A FREQUENCY APPROACH FOR STABILIZATION OF ONE-DIMENSIONAL DEGENERATE WAVE EQUATION

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Abstract. In this paper, we are concerned with the study of stabilization problem for the following strongly degenerate wave equation in one space dimension

\[ w_{tt}(x,t) - (x^\alpha w_x(x,t))_x = 0 \]

where \( \alpha \in [1, 2) \). Thus, using a frequency domain method inspired from [4], we prove the polynomial decays of its total energy with \( t^{-1/2} \) decay rate.

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1. Introduction

Control problems for degenerate PDE’s (and especially for parabolic equations) have received a lot of attention in the last few years, (see for instance [6, 7, 8]). So under Carleman estimates with suitable weighted functions, they obtained some observability inequality for the corresponding dual problems. Therefore, the purpose of this paper is to study stabilization issue for the following degenerate wave equation with $\alpha \in [1, 2)\) $w_{tt}(x, t) - (x^\alpha w_x(x, t))_x = 0 \quad \text{on} \ (0, 1) \times (0, \infty) \\
(x^\alpha w_x)(0, t) = w_t(0, t), \ w(1, t) = 0 \quad t \in (0, \infty), \quad (1.1) \\
w(x, 0) = w^0(x), \ w_t(x, 0) = w^1(x) \quad \text{on} \ (0, 1) \).\]

So, in a recent paper of Alabau-Cannarsa and Leugering [1], authors studied the same problem as (1.1) and they proved that exact observability inequality fails for $\alpha \in [1, 2)$ via the normal derivative $u_x(1, \cdot)$ and therefore they didn’t obtained a such exponentially decay for solutions of (1.1). More concisely, they studied the following degenerate wave equation

\[w_{tt}(x, t) - (a(x)w_x(x, t))_x = 0 \quad \text{on} \ (0, 1) \times (0, \infty) \quad (1.2)\]

where $a$ is positive function on $[0, 1]$ and vanishes at zero. So under the following linear feedback

\[w_x(t, 1) = -\beta w(t, 1) - w_t(t, 1), \quad (1.3)\]

they obtained exponential stability of solutions of (1.2).

In this paper, via a frequency domain approach due to Borichev-Tomilov [4], we show that system (1.1) is polynomially stable for $\alpha \in [1, 2)$.

Here we want to focus on he following remarks:

- System (1.1) under study is different from one studied on [1]. Indeed, the degeneracy is located at $x = 0$.

- The frequency domain method gives us a sharp polynomial decay rate, however in [1], stabilization is done under the classical energy method due to Komornik [12].

The outline of this paper as follows. In section 2, we introduce our notations, functional space and establish the well-posedness of system...
under study. In section 3, we set our main result concerning stability. In the last section, we give a numerical simulation of the transfer function for the control system.

2. The semigroup setting

We define the Hilbert space $H_{\alpha,r}^1(0,1)$ as

$$H_{\alpha,r}^1(0,1) = \{ u \in L^2(0,1) : x^{\alpha/2}u_x \in L^2(0,1) \text{ and } u(1) = 0 \}, \quad (2.4)$$

equipped with the following inner product

$$\langle f, g \rangle_{H_{\alpha,r}^1(0,1)} = \int_0^1 x^{\alpha/2} f_x x^{\alpha/2} \overline{g_x} dx + \int_0^1 f \overline{g} dx, \quad (2.5)$$

and its associated norm

$$\| f \|_{H_{\alpha,r}^1(0,1)}^2 = \| x^{\alpha/2} f_x \|_{L^2(0,1)}^2 + \| f \|_{L^2(0,1)}^2. \quad (2.6)$$

Moreover, we introduce the operator $A_{\alpha} : D(A_{\alpha}) \subset L^2(0,1) \rightarrow L^2(0,1)$ as

$$D(A_{\alpha}) = \{ u \in H_{\alpha,r}^1(0,1) : (x^{\alpha} u_x)_x \in L^2(0,1) \}, \quad A_{\alpha} u = -(x^{\alpha} u_x)_x, \quad \forall u \in D(A_{\alpha}), \quad (2.7)$$

One can easily check that $A_{\alpha}$ is self-adjoint positive operator with compact resolvent. Thus, there exists an orthonormal basis of eigenfunctions denoted by $(\Psi_n)_{n \in \mathbb{N}^*}$ in $L^2(0,1)$ and a real sequence of eigenvalues $(\mu_n)_{n \in \mathbb{N}^*}$ with $\mu_n > 0$ and $\mu_n \rightarrow \infty$ such that

$$A_{\alpha} \Psi_n = \mu_n \Psi_n, \quad \forall n \in \mathbb{N}^*. \quad (2.8)$$

Next, for $s \geq 0$, we introduce the following extrapolated spaces

$$H_{\alpha,r}^s(0,1) = D(A_{\alpha}^{s/2}) = \{ u = \sum_{n \geq 1} a_n \Psi_n \ | \ \| u \|_s^2 = \sum_{n \geq 1} \mu_n^s |a_n|^2 < \infty \} \quad (2.9)$$

and its dual

$$H_{\alpha,r}^{-s}(0,1) = \left( D(A_{\alpha}^{s/2}) \right)' . \quad (2.10)$$

Introducing the following Hilbert space

$$\mathcal{H}_\alpha = H_{\alpha,r}^1(0,1) \times L^2(0,1) \quad (2.11)$$
equipped with the scalar product

$$\langle (u, v)^T, (\tilde{u}, \tilde{v})^T \rangle_{\mathcal{H}_\alpha} = \int_0^1 x^{\alpha/2} u_x x^{\alpha/2} \overline{\tilde{u}_x} dx + \int_0^1 v \overline{\tilde{v}} dx \quad (2.12)$$
If we denote by \( Z(t) = (w(t), w'(t))^T \), then the solution of \( (1.1) \) can be written in the abstract Cauchy problem as

\[
\begin{align*}
\left\{ \begin{array}{l}
Z'(t) = A_\alpha Z(t) \\
Z(0) = Z_0,
\end{array} \right.
\end{align*}
\]

(2.13)

where \( Z_0 = (w^0, w^1)^T \) and \( A_\alpha \) is an unbounded operator of \( H_\alpha \) given by

\[
A_\alpha (u, v)^T = (v, -A_\alpha u)^T, \quad (u, v) \in D(A_\alpha)
\]

(2.14)

with

\[
D(A_\alpha) = \{(u, v) \in H^1_{\alpha,r}(0, 1) \times H^1_{\alpha,r}(0, 1), \quad u \in D(A_\alpha) \text{ and } (x^\alpha u_x)(0) = v(0)\}
\]

The well-posedness of \( (2.13) \) is given by the following proposition.

**Proposition 2.1.** For an initial data \( Z_0 \in H_\alpha \), there exists a unique solution \( Z \in C([0, \infty), H_\alpha) \) to system \( (2.13) \). Moreover, if \( Z_0 \in D(A_\alpha) \), then

\[
Z \in C([0, \infty), D(A_\alpha)) \cap C^1([0, \infty), H_\alpha).
\]

Moreover, the energy of system \( (1.1) \) is given by

\[
E_w(t) = \frac{1}{2} \int_0^1 \left(w_t^2 + x^\alpha w_x^2\right) dx, \quad t \geq 0,
\]

(2.15)

and satisfies

\[
E_w(0) - E_w(t) = \int_0^1 |w_t(0, t)|^2 dx.
\]

(2.16)

**Proof.** Using Lumer-Phillips theorem \[16\], it suffices to prove that \( A_\alpha \) is maximal-dissipative on \( X \). In fact, for all \((u, v)^T \in D(A_\alpha)\), we have

\[
\Re \langle A_\alpha (u, v)^T, (u, v)^T \rangle = \Re \langle (v, -A_\alpha u)^T, (u, v)^T \rangle
\]

\[
= \Re \left( \int_0^1 x^\alpha v_x u_x dx + \int_0^1 (x^\alpha u_x)_x v dx \right)
\]

\[
= \Re \left( \int_0^1 x^\alpha (v_x u_x - \bar{v} u_x) dx + \left[ (x^\alpha u_x) \bar{v} \right]_1 \right)
\]

\[
= -|v(0)|^2 \leq 0,
\]

which proves the dissipativeness of \( A_\alpha \).

Next, let \( \lambda > 1 \), \((f, g)^T \in X\) and we look for \((u, v)^T \in D(A_\alpha)\) such

\[
(\lambda - A_\alpha)(u, v)^T = (f, g)^T.
\]

(2.17)
That’s
\[
\begin{align*}
\lambda u - v &= f \\
A_{\alpha} u + \lambda v &= g.
\end{align*}
\tag{2.18}
\]
If we suppose that we have found \(u\) with an appropriate regularity, then we get
\[v = \lambda u - f \in H^{1}_{\alpha,r}.
\]
Inserting the previous expression in the second equation of (2.18) we find that \(u\) must satisfy
\[A_{\alpha} u + \lambda^2 u = g + \lambda f.
\]
Multiplying the previous identity by \(w \in H^{1}_{\alpha,r}\), we get
\[-\int_{0}^{1} \left( x^{\alpha} u_{x} \right)_{x} \overline{w} dx + \lambda^2 \int_{0}^{1} u \overline{w} dx = \int_{0}^{1} (g + \lambda f) \overline{w} dx. \tag{2.19}
\]
By a formal integrations by parts, we obtain
\[
\int_{0}^{1} x^{\alpha} u_{x} \overline{w_{x}} + \lambda^2 \int_{0}^{1} u \overline{w} dx = \int_{0}^{1} (g + \lambda f) \overline{w} dx. \tag{2.20}
\]
Thus, equation (2.20) becomes
\[
b(u, w) = F(w), \quad w \in V = D(A_{\alpha}), \tag{2.21}
\]
where \(b : H^{1}_{\alpha,r} \times H^{1}_{\alpha,r} \rightarrow \mathbb{R}\) is a bilinear form given by
\[b(u, w) = \int_{0}^{1} \left( x^{\alpha} u_{x} \overline{w_{x}} + \lambda^2 u \overline{w} \right) dx
\]
and \(F : H^{1}_{\alpha,r} \rightarrow \mathbb{R}\) is a linear form given by
\[F(w) = \int_{0}^{1} (g + \lambda f) \overline{w} dx.
\]
Since \(b\) is a continuous bilinear coercive form on \(H^{1}_{\alpha,r}\) (this follows immediately) and \(F\) is a continuous linear form on \(H^{1}_{\alpha,r}\), then by using the Lax-Milgram theorem, we conclude that problem (2.21) has a unique solution \(u \in H^{1}_{\alpha,r}\).

By an appropriate choice of
\[v = \lambda u - f,
\]
we ensured that \((u, v)^{T}\) is a solution of (2.17), and thus \((\lambda I - A_{\alpha})\) is surjective. Finally, the Lumer-Phillips theorem leads to the claim.

For the identity (2.16), it’s easy to check.  \(\square\)
3. Stability results

First of all, let us recall the following result due to Borichev and Tomilov\cite{4} which is will be needed later.

**Theorem 3.1.** (See\cite{4}) Let $A$ be the generator of a $C_0$-semigroup of contractions on a Hilbert space $X$. Then,

$$
\left\| e^{tA}U_0 \right\|_X \leq \frac{C}{t^{\gamma}} \left\| U_0 \right\|_{D(A)}, \quad t > 0
$$

(3.22)

for some constant $C > 0$, if and only if

$$
i\mathbb{R} \subset \rho(A)
$$

(3.23)

and

$$
\lim_{|\lambda| \to \infty} \sup \frac{1}{|\lambda|^2} \left\| (i\lambda - A)^{-1} \right\| < \infty.
$$

(3.24)

Now, we state our main result.

**Theorem 3.2.** Let $\alpha \in [1, 2)$. Then, the total energy of system\cite{2,13} decays to zero polynomially with the rate $t^{-1/2}$, that’s

$$
\left\| e^{tA_\alpha}U_0 \right\|_X \leq \frac{C}{t^{1/2}} \left\| U_0 \right\|_{D(A_\alpha)}, \quad t > 0
$$

(3.25)

**Proof.** In view of Theorem 3.1, we need firstly to identify the spectrum of $A_\alpha$ lying on the imaginary axis. We have then to show that :

1. $\ker (i\beta - A_\alpha) = \{0\}, \quad \forall \beta \in \mathbb{R},$ and
2. $R((i\beta - A_\alpha) = \mathcal{H}_\alpha, \quad \forall \beta \in \mathbb{R}.$

This is the aim of the two following lemmas.

**Lemma 3.3.** There is no eigenvalue of $A_\alpha$ on the imaginary axis.

**Proof.** We proceed by contradiction. Assume that there exists at least one $\tilde{\lambda} = i\beta \in \sigma(A_\alpha), \beta \in \mathbb{R}$ on the imaginary axis and $\tilde{Z} = (u, v)^T \in D(A_\alpha)$ such that

$$
A_\alpha \tilde{Z} = \tilde{\lambda} \tilde{Z}.
$$

(3.26)

Then, we have

$$
i\beta u - v = 0 \quad (3.27)
$$

$$
A_\alpha u + i\beta v = 0. \quad (3.28)
$$

By taking the inner product of (3.26) with $\tilde{Z}$ and using the dissipativity of $A_\alpha$, we have

$$
0 = \Re \langle (i\beta - A_\alpha) \tilde{Z}, \tilde{Z} \rangle = |v(0)|^2,
$$

(3.29)
which yields \( v(0) = 0 \). Next, according to (3.27)-(3.28), we have
\[
(x^\alpha u_x)_x + \beta^2 u = 0, \quad \beta \in \mathbb{R}. \tag{3.30}
\]
The solutions of (3.30) are given as follows:
\[
\beta_n = \kappa j_{\nu,n} \quad \text{and} \quad u_n(x) = \frac{\sqrt{2\kappa}}{|J'_\nu(j_{\nu,n})|} x^{\frac{\nu}{2}} J_\nu(j_{\nu,n}x^{\kappa}), \tag{3.31}
\]
where
\[
\nu = \frac{\alpha - 1}{2 - \alpha}, \quad \kappa = \frac{2 - \alpha}{2} \quad \text{and} \quad J_\nu(x) = \sum_{m \geq 0} \frac{(-1)^m}{m!\Gamma(m + \nu + 1)} \left(\frac{x}{2}\right)^{2m+\nu}, \quad x \geq 0. \tag{3.32}
\]
Here \( \Gamma(.) \) is the Gamma function, and \( (j_{\nu,n})_{n \geq 1} \) are the positive zeros of the Bessel function \( J_\nu \). See [18] for more details.

As \( u_n \in D(A_\alpha), \forall n \geq 1 \), then in particular \( u_n(1) = 0 \) which gives us \( u = 0 \), and taking account (3.27) we obtain \( v = 0 \) which contradicts the fact that \( \tilde{Z} = (u,v)^T = 0 \) is an eigenvector.

The desired result follows. \[Q.E.D.\]

**Lemma 3.4.** For all \( \beta \in \mathbb{R} \), one has
\[
R((i\beta - A_\alpha)\| = \mathcal{H}_\alpha.
\]

**Proof.** Similarly to the proof of second part of Proposition 2.1 so we omit. \[Q.E.D.\]

In order to complete the proof of Theorem 3.2 it remains to check condition (3.24) of Theorem 3.1. For this end, we proceed by using a contradiction argument. Thus, we assume that (3.24) does not hold, then there exist sequences \( (\beta_n)_{n}, \beta_n \in \mathbb{R}^+, \beta_n \to \infty \) and \( (U_n)_n \) with \( U_n = (u_n,v_n) \) in \( D(A_\alpha), \ n \in \mathbb{N} \), such that
\[
\|U_n\|_{\mathcal{H}_\alpha} = 1, \quad \forall n \in \mathbb{N} \tag{3.33}
\]
and
\[
\beta_n^l (i\beta_n - A_\alpha) U_n \to 0, \quad \text{in } \mathcal{H}_\alpha \text{ as } n \to \infty. \tag{3.34}
\]
This yields: As \( n \to \infty \)
\[
\beta_n^l (i\beta_n u_n - v_n) = f_n \to 0 \quad \text{in } H^1_{\alpha,r}(0,1) \tag{3.35}
\]
\[
\beta_n^l (A_\alpha u_n + i\beta_n v_n) = g_n \to 0 \quad \text{in } L^2(0,1).
\]
Taking into account the following
\[
\beta_n^l |v_n(0)|^2 = \Re \langle \beta_n^l (i\beta_n - A_\alpha) U_n, U_n \rangle \leq \|\beta_n^l (i\beta_n - A_\alpha)\|, \tag{3.36}
\]
we get
\[
\beta_n^l |v_n(0)|^2, \quad \text{as } n \to \infty. \tag{3.37}
\]
On the other hand, we can write
\[ \left| v_n^2(1) - v_n^2(0) \right| = \left| \int_0^1 v_{n,x}^2 \, dx \right| \leq \| v_{n,x} \|_{L^2}^2 \] (3.38)
which implies by invoking Poincaré’s inequality
\[ v_n \to 0 \quad \text{in} \quad L^2(0,1). \] (3.39)
Multiplying the first equation in (3.35) by \( i\beta_n \) and summing with the second equation to get
\[ -\beta_n (x^n u_{n,x})_x - \beta_n^{l+2} u_n = g_n + i\beta_n f_n. \] (3.40)
Now, setting \( l = 2 \) and taking the \( L^2 \)-inner product of (3.40) with \( (iu_n) \) we arrive after, taking real parts, at
\[ \| x^{\alpha/2} u_{n,x} \|_{L^2}^2 + \beta_n^2 \| u_n \|_{L^2}^2 \to 0. \]
Since \( \beta_n \to \infty \) as \( n \) tends to infinity, then it follows that \( 1 \leq \beta_n \) for sufficiently big \( n \), so we can write
\[ \| x^{\alpha/2} u_{n,x} \|_{L^2}^2 + \| u_n \|_{L^2}^2 \to 0, \]
that’s
\[ \| u_n \|_{H^{3/2}_\alpha} \to 0. \] (3.41)
Combining (3.39) and (3.41), we have a contradiction with (3.33). Thus, (3.24) is verified and the proof of Theorem 3.2 is complete. \( \square \)

Now, let us further show the lack of exponential decays for solutions of (1.1) by using a frequency domain estimate for exponential stability as described in [11, 17]. For this end, we state the following result.

**Lemma 3.5.** There exists at least one sequence \((\lambda_n, F_n)\) such that \( \lambda_n \to +\infty \) as \( n \to \infty \) and
\[ \| (i\lambda_n - \mathcal{A}_\alpha)^{-1} F_n \|_{H_\alpha} \to \infty \quad \text{as} \quad n \to \infty, \] (3.42)
with \( F_n \in H_\alpha \) and \( \| F_n \|_{H_\alpha} \) is bounded, \( \tilde{M} \) is positive constant.

**Proof.** Setting the following resolvent equation
\[ (i\lambda - \mathcal{A}_\alpha) U = F, \quad \lambda \in \mathbb{R}, \] (3.43)
where \( U = (u,v)^T \) and \( F = (f,g)^T \). That’s
\[ \begin{aligned}
   i\lambda u - v &= f \\
   Au + i\lambda v &= g.
\end{aligned} \] (3.44)
Choosing \( f = 0 \) and substituting \( v = i\lambda u \) into the second equation of (3.44), we get

\[
\begin{cases}
(x^\alpha u_x)_x + \lambda^2 u = g, \quad x \in (0, 1), \\
\text{with boundary conditions}
\end{cases}
\]

\[
(x^\alpha u_x)(0) = 0, \quad u(1) = 0.
\]

So, according to [18], the solutions of (3.45) are given as follows:

\[
\lambda_n = \kappa j_{\nu,n} \quad \text{and} \quad u_n(x) = \sqrt{\frac{2\kappa}{J'_\nu(j_{\nu,n})}} x^{1-\alpha} J_\nu(j_{\nu,n} x^\kappa),
\]

where

\[
\nu = \frac{\alpha - 1}{2 - \alpha}, \quad \kappa = \frac{2 - \alpha}{2} \quad \text{and} \quad J_\nu(x) = \sum_{m \geq 0} \frac{(-1)^m}{m! \Gamma(m + \nu + 1)} \left( \frac{x}{2} \right)^{2m+\nu}, \quad x \geq 0.
\]

Here \( \Gamma(.) \) is the Gamma function, and \( (j_{\nu,n})_{n \geq 1} \) are the positive zeros of the Bessel function \( J_\nu \).

Using asymptotic behavior of Bessel functions [15], we get

\[
u_n(x) = \sqrt{\frac{2\kappa}{J'_\nu(j_{\nu,n})}} x^{1-\alpha} J_\nu(j_{\nu,n} x^\kappa) \approx \sqrt{\frac{2\kappa}{J'_\nu(j_{\nu,n})}} x^{1-\alpha} \left( \frac{j_{\nu,n} x^\kappa}{2} \right)^\nu
\]

\[
\approx \frac{2^{-\nu} \sqrt{k\pi}}{\Gamma(\nu + 1)} (j_{\nu,n})^{\nu + \frac{1}{2}}.
\]

Now, evaluating

\[
\|(u_n, iu_n)\|_{H_\alpha}^2 = \int_0^1 x^\alpha u_{nx}^2 dx + \int_0^1 u_n^2 dx
\]

\[
= \int_0^1 \frac{2^{-2\nu} \sqrt{k\pi}}{\Gamma^2(\nu + 1)} (j_{\nu,n})^{2\nu + 1} dx
\]

\[
= M\lambda_n^{2\nu + 1}
\]

\[
\geq M\lambda_n \to \infty, \quad \text{as} \quad n \to \infty.
\]

4. Numerical simulation of transfer function

Here we begin by recalling some aspects on input-output systems (see [9] for more details). So, let us consider \( U, X \) be two Hilbert spaces and consider the abstract control problem

\[
\begin{cases}
\dot{z}(t) = Az(t) + Bu(t), \quad z(0) = z_0 \\
y(t) = B^*z(t)
\end{cases}
\]

(4.50)
where $A : D(A) \subset X \to X$ generates a $C_0$-semigroups of contractions $T(t)_{t \geq 0}$, $B \in \mathcal{L}(U, X)$ is an admissible control operator, $u(.) \in L^2_{loc}(0, +\infty; U)$ design the input (or control) function and $y(.)$ design the output (or observation) function. The transfer function of (4.50) is given by $H(\lambda) \in \mathcal{L}(U)$ such that

$$
\hat{y}(\lambda) = H(\lambda)\hat{u}(\lambda),
$$

where $\hat{.}$ denotes the Laplace transformation. For these concepts, see [19].

Now, we consider the following control system

$$
\begin{cases}
\omega_{tt}(x, t) - (x^\alpha \omega_x(x, t))_x = 0 \quad \text{on } (0, 1) \times (0, T) \\
(x^\alpha \omega_x)(0, t) = \theta(t), \quad \omega(1, t) = 0 \quad t \in (0, T) \\
\omega(x, 0) = 0, \quad \omega_t(x, 0) = 0 \quad \text{on } (0, 1)
\end{cases}
$$

(4.51)

where $\theta(.) \in L^2(0, T)$. Then, system (4.51) can be written on the abstract form as (4.50) with $B$ is an unbounded control operator. Hence admissibility of $B$ is not verified and as it was shown in [2], we replace this issue by proving the boundedness of its associated transfer function. More precisely, we have the following.

**Lemma 4.1.** Let $\gamma > 0$ and $C_\gamma = \{\lambda \in \mathbb{C}, \quad \Re\lambda = \gamma\}$. Then, the transfer function of (4.51) is given by

$$
\lambda \in C_\gamma \to H(\lambda) = \frac{2((\nu + 1)\lambda)^{\nu+1}}{\lambda \Gamma(\nu + 1)} \left( (\nu + 1)\lambda K_\nu((\nu + 1)\lambda) \frac{I_\nu((\nu + 1)\lambda)}{I_\nu((\nu + 1)\lambda)} - c_\nu \right)
$$

(4.52)

and is bounded on $C_\gamma$, where $I_\nu, K_\nu$ are the modified Bessel functions of first and second kind and $c_\nu$ is constant complex number.

**Proof.** Applying the Laplace transform to (4.51) with respect to time $t$ to get $\hat{\omega}(x, \lambda)$ where $\lambda = \gamma + ik$ and $\gamma > 0$. Then

$$
\begin{cases}
\lambda^2 \hat{\omega}(x, \lambda) - (x^\alpha \hat{\omega}(x, \lambda))_x = 0, \quad 0 < x < 1 \\
x^\alpha \hat{\omega}_x(0, \lambda) = \hat{\theta}(\lambda), \quad \hat{\omega}(1, \lambda) = 0.
\end{cases}
$$

(4.53)

So we obtain the following Sturm-Liouville problem

$$
x^2 \ddot{\omega}_{xx} + \alpha x \ddot{\omega}_x - \lambda^2 x^{2-\alpha} \dot{\omega} = 0,
$$
with a solution

\[
\hat{\omega}(x, \lambda) = \begin{cases} 
  x^{1-\alpha} \left[ A_1 I_\nu \left( \frac{2\lambda}{2-\alpha} x^{\frac{2-\alpha}{2}} \right) + B_1 K_\nu \left( \frac{2\lambda}{2-\alpha} x^{\frac{2-\alpha}{2}} \right) \right] & \text{if } \nu \in \mathbb{N}^*, \\
  x^{1-\alpha} \left[ A_2 I_{-\nu} \left( \frac{2\lambda}{2-\alpha} x^{\frac{2-\alpha}{2}} \right) + B_2 I_\nu \left( \frac{2\lambda}{2-\alpha} x^{\frac{2-\alpha}{2}} \right) \right] & \text{if } \nu \notin \mathbb{N}^*,
\end{cases}
\]

(4.54)

where \( A_i, B_i \) are complex numbers, \( I_\nu, K_\nu \) are Bessel functions of first and second kind and

\[ \nu = \frac{\alpha - 1}{2 - \alpha} > 0. \]

First case: \( \nu \in \mathbb{N}^* \).

We shall note that

\[ \nu = \frac{\alpha - 1}{2 - \alpha} \in \mathbb{N}^* \text{ if and only if } \alpha \in \left[ \frac{3}{2}, 2 \right]. \]

For instance, for \( 0 < x \ll \sqrt{\nu + 1} \) the following estimation holds

\[
\begin{align*}
  I_\nu(x) &\simeq \frac{1}{\Gamma(\nu + 1)} \left( \frac{x}{2} \right)^\nu \\
  K_\nu(x) &\simeq \frac{\Gamma(\nu)}{2} \left( \frac{2}{x} \right)^\nu.
\end{align*}
\]

(4.55)

The second equation in (4.53) gives us

\[
\begin{align*}
  A_1 I_\nu \left( \frac{2\lambda}{2-\alpha} \right) + B_1 K_\nu \left( \frac{2\lambda}{2-\alpha} \right) &= 0 \\
  B_1 c_1 &= \hat{\theta}(\lambda)
\end{align*}
\]

(4.56)

where

\[ c_2 = -\frac{\lambda}{2} \Gamma \left( \frac{1}{2 - \alpha} \right) \left( \frac{2 - \alpha}{\lambda} \right)^{\frac{1}{2 - \alpha}}. \]

(4.57)

Thus, constants \( A_1 \) and \( B_1 \) are determined by the following expressions

\[
\begin{align*}
  A_1 &= \frac{(-1)}{c_2} \frac{K_\nu \left( \frac{2\lambda}{2-\alpha} \right)}{I_\nu \left( \frac{2\lambda}{2-\alpha} \right)} \hat{\theta}(\lambda) \\
  B_1 &= \frac{\hat{\theta}(\lambda)}{c_2}
\end{align*}
\]

where \( c_2 \) as in (4.57). Hence, the Bessel functions \( K_\nu \) change its shape for each order \( \nu \), then using numerical calculations, we show that the solution \( \omega \) defined in (4.54) exists and well-defined in a neighborhood of the origin so that

\[ \hat{\omega}(0, \lambda) \simeq \frac{A_1}{\Gamma(\nu + 1)} \left( \frac{\lambda}{2 - \alpha} \right)^\nu + B_1 c_\nu, \quad |c_\nu| < \infty, \]
where \( c_\nu = \lim_{x \to 0} x^{1-\alpha} K_\nu \left( \frac{2\lambda}{2-\alpha} x^{\frac{2-\alpha}{2}} \right) \) is a complex number. By the relation

\[
\hat{\omega}(0, \lambda) = H(\lambda) \hat{\theta}(\lambda)
\]  

(4.58)

we can deduce the transfer function which is given by

\[
H(\lambda) = 2 \left( (\nu + 1) \lambda \right)^{\nu+1} \Gamma(\nu+1) \left( ((\nu + 1) \lambda)^\nu K_\nu((\nu + 1) \lambda) - c_\nu \right)
\]  

(4.59)

Let \( \lambda \in \mathbb{C} \setminus \mathbb{R}_- \), \( \lambda = \gamma + ik \), \( \gamma > 0 \). The principal determination of the logarithm of \( \lambda \) is defined as follows

\[
\log(\lambda) = \ln |\lambda| + i \arg(\lambda), \quad -\frac{\pi}{2} < \arg(\lambda) \leq \frac{\pi}{2},
\]

Since

\[
\lambda \mapsto (\nu + 1) \lambda^{\nu+1} = e^{(\nu+1) \log((\nu+1) \lambda)}
\]

\[
= \omega e^{(\nu+1) \log(\lambda)}, \quad \omega = (\nu + 1)^{\nu+1},
\]

we get

\[
|H(\lambda)| \leq 2 \omega^2 \nu \left| K_\nu((\nu + 1) \lambda) \right| I_\nu((\nu + 1) \lambda) + 2 \omega |\lambda^\nu | |c_\nu|.
\]

where \( |c_\nu| \) is finite and

\[
L = \left| \frac{K_\nu((\nu + 1) \lambda)}{I_\nu((\nu + 1) \lambda)} \right| \neq 0 \quad \text{and finite.}
\]

This shows that \( H(\lambda) \) is bounded on \( \mathbb{C} \setminus \mathbb{R}_- \).

**Second case:** \( \nu \notin \mathbb{N}^* \).

Since \( \nu \notin \mathbb{N}^* \), we have \( K_\nu(x) = I_{-\nu}(x) \). As a result, the solution of

(4.53)

becomes

\[
\hat{\omega}(x, \lambda) = x^{\frac{1-\alpha}{2}} \left[ 2A_2 I_\nu \left( \frac{2\lambda}{2-\alpha} x^{\frac{2-\alpha}{2}} \right) + B_2 I_{-\nu} \left( \frac{2\lambda}{2-\alpha} x^{\frac{2-\alpha}{2}} \right) \right].
\]  

(4.60)

The boundary conditions \( x^\alpha \hat{\omega}_x(0, \lambda) = \hat{\theta}(\lambda) \) and \( \hat{\omega}(1, \lambda) = 0 \) allows us to determine

\[
\begin{align*}
A_2 &= \frac{I_\nu \left( \frac{2\lambda}{2-\alpha} \right)}{(\alpha - 1) I_\nu \left( \frac{2\lambda}{2-\alpha} \right)} \left( \frac{\lambda}{2 - \alpha} \right)^{\frac{\alpha-1}{2-\alpha}} \hat{\theta}(\lambda), \\
B_2 &= \frac{1}{(1 - \alpha)} \left( \frac{\lambda}{2 - \alpha} \right)^{\frac{\alpha-1}{2-\alpha}} \hat{\theta}(\lambda).
\end{align*}
\]
Hence
\[ \hat{\omega}(x, \lambda) \simeq \left( \frac{A_2}{\Gamma(\nu + 1)} ((\nu + 1)\lambda)^\nu + \frac{B_2}{\Gamma(1 - \nu)} ((\nu + 1)\lambda)^{-\nu} x^{\frac{\nu}{\nu + 1}} \right). \]
As \( \hat{\omega}(0, \lambda) \to \infty \), then the transfer function is not bounded.

Using relation (4.58), we conclude that the transfer function is bounded on \( C_\gamma \) if \( \nu \in \mathbb{N}^* \). \( \square \)
Figure 1. value of $|c_\nu|$ with $\text{arg}(\lambda) = \frac{\pi}{3}$

Figure 2. value of $|c_\nu|$ with $\text{arg}(\lambda) = \frac{\pi}{4}$

Figure 3. value of $|c_\nu|$ with $\text{arg}(\lambda) = \frac{\pi}{6}$

Figure 4. value of $|c_\nu|$ with $\text{arg}(\lambda) = \frac{\pi}{2}$
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DEGENERATE WAVE EQUATION

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