Generalization of Rashmi-Shah-Kumar Minimum-Storage-Regenerating Codes
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Abstract

In this paper, we propose a generalized version of the Rashmi-Shah-Kumar Minimum-Storage-Regenerating (RSK-MSR) codes based on the product-matrix framework. For any \((n, k, d)\) such that \(d \geq 2k - 2\) and \(d \leq n - 1\), we can directly construct an \((n, k, d)\) MSR code without constructing a larger MSR code and shortening of the larger MSR code. As a result, the size of a finite field over which the proposed code is defined is smaller than or equal to the size of a finite field over which the RSK-MSR code is defined. In addition, the \(\{\ell, \ell'\}\) secure codes based on the generalized RSK-MSR codes can be obtained by applying the construction method of \(\{\ell, \ell'\}\) secure codes proposed by Shah, Rashmi and Kumar. Furthermore, the message matrix of the \((n, k, d)\) generalized RSK-MSR code is derived from that of the RSK-MSR code by using the construction method of the \(\{\ell = k, \ell' = 0\}\) secure code.

Index Terms

Distributed storage, regenerating codes, Minimum-Storage-Regenerating codes (MSR codes), Rashmi-Shah-Kumar MSR codes (RSK-MSR codes), generalized RSK-MSR codes, secure regenerating codes.

I. INTRODUCTION

Dimakis, Godfrey, Wu, Wainwright and Ramchandran introduced a concept of regenerating codes into distributed storage systems [1]. Under the concept, a regenerating code with six parameters \((n, k, d, \alpha, \beta, B)\) has two properties of reconstruction and regeneration as follows. A message consists of \(B\) message symbols over a finite field \(\mathbb{F}_q\) with \(q\) elements. The message is encoded to \(n\) shares in such a way that the message can be reconstructed from any \(k\) shares, and the \(n\) shares are stored across \(n\) storage nodes in a distributed storage system. Each share consists of \(\alpha\) symbols over \(\mathbb{F}_q\), i.e., we assume that the storage capacity of each node is \(\alpha\) symbols. A data collector is permitted to connect to any \(k\) active nodes to reconstruct the message, and downloads the \(k\) shares from the \(k\) nodes. Then, the data collector can reconstruct the message from the \(k\) shares. Furthermore, a failed node is permitted to connect to any \(d\) active nodes, which are called helper nodes, to regenerate the share as was stored in itself, and downloads data consisting of \(\beta\) symbols over \(\mathbb{F}_q\) from each helper node. As a result, the failed node obtains the

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downloaded data of total amount \(d\beta\), and can regenerate the share from the downloaded data, i.e., the failed node can be repaired. The total amount \(d\beta\) of the downloaded data for repair is called the repair-bandwidth.

Dimakis et al.\(^1\) also showed that there is a fundamental tradeoff between storage \(\alpha\) and repair-bandwidth \(\beta\). In the optimal tradeoff, there are Minimum Storage Regenerating(MSR) codes and Minimum Bandwidth Regenerating(MBR) codes as optimal regenerating codes. In particular, an \((n,k,d)\) MSR code satisfies the following optimal condition.

\[
(\alpha, \beta) = \left( \frac{B}{k}, \frac{B}{k(d - k + 1)} \right).
\]

(1)

When the value of \(\beta\) is equal to one, i.e., \(\beta = 1\), the parameters \(\alpha\) and \(B\) are uniquely decided as follows:

\[
\alpha = d - k + 1,
\]

(2)

\[
B = k\alpha = k(d - k + 1).
\]

(3)

In this paper, we focus on a construction of MSR codes. The constructions of MSR codes are proposed in [2]–[4]. Rashmi, Shah and Kumar proposed a construction of \((n,k,d = 2k - 2)\) MSR codes based on product-matrix framework, where \(d \leq n - 1\) and \(\beta = 1\) [2 Sec. V] (see Table I). Furthermore, for any \((n,k,d)\) such that \(d \geq 2k - 2\) and \(d \leq n - 1\), an \((n,k,d)\) MSR code with \(\beta = 1\) is derived from a larger \((n' = n + (d - 2k + 2), k' = k + (d - 2k + 2), d' = d + (d - 2k + 2))\) MSR code by using a procedure of shortening of the larger MSR code, where \(d' = 2k' - 2\) [4 Sec. V-C, Corollary 8] (see Table II). The larger \((n',k',d')\) MSR code can be constructed by [2] Sec. V]. In particular, in this paper, the MSR code is called a Rashmi-Shah-Kumar(RSK) MSR code, i.e., RSK-MSR code.

Suh and Ramchandran [3] and Shah, Rashmi, Kumar and Ramchandran [4] proposed MSR codes based on interference alignment techniques. Suh and Ramchandran proposed a construction of \((2k,k,d = 2k - 1)\) MSR codes, which are called Exact-Repair MDS codes, where \(q \geq 2k\) and \(\beta = 1\) [3 Sec. VI-A, Theorem 2] (see Table I). Moreover, for any \((n,k,d)\) such that \(n \geq 2k\) and \(d \geq 2k - 1\), an \((n,k,d)\) MSR code with \(\beta = 1\) is derived from a larger \((n' = 2(n - k), k' = n - k, d' = 2(n - k) - 1)\) MSR code by using procedures of removing nodes and pruning equations, where \(d' = 2k' - 1\) [3 Sec. VI-B, Theorem 3] (see Table II). The larger \((n',k',d')\) MSR code can be constructed by [3] Sec. VI-A, Theorem 2).

On the other hand, Shah et al. proposed a construction of \((2k,k,d = 2k - 1)\) MSR codes which are called MISER codes\(^1\), where \(\beta = 1\) [4 V-B] (see Table III). Furthermore, for any \((n,k,d)\) such that \(n \geq 2k\) and \(d = n - 1\), an \((n,k,d)\) MSR code with \(\beta = 1\) is derived from a larger \((n' = n + (n - 2k), k' = k + (n - 2k), d' = d + (n - 2k))\) MSR code by using a procedure of shortening of the larger MSR codes, where \(d' = 2k' - 1\) [4 Sec. V-C] (see Table III). The \((n',k',d')\) MSR code can be constructed by [4 Sec. V-B]. The relation between codes of [4] and [3] is written in [4] Sec. I-D and V-G and [3] Sec. II-C.

\(^1\)Short for MDS, Interference-aligning, Systematic, Exact-Regenerating codes

\(^2\)Under the additional constraint in regeneration, Shah et al. showed the extension of the MISER code to the case \(2k - 1 \leq d \leq n - 1\) [4 Sec. V-D].
TABLE I

Relations of parameters and conditions of \((n, k, d)\) MSR codes with \(\beta = 1\) over \(\mathbb{F}_q\) in [2]–[4].

| Parameters \((n, k, d)\) | Conditions |
|-------------------------|------------|
| Rashmi et al. (2 Sec. V) | \((n, 2k - 2)\) \(d = 2k - 2, d \leq n - 1, q \geq n(d - k + 1)\) |
| Suh et al. (3 Sec. VI-A) | \((2k, k, 2k - 1)\) \(d = 2k - 1, d = n - 1, q \geq 2k\) |
| Shah et al. (4 Sec. V-B) | \((2k, k, 2k - 1)\) \(d = 2k - 1, d = n - 1, q \geq 2k\) |
| Presented codes in this paper | \((n, k, d)\) \(d \geq 2k - 2, d \leq n - 1, q \geq n(d - k + 1)\) |

\(^\dagger\) The minimum field size is derived from [4 Eq.(37)] as follows: \(q \geq \alpha + n - k = n = 2k\).

TABLE II

Relations of parameters and conditions of the larger \((n', k', d')\) MSR codes to construct the target \((n, k, d)\) MSR codes with \(\beta = 1\) over \(\mathbb{F}_q\) in [2]–[4], where \(i_1 = d - 2k + 2\) and \(i_2 = n - 2k\).

| Parameters \((n', k', d')\) | Conditions |
|-------------------------|------------|
| Rashmi et al. (2 Sec. V-C) | \((n + i_1, k + i_1, d + i_1)\) \(d \geq 2k - 2, d \leq n - 1, q \geq (n + d - 2k + 2)(d - k + 1)\) |
| Suh et al. (3 Sec. VI-B) | \((2(n - k), n - k, 2(n - k) - 1)\) \(d \geq 2k - 1, n \geq 2k, q \geq 2(n - k)\) |
| Shah et al. (4 Sec. V-C) | \((n + i_2, k + i_2, d + i_2)\) \(n \geq 2k, d = n - 1, q \geq 2(n - k)\) |

In this paper, we propose a generalized version of RSK-MSR codes, which are based on product-matrix framework, proposed by Rashmi et al. [2]. For any \((n, k, d)\) such that \(d \geq 2k - 2\) and \(d \leq n - 1\), we can directly construct an \((n, k, d)\) MSR code with \(\beta = 1\) without using processes of constructing a larger MSR code and shortening of the larger MSR code (see Table I). We will call the presented code in this paper a generalized RSK-MSR code.

This paper is organized as follows: In section II a construction of generalized RSK-MSR codes is proposed. Furthermore, the reconstruction and regeneration of the code are described. In section III, examples of construction, reconstruction and regeneration are given. In section IV relations between generalized RSK-MSR codes and \(\{\ell, \ell'\}\) secure codes are described. Finally, a conclusion is given in section V.

II. GENERALIZED RSK-MSR CODES

In this section, we propose a construction of \((n, k, d)\) MSR code with \(\beta = 1\) over \(\mathbb{F}_q\) for any \((n, k, d)\) such that \(d \geq 2k - 2\) and \(d \leq n - 1\). Then, the remaining parameters \(\alpha\) and \(B\) are uniquely determined from \(k, d\) and \(\beta\) as follows: \(\alpha = d - k + 1\) and \(B = k\alpha = k(d - k + 1)\).
We assume that there are \( n \) storage nodes such as node \( i \), \( 1 \leq i \leq n \), in a distributed storage system. The storage capacity of each node is \( \alpha \) symbols over \( \mathbb{F}_q \). Furthermore, for each \( i \in \{1, \ldots, n\} \), assign an unique public symbol \( x_i \) in \( \mathbb{F}_q \) to node \( i \) in such a way that the following conditions are satisfied.

1) For any \( i \in \{1, \ldots, n\} \), \( x_i \neq 0 \),
2) For any \( i, j \in \{1, \ldots, n\} \), \( x_i^\alpha \neq x_j^\alpha \) if \( i \neq j \).

From the condition of parameters \( q, n \) and \( \alpha \) written in [2] Sec. V, if the condition, \( q \geq n\alpha \), is true, then there are at least \( n \) elements \( x_1, \ldots, x_n \) in \( \mathbb{F}_q \) such that \( x_i^\alpha \neq x_j^\alpha \) if \( i \neq j \). The condition is the sufficient condition of the existence of such \( n \) elements, but it is not the necessary condition of that (see Example in Section III).

In general, we can construct an \((n, k, d)\) generalized RSK-MSR code over \( \mathbb{F}_q \), where \( q \geq n(d - k + 1) \). On the other hand, in the construction method proposed by Rashmi et al. [2], the target \((n, k, d)\) RSK-MSR code can be constructed by shortening a larger \((n' = n + d - 2k + 2, k' = d - k + 2, d' = 2(d - k + 1))\) RSK-MSR code over \( \mathbb{F}_{q_0} \), where \( q_0 \geq n'(d' - k' + 1) = \{n + (d - 2k + 2)\}(d - k + 1) \). Thus, the size of the finite field \( \mathbb{F}_q \) is smaller than or equal to that of the finite field \( \mathbb{F}_{q_0} \), since \( d \geq 2k - 2 \).

A. Message Matrix \( M \)

Firstly, for given parameters \( k \) and \( d \), we define five types of \((d \times (d-k+1))\) message matrices \( M \) consisting of the following sub-matrices. Note that the message matrix is also represented as a \((d \times \alpha)\) matrix because \( \alpha = d - k + 1 \).

\[
T_1, T_2 : \quad ((k - 1) \times (k - 1)) \text{ symmetric matrices,}
\]

\[
U_1 : \quad \text{a } ((k - 1) \times 1) \text{ matrix, i.e., a column vector of length } k - 1,
\]

\[
U_2 : \quad \text{a } (1 \times 1) \text{ matrix, i.e., a scalar,}
\]

\[
V_1 : \quad \text{a } ((k - 1) \times (d - 2k + 1)) \text{ matrix,}
\]

\[
V_2 : \quad \text{a } (1 \times (d - 2k + 1)) \text{ matrix, i.e., a row vector of length } d - 2k + 1,
\]

\[
O_1 : \quad \text{a } ((d - 2k + 1) \times (d - 2k + 1)) \text{ square zero matrix,}
\]

\[
O_2 : \quad \text{a } ((k - 1) \times (d - 2k + 2)) \text{ zero matrix.}
\]

1) Type I : When \( k \geq 2 \) and \( d = 2k - 2 \), the message matrix is defined as

\[
M = \begin{bmatrix}
T_1 \\
U_1 \\
T_2
\end{bmatrix}
\]

The message matrix of Type I is identical with that of the RSK-MSR code [2]. The message matrix is a \((d \times (d-k+1))\) matrix because \( d = 2k - 2 \). Since \( T_1 \) and \( T_2 \) are \(((k - 1) \times (k - 1))\) symmetric matrices, each of the two matrices consists of \( k(k-1)/2 \) distinct entries. Thus, the message matrix consists of \( B = k(k-1) \) distinct message symbols because \( B = k(d - k + 1) = k(k - 1) \) when \( d = 2k - 2 \).

2) Type II : When \( k \geq 2 \) and \( d = 2k - 1 \), the message matrix is defined as

\[
M = \begin{bmatrix}
T_1 & U_1 \\
U_1^t & U_2 \\
T_2 & O_2
\end{bmatrix}
\]
where $U_1^t$ is the transpose of $U_1$. The message matrix of Type II is a $(d \times (d - k + 1))$ matrix because $d = 2k - 1$. The sub-matrix $\begin{bmatrix} T_1 & U_1 \\ U_1^t & U_2 \end{bmatrix}$ of $M$ is a $(k \times k)$ symmetric matrix and the $(k \times 1)$ sub-matrix $\begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$ consists of $k$ entries. Thus, the message matrix consists of $B = k^2$ distinct message symbols because $B = k(d - k + 1) = k^2$ when $d = 2k - 1$.

3) Type III: When $k \geq 2$ and $d \geq 2k$, the message matrix is defined as

$$M = \begin{bmatrix} T_1 & U_1 & V_1 \\ U_1^t & U_2 & V_2 \\ V_1^t & V_2^t & O_1 \\ T_2 & O_2 \end{bmatrix}. \quad (6)$$

The message matrix of Type III is a $(d \times (d - k + 1))$ matrix. Since the sub-matrix $\begin{bmatrix} T_1 & U_1 & V_1 \\ U_1^t & U_2 & V_2 \\ V_1^t & V_2^t & O_1 \end{bmatrix}$ of $M$ is a $(d - k + 1) \times (d - k + 1)$ symmetric matrix and the $(k \times (d - 2k + 2))$ sub-matrix $\begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$ consists of $k(d - 2k + 2)$ entries, the message matrix consists of $B = k(k - d + 1)$ distinct message symbols.

4) Type IV: When $k = 1$ and $d = 1$, the message matrix is defined as

$$M = \begin{bmatrix} U_2 \end{bmatrix}. \quad (7)$$

The message matrix of Type IV is a $(d \times (d - k + 1))$ matrix because $d - k + 1 = d = 1$ when $k = 1$ and $d = 1$. Thus, the message matrix consists of $B = 1$ message symbol because $B = k(d - k + 1) = d = 1$ when $k = 1$ and $d = 1$.

5) Type V: When $k = 1$ and $d \geq 2$, the message matrix is defined as

$$M = \begin{bmatrix} U_2 & V_2 \\ V_2^t & O_1 \end{bmatrix}. \quad (8)$$

The message matrix of Type V is a $(d \times (d - k + 1))$ matrix because $d - k + 1 = d$ when $k = 1$. The message matrix is a $(d \times d)$ symmetric matrix and the sub-matrix $O_1$ is a $((d - 1) \times (d - 1))$ square zero matrix. Thus, the message matrix consists of $B = d$ distinct message symbols because $B = k(d - k + 1) = d$ when $k = 1$.

From the above definition, these five types of the message matrices for $k$ and $d$ are coordinated in Table III. The message matrix of every type is a $(d \times (d - k + 1))$ matrix and consists of $B$ message symbols.

### Table III

| $k$  | $(d = 1)$ | $(d \geq 2)$ |
|------|-----------|---------------|
|      | IV        | V             |
| $k \geq 2$ |          |               |
| $(d = 2k - 2)$ | I        | II            |
| $(d = 2k - 1)$ |          |               |
| $(d \geq 2k)$ |          | III           |
B. Encoding and Share

For each $i$, $1 \leq i \leq n$, we define a coding vector $\rho_i$ associated with node $i$ as follows:

$$\rho_i = [1, x_i, x_i^2, \ldots, x_i^{d-1}] \in \mathbb{F}^d,$$

where $x_i$ is the element assigned to node $i$. The message consisting of $B$ message symbols is encoded to $n$ shares by the coding vectors $\rho_i$, $1 \leq i \leq n$, and the message matrix $M$. For each $i$, $1 \leq i \leq n$, the share $c_i$, which is stored in node $i$, is defined by

$$c_i = [c_{i,1}, \ldots, c_{i,\alpha}] = \rho_i M \in \mathbb{F}^\alpha.$$

C. Reconstruction

In this section, we describe a reconstruction of the generalized RSK-MSR code. Before describing the reconstruction, we write the coding vector and the share using the following two sub-vectors.

$$\omega_i = [1, x_i, \ldots, x_i^{k-2}] \in \mathbb{F}^{k-1},$$

$$\theta_i = [1, x_i, \ldots, x_i^{\alpha-k-1}] \in \mathbb{F}^{\alpha-k}.$$

1) In the case of Type I, the coding vector is represented as

$$\rho_i = [\underbrace{1, \ldots, x_i^{k-2}}_{\omega_i}, \underbrace{x_i^{k-1}, \ldots, x_i^{d-1}}_{\theta_i}] ,$$

where $\alpha = k - 1$ because $d = 2k - 2$, and then, the components of the share are represented as

$$[c_{i,1}, \ldots, c_{i,\alpha}] = [\omega_i, x_i^{\alpha-k}]^T.$$

2) In the case of Type II, the coding vector is represented as

$$\rho_i = [\underbrace{1, \ldots, x_i^{k-2}}_{\omega_i}, \underbrace{x_i^{k-1}}_{\theta_i}, \underbrace{x_i^{k}, \ldots, x_i^{d-1}}_{\theta_i}] ,$$

where $\alpha = k$ because $d = 2k - 1$, and then, the components of the share are represented as

$$[c_{i,1}, \ldots, c_{i,\alpha-1}] = [\omega_i, x_i^{\alpha-k}]^T + x_i^{k-1}U_1^T,$$

$$c_{i,\alpha} = [\omega_i, x_i^{k-1}]^T U_2.$$

3) In the case of Type III, the coding vector is represented as

$$\rho_i = [\underbrace{1, \ldots, x_i^{k-2}}_{\omega_i}, \underbrace{x_i^{k-1}}_{\theta_i}, \underbrace{x_i^{k}, \ldots, x_i^{\alpha-1}}_{\theta_i}, \underbrace{x_i^{\alpha}, \ldots, x_i^{d-1}}_{\theta_i}] ,$$
where \( \alpha \geq k + 1 \) because \( d \geq 2k \), and then, the components of the share are represented as

\[
[c_{i,1}, \ldots, c_{i,k-1}] = [\omega_i, x_i^{0} \omega_i] \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} + [x_i^{k-1}, x_i^k \theta_i] \begin{bmatrix} U_1^t \\ V_1^t \end{bmatrix},
\]

\[
c_{i,k} = [\omega_i, x_i^{k-1}] \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} + x_i^k \theta_i V_2^t,
\]

\[
[c_{i,k+1}, \ldots, c_{i,\alpha}] = [\omega_i, x_i^{k-1}] \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}.
\]

4) In the case of Type IV, the coding vector is represented as

\[
\rho_i = [1],
\]

where \( k = 1 \) and \( d = 1 \), and then, the components of the share are represented as

\[
c_{i,1} = U_2.
\]

5) In the case of Type V, the coding vector is represented as

\[
\rho_i = [1, x_i, \ldots, x_i^{d-1}],
\]

where \( k = 1 \) and \( d \geq 2 \), and then, the components of the share are represented as

\[
c_{i,1} = U_2 + x_i \theta_i V_2^t,
\]

\[
[c_{i,2}, \ldots, c_{i,\alpha}] = V_2.
\]

In the case of Type I, since the message matrix is identical with that of the RSK-MSR code, the data collector can reconstruct the message from any \( k \) shares by using the method of MSR Data-Reconstruction proposed by Rashmi et al. [2, Theorem 5].

In the case of Type IV, the data collector can reconstruct the message from any share \( \omega_i \) without computing because \( U_2 = c_{i,1} \). Moreover, in the case of Type V, the data collector can reconstruct the message from any share \( \omega_i \) because \( V_2 = [c_{i,2}, \ldots, c_{i,\alpha}] \) and \( U_2 = c_{i,1} - x_i \theta_i V_2^t \).

We show the reconstruction for Type III in the following theorem. The reconstruction for Type II is included in that for Type III.

**Theorem 1 (Reconstruction for Type III).** A data collector connecting any \( k \) nodes can reconstruct the \( B \) message symbols from the \( k \) shares of the \( k \) nodes.

**Proof.** The data collector connects to \( k \) nodes \( \{i_1, \ldots, i_k\} \) to reconstruct the message, and downloads the \( k \) shares \( \omega_{i_1}, \ldots, \omega_{i_k} \).
Decomposing \( k \) coding vectors \( \rho_{i_1}, \ldots, \rho_{i_k} \) associated with the \( k \) nodes, we define the following three matrices.

\[
\Omega_{\text{DC}} = \begin{bmatrix} \bar{\omega}_{i_1} \\ \vdots \\ \bar{\omega}_{i_k} \end{bmatrix}, \quad x_{\text{DC}} = \begin{bmatrix} x_{i_1}^{k-1} \\ \vdots \\ x_{i_k}^{k-1} \end{bmatrix}, \quad \Theta_{\text{DC}} = \begin{bmatrix} x_{i_1}^k \rho_{i_1} \\ \vdots \\ x_{i_k}^k \rho_{i_k} \end{bmatrix},
\]

(13)

where \( \Omega_{\text{DC}} \) is a \((k \times (k - 1))\) matrix, \( x_{\text{DC}} \) is a column vector of length \( k \) and \( \Theta_{\text{DC}} \) is a \((k \times (\alpha - k))\) matrix.

(Step 1: Finding \( V_1 \) and \( V_2 \)) Firstly, the data collector solves the following system of linear equations.

\[
\begin{bmatrix} c_{i_1,k+1} & \cdots & c_{i_1,\alpha} \\ \vdots & \vdots & \vdots \\ c_{i_k,k+1} & \cdots & c_{i_k,\alpha} \end{bmatrix} \varepsilon_{\text{DC}} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} \Omega_{\text{DC}} \varepsilon_{\text{DC}} \end{bmatrix} \begin{bmatrix} \Theta_{\text{DC}} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}.
\]

(14)

The left \((k \times k)\) matrix \([\Omega_{\text{DC}} \varepsilon_{\text{DC}}]\) in the right-hand side of Eq. (14) is nonsingular, because the determinant of the \((k \times k)\) matrix is the Vandermonde determinant from the condition of \( x_i \). Thus, the data collector can recover \( V_1 \) and \( V_2 \).

(Step 2: Finding \( U_1 \) and \( U_2 \)) Next, the data collector solves the following system of linear equations.

\[
\begin{bmatrix} c_{i_1,k} \\ \vdots \\ c_{i_k,k} \end{bmatrix} - \Theta_{\text{DC}} V_2' = \begin{bmatrix} \Omega_{\text{DC}} \varepsilon_{\text{DC}} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}.
\]

(15)

Since the data collector knows \( V_2 \) from the previous step and the \((k \times k)\) matrix \([\Omega_{\text{DC}} \varepsilon_{\text{DC}}]\) is nonsingular, the data collector can recover \( U_1 \) and \( U_2 \).

(Step 3: Finding \( T_1 \) and \( T_2 \)) Finally, the data collector recovers \( T_1 \) and \( T_2 \) in the right-hand side of Eq. (16) by using the method of MSR Data-Reconstruction [2] Theorem 5, that is, \( T_1 \) and \( T_2 \) are recovered by using the reconstruction method for Type I.

\[
\begin{bmatrix} c_{i_1,1} & \cdots & c_{i_1,k-1} \\ \vdots & \vdots & \vdots \\ c_{i_k,1} & \cdots & c_{i_k,k-1} \end{bmatrix} - \varepsilon_{\text{DC}} \begin{bmatrix} \Theta_{\text{DC}} \end{bmatrix} \begin{bmatrix} U_1^2 \\ V_1^2 \end{bmatrix} = \begin{bmatrix} \Omega_{\text{DC}} \Lambda_{\text{DC}} \Omega_{\text{DC}} \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix},
\]

(16)

where \( \Lambda_{\text{DC}} \) is a \((k \times k)\) diagonal matrix with diagonal components \( x_{\alpha i_1}^{\alpha}, \ldots, x_{\alpha i_k}^{\alpha} \). Note that the right-hand of Eq. (16) is corresponding to [2 Eq. (29)]. Since the data collector knows \( U_1 \) and \( V_1 \) from the previous steps, the data collector can obtain the values in the left-hand side of Eq. (16). Thus, the data collector can recover \( T_1 \) and \( T_2 \) in the right-hand side of Eq. (16) using the method of MSR Data-Reconstruction since the components in the right-hand side of Eq. (16) satisfy the following conditions.

- \( \Omega_{\text{DC}} \) is \((k \times (k - 1))\) matrix,
- \( \Lambda_{\text{DC}} \) is a nonsingular \((k \times k)\) matrix from the condition of \( x_i \),
- \( T_1 \) and \( T_2 \) are \(((k - 1) \times (k - 1))\) symmetric matrices,
- the \(((k - 1) \times (k - 1))\) matrix \([\bar{\omega}_{i_1}^{\prime}, \ldots, \bar{\omega}_{i_{k-1}}^{\prime}]\) consisting of the first \( k - 1 \) vectors \( \bar{\omega}_{i_1}, \ldots, \bar{\omega}_{i_{k-1}} \) is nonsingular from the condition of \( x_i \).
By the above three steps, the data collector can recover the message matrix $M$, that is, the data collector can reconstruct the message consisting of $B$ message symbols.

From Theorem 1 in the case of Type II, the data collector can reconstruct the message from any $k$ shares by the procedures of Step 2 and 3 in the proof of Theorem 1.

D. Regeneration

In this section, we describe a regeneration of the generalized RSK-MSR code, that is, a failed node $f$ connecting any $d$ helper nodes can regenerate the share as was stored in itself prior to failure.

Firstly, we describe the regeneration for Type I, II and III. For each type, we define the sub-vector $\varphi_i$ of the coding vector $\rho_i$ and the two sub-matrices $W_1$ and $W_2$ of the message matrix $M$, where $\varphi_i$ is a vector of length $\alpha$, and $W_1$ and $W_2$ are respectively an $(\alpha \times \alpha)$ square matrix and a $((k - 1) \times \alpha)$ matrix.

1) In the case of Type I, the sub-vector $\varphi_i$ and the two sub-matrices $W_1$, $W_2$ are defined as
   
   $\varphi_i = \omega_i^T \in \mathbb{F}^\alpha, \quad 1 \leq i \leq n,$
   
   $W_1 = T_1, W_2 = T_2.$

2) In the case of Type II, the sub-vector $\varphi_i$ and the two sub-matrices $W_1$, $W_2$ are defined as
   
   $\varphi_i = [\omega_i, x_i^{k-1}] \in \mathbb{F}^\alpha, \quad 1 \leq i \leq n,$
   
   $W_1 = \begin{bmatrix} T_1 & U_1 \\ U_1^T & U_2 \end{bmatrix}, W_2 = \begin{bmatrix} T_2 & O_2 \end{bmatrix}.$

3) In the case of Type III, the sub-vector $\varphi_i$ and the two sub-matrices $W_1$, $W_2$ are defined as
   
   $\varphi_i = [\omega_i, x_i^{k-1}, x_i^k \theta_i] \in \mathbb{F}^\alpha, \quad 1 \leq i \leq n,$
   
   $W_1 = \begin{bmatrix} T_1 & U_1 & V_1 \\ U_1^T & U_2 & V_2 \\ V_1^T & V_2^T & O_1 \end{bmatrix}, W_2 = \begin{bmatrix} T_2 & O_2 \end{bmatrix}.$

From the above definitions, the share $c_f$ of the failed node $f$ is represented as follows:

$$c_f = \varphi_f W_1 + x_f^\theta \omega_f W_2.$$  \hspace{1cm} (17)

The proof of the following theorem gives the regeneration method for Type I, II and III. The idea of the regeneration method is derived from the method of MSR Exact-Regeneration proposed by Rashmi, et al. [2, Theorem 4].

**Theorem 2 (Regeneration for Type I, II and III).** A failed node connecting any $d$ nodes can regenerate the share as was stored in itself prior to failure.
Proof. The failed node $f$ connects $d$ helper nodes $\{h_1, \cdots, h_d\}$ to regenerate the same share that was stored in the failed node prior to failure, and downloads the following data for repair from each helper node. Each helper node $h_p$ computes data $d_{f,h_p}$ for repair from $\omega_{h_p}$ and $\varphi_f$ as follows:

$$d_{f,h_p} = \omega_{h_p} \varphi_f^t \in \mathbb{F},$$

(18)

where $d_{f,h_p}$ is a scalar because $\beta = 1$, and sends it to the failed node $f$. As a result, the failed node obtains data of $d$ symbols $d_{f,h_1}, \ldots, d_{f,h_d}$ for repair such that

$$\begin{bmatrix}
  d_{f,h_1} \\
  \vdots \\
  d_{f,h_d}
\end{bmatrix} = M \varphi_f^t \in \mathbb{F}^d.$$

(19)

Since the determinant of the leftmost $(d \times d)$ matrix in the right-hand side is the Vandermonde determinant from the condition of $x_i$, the $(d \times d)$ matrix is nonsingular. Thus, the failed node can obtain the following equation from the above system.

$$\begin{bmatrix}
  \omega_{h_1} \\
  \vdots \\
  \omega_{h_d}
\end{bmatrix}^{-1} \begin{bmatrix}
  d_{f,h_1} \\
  \vdots \\
  d_{f,h_d}
\end{bmatrix} = M \varphi_f^t = \begin{bmatrix} W_1 \varphi_f^t \\ W_2 \varphi_f^t \end{bmatrix}$$

(20)

Since the matrix $W_1$ is symmetric, we have $(W_1 \varphi_f^t)^t = \varphi_f W_1$. Since $W_2 \varphi_f^t = T_2 \omega_f^t$ and the matrix $T_2$ is symmetric, we have $(W_2 \varphi_f^t)^t = (T_2 \omega_f^t)^t = \omega_f T_2$. Thus, the failed node can regenerate the share $\varphi_f$ as follows:

$$(W_1 \varphi_f^t)^t + x_f^t \left[ (W_2 \varphi_f^t)^t, 0_{\alpha-k+1} \right] = \varphi_f W_1 + x_f^t \omega_f T_2 O_2$$

(21)

$$= \varphi_f W_1 + x_f^t \omega_f W_2 = \varphi_f,$$

(22)

where $0_{\alpha-k+1}$ is a zero row vector of length $\alpha - k + 1$ and $\omega_f O_2 = 0_{\alpha-k+1}$. \hfill $\square$

In the case of Type I, the regeneration method in the proof of Theorem 2 is identical with the method of MSR Exact-Regeneration [2, Theorem 4], where the zero vector $0_{\alpha-k+1}$ and the zero matrix $O_2$ are removed from Eq.(21) because $\alpha = k - 1$.

Next, we describe the regeneration for the remaining types IV and V. In the case of Type IV, the reconstruction is trivial because $\varphi_i = [e_{i,1}] = U_2$ and $d = 1$. In the case of Type V, the failed node connecting any $d$ nodes can regenerate the share as was stored in itself prior to failure using the method of MBR Exact-Regeneration proposed by Rashmi et al. [2, Theorem 2] because the message matrix $M$ is symmetric.

III. EXAMPLE

In this section, we give an example to help understanding of the generalized RSK-MSR code and the reconstruction and the regeneration of the code.
Let \((n,k,d) = (10,2,4)\). Then \(\alpha = d - k + 1 = 3\) and \(B = k\alpha = 6\). From \((k,d) = (2,4)\), the message matrix \(M\) corresponds to Type III. We can construct a \((10,2,4)\) generalized RSK-MSR code over \(\mathbb{F}_{11}\) from the following table for \(\mathbb{F}_{11}\).

| \(i\) | 1 2 3 4 5 6 7 8 9 10 |
|-------|-----------------|
| \(x_i^3\) | 1 8 5 9 4 7 2 6 3 10 |
| \(\mathbb{F}_{11} \ni x_i\) | 1 2 3 4 5 6 7 8 9 10 |

For each \(i \in \{1, \ldots, 10\}\), we assign the element \(x_i \in \mathbb{F}_{11}\) to node \(i\).

Let

\[
M = \begin{bmatrix}
T_1 & U_1 & V_1 \\
U_1^T & U_2 & V_2 \\
V_1^T & V_2^T & \Theta_1 \\
T_2 & \Theta_2
\end{bmatrix}
= \begin{bmatrix}
1 & 2 & 3 \\
2 & 4 & 5 \\
3 & 5 & 0 \\
6 & 0 & 0
\end{bmatrix},
\]

that is, the \(B = 6\) message symbols are \(1, 2, 3, 4, 5\) and \(6 \in \mathbb{F}_{11}\). For each \(i \in \{1, \ldots, 10\}\), the coding vector is given as \(\rho_i = [1, x_i, x_i^2, x_i^3]\), and then, the share \(\xi = \rho_i \cdot M\) of node \(i\) is given as

\[
\begin{pmatrix}
\xi_1 \\
\xi_2 \\
\xi_3 \\
\xi_4 \\
\xi_5 \\
\xi_6 \\
\xi_7 \\
\xi_8 \\
\xi_9 \\
\xi_{10}
\end{pmatrix} =
M
= \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 0 & 8 \\
1 & 2 & 4 & 8 & 10 & 8 & 2 \\
1 & 3 & 9 & 5 & 9 & 4 & 7 \\
1 & 4 & 5 & 9 & 1 & 10 & 1 \\
1 & 5 & 3 & 4 & 0 & 4 & 6 \\
1 & 6 & 3 & 7 & 9 & 8 & 0 \\
1 & 7 & 5 & 2 & 9 & 0 & 5 \\
1 & 8 & 9 & 6 & 3 & 2 & 10 \\
1 & 9 & 4 & 3 & 5 & 3 & 4 \\
1 & 10 & 1 & 10 & 7 & 3 & 9
\end{bmatrix}.
\]

*(Reconstruction:)* A data collector connects to \(k = 2\) nodes \(\{1, 2\}\) to reconstruct the message, and downloads \(k = 2\) shares \(\xi_1\) and \(\xi_2\). Then, the matrices \(\Omega_{DC}, \Theta_{DC}\) and \(\Theta_{DC}\) are given as

\[
\Omega_{DC} = \begin{bmatrix}
\xi_1 \\
\xi_2
\end{bmatrix} = \begin{bmatrix}
1 \\
1
\end{bmatrix},
\Theta_{DC} = \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} = \begin{bmatrix}
1 \\
2
\end{bmatrix},
\Theta_{DC} = \begin{bmatrix}
x_1^2 \Theta_1 \\
x_2^2 \Theta_2
\end{bmatrix} = \begin{bmatrix}
1 \\
4
\end{bmatrix}.
\]

*(Step 1:)* Firstly, the data collector recovers \(V_1\) and \(V_2\) as follows:

\[
\begin{bmatrix}
V_1 \\
V_2
\end{bmatrix} = \begin{bmatrix}
\Omega_{DC} & \Theta_{DC}
\end{bmatrix}^{-1}
\begin{pmatrix}
\xi_{1,3} \\
\xi_{2,3}
\end{pmatrix} = \begin{bmatrix}
2 & 10 \\
10 & 1
\end{bmatrix}
\begin{pmatrix}
8 \\
2
\end{pmatrix} = \begin{bmatrix}
3 \\
5
\end{bmatrix}.
\]
Next, the data collector recovers $U_1$ and $U_2$ as follows:

$$
\begin{bmatrix}
U_1 \\
U_2
\end{bmatrix} = \left[ \Omega_{DC} \zeta_{DC} \right]^{-1} \left( \begin{bmatrix} c_{1,2} \\
1 \\
0 \end{bmatrix} - \Theta_{DC} \Theta_{DC}^t \right)
= \begin{bmatrix} 2 & 10 \\
10 & 1 \end{bmatrix} \begin{bmatrix} 0 \\
8 \end{bmatrix} - \begin{bmatrix} 1 \\
4 \end{bmatrix} \begin{bmatrix} 5 \\
7 \end{bmatrix} = \begin{bmatrix} 2 \\
4 \end{bmatrix}.
$$

Finally, the data collector has the following system of linear equations from $U_1$ and $V_1$.

$$
\begin{bmatrix}
\Omega_{DC} \Lambda_{DC} \Omega_{DC} \\
\Lambda_{DC}
\end{bmatrix} \begin{bmatrix} T_1 \\
T_2
\end{bmatrix} = \begin{bmatrix} c_{1,1} \\
c_{2,1} \end{bmatrix} - \begin{bmatrix} \Theta_{DC} \Omega_{DC} \end{bmatrix} \begin{bmatrix} U_1^t \\
V_1^t
\end{bmatrix}
= \begin{bmatrix} 1 \\
10 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\
2 & 4 \end{bmatrix} \begin{bmatrix} 2 \\
3 \end{bmatrix} = \begin{bmatrix} 7 \\
5 \end{bmatrix},
$$

where

$$
\Lambda_{DC} = \begin{bmatrix} x_1^3 & 0 \\
0 & x_2^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\
0 & 8 \end{bmatrix}.
$$

By using the method of MSR Data-Reconstruction [2, Theorem 5], the data collector can recover $T_1$ and $T_2$ as follows: $T_1 = [1]$ and $T_2 = [6]$.

From the above three steps, the data collector recovers the message matrix $M$, that is, the data collector reconstructs the $B = 6$ message symbols.

(Regeneration: ) Assume that node 1 is a failed node, and node 2, 3, 4 and 5 are $d = 4$ helper nodes for the failed node. The failed node obtains the following data of $d = 4$ symbols for repair from the $d = 4$ helper nodes.

$$
\begin{bmatrix}
d_{1,2} \\
d_{1,3} \\
d_{1,4} \\
d_{1,5}
\end{bmatrix} = \begin{bmatrix} \zeta_1 \\
\zeta_2 \\
\zeta_3 \\
\zeta_4 \end{bmatrix} \Phi_1^t = \begin{bmatrix} 9 \\
9 \\
1 \\
10 \end{bmatrix}.
$$

By computing the multiplication of two matrices in the left-hand side of Eq.(20), the failed node has the following equation.

$$
\begin{bmatrix}
W_{1,\Phi_1^t} \\
W_{2,\Phi_1^t}
\end{bmatrix} = \begin{bmatrix} \rho_2 & \rho_3 & \rho_4 & \rho_5 \\
\rho_2 & \rho_3 & \rho_4 & \rho_5
\end{bmatrix}^{-1} \begin{bmatrix} d_{1,2} \\
d_{1,3} \\
d_{1,4} \\
d_{1,5}
\end{bmatrix} = \begin{bmatrix} 10 & 2 & 4 & 7 \\
5 & 8 & 1 & 8 \\
2 & 0 & 5 & 4 \\
9 & 6 & 5 & 2
\end{bmatrix} \begin{bmatrix} 9 \\
9 \\
1 \\
10 \end{bmatrix} = \begin{bmatrix} 6 \\
0 \\
8 \\
6 \end{bmatrix}.
$$

Next, the failed node substitutes the values such that $W_{1,\Phi_1^t} = [6, 0, 8]^t$ and $W_{2,\Phi_1^t} = [6]$ for Eq.(21), and regenerate the share $\zeta_1$ as follows:

$$
\zeta_1 = (W_{1,\Phi_1^t})^t + x_1^3 \left( (W_{2,\Phi_1^t})^t, \Omega_2 \right)
= [6, 0, 8] + [6, 0, 0] = [1, 0, 8],
$$
IV. Relations Between \{\ell, \ell'\} Secure Codes and Generalized RSK-MSR Codes

In this section, we first describe the concept of the \{\ell, \ell'\} secure code and the construction method proposed by Shah, Rashmi and Kumar [5]. Next, we show how to apply the construction method of \{\ell, \ell'\} secure code to the generalized RSK-MSR code. Finally, we explain that the message matrix of the generalized RSK-MSR code is derived from that of the original RSK-MSR code by using the construction method of the \{\ell = k, \ell' = 0\} secure code.

A. \{\ell, \ell'\} Secure Codes based on Product-Matrix MSR Codes

Shah, Rashmi and Kumar [5] proposed the construction methods of the \{\ell, \ell'\} secure codes based on the \(n, k, d\) Product-Matrix (PM) MSR codes and the \(n, k, d\) PM MBR codes, which are proposed by Rashmi et al. [2]. Note that the \(n, k, d\) PM MSR code is identical with the \(n, k, d\) RSK-MSR codes with the \((2\alpha \times \alpha)\) message matrix in the case of Type I.

In [5], the \{\ell, \ell'\} secure codes are defined in the following threat model. An eavesdropper can gain read-access to the data stored in any set of at-most \(\ell(< k)\) storage nodes. The eavesdropper may also gain read-access to the data being downloaded during (possibly multiple instances of) repair of some \(\ell'(<\ell)\) of these \(\ell\) nodes. Note that the data downloaded by a replacement node during any instance of repair also contains the data that is eventually stored in that node. This is formalized in the following definition [5 Definition 1].

**Definition 3** (\{\ell, \ell'\} secure distributed storage system [5]). Consider a distributed storage system in which an eavesdropper gains access to the data stored in some \((\ell - \ell')\) nodes, and the data stored as well as the data downloaded during repair in some other \(\ell'\) nodes. An \{\ell, \ell'\} secure distributed storage system is one in which such an eavesdropper obtains no information about the message.

We describe the construction method of the \{\ell, \ell'\} secure code [5 Sec. IV-B]. The form of the message matrix of the \(n, k, d\) PM MSR code is a \((2\alpha \times \alpha)\) matrix \([S_1 \ S_2]\), where \(S_1\) and \(S_2\) are \((\alpha \times \alpha)\) symmetric matrices. To construct an \{\ell, \ell'\} secure code based on the \((n, k, d = 2k - 2)\) PM MSR code with \((\alpha = k - 1, \beta = 1, B = k\alpha)\) over \(F_q\), in the input to the \((n, k, 2k - 2)\) PM MSR code (without secrecy), a specific, carefully chosen set of \(R = \ell\alpha + (k - \ell)\ell'\) message symbols in the \((2\alpha \times \alpha)\) message matrix are replaced with \(R\) random symbols as follows. Each of these random symbols are chosen uniformly and independently from \(F_q\), and are also independent of the message symbols. Use these \(R\) random symbols to replace the following \(R\) symbols in the \((2\alpha \times \alpha)\) message matrix \([S_1 \ S_2]\), to obtain matrix \(M^{(s)}\) of the \{\ell, \ell'\} secure code:

- the \(\ell\alpha - \binom{\ell}{2}\) symbols in the first \(\ell\) rows (and hence the first \(\ell\) columns) of the \((\alpha \times \alpha)\) symmetric matrix \(S_1\),
- the \(\binom{\ell}{2}\) symbols in the intersection of the first \((\ell - 1)\) rows and the first \((\ell - 1)\) columns of the \((\alpha \times \alpha)\) symmetric matrix \(S_2\),
- the \((k - \ell)\ell'\) remaining symbols in the first \(\ell'\) rows (and hence the first \(\ell'\) columns) of \(S_2\).
For each $i, 1 \leq i \leq n$, the data stored in node $i$, i.e., the share $\mathcal{C}_i$ of node $i$, is given by $\mathcal{C}_i = [1, x_i, x_i^2, \cdots, x_i^{2^\alpha-1}]M^{(s)}$, where $[1, x_i, x_i^2, \cdots, x_i^{2^\alpha-1}]$ is identical with the coding vector $\rho$ in the case of Type I.

The $\{\ell, \ell'\}$ secure codes guarantee the following secrecy. Let $\mathcal{U}$ denote the collection of the $(B - R)$ message symbols, and let $\mathcal{R}$ denote the collection of $R$ random symbols. Further, let $\mathcal{E}$ denote the collection of symbols that an eavesdropper gains access to. And then, the eavesdropper obtains no information about the message from the data stored in some $(\ell - \ell')$ nodes and the data stored as well as the data downloaded during repair in some other $\ell'$ nodes, that is, the mutual information between the message symbols $\mathcal{U}$ and the symbols $\mathcal{E}$ obtained by the eavesdropper is zero, i.e., $I(\mathcal{U}; \mathcal{E}) = 0$, where $I(\mathcal{U}; \mathcal{E})$ denotes a mutual information between $\mathcal{U}$ and $\mathcal{E}$, and all logarithms are taken to the base $q$.

B. $\{\ell, \ell'\}$ Secure Codes based on Generalized RSK-MSR Codes

In the cases of Type II and III, we show how to apply the construction method of $\{\ell, \ell'\}$ secure code [Sec. IV-B] to the $(n, k, d)$ generalized RSK-MSR code with the $(d \times \alpha)$ message matrix $M$. To construct an $\{\ell, \ell'\}$ secure code based on the $(n, k, d)$ generalized RSK-MSR code, which has the message matrix of Type II and III, we use $R = \ell\alpha + (k - \ell)\ell'$ random symbols and $(B - R)$ message symbols. Use these $R$ random symbols to replace the following $R$ symbols in the $(d \times \alpha)$ message matrix $M$, to obtain matrix $M^{(s)}$ of the $\{\ell, \ell'\}$ secure code:

- the $\ell\alpha - \binom{\ell}{2}$ symbols in the first $\ell$ rows (and hence the first $\ell$ columns) of the $(\alpha \times \alpha)$ symmetric matrix $W_1$,
- the $\binom{\ell}{2}$ symbols in the intersection of the first $(\ell - 1)$ rows and the first $(\ell - 1)$ columns of the $((k - 1) \times (k - 1))$ symmetric matrix $T_2$,
- the $(k - \ell)\ell'$ remaining symbols in the first $\ell'$ rows (and hence the first $\ell'$ columns) of $T_2$.

For each $i, 1 \leq i \leq n$, the share $\mathcal{C}_i$ stored in node $i$ is given by $\mathcal{C}_i = \rho M^{(s)}$.

C. Relation between message matrices of $\{\ell = k, \ell' = 0\}$ secure code and $(n, k, d)$ Generalized RSK-MSR Code

We explain that the $(d \times \alpha)$ message matrix $M$ of an $(n, k, d)$ generalized RSK-MSR code is derived from that of the PM MSR code by using the construction method of an $\{\ell = k, \ell' = 0\}$ secure code.

To construct the $(d \times \alpha)$ message matrix $M$ of Type II, III, IV and V, we prepare the $(2\alpha \times \alpha)$ message matrix $\begin{bmatrix} S_1' \ S_2' \end{bmatrix}$ of an $(\tilde{n}, \tilde{k}, \tilde{d})$ PM MSR code with $(\tilde{\alpha}, \tilde{\beta} = 1, \tilde{B})$ such that $\tilde{d} = 2\tilde{k} - 2$, where $\tilde{k} = d - k + 2$. From the condition $\tilde{k} = d - k + 2$, it holds that $\tilde{k} = \alpha + 1$, $\tilde{d} = 2\alpha$ and $\tilde{B} = \tilde{k}\tilde{\alpha} = \alpha(\alpha + 1) \geq B$ because $\alpha = d - k + 1$ and $d \geq 2k - 2$, where $\alpha = d - k + 1$ and $B = k\alpha$. The $(\tilde{n}, \tilde{k}, \tilde{d})$ PM MSR code is the underlying code of the $\{\ell = k, \ell' = 0\}$ secure code.

Firstly, by using the construction method of $\{\ell = k, \ell' = 0\}$ secure code, we replace the following $B$ symbols in the $(2\alpha \times \alpha)$ matrix $\begin{bmatrix} S_1' \ S_2' \end{bmatrix}$ with $B$ message symbols as follows:

1) In the cases of Type II and III (i.e., $2 \leq k \leq \alpha$),
   - $B - \binom{k}{2}$ symbols in the first $k$ rows of the $(\alpha \times \alpha)$ symmetric matrix $S_1$,
   - the $\binom{k}{2}$ symbols in the intersection of the first $(k - 1)$ rows and the first $(k - 1)$ columns of the $(\alpha \times \alpha)$ symmetric matrix $S_2$. 
2) In the cases of Type IV and V (i.e., \( k = 1 \)), \( B \) symbols in the first row of the \( (\alpha \times \alpha) \) symmetric matrix \( S_1 \), where \( B = d = \alpha \).

Next, we replace the remaining \((\hat{B} - B)\) symbols in the \((2\alpha \times \alpha)\) matrix \([ \hat{S}_1 \]

\( S_2 \]) with \((\hat{B} - B)\) zeros.

As a result, we obtain the matrix \([ \hat{S}_1 \]

\( S_2 \]) such that the \(( (\alpha - k + 1) \times \alpha) \) lower sub-matrix of \([ \hat{S}_1 \]

\( S_2 \]) is a zero matrix and the \((d \times \alpha)\) upper sub-matrix of \([ \hat{S}_1 \]

\( S_2 \]) is identical with the \((d \times \alpha)\) message matrix \( M \) of the \((n, k, d)\) generalized RSK-MSR code.

V. CONCLUSION

We have proposed the construction of \((n, k, d)\) MSR codes for arbitrary \((n, k, d)\) such that \( d \geq 2k - 2 \) and \( d \leq n - 1 \). The proposed MSR code is the generalized version of the Rashmi-Shah-Kumar Minimum-Storage-Regenerating(RSK-MSR) code based on the product-matrix framework. In addition, we have described the relations between the generalized RSK-MSR codes and \{\ell, \ell'\} secure codes.

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