IRREDUCIBILITY OF THE HILBERT-BLUMENTHAL MODULI
SPACES WITH PARAHORIC LEVEL STRUCTURE

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Abstract. We determine the number of irreducible components of the reduction mod $p$ of any Hilbert-Blumenthal moduli space with a parahoric level structure, where $p$ is unramified in the totally real field.

1. Introduction

In their 1984 paper [1], Brylinski and Labesse computed the $L$-factors of Hilbert-Blumenthal moduli spaces for almost all good places. By that time the arithmetic minimal compactification was not known. In [2] Chai furnished the desired minimal compactification by observing that Rapoport’s arithmetic toroidal compactification [23] plays the crucial role. Thus, the results of Brylinski and Labesse have been improved for all good places (see [7, p. 137]). A next task is to treat the case where $p$ is unramified and the level group $K_p$ at $p$ is a standard Iwahori subgroup. This moduli space is studied in Stamm [28], following the works of Zink [35] and of Rapoport-Zink [24]. Several local properties on geometry as well as fine global descriptions of the surface case have been obtained in [28]. In this paper we settle a global problem concerning the irreducibility in this moduli space.

Let $p$ be a fixed rational prime. Let $F$ be a totally real number field of degree $g$ and $O_F$ the ring of integers. Let $n \geq 3$ be a prime-to-$p$ integer. Choose a primitive $n$th root $\zeta_n$ of unity in $\overline{\mathbb{Q}} \subset \mathbb{C}$ and an embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{Q}_p$. Let $(L, L^+)$ be a rank one projective $O_F$-module with a notion of positivity. Let $M_{(L, L^+), n}$ denote the moduli space over $\mathbb{Z}((\zeta_n))$ that parametrizes equivalence classes of objects $\mathcal{A} = (A, i, i, \eta)$ over a locally Noetherian $\mathbb{Z}((\zeta_n))$-scheme $S$, where

- $A$ is an abelian scheme of relative dimension $g$;
- $i : O_F \to \text{End}_S(A)$ is a ring monomorphism;
- $i : (L, L^+)S \to (\mathcal{P}(A), P(A)^\dagger)$ is a morphism of étale sheaves such that the induced morphism

$$L \otimes_{O_F} A \to A^\dagger$$

is an isomorphism, where $(\mathcal{P}(A), P(A)^\dagger)$ is the polarization sheaf of $A$ (see [5]):

- $\eta : (O_F/nO_F)_{\zeta_n}^2 \simeq A[n]$ is an $O_F$-linear isomorphism such that the pull back of the Weil pairing $e_i(\lambda_0)$ is the standard pairing on $(O_F/nO_F)^2$ with respect to $\zeta_n$, where $\lambda_0$ is any element in $L^+$ such that $|L/O_F \lambda_0|$ is prime to $pn$.

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It is proved in Rapoport [23] and Deligne-Pappas [5] that

**Theorem 1.1 (Rapoport, Deligne-Pappas).** The fibers of $M_{(L,L^+),n} \to \text{Spec } \mathbb{Z}[[\zeta_n]]$ are geometrically irreducible.

In the paper we consider the Iwahori level structure $M_{(L,L^+),\Gamma_0(p),n} \to \text{Spec } \mathbb{Z}[[\zeta_n]]$ over $M_{(L,L^+),n}$ where $M_{(L,L^+),n}$ has good reduction at $p$. The goal is to determine the set of irreducible components of the reduction $M_{(L,L^+),\Gamma_0(p),n} \otimes \bar{\mathbb{F}}_p$ modulo $p$. We write $\Pi_0(X)$ for the set of irreducible components of a Noetherian scheme $X$. Thus, if $X$ is a scheme of finite type over a field $K$, then $\Pi_0(X \otimes K)$ is in bijection with the set of geometrically irreducible components of $X$. The latter is the case considered in this paper.

Assume that $p$ is unramified in $F$. Let $M_{(L,L^+),\Gamma_0(p),n}$ denote the moduli space over $\mathbb{Z}_p[[\zeta_n]]$ that parametrizes equivalence classes of objects $(A,i,\iota,H,\eta)$, where

- $(A,i,\iota,\eta)$ is in $M_{(L,L^+),n}$, and
- $H \subset A[p]$ is a finite flat rank $p^g$ subgroup scheme which is invariant under the action of $O_F$ and maximally isotropic with respect to the Weil pairing $e_i(\lambda_0)$ as above.

Write $M := M_{(L,L^+),n} \otimes \bar{\mathbb{F}}_p$ and $M_{\Gamma_0(p)} := M_{(L,L^+),\Gamma_0(p),n} \otimes \bar{\mathbb{F}}_p$ throughout this paper. We will state our main results concerning the number $|\Pi_0(M_{\Gamma_0(p)})|$ in the next section. We describe them together with background and methods. See Theorem 2.9 and Theorem 5.5 for the precise statement.

The method in this paper is completely different from that used in [32] for the Siegel moduli spaces. In the previous paper the proof is based on the Faltings-Chai theorem on the $p$-adic monodromy for the ordinary locus [7] and a theorem proved by Ngô and Genestier [16] that the ordinary locus is dense in the parahoric level moduli spaces. The latter is obtained by analyzing the so called Kottwitz-Rapoport stratification introduced in [12] (cf. [16]).

For the present situation, the ordinary locus is no longer dense, as has been pointed out in Stamm [28] in the surface case. Thus Ribet’s $p$-adic monodromy result [25] can only conclude the irreducibility for ordinary components. One may need to establish the surjectivity result of the naive $p$-adic monodromy for smaller strata in $M$ defined by certain $p$-adic invariant, which is not available yet. However, even though we can prove these $p$-adic monodromy results, the standard $p$-adic monodromy argument is still insufficient to conclude the irreducibility of non-ordinary components. Indeed, the naive $p$-adic monodromy result implies the connectedness of covers of a stratum arising from the etale $p^m$-torsion part, and these covers are always finite. On the other hand, a possibly irreducible non-ordinary component of $M_{\Gamma_0(p)}$ is necessarily having positive dimensional fibers over a stratum in $M$, and hence cannot be dominated by a finite cover. Besides the non-density, we do not have yet geometric properties for Kottwitz-Rapoport strata of $M_{\Gamma_0(p)}$ along the direction of work of Ngô-Genestier [16].
To overcome these new difficulties, we stratify the moduli space by a suitable \( p \)-adic invariant:

\[
\mathcal{M}_{\Gamma_0(p)} = \prod_{\alpha} \mathcal{M}_{\Gamma_0(p),\alpha}.
\]

Then we study the corresponding \textit{discrete Hecke orbit problem}, namely asking whether the prime-to-\( p \) Hecke correspondences operate transitively on the set \( \Pi_0(\mathcal{M}_{\Gamma_0(p),\alpha}) \).

This discrete Hecke orbit problem, though itself does not have an affirmative answer, can be refined through the computation of the fibers of the stratified morphism \( f_\alpha : \mathcal{M}_{\Gamma_0(p),\alpha} \to \mathcal{M}_\alpha \), and is reduced to the discrete Hecke orbit problem for the set \( \Pi_0(M_\alpha) \) of irreducible components of the base. The former one can be done using Dieudonné calculus, for which the present computation (see Sections 3 and 4) is largely based on the work [30].

The next crucial ingredient is Chai’s monodromy theorem on Hecke invariant subvarieties. This is a global method which may be regarded as the counterpart of the \( p \)-adic monodromy method. Its original form for Siegel moduli spaces is developed by Chai [4]. Chai’s method works for good reductions of any modular variety of PEL-type, with the modification where the reductive group in the Shimura input data should be replaced by the simply-connected cover of its derived group [4, p. 291]. We supply the proof due to Chai in Section 6 for the reader’s convenience. This ingredient enables us to confirm the irreducibility in the non-supersingular contribution (components that contain non-supersingular points). To treat the remaining supersingular contribution (components that are entirely contained in the supersingular locus), the tool is essentially the result that the Tamagawa number is one for semi-simple, simply-connected algebraic groups [11]. The present cases heavily rely on the computations of the \textit{geometric mass formula} in [33], which are based on work [26] of Shimura.

The paper is organized as follows. In Section 2 we describe the main theorems and provide the methods and ingredients. In Section 3 we give the proofs of the theorems. In Section 4 we treat the supersingular contribution. To make the exposition clean and more accessible, we assume \( p \) inert in \( F \) in these sections. In Sections 5 we show how to establish the analogous results in the unramified situation from the inert case. Section 6 provides a proof of Chai’s result on Hecke invariant subvarieties in the Hilbert-Blumenthal moduli spaces. We attempted to write this as an independent section so that the reader can read this section alone together with Chai’s well written paper [4].

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2. Statements and methods

2.1. We keep the notation as in the previous section. Let \( k \) be an algebraically closed field of characteristic \( p \). We will assume in Sections 2-4 that \( p \) is inert in \( F \).

Write

\[
f : M_{\Gamma_0(p)} \to M
\]

for the forgetful morphism; this is a proper surjective morphism.

We recall the alpha stratification on the moduli space \( M \) introduced in Goren and Oort [8] and in [30] Section 3. Let \( W := W(k) \) be the ring of Witt vectors over \( k \) and \( \sigma \) the absolute Frobenius map on \( W \). Put \( O := O_F \otimes \mathbb{Z}_p \) and let \( J := \text{Hom}(O, W) = \{ \sigma_i \} \) be the set of embeddings, arranged in a way that \( \sigma_i = \sigma_{i+1} \) for \( i \in \mathbb{Z}/g\mathbb{Z} \). Let \( A = (A, \iota) \) be an abelian \( O_F \)-variety over \( k \), and let \( \overline{M} \) be the associated covariant Dieudonné \( O \)-module. The alpha type of \( \underline{A} \) is defined to be

\[
a(A) := a(M) := (a_i)_{i \in \mathbb{Z}/g\mathbb{Z}}, \quad \text{where} \quad a_i := \dim_k(M/(F,V)M)^i,
\]

Here \((M/(F,V)M)^i\) denotes the \( \sigma_i \)-component of the \( k \)-vector space \( M/(F,V)M \).

Since \( M \) is a free \( O_F \otimes W \)-module of rank two, each \( a_i \) lies in \( \{0, 1, 2\} \). If \( \underline{A} = (A, \iota, \eta) \) is a point in \( M(k) \), the alpha type \( a(\underline{A}) \) of \( \underline{A} \) is defined to be the alpha type of its underlying abelian \( O_F \)-variety \( (A, \iota) \). In this case each \( a_i \) is 0 or 1, since \( \text{Lie}A \) satisfies the Rapoport condition.

For each \( a \in \{0, 1\}^2 \), let \( M_a \) denote the reduced subscheme of \( M \) consisting of points with alpha type \( \underline{a} \). It is known [8] that every alpha stratum \( M_a \) is non-empty. Put \( \Delta := \{0, 1\}^2 \); this is exactly the set of alpha types that occur in \( M \). The partial order on \( \Delta \) is given by \( \underline{a}' \preceq \underline{a} \) if and only if \( a'_i \geq a_i \) for all \( i \in \mathbb{Z}/g\mathbb{Z} \). Write \( \sigma(\underline{a}) := \sum_{i \in \mathbb{Z}/g\mathbb{Z}} a_i \), the size of the alpha type \( \underline{a} \). It is proved in Goren and Oort [8] that each stratum \( M_a \) is smooth, quasi-affine, of pure dimension \( g - \sigma(\underline{a}) \), and that the Zariski closure of \( \overline{M_a} \) in \( M \) is smooth and

\[
\overline{M_a} = \bigcup_{\underline{a}' \preceq \underline{a}} M_{\underline{a}'}.
\]

Thus, we have a stratification of \( M \)

\[
M = \coprod_{\underline{a} \in \Delta} M_{\underline{a}}
\]

by locally closed smooth subschemes \( M_{\underline{a}} \), called the alpha stratification.

For each \( \underline{a} \in \Delta \), put \( M_{\Gamma_0(p), \underline{a}} := M_{\Gamma_0(p)} \times_M M_{\underline{a}} \) and let

\[
f_{\underline{a}} : M_{\Gamma_0(p), \underline{a}} \to M_{\underline{a}}
\]

be the restriction of \( f \) on \( M_{\Gamma_0(p), \underline{a}} \). We decompose the moduli space

\[
M_{\Gamma_0(p)} = \coprod_{\underline{a} \in \Delta} M_{\Gamma_0(p), \underline{a}}
\]

into locally closed subschemes.

In [30] Section 2] an alpha type \( \underline{a} = (a_i) \in \Delta \) is called generic if \( a_i a_{i+1} = 0 \) for all \( i \in \mathbb{Z}/g\mathbb{Z} \). For example, when \( g = 4 \), the generic alpha types are

\[
(0, 0, 0, 0), (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0),
\]

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\]
This notion was first introduced by Goren and Oort in which it is called spaced. It is used in order to describe the Newton points of maximal points of E-O strata (see [8, Theorem 5.4.11], also see Theorem 2.7). Conversely, it is proved in [30, Section 6] that the alpha type of any maximal point of a Newton stratum of \(M\) is generic. Let \(\Delta^\text{gen} \subset \Delta\) denote the set of generic alpha types.

In this paper we prove

**Theorem 2.1.** We have \(\dim M_{1_0(p),\underline{a}} = g\) if and only if \(\underline{a}\) is of generic type.

Since the moduli space \(M_{1_0(p)}\) is equi-dimensional of dimension \(g\) [28, Theorem 1, p. 407], each stratum \(M_{1_0(p),\underline{a}}\) has dimension less than or equal to \(g\). Theorem 2.1 asserts that the equality holds exactly when \(\underline{a}\) is of generic type. An immediate consequence is the following

**Corollary 2.2.** The image of the set of maximal points of \(M_{1_0(p)}\) under the morphism \(f\) (2.1) are exactly that of maximal points of all generic alpha strata \(M_{\underline{a}}\).

For each alpha type \(\underline{a} = (a_i) \in \Delta\), denote by \(\tau(\underline{a}) \subset \mathbb{Z}/g\mathbb{Z}\) the subset consisting of elements \(i\) such that \(a_i = 1\). The subset \(\tau(\underline{a})\) is called the alpha index corresponding to \(\underline{a}\). Write \(\tau(\underline{a}) = \{n_1, \ldots, n_a\}\) with \(0 \leq n_i < n_{i+1} < g\) and put \(n_{a+1} = g + n_1\).

Define a function \(w : \Delta \rightarrow \mathbb{Z}\) by

\[
(2.2) \quad w(\underline{a}) := w(\tau(\underline{a})) := \begin{cases} 2 & \text{if } \tau(\underline{a}) = \emptyset; \\ \prod_{j=1}^{a}(n_{j+1} - n_j - 1) & \text{otherwise.} \end{cases}
\]

It is clear that \(w(\underline{a}) > 0\) if and only if \(\underline{a} \in \Delta^\text{gen}\).

**Theorem 2.3.** Let \(\underline{a}\) be a generic alpha type.

1. For any point \(x \in M_{\underline{a}}(k)\), the fiber \(f^{-1}(x)\) has \(w(\underline{a})\) irreducible components of dimension \(|\underline{a}|\).
2. The subscheme \(M_{1_0(p),\underline{a}}\) has \(w(\underline{a})|\Pi_0(M_{\underline{a}})|\) irreducible components of dimension \(g\).

We remark that the fiber \(f^{-1}(x)\) is not equi-dimensional in general; see Example 3.2. Similarly, some component of \(M_{1_0(p),\underline{a}}\) may have dimension less than \(g\). By Theorems 2.1 and 2.3 (2), we get

\[
(2.3) \quad |\Pi_0(M_{1_0(p)})| = \sum_{\underline{a} \in \Delta^\text{gen}} w(\underline{a})|\Pi_0(M_{\underline{a}})|.
\]

In the next step we consider the \(\ell\)-adic Hecke correspondences operating on the set \(\Pi_0(M_{\underline{a}})\) of irreducible components, where \(\ell \neq p\) is a prime.

For any non-negative integer \(m \geq 0\), let \(H_{\ell,m}\) be the moduli space over \(\mathbb{F}_p\) that parametrizes equivalence classes of objects \((A_j = (A_j, i_j, \ell_j, \eta_j), j = 1, 2, 3; \varphi_1, \varphi_2)\) as the diagram

\[
A_1 \xrightarrow{\varphi_1} A_3 \xrightarrow{\varphi_2} A_2,
\]

where

- \(A_1\) and \(A_2\) are objects in \(\mathcal{M}\), and \(A_3\) is a \(g\)-dimensional abelian \(O_F\)-variety with a class of polarizations and a symplectic level-\(n\) structure as defined in Section 1 but the condition (1.1) is not required;
the morphisms $\varphi_1$ and $\varphi_2$ are $O_F$-linear isogenies of degree $\ell^m$ such that
$(\varphi_j)^*i_j = i_3$ (pull back the polarizations) and $(\varphi_j)_*\eta_3 = \eta_j$ (pushfoward the level structures) for $j = 1, 2$.

Let $\mathcal{H}_\ell := \bigcup_{m \geq 0} \mathcal{H}_{\ell,m}$. An $\ell$-adic Hecke correspondence is given by an irreducible component $\mathcal{H}$ of $\mathcal{H}_\ell$ together with natural projections $\text{pr}_1$ and $\text{pr}_2$. A subset $Z$ of $M$ is called $\ell$-adic Hecke invariant if $\text{pr}_2(\text{pr}_1^{-1}(Z)) \subset Z$ for any $\ell$-adic Hecke correspondence $(\mathcal{H}, \text{pr}_1, \text{pr}_2)$. If $Z$ is an $\ell$-adic Hecke invariant, locally closed subset of $M$, then the $\ell$-adic Hecke correspondences induce correspondences on the set $\Pi_0(Z)$ of irreducible components. We call $\Pi_0(Z)$ $\ell$-adic Hecke transitive if the $\ell$-adic Hecke correspondences operate transitively on $\Pi_0(Z)$. The discrete Hecke problem for any $\ell$-adic Hecke invariant subscheme $Z$ is asking whether $\Pi_0(Z)$ is $\ell$-adic Hecke transitive.

**Theorem 2.4** (Chai). Let $Z$ be an $\ell$-adic Hecke invariant subscheme of $M$. If the set $\Pi_0(Z)$ is $\ell$-adic Hecke transitive and maximal points of $Z$ are not supersingular, then $Z$ is irreducible.

Notice that the formulation of Theorem 2.4 does not require our assumption on $p$ and Theorem 2.4 remains valid without this assumption (see Section 6).

Since the alpha type is an invariant under isogenies of prime-to-$p$ degree, each alpha stratum $M_{\underline{a}}$ is $\ell$-adic Hecke invariant. The following result is due to Goren and Oort [8, Corollary 4.2.4]

**Theorem 2.5.** For any alpha type $\underline{a}$, the set $\Pi_0(M_{\underline{a}})$ is $\ell$-adic Hecke transitive.

An alpha stratum $M_{\underline{a}}$ is called supersingular if it is contained in the supersingular locus. This is equivalent to that all of its maximal points are supersingular. An alpha type $\underline{a}$ is called supersingular if the corresponding stratum $M_{\underline{a}}$ is so. It follows from Theorems 2.4 and 2.5 that

**Corollary 2.6.** Any non-supersingular stratum $M_{\underline{a}}$ is irreducible.

2.2. It remains to treat the supersingular contribution in (2.3). In the following we describe all supersingular strata $M_{\underline{a}}$, not just for generic ones. This is slightly more than what we need.

For $j = g/2$ or a non-negative integer with $0 \leq j \leq g/2$, write $s(j, g)$ for the slope sequence
$$\left\{ \frac{j}{g}, \ldots, \frac{j}{g}, \frac{g-j}{g}, \ldots, \frac{g-j}{g} \right\}$$
with each multiplicity $g$. The set of all such $s(j, g)$’s is the set of slope sequences (or Newton polygons) which are realized by points in $M$ (see [31] Lemma 3.1 and Theorem 7.4).

We recall the following result in Goren and Oort [8, Theorem 5.4.11].

**Theorem 2.7.** The generic point of each component of $M_{\underline{a}}$ has slope sequence $s(\lambda(\underline{a}), g)$ except when $g$ is odd and $|\underline{a}| = g$, where
$$\lambda(\underline{a}) := \{ |\underline{b}| : \underline{a} \leq \underline{b} \text{ and } \underline{b} \text{ is generic} \}.$$

In the except case where $g$ is odd and $|\underline{a}| = g$, the alpha stratum $M_{\underline{a}}$ is the superspecial locus. One can characterize easily from Theorem 2.7 when an alpha stratum $M_{\underline{a}}$ is supersingular. When $\underline{a}$ is of generic type, the stratum $M_{\underline{a}}$ is supersingular if and only if $g = 2d$ is even and $|\underline{a}| = d$. They correspond to alpha
Choose and fix a non-zero element \( \lambda_0 \) in \( L^+ \) so that \( (|L/O_L\lambda_0|, np) = 1 \). Let \( x \) be any point in \( \mathcal{M}(L,L^+,n)(\mathbb{C}) \). One associates a skew-Hermitian \( O_L \)-module \( H_1(A_x(\mathbb{C}), \mathbb{Z}) \) to \( (A_x, \iota_x(\lambda_0), \iota_x) \). The isomorphism class of the skew-Hermitian \( O_F \)-module \( H_1(A_x(\mathbb{C}), \mathbb{Z}) \) only depends on the moduli space \( \mathcal{M}(L,L^+,n) \), which we write \( (V_\mathbb{C}, \langle , , \rangle, \iota) \). Let \( G \) be the automorphism group scheme over \( \mathbb{Z} \) associated to the skew-Hermitian \( O_F \)-module \( (V_\mathbb{C}, \langle , , \rangle, \iota) \), and \( \Gamma(n) \) be the kernel of the reduction map \( G(\mathbb{Z}) \to G(\mathbb{Z}/n\mathbb{Z}) \). One has the complex uniformization

\[
\mathcal{M}(L,L^+,n)(\mathbb{C}) \cong \Gamma(n)\backslash G(\mathbb{R})/SO_2(\mathbb{R}).
\]

**Theorem 2.8.** Let \( \mathcal{M}_{\underline{a}} \) be a supersingular stratum.

1. If \( g \) is odd, then \( \mathcal{M}_{\underline{a}} \) consists of all superspecial points and

\[
|M_{\underline{a}}(k)| = [G(\mathbb{Z}) : \Gamma(n)] \cdot \left[ \frac{-1}{2} \right]^g \cdot \zeta_F(-1) \cdot (p^g - 1).
\]

2. If \( g \) is even and \( |\underline{a}| = g \), then \( \mathcal{M}_{\underline{a}} \) consists of all superspecial points and

\[
|M_{\underline{a}}(k)| = [G(\mathbb{Z}) : \Gamma(n)] \cdot \left[ \frac{-1}{2} \right]^g \cdot \zeta_F(-1) \cdot (p^g + 1).
\]

3. If \( g \) is even and \( |\underline{a}| \neq g \), then any irreducible component of \( \overline{\mathcal{M}_{\underline{a}}} \) is isomorphic to \( (\mathbb{P}^1)^{g-1} \underline{a} \) and

\[
|\Pi_0(\mathcal{M}_{\underline{a}})| = [G(\mathbb{Z}) : \Gamma(n)] \cdot \left[ \frac{-1}{2} \right]^g \cdot \zeta_F(-1).
\]

Let \( \Delta_{ss}^\text{gen} \subset \Delta_{ss}^\text{gen} \) denote the subset of supersingular generic alpha types. If \( g \) is odd, then \( \Delta_{ss}^\text{gen} \) is empty; if \( \underline{a} \in \Delta_{ss}^\text{gen} \), then \( w(\underline{a}) = 1 \). By Corollary 2.6, Theorem 2.8 (3) and (2.3), we get

**Theorem 2.9.** Notation as before. Assume that \( p \) is inert in \( F \). Then

\[
|\Pi_0(\mathcal{M}_{\Gamma(\underline{a}, p)})| = \begin{cases} 
\sum_{\underline{a} \in \Delta_{ss}^\text{gen}} w(\underline{a}) + 2[G(\mathbb{Z}) : \Gamma(n)] \cdot \left[ \frac{-1}{2} \right]^g \cdot \zeta_F(-1) & \text{if } g \text{ is even;} \\
2[G(\mathbb{Z}) : \Gamma(n)] \cdot \left[ \frac{-1}{2} \right]^g \cdot \zeta_F(-1) - 1 & \text{if } g \text{ is odd.}
\end{cases}
\]

The following is an elementary combinatorial result.

**Lemma 2.10.** For any subset \( \tau \) of \( \mathbb{Z}/g\mathbb{Z} \), let \( w(\tau) \) be as in (2.4). One has

\[
\sum_{\tau \subset \mathbb{Z}/g\mathbb{Z}} w(\tau) = 2^g.
\]

Since \( w(\underline{a}) = 0 \) for non-generic alpha types \( \underline{a} \) and \( w(\underline{a}) = 1 \) for \( \underline{a} \in \Delta_{ss}^\text{gen} \) (note that \( \Delta_{ss}^\text{gen} \) is empty if \( g \) is odd), we rewrite the formula in Theorem 2.9 as follows

\[
|\Pi_0(\mathcal{M}_{\Gamma(\underline{a}, p)})| = \sum_{\underline{a} \in \Delta} w(\underline{a}) + \sum_{\underline{a} \in \Delta_{ss}^\text{gen}} \left[ G(\mathbb{Z}) : \Gamma(n) \right] \cdot \left[ \frac{-1}{2} \right]^g \cdot \zeta_F(-1) - 1
\]

Using Lemma 2.10, Theorem 2.9 is rephrased as

\[
|\Pi_0(\mathcal{M}_{\Gamma(\underline{a}, p)})| = \sum_{\underline{a} \in \Delta} w(\underline{a}) + \sum_{\underline{a} \in \Delta_{ss}^\text{gen}} \left[ G(\mathbb{Z}) : \Gamma(n) \right] \cdot \left[ \frac{-1}{2} \right]^g \cdot \zeta_F(-1) - 1
\]
Theorem 2.11. Assume that $p$ is inert in $F$. Then
\begin{equation}
|\Pi_0(\mathcal{M}_{G_0(p)})| = 2^g + \sum_{\alpha \in \Delta_{ss}^{gen}} \left\{ [G(\mathbb{Z}) : \Gamma(n)] \cdot \left[ \frac{-1}{2} \right]^g \cdot \zeta_F(-1) - 1 \right\}.
\end{equation}

See a formula for $|\Pi_0(\mathcal{M}_{G_0(p)})|$ when $p$ is unramified in Section 5.

In the following, we determine the slope sequence (or Newton polygon) of the generic point of each irreducible component of $\mathcal{M}_{G_0(p)}$.

Theorem 2.12.

(1) Let $\eta$ be a maximal point of $\mathcal{M}_{G_0(p)}$. Put $j = |\mathbb{Z}(\eta)|$, the a-number of its image $f(\eta)$. Then the slope sequence of $\eta$ is equal to $(j, g)$.

(2) For each non-negative integer $j$ with $0 \leq j < \left[ \frac{g}{2} \right]$, the moduli space $\mathcal{M}_{G_0(p)}$ has exactly $2 \left( \frac{g}{2j} \right)$ irreducible components whose maximal point has slope sequence $(j, g)$.

(3) The moduli space $\mathcal{M}_{G_0(p)}$ has
\[ |\Delta_{ss}^{gen}| \cdot [G(\mathbb{Z}) : \Gamma(n)] \cdot \left[ \frac{-1}{2} \right]^g \cdot \zeta_F(-1) \]
supersingular irreducible components.

Note that statement (2) deals with non-supersingular slope sequences.

3. Proof of Theorems 2.1 and 2.3

3.1. Let $f : \mathcal{M}_{G_0(p)} \to \mathcal{M}$ be the forgetful morphism, and let $x = (A, i_A, \iota_A, \eta_A)$ be a point in $\mathcal{M}_0(k)$. Choose a separable $O_F$-linear polarization $\lambda_A = i_A(\lambda_0)$ on $A$. Each point in $f^{-1}(x)$ is given by an $O_F$-invariant finite subgroup scheme $H$ of $A$ of rank $p^g$ which is maximally isotropic with respect to the Weil pairing $e_{\lambda_A}$. Then there is an $O_F$-linear polarization $\lambda_B$, necessarily separable, on $B := A/H$ such that the pull back $\pi^* \lambda_B$ is equal to $p^g \lambda_A$. Denote by $M^*(A)$ the classical contravariant Dieudonné module of $A$. We have an $O$-invariant Dieudonné submodule $M^*(B)$ of $M^*(A)$ such that
\[ M^*(A)/M^*(B) \cong k \oplus \cdots \oplus k, \quad \text{and} \quad \langle \cdot, \cdot \rangle_{M^*(B)} = p \langle \cdot, \cdot \rangle_{M^*(A)}. \]

Note that $M^*(A)$ is canonically isomorphic to the dual $M(A)^t$ of the covariant Dieudonné module $M(A)$. We also know that $g(M(A)^t) = g(M(A))$ (see [50 Lemma 8.1]). Put $M_0 := M^*(A)$ and let $\tau := \tau(\underline{a})$ be corresponding alpha index as in Section 2. Let $\mathcal{X}_r$ be the space of Dieudonné $O$-submodules $M$ of $M_0$ such that
\[ M_0/M \cong k \oplus \cdots \oplus k. \]

We regard $\mathcal{X}_r$ as a scheme over $k$ with the reduced structure. For any point $M$ in $\mathcal{X}_r$, it is clear that the induced $k$-valued pairing $\langle \cdot, \cdot \rangle$ is trivial on $M/pM_0$. Therefore there is a polarized abelian $O_F$-variety $\mathcal{H} = (B, \lambda_B, \iota_B)$ and an $O_F$-linear isogeny $\pi : A \to B$ such that $\pi^* \lambda_B = p^g \lambda_A$ and $M^*(B) = M$. This establishes

Lemma 3.1. The map $\langle \underline{A}, \underline{H} \rangle \mapsto M^*(A/H)$ gives rise to an isomorphism $\xi_x : f^{-1}(x)_{\text{red}} \cong \mathcal{X}_r$, where $f^{-1}(x)_{\text{red}}$ is the reduced subscheme underlying the fiber $f^{-1}(x)$. 
Lemma 3.2. The scheme $X_\tau$ is isomorphic to the subscheme of $(\mathbb{P}^1)^g = \{([s_i : t_i])_{i \in \mathbb{Z}/g\mathbb{Z}} \}$ defined by the equations $t_i - s_i = 0$ for $i \notin \tau$ and $t_i - t_i = 0$ for $i \in \tau$.

Proof. A point in $X_\tau(k)$ is represented by a $k$-subspace $\overline{M}$ of $\overline{M}_0 := M_0/pM_0$ such that $F(\overline{M}) \subseteq \overline{M}$, $V(\overline{M}) \subseteq \overline{M}$, and $\dim_k \overline{M} = 1$ for each $i \in \mathbb{Z}/g\mathbb{Z}$. Hence $X_\tau$ is a closed subscheme of $(\mathbb{P}^1)^g$. Choose a basis $\{X_i, Y_i\}$ for $M$ [30] Proposition 4.2 such that

$$FX_i = \begin{cases} X_i & \text{if } i \notin \tau; \\ Y_i + pc_iX_i & \text{if } i \in \tau; \end{cases}$$

$$FY_i = \begin{cases} pY_i & \text{if } i \notin \tau; \\ pX_i & \text{if } i \in \tau; \end{cases}$$

where $c_i$ are some elements of $W(k)$ for $i \in \tau$. (There should be no confusion on our notation for the Frobenius map and the totally real field.) Let $P = ([s_i : t_i])_i$ be a point in $(\mathbb{P}^1)^g(k)$ and write $\overline{M}_P$ for the $k$-subspace of $\overline{M}_0$ generated by $s_iY_i + t_iX_i$ for $i \in \mathbb{Z}/g\mathbb{Z}$. We have

$$F(s_i - Y_i + t_iX_i - 1) = \begin{cases} t_i^pX_i & i \notin \tau; \\ t_i^pY_i & i \in \tau. \end{cases}$$

From the closed condition $F\overline{M}_P \subseteq \overline{M}_P$ we get the equations

$$t_i - s_i = 0 \text{ for } i \notin \tau, \quad \text{and } t_i - t_i = 0 \text{ for } i \in \tau.$$  

From the closed condition $V\overline{M}_P \subseteq \overline{M}_P$ we get the same equations as above. This finishes the computation.  

3.2. Examples. (1) If $g = 0$, then $X_\tau$ consists of two points: $([1 : 0], [1 : 0], \ldots, [1 : 0])$ and $([0 : 1], [0 : 1], \ldots, [0 : 1])$.

(2) If $g = (1, 0, 1, 0, 0)$, then $X_\tau$ is defined by the equations $t_4t_0, t_0s_1, t_1t_2, t_2s_3, t_3s_4$.

There are four irreducible components:

$$\mathbb{P}^1 \times [0 : 1] \times [1 : 0] \times [1 : 0] \times [1 : 0], \quad [1 : 0] \times [1 : 0] \times \mathbb{P}^1 \times [0 : 1] \times [0 : 1],$$

$$[1 : 0] \times \mathbb{P}^1 \times [1 : 0] \times \mathbb{P}^1 \times [0 : 1], \quad [1 : 0] \times \mathbb{P}^1 \times [1 : 0] \times [1 : 0] \times \mathbb{P}^1.$$

Notice that for maximally dimensional components, every $\mathbb{P}^1$ is placed at a position $i$ where $a_i = 0$.

(3) If $g = (1, 0, 1, 1, 0)$, then $X_\tau$ is defined by the equations $t_5t_0, t_0s_1, t_1t_2, t_2s_3, t_3s_4, t_4s_5$.

There are 3-dimensional component $[1 : 0] \times \mathbb{P}^1 \times [0 : 1] \times \mathbb{P}^1 \times [0 : 1] \times \mathbb{P}^1$, and four 2-dimensional components

$$\mathbb{P}^1 \times [0 : 1] \times [1 : 0] \times \mathbb{P}^1 \times [1 : 0] \times [1 : 0], \quad [1 : 0] \times \mathbb{P}^1 \times [1 : 0] \times [1 : 0] \times \mathbb{P}^1 \times [0 : 1],$$

$$[1 : 0] \times [1 : 0] \times \mathbb{P}^1 \times [1 : 0] \times \mathbb{P}^1 \times [0 : 1], \quad [1 : 0] \times [1 : 0] \times \mathbb{P}^1 \times [1 : 0] \times [1 : 0] \times \mathbb{P}^1.$$

Proposition 3.3.

(1) We have $\dim X_\tau \leq |g|$. Furthermore, $\dim X_\tau = |g|$ if and only if $g \in \Delta^\text{gen}$.

(2) For $g \in \Delta^\text{gen}$, the scheme $X_\tau$ has $w(g)$ irreducible components of dimension $|g|$.

Proof. We may assume that $|g| > 0$, as the case $g = 0$ is treated in Example 3.2.

(1). Since the defining equations are either $s_i = 0$ or $t_i = 0$, any irreducible component of $X_\tau$ is of the form $X = \bigcap_{i \in \mathbb{Z}/g\mathbb{Z}} X_i$, where

$$X_i = [1 : 0], [0 : 1], \text{ or } \mathbb{P}^1.$$
If \( i \notin \tau \), then we have \( t_{i-1}s_i = 0 \). This tells us that there are at least \( g - |a| \) zeros for \( s_i \) or \( t_i \) in the components \([s_i : t_i]\) for \( i \notin \tau \) or \( i - 1 \notin \tau \). So \( X_i = \mathbb{P}^1 \) for at most \(|a|\) numbers of \( i \). This shows that \( \dim X_{\tau} \leq |a| \).

If \( a \notin \Delta^{\text{gen}} \), then one can choose \( i \) such that \( i - 1 \notin \tau \), \( i \in \tau \) and \( i + 1 \in \tau \). It follows from the equation \( t_{i}t_{i+1} = 0 \) that there are at least \( g - |a| + 1 \) zeros for \( s_i \) or \( t_i \) in the components \([s_i : t_i]\) for \( i \notin \tau \). Thus, \( \dim X_{\tau} < |a| \). Suppose that \( a \in \Delta^{\text{gen}} \). Put \( s_i = 1 \) for all \( i \in \mathbb{Z}/g\mathbb{Z} \), then the defining equations become \( t_{i-1} = 0 \) for \( i \notin \tau \). Thus, \( \dim X_{\tau} = |a| \). This proves the statement (1).

(2) Let \( a \in \Delta^{\text{gen}} \) and \( X = \prod_{i \in \mathbb{Z}/g\mathbb{Z}} X_i \) be an irreducible component of \( X_{\tau} \). Write \( \tau = \{n_1, \ldots, n_a\} \). First notice that

(i) If \( X_{i_0} = \mathbb{P}^1 \) for some \( n_j \leq i_0 \leq n_{j+1} \), then \( X_i = [0 : 1] \) for \( i_0 < i < n_{j+1} \), and \( X_i = [1 : 0] \) for \( n_j \leq i < i_0 \) or \( i_0 < i = n_{j+1} \).

It follows that

(ii) There is at most one \( i \in \mathbb{Z} \) in each interval \([n_j, n_{j+1}]\) such that \( X_i = \mathbb{P}^1 \).

(iii) If \( X_i = \mathbb{P}^1 \) for some \( i \in \tau \), then \( \dim X < |a| \).

If \( \dim X = |a| \), then \( X_{i_j} = \mathbb{P}^1 \) for one \( i_j \) in each interval \( n_j < i_j < n_{j+1} \). Conversely, choose \( i_j \) in each interval \( n_j < i_j < n_{j+1} \). Then there is a unique irreducible component \( X \) such that \( X_{i_j} = \mathbb{P}^1 \) for each \( j \); this follows from (i). There are \( \prod_j (n_{j+1} - n_j - 1) \) such choices. Thus, the scheme \( X_{\tau} \) has \( w(a) \) irreducible components of dimension \(|a| \).

Theorem 2.1 follows from Lemma 3.1 and Proposition 3.3 (1).

3.3. **Proof of Theorem 2.2**. Part (1) follows from Lemma 3.1 and Proposition 5.3 (2). We prove the statement (2). We prove that irreducible components of \( X_{\tau} \) give rise to well-defined closed subvarieties in \( M_{\Gamma_0(p)}^{\text{gen}} \). Notice two isomorphisms between \( f^{-1}(\tau) \) and \( X_{\tau} \) (in Lemma 5.1) differ by an automorphism \( \beta \) of \( \mathbb{P}^1 \), which sends each factor of \((\mathbb{P}^1)^g\) to itself. If \( X = \prod_i X_i \) is an irreducible component of \( X_{\tau} \), then the \( i \)-th component \( \beta(X_i) \) of \( \beta(X) \) is equal to \( \mathbb{P}^1 \) whenever \( X_i = \mathbb{P}^1 \). By the property (i) in the proof of Proposition 5.3 we have shown that \( \beta(X) = X \). Therefore,

\[
\mathcal{M}_X := \{ y \in M_{\Gamma_0(p)}^{\text{gen}} \mid \xi_{f(y)}(y) \in X \}
\]

is a well-defined closed subvariety of \( M_{\Gamma_0(p)}^{\text{gen}} \). One has \( M_{\Gamma_0(p)}^{\text{gen}} = \bigcup_X \mathcal{M}_X \) as a union of components; any irreducible component of \( M_{\Gamma_0(p)}^{\text{gen}} \) is contained in \( \mathcal{M}_X \) for a unique \( X \). The morphism \( f_{\Delta} : \mathcal{M}_X \to \mathcal{M}_a \) is proper and surjective with fibers isomorphic to \( X \). Thus, \( \Pi_0(\mathcal{M}_X) = \Pi_0(\mathcal{M}_a) \) and \( \dim \mathcal{M}_X = \dim \mathcal{M}_a + \dim X \).

From this and Proposition 5.3 (2) the statement (2) then follows.

3.4. **Proof of Lemma 2.10**. If \( |a| = j > 0 \), then \( w(a) \) is the number of ways replacing a zero by 2 in \( a \) on each interval \([n_j, n_{j+1}]\). In other words, \( \sum_{|a|=j} w(a) \) is the number of ways of choosing \( 2j \) positions from \( \mathbb{Z}/g\mathbb{Z} \) and filling them with 1 and 2 alternatively. This gives \( \sum_{|a|=j} w(a) = 2 \binom{g}{2j} \). Thus

\[
\sum_{a \in \Delta} w(a) = 2 + \sum_{j>0} 2 \binom{g}{2j} = 2^g.
\]

This completes the proof.
3.5. **Proof of Theorem 2.12**  
(1) The point \( \eta \) lies in \( \mathcal{M}_{\Gamma_0(p),\mathcal{A}} \) for \( a = a(f(\eta)) \), which is generic. Since any maximal point of the generic alpha stratum \( \mathcal{M}_{\mathcal{A}} \) has slope sequence \( s(\mathcal{A}, g) \) (Theorem 2.7), the statement follows.

(2) Note that \( s(j, g) \) is a non-supersingular slope sequence. From (1) and Theorem 2.3 (2), the number of maximal points of \( \mathcal{M}_{\Gamma_0(p)} \) with slope sequence \( s(j, g) \) is equal to \( \sum_{|a|=j} w(a) \), which is \( 2 \left( \frac{g}{2j} \right) \).

(3) This follows from Subsection 2.2 Theorem 2.3 (2) and Theorem 2.8 (3). □

4. **Supersingular contribution**

Keep the notation and the assumption of \( p \) as before.

4.1. We recall the geometric mass formula for superspecial abelian varieties of HB-type in \[33\].

Let \( x_0 = \mathcal{A}_0 = (A_0, \lambda_0, \iota_0, \eta_0) \) be a superspecial (not necessarily separably) polarized abelian \( O_{\overline{p}} \)-variety over \( k \) of dimension \( g \) with symplectic level-\( n \) structure with respect to \( \zeta_n \). Let \( \mathcal{M}_0 = (M_0, (\cdot), \iota) \) be its covariant Dieudonné module with additional structures. As \( \mathcal{M}_0 \) is superspecial, the alpha type \( \mathcal{A} \) of \( M_0 \) has the form

\[
(e_1 + e_2, 2 - (e_1 + e_2), e_1 + e_2, \ldots)
\]

for some integers \( e_1, e_2 \) with \( 0 \leq e_1 \leq e_2 \leq 1 \); see \[31\] Section 2. When \( g \) is odd, it satisfies the additional condition \( e_1 + e_2 = 1 \). We say that \( \mathcal{M}_0 \) is of *superspecial type* \((e_1, e_2)\) if its alpha type is as above.

Let \( G_{x_0} \) denote the automorphism group scheme over \( \text{Spec} \mathbb{Z} \) associated to \( (A_0, \lambda_0, \iota_0) \); for any commutative ring \( R \), its group of \( R \)-points is

\[
G_{x_0}(R) = \{ \phi \in (\text{End}_{O_{\overline{p}}}(A_0) \otimes R)^\times; \phi^t \phi = 1 \},
\]

where the map \( \phi \mapsto \phi^t \) is the Rosati involution induced by \( \lambda_0 \).

Let \( \Lambda_{x_0,n} \) denote the set of isomorphism classes of polarized abelian \( O_{\overline{p}} \)-varieties \( \mathcal{A} = (A, \lambda, \iota, \eta) \) with level-\( n \) structure (w.r.t. \( \zeta_n \)) over \( k \) such that (c.f. (2.4) of \[33\])

(i) the Dieudonné module \( M(\mathcal{A}) \) is isomorphic to \( M(\mathcal{A}_0) \), compatible with \( O_{\overline{p}} \otimes \mathbb{Z}_p \)-actions and quasi-polarizations, and

(ii) the Tate module \( T_l(\mathcal{A}) \) is isomorphic to \( T_l(\mathcal{A}_0) \), compatible with \( O_{\overline{p}} \otimes \mathbb{Z}_l \)-actions and the Weil pairings, for all \( \ell \neq p \).

The condition (i) implies that \( A \) is superspecial and \( \text{dim } A = g \). Let \( K_n \) be the kernel of the reduction map \( G_{x_0}(\hat{\mathbb{Z}}) \to G_{x_0}(\hat{\mathbb{Z}}/n\hat{\mathbb{Z}}) \). There is a natural isomorphism

\[
\Lambda_{x_0,n} \cong G_{x_0}(\mathbb{Q})/G_{x_0}(A_f)/K_n;
\]

see \[30\] Theorem 10.5 and \[33\] Theorem 2.1 and Subsection 4.6. It is proved in \[33\] Theorem 3.7 and Subsection 4.6] that

\[
|\Lambda_{x_0,n}| = \left| G_{x_0}(\hat{\mathbb{Z}}) : K_n \right| \left[ \frac{-1}{2} \right]^g \zeta_{\overline{p}}(-1)c_p,
\]

where

\[
c_p := \begin{cases} 
1 & \text{if } g \text{ is even and } e_1 = e_2, \\
p^g + 1 & \text{if } g \text{ is even and } e_1 < e_2, \\
p^g - 1 & \text{if } g \text{ is odd},
\end{cases}
\]
and \((e_1, e_2)\) is the supersingular type of \(M_0\).

If \(T_{\ell}(A_0) \cong (V \otimes \mathbb{Z}_l, (\cdot, \cdot))\) (Subsection 2.2) for all \(\ell \neq p\), then it is easy to see that \([G_{\omega_0}(\hat{\omega}) : K]\) \(\sim\) \([G(\mathbb{Z}) : \Gamma(n)]\). In this case, the formula (4.2) becomes

\[
\zeta_p(-1)c_p,
\]

where \(c_p\) is as above.

4.2. If \(g\) is odd, then it follows from Theorem 2.7 that \(\mathcal{M}_a\) is supersingular if and only if \(|g| = g\), that is, \(\mathcal{M}_a\) consists of all superspecial points in \(\mathcal{M}\). By the formula (4.4), we get the equation (2.4).

If \(g\) is even, then it follows from Theorem 2.7 that \(\mathcal{M}_a\) is supersingular if and only if \(g \leq (1, 0, \ldots, 1, 0)\) or \(g \leq (0, 1, \ldots, 0, 1)\). If \(|g| = g\), then \(\mathcal{M}_a\) consists of all superspecial points in \(\mathcal{M}\). By the formula (4.4), we get the equation (2.5). This proves the statements (1) and (2) of Theorem 2.8.

4.3. We prove Theorem 2.8 (3). Suppose \(g = 2d\) is even and \(|g| \neq g\). Put \(a_0 := (1, 0, \ldots, 1, 0)\). We may assume that \(\omega \leq a_0\) due to symmetry. Let \(\mathcal{M}(p)\) be the moduli space over \(\overline{\mathbb{F}}_p\) of \(g\)-dimensional separably polarized abelian \(O_F\)-varieties with a symplectic level-\(n\) structure with respect to \(\zeta_n\). We may identify the moduli space \(\mathcal{M}\) with an irreducible component of \(\mathcal{M}(p)\) by choosing an suitable element \(\lambda_0 \in L^+\); see [33, Proposition 4.1].

Choose any point \(A_0\) in \(\mathcal{M}_{a_0}(k)\). Let \(\mathcal{M}_{a_i}\) be the covariant Dieudonné module of \(A_0\). Let \(\mathcal{N} := (F, V)M_0\), a Dieudonné \(O\)-submodule with the induced quasi-polardization. Then there is a tuple \(B = (B, \lambda_B, \eta_B)\) and an \(O_F\)-linear isogeny \(\varphi : B \to A_0\) of a \(p\)-power degree, compatible with additional structures, such that \(M(B) = N \subset M_0\).

One easily computes that \(\mathcal{N}\) has alpha type \((0, 2, \ldots, 0, 2)\). Then one can find a basis \(\{X_i, Y_i\}\) for \(N^i\) [30, Lemma 4.4] such that

\[
\begin{align*}
FX_i &= -pY_{i+1}, & FY_i &= pX_{i+1}, & \text{if } i \text{ is even,} \\
FX_i &= -Y_{i+1}, & FY_i &= X_{i+1}, & \text{if } i \text{ is odd.}
\end{align*}
\]

Let \(N_{-1} := (F, V)^{-1}N\); it is spanned by elements

\[
\frac{1}{p}X_{2i}, \frac{1}{p}Y_{2i}, \quad X_{2i+1}, \quad Y_{2i+1}, \quad i = 0, \ldots, d - 1.
\]

We have \(N_{-1}/N \cong k^2 \oplus \cdots \oplus k^2 \oplus 0\) as \(O \otimes_{\mathbb{Z}_p} k\)-modules. Let \(\mathcal{X}\) be the space of Dieudonné \(O\)-modules \(M\) such that

\[
N \subset M \subset N_{-1}, \quad M/N \cong k \oplus 0 \oplus k \oplus \cdots \oplus k \oplus 0.
\]

It is clear that \(\mathcal{X} \cong (\mathbb{P}^1)^d\).

Let \(A\) denote the set of isomorphism classes of objects \(B' = (B', \lambda', \eta')\) (with respect to \(\zeta_n\)) such that (cf. Subsection 1.1)

- the Dieudonné module \(M(B')\) is isomorphic to \(M(B)\), compatible with additional structures, and
- the Tate module \(T_{\ell}(B')\) is isomorphic to \(T_{\ell}(B)\), compatible with additional structures, for all \(\ell \neq p\).

**Proposition 4.1.** There is an isomorphism \(pr : \coprod_{\ell \in \Lambda} \mathcal{X} \to \mathcal{M}_{a_0}\).
Proof. We write the map set-theoretically first. For any member $\xi \in \Lambda$ and any point $x \in \mathcal{X}_\xi(k) := \mathcal{X}(k)$, we have $M(B_\xi) = N \subset M_x$. Then one gets a point $A_x$ together with a polarized $O_F$-linear isogeny $\varphi : B_\xi \to A_x$ of $p$-power degree such that $M(A_x) = M_x$. Define $\text{pr}(M_x) := A_x$. Then one can show that it gives a bijective map from $\prod_{\xi \in \Lambda} \mathcal{X}(k)$ onto $\overline{\mathcal{M}}_{\omega, \xi}(k)$. To see this map comes from a morphism of schemes, we need to construct a moduli space with a prescribed isogeny type a priori, and show that this map agrees with the natural projection. Since the construction is lengthy and is the same as [30, Lemma 9.1], we refer the reader to that and omit the details here. Finally using the tangent space calculation, we prove that the morphism $\text{pr}$ is étale and particularly separable; see the computation in Lemma 9.2 of [30]. Thus the morphism $\text{pr}$ is isomorphism and the proof is complete.

By definition $\Lambda$ is nothing but the set $\Lambda_{\omega, n}$ defined in Subsection 4.1. Note that the alpha type of $\overline{\mathcal{M}}$ is $(0, 2, \ldots, 0, 2)$. By the formula (4.2), we get

Lemma 4.2. $|\Lambda| = [G(\mathbb{Z}) : \Gamma(n)] \left(\frac{1}{\omega}\right)^g \zeta_F(-1)$.

Denote by $\overline{\mathcal{M}}_{\omega, \xi}$ the irreducible component corresponding to $\xi$ and write $\text{pr} : \mathcal{X} \to \overline{\mathcal{M}}_{\omega, \xi}$. Let $\mathcal{M}_{\leq \omega, \xi} \subset \overline{\mathcal{M}}_{\omega, \xi}$ be the closed subscheme consisting of points with alpha type $\leq \omega$.

Lemma 4.3. The scheme $\mathcal{M}_{\leq \omega, \xi}$ is isomorphic to $(\mathbb{P}^1)^{g-\omega}$.

Proof. For a point $P = ([x_0 : y_0], [x_2 : y_2], \cdots, [x_{2d-2} : y_{2d-2}]) \in (\mathbb{P}^1(k))^d$, the representing Dieudonné module is given by

$$M_P = N+ < \frac{1}{p} \tilde{x}_{2i-1} X_{2i} + \frac{1}{p} \tilde{y}_{2i} Y_{2i} >_{i=0,\ldots,d-1},$$

where $\tilde{x}_{2i}, \tilde{y}_{2i}$ are any liftings of $x_{2i}, y_{2i}$ in $W$, respectively.

We compute the defining equations for $\mathcal{M}_{\leq \omega, \xi}$ on an affine open subset. Let $V_{2i} := \frac{1}{p} X_{2i} + \frac{1}{p} Y_{2i}$, then

$$M_P = < X_{2i}, V_{2i}, X_{2i+1}, Y_{2i+1} >_{i=0,\ldots,d-1}, \text{ and } M_{P}^{2i+1} = < X_{2i+1}, Y_{2i+1} >_{i=0,\ldots,d-1}.$$

One computes that

$$((F, V)M_P)^{2i+1} \mod pM_P^{2i+1} = \langle -X_{2i+1} - x_{2i}^{p^2} \Gamma_{2i+1}, -X_{2i+1} + x_{2i+2}^{p-1} \Gamma_{2i+1} >.$$

Therefore, $a_{2i+1}(M_P) = 1$ if and only if $x_{2i}^{p^2} = x_{2i+2}$. Let $\tau = \tau(a) \subset \mathbb{Z}/g\mathbb{Z}$. We have shown that the subscheme $\mathcal{M}_{\leq \omega, \xi}$ of $\overline{\mathcal{M}}_{\omega, \xi} = (\mathbb{P}^1)^d = \{(x_2, \cdots, x_{2d})\}$ defined by the equations $x_{j-1}^{p^2} = x_{j+1}$ for all odd $j \in \tau$, and thus it is isomorphic to $(\mathbb{P}^1)^{g-\omega}$. This completes the proof.

By Proposition 4.1 and Lemmas 4.2 and 4.3, the statement (3) of Theorem 2.8 is proved.

5. Unramified setting

In this section we only assume that $p$ is unramified in $F$.  

5.1. Let $O := O_F \otimes \mathbb{Z}_p$, $\mathcal{J} := \text{Hom}(O, W)$, $\Delta := \{0, 1\}^3$ be the same as in Section 2. Let $\mathbb{P}$ be the set of primes of $O_F$ lying over $p$. For $v \in \mathbb{P}$, let $O_v$ be the completion of $O_F$ at $v$, $f_v$ its residue degree, $\mathcal{I}_v := \text{Hom}(O_v, W)$ and $\Delta_v := \{0, 1\}^3$. One has $O = \bigoplus_{v \in \mathbb{P}} O_v$, $\sum_{v \in \mathbb{P}} f_v = g$, $\mathcal{I} = \bigcoprod_{v \in \mathbb{P}} \mathcal{I}_v$, $\mathcal{I}_v \simeq \mathbb{Z}/f_v \mathbb{Z}$, and $\Delta = \prod_{v \in \mathbb{P}} \Delta_v$.

An alpha type $\underline{a} = (a_v) \in \Delta$ is called generic if every component $a_v$ is generic; it is called supersingular if the associated alpha stratum $\mathcal{M}_a$ is supersingular.

Let $\Delta^\text{gen} \subset \Delta$ be the subset of generic alpha types, and $\Delta^\text{ss} \subset \Delta^\text{gen}$ the subset of supersingular alpha types. The set $\Delta^\text{ss}$ is empty if and only if $f_v$ is odd for some $v$.

**Theorem 5.1.** For any alpha type $\underline{a}$, the set $\Pi_0(\mathcal{M}_a)$ is $\ell$-adic Hecke transitive.

This is essentially due to Goren and Oort (cf. [8, Corollary 4.2.4]). We provide suitable details to fit the present situation: $p$ is unramified and the objects $(A, \lambda, \iota, \eta)$ that $\mathcal{M}$ parametrizes may not be principally polarized abelian $O_F$-varieties.

**Proposition 5.2.**

1. Every alpha stratum $\mathcal{M}_a$ is non-empty.
2. Every alpha stratum $\mathcal{M}_a$ is quasi-affine.
3. The non-ordinary locus of $\mathcal{M}$ is proper.
4. The Zariski closure $\overline{\mathcal{M}_a}$ of each stratum $\mathcal{M}_a$ in $\mathcal{M}$ is smooth.
5. The set $\mathcal{M}_0$ of superspecial points is $\ell$-adic Hecke transitive.

**Proof.** (1) It is easy to construct a superspecial point in $\mathcal{M}$. Indeed, one constructs a separably polarized superspecial abelian $O_F$-variety, then one chooses a point within its prime-to-$p$ isogeny class so that it lies in $\mathcal{M}$. Then one constructs a deformation of this point so that the generic point has the given alpha type $\underline{a}$. Such a construction is a local problem, and it reduces to inert cases. This proves the statement (1).

(2) This is a global property; it does not follow directly from the result of inert cases. One can slightly modify the proof in [8] to make it work. Alternatively, consider the forgetful morphism $b : \mathcal{M} \to A_{g,d,n} \otimes \overline{\mathbb{F}}_p$, for some positive integer $d$ with $(d, p) = 1$. Then the image $b(\mathcal{M}_a)$ is contained in an Ekedahl-Oort stratum $S_\varphi$ of $A_{g,d,n} \otimes \overline{\mathbb{F}}_p$. Since $S_\varphi$ is quasi-affine [18], the image $b(\mathcal{M}_a)$ is also quasi-affine. Since the morphism $b$ is finite, the stratum $\mathcal{M}_a$ is quasi-affine.

(3) This follows from the semi-stable reduction theorem for abelian varieties due to Grothendieck [9].

(4) This is a local property, and hence follows directly from the results of inert cases.

(5) Let $x_0 = (A_0, \lambda_0, \iota_0, \eta_0)$ be a superspecial point in $\mathcal{M}$. Define $A_{x_0,n}$, $G_{x_0}$, $K_n$ as in Subsection 4.1. Note that any point in $\mathcal{M}_a$ satisfies the conditions (i) and
(ii) in Subsection 4.1 (see [30] Lemma 4.3). Therefore, we have \( \Lambda_{x_0, n} = \mathcal{M}_0 \) and have the double coset description
\[
\mathcal{M}_0 \simeq G_{x_0}(\mathbb{Q}) \backslash G_{x_0}(\mathbb{A}) / K_n
\]
as \((\mathbb{A})\). By the strong approximation \([21] \text{Theorem } 7.12, \text{p. } 427\]
the natural map \( G_{x_0}(\mathbb{Q}) \rightarrow G_{x_0}(\mathbb{Q}) \backslash G_{x_0}(\mathbb{A}) / K_n \) is surjective. This shows that the \( \ell \)-adic Hecke orbit \( \mathcal{H}_\ell(A_0) \) is equal to \( \mathcal{M}_0 \), and hence that the action of \( \ell \)-adic Hecke correspondences on the set \( \mathcal{M}_0 \) is transitive. \( \blacksquare \)

5.2. Proof of Theorem 5.1. Using (1), (2) and (3) of Proposition 5.2 one shows that the closure of any irreducible component \( W \) of \( \mathcal{M}_0 \) contains a point in \( \mathcal{M}_0 \).

By Proposition 5.2 (4), any point in \( \mathcal{M}_0 \) is contained in \( \mathcal{W} \) for a unique irreducible component \( W \) of \( \mathcal{M}_0 \). This shows that there is a surjective \( \ell \)-adic Hecke equivariant map
\[
i : \mathcal{M}_0 \rightarrow \Pi_0(\mathcal{M}_0) = \Pi_0(\mathcal{M}_0).
\]

By Proposition 5.2 (5), the set \( \Pi_0(\mathcal{M}_0) \) is \( \ell \)-adic Hecke transitive. \( \blacksquare \)

An immediate consequence of Theorems 2.4 and 5.1 is the following

Corollary 5.3. Any non-supersingular stratum \( \mathcal{M}_0 \) is irreducible.

5.3. Let \( a = (a_v)_v \in \Delta \) be a supersingular alpha type. If \( f_v \) is odd, then \( |a_v| = f_v \).

If \( f_v \) is even, then either \( a_v \leq (1, 0, \ldots, 1, 0) \) or \( a_v \geq (0, 1, \ldots, 0, 1) \).

Define
\[
P_1 := \{ v \in \mathbb{P} | f_v \text{ is odd} \}
\]
\[
P_2(a) := \{ v \in \mathbb{P} | f_v \text{ is even and } |a_v| = f_v \}
\]
\[
P_3(a) := \{ v \in \mathbb{P} | f_v \text{ is even and } |a_v| < f_v \}
\]

Theorem 5.4. Let \( a \in \Delta \) be a supersingular alpha type. Then any irreducible component of \( \mathcal{M}_0 \) is isomorphic to \((\mathbb{P}^1)^{9-|\Delta|}\) and
\[
|\Pi_0(\mathcal{M}_0)| = [G(\mathbb{Z}) : \Gamma(n)] \left[ \frac{-1}{2} \right]^g \zeta_\ell(-1) \prod_{v \in P} c_v,
\]
where
\[
c_v := \begin{cases} 
p^f_v - 1 & \text{if } v \in P_1; \\
p^f_v + 1 & \text{if } v \in P_2(a); \\
1 & \text{if } v \in P_3(a).
\end{cases}
\]

Proof. Define \( a_0 = (a_{0,v})_v \in \Delta \) by
\[
a_{0,v} = \begin{cases} 
(1, 0, \ldots, 1, 0) & \text{if } v \in P_2(a); \\
a_v & \text{otherwise}.
\end{cases}
\]

We may assume that \( a \leq a_0 \) due to symmetry. Choose any point \( A_0 \) in \( \mathcal{M}_{0}(k) \).

Let \( \mathcal{M}_0 \) be the covariant Dieudonné module of \( A_0 \). Define \( N = \oplus N_v \subset M_0 = \oplus M_{0,v} \)
the Dieudonné \( O \)-submodule with the induced quasi-polarization by
\[
N_v := \begin{cases} 
(F, V) M_{0,v} & \text{if } v \in P_2(a); \\
M_{0,v} & \text{otherwise}.
\end{cases}
\]

Then there is a tuple \( B = (B, \lambda_B, \iota_B, \eta_B) \) and an \( O_\mathbb{F} \)-linear isogeny \( \varphi : B \rightarrow A_0 \)
of \( p \)-power degree, compatible with additional structures, such that \( M(B) = N \subset \)
$M_0$. Let $\mathcal{X} = \prod_{v \in \mathbb{P}_3(a)} \mathcal{X}_v$, where $\mathcal{X}_v$ is defined as $\mathcal{X}$ in Subsection 4.3. One has $\mathcal{X}_v \simeq (\mathbb{P}^1)^{f_v/2}$. Define the set $\Lambda$ for $B$ as in Subsection 4.3. By Proposition 4.1 we have an isomorphism $\prod_{\xi \in \Lambda} \mathcal{X} \simeq (\mathbb{P}^1)^{f_{\xi}/2}$. Define the set $\Lambda$ for $B$ as in Subsection 4.3. By Proposition 4.1 we have an isomorphism $\prod_{\xi \in \Lambda} \mathcal{X} \simeq (\mathbb{P}^1)^{f_{\xi}/2}$. Define the set $\Lambda$ for $B$ as in Subsection 4.3. By Proposition 4.1 we have an isomorphism $\prod_{\xi \in \Lambda} \mathcal{X} \simeq (\mathbb{P}^1)^{f_{\xi}/2}$.

Therefore, we have

$$\Pi_0(\mathcal{M}_a) \simeq \Pi_0(\mathcal{M}_a) \simeq \Lambda.$$  

(5.4)

The alpha type of the factor $N_v$ is $(0, 2, \ldots, 0, 2)$ if $v \in \mathbb{P}_3(a)$ and $(1, 1, \ldots)$ otherwise. Hence $N_v$ has superspecial type $(e_1, e_2) = (0, 0)$ if $v \in \mathbb{P}_3(a)$ and $(e_1, e_2) = (0, 1)$ otherwise (Subsection 4.1). By the mass formula [33, Theorem 3.7 and Subsection 4.6] (cf. (4.4)), we get

$$|\Lambda| = \left[\frac{1}{2} \left(\frac{1}{2}\right)^g \cdot \zeta_F(-1) \prod_{v \in \mathbb{P}_3(a)} c_v\right],$$

where $c_v$ is as above. This completes the proof.

4.4. Define the function $w' : \Delta \to \mathbb{R}$ by

$$w'(\mathfrak{a}) := \begin{cases} [G(\mathbb{Z}) : \Gamma(n)] \left[\frac{1}{2}\right]^g \cdot \zeta_F(-1) \prod_{v \in \mathbb{P}_3(a)} \frac{1}{w_v(\mathfrak{a})} & \text{if } \mathfrak{a} \in \Delta_{\text{gen}}; \\
0 & \text{otherwise,} \end{cases}$$

where $w_v(\mathfrak{a})$ is the function as in (2.2). It is clear that $w'(\mathfrak{a}) \neq 0$ if and only if $\mathfrak{a} \in \Delta_{\text{gen}}$. It is rather unclear but indeed a fact that $w'(\mathfrak{a}) \in \mathbb{Z}_{\geq 0}$ (by (5.7)).

Theorem 5.5. Notation as above. We have

$$|\Pi_0(\mathcal{M}_{\Gamma_0(p)})| = \sum_{\mathfrak{a} \in \Delta_{\text{gen}}} w'(\mathfrak{a}),$$

(5.6)

**Proof.** Suppose that $\mathfrak{a}$ is a non-supersingular generic alpha type. It follows from the local computation in Section 3 and Corollary 5.3 that $\mathcal{M}_{\Gamma_0(p)}(\mathfrak{a})$ has $w'(\mathfrak{a})$ irreducible components of dimension $g$.

Suppose that $\mathfrak{a}$ is a supersingular generic alpha type. Every fiber of the map $f_{\mathfrak{a}}$ has one irreducible component of dimension $|\mathfrak{a}|$ (Section 3). Thus, $\mathcal{M}_{\Gamma_0(p)}(\mathfrak{a})$ has $|\Pi_0(\mathcal{M}_a)|$ irreducible components of dimension $g$. It follows from Theorem 5.4 that

$$|\Pi_0(\mathcal{M}_{\Gamma_0(p)})| = |\Pi_0(\mathcal{M}_a)| = w'(\mathfrak{a}).$$

(5.7)

This completes the proof.

We can rephrase Theorem 5.5 by an elementary combinatorial result (Lemma 2.10) as follows

Theorem 5.6. We have

$$|\Pi_0(\mathcal{M}_{\Gamma_0(p)})| = 2^g + \sum_{\mathfrak{a} \in \Delta_{\text{gen}}} (w'(\mathfrak{a}) - 1),$$

(5.8)

Similar to Theorem 2.12 we have the following

Theorem 5.7.
(1) Let \( \{s(j_v, f_v); v \in \mathbb{P}\} \) be a tuple of slope sequences indexed by \( \mathbb{P} \) and suppose that \( 0 \leq j_v < \left\lceil \frac{f_v}{2} \right\rceil \) for some \( v \in \mathbb{P} \). Then the moduli space \( \mathcal{M}_{\Gamma_0(p)} \) has exactly

\[
\prod_{v \in \mathbb{P}} 2^{\left( \frac{f_v}{2j_v} \right)}
\]

irreducible components whose maximal point has slope sequence \( \{s(j_v, f_v); v \in \mathbb{P}\} \).

(2) The moduli space \( \mathcal{M}_{\Gamma_0(p)} \) has

\[
|\Delta_{\text{ss}}^{\text{gen}}| \cdot [G(\mathbb{Z}) : \Gamma(n)] \cdot \left[ \frac{-1}{2} \right]^9 \cdot \zeta_F(-1)
\]

supersingular irreducible components.

Remark 5.8.

(1) The connection of supersingular strata with class numbers and special zeta values becomes a standard fact now. If the moduli space \( \mathcal{M}_{\Gamma_0(p)} \) contains supersingular irreducible components, then it is expected that the special zeta value \( \zeta_F(-1) \) occurs in the formula for \( |\Pi_0(\mathcal{M}_{\Gamma_0(p)})| \). However, the number of irreducible components of a supersingular stratum is also related to \( p \) in general. It is indeed unexpected that the number of supersingular irreducible components of \( \mathcal{M}_{\Gamma_0(p)} \) turns out to be independent of \( p \). As a result, the number \( |\Pi_0(\mathcal{M}_{\Gamma_0(p)})| \) of irreducible components is independent of \( p \). We do not know any direct proof of this fact without knowing the explicit formula (5.8).

(2) The \( p \)-adic invariant stratification used in this paper is nothing but the Ekedahl-Oort stratification (see [18] and [8]). Since a parahoric level structure on an abelian variety \( A \) is a flag of finite flat subgroup schemes of the \( p \)-torsion \( A[p] \) that satisfy certain conditions, this structure only depends on the isomorphism class of \( A[p] \), but not on \( A \). It would be interesting to know the relationship between the group-theoretic description of Kottwitz-Rapoport strata and that of Ekedahl-Oort strata by the works of Moonen [13, 14] and of Wedhorn [29].

(3) The irreducibility problem for PEL-type moduli spaces \( \mathcal{M}_{K_0} \) with parahoric level structure at \( p \) is, as suggested by this work, related to the same problem for Ekedahl-Oort strata in \( \mathcal{M} \) (without level at \( p \)), which is of interest in its own right. It seems plausible to expect that in any irreducible component of \( \mathcal{M} \), (i) any non-basic Ekedahl-Oort stratum is irreducible, and (ii) the number of irreducible components of a basic Ekedahl-Oort stratum is a single class number.

For Siegel moduli spaces, the statement (i) is confirmed in Ekedahl and van der Geer [6], and the statement (ii) is treated in Harashita [10].

For Hilbert-Blumenthal moduli spaces, the statement (i) is essentially due to Goren and Oort [8] and Chai [4] (Corollary 5.3), and the statement (ii) is confirmed by Theorem 5.4 (see (5.4)).

6. \( \ell \)-adic monodromy of Hecke invariant subvarieties

The goal of this section is to provide a proof of a theorem of Chai on Hecke invariant subvarieties for Hilbert-Blumenthal moduli spaces on which Theorem 5.6 relies. We follow the proof in Chai [4] where the Siegel case is proved. There is no novelty on the proof here and this is purely expository; the author is responsible
for any inaccuracies and mistakes. We write this as an independent section; some
setup and notation may be repeated and slightly modified.

6.1. Let \( F \) be a totally real number field of degree \( g \) and \( O_F \) be the ring of
integers in \( F \). Let \( V \) be a 2-dimensional vector space over \( F \) and \( \psi : V \times V \to \mathbb{Q} \)
be a \( \mathbb{Q} \)-bilinear non-degenerate alternating form such that \( \psi(ax,y) = \psi(x,ay) \)
for all \( x, y \in V \) and \( a \in F \). Let \( p \) be a fixed rational prime, not necessarily unramified
in \( F \). We choose and fix an \( O_F \)-lattice \( V_\mathbb{Z} \subset V \) so that \( V_\mathbb{Z} \otimes \mathbb{Z}_p \)
is self-dual with respect to \( \psi \). We choose a projective system of primitive prime-to-\( p \)-th roots of unity
\( \zeta = (\zeta_m)_{(m,p)=1} \subset \overline{\mathbb{Q}} \subset \mathbb{C} \). We also fix an embedding \( \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p \). For any prime-to-\( p \)
integer \( m \geq 1 \) and any connected \( \mathbb{Z}_p[\zeta_m] \)-scheme \( S \), the choice \( \zeta \)
determines an isomorphism \( \zeta_m : \mathbb{Z}/m\mathbb{Z} \cong \mu_m(S) \), or equivalently, a \( \pi_1(S, \bar{s}) \)-invariant \((1+m\hat{\mathbb{Z}}/(p))^{\times}\)
orbit of isomorphisms \( \tilde{\zeta}_m : \hat{\mathbb{Z}}(p) \to \hat{\mathbb{Z}}(p)(1)_s \), where \( \hat{\mathbb{Z}}(p) := \prod_{\ell \neq p} \hat{\mathbb{Z}}_\ell \) and \( \bar{s} \) is a
geometric point of \( S \).

Let \( G \) be the automorphism group scheme over \( \mathbb{Z} \) associated to the pair \((V_\mathbb{Z}, \psi)\); for
any commutative ring \( R \), the group of \( R \)-valued points is

\[
(6.1) \quad G(R) := \{ g \in \text{GL}_{ \mathbb{P}_F } (V_\mathbb{Z} \otimes \mathbb{Z} R) : \psi(g(x), g(y)) = \psi(x,y), \ \forall x,y \in V_\mathbb{Z} \otimes \mathbb{Z} R \}.
\]

Let \( n \geq 3 \) be a prime-to-\( p \) positive integer and \( \ell \) be a prime with \((\ell, pn) = 1\) and
\((\ell, \text{disc}(\psi)) = 1\), where \( \text{disc}(\psi) \) is the discriminant of \( \psi \) on \( V_\mathbb{Z} \). Let \( m \geq 0 \) be
a non-negative integer. Let \( U_{n,\ell^m} \) be the kernel of the reduction map \( G(\hat{\mathbb{Z}}(p)) \to
G(\hat{\mathbb{Z}}(p)/m\ell^m\hat{\mathbb{Z}}(p)) \); this is an open compact subgroup of \( G(\hat{\mathbb{Z}}(p)) \).

Let \( \mathcal{D} = (F, V, \psi, V_\mathbb{Z}, \zeta) \) be a list of data as above. Denote by \( \mathcal{M}_{\mathcal{D}, n,\ell^m} \) the moduli
space over \( \mathbb{Z}_p[\zeta_{n,\ell^m}] \) that parametrizes equivalence classes of objects \((A, \lambda, \iota, [\eta])_S \)
over a connected locally Noetherian \( \mathbb{Z}_p[\zeta_{n,\ell^m}] \)-scheme \( S \), where

- \((A, \lambda)\) is a \( p \)-principally polarized abelian scheme over \( S \) of relative dimen-
sion \( g \),
- \( \iota : O_F \to \text{End}_S(A) \) is a ring monomorphism such that \( \lambda \circ \iota(a) = \iota(a)^{t} \circ \lambda \)
for all \( a \in O_F \), and
- \([\eta]\) is a \( \pi_1(S, \bar{s}) \)-invariant \( U_{n,\ell^m} \)-orbit of \( O_F \)-linear isomorphisms

\[
(6.2) \quad \eta : V_\mathbb{Z} \otimes \hat{\mathbb{Z}}(p) \cong T(p)(A_\bar{s}) := \prod_{\ell \neq p} T_{\ell}(A_\bar{s})
\]
such that

\[
(6.3) \quad e_\lambda(\eta(x), \eta(y)) = \bar{\zeta}_{n,\ell^m}(\psi(x,y)) \pmod{(1+m\hat{\mathbb{Z}}/(p))^{\times}}, \ \forall x,y \in V_\mathbb{Z} \otimes \hat{\mathbb{Z}}(p),
\]

where \( e_\lambda \) is the Weil pairing induced by the polarization \( \lambda \) and \( \bar{s} \) is a
geometric point of \( S \).

We write \([\eta]_{U_{n,\ell^m}}\) for \([\eta]\) in order to specify the level. Let \( \mathcal{M}_{n,\ell^m} := \mathcal{M}_{\mathcal{D}, n,\ell^m} \otimes \mathbb{F}_p \)
be the reduction modulo \( p \) of the moduli scheme \( \mathcal{M}_{\mathcal{D}, n,\ell^m} \). We have a natural
morphism \( \pi_{m,m'} : \mathcal{M}_{n,\ell^m} \to \mathcal{M}_{n,m'} \), for \( m < m' \), which sends \((A, \lambda, \iota, [\eta])_{U_{n,\ell^m}}\) to
\((A, \lambda, \iota, [\eta])_{U_{n,m'}}\). Let \( \mathcal{M}_n := (\mathcal{M}_{n,m})_{m \geq 0} \) be the tower of this projective
system.

Let \((X, \lambda, \iota, \tilde{\eta}) : M_n \to M_n \) be the universal family. The cover \( \mathcal{M}_{n,\ell^m} \) over \( \mathcal{M}_n \)
represents the étale sheaf

\[
(6.4) \quad P_m := \mathcal{I}son_{\mathcal{M}_n}((V_\mathbb{Z}/\ell^mV_\mathbb{Z}, \psi), (X[\ell^m], e_\lambda); \zeta_{n,\ell^m})
\]
of \( O_F \)-linear symplectic level-\( \ell^m \) structures with respect to \( \zeta_{n,\ell^m} \). This is a \( \mathbb{G}(\mathbb{Z}/\ell^m\mathbb{Z}) \)-
torsor. Let \( \bar{x} \) be a geometric point in \( \mathcal{M}_n \). Choose an \( O_F \)-linear isomorphism
y : V ⊗ \mathbb{Z}_p \simeq T_{\ell}(X_{\overline{\mathbb{F}}_\ell}) that is compatible with the polarizations with respect to \zeta.

This amounts to choose a geometric point in \tilde{\mathcal{M}}_n over the point \tilde{x}. The action of the geometric fundamental group \pi_1(\mathcal{M}_n, \tilde{x}) on the system of fibers \( (X_{\overline{\mathbb{F}}_\ell})_m \) gives rise to the monodromy representation

\[
\rho_{\mathcal{M}_n, \ell} : \pi_1(\mathcal{M}_n, \tilde{x}) \to \text{Aut}_{\mathcal{O}_\ell}(T_\ell(X_{\overline{\mathbb{F}}_\ell}), e_\lambda)
\]

and to the monodromy representation (using the same notation), through the choice of y,

\[
\rho_{\mathcal{M}_n, \ell} : \pi_1(\mathcal{M}_n, \tilde{x}) \to G(\mathbb{Z}_\ell).
\]

**Lemma 6.1.** The map \( \rho_{\mathcal{M}_n, \ell} \) is surjective.

**Proof.** It is well-known that \( \mathcal{M}_{D, n\ell^m}(\mathbb{C}) \simeq \Gamma(n\ell^m) \backslash \Gamma(\mathbb{R})/SO(2, \mathbb{R})^\eta \), where \( \Gamma(n\ell^m) := \ker G(\mathbb{Z}) \to G(\mathbb{Z}/n\ell^m\mathbb{Z}) \). It follows that the geometric generic fiber \( \mathcal{M}_{D, n\ell^m} \otimes \overline{\mathbb{Q}} \) is connected. It follows from the arithmetic toroidal compactification constructed in Rapoport [23] that the geometric special fiber \( \mathcal{M}_{n\ell^m} \) is also connected. The connectedness of \( \tilde{\mathcal{M}}_n \) confirms the surjectivity of \( \rho_{\mathcal{M}_n, \ell} \).

6.2. The action of \( G(\mathbb{Z}_\ell) \) on \( \tilde{\mathcal{M}}_n \) extends uniquely a continuous action of \( G(\mathbb{Q}_\ell) \). Descending from \( \mathcal{M}_n \) to \( \tilde{\mathcal{M}}_n \), elements of \( G(\mathbb{Q}_\ell) \) induce algebraic correspondences on \( \mathcal{M}_n \), known as the \( \ell \)-adic Hecke correspondences on \( \mathcal{M}_n \). More precisely, to each \( g \in G(\mathbb{Q}_\ell) \) we associate an \( \ell \)-adic Hecke correspondence \( (\mathcal{H}_g, \rho_{1, \ell}, \rho_{2, \ell}) \) as follows. Extending isomorphisms \( \eta \) to isomorphisms

\[
\eta' : V \otimes \mathbb{A}_f^{(p)} \to V^{(p)}(A) := T^{(p)}(A) \otimes \mathbb{A}_f^{(p)},
\]

we see the class \([\eta]_{U_n} \) gives rise to a class \([\eta']_{U_n} \) in \( \text{Isom}(V \otimes \mathbb{A}_f^{(p)}, V^{(p)}(A))/U_n \) and \([\eta]_{U_n} \) is determined by \([\eta']_{U_n} \). The right translation \( \rho_g : (A, \lambda, \iota, [\eta']_{U_n}) \mapsto (A, \lambda, \iota, [\eta'g^{-1}U_n]) \) gives rise an isomorphism \( \rho_g : \mathcal{M}_n \simeq \mathcal{M}_{g^{-1}U_n} \). Let \( U_{n, g} := U_n \cap g^{-1}U_n g \) and \( \mathcal{H}_g \) be the étale cover of \( \mathcal{M}_n \) corresponding to the subgroup \( U_{n, g} \subset U_n \). Let \( \rho_{1, \ell} \) be the natural projection \( \mathcal{H}_g \to \mathcal{M}_n \) and \( \rho_{2, \ell} := \rho_{g^{-1}} \circ \rho_{1, \ell} : \mathcal{H}_g \to \mathcal{M}_n \) be the composition of the isomorphism \( \rho_g^{-1} \) with the natural projection \( \rho_{1, \ell} : \mathcal{H}_g \to \mathcal{M}_{g^{-1}U_n} \). This defines an \( \ell \)-adic Hecke correspondence \( (\mathcal{H}_g, \rho_{1, \ell}, \rho_{2, \ell}) \). For two \( \ell \)-adic Hecke correspondences \( \mathcal{H}_{g_1} = (\mathcal{H}_{g_1}, p_{11}, p_{12}) \) and \( \mathcal{H}_{g_2} = (\mathcal{H}_{g_2}, p_{21}, p_{22}) \), one defines the composition \( \mathcal{H}_{g_2} \circ \mathcal{H}_{g_1} \) by

\[
(\mathcal{H}_{g_2} \circ \mathcal{H}_{g_1}, p_{1}, p_{2}),
\]

where \( \mathcal{H}_{g_2} \circ \mathcal{H}_{g_1} := \mathcal{H}_{g_1} \times_{p_{12}, \mathcal{M}_n, p_{21}} \mathcal{H}_{g_2} \), \( p_1 \) is the composition \( \mathcal{H}_{g_2} \circ \mathcal{H}_{g_1} \to \mathcal{H}_{g_1} \to \mathcal{M}_n \), and \( p_2 \) is the composition \( \mathcal{H}_{g_2} \circ \mathcal{H}_{g_1} \to \mathcal{H}_{g_2} \to \mathcal{M}_n \). A correspondence \( (\mathcal{H}, \rho_{1, \ell}, \rho_{2, \ell}) \) generated by correspondences of the form \( \mathcal{H}_g \) is also called an \( \ell \)-adic Hecke correspondence.

A subset \( Z \) of \( \mathcal{M}_n \) is called \( \ell \)-adic Hecke invariant if \( \rho_{2, \ell}(pr_{1, \ell}^{-1}(Z)) \subset Z \) for any \( \ell \)-adic Hecke correspondence \( (\mathcal{H}, \rho_{1, \ell}, \rho_{2, \ell}) \). If \( Z \) is an \( \ell \)-adic Hecke invariant, locally closed subvariety of \( \mathcal{M}_n \), then the \( \ell \)-adic Hecke correspondences induce correspondences on the set \( \mathcal{P}_0(Z) \) of geometrically irreducible components. We say that \( \mathcal{P}_0(Z) \) is \( \ell \)-adic Hecke transitive if the \( \ell \)-adic Hecke correspondences operate transitively on \( \mathcal{P}_0(Z) \), that is, for any two maximal points \( \eta_1, \eta_2 \) of \( Z \) there is an \( \ell \)-Hecke Hecke correspondence \( (\mathcal{H}, \rho_{1, \ell}, \rho_{2, \ell}) \) so that \( \eta_1 \in \rho_{2, \ell}(pr_{1, \ell}^{-1}(\eta_2)) \). Let \( k \) be an algebraically closed field of characteristic \( p \). For a geometric point \( x \in \mathcal{M}_n(k) \),
denote by $\mathcal{H}_\ell(x)$ the $\ell$-adic Hecke orbit of $x$; this is the set of points generated by $\ell$-adic correspondences starting from $x$.

**Lemma 6.2.**

(1) For any point $x \in \mathcal{M}_n(k)$, the corresponding abelian variety $A_x$ is supersingular if and only if the $\ell$-adic Hecke orbit $\mathcal{H}_\ell(x)$ of $x$ is finite.

(2) Any closed $\ell$-adic Hecke invariant subscheme $Z$ of $\mathcal{M}_n$ contains a supersingular point.

**Proof.** (1) This is Lemma 7 in Chai [3]. (2) This is Proposition 6 in Chai [3].

6.3. Put $G_\ell := G \otimes \mathbb{Q}_\ell$ (Subsection 63). One has

$$G_\ell = \prod_{\lambda | \ell} G_\lambda, \quad G_\lambda = \text{Res}_{F_\lambda/k} \text{SL}_2(F_\lambda).$$

Let $\text{pr}_\lambda : G_\ell \to G_\lambda$ be the projection map. Let $Z$ be a smooth locally closed subscheme of $\mathcal{M}_n$ that is $\ell$-adic Hecke invariant. Let $Z^0$ be a connected component of $Z$, and $\eta$ be the generic point of $Z^0$. Let

$$\rho_{Z^0,\ell} : \pi_1(Z^0, \bar{\eta}) \to G(G_\ell)$$

be the associated $\ell$-adic monodromy representation, and $\rho_{Z^0,\lambda} := \text{pr}_\lambda \circ \rho_{Z^0,\ell}$ be its projection at $\lambda$.

**Lemma 6.3.**

(1) If the image $\text{Im}\rho_{Z^0,\lambda}$ is finite for one $\lambda | \ell$, then the image $\text{Im}\rho_{Z^0,\lambda}$ is finite for all $\lambda | \ell$.

(2) The abelian variety $A_\eta$ is not supersingular if and only if the image $\text{Im}\rho_{Z^0,\lambda}$ is infinite for all $\lambda | \ell$.

**Proof.** (1) Let $Z^0_0$ be a scheme over $\mathbb{F}_q$ such that $Z^0 = Z^0_0 \otimes \mathbb{F}_q$, and let $\bar{\eta}_0$ be the generic point of $Z^0_0$. Replacing by a finite surjective cover of $Z^0_0$ (thus of $Z^0$), we may assume that $\text{End}^0(A_{\eta}) = \text{End}^0(A_{\eta_0}) := \text{End}(A_{\eta_0}) \otimes \mathbb{Q}$ and that $\text{Im}\rho_{Z^0,\lambda} = 1$ whenever it is finite. Write the Tate module $V_\ell(A_{\bar{\eta}}) = \prod_{\lambda | \ell} V_\lambda$ into the decomposition with respect to the action of $F$, and let $\rho_{\lambda} : \text{Gal}(k(\bar{\eta}_0)/k(\eta_0)) \to \text{Aut}(V_\lambda)$ be associated $\lambda$-adic Galois representation. Let $E_\lambda$ be the $F_\lambda$-subalgebra of $\text{End}_{F_\lambda}(V_\lambda)$ generated by the image $\rho_{\lambda}(\text{Gal}(k(\bar{\eta}_0)/k(\eta_0)))$. By a theorem of Zarhin on endomorphisms of abelian varieties over function fields [44], the subalgebra $E_\lambda$ is semi-simple and the endomorphism algebra $\text{End}^0_F(A) \otimes F_\lambda$ is isomorphic to the commutant of $E_\lambda$ in $\text{End}_{F_\lambda}(V_\lambda)$. If $\text{Im}\rho_{Z^0,\lambda} = 1$ for some $\lambda$, then $\rho_{\lambda}$ factors through the quotient $\text{Gal}(\mathbb{F}_q/F_q)$, and thus $E_\lambda$ is commutative. In this case, $\dim F_\lambda \text{End}^0_F(A) \otimes F_\lambda$ is 2 or 4, and the same that $\dim F_\lambda \text{End}^0_F(A)$ is 2 or 4. This shows that the abelian variety $A_{\eta_0}$ is of CM-type. By a theorem of Grothendieck on CM abelian varieties in characteristic $p$ ([13] p. 220 and [17] Theorem 1.1), $A_{\eta_0}$ is isogenous to, over a finite extension of $k(\eta_0)$, an abelian variety that is defined over a finite field. This shows the image $\text{Im}\rho_{Z^0,\ell}$ is finite. Therefore, $\text{Im}\rho_{Z^0,\lambda}$ is finite for all $\lambda | \ell$.

(2) It is proved in [3] Corollary 3.5 that $A_\eta$ is not supersingular if and only if the image $\text{Im}\rho_{Z^0,\ell}$ is finite. The statement then follows from (1).

**Lemma 6.4.** Let $H$ be a connected normal subgroup of an algebraic group $G_1 \times \cdots \times G_r$ over a field of characteristic zero, where $G_i$ is a connected simple algebraic group. Then $H$ is of the form $H_1 \times \cdots \times H_r$ with $H_i$ is $\{1\}$ or $G_i$. 


Lemma 6.5. Notation as in Subsection 6.3, if the abelian variety $A_\eta$ is not supersingular, then the image $\text{Im } \rho_{Z^0,\ell}$ is an open subgroup of $G(\mathbb{Z}_\ell)$.

Proof. Replacing $Z$ by the orbit of the component $Z^0$ under all $\ell$-adic Hecke correspondences, we may assume that the set $\pi_0(Z)$ of connected components is $\ell$-adic Hecke transitive. Put $M := \text{Im } \rho_{Z^0,\ell}$ and let $H$ be the neutral component of the algebraic envelope of $M$. It is proved in [3, Proposition 4.1] that $M$ is open in $H(\mathbb{Q}_\ell)$ and $H$ is a connected normal subgroup of $G_\ell$. By Lemma 6.4, the group $H$ has the form $\prod_{\lambda|\ell} H_\lambda$ with $H_\lambda = \{1\}$ or $G_\lambda$. Since $A_\eta$ is not supersingular, it follows from Lemma 6.5 that $H = G$. This completes the proof.

Lemma 6.6. Let $G$ be a connected simply-connected semi-simple algebraic group over a local field $K$ such that each simple factor of $G$ is $K$-isotropic. Then $G(K)$ has no proper subgroup of finite index.

Lemma 6.6 follows immediately from the affirmative solution to the Kneser-Tits problem for local fields (See Platonov [20] for more details). This is proved by Platonov [20] for the characteristic zero case and by Prasad and Raghunathan [22] for the characteristic $p$ case. Only the characteristic zero case of Lemma 6.6 is needed.

Theorem 6.7 (Chai). Let $Z$ be an $\ell$-adic Hecke invariant, smooth locally closed subscheme of $M_n$. Let $\bar{\eta}$ be a geometric generic point of an irreducible component $Z^0$ of $Z$. Suppose that the abelian variety $A_{\bar{\eta}}$ corresponding to the point $\bar{\eta}$ is not supersingular, and that the set $\pi_0(Z)$ of connected components is $\ell$-adic Hecke transitive. Then the monodromy representation

$$\rho_{Z^0,\ell} : \pi_1(Z^0, \bar{\eta}) \to G(\mathbb{Z}_\ell)$$

is surjective and $Z$ is irreducible.

Proof. Let $\bar{Z}^0$ and $\bar{Z}$ be the preimage in $\bar{M}_n$ of the subschemes $Z^0$ and $Z$, respectively, under the morphism $\pi : \bar{M}_n \to M_n$. Let $Y$ be a connected component of $\bar{Z}^0$ and $M$ be the image $\text{Im } \rho_{Z^0,\ell}$. The group $\text{Aut}(Y/Z^0)$ of deck transformations is equal to $M$. Since the group $G(\mathbb{Q}_\ell)$ acts transitively on the fiber $\pi^{-1}(x)$ for any $x \in Z$ and $G(\mathbb{Q}_\ell)$ acts transitively on the set $\pi_0(Z)$, the group $G(\mathbb{Q}_\ell)$ acts transitively on the set $\pi_0(\bar{Z})$. This gives a homeomorphism (see [3, Lemma 2.8])

$$Q \setminus G(\mathbb{Q}_\ell) \sim \pi_0(\bar{Z}), \quad g \mapsto [Y],$$

where $Q$ is the stabilizer of the class $[Y]$ (in $\pi_0(\bar{Z})$). Clearly $Q \cap G(\mathbb{Z}_\ell) = M$ and we have $M \setminus G(\mathbb{Z}_\ell) \simeq \pi_0(\bar{Z}^0)$. It follows from Lemma 6.5 that $\pi_0(\bar{Z}^0) = M \setminus G(\mathbb{Z}_\ell)$ is finite. Write $Z = \bigsqcup_{i=0} M \setminus G(\mathbb{Z}_\ell)$ as a disjoint union of connected components. Since $G(\mathbb{Q}_\ell)$ acts transitively on $\pi_0(Z)$ and $\pi_0(\bar{Z}^0)$ is finite, each $\pi_0(\bar{Z}_i)$ is finite. We have

$$|\pi_0(\bar{Z})| < \infty \implies |Q \setminus G(\mathbb{Q}_\ell)| < \infty \implies Q = G(\mathbb{Q}_\ell) \text{ (by Lemma 6.6)} \implies M = G(\mathbb{Z}_\ell).$$

This shows the connectedness of $\bar{Z}$ and hence that of $Z$. This completes the proof.
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