Separation principle for discrete-time quasi-one-sided Lipschitz nonlinear systems

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Abstract

This paper is concerned with output stabilisation for a class of discrete-time quasi-one-sided Lipschitz nonlinear systems. Firstly, an observer is designed for estimating the state of the systems in terms of quasi-one-sided Lipschitz condition and quadratic inner-boundness condition. Then, a state feedback controller is proposed to stabilize the systems. Subsequently, it is shown that the separation principle holds for stabilisation of the discrete-time nonlinear systems based on the observer-based controller. Under the quasi-one-sided Lipschitz condition, the state observer and feedback controller can be designed separately, even though the parameter \((A, B)\) of discrete-time systems is not stabilisable and parameter \((A, C)\) is not detectable. Finally, two numerical examples are given to indicate the feasibility of the main results.

1 INTRODUCTION

During the last several decades, tremendous research activities have been developed to deal with the problem of observer design and feedback controller design for nonlinear dynamical systems. As we know, the information of system state plays a key role for stabilisation. However, in engineering practice, sometimes it is impossible to measure all the system state variables through sensors directly. Under those situations, a state observer is usually needed, and then the so-called observer-based control can be carried out using the estimated state. In recent years, many research efforts were mainly focused on the observer design problem of the Lipschitz nonlinear systems for continuous-time or discrete-time cases.

For the continuous-time case, the works on the study of observers include [1–17]. In 1973, the observer design of systems with the Lipschitz condition was firstly studied by Thau [1]. Subsequently, many researchers addressed the observer synthesis of Lipschitz non-linear systems by using various approaches, for instance, [2–9], where the non-linearities were restricted to the cases satisfying the Lipschitz condition. However, for Lipschitz non-linear systems, most of existing observer design techniques can only stabilise the error dynamics with small Lipschitz constants but fail to provide a solution when the Lipschitz constant is large, and the non-linear part of systems is thought as a perturbation. In 2006, the one-sided Lipschitz condition is first introduced by Hu [10] instead of the classical Lipschitz condition for observer design. Based on the one-sided Lipschitz condition, sufficient conditions for the observer design remain valid even if the parameter of systems \((A, C)\) is not detectable. This shows that the one-sided Lipschitz condition includes much more information of the non-linear part than the classical Lipschitz condition. Subsequently, the observer design schemes for non-linear systems in terms of one-sided Lipschitz condition have been reported by [11–17]. In [11], reduced-order observers for one-sided Lipschitz non-linear systems are given via using a similar method of Zhu and Han [6]. The existence condition of the observer is further discussed by Zhao et al. [12]. In [13], the design of sampled-data state observers for non-linear plants is presented under the effect of system and measurement disturbance signals. In [14], the controller design problem for a class of stochastic one-sided Lipschitz differential inclusion systems with time delay is addressed. In [15], by introducing a continuous frequency distributed equivalent model and using indirect Lyapunov approach, the sufficient condition for asymptotic stability of the full-order observer error dynamic system is presented. Subsequently, the proposed design method is extended to reduced-order observer design for fractional-order non-linear systems. In [16], the problem of observer design for one-sided Lipschitz non-linear systems by using the linear matrix inequality approach is investigated. Sufficient conditions that ensure the existence of observers are...
established, which are easily and numerically tractable via standard software algorithms. In [17], fault tolerant controllers for a class of one-sided Lipschitz non-linear systems with external disturbances are presented, and the problem is investigated in the presence of faults and disturbances simultaneously.

As the discrete-time counterpart, the design schemes about state observers for non-linear systems have been investigated in [18–23]. In [18], by introducing a new Lyapunov function, the existence conditions of observers have been derived for the discrete-time Lipschitz non-linear systems. In [19], the results in [18] are improved. In [20], the discrete-time non-linear observer is designed by using a circle criterion approach. In [21], the one-sided Lipschitz condition is applied to design observer of discrete-time non-linear systems. A simple and useful observer design method for a class of non-linear systems that satisfies the one-sided Lipschitz condition is proposed in [22]. In [23], a convex optimisation approach to observation of both constant-delay and time-varying delay non-linear systems is presented, which the observation problem is reduced to a stability problem of linear systems with structured known uncertainties.

In 2008, the so-called quasi-one-sided Lipschitz condition is originally presented by Hu [24] instead of the one-sided Lipschitz condition and Lipschitz condition for observer design of the non-linear system. It is shown that the quasi-one-sided Lipschitz condition is less conservative than the one-sided Lipschitz condition and Lipschitz condition. In [25], observer-based stabilisation of quasi-one-sided non-linear systems is considered, where the LMI-based design of the state observer and feedback controller are done separately. In [26], the observer-based controller design problem for tracking a constant reference input for quasi-one-sided Lipschitz non-linear systems is considered. In [27], the observer design problem for integer-order systems is studied, a particular form of observers for integer-order Lipschitz, one-sided Lipschitz and quasi-one-sided Lipschitz systems are extended to the fractional-order calculus. In the work of Hu et al. [28], the problem of output stabilisation for a class of non-linear time-delay systems is addressed. Under the quasi-one-sided Lipschitz condition, separation principle holds for stabilisation of the systems based on the observer-based controller.

To the best of our knowledge, until now, few results have been on the study of observer design and controller design of discrete-time non-linear systems with quasi-one-sided Lipschitz condition, not to mention the separation principle for the discrete-time non-linear systems. Due to the difference among the structure of Lyapunov function, the conditions required to fulfill non-linear function, and the asymptotic stability criterion of observer, our earlier results obtained in the case of continuous time in [29] can not be directly extended to the case of discrete time. Therefore, it is necessary to study the output stabilisation problem for a class of discrete-time non-linear systems directly. This motivates our research.

In this article, based on Lyapunov stability theory, the observer-based stabilisation for discrete-time quasi-one-sided Lipschitz non-linear systems is considered.

The paper is arranged as follows: In Section 2, the description of the system under consideration along with some preliminary definitions are presented. In Section 3, an observer design for the system is presented. In Section 4, a state feedback controller design is investigated. In Section 5, a separation principle is derived for stabilisation the system using the observer-based controller. That is to say, the observer and controller can be designed separately for discrete-time non-linear system. In Section 6, two numerical examples are proposed in order to show the validity of our results. Finally, some conclusions are given in Section 7.

Notations: The notations used throughout the paper are standard. $\mathbb{R}^n$ represents the set of $n \times 1$ real column vectors, $\mathbb{R}^{n \times m}$ denotes the set of $n \times m$ real matrices, $I_n$ denotes $n \times n$ identity matrix, $(\cdot)^T$ denotes transpose, $(\cdot)^{-1}$ denotes inverse, $(\cdot)$ is the inner product in $\mathbb{R}^n$, that is, given $x, y \in \mathbb{R}$, then $\langle x, y \rangle = x^T y$, $\| \cdot \|$ denotes the Euclidean norm in $\mathbb{R}^n$. For a symmetric matrix $S$, $S > 0$ ($S < 0$) means that the matrix is positive definite (negative definite). If $A$ and $B$ are symmetric matrices, $A > B$ means that $A - B$ is positive definite. In symmetric block matrices, an asterisk $\ast$ represents a term induced by symmetry.

\section{System Description and Preliminaries}

In this section, we begin by introducing some notations and definitions that will be used later in the following sections.

Consider the following discrete-time non-linear systems described as

$$
\begin{align*}
\dot{x}(k+1) &= Ax(k) + Bu(k) + \phi(x(k)), \\
\gamma(k) &= Cx(k),
\end{align*}
$$

where $x(k) \in \mathbb{R}^n$ is the state vector, $y(k) \in \mathbb{R}^p$ is the output vector, $u(k) \in \mathbb{R}^m$ is the control input vector, $A \in \mathbb{R}^{n \times n}$ and $C \in \mathbb{R}^{m \times n}$ are the constant matrices of appropriate dimensions. The vector-valued function $\phi(x(k)) : \mathbb{R}^n \to \mathbb{R}^m$ represents the non-linearity of the systems. Throughout this paper, without loss of generality, we assume that $\phi(0) = 0$.

**Definition 1.** [24] The non-linear function $\phi(x)$ is said to be quasi-one-sided Lipschitz, if there exist a real symmetric matrix $M$ such that

$$
\langle \phi(x) - \phi(\hat{x}), x - \hat{x} \rangle \leq (x - \hat{x})^T M (x - \hat{x})
$$

holds for any $x, \hat{x} \in \mathbb{R}^n$. Inequality (2) is called the quasi-one-sided Lipschitz condition and the matrix $M$ is called a quasi-one-sided Lipschitz constant matrix for $\phi(x)$ with respect to $x$.

**Remark 1.** The $M$ in the quasi-one-sided Lipschitz condition (2) need not be positive-or negative-definite matrix. Inequality (2) only requires that the matrix $M$ is symmetric.

**Definition 2.** [31] The non-linear function $\phi(x)$ is said to satisfy the quadratic inner-boundness condition, if there exist...
constants $\alpha, \beta \in \mathbb{R}$, such that
\[
(\phi(x) - \phi(\hat{x}))^T (\phi(x) - \phi(\hat{x})) \leq \alpha \|x - \hat{x}\|^2 + \beta (x - \hat{x}, \phi(x) - \phi(\hat{x}))
\]
holds for any $x, \hat{x} \in \mathbb{R}^n$.

Remark 2. It is clear that any Lipschitz function is also quadratically inner bounded with $\beta = 0, \alpha > 0$. Thus, the Lipschitz continuity implies quadratic inner-boundedness. However, the converse is not true. Note that constant $\beta \in \mathbb{R}$ can be positive, zero, or even negative. If $\beta$ is restricted to be positive, then it can be shown that $\phi$ must be Lipschitz.

Definition 3. [29] The non-linear function $\phi(x(k))$ is said to be weak quasi-one-sided Lipschitz, if there exist real symmetric matrices $M_0$ such that
\[
(\phi(x), x) \leq x^T M_0 x
\]
holds for any $x \in \mathbb{R}^n$. Inequality (4) is called the weak quasi-one-sided Lipschitz condition and matrix $M_0$ is called a weak quasi-one-sided-Lipschitz constant matrix for $\phi(x)$ with respect to $x$. The inequality
\[
(\phi(x), x) \leq v_0 \|x\|^2
\]
is called the weak one-sided Lipschitz condition for any $x \in \mathbb{R}^n$ and $v_0$ is the weak one-sided Lipschitz constant. The constant $v_0$ can be positive, zero, or even negative.

Definition 4. The non-linear function $\phi(x)$ is called weak quadratically inner-bounded, if there exist constants $\alpha_0, \beta_0 \in \mathbb{R}$, such that
\[
\phi^T(x) \phi(x) \leq \alpha_0 \|x\|^2 + \beta_0 (x, \phi(x))
\]
holds for any $x \in \mathbb{R}^n$.

Remark 3. According to the assumption $\phi(0) = 0$, let $\hat{x} = 0$, then the quasi-one-sided Lipschitz condition (2) reduces to the weak quasi-one-sided Lipschitz condition (4), the quadratically inner-bounded condition (3) reduces to the weak quadratically inner-bounded condition (6). Therefore, the weak quasi-one-sided Lipschitz condition (4) and weak quadratically inner-bounded condition (6) are the extension of the quasi-one-sided Lipschitz condition (2) and quadratically inner-bounded condition (3).

Remark 4. When $M_0 = v_0 I$, the weak quasi-one-sided Lipschitz condition (4) is reduced to the weak one-sided Lipschitz condition (5). The weak quasi-one-sided Lipschitz condition (4) is also an extension of the weak one-sided Lipschitz condition (5).

Lemma 1. (the Schur complement lemma, for instance, [30]). For a real symmetric matrix $X$, the following statements are equivalent:

1. $X = \begin{pmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{pmatrix} < 0$.
2. $X_{11} < 0$, and $X_{22} - X_{12}^T X_{11}^{-1} X_{12} < 0$.
3. $X_{22} < 0$, and $X_{11} - X_{12}X_{22}^{-1} X_{12}^T < 0$.

3 | OBSERVER DESIGN

In this section, we propose a sufficient condition for existence of the observer for discrete-time non-linear system (1) using the quasi-one-sided Lipschitz condition (2) and quadratic inner-boundedness condition (3).

For system (1), we consider an observer of the following form:
\[
\begin{align*}
\hat{x}(k+1) &= A\hat{x}(k) + Bu(k) + \phi(\hat{x}(k)) + L(y(k) - C\hat{x}(k)) \\
y(k) &= C\hat{x}(k),
\end{align*}
\]
where $L \in \mathbb{R}^{n \times p}$ is observer gain matrix to be determined later.

Define estimation error $e(k) = x(k) - \hat{x}(k)$, then the dynamics of the observer error is given by the equation
\[
e(k + 1) = (A - LC)e(k) + \phi(x(k)) - \phi(\hat{x}(k)).
\]

Utilising the above definition, the following theorem provide a sufficient condition that observer (7) is an asymptotically stable observer for system (1).

Theorem 1. Consider system (1) with quasi-one-sided Lipschitz condition (2) and quadratic inner-boundedness condition (3). If there exist some matrix $P > 0, L$ with appropriate dimensions and constants $\varepsilon_1, \varepsilon_2$ can be chosen such that the following matrix inequality holds:
\[
\begin{pmatrix}
N_{11} & (A - LC)^T P + \frac{1}{2} \varepsilon_1 \beta I - \frac{1}{2} \varepsilon_2 I \\
* & P - \varepsilon_2 I
\end{pmatrix} < 0,
\]
where
\[
N_{11} = (A - LC)^T P(A - LC) - P + \varepsilon_1 M I + \varepsilon_2 A I,
\]
then the observer (7) yields an asymptotically stable estimate for system (1), where the symmetric matrix $M$ satisfies the quasi-one-sided Lipschitz condition (2), constants $\alpha$ and $\beta$ satisfy quadratically inner-bounded condition (3).

Furthermore, let $L = P^{-1} R^T$, inequality (9) is equivalent to the following condition: there are symmetric positive-definite matrix $P$ and a real matrix $R$ satisfying the LMI
\[
\begin{pmatrix}
\Omega_{11} & \Omega_{12} \\ * & 0
\end{pmatrix} A^T P - C^T R < 0,
\]
\[
\begin{pmatrix}
\Omega_{11} & \Omega_{12} \\ * & -P
\end{pmatrix} < 0,
\]
where
\[
\begin{align*}
\Omega_{11} &= -P + \varepsilon_1 M I + \varepsilon_2 \alpha I, \\
\Omega_{12} &= A^T P - C^T R + \frac{1}{2} \varepsilon_2 \beta I - \frac{1}{2} \varepsilon_1 I.
\end{align*}
\] (12)

Proof. Consider the Lyapunov function candidate
\[
V(e(k)) = e^T(k) P e(k).
\] (13)

Then, the difference of \( V(e(k)) \) is given by
\[
\begin{align*}
\Delta V_k &= V(e(k + 1)) - V(e(k)) \\
&= e^T(k + 1) P e(k + 1) - e^T(k) P e(k) \\
&= e^T(k) [(A - LC)^T P(A - LC) - P] e(k) \\
&\quad + 2 e^T(k)(A - LC)^T P \Delta \phi_k + \Delta \phi_k^T P \Delta \phi_k,
\end{align*}
\] (14)

where \( \Delta \phi_k = \phi(x(k)) - \phi(\hat{x}(k)) \). From quasi-one-sided Lipschitz condition (2) and quadratic inner-boundedness condition (3), there exist \( \varepsilon_1, \varepsilon_2 > 0 \) such that
\[
\varepsilon_1 e^T(k) M e(k) - \varepsilon_1 e^T(k) \Delta \phi_k \geq 0
\] (15)
and
\[
\varepsilon_2 e^T(k) r(k) + \varepsilon_2 \beta e^T(k) \Delta \phi_k - \varepsilon_2 \Delta \phi_k^T \Delta \phi_k \geq 0
\] (16)
hold.

Adding the left side of equations (15) and (16) to the right side of equation (14) gives
\[
\begin{align*}
\Delta V_k &\leq e^T(k) [(A - LC)^T P(A - LC) - P + \varepsilon_1 M I + \varepsilon_2 \alpha I] r(k) \\
&\quad + 2 e^T(k)(A - LC)^T P + \frac{1}{2} \varepsilon_2 \beta I - \frac{1}{2} \varepsilon_1 I] \Delta \phi_k \\
&\quad + \Delta \phi_k^T (P - \varepsilon_2 I) \Delta \phi_k.
\end{align*}
\] (17)

By means of the Lyapunov stability theory and the dynamics of the estimation error, the observer (7) is asymptotically stable, if
\[
\Delta V_k \leq \eta^T(k) N \eta(k) < 0,
\] (18)

where
\[
N = \begin{pmatrix} N_{11} & (A - LC)^T P + \frac{1}{2} \varepsilon_2 \beta I - \frac{1}{2} \varepsilon_1 I \\ * & P - \varepsilon_2 I \end{pmatrix},
\] (19)

\[
N_{11} = (A - LC)^T P(A - LC) - P + \varepsilon_1 M I + \varepsilon_2 \alpha I,
\]

and
\[
\eta(k) = (e^T(k), \Delta \phi_k^T).
\] (20)

Set \( L = P^{-1} R^T \), according to the Schur complement lemma, \( N < 0 \) is equivalent to \( \Omega < 0 \), where
\[
\Omega = \begin{pmatrix} \Omega_{11} & \Omega_{12} \\ * & A^T P - C^T R \end{pmatrix} < 0
\] (21)
\[
\Omega_{11} = -P + \varepsilon_1 M I + \varepsilon_2 \alpha I,
\]
\[
\Omega_{12} = A^T P - C^T R + \frac{1}{2} \varepsilon_2 \beta I - \frac{1}{2} \varepsilon_1 I.
\]

If LMI \( \Omega < 0 \) holds, then a feasible solution exists, and \( \Delta V_k < 0 \) for all \( e(k) \neq 0 \), which implies that error dynamics (8) is asymptotically stable for system (1) with conditions (2) and (3). This ends the proof.

Remark 5. In [21], the existence of a gain matrix \( L \) is investigated for system (1) with the one-sided Lipschitz condition
\[
\langle \phi(x) - \phi(\hat{x}), x - \hat{x} \rangle \leq v \| x - \hat{x} \|^2,
\] (22)
where \( v \) is called the one-sided Lipschitz constant. The constant \( v \) can be positive, zero, or even negative. The sufficient condition that observer yields an asymptotically stable estimate for system (1) is \( Y < 0 \), where
\[
Y = \begin{pmatrix} Y_{11} & Y_{12} \\ * & A^T P - C^T R \end{pmatrix} < 0
\] (23)
\[
Y_{11} = -P + \varepsilon_1 M I + \varepsilon_2 \alpha I,
\]
\[
Y_{12} = A^T P - C^T R + \frac{1}{2} \varepsilon_2 \beta I - \frac{1}{2} \varepsilon_1 I.
\]

Assume that
\[
M \leq v I,
\] (24)
it is obvious that \( \Omega \leq Y \). That is to say, if the condition of Proposition 8 in [21] holds, then the condition of Theorem 1 holds. However, the converse is not true. Therefore, the quasi-one-sided Lipschitz condition is less conservative than the one-sided Lipschitz condition to design an observer for discrete-time non-linear system (1). In general, we define \( v = \lambda_{\text{max}}(M) \), Theorem 1 is an extension of the Proposition 8 of [21].

Remark 6. In [19], the existence of a gain matrix \( L \) is investigated for system (1) with the Lipschitz condition
\[
\| \phi(x) - \phi(\hat{x}) \| \leq \gamma \| x - \hat{x} \|
\] (25)
for any \( x, \dot{x} \in \mathbb{R}^n \), where \( \gamma > 0 \) is called the Lipschitz constant. The sufficient condition that observer yields an asymptotically stable estimate for system (1) is \( \Pi < 0 \), where

\[
\Pi = \begin{pmatrix}
-P + \gamma^2 I & A^T P - C^T R \\
O & P - \gamma I \\
O & O \\
\end{pmatrix},
\]

(26)

Note that, if \( \varepsilon_1 = 0, \beta = 0, \varepsilon_2 = \tau \) and \( \alpha = \gamma^2 \), \( \Omega < 0 \) reduces to \( \Pi < 0 \). Therefore, Theorem 1 is also an extension of the Theorem 1 of [19].

**Remark 7.** From Remark 5 and Remark 6, we know that the sufficient condition for existence of observers of discrete-time non-linear system (1) based on the quasi-one-sided Lipschitz condition is less conservative than the results based on the classical Lipschitz condition (25) and the one-sided Lipschitz condition (22).

As we know, the discrete-time non-linear system (1) consists of two parts, a linear part \( Ax(k) \) and a non-linear part \( \phi(x(k)) \), we hope to make use of the information of the non-linear part as far as possible to design an observer of system, it is important that how to evaluate the influence of the non-linear part on the system. When the non-linear part \( \phi(x(k)) = 0 \), the system can be described as

\[
\begin{align*}
x(k + 1) &= Ax(k) + Bu(k) \\
y(k) &= Cx(k).
\end{align*}
\]

(27)

In order to design an observer of system (27), we need to find a gain matrix \( L \) with purpose to guarantee \( A - LC \) is stable. By the Lyapunov stability theory, it is necessary to find a positive-definite matrix \( P \) such that

\[
(A - LC)^T P (A - LC) - P < 0.
\]

(28)

For non-linear system (1), it is obvious that the Lipschitz condition cannot be used to design the observer when the matrix \( A - LC \) is not stabilised by any gain matrix \( L \) in [19]. Therefore, when designing an observer by means of the classical Lipschitz condition, it requires the parameter \( (A, C) \) of system is detectable, along with ensuring the Lipschitz constant is small enough. In [19], it cannot design an observer if the Lipschitz constant \( \gamma \) is greater than one. This means that the non-linear part \( \phi(x(k)) \) has no contributions to design an asymptotically stable observer for system, the non-linear part of system is a perturbation. However, since quasi-one-sided Lipschitz constant \( M \) is unnecessarily positive definite and one-sided Lipschitz constant \( \gamma \) is unnecessarily positive, it is possible that quasi-one-sided Lipschitz condition as well as one-sided Lipschitz condition is used to design a synchronously stable observer for discrete-time non-linear system (1) even though the parameter \( (A, C) \) of system is not detectable, the non-linear part is beneficial to the system. Furthermore, the quasi-one-sided Lipschitz condition includes more useful information of the non-linear part than the one-sided Lipschitz condition.

### 4 | CONTROLLER DESIGN

In this section, based on the weak quasi-one-sided Lipschitz condition (4) and weak quadratically inner-bounded condition (6), we design a state feedback controller to stabilise the discrete-time non-linear system (1).

For system (1), let a state feedback controller be of the form

\[
u(k) = -Kx(k),
\]

(29)

where the gain matrix \( K \in \mathbb{R}^{m \times n} \). From (1) and (29), we obtain the discrete-time closed-loop system

\[
\begin{align*}
x(k + 1) &= (A - BK)x(k) + \phi(x(k)) \\
y(k) &= Cx(k).
\end{align*}
\]

(30)

For the feedback controller (29), a sufficient condition that the zero solution of the discrete-time closed-loop system (30) is asymptotically stable is as follows.

**Theorem 2.** Consider system (1) with weak quasi-one-sided Lipschitz condition (4) and the weak quadratic inner-bounded condition (6). If there exist some matrix \( P \geq 0 \) with appropriate dimensions and constants \( \varepsilon_3, \varepsilon_4 \) can be chosen such that the following matrix inequality holds

\[
\begin{pmatrix}
N_{01} & (A - BK)^T P_0 + 2\varepsilon_3 \beta_0 I - 2\varepsilon_3 I \\
O & P_0 - \varepsilon_4 I
\end{pmatrix} < 0
\]

(31)

where

\[
N_{01} = (A - BK)^T P_0 (A - BK) - P_0 + \varepsilon_3 \beta_0 I + \varepsilon_4 \alpha_0 I,
\]

(32)

then the zero solution of the closed-loop system (30) is asymptotically stable, where the symmetric matrix \( M_0 \) satisfies the weak quasi-one-sided Lipschitz condition (4), constants \( \alpha_0 \) and \( \beta_0 \) satisfy the weak quadratically inner-bounded condition (6).

Furthermore, set \( \Omega = P_0^{-1} \) and \( K = W \Omega^{-1} \), inequality (31) is equivalent to the following condition: there are a symmetric positive-definite matrix \( Q \) and a real matrix \( W \) satisfying the LMI

\[
\begin{pmatrix}
\Theta_{11} & \Theta_{12} \\
\Theta_{21} & \Theta_{22}
\end{pmatrix} = \begin{pmatrix}
Q & A^T - WB^T \\
A - BK & 0
\end{pmatrix} < 0,
\]

(33)

where

\[
\Theta_{11} = -Q + \varepsilon_3 Q M_0 Q + \varepsilon_4 \alpha_0 Q Q,
\]

\[
\Theta_{12} = QA^T - WB^T B^T + \frac{1}{2} \varepsilon_4 \beta_0 Q Q - \frac{1}{2} \varepsilon_3 Q Q.
\]

(34)

**Proof.** Consider the Lyapunov function candidate

\[
V(x(k)) = x^T(k)P_0 x(k).
\]

(35)
Then, the difference of $V(\mathbf{x}(k))$ is given by

$$
\Delta V_k = V(\mathbf{x}(k + 1)) - V(\mathbf{x}(k))
$$

$$
= \mathbf{x}^T(k + 1)P_0\mathbf{x}(k + 1) - \mathbf{x}^T(k)P_0\mathbf{x}(k)
$$

$$
= \mathbf{x}^T(k)(A - BK)^T P_0(A - BK) - P_0\mathbf{x}(k)
$$

$$
+ 2\mathbf{x}^T(k)(A - BK)^T P_0\phi_k + \phi_k^T P\phi_k,
$$

where $\phi_k = \phi(\mathbf{x}(k))$. From the weak quasi-one-sided Lipschitz condition (4) and the weak quadratic inner-boundedness condition (6), there exist $\epsilon_3, \epsilon_4 > 0$ such that

$$
\epsilon_3\mathbf{x}^T(k)M_0\mathbf{x}(k) - \epsilon_3\mathbf{x}^T(k)\phi_k \geq 0,
$$

and

$$
\epsilon_4\alpha_0\mathbf{x}^T(k)\mathbf{x}(k) + \epsilon_4\beta_0\mathbf{x}^T(k)\phi_k - \epsilon_4\phi_k^T\phi_k \geq 0
$$

hold.

Adding the left side of equations (37) and (38) to the right side of equation (36) gives

$$
\Delta V_k \leq \mathbf{x}^T(k)(A - BK)^T P_0(A - BK) - P_0 + \epsilon_3M_0I
$$

$$
+ \epsilon_4\alpha_0I\mathbf{x}(k) + 2\mathbf{x}^T(k)(A - BK)^T P_0 + \frac{1}{2}\epsilon_4\beta_0I
$$

$$
- \frac{1}{2}\epsilon_3I\phi_k + \phi_k^T (P_0 - \epsilon_4I)\phi_k.
$$

According to the Lyapunov stability theory, the discrete-time closed-loop system (30) is asymptotically stable, if

$$
\Delta V_k \leq \xi^T(k)N_0\xi(k) < 0,
$$

where

$$
N_0 = \begin{pmatrix}
N_{01} & (-A - BK)^T P_0 + \frac{1}{2}\epsilon_4\beta_0I - \frac{1}{2}\epsilon_3I \\
0 & P_0 - \epsilon_4I
\end{pmatrix},
$$

$$
N_{01} = (A - BK)^T P_0(A - BK) - P_0 + \epsilon_3M_0I + \epsilon_4\alpha_0I,
$$

and

$$
\xi(k) = (\mathbf{x}^T(k), \phi^T(k))^T.
$$

Set $Q = P_0^{-1}$, $K = WQ^{-1}$, by the Schur complement lemma and pre-and post multiplying matrix diag $(Q, Q, Q)$ yields

$$
\Theta < 0,
$$

where

$$
\Theta = \begin{pmatrix}
\Theta_{11} & \Theta_{12} & QAT - W^TB^T \\
\ast & Q - \epsilon_4QQ & 0 \\
\ast & \ast & -Q
\end{pmatrix},
$$

$$
\Theta_{11} = -Q + \epsilon_3MQ_0Q + \epsilon_4\alpha_0QQ,
$$

$$
\Theta_{12} = QAT - WB^T + \frac{1}{2}\epsilon_4\beta_0QQ - \frac{1}{2}\epsilon_3QQ.
$$

So $N_{01} < 0$ is equivalent to $\Theta < 0$.

If LMI (43) holds, then a feasible solution exists, and $\Delta V_k < 0$ for all $\mathbf{x}(k) \neq 0$, which implies that the zero solution of the discrete-time closed-loop system (30) is asymptotically stable. The proof is completed.

**Remark 8.** Since matrix $M_0$ is unnecessarily positive definite, it is possible to stabilize the discrete-time closed-loop system (30) based on the weak one-sided Lipschitz condition (5) even though the parameter $(A, B)$ of system is not stabilisable.

Based on the weak one-sided Lipschitz condition (5), Theorem 2 is reduced to the following corollary, which provides a design approach of feedback controller.

**Corollary 1.** Consider non-linear time-delay system (1) with the weak one-sided Lipschitz condition (5) and weak quadratic inner-boundedness condition (6). The sufficient condition that the zero solution of the closed-loop system (30) is asymptotically stable is $\Lambda < 0$, where

$$
\Lambda = \begin{pmatrix}
\Lambda_{11} & \Lambda_{12} & QAT - WB^T \\
\ast & Q - \epsilon_4QQ & 0 \\
\ast & \ast & -Q
\end{pmatrix},
$$

$$
\Lambda_{11} = -Q + \epsilon_3MQ_0Q + \epsilon_4\alpha_0QQ,
$$

$$
\Lambda_{12} = QAT - WB^T + \frac{1}{2}\epsilon_4\beta_0QQ - \frac{1}{2}\epsilon_3QQ.
$$

the constant $\nu_0$ satisfies the weak one-sided Lipschitz condition (3), constants $\alpha_0, \beta_0$ satisfy the weak quadratically inner-bounded condition (6), controller gain matrix is given by $K = WQ^{-1}$.

Based on the proof of Theorem 2, Corollary 1 can be obtained easily.

**Remark 9.** From Theorem 2 and Corollary 1, we suppose that

$$
M_0 \leq \nu_0I,
$$

it is not hard to find that $\Theta < \Lambda$. In other words, if the condition of Corollary 1 holds, then the condition of Theorem 2 holds. However, the converse is not true. Therefore, the weak quasi-one-sided Lipschitz condition (4) is less conservative than
the weak one-sided Lipschitz condition (5) for stabilising the discrete-time non-linear system (1).

5 | SEPARATION PRINCIPLE

In this section, a separation principle is derived for the observer-based controller. We emphasise that the state observer and feedback controller can be designed separately.

Combining the controller
\[ u(k) = -K\hat{x}(k) \tag{47} \]
with the observer
\[ \hat{x}(k+1) = A\hat{x}(k) + Bu(k) + \phi(\hat{x}(k)) + L(y(k) - C\hat{x}(k)). \tag{48} \]

For system (1), the discrete-time closed-loop system under the control law (47) is as follows:
\[ x(k+1) = Ax(k) - BK\hat{x}(k) + \phi(x(k)). \tag{49} \]

Because estimation error \( e(k) = x(k) - \hat{x}(k) \), the closed-loop system (49) can be written as
\[ x(k+1) = (A - BK)x(k) + \phi(x(k)) + BK\epsilon(k) \tag{50} \]
and the estimation error dynamics is given by
\[ e(k+1) = (A - LC)e(k) + \Delta\phi_k, \tag{51} \]
where \( \Delta\phi_k = \phi(x(k)) - \phi(\hat{x}(k)) \).

Notice that the non-singular transformation
\[ \begin{pmatrix} x(k) \\ \hat{x}(k) \end{pmatrix} = \begin{pmatrix} I & 0 \\ I & -I \end{pmatrix} \begin{pmatrix} x(k) \\ e(k) \end{pmatrix}. \tag{52} \]

Because the transformation is non-singular, the dynamics of the discrete-time closed-loop system (49) with the observer (48) is equivalent to that of the closed-loop system (50) with the estimation error (51).

The following separation principle holds for the observer-based controller.

**Theorem 3.** Consider the closed-loop system (50) with conditions (2), (3), (4) and (6). Assume that the inequalities (9) and (31) hold. Let the controller \( K = WD^{-1} \) and the observer \( L = P^{-1}Q \), then the zero solution of the discrete-time closed-loop system (50) with the observer (48) is asymptotically stable under the controller law (47).

**Proof.** Consider the Lyapunov function candidate
\[ V_k = ax^T(k)P_0x(k) + e^T(k)P_0\epsilon(k), \tag{53} \]
where constant \( a > 0 \) which is determined later, the matrices \( P_0 \) and \( P \) satisfy the inequalities (9) and (31). The difference of \( V_k \) is given by
\[ \Delta V_k = V_{k+1} - V_k \]
\[ = ax^T(k+1)P_0x(k+1) + e^T(k+1)P_0\epsilon(k+1) \]
\[ - ax^T(k)P_0x(k) - e^T(k)P_0\epsilon(k) \]
\[ = ax^T(k)((A - BK)^TP_0(A - BK) - P_0)x(k) \]
\[ + 2ax^T(k)(A - BK)^TP_0\phi_k + a\phi_kP_0\phi_k \]
\[ + 2a\phi_k^T(k)(A - BK)^TP_0BK\epsilon(k) + 2a\phi_k^T(k)P_0BK\epsilon(k) \]
\[ + a(BK\epsilon(k))^TP_0BK\epsilon(k) - a\phi^T(k)P_0\phi(k) \]
\[ + e^T(k)((A - LC)^TP(A - LC) - P)e(k) \]
\[ + 2e^T(k)(A - LC)^TP\Delta\phi_k + \Delta\phi_k^T P\Delta\phi_k. \tag{54} \]

Adding the left side of equations (2), (3), (4) and (6) to the right side of equation (54) gives
\[ \Delta V_k \leq ax^T(k)((A - BK)^TP_0(A - BK) - P_0 + \epsilon_3M_0I \]
\[ + \epsilon_2\alpha_0I)x(k) \]
\[ + 2a\phi_k^T(k)(A - BK)^TP_0 + \frac{1}{2}\epsilon_4\beta_0 - \frac{1}{2}\epsilon_3I\phi_k \]
\[ + a\phi_k^T(k)(P_0 - \epsilon_4I)\phi_k + 2ax^T(k)(A - BK)^TP_0BK\epsilon(k) \]
\[ + 2a\phi_k^T(k)P_0BK\epsilon(k) + e^T(k)((A - LC)^TP(A - LC) - P \]
\[ + \epsilon_1MI + \epsilon_3\alpha_1 + a(BK)^TP_0(BK))e(k) \]
\[ + 2e^T(k)((A - LC)^TP + \frac{1}{2}\epsilon_2\beta_1 - \frac{1}{2}\epsilon_1I)\Delta\phi_k \]
\[ + \Delta\phi_k^T P(\epsilon_3I)\Delta\phi_k. \tag{55} \]

By means of the Lyapunov stability theory, the zero solution of the closed-loop system (50) with the observer (48) is asymptotically stable under the controller law (47), if
\[ \Delta V_k \leq \theta^T(k)\Gamma\theta(k) < 0, \tag{56} \]
where
\[ \Gamma = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} & a(A - BK)^TP_0BK & 0 \\ \ast & a(P_0 - \epsilon_4I) & aP_0BK & 0 \\ \ast & \ast & \Gamma_{33} & \Gamma_{34} \\ \ast & \ast & \ast & P - \epsilon_3I \end{bmatrix}. \tag{57} \]
\[ \mathbf{\Theta}(k) = (x^T(k), \mathbf{\Phi}_k^T, e^T(k), \Delta \mathbf{\Phi}_k^T)^T, \]
\[ \Gamma_{11} = a[A - BK]^TP_0(A - BK) - P_0 + \varepsilon_3 M_0 I + \varepsilon_4 \alpha_0 I, \]
\[ \Gamma_{12} = a[A - BK]^TP_0 + \frac{1}{2} \varepsilon_4 \beta_0 - \frac{1}{2} \varepsilon_3 I, \]
\[ \Gamma_{33} = (A - LC)^TP(A - LC) - P + \varepsilon_1 M_t + \varepsilon_2 \alpha I \]
\[ + a(BK)^T R_t BK, \]
\[ \Gamma_{34} = (A - LC)^TP + \frac{1}{2} \varepsilon_2 \beta I - \frac{1}{2} \varepsilon_1 I. \]

Since \( \xi(k) = (x^T(k), \mathbf{\Phi}_k^T)^T \) and \( \eta(k) = (e^T(k), \Delta \mathbf{\Phi}_k^T)^T \), according to (56), we obtain
\[ \Delta V_k \leq a \xi^T(k) N_0 \xi(k) + 2a \xi^T(k) \eta(k) + \xi^T(k) N_0 \eta(k) \]
\[ + 2a e^T(k) (BK)^T R_t BK \xi(k), \]
where
\[ S = \begin{pmatrix} (A - BK)^T P_0 BK & 0 \\ P_0 BK & 0 \end{pmatrix}, \]

matrices \( N \) and \( N_0 \) are given by (19) and (41), respectively. From Theorem 2, we have
\[ N_0 = \begin{pmatrix} N_{01} & (A - BK)^T P_0 + \frac{1}{2} \varepsilon_4 \beta_0 I - \frac{1}{2} \varepsilon_3 I \\ \ast & P_0 - \varepsilon_2 I \end{pmatrix} < 0. \]

This means that
\[ \lambda_{\text{max}}(N_0) < 0. \]

If LMI (33) holds, the controller is obtained by \( K = WQ^{-1} \).

Similarly, utilising Theorem 1, we have
\[ N = \begin{pmatrix} N_{11} & (A - LC)^T P + \frac{1}{2} \varepsilon_3 \beta I - \frac{1}{2} \varepsilon_1 I \\ \ast & P - \varepsilon_2 I \end{pmatrix} < 0. \]

This means that
\[ \lambda_{\text{max}}(N) < 0. \]

If LMI (11) holds, the observer can be designed by \( L = P^{-1} R^T \).

According to (58), we obtain
\[ \Delta V_k \leq a \lambda_{\text{max}}(N_0) \| \xi(k) \| ^2 + 2a \| \xi(k) \| \| (A - BK)^T P_0 \]
\[ \cdot BK \| \| \eta(k) \| + 2a \| \xi(k) \| \| P_0 BK \| \| \eta(k) \| \]
\[ + a \lambda_{\text{max}}(N) \| \eta(k) \| ^2 \]
\[ + a \| \eta(k) \| \| (BK)^T P_0 BK \| \| \eta(k) \| \]
\[ = -\omega^T(k) \Sigma \omega(k), \]

where
\[ \Sigma = \begin{pmatrix} -a \lambda_{\text{max}}(N_0) & -a \| (A - BK)^T P_0 BK \| \\ -a \lambda_{\text{max}}(N) & -a \| (BK)^T P_0 BK \| \end{pmatrix}, \]

and
\[ \omega(k) = (\| \xi(k) \| , \| \eta(k) \| )^T. \]

The matrix \( \Sigma \) is positive definite if and only if
\[ -a \lambda_{\text{max}}(N_0) > 0 \]

and
\[ a^2 \lambda_{\text{max}}(N_0) \| BK \| P_0 BK \| + a \lambda_{\text{max}}(N_0) \lambda_{\text{max}}(N) \]
\[ - a^2 \| (A - BK)^T P_0 BK \| + \| P_0 BK \| ^2 > 0. \]

Since \( a > 0 \), \( \lambda_{\text{max}}(N_0) < 0 \) and \( \lambda_{\text{max}}(N) < 0 \), let \( G = (A - BK)^T P_0 BK \), we have
\[ 0 < a < \frac{\lambda_{\text{max}}(N_0) \lambda_{\text{max}}(N)}{-\lambda_{\text{max}}(N_0) \| BK \| P_0 BK \| + \| G \| + \| P_0 BK \| ^2} \]

for \( BK \neq 0 \) or \( a > 0 \) for \( BK = 0 \).

When (68) holds, \( \Delta V_k < 0 \) for all \( (x(k), e(k)) \neq 0 \), the zero solution of the closed-loop system is asymptotically stable. It is shown that the separation principle holds for the stabilisation of the discrete-time quasi-one-sided Lipschitz non-linear system based on the observer-based controller. The proof is completed. \( \square \)

6 | NUMERICAL EXAMPLES

To demonstrate the effectiveness of our results, two numerical examples are given in this section.

Example 6.1. Consider the non-linear system in continuous-time as follows:

\[ \begin{cases} \dot{x} = A_1 x + B_1 u + \phi_1(x), \\ y = C_1 x, \end{cases} \]

where
\[ A_1 = \begin{pmatrix} -10 & 2 & 0 \\ 0 & 0.05 & 0 \\ 0.7 & 0 & 0.01 \end{pmatrix}, \]
\[ B_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \]
\[ C_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \]

the non-linear function \( \phi_1(x) \) is given by
\[ \phi_1(x) = \begin{pmatrix} 0.25 \sin x_1 \\ -0.3 x_2 \end{pmatrix} \begin{pmatrix} 1 - 0.5 \varepsilon^{-1.5} \\ 1 - 0.5 \varepsilon^{-1.5} \end{pmatrix}^T. \]
By applying the Euler discretisation with sampling time \( T = 0.1 \) s, we can obtain the discrete-time non-linear system described under the form (1) with the following parameters: \( A = A_1 + EA_1, B = TB_1, C = C_1 \) and \( \phi(x(k)) = T\phi_1(x) \).

At first, we design an observer for estimating the states of system. By calculation, it is not difficult to find that no feasible solution \( L \) such that all eigenvalues of \( A - LC \) have magnitude strictly less than unity, the matrix pair \( A, C \) is not detectable. Therefore, it is impossible to design a state observer for non-linear system based on the Lipschitz condition (25) even if the non-linear term \( \phi(x(k)) \) is global Lipschitz. The observer design approach in [19] is invalid. However, using the quasi-one-sided Lipschitz condition (2), it is possible to design an observer for the system even though matrix pair \( A, C \) is not detectable. That is to say, the non-linear part \( \phi(x(k)) \) is beneficial to design an observer for evaluating the states of system.

Based on the quasi-one-sided Lipschitz condition (2), for any \( x(k) = (x_1(k), x_2(k), x_3(k))^T \in \mathbb{R}^3 \), by means of the mean-value theorem, we have

\[
\langle \phi(x(k)) - \phi(\hat{x}(k)), x(k) - \hat{x}(k) \rangle = (x(k) - \hat{x}(k))^T M (x(k) - \hat{x}(k)),
\]

where

\[
M = \begin{pmatrix}
0.25T & 0 & 0 \\
0 & -0.2T & 0 \\
0 & 0 & -0.2T
\end{pmatrix}.
\]

Then we can choose the quadratic inner-boundedness parameters

\[
\alpha = \frac{1}{4} T^2 \quad \text{and} \quad \beta = \frac{3}{4} T.
\]

By means of Theorem 1, solving inequality (11), we can obtain the following solution:

\[
\varepsilon_1 = 4.1487, \varepsilon_2 = 8.1446,
\]

\[
R = \begin{pmatrix}
-1.4039 & 0 & 0.1541 \\
0.9301 & 3.2447 & 0 \\
0 & 0 & 2.2008
\end{pmatrix},
\]

\[
P = \begin{pmatrix}
4.6506 & 0 & 0 \\
0 & 4.6217 & 0 \\
0 & 0 & 2.2008
\end{pmatrix},
\]

Then, the gain matrix of the observer is given by

\[
L = P^{-1}R^T = \begin{pmatrix}
-0.3019 & 0.2000 \\
0 & 0.7020 \\
0.07 & 0
\end{pmatrix}.
\]

However, when using the one-sided Lipschitz condition (5), the one-sided Lipschitz constant \( \nu = \lambda_{\max}(M) = 0.257 \), we cannot find a feasible solution \( P, R \) satisfying \( Y < 0 \). The observer design approach in [21] is also invalid. This shows that the quasi-one-sided Lipschitz condition (2) is also less conservative than the one-sided Lipschitz condition (22) to design an asymptotically stable observer for system.

Now, we design a controller for stabilisation of system. By calculation, we know that the parameter \( (A, B) \) of the system is not stabilisable. Therefore, we cannot design a controller for discrete-time non-linear system based on the Lipschitz condition (25). However, it is possible to stabilise the system based on the weak quasi-one-sided Lipschitz condition (4). The non-linear part \( \phi(x(k)) \) is no longer regarded as a disturbance, but a beneficial contribution to stabilise the closed-loop system.

Similar to the calculation of quasi-one-sided Lipschitz constant matrix \( M \) and quadratic inner-boundedness constants \( \alpha, \beta \), let \( P = I \), we can get

\[
\alpha_0 = \frac{1}{4} T^2 \quad \text{and} \quad \beta_0 = \frac{3}{4} T
\]

and

\[
M_0 = \begin{pmatrix}
0.25T & 0 & 0 \\
0 & -0.3T & 0 \\
0 & 0 & -0.3T
\end{pmatrix},
\]

where \( M_0 \) is the weak quasi-one-sided Lipschitz constant matrix and \( \alpha_0, \beta_0 \) are the weak quadratic inner-boundedness constants.
According to Theorem 2, solving inequality (33), we can obtain the following solution:

\[ \varepsilon_3 = 1.5437, \quad \varepsilon_4 = 2.5683, \]

and

\[ W' = (-3.3508, 2, 0). \]

Then, the gain matrix of the controller is given by

\[ K = (-3.3508, 2, 0). \]

Therefore, the zero solution of the closed-loop system is asymptotically stable under controller law \( u(k) = -K\hat{x}(k) \). However, when using the weak one-sided Lipschitz condition (5), the weak one-sided Lipschitz constant is given by \( \nu_0 = \lambda_{\text{max}}(M_0) = 0.25T \), we cannot find a feasible solution \( \varepsilon_3, \varepsilon_4, W' \) such that \( \Lambda < 0 \) holds. This shows that the weak quasi-one-sided Lipschitz condition (4) includes more useful information of the non-linear part than the Lipschitz condition (25) and the weak one-sided Lipschitz condition (5).

Let the initial condition \( \mathbf{x}(0) = (-4, -5.03, 1.04)^T \) and \( \hat{\mathbf{x}}(0) = (5.32, 3.73, -0.5)^T \), then the simulation results are shown in Figures 1–4. Figures 1–3 show the responses of the actual states \( x_i(k) \) and estimated states \( \hat{x}_i(k) \), where \( i = 1, 2, 3 \). The response of observer-based state feedback is demonstrated in Figure 4. Figures 1–4 show that the zero solution of discrete-time closed-loop system is asymptotically stable as expected.

**Example 6.2.** Consider the following continuous-time non-linear system (69) motivated from the dynamics of a moving object in [31], where

\[ A_1 = \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad C_1 = \begin{pmatrix} 0 & 1 \end{pmatrix}. \]
the non-linear function \( \phi_1(x) \) is given by

\[
\phi_1(x) = \left(\begin{array}{c}
-x_1^2 + x_2^2 \\
-x_2^2 + x_1^2
\end{array}\right).
\]

By applying the Euler discretisation on system (69) with sampling time \( T = 0.1 \) s, a discrete-time system is derived as follows:

\[
\begin{align*}
\dot{x}(k+1) &= Ax(k) + Bu(k) + \phi(x(k)), \\
y(k) &= Cx(k),
\end{align*}
\]

where \( A = I_2 + TA_1, B = TB_1, C = C_1 \) and \( \phi(x(k)) = T \phi_1(x) \).

Firstly, we design a state observer for estimating the states of the system. Let \( D = \{x = (x_1, x_2)^T \in \mathbb{R}^2 : \|x\| \leq r\} \) as verified in [31], the nonlinear function \( \phi(x(k)) \) is not globally Lipschitz, but locally Lipschitz with the Lipschitz constant \( 37r^2 \) in any set \( D \).

By calculation, we know that \( \phi(x(k)) \) satisfies globally quasi-one-sided Lipschitz condition with quasi-one-sided Lipschitz constant matrix \( M = 0 \). Let

\[
r = \min \left\{ \sqrt{-\alpha/4T}, \sqrt{\alpha/4T + \beta^2/4T^2} \right\}, \alpha \leq 0, \alpha + \beta^2/4 \geq 0,
\]

then \( \phi(x(k)) \) is quadratically inner-bounded with constants \( \alpha \) and \( \beta \) in \( D \) [31]. By choosing appropriate values for \( \alpha \) and \( \beta \), the region \( D \) can be made arbitrary large. Let \( \alpha = -100 \) and \( \beta = -50 \); we can get \( r = 11.1803 \).

Based on the quasi-one-sided Lipschitz condition and quadratic inner-boundedness condition, by means of Theorem 1, solving linear matrix inequality (11), we can obtain observer gain matrix:

\[
L = \begin{pmatrix} 0.0729 \\ -0.7021 \end{pmatrix}.
\]

However, for \( r = 11.1803 \), the Lipschitz constant is \( 37r^2 = 37.5 \). It is almost impossible to design a state observer with the Lipschitz condition for such a large Lipschitz constant. The observer design approach in [19] is also invalid. Therefore, the quasi-one-sided Lipschitz condition and quadratic inner-boundedness condition are much more appropriate for designing a state observer than the Lipschitz condition.

Similar to the calculation of quasi-one-sided Lipschitz constant matrix \( M \) and quadratic inner-boundedness constants \( \alpha, \beta \), when designing a controller for stabilisation of the system, we can choose weak quasi-one-sided Lipschitz constant matrix \( M_0 = 0 \) and weak quadratic inner-boundedness constants \( \alpha_0 = 1, \beta_0 = -50 \).

According to Theorem 2, solving matrix inequality (33), the controller gain matrix can be obtained:

\[
K = (6.1924, 1).
\]

Therefore, the zero solution of the closed-loop system is asymptotically stable under controller law \( u(k) = -Kx(k) \). Since the Lipschitz constant is too large, the Lipschitz condition is also not suitable for designing the controller of system. The simulation results are shown in Figures 5 and 6 with the initial values \( x(0) = (3.01, -1.23)^T \) and \( \hat{x}(0) = (-2.32, 1.47)^T \), where solid lines represent the actual state \( x(k) \) and dotted lines indicate estimated state \( \hat{x}(k) \). Figures 5 and 6 show that the zero solution of discrete-time closed-loop system is asymptotically stable as expected.

7 | CONCLUSION

In this work, we design an observer for estimating the states of the discrete-time nonlinear systems based on the quasi-one-sided Lipschitz condition and quadratic inner-boundedness condition. The quasi-one-sided Lipschitz condition is less con-
servative than the one-sided Lipschitz condition and Lipschitz condition. Moreover, we propose a state feedback controller to stabilise the discrete-time nonlinear systems based on the weak quasi-one-sided Lipschitz condition and weak quadratic inner-boundedness condition. Finally, it is shown that the separation principle holds for stabilisation of system. We emphasize the fact that the sufficient conditions of observer design and controller design remain valid in this paper even though the matrix pair \((A, C)\) is not detectable and the matrix pair \((A, B)\) is not stabilisable. The simulation results indicate the feasibility and superiority of the proposed methods. The main results in the paper extend those in the literature.

An interesting direction for further research is that whether the obtained results in this paper can be extended to the case of discrete-time nonlinear system with time-delay.

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