Reverse optimization and capital asset pricing model in the application of the Black Litterman portfolio

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Abstract. In the application of the Black Litterman model on portfolio construction, the terms of reverse optimization and Capital Asset Pricing Model (CAPM) are often associated with the Black Litterman process. This study explains the Black Litterman model from two perspectives: reverse optimization and the CAPM formula. Two portfolios are built with different starting points and without any lambda difference as a risk aversion coefficient, a scalar tau, and views return in the Black Litterman strategy. Both models are applied in the Indonesian stock market, and the empirical comparison of both portfolios are demonstrated. The result shows that both approaches can build a portfolio using the Black Litterman with a similar weight allocation.

1. Introduction

Many articles on portfolio optimization used the Markowitz model and its development. One of the research developments on how to form a portfolio was the model presented by Fischer Black and Robert Litterman in 1991 [1], [2]. The model offers flexibility, such as investment managers' subjectivity on the asset allocation based on Markowitz. This research becomes interesting because there is a potential to explore, learn, and develop from the emergence of Black Litterman to various interpretations of the model.

Capital Asset Pricing Model (CAPM) mentioned in the original and developed of Black Litterman article [3]–[9]. The starting point used is the same as CAPM, but the explanation showed two kinds of CAPM. Firstly, from the capitalization market, CAPM’s equilibrium return is built from the market return and its relation with risk-free assets. The original reference [2], and it is followed by [5], [7], [10], recommends using the market weight of capitalization towards reverse optimization into equilibrium return. On the other hand, the Black Litterman model can also be applied using the CAPM formula [4] and [11].

This study is motivated by the two possibilities of equilibrium in the Black Litterman model. From [5], data equilibrium returns with the market weight of capitalization and CAPM were provided explicitly. There is a note in [5] that stated the ambiguity could emerge from two perspectives. However, in many subsequent studies, this issue is not much given attention. This issue becomes potential in developing the Black Litterman model, one of which is the possibility of a new starting point in applying the Black Litterman model. Nocera [8] propose the new starting point in the Black Litterman model. Therefore, the application of both approaches is focused on the use of CAPM equilibrium for the Black Litterman model.

This paper is organized as follows. The basic theory of Markowitz is discussed in Section 2, while Section 3 explains the reverse optimization and CAPM formula as a starting point in Black Litterman. The empirical analysis and discussion are given in Section 4, and the conclusion is in Section 5.
2. Markowitz model

The Markowitz model proposed the relation between the portfolio’s expected return and its variance to construct a portfolio. There is flexibility in this model that an investor is allowed to choose the preference. It can be interpreted as maximizing the expected portfolio return in a given risk or minimizing the risk in a given expected return. The model can be written as follows:

Maximize \( \mathbf{w}^{'} \mu \) subject to \( \mathbf{w}^{'} \mathbf{\Sigma} \mathbf{w} = \sigma_p^2 \)  

(1)

or

Minimize \( \mathbf{w}^{'} \mathbf{\Sigma} \mathbf{w} \) subject to \( \mathbf{w}^{'} \mu = \mu_p \)  

(2)

In contrast, this model has been reviewed as a model with many lacks, such as the sensitivity, the actual problem in the financial market, and sufficient sample data [12]. However, through the Markowitz model, the investor can specify the return or risk in the model. As a pioneer model, the Markowitz model is still evolving with many considerations.

3. Black Litterman model

The model proposed in 1991 by Fischer Black and Robert Litterman has been known as Markowitz's evolution. Some of the disadvantages of the Markowitz model can be addressed through the BL model. Better diversification is derived from BL allocation because between assets is less extreme [12]–[15].

The Black Litterman model can also improve the portfolio's performance [16]–[20].

The popular feature in this model is the possibility of an investor input the prediction as a view component. Then, it is blended with the equilibrium portfolio from CAPM. The combination of CAPM and views investors resulted in the new expected return. An overview of the combination process of two components can be shown from the illustration below.

![Components in estimating of Black Litterman return](image)

Figure 1. Components in estimating of Black Litterman return

The component that is related to the equilibrium return (\( \mathbf{\pi} \)) is a covariance matrix \( \mathbf{\Sigma} \). The estimated return from the investor is denoted as \( \mathbf{Q} \) is related to \( \mathbf{\Omega} \) as a covariance matrix of views and \( \mathbf{P} \) as a link matrix. A more detailed of the Black Litterman model can be reviewed from [21]. There are two types of the use of equilibrium return in the application of Black Litterman. Firstly, the original authors' concept equilibrium is when the weight of market capitalization captures supply equal to demand. Secondly, Sharpe, Lintner, and Mossin formulated the expected return in the mid of 1960 [22].

The market capitalization of stocks is defined as the firm size. According to [2], the formula of market capitalization-weighted for asset \( i \) is written as follows:

\[
w_{mkt-i} = \frac{m_i}{\sum_{i=1}^{n} m_i} \]

(3)

where \( n \) is the number of assets in the portfolio, \( m_i \) is the market capitalization of asset \( i \) that is calculated by multiplying the price (\( P_i \)) and volume of asset \( i \) in trading (\( N_i \)). The volume means the number of shares. The formula for the weight of market capitalization in equation (3) can be rewritten as follows:

\[
w_{mkt-i} = \frac{P_i N_i}{\sum_{i=1}^{n} P_i N_i} \]

(4)

In the framework of the original Black Litterman, reverse optimization is used to obtain the equilibrium return. Reverse optimization means that the input and output are reversed. In a typical optimization problem, the returns are treated as input, and the output is the weight to optimize the goal; for example, finding the best allocation in the portfolio. Referred to the Markowitz Mean-Variance
model (2), the expected return is given, and the goal function is to minimize the risk. The problem can be solved by using Lagrange Multiplier.

\[ L = \frac{1}{2} w^T \Sigma w - \lambda w^T \mu \]  

The solution is achieved by setting the derivative of \( L \) to \( w \) with zero. Then, the optimal weight is

\[ w = (\lambda \Sigma)^{-1} \mu, \]

where \( \lambda \) is the risk aversion coefficient, \( \Sigma \) is a covariance matrix, and \( \mu \) is an expected return vector. Equation (6) is the solution to the problem in the model (2).

The result of the optimization in reverse is that the expectation of return will be calculated from the given weight. The process of reversing is then used to find the equilibrium return, which is called reverse optimization. The estimated return as a reverse optimization by utilizing equation (6) can be reformulated as follows:

\[ \mu = \lambda \Sigma w \]

For Black Litterman, with reverse optimization, the weight is treated as an input while the equilibrium return is denoted by \( \pi \) as an output. Thus, the formula can be rewritten by using the weight of market capitalization denoted by \( w_{\text{capit}} \) as follows:

\[ \pi = \lambda \Sigma w_{\text{capit}} \]

Then, this return is also called implied excess equilibrium [21], with the assumption to use the coefficient of risk-averse \( \lambda \) is 2.5 [21]. The determination of the \( \lambda \) can be derived from Sharpe ratio of portfolio market. In this study, the risk-aversion coefficient is assumed at 2.5 [16].

In contrast, based on the equilibrium return concept proposed by Sharpe, Lintner, and Mossin [6], the CAPM formula can be used to estimate the asset return. It is determined by involving the risk premium \( (E(r_M) - r_f) \) and risk-free rate. While in the original article [2], CAPM was also mentioned. Hence, the formula for the expected excess return is written as follows:

\[ E(r_i) - r_f = \beta_i(E(r_M) - r_f) \]

where \( E(r_i) - r_f \) is the excess equilibrium return, \( E(r_M) \) is an expected market return, \( r_f \) is a free-risk rate, \( \beta_i \) is estimated beta calculating by regression analysis, or it can be determined by the ratio of each stock's covariance to the variance of the market return.

The combining process in figure 1 into the new expected return has been explained with variations [23]. The expected return of Black Litterman is written as follows:

\[ \mu_{BL} = [(\tau \Sigma)^{-1} + P' \Omega^{-1} P]^{-1} [(\tau \Sigma)^{-1} \pi + P' \Omega^{-1} Q] \]

\[ \Sigma_{BL} = [(\tau \Sigma)^{-1} + P' \Omega^{-1} P]^{-1} \]

where \( \tau \) is a scalar that reflecting the confidence of investor in the prediction of equilibrium. This new expectation then becomes the new input in the process of optimization. Using the Markowitz model, the asset allocation of Black Litterman can be expressed by:

\[ w_{BL} = (\lambda \Sigma_{BL})^{-1} \mu_{BL} \]

4. Empirical result

In this study, the portfolio is built from five assets in LQ45: BSDE, EXCL, INCO, JPFA, and LPPF, which are different sectors such as property, telecommunication, mining, animal feed and retail, respectively. The monthly return was gathered from January 2015 to January 2019.

Table 1 presents the descriptive statistics for all five assets, including the p-value of the Jarque-Bera (JB) and Augmented Dickey-Fuller (ADF) test to check the distribution and stationarity. The return of all five assets are normally distributed and can be categorized as stationary from table 1.

The information of a covariance matrix of five returns, the link matrix \( P \) associated with the views and the prediction \( Q \) are provided in table 2. The information on market returns is also included.

| Table 1. Summary descriptive, normality, and stationarity test of five returns. |
|-----------------------------------------------|
| BSDE  | EXCL  | INCO  | JPFA  | LPPF  |
| Mean  | -0.00280 | -0.01004 | 0.01400 | 0.03561 | -0.00979 |
Table 2. The information related to parameter Black Litterman.

| Parameter          | Estimated value |
|--------------------|-----------------|
| A scalar $\tau$    | 0.1             |
| A risk-aversion $\lambda$ | 0.025          |
| Link/Pick matrix   | $P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ |
| Predictive return Q | $Q = \begin{bmatrix} q_{BSDE} \\ q_{INCO} \\ q_{JPFA} \\ q_{LPPF} \end{bmatrix} = [0.026308, 0.011587, 0.068856, 0.006404]$ |
| Covariance matrix $\Sigma$ | $\Sigma = \begin{bmatrix} 0.00692 & 0.00022 & 0.00022 & 0.01205 \\ 0.00022 & 0.01205 & 0.00022 & 0.00002 \\ 0.00022 & 0.00002 & 0.0799 & 0.0074 \\ 0.00002 & 0.00002 & 0.0074 & 0.00002 \end{bmatrix}$ |
| Covariance matrix $\Omega$ | $\Omega = \begin{bmatrix} 0.00069 & 0 & 0 & 0 \\ 0 & 0.00271 & 0 & 0 \\ 0 & 0 & 0.00270 & 0 \\ 0 & 0 & 0 & 0.00113 \end{bmatrix}$ |
| Mean of market return | 0.00511 |
| Std. dev of market return | 0.03328 |

The absolute views for the $Q$ component are obtained as the prediction from the simple moving average, while the error of views $\Omega$ is calculated by following [5]. Additionally, there is a market issue that the telecommunication sector is in a red zone at the end of the year 2018. Then, the return prediction of investor ($Q$) is given to all assets, excluding EXCL. The following table 3 and 4 provide the information related to calculating the market capitalization weight and estimated beta, respectively.

Table 3. Weight of market capitalization and the return equilibrium from reverse optimization.

| Asset | Price | Volume | $w_{capit}$ | $\pi$ |
|-------|-------|--------|-------------|------|
| BSDE  | 1330  | 516000000 | 0.1319 | 0.0000608 |
| EXCL  | 2170  | 485000000 | 0.2026 | 0.0001011 |
| INCO  | 3850  | 409000000 | 0.3026 | 0.0002375 |
| JPFA  | 2930  | 317000000 | 0.1784 | 0.0002525 |
| LPPF  | 7000  | 137000000 | 0.1845 | 0.0002724 |

Table 3 reports the weight of market capitalization using equation (4). INCO dominates as the largest proportion with 30%, and the smallest part is BSDE, around 13.2%. The capital market's weight is then used to calculate the portfolio's equilibrium return in the fifth column.

Table 4. Estimated beta and the return equilibrium CAPM.

| Asset | $\hat{\beta}$ | $\pi$ |
|-------|----------------|------|
| BSDE  | 1.61470        | 0.08766 |
| EXCL  | 0.85129        | 0.04653 |
| INCO  | 1.07933        | 0.06061 |
| JPFA  | 3.08554        | 0.16828 |
| LPPF  | 1.25661        | 0.06754 |
From Table 4, the estimated beta is used to estimate the excess equilibrium return. Then, each equilibrium return is applied to the Black Litterman model without any difference for a scalar $\tau$, $Q$, $P$, and $\lambda$ as a risk aversion coefficient. The portfolio is evaluated to verify reverse optimization and the CAPM formula in the Black Litterman model. Hence, two scenarios are created in this empirical study to examine the CAPM beta formula directly as a starting point in Black Litterman. The equilibrium return obtained from reverse optimization with market capitalization weight in Table 3 is used as the first portfolio. The other one is portfolio Black Litterman with equilibrium return from the CAPM formula in Table 4. The following table presents the result of the Black Litterman portfolio.

| Table 5. Black Litterman portfolio. |
|-----------------------------------|
| $E(R_{BL})$ | Portfolio I | Portfolio II |
| BSDE | 0.01603 | 0.05076 |
| EXCL | 0.00942 | 0.03557 |
| INCO | 0.01011 | 0.03231 |
| JPFA | 0.03852 | 0.10233 |
| LPPF | 0.00746 | 0.02992 |
| $w_{BL}$ | | |
| BSDE | 0.51683 | 0.50863 |
| EXCL | 0.00068 | 0.05962 |
| INCO | 0.05897 | 0.07228 |
| JPFA | 0.34605 | 0.22888 |
| LPPF | 0.07744 | 0.13057 |
| $E(R_F)$ | 0.02279 | 0.05761 |
| $Sd(R_F)$ | 0.01759 | 0.01511 |

In comparing the first and second portfolios, the correlation is calculated between starting points in Tables 3 and 4. The correlation coefficient is 52.67%, and it indicates a higher correlation. The use of the CAPM formula directly as a starting point in Black Litterman is rarely used because of CAPM’s weakness as a prediction [8].

Table 5 reports the result of the Black Litterman strategy, such as the expected return and the asset weight in both portfolios. The comparison of both asset allocation is shown in Figure 2.

At a glance, from Figure 2, BSDE dominates the first position almost over 50% in both portfolios, it is followed by JPFA, LPPF, INCO, and the smallest proportion is EXCL. The differences in proportion are seen in EXCL, JPFA and LPPF. EXCL still seems to be allocated 5% in portfolio-II, but the proportion is very small, even barely allocated in portfolio-I. JPFA is allocated 10% more in portfolio-I than in portfolio-II. In contrast, LPPF is slightly smaller than in portfolio-II.

Nevertheless, the order of proportions of each asset looks the same for both portfolios of Black Litterman. Portfolio-I is built using equilibrium returns from market capitalization weights. Portfolio II uses the equilibrium return starting point of the CAPM formula, both resulting in almost the same
diversification for each asset. While in table 5, it is shown that the expected return value of portfolio-II is greater with the portfolio risk is smaller than portfolio-I. These results demonstrated that the CAPM formula for equilibrium return as the starting value in the Black Litterman model could also be used directly.

5. Conclusion
It cannot be neglected that the use of different equilibrium returns is the ambiguity and confusion in practicing the Black Litterman model. There is an allegation that insignificant return equilibrium from CAPM and reverse optimization with the weight of market capitalization will result in a different portfolio in Black Litterman. For that reason, the Black Litterman portfolios with the two approaches are presented. In this empirical study, only five assets are selected in the portfolio from LQ45 Indonesia. Two portfolios of Black Litterman were built with different equilibrium returns. Firstly, the expected equilibrium return is obtained from reverse optimization using the market capitalization weight. Secondly, the expected equilibrium return is obtained by using the CAPM formula.

Both results of allocation are shown in table 5. It is reported that the equilibrium returns from both models in table 3 and table 4 are significantly different. However, there is no difference in both portfolios’ weight using the Black Litterman model from table 5. In this case, the different starting points, such as the CAPM formula, demonstrates a similar result with reverse optimization in the Black Litterman portfolio. Consequently, the weight of the Black Litterman using two approaches, in this case, can be considered.

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References
[1] Black F and Litterman R 1991 J. Fixed Income 1 7
[2] Black F and Litterman R 1992 Financ. Anal. J. 48 28
[3] Cheung W 2013 Quant. Financ. 13 301
[4] Cheung W 2009 J. Asset Manag. 11 229
[5] Idzorek T 2007 Forecasting expected returns in the financial markets ed. S Satchell (Academic Press)
[6] Duqi A, Franci L and Torlucchini G 2014 Appl. Financ. Econ. 24 1285
[7] Fang Y, Bo L, Zhao D and Wang S 2018 J. Syst. Sci. Complex. 31 975
[8] Nocera S E 2016 Portfolio Construction and Global Asset Allocation: A Practitioner Solution to a Black Litterman Flaw SSRN Electron. J. 2864954
[9] Mankert C and Seiler M J 2011 J. Real Estate Portf. Manag. 17 139
[10] Bertsimas D, Gupta V and Paschalidis I C 2012 Oper. Res. 60 1389
[11] Allaj E 2013 Financ. Mark. Portf. Manag. 27 217
[12] Bessler W and Wolff D 2013 A Theoretical and Empirical Analysis of the Black-Litterman Model Algorithms from Nat. Life (Switzerland: Springer, Cham) p 377
[13] Bessler W, Opfer H and Wolff D 2017 Eur. J. Financ. 23 1
[14] Ganikhodjaev N and Bayram K 2016 Malaysian J. Math. Sci. 10 193
[15] Bayram K, Abdullah A and Meera A K 2018 Int. J. Islam. Middle East. Financ. Manag. 11 334
[16] Geyer A and Lučivjanská K L 2016 J. Portf. Manag. 42 38
[17] Kocuk B 2020 Omega Int. J. Manag. Sci. 91 102008
[18] Subekti R, Ratna Sari E and Kusumawati R 2019 J. Phys. Conf. Ser. 1321 022051
[19] Martin K J and Sankaran H 2019 J. Invest. 28 112
[20] Chen L, Da Z and Schaumburg E 2015 J. Invest. 24 34
[21] He G and Litterman R 2005 The Intuition Behind Black-Litterman Model Portfolios SSRN Electron. J. 334304
[22] Brandimarte P 2018 An Introduction to Financial Markets: A Quantitative Approach (United States of America: John Wiley & Sons)
[23] Walters J 2014 The Black Litterman Model in Detail SSRN Electron. J. 1314585