Electromagnetic field and the chiral magnetic effect in the quark-gluon plasma

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Time evolution of electromagnetic field created in heavy-ion collisions strongly depends on the electromagnetic response of the quark-gluon plasma, which can be described by the Ohmic and chiral conductivities. The later is intimately related to the Chiral Magnetic Effect. I argue that a solution to the classical Maxwell equations at finite chiral conductivity is unstable due to the soft modes $k < \sigma_\chi$ that grow exponentially with time. In the kinematical region relevant for the relativistic heavy-ion collisions, I derive analytical expressions for the magnetic field of a point charge. I show that finite chiral conductivity causes oscillations of magnetic field at early times.

I. INTRODUCTION

Collision of relativistic heavy-ions produces hot nuclear matter that can be described using the relativistic hydrodynamics [1, 2]. I will refer to this matter as the Quark-Gluon Plasma (QGP) leaving aside the issues of its equilibration and thermalization. Valence electric charges of the colliding ions are not a part of the plasma, as they continue on the incident trajectory along the beam directions with very little deflection [3]. However, they create strong electromagnetic field (EMF) that influences the plasma behavior [4–9]. Electrically conducting plasma responds by generating induced EMF. The resulting EMF is a solution to a complicated magneto-hydrodynamic problem. As a first approximation, one can rely on slow time-dependence of the relevant kinetic coefficients on time to decouple the Maxwell equations from the time evolution of the QGP. Analytical solution to these equations shows that the EMF decreases with time much slower than in vacuum and is approximately collision energy independent; rather it depends only on the impact parameter and the electrical conductivity of the QGP [4, 10–12]. Numerical simulations that take into account the QGP expansion [13] qualitatively agree with this conclusion.∗

It has been recently realized that kinetic properties of the QGP reflect the nontrivial topological structure of the QCD. In particular, the QGP responds to the chirality imbalance by generating metastable parity-odd domains. In the presence of external magnetic field such a metastable

∗ A different strength of EMF in [13] and [11] is due to different initial time at which the plasma evolution starts.
domain induces a parallel to it electric field, which is known as the Chiral Magnetic Effect (CME) [9, 14–17]. Electric current generated by the CME is proportional to the external magnetic field, with the chiral conductivity $\sigma_\chi$ being the proportionality coefficient. In this paper, I derive the electromagnetic field generated by valence charges at finite chiral conductivity and determine the role of the Chiral Magnetic Effect (CME) in the electromagnetic field dynamics in the QGP.

I found a two-fold effect of the CME on the electromagnetic field evolution. Firstly, the field becomes unstable because soft modes with $k < \sigma_\chi$ grow exponentially with time. For the QGP this effects is of little importance since the largest wavelength $1/k$ that is allowed in QGP is much smaller than $1/\sigma_\chi$. However, in non-Abelian plasmas with large spatial extent this is an important phenomenon that may lead to a breakdown of electromagnetic field into a set of knots with non-trivial topology. Similar effect might have happened in the early universe [18–20]. Secondly, finite chiral conductivity induces magnetic field oscillations at early times after a heavy-ion collision. These oscillations may result in partial cancelation of the magnetic field effects, when averaged over time.

The paper is structured as follows: In Sec. II I describe the Maxwell-Chern-Simons (MCS) theory which is an elegant way to incorporate the topological effects in QED. In Sec. III I solve MCS equations away from charges and show that the dispersion relation of electromagnetic wave contains an unstable mode at $k < \sigma_\chi$. In Sec. IV I derive expressions for the electromagnetic field of a relativistic point charge and discuss is properties. Explicit analytical expressions for the magnetic field of a point charge is derived in Sec. V in the diffusion approximation, which is appropriate for the relativistic heavy-ion collisions. The main result shown in Fig. 2 indicates that at finite chiral conductivity, magnetic field components oscillate at early times. I discuss these results and conclude in Sec. VI.

II. MAXWELL-CHERN-SIMONS EQUATIONS

The Lagrangian of electrodynamics coupled to the topological charge carried by the gluon field, the so-called Maxwell-Chern-Simons theory, reads [17, 21–23]

$$L = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - A_\mu j^\mu - \frac{c}{4} g F^{\mu\nu} F_{\mu\nu},$$  \hspace{1cm} (1)
where $c = N_c \sum_f q_f^2 e^2 / 2\pi^2$. An external pseudo-scalar field $\theta$ depends on the medium properties and originates in the QCD Lagrangian. The corresponding field equations are given by
\begin{align*}
\nabla \cdot B &= 0, \quad (2) \\
\nabla \cdot E &= \rho + c \nabla \theta \cdot B, \quad (3) \\
\nabla \times E &= -\partial_t B, \quad (4) \\
\nabla \times B &= \partial_t E + j + c(\partial_t \theta B - \nabla \theta \times E), \quad (5)
\end{align*}

If $\theta$ is a slowly varying function of coordinates we can expand it as
\begin{equation}
\theta \approx \theta_0 + \mu_5 t + c^{-1} P \cdot r. \quad (6)
\end{equation}

where the axial chemical potential $\mu_5$ [16, 17], and vector $P/c$ are the components of a constant covariant four-vector. In this approximation, we obtain [24]
\begin{align*}
\nabla \cdot B &= 0, \quad (7) \\
\nabla \cdot E &= \rho + P \cdot B, \quad (8) \\
\nabla \times E &= -\partial_t B, \quad (9) \\
\nabla \times B &= \partial_t E + j + \sigma_\chi B - P \times E, \quad (10)
\end{align*}

where
\begin{equation}
\sigma_\chi = \mu_5 \frac{e^2}{2\pi^2} N_c \sum_f q_f^2 
\end{equation}

is the chiral conductivity induced by the QED anomaly.

Usually it is assumed that $\theta$-field is spatially homogeneous, so the term proportional to $P$ is negligible. In this case and in the absence of electric charges, the system of equations (7)-(10) has a non-trivial stationary solution with finite magnetic field and vanishing electric field [25–27]. In a context of the QGP physics, these solutions where recently reviewed in [28]. The corresponding magnetic field satisfies the following equations
\begin{align*}
\nabla \cdot B &= 0, \quad (12) \\
\nabla \times B &= \sigma_\chi B. \quad (13)
\end{align*}

\footnote{Possible fluctuations of $\mu_5$ in QGP [24] is not considered in this paper.}
Since $\sigma_\chi \neq 0$ only inside a domain of finite volume $D$, there is no outward current on its boundary. This implies the boundary condition
\[ \hat{r} \cdot B|_{\partial D} = 0. \] (14)

Solution to (12)-(14) is a system of magnetized knots of different sizes. In a simplest case of spherical boundary the possible values of its radius are
\[ R_n = \frac{\kappa_n}{\sigma_\chi}, \quad n = 0, 1, 2, \ldots, \] (15)
where $n$ enumerates zeros of spherical Bessel functions $\kappa_n$. The smallest of $\kappa$’s is $\kappa_0 \approx 4.5$, which for a realistic $\sigma_\chi$ yields $R_0 \approx 200$ fm. This is much larger than a characteristic transverse size of QGP $R_A \sim 6 - 10$ fm. Thus, magnetic knots are more relevant in astrophysics than in heavy-ion collisions [28].

Neglecting vector $P$ in (6) is not a boost invariant statement. We therefore should anticipate a possible appearance of acausal solutions to the MCS equations. Indeed, such solutions appear in the dispersion relation for the electromagnetic field of a point charge. I will address this problem in Sec. IV.

III. INSTABILITY OF ELECTROMAGNETIC WAVES IN INFINITE PLASMA

Consider first electromagnetic waves propagating in plasma far from any sources. In a conducting medium Maxwell equations for the electromagnetic field read
\[ \nabla \cdot B = 0, \] (16)
\[ \nabla \cdot D = 0, \] (17)
\[ \nabla \times E = -\partial_t B, \] (18)
\[ \nabla \times H = \partial_t D + \sigma_\chi B. \] (19)

$D$ is electric displacement vector. We will assume that $\mu = 1$. Fourier transformation
\[ E(r, t) = \int \frac{d^4k}{(2\pi)^4} e^{-ik \cdot x} E_{\omega, k}, \quad B(r, t) = \int \frac{d^4k}{(2\pi)^4} e^{-ik \cdot x} B_{\omega, k} \] (20)
where $x = (t, r)$, $k = (\omega, k)$ yields Maxwell equation in momentum space
\[ k \cdot B_{\omega, k} = 0, \] (21)
\[ \epsilon k \cdot E_{\omega, k} = 0, \] (22)
\[ k \times E_{\omega, k} = \omega B_{\omega, k}, \] (23)
\[ k \times B_{\omega, k} = -\omega \epsilon E_{\omega, k} - i \sigma_\chi B_{\omega, k}, \] (24)
where $D_{\omega,k} = \epsilon E_{\omega,k}$. In electrically conducting medium with the Ohmic conductivity $\sigma$ the permittivity is $\epsilon = 1 + i\sigma/\omega$. Taking vector product of (24) with $k$ and using (21) and (23) we get

$$B_{\omega,k}[\omega(\omega + i\sigma) - k^2] = -i\sigma k \times B_{\omega,k}.$$  \hspace{1cm} (25)

Taking another vector product with $k$ gives

$$(k \times B_{\omega,k})[\omega(\omega + i\sigma) - k^2] = i\sigma k^2 B_{\omega,k}.$$  \hspace{1cm} (26)

Equations (25) and (26) have a non-trivial solution only if the following dispersion relation is satisfied

$$[\omega(\omega + i\sigma) - k^2]^2 = \sigma^2 k^2.$$  \hspace{1cm} (27)

It has four solutions

$$\omega_{\lambda_1, \lambda_2} = -\frac{i\sigma}{2} + \lambda_1 \sqrt{k^2 + \lambda_2 \sigma \chi k - \sigma^2/4},$$  \hspace{1cm} (28)

where $\lambda_1, \lambda_2 = \pm 1$. These solutions determine the time dependence of electromagnetic wave as $\sim e^{-i\omega_{\lambda_1, \lambda_2} t}$.

Let $\kappa^2 = k^2 + \lambda_2 \sigma \chi k - \sigma^2/4$. When $\kappa^2 > 0$ the electromagnetic wave oscillates with frequency $\kappa$ and is damped over the distance $1/\sigma$. This corresponds to momenta

$$k > k_0 \equiv \frac{1}{2} \sqrt{\sigma^2 \chi^2 + \sigma^2} - \frac{\lambda_2 \sigma \chi}{2}.$$  \hspace{1cm} (29)

For $k < k_0$, $\kappa^2 < 0$, and all $\omega_{\lambda_1, \lambda_2}$'s become imaginary implying that electromagnetic wave is a monotonic function of time. At $\kappa^2 = -\sigma^2/4$, which occurs at $k = \sigma \chi$, $\lambda_2 = -1$, and $\lambda_1 = +1$, $\omega_{+,-}$ vanishes indicating a stationary mode. Finally, when $\kappa^2 < -\sigma^2/4$, i.e. $k < \sigma \chi$, $\lambda_2 = -1$, $\lambda_1 = +1$ there is an unstable mode with $\text{Im} \omega_{+,-} > 0$ which corresponds to exponentially increasing magnetic field.

Electromagnetic wave which at some initial time contains modes extending to the region $k < \sigma \chi$ is unstable. This is a usual situation in an infinite plasma. However, in a plasma of spatial size $R$ there are only modes $k \gtrsim 1/R$. Therefore, the instability affects the field evolution only if $R \gtrsim 1/\sigma \chi$. In the QGP this condition is not satisfied, except, perhaps, in a very rare fluctuations of the $\theta$-parameter, and hence can be ignored.
IV. ELECTROMAGNETIC FIELD OF A POINT CHARGE

In electrically conducting medium Maxwell equations for the electromagnetic field of a point charge moving along a straight line \( z = vt \) read

\[
\nabla \cdot B = 0 , \tag{30}
\]

\[
\nabla \cdot D = e \delta(z - vt) \delta(b) , \tag{31}
\]

\[
\nabla \times E = - \partial_t B , \tag{32}
\]

\[
\nabla \times H = \partial_t D + \sigma \chi B + ev \hat{\hat{z}} \delta(z - vt) \delta(b) . \tag{33}
\]

These equations in momentum space are

\[
k \cdot B_{\omega,k} = 0 , \tag{34}
\]

\[
\epsilon k \cdot E_{\omega,k} = -2\pi i e \delta(\omega - k_z v) , \tag{35}
\]

\[
k \times E_{\omega,k} = \omega B_{\omega,k} , \tag{36}
\]

\[
k \times B_{\omega,k} = -\omega \epsilon k \times E_{\omega,k} - i \sigma \chi B_{\omega,k} - 2\pi i e v \hat{\hat{z}} \delta(\omega - k_z v) . \tag{37}
\]

We repeat the algebraic manipulations of the previous section. Firstly, taking the vector product of (37) with \( k \) and using (34) and (36) we arrive at

\[
B_{\omega,k}[\omega(\omega + i\sigma) - k^2] = -i \sigma \chi k \times B_{\omega,k} - 2\pi i e v k \times \hat{\hat{z}} \delta(\omega - k_z v) . \tag{38}
\]

Secondly, we take another vector product with \( k \) to obtain

\[
(k \times B_{\omega,k})[\omega(\omega + i\sigma) - k^2] = i \sigma \chi k^2 B_{\omega,k} - 2\pi i e v k \times (k \times \hat{\hat{z}}) \delta(\omega - k_z v) . \tag{39}
\]

We are interested in a particular solution to equations (38),(39), namely the one that is generated by the electric charge \( e \). Solving (38) and (39) yields

\[
B_{\omega,k} = \frac{(k \times \hat{\hat{z}})[\omega(\omega + i\sigma) - k^2] - i \sigma \chi k \times (k \times \hat{\hat{z}})}{[\omega(\omega + i\sigma) - k^2]^2 - \sigma^2 k^2} (-2\pi i e v \delta(\omega - k_z v)) . \tag{40}
\]

Electric field follows from the Faraday law (36) upon taking its vector product with \( k \):

\[
k(k \cdot E_{\omega,k}) - k^2 E_{\omega,k} = \omega (k \times B_{\omega,k}) . \tag{41}
\]

Substituting (35) and (37) we find

\[
E_{\omega,k} = \frac{2\pi i e \delta(\omega - k_z v)[k/\epsilon - v \omega \hat{\hat{z}}] - i \omega \sigma \chi B_{\omega,k}}{\omega(\omega + i\sigma) - k^2} , \tag{42}
\]
with $B_{\omega,k}$ given by (40).

It will be suitable to write the cross products in (40) in cylindrical coordinates. Let $\psi$ be the angle between the vector $k_\perp$ and the $x$-axis, the corresponding unit vector is $\hat{\psi} = -\hat{x} \sin \psi + \hat{y} \cos \psi$. Then

$$k \times \hat{z} = -k_\perp \hat{\psi},$$

$$k \times (k \times \hat{z}) = k_z k_\perp - k_\perp^2 \hat{z}. \tag{43}$$

Using identities (43),(44) in (35), substituting the result into (20) and taking integral over $k_z$ we find

$$B = i e \int \frac{d\omega}{2\pi} \int \frac{d^2 k_\perp}{(2\pi)^2} \frac{k_\perp \psi [\omega(\omega + i\sigma) - k_\perp^2 - \omega^2]}{[\omega(\omega + i\sigma) - k_\perp^2 - \omega^2/v^2 - \sigma^2 (k_\perp^2 + \omega^2/v^2)]} e^{-i\omega x_- + i k_\perp \cdot b}. \tag{45}$$

where $x_- = t - z/v$.

Time dependence of magnetic field is determined by the poles of (40) in the plane of complex $\omega$. These poles are solutions of the following quartic equation

$$\left[\omega(\omega + i\sigma) - k_\perp^2 - \omega^2/v^2\right] - \sigma^2 \chi \left(k_\perp^2 + \omega^2/v^2\right) = 0. \tag{46}$$

Eq. (46) can be obtained from the dispersion relation (27) of a free wave by restricting it to the particle equation of motion $k_z = \omega/v$. Introducing $\gamma = (1 - v^2)^{-1/2}$ allows us to cast (46) in a more convenient form

$$\left(-\frac{\omega^2}{v^2 \gamma^2} + i\omega \sigma - k_\perp^2\right)^2 - \sigma^2 \chi \left(\frac{\omega^2}{v^2} + k_\perp^2\right) = 0. \tag{47}$$

Four solutions to this equation can be found using the standard algebraic methods. However, they are quite bulky, so I am not reproducing them here. Instead, I find it more illuminating to plot them at fixed $\sigma, \sigma \chi$ and $\gamma$ for different values of $k_\perp$ as shown in Fig. 1.

The fact that there are singularities at $\text{Im} \omega > 0$ implies existence of the field at $x_- < 0$, which contradicts causality. This is a consequence of the assumption that the $\theta$-field is a function of time but not space. Unlike the dispersion relation (27), which has only one branch $\omega_\pm$ with a positive imaginary part at $k < \sigma_\chi$, the dispersion relation (47) has three branches with a positive imaginary part of $\omega$: two upper and one lower branch. The two upper branches have $\text{Im} \omega > 0$ for any $k_\perp$. The minima values of $\text{Im} \omega$ can be found by taking $k_\perp \to 0$ in which case (47) has three distinct solutions $\omega = 0$ and $\omega = v^2 \gamma^2 (i\sigma \pm \sigma_\chi)$. The former corresponds to the minimum value of the lower branches, while the later to the minimum values of the upper branches. Thus, the upper branches
FIG. 1: Four solutions of (47) at $\sigma = 5.8$ MeV, $\sigma_{\chi} = 1$ MeV, $\gamma = 100$. Horizontal and vertical axes are in units of GeV. Each line is a unique function of $k_{\perp}$. Squares, circles and triangles indicate the positions of the poles at $k_{\perp} = 0.1, 0.6, 1.1$ GeV respectively.

are separated from the real axis by a gap $v^2\gamma^2\sigma$. The absolute value of the real part of the upper branches decreases monotonically with $k_{\perp}$. At $k_{\perp} \rightarrow \infty$

$$\omega \approx \pm iv\gamma k_{\perp} \pm \frac{1}{2}v\gamma\sigma_{\chi}\sqrt{\gamma^2 - 1},$$

(48)

Thus, the real value of $\omega$ of upper branches approaches a constant at large $k_{\perp}$, which indicates that a gap of size $\sim \gamma^2\sigma_{\chi}$ exists also between the upper branches and the imaginary axis.

The fact that $\omega$ does not vanish at $k_{\perp} \rightarrow 0$ is another indication that the upper branches are not physical. Fortunately, in heavy-ion physics they do not cause any problem because electromagnetic excitations in plasma occur at small $\omega$’s, as I discuss in the next section. However, in astrophysical and cosmological applications they may be troublesome.

V. DIFFUSION APPROXIMATION

At a given light-cone time $x_-$ the $\omega$-integral in (45) vanishes at $\omega \gg 1/x_-$ due to the rapid oscillation of the integrand. Therefore, at later times the terms in (47) that are quadratic in $\omega$ are suppressed. This correspond to the following “diffusion” approximation:

$$\omega \ll \sigma v^2\gamma^2, \quad \omega \ll v\gamma k_{\perp},$$

(49)

which is tantamount to

$$x_- \gg \frac{1}{\sigma v^2\gamma^2}, \quad x_- \gg \frac{b}{v\gamma},$$

(50)

where we estimated $k_{\perp} \sim 1/b$. Electrical conductivity of the quark-gluon plasma at the critical temperature is $\sigma = 5.8$ MeV [29–32]. For a heavy-ion collision at $\gamma = 100$ we estimate $1/\sigma v^2\gamma^2 \sim$
3 · 10^{-3} \text{ fm}. For b \sim 10 \text{ fm}, b/\gamma \sim 0.1 \text{ fm}. Taking into account that it takes about 1/Q_s \sim 0.2 \text{ fm} to release the color charges from the nuclei wave functions, it follows that approximation (49) applies to the entire lifetime of the QGP. The precise initial conditions do not play an important role in the electromagnetic field evolution.

Since the valence quarks are ultra-relativistic, i.e. \( \gamma \gg 1 \), we will approximate their velocity as \( v \approx 1 - 1/2\gamma^2 \). Then, the dispersion relation (47) in the diffusion approximation takes form

\[
(i\omega - k_\perp^2)^2 - \sigma^2 (\omega^2 + k_\perp^2) = 0.
\] (51)

The two solutions of (51) are

\[
\omega_{1,2} = \frac{-i\sigma k_\perp \pm k_\perp \sqrt{k_\perp^2 - \sigma^2 - \sigma^2 \chi}}{\sigma^2 + \sigma^2 \chi}. \tag{52}
\]

These are the poles of the Fourier component of magnetic field \( B_{\omega,k} \) in the complex \( \omega \)-plane. If \( k_\perp > \sqrt{\sigma^2 + \sigma^2 \chi} \), then both complex-conjugated poles lie in the lower half-plane. If \( \sigma_\chi < k_\perp < \sqrt{\sigma^2 + \sigma^2 \chi} \), then there are two poles on the imaginary axis in the lower half-plane. Finally, if \( k_\perp < \sigma_\chi \), then both poles lie on the imaginary axis, but \( \omega_1 \) is in the upper-half plane, while \( \omega_2 \) is still in the lower one. The value of \( \sigma_\chi \) probably does not exceed a few MeV at best, while typical \( k_\perp \) is in the range 20 – 200 MeV corresponding to \( b \)'s in the range 1 – 10 fm. Therefore, only the first case (viz. large \( k_\perp \)) has a practical significance.

In the diffusion approximation (45) reads

\[
B = -ie \int \frac{d\omega}{2\pi} \int \frac{d^2k_\perp}{(2\pi)^2} k_\perp \hat{\psi}(i\omega_\sigma - k_\perp^2) + i\sigma_\chi (k_\perp\omega - k_\perp^2 \hat{z}) e^{-i\omega_{x-}+ik_\perp b} \tag{53}
\]

\[
= \int \frac{d^2k_\perp}{(2\pi)^2} e^{ik_\perp b} \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \frac{f(\omega)}{(\omega - \omega_1)(\omega - \omega_2)} e^{-i\omega_{x-}}, \tag{54}
\]

where I denoted

\[
f(\omega) = -\frac{ie}{\sigma^2 + \sigma^2 \chi} \left[ k_\perp \hat{\psi}(i\omega_\sigma - k_\perp^2) + i\sigma_\chi (k_\perp\omega - k_\perp^2 \hat{z}) \right] \tag{55}
\]

The integration contour in (54) can be closed by an infinite semi-circle in the lower half-plane if \( x_- > 0 \), or in the upper half-plane if \( x_- < 0 \). The result of the integration is

\[
B = \int \frac{d^2k_\perp}{(2\pi)^2} e^{ik_\perp b} \left\{ \frac{i}{\omega_2 - \omega_1} \left[ e^{-i\omega_1 x_-} f(\omega_1) \theta(k_\perp - \sigma_\chi) - e^{-i\omega_2 x_-} f(\omega_2) \right] \theta(x_-) \right. \\
- \left. \frac{i}{\omega_2 - \omega_1} e^{-i\omega_1 x_-} f(\omega_1) \theta(\sigma_\chi - k_\perp) \theta(-x_-) \right\}. \tag{56}
\]

A finite contribution to the field at \( x_- < 0 \) contradicts causality and is a peculiar way in which the field instability manifests itself. This contribution arises at very small momenta \( k_\perp < \sigma_\chi \), which
has almost no relevance to the dynamics of heavy-ion collisions. In Fig. 1 one of the the lower branches has positive \( \text{Im} \omega \) at \( k_\perp < \sigma_\chi \). It is not readily seen there due its small value \( \text{Im} \omega \sim \sigma \).

Since \( k_\perp^2 \gg \sigma^2 + \sigma_\chi^2 \), we approximate the poles of (52) as follows

\[
\omega_{1,2} \approx \frac{k_\perp^2 (-i \sigma \pm \sigma_\chi)}{\sigma^2 + \sigma_\chi^2} = \frac{k_\perp^2}{i \sigma \pm \sigma_\chi}.
\]

(57)

Magnetic field at \( x_- > 0 \) becomes

\[
B \approx \int \frac{d^2 k_\perp}{(2\pi)^2} e^{ik_\perp \cdot b} \frac{i}{\omega_2 - \omega_1} \left[ e^{-i\omega_1 x_- f(\omega_1)} - e^{-i\omega_2 x_- f(\omega_2)} \right] \theta(x_-).
\]

(58)

Its polar component is given by

\[
B_\phi = \int \frac{d^2 k_\perp}{(2\pi)^2} e^{ik_\perp \cdot b} \frac{i}{\omega_2 - \omega_1} \hat{\psi} \cdot \left[ e^{-i\omega_1 x_- f(\omega_1)} - e^{-i\omega_2 x_- f(\omega_2)} \right] \theta(x_-),
\]

(59)

where \( \phi \) is the angle between the impact parameter \( b \) and the \( x \)-axis. Integration over the directions of \( k_\perp \) given by the polar angle \( \psi \) is done as follows:

\[
\int_0^{2\pi} e^{ik_\perp \cdot \hat{b}} \hat{\psi} d\psi = \int_0^{2\pi} e^{ik_\perp \cdot \hat{b}} \cos(\psi - \phi) (-\hat{x} \sin \psi + \hat{y} \cos \psi) d\psi = 2\pi i J_1(k_\perp b) \hat{\phi},
\]

(60)

Using (60) in (59) and substituting (55),(57) we have (from now on we assume \( x_- \geq 0 \)):

\[
B_\phi = -i \int_0^\infty \frac{dk_\perp}{2\pi} \frac{k_\perp}{2(\sigma^2 + \sigma_\chi^2)} \left[ (i \sigma - \sigma_\chi) e^{-i\frac{k_\perp^2 x_-}{\sigma^2 + \sigma_\chi^2}} + (i \sigma + \sigma_\chi) e^{-i\frac{k_\perp^2 x_-}{\sigma^2 + \sigma_\chi^2}} \right].
\]

(61)

The remaining integral can be done analytically yielding

\[
B_\phi = \frac{eb}{8\pi x_-^2} e^{-\frac{b^2 x_+}{4 x_-}} \left[ \sigma \cos \left( \frac{b^2 \sigma_\chi}{4 x_-} \right) + \sigma_\chi \sin \left( \frac{b^2 \sigma_\chi}{4 x_-} \right) \right].
\]

(62)

Turning to the component of magnetic field aligned along the \( b \)-direction we obtain:

\[
B_r = \int \frac{d^2 k_\perp}{(2\pi)^2} e^{ik_\perp \cdot b} \frac{i}{\omega_2 - \omega_1} \hat{k}_\perp \cdot \left[ e^{-i\omega_1 x_- f(\omega_1)} - e^{-i\omega_2 x_- f(\omega_2)} \right].
\]

(63)

Angular integration is done using

\[
\int_0^{2\pi} e^{ik_\perp \cdot \hat{b}} d\psi = \int_0^{2\pi} e^{ik_\perp \cdot \hat{b}} \cos(\psi - \phi) (-\hat{x} \sin \psi + \hat{y} \cos \psi) d\psi = 2\pi i J_1(k_\perp b) \hat{b}.
\]

(64)

Plugging the \( k_\perp \)-component of \( f \) from (55) and integrating over \( k_\perp \) we derive

\[
B_r = \frac{eb}{8\pi x_-^2} e^{-\frac{b^2 x_+}{4 x_-}} \left[ \sigma \sin \left( \frac{b^2 \sigma_\chi}{4 x_-} \right) - \sigma_\chi \cos \left( \frac{b^2 \sigma_\chi}{4 x_-} \right) \right].
\]

(65)

Finally, repeating the by now familiar procedure and using the integral

\[
\int_0^{2\pi} e^{ik_\perp \cdot \hat{b}} d\psi = 2\pi J_0(k_\perp b) \hat{b}
\]

(66)
we find for the longitudinal field component:

\[ B_z = \frac{eb}{4\pi x_-} e^{-\frac{b^2}{4x_-}} \left[ \sigma \sin\left(\frac{b^2 \sigma}{4x_-}\right) - \sigma \chi \cos\left(\frac{b^2 \sigma}{4x_-}\right) \right]. \] (67)

It is seen in (65) and (67) that the field components \( B_r \) and \( B_z \) are generated only at a finite chiral conductivity \( \sigma_\chi \).

Eqs. (62), (63) and (67) is the main result of this paper. It shows that at finite \( \sigma_\chi \), magnetic field of a point charge acquires two components that are absent in the chirally neutral medium: the radial and the longitudinal components. All field components oscillate at early times. This is clearly seen in Fig. 2. The \( B_z \) and \( B_r \) components change sign at light-cone times

\[ x_-^{(n)} = \frac{b^2 \sigma_\chi}{4[\arctan\frac{\sigma_\chi}{\sigma} + \pi n]}, \quad n = 0, 1, \ldots, \] (68)

while the \( B_\phi \) components changes sign at

\[ \tilde{x}_-^{(n)} = \frac{b^2 \sigma_\chi}{4[-\arctan\frac{\sigma_\chi}{\sigma} + \pi n]}, \quad n = 0, 1, \ldots, \] (69)

The latest oscillation corresponds to \( n = 0 \); it increases with \( \sigma_\chi \).

VI. DISCUSSION AND SUMMARY

There are two major results presented in this paper.

(i) I showed that solutions to the Maxwell equations are not stable in the presence of the chirality imbalance. It is possible that electromagnetic field collapses into a set of magnetic knots. This
problem certainly deserves a dedicated study and may be important in cosmology. However, as far as heavy-ion collisions are concerned, this instability has negligible impact on the QGP because it originates from soft modes $k < \sigma_\chi$ that do not exist in the QGP of realistic dimensions.

(ii) I derived an analytical solution for magnetic field at finite chiral conductivity $\sigma_\chi$. Its components are given by equations (62),(65) and (67), which indicate emergence of the radial $B_r$ and longitudinal $B_z$ components of magnetic field (as compared to the $\sigma_\chi = 0$ case). If $\sigma_\chi$ is not much smaller than $\sigma$, then all components perform oscillations at early times after the collision. Since magnetic field is strongest at early times, these oscillations should have important impact on heavy-ion phenomenology. In particular, they may weaken effects that depend on the magnetic field direction, such as the $B$-dependent elliptic flow \cite{33, 34} and charge separation effect \cite{9}. This is especially true for the charge separation effect that requires sufficiently large $\sigma_\chi$.

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