Shape of spinfoams encoded in graphs

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Abstract. As the EPRL model was evolving into its present form, a precise identification of the family of complexes (foams) used in the definition of the spinfoams has been abandoned. Most often mentioned possibilities were: simplicial complexes, linear complexes or locally linear complexes. We have solved that problem and we present a new compact characterization of a general foam. Our characterization leads also to a new type of diagrammatic notation directly from which a spinfoam amplitude can be written. This allows one to consider the diagrammatic notation as an alternative to standard 2-complex approach.

1. Motivation
The spinfoam models provide a covariant description of dynamics of Loop Quantum Gravity. The framework defines a way to calculate transition amplitudes between boundary spinnetwork states. Since the states are labeled by graphs, the key object in spinfoam theories are 2-complexes considered as time-histories of spinnetworks.

First of all, the only foams considered were 2-complexes dual to triangulations of a 4-manifold (see, e.g., [2]). Then became clear that the spinfoam amplitude can be calculated for arbitrary vertex [3]. It turned out that to calculate a vertex amplitude one needs data that can be expressed as a spinfoam, i.e. the intersection of a colored 2-complex with a little 3-sphere centered at the considered vertex.

To calculate the spinfoam amplitude for a given boundary spinnetwork state one has to sum the amplitudes given by all 2-complexes of a given boundary (or calculate the amplitude of a maximally refined 2-complex, which is equivalent [4]). However the precise class of 2-complexes that one should consider in this calculation has not been defined yet.

We propose such a class. Our idea is based on an observation: given any spinfoam one can decompose it into a set of spinnetworks (one per each vertex - see [3, 5]) with some relations between nodes and links of these spinnetworks - representing the way in which vertices are connected by edges and faces. Thus let us reverse this procedure and say that each relevant spinfoam is given by some set of graphs (spinnetworks) with their nodes and links connected by some relations. Our class will consist of all 2-complexes that can be obtained by such construction.

The key issue is now to prove that for every such initial data one can construct a 2-complex. The catch is in the relations between nodes and links: from spinfoams one can obtain some of them, but can one get all the 2-complexes? We will prove it by giving a detailed construction of a 2-complex from the set of (arbitrarily related) graphs, which is the core of the talk.
Figure 1. The idea of squid-graphs. The squid $\lambda_{n_2}$ is highlighted in blue at (c). Some edges must be reoriented when passing from (b) to (c), but since one considers spin-networks, the information is not lost - it passes to the coloring. (d) A one-vertex foam obtained by shrinking the boundary squid-graph (green).

The problem is treated in detail (and extended) in the paper of Marcin Kisielowski, Jerzy Lewandowski and JP [1], especially in section 3. Moreover, as one can find in section 2 of [1], our construction allows to rewrite the spinfoam amplitude in a transparent form of a contraction of some projection operators with some tensors (see also [3]), what is the subject of the complementary talk of Marcin Kisielowski.

2. Construction

Let us make a remark: the paper [5] defines some operations on spinfoams, i.e. edge- and face-splitting and edge/face reorientation. In what follow we consider equivalent any two 2-complexes that differ only by these operations.

2.1. Ingredients - graph diagrams

The construction is started from the following ingredients (called a graph diagram):

- A set of graphs $A = \{\gamma_1, \ldots, \gamma_N\}$.
- A family of homeomorphic maps $\phi_{n \rightarrow n'}: D_n \rightarrow D_{n'}$, where $D_n$ is a small neighborhood of a node $n$ of one of graphs. Each node appears at most once, either as $n$ or as $n'$ in the family.

The maps $\phi_{n \rightarrow n'}$ indicates the gluing of pieces of 2-complex. It is important which link $\ell$ incident to $n$ is glued to which link $\ell'$ incident to $n'$. That is why the maps are required to be homeomorphic.

2.2. Squid-graphs

The following technical definition of squid-graph makes further steps (gluings) more transparent. Let us define a squid as a graph consisting of a selected node, called head, and a number of nodes, called leg-nodes, where every one is connected only with the head by a single link, called leg - see Fig. 1(a).

A sufficiently small neighborhood of each node of a graph is a squid. Thus given a graph $\gamma$ we define its decomposition into set of squids $S_\gamma$ as follows:

(i) split each link $\ell$ of $\gamma$ into two links by putting a node $x_\ell$ on it.
(ii) Each node $n_i$ of the old graph $\gamma$ becomes a head of one squid $\lambda_{n_i}$.
(iii) Each link $\tilde{\ell}$ of the new graph $\tilde{\gamma}$ has one end $x_\ell$ and one end $n_i$. Each such link becomes a leg of the squid $\lambda_{n_i}$ - see Fig. 1(c).
The pair \((\bar{\gamma}, S_\gamma)\) will be called a squid-graph. As the first step of our construction let us construct a squid-graph \((\bar{\gamma}, S_\gamma)\) out of each graph \(\gamma \in A\). It is straightforward to see that each map \(\phi_{n \to n'}\) defines a morphism of squids \(\Phi_{\lambda_n, \lambda_{n'}}\), i.e. a map saying which leg of \(\lambda_n\) is identified with which leg of \(\lambda_{n'}\).

### 2.3. One-vertex foams

Given any graph \(\gamma\) one can construct a 2-complex \(\kappa_\gamma\) being a neighborhood of a vertex with structure given by the graph. The prescription is precisely the one used to obtain the spinnetwork used in evaluation of the vertex amplitude \([3]\) but reversed (see Fig. 1(d)):

1. Draw the graph \(\gamma\) on a 3-sphere of radius 1.
2. Shrink the sphere to a point. The tracks of links of \(\gamma\) give faces of \(\kappa_\gamma\), the tracks of nodes of \(\gamma\) are internal edges of \(\kappa_\gamma\), the middle point of the ball is the only internal vertex of \(\kappa_\gamma\).

We call \(\kappa_\gamma\) a one-vertex foam. The boundary 1-complex of \(\kappa_\gamma\) is the graph \(\gamma\) itself. Thus if \(\gamma\) has a squid-graph structure, the boundary of \(\kappa_\gamma\) inherit the squid-decomposition.

Let us construct a one-vertex foam \(\kappa_\gamma\) out for each \(\gamma \in A\) and take their disjoint union \(\kappa_0 := \bigsqcup_{\gamma \in A} \kappa_\gamma\).

### 2.4. Gluing along pair of squids

Given any 2-complex \(\kappa\) with boundary decomposed into squids and any map \(\Phi_{\lambda, \lambda'}\) being morphism between two boundary squids let us define gluing \(\kappa\) along \(\lambda, \lambda'\) as follows:

- All the components of \(\kappa\) (i.e. faces, edges, vertices, relations between them) remain unchanged, but,
- the edges and vertices that belong to \(\lambda\) or \(\lambda'\) are identified according to \(\Phi_{\lambda, \lambda'}\) - see Fig. 2,
- the set of boundary squids is reduced by its subset \(\{\lambda, \lambda'\}\).

The resulting 2-complex is called \(\kappa/\Phi_{\lambda, \lambda'}\).

### 2.5. Induction

Gluing along any two pairs of squids commutes (see App. A of [1]). Thus having a set of maps \(\{\Phi_{\lambda_1, \lambda'_1}, \ldots, \Phi_{\lambda_E, \lambda'_E}\}\) one may glue \(\kappa_0\) along them in arbitrary order obtaining the same unique 2-complex: \(\kappa_0 \rightarrow \kappa_0/\Phi_{\lambda_1, \lambda'_1} \rightarrow \ldots \rightarrow (\ldots (\kappa_0/\Phi_{\lambda_1, \lambda'_1})/\ldots)/\Phi_{\lambda_E, \lambda'_E} =: \kappa_{\text{final}}\) which is the one we are looking for.

### 3. Properties of obtained 2-complexes

Below we present a brief characterization of the 2-complexes that might be obtained using our prescription.
Figure 3. A procedure of building the boundary graph of a graph diagram. First remove a squid containing both glued nodes - (a)$\rightarrow$(b), then connect the remaining links according to the map $\Phi$ - (b)$\rightarrow$(c).

3.1. Edges
Edges of three types appear in our 2-complexes (see Fig. 2):

(i) $VN$ edges being traces of heads of squids. They are always internal edges, each of them is at least 2-valent (in general they are 3-valent or more).

(ii) $VX$ edges are traces of leg-nodes of squids. They are always internal and 2-valent edges, thus they are called removable (see face-splitting in [5]).

(iii) $NX$ edges are legs of squids. They are either boundary edges (if the squid to which they belong was not glued) or internal 2-valent (thus removable) edges (if the squid was glued to another squid).

3.2. Boundary
The gluing procedure induces the following simple algorithm to find the boundary graph of the 2-complex $\kappa_{\text{final}}$:

\begin{itemize}
  \item Given a map $\Phi_{\lambda,\lambda'}$ remove the squids $\lambda$ and $\lambda'$ from the graphs they belong.
  \item Glue the remaining links of the graphs according to the map $\Phi_{\lambda,\lambda'}$ - see Fig. 3.
  \item Repeat it for each of maps $\Phi_{\lambda,\lambda'}$.
\end{itemize}

3.3. Faces
Each face that appears in the complex $\kappa_{\text{final}}$ is a triangle with one vertex being the middle of one of the balls ($V$), the second one being a head of one of squids ($N$) and the third one being a leg-node of the squid ($X$). However, some edges are trivial and thus removable. Removing of this edges results in obtaining the so-called generalized faces being several triangles glued. Each of such generalized faces is topologically a disc glued onto a skeleton. However, in some cases gluing onto the skeleton may cause the face to have topology more complicated than a disc. An example is shown at Fig. 4.

4. Summary and outlook
We have proven that given any graph diagram one can reconstruct a 2-complex out of it. We claim that all such 2-complexes constitute the class that should be used when considering spinfoams. The class is wide enough to contain 2-complexes with faces of nontrivial topology. However, detailed study and characterization of the class is still needed. Moreover our construction is compatible with the decomposition of the spinfoam amplitude into projection...
Figure 4. An example of obtaining a face with topology of a projective plane. (a) A fragment of a graph diagram. (b) Fragments of one vertex foams, cut along edges for transparency. (c) The same fragments of one vertex foams, now without the cuts. (d) The face with all removable edges erased. The bold line $N_1V_1N_2V_2N_1$ is built of the edges (the 1-skeleton of 2-complex) onto which the face is glued.

operators and tensors called contractors, what may simplify some calculations based on complicated (e.g. knoted) spinfoams.

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