Closed range integral operators on Hardy, BMOA and Besov spaces

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ABSTRACT

If \( g \in H^\infty \), the integral operator \( S_g \) on \( \mathbb{H}^p \), BMOA and \( \mathbb{B}^p \) (Besov) spaces, is defined as
\[
S_g f(z) = \int_0^z f'(w) g(w) \, dw.
\]
In this paper, we prove three necessary and sufficient conditions for the operator \( S_g \) to have closed range on \( \mathbb{H}^p \) (1 \( \leq p < \infty \)), BMOA and \( \mathbb{B}^p \) (1 \( < p < \infty \)).

1. Introduction and preliminaries

Let \( \mathbb{D} \) denote the open unit disk in the complex plane, \( \mathbb{T} \) the unit circle, \( A \) the normalized area Lebesgue measure in \( \mathbb{D} \) and \( m \) the normalized length Lebesgue measure in \( \mathbb{T} \). For \( 1 \leq p < \infty \) the Hardy space \( \mathbb{H}^p \) is defined as the set of all analytic functions \( f \) in \( \mathbb{D} \) for which
\[
\sup_{0 \leq r < 1} \int_{\mathbb{T}} |f(r\zeta)|^p \, dm(\zeta) < +\infty
\]
and the corresponding norm in \( \mathbb{H}^p \) is defined by
\[
\|f\|_{\mathbb{H}^p}^p = \sup_{0 \leq r < 1} \int_{\mathbb{T}} |f(r\zeta)|^p \, dm(\zeta).
\]

When \( p = \infty \), we define \( H^\infty \) to be the space of bounded analytic functions \( f \) in \( \mathbb{D} \) and \( \|f\|_\infty = \sup \{|f(z)| : z \in \mathbb{D}\} \).

In this work we will mainly make use of the following equivalent norm (see Calderon’s theorem in [1, p.213]) in \( \mathbb{H}^p \), \( 1 \leq p < \infty \):
\[
\|f\|_{\mathbb{H}^p}^p = |f(0)|^p + \int_{\mathbb{T}} \left( \int_{\Gamma_p(\zeta)} |f'(z)|^2 \, dA(z) \right)^{\frac{p}{2}} \, dm(\zeta), \tag{1}
\]
where $\Gamma_\beta(\zeta)$ is the Stolz angle at $\zeta \in \mathbb{T}$, the conelike region with aperture $\beta \in (0, 1)$, which is defined as

$$\Gamma_\beta(\zeta) = \{ z \in \mathbb{D} : |z| < \beta \} \cup \bigcup_{|z|<\beta} [z, \zeta].$$

The BMOA space is defined as the set of all analytic functions $f$ in $\mathbb{D}$ for which

$$\sup_{\beta \in \mathbb{D}} \int \int_{\mathbb{D}} \frac{1 - |\beta|^2}{|1 - \beta z|^2} |f'(z)|^2 \log \frac{1}{|z|} \, dA(z) < \infty$$

and we may define the corresponding norm in BMOA by

$$\|f\|_{2}^2 = |f(0)|^2 + \sup_{\beta \in \mathbb{D}} \int \int_{\mathbb{D}} \frac{1 - |\beta|^2}{|1 - \beta z|^2} |f'(z)|^2 \log \frac{1}{|z|} \, dA(z).$$

For $1 < p < \infty$ the Besov space $B^p$ is defined as the set of all analytic functions $f$ in $\mathbb{D}$ for which

$$\int \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^{p-2} \, dA(z) < +\infty$$

and the corresponding norm in $B^p$ is defined by

$$\|f\|_{B^p}^p = |f(0)|^p + \int \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^{p-2} \, dA(z).$$

Let $g : \mathbb{D} \to \mathbb{C}$ be an analytic function. If $X$ is a space of analytic functions $f$ in $\mathbb{D}$ (in particular, in this paper, $X = H^p$ or $X = BMOA$ or $X = B^p$) then, the integral operator $S_g : X \to X$, induced by $g$, is defined as

$$S_g f(z) = \int_{0}^{z} f'(w) g(w) \, dw, \quad z \in \mathbb{D},$$

for every $f \in X$. $S_g$ is companion to the widely studied operator $T_g : X \to X$ which is defined as

$$T_g f(z) = \int_{0}^{z} f(w) g'(w) \, dw, \quad z \in \mathbb{D},$$

for every $f \in X$. If $g(z) = z$ or $g(z) = \log \frac{1}{1-z}$, then $T_g$ is the integration operator and the Cesáro operator respectively. Interest in $T_g$ arose originally from studying semigroups of analytic composition operators because, for certain $g$, $T_g$ are related to the resolvents of such semigroups (see [2]). Results, concerning the boundedness and compactness of $T_g$ on certain spaces of analytic functions, can be found in [3–6]. It can be easily seen (using integration by parts) that $T_g$ and its companion operator $S_g$ are related to the multiplication operator $M_g f(z) = g(z) f(z)$ by

$$M_g f(z) = f(0) g(0) + T_g f + S_g f.$$
Let \( \rho(z, w) \) denote the pseudo-hyperbolic distance between \( z, w \in \mathbb{D} \),

\[
\rho(z, w) = \frac{|z - w|}{1 - \overline{z}w},
\]

\( D_{\eta}(a) \) denote the pseudo-hyperbolic disk of center \( a \in \mathbb{D} \) and radius \( \eta < 1 \):

\[
D_{\eta}(a) = \{ z \in \mathbb{D} : \rho(a, z) < \eta \},
\]

and \( \Delta_{\eta}(\alpha) \) denote the euclidean disk of center \( \alpha \in \mathbb{D} \) and radius \( \eta(1 - |\alpha|) \), \( \eta < 1 \):

\[
\Delta_{\eta}(\alpha) = \{ z \in \mathbb{D} : |z - \alpha| < \eta(1 - |\alpha|) \}.
\]

In the following, \( C \) denotes a positive and finite constant which may change from one occurrence to another. Moreover, by writing \( K(z) \approx L(z) \) for the non-negative quantities \( K(z) \) and \( L(z) \) we mean that there are positive constants \( C_1 \) and \( C_2 \) independent of \( z \) such that

\[
C_1 K(z) \leq L(z) \leq C_2 K(z).
\]

2. Closed range integral operators on Hardy spaces

Let \( g : \mathbb{D} \to \mathbb{C} \) be an analytic function and, for \( c > 0 \), let \( G_c = \{ z \in \mathbb{D} : |g(z)| > c \} \). It is well known (see [7]) that the integral operator \( S_g : \mathcal{H}^p \to \mathcal{H}^p \) (\( 1 \leq p < \infty \)) is bounded if and only if \( g \in \mathcal{H}^\infty \).

We say that \( S_g \), on \( \mathcal{H}^p \), is bounded below, if there is \( C > 0 \) such that \( \| S_g f \|_{\mathcal{H}^p} > C \| f \|_{\mathcal{H}^p} \) for every \( f \in \mathcal{H}^p \). Since \( S_g \) maps every constant function to the 0 function, if we want to study the property of being bounded below for \( S_g \), we are obliged to consider spaces of analytic functions modulo the constants or, equivalently, spaces of analytic functions \( f \) such that \( f(0) = 0 \). Theorem 3.2 in [7] states that \( S_g \) is bounded below on \( \mathcal{H}^p / \mathbb{C} \) if and only if it has closed range on \( \mathcal{H}^p / \mathbb{C} \). Next, we denote \( \mathcal{H}^p / \mathbb{C} \) as \( \mathcal{H}^p_0 \).

Corollary 3.6 in [7] states that \( S_g : \mathcal{H}^2_0 \to \mathcal{H}^2_0 \) has closed range if and only if there exist \( c > 0, \delta > 0 \) and \( \eta \in (0, 1) \) such that

\[
A(G_c \cap D_{\eta}(a)) \geq \delta A(D_{\eta}(a))
\]

for all \( a \in \mathbb{D} \).

In the end of [7] A. Anderson posed the question, if the above condition for \( \mathcal{H}^2_0 \) holds also for all \( \mathcal{H}^p_0 \). In this paper, Theorem 2.2 gives an affirmative answer to this question, for the case \( 1 \leq p < \infty \). Although the answer in case \( p = 2 \) is an immediate consequence of D. Luecking’s theorem (see [7], Proposition 3.5), the answer in case \( 1 \leq p < \infty \) requires much more effort.

For \( \lambda \in (0, 1) \) and \( f \in \mathcal{H}^p \) we set

\[
E_{\lambda}(\alpha) = \{ z \in \Delta_{\eta}(\alpha) : |f'(z)|^2 > \lambda |f'(\alpha)|^2 \}
\]

and

\[
B_{\lambda}(\alpha) = \frac{1}{A(E_{\lambda}(\alpha))} \int \int_{E_{\lambda}(\alpha)} |f'(z)|^2 \, dA(z).
\]

Lemma 2.1 is due to D. Luecking (see [8], lemma 1).
Lemma 2.1: Let \( f \) analytic in \( \mathbb{D} \), \( a \in \mathbb{D} \) and \( \lambda \in (0, 1) \). Then

\[
\frac{A(E_\lambda(\alpha))}{A(\Delta_\eta(\alpha))} \geq \frac{\log \frac{1}{\lambda}}{\log \frac{B_\lambda f(\alpha)}{|f'(\alpha)|^2} + \log \frac{1}{\lambda}}.
\]

Moreover in [8], the following sentence is proved: If \( \alpha \in \mathbb{D} \) and \( \frac{2\eta}{1+\eta} \leq r < 1 \) then

\[
\Delta_\eta(\alpha) \subseteq D_r(\alpha).
\] (2)

We proceed with the main result of this section.

Theorem 2.2: Let \( 1 \leq p < \infty \) and \( g \in H^\infty \). Then the following are equivalent:

(i) \( S_g : H^p_0 \to H^p_0 \) has closed range

(ii) There exist \( c > 0 \), \( \delta > 0 \) and \( \eta \in (0, 1) \) such that

\[
A(G_c \cap D_\eta(a)) \geq \delta A(D_\eta(a))
\] (3)

for all \( a \in \mathbb{D} \).

(iii) There exist \( c > 0 \), \( \delta > 0 \) and \( \eta \in (0, 1) \) such that

\[
A(G_c \cap \Delta_\eta(a)) \geq \delta A(\Delta_\eta(a))
\] (4)

for all \( a \in \mathbb{D} \).

We first prove two lemmas which will play an important role in the proof of Theorem 2.2.

For \( \zeta \in \mathbb{T} \) and \( 0 < \beta < \beta' < 1 \) we consider the Stolz angles \( \Gamma_1(\beta)(\zeta) \) and \( \Gamma_1(\beta')(\zeta) \), where \( \beta' \) has been chosen so that \( \Delta_\eta(\alpha) \subseteq \Gamma_1(\beta')(\zeta) \) for every \( \alpha \in \Gamma_1(\beta)(\zeta) \).

Lemma 2.3: Let \( \varepsilon > 0 \), \( f \) analytic in \( \mathbb{D} \) and

\[
A = \left\{ \alpha \in \mathbb{D} : |f'(\alpha)|^2 < \frac{\varepsilon}{A(\Delta_\eta(\alpha))} \int_{\Delta_\eta(\alpha)} |f'(z)|^2 \, dA(z) \right\}.
\]

There is \( C > 0 \) depending only on \( \eta \) such that

\[
\int_{\Delta_\eta(\alpha)} \int_{\Delta_\eta(\alpha)} |f'(z)|^2 \, dA(z) \leq \varepsilon C \int_{\Gamma_1(\beta')(\zeta)} |f'(z)|^2 \, dA(z)
\]

Proof: Integrating

\[
|f'(\alpha)|^2 < \frac{\varepsilon}{A(\Delta_\eta(\alpha))} \int_{\Delta_\eta(\alpha)} |f'(z)|^2 \, dA(z)
\]

over \( \alpha \in A \cap \Gamma_1(\beta)(\zeta) \) and using Fubini’s theorem on the right side, we get

\[
\int_{\Delta_\eta(\alpha)} \int_{\Delta_\eta(\alpha)} |f'(\alpha)|^2 \, dA(\alpha) < \int_{\Gamma_1(\beta')(\zeta)} |f'(z)|^2 \left[ \int_{\Delta_\eta(\alpha)} \frac{\chi_{\Delta_\eta(\alpha)}(z)}{A(\Delta_\eta(\alpha))} \, dA(\alpha) \right] \, dA(z)
\]

Using (2) with \( r = \frac{2\eta}{1+\eta} \), we have \( \chi_{\Delta_\eta(\alpha)}(z) \leq \chi_{D_r(\alpha)}(z) = \chi_{D_r(z)}(\alpha) \). We have that \( A(D_r(z)) \asymp (1 - |z|^2) \) and, for \( \alpha \in D_\eta(z) \), we have \( (1 - |z|) \asymp (1 - |\alpha|) \), where the underlying constants in these relations depend only on \( \eta \). In addition, \( A(\Delta_\eta(\alpha)) = \eta^2 (1 - |\alpha|^2) \).
So,
\[
\int\int_{A \cap \Gamma_\beta(\zeta)} \frac{\chi_{\Delta_\eta(\alpha)}(z)}{A(\Delta_\eta(\alpha))} \, dA(\alpha) \leq \int\int_{A \cap \Gamma_\beta(\zeta)} \frac{\chi_{D_\alpha(z)}}{\eta^2(1 - |\alpha|)^2} \, dA(\alpha)
\]
\[
\leq C \int\int_{D_\alpha(z)} \frac{1}{\eta^2(1 - |z|)^2} \, dA(\alpha) = C \frac{A(D_\alpha(z))}{\eta^2(1 - |z|)^2} \leq C,
\]
where \( C > 0 \) depends only on \( \eta \).

\[\blacksquare\]

**Lemma 2.4:** Let \( 0 < \varepsilon < 1, f \) analytic in \( \mathbb{D} \), \( 0 < \lambda < \frac{1}{2} \) and
\[B = \left\{ \alpha \in \mathbb{D} : |f'(\alpha)|^2 < \varepsilon^3 B_\lambda f(\alpha) \right\}.
\]
There is \( C > 0 \) depending only on \( \eta \) such that
\[
\int\int_{B \cap \Gamma_\beta(\zeta)} |f'(z)|^2 \, dA(z) \leq \varepsilon C \int\int_{\Gamma_\beta(\xi)} |f'(z)|^2 \, dA(z)
\]

**Proof:** We write
\[
\int\int_{B \cap \Gamma_\beta(\zeta)} |f'(\alpha)|^2 \, dA(\alpha) = \int\int_{B \cap \Gamma_\beta(\zeta) \cap A} |f'(\alpha)|^2 \, dA(\alpha) + \int\int_{(B \cap \Gamma_\beta(\zeta)) \setminus A} |f'(\alpha)|^2 \, dA(\alpha),
\]
where \( A \) is as in Lemma 2.3. The first integral is estimated by Lemma 2.3, so it remains to show the desired result for the second integral. Integrating the relation
\[
|f''(\alpha)|^2 < \varepsilon^3 B_\lambda f(\alpha) = \varepsilon^3 \frac{1}{A(E_\lambda(\alpha))} \int_{E_\lambda(\alpha)} |f'(z)|^2 \, dA(z)
\]
over the set \((B \cap \Gamma_\beta(\zeta)) \setminus A\) and using Fubini’s theorem on the right side, we get
\[
\int\int_{(B \cap \Gamma_\beta(\zeta)) \setminus A} |f'(\alpha)|^2 \, dA(\alpha)
\]
\[
\leq \varepsilon^3 \int\int_{\Gamma_\beta(\xi)} |f'(z)|^2 \left[ \int\int_{(B \cap \Gamma_\beta(\zeta)) \setminus A} \frac{1}{A(E_\lambda(\alpha))} \chi_{E_\lambda(\alpha)}(z) \, dA(\alpha) \right] \, dA(z)
\]
\[
\leq \varepsilon^3 \int\int_{\Gamma_\beta(\xi)} |f'(z)|^2 \left[ \int\int_{(B \cap \Gamma_\beta(\zeta)) \setminus A} \frac{1}{A(E_\lambda(\alpha))} \chi_{\Delta_\eta(\alpha)}(z) \, dA(\alpha) \right] \, dA(z)
\]
(6)

where the last inequality is justified by \( E_\lambda(\alpha) \subseteq \Delta_\eta(\alpha) \). Let \( \alpha \not\in A \), i.e.
\[
|f''(\alpha)|^2 \geq \frac{\varepsilon}{A(\Delta_\eta(\alpha))} \int_{\Delta_\eta(\alpha)} |f'(z)|^2 \, dA(z).
\]
(7)

Set \( r = \eta(1 - |\alpha|) \) and suppose \( \lambda < \frac{1}{2} \) and \( |z - \alpha| < \frac{r}{4} \). We have that
\[
|f'(z)^2 - f'(\alpha)^2| = \frac{1}{2\pi} \left| \int_{|w - \alpha| = \frac{r}{4}} f'(w)^2 \left( \frac{1}{w - z} - \frac{1}{w - \alpha} \right) \, dw \right|
\]
\[ \frac{1}{2\pi} \left| \int_{|w-\alpha| = \frac{r}{2}} f'(w)^2 \frac{z-\alpha}{(w-z)(w-\alpha)} \, dw \right|. \] \hspace{1cm} (8)

For \(|w-\alpha| = \frac{r}{2}\), by the subharmonicity of \(|f'|^2\) we have

\[ |f'(w)|^2 < \frac{1}{r^2} \int \int_{|u-w| \leq \frac{r}{2}} |f'(u)|^2 \, dA(u) \leq \frac{C}{A(\Delta_\eta(\alpha))} \int \int_{\Delta_\eta(\alpha)} |f'(u)|^2 \, dA(u). \]

Since \(|w-z| > \frac{r}{4}\) when \(|w-\alpha| = \frac{r}{2}\), from (8) we get

\[ |f'(z)^2 - f'(\alpha)^2| \leq \frac{C|z-\alpha|}{r} \int \int_{\Delta_\eta(\alpha)} |f'(u)|^2 \, dA(u). \]

Combining (7) and (9), we get

\[ |f'(z)^2 - f'(\alpha)^2| \leq \frac{2A(\Delta_\eta(\alpha))}{2A(\Delta_\eta(\alpha))} \int \int_{\Delta_\eta(\alpha)} |f'(u)|^2 \, dA(u). \] \hspace{1cm} (9)

This means that if \(\Delta' = \{z \in \mathbb{D} : |z-\alpha| < \frac{\varepsilon r}{2C}\}\) then \(\Delta' \subset E_\lambda(\alpha)\) and

\[ A(E_\lambda(\alpha)) \geq A(\Delta') \geq \frac{\varepsilon^2}{4C^2} r^2 = \frac{\varepsilon^2}{4C^2} A(\Delta_\eta(\alpha)). \]

We finally use this last inequality in (6) and we complete the proof. \(\blacksquare\)

**Proof of Theorem 2.2:** (ii) \(\iff\) (iii) This is easy and it is proved in [8].

(iii) \(\implies\) (i) Let \(\alpha \in \mathbb{D} \setminus B\), where \(B\) is as in Lemma 2.4, where \(0 < \varepsilon < 1, 0 < \lambda < \frac{1}{2}\).

Then \(\frac{B_{\lambda f(\alpha)}}{|f'(\alpha)|^2} \leq \frac{1}{\varepsilon^3}\) and, if we choose \(\lambda < \varepsilon \frac{6}{2C}\), then, from Lemma 2.1, we get that

\[ \frac{A(E_\lambda(\alpha))}{A(\Delta_\eta(\alpha))} > \frac{\frac{2}{\delta} \log \frac{1}{\varepsilon^3}}{\log \frac{1}{\varepsilon^3} + \frac{2}{\delta} \log \frac{1}{\varepsilon^3}} > 1 - \frac{\delta}{2}. \] \hspace{1cm} (10)

Combining (4) and (10), we get

\[ A(G_c \cap E_\lambda(\alpha)) = A(G_c \cap \Delta_\eta(\alpha)) - A(G_c \cap (\Delta_\eta(\alpha) \setminus E_\lambda(\alpha))) \]
\[ \geq \delta A(\Delta_\eta(\alpha)) - A(\Delta_\eta(\alpha) \setminus E_\lambda(\alpha)) \]
\[ = \delta A(\Delta_\eta(\alpha)) - A(\Delta_\eta(\alpha)) + A(E_\lambda(\alpha)) \]
\[ \geq \delta A(\Delta_\eta(\alpha)) - A(\Delta_\eta(\alpha)) + A(\Delta_\eta(\alpha)) - \frac{\delta}{2} A(\Delta_\eta(\alpha)) \]
\[ = \frac{\delta}{2} A(\Delta_\eta(\alpha)). \]
Now let $f \in H^p_0, \zeta \in \mathbb{T}$ and $\alpha \in \Gamma_\beta(\zeta) \setminus B$. Then, using the last relation and $E_\lambda(\alpha) \subset \Delta_\eta(\alpha) \subset \Gamma_\beta'(\zeta)$, we get

$$\frac{1}{A(\Delta_\eta(\alpha))} \int_{G_c \cap \Gamma_\beta'(\zeta)} |f'(z)|^2 \, dA(z)$$

$$\geq \frac{\delta}{2A(G_c \cap E_\lambda(\alpha))} \int_{G_c \cap E_\lambda(\alpha)} |f'(z)|^2 \, dA(z)$$

$$= \frac{\delta}{2A(G_c \cap E_\lambda(\alpha))} \int_{G_c \cap E_\lambda(\alpha)} |f'(z)|^2 \, dA(z) \geq \frac{\delta \lambda}{2} |f'(\alpha)|^2.$$ 

Integrating the last relation over the set $\Gamma_\beta(\zeta) \setminus B$ and using Fubini’s theorem on the left side, we have

$$\int_{G_c \cap \Gamma_\beta'(\zeta)} |f'(z)|^2 \, dA(z) \geq \frac{\delta \lambda}{2} \int_{\Gamma_\beta(\zeta) \setminus B} |f'(\alpha)|^2 \, dA(\alpha).$$

With similar arguments as in relation (5), we can show that the integral in the brackets is bounded above from a constant $C > 0$ depending only on $\eta$. So, we have that

$$\int_{G_c \cap \Gamma_\beta'(\zeta)} |f'(z)|^2 \, dA(z) \geq \frac{C \delta \lambda}{2} \int_{\Gamma_\beta(\zeta) \setminus B} |f'(\alpha)|^2 \, dA(\alpha) - \frac{\varepsilon C \delta \lambda}{2} \int_{\Gamma_\beta'(\zeta) \cap B} |f'(\alpha)|^2 \, dA(\alpha).$$

Because of lemma 2.4 we have that

$$\int_{G_c \cap \Gamma_\beta'(\zeta)} |f'(z)|^2 \, dA(z) \geq \frac{C \delta \lambda}{2} \int_{\Gamma_\beta(\zeta) \setminus B} |f'(\alpha)|^2 \, dA(\alpha) - \frac{\varepsilon C \delta \lambda}{2} \int_{\Gamma_\beta'(\zeta) \cap B} |f'(\alpha)|^2 \, dA(\alpha)$$

and so

$$\int_{G_c \cap \Gamma_\beta'(\zeta)} |f'(z)|^2 \, dA(z) \geq \frac{C \delta \lambda}{2} \int_{\Gamma_\beta(\zeta)} |f'(\alpha)|^2 \, dA(\alpha) + \varepsilon \frac{C \delta \lambda}{2} \int_{\Gamma_\beta'(\zeta)} |f'(\alpha)|^2 \, dA(\alpha)$$

$$\geq \frac{C \delta \lambda}{2} \int_{\Gamma_\beta(\zeta)} |f'(\alpha)|^2 \, dA(\alpha).$$

Hence,

$$\left( \int_{G_c \cap \Gamma_\beta'(\zeta)} |f'(z)|^2 \, dA(z) \right)^{\frac{1}{2}} + \left( \frac{C \varepsilon \delta \lambda}{2} \right)^{\frac{1}{2}} \left( \int_{\Gamma_\beta'(\zeta)} |f'(\alpha)|^2 \, dA(\alpha) \right)^{\frac{1}{2}}$$
\[ \geq \left( \frac{C \delta \lambda}{2} \right)^{\frac{1}{2}} \left( \iint_{\Gamma_{\beta}(\xi)} |f'(\alpha)|^2 \, dA(\alpha) \right)^{\frac{1}{2}}. \]

Applying Minkowski's inequality, we get

\[
\left[ \int_T \left( \iint_{G_c \cap \Gamma_{\beta'}(\xi)} |f'(z)|^2 \, dA(z) \right)^{\frac{p}{2}} \, dm(\xi) \right]^{\frac{1}{p}}
+ \left( \frac{C' \delta \lambda}{2} \right)^{\frac{1}{2}} \left[ \int_T \left( \iint_{\Gamma_{\beta'}(\xi)} |f'(\alpha)|^2 \, dA(\alpha) \right)^{\frac{p}{2}} \, dm(\xi) \right]^{\frac{1}{p}}
\geq \left( \frac{C \delta \lambda}{2} \right)^{\frac{1}{2}} \left[ \int_T \left( \iint_{\Gamma_{\beta}(\xi)} |f'(\alpha)|^2 \, dA(\alpha) \right)^{\frac{p}{2}} \, dm(\xi) \right]^{\frac{1}{p}}
\]

and so

\[
\left[ \int_T \left( \iint_{G_c \cap \Gamma_{\beta'}(\xi)} |f'(z)|^2 \, dA(z) \right)^{\frac{p}{2}} \, dm(\xi) \right]^{\frac{1}{p}}
\geq \left( \frac{C \delta \lambda}{2} \right)^{\frac{1}{2}} \left[ \int_T \left( \iint_{\Gamma_{\beta}(\xi)} |f'(\alpha)|^2 \, dA(\alpha) \right)^{\frac{p}{2}} \, dm(\xi) \right]^{\frac{1}{p}}
\]

According to (1), both integrals at the right side of (11), represent equivalent norms in $H_0^p$. Due to the relation between $\beta$ and $\beta'$ there is $C'' > 0$ which depends only on $\eta$, such that

\[
\left[ \int_T \left( \iint_{\Gamma_{\beta'}(\xi)} |f'(\alpha)|^2 \, dA(\alpha) \right)^{\frac{p}{2}} \, dm(\xi) \right]^{\frac{1}{p}}
\leq C'' \left[ \int_T \left( \iint_{\Gamma_{\beta}(\xi)} |f'(\alpha)|^2 \, dA(\alpha) \right)^{\frac{p}{2}} \, dm(\xi) \right]^{\frac{1}{p}}. \]
If \( \in \zeta \) So the integral operator \( S_g \) and since

\[
\left| \int_{\mathbb{T}} \left( \int_{\Gamma^c_{\beta}(\zeta)} |f'(\alpha)|^2 dA(\alpha) \right)^{\frac{p}{2}} \right| \geq \frac{1}{\beta}
\]

Choosing \( \varepsilon \) small enough so that \( C - \varepsilon \frac{1}{2} C' \frac{1}{2} C'' > 0 \), we have that

\[
\left[ \int_{\mathbb{T}} \left( \int_{G \cap \Gamma^c_{\beta}(\zeta)} |f'(z)|^2 dA(z) \right)^{\frac{p}{2}} \right] \geq C\|f\|_{H^0_p},
\]

and since \( G_c = \{ z \in \mathbb{D} : |g(z)| > c \} \), we have

\[
\|S_g f\|_{H^0_p} \leq \left[ \int_{\mathbb{T}} \left( \int_{\Gamma^c_{\beta}(\zeta)} |(S_g f(z))'|^2 dA(z) \right)^{\frac{p}{2}} \right] \geq C\|f\|_{H^0_p}.
\]

So the integral operator \( S_g \) has closed range.

(i) \( \Rightarrow \) (ii) Let \( \alpha \in \mathbb{D}, \xi \in \mathbb{T} \), \( \eta \in (0, 1) \), \( E(z_0, r) = \{ z \in \mathbb{D} : |z - z_0| < r \} \), \( C(z_0, r) = \{ z \in \mathbb{D} : |z - z_0| = r \} \) and the arc \( I_\alpha = \{ \xi \in \mathbb{T} : \Gamma^c_{1}(\zeta) \cap D_\eta(\alpha) \neq \emptyset \} \). It’s easy to see that \( \zeta \in I_\alpha \) is equivalent to \( \alpha \in \Gamma'_{\eta}(\zeta) \), where \( \eta' \) depends only on \( \eta \). In fact, an elementary geometric argument shows that \( 1 - \eta' \sim 1 - \eta \), where the underlying constants are absolute.

Set \( R_0 = \frac{1 + \eta'}{2} \). We continue with the proof by considering two cases for \( \alpha \): (a) \( R_0 \leq |\alpha| < 1 \) and (b) \( 0 \leq |\alpha| \leq R_0 \).

Case (a) \( R_0 \leq |\alpha| \leq 1 \): At first, we consider the case \( p > 1 \).

Another simple geometric argument gives \( m(I_\alpha) \geq \frac{1 - |\alpha|}{(1 - \eta')^2} \) and hence:

\[
m(I_\alpha) \geq \frac{1 - |\alpha|}{(1 - \eta')^2} \geq (13)
\]

If \( S_g \) has closed range on \( H^0_p \) then there exists \( C > 0 \) such that for every \( f \in H^0_p \) we have

\[
C\|S_g f\|^p_{H^0_p} \geq \|f\|^p_{H^0_p}.
\]

(14)
Let
\[ \psi_\alpha(z) = \frac{\alpha - z}{1 - \bar{\alpha}z}. \]

Then, after some calculations, we get that \( \|\psi_\alpha - \alpha\|_{H^p}^p \sim (1 - |\alpha|). \)

Setting \( f = \psi_\alpha - \alpha \) in (14) and using \( (x + y)^p \leq 2^{p-1}(x^p + y^p) \), we get

\[
1 - |\alpha| \leq C\|S_p(\psi_\alpha - \alpha)\|_{H^p}^p = C \int_{\mathbb{T}} \left( \int_{\Gamma_{\frac{1}{2}}(\zeta)} |\psi_\alpha'(z)|^2 |g(z)|^2 \, dA(z) \right)^{\frac{p}{2}} \, dm(\zeta)
\]

\[
\leq C \int_{I_a} \left( \int_{\Gamma_{\frac{1}{2}}(\zeta) \cap G_c} |\psi_\alpha'(z)|^2 |g(z)|^2 \, dA(z) \right)^{\frac{p}{2}} \, dm(\zeta)
\]

\[
+ C \int_{I_a} \left( \int_{\Gamma_{\frac{1}{2}}(\zeta) \cap (D_\eta(\alpha) \setminus G_c)} |\psi_\alpha'(z)|^2 |g(z)|^2 \, dA(z) \right)^{\frac{p}{2}} \, dm(\zeta)
\]

\[
+ C \int_{I_a} \left( \int_{\Gamma_{\frac{1}{2}}(\zeta) \setminus D_\eta(\alpha)} |\psi_\alpha'(z)|^2 |g(z)|^2 \, dA(z) \right)^{\frac{p}{2}} \, dm(\zeta)
\]

\[
= C(I_1 + I_2 + I_3 + I_4). 
\]

Using \( A(D_\eta(\alpha)) = \frac{(1 - |\alpha|^2)^2}{(1 - \eta^2|\alpha|^2)^2} \leq \frac{(1 - |\alpha|^2)^2}{(1 - \eta^2)^2} \), we get

\[
I_1 \leq \|g\|_{\infty}^p \int_{I_a} \left( \int_{G_c \cap D_\eta(\alpha)} \frac{(1 - |\alpha|^2)^2}{|1 - \bar{\alpha}z|^4} \, dA(z) \right)^{\frac{p}{2}} \, dm(\zeta)
\]

\[
\leq \|g\|_{\infty}^p m(I_a) \left( \frac{A(G_c \cap D_\eta(\alpha))}{(1 - |\alpha|^2)^2} \right)^{\frac{p}{2}}
\]

\[
\leq \|g\|_{\infty}^p m(I_a) \frac{1}{(1 - \eta^2)^p} \left( \frac{A(G_c \cap D_\eta(\alpha))}{A(D_\eta(\alpha))} \right)^{\frac{p}{2}}.
\]

Using \( |g(z)| \leq c \) in \( \mathbb{D} \setminus G_c \) and making the change of variables \( w = \psi_\alpha(z) \), we get

\[
I_2 \leq c^p \int_{I_a} \left( \int_{\mathbb{D}} |\psi_\alpha'(z)|^2 \, dA(z) \right)^{\frac{p}{2}} \, dm(\zeta) = c^p \int_{I_a} \left( \int_{\mathbb{D}} dA(w) \right)^{\frac{p}{2}} \, dm(\zeta) = c^p m(I_a).
\]

We increase \( I_3 \) by extending it over \( \mathbb{D} \setminus D_\eta(\alpha) \) and then we make the change of variables \( w = \psi_\alpha(z) \) to get

\[
I_3 \leq \|g\|_{\infty}^p \int_{I_a} \left( \int_{\mathbb{D} \setminus D_\eta(\alpha)} \, dA(w) \right)^{\frac{p}{2}} \, dm(\zeta) = \|g\|_{\infty}^p m(I_a)(1 - \eta^2)^{\frac{p}{2}}.
\]
In order to estimate $I_4$ we have first to estimate $\iint_{\Gamma_1(\zeta)} |\psi'(z)|^2 \, dA(z)$, when $\zeta \in \mathbb{T} \setminus I_\alpha$.

Without loss of generality we may assume that $\alpha \in [R_0, 1)$. For $j \in \mathbb{N}$, $j \geq 2$, we define $r_j = 1 - \frac{1}{2^j}$ and consider the sets $\Omega_1 = E(0; \frac{1}{2})$ and $\Omega_j = (E(0; r_j) \setminus E(0; r_{j-1})) \cap \Gamma_1(\zeta)$.

Then we have that $\Gamma_1(\zeta) = \bigcup_{j=1}^{+\infty} \Omega_j$ and $A(\Omega_j) \asymp \frac{1}{4^j}$, when $j \geq 1$. We fix $z_j \in \Omega_j$ such that $\arg(z_j) = \arg(\zeta)$. Then, if $z \in \Omega_j$, we have $|1 - \alpha z| \asymp |\frac{1}{\alpha} - z| \asymp |\frac{1}{\alpha} - z_j|$. Also we have that $|\frac{1}{\alpha} - z_j| \asymp |\frac{1}{2^j} + 1 - |\alpha| + |\arg(\zeta)|$. In all these relations, the underlying constants are absolute. If $\zeta \in \mathbb{T} \setminus I_\alpha$, then $a \notin \Gamma_1(\zeta)$ which means that $1 - |\alpha| < |\arg(\zeta)|$, so we have that $|\frac{1}{\alpha} - z_j| \asymp |\frac{1}{2^j} + |\arg(\zeta)||$. There is some $j_0$ so that $\frac{1}{2^{j_0}} \leq |\arg(\zeta)| \leq \frac{1}{2^{j_0} - 1}$.

For $j < j_0$ we have $|\arg(\zeta)| < \frac{1}{2^j}$ which implies that $|\frac{1}{\alpha} - z_j| \asymp \frac{1}{2^j}$ and for $j > j_0$ we have $|\arg(\zeta)| > \frac{1}{2^j}$ which implies that $|\frac{1}{\alpha} - z_j| \asymp |\arg(\zeta)|$. Therefore

$$
\iint_{\Gamma_1(\zeta)} |\psi'(z)|^2 \, dA(z) = \int \int_{\Omega_1} (1 - |\alpha|^2)^2 \, dA(z) + \sum_{j=2}^{+\infty} \int \int_{\Omega_j} (1 - |\alpha|^2)^2 \, dA(z)
$$

$$
\asymp (1 - |\alpha|^2)^2 + \sum_{j=2}^{j_0} \int \int_{\Omega_j} (1 - |\alpha|^2)^2 \, dA(z) + \sum_{j=j_0}^{+\infty} \int \int_{\Omega_j} (1 - |\alpha|^2)^2 \, dA(z)
$$

$$
\asymp (1 - |\alpha|^2)^2 + \sum_{j=2}^{j_0} A(\Omega_j)(1 - |\alpha|^2)^2 (2j)^4 + \sum_{j=j_0}^{+\infty} A(\Omega_j) \frac{(1 - |\alpha|^2)^2}{|\arg(\zeta)|^4}
$$

$$
\asymp (1 - |\alpha|^2)^2 + (1 - |\alpha|^2)^2 \sum_{j=2}^{j_0} \frac{1}{4^j} 16^j + \frac{(1 - |\alpha|^2)^2}{|\arg(\zeta)|^4} \sum_{j=j_0}^{+\infty} \frac{1}{4^j}.
$$

But $\sum_{j=2}^{j_0} 4^j \asymp 4^{j_0} \asymp \frac{1}{|\arg(\zeta)|^2}$ and $\sum_{j=j_0}^{+\infty} \frac{1}{4^j} \asymp \frac{1}{4^{j_0}} \asymp |\arg(\zeta)|^2$. Therefore

$$
\iint_{\Gamma_1(\zeta)} |\psi'(z)|^2 \, dA(z) \asymp (1 - |\alpha|^2)^2 + \frac{(1 - |\alpha|^2)^2}{|\arg(\zeta)|^2}.
$$

(16)

Since $\alpha$ is positive, there is $\phi_0$ such that $\mathbb{T} \setminus I_\alpha = [\phi_0, 2\pi - \phi_0]$ and $\phi_0 \asymp m(I_\alpha)$. Therefore

$$
I_4 \leq C||g||_{\infty}^p \int_{\phi_0}^{\pi} (1 - |\alpha|^2)^p \, d\phi + C||g||_{\infty}^p \int_{\phi_0}^{\pi} \frac{(1 - |\alpha|^2)^p}{\phi^p} \, d\phi
$$

$$
\leq C||g||_{\infty}^p (1 - |\alpha|^2)^p + C||g||_{\infty}^p \frac{(1 - |\alpha|^2)^p}{\phi_0^{p-1}}
$$

$$
\leq C||g||_{\infty}^p (1 - |\alpha|^2)^p + C||g||_{\infty}^p \frac{(1 - |\alpha|^2)^p}{m(I_\alpha)^{p-1}}.
$$
Substituting the estimates for $I_1, I_2, I_3, I_4$ in (15), we get

$$1 - |\alpha| \leq C \left[ \|g\|_\infty^p m(I_\alpha) \frac{1}{(1 - \eta^2)^{\frac{p}{2}}} \left( \frac{A(G_c \cap D_\eta(\alpha))}{A(D_\eta(\alpha))} \right)^{\frac{p}{2}} + c^p m(I_\alpha) \right. $$
$$+ \left. \|g\|_\infty^p m(I_\alpha)(1 - \eta^2)^{\frac{p}{2}} + \|g\|_\infty^p (1 - |\alpha|^2)^p + \|g\|_\infty^p \frac{1 - |\alpha|^2}{m(I_\alpha)^{p-1}} \right].$$

Using (13) we get

$$1 - |\alpha| \leq C \left[ \|g\|_\infty^p \frac{1 - |\alpha|}{(1 - \eta)} \frac{1}{(1 - \eta^2)^{\frac{p}{2}}} \left( \frac{A(G_c \cap D_\eta(\alpha))}{A(D_\eta(\alpha))} \right)^{\frac{p}{2}} $$
$$+ c^p \frac{1 - |\alpha|}{(1 - \eta)^{\frac{p}{2}}} + \|g\|_\infty^p \frac{1 - |\alpha|}{(1 - \eta)^{\frac{p}{2}}} (1 - \eta^2)^{\frac{p}{2}} $$
$$+ \|g\|_\infty^p (1 - |\alpha|^2)(1 - \eta^2)^{\frac{p}{2}} + \|g\|_\infty^p (1 - |\alpha|^2)(1 - \eta)^{\frac{p-1}{2}} \right].$$

Thus

$$C \leq \|g\|_\infty^p \frac{1}{(1 - \eta)^{\frac{2p+1}{2}}} \left( \frac{A(G_c \cap D_\eta(\alpha))}{A(D_\eta(\alpha))} \right)^{\frac{p}{2}} + \frac{c^p}{(1 - \eta)^{\frac{1}{2}}} $$
$$+ \|g\|_\infty^p (1 - \eta)^{\frac{p-1}{2}} + \|g\|_\infty^p (1 - \eta)^{p-1} + \|g\|_\infty^p (1 - \eta)^{\frac{p-1}{2}}.$$

Choose $\eta$ close enough to 1 so that $\|g\|_\infty^p (1 - \eta)^{\frac{p-1}{2}} + \|g\|_\infty^p (1 - \eta)^{p-1} + \|g\|_\infty^p (1 - \eta)^{\frac{p-1}{2}} < \frac{C}{4}$ and then set $C_\eta = (1 - \eta)^{\frac{1}{2}}$. We have that

$$\frac{3C}{4} \leq \frac{\|g\|_\infty^p}{C_\eta^{2p+1}} \left( \frac{A(G_c \cap D_\eta(\alpha))}{A(D_\eta(\alpha))} \right)^{\frac{p}{2}} + \frac{c^p}{C_\eta}.$$

Choose $c$ small enough so that $\frac{c^p}{C_\eta} < \frac{C}{4}$. Then

$$\frac{C}{2} \leq \frac{\|g\|_\infty^p}{C_\eta^{2p+1}} \left( \frac{A(G_c \cap D_\eta(\alpha))}{A(D_\eta(\alpha))} \right)^{\frac{p}{2}}$$

and finally

$$\left( \frac{CC_\eta^{2p+1}}{2\|g\|_\infty^p} \right)^{\frac{2}{p}} \leq \frac{A(G_c \cap D_\eta(\alpha))}{A(D_\eta(\alpha))}$$

or

$$A(G_c \cap D_\eta(\alpha)) \geq \delta A(D_\eta(\alpha)),$$

for every $\alpha$ with $R_0 \leq |\alpha| < 1.$
Now, we consider the case $p = 1$. Let $\alpha \in \mathbb{D}$ and the functions

$$f_\alpha(z) = \frac{(1 - |\alpha|^2)^2}{3\alpha(1 - \overline{\alpha}z)^3} - \frac{(1 - |\alpha|^2)^2}{3\alpha}.$$ 

Obviously $f_\alpha \in H^1_0$. We define the sets $I_\alpha = \{ \zeta \in \mathbb{T} : \alpha \in \Gamma_{\frac{1}{2}}(\zeta) \}$ and it’s clear that $m(I_\alpha) \asymp 1 - |\alpha|$. Then we consider the integral

$$I = \int_{\mathbb{T} \setminus I_\alpha} \left( \iint_{\Gamma_{\frac{1}{2}}(\zeta)} |f'_\alpha(z)|^2 \, dA(z) \right)^{\frac{1}{2}} \, dm(\zeta).$$

If $\zeta \in \mathbb{T} \setminus I_\alpha$ then $\alpha \not\in \Gamma_{\frac{1}{2}}(\zeta)$. Using similar arguments as in the proof of (16), we get

$$\iint_{\Gamma_{\frac{1}{2}}(\zeta)} |f'_\alpha(z)|^2 \, dA(z) = \iint_{\Gamma_{\frac{1}{2}}(\zeta)} \frac{(1 - |\alpha|^2)^4}{|1 - \alpha z|^8} \, dA(z) \asymp (1 - |\alpha|^2)^4 + \frac{(1 - |\alpha|^2)^4}{|\text{Arg}(\zeta)|^6},$$

hence

$$I \asymp \int_{1 - |\alpha|}^{\pi} \frac{(1 - |\alpha|^2)^2}{\phi^3} \, d\phi \asymp 1. \quad (17)$$

If $F \in H^1_0$, it’s easy to show (integrating $F'$ over radii) that

$$\|F\|_{H^1_0} \leq C' \iint_{\mathbb{D}} |F'(z)| \, dA(z). \quad (18)$$

$S_g$ has been supposed to have closed range, so there exists $C' > 0$ such that $\|f_\alpha\|_{H^1_0} \leq C' \|S_gf_\alpha\|_{H^1_0}$ and using (17) and (18) (with $F = S_gf_\alpha$), we get

$$0 < C_0 \leq I \leq \|f_\alpha\|_{H^1_0} \leq C' \|S_gf_\alpha\|_{H^1_0} \leq C' C'' \iint_{\mathbb{D}} |(S_gf_\alpha)'(z)| \, dA(z).$$

Hence, observing that $|f'_\alpha(z)| = |\psi'_\alpha(z)|^2$, we have

$$C_1 \leq \iint_{\mathbb{D}} |(S_gf_\alpha)'(z)| \, dA(z) = \iint_{\mathbb{D}} |\psi'_\alpha(z)|^2 |g(z)| \, dA(z)$$

$$\leq \|g\|_{\infty} \iint_{G_c \cap D_\eta(\alpha)} \frac{(1 - |\alpha|^2)^2}{|1 - \alpha z|^4} \, dA(z) + c \iint_{D_\eta(\alpha) \setminus G_c} |\psi'_\alpha(z)|^2 \, dA(z)$$

$$+ \|g\|_{\infty} \iint_{D_\eta(\alpha) \setminus D_\eta(0)} |\psi'_\alpha(z)|^2 \, dA(z)$$

$$\leq \|g\|_{\infty} \iint_{G_c \cap D_\eta(\alpha)} \frac{1}{(1 - |\alpha|^2)^2} \, dA(z)$$

$$+ c \iint_{D_\eta(0)} |\psi'_\alpha(z)|^2 \, dA(z) + \|g\|_{\infty} \iint_{D_\eta(0) \setminus D_\eta(0)} \, dA(w)$$

$$\leq \frac{\|g\|_{\infty} A(G_c \cap D_\eta(\alpha))}{(1 - \eta^2)^2 \frac{A(D_\eta(\alpha))}{A(D_\eta(\alpha))}} + c \iint_{D_\eta(0)} \, dA(w) + \|g\|_{\infty} A(\mathbb{D} \setminus D_\eta(0)).$$
\[
\leq C_2 \left[ \frac{\|g\|_{\infty}}{(1 - \eta^2)^2} \frac{A(G_c \cap D_\eta(\alpha))}{A(D_\eta(\alpha))} + c + \|g\|_{\infty}(1 - \eta) \right].
\] (19)

Choosing \( c \) close enough to 0 and \( \eta \) close enough to 1 we get

\[
A(G_c \cap D_\eta(\alpha)) \leq C \|g\|_{\infty} A(D_\eta(\alpha)) = \delta A(D_\eta(\alpha)).
\]

**Case (b) \( 0 \leq |\alpha| \leq R_0 \):** There exists \( \eta_1 \), depending only on \( \eta \), such that \( D_\eta(R_0) \subseteq D_{\eta_1}(0) \). Take \( \alpha' \) so that \( |\alpha'| = R_0 \) and \( \text{Arg}(\alpha') = \text{Arg}(\alpha) \). Then \( D_\eta(\alpha') \subseteq D_{\eta_1}(\alpha) \). Set \( \eta_2 = \max\{\eta, \eta_1\} \). Then from case (a) for \( \alpha' \) we have

\[
A(G_c \cap D_{\eta_2}(\alpha)) \geq A(G_c \cap D_{\eta_1}(\alpha)) \geq A(G_c \cap D_{\eta}(\alpha'))
\]

\[
\geq \delta A(D_{\eta_1}(\alpha)) \geq C\delta A(D_{\eta_2}(\alpha)),
\]

where the constants \( C > 0 \) depend only on \( \eta \).

Moreover, when \( R_0 \leq |\alpha| < 1 \), we have

\[
A(G_c \cap D_{\eta_2}(\alpha)) \geq A(G_c \cap D_{\eta}(\alpha)) \geq \delta A(D_{\eta}(\alpha)) \geq C\delta A(D_{\eta_2}(\alpha)),
\]

where the constant \( C > 0 \) depends only on \( \eta \). So, we have proved that there are \( \eta_2 \in (0, 1) \), \( c > 0 \) and \( C > 0 \) such that

\[
A(G_c \cap D_{\eta_2}(\alpha)) \geq CA(D_{\eta_2}(\alpha)),
\]

for every \( \alpha \in \mathbb{D} \), which is what we had to prove. \( \square \)

### 3. Closed range integral operators on BMOA space

Let denote as \( BMOA_0 \) the space \( BMOA/\mathbb{C} \). In [7], A. Anderson posed the question of finding a necessary and sufficient condition for the operator \( S_g \) to have closed range on \( BMOA_0 \).

Next, we answer this question, proving that conditions (ii) and (iii) of Theorem 2.2, for \( H_0^p \), are also necessary and sufficient for the integral operator \( S_g \) to have closed range on \( BMOA_0 \).

Let \( z_0 \in \mathbb{D} \). The point evaluation functional of the derivative, on \( BMOA \), induced by \( z_0 \), is defined as \( \Lambda_{z_0}f = f'(z_0), f \in BMOA \). It is easy to check that \( \Lambda_{z_0} \) is bounded on \( BMOA \). Therefore, using Theorem 2.2 and Corollary 2.3 in [7], we conclude that the operator \( S_g : BMOA_0 \to BMOA_0 \) is bounded if and only if \( g \in H^\infty \). So, we consider \( g \in H^\infty \) and set again \( G_c = \{z \in \mathbb{D} : |g(z)| > c\} \).

The following theorem is the main result of this section.

**Theorem 3.1:** Let \( g \in H^\infty \). Then the following are equivalent:

(i) The operator \( S_g : BMOA_0 \to BMOA_0 \) has closed range

(ii) There exist \( c > 0, \delta > 0 \) and \( \eta \in (0, 1) \) such that

\[
A(G_c \cap D_\eta(a)) \geq \delta A(D_\eta(a))
\]

(20)

for all \( a \in \mathbb{D} \).
Recall that the weighted Bergman space $A_{p, \gamma}$, $\gamma > -1$, is defined as the set of all analytic functions $f$ in $\mathbb{D}$ such that

$$\iint_{\mathbb{D}} |f(z)|^p (1 - |z|^2)\gamma \ dA(z) < \infty.$$  

We will make use of the following theorem of D. Luecking (see [8]).

**Theorem 3.2:** Let $p \geq 1$, $\gamma > -1$ and measurable $G \subseteq \mathbb{D}$. The following assertions are equivalent.

(i) There exists $C > 0$ such that

$$\iint_{G} |f(z)|^p (1 - |z|^2)\gamma \ dA(z) \geq C \iint_{\mathbb{D}} |f(z)|^p (1 - |z|^2)\gamma \ dA(z)$$  

for every $f \in A_{p, \gamma}$.

(ii) There exist $c > 0$, $\delta > 0$ and $\eta \in (0, 1)$ such that

$$A(G \cap D_\eta(a)) \geq \delta A(D_\eta(a))$$  

for all $a \in \mathbb{D}$.

In the proof of Theorem 3.1, we will use the fact that $\log \frac{1}{|z|} \asymp 1 - |z|^2$, when $0 < \delta \leq |z| < 1$, where $\delta$ is fixed but arbitrary.

**Proof of Theorem 3.1:** (ii) $\Rightarrow$ (i) If (20) holds then, because of theorem 3.2, (21) also holds for $G = G_c$. For $\beta \in \mathbb{D}$, $z \in \mathbb{D}$ and $f \in BMOA_0$, we consider the function $h_\beta(z) = \frac{(1 - |\beta|^2)^{1/2}}{1 - \beta z} f'(z)$. It’s easy to see that if $f \in BMOA_0$ then $h_\beta \in A_{1, 1}$. Indeed

$$\|h_\beta\|^2_{A_{1, 1}} = \iint_{\mathbb{D}} \frac{1 - |\beta|^2}{|1 - \beta z|^2} |f'(z)|^2 (1 - |z|^2) \ dA(z)$$

$$\leq C \iint_{\mathbb{D}} \frac{1 - |\beta|^2}{|1 - \beta z|^2} |f'(z)|^2 \log \frac{1}{|z|} \ dA(z) \leq \|f\|^2_{BMOA_0} < \infty.$$  

Let $\beta \in \mathbb{D}$. We have that

$$\|S_g f\|^2_{BMOA_0} = \sup_{z_0 \in \mathbb{D}} \iint_{\mathbb{D}} \frac{1 - |z_0|^2}{|1 - \overline{z_0} z|^2} |(S_g f(z))'|^2 \log \frac{1}{|z|} \ dA(z)$$

$$= \sup_{z_0 \in \mathbb{D}} \iint_{\mathbb{D}} \frac{1 - |z_0|^2}{|1 - \overline{z_0} z|^2} |f'(z)|^2 |g(z)|^2 \log \frac{1}{|z|} \ dA(z)$$

$$\geq \iint_{\mathbb{D}} \frac{1 - |\beta|^2}{|1 - \beta z|^2} |f'(z)|^2 |g(z)|^2 \log \frac{1}{|z|} \ dA(z)$$

$$\geq c^2 \iint_{G_c} \frac{1 - |\beta|^2}{|1 - \beta z|^2} |f'(z)|^2 \log \frac{1}{|z|} \ dA(z)$$
\[ \|Sgf\|_{BMOA_0}^2 \geq C \int_\mathbb{D} \frac{1 - |\beta|^2}{|1 - \beta z|^2} |f'(z)|^2 \log \frac{1}{|z|} \, dA(z). \]

Taking the supremum over \( \beta \in \mathbb{D} \) in the last relation we get

\[ \|Sgf\|_{BMOA_0}^2 \geq C \|f\|_{BMOA_0}^2. \]

(i) \( \Rightarrow \) (ii) If \( S_\gamma \) has closed range then there exist \( C_1 > 0 \) such that for every \( f \in BMOA_0 \) we have

\[ \|Sgf\|_{BMOA_0}^2 \geq C_1 \|f\|_{BMOA_0}^2. \]

For \( \alpha \in \mathbb{D} \), if we set \( f = \psi_\alpha - \alpha \) in the last inequality, just as in the case of Hardy spaces and observe that \( \|\psi_\alpha - \alpha\|_{BMOA} \asymp 1 \) and \( \frac{(1-|\beta|^2)(1-|z|^2)}{|1-\beta z|^2} < 1 \), for every \( z, \beta \in \mathbb{D} \), then we have

\[ C_1 \leq \|Sg(\psi_\alpha - \alpha)\|_{BMOA_0}^2 \]

\[ = \sup_{\beta \in \mathbb{D}} \int_\mathbb{D} \frac{1 - |\beta|^2}{|1 - \beta z|^2} |(Sg(\psi_\alpha - \alpha)(z))'|^2 \log \frac{1}{|z|} \, dA(z) \]

\[ \leq C \sup_{\beta \in \mathbb{D}} \int_\mathbb{D} \frac{1 - |\beta|^2}{|1 - \beta z|^2} |\psi_\alpha'(z)|^2 |g(z)|^2 (1 - |z|^2) \, dA(z) \]

\[ \leq C \int_\mathbb{D} |\psi_\alpha'(z)|^2 |g(z)|^2 \, dA(z) \]

At this point we continue in the same manner as with (19), replacing \( |g(z)| \) with \( |g(z)|^2 \), and we get

\[ C_1 \leq C_2 \left[ \frac{\|g\|_\infty^2}{(1 - \eta^2)^2} \frac{A(G_c \cap D_\eta(\alpha))}{A(D_\eta(\alpha))} + c^2 + \|g\|_\infty^2 (1 - \eta) \right]. \]

Choosing \( c \) close enough to 0 and \( \eta \) close enough to 1 we get

\[ A(G_c \cap D_\eta(\alpha)) \geq \frac{C}{\|g\|_\infty^2} A(D_\eta(\alpha)) = \delta A(D_\eta(\alpha)), \]

where \( C \) depends only on \( \eta \).
Remark 3.1: The $Q_p$ space, $0 < p < \infty$, is defined as the set of all analytic functions $f$ in $\mathbb{D}$ for which
\[
\sup_{\beta \in \mathbb{D}} \iint_{\mathbb{D}} \frac{(1-|\beta|^2)^p}{|1-\beta z|^2} |f'(z)|^2 (1-|z|^2)^p \, dA(z) < \infty.
\]
Let denote as $Q_{p,0}$ the space $Q_p/\mathbb{C}$. For $\beta, z \in \mathbb{D}$ and $f \in Q_{p,0}$, we consider the functions
\[
h_\beta(z) = \frac{(1-|\beta|^2)^p}{(1-\beta z)^2} f'(z).
\]
It’s easy to see that if $f \in Q_{p,0}$, then $h_\beta \in A^2_p$ and using similar arguments as in the proof of Theorem 3.1, we can prove that (20) is also necessary and sufficient for $S_g$ to have closed range on $Q_{p,0}$ ($0 < p < \infty$).

4. Closed range integral operators on Besov spaces

Let denote as $B^p_0$ the space $B^p/\mathbb{C}$. With similar arguments as in the case of $BMOA$ space we can see that the operator $S_g : B^p_0 \to B^p_0$ ($1 < p < \infty$) is bounded if and only if $g \in H^\infty$. So, we consider $g \in H^\infty$ and $G_c = \{ z \in \mathbb{D} : |g(z)| > c \}$. We will prove that condition (20) is also necessary and sufficient for the operator $S_g$ to have closed range on $B^p_0$. For the sufficiency, we observe that, if $f \in B^p$ then $f' \in \mathcal{A}_p^{-1}$, the weighted Bergman space defined in the previous section, so we can use theorem 3.2. We have
\[
\|S_g f\|_{B^p_0}^p = \iint_{\mathbb{D}} |(S_g f(z))'|^p (1-|z|^2)^{p-2} \, dA(z)
\geq \iint_{G_c} |f'(z)|^p |g(z)|^p (1-|z|^2)^{p-2} \, dA(z)
\geq c^p \iint_{G_c} |f'(z)|^p (1-|z|^2)^{p-2} \, dA(z)
\geq C \iint_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{p-2} \, dA(z)
= C \|f\|_{B^p_0}^p,
\]
where the last inequality is justified by Theorem 3.2. So $S_g$ has closed range on $B^p_0$.

If $S_g$ has closed range on $B^p_0$ then there exist $C_1 > 0$ such that for every $f \in B^p_0$ we have
\[
\|S_g f\|_{B^p_0}^p \geq C_1 \|f\|_{B^p_0}^p.
\]
For $\alpha \in \mathbb{D}$, if we set $f = f_\alpha = \frac{(1-|\alpha|^2)^p}{|1-\alpha z|^p} - \frac{(1-|\alpha|^2)^p}{(1-\alpha z)^{2+p}}$ in the last inequality, just as in the case of $BMOA$, and observe that $\|f_\alpha\|_{B^p_0} \approx 1$ and $|f_\alpha'(z)| = \frac{(1-|\alpha|^2)^p}{|1-\alpha z|^{2+p}}$, then we have
\[
C_1 \leq \|S_g f_\alpha\|_{B^p_0}^p = \iint_{\mathbb{D}} |f_\alpha'(z)|^p |g(z)|^p (1-|z|)^{p-2} \, dA(z)
\leq \|g\|_\infty \iint_{G_c \cap D_\alpha(\alpha)} \frac{(1-|\alpha|^2)^p}{|1-\alpha z|^{2+p}} (1-|z|)^{p-2} \, dA(z)
\]
\[ + \int_{D_\eta(\alpha) \setminus G_c} |f'_\alpha(z)|^p (1 - |z|)^{p-2} dA(z) \]
\[ + \|g\|_\infty^p \int_{\mathbb{D} \setminus D_\eta(\alpha)} \frac{(1 - |\alpha|^2)^2}{|1 - \overline{\alpha} z|^2 + p} (1 - |z|)^{p-2} dA(z) \]
\[ \leq \|g\|_\infty^p \int_{G_c \cap D_\eta(\alpha)} \frac{1}{(1 - |\alpha|^2)^2} dA(z) + c^p \int_{\mathbb{D}} |f'_\alpha(z)|^p (1 - |z|)^{p-2} dA(z) \]
\[ + \|g\|_\infty^p \int_{\mathbb{D} \setminus D_\eta(\alpha)} \frac{(1 - |\alpha|^2)^2}{|1 - \overline{\alpha} \psi_\alpha(w)|^2 + p} (1 - |\psi_\alpha(w)|)^{p-2} |\psi'_\alpha(w)|^2 dA(w) \]
\[ = \|g\|_\infty^p \int_{G_c \cap D_\eta(\alpha)} \frac{1}{(1 - |\alpha|^2)^2} dA(z) + c^p \|f_\alpha\|_{B^p_0}^p \]
\[ + \|g\|_\infty^p \int_{\mathbb{D} \setminus D_\eta(\alpha)} \frac{(1 - |w|^2)^{p-2}}{|1 - \overline{\alpha} w|^{p-2}} dA(w) \]
\[ \leq \|g\|_\infty^p \frac{A(G_c \cap D_\eta(\alpha))}{(1 - |\alpha|^2)^2} + c^p \|f_\alpha\|_{B^p_0}^p + \|g\|_\infty^p \int_{\mathbb{D} \setminus D_\eta(\alpha)} dA(w) \]
\[ \leq C' \|g\|_\infty^p \frac{A(G_c \cap D_\eta(\alpha))}{A(D_\eta(\alpha))} + Cc^p + \|g\|_\infty^p (1 - \eta^2), \]

where \(C'\) depends only on \(\eta\) and \(C\) is absolute. So we have

\[ C_1 \leq C' \|g\|_\infty^p \frac{A(G_c \cap D_\eta(\alpha))}{A(D_\eta(\alpha))} + Cc^p + \|g\|_\infty^p (1 - \eta^2). \]

Choosing \(\eta\) close enough to 1 so that \(\|g\|_\infty^p (1 - \eta^2) < \frac{C_1}{4}\), and \(c\) small enough so that \(C c^p < \frac{C_1}{4}\), we get

\[ A(G_c \cap D_\eta(\alpha)) \geq \frac{C_1}{2C' \|g\|_\infty^p} A(D_\eta(\alpha)) = \delta A(D_\eta(\alpha)). \]

**Acknowledgements**

Many thanks to Prof. Michael Papadimitrakis for discussions about the mathematical content of this paper. His contribution was essential in order for it to take its final form. I also thank Prof. Petros Galanopoulos for helpful discussions.

**Disclosure statement**

No potential conflict of interest was reported by the author(s).

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