SYMPLECTIC LEAVES FOR RATIONAL LOOPS WITH YANGIAN R-MATRIX

ALEXANDER SHAPIRO

Abstract. We consider a variety $\mathcal{M}$ of monic matrix polynomials endowed with the Yangian $r$-matrix Poisson structure, and the Poisson-Lie group $\mathcal{G}$ of equivalence classes of these matrix polynomials. We give an explicit description of symplectic leaves on $\mathcal{M}$ and $\mathcal{G}$, together with coordinate “charts” on the leaves and Poisson transition functions between the charts. We also suggest Poisson-Lie pro-groups that serve as classical analogues of Yangians $Y(\mathfrak{gl}_m)$ and $Y(\mathfrak{sl}_m)$, and describe their symplectic leaves as the leaves of a certain distribution.

Introduction

One of the central problems in the theory of integrable systems is the description of the geometry of their phase spaces. A broad and well-studied class consists of the systems whose phase space is realized as a symplectic leaf of a Poisson-Lie group. The case of finite-dimensional complex Poisson-Lie groups with the standard Poisson-Lie structure is well understood. The symplectic leaves, corresponding integrable systems, and Poisson transformations leading to certain discrete integrable systems are described in [HKKR, KZ] and references therein. In [W] similar results were obtained for the symmetrizable Kac-Moody groups with the standard Poisson structure, the one defined by the trigonometric $r$-matrix. It is also known that the above mentioned Poisson transformations are cluster mutations, realized as compositions of transition maps between charts parameterized by double reduced words on a double Bruhat cell, see [GSTV, GK, FM].

In the present article we characterize symplectic leaves on the ind-variety $\mathcal{M}$ (see section 1.2) of monic matrix polynomials endowed with the rational $r$-matrix Poisson structure. The reason to study this object is two-fold. First, the classical $GL_m$ magnetic chain and its degeneration, the Gaudin model, live on generic symplectic leaves of the Poisson manifold under consideration [S, G]. Second, if we replace the variety $\mathcal{M}$ of monic matrix polynomials by the pro-variety $\tilde{\mathcal{M}}$ of monic matrix formal power series, we get a Poisson-Lie group that should be treated as a classical analogue of Yangian $Y(\mathfrak{g}_m)$. Indeed, the Yangian algebra $Y(\mathfrak{g}_m)$ is the quantization of the algebra of regular functions on $\mathcal{M}$, see [KWWY]. Thus, the description of symplectic leaves on $\mathcal{M}$ and $\tilde{\mathcal{M}}$ is important for both classical integrable systems and quantum integrable systems with the Yangian symmetry.
In Theorem 1 we provide an explicit description of the symplectic leaves on \( \mathcal{M} \), namely, the leaves are the connected components of the manifolds consisting of monic matrix polynomials of fixed degree and with a given Smith normal form. In particular, all the leaves on \( \mathcal{M} \) are finite-dimensional, unlike the leaves on \( \tilde{\mathcal{M}} \). We describe the symplectic leaves on \( \tilde{\mathcal{M}} \) as the leaves of a certain distribution, which (modulo minor technicalities) coincides with the tangent bundle to the dressing orbit on \( \tilde{\mathcal{M}} \). Unfortunately, at the moment we do not have a description of finite-dimensional leaves on \( \tilde{\mathcal{M}} \) in terms of Smith forms, however a trivial corollary from the treatment of the variety \( \mathcal{M} \) provides us with the family of finite-dimensional leaves on \( \tilde{\mathcal{M}} \) of the form \( f(z) \mathcal{S} \) where \( f(z) \) is some formal power series, and \( \mathcal{S} \) is a leaf on \( \mathcal{M} \). It looks plausible that the above family exhausts all the finite-dimensional symplectic leaves on \( \tilde{\mathcal{M}} \). Let \( \mathcal{G} \) and \( \tilde{\mathcal{G}} \) be the equivalence classes of elements of \( \mathcal{M} \) and \( \tilde{\mathcal{M}} \) modulo multiplication by respectively monic polynomials or monic power series. Then, \( \tilde{\mathcal{G}} \) serves as a classical analogue of Yangian \( Y(\mathfrak{sl}_m) \), and we obtain the classification of symplectic leaves on \( \mathcal{G} \) and a family of finite-dimensional symplectic leaves on \( \tilde{\mathcal{G}} \) as straightforward corollaries. Note, that the group \( \tilde{\mathcal{G}} \) can be realized as the subgroup of rational loops \( P(z) \in PGL_m(\mathbb{C}(z)) \) satisfying \( \lim_{z \to \infty} P(z) = 1 \). This justifies the title of the present paper.

Next, we focus on the polynomial case. It follows from Theorem 1 that the determinant of a matrix polynomial is constant on a symplectic leaf. Therefore, each Poisson subspace \( \mathcal{M}_n \subset \mathcal{M} \) of monic matrix polynomials of degree \( n \) is fibred over the space \( \mathbb{C}^{mn} \) of roots of the determinant with the fibers being unions of symplectic leaves. In the case when all the roots are distinct, the fiber consists of just one generic leaf. If the determinant has roots of multiplicity greater than 1, the fiber consists of a unique regular leaf of maximal dimension and a union of smaller leaves. Similar to the case of Poisson-Lie groups with standard structure, we show in Theorem 2 that a generic leaf is covered by a union of charts, isomorphic to the product of generic (co-)adjoint orbits in \( \mathfrak{gl}_m \), with Poisson birational transition functions. Explicit formulas for the transition functions are obtained in Propositions 3.10 and 3.11. Similar statements hold for regular leaves, see Theorem 3 although only an open proper dense subset of a regular leaf is covered by charts.

The main results can be summarized as follows:

- Symplectic leaves on the group of rational loops \( P(z) \in PGL_m(\mathbb{C}(z)) \) satisfying \( P(\infty) = 1 \) are the connected components of the variety of monic matrix polynomials with a given Smith Normal Form (modulo multiplication by an element of \( \mathbb{C}(z) \)).
- A generic symplectic leaf is covered by a finite union of charts with Poisson transition maps.
The outline of the paper is as follows. In Section 1 we provide some standard results about Poisson-Lie groups, define the ind- and pro-varieties and Poisson-Lie groups that we discuss in the paper, and establish some notations. Section 2 contains the classification of symplectic leaves on the above mentioned varieties. Finally, in Section 3 we discuss the charts and transition maps on symplectic leaves of maximal dimension on $\mathcal{M}_n$.

Acknowledgements

I would like to express deep gratitude to my advisor, Nicolai Reshetikhin, for suggesting the topic of the present publication and for providing generous support and advice throughout. I would also like to thank Gus Schrader and Piotr Achinger for many valuable discussions and comments. This research was supported by the NSF grant DGE-1106400 and by RFBR grant 14-01-00547.

1. Poisson-Lie structure

In this section we recall some basic facts from the theory of Poisson-Lie groups and their tangent Lie bialgebras. Then we define the Poisson-Lie ind-variety $\mathcal{M}$, the ind-group $\mathcal{G}$, and the pro-groups $\tilde{\mathcal{M}}$ and $\tilde{\mathcal{G}}$, the main subjects of the paper.

1.1. Finite-dimensional theory. Let us recall that a Poisson manifold is a manifold $M$ endowed with a bilinear antisymmetric operation on functions $\{,\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$, called the Poisson bracket, satisfying the Jacobi identity and the Leibnitz rule. For any function $f \in C^\infty(M)$, the map $C^\infty(M) \to C^\infty(M)$, $g \mapsto \{f, g\}$ is a derivation, and so defines a vector field $\xi_f$ by the formula $\langle \xi_f, dg \rangle = \{f, g\}$. Such vector fields are called Hamiltonian. In particular, we see that $\{f, g\}$ depends only on $df \wedge dg$ and there exists a bivector field $\Pi \in \Gamma(\Lambda^2 TM)$ uniquely defined by $\{f, g\} = df \otimes dg(\Pi)$.

A symplectic leaf on a Poisson manifold is an equivalence class of points, joined by a piecewise smooth Hamiltonian integral curve. Each symplectic leaf is an immersed Poisson submanifold bearing a symplectic structure, and any Poisson manifold is a disjoint union of its symplectic leaves.

A Poisson-Lie group is a Lie group $G$ equipped with a Poisson structure such that the product map $m : G \times G \to G$ is a map of Poisson manifolds. It is easy to show that a bivector $\Pi \in \Gamma(\Lambda^2 TG)$ defines a Poisson structure on a Lie group $G$ if and only if

$$\Pi(xy) = (d_x(\rho_y) \otimes d_x(\rho_y))\Pi(x) + (d_y(\lambda_x) \otimes d_y(\lambda_x))\Pi(y) \quad (1.1)$$

where $\rho_y : G \to G$, $g \mapsto gy$ and $\lambda_x : G \to G$, $g \mapsto xg$ are respectively the right and left multiplications in $G$. Let $H$ be a closed Lie subgroup of $G$. It follows from (1.1) that the map $\pi : G \to G/H$ defines a Poisson structure on $G/H$ if and only if $(d\pi|_h \otimes d\pi|_h)\Pi(h) = 0$. In particular, if $\Pi(h) = 0$ for all $h \in H$, then every point $h \in H$ is a symplectic leaf, and the symplectic leaves on $G/H$ are the images of symplectic leaves on $G$ under the projection $\pi$. 
Let $G$ be a Poisson-Lie group with a bivector $\Pi$ and a tangent Lie algebra $\mathfrak{g} = T_e G$. With the use of right translations on the tangent bundle $TG$, the bivector $\Pi \in \Gamma(\Lambda^2 TG)$ defines a map $\tilde{\Pi} : G \to \Lambda^2 \mathfrak{g}$ with a derivative $\delta = d_e \tilde{\Pi} : \mathfrak{g} \to \Lambda^2 \mathfrak{g}$. This yields the following definition. A Lie bialgebra is a Lie algebra $\mathfrak{g}$ equipped with a cobracket $\delta : \mathfrak{g} \to \Lambda^2 \mathfrak{g}$ such that $\delta_* : \Lambda^2 \mathfrak{g}^* \to \mathfrak{g}^*$ defines a Lie bracket on $\mathfrak{g}^*$ and the cocycle condition

$$\delta([a, b]) = (\text{ad}_a \otimes 1 + 1 \otimes \text{ad}_a) \delta(b) - (\text{ad}_b \otimes 1 + 1 \otimes \text{ad}_b) \delta(a)$$

is satisfied. A classical theorem, due to Drinfeld [D], asserts that the functor $G \to \text{Lie}(G)$ between the category of connected, simply connected Poisson-Lie groups and the category of finite-dimensional Lie bialgebras is an equivalence of categories.

Denote by $\sigma : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$, $\sigma(a \otimes b) = b \otimes a$ the permutation of tensor factors in $\mathfrak{g}^\otimes 2$. For $r = \sum_i a_i \otimes b_i \in \mathfrak{g} \otimes \mathfrak{g}$, define elements $r_{12}, r_{13}, r_{23} \in \mathfrak{g}^\otimes 3$ as follows, $r_{12} = r \otimes 1$, $r_{13} = (1 \otimes \sigma) r_{12}$, $r_{23} = 1 \otimes r$. We call an element $r \in \mathfrak{g} \otimes \mathfrak{g}$ an $r$-matrix if it satisfies the classical Yang-Baxter equation

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0.$$ 

Let $r \in \mathfrak{g} \otimes \mathfrak{g}$ be an $r$-matrix whose symmetric part $r + \sigma(r)$ is invariant under the adjoint action of $\mathfrak{g}$. Then, the Lie bialgebra $\mathfrak{g}$ with a cobracket defined by $\delta(a) = [1 \otimes a + a \otimes 1, r]$ is called quasitriangular. Now, consider a Lie group $G$ whose Lie algebra $\mathfrak{g}$ carries the structure of a quasitriangular Lie bialgebra with the $r$-matrix $r$. Then, the bivector

$$\Pi(x) = (d_e \lambda_x \otimes d_e \lambda_x) r - r(d_e \rho_x \otimes d_e \rho_x)$$

defines a Poisson-Lie structure on $G$. We refer the reader to [CP, ES] for more details.

1.2. Lie structure. The main objects of the present paper are the semigroups of matrix polynomials and matrix formal power series. These semigroups are infinite-dimensional, so in order to define a Poisson-Lie structure on them we must first specify the smooth structure. Two possible ways to do it are to use the Fréchet manifolds as in [PS, KW] or the theory of ind- and pro-groups, see [K]. Here we take the second approach. Recall that an ind-variety $X$ is the union of an increasing sequence of finite-dimensional varieties $X_n$ whose inclusions $X_n \hookrightarrow X_{n+1}$ are closed embeddings. Somewhat analogously, a pro-group is an algebraic group $G$ and a family $F$ (satisfying certain properties) of its normal subgroups, such that $G$ is isomorphic to an inverse limit $\lim_{\leftarrow H \in F} G/H$. On the other hand, all objects we consider in the present paper happen to be direct or inverse limits of smooth complex manifolds. This allows us, first, to avoid the general theory of ind- and pro-groups, and second, to immediately adapt many results from the finite-dimensional case to our situation without change. Thus, we refer the reader to [K] for details on ind- and pro-groups.
Consider the space $\text{Mat}_m(\mathbb{C}[z^{-1}])$ of matrices over the ring of $\mathbb{C}$-valued polynomials in $z^{-1}$. Elements of this space can be also looked at as matrix-valued polynomials

$$P(z) = \sum_{k=0}^{n} P_k z^{-k}$$

where $P_k \in \text{Mat}_m(\mathbb{C})$.

**Definition 1.1.** We call a (matrix) polynomial or formal power series in $z^{-1}$ monic if its constant term is the identity matrix $1 \in \text{Mat}_m(\mathbb{C})$.

Denote by $\mathcal{M} \subset \text{Mat}_m(\mathbb{C}[z^{-1}])$ the semigroup of monic matrix polynomials, and let

$$\mathcal{M} \leq n = \{1 + P_1 z^{-1} + \cdots + P_n z^{-n}\}$$

be the subset of monic matrix polynomials in $z^{-1}$ of degree less than or equal to $n$. As a set $\mathcal{M} = \lim \rightarrow \mathcal{M} \leq n$ under the natural inclusions

$$\iota_n: \mathcal{M} \leq n \hookrightarrow \mathcal{M} \leq (n+1), \quad \iota_n(P(z)) = P(z). \quad (1.3)$$

At the same time, all $\mathcal{M} \leq n$ are smooth complex manifolds, so we endow $\mathcal{M}$ with the direct limit topology, which turns $\mathcal{M}$ into an ind-variety.

Now, let us consider the group $\tilde{\mathcal{M}} \subset \text{Mat}_m(\mathbb{C}[z^{-1}])$ of formal monic power series, and let $\tilde{\mathcal{M}} > n$ be a normal subgroup consisting of elements

$$\{1 + P_{n+1} z^{-n-1} + P_{n+2} z^{-n-2} + \cdots\}.$$

Then $\tilde{\mathcal{M}}/\tilde{\mathcal{M}} > n \simeq \mathcal{M} \leq n$ as smooth manifolds, and $\tilde{\mathcal{M}} = \lim \rightarrow \mathcal{M} \leq n$ as a set under the natural projections

$$\pi_n: \mathcal{M} \leq (n+1) \to \mathcal{M} \leq n, \quad \pi_n \left( \sum_{k=0}^{n+1} P_k z^{-k} \right) = \sum_{k=0}^{n} P_k z^{-k}. \quad (1.4)$$

We endow $\tilde{\mathcal{M}}$ with the inverse limit topology, which defines a structure of a pro-group on $\mathcal{M}$.

One might expect that $\mathcal{M}$ is a subvariety in $\tilde{\mathcal{M}}$, however this is not exactly true. The reason is that the topologies of the direct and inverse limit are different. For example, the sequence $\{(1 + z^{-n})1\}$ converges to $1$ in $\tilde{\mathcal{M}}$ but diverges in $\mathcal{M}$. On the other hand, let us consider a system $\mathcal{A}_n$, $n = 1, 2, \ldots$ that is direct with respect to maps $i_n: \mathcal{A}_n \to \mathcal{A}_{n+1}$ and inverse with respect to maps $p_n: \mathcal{A}_{n+1} \to \mathcal{A}_n$ satisfying $\pi_n \circ i_n = \text{id}_{\mathcal{A}_n}$. Then, as follows from universal properties of direct and inverse limits, there exists a unique morphism $\lim \rightarrow \mathcal{A}_n \to \lim \leftarrow \mathcal{A}_n$. Applying this categorical consideration to our case, we get the system $\mathcal{M} \leq n$ of topological spaces which is inverse and direct with respect to maps $\iota_n$ and $\pi_n$ respectively, satisfying $\pi_n \circ i_n = \text{id}_{\mathcal{M} \leq n}$. Therefore, the natural injection $\mathcal{M} \hookrightarrow \tilde{\mathcal{M}}$ is continuous, moreover the image of $\mathcal{M}$ is dense in $\tilde{\mathcal{M}}$. 
Definition 1.2. We call an element $P(z)$ of $\mathcal{M}$ or $\widetilde{\mathcal{M}}$ scalar if $P(z) = p(z)I$ where $p(z)$ is an element of $\mathbb{C}[z^{-1}]$ or $\mathbb{C}[z^{-1}]$ respectively.

Define an equivalence relation $\sim$ on $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ as follows: $P_1(z) \sim P_2(z)$ if there exists a scalar $p(z)$ such that $P_1(z) = p(z)P_2(z)$.

Proposition 1.3. Sets $\mathcal{G} = \mathcal{M}/\sim$ and $\widetilde{\mathcal{G}} = \widetilde{\mathcal{M}}/\sim$ of equivalence classes of elements in $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ have a Lie group structure.

Proof. The case of $\widetilde{\mathcal{G}}$ is trivial. Now, the set $\mathcal{G}$ carries a quotient Lie monoid structure. The inverse to the class $[P(z)]$ of matrix polynomials is naturally defined as the class of matrices adjugate to $P(z)$. The smoothness is straightforward. □

Occasionally, it will be more convenient to work with matrix polynomials in $z$ rather than in $z^{-1}$. For that purpose, let us consider a subset $\mathcal{P} \subset \text{Mat}_m(\mathbb{C}[z])$ of monic (in usual sense) matrix polynomials $P(z) = z^n + P_1z^{n-1} + \cdots + P_n$.

Let

$$\mathcal{P}_n = \{P_0z^n + P_1z^{n-1} + \cdots + P_n\}$$

be the subset of monic matrix polynomials of degree $n$. Then $\mathcal{P} = \varinjlim \mathcal{P}_n$ with the maps

$$j_n : \mathcal{P}_n \hookrightarrow \mathcal{P}_{n+1}, \quad j_n(P(z)) = zP(z),$$

and we endow $\mathcal{P}$ with the direct limit topology. The isomorphism between $\mathcal{M}$ and $\mathcal{P}$ is given by maps

$$\mathcal{M}_{\leq n} \rightarrow \mathcal{P}_n, \quad P(z) \mapsto z^n P(z). \quad (1.5)$$

1.3. Poisson structure.

Definition 1.4. A Manin triple is a triple of Lie algebras $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ where $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ is equipped with an invariant nondegenerate bilinear form $(\, , \, )$ such that

1. $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are isotropic subalgebras,
2. the form $(\, , \, )$ induces an isomorphism $\mathfrak{g}_- \simeq \mathfrak{g}_+^\star$.

The following proposition, due to Drinfeld, relates Manin triples and Lie bialgebras. The proof can be found, for example, in [ES].

Proposition 1.5. Let $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ be a (possibly infinite-dimensional) Manin triple, then the Lie bracket on $\mathfrak{g}_- \simeq \mathfrak{g}_+^\star$ defines a map $\delta : \mathfrak{g}_+ \rightarrow \Lambda^2 \mathfrak{g}_+$ that turns $\mathfrak{g}_+$ into a Lie bialgebra.

Consider an example of a Manin triple with $\mathfrak{g} = \mathfrak{g}(z)$, $\mathfrak{g}_+ = z^{-1} \mathfrak{g}[z^{-1}]$, $\mathfrak{g}_- = \mathfrak{g}[z]$, and the bilinear form given by

$$(a(z), b(z)) = \text{Res}_{z=0} \text{tr}(a(z)b(z)).$$
It defines the dual Yangian bialgebra structure on $\mathfrak{g}_+$ with the cobracket

$$\delta(a z^{-n}) = \sum_{r=1}^{\dim \mathfrak{g}} \sum_{i=1}^{\dim \mathfrak{g}} [x_i, a] z^{-r} \otimes x_i z^{r-n+1}$$

(1.6)

where $(x_i)$ is an orthonormal basis of $\mathfrak{g}$. Let us use two different variables $z$ and $w$ to distinguish between $\mathfrak{g}_+ = z^{-1} \mathfrak{g}[z^{-1}]$ and $\mathfrak{g}_- = \mathfrak{g}[w]$. Then $\mathfrak{g}_+$ is a quasitriangular Lie bialgebra with the $r$-matrix

$$r = \sum_{n \geq 0} \sum_{i=1}^{\dim \mathfrak{g}} x_i z^{-n-1} \otimes x_i w^n = \frac{\Omega}{z - w}$$

where $\Omega$ is the Casimir element in $\mathfrak{g}$ and $(z - w)^{-1}$ is expanded as the series $\sum_{n \geq 0} z^{-n-1} w^n$. Let $\pi_x : \mathfrak{g}[z, z^{-1}] \to \mathfrak{g}$, $\pi_x (a z^n) = a x^n$ be the evaluation map at $x \in \mathbb{C}^\times$. Then $r(z_1 - z_2) = (\pi_{z_1} \otimes \pi_{z_2})(r)$ satisfies the classical Yang-Baxter equation

$$[r_{12}(z_1 - z_2), r_{13}(z_1 - z_3)] + [r_{12}(z_1 - z_2), r_{23}(z_2 - z_3)] +
[r_{13}(z_1 - z_3), r_{23}(z_2 - z_3)] = 0.$$

The tangent Lie algebra of the Lie group $\mathcal{G}$ is isomorphic to $z^{-1} \mathfrak{sl}_n[z^{-1}]$, therefore the quasitriangular Lie bialgebra structure described above, defines a Poisson-Lie structure on $\mathcal{G}$. Here we run into two technical problems. First, we need to check that there is a bijection between continuous $n$-derivations on $\mathcal{G}$ and $n$-vector fields, which follows from the fact that $\mathcal{G}$ is a direct limit of smooth complex varieties. However, this holds for ind-groups in general, see [W]. Second, the above $r$-matrix is not an element of $\mathfrak{g}_+ \otimes \mathfrak{g}_+$ but rather of its completion. Nevertheless, it defines a Poisson bivector on $\mathcal{G}$ which follows from the explicit formula for the cobracket (1.6).

Now, the manifold $\mathcal{M}$ is only a Lie semigroup, so we can not talk about its Lie algebra. On the other hand, $T_0 \mathcal{M} \simeq z^{-1} \mathfrak{sl}_n[z^{-1}]$ as a vector space, which is still enough to define a Poisson-Lie semigroup structure on $\mathcal{M}$. Monic scalar polynomials form a Lie subgroup in $\mathcal{M}$ and the Poisson bivector is zero on this subgroup. Recall that the group $\mathcal{G}$ is a factor of $\mathcal{M}$ modulo this subgroup. This implies

**Proposition 1.6.** The Poisson-Lie structure on the group $\mathcal{G}$ coincides with the quotient Poisson-Lie structure on $\mathcal{M}/\sim$, and the symplectic leaves on $\mathcal{G}$ are the equivalence classes of symplectic leaves on $\mathcal{M}$.

Let $t_{ij}^{(k)} \in C^\infty(\mathcal{M})$ be a function that evaluates to the $(ij)$-entry of the $k$-th coefficient $P_k$ on a monic matrix polynomial $P(z)$. Consider a generating series of functions

$$T(u) = T^{(0)} + T^{(1)} u^{-1} + T^{(2)} u^{-2} + \ldots$$

where $T^{(k)} = (t_{ij}^{(k)})_{i,j=1}^m$. 
It is easy to check that the Poisson bracket on $\mathcal{M}$ can be written as follows

$$\{T(u) \otimes T(v)\} = \left[ \frac{\Omega}{u - v}, T(u) \otimes T(v) \right]$$

where $\Omega = \sum_{i,j=1}^{n} E_{ij} \otimes E_{ji}$ is the Casimir element in $\mathfrak{gl}_n$, see [CP] for details.

Equivalently,

$$\{t_{ij}(u), t_{kl}(v)\} = \frac{1}{u - v} (t_{kj}(u)t_{il}(v) - t_{kj}(v)t_{il}(u)),$$

where $t_{ij}(u) = t_{ij}^{(0)} + t_{ij}^{(1)} u^{-1} + t_{ij}^{(2)} u^{-2} + \ldots$, or even more explicitly,

$$\left\{ t_{ij}^{(r)}, t_{kl}^{(s)} \right\} = \sum_{q = \max(r,s)}^{r+s-1} t_{kj}^{(r+s-q-1)} t_{il}^{(q)} - t_{kj}^{(r)} t_{il}^{(s-q-1)}.$$

Let $\mathcal{M}_n \subset \mathcal{M}$ be a subvariety of matrix polynomials in $z^{-1}$ of degree exactly $n$. With the above Poisson structure $\mathcal{M}$ becomes a disjoint union of finite dimensional Poisson subvarieties $\mathcal{M}_n$. Now, the Manin triple with $\mathfrak{g} = \mathfrak{gl}_m[\langle z^{-1} \rangle]$, $\mathfrak{g}_+ = z^{-1} \mathfrak{gl}_m[z^{-1}]$, and $\mathfrak{g}_- = \mathfrak{gl}_m[z]$ defines a structure of a (topological) Lie bialgebra on $\mathfrak{g}_+$. It in turn induces a Poisson-Lie structure on $\tilde{\mathcal{M}}$ given by the same formula (1.7). The variety $\tilde{\mathcal{M}}$ also contains Poisson subvarieties $\mathcal{M}_n$ but is not exhausted by their union. Analogously to the case of $\tilde{\mathcal{G}}$ we have

**Proposition 1.7.** The Poisson-Lie structure on the group $\tilde{\mathcal{G}}$ coincides with the quotient Poisson-Lie structure on $\tilde{\mathcal{M}}/\sim$, and the symplectic leaves on $\tilde{\mathcal{G}}$ are the equivalence classes of symplectic leaves on $\tilde{\mathcal{M}}$.

Let us note that neither Poisson bracket nor the tangent space (and the Lie bialgebra structure on it) depend on the topologies of the Poisson-Lie groups $\tilde{\mathcal{G}}$ and $\tilde{\mathcal{G}}$. Therefore, although $\tilde{\mathcal{G}}$ is not a Poisson-Lie subgroup of $\tilde{\mathcal{G}}$, one can consider it as such.

**Remark 1.8.** The Poisson structure (1.7) suggests that the Poisson-Lie groups $\tilde{\mathcal{M}}$ and $\tilde{\mathcal{G}}$ should be treated as classical counterparts of Yangians $\mathcal{Y}(\mathfrak{gl}_m)$ and $\mathcal{Y}(\mathfrak{sl}_m)$ respectively.

### 2. Symplectic leaves

In this section we first describe all the symplectic leaves on Poisson manifolds $\mathcal{M}$ and $\mathcal{G}$, and then present a family of finite-dimensional symplectic leaves on $\tilde{\mathcal{M}}$ and $\tilde{\mathcal{G}}$.

#### 2.1. Polynomial case

Let $R$ be a principal ideal domain. Then, any matrix $M \in \operatorname{Mat}_m(R)$ can be decomposed into a product

$$M = APB$$
where $A, B \in GL_m(R)$ are invertible matrices, and $P$ is a diagonal matrix with entries $P_{i,i} = p_i$ such that $p_i$ divides $p_{i+1}$ for $i = 1, \ldots, m - 1$. Elements $p_i$ are unique up to multiplication by a unit in $R$. Moreover, the product $p_1 \ldots p_r$ equals the greatest common divisor of all $r \times r$ minors of the matrix $M$. The matrix $P$ is called the Smith normal form of $M$ (see, for example, [GLR]). In case $R = \mathbb{C}[z]$ elements $p_i$ are called the invariant polynomials of the matrix $M$. Now we can formulate the main result of this section.

**Theorem 1.** Symplectic leaves on the manifold $M_n$ are the connected components of the variety of monic matrix polynomials of degree $n$ with a given Smith normal form.

**Proof.** Consider a simultaneous action of two copies of $GL(\mathbb{C}[z^{-1}])$ on $M$ by left and right multiplication

$$GL(\mathbb{C}[z^{-1}]) \times M \times GL(\mathbb{C}[z^{-1}]) \to M, \quad (A, P, B) \mapsto APB^{-1}. \quad (2.1)$$

Orbits of this action are classified by Smith normal forms of elements of $M$. We will show that the tangent space to the intersection of $M$ with orbits of the action (2.1) coincides with the space generated by Hamiltonian vector fields. This implies that the symplectic leaves are the connected components of the set of monic matrix polynomials of degree $n$ with a given Smith normal form.

Let $T = \text{Mat}_m(\mathbb{C}((z^{-1})))$ be the space of matrix Laurent series in $z^{-1}$. Then

$$T = T^+ \oplus T^- \quad \text{where} \quad T^+ = \text{Mat}_m(\mathbb{C}[z]) \quad \text{and} \quad T^- = \text{Mat}_m(\mathbb{C}[z^{-1}]).$$

Let $T^+_n \subset T^+ \subset T^\pm$ be the subspaces of matrix polynomials of degree at most $n$ in $z$ or $z^{-1}$ respectively. Then $T^-_n$ is the tangent space to $M_{\leq n}$ at the identity matrix. Consider the symplectic leaf $S \subset M_n$ containing $P(z) \in M_n$. Denote by $A_\pm$ the projection of an element $A \in T$ onto $T_{\pm}$. Then one can prove by direct calculation the following

**Proposition 2.1.** The tangent space $T_PS$ to the symplectic leaf $S \subset M_n$ at the point $P(z)$ can be described as the space of all matrix polynomials $X \in T^-_n$ of the form

$$X = (PA)_+P - P(AP)_+ \quad (2.2)$$

with $A \in T^+_n$.

**Proof.** Let $\xi^{(r)}_{ij}$ be a Hamiltonian vector field corresponding to the function $t^{(r)}_{ij}$ and $A = \sum_{r=0}^{n-1} A_{-r}z^r$ be an element of $T^+_n$. Set

$$\xi_A = \sum_{r=1}^{n} A^r_{1,-r} \xi^{(r)} \quad \text{where} \quad \xi^{(r)} = \left(\xi^{(r)}_{ij}\right)_{i,j=1}^{n}$$

where $A = \sum_{r=0}^{n-1} A_{-r}z^r$ be an element of $T^+_n$. Set
and $A_{1-r}^t$ stays for the transpose of the matrix $A_{1-r}$. The vector fields $\xi_A$ for various $A \in T_{n-1}$ exhaust all Hamiltonian vector fields on $\mathcal{M}_n$. One can see that
\[
\xi_A(P(z)) = \sum_{s=1}^{n} \left( \sum_{r=1}^{\max(r,s)}^{r+s-1} [P_{s+r-1-q}, A_{1-r}, P_q] \right) z^{-s}, \tag{2.3}
\]
where $P_0 = 1$ and
\[
[P_{s+r-1-q}, A_{1-r}, P_q] = P_{s+r-1-q}A_{1-r}P_q - P_qA_{1-r}P_{s+r-1-q}.
\]
Now, a straightforward calculation shows that expressions (2.2) and (2.3) coincide after cancelling out terms with opposite signs in (2.2). □

Remark 2.2. In the above proposition one can consider all $A \in \text{Mat}_m(\mathbb{C}[z])$ since the righthand side of the formula (2.2) vanishes for $A \in z^n \text{Mat}_m(\mathbb{C}[z])$.

The following proposition is an infinitesimal version of a theorem proved for finite-dimensional Poisson-Lie groups in [LW]. Namely, let $G$ be a finite-dimensional Poisson-Lie group, $G^*$ its dual, and $\mathcal{D}$ its double (see [ES] or [CP] for definitions). Then, the symplectic leaves of $G$ are connected components of its intersections with the double cosets of $G^*$ in $\mathcal{D}$. In [W] this theorem was adapted to the case of Kac-Moody groups. Although, one can establish Proposition 2.3 following the proofs of [LW] and [W], here we take a different approach and prove it by direct computation.

Proposition 2.3. The tangent space $T_P S$ can be written as a space of all matrix polynomials $X \in T_{n-1}^-$ of the form
\[
X = BP - PC \tag{2.4}
\]
where $B, C \in T_{n-1}^+$.

Proof. One needs to show that under the condition $X \in T_{n-1}^-$ formulas (2.4) and (2.2) coincide. An arbitrary element $B \in T_{n-1}^+$ can be written in the form
\[
B(z) = \sum_{k=0}^{n-1} B_{-k} z^k = \sum_{k=0}^{n-1} \sum_{i=0}^{n-1-k} (P_i A_{-k-i}) z^k,
\]
where $A_i \in \text{Mat}_m(\mathbb{C})$ for $i \leq 0$ and $P_0 = 1$. Then, the relation (2.4) implies
\[
C(z) = \sum_{k=0}^{n-1} C_{-k} z^k = \sum_{k=0}^{n-1} \sum_{i=0}^{n-1-k} (A_{-k-i} P_i) z^k,
\]
and
\[
BP - PC = \sum_{s=1}^{n} \sum_{r=s}^{n} (B_{s-r} P_r - P_r C_{s-r}) z^{-s}.
\]
Plugging in expressions for $B_k$ and $C_k$ and omitting zero summands one gets
\[
BP - PC = \sum_{s=1}^{n} \left( \sum_{r=s}^{n} \sum_{b=0}^{n-s-r} (P_b A_{s-r-b} P_r - P_r A_{s-r-b} P_b) \right) z^{-s} = \sum_{s=1}^{n} \left( \sum_{r=s}^{n} \sum_{b=0}^{s-1} (P_b A_{s-r-b} P_r - P_r A_{s-r-b} P_b) \right) z^{-s}.
\]

On the other hand, formula (2.2) can be written as
\[
\sum_{s=1}^{n} \left( \sum_{r=1}^{\min(r+s-1,n)} \sum_{q=\max(r,s)}^{\min(r+s-1,n)} (P_{s+r-1-q} A_{1-r} P_q - P_q A_{1-r} P_{s+r-1-q}) \right) z^{-s}.
\]

Now it is only left to show that
\[
\sum_{a=s}^{n} \sum_{b=0}^{s-1} (P_b A_{s-r-b} P_r - P_r A_{s-r-b} P_b) = \sum_{r=1}^{n} \sum_{q=\max(r,s)}^{\min(r+s-1,n)} (P_{r+s-1-q} A_{n-r} P_q - P_q A_{n-r} P_{r+s-1-q}).
\]

The latter equality can be easily verified if one sets $b = r + s - 1 - q$. \hfill \Box

Let $E_{ij}$, $i \neq j$ be the matrix whose $(ij)$-entry is 1 and all other entries are 0. Then,
\[
G_{ij} = \{ 1 + E_{ij}p(z) \mid p(z) \in \mathbb{C}[z^{-1}] \}
\]
is a commutative Lie subgroup of $GL(\mathbb{C}[z^{-1}])$ with the Lie algebra
\[
g_{ij} = \{ E_{ij}p(z) \mid p(z) \in \mathbb{C}[z^{-1}] \}
\]
and a well-defined exponential map $\exp : g_{ij} \to G_{ij}$. Consider differentials of the action (2.1) restricted to the subgroups $GL_n(\mathbb{C}) \times \mathbb{C}^2$ and $G_{ij} \times \mathbb{C}^2$, $i \neq j$, of $GL_n(\mathbb{C}[z^{-1}]) \times \mathbb{C}^2$. The intersection of $T_n^*$ with the images of these differentials evaluated at point $P(z)$ is of the form (2.4). At the same time, sub-

Now, it is only left to check that the infinitesimal action by the fields of the form (2.4) preserves the Smith Normal Form, if matrices $B$ and $C$ are diagonal. Indeed, let $B, C \in T_{n-1}^+$ be a pair of diagonal matrices with entries $B_{ii} = b_i(z)$ and $C_{ii} = c_i(z)$ respectively. Denote by $d_r(P)$ the greatest common divisor of all $r \times r$ minors of the matrix $P$. Then, it is easy to see that
\[
d_r(P + \varepsilon (BP - PC)) = d_r(P)(1 + \varepsilon K) + O(\varepsilon^2)
\]
where $K$ in turn is the greatest common divisor of the expressions of the form $\sum_{s=1}^{r} (b_s(z) - c_j(z))$. Therefore, the infinitesimal action under discussion do
not decrease, and thus leaves invariant, the powers of invariant polynomials of the Smith Normal Form of $P$. This finishes the proof.  

**Corollary 2.4.** Symplectic leaves on $\mathcal{G}$ are the connected components of the variety that consist of equivalence classes of monic matrix polynomials with a given equivalence class of Smith normal forms.

Theorem 1 and Corollary 2.4 allow us to give the following

**Definition 2.5.** The Smith normal form of a symplectic leaf $\mathcal{S}$ is the Smith normal form of any polynomial $P(z) \in \mathcal{S}$.

2.2. **Case of formal power series.** The following proposition is proved exactly as Proposition 2.1.

**Proposition 2.6.** Symplectic leaves on $\tilde{\mathcal{M}}_n$ are the leaves of the distribution described at point $P(z)$ as the space of all matrix polynomials $X \in \mathcal{T}^-$ of the form

$$X = (PA)_+ P - P(AP)_+$$

with $A \in \mathcal{T}^+$.  

As a consequence of the fact that the Poisson bivector vanishes on scalar power series, we derive

**Proposition 2.7.** The subvarieties of the form $p(z)S \subset \tilde{\mathcal{M}}$, where $p(z)$ is a scalar monic power series in $z^{-1}$ and $S$ is a symplectic leaf on $\mathcal{M}$ or $\mathcal{G}$, deliver a family of the finite-dimensional leaves on $\tilde{\mathcal{M}}$ or $\tilde{\mathcal{G}}$ respectively.

We do not know if the above proposition gives a description of all the finite-dimensional symplectic leaves on $\tilde{\mathcal{M}}$ and $\tilde{\mathcal{G}}$, although this sounds plausible.

3. **Factorization on symplectic leaves**

In this section we prove that a generic symplectic leaf is covered by open “charts” with the following properties: any element from a chart admits a factorization into linear factors, the transition functions between charts are birational and Poisson. Then we prove analogues of these statements for all symplectic leaves of the maximal dimension on $\mathcal{P}_n$.

3.1. **Generic leaves.** In what follows we replace $\mathcal{M}$ by the manifold $\mathcal{P}$ of monic matrix polynomials in $z$. Recall that $\mathcal{P} \simeq \mathcal{M}$ via maps (1.5).  

Theorem 1 and Corollary 2.4 hold without change.

Consider a monic matrix polynomial

$$P(z) = z^n + P_1 z^{n-1} + \cdots + P_n, \quad P_i \in \text{Mat}_m(\mathbb{C}).$$

**Definition 3.1.** The $mn$ roots of polynomial $\det P(z)$ are the eigenvalues of $P(z)$. For any eigenvalue $\lambda$, the matrix $P(\lambda)$ has a nonzero kernel and elements of this kernel are called the eigenvectors of $P(z)$ corresponding to the eigenvalue $\lambda$. 
A matrix polynomial \( z - A, A \in \text{Mat}_m(\mathbb{C}) \) is said to be a right divisor of \( P(z) \) if \( P(z) = Q(z)(z - A) \) for some polynomial \( Q(z) \in \mathcal{P} \) of degree \( n - 1 \). The following lemma is trivial.

**Lemma 3.2.** A matrix polynomial \( P(z) = z^n + P_1 z^{n-1} + \ldots + P_n \) has a right divisor \( z - A \) if and only if
\[
A^n + P_1 A^{n-1} + \ldots + P_n = 0.
\]

**Corollary 3.3.** Let \( \lambda_1, \ldots, \lambda_m \) be the eigenvalues of \( P(z) \) and \( v_1, \ldots, v_m \) be the corresponding eigenvectors. Assume that \( v_1, \ldots, v_m \) are linearly independent, and take \( A \in \text{Mat}_m(\mathbb{C}) \) such that \( A v_i = \lambda_i v_i \) for \( i = 1, \ldots, m \). Then \( z - A \) is a right divisor of \( P(z) \).

**Lemma 3.4.** Assume that a monic matrix polynomial \( P(z) \) admits a decomposition into linear factors
\[
P(z) = (z - A_1) \ldots (z - A_n).
\]
If in addition the spectra of the matrices \( A_i \) are disjoint, then the decomposition above is uniquely defined by the order of factors and the spectra of matrices \( A_i, i = 1, \ldots, n \).

Lemma 3.2 and Lemma 3.4 are taken from [GLR], one can find a proof of Corollary 3.3 in [B]. Now, let us endow the algebra of functions on the Lie algebra \( \mathfrak{gl}_m \) with the Kirillov-Kostant Poisson bracket. Consider the product Poisson structure on \( \mathfrak{gl}_m \times \mathbb{C}^n \). Proof of the following Lemma is given in [FT].

**Lemma 3.5.** The following map is Poisson
\[
\mathfrak{gl}_m \times \mathbb{C}^n \to \mathcal{P}_n, \quad (A_1, \ldots, A_n) \mapsto (z - A_1) \ldots (z - A_n).
\]
Eigenvalues of matrices \( A_i \) form the Poisson center of \( C^\infty(\mathfrak{gl}_m \times \mathbb{C}^n) \).

Therefore, eigenvalues of matrix polynomials \( P(z) \) are constant on symplectic leaves. This allows us to give the following

**Definition 3.6.** The spectrum \( \text{Sp}(S) \) and the determinant \( \det(S) \) of the symplectic leaf \( S \) are respectively the set of eigenvalues or the determinant of any monic matrix polynomial \( P(z) \in S \).

**Definition 3.7.** A symplectic leaf on \( \mathcal{M} \) or \( \mathcal{G} \) is called generic if all its eigenvalues are distinct.

Let \( P(z) \) be an element of a generic symplectic leaf \( S \). By definition, all invariant polynomials of the Smith normal form of \( P(z) \) have to be constant, except for the last one which equals the determinant of \( P(z) \). The following lemma proves that a generic symplectic leaf is defined by its determinant.

**Lemma 3.8.** Let \( S \) be a generic symplectic leaf with spectrum \( \text{Sp}(S) \), then there are no other symplectic leaves with the same spectrum.
Proof. Let $n$ be the degree of matrix polynomials in $\mathcal{S}$. By Theorem 1, it suffices to show that the set of monic matrix polynomial of degree $n$ and with spectrum $\text{Sp}(\mathcal{S})$ is connected. Let

$$
\Lambda = (\Lambda_1, \ldots, \Lambda_n), \quad \Lambda_i = \{\lambda_{i,1}, \ldots, \lambda_{i,m}\}, \quad \lambda_{i,j} \in \text{Sp}(\mathcal{S})
$$

be an ordered partition of the spectrum of $\mathcal{S}$, here $i = 1, \ldots, n$, $j = 1, \ldots, m$. Then there exists an open dense subset $\mathcal{S}_\Lambda$ of monic matrix polynomials that admit decomposition

$$P(z) = (z - A_1) \ldots (z - A_n)$$

with $\text{Sp}(A_i) = \Lambda_i$, $i = 1, \ldots, n$. Indeed, $\mathcal{S}_\Lambda$ is defined by the conditions that vectors $v_1, \ldots, v_m$ from Corollary 3.3 are linearly independent for each $\Lambda_i$, and therefore $\mathcal{S}_\Lambda$ is open and dense. Now it is only left to show that $\mathcal{S}_\Lambda$ is connected. Consider a pair of polynomials $P(z)$ and $Q(z)$ in $\mathcal{S}_\Lambda$. Then, $P(z)$ and $Q(z)$ admit decompositions

$$P(z) = (z - A_1) \ldots (z - A_n), \quad Q(z) = (z - B_1) \ldots (z - B_n),$$

with $\text{Sp}(A_i) = \text{Sp}(B_i)$ for all $i = 1, \ldots, n$. For any $i = 1, \ldots, n$ there exists a matrix $C_i \in GL_m(\mathbb{C})$ satisfying $B_i = C_i A_i C_i^{-1}$. Since $GL_m(\mathbb{C})$ is connected, there exists a path $C_i(t) \in GL_m(\mathbb{C}) \times [0, 1]$ such that $C_i(0)$ is the identity matrix and $C_i(1) = C_i$. Define

$$P_t(z) = (z - C_1(t) A_1 C_1(t)^{-1}) \ldots (z - C_n(t) A_n C_n(t)^{-1}),$$

then $P_0(z) = P(z)$ and $P_1(z) = Q(z)$. Since conjugation does not change the spectra of $A_i$ we have $\text{Sp}(P_t(z)) = \text{Sp}(P(z))$. Therefore, the path $P_t(z) \in \mathcal{S}_\Lambda \times [0, 1]$, which proves that $\mathcal{S}_\Lambda$ is connected. \qed

Now we show, that a generic symplectic leaf is covered by open subsets $\mathcal{S}_\Lambda$ defined in the previous lemma.

Lemma 3.9. Any matrix polynomial $P(z)$ in a generic symplectic leaf admits a decomposition into linear factors.

Proof. Indeed, $P(z)$ has a linear right divisor unless all the eigenvectors of $P(z)$ lie in some $d$-dimensional subspace of $\mathbb{C}^m$, $d < m$. For $x \in \mathbb{C}$, let $Q(x)$ be the restriction of $P(x)$ onto this subspace. This defines a monic matrix polynomial $Q(z)$ of degree at most $n$ and coefficients in $\text{Mat}_d(\mathbb{C})$. Its determinant $\det(Q(z))$ is a polynomial of degree at most $dn$. On the other hand, any of $mn$ distinct roots of $\det(P(z))$ is a root of $\det(Q(z))$, and we arrive at a contradiction. The proof is then finished by induction on the degree of $P(z)$. \qed

The main result of this section is

Theorem 2. Let $\mathcal{S}$ be a generic symplectic leaf on $\mathcal{M}$ or $\mathcal{G}$. Then

1) $\mathcal{S}$ is defined by its determinant, a polynomial $a(z) \in \mathbb{C}[z]$ of degree $d$. The dimension of $\mathcal{S}$ is $d(m - 1).$
2) For any ordered partition
\[ \Lambda = (\Lambda_1, \ldots, \Lambda_n), \quad \Lambda_i = \{\lambda_{i,1}, \ldots, \lambda_{i,m}\}, \quad \lambda_{i,j} \in \text{Sp}(\mathcal{S}) \]
of the spectrum of \( \mathcal{S} \), with \( i = 1, \ldots, n \) and \( j = 1, \ldots, m \), there exists an open subset \( \mathcal{S}_\Lambda \subset \mathcal{S} \) and a Poisson bijection
\[ \phi_\Lambda : \mathcal{S}_\Lambda \to \mathcal{O}_{\Lambda_1} \times \cdots \times \mathcal{O}_{\Lambda_n}, \]
where \( \mathcal{O}_{\Lambda_i} \) is the adjoint orbit on \( \mathfrak{gl}_m \) with the fixed spectrum \( \Lambda_i \). Map \( \phi_\Lambda \) is inverse to the product map given by the formula \( (3.7) \).
The subset \( \mathcal{S}_\Lambda \subset \mathcal{S} \) is described by conditions that vectors \( v_1, \ldots, v_m \) from Corollary \( 3.3 \) are linearly independent for each \( \Lambda_i, i = 1, \ldots, n \). The symplectic leaf \( \mathcal{S} \) is covered by the union of open subsets \( \mathcal{S}_\Lambda \).

3) For any pair \( \Lambda \) and \( \Lambda \) of ordered partitions of \( \text{Sp}(\mathcal{S}) \) there exists a birational Poisson map
\[ \tau_{\Lambda \Lambda} : \phi_\Lambda (\mathcal{S}_\Lambda \cap \mathcal{S}_\Lambda) \to \phi_\Lambda (\mathcal{S}_\Lambda \cap \mathcal{S}_\Lambda), \quad \tau_{\Lambda \Lambda} = \phi_\Lambda \circ (\phi_\Lambda)^{-1}. \]

**Proof.** Indeed, part 1) of the theorem follows from Theorem \( 1 \) Corollary \( 1 \) and Lemma \( 3.8 \) Parts 2) and 3) follow from Corollary \( 3.3 \) Lemmas \( 3.4, 3.5, 3.9 \) and the following consideration. If a Poisson map is bijective, then its inverse is also Poisson. Therefore, maps \( \phi_\Lambda, \phi_\Lambda, \) and \( \tau_{\Lambda \Lambda} \) are well defined and Poisson. 

Propositions \( 3.10 \) and \( 3.11 \) below provide a more explicit description of the map \( \tau_{\Lambda \Lambda} \) from Theorem \( 2 \). Assume that a matrix polynomial \( P(z) \) of degree 2 admits decompositions \( P(z) = (z - A)(z - B) \) and \( P(z) = (z - \tilde{A})(z - \tilde{B}) \), where
\[ \text{Sp}(A) = \{\lambda_1, \lambda_2, \ldots, \lambda_m\}, \quad \text{Sp}(B) = \{\mu_1, \mu_2, \ldots, \mu_m\}, \quad (3.2) \]
\[ \text{Sp}(\tilde{A}) = \{\mu_1, \lambda_2, \ldots, \lambda_m\}, \quad \text{Sp}(\tilde{B}) = \{\lambda_1, \mu_2, \ldots, \mu_m\}, \quad (3.3) \]
and all \( \lambda_i, \mu_j, i, j = 1, \ldots, m \) are distinct. In the rest of this section we set \( \lambda = \lambda_1 \) and \( \mu = \mu_1 \) for brevity.

**Proposition 3.10.** Consider vectors \( u, v \in \mathbb{C}^m \) such that \( Au = \lambda u \) and \( u^t B = \mu u^t \). Then their scalar product \( (u, v) \) is nonzero.

**Proof.** Let \( u_i \) be the eigenvectors of \( B \) with eigenvalues \( \mu_i \) respectively. Note that \( u \) is not an eigenvector of \( B \) but of \( B^t \). For \( i = 2, \ldots, m \) vector \( u \) is orthogonal to \( u_i \). Indeed,
\[ \mu(u, u_i) = \mu u^t u_i = u^t B u_i = \mu_i u^t u_i = \mu_i (u, u_i) \]
which yields \( (u, u_i) = 0 \). Assume that \( (u, v) = 0 \), then \( v \) is a linear combination of vectors \( u_i, i = 2, \ldots, m \).

Consider a matrix polynomial \( P(z) = (z - A)(z - B) \), then \( P(\lambda) \) has an eigenvector \( w = (\lambda - B)^{-1} v \) which is also a linear combination of vectors \( u_i, i = 2, \ldots, m \). Now, if \( P(z) = (z - \tilde{A})(z - \tilde{B}) \), the matrix \( \tilde{B} \) has eigenvectors \( w \) and \( u_i, i = 2, \ldots, m \). Then the restriction of \( \tilde{B} \) onto the \((m - 1)\)-dimensional
subspace generated by \( u_i, i = 2, \ldots, m \), has \( m \) distinct eigenvalues. Thus, we arrive at a contradiction and \((u, v) \neq 0\).

**Proposition 3.11.** One has
\[
\tilde{A} = A + (\mu - \lambda)T, \quad \tilde{B} = B + (\lambda - \mu)T
\]  
where \( T \) is a projector onto \( v \) along \( u^t \).

**Proof.** The projector \( T \) can be written as
\[
T = \frac{vu^t}{(u, v)}
\]  
and is well defined due to Proposition 3.10. Consider vectors \( v_i, u_i \in \mathbb{C}^m \) such that \( Av_i = \lambda_i v_i, \ u^t_i B = \mu_i u_i \) for \( i = 2, \ldots, m \). Note that unlike in Proposition 3.10, vectors \( u_i \) are not eigenvectors of \( B \) but of \( B^t \), as well as vector \( u \). A straightforward check shows that vectors \( \tilde{v}_i = v_i + \frac{\mu - \lambda}{\lambda_i - \mu} (u, v_i) \) and \( \tilde{u}_i = u_i + \frac{\lambda - \mu}{\mu_i - \lambda} (u, v_i) \) satisfy \( \tilde{A} \tilde{v}_i = \lambda_i \tilde{v}_i, \ \tilde{u}^t_i \tilde{B} = \mu_i \tilde{u}_i \) for \( i = 2, \ldots, m \). One also has \( \tilde{A}v = \mu v \) and \( u^t \tilde{B} = \lambda u^t \). Thus, spectra of \( \tilde{A} \) and \( \tilde{B} \) are as in (3.3).

To prove that \((z - A)(z - B) = (z - \tilde{A})(z - \tilde{B})\) one needs to check that \( A + B = \tilde{A} + \tilde{B} \) and \( AB = \tilde{A} = \tilde{B} \). The first equality is obvious from (3.4). Using that \( T^2 = T \) since \( T \) is a projector, and that \( AT = \lambda T \) and \( TB = \mu T \), which follows from formula (3.5), one has
\[
\tilde{A} \tilde{B} = AB + (\lambda - \mu)\lambda T + (\mu - \lambda)\mu T - (\lambda - \mu)^2 T^2 = AB.
\]  
This finishes the proof.

3.2. **Regular leaves.**

**Definition 3.12.** A symplectic leaf \( S \) is called regular if the only nontrivial invariant polynomials of its Smith normal form is the determinant \( \det(S) \). Equivalently, a symplectic leaf \( S \) is regular if
\[
\dim(\ker P(\lambda)) \leq 1 \quad \forall P(z) \in S \quad \text{and} \quad \forall \lambda \in \mathbb{C}. \tag{3.6}
\]

**Remark 3.13.** The regular leaves on \( \mathcal{P}_n \) are precisely the leaves of maximal dimension. Indeed, any nontrivial invariant polynomial of the Smith normal form of a symplectic leaf imply conditions that necessarily decrease the dimension of the leaf.

**Definition 3.14.** Let \( P(z) \in \text{Mat}_m(\mathbb{C}) \) be a monic matrix polynomial. A Jordan chain of length \( k + 1 \) for \( P(z) \) corresponding to the eigenvalue \( \lambda \) is the sequence of \( m \)-dimensional vectors \( v_0, v_1, \ldots, v_k \) satisfying
\[
\sum_{i=0}^r \frac{1}{r!} P^{(i)}(\lambda) v_{r-i} = 0 \quad \text{for} \quad r = 0, \ldots, k.
\]

The following proposition is an immediate corollary of Lemma 3.2.
Proposition 3.15. Let $\lambda_1, \ldots, \lambda_s$ be the distinct eigenvalues of $P(z)$, for each $i = 1, \ldots, s$ let $v_{i,j}$, $j = 0, \ldots, k_i$ be a Jordan chain corresponding to the eigenvalue $\lambda_i$, such that the lengths of these Jordan chains add up to $m$. Assume that all the vectors $v_{i,j}$ are linearly independent and consider a matrix $A \in \text{Mat}_m(\mathbb{C})$ satisfying $Av_{i,j} = \lambda_i v_{i,j} + v_{i,j-1}$ where $v_{i,-1} = 0$, $i = 1, \ldots, s$, $j = 0, \ldots, k_i$. Then $z - A$ is a right divisor of $P(z)$. Conversely, if $z - A$ is a right divisor of $P(z)$ then the vectors $v_{i,j}$, $j = 0, \ldots, k_i$, $i = 1, \ldots, s$ form Jordan chains of $P(z)$ corresponding to eigenvalues $\lambda_1, \ldots, \lambda_s$.

Proposition 3.16. Let $A(z), B(z), D(z) \in \text{Mat}_m(\mathbb{C}[z])$ be matrix polynomials (not necessarily monic), such that $A(z)$ and $B(z)$ are nonsingular for some value $\lambda \in \mathbb{C}$. Then $v_0, \ldots, v_k$ is a Jordan chain of the polynomial $A(z)B(z)$ corresponding to $\lambda$ if and only if the vectors

$$w_j = \sum_{i=0}^{j} \frac{1}{i!} B^{(i)}(\lambda)v_{j-i}, \quad j = 0, \ldots, k$$

form a Jordan chain of $D(z)$ corresponding to $\lambda$.

See [GLR] for the proof.

Lemma 3.17. Let $S$ be a regular symplectic leaf. Consider a matrix polynomial $P(z) \in S$ that admits a decomposition into linear factors

$$P(z) = (z - A_1) \ldots (z - A_n).$$

This decomposition is uniquely defined by the order of factors and the spectra of matrices $A_i$, $i = 1, \ldots, n$.

Proof. It is enough to show that the right divisor is defined by its spectrum. The latter follows from Proposition 3.16. Indeed, let $D(z)$ be the Smith normal form of the regular leaf $S$. Then, $D(z)$ is a diagonal matrix with 1 on diagonal everywhere except for the last entry. Therefore, for any eigenvalue $\lambda$ of $D(z)$, there exists a unique Jordan chain

$$w_0 = w_1 = \cdots = w_k = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \in \mathbb{C}^m$$

and $k+1$ is the multiplicity of $\lambda$. On the other hand, by Theorem 1 every element of $S$ is of the form $P(z) = A(z)D(z)B(z)$ where $A(z), B(z) \in GL_m(\mathbb{C})$. Therefore, for any eigenvalue $\lambda \in \text{Sp}(S)$ there exists a unique Jordan chain of $P(z)$ corresponding to $\lambda$. Now the lemma follows from Proposition 3.15.\qed

It follows from Theorem 1 that if the leaf $S$ is not generic, then there are many leaves with the same spectrum $\text{Sp}(S)$. On the other hand, the following lemma shows that there is still only one leaf of maximal dimension for any spectrum.
Lemma 3.18. Let $S$ be a regular symplectic leaf with spectrum $\text{Sp}(S)$, then there are no other regular symplectic leaves with the same spectrum.

Proof. The proof is very similar to the one of Lemma 3.8. Again, it suffices to show that the set of monic matrix polynomial of degree $n$ with the Smith normal form of $S$ is connected. Let

$$\Lambda = (\Lambda_1, \ldots, \Lambda_n), \quad \Lambda_i = \{\lambda_{i,1}, \ldots, \lambda_{i,m_i}\}, \quad \lambda_{i,j} \in \text{Sp}(S)$$

be an ordered partition of the spectrum of $S$, where $i = 1, \ldots, n$ and $j = 1, \ldots, m_i$. Then there exists an open dense subset $S_\Lambda$ of monic matrix polynomials that admit decomposition

$$P(z) = (z - A_1) \ldots (z - A_n)$$

with $\text{Sp}(A_i) = \Lambda_i$, $i = 1, \ldots, n$. Indeed, $S_\Lambda$ is defined by conditions that the vectors $v_{i,j}$ from Proposition 3.15 are linearly independent for each $\Lambda_i$, therefore $S_\Lambda$ is open and dense. Now it is only left to show that $S_\Lambda$ is connected. This is done exactly as in Lemma 3.8 but with one slight modification. The path $P_t(z)$ might not lie in the symplectic leaf $S$ anymore. However, the path lies in the union of symplectic leaves with spectrum $\text{Sp}(S)$. On the other hand, $S$ is the only leaf of maximal dimension in this union, therefore, all other leaves form an obstacle of (complex) codimension 2 (we note that even a codimension 1 would suffice, since the leaves are complex). This implies that there exists a path completely in the leaf of maximal dimension, hence $S$ is connected. □

Remark 3.19. It follows from the above lemma that the subset of monic matrix polynomials that admit a decomposition into linear factors is dense and open in a regular symplectic leaf. However, in any non-generic symplectic leaf $S$ there are matrix polynomials $P(z)$ that do not admit factorization into linear factors, for example

$$P(z) = \begin{pmatrix} (z - \lambda)^2 & 1 \\ 0 & (z - \lambda)^2 \end{pmatrix}.$$ 

For a classification of matrix polynomials that do admit decomposition into linear factors see [GLR]. Therefore, a non-generic symplectic leaf $S$ is not covered by open subsets $S_\Lambda$.

Now, we can state an analogue of the theorem 2 for regular leaves.

Theorem 3. Let $S$ be a regular symplectic leaf on $M$ or $G$. Then

1) $S$ is defined by its determinant, a polynomial $a(z) \in \mathbb{C}[z]$ of degree $d$. The dimension of $S$ is $d(m - 1)$.

2) For any ordered partition

$$\Lambda = (\Lambda_1, \ldots, \Lambda_n), \quad \Lambda_i = \{\lambda_{i,1}, \ldots, \lambda_{i,m_i}\}, \quad \lambda_{i,j} \in \text{Sp}(S)$$

of the spectrum of $S$, with $i = 1, \ldots, n$ and $j = 1, \ldots, m_i$, there exists an open subset $S_\Lambda \subset S$ and an open Poisson embedding

$$\phi_\Lambda : S_\Lambda \rightarrow O_{\Lambda_1} \times \cdots \times O_{\Lambda_n}.$$
where $O_{\Lambda_i}$ is the adjoint orbit on $\mathfrak{gl}_m$ with the fixed spectrum $\Lambda_i$ and all Jordan blocks of maximal dimension. The map $\phi_\Lambda$ is inverse to the product map given by the formula (3.7). The subset $S_{\Lambda \subset S}$ is described by conditions that vectors $v_{i,j}$ from Proposition 3.15 are linearly independent for each $\Lambda_i$, $i = 1, \ldots, n$. Its image under $\phi_\Lambda$ is described by conditions (3.6) that read as

$$\ker((\lambda - A_1) \ldots (\lambda - A_i)) \cap \text{Im}((\lambda - A_{i+1}) \ldots (\lambda - A_n)) = \emptyset$$

for all $\lambda \in \text{Sp}(S)$ and $i = 1, \ldots, n - 1$.

3) For any pair $\Lambda$ and $M$ of ordered partitions of $\text{Sp}(S)$ there exists a birational Poisson map

$$\tau_{\Lambda M}: \phi_M(S_{\Lambda \cap M}) \to \phi_\Lambda(S_{\Lambda \cap M}), \quad \tau_{\Lambda M} = \phi_\Lambda \circ (\phi_M)^{-1}.$$

We end this section with a remark that Theorem 3 takes care of all symplectic leaves on $\text{Mat}_2(\mathbb{C})/\sim$. Indeed, in the case $m = 2$ all the symplectic leaves are regular.

**Conclusion**

Although, we give an explicit description of symplectic leaves of maximal dimension on the variety $\mathcal{M}$ of monic matrix polynomials with the rational $r$-matrix Poisson bracket, we feel that the following questions still await careful treatment. First, formulas for dimensions of (non-regular) symplectic leaves, and the number of (non-regular) symplectic leaves with the same Smith normal form are lacking, as well as the classification of finite-dimensional symplectic leaves on $\mathcal{M}$. Next, we find it interesting to understand the meaning of the transition functions, written in Proposition 3.11, for the Gaudin model. We also look forward to obtain quantum analogues of these transition functions and relate them to the representation theory of Yangians. Third, the provided families of symplectic leaves on $\mathcal{M}$ and $\tilde{\mathcal{G}}$ make it possible to ask for an analogue of the geometric quantization for the Yangians $Y(\mathfrak{gl}_m)$ and $Y(\mathfrak{sl}_m)$. Finally, we believe that the results of this paper should be generalized from $\mathfrak{gl}_m$ to other classical Lie algebras. We return to these questions in the forthcoming publications.

**References**

[B] A. Borodin. *Isomonodromy Transformations of linear Systems of Difference Equations*. Ann. of Math. 160(3), 2004, 1141-1182.

[CP] V. Chari, A. Pressley. *A guide to quantum groups*. Cambridge University Press, Cambridge, 1994.

[D] V. Drinfeld. *Quantum groups*. Proceedings of ICM. 1, 1986, 798-820.

[ES] P. Etingof, O. Schiffman. *Lectures on Quantum Groups*. International Press, Cambridge, 1998.

[FT] L. Faddeev, L. Takhtajan. *Hamiltonian Methods in the Theory of Solitons*. Berlin, Heidelberg, New York: Springer, 1987.

[FM] V. Fock, A.Marshakov. *Loop groups, Clusters, Dimers and Integrable systems*. arXiv:1401.1606.
[G] M. Gekhtman. Separation of Variables in the Classical Integrable SL(N) Magnetic Chain. Commun. Math. Phys. 167, 1995, 593-605.

[GK] A. Goncharov, R. Kenyon. Dimers and cluster integrable systems. arXiv:1107.5588.

[GLR] I. Gohberg, P. Lancaster, L. Rodman. Matrix Polynomials. New York: Academic Press, 1982.

[GSTV] M. Gekhtman, M. Shapiro, S. Tabachnikov, A. Vainshtein. Higher pentagram maps, weighted directed networks, and cluster dynamics. Electron. Res. Announc. Math. Sci. 19, 2012, 117.

[HKKR] T. Hoffmann, J. Kellendonk, N. Kutz, N. Reshetikhin. Factorization dynamics and Coxeter-Toda lattices. Comm. Math. Phys. 212(2), 2000, 297-321.

[K] S. Kumar. Kac-Moody Groups, Their Flag Varieties, and Representation Theory. Progr. Math., 204, Birkhauser, Boston, MA, 2002.

[KZ] M. Kogan, A. Zelevinsky. On symplectic leaves and integrable systems in standard complex semisimple Poisson-Lie groups. Intern. Math. Res. Notices, 2002, 1685-1702.

[KW] B. Khesin, R. Wendt. The Geometry of Infinite-Dimensional Groups. Springer-Verlag, New York, 2009.

[KWWY] J. Kamnitzer, B. Webster, A. Weekes, O. Yacobi. Yangians and quantizations of slices in the affine Grassmannian. arXiv:1209.0349.

[LW] J.-H. Lu, A. Weinstein. Poisson Lie groups, dressing transformations, and Bruhat decompositions. J. Differential Geom. 31(2), 1990, 501-526.

[PS] A. Pressley, G. Segal. Loop Groups. Oxford University Press, Oxford, 1986.

[S] E. Sklyanin. Separation of Variables in the Classical Integrable SL(3) Magnetic Chain. Commun. Math. Phys. 150, 1992, 181-191.

[W] H. Williams. Double Bruhat Cells in Kac-Moody Groups and Integrable Systems. Lett. Math. Phys. 103, 2013, 389-419.

ALEXANDER SHAPIRO

UNIVERSITY OF CALIFORNIA, BERKELEY,
DEPARTMENT OF MATHEMATICS,
BERKELEY, CA, 94720, USA;

INSTITUTE OF THEORETICAL & EXPERIMENTAL PHYSICS,
117259, MOSCOW, RUSSIA;

E-mail address: shapiro@math.berkeley.edu