INTEGRABILITY IN DIFFERENTIAL COVERINGS

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Abstract. Let $\tau: \tilde{E} \to E$ be a differential covering of a PDE $\tilde{E}$ over $E$. We prove that if $E$ possesses infinite number of symmetries and/or conservation laws then $\tilde{E}$ has similar properties.

Introduction

The notion of a covering (or, better, differential covering) was introduced by A. Vinogradov in [10] and elaborated in detail later in [7] and [8]. Coverings, explicitly or implicitly, provide an adequate background to deal with nonlocal aspects in the geometry of PDEs (nonlocal symmetries and conservation laws, Wahlquist-Estabrook prolongation structures, Lax pairs, zero-curvature representations, etc.). Coverings of a special type (the so-called tangent and cotangent one) are efficient in analysis and construction of Hamiltonian structures and recursion operators, see [6]. A very interesting development in the theory of coverings can also be found in [2].

In this paper, we solve the following naturally arising problem: let a covering $\tau: \tilde{E} \to E$ be given and the equation $E$ is known to possess infinite number of symmetries and/or conservation laws. Is $\tilde{E}$ endowed with similar properties? The answer, under reasonable assumptions, is positive.

In Section 1, we present a short introduction to the theory of coverings based mainly on [8] and formulate and prove necessary auxiliary facts. Section 2 contains the proof of the main result for the case of Abelian coverings. Finally, the non-Abelian case is discussed in Section 3.

1. Basic notions

For a detailed exposition of the geometrical approach to PDEs we refer the reader to the books [5] and [1]. Coverings are discussed in [8].

Equations. Let $\pi: E \to M$, $\dim M = n$, $\dim E = m+n$, be a locally trivial vector bundle and $E \subset J^\infty(\pi)$ be an infinitely prolonged differential equation embedded to the space of infinite jets. One has the surjection $\pi_\infty: E \to M$. The main geometric structure on $E$ is the Cartan connection $C: Z \mapsto \mathcal{C}_Z$ that takes vector fields on $M$ to those on $E$. Vector fields of the form $\mathcal{C}_Z$ are called Cartan fields. The connection is flat, i.e., $C[Z,Y] = [C_Z,C_Y]$ for any vector fields on $M$. The corresponding horizontal distribution (the Cartan distribution) on $E$ is integrable and its maximal integral manifolds

2010 Mathematics Subject Classification. 37K05, 37K10, 37K35.

Key words and phrases. Geometry of differential equations, integrability, symmetries, conservation laws, differential coverings.

I am grateful to the Mathematical Institute of the Silesian University in Opava for support and comfortable working condition.
are solutions of $\mathcal{E}$. We always assume $\mathcal{E}$ to be differentially connected which means that for any set of linearly independent vector fields $Z_1, \ldots, Z_n$ on $M$ the system

$$\mathcal{C}_{Z_i}(h) = 0, \quad i = 1, \ldots, n,$$

has constant solutions only.

If $x^1, \ldots, x^n$ are local coordinates on $M$ then the Cartan connection takes the partial derivatives $\partial/\partial x^i$ to the total derivatives $D_{x^i}$ on $\mathcal{E}$. Flatness of $\mathcal{C}$ amounts to the fact that the total derivatives pair-wise commute,

$$[D_{x^i}, D_{x^j}] = 0.$$

A $\pi_\infty$-vertical vector field $S$ is a symmetry of $\mathcal{E}$ if it commutes with all Cartan fields, i.e., $[S, \mathcal{C}_Z] = 0$ for all $X$. The set of symmetries is a Lie algebra over $\mathbb{R}$ denoted by $\text{sym}\, \mathcal{E}$.

A differential $q$-form $\omega$ on $\mathcal{E}$, $q = 0, 1, \ldots, n$, is horizontal if $i_V \omega = 0$ for any $\pi_\infty$-vertical field $V$. The space of these forms is denoted by $\Lambda^q_h(\mathcal{E})$. Locally, horizontal forms are

$$\omega = \sum a_{i_1, \ldots, i_q} \, dx^{i_1} \wedge \cdots \wedge dx^{i_q}, \quad a_{i_1, \ldots, i_q} \in \mathcal{F}(\mathcal{E}).$$

The horizontal de Rham differential $d_h: \Lambda^q_h(\mathcal{E}) \to \Lambda^{q+1}_h(\mathcal{E})$ is defined, whose action locally is presented by

$$d_h(a_{i_1, \ldots, i_q} \, dx^{i_1} \wedge \cdots \wedge dx^{i_q}) = \sum_{i=1}^n D_{x^i}(a_{i_1, \ldots, i_q}) \, dx^i \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_q}.$$

A closed horizontal $(n - 1)$-form is called a conservation law of $\mathcal{E}$. Thus, conservation laws are defined by $d_h\omega = 0$, $\omega \in \Lambda^{n-1}_h(\mathcal{E})$. A conservation law is trivial if $\omega = d_h\rho$ for some $\rho \in \Lambda^{n-2}_h(\mathcal{E})$. The quotient space of all conservation laws modulo trivial ones is denoted by $\text{cl}\, \mathcal{E}$.

If $S \in \text{sym}\, \mathcal{E}$ and $\omega$ is a conservation law then the Lie derivative $L_S \omega$ is a conservation law as well and trivial conservation laws are taken to trivial ones. Thus we have a well-defined action $L_S: \text{cl}\, \mathcal{E} \to \text{cl}\, \mathcal{E}$.

**Coverings.** Let us now give the main definition. Consider a locally trivial vector bundle $\tau: \tilde{\mathcal{E}} \to \mathcal{E}$ of rank $r$ and denote by $\mathcal{F}(\mathcal{E})$ and $\mathcal{F}(\tilde{\mathcal{E}})$ the algebras of smooth functions on $\mathcal{E}$ and $\tilde{\mathcal{E}}$, respectively. We have the embedding $\tau^*: \mathcal{F}(\mathcal{E}) \hookrightarrow \mathcal{F}(\tilde{\mathcal{E}})$.

**Definition 1.** We say that $\tau$ carries a covering structure (or is a differential covering) over $\mathcal{E}$ if: (a) there exists a flat connection $\tilde{\mathcal{C}}$ in the bundle $\pi_\infty \circ \tau: \tilde{\mathcal{E}} \to M$ and (b) this connection enjoys the equation

$$\tilde{\mathcal{C}}_Z|_{\mathcal{F}(\mathcal{E})} = \mathcal{C}_Z$$

for all vector fields $Z$ on $M$.

In local coordinates, any covering is determined by a system of vector fields

$$\tilde{D}_{x^i} = D_{x^i} + X_i, \quad i = 1, \ldots, n,$$

on $\tilde{\mathcal{E}}$, where $X_i$ are $\tau$-vertical fields that satisfy the relations

$$D_{x^i}(X_j) - D_{x^j}(X_i) + [X_i, X_j] = 0, \quad 1 \leq i < j \leq n.$$
Let $w^1, \ldots, w^r$ be local coordinates in the fiber of $\tau$ (the nonlocal variables in $\tau$) and $X_i = X^1_i \partial/\partial w^1 + \cdots + X^r_i \partial/\partial w^r$. Then $\tilde{E}$, endowed with $\tilde{C}$, is equivalent to the overdetermined system of PDEs

$$\frac{\partial w^\alpha}{\partial x^i} = X^\alpha_i, \quad i = 1, \ldots, n, \quad \alpha = 1, \ldots, r,$$

compatible by virtue of $E$.

Two coverings $\tau_i: \tilde{E}_i \to E$, $i = 1, 2$, are equivalent if there exists a diffeomorphism $f: \tilde{E}_1 \to \tilde{E}_2$ such that the diagram

$$\begin{array}{ccc}
\tilde{E}_1 & \xrightarrow{f} & \tilde{E}_2 \\
\downarrow{\tau_1} & & \downarrow{\tau_2} \\
\tau & & \\
\end{array}$$

is commutative and $f_* \circ \tilde{C}_2 = \tilde{C}_2$ for all fields $Z$ on $M$, where $\tilde{C}_i$ is the Cartan connection on $\tilde{E}_i$ and $f_*$ is the differential of $f$.

Again, having two coverings $\tau_1$ and $\tau_2$, consider the Whitney product of fiber bundles

$$\begin{array}{ccc}
\tilde{E}_1 \times_F \tilde{E}_2 & \xrightarrow{\tau_1 \times_F \tau_2} & \tilde{E}_1 \times E \tilde{E}_2 \\
\downarrow{\tau_1} & & \downarrow{\tau_2} \\
\tau & & \\
\end{array}$$

Since the tangent plane to $\tilde{E}_1 \times_F \tilde{E}_2$ at any point splits naturally into direct sum of tangent planes to $\tilde{E}_1$ and $\tilde{E}_2$, we can define a connection in the bundle $\tau_1 \times_F \tau_2$ by setting

$$\tilde{C}_{12}^\phi(\varphi_1, \varphi_2) = \tilde{C}_2^\phi(\varphi_1) \cdot \varphi_2 + \varphi_1 \cdot \tilde{C}_2^\phi(\varphi_2), \quad \varphi_1 \in F(\tilde{E}_1), \quad \varphi_2 \in F(\tilde{E}_2).$$

This is a covering structure in $\tau_1 \times_F \tau_2$ which is called the Whitney product of $\tau_1$ and $\tau_2$. Note that the maps $\tau_1^\phi(\tau_2)$ and $\tau_2^\phi(\tau_1)$ are coverings as well (they are called pull-backs).

Assume that in local coordinates the coverings $\tau_1$ and $\tau_2$ are given by the vector fields

$$\tilde{D}_x^k = D_x^1 + \sum_{a=1}^{\dim \tau_k} X^{k,a}_{x} \frac{\partial}{\partial w^a_k}, \quad k = 1, 2.$$

Then the vector fields defining the Whitney product are of the form

$$\tilde{D}_x^{1,2} = D_x^1 + \sum_{a=1}^{\dim \tau_1} X^{1,a}_x \frac{\partial}{\partial w^a_1} + \sum_{a=1}^{\dim \tau_2} X^{2,a}_x \frac{\partial}{\partial w^a_2}.$$

**Remark 1.** From now on we shall assume all the coverings under consideration to be finite-dimensional. It is known that, see [3, 9], in the multidimensional case (i.e., $n > 2$) non-overdetermined equations do not possess finite-dimensional coverings. So, we restrict ourselves to the case $n = 2$. 
Definition 2. A covering $\tau: \tilde{E} \to E$ is called irreducible if the covering equation $\tilde{E}$ is differentially connected, i.e., if for any set $Z_1, \ldots, Z_n$ of independent vector fields on $M$ the system
\[
\tilde{C}_{Z_i}(h) = 0, \quad i = 1, \ldots, n,
\] (4)
possesses constant solutions only. Otherwise we say that $\tau$ is reducible.

Equivalently, one can study the equations
\[
\tilde{D}_{x_i}(h) = 0, \quad i = 1, \ldots, n,
\]
instead of System (4).

Reducibility can be ‘measured’ by the maximal number of functionally independent integrals of Equation (4), which cannot exceed $r = \dim \tau$. Maximally reducible coverings are called trivial. Triviality of a covering means that it is locally equivalent to the one with $\tilde{D}_{x_i} = D_{x_i}$ for all $i = 1, \ldots, n$.

Also, directly from the definition one has

Proposition 1. Any finite-dimensional covering $\tau$, in a neighborhood of a generic point, splits into the Whitney product $\tau = \tau_{\text{triv}} \times \tau_{\text{irr}}$, where $\tau_{\text{triv}}$ is trivial and $\tau_{\text{irr}}$ is irreducible.

Abelian coverings. Let us introduce an important class of coverings.

Definition 3. A covering $\tau$ is called Abelian if for any vector field $X$ on $M$ one has $\tilde{C}_X(f) \in \mathcal{F}(E)$ for any fiber-wise linear function $f \in \mathcal{F}(\tilde{E})$. Coverings locally equivalent to such ones are also called Abelian.

Locally this means that coordinates in the fibers of $\tau$ may be chosen in such a way that coefficient of the vertical fields $X_i$ in Equations (1) are independent of the nonlocal variables $w^\alpha$. Consider such a choice and recall that we are in the two-dimensional situation (see Remark 1). Set $x^1 = x$, $x^2 = y$ and $X_1 = X$, $X_2 = Y$. Then, by Equations (2), one has
\[
D_x(Y) - D_y(X) + [X, Y] = \sum_{\alpha=1}^r \left( D_x(Y^\alpha) - D_y(X^\alpha) \right) \frac{\partial}{\partial w^\alpha},
\]
since $[X, Y] = 0$. Thus, $D_x(Y^\alpha) - D_y(X^\alpha) = 0$ for all $\alpha$ and all the forms
\[
\omega^\alpha = X^\alpha \, dx + Y^\alpha \, dy, \quad \alpha = 1, \ldots, r,
\] (5)
are conservation laws of $E$.

Using the result of this simple computation, let us give a complete description of finite-dimensional Abelian coverings:

Theorem 1. There locally exists a one-to-one correspondence between equivalence classes of $r$-dimensional irreducible Abelian coverings over $E$ and $r$-dimensional vector $\mathbb{R}$-subspaces in $\text{cl} E$.

Let $\tau$ be an Abelian covering of finite dimension $r$. Then we can construct conservation laws $\{\omega^\alpha\}$ like in Equation (5) and consider the space $\mathcal{L}_\tau \subset \text{cl} E$ that spans the set $\{[\omega^\alpha]\}$ of their equivalence classes. Vice versa, let $\mathcal{L} \subset \text{cl} E$ be an $r$-dimensional subspace. Take its basis $e^1, \ldots, e^r$ and choose a
representative $\omega^\alpha = X^\alpha dx + Y^\alpha dy$ in each class $e^\alpha$. Consider $E \times \mathbb{R}^r$ and set

$$\tilde{D}_x = D_x + \sum_{\alpha} X^\alpha \frac{\partial}{\partial \bar{w}^\alpha}, \quad \tilde{D}_y = D_y + \sum_{\alpha} Y^\alpha \frac{\partial}{\partial \bar{w}^\alpha}.$$ 

This obviously defines a covering structure and we denote it by $\tau_L$. We are to prove that

1. if $\tau$ is irreducible of rank $r$ then $\dim L_\tau = r$;
2. $\tau_L$ is irreducible of rank $r$;
3. if $\tau$ and $\bar{\tau}$ are equivalent then $L_\tau = L_{\bar{\tau}}$;
4. equivalence class of $\tau_L$ is independent of a basis choice in $L$.

**Proof of Theorem**

Let us do it.

1. Take the forms (5) and assume that $\lambda_1[\omega]_1 + \cdots + \lambda_r[\omega]_r = 0$ for some nontrivial set of $\lambda_\alpha \in \mathbb{R}$. This means that

$$\lambda_1 \omega^1 + \cdots + \lambda_r \omega^r = dh, \quad P \in \mathcal{F}(E),$$ 

or

$$\sum_{\alpha} \lambda_\alpha X^\alpha = D_x P, \quad \sum_{\alpha} \lambda_\alpha Y^\alpha = D_y P.$$ 

Hence, the function

$$h = \lambda_1 w^1 + \cdots + \lambda_r w^r - P$$

is a nontrivial integral for the fields $\tilde{D}_x, \tilde{D}_y$ and the covering is reducible. Contradiction.

2. Consider the system

$$\tilde{D}_x(h) = D_x(h) + \sum_{\alpha} X^\alpha \frac{\partial h}{\partial \bar{w}^\alpha} = 0, \quad \tilde{D}_y(h) = D_y(h) + \sum_{\alpha} Y^\alpha \frac{\partial h}{\partial \bar{w}^\alpha} = 0.$$ 

Note that when the partial derivatives $\partial h/\partial w^\alpha$ vanish everywhere, the function $h$ is constant, since $E$ is differentially connected.

On the other hand, assume that there exists a point $\theta \in \tilde{E}$ such that at least one derivative, say $\partial h/\partial w^1|_{\theta} \neq 0$. Hence, in a neighbor of $\theta$ we can choose a new fiber coordinate $\bar{w}^1 = h$ and immediately make sure that the conservation law $\omega^1$ is trivial.

3. Let $\tau$ and $\bar{\tau}$ be two equivalent irreducible Abelian coverings and

$$\bar{w}^\alpha = f^\alpha(\theta, w^1, \ldots, w^r), \quad \theta \in E, \quad \alpha = 1, \ldots, r,$$

be their equivalence. Then

$$\bar{X}^\alpha = \tilde{D}_x(f^\alpha), \quad \bar{Y}^\alpha = \tilde{D}_y(f^\alpha), \quad \alpha = 1, \ldots, r,$$

from where it follows that

$$\tilde{D}_x \left( \frac{\partial f^\alpha}{\partial \bar{w}^\beta} \right) = 0, \quad \tilde{D}_y \left( \frac{\partial f^\alpha}{\partial \bar{w}^\beta} \right) = 0$$

for all $\alpha, \beta = 1, \ldots, r$. But the coverings are irreducible and consequently

$$f^\alpha = \sum_{\beta} a^\alpha_\beta \bar{w}^\beta + a^\alpha, \quad a^\alpha_\beta \in \mathbb{R}, \quad a^\alpha \in \mathcal{F}(E),$$

(6)
where \( \det a^\alpha_\beta \neq 0 \). Thus
\[
\bar{X}^\alpha = \sum_\beta a^\alpha_\beta X^\beta + a^\alpha,
\]
which means that \( \mathcal{L}(\tau) = \mathcal{L}(\bar{\tau}) \).

(4) Let \( \{[\omega^\alpha]\} \) and \( \{[\bar{\omega}^\beta]\} \) be two bases in \( \mathcal{L} \). Then
\[
\bar{\omega}^\alpha = \sum_\beta a^\alpha_\beta \omega^\beta + d h a^\alpha,
\]
and (4) is the needed equivalence.

\[ \square \]

2. THE MAIN RESULT (ABELIAN CASE)

Using the above results, we study here some relations between symmetries and conservation laws of the equations \( \mathcal{E} \) and \( \hat{\mathcal{E}} \) in an irreducible Abelian covering \( \tau : \hat{\mathcal{E}} \to \mathcal{E} \).

**Lifting conservation laws.** Consider a nontrivial conservation law \( \omega \) of \( \mathcal{E} \). Then the pull-back \( \tau^* \omega \) is a conservation law of \( \hat{\mathcal{E}} \).

**Proposition 2.** The form \( \tau^* \omega \) is a trivial conservation law of the equation \( \hat{\mathcal{E}} \) if and only if \( [\omega] \in \mathcal{L}_\tau \).

**Proof.** Consider the covering \( \tau \times \tau_\omega \), where \( \tau_\omega \) is the one-dimensional covering associated with the conservation law \( \omega \). Obviously, triviality of \( \tau^* \omega \) amounts to reducibility of \( \tau \times \tau_\omega \). But by Theorem [1] the covering \( \tau \times \tau_\omega \) is irreducible if and only if \( [\omega] \notin \mathcal{L}_\tau \). \[ \square \]

**Corollary 1.** Let \( \tau : \hat{\mathcal{E}} \to \mathcal{E} \) be a finite-dimensional Abelian covering and assume that \( \dim \mathcal{R} \) cl \( \mathcal{E} = \infty \). Then \( \dim \mathcal{R} \) cl \( \hat{\mathcal{E}} = \infty \) as well.

**Example 1.** Consider the Korteweg-de Vries equation
\[
\frac{du}{dt} = uu_x + u_{xxx}
\]
and its first conservation law \( \omega^1 = u \, dx + \left( \frac{u^2}{2} + u_{xx} \right) \, dt \). In the corresponding covering \( \tau : \hat{\mathcal{E}} \to \mathcal{E} \), the covering equation \( \hat{\mathcal{E}} \) is the potential KdV
\[
\frac{du}{dt} = \frac{1}{2} u_x^2 + u_{xxx}.
\]
All conservation laws of the KdV survive in pKdV except for \( \omega^1 \).

**Lifting symmetries.** Let \( \tau : \hat{\mathcal{E}} \to \mathcal{E} \) be an arbitrary covering and \( S \in \text{sym} \mathcal{E} \) be a symmetry of the equation \( \mathcal{E} \). We say that \( S \) lifts to \( \hat{\mathcal{E}} \) if there exists a symmetry \( \bar{S} \in \text{sym} \hat{\mathcal{E}} \) such that \( \bar{S} \big|_{\mathcal{F}(\mathcal{E}) \subset \mathcal{F}(\hat{\mathcal{E}})} = S \).

For any conservation law \( \omega \) of \( \mathcal{E} \), the Lie derivative \( L_S(\omega) \) is a conservation law as well and if two conservation laws are equivalent then their Lie derivatives are also equivalent. So, the action
\[
L_S : \text{cl} \mathcal{E} \to \text{cl} \mathcal{E}
\]
is well defined.

**Proposition 3.** Action (7) is \( \mathbb{R} \)-linear.
Proof. Choose a basis in \( \text{cl} \mathcal{E} \) and let \( \omega^\alpha = X^\alpha \, dx + Y^\alpha \, dy \), \( \alpha = 1, 2, \ldots \), be the corresponding conservation laws. Then, by \([3]\), we have nonlocal variables defined by the equations

\[
\frac{\partial w^\alpha}{\partial x} = X^\alpha, \quad \frac{\partial w^\alpha}{\partial y} = Y^\alpha
\]

for all possible values of \( \alpha \). Consequently,

\[
S(X^\alpha) = \tilde{D}_x(\tilde{S}(w^\alpha)), \quad S(Y^\alpha) = \tilde{D}_y(\tilde{S}(w^\alpha)),
\]

where

\[
\tilde{D}_x = D_x + \sum_\alpha X^\alpha \frac{\partial}{\partial w^\alpha}, \quad \tilde{D}_y = D_x + \sum_\alpha Y^\alpha \frac{\partial}{\partial w^\alpha},
\]

The right-hand sides in \((8)\) are independent of \( w^\beta \) while

\[
[\frac{\partial}{\partial w^\beta}, \tilde{D}_x] = [\frac{\partial}{\partial w^\beta}, \tilde{D}_y] = 0
\]

for all \( \beta \). Hence,

\[
\tilde{D}_x \left( \frac{\partial \tilde{S}(w^\alpha)}{\partial w^\beta} \right) = 0, \quad \tilde{D}_y \left( \frac{\partial \tilde{S}(w^\alpha)}{\partial w^\beta} \right) = 0,
\]

from where it follows (Theorem \([4]\)) that

\[
\frac{\partial \tilde{S}(w^\alpha)}{\partial w^\beta} = a^\alpha_\beta \in \mathbb{R},
\]

or

\[
\tilde{S}(w^\alpha) = \sum_\beta a^\alpha_\beta w^\beta + a^\alpha, \quad a^\alpha \in \mathcal{F}(\mathcal{E}) \quad (9)
\]

(the sum above is taken over finite number of \( \beta \)'s).

Remark 2. Equations \((9)\) mean that \( L_S \omega^\alpha = \sum_\beta a^\alpha_\beta \omega^\beta + d_h a^\alpha \).

An immediate consequence of this result is

**Proposition 4.** Let \( \mathcal{L} \subset \text{cl} \mathcal{E} \) be an \( r \)-dimensional subspace and \( \tau_\mathcal{L} \) be the corresponding irreducible Abelian covering. Then a symmetry \( S \) lifts to \( \tau_\mathcal{L} \) if and only if \( L_S(\mathcal{L}) \subset \mathcal{L} \).

Assume now that \( \mathcal{E} \) admits an infinite-dimensional symmetry algebra \( S = \text{sym} \mathcal{E} \). Consider a finite-dimensional irreducible Abelian covering \( \tau: \bar{\mathcal{E}} \to \mathcal{E} \) associated to conservation laws \( \omega^1, \ldots, \omega^r \) and the subspace \( \mathcal{L}_\tau \subset \text{cl} \mathcal{E} \). Then, by Proposition\([4]\) \( S \) lifts to the covering \( S \tau \) associated to the space \( L_S \mathcal{L}_\tau \) that spans all the the conservation laws

\[
\omega^0_0 = \omega^\alpha, \quad \omega^0_1 = L_S \omega^\alpha, \quad \omega^0_2 = L_S \omega^1, \ldots, \quad S \in \mathcal{S}, \alpha = 1, \ldots, r. \quad (10)
\]

If the space \( L_S \mathcal{L}_\tau \) is finite-dimensional a stronger result is valid which is based on the following

**Proposition 5.** Let

1. \( \mathcal{E} \) possess an infinite dimensional symmetry algebra \( S \);
2. \( \tau \) be a finite-dimensional irreducible Abelian covering over \( \mathcal{E} \);
3. \( \bar{\tau} \) be another finite-dimensional Abelian covering over the equation \( \mathcal{E} \) such that \( \tau \times \bar{\tau} \) is irreducible and any \( S \in \mathcal{S} \) lifts to \( \tau \times \bar{\tau} \).
Then the exist infinite number of symmetries in \( S \) that lift to \( \tau \).

**Proof.** Due to Proposition 3 this is a fact from linear algebra. Choose bases \( w^1, \ldots, w^r \) and \( \tilde{w}^1, \ldots, \tilde{w}^\tilde{r} \) in \( \mathcal{L}_\tau \) and \( \mathcal{L}_{\tilde{\tau}} \), respectively. Then

\[
S_k(w^\alpha) = \sum_{\beta=1}^{r} \lambda_{k,\beta}^\alpha w^\beta + \sum_{\tilde{\beta}=1}^{\tilde{r}} \tilde{\lambda}_{k,\tilde{\beta}}^\alpha \tilde{w}^{\tilde{\beta}},
\]

for all \( \alpha = 1, \ldots, r \) and \( k \geq 1 \). Thus, the action of \( S_k \) on \( \mathcal{L}_\tau \) is determined by two matrices

\[
\Lambda_k = (\lambda_{k,\beta}^\alpha)_{\alpha=1,\ldots,r, \beta=1,\ldots,r}, \quad \tilde{\Lambda}_k = (\tilde{\lambda}_{k,\tilde{\beta}}^\alpha)_{\alpha=1,\ldots,r, \tilde{\beta}=1,\ldots,\tilde{r}}.
\]

Then the space that spans the matrices \( \tilde{\Lambda}_k \) is of dimension \( d \leq r \cdot \tilde{r} \) at most. Choose its basis \( \tilde{\Lambda}_{k_1}, \ldots, \tilde{\Lambda}_{k_d} \). Then

\[
\tilde{\Lambda}_k = \mu_{k_1}^{1} \tilde{\Lambda}_{k_1} + \cdots + \mu_{k_d}^{d} \tilde{\Lambda}_{k_d}, \quad k \geq 1,
\]

and consequently the space \( \mathcal{L}_\tau \) is invariant with respect to all symmetries of the form \( \tilde{S}_k = \tilde{S}_k - \sum_{i=1}^{d} \mu_{k}^{i} \tilde{S}_{k_i} \).

Denote by \( S_\tau \) the space of all conservation laws generated from \( \mathcal{L}_\tau \) by the iterated action of \( S \), see Equations (10).

**Corollary 2.** Let \( S_\tau \) be finite-dimensional. Then there exists infinite number of independent symmetries in \( S \) that lift to \( \tau \).

**Remark 3.** Actually the proof of Proposition 3 shows that the symmetries \( \tilde{S}_k \) may be chosen in such a way that they will act on all nonlocal variables trivially.

**Example 2.** Consider the Burgers equation

\[
u_t = u \nu_x + u_{xx}.
\]

It possesses only one conservation law \( \omega^1 = u \, dx + \left( \frac{1}{2} \nu^2 + u_x \right) \, dt \) which has to be invariant with respect to all symmetries. Consequently, all these symmetries lift to the covering equation, which is the heat equation.

**The main result.** Gathering the results obtained above, we obtain the following

**Theorem 2.** Let \( \mathcal{E} \) be a differentially connected equation and \( \tau: \tilde{\mathcal{E}} \rightarrow \mathcal{E} \) be an irreducible finite-dimensional Abelian covering. Then:

1. if \( \mathcal{E} \) possesses infinite number of conservation laws the same is valid for \( \tilde{\mathcal{E}} \);
2. if \( \mathcal{E} \) possesses infinite number of symmetries then \( \tilde{\mathcal{E}} \) either has the same property or admits infinite number of conservation laws or both.

**Proof.** Statement (1) is Corollary 1 exactly.

To prove Statement (2), consider the space generated from \( \text{cl} \mathcal{E} \) by \( \text{sym} \mathcal{E} \). There are two options: (a) the space is finite-dimensional and we find ourselves in the situation of Corollary 2; (b) otherwise we come back to Corollary 1.
3. Non-Abelian case

The non-Abelian case is more complicated, and the first thing to be done is to narrow the universum of non-Abelian coverings.

**Definition 4.** A finite-dimensional covering \( \tau: \tilde{E} \to E \) is called **strictly non-Abelian** if it is not equivalent to a composition of coverings \( \tilde{E} \xrightarrow{\tau_1} E' \xrightarrow{\tau_2} E \), where \( \tau_2 \) is Abelian.

**Proposition 6.** Let \( \tau: \tilde{E} \to E \) be a finite-dimensional strictly non-Abelian covering and \( \omega \) be a nontrivial conservation law of the equation \( E \). Then the conservation law \( \tau^*(\omega) \in \text{cl}(\tilde{E}) \) is nontrivial as well.

**Proof.** Let \( \omega = X \, dx + Y \, dy \), where \( X \) and \( Y \) are functions on \( E \). Assume that \( \tau^*(\omega) \) is trivial. Then there exists a function \( f \) on \( \tilde{E} \) such that \( X = \tilde{D}_x(f) \), \( Y = \tilde{D}_y(f) \), where \( \tilde{D}_x \), \( \tilde{D}_y \) are the total derivatives on \( \tilde{E} \). Since \( \omega \) is nontrivial, at least one of the partial derivatives \( \partial f/\partial w_i \), say \( \partial f/\partial w_1 \), does not vanish, where \( w_1, \ldots, w_r \) are nonlocal variables in \( \tau \). Then, by choosing new coordinates \( \bar{w}_1 = f, \bar{w}_2 = w_2, \ldots, \bar{w}_r = w_r \) in the fiber, we see that \( \tau \) is not strictly non-Abelian. \( \square \)

**Corollary 3.** One has \( \dim_{\mathbb{R}} \ker \tau^* < \infty \) for any finite-dimensional covering \( \tau \).

**Proof.** The result follows from the proof of Proposition 6. \( \square \)

**Corollary 4.** If \( \tau: \tilde{E} \to E \) is a strictly non-Abelian covering then the map \( \tau^*: \text{cl}(E) \to \text{cl}(\tilde{E}) \) is an embedding. In particular, if \( \dim \text{cl}(E) = \infty \) the same holds for \( \text{cl}(\tilde{E}) \).

Consider now a symmetry \( S \in \text{sym} E \). Let try to formulate an analogue of Proposition 4 in the non-Abelian case. First of all, recall an old result from [8]:

**Proposition 7.** Let \( \tau: \tilde{E} \to E \) be a finite-dimensional covering and \( S \in \text{sym} E \) be a symmetry that possesses a one-parameter group \( A_\lambda \) of transformations (e.g., a contact symmetry). Then:

- either \( S \) can be lifted to a symmetry \( \tilde{S} \in \text{sym} \tilde{E} \) that is projectible to \( S \) by \( \tau_* \),
- or the action of the group \( A_\lambda \) gives rise to a one-parameter family of coverings \( \tau_\lambda: \tilde{E} \to E, \tau_0 = \tau \), with a nonremovable parameter \( \lambda \in \mathbb{R} \).

To formulate a counterpart to Proposition 7 in the case when \( A_\lambda \) does not exist, let us recall the basic constructions from [9]. Let \( E \subset J^\infty(\pi) \) be an equation with a set of internal coordinates \( \text{int}(E) \). The **structural element** of \( E \) is the vector-valued differential one-form

\[
U_E = \sum_{u^j_{\sigma} \in \text{int}(E)} \omega^j_{\sigma} \otimes \frac{\partial}{\partial u^j_{\sigma}}, \tag{11}
\]
where \( \omega^j = u^j - \sum_i u_{\sigma i} dx^i \) are the Cartan forms corresponding to \( u^j \).

Denote by \( D^\nu(\Lambda^*(\mathcal{E})) = \bigoplus_i D^\nu(\Lambda^i(\mathcal{E})) \) differential forms on \( \mathcal{E} \) with values in \( \pi_\infty \)-vertical vector fields (or, in other words, vertical form-valued derivations of the function algebra \( \mathcal{F}(\mathcal{E}) \)). Then \( D^\nu(\Lambda^*(\mathcal{E})) \) is a super Lie algebra with respect to the Nijenhuis bracket
\[
[\cdot, \cdot] : D^\nu(\Lambda^i(\mathcal{E})) \times D^\nu(\Lambda^j(\mathcal{E})) \to D^\nu(\Lambda^{i+j}(\mathcal{E})).
\]

The element \( U_\mathcal{E} \) from (11) can be understood as a derivation \( U_\mathcal{E} \in D^1_{\nu}(\Lambda^1(\mathcal{E})) \).

Then \( \partial C = [U_\mathcal{E}, \cdot] \). Its cohomology is called the \( C \)-cohomology of \( \mathcal{E} \) and is denoted by \( H^i_C(\mathcal{E}) \).

**Theorem 3** (see [4]). Let \( \mathcal{E} \subset J^\infty(\pi) \) be an equation. Then:

1. \( H^0_C(\mathcal{E}) = \text{sym} \mathcal{E} \),
2. \( H^1_C(\mathcal{E}) \) consists of equivalence classes of infinitesimal deformations of the equation structure modulo trivial ones,
3. \( H^2_C(\mathcal{E}) \) contains obstructions to continuation of infinitesimal deformations to formal ones.

Let \( \tau : \tilde{\mathcal{E}} \to \mathcal{E} \) be a covering with local coordinates \( w^\alpha \) in its fibers. The structural element of \( \tilde{\mathcal{E}} \) is
\[
U_{\tilde{\mathcal{E}}} = U_\mathcal{E} + \sum \theta^\alpha \otimes \frac{\partial}{\partial w^\alpha},
\]
where
\[
\theta^\alpha = dw^\alpha - \sum_i X_i^\alpha dx^i
\]
and the functions \( X_i^\alpha \) are from Equation (3).

Denote the second summand in (13) by \( U_\tau \). Then the covering structure in \( \tau \) is governed by the Maurer-Cartan type equations
\[
\partial C(U_\tau) + \frac{1}{2}[U_\tau, U_\tau] = 0.
\]

Consider complex (12) for the covering equation \( \tilde{\mathcal{E}} \) and its subcomplex
\[
0 \to D^\nu(\tilde{\mathcal{E}}) \xrightarrow{\partial_C} D^\nu(\Lambda^1(\tilde{\mathcal{E}})) \xrightarrow{\partial_C} \cdots \to D^\nu(\Lambda^i(\tilde{\mathcal{E}})) \xrightarrow{\partial_C} \cdots,
\]
where
\[
D^\nu(\Lambda^i(\mathcal{E})) = \{ Z \in D^\nu(\Lambda^i(\mathcal{E})) | Z|_{\mathcal{E}} = 0 \}
\]
(recall that the algebra \( \mathcal{F}(\mathcal{E}) \) is embedded into \( \mathcal{F}(\tilde{\mathcal{E}}) \) by \( \tau^* \)). Denote the corresponding cohomology groups by \( H^i_g(\tau) \). Then we have the following analogue of Theorem 3:

**Theorem 4.** Let \( \tau : \tilde{\mathcal{E}} \to \mathcal{E} \) be a covering. Then:

1. \( H^0_g(\tau) \) consists of gauge symmetries (infinitesimal equivalences) of \( \tau \),
2. \( H^1_g(\tau) \) consists of equivalence classes of the covering structure infinitesimal deformations modulo trivial ones,
(3) \( H_g^2(\tau) \) is the set of obstructions to continuation of infinitesimal deformations to formal ones.

To formulate the last result of this paper, let us give the following

**Definition 5.** Let \( \mathcal{E} \) be an equation and \( \tau: \tilde{\mathcal{E}} \to \mathcal{E} \) be a trivial vector bundle. We say that \( \tau \) carries a formal covering structure \( \tau_\varepsilon \), where \( \varepsilon \) is a formal parameter, if there exist formal \( \tau \)-vertical vector fields \( X_i = \sum_{i=0}^{\infty} \varepsilon^i X_{i\varepsilon} \) such that the equalities

\[
D_{x_i}(X_j) - D_{x_j}(X_i) + [X_i, X_j] = 0
\]

hold formally for all \( i, j = 1, \ldots, n \).

Let now \( S \) be a symmetry of \( \mathcal{E} \). Then (locally) \( S \) can be lifted to a vector field \( \tilde{S} \) on \( \tilde{\mathcal{E}} \). Two options are possible: (1) \( \tilde{S} \) is a symmetry of \( \tilde{\mathcal{E}} \); (2) \( \tilde{S} \) is not a symmetry. In the latter case, consider the element \( L_{\tilde{S}}(U_{\tilde{\mathcal{E}}}) = [\tilde{S}, U_{\tilde{\mathcal{E}}}] \).

One has

**Proposition 8.** The element \( L_{\tilde{S}}(U_{\tilde{\mathcal{E}}}) \) is a 1-cocycle in \( (12) \).

**Proof.** The result immediately follows from the two facts:

1. For any elements \( \Omega \in D^v(\mathcal{E}) \) and a vertical vector field \( Z \) one has

\[
(L_Z(\Omega))(\varphi) = [Z, \Omega](\varphi) = L_Z(\Omega(\varphi)) - (-1)^i \Omega(Z(\varphi)),
\]

where \( \varphi \) is an arbitrary smooth function on \( \tilde{\mathcal{E}} \).

2. \( L_{\tilde{S}}(U_{\tilde{\mathcal{E}}}) = [\tilde{S}, U_{\tilde{\mathcal{E}}}] = [S, U_{\tau}] + [S_g, U_{\mathcal{E}} + U_{\tau}] \), where \( S_g \in D^g(\tilde{\mathcal{E}}) \) is locally defined by \( S_g = \tilde{S} - S \).

Hence,

\[
\tilde{U}_\tau = U_{\tau} + \varepsilon : L_{\tilde{S}}(U_{\tilde{\mathcal{E}}})
\]

is an infinitesimal deformation of the covering structure in \( \tau \). This deformation is trivial if and only if the lift \( \tilde{S} \) is a symmetry of \( \tilde{\mathcal{E}} \).

**Theorem 5.** Let \( \tau: \tilde{\mathcal{E}} \to \mathcal{E} \) be a covering such that \( H_g^2(\tau) = 0 \) and \( S \in \text{sym} \mathcal{E} \). Then there exists a formal covering structure \( \tau_\varepsilon \) in \( \tau \) such that \( \tau_0 = \tau \).

**Proof.** The result immediately follows from Theorem 3. \( \square \)

**Remark 4.** This result is a weaker analogue of Proposition 7. It may have stronger consequences with additional assumptions. For example, if we assume that the vector fields \( X_i \) depend on the nonlocal variables polynomially then, expanding these variables in formal series with respect to the deformation parameter \( \varepsilon \), we shall obtain an infinite-dimensional covering (genuine, not formal) over \( \mathcal{E} \).

Moreover, if the covering equation \( \tilde{\mathcal{E}}_\varepsilon \) is in the divergent form there is a hope to construct infinite number of conservation laws for \( \mathcal{E} \), similar to the classical construction applied to the Korteweg-de Vries equation using the Miura transform.

**Acknowledgments**

I am grateful to Artur Sergyeyev who asked the question which is, hopefully, answered in this paper. My thanks are also due to Alik Verbovetsky for criticism and fruitful discussions.
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