies below the quintessence case indicating that Slotheon case satisfy the PLANCK limit better even if $f$ values are further lowered. For $m = 1$ TeV (higher value) the PLANCK range is generally
satisfied for both the Slotheon and quintessence cases when $f$ is high but in this case also the lower range of $f$ can be more explored for Slotheon model results than those for quintessence. From Fig. 6(b) also, similar conclusions can be drawn by observing how the variations of $\Omega_0^{DE}$ with $f$ obey the PLANCK range. Therefore from Fig. 6(a) and Fig. 6(b) it is observed that the Slotheon field dark energy when coupled to dark matter of “bare” mass $\sim 1$ keV, can approach the high-$f$ problem most effectively.

Figure 6: (a) (left panel) Comparisons of evolutions of the present epoch value of dark energy equation of state parameter with $f/M_{pl}$ for quintessence dark energy model (solid lines) and Slotheon field dark energy model (dashed lines) for two chosen dark matter “bare” masses namely $m = 1$ keV and $m = 1$ TeV. (b) (right panel) Same as Fig. 6(a) but for present value of dark energy density parameter $\Omega_0^{DE}$. See text for discussion.

6 Summary and Conclusions

In this work we explore the dark energy equation of state and dark energy density parameter in case of a dark matter dark energy interaction where the dark energy is considered to have driven by a pNGB scalar field $\alpha$ with potential having a form $V(\alpha) = \mu^4(1 + \cos(\frac{\alpha}{f}))$. A pNGB is produced when the Nambu Goldstone boson that arises due to spontaneous breaking of a global symmetry, acquires a mass at soft explicit symmetry breaking scale lower than the spontaneous symmetry breaking scale $f$. We address in this work the high-$f$ problem of such dark energy models. The high-$f$ problem arises from the consideration that although large value of $f$ leads to a flatter potential, but it is not compatible with the field theory limits and quantum gravity corrections. In this work we show that when dark matter dark energy interaction is taken into account, the calculated cosmological parameters agree much better with the observational results of PLANCK 2018 experiment for lower values of $f$ and then the high-$f$ problem can be averted. We show this for both the quintessence dark energy model and for another model namely the
Slotheon dark energy model. The latter is inspired by the theories of extra dimensions such as the DGP theory. We also find that Slotheon dark energy model with the DM-DE interaction addresses better the high-$f$ problem than the quintessence model.

We also explore in the present framework of dark matter dark energy interaction, the mass limits of dark matter that would produce dark energy equation of state and dark energy density parameters for low values of $f$. We found this limit to lie between $\sim 1$ keV and $\sim 1$ TeV. The PLANCK limit is found to be better satisfied for lower values of the mass of the dark matter (the lower limit being $\sim 1$ keV). Here too the Slotheon dark energy consideration appears to be better than the quintessence dark energy.

From the present analyses and calculations, this may be concluded that the high-$f$ value problem for pNGB potential can be better addressed if dark matter dark energy interaction is present. Also the Slotheon dark energy model is better suited than the quintessence model for the purpose. A dark matter mass limit for such a scenario is also estimated.

Acknowledgements
The authors would like to thank A. Biswas and A. Dutta Banik for some useful suggestions. One of the authors (U.M.) receives her fellowship (as a graduate student leading to Ph.D. degree) grant from Council of Scientific & Industrial Research (CSIR), Government of India as Junior Research Fellow (JRF) having the fellowship Grant No. 09/489(0106)/2017-EMR-I.

Appendix
Equation of motion (EOM) of the dark matter field $\chi$ (eq. (7)) is as follows

$$\ddot{\chi} + 3H\dot{\chi} + \frac{\partial V(\phi, \chi)}{\partial \chi} = 0.$$  

(61)

Multiplying both sides of the above EOM by $\dot{\chi}$ and with $V(\phi, \chi) = V_{DE}(\phi) + V_{DM}(\phi, \chi)$ ($V_{DE}(\phi)$ has no $\chi$ dependence (eq. (10)), we obtain,

$$\dot{\chi} \ddot{\chi} + 3H\dot{\chi}^2 + \dot{\chi} \frac{\partial V_{DM}(\phi, \chi)}{\partial \chi} + \dot{\phi} \frac{\partial V_{DM}(\phi, \chi)}{\partial \phi} - \dot{\phi} \frac{\partial V_{DM}(\phi, \chi)}{\partial \phi} = 0,$$

$$\frac{d}{dt} \left(\frac{1}{2} \dot{\chi}^2 + V_{DM}(\phi, \chi)\right) + 3H\dot{\chi}^2 - \dot{\phi} \frac{\partial V_{DM}(\phi, \chi)}{\partial \phi} = 0.$$  

(62)
Again with eqs. (10 - 16) we get,

\[
\frac{d\langle\dot{\chi}^2\rangle}{dt} + 3H\langle\dot{\chi}^2\rangle - \frac{1}{2}\dot{\phi}\frac{\partial M^2(\phi)}{\partial \phi}\langle\chi^2\rangle = 0 ,
\]

\[
\frac{d\langle\rho_{DM}\rangle}{dt} + 3H\langle\rho_{DM}\rangle - \frac{1}{2}\dot{\phi}\frac{1}{M^2(\phi)}\frac{\partial M^2(\phi)}{\partial \phi}\langle\rho_{DM}\rangle = 0 .
\]

(63)

It is noted that to a good approximation, density is not sensitive to the oscillations and hence it remains unaffected by a cyclic average [60]. Therefore replacing \(\langle\rho_{DM}\rangle\) with \(\rho_{DM}\) and \(\langle\dot{\rho}_{DM}\rangle\) with \(\dot{\rho}_{DM}\) we get

\[
\dot{\rho}_{DM} + 3H\rho_{DM} = \frac{1}{2}\dot{\phi}\frac{1}{M^2(\phi)}\frac{\partial M^2(\phi)}{\partial \phi}\rho_{DM} = -Q .
\]

Moreover the solution of this equation is,

\[
\rho_{DM}(\phi, a) = n_o a^{-3} M(\phi) .
\]

(64)

Therefore, the effective potential of the field \(\phi\) is now given as,

\[
V_{eff}(\phi, a) = V_{DE}(\phi) + \rho_{DM}(\phi, a) .
\]

(65)

It is useful to check whether this field theory approach of DM-DE interactions yields equations that are of the same form as eqs. (2 - 3) which are obtained from the energy-momentum conservation equations. From eq. (6) we have,

\[
\ddot{\phi} + 3H\dot{\phi} + \frac{\partial V_{eff}(\phi, a)}{\partial \phi} = 0 .
\]

(66)

Multiplying throughout by \(\dot{\phi}\),

\[
\dot{\phi}\ddot{\phi} + 3H\dot{\phi}^2 + \frac{\partial V_{eff}(\phi, a)}{\partial \phi}\dot{\phi} = 0 ,
\]

\[
\frac{d}{dt}\left(\frac{1}{2}\dot{\phi}^2 + V_{DE}(\phi)\right) + 3H(p_{DE} + \rho_{DE}) = -\frac{\partial \rho_{DM}}{\partial \phi}\dot{\phi} .
\]

(67)

The energy and pressure densities for dark energy (DE) (\(\rho_{DE}\) and \(p_{DE}\) respectively) introduced in the above are given by

\[
\rho_{DE} = \frac{1}{2}\dot{\phi}^2 + V_{DE}(\phi) ,
\]

(68)

\[
p_{DE} = \frac{1}{2}\dot{\phi}^2 - V_{DE}(\phi) .
\]

(69)
Now the right hand side (R.H.S.) of eq. (67) can be calculated as,

$$\frac{\partial \rho_{DM}}{\partial \phi} \dot{\phi} = \frac{1}{2} \dot{\phi} \rho_{DM} \left( \frac{2}{\rho_{DM}} \frac{\partial \rho_{DM}}{\partial \phi} \right)$$

$$= \frac{1}{2} \dot{\phi} \rho_{DM} \left( \frac{2}{M(\phi)} \frac{\partial M(\phi)}{\partial \phi} \right)$$

$$= \frac{1}{2} \dot{\phi} \rho_{DM} \frac{\partial}{\partial \phi} \ln M^2(\phi)$$

$$= Q.$$ \hspace{1cm} (70)

Hence,

$$\dot{\rho}_{DE} + 3H(\rho_{DE} + p_{DE}) = Q.$$ \hspace{1cm} (71)

Therefore the field theory approach of DM-DE interaction leads us to eqs. (2 - 3) and hence obeys the total energy-momentum conservation of the Universe.

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