A note on the Turán number of a Berge odd cycle

Dániel Gerbner
Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences
P.O.B. 127, Budapest H-1364, Hungary.
gerbner@renyi.hu

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Abstract

In this note we obtain upper bounds on the number of hyperedges in 3-uniform hypergraphs not containing a Berge cycle of given odd length. We improve the bound given by Füredi and Özkahya in 2017. The result follows from a more general theorem. We also obtain some new results for Berge cliques.

Keywords: Berge, hypergraph, cycle, Turán number

1 Introduction

We say that a hypergraph \( H \) is a Berge copy of a graph \( F \) (in short: \( H \) is a Berge-\( F \)) if \( V(F) \subseteq V(H) \) and there is a bijection \( f : E(F) \to E(H) \) such that for any \( e \in E(F) \) we have \( e \subseteq f(e) \). This definition was introduced by Gerbner and Palmer [11], extending the well-established notion of Berge cycles and paths. Note that there are several non-uniform Berge copies of \( F \), and a hypergraph \( H \) is a Berge copy of several graphs. A particular copy of \( F \) defining a Berge-\( F \) is called its core. Note that there can be multiple cores in a Berge-\( F \).

We denote by \( ex_r(n, \text{Berge-}F) \) the largest number of hyperedges in an \( r \)-uniform Berge-\( F \)-free hypergraph on \( n \) vertices. There are several papers dealing with \( ex_r(n, \text{Berge-}C_k) \) (e.g. [8, 14, 15, 16]) or \( ex_r(n, \text{Berge-}F) \) in general (e.g. [9, 10, 11, 12, 20]). For a short survey on this topic see Subsection 5.2.2 in [13].

In this note we consider \( ex_3(n, \text{Berge-}C_k) \). In the case \( k = 5 \), this was first studied by Bollobás and Győri [2]. They showed \( ex_3(n, \text{Berge-}C_5) \leq \sqrt{2n^{3/2}} + 4.5n \). This bound was improved to \((0.254 + o(1))n^{3/2}\) by Ergemlidze, Győri and Methuku [5]. For cycles of any length, Győri and Lemons [15, 16] proved \( ex_r(n, \text{Berge-}C_k) = O(n^{1+1/(k/2)}) \). The constant factors were improved by Jiang and Ma [18], and in the case \( k \) is even by Gerbner, Methuku and Vizer [10]. In the 3-uniform case, Füredi and Özkahya [8] obtained better constant factors (depending on \( k \)). In the case \( k \) is even, further improvements were obtained by Gerbner, Methuku and Vizer [10] and by Gerbner, Methuku and Palmer [9].

A closely related area is counting triangles in \( C_k \)-free graphs. More generally, let \( ex(n, H, F) \) denote the maximum number of copies of \( H \) in an \( F \)-free graph on \( n \) vertices. After some sporadic results, the systematic study of these problems (often called generalized Turán problems) was initiated by Alon and Shikhelman [1]. Their connection to Berge hypergraphs was established by Gerbner and Palmer [12], who proved

\[
ex(n, K_r, F) \leq ex_r(n, \text{Berge-}F) \leq ex(n, K_r, F) + ex(n, F)\]

for any \( r, n \) and \( F \).

Counting triangles in \( C_k \)-free graphs and counting hyperedges in Berge-\( C_k \)-free 3-uniform hypergraphs was handled together already by Bollobás and Győri [2] for \( C_5 \), and by Füredi and Özkahya [8], who proved \( ex(n, K_3, C_{2k}) \leq \frac{2k-3}{k} ex(n, C_{2k}) \) and \( ex_3(n, \text{Berge-}C_{2k}) \leq \frac{4k}{k} ex(n, C_{2k}) \). Their upper bound for \( ex(n, K_3, C_{2k}) \) is still the best known bound, but their other upper bound was improved to \( ex_3(n, \text{Berge-}C_{2k}) \leq \frac{2k-3}{k} ex(n, C_{2k}) \) by Gerbner, Methuku and Vizer [10] in the case \( k \geq 5 \) and by Gerbner, Methuku and Palmer [9] in the case \( k = 3, 4 \).
In the case of forbidden cycles of any odd length, the number of triangles was first studied by Györi and Lü [17], who proved \( ex(n, K_3, C_{2k+1}) \leq \frac{(2k-2)(16k-1)}{3}ex(n, C_k) \). It was improved independently by Füredi and Özkahya [8] and by Alon and Shikhelman [1]. The latter had the stronger bound \( ex(n, K_3, C_{2k+1}) \leq \frac{16(k-1)}{3}ex([n/2], C_k) \). In the case \( k = 2 \), the current best bound \( ex(n, K_3, C_5) \leq 0.231975n^{3/2} \) is due to Ergemlidze and Methuku [6].

Füredi and Özkahya [8] obtained the currently best upper bound on the Berge version by showing

\[
ex_3(n, \text{Berge-}C_{2k+1}) \leq ex(n, K_3, C_{2k+1}) + 4ex(n, C_{2k}) + 12ex^3(n, \text{Berge-}C_{2k+1}),
\]

where \( ex^r(n, \text{Berge-F}) \) denotes the largest number of hyperedges in an \( r \)-uniform Berge-\( F \)-free linear hypergraph on \( n \) vertices. Recall that a linear hypergraph is one in which any two hyperedges share at most one vertex.

In this note we improve the bound \((1.1)\). Recall that we have \( ex_3(n, \text{Berge-}C_{2k+1}) \geq ex(n, K_3, C_{2k+1}) \), thus we cannot hope for a huge improvement, especially as \( ex(n, K_3, C_{2k+1}) \) might be the largest of the three terms. Indeed, the best upper bound currently known is \( O(n^{1+1/k}) \) for all the three terms, but the dependence of the known upper bound in \( k \) is the largest for \( ex(n, K_3, C_{2k+1}) \) (we will state these bounds after Theorem 1.2).

Recall that in case of \( C_{2k} \), the two upper bounds obtained by Füredi and Özkahya [8] were \( ex(n, K_3, C_{2k}) \leq \frac{2k-2}{3}ex(n, C_{2k}) \) and \( ex_3(n, \text{Berge-}C_{2k}) \leq \frac{2k}{3}ex(n, C_{2k}) \), and the Berge bound was improved in [10, 9] to match the generalized Turán bound. Our goal would be to do the same here and get rid of the terms \( 4ex(n, C_{2k+1}) + 12ex^3(n, \text{Berge-}C_{2k+1}) \) in \((1.1)\). We cannot achieve that, but we decrease these additional terms. Recall that the currently best bound for the generalized Turán problem is \( ex(n, K_3, C_{2k+1}) \leq \frac{16(k-1)}{3}ex([n/2], C_k) \) by Alon and Shikhelman [1]. Our new upper bound on \( ex_3(n, \text{Berge-}C_{2k+1}) \) is larger than that bound by \( ex^3(n, \text{Berge-}C_{2k+1}) \).

We wonder if it is an example of a more general phenomenon and similar bounds could be obtained for other graphs.

The way we use the linearity involves subdividing an edge \( uv \), i.e. deleting it and adding \( uw \) and \( vw \) for a new vertex \( w \). Our method uses only the following two properties of \( C_{2k+1} \): it can be obtained from \( C_{2k} \) by subdividing an edge and deleting a vertex from \( C_{2k+1} \) we obtain a path. In the next theorem we state our result in the most general form.

**Theorem 1.1.** Let \( F \) be a connected graph obtained from \( F_0 \) by subdividing an edge and \( F' \) be obtained from \( F \) by deleting a vertex. Let \( c = c(n) \) be such that \( ex(n, K_{r-1}, F') \leq cn \) for every \( n \).

Then we have

(i) \( ex_3(n, \text{Berge-}F) \leq ex(n, K_r, F) + 2^{r-1}ex(n, F_0) + ex^r(n, \text{Berge-}F) \),

(ii) \( ex_3(n, \text{Berge-}F) \leq \max \left\{ 1, \frac{2k}{3} \right\} 2^{r-1}ex(n, F_0) + ex^r(n, \text{Berge-}F) \).

In the case \( F = C_{2k+1} \) we have \( F_0 = C_{2k} \) and \( F' = P_{2k} \), the path on \( 2k \) vertices. A theorem of Luo [19] shows \( ex(n, K_{r-1}, P_{2k}) \leq \frac{2k}{r} \left( \frac{2k-1}{r-1} \right) \), but what we need for the 3-uniform case is the Erdős-Gallai theorem [4] showing \( ex(n, P_{2k}) \leq (k - 1)n \). Using this, (ii) of Theorem 1.1 gives \( ex_3(n, \text{Berge-}C_{2k+1}) \leq \frac{8k-3}{3}ex(n, C_{2k}) + ex^3(n, \text{Berge-}C_{2k+1}) \) if \( k > 2 \). We can improve this a little bit.

**Theorem 1.2.** If \( k > 2 \), then \( ex_3(n, \text{Berge-}C_{2k+1}) \leq \frac{168k-16}{3}ex([n/2], C_k) + ex^3(n, \text{Berge-}C_{2k+1}) \)

\[
\leq \left( \frac{1280k - 1280}{3} \sqrt{k} \log k \right) \left[ \frac{n}{2} \right]^{1+1/k} + 2kn^{1+1/k} + 9kn + \frac{16k-16}{3}10k^2 [n/2].
\]

The bound in Theorem 1.2 is currently stronger than the bound given by (i) of Theorem 1.1 for \( F = C_{2k+1} \) and \( r = 3 \). However, an improvement on \( ex(n, K_3, C_{2k+1}) \) would immediately improve the bound in (i). Any significant improvement would make (i) stronger than Theorem 1.2 for \( F = C_{2k+1} \).

The second inequality in Theorem 1.2 follows from known results. Füredi and Özkahya [8] proved \( ex^3(n, \text{Berge-}C_{2k+1}) \leq 2kn^{1+1/k} + 9kn \), and Bukh and Jiang [5] obtained the strongest bound on the Turán number of even cycles by showing \( ex(n, C_{2k}) \leq 80\sqrt{k} \log k n^{1+1/k} + 10k^2 n \).

As we do not have good lower bounds on \( ex(n, C_{2k}) \), we cannot be sure that the first term is actually the larger term. However, if \( ex^3(n, \text{Berge-}C_{2k+1}) \) is the larger term, then our improvement on the upper bound of \( ex_3(n, \text{Berge-}C_{2k+1}) \) is more significant, as we changed the constant factor of

\[\text{We note that the bound is incorrectly stated in their paper [17].} \]

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that term from 12 to 1. Obviously we have \( ex_{2k}^3(n, Berge-C_{2k+1}) \leq ex_3(n, Berge-C_{2k+1}) \), hence further improvement is impossible here.

We prove Theorem 1.1 by combining the ideas of [8] and [1] with the methods developed in [9, 10]. In the next section we state some lemmas needed for the proof. We give a new proof of a lemma by Gerbner, Methuku and Palmer [9], and we strengthen the lemma a little bit. This strengthens results on \( ex_r(n, Berge-K_k) \) for some values of \( r, k \) and \( n \). In Section 3 we prove Theorems 1.1 and 1.2.

2 Lemmas

We say that a graph \( G \) is red-blue if each of its edges is colored with one of the colors red and blue. For a red-blue graph \( G \), we denote by \( G_{\text{red}} \) the subgraph spanned by the red edges and \( G_{\text{blue}} \) the subgraph spanned by the blue edges. For two graphs \( H \) and \( G \) we denote by \( N(H, G) \) the number of subgraphs of \( G \) that are isomorphic to \( H \). Let \( g_r(G) = |E(G_{\text{red}})| + N(K_r, G_{\text{blue}}) \).

Lemma 2.1 (Gerbner, Methuku, Palmer [9]). For any graph \( F \) and integers \( r \) and \( n \), there is a red-blue \( F \)-free graph \( G \) on \( n \) vertices, such that \( ex_r(n, Berge-F) \leq g_r(G) \).

Note that an essentially equivalent version was obtained by Füredi, Kostochka and Luo [7]. The proof of Lemma 2.1 relies on a lemma about bipartite graphs (hidden in the proof of Lemma 2 in [9]). If \( M \) is a matching and \( ab \) is an edge in \( M \), then with a slight abuse of notation we say \( M(a) = b \) and \( M(b) = a \).

Lemma 2.2. Let \( \Gamma \) be a finite bipartite graph with parts \( A \) and \( B \) and let \( M \) be a largest matching in \( \Gamma \). Let \( B' \) denote the set of vertices in \( B \) that are incident to \( M \). Then we can partition \( A \) into \( A_1 \) and \( A_2 \) and partition \( B' \) into \( B_1 \) and \( B_2 \) such that for \( a \in A_1 \) we have \( M(a) \in B_1 \), and every neighbor of the vertices of \( A_2 \) is in \( B_2 \).

Here we present a proof that is built on the same principle, but is somewhat simpler than the proof found in [9]. Before that, let us recall the well-known notion of alternating paths. Given a bipartite graph \( \Gamma \) and a matching \( M \) in it, a path \( P \) in \( \Gamma \) is called alternating if its first edge is not in \( M \), and then it alternates between edges in \( M \) and edges not in \( M \), finishing with an edge not in \( M \). It is well-known and easy to see that deleting the edges of \( P \) from \( M \) and replacing them with the edges of \( P \) that were not in \( M \), we obtain another matching, that is larger than \( M \).

Proof. First we build a set \( V' \subset V(\Gamma) \) in the following way. Let \( V_0 \) be the set of vertices in \( A \) that are not incident to any edges of \( M \). Then in the first step we add to \( V_0 \) the set of vertices in \( B \) that are neighbors of a vertex in \( V_0 \), to obtain \( V_1 \). In the second step we add to \( V_1 \) the vertices in \( A \) that are connected to a vertex in \( V_1 \) by an edge in \( M \), to obtain \( V_2 \). Similarly, in the ith step, if \( i < n \), we add to \( V_{i-1} \) the set of vertices in \( B \) that are neighbors of a vertex in \( V_{i-1} \), while if \( i \geq n \), we add to \( V_{i-1} \) the vertices in \( A \) that are connected to a vertex in \( V_{i-1} \) by an edge in \( M \) (i.e. \( M(b) \) for some \( b \in B \) not in \( V_{i-1} \)), to obtain \( V_i \). After finitely many steps, \( V_i \) does not increase anymore, let \( V' \) be the resulting set of vertices.

We claim that no vertex from \( B \setminus B' \) can be in \( V' \). Indeed, such a vertex could be reached by an alternating path from a vertex in \( A \) that is not incident to \( M \), thus \( M \) is not a largest matching, a contradiction.

Then let \( A_2 = A \cap V' \), \( A_1 = A \setminus A_2 \), \( B_2 = B' \setminus V' \) and \( B_1 = B' \setminus B_2 \). A vertex in \( A_2 \) cannot be connected to a vertex \( v \) not in \( B_2 \), as \( v \) could be added to \( V' \) then. Similarly, for a vertex \( u \in A_1 \), \( M(u) \) has to be in \( B_1 \), otherwise \( M(u) \) is in \( B_2 \) and then \( u \) can be added to \( V' \).

Let us briefly describe how we can apply this lemma to obtain Lemma 2.1. We take a Berge-\( F \)-free \( r \)-uniform hypergraph \( \mathcal{H} \) on \( n \) vertices. Let \( A \) be the set of hyperedges in \( \mathcal{H} \) and \( B \) be the set of sub-edges of these hyperedges (by edge and sub-edge we always mean an edge of size two, i.e. a pair of vertices). We connect \( a \in A \) to \( b \in B \) if \( a \supset b \). Let \( \Gamma' \) denote this auxiliary bipartite graph. Let \( M \) be an arbitrary largest matching and \( B' \) be the vertices of \( B \) incident to the edges in \( M \). It is easy to see that the elements of \( B' \) form an \( F \)-free graph which we call \( G \). Indeed, otherwise \( M \) defines the bijection between a copy of \( F \) and hyperedges in \( \mathcal{H} \) to form a Berge-\( F \).

Now we apply Lemma 2.2 to \( \Gamma' \) and \( M \). We define a red-blue coloring of \( G \) by taking the edges of \( G \) in \( B_1 \) to be the red edges, and the edges of \( G \) in \( B_2 \) to be the blue edges. We have
\(|H| = |A_1| + |A_2| = |B_1| + |A_2| = |E(G_{red})| + |A_2|\). As hyperedges in \(A_2\) have all their neighbors in \(B_2\), they each contain a blue \(K_r\), which is distinct from the other blue \(r\)-cliques obtained this way, showing \(|A_2| \leq N(K_r, G_{blue})\).

Let us remark here that Lemma 2.2 also gives some information on the structure of \(G\). If there is \(a \in A_1\) that has a neighbor \(b \in B \setminus B'\), then we could obtain another matching \(M'\) by changing the neighbor of \(a\) to \(b\), i.e. \(M'(a) = b \) and if \(a' \neq a\), then \(M'(a') = M(a')\). Then \(B'\) is replaced by \(B'' = B' \setminus \{M(a)\} \cup \{b\}\). In this case the same partition of \(A\) into \(A_1\) and \(A_2\), and the partition of \(B''\) into \(B_2\) and \(B'' \setminus B_2\) satisfies Lemma 2.2. This means for \(G\) that we can delete the (red) edge \(M(a)\) and replace it with the edge \(b\), to obtain another \(F\)-free graph.

If on the other hand the vertices in \(A_1\) have all their neighbors in \(B'\), then we could recolor the red edges to blue. Therefore, in \(G\) we can delete an edge and add another edge so that the resulting graph is still \(F\)-free. Let \(\alpha = \alpha_{F,n}\) be the largest value of \(g_r(G')\), where \(G'\) is an \(n\)-vertex \(F\)-free blue-red graph. Assume that each \(n\)-vertex \(F\)-free blue-red graph \(G'\) with \(g_r(G') = \alpha\) is not monoblue and we cannot delete an edge and add another edge to \(G'\) so that the resulting graph is still \(F\)-free. Then by the above, \(G\) cannot be one of these graphs, thus \(ex_r(n, Berge-F) \leq g_r(G) < \alpha\). This is usually a negligible improvement, as we often do not even know the order of magnitude.

However, if \(F = K_k\), Gerbner, Methuku and Palmer [9] proved that \(\alpha_{K_k,n} = \max\{g_r(T_B(n, k - 1)), g_r(T_P(n, k - 1))\}\), where \(T_B(n, k - 1)\) is the monoblue Turán graph \(T(n, k - 1)\) and \(T_P(n, k - 1)\) is the monored Turán graph \(T(n, k - 1)\). We mention without going into the details that their proof also shows that for any other graphs \(G\) we have \(g_r(G) < \alpha_{K_k,n}\). As we cannot delete an edge from \(T(n, k - 1)\) and add another edge to obtain a \(K_k\)-free graphs, we do have an improvement. For example, if \(r = 4\) and \(k = 5\), then the result in [9] determines \(ex_4(n, Berge-K_k)\) for \(n \geq 11\). For \(n = 10\), \(T(10, 4)\) has 36 copies of \(K_4\) and 37 edges. Therefore, \((ex(n, K_r, F)\) is a lower bound on \(ex_r(n, Berge-F)\), we have \(36 \leq ex_4(n, Berge-K_k) \leq 37\). With our new observation, we know \(ex_4(n, Berge-K_k) = 36\).

### 3 Proof of Theorems 1.1 and 1.2

Let \(\cal H\) be a Berge-\(F\)-free \(r\)-graph on \(n\) vertices. We say that an edge \(uv\) with \(u, v \in V(\cal H)\) is \(t\)-heavy if \(u, v\) are contained together in exactly \(t\) hyperedges. First we will build a linear subhypergraph \(\cal H_1\) in a greedy way: if we can find a hyperedge \(H\) that does not share an edge with any hyperedge in \(\cal H_1\), we add \(H\) to \(\cal H_1\), and then repeat this procedure. By definition, \(\cal H_1\) is linear. Let \(\cal H_2\) consist of the remaining hyperedges. Note that \(|\cal H| = |\cal H_1| + |\cal H_2| \leq ex^{lin}_r(n, Berge-F) + |\cal H_2|\), and the remainder of the proof is for proving the needed upper bound on \(|\cal H_2|\).

We build an auxiliary bipartite graph \(\Gamma\) in the usual way: let \(A\) be the set of hyperedges in \(\cal H_2\) and \(B\) be the set of sub-edges of these hyperedges. We connect \(a \in A\) to \(b \in B\) if \(a \supset b\). We will let \(M\) be a largest matching in \(\Gamma\), however, we do not choose \(M\) arbitrarily. Let \(M_0\) be an arbitrary largest matching in \(\Gamma\). Let \(B'\) be the set of vertices in \(B\) that are incident to some edge of \(M_0\) and \(A_0\) denote the set of vertices in \(A\) that are incident to some edge of \(M_0\). Now a hyperedge \(a \in A_0\) contains a sub-edge \(M_0(a)\), at least one sub-edge \(b_0\) shared with a hyperedge in \(\cal H_1\), maybe some sub-edges that are matched to some other \(a' \in A\), and maybe some other sub-edges \(b \in B \setminus B'\).

We have the option to replace in \(M_0\) the edge between \(a\) and \(M_0(a)\) with any of the edges of \(\Gamma\) between \(a\) and an unused sub-edge of \(a\), to obtain another largest matching. We will build a largest matching \(M\), that contains the same vertices \(A_0\) from \(M_0\).

For \(a \in A_0\), we pick \(M(a)\) to be one of the sub-edges \(b \in B\) of \(a\) (potentially we let \(M(a) = M_0(a)\)) in the following way: \(M(a)\) should share exactly one vertex with \(b_0\) (where \(b_0\) is a sub-edge that is also a sub-edge of a hyperedge in \(\cal H_1\)) if possible. We go through the hyperedges greedily; as long as there is a hyperedge \(a \in A_0\) such that \(M_0(a)\) can be changed in this way, we execute the change (it is possible that \(M_0(a)\) cannot be changed originally, but later a sub-edge of \(a\) that is \(M_0(a')\) becomes free to use, when \(M(a')\) is chosen to be different from \(M_0(a')\)). This process finishes after finitely many (at most \(|A_0|\)) steps, as we change \(M_0(a)\) to \(M(a)\) at most once for every \(a \in A_0\). After this, we rename the unchanged \(M_0(a)\) to \(M(a)\).

The resulting matching \(M\) has the following property: for every \(a \in A_0\), \(a\) shares a sub-edge \(b_0\) with a hyperedge in \(\cal H_1\), such that that either \(M(a)\) shares exactly one vertex with \(b_0\), or all the sub-edges of \(a\) sharing exactly one vertex with \(b_0\) are \(M(a')\) for some \(a' \in A_0\).

Now we can apply Lemma 2.2 to \(\Gamma\) and \(M\) to obtain \(A_1, A_2, B_1, B_2\). Let us call the elements of \(B_1\) red edges and the elements of \(B_2\) blue edges. Let \(G\) be the graph consisting of all the red and blue edges. Then \(G\) is obviously \(F\)-free.
Let us now take a random partition of $V(H)$ into $V_1$ and $V_2$. For every $a \in A_0$, we look at $b = M(a)$. If the two vertices of $b$ are in one part, and all the other vertices of $a$ are in the other part, we keep $a$, otherwise we delete it. Let $A^*$ denote the set of elements in $A$ that are not deleted (note that elements in $A \setminus A_0$ are never deleted, thus are in $A^*$). Let $G'$ be the graph consisting of the elements of $B'$ that are connected by an edge in $M$ to an element of $A^*$. Then $G'$ is obviously $F$-free, as it is a subgraph of $G$.

**Claim 3.1.** $G'$ is $F_0$-free, where $F_0$ is any graph for which $F$ can be obtained from $F_0$ by subdividing an edge of $F_0$.

**Proof.** Let us assume we are given a copy $Q$ of $F_0$ in $G'$ such that $uv$ is the edge that needs to be subdivided to obtain $F$. Observe that there is no edge between $V_1$ and $V_2$ in $G'$, thus $Q$ is in one of them, say $V_1$. Let $w$ be a vertex of $M(uw)$ with $u \neq w \neq v$, then $w \in V_2$, thus $w$ is not in $Q$.

We say that a hyperedge $H$ in $H$ is good if $H$ contains $u$ and $w$ for some $w \in M(uw) \setminus \{u, v\}$ and $H$ is not $M(e)$ for any edge $e$ of $Q$. If there is a good hyperedge, then we build a Berge-$F$ with the following core: we subdivide $uw$ with $w$. For each edge $e$ of this core we assign $M(e)$ except for $uw$ (where we assign $H$) and $uv$ (where we assign $M(uw)$). This way we obtain a Berge-$F$, a contradiction.

$M(uw)$ shares at least one sub-edge with a hyperedge $H \in H_1$. If the sub-edge shares exactly one vertex with $uw$, then $H$ is good and we are done. Thus every sub-edge of $M(uw)$ shared with a hyperedge in $H_1$ has to contain none or both of $u$ and $v$. In both cases, when we tried to change $M_0(M(uw))$ when constructing $M$, we failed, because all such edges are matched to some other hyperedges of $H_2$. In particular, $uw$ is $M(a)$ for some $a \in A_0$ and for some $w \in M(u, v) \setminus \{u, v\}$. Observe that $w$ is in $V_2$, thus $M(a)$ has vertices from both parts $V_1$ and $V_2$, hence $a$ cannot be in $A^*$ by the definition of $A^*$. This implies $a$ is good, finishing the proof.

The above claim implies $G'$ has at most $ex(n, F_0)$ edges. For an arbitrary $a \in A$, the probability that $a$ is in $A^*$ is at least $1/2^r - 1$. Let $S$ be any subset of $A$, then we have that the expected value of the number of hyperedges in $A^* \cap S$ is at least $|S|/2^r - 1$, thus there is a partition with $|A^* \cap S| \geq |S|/2^r - 1$.

There are $|B_1| = |A_1|$ red edges in $G$, and there is a random partition where at least $|A_1|/2^r - 1$ elements of $A_1$ are undeleted, hence there are at least $|A_1|/2^r - 1$ red edges in $G'$. This implies $|A_1|/2^r - 1 \leq ex(n, F_0)$. Hence there are at most $2^r - 1 \text{ex}(n, F_0)$ red edges altogether. For the total number of edges in $G$ we can use the same argument: there is a random partition where at least $|A_0|/2^r - 1$ hyperedges in $A_0$ are undeleted, thus for the $G'$ defined by that partition, we have $|A_0| = |E(G)| \leq 2^r - 1 |E(G')| \leq 2^r - 1 \text{ex}(n, F_0)$.

Observe that we have $|H_2| = |A_1| + |A_2| \leq |A_1| + N(K_r, G_{blue}) \leq |A_1| + \text{ex}(n, K_r, F)$, hence we are done with the proof of (i).

Note that $G$ is not necessarily $F_0$-free, but it is $F$-free. Let $m$ be the number of blue edges in $G$, then $G$ has at most $2^r - 1 \text{ex}(n, F_0) - m$ red edges. An argument of Gerbner, Methuku and Vizer [10] bounds the number of $r$-cliques in $F$-free graphs with the given number of vertices and edges. For sake of completeness, we include the argument here.

Let $d(v)$ be the degree of $v$ in $G_{blue}$. Obviously the neighborhood of every vertex in $G_{blue}$ is $F'$-free. An $F'$-free graph on $d(v)$ vertices contains at most $ex(d(v), K_{r-1}, F') \leq c d(v)$ copies of $K_{r-1}$. Thus $v$ is contained in at most $cd(v)$ copies of $K_r$ in $G_{blue}$. If we sum, for each vertex, the number of $K_r$’s containing a vertex, then each $K_r$ is counted $r$ times. On the other hand as $\sum_{v \in V(G_{blue})} d(v) = 2 |E(G_{blue})| = 2m$, we have $\sum_{v \in V(G_{blue})} c d(v) = 2cm$. This gives that the number of blue $K_r$’s is at most $2cm/r$. Thus we have

$$g_r(G) \leq 2^r - 1 \text{ex}(n, F_0) - m + 2cm/r \leq \max \left\{ 1, \frac{2c}{r} \right\} \left( 2^r - 1 \text{ex}(n, F_0) - m + m \right) = \max \left\{ 1, \frac{2c}{r} \right\} 2^r - 1 \text{ex}(n, F_0).$$

The above inequality, together with Lemma 2.1 implies that $|H_2| \leq \max \left\{ 1, \frac{2c}{r} \right\} 2^r - 1 \text{ex}(n, F_0)$, finishing the proof of (ii).

Now we show how to obtain the small improvement needed to prove Theorem 1.2. It is based on the proof of the upper bound on $\text{ex}(n, K_3, C_{2k+1})$ in [1]. If $n$ is odd, replace it by $n + 1$. As
the stated upper bound is the same in both cases, obvious mononicity conditions show we can do this. Thus we can assume \( n \) is even. When we take the random partition into \( n/2 \) sets \( U_1, \ldots, U_{n/2} \) of size 2, and then randomly put one vertex into \( V_1 \) and the other into \( V_2 \). The obtained graph \( G' \) will be \( C_{2k} \)-free, and it is divided into two components, hence it has at most \( \text{ex}(|V_1|, C_{2k}) + \text{ex}(|V_2|, C_{2k}) \) edges. The way we chose \( V_1 \) ensures the above sum is \( 2\cdot \text{ex}(\lceil n/2 \rceil, C_{2k}) \). Then we can go through every step of the remaining part of the proof to obtain the result we need, if for an arbitrary \( a \in A \), the probability that \( a \) is in \( A^* \) is still at least \( 1/4 \). We will separate into cases according to the intersection of \( a \) with the parts \( U_i \). In case the three vertices of \( a \) are in three different \( U_i \)’s, the probability is 1/4. In case \( a \) contains \( U_i \) for some \( i \), there are two cases. If \( M(a) = U_i \), then the probability is 0, otherwise it is 1/2. As \( M(a) = U_i \) happens with probability 1/3 (having the condition that \( a \) contains \( U_i \)), for every \( i \) we have that the probability of \( a \) being in \( A^* \) if \( a \) contains \( U_i \) is \( 2/3 \cdot 1/2 \geq 1/4 \).

This gives the first inequality of Theorem 1.2. As we have mentioned after the statement, the second inequality follows from earlier results, stated there.

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