Granular Solid Hydrodynamics: Dense Flow, Fluidization and Jamming

Stefan Mahle,1 Yimin Jiang,2 and Mario Liu1

1Theoretische Physik, Universität Tübingen, Germany
2Central South University, Changsha 410083, China

(Received: October 27, 2010)

Granular solid hydrodynamics, constructed to describe quasi-elastic and plastic motion of granular solid, is shown also capable of accounting for the rheology of granular dense flow. This makes it a unified, though still qualitative, hydrodynamic description, enabling one to tackle fluidization and jamming, the hysteretic transition between elasto-plastic motion and uniform dense flow.

PACS numbers: 45.70.-n, 83.60.La , 83.50.-v, 45.70.Ht

A potentially catastrophic feature of granular media is their variable capability to sustain external stresses. As the mechanical stability of any structure relies on this capability, it is important to have a thorough understanding when and why it weakens, and how it recovers. The transition from a solid-like response to a liquid-like one is \( \sigma \) experiments and simulations, for the density

\[ \frac{1}{2} \frac{\sigma}{\rho} \]

\[ \sigma = \mu_3 \frac{1}{\sqrt{\rho}} \rho \]

The authors took \( \frac{1}{2} \frac{\sigma}{\rho} \), where, with \( \rho_p \), the closed-packed density, \( \rho_r \equiv \rho/\rho_p \) is the relative one. \( \mu_1, \mu_2 \) denote, respectively, the friction angle for \( \gamma \rightarrow 0, \infty \). The authors took \( n = 1 \), though the difference to \( n = 2 \) is subtle, as both describe a gentle change from \( \mu_1 \) to \( \mu_2 \) with \( \gamma \).

Earlier, Boquet et al. [3] developed a continuum theory to account for their experiment. Starting from the results of the kinetic theory for inelastic hard spheres, they modified the density dependence of the pressure \( P \), viscosity \( \eta \) and relaxation rate \( \gamma \) to accommodate the higher density in their system. They employed the Cauchy stress \( \sigma_{ij} \), and a balance equation for the granular temperature \( T_G \):

\[ \sigma_{ij} = P - \eta v^0_{ij}, \quad \partial T_G/\partial t \sim \eta v^2_{ij} - \gamma T_G, \text{ with } (2) \]

\[ P \sim \frac{T_G}{1 - \rho_r}, \quad \eta \sim \frac{\sqrt{T_G}}{(1 - \rho_r)^{\beta}}, \quad \gamma \sim \frac{\sqrt{T_G}}{(1 - \rho_r)^{\alpha}}, \text{ with } (3) \]

where \( \alpha = 1, \beta \) between 1 and 2.5. \( v_{ij} \equiv \frac{1}{2} (\nabla_i v_j + \nabla_j v_i) \) denotes the strain rate, with \( v^0_{ij} \) its traceless part, and \( v^2_{ij} \equiv \frac{1}{2} v^0_{ij} v^0_{ij} \) the scalarized shear rate. (The notation \( ^0 \) and \( _s \) are used also for other tensors below, such as the strain and stress.) Solving both equations for a Couette cell, the solution was found to agree well with their data. This theory II does not consider any elastic contributions.

Considering shallow flows on an inclined plane and rotating drums, Aranson and Tsimring identified the hysteresis of transition, or the delay between jamming and fluidization, as a key feature of granular behavior [6]. Their theory III treats the Cauchy stress as the sum of two parts, a solid-like, possibly elastic contribution \( \tilde{\sigma}_{ij} \), and a rate-dependent fluid one. A crucial variable is an order parameter \( \tilde{\rho} \) that is 1 for granular solid, and 0 for dense flow. The authors take the friction angle \( \phi \), differently than above, as the ratio of the solid stress components, and postulate a free energy \( f(\tilde{\rho}) \) such that granular solid, \( \tilde{\rho} = 1 \), is unstable for large shear stresses, \( \phi > \phi_0 \); while dense flow, \( \tilde{\rho} = 0 \), is unstable for small ones, \( \phi < \phi_0 \). Both are stable in the intermediate region, \( \phi_1 > \phi > \phi_0 \). This theory does not consider variations in the density \( \rho \), or in \( T_G \), and takes \( \sigma_{ij}^v \) as an input from some other theory. But its success provides a pivotal insight: The viability, even appropriateness, of using a partially bistable energy to account for the hysteresis.

GSH starts from the basic fact that grains with enduring contacts are elastically deformed. Its essential idea...
is that this deformation is slowly lost when grains jiggles, as they briefly lose contact with one another. Granular solid’s complex elasto-plastic behavior was shown to be a result of this simple physics, assuming the dominance of the elastic energy. Kinetic energy, or granular heat, is what underlies the behavior of granular gas. So it seems obvious that the behavior of dense flow results when both energies are comparable, when the contribution to the stress from granular temperature becomes equally important as that from deformation.

GSH was first employed to calculate static stress distribution for various geometries, including sand piles, silos, and point load, achieving results in agreement with observation [8]. It was then employed to consider slowly strained granular solid, and found to yield response envelopes similar to those from modern hypoplastic theory [8]. Recently, the critical state – generally considered important as that from deformation. Stress from granular temperature becomes equally important as the elastic energy. Kinetic energy, or granular heat, is a result of this simple physics, assuming the dominance of the granular material, or the specific form of the elastic stress, $g_{ij}$, and the true entropy, $s$. Two entropies are necessary, because granular media display a two-stage irreversibility: Macroscopic energy, kinetic and elastic, dissipates into mesoscopic, inter-granular degrees of freedom, mainly granular jiggling and the collision-induced, fluctuating elastic deformation. After a characteristic time, the energy degrades further into microscopic, inner-granular degrees of freedom, especially phonons. The granular and the true entropy, $s_g$, $s$, account respectively for the energy of the meso- and macroscopic degrees of freedom. The elastic strain $u_{ij}$ is the portion of the total strain $\varepsilon_{ij}$ that deforms the grains and leads to reversible storage of elastic energy. The rest-frame energy density $w_0$ is a function of $s_g$, $s$, $\rho$, $u_{ij}$ (though we shall neglect $s$, as we are not interested in effects such as thermal expansion at present). The conjugate variables are: Granular temperature $T_g \equiv \partial w_0/\partial s_g$, chemical potential $\mu \equiv \partial w_0/\partial \rho$, elastic stress, $\pi_{ij} \equiv -\partial \varepsilon_{ij}/\partial \sigma_{ij}$, and the gaseous pressure $P_T \equiv \rho^2 \partial (w_0/\rho)/\partial \sigma_{ij}$. The elastic stress $\pi_{ij}$ derives from granular deformation, while $P_T$ is generated by granular temperature – similar to the temperature generated pressure in a gas. All conjugate variables: $(T_g, \mu, \pi_{ij}, P_T)$ are given once $w_0$ is.

In GSH, the Cauchy stress $\sigma_{ij}$ [given by momentum conservation, $\partial (\rho v_i)/\partial t + \nabla_j (\sigma_{ij} + \rho \pi_{ij}) = 0$] and the balance equation for $s_g$ are given as

$$\sigma_{ij} = (1 - \alpha) \pi_{ij} + P_T \delta_{ij} - \eta_g v_{ij}^0,$$  \hspace{1cm} (4)

$$\partial s_g/\partial t = (\eta_g v_s^2 - \gamma T_g^2)/T_g.$$ \hspace{1cm} (5)

Although a result of general principles, the expression for $\sigma_{ij}$ is, remarkably, a simple sum of the elastic stress, the gaseous pressure, and the viscous stress, with $\eta_g$ the shear viscosity. (Compressional flow is usually negligible. If not, one needs to include the bulk viscosity.) For elasto-plastic motion, only $(1 - \alpha) \pi_{ij}$ is important; granular gas is well accounted for by $P_T \delta_{ij} - \eta_g v_{ij}^0$; dense flow needs all three terms. $\alpha \approx 0.8$ is a softening coefficient that remains constant for all shear rates considered in the present context. It becomes smaller only for ultra low shear rates, in ratcheting or elastic waves. Note the similarity of Eq (4) to the above cited theories, with the difference that theory II ignores $\pi_{ij}$, and theory III takes it as given. In Eq (5), $\gamma$ is the relaxation rate of $s_g$, accounting for the inelastic collisions that occur when grains jiggles. The positive term $\eta_g v_s^2 \equiv \eta_g v_0^2 v_0^0$ describes how grains, being sheared past one another, start to jiggles in the process, leading to an increase of $s_g$. From a more general point of view, this term describes how the kinetic energy dissipates into granular heat. In the stationary limit, for $\partial s_g/\partial t = 0$, we have $T_g = v_s \sqrt{\eta_g/\gamma}$. (Only a uniform $T_g$ is considered – more terms exist otherwise.) See [12] for derivation and detailed explanation.

Eqs (4,5) hold for the given set of variables, independent of the granular material, or the specific form of $w_0$. Material-specific properties are encoded in $w_0(\rho, u_{ij}, s_g)$, also the transport coefficients: $\eta_g, \gamma$. We obtain them from qualitative consideration, also comparison to experiments and existing theories. (More puristically, one would of course like to obtain them from simulation or microscopic calculations.) For dry sand and glass beads, a simple energy expression, the sum of the elastic energy $w_1(u_{ij}, \rho)$ and granular heat $w_2(s_g, \rho)$: $w_0 = w_1 + w_2$, has turned out to be quite adequate as a first approximation. Then $\pi_{ij} = -\partial w_1/\partial u_{ij}$, $P_T \approx \rho^2 \partial (w_0/\rho)/\partial \rho$ (if one assumes $u_{ij} \ll 1$, see [12]), which is why stress and energy contributions are simply linked: If $w_1$ dominates, only $\pi_{ij}$ is important; while $P_T$ hinges on a sufficiently large $w_2$. We have $w_1 = B(\rho) \sqrt[2]{\Delta [2^2 + u_s^2]}$, with $\Delta = -u_{ij} u_{ij}$, $u_s^2 = u_{ij} u_{ij}$. For a granular system at rest, $w_1$ is the only energy, and the elastic stress is the total stress. The mentioned calculation of static stress distributions was carried out using $w_1$ [13]. Granular heat $w_2$ is the lowest order expansion in $s_g$,

$$w_2 = \frac{s_g^2}{2 \rho b}, \hspace{1cm} T_g = \frac{s_g}{\rho b}, \hspace{1cm} P_T = -\frac{T_g^2 \rho^2}{2} \frac{\partial b(\rho)}{\partial \rho}.$$ \hspace{1cm} (6)

The linear term vanishes because granular jiggling dissipates and decreases toward zero, implying $w_2(s_g)$ is minimal for $s_g = 0$. Expanding also the transport coefficients,

$$\eta_g = \eta_0 + \eta_1 T_g, \hspace{0.5cm} \gamma = \gamma_0 + \gamma_1 T_g, \hspace{0.5cm} T_g = v_s \sqrt{\eta_0/\gamma_1},$$ \hspace{1cm} (7)

we take $\eta_0 \ll \eta_1 T_g$, $\gamma_0 \ll \gamma_1 T_g$, as is appropriate for any $T_g$, typical of elasto-plastic motion and dense flow, see [12]. Then the last of Eqs (7) holds, for $\partial s_g/\partial t = 0$. 

$\pi_{ij} \equiv$...
In theory II, $T_g$ is the energy per degree of freedom, $\omega_2 \sim T_g$, while $\omega_3 \sim \rho_g^2 \sim T^2_g$. We therefore identify $T_g \sim \sqrt{T_g}$, and note the perfect agreement between Eqs (2,3) and Eqs (4,5,6,7). This is important, because the $T_g$-dependence (in contrast to the density dependence discussed below) is rather fixed – it is the result of the kinetic theory on one hand, and the general consideration rendered above on the other. (Note taking $\omega_2 \sim T_g$ would disregard the fact that $T_g$ dissipates, and $\omega_2$ is minimal for $\omega_g$, $T_g = 0$ in an adiabatic system. It is more appropriate for ideal gas than the granular one.)

To consider the density dependence, we focus on $1 - \rho_r \equiv 1 - \rho/\rho_{cp}$, which represents a stronger dependency than $\rho$ if the sand is dense, $\rho_r \approx 1$. We take

$$P_T = \frac{ab\rho_c\rho_r^2T_g^2}{2(1 - \rho_r)}, \quad \eta_1 = \frac{h_1\rho_c}{(1 - \rho_r)^3}, \quad \gamma_1 = \frac{g_1\rho_c}{(1 - \rho_r)^3}, \quad (8)$$

with $\alpha = \frac{1}{2}$, $\beta = \frac{3}{2}$ to fit the experimental results of $\rho$ with respect to the polystyrene beads. The expression for $P_T$ derives from $b = b_0(1 - \rho_r)^a$, cf Eq (3). Taking $b \sim \ln(1 - \rho_r)$ would yield $P_T \sim 1/(1 - \rho_r)$ exactly, as in Eqs (3), but also leads to a divergent $s_g$, see [12]. Assuming $a \approx 0.1$ approximates the result, yet avoids the problem. The coefficients $b_0, h_1, g_1$ are (material dependent) numbers. Combining Eq (1, 6, 7, 8) and denoting the shear rate as $\dot{\gamma} = \nabla \cdot v_y$ (hence $\dot{\gamma} = \sqrt{2}v_y$ in simple-shear geometry), we arrive at the final expressions for the pressure $P = \sigma_{\text{eff}}/3$ and the shear stress $\sigma_s = (\sigma_{ij}\sigma_{ij})^{1/2}$,

$$P = P_c + C_1 \frac{\dot{\gamma}^2}{(1 - \rho_r)^2}, \quad \sigma_s = \Pi_c + C_2 \frac{\dot{\gamma}^2}{(1 - \rho_r)^2}, \quad (9)$$

where $C_1 = \frac{1}{2}ab\rho_c\rho_r^2h_1/g_1$, $C_2 = \frac{1}{2}\rho_c\sqrt{h_1^3/2g_1}$. $P_c, \Pi_c$ denote the rate-independent, elastic contributions, with $\Pi_c/P$ independent of $1 - \rho_r$, see the explanation below.

The first of Eq (9) may be written as $1 - \rho_r \sim \dot{\gamma}/\sqrt{P - P_c} = \dot{\gamma}/\sqrt{P_T}$. It is the same as Eq 1 if $P_c$ is neglected – an understandable mismatch, because the consideration of theory I involves inertia and confining pressure, but neglects elasticity. The friction angle $\sigma_s/P$ is given by $\Pi_c/P_c$ for $\dot{\gamma} \to 0$, and $C/C_1$ for $\dot{\gamma} \to \infty$, which we may respectively identify as $\mu_1, \mu_2$. The number $n$ of Eq (1) is 2 in GSH. As mentioned, the difference to $n = 1$ is subtle for the friction angle – but less so if one look at the pressure or shear stress individually. That both grow with $\dot{\gamma}^2$ is in fact a behavior that already Bagnold observed [14]. Finally, a note on pressure versus pressure control: Yielding $P, \sigma_s$ for given $\dot{\gamma}, 1 - \rho_r$, Eqs (9) are directly appropriate for experiments performed under constant volume. If $P$ is fixed, one uses the first calculate $\rho$, and rewrite the second as $\sigma_s - \Pi_c = (P - P_c)C/C_1$, with a coefficient $C/C_1$ that does not depend on $1 - \rho_r$ – though still on $\rho$, a weaker function of $P$ and $\dot{\gamma}$.

Returning to the exponent of $\dot{\gamma}, \eta_1$, we note $\alpha + \beta = 2$ if $(\sigma_s - \Pi_c)/(P - P_c)$ is to be independent of $1 - \rho_r$; and $\beta = 1/2$, $\alpha = 3/2$, as given above, see Fig 1. However, for glass beads of the same experiment [3], $P_T \sim \dot{\gamma}^2/(1 - \rho_r)$, or $\beta = \alpha = 1$ is more appropriate. In addition, the friction angle decreases for increasing $\dot{\gamma}$ here, implying $\Pi_c/P_c > C/C_1$, without contradicting any general principle.

Next we discuss the elastic contributions: $P_c, \Pi_c$. Applying a constant shear rate $v_y$ to an elastic body, the shear stress will monotonically increase – until the point of breakage. Sand is different and can maintain a constant stress, $\sigma_{ij}^c$. This is the famous critical state [1, 2] that has, for given density, a unique, rate-independent stress value. Employing GSH, this is easy to understand: Because elastic deformation $u_{ij}$ is slowly lost if the grains joggle, and because grains indeed joggle when forced to shear past one another, a shear rate $v_y$ not only in-

![FIG. 1: Comparison of GSH to the polystyrene data of Savage and Sayed [3], with Pressure $P$, shear stress $\sigma_s$, and $\sigma_s/P$ given as functions of the shear rate $\dot{\gamma}$. The first two figures show the $\dot{\gamma}^2$-dependence, and the third the convergence onto the weakly density-dependent, high-rate limit $\mu_2 = C/C_1$. Diamonds, squares and circles are the experimental points at the specified densities $\rho$. (Data for the largest $\rho$ are not used, because the authors believe they may be plauged by “finite-particle-size effects.”) The curves render Eqs (9), with $h_1 = 3.1 \cdot 10^{-4} \sqrt{a/\rho_0}, g_1 = 121.7 \sqrt{a/\rho_0}, a = 0.1, \rho_{cp} = 0.64 \rho_{bulk}$.](image-url)
creases $u_{ij}$, as in any elastic medium, but also decreases it. The critical state is the steady state in which both processes balance each other, such that the elastic deformation remains constant over time, in spite of a finite $v_s$. As shown in [2], the stationary solution $u^c_{ij}$ depends on the density, but not on $v_s$. The associated elastic stress $\pi^c_{ij} = \pi^c_{ij}(u^c_{ij}, \rho) = \pi^c_{ij} \delta_{ij}/3 - \pi^c_{ij} v^0_{ij}/v_s$, characterized by two scalars, $\pi^c_{ij}$ and $\pi^c_{ij}$, is also independent of $v_s$. Its contributions in Eq (9) are: $P_c = \frac{1}{2} (1 - \alpha) \pi^c_{ij}$, $\Pi = (1 - \alpha) \pi^c_{ij}$. Although both $P_c$, $\Pi$, depend on $1 - \rho_r$, the ratio $\Pi/P_c = \pi^c_{ij}/\pi^c_{ij}$ does not.

We did not find any independent data on the critical state of polystyrene beads, though that from [2] indicate $P_c \approx 50$ Pa, $\sigma_c/P_c \approx 0.25$, implying that the softer polystyrene beads have a $B \approx 5 \times 10^9$ Pa, while the other coefficients retain their orders of magnitude as given in [2]. (Note: $\pi_{ij} \sim B$, and $B \approx 5 \times 10^9$ Pa for sand.)

At lower shear rates, say for $v_s \lesssim 10$ s$^{-1}$, the rate-dependent terms of $\sigma_{ij}$ are quadratically small, $P_r \sim T_g \sim v^2_s$, $\eta_T v^0_{ij} \sim v^2_s$, and may be neglected. This is the reason the total stress is given by the rate-independent critical state, $\sigma_{ij} = (1 - \alpha) \pi^c_{ij}$, for a fairly broad range of shear rates, and why soil mechanic textbooks emphasize the rate-independence of granular behavior.

We note that fluidization, as considered above, is uniform and continuous, without anything resembling “failure” or “yield.” Starting from a state of isotropic stress, a sheared granular system will approach the critical state, in the continuous way as calculated in [2]. The end state is, more generally, given by Eqs (9), though the differences to the critical state is evident only at higher shear rates. There is an alternative path that goes through an energetic instability, eg. the Coulomb yield contained in $w_1(u_{ij})$, see [12], which sets in when the ratio $\sigma_s/\pi^c_{ij}$ becomes too large. This transition is discontinuous, non-uniform, and shear bands necessarily appear. We shall consider it in a forthcoming paper.

Jamming, the reverse transition – a drop of the shear rate $v_s$ from a finite value to zero at given stress – is necessarily discontinuous. In contrast to the authors of theory III, however, we do not believe this instability is marked by a lower bound of $\pi_s/\pi^c_{ij}$, as elastic solutions are perfectly stable at isotropic stresses, $\pi_s = 0$. Rather, jamming seems an instability that sets in when the density is too high to enable a shear flow $v_s$. Although $v_s$ is not a state variable, $T_g$ is, and we have $T_g \sim v_s$ [cf. Eq (2)] for any processes slow enough for Eq (5) to have reached its stationary limit. The appropriate instability must therefore be in $w_2(s_g, \rho) = s_g^2/2\rho b$. If we substitute $\hat{b}$ for $b$, we have $\hat{P}_T$ instead of $P_T$,

$$\frac{\hat{b}}{b} = \left[1 + \frac{b_1}{1 - \rho_r}\right], \quad \frac{\hat{P}_T}{P_T} = 1 - \frac{(1 - a) b_1}{a (1 - \rho_r)}.$$  \hspace{1cm} (10)

With $b_1$ small, we may neglect the correction term as long as $\rho_r$ is away from 1, and all results above remain valid. For $\rho$ equal to

$$\rho_{jam} = \rho_c (1 - 2b/a),$$  \hspace{1cm} (11)

however, the convexity of $w_2$ with respect to $\rho$ is lost, and no finite value of $T_g \sim v_s$ is stable. $\rho_{jam}$ is obtained from the condition: $\partial^2 w_2/w^2 v_s = 0$, or equivalently, from $\partial^2 f_2/\partial^2 |T_g \sim \alpha_s \left(\frac{v^2}{\rho b} \right)T_g = 0$, where $f_2 \equiv w_2 - T_g s_g = -\rho b T^2_g/2$. Eq (11) is the result to lowest order in $b_1$ and $a$. Note $\rho < \rho_{jam}$ implies a lower bound for $v_s$ if the pressure is given instead of the density, as a smaller $v_s$ will imply a larger $\rho$, see the first of Eq (9).

On a plane inclined by the angle $\varphi$, the friction angle is $\tan \varphi = \sigma_s/P$, with the angle of repose given by $\varphi_r = \varphi(\rho_{jam})$. Since the two terms $\sim \hat{b}^2$ are negligible by then, the angle of repose is given by the critical angle at $\rho_{jam}$: $\tan \varphi_r = \Pi/(\rho_{jam})/P(\rho_{jam})$. This is consistent with observation, because the critical angle is necessarily smaller than the angle at which Coulomb yield sets in. All these statements are independent of the specific form of $\hat{b}$, which may possibly prove inappropriate – though the case for an instability in $b(\rho)$ seems watertight.

Summary: Because GSH is capable of accounting for elasto-plastic motion, including the critical state, and also for dense flow, fluidization and jamming, we believe that this hydrodynamic theory, conventionally based on conservation laws and thermodynamics, is a viable candidate for a unified theory of granular media.

[1] A. Schofield and P. Wroth. *Critical State Soil Mechanics*. McGraw-Hill, London, 1968.
[2] G. Gudehus. *Physical Soil Mechanics*. Springer SPIN, 2010
[3] S.B. Savage, M. Sayed. J. Fluid Mech, 142:391, 1984.
[4] P. Jop, Y. Forterre, O. Pouliquen. Nature, 414:727, 2006. GDR MiDi. EJP, E14(4):341, 2004. Y. Forterre, O. Pouliquen, An. Rev. Fluid Mech, 40(1):1, 2008.
[5] L. Bocquet, W. Losert, D. Schalk, T. C. Lubensky, and J. P. Gollub. Phys. Rev. E, 65(1):011307, 2001.
[6] I. S. Aranson and L. S. Tsimring. Phys. Rev. E, 65:061303, 2002. Rev. Mod. Phys., 78:641, 2006.
[7] D. O. Krimer, M. Pitzinger, K. Bräuer, Y. Jiang, and M. Liu. Phys. Rev. E, 74(6):061310, 2006.
[8] Y. Jiang, M. Liu. Phys. Rev. Lett., 99(10):105501, 2007.
[9] S. Mahle, Y.M. Jiang, and M. Liu. arXiv:1006.5131v1
[10] M. Mayer, M. Liu. Phys. Rev. E, 82:042201, 2010.
[11] Y. Khidas, X. Jia. Phys. Rev. E, 81:021303, 2010.
[12] Y. Jiang and M. Liu. Granular Matter, 11:139, 2009.
[13] D. Kolyambas and G. Viggiani, editors. *Mechanics of Natural Solids*, pages 27–46. Springer, 2009.
[14] R.A.Bagnold. Proc.R.Soc.London, Ser.A, 225(1160):49, 1954.