OPTIMAL SMALL DATA SCATTERING FOR THE GENERALIZED DERIVATIVE NONLINEAR SCHRÖDINGER EQUATIONS

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Abstract. In this work, we consider the following generalized derivative nonlinear Schrödinger equations
\[
i\partial_t u + \partial_{xx} u + i|u|^{2\sigma} \partial_x u = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}.
\]
We prove that when \(\sigma \geq 2\), the solution is global and scattering when the initial data is small in \(H^s(\mathbb{R})\), \(s \geq \frac{1}{2}\). Moreover, we show that when \(0 < \sigma < 2\), there exist a class of solitary wave solutions \(\{\phi_c\}\) satisfy
\[
\|\phi_c\|_{H^1(\mathbb{R})} \to 0,
\]
when \(c\) tends to some endpoint, which is against the small data scattering statement. Therefore, the restriction \(\sigma \geq 2\) is optimal for scattering.

1. Introduction

In this paper, we consider the small data scattering of the Cauchy problem for the following generalized derivative nonlinear Schrödinger equation (gDNLS)
\[
\begin{cases}
i\partial_t u + \partial_x^2 u + i|u|^{2\sigma} \partial_x u = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}, \\
u(x, 0) = \varphi(x).
\end{cases}
\tag{1.1}
\]
Here \(\sigma > 0\), \(u : \mathbb{R}^n \to \mathbb{C}\) is an unknown function.

The generalized derivative nonlinear Schrödinger equations describe the physical phenomenon of Alfvén waves with small but finite amplitude propagating along the magnetic field in cold plasmas (see for example [56]).

After suitable gauge transformation, the equation in (1.1) can be regarded as a generalization of the following derivative nonlinear Schrödinger equation (DNLS)
\[
i\partial_t u + \partial_x^2 u + i\partial_x(|u|^2 u) = 0.
\tag{1.2}
\]
The well-posedness theory and the long time behavior of the solution for the equation (1.2) has been widely considered by many researchers. For a local well-posedness result, Hayashi and Ozawa [31, 32] proved that equation (1.2) is locally well-posed in the Sobolev space \(H^1(\mathbb{R})\), see also the previous works [19, 73]. Very recently, Mosincat and Yoon [60] proved the unconditional well-posedness in \(H^s(\mathbb{R})\), \(s > \frac{1}{2}\), see also Dan, Li and Ning [13] for the previous work in \(H^s(\mathbb{R})\), \(s > \frac{3}{2}\). With regard to the theory of global well-posedness, Hayashi and Ozawa [31] proved that it is globally well-posed in \(H^1(\mathbb{R})\) under the condition that the
initial data satisfies $\|u_0\|_{L^2} < \sqrt{2\pi}$. Wu [75, 76] showed that it is globally well-posed in $H^1(\mathbb{R})$ under the condition $\|u_0\|_{L^2} < 2\sqrt{\pi}$. Guo and Wu [21] later proved that it is globally well-posed in $H^\frac{1}{2}(\mathbb{R})$ under the same condition of initial data, see also [29, 10, 61] for the previous results on the low regularity. The same results also hold in the periodic case, see Mosincat and Oh [59] in $H^1(\mathbb{T})$, and Mosincat [68] in $H^\frac{3}{2}(\mathbb{T})$. More recently, Jenkins, Liu, Perry and Sulem [38] proved that the Cauchy problem (1.2) is globally well-posed in the weighted Sobolev space $H^{2,2}(\mathbb{R})$.

The equation in (1.1) in the case of $\sigma \neq 1$ also attracts a lot of researchers in recent years. Firstly, for the local well-posedness result, when $0 < \sigma < \frac{1}{2}$, Linares, Ponce and Santos [47, 48] proved the local well-posedness for a class of data of arbitrary size in an appropriate weighted Sobolev space. When $\frac{1}{2} \leq \sigma < 1$, Hayashi and Ozawa [34] proved that (gDNLS) is locally well-posed in $H^2(\mathbb{R})$, and Santos [67] showed the local well-posedness in a weighted space. When $\sigma > 1$, Hayashi and Ozawa [34] proved that (gDNLS) is locally well-posed in energy space $H^1(\mathbb{R})$. Hao [29] proved that it is locally well-posed in $H^{\frac{3}{2}}(\mathbb{R})$, when $\sigma \geq \frac{5}{2}$. Santos [67] proved that it is locally well-posed in $H^{\frac{3}{2}}(\mathbb{R})$ with small initial data when $\sigma > 1$. Secondly, compared with the local well-posedness, there are only a few results of global well-posedness. When $0 < \sigma < 1$, Hayashi and Ozawa [34] showed the global existence without uniqueness of (gDNLS) in $H^1(\mathbb{R})$. For $\sigma > 1$, Fukaya, Hayashi and Imi [15] gave a sufficient condition of initial data for global well-posedness in $H^1(\mathbb{R})$. Some other results related to the stability theory and inverse scattering theory can be found in [7, 8, 16, 18, 20, 37, 39, 44, 45, 50, 51, 52, 53, 54, 55, 63, 64, 69] and the references therein.

This equation also has its independent interest, which obeys the form of

$$i\partial_t u + \Delta u = P(u, \bar{u}, \partial_x u, \partial_x \bar{u}).$$

The well-posedness theory of equation (1.3) has been studied by many researchers, which we only involve a few of them and readers can find more of them from their references, when $P$ is a polynomial of the form $P(z) = \sum_{d \leq |\alpha| \leq l} C_{\alpha} z^\alpha$ and $l, d$ are integers with $l \geq d$. For general cases with $d \geq 3$, Kenig, Ponce and Vega [42] showed that the equation (1.3) is locally well-posed with small initial data in $H^{\frac{d}{2}}(\mathbb{R})$. Some further results have been acquired when $P$ is only composed of $\bar{u}$ and $\partial_x \bar{u}$ under some suitable assumption. Grünrock [28] proved that the equation (1.3) is locally well-posed for $s > \frac{1}{2} - \frac{1}{d-1}$ when $P = \partial_x (\bar{u}^d)$ and $s > \frac{3}{2} - \frac{1}{d-1}$ when $P = (\partial_x \bar{u})^d$ respectively. Hirayama [30] later extended Grünrock’s results to the small data global well-posedness for $s \geq \frac{1}{2} - \frac{1}{d-1}$ when $P = \partial_x (\bar{u}^d)$. Recently, Pornnopparath [65] proved that when each term in $P$ contains only one derivative, the equation (1.3) is locally well-posed in $H^{\frac{d}{2}}(\mathbb{R})$, and when a term in $P$ has more than one derivative, the equation (1.3) is locally well-posed in $H^{\frac{d}{2}}(\mathbb{R})$. Moreover, Pornnopparath also proved that when $d \geq 5$, (1.3) is almost globally well-posed in $H^s(\mathbb{R})$ when $P$ has only one derivative and $s \geq \frac{1}{2}$, or when $P$ has more than one derivative and $s > \frac{3}{2}$. Some more results on nonlinearity $\partial_x P(u, \bar{u})$ are also contained. For higher dimension and more related theories, see [11, 2, 11, 42, 74] and the references therein.

All of the results above are related to the theories of local and global well-posedness. To our knowledge, there is no scattering result yet to (gDNLS). The related result on modified scattering can be found in [23, 33] and the references therein.

One of the motivation to prove scattering is as a serve of our further study. In order to consider the long-time behavior of the solution to (gDNLS), the small data scattering theory
is initially needed in some situation, for example, long-time perturbation theory when we use the concentration-compactness argument.

Moreover, it was known that when $\sigma = 1$, there exist solitary wave solutions which can be arbitrarily close to zero. This implies that the small data scattering is not true when $\sigma = 1$. So one may wander the optimal value of $\sigma$ such that the scattering statement holds when the initial data is small enough in some Sobolev space. This is another motivation in the present paper. In particular, for semilinear Schrödinger equation, there are two important exponents named short range exponent and the Strauss exponent. When the nonlinear power is larger than the short range exponent $3 \left( 1 + \frac{2}{d} \right)$ for general dimensions), one has the global well-posedness and the existence of the wave operator for small data, see for examples [6, 17, 62]; when the nonlinear power is larger than the Strauss exponent $\sqrt{\frac{17+3}{2}} \approx 3.56 \left( \frac{\sqrt{d^2+12d+4}}{2} + \frac{d}{2} \right)$ for general dimensions), one has the scattering for small data, see [68]. According to these, especially because of the short range exponent, one may ask whether $\sigma = 1$ is the optimal exponent for scattering. However, there is no such general result for non-semilinear Schrödinger equation, related results see [12, 14, 25, 30] and the references therein. In fact, it is of much model dependence when the nonlinearity contains derivatives. In the present paper, as what we will shown in the following, the situation for the generalized derivative nonlinear Schrödinger equation (1.1) is of much difference, compared with the semilinear Schrödinger equations, and the models mentioned in the references above.

For all $\sigma > 0$, the equation (1.1) has a two-parameter family of solitary waves,

$$u_{\omega,c}(t) = e^{i\omega t} \phi_{\omega,c}(x - ct),$$

where the parameters $c^2 < 4\omega$ or $0 < c = 2\sqrt{\omega}$, and $\phi_{\omega,c}$ is the solution of the form

$$\phi_{\omega,c}(x) = \varphi_{\omega,c}(x) \exp \left\{ \frac{c}{2}ix - \frac{i}{2\sigma + 2} \int_{-\infty}^{x} \varphi_{\omega,c}^{2\sigma}(y) dy \right\}, \quad (1.4)$$

with

$$\varphi_{\omega,c}(x) = \left\{ \frac{(\sigma + 1)(4\omega - c^2)}{2\sqrt{\omega} \cosh(\sigma \sqrt{4\omega - c^2} - c)} \right\}^{\frac{1}{2\sigma}}.$$

Shown in Appendix, we prove that when $0 < \sigma < 2$,

$$\|\phi_{\omega,c}\|_{H^1(\mathbb{R})} \to 0, \quad \text{when } c \to -2\sqrt{\omega}.$$

However, when $\sigma \geq 2$, there exists a positive constant $c_0$, such that

$$\|\phi_{\omega,c}\|_{L^2(\mathbb{R})} \geq c_0, \quad \text{for any } (\omega, c) \in \left\{ (\omega, c) : c^2 < 4\omega, \text{or } 0 < c = 2\sqrt{\omega} \right\}.$$

Hence, the small data scattering is not true for all $0 < \sigma < 2$, but it is reasonable to claim that the small data scattering holds when $\sigma \geq 2$. Our paper aims to show this assertion.

Now we state our main result.

**Theorem 1.1.** Let $\sigma \geq 2$, $s \geq \frac{1}{2}$ and $\varphi \in H^s(\mathbb{R})$. There exists a constant $\delta_0 > 0$, such that if $\|\varphi\|_{H^s(\mathbb{R})} \leq \delta_0$, then the corresponding solution $u$ is global, and

$$\left\| D^{s-\frac{1}{2}} \partial_x u \right\|_{L^2_t L^2_x(\mathbb{R} \times \mathbb{R})} + \left\| D^{s-\frac{1}{2}} u \right\|_{L^1_t L^\infty_x(\mathbb{R} \times \mathbb{R})} \lesssim \|\varphi\|_{H^s(\mathbb{R})}.$$

Moreover, there exists a unique $u_{\pm}$ such that for any $s \geq \frac{1}{2}$,

$$\|u(t) - e^{it\Delta} u_{\pm}\|_{H^s(\mathbb{R})} \to 0 \quad \text{as } t \to \pm\infty.$$
Remark 1.2. The same result is also true when we consider the nonlinearity $P(u, \bar{u}, \partial_x u, \partial_x \bar{u})$ and $d \geq 5$ in (1.3) with $P$ has only one derivative. As a comparable result, Pornnopparath [65] proved that when $\sigma \geq 5$, and is an integer, the equation in (1.1) is almost globally well-posed in $H^s(\mathbb{R})$, $s > \frac{1}{2}$. Here “almost” is in the sense that given an arbitrary large $T > 0$, there exists a constant $C = C(T) > 0$, such that for any initial data $u_0 : \|u_0\|_{H^s(\mathbb{R})} \leq C$, the corresponding solution is in $[0, T]$. Theorem 1.1 improves Pornnopparath’s result. On one hand, we do not restrict that $\sigma$ is an integer. On the other hand, as a byproduct of scattering, we prove the global well-posedness in $H^4(\mathbb{R})$, $s \geq \frac{1}{2}$, which contains the “endpoint” case $s = \frac{1}{2}$ and the global well-posedness in the general sense. We believe that the index $s = \frac{1}{2}$ is optimal for local well-posedness in the sense of uniform continuity of the solution flow. However, it is not proved in this paper and leaves us an interesting problem to pursue later.

We use the bootstrap argument to prove Theorem 1.1. More precisely, after suitably defining the working space $X_T$ with any fixed time $T$, our purpose is to show the uniform-in-time estimate:

$$\|u\|_{X_T} \leq C_1 \|\varphi\|_{H^s(\mathbb{R})} + C_2 \|u\|_{X_T}^{2s+1}.$$ 

Here $C_1, C_2$ are the constants independent of $T$. Hence, as another byproduct, we can prove the local well-posedness for large initial data in $H^s(\mathbb{R})$, $s \geq \frac{1}{2}$, by using the standard fixed point argument. This improves the previous work of Hao [29]. The tools we use in the present paper are the smoothing effects and the maximal function estimates. Compared with the low power case $\sigma < 2$, the maximal function estimates in the case of $\sigma \geq 2$ provide many benefits. This enables us to handle the nonlinearity properly and establish the uniform-in-time estimate. However, since our desired result is stronger than the previous ones, the situation here has more obstacles, the key ingredients in our proofs are presented below.

(1) A suitable working space is constructed. In order to establish the uniform estimation on time $T$, a related complicated working space need to be constructed. We define the working space with its norm as

$$\|u\|_{X_T} = \|u\|_{L^s_t H^s_x([0, T] \times \mathbb{R})} + \|\partial_x u\|_{L^2_t L^2_x([0, T] \times \mathbb{R})} + \sup_{q \in [2, \infty)} \|u\|_{L^q_t L^\infty_x(\mathbb{R} \times [0, T])} + \sup_{r \in [2, \infty)} \|D^s - \frac{1}{2} u\|_{L^r_t L^\infty_x(\mathbb{R} \times [0, T])} + \|D^s - \frac{1}{2} \partial_x u\|_{L^2_t L^2_x([0, T] \times \mathbb{R})} + \|D^s - \frac{1}{2} u\|_{L^2_t L^\infty_x([0, T] \times \mathbb{R})}.$$ 

Here $\varepsilon$ is a fixed small parameter. We shall prove that the estimation of each norm in $X_T$ is closed. The selection of norms plays an important role in our paper.

(2) A key split on the terms involved the fractional derivatives is carried out. The endpoint Kato-Ponce inequality recently proved by Bourgain and Li [4] shall be used to deal with some $L^\infty - L^\infty$ type Leibniz rule for fractional derivatives. Moreover, a regular process using Hölder’s inequality fails to control these terms by $\|u\|_{X_T}$, since most of the mixed norms like $\sup_{q \in [2, \infty)} \|u\|_{L^q_t L^\infty_x(\mathbb{R} \times [0, T])}$ are the norm of time ahead. So the subtle split is established, thus we are able to change the order of the mixed norm in some applicable way. This idea has significant influence to obtain our whole estimation on the form of $\|u\|_{X_T}$.

The rest of the paper is organized as follows. In Section 2 we give some basic notations and some preliminary estimates that will be used throughout in our paper. In Section 3 we prove scattering result for (gDNLS) in $H^s(\mathbb{R})$ with small initial datum.
2. Notation and Preliminary

2.1. Notation. We write $X \lesssim Y$ or $Y \gtrsim X$ to indicate $X \leq CY$ for some constant $C > 0$. The notation $a + \varepsilon$ denotes $a + \epsilon$ for any small $\varepsilon$, and also $a - \varepsilon$ for $a - \varepsilon$. Denote $(\cdot) = (1 + |\cdot|^2)^{\frac{1}{2}}$ and $D^\alpha = (-\partial_x^2)^{\frac{\alpha}{2}}$. The Hilbert space $H^s(\mathbb{R})$ is a Banach space of elements such that $(\xi)^s \hat{u} \in L^2(\mathbb{R})$, where $\mathcal{F}$ denotes the Fourier transform $\mathcal{F}u(\xi) = \hat{u}(\xi) = \int_\mathbb{R} e^{-2\pi ix \xi} u(x) \, dx$, and equipped with the norm $\|u\|_{H^s} = \|(\xi)^s \hat{u}(\xi)\|_{L^2}$. An usual property of the Fourier transform is the Plancherel equality, that is, $\|f\|_{L^2} = \|\hat{f}\|_{L^2}$. We also have an embedding theorem that $\|u\|_{H^{s_1}} \lesssim \|u\|_{H^{s_2}}$ for any $s_1 \leq s_2$, $s_1, s_2 \in \mathbb{R}$. Throughout the whole paper, the letter $C$ will denote various positive constants which are of no importance in our analysis. We use the following norms to denote the mixed spaces $L_t^q L_x^r([0, T] \times \mathbb{R})$ and $L_t^\infty L_x^\infty(\mathbb{R} \times [0, T])$, that is,

$$\|u\|_{L_t^q L_x^r([0, T] \times \mathbb{R})} = \left( \int_0^T \|u\|_{L_x^r(\mathbb{R})}^q \, dt \right)^{\frac{1}{q}}$$

and

$$\|u\|_{L_t^\infty L_x^\infty(\mathbb{R} \times [0, T])} = \left( \int_\mathbb{R} \|u\|_{L_t^\infty(\mathbb{R} \times [0, T])}^r \, dx \right)^{\frac{1}{r}}.$$

2.2. Preliminary. In this section, we state some preliminary estimates of the linear Schrödinger operator $e^{it\Delta}$ which will be used in our later sections. Firstly, we recall the well-known Strichartz estimates.

Lemma 2.1. (Strichartz’s estimates, see [5]). Let $I \subset \mathbb{R}$ be an interval. For all admissible pair $(q, r)$ satisfying

$$2 \leq r \leq \infty \quad \text{and} \quad \frac{2}{q} = \frac{1}{2} - \frac{1}{r},$$

the following estimates hold:

$$\|e^{it\Delta} f\|_{L_t^q L_x^r(I \times \mathbb{R})} \lesssim \|f\|_{L^2(\mathbb{R})} \quad (2.1)$$

and

$$\left\| \int_0^t e^{i(t-t')\Delta} F(x, t') \, dt' \right\|_{L_t^q L_x^r(I \times \mathbb{R})} \lesssim \|F\|_{L_t^p L_x^\infty(I \times \mathbb{R})} \quad (2.2)$$

where $\frac{1}{p} + \frac{1}{p'} = \frac{1}{r} + \frac{1}{r'} = 1$.

The next lemma is the smoothing effects.

Lemma 2.2. (Smoothing effects, see [41], [49]). Let $I \subset \mathbb{R}$ be an interval. Then

1) $\|D_x^{\frac{1}{2}} e^{it\Delta} f\|_{L_t^\infty L_x^2(I \times \mathbb{R})} \lesssim \|f\|_{L^2(\mathbb{R})}$

(2.3)

for all $f \in L^2(\mathbb{R})$; and

2) $\left\| D_x^{\frac{1}{2}} \int_0^t e^{i(t-t')\Delta} F(x, t') \, dt' \right\|_{L_t^\infty L_x^2(I \times \mathbb{R})} \lesssim \|F\|_{L_t^1 L_x^2(I \times \mathbb{R})} \quad (2.4)$

3) \[
\left\| \partial_x \int_0^t e^{i(t-t')\Delta} F(x, t') \, dt' \right\|_{L_\infty^p L_{q, q}^p (\mathbb{R} \times I)} \lesssim \| F \|_{L^p_1 L^p_2 (\mathbb{R} \times I)};
\]
for all \( F \in L^p_1 L^p_2 (\mathbb{R} \times I) \).

Next, we introduce the following maximal function estimates for the linear Schrödinger equation.

**Lemma 2.3.** (Maximal function estimates, see \[10, 13, 57, 60, 67\].) Let \( I \subset \mathbb{R} \) be an interval. Let \( 2 < p < \infty \) and \( s \geq \max \{ \frac{1}{2} - \frac{1}{p}, \frac{1}{p}, \frac{1}{p} \} \). Then we have
\[
\| e^{it\Delta} f \|_{L^p_1 L^p_2 (\mathbb{R} \times I)} \lesssim \| f \|_{H^s (\mathbb{R})};
\]
and
\[
\left\| \int_0^t e^{i(t-t')\Delta} F(x, t') \, dt' \right\|_{L^p_1 L^p_2 (\mathbb{R} \times I)} \lesssim \| F \|_{L^1_1 L^1_2 (\mathbb{R} \times I)} + \| F \|_{L^1_1 L^1_2 (I \times \mathbb{R})}.
\]

Next, we show the Leibniz and chain rule for fractional derivatives, see \[4, 41, 46\] and the references therein.

**Lemma 2.4.** (Leibniz and chain rule for fractional derivatives.) Let \( I \subset \mathbb{R} \) be an interval. Then

1) Let \( s \in (0, 1) \), \( 1 < p \leq \infty \), and \( 1 < p_1, p_2, p_3, p_4 \leq \infty \) with \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \), \( \frac{1}{p} = \frac{1}{p_3} + \frac{1}{p_4} \), and let \( f, g \in S(\mathbb{R}) \), then
\[
\| D^s (fg) \|_{L^p_1 (\mathbb{R})} \lesssim \| D^s f \|_{L^{p_1}_1 (\mathbb{R})} \| g \|_{L^{p_2}_1 (\mathbb{R})} + \| D^s g \|_{L^{p_3}_1 (\mathbb{R})} \| f \|_{L^{p_4}_1 (\mathbb{R})}.
\]

2) Let \( s \in (0, 1) \) and \( p, q, p_1, p_2, q_1, q_2 \in (1, \infty), q_1 \in (1, \infty] \) such that
\[
\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \quad \text{and} \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}.
\]
Then
\[
\| D^s F (f) \|_{L^p_2 L^q_2 (\mathbb{R} \times I)} \lesssim \| F (f) \|_{L^{p_1}_2 L^{q_1}_2 (\mathbb{R} \times I)} \| D^s f \|_{L^{p_2}_2 L^{q_2}_2 (\mathbb{R} \times I)}.
\]

3) Let \( s \in (0, 1), s_1, s_2 \in [0, s] \) with \( s = s_1 + s_2 \). Let \( p, q, p_1, p_2, q_1, q_2 \in (1, \infty) \) be such that
\[
\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \quad \text{and} \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}.
\]
Then
\[
\| D^s (fg) - f D^s g - g D^s f \|_{L^p_2 L^q_2 (\mathbb{R} \times I)} \lesssim \| D^{s_1} f \|_{L^{p_1}_2 L^{q_1}_2 (\mathbb{R} \times I)} \| D^{s_2} g \|_{L^{p_2}_2 L^{q_2}_2 (\mathbb{R} \times I)}.
\]
Moreover, for \( s_1 = 0 \) the value \( q_1 = \infty \) is allowed.

4) Let \( s \in (0, 1), s, s_2 \in [0, s] \) with \( s = s_1 + s_2 \). Let \( p_1, p_2, q_1, q_2 \in (1, \infty) \) with \( 1 = \frac{1}{p_1} + \frac{1}{p_2} \) and \( \frac{1}{2} = \frac{1}{q_1} + \frac{1}{q_2} \). Then
\[
\| D^s (fg) - f D^s g - g D^s f \|_{L^p_2 L^q_2 (\mathbb{R} \times I)} \lesssim \| D^{s_1} f \|_{L^{p_1}_2 L^{q_1}_2 (\mathbb{R} \times I)} \| D^{s_2} g \|_{L^{p_2}_2 L^{q_2}_2 (\mathbb{R} \times I)}.
\]
3. Proof of Theorem 1.1

In this section, we give the proof of Theorem 1.1. Given $s \geq \frac{1}{2}$ and $\varphi \in H^s(\mathbb{R})$. Fixing $T > 0$, $\varepsilon > 0$, we define the working space with the norm

$$
\|u\|_{X_T} = \|u\|_{L^\infty_t H^s_x([0,T] \times \mathbb{R})} + \|\partial_x u\|_{L^\infty_t L^2_x([0,T] \times \mathbb{R})} + \sup_{q \in [2,\varepsilon, +\infty)} \|u\|_{L^q_t L^\infty_x([0,T] \times \mathbb{R})}
+ \|u\|_{L^4_t L^\infty_x([0,T] \times \mathbb{R})} + \sup_{r \in [2,\varepsilon, +\infty)} \|D^{s-\frac{1}{2}} u\|_{L^r_t L^\infty_x([0,T] \times \mathbb{R})}
+ \|D^{s-\frac{1}{2}} \partial_x u\|_{L^\infty_t L^2_x([0,T] \times \mathbb{R})} + \|D^{s-\frac{1}{2}} u\|_{L^4_t L^\infty_x([0,T] \times \mathbb{R})}.
$$

(3.1)

With the local well-posed result in Santos [67] in hand, we only need to show that for any fixed $T \in \mathbb{R}$,

$$
\|u\|_{X_T} \lesssim \|\varphi\|_{H^s(\mathbb{R})} + \|u\|_{X_T}^{2\sigma+1},
$$

(3.2)

where the implicit constant is independent of $T$. Then the bootstrap argument yields that there exists $\delta_0 > 0$, such that when $\|\varphi\|_{H^s(\mathbb{R})} \leq \delta_0$, $\|u\|_{X_T} \lesssim \|\varphi\|_{H^s(\mathbb{R})}$ for any $T \in \mathbb{R}$. In the following, we only consider the positive time. Since the negative time direction can be obtained in the same way.

To show (3.2), according to the definition of $\|u\|_{X_T}$, we control the norms in the right-hand side of (3.1) one by one.

3.1. Estimates on $\|u\|_{L^\infty_t H^s_x([0,T] \times \mathbb{R})}$. In this section, we will show that

$$
\|u\|_{L^\infty_t H^s_x([0,T] \times \mathbb{R})} \lesssim \|\varphi\|_{H^s(\mathbb{R})} + \|u\|_{X_T}^{2\sigma+1}.
$$

(3.3)

We prove (3.3) by the following two steps.

**Step 1.** $\|u\|_{L^\infty_t L^2_x([0,T] \times \mathbb{R})} \lesssim \|\varphi\|_{L^2(\mathbb{R})} + \|u\|_{X_T}^{2\sigma+1}$.

Using the Duhamel formula

$$
u(t) = e^{it\Delta} \varphi - \int_0^t e^{i(t-t')\Delta} \left(|u|^{2\sigma} \partial_x u\right)(t') \, dt',
$$

(3.4)

and the Strichartz estimates (2.1) and (2.2), we get

$$
\|u\|_{L^\infty_t L^2_x([0,T] \times \mathbb{R})} \lesssim \left\|e^{it\Delta} \varphi\right\|_{L^\infty_t L^2_x([0,T] \times \mathbb{R})} + \left\|\int_0^t e^{i(t-t')\Delta} \left(|u|^{2\sigma} \partial_x u\right)(t') \, dt'\right\|_{L^\infty_t L^2_x([0,T] \times \mathbb{R})}
\lesssim \|\varphi\|_{L^2(\mathbb{R})} + \left\|u\right\|_{L^1_t L^2_x([0,T] \times \mathbb{R})}^{2\sigma} \left\|\partial_x u\right\|_{L^1_t L^2_x([0,T] \times \mathbb{R})}.
$$

(3.5)

Next we consider the term $\left\|u\right\|_{L^1_t L^2_x([0,T] \times \mathbb{R})}^{2\sigma} \left\|\partial_x u\right\|_{L^1_t L^2_x([0,T] \times \mathbb{R})}$. We claim that

$$
\left\|u\right\|_{L^1_t L^2_x([0,T] \times \mathbb{R})}^{2\sigma} \left\|\partial_x u\right\|_{L^1_t L^2_x([0,T] \times \mathbb{R})} \lesssim \|u\|_{X_T}^{2\sigma+1}.
$$

(3.6)

Now we write

$$
\left\|u\right\|_{L^1_t L^2_x([0,T] \times \mathbb{R})}^{2\sigma} \left\|\partial_x u\right\|_{L^1_t L^2_x([0,T] \times \mathbb{R})} = \left|u\right|^2 \left|u\right|^{2\sigma-2} \left\|\partial_x u\right\|_{L^1_t L^2_x([0,T] \times \mathbb{R})}.
$$

(3.7)
We consider the inner integration $L^2_x$ first. By Hölder’s inequality, we have
\[
\left\| u |^2 \cdot |u|^{2\sigma - 2} \partial_x u \right\|_{L^2_x(\mathbb{R})} \lesssim \|u\|_{L^\infty_x(\mathbb{R})} L^2 \left\| |u|^{2\sigma - 2} \partial_x u \right\|_{L^2_x(\mathbb{R})}.
\]
Hence,
\[
\left\| u |^{2\sigma} \partial_x u \right\|_{L^1_t L^2_x([0,T] \times \mathbb{R})} \lesssim \left\| u \right\|^2_{L^\infty_x([0,T])} L^2 \left\| |u|^{2\sigma - 2} \partial_x u \right\|_{L^2_x(\mathbb{R})} \lesssim \left\| u \right\|^2_{L^1_t L^\infty_x([0,T])} \left\| |u|^{2\sigma - 2} \partial_x u \right\|_{L^2_x L^2_\mathbb{R}[0,T]}.
\] (3.8)

For the term $\left\| u |^{2\sigma - 2} \partial_x u \right\|_{L^2_x L^2_\mathbb{R}[0,T]}$, noting that $2(2\sigma - 2) \geq 4$, by Hölder’s inequality again we obtain
\[
\left\| u |^{2\sigma - 2} \partial_x u \right\|_{L^2_x L^2_\mathbb{R}[0,T]} \lesssim \left\| u \right\|_{L^2_x L^\infty_x([0,T])} \left\| \partial_x u \right\|_{L^2_x L^2_\mathbb{R}[0,T]} \lesssim \left\| u \right\|^{\frac{2\sigma - 1}{X_T}}.
\] (3.9)

Putting this result into (3.8), we get
\[
\left\| u \right\|^{2\sigma}_{L^1_t L^2_x([0,T] \times \mathbb{R})} \lesssim \left\| u \right\|^{2\sigma - 1}_{L^2(\mathbb{R})} + \left\| \partial_x u \right\|_{H^2(\mathbb{R})} + \left\| u \right\|^{2\sigma + 1}_{X_T}.
\] (3.10)

Thus we have finished the proof on Step 1.

**Step 2**, \(D^s u \|_{L^\infty_t L^2_x([0,T] \times \mathbb{R})} \lesssim \| \varphi \|_{L^2(\mathbb{R})} + \left\| u \right\|^{2\sigma + 1}_{X_T}.

Using the Duhamel formula (3.4), the Strichartz estimate (2.1) and the smoothing effect (2.4), we have
\[
\|D^s u\|_{L^\infty_t L^2_x([0,T] \times \mathbb{R})} \lesssim \left\| e^{it \Delta} D^s \varphi \right\|_{L^\infty_t L^2_x([0,T] \times \mathbb{R})} + \left\| D^s \int_0^t e^{it \Delta} D^{s - \frac{1}{2} \left\| u \right\|^{2\sigma}_{L^2_x} \partial_x u \right\|_{L^1_t L^2_x([0,T] \times \mathbb{R})} \lesssim \left\| D^s \varphi \right\|_{L^2_x} + \left\| D^{s - \frac{1}{2} \left\| u \right\|^{2\sigma}_{L^2_x} \partial_x u \right\|_{L^1_t L^2_x([0,T])} \lesssim \left\| \varphi \right\|_{H^2(\mathbb{R})} + \left\| D^{s - \frac{1}{2} \left\| u \right\|^{2\sigma}_{L^2_x} \partial_x u \right\|_{L^1_t L^2_x([0,T])}.
\] (3.11)

Next we claim that
\[
\left\| D^{s - \frac{1}{2} \left\| u \right\|^{2\sigma}_{L^2_x} \partial_x u \right\|_{L^1_t L^2_x([0,T])} \lesssim \left\| u \right\|^{2\sigma + 1}_{X_T}.
\] (3.12)

To prove this claim, we split it into two cases: \( s = \frac{1}{2} \) and \( s > \frac{1}{2} \).

**Case 1**: \( s = \frac{1}{2} \).

By Hölder’s inequality, we have
\[
\left\| u \right\|^{2\sigma}_{L^2_x L^2_\mathbb{R}[0,T]} \lesssim \left\| u \right\|^{2\sigma}_{L^2_x([0,T])} \left\| \partial_x u \right\|_{L^2_x([0,T])} \lesssim \left\| u \right\|^{2\sigma + 1}_{L^2_x([0,T])} \left\| \partial_x u \right\|_{L^2_x L^2_\mathbb{R}[0,T]} \lesssim \left\| u \right\|^{2\sigma + 1}_{X_T}.
\] (3.13)
Case 2: $s > \frac{1}{2}$.

By the Leibniz rule for fractional derivative, we get

$$
\| D^{s-\frac{1}{2}} (|u|^{2\sigma} \partial_x u) \|_{L^1_x L^2_t (\mathbb{R} \times [0,T])} \leq \| D^{s-\frac{1}{2}} (|u|^{2\sigma}) \|_{L^1_x L^2_t (\mathbb{R} \times [0,T])} \cdot \| \partial_x u \|_{L^{2s}_x L^{2s}_t (\mathbb{R} \times [0,T])} + \| u^{2\sigma} \cdot D^{s-\frac{1}{2}} \partial_x u \|_{L^1_x L^2_t (\mathbb{R} \times [0,T])}
$$

$$
+ \| D^{s-\frac{1}{2}} (|u|^{2\sigma}) \|_{L^1_x L^\infty_t (\mathbb{R} \times [0,T])} \cdot \| \partial_x u \|_{L^{2s}_x L^{2s}_t (\mathbb{R} \times [0,T])}.
$$

(3.14)

We estimate on terms above one by one.

For the first term $\| D^{s-\frac{1}{2}} (|u|^{2\sigma}) \cdot \partial_x u \|_{L^1_x L^2_t (\mathbb{R} \times [0,T])}$ in (3.14), by Hölder’s inequality, we have

$$
\| D^{s-\frac{1}{2}} (|u|^{2\sigma}) \cdot \partial_x u \|_{L^1_x L^2_t ([0,T])} \lesssim \| D^{s-\frac{1}{2}} (|u|^{2\sigma}) \|_{L^{2s}_x (0,T)} \cdot \| \partial_x u \|_{L^{2s}_x (0,T)}.
$$

Then

$$
\| D^{s-\frac{1}{2}} (|u|^{2\sigma}) \cdot \partial_x u \|_{L^1_x L^2_t (\mathbb{R} \times [0,T])} \lesssim \| D^{s-\frac{1}{2}} (|u|^{2\sigma}) \|_{L^{2s}_x L^\infty_t (\mathbb{R} \times [0,T])} \cdot \| \partial_x u \|_{L^{2s}_x L^{2s}_t (\mathbb{R} \times [0,T])}.
$$

(3.15)

To the term $\| D^{s-\frac{1}{2}} (|u|^{2\sigma}) \|_{L^1_x L^\infty_t (\mathbb{R} \times [0,T])}$ in (3.15), using (2.9), we get

$$
\| D^{s-\frac{1}{2}} (|u|^{2\sigma}) \|_{L^1_x L^\infty_t (\mathbb{R} \times [0,T])} \lesssim \| u^{2\sigma-1} \|_{L^2_x L^\infty_t (\mathbb{R} \times [0,T])} \cdot \| D^{s-\frac{1}{2}} u \|_{L^2_x L^\infty_t (\mathbb{R} \times [0,T])} \lesssim \| u \|_{X_T}^{2\sigma-1} \cdot \| D^{s-\frac{1}{2}} u \|_{L^2_x L^\infty_t (\mathbb{R} \times [0,T])}.
$$

By interpolating between $\| D^{s-\frac{1}{2}} \partial_x u \|_{L^2_x L^2_t (\mathbb{R} \times [0,T])}$ and $\| u \|_{L^2_x L^2_t (\mathbb{R} \times [0,T])}$, we have that for some $\theta_1 \in (0,1)$,

$$
\| D^{s-\frac{1}{2}} u \|_{L^2_x L^\infty_t (\mathbb{R} \times [0,T])} \lesssim \| D^{s-\frac{1}{2}} \partial_x u \|_{L^2_x L^2_t (\mathbb{R} \times [0,T])} \cdot \| u \|_{L^2_x L^\infty_t (\mathbb{R} \times [0,T])}^{1-\theta_1} \cdot \| u \|_{L^{2\theta_1}_x L^{2\theta_1}_t (\mathbb{R} \times [0,T])} \lesssim \| u \|_{X_T}.
$$

(3.16)

Then

$$
\| D^{s-\frac{1}{2}} (|u|^{2\sigma}) \|_{L^1_x L^\infty_t (\mathbb{R} \times [0,T])} \lesssim \| u \|_{X_T}^{2\sigma}.
$$

(3.17)

To the term $\| \partial_x u \|_{L^\infty_t L^2_x (\mathbb{R} \times [0,T])}$ in (3.15), it follows from the interpolation between $\| D^{s-\frac{1}{2}} \partial_x u \|_{L^\infty_t L^2_x (\mathbb{R} \times [0,T])}$ and $\| D^{s-\frac{1}{2}} u \|_{L^2_x L^2_t (\mathbb{R} \times [0,T])}$, that is, there exists $\theta_2 \in (0,1)$,

$$
\| \partial_x u \|_{L^\infty_t L^2_x (\mathbb{R} \times [0,T])} \lesssim \| D^{s-\frac{1}{2}} \partial_x u \|_{L^\infty_t L^2_x (\mathbb{R} \times [0,T])} \cdot \| D^{s-\frac{1}{2}} u \|_{L^2_x L^2_t (\mathbb{R} \times [0,T])} \lesssim \| u \|_{X_T}.
$$

(3.18)

Inserting (3.17) and (3.18) into (3.15), we have

$$
\| D^{s-\frac{1}{2}} (|u|^{2\sigma}) \cdot \partial_x u \|_{L^1_x L^2_t (\mathbb{R} \times [0,T])} \lesssim \| u \|_{X_T}^{2\sigma+1}.
$$

(3.19)
Finally, we get then

Thus the estimate on the second term of (3.14) is also completed.

Owing to the above two cases, we finish the proof of claim (3.12). Putting (3.22) into (3.11),

Inserting (3.19), (3.20) and (3.21) into (3.14), we have

Estimates on \( \|u\|_{L^2_x} \) in (3.14), using (3.17) and (3.18), we have

Inserting (3.19), (3.20) and (3.21) into (3.14), we have

Owing to the above two cases, we finish the proof of claim (3.12). Putting (3.22) into (3.11),
we finish the proof on Step 2.

3.2. Estimates on \( \|\partial_x u\|_{L^\infty_x L^2_t(\mathbb{R} \times [0,T])} \). Using the Duhamel formula (3.3) and the smoothing effects (2.3) and (2.5), we have

Next we estimate on the term \( \|u\|_{L^2_x} \). By Hölder’s inequality, we have

Finally, we get

\[ \|\partial_x u\|_{L^\infty_x L^2_t(\mathbb{R} \times [0,T])} \lesssim \|\varphi\|_{H^4_x} + \|u\|^2 \|\partial_x u\|_{L^2_x L^2_t(\mathbb{R} \times [0,T])} \lesssim \|\varphi\|_{H^4_x} + \|u\|^2 \|\partial_x u\|_{L^2_x L^2_t(\mathbb{R} \times [0,T])} \lesssim \|u\|^2_{e^{\cdot^4}} \]
3.3. Estimates on $\sup_{q \in [2+\varepsilon, +\infty)} \|u\|_{L^q_x L^\infty_T(\mathbb{R} \times [0,T])}$: By Duhamel’s formula (3.4) and the maximal function estimates (2.6) and (2.7), we have
\[
\|u\|_{L^q_x L^\infty_T(\mathbb{R} \times [0,T])} \lesssim \left\| e^{it\Delta} \varphi \right\|_{L^q_x L^\infty_T(\mathbb{R} \times [0,T])} + \left\| \int_0^t e^{i(t-t')\Delta} \left( |u|^{2\sigma} \partial_x u \right)(t') \, dt' \right\|_{L^q_x L^\infty_T(\mathbb{R} \times [0,T])}
\lesssim \|\varphi\|_{H^s_x(\mathbb{R})} + \left\| |u|^{2\sigma} \partial_x u \right\|_{L^q_x L^2_T(\mathbb{R} \times [0,T])},
\]
where we have used the condition $s \geq \frac{1}{2} > \max\{\frac{1}{2} - \frac{1}{q}, \frac{1}{q} \}$ in Lemma 2.3. By (3.6) and (3.23), we obtain
\[
\sup_{q \in [2+\varepsilon, +\infty)} \|u\|_{L^q_x L^\infty_T(\mathbb{R} \times [0,T])} \lesssim \|\varphi\|_{H^s_x(\mathbb{R})} + \|u\|^{2s+1}_{X^s_T}.
\]

3.4. Estimates on $\|u\|_{L^q_x L^\infty_T([0,T] \times \mathbb{R})}$. By Duhamel’s formula (3.4) and the Strichartz estimates (2.1) and (2.2), we get
\[
\|u\|_{L^q_x L^\infty_T([0,T] \times \mathbb{R})} \lesssim \left\| e^{it\Delta} \varphi \right\|_{L^q_x L^\infty_T([0,T] \times \mathbb{R})} + \left\| \int_0^t e^{i(t-t')\Delta} \left( |u|^{2\sigma} \partial_x u \right)(t') \, dt' \right\|_{L^q_x L^\infty_T([0,T] \times \mathbb{R})}
\lesssim \|\varphi\|_{L^2_x(\mathbb{R})} + \left\| |u|^{2\sigma} \partial_x u \right\|_{L^q_x L^2_T([0,T] \times \mathbb{R})}.
\]
By (3.6), we have
\[
\|u\|_{L^q_x L^\infty_T([0,T] \times \mathbb{R})} \lesssim \|\varphi\|_{L^2_x(\mathbb{R})} + \|u\|^{2s+1}_{X^s_T}.
\]

3.5. Estimates on $\sup_{q \in [2+\varepsilon, +\infty)} \|D^{-\frac{1}{2}} u\|_{L^q_x L^\infty_T(\mathbb{R} \times [0,T])}$. Using Duhamel’s formula (3.4) and the maximal function estimates (2.6) and (2.7), we get
\[
\left\| D^{-\frac{1}{2}} u \right\|_{L^q_x L^\infty_T(\mathbb{R} \times [0,T])} \lesssim \left\| e^{it\Delta} D^{-\frac{1}{2}} \varphi \right\|_{L^q_x L^\infty_T(\mathbb{R} \times [0,T])}
+ \left\| \int_0^t e^{i(t-t')\Delta} D^{-\frac{1}{2}} \left( |u|^{2\sigma} \partial_x u \right)(t') \, dt' \right\|_{L^q_x L^\infty_T(\mathbb{R} \times [0,T])}
\lesssim \|D^{-\frac{1}{2}} \varphi\|_{H^s_x(\mathbb{R})} + \left\| D^{-\frac{1}{2}} \left( |u|^{2\sigma} \partial_x u \right) \right\|_{L^q_x L^2_T(\mathbb{R} \times [0,T])}
+ \left\| \int_0^t e^{i(t-t')\Delta} D^{-\frac{1}{2}} \left( |u|^{2\sigma} \partial_x u \right) \, dt' \right\|_{L^q_x L^2_T(\mathbb{R} \times [0,T])}
\lesssim \|\varphi\|_{H^s_x(\mathbb{R})} + \left\| D^{-\frac{1}{2}} \left( |u|^{2\sigma} \partial_x u \right) \right\|_{L^q_x L^2_T(\mathbb{R} \times [0,T])}
+ \left\| D^{-\frac{1}{2}} \left( |u|^{2\sigma} \partial_x u \right) \right\|_{L^q_x L^2_T(\mathbb{R} \times [0,T])}. \tag{3.24}
\]

Recall that the term $\left\| D^{-\frac{1}{2}} \left( |u|^{2\sigma} \partial_x u \right) \right\|_{L^q_x L^2_T(\mathbb{R} \times [0,T])}$ is already estimated in (3.12). So we only need to consider the term $\left\| D^{-\frac{1}{2}} \left( |u|^{2\sigma} \partial_x u \right) \right\|_{L^q_x L^2_T(\mathbb{R} \times [0,T])}$. Now we claim that
\[
\left\| D^{-\frac{1}{2}} \left( |u|^{2\sigma} \partial_x u \right) \right\|_{L^q_x L^2_T(\mathbb{R} \times [0,T])} \lesssim \|u\|^{2s+1}_{X^s_T}. \tag{3.25}
\]
Again, we split it into two cases: $s = \frac{1}{2}$ and $s > \frac{1}{2}$.

Case 1: $s = \frac{1}{2}$.
The term \( \|u^{2\sigma} \partial_x u\|_{L^1_t L^2_x([0,T] \times \mathbb{R})} \) is already estimated in (3.6).

Case 2: \( s > \frac{1}{2} \).

Using a similar treatment as (3.7), we have

\[
\left\| D^{s-\frac{1}{2}} (|u|^{2\sigma} \partial_x u) \right\|_{L^1_t L^2_x([0,T] \times \mathbb{R})}.
\]

Further, using the Leibniz rule for fractional derivative (2.8), we have

\[
\left\| D^{s-\frac{1}{2}} (|u|^2 \cdot |u|^{2\sigma-2} \partial_x u) \right\|_{L^2_x(\mathbb{R})} \leq \left\| D^{s-\frac{1}{2}} (|u|^2) \right\|_{L^\infty_x(\mathbb{R})} \cdot \left\| |u|^{2\sigma-2} \partial_x u \right\|_{L^2_x(\mathbb{R})}.
\]

Note that the term \( \left\| |u|^{2\sigma-2} \partial_x u \right\|_{L^2_x(\mathbb{R})} \) has been considered in (3.9), so we only need to deal with the terms \( \left\| D^{s-\frac{1}{2}} (|u|^2) \right\|_{L^2_x(\mathbb{R})} \) and \( \left\| D^{s-\frac{1}{2}} (|u|^{2\sigma-2} \partial_x u) \right\|_{L^2_x(\mathbb{R})} \) respectively.

For the term \( \left\| D^{s-\frac{1}{2}} (|u|^2) \right\|_{L^2_x(\mathbb{R})} \), by (2.8) and the Hölder inequality, we have

\[
\left\| D^{s-\frac{1}{2}} (|u|^2) \right\|_{L^2_x(\mathbb{R})} \leq \|u\|_{L^4_t L^\infty_x([0,T] \times \mathbb{R})} \cdot \left\| D^{s-\frac{1}{2}} u \right\|_{L^4_t L^\infty_x([0,T] \times \mathbb{R})} \leq \|u\|_{X_T}^2.
\]

For the term \( \left\| D^{s-\frac{1}{2}} (|u|^{2\sigma-2} \partial_x u) \right\|_{L^2_t L^2_x(\mathbb{R} \times [0,T])} \), we claim that

\[
\left\| D^{s-\frac{1}{2}} (|u|^{2\sigma-2} \partial_x u) \right\|_{L^2_t L^2_x(\mathbb{R} \times [0,T])} \leq \|u\|_{X_T}^{2\sigma-1}. \tag{3.26}
\]

Indeed, using the Leibniz rule for fractional derivative (2.10), we obtain

\[
\left\| D^{s-\frac{1}{2}} (|u|^{2\sigma-2} \partial_x u) \right\|_{L^2_t L^2_x(\mathbb{R} \times [0,T])} \leq \left\| D^{s-\frac{1}{2}} (|u|^{2\sigma-2}) \cdot \partial_x u \right\|_{L^2_x(\mathbb{R} \times [0,T])} + \left\| |u|^{2\sigma-2} \cdot D^{s-\frac{1}{2}} \partial_x u \right\|_{L^2_x(\mathbb{R} \times [0,T])} + \left\| D^{s-\frac{1}{2}} (|u|^{2\sigma-2}) \cdot \partial_x u \right\|_{L^2_t L^2_x(\mathbb{R} \times [0,T])}.
\]
To the term $\|D^{s-\frac{1}{2}}(|u|^{2\sigma-2}) \cdot \partial_x u\|_{L^2 L_t^2(\mathbb{R} \times [0,T])}$, using the Hölder inequality, (2.9) and (3.18), we have

\[
\|D^{s-\frac{1}{2}}(|u|^{2\sigma-2}) \cdot \partial_x u\|_{L^2 L_t^2(\mathbb{R} \times [0,T])} \lesssim \|u\|^{2\sigma-1}_{X_T}.
\]  

To the term $\|u|^{2\sigma-2} \cdot D^{s-\frac{1}{2}} \partial_x u\|_{L^2 L_t^2(\mathbb{R} \times [0,T])}$, using Hölder’s inequality, we get

\[
\|u|^{2\sigma-2} \cdot D^{s-\frac{1}{2}} \partial_x u\|_{L^2 L_t^2(\mathbb{R} \times [0,T])} \lesssim \|u\|^{2\sigma-2}_{L^{2/(2\sigma-2)} L_t^\infty(\mathbb{R} \times [0,T])} \cdot \|D^{s-\frac{1}{2}} \partial_x u\|_{L^2 L_t^2(\mathbb{R} \times [0,T])} \lesssim \|u\|^{2\sigma-1}_{X_T}.
\]  

To the term $\|D^{s-\frac{1}{2}}(|u|^{2\sigma-2}) \cdot \partial_x u\|_{L^2 L_t^2(\mathbb{R} \times [0,T])}$, as the same estimation in (3.27), we have

\[
\|D^{s-\frac{1}{2}}(|u|^{2\sigma-2}) \cdot \partial_x u\|_{L^2 L_t^2(\mathbb{R} \times [0,T])} \lesssim \|\partial_x u\|_{L^{\infty} - L^2(\mathbb{R} \times [0,T])} \lesssim \|u\|^{2\sigma-1}_{X_T}.
\]  

Thus we finish the proof of claim (3.26) and (3.25) and obtain

\[
\sup_{r \in [2+\varepsilon, +\infty)} \|D^{s-\frac{1}{2}} u\|_{L^2_t L^\infty_x(\mathbb{R} \times [0,T])} \lesssim \|\varphi\|_{H^s_x(\mathbb{R})} + \|u\|^{\sigma+1}_{X_T}.
\]  

### 3.6. Estimates on $\|D^{s-\frac{1}{2}}(\partial_x u)\|_{L^\infty_t L^2_x(\mathbb{R} \times [0,T])}$.

Using Duhamel’s formula (3.31) and the smoothing effects (2.2) and (2.3), we get

\[
\|D^{s-\frac{1}{2}} \partial_x u\|_{L^\infty_t L^2_x(\mathbb{R} \times [0,T])} \lesssim \|e^{it\Delta} D^{s-\frac{1}{2}} \partial_x u\|_{L^\infty_t L^2_x(\mathbb{R} \times [0,T])} + \|D^s \partial_x u\|_{L^\infty_t L^2_x(\mathbb{R} \times [0,T])} + \|D^{s-\frac{1}{2}}(|u|^{2\sigma} \partial_x u)\|_{L^2_t L^2_x(\mathbb{R} \times [0,T])}
\]  

\[
\lesssim \|u\|_{H^s_x(\mathbb{R})} + \|D^{s-\frac{1}{2}}(|u|^{2\sigma} \partial_x u)\|_{L^2_t L^2_x(\mathbb{R} \times [0,T])}.
\]  

Note that we already have the estimate on $\|D^{s-\frac{1}{2}}(|u|^{2\sigma} \partial_x u)\|_{L^2_t L^2_x(\mathbb{R} \times [0,T])}$ in (3.22). Then

\[
\|D^{s-\frac{1}{2}} \partial_x u\|_{L^\infty_t L^2_x(\mathbb{R} \times [0,T])} \lesssim \|u\|_{H^s_x(\mathbb{R})} + \|u\|^{\sigma+1}_{X_T}.
\]
3.7. Estimates on $\left\|D^{s-\frac{1}{2}}u\right\|_{L_t^4L_x^\infty([0,T] \times \mathbb{R})}$ By Duhamel’s formula (3.21) and the Strichartz estimates (2.1) and (2.2), we have

$$\left\|D^{s-\frac{1}{2}}u\right\|_{L_t^4L_x^\infty([0,T] \times \mathbb{R})} \lesssim \left\|e^{it\Delta} D^{s-\frac{1}{2}} \varphi\right\|_{L_t^4L_x^\infty([0,T] \times \mathbb{R})}$$

$$+ \left\| \int_0^t e^{i(t-t') \Delta} D^{s-\frac{1}{2}} \left(|u|^{2\sigma} \partial_x u\right)(t') dt' \right\|_{L_t^4L_x^\infty([0,T] \times \mathbb{R})}$$

$$\lesssim \left\|D^{s-\frac{1}{2}} \varphi\right\|_{L_2^2(\mathbb{R})} + \left\|D^{s-\frac{1}{2}} \left(|u|^{2\sigma} \partial_x u\right)\right\|_{L_t^4L_x^\infty([0,T] \times \mathbb{R})}$$

$$\lesssim \|\varphi\|_{H^2(\mathbb{R})} + \left\|D^{s-\frac{1}{2}} \left(|u|^{2\sigma} \partial_x u\right)\right\|_{L_t^4L_x^\infty([0,T] \times \mathbb{R})}.$$ Note that the estimation on $\left\|D^{s-\frac{1}{2}} \left(|u|^{2\sigma} \partial_x u\right)\right\|_{L_t^4L_x^\infty([0,T] \times \mathbb{R})}$ is obtained in (3.25). Then we have

$$\left\|D^{s-\frac{1}{2}}u\right\|_{L_t^4L_x^\infty([0,T] \times \mathbb{R})} \lesssim \|\varphi\|_{H^2(\mathbb{R})} + \left\|u\right\|^{2\sigma+1}_{X_T}.$$ Finally, all the estimates on $\|u\|_{X_T}$ are obtained and we have

$$\|u\|_{X_T} \lesssim \|\varphi\|_{H^2(\mathbb{R})} + \|u\|^{2\sigma+1}_{X_T}$$ uniformly on $T$. Hence we get $\|u\|_{X_\infty} \lesssim \|\varphi\|_{H^2(\mathbb{R})}$. This proves Theorem 1.1.

APPENDIX

In this appendix, we consider the solitary wave solutions described in Introduction, and prove the following lemma. Let $\Omega = \{(\omega, c) : c^2 < 4\omega, \text{or} \ 0 < c = 2\sqrt{\omega}\}$.

Lemma 3.1. Let $\phi_{\omega,c}$ be defined in (1.4), and $(\omega, c) \in \Omega$, then

1. when $\sigma \in (0, 2)$,

$$\|\phi_{\omega,c}\|_{H^1(\mathbb{R})} \to 0, \ \text{when} \ \ c \to -2\sqrt{\omega};$$

2. when $\sigma \geq 2$, there exists a constant $c_0 > 0$, such that for any $(\omega, c) \in \Omega$,

$$\|\phi_{\omega,c}\|_{L^2(\mathbb{R})} \geq c_0,$$

Proof. Note that $\phi_{\omega,c}$ is the solution of the following equation

$$-\partial_x^2 \phi + \omega \phi + ci \partial_x \phi - i|\phi|^{2\sigma} \partial_x \phi = 0.$$ Multiplying on both sides with $\overline{x \partial_x \phi_{\omega,c}}$, taking the real part and integrating over $x$, we obtain that for any $(\omega, c) \in \Omega$,

$$\|\partial_x \phi_{\omega,c}\|^2_{L^2} = \omega \|\phi_{\omega,c}\|^2_{L^2}.$$ Hence, for the statement (1), we only need to consider $\|\phi_{\omega,c}\|_{L^2(\mathbb{R})}$.

Now we fix $\omega > 0$ and denote $\alpha = \sqrt{4\omega - c^2}$. From (1.4), we find that for some $C_{\omega,\sigma} > 0$ which may vary line to line,

$$\int_{\mathbb{R}} |\phi_{\omega,c}|^2 dx = C_{\omega,\sigma} \alpha \int_{\mathbb{R}} \left(\frac{1}{\cosh(\sigma \alpha x)} - \frac{c}{2\sqrt{\omega}}\right)^\frac{1}{2} dx$$

$$= C_{\omega,\sigma} \alpha \int_0^\infty \left(\frac{1}{\cosh x - \frac{c}{2\sqrt{\omega}}}\right)^\frac{1}{2} dx.$$ (3.28)
For convenience, we denote
\[ I(c) = \int_0^\infty \left( \frac{1}{\cosh x - \frac{c}{2\sqrt{\omega}}} \right)^{\frac{1}{2}} \, dx. \]

Moreover, we denote \( c_\sigma \) as
\[ c_\sigma = I(-2\sqrt{\omega}) = \int_0^\infty \left( \frac{1}{\cosh x + 1} \right)^{\frac{1}{2}} \, dx, \]
which makes sense since the last integral above is finite. Note that \( I(c) \) is an increasing function, thus we have that for any \( c : -2\sqrt{\omega} < c \leq 0 \),
\[ c_\sigma \leq I(c) \leq I(0). \]

This combining with (3.28) yields that
\[ \int_{\mathbb{R}} |\phi_{\omega,c}|^2 \, dx \leq C_{\omega,\sigma} I(0) \alpha^{\frac{3}{2} - 1} \to 0, \quad \text{when } c \to -2\sqrt{\omega}. \]
This proves the statement (1).

For the statement (2), similarly as above, using the monotonicity of \( I(c) \), we have
\[ I(c) \geq c_\sigma. \]

Hence,
\[ \int_{\mathbb{R}} |\phi_{\omega,c}|^2 \, dx = C_{\omega,\sigma} \alpha^{\frac{3}{2} - 1} I(c) \geq C_{\omega,\sigma} c_\sigma \alpha^{\frac{3}{2} - 1}, \]
when \( 0 < c = 2\sqrt{\omega} \) and \( \sigma > 2 \), \( \phi_{\omega,c} \notin L^2(\mathbb{R}) \). Since \( \alpha \leq 2\sqrt{\omega} \), we obtain
\[ \int_{\mathbb{R}} |\phi_{\omega,c}|^2 \, dx \geq C_{\omega,\sigma} c_\sigma (2\sqrt{\omega})^{\frac{3}{2} - 1}. \]
This proves the lemma. \( \square \)

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