THE UNIT TANGENT SPHERE BUNDLE WHOSE CHARACTERISTIC JACOBI OPERATOR IS PSEUDO-PARALLEL

JONG TAEK CHO AND SUN HYANG CHUN

ABSTRACT. We study the characteristic Jacobi operator $\ell = \bar{R}(\cdot, \xi)\xi$ (along the Reeb flow $\xi$) on the unit tangent sphere bundle $T_1M$ over a Riemannian manifold $(M^n, g)$. We prove that if $\ell$ is pseudo-parallel, i.e., $\hat{R} \cdot \ell = L\mathcal{Q}(\bar{g}, \ell)$, by a non-positive function $L$, then $M$ is locally flat. Moreover, when $L$ is a constant and $n \neq 16$, $M$ is of constant curvature 0 or 1.

1. Introduction

It is intriguing to study the interplay between Riemannian manifolds and their unit tangent sphere bundles. In particular, we are interested in the standard contact metric structure $(\eta, \bar{g}, \phi, \xi)$ of a unit tangent sphere bundle $T_1M$ over a given Riemannian manifold $(M, g)$. It is remarkable that the characteristic vector field $\xi$ on $T_1M$ contains crucial information about $M$. In fact, all the geodesics in $M$ are controlled by the geodesic flow on $T_1M$ which is precisely given by $\xi$. Apart from the defining structure tensors $\eta$, $\bar{g}$, $\phi$ and $\xi$, the so-called characteristic Jacobi operator $\ell = \bar{R}(\cdot, \xi)\xi$ plays a fundamental role in contact Riemannian geometry, especially in the unit tangent sphere bundle (cf. [2]). Here, $\bar{R}$ denotes the Riemannian curvature tensor determined by $\bar{g}$. In Section 3, we prove that the characteristic Jacobi operator $\ell$ vanishes if and only if $M$ is locally flat (Proposition 2).

On the other hand, for a Riemannian manifold $(\hat{M}, \hat{g})$ a tensor field $F$ of type $(1, 3);

$$F : \mathfrak{X}(\hat{M}) \times \mathfrak{X}(\hat{M}) \times \mathfrak{X}(\hat{M}) \to \mathfrak{X}(\hat{M})$$

is said to be curvature-like provided that $F$ has the symmetric properties of $\bar{R}$. Here $\mathfrak{X}(\hat{M})$ is the Lie algebra of all vector fields on $\hat{M}$. For example,
Let $(\bar{X} \wedge \bar{Y})\bar{Z} = \bar{g}(\bar{Y}, \bar{Z})\bar{X} - \bar{g}(\bar{Z}, \bar{X})\bar{Y}$. Note that a Riemannian manifold $(M, \bar{g})$ of constant curvature $c$ satisfies the formula $\bar{R}(\bar{X}, \bar{Y}) = c(\bar{X} \wedge \bar{Y})$.

As is well-known, a curvature-like tensor field $F$ acts on the algebra $T^1_s(M)$ of all tensor fields on $M$ of type $(1, s)$ as a derivation (cf. [5]). Then $P$ is said to be semi-parallel if $\bar{R} \cdot P = 0$, where $\cdot$ means that $\bar{R}$ acts as a derivation on $P$. Pseudo-parallelism is defined as the natural generalization. Namely, $P$ is said to be pseudo-parallel if $\bar{R} \cdot P = LQ(\bar{g}, P)$ for some function $L$, where $Q(\bar{g}, P)$ is defined by

$$Q(\bar{g}, P)(X_1, \ldots, X_s; Y, X) = (X \wedge Y)P(X_1, \ldots, X_s) - \sum_{j=1}^s P(X_1, \ldots, (X \wedge Y)X_j, \ldots, X_s).$$

In the present paper, we study pseudo-parallelism of the characteristic Jacobi operator $\ell$ on the unit tangent sphere bundle $T_1M$: $\bar{R} \cdot \ell = LQ(\bar{g}, \ell)$ for a function $L$ on $T_1M$. Then we easily see that vanishing $\ell$ implies pseudo-parallel $\ell$. Moreover, pseudo-parallel $\ell$ includes the case of semi-parallel $\ell$ ($L = 0$). The main purpose of the present paper is to prove the following.

**Main Theorem.** Let $(M, \bar{g})$ be an $n$-dimensional Riemannian manifold and $T_1M$ be the unit tangent sphere bundle over $M$ with the standard contact metric structure $(\eta, \bar{g}, \phi, \xi)$. Suppose that the characteristic Jacobi operator $\ell$ of $T_1M$ is pseudo-parallel by a function $L$ on $T_1M$. Then we have the following results:

(i) if $L \leq 0$, then $M$ is locally flat,

(ii) if $L$ is constant and $n \neq 16$, then $M$ is of constant curvature 0 or 1.

Conversely, for the unit tangent sphere bundle over a space of constant curvature $c = 0$ or $c = 1$, the characteristic Jacobi operator $\ell$ is pseudo-parallel with $L = 0$ or $L = 1$, respectively.

### 2. Preliminaries

All manifolds in the present paper are assumed to be connected and of class $C^\infty$. We start by collecting some fundamental material about contact metric geometry. We refer to [1] for further details. A $(2n + 1)$-dimensional manifold $M^{2n+1}$ is said to be a contact manifold if it admits a global 1-form $\eta$ such that $\eta \wedge (d\eta)^n \neq 0$ everywhere. Given a contact form $\eta$, we have a unique vector field $\xi$, the characteristic vector field, satisfying $\eta(\xi) = 1$ and $d\eta(\xi, X) = 0$ for any vector field $X$ on $M$. It is well-known that there exists a Riemannian metric $\bar{g}$ on $M$ and a $(1, 1)$-tensor field $\phi$ such that

1. $\eta(X) = g(X, \xi), \ d\eta(X, Y) = g(X, \phi Y), \ \phi^2 X = -X + \eta(X)\xi,$

where $\bar{X}$ and $\bar{Y}$ are vector fields on $M$. From (1) it follows that

2. $\phi \xi = 0, \ \eta \circ \phi = 0, \ g(\phi \bar{X}, \phi \bar{Y}) = \bar{g}(\bar{X}, \bar{Y}) - \eta(\bar{X})\eta(\bar{Y}).$
A Riemannian manifold $M$ equipped with structure tensors $(\eta, \bar{g}, \phi, \xi)$ satisfying (1) is said to be a contact metric manifold and is denoted by $M = (M; \eta, \bar{g}, \phi, \xi)$. Given a contact metric manifold $M$, we define the structural operator $h$ by $h = \frac{1}{2}\mathcal{L}\phi$, where $\mathcal{L}$ denotes Lie differentiation. Then we may observe that $h$ is symmetric and satisfies
\begin{align*}
h\xi &= 0 \quad \text{and} \quad h\phi = -\phi h, \\
\nabla_X\xi &= -\phi X - \phi hX,
\end{align*}
where $\nabla$ is the Levi-Civita connection. From (3) and (4) we see that each trajectory of $\xi$ is a geodesic. We denote by $R$ the Riemannian curvature tensor defined by
\[ R(X, Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}Z \]
for all vector fields $X, Y$ and $Z$. Along a trajectory of $\xi$, the Jacobi operator $t = R(\cdot, \xi)\xi$ is a symmetric $(1,1)$-tensor field. We call it the characteristic Jacobi operator. A contact metric manifold for which $\xi$ is Killing is called a $K$-contact manifold. For a contact Riemannian manifold $M$ one may define naturally an almost complex structure $J$ on $M \times \mathbb{R}$:
\[ J(\bar{X}, f\frac{dt}{dt}) = (\varphi \bar{X} - f\xi, \eta(\bar{X})\frac{dt}{dt}), \]
where $\bar{X}$ is a vector field tangent to $M$, $t$ the coordinate on $\mathbb{R}$ and $f$ a function on $M \times \mathbb{R}$. If the almost complex structure $J$ is integrable, $M$ is said to be normal or Sasakian. It is known that a contact metric manifold $M$ is normal if and only if $M$ satisfies
\[ \{\varphi, \varphi\} + 2d\eta \otimes \xi = 0, \]
where $[\varphi, \varphi]$ is the Nijenhuis torsion of $\varphi$. A Sasakian manifold is also characterized by the condition $(\nabla_X \varphi)Y = \bar{g}(X, Y)\xi - \eta(Y)\bar{X}$ and this is equivalent to
\[ R(\bar{X}, \bar{Y})\xi = \eta(\bar{Y})\bar{X} - \eta(\bar{X})\bar{Y} \]
for all vector fields $\bar{X}$ and $\bar{Y}$.

**Proposition 1.** For a Sasakian manifold, the characteristic Jacobi operator $t$ is pseudo-parallel with $L = 1$.

**Proof.** Let $M = (M; \eta, \bar{g}, \phi, \xi)$ be a Sasakian manifold. Then, from (5) we get
\[ t\bar{X} = \bar{X} - \eta(\bar{X})\xi \]
for any vector field $\bar{X}$ on $M$. Using (6) we compute
\begin{align*}
(R(\bar{X}, \bar{Y}) \cdot t)\bar{Z} \\
&= R(\bar{X}, \bar{Y})t\bar{Z} - t(R(\bar{X}, \bar{Y})\bar{Z}) \\
&= \eta(\bar{X})\bar{g}(\bar{Y}, \bar{Z})\xi - \eta(\bar{Y})\bar{g}(\bar{X}, \bar{Z})\xi + \eta(\bar{X})\eta(\bar{Z})\bar{Y} - \eta(\bar{Y})\eta(\bar{Z})\bar{X}.
\end{align*}
\[ L((\bar{X} \wedge \bar{Y}) \cdot t) \bar{Z} \]
\[ = L((\bar{X} \wedge \bar{Y})t\bar{Z} - t((\bar{X} \wedge \bar{Y})\bar{Z})) \]
\[ = L\{g(\bar{Y}, \ell \bar{Z})\bar{X} - g(\bar{X}, \ell \bar{Z})\bar{Y} - g(\bar{Y}, \bar{Z})\ell \bar{X} + g(\bar{X}, \bar{Z})\ell \bar{Y} \} \]
\[ = L\{\eta(\bar{X})g(\bar{Y}, \bar{Z})\xi - \eta(\bar{Y})g(\bar{X}, \bar{Z})\xi + \eta(\bar{X})\eta(\bar{Z})\bar{Y} - \eta(\bar{Y})\eta(\bar{Z})\bar{X} \}. \]

Then from (7) and (8), we can see that \( \ell \) is pseudo-parallel and \( L = 1 \). \( \square \)

3. The contact metric structure of the unit tangent sphere bundle

The basic facts and fundamental formulae about tangent bundles are well-known (cf. [6], [9], [14]). We only briefly review some notations and definitions. Let \( M = (M, g) \) be an \( n \)-dimensional Riemannian manifold and let \( TM \) denote its tangent bundle with the projection \( \pi : TM \to M \), \( \pi(p,u) = p \). For a vector field \( X \) on \( M \), its vertical lift \( X^v \) on \( TM \) is the vector field defined by \( X^v = \omega(X) \circ \pi \), where \( \omega \) is a 1-form on \( M \). For the Levi Civita connection \( \nabla \) on \( M \), the horizontal lift \( X^h \) of \( X \) is defined by \( X^h = \nabla_X \omega \). The tangent bundle \( TM \) can be endowed in a natural way with a Riemannian metric \( \bar{g} \), the so-called Sasaki metric, depending only on the Riemannian metric \( g \) on \( M \). It is determined by

\[ \bar{g}(X^h, Y^h) = \bar{g}(X^v, Y^v) = g(X,Y) \circ \pi, \quad \bar{g}(X^h, Y^v) = 0 \]

for all vector fields \( X \) and \( Y \) on \( M \). Also, \( TM \) admits an almost complex structure tensor \( J \) defined by \( JX^h = X^v \) and \( JX^v = -X^h \). Then \( \bar{g} \) is a Hermitian metric for the almost complex structure \( J \).

The unit tangent sphere bundle \( \bar{p} : T_1 M \to M \) is a hypersurface of \( TM \) given by \( \bar{p}(u,u) = 1 \). Note that \( \bar{p} = \pi \circ i \), where \( i \) is the immersion of \( T_1 M \) into \( TM \). A unit normal vector field \( N = u^v \) to \( T_1 M \) is given by the vertical lift of \( u \) for \( (p,u) \). The horizontal lift of a vector is tangent to \( T_1 M \), but the vertical lift of a vector is not tangent to \( T_1 M \) in general. So, we define the tangential lift of \( X \) to \( (p,u) \in T_1 M \) by

\[ X^t_{(p,u)} = (X - g(X,u)u)^v. \]

Clearly, the tangent space \( T_{(p,u)} T_1 M \) is spanned by vectors of the form \( X^h \) and \( X^v \), where \( X \in T_p M \).

We now define the standard contact metric structure of the unit tangent sphere bundle \( T_1 M \) over a Riemannian manifold \( (M, g) \). The metric \( g' \) on \( T_1 M \) is induced from the Sasaki metric \( \bar{g} \) on \( TM \). Using the almost complex structure \( J \) on \( TM \), we define a unit vector field \( \xi' \), a 1-form \( \eta' \) and a \((1,1)\)-tensor field \( \phi' \) on \( T_1 M \) by

\[ \xi' = -JN, \quad \phi' = J - \eta' \otimes N. \]

Since \( g'(\bar{X}, \phi' Y) = 2d\eta'(\bar{X}, \bar{Y}) \), \((\eta', g', \phi', \xi') \) is not a contact metric structure. If we rescale this structure by

\[ \xi = 2\xi', \quad \eta = \frac{1}{2} \eta', \quad \phi = \phi', \quad \bar{g} = \frac{1}{4} g', \]

we get the standard contact metric structure \((\eta, \bar{g}, \phi, \xi)\). The tensors \(\xi\) and \(\phi\) are explicitly given by

\[
\xi = 2u^h, \quad \phi X^t = -X^h + \frac{1}{2}g(X, u)\xi, \quad \phi X^h = X^t,
\]

where \(X\) and \(Y\) are vector fields on \(M\).

From now on, we consider \(T_1 M = (T_1 M; \eta, \bar{g}, \phi, \xi)\) with the standard contact metric structure. Then the Levi-Civita connection \(\bar{\nabla}\) of \(T_1 M\) is described by

\[
\bar{\nabla}_{X^t} Y^t = -g(Y, u)X^t,
\]

\[
\bar{\nabla}_{X^t} Y^h = \frac{1}{2}(R(u, X)Y)^h,
\]

\[
\bar{\nabla}_{X^t} Y^t = (\nabla_X Y)^t + \frac{1}{2}(R(u, Y)X)^h,
\]

\[
\bar{\nabla}_{X^h} Y^t = (\nabla_X Y)^t - \frac{1}{2}(R(X, Y)u)^t
\]

for all vector fields \(X\) and \(Y\) on \(M\).

Also the Riemann curvature tensor \(\bar{R}\) of \(T_1 M\) is given by

\[
\bar{R}(X^t, Y^t)Z^t = -(g(X, Z) - g(X, u)g(Z, u))Y^t + (g(Y, Z) - g(Y, u)g(Z, u))X^t,
\]

\[
\bar{R}(X^t, Y^t)^h = \left\{ R(X - g(X, u)Y, Y - g(Y, u)u)Z \right\}^h + \frac{1}{4}\left\{ [R(u, X), R(u, Y)]Z \right\}^h,
\]

\[
\bar{R}(X^h, Y^t)Z^t = \frac{1}{2}\left\{ R(Y - g(Y, u)u, Z - g(Z, u)u)X \right\}^h - \frac{1}{4}\left\{ R(X, R(u, Y)Z)u \right\}^t + \frac{1}{2}\left\{ (\nabla_X R)(u, Y)Z \right\}^h,
\]

\[
\bar{R}(X^h, Y^h)Z^t = \left\{ R(X, Y)(Z - g(Z, u)u) \right\}^t + \frac{1}{4}\left\{ R(Y, R(u, Z)X)u - R(X, R(u, Z)Y)u \right\}^t + \frac{1}{2}\left\{ (\nabla_X R)(u, Z)Y - (\nabla_Y R)(u, Z)X \right\}^h,
\]

\[
\bar{R}(X^h, Y^h)^h = (R(X, Y)Z)^h + \frac{1}{2}\left\{ R(u, R(X, Y)u)Z \right\}^h - \frac{1}{4}\left\{ R(u, R(Y, Z)u)X - R(u, R(X, Z)u)Y \right\}^h + \frac{1}{2}\left\{ (\nabla_Z R)(X, Y)u \right\}^t
\]
for all vector fields $X$, $Y$ and $Z$ on $M$. Using the formulae (11), we get

\begin{equation}
\ell X^i = (R^2_u X)^i + 2(R^i_u X),
\end{equation}

(12)

\begin{equation}
\ell X^h = 4(R^i_u X)^h - 3(R^2_u X)^h + 2(R^i_u X)^i,
\end{equation}

where $R_u = R(\cdot, u)u$, $R^i_u = (\nabla_u R)(\cdot, u)u$ and $R^2_u = R(R(\cdot, u)u, u)u$. We can refer to [2, 3, 4] for the formulas (10) $\sim$ (12). From (12), we have the following proposition.

**Proposition 2.** The characteristic Jacobi operator $\ell$ of $T_1 M$ vanishes if and only if $M$ is locally flat.

**Proof.** Suppose that the characteristic Jacobi operator $\ell$ vanishes. Then we get from (12) $R^i_u X = 0$ and $R^2_u X = 0$. The former implies that $(M, G)$ is a locally symmetric space (\cite{8}, \cite{13}) and the latter does that the eigenvalues of $R_u$ are constant and equal to 0, i.e., $(M, G)$ is a globally Osserman space (i.e., the eigenvalues of $R_u$ do not depend on the point $p$ and not on the choice of unit vector $u$ at $p$). However, a locally symmetric globally Osserman space is locally flat or locally isometric to a rank one symmetric space (\cite{7}). Therefore, we conclude that $M$ is a space of constant curvature 0. \hfill $\square$

\section{4. Proof of Main Theorem}

Suppose that the characteristic Jacobi operator $\ell$ of $T_1 M$ is pseudo-parallel by a function $L$ on $T_1 M$. Then $T_1 M$ satisfies

\begin{equation}
\hat{R}(X, Y)\ell Z - \ell(\hat{R}(X, Y)Z) = L(\hat{g}(Y, \ell Z)X - \hat{g}(X, \ell Z)Y - \hat{g}(\ell Y, \ell Z)X) + \hat{g}(\ell X, \ell Z).
\end{equation}

We put $\ell Y = \xi$ in (13). Then we have

\begin{equation}
\hat{R}(X, \xi)\ell Z - \ell(\hat{R}(X, \xi)Z) = L(-\hat{g}(X, \ell Z)\xi - \eta(\ell X)).
\end{equation}

Setting $X = X^i$, $Z = Z^j$ in (14), and applying the Riemannian metric $\hat{g}$ on $T_1 M$ for $Y^h$ on both sides, then we have the following equation:

\begin{equation}
\frac{1}{2}\hat{g}(R(X, R^2_u Z)u, Y) + \frac{1}{2}\hat{g}(X, u)\hat{g}(R_u Z, Y) + \frac{1}{4}\hat{g}(R(X, u)R_u Z, Y)
\end{equation}

\begin{equation}
- \hat{g}((\nabla_u R)(u, X)R_u Z, Y) = -\frac{1}{4}L\hat{g}(X, R_u Z)\hat{g}(Y, u).
\end{equation}

We put $Y = u$ in (15). Then we have

\begin{equation}
\hat{g}(-\frac{1}{4}R_u^2 X - R_u^2 X, Z) = -\frac{1}{4}L\hat{g}(R_u^2 X, Z)
\end{equation}

for any vector fields $X$ and $Z$ on $M$, that is, it holds

\begin{equation}
R_u^2 X + 4R_u^2 X = LR_u^2 X.
\end{equation}

Since $R_u$ is symmetric operator, if $L \leq 0$, from (16) we have $R_u^2 = 0$ and $R_u = 0$. Therefore, using the similar arguments in the proof of Proposition 2 we see that $M$ is locally flat. This completes the proof of (i).
Next, in order to prove the second part of Main Theorem we prepare the following lemma.

**Lemma 3.** Let \((M, g)\) be a locally symmetric space. Then the characteristic Jacobi operator \(\ell\) of \(T_1M\) is pseudo-parallel by a function \(L\) on \(T_1M\) if and only if \(M\) is of constant curvature \(0\) or \(1\).

**Proof.** If we set \(X = X^h\), \(Z = Z^h\) in (14), and apply the Riemannian metric \(\bar{g}\) on \(T_1M\) for \(Y^h\) on both sides, then we have the following equation:

\[
(17) \quad 4g(R(X, u)R_u Z, Y) + 2g(R(u, R_u X)R_u Z, Y) - g(R(R_u^2 Z, u)X, Y)
- g(R(X, R_u Z)u, R_u Y) - 3g(R(X, u)R_u^2 Z, Y) - \frac{3}{2}g(R(u, R_u X)R_u^2 Z, Y)
+ \frac{3}{4}g(R(R_u^2 Z, u)X, Y) - \frac{3}{4}g(R(R_u^2 Z, X)u, R_u Y) + g((\nabla_u R)(u, R_u^c Z)u, Y)
- g((\nabla_u R)(u, R_u^c Z)X, Y) - 4g(R(X, u)Z, R_u Y) + 3g(R(X, u)Z, R_u^c Y)
- 2g(R(u, R_u X)Z, R_u Y) + \frac{3}{2}g(R(u, R_u X)Z, R_u^c Y) + g(R(R_u X, u)Z, R_u Y)
- \frac{3}{4}g(R(R_u Z, u)X, R_u^2 Y) + g(R(X, Z)u, R_u^2 Y) - \frac{3}{4}g(R(X, Z)u, R_u^2 Y)
- g((\nabla_Z R)(X, u)u, R_u^c Y)
= \frac{1}{4}L(-4g(X, R_u Z)g(Y, u) + 3g(X, R_u^2 Z)g(Y, u) - 4g(R_u X, Y)g(Z, u)
+ 3g(R_u^2 X, Y)g(Z, u)).
\]

Putting \(Y = u\) in (17), we have

\[
(18) \quad - \frac{9}{4}R_u^4 X + 6R_u^3 X - 4R_u^2 X - R_u^2 X = \frac{1}{4}L(-4R_u X + 3R_u^2 X).
\]

We suppose that \(M\) is locally symmetric. Then from (16) and (18), we obtain

\[
(19) \quad R_u^4 X = LR_u^2 X,
\]

\[
(20) \quad - 9R_u^4 X + 24R_u^3 X - 16R_u^2 X = L(-4R_u X + 3R_u^2 X).
\]

We assume that \(R_u X = \lambda X\) for a function \(\lambda\) on \(M\). Then from (19) and (20), we have

\[
(21) \quad \lambda^4 = L\lambda^2,
\]

\[
(22) \quad 9\lambda^4 - 24\lambda^3 + 16\lambda^2 - 4L\lambda + 3L\lambda^2 = 0.
\]

From (21), we have \(\lambda = 0\) or \(L = \lambda^2\). If \(L = \lambda^2\) and \(\lambda \neq 0\), from (22), we have

\[
(3\lambda - 4)(\lambda - 1) = 0.
\]

Hence, \(\lambda = 0, 1\) or \(\frac{4}{3}\), and then \((M, g)\) is a globally Osserman space. But, it is also locally symmetric, and then it is locally isometric to a rank one symmetric space. However, we can easily check that \(T_1M\) of a space of constant curvature
\( \frac{4}{3} \) does not satisfy pseudo-parallelism of \( \ell \). Therefore, we conclude that \((M, g)\) is of constant curvature 0 or 1. By Propositions 1 and 2, the converse is easily proved. \( \square \)

Now we assume that \( L \) is constant. Then, from (16) and (18), we have

\[
2R^2_uX - 6R^3_uX + 4R^4_uX = L(R_uX - R^2_uX).
\]

If we put \( R_uX = \lambda X \), we get

\[
\lambda(\lambda - 1)(2\lambda^2 - 4\lambda + L) = 0.
\]

Here, we use Nikolayevsky’s results ([10, 11, 12]) on the Osserman conjecture. Then we find that \((M^n, g)\) is locally isometric to a rank one symmetric space, when \( n \neq 16 \). Thus, by Lemma 3 we conclude that \((M, g)\) is of constant curvature 0 or 1, when \( n \neq 16 \). Conversely, by Propositions 1 and 2, we see that for the unit tangent sphere bundle over a space of constant curvature \( c = 0 \) or \( c = 1 \), the characteristic Jacobi operator \( \ell \) is pseudo-parallel with \( L = 0 \) or \( L = 1 \), respectively. This completes the proof of Main Theorem.

**Corollary 4.** If \( \ell \) of \( T_1M \) is semi-parallel, that is, \( L = 0 \), then \( M \) is locally flat.

**References**

[1] D. E. Blair, *Riemannian Geometry of Contact and Symplectic Manifolds*, Second edition, Progr. Math. 203, Birkhäuser Boston, Inc., Boston, MA, 2010.

[2] E. Boeckx, J. T. Cho, and S. H. Chun, *Flow-invariant structures on unit tangent bundles*, Publ. Math. Debrecen 70 (2007), no. 1-2, 167–178.

[3] E. Boeckx and L. Vanhecke, *Characteristic reflections on unit tangent sphere bundles*, Houston J. Math. 23 (1997), no. 3, 427–448.

[4] J. T. Cho and S. H. Chun, *On the classification of contact Riemannian manifolds satisfying the condition (C)*, Glasg. Math. J. 45 (2003), no. 3, 475–492.

[5] J. T. Cho and J.-I. Inoguchi, *Pseudo-symmetric contact 3-manifolds*, J. Korean Math. Soc. 42 (2005), no. 5, 913–932.

[6] P. Dombrowski, *On the geometry of the tangent bundle*, J. Reine Angew. Math. 210 (1962), 73–88.

[7] P. Gilkey, A. Swann, and L. Vanhecke, *Isoparametric geodesic spheres and a conjecture of Osserman concerning the Jacobi operator*, Quart. J. Math. Oxford Ser. (2) 46 (1995), no. 183, 299–320.

[8] A. Gray, *Classification des variétés approximativement kählériennes de courbure sectionnelle holomorphe constante*, J. Reine Angew. Math. 279 (1974), 797–800.

[9] O. Kowalski, *Curvature of the induced Riemannian metric on the tangent bundle of a Riemannian manifold*, J. Reine Angew. Math. 250 (1971), 124–129.

[10] Y. Nikolayevsky, *Osserman manifolds of dimension 8*, Manuscr. Math. 115 (2004), no. 1, 31–53.

[11] Y. Nikolayevsky, *Osserman conjecture in dimension \( n \neq 8, 16 \)*, Math. Ann. 331 (2005), no. 3, 505–522.

[12] Y. Nikolayevsky, *On Osserman manifolds of dimension 16*, Contemporary geometry and related topics, 379–398, Univ. Belgrade Fac. Math., Belgrade, 2006.

[13] L. Vanhecke and T. J. Willmore, *Interactions of tubes and spheres*, Math. Anal. 21 (1983), no. 1, 31–42.

[14] K. Yano and S. Ishihara, *Tangent and Cotangent Bundles*, M. Dekker Inc., 1973.
JONG TAEK CHO
DEPARTMENT OF MATHEMATICS
CHONNAM NATIONAL UNIVERSITY
GWANGJU 61186, KOREA
E-mail address: jtcho@chonnam.ac.kr

SUN HYANG CHUN
DEPARTMENT OF MATHEMATICS
CHOSUN UNIVERSITY
GWANGJU 61452, KOREA
E-mail address: shchun@chosun.ac.kr