On purely real surfaces in Kaehler surfaces
and Lorentz surfaces in Lorentzian Kaehler surfaces

Bang-Yen Chen

Abstract

An immersion $\phi: M \rightarrow \tilde{M}$ of a manifold $M$ into an indefinite Kaehler manifold $\tilde{M}$ is called purely real if the almost complex structure $J$ on $\tilde{M}$ carries the tangent bundle of $M$ into a transversal bundle. In this article we survey some recent results on purely real surfaces in Kaehler surfaces as well as on Lorentz surfaces in Lorentzian Kaehler surfaces.

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1 Introduction.

Let $\tilde{M}^n$ be a complex $n$-dimensional indefinite Kaehler manifold, that means $\tilde{M}^n$ is endowed with an almost complex structure $J$ and with an indefinite Riemannian metric $\tilde{g}$, which is $J$-Hermitian, i.e., for all $p \in \tilde{M}^n$, we have

$$\tilde{g}(JX, JY) = \tilde{g}(X, Y), \; \forall X, Y \in T_p\tilde{M}^n,$$

$$\tilde{\nabla}J = 0,$$

where $\tilde{\nabla}$ is the Levi-Civita connection of $\tilde{g}$. It follows that $J$ is integrable. The complex index of $\tilde{M}^n$ is defined as the complex dimension of the largest complex negative definite subspace of the tangent space.

When the complex index is one, the indefinite Kaehler manifold $\tilde{M}^n$ is called a Lorentzian Kaehler manifold.

Key words and phrases. Purely real surfaces; fundamental equations; minimal surface; Lorentz surface; Kaehler surface; Lorentzian Kaehler surface; parallel surface; optimal inequality.

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Let \( \tilde{M}_n^{(4c)} \) denote a complete simply-connected indefinite complex space form of constant holomorphic sectional curvature \( 4c \). The curvature tensor \( \tilde{R} \) of \( \tilde{M}_n^{(4c)} \) is given by

\[
\tilde{R}(X, Y)Z = c\{\langle Y, Z \rangle X - \langle X, Z \rangle Y + \langle JY, Z \rangle JX - \langle JX, Z \rangle JY + 2\langle X, JY \rangle JZ\}.
\] (1.3)

An immersion \( \phi: M \rightarrow \tilde{M} \) of a manifold \( M \) into an indefinite Kaehler manifold \( \tilde{M} \) is called purely real if the almost complex structure \( J \) on the ambient space \( \tilde{M} \) carries the tangent bundle of \( M \) into a transversal bundle, that is \( J(TM) \cap TM = \{0\} \). Thus, if an immersion \( \phi: M \rightarrow \tilde{M} \) is purely real, it contains no complex points.

The simplest examples of purely real submanifolds are Lagrangian surfaces and proper slant surfaces.

In this article we survey recent results on purely real surfaces in Kaehler surfaces as well as on Lorentz surfaces in Lorentzian Kaehler surfaces.

We divide this article into sixteen sections as follows:
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2. Basic formulas and fundamental equations.
3. Basics on purely real surfaces.
4. Dependence of fundamental equations
5. Purely real minimal surfaces.
6. A general optimal inequality for purely real surfaces.
7. Purely real surfaces in \( \mathbb{C}^2 \) satisfying the equality.
8. Purely real surfaces in \( \mathbb{C}P^2 \) and \( \mathbb{C}H^2 \) satisfying the equality.
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11. Minimal Lorentz surfaces in \( \mathbb{C}^2 \).
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15. Parallel Lorentz surfaces in \( \mathbb{C}P^2 \).
16. Parallel Lorentz surfaces in \( \mathbb{C}H^2 \).

2 Basic formulas and fundamental equations.

Let \( \tilde{M}^2 \) be an indefinite Kaehler surface. Denote by \( \tilde{R} \) the Riemann-Christoffel curvature tensor of \( \tilde{M}^2 \). Assume that \( M \) is a non-degenerate surface in \( \tilde{M}^2 \) in the sense that the induced metric on \( M \) is non-degenerate. Thus, \( M \) is either space-like or Lorentzian.
Denote by $\nabla$ and $\tilde{\nabla}$ the Levi Civita connections on $M$ and $\tilde{M}^2$, respectively. The formulas of Gauss and Weingarten are given by (cf. [6, 45])

$$\tilde{\nabla} X Y = \nabla X Y + h(X,Y), \quad (2.1)$$

$$\tilde{\nabla} X \xi = -A \xi X + D X \xi \quad (2.2)$$

for tangent vector fields $X, Y$ and a normal vector field $\xi$, where $h$, $A$ and $D$ are the second fundamental form, the shape operator and the normal connection.

For each $\xi \in T^\perp p M$, the shape operator $A\xi$ is a symmetric endomorphism of the tangent space $T_p M$ at $p \in M$.

The shape operator and the second fundamental form are related by

$$\langle h(X,Y), \xi \rangle = \langle A\xi X, Y \rangle \quad (2.3)$$

for $X, Y$ tangent to $M$ and $\xi$ normal to $M$. The mean curvature vector is defined by

$$\vec{H} = \frac{1}{2} \text{trace} h. \quad (2.9)$$

The ellipse of curvature of a surface $M$ in a Kaehler surface $\tilde{M}^2$ is the subset of the normal plane defined as

$$\{ h(v,v) \in T^\perp_p M : |v| = 1, v \in T_p M, p \in M \}. \quad (2.11)$$

To see that it is an ellipse, we consider an arbitrary orthogonal tangent frame $\{e_1, e_2\}$. Put $h_{ij} = h(e_i, e_j), i, j = 1, 2$, and look at the following formula

$$h(v,v) = \vec{H} + \frac{h_{11} - h_{22}}{2} \cos 2t + h_{12} \sin 2t \quad (2.11)$$
for \( v = \cos te_1 + \sin te_2 \). As \( v \) goes once around the unit tangent circle, \( h(v, v) \) goes twice around the ellipse. The ellipse of curvature could degenerate into a line segment or a point.

The center of the ellipse is \( \overrightarrow{H} \). The ellipse of curvature is a circle if and only if the following two conditions hold:

\[
|h_{11} - h_{22}|^2 = 4|h_{12}|^2, \quad \langle h_{11} - h_{22}, h_{12} \rangle = 0.
\] (2.12)

The property that the ellipse of curvature is a circle is a conformal invariant.

3 Basics on purely real surfaces.

An immersion \( \phi: M \rightarrow \tilde{M}^2 \) of a surface \( M \) into a Kaehler surface is called purely real if the almost complex structure \( J \) on \( \tilde{M}^2 \) carries the tangent bundle of \( M \) into a transversal bundle (cf. [12], see also [45]).

A point \( p \) on a purely real surface \( M \) is called a Lagrangian point if \( J \) carries the tangent space \( T_p M \) into its normal space \( T_p^\perp M \). A purely real surface \( M \) is called Lagrangian (or totally real in the sense of [30]) if the almost complex structure \( J \) on \( \tilde{M}^2 \) carries the tangent bundle of \( M \) into its normal bundle.

For each tangent vector \( X \) of a purely real surface \( M \), we put

\[
JX = PX + FX,
\] (3.1)

where \( PX \) and \( FX \) are the tangential and the normal components of \( JX \).

For an oriented orthonormal frame \( \{e_1, e_2\} \), it follows from (3.1) that

\[
P e_1 = (\cos \alpha) e_2, \quad P e_2 = -(\cos \alpha) e_1
\] (3.2)

for some function \( \alpha \). This function \( \alpha \) is known as the Wirtinger angle. It is easy to see that the Wirtinger angle is independent of the choice of \( e_1, e_2 \) which preserves the orientation.

A purely real surface is called a slant surface if its Wirtinger angle \( \alpha \) is constant. For slant surfaces the Wirtinger angle \( \alpha \) is called the slant angle (cf. [7]).

4 Dependence of fundamental equations.

The three fundamental equations of Gauss, Codazzi and Ricci provide the necessary conditions for local isometric embeddability. These three fundamental equations also play some important roles in physics; in particular in the Kaluza-Klein theory in general relativity theory (cf. [39, 42, 47]).

The three fundamental equations of Gauss, Codazzi and Ricci are independent in general. Recently, I am able to prove the following general property for purely real surfaces in an arbitrary Kaehler surface [21].
Theorem 4.1. The equation of Ricci is a consequence of the equations of Gauss and Codazzi for any purely real surface in any Kaehler surface.

We also have the following.

Theorem 4.2. The equation of Gauss is a consequence of the equations of Codazzi and Ricci for purely real surfaces in any Kaehler surfaces.

5 Purely real minimal surfaces in Kaehler surfaces.

The following result from [21] provides a necessary condition for purely real surfaces in complex space forms to be minimal.

Theorem 5.1. Let \( M \) be a purely real surface in a complex space form \( \tilde{M}^2(4c) \) of constant holomorphic sectional curvature \( 4c \). If \( M \) is minimal, then the Wirtinger angle \( \alpha \) of \( M \) satisfies

\[
\Delta \alpha = \left\{ \| \nabla \alpha \|^2 + 6c \sin^2 \alpha \right\} \cot \alpha,
\]

where \( \Delta \) is the Laplace operator of \( M \) and \( \nabla \alpha \) is the gradient of \( \alpha \).

Some easy consequences of Theorem 5.1 are the following (see [21]).

Corollary 5.1. Every slant surface in a complex space form \( \tilde{M}^2(4c) \) with \( c \neq 0 \) is non-minimal unless it is either Lagrangian or complex.

Corollary 5.2. Every compact oriented minimal purely real surface in the complex projective plane \( \mathbb{C}P^2(4) \) contains some Lagrangian points.

Corollary 5.3. Let \( M \) be a purely real minimal surface in \( \mathbb{C}^2 \). If the Wirtinger angle \( \alpha \) is a harmonic function, then \( M \) is slant.

Corollary 5.4. Let \( M \) be a purely real minimal surface in \( \mathbb{C}P^2(4) \). If the Wirtinger angle \( \alpha \) is a harmonic function, then \( M \) is Lagrangian.

A function \( f \) on \((M, g)\) is called subharmonic if \( \Delta f \geq 0 \) holds everywhere on \( M \). The surface \( M \) is called parabolic if there exists non non-constant negative subharmonic function.

From Theorem 5.1 we also have the following.

Corollary 5.5. Let \( M \) be an oriented minimal purely real surface in \( \mathbb{C}P^2(4) \). If \( M \) is parabolic, then \( M \) contains some Lagrangian points.
6 A general optimal inequality for purely real surfaces.

For purely real surfaces in complex space forms, we have the following general optimal inequality proved in [21].

**Theorem 6.1.** Let $M$ be a purely real surface in a complex space form $\tilde{M}^2(4c)$. Then we have

$$H^2 \geq 2\{K - \|\nabla \alpha\|^2 - (1 + 3 \cos^2 \alpha)c\} + 4 \langle \nabla \alpha, Jh(e_1, e_2) \rangle \csc \alpha$$

(6.1)

with respect to any orthonormal frame $\{e_1, e_2\}$ satisfying $\langle \nabla \alpha, e_2 \rangle = 0$, where $H^2$ and $K$ denote the squared mean curvature and the Gauss curvature of $M$, respectively.

The equality case of (6.1) holds at $p$ if and only if, with respect a suitable adapted orthonormal frame $\{e_1, e_2, e_3, e_4\}$, the shape operators of $M$ at $p$ take the following forms:

$$A_{e_3} = \begin{pmatrix} 3\varphi & \delta \\ \delta & \varphi \end{pmatrix}, \quad A_{e_4} = \begin{pmatrix} \delta + e_1\alpha & \varphi \\ \varphi & 3\delta + 3e_1\alpha \end{pmatrix}.$$  

(6.2)

When $M$ is a slant surface in $\tilde{M}^2(4c)$, Theorem 6.1 reduces to the following.

**Corollary 6.1.** [11] If $M$ is a slant surface in a complex space form $\tilde{M}^2(4c)$ with slant angle $\theta$, then we have

$$H^2 \geq 2\{K - (1 + 3 \cos^2 \theta)c\}.$$  

(6.3)

The equality case of (6.3) holds at $p$ if and only if, with respect a suitable adapted orthonormal frame $\{e_1, e_2, e_3, e_4\}$, the shape operators of $M$ at $p$ take the following forms:

$$A_{e_3} = \begin{pmatrix} 3\varphi & \delta \\ \delta & \varphi \end{pmatrix}, \quad A_{e_4} = \begin{pmatrix} \delta & \varphi \\ \varphi & 3\delta \end{pmatrix}.$$  

(6.4)

The following lemma follows easily from Corollary 6.1 and (2.11).

**Lemma 6.1.** If a slant surface in a complex space form $\tilde{M}^2(4c)$ satisfies the equality case of inequality (6.3), then $M$ has circular ellipse of curvature.

When the ambient space is $\mathbb{C}^2$, Theorem 6.1 reduces to the following.

**Corollary 6.2.** Let $M$ be a purely real surface in $\mathbb{C}^2$. Then we have

$$H^2 \geq 2\{K - \|\nabla \alpha\|^2 + 2 \langle \nabla \alpha, Jh(e_1, e_2) \rangle \csc \alpha\}$$

with respect to an orthonormal frame $\{e_1, e_2\}$ satisfying $\langle \nabla \alpha, e_2 \rangle = 0$.

The equality case of (6.5) holds if and only if, with respect a suitable adapted orthonormal frame $\{e_1, e_2, e_3, e_4\}$, the shape operators of $M$ take the following forms:

$$A_{e_3} = \begin{pmatrix} 3\varphi & \delta \\ \delta & \varphi \end{pmatrix}, \quad A_{e_4} = \begin{pmatrix} \delta + e_1\alpha & \varphi \\ \varphi & 3\delta + 3e_1\alpha \end{pmatrix}.$$  

(6.6)
Example 6.1. Let $\alpha(x)$ and $f(y)$ be non-constant real-valued functions and $b$ is a nonzero real number. Consider the map:

$$L(x, y) = \left(be^{-ib^{-1}f(y)} \cot \alpha(x), f(y)\right). \quad (6.7)$$

Then the induced metric via (6.7) is given by

$$g = b^2 \alpha'^2 \csc^4 \alpha dx \otimes dx + f'^2(y) \csc^2 \alpha dy \otimes dy,$$

and whose Gauss curvature is $K = -b^{-2}\sin^4 \alpha$. Moreover, it is direct to show that (6.7) defines a purely real minimal surface with Wirtinger angle $\alpha$ which satisfies the equality case of (6.5).

Example 6.2. Let $\alpha(x)$ and $f(y)$ be non-constant real-valued functions defined on open intervals $I_1, I_2$ respectively and $b$ is a nonzero real number. Consider $M = (I_1 \times I_2, g)$ equipped with metric:

$$g = b^2 \alpha'^2(x) \sin^4 \alpha(x) dx \otimes dx + f'^2(y) \sin^6 \alpha(x) dy \otimes dy. \quad (6.8)$$

Then the map:

$$\phi(x, y) = \frac{b}{12} \left(4e^{3ib^{-1}f(y)} \sin^2 \alpha(x), \cos(3\alpha(x)) - 9 \cos \alpha(x)\right), \quad (6.9)$$

defines a purely real isometric immersion of $M$ into $\mathbb{C}^2$ whose Wirtinger angle is $\alpha$. Moreover, it is direct to show that the squared mean curvature $H^2$, Gauss curvature $K$, the gradient of $\alpha$, and the second fundamental form $h$ of $M$ satisfy

$$H^2 = \frac{4}{b^2 \sin^4 \alpha}, \quad K = \frac{3}{b^2 \sin^4 \alpha}, \quad ||\nabla \alpha||^2 = \frac{1}{b^2 \sin^4 \alpha}, \quad h \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = 0. \quad (6.10)$$

Hence, this purely real surface satisfies the equality case of (6.5).

Example 6.3. Let $w : S^2 \to \mathbb{C}^2$ be the map defined by

$$w(y_0, y_1, y_2) = \frac{1 + iy_0}{1 + y_0^2} (ry_1, ry_2), \quad y_0^2 + y_1^2 + y_2^2 = 1,$$

where $r$ is a positive real number. Then $w$ is a Lagrangian immersion of the 2-sphere $S^2$ into $\mathbb{C}^2$ which is called the Whitney sphere. It is well-known that, up to rigid motions, the Whitney sphere is the only Lagrangian surface in $\mathbb{C}^2$ satisfying the equality case of (6.5) (cf. [4, 9]).
7 Purely real surfaces in $\mathbb{C}^2$ satisfying the equality.

A purely real surface $M$ in a Kaehler surface is said to have full second fundamental form if its first normal space, $\text{Im} h$, satisfies $\dim (\text{Im} h) = 2$ at each point in $M$. It is said to have degenerate second fundamental form if $\dim (\text{Im} h) < 2$ holds at each point in $M$.

The following result from [21] classifies purely real minimal surfaces in $\mathbb{C}^2$ which satisfy the equality case of the inequality (6.5).

**Theorem 7.1.** If $M$ is a purely real minimal surface in $\mathbb{C}^2$ satisfying the equality case of (6.5), then either $M$ is an open portion of a totally geodesic slant plane or it is congruent to an open portion of a negatively curved surface defined by

$$L(x, y) = \left( be^{-ib^{-1}f(y)} \cot \alpha(x), f(y) \right),$$  \hspace{1cm} (7.1)

where $\alpha$ is the Wirtinger angle, $f$ is a non-constant real-valued function, and $b$ is a nonzero real number.

Also, we have following two classification theorems from [21].

**Theorem 7.2.** Let $M$ be a purely real surface in $\mathbb{C}^2$ satisfying the equality case of (6.5). If $M$ has circular ellipse of curvature, then $M$ is either an open portion of a totally geodesic slant plane or an open portion of a Whitney sphere.

**Theorem 7.3.** Let $M$ be a purely real surface in $\mathbb{C}^2$ satisfying the equality case of (6.5). If $M$ has degenerate second fundamental form, then $M$ is congruent to an open portion of one of the following three types of surfaces:

1. A totally geodesic slant plane.
2. A positively curved surface with Wirtinger angle $\alpha$ defined by

$$L(x, y) = \left( b \frac{1}{12} \left( 4e^{3ib^{-1}f(y)} \sin^2 \alpha(x), \cos(3\alpha(x)) - 9 \cos \alpha(x) \right) \right),$$

where $\alpha(x)$ and $f(y)$ are non-constant real-valued functions and $b$ is a nonzero real number.

3. A negatively curved surface with Wirtinger angle $\alpha$ defined by

$$L(x, y) = \left( be^{-ib^{-1}f(y)} \cot \alpha(x), f(y) \right),$$

where $\alpha(x)$ and $f(y)$ are non-constant real-valued functions and $b$ is a nonzero real number.
Purely real surfaces in $CP^2$ and $CH^2$ satisfying the equality.

For purely real surfaces in non-flat complex space form $\tilde{M}^2(4c)$, $c = 1$ or $-1$, we have the following three classification results of B. Y. Chen, A. Mihai and I. Mihai [28].

**Theorem 8.1.** Let $M$ be a purely real minimal surface of a complex space form $\tilde{M}^2(4c)$ with $c = 1$ or $-1$. If $M$ satisfies the basic equality, then we have either

1. $c = 1$ and $M$ is a totally geodesic Lagrangian surface, or
2. $c = -1$ and $M$ is a totally geodesic Lagrangian surface in $CH^2(-4)$, or
3. $c = -1$ and $M$ is congruent to an open portion of a non-slant surface in $CH^2(-4)$ given by the composition $\pi \circ \phi$, where $\pi : H^5_1 \to CH^2(-4)$ is the hyperbolic Hopf fibration and $\phi : R^3 \to H^5_1 \subset C^3$ is

$$
\phi(x, v, t) = e^{i\frac{1+iv}{\sqrt{3}}} \left(3i\sqrt{2}\sinh\left(\frac{\sqrt{3}u}{\sqrt{2}}\right) + \sqrt{3}\left(\sqrt{2} + 2e^{2x}\right)\cosh\left(\frac{\sqrt{3}u}{\sqrt{2}}\right),
3e^{2x}\cosh\left(\frac{\sqrt{3}u}{\sqrt{2}}\right) + i\sqrt{3}\left(2\sqrt{2} + e^{2x}\right)\sinh\left(\frac{\sqrt{3}u}{\sqrt{2}}\right), \frac{\sqrt{3}}{e^{3i(u)/\sqrt{2}}}\left(\sqrt{2} - e^{2x}\right) \right).
$$

This purely real surface has Wirtinger angle $\alpha = \arctan(e^{3x})$.

**Theorem 8.2.** Let $M$ be a purely real surface with circular ellipse of curvature in a complex space form $\tilde{M}^2(4c)$, $c = 1$ or $-1$. If $M$ satisfies the basic equality, then we have either

1. $M$ is a Lagrangian surface satisfying the equality

$$
H^2 = 2K - 2c
$$

identically, or
2. $c = -1$ and $M$ is congruent to an open portion of a proper slant surface in $CH^2(-4)$ given by the composition $\pi \circ \phi$, where $\pi : H^5_1 \to CH^2(-4)$ is the Hopf fibration and $\phi : M \to H^5_1 \subset C^3$ is

$$
\phi(u, v, t) = e^{it} \left(\frac{3}{2}\cosh av + \frac{1}{6}u^2e^{-av} - \frac{i}{6}\sqrt{6u}(1 + e^{-av}) - \frac{1}{2},
\frac{1}{3}(1 + 2e^{-av})u + i\sqrt{6}\left(-\frac{1}{3} + \frac{1}{4}e^{av} + e^{-av}\left(\frac{1}{12} + \frac{1}{18}u^2\right)\right),
\frac{\sqrt{3}}{6}(1 - e^{-av})u + i\sqrt{3}\left(\frac{1}{6} + \frac{1}{4}e^{av} + e^{-av}\left(-\frac{5}{12} + \frac{1}{18}u^2\right)\right) \right),
$$

with $a = \sqrt{2/3}$. 

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Theorem 8.3. Let $M$ be a purely real surface satisfying the basic equality in a complex space form $\tilde{M}^2(4c)$ with $c = \pm 1$. If $M$ has degenerate second fundamental form, then either

(i) $M$ is a totally geodesic Lagrangian surface or
(ii) $c = -1$ and $M$ is congruent to an open portion of the surface in $CH^2(-4)$ given by $\pi \circ \phi$, where $\pi : H^5_1 \to CH^2(-4)$ is the hyperbolic Hopf fibration and $\phi : \mathbb{R}^3 \to H^5_1 \subset \mathbb{C}^3$ is

$$
\phi(u,v,t) = \frac{e^{it+iv/\sqrt{2}}}{3\sqrt{4e^u}} \left( 3i\sqrt{2}\sinh\left(\frac{\sqrt{3}u}{\sqrt{2}}\right) + \sqrt{3}(\sqrt{2} + 2e^{2u}) \cosh\left(\frac{\sqrt{3}u}{\sqrt{2}}\right),
3e^{2u} \cosh\left(\frac{\sqrt{3}u}{\sqrt{2}}\right) + i\sqrt{3}(2\sqrt{2} + e^{2u}) \sinh\left(\frac{\sqrt{3}u}{\sqrt{2}}\right),
\frac{\sqrt{3}}{e^{3iv/\sqrt{2}}} \left(\sqrt{2} - e^{2u}\right) \right).
$$

Remark 8.1. Lagrangian surfaces in complex space form $\tilde{M}^2(4c)$ with $c = 1$ or $c = -1$ satisfying equality (8.1) have been completely classified in [4, 8, 33]. It follows from Lemma 6.1 that such surfaces have circular ellipse of curvature.

9 Two general properties for Lorentz surfaces.

For arbitrary Lorentz surfaces in arbitrary Lorentzian Kaehler surfaces we have the following two general results.

Theorem 9.1. [16] Every Lorentz surface in an arbitrary Lorentzian Kaehler surface is purely real.

Theorem 9.2. [22] The equation of Ricci is a consequence of the equations of Gauss and Codazzi for any Lorentz surface in any Lorentzian Kaehler surface.

Remark 9.1. The converse of Theorem 9.1 is false. In fact, not every purely real surface in a Lorentzian Kaehler surface is Lorentzian. A simple such example is the slant plane in $C^2_1$ with slant angle $\theta = \cosh^{-1} \sqrt{1 + b^2}$ defined by

$$
L(x,y) = \left( by, x + i\sqrt{1 + b^2}y \right).
$$

where $b$ is a nonzero real number. It is easy to see that this slant plane is a space-like surface whose metric tensor is given by $g = dx^2 + dy^2$.

Remark 9.2. Space-like surfaces in Lorentzian Kaehler surfaces may contain complex points. For instance,

$$
L(x,y) = (0, x + iy)
$$

is a space-like complex plane in $C^2_1$.

Remark 9.3. Theorem 9.2 is false in general if the Lorentz surface were replaced by a spatial surface in a Lorentzian Kaehler surface.
10 Lorentzian complex space forms and Legendre curves.

10.1 Lorentzian complex space forms

Let $\mathbb{C}^n$ denote the complex $n$-plane with complex coordinates $z_1, \ldots, z_n$. The $\mathbb{C}^n$ endowed with $g_{i,n}$, i.e., the real part of the Hermitian form

$$b_{i,n}(z,w) = -\sum_{k=1}^{i} \bar{z}_k w_k + \sum_{j=i+1}^{n} \bar{z}_j w_j, \quad z, w \in \mathbb{C}^n,$$

defines a flat indefinite complex space form with complex index $i$. We simply denote the pair $(\mathbb{C}^n, g_{i,n})$ by $\mathbb{C}_i^n$.

Consider the differentiable manifold:

$$S_{2}^{2n+1} = \{ z \in \mathbb{C}_{n+1}^{n+1} ; b_{1,n+1}(z,z) = 1 > 0 \},$$

which is an indefinite real space form of constant sectional curvature one. The Hopf fibration

$$\pi : S_{2}^{2n+1} \rightarrow CP_1^n(4) : z \mapsto z \cdot \mathbb{C}^*$$

is a submersion and there exists a unique pseudo-Riemannian metric of complex index one on $CP_1^n(4)$ such that $\pi$ is a Riemannian submersion.

The pseudo-Riemannian manifold $CP_1^n(4)$ is a Lorentzian complex space form of constant holomorphic sectional curvature 4.

Analogously, consider

$$H_{2}^{2n+1} = \{ z \in \mathbb{C}_{n+1}^{n+1} ; b_{2,n+1}(z,z) = -1 < 0 \},$$

which is an indefinite real space form of constant sectional curvature $-10$. The Hopf fibration

$$\pi : H_{2}^{2n+1} \rightarrow CH_1^n(-4) : z \mapsto z \cdot \mathbb{C}^*$$

is a submersion and there exists a unique pseudo-Riemannian metric of complex index 1 on $CH_1^n(-4)$ such that $\pi$ is a Riemannian submersion.

The pseudo-Riemannian manifold $CH_1^n(-4c)$ is a Lorentzian complex space form of constant holomorphic sectional curvature $-4$.

It is well-known that a complete simply-connected Lorentzian complex space form $\tilde{M}_1^n(4c)$ is holomorphically isometric to $\mathbb{C}_i^n$, $CP_1^n(4)$, or $CH_1^n(-4)$, according to $c = 0$, $c = 1$ or $c = -1$, respectively (cf. [2]).

10.2 Special Legendre curves in light cone

A vector $v$ is called space-like (respectively, time-like) if $\langle v, v \rangle > 0$ (respectively, $\langle v, v \rangle < 0$). A vector $v$ is called null or light-like if it is a nonzero vector and it satisfies $\langle v, v \rangle = 0$. 
The light cone $L_C$ in $\mathbb{C}_i^n$ ($n \geq 3$, $i = 1, 2$) is defined by

$$L_C = \{ z \in \mathbb{C}_i^n : \langle z, z \rangle = 0 \}.$$ 

A unit speed curve $z(s)$ lying in $L_C$ is called Legendre if $\langle iz', z \rangle = 0$ holds identically. For a unit speed Legendre curve $z$ in $L_C$, we have

$$\langle z, z \rangle = \langle z, iz' \rangle = \langle iz, z'' \rangle = \langle z', z'' \rangle = 0.$$ 

The Legendre curve $z$ in $L_C$ is called special Legendre if $\langle iz', z'' \rangle = 0$ holds.

The squared curvature $\kappa^2$ of a unit speed special Legendre curve $z$ is defined by

$$\kappa^2 = \langle z'', z'' \rangle$$

and its Legendre torsion $\hat{\tau}$ is defined by

$$\hat{\tau} = \epsilon_z \langle z'', iz''' \rangle,$$

where $\epsilon_z = 1$ or $-1$ according to $z$ is space-like or time-like (cf. [13, 14]).

11 Minimal Lorentz surfaces in $\mathbb{C}^2_1$.

Lagrangian surfaces in $\mathbb{C}^2_1$ are Lorentz surfaces automatically. Minimal flat Lagrangian surfaces in the Lorentzian complex plane $\mathbb{C}^2_1$ have been classified by Chen and L. Vrancken in [34]. On the other hand, Vrancken proved in [49] that every Lagrangian minimal surface of constant curvature in $\mathbb{C}^2_1$ is a flat surface (see, also [40]).

For minimal flat Lorentzian surfaces in the Lorentzian complex plane $\mathbb{C}^2_1$, we have the following result from [16], which can be consideration of an extension of Chen-Vrancken’s result in [34].

**Theorem 11.1.** Let $\alpha(y)$ and $f(y)$ be two arbitrary differentiable functions of single variable defined on an open interval $I \ni 0$. Then

$$\psi(x, y) = \left( x + iyf(y) + \frac{1}{2} \int_0^y \cosh^2 ady - \int_0^y f'(y) \sinh ady, \right.$$ 

$$x - y + iyf(y) + \frac{1}{2} \int_0^y \cosh^2 ady - \int_0^y f'(y) \sinh ady - i \int_0^y \sinh ady \right)$$

defines a minimal flat Lorentzian surface in the Lorentzian complex plane $\mathbb{C}^2_1$ with $\alpha$ as its Wirtinger angle.

Conversely, every minimal flat Lorentzian surface in $\mathbb{C}^2_1$ is either an open portion of a totally geodesic Lorentzian plane or congruent to the Lorentzian surface described above.
The following result from \[17\] determines all Lagrangian minimal surfaces in \(\mathbb{C}^2_1\) which are free from flat points.

**Theorem 11.2.** Let \(M\) be a Lagrangian minimal surface in the Lorentzian complex plane \(\mathbb{C}^2_1\). If \(M\) contains no flat points, then \(M\) is congruent to an open portion of the Lagrangian surface defined by

\[
L(x, y) = \left( \int_{x_0}^{x} \frac{f(x)dx}{\sqrt{f'(x)}} - \frac{i}{2} \int_{x_0}^{x} \frac{dx}{\sqrt{f'(x)}} - i b \int_{y_0}^{y} \frac{k(y)dy}{\sqrt{k'(y)}} - \frac{1}{2b} \int_{y_0}^{y} \frac{dy}{\sqrt{k'(y)}} \right),
\]

where \(b\) is a nonzero real number, \(f(x)\) is a differentiable function with \(f'(x) > 0\) on an open interval \(I_1 \ni x_0\), and \(k(y)\) is a differentiable function with \(k'(y) > 0\) on an open interval \(I_2 \ni y_0\).

The following result from \[19\] completely classifies Lorentzian minimal surfaces in \(\mathbb{C}^2_1\).

**Theorem 11.3.** Let \(z(x)\) and \(w(y)\) be two null curves defined on open intervals \(I_1\) and \(I_2\) respectively in the Lorentzian complex plane \(\mathbb{C}^2_1\). If \(\langle z(x), w(y) \rangle \neq 0\) for \((x, y) \in I_1 \times I_2\), then

\[
\psi(x, y) = z(x) + w(y) \tag{11.1}
\]

defines a Lorentzian minimal surface in \(\mathbb{C}^2_1\).

Conversely, locally every Lorentzian minimal surface in \(\mathbb{C}^2_1\) is congruent to a translation surface defined above.

It was proved by Chen and J.-M. Morvan \[29\] that an orientable minimal surface in \(\mathbb{C}^2 = (\mathbb{E}^4, J)\) is a Lagrangian surface if and only if it is a holomorphic curve with respect to some other orthogonal almost complex structure on \(\mathbb{E}^4\) (for a simple alternate proof of this fact, see \[1\]). Using this fact, we know from Theorem 11.3 that the situation of Lagrangian minimal surfaces in the Lorentzian complex plane \(\mathbb{C}^2_1\) is quite different from Lagrangian minimal surface in the complex Euclidean plane \(\mathbb{C}^2\).

## 12 Two existence results on minimal Lorentz surfaces.

We have the following two existence results obtained in \[19\].
Proposition 12.1. Let $F$ be a nonconstant real-valued function defined on a simply-connected open subset $U$ of $\mathbb{R}^2$ which satisfies the following nonlinear Klein-Gordon equation:

$$\left(\ln F\right)_{uv} = -\frac{1}{F} - F^2. \quad (12.1)$$

Put

$$g_F = -F^{-1}(du \otimes dv + dv \otimes du).$$

Then, up to rigid motions, there exists a unique Lagrangian minimal immersion $L_F : (U, g_F) \rightarrow \mathbb{C}P_1^2(4)$ whose second fundamental form satisfies

$$h \left( \frac{\partial}{\partial u}, \frac{\partial}{\partial u} \right) = FJ \frac{\partial}{\partial v}, \quad h \left( \frac{\partial}{\partial u}, \frac{\partial}{\partial v} \right) = 0, \quad h \left( \frac{\partial}{\partial v}, \frac{\partial}{\partial v} \right) = FJ \frac{\partial}{\partial u}.$$

We call such a Lagrangian minimal surface associated with a solution of the nonlinear Klein-Gordon equation (12.1) a Lagrangian minimal surface of Klein-Gordon type in $\mathbb{C}P_1^2(4)$.

Proposition 12.2. Let $P(u,v)$ be a nonconstant real-valued function on a simply-connected open subset $U$ of $\mathbb{R}^2$ which satisfies the nonlinear Klein-Gordon equation:

$$\left(\ln P\right)_{uv} = \frac{1}{P} - P^2. \quad (12.2)$$

Put

$$g_P = -P^{-1}(du \otimes dv + dv \otimes du).$$

Then, up to rigid motions, there exists a unique Lagrangian minimal immersion $L_P : (U, g_P) \rightarrow \mathbb{C}H_1^2(-4)$ whose second fundamental form satisfies

$$h \left( \frac{\partial}{\partial u}, \frac{\partial}{\partial u} \right) = PJ \frac{\partial}{\partial v}, \quad h \left( \frac{\partial}{\partial u}, \frac{\partial}{\partial v} \right) = 0, \quad h \left( \frac{\partial}{\partial v}, \frac{\partial}{\partial v} \right) = PJ \frac{\partial}{\partial u}.$$

Similarly, we call a Lagrangian minimal surface in $\mathbb{C}H_1^2(-4)$ associated with a solution of the nonlinear Klein-Gordon equation (12.2) a Lagrangian minimal surface of Klein-Gordon type in $\mathbb{C}H_1^2(-4)$.

Remark 12.1. The two nonlinear Klein-Gordon equations (12.1) and (12.2) admit infinitely many solutions. Consequently, there are infinitely many Lagrangian minimal surfaces of Klein-Gordon type in $\mathbb{C}P_1^2(4)$ and in $\mathbb{C}H_1^2(-4)$.

13 Minimal slant surfaces in $\mathbb{C}P_1^2$ and in $\mathbb{C}H_1^2$.

The following two theorems classify minimal slant surface in the Lorentzian complex projective and complex hyperbolic planes.
Theorem 13.1. Let $L : M \to \mathbb{CP}^2_1(4)$ be a minimal slant surface in the Lorentzian complex projective plane $\mathbb{CP}^2_1(4)$. Then we have:

1. If $M$ is of constant curvature, then $M$ is congruent to one of the following three types of surfaces:
   - (1.a) a totally geodesic Lagrangian surface of $\mathbb{CP}^2_1(4)$;
   - (1.b) a curvature one Lagrangian minimal surface defined by $\pi \circ \tilde{L}$ with
     $$\tilde{L}(x, y) = z'(x) - \frac{2z(x)}{x + y},$$
     where $z(x), x \in I$, is a unit speed space-like special Legendre curve lying in the light cone $LC \subset \mathbb{C}^3_1$ with null squared curvature $\kappa^2(s)$, i.e., $\langle z''(x), z''(x) \rangle = 0$ on $I$;
   - (1.c) a flat Lagrangian minimal surface defined by $\pi \circ \tilde{L}$ with
     $$\tilde{L}(x, y) = \frac{1}{\sqrt{3}} \left( \sqrt{2} e^{\frac{1}{2a}}(x-a^2y) \cosh \left( \frac{\sqrt{3}}{2a} (x + a^2 y) \right), e^{\frac{1}{2a}}(a^2y-x) \right),$$
     $$\sqrt{2} e^{\frac{1}{2a}}(a^2y-x) \sinh \left( \frac{\sqrt{3}}{2a} (x + a^2 y) \right),$$
     where $a$ is a nonzero real number.
2. If $M$ contains no open subset of constant curvature, then $M$ is a Lagrangian minimal surface of Klein-Gordon type in $\mathbb{CP}^2_1(4)$.

Example 13.1. There exist infinitely many unit speed space-like special Legendre curve lying in the light cone $LC \subset \mathbb{C}^3_1$ with null squared curvature. The simplest such examples are the following.

$$z(x) = \left( a + \left( \frac{1}{4a} + i b \right) s^2, a - \left( \frac{1}{4a} - i b \right) s^2, s \right),$$

where $a, b$ are nonzero real numbers. It is easy to check that this special Legendre curve has null Legendre torsion, i.e., $\hat{\tau} = 0$.

Example 13.2. Another example of unit speed space-like special Legendre curve lying in the light cone $LC \subset \mathbb{C}^3_1$ with null squared curvature is the following.

$$\frac{1}{\sqrt{3}} \left( e^{\frac{i}{2s}} \cosh \left( \frac{\sqrt{3}s}{2} \right) - i \sqrt{3} e^{\frac{i}{2s}} \sinh \left( \frac{\sqrt{3}s}{2} \right), 2 e^{\frac{i}{2s}} \sinh \left( \frac{\sqrt{3}s}{2} \right), \right), e^{-is}.$$

This special Legendre curve has constant Legendre torsion $\hat{\tau} = -1$.

Theorem 13.2. Let $L : M \to CH^2_1(-4)$ be a minimal slant surface in the Lorentzian complex hyperbolic plane $CH^2_1(-4)$. Then we have:
If $M$ is of constant curvature, then $M$ is congruent to one of the following three types of surfaces:

1. a totally geodesic Lagrangian surface of $CH^2_1(-4)$;
2. a Lagrangian minimal surface of constant curvature $-1$ given by $\pi \circ \bar{L}$ with
   \[ \bar{L}(x, y) = z'(x) - \sqrt{2}z(x) \tanh \left( \frac{x + y}{\sqrt{2}} \right), \]
   where $z(x), x \in I$, is a unit speed time-like special Legendre curve in the light cone $\mathcal{LC} \subset C^3_2$ with constant squared curvature $\kappa^2 = 2$;
3. a flat Lagrangian minimal surface defined by $\pi \circ \tilde{L}$ with
   \[ \tilde{L}(x, y) = \frac{1}{\sqrt{3}} \left( \sqrt{2}e^{-\frac{1}{2a}(x+a^2y)} \cosh \left( \frac{\sqrt{3}}{2a}(x - a^2y) \right), e^{i(ay + \frac{\pi}{4})}, \right) \sqrt{2}e^{-\frac{1}{2a}(x+a^2y)} \sinh \left( \frac{\sqrt{3}}{2a}(x - a^2y) \right), \]
   where $a$ is a nonzero real number.

If $M$ contains no open subset of constant curvature, then $M$ is a Lagrangian minimal surface of Klein-Gordon type in $CH^2_1(-4)$.

Example 13.3. There exist many unit speed time-like special Legendre curve in the light cone $\mathcal{LC} \subset C^3_2$ with constant squared curvature $\kappa^2 = 2$. The simplest such examples are the following.

\[ z(x) = \left( \frac{1}{\sqrt{2}}, ae^{\sqrt{2}x} - \left( \frac{1}{8a} - ie \right) e^{-\sqrt{2}x}, ae^{\sqrt{2}x} + \left( \frac{1}{8a} + ie \right) e^{-\sqrt{2}x} \right), \]

where $a$ is a nonzero real number.

14 Parallel Lorentz surfaces in $C^2_1$.

14.1 Lorentzian real space forms.

Let $R^s_n$ denote the pseudo-Euclidean $n$-space with metric tensor given by

\[ g_0 = - \sum_{i=1}^{s} \, dx_i^2 + \sum_{j=s+1}^{n} \, dx_j^2, \quad (14.1) \]

where $\{x_1, \ldots, x_n\}$ is the rectangular coordinate system of $E^s_n$. Then $(R^s_n, g_0)$ is a flat semi-Riemannian manifold with index $s$.

We put

\[ S^s_n = \{ x \in R^{n+1}_s | \langle x, x \rangle = 1 \}, \]
\[ H^s_n = \{ x \in R^{n+1}_s | \langle x, x \rangle = -1 \}, \quad (14.2) \]

\[ \langle x, x \rangle = -1 \]
where $\langle \cdot, \cdot \rangle$ is the indefinite inner product on the pseudo-Euclidean space. It is well-known that $S^n_1$ and $H^n_1$ are complete semi-Riemannian manifolds with index $s$ of constant sectional curvature $1$ and $-1$, respectively.

The three semi-Riemannian manifolds $\mathbb{R}^n_1, S^n_1$ and $H^n_1$ are known as the Minkowski space-time, the de Sitter space-time and the anti-de Sitter space-time, respectively. These spaces with index $1$ are called Lorentzian real space forms.

### 14.2 B-scroll over null cubic.

In $\mathbb{R}^3_1$ take a null curve $\gamma(v)$ with a null frame, i.e., a set of vector fields $A(v), B(v), C(v)$ such that

\[ \gamma'(v) = A(v), \quad \langle A, A \rangle = \langle B, B \rangle = \langle A, C \rangle = \langle B, C \rangle = 0 \]

and $\langle A, B \rangle = \langle C, C \rangle = 1$. If these satisfy the following system of equations:

\[
\begin{align*}
A'(v) &= \kappa_1(v)A(v) + \kappa_2(v)C(v), \\
B'(v) &= \kappa_1(v)B(v), \\
C'(v) &= -\kappa_2(v)B(v),
\end{align*}
\]

then

\[ P(u, v) = \gamma(v) + uB(v) \]

is a Lorentz surface in $\mathbb{R}^3_1$ which called a B-scroll over $\gamma$.

If $\kappa_1 \equiv 0$ and $\kappa_2 \equiv 1$ for $\gamma$, then the curve is called the null cubic $C$. In this case, we have the $B$-scroll over the null cubic $C$ (cf. [11]).

### 14.3 Parallel submanifolds.

A submanifold of a pseudo-Riemannian manifold (in particular, in a Riemannian manifold) is called parallel if it has parallel second fundamental form. Parallel submanifolds are one of the most fundamental submanifolds.

Parallel submanifolds in real and complex space forms have been classified in [36, 48] and in [43, 44], respectively.

### 14.4 Classification of parallel Lorentz surfaces.

Some special classes of parallel submanifolds in Lorentzian space forms have been studied in [3, 37, 38, 41]. Complete classification of parallel space-like surfaces and parallel Lorentz surfaces in 4-dimensional Lorentzian real space forms have been obtained in [32].

Recently, together with F. Dillen and J. Van der Veken, we have completely classified parallel Lorentz surfaces in Lorentzian complex space forms.

The following results are obtained in [25].
Proposition 14.1. If $M$ is a parallel Lorentz surface in $\mathbb{C}^2_1$, then either $M$ is a flat surface or $M$ is totally umbilical in $\mathbb{C}^2_1$.

Proposition 14.2. If $M$ is a parallel Lorentz surface in $\mathbb{C}P^2(4)$, then either $M$ is a flat surface or $M$ is a totally geodesic Lagrangian surface.

Proposition 14.3. If $M$ is a parallel Lorentz surface in $\mathbb{C}H^2_1(4)$, then either $M$ is a flat surface or $M$ is a totally geodesic Lagrangian surface.

The following result of Chen, Dillen and Van der Veken completely classify parallel Lorentz surfaces in $\mathbb{R}^4_2$ and also in $\mathbb{C}^2_1$ (see [25] for details).

Theorem 14.1. Let $L : M \to \mathbb{R}^4_2$ be a parallel Lorentz surface in $\mathbb{R}^4_2$. Then, up to dilations and complex Lorentzian transformations on $\mathbb{R}^4_2$, $L$ is an open portion of one of the following eleven types of surfaces:

(a) A totally geodesic Lorentz plane;
(b) The product surface $R^1_1 \times S^1 \subset R^2_2 \times R^2_2 = \mathbb{R}^4_2$, where $S^1 \subset R^2$ is a circle;
(c) The product surface $H^1_1 \times R^1_2 \subset R^2_2 \times R^2_2$;
(d) The product surface $S^1_1 \times R^1_2 \subset R^2_2 \times R^2_2$ defined by
   $L = (\sinh y, x, 0, \cosh y)$;
(e) The product surface $H^1_1 \times R^1_2 \subset R^2_2 \times R^2_2$ defined by
   $L = (\cosh x, \sinh x, y, 0)$;
(f) A complex circle given by
   $L = (c + id)(\cos z, \sin z) \in C^2_1 = \mathbb{R}^4_2$ with $z = x+iy$ and $0 \neq c + id \in C$;
(g) A $B$-scroll over a null cubic;
(h) A flat surface given by
   $L = \frac{1}{\sqrt{2}} \left( (1 + c) \sin y - (x + cy) \cos y, (1 + c) \cos y + (x + cy) \sin y, 
   (1 - c) \sin y + (x + cy) \cos y, (1 - c) \cos y - (x + cy) \sin y \right), c \in \mathbb{R}$;
(i) A marginally trapped flat surface given by
   $L = (q(x, y), x, y, q(x, y))$,
   where $q = ax^2 + bxy + cu^2 + dx + ey + f$ for some real numbers $a, b, c, d, e, f$ with $a^2 + b^2 + c^2 \neq 0$;
(j) A totally umbilical imbedding of $S^2$ in $\mathbb{R}^4_2$ defined by

$$L = \left(0, \sinh x, \cosh x \cos y, \cosh x \sin y\right);$$

(k) A totally umbilical imbedding of $H^2$ in $\mathbb{R}^4_2$ defined by

$$L = \left(\sin x, \cos x \cosh y, \cos x \sinh y, 0\right).$$

Remark 14.1. By a marginally trapped surface in a Lorentzian Kaehler surface we mean a surface whose mean curvature vector is light-like at each point. Marginally trapped Lagrangian surfaces in Lorentzian complex space forms have been completely classified by Chen and F. Dillen in [24]. Marginally trapped proper slant surfaces in Lorentzian complex space forms were classified by Chen and I. Mihai in [27]. Furthermore, marginally trapped surfaces of constant curvature in Lorentzian complex space forms were completely classified in [15, 20].

For a recent survey on marginally trapped surfaces and Kaluza-Klein theory in general relativity, see [18].

Remark 14.2. To obtain the classification of parallel Lorentz surfaces in $C^2_1$, we only need to equip $\mathbb{R}^4_2$ with any compatible almost complex structure $J$ on $\mathbb{R}^4_2$.

15 Parallel Lorentz surfaces in $CP^2_1$.

The following result of Chen, F. Dillen and J. Van der Veken completely classifies parallel Lorentz surfaces in Lorentzian complex projective plane $CP^2_1$ (see [25] for details).

Theorem 15.1. Let $L : M \to CP^2_1(4)$ be a parallel Lorentz surface in $CP^2_1(4)$. Then either

(a) $M$ is an open portion of the real projective plane $RP^2_1(1)$ of constant curvature one immersed as a totally geodesic Lagrangian surface in $CP^2_1(4)$, or

(b) $M^2$ is a flat surface and up to rigid motions, $L$ is an open portion of the composition $\pi \circ \phi$, where $\pi : S^5_2 \to CP^2_1(4)$ is the Hopf fibration and $\phi$ is one of the fourteen types of immersions:

(b.1) $\phi : M \to S^5_2 \subset C^3_1$ is given by

$$\phi = \frac{1}{\sqrt{3}} \left(\sqrt{2}e^{\frac{x}{2}} \sinh \left(\frac{\sqrt{3}}{2} y\right), \sqrt{2}e^{\frac{x}{2}} \cosh \left(\frac{\sqrt{3}}{2} y\right), e^{-ix}\right);$$

(b.2) $\phi : M \to S^5_2 \subset C^3_1$ is given by

$$\phi = \frac{e^{\frac{1}{2}(2x+y+\sqrt{1+4by})}}{(1+4b)^\frac{1}{4}}, \frac{e^{\frac{1}{2}(2x+y-\sqrt{1+4by})}}{(1+4b)^\frac{1}{4}}, e^{iy}, b > \frac{1}{4};$$
(b.3) \( \phi : M \to S^5_2 \subset \mathbb{C}^3 \) is given by

\[
\phi = \frac{1}{\sqrt{2}} \left( e^{i(x+\frac{\phi}{2})} (1 + iy), e^{i(x+\frac{\phi}{2})} (1 - iy), \sqrt{2}e^{iy} \right);
\]

(b.4) \( \phi : M \to S^5_2 \subset \mathbb{C}^3 \) is given by

\[
\phi = \left( \frac{e^{i(x+\frac{\phi}{2})}}{4b - 1} \frac{\left( \cosh \left( \frac{\sqrt{4b - 1}}{2} y \right) + i \sinh \left( \frac{\sqrt{4b - 1}}{2} y \right) \right)}{b} \right), \quad b > \frac{1}{4};
\]

(b.5) \( \phi : M \to S^5_2 \subset \mathbb{C}^3 \) is given by

\[
\phi = \left( \frac{e^{i(\frac{\phi}{2})}}{\sqrt{b - 2}}, \frac{e^{i(\frac{\phi}{2})}}{\sqrt{b - 2}}, \frac{e^{i(\frac{\phi}{2})}}{\sqrt{b^2 - 4}} \right), \quad 0 < b < 2;
\]

(b.6) \( \phi : M \to S^5_2 \subset \mathbb{C}^3 \) is given by

\[
\phi = \left( \frac{b e^{i(\frac{\phi}{2})}}{\sqrt{4 - b^2}}, \frac{e^{i(\frac{\phi}{2})}}{\sqrt{2 - b}}, \frac{e^{i(\frac{\phi}{2})}}{\sqrt{2 - b}} \right), \quad b > 2;
\]

(b.7) \( \phi : M \to S^5_2 \subset \mathbb{C}^3 \) is given by

\[
\phi = \left( \frac{\sqrt{a(2 - a - b)} e^{i(bx + \frac{(1 - b)y}{a(2 - a - b)})}}{\sqrt{(a-b)(a+2b-2)}}, \frac{\sqrt{b(2 - a - b)} e^{i(ax + \frac{(1 - a)y}{a(2 - a - b)})}}{\sqrt{(a-b)(2a + b - 2)}}, \frac{\sqrt{ab e^{i((2-a-b)x+\frac{2 + b - 1}{ab}y)}}}{\sqrt{(a+2b-2)(2a+b-2)}} \right)
\]

with \( a > b > 0, a + b < 2; \)

(b.8) \( \phi : M \to S^5_2 \subset \mathbb{C}^3 \) is given by

\[
\phi = \left( \frac{\sqrt{b(a + b - 2)} e^{i(ax + \frac{(1 - a)y}{a(2 - a - b)})}}{\sqrt{(a-b)(2a + b - 2)}}, \frac{\sqrt{a(a + b - 2)} e^{i(bx + \frac{(1 - b)y}{a(2 - a - b)})}}{\sqrt{(a-b)(2a + b - 2)}}, \frac{\sqrt{ab e^{i((2-a-b)x+\frac{2 + b - 1}{ab}y)}}}{\sqrt{(a+2b-2)(2a+b-2)}} \right)
\]

with \( a > b > 0, a + b > 2; \)
(b.9) $\phi : M \to S_2^5 \subset \mathbb{C}_1^3$ is given by

$$\phi = \left( \frac{\sqrt{-abe^{i((2-a-b)x+\frac{a+b+1}{ab})}}}{\sqrt{(a+2b-2)(2a+b-2)}}, \frac{\sqrt{b(2-a-b)}e^{i(ax+\frac{1-a}{a(2-a-b)})}}{\sqrt{(a-b)(2a+b-2)}}, \frac{\sqrt{a(a+b-2)}e^{i(bx+\frac{1-b}{a(2-a-b)})}}{\sqrt{(a-b)(a+2b-2)}} \right)$$

with $a > 2$, $b < 0$;

(b.10) $\phi : M \to S_2^5 \subset \mathbb{C}_1^3$ is given by

$$\phi = \left( \frac{2i\sqrt{(2r-1)(1-r)}}{2-3r} + \frac{2r^2(r-1)x + (2r-1)y}{2r\sqrt{(2r-1)(1-r)}} e^{i(rx+\frac{y}{2r})}, \frac{2r^2(r-1)x + (2r-1)y}{2r\sqrt{(2r-1)(1-r)}} e^{i(rx+\frac{y}{2r})}, \frac{re^{i(2(1-r)x+\frac{y}{2r})}}{3r-2} \right)$$

with $\frac{2}{3} < r < 4$;

(b.11) $\phi : M \to S_2^5 \subset \mathbb{C}_1^3$ is given by

$$\phi = \frac{e^{\left(8x+9y\right)}}{288} \left( (8x - 9y)^2 + 24(8x + 9y), 24(8x - 9y), \right.$$

$$\left. (8x - 9y)^2 + 24i(8x + 9y + 12i) \right);$$

(b.12) $\phi : M \to S_2^5 \subset \mathbb{C}_1^3$ is given by

$$\phi = \left( \frac{e^{-irx}}{r(3r-2)} \left\{ 2r^2(2 + 3r^2 - 5r)x + (2r^2(2 - 7r + 6r^2)t - ir \right.$$

$$\left. - \frac{r^2}{5 - 12r + 8r^2} e^{i(2x+\frac{y}{2r})} \right\}, \frac{ir\sqrt{5 - 12r + 8r^2}}{2 - 3r} e^{i(2(1-r)x+\frac{y}{2r})} + (2r(r-1)x + (2 - r^{-1})y)e^{i(rx+\frac{y}{2r})}, \right.$$

$$\left. \frac{re^{i(2(1-r)x+\frac{y}{2r})}}{2 - 3r} + \frac{\sqrt{5 - 12r + 8r^2}}{2 - 3r} e^{i(rx+\frac{y}{2r})} \right)$$

with $r \neq 0,1,\left[\frac{2}{3},4\right]$;
\begin{align*}
\phi : M \to S^5_2 \subset C^3_1 & \text{ is given by} \\
\phi = \left( \sqrt{a - 1} e^{i(ax + \frac{(a^2 - b^2 - a)y}{2(a-1)(a^2 + b^2)}} \right) \\
\frac{i(3a^2 - b^2 - 2a) \sinh (bx + ky) + (2 - 4a)b \cosh (bx + ky) }{b \sqrt{1 - 2a} \sqrt{a^2 + b^2 \sqrt{(2-3a)^2 + b^2}} e^{i(2(1-a)x + \frac{2a-1}{a^2 + b^2})}} \\
\frac{\sqrt{a^2 + b^2 \sqrt{(2-3a)^2 + b^2}} e^{i(2(1-a)x + \frac{2a-1}{a^2 + b^2})}}{b \sqrt{1 - 2a} \sqrt{a^2 + b^2 \sqrt{(2-3a)^2 + b^2}} \cosh (bx + ky)}
\end{align*}

with
\[ k = \frac{(2a - 1)b}{2(a - 1)(a^2 + b^2)}, \quad a \in \left( \frac{1}{2}, 1 \right); \]

(b.13) \( \phi : M \to S^5_2 \subset C^3_1 \) is given by

\begin{align*}
\phi = \left( \sqrt{a - 1} e^{i(ax + \frac{(a^2 - b^2 - a)y}{2(a-1)(a^2 + b^2)}} \right) \\
\frac{i(3a^2 - b^2 - 2a) \cosh (bx + ky) + (2 - 4a)b \sinh (bx + ky) }{b \sqrt{2a - 1} \sqrt{(2-3a)^2 + b^2}} \\
\frac{\sqrt{a^2 + b^2 \sqrt{(2-3a)^2 + b^2}} e^{i(2(1-a)x + \frac{2a-1}{a^2 + b^2})}}{b \sqrt{2a - 1} \sqrt{a^2 + b^2 \sqrt{(2-3a)^2 + b^2}} \cosh (bx + ky)}
\end{align*}

with
\[ k = \frac{(2a - 1)b}{2(a - 1)(a^2 + b^2)}, \quad a \notin \left[ \frac{1}{2}, 1 \right]. \]

16 Parallel Lorentz surfaces in \( CH^2_1 \).

The following result of Chen, F. Dillen and J. Van der Veken completely classifies parallel Lorentz surfaces in Lorentzian complex hyperbolic plane \( CH^2_1 \) (see [25] for details).

**Theorem 16.1.** Let \( L : M \to CH^2_1(-4) \) be a parallel Lorentz surface in \( CH^2_1(-4) \). Then either

(i) \( M \) is an open portion of the real hyperbolic plane \( H^2_1(-1) \) of constant curvature \(-1\) immersed as a totally geodesic Lagrangian surface in \( CH^2_1(-4) \), or
(ii) $M^2_1$ is a flat surface and up to rigid motions, $L$ is an open portion of the composition $\pi \circ \phi$, where $\pi : H^5_2 \subset C H^1_2(-4)$ is the Hopf fibration and $\phi$ is one of the thirteen types of immersions:

(iii.1) $\phi : M \to H^5_2 \subset C^3_2$ is given by $\phi = (e^{iy}, e^{ix}, e^{i(x+y)})$;

(iii.2) $\phi : M \to H^5_2 \subset C^3_2$ is given by

$$
\phi = \left( \left( y - \frac{1}{2} \right) e^{i\left( x - \frac{3}{2}\right)}, e^{-iy}, \left( y + \frac{1}{2} \right) e^{i\left( x - \frac{3}{2}\right)} \right);
$$

(iii.3) $\phi : M \to H^5_2 \subset C^3_2$ is given by

$$
\phi = \frac{1}{\sqrt{6}} \left( e^{\sqrt{3}x+y}, e^{\sqrt{3}x-y}, e^{\sqrt{3}(x+y)}, \sqrt{2} e^{i(x+y)} \right);
$$

(iii.4) $\phi : M \to H^5_2 \subset C^3_2$ is given by

$$
\phi = \left( e^{-iy}, e^{i\left( x + \frac{3}{2} \right)(1+4b-1)}, e^{i\left( x - \frac{3}{2} \right)(1+4b) + i \sqrt{2} e^{i(x+y)}} \right), \quad b > -\frac{1}{4};
$$

(iii.5) $\phi : M \to H^5_2 \subset C^3_2$ is given by

$$
\phi = \left( e^{-iy}, \frac{2 e^{\sqrt{4b-1}y - i} e^{i\left( x - \frac{3}{2} \right)(1+4b-1)}}{2(4b-1)^{\frac{1}{2}}}, \frac{2 e^{\sqrt{4b-1}y + i} e^{i\left( x - \frac{3}{2} \right)(1+4b-1)}}{2(4b-1)^{\frac{1}{2}}} \right)
$$

with $b > -\frac{1}{4}$;

(iii.6) $\phi : M \to H^5_2 \subset C^3_2$ is given by

$$
\phi = \left( \sqrt{b(2a-b)} e^{i\left( \frac{a(x+\frac{(a-1)y)}{a-b})} \right) e^{i\left( \frac{(a-x+\frac{1}{a-b})y}{(a+b-2)} \right)}, \sqrt{abe} e^{i\left( \frac{(2-a-b)x+\frac{1}{a-b})y}{(a+2b-2)(2a+b-2)} \right)}, \sqrt{\overline{a(2a-b)} e^{i\left( \frac{b(x+\frac{(b-1)y}{a(2a-b)})} {(a-b)(2a-b-2)} \right)}, a > b > 0, a + b < 2;
$$

(iii.7) $\phi : M \to H^5_2 \subset C^3_2$ is given by

$$
\phi = \left( \sqrt{a(2a-b)} e^{i\left( \frac{(b-x+\frac{(b-1)y}{a(2a-b)})} {(a-b)(2a-b-2)} \right)}, \sqrt{abe} e^{i\left( \frac{(2-a-b)x+\frac{1}{a-b})y}{(a+2b-2)(2a+b-2)} \right)}, \sqrt{b(2-a-b)} e^{i\left( \frac{a(x+\frac{(a-1)y}{b(2a-b)})} {(a-b)(2a-b)} \right)}, a > b > 0, a + b > 2;
$$
(ii.8) $\phi : M \to H_2^5 \subset C_2^3$ is given by

$$\phi = \left( \frac{\sqrt{b(2-a-b)} e^{i(ax + \frac{(a-1)y}{a-a-b})}}{\sqrt{(a-b)(2a+b-2)}}, \frac{\sqrt{a(a+b-2)} e^{i(bx + \frac{(b-1)y}{a-2a-b})}}{\sqrt{(a-b)(a+2b-2)}}, \frac{-abe^{i((2-a-b)x + \frac{1-a-b}{a})}}{\sqrt{(a+2b-2)(2a+b-2)}} \right)$$

with $a > 0, b < 0, a + 2b > 2$;

(ii.9) $\phi : M \to H_2^5 \subset C_2^3$ is given by

$$\phi = \left( e^{i(2(1-r)x + \frac{1-2y}{r})}, \frac{2r^2(r-1)x + (1-2r)y}{2r\sqrt{(2r-1)(1-r)}}, \frac{2i\sqrt{(2r-1)(1-r)}}{2r\sqrt{(2r-1)(1-r)}} e^{i(rx - \frac{y}{r})} \right)$$

with $r \in (\frac{1}{2}, 1)$ and $r \neq \frac{2}{3}$;

(ii.10) $\phi : M \to H_2^5 \subset C_2^3$ is given by

$$\phi = \left( \frac{2r(r-1)(2(2r-1)x + (7r-2-6r^2)y)}{2r(2-3r)\sqrt{1-3r+2r^2}}, \frac{2r(2-3r)\sqrt{1-3r+2r^2}}{r}, \frac{2r^2(r-1)x + 2r(2r-1)y}{2r\sqrt{1-3r+2r^2}} e^{i(rx - \frac{y}{r})} \right)$$

with $r \notin [\frac{1}{2}, 1]$ and $r \neq 2$;

(ii.11) $\phi : M \to H_2^5 \subset C_2^3$ is given by

$$\phi = \frac{e^{\frac{1}{12}(8x-9y)}}{24} \left( (8x + 9y)^2 - 432iy - 1, 16x + 18y + 24i, (8x - 9y)^2 - 431iy + 1 \right)$$

(ii.12) $\phi : M \to H_2^5 \subset C_2^3$ is given by

$$\phi = \left( \sqrt{a-1} e^{i(ax + \frac{(a-a^2+2b^2)y}{2(a-1)(a^2+b^2)})}, \frac{\sqrt{a^2+b^2}}{b\sqrt{2a-1}\sqrt{(2-3a)^2+b^2}} e^{i(2(1-a)x + \frac{1-b}{a-2a+b}y)}, \frac{\sqrt{a-1}\sqrt{a^2+b^2} e^{i(ax + \frac{(a-a^2+b^2)y}{2(a-1)(a^2+b^2)})}}{b\sqrt{2a-1}} \sinh(bx + ky) \right)$$
with

\[ k = \frac{(1 - 2a)b}{2(a - 1)(a^2 + b^2)}, \quad a \notin \left[\frac{1}{2}, 1\right); \]

(ii.13) \( \phi : M \to H_2^3 \subset \mathbb{C}_2^3 \) is given by

\[
\phi = \left( \frac{\sqrt{a^2 + b^2}}{\sqrt{(2 - 3a)^2 + b^2}}, \frac{e^{i(2(1-a)x + \frac{1-2a}{a^2+b^2}y)}}{\sqrt{2a - 1}} \right)
\]

with

\[ k = \frac{(2a - 1)b}{2(a - 1)(a^2 + b^2)}, \quad a \in \left(\frac{1}{2}, 1\right). \]

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Department of Mathematics, Michigan State University, East Lansing, Michigan 48824-1027, U.S.A.

Email address: bychen@math.msu.edu