Fragility of a class of highly entangled states of many quantum-bits

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We consider a Quantum Computer with \( n \) quantum-bits (‘qubits’), where each qubit is coupled independently to an environment affecting the state in a dephasing or depolarizing way. For mixed states we suggest a quantification for the property of showing quantum uncertainty on the macroscopic level. We illustrate in which sense a large parameter can be seen as an indicator for large entanglement and give hypersurfaces enclosing the set of separable states. Using methods of the classical theory of maximum likelihood estimation we prove that this parameter is decreasing with \( 1/\sqrt{n} \) for all those states which have been exposed to the environment.

Furthermore we consider a Quantum Computer with perfect 1-qubit gates and 2-qubit gates with depolarizing error and show that any state which can be obtained from a separable initial state lies inbetween a family of pairs of certain hypersurfaces parallel to those enclosing the separable ones.

I. INTRODUCTION

The sensitivity of quantum systems to interactions with the environment is one of the challenging problems for the realization of Quantum Computers \(^1\). But apart from this motivation the effect of the environment to quantum states has been subject of pure research for many decades (see \(^2\) and references therein). After all the decoherence caused by the environment is commonly accepted to be the explanation for classical behavior of physical systems in everyday life \(^3\), i.e., on the macroscopic level of physics \(^4\). Although there is no precise definition of the word ‘macroscopic’, most of those explanations contain (explicitly or implicitly) the statement that the destruction of coherence takes place on a very small time scale particularly for superpositions of ‘macroscopically distinct’ states, i.e., States showing quantum uncertainty on the macroscopic level. Despite the fact, that this statement cannot be maintained without taking into account the way of coupling to the environment (see \(^5\)), it has served as an intuitive motivation for our investigation of the sensitivity of many particle quantum states with respect to a coupling to independent environments. For this we introduce a function \( e: \rho \mapsto e_\rho \) from the set of states to the positive numbers quantifying the property of showing quantum uncertainty on the macroscopic level and prove quantitative statements about the sensitivity of those states \( \rho \) having large values \( e_\rho \).

We show that large values for \( e_\rho \) require large-scale-entanglement in the sense that there cannot be small clusters of entangled qubits without entanglement between qubits in different clusters. Therefore our investigations should be considered in the context of the fragility of entanglement, which is an important subject since it is decisive for the computational power of decohered Quantum Computers \(^6\).

The connection between the sensitivity of a state with respect to disturbances of the environment and the property of being a superposition of macroscopically distinct states can be illustrated by the following straightforward example: Take an \( n \)-qubit Quantum Computer, that is a quantum system with the Hilbert space

\[
\mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2.
\]

Denote the canonical basis states by the binary words of length \( n \). Furthermore we use the following terminology: The Hamming weight of a binary word is the number of characters ‘1’. The Hamming distance of two words is the Hamming weight of their difference. Now we take a superposition of two arbitrary basis states

\[
|\psi\rangle := \frac{1}{\sqrt{2}} (|a\rangle + |b\rangle).
\]

If \( a \) and \( b \) differ at many positions, i.e., have a large Hamming distance, we call \( |a\rangle \) and \( |b\rangle \) macroscopically distinct states. Now we perform a measurement of \( |\psi\rangle \) in the canonical basis \( |0\rangle, |1\rangle \) of \( \mathbb{C}^2 \) on one randomly chosen qubit \( i \). Obviously this state collapses to \( |a\rangle \) or \( |b\rangle \) if and only if the words \( a \) and \( b \) differ at the position \( i \). Hence the superposition state is more fragile if \( a \) and \( b \) have a large Hamming distance. At the first sight it seems to be straightforward to characterize the fragility of a state (with respect to dephasing) by the probability that it is changed by a measurement of a randomly chosen qubit. But this probability is 1 in the generic case: Even the unentangled state

\[
|\pi\rangle := \left( \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \right)^{\otimes n}
\]
is changed by a measurement of any qubit\(^1\). Nevertheless we want to consider this state as much less fragile than the ‘cat-state’

\[
|\gamma\rangle = \frac{1}{\sqrt{2}}(|0\ldots 0| + |1\ldots 1|)
\]

since the ‘error’ caused by the single-qubit-measurement can be corrected by a single-qubit-operation in the case of the state $|\pi\rangle$ whereas the local disturbance of the cat state $|\gamma\rangle$ requires a much more complicated procedure restoring the entanglement. Therefore we consider fragility as a property of a class of states rather than of a single state. We prove a class of highly entangled states to be fragile in the sense that every mixed state obtained by small independent disturbances of each qubit lies outside this set.

### II. TWO SUFFICIENT CONDITIONS FOR ENTANGLEMENT IN MANY PARTICLE SYSTEMS

As usual we call a state of the Quantum Computer a product state if its density matrix is an \(n\)-fold tensor product of the form

\[
\rho_1 \otimes \rho_2 \otimes \ldots \otimes \rho_n
\]

where each $\rho_i$ is the density matrix of the qubit $i$. A separable state is a convex combination of arbitrary many product states. For any operator $a$ acting on an Hilbert space we denote its operator norm by $\|a\|$. By an ‘1-qubit-operator at the qubit $i$’ (or ‘acting on the qubit $i$’) we mean an operator acting on the $n$-fold tensor product of $\mathbb{C}^2$ which is of the form

\[
1 \otimes 1 \otimes \ldots \otimes a \otimes \ldots 1 \otimes 1
\]

for any $a \neq 1$ acting on $\mathbb{C}^2$.

Furthermore we introduce the following type of observables: Let $(a_i)_{1 \leq i \leq n}$ with $\|a_i\| \leq 1$ be a family of selfadjoint operators where each $a_i$ is acting on the qubit $i$. Then we define the averaging observable

\[
\bar{a} := \frac{1}{n} \sum_i a_i.
\]

In the case that the difference of the lowest and the greatest eigenvalue is the same for every $a_i$ we call $\bar{a}$ an equally weighted average of 1-qubit observables. Despite the fact, that there is no precise distinction between the macroscopic and microscopic level in an $n$-qubit system, we have good reasons for considering the equally weighted averages as the ‘most macroscopic ones’. An easy example might illustrate this point of view: If the qubits are represented as spin-1/2-particles, the magnetization of the system in $z$-direction is given by

\[
\bar{a}_z := \frac{1}{n} \sum_i \sigma_z^i
\]

where $\sigma_z^i$ is the copy of the Pauli matrix $\sigma_z$ acting on the qubit $i$. If we would define

\[
a_i := \lambda_i \sigma_z^i
\]

with arbitrary $\lambda_i$ we get a less macroscopic observable in general since $\bar{a}$ is dominated by the spins of those $i$ with large $|\lambda_i|$.

In a product state there is no correlation between the values of two 1-qubit observables at different qubits. Hence we conclude the following from classical probability theory:

For any observable $a$ and any density matrix $\nu$ let $s_{a,\nu}$ be the standard deviation of $a$ in the state $\nu$, i.e.,

\[
s_{a,\nu} := \sqrt{\text{tr}(a^2\nu) - (\text{tr}(a\nu))^2}.
\]

Let $\rho$ be a product state. Then for the standard deviation of any averaging observable $\bar{a}$ in the state $\rho$ the inequality

\[
s_{\bar{a},\rho} \leq \frac{1}{\sqrt{n}}
\]

holds since the variance of a sum of independent random variables is the sum of their variances and the variance of any $a_i$ cannot exceed 1 due to its operator norm. We conclude, that every separable state has a decomposition into states fulfilling inequality (1) for every family $(a_i)$. If we want to use this result for showing that a given state is not separable one would have to check every possible decomposition into pure states. Hence one might doubt its practical importance. However, we can derive another sufficient condition for entanglement which does not require to check every decomposition: The question, as to which extent a state can be decomposed into those pure states with small standard deviations with respect to any given observable $a$, is closely related to the question as to which extent its density matrix (written in any basis diagonalizing $a$) is dominated by ‘strongly’ off-diagonal positions, i.e., those positions where row and column correspond to rather different eigenvalues of $a$. The convex function

\[1\] In [2] this state is taken as an example for the difficulty of maintaining coherence in large Quantum Computers.
\[
\rho \mapsto \sup_{\|b\| \leq 1} |\text{tr}(\rho[a, b])|
\]
can be considered as a measure for the ‘dominance of the
strongly off-diagonal’ terms since it vanishes for every \(\rho\)
commuting with \(a\).

To be precise, we have the following lemma:

**Lemma 1** Let \(A\) be an arbitrary (finite dimensional)
matrix algebra. Let \(b, c \in A\) with \(b\) selfadjoint and
\(\|c\| \leq 1\). Let \(\nu \in A\) be an arbitrary density matrix. Then
we have the following inequality:

\[
|\text{tr}(\nu[b, c])| \leq 2s_{b,\nu}
\]

**Proof:** Since the standard deviation is a concave
function on the set of probability measures on \(R\), we have
\(s_{b,\nu} \geq \sum_j \lambda_j s_{b,\nu_j}\) if \(\nu\) is the convex sum
\(\nu := \sum_j \nu_j\). Therefore we can assume \(\nu\) to be a density matrix of
a pure state, i.e. \(\nu = |\psi\rangle\langle\psi|\). Let us choose an eigenvector
basis of \(b\) and expand \(c\) with respect to this basis. Let \(\lambda_1, \ldots, \lambda_k\)
be the set of eigenvalues of \(b\) with their corresponding multiplicities
and \(\psi_j\) to be the coordinates of \(|\psi\rangle\). We have:

\[
|\langle\psi|b\psi\rangle| = |\sum_{ij} \psi_i (\lambda_i - \lambda_j) \pi_{ij} |\psi_j| |
\leq \sum_{ij} |\psi_i (\lambda_i - \lambda_j - \mu) \pi_{ij} |\psi_j| + \sum_{ij} |\psi_i \pi_{ij} (\lambda_j - \mu)| |\psi_j|,
\]
for any \(\mu \in R\).

Defining the vector \(\kappa\) with \(\kappa_i := (\lambda_i - \mu)\psi_i\), the
inequality (2) reads

\[
|\langle\psi|b\psi\rangle| \leq |\langle\kappa|c\psi\rangle| + |\langle\psi|c\kappa\rangle|,
\]
which gives us an upper bound by the Cauchy Schwartz
inequality

\[
|\langle\psi|b\psi\rangle| \leq 2\|\kappa\| \|c\psi\| \leq 2\|\kappa\|
\]

where the last estimation holds due to the operator
norm of \(c\). With the definition \(\mu := \langle\psi|b\psi\rangle\) the vector
norm \(\|\kappa\|\) is the standard deviation of the observable \(b\).

By the triangle inequality we conclude:

**Corollary 1** Let \(\rho\) be a density matrix of a finite
-dimensional quantum system. Let \(b\) be an arbitrary selfadjoint
operator and \(c\) an arbitrary operator with \(\|c\| \leq 1\). Assume \(\rho\) to have a decomposition into pure states of the
form

\[
\rho = \sum_j \lambda_j \rho_j,
\]
where \(\lambda_j > 0\) and \(\sum_j \lambda_j = 1\). The states \(\rho_j\) are
arbitrary pure states. Then we call \(s_{b,\rho_j}\) ‘the mean
standard deviation of the observable \(b\) in the state \(\rho\) with
respect to the decomposition (4)’ and have the following inequality:

\[
\frac{1}{2} |\text{tr}(\rho[b, c])| \leq \sigma
\]

Note that for any pair of selfadjoint operators \(b, c\) and
any state \(\nu\) the expectation value \(i \text{tr}[\nu[b, c]]\) is real. For
any such pair \(b, c\) we define the hypersurface

\[
H_{b,c,r} := \{ \rho \in S | i \text{tr}(\rho[b, c]) = r \}
\]
where \(S\) is the set of density matrices of the \(n\)-qubit
system. We conclude from Corollary 1:

**Corollary 2** (‘Hypersurface-Criterion’) Let \(\pi\) be an
averaging observable as in the beginning of this section and
c a selfadjoint operator with \(\|c\| \leq 1\). Then every
separable state \(\rho\) lies between the hypersurfaces

\[
H_{\pi,c,\pm r},
\]
i.e.,

\[
-\frac{2}{\sqrt{n}} \leq i \text{tr}(\rho[\pi, c]) \leq \frac{2}{\sqrt{n}}
\]

This gives us a sufficient condition for entanglement
which is easy to handle since it can be verified by finding
just one pair \(\pi, c\) such that inequality (3) is violated.

Despite the fact, that there is no commonly accepted
quantification of entanglement (see [11–13]), we will con-
sider a large value (compared to \(1/\sqrt{n}\)) of the term
\(\text{tr}(\rho[\pi, b])\) for any such family \((\pi_i)\) and any such \(b\)
as a sufficient condition for \(\rho\) to be ‘highly entangled’. The
term highly entangled might be interpreted in two differ-
ent ways: Firstly a large value shows that the state has
a great distance from the set of separable states in the
traces norm. Secondly it shows that there is entanglement
between many qubits:

**Lemma 2** Take a partition of the qubits \(\{1, \ldots, n\}\) into
subsets (‘clusters’) of size \(l_1, \ldots, l_k\) with the property that
the state \(\rho\) has no entanglement between qubits of differ-
ent clusters. Let \(\pi\) be an averaging observable and \(b\) be
a selfadjoint operator with \(\|b\| \leq 1\). Then the following
inequality holds:

\[
|\text{tr}(\rho[b, c])| \leq \frac{2}{n} \sqrt{\sum_{i \leq k} l_i^2}
\]

**Proof:** Denote the clusters by \(S_1, \ldots, S_k \subset \{1, \ldots, n\}\). It is sufficient to take a state which is fac-
toring with respect to this partition into clusters since
the set of states fulfilling inequality (3) is convex. For
such a ‘partial product state’ \(\rho\) the standard deviation of \(\pi\) is given by

\[
s_{\pi,\rho} = \frac{1}{n} \sqrt{\sum_{i \leq k} S_i}
\]
where \( s_i \) denotes the standard deviation of the observable \( \sum_{j \in S} a_j \) which is less or equal to \( l_i \). Lemma \( \Box \) completes the proof. \( \Box \)

In the following, we will restrict ourselves to the equally weighted averaging observables. Up to a constant factor and a constant summand they can be obtained by taking every \( a_i \) as a projection.

Therefore, we shall consider the parameter
\[
e_\rho := \sup_{\mathcal{Q},b} |\text{tr}(\rho(\mathcal{Q}, b))|
\]
where every \( \mathcal{Q} \) is the average \( \mathcal{Q} \) over the 1-qubit projections \( Q_k \) and \( \|b\| \leq 1 \), as a reasonable quantification for the property of showing quantum uncertainty on the macroscopic level.

**Example 1** In order to give more intuition about the states with large \( e_\rho \), we assume \( \rho \) to be the density matrix of a coherent superposition of two distinct basis states \(|f\rangle\) and \(|g\rangle\), i.e., \( \rho = \frac{1}{2}(|f\rangle + |g\rangle)(\langle f| + \langle g|) \). Let \( f \) and \( g \) be binary words with Hamming weights \( \text{wgt}(f) \) and \( \text{wgt}(g) \). Let \( P_k \) be the projection onto the state \(|1\rangle\) for the \( k \)-th qubit. With the definition
\[
b := i|f\rangle\langle g| - i|g\rangle\langle f|
\]
we get
\[
\text{tr}(\rho(P, b)) = \frac{i}{n} (\text{wgt}(f) - \text{wgt}(g)),
\]
and hence we have \( e_\rho \geq \frac{1}{n} |\text{wgt}(f) - \text{wgt}(g)| \).

We prove the following more general statement:

**Lemma 3** Let \( f_1, \ldots, f_j, g_1, \ldots, g_j \) be a set of \( 2j \) distinct binary words. Let \( \rho \) be the density matrix given by
\[
\rho := \sum_k \lambda_k |\psi_k\rangle\langle \psi_k|
\]
where \( \lambda_k \) is the probability of the pure superposition state
\[
|\psi_k\rangle := \frac{1}{\sqrt{2}} (|f_k\rangle + |g_k\rangle).
\]

Define \( b \) by
\[
b := \sum_k (i|f_k\rangle\langle g_k| - i|g_k\rangle\langle f_k|).
\]

Then we have the following equation:
\[
\text{tr}(\rho(P, b)) = \frac{i}{n} \sum_k \lambda_k (\text{wgt}(f_k) - \text{wgt}(g_k)).
\]

**Proof:** Using
\[
\mathcal{P}(f) = \frac{1}{n} \text{wgt}(f)|f\rangle
\]
for every binary word \( f \), the statement follows by easy calculations. \( \Box \)

Note that the operator \( b \) in the definition above fulfills the requirement \( \|b\| = 1 \) since the operators
\[
i|f_k\rangle\langle g_k| - i|g_k\rangle\langle f_k|
\]
have operator norm 1 and act on mutually orthogonal subspaces.

The states \(|\psi_k\rangle\) are superpositions of macroscopic distinct states if the difference \( \text{wgt}(f_k) - \text{wgt}(g_k) \) has the order of \( n \) rather than the order of 1. In this case we say the state shows quantum uncertainty on the macroscopic level. The sum \( \sum_k \lambda_k (\text{wgt}(f_k) - \text{wgt}(g_k)) \) measures to what extend the mixture \( \rho \) consists of pure states with a large uncertainty of the observable \( \mathcal{P} \). Note that the mixture of two states with large parameter \( e \) can have small \( e \) due to the fact that the mixture of two highly entangled states can be separable. Therefore the assumption that the \( 2j \) binary words \( f_1, \ldots, f_j, g_1, \ldots, g_j \) are mutually distinct is essential and turns up not to be just a technical requirement for the proof: Take the mixture given by
\[
\rho := \frac{1}{2} (|\psi_1\rangle\langle \psi_1| + |\psi_2\rangle\langle \psi_2|)
\]
with
\[
|\psi_{1/2}\rangle := \frac{1}{\sqrt{2}} (|0\ldots0\rangle \pm |1\ldots1\rangle).
\]

Easy calculation in the canonical basis shows that \( \text{tr}(\rho(P, b)) \) vanishes for every operator \( b \) since \( \rho \) and \( \mathcal{P} \) are diagonal in this basis. Actually, we can get this result using Lemma \( \Box \) as well: The state \( \rho \) has another decomposition into the pure states \( \rho_0 := |0\ldots0\rangle\langle 0\ldots0| \) and \( \rho_1 := |1\ldots1\rangle\langle 1\ldots1| \). Both states do not show any uncertainty with respect to the observable \( \mathcal{P} \), i.e., the standard deviations \( \mathcal{P}_{\rho_0} \) and \( \mathcal{P}_{\rho_1} \) vanish.

We can make general statements about the range of the convex function \( \rho \mapsto e_\rho \) on the set of density matrices:

**Lemma 4** For every \( n \) the range of the function \( \rho \mapsto e_\rho \) is the interval \([0, 1]\).

**Proof:** If \( \rho \) is the maximally mixed state, i.e., \( \rho \) is the identity matrix up to a constant factor, we have \( e_\rho = 0 \) since \( \text{tr}(\rho(a, b)) = \text{tr}(a, b) = 0 \) for every pair of operators \( a, b \). In general we have \( e_\rho \leq 1 \) due to Lemma \( \Box \) since the standard deviation of \( \mathcal{P} \) cannot exceed 1/2 due to the fact that its spectrum is contained in \([0, 1]\). For the cat state as defined in the introduction we can conclude \( e_\rho \geq 1 \) by setting \( f := 0 \ldots0 \) and \( g := 1 \ldots1 \) in Example \( \Box \). Hence for the cat state we have \( e_\rho = 1 \). We can obtain any value between 0 and 1 by a mixture of the cat state and the maximally mixed state with the corresponding weight. \( \Box \)
III. THE ERROR MODELS

In many cases it is well-justified from a physical point of view to assume errors acting independently on every qubit [10]. One kind of these 1-qubit-error which seems reasonable is a random dephasing with respect to the canonical basis. Describing this on the set of density matrices this error affects the state like a measurement environment by a map \( G \) on\(^2 \) the set of \( n \)-qubit density matrices as follows:

Let \( P_i \) be as in Example [10] and \( M_i \) the instrument performing a measurement of the qubit \( i \) in the canonical basis of \( \mathbb{C}^{2} \), i.e.,

\[
M_i(\rho) = P_i \rho P_i + (1 - P_i) \rho (1 - P_i)
\]

for every density matrix \( \rho \). Then our first error model will be an instrument \( G \) which acts on each qubit independently as \( wM_i + (1 - w)id \), i.e. \( G \) acts as:

\[
G := \prod_i (wM_i + (1 - w)id)
\]

where \( id \) denotes the identity map and \( w \) is the error probability.

Our second error model is given by depolarizing channels acting on each qubit independently: Let \( I \) be this map on the set of 1-qubit density matrices which maps every state to the maximally mixed one. Let \( I_i \) be the canonical extension of this map from the state space of the qubit \( i \) to the \( n \)-qubit-system, i.e.,

\[
I_i := id \otimes \ldots \otimes id \otimes I \otimes \ldots \otimes id.
\]

Define the instrument \( D \) by:

\[
D := \prod_i (wI_i + (1 - w)id).
\]

In the following chapter we shall study the images of the maps \( D \) and \( G \) according to the error probability \( w \) and the size \( n \).

IV. QUANTITATIVE STATEMENTS ABOUT FRAGILITY

In order to investigate the way in which the instrument \( G \) affects a state we introduce a family of instruments \( \{G_l\}_{1 \leq l \leq n} \) which is defined as follows: Let \( \mathcal{L}_l \) be the set of \( l \)-element subset of \( \{1, \ldots, n\} \). Set

\[
G_l := \frac{1}{\binom{n}{l}} \sum_{L \in \mathcal{L}_l} \prod_{i \in L} M_i.
\]

Then easy calculation shows that \( G \) can be written as the convex combination of all the \( G_l \) with binomial coefficients:

\[
G = \sum_{l=0}^{n} B_{nw}(l) G_l. \tag{8}
\]

where we use the abbreviation

\[
B_{nw}(l) := \binom{n}{l} w^l (1 - w)^{n-l}.
\]

The instrument \( G_l \) is a random machine performing a measurement on every qubit in a randomly selected \( l \)-element subset of qubits. It can map a pure state to a mixed one for two reasons: Firstly we do not know, which \( l \) qubits are measured, i.e., which \( L \) was selected, and secondly we do not know the measured result. If we knew both, we would get a certain pure state obtained from the original one by a partial collapse of the wavefunction. In the following we show, that most likely this collapse leads to a state in which the observable \( \mathcal{P} = \frac{1}{n} \sum P_i \) has small standard deviation provided that \( l \gg 1 \). Intuitively this is not astonishing since the measurement of \( l \) qubits allows a prediction of the values of the average observable \( \mathcal{P} \) with a high ‘confidence level’. This analogy to the theory of maximum likelihood estimation motivates the idea of the proof in the following quantitative analysis:

**Theorem 1** Let \( \rho \) be an arbitrary density matrix of an \( n \) qubit system. Let \( G_l \) and \( \mathcal{P} \) as described above. For \( b \) an arbitrary operator on \( (\mathbb{C}^2)^{\otimes n} \) with \( \|b\| \leq 1 \) we have

\[
|\text{tr}(G_l(\rho) [\mathcal{P}, b])| \leq \frac{1}{\sqrt{l}} \sqrt{\frac{n - l}{n - 1}}.
\]

Before we prove the theorem we draw some conclusions. In order to get statements about \( G_l(\rho) \) instead of \( G_l(\rho) \) we can use the decomposition \( [8] \). For this we have to consider the case \( l = 0 \) separately. The instrument \( G_0 \) is the identity map and occurs in the sum \( [8] \) with the weight \( (1 - w)^n \). The standard deviation of \( \mathcal{P} \) can never exceed 1/2. Therefore, \( |\text{tr}(G_0(\rho) [\mathcal{P}, b])| = |\text{tr}(\mathcal{P} [\mathcal{P}, b])| \leq 1 \) and we conclude:

2\( G \) is a completely positive trace preserving map (see [13]), since every manipulation of a quantum state can be described by a map of this type.
Corollary 3 Let \( G \) be as in equation \([8]\). Then we have for any arbitrary density matrix \( \rho \)
\[
|tr(G(\rho)(\mathcal{P}, b))| \leq \sum_{l=1}^{n} \frac{1}{\sqrt{n-l}} \sqrt{\frac{n-1}{n-1}} B_{\text{nn}}(l) \tag{9}
\]
\[+ (1-w)^n \]
\[=: r_{wn}.
\]

Proof (of the Theorem): Since the set of density matrices fulfilling the inequality is convex, we may restrict the proof to the case of \( \rho \) being a pure state, i.e. \( \rho = |\psi\rangle\langle\psi| \) with \( |\psi\rangle \in (\mathbb{C}^2)^n \). Let us assume a measurement which has been performed on the qubits in \( \mathcal{L} \), where \( \mathcal{L} \) is an arbitrary \( l \)-element subset of \( \{1, \ldots, n\} \). Let \( R_L \) be the map \( R_L : \{0, 1\}^n \to \{0, 1\}^l \) restricting a binary word to the set \( L \). Let \( P_{L,g} \) be the projector onto the linear span of those basis vectors given by the binary words in \( R_L^{-1}(g) \) for any \( g \in \{0, 1\}^l \).

Then \( |\psi\rangle\langle\psi| \) is transduced to the mixed state
\[
\sum_{g \in \{0,1\}^l} P_{L,g}|\psi\rangle\langle\psi|P_{L,g}.
\]

On the set \( \mathcal{L}_l \) we introduce the measure \( r_l \) as the equally distributed probability measure, i.e.
\[
\forall L \in \mathcal{L}_l : r_l(L) = \frac{1}{n \\ l}.
\]

Then \( G_l \) transduces \( |\psi\rangle\langle\psi| \) to the state
\[
G_l(|\psi\rangle\langle\psi|) = \sum_{L \in \mathcal{L}_l} r_l(L) \sum_{g \in \{0,1\}^l} P_{L,g}|\psi\rangle\langle\psi|P_{L,g}. \tag{10}
\]

With the definition \( |\psi_{L,g}\rangle := |P_{L,g}|\psi\rangle|P_{L,g}\psi\rangle \) and
\[
p(L,g) := r_l(L)\|P_{L,g}\psi\|^2. \tag{11}
\]

we obtain:
\[
G_l(|\psi\rangle\langle\psi|) = \sum_{L \in \mathcal{L}_l, g \in \{0,1\}^l} p(L,g)|\psi_{L,g}\rangle\langle\psi_{L,g}|.
\]

Note that \( p(L,g) \) is the probability for the event ‘a measurement has been performed on the qubits in \( L \) and the result \( (g_1, \ldots, g_l) \) (in an ascending order) has been obtained.’ We denote this event by \((L,g)\).

Let \( s_{L,g} \) be the standard deviation of \( \mathcal{P} \) in the state \( |\psi_{L,g}\rangle\langle\psi_{L,g}| \).

In order to prove the theorem it is sufficient (see Corollary \([8]\)) to show
\[
\sum_{L,g} p(L,g)s_{L,g} \leq \frac{1}{2\sqrt{l}} \sqrt{\frac{n-l}{n-1}}. \tag{12}
\]

For doing so we introduce the probability space \( \mathcal{L}_l \times \{0, 1\}^n \)

\[
\text{endowed with the product measure } r_l \otimes q_\psi, \text{ where } q_\psi \text{ assigns to every binary word the probability given by the square of the probability amplitudes of } \psi. \text{ All the random variables which will be introduced below are defined on this product space. In the formal framework of this space the formal correct notation of } (L,g) \text{ is } \{L \times R^{-1}_L(g)\}. \text{ We will keep the less formal notation } (L,g) \text{ for reasons of convenience. Now we introduce the random variable}
\]
\[
U : \mathcal{L}_l \times \{0, 1\}^n \to \mathbb{R}
\]

by
\[
U(L,g) := \frac{1}{n} \text{wgt}(b).
\]

Then we can write the standard deviation in inequality \([13]\) as
\[
s_{L,g} = \sqrt{E((U - E(U|L,g))^2|L,g)},
\]

where \( E(|L,g) \) denotes the expectation value of a random variable with the conditional probability measure given the event \((L,g)\). Furthermore we define the random variable \( S \) by
\[
S(L,b) := \frac{1}{l} \text{wgt}(R_L(b)).
\]

Due to the fact, that \( S \) has a constant value on every subset \((L,g)\), we can give the following upper bound:
\[
s_{L,g} \leq \sqrt{E((U - S)^2|L,g)},
\]

because for any arbitrary random variable \( X \) and \( \mu \in \mathbb{R} \) the expectation value \( E((X - \mu)^2) \) is minimized by \( \mu = E(X) \).

Since the square root function is concave we get
\[
\sum_{L,g} p(L,g)s_{L,g} \leq \sqrt{\sum_{L,g} p(L,g)E((U - S)^2|L,g)} \tag{13}
\]
\[
= \sqrt{\sum_{b \in \{0,1\}^n} p(b)E((U - S)^2|b)} \tag{14}
\]

The last equation holds since the family of sets
\[
(L,g)_{L \in \mathcal{L}_l, g \in \{0,1\}^l}
\]

as well as \( b \in \{0,1\}^n \) define different partitions of the probability space \( \mathcal{L}_l \times \{0, 1\}^n \). We conclude
\[
\sum_{L,g} p(L,g)s_{L,g} \leq \sup_{b \in \{0,1\}^n} \sqrt{E((U - S)^2|b)}. \tag{15}
\]

Note that for any fixed \( b \) the product \( SL \) is a random variable with a hypergeometric distribution, since it measures the hamming weight of the restriction of \( b \) to a
randomly chosen \( L \)-element subset of \( \{1, \ldots, n\} \). Furthermore \( U \) is a constant for any given \( b \) and \( U \) is the expectation value of the random variable \( S \) with respect to the conditional probability measure given the event \( b \). Therefore the square root in the right hand term is the standard deviation \( \sigma \) of \( S \) and is given by (\[3\], Sec. 2.3):

\[
\sigma = \sqrt{\frac{n - l}{l(n - 1)}} pq
\]

with \( p := \text{wgt}(b)/n \) and \( q := 1 - p \). Since \( pq < 1/4 \) independent of \( \text{wgt}(b) \) we can estimate the term \( \[15\] \) by

\[
\sum_{L,g} p(L,g)s_{L,g} \leq \frac{1}{2}\sqrt{\frac{n - l}{n - 1}}.
\]

\( \square \)

Since \( G \) and \( G_i \) describe error models which are not invariant with respect to local unitary transformations, there is no evident generalization for any other family \( (Q_i) \) of projections instead of \( (F_i) \). In contrast, the error model defined by the map \( D \) (see end of section \[11\]) is symmetric with respect to the group \( SU_2 \otimes SU_2 \otimes \ldots \otimes SU_2 \) of independent local unitary transformations on every qubit. Therefore we obtain estimations for the standard deviations of any observable obtained by averaging over an arbitrary family \( (Q_i) \) of 1-qubit-projections and get:

**Theorem 2** Let \( \rho \) be an arbitrary state of an \( n \)-qubit Quantum Computer. Let \( r_{wn} \) as in \( \[3\] \). Then we have the following inequality:

\[
e_{D(\rho)} \leq r_{wn}.
\]

**Proof:** Let \( \sigma_x^{(i)} \) be the operator representing the Pauli matrix \( \sigma_x \) acting on the qubit \( i \). Let \( M_i \) be as in section \[11\] and \( F_i \) be the instrument performing the bit-flip \( \nu \mapsto \sigma_x^{(i)} \nu \sigma_x^{(i)} \) on every density matrix \( \nu \).

Then \( I_i \) of section \[11\] is given by:

\[
I_i = \left( \frac{1}{2} (F_i + id) \circ M_i \right).
\]

Hence \( D \) can be written as the product

\[
D = \prod_{i \leq n} D_i,
\]

where \( D_i \) is defined by:

\[
D_i = \left( \frac{1}{2} (F_i + id) \circ M_i \right) + (1 - w) id.
\]

In analogy to equation \( \[8\] \) we can decompose \( D \) into a convex sum of instruments \( D_i \) where \( D_i \) is a machine performing a depolarizing error on a randomly chosen \( l \)-element subset \( L \subset \{1, \ldots, n\} \) of qubits. Hence we have:

\[
D_i(\langle \psi | \psi \rangle) = \sum_{L \in \mathcal{L}_i} r_i(L) \prod_{i \in L} \frac{1}{2} (F_i + id) \circ M_i(\langle \psi | \psi \rangle) = \sum_{L \in \mathcal{L}_i} r_i(L) \prod_{i \in L} \frac{1}{2} (F_i + id) \prod_{i \in L} M_i(\langle \psi | \psi \rangle).
\]

Using

\[
\prod_{i \in L} M_i(\langle \psi | \psi \rangle) = \sum_{g \in \{0,1\}^l} \| P_{L,g} \psi \|^2 |\psi_{L,g} \rangle
\]

and the definition \( \[11\] \) we obtain

\[
D_i(\langle \psi | \psi \rangle) = \sum_{L \in \mathcal{L}_i, g \in \{0,1\}^l} p(L,g) \prod_{i \in L} \frac{1}{2} (F_i + id) |\psi_{L,g} \rangle \langle \psi_{L,g} | \]

\[
= \sum_{L \in \mathcal{L}_i, g \in \{0,1\}^l} p(L,g) \prod_{i \in L} \frac{1}{2} (F_i + id) \prod_{i \in T} F_i(\langle \psi_{L,g} | \psi_{L,g} \rangle) \]

This completes the proof: For any \( T \) the standard deviation of \( \mathcal{P} \) in the pure state \( \prod_{i \in T} F_i(\langle \psi_{L,g} | \psi_{L,g} \rangle) \) is the same as in the state \( |\psi_{L,g} \rangle \langle \psi_{L,g} | \) since every \( F_i \) is only a permutation of the states \( |0 \rangle \) and \( |1 \rangle \) in the qubit \( i \).

Therefore we have shown, that \( D_i(\langle \psi | \psi \rangle) \) has a decomposition such that the corresponding mean standard deviation of \( \mathcal{P} \) is less or equal to

\[
\frac{1}{2} \sqrt{\frac{n - l}{l(n - 1)}}.
\]

Therefore the mean standard deviation of \( D(\langle \psi | \psi \rangle) \) is less or equal to \( r_{wn}/2 \) (see the Definition of \( r_{wn} \) in \( \[1\] \) and the decomposition of \( D \) into a convex sum of \( D_i \)) and hence

\[
|\text{tr}(D(\rho) |\mathcal{P}, b|) | \leq r_{wn}
\]

by Corollary \[1\]. Due to the symmetry of the error model with respect to local unitary transformations we can substitute \( \mathcal{P} \) by any other average \( \mathcal{Q} \) over 1-qubit-projections. \( \square \)

The theorem shows, that the ‘entanglement-parameter’ \( e_\rho \) of any state \( \rho \) is extremely sensitive to small depolarizing perturbations acting on every qubit independently.

At the first sight, the only quintessence of these results seems to be that they support the well-known fragility of entanglement by a quantitative analysis without taking into account the possibility of error correction \( \[16\] \). However, we can make easy conclusions for the following model which is so general as to include every possible error correction procedure. We take a Quantum Computer which allows only an imperfect implementation of gates. Taking into account that every error correction has to rely on these gates producing new errors, we see that there are states which never can be obtained.
The following theorem shows this statement quantitatively. Therefore we assume that our Quantum Computer is endowed with the following set of operations on the set of density matrices:

- perfect 1-qubit gates, i.e., maps of the type \( \rho \mapsto upu^* \) where \( u \) is an arbitrary unitary operator acting on 1 qubit and \( \rho \) is the density matrix of the Quantum Computer.
- imperfect 2-qubit gates \( g \) of the following type:
  \[ g(\rho) := (1 - w)upu^* + w(I_i \circ I_j)(\rho), \]
where \( w \) is the error probability and \( u \) is an arbitrary unitary operator acting on the qubits \( i \) and \( j \).

**Theorem 3** Let the Quantum Computer (endowed with the basic operations above) be initialized in a separable state. Let \( (Q_i) \) be a family of projections where \( Q_i \) is acting on the qubit \( i \). Let \( b \) with \( \|b\| \leq 1 \) be a selfadjoint operator and \( r_{\text{w}n} \) as in Corollary 3.

Then it is not possible to prepare a state outside the slice described by the pair of hypersurfaces

\[ H_{Q,b,\pm x} \]

with

\[ x := \frac{1}{n} \sup_{k \leq n} \{ r_{\text{w}k} k + \sqrt{n - k} \}, \]

i.e., it is not possible to prepare a state \( \rho \) with \( e_\rho > x \).

**Proof:** Let the initial state be a pure product state. Let \( k \) be the number of those qubits which are accessed by 2-qubit-gates during the preparation procedure. Without loss of generality assume them to be the qubits \( 1, \ldots, k \). Let \( \rho \) be the density matrix obtained by the preparation. Then we have:

\[ |\text{tr}(\rho[\tilde{P}, b])| \leq \frac{k}{n} |\text{tr}(\rho[\frac{1}{k} \sum_{i \leq k} P_i, b])| + \frac{1}{n} |\text{tr}(\rho[\sum_{i = k + 1} P_i, b])|. \]

Firstly we can show

\[ |\text{tr}(\rho[\frac{1}{k} \sum_{i \leq k} P_i, b])| \leq r_{\text{w}k} \quad (20) \]
by a slight modification of the arguments in the proof of Theorem 1. Due to the fact, that for any qubit \( i \) with \( i \in \{1, \ldots, k\} \) there is a step in the algorithm which is the last access on \( i \) by 2-qubit-gates, every of those qubits is exposed to the depolarizing channel, i.e., the state \( \rho \) can be written as \( \rho = \tilde{D}(\nu) \otimes \mu \), where \( \nu \) and \( \mu \) are states of the qubits \( 1, \ldots, k \) and \( k + 1, \ldots, n \), respectively and \( \tilde{D} \) is the \( k \)-fold depolarizing channel on the qubits \( 1, \ldots, k \). The map \( \tilde{D} \) is given by the restriction of \( \prod_{i \leq k} D_i \) to this subsystem. By convexity arguments, it is sufficient to show

\[ |\text{tr}((\tilde{D} \nu) \otimes \mu)[\frac{1}{k} \sum_{i \leq k} P_i, b])| \leq r_{\text{w}k} \]

for every pure state \( \nu \). Since \( \tilde{D} \) is the analogue to \( D \) for the \( k \)-qubit-system \( 1, \ldots, k \), the mean standard deviations of the mixture \( \nu \) can be estimated in analogy to the proof of Theorem 1. Using Corollary 2 shows inequality (20).

Since the qubits \( k + 1, \ldots, n \) are still in a product state after the preparation procedure, the observables \( \{P_i\}_{i \geq k + 1} \) are stochastically independent. Therefore the variance of the sum \( \sum_{i \geq k + 1} P_i \) is the sum of the variances of \( P_i \). Hence the standard deviation of \( \frac{1}{n} \sum_{i \geq k + 1} P_i \) cannot exceed \( \frac{1}{\sqrt{n - k}} \). Hence we have

\[ |\text{tr}(\rho[\frac{1}{n} \sum_{i \geq k + 1} P_i, b])| \leq \frac{1}{n} \sqrt{n - k}. \]

The extension of the proof from a pure to a separable initial state is given by standard convexity arguments. \( \square \)

**Corollary 4** Modify the assumptions of Theorem 3 as follows: Instead of 2-qubit-gates with depolarizing error we assume a dephasing error, that is, we have imperfect 2-qubit-gates \( g \) of the form

\[ g(\rho) = (1 - w)upu^* + w(M_i \circ M_j)(\rho), \]
where \( u \) is an arbitrary unitary acting on the qubits \( i \) and \( j \).

Then it is not possible to prepare a state outside the slice defined by the hypersurfaces

\[ H_{P,b,\pm x} \]

with \( b \) and \( x \) as in Theorem 3.

**Proof:** In analogy to the proof of Theorem 3 Those qubits \( 1, \ldots, k \), which are accessed at all, are subjected to the dephasing error, i.e., to the instrument \( G \) on the \( k \)-qubit Quantum Computer \( 1, \ldots, k \). \( \square \)

Theorem 2 shows up to which values the error rate \( w \) of a depolarizing channel has to be reduced in order to maintain macroscopic superpositions in the sense discussed here: In order to investigate the asymptotic behaviour of

\[ r_{\text{w}n} = \sum_{l = 0}^{n} \frac{1}{\sqrt{l}} \sqrt{\frac{n - l}{n - 1}} B_{nw}(l) + (1 - w)^n \]
we choose an arbitrary \( 0 < \alpha < 1 \) and split the summation over \( l \) into 2 parts and get
\[ r_{wn} = \sum_{1 \leq l \leq wn} \frac{1}{\sqrt{l}} B_{nw}(l) \quad (21) \]
\[ + \sum_{\alpha wn < l \leq n} \frac{1}{\sqrt{l}} B_{nw}(l) + (1-w)^n \quad (22) \]
\[ \leq \sum_{1 \leq l \leq wn} B_{nw}(l) + \frac{1}{\sqrt{nw \alpha}} + (1-w)^n \quad (23) \]
\[ = \sum_{0 \leq l \leq wn} B_{nw}(l) + \frac{1}{\sqrt{nw \alpha}} \quad (24) \]

Due to the Tschebyscheff inequality the sum over the binomial coefficients from 0 to \( wn \alpha \) is less or equal to 1. Hence we conclude for any 0 < \( \alpha < 1 \):

\[ r_{wn} \leq \frac{1}{wn(1-\alpha)^2} + \frac{1}{\sqrt{nw \alpha}}. \]

V. CONCLUSIONS

We have shown, in which way the parameter \( e_\rho \) and the ‘hypersurface-criterion’ can be used for detecting many-particle-entanglement. States with large \( e_\rho \) are fragile in the following sense: We see that \( r_{wn} \) is decreasing with \( O(\sqrt{wn}) \) for \( n \) going to infinity. For a fixed error probability \( w \) we conclude that the maximum of \( e_\rho \) which can be attained by states subjected to the \( n \)-fold depolarizing channel is decreasing with \( O(1/\sqrt{n}) \).

Furthermore we see that the preparation of states \( \rho \) with a given \( e_\rho \) by imperfect gates requires an error probability decreasing with \( 1/n \) or faster. Similarly, it requires an error probability not greater than \( O(1/n) \) in order to have a state with a fixed \( e_\rho \) after one time step if the decohering effect of the environment during one time step is described by the map \( D \). Note that these are statements about the physical states of the Quantum Computer in contrast to the logical states which may be defined by a certain Quantum Code.

At the moment, we cannot see whether stronger bounds can be given. However, it should be emphasized that our bounds for the fragility of macroscopic superpositions are much weaker than those which are suggested by simple but non-generic examples: The instrument \( D \) transduces the density matrix of the cat state \( \frac{1}{\sqrt{2}}(|0\ldots0\rangle + |1\ldots1\rangle) \) to a density matrix \( \rho \) containing the cat state with probability \((1-w)^n\) and containing unentangled states with probability \( 1 - (1-w)^n \). Therefore we get exponential decay of \( e_\rho \) for increasing \( n \).

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