Nonlocal Quantum Electrodynamics Admits a Finitely Induced Gauge Field Action

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Abstract

The Letter reconsiders a result obtained by Chrétien and Peierls in 1954 within nonlocal QED in 4D [Proc. Roy. Soc. London A 223, 468]. Starting from secondly quantized fermions subject to a nonlocal action with the kernel $[i \partial_x a(x) - m b(x)]$ and gauge covariantly coupled to an external U(1) gauge field they found that for $a = b$ the induced gauge field action cannot be made finite irrespectively of the choice of the nonlocality $a (= b)$. But, the general case $a \neq b$ naturally to be studied admits a finitely induced gauge field action, as the present Letter demonstrates.

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Despite its inherent difficulties nonlocal quantum field theory has received continued although changing attention in past decades (for a review and references see [1]–[3]). Restricting ourselves to gauge theories let us mention that in recent years work in nonlocal theories has been performed within quantum field theory [4]–[8] as well as in the neighbouring field of the study of effective Lagrangians in nuclear theory [9]–[11]. The conceptual intentions in the few references selected are quite diverse, but they have in common to refer and to relate to earlier work of Chrétien and Peierls dealing with nonlocal QED in 4D [12]. This investigation explored a special Ansatz for nonlocal QED with the aim to remove the UV divergencies present in standard, local QED (for the Ansatz see Eq. (1) below, the special choice of Chrétien and Peierls was $a = b$, $a(= b)$ arbitrary else). But, the attempt failed (see also [13]). In particular, this negative result is believed to have closed the door to a finitely induced gauge field action. However, because Chrétien and Peierls extended their consideration not to the most general setting possible ($a \neq b$), this belief is not correct in general. Contrary to this belief, in this Letter we will demonstrate that nonlocal QED in 4D admits a finitely induced gauge field action under appropriate choice of the nonlocal Ansatz applied. The purpose of the present Letter therefore is a purely technical one. Conceptual questions the calculation below grew from are dealt with in a separate comprehensive paper [14].

Consider within QED in 4D Minkowski space the following gauge covariant fermion action.

$$
\Gamma_F[A, \psi, \bar{\psi}] = \int d^4x \; d^4x' \; \bar{\psi}(x) \; e^{ie \int_x^{x'} dy_{\mu} A^\mu(y)} \; \cdot \; \left[ a \; (x - x') \; \left( i \; \vec{\phi}_{x'} - e \; A(x') \right) - m \; b \; (x - x') \right] \psi(x')
$$

(1)

$a, b$ are functions (distributions) arbitrary for the moment. The line integration in the phase factor is understood to be performed along a straight line connecting starting and end point. Eq. (1) is written in such shape as to keep contact with standard (local) QED ($\bar{a} = b \equiv 1$) as close as possible (When referring below

\footnote{Fourier transforms are defined for $f(x)$ by ($f$ stands for $a, b$.)}

$$
\hat{f}(p) = \int d^4x \; e^{-ipx} \; f(x).
$$

For convenience, we always write $\hat{f}(s)$ instead of $\hat{f}(p)$ for one and the same function $\hat{f}$ because all functions $f(x)$ we are studying depend on $x$ via $x^2$ only ($s = -p^2/m^2 = p_E^2/m^2$; the subscript $E$ refers to the (Wick rotated) Euclidean momentum variable).
The object of the study is the induced gauge field action $\Gamma_G[A]$ given by the formula

$$ e^{i\Gamma_G[A]} = \int D\psi D\bar{\psi} \ e^{i\Gamma_F[A, \psi, \bar{\psi}]} . $$

(2)

We here report on the calculation of the coefficients of the first two quadratic terms of the derivative expansion of $\Gamma_G[A]$, i.e., the coefficient of the mass term $A_\mu A^\mu$ and the coefficients of $(\partial_\mu A^\mu)^2$ and $\partial_\mu A_\nu \partial^\mu A^\nu$, which are divergent in standard QED. All further terms which from standard QED are known to be finite even in the local limit $\tilde{a} = \tilde{b} \equiv 1$ are considered beyond present interest and will be commented at the end of the Letter only. We will deliberately employ a gauge non-invariant regularization in order to also study the behaviour of the mass term coefficient when lifting the regularization.

For the purpose of the explicit calculation we rewrite $\Gamma_F[A, \psi, \bar{\psi}]$ in the following symmetrized form.

$$ \Gamma_F[A, \psi, \bar{\psi}] = $$

$$ = \frac{1}{2} \int d^4x \ d^4x' \ \bar{\psi}(x) \ e^{ie \int_x^{x'} dy_\mu A^\mu(y)} \ . $$

$$ \cdot \left[ a(x - x') \left( i \Phi_x - e A(x') \right) - m b(x - x') \right] \psi(x') + $$

$$ + \frac{1}{2} \int d^4x \ d^4x' \ \bar{\psi}(x) \left[ - \left( i \Phi_x + e A(x) \right) a(x - x') - m b(x - x') \right] \cdot $$

$$ e^{ie \int_x^{x'} dy_\mu A^\mu(y)} \psi(x') $$

(3)

We then expand the right hand side of Eq. (3) in powers of $A_\mu$ up to $O(A^2)$ (i.e., $O(e^2)$) and insert following expansions (the upper obtained by using $y_\mu(\tau) = (x' - x)_\mu \cdot \tau + x_\mu$, $\tau \in [0, 1]$).

$$ \int_x^{x'} dy_\mu A^\mu(y) = (x' - x)^\mu \left\{ A_\mu(y) + $$
\[ A_\mu(x) + A_\mu(x') = \]
\[ = 2 \left\{ A_\mu(y) + \frac{1}{8} (x' - x)^\nu(x' - x)^\lambda \partial_\nu \partial_\lambda A_\mu(y) + \ldots \right\}_{y = \frac{(x + x')}{2}} \quad \text{(5)} \]

For calculating the coefficients of \( A_\mu A^\mu \), \( (\partial_\mu A^\mu)^2 \), and \( \partial_\mu A_\nu \partial^\mu A^\nu \) in \( \Gamma_G[A] \) it is sufficient to keep at most two derivatives acting on the gauge potentials in \( \Gamma_F[A, \psi, \bar{\psi}] \). The expression obtained this way for \( \Gamma_F \) (we will not give this rather long expression) now serves as the starting point for deriving Feynman rules and calculating the effective action terms desired. One should take notice that \( \Gamma_F \) also contains terms quadratic in \( A_\mu \) what leads to the situation that besides the standard photon polarization diagram also a tadpole contribution to the photon self-energy is to be taken into account.

The explicit calculation of the terms we are aiming at is quite tedious and shall not be displayed here. We only comment few points of the calculation. Coordinate differences as occurring in Eqs. (4), (5) are translated into momentum space as derivatives with respect to a corresponding momentum variable acting on certain functions in momentum space. This of course involves partial integrations in momentum space for which as usual boundary contributions are assumed not to occur. The photon polarization function is a nonlocal distribution. Therefore, from the formal expression derived by the Feynman rules the local structures we are interested in have to be extracted. In order to properly define this procedure we apply a (radial) momentum space UV cut-off at \( \Lambda \) for the loop integration. This regularization is most suited for our purposes. The final result will be given within this gauge non-invariant cut-off regularization. Furthermore, a Wick rotation for the loop integration is performed and such equivalences like (12), (13) further below are used. Then, the final result reads

\[ \Gamma_G[A] = \text{const.} + \frac{e^2}{16\pi^2} \int d^4x \left\{ C_0 m^2 A_\mu(x) A^\mu(x) + \right. \]
\[ + \left[ C_{1s} [g_\mu g_\alpha + g_\mu g_\nu] + C_{1a} [g_\mu g_\alpha - g_\mu g_\nu] \right] A^\mu(x) \partial^\rho \partial^\beta A^\nu(x) + \]
\[ + \ldots \right\} \quad \text{(6)} \]

where \( f' = d/ds f \)
\( C_0 = -s^2 h' \left. \right|_0^{\Lambda^2 \over m^2} \), \hspace{1cm} (7)

\[
\begin{align*}
C_{1s} &= -\frac{1}{6} s^3 h''' - \frac{1}{2} s^2 h'' + \\
&+ \frac{1}{2} \left( e^{-h} \left[ s^4 \tilde{a} \tilde{a}'' + 2 s^3 \tilde{a} \tilde{a}' + s^3 \tilde{b} \tilde{b}'' + s^2 \tilde{b} \tilde{b}' \right] \right)' - \\
&- e^{-h} \left[ \frac{1}{3} s^4 \tilde{a} \tilde{a}''' + 2 s^3 \tilde{a} \tilde{a}'' + \frac{1}{3} s^3 \tilde{b} \tilde{b}'' + 2 s^2 \tilde{a} \tilde{a}' + \\
&+ \frac{3}{2} s^2 \tilde{b} \tilde{b}'' + s \tilde{b} \tilde{b}' \right] \left. \right|_0^{\Lambda^2 \over m^2} , \hspace{1cm} (8)
\end{align*}
\]

\( C_{1a} = \frac{1}{18} s^3 h''' - \frac{1}{6} s^2 h'' - \frac{2}{3} s h' + \frac{2}{3} h + \\
+ \frac{1}{2} \left( e^{-h} \left[ -\frac{1}{3} s^4 \tilde{a} \tilde{a}'' + \frac{2}{3} s^3 \tilde{a} \tilde{a}' - \frac{1}{3} s^3 \tilde{b} \tilde{b}'' + s^2 \tilde{b} \tilde{b}' \right] \right)' + \\
+ e^{-h} \left[ \frac{1}{9} s^4 \tilde{a} \tilde{a}''' + \frac{4}{3} s^3 \tilde{a} \tilde{a}'' + \frac{1}{9} s^3 \tilde{b} \tilde{b}''' + 2 s^2 \tilde{a} \tilde{a}' + \\
+ \frac{7}{6} s^2 \tilde{b} \tilde{b}'' + 2 s \tilde{b} \tilde{b}' \right] \left. \right|_0^{\Lambda^2 \over m^2} - \\
- \int_0^{\Lambda^2 \over m^2} ds \left( \frac{1}{s \tilde{a}^2 + \tilde{b}^2} \left[ \frac{s \tilde{a}^2}{s \tilde{a}^2 + \tilde{b}^2} \left[ s \tilde{a} \tilde{a}' + \tilde{b} \tilde{b}' \right] + \\
+ \frac{2}{3} s^3 \tilde{a} \tilde{a}''' + 3 s^2 \tilde{a} \tilde{a}'' + \frac{2}{3} s^2 \tilde{b} \tilde{b}'' + \\
+ 2 s \tilde{a} \tilde{a}' + 3 s \tilde{b} \tilde{b}'' - s \left( \tilde{b}' \right)^2 + 3 \tilde{b}' \right] \right) , \hspace{1cm} (9)
\]

\[
\begin{align*}
\frac{\alpha}{\pi} &= h(s) = \ln \left[ s \tilde{a}^2 + \tilde{b}^2 \right] , \hspace{1cm} \tilde{a} = \tilde{a}(s) , \hspace{1cm} \tilde{b} = \tilde{b}(s) .
\end{align*}
\]

The displayed result is exact for any value of the cut-off \( \Lambda \), so far no term vanishing at removing the cut-off has been neglected. For \( \tilde{a} = \tilde{b} \equiv 1 \) the standard QED result is reproduced (cf. [13]; [14], Eq. (9-64), for \( \Lambda \to \infty \) the coefficient \( C(0) \) there is related to our expressions by the equation \( C(0) = -e^2 (5C_{1s} + 3C_{1a})/24\pi^2) \).

The expression for the mass term coefficient \( \tilde{\bar{C}} \) can easily be rederived by an
independent method. In order to look for a mass term of the gauge field $A_\mu$ we can restrict ourselves to the class of constant gauge potentials $A_\mu(x) = e^{-1}k_\mu \equiv \text{const.}$ the consideration of which is sufficient for this purpose. For this simple background $\Gamma_G[A]$ is given by the determinant of the kernel of the fermion action $\Gamma_F$ in the presence of the constant background $k_\mu$ which can be viewed in momentum space representation as a constant external momentum. The induced gauge field action reads then $(h = \ln \left[ s\tilde{a}^2 + \tilde{b}^2 \right])$

$$\Gamma_G[e^{-1}k] = -2i V_4 \int_\Lambda \frac{d^4 p}{(2\pi)^4} h \left( \frac{-(p+k)^2}{m^2} \right). \quad (10)$$

The subscript $\Lambda$ in Eq. (10) again indicates that we apply a cut-off regularization with a (radial) momentum space UV cut-off at $\Lambda$. Because we cannot assume from the very beginning that the result in Eq. (10) will be finite (this is related to the vacuum energy problem which we will not consider) we are barred from simply using a shift $p \rightarrow p - k$ (which would make vanish the dependence on $k$ at once; this would only be applicable in a gauge invariant regularization).

Let us further transform the integral appearing in Eq. (10). First, we perform a Wick rotation and then we expand the integrand in powers of $k$ (up to $O(k^4)$).

$$\int_\Lambda \frac{d^4 p_E}{m^2} h \left( \frac{(p_E + k_E)^2}{m^2} \right) = \int_\Lambda \frac{d^4 p}{m^2} \left\{ h(s) + 2 \frac{pk}{m^2} h'(s) + \frac{k^2}{m^2} h''(s) + \right.$$  

$$+ 2 \frac{(pk)^2}{m^4} h'''(s) + 2 \frac{k^2 pk}{m^4} h''(s) +$$  

$$+ \frac{4}{3} \frac{(pk)^3}{m^6} h''''(s) + 2 \frac{2}{3} \frac{(pk)^4}{m^8} h''''(s) + \ldots \right\} \quad (11)$$

For convenience, we have omitted the subscript $E$ on the right hand side. Deleting in the integrand terms antisymmetric with respect to $p \rightarrow -p$ and applying following equivalences (valid under the 4D integral)

$$(pk)^2 \equiv \frac{1}{4} k^2 p^2, \quad (12)$$

$$(pk)^4 \equiv \frac{1}{8} (k^2)^2 (p^2)^2. \quad (13)$$

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we find after some manipulations

\[ \Gamma_G[e^{-1}k] = \frac{V_4}{8\pi^2} m^4 \left\{ \int_0^{\frac{\lambda^2}{m^2}} ds s h(s) - \frac{1}{2} \frac{k^2}{m^2} \left[ \frac{s^2}{m^4} \right]_{\frac{\lambda^2}{m^2}} + \right. \\
\left. + \frac{1}{12} \frac{(k^2)^2}{m^4} \left[ 3 s^2 h''(s) + s^3 h'''(s) \right]_{\frac{\lambda^2}{m^2}} + \ldots \right\} \quad (14) \]

where \( k_\mu \) denotes the constant (Minkowski space) gauge potential. A comparison of the second term with Eq. (7) shows that both mass term results although obtained by different methods agree as expected. Also the first two terms of Eq. (8) can be re-identified in Eq. (14).

We may now ask ourselves which conditions are to be placed on \( a \) and \( b \) in order to make the mass term vanish when lifting the regularization. From Eq. (7) (and the second term in Eq. (14)) we see that the requirement of gauge invariance (i.e., vanishing of any mass term) yields that \( h(s) \) should behave for \( s \rightarrow \infty \) like

\[ h(s) \sim \frac{\text{const.}}{} + O(s^\kappa) \quad , \quad \kappa < -1 \quad . \quad (15) \]

Above condition obviously is also appropriate to make vanish all higher (in powers of \( k \)) gauge non-invariant structures in Eq. (14). By translating information contained in (15) one finds following conditions sufficient to obey it \[ \]

\[ \tilde{a}(s) \sim O(s^\kappa) \quad , \quad \kappa < -1 \quad (16) \]

\[ \tilde{b}(s) \sim \text{const.} + O(s^\kappa) \quad , \quad \text{const.} \neq 0 \quad , \quad \kappa < -1 \quad (17) \]

From these relations one recognizes that \( \tilde{a} \) and \( \tilde{b} \) should behave differently for \( s \rightarrow \infty \), i.e., they cannot be identical. Above consideration explains (in part) the no-go result obtained by CHRÉTIEN and PEIERLS [12] which is caused by the inappropriate factorization property of the kernel of the fermion action in the case of \( \tilde{a} = \tilde{b} \). Such an Ansatz is also in contradiction to results for the fermion self-energy calculated in lowest order of standard QED perturbation theory where \( \tilde{a} \) and \( \tilde{b} \) already differ (see, e.g., [16]).

\[ 2 \] We disregard here the somewhat weaker condition

\[ \tilde{a}(s) \sim -\frac{3}{2} + O(s^\kappa) \quad , \quad \kappa < \frac{-3}{2} \quad , \]

and all other variants requiring some fine tuning between \( \tilde{a} \) and \( \tilde{b} \).
Now, if conditions (16), (17) are fulfilled the expression for the induced gauge field action (6) significantly simplifies. Then, the UV cut-off can be lifted without any problem ($\Lambda \rightarrow \infty$), the coefficients $C_0$ and $C_1$ connected with terms spoiling gauge invariance are vanishing and the completely gauge invariant result finally reads

$$\Gamma_G[A] = \text{const.} +$$

$$+ C_{1a} \frac{e^2}{16\pi^2} \int d^4x \ A^\mu(x) [g_{\mu\nu} \Box - \partial_\mu \partial_\nu] A^\nu(x) + \ldots , \quad (18)$$

with

$$C_{1a} = \frac{2}{3} \ln \left[ \frac{\tilde{b}(\infty)}{\tilde{b}(0)} \right]^2 -$$

$$- \int_0^\infty ds \ \frac{1}{s\tilde{a}^2 + \tilde{b}^2} \left[ \frac{s}{s\tilde{a}^2 + \tilde{b}^2} \left[ s \tilde{a} \tilde{a}' + \tilde{b} \tilde{b}' \right] +$$

$$+ \frac{2}{3} s \tilde{a} \tilde{a}''' + 3 s^2 \tilde{a} \tilde{a}'' + \frac{2}{3} s^2 \tilde{b} \tilde{b}''' +$$

$$+ 2 s \tilde{a} \tilde{a}' + 3 s \tilde{b} \tilde{b}'' - s (\tilde{b}')^2 + 3 \tilde{b}^2 \right] . \quad (19)$$

It is worth noting that the coefficient $C_{1a}$ is finite due to conditions (16), (17). Gauge invariance and UV finiteness are closely related here. All further vacuum polarization terms are of course finite as in standard QED. To see this note that all coefficients of these terms have representations (so far not calculated explicitly yet) analogous to Eqs. (7)–(9). However, for purely dimensional reasons their ingredients decay faster for $s \rightarrow \infty$ than those of the latter. This is the cause that they are finite even in standard QED. The same argument applies to interaction terms in the induced gauge field action $\Gamma_G[A]$. This can easily be inferred from the third term (and all further terms) in Eq. (14). Consequently, if $\tilde{a}$, $\tilde{b}$ obey conditions (16), (17) the induced gauge field action $\Gamma_G[A]$ is completely finite. It seems plausible that the present result will generalize to non-abelian gauge theories as well.

Let us finally further comment the no-go result of Chrétien and Peierls [12]. If one chooses $\tilde{a} = \tilde{b}$ (as done in Ref. [12]), one immediately recognizes from Eq. (8)

\[\text{Were it not for the first term (~} (s\tilde{a}^2 + \tilde{b}^2)^{-2} \text{~in the integral in Eqs. (9), (19), also the weaker condition given in the footnote on p. 7 then replacing (16) would lead to gauge invariance and UV finiteness at the same time.}\]
that the photon polarization function is logarithmically (at least) divergent then irrespectively of the particular choice of $\tilde{a} (= \tilde{b})$ made. If one chooses $\tilde{a} = \tilde{b} \sim \exp[-s]$, e.g., the divergency problem becomes even worse compared with standard QED. In contradistinction to standard QED the mass term in $\Gamma_G[A]$ then would even be less divergent ($\sim \Lambda^4$) than the kinetic term ($\sim \Lambda^6$) and all further terms finite in the local limit ($\tilde{a} = \tilde{b} \equiv 1$) would likely acquire a divergency as well. However, while this definitely rules out the Ansatz $\tilde{a} = \tilde{b}$ it does by far not rule out, as we have seen above, any finite nonlocal quantum electrodynamics in general.

The author thanks D. Robaschik for reading a draft version of the Letter and for helpful advice, and C. Eberlein for support in computerized reference retrieval.
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