We consider the sensitivity of semidefinite programs (SDPs) under perturbations. It is well known that the optimal value changes continuously under perturbations on the right hand side if the primal problem is strictly feasible. In this manuscript, we observe by investigating a concrete SDP that the optimal value could change discontinuously if the coefficient matrices are perturbed. We show that the optimal value of such an SDP changes continuously if the perturbations move the minimal face of the dual problem continuously and preserve the dimension of the space spanned by submatrices of the coefficient matrices. In addition, we determine what kinds of perturbations make the minimal faces invariant, by using the reducing certificates which are produced in the facial reduction algorithm. Our results allow us to classify the behavior of the minimal face of an SDP obtained from a control problem if the perturbations preserve matrix structures that appear in the associated dynamical system.

Keywords: Semidefinite programming, sensitivity, facial reduction, minimal face, H-infinity feedback control problem

2010 Mathematical subject classification: 90C31, 90C22, 90C51, 93D15

1. Introduction

1.1. A singular SDP and its perturbation

A semidefinite program (SDP) is the problem of minimizing a linear objective function over the intersection of the positive semidefinite cone and an affine space over symmetric matrices. The primal SDP \( (P) \) and its dual \( (D) \) are formulated as follows:

\[
\sup_{y, Z} \left\{ b^T y : A_0 - \sum_{k=1}^m y_k A_k = Z, y \in \mathbb{R}^m, Z \in \mathcal{S}_+^n \right\},
\]

\[
\inf_X \left\{ A_0 \cdot X : A_k \cdot X = b_k (k \in [m]), X \in \mathcal{S}_+^n \right\},
\]

\( (P) \)

\( (D) \)
where $A_0, A_1, \ldots, A_m$ are symmetric matrices, $b \in \mathbb{R}^m$, $[m] := \{1, \ldots, m\}$, $A \cdot B = \sum_{i,j=1}^n A_{ij}B_{ij}$ for $n \times n$ symmetric matrices $A$ and $B$, and $S^n_+$ stands for the cone of $n \times n$ positive semidefinite matrices.

We say that (7) is strictly feasible if there exists a feasible point $(y, Z)$ such that $Z$ is positive definite. Strict feasibility of (12) is defined similarly. It is well known that the strong duality for (12) and (11) holds when one is strictly feasible and the other is feasible. An SDP is said to be nonsingular if both of (12) and (11) are strictly feasible and $\{A_k\}_{k \in [m]}$ is linearly independent; otherwise, it is said to be singular. Interior-point methods for nonsingular SDPs are guaranteed to obtain an approximation to a solution in polynomial time. Various variants of interior-point methods are implemented in software for solving SDPs, such as SDPA [10], SDPT3 [30], SeDuMi [27] and Mosek [17].

The $H_\infty$ control problem is one of the most successful applications of SDP and is the problem for designing a controller that achieves stabilization with some guaranteed performance. In particular, the $H_\infty$ state feedback control problem is a special case of the $H_\infty$ control problem. We consider the following linear time-invariant dynamical system.

$$
\begin{align*}
\dot{x} &= Ax + B_1 w + B_2 u, \\
\dot{z} &= C_1 x + D_{11} w + D_{12} u,
\end{align*}
$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^{m_2}$, $z \in \mathbb{R}^p$ and $w \in \mathbb{R}^{m_1}$ are called the state, control input, evaluated output, measurement output and plant disturbance variables, respectively. The constant matrices $A, B_1, B_2, C, D_{11}$ and $D_{12}$ are given with the appropriate sizes. Let $u = K x$ for a state feedback gain $K \in \mathbb{R}^{n \times m_2}$ to apply a state feedback in (11). Then the state space representation of the closed loop system with (11) and $u = K x$ is given by

$$
\begin{align*}
\dot{x} &= (A + B_2 K)x + B_1 w, \\
\dot{z} &= (C_1 + D_{12} K)x + D_{11} w,
\end{align*}
$$

The purpose of the $H_\infty$ state feedback control problem is to design a controller $u = K x$ which stabilizes (2) and minimizes the $H_\infty$ norm of (2). The following SDP problem can often be used to find such a matrix $K$. See e.g., [13, 26].

$$
\sup \left\{ -\gamma : \begin{pmatrix}
-\text{He}(AY_1 + B_2 Y_2) & -C_1 Y_1 - D_{12} Y_2 \\
-C_1 Y_1 - D_{12} Y_2 & \gamma I_p
\end{pmatrix}, \begin{pmatrix}
\gamma I_p \\
-D_{11} \\
\gamma I_{m_1}
\end{pmatrix} \right\} \in S^n_+, \quad Y_1 \in S^n_+, Y_2 \in \mathbb{R}^{m_2 \times n}, \quad \gamma \in \mathbb{R}
$$

where $N = n + m_1 + p$, $\text{He}(X) = X + X^T$ for $X \in \mathbb{R}^{n \times n}$ and blanks in matrices stand for the transpose of the lower triangular block part. Its dual can be formulated as follows:

$$
\inf \left\{ -\begin{pmatrix}
O & O \\
O & O
\end{pmatrix}, \begin{pmatrix}
\text{He}(A^T Z_{11} + C_1^T Z_{21}) \\
B_1^T Z_{11} + D_{12}^T Z_{21}
\end{pmatrix} \in S^n_+, \quad Z = \begin{pmatrix}
Z_{11} & Z_{21} & Z_{22} \\
Z_{31} & Z_{32} & Z_{33}
\end{pmatrix} \in S^n_+ight\}.
$$

Now we consider the following dynamical system

$$
\begin{align*}
\dot{x} &= \begin{pmatrix}
-1 & -1 \\
1 & 0
\end{pmatrix} x + \begin{pmatrix}
-1 & -1 \\
-1 & 0
\end{pmatrix} w + \begin{pmatrix}
0 \\
1
\end{pmatrix} u, \\
\dot{z} &= \begin{pmatrix}
2 & -1 \\
-1 & 2
\end{pmatrix} x + \begin{pmatrix}
0 & -1 \\
-1 & 0
\end{pmatrix} w + \begin{pmatrix}
2 \\
-1
\end{pmatrix} u.
\end{align*}
$$

Substituting the following matrices to (3), we obtain the SDP problem for $H_\infty$ state feedback control associated with (5).

$$
\begin{pmatrix}
Y_1 \\
Y_2
\end{pmatrix} = \begin{pmatrix}
y_1 & y_2 \\
y_2 & y_3 \\
y_4 & y_5
\end{pmatrix}, y_6 = \gamma, \begin{pmatrix}
A & B_1 \\
C_1 & D_{11}
\end{pmatrix}, B_2 = \begin{pmatrix}
-1 & -1 & -1 & 0 \\
1 & -1 & -1 & 0 \\
2 & -1 & -1 & 0 \\
-1 & 2 & -1 & 0
\end{pmatrix}.
$$

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The SDP problem can be formulated as follows.

\[
\sup_{y_1,\ldots,y_6} \left\{ -y_6 : \begin{pmatrix}
2y_1 + 2y_2 \\
-y_1 + y_2 + y_3 - y_4 \\
y_1 - 2y_3 + y_4
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 1 & 0 & 0 & 0 \\
y_1 - 2y_3 + y_4 & y_2 - 2y_5 + y_6
\end{pmatrix}
\right\} \in \mathbb{S}_+^6.
\] (7)

To rewrite (7) into the form of \(P\), we define the coefficient matrices \(A_k\) \((k \in [m] \cup \{0\})\) and vector \(b\) by

\[
A_k = \begin{pmatrix} A_{k,1} & O \\ O & A_{k,2} \end{pmatrix}, \quad b = (0 \ 0 \ 0 \ 0 \ 0 \ -1)^T,
\]

where the matrices \(A_{k,j}\) \((k = 0, 1, \ldots, 6, j = 1, 2)\) are given by

\[
\begin{align*}
A_{1,1} &= \begin{pmatrix} -2 & 0 \\ -2 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, & A_{2,1} &= \begin{pmatrix} 0 & -2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, & A_{3,1} &= \begin{pmatrix} -2 & 0 \\ -2 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \\
A_{4,1} &= \begin{pmatrix} 0 & -2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, & A_{5,1} &= \begin{pmatrix} 0 & -2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, & A_{6,1} &= \begin{pmatrix} 0 & -2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \\
A_{1,2} &= \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, & A_{2,2} &= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, & A_{3,2} &= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad A_{0,1} = \begin{pmatrix} O & O \\ -B^T & -D^T \end{pmatrix}.
\]

and \(A_{4,2} = A_{5,2} = A_{6,2} = A_{0,2} = O\). Then (7) can be rewritten as the form of \(P\) with \(n = 8\) and \(m = 6\), and we can write the dual problem of (7) as follows.

\[
\begin{align*}
\inf_{z_{ij}} & \quad -2(z_{51} + z_{61} + z_{52} + z_{62} + z_{53} + z_{63} + z_{54} + z_{64}) \\
\text{s.t.} \quad & \quad \text{He}
\begin{pmatrix}
-z_{11} + z_{21} & 2z_{31} - z_{41} & -z_{21} + z_{22} + 2z_{32} - z_{42} \\
-z_{11} - z_{31} + 2z_{41} & -z_{21} - z_{32} + 2z_{42} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\right\} \in \mathbb{S}_+^2, \\
& \quad z_{21} + 2z_{31} - z_{41} = 0, \quad z_{22} + 2z_{32} - z_{42} = 0, \quad \sum_{i=3}^{6} z_{ii} = 1, \quad (z_{ij})_{1 \leq i,j \leq 6} \in \mathbb{S}_+^6.
\] (8)

It follows from [36] Theorems 3.3 and 3.5 that (7) is strictly feasible but its dual problem (8) is not strictly feasible. Thus we can say that the SDP is singular. This will be discussed in Remark 4.10.

We compare computational results on (7) with the following three perturbed SDPs for (7). For \(\epsilon = 1.0e-16\),

(P1) SDP obtained by perturbing the (2, 2)nd element of \(A_{5,1}\) in (7) into \(-2(1 + \epsilon)\),

(P2) SDP obtained by perturbing the (2, 3)rd and (3, 2)nd elements in \(A_{5,1}\) of (7) into \(-2(1 + \epsilon)\), and

(P3) SDP obtained by perturbing the (2, 4)th and (4, 2)nd elements of \(A_{5,1}\) in (7) into \(1 + \epsilon\).

As it is reported in e.g., [12, 18, 33, 35] that the standard floating point computation may provide wrong results for singular SDPs, we apply SDPA-GMP [10] to solve (7) with stopping tolerances \(\delta = 1.0e-10, 1.0e-30\) and 1.0e-50) and set the floating point computation to approximately 300 significant digits; otherwise one may encounter strange behavior for SDP software. We provide other parameters used for SDPA-GMP in Table 1. See [10] for more details on parameters. Table 2 shows the numerical results.

We observe the following from Table 2.
1. Introduction

- The computed values of (7) are almost the same for all $\delta$, whereas the values for perturbed problems (P1), (P2) and (P3) are different. In fact, we can prove that the optimal values of (7) and (P1) are $-\sqrt{5}$ and $-\sqrt{2}$ respectively. We provide the proofs in Appendices A and B. These significant differences imply that one needs to choose suitable tolerances $\delta$ in order to use the floating point computation with longer significant digits for singular SDPs.

- The optimal value of (7) is $-\sqrt{5}$, while the optimal values of the perturbed problems are $-\sqrt{2}$ and $-2$. These differences show that a small perturbation of coefficient matrices $A_k$ in (7) may yield a significant change of the optimal value of (7).

### Table 1: Details on parameters used for solving (7) and its perturbed problems

| parameter       | value | parameter       | value | parameter       | value |
|-----------------|-------|-----------------|-------|-----------------|-------|
| maxIteration    | 10000 | lowerBound      | -1.0e+5 | gammaStar       | 0.5   |
| epsilonStar     | $\delta$ | upperBound      | 1.0e+5 | epsilonDash     | $\delta$ |
| lambdaStar      | 1.0e+4 | betaStar        | 0.5   | precision       | 1024  |
| omegaStar       | 2.0   | betaBar         | 0.5   |                 |       |

### Table 2: Computed values for (7), its perturbed problems (P1), (P2) and (P3) by SDPA-GMP

| Problem | $\delta = 1.0e-10$ | $\delta = 1.0e-30$ | $\delta = 1.0e-50$ |
|---------|-------------------|-------------------|-------------------|
| (7)     | -2.2360679775444764 | -2.2360679774997897 | -2.2360679774997897 |
| (P1)    | -2.2360072694172055 | -2.0000000000000000 | -2.0000000000000000 |
| (P2)    | -2.2360072694172072 | -2.0000000000000000 | -2.0000000000000000 |
| (P3)    | -2.2360072665294605 | -1.4142135623730950 | -1.4142135623730950 |

1.2. Contribution and literature

The main contribution of this manuscript is to determine and analyze the kinds of perturbations of coefficient matrices that change the optimal values continuously. To this end, we consider any perturbations of coefficient matrices $A_k$ and $b_k$ in (P) and (D). More precisely, we analyze the following perturbed SDP and its dual:

$$
\sup_{y,Z} \left\{ b(t)T y : \sum_{k=m} y_k A_k(t) + Z = A_0(t), \ y \in \mathbb{R}^m, \ Z \in \mathbb{S}_+^n \right\}, \quad (P_1)
$$

$$
\inf_X \left\{ A_0(t) \bullet X : A_k(t) \bullet X = b_k(t) (k \in [m]), \ X \in \mathbb{S}_+^n \right\}, \quad (D_1)
$$

where $t \geq 0$, $A_k(t) \in \mathbb{S}_n$, $b(t) \in \mathbb{R}^m$ are continuous at $t = 0$ and $A_k(0) = A_k$, $b(0) = b$.

The first contribution is to provide a result on continuity of the set of optimal solutions of singular SDPs under perturbations on any data. In Theorem 3.1, we show that the optimal value of (P) changes continuously under any perturbation of (D) if (D) is feasible and the perturbation changes continuously the minimal face of (D) and satisfies a rank condition on the coefficient matrices. Here the minimal face is the intersection of all faces of $\mathbb{S}_+^n$ that contains the feasible region of (D). As a corollary, we show the continuity of the set of optimal solutions of (P) and (D) for nonsingular SDPs. This result has been shown by Gol’shtein [11] for general convex programs which satisfy some regularity conditions. Similar results for nonsingular SDPs can be obtained if one assumes that the problem and its perturbations satisfy the inf-compactness condition [2]. Some of classical sensitivity analysis for SDPs are discussed in [21] [22]. Several characterizations of the continuity of the set of
optimal solutions, or optimal values are given via concepts from variational analysis [16, 22]. An approach based on the minimal face is proposed for optimal partitions of SDPs in [15].

Although an individual optimal solution rarely moves continuously under perturbations as in the case of linear programming, we can extract sufficient conditions for continuity of an optimal solution as in Alizadeh, Haeblerly and Overton [1]. Namely, suppose that \((X, Z)\) is a pair of optimal solutions for \((\mathcal{P})\) and \((\mathcal{D})\) respectively. Then \((X, Z)\) moves continuously if both \((\mathcal{P})\) and \((\mathcal{D})\) are strictly feasible, \((X, Z)\) satisfies the strict complementarity condition, \((X, Z)\) is nondegenerate and positive eigenvalues of \(X\) and \(Z\) are all distinct.

On the other hand, it is well known in [3] from the general theory in convex analysis that the optimal value changes continuously if one of \((\mathcal{P})\) and \((\mathcal{D})\) is strictly feasible in the case of perturbing only \(A_0(t)\).

However if we perturb \(A_0, A_k\) and \(b_k\) in both \((\mathcal{P})\) and \((\mathcal{D})\), the behavior of the optimal value may change enormously. In fact, the continuity of the optimal value is not guaranteed when exactly one of \((\mathcal{P})\) and \((\mathcal{D})\) is strictly feasible, as presented in Table 2. Recall that the original SDP \((\mathcal{P})\) is strictly feasible and the dual problem \((\mathcal{D})\) of \((\mathcal{P})\) is non-strictly feasible. Thus the minimal face of \((\mathcal{P})\) is a proper subset of the positive semidefinite cone. The reason for \((\mathcal{P})\) having distinct optimal values is that their perturbations change the minimal faces of their duals significantly. In fact, we prove in Appendix C that the minimal face of the dual of the perturbed SDP \((\mathcal{P})\) is different from that of \((\mathcal{D})\).

On the other hand, Theorem 3.1 tells us that some continuous behavior of the minimal face under perturbations, together with the rank condition on the coefficient matrices, implies continuity of the optimal value.

The second contribution is to use the reducing certificates to give sufficient conditions that the perturbations do not change the minimal face. Using these conditions, we show that the minimal face of \((\mathcal{P})\) does not change or changes into the full-dimensional cone if the perturbations preserve matrix structures that appear in the \(H_\infty\) state feedback control problem obtained for the dynamical system [3]. The reducing certificates are produced in the facial reduction, which is a procedure to find the minimal face for a given SDP. However we remark that the reducing certificates are often obtained without solving SDPs if the problems are generated from combinatorial optimization problems, matrix completion problems, sums of squares problems or \(H_\infty\) control problems.

The organization of this manuscript is as follows: preliminaries on the minimal face and facial reduction are introduced in Section 2. In Section 3, we show the main results on the continuity of the optimal values of \((\mathcal{P})\) for singular SDPs. In Section 4, we give conditions on the perturbations under which the minimal face does not change. We devote Section 5 to conclusions of this manuscript. We provide detailed analyses of \((\mathcal{P})\) and \((\mathcal{P})\) in Appendices A-B and C.

2. Preliminaries on face, minimal face and facial reduction

We give a brief introduction to define a face for a convex set and the minimal face for an SDP. These definitions are described in [7, 19, 21] in detail.

For a convex subset \(C\) of \(\mathbb{R}^n\) and a convex subset \(F\) of \(C\), we say that \(F\) is a face of \(C\) if for all \(x_1, x_2 \in C\), nonemptyness of the intersection of the open line segment \((x_1, x_2)\) and \(F\) implies that \(x_1\) and \(x_2\) are both in \(F\). For a nonempty convex subset \(S\) of \(C\), the minimal face of \(C\) containing \(S\) is defined as the intersection of all faces of \(C\) that contain \(S\).

The following results on a facial structure of \(S^n_+\) are known in e.g., [1, 4, 32].

Lemma 2.1. 1. Any face of \(S^n_+\) is either the empty set, \(\{O_{n \times n}\}\), \(S^n_+\) or

\[
\left\{ Q \left( \begin{array}{cc} O_{(n-r) \times (n-r)} & O_{(n-r) \times r} \\ O_{r \times (n-r)} & M \end{array} \right) Q^T : M \in S^n_+ \right\},
\]

where \(Q\) is an \(n \times n\) nonsingular matrix and \(O_{k \times k'}\) stands for the \(k \times k'\) zero matrix for positive integers \(k\) and \(k'\). We call a \(Q\) nonsingular matrix associated to the face. It follows from this...
2. Preliminaries on face, minimal face and facial reduction

property that for any \( U \in S^*_n \), the set \( S^*_n \cap \{U\}^\perp \) is a face of \( S^*_n \), where \( \{U\}^\perp \) stands for the set of the symmetric matrices which are orthogonal to \( U \), i.e., \( \{U\}^\perp = \{X \in S^n : X \cdot U = 0\} \).

2. The set \( S^*_n + F^\perp \) is closed for all faces \( F \) of \( S^*_n \), where \( \{U\}^\perp \) stands for the set \( \{Z \in S^n : Z \cdot X = 0 (\forall X \in F)\} \). This property is called the niceness. The niceness property implies that \( F^* = S^*_n + F^\perp \) for all faces \( F \) of \( S^*_n \), where \( F^* \) is the dual cone of \( F \), i.e., \( F^* = \{Z \in S^n : Z \cdot X \geq 0 (\forall X \in F)\} \).

We define the minimal face and facial reduction for only \( D \) because the dual problem \( S^* \) of \( D \), which is the motivation of this manuscript, is non-strictly feasible. One can discuss the minimal face and the facial reduction for \( D \) in a similar manner. The minimal face for \( D \) is defined as the minimal face of \( S^*_n \) containing the feasible region of \( D \). We denote the minimal face by \( F_{\text{min}} \). The following result on the minimal face is obtained by \[19\] and 2 in Lemma 2.1.

Lemma 2.2. \[19\] SDP version of Section 28.2.6 and Lemma 28.4] Assume that \( F \) and \( D \) are feasible. Let \( F \) be a face of \( S^*_n \) that contains \( F_{\text{min}} \) and rint \( F \) be its relative interior. Then the following are equivalent;

1. \( F \neq F_{\text{min}} \);

2. There exists \( (y, U, V) \in \mathbb{R}^m \times S^*_n \times F^\perp \) such that

\[
b^T y = 0, \quad - \sum_{k \in [m]} y_k A_k = U + V \text{ and } U + V \not\subset F^\perp;
\]

3. \( \{X \in \text{rint } F : A_k \cdot X = b_k (k \in [m])\} = \emptyset \).

If \( U \) satisfies the system in 2, we have \( F_{\text{min}} \subseteq F \cap \{U\}^\perp \subset F \).

We call the above system \( [4, 20, 19] \) the discriminant system for the facial reduction for \( D \) and a solution \( (y, U, V) \) a reducing certificate.

The facial reduction in \[14, 20, 19\] is a procedure based on Lemma 2.2. It generates a sequence \( \{F_i\}_{i=0}^s \) of faces of \( S^*_n \) such that

\[ S^*_n = F_0 \supseteq F_1 \supseteq F_2 \supseteq \cdots \supseteq F_s = F_{\text{min}}, \]

where \( F_{\text{min}} \) is the minimal face for \( D \). We describe the facial reduction for \( D \) in Algorithm 1 below. It is proven in e.g., \[19, 31, 35\], that the facial reduction terminates in finitely many iterations. At the \( i \)th iteration of the facial reduction, if a face \( F_i \) is not the minimal face \( F_{\text{min}} \), then we obtain a proper face \( F_{i+1} \) of \( F_i \) by \( F_{i+1} = F_i \cap \{U\}^\perp \), where \( (y, U, V) \) is a reducing certificate in \( [4, 20, 19] \) and \( F_{i+1} \) contains \( F_{\text{min}} \). The process can be represented as

\[
[D] \quad S^*_n = F_0 \longrightarrow F_1 \longrightarrow F_2 \longrightarrow \cdots \longrightarrow F_s = F_{\text{min}}.
\]

Here we call \( \{(y^i, U^i, V^i)\}_{i=1}^s \) a facial reduction sequence for \( D \).

A solution of the discriminant system \( [4, 20, 19] \) is not unique. For this, we have flexibility in choosing a facial reduction sequence for \( D \). Cheung and Wolkowicz \[7, Proposition B.1\] prove that any two minimal facial reduction sequence must be of the same length when reducing certificates whose \( U \) has the maximal rank are selected in each iteration. The length is called the degree of singularity for \( D \). For instance, the degrees of singularity for \( S \) and dual of \( [P1] \) are 1 and 2, respectively. The degree
of singularity is used in [7] for the sensitivity analysis of SDPs and in [28] for the error bounds. The backwards stability result for SDPs with singularity degree one is obtained in [6].

```
Algorithm 1: Facial reduction for (D)
Input: A feasible SDP (D)
Output: Minimal face $F_{\text{min}}$ of (D)

\[
F \leftarrow S^0_{+}; \\
\text{while } \exists \text{ reducing certificate } (y, U, V) \text{ that satisfies } (9) \text{ do} \\
\quad F \leftarrow F \cap \{U\}^\perp; \\
\text{end} \\
\text{return } F;
\]
```

One of the numerical difficulties in the facial reduction is to find reducing certificates $(y, U, V)$ numerically. A straightforward computation of $(y, U, V)$ is to convert $F$ into an SDP. However this may cause the numerical instability if the SDP problem or its dual is not strictly feasible. Instead of solving the SDP problem, partial but robust facial reductions are proposed by using properties and structures in the original problems. For instance, see [38, 37] for semidefinite programming relaxation of combinatorial optimization problems, [14] for Euclidean distance matrix completion problems, [34] for sum-of-square problems and [36] for $H_{\infty}$ state feedback control problems. Variants of facial reductions are executed in their work without solving any SDP problems to find reducing certificates numerically.

We present the facial reduction for (8) as an example of the facial reduction. More elementary examples can be seen in [19] Examples 28.2 and 28.3 continued.

**Example 2.3.** We apply the facial reduction (Algorithm 1) to the dual problem (8). Let us recall that (7) is strictly feasible and its dual problem (8) is non-strictly feasible. Since all coefficient matrices of (8) are block-diagonal with the sizes $6 \times 6$ and $2 \times 2$, we can simplify the facial reduction. In fact, since $U$ and $V$ in (9) are also block-diagonal with the sizes $6 \times 6$ and $2 \times 2$, the matrix $-\sum_{k=1}^{6} y_k A_k = U + V$ in (9) can be partitioned into

\[
-\sum_{k=1}^{6} y_k A_{k,1} = U_1 + V_1, \quad -\sum_{k=1}^{6} y_k A_{k,2} = U_2 + V_2.
\]

In addition, $U + V \notin F^\perp$ in (9) can be partitioned into $(U_1 + V_1, U_2 + V_2) \notin F^\perp_1 \times F^\perp_2$, where $F_1 \subset S^6_{+}$ and $F_2 \subset S^2_{+}$ are faces of $S^6_{+}$ and $S^2_{+}$, respectively.

We apply the first iteration of the facial reduction to (8). Since $(S^6_{+})^\perp = \{O_{p \times p}\}$ for a positive integer $p$, we will find a reducing certificate $(y, U, V)$ such that

\[
\left(\begin{array}{cccccc}
2g_1+2g_2 & -2g_2 & -2g_2 & -2g_2 & -2g_2 & -2g_2 \\
-2g_2 & y_1+y_2+y_3-y_4 & -2g_2 & -2g_2 & -2g_2 & y_6 \\
y_1 & y_2 & 0 & y_4 & 0 & y_6 \\
& 0 & y_2 & 0 & y_4 & y_6 \\
& & 0 & 0 & 0 & y_6 \\
& & & 0 & 0 & 0 \\
\end{array}\right) = U_1 + V_1 \in S^6_{+}, \ y_6 = 0,
\]

\[
\left(\begin{array}{c}
y_1 \\
y_2 \\
y_3
\end{array}\right) = U_2 + V_2 \in S^2_{+}, \ U_1 \in S^6_{+}, U_2 \in S^2_{+},
\]

\[
V_1 = O_{6 \times 6}, V_2 = O_{2 \times 2}, (U_1 + V_1, U_2 + V_2) \neq (O_{6 \times 6}, O_{2 \times 2}).
\]

Then the following $(y, U, V)$ satisfies (10).

\[
\begin{align*}
y &= (1, 0, 0, -1, 0, 0)^T, \\
U &= \begin{pmatrix}
U_1 \\
O_{2 \times 6}
\end{pmatrix}, \\
V &= \begin{pmatrix}
V_1 \\
O_{2 \times 6}
\end{pmatrix}, \\
V_1 &= O_{6 \times 6}, \ V_2 = O_{2 \times 2}, \ U_1 = \begin{pmatrix}
1 \\
O_{5 \times 1}
\end{pmatrix}, \ U_2 = \begin{pmatrix}
1 \\
0
\end{pmatrix}.
\end{align*}
\]
The obtained faces are

\[
F_1^1 = S^6_+ \cap \{U_1\}^\perp = \left\{ \begin{pmatrix} 0 & O_{1 \times 5} \\ O_{5 \times 1} & X_1 \end{pmatrix} : X_1 \in S^5_+ \right\} \quad \text{and} \\
F_2^1 = S^2_+ \cap \{U_2\}^\perp = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & X_2 \end{pmatrix} : X_2 \geq 0 \right\}. 
\]

(12)

At the second iteration, we consider a reducing certificate \((y, U, V)\) such that

\[
\begin{cases}
- \sum_{k=1}^6 y_k A_{k,1} = U_1 + V_1, & U_1 + V_2 \in S^6_+ + (F_1^1)^\perp, \\
- \sum_{k=1}^6 y_k A_{k,2} = U_2 + V_2, & U_2 + V_2 \in S^2_+ + (F_2^1)^\perp, \\
U_1 \in S^6_+, & y_6 = 0, \\
U_1 + V_1, U_2 + V_2 \notin (F_1^1)^\perp \times (F_2^1)^\perp,
\end{cases} 
\]

(13)

where the sets \((F_1^1)^\perp\) and \((F_2^1)^\perp\) can be explicitly described as follows:

\[
(F_1^1)^\perp = \left\{ \begin{pmatrix} Z_1 & Z_2^T \\ Z_2 & O_5 \end{pmatrix} : Z_1, Z_2 \in \mathbb{R}^5 \right\} \quad \text{and} \quad (F_2^1)^\perp = \left\{ \begin{pmatrix} Z_1 & Z_2 \\ Z_2 & 0 \end{pmatrix} : Z_1, Z_2 \in \mathbb{R} \right\}. 
\]

Therefore the sets in the right-hand sides of (13) are described by

\[
S^6_+ + (F_1^1)^\perp = \left\{ \begin{pmatrix} Z_1 & Z_2^T \\ Z_2 & Z_3 \end{pmatrix} : Z_1, Z_2, Z_3 \in S^5_+ \right\} \quad \text{and} \\
S^2_+ + (F_2^1)^\perp = \left\{ \begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \end{pmatrix} : Z_1, Z_2 \in \mathbb{R}, Z_3 \geq 0 \right\}.
\]

We will show that (13) has no solutions. In other words, the facial reduction for (8) terminates in one iteration and \(F_1^1 \times F_2^1\) is the minimal face of (8). This implies that the degree of singularity of (8) is one. For this, we consider \(y\) that satisfies

\[
- \sum_{k=1}^6 y_k A_{k,1} \in S^6_+ + (F_1^1)^\perp, \quad - \sum_{k=1}^6 y_k A_{k,2} \in S^2_+ + (F_2^1)^\perp, \quad y_6 = 0.
\]

(14)

We prove that any solution \(y\) of (14) satisfies \(- \sum_{k=1}^6 y_k A_{k,1} \in (F_1^1)^\perp\) and \(- \sum_{k=1}^6 y_k A_{k,2} \in (F_2^1)^\perp\). This means (13) has no solutions. Since we have \(y_6 = 0\), positive semidefiniteness of the first constraint in (14) implies that \(- 2y_2 + y_3 - 2y_5 = y_2 - 2y_3 + y_5 = 0\) and hence that \(y_6 = y_3 = y_2 + y_5 = 0\). Substituting these equations into \(- \sum_{k=1}^6 y_k A_{k,1}\) and \(- \sum_{k=1}^6 y_k A_{k,2}\), then we have

\[
- \sum_{k=1}^6 y_k A_{k,1} = \begin{pmatrix} 2y_1 + 2y_2 \\ -y_1 + y_2 - y_4 \\ -2y_1 + y_2 - 2y_4 \\ y_1 - 2y_2 + y_4 \\ 0 \\ 0 \end{pmatrix}, \quad - \sum_{k=1}^6 y_k A_{k,2} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.
\]

Hence any solution \(y\) of the system (14) satisfies \(- \sum_{k=1}^6 y_k A_{k,1} \in (F_1^1)^\perp\) and \(- \sum_{k=1}^6 y_k A_{k,2} \in (F_2^1)^\perp\).

3. Main results

3.1. Stability of singular SDPs

In this subsection, we consider the following conditions on an SDP:
3. Main results

Condition 1.

(C1) \( \{D\} \) is feasible and \( \{D\} \) is strictly feasible;

(C2) \( A_1, \ldots, A_m \) are linearly independent.

Then by applying the facial reduction in Algorithm 1 to \( \{D\} \), there exist a nonsingular matrix \( Q \) and \( r \in \mathbb{N} \) such that

\[
\inf_{X_3} \left\{ Q^T A_0 Q \cdot \begin{pmatrix} O & O \\ O & X_3 \end{pmatrix} : Q^T A_0 Q \cdot \begin{pmatrix} O & O \\ 0 & X_3 \end{pmatrix} = b_k \ (k \in [m]), \ X_3 \in S_+^r \right\}
\]  
\((F(D)_0)\)

has the same optimal value as \( \{D\} \) and \( F(D)_0 \) is strictly feasible due to Lemma 2.2. Here for \( n \times n \) matrix \( M \), we denote by \( M_3 \) the right bottom block of the partitioning

\[
M = \begin{pmatrix} M_1 & M_2' \\ M_2 & M_3 \end{pmatrix}
\]

where for \( t \in \mathbb{N} \), the partitioning is uniquely determined by Lemma 2.1 for the minimal face of \( \{D\} \) with \( M_1 \in S^{n-r}, M_2 \in \mathbb{R}^{r \times (n-r)}, M_3 \in S^r \). We call \( M_3 \) the third block of \( M \) associated to the minimal face of \( \{D\} \). Then we can rewrite \( F(D)_0 \) as follows:

\[
\inf_X \left\{ (Q^T A_0 Q) \cdot X : (Q^T A_0 Q) \cdot X = b_k \ (k \in [m]), X \in S_+^r \right\}.
\]  
\((F(D))\)

For \( A = (a_{ij})_{1 \leq i, j \leq n} \in S^n \), we define vec\( (A) \) as the vectorization of \( A \), i.e.,

\[
\text{vec}(A) = (a_{11}, a_{12}, \ldots, a_{1n}, a_{21}, a_{22}, \ldots, a_{n1}, \ldots, a_{nn})^T.
\]

Define the rank of the matrix \( \text{vec}(A_1), \text{vec}(A_2), \ldots, \text{vec}(A_m) \) by \( r(A_1, A_2, \ldots, A_m) \). The following theorem is one of the main results of this manuscript.

**Theorem 3.1.** Under Condition 1, suppose that the minimal face \( F_{\min} \) of \( \{D\} \) can be written as

\[ F_{\min} = \left\{ Q \begin{pmatrix} O_{(n-r) \times (n-r)} & O_{(n-r) \times r} \\ O_{r \times (n-r)} & X \end{pmatrix} Q^T : X \in S_+^r \right\} \]

for some nonsingular matrix \( Q \in \mathbb{R}^{n \times n} \) and \( r \in \mathbb{N} \). In addition we suppose that the set \( \{(A_0(t), \ldots, A_m(t), b(t)) : 0 \leq t \leq \delta\} \) satisfies the following assumptions for some \( \delta > 0 \):

1. \( \{D_t\} \) is feasible for each \( t \in [0, \delta] \);
2. For each \( t \in [0, \delta] \), there exists a nonsingular matrix \( Q(t) \) such that \( \lim_{t \rightarrow 0} Q(t) = Q \) and the minimal face of \( \{D_t\} \) can be written as

\[ \left\{ Q(t) \begin{pmatrix} O_{(n-r) \times (n-r)} & O_{(n-r) \times r} \\ O_{r \times (n-r)} & X \end{pmatrix} Q(t)^T : X \in S_+^r \right\} ; \]

3. For each \( t \in [0, \delta] \), we have

\[
r\ (Q(t)^T A_1(t) Q(t))_3, \ldots, (Q(t)^T A_m(t) Q(t))_3) = r\ ((Q^T A_1 Q)_3, \ldots, (Q^T A_m Q)_3),
\]

where for \( M \in S^n \), \( M_3 \) is the third block associated with the minimal face of \( \{D\} \).

Then the optimal value of \( \{D_t\} \) varies continuously at \( t = 0 \).

If we can choose the matrices \( Q(t) \) as \( Q(t) = Q \) for all \( t \in [0, \delta] \) in the assumptions 2 and 3 of Theorem 3.1, then we obtain the following corollary from Theorem 3.1.
Corollary 3.2. Under Condition 1, suppose that there exists \( \delta > 0 \) such that \( \{D_t\} \) has a nonempty feasible set and the same minimal face as \( \{D\} \), and

\[
\begin{align*}
r ((Q^T A_1(t)Q)_3, \ldots, (Q^T A_m(t)Q)_3) &= r ((Q^T A_1 Q)_3, \ldots, (Q^T A_m Q)_3) 
\end{align*}
\]

for \( t \in [0, \delta] \). Then the optimal value of \( \{D_t\} \) varies continuously at \( t = 0 \).

Before proceeding to the proof, we investigate examples and show that the rank condition or the condition on the face cannot be removed from Theorem 3.1 and Corollary 3.2.

Example 3.3. The following example satisfies the condition on the face but does not satisfy the rank condition. We set \( b = (2, 2, 2)^T \) and

\[
\begin{align*}
A_0 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\
A_2 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\end{align*}
\]
in \( \{P\} \) and \( \{D\} \). Then \( A_1, A_2, A_3 \) are linearly independent, \( \{P\} \) is strictly feasible and \( \{D\} \) is non-strictly feasible. The optimal value is 0 and an optimal pair is \( X = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad y = (0, 0, 0), \quad Z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \).

The minimal face of \( \{D\} \) is

\[
F_{\min} = \left\{ \begin{pmatrix} O_{2 \times 2} & O_{2 \times 2} \\ X_3 \end{pmatrix} \in \mathbb{S}^1_+ : X_3 \in \mathbb{S}^2_+ \right\}.
\]

If we perturb the matrices as

\[
\begin{align*}
A_i(t) &= A_i \quad (i = 0, 1, 2), \quad A_3(t) &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\
\end{align*}
\]

then \( \{D_t\} \) remains feasible for each \( t > 0 \). In fact the feasible points of \( \{D_t\} \) can be written as

\[
X = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad y = (\beta, 1+t, -1, 0), \quad Z = \begin{pmatrix} -\beta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (\beta \leq -\frac{1}{2t}) \text{ for each } t > 0.
\]

Thus the optimal value changes discontinuously.

Example 3.4. The following example satisfies the rank condition but does not satisfy the condition on the face. We set \( b = (2, 2, 2, 0)^T \) and

\[
\begin{align*}
A_0 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
A_2 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
A_4 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\end{align*}
\]
in \( \{P\} \) and \( \{D\} \). Then \( A_1, \ldots, A_4 \) are linearly independent, \( \{P\} \) is strictly feasible and \( \{D\} \) is non-strictly feasible. The optimal value is \( \frac{1}{2} \) and the optimal pairs are

\[
X = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad y = (-\frac{1}{2}, \frac{1}{2}, 0, 0), \quad Z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\]

The minimal face of \( \{D\} \) is

\[
F_{\min} = \left\{ \begin{pmatrix} O_{2 \times 1} & O_{1 \times 2} \\ X_3 \end{pmatrix} \in \mathbb{S}^3_+ : X_3 \in \mathbb{S}^2_+ \right\}.
\]

If we perturb the matrices as

\[
\begin{align*}
A_i(t) &= A_i \quad (i = 0, 1, 2), \quad A_3(t) &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\
A_4(t) &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},
\end{align*}
\]
then \( D_t \) is strictly feasible for each \( t > 0 \). In fact, \( X = \begin{pmatrix} 2^{\frac{t^2}{2}} & 0 & 0 \\ 0 & 2t & 1 \\ 0 & 1 & 3 \end{pmatrix} \) are strict feasible points of \( D_t \).

Thus the minimal face of \( D_t \) is \( S^3_t \) for each \( t > 0 \). Since the span of the third blocks of the matrices \( A_1(t), \ldots, A_4(t) \) have the same basis as that of \( A_1, \ldots, A_4 \) for each \( t > 0 \), the rank condition is satisfied.

However the optimal value of \( D_t \) is 2 with \( X = \begin{pmatrix} 2^{\frac{t^2}{2}} & 0 & 0 \\ 0 & 2t & 1 \\ 0 & 1 & 3 \end{pmatrix} \), \( y = (2, \frac{1}{2T} - 1, -\frac{1}{2T}, 1) \), \( Z = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \) being the unique optimal pair for each \( t > 0 \). Thus the optimal value changes discontinuously.

**Example 3.5.** Consider the same SDP as in Example 3.4. If we perturb the matrices as

\[ A_i(t) = A_i (i = 0, 1, 4), A_2(t) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad A_3(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \]

the minimal face of each of the perturbed problems is equal to \( F_{\text{min}} \) in Example 3.3. Here the condition on the face and the rank condition are satisfied for sufficiently small \( t > 0 \). Thus Theorem 3.1 guarantees the continuity of the optimal value. In fact, the optimal value of \( D_t \) is \( \frac{2t+4-4\sqrt{t+1}}{t^2} \) and converges to \( \frac{1}{2} \) as \( t \to 0 \). The optimal pairs are

\[
X = \begin{pmatrix} 0 & 0 & 2^{\frac{t^2}{2t-1}} \\ 0 & 0 & 2^{\frac{t^2}{2t-1}} \\ 0 \end{pmatrix}, \quad y = \begin{pmatrix} -\sqrt{t^2+1}t^2(1-t) \frac{1}{t^2+2} \\ 0 \end{pmatrix}, \quad \beta t^2 (1-t) \begin{pmatrix} 2^{\frac{t^2}{2t-1}} \\ 0 \end{pmatrix}, \quad Z = \begin{pmatrix} \sqrt{t^2+1}t^2(1-t) \frac{1}{t^2+2} \\ 0 \end{pmatrix},
\]

for all \( \beta \) such that the 1st element of \( Z \) is nonnegative.

**Proof of Theorem 3.1.** By the assumptions 1 and 2 in Theorem 3.1, the optimal value of \( D_t \) is equal to

\[
\inf_X \{(Q(t)^TA_0(t)Q(t))_3 \bullet X : (Q(t)^TA_k(t)Q(t))_3 \bullet X = b_k(t) (k \in [m]), X \in S^+_t \} \quad (F(D_t))
\]

and \( F(D_t) \) has a nonempty feasible set for each \( t \in [0, \delta] \). Thus if the continuity of the optimal value of \( F(D_t) \) at \( t = 0 \) is shown, then that of the optimal value of \( D_t \) is also shown. Now the dual of \( F(D_t) \) is

\[
\sup_{y,Z} \left\{ b(t)^T y : \sum_{k \in [m]} y_k (Q(t)^TA_k(t)Q(t))_3 + Z = (Q(t)^T A_0(t) Q(t))_3, Z \in S^+_t \right\} \quad (F(D_t)')
\]

for each \( t \in [0, \delta] \). Then \( F(D_t) \) have the same optimal values as \( F(D_t)' \) because \( F(D_t) \) and \( F(D_t)' \) are strictly feasible. The strict feasibility of \( F(D_t) \) follows from the properties of the facial reduction algorithm. The strict feasibility of \( F(D_t)' \) follows from that of \( F \). In fact, if \( (\tilde{y}, \tilde{Z}) \) is a strictly feasible point in \( F \), then \( (\tilde{y}, (Q(t)^T \tilde{Z} Q(t))_3) \) is also a strictly feasible point in \( F(D_t)' \). Therefore the proof is done by showing Theorem 3.6.

We note that \( A_1, \ldots, A_m \) in the following theorem can be linearly dependent.

**Theorem 3.6.** If both \( P \) and \( D \) are strictly feasible, \( D_t \) is feasible and \( r(A_1(t), \ldots, A_m(t)) = r(A_1, \ldots, A_m) \) for each sufficiently small \( t > 0 \), then the optimal value of \( D_t \) varies continuously at \( t = 0 \).

We will prove Theorem 3.6 in Subsection 3.2.
3. Main results

Remark 3.7. The coefficient matrices $A_1, \ldots, A_m$ in $[\mathcal{P}]$ are usually assumed to be linearly independent in the literature. However the coefficient matrices in $[\mathcal{F}(D)]$ can be linearly dependent even if the initial SDP has linearly independent constraints. In fact, the coefficient matrices of the reduced SDPs are linearly dependent in Examples 3.3, 3.4, and 3.5. Thus we need to consider SDPs with linearly dependent coefficient matrices in Theorem 3.6.

The optimal value of $|[\mathcal{D}]|$ can vary discontinuously when $r(A_1(t), \ldots, A_m(t)) = r(A_1, \ldots, A_m)$ and $|[\mathcal{D}]|$ is merely non-strictly feasible as in Example 3.3 and 3.4. In addition, we present an example and show that feasibility condition on $|[\mathcal{D}]|$ or the rank condition can not be removed from Theorem 3.6.

Example 3.8. In $[\mathcal{P}]$ and $|[\mathcal{D}]|$, we set $b = (2, 2)^T$,

$$A_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \; A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$ 

Then $[\mathcal{P}]$ and $|[\mathcal{D}]|$ are strictly feasible. The optimal value is 0 and the optimal pairs are $X = (\frac{3}{2} 0)$, $y = (\alpha, -\alpha)$, $Z = (0 1)$ for any $\alpha \in \mathbb{R}$. However, if we take $A_2(t) = \left(1 + t \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right)$, then $r(A_1(t), A_2(t)) = r(A_1, A_2) = 1$ but $|[\mathcal{D}]|$ is infeasible. Therefore feasibility of $|[\mathcal{D}]|$ can not be derived from the rank condition and needs to be assumed.

On the other hand if we take $A_2(t) = \left(1 + t \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right)$, then $|[\mathcal{D}]|$ is feasible and $r(A_1(t), A_2(t)) = 2$ for all $t > 0$. The optimal value is 1 and the optimal pair is $X = \begin{pmatrix} 1 \\ \beta \end{pmatrix}$, $y = \begin{pmatrix} 1 + \frac{1}{2t} \\ -\frac{1}{2t} \end{pmatrix}$, $Z = \begin{pmatrix} 0 & 0 \end{pmatrix}$. Thus the optimal value varies discontinuously without the rank condition.

3.2. Proof of Theorem 3.6

First, we recall an existence theorem for optimal solutions to an SDP with a focus on the linear independence of the coefficient matrices.

Theorem 3.9. [29] Theorem 4.1 and Corollary 4.1] Suppose $[\mathcal{P}]$ is strictly feasible and $|[\mathcal{D}]|$ is feasible. Then $|[\mathcal{D}]|$ has a nonempty compact optimal set and the same optimal value as $[\mathcal{P}]$. Also, suppose that $[\mathcal{P}]$ is feasible and $|[\mathcal{D}]|$ is strictly feasible. If the coefficient matrices $A_1, \ldots, A_m$ are linearly independent, then $[\mathcal{P}]$ has a nonempty compact optimal set and the same optimal value as $|[\mathcal{D}]|$.

Remark 3.10. 1. Suppose that $[\mathcal{P}]$ is feasible and $|[\mathcal{D}]|$ is strictly feasible. However we do not assume that the coefficient matrices $A_1, \ldots, A_m$ are linearly independent. Then $|[\mathcal{D}]|$ still has a nonempty compact optimal set and the same optimal value as $[\mathcal{P}]$, and easy arguments show that $[\mathcal{P}]$ has a nonempty optimal set. Here we lost the compactness of the optimal set of $|[\mathcal{D}]|$.

2. The set of optimal solutions $(y, Z)$ of $|[\mathcal{D}]|$ is unbounded when the matrices $A_1, \ldots, A_m$ are linearly dependent. However Lemma 3.13 below tells that the image of optimal solutions under the projection $(y, Z) \mapsto Z$ is bounded if $[\mathcal{P}]$ and $|[\mathcal{D}]|$ are strictly feasible.

We note that we do not assume the linear independence of the coefficient matrices $A_1, \ldots, A_m$ in the following arguments. First we prove that if $|[\mathcal{D}]|$ and $[\mathcal{P}]$ have strictly feasible points, so do $[\mathcal{D}_t]$ and $[\mathcal{P}_t]$ for all sufficiently small $t > 0$. Here we will use the symbol $S(t) = (\text{vec} A_1(t), \ldots, \text{vec} A_m(t)) \in \mathbb{R}^{n^2 \times m}$ and the symbol $(S(t)^T)^\dagger$ for the pseudo-inverse of $S(t)^T$.

Lemma 3.11. Suppose $X_0$ is a strictly feasible point of $[\mathcal{D}]$. If for each $t \in [0, \delta]$, $[\mathcal{D}_t]$ is feasible and $r(A_1(t), \ldots, A_m(t)) = r(A_1, \ldots, A_m)$, then there exist strictly feasible points $X_t$ of $[\mathcal{D}_t]$ for all sufficiently small $t > 0$ such that $X_t \to X_0$ as $t \to 0$.

Proof. We can write the equality constraints of $[\mathcal{D}]$ and $[\mathcal{D}_t]$ by $S(0)^T \text{vec}(X) = b$ and $S(t)^T \text{vec}(X) = b(t)$, respectively. Note that $A_k(0) = A_k$ (for $k \in [m]$), $b = b(0)$. We set

$$\text{vec}(X_0) = (I - (S(0)^T)^\dagger S(0)^T) \text{vec}(X_0) + (S(0)^T)^\dagger b(0)$$

and

$$\text{vec}(X_t) = (I - (S(t)^T)^\dagger S(t)^T) \text{vec}(X_0) + (S(t)^T)^\dagger b(t).$$
Then we can check that $S(0)^T \text{vec}(X_0) = b$ and $S(t)^T \text{vec}(X_t) = b(t)$ by using the fact that $S(t)^T (S(t))^\dagger v = v$ if and only if $v \in \text{Im} S(t)^T$. Since we have rank$(S(t)) = r(A_1(t), \ldots, A_m(t))$ for all $t \geq 0$, it follows from the assumption on the rank and \cite[Theorem 5.2]{25} that $(S(t)^T) \to (S(0)^T)^\dagger$ as $t \to 0$. Therefore $X_t \to X_0$ as $t \to 0$.

\textbf{Remark 3.12.} Unlike (\textit{D}_2), we can prove that (\textit{P}) have strictly feasible points $(y_t, Z_t)$ for all sufficiently small $t \geq 0$ without assuming the rank condition. Let $S^m_{n+}$ be the set of $n \times n$ positive definite matrices. If (\textit{P}) is strictly feasible, there exists $y_0 \in \mathbb{R}^m$ such that $A_0 - \sum_k y_{0,k} A_k \in S^m_{n+}$. Then we have that $Z_t := A_0(t) - \sum_k y_{0,k} A_k(t) \in S^m_{n+}$ for all sufficiently small $t \geq 0$. For each $t > 0$, $(y_0, Z_t)$ is a strictly feasible point of (\textit{P}) and converges to a strict feasible point of (\textit{P}).

Let $\mathcal{U}(t)$ be the set of optimal solutions of (\textit{D}_1) and $\mathcal{V}(t) = \{ Z \in \mathbb{R}^m : \mathcal{V}(t) \text{ is optimal to (\textit{P}) for some } y \in \mathbb{R}^m \}$.

If (\textit{P}) and (\textit{D}_1) have strictly feasible points, then it follows from Remark 3.10 that the sets $\mathcal{U}(t)$ and $\mathcal{V}(t)$ are nonempty. In addition, we can prove that these sets are uniformly bounded.

\textbf{Lemma 3.13.} Suppose that (\textit{P}) is strictly feasible. If there exist strictly feasible points $X_t$ of (\textit{D}_1) for all sufficiently small $t \geq 0$ such that $X_t \to X_0$ as $t \to 0$, then both sets $\mathcal{U}(t)$ and $\mathcal{V}(t)$ are nonempty and uniformly bounded; i.e., there exist $\delta > 0$ and compact sets $C_1, C_2$ such that

$$\mathcal{U}(t) \subset C_1, \mathcal{V}(t) \subset C_2 \quad (0 \leq t \leq \delta).$$

\textbf{Proof.} Since (\textit{D}_1) and (\textit{P}) have strictly feasible points, Remark 3.10 ensures that they have the same optimal value and that $\mathcal{U}(t)$ and $\mathcal{V}(t)$ are nonempty for all sufficiently small $t \geq 0$. Let $X$ and $(y, Z)$ be arbitrary optimal solutions to (\textit{D}_1) and (\textit{P}) respectively. For a strictly feasible point $(y_0, Z_0)$ of (\textit{D}_1), we set $y_t = y_0, Z_t = A_0(t) - \sum_k y_{0,k} A_k(t)$. Then $(y_t, Z_t)$ is a strictly feasible point of (\textit{P}) for each small $t \geq 0$ as explained in Remark 3.12. Since $X_t$ and $(y_t, Z_t)$ are feasible points, we have

$$A_k(t) \cdot (X - X_t) = 0, \quad \sum_{k \in [m]} (y_k - y_{t,k}) A_k(t) + Z - Z_t = 0.$$ 

Then it follows that $(X - X_t) \cdot (Z - Z_t) = 0$ and hence that $X \cdot Z_t + X_t \cdot Z = X_t \cdot Z_t$. Moreover, positive semidefiniteness of $X_t$ and $Z$ guarantees that $X \cdot Z_t \leq X_t \cdot Z_t$. Thus, by positive definiteness of $Z_t$, there exists $\epsilon > 0$ such that for all sufficiently small $t > 0$, we have

$$\|X\| \leq \frac{X_t \cdot Z_t}{\lambda_{\min}(Z_t)} \leq \frac{X_0 \cdot Z_0 + \epsilon}{\lambda_{\min}(Z_0) - \epsilon},$$

where $\lambda_{\min}(M)$ is the smallest eigenvalue of a matrix $M$. Therefore $\mathcal{U}(t)$ is uniformly bounded for all sufficiently small $t > 0$. Similar arguments are applied to $\mathcal{V}(t)$. \hfill $\square$

The following lemma is well known and the proof is omitted.

\textbf{Lemma 3.14.} Suppose that (\textit{D}) has the same optimal value as (\textit{P}) and both of (\textit{D}) and (\textit{P}) have optimal solutions. We define the function $L : \mathbb{S}^n \times \mathbb{R}^m \to \mathbb{R}$ as follows:

$$L(X, y) = A_0 \cdot X + \sum_{k \in [m]} y_k (b_k - A_k \cdot X).$$

Then $\tilde{X}$ and $(\tilde{y}, A_0 - \sum_k \tilde{y}_k A_k)$ are optimal solutions of (\textit{D}) and (\textit{P}) respectively if and only if $(\tilde{X}, \tilde{y}) \in \mathbb{S}^n_+ \times \mathbb{R}^m$ satisfies

$$L(\tilde{X}, y) \leq L(\tilde{X}, \tilde{y}) \leq L(X, \tilde{y}), \quad \forall (X, y) \in \mathbb{S}^n_+ \times \mathbb{R}^m.$$
3. Main results

Lemma 3.16 plays an essential role in the proof of Theorem 3.6 and ensures a kind of continuity of the set of optimal solutions. Before we prove it, we present an easy lemma.

**Lemma 3.15.** Let $S = (\text{vec}(A_1) \ldots \text{vec}(A_m)) \in \mathbb{R}^{n \times m}$. If $(P)$ has an optimal solution $(\bar{y}, \bar{Z})$, then $(y, \bar{Z})$ is also optimal to $(P)$ where $y = S^\dagger(\text{vec}(A_0) - \text{vec}(\bar{Z}))$.

**Proof.** By feasibility of $(\bar{y}, \bar{Z})$, we have $S \bar{y} = \text{vec}(A_0) - \text{vec}(\bar{Z})$. Since $SS^\dagger v = v$ if and only if $v \in \text{Im} S$, we see that $S y = \text{vec}(A_0) - \text{vec}(\bar{Z})$. Then we obtain $y, \bar{Z}$ are strictly feasible and hence $(y, \bar{Z})$ is optimal.

In the following, $B$ denotes the unit ball in $S^n$. We define, for $X \in S^n$ and $C \subset S^n$,

$$d(X, C) = \inf\{\|X - Y\| : Y \in C\}.$$

**Lemma 3.16.** Suppose that $(P)$ is strictly feasible. If there exist strictly feasible points $X_t$ of $(P)$ for all sufficiently small $t \geq 0$ such that $X_t \to X_0$ as $t \to 0$, then for any $\epsilon > 0$, there exists $\eta > 0$ such that

$$U(t) \subset U(0) + \epsilon B, \ V(t) \subset V(0) + \epsilon B \quad (0 \leq t \leq \eta).$$

**Proof.** By Remark 3.10 $(P_t)$ and $(P)$ have optimal solutions and the same optimal value. Suppose that the conclusion is false. Then there exist $\epsilon > 0, \{t_j\}$ and $X(t_j) \in U(t), Z(t_j) \in V(t)$ such that $t_j \to 0$ and

$$d(X(t_j), U(0)) \geq \epsilon, \ d(Z(t_j), V(0)) \geq \epsilon,$$

for all $j$. Recall $S(t) = (\text{vec}(A_1(t)) \ldots \text{vec}(A_m(t)))$ and let $y(t_j) = S(t_j)^\dagger(\text{vec}(A_0) - \text{vec}(Z(t_j)))$. Then Lemma 3.15 implies that $(y(t_j), Z(t_j))$ is optimal for $(P_t)$ for each $j$. Let $L(X, y, t) = A_0(t) \cdot X + \sum_{k \in [m]} y_k(b_k(t) - A_k(t) \cdot X)$. Note that $L(X, y, 0)$ is equal to $L(X, y)$ defined in Lemma 3.14. By Lemma 3.14 we have

$$L(X(t_j), y(t_j), t_j) \leq L(X(t_j), y(t_j), t_j) \leq L(X, y(t_j), t_j), \ \forall(X, y) \in S^n \times \mathbb{R}^m.$$

Since Lemma 3.13 ensures that $\{(X(t), Z(t))\}$ is uniformly bounded, we may assume that

$$(X(t_j), y(t_j), Z(t_j)) \to (\bar{X}, \bar{y}, \bar{Z})$$

as $j \to \infty$ for some $(\bar{X}, \bar{y}, \bar{Z})$. Thus we have

$$L(\bar{X}, y, 0) \leq L(\bar{X}, \bar{y}, 0) \leq L(X, \bar{y}, 0), \ \forall(X, y) \in S^n \times \mathbb{R}^m.$$

By applying Lemma 3.14 again, $\bar{X}$ and $(\bar{y}, \bar{Z})$ are optimal for $(P)$ and $(D)$ respectively. This contradicts the inequalities 16.

**Proof of Theorem 3.6.** By Lemma 3.11 and 3.16 we have that for any $\epsilon > 0$ and $X(t) \in U(t)$, there exist $\eta > 0$ and $\bar{X}^t \in U(0)$ such that for $t \in [0, \eta],$

$$|A_0(t) \cdot X(t) - A_0 \cdot \bar{X}^t| \leq k_1\|X(t) - \bar{X}^t\| + k_2\|A_0(t) - A_0(0)\| < \epsilon$$

for some $k_1, k_2 > 0$. This completes the proof of Theorem 3.6.

**Corollary 3.17.** If both $(P)$ and $(D)$ are strictly feasible and $A_1, \ldots, A_m$ are linearly independent, the optimal value of $(D)$ varies continuously at $t = 0$.

**Proof.** By the strict feasibility and the linear independence condition, $(P)$ and $(D)$ are feasible and the rank condition is satisfied for all sufficiently small $t$.

\[ \square \]
4. Behavior of a minimal face under perturbations

In this section, we focus on the behavior of a minimal face under perturbations. In particular, we determine the kinds of perturbations that make the minimal face invariant. Then we deal with the perturbations for (7) which are obtained by the matrix-wise perturbations for (5). If we first perturb the coefficient matrices in the dynamical system (5) and then construct the associated SDP, the obtained SDP has a special structure in the perturbation. It is numerically confirmed that such a perturbation changes the optimal value of the SDP problem continuously as in Example 4.9. We provide a result on what kinds of matrix-wise perturbations for (5) make the minimal face invariant, which is presented in Example 4.9.

If only $A_0$ in SDP (2) is perturbed, the behavior of the minimal face has been already investigated by Cheung and Wolkowicz in [7]. However, if the coefficient matrices $A_1, \ldots, A_m$ are also perturbed, the behavior of the minimal face becomes far more complicated. In this section, we focus on the similar types of perturbations in the numerical results presented in the introduction. In addition, as explained in Example 4.9, the matrix-wise perturbations in (5) correspond to perturbations on only the coefficient matrices in (7). Therefore we slightly simplify the situations and consider the following perturbed problem:

$$\inf \left\{ A_0 \bullet X : (A_k + E_k(t)) \bullet X = b_k \ (k \in [m]), \ X \in S^+_r \right\}, \tag{D_t}$$

where $E_k(t) = A_k(t) - A_k$ for all $k \in [m]$. We note $E_k(t) \to 0$ as $t \to 0$ since we assume $A_k(t)$ are continuous at 0 and $A_k(0) = A_k$. Throughout this section, we assume the following conditions:

**Condition 2.**

1. (D) is feasible and (Pa) is strictly feasible;
2. $A_1, \ldots, A_m$ are linearly independent;
3. (D_t) is feasible for each sufficiently small $t > 0$.

**Remark 4.1.** $3$ of Condition 2 is not implied by $1$ and $2$. For instance, we set $b = (2, 0, 0)^T$, $A_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, $A_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $E_3(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

and $E_1(t) = E_2(t) = O_{3 \times 3}$ in (D), (Pa) and (D_t). Then $1$ of Condition 2 are satisfied, but $3$ is not.

We say that $\{D_{t}\}_{t \geq 0}$ satisfies the rank condition if there exist an associated nonsingular matrix $Q$ to the minimal face of (D) and $\delta > 0$ such that for all $t \in [0, \delta]$,

$$r \left( (Q^T (A_1+E_1(t)) Q)_3, \ldots, (Q^T (A_m+E_m(t)) Q)_3 \right) = r \left( (Q^T A_1 Q)_3, \ldots, (Q^T A_m Q)_3 \right),$$

where the submatrix $M_3$ for $M \in S^n$ is determined by the minimal face of (D) as in (15). We start with the following lemma.

**Lemma 4.2.** Let $F_{min}$ and $F^t_{min}$ be the minimal faces of (D) and (D_t) respectively. Suppose $\{D_{t}\}_{t \geq 0}$ satisfies the rank condition. If there exists $\delta > 0$ such that $F^t_{min} \subset F_{min}$ for all $t \in [0, \delta]$, we have $F^\delta_{min} = F_{min}$ for all sufficiently small $t > 0$.

**Proof.** By Lemma 2.2, the reduced problem $F(D)$ of (D) has a strictly feasible point which solves $(Q^T A_k Q)_3 \bullet X = b_k \ (k \in [m]), \ X \in S^+_r$ for some $r > 0$, where $Q$ is an associated nonsingular matrix to the minimal face of (D). For each $t \in [0, \delta]$, feasibility of (D) and $F^t_{min} \subset F_{min}$ imply that there exists $X \in F_{min}$ such that $(A_k + E_k(t)) \bullet X = b_k \ (k \in [m])$. It follows from the definition of $F_{min}$ that

$$(Q^T (A_k+E_k(t)) Q)_3 \bullet X = b_k \ (k \in [m]), \ X \in S^+_r$$
is feasible. Consider the following problem obtained by perturbing \( F(D) \)

\[
\inf_X \left\{ (Q^T A_0 Q)_3 \cdot X : (Q^T (A_k + E_k(t)) Q)_3 \cdot X = b_k \ (k \in [m]), \ X \in S^n_+ \right\} . \tag{17}
\]

Here \( F(D) \) has a strictly feasible point, \( (17) \) is feasible and the rank condition is satisfied. Thus Theorem 3.11 implies that for each sufficiently small \( t > 0 \), \( (17) \) has a strictly feasible point. It means that \( \{X \in \text{int} F_{\text{min}} : (A_k + E_k(t)) \cdot X = b_k \ (k \in [m])\} \neq \emptyset \) for each sufficiently small \( t > 0 \). Since \( F_{\text{min}} \) is a face of \( S^n_+ \) containing \( F_{\text{min}}^t \), we have \( F_{\text{min}} = F_{\text{min}}^t \) by Lemma 2.2.

**Example 4.3.** Lemma 4.2 does not hold without the assumption \( F_{\text{min}}^t \subset F_{\text{min}} \). The perturbation in Example 4.3 is of the same type as this section is considering. Condition 2 and the rank condition are satisfied, but the minimal faces \( F_{\text{min}}^t \) of \( (D_t) \) are not equal to \( F_{\text{min}} \). Here \( F_{\text{min}}^t \) are not included in \( F_{\text{min}} \). On the other hand, we note that the perturbed SDP \( (P1) \) in Section II satisfies Condition 2 and \( F_{\text{min}}^t \subset F_{\text{min}} \) for each sufficiently small \( t > 0 \). But its minimal face is smaller than that of \( (D) \) as in Appendix C. There the rank condition is not satisfied.

**Proposition 4.4.** For a facial reduction sequence \( \{(\hat{y}^i, \hat{U}^i, \hat{V}^i)\}_{i=1}^s \) of \( (D) \), let the minimal face of \( (D) \) be \( F_{\text{min}} \) and \( \hat{K} = \{k : \hat{y}^i_k = 0 \ (\forall i = 1, \ldots, s)\} \). Suppose that \( \{(D_t)\}_{t \geq 0} \) satisfies the rank condition and \( E_k(t) = O_{n \times n} \ (k \notin \hat{K}) \). Then the minimal faces of \( (D_t) \) are equal to \( F_{\text{min}} \) for all sufficiently small \( t > 0 \).

**Proof.** As \( E_k(t) = O_{n \times n} \) for all \( k \notin \hat{K} \), it is obvious that \( \{(\hat{y}^i, \hat{U}^i, \hat{V}^i)\}_{i=1}^s \) is a reducing certificate sequence up to the \( s \)-th loop of the facial reduction for \( (D_t) \) and that they generate the same faces. It is summarized as

\[
\begin{align*}
(D_t) & \Rightarrow S^n_+((\hat{y}^1, \hat{U}^1, \hat{V}^1)) \Rightarrow F_1 \Rightarrow (\hat{y}^2, \hat{U}^2, \hat{V}^2) \Rightarrow F_2 \Rightarrow (\hat{y}^3, \hat{U}^3, \hat{V}^3) \Rightarrow \cdots \Rightarrow (\hat{y}^r, \hat{U}^r, \hat{V}^r) \Rightarrow F_s = F_{\text{min}}. 
\end{align*}
\]

Thus the minimal faces of \( (D_t) \) are contained in \( F_s \). In addition, since \( \{(D_t)\}_{t \geq 0} \) satisfies the rank condition, it follows from Lemma 4.2 that \( F_s \) is the minimal face of \( (D_t) \) for each sufficiently small \( t > 0 \).

As a corollary, we obtain a simple geometric condition, which is easier to be verified.

**Corollary 4.5.** For a facial reduction sequence \( \{(\hat{y}^i, \hat{U}^i, \hat{V}^i)\}_{i=1}^s \) of \( (D) \), let the minimal face of \( (D) \) be \( F_{\text{min}} \) and \( \hat{K} = \{k : \hat{y}^i_k = 0 \ (\forall i = 1, \ldots, s)\} \). If \( \{(D_t)\}_{t \geq 0} \) satisfies the rank condition and \( E_k(t) = O_{n \times n} \ (k \notin \hat{K}) \), then the minimal faces of \( (D_t) \) are equal to \( F_{\text{min}} \) for all sufficiently small \( t > 0 \).

**Proof.** Suppose we have

\[
F_{\text{min}} = \left\{ Q \left( O_{(n-r) \times (n-r)} \ 0_{r \times (n-r)} \right) \ 0_{(n-r) \times r} \left( X \right) Q^T : X \in S^n_+ \right\} ,
\]

where \( Q \in \mathbb{R}^{n \times n} \) is a nonsingular matrix. Then \( E_k(t) \in F_{\text{min}}^t \) means that

\[
E_k(t) = \left\{ Q^{-T} \begin{pmatrix} Y_1 & Y_2 \end{pmatrix} \left( Y_1 \ 0_{r \times (n-r)} \right) Q^{-1} : Y_1 \in \mathbb{R}^{n-r}, Y_2 \in \mathbb{R}^{r \times (n-r)} \right\}.
\]

Thus we have that \( (Q^T (A_k + E_k(t)) Q)_3 = (Q^T A_k Q)_3 + (Q^T E_k(t) Q)_3 = (Q^T A_k Q)_3 \) and that the rank condition is satisfied. Therefore we can apply Proposition 4.4.

**Example 4.6.** Let \((y, U, V)\) be as in (11) of Example 2.3. Then \( \hat{K} = \{2, 3, 5, 6\} \). By Corollary 4.5, if \( \{E_k(t)\} \) has the following form:

\[
E_1(t) = E_4(t) = (O_{6 \times 6}, O_{2 \times 2}), \ E_2(t) = \begin{pmatrix} * & O_{5 \times 5} & * \\ * & * & 0 \end{pmatrix}
\]

for \( k = 2, 3, 5, 6 \), then the minimal face of \( (D) \) does not change under the perturbation with \( \{E_k(t)\} \). Here the symbol * indicates that we can choose an arbitrary real number and vector for each *.
Next, we will use positive eigenvectors of reducing certificates to give conditions for minimal face to be invariant.

**Proposition 4.7.** Let \( \{(\hat{y}^i, \hat{U}^i, \hat{V}^i)\}_{i=1}^s \) be a facial reduction sequence of \([D]\), \( F_0 = S_n^+ \) and \( F_1, \ldots, F_s \) be the generated faces. In addition, let \( L_i = \text{Span}\{qq^T : q \text{ is an eigenvector of } \hat{U}^i \text{ associated with a positive eigenvalue}\} \).

Suppose that \( \{[D_t]\}_{t \geq 0} \) satisfies the rank condition and for each \( i = 1, \ldots, s \),

\[
\sum_{k \in [m]} \hat{y}^i_k E_k(t) + v_i(t) \in L_i
\]

for some \( v_i(t) \in F_{i-1}^\perp \) with \( v_i(t) \to O_{n \times n} \) as \( t \to 0 \). Then \([D_t]\) have the same minimal face as \([D]\) for all sufficiently small \( t > 0 \).

**Proof.** By the definition of the facial reduction sequence, we have that \( \hat{U}^i \in S_n^+, \hat{V}^i \in F_{i-1}^\perp \) and that \(- \sum_k \hat{y}^i_k A_k = \hat{U}^i + \hat{V}^i \). Let \( \{q_i\} \) be the set of the eigenvectors of \( \hat{U}^i \) which are associated with positive eigenvalues, orthogonal to each other and \( ||q_i|| = 1 \). Then every matrix in \( L_i \) can be written as a linear combination of \( qq^T \). By the assumption, for each \( i = 1, \ldots, s \), there exist \( \alpha_i(t) \in \mathbb{R} \) and \( v(t) \in F_{i-1}^\perp \) such that \(- \sum_k \hat{y}^i_k E_k(t) = \sum_i \alpha_i(t) qq^T + v(t) \). Since \( \sum_k \hat{y}^i_k E_k(t) \to O_{n \times n}, v(t) \to O_{n \times n} \) as \( t \to 0 \) and \( \{qq^T\} \) is linearly independent, we have \( \alpha_i(t) \to 0 \) for each \( i \). We set

\[
U^i = \hat{U}^i + \sum \alpha_i(t) qq^T, \quad V^i = \hat{V}^i + v(t)
\]

for \( i = 1, \ldots, s \). Then \( V^i \in F_{i-1}^\perp \). Since \( \hat{U}^i \) can be written as \( \sum \lambda_i qq^T \) where \( \lambda_i \) is the positive eigenvalue of \( \hat{U}^i \) corresponding to \( q_i \), we see that \( U^i \in S_n^+ \) for all sufficiently small \( t > 0 \). Thus we have

\[
- \sum_k \hat{y}^i_k (A_k + E_k(t)) = \hat{U}^i + \hat{V}^i + \sum \lambda_i(t) qq^T + v(t) = U^i + V^i \in S_n^+ + F_{i-1}^\perp.
\]

Since \( \hat{U}^i + \hat{V}^i \notin F_{i-1}^\perp \) by the definition of the facial reduction sequence and \( F_{i-1}^\perp \) is closed, we also have \( U^i + V^i \notin F_{i-1}^\perp \) for all sufficiently small \( t > 0 \). Thus we see that \( U^i + V^i \in (S_n^+ + F_{i-1}^\perp) \setminus F_{i-1}^\perp \) and hence that \( \{(\hat{y}^i, U^i, V^i)\}_{i=1}^s \) is a facial reduction sequence of \([D]\) up to \( s \). In addition, we have

\[
F_{i-1} \cap \{U^i\}^\perp = F_{i-1} \cap \{\hat{U}^i + \sum \lambda_i(t) qq^T\}^\perp
\]

\[
= F_{i-1} \cap \{\sum (\lambda_i + \alpha_i(t)) qq^T\}^\perp = F_{i-1} \cap \{\hat{U}^i\}^\perp = F_i
\]

for \( i = 1, \ldots, s \). This shows that \( \{(\hat{y}^i, U^i, V^i)\}_{i=1}^s \) also generates faces \( F_1, \ldots, F_s \) and that \( F_s \) contains the minimal face of \([D]\) for each sufficiently small \( t > 0 \). In addition, since \( \{[D_t]\}_{t \geq 0} \) satisfies the rank condition, Lemma 4.2 implies that the minimal face of \([D]\) is equal to \( F_s \) for each sufficiently small \( t > 0 \).

**Remark 4.8.** In particular, the inclusion in Proposition 4.7 holds if we have

\[
- \sum_{k \in [m]} \hat{y}^i_k E_k(t) \in \alpha_i(t) \hat{U}^i + F_{i-1}^\perp,
\]

with \( \alpha_i(t) \to 0 \) as \( t \to 0 \) for each \( i = 1, \ldots, s \).
Example 4.9. Consider the singular SDP (7). Since (5) is stabilizable, i.e., for any complex number \( \lambda \) with the nonnegative real part, \( \text{rank}(A - \lambda I_2, B_2) = 2 \) and (7) is strictly feasible. See [36] for the detail. We show that matrix-wise perturbations make the minimal face of the dual problem (8) of (7) invariant or full-dimensional, i.e., \( S_+^6 \times S_+^2 \).

Let \( A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \), \( B_2 = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \), \( C_1 = \begin{pmatrix} c_{11} & c_{12} \end{pmatrix} \), \( D_{12} = \begin{pmatrix} d_{11} \\ d_{21} \end{pmatrix} \), and let \( B_1 \) and \( D_{11} \) be the same matrices as in (6). Then the first constraint in (7) means that

\[
\begin{pmatrix}
-2a_{21}y_1 - 2a_{12}y_2 - 2b_1y_4 \\
-a_{21}y_1 - (a_{11} + a_{22})y_2 - a_{12}y_3 - b_2y_4 - b_1y_5 \\
-c_{11}y_1 - c_{12}y_2 - d_1y_4 \\
-c_{21}y_1 - c_{22}y_2 - d_2y_4 \\
1 \\
1
\end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}
\]

is contained in \( S_+^6 \). The related part with \( a_{11} \) in the above matrix can be extracted as

\[
a_{11} \begin{pmatrix} -2y_1 - y_2 & 0 & 0 & 0 & 0 & 0 \\ -y_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = a_{11}(y_1E_{1,1} + y_2E_{2,1}),
\]

where

\[
E_{1,1} = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}
\]

and

\[
E_{2,1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.
\]

Then the perturbation on \( a_{11} \) corresponds to perturbing matrices \( E_1(t) = (tE_{1,1}, O_{2 \times 2}) \), \( E_2(t) = (tE_{2,1}, O_{2 \times 2}) \) and \( E_k(t) = (O_{6 \times 6}, O_{2 \times 2}) \) \((k = 3, \ldots, 6)\). Consider problem (P_k) perturbed with \( \{E_k(t)\} \). Then the facial reduction sequence for (P_k) is \{\(y, U, V\)\} as given in (11). Example 2.8. Let \( e_i \in \mathbb{R}^6 \) and \( f_i \in \mathbb{R}^2 \) be the unit vectors whose ith entry is 1 and others are zero. Then the positive eigenvalues of \( U \) are 2, 1 and the associated eigenvectors are \((e_1, 0^T_2, (0^T_6, f_1) \} \) respectively. Here \( 0_p \) is the \( p \)-dimensional zero vector for a given positive integer \( p \). Since we have that

\[-(1 \cdot E_1(t) + 0 \cdot E_2(t)) \in \text{Span}\{e_1e_1^T, O_{2 \times 2}, (O_{6 \times 6}, O_{2 \times 2})\}\]

and that \( \{D_i\}_{i \geq 0} \) satisfies the rank condition, Proposition 4.7 implies that this perturbation does not change the minimal face of \( \mathbb{S} \).

On the other hand, the related part with \( a_{21} \) is

\[
a_{21} \begin{pmatrix} 0 & -y_1 & 0 & 0 & 0 & 0 \\ -y_1 & -2y_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = a_{21}(y_1E_{1,1} + y_2E_{2,1}),
\]

where

\[
E_{1,1} = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}
\]

and

\[
E_{2,1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.
\]

It is easily verified that the perturbation with respect to \( a_{21} \), i.e., \( E_1(t) = (tE_{1,1}, O_{2 \times 2}) \), \( E_2(t) = (tE_{2,1}, O_{2 \times 2}) \) and \( E_k(t) = (O_{6 \times 6}, O_{2 \times 2}) \) \((k = 3, \ldots, 6)\), makes the discriminant system of the first loop of the facial reduction infeasible. Thus the perturbed problem (P_k) with this \( \{E_k(t)\} \) is strictly feasible for any sufficiently small \( t > 0 \). Similar arguments provide the results in Table 3. “Invariant” in Table 3 means that the corresponding perturbation makes the minimal face of \( \mathbb{S} \) invariant. “Full-dimensional” in Table 3 means that the corresponding perturbation makes the minimal face of \( \mathbb{S} \) to be \( S_+^6 \times S_+^2 \), which implies that the perturbed problem is strictly feasible. Here we observe that if we perturb matrices \( A, B_2, C_1 \) and \( D_{12} \) in the structured form, the minimal face can still be different, but can not be smaller.
4. Behavior of a minimal face under perturbations

Table 3: Behavior of the minimal face under the matrix-wise perturbations

| Pert. | Face     | Pert. | Face     | Pert. | Face     |
|-------|----------|-------|----------|-------|----------|
| $a_{11}$ | Invariant | $c_{11}$ | Full-dimensional | $b_1$ | Invariant |
| $a_{12}$ | Invariant | $c_{12}$ | Invariant | $b_2$ | Full-dimensional |
| $a_{21}$ | Full-dimensional | $c_{21}$ | Full-dimensional | $d_1$ | Full-dimensional |
| $a_{22}$ | Invariant | $c_{22}$ | Invariant | $d_2$ | Full-dimensional |

Figure 1: The changes of the optimal values via the matrix-wise perturbation

Figure 1 displays the differences between optimal values of the original SDP (7) and SDPs obtained by matrix-wise perturbations. The circles in Figure 1 stand for the differences between the optimal values of (7) and SDP obtained by perturbing $a_{11}$ with $a_{11} + \epsilon$ for $\epsilon = \pm j \times 5.0 \times 10^{-3}$ ($j = 1, \ldots, 10$), while the asterisks in Figure 1 stand for the differences between the optimal values of (7) and SDP obtained by perturbing $a_{21}$ in a similar manner to $a_{11}$. All SDPs are solved by SDPA-GMP with the same parameters in Table 1 and the stopping tolerance $\delta = 1.0 \times 10^{-50}$.

We see from Figure 1 that (i) the optimal values of SDPs obtained by perturbing $a_{11}$ are the same as the that of the original, and (ii) the optimal values of SDPs obtained by perturbing $a_{21}$ change continuously although the dual minimal face changes into $S_+^6 \times S_+^2$.

Remark 4.10. 1. We discuss non-strict feasibility under perturbations by using a result in [36]. They obtained a necessary and sufficient condition for the dual problem of SDP associated with $H_\infty$ state feedback control problem to be non-strictly feasible. To introduce the condition, we consider the dynamical system (1). Then the dual problem of the SDP obtained from $H_\infty$ state feedback control problem for (1) is non-strictly feasible if and only if $D_{12}$ is not of full column rank or there exists $\lambda \in \mathbb{C}$ such that

$$\Re(\lambda) \leq 0 \text{ and } \text{rank} \begin{pmatrix} A - \lambda I_n & B_2 \\ C_1 & D_{12} \end{pmatrix} < n + m_2,$$

where $\Re(\lambda)$ is the real part of $\lambda \in \mathbb{C}$ and $I_n$ is the $n \times n$ identity matrix. Let us consider the above condition for the control problem (5); see (6) for the entries of $A, B_2, C_1$ and $D_{12}$. Recall that the associated primal problem is (7) and that the dual problem is (8). For this case, we see that $\lambda = -1$ satisfies (18) and then it follows from the above condition that (8) is non-strictly feasible. Moreover, let us perturb $a_{11}$ to $a_{11} + \epsilon$, where $a_{11}$ is the $(1,1)$st entry of $A$. Then we
5. Conclusions

can see that the following linear system with $\lambda = -1 - \epsilon$ has a nonzero solution $(u_1, u_2, v)$:

\[
\begin{pmatrix}
-1 - \epsilon - \lambda & -1 & 0 \\
1 & -\lambda & 1 \\
2 & -1 & 2 \\
-1 & 2 & -1
\end{pmatrix}
\begin{pmatrix}
u_1 \\
u_2 \\
v
\end{pmatrix} = 0.
\]

This means that (18) also holds in the perturbed dynamical system, and thus the dual of the corresponding SDP is non-strictly feasible. As we can expect from Table 3 we see that (18) holds for the problem obtained by perturbing (8) on each of $a_{11}, a_{12}, a_{22}, b_1, c_{12}$ and $c_{22}$.

2. The optimal value of $\{D_t\}$ changes continuously at $t = 0$ due to Theorem 3.1 in the case where the perturbations preserve the minimal face, i.e., perturbations with $a_{11}, a_{12}, a_{22}, b_1, c_{12}$ and $c_{22}$. In the other cases, perturbations change the minimal face into $S^6_+ \times S^2_+$, i.e., perturbations with $a_{21}, b_2, c_{11}, c_{21}, d_1$ and $d_2$. Discontinuity of the optimal value might occur by perturbations which change the minimal face into the whole positive semidefinite cone as in Example 3.4. However, we have numerically confirmed that the optimal value of $\{D_t\}$ also varies continuously in the latter cases. See Figure 1 for $a_{11}$ and $a_{21}$. It is a future study to find other structures in $\{D_t\}$ than those investigated in this paper which ensure the continuity of the optimal value under any matrix-wise perturbations.

5. Conclusions

We begin this study with the analysis of the numerical results in Table 2. It is known that the strict feasibility of either the primal or the dual SDP is sufficient for the optimal value to be continuous if one perturbs only data on the right hand side. However, Table 2 shows that if one perturbs the coefficient matrices on the left hand side, the optimal value can be discontinuous. Table 2 also provides a guideline for solving singular SDPs. In particular, when we use SDPA-GMP to solve singular SDPs, it is important not only to use the floating point computation with longer significant digits, but also to choose the appropriate tolerance for the stopping criteria of computation in this case.

We first provide the result on the continuity of the optimal values of singular SDPs in Theorem 3.1. It is proven that the continuous behavior of the minimal face and the rank condition ensure the continuity of optimal value. A detailed analysis on numerical results are given in Table 2 based on Theorem 3.1 and Example 2.3. Furthermore, we use the reducing certificates to give sufficient conditions for a perturbation to preserve the minimal face in Section 4. It should be noted that the reducing certificates are obtained without solving SDPs for several concrete problems, such as $H_\infty$ control problems. Then the behavior of the minimal face under the matrix-wise perturbations of the dynamical system (5) is completely determined as in Example 4.9. In the future work, we could use these structures to obtain sharper criteria for perturbations to make minimal faces invariant. In addition, it may be interesting to try to find combinatorial structures in elements of matrices which represent the perturbations that preserve minimal faces.

**A. On the optimal value of SDP (7)**

Let $\gamma^* = -\sqrt{5}$. Define the sequence $\{(\hat{y}_{1n}, \ldots, \hat{y}_{6n})\}_{n=1}^\infty$ by $\hat{y}_{1n} = n$, $\hat{y}_{2n} = \hat{y}_{3n} = 0$, $\hat{y}_{4n} = -n$, $\hat{y}_{5n} = \gamma^*/4$ and $\hat{y}_{6n} = -\gamma^* + 1/n$ for all $n \geq 1$. It is not difficult to prove that the sequence consists of feasible solutions of (7) with the objective value $\gamma^* - \frac{4}{n}$ and that the objective value converges to
A. On the optimal value of SDP (7)

\[ \gamma^* = -\sqrt{5}. \] In fact, we have

\[
- \sum_{k=1}^{6} y_k^* A_{k,1} = \begin{pmatrix}
2n & 0 & 0 & 0 & 1 & 1 \\
0 & -\gamma^*/2 & -\gamma^*/2 & \gamma^*/4 & 1 & 0 \\
0 & -\gamma^*/2 & -\gamma^* + 1/n & 0 & 1 & 0 \\
0 & \gamma^*/4 & 0 & -\gamma^* + 1/n & 1 & 0 \\
1 & 1 & 1 & 1 & -\gamma^* + 1/n & 0 \\
0 & 0 & 0 & 0 & 0 & -\gamma^* + 1/n
\end{pmatrix}
\]

and

\[
- \sum_{k=1}^{6} y_k^* A_{k,2} = \begin{pmatrix}
n & 0 \\
0 & 0
\end{pmatrix}.
\]

The second matrix is clearly positive semidefinite. To see the positive semidefiniteness of the first matrix, we apply Schur complement to the matrix. Then we obtain

\[
\begin{pmatrix}
-\gamma^*/2 & -\gamma^*/2 & \gamma^*/4 & 1 & 0 \\
-\gamma^*/2 & -\gamma^* & 0 & 1 & 0 \\
\gamma^*/4 & 0 & -\gamma^* & 1 & 0 \\
1 & 1 & 1 & -\gamma^* & 0 \\
0 & 0 & 0 & 0 & -\gamma^*
\end{pmatrix} + \frac{1}{2n} \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix} \]

\begin{equation}
(19)
\end{equation}

The first and second matrices in above are positive semidefinite, respectively. In fact, by using Schur complement repeatedly, we have

\[
\begin{pmatrix}
-\gamma^*/2 & -\gamma^*/2 & \gamma^*/4 & 1 & 0 \\
-\gamma^*/2 & -\gamma^* & 0 & 1 & 0 \\
\gamma^*/4 & 0 & -\gamma^* & 1 & 0 \\
1 & 1 & 1 & -\gamma^* & 0 \\
0 & 0 & 0 & 0 & -\gamma^*
\end{pmatrix} \in S^4_+ \iff \begin{pmatrix}
-\gamma^*/2 & -\gamma^*/2 & \gamma^*/4 & 1 & 0 \\
-\gamma^*/2 & -\gamma^* & 0 & 1 & 0 \\
\gamma^*/4 & 0 & -\gamma^* & 1 & 0 \\
1 & 1 & 1 & -\gamma^* & 0 \\
0 & 0 & 0 & 0 & -\gamma^*
\end{pmatrix} - \frac{1}{\gamma^*} \begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix}
\]

\[
\iff \frac{\sqrt{5}}{10} \begin{pmatrix}
3 & 3 & -9/2 \\
3 & 8 & -2 \\
-9/2 & -2 & 8
\end{pmatrix} \in S^3_+.
\]

Since the eigenvalues of the last matrix are 0 and \((19 \pm \sqrt{46})/2\), the last matrix is positive semidefinite. Thus, the first matrix in (19) is positive semidefinite. The second matrix in (19) is clearly positive semidefinite. These imply that the matrix (19) is positive semidefinite. Thus the sequence is feasible in (7) and the optimal value of (7) is greater than or equal to \(\gamma^* = -\sqrt{5}\).

Next, we prove the optimality. To this end, we check that the dual of (7) has a feasible solution with the objective value \(-\sqrt{5}\). The dual is given in (8). It is easy to see that the following matrix is feasible in (8) with the objective value \(-\sqrt{5}\):

\[
(z_{ij})_{1 \leq i,j \leq 6} = \begin{pmatrix}
0 & 1.6 \\
0 & -0.4 \\
0 & 0.8 \\
0 & -4\delta^* \\
0 & 0 \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{pmatrix}
= \frac{1}{10} \begin{pmatrix}
-4 & 0 \\
1 & 0 \\
0 & 0 \\
-2 & 0 \\
0 & 0 \\
0 & 0
\end{pmatrix} \begin{pmatrix}
-4 \\
1 \\
0 \\
-2 \\
0 \\
0
\end{pmatrix}^T,
\]

where \(\delta^* = \sqrt{5}/10\). Therefore by weak duality, the optimal value of (7) is \(\gamma^* = -\sqrt{5}\).

Finally, we prove that (7) does not have any optimal solutions, i.e., the optimal value \(-\sqrt{5}\) is not attained. To this end, we suppose that (7) has an optimal solution \((y_1^*, \ldots, y_6^*)\). Since (7) and (8) have the same optimal value, a dual pair of optimal solutions satisfies the complementarity condition;
see e.g., [8, Section 2.4]. Thus, together with positive semidefiniteness of the solutions, we obtain that $y_2^* = y_3^* = 0$ and

$$
\begin{pmatrix}
2y_1^* + 2y_2^* \\
y_1^* + y_2^* - y_3^* \\
y_1^* - y_2^* + y_4^* \\
1
\end{pmatrix}
\begin{pmatrix}
y_1^* \\
y_2^* \\
y_4^* \\
1
\end{pmatrix}
= 0.
$$

We can see that these equations have no solutions. Therefore (7) does not have any optimal solutions.

**B. On the optimal value of the perturbed SDP (P1)**

As the perturbed SDP (P1) is strictly feasible, it follows from Theorem 3.9 that the optimal value of (P1) is equal to the optimal value of the dual problem. In fact, as mentioned in Example 4.9, the original SDP (7) is strictly feasible. Hence from Remark 3.12 we see that (P1) is strictly feasible.

The dual problem of (P1) can be formulated as follows:

$$
\begin{align*}
\inf_{z_{ij}} & \quad -2(z_{51} + z_{61} + z_{52} + z_{53} + z_{54}) \\
\text{s.t.} & \quad \text{He} \left( \begin{pmatrix}
-z_{11} + 2z_{31} - z_{41} & -z_{21} + 2z_{22} + 2z_{32} - z_{42} \\
-z_{11} - 2z_{31} + 2z_{41} & -z_{21} - z_{32} + 2z_{42}
\end{pmatrix} \right) \in S^2_+, \sum_{i=3}^6 z_{ii} = 1, \\
2z_{21} + 2z_{31} - z_{41} = 0, (1 + \epsilon)2z_{22} + 2z_{32} - z_{42} = 0, (z_{ij})_{1 \leq i,j \leq 6} \in S^6_+.
\end{align*}
$$

(20)

From the first, second and last constraints, we obtain that $z_{11} = 0$, and thus that $z_{11} = z_{ii} = 0$ hold for $i = 1, \ldots, 6$ due to the positive semidefiniteness of the matrix $(z_{ij})_{1 \leq i,j \leq 6}$. Substituting them into (20), we obtain

$$
\begin{align*}
\inf_{z_{ij}} & \quad \begin{cases}
-2z_{22} + 2z_{32} & \geq 0, z_{22} + 2z_{32} - z_{42} = 0, \sum_{i=3}^6 z_{ii} = 1, \\
(1 + \epsilon)2z_{22} + 2z_{32} - z_{42} = 0, (z_{ij})_{2 \leq i,j \leq 6} \in S^5_+
\end{cases}
\end{align*}
$$

(21)

Moreover we obtain $z_{22} = 0$ from the second and third constraints in (21), and thus $z_{12} = z_{1i} = 0$ for all $i = 2, \ldots, 6$. Then (20) is equivalent to the following problem:

$$
\begin{align*}
\inf_{z_{ij}} & \quad -2(z_{53} + z_{54}) : \sum_{i=3}^6 z_{ii} = 1, (z_{ij})_{3 \leq i,j \leq 6} \in S^4_+
\end{align*}
$$

(22)

(22) is the minimization problem of the eigenvalues of the matrix $\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}$. Since the minimum of the eigenvalues of the matrix is $-\sqrt{2}$, it follows from Theorem 3.9 that the optimal value of (P1) is $-\sqrt{2}$.

**C. Minimal face of the dual of SDP (P1)**

We compare the dual problem (8) of SDP (7) with the dual problem (20) of its perturbed SDP (P1) from the viewpoint of the minimal faces. In fact, we prove here that the minimal face of (20) of its perturbed SDP (P1) is smaller than the minimal face of (8).
As we have seen in Example 2.3, the minimal face of (8) is $F_1 \times F_2$ in (12). Now we see that $(y, U, V)$ in (11) is a reducing certificate at the first iteration of the facial reduction for (20). The discriminant system (9) at the second iteration is equivalent to $y_6 = 0$, $(\frac{y_1}{y_2, y_3}) \in S_+^2 + (F_1^2)^\perp$ and

\[
\begin{pmatrix}
2y_1 + 2y_2 \\
-y_1 + y_2 + y_3 - y_4 \\
-2y_1 + y_2 - 2y_4 \\
y_1 - 2y_2 + y_4 \\
0 \\
0
\end{pmatrix}
\begin{pmatrix}
y_1 \\
y_2 \\
y_3 \\
y_4 \\
y_5 \\
y_6
\end{pmatrix}
\in S_+^6 + (F_1^4)^\perp.
\]

Then the following $(y^2, U^2, V^2)$ is a reducing certificate at the second iteration:

\[
y^2 = \begin{pmatrix}
-1 \\
0 \\
2 \\
-1
\end{pmatrix},
U^2 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
O_{2x2},
V^2 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix},
O_{2x2}.
\]

The obtained faces are

\[
F_1^2 = \left\{ \begin{pmatrix}
O_{2x2} & O_{2x4} \\
O_{4x2} & X_2
\end{pmatrix} : X_2 \in S_+^4 \right\}
\text{ and } F_2^2 = F_2^1.
\]

We obtain the minimal face $F_1^2 \times F_2^2$ of (20) with two iterations. We see that the degree of singularity of (20) is two and that this face is smaller than the minimal face of (8).

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