Quantum gravity and mass of gauge field: a four-dimensional unified quantum theory

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We present in detail a four-dimensional unified quantum theory. In this theory, we identify three classes of parameters, coordinate-momentum, spin and gauge, as all and as the only fundamental parameters to describe quantum fields. The coordinate-momentum is formulated by the general relativity in four-dimensional space-time. This theory satisfies the general covariance condition and the general covariance derivative operator is given. In a unified and combined description, the matter fields, gravity field and gauge fields satisfy Dirac equation, Einstein equation and Yang-Mills equation in operator form. In the framework of our theory, we mainly realize the following aims: (1) The gravity field is described by a quantum theory, the graviton is massless, it is spin-2; (2) The mass problem of gauge theory is solved. Mass arises naturally from the gauge space and thus Higgs mechanism is not necessary; (3) Color confinement of quarks is explained; (4) Parity violation for weak interactions is obtained; (5) Gravity will cause CPT violation; (6) A dark energy solution of quantum theory is presented. It corresponds to Einstein’s cosmological constant. We propose that the candidate for dark energy should be gluon which is one of the elementary particles.

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Quantum mechanics and the relativity theory \[1, 2\] are well-known great discoveries in the last century. Both of them are so successful that they play the central roles in modern science and even in our life. In quantum mechanics, in order to explain a finer structure in the spectrum of hydrogen atom due to the spin of the electron, Dirac \[3\] introduced his equation, the Dirac equation, by adding spin corrections to the Schrödinger equation \[4\]. Dirac equation is a quantum theory while satisfying invariance under special relativity. It describes a single-particle obeying both relativity and quantum mechanics and thus unifies the theory of special relativity and quantum mechanics.

Dirac equation is extended as a quantum field theory which provides an unified description for both fields and particles \[5, 6, 7, 8, 9\]. Later, Yang and Mills \[10\] introduced gauge field theories. Then the Standard Model and the related theories \[11, 12, 13, 14, 15, 16, 17, 18\] are proposed for elementary particles and the interactions between them, see Ref.\[19\] for complete references. Up to then, the electromagnetic force, weak and strong nuclear forces are merged into a unified model. While only gravity which is one of the four basic forces in nature remains outside of this unified framework.

In the past decades, much effort has been put into studying how to combine quantum mechanics with general relativity into a quantum theory of gravity. This is because that on the one hand, we would like to find a unified final theory \[20\], and on the other hand, this theory can describe systems where both quantum mechanics and general relativity are important, for example in black hole or the early stage of universe.

Our start point is the general relativity, particles and fields are set of “events” which is a concept coined by Einstein to emphasize space-time. We propose a quantum theory of general relativity, an “event” is described by not only a complete set of canonical coordinates and canonical momentum of 4D space in operator form \(\hat{x}^a, \hat{p}_a\), but also by spin operators \(\hat{s}_\mu\) and gauge charge operators \(T_\alpha\). These operators are covariance and constitute a Lie algebra. In our work, events are quantum states of elementary particles. We then define a covariance derivative operator in the sense of event with vierbein and connections of general relativity. Thus Einstein equation of general relativity, Dirac equation and Yang-Mills equation are all reformulated by the covariant operators. This provides a unified framework of quantum mechanics and general relativity.

We next clarify four questions concerning about our theory. Firstly, what is the structure of the theory? Secondly, what is new in the theory? Thirdly, why is the theory correct? Fourthly, any results from this theory and is this theory necessary?

Our theory can be simply summarized as (1) representations, (2) equations and (3) observable quantities. Consequently the structure of our theory can be described similarly as: (1) Representations: In the language of quantum field theory, “particles” are dealt like “fields”. Fields are represented by quantum states with three class of independent parameters, coordinate-momentum, spin and gauge. The forces are represented as operators. (2) Equations: All fields, gravity field, matter fields and gauge fields, which already have the representations, satisfy three fundamental equations, Einstein equation, Dirac equation and Yang-Mills equation, all in operator form which is the standard of quantum theory. (3) Observable quantities: With equations and representations, we may construct quantities which are related with observable interactions and observable physical quantities such as current densities, particle production rate density and scattering of particles.

As is well-known, it is obvious not easy to find a unified quantum theory. Then what is new in our theory?

In the framework of general relativity, there is no absolute space and time. The space and time are dealt symmetrically and constitute together to be a four-dimensional space-time. In comparison, in conventional theory, time is generally special which is different from the three-dimensional space. For example, the evolution of particles is depicted by a world-line in three-dimension space as time passes. In order to propose a unified theory of general relativity and quantum mechanics, we should start directly from the four-dimensional space-time. In our theory, coordinate-momentum space is the four-dimensional space of the general relativity. In this sense, we mean that our theory is a four-dimensional theory.

A quantum theory compatible with the general relativity should satisfy the general covariance condition. Our theory is a general covariant theory where three fundamental equations, quantum states, operators, current densities and all other quantities satisfy the general covariance condition. So instead of gauge invariance condition for conventional quantum field theory, one new feature of our theory is that it satisfies the general gauge covariance condition. Due to the general covariance condition, mass can not be defined through momentum like \(\hat{p}_\mu \hat{p}^\mu = m^2\). Mass-matrices \(\hat{m}, \hat{M}\) are defined in the gauge space which is another new feature of our theory. Since mass naturally arises from the
gauge space, the Higgs mechanism seems not necessary in our theory. In particular, for massless particles, the gauge
condition is arbitrary such as for electromagnetic field, but for massive particles, the gauge will not be arbitrary.

Next, we would like to show the evidences that our theory is correct.

The most important point that our theory is correct is that all fundamentals of our theory, more or less, are well
accepted while all results of our theory agree well with the basic physics facts and are reasonable. The key result
of our theory is that with suitable representations, three fundamental equations, Dirac equation, Einstein equation
and Yang-Mills equation in operator form are all self-consistently presented. This is not a coincidence. We all agree
that those three equations are the cornerstones of modern physics. We do not change the spirit of those fundamental
equations, but present them in a unified and combined form. From our theory the graviton is found to be massless
which can be considered as the benchmark to test a quantum theory with general relativity. The gauge charge
representations of our theory agree with the basic results of the elementary particles scattering process obtained by
Feynman diagram method. Our results agree with the Standard Model except that we do not have Higgs mechanism.

Finally, we would like to point out that our theory is powerful. Some important results can be explained and
obtained easily in this theory such as color confinement, parity violation for weak interactions, gravity can cause CPT
violation, a dark energy of universe solution. With a completely new platform, we expect that answers may be found
for some other unsolvable problems. So our theory is not only necessary, but also we hope that it may provide a new
foundation for quantum physics.

A. Brief explanation of the concepts in this work

In quantum field theory, particles and fields are generally dealt in an unified description. In principle, we accept
this framework. However, there are also some subtle differences for particles and fields in this work. We consider that
fields are described as the vectors, while particles are the quantum states acting as basis of the vectors. So fields are
the expansions in terms of the particle states in the superposition form like $|\Psi\rangle = \int \tilde{\Psi}^\dagger(p)|e_{st}(p)d^4p$. Particles as
the elements constitute the fields as the whole. The motion equations of fields are described by the fields equations
in operator form, in this work these equations are three fundamental equations, Dirac equation, Yang-Mills equation
and Einstein equation. The evolution of fields can be explained as propagation, scattering, creation and annihilation
of particles while all of these processes should rely on those three fundamental equations.

The quantization of fields generally means finding a complete set of canonical coordinate-momentums and changing
them as the corresponding operators. In this work, we still keep the spirit of this concept, additionally, the three
fundamental equations are in operator form as generally expected.

II. COORDINATE AND MOMENTUM

A. General covariance and commutation relation

In the framework of general relativity, the space-time is in 4D space. We consider a 4D quantum mechanics, the
coordinate and momentum are denoted as $\hat{x}^\mu$ and $\hat{p}_\nu$, $(\mu, \nu = 0, 1, 2, 3)$, which as usual satisfy the relations

$$[\hat{x}^\mu, \hat{p}_\nu] = -i\delta^\mu_\nu,$$  \hspace{1cm} (1)

$$[\hat{x}^\mu, \hat{x}^\nu] = [\hat{p}_\mu, \hat{p}_\nu] = 0.$$  \hspace{1cm} (2)

Suppose $F(\hat{x})$ is an analytic function of 4D space-time coordinate $\hat{x}$, i.e., $F(\hat{x})$ can be expanded in terms of $\hat{x}$ by
Taylor expansion. We can prove that $F(\hat{x})$ satisfies the commutation relation

$$[\hat{p}_\mu, F(\hat{x})] = i\frac{\partial}{\partial \hat{x}^\mu} F(\hat{x}).$$  \hspace{1cm} (3)

The coordinate-momentum algebra is constituted by coordinate $\hat{x}^\mu$, the momentum $\hat{p}_\mu$ and the unit operator 1,

$$A_{xp} = \{\hat{Z}_{xp} : \hat{Z}_{xp} = a_\mu \hat{x}^\mu + b^\nu \hat{p}_\nu + \alpha; a_\mu, b^\nu, \alpha \in R\}.\hspace{1cm} (4)$$

This algebra is a 9D Lie algebra.

The coordinate-momentum group $G_{xp}$ is the Lie group corresponding to Lie algebra $A_{xp},$

$$G_{xp} = \{\hat{U}_{xp} : \hat{U}_{xp} = \exp[i(a_\mu \hat{x}^\mu + b^\nu \hat{p}_\nu + \alpha)]; a_\mu, b^\nu, \alpha \in R\}$$  \hspace{1cm} (5)
The group $G_{xp}$ is a 9D non-Abelian Lie group manifold. Four elements of coordinate $\hat{x}$, four elements of momentum $\hat{p}$ and the unit identity 1 constitute the generators of this group. The geometry structure of coordinate-momentum group $G_{xp}$ can provide the geometry structures for quantum space-time and quantum energy-momentum.

We next define the general covariance law in quantum case. In our quantum theory, $\hat{x}^\mu$ and $\hat{p}_\mu$ are not orthogonal basis for coordinate and momentum, respectively. They are general covariance coordinate and momentum. Coordinate $\hat{x}^\mu$ should satisfy the quantum general coordinate transformation and inversion transformation which are written as,

$$\hat{x}'^\mu = \hat{x}'^\mu(\hat{x}),$$
$$\hat{x}^\mu = \hat{x}^\mu(\hat{x}') .$$

(6)

Corresponding to general coordinate transformation, the transformation and its inversion for general momentum are,

$$\hat{p}_\mu' = \frac{\partial \hat{x}^\nu}{\partial \hat{x}'^\mu} \hat{p}_\nu ,$$
$$\hat{p}_\mu = \frac{\partial \hat{x}^\nu}{\partial \hat{x}_\mu} \hat{p}_\nu .$$

(7)

(8)

We can find that for general coordinate transformation, the commutation relations of coordinate and momentum are invariant,

$$[\hat{x}^\mu , \hat{p}_\nu ] = -i \delta^\mu_\nu ,$$

(9)

$$[\hat{x}^\mu , \hat{x}'^\nu ] = [\hat{p}_\mu , \hat{p}_\nu ] = 0 .$$

(10)

The proof of this result can be found by simple calculations,

$$[\hat{x}^\mu , \hat{p}_\nu ] = \left[ \hat{x}^\mu (\hat{x}) , \frac{\partial \hat{x}^\lambda}{\partial \hat{x}^\nu} \hat{p}_\lambda \right]$$
$$= \frac{\partial \hat{x}^\lambda}{\partial \hat{x}^\nu} \left[ \hat{x}^\mu (\hat{x}) , \hat{p}_\lambda \right]$$
$$= -i \delta^\nu_\lambda \frac{\partial \hat{x}^\lambda}{\partial \hat{x}^\nu}$$
$$= -i \delta^\nu_\lambda .$$

(11)

$$[\hat{p}_\mu , \hat{p}_\nu ] = \left[ \frac{\partial \hat{x}^\kappa}{\partial \hat{x}_\mu} \hat{p}_\kappa , \frac{\partial \hat{x}^\lambda}{\partial \hat{x}_\nu} \hat{p}_\lambda \right]$$
$$= \frac{\partial \hat{x}^\lambda}{\partial \hat{x}^\nu} \left[ \frac{\partial \hat{x}^\kappa}{\partial \hat{x}_\mu} \hat{p}_\kappa + \frac{\partial \hat{x}^\kappa}{\partial \hat{x}_\nu} \hat{p}_\kappa \right] \hat{p}_\lambda$$
$$= \frac{\partial \hat{x}^\lambda}{\partial \hat{x}^\nu} \left[ \frac{\partial \hat{x}^\kappa}{\partial \hat{x}_\mu} \hat{p}_\kappa + \frac{\partial \hat{x}^\kappa}{\partial \hat{x}_\nu} \hat{p}_\kappa \right]$$
$$= 0 .$$

(12)

When an equation is unchanged under the quantum general coordinate transformation, we say this equation is quantum general covariance. In our theory, we demand that all physical equations, operators and quantities be quantum general covariance. This can be looked as a generalization of the classical general covariance to quantum case. The quantum general covariance is a condition to has an unified description of general relativity and quantum mechanics.

The three types of particles, matter particles, gauge particles and graviton, all have their coordinate-momentum representations. Next, we will consider their coordinate-momentum properties.

B. Coordinate-momentum representation of the matter particles

The coordinate-momentum basis for matter particles can be denoted as

$$|x\rangle = |x^0\rangle \otimes |x^1\rangle \otimes |x^2\rangle \otimes |x^3\rangle ,$$

(13)
where $|x^0\rangle$, $|x^1\rangle$, $|x^2\rangle$, $|x^3\rangle$ are eigenvectors of $\hat{x}^0$, $\hat{x}^1$, $\hat{x}^2$, $\hat{x}^3$, and the eigenvalues are $x^0$, $x^1$, $x^2$, $x^3 \in \mathbb{R}$, respectively.

$$\hat{x}^\mu |x\rangle = x^\mu |x\rangle. \quad (14)$$

For momentum operator, we have,

$$\hat{p}_\mu |x\rangle = \int_{\mathbb{R}^4} i \frac{\partial}{\partial x^\mu} \delta^4(x - x') |x\rangle d^4 x'. \quad (15)$$

The adjoint of basis $|x\rangle$ is denoted as $\langle x| = |x\rangle^*$, and they satisfy the relations,

$$\langle x'|x\rangle = \delta^4(x - x'), \quad (16)$$

$$\int_{\mathbb{R}^4} |x\rangle \langle x| d^4 x = 1. \quad (17)$$

Similarly, for momentum basis, we have,

$$|p\rangle = |p_0\rangle \otimes |p_1\rangle \otimes |p_2\rangle \otimes |p_3\rangle, \quad (18)$$

where $|p_\mu\rangle$ are eigenvectors of $\hat{p}_\mu$, and the eigenvalues are $p_\mu \in \mathbb{R}$, $(\mu = 0, 1, 2, 3)$, respectively,

$$\hat{p}_\mu |p\rangle = p_\mu |p\rangle, \quad (19)$$

$$\hat{x}^\mu |p\rangle = \int_{\mathbb{R}^4} i \frac{\partial}{\partial p_\mu} \delta^4(p - p') |p'\rangle d^4 p'. \quad (20)$$

The adjoint basis $\langle p| = |p\rangle^*$ satisfies the relations,

$$\langle p'|p\rangle = \delta^4(p - p'), \quad (21)$$

$$\int_{\mathbb{R}^4} |p\rangle \langle p| d^4 p = 1. \quad (22)$$

The transformation between coordinate and momentum takes the form

$$\langle x|p\rangle = (2\pi)^{-\frac{3}{2}} \exp(-ip \cdot \vec{x}). \quad (23)$$

### C. Comparison between the quantum mechanics and the unified theory

The main difference between our unified theory and the quantum mechanics is that the time parameter is dealt in a different way. In our theory, space and time are symmetric and should satisfy the general covariance condition in the framework of general relativity. While in quantum mechanics, the time is used to describe the motion of particles, in principle, the time is in a different position from space parameters. Here for completeness, we briefly list some results of quantum mechanics.

In quantum mechanics, the 3D coordinate and 3D momentum parameters are in symmetric positions, and should take the following form

$$[\hat{x}_i, \hat{p}_j] = i \delta_{ij}, \quad (24)$$

$$\hat{x}_i |\vec{x}\rangle = x_i |\vec{x}\rangle, \quad (25)$$

$$\hat{p}_i |\vec{p}\rangle = p_i |\vec{p}\rangle, \quad (26)$$

$$\langle \vec{x}|\vec{x}\rangle = \delta^3(\vec{x} - \vec{x}), \quad (27)$$

$$\langle \vec{p}|\vec{p}\rangle = \delta^3(\vec{p} - \vec{p}), \quad (28)$$

$$\int |\vec{x}\rangle \langle \vec{x}| d^3 \vec{x} = 1, \quad (29)$$

$$\int |\vec{p}\rangle \langle \vec{p}| d^3 \vec{p} = 1, \quad (30)$$

$$\langle \vec{x}|\vec{p}\rangle = (2\pi)^{-\frac{3}{2}} \exp(-i \vec{p} \cdot \vec{x}). \quad (31)$$
If we consider the role of time, we need to introduce the moving pictures depending on time, Heisenberg picture, or instead the Schödinger picture. The time-translation operator is introduced as

$$u = \exp(-i\hat{H}t),$$

(32)

where $\hat{H}$ is the Hamiltonian. The time evolution of the operators such as $\hat{x}_i(t), \hat{p}_i(t)$ and the quantum states $|\vec{x}, t\rangle, |\vec{p}, t\rangle$ are defined as

$$\hat{x}_i(t) = \exp(i\hat{H}t)\hat{x}_i \exp(-i\hat{H}t),$$

$$\hat{p}_i(t) = \exp(i\hat{H}t)\hat{p}_i \exp(-i\hat{H}t),$$

$$|\vec{x}, t\rangle = \exp(-i\hat{H}t)|\vec{x}\rangle,$$

$$|\vec{p}, t\rangle = \exp(-i\hat{H}t)|\vec{p}\rangle.$$  

(33)

(34)

When the time $t$ are the same, operator $\hat{x}_i(t), \hat{p}_i(t)$ and the quantum state $|\vec{x}, t\rangle, |\vec{p}, t\rangle$ satisfy the relations,

$$[\hat{x}_i(t), \hat{p}_j(t)] = i\delta_{ij},$$

$$\hat{x}_i(t)|\vec{x}, t\rangle = x_i|\vec{x}, t\rangle,$$

$$\hat{p}_i(t)|\vec{p}, t\rangle = p_i|\vec{p}, t\rangle,$$

$$\langle \vec{x}, t|\vec{x}'\rangle = \delta^3(\vec{x} - \vec{x}'),$$

$$\langle \vec{p}, t|\vec{p}', t\rangle = \delta^3(\vec{p} - \vec{p}'),$$

$$\int |\vec{x}, t\rangle \langle \vec{x}, t|d^3\vec{x} = 1,$$

$$\int |\vec{p}, t\rangle \langle \vec{p}, t|d^3\vec{p} = 1,$$

$$\langle \vec{x}, t|\vec{p}, t\rangle = (2\pi)^{-\frac{3}{2}} \exp(-i\vec{p} \cdot \vec{x}).$$  

(35)

(36)

(37)

(38)

(39)

(40)

(41)

(42)

In quantum mechanics, the transformation between eigenvectors with different time can be represented as the path-integral:

$$\langle \vec{x}', t'|\vec{x}, t\rangle = \langle \vec{x}'| \exp[-i\hat{H}(t - t')]|\vec{x}\rangle = \int D\vec{x} \exp\left[i \int L(\vec{x}, \dot{\vec{x}}, t)dt\right].$$

(43)

where $L(\vec{x}, \dot{\vec{x}}, t)$ is the Lagrangian of the system; $D\vec{x}$ means to make integral for all possible paths.

Here let us list the differences between our theory and the standard quantum mechanics,

1. Time is dealt differently, as we already mentioned.

2. Energy is dealt differently. In our theory, the energy is dealt as an element of 4D energy-momentum tensor. Thus the energy and momentum are in symmetric positions. In quantum mechanics, energy is used for the Hamiltonian which is a function of coordinate, momentum and time. The Hamiltonian determines the motion property of the system. Hamiltonian is symmetric with the 3D momentum.

3. In our theory, we have an unified orthogonal equation

$$\langle x'|x\rangle = \delta^4(x - x').$$

(44)

For comparison, in quantum mechanics, the orthogonal equations are satisfied when time are equal. For different time, the path-integral are used,

$$\langle \vec{x}', t'|\vec{x}, t\rangle = \begin{cases} 
\delta^3(\vec{x} - \vec{x}'), & t' = t \\
\int D\vec{\vec{x}} \exp\left[i \int L(\vec{x}, \dot{\vec{x}}, t)dt\right], & t' \neq t
\end{cases}$$

(45)

4. In our theory, we have the general covariance condition.
D. Coordinate-momentum state of the matter particles

A coordinate-momentum state of matter particles can be expanded in terms of coordinate $|x\rangle$ or momentum $|p\rangle$,

$$|\Psi\rangle = \int_{R^4} \Psi(x)|x\rangle d^4x = \int_{R^4} \tilde{\Psi}(p)|p\rangle d^4p,$$

(46)

where $\Psi(x)$ and $\tilde{\Psi}(p)$ are coordinate state functions and momentum state functions. They are complex functions and connected through Fourier transformation,

$$\Psi(x) = \langle x|\Psi \rangle = (2\pi)^{-2} \int_{R^4} \tilde{\Psi}(p) \exp(-ipx) d^4p,$$

(47)

$$\tilde{\Psi}(p) = \langle p|\Psi \rangle = (2\pi)^{-2} \int_{R^4} \Psi(x) \exp(ipx) d^4x.$$

(48)

The action of coordinate and momentum operators on the state take the form

$$\hat{x}^{\mu}|\Psi\rangle = \int_{R^4} x^{\mu}\Psi(x)|x\rangle d^4x = \int_{R^4} -i \frac{\partial}{\partial p^{\mu}} \tilde{\Psi}(p)|p\rangle d^4p,$$

(49)

$$\hat{p}_{\mu}|\Psi\rangle = \int_{R^4} i \frac{\partial}{\partial x^{\mu}} \Psi(x)|x\rangle d^4x = \int_{R^4} p_{\mu} \tilde{\Psi}(p)|p\rangle d^4p.$$

(50)

The calculations concerning about adjoint state $\langle \Psi |$ of state $|\Psi\rangle$ are standard, the results are listed below:

$$\langle \Psi | = |\overline{\Psi}\rangle = \int_{R^4} \Psi^{\ast}(x)\langle x|d^4x$$

$$= \int_{R^4} \tilde{\Psi}^{\ast}(p)|p\rangle d^4p,$$

(51)

$$\Psi^{\ast}(x) = \langle \Psi | x \rangle = (2\pi)^{-2} \int_{R^4} \tilde{\Psi}^{\ast}(p) \exp(ipx) d^4p,$$

(52)

$$\tilde{\Psi}^{\ast}(p) = \langle \Psi | p \rangle = (2\pi)^{-2} \int_{R^4} \Psi^{\ast}(x) \exp(-ipx) d^4x.$$

(53)

The action of the coordinate and momentum operators on the adjoint state can be represented as

$$\langle \Psi | \hat{x}^{\mu} = \int_{R^4} x^{\mu}\Psi^{\ast}(x)\langle x|d^4x = \int_{R^4} -i \frac{\partial}{\partial p^{\mu}} \tilde{\Psi}^{\ast}(p)|p\rangle d^4p,$$

(54)

$$\langle \Psi | \hat{p}_{\mu} = \int_{R^4} -i \frac{\partial}{\partial x^{\mu}} \Psi^{\ast}(x)\langle x|d^4x = \int_{R^4} p_{\mu} \tilde{\Psi}^{\ast}(p)|p\rangle d^4p.$$

(55)

The inner product can be defined as

$$\langle \Psi | \Phi \rangle = \langle \Phi | \Psi \rangle^{\ast} = \int_{R^4} \Psi^{\ast}(x)\Phi(x)d^4x = \int_{R^4} \tilde{\Psi}^{\ast}(p)\tilde{\Phi}(p)d^4p.$$

(56)

The representation space of coordinate-momentum for matter particles is a linear space in support of coordinate basis $|x\rangle$ and momentum basis $|p\rangle$. So the space of the representation $V_{xp}(M)$ of coordinate and momentum for matter particles is

$$V_{xp}(M) = \{ |\Psi\rangle : \langle \Psi | = \int_{R^4} \Psi(x)|x\rangle d^4x \}$$

$$= \{ |\Psi\rangle : \langle \Psi | = \int_{R^4} \tilde{\Psi}(p)|p\rangle d^4p \}$$

(57)

Similarly for adjoint representation, we have

$$\mathcal{V}_{xp}(M) = \{ \langle \Psi | : \langle \Psi | = \int_{R^4} \Psi^{\ast}(x)\langle x|d^4x \}$$

$$= \{ \langle \Psi | : \langle \Psi | = \int_{R^4} \tilde{\Psi}^{\ast}(p)\langle p|d^4p \}$$

(58)
E. Matrix representation of coordinate-momentum

The matrix representation of coordinate-momentum operators on their basis can be written respectively as

\[
\hat{x}^\mu = \int_{R^4} \int_{R^4} x^\mu \delta^4(x - x')|x'|\langle x|d^4x'd^4x
\]

\[
= \int_{R^4} \int_{R^4} -i \frac{\partial}{\partial p^\mu} \delta^4(p - p')|p'|\langle p|d^4p'd^4p;
\]

\[\text{(59)}\]

\[
\hat{p}_\mu = \int_{R^4} \int_{R^4} i \frac{\partial}{\partial x^\mu} \delta^4(x - x')|x'|\langle x|d^4x'd^4x
\]

\[
= \int_{R^4} \int_{R^4} p_\mu \delta^4(p - p')|p'|\langle p|d^4p'd^4p;
\]

\[\text{(60)}\]

The adjoint matrices are

\[
\bar{x}^\mu = (\hat{x}^\mu)^\dagger = \hat{x}^\mu, \quad \bar{p}_\mu = (\hat{p}_\mu)^\dagger = \hat{p}_\mu,
\]

\[\text{(61)}\]

\[\text{(62)}\]

We now consider the general operator, its matrix representation and the corresponding representation space. A linear operator on coordinate-momentum space can be represented as,

\[
\hat{A} = \int_{R^4} \int_{R^4} A(x', x)|x'|\langle x|d^4x'd^4x = \int_{R^4} \int_{R^4} \hat{A}(p', p)|p'|\langle p|d^4p'd^4p;
\]

\[\text{(63)}\]

\[
A(x', x) = \langle x'|\hat{A}|x \rangle = (2\pi)^{-4} \int_{R^4} \int_{R^4} \hat{A}(p', p) \exp[ipx - ip'x']d^4p'd^4p,
\]

\[\text{(64)}\]

\[
\hat{A}(p', p) = \langle p'|\hat{A}|p \rangle = (2\pi)^{-4} \int_{R^4} \int_{R^4} A(x', x) \exp[-ipx + ip'x']d^4x'd^4x,
\]

\[\text{(65)}\]

The space of the representation is the space in support of \(|x'|\langle x|\) and \(|p'|\langle p|\),

\[
O_{xp} = \left\{ \hat{A} : \hat{A} = \int_{R^4} \int_{R^4} A(x', x)|x'|\langle x|d^4x'd^4x \right\}
\]

\[
= \left\{ \hat{A} : \hat{A} = \int_{R^4} \int_{R^4} \hat{A}(p', p)|p'|\langle p|d^4p'd^4p \right\}.
\]

\[\text{(66)}\]

The space \(O_{xp}\) can be represented as the direct product of \(V_{xp}(M)\) and \(\nabla_{xp}(M)\),

\[
O_{xp} = V_{xp}(M) \otimes \nabla_{xp}(M)
\]

\[\text{(67)}\]

F. Representation of coordinate-momentum of the gauge particles and the graviton

The gauge particles and the graviton have property of coordinate-momentum. Since they are related with forces, there are operators related to them. For operators, the representation basis is

\[
\tilde{\varepsilon}(x) = \delta^4(\hat{x} - x) = \int_{R^4} \delta^4(x' - x)|x'|\langle x|d^4x' = |x\rangle\langle x|.
\]

\[\text{(68)}\]
The coordinate basis $\hat{\epsilon}(x)$ has the following properties:

$$\hat{\epsilon}(x) = \hat{\epsilon}(x)$$

$$[\hat{\epsilon}^\mu, \hat{\epsilon}(x)] = 0,$$

$$[\hat{p}_\mu, \hat{\epsilon}(x)] = \int_{R^4} i \frac{\partial}{\partial x^\nu} \delta^4(x' - x) \hat{\epsilon}(x) d^4x',$$

$$\hat{\epsilon}(x) \hat{\epsilon}(x') = \hat{\epsilon}(x') \hat{\epsilon}(x) = \delta^4(x - x') \hat{\epsilon}(x),$$

$$\text{tr} \hat{\epsilon}(x) = \text{tr} |(x)\rangle = 1,$$

$$\langle \hat{\epsilon}(x), \hat{\epsilon}(x') \rangle = \text{tr} [\hat{\epsilon}(x) \hat{\epsilon}(x')] = \delta^4(x - x'),$$

$$\hat{\epsilon}(x) |x'\rangle = \delta^4(x - x') |x'\rangle,$$

$$\langle \Psi | \hat{\epsilon}(x) | \Phi \rangle = \Psi^*(x) \Phi(x). \tag{69}$$

The definition of the momentum basis and its properties are listed below,

$$\hat{\epsilon}(p) = (2\pi)^{-2} \exp(-ip\hat{x}) = (2\pi)^{-2} \int_{R^4} \exp(-ip\hat{x}) |p'\rangle | \Psi\rangle |d^4p'\rangle$$

$$= (2\pi)^{-2} \int_{R^4} |(p + p')\rangle | \Psi\rangle | \Psi\rangle |d^4p'\rangle.$$

$$\hat{\epsilon}(p) = \hat{\epsilon}(-p)$$

$$[\hat{\epsilon}^\mu, \hat{\epsilon}(p)] = 0,$$

$$[\hat{p}_\mu, \hat{\epsilon}(p)] = p_\mu \hat{\epsilon}(p),$$

$$\hat{\epsilon}(p) \hat{\epsilon}(p') = \hat{\epsilon}(p') \hat{\epsilon}(p) = (2\pi)^{-2} \hat{\epsilon}(p' + p),$$

$$\text{tr} \hat{\epsilon}(p) = (2\pi)^2 \delta^4(p),$$

$$\langle \hat{\epsilon}(p'), \hat{\epsilon}(p) \rangle = \text{tr} [\hat{\epsilon}(p') \hat{\epsilon}(p)] = \delta^4(p - p'),$$

$$\hat{\epsilon}(p) |p\rangle = (2\pi)^{-2} |p\rangle | \Psi\rangle,$$

$$\langle \Psi | \hat{\epsilon}(p) | \Phi \rangle = \Phi^*(p) \Phi(p), \tag{70}$$

where * means convolution,

$$\Psi^*(p) \Phi(p) = (2\pi)^{-2} \int_{R^4} \Psi^*(p') \Phi(p - p') d^4p'.$$  \tag{71}

The transformations between basis of coordinate and momentum are

$$\langle \hat{\epsilon}(x), \hat{\epsilon}(p) \rangle = \langle \hat{\epsilon}(p), \hat{\epsilon}(x) \rangle^* = (2\pi)^{-2} \exp(-ip\hat{x}),$$

$$\hat{\epsilon}(x) = (2\pi)^{-2} \int_{R^4} \exp(ip\hat{x}) \hat{\epsilon}(p) d^4p;$$

$$\hat{\epsilon}(p) = (2\pi)^{-2} \int_{R^4} \exp(-ip\hat{x}) \hat{\epsilon}(x) d^4x. \tag{72}$$

For gauge particles and the graviton, their coordinate-momentum operator $\hat{F}$ can be expanded by the basis of $\hat{\epsilon}(x)$ and $\hat{\epsilon}(p),$

$$\hat{F} = \int_{R^4} F(x) \hat{\epsilon}(x) d^4x = \int_{R^4} \bar{F}(p) \hat{\epsilon}(p) d^4p, \tag{73}$$

where $F(x)$ and $\bar{F}(x)$ are coordinate and momentum functions, they are related by the Fourier transformation

$$F(x) = (2\pi)^{-2} \int_{R^4} \bar{F}(p) \exp(-ip\hat{x}) d^4p,$$

$$\bar{F}(p) = (2\pi)^{-2} \int_{R^4} F(x) \exp(ip\hat{x}) d^4x. \tag{74}$$

The adjoint operator of $\hat{F}$ is,

$$\overline{\hat{F}} = \int_{R^4} F^*(x) \hat{\epsilon}(x) d^4x = \int_{R^4} \bar{F}^*(p) \hat{\epsilon}(p) d^4p. \tag{75}$$

Next let us list some properties of the coordinate-momentum operators for gauge particles and graviton:
1. Commutation relations,
\[ [\hat{x}^\mu, F(\hat{x})] = 0, \]
\[ [\hat{p}_\mu, F(\hat{x})] = i \frac{\partial}{\partial x^\mu} F(\hat{x}) = \int_{R^4} i \frac{\partial}{\partial x^\mu} F(x) \hat{\epsilon}(x) d^4 x \]
\[ = \int_{R^4} p_\mu \hat{F}(p) \hat{\epsilon}(p) d^4 p \]  
(77)

2. Product between operators of gauge particles and graviton

\[ \hat{F} \hat{G} = \hat{G} \hat{F} = \int_{R^4} F(x) G(x) \hat{\epsilon}(x) d^4 x = \int_{R^4} \hat{F}(p) \hat{G}(p) \hat{\epsilon}(p) d^4 p, \]  
(78)

where the convolution takes the form

\[ \hat{F}^*(p) \hat{G}(p) = (2\pi)^{-2} \int_{R^4} \hat{F}^*(p') \hat{G}(p - p') d^4 p'. \]  
(79)

3. Trace.

\[ \text{tr} \hat{F} = \int_{R^4} F(x) d^4 x = (2\pi)^2 \hat{F}(0) \]  
(80)

4. Inner product.

\[ \langle F(\hat{x}), G(\hat{x}) \rangle = \int_{R^4} F^*(x) G(x) d^4 x = \int_{R^4} \hat{F}^*(p) \hat{G}(p) d^4 p, \]  
(81)

5. Action of operators on the state.

\[ \hat{F} |\Psi\rangle = \int_{R^4} F(x) \Psi(x) |x\rangle d^4 x = \int_{R^4} \hat{F}(p) \hat{\Psi}(p) |p\rangle d^4 p, \]  
(82)

where convolution is used. Similarly, we have equation for adjoint case,

\[ \langle \Psi |\hat{F} = \int_{R^4} \Psi^*(x) F(x) |x\rangle d^4 x = \int_{R^4} \Psi^*(p) \hat{F}(p) |p\rangle d^4 p, \]  
(83)

6. Representation space for the coordinate-momentum of gauge particles and graviton,

\[ V_{xp}(A) = V_{xp}(G) = \{ \hat{F} : \hat{F} = \int_{R^4} F(x) \hat{\epsilon}(x) d^4 x \} \]
\[ = \{ \hat{F} : \hat{F} = \int_{R^4} \hat{F}(p) \hat{\epsilon}(p) d^4 p \} \]  
(84)

7. Action multiplication of two matter fields is defined as

\[ |\Psi\rangle \circ \langle \Phi| = \int_{R^4} \Phi^*(x) \Psi(x) \hat{\epsilon}(x) d^4 x = \int_{R^4} \Phi^*(p) \Phi(p) \hat{\epsilon}(p) d^4 p. \]  
(85)

It defines the action fields. There are several properties for this action multiplication,

\[ |\Psi\rangle \circ \langle \Phi| = |\Phi\rangle \circ \langle \Psi|, \]  
(86)

\[ \text{tr}(|\Psi\rangle \circ \langle \Phi|) = \langle \Phi| \circ \langle \Psi|, \]  
(87)

\[ F(\hat{x})(|\Psi\rangle \circ \langle \Phi|) = (F(\hat{x}) |\Psi\rangle) \circ \langle \Phi| \]
\[ = |\Psi\rangle \circ \langle \Phi| F(\hat{x}) \]
\[ = (|\Psi\rangle \circ \langle \Phi|) F(\hat{x}), \]  
(88)

\[ [\hat{p}_\mu, |\Psi\rangle \circ \langle \Phi|] = (\hat{p}_\mu |\Psi\rangle) \circ \langle \Phi| - |\Psi\rangle \circ \langle \Phi| (\hat{p}_\mu). \]  
(89)
The partial derivation operator $\hat{\partial}_\mu$ is defined as,

$$\hat{\partial}_\mu = -i\tilde{\partial}_\mu. \tag{90}$$

It's action on a state and its adjoint can be calculated as,

$$\hat{\partial}_\mu |\Psi\rangle = \int_{R^4} \frac{\partial}{\partial x^\mu} \Psi(x)|x\rangle d^4x = \int_{R^4} -ip_\mu \Psi(p)|p\rangle d^4p,$$

$$\langle \Psi |\hat{\partial}_\mu = - \int_{R^4} \frac{\partial}{\partial x^\mu} \Psi^*(x)|x\rangle d^4x = \int_{R^4} -ip_\mu \tilde{\Psi}^*(p)|p\rangle d^4p. \tag{91}$$

The commutation relations for gauge particles and the graviton and the Leibniz rule are presented as,

$$[\hat{\partial}_\mu, F(\xi)] = \frac{\partial}{\partial \xi^\mu} F(\xi) = \int_{R^4} \frac{\partial}{\partial \xi^\mu} F(x) \tilde{\xi}(x) d^4x$$

$$= \int_{R^4} -ip_\mu \tilde{F}(p)\tilde{\xi}(p)d^4p \tag{92}$$

$$[\hat{\partial}_\mu, \tilde{F}\tilde{G}] = [\hat{\partial}_\mu, \tilde{F}]\tilde{G} + \tilde{F}[\hat{\partial}_\mu, \tilde{G}] \tag{93}$$

$$\hat{\partial}_\mu \langle \Psi |\tilde{F} \rangle = [\hat{\partial}_\mu, \tilde{F}] \langle \Psi | + \tilde{F} [\hat{\partial}_\mu, |\Psi\rangle, \tag{94}$$

$$\left( \langle \Psi |\tilde{F} \right) \hat{\partial}_\mu = \left( \langle \Psi |\hat{\partial}_\mu \right) \tilde{F} - \langle \Psi | [\hat{\partial}_\mu, \tilde{F}], \tag{95}$$

$$[\hat{\partial}_\mu, \langle \Phi |\Psi\rangle] = \langle \Phi | \left( \hat{\partial}_\mu \right) |\Psi\rangle - \left( \langle \Phi |\hat{\partial}_\mu \right) |\Psi\rangle = 0 \tag{96}$$

III. SPIN

A. Spin algebra

In our theory, the spin is one of the fundamental parameters of fields and operators. In 4D space-time, we use notations $s_{\alpha\beta}$, $(\alpha, \beta = 0, 1, 2, 3)$ denote the spin operators which are antisymmetric $\hat{s}_{\alpha\beta} = -\hat{s}_{\beta\alpha}$ and satisfy the commutation relation

$$[\hat{s}_{\alpha\beta}, \hat{s}_{\rho\sigma}] = -i(\eta_{\alpha\rho} \hat{s}_{\beta\sigma} - \eta_{\beta\rho} \hat{s}_{\alpha\sigma} + \eta_{\alpha\sigma} \hat{s}_{\rho\beta} - \eta_{\beta\sigma} \hat{s}_{\rho\alpha}), \tag{97}$$

where $\eta_{\alpha\beta}$ is the Minkowski metric defined as

$$\eta_{00} = 1, \quad \eta_{ij} = -\delta_{ij}, \quad \eta_{0i} = \eta_{i0} = 0, \quad \eta^{\alpha\beta} = \eta_{\alpha\beta} \tag{98}$$

The spin algebra is represented as the direct summation of two Lie $su(2)$ algebras. The new-defined 6 operators constitute a basis for spin algebra $A_i$

$$\hat{M}_i = \frac{1}{4} \varepsilon_{ijk} \hat{s}_{jk} + \frac{i}{2} \hat{s}_{0i},$$

$$\hat{N}_i = \frac{1}{4} \varepsilon_{ijk} \hat{s}_{jk} - \frac{i}{2} \hat{s}_{0i}, \tag{99}$$

where $i, j, k = 1, 2, 3, \varepsilon_{ijk}$ is the Levi-Civita symbol. $\hat{M}_i$ and $\hat{N}_i$ satisfy the commutation relations

$$[\hat{M}_i, \hat{M}_j] = i\varepsilon_{ijk} \hat{M}_k \tag{100}$$

$$[\hat{N}_i, \hat{N}_j] = i\varepsilon_{ijk} \hat{N}_k \tag{101}$$
[\hat{M}_i, \hat{N}_j] = 0. \quad (102)

One can find that three \( \hat{M}_i \) and three \( \hat{M}_i \) constitute independently a \( su(2) \) algebra, named usually as left-spin algebra and right-spin algebra, respectively. Spin algebra can be represented as the direct summation of two algebras \( su(2)_L \) and \( su(2)_R \).

\[ A_S = su(2)_L \oplus su(2)_R \quad (103) \]

The spin angular-momentum operator \( \hat{s}_{\alpha\beta} \) can be represented in terms of \( \hat{M}_i \) and \( \hat{N}_i \) as

\[ \hat{s}_{ij} = \varepsilon_{ijk}(\hat{M}_k + \hat{N}_k), \]

\[ \hat{s}_{0i} = -i(\hat{M}_i - \hat{N}_i) \quad (104) \]

**B. Irreducible representations of left and right-spin algebras**

Left-spin algebra is a \( su(2) \) algebra, the irreducible representation space is \( V_L(j_1) \), where \( j_1 = 0, \frac{1}{2}, 1, \frac{3}{2}, ..., \) the dimension is

\[ \dim V_L(j_1) = 2j_1 + 1. \quad (105) \]

The basis of the representation is denoted as \( |j_1, m_1\rangle \), \( (m_1 = -j_1, -j_1 + 1, ..., j_1) \), it is the eigenvector of the operator \( \hat{M}_3 \),

\[ \hat{M}_3 |j_1, m_1\rangle = m_1 |j_1, m_1\rangle. \quad (106) \]

Similarly for right-spin algebra, the irreducible representation space is denoted as \( V_R(j_2) \), \( (j_2 = 0, \frac{1}{2}, 1, \frac{3}{2}, ...), \) and the dimension is \( \dim V_L(j_1) = 2j_1 + 1 \). For representation basis \( |j_2, m_2\rangle \), \( (m_2 = -j_2, -j_2 + 1, ..., j_2) \), we have

\[ \hat{N}_3 |j_2, m_2\rangle = m_2 |j_2, m_2\rangle. \quad (107) \]

The irreducible representation of spin algebra \( A_S \) can be represented as the direct product of the irreducible representations of left-spin algebra and right-spin algebra,

\[ V_S(j_1, j_2) = V_L(j_1) \otimes V_R(j_2). \quad (108) \]

Apparently, the dimension of the irreducible representation is

\[ \dim V_S(j_1, j_2) = (2j_1 + 1)(2j_2 + 1). \quad (109) \]

The basis of the representation space of spin algebra can be simply taken as

\[ |j_1, m_1; j_2, m_2\rangle = |j_1, m_1\rangle \otimes |j_2, m_2\rangle. \quad (110) \]

It is known that those basis are the eigenvectors for \( \hat{s}_{12}, \hat{s}_{03} \)

\[ \hat{s}_{12} |j_1, m_1; j_2, m_2\rangle = (m_1 + m_2) |j_1, m_1; j_2, m_2\rangle, \]

\[ \hat{s}_{03} |j_1, m_1; j_2, m_2\rangle = -i(m_1 - m_2) |j_1, m_1; j_2, m_2\rangle. \quad (111) \]

**C. Spin of matter particles**

Representation space for spin of matter particles \( V_S(M) \) is the Dirac spinor space, or simply spinor-space. The spinor space can be represented as the direct summation of two irreducible representation space:

\[ V_S(M) = V_S(\frac{1}{2}, 0) \otimes V_S(0, \frac{1}{2}), \quad (112) \]

where \( V_S(\frac{1}{2}, 0) \) is the left-spin space, and \( V_R(\frac{1}{2}, 0) \) is the right-spin space. The total dimension of spinor space is

\[ \dim V_S(M) = \dim V_S(\frac{1}{2}, 0) + \dim V_R(0, \frac{1}{2}) = 4 \quad (113) \]
The basis for spin of matter particles is denoted as \(|e_s\rangle\) (s=1,2,3,4), the definition for this basis by standard basis \(|j_1, m_1; j_2, m_2\rangle\) is

\[
\begin{align*}
|e_1\rangle &= \left|\frac{1}{2}, \frac{1}{2}; 0, 0\right>, \\
|e_2\rangle &= \left|\frac{1}{2}, -\frac{1}{2}; 0, 0\right>, \\
|e_3\rangle &= |0, 0; \frac{1}{2}, \frac{1}{2}\rangle, \\
|e_4\rangle &= |0, 0; \frac{1}{2}, -\frac{1}{2}\rangle,
\end{align*}
\]

(114)

where \(|e_1\rangle, |e_2\rangle\) are the basis of left-spin space \(V_S(\frac{1}{2}, 0)\), and similarly \(|e_3\rangle, |e_4\rangle\) are the basis of right-spin space \(V_R(0, \frac{1}{2})\). The spin state of the matter particle is represented as

\[
V_S(M) = \{|\Psi\rangle: |\Psi\rangle = \Psi^*|e_s\rangle\},
\]

(115)

where \(\Psi^* = \langle e_s|\Psi\rangle\), and \(\langle e_s|\rangle\) is the adjoint of \(|e_s\rangle\).

Here the adjoint of \(|e_s\rangle\) is defined as

\[
\langle e_s| = \overline{|e_s\rangle}.
\]

(116)

The inner-product for \(|e_s\rangle\) and its adjoint is defined as

\[
\langle e_s|e_{s'}\rangle = (\gamma^0)_{ss'}.
\]

(117)

Thus the spinor metric defined by the inner product of basis, which is the parity matrix, takes the form

\[
\hat{g} = \gamma^0 = \hat{\sigma}_1 \otimes \hat{\sigma}_0 = \begin{pmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{pmatrix},
\]

(118)

where \(\sigma_0\) is the identity in 2D. Also please note the lowering or rising of the indices can be realized by the metric,

\[
\langle e^s| = g^{ss'}\langle e_{s'}| = g^{ss'}\overline{|e_s\rangle},
\]

(119)

and

\[
\langle e^s|e_{s'}\rangle = \delta^s_{s'}, \quad |e_s\rangle \otimes \langle e^s| = \hat{I}.
\]

(120)

The spin of matter particles is the spinor in spinor space \(V_S(M)\) which can be expanded in basis of spinor,

\[
|\Psi\rangle = \Psi^*|e_s\rangle,
\]

(122)

where \(\Psi^* = \langle e_s|\Psi\rangle\). The adjoint of \(|\Psi\rangle\) is \(\overline{|\Psi\rangle}\) which has the form,

\[
\langle \Psi| = \overline{|\Psi\rangle} = \Psi^*_s\langle e^s|,
\]

(123)

where \(\Psi^*_s = \langle \Psi|e_s\rangle = \langle e_s|\Psi\rangle^* = g_{ss'}\Psi^{s*}.\) The inner product of the two spinors \(|\Psi\rangle\) and \(|\Phi\rangle\) is defined as,

\[
\langle \Psi|\Phi\rangle = \langle \Phi|\Psi\rangle^* = \Psi^*_s\Phi^{s*} = g_{ss'}\Psi^{s*}\Phi^{s*}.
\]

(124)

**D. Matrix representation of the spin**

Spin operator \(\hat{s}_{\alpha\beta}\) with basis \(|e_s\rangle\) can be represented by matrix as

\[
\begin{align*}
\hat{s}_{\alpha\beta} &= (\hat{s}_{\alpha\beta})^s_{s'}|e_s\rangle \otimes \langle e^s|, \\
(\hat{s}_{\alpha\beta})^s_{s'} &= \langle e^s|\hat{s}_{\alpha\beta}|e_{s'}\rangle.
\end{align*}
\]

(125)
Explicitly, the spin angular-momentum operators are $4 \times 4$ matrices with the form:

\[
\hat{s}_{ij} = \frac{1}{2} \varepsilon_{ijk} \left( \sigma^k 0 \\ 0 \sigma^k \right),
\]
\[
\hat{s}_{0k} = -\frac{i}{2} \varepsilon_{ijk} \left( \sigma^k 0 \\ 0 -\sigma^k \right),
\]

(126)

where $\sigma^k$ are the $2 \times 2$ Pauli matrices.

The Hermitian conjugate of the spin operators are

\[
\hat{s}^\dagger_{ij} = \hat{s}_{ij},
\]
\[
\hat{s}^\dagger_{0k} = -\hat{s}_{0k}.
\]

(127)

Also we have

\[
\{\hat{g}, \hat{s}_{ij}\} = 0,
\]
\[
\{\hat{g}, \hat{s}_{0k}\} = 0.
\]

(128)

The adjoint of these operators are

\[
\bar{s}_{ij} = \hat{g}^{-1} \hat{s}_{ij} \hat{g} = \hat{s}_{ij},
\]
\[
\bar{s}_{0k} = \hat{g}^{-1} \hat{s}_{0k} \hat{g} = \hat{s}_{0k},
\]

(129)

those equation can be written as a concise form

\[
\bar{s}_{\alpha\beta} = \hat{g}^{-1} \hat{s}_{\alpha\beta} \hat{g} = \hat{s}_{\alpha\beta}.
\]

(130)

The chiral charge $\hat{\gamma}^5$ in basis $|e_s\rangle$ is defined as

\[
\hat{\gamma}^5 = \left( \begin{array}{cc} \hat{\sigma}_0 & 0 \\ 0 & \hat{\sigma}_0 \end{array} \right).
\]

(131)

We also have the properties

\[
(\hat{\gamma}^5)^\dagger = \hat{\gamma}^5,
\]
\[
\bar{\gamma}^0 = \hat{g}^{-1} (\hat{\gamma}^5)^\dagger \hat{g} = -\hat{\gamma}^5,
\]
\[
(\hat{\gamma}^0)^\dagger = \hat{\gamma}^0,
\]
\[
\bar{\gamma}^0 = \hat{g}^{-1} (\hat{\gamma}^0)^\dagger \hat{g} = \hat{\gamma}^0
\]

(132)

The left-chiral projector and the right-chiral projector are defined as

\[
\hat{P}_L = \frac{1}{2} (1 + \hat{\gamma}^5),
\]
\[
\hat{P}_R = \frac{1}{2} (1 - \hat{\gamma}^5),
\]

(133)

These two operators project a 4D space to 2D chiral left or right spaces, respectively, and they have the properties,

\[
\hat{P}_L + \hat{P}_R = 1,
\]
\[
\hat{P}_L \hat{P}_R = \hat{P}_L,
\]
\[
\hat{P}_R \hat{P}_L = \hat{P}_R,
\]
\[
\hat{P}_L \hat{P}_R = \hat{P}_R \hat{P}_L = 0.
\]

(134)
E. Dirac matrices

There are 16 Dirac matrices with $4 \times 4$ acting on 4D space. Those Dirac matrices are divided into 5 groups. 1 rank-0 antisymmetric metric $\hat{\gamma}$, 4 rank-1 antisymmetric metrics $\hat{\gamma}^{\alpha}$, 6 rank-2 antisymmetric metrics $\hat{\gamma}^{\alpha_1 \alpha_2}$, 4 rank-3 antisymmetric metrics $\hat{\gamma}^{\alpha_1 \alpha_2 \alpha_3}$ and 1 rank-4 antisymmetric metric $\hat{\gamma}^{\alpha_1 \alpha_2 \alpha_3 \alpha_4}$. Those 16 matrices can be represented as the following by basis $|e_s\rangle$,

\[ \hat{\gamma} = \sigma_0 \otimes \sigma_0 = \begin{pmatrix} \hat{\sigma}_0 & 0 \\ 0 & \hat{\sigma}_0 \end{pmatrix}, \]

\[ \hat{\gamma}^0 = \sigma_1 \otimes \sigma_0 = \begin{pmatrix} 0 & \hat{\sigma}_0 \\ \hat{\sigma}_0 & 0 \end{pmatrix}, \]

\[ \hat{\gamma}^i = i\sigma_2 \otimes \sigma_i = \begin{pmatrix} 0 & \hat{\sigma}_i \\ -\hat{\sigma}_i & 0 \end{pmatrix}, \]

\[ \hat{\gamma}^{0i} = i\sigma_3 \otimes \sigma_i = \begin{pmatrix} \hat{\sigma}_i & 0 \\ 0 & -\hat{\sigma}_i \end{pmatrix}, \]

\[ \hat{\gamma}^{ij} = \varepsilon_{ijk}\sigma_0 \otimes \sigma_k = \varepsilon_{ijk} \begin{pmatrix} \hat{\sigma}_k & 0 \\ 0 & \hat{\sigma}_k \end{pmatrix}, \]

\[ \hat{\gamma}^{0ij} = \varepsilon_{ijk}\sigma_1 \otimes \sigma_k = \varepsilon_{ijk} \begin{pmatrix} 0 & \hat{\sigma}_k \\ \hat{\sigma}_k & 0 \end{pmatrix}, \]

\[ \hat{\gamma}^{123} = -i\sigma_2 \otimes \sigma_0 = \begin{pmatrix} 0 & -\hat{\sigma}_0 \\ \hat{\sigma}_0 & 0 \end{pmatrix}, \]

\[ \hat{\gamma}^{0123} = -i\sigma_3 \otimes \sigma_0 = -i \begin{pmatrix} \hat{\sigma}_0 & 0 \\ 0 & -\hat{\sigma}_0 \end{pmatrix}, \]

where $\hat{\sigma}_0$ is a $2 \times 2$ identity, $\hat{\sigma}_i$, $(i=1,2,3)$ are Pauli matrices.

We next introduce an unified notations to denote Dirac matrices $\hat{\gamma}^\alpha$,

\[ \hat{\gamma}^{\alpha_1 \cdots \alpha_p} = \frac{1}{2^p p!} {\delta^\alpha_{\rho_1 \cdots \rho_p}} \hat{\gamma}_{\rho_1 \cdots \rho_p}, \tag{136}\]

where $p = 0, 1, 2, 3, 4$, $\delta^\alpha_{\rho_1 \cdots \rho_p}$ are the generalized Kronecker matrices defined as

\[ \delta^\alpha_{\rho_1 \cdots \rho_p} = \begin{vmatrix} \delta^\alpha_{\rho_1} & \cdots & \delta^\alpha_{\rho_p} \\ \vdots & \ddots & \vdots \\ \delta^\alpha_{\rho_p} & \cdots & \delta^\alpha_{\rho_p} \end{vmatrix} \tag{137}\]

It will be 1 when the upper indices and the lower indices are even permutations, -1 for odd permutations, and 0 for other cases.

The Dirac matrices with upper indices are called inversion Dirac matrices. By spin metric $\eta_{\alpha \beta}$, the covariance Dirac matrices can be defined as

\[ \hat{\gamma}^{\alpha_1 \cdots \alpha_p} = \eta_{\alpha_1 \beta_1} \cdots \eta_{\alpha_p \beta_p} \hat{\gamma}^{\beta_1 \cdots \beta_p}, \tag{138}\]

Compared with the representation matrices of $\hat{s}_{\alpha \beta}$, we can find

\[ \hat{s}_{\alpha \beta} = \frac{1}{2} \eta_{\alpha \rho} \eta_{\beta \sigma} \hat{\gamma}^{\rho \sigma} = \frac{1}{2} \hat{\gamma}_{\alpha \beta}. \tag{139}\]

We next present some properties of the matrices $\hat{\gamma}$:

1. Anti-commuting relation

\[ \hat{\gamma}^\alpha \hat{\gamma}^\beta + \hat{\gamma}^\beta \hat{\gamma}^\alpha = 2 \eta^{\alpha \beta} \hat{I} \tag{140}\]

2. Commuting relation

\[ \hat{\gamma}^\alpha \hat{\gamma}^\beta - \hat{\gamma}^\beta \hat{\gamma}^\alpha = -4i \delta^{\alpha \beta} \tag{141}\]
3. Product relation

\[\hat{\gamma}^\alpha \hat{\gamma}^\beta = g^{\alpha \beta} - 2i\hat{s}^{\alpha \beta},\]

(142)

4. Unitary relation

\[(\hat{\gamma}^\alpha)^\dagger = (\hat{\gamma}^\alpha)^{-1} = \hat{\gamma}_\alpha = \eta_{\alpha \beta} \hat{\gamma}^\beta\]

(143)

5. Adjoint matrices

\[\overline{\gamma}^\alpha = \gamma^0 (\gamma^\alpha)^\dagger (\gamma^0)^{-1} = -\gamma^\alpha\]

(144)

6. Commuting relation with spin operator \(\hat{s}_{\alpha \beta}\),

\[\commutator{\hat{s}_{\alpha \beta}}{\hat{\gamma}^\gamma} = i(\delta^\gamma_\beta \hat{\gamma}_\alpha - \delta^\gamma_\alpha \hat{\gamma}_\beta) = i\eta_{\rho \tau} \delta_{\alpha \beta}^\rho \delta_{\gamma}^\tau.\]

(145)

7. Trace is zero

\[\text{tr} \hat{\gamma}^\alpha = 0.\]

(146)

8. Orthogonal condition

\[\langle \hat{\gamma}_\alpha, \hat{\gamma}^\beta \rangle = \text{tr} [\gamma_\alpha \gamma^\beta] = -4\delta^\beta_\alpha.\]

(147)

The properties of the Dirac matrices:

1. Unitary

\[(\hat{\gamma}^{\alpha_1 \cdots \alpha_p})^\dagger = (\hat{\gamma}^{\alpha_1 \cdots \alpha_p})^{-1} = \hat{\gamma}_{\alpha_1 \cdots \alpha_p} = \eta_{\alpha_1 \beta_1} \cdots \eta_{\alpha_p \beta_p} \hat{\gamma}^\beta_1 \cdots \hat{\gamma}^\beta_p.\]

(148)

2. Adjoint matrices

\[\overline{\gamma}^{\alpha_1 \cdots \alpha_p} = \gamma^0 (\gamma^{\alpha_1 \cdots \alpha_p})^\dagger (\gamma^0)^{-1} = (-1)^p \gamma^{\alpha_1 \cdots \alpha_p},\]

(149)

\[\overline{\gamma}_{\alpha_1 \cdots \alpha_p} = \gamma^0 (\gamma_{\alpha_1 \cdots \alpha_p})^\dagger (\gamma^0)^{-1} = (-1)^p \gamma_{\alpha_1 \cdots \alpha_p}.\]

(150)

3. Commutation relation with spin \(\hat{s}_{\alpha \beta}\),

\[\commutator{\hat{s}_{\alpha \beta}}{\hat{\gamma}^{\alpha_1 \cdots \alpha_p}} = i\eta_{\rho \tau} \delta^{\rho \sigma}_{\alpha \beta} (\delta_\sigma^{\alpha_1} \hat{\gamma}^{\tau \cdots \alpha_p} + \cdots + \delta_\sigma^{\alpha_p} \hat{\gamma}^{\alpha_1 \cdots \tau}),\]

(151)

\[\commutator{\hat{s}_{\alpha \beta}}{\hat{\gamma}_{\alpha_1 \cdots \alpha_p}} = -i\eta_{\rho \tau} \delta^{\rho \sigma}_{\alpha \beta} (\delta_\sigma^{\alpha_1} \hat{\gamma}_{\tau \cdots \alpha_p} + \cdots + \delta_\tau^{\alpha_p} \hat{\gamma}_{\alpha_1 \cdots \tau}).\]

(152)

4. Trace

\[\text{tr} \hat{\gamma} = 4, \quad \text{tr} \hat{\gamma}^{\alpha_1 \cdots \alpha_p} = 0, \quad (p \neq 0)\]

(153)

5. Orthogonality

\[\langle \hat{\gamma}_{\alpha_1 \cdots \alpha_p}, \hat{\gamma}^{\beta_1 \cdots \beta_q} \rangle = \text{tr} [\overline{\gamma}_{\alpha_1 \cdots \alpha_p} \overline{\gamma}^{\beta_1 \cdots \beta_q}] = (-1)^p 4\delta_{\rho \tau} \delta^{\alpha_1 \cdots \alpha_p} \delta_{\beta_1 \cdots \beta_q}.\]

(154)
F. Tensor representation of the spin algebra

The tensor representation of the spin algebra can be listed from rank-0 to rank-4, we then use a general form to summarize these representations.

The rank-0 anti-symmetric tensor is actually a scalar representation, the representation space is denoted as \( AT_S(0) = V_S(0, 0) \), the dimension of the space is \( \dim[AT_S(0)] = \dim[V_S(0, 0)] = 1 \). The representation basis is \( \hat{\gamma} \), so the 0-order antisymmetric tensor is represented as

\[
\hat{K} = K \hat{\gamma}
\]  

Similarly for rank-1, we have

\[
AT_S(1) = V_S(\frac{1}{2}, \frac{1}{2})
\]

\[
\dim[AT_S(1)] = \dim[V_S(\frac{1}{2}, \frac{1}{2})] = 4
\]

\[
\hat{K} = K_\alpha \hat{\gamma}^\alpha.
\]

And further for rank-2, the results are,

\[
AT_S(2) = A_S = V_S(0, 1) \oplus V_S(1, 0)
\]

\[
\dim[AT_S(2)] = \dim[A_S] = \dim[V_S(0, 1) \oplus V_S(1, 0)] = 6
\]

\[
\hat{K} = \frac{1}{2!} K_{\alpha\beta} \hat{\gamma}^{\alpha\beta}.
\]

The rank-3 results are,

\[
AT_S(3) = V_S(\frac{1}{2}, \frac{1}{2})
\]

\[
\dim[AT_S(3)] = \dim[V_S(\frac{1}{2}, \frac{1}{2})] = 4
\]

\[
\hat{K} = \frac{1}{3!} K_{\alpha\beta\gamma} \hat{\gamma}^{\alpha\beta\gamma}.
\]

And for rank-4, we have

\[
AT_S(4) = V_S(0, 0)
\]

\[
\dim[AT_S(4)] = \dim[V_S(0, 0)] = 1
\]

\[
\hat{K} = \frac{1}{4!} K_{\alpha\beta\gamma\delta} \hat{\gamma}^{\alpha\beta\gamma\delta}.
\]

The summarized form for rank-\( p \) tensor can be written in the following, the space is \( AT_S(p) \),

\[
\dim[AT_S(p)] = \frac{4!}{p!(4-p)!}
\]

\[
\hat{K} = \frac{1}{p!} K_{\alpha_1...\alpha_p} \hat{\gamma}^{\alpha_1...\alpha_p}.
\]

The anti-symmetric tensor representation of the spin algebra can be represented as a direct summation of the rank-0 to rank-4 anti-symmetric tensor.

\[
AT_S = \bigoplus_{p=0}^{4} AT_S(p).
\]

The dimension of the space can be calculated as

\[
\dim[AT_S] = \sum_{p=0}^{4} \dim[AT_S(p)] = \sum_{p=0}^{4} \frac{4!}{p!(4-p)!} = 16
\]
The representation space is the direct summation of spinor space and the adjoint spinor space:

\[ AT_S = V_S \otimes \overline{V}_S \]  \hspace{1cm} (172)

We should note that the space for spin of the gauge particles is the 16D space for spin antisymmetric tensor,

\[ V_S(A) = AT_S. \]  \hspace{1cm} (173)

G. Mixed tensor representation of the spin algebra

The \((p,q)\) mixed tensor representation of the spin algebra is the direct product of the rank-\(p\) and rank-\(q\) antisymmetric tensor representations,

\[ MT_S(p, q) = AT_S(p) \otimes AT_S(q). \]  \hspace{1cm} (174)

The dimension of the representation space is

\[ \dim[MT_S(p, q)] = \dim[AT_S(p)] \times \dim[AT_S(q)] = \frac{4!4!}{p!(4-p)!q!(4-q)!}. \]  \hspace{1cm} (175)

The basis of the representation is

\[ \hat{\gamma}^{\alpha_1...\alpha_p} = \hat{\gamma}^{\alpha_1...\alpha_p} \otimes \hat{\gamma}_{\beta_1...\beta_q}, \]  \hspace{1cm} (176)

which is a \(16 \times 16\) matrix. The commutation relation takes the form

\[ [\hat{s}_{\alpha\beta}, \hat{\gamma}^{\alpha_1...\alpha_p}] = i\eta_{\rho\sigma} \hat{\delta}_{\alpha\beta}^{\sigma\rho}(\hat{\gamma}^{\alpha_1...\alpha_p\sigma} + \cdot \cdot \cdot + \hat{\delta}_{\alpha\beta}^{\sigma\rho}(\hat{\gamma}^{\alpha_1...\alpha_p\sigma} - \hat{\gamma}^{\alpha_1...\alpha_p\sigma} - \cdot \cdot \cdot)), \]  \hspace{1cm} (177)

This \((p, q)\) mixed tensor representation includes all tensor representation of spin algebra, while rank-\(p\) antisymmetric tensor representation can be dealt as a \((p, 0)\) or \((0, p)\) form mixed tensor representation. The \((p, q)\) mixed tensor takes the form

\[ \hat{K} = \frac{1}{p!q!} K^{\beta_1...\beta_q}_{\alpha_1...\alpha_p} \hat{\gamma}^{\alpha_1...\alpha_p} \otimes \hat{\gamma}_{\beta_1...\beta_q}. \]  \hspace{1cm} (178)

The mixed tensor representation of the spin algebra is represented as the direct summation of all \((p, q)\) mixed tensor representations,

\[ MT_S = \sum_{p,q=0}^{4} \oplus MT_S(p, q). \]  \hspace{1cm} (179)

The dimension is

\[ \dim[MT_S] = \sum_{p,q=0}^{4} \dim[MT_S(p, q)] = \sum_{p,q=0}^{4} \frac{4!4!}{p!(4-p)!q!(4-q)!} = 256. \]  \hspace{1cm} (180)

The basis is \(\hat{\gamma}^{\alpha_1...\alpha_p}_{\beta_1...\beta_q}\), \((p, q = 0, 1, 2, 3, 4)\). The mixed tensor space is a direct summation of two antisymmetric tensor space,

\[ V_S(G) = MT_S \]  \hspace{1cm} (181)

We should note that the space for spin of the gravity particle is the 256D mixed spin tensor space.

\[ V_S(G) = MT_S \]  \hspace{1cm} (182)
H. Some fundamental calculations of the spin tensor

1. Exterior product of the antisymmetric spin tensor. Similar as in differential geometry, the exterior product of two basis can be defined as

\[ \gamma^{\alpha_1 \cdots \alpha_p} \wedge \gamma^{\beta_1 \cdots \beta_q} = \gamma^{\alpha_1 \cdots \alpha_p \beta_1 \cdots \beta_q} \]  

(183)

So the exterior product of two antisymmetric tensors will take the form

\[ \hat{K} \wedge \hat{H} = \frac{1}{pl!} K_{\alpha_1 \cdots \alpha_p} \gamma^{\alpha_1 \cdots \alpha_p} \wedge \frac{1}{ql!} H_{\beta_1 \cdots \beta_q} \gamma^{\beta_1 \cdots \beta_q} = \frac{1}{pl!q!} K_{\alpha_1 \cdots \alpha_p} H_{\beta_1 \cdots \beta_q} \gamma^{\alpha_1 \cdots \alpha_p \beta_1 \cdots \beta_q} \]  

(184)

Thus the exterior product of a \( p \)-form antisymmetric tensor and a \( q \)-form antisymmetric tensor is \( p + q \)-order antisymmetric tensor. The 16D antisymmetric spin tensor space is closed for exterior product.

2. Tensor product for the antisymmetric spin tensor.

The tensor product obeys the following role,

\[ \gamma^{\alpha_1 \cdots \alpha_p} \otimes \gamma_{\beta_1 \cdots \beta_q} = \gamma^\alpha_{\beta_1 \cdots \beta_q} \]  

(185)

\[ \hat{K} \otimes \hat{H} = \frac{1}{pl!} K_{\alpha_1 \cdots \alpha_p} \gamma^{\alpha_1 \cdots \alpha_p} \otimes \frac{1}{ql!} H^{\beta_1 \cdots \beta_q} \gamma_{\beta_1 \cdots \beta_q} = \frac{1}{pl!q!} K_{\alpha_1 \cdots \alpha_p} H^{\beta_1 \cdots \beta_q} \gamma_{\alpha_1 \cdots \alpha_p \beta_1 \cdots \beta_q} \]  

(186)

3. Scalar product of rank-1 spin tensor. The scalar product of two rank-1 Dirac matrices is defined as,

\[ \gamma^\alpha \gamma^\beta = \eta^{\alpha \beta} \]  

(187)

It is known that \( \eta^{\alpha \beta} \) is the Minkowski vierbein, also named as free vierbein, so \( \gamma^\alpha \) are orthogonal. The scalar product of two 1-form spin tensor is written as:

\[ K \cdot L = \eta^{\alpha \beta} K_\alpha L_\beta \]  

(188)

4. Contraction calculation for mixed tensor. With the form of free vierbein \( \eta^{\alpha \beta} \), we can define the contraction calculation for mixed tensor. First, the contraction of the \((p, q)\) mixed tensor basis \((p, q \geq 1)\) is defined as

\[ \text{con} \gamma^{\alpha_1 \cdots \alpha_p} = \delta^{\alpha_p}_{\beta_1} \gamma^{\alpha_1 \cdots \alpha_{p-1}} \]  

(189)

So we can define the contraction of \((p, q)\) mixed tensor, \((p, q \geq 1)\),

\[ \text{con} \hat{K} = \text{con} \left( \frac{1}{pl!} K_{\alpha_1 \cdots \alpha_p} \gamma^{\alpha_1 \cdots \alpha_p} \right) = \left( \frac{1}{pl!q!} K_{\alpha_1 \cdots \alpha_p} \delta^{\alpha_p}_{\beta_1} \gamma^{\alpha_1 \cdots \alpha_{p-1}} \right) \]  

(190)

After the contraction calculation, the \((p, q)\) mixed tensor is reduced as a \((p-1, q-1)\)-tensor.

5. Lowering and rising of the indices of the spin tensor. By using the product of the inversion Minkowski vierbein \( \eta^{\alpha \beta} \) and covariance Minkowski vierbein \( \eta_{\alpha \beta} \), we can realize the lowering and rising of the indices for spin tensor, we use the antisymmetric tensor as the examples,

\[ \gamma_{\alpha_1 \cdots \alpha_p} = \eta_{\alpha_1 \beta_1} \cdots \eta_{\alpha_p \beta_p} \gamma^{\beta_1 \cdots \beta_p} \]  

(191)

\[ K^{\alpha_1 \cdots \alpha_p} = \eta^{\alpha_1 \beta_1} \cdots \eta^{\alpha_p \beta_p} K_{\beta_1 \cdots \beta_p} \]  

(192)

\[ \hat{K} = \frac{1}{pl!} K_{\alpha_1 \cdots \alpha_p} \gamma^{\alpha_1 \cdots \alpha_p} = \frac{1}{pl!} K^{\alpha_1 \cdots \alpha_p} \gamma_{\alpha_1 \cdots \alpha_p} \]  

(193)

For lowering or rising of the indices, the rank of the indices remains unchanged. One spin tensor may have different indices representation, they may be used for different purposes.
IV. GAUGE CHARGES

As we pointed out that gauge is one fundamental parameter in our theory, we next consider the properties of gauge charges. Also we would like to emphasize that mass matrices are defined in gauge space in our theory. The conventional gauge theory generally satisfies the gauge invariant condition. Our unified theory satisfies the general covariance condition which releases the previous restriction. Thus mass arises naturally.

What we study is the all elementary particles and the interactions between them. The elementary particles are divided into three families as matter particles (Ψ), gauge particles (A) and graviton (g). They satisfy Dirac equation, Yang-Mills equation and Einstein equation, respectively. In this work, we will present an unified quantum theory for those equations

We can define square of the isospin charges ˆI (1) and color charges ˆλ, (p = 1, 2, ..., 8). Hypercharge ˆY is the generator of gauge group U(1). Three isospin charges ˆI_i, (i = 1, 2, 3) are generators of gauge group SU(2), they constitute a basis for algebra su(2), and eight color charges ˆλ_p, (p = 1, 2, ..., 8) are generators of gauge group SU(3) and constitute a set of basis.

Those generators satisfy the commutation relations

\[ [\hat{I}_i, \hat{I}_j] = i\epsilon_{ijk} \hat{I}_k, \]  
\[ [\hat{\lambda}_p, \hat{\lambda}_q] = i f_{pq}^{\tau} \hat{\lambda}_\tau, \]  
\[ [\hat{Y}, \hat{I}_i] = [\hat{Y}, \hat{\lambda}_p] = [\hat{I}_i, \hat{\lambda}_p] = 0, \]

where \( \epsilon_{ijk} \) is Levi-Civita symbol and is the structure constant for algebra su(2), \( f_{pq}^{\tau} \) are structure constants of algebra su(3) group. They are completely symmetric for three indices, the non-zero elements are

\[ f_{12}^{\lambda} = 1, \]  
\[ f_{14}^{\lambda} = -f_{15}^{\lambda} = f_{24}^{\lambda} = f_{25}^{\lambda} = f_{34}^{\lambda} = -f_{36}^{\lambda} = 1, \]  
\[ f_{45}^{\lambda} = f_{67}^{\lambda} = \frac{\sqrt{3}}{2}. \]

We can define square of the isospin charges \( \hat{I}^2 \) and the color charges \( \hat{\lambda}^2 \):

\[ \hat{I}^2 = \sum_{i=1}^{3} \hat{I}_i^2, \]  
\[ \hat{\lambda}^2 = \sum_{p=1}^{8} \hat{\lambda}_p^2. \]

It can be checked that \( \hat{I}^2 \) and \( \hat{\lambda}^2 \) commute with all elements of gauge charge

\[ [\hat{I}^2, \hat{Y}] = [\hat{I}^2, \hat{I}_i] = [\hat{I}^2, \hat{\lambda}_p] = 0, \]  
\[ [\hat{\lambda}^2, \hat{Y}] = [\hat{\lambda}^2, \hat{I}_i] = [\hat{\lambda}^2, \hat{\lambda}_p] = 0, \]
We use notation $\hat{t}_a, (a = 1, 2, ..., 12)$ to represent those 12 gauge charges as

$$\hat{t}_1 = \frac{\hat{Y}}{2},$$

$$\hat{t}_{1+i} = g_2 \hat{I}_i, \quad (i = 1, 2, 3),$$

$$\hat{t}_{4+p} = g_3 \hat{\lambda}_p, \quad (p = 1, 2, ..., 8),$$

where $g_1, g_2$ and $g_3$ are coefficients of hypercharge, isospin charges and color charges, respectively. So the commutation relations take a concise form as

$$[\hat{t}_a, \hat{t}_b] = id^{c}_{ab} \hat{t}_c,$$

where coefficients $d^{c}_{ab}$ are defined as

$$d^{1+k}_{1+i,1+j} = g_2 \epsilon_{ijk}, \quad (i, j, k = 1, 2, 3)$$

$$d^{4+r}_{4+p,4+q} = g_3 f^{r}_{pq}, \quad (p, q, r = 1, 2, ..., 8)$$

and $d^{c}_{ab} = 0$ elsewhere. Since $\epsilon_{ijk}$ and $f^{r}_{pq}$ are completely antisymmetric, $d^{c}_{ab}$ are also completely antisymmetric

$$d^{c}_{ab} = d^{c}_{ba} = d^{b}_{ac} = -d^{a}_{cb} = -d^{b}_{ca}$$

Those gauge charges $\hat{t}_a$ can be denoted as gauge bosons in Cartan-Weyl basis $\hat{T}_a, (a=1,2,...,12)$,

$$\hat{T}_1 = \hat{t}_1 \cos \theta_w + \hat{t}_4 \sin \theta_w,$$

$$\hat{T}_2 = -\hat{t}_1 \sin \theta_w + \hat{t}_4 \cos \theta_w,$$

$$\hat{T}_3 = \frac{1}{\sqrt{2}} (\hat{t}_2 + i \hat{t}_3),$$

$$\hat{T}_4 = \frac{1}{\sqrt{2}} (\hat{t}_2 - i \hat{t}_3),$$

$$\hat{T}_5 = \hat{t}_7,$$

$$\hat{T}_6 = \hat{t}_{12},$$

$$\hat{T}_7 = \frac{1}{\sqrt{2}} (\hat{t}_5 + i \hat{t}_6),$$

$$\hat{T}_8 = \frac{1}{\sqrt{2}} (\hat{t}_5 - i \hat{t}_6),$$

$$\hat{T}_9 = \frac{1}{\sqrt{2}} (\hat{t}_8 + i \hat{t}_9),$$

$$\hat{T}_{10} = \frac{1}{\sqrt{2}} (\hat{t}_8 - i \hat{t}_9),$$

$$\hat{T}_{11} = \frac{1}{\sqrt{2}} (\hat{t}_{10} + i \hat{t}_{11}),$$

$$\hat{T}_{12} = \frac{1}{\sqrt{2}} (\hat{t}_{10} - i \hat{t}_{11}),$$

where $\theta_w$ is the Weinberg angle. The transformation from orthogonal gauge charges $\hat{t}_a$ to eigen-gauge charges $\hat{T}_a$ is unitary,

$$\hat{T}_a = L^b_a \hat{t}_b,$$

as we can find that the transformation matrix satisfy

$$L^{-1} = L^\dagger,$$

where super-indices $\dagger$ means the hermitian conjugation (complex conjugation plus matrix transposition). So the inverse transformation takes the form

$$\hat{t}_a = L^{b\dagger}_a \hat{T}_b.$$
The eigen-gauge charges satisfy the relation,

\[ [\hat{T}_a, \hat{T}_b] = C^{c}_{ab} \hat{T}_c. \]  

(213)

The coefficients can be found to be

\[
C^3_{1,3} = -C^4_{1,4} = C^4_{3,4} = g_2 \sin \theta_w, \quad C^3_{2,3} = -C^4_{2,4} = C^2_{3,4} = g_2 \cos \theta_w,
\]

\[
C^7_{5,7} = -C^8_{5,8} = C^5_{7,8} = g_3, \quad C^7_{6,7} = -C^8_{6,8} = C^6_{7,8} = 0,
\]

\[
C^9_{5,9} = -C^{10}_{5,10} = C^5_{9,10} = -\frac{1}{2}g_3, \quad C^0_{6,9} = -C^{10}_{6,10} = C^6_{9,10} = \frac{\sqrt{3}}{2}g_3,
\]

\[
C^{11}_{5,11} = -C^{12}_{5,12} = C^5_{11,12} = -\frac{1}{2}g_3, \quad C^{11}_{6,11} = -C^{12}_{6,12} = C^6_{11,12} = -\frac{\sqrt{3}}{2}g_3,
\]

\[
C^{12}_{7,9} = -C^9_{7,11} = C^{10}_{11,7} = \frac{\sqrt{2}}{2}g_3, \quad C^{11}_{8,10} = C^{10}_{11,12} = C^9_{12,8} = -\frac{\sqrt{2}}{2}g_3.
\]

(214)

Those 12 gauge charges \( \hat{t}_a, a = 1, \cdots, 12 \), constitute the gauge algebra,

\[ A_g = \{ \hat{Z}_g : \hat{Z}_g = \theta^a \hat{t}_a \} = u(1) \oplus su(2) \oplus su(3). \]  

(215)

The gauge algebra is a 12-dimension Lie algebra, it is constituted by direct summation of an Abel algebra \( u(1) \) and two simple Lie algebras \( su(2) \) and \( su(3) \).

Consequently, corresponding to gauge algebra \( A_g \), we have the gauge group \( G_g \) which can be obtained by the exponential of the gauge algebra,

\[ G_g = \{ \hat{U}_g : \hat{U}_g = \exp(i \theta_a \hat{t}_a) \} = U(1) \otimes SU(2) \otimes SU(3). \]  

(216)

For gauge algebra \( A_g \), we introduce the gauge metric tensor:

\[ \hat{G} = g^{ab} \hat{t}_a \otimes \hat{t}_b = G^{ab} \hat{T}_a \otimes \hat{T}_b. \]  

(217)

Gauge metric tensor is a generalization of Lie algebra Cartan metric tensor. For simplicity, we introduce the orthonormal metric for orthogonal gauge charges \( \hat{t}_a \),

\[ g_{ab} = g^{ab} = \delta_{ab}. \]  

(218)

Due to the transformation for \( \hat{t}_a \) to \( \hat{T}_a \) and the explicit form of \( g^{ab} \), we can find that the gauge metric tensors \( G^{ab} \) and \( G_{ab} \) take the form

\[
G^{ab} = G_{ab}, \\
G_{1,1} = G_{2,2} = G_{3,4} = G_{4,3} = 1, \\
G_{5,5} = G_{6,6} = G_{7,8} = G_{8,7} = G_{9,10} = G_{10,9} = G_{11,12} = G_{12,11} = 1, \\
G_{ab} = 0, \text{ elsewhere.}
\]

(219)

According to the gauge metric tensor, the scalar product of the gauge charges are defined as:

\[ \hat{t}_a \cdot \hat{t}_b = \hat{t}_b \cdot \hat{t}_a = g_{ab} = \delta_{ab}; \]

\[ \hat{T}_a \cdot \hat{T}_b = \hat{T}_b \cdot \hat{T}_a = G_{ab} \]

\[ \hat{Y} \cdot \hat{Y} = 4g_{1}^{-2}, \]

\[ \hat{I}_i \cdot \hat{I}_j = g_{2}^{-2} \delta_{ij}, \]

\[ \hat{\lambda}_p \cdot \hat{\lambda}_q = g_{3}^{-2} \delta_{pq}, \]

\[ \hat{Y} \cdot \hat{I}_i = \hat{Y} \cdot \hat{\lambda}_p = \hat{I}_i \cdot \hat{\lambda}_p = 0. \]

(220)

The rising and lowering the indices by the gauge metric tensor can be written as, for example,

\[ \hat{T}^a = g^{ab} \hat{t}_b = \hat{t}_a, \]

\[ \hat{T}^a = G^{ab} \hat{T}_b. \]  

(221)
B. Irreducible representation of the gauge algebra

The irreducible representation space of algebra $u(1)$ is denoted as $V_1(Y)$, where $Y$ is the eigenvalue of the hypercharge $Y$. Here $Y$ can be arbitrary real number, and the dimension of space $V_1(Y)$ is,

$$\dim V_1(Y) = 1.$$ (222)

We denote the irreducible representation space of algebra $su(2)$ as $V_2(I)$. $I$ is the maximal eigenvalue of the third isospin operator $I_3$. $I$ can be non-negative integer and half-integer. The dimension of space $V_2(I)$ is

$$\dim V_2(I) = 2I + 1.$$ (223)

Similarly the space of the irreducible representation of algebra $su(3)$ is denoted as $V_3(m,n)$, where $m,n$ are non-negative. The dimension of space $V_3(m,n)$ is

$$\dim V_3(m,n) = \frac{1}{2}(m+1)(n+1)(m+n+2).$$ (224)

Gauge algebra $A_g$ is the direction summation of algebras $u(1)$, $su(2)$ and $su(3)$, and its irreducible representation is the tensor product of the irreducible representations of $u(1)$, $su(2)$ and $su(3)$. Denote the irreducible representation space of algebra $A_g$ as $V_g(Y, I, m, n)$ and it takes the form

$$V_g(Y, I, m, n) = V_1(Y) \otimes V_2(I) \otimes V_3(m, n).$$ (225)

Thus the dimension of space $V_g(Y, I, m, n)$ is

$$\dim V_g(Y, I, m, n) = \dim V_1(Y) \times \dim V_2(I) \times \dim V_3(m, n) = \frac{1}{2}(2I+1)(m+1)(n+1)(m+n+2)$$ (226)

C. Gauge representations of the matter particles, representation spaces and gauge states

Matter particles include leptons and quarks. We next consider their gauge representations respectively.

(i). Space of the gauge representations for leptons. Leptons can be divided into three generations, the gauge representation space for each generation is the same. The representation space is

$$V_g(-1, \frac{1}{2}, 0, 0) \oplus V_g(0, 0, 0, 0) \oplus V_g(-2, 0, 0, 0).$$ (227)

The dimension for each generation of leptons in the gauge representation is $2+1+1 = 4$. It is corresponding to the fact that each generation of leptons includes four states as: isospin doublet state for leptons with electric-charge, isospin singlet state of leptons with electric-charge, isospin doublet state of neutrino and isospin singlet state of neutrino.

Gauge representation space for all leptons can be denoted as the direct summation of the gauge representation spaces for three generations of lepton:

$$V_g(l) = [V_g(-1, \frac{1}{2}, 0, 0) \oplus V_g(0, 0, 0, 0) \oplus V_g(-2, 0, 0, 0)] \oplus [V_g(-1, \frac{1}{2}, 0, 0) \oplus V_g(0, 0, 0, 0) \oplus V_g(-2, 0, 0, 0)] \oplus [V_g(-1, \frac{1}{2}, 0, 0) \oplus V_g(0, 0, 0, 0) \oplus V_g(-2, 0, 0, 0)]$$ (228)

The total dimension is

$$\dim V_g(l) = (2+1+1) \times 3 = 12.$$ (229)

The total dimension 12 corresponds to 12 different leptons.

(ii). Space of the gauge representations for quarks. Quarks also have three generations, the representation space for each generation of quarks is the same. Each generation can be represented as

$$V_g(\frac{1}{3}, \frac{1}{2}, 1, 0) \oplus V_g(-\frac{2}{3}, 0, 1, 0).$$ (230)
One can check that the dimension is \(6 + 3 + 3 = 12\), it corresponds to that there are 12 gauge states for each generation of quark.

The gauge representation space for three generations of quarks is denoted as the direction summation as

\[
V_g(q) = [V_g(\frac{1}{3}, \frac{1}{2}, 1, 0) \oplus V_g(\frac{4}{3}, 0, 1, 0) \oplus V_g(-\frac{2}{3}, 0, 1, 0)] \\
\oplus[V_g(\frac{1}{3}, \frac{1}{2}, 1, 0) \oplus V_g(\frac{4}{3}, 0, 1, 0) \oplus V_g(-\frac{2}{3}, 0, 1, 0)] \\
\oplus V_g(\frac{1}{3}, \frac{1}{2}, 1, 0) \oplus V_g(\frac{4}{3}, 0, 1, 0) \oplus V_g(-\frac{2}{3}, 0, 1, 0).
\]

(231)

The dimension of quark gauge representation space is

\[
\dim V_g(q) = (6 + 3 + 3) \times 3 = 36.
\]

(232)

The dimension 36 corresponds to 36 kind of quarks. 

So in together, the gauge representation space of matter particles is denoted as

\[
V_g(M) = V_g(l) \oplus V_g(q),
\]

(233)

The total dimension is

\[
\dim V_g(M) = 12 + 36 = 48.
\]

(234)

The total dimension corresponds to 48 gauge states. 

The gauge state basis of matter particles is represented as:

\[
|e_t\rangle,
\]

(235)

where \(t = 1, 2, ..., 48\) is the indices of matter particles. For each kind of matter particles, there is a corresponding gauge basis.

Matter particles in gauge representation space \(V_g(M)\) can be represented by gauge state basis as

\[
V_g(M) = \{ |\Psi\rangle : |\Psi\rangle = \Psi^t|e_t\rangle \}.
\]

(236)

The adjoint representation of matter particles takes the form

\[
\langle e_t| = \overline{|e_t\rangle}.
\]

(237)

The inner product of basis and its adjoint is,

\[
\langle e_t|e_{t'}\rangle = \delta_{tt'}.
\]

(238)

And also,

\[
|e_t\rangle\langle e_t'| = \hat{I}.
\]

(239)

This is the property of the metric for the gauge state space. 

The gauge state \(|\Psi\rangle\) of matter particles can be represented as the the superposition of the gauge basis

\[
|\Psi\rangle = \Psi^t|e_t\rangle,
\]

(240)

where

\[
\Psi^t = \langle e_t|\Psi\rangle.
\]

(241)

The adjoint of \(|\Psi\rangle\) is \(\langle \Psi|\), it has the form

\[
\langle \Psi| = \overline{|\Psi\rangle} = \Psi^*_t\langle e^t|.
\]

(242)

The inner product has the form,

\[
\langle \Psi|\Phi\rangle = \langle \Phi|\Psi\rangle^* \\
= \Psi^*_t \Phi^t = \delta_{tt'} \Psi^*_t \Phi^t.
\]

(243)
D. Elementary particles and their classification

Here we list the properties of electric-charge and the mass of matter particles:

| generations | lepton | electric-charge | mass | quark | electric-charge | mass |
|-------------|--------|-----------------|------|-------|-----------------|------|
| first generation | e      | -1              | m_e  | d     | -1              | m_d  |
|               | ν_e    | 0               | m_ν_e| u     | +2              | m_u  |
| second generation | μ      | -1              | m_μ  | s     | -1              | m_s  |
|               | ν_μ    | 0               | m_ν_μ| c     | +2              | m_c  |
| third generation | τ      | -1              | m_τ  | b     | -1              | m_b  |
|               | ν_τ    | 0               | m_ν_τ| t     | +2              | m_t  |

As we mentioned, in our theory, in order to consider properties of the involvements of the matter particles into the action force, the matter particles are divided into 48 classes. There are 6 classes of leptons according to the above table, besides mass and electric-charge, there are isospin singlet and isospin doublet, so there are 12 classes of leptons. There are 6 classes of quarks in the above table, we can also consider the isospin singlet and isospin doublet and three colors for quarks. So quarks are divided into 36 types. So matter particles have 48 classes. The explicit classification of those particles are presented explicitly in the following tables.

Color charge represents the quantum number of the involvement of particles into color interactions. Color charges of particles are represented by two parameters ($\lambda_3, \lambda_8$). According to color charge, matter particles are divided as leptons ($l$) and quarks ($q$). The color charge of leptons is (0,0), that means the color charge of them are zeroes, and they are not involved into the color force. The color charges of quarks are three types, ($\frac{1}{3}$, $\sqrt{6}$), (−$\frac{1}{3}$, $\sqrt{6}$), (0, $\sqrt{3}$). Usually they are called red, green and blue colors, respectively. So quarks can be red quark ($q_r$), green quark ($q_g$) and blue quark ($q_b$).

Electric charge $Q$ represents the quantum number of particles in interaction of electromagnetism. The leptons are divided into neutral-leptons ($l_0$) and electric-charged leptons ($l_{\pm 1}$). The neutral-lepton is the neutrino. The electric-charge of the $l_{-1}$ is $-1$. Quarks can be divided into positive-electric-charge quarks ($q_{+\frac{2}{3}}$) and negative-electric-charge quarks ($q_{-\frac{1}{3}}$) according to their electric charge. The positive electric-charge is $+\frac{2}{3}$, the negative electric-charge is $-\frac{1}{3}$.

The isospin charge $I_3$ represents the quantum number of the particles in weak interactions. The isospin charge can take three values $\frac{1}{2}, \frac{-1}{2}$ and 0, where $I_3 = \pm \frac{1}{2}$ represents the isospin doublet ($\Psi_D$), and $I_3 = 0$ represent the isospin singlet. The matter particles with same color charge and electric-charge can have isospin singlet and isospin doublet. The isospin doublet are in the weak interactions, while isospin singlet is not involved in the weak interactions.

The hypercharge and the weak-charge can be represented by electric-charge and the isospin charge as,

\[
Y = 2(Q - I_3),
\]
\[
Z = I_3 - Q \sin^2 \theta_w, \tag{244}
\]

where $\sin \theta_w = \frac{g_1}{\sqrt{g_1^2 + g_2^2}}$ and $g_1, g_2$ are the coupling constants between hypercharge and the isospin charges, $\theta_w$ is the Weinberg angle. Weak-charge $Z$ represents the quantum number of particles involved in the interactions of $Z_0$ particles.

Note there are only two independent parameters in four quantum numbers, electric-charge $Q$, isospin charge $I_3$, hypercharge $Y$ and the weak-charge $Z$. Usually the hypercharge $Y$ and isospin charge $I_3$ are chosen as the independent parameters, and electric-charge $Q$ and weak-charge $Z$ are represented as,

\[
Q = I_3 + \frac{1}{2} Y, \tag{245}
\]

\[
Z = I_3 \cos^2 \theta_w - \frac{1}{2} Y \sin^2 \theta_w.
\]

Masses of the particles are the quantum numbers representing the property of the symmetry breaking of the isospin in weak interactions. Matter particles with the same color-charge, electric-charge and isospin charge can be divided into three generations according to their masses.

Three generations of neutrinos are called respectively the electric-neutrino, $\mu$ neutrino and $\tau$ neutrino represented as $\nu_e, \nu_\mu, \nu_\tau$. The masses are $m_{\nu_e}, m_{\nu_\mu}$ and $m_{\nu_\tau}$. It is believed previously that the masses of the three generations of neutrinos are zeroes. Later experiments showed that the masses are small but not zeroes. Three generations
of electric-leptons are named respectively electron, $\mu$ particle and $\tau$ particle represented as $e, \mu, \tau$. The masses are $m_e, m_\mu, m_\tau$.

Three generations of positive-electric quarks are $u, c, t$ with masses $m_u, m_c, m_t$. The negative-electric quarks are $d, s, b$ with masses $m_d, m_s, m_b$. The masses of matter particles differ only depending on electrical-charge and generation, but not depending on color charge and isospin charge. That means matter particles with the same electrical-charge and generation but different color charges and isospin charges, their masses will be the same.

Depending on color-charge, electrical-charge, isospin-charge and mass, the matter particles are divided into 48 classes. The representation of those particles constitute a 48-dimensional gauge space. The gauge numbers of those particles can be found in the table next.

| elementary particle          | gauge charge $(Y, I_3, \lambda_3, \lambda_8)$ | first generation | second generation | third generation |
|------------------------------|---------------------------------------------|------------------|-------------------|------------------|
| neutrino doublet             | $(-1, \frac{1}{3}, 0, 0)$                   | $\nu_e(1)$       | $\nu_\mu(2)$      | $\nu_\tau(3)$   |
| neutrino singlet             | $(0, 0, 0, 0)$                             | $\nu_e(4)$       | $\nu_\mu(5)$      | $\nu_\tau(6)$   |
| electrical-lepton doublet    | $(-1, -\frac{2}{3}, 0, 0)$                 | $\nu_D(7)$       | $\mu_D(8)$        | $\tau_D(9)$     |
| electrical-lepton singlet    | $(-2, 0, 0, 0)$                            | $\nu_S(10)$      | $\mu_S(11)$       | $\tau_S(12)$    |
| red-$(+)$-quark doublet      | $(\frac{1}{3}, \frac{1}{3}, -\frac{1}{3})$ | $u_r(13)$        | $c_r(14)$         | $t_r(15)$       |
| red-$(+)$-quark singlet      | $(\frac{2}{3}, 0, -\frac{1}{3})$          | $u_r(16)$        | $c_r(17)$         | $t_r(18)$       |
| red-$(-)$(quark) doublet     | $(\frac{1}{3}, -\frac{1}{3}, \frac{2}{3})$ | $d_r(19)$        | $s_r(20)$         | $b_r(21)$       |
| red-$(-)$(quark) singlet     | $(\frac{1}{3}, 0, \frac{2}{3})$           | $d_r(22)$        | $s_r(23)$         | $b_r(24)$       |
| green-$(+)$-quark doublet    | $(\frac{2}{3}, \frac{1}{3}, -\frac{1}{3})$| $u_g(25)$        | $c_g(26)$         | $t_g(27)$       |
| green-$(+)$-quark singlet    | $(\frac{1}{3}, 0, -\frac{1}{3})$          | $u_g(28)$        | $c_g(29)$         | $t_g(30)$       |
| green-$(-)$(quark) doublet   | $(\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3})$| $d_g(31)$        | $s_g(32)$         | $b_g(33)$       |
| green-$(-)$(quark) singlet   | $(\frac{1}{3}, 0, -\frac{1}{3})$          | $d_g(34)$        | $s_g(35)$         | $b_g(36)$       |
| blue-$(+)$-quark doublet     | $(\frac{1}{3}, \frac{1}{3}, -\frac{2}{3})$| $u_b(37)$        | $c_b(38)$         | $t_b(39)$       |
| blue-$(+)$-quark singlet     | $(\frac{1}{3}, 0, \frac{2}{3})$           | $u_b(40)$        | $c_b(41)$         | $t_b(42)$       |
| blue-$(-)$(quark) doublet    | $(\frac{1}{3}, -\frac{1}{3}, -\frac{2}{3})$| $d_b(43)$        | $s_b(44)$         | $b_b(45)$       |
| blue-$(-)$(quark) singlet    | $(\frac{1}{3}, 0, -\frac{2}{3})$          | $d_b(46)$        | $s_b(47)$         | $b_b(48)$       |

The spin of the matter particles is $\frac{1}{2}$ corresponding to fermions. The spin is represented by $s_3$ which take values $\pm1/2$.

Gauge particles are spin-1 corresponding to bosons. The gauge forces are divided as electromagnetic force, weak-interaction force and color force, correspondingly the gauge particles are photons $\gamma$, weak interaction bosons $W_{\pm}, Z_0$ and gluons $g_{i(1)}$, $i = 1, 2, \cdots, 8$. One photon and three weak bosons corresponds to 4 generators of electric-weak gauge group $U(1) \otimes SU(2)$. Gluons are responsible for color force, eight gluons corresponds to 8 generators of group $SU(3)$. The following table shows the quantum numbers of gauge particles, where masses of gluons are equal and take value $m$.

| $\gamma$ | $Z_0$ | $W_+$ | $W_-$ | $g_{(1)}$ | $g_{(2)}$ | $g_{(3)}$ | $g_{(4)}$ | $g_{(5)}$ | $g_{(6)}$ | $g_{(7)}$ | $g_{(8)}$ |
|----------|-------|-------|-------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| mass 0   | $m_Z$ | $m_{W_+}$ | $m_{W_-}$ | $m$ | $m$ | $m$ | $m$ | $m$ | $m$ | $m$ | $m$ |
| $f_{(3)}$ | $0$ | $0$ | $0$ | $-1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $f_{(5)}$ | $0$ | $0$ | $+1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $f_{(7)}$ | $0$ | $0$ | $0$ | $0$ | $0$ | $+1$ | $1$ | $-\frac{1}{2}$ | $+\frac{1}{2}$ | $-\frac{1}{2}$ | $+\frac{1}{2}$ |
| $f_{(8)}$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $+\frac{2}{3}$ | $-\frac{2}{3}$ | $\frac{2}{3}$ | $+\frac{2}{3}$ |

There is one kind of graviton, it has complicated spin representation, its mass, electric-charge and color charge are all zeroes. Also we do not assume there exist Higgs particles.
E. Matrix representations of the gauge charges

In gauge basis \(|e_i\rangle\) of matter particles, gauge charges are represented as \(48 \times 48\) matrices. As we know, those gauge charges are hypercharge \(\hat{Y}\), isospin charges \(\hat{I}_i\) \((i = 1, 2, 3)\) and color charges \(\hat{\lambda}_p\), \((p = 1, 2, ..., 8)\). We will present explicitly the matrix representations of those gauge charges.

1. Matrix representation of hypercharge, isospin charges and color charges

We will next use the following notations: \(\hat{0}\) is a \(3 \times 3\) matrix with all elements zeros, \(\hat{I}\) is an \(3 \times 3\) identity matrix, \(\hat{U}_l\) and \(\hat{U}_q\) are the Kobayashi-Maskawa mixed matrices of leptons and quarks which will be presented later.

\[
\hat{Y} = \left( \begin{array}{ccc} -\hat{I} & \hat{0} & \hat{0} \\ 0 & \hat{0} & \hat{0} \\ 0 & \hat{0} & \hat{0} \end{array} \right) \oplus \hat{I} \otimes \left( \begin{array}{ccc} \frac{1}{3}\hat{I} & \hat{0} & \hat{0} \\ 0 & \frac{1}{3}\hat{I} & \hat{0} \\ 0 & \hat{0} & -\frac{2}{3}\hat{I} \end{array} \right) \tag{246}
\]

\[
\hat{I}_1 = \frac{1}{2} \left\{ \left( \begin{array}{ccc} 0 & \hat{0} & \hat{0} \\ \hat{0} & \hat{0} & \hat{0} \\ \hat{0} & \hat{0} & \hat{0} \end{array} \right) \oplus \hat{I} \otimes \left( \begin{array}{ccc} \hat{0} & \hat{0} & \hat{0} \\ \hat{0} & \hat{0} & \hat{0} \\ \hat{0} & \hat{0} & \hat{0} \end{array} \right) \right\}, \tag{247}
\]

\[
\hat{I}_2 = \frac{1}{2} \left\{ \left( \begin{array}{ccc} \hat{I} & \hat{0} & \hat{0} \\ 0 & \hat{0} & \hat{0} \\ 0 & \hat{0} & \hat{0} \end{array} \right) \oplus \hat{I} \otimes \left( \begin{array}{ccc} \hat{0} & \hat{0} & \hat{0} \\ \hat{0} & \hat{0} & \hat{0} \\ \hat{0} & \hat{0} & \hat{0} \end{array} \right) \right\}, \tag{248}
\]

\[
\hat{I}_3 = \frac{1}{2} \left\{ \left( \begin{array}{ccc} \hat{0} & \hat{0} & \hat{0} \\ \hat{0} & \hat{0} & \hat{0} \\ \hat{0} & \hat{0} & \hat{0} \end{array} \right) \oplus \hat{I} \otimes \left( \begin{array}{ccc} \hat{I} & \hat{0} & \hat{0} \\ 0 & \hat{0} & \hat{0} \\ 0 & \hat{0} & \hat{0} \end{array} \right) \right\}, \tag{249}
\]

\[
\hat{\lambda}_p = \left( \begin{array}{ccc} \hat{0} & \hat{0} & \hat{0} \\ \hat{0} & \hat{0} & \hat{0} \\ \hat{0} & \hat{0} & \hat{0} \end{array} \right) \oplus (\hat{\lambda}_p)_{3 \times 3} \otimes \left( \begin{array}{ccc} \hat{I} & \hat{0} & \hat{0} \\ \hat{0} & \hat{0} & \hat{0} \\ \hat{0} & \hat{0} & \hat{0} \end{array} \right), \tag{250}
\]

where \((\hat{\lambda}_p)_{3 \times 3}\) are \(3 \times 3\) Gell-Mann matrices which take the forms,

\[
\hat{\lambda}_1 = \frac{1}{2} \left[ \begin{array}{ccc} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right], \quad \hat{\lambda}_2 = \frac{1}{2} \left[ \begin{array}{ccc} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{array} \right],
\]

\[
\hat{\lambda}_3 = \frac{1}{2} \left[ \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right], \quad \hat{\lambda}_4 = \frac{1}{2} \left[ \begin{array}{ccc} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{array} \right],
\]

\[
\hat{\lambda}_5 = \frac{1}{2} \left[ \begin{array}{ccc} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{array} \right], \quad \hat{\lambda}_6 = \frac{1}{2} \left[ \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{array} \right],
\]
\[ \hat{\lambda}_7 = \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \quad \hat{\lambda}_8 = \frac{1}{2\sqrt{3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}. \]

We can find the matrices of \( \hat{J}^2 \) and \( \hat{\lambda}^2 \) take the form

\[ \hat{J}^2 = \frac{3}{4} \left\{ \left( \begin{array}{ccc} \hat{I} & 0 & 0 \\ 0 & \hat{I} & 0 \\ 0 & 0 & -\hat{I} \end{array} \right) \otimes \hat{I} \otimes \left( \begin{array}{ccc} \hat{I} & 0 & 0 \\ 0 & \hat{I} & 0 \\ 0 & 0 & -\hat{I} \end{array} \right) \right\}, \quad (251) \]

\[ \hat{\lambda}^2 = \frac{4}{3} \left\{ \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) \otimes \hat{I} \otimes \left( \begin{array}{ccc} \hat{I} & 0 & 0 \\ 0 & \hat{I} & 0 \\ 0 & 0 & \hat{I} \end{array} \right) \right\}. \quad (252) \]

The operators \( \hat{Y}, \hat{I}, \hat{\lambda}_p \) are unitary, and the following properties are satisfied,

\[
\hat{Y}^\dagger = \hat{Y}, \quad \hat{I}^\dagger = \hat{I}, \quad \hat{\lambda}_p^\dagger = \hat{\lambda}_p,
\]

\[
\langle \hat{Y}, \hat{Y} \rangle = \text{tr}(\hat{Y} \hat{Y}) = 40, \quad \langle \hat{I}_i, \hat{I}_j \rangle = \text{tr}(\hat{I}_i \hat{I}_j) = 6\delta_{ij},
\]

\[
\langle \hat{\lambda}_p, \hat{\lambda}_q \rangle = \text{tr}(\hat{\lambda}_p \hat{\lambda}_q) = 6\delta_{pq}, \quad \langle \hat{Y}, \hat{I}_i \rangle = \langle \hat{Y}, \hat{\lambda}_p \rangle = \langle \hat{I}_i, \hat{\lambda}_p \rangle = 0. \quad (253)
\]

2. Matrix representation of the eigen-gauge charges

Due to the transformation between orthogonal gauge charges \( \hat{t}_a \) and the eigen-gauge charges \( \hat{T}_a \) and the matrices representations of hypercharge \( \hat{Y} \), isospin charges \( \hat{I}_i \) and the color charges \( \hat{\lambda}_p \), the matrix representations of the eigen-gauge charges take the form

\[ \hat{T}_1 = e \left\{ \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\hat{I} \end{array} \right) \otimes \hat{I} \otimes \left( \begin{array}{ccc} 2\hat{I} & 0 & 0 \\ 0 & \frac{2}{3}\hat{I} & 0 \\ 0 & 0 & -\frac{2}{3}\hat{I} \end{array} \right) \right\}, \quad (254) \]

\[ \hat{T}_2 = \frac{\sqrt{g_1^2 + g_2^2}}{2} \left\{ \left( \begin{array}{ccc} \hat{I} & 0 & 0 \\ 0 & \hat{I} & 0 \\ 0 & 0 & -1 + 2\sin^2 \theta_w \hat{I} \end{array} \right) \right. \]

\[ \left. \left. \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & -\frac{4}{3}\sin^2 \theta_w \hat{I} \end{array} \right) \right) \right\} \otimes \hat{I} \otimes \left( \begin{array}{ccc} 1 - \frac{4}{3}\sin^2 \theta_w \hat{I} & 0 & 0 \\ 0 & -\frac{4}{3}\sin^2 \theta_w \hat{I} & 0 \\ 0 & 0 & -\frac{4}{3}\sin^2 \theta_w \hat{I} \end{array} \right) \right\}, \quad (255) \]

\[ \hat{T}_3 = \frac{g_2}{\sqrt{2}} \left\{ \left( \begin{array}{ccc} 0 & 0 & \hat{U}_1 \hat{I} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) \right. \]

\[ \left. \left. \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) \right) \right\} \otimes \hat{I} \otimes \left( \begin{array}{ccc} 0 & 0 & \hat{U}_1 \hat{I} \hat{I} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) \right\}. \quad (256) \]
\[
\hat{T}_4 = \frac{g_2}{\sqrt{2}} \left\{ \left( \begin{array}{ccc} \hat{0} & \hat{0} & \hat{0} \\ \hat{0} & \hat{0} & \hat{0} \\ \hat{U}_q^T & \hat{0} & \hat{0} \\ \hat{U}_q & \hat{0} & \hat{0} \\ \hat{0} & \hat{0} & \hat{0} \end{array} \right) \oplus \hat{I} \otimes \left( \begin{array}{ccc} \hat{0} & \hat{0} & \hat{0} \\ \hat{0} & \hat{0} & \hat{0} \\ \hat{0} & \hat{0} & \hat{0} \end{array} \right) \right\}, \tag{257}
\]

\[
\hat{T}_{4+p} = g_3 \left\{ \left( \begin{array}{ccc} \hat{0} & \hat{0} & \hat{0} \\ \hat{0} & \hat{0} & \hat{0} \\ \hat{0} & \hat{0} & \hat{0} \end{array} \right) \oplus (\hat{\Lambda}_p)_{3 \times 3} \otimes \left( \begin{array}{ccc} \hat{I} & \hat{0} & \hat{0} \\ \hat{0} & \hat{I} & \hat{0} \\ \hat{0} & \hat{0} & \hat{I} \end{array} \right) \right\} \tag{258}
\]

Here matrix \((\hat{\Lambda}_p)_{3 \times 3}\) are the representation of algebra \(su(3)\) in Cartan-Weyl basis:

\[
\hat{\Lambda}_1 = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \hat{\Lambda}_2 = \frac{1}{2\sqrt{3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix},
\]

\[
\hat{\Lambda}_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \hat{\Lambda}_4 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},
\]

\[
\hat{\Lambda}_5 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \hat{\Lambda}_6 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},
\]

\[
\hat{\Lambda}_7 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \hat{\Lambda}_8 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{259}
\]

One can check that we have the following property:

\[
\hat{T}_{4}^a = G^{ab}\hat{T}_{b}^c = \hat{T}_{a}, \tag{260}
\]

\[
\overline{T}_{a}^c = \hat{T}_{a}^c. \tag{261}
\]

\section*{F. Mass matrices of the matter particles}

Mass is of fundamental for us. One feature of our theory is that the mass matrices are defined in gauge space. Conventionally, the gauge invariant is necessary for gauge theory, thus mass can only be created by gauge symmetry breaking caused by Higgs mechanism. In our theory, weak interaction gauge space does not satisfy gauge invariant, but satisfy the general covariance condition. Thus mass can be represented as covariance matrix in gauge space. We denote the mass matrix in gauge basis \(\ket{e_i}\) as:

\[
\hat{m} = \left( \begin{array}{ccc} \hat{0} & \hat{m}_{l_0} & \hat{0} \\ \hat{m}_{l_0} & \hat{0} & \hat{0} \\ \hat{0} & \hat{0} & \hat{m}_{l_-} \end{array} \right) \oplus \hat{I} \otimes \left( \begin{array}{ccc} \hat{0} & \hat{m}_{g^+} & \hat{0} \\ \hat{0} & \hat{0} & \hat{0} \\ \hat{0} & \hat{0} & \hat{m}_{g^-} \end{array} \right), \tag{262}
\]

where \(\hat{m}_{l_0}, \hat{m}_{l_-}, \hat{m}_{g^+}\) and \(\hat{m}_{g^-}\) are mass matrices of neutral-lepton, electric-lepton, positive-electric-quark and negative-electric-quark, respectively.
Suppose the masses of three neutrinos are \( m_{\nu_e}, m_{\nu_\mu}, m_{\nu_\tau} \), the mass matrices take the form

\[
\hat{m}_{\nu 0} = \begin{pmatrix} m_{\nu_e} & 0 & 0 \\ 0 & m_{\nu_\mu} & 0 \\ 0 & 0 & m_{\nu_\tau} \end{pmatrix}
\] (263)

The three electric-leptons have masses \( m_e, m_\mu, m_\tau \), we denote

\[
\hat{m}_{l-} = \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix}.
\] (264)

Also we denote three positive-electric-quarks have masses \( m_u, m_c, m_t \), three negative-electric-quarks have masses \( m_d, m_s, m_b \), and the matrices as the following forms

\[
\hat{m}_{q+} = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_e & 0 \\ 0 & 0 & m_t \end{pmatrix}.
\] (265)

\[
\hat{m}_{q-} = \begin{pmatrix} m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{pmatrix}.
\] (266)

The coupling constant of the electric-charge is

\[
e = \frac{g_1 g_2}{\sqrt{g_1^2 + g_2^2}}
\] (267)

The coupling constant of the weak charges is

\[
g_Z = \sqrt{g_1^2 + g_2^2}
\] (268)

The Weinberg angle takes the form

\[
\cos \theta_w = \frac{g_2}{\sqrt{g_1^2 + g_2^2}}
\] (269)

\[
\sin \theta_w = \frac{g_1}{\sqrt{g_1^2 + g_2^2}}
\] (270)

The angles between weak interactions of three generations of quarks are denoted by \( \theta_1, \theta_2, \theta_3 \), the PC broken symmetry factor of weak interaction is \( \delta_1 \). And the Kobayashi-Maskawa matrices [17] for weak interaction are defined as

\[
\hat{U}_q = \begin{pmatrix} c_1 & -s_1 c_2 & -s_1 c_2 \\ s_1 c_3 & c_1 c_2 c_3 + s_2 s_3 e_1 & c_1 s_2 c_3 - c_2 s_3 e_1 \\ s_1 c_3 & c_1 c_2 s_3 - s_2 c_3 e_1 & c_1 s_2 s_3 + c_2 c_3 e_1 \end{pmatrix}
\] (271)

where \( s_i = \sin \theta_i, c_i = \cos \theta_i, (i = 1, 2, 3), e_1 = \exp(-i \delta_1) \).

Similarly for three generations of leptons, three angles of weak interaction are denoted as \( \theta_4, \theta_5, \theta_6 \), the broken symmetry factor is \( \delta_2 \), and the Kobayashi-Kaskawa matrices take the form

\[
\hat{U}_l = \begin{pmatrix} c_4 & -s_4 c_5 & -s_4 c_5 \\ s_4 c_6 & c_4 c_5 c_6 + s_5 s_6 e_2 & c_4 s_5 c_6 - c_5 s_6 e_2 \\ s_4 c_6 & c_4 c_5 s_6 - s_5 c_6 e_2 & c_4 s_5 s_6 + c_5 c_6 e_2 \end{pmatrix}
\] (272)

where similarly, \( s_j = \sin \theta_j, c_j = \cos \theta_j, (j = 4, 5, 6), e_2 = \exp(-i \delta_2) \).

The commutation relation between gauge charges \( \hat{T}_a \) and the mass matrices of matter particles is

\[
[\hat{T}_a, \hat{m}] = \hat{m}_a
\] (273)
where

\[ \hat{m}_1 = 0, \]  \tag{274}  

\[ \hat{m}_2 = \frac{g_2}{2} \left\{ \begin{pmatrix} 0 & \hat{m}_{l0} & 0 & 0 \\ -\hat{m}_{l0} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\hat{m}_{l-} \\ 0 & 0 & \hat{m}_{l-} & 0 \end{pmatrix} \right\} \oplus \hat{\hat{i}} \otimes \begin{pmatrix} 0 & \hat{m}_{q+} & 0 & 0 \\ -\hat{m}_{q+} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\hat{m}_{q-} \\ 0 & 0 & \hat{m}_{q-} & 0 \end{pmatrix}, \]  \tag{275}  

\[ \hat{m}_3 = \frac{g_2}{\sqrt{2}} \left\{ \begin{pmatrix} 0 & 0 & 0 & \hat{\bar{U}}_l \hat{m}_{l0} \\ 0 & -\hat{m}_{l-} \hat{\bar{U}}_q & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\} \oplus \hat{\hat{i}} \otimes \begin{pmatrix} 0 & 0 & 0 \hat{\bar{U}}_q \hat{m}_{q+} \\ 0 & 0 & -\hat{m}_{q-} \hat{\bar{U}}_q & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \]  \tag{276}  

\[ \hat{m}_4 = \frac{g_2}{\sqrt{2}} \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \hat{\bar{U}}_l \hat{m}_{l-} & 0 & 0 \\ -\hat{m}_{l0} \hat{\bar{U}}_l & 0 & 0 & 0 \end{pmatrix} \right\} \oplus \hat{\hat{i}} \otimes \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 \hat{\bar{U}}_q \hat{m}_{q-} & 0 \\ 0 & 0 & 0 & 0 \\ -\hat{m}_{q+} \hat{\bar{U}}_q & 0 & 0 & 0 \end{pmatrix}, \]  \tag{277}  

\[ \hat{m}_{4+p} = 0 \]  \tag{278}  

In the above commutation relations, mass matrices commute with electric-charge and color charge, while does not commute with weak charges.

Also we have \( \hat{m}^\dagger = \hat{m} \) and \( \hat{m} = -\hat{m} \).

Here are some comments about the mass matrix. On gauge basis of matter particles \( |e_i\rangle \), the matrix representation of the following operators are diagonal: hypercharge \( \hat{Y} \), the third element of the isospin \( \hat{I}_3 \), the third element of the color charge \( \hat{\lambda}_8 \), the eighth element of the color charge \( \hat{\lambda}_8 \). So basis \( |e_i\rangle \) is the common eigenvector of those operators. The mass matrix \( \hat{m} \) is quasi-diagonal. The eigenvalues are \( \hat{Y}, \hat{I}_3, \hat{\lambda}_3, \hat{\lambda}_8, \hat{m} \), respectively. Those quantities are hypercharge, third element of the isospin, third and eighth elements of the color charge and mass of the matter particles.

### G. Projection operators

Here let us define some projectors of gauge space related with isospin singlet and isospin doublet.

1) Projection operators \( \hat{P}_D, \hat{P}_S \) for isospin singlet and doublet are written as

\[ \hat{P}_D = \frac{4}{3} \hat{J}^2 \]  \tag{279}  

\[ \hat{P}_S = 1 - \frac{4}{3} \hat{J}^2. \]  \tag{280}  

The singlet projector and the doublet projector satisfy the equations,

\[ \hat{P}_D + \hat{P}_S = 1 \]
\[ \hat{P}_D \hat{P}_D = \hat{P}_D \]
\[ \hat{P}_S \hat{P}_S = \hat{P}_S \]
\[ \hat{P}_D \hat{P}_S = 0 \]  \tag{281}  

2) Similarly, lepton projector and quark projector are written as the following and have the properties,

\[ \hat{P}_l = 1 - \frac{3}{4} \hat{\lambda}_3^2 \]  \tag{282}  

\[ \hat{P}_q = \frac{3}{4} \hat{\lambda}_8^2, \]  \tag{283}  

\[ \hat{P}_l + \hat{P}_q = 1 \]
\[ \hat{P}_l \hat{P}_l = \hat{P}_l \]
\[ \hat{P}_q \hat{P}_q = \hat{P}_q \]
\[ \hat{P}_l \hat{P}_q = 0 \]  \tag{284}  

Those projectors will be useful when we consider later Dirac equation.
H. Eigenvectors of the weak interaction

Gauge charges of the weak interaction are $\hat{T}_3$ and $\hat{T}_4$. On the gauge charge basis of matter particles, $\hat{T}_3$ and $\hat{T}_4$ include the Kobayashi-Maskawa matrices $\hat{U}_1$ and $\hat{U}_q$, so $\{e_t\}$ is not the eigenvector for weak interaction. The basis can be changed by unitary transformation to the eigenvector of the weak interaction,

$$|e'_{t'}\rangle = \hat{U}_{t'}^\dagger |e_t\rangle,$$

(285)

where this $\hat{U}$ is a $48 \times 48$ unitary matrix and can be represented as

$$\hat{U} = \left( \begin{array}{c} \hat{U}_1^\dagger \hat{0} \hat{0} \hat{0} \\ \hat{0} \hat{U}_1^\dagger \hat{0} \hat{0} \\ \hat{0} \hat{0} \hat{I} \hat{0} \\ \hat{0} \hat{0} \hat{0} \hat{I} \end{array} \right) \oplus \hat{I} \otimes \left( \begin{array}{c} \hat{I} \hat{0} \hat{0} \hat{0} \\ \hat{0} \hat{I} \hat{0} \hat{0} \\ \hat{0} \hat{0} \hat{U}_q^\dagger \hat{0} \\ \hat{0} \hat{0} \hat{0} \hat{U}_q^\dagger \end{array} \right),$$

(286)

On the basis of the eigenvector of weak interaction $|e'_{t'}\rangle, (t' = 1, 2, ..., 48)$, operators $\hat{I}_1, \hat{I}_2, \hat{T}_3, \hat{T}_4$ and $\hat{m}$ can be represented as

$$\hat{I}_1 = \hat{U} \hat{I}_1 \hat{U}^\dagger = \frac{1}{2} \left\{ \begin{array}{c} \hat{0} \hat{0} \hat{I} \hat{0} \\ \hat{0} \hat{0} \hat{0} \hat{0} \\ \hat{I} \hat{0} \hat{0} \hat{0} \\ \hat{0} \hat{0} \hat{0} \hat{0} \end{array} \right\} \oplus \hat{I} \otimes \left( \begin{array}{c} \hat{0} \hat{0} \hat{I} \hat{0} \\ \hat{0} \hat{0} \hat{0} \hat{0} \\ \hat{I} \hat{0} \hat{0} \hat{0} \\ \hat{0} \hat{0} \hat{0} \hat{0} \end{array} \right),$$

(287)

$$\hat{I}_2 = \hat{U} \hat{I}_2 \hat{U}^\dagger = \frac{1}{2} \left\{ \begin{array}{c} \hat{0} \hat{0} \hat{i} \hat{0} \\ \hat{0} \hat{0} \hat{0} \hat{0} \\ \hat{i} \hat{0} \hat{0} \hat{0} \\ \hat{0} \hat{0} \hat{0} \hat{0} \end{array} \right\} \oplus \hat{I} \otimes \left( \begin{array}{c} \hat{0} \hat{0} \hat{i} \hat{0} \\ \hat{0} \hat{0} \hat{0} \hat{0} \\ \hat{i} \hat{0} \hat{0} \hat{0} \\ \hat{0} \hat{0} \hat{0} \hat{0} \end{array} \right),$$

(288)

$$\hat{T}_3 = \hat{U} \hat{T}_3 \hat{U}^\dagger = \frac{g_2}{\sqrt{2}} \left\{ \begin{array}{c} \hat{0} \hat{0} \hat{I} \hat{0} \\ \hat{0} \hat{0} \hat{0} \hat{0} \\ \hat{I} \hat{0} \hat{0} \hat{0} \\ \hat{0} \hat{0} \hat{0} \hat{0} \end{array} \right\} \oplus \hat{I} \otimes \left( \begin{array}{c} \hat{0} \hat{0} \hat{I} \hat{0} \\ \hat{0} \hat{0} \hat{0} \hat{0} \\ \hat{I} \hat{0} \hat{0} \hat{0} \\ \hat{0} \hat{0} \hat{0} \hat{0} \end{array} \right),$$

(289)

$$\hat{T}_4 = \hat{U} \hat{T}_4 \hat{U}^\dagger = \frac{g_2}{\sqrt{2}} \left\{ \begin{array}{c} \hat{0} \hat{0} \hat{0} \hat{0} \\ \hat{0} \hat{0} \hat{0} \hat{0} \\ \hat{I} \hat{0} \hat{0} \hat{0} \\ \hat{0} \hat{0} \hat{0} \hat{0} \end{array} \right\} \oplus \hat{I} \otimes \left( \begin{array}{c} \hat{0} \hat{0} \hat{0} \hat{0} \\ \hat{0} \hat{0} \hat{0} \hat{0} \\ \hat{I} \hat{0} \hat{0} \hat{0} \\ \hat{0} \hat{0} \hat{0} \hat{0} \end{array} \right),$$

(290)

$$\hat{m} = \hat{U} \hat{m} \hat{U}^\dagger = \left( \begin{array}{c} \hat{0} \hat{0} \hat{0} \hat{0} \\ \hat{U}_1^\dagger \hat{m}_q \hat{U}_i \hat{0} \hat{0} \hat{0} \\ \hat{0} \hat{0} \hat{0} \hat{0} \hat{m}_{i+} \hat{0} \\ \hat{0} \hat{0} \hat{0} \hat{0} \hat{m}_{i-} \hat{0} \end{array} \right) \oplus \hat{I} \otimes \left( \begin{array}{c} \hat{0} \hat{m}_q^+ \hat{0} \hat{0} \\ \hat{0} \hat{0} \hat{0} \hat{0} \hat{0} \hat{0} \hat{0} \hat{0} \hat{0} \hat{0} \end{array} \right),$$

(291)

Operators $\hat{m}, \hat{I}_3, \hat{\lambda}_p, \hat{T}_1, \hat{T}_2, \hat{T}_4, \hat{m}_{i+}, \hat{m}_{i-}$ are invariant under the unitary transformation $\hat{U}$.

I. The correspondence between gauge particles and gauge charges

The 12 gauge particles corresponds to 12 elements of the gauge charge $\hat{T}_3$. Photon $\gamma$ corresponds to gauge charge $\hat{T}_1$, weak interaction bosons $Z_0, W^+, W^-$ correspond to gauge charges $\hat{T}_2, \hat{T}_3, \hat{T}_4$, gluons $g(p), (p = 1, 2, ..., 8)$ correspond to gauge charges $\hat{T}_{4+p}$. 
The representation of the gauge basis can take the form of 12 eigen-gauge charges, they satisfy the equation,

\[ [\hat{T}_a, \hat{T}_b] = C_{ab}^c \hat{T}_c, \]  

(292)

The elements of the gauge charges can be the structure constants

\[ (\hat{T}_a)_b^c = C_{ab}^c. \]  

(293)

The following properties can be checked by direct calculations,

\[ [\hat{Y}, \hat{T}_a] = Y \hat{T}_a, \]  

(294)

\[ [\hat{I}_3, \hat{T}] = I_3 \hat{T}_a, \]  

(295)

\[ [\hat{\lambda}_3, \hat{T}_a] = \lambda_3 \hat{T}_a, \]  

(296)

\[ [\hat{\lambda}_8, \hat{T}_a] = \lambda_8 \hat{T}_a, \]  

(297)

**J. Mass matrix of the gauge particles**

The mass matrix of the gauge particles is defined as gauge tensor in eigen-gauge charges \( \hat{T}_a \),

\[ \check{M} = M^a_b \hat{T}_a \otimes \hat{T}_b. \]  

(298)

We consider respectively the weak interaction bosons part \( \check{M}_{weak} \) and the gluon part \( \check{M}_{gluon} \), so

\[ \check{M} = \check{M}_{weak} + \check{M}_{gluon}. \]  

(299)

The mass matrix elements concerning about weak interaction bosons can be written as,

\[ \check{M}_{weak} = m_Z^2 \hat{T}_2 \otimes \hat{T}_2 + m_W^2 \hat{T}_3 \otimes \hat{T}_3 + m_W^2 \hat{T}_4 \otimes \hat{T}_4, \]  

(300)

where \( m_Z = 91188 \text{MeV} \), is the mass of particle \( Z_0 \), \( m_W = 80398 \text{MeV} \), is the mass of particles \( W^+, W^- \), they are connected through the Weinberg angle as

\[ m_W = m_Z \cos \theta_W. \]  

(301)

Due to the transformation between \( \hat{t}_a \) and \( \hat{T}_a \), we have

\[ \check{M}_{weak} = m^{ab} \hat{t}_a \otimes \hat{t}_b \]
\[ = m_Z^2 (\sin^2 \theta_w \hat{t}_1 \otimes \hat{t}_1 - 2 \sin \theta_w \cos \theta_w \hat{t}_1 \otimes \hat{t}_4 + \cos^2 \theta_w \hat{t}_4 \otimes \hat{t}_4)
+ m_W^2 \cos^2 \theta_w (\hat{t}_2 \otimes \hat{t}_2 + \hat{t}_3 \otimes \hat{t}_3)
\]
\[ = m_Z^2 \cos^2 \theta_w (\tan^2 \theta_w \hat{t}_1 \otimes \hat{t}_1 - 2 \tan \theta_w \hat{t}_1 \otimes \hat{t}_4 + \hat{t}_2 \otimes \hat{t}_2 + \hat{t}_3 \otimes \hat{t}_3 + \hat{t}_4 \otimes \hat{t}_4). \]  

(302)

Note if \( \hat{t}_1 = 0 \), operators \( \hat{t}_2, \hat{t}_3, \hat{t}_4 \) are symmetric for \( \check{M} \), that means \( \check{M} \) are symmetric for isospin group \( SU(2) \), and also \( m_W = m_Z \cos \theta_W \) is necessary for this symmetry.

In this work, we assume the masses of gluons are all \( m \), thus we have

\[ \check{M}_{gluon} = M_{(4+p)}^a \hat{T}_{4+p} \otimes \hat{T}_{4+p} \]
\[ = m^2 \hat{T}_{4+p} \otimes \hat{T}_{4+p} \]  

(303)

where \( p = 1, 2, \ldots, 8 \), \( M_{(4+p)}^a = m^2 \), and \( \hat{T}_{4+p} \) corresponds to color charges. This result will be useful in studying the dark energy. Note that the \( SU(3) \) symmetry does not be broken by this mass matrix.
K. Gauge representation for gravity field

Gauge representation of gravity is 1D identity denoted as

\[ V_g(G) = T_g(0) = V_g(0, 0, 0, 0) \]  \hspace{1cm} (304)

We can choose the gauge basis for graviton as 1, gauge charges \( T_α \) commute with 1, thus we can say the gauge charge of graviton is zero.

V. REPRESENTATION THEORY

In this work, particles or fields are defined by three class of parameters: coordinate-momentum, spin and gauge charges. The coordinates-momentum are independent with spin and gauge bosons. Fields or particles and the interactions are represented as a vector for matter fields or operator for force fields in a direct product space by three class of spaces: coordinate-momentum space, spin space and gauge space, \( V(M) = V_{xp} \otimes V_S(M) \otimes V_g(M) \). The quantum state \( |e_{st}\rangle \) of a matter particle is defined as the direct (tensor) product in coordinate-basis, spin-basis and gauge-basis, \( |e_{st}(x)\rangle = |x\rangle \otimes |s\rangle \otimes |t\rangle \), where \( x \in \mathbb{R}^4, s = 1, 2, 3, 4 \) and \( t = 1, 2, \ldots, 48 \). It is the common-eigenstate of 10 operators \( \hat{x}^0, \hat{x}^1, \hat{x}^2, \hat{x}^3, \gamma_5, \hat{\gamma}_{12}, \hat{Y}, \hat{I}_3, \hat{\lambda}_3 \) and \( \hat{\lambda}_8 \). In this representation, coordinate state can be changed to momentum state \( |x\rangle \rightarrow |p\rangle \) and we have \( |e_{st}(p)\rangle = |p\rangle \otimes |s\rangle \otimes |t\rangle \). The quantum state of matter-particle \( |\psi\rangle \) can then be expanded in either coordinate state \( |e_{st}(x)\rangle \) or momentum state \( |e_{st}(p)\rangle \), Please note in our representation, spin, gauge and general coordinate-momentum are dealt in the same positions.

Before proceed, we would like to briefly summarize our representation results. The framework of this theory is that all fields and particles are described by their representations with three properties: coordinate-momentum, spin and gauge. Their properties will be governed by three fundamental equations.

The quantum parameters are coordinate \( \hat{x}^\mu \), momentum \( \hat{p}_\mu \), spin \( \hat{s}_{\alpha\beta} \) and gauge charges \( \hat{T}_a \) (or \( \hat{t}_a \)), altogether there are 26 parameters. The commutation relations for those parameters are:

\[ [\hat{x}^\mu, \hat{p}_\nu] = -i \delta^\mu_\nu, \]  \hspace{1cm} (305)

\[ [\hat{x}^\mu, \hat{x}^\nu] = [\hat{p}_\mu, \hat{p}_\nu] = 0, \]  \hspace{1cm} (306)

\[ [\hat{s}_{\alpha\beta}, \hat{s}_{\rho\sigma}] = -i(\eta_{\alpha\rho} \hat{s}_{\beta\sigma} - \eta_{\beta\rho} \hat{s}_{\alpha\sigma} + \eta_{\alpha\sigma} \hat{s}_{\beta\rho} - \eta_{\beta\sigma} \hat{s}_{\alpha\rho}), \]  \hspace{1cm} (307)

\[ [\hat{T}_a, \hat{T}_b] = C_{ab}^c \hat{T}_c, \]  \hspace{1cm} (308)

\[ [\hat{x}^\mu, \hat{s}_{\alpha\beta}] = [\hat{x}^\mu, \hat{T}_a] = [\hat{p}_\mu, \hat{s}_{\alpha\beta}] = [\hat{p}_\mu, \hat{T}_a] = [\hat{s}_{\alpha\beta}, \hat{T}_a] = 0. \]  \hspace{1cm} (309)

The 26 quantum parameters and the unit constitute a Lie algebra \( A \),

\[ A = \left\{ \hat{Z} : \hat{Z} = a_\mu \hat{x}^\mu + b_\rho \hat{p}_\rho + \alpha + \frac{1}{2} \Gamma^{\alpha\beta} \hat{s}_{\alpha\beta} + \theta^a \hat{T}_a \right\}. \]  \hspace{1cm} (310)

This algebra is a direct summation of three algebras, coordinate-momentum algebra \( A_{xp} \), spin algebra \( A_S \) and gauge algebra \( A_g \),

\[ A = A_{xp} \oplus A_S \oplus A_g. \]  \hspace{1cm} (311)

The group corresponding to this Lie algebra is written as

\[ G = \left\{ \hat{U} : \hat{U} = \exp[i(a_\mu \hat{x}^\mu + b_\rho \hat{p}_\rho + \alpha + \frac{1}{2} \omega^{\alpha\beta} \hat{s}_{\alpha\beta} + \xi^a \hat{T}_a)] \right\}. \]  \hspace{1cm} (312)

This group is a direct product of groups of coordinate-momentum, spin and gauge

\[ G = G_{xp} \otimes G_S \otimes G_g. \]  \hspace{1cm} (313)

The representation space is simply the direct product of three representation spaces corresponding to coordinate-momentum, spin and gauge respectively.

We next list respectively the representation of (i) matter particles, (ii) the gauge particles and (iii) the graviton.
A. Representation of matter particles

The representation of matter particles is in the total direct product space

\[ V(M) = V_x p(M) \otimes V_s(M) \otimes V_g(M). \]  \hspace{1cm} (314)

The adjoint has similar representation,

\[ \overline{V}(M) = \overline{V}_x p(M) \otimes \overline{V}_s(M) \otimes \overline{V}_g(M). \]  \hspace{1cm} (315)

The operators are acting on those spaces,

\[ O(M) = V(M) \otimes \overline{V}(M). \]  \hspace{1cm} (316)

The basis of matter particles is a direct product of three class of basis, coordinate, spin and gauge

\[ |e_{st}(x)\rangle = |x\rangle \otimes |e_s\rangle \otimes |e_t\rangle, \]  \hspace{1cm} (317)

as we already know that \( x \in \mathbb{R}^4; s = 1, 2, 3, 4; t = 1, 2, ..., 48 \). Recall the metrics of spin space and gauge space, the adjoint of the basis takes the form

\[ \langle e_{st}(x)| = (\hat{\gamma}_0)^{st}\overline{|e_{s't'}(x')\rangle} = \langle x| \otimes \langle e^s| \otimes e^{t'}. \]  \hspace{1cm} (318)

The normalization and the complete conditions are

\[ \langle e_{st}(x)|e_{s't'}(x')\rangle = \delta_s^{s'}\delta_t^{t'}\delta^4(x-x'), \]
\[ \int_{R^4} |e_{st}(x)\rangle\langle e_{st}(x)|d^4x = 1. \]  \hspace{1cm} (319)

Similarly for momentum representation, we also have

\[ |e_{st}(p)\rangle = |p\rangle \otimes |e_s\rangle \otimes |e_t\rangle. \]  \hspace{1cm} (320)
\[ \langle e_{st}(p)| = \langle p| \otimes \langle e^s| \otimes e^{t'}. \]  \hspace{1cm} (321)

The normalization and the complete conditions are

\[ \langle e_{st}(p)|e_{s't'}(p')\rangle = \delta_s^{s'}\delta_t^{t'}\delta^4(p-p'), \]
\[ \int_{R^4} |e_{st}(p)\rangle\langle e_{st}(p)|d^4p = 1. \]  \hspace{1cm} (322)

The transformation elements between coordinate and momentum take the form

\[ \langle e_{st}(x)|e_{s't'}(p)\rangle = (2\pi)^{-2}\delta_s^{s'}\delta_t^{t'}\exp(-ipx). \]  \hspace{1cm} (323)

The general quantum state of the matter particles can be expanded in terms of those basis, either in coordinate or in momentum basis,

\[ |\Psi\rangle = \int_{R^4} \Psi_{st}(x)|e_{st}(x)\rangle d^4x = \int_{R^4} \overline{\Psi}_{st}(p)|e_{st}(p)\rangle d^4p, \]  \hspace{1cm} (324)

where coefficients \( \Psi_{st}(x) \) in the expansion are defined as

\[ \Psi_{st}(x) = \langle e_{st}(x)|\Psi\rangle = (2\pi)^{-2} \int_{R^4} \overline{\Psi}_{st}(p)\exp(-ipx)d^4p, \]
\[ \overline{\Psi}_{st}(p) = \langle e_{st}(p)|\Psi\rangle = (2\pi)^{-2} \int_{R^4} \Psi_{st}(x)\exp(ipx)d^4x. \]  \hspace{1cm} (325)
The adjoint representation space is constituted by the adjoint states \( \langle \Phi | \Psi \rangle = \int_{R^4} \Psi^* e^{\text{i} \theta(x)} e_{\text{st}}(x) d^4 x \) where \( e_{\text{st}}(x) \) are the basis of coordinate-momentum representation space, spin representation space and the gauge representation space.

Thus all quantum states of matter particles constitute the representation space \( V(M) \) which is simply written as,

\[
V(M) = \{ |\Psi \rangle : |\Psi \rangle = \int_{R^4} \Psi^* e^{\text{i} \theta(x)} e_{\text{st}}(x) d^4 x \} = \{ |\Psi \rangle : |\Psi \rangle = \int_{R^4} \Psi^* e^{\text{i} \theta(p)} e_{\text{st}}(p) d^4 p \}. \tag{326}
\]

The adjoint states \( |\Psi \rangle \) can be similarly written as,

\[
|\Psi \rangle = |\Psi \rangle = \int_{R^4} \Psi^* e^{\text{i} \theta(x)} e_{\text{st}}(x) d^4 x = \int_{R^4} \Psi^* e^{\text{i} \theta(p)} e_{\text{st}}(p) d^4 p, \tag{327}
\]

Note that the coefficients have the metric of spin space,

\[
\Psi^* e^{\text{i} \theta(x)} e_{\text{st}}(x) = (\hat{\gamma}^0)_{ss'} \Psi^* e_{ss'}(x), \tag{328}
\]

\[
\Psi^* e^{\text{i} \theta(p)} e_{\text{st}}(p) = (\hat{\gamma}^0)_{ss'} \Psi^* e_{ss'}(p). \tag{329}
\]

The adjoint representation space is constituted by the adjoint states \( \overline{V}(M) = \{ |\Psi \rangle \} \).

The operator representation space can be considered to be constructed by two spaces \( V(M) \otimes \overline{V}(M) \) and is denoted as,

\[
O(M) = \left\{ \hat{A} : \hat{A} = \int_{R^4} \int_{R^4} A_{s't'}(x', x)|e^{\text{i} \theta(x')} e_{\text{st}}(x)|d^4 x' d^4 x \right\} = \left\{ \hat{A} : \hat{A} = \int_{R^4} \int_{R^4} A_{s't'}(p', p)|e^{\text{i} \theta(p')} e_{\text{st}}(p)|d^4 p' d^4 p \right\}. \tag{330}
\]

The inner product of two vectors is represented as

\[
\langle \Phi | \Psi \rangle = \text{tr} (|\Phi \rangle \langle \Phi |) = \int_{R^4} \Phi^* e^{\text{i} \theta(x)} e_{\text{st}}(x) d^4 x = \int_{R^4} \Phi^* e^{\text{i} \theta(p)} e_{\text{st}}(p) d^4 p. \tag{331}
\]

### B. Representation of gauge particles

Representation theory for gauge particles: The representation space of gauge particles is also a direct product of coordinate-momentum representation space, spin representation space and the gauge representation space,

\[
V(A) = V_{zp}(A) \otimes V_{S}(A) \otimes V_{g}(A). \tag{332}
\]

The basis of \( V_{zp}(A) \) for coordinate and momentum are \( \hat{\epsilon}(x) \) and \( \hat{\epsilon}(p) \), where \( x \in R^4, p \in R^4 \). The basis \( V_{S}(A) \) of spin are \( \hat{\gamma}^{0,1,2,3} \), where \( \alpha_i = 0, 1, 2, 3 \) are orthogonal space-time indices, \( p = 0, 1, 2, 3 \) is the rank of the tensor. \( T_{\alpha} \) is the basis of \( V_{g}(A) \), where \( \alpha = 1, 2, \cdots, 12 \). So the total basis can be written as

\[
\hat{\epsilon}^{0,1,2,3}_a(x) = \hat{\epsilon}(x) \otimes \hat{\gamma}^{0,1,2,3}_a \otimes T_{\alpha}, \tag{333}
\]

\[
\hat{\epsilon}^{0,1,2,3}_a(p) = \hat{\epsilon}(p) \otimes \hat{\gamma}^{0,1,2,3}_a \otimes T_{\alpha}. \tag{334}
\]

The operator of gauge particles is represented as,

\[
\hat{X} = \frac{1}{p^4} \hat{X}^{a,\alpha_1,\cdots,\alpha_p} \otimes \hat{\gamma}^{0,1,2,3}_a \otimes T_{\alpha}
\]

\[
= \frac{1}{p^4} \int_{R^4} \hat{X}^{a,\alpha_1,\cdots,\alpha_p}(x) \hat{\gamma}^{0,1,2,3}_a(x) d^4 x
\]

\[
= \frac{1}{p^4} \int_{R^4} \hat{X}^{a,\alpha_1,\cdots,\alpha_p}(p) \hat{\gamma}^{0,1,2,3}_a(p) d^4 p. \tag{335}
\]
The coordinate-momentum functions $X_{\alpha_1...\alpha_p}^a(x)$ and $\tilde{X}_{\alpha_1...\alpha_p}^a(p)$ are related through Fourier transformation,

$$
X_{\alpha_1...\alpha_p}^a(x) = (2\pi)^{-2} \int_{R^4} \tilde{X}_{\alpha_1...\alpha_p}^a(p) \exp(-ixp) d^4p, \quad (336)
$$

$$
\tilde{X}_{\alpha_1...\alpha_p}^a(p) = (2\pi)^{-2} \int_{R^4} X_{\alpha_1...\alpha_p}^a(x) \exp(ixp) d^4x, \quad (337)
$$

### C. Representation of graviton

The representation space of graviton also includes coordinate-momentum, spin and gauge,

$$
V(G) = V_{xp}(G) \otimes V_\Sigma(G) \otimes V_g(G). \quad (338)
$$

Here we would like to point out that $V_{xp}$ is an infinite dimensional space, $V_\Sigma(G)$ is a 256 dimensional mixed tensor space, $V_G(G)$ is a 1-dimensional gauge representation space.

$V_{xp}(G)$ has the basis $\hat{\epsilon}(x)$ and $\hat{\epsilon}(p)$. The basis of $V_\Sigma(G)$ is $\hat{\gamma}^{\alpha_1...\alpha_p}_{\beta_1...\beta_q}$, where $\alpha_i, \beta_i = 0, 1, 2, 3$ are indices of the orthogonal space-time, $p, q = 0, 1, 2, 3, 4$ are ranks of the spin mixed tensor.

So the basis of graviton for coordinate and momentum are,

$$
\hat{\epsilon}^{\alpha_1...\alpha_p}_{\beta_1...\beta_q}(x) = \hat{\epsilon}(x) \otimes \hat{\gamma}^{\alpha_1...\alpha_p}_{\beta_1...\beta_q}, \quad (339)
$$

$$
\hat{\epsilon}^{\alpha_1...\alpha_p}_{\beta_1...\beta_q}(x) = \hat{\epsilon}(p) \otimes \hat{\gamma}^{\alpha_1...\alpha_p}_{\beta_1...\beta_q}. \quad (340)
$$

The operator representation for graviton is

$$
\hat{Y} = \frac{1}{p!q!} \hat{Y}_{\alpha_1...\alpha_p}^{\beta_1...\beta_q} \otimes \hat{\gamma}^{\alpha_1...\alpha_p}_{\beta_1...\beta_q}
$$

$$
= \frac{1}{p!q!} \int_{R^4} Y_{\alpha_1...\alpha_p}^{\beta_1...\beta_q}(x) \hat{\epsilon}^{\alpha_1...\alpha_p}_{\beta_1...\beta_q}(x) d^4x
$$

$$
= \frac{1}{p!q!} \int_{R^4} \tilde{Y}_{\alpha_1...\alpha_p}^{\beta_1...\beta_q}(p) \hat{\epsilon}^{\alpha_1...\alpha_p}_{\beta_1...\beta_q}(p) d^4p. \quad (341)
$$

The Fourier transformation connects the functions for coordinate and momentum together,

$$
\tilde{Y}_{\alpha_1...\alpha_p}^{\beta_1...\beta_q}(x) = (2\pi)^{-2} \int_{R^4} \tilde{Y}_{\alpha_1...\alpha_p}^{\beta_1...\beta_q}(p) \exp(-ixp) d^4p, \quad (342)
$$

$$
\tilde{Y}_{\alpha_1...\alpha_p}^{\beta_1...\beta_q}(p) = (2\pi)^{-2} \int_{R^4} Y_{\alpha_1...\alpha_p}^{\beta_1...\beta_q}(x) \exp(ixp) d^4x. \quad (343)
$$

In quantum mechanics, the time evolution of a quantum state or an operator are described by Schödinger representation or Heisenberg representation, respectively. We may notice that time and space are not symmetric. In comparison for our work, there is no absolute time and space in general relativity, thus coordinates are dealt symmetrically. The state $|\psi_{\mu\nu}\rangle$ is an event.

### VI. DIFFERENTIAL GEOMETRY, REPRESENTATION OF GRAVITY FIELD AND GAUGE FIELDS

For a manifold, the geometry is characterized by vierbeins and the corresponding connections. We consider next the differential geometry properties related with gravity field and gauge fields. Our work, however, is actually an algebraic realization of the differential geometry by a proper representations.

#### A. The spin vierbein

The spin vierbein formalism takes the form

$$
\hat{\theta} = \hat{\theta}_a^\mu \otimes \hat{\gamma}^\alpha \otimes \hat{\theta}_\mu, \quad (344)
$$

$$
\hat{\theta}_a^\mu = \int_{R^4} \hat{\theta}_a^\mu(x) \hat{\epsilon}(x) d^4x \quad = \int_{R^4} \hat{\theta}_a^\mu(p) \hat{\epsilon}(p) d^4p.
$$
where \( \hat{\theta}^\mu_\alpha \) are the spin vierbein coefficients. \( \theta^\mu_\alpha(x) \) and \( \tilde{\theta}^\nu_\beta(x) \) are coordinate functions and momentum functions in spin vierbein formalism, they satisfy relations

\[
\tilde{\theta}^\nu_\beta(p) = (2\pi)^{-2} \int_{R^4} \theta^\mu_\alpha(x) \exp(ipx) d^4x, \\
\theta^\mu_\alpha(x) = (2\pi)^{-2} \int_{R^4} \tilde{\theta}^\nu_\beta(p) \exp(-ipx) d^4p.
\]

(345)

The adjoint of the spin vierbein coefficient is defined as

\[
\overline{\theta}^\mu_\alpha = \hat{\epsilon}^\mu_\alpha.
\]

(346)

Due to the spin vierbein coefficients \( \hat{\theta}^\mu_\alpha \), we can define the momentum metric as

\[
\hat{g}^{\mu\nu} = \hat{\eta}^{\alpha\beta} \hat{\theta}_{\alpha\beta}, \\
\hat{g} = \hat{g}^{\mu\nu} \otimes \hat{p}_\mu \otimes \hat{p}_\nu,
\]

(347)

where \( \hat{\eta}^{\alpha\beta} \) is the free metric, i.e., Minkowski metric. \( \hat{g}^{\mu\nu} \) satisfies,

\[
\overline{\hat{g}}^{\mu\nu} = \hat{g}^{\mu\nu}, \\
\overline{\hat{g}}^{\mu\nu} = \hat{g}^{\mu\nu}.
\]

(348)

\( \hat{g}^{\mu\nu} \) is the contra-variance metric tensor, the covariance metric tensor \( \hat{g}_{\mu\nu} \) can be defined as the inverse of the contra-variance metric tensor \( \hat{g}^{\mu\nu} \),

\[
\hat{g}_{\mu\lambda} \hat{g}^{\lambda\nu} = \delta^\nu_\nu, \\
\hat{g}_{\mu\nu} = \hat{A}_{\mu\nu} / \det[\hat{g}],
\]

(349)

where \( \hat{A}_{\mu\nu} \) is the algebraic complementary minor of elements \( \hat{g}^{\mu\nu} \). The covariance metric takes the form

\[
\eta_{\alpha\beta} = \hat{g}_{\mu\nu} \hat{\theta}^{\mu\nu}_{\alpha\beta}.
\]

(350)

The lowering or rising of the indices in momentum tensor can be realized by multiplying momentum metric \( \hat{g}^{\mu\nu} \) and \( \hat{g}_{\mu\nu} \), while for indices in spin tensor, the spin metrics \( \hat{\eta}^{\alpha\beta} \) and \( \eta_{\alpha\beta} \) should be used. For example, the case of orthogonal vierbein we have,

\[
\hat{\epsilon}^\alpha_\mu = \eta^{\alpha\beta} \hat{g}_{\mu\nu} \hat{\theta}^\nu_\beta, \\
\hat{\gamma}_\alpha = \eta_{\alpha\beta} \hat{\gamma}^\beta, \\
\hat{\epsilon} = \hat{\epsilon}^\alpha_\mu \otimes \hat{\gamma}_\alpha \otimes dx^\mu,
\]

(351)

where the orthogonal vierbein operator \( \hat{\theta}^\mu_\alpha \) and the dual orthogonal vierbein operator \( \hat{\theta}^{\alpha}_\mu \) satisfy the relation,

\[
\hat{\theta}^\mu_\alpha \hat{\epsilon}^\beta_\mu = \delta^\beta_\alpha, \\
\hat{\theta}^{\alpha}_\mu \hat{\epsilon}^{\beta}_\mu = \delta^\alpha_\beta.
\]

(354)

The spin of the vierbein formalism \( \hat{\theta}_\alpha = \hat{\theta}^\mu_\alpha \otimes \hat{p}_\mu \) satisfies the commutation relations

\[
[\hat{\theta}_\alpha, \hat{\theta}_\beta] = i \hat{f}_\beta^{\gamma} \hat{\theta}^\gamma_\alpha,
\]

(355)

where \( \hat{f}_\beta^{\gamma} \) are the structure coefficients in spin vierbein formalism and are represented as

\[
\hat{f}_\beta^{\gamma} = (\hat{\theta}^{\rho}_\alpha \partial_\rho \hat{\theta}^\gamma_\beta - \hat{\theta}^{\rho}_\beta \partial_\rho \hat{\theta}^\gamma_\alpha) \hat{\epsilon}^\rho_\mu.
\]

(356)

Note that the structure coefficients here may not be confused as the structure constants of \( su(3) \) appeared previously. The structure coefficients are antisymmetric, have adjoint, and satisfy Jacobi equation:

\[
\hat{f}_\beta^{\gamma} = -\hat{f}_\beta^{\gamma}, \\
\overline{\hat{f}}^{\gamma}_\alpha\beta = \hat{f}^{\gamma}_\alpha\beta, \\
\hat{\theta}^\mu_\alpha \partial_\mu \hat{f}_\beta^{\gamma} + \hat{\theta}^\mu_\beta \partial_\mu \hat{f}_\alpha^{\gamma} + \hat{\theta}^\mu_\gamma \partial_\mu \hat{f}_\alpha^{\beta} + \hat{f}_\beta^{\rho} \hat{f}_\gamma^{\rho} + \hat{f}_\gamma^{\rho} \hat{f}_\alpha^{\rho} + \hat{f}_\alpha^{\rho} \hat{f}_\gamma^{\rho} = 0,
\]

(357)

where we have used the equation,

\[
[\hat{\theta}_\alpha, [\hat{\theta}_\beta, \hat{\theta}_\gamma]] + [\hat{\theta}_\beta, [\hat{\theta}_\gamma, \hat{\theta}_\alpha]] + [\hat{\theta}_\gamma, [\hat{\theta}_\alpha, \hat{\theta}_\beta]] = 0.
\]

(360)
B. Connection of gravity

The gravity connection is defined as

\[ \hat{\Gamma}_\alpha^\rho\sigma = \frac{1}{2} \hat{\gamma}_\alpha^\rho\sigma, \]

\[ \hat{\Gamma}^\rho\sigma_\alpha = \frac{1}{2} ( \hat{f}_\rho^\sigma_\alpha + \hat{f}_\rho_{\sigma\alpha} - \hat{f}_\sigma^\rho_\alpha ), \]

where we have used the notations

\[ \hat{f}_\rho^\sigma_\tau = \eta_\rho^\alpha \eta_\sigma^\beta \eta_\tau^\gamma \hat{\gamma}^\alpha_{\beta\gamma}, \]

\[ \hat{f}^\rho_{\sigma\tau} = \eta_\sigma^\beta \hat{f}^\rho_{\tau\beta}. \]

The gravity connections are antisymmetric, have adjoint and satisfy no-torsion condition,

\[ \hat{\Gamma}_{\gamma\beta}^\alpha = -\hat{\Gamma}_{\beta\gamma}^\alpha, \]

\[ \hat{\Gamma}_{\gamma}^\alpha = \hat{\Gamma}_\gamma^\alpha, \]

\[ \hat{\Gamma}^\alpha_{\gamma,\beta} - \hat{\Gamma}^\alpha_{\beta,\gamma} = \hat{f}^\alpha_{\gamma\beta}, \]

the last equation can lead to the result that the torsion is zero,

\[ \hat{T}^\gamma_{\alpha,\beta} = \hat{\Gamma}^\gamma_{\alpha,\beta} - \hat{\Gamma}^\gamma_{\beta,\alpha} - \hat{f}^\gamma_{\alpha\beta} = 0. \]

Here let us discuss the spin of graviton defined by \( \hat{\gamma}_\alpha \otimes \hat{s}_{\rho\sigma} \). As we know the representation space of \( \hat{\gamma}_\alpha \) is 4-dimensional \( V_\gamma(\frac{1}{2}, \frac{1}{2}) \), the representation space of \( \hat{s}_{\rho\sigma} \) is 6-dimensional \( V_S(0, 1) \oplus V_S(1, 0) \). So the spin representation space of graviton is expressed as,

\[ V_S(\frac{1}{2}, \frac{1}{2}) \otimes [V_S(0, 1) \oplus V_S(1, 0)] \]

\[ = V_S(\frac{1}{2}, \frac{3}{2}) \oplus V_S(\frac{1}{2}, 0) \oplus V_S(\frac{1}{2}, \frac{1}{2}) \oplus V_S(\frac{3}{2}, \frac{1}{2}). \]

The spin is defined as the maximal eigenvalues of \( \hat{s}_3 \). Here we can find \( \max(s_3) = \frac{1}{2} + \frac{3}{2} = 2 \), thus the graviton is spin-2.

C. Gauge connection

The gauge connection is defined as

\[ \hat{A} = \hat{A}_a^\alpha \otimes \hat{\gamma}^\alpha \otimes \hat{T}_a, \]

similarly we have

\[ \hat{A}_a^\alpha = \int_{R^4} A_a^\alpha(x) \hat{\epsilon}(x) d^4 x = \int_{R^4} \hat{\bar{A}}_a^\alpha(p) \hat{\epsilon}(p) d^4 p, \]

where \( A_a^\alpha(x) \) and \( \hat{A}_a^\alpha(x) \) are coordinate and momentum functions, respectively. The relation is

\[ \hat{\bar{A}}_a^\alpha(p) = (2\pi)^{-2} \int_{R^4} A_a^\alpha(x) \exp(ipx) d^4 x, \]

\[ A_a^\alpha(x) = (2\pi)^{-2} \int_{R^4} \hat{\bar{A}}_a^\alpha(p) \exp(-ipx) d^4 p, \]

The adjoint of the gauge connection is,

\[ \bar{A}_a^\alpha = G_{ab} \hat{A}_b^\alpha. \]
D. Definition of the general covariance derivative operator

Now we define the covariance derivative operator as

\[
\hat{D}_\alpha = -i\hat{\theta}_\alpha^\mu \otimes \hat{p}_\mu + \frac{i}{2} \hat{\Gamma}_\alpha^{\rho\sigma} \otimes \hat{s}_{\rho\sigma} - i\hat{A}_\alpha^a \otimes \hat{T}_a. \tag{373}
\]

This operator has connections of gravity, connections of gauge and spin vierbein. Thus it can describe all force fields. As in quantum mechanics, when acting on operators, it is represented in the form of commutating calculation, when acting on matter fields, it is represented as an operator acting on quantum states.

Here we list the properties of this newly defined general covariance derivative operator,

\[
\overline{\hat{D}}_\alpha = -\hat{D}_\alpha, \tag{374}
\]

\[
\hat{D}_\alpha|_{e_{st}(x)} = \int_{R^4} \left[ \theta_\alpha^\mu(x') \frac{\partial}{\partial x^\mu} + \frac{i}{2} \Gamma_\alpha^{\rho\sigma}(x') \hat{s}_{\rho\sigma} - i A_\alpha^a(x') \hat{T}_a \right] \delta^4(x - x')|_{e_{st}(x')}d^4x', \tag{375}
\]

\[
\langle e^{st}(x)|\hat{D}_\alpha = \int_{R^4} d^4x' \langle e^{st}(x') \left[ -\theta_\alpha^\mu(x') \frac{\partial}{\partial x^\mu} + \frac{i}{2} \Gamma_\alpha^{\rho\sigma}(x') \hat{s}_{\rho\sigma} - i A_\alpha^a(x') \hat{T}_a \right] \delta^4(x - x'), \tag{376}
\]

\[
\hat{D}_\alpha|_{e_{st}(p)} = (2\pi)^{-2} \int_{R^4} \left[ -\theta_\alpha^\mu(p') \frac{\partial}{\partial p^\mu} + \frac{i}{2} \Gamma_\alpha^{\rho\sigma}(p') \hat{s}_{\rho\sigma} - i A_\alpha^a(p') \hat{T}_a \right] |_{e_{st}(p + p')}d^4p', \tag{377}
\]

\[
\langle e^{st}(p)|\hat{D}_\alpha = (2\pi)^{-2} \int_{R^4} d^4p' \langle e^{st}(p + p') \left[ -\theta_\alpha^\mu(p') \frac{\partial}{\partial p^\mu} + \frac{i}{2} \Gamma_\alpha^{\rho\sigma}(p') \hat{s}_{\rho\sigma} - i A_\alpha^a(p') \hat{T}_a \right]\rangle_{e_{st}(p + p')}, \tag{378}
\]

\[
[\hat{D}_\alpha, \hat{\varepsilon}(x)] = \int_{R^4} \theta_\alpha^\mu(x') \frac{\partial}{\partial x^\mu} \delta^4(x - x') \hat{\varepsilon}(x')d^4x', \tag{379}
\]

\[
[\hat{D}_\alpha, \hat{\varepsilon}(p)] = (2\pi)^{-2} \int_{R^4} -\theta_\alpha^\mu(p') \frac{\partial}{\partial p^\mu} \hat{\varepsilon}(p + p')d^4p', \tag{380}
\]

\[
[\hat{D}_\alpha, \hat{\gamma}_\alpha^{\rho\cdots\sigma\rho_\cdots\sigma}] = -\Gamma_{\alpha,\rho}^{\alpha,\rho_\cdots\sigma} \hat{\gamma}_\alpha^{\rho_\cdots\sigma} - \cdots - \Gamma_{\alpha,\rho}^{\alpha,\rho_\cdots\sigma} \hat{\gamma}_\alpha^{\rho_\cdots\sigma}, \tag{381}
\]

\[
[\hat{D}_\alpha, \hat{\gamma}_\alpha^{\rho\cdots\sigma} \rho_\cdots\sigma] = \Gamma^\rho_{\alpha,\alpha_1} \hat{\gamma}_\rho^{\rho_\cdots\sigma} \cdots + \Gamma^\rho_{\alpha,\alpha_\rho} \hat{\gamma}_\rho^{\rho_\cdots\sigma}, \tag{382}
\]

\[
[\hat{D}_\alpha, \hat{T}_b] = -ic_{ab} A_\alpha^a \hat{T}_c, \tag{383}
\]

\[
\hat{D}_\alpha(a|\Psi) + b|\Phi) = a\hat{D}_\alpha|\Psi) + b\hat{D}_\alpha|\Phi), \tag{384}
\]

\[
[\hat{D}_\alpha, a\hat{A} + b\hat{B}] = a[\hat{D}_\alpha, \hat{A}] + b[\hat{D}_\alpha, \hat{B}], \tag{385}
\]

\[
\hat{D}_\alpha(\hat{A}|\Psi) = [\hat{D}_\alpha, \hat{A}]|\Psi) + \hat{A}(\hat{D}_\alpha|\Psi)), \tag{386}
\]

\[
[\hat{D}_\alpha, \hat{A}\hat{B}] = [\hat{D}_\alpha, \hat{A}]\hat{B} + \hat{A}[\hat{D}_\alpha, \hat{B}], \tag{387}
\]

\[
[\hat{D}_\alpha, \hat{A} \otimes \hat{B}] = [\hat{D}_\alpha, \hat{A}] \otimes \hat{B} + \hat{A} \otimes [\hat{D}_\alpha, \hat{B}], \tag{388}
\]

\[
[\hat{D}_\alpha, \hat{A} \cdot \hat{B}] = [\hat{D}_\alpha, \hat{A}] \cdot \hat{B} + \hat{A} \cdot [\hat{D}_\alpha, \hat{B}], \tag{389}
\]

\[
[\hat{D}_\alpha, \hat{A} \wedge \hat{B}] = [\hat{D}_\alpha, \hat{A}] \wedge \hat{B} \wedge \hat{B} \wedge [\hat{D}_\alpha, \hat{B}], \tag{390}
\]

\[
[\hat{D}_\alpha, \text{con}(\hat{A})] = \text{con}([\hat{D}_\alpha, \hat{A}]), \tag{391}
\]

\[
[\hat{D}_\alpha, [\hat{A}, \hat{B}]] = [[\hat{D}_\alpha, \hat{A}], \hat{B}] + [\hat{A}, [\hat{D}_\alpha, \hat{B}]], \tag{392}
\]

\[
\langle \Phi | \left( \hat{D}_\alpha |\Psi) - \left( \langle \Phi | \hat{D}_\alpha \right) |\Psi), \tag{393}
\]

\[
\langle \Phi | \hat{\varepsilon}(x) \left( \hat{D}_\alpha |\Psi) - \left( \langle \Phi | \hat{D}_\alpha \right) \hat{\varepsilon}(x) |\Psi) = \theta_\alpha^\mu(x) \partial_\mu \langle \Phi | \hat{\varepsilon}(x) |\Psi), \tag{394}
\]

\[
\langle \Phi | \hat{\varepsilon}(p) \left( \hat{D}_\alpha |\Psi) - \left( \langle \Phi | \hat{D}_\alpha \right) \hat{\varepsilon}(p) |\Psi) = \theta_\alpha^\mu(p) \partial_\mu \langle \Phi | \hat{\varepsilon}(x) |\Psi), \tag{395}
\]

\[
[[\hat{D}_\alpha, \hat{A}], \hat{B}] + [[\hat{A}, [\hat{D}_\alpha, \hat{B}]] = 0. \tag{396}
\]

The general covariance derivative operator \( \hat{D}_\alpha \) can act on the matter fields. Let’s next see how the calculation can be made explicitly.

As we have already seen, the matter fields can be described in several forms below, respectively,

\[
|\Psi) = |e^{st}) \otimes |e_s) \otimes |e_t) = \int_{R^4} \Psi^{st}(x)|e_{st}(x))d^4x = \int_{R^4} \tilde{\Psi}^{st}(p)|e_{st}(x))d^4p. \tag{397}
\]
By applying the operator $\hat{D}_\alpha$, we have,

$$\hat{D}_\alpha |\Psi\rangle = |D_\alpha \Psi^{st}\rangle \otimes |e_s\rangle \otimes |e_t\rangle$$

$$= \int_{R^4} D_\alpha \Psi^{st}(x)|e_s(x)\rangle d^4x$$

$$= \int_{R^4} D_\alpha \tilde{\Psi}^{st}(p)|e_t(x)\rangle d^4p,$$

(398)

where the calculation should be in form

$$\hat{D}_\alpha |\Psi\rangle = (-i\hat{\theta}_\mu^s \hat{\mu}_\mu + \frac{i}{2} \hat{\Gamma}_\alpha^\rho_\sigma \hat{s}_\rho_\sigma - i\hat{A}_\alpha^a \hat{T}_a)|\Psi\rangle,$$

(399)

$$|D_\alpha \Psi^{st}\rangle = \hat{\theta}_\mu^s |\partial_\mu \Psi^{st}\rangle + \frac{i}{2} \hat{\Gamma}_\alpha^\rho_\sigma (\hat{s}_\rho_\sigma)^* |\Psi^{st}\rangle - i\hat{A}_\alpha^a (\hat{T}_a)_t^p |\Psi^{st}\rangle,$$

(400)

$$D_\alpha \Psi^{st} = \theta^s_{\mu} \partial_\mu \Psi^{st} + \frac{i}{2} \hat{\Gamma}_\alpha^\rho_\sigma (\hat{s}_\rho_\sigma)^* \Psi^{st} - i\hat{A}_\alpha^a (\hat{T}_a)_t^p \Psi^{st},$$

(401)

$$D_\alpha \tilde{\Psi}^{st} = -i\hat{\theta}_\mu^s * [p_\mu \tilde{\Psi}^{st}] + \frac{i}{2} \hat{\Gamma}_\alpha^\rho_\sigma [ (\hat{s}_\rho_\sigma)^* \tilde{\Psi}^{st} ] - i\hat{A}_\alpha^a * [ (\hat{T}_a)_t^p \tilde{\Psi}^{st}].$$

(402)

Similar results are for the adjoint state of matter fields when we apply the operator $\hat{D}_\alpha$,

$$\langle \Psi | \hat{D}_\alpha = \langle e^t | \otimes \langle e^s | \otimes |D_\alpha \Psi^{st}\rangle$$

$$= \int_{R^4} d^4x (|e^t(x)\rangle D_\alpha \Psi^{st}_t(x))$$

$$= \int_{R^4} d^4p (|e^t(x)\rangle D_\alpha \tilde{\Psi}^{st}_t(p)),$$

(404)

where

$$\langle \Psi | \hat{D}_\alpha = \langle \Psi | (-i\hat{\theta}_\mu^s \hat{\mu}_\mu + \frac{i}{2} \hat{\Gamma}_\alpha^\rho_\sigma \hat{s}_\rho_\sigma - i\hat{T}_a \hat{A}_\alpha^a),$$

(405)

$$\langle D_\alpha \Psi^{st}\rangle = \langle \partial_\mu \Psi^{st} \hat{\mu}_\mu + \frac{i}{2} \hat{\Gamma}_\alpha^\rho_\sigma (\hat{s}_\rho_\sigma)^* \Psi^{st} - i\hat{A}_\alpha^a (\hat{T}_a)_t^p \Psi^{st},$$

(406)

$$D_\alpha \Psi^{st} = -\theta^s_{\mu} \partial_\mu \Psi^{st} + \frac{i}{2} \hat{\Gamma}_\alpha^\rho_\sigma (\hat{s}_\rho_\sigma)^* \Psi^{st} - i\hat{A}_\alpha^a (\hat{T}_a)_t^p \Psi^{st},$$

(407)

$$D_\alpha \tilde{\Psi}^{st} = i\hat{\theta}_\mu^s * (p_\mu \tilde{\Psi}^{st}) + \frac{i}{2} \hat{\Gamma}_\alpha^\rho_\sigma [(\hat{s}_\rho_\sigma)^* \tilde{\Psi}^{st}] - i\hat{A}_\alpha^a * [(\hat{T}_a)_t^p \tilde{\Psi}^{st}].$$

(408)

For gauge operator $\hat{X}$, the covariance derivation is

$$\hat{X} = \frac{1}{p!} \hat{X}_a^{a_1\cdots a_p} \otimes \hat{\gamma}^{a_1\cdots a_p} \otimes \hat{T}_a,$$

(409)

$$[\hat{D}_\alpha, \hat{X}] = \frac{1}{p!} (D_\alpha \hat{X}_a^{a_1\cdots a_p}) \otimes \hat{\gamma}^{a_1\cdots a_p} \otimes \hat{T}_a,$$

(410)

$$D_\alpha \hat{X}_a^{a_1\cdots a_p} = \hat{\theta}_{a_1}^\rho \hat{\rho}_a \hat{X}_a^{a_2\cdots a_p} - i\hat{\gamma}_a^{bc} \hat{A}_b \hat{X}_a^{c} \hat{\gamma}_{a_2\cdots a_p} - \hat{\Gamma}_{a_1\cdots a_p}^\rho_a \hat{X}_a^{a_2\cdots a_p} - \cdots - \hat{\Gamma}_{a_1\cdots a_p}^a a \hat{X}_a^{a_2\cdots a_p},$$

(411)

where we have used the notation $\hat{\Gamma}_{a_1\cdots a_p}^\rho = \eta_{\beta_3\cdots \beta_q} \hat{\Gamma}_{a_1\cdots a_p}^{\beta_3\cdots \beta_q}$.

The calculation of the derivation of the gravity field takes the form

$$\hat{Y} = \frac{1}{p!q!} \hat{Y}_{\beta_1\cdots \beta_q} \otimes \hat{\gamma}^{a_1\cdots a_p},$$

(412)

$$[\hat{D}_\alpha, \hat{Y}] = \frac{1}{p!q!} (D_\alpha \hat{Y}_{\beta_1\cdots \beta_q}) \otimes \hat{\gamma}^{a_1\cdots a_p},$$

(413)

$$D_\alpha \hat{Y}_{\beta_1\cdots \beta_q} = \hat{\theta}_{\beta_1}^\rho \hat{\rho}_{a} \hat{Y}_{\beta_2\cdots \beta_q} - \hat{\gamma}_{b}^{bc} \hat{A}_b \hat{Y}_{\beta_2\cdots \beta_q} - \cdots - \hat{\Gamma}_{\beta_1\cdots \beta_q}^\rho a \hat{Y}_{\alpha_1\cdots \alpha_p}$$

$$+ \hat{\Gamma}_{\beta_1}^\rho \hat{\rho}_a \hat{Y}_{\beta_2\cdots \beta_q} + \cdots + \hat{\Gamma}_{\beta_1}^a a \hat{Y}_{\beta_2\cdots \beta_q}.$$  

(414)
Based on the general covariation derivation, the generalized divergence and curl can be defined. Suppose $\hat{K}$ is a $p$-form antisymmetric spin tensor, we define

\begin{align}
\hat{D} \wedge \hat{K} &= \hat{\gamma} \wedge [\hat{D}_\alpha, \hat{K}], \\
\hat{D} \cdot \hat{K} &= \hat{\gamma} \cdot [\hat{D}_\alpha, \hat{K}],
\end{align}

where the covariance differential takes the form $\hat{D} = \hat{\gamma}^\alpha \otimes \hat{D}_\alpha$. We can find that $\hat{D} \wedge \hat{K}$ is a $(p+1)$-form antisymmetric spin tensor, and $\hat{D} \cdot \hat{K}$ is a $(p-1)$-form tensor. So these calculations can be considered as a generalization of divergence and curl.

### E. Curvatures

We then define the interaction curvature as

\begin{align}
\hat{\Omega}_{\alpha\beta} &= i[\hat{D}_\alpha, \hat{D}_\beta] - i f^\gamma_{\alpha\beta} \hat{D}_\gamma, \\
\hat{\Omega} &= \hat{\Omega}_{\alpha\beta} \otimes \hat{s}^{\alpha\beta}
\end{align}

The interaction curvature has adjoint, is antisymmetric and satisfies the Bianchi identity,

\begin{align}
\hat{\Omega}_{\alpha\beta} &= -\hat{\Omega}_{\beta\alpha} \\
\overline{\hat{\Omega}_{\alpha\beta}} &= \hat{\Omega}_{\alpha\beta} \\
\hat{D}_\alpha \hat{\Omega}_{\beta\gamma} + \hat{D}_\beta \hat{\Omega}_{\gamma\alpha} + \hat{D}_\gamma \hat{\Omega}_{\alpha\beta} &= 0.
\end{align}

The Bianchi identity can be represented as the covariance curl as

\begin{align}
\hat{D} \wedge \hat{\Omega} &= 0.
\end{align}

The proof that the interaction curvature satisfies the Bianchi identity is presented as follows. We start from the Jacobi relation,

\begin{align}
[\hat{D}_\alpha, [\hat{D}_\beta, \hat{D}_\gamma]] + [\hat{D}_\beta, [\hat{D}_\gamma, \hat{D}_\alpha]] + [\hat{D}_\gamma, [\hat{D}_\alpha, \hat{D}_\beta]] = 0.
\end{align}

With the help of the definition of the interaction curvature, we have

\begin{align}
[\hat{D}_\alpha, [\hat{D}_\beta, \hat{D}_\gamma]] &= i[\hat{D}_\alpha, [\hat{D}_\beta, \hat{D}_\gamma]] - i[\hat{D}_\alpha, f^p_{\beta\gamma}] \hat{D}_p - i f^p_{\beta\gamma} [\hat{D}_\alpha, \hat{D}_p] \\
&= i[\hat{D}_\alpha, [\hat{D}_\beta, \hat{D}_\gamma]] - i(\hat{\theta}^p_{\alpha\beta} [\hat{D}_\gamma, \hat{f}^p_{\beta\gamma} - \hat{\Gamma}^p_{\alpha\beta} \hat{f}^p_{\gamma\rho}] - \hat{f}^p_{\beta\gamma} \hat{\Omega}_{\alpha\rho})
\end{align}

also

\begin{align}
\hat{D}_\alpha \hat{\Omega}_{\beta\gamma} &= [\hat{D}_\alpha, \hat{\Omega}_{\beta\gamma}] - \hat{\Gamma}^p_{\alpha,\beta} \hat{\Omega}_{\gamma\rho} - \hat{\Gamma}^p_{\alpha,\gamma} \hat{\Omega}_{\rho\beta} \\
&= i[\hat{D}_\alpha, [\hat{D}_\beta, \hat{D}_\gamma]] - i[\hat{D}_\alpha, f^p_{\beta\gamma}] \hat{D}_p - i f^p_{\beta\gamma} [\hat{D}_\alpha, \hat{D}_p] - \hat{\Gamma}^p_{\alpha,\beta} \hat{\Omega}_{\gamma\rho} - \hat{\Gamma}^p_{\alpha,\gamma} \hat{\Omega}_{\rho\beta} \\
&= i[\hat{D}_\alpha, [\hat{D}_\beta, \hat{D}_\gamma]] - i(\hat{\theta}^p_{\alpha\beta} [\hat{D}_\gamma, \hat{f}^p_{\beta\gamma} + \hat{\Gamma}^p_{\alpha,\gamma} \hat{f}^p_{\rho\gamma}] - \hat{f}^p_{\beta\gamma} \hat{\Omega}_{\alpha\rho} + \hat{\Gamma}^p_{\alpha,\beta} \hat{\Omega}_{\gamma\rho} - \hat{\Gamma}^p_{\alpha,\gamma} \hat{\Omega}_{\rho\beta})
\end{align}

Substituting the definition of the spin connection and structure functions into the Jacobi identity, we have

\begin{align}
\hat{D}_\alpha \hat{\Omega}_{\beta\gamma} + \hat{D}_\beta \hat{\Omega}_{\gamma\alpha} + \hat{D}_\gamma \hat{\Omega}_{\alpha\beta} &= i \left( [\hat{D}_\alpha, [\hat{D}_\beta, \hat{D}_\gamma]] + [\hat{D}_\beta, [\hat{D}_\gamma, \hat{D}_\alpha]] + [\hat{D}_\gamma, [\hat{D}_\alpha, \hat{D}_\beta]] \right) \\
&- (\hat{\theta}^p_{\alpha\beta} [\hat{D}_\gamma, \hat{f}^p_{\beta\gamma} + \hat{\Gamma}^p_{\alpha,\gamma} \hat{f}^p_{\rho\gamma}] - \hat{f}^p_{\beta\gamma} \hat{\Omega}_{\alpha\rho} + \hat{\Gamma}^p_{\alpha,\beta} \hat{\Omega}_{\gamma\rho} - \hat{\Gamma}^p_{\alpha,\gamma} \hat{\Omega}_{\rho\beta}) \\
&= 0
\end{align}
Note that the interaction curvature can be represented as the summation of gravity curvature and the gauge curvature,

\[ \Omega_{\alpha\beta} = \frac{1}{2} \hat{R}_{\alpha\beta}^{\rho\sigma} \otimes \delta_{\rho\sigma} + \hat{F}_{\alpha\beta}^{\rho} \otimes T_{\rho}. \]  \hspace{1cm} (428)

We next consider respectively the gravity curvature and the gauge curvature. The gravity curvature takes the form

\[ \hat{R}_{\alpha\beta}^{\rho\sigma} = \hat{\theta}_{\alpha}^{\mu} \partial_{\mu} \hat{\Gamma}_{\beta}^{\rho\sigma} - \hat{\theta}_{\beta}^{\mu} \partial_{\mu} \hat{\Gamma}_{\alpha}^{\rho\sigma} + \hat{\Gamma}_{\gamma,\alpha}^{\rho} \hat{\Gamma}_{\beta}^{\gamma\sigma} - \hat{\Gamma}_{\gamma,\beta}^{\rho} \hat{\Gamma}_{\alpha}^{\gamma\sigma} - \hat{f}_{\alpha\beta}^{\rho} \hat{f}_{\gamma}^{\gamma\sigma}, \]  \hspace{1cm} (429)

\[ \hat{R} = \frac{1}{2} \hat{R}_{\alpha\beta}^{\rho\sigma} \otimes \hat{s}_{\rho\sigma}, \]  \hspace{1cm} (430)

\[ \hat{\nabla}_{\alpha} = -i \hat{\theta}_{\alpha}^{\mu} \hat{p}_{\mu} \otimes \frac{1}{2} \hat{\Gamma}_{\rho\sigma}^{\alpha}, \]  \hspace{1cm} (431)

\[ \hat{R}_{\mu\nu} = i [\hat{\nabla}_{\mu}, \hat{\nabla}_{\nu}] - i f_{\mu\nu} \hat{\nabla}. \]  \hspace{1cm} (432)

The properties of the gravity curvature are listed in the following,

\[ \hat{R}_{\alpha\beta}^{\rho\sigma} = \hat{R}_{\alpha\beta}^{\rho\sigma}, \]  \hspace{1cm} (433)

\[ \hat{R}_{\alpha\beta}^{\rho\sigma} = - \hat{R}_{\alpha\beta}^{\rho\sigma}, \]  \hspace{1cm} (434)

\[ \hat{R}_{\alpha\beta}^{\rho\sigma} = - \hat{R}_{\beta\alpha}^{\rho\sigma}, \]  \hspace{1cm} (435)

\[ \hat{R}_{\sigma,\alpha\beta}^{\rho} + \hat{R}_{\alpha,\beta\sigma}^{\rho} + \hat{R}_{\beta,\sigma\alpha}^{\rho} = 0, \]  \hspace{1cm} (436)

\[ \hat{R}_{\rho\sigma,\alpha\beta} = \hat{R}_{\alpha\beta,\rho\sigma}. \]  \hspace{1cm} (437)

The Bianchi identity for gravity curvature and the contraction take the form,

\[ \hat{D}_{\alpha} \hat{R}_{\beta\gamma}^{\rho\sigma} + \hat{D}_{\beta} \hat{R}_{\gamma\alpha}^{\rho\sigma} + \hat{D}_{\gamma} \hat{R}_{\alpha\beta}^{\rho\sigma} = 0, \]  \hspace{1cm} (438)

\[ D_{\alpha}(\hat{R}_{\alpha}^{\rho} - \frac{1}{2} \hat{R}) = 0, \]  \hspace{1cm} (439)

where we used the notations \( \hat{R}_{\beta}^{\rho} = \hat{R}_{\beta}^{\rho\gamma}, \hat{R} = \hat{R}_{\beta}^{\rho}, \) \( \hat{R}_{\rho\sigma} = \hat{R}_{\rho\sigma}^{\beta\gamma}. \) This means

\[ \hat{D} \wedge \hat{R} = 0. \]  \hspace{1cm} (440)

The proof that the summation of circular of indices is zero can be like the follows,

\[ \hat{R}_{\sigma,\alpha\beta}^{\rho} + \hat{R}_{\alpha,\beta\sigma}^{\rho} + \hat{R}_{\beta,\sigma\alpha}^{\rho} = \hat{\theta}_{\sigma}^{\mu} \partial_{\mu} \hat{f}_{\alpha\beta}^{\rho} + \hat{\theta}_{\alpha}^{\mu} \partial_{\mu} \hat{f}_{\beta\sigma}^{\rho} + \hat{\theta}_{\beta}^{\mu} \partial_{\mu} \hat{f}_{\sigma\alpha}^{\rho} + \hat{f}_{\alpha\beta}^{\rho} \hat{f}_{\gamma}^{\gamma\sigma} + \hat{f}_{\sigma\alpha}^{\rho} \hat{f}_{\beta\gamma}^{\beta}, \]  \hspace{1cm} (441)
The gauge curvature takes the form
\[ \hat{F}_{a\beta} = \hat{\theta}^\gamma_{\alpha} \partial_{\alpha} \hat{A}_a - \hat{\theta}^\beta_{\alpha} \partial_{\alpha} \hat{A}_a - i C^\gamma_{bc} \hat{A}_b^\alpha \hat{A}_c^\beta - \hat{\gamma}_a^\alpha \hat{A}_p^\beta. \] (442)
\[ \hat{F} = \hat{F}_{a\beta} \otimes s^{a\beta} \otimes \hat{T}_a. \] (443)

The properties of the gauge curvature are listed in the following,
\[ \bar{F}_{a\beta} = G_{ab} \hat{F}_{b\beta}, \] (444)
\[ \hat{F}_{a\beta} = -\hat{F}_{b\alpha}, \] (445)
\[ \hat{\Theta}_a \hat{F}_{\beta\gamma} + \hat{\Theta}_b \hat{F}_{a\gamma} + \hat{\Theta}_c \hat{F}_{\alpha\beta} = 0. \] (446)

In a concise form, we have
\[ \hat{D} \wedge \hat{F} = 0. \] (447)

We may also have the square of the general covariance derivation. Some results are,
\[ [\hat{D}_\alpha, \hat{D}_\beta] |\Psi\rangle = (-i \hat{\Theta}_{\alpha\beta} + \hat{\gamma}_{\alpha\beta} \hat{D}_\gamma) |\Psi\rangle. \] (448)

For force field represented as operator \( \hat{H} \), the square of the general covariance derivation has properties,
\[ \hat{D}_\alpha (\hat{D}_\beta \hat{H}) - \hat{D}_\beta (\hat{D}_\alpha \hat{H}) = -i [\hat{\Theta}_{\alpha\beta}, \hat{H}]. \] (449)

The proof of this equation is presented in the following. The Jacobi equation takes the form
\[ [\hat{D}_\alpha, [\hat{D}_\beta, \hat{H}]] - [\hat{D}_\beta, [\hat{D}_\alpha, \hat{H}]] - [\hat{H}, [\hat{D}_\alpha, \hat{D}_\beta]] = 0 \] (450)

With the help of the equations,
\[ [\hat{D}_\alpha, [\hat{D}_\beta, \hat{H}]] - [\hat{D}_\beta, [\hat{D}_\alpha, \hat{H}]] = [[\hat{D}_\alpha, \hat{D}_\beta], \hat{H}] = -i [\hat{\Theta}_{\alpha\beta}, \hat{H}] + \hat{\gamma}_{\alpha\beta} [\hat{D}_\gamma, \hat{H}] \] (451)
we can find
\[ \hat{D}_\alpha (\hat{D}_\beta \hat{H}) - \hat{D}_\beta (\hat{D}_\alpha \hat{H}) = [\hat{D}_\alpha, [\hat{D}_\beta, \hat{H}]] + \hat{\gamma}_{\alpha\beta} [\hat{D}_\gamma, \hat{H}] = [\hat{D}_\alpha, [\hat{D}_\beta, \hat{H}]] - [\hat{D}_\beta, [\hat{D}_\alpha, \hat{H}]] + \hat{\gamma}_{\alpha\beta} [\hat{D}_\gamma, \hat{H}] = -i [\hat{\Theta}_{\alpha\beta}, \hat{H}] \] (452)

VII. DIRAC EQUATION, YANG-MILLS EQUATION AND EINSTEIN EQUATION

A. Dirac equation

The Dirac equation for matter fields can be written as:
\[ (i \hat{\gamma}^\alpha \hat{D}_\alpha - \hat{m}) |\Psi\rangle = 0, \] (453)
where \( \hat{m} \) is the mass matrix in gauge space, its eigenvalues are masses of the corresponding elementary particles. Similarly the adjoint matter fields also satisfy Dirac equation,
\[ |\Psi\rangle (i \hat{D}_\alpha \hat{\gamma}^\alpha - \hat{m}) = 0, \] (454)

The square differential of the matter fields can be represented as
\[ (\hat{D}_\alpha \hat{D}^\alpha - s^{a\beta} \hat{\Theta}_{\alpha\beta} + \hat{\gamma}_{\alpha\beta} \hat{D}^\beta + A_{\alpha}^a \hat{\gamma}^\alpha \hat{m}_a + \hat{m}^2) |\Psi\rangle = 0, \] (455)
or
\[ (\hat{D}_\alpha \hat{D}^\alpha - \hat{R} - \hat{F} + \hat{\gamma}_{\alpha\beta} \hat{D}^\beta + A_{\alpha}^a \hat{\gamma}^\alpha \hat{m}_a + \hat{m}^2) |\Psi\rangle = 0. \] (456)
The proof of this equation starts from the Dirac equation by applying operator \((-i\gamma^\alpha \hat{D}_\alpha - \hat{m})\) on both sides,

\[
(-i\gamma^\alpha \hat{D}_\alpha - \hat{m})(i\gamma^\beta \hat{D}_\beta - \hat{m})|\Psi\rangle = 0. \tag{457}
\]

We have

\[
(-i\gamma^\alpha \hat{D}_\alpha - \hat{m})(i\gamma^\beta \hat{D}_\beta - \hat{m}) = \gamma^\alpha \hat{D}_\alpha \gamma^\beta \hat{D}_\beta + i\gamma^\alpha [\hat{D}_\alpha, \gamma^\beta] \hat{D}_\beta + \hat{m}^2,
\]

where

\[
\begin{align*}
\hat{D}_\alpha \gamma^\beta \hat{D}_\beta &= \gamma^\alpha \gamma^\beta \hat{D}_\alpha \hat{D}_\beta + \gamma^\alpha [\hat{D}_\alpha, \gamma^\beta] \hat{D}_\beta \\
&= \hat{D}_\alpha \hat{D}_\alpha - i\gamma^\alpha \hat{D}_\alpha \hat{D}_\beta - \hat{D}_\alpha \gamma^\alpha \gamma^\beta \hat{D}_\beta \\
&= \hat{D}_\alpha \hat{D}_\alpha - i\gamma^\alpha \hat{D}_\alpha \hat{D}_\beta + \hat{D}_\alpha \gamma^\alpha \gamma^\beta \hat{D}_\beta \\
&= \hat{D}_\alpha \hat{D}_\alpha - i\gamma^\alpha \hat{D}_\alpha \hat{D}_\beta + \hat{D}_\alpha \gamma^\alpha \gamma^\beta \hat{D}_\beta
\end{align*}
\]

Thus the squared differential equation of the matter fields is obtained. Similarly, we have

\[
i\gamma^\alpha [\hat{D}_\alpha, \hat{m}] = \hat{D}_\alpha \hat{m} = \hat{A}_\alpha \gamma^\alpha \hat{m}_a. \tag{460}
\]

Then

\[
(-i\gamma^\alpha \hat{D}_\alpha - \hat{m})(i\gamma^\beta \hat{D}_\beta - \hat{m}) = \gamma^\alpha \gamma^\beta \hat{D}_\alpha \hat{D}_\beta + i\gamma^\alpha [\hat{D}_\alpha, \gamma^\beta] \hat{D}_\beta + \hat{m}^2
\]

where we have used the equation,

\[
\gamma^\alpha \gamma^\beta = g^\alpha \beta - 2i\gamma^\alpha \beta.
\]

Thus the squared differential equation of the matter fields is obtained. Similarly, we have

\[
\langle \Psi | (\hat{D}_\alpha \hat{D}_\alpha - \Omega_{\alpha\beta} s^{\alpha\beta} - \hat{D}_\alpha \hat{D}_\alpha - A^\alpha \gamma^\alpha \hat{m}_a + \hat{m}^2) = 0. \tag{463}
\]

We define energy-momentum tensor of matter fields as

\[
t^\alpha_\beta (x) = \frac{i}{4} \{\langle \Psi | \hat{\varepsilon} (x) | (\gamma^\alpha \hat{D}_\alpha + \gamma^\beta \hat{D}_\beta) | \Psi \rangle \} + \{\langle \Psi | (\hat{D}_\alpha \gamma^\alpha + \hat{D}_\alpha \gamma^\beta) \hat{\varepsilon} (x) | \Psi \rangle \}
\]

B. Yang-Mills equation and gauge condition

The gauge fields Yang-Mills equation takes the form

\[
\hat{D} \cdot \hat{F} = \hat{J}, \tag{464}
\]

or explicitly it can be written as,

\[
\hat{D}_\alpha \hat{F}^\alpha_\beta = \hat{j}^\beta_\alpha. \tag{465}
\]

Here the total current is written as,

\[
\hat{j}^\beta_\alpha = j^\beta_a + M^\beta_\alpha \hat{A}^\beta_b, \tag{466}
\]

\(M^\beta_\alpha\) are mass tensor of gauge bosons with \(M^2_2 = m^2_2, M^3_3 = M^4_4 = m^2_W, M^{4+p}_4 = m^2, p = 1, 2, \cdots, 8\) for gluons and \(M^0_0 = 0\) elsewhere, \(j^\beta_a\) is the gauge current density defined as,

\[
j^\beta_a (x) = \langle \Psi | \hat{\varepsilon} (x) \gamma^\beta \hat{T}_a | \Psi \rangle. \tag{467}
\]
We have made the calculations,
\[
\hat{D}_\alpha \hat{F}^{\alpha \beta} = \hat{\theta}^\mu_{\alpha, \mu} \hat{F}^{\alpha \beta} + C_{ab} \hat{A}_a^b \hat{F}^{\alpha \beta} + \hat{\Gamma}^{\alpha}_{\alpha, \rho} \hat{F}^{\rho \beta} + \hat{\Gamma}^{\beta}_{\alpha, \rho} \hat{F}^{\alpha \rho}
\]
\[
= \hat{j}_a^\beta. \tag{468}
\]

The conservation law for the gauge charges is
\[
\hat{D}_\beta \hat{j}_a^\beta = 0, \tag{469}
\]
or
\[
\hat{D} \cdot \hat{j} = 0.
\]

The proof of the conservation law of gauge charges is like the following. Due to the Yang-Mills equation,
\[
\hat{D}_\beta \hat{j}_a^\beta = \hat{D}_\beta \hat{D}_\alpha \hat{F}^{\alpha \beta} = \frac{1}{2} (\hat{D}_\beta \hat{D}_\alpha - \hat{D}_\alpha \hat{D}_\beta) \hat{F}^{\alpha \beta} \tag{470}
\]
with the help of the squared covariance derivation, we have
\[
(\hat{D}_\beta \hat{D}_\beta - \hat{D}_\beta \hat{D}_\beta) \hat{F}^{\alpha \beta} = C_{abc} \hat{F}_b^{\beta \alpha} \hat{F}_c^{\alpha \beta} + 2 \hat{R}_a^{\alpha, \beta} \hat{F}^{\alpha \beta} = 0, \tag{471}
\]
where we have used the properties that \(C_{abc}\) are antisymmetric while \(\hat{R}_a^{\alpha, \beta}\) are symmetric.

We next consider the gauge condition and the mass commutation relation. Since the conservation law for gauge charges and the divergence of the gauge current, we can find that the gauge connections satisfy the condition,
\[
\hat{D}_\alpha (M_b^a \hat{A}_b^{\alpha}) = -\hat{u}_a = i \left( [\hat{T}_a, \hat{m}] \right) \langle \Psi \rangle \langle \Psi \rangle. \tag{472}
\]
This is the gauge condition. We can see that in the present work this condition is not artificial but a necessary condition for Yang-Mills equation. On the other hand, this equation is a restriction which indicates the relation between gauge mass and the mass of matter fields.

Since mass of photon is zero, \(M_1^1 = 0\), we also have
\[
[\hat{Q}, \hat{m}] = 0. \tag{473}
\]
The gauge condition \(472\) is always satisfied, so gauge of electromagnetic is arbitrary.

The masses of weak charges \(\hat{W}_+, \hat{W}_-, \hat{Z}_0\) are not zeroes, also they do not commute with the mass matrices of matter fields,
\[
[\hat{W}_\pm, \hat{m}] \neq 0, \tag{474}
\]
\[
[\hat{Z}_0, \hat{m}] \neq 0. \tag{475}
\]
From Eq.(472), we know that \(M_a^b\) can not be zero for weak charges \(\hat{W}_\pm, \hat{Z}_0\), that means weak bosons have masses. Additionally, the gauge condition \(472\) is a constraint.

The masses of gluons are all \(m\), and we know that,
\[
[\hat{\lambda}_p, \hat{m}] = 0, \tag{476}
\]
where \(p = 1, 2, \cdots, 8\). The gauge condition becomes as,
\[
\hat{D}_\alpha (M_a^b \hat{A}_b^{\alpha}) = \hat{D}_\alpha (m^2 \hat{A}_b^{\alpha}) = 0, \tag{477}
\]
that means,
\[
\hat{D}_\alpha \hat{A}_b^{\alpha} = 0. \tag{478}
\]
This is the gauge condition for gluon field.

The energy-momentum tensor of the gauge fields is
\[
\hat{\tau}^{\alpha \beta} = \hat{F}^{\alpha \rho} \hat{F}_{\beta \rho}^{\beta} - \frac{1}{4} \delta^{\alpha \beta} \hat{F}^{\rho \sigma} \hat{F}_{\rho \sigma} + M_b^a \hat{A}_b^{\alpha} \hat{A}_b^{\beta} - \frac{1}{2} \delta^{\alpha \beta} M_b^a \hat{A}_b^{\alpha} \hat{A}_b^{\beta}. \tag{479}
\]
C. Einstein equation and energy-momentum conservation law

The Einstein equation takes the form,
\[ \hat{R}_\beta^\alpha - \frac{1}{2} \delta_\beta^\alpha \hat{R} = -8\pi G \hat{T}_\beta^\alpha, \] (480)
where \( G \) is the gravity constant, \( \hat{T}_\beta^\alpha \) is the total energy-momentum tensor, \( \hat{R}_\beta^\alpha = \hat{R}_\beta^\gamma, \beta \) and \( \hat{R} = \hat{R}_\alpha^\alpha \). There is no mass of graviton appeared in Einstein equation, we thus mean that graviton is massless. From Einstein equation and the contracted tensor of Bianchi identity of gravity, we can find,
\[ \hat{D}_\alpha (\hat{R}_\beta^\alpha - \frac{1}{2} \delta_\beta^\alpha \hat{R}) = 0. \] (481)
This turns out to be the energy-momentum conservation law,
\[ \hat{D}_\alpha \hat{T}_\beta^\alpha = 0. \] (482)
The energy-momentum tensor is defined as the total energy-momentum tensors of gauge fields \( \hat{T}_\beta^\alpha \) and matter fields \( \hat{t}_\beta^\alpha \),
\[ \hat{T}_\beta^\alpha = \hat{t}_\beta^\alpha + \hat{\tau}_\beta^\alpha. \] (483)
We will later show that the energy momentum conservation law can also be proved by a direct calculations from Dirac equation and Yang-Mills equation.

D. The unification properties of Dirac equation, Yang-Mills equation and Einstein equation

Here we would like to present the relationships for the three fundamental equations. For convenience, we list all the related relations here:
1. Dirac equation:
\[ (i\hat{\gamma}^\alpha \hat{D}_\alpha - \hat{m})|\Psi\rangle = 0. \]
2. Yang-Mills equation:
\[ \hat{D}_\alpha \hat{F}_\alpha^\beta = \hat{J}_\beta^\alpha. \]
3. Einstein equation:
\[ \hat{R}_\beta^\alpha - \frac{1}{2} \delta_\beta^\alpha \hat{R} = -8\pi G \hat{T}_\beta^\alpha. \]
4. Total gauge current density:
\[ \hat{J}_\beta^\alpha = \hat{\gamma}_\beta^\alpha + \hat{t}_\beta^\alpha. \]
5. Gauge current density of matter fields:
\[ \hat{t}_\beta^\alpha (x) = \langle \Psi | \hat{\epsilon}(x) \gamma^\beta \hat{T}_\alpha | \Psi \rangle. \]
6. Total energy-momentum tensor:
\[ \hat{T}_\beta^\alpha = \hat{\tau}_\beta^\alpha + \hat{t}_\beta^\alpha. \]
7. Energy-momentum tensor of matter fields:
\[ \hat{t}_\beta^\alpha (x) = \hat{t}_\beta^\alpha = \frac{i}{4} \{ \langle \Psi | \hat{\epsilon}(x) \gamma^\alpha \hat{D}_\beta + \gamma_\beta \hat{D}^\alpha \rangle | \Psi \rangle \} + \{ \langle \Psi | (\hat{D}_\beta \gamma^\alpha + \hat{D}^\alpha \gamma_\beta) \hat{\epsilon}(x) | \Psi \rangle \} \]
8. Energy-momentum tensor of gauge fields:

\[ \hat{T}_\beta^\alpha = \hat{F}_a^{\alpha\rho} \hat{F}_b^\beta_{\rho} - \frac{1}{4} \delta^\alpha_\beta \hat{F}_a^{\rho\sigma} \hat{F}_a^{\sigma\rho} + M_b^a \hat{A}_a^\alpha \hat{A}_b^\beta - \frac{1}{2} \delta^\alpha_\beta M_b^a \hat{A}_a^\rho \hat{A}_b^\rho. \]

The general covariance derivative operator has the elements of the gravity field, gauge fields. It corresponds to force. The Dirac equation means that the gravity force and the gauge forces can act on the matter fields. For Yang-Mills equation, we find that the gravity force has effect on gauge fields, and gauge fields have effects on themselves. For Einstein equation, gravity can acts on itself. From the representation of gravity field, we find that gauge charge of graviton is zero. So gauge force does not have effect on gravity field. We can thus find that all three fundamental equations are involved together. One key point that those equations are compatible is that from Einstein equation, we can prove the energy-momentum conservation law. This result can also be obtained from Dirac equation and Yang-Mills equation. The total energy-momentum of matter fields and gauge fields in Einstein equation can be understood as the source of gravity.

VIII. ENERGY-MOMENTUM CONSERVATION LAW AND PHYSICAL QUANTITIES

Based on the representations and the fundamental equations. We next consider some physical quantities of the unified theory. By tremendous calculations, we find one important result of our theory: the energy-momentum conservation law. With this result, we confirm that our theory is a compatible and combined form for three fundamental equations.

A. Particle current density and spin current density

By using matter field \(|\Psi\rangle\), its adjoint \langle\Psi|\) and the antisymmetric matrix \(\hat{\gamma}^{\alpha_1\cdots\alpha_p}\) (p=0,1,2,3,), one can construct two antisymmetric tensors listed as

\[ \rho^\alpha(x) = \langle \Psi|\hat{\varepsilon}(x)\hat{\gamma}^\alpha|\Psi\rangle, \quad (484) \]

\[ \rho^{\alpha\beta\gamma}(x) = \langle \Psi|\hat{\varepsilon}(x)\hat{\gamma}^{\alpha\beta\gamma}|\Psi\rangle, \quad (485) \]

The first one \(\rho^\alpha(x)\) is the particle current density, the second one \(\rho^{\alpha\beta\gamma}(x)\) is related with the spin current density. The antisymmetric tensor fields can be represented in an unified form,

\[ \rho^{\alpha_1\cdots\alpha_p}(x) = \langle \Psi|\hat{\varepsilon}(x)\hat{\gamma}^{\alpha_1\cdots\alpha_p}|\Psi\rangle. \quad (486) \]

This antisymmetric tensor field and its adjoint are the same due to the properties of \(\hat{\varepsilon}\) and \(\hat{\gamma}^{\alpha_1\cdots\alpha_p}\),

\[ \overline{\rho^{\alpha_1\cdots\alpha_p}(x)} = \rho^{\alpha_1\cdots\alpha_p}(x). \quad (487) \]

It is also antisymmetric.

The results of covariance differential are presented as,

\[ \hat{D}_\alpha \rho^\alpha(x) = 0, \quad (488) \]

\[ \hat{D}_\alpha \rho^{\alpha\beta\gamma}(x) = 0. \quad (489) \]

The proof of the first one is like the following,

\[ \theta^\mu_\alpha(x) \partial_\mu \rho^\alpha = \theta^\mu_\alpha(x) \partial_\mu \langle \Psi|\hat{\epsilon}(x)\hat{\gamma}^\alpha|\Psi\rangle \]

\[ = \langle \Psi|\hat{\epsilon}(x) \left( \hat{D}_\alpha \hat{\gamma}^\alpha \right) |\Psi\rangle - \left( \langle \Psi|\hat{D}_\alpha \hat{\epsilon}(x) \hat{\gamma}^\alpha |\Psi\rangle \right) \]

\[ = \langle \Psi \left| \left( \hat{D}_\alpha \hat{\epsilon}(x) \hat{\gamma}^\alpha \right) \right| \hat{\epsilon}(x) \hat{\gamma}^\alpha |\Psi\rangle \right) + i \left( \langle \Psi|\hat{D}_\alpha \hat{\gamma}^\alpha \right) \hat{\epsilon}(x)|\Psi\rangle \]

\[ = -\hat{\Gamma}^{\alpha\beta}(x) \langle \Psi|\hat{\epsilon}(x)\hat{\gamma}^\beta |\Psi\rangle - i \langle \Psi|\hat{\epsilon}(x) \hat{n}|\Psi\rangle + i \left( \langle \Psi|\hat{n} \hat{\epsilon}(x) \right) |\Psi\rangle \]

\[ = -\hat{\Gamma}^{\alpha\beta}(x) \rho^\beta(x). \quad (490) \]
We thus have
\[ \hat{D}_a \rho^a(x) = \theta^{\alpha}_a(x) \partial_\mu \rho^\alpha(x) + \hat{\Gamma}^{\alpha}_{\alpha,\beta}(x) \rho^\beta(x) = 0. \] (491)

The proof of the second one is like the proof of the first one and is given as,
\[ \theta^{\alpha}_a(x) \partial_\mu \rho^{\alpha\beta\gamma}(x) = \theta^{\alpha}_a(x) \partial_\mu \langle \Psi | \hat{\g}^\alpha \hat{\gamma}^{\beta\gamma} | \Psi \rangle 
   = \langle \Psi | \hat{\g}^{\alpha} \hat{\g}^{\beta\gamma} \hat{\g}^{\beta\gamma} | \Psi \rangle - \langle \Psi | \hat{\g}^{\alpha} \hat{\g}^{\beta\gamma} \hat{\g}^{\beta\gamma} | \Psi \rangle 
   = -\hat{\Gamma}^{\alpha}_{\alpha,\beta}(x) \rho^{\beta\gamma}(x) - \hat{\Gamma}^{\alpha}_{\alpha,\beta}(x) \rho^{\gamma\beta}(x) - \hat{\Gamma}^{\alpha}_{\alpha,\beta}(x) \rho^{\beta\gamma}(x) 
   = -\hat{\Gamma}^{\alpha}_{\alpha,\beta}(x) \rho^{\beta\gamma}(x) + \hat{\Gamma}^{\alpha}_{\alpha,\beta}(x) \rho^{\alpha\beta}(x) \rho^{\beta\gamma}(x) 
   = -\hat{\Gamma}^{\alpha}_{\alpha,\beta}(x) \rho^{\beta\gamma}(x) + \hat{\Gamma}^{\alpha}_{\alpha,\beta}(x) \rho^{\alpha\beta}(x) \rho^{\beta\gamma}(x). \] (492)

We thus have
\[ \hat{D}_a \rho^{\alpha\beta\gamma}(x) = \theta^{\alpha}_a(x) \partial_\mu \rho^{\alpha\beta\gamma}(x) + \hat{\Gamma}^{\alpha}_{\alpha,\beta}(x) \rho^{\beta\gamma}(x) + \hat{\Gamma}^{\alpha}_{\alpha,\beta}(x) \rho^{\gamma\beta}(x) + \hat{\Gamma}^{\alpha}_{\alpha,\beta}(x) \rho^{\beta\gamma}(x) + \hat{\Gamma}^{\alpha}_{\alpha,\beta}(x) \rho^{\alpha\beta}(x) \rho^{\beta\gamma}(x) = 0. \] (493)

The particle current density in 4D is defined by \( \rho^a(x) \), where \( \rho^i(x) \) is the particle number density, \( \rho^i(x), (i = 1, 2, 3) \) are the particle current density in 3D. So equation
\[ \hat{D}_a \rho^a(x) = 0 \] (494)

means that the number of particles is conserved. The 4D current density of particles with gauge indices \( t \) can be defined as,
\[ \rho^a_t = \langle \Psi | \hat{\g}^{t} \hat{\gamma}^{\alpha} | \Psi \rangle, \] (495)

where \( | \Psi \rangle \) is the \( t \) gauge element of matter fields, note that no summation is assumed in the above equation.

The spin of the matter particles is related with \( S^{\alpha\beta\gamma}(x) \equiv \frac{1}{2} \rho^{\alpha\beta\gamma}(x) \), so \( S^{\alpha\beta\gamma}(x) \) can be considered as the 4D spin current density. By definition, the spin current density is
\[ S^{\alpha\beta\gamma} = \frac{1}{2} \langle \Psi | \hat{\g}^{\alpha} \hat{\gamma}^{\beta\gamma} \hat{\g}^{\beta\gamma} | \Psi \rangle \]
\[ = \frac{1}{2} \langle \Psi | \hat{\g}^{\alpha} \hat{\gamma}^{\alpha\beta\gamma} | \Psi \rangle \] (496)

We can find that this spin current density is antisymmetric,
\[ S_{\alpha\beta\gamma} = S_{\beta\gamma\alpha} = S_{\gamma\alpha\beta} = -S_{\gamma\beta\alpha} = -S_{\beta\alpha\gamma} = -S_{\alpha\gamma\beta} \] (497)

The divergence equations of the spin current is zero as we have proved,
\[ \hat{\Gamma}_a S^{\alpha\beta\gamma} = 0. \] (498)

B. The gauge current density of matter fields

The gauge current density and gauge charge production rate density of matter fields are defined respectively as,
\[ j^{\beta}_a(x) = \langle \Psi | \hat{\g}^{\beta} \hat{\g}^{\alpha} \hat{\g}^{\alpha} | \Psi \rangle, \]
\[ u^{\alpha}_a(x) = -i \langle \Psi | \hat{\g}^{\alpha} \hat{\g}^{\alpha\beta\gamma} \hat{\g}^{\beta\gamma} | \Psi \rangle 
   = -i \langle \Psi | \hat{\g}^{\alpha} \hat{\g}^{\alpha\beta\gamma} | \Psi \rangle. \] (499)
Recall the definition of the gauge current density of matter fields [107], we have an equation,

\[ \hat{D}_\alpha j^\alpha_a = u_a. \]  

It means that for matter fields, the changing of the gauge current density equals to the gauge charge production rate density.

The proof is presented below.

\[ \theta^\mu_a \partial_\mu j^\alpha_a (x) = \hat{\theta}^\mu(x) \partial_\mu (\Psi|\hat{\varepsilon}(x)|\hat{\gamma}^\alpha \hat{T}_a|\Psi) \]
\[ = \langle \Psi|\hat{\varepsilon}(x) \left( \hat{D}_\alpha \hat{\gamma}^\alpha \hat{T}_a |\Psi \right) \rangle - \langle \langle \Psi|\hat{\varepsilon}(x) \right) \left( \hat{\gamma}^\alpha |\hat{D}_\alpha, \hat{T}_a )|\Psi \rangle \rangle \]
\[ = \langle \Psi|\hat{\varepsilon}(x) \left( [\hat{D}_\alpha, \hat{\gamma}^\alpha ]\hat{T}_a |\Psi \right) \rangle + \langle \langle \Psi|\hat{\varepsilon}(x) \right) \left( \hat{\gamma}^\alpha |\hat{D}_\alpha, \hat{T}_a )|\Psi \rangle \rangle \]
\[ -i\langle \Psi|\hat{\varepsilon}(x) \hat{T}_a \left( i\hat{\gamma}^\alpha \hat{D}_\alpha |\Psi \right) \rangle + i \langle \langle \Psi|\hat{\varepsilon}(x) \right) \hat{T}_a \hat{\gamma}^\alpha |\Psi \rangle \rangle \]
\[ = -\Gamma_{\alpha, \beta}^a (x) \langle \Psi|\hat{\varepsilon}(x) \hat{\gamma}^\beta |\Psi \rangle + i C_{\alpha \beta}^c \hat{A}_a^b (x) \langle \Psi|\hat{\varepsilon}(x) \hat{\gamma}^\beta |\Psi \rangle \]
\[ -i\langle \Psi|\hat{\varepsilon}(x) \hat{T}_a \hat{\gamma}^\alpha |\Psi \rangle \rangle + i \langle \langle \Psi|\hat{\varepsilon}(x) \right) \hat{T}_a \hat{\gamma}^\alpha |\Psi \rangle \rangle \]
\[ = -\Gamma_{\alpha, \beta}^a (x) j^\beta_a (x) + i C_{\alpha \beta}^c \hat{A}_a^b (x) j^\beta_a (x) + u_a (x) \] 

Thus we can find,

\[ \hat{D}_\alpha j^\alpha_a = \theta^\mu_a \partial_\mu j^\alpha_a + \Gamma_{\alpha, \beta}^a j^\beta_a - i C_{\alpha \beta}^c \hat{A}_a^b j^\beta_a = u_a. \]

C. Energy-momentum conservation law of matter fields and gauge fields

Recall the definition of the energy-momentum tensor of gauge fields,

\[ \hat{T}_{\beta}^\alpha = \hat{F}_{\alpha \beta}^a \hat{F}^\alpha_a - \frac{1}{4} \delta_\alpha^\beta \hat{F}_{\rho \sigma}^a \hat{F}^{\rho \sigma}_a + M^a_b \hat{A}_a^b \hat{A}^\beta_b - \frac{1}{2} \delta_\beta^\alpha M^a_b \hat{A}_a^b \hat{A}^\beta_b. \]

We can find the following important property of the energy-momentum tensor of gauge fields with the help of Yang-Mills equation and Eq. [107],

\[ \hat{D}_\alpha \hat{T}^\alpha_\beta = -\hat{j}^\alpha_a \hat{F}_{\alpha \beta}^a - \hat{A}_\beta^a \hat{u}_a. \]

The proof is presented below,

\[ \hat{D}_\alpha (\hat{F}_{\alpha \beta}^a \hat{F}^\alpha_a - \frac{1}{4} \delta_\beta^\alpha \hat{F}_{\rho \sigma}^a \hat{F}^{\rho \sigma}_a ) \]
\[ = (\hat{D}_\alpha \hat{F}_{\alpha \beta}^a )\hat{F}^\alpha_a + \hat{F}_{\alpha \beta}^a (\hat{D}_\alpha \hat{F}^\alpha_a ) - \frac{1}{2} \hat{F}^{\rho \sigma}_a (\hat{D}_\beta \hat{F}_{\rho \sigma}^a ) \]
\[ = (\hat{D}_\alpha \hat{F}_{\alpha \beta}^a )\hat{F}^\alpha_a + \frac{1}{2} \hat{F}_{\alpha \beta}^a (\hat{D}_\alpha \hat{F}^\alpha_a + \hat{D}_\beta \hat{F}^\alpha_a ) \]
\[ = -\hat{j}^\alpha_a \hat{F}_{\alpha \beta}^a. \]

From the definition of gauge curvature and the relation between masses of gauge fields and matter fields, also \( M_{ab} = M_{ba}, M_{ab} = G_{kk} M^b_a \), we have

\[ \hat{D}_a (M^b_a \hat{A}^a_b \hat{A}_\beta^b - \frac{1}{2} \delta_\beta^\alpha M^b_a \hat{A}^a_b \hat{A}_\alpha^b ) \]
\[ = \hat{D}_a (M^b_a \hat{A}^a_b \hat{A}_\beta^b ) \hat{A}^\beta_a + M^b_a \hat{A}^a_b \hat{D}_a \hat{A}_\beta^b - M^b_a \hat{A}^a_b \hat{D}_\beta \hat{A}_\alpha^a + i C_{cd}^a M^b_a \hat{A}^a_b \hat{A}_\alpha^c \hat{A}_d^\beta \]
\[ = -\hat{A}_\beta^a \hat{u}_a + M^a_b \hat{A}_a^b \hat{F}_{\alpha \beta}^a. \]

Thus we end the proof.

As we presented, the energy-momentum tensor of matter fields takes the form

\[ t_{\beta}^\alpha (x) = \frac{i}{4} \{ \langle \Psi|\hat{\varepsilon}(x) |(\hat{\gamma}^\alpha \hat{D}_\beta + \hat{\gamma}_\beta \hat{D}^\alpha)|\Psi \rangle \} + \{ \langle \langle \Psi| \hat{\varepsilon}(x) \right) \left( \hat{\gamma}^\alpha \hat{D}_\beta + \hat{\gamma}_\beta \hat{D}^\alpha \right) |\Psi \rangle \} \]
\[ = \frac{1}{2} (\hat{\theta}^\mu_a \theta^\alpha_{\alpha^\mu} + \hat{\theta}^\alpha_{\mu} \theta^\mu_{\mu^\alpha} + \hat{\gamma}^\beta \hat{\gamma}_{\beta} + \hat{\gamma}^\alpha_{\alpha^\beta} + \hat{\gamma}^\beta_{\beta} + \hat{\gamma}^\alpha_{\alpha^\beta} + \hat{\gamma}^\beta_{\beta^\alpha}). \]
We can find that the energy-momentum tensor is symmetric,

\[ t_{\alpha \beta} = t_{\beta \alpha}. \]  

(508)

By Dirac equation, the divergence equation for the energy-momentum tensor can be calculated as,

\[ \hat{D}_{\alpha} t_{\beta}^{\alpha} = F_{\alpha \beta} J_{\alpha}^{\alpha} + A_{\beta}^{a} u_{a}. \]  

(509)

The proof is presented in the following. We have used the following notations,

\[ t_{\beta}^{\alpha} (x) = \frac{i}{2} \left\{ \langle \Psi | \hat{\varepsilon} (x) (\hat{\gamma}^{\alpha} \hat{D}_{\beta}) | \Psi \rangle \right\} + \frac{i}{2} \left\{ \langle \Psi | \hat{\varepsilon} (x) (\hat{\gamma}^{\alpha} \hat{D}_{\beta}) | \Psi \rangle \right\} \]

(510)

So we have the expression,

\[ t_{\beta}^{\alpha} = \frac{1}{2} (t_{\beta}^{\alpha} + t_{\beta}^{\alpha}). \]  

(511)

We next consider terms \( t_{\beta}^{\alpha} \) and \( t_{\beta}^{\alpha} \), respectively.

\[
\begin{align*}
\theta_{\alpha}^{\mu} \partial_{\mu} t_{\beta}^{\alpha} (x) &= 
\frac{i}{2} \left\{ \theta_{\alpha}^{\mu} (x) \partial_{\mu} \langle \Psi | \hat{\varepsilon} (x) (\hat{\gamma}^{\alpha} \hat{D}_{\beta}) | \Psi \rangle \right\} + \theta_{\mu}^{\alpha} (x) \partial_{\mu} \left\{ \langle \Psi | \hat{D}_{\beta} (\hat{\gamma}^{\alpha}) \hat{\varepsilon} (x) | \Psi \rangle \right\} \\
&= \frac{i}{2} \left\{ \langle \Psi | \hat{\varepsilon} (x) \hat{D}_{\alpha} \hat{\gamma}^{\alpha} (\hat{D}_{\beta}) | \Psi \rangle \right\} - \left\{ \langle \Psi | \hat{\varepsilon} (x) \hat{\gamma}^{\alpha} \hat{D}_{\beta} | \Psi \rangle \right\} \\
&\quad + \left\{ \langle \Psi | \hat{D}_{\alpha} \hat{\gamma}^{\alpha} \hat{\varepsilon} (x) | \Psi \rangle \right\} - \left\{ \langle \Psi | \hat{\gamma}^{\alpha} \hat{D}_{\beta} \hat{\varepsilon} (x) | \Psi \rangle \right\} \\
&= \frac{i}{2} \left\{ \langle \Psi | \hat{\varepsilon} (x) \left[ (\hat{D}_{\alpha} \hat{\gamma}^{\alpha} \hat{D}_{\beta} + \hat{\gamma}^{\alpha} (\hat{D}_{\alpha} \hat{D}_{\beta} - \hat{\gamma}^{\alpha} (\hat{D}_{\beta} \hat{\gamma}^{\alpha} \hat{D}_{\beta}) \right) | \Psi \rangle \right\} \\
&\quad - \left\{ \langle \Psi | \hat{\gamma}^{\alpha} \hat{\varepsilon} (x) \hat{D}_{\beta} | \Psi \rangle \right\} + \left\{ \langle \Psi | \hat{D}_{\alpha} \hat{\gamma}^{\alpha} \hat{\varepsilon} (x) | \Psi \rangle \right\} \\
&\quad - \left\{ \langle \Psi | \hat{\gamma}^{\alpha} \hat{D}_{\beta} \hat{\varepsilon} (x) | \Psi \rangle \right\} + \left\{ \langle \Psi | \hat{D}_{\alpha} \hat{\gamma}^{\alpha} \hat{\varepsilon} (x) | \Psi \rangle \right\} \\
&= \frac{i}{2} \left\{ \langle \Psi | \hat{\varepsilon} (x) \left[ (\hat{D}_{\alpha} \hat{\gamma}^{\alpha} \hat{D}_{\beta} + \hat{\gamma}^{\alpha} (\hat{D}_{\alpha} \hat{D}_{\beta}) \right) | \Psi \rangle \right\} \\
&\quad + \left\{ \langle \Psi | \hat{\gamma}^{\alpha} \hat{\varepsilon} (x) \hat{D}_{\beta} | \Psi \rangle \right\} + \left\{ \langle \Psi | \hat{D}_{\alpha} \hat{\gamma}^{\alpha} \hat{\varepsilon} (x) | \Psi \rangle \right\} \\
&\quad + \left\{ \langle \Psi | \hat{\gamma}^{\alpha} \hat{D}_{\beta} \hat{\varepsilon} (x) | \Psi \rangle \right\} + \left\{ \langle \Psi | \hat{D}_{\alpha} \hat{\gamma}^{\alpha} \hat{\varepsilon} (x) | \Psi \rangle \right\} \\
&= \frac{i}{2} R_{\alpha \beta}^{\alpha} (x) \langle \Psi | \hat{\varepsilon} (x) (\hat{\gamma}^{\alpha} \hat{\gamma}^{\beta} + \hat{\gamma}^{\alpha} \hat{\gamma}^{\beta} + \hat{\gamma}^{\alpha} \hat{\gamma}^{\beta} + \hat{\gamma}^{\alpha} \hat{\gamma}^{\beta}) | \Psi \rangle \\
&\quad + \left\{ \langle \Psi | \hat{\gamma}^{\alpha} \hat{\varepsilon} (x) \hat{D}_{\beta} | \Psi \rangle \right\} + \left\{ \langle \Psi | \hat{D}_{\alpha} \hat{\gamma}^{\alpha} \hat{\varepsilon} (x) | \Psi \rangle \right\} \\
&\quad + \left\{ \langle \Psi | \hat{\gamma}^{\alpha} \hat{D}_{\beta} \hat{\varepsilon} (x) | \Psi \rangle \right\} + \left\{ \langle \Psi | \hat{D}_{\alpha} \hat{\gamma}^{\alpha} \hat{\varepsilon} (x) | \Psi \rangle \right\} \\
&= -\frac{1}{2} R_{\alpha \beta}^{\alpha} (x) \langle \Psi | \hat{\varepsilon} (x) \hat{D}_{\beta} | \Psi \rangle \\
&\quad + \Gamma_{\alpha \beta}^{\alpha} (x) t_{\beta}^{\alpha} (x) - F_{\alpha \beta}^{\alpha} (x) t_{\alpha}^{\beta} (x) \}
\end{align*}
\]

Thus we have

\[ \hat{D}_{\alpha} t_{\beta}^{\alpha} (x) = \theta_{\alpha}^{\mu} (x) \partial_{\mu} t_{\beta}^{\alpha} (x) + \Gamma_{\alpha \beta}^{\alpha} (x) t_{\beta}^{\alpha} (x) - \Gamma_{\alpha \beta}^{\alpha} (x) t_{\beta}^{\alpha} (x) \]

(512)

Similarly we have,

\[ \hat{D}_{\alpha} t_{\beta}^{\alpha} (x) = \theta_{\alpha}^{\mu} (x) \partial_{\mu} t_{\beta}^{\alpha} (x) + \Gamma_{\alpha \beta}^{\alpha} (x) t_{\beta}^{\alpha} (x) - \Gamma_{\alpha \beta}^{\alpha} (x) t_{\beta}^{\alpha} (x). \]  

(513)
\[ \theta^\mu_\alpha \partial_\mu \tilde{t}^\alpha_\beta (x) \]
\[ = \frac{i}{2} \left\{ \theta^\mu_\alpha (x) \partial_\mu (\Psi | \tilde{\xi} (x) (\gamma_\beta \hat{D}^\alpha | \Psi) + \theta^\nu_\alpha (x) \partial_\nu (\langle \Psi | \hat{D}^\alpha \gamma_\beta \hat{D}^\alpha | \Psi \rangle) \right\} \]
\[ = \frac{i}{2} \left\{ \langle \Psi | \tilde{\xi} (x) \left( \hat{D}_\alpha \gamma_\beta \hat{D}^\alpha | \Psi \rangle \right) \left( \langle \Psi | \hat{D}_\alpha \gamma_\beta \hat{D}^\alpha | \Psi \rangle \right) \right\} + \left\{ \langle \Psi | \tilde{\xi} (x) (\hat{D}_\alpha \gamma_\beta \hat{D}^\alpha | \Psi \rangle) \right\}
\[ = \frac{i}{2} \left\{ \langle \Psi | \tilde{\xi} (x) \left( \langle \hat{D}_\alpha \gamma_\beta \hat{D}^\alpha | \Psi \rangle \right) \right\} + \left\{ \langle \Psi | \tilde{\xi} (x) (\hat{D}_\alpha \gamma_\beta \hat{D}^\alpha | \Psi \rangle) \right\}
\[ = \frac{i}{2} \left\{ \Gamma_{\alpha, \beta} (\Psi | \tilde{\xi} (x) (\gamma_\gamma \hat{D}^\alpha | \Psi) \rangle + \langle \Psi | \tilde{\xi} (x) [\gamma_\beta (\hat{R} + \hat{F} - \Gamma_{\alpha, \gamma} \hat{D}^\gamma - \hat{\Lambda}^a_\alpha \gamma_\alpha \hat{m}_a - \hat{m}^2] | \Psi \rangle) \right\}
\[ + \Gamma_{\alpha, \beta} (\langle \Psi | \hat{D}^\alpha \gamma_\gamma \hat{D}^\alpha | \Psi \rangle) - \left\{ \langle \Psi | (\hat{R} + \hat{F} + \hat{D}^\gamma \Gamma_{\alpha, \gamma} + \hat{\Lambda}^a_\alpha \gamma_\alpha \hat{m}_a - \hat{m}^2 \rangle \tilde{\xi} (x) | \Psi \rangle \right\} \]
\[ = \frac{i}{2} \left\{ \Gamma_{\alpha, \beta} (\Psi | \tilde{\xi} (x) (\gamma_\gamma \hat{D}^\alpha | \Psi) \rangle + \langle \Psi | \tilde{\xi} (x) (\hat{D}^\alpha \gamma_\gamma \hat{D}^\alpha | \Psi \rangle \right\}
\[ = \frac{i}{2} \left\{ \Gamma_{\alpha, \beta} (\Psi | \tilde{\xi} (x) (\gamma_\gamma \hat{D}^\alpha | \Psi) \rangle + \langle \Psi | \tilde{\xi} (x) (\hat{D}^\alpha \gamma_\gamma \hat{D}^\alpha | \Psi \rangle \right\}
\[ = \frac{i}{2} \left\{ \tilde{\Gamma}_{\alpha, \beta} (\Psi | \tilde{\xi} (x) (\gamma_\gamma \hat{D}^\alpha | \Psi) \rangle + \langle \Psi | \tilde{\xi} (x) (\hat{D}^\alpha \gamma_\gamma \hat{D}^\alpha | \Psi \rangle \right\}
\[ + \frac{1}{4} \tilde{\Gamma}_{\alpha, \beta} (\Psi | \tilde{\xi} (x) (\gamma_\gamma \hat{D}^\alpha | \Psi) \rangle + \langle \Psi | \tilde{\xi} (x) (\hat{D}^\alpha \gamma_\gamma \hat{D}^\alpha | \Psi \rangle - i \hat{A}_\alpha (x) | \Psi \rangle \right\}
\[ = -\tilde{\Gamma}_{\alpha, \beta} (x) t_{\beta}^{\mu_\alpha} (x) + \tilde{\Gamma}_{\alpha, \beta} (x) t_{\gamma}^{\mu_\beta} (x) + \frac{1}{2} R_{\alpha, \beta} (x) S_{\rho_\sigma} (x) + F_{\alpha, \beta} (x) j_{\alpha} (x) + A_{\alpha} (x) u_a (x) \]

And we have
\[ \hat{D}_\alpha t_{\beta}^{\mu_\alpha} (x) = \theta^\mu_\alpha (x) \partial_\mu t_{\beta}^{\mu_\alpha} (x) + \tilde{\Gamma}_{\alpha, \beta} (x) t_{\gamma}^{\mu_\alpha} (x) - \tilde{\Gamma}_{\alpha, \beta} (x) t_{\gamma}^{\mu_\beta} (x) \]
\[ = \frac{1}{2} R_{\alpha, \beta} (x) S_{\rho_\sigma} (x) + F_{\alpha, \beta} (x) j_{\alpha} (x) + A_{\alpha} (x) u_a (x). \]

Spin current density \( S_{\rho_\sigma} \) is completely anti-symmetric tensor, the summation of gravity curvature \( R_{\beta}^{\rho_\sigma} \) for cyclic indices is zero, so we have
\[ R_{\alpha, \beta} S_{\rho_\sigma} = R_{\beta}^{\rho_\sigma} S_{\rho_\sigma} = \frac{1}{3} (R_{\beta}^{\rho_\sigma} + R_{\beta}^{\sigma_\rho} + R_{\beta}^{\rho_\sigma}) S_{\rho_\sigma} \]
\[ = 0 \]

Summarize the above three equations, we have
\[ D_{\alpha} t_{\beta}^{\mu_\alpha} = \theta^\mu_\alpha \partial_\mu t_{\beta}^{\mu_\alpha} + \tilde{\Gamma}_{\alpha, \beta} t_{\gamma}^{\mu_\alpha} - \tilde{\Gamma}_{\alpha, \beta} t_{\gamma}^{\mu_\beta} \]
\[ = F_{\alpha, \beta} j_{\alpha} + A_{\alpha} u_a. \]

Comparing the Eq.[514] of gauge fields and Eq.[514] of matter fields, we immediately find that for the total energy-momentum tensor \( T_{\beta}^{\alpha} = \tilde{\xi}^{\alpha} + t_{\beta}^{\alpha} \), there is the energy-momentum conservation law:
\[ \hat{D}_\alpha T_{\beta}^{\alpha} = 0. \]

D. Leptons and quarks and related projectors

We can define two projectors \( \hat{P}_l \) and \( \hat{P}_q \) which can project a quantum state onto special states for leptons or quarks, respectively.
\[ \hat{P}_l = 1 - \frac{3}{4} \hat{\lambda}_l^2 \]
\[ \hat{P}_q = \frac{3}{4} \hat{\lambda}_q^2 \]
And the matter fields are decomposed as the superposition of two states for leptons and quarks. The lepton field and quark field are defined as
\[ |\Psi_{(l)}\rangle = \hat{\rho}_{(l)} |\Psi\rangle, \] 
\[ |\Psi_{(q)}\rangle = \hat{\rho}_{(q)} |\Psi\rangle. \] (521)

There is no overlaps for lepton field and quark field, so we have the form of superposition,
\[ |\Psi\rangle = |\Psi_{(l)}\rangle + |\Psi_{(q)}\rangle. \] (523)

Here we note that
\[ \lambda^2 |\Psi_{(l)}\rangle = 0 |\Psi_{(l)}\rangle, \] 
\[ \lambda^2 |\Psi_{(q)}\rangle = \frac{4}{3} |\Psi_{(q)}\rangle. \] (524)

Some properties of the lepton and quark projectors can be found as
\[ [\hat{P}_{(l)}, \hat{D}_\alpha] = 0, \] 
\[ [\hat{P}_{(q)}, \hat{D}_\alpha] = 0, \] 
\[ [\hat{P}_{(l)}, \hat{\gamma}_\alpha] = 0, \] 
\[ [\hat{P}_{(q)}, \hat{\gamma}_\alpha] = 0, \] 
\[ \hat{P}_{(l)}, \hat{m} \] 
\[ \hat{P}_{(q)}, \hat{m} \] (526)

From the Dirac equation for matter fields, with the help of the properties of the projectors \( \hat{P}_{(l)} \) and \( \hat{P}_{(q)} \), we can find that the lepton field and the quark field satisfy the Dirac equation, respectively.
\[ (i\hat{\gamma}_\alpha \hat{D}_\alpha - \hat{m}) |\Psi_{(l)}\rangle = 0, \] 
\[ (i\hat{\gamma}_\alpha \hat{D}_\alpha - \hat{m}) |\Psi_{(q)}\rangle = 0, \] 
\[ (\hat{D}_\alpha \hat{D}_\alpha - \hat{R} - \hat{F} + \hat{\Gamma}_{\alpha,\beta} \hat{D}_\beta + \hat{A}_\alpha \hat{\gamma}_\alpha \hat{m}_a + \hat{m}^2) |\Psi_{(l)}\rangle = 0, \] 
\[ (\hat{D}_\alpha \hat{D}_\alpha - \hat{R} - \hat{F} + \hat{\Gamma}_{\alpha,\beta} \hat{D}_\beta + \hat{A}_\alpha \hat{\gamma}_\alpha \hat{m}_a + \hat{m}^2) |\Psi_{(q)}\rangle = 0. \] (527)

Similarly for the adjoint states of lepton and quark, we also have the following results,
\[ \langle \Psi_{(l)} | (i\hat{D}_\alpha \hat{\gamma}_\alpha - \hat{m}) = 0, \] 
\[ \langle \Psi_{(q)} | (i\hat{D}_\alpha \hat{\gamma}_\alpha - \hat{m}) = 0, \] 
\[ \langle \Psi_{(l)} | (\hat{D}_\alpha \hat{D}_\alpha - \hat{R} - \hat{F} - \hat{\Gamma}_{\alpha,\beta} \hat{D}_\beta - \hat{A}_\alpha \hat{\gamma}_\alpha \hat{m}_a + \hat{m}^2) = 0, \] 
\[ \langle \Psi_{(q)} | (\hat{D}_\alpha \hat{D}_\alpha - \hat{R} - \hat{F} - \hat{\Gamma}_{\alpha,\beta} \hat{D}_\beta - \hat{A}_\alpha \hat{\gamma}_\alpha \hat{m}_a + \hat{m}^2) = 0. \] (528)

We then can discuss the current densities for leptons and quarks, respectively. The lepton current density is defined as
\[ \rho^{l}_{(l)}(x) = \langle \Psi_{(l)} | \hat{\epsilon}(x) \hat{\gamma}_\alpha |\Psi_{(l)}\rangle. \] (529)

We can find the the number of leptons is conserved which is expressed as the following,
\[ \hat{D}_\alpha \rho^{l}_{(l)} = 0. \] (530)

Similarly for quarks, the current density takes the form
\[ \rho^{q}_{(q)}(x) = \langle \Psi_{(q)} | \hat{\epsilon}(x) \hat{\gamma}_\alpha |\Psi_{(q)}\rangle. \] (531)

Also the number of quarks is conserved,
\[ \hat{D}_\alpha \rho^{q}_{(q)} = 0. \] (532)

The total current density of matter fields can be expressed as the summation of lepton current density and the quark current density,
\[ \rho^\alpha = \rho^{l}_{(l)} + \rho^{q}_{(q)}. \] (533)
E. Gauge current densities for leptons and quarks

The current densities with different gauges for leptons and quarks are also important properties. The definition can be simply realized by gauge operators $\hat{T}_a$. We now define the gauge current density of lepton field as, 

$$ j_{(l)a}^\alpha(x) = \langle \Psi_{(l)} | \hat{\varepsilon}(x) \gamma^\alpha \hat{T}_a | \Psi_{(l)} \rangle. \tag{534} $$

The gauge charge production rate density for lepton field takes the form, 

$$ u_{(l)a}(x) = -i \langle \Psi_{(l)} | \hat{m}_a \hat{\varepsilon}(x) | \Psi_{(l)} \rangle. \tag{535} $$

One can find that for lepton field, the divergence of the gauge current density is equal to the gauge charge production rate density, 

$$ \hat{D}_\alpha j_{(l)a}^\alpha = u_{(l)a}. \tag{536} $$

Similar properties are also satisfied for quark field.

$$ j_{(q)a}^\alpha(x) = \langle \Psi_{(q)} | \hat{\varepsilon}(x) \gamma^\alpha \hat{T}_a | \Psi_{(q)} \rangle, $$

$$ u_{(q)a}(x) = -i \langle \Psi_{(q)} | \hat{m}_a \hat{\varepsilon}(x) | \Psi_{(q)} \rangle, $$

$$ \hat{D}_\alpha j_{(q)a}^\alpha = u_{(q)a}. \tag{537} $$

The total gauge current density is the summation of lepton gauge current density and the quark gauge current density. The total gauge charge production rate density is the summation of lepton and quark gauge charges production rate densities,

$$ j_a^\alpha(x) = j_{(l)a}^\alpha(x) + j_{(q)a}^\alpha(x), $$

$$ u_a = u_{(l)a} + u_{(q)a}. \tag{538} $$

The energy-momentum tensor of lepton field takes the form

$$ t_{(l)\beta}^\alpha(x) = \frac{i}{4} \{ \langle \Psi_{(l)} | \hat{\varepsilon}(x) | [\hat{\gamma}^\alpha \hat{D}_\beta + \hat{\gamma}_\beta \hat{D}^\alpha] | \Psi_{(l)} \rangle \} + \{ [\langle \Psi_{(l)} | (\hat{D}_\beta \hat{\gamma}^\alpha + \hat{D}^\alpha \hat{\gamma}_\beta) \hat{\varepsilon}(x) | \Psi_{(l)} \rangle \} \}. \tag{539} $$

The lepton energy-momentum tensor satisfies the equation,

$$ \hat{D}_\alpha t_{(l)\beta}^\alpha = -F_{\alpha\beta}^a j_{(l)a}^\alpha - A_\beta^a u_{(l)a}. \tag{540} $$

Similarly the energy-momentum tensor of quarks is

$$ t_{(q)\beta}^\alpha(x) = \frac{i}{4} \{ \langle \Psi_{(q)} | \hat{\varepsilon}(x) | [\hat{\gamma}^\alpha \hat{D}_\beta + \hat{\gamma}_\beta \hat{D}^\alpha] | \Psi_{(q)} \rangle \} + \{ [\langle \Psi_{(q)} | (\hat{D}_\beta \hat{\gamma}^\alpha + \hat{D}^\alpha \hat{\gamma}_\beta) \hat{\varepsilon}(x) | \Psi_{(q)} \rangle \} \}. \tag{541} $$

The lepton energy-momentum tensor satisfies the equation,

$$ \hat{D}_\alpha t_{(q)\beta}^\alpha = -F_{\alpha\beta}^a j_{(q)a}^\alpha - A_\beta^a u_{(q)a}. \tag{542} $$

The total energy-momentum tensor is the summation of lepton energy-momentum and the quark energy-momentum,

$$ t_\beta^\alpha = t_{(l)\beta}^\alpha + t_{(q)\beta}^\alpha. \tag{543} $$

IX. QUANTUM SYSTEM WITH DISCRETE SPACE-TIME, OBSERVABLE QUANTITIES AND PARTICLES SCATTERING

We have presented above the unified description of the three fundamental equations. In this section, we hope to discuss conceptually the quantum system, and how to define the observable physical quantities in 3D space from the present 4D theory. We will also present the particle scattering results from the representations in this work.
A. Quantum system

We next present some features of our unified quantum theory. The quantum field we discussed is generally in all 4D coordinate space or in all 4D momentum space. This means that the coordinate and momentum take values in all 4D spaces, \( x \in \mathbb{R}^4 \) and \( p \in \mathbb{R}^4 \). In our formulas, the integral area should be all 4D space which sometimes is omitted in the presentation. Here we would like to point out that the integral area should be in the following form,

\[
|\Psi\rangle = \int_{\mathbb{R}^4} \Psi^{st}(x|x_{st})d^4x = \int_{\mathbb{R}^4} \Psi^{st}(p|p_{st})d^4p
\]

\[
\dot{\epsilon} = \int_{\mathbb{R}^4} \tilde{\theta}^\alpha(x)\tilde{\varepsilon}^{\alpha}(x)d^4x = \int_{\mathbb{R}^4} \tilde{\theta}^\alpha(p)\tilde{\varepsilon}^{\alpha}(p)d^4p
\]

\[
\dot{A} = \int_{\mathbb{R}^4} A^\alpha_a(x)\tilde{\varepsilon}^{\alpha}(x)d^4x = \int_{\mathbb{R}^4} A^\alpha_a(p)\tilde{\varepsilon}^{\alpha}(p)d^4p.
\]

(544)

In application of our theory, what we consider are 4D coordinate space \( M_x \) with a boundary or 4D momentum space \( M_p \) with a boundary. The quantum systems in 4D coordinate space with boundaries and the 4D momentum space with boundaries all are systems we study.

For boundary 4D coordinate or momentum spaces, the matter fields \(|\Psi\rangle\), gravity orthogonal vierbein \( \dot{\epsilon} \) and the gauge connection \( \dot{A} \) can be expressed as,

\[
|\Psi\rangle = \int_{M_x} \Psi^{st}(x|x_{st})d^4x = \int_{M_p} \Psi^{st}(p|p_{st})d^4p
\]

\[
\dot{\epsilon} = \int_{M_x} \tilde{\theta}^\alpha(x)\tilde{\varepsilon}^{\alpha}(x)d^4x = \int_{M_p} \tilde{\theta}^\alpha(p)\tilde{\varepsilon}^{\alpha}(p)d^4p
\]

\[
\dot{A} = \int_{M_x} A^\alpha_a(x)\tilde{\varepsilon}^{\alpha}(x)d^4x = \int_{M_p} A^\alpha_a(p)\tilde{\varepsilon}^{\alpha}(p)d^4p.
\]

(545)

The 4D quantum theory, in which the space-time is in the sense of general relativity, can be considered as a generalization of 3D space plus time quantum theory. All physical quantities are considered in the framework of 4D theory.

Here we have some remarks about the 4D unified quantum theory. (1). A 4D quantum system can be described completely by the matter wave function \( \Psi^{st}(x) \), orthogonal vierbein function \( \epsilon^\alpha(x) \) and the gauge connection function \( A^\alpha_a \). The Dirac equation for matter fields, Einstein equation of gravity and the Yang-Mill gauge equation constitute a complete description of this quantum system. If we know the boundary condition \( \partial M_x \), the solution of these equations are fixed then. (2). The 4D unified quantum theory is a local theory in the sense that the interactions are local which just depend on their neighbors. (3). For a 4D coordinate space \( M_x \) with a boundary, the matter fields \(|\Psi\rangle\) and the interactions \( \hat{F} \) can be expanded discretely in 4D momentum space,

\[
|\Psi\rangle = \int_{M_x} \Psi(x|x)d^4x = \sum_{n=1}^{\infty} \Psi(p_n)|p_n\rangle.
\]

\[
\hat{F} = \int_{M_x} F(x)\varepsilon(x)d^4x = \sum_{n=1}^{\infty} F(p_n)\varepsilon(p_n).
\]

(546)

And vise versa, the matter fields \(|\Psi\rangle\) and the interaction fields \( \hat{F} \) in 4D momentum space with a boundary, the coordinate expansion is discrete,

\[
|\Psi\rangle = \sum_{m=1}^{\infty} \tilde{\Psi}(x_m|x_m) = \int_{M_p} \tilde{\Psi}(p)|p\rangle d^4p
\]

\[
\hat{F} = \sum_{m=1}^{\infty} F(x_m)\varepsilon(x_m) = \int_{M_p} \tilde{F}(p)\varepsilon(p)d^4p.
\]

(547)

For a matter field \(|\Psi\rangle\) and interaction field \( \hat{F} \) existing simultaneously in 4D coordinate space \( M_x \) with a boundary and in 4D momentum space \( M_p \) with a boundary, the expansions are discrete at the same time in coordinate space
and the momentum space, and the terms in the expansion are finite,

\[ |\Psi\rangle = \sum_{m=1}^{M} \Psi(x_m) |x_m\rangle = \sum_{n=1}^{N} \bar{\Psi}(p_n) |p_n\rangle, \]

\[ \hat{F} = \sum_{m=1}^{M} F(x_m) \hat{\varepsilon}(x_m) = \sum_{n=1}^{N} \bar{F}(p_n) \hat{\varepsilon}(p_n), \]

\[ M = N = \int_{M_x} d^4x \int_{M_p} d^4p. \]  \((548)\)

There are only finite number of quantum events for finite coordinate space \(M_x\) and finite momentum space \(M_p\). When the term quantum was first proposed, it means that the energy has a smallest unit and is discrete. In our 4D unified theory, we can find that the energy, momentum, space-time are all discrete.

### B. Observable quantities

We next consider the real world observable quantities from this theory.

In our unified theory, we can define several current densities in which, three important quantities are the densities of particle current \(\hat{\rho}\), total gauge current \(\hat{J}\) defined in Eq. \((466)\) and the energy-momentum tensor \(\hat{T}\). The particle current density is related with the number of particles, the gauge current density can be considered as the source of gauge fields, the energy-momentum tensor can be considered as the source of gravity. For 4D coordinate space \(M_x\) with a boundary and 4D momentum space \(M_p\) with a boundary, the density of particle current, the gauge current density, the energy-momentum tensor are defined respectively as,

\[ \hat{\rho} = \rho^\alpha(\hat{x}) \hat{\gamma}_\alpha \]

\[ = \int_{M_x} \rho^\alpha(x) \hat{\gamma}_\alpha \hat{\varepsilon}(x) d^4x \]

\[ = \int_{M_p} \bar{\rho}^\alpha(p) \hat{\gamma}_\alpha \hat{\varepsilon}(p) d^4p. \]  \((549)\)

\[ \hat{J} = J_\alpha^a(\hat{x}) \hat{\gamma}_\alpha \hat{T}^a \]

\[ = \int_{M_x} J_\alpha^a(x) \hat{\gamma}_\alpha \hat{T}^a \hat{\varepsilon}(x) d^4x \]

\[ = \int_{M_p} \bar{J}_\alpha^a(p) \hat{\gamma}_\alpha \hat{T}^a \hat{\varepsilon}(p) d^4p. \]  \((550)\)

\[ \hat{T} = T_\beta^\gamma \hat{\gamma}_\alpha \otimes \hat{\gamma}_\beta \]

\[ = \int_{M_x} T_\beta^\gamma(x) \hat{\gamma}_\alpha \otimes \hat{\gamma}_\beta \hat{\varepsilon}(x) d^4x \]

\[ = \int_{M_p} \bar{T}_\beta^\gamma(p) \hat{\gamma}_\alpha \otimes \hat{\gamma}_\beta \hat{\varepsilon}(p) d^4p. \]  \((551)\)

We then define three production rate densities related with particle numbers, gauge charges and energy-momentum

\[ \hat{\theta}_\alpha^\mu (\hat{x}) \partial_\mu \hat{\rho}^\alpha (\hat{x}) + \hat{\Gamma}_\beta^\alpha_{\beta,\alpha} \hat{\rho}^\alpha (x) = \hat{n}(x) \]  \((552)\)

\[ \hat{\theta}_\alpha^\mu (\hat{x}) \partial_\mu \hat{J}_\alpha^a (\hat{x}) + \hat{\Gamma}_\beta^\alpha_{\beta,\alpha} \hat{J}_\alpha^a (\hat{x}) = \hat{q}_a (x) \]  \((553)\)

\[ \hat{\theta}_\alpha^\mu (\hat{x}) \partial_\mu \hat{T}_\beta^\gamma (\hat{x}) + \hat{\Gamma}_\sigma^\alpha_{\sigma,\alpha} \hat{T}_\beta^\gamma = \hat{p}_\beta (x) \]  \((554)\)
We define an operator on manifold corresponding to coordinate 4D space $M_x$ and momentum 4D space $M_p$,

$$\hat{\omega} = \det[\hat{e}_\mu^\alpha(x)]$$

$$= \int_{M_x} \omega(x)\hat{e}(x)d^4x$$

$$= \int_{M_p} \tilde{\omega}(p)\hat{e}(p)d^4p.$$ (555)

We may notice vierbein $\hat{e}_\mu^\alpha$ appears here, this is an indication of this theory is in general relativity curved space-time. The trace of this operator is the 4D volume of $M_x$,

$$\omega = \text{tr}\hat{\omega}. \quad (556)$$

In 4D space $M_x$, the number of particles created is denoted as $N$, it takes the form

$$N = \langle \hat{n}, \hat{\omega} \rangle = \text{tr}(\hat{n}\hat{\omega})$$

$$= \int_{M_x} n(x)\omega(x)d^4x$$

$$= (2\pi)^2 \tilde{n}(0)*\tilde{\omega}(0) \quad (557)$$

Similarly, the number of gauge charges created, the energy-momentum created in 4D space $M_x$ can be written as,

$$Q_a = \langle \hat{q}_a, \hat{\omega} \rangle,$$

$$P_\beta = \langle \hat{p}_\beta, \hat{\omega} \rangle. \quad (558)$$

Generally if we would like to describe a physical quantities in the real world 3D space with time, for example, in order to find the total particle numbers in 3D area $V$, we need to make the integral of the 3D area $V$ at a fixed time point $t_0$,

$$N_{3D,t_0} = \int_V \rho^0(x)d^3\vec{x}. \quad (559)$$

However, the problems of this equation are: first, one element of particle current density is not general covariance; second, the 3D manifold $V$ is not covariance and a fixed $t_0$ is not allowed in general relativity. So the calculation for quantity $N_{3D,t_0}$ is not allowed in the 4D unified theory.

The proper method to find the observable quantities of this theory should be in covariance condition. We can consider to construct a covariance 4D space $M_x$ by using the real world 3D space $V$ and a real world short time period $[t_0, t_0 + \Delta t]$,

$$M_x = V \otimes [t_0, t_0 + \Delta t]. \quad (560)$$

The observable quantity like particle number takes the form,

$$\langle N \rangle = \frac{\int_{M_x} \rho^0(x)\omega(x)d^4x}{\Delta t}. \quad (561)$$

Thus from 4D unified theory, all observable quantities in the real world can be understood as the average quantities by a short time period $\Delta t$. Similarly, the gauge charge and energy-momentum in 3D space $V$ can be written as,

$$\langle Q_a \rangle = \frac{\int_{M_x} J_a^0(x)\omega(x)d^4x}{\Delta t},$$

$$\langle P_\beta \rangle = \frac{\int_{M_x} T_\beta^0(x)\omega(x)d^4x}{\Delta t}. \quad (562)$$

C. Particles scattering

We next will first briefly present a summary of the representations of matter particles, gauge particles and graviton. Based on those representations, we will then consider the results of scattering of those particles. As we all know the particle scattering process can be described well by Feynman diagram method, our representations are based on those results and on the other hand those representation can deduce easily the particles scattering results. Those results agree with the results by Feynman diagram method.

The following is a summary of the representations of particles.
1. The state of matter particles takes the form,
\[ |e_{st}(p)\rangle = |p\rangle \otimes |s\rangle \otimes |t\rangle, \]
where \( p \in R^4 \) is the momentum, \( s = 1, 2, 3, 4 \) is the spin, the spin representation space is \( V_S(\frac{1}{2}, 0) \oplus V_S(0, \frac{1}{2}) \), so the matter particles are spin-\( \frac{1}{2} \), \( t = 1, 2, \cdots, 48 \) corresponds to 48 classes of matter particles.

2. The gauge particles are denoted as,
\[ \hat{\mathbb{e}}\mathbb{^a}(p) = \hat{\mathbb{e}}(p) \otimes \hat{\gamma}^a \otimes \hat{T}_a, \]
where \( p \in R^4 \) is the momentum, \( \alpha = 0, 1, 2, 3 \), the spin representation space is \( V_S(\frac{1}{2}, \frac{1}{2}) \) and so they are spin-1, \( a = 1, 2, \cdots, 12 \), this means that there are 12 classes of gauge particles.

3. The graviton is denoted as,
\[ \hat{\mathbb{g}}_{\alpha \rho \sigma}(p) = \hat{\mathbb{g}}(p) \otimes \hat{\gamma}^\alpha \otimes \hat{s}_{\rho \sigma}, \]
where \( p \in R^4 \), \( \alpha, \rho, \sigma = 0, 1, 2, 3 \), the graviton is spin-1. Here we have a particle corresponding to vierbein, its representation is written as,
\[ \hat{\mathbb{e}}\mathbb{^\alpha}_\mu(p) = \hat{\mathbb{e}}(p) \otimes \hat{\gamma}^\alpha \otimes \hat{s}_{\mu}, \]
where \( \alpha = 0, 1, 2, 3 \), the spin representation space is \( V_S(\frac{1}{2}, -\frac{1}{2}) \), the gauge charge of this vierbein particle are 0, it is spin-1.

With those representations, we next present the scattering matrices. The 3-vertex scattering matrices for matter particles and action particles, which include gauge particles, graviton and vierbein particles, are written as,
\[ \langle \hat{e}^{s_1t_1}(p_2)|\hat{\mathbb{e}}_{a}(p_3)|e_{s_1t_1}(p_1) \rangle = (\hat{\gamma}^a)^s_1\hat{s}_s_1\hat{\delta}^4(p_1 + p_3 - p_2). \]
This is the case of scattering between matter particles which are fermions with the gauge bosons. Explicitly, we can see the initial state is \( |e_{s_1t_1}(p_1)\rangle \) defined in (563) which has 4 \( \times \) 48 choices for gauge and spin altogether, the gauge boson \( \hat{\mathbb{e}}^a \) has 12 \( \times \) 4 choices, the final state is still a matter particle. The right hand side of the equation is the scattering result. Also we have,
\[ \langle \hat{e}^{s_2t_2}(p_2)|\hat{\mathbb{g}}_{\alpha \rho \sigma}(p_3)|e_{s_2t_2}(p_1) \rangle = (\hat{\gamma}^\rho)^s_2\hat{s}_s_2\hat{\delta}^4(p_1 + p_3 - p_2), \]
This is the case of scattering between matter particles with the graviton.
\[ \langle \hat{e}^{s_2t_2}(p''')(p''')|\hat{\mathbb{g}}_{\alpha \rho \sigma}(p''')|e_{s_2t_2}(p') \rangle = (\hat{\gamma}^\rho)^{s_2}_3\hat{s}_{s_3}\hat{\delta}^4(p' + p''' - p''), \]
The case describes the scattering between matter particles with the vierbein particle related with the propagation of matter particles in curved space-time.

The 4-vertex scattering matrices represent the scattering between action particles and action particles,
\[ \langle \hat{e}^{s_4}(p_4), \hat{\mathbb{e}}_{\alpha a_3}(p_3), \hat{\mathbb{e}}_{\alpha a_2}(p_2), \hat{\mathbb{e}}_{\alpha a_1}(p_1) \rangle = \eta^{s_3 a_4} \eta^{s_2 a_3} G_{a_4 c} C_{a_3 c} a_{a_2} C_{a_2 b} a_{a_1} \delta^4(p_1 + p_2 + p_2 - p_4). \]
The middle two particles \( \hat{\mathbb{e}}_{\alpha a_3}(p_3), \hat{\mathbb{e}}_{\alpha a_2}(p_2) \) are two gauge particles, each of them can also be graviton and vierbein particle, so altogether there are 9 cases. For example, we can change the second particle as vierbein particle, the 4-vertex scattering matrix is,
\[ \langle \hat{\mathbb{e}}_{\alpha a_4}(p_1), \hat{\mathbb{e}}_{\alpha a_3}(p_3), \hat{\mathbb{e}}_{\mu a_2}(p_2), \hat{\mathbb{e}}_{\alpha a_1}(p_1) \rangle = \eta^{s_4 a_1} \eta^{s_3 a_4} G_{a_4 c} C_{a_3 c} a_{a_2} p_{12} \delta^4(p_1 + p_2 + p_2 - p_4). \]
Those are the basic scattering matrices, other cases can be deduced from those results.

X. SYMMETRIES AND SYMMETRIES BROKEN AND DARK ENERGY

The energy-momentum conservation law provides us a concrete foundation to confirm our unified theory. Based on our theory, some fundamental problems may be studied. We next consider several questions.

Symmetries and symmetries broken play a key role in modern physics [19]. They are also important in studying the physical implications of our theory. In this work, the key parameters are those involved in the three fundamental equations (533, 563, 580). Explicitly they are: \( |\Psi\rangle, D, M, \hat{m} \). For definitions, see \( |\Psi\rangle \) in (524), derivative operator \( D \) in (573), and mass related operators \( \hat{m} \) for matter particles in (202), \( M \) for gauge fields in (208).
A. Unitary transformation

According to the representations, we can define the general transformations as

$$\hat{U} = \exp[-i(a_\mu \hat{x}^\mu + b_\mu \hat{p}_\mu + \alpha + \frac{1}{2} \omega^{\alpha\beta} \hat{s}_{\alpha\beta} + \xi^\alpha T_\alpha)].$$

(572)

Under this transformation, $|\Psi\rangle$, $\hat{D}$, $\hat{M}$, $\hat{m}$ will be changed as,

$$|\Psi\rangle \rightarrow \hat{U}|\Psi\rangle,$$

$$\hat{D} \rightarrow \hat{U} \hat{D} \hat{U}^{-1},$$

$$\hat{M} \rightarrow \hat{U} \hat{M} \hat{U}^{-1},$$

$$\hat{m} \rightarrow \hat{U} \hat{m} \hat{U}^{-1}.$$  

(573)

Here we define the unitary condition as

$$\hat{U}^{-1} = \overline{U}.$$  

(574)

With this unitary transformation, the three fundamental equations remains the same.

B. Mass matrices of elementary particles, color confinement

For gauge theory, physical quantities should be gauge invariance. For the unified theory of this work, all fields represented by vectors and operators are gauge covariance. Still by proper representations, physical process and physical quantities are gauge invariance.

A key feature of our theory is that the general mass matrices are defined in gauge space as,

$$\hat{m} = m_t |e_t\rangle \otimes |e_t\rangle^t,$$

$$\hat{M} = M^a_b \hat{T}_a \otimes \hat{T}_b.$$  

(575)

We do not need each element of $\hat{m}$ and $\hat{M}$ be fixed. However, the observable physical quantities including mass are still gauge invariance. By comparison in conventional gauge theory, each element $m^t_a$ and $M^a_b$ of the mass matrices are gauge invariance and mass in gauge theory is introduced by Higgs mechanism by gauge symmetries breaking.

Mass matrices are representations in gauge space and the only restrictions are three fundamental equations. Thus in principle, any kind of mass matrices are allowed if no experimental facts are violated. In this sense, no gauge symmetries breaking is necessary to create mass. So we mean Higgs mechanism is not necessary in this theory. We remark that the mass matrix itself is gauge covariant while the observable mass which corresponding to eigenvalues of mass matrix are of course should be gauge invariance. From the proof of the energy-momentum conservation law, what we use is only the conditions $M_{ab} = M_{ba}$, $\hat{m}^\dagger = \hat{m}$, no Higgs particles are necessary to play a role in the energy-momentum tensor.

From the form of mass matrices, we may notice that there exist a symmetry in color charge space. Explicitly, the mass matrix is $SU(3)$ invariant, for color charges $\hat{T}_a, a = 5, 6, \cdots, 12$ or $\hat{\lambda}_p, p = 1, 2, \cdots, 8$, as we already seen that,

$$[\hat{T}_a, \hat{m}] = 0.$$  

(576)

Thus the solution of the three fundamental equations has this $SU(3)$ symmetry for free boundary condition. Suppose we have a solution presented as,

$$\hat{A} = A^a_\alpha (x) \hat{T}_\alpha \otimes \hat{\gamma}_a,$$

$$|\Psi\rangle = \int_{R^4} \Psi^{st}(x) |e_{st}\rangle d^4 x.$$  

(577)

We know that the unitary transformations $\hat{U}$ by the color charges on the solution is still a solution,

$$\hat{A'} = \hat{U} \hat{A} \hat{U}^{-1},$$

$$|\Psi'\rangle = \hat{U} |\Psi\rangle.$$  

(578)
Since the existing of this SU(3) symmetry in color charge space, we will never be able to distinguish a single color state by only mass since this will break this SU(3) symmetry. This also means, a single color quantum state which corresponds to a single color quark will never been separated and observed. What we observed is always a coherence of all three color states. This is the result of color confinement. The color confinement is due to the symmetry of mass matrices. While for weak interactions, no such symmetry exists so there is no confinement. The reason of confinement is based on the symmetry of mass matrices whose definitions are based on experimental results in gauge space as in Eq. (262) and in Eq. (298), and also no Higgs mechanism is necessary, it is thus easy to understand why there is confinement in color space but no confinement in weak space. Gluon is connected only with quarks, the color confinement also implies the confinement of gluon. Explicitly, in the above equations, $A' = UAU^{-1}$ implies the confinement of gluons, and $|\Psi'\rangle = U|\Psi\rangle$ implies the confinement of quarks.

Mass is always a basic question in physics. Einstein’s mass-energy equation $E = mc^2$ is well known. In quantum field theory, we have $\hat{p}_\mu \hat{p}^\mu = m^2$. By this equation, we may understand that mass is defined in momentum space. However, this form is in general not true in the present theory. What we have is due to the square differential of Dirac equation as in Eq. (580), it reduces to form $\hat{p}_\mu \hat{p}^\mu = m^2$ only in special case for a free field.

C. Parity violation for weak interactions

The left chiral projector and the right chiral projector are defined respectively as,

$$\hat{P}_L = \frac{1}{2} (1 + \gamma^5),$$
$$\hat{P}_R = \frac{1}{2} (1 - \gamma^5). \quad (579)$$

Those two projectors satisfy the relations,

$$\hat{P}_L |\Psi\rangle = \hat{P}_L |\Psi\rangle,$$
$$\hat{P}_R |\Psi\rangle = \hat{P}_L |\Psi\rangle,$$
$$\hat{P}_L \gamma^\alpha |\Psi\rangle = \gamma^\alpha \hat{P}_L |\Psi\rangle,$$
$$\hat{P}_L \gamma^\alpha = \gamma^\alpha \hat{P}_L,$$
$$|\hat{P}_L, \hat{D}_\alpha\rangle = |\hat{P}_R, \hat{D}_\alpha\rangle = 0,$$
$$|\hat{P}_L, \hat{m}\rangle = |\hat{P}_R, \hat{m}\rangle = 0. \quad (580)$$

So the left chiral field and the right chiral field can then be defined as,

$$|\Psi_{(L)}\rangle = \hat{P}_{(L)} |\Psi\rangle,$$
$$|\Psi_{(R)}\rangle = \hat{P}_{(R)} |\Psi\rangle. \quad (581)$$

The properties can be found as,

$$\tilde{\gamma}^5 |\Psi_{(L)}\rangle = (+1) |\Psi_{(L)}\rangle,$$
$$\tilde{\gamma}^5 |\Psi_{(R)}\rangle = (-1) |\Psi_{(L)}\rangle. \quad (582)$$

The adjoint states have following relations, note that $L$ and $R$ are exchanged,

$$|\Psi_{(L)}\rangle = \langle \Psi | \hat{P}_{(R)} \rangle,$$
$$|\Psi_{(R)}\rangle = \langle \Psi | \hat{P}_{(L)} \rangle,$$
$$\tilde{\gamma}^5 |\Psi_{(L)}\rangle = \langle \Psi_{(L)} | (-1),$$
$$\tilde{\gamma}^5 |\Psi_{(R)}\rangle = \langle \Psi_{(R)} | (+1). \quad (583)$$

When applying those projectors onto the Dirac equation, we have the equations,

$$i \tilde{\gamma}^\alpha \hat{D}_\alpha |\Psi_{(R)}\rangle = \hat{m} |\Psi_{(L)}\rangle,$$
$$i \tilde{\gamma}^\alpha \hat{D}_\alpha |\Psi_{(L)}\rangle = \hat{m} |\Psi_{(R)}\rangle. \quad (584)$$
For adjoint states, we have
\[ \langle \Psi_{(R)} | \hat{D}_\alpha \hat{\gamma}^\alpha i | = \langle \Psi_{(L)} | \hat{m}, \] (586)
\[ \langle \Psi_{(L)} | \hat{D}_\alpha \hat{\gamma}^\alpha i | = \langle \Psi_{(R)} | \hat{m}. \] (587)

The isospin doublet projector and the isospin singlet projector are defined as,
\[ \hat{P}_D = \frac{4}{3} \hat{I}_2, \]
\[ \hat{P}_S = 1 - \frac{4}{3} \hat{I}_2, \] (588)

here, \( \hat{I}_2 \) is the summation of squares of three elements of isospin. We can prove that,
\[ \hat{P}_D \hat{\gamma}_\alpha = \hat{\gamma}_\alpha \hat{P}_S, \]
\[ \hat{P}_S \hat{\gamma}_\alpha = \hat{\gamma}_\alpha \hat{P}_D, \]
\[ \hat{P}_D \hat{m} = \hat{m} \hat{P}_S, \]
\[ \hat{P}_S \hat{m} = \hat{m} \hat{P}_D. \] (589)

The isospin doublet state and the isospin singlet state are
\[ |\Psi_D\rangle = \hat{P}_D |\Psi\rangle, \]
\[ |\Psi_S\rangle = \hat{P}_S |\Psi\rangle. \] (590)

Applying the projection onto the Dirac equation, we can find two equations,
\[ i \hat{\gamma}^\alpha \hat{D}_\alpha |\Psi_D\rangle = \hat{m} |\Psi_S\rangle, \]
\[ i \hat{\gamma}^\alpha \hat{D}_\alpha |\Psi_S\rangle = \hat{m} |\Psi_D\rangle. \] (591)

Next we define two new projectors corresponding to normal particles and anomaly particles, respectively,
\[ \hat{P}_{(a)} = \hat{P}_{(D)} + \hat{P}_{(R)} \hat{P}_{(S)} = \frac{1}{2} (1 + \hat{\gamma}^5 \hat{H}), \]
\[ \hat{P}_{(n)} = \hat{P}_{(R)} \hat{P}_{(D)} + \hat{P}_{(L)} \hat{P}_{(S)} = \frac{1}{2} (1 - \hat{\gamma}^5 \hat{H}). \] (593)

We can find the following properties for those two projectors,
\[ \hat{P}_{(a)} = \hat{P}_{(n)}, \]
\[ \hat{P}_{(n)} = \hat{P}_{(a)}, \]
\[ \hat{P}_{(a)} \hat{\gamma}_\alpha = \hat{\gamma}_\alpha \hat{P}_{(n)}, \]
\[ \hat{P}_{(n)} \hat{\gamma}_\alpha = \hat{\gamma}_\alpha \hat{P}_{(a)}, \]
\[ [\hat{P}_{(a)}, \hat{D}_\alpha] = [\hat{P}_{(n)}, \hat{D}_\alpha] = 0, \]
\[ \hat{P}_{(a)} \hat{m} = \hat{m} \hat{P}_{(n)}, \]
\[ \hat{P}_{(n)} \hat{m} = \hat{m} \hat{P}_{(a)}. \] (595)

The quantum states of normal particles and anomaly particles are,
\[ |\Psi_{(a)}\rangle = \hat{P}_{(a)} |\Psi\rangle, \]
\[ \langle \Psi_{(a)} | = \langle \Psi | \hat{P}_{(a)}, \]
\[ |\Psi_{(n)}\rangle = \hat{P}_{(n)} |\Psi\rangle, \]
\[ \langle \Psi_{(n)} | = \langle \Psi | \hat{P}_{(n)}. \] (596)
We can find that the normal particle state and the anomaly particle state satisfy the Dirac equation, respectively,
\begin{align}
i\hat{\gamma}^\alpha \hat{D}_\alpha |\Psi_{(a)}\rangle &= \hat{m}_a |\Psi_{(a)}\rangle, \\
i\hat{\gamma}^\alpha \hat{D}_\alpha |\Psi_{(n)}\rangle &= \hat{m}_n |\Psi_{(n)}\rangle.
\end{align}
(598)
(599)
Since normal particles and the anomaly particles independently satisfy their Dirac equations, the type of normal particles and the type of anomaly particles do not evolve into each other. The particles in our world are all normal particles, there does not exist anomaly particles. We know that chiral left state only corresponds to isospin doublet state, while chiral right state only corresponds to isospin singlet state. Since the chiral left state involves into the weak interactions, chiral right state does not involves into the weak interactions, the consequence is that for weak interactions, there is only chiral left state. This is the conclusion that there is no parity conservation law for weak interactions [21]. Here we can see that the parity violation can be explained naturally in our theory.

D. Gravity and CPT violation

CPT is a combined transformation of charge conjugation, space reversal and time reversal. It is known that the Standard Model are symmetric under CPT transformation. Here we would like to discuss the relationships between gravity and CPT violation.

Since the theory of this work is in the framework of general relativity, there is no independent space reversal and time reversal. What we have is a combined PT reversal, the momentum operator transforms as, \(\hat{p}_\mu \rightarrow -\hat{p}_\mu, \mu = 0, 1, 2, 3\). We define the charge conjugation transformation as,
\[\hat{T}_a \rightarrow -\hat{T}_a.\]
(600)
Under CPT transformation, we have \(\hat{\gamma}^\alpha \rightarrow -\hat{\gamma}^\alpha\), while spin operators remain the same \(\hat{s}_{\alpha\beta} \rightarrow \hat{s}_{\alpha\beta}\). Recall the definition of the general covariance derivative operator in (373), \(\hat{D}_a = -i\hat{\theta}_a^\mu \otimes \hat{p}_\mu + \frac{1}{2} \hat{\Gamma}_a^\mu \otimes \hat{s}_{\rho\sigma} - i\hat{\Lambda}_a^\mu \otimes \hat{T}_a\), and also consider that always the form \(\hat{\gamma}^\alpha \hat{D}_\alpha\) appears in fundamental equations, it transforms as
\[\hat{\gamma}^\alpha \hat{D}_\alpha \rightarrow -i\hat{\theta}_a^\mu \hat{\gamma}^\alpha \otimes \hat{p}_\mu - \frac{1}{2} \hat{\Gamma}_a^\mu \hat{\gamma}^\alpha \otimes \hat{s}_{\rho\sigma} - i\hat{\Lambda}_a^\mu \hat{\gamma}^\alpha \otimes \hat{T}_a.\]
(601)
One may notice that the first and the third terms are invariant under CPT transformation, while the second term which is related with gravity field changes its symbol from positive to negative. Also we know that \(\hat{m}, M\) are invariant. Consider that \(\hat{\gamma}^\alpha \hat{D}_\alpha\) involves into Dirac equation and Yang-Mills equation, we can conclude that the gravity field will cause CPT violation. In case \(\hat{\Gamma}_a^\rho = 0\), which means there is no gravity field, the CPT symmetry remains.

In general it is considered that the CPT symmetry represents the symmetry between particle and antiparticle. The result that gravity will cause CPT violation means that gravity is not symmetric for particles and antiparticles. This is understandable since gravity force is always attractive for particle and antiparticle.

E. A dark energy solution

The observed evidences [22, 23] show that the expansion of the universe is accelerating. Also, by the data from the Wilkinson Microwave Anisotropy Probe (WMAP) [24] and other teams, the universe is flat, homogeneous, and isotropic and the distribution of baryonic matter and radiation, dark matter and dark energy is approximately: 4.5%, 23% and 72%. The dark energy is homogeneous, nearly independent of time and the density is very small which is around \(\rho_\Lambda = 10^{-29} g/cm^3\). The dark energy is not known to interact through any of the fundamental forces except gravity. The gravitational effect of dark energy approximates that of Einstein’s cosmological constant, it has a strong negative pressure which can explain the observed accelerating universe. Still there are some other models about it. Presently it seems that there is no general accepted quantum theory of dark energy, actually the Planck energy density which is the expected candidate is about 120 orders of magnitude larger than the dark energy.

Based on the unified theory of the present work, here we propose a dark energy solution. We assume: (a) There is no matter or matter particles, \(|\Phi\rangle = 0, j_\alpha^a = 0\); (b) There is no gravity \(\hat{\Gamma}_a^\rho = 0\); (c) The gauge potential is constant, \(\partial_\mu A^\mu_\alpha = 0\). Recall the definition in Eq. (442), we find the gauge curvature now takes the form
\[F^a_{\alpha\beta} = -iC^a_{bc} A^b_\alpha A^c_\beta.\]
(602)
Substitute the gauge curvature into Yang-Mills equation in (443), we have,
\[D^a F^a_{\alpha\beta} = M^a_\beta A^b_\beta,\]
(603)
\[ -iC^a_{bc}A^{b,\alpha}F^c_{\alpha\beta} = M^a_{\beta}A^b_{\beta}. \]  

(604)

Thus the equation is,

\[ -C^a_{bc}C^b_{de}A^{b,\alpha}A^d_{\alpha}A^e_{\beta} = M^e_{\alpha}A^a_{\beta}. \]  

(605)

One solution of this equation is \( A^a_{\beta} = 0 \), it is trivial and we do not discuss it. Next, we shall consider the solution,

\[ -C^a_{bc}C^c_{de}A^{b,\alpha}A^d_{\alpha} = M^a_{\beta}. \]  

(606)

Consider that,

\[ F^{a,\rho}_{\alpha\beta}F^a_{\beta\rho} = -C^{a,\rho}_{\alpha\beta}A^{b,\alpha}A^c_{\beta}A^d_{\alpha}A^e_{\rho} \]

\[ = (C^{a,\rho}_{\alpha\beta}A^{b,\alpha}A^c_{\beta})A^d_{\alpha}A^e_{\rho} \]

\[ = -M^f_{\rho}A^a_{\rho}. \]  

(607)

where the solution (606) is used in the last equation. Also we have,

\[ F^{a,\rho}_{\alpha\beta}F^a_{\beta\rho} = -M^b_{\rho}A^a_{\rho}, \]  

(608)

Substituting those results to the energy-momentum tensor of gauge fields (479), we can find that,

\[ \tau^\alpha_{\beta} = -\frac{1}{4} \delta_{\beta}^\alpha m^4 g^2 G_{\beta\rho}A^a_{\rho}. \]  

(609)

The explicit form of this energy-momentum tensor depends on the solution of equation (606) if the exact mass matrix \( M^a_{\beta} \) is given. Please note here this energy-momentum tensor will provide a cosmological constant.

Next, we assume: (d) The candidate of dark energy is related with the elementary particle gluon. For this case, the mass matrix of gluon takes the form,

\[ M^a_{\beta} = m^2 \delta^a_{\beta}, \]  

(610)

where \( m \) is the mass of gluon, with the help of the solution of Yang-Mills equation (606), we have,

\[ m^2 = -C^a_{bc}C^c_{de}A^{b,\alpha}A^d_{\alpha} \]

\[ = -g^2_3 G_{bd}A^{b,\alpha}A^d_{\alpha} \]

\[ = -g^2_3 A^{d}_{\alpha}A^d_{\alpha}. \]  

(611)

where \( g^3_3 \) is the coupling constant as presented in (214). Consider also the form of mass matrix of gluon, the energy-momentum tensor of gauge field is,

\[ \tau^\alpha_{\beta} = \delta_{\beta}^\alpha m^4 \frac{4g^3_3}{4g^3_3}. \]  

(612)

Recall our assumption (a) which means the energy-momentum tensor of matter fields is zero, the total energy-momentum tensor is

\[ T_{\alpha,\beta} = \eta_{\alpha,\beta} \frac{m^4}{4g^3_3}. \]  

(613)

where \( \eta_{\alpha,\beta} \) is the Minkowski metric. Please note that no boundary condition is used to obtain this solution, and it can be assumed to be valid for universe. Recall the Einstein equation in (450), it is now clear that the energy-momentum tensor here (613) corresponds to Einstein’s cosmological constant. So we conclude that the density of the dark energy \( \rho_\Lambda \) is,

\[ \rho_\Lambda = \frac{m^4}{4g^3_3}. \]  

(614)

In Standard Model and also in our theory, \( g_3 \approx 1.22 \), consider that the density of the dark energy \( \rho_\Lambda \approx 10^{-29} g/cm^3 \), we can estimate that the mass of gluon is around \( 10^{-3} eV \sim 10^{-2} eV \). It is theoretically accepted and experimentally confirmed that the mass of gluon is zero, here we can see that the estimated mass of gluon is very small thus it should not have detectable effects on present experiments.

Let’s list some properties of the present theory of dark energy and show that this result is reasonable.
1. The elementary particle gluon does not interact through electrical force, does not have weak interactions, its mass is very small. And further there is the confinement of gluon due to color confinement, A free gluon is thus impossible or hard to be detected directly in experiments. The solution of gluon is a 0-mode solution, the momentum is zero since of the constant gauge potential (c) and correspondingly it does not change in 4D coordinate space. That means it is invariant in 4D space-time and the gluon will not cause any energy excitation, it is like the vacuum state. Except gravitational effects, our theory shows that there will be no interactions available now and in the future. All of those agree with the properties of dark energy. Here we would like to emphasize again that the unified theory of the present work itself does not assume that the gluon is the only candidate for dark energy since Eq. (609) always provides a cosmological constant like energy-momentum tensor. However, the dark energy is generally not assumed to be any matter particles including neutrino, it seems not photon by observation. We may consider the weak interaction bosons W±, Z0, but they seem too heavy to be related with the dark energy. We may roughly estimate that \( \rho_\Lambda \approx \frac{m_W^2}{4g_\ast^2} \approx 10^{43} \text{eV}^4 \), it is around \( 10^{25} \text{g/cm}^3 \) which could be a candidate for black hole or the early state of universe but not the dark energy, where \( g_\ast \approx 0.65 \). Because of the above reasons, we consider that gluon should be the most possible candidate.

2. We can find that the energy-momentum tensor with our assumptions plays the role of Einstein’s cosmological constant. The present observations for dark energy, for example in WMAP, prefer to the cosmos model of cosmological constant with equation of state parameter \( w = -1 \) within 14% level. Our theory of dark-energy explains well the origin of the cosmological constant.

3. The mass of gluon in Standard Model is generally assumed to be zero, here we find that the estimated mass of gluon is very small which is reasonable for present particle experiments. The Standard Model which is very successful theoretically and experimentally still remains correct. Of course as we mentioned, a massive gluon is allowed in our 4D unified theory.

4. We would like to remark that our solution of Yang-Mills equation does not use any boundary condition thus it can be applied to case of universe. It is a fixed constant solution of an equation. On the other hand, for this existing solution of Yang-Mills equation, the cosmological constant is a reasonable explanation.

5. To find a solution from Yang-Mills equation, we make several assumptions based on the observable facts of universe. The fact that there is no matter available corresponds to assumption (a). The universe is flat, homogeneous and isotropic, this is corresponding to assumptions (b) and (c). Those assumptions also lead to the fact that cosmological constant has an origin from quantum effects and is independent of gravity.

6. We finally comment that the present theory of dark energy shows that our 4D unified theory which is a unification of quantum mechanics and general relativity can simply explain the dark energy well. This theory should be a complete theory. Let us emphasize again the structure of our theory: we first have representations, then we use three fundamental equations to find the results.

XI. CONCLUSIONS AND DISCUSSIONS

Our theory based on the cornerstones of physics, such as the Dirac equation of quantum mechanics, Einstein general relativity of gravity and Yang-Mills gauge theory. The results deduced from our theory agree well with many basic facts of physics: such as those already presented in abstract of this work: (1) Graviton is massless; (2) Mass problem of gauge theory is explained well; (3) Color confinement; (4) Parity violation; (5) CPT violation; (6) Dark energy can be explained well. Also please note that our gauge representations agree with the basic results of the Feynman diagram quantum field theory.

Our theory is a unified theory which combines the general relativity and quantum mechanics. This is not only a long dream in physics, but also it provides a theory for things like black hole or early stage of universe where both general relativity and quantum mechanics are important. One may already see, as we have presented, the dark energy is a large scale phenomenon of universe where the general relativity is important, we provide a reasonable explanation from quantum scale of Yang-Mills equation. The origin of cosmological constant is from the elementary particle.

Two foundations of our theory are; first we formulate three fundamental equations in the framework of general relativity; second we provide a proper definition of mass; the problems solved in this work generally depends on a correct concept of mass. We may also notice that gauge condition in gauge theory is in general not arbitrary except for massless particles such as photon for electromagnetic field.

Here let us discuss further the problem of mass. Our opinion is that the Higgs mechanism is not necessary. The reasons are the following: (1). From our theory, we may find that mass is defined naturally in gauge space instead of defining by momentum. In particular, no Higgs particle is introduced, actually from the proof of energy-momentum

\[ m \approx \frac{Z_1^2}{4g_\ast^2} \approx 10^{25} \text{eV}, \]

\[ \rho_{\Lambda} \approx \frac{1}{4g_\ast^2} \approx 10^{43} \text{eV}^4, \]

\[ \rho_{\Lambda} \approx \frac{m_W^2}{4g_\ast^2} \approx 10^{43} \text{eV}^4, \]

\[ \rho_{\Lambda} \approx \frac{m_W^2}{4g_\ast^2} \approx 10^{43} \text{eV}^4, \]

\[ \rho_{\Lambda} \approx \frac{m_W^2}{4g_\ast^2} \approx 10^{43} \text{eV}^4, \]
conservation law, as we mentioned, there is no role of Higgs particles. (2). Our theory does not allow a spin-0 elementary particle since it does not interact through gravity, while the Higgs particle is spin-0. (3). We have three kind of gauge conditions respectively for massless photons, massive weak interaction bosons, and gluons. The gauge is arbitrary only for massless photons case of electromagnetic field, while the gauge conditions are equations which should be satisfied for weak interaction bosons and gluons, the necessity of Higgs mechanism does not exist. We thus prefer the idea that Higgs particle does not exist. On the other hand, one may argue that the mass matrices $\hat{m}, \hat{M}$ defined in our theory might also be explained as from the Higgs mechanism. Our idea is that since mass matrices are defined in gauge space depending on experiments, if still an explanation of those matrices is necessary, or further to discuss the origin of mass, Higgs mechanism is now not a must. Let us note that the experiments for searching of Higgs particles are on going at Fermilab and will soon become operational in Large Hadron Collider (LHC) at CERN. As we know the masses of Higgs bosons below $114.4 GeV$ were previously excluded. The result of Fermilab excludes Higgs bosons between $160 GeV \sim 170 GeV$ [25]. The LHC is designed to make proton-proton collisions at an energy of $7 TeV$ per beam which can scan $10^{11} eV$ of Higgs particles in electroweak theory thus it should provide a definite answer whether Higgs particles exist or not. If no Higgs particle is found in LHC while we still believe in it, one might go to $10^{25} eV$ or higher to test grand-unification-theory or super-symmetry theory with gravity.

Our theory provides a new framework of quantum theory. We expect there may be two directions for the future of our theory. On the one hand, we need to check whether our theory agrees well with all well-established physical facts, experimentally and theoretically. On the other hand, while our theory is proved to be powerful, we still need to find some new predictions from our theory. Our comment is that with a new foundation of quantum physics, we expect that there will be a lot of new results.

We understand that quantum field theory has already been well established. However, its problem is also well-accepted, the main results are in general relies on the approximation methods. Our theory is exact: we have the representations of fields and interactions, their properties are governed by three equations, Dirac equation, Einstein equation and Yang-Mills equation, then we are led to observable quantities. The equations themselves are exact. In this sense, our theory is quite simple. This theory turns us to the similar route as the Newton classical mechanics: all are governed by equations.

A brief introduction of the quantization of gravity is presented in Ref.[26]. The present work is a detailed presentation of the whole theory. A brief presentation about mass of gauge field will be published elsewhere [27].

Finally let us recall that the proposed theory provides an unified framework for three fundamental equations. It could be falsified, but it could not be fudged. To end this paper, we comment that we intend to use the little to get the big, in Chinese, it is to throw out a brick to attract a jade.

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APPENDIX A: APPENDIX

For self-consistent and convenient, we present the notations, foundations and some detailed calculations of this work in this appendix.

1. Tensor representation

The 1st-order tensor space $T^{(1)}$ is a $n$-dimensional space spanned by basis $\{e_i\}, (i = 1, 2, \cdots, n)$. A 1st-order tensor $K$ can be expanded by this basis as

$$K = K^i e_i,$$  \hspace{1cm} (A1)

where summation over repeated lower-upper indices is assumed.

The $r$-th order tensor space $T^{(r)}$ is the tensor product of $r$ 1st-order tensor spaces,

$$T^{(r)} = \prod_r^{\otimes} T^{(1)}$$ \hspace{1cm} (A2)

The space $T^{(r)}$ is spanned by basis $\{e_{i_1, \cdots, i_r}\}$ expressed as,

$$e_{i_1, \cdots, i_r} = e_{i_1} \otimes \cdots \otimes e_{i_r},$$ \hspace{1cm} (A3)

where $i_1, \cdots, i_r = 1, 2, \cdots, n$. The $r$-th order tensor can be represented by the this basis as,

$$K = K^{i_1, \cdots, i_r} e_{i_1, \cdots, i_r}.$$ \hspace{1cm} (A4)

The tensor product of a $r$-th tensor $K$ and a $s$-th tensor $M$ is a $r + s$-th tensor which takes the form

$$K \otimes M = K^{i_1, \cdots, i_r} e_{i_1, \cdots, i_r} \otimes M^{j_1, \cdots, j_s} e_{j_1, \cdots, j_s} = K^{i_1, \cdots, i_r} M^{j_1, \cdots, j_s} e_{i_1, \cdots, i_r, j_1, \cdots, j_s}. $$ \hspace{1cm} (A5)

The tensor $K$ is symmetric if the tensor remains invariant by permuting the indices of its elements $K^{i_1, \cdots, i_r} e_{i_1, \cdots, i_r}$. It is antisymmetric if there is a negative symbol by odd permuting of the indices and remains invariant by even permutation of the indices. The basis for anti-symmetric tensor can be expressed as

$$\theta_{i_1, \cdots, i_r} = \delta^{k_1, \cdots, k_r}_{i_1, \cdots, i_r} e_{k_1, \cdots, k_r},$$ \hspace{1cm} (A6)

where please note the definition of the generalized Kronecker symbol $\delta^{k_1, \cdots, k_r}_{i_1, \cdots, i_r}$ takes the form

$$\delta^{k_1, \cdots, k_r}_{i_1, \cdots, i_r} = \begin{vmatrix} \delta^k_{i_1} & \cdots & \delta^k_{i_r} \\ \vdots & \ddots & \vdots \\ \delta^k_{i_1} & \cdots & \delta^k_{i_r} \end{vmatrix}. $$ \hspace{1cm} (A7)

When the upper indices is an even permutation of the lower indices, $\delta^{k_1, \cdots, k_r}_{i_1, \cdots, i_r}$ is 1, it is $-1$ for odd permutations and 0 for other cases. Any antisymmetric tensor can be expressed as,

$$a_r = \frac{1}{r!} \theta_{i_1, \cdots, i_r} \theta_{i_1, \cdots, i_r},$$ \hspace{1cm} (A8)

where parameter $a^{i_1, \cdots, i_r}$ is antisymmetric. For $n$-dimensional spaces tensor product together, the highest antisymmetric tensor is rank-$n$ antisymmetric tensor. The direct summation of all rank antisymmetric spaces constitute a antisymmetric space. The summation of the ranks is $2^n$,

$$\Lambda = \Lambda^0 \oplus \cdots \oplus \Lambda^n,$$

$$\dim \Lambda = \dim \Lambda^0 + \cdots + \dim \Lambda^n$$

$$= C_0^n + C_1^n + \cdots + C_n^n$$

$$= 2^n.$$ \hspace{1cm} (A9)
Besides computing multiplication and summation, we can define wedge in this antisymmetric space $\Lambda$,

$$\alpha_p \wedge \beta_q = \frac{(p+q)!}{p!q!} A_{p+q}(\alpha_p \otimes \beta_q),$$  \hfill (A10)

where

$$A_{p+q} = \frac{1}{(p+q)!} \sum_{\sigma \in P(p+q)} \text{sgn}(\sigma) \sigma$$  \hfill (A11)

is the anti-symmetrized operator, for odd permutation $\sigma$, $\text{sgn}(\sigma) = -1$; for even permutation $\sigma$, $\text{sgn}(\sigma) = 1$. Wedge has the property,

$$\alpha_p \wedge \beta_q = (-1)^{pq} \beta_q \wedge \alpha_p$$  \hfill (A12)

2. **Metric, covariance tensor and contra-variance tensor**

Suppose $K, M$ are the rank-1 tensors in space $T(1)$, the scalar product of those two tensors is a rank-0 tensor, i.e., a number, and satisfy the equation,

$$K \cdot M = M \cdot K.$$  \hfill (A13)

The scalar products between $n$ rank-1 tensor basis in $T(1)$ can constitute a $n \times n$ matrix which is named metric:

$$g_{ij} = e_i \cdot e_j.$$  \hfill (A14)

It is obvious that $g_{ij}$ is symmetric,

$$g_{ij} = g_{ji}.$$  \hfill (A15)

Generally, we demand that the rank of the metric matrix is full, that is its determinant is non-zero,

$$\det\{g_{ij}\} \neq 0.$$  \hfill (A16)

With the help of the metric, the scalar product of two rank-1 tensors can be represented as,

$$K \cdot M = g_{ij} K^i M^j.$$  \hfill (A17)

For a tensor, there is two different indices, the upper indices and the lower indices. The lower indices is called covariance indices, and the upper indices is called contra-variance indices. If a tensor only has covariance indices, it is called covariance tensor. A tensor which has only contra-variance indices is called contra-variance indices. If a tensor has both covariance indices and the contra-variance indices, it is called mixed tensor. The metric $g_{ij}$ is called covariance metric, $g^{ij}$ is called contra-variance metric. They satisfy the equations,

$$g_{ik} g^{jk} = g^{jk} g_{ki} = \delta_i^j.$$  \hfill (A18)

Metric can be represented as a rank-2 symmetric tensor as,

$$G = g^{ij} e_{ij}.$$  \hfill (A19)

By using contra-variance metric $g^{ij}$ and the covariance metric $g_{ij}$, we can realize the rising or lowering the indices of a tensor. For example, for a rank-1 tensor, we have,

$$e^i = g^{ij} e_j, \quad K_i = g_{ij} K^j, \quad K = K^i e_i = K_i e^i.$$  \hfill (A20, A21, A22)

And for a rank-2 tensor, we have,

$$e_{i'} = g^{i'j} e_{j'}, \quad K_{i'} = g_{i'j} K^{j'}, \quad e^{i'} = g^{i'j} g^{j'i'} e_{j'}, \quad K_{i'} = g_{i'j} g^{j'i'} K^{j'}, \quad K = K^{i'i'} e_{i'i'} = K_{i'i'} e^{i'i'}.$$  \hfill (A23, A24, A25)
If the rank of a tensor is larger than 2, the contraction calculation can be made to this tensor and the rank will decrease 2. For example, the contraction of a rank-2 tensor may be performed as,

$$\text{con}K = g_{ij}K^{ij} = K^i_i = g^{ij}K_{ij}. \quad (A26)$$

The contraction for a rank-3 tensor may be performed as,

$$\text{con}K = g_{ijk}e_k = K^{ijk}e_k = g^{ij}K_{ijk}e_k. \quad (A27)$$

In general, the contraction for a rank-$$r$$ ($$r \geq 3$$) can be calculated as,

$$\text{con}K = \text{con}(K_{ij1}e_i e_j e_1) = g_{ij1}e_i e_j e_1. \quad (A28)$$

The tensor transformation is generally defined by the basis transformation. Suppose there are two sets of tensor basis \(\{e_i\}\) and \(\{e'_i\}\), \((i, i' = 1, 2, \cdots, n)\). The transformation between these two sets basis is defined by the transformation matrix \(\{L_{ij}'\}\),

$$e_i = L_{ij}' e'_j. \quad (A29)$$

The inverse transformation can be found to be

$$e'_i = L^{-1}_{ij} e_i, \quad (A30)$$

where \(L^{-1}_{ij}\) is the inverse transformation matrix and satisfy the equation,

$$L^{-1}_{ij} L_{kj} = \delta_{ik}. \quad (A31)$$

A general tensor can be expanded in two sets of basis,

$$K = K^{i'_1 \cdots i'_r \cdots} e_{i'_1} \cdots e_{i'_r} = K'^{i''_1 \cdots i''_r \cdots} e_{i''_1} \cdots e_{i''_r}. \quad (A32)$$

The transformation between two basis takes the form

$$e'^{i''_1 \cdots i''_r} = L^{-1}_{i'k} e^i_k \cdots e_{i''_1} \cdots e_{i''_r}. \quad (A33)$$

So the transformation between the elements of the tensor is,

$$K'^{i''_1 \cdots i''_r} = L^i_k L^{-1}_{i'k} \cdots K^{i'_1 \cdots i'_r}. \quad (A34)$$

### 3. Matrix

The matrix calculation obey the standard law of mathematics. Suppose \(\hat{A}, \hat{B}\) are two \(n \times n\) matrices, the inner product of the two matrices is defined as,

$$\langle \hat{A}, \hat{B} \rangle = \text{tr} \left( \hat{A}^\dagger \hat{B} \right) = a^*_{ij} b_{ij} \quad (A35)$$

where the upper index \(\dagger\) means to take a matrix transposition plus complex conjugation. The inner product of matrices has the properties:

$$\langle \hat{A}, \hat{A} \rangle \geq 0, \quad \langle \hat{A}, \hat{B} \rangle = \langle \hat{B}, \hat{A} \rangle^*. \quad (A36)$$

If the inner product of two matrices is zero, \(\langle \hat{A}, \hat{B} \rangle = 0\), we say these two matrices are orthogonal.

In \(n \times n\) matrix representation space, we can introduce \(n^2\) ortho-normal basis,

$$\hat{e}_l, \quad l = 1, 2, \cdots, n^2; \quad \langle \hat{e}_l, \hat{e}_{l'} \rangle = \text{tr}(\hat{e}_l^\dagger \hat{e}_{l'}) = \delta_{ll'}. \quad (A37)$$
An arbitrary $n \times n$ matrices can be represented in terms of this basis as,

$$\hat{A} = A_l \hat{e}_l,$$  \hspace{1cm} (A38)

where the coefficients in the expansion can be calculated as,

$$A_l = \langle \hat{e}_l, \hat{A} \rangle = \text{tr}(\hat{e}_l \hat{A}).$$  \hspace{1cm} (A39)

Suppose $F(x)$ is an analytical function, that is $F(x)$ can be expanded by Taylor expansion,

$$F(x) = \sum_{l=0}^{\infty} f(l) x^l.$$  \hspace{1cm} (A40)

the matrix function $F(\hat{A})$ is defined as,

$$F(\hat{A}) = \sum_{l=0}^{\infty} f(l) \hat{A}^l.$$  \hspace{1cm} (A41)

The matrix similarity transformation is defined as,

$$\hat{A}' = \hat{L} \hat{A} \hat{L}^{-1}.$$  \hspace{1cm} (A42)

The matrix unitary transformation is defined as,

$$\tilde{\hat{A}} = \hat{U} \hat{A} \hat{U}^\dagger,$$  \hspace{1cm} (A43)

where $\hat{U}$ is an unitary matrix $\hat{U} \hat{U}^\dagger = I$, $I$ is the identity.

4. Pauli matrices, Gell-Mann matrices

The Pauli matrices are defined as,

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\sigma_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$  \hspace{1cm} (A44)

Some properties of Pauli matrices can be listed, for example, as,

$$\text{tr}(\sigma_\mu \sigma_\nu) = 2\delta_{\mu\nu}, \quad \text{tr} \sigma_i = 0,$$

$$[\sigma_i, \sigma_j] = 2i \varepsilon_{ijk} \sigma_k, \quad \{\sigma_i, \sigma_j\} = 2i \delta_{ij} \sigma_0,$$

$$\sigma_i \sigma_j = \delta_{ij} \sigma_0 + i \varepsilon_{ijk} \sigma_k, \quad \sigma_1 \sigma_2 \sigma_3 = i \sigma_0.$$  \hspace{1cm} (A45)

The $3 \times 3$ Gell-Mann matrices take the forms,

$$\lambda_0 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \lambda_1 = \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_2 = \frac{1}{2} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_4 = \frac{1}{2} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \frac{1}{2} \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix},$$

$$\lambda_6 = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \lambda_7 = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix},$$

$$\lambda_8 = \frac{1}{2\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$  \hspace{1cm} (A46)
Note that for convenience, the Gell-Mann matrices here is slightly different from the standard Gell-Mann matrices by a whole factor $\frac{1}{2}$.

The properties of the Gell-Mann matrices can be listed as,

\[
\text{tr} \left( \hat{\lambda}_p \hat{\lambda}_q \right) = \frac{1}{2} \delta_{pq}, \quad (A47)
\]

\[
\text{tr} \hat{\lambda}_p = 0, \quad (p \neq 0), \quad (A48)
\]

\[
[\hat{\lambda}_p, \hat{\lambda}_q] = i f_{pqr} \hat{\lambda}_r, \quad (A49)
\]

\[
\{ \hat{\lambda}_p, \hat{\lambda}_q \} = d_{pqr} \hat{\lambda}_r + \frac{\sqrt{6}}{3} \delta_{pq} \hat{\lambda}_0, \quad (A50)
\]

where $f_{pqr}$ and $d_{pqr}$ are completely anti-symmetric, the non-zero elements are,

\[
f_{123} = 1,
\]

\[
f_{458} = f_{678} = \frac{\sqrt{3}}{2},
\]

\[
f_{147} = -f_{156} = f_{246} = f_{257} = f_{345} = -f_{367} = \frac{1}{2}, \quad (A51)
\]

and

\[
d_{118} = d_{228} = d_{338} = -d_{888} = \frac{1}{\sqrt{3}},
\]

\[
d_{146} = d_{157} = -d_{247} = d_{256} = d_{344} = d_{355} = -d_{366} = -d_{377} = \frac{1}{2},
\]

\[
d_{448} = d_{558} = d_{668} = -d_{778} = -\frac{1}{2\sqrt{3}}. \quad (A52)
\]

5. Vector and its adjoint

A vector $|a\rangle$ has an adjoint $\langle a|$ represented as,

\[
\langle a| = |a\rangle^\dagger. \quad (A53)
\]

It has the property,

\[
\alpha_1 |a_1\rangle + \alpha_2 |a_2\rangle = \alpha_1^* \langle a_1| + \alpha_2^* \langle a_2|. \quad (A54)
\]

Each vector of basis $\{|e_i\rangle\}$ has its adjoint and they all constitute a basis $\{\langle e_i|\}$ for the adjoint space $\overline{V}$, so the adjoint of $|a\rangle = a^i |e_i\rangle$ takes the form

\[
\langle a| = a^i \langle e_i|. \quad (A55)
\]

The metric of a linear space $V$ is defined by a basis $\{|e_i\rangle\}$ and the adjoint basis $\{\langle e_i|\}$ by their inner products,

\[
g_{ij} = \langle e_i| e_j\rangle. \quad (A56)
\]

We know that $g_{ij} = g_{ji}^*$, and also we demand that the metric matrix is non-degenerate,

\[
\det(g_{ij}) \neq 0. \quad (A57)
\]

If all eigenvalues of metric $g_{ij}$ are positive, it is the positive metric, and all eigenvalues are negative, it is negative metric. If the eigenvalues have both positive and negative symbols, it is the indefinite metric. If the metric can be expressed as $g_{ij} = \pm \delta_{ij}$, we say the metric is normalized.

The inner product of vectors $|a\rangle = a^i |e_i\rangle, |b\rangle = b^j |e_j\rangle$ can be expressed by the metric matrix,

\[
\langle a|b\rangle = a^i \langle e_i| e_j\rangle b^j = a^i g_{ij} b^j. \quad (A58)
\]
As usual, the metric with lower indices is called the covariance metric, the metric with upper indices is called contravariance metric which is defined below,

\[ \{ g^{ij} \} = \{ g_{ij} \}^{-1} = \hat{g}^{-1}. \quad (A59) \]

So we have

\[ g_{ik} g^{kj} = g^{jk} g_{ki} = \delta^j_i. \quad (A60) \]

One may find that

\[ \langle a | \hat{g}^{-1} | a \rangle \geq 0, \quad (A61) \]

this is because,

\[ \langle a | \hat{g}^{-1} | a \rangle = g_{ik} g^{kj} a^*_i a^j \geq 0 \quad (A62) \]

The metric \( \hat{g} \) and its inverse can be used to realize the rising and lowering the lower and upper indices. For simplicity, we denote the vector is expressed in terms of covariance basis \( \{|e_i]\} \), the adjoint vector is expressed in terms of the contravariance basis \( \langle e^j| \} \), so we have

\[ \langle e^i| = g^{ij} \langle e_j|, \]
\[ \langle e^i|e_j\rangle = \delta^i_j, \]
\[ \langle a| = a^*_i \langle e_i| = a^*_i \langle e^i|, \]
\[ a^*_i = a^*_j g_{ji}, \]
\[ \langle a|b\rangle = a^*_i b^j. \quad (A63) \]

6. Operator space

A \( m \)-dimensional linear space \( V \) tensor product with its adjoint space \( \overline{V} \) constitute an operator space. The basis of the operator space can be constructed by the basis \( \{|e_i]\} \) and the adjoint basis \( \langle e^j| \} \) as the following,

\[ |e_i\rangle \otimes \langle e^j|, \quad (i, j = 1, 2, \ldots, m), \quad (A64) \]

where \( \langle e^i| = g^{ij} \langle e_j| = g^{ij}\overline{e_j} \). An operator \( A \) can be expanded in terms of this basis,

\[ \hat{A} = A^j_i |e_i\rangle \otimes \langle e^j|. \quad (A65) \]

The behaviors of the operator is the same as the matrix. The follows are some of the properties and some notations,

\[ |a\rangle \otimes |b\rangle = a^* b^*_j |e_i\rangle \otimes \langle e^j|, \]
\[ \hat{A} \hat{B} = (A^j_k B^k_i) |e_i\rangle \otimes \langle e^j|, \]
\[ [\hat{A}, \hat{B}] = \hat{A} \hat{B} - \hat{B} \hat{A}, \]
\[ \{ \hat{A}, \hat{B} \} = \hat{A} \hat{B} + \hat{B} \hat{A}. \quad (A66) \]

The adjoint of the basis of operator space takes the form,

\[ \overline{|e_i\rangle} \otimes \langle e^j| = |e^j\rangle \otimes \langle e_i| = g^{ij} g_{ik} |e_i\rangle \otimes \langle e^k|. \quad (A67) \]

The adjoint of an operator can be expressed as,

\[ \overline{A} = g^{ik} A^*_{ik} g_{lj} |e_i\rangle \otimes \langle e^j| = \overline{A^j_i} |e_i\rangle \otimes \langle e^j|, \quad (A68) \]

where we define,

\[ \overline{A^j_i} = g^{ik} A^*_{ik} g_{lj}. \quad (A69) \]
The adjoint of operators satisfy the properties,
\[
\begin{align*}
\bar{A} &= \hat{A}, \\
A|a\rangle &= \langle a|\bar{A}, \\
\alpha A &= a^\ast \bar{A}, \\
\bar{A}\bar{B} &= \bar{B}\bar{A}.
\end{align*}
\] (A70)

We can find the adjoint of an operator is related with the hermitian conjugation of this operator by the metric matrix as,
\[
\bar{A} = \hat{g}^{-1}\hat{A}^\dagger\hat{g}.
\] (A71)

The inner product of the operators is defined as,
\[
\langle \hat{A}, \hat{B} \rangle = \text{tr} (\hat{A}\hat{B}^\dagger) = \bar{A}_i B_i^j
\]
\[= g^{ik} g_{ij} A_i^\ast B_i^j.\] (A72)

The properties of the operator inner product calculation are listed as,
\[
\langle \hat{A}, \hat{B} \rangle = \langle \hat{B}, \hat{A} \rangle^\ast, \\
\langle \hat{A}, \beta \hat{B} + \gamma \hat{C} \rangle = \beta \langle \hat{A}, \hat{B} \rangle + \gamma \langle \hat{A}, \hat{C} \rangle, \\
\langle \alpha \hat{A} + \beta \hat{B}, \hat{C} \rangle = \alpha^\ast \langle \hat{A}, \hat{C} \rangle + \beta^\ast \langle \hat{B}, \hat{C} \rangle.
\] (A73)

Operators \(\hat{A}, \hat{B}\) are orthogonal if there inner product is zero, \(\langle \hat{A}, \hat{B} \rangle = 0\).

The identity operator is denoted as,
\[
\hat{I} = |e_i\rangle \otimes \langle e_i| = |e^i\rangle \otimes \langle e_i| = g^{ij} |e_i\rangle \otimes \langle e_j|
\]
\[= g_{ij} |e^i\rangle \otimes \langle e^j| = \hat{g}.
\] (A74)

The inverse of an operator is defined as,
\[
\hat{A}^{-1} \hat{A} = \hat{I}.
\] (A75)

And as usual the power zero of an operator is the identity,
\[
\hat{A}^0 = \hat{I}.
\] (A76)

Suppose function \(F(x)\) is analytic, the function of an operator is defined through Taylor expansion as,
\[
F(\hat{A}) = \sum_{l=0}^{\infty} f(l) \hat{A}^l
\] (A77)

By applying the identity operator \(I = |e_i\rangle \otimes \langle e_i|\) on a vector \(|a\rangle\)and on its adjoint, or on an operator, they will be expanded by basis \(|e_i\rangle\),
\[
|a\rangle = (|e_i\rangle \otimes \langle e^i|)|a\rangle = a^i|e_i\rangle, \quad a^i = \langle e^i|a\rangle,
\]
\[
\langle a| = \langle a|(|e_i\rangle \otimes \langle e^i|) = a_i^\ast \langle e_i|, \quad a_i^\ast = \langle a|e_i\rangle = a^\ast g_{ji},
\]
\[
\hat{A} = (|e_i\rangle \otimes \langle e^i|)\hat{A}(|e_j\rangle \otimes \langle e^j|) = A_i^j |e_i\rangle \otimes \langle e^j|, \quad A_i^j = \langle e^j|\hat{A}|e_i\rangle.
\] (A78)

For the identity operator in another basis \(I = |e'_i\rangle \otimes \langle e'^i|\), we have similar results. The transformation between those two basis can be obtained as,
\[
|e_i\rangle = (|e'_j\rangle \otimes \langle e'^j|)|e_i\rangle = L_i^j |e'_j\rangle,
\] (A79)

where \(L_i^j = \langle e^j|e_i\rangle\) gives the definition of the transformation elements, and also we have
\[
\hat{L} = \hat{I}^\dagger \hat{I} = L_i^j |e'_i\rangle \otimes \langle e^j|.
\] (A80)
The inverse of the basis transformation takes the form,

\[ |e'_i\rangle = (|e_j\rangle \otimes \langle e^j|)|e'_j\rangle = L^{-1}_{ij}|e_j\rangle, \tag{A81} \]

where \( L^{-1}_{ij} = \langle e^j|e'_i\rangle \) and can be represented as,

\[ \hat{L}^{-1} = \hat{I}\hat{I}' = L^{-1}_{ij}|e_i\rangle \otimes \langle e^j|. \tag{A82} \]

The transformation matrix and its inverse have the relations,

\[ L^{-1}_{ij}L^i_k = \langle e^i|e'_j\rangle \otimes \langle e^j|e_k\rangle = \langle e^i|e_k\rangle = \delta^{i}_{k}, \tag{A83} \]

that is \( \hat{L}^{-1}\hat{L} = \hat{I} \). We also have,

\[ L^{-1}_{ij} = \langle e^i|e'_j\rangle = \langle e'_j|e^i\rangle^* = g^i_jg^{kl}\langle e^i|e_k\rangle = g^{ik}L^i_kg^i_k = \overline{\mathbf{T}}_j, \tag{A84} \]

\[ \overline{\mathbf{L}} = \hat{L}^{-1}. \tag{A85} \]

### 7. Metric transformation

With basis \( \{|e'_i\rangle\} \), the metric can be defined as,

\[ g^{ij} = \langle e^i|e^j\rangle. \tag{A86} \]

By inserting an identity operator \( \hat{I} = g^{kl}\langle e_k|\otimes\langle e_l| \) into this equation, we can have,

\[ g^{ij} = g^{kl}\langle e'_i|e_k\rangle \otimes \langle e^j|e_l\rangle = g^{kl}(\langle e^i|e_k\rangle \otimes \langle e^j|e_l\rangle)^*. \tag{A87} \]

For different basis, the metric transformation takes the form,

\[ g^{ij} = L^i_kg^{kl}L^j_k. \tag{A88} \]

In a concise form, it can be rewritten as,

\[ \hat{g}' = \hat{L}\hat{g}\hat{L}^\dagger. \tag{A89} \]

In a different basis, a vector can be expressed as the following,

\[ |a\rangle = a^i|e_i\rangle = a'^i|e'_i\rangle = |a'\rangle, \tag{A90} \]

where the coefficients \( a^i \) and \( a'^i \)

\[ a'^i = \langle e'^i|a\rangle = \langle e'^i|e_j\rangle \otimes \langle e^j|a\rangle = L^i_ja^j, \tag{A91} \]

\[ a^i = \langle e^i|a\rangle = \langle e^i|e'_j\rangle \otimes \langle e^j|a\rangle = L^{-1}_{ij}a'^j. \tag{A92} \]

Thus the vector transformation has the form,

\[ |a'\rangle = \hat{L}|a\rangle. \tag{A93} \]

Please note that vectors \( |a\rangle \) and \( |a'\rangle \) are actually one vector in different basis. Similarly for adjoint vector \( \langle a| \), we have

\[ \langle a| = a'^*_i\langle e^i| = a'^*_i\langle e'^i| = \langle a'|, \tag{A94} \]

where \( a'^*_i = a_j^*L^{-1}_{ij}, \) \( a'^*_i = a_j^*L^j_i \), and

\[ \langle a'| = \langle a|\hat{L}^{-1}. \tag{A95} \]
The operator transformation has the following results,

\[
\hat{A} = A^i_j |e_i⟩ ⊗ ⟨e_j| = A^i_j |e_i⟩ ⊗ ⟨e_j| = \hat{A}'.
\] (A96)

We can find that

\[
A^i_j = L^i_k A^k_l L^{-1}_j,
\]

\[
A^i_j = L^{-1}_k A^k_l L^i_j,
\]

\[
\hat{A}' = LAL^{-1}.
\] (A97)

Next we will present some results concerning about the transformation for some calculations,

\[
(α|a⟩ + β|b⟩)' = α|a⟩' + β|b⟩';
\]

\[
(α|a⟩ + β|b⟩)' = α|a⟩' + β|b⟩';
\]

\[
(a'|b')\rangle = \langle a|b⟩;
\]

\[
(α\hat{A} + β\hat{B})' = α\hat{A}' + β\hat{B}',
\]

\[
(\hat{A}|α⟩)' = \hat{A}'|α⟩;
\]

\[
(\langle a|\hat{A}⟩)' = \langle a'|\hat{A}'⟩;
\]

\[
(\hat{A}B)' = \hat{A}'\hat{B}',
\]

\[
[A, B]' = [\hat{A}', \hat{B}'],
\]

\[
\{A, B\}' = \{\hat{A}', \hat{B}'\},
\]

\[
\text{det}\hat{A}' = \text{det}\hat{A},
\]

\[
\text{tr}\hat{A}' = \text{tr}\hat{A},
\]

\[
(\hat{A}^{-1})' = (\hat{A}')^{-1},
\]

\[
[F(\hat{A})]' = F(\hat{A}'),
\]

\[
(\overline{A})' = \overline{A}',
\]

\[
\langle \hat{A}', \hat{B}'\rangle = \langle \hat{A}, \hat{B}\rangle.
\] (A98)

Those results can be easily proved, for example,

\[
\langle a'|b'⟩ = \langle a|L^{-1} \hat{L}|b⟩ = \langle a|b⟩;
\]

\[
\overline{A'} = \overline{L^{-1}AL} = \overline{LAL^{-1}} = \overline{(A)}',
\]

\[
\langle \hat{A}', \hat{B}'\rangle = \text{tr}(\overline{A'}\overline{B}') = \text{tr}[\overline{LAL^{-1} \hat{L}B \hat{L}^{-1}}] = \text{tr}(\overline{AB}) = \langle \hat{A}, \hat{B}\rangle.
\] (A99)

### 8. Direct sum and direct product

The direct sum of the \(m\)-dimensional space \(V_1\) and the \(n\)-dimensional space \(V_2\) is a \((m + n)\)-dimensional space,

\[
V = V_1 ⊕ V_2.
\] (A100)

Similarly the adjoint can have the same result,

\[
\overline{V} = \overline{V_1} ⊕ \overline{V_2}.
\] (A101)

For operator space \(O_1 = V_1 ⊗ \overline{V_1}\) and \(O_2 = V_2 ⊗ \overline{V_2}\), the operator space \(O = V ⊗ \overline{V}\) has the decomposition,

\[
O = O_1 ⊕ O_2.
\] (A102)

The vector space \(V_1\) has basis \(\{|e_i⟩\}\), the metric is denoted as \(g_{i'i''}\), \((i, i' = 1, 2, \cdots, m)\), additionally vector space \(V_2\) has basis \(\{|E_j⟩\}\), the metric is \(G_{jj'}\), \((j, j' = 1, 2, \cdots, n)\). The basis for the direct sum space \(V = V_1 ⊕ V_2\) should be,

\[
\{|ε_k⟩\} = \{|e_i⟩\} ∪ \{|E_j⟩\} = \{|ε_k⟩; |ε_i⟩ = |e_i⟩, |ε_{m+j}⟩\} = \{|E_j⟩\},
\] (A103)
where $k = 1, 2, \cdots, m, m + 1, \cdots, m + n$. The metric $\{\eta_{kk'}\}$ of direct sum space $V$ can be found to be,

$$\eta_{kk'} = \{g_{ii'}\} \oplus \{G_{jj'}\}. \quad (A104)$$

The basis of the adjoint space $\mathbf{V}$ can be written as,

$$\langle \varphi^k \rangle = \eta^{kk'} \langle \varphi_k \rangle. \quad (A105)$$

The operator basis of the operator space $O$ takes the form $|\varphi^k\rangle \otimes |\varphi_k\rangle$.

The direct product calculations for spaces and operators obey the standard method, like the following,

$$O = O_1 \otimes O_2,$$

$${\{|\varphi_k\rangle\}} = \{\{|\varphi_k\rangle\}; |\varphi_n(i-1)+j\rangle = |e_i\rangle \otimes |E_j\rangle\},$$

$$\eta_{kk'} = \{g_{ii'}\} \otimes \{G_{jj'}\}. \quad (A106)$$

9. Lie algebra

We next present some basics of Lie algebra and some particular concepts and calculations which have been used in this paper.

Suppose $A$ is a finite-dimensional linear space over $F$, and we also defined the commutation calculations on $A$, then $A$ is a Lie algebra. The dimension of the Lie algebra is the dimension of the linear space, denoted as $\text{dim} A$.

We can define three kind of calculations, plus, number multiplication and commutation. This defined plus and the number multiplication constitute the linear space. The commutator of two elements of $A$ is denoted as $[X, Y]$. For arbitrary $X, Y, Z \in A$ and arbitrary numbers $a, b \in F$, the commutator calculations have the properties, respectively listed are closed condition, linear and anti-commuting,

$$[X, Y] \in A,$$

$$[aX + bY, Z] = a[X, Z] + b[Y, Z],$$

$$[X, Y] = -[Y, X]. \quad (A107)$$

The Jacobi equation takes the form,

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0. \quad (A108)$$

If all elements of the Lie algebra are commuting, then $A$ is Abel algebra.

$A_1$ is a subspace of Lie algebra $A$, and the commutation calculation of elements in $A_1$ is closed, then $A_1$ is a subalgebra of $A$.

If $A_1$ is a subalgebra of $A$, and for arbitrary $X_1 \in A_1$ and $X \in A$, we have $[X_1, X] \in A_1$, then $A_1$ is called the ideal of $A$. Lie algebra always has two ideals, $\{0\}$ and itself $A$.

If $A$ has no fixed ideal, it is called simple Lie algebra.

If there is no non-Abel ideal which is not $\{0\}$, this Lie algebra is called semi-simple Lie algebra. The semi-simple Lie algebra can be decomposed as the direct sum of simple Lie algebras.

Given two Lie algebras $A_1$ and $A_2$, suppose $A_1 \cap A_2 = \{0\}$ and for arbitrary $X_1 \in A_1$, $X_2 \in A_2$, we have $[X_1, X_2]$, the direct sum can thus take the form,

$$A = A_1 \oplus A_2 = \{X, \hat{X} = X_1 + \hat{X}_2, \hat{X}_1 \in A_1, \hat{X}_2 \in A_2\}. \quad (A109)$$

We can find that $A$ is a Lie algebra, the dimension of $A$ is the sum of dimensions $A_1$ and $A_2$, $\text{dim} A = \text{dim} A_1 + \text{dim} A_2$, both $A_1$ and $A_2$ are ideals of Lie algebra $A$.

Suppose $A$ and $A'$ are two Lie algebras, $f$ maps $A$ to $A'$, and for arbitrary $\hat{X}, \hat{Y} \in A$, there exist $f(\hat{X}), f(\hat{Y}) \in A'$ and satisfy

$$f(a\hat{X} + b\hat{Y}) = af(\hat{X}) + bf(\hat{Y}),$$

$$f([\hat{X}, \hat{Y}]) = [f(\hat{X}), f(\hat{Y})], \quad (A110)$$

we mean $A$ and $A'$ are homomorphism, $f$ is a homomorphism map. If there exists an one to one homomorphism map, $A$ and $A'$ are isomorphism.
For a given Lie algebra $A$, we can introduce a complete set basis, 
\[ \{ \hat{t}_a \}, a = 1, 2, \cdots, A. \] (A111)

An arbitrary element $X$ can be expanded by this basis, 
\[ \hat{X} = x^a \hat{t}_a, \] (A112)
where $x^a \in F$ are the coefficients in the expansion. The commutation relation of the basis can be represented as, 
\[ [\hat{t}_a, \hat{t}_b] = C_{ab}^c \hat{t}_c, \] (A113)
where $C_{ab}^c$ are the structure constants. There are altogether $(\text{dim}A)^3$ structure constants. One can check that the structure constants satisfy the relations,
\[ C_{ab}^c = C_{ba}^c, \]
\[ C_{ab}^d C_{cd}^e + C_{bc}^d C_{ad}^e + C_{ca}^d C_{bd}^e = 0. \] (A114)

For a given basis, the structure constants are fixed. A given Lie algebra $A$ can be defined by a set of basis $\{ \hat{t}_a \}$ and the structure constants.

With the help of the structure constants, the Cartan metric of a Lie algebra can be defined as,
\[ g_{ab} = g_{ba} = C_{ac}^d C_{bd}^c = C_{bd}^c C_{ac}^d. \] (A115)

A Lie algebra is semi-simple, the necessary and sufficient condition is that the Cartan metric is non-degenerate, \( \det \{ g_{ab} \} \neq 0 \). For an Abel algebra, Cartan metric is always zero, \( g_{ab} = 0 \).

In our 4D unified quantum theory, we can define a generalized metric $G_{ab}$ for Lie algebra.

1. For a simple algebra, the generalized metric $G_{ab}$ is defined as,
\[ G_{ab} = G_{ba} = g g_{ab} = g C_{ac}^d C_{bd}^c, \] (A116)
where $g \in R, \neq 0$ is a non-zero real constant, $g_{ab}$ is the Cartan metric of simple algebra.

2. For a semi-simple algebra, if it can be constitute as the direct sum of $n$ semi-simple algebras, and the Cartan metrics for each simple algebra are $g_{a_1 b_1}, \cdots, g_{a_nb_n}$, the generalized metric is defined as,
\[ G_{a_1 b_1} = g_1 g_{a_1 b_1}, \]
\[ \cdots \cdots \]
\[ G_{a_nb_n} = g_n g_{a_nb_n}, \] (A117)
where $g_1, \cdots, g_n \in R, \neq 0$.

3. For an Abel algebra, the generalized metric can be defined as,
\[ G_{ab} = G_{ba}, \]
\[ \det \{ G_{ab} \} \neq 0, \] (A118)
where $G_{ab}$ can be arbitrary real number.

Thus, for arbitrary Lie algebra including semi-simple Lie algebra and the Abel Lie algebra, the generalized metric is non-degenerate, \( \det \{ G_{ab} \} \neq 0 \). Thus we can introduce an inverse for this generalized metric,
\[ G^{ab} G_{bc} = \delta^a_c, \]
\[ G^{ab} = \frac{A^{ab}}{\det \{ G_{ab} \}}, \] (A119)
where $A^{ab}$ is the algebraic complement of element $G_{ab}$.

We define the covariance structure constant as,
\[ C_{abc} = G_{ad} C_{bd}^c. \] (A120)
We can find that the covariance structure constants are completely antisymmetric,
\[ C_{abc} = C_{bca} = C_{cab} = -C_{cba} = -C_{bac} = -C_{acb}. \] (A121)

The contra-variance algebraic basis is defined as,
\[ \hat{t}^a = G^{ab} \hat{t}_b. \] (A122)

The rank-\( n \), \( (n \geq 2) \), Casimir operators take the form,
\[ \hat{C}_n = C^{b_2}_{a_1 b_1} C^{b_3}_{a_2 b_2} \cdots C^{b_1}_{a_n b_n} \hat{t}^{a_1} \hat{t}^{a_2} \cdots \hat{t}^{a_n}. \] (A123)

In particular, for \( n = 2 \), the Casimir operator is,
\[ \hat{C}_2 = C^{b_2}_{a_1 b_1} C^{b_1}_{a_2 b_2} \hat{t}^{a_1} \hat{t}^{a_2}. \] (A124)

One can prove that,
\[ [\hat{C}_n, \hat{t}_a] = 0. \] (A125)

The number of independent Casimir operators is equal to the rank of this Lie algebra.

10. Vectors and operators associated with Lie algebra

Lie algebra \( A \) is defined as a linear space, it can be considered as 1-form Lie algebra tensor space,
\[ T(1) = A. \] (A126)

The tensor basis is \( \{ \hat{t}_a \} \) in this space.

We then can define the direct product space of \( r \) Lie algebra spaces \( A \) as the \( r \)-form Lie algebra tensor space,
\[ T(r) = \prod_r A. \] (A127)

The tensor basis takes the form,
\[ \hat{t}_{a_1 \cdots a_r} = \hat{t}_{a_1} \otimes \cdots \otimes \hat{t}_{a_r}. \] (A128)

The direct sum of all \( r \)-form \( (r = 0, 1, \cdots) \) spaces constitute the Lie algebra tensor space,
\[ T = \sum_{r=0}^{\infty} \oplus T(r). \] (A129)

We next present several examples of the Lie algebra tensor:

A constant \( \alpha \) can be considered as the 0-form Lie algebra tensor.

An element \( \hat{X} \) of Lie algebra can be considered as the form-1 Lie algebra tensor, \( \hat{X} = x^a \hat{t}_a. \)

The metric of a Lie algebra can be considered as the form-2 Lie algebra tensor,
\[ \hat{G} = G^{ab} \hat{t}_a \otimes \hat{t}_b = G_{ab} \hat{t}^a \otimes \hat{t}^b. \] (A130)

As we already known, the rising and lowering of the indices can be realized by the metric matrix.

Structure constant can be considered as the type-3 Lie algebra tensor,
\[ \hat{C} = C^{c}_{ab} \hat{t}^a \otimes \hat{t}^b \otimes \hat{t}_c. \] (A131)

The Lie algebra tensor is similar as the convention tensor, and we can define calculations like plus, number multiplication, tensor product, contraction. Additionally, we can define the commuting calculation for the Lie algebra tensor the convention tensor. The commutation relation for \( \hat{t}_a \) and \( \hat{t}_{a_1 \cdots a_r} \) is,
\[ [\hat{t}_a, \hat{t}_{a_1 \cdots a_r}] = C^{c}_{aa_1} \hat{t}_{c \cdots a_r} + \cdots + C^{c}_{aa_r} \hat{t}_{a_1 \cdots c}. \] (A132)
Two different basis of the Lie algebra \( A \) can be transformed to each other as,

\[
\hat{t}_a = \Lambda^b_{ab} \hat{t}_b, \\
\hat{t}_a' = \Lambda^{-1b}_{ab} \hat{t}_b. \tag{A133}
\]

By the transformation between two different basis, the Lie algebra expansion coefficients, structure constants and metric changed as the conventional tensor transformations,

\[
\begin{align*}
\hat{X} &= x^a \hat{t}_a = x^a \hat{t}_a', \\
x^a &= \Lambda^b_{ab} x^b, \\
C^a_{bc} &= \Lambda^{-1d}_{ad} \Lambda^{-1e}_{be} C^f_{de}, \\
G^a_{ab} &= \Lambda^{-1d}_{ad} \Lambda^{-1e}_{be} G_{cd}, \\
G^{ab} &= \Lambda^a_{a} \Lambda^b_{b} G^{cd}. \tag{A134}
\end{align*}
\]

Under this transformation, the calculations of plus, number multiplication and commuting are invariant. The definition forms of metric and the Casimir operator are invariant. That means that the calculations of Lie algebra and the properties are independent of the choice of basis.

11. The representation theory of Lie algebra

Suppose \( M \) is a set of \( n \times n \) matrices on \( F \), with the definition of the matrix plus and number multiplication, \( M \) constitute a \( n^2 \) dimension linear space. For \( \hat{X}, \hat{Y} \in M \), the matrices commuting calculation can be defined as,

\[
[\hat{X}, \hat{Y}] = \hat{X}\hat{Y} - \hat{Y}\hat{X}. \tag{A135}
\]

The commutation calculation defined above satisfy the conditions like closed, linear, antisymmetric and Jacobi relation, and thus \( M \) is a \( n^2 \)-dimension Lie algebra. It can be named as the matrix algebra.

Given a Lie algebra \( A \), if we can find a homomorphism map \( f \) from \( A \) to \( n \times n \) matrix algebra \( M \), this map \( f \) can be considered as a linear representation of \( A \) if additionally the following conditions are satisfied: consider for arbitrary \( \hat{X}, \hat{Y} \in A \), there exist \( f(\hat{X}), f(\hat{Y}) \in M \), and

\[
\begin{align*}
f(a \hat{X} + b \hat{Y}) &= af(\hat{X}) + bf(\hat{Y}), \\
f([\hat{X}, \hat{Y}]) &= [f(\hat{X}), f(\hat{Y})]. \tag{A136}
\end{align*}
\]

\( f(\hat{X}) \) is the matrix representation of \( \hat{X} \). The linear space \( V \) of the matrix algebra \( M \) is the representation space of Lie algebra \( A \). The dimension \( n \) of space \( V \) is the representation dimension of Lie algebra \( A \). The basis of \( V \) is the representation basis. Lie algebra \( A \) can be denoted as \( V(f) \) or simply \( V \).

In order to give Lie algebra \( A \) a representation \( V(f) \), we need three conditions:

1. There is a representation space \( V \).
2. The representation matrix \( f(\hat{X}) \) can be given.
3. The representation can be found to be a homomorphism.

Suppose \( V(f) \) is a representation of Lie algebra \( A \), for arbitrary different two elements of \( A \), \( M \) has two different corresponding matrices, this representation is called faithful. Otherwise it is not a faithful representation.

Suppose \( V_1(f_1) \) and \( V_1(f_1) \) are two representations of Lie algebra \( A \), the direct sum of those two representations can be defined as: First, the representation space \( V(f) \) of the direct sum is the direct sum of the two representation spaces of \( V_1(f_1) \) and \( V_2(f_2) \),

\[
V = V_1 \oplus V_2. \tag{A137}
\]

Correspondingly, the dimension of the direct sum representation space is the sum of the two dimensions \( V_1(f_1) \) and \( V_2(f_2) \),

\[
\dim V(f) = \dim V_1(f_1) + \dim V_2(f_2). \tag{A138}
\]

The representation matrix is the direct sum of two representation matrices,

\[
f(\hat{X}) = f_1(\hat{X}) \oplus f_2(\hat{X}). \tag{A139}
\]
The homomorphism of the representation $V(f)$ can be proved in following,

$$\begin{align*}
[f(\hat{X}),f(\hat{Y})] &= f(\hat{X})f(\hat{Y}) - f(\hat{Y})f(\hat{X}) \\
&= \left( f_1(\hat{X}) \oplus f_2(\hat{X}) \right) \left( f_1(\hat{Y}) \oplus f_2(\hat{Y}) \right) - \left( f_1(\hat{Y}) \oplus f_2(\hat{Y}) \right) \left( f_1(\hat{X}) \oplus f_2(\hat{X}) \right) \\
&= \left( f_1(\hat{X})f_1(\hat{Y}) \right) \oplus \left( f_2(\hat{X})f_2(\hat{Y}) \right) - \left( f_1(\hat{Y})f_1(\hat{X}) \right) \oplus \left( f_2(\hat{Y})f_2(\hat{X}) \right) \\
&= [f_1(\hat{X}),f_1(\hat{Y})] \oplus [f_2(\hat{X}),f_2(\hat{Y})] \\
&= f_1[\hat{X},\hat{Y}] \oplus f_2[\hat{X},\hat{Y}] \\
&= f[\hat{X},\hat{Y}].
\end{align*}$$

(A140)

Here we have used the following relations,

$$\begin{align*}
[f_1(\hat{X}),f_1(\hat{Y})] &= f_1[\hat{X},\hat{Y}], \\
[f_2(\hat{X}),f_2(\hat{Y})] &= f_2[\hat{X},\hat{Y}], \\
V(f) &= V_1(f_1) \oplus V_2(f_2). \tag{A141}
\end{align*}$$

If a representation can not be expressed as the direct sum of two representations, this representation is a irreducible representation, otherwise it is a reducible representation. If all irreducible representations are given for a Lie algebra, it is equivalent that all representation of this Lie algebra are given.

Suppose $V_1(f_1)$ and $V_2(f_2)$ are two representations of Lie algebra $A$, the direct product $V(f)$ can be defined as follows:

The direct product representation space is the direct product of two representation spaces of $V_1(f_1)$ and $V_2(f_2),

$$V = V_1 \otimes V_2. \tag{A142}$$

The dimension of the multiplication of two dimensions,

$$\dim V(f) = \dim V_1(f_1) \times \dim V_2(f_2). \tag{A143}$$

The matrix representation is defined as,

$$f(\hat{X}) = f_1(\hat{X}) \otimes \hat{I}_2 + \hat{I}_1 \otimes f_2(\hat{X}), \tag{A144}$$

where $\hat{I}_1$ and $\hat{I}_2$ are identity operators on space $V_1$ and $V_2$, respectively.

The homomorphism of the direct product representation can be proved as follows,

$$\begin{align*}
[f(\hat{X}),f(\hat{Y})] &= f(\hat{X})f(\hat{Y}) - f(\hat{Y})f(\hat{X}) \\
&= \left( f_1(\hat{X}) \otimes \hat{I}_2 + \hat{I}_1 \otimes f_2(\hat{X}) \right) \left( f_1(\hat{Y}) \otimes \hat{I}_2 + \hat{I}_1 \otimes f_2(\hat{Y}) \right) - \left( f_1(\hat{Y}) \otimes \hat{I}_2 + \hat{I}_1 \otimes f_2(\hat{Y}) \right) \left( f_1(\hat{X}) \otimes \hat{I}_2 + \hat{I}_1 \otimes f_2(\hat{X}) \right) \\
&= \left( [f_1(\hat{X})f_1(\hat{Y})] \otimes \hat{I}_2 + f_1(\hat{X}) \otimes f_2(\hat{Y}) + f_1(\hat{Y}) \otimes f_2(\hat{X}) + \hat{I}_1 \otimes [f_2(\hat{X})f_2(\hat{Y})] \right) \\
&\quad - \left( [f_1(\hat{Y})f_1(\hat{X})] \otimes \hat{I}_2 - f_1(\hat{Y}) \otimes f_2(\hat{X}) - f_1(\hat{X}) \otimes f_2(\hat{Y}) - \hat{I}_1 \otimes [f_2(\hat{Y})f_2(\hat{X})] \right) \\
&= [f_1(\hat{X}),f_1(\hat{Y})] \otimes \hat{I}_2 + \hat{I}_1 \otimes [f_2(\hat{X}),f_2(\hat{Y})] \\
&\quad = f_1[\hat{X},\hat{Y}] \otimes \hat{I}_2 + \hat{I}_1 \otimes f_2[\hat{X},\hat{Y}] \\
&= f[\hat{X},\hat{Y}], \tag{A145}
\end{align*}$$

where we have used,

$$\begin{align*}
[f_1(\hat{X}),f_1(\hat{Y})] &= f_1[\hat{X},\hat{Y}], \\
[f_2(\hat{X}),f_2(\hat{Y})] &= f_2[\hat{X},\hat{Y}], \\
V(f) &= V_1(f_1) \otimes V_2(f_2). \tag{A146}
\end{align*}$$

Suppose a Lie algebra is a direct sum of two Lie algebras $A = A_1 \oplus A_2$, $V_1(f_1)$ $V_2(f_2)$ are representations of Lie algebra $A_1$ and $A_2$, respectively,

$$V(f) = V_1(f_1) \otimes V_2(f_2). \tag{A147}$$
The commutation relations can be expressed as,
\[ \dim V(f) = \dim V_1(f_1) \times \dim V_2(f_2). \] (A148)

Suppose \( \hat{X}_1 \in A_1 \) and has the matrix representation \( f_1(\hat{X}_1) \), similarly \( \hat{X}_2 \in A_2 \), the matrix representation is \( f_2(\hat{X}_2) \), we can find that \( \hat{X} = \hat{X}_1 + \hat{X}_2 \in A \), the matrix representation takes the form,
\[ f(\hat{X}) = f_1(\hat{X}_1) \otimes \hat{I}_2 + \hat{I}_1 \otimes f_2(\hat{X}_2), \] (A149)

where \( \hat{I}_1 \) and \( \hat{I}_2 \) are identity operators of spaces \( V_1 \) and \( V_2 \), respectively. The homomorphism of this representation can be proved in the following,
\[
[f(\hat{X}), f(\hat{Y})] = f(\hat{X})f(\hat{Y}) - f(\hat{Y})f(\hat{X})
\]
\[
= \left( f_1(\hat{X}_1) \otimes \hat{I}_2 + \hat{I}_1 \otimes f_2(\hat{X}_2) \right) \left( f_1(\hat{Y}_1) \otimes \hat{I}_2 + \hat{I}_1 \otimes f_2(\hat{Y}_2) \right) - \left( f_1(\hat{Y}_1) \otimes \hat{I}_2 + \hat{I}_1 \otimes f_2(\hat{Y}_2) \right) \left( f_1(\hat{X}_1) \otimes \hat{I}_2 + \hat{I}_1 \otimes f_2(\hat{X}_2) \right)
\]
\[
= \left[ f_1(\hat{X}_1)f_1(\hat{Y}_1) \right] \otimes \hat{I}_2 + f_1(\hat{Y}_1) \otimes f_2(\hat{X}_2) + f_1(\hat{X}_1) \otimes f_2(\hat{Y}_2) + \hat{I}_1 \otimes \left[ f_2(\hat{X}_2)f_2(\hat{Y}_2) \right] - \left[ f_1(\hat{Y}_1)f_1(\hat{X}_1) \right] \otimes \hat{I}_2 - f_1(\hat{Y}_1) \otimes f_2(\hat{X}_2) - f_1(\hat{X}_1) \otimes f_2(\hat{Y}_2) - \hat{I}_1 \otimes \left[ f_2(\hat{X}_2)f_2(\hat{Y}_2) \right]
\]
\[
= \left[ f_1(\hat{X}_1), f_1(\hat{Y}_1) \right] \otimes \hat{I}_2 + \hat{I}_1 \otimes \left[ f_2(\hat{X}_2), f_2(\hat{Y}_2) \right]
\]
\[
= f_1[\hat{X}_1, \hat{Y}_1] \otimes \hat{I}_2 + \hat{I}_1 \otimes f_2[\hat{X}_2, \hat{Y}_2]
\]
\[
= f[\hat{X}, \hat{Y}],
\]

where
\[
[f_1(\hat{X}_1), f_1(\hat{Y}_1)] = f_1[\hat{X}_1, \hat{Y}_1],
\]
\[
[f_2(\hat{X}_2), f_2(\hat{Y}_2)] = f_2[\hat{X}_2, \hat{Y}_2].
\] (A151)

### 12. Vector representation and operator representation

Suppose \( A \) is a Lie algebra, the algebraic basis is \( \{ \hat{a}_i \} \), the structure constant is \( C_{ab}^c \), consider that \( V(f) \) is a representation of \( A \), \( V \) can be considered as a vector space, and thus \( V(f) \) can always be considered as a vector representation. We can introduce a set of vectors as a basis of space \( V \),
\[
\{ |e_i\rangle \}, i = 1, 2, \cdots, \dim V.
\] (A152)

With this vector basis, the algebraic basis have the corresponding matrix representations,
\[
f(\hat{a}_i) = (\hat{a}_i)_j^i |e_i\rangle \otimes |e_j\rangle,
\]
\[
(\hat{a}_i)_j^i = \langle e_i | \hat{a}_i | e_j \rangle.
\] (A153)

The commutation relations can be expressed as,
\[
[f(\hat{a}_a), f(\hat{a}_b)] = C_{ab}^c f(\hat{a}_c),
\]
\[
(\hat{a}_a)_j^i (\hat{a}_b)_k^j - (\hat{a}_b)_j^i (\hat{a}_a)_k^j = C_{ab}^c (\hat{a}_c)_k^j,
\] (A154)

The relation between the algebraic basis and the vector basis takes the form,
\[
\hat{a}_a |e_i\rangle = (\hat{a}_a)_j^i |e_j\rangle.
\] (A155)

We can have the adjoint vector representation \( \nabla (f) \). The adjoint basis in space \( \nabla \) takes the form,
\[
\{ |e^i\rangle \}, i = 1, 2, \cdots, \dim V.
\] (A156)

In the adjoint basis, the algebraic basis has a matrix representation as,
\[
\nabla (\hat{a}_a) = (\hat{a}_a)_j^i |e_i\rangle \otimes |e^j\rangle,
\]
\[
(\hat{a}_a)_j^i = \langle e^i | \hat{a}_a | e_j \rangle.
\] (A157)
The commutation relation takes the form,
\[
[f(\hat{t}_a), f(\hat{t}_b)] = C_{ab} f(\hat{t}_c),
\]
\[
(\hat{t}_a)_j^i (\hat{t}_b)_k^j - (\hat{t}_b)_j^i (\hat{t}_a)_k^j = C_{ab}^c (\hat{t}_c)_k^j.
\]  \hfill (A158)

The algebraic basis and the vector basis has a relation,
\[
\langle e^i | \hat{t}_a = \langle e^j | (\hat{t}_a)_j^i.
\]  \hfill (A159)

We can find that actually the vector representation matrix \( f(\hat{t}_a) \) and the adjoint vector representation matrix \( \overline{f}(\hat{t}_a) \) are actually the same. For simplicity, we may directly denote he matrix representations and the operator by the same notations,
\[
f(\hat{t}_a) = \overline{f}(\hat{t}_a) = \hat{t}_a = (\hat{t}_a)_j^i |e_i \rangle \otimes \langle e^j|.
\]  \hfill (A160)

The operator representation \( \hat{O}(\hat{f}) \) is in the representation space \( \hat{O} = V \otimes \overline{V} \) constituted by the vector representation space \( V \) and the adjoint vector representation space \( \overline{V} \). The basis of the operator space is denoted as,
\[
\{ |e_i \rangle \otimes \langle e^j|, \quad i, j = 1, 2, \cdots, \dim V.
\]  \hfill (A161)

For operator representation \( \hat{O}(\hat{f}) \), the action of algebraic basis representation \( \hat{f}(\hat{t}_a) \) is defined by the following commutation relation,
\[
\hat{f}(\hat{t}_a)|e_i \rangle \otimes \langle e^j| = [\hat{t}_a, |e_i \rangle \otimes \langle e^j|]
\]
\[
\quad = \hat{t}_a|e_i \rangle \otimes \langle e^j| - |e_i \rangle \otimes \langle e^j| \hat{t}_a
\]
\[
\quad = (\hat{t}_a)_k^i |e_k \rangle \otimes \langle e^j| - |e_i \rangle \otimes \langle e^j|(\hat{t}_a)_k^i
\]
\[
\quad = [(\hat{t}_a)_k^i \delta_{j}^{k} - \delta_{i}^{k}(\hat{t}_a)_k^j]|e_k \rangle \otimes \langle e^j|.
\]  \hfill (A162)

So we can find the algebraic basis matrix representation takes the form,
\[
\hat{f}(\hat{t}_a) = [(\hat{t}_a)_k^i \delta_{j}^{k} - \delta_{i}^{k}(\hat{t}_a)_k^j][|e_k \rangle \otimes \langle e^j|] \otimes [|e_j \rangle \otimes \langle e^i|]
\]
\[
\quad = \hat{t}_a \otimes I - I \otimes \hat{t}_a^T,
\]  \hfill (A163)

where the upper indices \( ^T \) means the matrix transposition. Note that for matrix representation, the operator representation and the basis representation is not the same.

With help of the Jacobi relation deduced from the commutation relation, we can find that,
\[
[\hat{t}_a, [\hat{t}_b, |e_i \rangle \otimes \langle e^j|]] - [\hat{t}_b, [\hat{t}_a, |e_i \rangle \otimes \langle e^j|]] = [[\hat{t}_a, \hat{t}_b], |e_i \rangle \otimes \langle e^j|]
\]
\[
\quad = C_{ab}^c \hat{t}_c \hat{t}_c |e_k \rangle \otimes \langle e^j|.
\]  \hfill (A164)

Thus we can find that \( \hat{f}(\hat{t}_a) \) satisfy the homomorphism relation,
\[
\hat{f}(\hat{t}_a) \hat{f}(\hat{t}_b) - \hat{f}(\hat{t}_b) \hat{f}(\hat{t}_a) = [\hat{f}(\hat{t}_a), \hat{f}(\hat{t}_b)]
\]
\[
\quad = \hat{f}[\hat{t}_a, \hat{t}_b]
\]
\[
\quad = C_{ab}^c \hat{f}(\hat{t}_c).
\]  \hfill (A165)

The homomorphism can be proved directly by the matrix representation,
\[
[\hat{f}(\hat{t}_a), \hat{f}(\hat{t}_b)] = \hat{f}(\hat{t}_a) \hat{f}(\hat{t}_b) - \hat{f}(\hat{t}_b) \hat{f}(\hat{t}_a)
\]
\[
\quad = (\hat{t}_a \otimes I - I \otimes \hat{t}_a^T)(\hat{t}_b \otimes I - I \otimes \hat{t}_b^T) - (\hat{t}_b \otimes I - I \otimes \hat{t}_b^T)(\hat{t}_a \otimes I - I \otimes \hat{t}_a^T)
\]
\[
\quad = (\hat{t}_a \hat{t}_b) \otimes I - \hat{t}_b \hat{t}_a + \hat{t}_a \hat{t}_b^T + I \otimes (\hat{t}_a^T \hat{t}_b^T)
\]
\[
\quad = (\hat{t}_a \hat{t}_b - \hat{t}_b \hat{t}_a)(I + \hat{t}_a \otimes \hat{t}_b)
\]
\[
\quad = [\hat{t}_a, \hat{t}_b] \otimes I + I \otimes [\hat{t}_a, \hat{t}_b]^T
\]
\[
\quad = [\hat{t}_a, \hat{t}_b] \otimes I - I \otimes [\hat{t}_a, \hat{t}_b]^T
\]
\[
\quad = \hat{f}[\hat{t}_a, \hat{t}_b]
\]
\[
\quad = C_{ab}^c \hat{f}(\hat{t}_c).
\]  \hfill (A166)
The adjoint representation space of Lie algebra is the Lie algebra space itself,

$$V = A.$$  \hfill (A176)

The basis of the adjoint representation is the algebraic basis \( \hat{t}_b \),

$$[\hat{t}_a, \hat{t}_b] = C_{ab}^c \hat{t}_c.$$  \hfill (A168)

The matrix representation of the adjoint representation is the structure constant,

$$(\hat{t}_a)^c_b = C_{ab}^c,$$

$$\hat{t}_a = \{ (\hat{t}_a)_b \}.$$  \hfill (A169)

Since the structure constants satisfy the Jacobi relation, we can find the adjoint representation satisfy the homomorphism condition,

$$(\hat{t}_a)^d_c (\hat{t}_b)^e_d - (\hat{t}_b)^d_c (\hat{t}_a)^e_d = C_{ae}^d C_{bc}^e - C_{be}^d C_{ac}^e = C_{ab}^e (\hat{t}_c)^d_c.$$  \hfill (A170)

The direct product of \( r \) Lie algebra representation space of \( A \) constitute \( r \)-form tensor space,

$$T^{(r)} = \prod_r \otimes A.$$  \hfill (A171)

The tensor basis of \( T^{(r)} \) takes the form,

$$\hat{t}_{a_1 \cdots a_r} = \hat{t}_{a_1} \otimes \cdots \otimes \hat{t}_{a_r}.$$  \hfill (A172)

The action of \( \hat{t}_a \) on \( \hat{t}_{a_1 \cdots a_r} \) is defined by the commutation relation,

$$[\hat{t}_a, \hat{t}_{a_1 \cdots a_r}] = C_{aa_1}^c \hat{t}_{c \cdots a_r} + \cdots + C_{aa_r}^c \hat{t}_{a_1 \cdots c}.$$  \hfill (A173)

The representation matrix \( \hat{t}_a \) by tensor representation,

$$\hat{t}_a = \{ (\hat{t}_a)^a_b \} \otimes \hat{I}_2 \otimes \cdots \otimes \hat{I}_r + \hat{I}_1 \otimes \{ (\hat{t}_a)^a_b \} \otimes \cdots \otimes \hat{I}_r + \cdots + \hat{I}_1 \otimes \hat{I}_2 \otimes \cdots \otimes \{ (\hat{t}_a)^a_b \},$$  \hfill (A174)

where \( \hat{I}_1, \hat{I}_2, \cdots \hat{I}_r \) are \( \text{dim} A \times \text{dim} A \) identity matrices.

The relationships between Lie group and Lie algebra can be found in textbooks of group theory, generally speaking, the behaviors of Lie group \( G \) near the unit identity \( e \) is described by the corresponding Lie algebra,

$$A \rightarrow G : g = \exp (i \hat{X}), \quad \hat{X} \in A, g \in G.$$  \hfill (A175)

13. The Lie algebra used in 4D unified quantum theory

In 4D unified quantum theory, we have used three Lie algebras, the coordinate-momentum algebra \( A_{xp} \), the spin algebra \( A_S \) and the gauge algebra \( A_g \).

The coordinate-momentum algebra \( A_{xp} \) is a 9-dimensional Lie algebra, the algebraic basis is constituted by 4D coordinate \( \hat{x}^\mu \), 4D momentum \( \hat{p}_\nu \) and the identity. The coordinate-momentum algebra is a very special algebra, its not Abel, cannot be represented as the direct sum of simple Lie algebras.

The spin algebra \( A_S \) is a 6D Lie algebra. Its algebraic basis is the 6 spin elements \( \hat{s}_{\alpha \beta} \), it has the structure of Lie algebra \( D_2 \). The spin algebra is a semi-simple algebra and can be represented as a direct sum of two algebras \( A_1 \),

$$A_S = A_1 \oplus A_2.$$  \hfill (A176)

The gauge algebra \( A_g \) is a 12D Lie algebra. Its algebraic basis is constituted by hypercharge \( \hat{Y} \), isospin charge \( \hat{I}_I \) and color charge \( \hat{\lambda}_p \). The hypercharge \( \hat{Y} \) is the basis for a \( u(1) \) algebra, three isospin charges \( \hat{I}_I \) provide a basis for \( su(2) \) algebra, eight color charges \( \hat{\lambda}_p \) constitute a basis for \( su(3) \) algebra. The gauge algebra is the direct sum of \( u(1), su(2) \) and \( su(3) \),

$$A_g = u(1) \oplus su(2) \oplus su(3).$$  \hfill (A177)
14. 4D δ-function and convolution

The 4D δ-function is defined as,

$$\delta^4(x) = \delta(x^0)\delta(x^1)\delta(x^2)\delta(x^3).$$  \hfill (A178)

Its properties have the following,

$$\int \Psi(x')\delta^4(x-x')d^4x' = \Psi(x)$$

$$\int \delta^4(x)d^4x = 1,$$

$$\delta^4(-x) = \delta(x).$$  \hfill (A179)

The definition of the partial differential of the δ-function and its properties can be found as,

$$\delta^{(1)}(x-x') = \frac{\partial}{\partial x^\mu}\delta(x-x'),$$

$$\int \Psi(x')\frac{\partial}{\partial x^\mu}\delta^4(x-x')d^4x' = -\frac{\partial}{\partial x^\mu}\Psi(x),$$

$$\int \frac{\partial}{\partial x^\mu}\delta^4(x)d^4x = 0,$$

$$\frac{\partial}{\partial x^\mu}\delta^4(-x) = -\frac{\partial}{\partial x^\mu}\delta^4(x).$$  \hfill (A180)

We also have,

$$\exp(ip\cdot x) = \exp(ip_\mu x^\mu),$$

$$\int \exp(ip\cdot x)d^4p = (2\pi)^4\delta^4(x),$$

$$\int \exp(ip\cdot x)d^4x = (2\pi)^4\delta^4(p).$$  \hfill (A181)

The 4D Fourier transformations take the form

$$\tilde{\Phi}(p) = (2\pi)^{-2} \int \Phi(x)\exp(ip\cdot x)d^4x,$$

$$\Phi(x) = (2\pi)^{-2} \int \tilde{\Phi}(p)\exp(-ip\cdot x)d^4p.$$  \hfill (A182)

The derivative of the Fourier transformation and its multiplication can be written as,

$$\frac{\partial}{\partial p_\mu}\tilde{\Phi}(p) = (2\pi)^{-2} \int ix^\mu\Phi(x)\exp(ip\cdot x)d^4x,$$

$$\frac{\partial}{\partial x^\mu}\Phi(x) = (2\pi)^{-2} \int -ip_\mu\tilde{\Phi}(p)\exp(-ip\cdot x)d^4p,$$

$$p_\mu\tilde{\Phi}(p) = (2\pi)^{-2} \int i\frac{\partial}{\partial x^\mu}\Phi(x)\exp(ip\cdot x)d^4x,$$

$$x^\mu\Phi(x) = (2\pi)^{-2} \int -i\frac{\partial}{\partial p_\mu}\tilde{\Phi}(p)\exp(-ip\cdot x)d^4p.$$  \hfill (A183)

The Fourier transformations of the δ-function take the following form,

$$\exp(ip\cdot x') = \int \delta^4(x-x')\exp(ip\cdot x)d^4x,$$

$$\delta^4(x-x') = (2\pi)^{-4} \int \exp(ip\cdot x')\exp(-ip\cdot x)d^4p,$$

$$\delta^4(p-p') = (2\pi)^{-4} \int \exp(-ip'\cdot x)\exp(ip\cdot x)d^4x,$$

$$\exp(-ip'\cdot x) = \int \delta^4(p-p')\exp(-ip\cdot x)d^4p.$$  \hfill (A184)
The 4D convolution is defined as,

\[ F(p) * G(p) = \int F(p')G(p-p')d^4p'. \] (A185)

The properties of the 4D convolution are listed as,

\[ F(p) * G(p) = G(p) * F(p) \]
\[ F(p) * [G(p) * H(p)] = [G(p) * F(p)] * H(p) \]
\[ F(p) * [G(p) + H(p)] = F(p) * G(p) + F(p) * H(p) \] (A186)

Suppose \( F(p) * G(p) = K(p) \), we can find that,

\[ F(p - p') * G(p) = F(p) * G(p - p') = K(p - p'). \] (A187)

The derivative of the convolution is,

\[ \frac{\partial}{\partial p_\mu}[F(p) * G(p)] = \left[ \frac{\partial}{\partial p_\mu}F(p) \right] * G(p) = F(p) * \left[ \frac{\partial}{\partial p_\mu}G(p) \right]. \] (A188)

The convolution of the 4D \( \delta \)-function take the form,

\[ F(p) * \delta(p) = F(p), \]
\[ F(p) * \frac{\partial}{\partial p_\mu}\delta(p) = \frac{\partial}{\partial p_\mu}F(p). \] (A189)

The Fourier transformation of the convolution,

\[ F(x)G(x) = (2\pi)^{-2} \int_{R^4} \tilde{F}(p) \ast \tilde{G}(p) \exp(-ip \cdot x)d^4p, \]
\[ \tilde{F}(p) \ast \tilde{G}(p) = (2\pi)^{-2} \int_{R^4} [F(x)G(x)] \exp(ip \cdot x)d^4x, \]
\[ F(x) \ast G(x) = (2\pi)^{-2} \int_{R^4} \tilde{F}(p)\tilde{G}(p) \exp(-ip \cdot x)d^4p, \]
\[ \tilde{F}(p)\tilde{G}(p) = (2\pi)^{-2} \int_{R^4} [F(x) \ast G(x)] \exp(ip \cdot x)d^4x. \] (A191)

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