Torus actions on compact quotients

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Introduction

Let $G$ denote a Lie group and $\Gamma$ a uniform lattice in $G$. We fix a maximal torus $T$ in $G$ and consider the action of $T$ on the compact quotient $\Gamma/G$. Assuming $T$ to be noncompact we will prove a Lefschetz formula relating compact orbits as local data to the action of the torus $T$ on a global cohomology theory (tangential cohomology). Modulo homotopy, the compact orbits are parametrized by those conjugacy classes $[\gamma]$ in $\Gamma$ whose $G$-conjugacy classes meet $T$ in points which are regular in the split component. Having a bijection between homotopy classes and conjugacy classes in the discrete group we will identify these two. For a class $[\gamma]$ let $X_\gamma$ be the union of all compact orbits in that class. Then it is known that $X_\gamma$ is a smooth submanifold and with $\chi_r(X_\gamma)$ we denote its de-twisted Euler characteristic (see sect. 2). Note that $\chi_r(X_\gamma)$ is local, i.e. it can be expressed as the integral over $X_\gamma$ of a canonical differential form (generalized Euler form). On the other hand $\chi_r(X_\gamma)$ can be expressed as a simple linear combination of Betti numbers (see sect. 2). Next, $\lambda_\gamma$ will denote the volume of the orbit and $P_\gamma$
the stable part of the Poincaré map around the orbit. Then the number

\[ L(\gamma) := \frac{\lambda_\gamma \chi_\gamma (X_\gamma)}{\det(1 - P_s)} \]

will be called the Lefschetz number of \( [\gamma] \) (compare [8]). The class \( [\gamma] \) defines a point \( a_\gamma \) in the split part \( A \) of the torus \( T \) modulo the action of the Weyl group. In the case when the Weyl group has maximal size (for example when \( T \) is maximally split) our Lefschetz formula is an equality of distributions:

\[ \sum_{[\gamma]} L(\gamma) \delta_{a_\gamma} = \text{tr}(\cdot | H^*(F)), \]

where \( H^* \) is the tangential cohomology of the unstable/neutral foliation \( F \) induced by the torus action. In [3] a similar formula is proven to hold up to a smooth function in the case of a flow. The present paper extends results of Andreas Juhl [10], [13] in the real rank one case. See also [11], [12].

1 Euler-Poincaré functions

In this section and the next we list some technical results for the convenience of the reader. Let \( G \) denote a real reductive group of inner type [14] and fix a maximal compact subgroup \( K \). Let \((\tau, V_\tau)\) be a finite dimensional unitary representation of \( K \) and write \((\check{\tau}, V_{\check{\tau}})\) for the dual representation. Assume that \( G \) has a compact Cartan subgroup \( T \subset K \). Let \( g_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0 \) be the polar decomposition of the real Lie algebra \( g_0 \) of \( G \) and write \( g = \mathfrak{t} + \mathfrak{p} \) for its complexification. Choose an ordering of the roots \( \Phi(g, \mathfrak{t}) \) of the pair \( (g, \mathfrak{t}) \). This choice induces a decomposition \( \mathfrak{p} = \mathfrak{p}_- \oplus \mathfrak{p}_+ \).

**Proposition 1.1** For \((\tau, V_\tau)\) a finite dimensional representation of \( K \) there is a compactly supported smooth function \( f_\tau \) on \( G \) such that for every irreducible unitary representation \((\pi, V_\pi)\) of \( G \) it holds:

\[ \text{tr} \pi(f_\tau) = \sum_{p=0}^{\dim(p)} (-1)^p \dim(V_\pi \otimes \wedge^p \mathfrak{p} \otimes V_{\check{\tau}})^K. \]

**Proof:** [3].
Proposition 1.2 Let $g$ be a semisimple element of the group $G$. If $g$ is not elliptic, then the orbital integral $O_g(f_\tau)$ vanishes. If $g$ is elliptic we may assume $g \in T$, where $T$ is a Cartan in $K$ and then we have

$$O_g(f_\tau) = \frac{\text{tr } \tau(g)|W(t, g)| \prod_{\alpha \in \Phi^+_g}(\rho_g, \alpha)}{|G_g : G^0_g|c_g},$$

where $c_g$ is Harish-Chandra’s constant, it does only depend on the centralizer $G_g$ of $g$. Its value is given for example in [3].

Proof: [5]. □

Proposition 1.3 For the function $f_\sigma$ we have for any $\pi \in \hat{G}$:

$$\text{tr } \pi(f_\sigma) = \sum_{p=0}^{\dim \ g/\mathfrak{t}} (-1)^p \dim \text{Ext}^p_{(\mathfrak{g}, K)}(V_\sigma, V_\pi),$$

i.e. $f_\sigma$ gives the Euler-Poincaré numbers of the $(\mathfrak{g}, K)$-modules $(V_\sigma, V_\pi)$, this justifies the name Euler-Poincaré function.

Proof: [5]. □

2 De-twisted Euler characteristics

Let $\mathcal{C}^+$ denote the category of complexes of $\mathbb{C}$-vector spaces which are zero in negative indices and have degreewise finite dimensional cohomology, i.e. the dimension of $H^j(E)$ is finite for all $j$. Let $\mathcal{K}^+$ denote the weak Grothendieck group of $\mathcal{C}^+$, i.e. $\mathcal{K}^+$ is the abelian group generated by all isomorphism classes of objects modulo the relations $A = B + C$, whenever any object in $A$ is isomorphic to the direct sum of an object in $B$ and one in $C$. An element $E = E_+ - E_-$ of $\mathcal{K}^+$ is called a virtual complex. Define the de-twist of an element $E$ of $\mathcal{K}^+$ as $E' = \sum_{k=0}^{\infty} E[-k]$, where $E[k]_j = E_{k+j}$. Since the sum is degreewise finite this defines a new element of $\mathcal{K}^+$. The higher de-twists are defined inductively, so $E^{(0)} = E$ and $E^{(r+1)} = E^{(r)'}$. 
We need to extend the notion of an **Euler characteristic** to infinite virtual complexes by

$$\chi(E) = \sum_{k=0}^{\infty} (-1)^k \dim H^k(E),$$

provided $\dim H^k(E) = \dim H^k(E_+) - \dim H^k(E_-)$ vanishes for almost all $k$.

Call a virtual complex **cohomologically finite** if $\dim H^j(E) = 0$ for large $j$, in other words, the total cohomology $\text{H}(E)$ is finite dimensional.

**Observation:** Let the virtual complex $E$ be cohomologically finite and assume that the Euler characteristic $\chi(E)$ vanishes. Then the de-twist $E'$ is cohomologically finite.

So start with a cohomologically finite virtual complex $E$. If $E^{(1)}, \ldots, E^{(r)}$ are cohomologically finite we have

$$\chi(E^{(0)}) = \ldots = \chi(E^{(r-1)}) = 0$$

and

$$\chi(E^{(r)}) = (-1)^r \sum_{j=0}^{\infty} \binom{j}{r} (-1)^j \dim H^j(E).$$

This is easily proven by induction on $r$. This motivates the following Definition: The **$r$-th de-twisted Euler characteristic** of a cohomologically finite virtual complex $E$ is defined by

$$\chi_r(E) := (-1)^r \sum_{j=0}^{\infty} \binom{j}{r} (-1)^j \dim H^j(E).$$

To every compact manifold $M$ we now can attach a sequence of Euler numbers

$$\chi_0(M), \ldots, \chi_n(M),$$

where $n$ is the dimension of $M$. The most significant of these is, as we shall see, the first nonvanishing one, so define the **generic Euler number** of $M$ as

$$\chi_{\text{gen}}(M) = \chi_r(M),$$

where $r$ is the least index with $\chi_r(M) \neq 0$. 
Proposition 2.1 Let $M, N$ be compact manifolds. We have

$$\chi_{\text{gen}}(M \times N) = \chi_{\text{gen}}(M)\chi_{\text{gen}}(N).$$

Proof: See [4].

To give another example of a situation in which higher Euler characteristics occur we will describe a situation in Lie algebra cohomology which will show up later.

We consider a short exact sequence

$$0 \to n \to l \to a \to 0$$

of finite dimensional complex Lie algebras where $a$ is abelian. In such a situation a $l$-module $V$ is called acceptable, if the $a$-module $H^q(n, V)$ is finite dimensional. Note that $V$ itself needn’t be finite dimensional.

Example 1.: Any finite dimensional $l$-module will be acceptable.

Example 2.: Let $g_0$ denote the Lie algebra of a semisimple Lie group $G$ of the Harish-Chandra class, i.e. $G$ is connected and has a finite center. Let $K$ be a maximal compact subgroup of $G$ and let $G = KAN$ be an Iwasawa decomposition of $G$. Write the corresponding decomposition of the complexified Lie algebra as $g = t \oplus a \oplus n$. Now let $l = a \oplus n$ with the structure of a subalgebra of $g$. Consider an admissible $(g, K)$-module $V$. A theorem of [HeSchm] assures us that $V$ then is an acceptable $l$-module.

Proposition 2.2 Let

$$0 \to n \to l \to a \to 0$$

be an exact sequence of finite dimensional complex Lie algebras. Assume that the Lie algebra $a$ is abelian. Let $V$ be an acceptable $l$-module then with $r = \dim(a)$ we have

$$\chi_0(H^s(l, V)) = \ldots = \chi_{r-1}(H^s(l, V)) = 0,$$

and

$$\chi_r(H^s(l, V)) = \chi_0(H^s(n, V)^a),$$

where $H^s(n, V)^a$ denotes the $a$-invariants in $H^s(n, V)$.

Proof: [5].
3 The Lefschetz formula

Let $G$ be a connected Lie group and $\Gamma \subset G$ a uniform lattice. Note that the existence of $\Gamma$ forces $G$ to be unimodular. Fix a Haar measure on $G$ and consider the representation of $G$ on the Hilbert space $L^2(\Gamma \backslash G)$ given by $R(g)\varphi(x) = \varphi(xg)$. For any smooth compactly supported function $f$ on $G$ define $R(f)\varphi(x) := \int_G f(y)\varphi(xy)dy$, then a calculation shows that $R(f)$ is an integral operator with smooth kernel $k(x,y) = \sum_{\gamma \in \Gamma} f(x^{-1}\gamma y)$. From this it follows that $R(f)$ is a trace class operator. Since this holds for any $f$ we conclude that $L^2(\Gamma \backslash G)$ decomposes under $G$ as a discrete sum of irreducibles with finite multiplicities:

$$L^2(\Gamma \backslash G) = \bigoplus_{\pi \in \hat{G}} N_\Gamma(\pi)\pi.$$  

It follows that $\text{tr} R(f) = \sum_{\pi \in \hat{G}} N_\Gamma(\pi)\text{tr} \pi(f)$. On the other hand, the trace of $R(f)$ equals the integral over the diagonal of the kernel, so

$$\text{tr} R(f) = \int_{\Gamma \backslash G} k(x,x)dx = \sum_{[\gamma]} \text{vol}(\Gamma_\gamma \backslash G_\gamma) O_\gamma(f),$$

where $O_\gamma(f) := \int_{G_\gamma \backslash G} f(x^{-1}\gamma x)dx$ is the orbital integral. Note that this expression depends on the choice of a Haar measure on $G_\gamma$. So we state the Selberg trace formula as

$$\sum_{\pi \in \hat{G}} N_\Gamma(\pi)\text{tr} \pi(f) = \sum_{[\gamma]} \text{vol}(\Gamma_\gamma \backslash G_\gamma) O_\gamma(f).$$

From now on we will assume:

(A1) $G$ is a direct product: $G \cong H \times R$

of an abelian Lie group $R$ and a semisimple connected Lie group $H$ with finite center.

For the following fix a maximal torus $T$ of $H$, write $T = AB$, where $A$ is the split component and $B$ is compact. Let $P = MAN$ a parabolic then $B \subset M$. Let $A^{reg}$ be the set of regular elements of the split torus $A$. Since $H$ acts on $R$ it acts on the unitary dual $\hat{R}$. Our second assumption is
(A2) Any element of $A^{reg}M$ acts freely on $R - \{0\}$ and on $\hat{R} - \{\text{triv}\}$.

For any $\tau \in \hat{R}$ let $H_\tau$ be its stabilizer in $H$. For the trivial representation we clearly have $H_{\text{triv}} = H$. The condition (A2) says that for any nontrivial $\tau \in \hat{R}$ we have $H_\tau \cap A^{reg}M = \emptyset$.

Example: Clearly any semisimple connected $G$ with finite center would give an example but there are also a lot on nonreductive examples such as the following: Let $R := \text{Mat}_2(\mathbb{R})$ with the addition, $H := \text{SL}_2(\mathbb{R})$ and let $H$ act on $R$ by matrix multiplication from the left. Let $G := H \times N$ and $T := \left\{ \begin{pmatrix} a & \ 0 \\ a^{-1} & 1 \end{pmatrix} \right\}$. It is easily seen that our assumptions are satisfied in this case.

We will only consider uniform lattices of the form $\Gamma = \Gamma_H \times \Gamma_R$, where $\Gamma_H$ and $\Gamma_R$ are uniform lattices in $H$ and $R$. We will further assume $\Gamma_H$ to be weakly neat, this means, $\Gamma_H$ is a cocompact torsion free discrete subgroup of $H$ which is such that for any $\gamma \in \Gamma_H$ the adjoint $\text{Ad}(\gamma)$, acting on the Lie algebra of $H$ does not have a root of unity $\neq 1$ as an eigenvalue. Any arithmetic group has a weakly neat subgroup of finite index \[1\].

Example: Take up the above example and let $D$ denote a quaternion division algebra over $\mathbb{Q}$ which splits over $\mathbb{R}$. So we have $D \hookrightarrow GL_2(\mathbb{R})$ and $D^1 \hookrightarrow SL_2(\mathbb{R})$, where $D^1$ is the set of elements of reduced norm 1. Let $O$ denote an order in $D$ and $O^1 := O \cap D^1$. Then $\Gamma := O^1 \times O$ is a uniform lattice in $SL_2(\mathbb{R}) \times \text{Mat}_2(\mathbb{R})$.

Write the real Lie algebras of $G, H, M, A, N, R$ as $\mathfrak{g}_0, \mathfrak{h}_0, \mathfrak{m}_0, \mathfrak{a}_0, \mathfrak{n}_0, \mathfrak{r}_0$ and their complexifications as $\mathfrak{g}, \mathfrak{h}, \mathfrak{m}, \mathfrak{a}, \mathfrak{n}, \mathfrak{r}$. Let $\Phi(\mathfrak{h}, \mathfrak{a})$ denote the set of roots of the pair $(\mathfrak{h}, \mathfrak{a})$. The choice of the parabolic $P$ amounts to the same as a choice of a set of positive roots $\Phi^+(\mathfrak{h}, \mathfrak{a})$. Let $A^- \subset A$ denote the negative Weyl chamber corresponding to that ordering, i.e. $A^-$ consists of all $a \in A$ which act contractingly on the Lie algebra $\mathfrak{n}$. Further let $\overline{A^-}$ be the closure of $A^-$ in $G$, this is a manifold with boundary. Let $K_M$ be a maximal compact subgroup of $M$. We may suppose that $K_M$ contains $B$. Fix an irreducible unitary representation $(\tau, V_\tau)$ of $K_M$. Let $K$ be a maximal compact subgroup of $H$. We may assume $K \supset K_M$.

Since $\Gamma_H$ is the fundamental group of the Riemannian manifold $X_{\Gamma_H} = \Gamma_H \backslash X = \Gamma_H \backslash H / K$
it follows that we have a canonical bijection of the homotopy classes of loops:

\[ [S^1 : X_{\Gamma_H}] \to \Gamma_H/\text{conjugacy}. \]

For a given class \([\gamma]\) let \(X_\gamma\) denote the union of all closed geodesics in the corresponding class in \([S^1 : X_{\Gamma}]\). Then \(X_\gamma\) is a smooth submanifold of \(X_{\Gamma_H}\). Let \(\chi_r(X_\gamma)\) denote the \(r\)-fold de-twisted Euler characteristic of \(X_\gamma\), where \(r = \dim A\).

Let \(E_P(\Gamma)\) denote the set of all conjugacy classes \([\gamma]\) in \(\Gamma\) such that \(\gamma_H\) is in \(H\) conjugate to an element \(a_\gamma b_\gamma\) of \(A^- B\).

Take a class \([\gamma]\) in \(E_P(\Gamma)\). Modulo conjugation assume \(\gamma \in T = AB\), then the centralizer \(\Gamma_{H,\gamma}\) projects to a lattice \(\Gamma_{A,\gamma}\) in the split part \(A\). Let \(\lambda_\gamma\) be the covolume of this lattice. Normalize the measure on \(R\) such that \(\text{vol}(\Gamma_R \backslash R) = 1\).

**Theorem 3.1 (Lefschetz formula, first version)** Let \(\phi\) be compactly supported on \(A^-\), \(\dim G\)-times continuously differentiable and suppose \(\phi\) vanishes on the boundary to order \(\dim G + 1\). Then we have that the expression

\[
\sum_{\pi \in \hat{G}} N_{\Gamma}(\pi) \sum_{p,q} (-1)^{p+q} \int_{A^-} \varphi(a) \text{tr}(a)(H^p(n, \pi) \otimes \wedge^q \mathfrak{p}_M \otimes V_\tau)^{K_M}) da
\]

equals

\[
(-1)^{\dim(N)} \sum_{[\gamma] \in E_P(\Gamma)} \lambda_\gamma \chi_r(X_\gamma) \frac{\varphi(a) \text{tr}(b_\gamma)}{\det(1 - a_\gamma b_\gamma | n) \det(1 - \gamma | r)}. 
\]

**Proof:** Let \(H\) act on itself by conjugation, write \(h.x = hxh^{-1}\), write \(H.x\) for the orbit, so \(H.x = \{hxh^{-1} | h \in H\}\) as well as \(H.S = \{hs^{-1}h^{-1} | s \in S, h \in H\}\) for any subset \(S\) of \(H\). We are going to consider functions that are supported on the closure of the set \(H.(MA^-)\). At first let \(f_\tau\) be the Euler-Poincaré function defined on \(M\) attached to the representation \((\tau, V_\tau)\) of \(K_M\). Next fix a smooth function \(\eta\) on \(N\) which has compact support, is positive, invariant under \(K_M\) and satisfies \(\int_N \eta(n) dn = 1\). Given these data let \(\phi = \phi_{\eta, \tau, \varphi} : H \to \mathbb{C}\) be defined by

\[
\phi(knma(kn)^{-1}) := \eta(n)f_\tau(m)\frac{\varphi(a)}{\det(1 - (ma)|n)},
\]
for \( k \in K, n \in N, m \in M, a \in A^\perp \). Further \( \phi(h) = 0 \) if \( h \) is not in \( H.(M.A^\perp) \).

Next choose any compactly supported positive function \( \psi \) on \( R \) with \( \int \psi = 1 \). Let \( \Phi(h,r) := \phi(h)\psi(r) \). We will plug \( \Phi \) into the trace formula. For the geometric side let \( \gamma = (\gamma_H, \gamma_R) \in \Gamma \). We have to calculate the orbital integral:

\[
O_\gamma(\Phi) = \int_{G_\gamma \backslash G} \Phi(x^{-1}_\gamma x) dx.
\]

Now let \( x = (h,r) \in G \) and compute

\[
x^{-1}_\gamma x = (h^{-1}_\gamma H h, r + h^{-1}_\gamma H - h^{-1}_\gamma H h r).
\]

So \((h,r)\) lies in the centralizer \( G_\gamma \) iff \( h \in H_\gamma \) and \( r \in R \) satisfies

\[
(1 - h^{-1}_\gamma)\gamma_R = (1 - h^{-1}_\gamma) r.
\]

Note that by (A2) to any \( \gamma \) such that \( \gamma_H \) is conjugate to an element of \( A^{reg} M \), and to any \( h \in H_\gamma \) such an \( r \) exists and is unique. But this condition on \( \gamma \) is satisfied if \( \varphi(h^{-1}_\gamma H h) \neq 0 \). So suppose \( \gamma_H \) is in \( H.(A^{reg} M) \). In this case we have the integration rule

\[
\int_{G_\gamma \backslash G} f(g) dg = \int_{H_\gamma \backslash H} \int_R f(h, r) dr dh.
\]

This is proven by showing that the right hand side is in fact \( G \)-invariant. We compute

\[
\int_R \Phi((h, r)^{-1}_\gamma (h, r)) dr = \frac{\varphi(h^{-1}_\gamma H h)}{\det(1 - \gamma_H | r)}
\]

from which we see that the geometric side of the trace formula coincides with our claim.

Now for the spectral side let \( \pi \in \hat{G} \) then the restriction of \( \pi \) to \( R \) is a direct integral over \( \hat{R} \). The irreducibility of \( \pi \) implies that the corresponding measure is supported on a single orbit \( o \) of the \( H \)-action on \( \hat{R} \). So we have

\[
\pi|_R = \int_o V_\pi(\tau) dm(\tau),
\]

where \( m \) is a scalar valued measure and \( V_\pi(\tau) \) is a multiple of \( \tau \). Fix \( \tau_0 \in o \) then the stabilizer \( H_{\tau_0} \) acts trivially on \( \tau_0 \) and not only its class since by \( \dim \tau_0 = 1 \) these two notions coincide. It follows that as \( H_{\tau_0} \)-modules we have \( V_\pi(\tau) \cong \eta \otimes \tau \) for some representation \( \eta \) of \( H_{\tau_0} \). The measure \( m \) induces a measure on \( G_{\tau_0} \backslash G \) also denoted \( m \) which is quasi-invariant. It follows that
\( \pi = \text{ind}^G_{H_G \ltimes R}(\eta \otimes \tau) \) and hence \( \eta \) must be irreducible since \( \pi \) is. Let \( \lambda(x, y) \) denote the Radon-Nikodym derivative of the translate \( m_x \) with respect to \( m \). We conclude that \( \pi(\Phi) \) is given as an integral operator on \( G_{\tau_0} \backslash G \) with kernel
\[
k(x, y) = \int_{G_{\tau_0}} \Phi(x^{-1}zy)\lambda(x^{-1}zy, x)^{1/2} (\eta \otimes \tau_0)(z)dz.
\]
From this we get
\[
\text{tr} \, \pi(\Phi) = \int_{G_{\tau_0}} \text{tr} \left( \int_{G_{\tau_0}} \Phi(x^{-1}zx)\lambda(x^{-1}zx, x)^{1/2} (\eta \otimes \tau_0)(z)dz \right)dx.
\]
Consider the term \( \Phi(x^{-1}zx) = \varphi(x_H^{-1}zxH)\psi(\ldots) \). By (A2) this expression vanishes unless \( \tau_0 \) is the trivial character. In the case \( \tau_0 = \text{triv} \) it follows that \( \pi(R) = 1 \), so \( \pi \) may be viewed as an element of \( \hat{H} \).

To evaluate \( \text{tr} \, \pi(\Phi) \) further we will employ the Hecht-Schmid character formula [9]. For this let \((MA)^- = \text{interior in } MA \text{ of the set} \\{ g \in MA | \det(1 - ga|n) \geq 0 \text{ for all } a \in A^- \} \). The character \( \Theta^G_\pi \) of \( \pi \in \hat{G} \) is a locally integrable function on \( G \). In [9] it is shown that for any \( \pi \in \hat{H} \), denoting by \( \pi^0 \) the underlying Harish-Chandra module we have that all Lie algebra cohomology groups \( H^p(n, \pi^0) \) are Harish-Chandra modules for \( MA \). The main result of [9] is that for \( ma \in (MA)^- \cap H^{reg} \), the regular set, we have
\[
\Theta^{H}_\pi(ma) = \sum_{p=0}^{\dim n} (-1)^p \frac{\Theta^{MA}_{H, (n, \pi^0)}(ma)}{\det(1 - ma|n)}.
\]
Let \( f \) be supported on \( H(MA^-) \), then the Weyl integration formula states that
\[
\int_H f(x)dx = \int_{H/MA} \int_{MA^-} f(hmah^{-1})|\det(1 - ma|n \oplus \bar{n})|dadm dh.
\]
So that for \( \pi \in \hat{H} \):
\[
\text{tr} \, \pi(\phi) = \int_H \Theta^H_\pi(x)\phi(x)dx
= \int_{MA^-} \Theta^H_\pi(ma)f_\tau(m)\varphi(a)|\det(1 - ma|n)|dadm
= (-1)^{\dim N} \int_{MA^-} f_\tau(m)\Theta^{MA}_{H, (n, \pi^0)}(ma)\varphi(a)dadm,
\]
where we have used the isomorphism $H_p(n, \pi^0) \cong H^{\dim N - p}(n, \pi^p) \otimes \wedge^p n$. This gives the claim.

In the second version of the Lefschetz formula we want to substitute the character of the representation $\tau$ by an arbitrary central function on $K_M$. A smooth function $f$ on $K_M$ is called central if $f(kk_1k^{-1}) = f(k)$ for all $k, k_1 \in K_M$. Since $B$ is a Cartan subgroup of the compact group $K_M$, any $k \in K_M$ is conjugate to some element of $B$ so the restriction gives an isomorphism from the space of smooth central functions on $K_M$ to the space of smooth functions on $B$, invariant under the Weyl group. Hence we are led to consider Weyl group invariant functions on $T$.

Let $A$ denote the convolution algebra of all $W(H, T)$-invariant smooth functions on $T$ with compact support. Let $S \subset T$ be the set of all $ab$ with singular $a$-part.

For any $t = ab$ in $T$ let $n_t$ be the space of all $X \in \text{ad}(t)g$ on which $t$ acts contractingly. Then $n_t$ is a nilpotent Lie subalgebra of $g$.

**Theorem 3.2** (Lefschetz formula, second version) Let $\varphi \in A$ and suppose $\varphi$ vanishes on the singular set to order $\dim G + 1$ then the expression

$$
\sum_{\pi \in \hat{G}} N_{\Gamma}(\pi) \sum_q (-1)^q \int_{T/W(H, T)} \varphi(t) \text{tr}(t|H^q(n_t, \pi)) dt
$$

equals

$$
(-1)^{\dim(N)} \sum_{[\gamma] \in \mathcal{E}_F(T)} \lambda_{\gamma} \chi_\gamma(X_\gamma) \frac{\varphi(t_\gamma)}{\det(1 - t_\gamma|p_M \oplus n_{h_\gamma})\det(1 - \gamma|t)}. \tag{1}
$$

**Proof:** Extend $b \mapsto \varphi(ab)$ to a central function on $K_M$. Then expand $\varphi$ into $K_M$-types:

$$
\varphi(ab) = \sum_{\tau \in K_M} c\tau \text{tr}(\tau(b)\varphi_\tau(a)), \tag{2}
$$

since $\varphi$ is smooth the coefficients $c\tau$ are rapidly decreasing so the expressions of Theorem 3.1 when plugging in $\varphi_\tau|_{A^+}$ converge to

$$
\sum_{\pi \in \hat{G}} N_{\Gamma}(\pi) \sum_{p, q} (-1)^{p+q} \int_{T/W(H, T)} \varphi(t) \text{tr}(t|H^q(n_t, \pi) \otimes \wedge^p p_M) dt, \tag{3}
$$

where $\varphi$ is smooth the coefficients $c\tau$ are rapidly decreasing.
which equals

\[ (-1)^{\dim(N)} \sum_{[\gamma] \in \mathcal{E}_H(\Gamma)} \lambda_{\gamma} \chi_{r}(X_{\gamma}) \frac{\varphi(t_{\gamma})}{\det(1 - t_{\gamma}|n_{h_{\gamma}}) \det(1 - \gamma|r)}. \]

Now replace \( \varphi(t) \) by \( \varphi(t) / \det(1 - t|p_M) \) which gives the claim. \( \square \)

At last we also mention a reformulation in terms of relative Lie algebra cohomology. Again, fix a parabolic \( P = MAN \) and now fix also a finite dimensional irreducible representation \((\sigma, V_\sigma)\) of \( M \).

**Theorem 3.3 (Lefschetz formula, third version)** Let \( \varphi \) be compactly supported on \( A^- \), \( \dim G \)-times continuously differentiable and suppose \( \varphi \) vanishes on the boundary to order \( \dim G + 1 \). Then we have that the expression

\[
\sum_{\pi \in \hat{G}} N_\Gamma(\pi) \sum_q (-1)^q \int_{A^-} \varphi(a) \text{tr}(a|H^q(m \oplus n, K_M, \pi \otimes V_\sigma))
\]

equals

\[
(-1)^{\dim(N)} \sum_{[\gamma] \in \mathcal{E}_P(\Gamma)} \lambda_{\gamma} \chi_{r}(X_{\gamma}) \frac{\varphi(a_{\gamma}) \text{tr}(\sigma(b_{\gamma}))}{\det(1 - a_{\gamma}b_{\gamma}|n) \det(1 - \gamma|r)}.
\]

**Proof:** Extend \( V_\sigma \) to a \( m \oplus n \)-module by letting \( n \) act trivially. We then get

\[
H^p(n, \pi^0) \otimes V_\sigma \cong H^p(n, \pi^0 \otimes V_\sigma).
\]

The \((m, K_M)\)-cohomology of the module \( H^p(n, \pi^0 \otimes V_\sigma) \) is the cohomology of the complex \((C^*)\) with

\[
C^q = \text{Hom}_{K_M}(\wedge^q p_M, H^p(n, \pi^0) \otimes V_\sigma) = (\wedge^q p_M \otimes H^p(n, \pi^0) \otimes V_\sigma)^{K_M},
\]

since \( \wedge^q p_M \) is a self-dual \( K_M \)-module. Therefore we have an isomorphism of virtual \( A \)-modules:

\[
\sum_q (-1)^q (H^p(n, \pi^0) \otimes \wedge^q p_M \otimes V_\sigma)^{K_M} \cong \sum_q (-1)^q H^q(m, K_M, H^p(n, \pi^0 \otimes V_\sigma)).
\]
Now one considers the Hochschild-Serre spectral sequence in the relative case for the exact sequence of Lie algebras

$$0 \to n \to m \oplus n \to m \to 0$$

and the $\langle m \oplus n, K_M \rangle$-module $\pi \otimes V_{\theta}$. We have

$$E_2^{p,q} = H^q(m, K_M, H^p(n, \pi^0 \otimes V_{\theta}))$$

and

$$E_\infty^{p,q} = \text{Gr}^q(H^{p+q}(m \oplus n, K_M, \pi^0 \otimes V_{\theta})).$$

Now the module in question is just

$$\chi(E_2) = \sum_{p,q} (-1)^{p+q} E_2^{p,q}.$$

Since the differentials in the spectral sequence are $A$-homomorphisms this equals $\chi(E_\infty)$. So we get an $A$-module isomorphism of virtual $A$-modules

$$\sum_{p,q} (-1)^{p+q} (H^p(n, \pi^0) \otimes \wedge^q p_M \otimes V_{\theta})^K_M \cong \sum_j (-1)^j H^j(m \oplus n, K_M, \pi^0 \otimes V_{\theta}).$$

The claim follows.

4 Geometric interpretation

Now consider the first version of the Lefschetz formula in the case $R = 0$. The representation $\tau$ defines a homogeneous vector bundle $E_\tau$ over $G/K_M$ and by homogeneity this pushes down to a locally homogeneous bundle over $\Gamma\backslash G/K_M = M X_\Gamma$. The tangent bundle $T(M X_\Gamma)$ can be described in this way as stemming from the representation of $K_M$ on

$$\mathfrak{g}/\mathfrak{t}_M \cong \mathfrak{a} \oplus p_M \oplus n \oplus \bar{n}.$$
acts contractingly on $T_s$. On the **unstable part** $T_u$ the opposite chamber $A^-$ acts contractingly. $T_c$, the **central part** is spanned by the ”flow” $A$ itself and $T_n$ is an additive **neutral part**. Note that $T_n$ vanishes if we choose $H$ to be the maximal split torus. The bundle $T_n \oplus T_u$ is integrable, so it defines a foliation $\mathcal{F}$. To this foliation we have the tangential cohomology $H^*(\mathcal{F})$ and also for its $\tau$-twist: $H^*(\mathcal{F} \otimes \tau)$. The flow $A$ acts on the tangential cohomology whose alternating sum we will consider as a virtual $A$-module. For any $\varphi \in C^\infty_c(A^-)$ we define $L_\varphi = \int_{A^-} \varphi(a)(a|H^*(\mathcal{F} \otimes \tau))da$ as a virtual operator on $H^*(\mathcal{F} \otimes \tau)$. Then we have

**Proposition 4.1** Under the assumptions of theorem 3.1 the virtual operator $L_\varphi$ is of trace class and the RHS of Theorem 3.1 can be written as

$$\sum_q (-1)^q \text{tr}(L_\varphi H^q(\mathcal{F} \otimes \tau)).$$

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