CLASSIFICATION OF SOLVABLE 3-DIMENSIONAL LIE TRIPLE SYSTEMS

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Abstract. We give the classification of solvable and splitting Lie triple systems and it turn that, up to isomorphism there exist 7 non isomorphic canonical Lie triple systems and, 6 non isomorphic splitting canonical Lie triple systems and find the solvable Lie algebras associated.

1. INTRODUCTION

A Lie triple system (LTS), is a space where is defined a ternary operation, verifying some conditions, namely the Jacobi identity and the derivation identity. They where first introduce by Jacobson [19]. Later on Lister [25] gave a structure theory and the classification of simple LTS. Yamaguty [49] obtain from a total geodesic space triple algebras which where the generalization of LTS. Loos [27] show that a symmetric space can be seen as a quasigroup, and Sabinin [38, 39] show that a quasigroup can be seen as a homogeneous space. In particular, any Bol loop under the left action derivative give a LTS i.e. the description of the infinitesimal structure of a smooth Bol loop contain a LTS. This fact give the idea of investigation of LTS since also the use of LTS appear in the ordinary differential equation functional analysis...In this paper our main object is to give the classification of solvable and splitting LTS up to isomorphism our approach is based on the enveloping Lie algebras of a LTS.Since the Lie algebras obtain from the standard embedding of a LTS it is an enveloping Lie algebra i.e. if a LTS is solvable his enveloping Lie algebra is solvable, conversely if a Lie algebra is solvable the the LTS obtain is solvable. Considering the classification of solvable Lie algebra, we will carry out the classification of LTS of small dimension.

We will organize this paper as follows: The first part is the introduction, the second part we give the definition and some result about LTS. In the third part we give the classification of LTS of dimension two, the forth part we give the classification of solvable LTS and finally the last part we give the classification of splitting LTS.

2. ABOUT LIE TRIPLE SYSTEMS

Definition 2.1 The vector space $\mathfrak{M}$ (finite over the field of real numbers $\mathbb{R}$) with trilinear operation $(x, y, z)$ is called a LTS if the following identities are verify:

$$(x, x, y) = 0$$
Let $M$ be a LTS, a subspace $D \subset M$ is called a subsystem if $(D, D, D) \subset D$, and is called an ideal, if $(D, M, M) \subset D$. The ideals are the Kernel of the homomorphism of the LTS see [27, 46].

**Example** For a typical way of construction of a LTS see in [27, 46].

Let $G$ be a Lie algebra (finite over the field of real numbers $\mathbb{R}$) and $\sigma$-an involutive automorphism, then

$$G = G^+ + G^-$$

where $\sigma|G^+ = Id$ and $\sigma|G^- = -Id$, as any element $x$ from $G$ can be written in the form:

$$x = \frac{1}{2}(x + \sigma x) + \frac{1}{2}(x - \sigma x),$$

where $x + \sigma x \in G^+$, $x - \sigma x \in G^-$ and $G^+ \cap G^- = 0$.

The following inclusions hold:

$$[G^+, G^+] \subset G^+, [G^+, G^-] \subset G^-, [G^-, G^-] \subset G^-.$$  

Then the subspace $G^-$ turns into a LTS relatively under the operation $(x, y, z) = [x, y, z]$.

The inverse construction [27].

Let $M$ be a LTS and define by

$$h(X, Y) : z \rightarrow (X, Y, Z)$$

a linear transformation of the space $M$ into itself where $X, Y, Z \in M$.

Let $H$ be a subspace of the space of linear transformations of the LTS $M$ whose elements are the transformations of the form $h(X, Y)$. The vector space $G = M + H$, become a Lie algebra relatively to the commutator $[A, B] = AB - BA$, $[A, X] = -[X, A] = AX$; $[X, Y] = h(X, Y)$ where $A, B \in H$, $X, Y \in M$.

Let us define the mapping $\sigma$ with the condition $\sigma(A) = A$, if $A \in H$ and $\sigma(X) = -X, X \in M$, then $\sigma$ is an involutive automorphism of a Lie algebra $G = M + H$.

The algebra $G$ constructed above from the LTS, is called universal enveloping Lie algebra of the LTS $M$.

**Definition 2.2** The derivation of the LTS $M$, is called the linear transformation $d : M \rightarrow M$ such that

$$(X, Y, Z)d = (Xd, Y, Z) + (X, Yd, Z) + (X, Y, Zd).$$

One can verify that, the set $d(M)$ of all the derivation of the LTS $M$ is a Lie algebra of the linear transformations acting on $M$.

**Definition 2.3** The embedding of a LTS $M$ into a Lie algebra $G$ is called the linear injection $R : M \rightarrow G$ such that $(X, Y, Z) = [[X^R, Y^R], Z^R]$.

The embedding $R$ of the LTS $M$ into the Lie algebra $G$ is called canonical, if the envelope of the image of the set $M^R$ in the Lie algebra $G$ coincide with $G$ and $h$ does not contain trivial ideals of Lie algebra $G$. Let us note that if the LTS $M$ is a subset of the Lie algebra $G$, then $(X, Y, Z) = [[X, Y, Z]$ and $[M, M]$ is a subalgebra of the Lie algebra $G$ hence $M + [M, M]$ is a Lie
subalgebra of $\mathfrak{g}$ and the initial embedding $\mathcal{R}$ can be consider as canonical in $\mathfrak{M}^R + [\mathfrak{M}^R, \mathfrak{M}^R]$; this lead us to formulate the following proposition:

**Proposition 2.1** For any finite LTS $\mathfrak{M}$ over $\mathbb{R}$, there exist one and only up to automorphism accuracy, one canonical embedding to the Lie algebra.

2.1. **SOLVABLE AND SEMISIMPLE LIE TRIPLE SYSTEM.** Following [25], let $\Omega$ be an ideal of the LTS $\mathfrak{M}$, we assume $\Omega^{(1)} = (\mathfrak{M}, \Omega, \Omega)$ and, $\Omega^{(k)} = (\mathfrak{M}, \Omega^{(k-1)}, \Omega^{(k-1)})$

**Proposition 2.2** [25] For all natural number $k$, the subspace $\Omega^{(k)}$ is an ideal of $\mathfrak{M}$ and we have the following inclusions:

$$\Omega \supseteq \Omega^{(1)} \supseteq ...... \supseteq \Omega^{(k)}$$

**Proof**

$$(\Omega^{(1)}, \mathfrak{M}, \mathfrak{M}) = ((\mathfrak{M}, \Omega, \Omega), \mathfrak{M}, \mathfrak{M}) \subseteq ((\mathfrak{M}, \Omega, \mathfrak{M}, \Omega, \mathfrak{M}) + [[[\mathfrak{M}, \Omega, [\Omega]], \mathfrak{M}])$$

according to the definition of a LTS

$$(\Omega^{(1)}, \mathfrak{M}, \mathfrak{M}) \subseteq (\Omega, \Omega, \Omega)+(\mathfrak{M}, \Omega, \mathfrak{M}), [\mathfrak{M}, \Omega, \mathfrak{M}]) \subseteq (\mathfrak{M}, \Omega, \Omega)+(\mathfrak{M}, \Omega, \Omega) = \Omega^{(1)}$$

that means $\Omega^{(1)}$ is an ideal of $\mathfrak{M}$ further more $\Omega^{(k)} = (\Omega^{(k-1)})^{(1)}$ hence each $\Omega^{(k)}$ is an ideal in $\mathfrak{M}$.

**Definition 2.4** The ideal $\Omega$ of a LTS $\mathfrak{M}$ is called solvable, if there exist a natural number $k$ such that $\Omega^{(k)} = 0$.

**Proposition 2.3** [25] If $\Omega$ and $\Theta$ are two solvable ideals of a LTS $\mathfrak{M}$ then $\Omega + \Theta$ is also a solvable ideal in $\mathfrak{M}$.

**Proof**

using the definition of a LTS, the following inclusion hold: $(\Theta + \Omega)^{(1)} \subseteq (\mathfrak{M}, \Theta, \Theta) + (\mathfrak{M}, \Omega, \Omega) + (\mathfrak{M}, \Theta, \Omega) + (\mathfrak{M}, \Omega, \Theta) \subseteq \Theta^{(1)} + \Omega^{(1)} + \Theta \cap \Omega$.

Assume for every natural number $k$ the following inclusion holds:

$$(\Theta + \Omega)^{(k)} \subseteq \Theta^{(k)} + \Omega^{(k)} + \Theta \cap \Omega$$

by induction let’s prove that its holds for $(k + 1)$

$$(\Theta + \Omega)^{(k+1)} = (\mathfrak{M}, (\Theta + \Omega)^{(k)} + (\Theta + \Omega)^{(k)}) \subseteq (\mathfrak{M}, \Theta^{(k)} + \Omega^{(k)} + \Theta \cap \Omega, (\Theta \cap \Omega)) \subseteq \Theta^{(k+1)} + \Omega^{(k+1)} + \Theta \cap \Omega$$

hence the result

**Definition 2.5** The radical of a LTS denoted by $\mathcal{R}(\mathfrak{M})$, is called the maximal solvable ideal of the LTS $\mathfrak{M}$.

A LTS $\mathfrak{M}$ is called semi-simple if $\mathcal{R}(\mathfrak{M}) = 0$.

**Theorem 2.1** [25] If $\mathcal{R}$ is a radical in $\mathfrak{M}$ then $(\mathfrak{M} \setminus \mathcal{R})$ is semisimple. And if $\Omega$ is an ideal in $\mathfrak{M}$ such that $(\mathfrak{M} \setminus \mathcal{R})$ is semisimple then $\Omega \supseteq \mathcal{R}$.

**Proposition 2.4** [25] The enveloping Lie algebra, of a solvable LTS is solvable. And if a LTS has some solvable enveloping Lie algebra, it is solvable.

**Theorem 2.2** If $\mathfrak{M}$ is a semisimple LTS, then the universal enveloping Lie algebra $\mathfrak{g}$ is semisimple.

**Theorem 2.3** [2] Let $\mathfrak{M}$ be a LTS and $\mathcal{G} = \mathfrak{M} + \mathfrak{h}$ his canonical enveloping Lie algebra and $\mathfrak{r}$ the radical of the Lie algebra $\mathcal{G}$. In $\mathcal{G}$ there exist a subalgebra $\mathcal{P}$ semisimple supplementary to with $\mathfrak{r}$ such that:

$$\mathfrak{M} = \mathfrak{M}^{\prime} + \mathfrak{M}^{\prime\prime}$$ (direct sum of vectors spaces)
where
\[ M' = M \cap r \quad \text{radical of the LTS } M \]
\[ M'' = M \cap P \quad \text{semisimple subalgebra of LTS } M \]
\[ h = h' + h'' \quad \text{(direct sum of vectors spaces)} \]
\[ h' = h \cap r \]
and
\[ h'' = h \cap P \quad \text{are subalgebra in } h \]
\[ r = M' + h' \]
\[ P = M'' + h''. \]

2.2. PROBLEM SETTING. Let \( M \) be a LTS and \( \dim M = 3 \). To be consistent with the above Theorem the following cases are possible:

(1) **Semisimple case**
\( M \)- semisimple LTS (in fact simple). About the classification of such LTS see [1, 12, 25]

(2) **Solvable case**
\( M \) is a solvable LTS. The classification of such system is given section 4.

(3) **Splitting case**
\[ M = M_1 + M_2 \]
where \( M \equiv \mathbb{R} \)- solvable ideal of dimension 1 in \( \mathbb{R} \) and \( M_2 \)-semisimple LTS of dimension 2 This type of LTS is considered at the last section.

3. CLASSIFICATION OF LIE TRIPLE SYSTEM OF DIMENSION 2

For a better survey of such LTS, we will write their trilinear operation in a special form.

Let \( M \) be a 2-dimensional LTS we write the trilinear operation \( (X, Y, Z) = \beta(X, Y)Y - \beta(Y, Z)X \) where \( \beta : V \times V \rightarrow \mathbb{R} \) is a symmetric form. The choice of the basis \( V \equiv e_1, e_2 > \) one can reduce the symmetric form to the view:
\[
\begin{pmatrix}
\alpha & 0 \\
0 & \nu
\end{pmatrix},
\]
where \( \alpha, \nu = \pm 1; 0 \).

By introducing the notation of the derivation
\[ D_{x,y} : M \rightarrow M \]
\[ z \rightarrow (x, y, z) \]
\[ h = \{ D_{x,y} \}_{x,y \in \mathbb{M}}. \]

And
\[ G = M + h \quad \text{canonical enveloping Lie algebra of the LTS } M. \]
Let \( M = \langle e_1, e_2 > \) then,
\[ h = \{ tD_{x,y} \}_{t \in \mathbb{R}}, \]
\[ e_1D = (e_1, e_2, e_1) = \beta(e_1, e_1)e_2 \]
\[ e_2D = (e_1, e_2, e_2) = -\beta(e_2, e_2)e_1 \]
$\mathfrak{g} = \langle e_1, e_2, e_3 \rangle$
where $[e_1, e_2] = e_3$, $[e_1, e_3] = -e_1 D$, $[e_2, e_3] = -e_2 D$

Therefore we can have the up to isomorphism accuracy the following five cases:

1. **(Spherical Geometry)**

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$\mathfrak{g}/\mathfrak{h} \cong \text{so}(3)/\text{so}(2)$

2. **(Lobatchevski Geometry)**

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

$\mathfrak{g}/\mathfrak{h} \cong \text{sl}(2, \mathbb{R})/\text{so}(2)$

3. **LTS with non compact subalgebra $\mathfrak{h}$**

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$\mathfrak{g}/\mathfrak{h} \cong \text{sl}(2, \mathbb{R})/\mathbb{R}$

4. **Solvable case**

   - a)

     $$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

     $e_1 \cdot e_2 = e_3$, $e_1 \cdot e_3 = e_2$

     (This is a Lie algebra $\mathfrak{g}$ of type $g_{3,5}(p = 0)$ in \cite{31})

   - b)

     $$\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix},$$

     $e_1 \cdot e_2 = e_3$, $e_1 \cdot e_3 = -e_2$

     (This is a Lie algebra $\mathfrak{g}$ of type $g_{3,4}(h = -1)$ in \cite{31})

5. **Abelian case**

   $\beta = 0$ $\mathfrak{g}/\mathfrak{h} \cong (\mathbb{R})^2 / \{0\}$

4. **CLASSIFICATION OF SOLVABLE LIE TRIPLE SYSTEMS OF DIMENSION 3**

Let $\mathfrak{m}$ be a solvable LTS of dimension 3, and $\mathfrak{g} + \mathfrak{h}$ its canonical enveloping Lie algebra then $\mathfrak{g}$ is solvable in particular $\mathfrak{g}$ posses a characteristic ideal $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}] \triangleright \mathfrak{g}$,

$\sigma \mathfrak{g}' = \mathfrak{g}'$, $\mathfrak{g}' \cap \mathfrak{m} = \mathfrak{g}' = (\mathfrak{m}, \mathfrak{m}, \mathfrak{m})$ further more $\mathfrak{h} \subset \mathfrak{g}$ since $\mathfrak{h} = [\mathfrak{m}, \mathfrak{m}, \mathfrak{m}]$

then $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}] = \mathfrak{m}' + \mathfrak{h}$ where $\mathfrak{m}' \subset \mathfrak{m}$

Possible situations:
(1) \( \dim \mathfrak{m} = 0 \). Then \( \mathfrak{h}, \mathfrak{m} = \mathfrak{m} = \{0\} \), that means \( \mathfrak{h} \triangleright \mathfrak{g} \)- ideal, that is why \( \mathfrak{h} = \{0\} \) (since \( \mathfrak{g} \) is an enveloping Lie algebra) and \( \mathfrak{m} = \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} \). In this case, the LTS is Abelian and we denote it (type I).

(2) \( \dim \mathfrak{m} = 1 \). Choosing the base \( e_1, e_2, e_3 \) in \( \mathfrak{m} \) such that, \( \mathfrak{m}' = < e_1 > \) and \( \mathfrak{m} = \mathfrak{m} + < e_2, e_3 > \).

We will introduce in consideration the linear transformation \( A, B, C : \mathfrak{m} \rightarrow \mathfrak{m} \), define as:

\[
A = (e_1, e_2, -) = \begin{pmatrix} a & b & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = (e_2, e_3, -) = \begin{pmatrix} \alpha & \beta & \gamma \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]

\[
C = (e_3, e_1, -) = \begin{pmatrix} x & -\alpha - c & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

And if a skew symmetric form defined as \( \Phi(-, -) : \mathfrak{m} \times \mathfrak{m} \rightarrow \mathbb{R} \), such that \( (x, y, e_1) = \Phi(x, y)e_1 \). The dimension of \( \mathfrak{m} \) is 3, that is why there exists \( z \in \mathfrak{m} \), \( z \neq 0 \), such that \( \Phi(-, z) = 0 \). The following cases are possible:

- **b.I.** The skew-symmetric form \( \Phi \) is non zero and \( z \) is parallel to \( e_1 \) \( (z \parallel e_1) \), then in the base \( e_1, e_2, e_3 \) the skew-symmetric form \( \Phi \) has the corresponding matrix:

\[
\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \delta \\ 0 & -\delta & 0 \end{pmatrix},
\]

where \( \delta \neq 0 \). Adjusting \( e_3 \) to \( 1 \mid \delta e_3 \), then \( \Phi(e_2, e_3) = 1 \), \( \Phi(e_3, e_2) = -1 \), so that \( \alpha = 1, a = x = 0 \) and

\[
A = (e_1, e_2, -) = \begin{pmatrix} 0 & b & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = (e_2, e_3, -) = \begin{pmatrix} 1 & \beta & \gamma \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]

\[
C = (e_3, e_1, -) = \begin{pmatrix} 0 & -1 - c & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

The verification of the defined relation of LTS shows that, with accuracy to the choice of the vector basis \( e_2 \) and \( e_3 \), it is possible to afford the following realization of the operators \( A, B, C \) as:

\[
A = 0, \quad B = (e_2, e_3, -) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]

\[
C = (e_3, e_1, -) = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

(type VII)
• b.II. The skew-symmetric form $\Phi$ is non zero and $z$ is not parallel to $e_1$, let $z = e_2$, then

\[
A = (e_1, e_2, -) = \begin{pmatrix} 0 & b & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 
B = (e_2, e_3, -) = \begin{pmatrix} 0 & \beta & \gamma \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 
\]

\[
C = (e_3, e_1, -) = \begin{pmatrix} -1 & -c & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

The verification of the defined relations of LTS, show that the indicated case has no realization.

• b.III. The skew-symmetric form $\Phi$ is trivial. By completing the vector $e_1$ with the arbitrary choose vector $e_2$ and $e_3$ up to the base, it is possible to realize the operator $A, B, C$:

\[
A = (e_1, e_2, -) = \begin{pmatrix} 0 & b & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 
B = (e_2, e_3, -) = \begin{pmatrix} 0 & \beta & \gamma \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 
\]

\[
C = (e_3, e_1, -) = \begin{pmatrix} 0 & -c & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

The verification of the defined relations of LTS, show that by a suitable choice of basis vectors $e_2, e_3$ the following realization of operators $A, B, C$ is possible:

- Abelian Type (Type above)
  $\quad A = C = 0,$

\[
B = (e_2, e_3, -) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.
\]

(Type II)

This LTS, is obtained by a direct multiplication of a LTS of dimension two $< e_1, e_2 >$, by an Abelian one dimensional $< e_3 >$.

- $A = (e_1, e_2, -) = \begin{pmatrix} \pm 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$

\[
B=C=0. (\text{Type III})
\]

$A = (e_1, e_2, -) = \begin{pmatrix} \pm 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$
\[ C = (e_3, e_1, -) = \begin{pmatrix} 0 & -1 & \mp 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \]

(Type IV)

(3) \( \dim \mathcal{M}' = 2 \) in particular, \( \mathcal{M}' \) is a subsystem of dimension two in \( \mathcal{M} \).

one can consider (refer to Section 3) \( \forall a, b, c \in \mathcal{M} \)

\[ (a, b, c) = \beta(a, c)b - \beta(b, c)a \]

where

\[ \beta = \begin{pmatrix} \pm 1 & 0 \\ 0 & 0 \end{pmatrix} \]

and \( \mathcal{M}' \) is a two-dimensional Abelian ideal in \( \mathcal{M} \). In the first case the choice of the base \( \mathcal{M} = \langle e_1, e_2, e_3 \rangle \) such that \( \mathcal{M}' = \langle e_1, e_2 \rangle \), the operations of the LTS are reduced to:

\[ A = (e_1, e_2, -) = \begin{pmatrix} 0 & \pm 1 & x \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}, \quad B = (e_2, e_3, -) = \begin{pmatrix} \alpha & \gamma & \mu \\ \beta & \delta & \nu \\ 0 & 0 & 0 \end{pmatrix}, \]

\[ C = (e_3, e_1, -) = \begin{pmatrix} \kappa & -x - \alpha & \xi \\ \chi & -y - \beta & \beta \\ 0 & 0 & 0 \end{pmatrix}. \]

The verification of the defined relation of LTS, leads to the contradiction of the condition that \( \dim \mathcal{M}' = 2 \).

Let \( \mathcal{M}' = \langle e_1, e_2 \rangle \)-be a two-dimensional Abelian ideal and \( e_3 \)-the vector completing \( e_1, e_2 \) up to the basis. Then:

\[ A = (e_1, e_2, -) = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}, \quad B = (e_2, e_3, -) = \begin{pmatrix} \alpha & \gamma & \mu \\ \beta & \delta & \nu \\ 0 & 0 & 0 \end{pmatrix}, \]

\[ C = (e_3, e_1, -) = \begin{pmatrix} \kappa & -a - \alpha & \xi \\ \chi & -b - \beta & \beta \\ 0 & 0 & 0 \end{pmatrix}. \]

Deforming the vector \( e_1 \) in the limit of the subspace \( \langle e_1, e_2 \rangle \), the matrix \( A \) can be reduced to the form \( a = b = 0 \) or \( a = 1, b = 0 \).

The verification of the defined relation of the LTS, in the second case leads to the following realization of the operators \( A, B, C \):

\[ A = 0, B = (e_2, e_3, -) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \]

\[ C = (e_3, e_1, -) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \]

(type V)
\[
A = 0, B = (e_2, e_3, \cdot ) = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & \pm 1 \\
0 & 0 & 0
\end{pmatrix}, C = 0.
\]

(type VI)

In conclusion to the conducted examination we have the following theorem:

**Theorem 4.1.** Let \( \mathfrak{M} = \langle e_1, e_2, e_3 \rangle \) be a solvable LTS of dimension 3, \( \mathfrak{G} \) - its canonical enveloping Lie algebra(solvable), and let \( A, B, C : \mathfrak{M} \to \mathfrak{M} \) the linear transformations of the view: \( A = (e_1, e_2, \cdot ) \), \( A = (e_1, e_2, \cdot ) \), \( B = (e_2, e_3, \cdot ) \), \( C = (e_3, e_1, \cdot ) \): with isomorphism accuracy, one can find the possibility of the following types:

- **Type I.** \( \mathfrak{M} \) - Abelian Lie triple system.
- **Type II.**

\[
A = 0, C = 0, B = (e_2, e_3, \cdot ) = \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

\( \mathfrak{G} = \langle e_1, e_2, e_3, e_4 \rangle \) - four-dimensional non-decomposable nilpotent Lie algebra with defined relations

\([e_2, e_3] = e_4, [e_3, e_4] = -e_1\)  
(this is \( g_{4,1} \) algebra in Mubaraczyanov classification \[31\]).

- **Type III.** \( \mathfrak{M} \) is a direct product of a two-dimensional solvable LTS \( \langle e_1, e_2 \rangle \), and a one-dimensional Abelian \( \langle e_3 \rangle \):

\[
A = (e_1, e_2, \cdot ) = \begin{pmatrix}
0 & \pm 1 & 1 \\
0 & 0 & b \\
0 & 0 & 0
\end{pmatrix}, B = 0, C = 0
\]

\( \mathfrak{G} = \langle e_1, e_2, e_3, e_4 \rangle \) - four-dimensional solvable and decomposable Lie algebra, with defined relations:

\([e_1, e_2] = e_4, [e_2, e_4] = \pm e_1\)

moreover \( \mathfrak{G} = \langle e_1, e_2, e_4 \rangle \oplus < e_3 > \), where \( \langle e_1, e_2, e_4 \rangle \) - three-dimensional solvable Lie (algebra \( g_{3,4,5} \) in Mubaraczyanov classification \[31\]).

- **Type IV.**

\[
A = (e_1, e_2, \cdot ) = \begin{pmatrix}
0 & \pm 1 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, B = 0, C = (e_3, e_1, \cdot ) = \begin{pmatrix}
0 & -1 & \pm 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

\( \mathfrak{G} = \langle e_1, e_2, e_3, e_4 \rangle \) - four-dimensional solvable and non-decomposable Lie algebra, with defined relations:

\([e_1, e_2] = e_4, [e_2, e_4] = \pm e_1\)

\([e_1, e_3] = \pm e_4, [e_3, e_4] = -e_1\)

(algebra \( g_{4,5,6} \) in Mubaraczyanov classification \[31\]).
• Type V

\[ B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \pm 1 \\ 0 & 0 & 0 \end{pmatrix}, \ A = C = 0 \]

\( \mathfrak{g} = \langle e_1, e_2, e_3, e_4 \rangle \) - four-dimensional solvable non-decomposable Lie algebra with defined relations:

\[ [e_2, e_3] = e_4, [e_2, e_4] = -e_1 \]
\[ [e_3, e_4] = \mp e_2 \]

(algebra \( g_{8\backslash 9} \) in Mubarakzyanov classification \[\mathfrak{3}\].)

• Type VI

\[ A = 0, B = (e_2, e_3, -) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, C = (e_3, e_1, -) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \]

\( \mathfrak{g} = \langle e_1, e_2, e_3, e_4, e_5 \rangle \) - five-dimensional solvable non-decomposable Lie algebra, with defined relations:

\[ [e_1, e_2] = e_4, [e_1, e_3] = -e_5 \]
\[ [e_3, e_4] = -e_1, [e_3, e_5] = -e_2 \]

(as a result we obtain an extension of four-dimensional Abelian ideal

\( \mathfrak{g} = \langle e_1, e_2, e_4, e_5 \rangle \) by means of \( \langle e_3 \rangle \), algebra \( g_{4,13} \) in Mubarakzyanov classification \[\mathfrak{3}\].)

• Type VII.

\[ A = 0, B = (e_2, e_3, -) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, C = (e_3, e_1, -) = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \]

Lie algebra \( \mathfrak{g} = \langle e_1, e_2, e_3, e_4, e_5 \rangle \) - five-dimensional solvable non-decomposable Lie algebra, with defined relations:

\[ [e_2, e_3] = e_4, [e_1, e_3] = e_5 \]
\[ [e_1, e_4] = -e_1, [e_2, e_5] = -e_1, [e_4, e_5] = e_5 \]

(algebra \( g_{4,11} \) in Mubarakzyanov classification \[\mathfrak{32}, \mathfrak{33}\].)

5. CLASSIFICATION OF SPLITTING 3-DIMENSIONAL LIE TRIPLE SYSTEMS

Let \( \mathfrak{M} = \mathfrak{M}_1 + \mathfrak{M}_2 \) - be a splitting 3-dimensional LTS, where \( \mathfrak{M}_1 \cong \mathbb{R} \) - is a one dimensional solvable ideal in \( \mathfrak{M} \) and \( \mathfrak{M}_2 \) be a 2-dimensional simple LTS. Introduce in consideration a basis \( (e_1, e_2, e_3) \) in \( \mathfrak{M} \) such that \( \mathfrak{M}_1 = \langle e_1 \rangle \) and \( \mathfrak{M}_2 = \langle e_2, e_3 \rangle \) and linear operators \( A, B, C : \mathfrak{M} \rightarrow \mathfrak{M} \) such that \( A = (e_1, e_2, -), A = (e_1, e_2, -), B = (e_2, e_3, -), C = (e_3, e_1, -) \) using the process apply in the previous case one can obtain the following theorem:

**Theorem 5.1.** The following situation are possible and non isomorphic:

• **Type 1.** \( \mathfrak{M} = \mathbb{R} \oplus \mathfrak{M}_2 \) - direct sum of one dimensional Abelian ideal and 2-dimensional simple ideal in \( \mathfrak{M} \) where \( \mathfrak{M}_2 \) is a simple a simple 2-dimensional LTS of the view \( \text{so}(3) / \text{so}(2), \text{sl}(2, \mathbb{R}) / \text{so}(2), \text{sl}(2, \mathbb{R}) / \mathbb{R} \).
• Type 2.
\[ A = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, C = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \]

\( M_2 \) is a simple 2-dimensional LTS of the view \( so(3)/so(2) \)

• Type 3.
\[ A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, C = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \]

\( M_2 \) is a simple 2-dimensional LTS of the view \( \text{sl}(2, \mathbb{R})/so(2) \)

• Type 4.
\[ A = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, C = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \]

\( M_2 \) is a simple 2-dimensional LTS of the view \( \text{sl}(2, \mathbb{R})/so(2) \)

• Type 5.
\[ A = -C = \begin{pmatrix} 0 & -\frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \]

\( M_2 \) is a simple 2-dimensional LTS of the view \( \text{sl}(2, \mathbb{R})/so(2) \)

• Type 6.
\[ A = C = \begin{pmatrix} 0 & -\frac{1}{4} & -\frac{1}{4} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \]

\( M_2 \) is a simple 2-dimensional LTS of the view \( \text{sl}(2, \mathbb{R})/so(2) \)

The proof is somewhat intricate calculation as done in the section above.

Acknowledgment: This paper was able to be achieved, thanks to the scholarship obtain from the Agence Universitaire de la Francophonie.

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