Upper limits on the size of a primordial black hole

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We provide precise constraints on the size of any black holes forming in the early Universe for a variety of formation scenarios. In particular, we prove that the size of the apparent horizon of a primordial black hole formed by causal processes in a flat Friedmann universe is considerably smaller than the cosmological apparent horizon size for an equation of state \( p = k \rho \) (\( 1/3 < k < 1 \)). This also applies for a stiff equation of state (\( k = 1 \)) or for a massless scalar field. The apparent horizon of a primordial black hole formed through hydrodynamical processes is also considerably smaller than the cosmological apparent horizon for \( 0 < k \leq 1 \). We derive an expression for the maximum size which an overdense region can have without being a separate closed universe rather than part of our own. Newtonian argument shows that a black hole smaller than the cosmological horizon can never accrete much.

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I. INTRODUCTION

In the early stage of the universe, black holes may have formed from cosmological density perturbations \[1\]. Such “primordial” black holes (PBHs) may produce \( \gamma \)-ray background radiation and cosmic rays through quantum evaporation or contribute to the density of the universe, and this leads to observational constraints on their number density \[2\]. Studying the formation and evolution of PBHs therefore leads to important constraints on models of the early universe. However, to calculate these constraints, one needs to know how large PBHs were at their formation epoch and how much they subsequently increased their mass through accretion.

A simple physical argument suggests that the size of a PBH will initially be comparable with the cosmological particle horizon scale for a hard equation of state \[3\]. This is because an overdense region at maximum expansion must be larger than the Jeans scale in order to collapse against the pressure but not so large that it forms a separate closed universe. Since both these scales are comparable to the particle horizon size, with the first being only slightly below the second, the size at maximum expansion must be finely tuned if a PBH is to form. (For a soft equation of state, the Jeans scale is much smaller than the cosmological horizon scale, but the collapse is likely to lead to pancake formation and fragmentation unless the region is very spherically symmetric \[4\], so one again needs fine-tuning.) Since the size of a region at maximum expansion is determined by its initial overdensity (compared to the flat Friedmann background), only density fluctuations with a narrow range of amplitudes can produce PBHs.

The limits on the size of a PBH have only been estimated rather roughly in previous papers. However, the fraction of the Universe going into PBHs is very sensitive to these limits \[3\], especially if the fluctuations on a given scale have a Gaussian distribution, so one would like to calculate them more precisely. In order to do this, we first need to determine the various cosmological scales that arise in the early Universe: in particular, the cosmological particle horizon, the cosmological apparent horizon, the cosmological sonic horizon and the Jeans length. Depending on the formation scenario, these may all be relevant to PBH production. They all have the same order of magnitude but they also depend on the equation of state parameter \( k \) (where \( p = k \rho \)) and one purpose of this paper is to calculate these dependencies for an exact Friedmann model very precisely.

The region undergoing collapse will not itself be part of a flat Friedmann model, so in order to discuss the size of a PBH, one needs to make some assumption about the form of the overdensity. One approach is to assume that the universe is exactly Friedmann beyond some matching radius. In this case, the black hole apparent horizon is necessarily within the matching radius but its precise position (relative to some cosmological scale) depends on the density profile in the overdense region. The simplest models have constant overdensity, in which case the region can be regarded as part of a closed Friedmann model. However, more complicated possibilities can be envisaged, so it is important to obtain more general results.

In this paper we will derive upper limits on the black hole size which are independent of any assumption about the density profile. In particular, we prove that the black hole apparent horizon size is considerably smaller than...
the cosmological apparent horizon scale for an equation of state parameter in the range $1/3 < k \leq 1$. An important tool in proving these results is the use of double-null coordinates. This is partly because, with the highly non-linear fluctuations required for PBH formation, it is not always clear how to construct spatial hypersurfaces of constant time. It is also because double-null coordinates are convenient for numerical calculations. Although we do not present such calculations here, we do present them in an accompanying paper \[5\] for the case in which the universe contains a scalar field.

If the fluctuations which generate the PBHs derive from inflation or quantum gravity effects, they may effectively extend to infinity (or at least well beyond the usual Friedmann particle horizon) rather than being confined to a finite region. In this case, the above results are inapplicable and the only firm upper limit on the size of the PBH comes from the separate universe condition. Although this constraint has been discussed in general terms \[6\], it cannot be formulated precisely without specifying the density profile within the overdense region. However, we will show that it can be calculated exactly in terms of the equation of state parameter $k$ if the overdense region is assumed to be homogeneous.

The amount of accretion by a PBH is also sensitive to its initial size. A simple Newtonian calculation \[7\] suggests that accretion is unimportant if the PBH is much smaller than the cosmological horizon at formation but that one could have self-similar evolution, with the black hole growing at the same rate as the universe, if the size is initially comparable with the cosmological horizon scale. The standard formation criterion implies that this is indeed the case, which suggests that there could be a large amount of accretion for a hard equation of state. However, self-similar PBH solutions have been studied extensively \[8, 9, 10, 11, 12\] and it has been found that no such solution contains a black hole inside an exact flat Friedmann solution for a perfect fluid with $0 \leq k \leq 1$. This applies even for a stiff fluid with $k = 1$, although this has been the subject of some controversy \[9, 10\]. These results have been confirmed by numerical calculations \[13, 14, 15, 16\]. Indeed simulations suggest that, the harder the equation of state, the more difficult it will be for a PBH to form and the smaller the mass accretion rate will be. Also one can show that a relativistic effect suppresses the accretion when the PBH has a size very close to that of the cosmological apparent horizon \[5\].

II. DOUBLE-NULL FORMULATION OF EINSTEIN’S FIELD EQUATIONS

If we adopt units in which $G = c = 1$ and the abstract index notation of \[17\], the Einstein equations

$$R_{ab} - \frac{1}{2} g_{ab} R = 8\pi T_{ab}$$  \hspace{1cm} (2.1)

can be written as

$$R_{ab} = 8\pi \bar{T}_{ab},$$  \hspace{1cm} (2.2)

where

$$\bar{T}_{ab} \equiv T_{ab} - \frac{1}{2} g_{ab} T,$$

$$T \equiv T_{aa}. \hspace{1cm} (2.3)$$

We consider a spherically symmetric system, for which the line element can be written in the form

$$ds^2 = -a^2(u, v) du dv + r^2(u, v) (d\theta^2 + \sin^2 \theta d\phi^2), \hspace{1cm} (2.5)$$

where $u$ and $v$ are advanced and retarded times, respectively. Then, from Eq. \[2.2\], we obtain the following partial differential equations:

$$\frac{r_{,uu}}{r} = 2 \frac{a_{,u}}{a} \frac{r_{,u}}{r} - 4\pi \bar{T}_{uu},$$  \hspace{1cm} (2.6)

$$\frac{r_{,uv}}{r} = \frac{a_{,u}}{a^2} \frac{r_{,v}}{r} - \frac{a_{,uv}}{a} - 4\pi \bar{T}_{uv},$$  \hspace{1cm} (2.7)

$$\frac{r_{,vv}}{r} = 2 \frac{a_{,v}}{a} \frac{r_{,v}}{r} - 4\pi \bar{T}_{vv},$$  \hspace{1cm} (2.8)

$$\frac{r_{,uv}}{r} = -\frac{a^2}{4r^2} - \frac{r_{,v} r_{,u}}{r^2} + \frac{a^2}{2r^2} 4\pi \bar{T}_{\theta \theta}. \hspace{1cm} (2.9)$$

There are no other independent components of the stress-energy tensor: spherical symmetry requires $\bar{T}_{\theta \phi} = \sin^2 \theta \bar{T}_{\phi \phi}$ and $\bar{T}_{\theta \phi} = 0$. The initial data for the above equations are most naturally given on two initial null surfaces, $u = u_0$ and $v = v_0$. On each null surface the geometry is described by the two functions $a$ and $r$. 

The Hawking mass \( m \), which is a well-behaved quasi-local mass in spherically symmetric spacetimes, is defined as
\[
m \equiv \frac{r}{2} \left(1 + \frac{4r_u r_v}{a^2}\right).
\] (2.10)

Equations (2.10)–(2.12) then imply (see also [18])
\[
m_{u} = \frac{8\pi r^2}{a^2}(T_{uv}r_{u,v} - T_{uu}r_{v})
\] (2.11)
\[
m_{v} = \frac{8\pi r^2}{a^2}(T_{uv}r_{v,u} - T_{uv}r_{u})
\] (2.12)

We adopt the double null coordinate system as a natural choice of coordinates because we will be dealing with highly nonlinear perturbations in the cosmological background, so it is not always guaranteed that there exists a “cosmologically preferred” comoving time slicing.

### III. UPPER LIMIT ON BLACK HOLE APPARENT HORIZON SIZE

In the following, we focus on an ingoing null surface \( v = v_0 \), as shown schematically in Fig. [1]. For a given matter stress-energy tensor, \( a \) can be chosen arbitrarily on this surface and then \( r \) is determined by Eq. (2.10). This implies that the form of the function \( a \) on \( v = v_0 \) can be understood as a gauge freedom.

At the black hole apparent horizon the expansion along the outgoing null congruence vanishes, i.e. \( r_v = 0 \), while at the cosmological apparent horizon the expansion along the ingoing null congruence vanishes, i.e. \( r_u = 0 \). The cosmological apparent horizon is distinct from the cosmological particle horizon and defined in terms of the quasi-local properties of spacetime. The radius of the cosmological apparent horizon is the Hubble radius \( R^{-1} \) in a flat Friedmann universe, as shown later. From the definition (2.10) of the Hawking mass, it is clear that \( r = 2m \) at these apparent horizons.

**Lemma 1** The condition \( 2m/r < 1 \) holds if and only if \( r_u r_v < 0 \). The condition \( 2m/r = 1 \) holds if and only if \( r_u r_v = 0 \). The condition \( 2m/r > 1 \) holds if and only if \( r_u r_v > 0 \).

*Proof.* Trivial from Eq. (2.10). \( \square \)

**Theorem 1** Suppose there exists \( u_0 \) such that \( r > 0 \), \( r_v > 0 \), and \( r_u < 0 \) at \((u_0,v_0)\). Then if \( T_{uu} \geq 0 \), there exists a centre \( u_c \) such that \( r(u_c,v_0) = 0 \) and \( r(u,v_0) \) is a strictly decreasing function of \( u \) for \( u_0 < u < u_c \).

*Proof.* Equation (2.10) can be rewritten as
\[
\left(\frac{r}{a^2}\right)_u = -4\pi \frac{r}{a^2} T_{uu}.
\]

The condition \( r > 0 \) at \((u_0,v_0)\) implies that \( r \) is non-zero there, so the left-hand side is negative on the null surface \( v = v_0 \). Since \( r > 0 \) and \( r_u < 0 \) at \((u,v_0)\), we have \((r_u/a^2)(u,v_0) \leq (r_u/a^2)(u_0,v_0) < 0 \) for \( u > u_0 \) and \( r > 0 \). Since \( a^2(u,v_0) \) is subject to the gauge choice, we can always assume that \( a^2(u,v_0) \) is bounded from below by a positive value. Thus \( r_u \) is bounded from above by a negative value. Therefore, there exists \( u_c \) such that \( r = 0 \) at \((u_c,v_0)\) and such that \( r(u,v_0) \) is a strictly decreasing function of \( u \) for \( u_0 < u < u_c \). \( \square \)

**Lemma 2** Suppose there exists \( u_0 \) such that \( r > 0 \), \( r_v > 0 \), and \( r_u < 0 \) at \((u_0,v_0)\). If \( T_{uu} \geq 0 \) and there exist roots \( \lambda \) such that \( 2m/r = 1 \) at \((\lambda,v_0)\) for \( u_0 < \lambda < u_c \), then \( r_u < 0 \) and \( r_v > 0 \) for \( u_0 < u < u_{BH} \), where \( u = u_{BH} \) is the smallest root and corresponds to the black hole apparent horizon. If there is no such root, then \( r_v > 0 \) and \( r_u < 0 \) for \( u_0 < u < u_c \).

*Proof.* Trivial from Lemma 1 and Theorem 1. \( \square \)

**Theorem 2** Suppose there exists \( u_0 \) such that \( r > 0 \), \( r_v > 0 \), and \( r_u < 0 \) at \((u_0,v_0)\). If \( T_{uu} \geq 0 \) and \( T_{uv} \geq 0 \), then \( m(u,v_0) \) is a decreasing function of \( u \) for \( u_0 < u < u_{BH} \) if a black hole apparent horizon exists at \( u = u_{BH} \) and for \( u_0 < u < u_c \) otherwise, where \( r(u_c,v_0) = 0 \). \( m(u,v_0) \) is constant if and only if \( T_{uu} = T_{uv} = 0 \).

*Proof.* Since \( r_u < 0 \) and \( r_v > 0 \) in the relevant interval, equation (2.11) implies \( m_u \leq 0 \). \( \square \)
Theorem 3 Suppose that $T_{uu} \geq 0$ and $T_{uv} \geq 0$ and that there exists $u_0$ such that $r > 0$, $r_v > 0$ and $r_u < 0$ at $(u_0, v_0)$. If a black hole apparent horizon exists at $u_{BH} > u_0$, then its area radius is less than or equal to $2m(u_0, v_0)$. Equality holds if and only if $T_{uu} = T_{uv} = 0$.

Proof. From Lemma 1 and Theorem 2 we have $r(u_{BH}, v_0) = 2m(u_{BH}, v_0) \leq 2m(u_0, v_0)$. □

Here we comment on the energy conditions assumed above. For a perfect fluid, we have

$$T^{ab} = \rho u^a u^b + p (g^{ab} + u^a u^b),$$

where $u^a$, $\rho$ and $p$ are the four velocity, energy density and pressure, respectively. The condition $T_{uu} \geq 0$ is equivalent to requiring $\rho \geq -p$. The condition $T_{uv} \geq 0$ is equivalent to requiring $\rho \geq p$. These two conditions together are equivalent to the dominant energy condition $\rho \geq |p|$. If we assume an equation of state of the form $p = k \rho$, the dominant energy condition requires $-1 \leq k \leq 1$ and $\rho \geq 0$. For a massless scalar field $\Psi$, we have

$$T^{ab} = \Psi \cdot a \Psi \cdot b - \frac{1}{2} g^{ab} \Psi \cdot c \Psi \cdot c,$$

and the conditions $T_{uu} \geq 0$ and $T_{uv} = 0$ automatically hold. Recall that a massless scalar field can be identified with a stiff fluid (with the equation of state $p = \rho$) if it is vorticity-free and has a timelike gradient [19]. However, we do not need this equivalence in the following discussion.

IV. APPLICATION OF UPPER LIMIT TO A PBH

The above considerations are completely general, but we now focus on the situation in which one has a black hole in a cosmological background. To make the discussion precise, we adopt a “compensated” PBH model, in which the black hole apparent horizon is contained within a perturbed region, which is surrounded by an exact flat Friedmann solution beyond some (generally finite) radius. Various approaches have been used in this context. One is to model the black hole as part of a closed Friedmann model, surrounded by a Schwarzschild void, so that the total mass at the edge of the void corresponds to that within a flat Friedmann model at the same radius [13]. Another is to match the collapsing region to the flat Friedmann background by a shock-wave or a sound-wave, since this allows a discontinuity in the density or density gradient [3]. If the PBH forms from density perturbations generated in a preceding inflationary phase, then the black hole could have a size larger than the particle horizon, but we do not consider such a situation in this section.

We therefore have various possible situations, depending on the size of the matching radius. These possibilities correspond to the following definitions, in which the upper limit on the matching radius is progressively reduced.

Definition 1 If the matching between the perturbed region containing the PBH and the flat Friedmann region is made at some finite radius, then we describe the PBH as “locally produced”.

If the perturbation results from a causal process, it is natural to assume that the perturbed region is inside the particle horizon of the surrounding flat Friedmann region. It should be noted that “causal”, in this sense, is distinct from “physical”, since there exist physical mechanisms (such as inflation) for producing non-causal perturbations.

Definition 2 If the matching between the perturbed region containing the PBH and the flat Friedmann region is made inside the cosmological particle horizon of the flat Friedmann solution, then we describe the PBH as “causally produced”.

If the perturbation is produced by some hydrodynamical process, it is natural to assume that the perturbation is confined within the cosmological sonic horizon, which is the distance traversed by a sound-wave since the big bang. In a flat Friedmann universe containing a perfect fluid with the equation of state $p = k \rho$ ($0 < k < 1$), this is just $\sqrt{k}$ times the particle horizon size. For a stiff equation of state $k = 1$ or a massless scalar field, the cosmological sonic horizon coincides with the particle horizon.

Definition 3 If the matching between the perturbed region containing the PBH and the flat Friedmann region is made inside the cosmological sonic horizon of the flat Friedmann solution, then we describe the PBH as “hydrodynamically produced”.

In fact, this is a very natural value for the matching radius, since the existence of a sound-wave allows for a discontinuity in the pressure and density gradients [8]. Note that the cosmological sonic horizon is somewhat smaller than the Jeans length, which is the scale on which perturbations oscillate as sound-waves rather than grow. As discussed in the Introduction, an overdense region needs to be bigger than the Jeans length to collapse against the pressure [3] and this might seem to be incompatible with the last definition. However, in this argument, the relevant Jeans length is measured at the time when the overdense region has its maximum expansion and this is somewhat before the black hole itself forms.

We now calculate these scales explicitly for a flat Friedmann universe with

\[ ds^2 = -dt^2 + a^2(t)[d\chi^2 + \chi^2(d\theta^2 + \sin^2 \theta d\phi^2)]. \]  

(4.1)

In this case, the null coordinates are related to the conformal time and comoving radial coordinates \( \eta \) and \( \chi \) by

\[ u = \eta - \chi, \]  

(4.2)

\[ v = \eta + \chi, \]  

(4.3)

where \( ad\eta = dt \). It is most convenient to derive expressions in terms of \( u \) and \( v \) since this is more suitable for the application of the results obtained in the previous section. In such coordinates the flat Friedmann spacetime for a perfect fluid with \( p = k \rho \) is given by

\[ a = C\eta^{2/(1+3k)} = C \left( \frac{u+v}{2} \right)^{2/(1+3k)}, \]  

(4.4)

\[ r = a\chi = C \left( \frac{u+v}{2} \right)^{2/(1+3k)} \frac{v-u}{2}, \]  

(4.5)

where \( C(>0) \) is a constant of integration. In the following we assume \( 0 \leq k \leq 1 \).

The cosmological particle horizon is given by

\[ u_{\text{CPH}}(v) = 0, \]  

(4.6)

(i.e. \( \eta = \chi \)). Using Eqs. (2.10), (4.4) and (4.5), we obtain

\[ \frac{2m}{r} = \left( \frac{2}{1+3k} \right)^2 \left( \frac{v-u}{v+u} \right)^2, \]  

(4.7)

so the cosmological apparent horizon has the retarded time coordinate

\[ u_{\text{CAH}}(v) = -\left( \frac{k-1/3}{1+k} \right) v. \]  

(4.8)

Therefore, the cosmological apparent horizon is spacelike for \( 1/3 < k \leq 1 \), null for \( k = 1/3 \) and timelike for \( 0 < k < 1/3 \), as seen in Fig. 2. For \( 0 < k \leq 1 \), the cosmological sonic horizon has the retarded coordinate

\[ u_{\text{CSH}}(v) = \left( \frac{\eta - \chi}{\eta + \chi} \right) v = \frac{1-\sqrt{k}}{1+\sqrt{k}} v, \]  

(4.9)

since the sound-speed is \( \sqrt{dp/d\rho} = \sqrt{k} \). As seen in Fig. 2, the cosmological sonic horizon is timelike and inside the cosmological particle horizon for \( 0 < k < 1 \), while it is null and coincides with the cosmological particle horizon for \( k = 1 \). It is also null for a flat Friedmann spacetime with a massless scalar field, so \( u_{\text{CSH}} = 0 \) in these cases. A similar figure has been presented in [20].

The area radii for these three cosmological horizons are written in terms of the advanced time coordinate \( v \) as

\[ r_{\text{CPH}}(v) = C \left( \frac{v}{2} \right)^{\frac{3(1+k)}{3+1+k}}, \]  

(4.10)

\[ r_{\text{CAH}}(v) = C \left( \frac{1+3k}{2} \right)^{\frac{2v}{3(1+k)}} \]  

(4.11)

\[ r_{\text{CSH}}(v) = C\sqrt{k} \left( \frac{v}{1+\sqrt{k}} \right)^{\frac{3(1+k)}{3+1+k}}, \]  

(4.12)
where we use the abbreviation $r_{\text{CAH}}(v) = r(u_{\text{CAH}}(v), v)$. It is also convenient to express these scales in terms of the Hubble parameter $H$. Since this is defined as

$$H = \frac{1}{a^2} \frac{da}{d\eta} = \frac{2}{1 + 3k} \frac{1}{a \eta},$$

we have

$$aH = \frac{1}{(1 + 3k)(u + v)}.$$  \hspace{1cm} (4.14)

The area radii of the cosmological particle horizon, apparent horizon and sonic horizon are then given by

$$r_{\text{CPH}}(v) = \frac{2}{1 + 3k} H^{-1},$$

$$r_{\text{CAH}}(v) = H^{-1},$$

$$r_{\text{CSH}}(v) = \frac{2\sqrt{k}}{1 + 3k} H^{-1}$$

in a flat Friedmann universe. Note that $H$ is here the Hubble parameter at the cosmological apparent horizon of the exact Friedmann background and this may not be the same as the Hubble parameter within the perturbed region itself.

Another important quantity is the Jeans scale, which is given by

$$\lambda_J = \sqrt{\frac{\pi k}{\rho + p}},$$

although this expression is strictly only valid for $\lambda_J$ less than the cosmological apparent horizon size. Below this scale density fluctuations oscillate as sound-waves rather than growing. However, it should be stressed that this is the Jeans length in the background universe, which is different from the Jeans length in an overdense region. It is usually assumed that a region can collapse to a black hole only if it is larger than Jeans length at maximum expansion but, as discussed later, this is somewhat less than the value given by above. Since $\lambda_J$ is a wavelength, the radius corresponding to Eq. (4.18) is

$$r_J = \frac{1}{2} \lambda_J = \pi \sqrt{\frac{2k}{3(1 + k)}} H^{-1},$$

where we have used the Friedmann relation

$$H^2 = \frac{8\pi}{3} \rho.$$  \hspace{1cm} (4.20)

However, we note that Coles gives a different equation for the Jeans length in a fluid with equation of state parameter $k$ and this leads to the expression

$$r_J = \pi \frac{4\sqrt{k}}{5 + 9k} H^{-1},$$

which we henceforth adopt. This is always of order the sonic horizon but it is not exactly the same. More precisely, the ratio of the physical scales given by Eqs. (4.17) and (4.21) is

$$\frac{r_{\text{CSH}}}{r_J} = \frac{5 + 9k}{2\pi(1 + 3k)}.$$  \hspace{1cm} (4.22)

As $k$ goes from 0 to 1, this decreases monotonically from 0.796 to 0.477. We should note that the nature of the Jeans instability will be considerably modified for $k \gtrsim 0.1$ because of the cosmological expansion. These scales are depicted in Fig. 3. It is interesting to note that $r_{\text{CPH}} = r_{\text{CAH}}$ at $k = 1/3$ and $r_{\text{CSH}}$ has a maximum there.

At the cosmological particle horizon, equation (4.7) with $u = 0$ implies

$$\frac{2m}{r} = \left(\frac{2}{1 + 3k}\right)^2.$$  \hspace{1cm} (4.23)
Therefore $2m/r < 1$ for $1/3 < k \leq 1$, $2m/r = 1$ for $k = 1/3$ and $2m/r > 1$ for $0 < k < 1/3$. This means that, for $1/3 < k \leq 1$, the cosmological particle horizon is inside the cosmological apparent horizon, i.e. $u_{\text{CPH}} > u_{\text{CAH}}$. In this case, $r > 0$, $r_u < 0$ and $r_v > 0$ at the cosmological particle horizon. Since a causally-produced PBH is associated with a perturbed region inside the cosmological particle horizon, Theorem 3 immediately implies the following:

**Theorem 4** For a PBH causally produced from a perfect fluid with $\rho \geq 0$ and $p = k \rho$ ($1/3 < k \leq 1$), the ratio of the area radii of the black hole and cosmological apparent horizons on the same ingoing null surface is less than

$$\left(\frac{2}{1+3k}\right)^3 \left[\frac{3(1+k)}{4}\right]^{3(1+k)/(1+3k)}.$$  

*Proof.* For $1/3 < k \leq 1$, we have $r > 0$, $r_v > 0$ and $r_u < 0$ at $(u_{\text{CPH}}(v), v)$. Theorem 3 then gives

$$r_{\text{BH}}(v) < 2m_{\text{CPH}}(v) = \frac{2m_{\text{CPH}}(v_0)}{r_{\text{CPH}}(v)} \frac{r_{\text{CPH}}(v)}{r_{\text{CAH}}(v)} r_{\text{CAH}}(v)$$

$$= \left(\frac{2}{1+3k}\right)^2 \left(\frac{2}{1+3k}\right) \left[\frac{3(1+k)}{4}\right]^{3(1+k)/(1+3k)} r_{\text{CAH}}(v),$$

where we have used Eqs. (4.11) and (4.12). \(\Box\)

This upper limit is plotted in Fig. 3 and is always less than the cosmological particle horizon for $k > 1/3$. Note that the ratio in Theorem 4 peaks with a value of 1 for a radiation equation of state and this is the most natural one for the early Universe.

**Theorem 5** For a PBH causally produced from a massless scalar field, the ratio of the area radii of the black hole and cosmological apparent horizons on the same ingoing null surface is less than $3\sqrt{6}/32$.

*Proof.* This follows trivially by putting $k = 1$ in Theorem 4 \(\Box\)

For $1/3 < k \leq 1$ or a massless scalar field, once we have fixed the matching coordinate $u_{\text{match}}$, the radius of the black hole apparent horizon is maximized if there is a vacuum outside this horizon. If we also change $u_{\text{match}}$, the upper limit on the black hole size is attained by the vacuum solution in the limit $u_{\text{match}} \rightarrow u_{\text{CPH}}$. For $0 < k \leq 1/3$, the radius of a causally produced PBH could be as large as the cosmological particle horizon size.

At the cosmological sonic horizon, we have

$$\frac{2m}{r} = \left(\frac{2}{1+3k}\right)^2 k$$

and this is necessarily less than 1 for $0 < k \leq 1$. In this case, $r_u < 0$ and $r_v > 0$ at the cosmological sonic horizon. For $k = 1$, the cosmological sonic horizon coincides with the cosmological particle horizon. Therefore, for a hydrodynamically-produced PBH, we can prove the following:

**Theorem 6** For a PBH produced hydrodynamically from a perfect fluid with $\rho \geq 0$ and $p = k \rho$ ($0 < k \leq 1$), the ratio of the area radii of the black hole and cosmological apparent horizons on the same ingoing null surface is less than:

$$\left(\frac{2}{1+3k}\right)^3 \left[\frac{3(1+k)}{2(1+\sqrt{k})}\right]^{3(1+k)/(1+3k)}.$$  

(4.25)

*Proof.* For $0 < k \leq 1$, we have $r > 0$, $r_v > 0$ and $r_u < 0$ at $(u_{\text{CSH}}(v), v)$. Then we have

$$r_{\text{BH}}(v) < 2m_{\text{CSH}}(v) = \frac{2m_{\text{CSH}}(v)}{r_{\text{CSH}}(v)} \frac{r_{\text{CSH}}(v)}{r_{\text{CAH}}(v)} r_{\text{CAH}}(v)$$

$$= \left(\frac{2}{1+3k}\right)^2 k \left(\frac{2\sqrt{k}}{1+3k}\right) \frac{3(1+k)}{2(1+\sqrt{k})} \frac{3(1+k)/(1+3k)}{r_{\text{CAH}}(v)},$$

where we have used Eqs. (4.11) and (4.12). \(\Box\)

This upper limit is also plotted in Fig. 3. It is noted that the upper limit (4.25) takes a maximum value of 0.309 at $k \approx 0.343$. 

V. NON-LOCALISED PERTURBATIONS AND SEPARATE UNIVERSE CONDITION

We have considered PBHs formed from localised perturbations. However, since the Friedmann universe will be perturbed everywhere in some situations (e.g. if the perturbations derive from inflation), the above upper limit analysis may not always apply. In this section we derive the condition that a PBH is part of our universe rather than being a separate closed universe. This argument does not depend on whether the perturbation is localised or not.

For clarity we include factors $c$ and $G$ explicitly. To make the discussion more precise, we assume that the overdense region which evolves to the PBH is part of a homogeneous closed Friedmann model with
\[ ds^2 = -c^2dt^2 + a^2(t)[d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2)]. \tag{5.1} \]

The cosmic scale factor within this region then evolves according to
\[ \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} - \frac{c^2}{a^2}, \tag{5.2} \]
where a dot denotes the derivative with respect to $t$. This implies that the scale factor at maximum expansion is
\[ a_{\text{max}} = \left(\frac{3c^2}{8\pi G\rho}\right)^{1/2}. \tag{5.3} \]

For an equation of state parameter $k$, the background Friedmann density is
\[ \rho_b = \frac{1}{6\pi G(1+k)^2t^2}. \tag{5.4} \]

If we denote the overdensity at maximum expansion by $\Delta \equiv (\rho_{\text{max}}/\rho_b$, equation (5.3) then gives
\[ a_{\text{max}} = \frac{3}{2}(1+k)\Delta^{-1/2}ct = \Delta^{-1/2}cH^{-1}. \tag{5.5} \]

Since the maximum separation between two points on a spatial hypersphere is $\pi$ times the radius of the hypersphere, the largest possible radius for a region at maximum expansion is $r_{\text{max}} = \pi a_{\text{max}}$.

The overdensity parameter $\Delta$ is expressed in terms of $k$ as
\[ \Delta = \frac{\pi}{4} \left[ \frac{3(1+k)}{1+3k} \right]^2 \frac{\Gamma\left(\frac{3(1+k)}{2(1+3k)}\right)^2}{\Gamma\left(\frac{2+3k}{1+3k}\right)^2}, \tag{5.6} \]
where the derivation is given in Appendix. From Eq. (5.6) our final expression for the separate universe scale in units of $H^{-1}$ is
\[ Hr_{\text{max}} = \pi\Delta^{-1/2} = 2\sqrt{\pi} \frac{(1+3k)}{3(1+k)} \frac{\Gamma\left(\frac{2+3k}{1+3k}\right)}{\Gamma\left(\frac{3(1+k)}{2(1+3k)}\right)}. \tag{5.7} \]

In a dust universe ($k=0$), this gives $4/3$. In a radiation universe ($k=1/3$), it gives $\pi/2$. In an (unphysical) universe with $k = \infty$, it gives $2$. This expression is shown in Figure 3. Note that $r_{\text{min}}$ exceeds the cosmological particle horizon for $k \gtrsim 0.126$. It should be stressed that, although we have presented our analysis in terms of the evolution of the overdense region, such a region does not evolve into separate universe. If it is a separate universe at maximum expansion, it is always separate.

We have noted that the Jeans length within an overdense region at maximum expansion is less than the background Jeans length. Since the Jeans length depends upon the inverse square root of the density, one might expect the reduction to be of order $\Delta^{1/2}$, although this is not an exact result since the expression given by Eq. (4.18) only applies in the linear regime. The important point is that the usual lower limit on the size of a black hole at formation is likely to be below the upper limits shown in Fig. 3. Furthermore the existence of critical phenomena may circumvent the Jeans limit anyway, since this permits black holes of arbitrarily small mass.
VI. PBH ACCRETION

If we neglect the background expansion and assume a quasi-stationary flow, a simple Newtonian treatment for a general fluid suggests that the accretion rate for a black hole of mass $M$ should be given approximately by

$$\frac{dM}{dt} = 4\pi \alpha \rho R_A^2 v_s,$$  \hspace{1cm} (6.1)

where $R_A = GM/v_s^2$ is the accretion radius, $v_s$ is the sound-speed in the accreted fluid, $\alpha$ is a factor of order unity which depends upon other physical features of the flow, and factors $G$ and $c$ are now written explicitly. For a flat Friedmann universe containing fluid with equation of state $p = k \rho$, we have

$$v_s = \sqrt{kc},$$  \hspace{1cm} (6.2)

$$R_A = \frac{GM}{ kc^2},$$  \hspace{1cm} (6.3)

$$\rho = \frac{1}{6\pi G (1 + k)^2 t^2}.$$  \hspace{1cm} (6.4)

Therefore the accretion rate is

$$\frac{dM}{dt} = \frac{2\alpha GM^2}{3k^{3/2}(1 + k)^2 c^3 t^2}.$$  \hspace{1cm} (6.5)

This can be integrated to give

$$M = \frac{\alpha t}{1 + \frac{1}{\alpha t} \left( \frac{Mt}{M_f} - 1 \right)},$$  \hspace{1cm} (6.6)

$$A = \frac{3k^{3/2}(1 + k)^2 c^3}{2\alpha G},$$  \hspace{1cm} (6.7)

where $M_f$ is the black hole mass at the time $t_f$ of formation. If we assume that $M_f = \xi At_f$ with $\xi < 1$, then Eq. (6.6) implies

$$M \rightarrow M_f(1 - \xi)^{-1} \text{ as } t \rightarrow \infty.$$  \hspace{1cm} (6.8)

If $\xi \ll 1$, the black hole could not grow very much. However, if $\xi$ is close to 1, then the black hole could grow significantly. In other words, if the black hole mass is initially very close to $At_f$, which is approximately the same as the mass contained within the cosmological sonic horizon, the black hole can grow considerably. In particular, in the limit $\xi = 1$, Eq. (6.6) implies $M \propto t$, so the black hole grows at the same rate as the universe. The physical scale corresponding to the mass for which this analysis predicts self-similar growth is given by

$$r_f H = 2\alpha^{-1}(1 + k)k^{3/2}.$$  \hspace{1cm} (6.9)

This scale is also indicated in Fig. 3 (for $\alpha = 1$) and one sees that in this case it violates the separate universe condition for sufficiently large values of $k$. This is another reason for regarding the Newtonian prediction as suspect.

Since the above calculation neglects the effects of the cosmological expansion, one needs a relativistic calculation to check this. The Newtonian result suggests that one should look for a spherically symmetric self-similar solution, in which every dimensionless variable is a function of $z = r/t$, so that it is unchanged by the transformation $t \rightarrow at$, $r \rightarrow ar$ for any constant $a$. This problem has an interesting but rather convoluted history. By looking for a black hole solution attached to an exact Friedmann solution via a sonic point, Carr & Hawking first showed that there is no such solution for a radiation fluid [8] and the argument was later extended [10] to a general $p = k \rho$ fluid with $0 < k < 1$. Lin et al. [9] subsequently claimed that there is such a solution in the special case $k = 1$. However, Bicknell & Henriksen [12] then showed that this solution is unphysical, in that the density gradient diverges at the event horizon. The solution can be completed only by attaching a Vaidya ingoing radiation solution interior to some surface (i.e. the scalar field has to turn into a null fluid). This is rather contrived, which suggests that the black hole must soon become much smaller than the particle horizon, after which Eq. (6.1) implies there will be very little further accretion.

The present results are qualitatively consistent with numerical calculations [13, 14, 15, 16]. In particular, simulations [14] suggest that, the harder the equation of state, the more difficult it will be for a PBH to form and the
smaller the mass accretion rate will be. This may be understood in the context of the present analysis. Even if the initial perturbation is larger than the cosmological apparent horizon, as may arise in the inflationary scenario, it soon becomes comparable with the cosmological particle horizon, since the propagation speed of the perturbation is the sound speed, and this is slower than the light speed if \( k < 1 \). This means that the constraint on causally produced PBHs obtained also applies in this case. We conclude that the maximum mass and radius of these PBHs will decrease as the equation of state becomes harder if \( k > 1/3 \).

VII. CONCLUSION

We have proved that a causally or hydrodynamically produced PBH cannot be as large as the cosmological apparent horizon for either a perfect fluid with a hard equation of state or a massless scalar field. It is clear that the result can be extended to a wider class of matter fields or to more general cosmological solutions. We have also derived an exact expression for the maximum size that an overdense region can have without being a separate universe. This is always larger than the cosmological apparent horizon and larger than the particle horizon size for a sufficiently hard equation of state. Finally, we have considered accretion by PBHs but numerical simulations will be necessary to determine this precisely. For the perfect fluid case, such simulations indicate a small accretion rate. For a massless scalar field, work is in progress by the present authors \[3\].

APPENDIX: CALCULATION OF THE OVERDENSITY \( \Delta \)

We here calculate the parameter \( \Delta \) appearing in Eq. (5.5). In the dust case this is well known to be \((3\pi/4)^2\) but we need to generalize this result. For general \( k \), we have \( \rho \propto a^{-3(1+k)} \) and so Eq. (5.2) can be written as

\[
\dot{a}^2 = Aa^{-(1+3k)} - B
\]

where

\[
A \equiv \dot{a}_o^2 \Omega_o a_o^{1+3k}, \quad B \equiv \dot{a}_o^2 (\Omega_o - 1) = c^2.
\]

Here the subscript “\( o \)” indicates some initial epoch at which the overdensity is small, so that the density parameter \( \Omega_o \) is close to unity.

By defining a new scale factor \( b \) and a new time coordinate \( \tau \) such that

\[
b = a^{1+3k}, \quad d\tau = (1 + 3k) b^{-3k} dt,
\]

we can transform Eq. (A.1) into the dust form:

\[
\left( \frac{db}{d\tau} \right)^2 = \frac{A}{b} - c^2.
\]

This has the well-known parametric solution:

\[
b = \frac{1}{2} b_{\text{max}} (1 - \cos \eta), \quad \tau = \frac{1}{\pi} \tau_{\text{max}} (\eta - \sin \eta)
\]

where \( \eta = \pi \) corresponds to the epoch of maximum expansion. Equations (A.2) and (A.4) then imply

\[
t = \frac{1}{1 + 3k} \int_0^\tau b^{-3k} d\tau
\]

\[
= \frac{1}{1 + 3k} \left( \frac{b_{\text{max}}}{2} \right)^{1+3k} \left( \frac{\tau_{\text{max}}}{\pi} \right) \int_0^\eta (1 - \cos \eta)^{1+3k} d\eta.
\]

Equations (A.1) and (A.2) give

\[
a_{\text{max}} \quad a_o = \left( \frac{\Omega_o}{\Omega_o - 1} \right)^{1+3k}, \quad b_{\text{max}} \quad b_o = \left( \frac{\Omega_o}{\Omega_o - 1} \right),
\]

\[
0 \quad \Omega_o = 0 \quad \Omega_o - 1,
\]

\[
\Omega_o \quad \Omega_o \quad \Omega_o - 1.
\]

\[
\Omega_o \quad \Omega_o \quad \Omega_o - 1.
\]
so we have
\[
\frac{\rho_{\text{max}}}{\rho_o} = \left( \frac{a_{\text{max}}}{a_o} \right)^{-3(1+k)} = \left( \frac{\Omega_o - 1}{\Omega_o} \right)^{\frac{3(1+k)}{1+3k}}.
\]  \hspace{1cm} (A.8)

Since Eq. (5.4) implies that the background density is given by
\[
\frac{\rho_b}{\rho_o} = \left( \frac{t_o}{t_{\text{max}}} \right)^2,
\]  \hspace{1cm} (A.9)

where \(t_o\) and \(t_{\text{max}}\) are given by Eq. (A.6), the overdensity can be expressed as
\[
\Delta = \left( \frac{\Omega_o - 1}{\Omega_o} \right)^{\frac{3(1+k)}{1+3k}} \left[ \frac{\left( \int_0^\eta (1 - \cos \eta) \frac{1}{\eta^{1+3k}} d\eta \right)^2}{\left( \int_0^\etao (1 - \cos \eta) \frac{1}{\eta^{1+3k}} d\eta \right)^2} \right].
\]  \hspace{1cm} (A.10)

Equation (A.9) implies that the integral limit \(\eta_o\), corresponding to the time \(t_o\), is given by
\[
\eta_o = \cos^{-1} \left( \frac{2 - \Omega_o}{\Omega_o} \right) \approx 2 \sqrt{\frac{\Omega_o - 1}{\Omega_o}}.
\]  \hspace{1cm} (A.11)

The top integral in Eq. (A.10) can be expressed as
\[
\int_0^\pi (1 - \cos \eta) \frac{1}{\eta^{1+3k}} d\eta = 2 \Gamma \left( \frac{3(1+k)}{2(1+3k)} \right) \frac{\Gamma \left( \frac{2+3k}{1+3k} \right)}{\Gamma \left( \frac{2+3k}{1+3k} \right)}
\]  \hspace{1cm} (A.12)

where \(\Gamma(z)\) denotes the gamma function. By using the fact that \(\eta_o\) is small, the lower integral in Eq. (A.10) can be simplified to
\[
\int_0^\etao \left( \frac{\eta^2}{2} \right) \frac{1}{\eta^{1+3k}} d\eta = 2 \Gamma \left( \frac{2+3k}{1+3k} \right) \left( \frac{1 + 3k}{3(1+k)} \right) \left( \frac{\Omega_o - 1}{\Omega_o} \right)^{\frac{3(1+k)}{2(1+3k)}},
\]  \hspace{1cm} (A.13)

where we have used Eq. (A.11). One observes that the terms involving \(\Omega_o\) in the expression for \(\Delta\) necessarily cancel to give
\[
\Delta = \frac{\pi}{4} \left[ \frac{3(1+k)}{1+3k} \right]^2 \frac{\Gamma \left( \frac{3(1+k)}{2(1+3k)} \right)}{\Gamma \left( \frac{2+3k}{1+3k} \right)^2}.
\]  \hspace{1cm} (A.14)

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FIG. 1: Schematic figure for proofs of Theorems 1, 3 and Lemma 2. Here $u_c$ and $u_{BH}$ denote the centre and the black hole apparent horizon, respectively.
FIG. 2: The conformal diagrams for flat Friedmann spacetimes with a perfect fluid with the equation of state $p = k \rho$ for (a) $0 < k < 1/3$, (b) $k = 1/3$, (c) $1/3 < k < 1$ and (d) $k = 1$ or a massless scalar field. CAH, CPH, CSH denote the cosmological apparent horizon, particle horizon and sonic horizon, respectively.
FIG. 3: The characteristic physical scales in units of $H^{-1}$ for different values of $k$, where $H$ is the Hubble parameter at the cosmological apparent horizon of the exact Friedmann background. CPH, CAH and CSH denote the cosmological particle horizon, apparent horizon and sonic horizon, respectively. “Jeans” denotes the Jeans length implied by Coles [22]. These four scales are plotted as thin lines. Causal and Hydro denote the upper limits for causally and hydrodynamically produced PBHs, respectively. SU denotes the upper limit which comes from the separate universe condition. These three scales are plotted as thick lines. SS denotes the black hole size which the Newtonian argument of self-similar growth suggests.