Portfolio Liquidation Games with Self-Exciting Order Flow

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Overview

1. Portfolio liquidation
2. Feedback effect and child order flow
3. The liquidation game
4. Deterministic benchmark games
5. Conclusion
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1. Portfolio liquidation
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Models of optimal block trading have long been studied in the economics literature (e.g. Kyle ’85, Easley and O’Hara ’87...) focus is on deriving endogenous impact functions from information asymmetries

Renewed attention in the financial mathematics literature (e.g. Bertsimas & Lo ’98, Almgren & Chriss ’01...) focus is on structural models within which to derive optimal portfolio strategies for endogenously given impact functions models give rise to novel stochastic control problems:

(‘Liquidation’) constraint on the terminal state
singular terminal condition on the associated HJB equation
unknown terminal condition on the associated adjoint equation
The single player benchmark model: Graewe & H. ’17

The large investor’s stochastic control problem is given by

$$\text{ess inf}_{\xi \in L^2_{\mathcal{F}}(0,T;\mathbb{R})} \mathbb{E} \left[ \int_0^T \left\{ \eta \xi_s^2 + \xi_s Y_s + \lambda_s X_s^2 \right\} ds \right]$$

subject to the state dynamics

$$\begin{cases} 
X_t = x - \int_0^t \xi_s ds, & t \in [0, T], \\
X_T = 0, \\
Y_t = \int_0^t \{ -\rho_s Y_s + \gamma \xi_s \} ds, & t \in [0, T].
\end{cases}$$
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Hawkes process

- Market order flow follows Hawkes processes whose base intensities depend on the large investor’s trading activities
  - Hawkes processes and stock price volatility: Bacry et al. ’13, ’15; El Euch et al. ’18; Jaisson & Rosenbaum ’15; H. & Xu ’19
  - *exogenous* Hawkes processes and liquidation: Alfonsi & Blanc’16; Amaral & Papanicolaou ’19; Cartea et al. ’18
- in our model: Hawkes processes are *endogenously controlled*
Feedback effect and child order flow

Hawkes market model

- market order dynamics follow Hawkes processes with rates:
  \[ \zeta_t^\pm := \mu_t + \xi_t^\pm + \alpha \int_0^t e^{-\beta(t-s)} dN_s^\pm \]

- expected number of (net) sell orders
  \[ \tilde{Z}_t = \mathbb{E}[\tilde{Z}_t^+ - \tilde{Z}_t^-] = \int_0^t \mathbb{E}[\xi_s] ds + \alpha \int_0^t e^{-\beta(t-s)} \tilde{Z}_s ds \]
  \[= C_t \]

- expected number \( C_t \) of (net) sell child orders satisfies
  \[ dC_t = (-(\beta - \alpha)C_t + \alpha(\mathbb{E}[X] - \mathbb{E}[X_t]))dt, \quad C_0 = 0. \]
The mean-field type control problem

The mean-field type control problem for our large investor:

$$\operatorname{ess} \inf_{\xi \in L^2_{\mathcal{F}}(0,T;\mathbb{R})} \mathbb{E} \left[ \int_0^T \left\{ \eta_s \xi_s^2 + \xi_s Y_s + \lambda_s X_s^2 \right\} ds \right]$$

subject to the following state dynamics on $[0, T]$:

$$\begin{cases} 
    dX_t = -\xi_t \, dt, \\
    dY_t = (-\rho_t Y_t + \gamma_t (\xi_t - (\beta - \alpha)C_t + \alpha (\mathbb{E}[X] - \mathbb{E}[X_t]))) \, dt, \\
    dC_t = (- (\beta - \alpha)C_t + \alpha (\mathbb{E}[X] - \mathbb{E}[X_t])) \, dt, \\
    X_0 = \mathcal{X}, \ X_T = 0, \ Y_0 = 0, \ C_0 = 0.
\end{cases}$$

This is a non-convex optimization problem (convex for the MFG).
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Many player models

The optimization problem for player $i$ given the strategies $\xi^{-i}$ reads:

$$\text{ess inf}_{\xi^i \in L^2_{\mathcal{F}}(0,T;\mathbb{R})} \mathbb{E} \left[ \int_0^T \eta^i_s (\xi^i_s)^2 + \xi^i_s Y^i_s + \lambda^i_s (X^i_s)^2 \, ds \right]$$

subject to the state dynamics ($\bar{\xi}_t$ etc. denotes average quantities)

$$\begin{cases}
    dX^i_t = -\xi^i_t \, ds, \\
    dY^i_t = \left( -\rho^i_t Y^i_t + \gamma^i_t (\bar{\xi}_t - (\beta^i_t - \alpha^i_t)C^i_t + \alpha^i_t (\mathbb{E}[\bar{X}] - \mathbb{E}[\bar{X}_t])) \right) \, dt, \\
    dC^i_t = \left( -(\beta^i_t - \alpha^i_t)C^i_t + \alpha^i_t (\mathbb{E}[\bar{X}] - \mathbb{E}[\bar{X}_t]) \right) \, dt, \\
    X^i_0 = X^i, \quad X^i_T = 0, \quad Y^i_0 = 0, \quad C^i_0 = 0.
\end{cases}$$
The mean field game

The corresponding MFG is then given by

$$\text{ess inf}_{\xi \in L^2_{\mathcal{F}}(0,T;\mathbb{R})} \mathbb{E} \left[ \int_0^T \left\{ \eta_t(\xi_t)^2 + \xi_t Y_t + \lambda_t(X_t)^2 \right\} dt \right]$$

subject to the state dynamics

$$\begin{cases}
\frac{dX_t}{dt} = -\xi_t ds,
\frac{dY_t}{dt} = (-\rho_t Y_t + \gamma_t(\mu_t - (\beta_t - \alpha_t)C_t + \alpha_t(\mathbb{E}[X] - \nu_t)) ) dt \\
\frac{dC_t}{dt} = -(\beta_t - \alpha_t)C_t + \alpha_t(\mathbb{E}[X] - \nu_t)) dt,
\end{cases}$$

$$X_0 = \mathcal{X}, \quad X_T = 0, \quad Y_0 = 0, \quad C_0 = 0,$$

and the equilibrium condition

$$\mathbb{E}[\xi^*_t(\mu, \nu)] = \mu_t, \quad \mathbb{E}[X^*_t(\mu, \nu)] = \nu_t.$$
The mean field FBSDE for open-loop equilibria

The following MF FBSDE system covers $N$-player and MF game.

\[
\begin{aligned}
    dX_t^i &= - \frac{M_t^i - \frac{1}{N} \langle \widehat{B}_t^{i,(1)}, P_t^i \rangle}{2\eta_t^i} \, dt, \\
    dS_t^i &= \left( -A_t^i S_t^i + K_t^i X_t + R_t^i \right) \, dt, \\
    -dM_t^i &= \left( 2\lambda_t^i X_t^i + \frac{1}{N} \mathbb{E} \left[ \langle \widehat{B}_t^{i,(2)}, P_t^i \rangle \right] + \left( \Theta, -A_t^i S_t^i + K_t^i X_t + R_t^i \right) \right) \, dt \\
    &\quad - Z_t^{M^i} \, dW_t, \\
    -dP_t^i &= \left( -(A_t^i)^\top P_t^i + \Theta \frac{M_t^i - \frac{1}{N} \langle \widehat{B}_t^{i,(1)}, P_t^i \rangle}{2\eta_t^i} \right) \, dt - Z_t^{P^i} \, dW_t, \\
    X_0^i &= X^i, \quad X_T^i = 0, \quad S_0^i = (0, 0)^\top, \quad P_T^i = (0, 0)^\top, \quad M_T^i = ?
\end{aligned}
\]

with

\[
\xi^j := \frac{M^j - \frac{1}{N} \langle \widehat{B}_t^{j,(1)}, P_t^j \rangle}{2\eta^j}, \quad \chi := (\xi, \mathbb{E}[X], \mathbb{E}[\xi])^\top.
\]
(No) Symmetry

Remark

- no symmetry is assumed for the $N$-player game
- symmetry is required for the convergence to the MFG solution
Existence

Theorem

Under a weak interaction condition, there exists a unique solution

\[(X^i, S^i, M^i, P^i, Z^{M^i}, Z^{P^i}) \in \mathcal{H}_{a,\mathcal{F}} \times \mathcal{S}_{\mathcal{F}}^2 \times L^2_{\mathcal{F}} \times \mathcal{H}_{\iota,\mathcal{F}} \times L^{2,\iota} \times L^2_{\mathcal{F}}\]

to the above FBSDE system for positive constants \(a < 1, \iota < 1/2\).

- \(\mathcal{H}_{a,\mathcal{F}}\): the subspace of \(\mathcal{S}_{\mathcal{F}}^2\) s.t. \(\|y\|_{a} := (\mathbb{E}[\sup_{0 \leq t \leq T} (\frac{|y_t|}{(T-t)^a})^2])^{\frac{1}{2}} < \infty\)

- \(L^{2,\iota}_{\mathcal{F}}\): the space of all progressive processes s.t. for each \(\epsilon > 0\),
  \[\mathbb{E} \left[ \int_{0}^{T-\epsilon} |y_t|^2 \, dt \right] < \infty\]

- \(\mathcal{S}_{\mathcal{F}}^{2,\iota}\): the space of all progressive continuous processes s.t.
  \[\|y\|_{\mathcal{S}^{2,\iota}} := (\sup_{\epsilon \geq 0} \mathbb{E}[\sup_{0 \leq t \leq T-\epsilon} |y_t|^2])^{\frac{1}{2}} < \infty\]
The proof

The proof uses a continuation method (cf. FGHP ’20). First, decouple the FBSDE system and make the ansatz

\[ \tilde{M}^i = A^i \tilde{X}^i + B^i. \]

where \( A^i \) satisfies the singular BSDE

\[
\begin{align*}
    dA_t^i &= \left(2 \lambda_t^i - \frac{(A_t^i)^2}{2\eta_t^i} \right) dt - Z_t^A dW_t^i, \\
    \lim_{t \to T} A_t^i &= +\infty
\end{align*}
\]

and \( B^i \) satisfies the linear BSDE

\[
\begin{align*}
    -dB_t^i &= \left( -\frac{A_t^i B_t^i}{2\eta_t^i} + \frac{A_t^i}{2N\eta_t^i} \langle \hat{B}_t^{i,(1)}, \tilde{P}_t^i \rangle + \frac{1}{N} \mathbb{E} \left[ \langle \hat{B}_t^{i,(2)}, \tilde{P}_t^i \rangle \right] \\
    &\quad + \left\langle \Theta, -A_t^i \tilde{S}_t^i + K_t \tilde{X}_t + \mathcal{R}_t^i \right\rangle \right) dt - Z_t^B dW_t
\end{align*}
\]

on \([0, T)\). One need to prove that \( B^i \in \mathcal{S}^{2,-}_\mathcal{F} \).
The continuation step requires a mapping

\[ \Phi : (X_i, M^i)_{i=1,\ldots,N} \rightarrow (\tilde{X}_i, \tilde{M}^i)_{i=1,\ldots,N} \]

where

\[ f^i(M) := \sigma \frac{M^i}{2 \eta^i} + f^i \]

\[ g^i(X) := \sigma X^i + g^i \]

is to be a contraction in the right space. This holds under a weak interaction condition:

\[ \lambda^i - \ldots - \frac{1}{N} * - \ldots > 0, \quad \eta^i - \ldots - \frac{1}{N} * - \ldots > 0. \]
Theorem

Let

\[(X^i, S^i, M^i, P^i, Z^{M^i}, Z^{P^i}) \in \mathcal{H}_{a,\mathcal{F}} \times S^2_{\mathcal{F}} \times L^2_{\mathcal{F}} \times \mathcal{H}_{\nu,\mathcal{F}} \times L^2_{\mathcal{F}} \times L^2_{\mathcal{F}}^{-} \times L^2_{\mathcal{F}}\]

be the unique solution to the \(N\)-player FBSDE system. Then, the processes

\[\xi^* = (\xi^*, 1, \ldots, \xi^*, N)\]

forms an open-loop Nash equilibrium, where

\[\xi^*,i = \frac{M^i - \frac{1}{N} \langle B^i,1, P^i \rangle}{2\eta^i}.\]
The liquidation game

Verification - without convexity

Theorem

Under our weak interaction condition, for any admissible strategy $\xi^i$ the cost $J^i(\xi^i, \xi^*, -i)$ can be decomposed into the equilibrium cost plus the cost of a round-trip strategy as

$$J^i(\xi^i, \xi^*, -i) = J^i(\xi^*, i, \xi^*, -i) + \mathbb{E} \left[ \int_0^T \eta^i_t \left( \xi^i_t - \xi^*_t, i \right)^2 + \lambda^i_t \left( X^i_t - X^*_t, i \right)^2 ight.$$

$$+ \left( X^i_t - X^*_t, i \right) \left( \Theta, -A^i_t (S^i_t - S^*_t, i) + B^i_t (\chi_t - \chi^*_t) \right) dt \right].$$

Moreover, the cost of the additional round-trip is non-negative.
From many player games to mean-field games

- Let the cost functions be homogeneous; for any coefficient $\varphi^i$

$$\varphi^i = \varphi (x^i, W^i)$$

for independent Brownian motions $W^1, W^2, \ldots$.

- Using the Yamada-Watanabe result for mean-field FBSDE, there exists a measurable function $\Sigma$ independent of $i$ such that the solution to the mean-field FBSDE satisfies that

$$(X^i, S^i, M^i, P^i) = \Sigma(x^i, W^i).$$
Theorem

The following convergence holds:

\[ \mathbb{E} \left[ \int_0^T |M_t^i - \overline{M}_t^i|^2 dt \right] + \mathbb{E} \left[ \sup_{0 \leq t \leq T} |X_t^i - \overline{X}_t^i|^2 dt \right] \xrightarrow{N \to \infty} 0. \]

As a result, the optimal strategy of player \( i \) in the \( N \)-player game converges to the one in MFG, i.e.,

\[ \mathbb{E} \left[ \int_0^T |\xi_{\ast,i}^*,i,N - \overline{\xi}_{\ast,i}^*|^2 dt \right] \to 0. \]
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The mean-field game

Figure: Dependence of equilibrium portfolio process on the market impact parameter $\alpha$, for small (left) and larger (right) $\gamma$. 
The single player mode with risk aversion.

**Figure:** Dependence of optimal portfolio process on the market impact parameters $\alpha$ and $\gamma$, $\gamma = 1$ (left) and $\alpha = 1$ (right).
The two player model

**Figure:** Equilibrium portfolio process in the two player game for small (left) and large (right) $\gamma$
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Conclusion

• novel portfolio liquidation games with self-exciting order flow
• existence and uniqueness of solutions result for a novel mean-field FBSDE system with unknown terminal condition
• sufficient maximum principle and existence of open-loop equilibria
• quantitative analysis of equilibrium strategies
The end.

Thank you!