DIAMETERS OF 3-SPHERE QUOTIENTS

WILLIAM D. DUNBAR, SARAH J. GREENWALD*, JILL MCGOWAN, AND CATHERINE SEARLE**

ABSTRACT. Let $G \subset O(4)$ act isometrically on $S^3$. In this article we
calculate a lower bound for the diameter of the quotient spaces $S^3/G$.
We find it to be $\frac{1}{2} \arccos(\frac{\tan(\frac{3\pi}{10})}{\sqrt{3}})$, which is exactly the value of the
lower bound for diameters of the spherical space forms. In the process,
we are also able to find a lower bound for diameters for the spherical
Aleksandrov spaces, $S^n/G$, of cohomogeneities 1 and 2, as well as for
cohomogeneity 3 (with some restrictions on the group type). This leads
us to conjecture that the diameter of $S^n/G$ is increasing as the coho-
mogeneity of the group $G$ increases.

1. INTRODUCTION

Diameter is one of the most basic geometric invariants. Knowing its
lower bound not only provides information about the orbit space $X =
S^n/G$, but also leads to other interesting results. And, while representa-
tions of compact Lie groups are well understood, the geometry of the cor-
responding spherical quotients is virtually unknown and is potentially very
important. Let $X^k = S^n/G$, where $G$ is a closed, nontransitive subgroup of
$O(n)$, and examine the diameter of $X^k$. When $G$ is finite, $k = n$ and $X^n$
is a manifold, there exits an explicit lower bound on the diameter that depends
only on the dimension $n$, and a global lower bound that is independent of
dimension also exists [Mc]. (Throughout this paper, unless otherwise ex-
plicitly stated, $S^n$ is taken to be the round unit sphere of dimension $n$.) For
other closed, nontransitive groups, a lower bound on the diameter also exists
[Gr1], but it is not always given explicitly. This paper examines the diam-
eter of quotients of the three-dimensional sphere to find an optimal lower
bound. We also find descriptions of many of these orbit spaces and learn
about their geometry.

Often, given a specific compact Lie group, $G$, acting isometrically and
(almost) effectively on $M$, with $\sec(M) > 0$, we can recover the manifold

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from the possible orbit space decompositions of the action. For example, a classic theorem of Hsiang and Kleiner [HK] states:

**Theorem.** Let $M^4$ be a 1-connected, strictly positively curved closed Riemannian manifold which admits an effective, isometric $S^1$ action. Then $M^4$ is homeomorphic to $S^4$ or $\mathbb{C}P^2$.

Here, the essential point of the proof lies in understanding the fixed point set of the circle action. By work of Freedman [Fr], one can recover the manifold merely by knowing the Euler characteristic, $\chi(M)$, which in this case turns out to be equal to $\chi(\text{Fix}(M; S^1))$. In particular, in the case where $\text{Fix}(M; S^1)$ consists of isolated points, these points are singular orbits of the action, and we can bound the total number of such orbits via the diameter of the orbit space of the normal sphere to any such point of isotropy. Here the normal sphere is an $S^3$ and the upper bound on the diameter of $S^3/S^1$ tells us that there are no more than 3 such points. We note as well, that a theorem of Rong [R] uses the same technique to show that a 1-connected, strictly positively curved, closed Riemannian 5-manifold admitting a $T^2$ isometric and effective action is homeomorphic to $S^5$. In particular, this is part of a more general phenomenon where a bound on the $q$-extent of a space allows us to limit the number of singular points of a given action, and we have ([GM], [GS]):

**Equivariant Sphere Theorem.** Let $M$ be a closed manifold with $\text{sec}(M) > 0$ on which $G$ acts (almost) effectively by isometries. Suppose $p_0, p_1 \in M$ are points such that $\text{diam} S_{\bar{p}_i} \leq \pi/4$, $i = 0, 1$, where $S_{\bar{p}_i}$ is the space of directions at $\bar{p}_i$ in $M/G$. Then $M$ can be exhibited as

$$M = D(G(p_0)) \bigcup E$$

where $D(G(p_i))$, $i = 0, 1$ are tubular neighborhoods of the $p_i$-orbits and $E = \partial D(G(p_0)) = \partial D(G(p_1))$. In particular, $M$ is homeomorphic to the sphere if $G(p_i) = p_i$, i.e., if $p_i$, $i = 0, 1$ are isolated fixed points of $G$ and $\text{diam} S_{\bar{p}_i} \leq \pi/4$.

Thus, local diameter information gives global results about the structure of the manifold.

When $G$ is finite in $O(n + 1)$, then $S^n/G$ is a good orbifold, that is, the global quotient of a Riemannian manifold by a discrete subgroup of its isometry group [B]. Finite subgroups of $O(4)$ are classified in [DuV] and various methods from McGowan [Mc] and Dunbar [Du] are used to find a lower bound on the diameter of the resulting spherical quotients. When $G$ is infinite, $S^n/G$ is a spherical Alexandrov space with curvature bounded below. This is a length space with Riemannian notions such as distance and
curvature obtained by comparison with $S^n$ via Toponogov [BGP]. Possible groups in $SO(4)$ are classified in [S1] and [S2]. Extensions of these groups in $O(4)$ are examined along with the diameters of the resulting spherical quotients.

**Theorem A.** If $G$ is a closed, non-transitive subgroup in $O(4)$ then

$$\text{diam}(S^3/G) \geq \frac{\alpha}{2}$$

where $\alpha = \arccos\left(\frac{\tan\left(\frac{\pi}{10}\right)}{\sqrt{3}}\right)$.

This diameter is approximately $\frac{\pi}{9.63}$ and is achieved by $S^3/\eta(S^1 \times I)$, where $I$ is the binary icosahedral group, and $\eta : Sp(1) \times Sp(1) \to SO(4)$ is defined by first noting that the unit quaternions may be identified with $S^3$ by $\phi(p_1 + ip_2 + jp_3 + kp_4) = (p_1, p_2, p_3, p_4)$, where $p_1^2 + p_2^2 + p_3^2 + p_4^2 = 1$. With this identification, $\eta$ maps $(a, b)$ to $A$, if for every $x \in Sp(1)$, $\phi(axb^{-1}) = A\phi(x)$. The map $\eta$ is a surjective homomorphism with kernel \{(1, 1), (-1, -1)\}.

If $G$ is finite then $S^3/\eta(C_{2m} \times I)$, in the limit as $m \to \infty$, achieves the smallest diameter, where $C_{2m}$ is the binary cyclic group. The orbit space is a manifold only if $\gcd(m, 30) = 1$ [W] and otherwise it is a Seifert-fibered orbifold, foliated by circles and intervals. Among nonfibering orbifolds, $S^3/\eta(O \times I)$ achieves the smallest diameter, where $O$ is the binary octahedral group. A lower bound estimate for this diameter is $\arccos\left(1/(\sqrt{40 + 12\sqrt{2} - 8\sqrt{5} - 12\sqrt{10}})\right)$, which is approximately $\frac{\pi}{8.93}$.

Note that the cohomogeneity of a connected $G$-action is the codimension of its principal orbit, or equivalently, the dimension of the orbit space. We extend this definition to include non-trivial disconnected actions. With respect to cohomogeneity one actions, we not only examine those actions on $S^3$, but also on a round sphere of any dimension, and we find that the smallest diameter for a non-trivial disconnected cohomogeneity one action on any $S^n$ is $\pi/6$. We show that only certain actions of cohomogeneity one admit a finite extension of the group that halves the diameter of the corresponding orbit space.

Using this result and results from [MS1] and [MS2] we are able to prove the following:

**Theorem B.** Let $G$ act by cohomogeneity 1, 2, or 3 on $S^n$. Further, suppose that if $G$ is connected, the action is a classical connected polar action. Then

$$\min(\text{diam}(S^n/G)) = \begin{cases} \frac{\pi}{6} & \text{for cohomogeneity 1} \\ \frac{\alpha}{2} & \text{for cohomogeneity 2} \\ \frac{\alpha}{4} & \text{for cohomogeneity 3} \end{cases}$$
Here we define a classical polar action to be one which corresponds to a symmetric space $G/H$ where both $G$ and $H$ are classical Lie groups. Recall that all polar actions on spheres correspond to the isotropy subgroup of a symmetric space $[D]$, i.e., they correspond to the natural action of the isotropy subgroup $H$ of $G/H$ acting on $T_{G(e)}G/H$.

This theorem and other work ($[MS1]$, $[MS2]$) lead us to the following conjecture:

**Conjecture C.** Let $G$ act irreducibly on $S^n$ by cohomogeneity $k$, where $n \in 2\mathbb{Z}$. Then for all $\epsilon > 0$ and for all sufficiently large $k$ (and all $n > k$), $\text{diam}(S^n/G)$ is within $\epsilon$ of $\pi/2$.

We break the remainder of the paper into three sections, which each consider the action of $G$ on $S^3$ of a specific cohomogeneity, plus a section giving our conclusions.

### 2. Cohomogeneity One

**Proposition 2.1.** Let $G$ be an action on $S^3$ whose orbit space is dimension 1, then the minimal diameter of $S^3/G$ is $\pi/4$.

**Proof.** The classification of low cohomogeneity actions on spheres ($[S1]$, $[S2]$, $[HL]$) tells us that the only two possible connected groups which can act effectively on $S^3$ by cohomogeneity one are $SO(3)$ and $T^2$.

The action by $SO(3)$ has principal orbit $SO(3)/SO(2) \simeq S^2$, and singular orbits equal to points $(SO(3)/SO(3))$. Its orbit space is an interval of length $\pi$. The second action is $T^2$ acting with principal orbit $T^2$ and singular orbits $T^1$ (each singular orbit is a different $T^1$). Its orbit space is an interval of length $\pi/2$.

**Observation:** Let $H'$ be a finite extension of $H$, a connected subgroup of $G$, then $H' \subset N_G(H)$.

Note that conjugation is a group isomorphism and that given any $g \in H'$, $gHg^{-1}$ is isomorphic to $H$, the connected component of the identity of $H'$. Further, since $gHg^{-1}$ is also a subgroup of $H'$, $gHg^{-1} = H$, and $H' \subset N_G(H)$.

We may write the action of $SO(3)$ acting by cohomogeneity one on $S^3$, as follows. Let $S^3$ be the standard sphere in $\mathbb{R}^4$; we represent its points as $(x, y, z, w)$ and the action is represented by the matrix

$$A = \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix},$$
where $B \in SO(3)$. It is clear that the points $(0, 0, 0, 1)$ and $(0, 0, 0, -1)$ are fixed by this action. As well, it is clear that the matrix

$$C = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}$$

is an element of $O(4)$, and that $C$ commutes with $A$ and acts on $S^3$ by interchanging the $w$ coordinate of any point with its negative. One easily sees that this action sends any given orbit with a specified $w$ coordinate to the corresponding orbit with $-w$ coordinate. This action “folds” the orbit space of $S^3/\text{SO}(3)$ (an interval of length $\pi$) in half to obtain an interval of length $\pi/2$. Moreover, the normalizer of $\text{SO}(3)$ in $O(4)$ is exactly the group generated by $\text{SO}(3)$ and $C$.

Now $T^2$ is the maximal torus in $O(4)$ and we will denote it by

$$T^2 = \begin{pmatrix}
\cos(\theta_1) & \sin(\theta_1) & 0 & 0 \\
-\sin(\theta_1) & \cos(\theta_1) & 0 & 0 \\
0 & 0 & \cos(\theta_2) & \sin(\theta_2) \\
0 & 0 & -\sin(\theta_2) & \cos(\theta_2)
\end{pmatrix}.$$  

The Weyl group of $\text{SO}(4)$ is isomorphic to $C_2 \times C_2$, where $C_n$ is the cyclic group of order $n$, and the order 2 element

$$(0 0 1 0)$$

identifies the two singular orbits $T^2/S^1$ to each other, and acts upon the principal orbit $T^2$ at distance $\pi/4$ from the singular orbits, identifying the remaining principal orbits in pairs according to their respective distances from the singular orbits. Thus, the diameter of the resulting orbit space is $\pi/4$.

We claim that any other finite extension cannot further reduce the diameter of the orbit space due to the following lemma, which will complete the proof. ∎

**Lemma 2.2.** Suppose $G$ acts isometrically and (almost) effectively on $M^n$ by cohomogeneity one, where $M^n$ is a closed Riemannian manifold of strictly positive sectional curvature. Then any finite extension of $G$ in $\text{Isom}(M^n)$ can reduce the diameter of $M^n/G$ by at most one-half.

**Proof.** Topologically, there are 4 possibilities for the orbit space of a manifold by a cohomogeneity one action. They are $\mathbb{R}$, $\mathbb{R}^+$, $S^1$, and an interval
However, the additional restriction of strictly positive sectional curvature eliminates the first three possibilities, since such a manifold will be compact and have finite fundamental group. Over the interval, the manifold decomposes into principal orbits over its interior and 2 singular orbits over each endpoint, and the diameter of the orbit space is given by the length of the interval.

Now, any finite isometric action on such an interval can only “fold” the interval in half, identifying the endpoints to each other and corresponding pairs of principal orbits at a given distance from one of the corresponding pairs of endpoints. The principal orbit equidistant from both endpoints is not identified to any other, and itself is acted nontrivially. Clearly such an action halves the diameter. Note that any other action would be discontinuous or not isometric. Since any finite group of order greater than 2 will have a cyclic element of order greater than or equal to 3, or a subgroup isomorphic to $C_2 \times C_2$, it suffices to understand these two cases.

In the case where we have a subgroup of order 3 acting effectively, the action would identify three distinct points in the interval. However, it would fail to be isometric (suppose $a \neq b$, $b \neq c$, $c \neq a$, and $f(a) = b$, $f(b) = c$, and $f(c) = a$, then we must have $d(a, b) = d(f(a), f(b)) = d(f(b), f(c))$ in order for it to be an isometry, but this is clearly impossible in any interval). The argument is similar if we have a cyclic group of order greater than three.

For the case where we have an effective action by a subgroup isomorphic to $C_2 \times C_2$, if the action decreases the diameter, then it must fold the interval in half two times. For the action to be isometric, this means that the corresponding singular orbits of the original interval and of the once-folded interval must respectively be isometric themselves. We observe however that even if this is true at first, it cannot hold for the once-folded interval, since one singular orbit of this interval will be one of the original singular orbits, and the other the result of a $C_2$ action on a principal orbit. A quick inspection of Table [1] of cohomogeneity one actions on spheres shows that in all cases, the principal orbit is of strictly larger dimension than its singular orbits and thus it is impossible to make any further identifications, that is, an effective action by a finite subgroup isomorphic to $C_2 \times C_2$ is not allowed.

\begin{proposition}
The smallest diameter one can obtain for the orbit space of a (non-trivial, disconnected) cohomogeneity one action on a sphere is $\pi/6$.
\end{proposition}

\begin{proof}
We will show that the cohomogeneity one actions of diameter $\pi$ and $\pi/2$ both admit finite extensions which fold the corresponding interval in half. As well, there are two actions of diameter $\pi/3$ (of the total four) which also admit finite extensions. The $\pi$ and $\pi/2$ diameter actions are exactly the
reducible actions. The remaining irreducible actions, other than the two of diameter \( \pi/3 \) we mentioned previously, do not admit finite extensions. We note that all the orbit spaces of diameter \( \pi/4 \) have non-isometric singular orbits and thus there is no isometric action which can reduce their respective diameters (see proof of Lemma 2.2). Thus we need only show that the remaining \( \pi/3 \) and the \( \pi/6 \) actions do not admit finite extensions which halve the diameters of their orbit spaces.

In Table 1 (cf. [MS1], [S2]), we give a list of the cohomogeneity one actions on spheres. Note that in order for an isometric action to actually fold the orbit space in half, the two singular orbits (as mentioned in the proof of Lemma 2.2) must be isometric themselves, since they will be identified to each other via an isometry. In the list of cohomogeneity one spherical orbit spaces, only the following have isometric singular orbits: numbers 1, 2, 3, 4, 5, 6, 7 (with \( k = m \)), 8, 9, 10, 11, 17, 18, and 19. The first three all have diameters which reduce to \( \pi/2 \) with the addition of an antipodal action. For example, in case 1, where \( SO(k) \) acts on a sphere of dimension \( k \), we see

## Table 1. Connected Spherical Cohomogeneity One Actions

| Group(\( G \)) | Representation(\( \Phi \)) | \( dim(\Phi) \) | Length |
|----------------|-----------------|-----------------|--------|
| 1) \( SO(k) \) | \( \rho_k + 1, k \geq 2 \) | \( k + 1 \) | \( \pi \) |
| 2) \( U(k) \) | \( \mu_k + 1, k \geq 1 \) | \( 2k + 1 \) | \( \pi \) |
| 3) \( Sp(k) \) | \( \nu_k + 1, k \geq 1 \) | \( 4k + 1 \) | \( \pi \) |
| 4) \( G_2 \) | \( \psi_1 + 1 \) | 7 | \( \pi \) |
| 5) \( Spin(7) \) | \( \Delta_7 + 1 \) | 8 | \( \pi \) |
| 6) \( Spin(9) \) | \( \Delta_9 + 1 \) | 16 | \( \pi \) |
| 7) \( SO(k) \times SO(m) \) | \( \rho_k + \rho_m, k, m \geq 2 \) | \( k + m \) | \( \pi/2 \) |
| 8) \( SO(3) \) | \( S^2 \rho_3 - 1 \) | 5 | \( \pi/3 \) |
| 9) \( SU(3) \) | \( \text{Ad} \) | 8 | \( \pi/3 \) |
| 10) \( Sp(3) \) | \( \wedge^2 \nu_3 - 1 \) | 14 | \( \pi/3 \) |
| 11) \( F_4 \) | \( \phi_1 \) | 26 | \( \pi/3 \) |
| 12) \( SO(2) \times SO(k) \) | \( \rho_2 \otimes \mathbb{R} \rho_k, k \geq 3 \) | \( 2k \) | \( \pi/4 \) |
| 13) \( U(2) \times SU(k) \) | \( \mu_2 \otimes_{\mathbb{C}} \mu_k, k \geq 2 \) | \( 4n \) | \( \pi/4 \) |
| 14) \( Sp(2) \times Sp(k) \) | \( \nu_2 \otimes_{\mathbb{H}} \nu_k, k \geq 2 \) | \( 8n \) | \( \pi/4 \) |
| 15) \( U(5) \) | \( [\wedge^2 \mu_5]_{\mathbb{R}} \) | 20 | \( \pi/4 \) |
| 16) \( Sp(2) \) | \( \text{Ad} \) | 10 | \( \pi/4 \) |
| 17) \( U(1) \times Spin(10) \) | \( [\mu_1 \otimes_{\mathbb{C}} \Delta_{10}^\pm]_{\mathbb{R}} \) | 32 | \( \pi/4 \) |
| 18) \( G_2 \) | \( \text{Ad} \) | 14 | \( \pi/6 \) |
| 19) \( SO(4) \) | \( \nu_1 \otimes_{\mathbb{H}} S^3 \nu_1 \) | 8 | \( \pi/6 \) |
that adding in the element
\[
\begin{pmatrix}
I_{k \times k} & 0 \\
0 & -1
\end{pmatrix}
\]
which belongs to \(O(k + 1) = \text{Isom}(S^k)\) identifies the two singular orbits (which are points here) to each other, each of the two principal orbits equidistant from each of the singular orbits will identify and the action on the principal orbit halfway between the two singular orbits is antipodal. Note that the action here is reducible and not maximal. Clearly this element also conjugates the two singular isotropy subgroups (which in this case are the entire group \(SO(k)\)) to one another. The remaining cases 2–6 with diameter \(\pi\) proceed in a similar fashion.

In number 7, we may add in the element
\[
\begin{pmatrix}
0 & I_{k \times k} \\
I_{k \times k} & 0
\end{pmatrix}
\]
which is an element of \(O(2k) = \text{Isom}(S^{2k-1})\). This action interchanges the two singular orbits (here they are \(S^{k-1}\)’s) and corresponding principal orbits. The principal orbit equidistant from the 2 singular orbits is acted upon antipodally and we obtain an orbit space of diameter \(\pi/4\). Note that this action is maximal, but it is reducible. Further note that both singular istropy subgroups are conjugate to each other precisely via this order 2 element.

Numbers 8, 9, 10, 11 all have diameter \(\pi/3\) and the principal orbits are flag manifolds and the corresponding singular orbits are projective spaces (respectively real, complex, hyperbolic and Cayley). Numbers 18 and 19 both have diameter \(\pi/6\). Of these actions, only numbers 8 and 9 admit a finite extension which will fold the interval in half. Note that all these actions correspond to maximal inclusions of their respective groups in the corresponding isometry group of the sphere on which they act by cohomogeneity one.

In the particular case of number 8, the action may be described as follows: \(SO(3)\) acts on \(S^4\) realized as symmetric \(3 \times 3\) real matrices of trace zero by conjugation. If we require that the symmetric matrix \(A\) also satisfies \(\|A\|^2 = \text{tr}(A^tA) = 1\), then this is an action on a subset of \(S^8\). These matrices can be diagonalized by the action, thus the orbits can be represented by the diagonal matrices with the appropriate eigenvalues whose sum is zero. Further, conjugation by the matrices
\[
\begin{pmatrix}
0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}
\text{ and }
\begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & -1 \\
0 & 1 & 0
\end{pmatrix},
\]
allows us to arrange the eigenvalues in descending order: $x \geq y \geq z$. The resulting orbit space is then the subset of the intersection of $S^2$ with the plane $\{(x, y, z) \in S^2 : x + y + z = 0\}$ with $x \geq y \geq z$, i.e., the segment of the great circle in $S^2$ with endpoints $(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}})$ and $(\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-1}{\sqrt{6}})$. No further identifications can be made by conjugation, since conjugation of matrices does not change their eigenvalues.

However, multiplying on the left and on the right by the matrix

$$A = \begin{pmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & i \end{pmatrix},$$

gives us an antipodal map, which interchanges the endpoints of the interval and effectively halves the diameter of the resulting orbit space.

The action of $SU(3)$ is similar and we may likewise conjugate by the matrix

$$A = \begin{pmatrix} j & 0 & 0 \\ 0 & j & 0 \\ 0 & 0 & j \end{pmatrix}.$$

Note that this map is not the antipodal map on the corresponding sphere, in fact it is not even fixed point free. However, it does map orbits to orbits (one can see this via the corresponding eigenvalues of a given orbit) and acts antipodally on the geodesic circle containing the orbit space, and thus on the orbit space itself. Therefore this action also will halve the orbit space diameter.

Now, for the remaining cases 10, 11, 18 and 19, we note that all of the actions $G$ correspond to a maximal inclusion of $G$ in the corresponding isometry group of the sphere upon which they act. That is $Sp(3) \subset O(14)$, $F_4 \subset O(28)$, $SO(4) \subset O(8)$ and $G_2 \subset O(14)$. Thus, if there exists such an element $g$ which finitely extends the group $G$, $g \in O(k) \setminus G$, then we claim that $g \in Aut(G)$. We see this as follows: we know that $g(G)$ must also be a subgroup of $O(k)$ since it must take isotropy subgroups of $G$ to each other. Further, since $G$ is maximally included, then $g(G) = G$, for if they were not equal, then by maximality we know that their union would be all of $O(k)$, which clearly contradicts the assumption that $g$ was an element that extends the group finitely, and Thus, $g \in Aut(G)$, that is, $g$ must be contained in the automorphism group of the action $G$. In particular, the inner automorphisms will not give us extensions (finite or otherwise), whereas the outer ones might. Thus those groups which have trivial outer automorphism group will not admit finite extensions halving the diameter. In the remaining cases, it is well-known that only for $SO(4)$ the outer automorphism group is non-trivial. We will now show that $SO(4)$ does not admit such a finite extension.
From Uchida [U], we have a description of the action on $S^7 \subset \mathbb{R}^8 \simeq \mathbb{H}^2$. We begin with the homomorphism $\sigma : SU(2) \rightarrow Sp(2)$ defined by

$$\sigma \left( \begin{array}{cc} a & -\bar{b} \\ b & \bar{a} \end{array} \right) = \left( \begin{array}{cc} a^3 + jb^3 & -\sqrt{3}(a^2\bar{b} - j\bar{ab}^2) \\ \sqrt{3}(a^2b - jab^2) & a^2\bar{a} - 2ab\bar{b} + jb^2\bar{b} - 2ja\bar{ab} \end{array} \right),$$

where $j$ is a quaternion such that $j^2 = -1$ and $aj = j\bar{a}$ for each complex number $a$.

Next, we let $A \in \sigma(SU(2))$, $q \in Sp(1)$, $X \in M(2,1;\mathbb{H})$, then the action is defined by $(A,q) \times X \mapsto AX\bar{q}$.

Now $Out(SO(4)) \simeq C_2$ and can be generated via the action $\tau : Sp(1) \times Sp(1) \rightarrow Sp(1) \times Sp(1)$, where $\tau(q,r) = (r,q)$. Since Uchida describes the action using the double cover $Sp(1) \times Sp(1)$ of $SO(4)$, we will work with the double cover as well. (See the description of the double cover in the introduction.) We will suppose that an extension by $\tau$ exists and derive a contradiction. Suppose there exists $\beta \in Sp(1) \times Sp(1)$ such that $\beta\tau$ interchanges orbits as desired (it suffices to consider this case since $\tau\beta^{-1} \in Sp(1) \times Sp(1)$ implies that $\tau\beta = \gamma\tau$ for some $\gamma \in Sp(1) \times Sp(1)$). Now given that $\beta\tau$ is such an extension, in particular it maps elements of $G(0,j)$ to elements of $G(\frac{1}{\sqrt{3}},j)$. Observe that

$$G_{(0,j)} = (\sigma \left( \left( \begin{array}{cc} a & 0 \\ 0 & \bar{a} \end{array} \right) \right), \bar{a}),$$

and that $\tau G_{(0,j)}\tau^{-1} = G_{(0,j)}$

By a direct calculation, one can see that the orbit of $(\frac{1}{\sqrt{3}},j)$ contains no elements of the form $(0,bj)$, $b \in \mathbb{C}$, which are the only elements in $S^7$ that are fixed by $G_{(0,j)}$. Thus $\tau$ cannot send the point $(0,j)$ to any point of the other singular orbit $G(\frac{1}{\sqrt{3}},j)$ (if it did then the isotropy subgroup would be $G_{(0,j)}$, since its image under conjugation by $\tau$ is itself.) Thus $\tau(0,j) \notin G_{(0,j)}$. Likewise $\beta\tau(0,j) \notin G_{(0,j)}$. Thus there exists no finite extension that folds the interval in half.

\[ \square \]

3. COHOMOGENEITY TWO

In this section we compute a lower bound on the diameters of $S^3/G$ of dimension two. We begin by proving two lemmas.

**Lemma 3.1.** If $G$ is a topological group acting by isometries on the $n$-sphere $S^n$, and $G_0$ denotes the connected component of the identity element, then $G/G_0$ acts on $S^n/G_0$ by isometries.
Proof. First, note that $S^n/G_0$ has a metric that is well defined and the distance between two orbits is given by
\[ d(G_0x, G_0y) = \min_{g,h \in G_0} d(gx, hy) = d(g_0x, h_0y), \text{for some } g_0, h_0 \in G_0 \]
\[ = d(h_0^{-1}g_0x, y) \geq d(G_0x, y) \geq d(G_0x, G_0y) \]
Now, let $G/G_0$ act on $S^n/G_0$. Let $hG_0 \in G/G_0$. We want to determine $d(hG_0(G_0x), hG_0(G_0y))$. But this is equal to
\[ d(hG_0x, hG_0y) = d(G_0hx, G_0hy) = d(G_0G_0hx, hy) = d(G_0hx, hy) \]
\[ = d(h^{-1}G_0hx, y) = d(G_0x, y) = d(G_0x, G_0y) \]
So the action by $G/G_0$ preserves distance. □

Hence, in order to examine the diameter of $S^n/G$, we will look at the diameter of $(S^n/G_0)/(G/G_0)$, which is isometric to $S^n/G$ [Mc]. In fact, Lemma 3.2 proves more than we actually need for Proposition 3.3, since it considers the question of any closed subgroup of $O(3)$ acting on $S^2$.

**Lemma 3.2.** If $G$ is a closed, non-transitive subgroup in $O(3)$ then
\[ \text{diam}(S^2/G) \geq \alpha, \]
where $\alpha = \arccos\left(\frac{\tan(\frac{3\pi}{10})}{\sqrt{3}}\right)$.

Proof. If $G$ acts by cohomogeneity one, from Table 1 we see that the only possible connected action on $S^2$ is by $SO(2)$ and gives diameter $\pi$. From the proof of Proposition 2.3 we see that the smallest diameter we can obtain with a finite extension of $SO(2)$ is $\frac{\pi}{2}$ and this occurs for $G = O(2)O(1)$.

If $G$ acts by cohomogeneity 2, then $G$ is finite, and the possibilities for $G$ and the corresponding diameters are listed in the following Table 2. For each group, we also provide several standard notations. The smallest diameter is achieved by $S^2/I$ and $S^2/I^h$ as $\arccos\left(\frac{\tan(\frac{3\pi}{10})}{\sqrt{3}}\right)$ [Gr2], where $I$ is the icosahedral group in $SO(3)$, and $I^h$ is the orientation-reversing extension in $O(3)$. This diameter has also been computed in [Mc] and [Gr1]; the details (which are not included in [Mc] and [Gr1]) are presented here for the sake of completeness and because they are important in the calculation of the optimal lower bound for quotients of $S^3$ and also give an idea of the technique for the diameter of the quotients of $S^2$. The spherical icosahedron has 20 spherical triangular faces, 12 vertices, and 30 edges. The generators of $I$ are a $\frac{2\pi}{5}$ rotation of a pentagon formed by the outer edges of five adjacent faces, a $\frac{2\pi}{3}$ rotation about the center of a given face, and a $\pi$ rotation about a line through the midpoints of opposite edges. The group has order 60. Any triangle can be rotated into any other triangle by a combination of rotations,
so examine one of these triangles. A rotation of $\frac{2\pi}{3}$ about its center $c$ self-identifies this triangle. The fundamental domain is shaded in Figure 1 and the diameter is achieved as the length of the spherical segment from vertex $v$ to $c$. $S^2/I$ is a triangular shaped inflatable pillow. The space has isotropy $C_3$ at $c$, isotropy $C_2$ at $e$ and isotropy $C_5$ at $v$. This fundamental domain can be cut in half with a mirror reflection in a spherical geodesic beginning at $c$ and ending at $e$. This group $I^h$ is also a Coxeter group, which can be generated purely by reflections. The fundamental domain for $S^2/I^h$ is a spherical triangle with vertex angles $\frac{\pi}{2}$ at $e$, $\frac{\pi}{3}$ at $c$, and $\frac{\pi}{5}$ at $v$. By applying the spherical trigonometry formula $\cos A = -\cos B \cos C + \sin A \sin B \cos a$, where $A, B, C$ are vertex angles and $a, b, c$ are spherical lengths of opposite edges, one can obtain the lower bound.

**Proposition 3.3.** Let $G$ be a closed subgroup of $O(4)$ acting on $S^3$ by cohomogeneity two. Then

$$\text{diam}(S^3/G) \geq \frac{1}{2} \alpha,$$

where $\alpha = \arccos\left(\frac{\tan\left(\frac{3\pi}{10}\right)}{\sqrt{3}}\right)$.

**Proof.** The only connected group that can act effectively by cohomogeneity two on $S^3$ is $T^1 = \{e^{i\theta} | \theta \in \mathbb{R}\}$. In the case where $G$ is connected, we see that $T^1$ can act under any of the various possible guises as $(z, w) \to (e^{ik\theta} z, e^{im\theta} w)$. As such, these various group actions are designated by $T_{k,m}$. 

**Figure 1. Fundamental Domain of $I$ on the Spherical Icosahedron**
Note that this action is effective exactly when \( \gcd(k, m) = 1 \), so we only consider these actions.

If \( G_0 = T^1 = T_{1,1} \), then \( S^3/G_0 \) is isometric to the 2-sphere of radius 1/2, denoted \( S^2(1/2) \). By Lemma 3.1, \( S^3/G = (S^3/G_0)/(G/G_0) = S^2(1/2)/(G/T_{1,1}) \). In addition, the action of \( G/T_{1,1} \) on \( S^3/G_0 \) is conjugate to the action of a finite subgroup \( K \) of \( O(3) \) on \( S^2(1/2) \). We obtain the desired lower bound by applying Lemma 3.2 since

\[
\text{diam}(S^3/G) = \text{diam}(S^2(1/2)/K) \geq \frac{1}{2} \arccos\left(\frac{\tan\left(\frac{3\pi}{10}\right)}{\sqrt{3}}\right)
\]

(see also Table 2 below).

Now, for the spaces \( X_{k,m} = S^3/G_{k,m} \), with \( k \neq m \) and \( \gcd(k, m) = 1 \), the orbits of \((1, 0)\) and \((0, 1)\) cannot be interchanged via an isometry. This is clear from the previous discussion, since the two exceptional orbits are not isometric, having non-isomorphic isotropy subgroups. The isotropy subgroup at \((1, 0)\) is \( C_k \); at \((0, 1)\), it is \( C_m \). The diameter is realized by a path of length \( \pi/2 \) from \((1, 0)\) to \((0, 1)\). Notice that \( X_{k,m} = S^3/G_{k,m} \) with \( k \neq m \) and \( \gcd(k, m) = 1 \) is a bad orbifold that is topologically a 2-sphere. It has one singular point if \( k \) or \( m \) is 1, and two singular points otherwise. Any group of isometries of \( X_{k,m} \) with \( k \neq m \) must fix the orbits of \((1, 0)\) and \((0, 1)\), and so quotients of \( X_{k,m} \) must always have diameter \( \pi/2 \).

If \( G \) is infinite but not connected, we rely on Lemma 3.1 to reduce the calculation of the diameter of \( S^3/G \) to that of \( S^2(1/2)/\Gamma \), where \( \Gamma \) is a finite subgroup of \( O(3) \) and then apply Lemma 3.2 to obtain the desired lower bound.

In Table 2, the second [S] column describes the geometry of the group actions. The third [TH] and fourth [CS] columns list the resulting topological orbit spaces (see pictures in [M]). The fifth column [Y], is given for better understanding of the algebraic structure of the groups; both this and the last column [H] focus on inversions instead of reflections. For the “C” groups, \( n \geq 1 \); for the “D” groups, \( n \geq 2 \). The special case “2mmm” is written “mmm”. “N” stands for the number \( 2n \). “x” stands for the antipodal map. When \( H \) is a subgroup of index two of \( G \subseteq SO(3) \), define \( H \mathbin{\mid} G := H \cup x(G - H) \).

4. COHOMOGENEITY THREE

In this section we compute a lower bound on the diameters of \( S^3/G \) where \( G \) is finite in \( O(4) \).

4.1. Classification of finite subgroups of \( O(4) \). While this discussion will basically follow the treatment in Du Val [DuV], the reader should note that Threlfall and Seifert [TS1] [TS2] classify finite subgroups of \( SO(4) \).
Conway and Smith [CS, Chapter 4] also have a classification of subgroups of $O(4)$.

The central idea in the classification of finite subgroups of $O(4)$, up to conjugacy, is that $SO(4)$ is “almost a product”. More precisely, there are 2-to-1 homomorphisms $S^3 \times S^3 \to SO(4) \to SO(3) \times SO(3)$ (the former homomorphism was called $\eta$ in section I).

They are defined by thinking of $S^3$ as the set of unit quaternions:

$$(p_1, p_2) \mapsto (q \mapsto p_1 qp_2^{-1}) \mapsto ((\bar{q} \mapsto p_1 \bar{q} p_1^{-1}), (\bar{q} \mapsto p_2 \bar{q} p_2^{-1})),$$

where $q := q_1 + q_2 i + q_3 j + q_4 k$ and $\bar{q} := q_2 i + q_3 j + q_4 k$.

### Table 2. Subgroups of $O(3)$ and Corresponding Diameters

| $n$ | $[S]$ | $[Th]$ | $[CS]$ | $[Y]$ | $[H]$ | Diameter |
|-----|-------|--------|--------|-------|-------|----------|
| odd $S_N, C_n$ | $D_n(n) \times C_n \cup xC_n$ | $\bar{n}$ | $\pi/2$ |
| even $S_N$ | $D_n(n) \times C_n \cap C_n$ | $\bar{N}$ | $\pi/2$ |
| odd $C_n^h$ | $D_n^2(n)$ | $n^*$ | $C_n \cap C_n$ | $\bar{N}$ | $\pi/2$ |
| even $C_n^h$ | $D_n^2(n)$ | $n^*$ | $C_n \cup xC_n$ | $n/m$ | $\pi/2$ |
| odd $C_n^v$ | $D_n^2(n,n)$ | $*nn$ | $C_n \cap C_n$ | $nm$ | $\pi$ |
| even $C_n^v$ | $D_n^2(n,n)$ | $*nn$ | $C_n \cap C_n$ | $nm$ | $\pi$ |

| $n$ | $S^2(2,2,n)$ | $T^d$ | $O^h$ | $I^h$ | Diameter |
|-----|---------------|-------|-------|-------|----------|
| odd $D_n$ | $S^2(2,2,n)$ | 22 | 432 | 532 | $\arccos \frac{1}{3}$ |
| even $D_n$ | $S^2(2,2,n)$ | 22 | 432 | 532 | $\arccos \frac{1}{3}$ |
| odd $D_n^h$ | $S^2(2,2,n)$ | 22 | $O \cap O$ | 532 | $\arccos \frac{1}{3}$ |
| even $D_n^h$ | $S^2(2,2,n)$ | 22 | $O \cap O$ | 532 | $\arccos \frac{1}{3}$ |
| odd $D_n^d$ | $D_n^2(2,2)$ | 2 | $O \cap O$ | $O$ | $\arccos \frac{1}{3}$ |
| even $D_n^d$ | $D_n^2(2,2)$ | 2 | $O \cap O$ | $O$ | $\arccos \frac{1}{3}$ |

Conway and Smith [CS, Chapter 4] also have a classification of subgroups of $O(4)$.
Finite subgroups of $SO(4)$ are classified by combining the well-known classification of finite subgroups of $SO(3)$ (cyclic, dihedral, tetrahedral, octahedral, icosahedral — see, e.g., [Y]) and the less-well-known, but elementary classification of subgroups of product groups (sketched below, but see also [Hal] pages 63–64).

If $G$ denotes a finite subgroup of $SO(4)$, then let $\hat{G}$ denote its inverse image $\eta^{-1}(G)$ in $S^3 \times S^3$. We define the following subgroups of $S^3$:

\[
\begin{align*}
L &:= \{ \ell : (\ell, r) \in \hat{G} \text{ for some } r \} \\
R &:= \{ r : (\ell, r) \in \hat{G} \text{ for some } \ell \} \\
L' &:= \{ \ell : (\ell, 1) \in \hat{G} \} \\
R' &:= \{ r : (1, r) \in \hat{G} \}
\end{align*}
\]

It can be shown that

\[L = \ker(\lambda : L \to \hat{G}/(L \times R)) \text{ and } R = \ker(\rho : R \to \hat{G}/(L \times R)),\]

inducing isomorphisms $\lambda : L/\ell \to \hat{G}/(L \times \ell)$ and $\rho : R/\ell \to \hat{G}/(L \times \ell)$ which, when composed back-to-back, give an isomorphism $\phi = \rho^{-1} \circ \lambda$ from $L/\ell$ to $R/\ell$. The group $G$ is denoted $(L/\ell; R/\ell; \rho)$. Often $\phi$ is omitted if the isomorphism is “obvious”; compare [DuV] page 54. The order of $G$ will equal $|R||L|/(|L||L|)/2 = |R||L|/2$.

The only possibilities for $L$ and $R$ are the finite subgroups of $S^3$ which are inverse images under the 2-to-1 homomorphism $S^3 \to SO(3)$ of finite subgroups of $SO(3)$, or in other words, finite subgroups of $S^3$ containing the kernel $\{ \pm 1 \}$ of that homomorphism. Hence they are conjugate to (exactly) one of the following “binary” groups:

\[
\begin{align*}
C_{2n} &:= \{ \cos(2m\pi/2n) + \sin(2m\pi/2n)k : m = 0, 1, \ldots, 2n - 1 \} \\
&\quad (n \geq 1) \\
D_n &:= C_{2n} \cup \{ \cos(2m\pi/2n)i + \sin(2m\pi/2n)j : m = 0, 1, \ldots, 2n - 1 \} \\
&\quad (n \geq 2) \\
T &:= D_2 \cup \{ \frac{1}{2} \pm \frac{1}{2}i \pm \frac{1}{2}j \pm \frac{1}{2}k \} \\
O &:= T \cup \{ \pm 1/\sqrt{2} \pm (1/\sqrt{2})i \} \cup \{ \pm 1/\sqrt{2} \pm (1/\sqrt{2})j \} \\
&\quad \cup \{ \pm 1/\sqrt{2} \pm (1/\sqrt{2})k \} \cup \{ \pm (1/\sqrt{2})i \pm (1/\sqrt{2})j \} \\
&\quad \cup \{ \pm (1/\sqrt{2})i \pm (1/\sqrt{2})k \} \cup \{ \pm (1/\sqrt{2})j \pm (1/\sqrt{2})k \} \\
I &:= T \cup (1/2)((\tau - 1) + \tau i + j)T \cup (1/2)(-\tau + i + (\tau - 1)j)T \\
&\quad \cup (1/2)(-\tau - i + (1 - \tau)j)T \cup (1/2)((\tau - 1) - \tau i - j)T,
\end{align*}
\]
where \( \tau := (\sqrt{5} + 1)/2 \). These groups have orders \( 2n, 4n, 24, 48 \), and 120, respectively. It may be worth noting that the given coset representatives for \( \mathbf{I} \) form a cyclic group of order five. If the signs of all the coefficients of \( \sqrt{5} \) in elements of \( \mathbf{I} \) are reversed, then a group \( \mathbf{I}^\dagger \) is obtained, which is conjugate to \( \mathbf{I} \) in \( S^3 \), and has the property that \( \mathbf{I} \cap \mathbf{I}^\dagger = T \[DuV\] page 55]. “Sign-reversal” defines an isomorphism \( \phi : \mathbf{I} \to \mathbf{I}^\dagger \), whose inverse is also accomplished by sign-reversal (i.e., by a different sort of “conjugation”, in the field \( \mathbb{Q}(\sqrt{5}) \)).

We arrive at 41 families of finite subgroups of \( SO(4) \), 33 of which contain the central element (-1 times the identity matrix) and which therefore equal the inverse image of their projections to \( SO(3) \times SO(3) \). The numbering convention follows Du Val \[DuV\] and goes back to Goursat \[G\], who classified the finite subgroups of \( Isom(\mathbb{R}P^3) \cong SO(3) \times SO(3) \), though in some places we are forced to interpolate extra families to cover gaps in that enumeration. The first 33 families are listed in Table 3, where \( m, n, r \geq 1 \), \( gcd(s, r) = 1 \), and \( 0 < s < r/2 \[DuV\] page 55]. Furthermore, \( \phi_s : C_{2mr}/C_{2m} \to C_{2nr}/C_{2n} \) is the isomorphism which takes the coset \( (\cos(2\pi/2mr) + \sin(2\pi/2mr)k)C_{2m} \) to the coset \( (\cos(2s\pi/2nr) + \sin(2s\pi/2nr)k)C_{2n} \). Similarly, \( \psi_s : D_{mr}/C_{2m} \to D_{nr}/C_{2n} \) is the isomorphism mapping cosets with representatives in \( C_{2mr} \) as above, while taking the coset \( iC_{2m} \) to the coset \( iC_{2n} \). Finally, \( \phi^\dagger : \mathbf{I} \to \mathbf{I}^\dagger \) induces an isomorphism \( \tilde{\phi} : \mathbf{I}/C_2 \to \mathbf{I}^\dagger/C_2 \). As noted in \[TS2\] page 585] and \[CS\] page 50], Goursat and Du Val omit a family of the form \( (m, n \geq 2) \)

\[11a. \quad (D_{2m}/C_{2m}; D_{2n}/C_{2n}; \psi^\#) \]

where the common quotient group is the Klein four-group, and \( \psi^\# \) is the isomorphism which takes the coset \( (\cos(\pi/m) + \sin(\pi/m)k)C_{2n} \) to the coset \( iC_{2n} \) and conversely takes the coset \( iC_{2m} \) to the coset \( (\cos(\pi/n) + \sin(\pi/n)k)C_{2n} \) (which does not respect the cyclic subgroups of index two in these binary dihedral groups).

There are also 8 families of subgroups of \( SO(4) \) which do not contain the central element. It follows that \( L \) and \( R \) for such groups must be cyclic of odd order, since each of the other subgroups of \( S^3 \) contains the quaternion \(-1\); we therefore extend the notation \( C_{2n} \) to allow odd subscripts. In addition to the conditions on \( m, n, r, s \) given above, in Table 4, \( m \) and \( n \) are both odd integers. The automorphism in \#26, \( \xi : O \to O \), is the identity on \( T \), and multiplies all other elements by -1. It cannot be induced by conjugation in \( S^3 \), and hence groups \#26 and \#26 are not conjugate in \( SO(4) \).

It remains to list, up to conjugacy, the finite subgroups of \( O(4) \) which contain orientation-reversing transformations. We can write an arbitrary
TABLE 3. Finite Subgroups of $O(4)$, part I

|   |                                                                 |   |                                                                 |
|---|-----------------------------------------------------------------|---|-----------------------------------------------------------------|
| 1 | $(C_{2mr}/C_{2m}; C_{2nr}/C_{2n}; \phi_s)$                      | 17| $(D_{2m}/D_m; O/T)$                                             |
| 2 | $(C_{2m}/C_{2m}; D_n/D_n)$                                       | 18| $(D_{6m}/C_{2m}; O/D_2)$                                        |
| 3 | $(C_{4m}/C_{2m}; D_n/C_{2n})$                                    | 19| $(D_m/D_m; I/I)$                                               |
| 4 | $(C_{4m}/C_{2m}; D_{2n}/D_n)$                                    | 20| $(T/T; T/T)$                                                    |
| 5 | $(C_{2m}/C_{2m}; T/T)$                                           | 21| $(T/C_2; T/C_2)$                                               |
| 6 | $(C_{6m}/C_{2m}; T/D_2)$                                         | 22| $(T/D_2; T/D_2)$                                               |
| 7 | $(C_{2m}/C_{2m}; O/O)$                                           | 23| $(T/T; O/O)$                                                    |
| 8 | $(C_{4m}/C_{2m}; O/T)$                                           | 24| $(T/T; I/I)$                                                    |
| 9 | $(C_{2m}/C_{2m}; I/I)$                                           | 25| $(O/O; O/O)$                                                    |
| 10| $(D_m/D_m; D_n/D_n)$                                             | 26| $(O/C_2; O/C_2)$                                               |
| 11| $(D_{mr}/C_{2m}; D_{nr}/C_{2n}; \psi_s)$                        | 27| $(O/D_2; O/D_2)$                                               |
| 11a| $(D_{2m}/C_{2m}; D_{2n}/C_{2n}; \psi_q)$                        | 28| $(O/T; O/T)$                                                    |
| 12| $(D_{2m}/D_m; D_{2n}/D_n)$                                       | 29| $(O/O; I/I)$                                                    |
| 13| $(D_{2m}/D_m; D_n/C_{2n})$                                       | 30| $(I/I; I/I)$                                                    |
| 14| $(D_m/D_m; T/T)$                                                 | 31| $(I/C_2; I/C_2)$                                               |
| 15| $(D_m/D_m; O/O)$                                                 | 32| $(I/1/C_2; I/C_2; \phi^{-1})$                                 |
| 16| $(D_m/C_{2m}; O/T)$                                              |   |                                                                 |

TABLE 4. Finite Subgroups of $O(4)$, part II

|   |                                                                 |   |                                                                 |
|---|-----------------------------------------------------------------|---|-----------------------------------------------------------------|
| 1’| $(C_{2mr}/C_{m}; C_{2nr}/C_{n}; \phi_s)$                        | 26’| $(O/C_1; O/C_1; id)$                                           |
| 11’| $(D_{mr}/C_{m}; D_{nr}/C_{n}; \psi_s)$                          | 26”| $(O/C_1; O/C_1; \zeta)$                                        |
| 11a’| $(D_{2m}/C_{m}; D_{2n}/C_{n}; \psi_q)$                         | 31’| $(I/C_1; I/C_1)$                                               |
| 21’| $(T/C_1; T/C_1)$                                                | 32’| $(I/1/C_1; I/C_1; \phi^{-1})$                                 |

element of $O(4) - SO(4)$ as the composition of the particular orientation-reversing map $q \mapsto \overline{q}$ (linear, mapping $1 \mapsto 1$, $i \mapsto -i$, $j \mapsto -j$, and $k \mapsto -k$), followed by an arbitrary orientation-preserving map $q \mapsto aq^b$, hence in the form $q \mapsto aq^b$ (where $a, b \in S^3$). This representation is unique up to multiplying both $a$ and $b$ by $-1$.

It follows from the identity $\overline{q_1 q_2} = \overline{q_2} \overline{q_1}$ that conjugation by the orientation-reversing map $q \mapsto \overline{q}$ takes an element of $SO(4)$ covered by $(\ell, r) \in S^3 \times S^3$ to one covered by $(r, \ell)$. Hence, more generally, for a finite subgroup $G$ of $O(4)$ containing orientation-reversing elements, the groups $L$ and $R$ describing $G \cap SO(4)$ must be conjugate. Indeed, $L = R$, except when $G \cap SO(4)$ equals group #32 or group #32'.
Following Du Val’s classification, we start with those subgroups of \( G \) of \( O(4) \) which contain the central element (and a few which do not, namely families \#33, 35, and 36, when \( n \) is odd). Du Val’s notation adds a superscript asterisk to the symbol for the orientation-preserving subgroup, in order to indicate the presence of orientation-reversing elements. In most cases, we specify the extension by describing \( \{(a, b) : q \mapsto aq^{-1}b \in G\} \).

The basic conditions on the integers \( n, r, s, h, k \) in Table 5 are \( n, r \geq 1, 0 \leq s, h, k < r, \gcd(s, r) = 1 \) and \( rn \) is even; further conditions are as follows:

1. \( s^2 = 1 \), \( h(s - 1) \equiv 0 \pmod{r} \); extending element \( q \mapsto p^h\overline{q} \), where \( p := \cos \frac{2\pi}{nr} + \sin \frac{2\pi}{nr}k \).
2. \( s^2 \equiv 1 \pmod{r} \), \( h \equiv k \pmod{2} \), \( (h - k)(s - 1) \equiv (h + k)(s + 1) \equiv 0 \pmod{2r} \); extending element \( q \mapsto p^{h^2}\overline{q}p^{k^2} \), using \( p^\pm \) to denote \( \cos \frac{\pi}{nr} + \sin \frac{\pi}{nr}k \).
3. \( s^2 + 1 \equiv 0 \pmod{r} \), \( h \equiv k \pmod{2} \), \( h + k \equiv s(h - k), k - h \equiv s(k + k) \pmod{2r} \); extending element \( q \mapsto ip^{h^2}\overline{q}p^{k^2} \), with \( p^\pm \) as in note 2).
4. \( a = p^t b', b = \pm(p^t)^{-1} \), with \( p \in I \), and \( t' \in O - T \). It suffices to let \( t' \) be any fixed element of \( O - T \), such as \((1/\sqrt{2})i + (1/\sqrt{2})j \).

In addition, a few of these groups have subgroups of index two which contain orientation-reversing elements, but do not contain the central element. These are listed in Table 6. Groups \#44pm and \#44mp do not appear in [DuV] page 61, but are listed as \( \pm\frac{1}{24}[O \times O] \cdot 2_3 \) and \( \pm\frac{1}{24}[O \times O] \cdot 2_1 \), respectively, in [CS] page 47. Extra conditions on the groups in this table are as follows:

5. \( (a, b \in C_n) \) or \( (a \in -kC_n, b \in iC_n) \) or \( (a \in iC_n, b \in -kC_n) \) or \( (a, b \in jC_n) \).
6. \( (a, -b \in C_n) \) or \( (a \in kC_n, b \in iC_n) \) or \( (a \in iC_n, b \in kC_n) \) or \( (a, -b \in jC_n) \).
7. \( a = p^t b', b = (pt')^{-1} \), with \( p \in I \), and \( t' \in O - T \).
8. \( a = p^t b', b = -(pt')^{-1} \), with \( p \in I \), and \( t' \in O - T \).

For each of these subgroups \( G \) of \( O(4) \), we will obtain a lower bound on the diameter of the orbifold \( S^3/G \) by maximizing the distance from the orbit of the quaternion 1, under the group action, to another orbit. To that end, we’ll define the pre-fundamental domain of a finite group action on \( S^3 \) to be the intersection of half-spheres formed by the set of points which are closer to the quaternion 1 than to any other image of 1 (equivalent to the Voronoi cell of 1 with respect to its \( G \)-orbit; when, in addition, 1 is fixed only by the identity element of \( G \), it is a Dirichlet domain). The geodesic segment from 1 to any point in the pre-fundamental domain realizes the distance in
the orbifold between the equivalence classes represented by those points. So the distance from 1 to the farthest vertex of the pre-fundamental domain gives a lower bound for the diameter of the orbifold. We expect that this lower bound will be sharp in most cases, based on the fact that a hyperbolic the orbifold”. This assertion is motivated by the fact that singular points in of a “sharp” cone, which should tend to “push it away” from “the rest of the orbifold”. This intuition is quite rough, but explains our choice of a large cyclic group acting by rotation) are contained in the middle of fatMargulis tubes, hence are far from the “thick part” of the orbifold [Me]. This intuition is quite rough, but explains our choice of 1 for one end of the geodesic segment which realizes our lower bound for the diameter. In some cases, noted below, we can prove that our bound is sharp. The coordinates of the vertices of a pre-fundamental domain can be calculated by linear algebra once triples of points in the orbit of 1 are found which are both

| Table 5. Finite Subgroups of $O(4)$, part III |
|-----------------------------------------------|
| 33. $(C_{nr}/C_n; C_{nr}/C_n; \phi_s)_h^*$; $\phi_s$ | [1] |
| 34. $(D_n/D_n; D_n/D_n)^*$; $a, b \in D_n$ | [2] |
| 35a. $(D_{2n}/C_{2n}; D_{2n}/C_{2n}; \psi^\#)^*$; $a, b \in D_{2n}, bC_{2n} = \psi^\#(aC_{2n})$ | [3] |
| 36. $(D_{2nr}/C_n; D_{2nr}/C_n; \phi_s)_{h,k}^*$; | [4] |
| 37. $(D_{2n}/D_n; D_{2n}/D_n)^*$; $a, b \in D_{2n}, aD_n = bD_n$ |
| 38. $(D_{2n}/D_n; D_{2n}/D_n)^*$; $a, b \in D_{2n}, aD_n \neq bD_n$ |
| 39. $(T/C_2; T/C_2)^*; a \in T, b = \pm a^{-1}$ |
| 40. $(T/C_2; T/C_2)^*; a \in O - T, b = \pm a^{-1}$ |
| 41. $(T/D_2; T/D_2)^*; a, b \in T, ab \in D_2$ |
| 42. $(T/D_2; T/D_2)^*; a, b \in O - T, ab^{-1} \in D_2$ |
| 43. $(T/T; T/T)^*; a, b \in T$ |
| 44. $(O/C_2; O/C_2)^*; a \in O, b = \pm a^{-1}$ |
| 45. $(O/T; O/T)^*; a, b \in O, aT = bT$ |
| 46. $(O/T; O/T)^*; a, b \in O, aT \neq bT$ |
| 47. $(O/D_2; O/D_2)^*; a, b \in O, ab \in D_2$ |
| 48. $(O/O; O/O)^*; a, b \in O$ |
| 49. $(I/C_2; I/C_2)^*; a \in I, b = \pm a^{-1}$ |
| 50. $(I/I; I/I)^*; a, b \in I$ |
| 51. $(I^\dagger/C_2; I/C_2; \phi^{-1})^*$; | [4] |
close to 1 and close to each other; there are three linear constraints since the vertex must be equidistant from 1 and each point in the triple, and also the vertex must have unit length (and make an acute angle with 1). We used Maple\textsuperscript{TM} software to handle the messier situations [Ma]. See [Du] for more on pre-fundamental domains; in particular, a fundamental domain for \( S^3/G \) can be obtained by intersecting the pre-fundamental domain of \( G \) with a cone which is a fundamental domain for the subgroup of \( G \) which fixes 1.

### 4.2. Diameters for the fibering subgroups of \( SO(4) \)

The subgroups \( G \) of \( SO(4) \) for which the corresponding orbifolds \( S^3/G \) admit a fibering over a 2-orbifold are precisely those groups for which at least one of the groups \( L \) and \( R \) belong to the set \( \{ C_{2n}, D_n \} \). In Du Val’s enumeration, these are the families \#1–19 (including \#11a, \#11', \#11a'). Among fibering subgroups of \( SO(4) \), families \#10, \#15, and \#19 are maximal; all other groups are subgroups of some member of these families. Since we are looking for a lower bound on the diameters of the orbifolds arising from these families, it suffices to examine the maximal families, as follows.

### 10 (\( D_m/D_m; D_n/D_n \))

The orbit of the quaternion 1 in \( S^3 \) is the union of \( C_{2L} \) and the coset \( iC_{2L} \), where \( L = \text{lcm}(m, n) \). The pre-fundamental domain is the same as for \( (D_L/D_L; D_L/D_L) \), a 2L-prism with vertices \( \frac{1}{\sqrt{2}} \left( \cos \left( \frac{\pi}{2L} \right) \pm \sin \left( \frac{\pi}{2L} \right) k + \cos \left( \frac{\pi t}{2L} \right) i + \sin \left( \frac{\pi t}{2L} \right) j \right) \) where \( t = 1, 3, ..., 4L - 1 \).
A lower bound for the diameter is \( \arccos \left( \frac{\cos(\frac{\pi}{2}L)}{\sqrt{2}} \right) \), which is always greater than \( \arccos(1/\sqrt{2}) = \pi/4 \). The diameter approaches \( \pi/4 \) as \( L \to \infty \) and the group approaches the corresponding cohomogeneity one action.

15 \( (D_m/D_m; O/O) \): Let \( m \to \infty \). The limit group \( G \) contains every group in this family. Its identity component \( G_0 \) can be described as

\[
A(t) = \begin{pmatrix}
\cos(t) & 0 & 0 & -\sin(t) \\
0 & \cos(t) & -\sin(t) & 0 \\
0 & \sin(t) & \cos(t) & 0 \\
\sin(t) & 0 & 0 & \cos(t)
\end{pmatrix},
\]

where \( t \in \mathbb{R} \). In addition, \( S^3/G_0 = S^2(\frac{1}{\sqrt{3}}) \) and \( G/G_0 = O^h \), and so a lower bound on the diameter is \( \frac{1}{2} \arccos \left( \frac{1}{\sqrt{3}} \right) \) (see Table 2).

19 \( (D_m/D_m; I/I) \): Similarly, \( S^3/G_0 = S^2(\frac{1}{\sqrt{7}}) \) and \( G/G_0 = I^h \), and so a lower bound on the diameter is \( \frac{1}{2} \arccos \left( \frac{\tan(\frac{\pi}{7})}{\sqrt{3}} \right) \).

4.3. **Diameters for the remaining fibering subgroups of \( O(4) \).** The remaining fibering subgroups belong to Du Val's families \#33–38. Each of these groups is contained in some member of family \#34, since \( (L/L; R/R; \phi) \) will be contained in \( (D_n/D_n; D_n/D_n)^* \) if \( L \) and \( R \) are contained in \( D_n \). Hence, to find a lower bound on the diameter, it suffices to examine this family.

34 \( (D_n/D_n; D_n/D_n)^* \): The orientation-preserving subgroup belongs to family \#10 (with \( m = n \)). The number of points in the orbit of the quaternion 1 under a finite subgroup of \( O(4) \) equals the order of the group divided by the order of the isotropy subgroup at 1. The full group has twice the order of the orientation-preserving subgroup, but the order of the isotropy subgroup also doubles, since \( q \mapsto -q \) is an orientation-reversing element of any group in family \#34, and fixes 1. Hence the orbit of 1 remains the same after extension, so the pre-fundamental domain remains the same. Consequently, the diameter remains at least \( \arccos \left( \frac{\cos(\frac{\pi}{2}L)}{\sqrt{2}} \right) \); see section 4.2.

4.4. **Diameters for the nonfibering subgroups of \( SO(4) \).** The subgroups \( G \) of \( SO(4) \) for which the corresponding orbifolds \( S^3/G \) do not admit a fibering over a 2-orbifold are precisely those groups for which both groups \( L \) and \( R \) belong to the set \( \{ T, O, I \} \). In Du Val’s enumeration, these are groups \#20–32 (including 21', 26', 26'', 31', 32'). The diameter bound from 1 is sharp when it equals \( \pi/2 \) or \( \pi \), since the diameter is greater than \( \pi/2 \) exactly when there is a point fixed by the entire group [B], [GM]. Among nonfibering subgroups of \( SO(4) \), \#29 is the only group which is maximal with respect to inclusion among finite subgroups of \( O(4) \); its orbit space is a natural candidate for the minimal diameter spherical orbifold. The
other groups are either subgroups of it or are contained in a subgroup of $O(4)$ which contains orientation-reversing transformations. We present the groups in order of decreasing diameter, first considering those whose diameter is a rational multiple of $\pi$ and then those whose whose diameter is an irrational multiple of $\pi$.

21' $(T/C_1; T/C_1)$, 26' $(O/C_1; O/C_1; id)$, 31' $(I/C_1; I/C_1)$: In all these cases, the quaternion 1 is fixed by the entire group. The diameter is $\pi$.

21 $(T/D_2; T/D_2)$, 26 $(O/D_2; O/D_2)$, 31 $(I/D_2; I/D_2)$: The pre-fundamental domain is a truncated regular tetrahedron with vertices (antipodal to these images) (23) (The pre-fundamental domain is bounded by the great sphere perpendicular to 1. So the diameter is $\pi/2$.

22 $(T/D_2; T/D_2)$, 27 $(O/D_2; O/D_2)$: The closest images of the quaternion 1 are $\pm i, \pm j, \pm k$; indeed, the entire orbit of 1 is $D_2$. The pre-fundamental domain is a cube with vertices $(1 \pm i \pm j \pm k)/2$. So a lower bound for the diameter is $\arccos(1/2) = \pi/3$.

20 $(T/T; T/T)$, 28 $(O/T; O/T)$: The closest images of the quaternion 1 under these groups are $(1 \pm i \pm j \pm k)/2$ (their isotropy subgroups are, respectively, tetrahedral and octahedral). The pre-fundamental domain is a regular octahedron with vertices $(1 \pm i)/\sqrt{2}, (1 \pm j)/\sqrt{2}, (1 \pm k)/\sqrt{2}$. So a lower bound for the diameter is $\arccos(1/\sqrt{2}) = \pi/4$.

32' $(I'/C_1; I'/C_1; \phi^{-1}_{I'})$: The closest images of the quaternion 1 are $(-1 + \sqrt{5}i + \sqrt{5}j + \sqrt{5}k)/4$ plus the 3 additional points obtained by changing two plus signs to minus signs. The pre-fundamental domain is a regular tetrahedron with vertices (antipodal to these images) $(1 - \sqrt{5}i - \sqrt{5}j - \sqrt{5}k)/4$ plus the 3 additional points obtained by changing two minus signs to plus signs. So a lower bound for the diameter is $\arccos(1/4) \approx \pi/2.38$.

32 $(I'/C_2; I'/C_2; \phi^{-1}_{I'})$: The closest images of the quaternion 1 are $(1 - \sqrt{5}i - \sqrt{5}j - \sqrt{5}k)/4$ together with $(1 - \sqrt{5}i + \sqrt{5}j + \sqrt{5}k)/4$, plus the 2 additional points obtained by cyclically permuting $i, j, k$ in the latter expression. The next closest images are the 4 points antipodal to these. The pre-fundamental domain is a truncated regular tetrahedron with vertices $(\sqrt{5} + 3i + j + k)/4$, plus the 11 additional points obtained by changing two plus signs to minus signs and/or cyclically permuting $i, j, k$. So a lower bound for the diameter is $\arccos(\sqrt{5}/4) \approx \pi/3.21$.

23 $(T/T; O/O)$, 25 $(O/O; O/O)$: The closest images of the quaternion 1 under this group are $(1 \pm i)/\sqrt{2}, (1 \pm j)/\sqrt{2}, (1 \pm k)/\sqrt{2}$; next closest are $(1 \pm i \pm j \pm k)/2$ (from the subgroup #20). The pre-fundamental domain is a truncated cube, with vertices $((\sqrt{2} + 1) \pm (\sqrt{2} - 1)i \pm j \pm k)/2\sqrt{2}$, plus
the 16 additional points obtained by cyclically permuting i, j, k. So a lower bound for the diameter is arccos((√2 + 1)/2√2) ≈ π/5.73.

24 (T/T: I/I), 30 (I/I: I/I): For group #24, the closest images of the quaternion 1 under this group are ((√5 + 1) + 2i + (√5 − 1)j + 0k)/4, plus the eleven additional points in the orbit of this point under the action of the tetrahedral group (the image of T in SO(3)). The same is true for group #30, except that instead of 11 additional points we have 8 obtained by cyclically permuting i, j, and k. The pre-fundamental domain is a dodecahedron with vertices ((3√2 + √10) + (3 + √5)(√10 − 2√2)i + (3 + √5)(√10 − 2√2)j + (3 + 5√2)(√10 − 2√2)k)/8, plus the 7 additional points obtained by reflections in the three great spheres orthogonal to i, j, and k. In addition, ((6√2 + 2√10) + 0i + (7 − 3√5)(3√2 + √10)j + 4√2k)/16 is a vertex, as well as the 11 additional points in the orbit of this point under the action of the tetrahedral group (the image of T in SO(3)). So a lower bound for the diameter is arccos((3√2 + √10)/8) ≈ π/8.10.

29 (O/O: I/I): The closest images of the quaternion 1 under this group are ((3√2 + √10) + (3 + √5)(√10 − 2√2)i + (3 + 5√2)(√10 − 2√2)j + (3 + √5)(√10 − 2√2)k)/8, ((3√2 + √10) + (3 + √5)(√10 − 2√2)i + (3 + √5)(√10 − 2√2)j + (3 + 5√2)(√10 − 2√2)k)/8, plus the two additional points obtained by cyclically permuting i, j, and k in the latter expression. The next closest images are ((√5 + 1) + 2i + (√5 − 1)j + 0k)/4, plus the eleven additional points in the orbit of this point under the action of the tetrahedral group (the image of T in SO(3)). The third layer of images are (√10 − √2i − √2j − √2k)/4 together with (√10 − √2i + √2j + √2k)/4, plus the two additional points obtained by cyclically permuting i, j, and k in the latter expression.

The pre-fundamental domain has 4 twelve-sided faces, 4 six-sided faces, and 12 faces which are isosceles triangles. It can also be described as the intersection of a smaller tetrahedron with a larger tetrahedron in dual position, all the vertices of which are then truncated by the intersection with a dodecahedron. Refer to Figure 9 in [Du] for details. The vertices of one of the isosceles triangles are as follows:

\[
\frac{1 + (3 - \sqrt{10})i + (2 + \frac{3}{2}\sqrt{2} - \sqrt{5} - \frac{1}{2}\sqrt{10})j + (1 + \frac{1}{2}\sqrt{2} - \frac{1}{2}\sqrt{10})k}{\sqrt{40 + 12\sqrt{2} - 8\sqrt{5} - 12\sqrt{10}}}
\]

\[
\frac{1 + ((-1 - \frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{10})i + (4 - 2\sqrt{2} + \sqrt{5} - \sqrt{10})j + (-5 + \frac{7}{2}\sqrt{2} - 2\sqrt{5} + \frac{3}{2}\sqrt{10})k}{\sqrt{136 - 90\sqrt{2} + 56\sqrt{5} - 42\sqrt{10}}}
\]

\[
\frac{1 + ((-3 + 3\sqrt{2} - 2\sqrt{5} + \sqrt{10})i + ((-2 + \sqrt{2} - \sqrt{5} + \sqrt{10})j + (3 - 3\sqrt{2} + 2\sqrt{5} - \sqrt{10})k}{\sqrt{136 - 90\sqrt{2} + 56\sqrt{5} - 42\sqrt{10}}}
\]

The first vertex is further from 1 than the other two vertices, which have the same distance from 1. The remaining 33 vertices are obtained by letting
the tetrahedral group act on these three vertices, which leaves invariant the distances to 1. Hence a lower bound on the diameter is
\[
\arccos\left(\frac{1}{\sqrt{40 + 12\sqrt{2} - 8\sqrt{5} - 12\sqrt{10}}}\right) \approx \pi/8.93.
\]

The group is maximal and the diameter is the smallest achieved by a nonfibering subgroup.

4.5. **Diameters for the remaining nonfibering subgroups of** \(O(4)\). For many of these groups, the orbit of the quaternion 1 is the same as its orbit under the orientation-preserving subgroup of index 2. This occurs if and only if some orientation-reversing element fixes 1, which in turn is equivalent to the condition \(b = a^{-1}\) in the description of the group, and hence is often easy to verify by inspection. In fact, it turns out that whenever an orientation-reversing element fixes 1, either there is an element with \(a = 1 = b\) or there is one with \(a = 1/\sqrt{2} + (1/\sqrt{2})k, b = 1/\sqrt{2} - (1/\sqrt{2})k\). In these cases, the analysis repeats that of the subgroup, so we refer the reader back to that subgroup for information about the images of 1 and the pre-fundamental domain. The only groups we need to consider then are \#39m, 40m, 44m, 46, 49m and 51m.

**39m** (\(T/C_1; T/C_1\), \(T/C_1; T/C_1\), \(O/C_1; O/C_1\), \(id\)_): The quaternion 1 is mapped to itself by all elements of the orientation-preserving subgroup (\#21’ in the first two cases, \#26’ in the third case, \#31’ in the fourth case) and mapped to \(-1\) by all other elements. The pre-fundamental domain is bounded by the great sphere perpendicular to 1, so the diameter is \(\pi/2\) in all four cases.

**46** (\(O/T; O/T\)_): The orientation-preserving subgroup is \#28. The orbit of the quaternion 1 is \(O\), and hence the pre-fundamental domain is the same truncated cube as for group \#25. So a lower bound for the diameter is \(\arccos((\sqrt{2} + 1)/2\sqrt{2}) \approx \pi/5.73\).

**51m** (\(I^1/C_1; I/C_1; \phi^{-1}_1\)_): The orientation-preserving subgroup is \#32’. The orbit of the quaternion 1 is the same as that of \#32, and hence the pre-fundamental domain is the same truncated tetrahedron. So a lower bound for the diameter is \(\arccos(\sqrt{5}/4) \approx \pi/3.21\).

In addition, we include descriptions of reflection groups and their Coxeter graphs in Table [7] cf. [GB]. For these groups, the diameter equals the minimum distance between vertices of the fundamental domain, a spherical polyhedron, so we can supply the exact diameter, not just a lower bound. The appearance of \(\Sigma\) in the column for the Coxeter graph signifies that the group is a suspension to \(O(4)\) of the group of reflections in \(O(3)\) which follows (in other words, the action is reducible and acts trivially on the extra dimension, as in the \(SO(3)\) action on \(S^3\) given near the start of section [2]).
TABLE 7. Reflection Subgroups of $O(4)$

| Du Val # | Coxeter graph | Diameter       |
|----------|---------------|----------------|
| $40p$    | $\Sigma$     | $\pi$          |
| $44p$    | $\Sigma$     | $\pi$          |
| $49p$    | $\Sigma$     | $\pi$          |
| $44mp$   | $\bullet$    | $\pi/2$        |
| $44$     | $\bullet$    | $\pi/2$        |
| $49$     | $\bullet$    | $\pi/2$        |
| $47$     | $\bullet$    | $\pi/3$        |
| $42$     | $\bullet$    | $\pi/3$        |
| $51p$    | $\bullet$    | $\pi/4$        |
| $45$     | $\bullet$    | $\pi/4$        |
| $50$     | $\bullet$    | $\arccos\left(\frac{3+\sqrt{5}}{4\sqrt{2}}\right)$ |

5. Conclusions

We summarize the results for cohomogeneity three in three tables: Table 8 for the fibering groups, Table 9 for the nonfibering groups with diameters a rational multiple of $\pi$, and Table 10 for the remaining nonfibering groups.
### Table 8. Diameters for Finite, Fibering Groups

| Du Val # | Lower Bound for Diameter |
|----------|--------------------------|
| 10       | $\frac{\pi}{4}$         |
| 15       | $\frac{1}{2} \arccos\left(\frac{1}{\sqrt{3}}\right)$ |
| 19       | $\frac{1}{2} \arccos\left(\tan\left(\frac{3\pi}{10}\right)\right)$ |
| 34       | $\frac{\pi}{4}$         |

All the normal subgroups in these tables have index 2. For the remaining subgroups we note that the inclusions of #22 in #20 and #27 in #28 are both index 3, while the inclusions of #21 in #22 and #26 in #27 are both index 4, the inclusion of #20 in #24 is index 5, and the inclusions of #31 and a conjugate in $SO(4)$ of #32 in #30 are both index 60.

We summarize this information in the following theorem:

**Theorem 5.1.** Let $G$ be a non-trivial finite subgroup of $O(4)$. Then

$$ \min(\text{diam}(S^n/G)) = \begin{cases} \frac{\alpha}{2} & \text{for fibering groups} \\ \beta & \text{for nonfibering groups} \end{cases} $$

where $\alpha = \arccos\left(\frac{\tan\left(\frac{3\pi}{8}\right)}{\sqrt{3}}\right)$, $\beta = \arccos\left(\frac{\sqrt{40 + 12\sqrt{2} - 8\sqrt{5} - 12\sqrt{10}}}{4}\right)$.  

Note that $\alpha/2 \approx \pi/9.63$ is strictly smaller than $\beta \approx \pi/8.93$. We further note that there seems to be no relationship whatsoever between the index of an extension and any subsequent change in diameter. In particular, there are many extensions of index 2 where the diameter remains unchanged, others where the diameter is reduced by half, others where the diameter is reduced instead by $\frac{3}{4}$ and still others by $\beta/(\arccos(\frac{3\sqrt{2}\pm\sqrt{10}}{8}))$.

We observe that the nonfibering groups giving “nice” diameters, i.e., rational multiples of $\pi$, are mainly of one type. The majority are of the form $(L/L; R/R)$ where $L = R \in \{T, O, I\}$ and $L = R \in \{C_2, D_2\}$. There are two exceptions, groups #20 and #28, which are of the form $(T/T; T/T)$ and $(O/T; O/T)$. The former groups with $L = R = C_2$ are “diagonal” groups; that is, they project to diagonal subgroups of $SO(3) \times SO(3)$. These groups are characterized by having very small orbits of the quaternion $1$ and hence very large pre-fundamental domains. If the group contains the non-trivial central element then the orbit of $1$ is $\pm 1$, and if not, its orbit is simply $1$. Thus these groups will have pre-fundamental domain the entire 3-sphere or the half sphere and corresponding diameters of $\pi$ and $\frac{\pi}{2}$. The bigger $L$ and $R$ become the larger the orbit of $1$ becomes (for fixed $L$, $R$). These will be points at rational multiples of $\pi$ away from $1$, generating faces of the pre-fundamental domain (halfway between each point and 1). However, there is no easy way to predict in general whether or not these faces will happen to
### Table 9. Diameters in $\pi \mathcal{Q}$ for Finite, Nonfibering Groups

| Du Val # | Maximal Inclusions | Diameter |
|----------|--------------------|----------|
| 21'      | $21' \prec 26'$    | $\pi$    |
| 26'      | $21' \prec 26'$    | $\pi$    |
| 31'      | $21' \prec 39p$    | $\pi$    |
| 39p      | $21' \prec 40$     | $\pi$    |
| 40       | $21' \prec 40p$    | $\pi$    |
| 44       | $26 \prec 44$      | $\pi$    |
| 44p      | $26' \prec 44p$    | $\pi$    |
| 49p      | $31' \prec 49p$    | $\pi$    |
| 21       | $21' \prec 21$     | $\pi$    |
| 26       | $21' \prec 26$     | $\pi$    |
| 26'      | $21' \prec 26$     | $\pi$    |
| 26''     | $21' \prec 26'$    | $\pi$    |
| 31       | $31' \prec 31$     | $\pi$    |
| 39       | $31' \prec 39$     | $\pi$    |
| 39m      | $21' \prec 39m$    | $\pi$    |
| 40m      | $21' \prec 40m$    | $\pi$    |
| 44m      | $26' \prec 44m$    | $\pi$    |
| 44pm     | $26' \prec 44pm$   | $\pi$    |
| 44mp     | $26' \prec 44mp$   | $\pi$    |
| 49       | $31' \prec 49$     | $\pi$    |
| 49m      | $31' \prec 49m$    | $\pi$    |
| 22       | $21 \subset 22$    | $\pi$    |
| 27       | $26 \subset 27$    | $\pi$    |
| 41       | $22 \subset 41$    | $\pi$    |
| 42       | $22 \subset 42$    | $\pi$    |
| 47       | $27 \subset 47$    | $\pi$    |
| 20       | $22 \subset 20$    | $\pi$    |
| 28       | $27 \subset 28$    | $\pi$    |
| 43       | $20 \subset 43$    | $\pi$    |
| 45       | $28 \subset 45$    | $\pi$    |
### Table 10. Diameters Not in $\pi \mathbb{Q}$ for Finite, Nonfibering Groups

| Du Val # | Maximal Inclusions | Diameter |
|----------|--------------------|----------|
| $32'$    |                  | $\arccos\left(\frac{1}{4}\right)$ |
| $51p$    | $32' \triangleleft 51p$ | $\arccos\left(\frac{1}{4}\right)$ |
| $32$     | $32' \triangleleft 32$ | $\arccos\left(\frac{\sqrt{5}}{4}\right)$ |
| $51$     | $32 \triangleleft 51$ | $\arccos\left(\frac{\sqrt{5}}{4}\right)$ |
| $51m$    | $32 \triangleleft 51m$ | $\arccos\left(\frac{\sqrt{5}}{4}\right)$ |
| $23$     | $20 \triangleleft 23$ | $\arccos\left(\frac{\sqrt{2}+1}{2}\right)$ |
| $25$     | $23 \triangleleft 25$ | $\arccos\left(\frac{\sqrt{2}+1}{2}\right)$ |
| $46$     | $25 \triangleleft 46$ | $\arccos\left(\frac{\sqrt{10}}{2}\right)$ |
| $48$     | $25 \triangleleft 48$ | $\arccos\left(\frac{\sqrt{10}}{2}\right)$ |
| $24$     | $20 \subset 24$ | $\arccos\left(\frac{\sqrt{2}+1}{2}\right)$ |
| $30$     | $31 \subset 30$, $g32g^{-1} \subset 30$, where $g \in SO(4)$ | $\arccos\left(\frac{3\sqrt{2}+\sqrt{10}}{8}\right)$ |
| $50$     | $30 \triangleleft 50$ | $\arccos\left(\frac{3\sqrt{2}+\sqrt{10}}{8}\right)$ |
| $29$     | $24 \triangleleft 29$ | $\arccos\left((\sqrt{40} + 12\sqrt{2} - 8\sqrt{5} - 12\sqrt{10})^{-1}\right)$ |

Intersect at a point which is at a distance from 1 which is a rational multiple of $\pi$.

The remaining nonfibering groups giving diameters that are irrational multiples of $\pi$ are generally of the form $(L/L; R/R)$, where $L, R \in \{T, O, I\}$ and $L, R \in \{T, O, I\}$ or of the form $(I/C_1; I/C_1)$ or an extension of the same.

Note that $(T/T; T/T) \triangleleft (T/T; O/O) \triangleleft (O/O; O/O)$ and $(O/T; O/T) \triangleleft (O/O; O/O)$ but $(T/T; T/T)$ and $(O/T; O/T)$ both have diameters that are rational multiples of $\pi$ and $(T/T; O/O)$, while $(O/O; O/O)$ both have diameters that are irrational multiples of $\pi$. Thus a general theorem relating these finite groups with diameters that are rational or irrational multiples of $\pi$ seems elusive.

We are interested in finding a global lower bound for isometric group actions on spheres. We note that the lower bound for any polar action arising from a symmetric space $G/H$ where either $G$ or $H$ (or both) is a product of classical Lie groups only, which we will define as a *classical polar* action,
approaches $\pi/2$ as the cohomogeneity of the action increases (cf. [MS2]). Those polar actions arising from symmetric spaces for which $G$ or $H$ (or both) is a product of classical Lie groups and exceptional Lie groups are to be called *exceptional polar* actions. The result holds for many of these groups as well, but for many of the groups in this list, the orbit space is yet to be calculated (given that these admit no “easy” matrix representation, other methods must be used). We note as well, that in the spherical cohomogeneity 2 case, the possible orbit spaces are $S^2$, $D^2$ (with 0, 1, 2 or 3 exceptional singular points corresponding to an isolated singular orbit). For the disk cases, those with exceptional singular orbits are limited by the corresponding isolated singular orbits as to further possible identifications, just as in the case of the interval for cohomogeneity 1). In particular, there are very few cases where the diameter will actually be changed after an identification and in none of these cases does the overall minimum diameter change. For the disk with no exceptional singular points, the same holds true. Given the work done here and work from [MS1] and [MS2] on classical connected polar actions of cohomogeneity 3, we can prove the following theorem:

**Theorem 5.2.** Let $G$ act irreducibly by cohomogeneity 1, 2, or 3 on $S^n$. Further suppose that the action is classical connected polar or non-trivial disconnected for the cohomogeneity 3 cases when $n = 3$. Then

$$\min(\text{diam}(S^n/G)) = \begin{cases} 
\frac{\pi}{2} & \text{for cohomogeneity 1} \\
\alpha^2 & \text{for cohomogeneity 2} \\
\alpha^2 & \text{for cohomogeneity 3}
\end{cases}$$

where $\alpha = \arccos\left(\frac{\tan(\frac{3\pi}{10})}{\sqrt{3}}\right)$

We further include the following conjecture:

**Conjecture 5.3.** Let $G$ act irreducibly on $S^n$ by cohomogeneity $k$, where $n \in 2\mathbb{Z}$. Then for all $\epsilon > 0$ and for all sufficiently large $k$ (and all $n > k$), $\text{diam}(S^n/G)$ is within $\epsilon$ of $\pi/2$.

We base this conjecture on the following: in [MS1] and [MS2], we can show that no further finite identifications can be made on the orbit spaces in cohomogeneity 3. Thus, there are no finite extensions of the classical connected polar actions which will decrease diameter. Furthermore, for a cohomogeneity $k$ action on $S^n$, where $3 \leq k < n$ and additionally $k \neq 3, 7$ when $n = 7, 15$ respectively, $S^k$ does not appear as a quotient space, in contrast to the $S^2$ which results from a cohomogeneity 2 circle action on $S^3$. Thus we can reasonably expect diameters to be strictly larger than those found in cohomogeneities 2 and 3, for sufficiently large $k$. We exclude the odd-dimensional spheres, because they all admit cohomogeneity $n - 1$ free circle actions with quotient spaces a complex projective space of diameter
and hence are likely to have quotients by finite extensions with diameters approaching $\frac{\pi}{4}$.

When considering cohomogeneity $n$ actions on $S^n$, $n \geq 4$, one might expect these diameters to increase toward $\frac{\pi}{2}$ as well, at least for $n$ even, since for $n$ uneven, we will have once again, as in section 4.2, finite actions converging to a free circle action at least halving the diameter of the corresponding spherical quotient space. To illustrate this tendency towards $\pi/2$, consider the orbifold $S^n/G$, where $G$ is the full group of symmetries of the cubic tessellation of $S^n$ (i.e., the radial projection of a hypercube in $\mathbb{R}^{n+1}$ onto $S^n$). Its diameter is $\arccos(1/\sqrt{n+1})$, which converges to $\pi/2$ as $n$ goes to infinity.

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(Dunbar) Department of Mathematics, Simon’s Rock College, Great Barrington, MA 01230, U. S. A.
E-mail address: wdunbar@simons-rock.edu

(Greenwald) Department of Mathematics, Appalachian State University, Boone, NC 28608, U. S. A.
E-mail address: greenwaldsj@appstate.edu

(McGowan) Department of Mathematics, Howard University, Washington, DC 20059, U. S. A.
E-mail address: jmcgowan@fac.howard.edu

(Searle) Institute of Mathematics, University Nacional Autonoma de Mexico, Cuernavaca, Morelos, Mexico
E-mail address: csearle@matcuer.unam.mx