Geometry of almost Cliffordian manifolds: Nijenhuis tensor

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GEOMETRY OF ALMOST CLIFFORDIAN MANIFOLDS: NIJENHUIS TENSOR

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Abstract. We generalize some classical results on Nijenhuis tensor for an almost Cliffordian manifold based on arbitrary Clifford algebra and suggest its relations with the integrability of the corresponding G-structure. We prove the set of properties for Nijenhuis tensors with respect to arbitrary Clifford algebra.

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1. Almost Cliffordian Manifolds

Let $\mathcal{O} = \mathcal{C}\ell(s,t)$ be a Clifford algebra. If $M$ is an $km$-dimensional manifold, where $k = 2^{s+t}$ and $m \in \mathbb{N}$, then an almost Clifford manifold is given by a reduction of the structure group $\text{GL}(km, \mathbb{R})$ of the principal frame bundle of $M$ to

$$\text{GL}(m, \mathcal{O}) = \{ A \in \text{GL}(km, \mathbb{R}) | AI = IA, I \in \mathcal{O} \},$$

where $\mathcal{O}$ is arbitrary Clifford algebra. In other words, an almost Clifford manifold is a smooth manifold equipped by the set of anti commuting and commuting affinors $I_i, i = 1, \ldots, t$, $J_j^2 = -E$ and $J_j, j = 1, \ldots, s$, $J_j^2 = E$ such that the free associative unitary algebra generated by $(I_i, J_j, E)$ is isomorphically equivalent to $\mathcal{O}$. In particular, on the elements of this reduced bundle, one can define affinors in the form $F_1, \ldots, F_k$ globally.

Definition 1. Let $M$ be a smooth manifold such that $\text{dim}(M) = m$. Let $A$ be a smooth $\ell$-dimensional ($\ell < m$) vector subbundle in $T^* M \otimes TM$ such that the identity affinor $E = \text{id}_{TM}$ restricted to $T_xM$ belongs to $A_xM \subset T^*_xM \otimes T_xM$ at each point $x \in M$. We say that $M$ is equipped with an $\ell$-dimensional $A$-structure.

It is easy to see that an almost Clifford structure is not an $A$-structure because the affinors in the form $F_0, \ldots, F_\ell \in A$ have to be defined only locally.

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Definition 2. The $A$-structure, where $A$ is a Clifford algebra $\mathcal{O}$, is called an almost Cliffordian manifold.

In particular, the almost Clifford and almost Cliffordian structures are $G$-structures based on Clifford algebras. Two most important examples are an almost hypercomplex geometry and an almost quaternionic geometry, which are based on Clifford algebra $Cl(0, 2)$. Note that the geometric property of an almost hypercomplex structure reads that there is no nontrivial $G$-invariant subspace $\mathcal{D}$ in $\nabla \otimes \wedge^2 \nabla^*$ because the first prolongation $g^{(1)}$ of the Lie algebra $\mathfrak{g}$ vanishes. For an almost quaternionic structure, the situation is more complicated, because $g^{(1)} = \nabla^*$ and there is a class of these structures indexed by $\nabla$, see [1]. Note that for Cliffordian structures based on $Cl(0, 3)$ is $g^{(1)} = \nabla^*$ and there is a class of these structures indexed by $\nabla$ too, see [2].

2. Nijenhuis tensor

The Nijenhuis tensor plays an important role in the theory of integrability. As a classical concept, Nijenhuis introduced $N_J \in \wedge^2 T^* M \otimes TM$ of an almost complex structure $J \in T^* M \otimes TM$. This tensor is an obstruction for an almost complex structure which distinguished it from the complex structure, i.e. their integrability. Recall that Nijenhuis tensor $N(P, Q) \in \wedge^2 T^* M \otimes TM$ for a pair of tensors $P, Q \in T^* M \otimes TM$ is given by the expression

$$N(P, Q)(X, Y) = [PX, QY] - P[QX, Y] - Q[X, PY] + [QX, PY] - Q[PX, Y] - P[X, QY] + (PQ + QP)[X, Y].$$

An almost quaternionic manifold $M$ is integrable if and only if the Nijenhuis tensors $N(I, I)$ and $N(J, J)$ vanish, where $I, J, IJ \in T^* M \otimes TM$ is a quaternionic structure. Let us finally note that in [3], the author proved a similar fact for Clifford algebra $Cl(0, 3)$.

If $P = Q$, then, by straightforward computing,

$$N(P, P)(X, Y) = 2(PX, PY) - P[PX, Y] - P[X, PY] + P^2 [X, Y]$$

and if $P = E$, where $E$ is an identity, then

$$N(E, Q)(X, Y) = 0.$$
and
\[(P \tilde{\wedge} S)(X, Y) := PS(X, Y)\]  
(2)

where \(S \in \otimes^2 \mathcal{V} \otimes \mathcal{V}^*\) and \(N \in \mathcal{V} \otimes \mathcal{V}^*\). Now, one can easily check the following identities
\[N(L, QP) + N(Q, LP) = N(L, QP)\tilde{\wedge}P + L\tilde{\wedge}N(Q, P)\]
\[+ Q\tilde{\wedge}N(L, P)\]
(3)
\[(S\tilde{\wedge}Q)\tilde{\wedge}P - (S\tilde{\wedge}P)\tilde{\wedge}Q = S\tilde{\wedge}QP - S\tilde{\wedge}PQ\]
(4)
\[(L\tilde{\wedge}S)\tilde{\wedge}P = L\tilde{\wedge}(S\tilde{\wedge}P)\]
(5)

**Lemma 1.** Let \(\mathcal{O} = \mathcal{C}l(s, t)\) be a Clifford algebra. If \(F, G \in \mathcal{O}\) such that \(F \neq G\), then the following identities hold:
\[N(FG, F) = -\frac{1}{2}N(F, G)\tilde{\wedge}F - \frac{1}{2}G\tilde{\wedge}N(F, F) + \frac{1}{4}N(F, F)\tilde{\wedge}G\]
(6)
\[0 = N(F, G)\tilde{\wedge}F + 2F\tilde{\wedge}N(F, G) + G\tilde{\wedge}N(F, F) + \frac{1}{2}N(F, F)\tilde{\wedge}G\]
(7)
\[N(G, H) = N(F, G)\tilde{\wedge}G + G\tilde{\wedge}N(F, G) + F\tilde{\wedge}N(G, G)\]
(8)

**Proof.** Putting \(L = Q = F, P = G\) in (3), we find
\[N(F, FG) + N(F, F) = N(F, F)\tilde{\wedge}G + F\tilde{\wedge}N(F, G) + F\tilde{\wedge}N(F, G),\]
that is,
\[N(FG, F) = \frac{1}{2}N(F, F)\tilde{\wedge}G + F\tilde{\wedge}N(F, G).\]
(9)

Putting \(L = G, P = Q = F\) in (3), we find
\[N(G, F^2) + N(F, GF) = N(G, F)\tilde{\wedge}F + G\tilde{\wedge}N(F, F) + F\tilde{\wedge}N(G, F),\]
that is,
\[N(F, GF) = N(G, F)\tilde{\wedge}F + G\tilde{\wedge}N(F, F) + F\tilde{\wedge}N(G, F).\]
(10)

Adding (9) and (10) and dividing the sum by 2, we find
\[N(FG, F) = -\frac{1}{2}N(F, G)\tilde{\wedge}F - \frac{1}{2}G\tilde{\wedge}N(F, F) + \frac{1}{4}N(F, F)\tilde{\wedge}G\]
and subtracting (10) from (9), we find
\[N(F, G)\tilde{\wedge}F + 2F\tilde{\wedge}N(F, G) + G\tilde{\wedge}N(F, F) + \frac{1}{2}N(F, F)\tilde{\wedge}G = 0\]
(11)

Finally, putting \(L = P = G, Q = F\) in (3), we find
\[N(F, G^2) + N(G, FG) = N(F, G)\tilde{\wedge}G + F\tilde{\wedge}N(G, F) + G\tilde{\wedge}N(F, G),\]
that is,
\[N(G, H) = N(F, G)\tilde{\wedge}G + G\tilde{\wedge}N(F, G) + F\tilde{\wedge}N(G, G).\]
Lemma 2. Let $\mathcal{O} = \mathcal{C} l(s,t)$ be a Clifford algebra. If $F, G \in \mathcal{O}$ such that $F \neq G$, then the following identity holds:

$$
\epsilon_1 N(F, F) - \epsilon_2 N(G, G) + N(H, F)\tilde{\gamma}G + G\tilde{\gamma}N(H, F) - N(G, H)\tilde{\gamma}F + F\tilde{\gamma}N(G, H) + 2H\tilde{\gamma}N(F, G) = 0,
$$

where $H = FG$ and $\epsilon_i = 1$ for $K^2 = -1$ and $\epsilon_i = -1$ for $K^2 = 1$.

Proof. Putting $L = FG$, $Q = F$, $P = G$ in (3), we find

$$
N(FG, FG) = N(F, FG) + FG\tilde{\gamma}N(F, G) + F\tilde{\gamma}N(FG, G),
$$

that is,

$$
N(H, H) = \epsilon N(F, F) + N(FG, F)\tilde{\gamma}G + FG\tilde{\gamma}N(F, G) + F\tilde{\gamma}N(FG, G),
$$

where $\epsilon = 1$ for $G^2 = -1$ and $\epsilon = -1$ for $G^2 = 1$.

Proof. One can easily check that putting $L = P = Q = F$ gives $N(F, F^2) + N(F, F^2) = N(F, F)\tilde{\gamma}F + F\tilde{\gamma}N(F, F) + F\tilde{\gamma}N(F, F)$.

Lemma 3. Let $\mathcal{O} = \mathcal{C} l(s,t)$ be a Clifford algebra. If $F \in \mathcal{O}$, then the following identity holds:

$$
N(F, F)\tilde{\gamma}F = -2F\tilde{\gamma}N(F, F).
$$

Proof. One can easily check that putting $L = P = Q = F$ gives $N(F, F^2) + N(F, F^2) = N(F, F)\tilde{\gamma}F + F\tilde{\gamma}N(F, F) + F\tilde{\gamma}N(F, F)$.

□
Theorem 1. Let $\mathcal{O}$ be a Clifford algebra $\mathcal{C}l(s,t)$ and let $F, G \in \mathcal{O}$ such that $F \neq G$. If the Nijenhuis tensors $N(F, F)$ and $N(G, G)$ vanish, then $N(FG, FG)$ vanishes.

Proof. Since $N(F, F) = 0$, we have from (9) $N(H, F) = \hat{F} \wedge N(F, G)$, and from (11),
\begin{equation}
N(F, G) \hat{F} = -2F \wedge N(F, G). \tag{15}
\end{equation}
Since $N(G, G) = 0$, we have from (9), where we changed $F$ and $G$,
\begin{equation}
N(F, G) \hat{G} = -2G \wedge N(F, G) \tag{16}
\end{equation}
and from (11), where we changed $F$ and $G$,
\begin{equation}
N(F, G) \hat{G} = -2G \wedge N(F, G). \tag{17}
\end{equation}
Now, if we substitute $N(F, F) = 0, N(G, G) = 0$, and (16) into (12), then the part containing $\epsilon_i$ vanishes and the proof is correct for any Clifford algebra, i.e., we find
\begin{equation}
(F \wedge N(F, G)) \hat{G} + G \wedge (F \wedge N(F, G))
- (G \wedge N(F, G)) \hat{F} - F \wedge (G \wedge N(F, G)) + 2H \wedge N(F, G) = 0,
\end{equation}
from which
\begin{equation}
(F \wedge N(F, G)) \hat{G} - (G \wedge N(F, G)) \hat{F} = 0, \tag{18}
\end{equation}
since
\begin{equation}
G \wedge (F \wedge N(F, G)) = -F \wedge (G \wedge N(F, G)) = -H \wedge N(F, G)
\end{equation}
by virtue of $GF = -FG = -H$. Now, using (5), (15) and (17), we find, from (18)
\begin{align*}
F \wedge (N(F, G) \hat{G}) - G \wedge (N(F, G) \hat{F}) &= 0, \\
-2F \wedge (G \wedge N(F, G)) + 2G \wedge (F \wedge N(F, G)) &= 0, \\
-4FG \wedge N(F, G) &= 0,
\end{align*}
that is,
\begin{equation}
H \wedge N(F, G) = 0. \tag{19}
\end{equation}
Since $H^2 = -1$, we have from (19) $N(F, G) = 0$. \qed

Corollary 1. Let $\mathcal{O}$ be a Clifford algebra $\mathcal{C}l(s,t)$. If the Nijenhuis tensors $N(I_i, I_i)$ vanish, where $I_i$ are the algebra generators of $\mathcal{O}$, then
\begin{equation}
N(F_i, F_j) = 0,
\end{equation}
where $F_i$ are vector space generators.
3. CLASSES OF SUBORDINATED CONNECTIONS

Recall the concept of $A$-planar curves on $A$-structures equipped with the linear connection $\nabla$. For any tangent vector $X \in T_X M$, we shall write $A_X(X)$ for the vector subspace

$$A_X(X) = \{F_i(X) | F_i \in A_X M\} \subset T_X \nabla \nabla$$

and call it the $A$-hull of the vector $X$. Similarly, $A$-hull of a vector field is a subbundle in $TM$ obtained pointwise. Let $M$ be a smooth manifold equipped with an $A$-structure and a linear connection $\nabla$. A smooth curve $c : \mathbb{R} \to M$ is said to be $A$-planar if

$$\nabla c \in A(\dot{c})$$

In [5], the authors proved a set of facts about the class of $D$-connections. The theorems below, about Cliffordian structures, are proved in paper [5] and some examples of this concept can be found in papers [2,4]. The theorems about $D$-connections can be found in [1].

Following [4, 5], we have a set of results on Clifford and Cliffordian manifolds.

**Corollary 2.** Let $M$ be a smooth manifold equipped with a $G$-structure, where $G = \text{GL}(n, \mathcal{O})$, $\mathcal{O} = \mathcal{C} \ell(s, t)$, $s + t > 1$, i.e. an almost Clifford manifold. Then the $G$-structure is of type 1 and there exists a unique $D$-connection.

One can see that an almost Cliffordian manifold $M$ is given as a $G$-structure provided that there is a reduction of the structure group of the principal frame bundle of $M$ to $G := \text{GL}(m, \mathcal{O}) \otimes \text{GL}(1, \mathcal{O})$ by $\text{GL}(m, \mathcal{O}) \times \text{GL}(1, \mathcal{O})$, the action of $G$ on $T_X M$ looks like $QXq$, where $Q \in \text{GL}(m, \mathcal{O}), q \in \text{GL}(1, \mathcal{O})$, where the right action of $\text{GL}(1, \mathcal{O})$ is blockwise. In this case the tensor fields in the form $F_1, \ldots, F_k$ can be defined only locally. It is easy to see that the Lie algebra $\mathfrak{gl}(m, \mathcal{O})$ of a Lie group $\text{GL}(m, \mathcal{O})$ is of the form

$$\mathfrak{gl}(m, \mathcal{O}) = \{A \in \mathfrak{gl}(km, \mathbb{R}) | AI_i = I_i A, AJ_j = J_j A\}$$

and the Lie algebra $\mathfrak{g}$ of a Lie group $\text{GL}(m, \mathcal{O}) \otimes \text{GL}(1, \mathcal{O})$ is of the form $\mathfrak{g} = \mathfrak{gl}(m, \mathcal{O}) \otimes \mathfrak{gl}(1, \mathcal{O})$.

Let us note that the cases of $\mathcal{C} \ell(s, t)$, where $s + t = 2$, were studied in [6,7] and the case of $\mathcal{C} \ell(0, 3)$ was studied in a detailed way in [2].

**Corollary 3.** Let $M$ be an almost Cliffordian manifold based on Clifford algebra $\mathcal{O} = \mathcal{C} \ell(s, t)$, where $\text{dim}(M) \geq 2(s + t)$, i.e., a smooth manifold equipped with a $G$-structure, where $G = \text{GL}(n, \mathcal{O}) \otimes \text{GL}(1, \mathcal{O})$ or equivalently an $A$-structure, where $A = \mathcal{O}$. Then the class of $D$-connections preserves $A$ and shares the same $A$-planar curves, which are isomorphic to $(\mathbb{R}^k m)^*$. 
4. Conclusion

From the classical theory, the Nijenhuis tensor is a part of a torsion of any almost complex connection $\nabla$ and is $J$-antilinear in each argument,

$$N_J(X, Y) = T \nabla(X, Y) + JT \nabla(JX, Y) + JT \nabla(X, JY) - T \nabla(JX, JY)$$

and the connection $\nabla$ called minimal such that $N_J = 4T \nabla$, i.e. the structure is integrable if and only if $N_J$ vanishes. Let $M$ be an almost quaternionic manifold or an almost quaternionic manifold of the second kind (paraquaternionic). In [6] and [7], the authors proved that the structure is integrable if and only if the structure tensor $T_Q = N(I, I) + N(J, J) + N(K, K)$ vanishes. In this case, there is the class of $\mathcal{D}$-connections without torsion.

A similar fact was proved for an almost Cliffordian manifold, where $\mathcal{C}l(0, 3)$, see [3]. In this article, the authors proved that the structure tensor which is locally generated by $F_i$ is given by

$$T_Q = \sum_{i=1}^{6} N(F_i, F_i) + \sum_{i=1}^{6} N(F_i, F_i) \partial(a \otimes F_a),$$

where $\partial$ denotes Spencer’s operator of alternation. The following step is to find a description of the structure preserving connection based on the Nijenhuis tensor for any Cliffordian manifold.

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