Upper bound about cross-sections inside black holes and complexity growth rate

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ABSTRACT: This paper proposes a new universal geometrical inequality between the spacelike cross-section inside event horizon and the total mass for static asymptotically AdS black holes. This inequality implies that complexity growth rate in “complexity-volume conjecture” satisfies the upper bound argued by quantum information theory and the planar Schwarzschild-AdS black hole has fastest complexity growth at late-time limit. This offers a new universal constraint for the inner geometry of black holes as well as makes a first step toward the proof of conjecture that black hole is the fastest quantum “computer” in nature.
1 Background and motivation

Black holes, as ultra dense objects in universe, exhibit many fascinating physical and mathematical properties. Particularly, many such properties can be presented by some universal inequalities, such as the positive mass theorem [1, 2], the second law of black holes [3, 4], the Penrose inequality [5–9] and so on. These universal inequalities do not only show some beautiful mathematical aspects of general relativity but also deepen the physical understanding about foundations of gravity.

From classical viewpoint of general relativity, the inner regions of event horizon are non-observable. Therefore, most of universal inequalities focus on the quantities which involve only horizon and outside black holes. The universal inequalities about the inner structures of black holes are still lack of exploring. However, the recent developments of holographic duality show that the inner structures of black holes may also have boundary correspondences and can play important roles in considering the boundary physics. For example, in the studies of holographic computational complexity [10–15], it has been showed that the complexity growth rate in a Schwarzschild-AdS black hole is determined completely by the inner geometry [16].

This paper makes a first step to explore a universal inequality about inner geometry of static black holes and exhibits an interesting application in holographic duality. The inequality arises from following simple question. Consider the \((d + 1)\)-dimensional Schwarzschild-AdS black hole with the metric
\[
ds^2 = \frac{1}{z^2} \left[ -f(z)dt^2 + \frac{dz^2}{f(z)} + \delta_{ij}dx^i dx^j \right].
\] (1.1)

A mathematical identity

\[
A \text{ About Eq. (2.3)}
\]
Here $f(r) = 1/\ell^2_{\text{AdS}} - f_0 z^d$ and $\ell_{\text{AdS}}$ is the AdS radius. Inside horizon, $z$ is time but $t$ is spatial coordinate. For a class of special slices which are fixed “time” $z$, the volume reads $V = \int dt \Sigma$ with

$$\Sigma = V_{0,d-1} \sqrt{-f} z^{-d}.$$  \hspace{1cm} (1.2)

Here $V_{0,d-1} := \int d^{d-1}x$. Geometrically, $\Sigma$ can be interpreted as the “size” of a cross-section as it integration with respective to $t$ gives us the volume of the slice. Different $z$ will give us different size of cross-section. The directly computation shows that $\max \Sigma = 8\pi E \ell_{\text{AdS}}/(d-1)$. Here $E$ is the total energy/mass of the spacetime. This shows an inequality for these special cross-sections inside a Schwarzschild-AdS black hole

$$0 \leq \Sigma \leq 8\pi E \ell_{\text{AdS}}/(d-1).$$  \hspace{1cm} (1.3)

After a few of straightforward computations, one can verify that above inequality is still true for non-extreme Reissner-Nordström (RN)-AdS black holes.

This paper conjectures that inequality (1.3) is still true for general spatial slices of general static black holes. More detailed, it conjectures that: if (i) outermost horizon is a connected Killing horizon with positive surface gravity and nonnegative sectional curvature, (ii) the spacetime is asymptotically Schwarzschild-AdS \(^1\) and (iii) weak energy condition and Einstein equation are satisfied, then inequality (1.3) is always true and the saturation may happen only in planar Schwarzschild black holes. To support this conjecture and as a very primary study, this paper gives the proof in planar/spherically symmetric cases. It also considers non-planar/non-spherical perturbations in planar/spherically symmetric backgrounds and shows that inequality (1.3) is still true.

Though it is not the original motivation of inequality (1.3), this paper finds that the inequality (1.3) has important application in holographic duality. It has been argued from quantum information theory that the complexity growth rate satisfies Lloyd’s bound \([18]\]

$$\dot{C}(\tau) \leq 2E/\pi.$$  \hspace{1cm} (1.4)

This bound describes the ultimate speed of quantum computations \([18]\). We will show that, if the inequality (1.3) is true, then the complexity growth rate in “complexity-volume” (CV) conjecture \([10, 11]\) always satisfies the Lloyd’s bound (1.4) and the upper bound of Eq. (1.4) is saturated only in planar Schwarzschild-AdS black holes when $\tau \to \infty$. Any regular matter (satisfies weak energy condition) in the bulk always slows complexity growth. This supports the conjecture that black holes may be fastest “computers” in nature \([13, 14]\). The similar growth rate bound was once conjectured in “complexity=action” (CA) conjecture \([13, 14]\) but has been found to be false even in Schwarzschild black holes \([16, 19–21]\). As inequality (1.4) is an important property of complexity in quantum information theory, inequality (1.3) gives us a new viewpoint to compare CV and CA conjectures.

\(^1\)The requirement “asymptotically Schwarzschild-AdS” is stronger than “asymptotically AdS”, see Ref. \([17]\) for a rigours mathematical definition about what is “asymptotically Schwarzschild-AdS”.\(
2 Cross-section inside the black hole

Let us begin our discussion from giving the precise definition about “size of cross-section”. Consider a \((d+1)\)-dimensional static black hole with an outermost non-degenerated connected Killing horizon. Assume \(\xi^I = (\partial/\partial t)^I\) is Killing vector field and timelike outside outermost Killing horizon. A cross-section \(S_{d-1}\) is an arbitrary spacelike \((d-1)\)-dimensional submanifold inside the black hole (If there are inner horizons, then “inside black hole” means the region between the outermost horizon and next-outermost horizon). The size of this cross-section is defined as

\[
\Sigma[\xi^I, S_{d-1}] := -\int_{S_{d-1}} \xi^I n^J d\Sigma_{IJ}.
\]

Here \(d\Sigma_{IJ}\) is the directed surface element 2-form of \(S_{d-1}\) and \(n^I\) is a unit normal covector of \(S_{d-1}\) which satisfies \(n^I \xi_I = 0\). Geometrically, \(\Sigma\) stands for the projected area of \(S_{d-1}\) along the direction \(\xi^I\), see Fig. 1 for an intuitive explanation. The cross-section \(S_{d-1}\) is trivial if \(\xi^I\) tangent to \(S_{d-1}\) as the size is zero. For nontrivial cross-section, normal covector \(n^I\) is timelike and unique upon the origination. This paper conjectures that the integration (2.1) always satisfies inequality (1.3) under the conditions mentioned above.

In a general static \((d+1)\)-dimensional spacetime, we can always choose local coordinates so that the metric reads

\[
ds^2 = \frac{1}{z^d} \left[ -f e^{-\chi} dt^2 + f^{-1} dz^2 + h_{ij} dx^i dx^j \right].
\]

Here \(f = f(z)\), \(h_{ij}\) and \(\chi\) are functions of \(\{z, x^i\}\). We assume that the outermost horizon is connected and locates at \(z = z_h\). A nontrivial cross-section \(S_{d-1}\) inside black hole can be parameterized by \(z = z_S(x^i)\) and \(t = t_S(x^i)\). Then size of \(S_{d-1}\) is

\[
\Sigma[\xi^I, S_{d-1}] = \int_{z = z_S(x^i)} z^{-d} \sqrt{-f e^{-\chi}/2} \sqrt{\tilde{h}} d^{d-1} x.
\]

Here \(\tilde{h}_{ij} := h_{ij}(x^i) + f^{-1} \partial_i z \partial_j z\), \(\tilde{h}^{ij}\) is the inverse of \(\tilde{h}_{ij}\) and

\[
\tilde{h} := \det(\tilde{h}_{ij}) = (1 + f^{-1} h^{ij} \partial_i z \partial_j z) \tilde{h} > 0.
\]

See appendixes A and B for mathematical details.

3 Relationship to the complexity growth rate

Though it is completely based on geometrical considerations, the “size of cross-section” has directly relationship to the complexity growth rate in CV conjecture. The CV conjecture (see Refs. [10–12] for more details) states that the complexity of a boundary CFT state is proportional to the maximal volume of space-like codimension-one surfaces \(W_d\) connecting boundary time slices \(t_L\) and \(t_R\), i.e.,

\[
C = \max_{\partial W_d = t_L \cup t_R} \frac{V[W_d]}{G_N \ell},
\]

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- 3 -
Here $G_N$ is the Newton constant and we set $G_N = 1$ for convenience, $\ell$ is a length scale associated with the bulk geometry such as the horizon radius or AdS radius and so on. We take $\ell = 4\pi^2 \ell_{\text{AdS}}/(d - 1)$.

Globally, the time coordinate $t$ in metric (2.2) is ambiguous when we compute the volume of $W_d$ as a same $t$ can stand for the region inside horizon or outside horizon. This ambiguous can be cured by introduce a new parameter $s$ such that $W_d$ is parameterized by $z = z(s, x^i)$ and $t = t(s)$ with $\{t(\infty) = -t_L, t(\infty) = t_R\}$. The volume functional then is given by $V[W_d] = \int L \, ds$ and

$$L = \sqrt{-f e^{-\chi t^2} + [f^{-1} - f^{-2} \tilde{h}^{ij} \partial_i z \partial_j z] z'^2} \sqrt{\tilde{h}}. \tag{3.2}$$

Here the “$'$” stands for the partial derivative with respective to $s$.

As the spacetime is static, the maximal volume depends on only the value of $t_L + t_R$. Note the time direction of $t_L$ is opposite to the direction of bulk time $t$ at the left boundary. For two symmetric time slices, i.e., $t_L = t_R$, the extremal surface contains two parts: $t > 0$ and $t < 0$. One of them is just the mirror symmetry of the other. Their intersection is denoted by $A$, which is given by $t(s_A) = 0$ and $z = z(s_A, x^i) = z_A(x^i)$. See Fig. 1 for a schematic explanation. Due to mirror symmetry between left ($t < 0$) and right parts ($t > 0$), we have $\partial_t z|_A = 0$.

The extremal surface is obtained by variation on volume functional. The maximal volume $V$, i.e., the on-shell value of $V$, is only the function $t_L + t_R = \tau$ and so we have $V_{\text{on-shell}} = V_{\text{on-shell}}(\tau)$. The growth rate of complexity then reads

$$\ell \dot{C} = \frac{dV_{\text{on-shell}}}{d\tau} = \frac{\partial V_{\text{on-shell}}}{\partial t_R}. \tag{3.3}$$

Recall Hamilton-Jacobi equation in classical mechanics that the partial derivative of on-shell action with respective to canonical coordinate gives us canonical momentum. Thus,
we obtain \( \ell \dot{C} = P \) with

\[
P = \int_{s=\infty}^{s_0} \frac{-fe^{-\chi}z^{-d}\sqrt{h}d^{d-1}x}{-fe^{-\chi} + [f^{-1} - f^{-2}h^{ij}\partial_i z \partial_j z](\partial_t z)^2} \tag{3.4}
\]

Here \( \partial_t z = z'/t' \). As the volume functional does not depend on canonical coordinate \( t \) explicitly, its “momentum” will be independent of \( s \). The integration in Eq. (3.4) will be same if we compute it in the surface of \( s = s_A \) (i.e., surface \( A \)), so we have

\[
\ell \dot{C} = P = \int_{A} \sqrt{-fe^{-\chi}z^{-d}\sqrt{h}d^{d-1}x} = \Sigma[\xi^I, A]. \tag{3.5}
\]

Here we have used the fact \( \partial_t z|_A = 0 \). The complexity growth rate is given by size of cross-section \( A \). In fact Eq. (3.5) will be still true even if the coordinates (2.2) can only cover a neighborhood of \( A \). In Schwarzschild-AdS and RN-AdS black holes, it recovers the result reported by Ref. [16] after we setting \( z(s, x^i) = z(s) \) and \( \chi = 0 \).

We see that inequality (1.3) implies inequality (1.4) and only the planner AdS black hole may saturated Lloyd’s bound (1.4). All regular matters (satisfy weak energy condition) will slow the complexity growth of black holes.

4 Proof in planar/spherically symmetric cases

We assume that \( \chi \) is only the function of \( z \) and \( h_{ij}dx^i dx^j = d\Omega_{k,d-1}^2 \) is the line element at \((d-1)\)-dimensional unite sphere \((k = 1)\) or \((d-1)\)-dimensional flat space \((k = 0)\). We also require that \( \chi = \mathcal{O}(z^d) \) and \( f(z)e^{-\chi/2} = k z^2 + 1/\ell_{AdS}^2 - f_0 z^d + \mathcal{O}(z^{d+1}) \) as \( z \to 0 \) so that the spacetime asymptotically Schwarzchild-AdS. The total energy/mass and entropy \( S \) reads

\[
E = \frac{(d-1)V_k,d-1}{16\pi} f_0, \quad S = \frac{V_{k,d-1}}{4z_h^d}. \tag{4.1}
\]

Here \( V_{k,d-1} := \int \sqrt{h}d^{d-1}x \). The proof contains 3 steps.

In the first step, we note that in the inner region \( 0 < \tilde{h} \leq h \). This shows

\[
\Sigma \leq \int_{z = z(s)} \sqrt{-fe^{-\chi/2}z^{-d}\sqrt{h}d^{d-1}x}. \tag{4.2}
\]

Note that \( z \) in above integration is the function of \( x^i \). Let us assume that \( z = \bar{z} \) maximizes the function \( \sqrt{-fe^{-\chi/2}z^{-d}} \). Then we find

\[
\Sigma \leq V_{k,d-1} \sqrt{R(z)}. \tag{4.3}
\]

and \( R(z) := -f(z)e^{-\chi(z)}z^{-2d} \). We now need to find upper bound of \( R \).

In the second step, we use Einstein’s equations. The equation for \( \chi \) reads

\[
\frac{d}{dz} \chi = 16\pi z e^{-\chi/2} T_{I,J} w^I w^J / (d-1). \tag{4.4}
\]
Here \( T_{ij} \) is the energy-momentum tensor and \( w^I = f^{-1}e^\chi/2 (\partial_t)^I + (\partial_I)^I \) is a future-directed null vector. The weak energy condition contains null energy condition and so insures \( \chi' \geq 0 \). Because boundary conditions show \( \chi(0) = 0 \), we find \( \chi \geq 0 \) and so \( e^\chi \geq 1 \).

On the other hand, we use Einstein’s equations for \( f \) and \( \chi \) and obtain

\[
\frac{d}{dz}(z^d R) = \frac{(d - 2)\ell^2_{\text{AdS}} k z^2 + d}{z^{d+1} \ell^2_{\text{AdS}}} + 16\pi e^{-\chi} T_{zz} f \left( \frac{d}{d-1} \right)^2 . \tag{4.5}
\]

When \( z > z_h \), \( (\partial/\partial z)^I \) is timelike and weak energy condition insures \( T_{zz} \geq 0 \) and so \( T_{zz} f \leq 0 \). Consider the partner of Eq. (4.5) in vacuum case with same horizon radius

\[
\frac{d}{dz}(z^d \tilde{R}) = \frac{(d - 2)\ell^2_{\text{AdS}} k z^2 + d}{z^{d+1} \ell^2_{\text{AdS}}} \tag{4.6}
\]

and \( \tilde{R}(z_h) = R(z_h) = 0 \). The solution reads

\[
\tilde{R}(z) = -[k z^2 + 1/\ell^2_{\text{AdS}} - \xi(z_h) z^d] z^{-2d} . \tag{4.7}
\]

Here \( \xi(z_h) \) is the solution of \( k z_h^2 + 1/\ell^2_{\text{AdS}} - \xi z_h^d = 0 \). One can verify \( \tilde{R}(z) \leq \xi(z_h)^2 \ell^2_{\text{AdS}}/4 \).

The weak energy condition and \( k \geq 0 \) can insure \( (z^d R)' \leq (z^d \tilde{R})' \) for \( z \geq z_h \), which implies \( R(z) \leq \tilde{R}(z) \) when \( z > z_h \). Thus we obtain

\[
\Sigma \leq \max \Sigma \leq V_{k,d-1} \xi(z_h) \ell_{\text{AdS}}/2 . \tag{4.8}
\]

The last inequality is saturated only if \( k = 0 \).

At the third step, we refer to Penrose inequality, which exhibits how the horizon area is bounded by the total energy of the black hole. It was first proposed for asymptotically flat spacetime \([5–9]\) and has been generalized into \((d + 1)\)-dimensional asymptotically AdS spacetime

\[
E \geq \frac{(d-1) V_{k,d-1}}{16\pi} \left[ k a(S) \frac{\ell_{AdS}^d}{\ell_{AdS}^d} + a(S) \frac{\ell_{AdS}^d}{\ell_{AdS}^d} \right] . \tag{4.9}
\]

and has been proven in planar/spherical cases (see, e.g., Refs. \([22–26]\)). Here \( a(S) := 4S/V_{k,d-1} = 1/\ell_{AdS}^{d-1} \) is the entropy density. It is saturated only in AdS black holes. Eq. (4.8) implies that \( \Sigma \leq V_{k,d-1} \xi(z_h) \ell_{AdS}/2 = V_{k,d-1} [k z_h^2 + 1/\ell_{AdS}^{d-1}] \ell_{AdS}/2 \), i.e.,

\[
\Sigma \leq \frac{\ell_{AdS} V_{k,d-1}}{2} \left[ k a(S) \frac{\ell_{AdS}^d}{\ell_{AdS}^d} + a(S) \frac{\ell_{AdS}^d}{\ell_{AdS}^d} \right] . \tag{4.10}
\]

The inequality (1.3) is obtained if we combine inequalities (4.9) and (4.10). The saturation may appear only if inequalities (4.10) and (4.9) are both saturated, which will happen only if the spacetime is Schwarzschild black hole with \( k = 0 \).

## 5 Non-planar/non-spherical perturbations

The proof about inequality (1.3) in general cases is high technical and till open. To support inequality (1.3), let us consider a planar/spherical background with small non-planar/non-spherical perturbations. As the upper bound of (1.3) may be saturated only in the planar
Schwarzschild-AdS black hole, we only need to check if the non-planar/non-spherical perturbations will break inequality (1.3) in planar Schwarzschild-AdS black hole background.

Under non-planar/non-spherical static perturbations the metric reads,

$$ds^2 = \frac{1}{z^2}[-fe^{-\epsilon x}dt^2 + f^{-1}dz^2 + h_{ij}dx^i dx^j]. \quad (5.1)$$

Here \( f = f^{(0)}(z) + \epsilon f^{(1)} \), \( h_{ij} = \delta_{ij} + \epsilon h^{(1)}_{ij} \) and \( f^{(0)}(z) = 1/\ell_{\text{AdS}}^2 - f_0 z^d \). We assume \( \epsilon \) is infinitesimal and all the functions are independent of \( t \). \( h^{(1)}_{ij}, f^{(1)} \) and \( \chi \) in general will be functions of \( (z,x^i) \). We also impose a gauge \( \text{tr}(h^{(1)}_{ij}) = 0 \), which leads that \( \det(h_{ij}) = 1 + \mathcal{O}(\epsilon^2) \). Near boundary \( z = 0 \), we assume \( f^{(1)} \sim \mathcal{O}(\epsilon^{d+1}) \) and \( \chi \sim \chi_0(x^i)z^d \).

In the planar Schwarzschild-AdS black hole, we have known that maximal size of cross-section is given by \( z(x^i) = \bar{z} \). With the perturbation, the maximal size of cross-section is given by the cross-section \( z(x^i) = \bar{z} + \epsilon \zeta^{(1)}(x^i) \). Taking them into Eq. (2.3) and only considering linear order of \( \epsilon \), one can find

$$\max \Sigma = \frac{V_{0,d-1}f_0\ell_{\text{AdS}}}{2} \left[ 1 - \int \frac{\epsilon \chi(z, x^i)}{2V_{0,d-1}}d^{d-1}x \right]. \quad (5.2)$$

Assume the energy-momentum tensor of matters fields to be \( \epsilon T_{IJ} \). The Einstein equation for \( \chi \) in the linear order reads

$$\chi' = \frac{16\pi zT_{IJ}w^I w^J}{d - 1} + \frac{z\partial^2 \chi}{(d - 1)f}. \quad (5.3)$$

Here \( \partial^2 := \partial_i \partial^i \). Integrating it and using the fact \( \epsilon T_{IJ}w^I w^J \geq 0 \), then we obtain

$$\partial_1 \int \frac{\epsilon \chi(z, x^i)}{V_{0,d-1}}d^{d-1}x \geq \frac{z\epsilon}{(d - 1)f} \int \frac{\partial^2 \chi}{V_{0,d-1}}d^{d-1}x = 0 \quad (5.4)$$

as the second integration can be converted into a boundary integration and \( V_{0,d-1} = \infty \).

This tells us that \( \forall \bar{z} > 0 \)

$$V_{0,d-1}^{-1} \int \frac{\epsilon \chi(z, x^i)}{V_{0,d-1}}d^{d-1}x \geq V_{0,d-1}^{-1} \int \epsilon \chi(0, x^i)d^{d-1}x = 0 \quad (5.5)$$

and so \( \max \Sigma \leq V_{0,d-1}f_0\ell_{\text{AdS}}/2 \). With the perturbation, the total energy/mass now reads

$$E = \frac{(d - 1)V_{0,d-1}^{-1}}{16\pi} \left[ f_0 + \int \frac{\epsilon \chi_0(x^i)}{2V_{0,d-1}\ell_{\text{AdS}}^2}d^{d-1}x \right]. \quad (5.6)$$

Eq. (5.5) also implies \( V_{0,d-1}^{-1} \int \epsilon \chi_0(x^i)d^{d-1}x \geq 0 \) (because \( \chi(z, x^i) \sim \chi_0(x^i)z^d \) when \( z \to 0 \)) and so \( 8\pi E_{\text{AdS}}/(d - 1) \geq V_{0,d-1}f_0\ell_{\text{AdS}}/2 \). Thus, we conclude that the inequality (1.3) is still satisfied under non-planar/non-spherical static perturbations. This result combining with the proof in planar/spherical symmetric cases offer us strong evidence for the inequality (1.3).
6 Discussion

To conclude, this paper proposed and discussed a new universal inequality for the inner geometry of black holes and found an interesting application in holographic duality. Except for seeking the proof about Eq. (1.3) in more general cases, there are many aspects which are worthy of exploring in detailed in the future.

It is interesting to check if we can use stationary spacetime to replace the static spacetime. An example is BTZ black hole with nonzero angular momentum [27, 28].

\[ ds^2 = \frac{1}{z^2} \left[ -f(z)dt^2 + \frac{dz^2}{f(z)} + (d\phi - J z^2 dt/2)^2 \right] \]  

with \( f(z) = 1/\ell_{\text{AdS}}^2 - f_0 z^2 + J^2 z^4/4 \). A direct computation shows that

\[ \max \Sigma = V_{1,2} \sqrt{f_0^2 \ell_{\text{AdS}}^2 - J^2}/2. \]

One can verify that \( \max \Sigma \leq 8\pi E \ell_{\text{AdS}} \) and the saturation appears only if \( J = 0 \). This implies that we may only need the spacetime to be stationary. On the other hand, the stationary condition seems to be necessary as we need Killing vector \( \xi^I \) to define \( \Sigma \), and it has been reported that the complexity growth rate in CV conjecture can violate Lloyd’s bound in dynamic spacetimes, e.g. see Ref. [29].

Consider that there is a next-outermost Killing horizon at \( z = \tilde{z}_H > z_h \) with nonzero surface gravity. In the limit \( z_h \to \tilde{z}_H \), i.e., the Hawking temperature \( T_H \to 0 \), we can find \( \max \Sigma \to 0 \) but the total energy and entropy can be arbitrarily large. This implies that, in low temperature limit, there may be an upper bound which is controlled by temperature and much tighter than inequality (1.3). For example, in BTZ black hole (6.1) one can verify \( \Sigma \leq \max \Sigma = 4\pi T_H S \ell_{\text{AdS}} \), which is much smaller then \( E \) in low temperature limit. It is interesting to check if a similar bound is also true in general cases.

Our proof is invalid for cases with hyperbolic AdS boundary, i.e., \( k = -1 \). Our discussion is still correct up to Eq. (4.6). However, as \((d-2)\ell_{\text{AdS}}^2 k z^2 + d \) may be negative for large \( z \), we cannot insure that \( \tilde{R}(z) \geq R(z) \) for all \( z \geq z_h \). This is the reason why the conjecture in this paper requires that horizon has nonnegative sectional curvature. In fact hyperbolically AdS black hole can have negative energy so Eq. (1.3) is not true. It is interesting to seek if there is other suitable upper bound for hyperbolic cases.

Finally, the bound can be generalized into asymptotic flat spacetimes. It only needs to set \( k = 1 \) and \( \ell_{\text{AdS}} \to \infty \). The upper bound in Eq. (4.8) should be replaced by \( a_d z_h^{-1-d} \) with a positive number \( a_d \). This leads to a different bound \( \Sigma \leq 4a_d S \leq c_d V_{1,d-1}(E/V_{1,d-1})^{(d-1)/(d-2)} \) with a positive number \( c_d \). The inequality may be saturated only in Schwarzschild black holes. It is worth exploring this result in detail in the future.

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A  About Eq. (2.3)

Let us first explain how to obtain Eq. (2.3). For the nontrivial cross-section $S_{d-1}$, we can always embed it into the a $d$-dimensional spacelike manifold $M_d$ of which unite normal covector $\tilde{n}_I$ satisfies $\tilde{n}_I|_{S_{d-1}} = n_I$. We parameterize the surface $M_d$ to be $z = z(t, x^i)$. The normal covector reads

$$\tilde{n}_I \propto (dz)_I - \dot{z}(dt)_I - \partial_i z(dx^i)_I. \quad (A.1)$$

The condition $\tilde{n}_I|_{S_{d-1}} = n_I$ and $\xi^I n_I = (\partial/\partial t)^I n_I = 0$ implies

$$\dot{z}|_{S_{d-1}} = 0. \quad (A.2)$$

Here dot means the partial derivative with respective to $t$. The induced metric on this surface then reads,

$$ds^2 = \frac{1}{z^2} \left[ (f e^{-x} + f^{-1}\dot{z}^2)dt^2 + 2f^{-1}\dot{z}\dot{t}dx^i \right]. \quad (A.3)$$

It can be rewritten in following form

$$ds^2 = \frac{1}{z^2} \left[ N^2 dt^2 + \tilde{h}_{ij}(dx^i + N_i dt)(dx^j + N^j dt) \right]. \quad (A.4)$$

Here $\tilde{h}_{ij} := h_{ij} + f^{-1}\partial_i z \partial_j z$ is the induced metric of co-dimensional 2 surface and it defines

$$N^i := \frac{1}{z} (1 + f^{-1} h_{ij} \partial_i t \partial_j z) \dot{t}^2. \quad (A.5)$$

As $M_d$ is spacelike, the metric $\tilde{h}_{ij}$ is positive-definite and $N^2 > 0$. The volume of $M_d$ then reads

$$V[M_d] = \int N \sqrt{h} z^{-d} d^{d-1} x dt. \quad (A.6)$$

Using Eq. (B.1) and setting $w_i = 0$, we find that

$$\tilde{h} = (1 + f^{-1} h_{ij} \partial_i t \partial_j z) h. \quad (A.7)$$

Assume the cross-section $S_{d-1}$ is given by $t = t(x^i)$, then we have following $(d-1)$-dimensional induced metric

$$ds^2_{d-1} = \frac{1}{z^2} \left[ \tilde{h}_{ij} + (N^2 + N_i N^i) \partial_t t \partial_j t \right. \quad (A.8)$$

The normal covector of $S_{d-1}$ immersed in $M_d$ is $r_I = \alpha[(\dot{t})_I - (\partial_t)(dx^i)_I]$. The normalized factor $\alpha$ is

$$\alpha = \frac{N z^{-1}}{\sqrt{(1 + \partial_t t N^i)^2 + N^2 h_{ij} \partial_i t \partial_j t}}. \quad (A.9)$$
Here
\((\hat{d}t)_I := (dt)_I + [\tilde{n}^J (dt)_J]\tilde{n}_I\)

and
\((\hat{d}x^i)_I := (dx^i)_I + [\tilde{n}^J (dx^i)_J]\tilde{n}_I\)

are the projections of \((dt)_I\) and \((dx^i)_I\) into the submanifold \(M_d\).

The directed surface element 2-form of cross-section \(S_{d-1}\) is
\(d\Sigma_{IJ} = n_I \wedge r_J \sqrt{\Xi_{d-1}} d^{d-1}x\),
where \(\Xi_{d-1}\) is the determinant of metric \((A.8)\). Based on the mathematical proposition \((B.1)\), we find
\[\Xi_{d-1} = \frac{\tilde{h}}{2^{n-2d}} [(1 + \partial_t N^i)^2 + N^2 \tilde{h}^{ij} \partial_t \partial_j t].\]  \hfill (A.10)

Note that \(\tilde{n}^I \tilde{n}_I = -1\) and, we have \(\tilde{n}^I r_I = 0\) and so
\[\xi^I n^J d\Sigma_{IJ} = -[\alpha (\hat{d}t)_I + (\hat{d}x^i)_I]\xi^I \sqrt{\Xi_{d-1}} d^{d-1}x.\]

Using the fact \(\xi^I = (\partial / \partial t)^I\), we have \((dt)_I \xi^I = 1\) and \((dx^i)_I \xi^I = 0\). Thus, we have
\[(\hat{d}t)_I \xi^I = 1 + [\tilde{n}^I (dt)_J]\tilde{n}_I \xi^I,\]
and
\[(\hat{d}x^i)_I \xi^I = [\tilde{n}^J (dx^i)_J]\tilde{n}_I \xi^I.\]

On the other hand, we have \(\tilde{n}_I |_{S_{d-1}} = n_I\) and so we have \(\tilde{n}_I \xi^I |_{S_{d-1}} = 0\). This shows
\[(\hat{d}x^i)_I \xi^I |_{S_{d-1}} = 0, \ (\hat{d}t)_I \xi^I |_{S_{d-1}} = 1.\]

Then we obtain
\[\xi^I n^J d\Sigma_{IJ} |_{S_{d-1}} = -\alpha \sqrt{\Xi_{d-1}} |_{S_{d-1}} d^{d-1}x.\]

Finally, we find that the size of cross-section then reads
\[\Sigma = - \int_{S_{d-1}} \xi^I n^J d\Sigma_{IJ} = \int_{S_{d-1}} \alpha \sqrt{\Xi_{d-1}} d^{d-1}x = \int_{S_{d-1}} N z^{-d/2} \tilde{h} d^{d-1}x.\]  \hfill (A.11)

Using Eq. \((A.2)\), we find that \(N |_{S_{d-1}} = \sqrt{-f} e^{-\chi/2}\) and then obtain Eq. \((2.3)\).

**B A mathematical identity**

Let us present a mathematical proposition: for a \(n\)-dimensional non-singular metric \(B_{ij}\), two \(n\)-dimensional vectors \(v_i\) and \(w_i\), a number \(c\), we define \(w^i = B^{ij} w_j\) and
\[W_{ij} = B_{ij} + (c + w^i w_i)v_j + (w_i v_j + v_i w_j),\]

Then we have following relationship
\[W = [(1 + v_i w^i)^2 + c B^{ij} v_i v_j] B.\]  \hfill (B.1)
Here $W$ and $B$ are the determinants of $W_{ij}$ and $B_{ij}$, respectively. The proof is as follow.

Let us define $H_{ij}(J) := J^{2}(c + w^{k}w_{k})v_{i}v_{j} + J(w_{i}v_{j} + v_{i}w_{j})$ and
\[ \hat{B}_{ij}(J) = B_{ij} + H_{ij}(J). \]

Then we see that $W = \hat{B}(1)$. Here $\hat{B}(J)$ is the determinant of $\hat{B}_{ij}(J)$. One can verify following three equations
\[ \frac{d\hat{B}}{dJ} = \hat{B} \hat{B}^{ij} H'_{ij}, \quad \frac{d\hat{B}^{ij}}{dJ} = -\hat{B}^{ik} \hat{B}^{jk} \frac{d\hat{B}_{kl}}{dJ}, \quad \frac{d\hat{B}_{ij}}{dJ} = H'_{ij}. \] (B.2)

Here $\hat{B}^{ij}$ is the inverse of $\hat{B}_{ij}$. By combining these three equations, one can find
\[ \frac{d^{2}\hat{B}}{dJ^{2}} = \frac{d\hat{B}}{dJ} \hat{B}^{ij} H'_{ij} + \hat{B} \frac{d\hat{B}^{ij}}{dJ} H'_{ij} + \hat{B} \hat{B}^{ij} H''_{ij}. \] (B.3)

and
\[ \frac{d^{3}\hat{B}}{dJ^{3}} = 0. \] (B.4)

This shows that $\hat{B}$ is the quadratic function of $J$. Thus we have
\[ \hat{B}(J) = \hat{B}_{|J=0} + J \frac{d\hat{B}}{dJ}_{|J=0} + \frac{J^{2}}{2} \frac{d^{2}\hat{B}}{dJ^{2}}_{|J=0}. \] (B.5)

Take $J = 1$ and one will obtain the desired equation (B.1).

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