BEHAVIOR OF WEAK TYPE BOUNDS FOR HIGH DIMENSIONAL MAXIMAL OPERATORS DEFINED BY CERTAIN RADIAL MEASURES

J. M. ALDAZ AND J. PÉREZ LÁZARO

Abstract. As shown in [A1], the lowest constants appearing in the weak type \((1,1)\) inequalities satisfied by the centered Hardy-Littlewood maximal operator associated to certain finite radial measures, grow exponentially fast with the dimension. Here we extend this result to a wider class of radial measures and to some values of \(p > 1\). Furthermore, we improve the previously known bounds for \(p = 1\). Roughly speaking, whenever \(p \in (1,1.03]\), if \(\mu\) is defined by a radial, radially decreasing density satisfying some mild growth conditions, then the best constants \(c_{p,d,\mu}\) in the weak type \((p,p)\) inequalities satisfy \(c_{p,d,\mu} \geq 1.005^d\) for all \(d\) sufficiently large. We also show that exponential increase of the best constants occurs for certain families of doubling measures, and for arbitrarily high values of \(p\).

1. Introduction

Given a Borel measure \(\mu\) on \(\mathbb{R}^d\) and a locally integrable function \(g\), the Hardy-Littlewood maximal operator \(M_\mu\) is given by

\[
M_\mu g(x) := \sup_{\{r > 0 : \mu(B(x,r)) > 0\}} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |g| \, d\mu,
\]

where \(B(x,r)\) denotes the euclidean closed ball of radius \(r > 0\) centered at \(x\). As is well known, \(M_\mu\) is a positive, sublinear operator, acting on the cone of positive, locally integrable functions (\(M_\mu\) is defined by using \(|g|\) rather than \(g\)). The Hardy-Littlewood maximal operator admits many variants: Instead of averaging \(|g|\) over balls centered at \(x\) (the centered operator) as in (1), it is possible to consider all balls containing \(x\) (the uncentered operator) or average over convex bodies more general than euclidean balls (and even over more general sets). And as part of the current effort to develop a calculus on metric spaces, the Hardy-Littlewood maximal operator has been studied in settings far more general than \(\mathbb{R}^d\). Here we work with the centered operator defined using euclidean balls in \(\mathbb{R}^d\), associated to certain radial measures \(\mu\) given by \(\mu(A) := \int_A f(\|y\|_2) \, d\lambda^d(y)\), where \(f : (0, \infty) \to [0, \infty)\) is nonincreasing (possibly unbounded) and not zero almost everywhere, and \(f(t)t^{d-1} \in L^1_{\text{loc}}((0, \infty), dt)\). We emphasize that the function \(f\) defining \(\mu\) is allowed to vary with the dimension \(d\). Additional hypotheses, regarding the growth at 0 of \(f\) and its decay at \(\infty\), are given below.

The Hardy-Littlewood maximal operator is an often used tool in Real and Harmonic Analysis, mainly (but not exclusively) due to the fact that while \(|g| \leq M_\mu g\) a.e., \(M_\mu g\) is
not too large (in an $L^p$ sense) since it satisfies the following strong type $(p, p)$ inequality:

$$\|M_\mu g\|_p \leq C_p \|g\|_p$$

for $1 < p \leq \infty$. For $p = 1$, $M_\mu$ satisfies instead the weak type $(1,1)$ inequality:

$$\sup_{\alpha > 0} \alpha \mu(\{M_\mu g \geq \alpha\}) \leq c_1 \|g\|_1.$$  

Another aspect of the maximal operator that is receiving increasing attention, but not touched upon here, is that of its regularity properties, cf. for instance [AlPe1], [AlPe2], [AlPe3] and the references contained therein. When $\mu = \lambda^d$, the $d$-dimensional Lebesgue measure, we often simplify notation, by writing $M$ rather than $M_\mu$ and $dx$ instead of $d\lambda^d(x)$. Considerable efforts have gone into determining how changing the dimension of $\mathbb{R}^d$ modifies the best constants appearing in the weak and strong type inequalities. When $p = \infty$, we can take $C_p = 1$ in every dimension $d$, since averages never exceed a supremum. Quite remarkably, E. M. Stein showed that for $M$, there exist bounds for $C_p$ that are independent of $d$ ([St1], [St2], [StSt], see also [St3]). Stein’s result was generalized to the maximal function defined using an arbitrary norm by J. Bourgain ([Bou1], [Bou2], [Bou3]) and A. Carbery ([Ca]) when $p > 3/2$. For $\ell_q$ balls, $1 \leq q < \infty$, D. Müller [Mu] showed that uniform bounds again hold for every $p > 1$ (given $1 \leq q < \infty$, the $\ell_q$ balls are defined using the norm $\|x\|_q := (x_1^q + x_2^q + \cdots + x_d^q)^{1/q}$). It is still an open question whether the maximal operator associated to cubes and Lebesgue measure is uniformly bounded for $1 < p \leq 3/2$.

When $p = 1$, the maximal operator is (typically) unbounded, so one considers weak type $(1,1)$ inequalities instead. In [StSt], E. M. Stein and J. O. Strömberg proved that the smallest constants in the weak type $(1,1)$ inequality satisfied by $M$ grow at most like $O(d)$ for euclidean balls, and at most like $O(d \log d)$ for more general balls. They also asked if uniform bounds could be found, a question still open for euclidean balls. But for cubes the answer is negative, cf. [A2]. In [A1], G. Aubrun refined the result from [A2] by showing that $c_{1,d} \geq \Theta(\log^{1-\varepsilon} d)$, where $\Theta$ denotes the exact order and $\varepsilon > 0$ is arbitrary. A very significant extension of the Stein and Strömberg’s $O(d \log d)$ result, beyond the euclidean setting, has recently been obtained by A. Naor and T. Tao, cf. [NaTa].

The weak type $(1,1)$ case for integrable radial densities defined via bounded decreasing functions was studied in [Al]. It was shown there that the best constants $c_{1,d}$ satisfy $c_{1,d} \geq \Theta(1) \left(2/\sqrt{3}\right)^{d/6}$, in strong contrast with the linear $O(d)$ bounds known for $M$. This suggests that for these measures and sufficiently small values of $p > 1$, lack of uniform bounds in $d$ should also hold. We show here that this is indeed the case, and for a wider class of measures than those considered in [Al]. We shall remove the assumption of boundedness on densities and the assumption of finiteness on measures, replacing these hypotheses with milder growth conditions on the relative size of balls centered at the origin (a possibility suggested in [Al] Remark 2.6). Instead of working directly with norm (or strong type) inequalities when $p > 1$, we shall consider the weak type $(p, p)$ inequalities. This allows us to treat the cases $p = 1$ and $p > 1$ simultaneously. Needless to say, lower bounds for weak type constants immediately imply the same bounds for strong type constants. In Theorem 3.4 we show that if balls centered at zero grow sufficiently fast for some given radius, and this growth experiences a certain rate of decay at infinity (cf. the theorem for the exact technical conditions) then there is exponential increase of the best constants $c_{p,d}$ in the weak type $(p, p)$ inequalities, for every
$p \in [1, p_0)$, where $p_0 \approx 1.0378$. The proof follows the lines of \cite{AlPe}, but replacing the Dirac delta $\delta_0$ with $\chi_{B(0,v)}$ for some suitably chosen radius $v$, and using a better ball decomposition. This allows us to improve the bound on $c_{1,d}$ from \cite{AlPe} to $c_{1,d} \geq \Theta(1) \left(\frac{2}{55^{1/6}}\right)^d$ for $p = 1$, even though we are considering the characteristic function of a ball, rather than the more efficient $\delta_0$. Of course, Dirac Deltas cannot be used when $p > 1$, since the only reasonable definition of $\|\delta\|_p$ for $p > 1$ is $\|\delta\|_p = \infty$.

Exponential dependency on the dimension of the best constants also holds for certain collections of doubling measures and arbitrarily high values of $p$, cf. Theorem 3.12 below. Thus, Stein result on uniform $L^p$ bounds for $M$ does not extend to arbitrary doubling measures on $\mathbb{R}^d$, even though the class of doubling measures represents a natural generalization of $\lambda^d$.

To highlight the difference between $\lambda^d$ and the measures considered here, we point out that when $M$ acts on radial, radially decreasing $L^p$ functions, the best weak type $(p, p)$ constants $c_{p,d}$ equal 1 in every dimension, see \cite{AlPe} Theorem 2.6] (actually, the result is stated there for $c_{1,d}$, but $c_{1,d} \geq c_{p,d}$, cf. (5) below and the explanations afterwards), while the best strong type constants satisfy $C_{p,d} \leq 2^{1/q}q^{1/p}$, where $q = p/(p - 1)$, see \cite{AlPe} Corollary 2.7.

Professor Fernando Soria informs us that he and Alberto Criado have also extended the results from \cite{AlPe} to some values of $p > 1$, cf. \cite{Cr}; we mention that where \cite{Cr} and this paper overlap, the results presented here are more general and give better bounds.

2. Notation and background results.

The restriction of $\mu$ to a measurable set $A$ is denoted by $\mu|_A$; that is, $\mu|_A(B) = \mu(A \cap B)$. We always assume that $\mu(\mathbb{R}^d) > 0$ and $\mu(B(x, r)) < \infty$, i.e., measures are nontrivial and locally finite. The maximal function of a locally finite measure $\nu$ is defined by

$$M_\mu \nu(x) := \sup_{\{r > 0: \mu(B(x, r)) > 0\}} \frac{\nu(B(x, r))}{\mu(B(x, r))}.$$  

Note that formula (1) is simply (2) in the special case $\nu << \mu$. Our choice of closed balls in (1) and (2) is mere convenience; using open balls instead does not change the value of the maximal operator at $x$, since each closed ball is a countable intersection of open balls. The boundary of $B(x, r)$ is the sphere $S(x, r)$. Sometimes we use $B^d(x, r)$ and $S^{d-1}(x, r)$ to make their dimensions explicit. If $x = 0$ and $r = 1$, we use the abbreviations $B^d$ and $S^{d-1}$. Balls are defined using the $\ell_2$ or euclidean distance $\|x\|_2 := \sqrt{x_1^2 + \cdots + x_d^2}$. The Lebesgue measure on $\mathbb{R}^d$ is denoted by $\lambda^d$, and area measure on a $d - 1$ sphere, by $\sigma^{d-1}$. Sometimes it is convenient to use normalized versions of these measures, so balls and spheres have total mass 1; we use $N$ as a subscript to denote these normalizations. Thus, $\lambda_N^d(B^d) = 1$ and $\sigma_N^{d-1}(S^{d-1}) = 1$.

Regarding the relationships between different constants, let us recall that by the Besicovitch Covering Theorem, for every locally finite Borel measure $\mu$ on $\mathbb{R}^d$, and every $p$ with $1 \leq p < \infty$, the maximal operator satisfies the following weak type $(p, p)$ inequality:

$$\mu(\{M_\mu g \geq \alpha\}) \leq \left(\frac{c\|g\|_p}{\alpha}\right)^p,$$  

where $c_0$ is independent of $\mu$. The usual weak type inequalities have the form

$$\mu(\{M_\mu g \geq \alpha\}) \leq \left(\frac{\alpha/k}{\|g\|_p}\right)^p,$$  

where $k$ is independent of $\mu$ and $g$.
where \( c = c(p, d, \mu) \) depends neither on \( g \in L^p(\mathbb{R}^d, \mu) \) nor on \( \alpha > 0 \). The constant \( c \) can also be taken to be independent of \( \mu \) and of \( p \). Set \( q := p/(p - 1) \). Using the quantitative version of the Besicovitch Covering Theorem given in [Su, p. 227], we have

\[
\mu(\{M_\mu g \geq \alpha\}) \leq \frac{(2.641 + o(1))^d \|g\|_1}{\alpha}.
\]

Thus, if \( g \in L^p(\mathbb{R}^d, \mu) \), then \( \|g\|^p \in L^1(\mathbb{R}^d, \mu) \), and it follows from Jensen’s inequality that

\[
\mu(\{M_\mu g \geq \alpha\}) = \mu(\{(M_\mu g)^p \geq \alpha^p\}) \leq \mu(\{M_\mu g \geq \alpha\}) \leq \frac{(2.641 + o(1))^d \|g\|^p}{\alpha^p}.
\]

Letting \( c_{p,d,\mu} \) be the best constant \( c \) in (3), we have \( c_{p,d,\mu} \leq (2.641 + o(1))^d/p \). This bound is uniform in \( \mu \), and setting \( p = 1 \) in the exponent \( d/p \), it can be made uniform in \( p \) also. Replacing \( (2.641 + o(1))^d/p \) by \( c_{1,d,\mu} \) in the right hand side of (5), we also obtain \( c_{p,d,\mu} \leq c_{1,d,\mu} \). Let \( C_{p,d,\mu} \) be the lowest constant in

\[
\|M_\mu g\|_{L^p(\mathbb{R}^d, \mu)} \leq C_{p,d,\mu}\|g\|_{L^p(\mathbb{R}^d, \mu)}.
\]

It is an immediate consequence of Chebyshev’s inequality that \( c_{p,d,\mu} \leq C_{p,d,\mu} \), since

\[
\alpha^p \mu(\{(M_\mu g)^p \geq \alpha^p\}) \leq \|M_\mu g\|_p^p \leq C_{p,d,\mu}\|g\|_p^p.
\]

When \( p \) is small, lower bounds for \( c_{p,d,\mu} \) are quite often not just formally stronger, but substantially stronger than lower bounds for \( C_{p,d,\mu} \), since it is well-known that for many measures \( C_{1,d,\mu} = \infty \) and \( \lim_{p \to 1} C_{p,d,\mu} = \infty \), while \( c_{p,d,\mu} \leq C_{1,d,\mu} \leq (2.641 + o(1))^d \).

Let \( d >> 1 \), and consider Lebesgue measure restricted to the unit ball. Most of its mass is concentrated near \( S^{d-1}(0,1) \), since volume scales like \( R^d \), so the ball “looks” very much like the sphere. The main idea in [Al1] and here is to realize that this is a rather general phenomenon: Rotationally invariant measures with a certain decay at infinity, will often be very similar in a certain region to area on some sphere \( S^{d-1}(0,R_1) \). Hence, the size of balls in that region can be estimated by intersecting them with \( S^{d-1}(0,R_1) \) and then using the area of the spherical caps resulting from such intersections. Given a unit vector \( v \in \mathbb{R}^d \) and \( s \in [0,1) \), the \( s \) spherical cap about \( v \) is the set \( C(s,v) := \{ \theta \in S^{d-1} : \langle \theta, v \rangle \geq s \} \). Spherical caps are just geodesic balls \( B_{S^d-1}(x,r) \) in \( S^{d-1} \). For spheres other than \( S^{d-1} \), spherical caps are defined in an entirely analogous way. If \( v = e_1 = (1,0,\ldots,0) \) and \( s = 2^{-1} \), then

\[
B_{S^d-1}(e_1, \pi/3) = C(2^{-1}, e_1) = S^{d-1} \cap B(e_1,1).
\]

More generally, given any angle \( r \in (0,\pi/2) \), writing \( s = \cos r \) and \( t = \sin r \), we have

\[
B_{S^d-1}(e_1, r) = C(s, e_1) \subset B(se_1, t).
\]

The following lemma shows that \( \sigma_{N-1}^{-1}(C(s,e_1)) = t^d/\Theta(\sqrt{d}) \), where \( \Theta \) stands for exact order (i.e., \( g = \Theta(h) \) if and only if \( g = O(h) \) and \( h = O(g) \)); the special case \( r = \pi/3 \) is used in the proof of [Al1]. Theorem 2.3]. We recall the following results on volumes and areas: i) \( \lambda^d(B^d) = \frac{\pi^{d/2}}{\Gamma(d/2+1)} \); ii) \( \sigma^{-1}(S^{d-1}) = d\lambda^d(B^d) \); iii) \( \sigma^{-1}(B_{S^d-1}(x,r)) = \sigma^{-2}(S^{d-2}) \int_0^r \sin^{d-2} t dt \) (cf. for instance [Gr, (A.11) pg. 259]).
Lemma 2.1. Let \( r \in (0, \pi/2) \), let \( \sigma^d_N \) be normalized area on the sphere \( \mathbb{S}^{d-1}(0, R) \), and let \( s = \cos r \), \( t = \sin r \), so with this notation, \( \sigma^d_N(B_{\mathbb{S}^{d-1}}(0, R)) = \sigma^d_N(C(R, Re_1)). \)

Then

\[
\frac{t^{d-1}}{\sqrt{2\pi d}} \leq \sigma^d_N(C(R, Re_1)) \leq \frac{t^{d-1}}{s\sqrt{2\pi d}} \sqrt{1 + \frac{1}{d}}.
\]

Proof. Observe first that the relative size of caps depends neither on the center of the ball nor on the radius. In particular, since we are dealing with normalized area, we may assume that \( R = 1 \). We use the following Gamma function estimate (an immediate consequence of the log-convexity of \( \Gamma \) on \( (0, \infty) \), cf. Exercise 5, pg. 216 of [Web]):

\[
\left(\frac{d}{2}\right)^{1/2} \leq \frac{\Gamma(1 + d/2)}{\Gamma(1/2 + d/2)} \leq \left(\frac{d + 1}{2}\right)^{1/2}.
\]

From i), ii), iii), (11) and the fact that \( \cos u \geq s \) on \([0, r]\), we get:

\[
\sigma^d_N(C(s, e_1)) \leq \frac{\sigma^{d-2}(S^{d-2})}{s\sigma^{d-1}(S^{d-1})} \int_0^r \sin^{d-2} u \cos u \, du
\]

\[
= \frac{1}{sd} \left(\frac{\lambda^{d-1}(B^{d-1})}{\lambda^d(B^d)}\right) t^{d-1} \leq \frac{t^{d-1}}{s\sqrt{2\pi d}} \sqrt{1 + \frac{1}{d}}.
\]

Likewise, since \( \cos u \leq 1 \),

\[
\sigma^d_N(C(s, e_1)) \geq \frac{\sigma^{d-2}(S^{d-2})}{\sigma^{d-1}(S^{d-1})} \int_0^r \sin^{d-2} u \cos u \, du = \frac{1}{d} \left(\frac{\lambda^{d-1}(B^{d-1})}{\lambda^d(B^d)}\right) t^{d-1} \geq \frac{t^{d-1}}{\sqrt{2\pi d}}.
\]

\[ \square \]

3. Weak type \((p, p)\) bounds for rotationally invariant measures

Fix \( d \in \mathbb{N} \setminus \{0\} \), and let \( f : (0, \infty) \to [0, \infty) \) be a nonincreasing (possibly unbounded) function, not zero almost everywhere, such that \( f(t)t^{d-1} \in L^1_{\text{loc}}((0, \infty), dt) \). Then the function \( f \) defines a locally integrable, rotationally invariant (or radial) measure \( \mu \) on \( \mathbb{R}^d \) via

\[
\mu(A) := \int_A f(\|y\|_2) \, d\lambda^d(y).
\]

Observe that the local integrability of \( f(t)t^{d-1} \) is assumed for a fixed \( d \), not for all values of \( d \) simultaneously. Note also that \( f \) can depend on \( d \). When \( A = B(0, R) \), integration in polar coordinates yields the well known expression \( \mu(B(0, R)) = \sigma^{d-1}(S^{d-1}) \int_0^R f(t)t^{d-1} \, dt. \) Since (unlike [Al]) finiteness of measures and boundedness of densities are not assumed in the present paper, we need to impose some conditions on the rate of growth of balls centered at zero. To this end, we define, for all \( u \in (0, 1] \) and all \( R > 0 \) such that \( \mu(B(0, uR)) > 0 \),

\[
h_u(R) := \frac{\mu(B(0, R))}{\mu(B(0, uR))}.
\]
In the extreme case $\mu = \delta_0$, $h_u(R) = 1$ always, and for every $g$ with $g(0) < \infty$, we have $M_\mu g = g = g(0)$ a.e. with respect to $\delta_0$. Thus, for all $p \geq 1$ and all $d \geq 1$, $c_{p,d,\delta_0} = C_{p,d,\delta_0} = 1$.

Of course, in this case there is no relationship between $\delta_0$ and $d$. For the measures considered in [A1, Theorem 2.3], $\lim_{R \to 0} h_u(R) = u^{-d}$ and $\lim_{R \to \infty} h_u(R) = 1$; we present this fact, which appears within the proof of [A1, Theorem 2.3], as part of the next proposition.

**Proposition 3.1.** Fix $d \in \mathbb{N} \setminus \{0\}$. Let $f : (0, \infty) \to [0, \infty)$ be a nonincreasing function with $f > 0$ on some interval $(0, a)$ and $f(t)t^{d-1} \in L^1_{loc}(0, \infty)$. If $\mu$ is the measure defined by (16), then for every $u \in (0, 1)$ and every $R > 0$ we have $h_u(R) \leq u^{-d}$. If additionally $f$ is bounded, then for every $u \in (0, 1)$, $\sup_{R>0} h_u(R) = \lim_{R \to 0} h_u(R) = u^{-d}$. Regardless of whether $f$ is bounded or not, if $\mu$ is finite, then for every $u \in (0, 1)$ we have $\lim_{R \to \infty} h_u(R) = 1$.

**Proof.** The fact that $\sup_{R>0} h_u(R) \leq u^{-d}$ is obvious since $f$ is nonincreasing, so the case where $f$ is constant yields the largest possible growth, and then we just have a multiple of Lebesgue measure. Or, more formally:

\[
\frac{\mu(B(0, R))}{\mu(B(0, uR))} = \frac{\mu(B(0, uR)) + \mu(B(0, R) \setminus B(0, uR))}{\mu(B(0, uR))} = 1 + \frac{\sigma^{d-1}(S^{d-1}) \int_0^R f(t)t^{d-1}dt}{\sigma^{d-1}(S^{d-1}) \int_u^R f(t)t^{d-1}dt} = 1 + \frac{f(uR) \int_0^R t^{-d}dt}{f(uR) \int_0^R t^{-d}dt} = u^{-d}. \tag{17}
\]

Suppose next that in addition to being nonincreasing, $f$ is bounded. Then the averages $\frac{1}{\lambda^d(B(0, R))} \int_{B(0, R)} f(\|x\|_2)dx$ are bounded and nonincreasing with respect to $R$. Thus, $\lim_{R \to 0} \frac{1}{\lambda^d(B(0, R))} \int_{B(0, R)} f(\|x\|_2)dx = L$ exists, and

\[
\lim_{R \to 0} \frac{\mu(B(0, R))}{\mu(B(0, uR))} = \lim_{R \to 0} \frac{\int_{B(0, R)} f(\|x\|_2)dx}{\int_{B(0, uR)} f(\|x\|_2)dx} = \lim_{R \to 0} \frac{L\lambda^d(B(0, R))}{L\lambda^d(B(0, uR))} = u^{-d}. \tag{19}
\]

The last assertion about finite measures is obvious. \hfill $\square$

**Remark 3.2.** The condition $\lim_{R \to 0} h_u(R) = u^{-d}$ can be satisfied by unbounded densities with a mild singularity a $0$. Consider, for instance, $f(x) = \log(x)\chi_{[0,1]}(x)$, for every $d \geq 1$.

**Lemma 3.3.** Let $\mu$ be a rotationally invariant measure on $\mathbb{R}^d$, let $1 \leq p < \infty$, and let $q := p/(p-1)$. For $0 < R$ and $0 < v < 1$, write $H := \sqrt{R^2 + v^2R^2}$. If the pair $(v, R)$ is such that $\mu(B(0, vR)) > 0$, then

\[
c_{p,d,\mu} \geq \frac{\mu(B(0, vR))^{1/q} \mu(B(0, R))^{1/p}}{2\mu(B(Re_1, H))}. \tag{20}
\]

If additionally there exist $T$, $t_0 > 0$ and $v_0 \in (0, 1)$ such that $\sup_{\{R > 0 : v_0 R \geq T\}} h_{v_0}(R) \leq v_0^{-t_0d}$, then

\[
c_{p,d,\mu} \geq \sup_{\{R > 0 : v_0 R \geq T\}} \frac{v_0^{t_0d/q} \mu(B(0, R))}{2\mu(B(Re_1, H))}. \tag{21}
\]
Theorem 2.3, we look for a radius $R$ and let $c_R$.

Proof. Note that

$$M_\mu \chi_{B(0,vR)}(Re_1) \geq \frac{\|\chi_{B(0,vR)}\|_1}{2\mu(B(Re_1,H))} =: \alpha.$$  \hfill (22)

By rotational invariance of $\mu$, we have $B(0,R) \subset \{M_\mu \chi_{B(0,vR)} \geq \alpha\}$. And since $\chi_{B(0,vR)} = \chi_{B(0,vR)^p}$, it follows that $\|\chi_{B(0,vR)}\|_1 = \|\chi_{B(0,vR)}\|_{p}^p$. Using (23) we see that

$$c_{p,d,\mu} \geq \alpha \left(\frac{\|\chi_{B(0,vR)}\|_1}{\|\chi_{B(0,vR)}\|_p}\right) \geq \frac{\|\chi_{B(0,vR)}\|_1}{\mu(B(Re_1,H))} \frac{\mu(B(0,R))^{1/p}}{\mu(B(0,R))^{1/p}}.$$

Specializing to $v = v_0$ and using the hypothesis on $T$ and $t_0$ we obtain

$$c_{p,d,\mu} \geq v_0^{t_0/d} \frac{\mu(B(0,R))}{\mu(B(Re_1,H))} \geq \frac{\mu(B(0,R))}{\mu(B(0,R))^{1/p}}.$$  \hfill (24)

for every $R > 0$ such that $v_0R \geq T$.

The preceding Lemma is more general than needed in the present paper, since we are not assuming that $\mu$ is of the form given by (15); this greater generality will be useful in future work. If $\mu$ is given by (15), then by Proposition 3.1 the condition on $h_u(R)$ is satisfied for some $t_0 \leq 1$, all $v \in (0,1)$, and all $T > 0$. So the Lemma is applicable and furthermore, any $v_0 \in (0,1)$ can be used (in the next Theorem we take $v_0 = 1/2$). The idea of the proof is to choose $R_1$ so $\mu(B(R_1 e_1,H))$ is exponentially small (in $d$) when compared with $\mu(B(0,R_1))$, and then to adjust $q$ in (21) so $v_0^{-1/q}$ is sufficiently close to 1. This yields exponential growth of the constants for $p > 1$ small enough. Recall that $C_{p,d,\mu}$ denotes the best constant in the strong type $(p,p)$ inequalities. We emphasize that in the next result, we can have different functions $f$ associated to different dimensions $d$.

**Theorem 3.4.** Fix $d \in \mathbb{N} \setminus \{0\}$, and set $u = \sqrt{2/3}$. Let $f : (0,\infty) \to [0,\infty)$ be a nonincreasing function and let $\mu$ be the radial measure defined via (14). Assume $\mu$ satisfies

$$\sup_{R > 0} h_u(R) \geq \frac{64}{55} \frac{1}{d} \limsup_{R \to \infty} h_u(R).$$  \hfill (26)

Then for every $p$ such that $1 \leq p < \frac{6 \log 2}{\log 55} \approx 1.0378$, we have

$$55^{-1/6}2^{1/p} > 1 \quad \text{and} \quad C_{p,d,\mu} \geq c_{p,d,\mu} \geq \frac{1}{4 + \Theta \left(\frac{1}{\sqrt{d}}\right)} \left(\frac{2^{1/p}}{55^{1/6}}\right)^d.$$  \hfill (27)

Proof. Assume that $d \geq 2$, and set $H = \sqrt{R_1^2 + v^2R_1^2}$, as in Lemma 3.3. Arguing as in [A1, Theorem 2.3], we look for a radius $R_1$ such that $B(R_1 e_1,H)$ has very small measure compared to $B(0,R_1)$. Fix $0 < \varepsilon < 1/10$. Define $A = \{R > 0 : h_u(R) \geq (1 - \varepsilon)(64/55)^d\}$. By the continuity in $R$ of $h_u$ and the hypotheses in (26) $A$ is a nonempty closed set. If $A$ is unbounded,
we choose \( R_1 \in A \) so large that \( h_u(R_1) \geq (1 - \varepsilon)(64/55)^{d/6} \) and \( h_u(u^{-1}R_1), h_u(u^{-2}R_1) < (1 + \varepsilon)(64/55)^{d/6} \). If \( A \) is bounded, then \( R_1 := \max A \) automatically satisfies the preceding conditions on \( h_u(R_1), h_u(u^{-1}R_1) \), and \( h_u(u^{-2}R_1) \). Set \( v = 1/2 \). Then \( H = R_1\sqrt{5}/2 \). Write \( T = \sqrt{R_1^2 + H^2} = 3R_1/2 \), and observe that \( T = u^{-2}R_1 \), so \( B(R_1e_1, H) \cap \{ x_1 \leq R_1 \} \subset B(0, 3R_1/2) \). Since the density of \( \mu \) is radially decreasing,

\[
\mu(B(R_1e_1, H) \cap \{ x_1 \geq R_1 \}) \leq \mu(B(R_1e_1, H) \cap \{ x_1 \leq R_1 \}).
\]

Thus \( \mu(B(R_1e_1, H) \leq 2\mu(B(R_1e_1, H) \cap \{ x_1 \leq R_1 \}) \), so is enough to control this latter term.

To this end, we split \( B(R_1e_1, H) \cap \{ x_1 \leq R_1 \} \) into the following three pieces and estimate the measure of each one: \( B(0, uR_1) \cap \{ x_1 \leq R_1 \} \) into the following three pieces and estimate the measure of each one: \( B(0, uR_1) \cap \{ x_1 \leq R_1 \} \), \( B(0, R_1^c) \cap \{ x_1 \leq R_1 \} \), and \( (B(0, R_1) \setminus B(0, \sqrt{2/3R_1}) \cap B(R_1e_1, H) \). First we bound the part containing the origin:

\[
\mu(B(0, uR_1) \cap B(R_1e_1, H)) \leq \mu(B(0, uR_1)) \leq \frac{\mu(B(0, R_1))}{1 - \varepsilon} \left( \frac{55}{64} \right)^{d/6}.
\]

The other two parts are contained inside certain cones, whose radial projections into the unit sphere are spherical caps. So we apply Lemma 2.1. To control \( \mu(B(0, R_1)^c \cap B(R_1e_1, H) \cap \{ x_1 \leq R_1 \}) \), we define \( \nu \) on \( S^{d-1}(0, 1) \) as the pushforward (via the radial projection map) of \( \mu \) restricted to \( B(0, u^{-2}R_1) \setminus B(0, R_1) \). Now \( \nu \) is a rotationally invariant measure on \( S^{d-1} \), so it must be a multiple \( m\sigma_N^{d-1} \) of normalized area. Since \( \nu(S^{d-1}) = m \), we have \( m = \mu(B(0, u^{-2}R_1) \setminus B(0, R_1)) \) and thus \( \nu = \mu(B(0, u^{-2}R_1) \setminus B(0, R_1)) \sigma_N^{d-1} \). We use symmetry to find the spherical cap \( C \) determined by the intersection of \( S(0, R_1) \) with \( B(R_1e_1, H) \), restricting ourselves to the \( x_1x_2 \)-plane. Simultaneously solving \( x_1^2 + x_2^2 = R_1^2 \) and \( (x_1 - R_1)^2 + x_2^2 = H^2 \), we find that the radial projection of \( C \) into \( S^{d-1} \) is \( C(3/8, e_1) \). Now

\[
\mu(B(0, R_1)^c \cap B(R_1e_1, H)) = \mu(B(0, R_1)^c \cap B(R_1e_1, H) \cap \{ x_1 \leq R_1 \}) + \mu(B(R_1e_1, H) \cap \{ x_1 > R_1 \}).
\]

By Lemma 2.1 with \( \cos r = 3/8 \) (so \( \sin r = \sqrt{55}/8 \)) and by the choice of \( R_1 \),

\[
\mu(B(0, R_1)^c \cap B(R_1e_1, H) \cap \{ x_1 \leq R_1 \}) \leq \mu(B(0, R_1))(1 + \varepsilon)^2 \left( \frac{64}{55} \right)^{d/6} \sigma_N^{d-1}(C(3/8, e_1))
\]

\[
\leq \mu(B(0, R_1)) \left( \frac{55}{64} \right)^{d/6} \Theta \left( \frac{1}{\sqrt{d}} \right).
\]

Regarding the measure of \( (B(0, R_1) \setminus B(0, \sqrt{2/3R_1}) \cap B(R_1e_1, H) \), this set is contained in the (positive) cone subtended by the cap \( C \) resulting from the intersection of \( S^{d-1}(0, \sqrt{2/3R_1}) \) with \( B(R_1e_1, H) \). The said cone is formed by all rays starting at 0 and crossing \( C \). Let \( r \) be the maximal angle between a vector in this cap and the \( x_1 \)-axis. We consider the intersection of \( C \) with the \( x_1x_2 \)-plane, in order to determine \( s := \cos r \) and \( t := \sin r \). Solving \( (x_1 - R_1)^2 + x_2^2 = H^2 \) and \( x_1^2 + x_2^2 = (\sqrt{2/3R_1})^2 \), we obtain \( t = \sqrt{1077}/(24\sqrt{2}) \). Projecting
radially \( \mu|_{B(0,R_1)} \) to \( \nu = \mu(B(0,R_1))^{d-1} \) on \( S^{d-1} \), and likewise projecting radially \( C \) onto \( C(s,e_1) \), from Lemma 2.1 we obtain

\[
(34) \quad \mu((B(0,R_1) \setminus B(0,uR_1)) \cap B(R_1e_1,H)) \leq \nu(C(s,e_1)) \leq \mu(B(0,R_1))t^dO\left( \frac{1}{\sqrt{d}} \right).
\]

The preceding estimates, together with \( t = \sqrt{1077/(24\sqrt{2})} < (55/64)^{1/6} \), entail that

\[
(35) \quad \mu(B(R_1e_1,H) \cap \{ x_1 \leq R_1 \}) \leq \mu(B(0,R_1)) \left( \frac{55}{64} \right)^{d/6} \left( \frac{1}{1 - \varepsilon} + \Theta \left( \frac{1}{\sqrt{d}} \right) \right),
\]

and we already know from (28) that \( \mu(B(R_1e_1,H)) \) is at most twice as large. Since by Proposition 3.1, \( \mu(B(0,R)) \leq \nu^{-d}\mu(B(0,vR)) \) for every \( v \in (0,1) \) and every \( R > 0 \), we can apply Lemma 3.3 with \( t_0 = 1 \), \( R = R_1 \) and \( v_0 = 1/2 \). This yields

\[
(36) \quad c_{p,d,\mu} \geq \frac{(2^{1/p}55^{-1/6})^d}{1 - \varepsilon} + \Theta \left( \frac{1}{\sqrt{d}} \right).
\]

Setting \( 2^{1/p}55^{-1/6} = 1 \), we find the solution \( p_0 = (6 \log 2)/\log 55 \approx 1.03782 \). Observing that \( c_{p,d,\mu} \) does not depend on our choice of \( \varepsilon \), the result follows by letting \( \varepsilon \to 0 \).

\[ \square \]

**Remark 3.5.** For \( p \leq 1.03 \), \( 2^{1/p}55^{-1/6} > 1.005 \). Thus, if \( d \) is “high”, \( c_{p,d,\mu} \geq 1.005^d \). How high must \( d \) be can be explicitly determined from the proof, by keeping track of the constants in Lemma 2.1 instead of writing \( \Theta(1/\sqrt{d}) \). Note also that in the specific case \( p = 1 \), the preceding theorem is more general and gives a better bound (since \( 55^{-1/6} > (2/\sqrt{3})^{1/6} \)) than [A1] Theorem 2.3], even though \( \chi_{B(0,R_1/2)} \) is a very poor choice when \( p = 1 \) (using \( \delta_0 \) is much more efficient). We shall explore the case \( p = 1 \) in more detail elsewhere.

**Remark 3.6.** The hypotheses contained in (26) are selected so that all finite, radial, radially decreasing measures with bounded densities are included, and still a concrete range for \( p \) is obtained. Numerically, \( t_0 := \frac{6 \log 2 - \log 55}{3 \log 3 - 3 \log 2} \approx 0.1246 \). Provided that the singularity at 0 is not too strong, Theorem 3.4 also applies to measures with unbounded densities. In particular, it applies to all measures defined via (15), with \( f_i(r) = r^{-td}\chi_{[0,1]}(r) \) and \( t \in (0,1 - t_0] \). This last condition comes from the fact that for these measures, \( h_u(R) = u^{-(1-t)d} \) when \( R \leq 1 \).

For infinite measures, however, (26) can be rather restrictive. Define \( \mu_{t,d} \) as in the preceding remark but without truncation, i.e., using \( f_i(r) = r^{-td} \). Then the theorem applies only to \( t = 1 - t_0 \). Observe, however, that values different from \( t_0 \) and \( \sqrt{2/3} \) could have been used, with the same qualitative results. Thus, a simple way to obtain a theorem covering an infinite subfamily of the measures \( \mu_{t,d} \) is to assume different rates of growth for the sup and the lim sup in (26). The proof of the next result is essentially identical to that of Theorem 3.4, so it will be omitted. We use \( u = \sqrt{2/3} \) to be able to apply the same splitting of the ball centered at \( R_1e_1 \), but other values are possible. Also, the upper bound given below for \( r_1 \) can be modified, by suitably choosing a different value for \( u \). Recall that \( f : (0,\infty) \to [0,\infty) \) is nonincreasing and that \( \mu \) is defined by \( f \) via (15).
Theorem 3.7. Fix $d \in \mathbb{N} \setminus \{0\}$, choose $t_0 \in (0,1)$, $t_1 \in (0, \log(64/55)/\log(9/4))$, and set $u = \sqrt{2/3}$. Then there exists a $p_0 = p_0(t_0,t_1) > 1$ with the following property: For all $p \in [1,p_0)$ we can find a $b(p,t_0,t_1) > 1$, such that for every measure $\mu$ satisfying $\sup_{R>0} h_u(R) \geq u^{-t_1 d}$ and $\limsup_{R \to \infty} h_u(R) \leq u^{-t_1 d}$, we have $C_{p,d,\mu} \geq c_{p,d,\mu} \geq \Theta(1)$.

Remark 3.8. If $t_0 < t_1$, then the preceding result covers all the measures $\mu_{t,d}$ defined by $f_t(r) = r^{-td}$ such that $t_0 \leq 1 - t \leq t_1$.

Returning to Theorem 3.4, it admits a simpler statement when $f$ is bounded and $f(x)x^{d-1} \in L^1(0,\infty)$, so $\mu$ is finite. By Proposition 3.1, the conditions $\sup_{R>0} h_u(R) \geq u^{-t_1 d}$ and $\limsup_{R \to \infty} h_u(R) \leq u^{-t_1 d}$ are automatically satisfied for all $t_0, t_1, u \in (0,1)$.

Corollary 3.9. Fix $d \in \mathbb{N} \setminus \{0\}$. Suppose $f$ is bounded and $f(x)x^{d-1} \in L^1(0,\infty)$. If $\mu$ is the finite measure defined via (15), then for every

$$p \in \left[\frac{1.6 \log 2}{\log 55}, \frac{1}{4 + \Theta(\frac{1}{\sqrt{d}})} \right]$$

we have $C_{p,d,\mu} \geq c_{p,d,\mu} \geq \frac{1}{4 + \Theta(\frac{1}{\sqrt{d}})} \left(\frac{2^{1/p}}{\sqrt{55}}\right)^d$.

Example 3.10. When dealing with concrete families of measures it is possible to obtain tighter bounds. We revisit the example from [A1, Remark 2.7], adapting the arguments given there to $p > 1$. Let $\nu_d(A) := \lambda^d(A \cap B^d)$ be Lebesgue measure restricted to the unit ball. We apply Lemma 3.3 with $R = 1$ and $v = 1/2$, so $H = 2^{-1}\sqrt{5}$, and $B^d \cap B(e_1, H)$ is the union of two solid spherical caps, the largest of which is $B^d \cap \{x_1 \geq 3/8\}$ (since the smaller sphere has larger curvature). Solving $x_1^2 + x_2^2 = 1$ and $(x_1 - 1)^2 + x_2^2 = H^2$ we obtain

$$\nu_d(B(e_1, 2^{-1}\sqrt{5})) \leq 2\lambda^d(B^d \cap \{x_1 \geq 3/8\}) = 2\lambda^{d-1}(B^{d-1}) \int_{3/8}^1 \left(\sqrt{1 - x_1^2}\right)^{d-1} dx_1$$

$$\leq \frac{16\lambda^{d-1}(B^{d-1})}{3} \int_{\arcsin(3/8)}^{\pi/2} \cos^d t \sin t dt = \frac{16}{3(d+1)} \left(\frac{\sqrt{55}}{8}\right)^{d+1} \lambda^{d-1}(B^{d-1}).$$

By Lemma 3.3

$$c_{p,d,\nu_d} \geq \left(\frac{1}{2}\right)^{d/q} \frac{\lambda^d(B^d)}{2\lambda^{d-1}(B^{d-1})} \frac{3(d+1)}{16} \left(\frac{8}{\sqrt{55}}\right)^{d+1} = \Theta(\sqrt{d}) \left(\frac{2^{1/p}}{\sqrt{55}}\right)^d.$$ 

Setting $2^{1/p} = \sqrt{55}$ and solving for $p$ we obtain that $c_{p,d}$ grows exponentially fast with $d$ whenever

$$p < \left(\frac{\log 55}{2\log 2 - 2}\right)^{-1} \approx 1.1227.$$ 

Remark 3.11. It is possible to present more involved arguments in Theorem 3.4 and in Example 3.10 by trying to optimize in $v \in (0,1]$ instead of simply using $v = 1/2$. But this does not seem to significantly improve the value of $p$. Specifically, using $B(e_1, H)$ with $H = \sqrt{1 + v^2}$ in Example 3.10, the same steps followed above lead us to maximize
Assume that \( p \) increase holds even when \( q \) is large and all \( v \in [0,1] \), \( g(v,q) < 1 \). Thus, with the methods of the present paper we cannot get exponential increase in Example 3.10 for any \( p \geq 9/8 = 1.125 \), which is very close to \( p_0 \approx 1.1227 \). Even the general bound \( p_0 \approx 1.0378 \) from Theorem 3.4 is not far from 1.125.

We mention that although for small values of \( q \), say, \( q \approx 10 \) it is better to consider as our \( L^p \) function \( \chi_{B(0,v)} \) with \( v \approx 1/2 \), in order to maximize \( g(v,q) \) as \( q \to \infty \), we must let \( v \to 0 \). Of course, at the endpoint value \( p = 1 \), the Dirac delta measure \( \delta_0 \) is a better choice than all the functions \( \chi_{B(0,v)}, v \in (0,1) \).

In general, good upper bounds for \( C_{p,d} \) and \( C_{p,d} \) are easier to establish when \( \mu \) is doubling, that is, when there exists an absolute constant \( C \) such that for all \( x \in \mathbb{R}^d \) and all \( R > 0 \),

\[
\mu(B(x,2R)) \leq C \mu(B(x,R)).
\]

The doubling condition captures the property of Lebesgue measure that yields weak type bounds via covering lemmas of Vitali type. It might be expected that arbitrary doubling measures would behave like Lebesgue measure, but in our context this is not the case: There is a collection of doubling measures for which exponential increase holds even when \( p \) is high. For all \( t \in (0,1) \), let \( \mu_{t,d} \) be defined on \( \mathbb{R}^d \) by \( d\mu_{t,d} := ||x||^{-td}dx \), and consider the families \( \mathcal{M}_t := \{ \mu_{t,d} : d \in \mathbb{N} \setminus \{0\} \} \). It is well known that the measures \( \mu_{t,d} \) are indeed doubling, cf. for instance [St3, 2.7, p. 12]. For simplicity, instead of \( C_{p,d,\mu_{t,d}} \) and \( C_{p,d,\mu_{t,d}} \) we shall write \( C_{p,d} \) and \( C_{p,d} \) to denote the best strong type and weak type \((p,p)\) constants for the measures in \( \mathcal{M}_t \).

**Theorem 3.12.** Fix \( p_0 \in [1,\infty) \). Then there exist constants \( t_0 = t_0(p_0) \in (0,1) \) and \( b_0 = b_0(p_0) > 1 \) such that for all \( p \in [1,p_0] \) and all \( t \in [t_0,1) \), we have \( C_{p,d} \geq b_0 \).
Solving $x_1^2 + x_2^2 = 1$ and $(x_1 - 1)^2 + x_2^2 = 5/4$ shows that $B(0,1)^c \cap B(e_1,H)$ is contained in the cone subtended by the cap $C(3/8,e_1)$. The distance from the boundary of this cap to the $x_1$ axis is $x_2 = \sqrt{55}/8$. Thus, Lemma 2.1, together with integration in polar coordinates from 1 to $1+2^{-1}\sqrt{5}$ in the radial variable and over the said cap in the angular variable, yield

\[ \mu_{t,d}(B(0,1)^c \cap B(e_1,H)) \leq \left( \frac{\sqrt{55}}{8} \right)^{1/(1-t)} \left( 1 + 2^{-1}\sqrt{5} \right)^{(1-t)d} \frac{\sigma^{d-1}(S^{d-1})}{(1-t)d\Theta(d)}. \]

Select $d_0 = d_0(c)$ such that if $d \geq d_0$, the expressions $1/\Theta(\sqrt{d})$ in (40) and (41) coming from Lemma 2.1 are bounded above by 1. As $t \to 1$, both $x_2(c)^{1/(1-t)} \to 0$ and $(\sqrt{55}/8)^{1/(1-t)} \to 0$, so by choosing $t_0$ close enough to 1, we can make the term in (39) larger than those in (40) and (41) for every $d \geq d_0$ and all $t \in [t_0,1)$. Hence,

\[ \mu_{t,d}(B(e_1,H)) \leq 3\mu_{t,d}(B(0,c/2)) = \frac{3\sigma^{d-1}(S^{d-1})c^{(1-t)d}}{(1-t)d2^{(1-t)d}}. \]

By Lemma 3.3 we obtain $c_{p,d,t} \geq \frac{1}{6} \left( \frac{1}{c} \right)^{(1-t)d/q} \left( \frac{2}{c} \right)^{(1-t)d} = \frac{1}{6} \left( \frac{2^{1/p}}{c} \right)^{(1-t)d} \geq \frac{1}{6} \left( \frac{2^{1/p_0}}{c} \right)^{(1-t)d}$ for every $p \in [1,p_0]$. Finally, by the choice of $c$, the inequality $b_0 := \min \{ 6^{1/d_0}, c^{-1}2^{1/p_0} \} > 1$ holds, so we have exponential increase of the best constants in $d$ (the term $6^{1/d_0}$ has been included to account for small values of $d$).

**Example 3.13.** Define $\mu_v$ on $\mathbb{R}^d$ as the sum of area measure on $S$ plus Lebesgue on $B^d(0,v)$. If $v = 1$, then the arguments used in this paper apply and we do get exponential growth of $c_{p,d}$ for sufficiently small values of $p > 1$. However, suppose we let $v \to 0$; taking $\chi_{B(0,r)}$ as our $L^p$ function, we see that having $v < r < 1$ offers no advantage over $0 < r \leq v$, so $r \to 0$ as $v \to 0$. This forces us to let $q = q(r) \to \infty$ in Lemma 3.3 and we do not obtain a uniform value of $p$ for this family. While the measures $\mu_v$ are not absolutely continuous, by taking $f_v = \chi_{[0,v]} + \chi_{[1-1/d,1]}$ we observe the same phenomenon for densities. Thus, additional hypotheses are needed in order to go beyond radially decreasing densities.

**References**

[A1] Aldaz, J. M. *Dimension dependency of the weak type $(1,1)$ bounds for maximal functions associated to finite radial measures*. Bull. Lond. Math. Soc. 39 (2007) 203–208. Available at the Math. ArXiv.

[A2] Aldaz, J. M. *The weak type $(1,1)$ bounds for the maximal function associated to cubes grow to infinity with the dimension*. Available at the Math. ArXiv.

[AlPe1] Aldaz, J. M., Pérez Lázaro, J. *Functions of bounded variation, the derivative of the one dimensional maximal function, and applications to inequalities*. Trans. Amer. Math. Soc. 359 (5) (2007), 2443–2461. Available at the Math. ArXiv.

[AlPe2] Aldaz, J. M.; Pérez Lázaro, J. *Boundedness and unboundedness results for some maximal operators on functions of bounded variation*. J. Math. An. Appl. Volume 337, Issue 1, (2008) 130–143. Available at the Math. ArXiv.

[AlPe3] Aldaz, J. M.; Pérez Lázaro, J. *Regularity of the Hardy-Littlewood maximal operator on block decreasing functions*. Studia Math. 194 (3) (2009) 253–277. Available at the Math. ArXiv.
Behavior of weak type bounds

[AIPe4] Aldaz, J. M.; Pérez Lázaro, J. The best constant for the centered maximal operator on radial functions. To appear, Math. Ineq. Appl.; available at the Math. ArXiv.

[Au] Aubrun, G. Maximal inequality for high-dimensional cubes. Confluentes Mathematici, Volume 1, Issue 2, (2009) pp. 169–179, DOI No: 10.1142/S1793744209000067. Available at the Math. ArXiv.

[Bou1] Bourgain, J. On high-dimensional maximal functions associated to convex bodies. Amer. J. Math. 108 (1986), no. 6, 1467–1476.

[Bou2] Bourgain, J. On the $L^p$-bounds for maximal functions associated to convex bodies in $R^n$. Israel J. Math. 54 (1986), no. 3, 257–265.

[Bou3] Bourgain, J. On dimension free maximal inequalities for convex symmetric bodies in $R^n$. Geometrical aspects of functional analysis (1985/86), 168–176, LNM, 1267, Springer, Berlin, 1987.

[Ca] Carbery, A. An almost-orthogonality principle with applications to maximal functions associated to convex bodies. Bull. Amer. Math. Soc. (N.S.) 14 (1986), no. 2, 269–273.

[Cr] Criado, A. On the lack of dimension free estimates in $L^p$ for maximal functions associated to radial measures. Available at the Math. ArXiv.

[Gra] Gray, A. Tubes (Addison-Wesley Publishing Company, 1990).

[Mu] Müller, D. A geometric bound for maximal functions associated to convex bodies. Pacific J. Math. 142 (1990), no. 2, 297–312.

[NaTa] Naor, A.; Tao, T. Random martingales and localization of maximal inequalities. To appear in J. Func. Anal. Available at the Math. ArXiv.

[St1] Stein, E. M. The development of square functions in the work of A. Zygmund. Bull. Amer. Math. Soc. (N.S.) 7 (1982), no. 2, 359–376.

[St2] Stein, E. M. Three variations on the theme of maximal functions. Recent progress in Fourier analysis (El Escorial, 1983), 229–244, North-Holland Math. Stud., 111, North-Holland, Amsterdam, 1985.

[St3] Stein, E. M. Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals. Princeton University Press, Princeton, NJ, 1993.

[StSt] Stein, E. M.; Strömberg, J. O. Behavior of maximal functions in $R^n$ for large $n$. Ark. Mat. 21 (1983), no. 2, 259–269.

[Su] Sullivan, John M. Sphere packings give an explicit bound for the Besicovitch covering theorem, J. Geom. Anal. 4 (1994), no. 2, 219–231.

[Web] Webster, R. J. Convexity (Oxford University Press, 1997).

Departamento de Matemáticas, Universidad Autónoma de Madrid, Cantoblanco 28049, Madrid, Spain.

E-mail address: jesus.munarriz@uam.es

Departamento de Matemáticas y Computación, Universidad de La Rioja, 26004 Logroño, La Rioja, Spain.

E-mail address: javier.perezl@unirioja.es