Structural Physical Approximation make possible to realize the optimal singlet fraction with two measurements

Satyabrata Adhikari¹

¹Delhi Technological University, Delhi-110042, Delhi, India

Structural physical approximation (SPA) has been exploited to approximate non-physical operation such as partial transpose. It has already been studied in the context of detection of entanglement and found that if the minimum eigenvalue of SPA to partial transpose is less than $\frac{1}{4}$ then the two-qubit state is entangled. We find application of SPA to partial transpose in the estimation of optimal singlet fraction. We show that optimal singlet fraction can be expressed in terms of minimum eigenvalue of SPA to partial transpose. We also show that optimal singlet fraction can be realized using Hong-Ou-Mandel interferometry with only two detectors. Further we have shown that the generated hybrid entangled state between a qubit and a binary coherent state can be used as a resource state in quantum teleportation.

PACS numbers: 03.67.Hk, 03.67.-a

I. INTRODUCTION

Entanglement is a non-classical correlation [1] and is a necessary ingredient to build a quantum computer that can outperform the classical computer. It has also been used as a quantum resource in various quantum communication tasks such as teleportation [2], superdense coding [3], secret sharing [4] and quantum-key distribution (QKD) [5]. To perform the classical computer. It has also been used as a necessary ingredient to build a quantum computer that can outperform the classical computer. To perform the classical

\[ \rho = (1 - q^*) (id \otimes T) + \frac{q^*}{4} I_A \otimes I_B \]  

Partial transposition (PT) is another strong entanglement detection criterion given by Peres [12]. Later, Horodecki [13] proved that the PT criterion is necessary and sufficient for $2 \times 2$ and $2 \times 3$ system. Although partial transposition criterion works well in qubit-qubit and qubit-qutrit system but it cannot be implemented in a laboratory for the detection of entanglement as it is a non-physical operation. Therefore, to make partial transposition map a physical operation we can approximate it in such a way that it would be a completely positive map. Let us consider that the partial transposition operation act on the second subsystem and is given by $id \otimes T$, where $T$ denotes the positive transposition map and $id$ represent the identity operator. In general, partial transposition operation $id \otimes T$ can be approximated as [8]

\[ \text{id} \otimes T = \left(1 - q^* \right) \left(id \otimes T \right) + \frac{q^*}{4} I_A \otimes I_B \]  

where $q^* = \frac{16\nu}{1 + 16\nu}$ and $\nu = -\min_{Q > 0} \text{Tr}[Q(id \otimes T)|\psi^\text{+} \rangle \langle \psi^\text{+}|]$, $|\psi^\text{+}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. The map $id \otimes T$ is a non-physical map but its approximate map $id \otimes T$ is a completely positive map corresponds to a quantum channel that can be experimentally implementable [3,11]. Let $\sigma_{12}$ be a two qubit-state and the task is to determine whether it is an entangled state or separable state. Since it is a two qubit system so we can apply partial transposition detection criterion. PT criterion states that if $\sigma_{12}^{T_B}$ ($T_B$ denotes partial transposition with respect to the second subsystem B) has a negative eigenvalue then the state $\sigma_{12}$ is an entangled state. But partial transposition is not a physical operation so apply SPA-PT operation [2] on $\sigma_{12}$ and at the output, we have $\tilde{\sigma}_{12}$. Since SPA-PT operation is completely positive so the output $\tilde{\sigma}_{12}$ also represents a state. Therefore, the practical problem of finding the eigenvalue of $\sigma_{12}^{T_B}$ ($T_B$ denotes partial transposition with respect to the second subsystem B) reduces to determine the eigenvalue of $\tilde{\sigma}_{12}$. Hence PT criterion modified as SPA-PT criterion which states that if the minimum eigenvalue of $\tilde{\sigma}_{12}$ is less than $\frac{1}{4}$ then the state $\sigma_{12}$ is entangled and vice-versa [8]. The minimum eigenvalue can be estimated by the procedure given in [14,15].

Recently, H-T Lim et al. [16] have demonstrated the experimental realization of SPA-PT for photonic two qubit photonic

*Electronic address: satyabrata@dtu.ac.in
system using single-photon polarization qubits and linear optical devices. They provided the decomposition of SPA-PT for a two-qubit state $\sigma_{12}$ as

$$\tilde{\sigma}_{12} = \tilde{T} \otimes \tilde{T}(\sigma_{12}) = \left[\frac{1}{3}(I \otimes \tilde{T}) + \frac{2}{3}(\tilde{\Theta} \otimes D)\right] \sigma_{12}$$ (3)

where $\tilde{T}$ denote SPA for transpose operation, $\tilde{\Theta}$ denotes the inversion map and works as $\tilde{\Theta}(\sigma) = -\sigma$, $\Theta$ denote its SPA and can be constructed by the prescription given in (1) and $D(\sigma) = \frac{1}{2}$ denote the polarization. Since $I \otimes \tilde{T}$ and $\tilde{\Theta} \otimes D$ are local operations and are completely positive operators so $\tilde{T} \otimes \tilde{T}$ is a physically realizable operators.

The motivation of this work is two fold: Firstly, the method of finding the eigenvalues in [14, 15] require more than one copy of the given state and the method described in [14] for estimating the eigenvalues works well asymptotically. Therefore, to circumvent these problems we take the approach of witness operator to determine the minimum eigenvalue, which require a single copy of SPA-PT of the given state. Also, we show that the minimum eigenvalue determined by our method require a set up that need only two measurements. Secondly, Verstraete and Vershelde [17] have established a relationship between the optimal singlet fraction and partial transpose of a given state and using the derived relation they have shown that the two-qubit state is useful as a resource state for teleportation if and only if the optimal singlet fraction is greater than $\frac{1}{2}$. But the partial tranposition is an non-physical operation and cannot be implemented in a laboratory so it would be not easy to realize the optimal singlet fraction. Also the filtering operation used in [17] to achieve the optimal singlet fraction depends on the quantum state under investigation. Thus information about the state under investigation is needed. To overcome these problems, we apply SPA-PT method and show that optimal singlet fraction does not depend on the state under investigation and also can be realized in experiment.

This paper is organized as follows: In section-II, we have constructed the witness operator to determine the minimum eigenvalue of SPA-PT of a given state $\rho_{12}$. In section-III, we have shown that the number of measurements needed to determine the minimum eigenvalue is two. In section-IV, we show that the teleportation fidelity can be determined experimentally using Hong-Ou-Mandel interferometry with only two detectors. In section-V, we have studied the hybrid entangled state between a qubit and binary coherent state and have shown that the mixed hybrid entangled state can be used as a resource state for teleportation and lastly, we conclude in section-VI.

II. WITNESS OPERATOR THAT DETERMINE THE MINIMUM EIGENVALUE OF SPA-PT OF TWO QUBIT STATE

Any arbitrary two qubit density operator in the computational basis is given by

$$\rho_{12} = \left(\begin{array}{cccc}
t_{11} & t_{12} & t_{13} & t_{14} \\
t_{12}^* & t_{22} & t_{23} & t_{24} \\
t_{13}^* & t_{23}^* & t_{33} & t_{34} \\
t_{14}^* & t_{24}^* & t_{34}^* & t_{44}
\end{array}\right), \sum_{i=1}^{4} t_{ii} = 1$$ (4)

where $(*)$ denotes the complex conjugate.

SPA-PT of $\rho_{12}$ is given by

$$\tilde{\rho}_{12} = \left[\frac{1}{3}(I \otimes \tilde{T}) + \frac{2}{3}(\tilde{\Theta} \otimes D)\right] \rho_{12}$$

where

$$E_{11} = \frac{1}{9}((2 + t_{11}), E_{12} = \frac{1}{9}((-it_{12} + t_{12}^*),$$

$$E_{13} = \frac{1}{9}(t_{13} - it_{13}^* t_{24}), E_{14} = \frac{1}{9}((-it_{14} + t_{23}),$$

$$E_{22} = \frac{1}{9}(2 + t_{22}), E_{23} = \frac{1}{9}(t_{14} + it_{23}),$$

$$E_{24} = \frac{1}{9}(-it_{13} + t_{24}), E_{33} = \frac{1}{9}(2 + t_{33},$$

$$E_{34} = \frac{1}{9}(-it_{34} + t_{34}^*), E_{44} = \frac{1}{9}(2 + t_{44})$$ (5)

We note that the matrix $\tilde{\rho}_{12}$ is not only a Hermitian matrix but also has negative eigenvalues. The trace of the matrix is equal to unity. So, it possesses all the properties of a matrix and thus the matrix can be regarded as a density matrix $\tilde{\rho}_{12}$. The minimum eigenvalue of $\tilde{\rho}_{12}$ detect whether the state $\rho_{12}$ is entangled or not? Therefore, our task is to construct the witness operator that detect whether the minimum eigenvalue of $\tilde{\rho}_{12}$ is less than $\frac{2}{9}$?

To start, we consider the operator $O = \tilde{\rho}_{12} - \frac{1}{9}\rho_{12}^T$, $T_2$ denote the partial transpose with respect to the second subsystem. The expectation value of the operator $O$ in the state $|\phi\rangle = \alpha|00\rangle + \beta|11\rangle(\alpha^2 + \beta^2 = 1)$ is given by

$$\langle \phi|O|\phi\rangle = Tr[(\tilde{\rho}_{12} - \frac{1}{9}\rho_{12}^T)|\phi\rangle\langle \phi|] = \frac{2}{9}$$

$$\Rightarrow Tr[|\phi\rangle\langle \phi|\tilde{\rho}_{12}] - \frac{1}{9}Tr[|\phi\rangle\langle \phi|\rho_{12}^T] = \frac{2}{9}$$

$$\Rightarrow Tr[W^opt \rho_{12}] = Tr[|\phi\rangle\langle \phi|\tilde{\rho}_{12}|\phi\rangle - \frac{2}{9}$$ (7)

where $W^opt = \frac{1}{9}|\phi\rangle\langle \phi|T_2$ is the optimal witness operator that detect whether the state $\rho_{12}$ is entangled or not. Let $\lambda_{min}$ be the minimum eigenvalue of $\tilde{\rho}_{12}$ and $|\phi\rangle$ be the
Proof: The separable state
\[ \tilde{\rho}_{12} | \phi \rangle = \lambda_{\text{min}} | \phi \rangle \quad (8) \]

Using (8) in (7), we have
\[ \text{Tr}(W^{\text{opt}} \rho_{12}) = \lambda_{\text{min}} - \frac{2}{9} \quad (9) \]

For all separable state \( \rho_{12} \), we have [8]
\[ \text{Tr}(W^{\text{opt}} \rho_{12}) \geq 0 \Rightarrow \lambda_{\text{min}} \geq \frac{2}{9} \quad (10) \]

If \( \rho_{12} \) is an entangled state and \( W^{\text{opt}} \) detect that entangled state then
\[ \text{Tr}(W^{\text{opt}} \rho_{12}) < 0 \Rightarrow \lambda_{\text{min}} < \frac{2}{9} \quad (11) \]

The inequality (11) gives us the condition that when \( \rho_{12} \) is an entangled state.

Since the above condition is a purely mathematical condition so naturally one can ask a question that can we achieve this inequality experimentally? To investigate this, let us again recall (7) and write it in a different form as
\[ \text{Tr}(W^{\text{opt}} \rho_{12}) = \text{Tr}((| \phi \rangle \langle \phi | - \frac{2}{9} I) \tilde{\rho}_{12}) \quad (12) \]

When \( W^{\text{opt}} \) detect an entangled state \( \rho_{12} \) then \( \text{Tr}(W^{\text{opt}} \rho_{12}) < 0 \) and hence we arrive at a condition given by
\[ \text{Tr}(| \phi \rangle \langle \phi | - \frac{2}{9} I) \tilde{\rho}_{12} < 0 \quad (13) \]

The above condition (13) is equivalent form of the condition (11) and hence the inequality implies that the eigenvalues of \( \rho_{12} \) is less than \( \frac{2}{9} \).

Let \( V \equiv | \phi \rangle \langle \phi | - \frac{2}{9} I \). Then the inequality (13) can be re-expressed as
\[ \text{Tr}(V \tilde{\rho}_{12}) < 0 \quad (14) \]

Next we investigate few properties of the operator \( V \).

**P1.** The expectation value of \( V \) for all separable state \( \rho_{12}^{\text{sep}} \) is non-negative i.e. \( \text{Tr}(V \tilde{\rho}_{12}^{\text{sep}}) \geq 0 \).

Proof: The separable state \( \rho_{12}^{\text{sep}} \) is given by
\[ \tilde{\rho}_{12}^{\text{sep}} = \begin{pmatrix} E_{11}^s & E_{12}^s & E_{13}^s & E_{14}^s \\ (E_{12}^s)^* & E_{22}^s & E_{23}^s & E_{24}^s \\ (E_{13}^s)^* & (E_{23}^s)^* & E_{33}^s & E_{34}^s \\ (E_{14}^s)^* & (E_{24}^s)^* & (E_{34}^s)^* & E_{44}^s \end{pmatrix} \quad (15) \]

where \( E_{11}^s = \frac{1}{2} (2 + t_{11}), E_{12}^s = \frac{1}{2} (-it_{12} + (t_{12}^*)), E_{13}^s = \frac{1}{2} (t_{13}^* - it_{23}), E_{14}^s = \frac{1}{2} (t_{14}^* + t_{24}), E_{22}^s = \frac{1}{2} (2 + t_{22}), E_{23}^s = \frac{1}{2} (t_{14}^* + t_{23}), E_{24}^s = \frac{1}{2} (t_{14}^* + (t_{24}^*)), E_{33}^s = \frac{1}{2} (2 + t_{33}), E_{34}^s = \frac{1}{2} (-it_{34} + (t_{34}^*)^*), E_{44}^s = \frac{1}{2} (2 + t_{44}) \).

Here, \( t_{ij} \) denote the elements of a separable state \( \rho_{12} \).

(12) can be re-expressed for any separable state \( \rho_{12}^s \) as
\[ \text{Tr}(W^{\text{opt}} \tilde{\rho}_{12}^{\text{sep}}) = \text{Tr}(V \tilde{\rho}_{12}^{\text{sep}}) \quad (16) \]

Since \( W^{\text{opt}} \) is a witness operator so the expectation value of \( W^{\text{opt}} \rho_{12}^{\text{sep}} \) is non-negative. Thus
\[ \text{Tr}(W^{\text{opt}} \tilde{\rho}_{12}^{\text{sep}}) \geq 0 \quad (17) \]

Using (16) and (17), we have
\[ \text{Tr}(V \tilde{\rho}_{12}^{\text{sep}}) \geq 0 \quad (18) \]

**P2.** It can be easily shown that \( V \) has at least one negative eigenvalues.

Thus, the operator \( V \) possess all the properties of a witness operator and hence it detect whether the eigenvalue of \( \rho_{12} \) is less than \( \frac{2}{9} \) or not. If \( V \) detect that the eigenvalue of \( \rho_{12} \) is less than \( \frac{2}{9} \) then we can say that the state described by the density operator \( \rho_{12} \) is entangled.

Since \( V \) is a hermitian operator so it is an observable and can be implemented experimentally. Therefore the inequality (13) is useful to detect entangled state experimentally.

**III. NUMBER OF MEASUREMENTS NEEDED TO DETERMINE THE VALUE OF Tr(V\tilde{\rho}_{12})**

The operator \( V \) can be expressed in terms of local Pauli matrices as
\[
V = \frac{9}{28} I \otimes I + (\alpha^2 - \beta^2) (I \otimes \sigma_z + \sigma_z \otimes I) + 2 \alpha \beta (\sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z) \quad (19)
\]

We find that the decomposition of the operator \( V \) in terms of local Pauli observables need more than two measurements to realize it. So in this section, our task is to show that it is possible to realize the operator \( V \) with just two measurements.

To achieve our goal, we approximate the entanglement witness operator \( V \) in a way we approximate the positive but not completely positive operator. Therefore, approximate entanglement witness operator \( \tilde{V} \) can be expressed as
\[
\tilde{V} = p V + (1 - p) I, 0 \leq p \leq 1 \quad (20)
\]

Choose the minimum value of \( p \) in such a way that the operator \( \tilde{V} \) will become a positive semi-definite operator. The Hermitian operator \( \tilde{V} \) is positive semi-definite if \( p_{\text{min}} = \frac{8}{15} \).

Therefore, \( \tilde{V} \) can be re-expressed as
\[
\tilde{V} = \frac{8}{15} V + \frac{7}{15} I \quad (21)
\]

We can observe that the operator \( \tilde{V} \) is not a normalized operator so it can be expressed after normalization as
\[
\tilde{V} = \frac{2}{9} V + \frac{7}{36} I \quad (22)
\]

Since \( \tilde{V} \) is positive semi-definite and \( \text{Tr}(\tilde{V}) = 1 \) so the operator \( \tilde{V} \) can be treated as a quantum state.

Again, \( \text{Tr}(V \tilde{\rho}_{12}) \) can be written in terms of \( \text{Tr}(\tilde{V} \tilde{\rho}_{12}) \) as
\[
\text{Tr}(V \tilde{\rho}_{12}) = \frac{15}{8} \text{Tr}(\tilde{V} \tilde{\rho}_{12}) - \frac{7}{8} \quad (23)
\]
It can be shown that \( Tr(\tilde{V}\tilde{\rho}_{12}) \) is equal to the average fidelity for two mixed quantum states \( \tilde{V} \) and \( \tilde{\rho}_{12} \) \cite{18}.

\[
Tr(\tilde{V}\tilde{\rho}_{12}) = F_{\text{avg}}(\tilde{V}, \tilde{\rho}_{12})
\]  
(24)

Using (23) and (24), we have

\[
Tr(V\rho_{12}) = \frac{15}{8} F_{\text{avg}}(V, \rho_{12}) - \frac{7}{8}
\]  
(25)

C. J. Kwong et al. \cite{18} have shown that the average fidelity between two mixed quantum states can be estimated experimentally by Hong-Ou-Mandel interferometry with only two detectors. Thus the quantity \( Tr(V\rho_{12}) \) needs only two measurements to estimate it in experiment.

Again, the minimum eigenvalue can be expressed in terms of \( F_{\text{avg}}(V, \rho_{12}) \) as

\[
\lambda_{\text{min}} = \frac{15}{8} F_{\text{avg}}(V, \rho_{12}) - \frac{47}{72}
\]  
(26)

Since the minimum eigenvalue of the quantum state \( \tilde{\rho}_{12} \) can be determined using two measurements and minimum eigenvalue is responsible for the detection of entanglement so we can say that the presence of entanglement in \( \rho_{12} \) can be detected using two measurements only.

### IV. REALIZATION OF OPTIMAL SINGLET FRACTION WITH ONLY TWO MEASUREMENTS

The singlet fraction is defined as

\[
F(\rho_{12}) = \max [\langle \phi^+ | \rho_{12} | \phi^+ \rangle \langle \phi^- | \rho_{12} | \phi^- \rangle]
\]  
(27)

\[
\langle \psi^+ | \rho_{12} | \psi^+ \rangle \langle \psi^- | \rho_{12} | \psi^- \rangle
\]  
(28)

where \( \{|\phi^+\rangle, |\phi^-\rangle, |\psi^+\rangle, |\psi^-\rangle\} \) are the maximally entangled Bell states.

Verstraete and Ventselde \cite{17} suggested the optimal trace preserving protocol for maximizing the singlet fraction of a given state. The optimal singlet fraction is given by

\[
F^{\text{opt}}(\rho_{12}) = \frac{1}{2} - Tr(X^{\text{opt}}_p \rho_{12}^{T_p})
\]  
(29)

\( X^{\text{opt}} \) is given by

\[
X^{\text{opt}} = (A \otimes I_2)|\psi^+\rangle\langle \psi^+| (A^\dagger \otimes I_2),
\]  
(30)

where \( I_2 \) represent an identity matrix of order 2, \( |\psi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \) and \( A = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, -1 \leq a \leq 1 \).

We note that if the state \( \rho_{12} \) is entangled then \( \rho_{12}^{T_p} \) has at least one negative eigenvalue and hence \( F^{\text{opt}}(\rho_{12}) \) is greater than \( \frac{1}{2} \). Thus every entangled two qubit state is useful for teleportation. But there are two problems in estimating the quantity \( F^{\text{opt}}(\rho_{12}) \) given in (29): (i) \( \rho_{12}^{T_p} \) cannot be realized in the laboratory and (ii) the parameter in \( X^{\text{opt}} \) is state dependent and hence to construct \( X^{\text{opt}} \), we need to know the state under investigation.

In this section, we obtain the optimal singlet fraction in terms of minimum eigenvalue \( \lambda_{\text{min}} \) of SPA-PT of the state \( \rho_{12} \) using local filtering operation but in this scenario the parameter of the local filtering operation does not depend on the state under investigation. Since we found in the previous section that \( \lambda_{\text{min}} \) can be estimated for any arbitrary state with two measurements so we can say that optimal singlet fraction of any arbitrary states can be calculated using two measurements.

#### A. Optimal singlet fraction in terms of \( \lambda_{\text{min}} \)

To start, let us consider the operator \( X^{\text{opt}}(\tilde{\rho}_{12} - \frac{1}{2} \rho_{12}^{T_p}) \), where \( X_{\text{opt}} \) is given in (20). Calculate the trace of it and after some simple algebra, (29) reduces to

\[
F^{\text{opt}}(\rho_{12}) = \frac{1}{2} - [9 Tr(X^{\text{opt}}_p \tilde{\rho}_{12}) - (a^2 + 1)]
\]  
(31)

It can be easily seen from (31) that \( F^{\text{opt}}(\rho_{12}) > \frac{1}{2} \) if and only if \( 9 Tr(X^{\text{opt}}_p \tilde{\rho}_{12}) - (a^2 + 1) < 0 \). This condition leads to

\[
Tr(X^{\text{opt}}_p \tilde{\rho}_{12}) < \frac{a^2 + 1}{9}, -1 \leq a \leq 1
\]  
(32)

In \( \{00\rangle, |11\rangle \} \) subspace, the operator \( \frac{1}{a^2 + 1}X \) can be expressed as

\[
\frac{2}{a^2 + 1}X^{\text{opt}} = |\chi\rangle\langle \chi|
\]  
(33)

where \( |\chi\rangle = \frac{1}{\sqrt{a^2 + 1}}(a|00\rangle + |11\rangle) \).

We note that the vector \( |\chi\rangle \) and the eigenvector \( |\phi\rangle \) of the operator \( \tilde{\rho}_{12} \) are parallel vectors and thus there exist a real scalar \( k \) such that

\[
|\chi\rangle = k|\phi\rangle
\]  
(34)

Using (34), it can be easily shown that the vector \( |\chi\rangle \) is also a eigenvector corresponding to the minimum eigenvalue \( \lambda_{\text{min}} \). Hence, the resource state \( \rho_{12} \) is useful for teleportation iff

\[
\lambda_{\text{min}} < \frac{2}{9}
\]  
(35)

The singlet fraction \( F^{\text{opt}}(\rho_{12}) \) given by (31) can be re-expressed in terms of the minimum eigenvalue \( \lambda_{\text{min}} \) as

\[
F^{\text{opt}}_{(a, \lambda_{\text{min}})}(\rho_{12}) = \frac{1}{2} - \frac{9(a^2 + 1)}{2} |\lambda_{\text{min}} - \frac{2}{9}|, -1 \leq a \leq 1
\]  
(36)

Further, we find that if the minimum eigenvalue \( \lambda_{\text{min}} \) is restricted to lie in the interval \( \frac{1}{9}, \frac{2}{9} \) then the singlet fraction \( F^{\text{opt}}_{(a, \lambda_{\text{min}})}(\rho_{12}) \) lies in the interval

\[
\frac{1}{2} < F^{\text{opt}}_{(a, \lambda_{\text{min}})}(\rho_{12}) < \frac{1}{2} + \frac{a^2 + 1}{4}, -1 \leq a \leq 1
\]  
(37)

The optimal singlet fraction can be achieved by putting \( a = \pm 1 \) in (36) and it is given by

\[
F^{\text{opt}}_{(\pm 1, \lambda_{\text{min}})}(\rho_{12}) = \frac{1}{2} - 9|\lambda_{\text{min}} - \frac{2}{9}|, \frac{1}{9} \leq \lambda_{\text{min}} < \frac{2}{9}
\]  
(38)
Without any loss of generality, we can take $a = 1$ and thus we have $X^{opt} = |\psi^+\rangle\langle\psi^+|$. Afterward, we denote $F^{opt}_{\lambda(\pm 1, \lambda_{min})}(\rho_{12})$ as simply $F^{opt}_{\lambda_{min}}(\rho_{12})$. Therefore,

$$F^{opt}_{\lambda_{min}}(\rho_{12}) = \frac{1}{2} - 9[\lambda_{min} - \frac{2}{9}], \frac{1}{6} \leq \lambda_{min} < \frac{2}{9} \quad (39)$$

We note here that the problem of finding the optimal singlet fraction reduces to finding the minimum eigenvalue of SPA-PT of any arbitrary state $\rho_{12}$.

B. Optimal singlet fraction in terms of average fidelity $F_{avg}(\tilde{V}, \tilde{\rho}_{12})$

Now, we are in a position to give the expression for optimal singlet fraction in terms of average fidelity $F_{avg}(\tilde{V}, \tilde{\rho}_{12})$. The optimal singlet fraction in terms of $\lambda_{min}$ can be re-written as

$$F^{opt}_{\lambda_{min}}(\rho_{12}) = \frac{1}{2} - 9[\lambda_{min} - \frac{2}{9}], \frac{1}{6} \leq \lambda_{min} < \frac{2}{9} \quad (40)$$

Using (26) in (40), we get

$$F^{opt}_{\lambda_{min}}(\rho_{12}) = \frac{1}{2} - \frac{135}{8}[F_{avg}(\tilde{V}, \tilde{\rho}_{12}) - \frac{7}{15}], \frac{59}{135} \leq F_{avg}(\tilde{V}, \tilde{\rho}_{12}) < \frac{7}{15} \quad (41)$$

Since $F_{avg}(\tilde{V}, \tilde{\rho}_{12})$ can be determined experimentally [18] so $F^{opt}_{\lambda_{min}}(\rho_{12})$ can be realized using Hong-Ou-Mandel interferometry with only two detectors.

C. Optimal Teleportation Fidelity for Two Qubit System

For two qubit system, optimal teleportation fidelity $f^{opt}(\rho_{12})$ and optimal singlet fraction $F^{opt}_{\lambda_{min}}(\rho_{12})$ are related by [19]

$$f^{opt}(\rho_{12}) = \frac{2F^{opt}_{\lambda_{min}}(\rho_{12}) + 1}{3} = \frac{2}{3} - \frac{135}{12}[F_{avg}(\tilde{V}, \tilde{\rho}_{12}) - \frac{7}{15}], \frac{59}{135} \leq F_{avg}(\tilde{V}, \tilde{\rho}_{12}) < \frac{7}{15} \quad (42)$$

We can say a teleportation scheme is quantum if teleportation fidelity is greater than $\frac{2}{3}$. We can find that the teleportation fidelity given in (42) is always greater than $\frac{2}{3}$. Since $f^{opt}(\rho_{12})$ depends only on $F_{avg}(\tilde{V}, \tilde{\rho}_{12})$ so again the optimal teleportation fidelity can be realized by Hong-Ou-Mandel interferometry with only two detectors.

V. APPLICATION

In this section, we will study a particular type of hybrid entangled system prepared with qubit and binary coherent state (BCS) in the context of quantum teleportation. Although coherent state is described by infinite dimensional Hilbert space but binary coherent state can be described by two dimensional Hilbert space [20]. The states $|+\alpha\rangle$ and $|-\alpha\rangle$ are called BCS and the set $\{|+\alpha\rangle, |-\alpha\rangle\}$ forms a non-orthogonal BCS basis. The state $|\alpha\rangle$ is given by

$$|\alpha\rangle = \exp\left(-\frac{|\alpha|^2}{2}\right)\sum_{n} \frac{\alpha^n}{n!} |n\rangle \quad (43)$$

The BCS basis can be expressed in terms of computational basis as [20]

$$|+\alpha\rangle = \cos(\theta)|0\rangle + \sin(\theta)|1\rangle,$$
$$|-\alpha\rangle = \sin(\theta)|0\rangle + \cos(\theta)|1\rangle, \quad 0 < \theta \leq \frac{\pi}{4} \quad (44)$$

The parameter $\theta$ can be determined by the overlapping between the non-orthogonal states $|+\alpha\rangle$ and $|-\alpha\rangle$ and it is given by

$$|+\alpha| - |\alpha| = \sin(2\theta) \quad (45)$$

A. Generation of Hybrid Entangled State between a qubit and BCS

We now describe the method suggested in [21] for the generation of hybrid entangled state between a qubit and BCS. We should note that strong Kerr nonlinear media can be used to generate hybrid entanglement but this nonlinear effects in existing media are extremely weak. Thus it would be better to use weak Kerr nonlinearity to generate entanglement. Weak Kerr nonlinearity interaction Hamiltonian is given by $H_k = \hbar\chi a_1^\dagger a_1 a_2 a_2$. The interaction between a single-photon qubit $|\psi\rangle_1 = c|0\rangle_1 + d|1\rangle_1, |c|^2 + |d|^2 = 1$ and a coherent state $|\alpha\rangle_2$ under interaction Hamiltonian $H_k$ is described as

$$|\Psi(\vartheta)\rangle_2 = \exp\left(i\frac{\hbar}{\hbar} H_k\vartheta\right)|\psi\rangle_1|\alpha\rangle_2$$
$$= c|0\rangle_1|\alpha\rangle_2 + d|1\rangle_1|\alpha \exp(i\vartheta)\rangle_2 \quad (46)$$

Taking $\vartheta = \pi$ in (46), the state $|\Psi(\vartheta)\rangle_2$ reduces to

$$|\Psi(\pi)\rangle_2 = c|0\rangle_1|\alpha\rangle_2 + d|1\rangle_1|-\alpha\rangle_2 \quad (47)$$

(47) represent a hybrid entangled system between a qubit and BCS. We may note here that an optical fibre of about 3000 km is required for $\vartheta = \pi$ for an optical frequency of $\omega = 5 \times 10^{14}$ rad/sec using currently available Kerr nonlinearity [22].

B. Qubit-BCS Hybrid Entangled State as a Non-Maximally Two Qubit entangled State

We will now show that the hybrid entangled state (47) can be used as a resource state for quantum teleportation. To accomplish our task, we first express BCS in computational basis as in (44) and then treat the hybrid entangled state as an
entangled state in four dimensional Hilbert space. Therefore, qubit-BCS hybrid entangled state \( \ket{\Psi_{12}} \) can be expressed in the computational basis as
\[
|\Psi_{12}\rangle = c\cos(\theta)|00\rangle_{12} + csin(\theta)|01\rangle_{12} + d\sin(\theta)|10\rangle_{12} + dcos(\theta)|11\rangle_{12}
\]
(48)

Let us assume that Alice have generated the hybrid entangled state \( \ket{\Psi_{12}} \). She attach an ancilla prepared in \( |0\rangle_a \) to the state \( |\Psi_{12}\rangle \). Therefore, the resulting three qubit state \( |\Psi_{12a}\rangle \) is given by
\[
|\Psi_{12a}\rangle = |c\cos(\theta)|00\rangle_{12} + csin(\theta)|01\rangle_{12} + d\sin(\theta)|10\rangle_{12} + dcos(\theta)|11\rangle_{12} \otimes |0\rangle_a
\]
(49)

Alice then apply two qubit CNOT-gate on qubit ‘2’ and qubit ‘a’. The state \( |\Psi_{12a}\rangle \) reduces to
\[
|\Phi_{12a}\rangle = c|0\rangle_1 \otimes (\cos(\theta)|00\rangle_{2a} + \sin(\theta)|11\rangle_{2a}) + d|1\rangle_1 \otimes (\sin(\theta)|00\rangle_{2a} + \cos(\theta)|11\rangle_{2a})
\]
(50)

She performs a single qubit measurement in \( \{|0\rangle_1,|1\rangle_1\} \) basis. If the measurement result is \( |0\rangle_1 \) then with probability \( |c|^2 \), she prepare the state
\[
|\Phi_{2a}^{(1)}\rangle = \cos(\theta)|00\rangle_{2a} + \sin(\theta)|11\rangle_{2a}
\]
(51)

Again, if the measurement result is \( |1\rangle_1 \) then with probability \( |d|^2 \), she prepare the state
\[
|\Phi_{2a}^{(2)}\rangle = \sin(\theta)|00\rangle_{2a} + \cos(\theta)|11\rangle_{2a}
\]
(52)

C. Mixed Qubit-BCS Entangled System Shared Between Two Distant Parties As A Resource State For Quantum Teleportation

Case-I: Let us assume that Alice have succeeded to generate the non-maximally entangled state \( |\Phi_{2a}^{(i)}\rangle \) given by (51). She now want to share a subsystem with her distant partner Bob so that they can use the shared entangled state in sending the quantum information. To achieve this, Alice send the subsystem ‘2’ to Bob through the memoryless amplitude damping channel. The transformation under memoryless amplitude damping channel with parameter \( p \) \((0 \leq p \leq 1)\) that governs the evolution of the system and the environment is given by
\[
|0\rangle_2 \otimes |0\rangle_E \rightarrow |0\rangle_2 \otimes |0\rangle_E
\]
\[
|1\rangle_2 \otimes |0\rangle_E \rightarrow \sqrt{1 - p}|1\rangle_2 \otimes |0\rangle_E + \sqrt{p}|0\rangle_2 \otimes |1\rangle_E
\]
(53)

When the qubit in mode ‘2’ of the hybrid system \( |\Phi_{2a}^{(i)}\rangle \) passes through the memoryless amplitude damping channel then the system evolve as a mixed state and it is given by
\[
\rho_{2a} = \sum_{i=0}^{1} (K_i \otimes I)|\Phi_{2a}^{(i)}\rangle \langle \Phi_{2a}^{(i)}| (K_i^\dagger \otimes I)
\]
\[
= \begin{pmatrix}
    \cos^2(\theta) & 0 & 0 & \sqrt{\frac{2 - \sqrt{2}}{2}}psin(2\theta) \\
    0 & psin^2(\theta) & 0 & 0 \\
    0 & 0 & 0 & 0 \\
    \sqrt{\frac{2 - \sqrt{2}}{2}}sin(2\theta) & 0 & 0 & (1 - p)sin^2(\theta)
\end{pmatrix}
\]
(54)

where the Kraus operators \( K_0 \) and \( K_1 \) are given by
\[
K_0 = \begin{pmatrix}
    1 & 0 & 0 \\
    0 & \sqrt{1 - p} & 0 \\
    0 & 0 & 0
\end{pmatrix}, \quad K_1 = \begin{pmatrix}
    1 & 0 \\
    0 & \sqrt{p}
\end{pmatrix}
\]
(55)

Now our task is to investigate whether the mixed state \( \rho_{2a} \) shared between Alice and Bob is still entangled and if it is entangled then under what condition? If we find that the state \( \rho_{2a} \) is entangled under certain conditions then the state \( \rho_{2a} \) can be used as a resource state for quantum teleportation. To probe the above question, we calculate the optimal singlet fraction given in (52). The optimal singlet fraction for the state \( \rho_{2a} \) is given by
\[
F_{(p, \theta)}^{\text{opt}}(\rho_{2a}) = \frac{1}{2} + \frac{1}{2}\sqrt{(1 - p)\sin^2(2\theta) + p^2\sin^4(\theta)} - psin^2(\theta), \quad 0 < \theta \leq \frac{\pi}{4}
\]
(56)

For \( 0 < p < 1 \) and \( 0 < \theta \leq \frac{\pi}{4} \), we have
\[
F_{(p, \theta)}^{\text{opt}}(\rho_{2a}) > \frac{1}{2}
\]
(57)

Hence, the hybrid system described by the density matrix \( \rho_{2a} \) is useful as a resource state for the teleportation of a single qubit. Further we observe the following points:
(i) For any value of the non-orthogonal parameter \( \theta \) \((0 < \theta < 1)\), the value of the quantity \( F_{(p, \theta)}^{\text{opt}}(\rho_{2a}) \) decreases as the noise parameter \( p \) increases from zero to unity.
(ii) For any value of the noise parameter \( p \) \((0 < p < 1)\), the value of the quantity \( F_{(p, \theta)}^{\text{opt}}(\rho_{2a}) \) increases as the non-orthogonal parameter \( \theta \) increases from zero to \( \frac{\pi}{4} \).

Case-II: If Alice have succeeded to generate the non-maximally entangled state \( |\Phi_{2a}^{(2)}\rangle \) given by (52) then also result remains the same as in case-I.

VI. CONCLUSION

To summarize, we have constructed the witness operator by approximating the partial transposition operation, that can detect the entangled state in a bipartite system. The constructed witness operator can be decomposed into Pauli matrices and therefore, we find that it need more than two measurements to realize the witness operator. To minimize the number of measurements, we approximate the entanglement witness, which is a positive semi-definite operator. Then we express the minimum eigenvalue of SPA-PT of the state under investigation in terms of average fidelity between the approximated entanglement witness and SPA-PT of the state. The latter one can be realized by Hong-Ou-Mandel interferometry with only two detectors. So we infer that minimum eigenvalue of SPA-PT of the state can be realized by Hong-Ou-Mandel interferometry with only two detectors. Further, we have obtained the minimum eigenvalue \( \frac{1}{2} \) for a large class of states and then we claim that it holds for any two qubit states. We derived a relation between the optimal
singlet fraction and the minimum eigenvalue of SPA-PT of the state under investigation. Therefore, optimal singlet fraction can also be realized by Hong-Ou-Mandel interferometry with only two detectors. Lastly, we have shown that the hybrid entangled state between a qubit and a binary coherent state can be used as a resource state in quantum teleportation.

[1] M. Piani, S. Gharibian, G. Adesso, J. Calsamiglia, P. Horodecki, and A. Winter, Phys. Rev. Lett. 106, 220403 (2011); R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Rev. Mod. Phys. 81, 865 (2009).
[2] C. H. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres, and W. K. Wootters, Phys. Rev. Lett. 70, 1895 (1993).
[3] C. H. Bennett and S. Wiesner, Phys. Rev. Lett. 69, 2881 (1992).
[4] M. Hillery, V. Buzek and A. Berthiaume, Phys. Rev. A 59, 1829 (1999); S. Adhikari, I. Chakrabarty, P. Agrawal, Quant. Inf. and Comp. 12, 0253 (2012).
[5] C. H. Bennett, and G. Brassard, in Proceedings of the IEEE International Conference on Computers, Systems and Signal Processing, Bangalore, India, (IEEE, New York), 175 (1984); A. K. Ekert, Phys. Rev. Lett. 67, 661 (1991); N. Gisin, G. Ribordy, W. Tittel, and H. Zbinden, Rev. Mod. Phys. 74, 145 (2002).
[6] B. Schumacher, Phys. Rev. A 54, 2614 (1996).
[7] J. Fiurasek, Phys. Rev. A 64, 062310 (2001).
[8] P. Horodecki and A. Ekert, Phys. Rev. Lett. 89, 127902-1 (2002).
[9] J. Fiurasek, Phys. Rev. A 66, 052315 (2002).
[10] J. K. Korbicz, M. L. Almeida, J. Bae, M. Lewenstein, and A. Acn, Phys. Rev. A 78, 062105 (2008).
[11] J. Bae, Rep. Prog. Phys. 80, 104001 (2017).
[12] A. Peres, Phys. Rev. Lett. 77, 1413 (1996).
[13] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Lett. A 223, 1 (1996).
[14] M. Keyl and R. F. Werner, Phys. Rev. A 64, 052311 (2001).
[15] T. Tanaka, Y. Ota, M. Kanazawa, G. Kimura, H. Nakazato, and F. Nori, Phys. Rev. A 89, 012117 (2014).
[16] H-T. Lim, Y-S. Kim, Y-S. Ra, J. Bae, and Y-H. Kim, Phys. Rev. Lett. 107, 160401 (2011).
[17] F. Verstraete, and H. Verschelde, Phys. Rev. Lett. 90, 097901 (2003).
[18] C. J. Kwong, S. Felicetti, L. C. Kwek, J. Bae, quant-ph/arXiv1606.00427.
[19] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. A 60, 1888 (1999).
[20] C. R. Muller, G. Leuchs, C. Marquardt, and U. L. Andersen, Phys. Rev. A 96, 042311 (2017).
[21] B. Yurke, and D. Stoler, Phys. Rev. Lett. 57, 13 (1986).
[22] B. C. Sanders, and G. J. Milburn, Phys. Rev. A 45, 1919 (1992).