Matrices and finite Alexander quandles

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Abstract

We study the question of whether a finite quandle specified by a matrix is isomorphic to an Alexander quandle. We first collect some standard observations about necessary conditions for a finite quandle to be Alexander. We then describe an algorithm for determining finding all possible Alexander presentations of a finite quandle given its matrix and provide a URL for Maple code implementing this algorithm.

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1 Introduction

A quandle is a set $Q$ with a binary operation $\triangleright : Q \times Q \to Q$ satisfying the three axioms

\begin{enumerate}
  \item For all $a \in Q$, $a \triangleright a = a$,
  \item For all $a, b \in Q$, there exists a unique $c \in Q$ such that $a = c \triangleright b$, and
  \item For all $a, b, c \in Q$, we have $(a \triangleright b) \triangleright c = (a \triangleright c) \triangleright (b \triangleright c)$.
\end{enumerate}

The uniqueness of $c$ in axiom (ii) implies that the map $f_b : Q \to Q$ defined by $f_b(a) = a \triangleright b$ is a bijection for all $b \in Q$; we denote the inverse $f_b^{-1}(a)$ by $a \triangleleft b$. Then $Q$ forms a quandle under the operation $\triangleleft$, called the dual of $(Q, \triangleright)$; in addition to satisfying the analogs of the above axioms, $\triangleleft$ also distributes over $\triangleright$ and vice-versa.

Quandles have been studied (indeed, rediscovered) numerous times by various authors including Conway and Wraith, Brieskorn [2], Mateev [16] and Fenn and Rourke [6]. The definition and notation above were introduced by David Joyce in [11].

Quandles and finite quandles in particular are of interest to knot theorists since associated to every knot there is a quandle, the knot quandle, which is a complete invariant of knot type up to homeomorphism of topological pairs. Finite quandles then give us a convenient way to distinguish knots, since if two knot quandles $Q_1$ and $Q_2$ are isomorphic, the sets $\text{Hom}(Q_1, Q)$ and $\text{Hom}(Q_2, Q)$ must have the same number of elements. More sophisticated knot invariants involving counting homomorphisms to a finite quandle weighted by cocycles in various quandle cohomology theories are studied in various recent papers such as [3] and [4].
One standard example of a quandle structure is the conjugation quandle of a group $G$. Specifically, $\text{Conj}(G)$ has the same underlying set as the group $G$ with quandle operation given by

$$a \triangleright b = b^{-1}ab.$$  

Moreover, a subset of a group need not be a subgroup to form a quandle under conjugation; any union of conjugacy classes in a group forms a quandle. If the group or collection of conjugacy classes is finite, then we have a finite quandle. Other standard examples of finite quandles include the cyclic quandle $Q = \{1, 2, \ldots, n\}$ with quandle operation defined by

$$i \triangleright j = 2j - i \pmod{n}$$  

and the trivial quandle $T_n = \{1, 2, \ldots, n\}$ with quandle operation given by

$$i \triangleright j = i \quad \forall i, j \in T_n.$$  

The conjugation quandle of any abelian group is trivial.

Another example of a useful quandle structure is the Alexander quandle construction described in section 2. Alexander quandles have been studied in various papers ([7], [18], [15], [17]), and most of the computations of quandle cohomology and counting invariants in recent papers have used finite Alexander quandles. In [10], the counting invariant is shown to depend on the classical Alexander invariants when the target quandle is Alexander, and in [12] the quandle cohomology invariants for quadratic Alexander quandles are shown to be determined by the Alexander invariants for torus knots. Moreover, methods for computing the second cohomology groups for Alexander quandles are given in [10], which permit computation of the 2-cocyle invariants when the target quandle is Alexander. Hence, when studying knots using quandle counting invariants, it is useful to know whether the target quandle is isomorphic to an Alexander quandle.

In [9], a method was described for representing finite quandles as square matrices. These matrices can then be used to find all possible quandle structures of a given cardinality $n$. Specifically, for a quandle $Q = \{x_1, \ldots, x_n\}$, the matrix of $Q$, $M_Q$, is the matrix abstracted from the quandle operation table by dropping the $x$s and keeping only the subscripts. That is, $M_{ij} = k$ where $x_i \triangleright x_j = x_k$. This matrix notation was used in [9] to determine all quandle structures with up to 5 elements. An improved algorithm for finding all quandle matrices together with URLs for the (rather large) files containing the results for $n = 6$, 7 and 8 as well as a method for computing the counting invariant using a target quandle given by a matrix are given in [8]. An independently derived list of quandles of order less than or equal to six can be found in [5].

In this paper, we describe a method of determining whether a quandle defined by a matrix is isomorphic to an Alexander quandle. In section 2 we collect definitions, examples and necessary conditions for a finite quandle to be Alexander, as well as some results which are useful for the following section. We then describe an algorithm for taking a finite quandle matrix and finding all possible Alexander presentations of the given quandle, or determining when none exist. In section 3 we describe our Maple implementation of this algorithm and provide a URL for this implementation.

## 2 Alexander quandles

We begin with a definition.
**Definition 1** Let $\Lambda = \mathbb{Z}[t^{\pm 1}]$ be the ring of Laurent polynomials in one variable with integer coefficients. Let $M$ be a module over $\Lambda$. Then $M$ is a quandle, called an *Alexander quandle*, with quandle operation given by

$$a \triangleright b = ta + (1 - t)b.$$ 

**Example 1** The trivial quandle $T_n$ is an Alexander quandle, namely the quotient module $T_n = \Lambda/(n, 1 - t)$:

$$a \triangleright b = t(a) + (1 - t)b = 1(a) + (1 - 1)b = a.$$ 

**Example 2** Let $n \in \mathbb{Z}_+$ and $h \in \Lambda$. Then the quotient ring $\Lambda/(n, h)$ of Laurent polynomials modulo the ideal generated by $n$ and $h$ is an Alexander quandle. More generally, an Alexander quandle may be a direct sum of such quotients or have a more complicated $\Lambda$-module structure. See [18] for more examples.

The structure of Alexander quandles has been explored in [7] and [18]. In particular, in [18] we find

**Theorem 1** If $M$ and $N$ are finite Alexander quandles, then there is an isomorphism of Alexander quandles $\phi : M \to N$ iff there is an isomorphism of $\Lambda$-modules $f : (1 - t)M \to (1 - t)N$.

See also [11] lemma 1.23.

This theorem tells us when two Alexander quandles are isomorphic, but how do we know whether a quandle given, say, by a matrix, might be secretly Alexander?

**Definition 2** Let $Q$ be a finite quandle. The *Alexanderization* of $Q$, denoted $A(Q)$, is the free $\Lambda$-module on $Q$ modulo the submodule spanned by elements of the form

$$tx_i + (1 - t)x_j - x_i \triangleright x_j$$

for all $i, j = 1, \ldots, |Q|$.

Now suppose there is an isomorphism of quandles $\phi : Q \to M$ where $M$ is a finite Alexander quandle. Then $\phi$ must factor through $A : Q \to A(Q)$, so the diagram

$$\begin{array}{ccc}
Q & \rightarrow & A(Q) \\
\downarrow_{\phi} & & \downarrow \\
M & & \\
\end{array}$$

must commute. In particular, injectivity of $\phi$ implies $A$ must also be injective. Thus we have

**Proposition 2** If a finite quandle $Q$ is isomorphic to an Alexander quandle, then $A : Q \to A(Q)$ is injective.

Proposition 2 gives us a way of identifying certain quandles as non-Alexander. For example, in the Alexanderization of the quandle with matrix

$$Q = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 1 \\ 3 & 3 & 3 \end{bmatrix}$$
we have \(tx_1 + (1-t)x_3 = x_1 \Rightarrow (1-t)x_1 = (1-t)x_3\) from entry \(Q_{32}\). Then \(Q_{13}\) says

\[x_2 = tx_1 + (1-t)x_3 = tx_1 + (1-t)x_1 = x_1\]

and \(A\) is not injective; hence \(Q\) is not Alexander.

**Definition 3** A quandle \(Q\) is **abelian** if for all \(a, b, c, d \in Q\) we have

\[(a \triangleright b) \triangleright (c \triangleright d) = (a \triangleright c) \triangleright (b \triangleright d).\]

**Proposition 3** Alexander quandles are abelian.

**Proof.** Let \(Q\) be Alexander. Then

\[
(a \triangleright b) \triangleright (c \triangleright d) = t(ta + (1-t)b + (1-t)(tc + (1-t)d)
= t^2a + t(1-t)(b + c) + (1-t)^2d
= (a \triangleright c) \triangleright (b \triangleright d).
\]

\[
\Box
\]

**Proposition 4** If \(Q\) is abelian, then \(\triangleright\) is left-distributive. That is,

\[a \triangleright (b \triangleright c) = (a \triangleright b) \triangleright (a \triangleright c).\]

**Proof.** Let \(Q\) be an abelian quandle. Then for any \(a, b, c \in Q\),

\[(a \triangleright b) \triangleright (a \triangleright c) = (a \triangleright a) \triangleright (b \triangleright c) = a \triangleright (b \triangleright c).
\]

\[
\Box
\]

**Corollary 5** Alexander quandles are left-distributive. (See also [14].)

These observations give us ways of testing whether a quandle is Alexander, but they do not give any information about what the Alexander structure(s) on \(Q\) might be. To solve this problem, we need a more constructive approach.

Let \(M\) be an Alexander quandle. The facts that \(t^{-1} \in \Lambda\) and \(t(x + y) = tx + ty \forall x, y \in M\) show that multiplication by \(t\) is an additive automorphism of \(M\). Conversely, given any abelian group \(A\) and automorphism \(\phi \in \text{Aut}_Z(A)\), \(A\) has the structure of a \(\Lambda\)-module, and hence an Alexander quandle, by defining \(tx = \phi(x) \forall x \in M\).

**Definition 4** For any Alexander quandle \(Q\), an Alexander presentation consists of an abelian group structure on \(Q\) together with an additive automorphism \(\phi\) of this abelian group structure such that

\[a \triangleright b = \phi(a) + (1-\phi)(b) \quad \forall a, b \in Q,\]

that is, such that the induced Alexander quandle structure on \(Q\) agrees with the original quandle structure.
Definition 5 Let \( Q = \{x_1, x_2, \ldots, x_n\} \) be a finite quandle. The (standard form) matrix of \( Q \), \( M_Q \), is the matrix \( M_Q \) whose entry in row \( i \) column \( j \) is \( k \) where \( x_i \triangleright x_j = x_k \). A map \( \phi : Q \rightarrow Q \) may be specified by a vector \( v \in Q^n \) such that

\[
v = \begin{bmatrix}
  \phi(1) \\
  \phi(2) \\
  \vdots \\
  \phi(n)
\end{bmatrix}.
\]

Example 3 Let \( Q \) be the Alexander quandle \( \Lambda/(2, t^2 + 1) = \{x_1 = 0, x_2 = 1, x_3 = t, x_4 = 1 + t\} \). Then \( Q \) has quandle matrix

\[
\begin{array}{cccc}
x_1 & x_2 & x_3 & x_4 \\
\hline
x_1 & x_1 & x_4 & x_1 \\
x_2 & x_4 & x_4 & x_1 \\
x_3 & x_2 & x_2 & x_3 \\
x_4 & x_3 & x_4 & x_1
\end{array} \quad \rightarrow \quad M_Q = \begin{bmatrix}
1 & 4 & 4 & 1 \\
3 & 2 & 2 & 3 \\
2 & 3 & 3 & 2 \\
4 & 1 & 1 & 4
\end{bmatrix}.
\]

The matrix of a quandle is just the operation table of the quandle with elements of the quandle replaced by their subscripts. Suppose we are given a quandle matrix \( M_Q \); we would like to determine whether \( Q \) is an Alexander quandle and, if it is, to find all Alexander presentations of \( Q \) from its quandle structure.

We can represent finite abelian (or non-abelian) groups using a matrix notation very similar to our matrix notation for quandles.

Definition 6 Let \( G = \{x_1, x_2, \ldots, x_n\} \) be a finite group. The (standard form) Cayley matrix of \( G \), \( C_G \), is the matrix \( C_G \) whose entry in row \( i \) column \( j \) is \( k \) where \( x_i x_j = x_k \) and \( x_1 \) is the identity element of \( G \).

Example 4 The Alexander quandle in example 2 has abelian group structure \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) with automorphism \( \phi(a, b) = (b, a) \). An Alexander presentation for this quandle is

\[
C_Q = \begin{bmatrix}
1 & 2 & 3 & 4 \\
2 & 1 & 4 & 3 \\
3 & 4 & 1 & 2 \\
4 & 3 & 2 & 1
\end{bmatrix}, \quad \phi = \begin{bmatrix}
1 \\
3 \\
2 \\
4
\end{bmatrix}.
\]

We check that \( \phi \) is an automorphism of \( C_Q \) by checking that applying the permutation \( \phi \) to each element of \( C_Q \), then un-permuting the rows and columns by conjugating by the matrix of the permutation \( \phi \) yields the original matrix.

\[
\begin{bmatrix}
1 & 2 & 3 & 4 \\
2 & 1 & 4 & 3 \\
3 & 4 & 1 & 2 \\
4 & 3 & 2 & 1
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
1 & 3 & 2 & 4 \\
3 & 1 & 4 & 2 \\
2 & 4 & 1 & 3 \\
4 & 2 & 3 & 1
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}.
\]
If $Q$ is a finite Alexander quandle, then one of the elements of $Q$ is the additive identity of $Q$ regarded as an abelian group. Then since $a \triangleright 0 = ta + (1-t)0 = ta$, the column corresponding to the additive identity tells us the action of $t$ on $Q$. We can use this information to either recover the additive structure(s) of $Q$ or show that none is possible with the help of the following observations.

**Lemma 6** If $M$ is an Alexander quandle, then for all $a, b \in M$ we have

$$a \triangleright b + b \triangleright a = a + b.$$  

**Proof.** If $M$ is Alexander, then for any $a, b \in M$

$$a \triangleright b + b \triangleright a = ta + (1-t)b + tb + (1-t)a$$

$$= ta + b - tb + tb + a - ta$$

$$= a + b.$$  

**Lemma 7** Let $M$ be an Alexander quandle. Then for every $a, b, c \in M$ we have

$$a \triangleright b + b \triangleright c = a \triangleright c + b.$$  

**Proof.** If $M$ is Alexander, then for every $c \in M$, we have $a \triangleright c = ta + (1-t)c$. Then

$$a \triangleright b + b \triangleright c = ta + (1-t)b + tb + (1-t)c$$

$$= ta + b - tb + tb + (1-t)c$$

$$= a \triangleright c + b.$$  

**Lemma 8** If $M$ is a finite Alexander quandle, then for a fixed $b \in M$ the map $g_b : M \rightarrow M$ defined by $g_b(x) = x + b$ is an automorphism of quandles (though not of modules). (See also [7], [13]).

**Proof.** By definition,

$$g_b(x \triangleright y) = x \triangleright y + b$$

$$= tx + (1-t)y + b$$

$$= tx + y - ty + b.$$  

On the other hand,

$$g_b(x) \triangleright g_b(y) = (x + b) \triangleright (y + b)$$

$$= t(x + b) + (1-t)(y + b)$$

$$= tx + tb + y + b - ty - tb$$

$$= tx + y + b - ty.$$
and \( g_b \) is a homomorphism of quandles. Setwise, \( g_b \) is a cyclic permutation of the finite set \( M = \{x_1, \ldots, x_n\} \), and hence is bijective. Thus, \( g_b \) is a quandle automorphism of \( M \).

Applying lemmas 7 and 6, every entry of a finite quandle matrix tells us several equations that any Alexander structure on \( Q \) must satisfy, each of which says that one entry in \( C_Q \) is the same as another. In order to fill in the Cayley matrix, we need to have some starting values already filled in. If we assume that the additive identity element in \( Q \) is \( x_1 \), then we can start the Cayley matrix with \( C_Q[1,i] = C_Q[i,1] = i \) for each \( i = 1, \ldots, n \). This assumption results in no loss of generality since by lemma 8 if \( Q \) is Alexander and \( x_1 \) is not the additive identity in \( Q \), we simply apply the quandle automorphism \( f(x_i) = x_i - x_1 \) to obtain the same quandle operation table. With this assumption, multiplication by \( t \) in the Alexander structure on \( Q \) is given by the first column of \( Q \), that is, \( tx_i = x_v[i] \) where \( v \) is the first column of \( Q \).

Then for every element of the quandle matrix we compare the entries in \( C_Q \) for each of the equations in lemmas 6 and 7 if the corresponding entries in \( C_Q \) are different then \( Q \) cannot be Alexander. If the entries are equal, or if both are blank, we move on; if one entry is known and the other blank, we fill in the blank with the known value. While going through this procedure, we exploit the facts that abelian groups are both commutative and associative to fill in the table and find contradictions more rapidly.

In this way, we fill in as much of the Cayley matrix as possible. There will generally be some entries which are left blank, since Alexander quandles may be isomorphic as quandles but distinct as \( \Lambda \)-modules. Thus, to find all possible Alexander structures on a given quandle, we systematically consider all possible ways of filling in the remaining blanks to obtain the Cayley matrix of an abelian group. Having found all such matrices, it only remains to verify that the bijection given by the first column of the quandle matrix \( Q \) is an automorphism of the abelian group structure so defined. If it is, then we have an Alexander presentation of the given quandle.

Our implementation of this algorithm in Maple, available from \url{www.esotericka.org/quandles}, is described in the following section.

### 3 Maple Implementation

In this section we describe an implementation of the algorithm described in section 2 in Maple. This code is available for download at \url{http://www.esotericka.org/quandles}; it uses the file \texttt{quandles-maple.txt} also available from the same website. Improvements and bugfixes will be made as necessary.

We begin with some basic programs for working with abelian groups represented by Cayley matrices. \texttt{assoctest} tests a matrix for associativity, \texttt{commtest} tests a matrix for commutativity, and \texttt{invtest} tests for the presence of inverses by checking that every row and column contains the identity element 1.

To implement the algorithm described in section 2 we start with \texttt{abgroupfill}, which uses the equations of lemmas 6 and 7 to fill in entries in a standard form Cayley matrix, with zeroes representing unknown entries. The program compares the entries in the table which should be equal according to these equations, either replacing a zero with the nonzero value, doing nothing if both values are zero, or setting a “contradiction” counter if it finds two different nonzero values, which results in quitting and reporting “false”.

\texttt{abgroupfill} makes use of \texttt{cafill}, which uses associativity and commutativity to fill in zeroes or find contradictions. The program runs through all triples, checking for associativity, then runs
through all pairs, checking for commutativity. When an entry is changed, a "continue" counter is set to true, so that the loop continues until no more zeroes can be filled in, again exiting if a contradiction is found.

The program `zerofill` uses `findzero` to find the first zero entry in the matrix. It then fills in the zero with each possible nonzero entry from 1 to $n$, propagating these values through with `cafill`. Any resulting matrices are checked for rows or columns without the identity (which fail to be valid Cayley matrices); if every row and column contains either 1 or 0, then the matrix is added to the working list and the "continue" counter is set. When no matrices in the working list have any zeroes, the loop is exited and we have a (possibly empty) list of Cayley matrices representing abelian groups.

Our main program, `alextest`, first checks whether the input quandle is abelian using `abqtest`. If it is, it then uses `abgroupfill` and `zerofill` to find all Cayley matrices corresponding to possible Alexander presentations of $Q$. The final step is to check whether the first column of $Q$ gives an automorphism of the Cayley matrix in question; if it does, we have an Alexander presentation. We use the program `homtest` from the file `quandles-maple.txt` to check this. If the list of Alexander presentations is empty, `alextest` returns "false."

Next, we give some basic tools for constructing quandle matrices for Alexander quandles. `cayley` returns the Cayley matrix for $Z_n$. Many of the quandle tools in `quandles-maple.txt` apply without modification to Cayley matrices; for example, `cprod` gives the Cayley matrix of a cartesian product of two abelian groups, `autlist` finds the automorphism group of an abelian group given the Cayley matrix of the group, etc. `alexquandle` takes a Cayley matrix and a vector representing an automorphism of $CQ$ and returns the matrix of the resulting Alexander quandle structure.

Finally, we include a short program `conjquachange` which takes a Cayley matrix for any group structure and returns the matrix of the conjugation quandle.

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