Exponentiation of soft photons in a process involving hard photons

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\textbf{Abstract}

We present a simple method of removing the singularities associated with soft photon emission to all orders in perturbation theory through exponentiation, while keeping a consistent description of hard photon emission. We apply this method to the process \(e^+e^- \rightarrow \mu^+\mu^- + n\gamma\) where we include both \(Z^0\) and \(\gamma\) exchange and retain the muon mass dependence. The photonic radiation is allowed to be radiated off any charged leg, and so we include all initial and final state radiation, as well as all interference effects. The effect of exponentiation is to suppress soft photon emission over the cross-section you would obtain from working at strictly leading order. We also show how one would extend the method to treat the collinear singularity; and remove the associated leading mass logarithms.

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1 Introduction

Photonic radiation off charged particles plays a very important part in the physics of high energy particle colliders. At LEP initial state photonic radiation is responsible for shifting the $Z^0$ peak and altering its measured width, and hence an accurate description of this radiation is necessary to extract the $Z^0$ boson mass and width in a meaningful way. At hadron colliders photonic radiation often forms a background (or indeed a signal) to new processes; as, for example, in the future search for an intermediate mass Higgs boson; or the testing of anomalous gauge boson couplings. Clearly it is important to understand this radiation.

If we calculate this photonic radiation at tree level in perturbation theory as the radiation becomes either soft, or collinear to a massless charged object, we encounter logarithms in $E_\gamma/\sqrt{s}$ and $\theta$ respectively. When these logarithms become large the probability that additional photons are also radiated becomes large; and so the tree level description breaks down. The situation can be improved by going to higher orders in perturbation theory, however this does not cure the problem. If we work at next to leading order then we include 1 additional photon and so this is an improvement over leading order, but when additional photonic radiation becomes important then this next to leading order description breaks down, clearly this happens at each finite order in perturbation theory. The solution to this is to go to fully infinite order in perturbation theory; and this is possible in the soft and collinear limits as in these limits the matrix element can be well approximated by the soft approximation and Altarelli–Parisi splitting functions. This means that all the logarithms associated with soft emission can be resummed which leads to an exponential series [1, 2, 3, 4]. The leading logs or next–to–leading logs associated with collinear emission can be resummed and this leads to a “parton distribution” for the charged particle [1, 5]. These two methods each respectively give an excellent description of soft and collinear radiation. The resummation of the soft logarithms can be framed as a reordering of the terms of perturbation theory in rigours way as was first done by Yennie Frautschi and Suura (YFS) [6]. This method provides an excellent description of both soft and hard radiation simultaneously. However there are few Monte Carlo programs that incorporate YFS exponentiation and the best (YFS3[3],BHLUMI4.0[4]) only describe 1 hard photon exactly (with a 2nd included through the leading log splitting function). On the other hand tree level Monte Carlos at a fixed order perturbation theory can describe
arbitrarily many hard photons [7, 8]. In this paper we take our previous tree level Monte Carlo [8] that calculated $e^+e^- \rightarrow \mu^+\mu^- + n\gamma$ and exponentiate the soft photons. We use a less rigorous, but more simple, form of exponentiation than full YFS exponentiation. Our form of exponentiation is equivalent to YFS exponentiation except that the effects of the virtual loop Feynman diagrams are not explicitly included, but only added using an ad hoc method. We show how one would go about resumming the collinear radiation to remove the large mass logarithms that occur, however as our primary interest in this paper is the interface between hard and soft radiation we make no attempt to include this resumed collinear radiation in our Monte Carlo program. Now when we calculate radiation at tree level we are forced to include cuts that keep us separate from both the soft and collinear region so we do not encounter the singularities that are present there; in fact, as we have suggested above these cuts should be large enough that we do not approach the areas of phase space where the soft and collinear logarithms become large. Now in our work as we resum the soft logarithms and remove the soft singularity, and so we are no longer required to keep the cut that keeps us separate from the soft corner of phase space, however we are forced to retain the cut that keeps us separate from the collinear singularity as we have not dealt with the singularities that are there. As with tree level calculations this cut should be viewed as an experimental cut, that should be chosen large enough to keep us away from the large collinear logarithms, that is imposed on all photons.

2 Exponentiation of soft photons with other hard photons present

In any process in which charged particles are accelerated the soft photon approximation tells us that soft photons are predominately radiated off external legs, and that the matrix element for the Feynman diagram with a soft photon radiated off external leg $i$ with charge $e$ is,

$$\mathcal{M} e \frac{\epsilon \cdot p_i}{k \cdot p_i}$$

where $\mathcal{M}$ is the matrix element for the process without a photon, and $\epsilon$ and $k$ are respectively the photon polarisation vector and momentum.

So if the matrix element for the process,

$$\emptyset \rightarrow e^+(p_1) \ e^-(p_2) \ \mu^+(p_3) \ \mu^-(p_4) + n\gamma$$

(2.2)
is $\mathcal{M}_n$ the matrix element squared for the process,
\[
\emptyset \rightarrow e^+(p_1) e^-(p_2) \mu^+(p_3) \mu^-(p_4) + n\gamma + \gamma_s(k)
\] (2.3)
in the soft photon approximation is given by,
\[
|M_{n,1}|^2 = -e^2 |M_n|^2 \left( \frac{p_1}{p_1 \cdot k} - \frac{p_2}{p_2 \cdot k} + \frac{p_3}{p_3 \cdot k} - \frac{p_4}{p_4 \cdot k} \right)^2
\] (2.4)
after we sum over the different spin states of the photon, where we define all particles in the final state. This means that initial state particles will have negative energy.

If the soft photon and the $n$ original photons are in mutually exclusive areas of phase space then there is no symmetry factor between the two. We can force this to happen by only considering the original $n$ photons with energy, $E_h$, larger than some cut, and soft photons with energy, $E_s$, less than that cut, i.e.,
\[
E_h > E_{\text{cut}} > E_s
\] (2.5)
then differential cross-section is given by,
\[
\frac{d\sigma_{n+1}}{d(LIPS)_{n+3}} = -e^2 \frac{1}{\text{Flux}} |M_n|^2 \left( \frac{p_1}{p_1 \cdot k} - \frac{p_2}{p_2 \cdot k} + \frac{p_3}{p_3 \cdot k} - \frac{p_4}{p_4 \cdot k} \right)^2 d(LIPS)_{n+3}
\] (2.6)
\[
\simeq \frac{1}{\text{Flux}} |M_n|^2 d(LIPS)_{n+2} \left(\text{Eikonal Factor}\right) \frac{E^2 dE d\Omega}{2E(2\pi)^3}
\] (2.7)
where,
\[
(\text{Eikonal Factor}) = -e^2 \left( \frac{p_1}{p_1 \cdot k} - \frac{p_2}{p_2 \cdot k} + \frac{p_3}{p_3 \cdot k} - \frac{p_4}{p_4 \cdot k} \right)^2
\] (2.8)
\[
\equiv e^2 \frac{f(\Omega)}{E^2}
\] (2.9)
and integrating over the soft photon phase space for photon energies $E_{\text{min}} < E_{\gamma} < E_{\text{cut}}$ gives,
\[
\frac{d\sigma_{n+1}}{d(LIPS)_{n+2}} = \frac{\alpha}{4\pi^2} \ln \left( \frac{E_{\text{cut}}}{E_{\text{min}}} \right) g(\Omega_c)
\] (2.10)
with,\(^2\)
\[
g(\Omega_c) = \int_{\Omega_c} f(\Omega) d\Omega
\] (2.11)
\(^2\) We give the explicit form of $g$ in the appendix.

- 3 -
and so we see from (2.10) that the probability to emit a soft photon just factorises the lower order cross-section. However in the factorising term we have a contribution from
\[ \ln(E_{\text{min}}) \]
and this diverges as \( E_{\text{min}} \) goes to zero, this is the soft singularity.

Now for fixed particle momenta in \( d\sigma_n \) unitarity tells us that when we integrate over the soft photon momenta that \( d\sigma_{n,1} \) is finite. As the soft, non collinear, singularity contains no electron or muon mass terms this tells us that the soft logarithm must cancel in the full calculation including virtual diagrams\[9\]. We can achieve the same effect as the virtual diagrams by imposing an effective lower energy cut off on the photon energy, \( E_{\text{reg}} \), and for the higher order corrections corrections to be \( \mathcal{O}(\alpha) \) we require,
\[ \ln \left( \frac{\sqrt{s}/2}{E_{\text{reg}}} \right) = \mathcal{O}(1) \] (2.12)
where \( \sqrt{s}/2 \) is the maximum photon energy. So the cross-section, differential in \( n \) photons with energy larger than \( E_{\text{cut}} \) and integrated over 1 soft photon with energy smaller than \( E_{\text{cut}} \) is given by,
\[ d\sigma_{n,1} = d\sigma_n \frac{\alpha}{4\pi^2} \ln \left( \frac{E_{\text{cut}}}{E_{\text{reg}}} \right) g(\Omega_c) \] (2.13)
In doing this we improve the accuracy of the differential cross-section calculation from \( \mathcal{O}(\alpha \ln(E)) \) to \( \mathcal{O}(\alpha) \). The lack of knowledge of \( E_{\text{reg}} \) beyond (2.12) is exactly the lack of knowledge that we have from doing a tree level calculation without calculating the virtual diagrams. Also note that if we consider the differential cross-section for values of \( E_{\text{cut}} < E_{\text{reg}} \) then this appears to go negative! This is just an artifact of working, at least as far as the soft logs go, beyond tree level.

If we now consider additional soft photonic radiation then for the process,
\[ \emptyset \rightarrow e^+(p_1) e^-(p_2) \mu^+(p_3) \mu^-(p_4) + n\gamma + \gamma_s(k_1) \cdots \gamma_s(k_m) \] (2.14)
then the soft photon approximation gives the matrix element squared as,
\[ |\mathcal{M}_{n,m}|^2 = |\mathcal{M}_n|^2 \prod_{i=1}^{m} -e^2 \left( \frac{p_1}{p_1 \cdot k_i} - \frac{p_2}{p_2 \cdot k_i} + \frac{p_3}{p_3 \cdot k_i} - \frac{p_4}{p_4 \cdot k_i} \right)^2 \] (2.15)
\(^3\) Also note that the term \( g(\Omega_c) \) diverges in the limit of massless fermions when the photon becomes collinear to any fermion. This is the collinear singularity, isolating this singularity and removing it leads to collinear description of photon radiation. In this work we will apply angular cuts to all photons (be they hard or soft) to keep the photons separate from the fermions, and as such we work in an area of phase space where these collinear singularities are not important.
The phase space for these soft photons also approximately factorises,

\[ d(\text{LIPS})_{2+n+m} \approx d(\text{LIPS})_{2+n} \prod_{i=1}^{m} \frac{E_i^2 dE_i d\Omega_i}{2E_i(2\pi)^3} \]  

(2.16)

and so as before we can write the differential cross-section for the process with \( n \) photons with energy larger than \( E_{\text{cut}} \) and integrated over \( m \) soft photons with energy less that \( E_{\text{cut}} \) as,

\[ d\sigma_{n,m} = d\sigma_{n,1} m! \left( \frac{\alpha}{4\pi^2} \ln \left( \frac{E_{\text{cut}}}{E_{\text{reg}}} \right) g(\Omega_c) \right)^m \]  

(2.17)

where the \( 1/m! \) term is the symmetry factor that we get from integrating the \( m \) identical soft photons over the same regions of phase space.

If we now ask what the differential cross-section, \( d\sigma_n^{\text{exp}} \), for process where we produce an arbitrary number of photons satisfying the cut \( \Omega_c \), where we stay differential in \( n \) photons with energy larger than \( E_{\text{cut}} \) and integrate over an arbitrary number of photons with energy less than \( E_{\text{cut}} \), we find,

\[ d\sigma_n^{\text{exp}} = d\sigma_{n,0} + d\sigma_{n,1} + d\sigma_{n,2} + \ldots \]  

(2.18)

\[ = d\sigma_n \left( 1 + \left( \frac{\alpha}{4\pi^2} \ln \left( \frac{E_{\text{cut}}}{E_{\text{reg}}} \right) g(\Omega_c) \right) + \frac{1}{2!} \left( \frac{\alpha}{4\pi^2} \ln \left( \frac{E_{\text{cut}}}{E_{\text{reg}}} \right) g(\Omega_c) \right)^2 + \ldots \right) \]  

(2.19)

\[ = d\sigma_n \exp \left( \frac{\alpha}{4\pi^2} \ln \left( \frac{E_{\text{cut}}}{E_{\text{reg}}} \right) g(\Omega_c) \right) \]  

(2.20)

\[ = d\sigma_n \left( \frac{E_{\text{cut}}}{E_{\text{reg}}} \right)^{\alpha g(\Omega_c)/4\pi^2} \]  

(2.21)

\( d\sigma_n^{\text{exp}} \) has the sum over an infinite number of soft photons, and this means that it has all the real soft logarithms resummed in it, each with the soft singularity removed; this means it will provide an accurate description of arbitrarily soft photons. Notice that in this “exponentiated” form we have no problems with cross-sections going negative for values of \( E_{\text{cut}} < E_{\text{reg}} \). \( d\sigma_n^{\text{exp}} \) has no knowledge of photons that fail the angular cut \( \Omega_c \) and with respect to those photons \( d\sigma_n^{\text{exp}} \) is a strictly leading order quantity. This means \( d\sigma_n^{\text{exp}} \) does not describe this collinear radiation, and in particular when large collinear logs appear in \( g(\Omega_c) \) or \( d\sigma_n \), \( d\sigma_n^{\text{exp}} \) will not give a good description of the radiation.

Now we can use the form of (2.21) to arrive at a workable Monte Carlo strategy;

- Choose the number of photons \( n \) to be produced.
• Calculate a point in phase space for the process (2.2) where the \(n\) photons have energy larger than \(E_{\text{cut}}\).

• Calculate the hard matrix element for the process (2.2) and weight it by,

\[
\left( \frac{E_{\text{cut}}}{E_{\text{reg}}} \right)^{\left( \frac{\alpha g(\Omega_c)}{4\pi^2} \right)} \tag{2.22}
\]

• Sum over the events to perform the integration over \(n\) and the hard phase space for the process (2.2).

Notice that we can only calculate the expression (2.22) after calculating the point in the hard phase space as the \(g(\Omega_c)\) term depends upon the orientations of the charged particles in the process.

In this prescription we appear to have introduced two extra parameters, \(E_{\text{reg}}\) and \(E_{\text{cut}}\), over the tree level calculation. Now the value of \(E_{\text{reg}}\) is introduced to cancel the soft singularity; and as in a tree level calculation we don’t know how much of the singularity cancels, the value of \(E_{\text{reg}}\) is unknown beyond what we learn from equation (2.12). Thus the dependence of our calculation on \(E_{\text{reg}}\) is a measure of our uncertainty in only calculating the real photon emission diagrams without calculating the virtual photon loop diagrams that cancel the soft singularities. The same level of uncertainty is also present at tree level although there is no parameter that displays it.

Our apparent dependence upon \(E_{\text{cut}}\) is more worrying. \(E_{\text{cut}}\) was just introduced as a parameter to distinguish the “hard” photons from the “soft” photons, however there is no physical meaning to the words “hard” and “soft”; there is no magical energy where photons suddenly become “hard”. Consider what happens as we decrease \(E_{\text{cut}}\); for the hard matrix elements we approach the soft singularity and the cross-section grows rapidly; now this increase in cross-section comes about from low energy photons, as these photons have low energy they do not change the physical signature of the process. However as the hard matrix elements are growing the eikonal factor (2.22) decreases and if the value of \(E_{\text{cut}}\) is to have no physical significance then these two effects should cancel each other. This will give us a very strong test of our results, if we have a result independent of \(E_{\text{cut}}\) we are practically required to a correct calculation of both the hard matrix element and the soft eikonal factor. Notice that as we include an arbitrary number of soft photons in the soft exponentiating factor we are required to consider an arbitrary number of hard photons in order for the \(E_{\text{cut}}\) cancellation to occur. This is true even if those hard photons
would not be experimentally observed on energy grounds alone. This is the opposite of what we do in a strictly LO calculation; there if we require say the 2 photon differential cross-section it is essential that we generate exactly 2 photons, here it is essential that we also calculate the $n$ photon rate for all $n \geq 2$. Also notice that we will only get this cancelation between the soft photon exponentiation and the hard photon cross-section if we treat both the soft and hard photons identically, in particular this means that the angular cuts that we apply to the hard photons must also be applied to the soft photons.

Comparing the naive form of exponentiation considered here with the more rigorous YFS exponentiation, then YFS exponentiation explicitly includes both hard photon corrections and virtual photon corrections to the exponentiating factor. The virtual photon corrections are totally ignored in this work, as we consider no virtual diagrams; this means that this work has uncertainties of $O(\alpha) \approx 1\%$ – these uncertainties manifest themselves as a lack of knowledge in the value of $E_{\text{reg}}$. The hard photon corrections give only a finite contribution to the matrix element in the soft corner of phase space, and as the volume of this phase space tends to zero as $E_{\text{cut}} \rightarrow 0$ in this limit the hard photon corrections have vanishing effect. In practice this means that we should take $E_{\text{cut}}$ small enough that our results have no dependance upon $E_{\text{cut}}$. In YFS exponentiation the total energy of all radiated soft photons is forced to be less than some cut; whereas in our form of exponentiation it is each soft photon energy individually that has energy less than some cut; this means that there is a worry that although each soft photon individually only carries away a small amount of energy, as there are in infinite number of soft photons they may carry away a sizable amount of energy in total. If this were to happen then our results would be incorrect because energy would not be conserved in the $E_{\text{cut}} \rightarrow 0$ limit; however if this were to happen our results would not be independent of the cut $E_{\text{cut}}$, thus independance of our results upon this cut check that our energy cut on the soft photons is valid.

3 Collinear radiation

In this paper we are primarily interested in the interaction between soft and hard radiation, as such we have only included the effects of resumming soft radiation in our Monte Carlo program. This means that we are always forced to include an “experimental” angular cut on all photons, $\Omega_c$, that keeps us separate from the singularity when radiation becomes collinear to a massless charged object. As we have this “experimental cut” our
method has nothing to say about photons that fail the cut $\Omega_c$, or the physics that these photons generate (like the shift in the measured $Z^0$ mass and width). In this section we show how such collinear radiation could be included in our method at the leading collinear log level; however we make no attempt to include this collinear radiation in a Monte Carlo set up.

In principle we would like to proceed in a similar way to the soft radiation, that is we evaluate the matrix element for a charged particle radiating an arbitrary number of collinear photons, then integrate the photon momenta over the collinear region of phase space; and finally sum over an arbitrary number of collinear photons. In practice the matrix element for a charged particle radiating an arbitrary number of collinear photons is not known and so this method fails. This means that we can not calculate all real collinear logarithms in the way that we calculated all real soft logarithms; however we can calculate the leading logs or the next to leading logs through an evolution equation. If we have 2 collinear photons then if one photon is far more collinear than the other we can approximate the matrix element as two independent collinear emissions of photons. Usually the evolution equation is in terms of the virtuality of the initial or final charged particle, or the maximum $p_T$ that the radiation can have; however for our case it is far more convenient for the evolution to be done in terms of the angle of the emitted radiation.

If we have a massless charged particle, $p$, that radiates a collinear photon, $k$, at an angle $\theta$; so the final charged particle, $p'$, has a fraction $z$ of the initial particle energy, then the lowest order matrix element is multiplied by,

$$|\mathcal{M}_\text{split}|^2 = 2e^2 \frac{1}{p'.k} \mathcal{P}(z) = 2e^2 \frac{1}{E'E_\gamma(1 - \cos \theta)} \mathcal{P}(z)$$

(3.1)

where,

$$\mathcal{P}(z) = \left( \frac{1 + z^2}{1 - z} \right)_+$$

(3.2)

and in the collinear region the phase space is given by,

$$d(\text{LIPS}) = \frac{\pi}{2} \ dz \ d(2p'.k) = \pi E'E_\gamma \ dz \ d\cos \theta$$

(3.3)

where we have integrated over the unimportant azimuthal angle. So the differential cross-section to emit an extra single collinear photon is given by,

$$|\mathcal{M}_\text{split}|^2 d(\text{LIPS}) = (2\pi e^2) \ d\left(-\ln(1 - \cos \theta)\right) \mathcal{P}(z)dz$$

(3.4)
If we now consider the cross-section to emit an arbitrary number of collinear photons up to some angle $\theta$ then the lowest order cross-section is multiplied by a function, $D$; for this to be useful we need to know the energy fraction of the final charged particle, $z$, and so it is convenient to define,

$$ D(z) = \frac{dD}{dz} \quad \text{where} \quad D = \int_0^1 dz \, D(z) \quad (3.5) $$

then the evolution of $D(z)$ is given by,

$$ \frac{dD(z)}{d(ln(1 - \cos \theta))} = (2\pi e^2) \int_z^1 \frac{dx}{x} P(x) D(z/x) \quad (3.6) $$

Where to derive this equation we have assumed that the splitting function for several angular ordered photons is given by a production of splitting functions, this is strictly only true when the photons are strongly ordered. This means that this evolution equation only sums the leading collinear logarithms.

If we integrate (3.6) over $z$ we find,

$$ D = \lambda (1 - \cos \theta)^2 \pi e^2 P \quad (3.7) $$

where $P = \int_0^1 dz \, P(z)$. Usually within the $+$ prescription $P \equiv 0$, and so $D$ is constant when evolved in $\theta$. This means that as we vary the angular cut around the charged particles $\Omega_c$ the cross-section in the collinear region does not change. However the cross-section for the noncollinear radiation does depend upon the angular cut $\Omega_c$, and this means that the total cross-section is not independent of this cut. In order to restore independence of the cross-section on $\Omega_c$ we define $\tilde{P}$,

$$ \tilde{P}(z) = P(z) + \delta(z - 1) \int_0^{1-E_{\text{reg}}/E} dy \, P(y) \quad (3.8) $$

where $E$ is the energy of the charged particle, and then use $\tilde{P}$ in place of $P$.

Equation (3.6) being an evolution equation only tells us how radiation up to some angle from the charged particle is related to radiation at any other angle, it does not tell us where to start the evolution. Now if the charged particle were indeed massless then the starting point of the evolution would be uncalculable, as in QCD parton distributions for the proton and fragmentation functions; however for massive charged particles radiation is suppressed in a dead cone surrounding the charged particle of angle defined by $\sin \theta_{\text{DC}} \approx$
where \( m \) and \( E \) are respectively the mass and original energy of the charged particle. This suggests the starting point of the evolution as,

\[
D(z, \theta_{DC}) = \delta(z - 1)
\]

(3.9)

This gives that,

\[
D(\theta_{DC}) = 1
\]

(3.10)

\( i.e., \lambda \simeq \left( \frac{2E^2}{m^2} \right) \frac{2\pi e^2 P}{m^2} \)

this means that,

\[
D(\cos \theta \approx 0) \simeq \left( \frac{2E^2}{m^2} \right) \frac{2\pi e^2 P}{m^2}
\]

(3.11)

Now in the limit \( m \to 0 \), \( D \) diverges; this, at least for final state radiation, is unphysical[9]. The origin of this logarithmic divergence is in the integral over \( \cos \theta \) in the differential cross-section, this takes the form,

\[
\int_{\cos \theta \approx 0}^{\cos \theta \approx \sqrt{1 - m^2/E^2}} d(- \ln(1 - \cos \theta)) \simeq \ln \left( \frac{2E^2}{m^2} \right)
\]

(3.12)

This is the collinear logarithm that, like the soft logarithm, also cancels (for final state radiation [10]) on the virtual diagrams. As with the soft singularity we have not calculated the virtual diagrams we again do not know how much of the mass logarithm cancels, and so we again introduce an extra parameter that quantifies our lack of knowledge in this cancellation. For the mass logarithms to be absent in the total cross-section we require,

\[
D(\cos \theta \approx 0) = 1 + \mathcal{O}(\alpha)
\]

(3.13)

and for this to be true we choose \( \lambda \) in equation (3.7) to have the value,

\[
\lambda = 1 + \mu \alpha \quad \text{where} \quad \mu \sim 1
\]

(3.14)

This means that,

\[
D(\theta_{DC}) \simeq (1 + \mu \alpha) \left( \frac{m^2}{2E^2} \right) \frac{2\pi e^2 P}{m^2}
\]

(3.15)

and so we choose the starting point for the evolution (3.6) to be,

\[
D(z, \theta_{DC}) = \delta(z - 1) \left(1 + \mu \alpha \right) \left( \frac{m^2}{2E^2} \right) \frac{2\pi e^2 P}{m^2}
\]

(3.16)
This ensures that the total cross-section is not enhanced by logarithms in \( m \). For angles less than \( \theta_{\text{DC}} \) the fragmentation function \( D \) no longer evolves in \( \cos \theta \) but stays at the value of \( D(\theta_{\text{DC}}) \). It should be noted that we require a different value for \( \mu \) for initial and final state radiation; indeed whereas for final state radiation we know that \( \mu \sim 1[10] \), we know no such thing for initial state radiation where the cross-section may contain logarithms in the mass. Also note that \( D(z) \) plays a very different role for initial state radiation than for final state radiation. For initial state radiation the charged particle fragments into a charged particle and photons before the hard scattering, and so the charged particle’s momentum is degraded and the hard scattering takes place at a lower \( \sqrt{s} \); whereas for final state radiation the fragmentation of the charged particle happens after the hard scattering and so does not change the hard scattering.

Having set up the fragmentation functions \( D \) and \( D \) the hard differential cross-section, where we have summed over an arbitrary number of soft and/or collinear photons, is given by,

\[
\begin{align*}
\left. d\sigma_n^{\text{col}-\text{exp}} \right|_{z_1, z_2} &= d\sigma_n \bigg|_{z_1, z_2} \\
&\times \frac{E_{\text{cut}}}{E_{\text{reg}}} \left( \frac{\alpha g(\Omega_c)}{4\pi^2} \right) \\
&\times D(\Omega_c, z_1) \, dz_1 \, D(\Omega_c, z_2) \, dz_2 \, D(\Omega_c, z_3) \, dz_3 \, D(\Omega_c, z_4) \, dz_4
\end{align*}
\]  

(3.17)

where \( d\sigma_n \bigg|_{z_1, z_2} \) is the hard differential cross-section where the incoming electrons have fractions \( z_1 \) and \( z_2 \) of the beam energy; \( z_3 \) and \( z_4 \) are the fractional energy that the final state muons have.

As the soft radiation is radiated off particles with a similar virtuality as the collinear radiation, or alternatively they are radiated at the same timescale, it is not clear whether \( g(\Omega_c) \) should be calculated before or after the collinear radiation has been emitted in the \( D \) functions. However recall that \( g \) is the integral over the matrix element squared, where the matrix element is the sum of terms that go like \( p/ p \cdot k \); and this depends only on the direction of \( p \) and not its energy in the massless limit. So \( g \) only depends on the directions of the charged particles, and not on the amount of energy they carry. This means that \( g \) is independent, in the massless collinear limit, of whether it is calculated before or after collinear radiation is emitted.

With the differential cross-section in the form (3.17) it is easy to give a Monte Carlo mechanism for generating hard photons.
• Choose the number of photons \( n \) to be produced.

• Choose \( z_1 \) and \( z_2 \) and hence the initial electron energies.

• Generate the hard phase space for \( n \) photons where the initial electrons have fraction \( z_1 \) and \( z_2 \) of the beam energy.

• Choose \( z_3 \) and \( z_4 \) that give the fractional energy that the observed muons have over the muons that emerge from the hard scattering.

• Calculate the hard matrix element for hard phase space and weight it by,

\[
\left( \frac{E_{\text{cut}}}{E_{\text{reg}}} \right)^{\left(\alpha g(\Omega_c)/4\pi^2\right)} D(\Omega_c, z_1) D(\Omega_c, z_2) D(\Omega_c, z_3) D(\Omega_c, z_4)
\]

• Sum over events to perform the integration over the hard phase space, \( z_1 \) through \( z_4 \) and \( n \) in the usual Monte Carlo way.

In the soft case the dependance upon \( E_{\text{cut}} \) cancels exactly between the hard cross-section and the soft exponential factor. In this collinear case we do not expect exact cancelation of \( \Omega_c \) dependance between the collinear factor \( D \) and the hard cross-section, as we have only resummed the leading logs in \( D \) rather than all the logs as in the soft case. Nonetheless there should only be rather mild dependance on \( \Omega_c \) arising from the sub leading logs that occur in the hard cross-section, but not in the factor \( D \).

The resummation of these collinear logs has no analogy in YFS exponentiation, which makes no attempt to resum collinear logs. Within YFS exponentiation the full collinear logs can only be added as hard corrections to the exponentiating factor. This means that our collinear resummation should give a more accurate description of radiation in the collinear region than YFS exponentiation.

4 Numerical Results

We have applied the procedure from section 2 to the process,

\[ e^+ e^- \rightarrow \mu^+ \mu^- + n \gamma \] \hspace{1cm} (4.1)

for arbitrary \( n \). To do this we need the hard matrix element for (4.1), this we have calculated in a previous paper [8]. In that calculation we retained the full mass dependence
of the muons, and allow an arbitrary number of hard photons radiated from both the initial
and final state fermions. We use the approximation that the electron is massless, however
as we have angular cuts to keep the photons separated from the electrons we do not work
in an area of phase space where the electron mass terms are important. We include both
s channel $Z^0$ and photon exchange.

For the physical constants we use,

\[
\alpha_{\text{em}}^{\text{int}} = \frac{1}{128} \\
\alpha_{\text{em}}^{\text{ext}} = \frac{1}{137} \\
\sin \theta_W = 0.23 \\
M_{Z^0} = 91.175 \text{ GeV} \\
m_\mu = 105.6584 \text{ MeV}
\]  

where we use $\alpha_{\text{em}}(M_Z) \equiv \alpha_{\text{em}}^{\text{int}}$ as the coupling for the s channel $Z^0$ and photon, and $\alpha_{\text{em}}(0) \equiv \alpha_{\text{em}}^{\text{ext}}$ as the coupling for all external photons [11].

As we have not removed the large logs associated with collinear radiation in the Monte
Carlo we are required to keep away from the regions of phase space where these logs be-
come important, this means that we must keep all photons (be they exponentiated or not)
separate from all charged particles. This we do by imposing the following experimental
cuts,

\[
\theta_{\mu\gamma} > 5^\circ \\
| \cos \theta_\gamma | < 0.9
\]  

where $\theta_{\mu\gamma}$ is the angle between a muon and photon, and $\theta_\gamma$ is the angle of a photon from
the beam pipe. In order for the final state particles to be observed we impose,

\[
\theta_{\mu\mu} > 20^\circ \\
35^\circ < \cos \theta_\mu < 145^\circ
\]  

where $\theta_{\mu\mu}$ is the angle between the two muons and $\theta_\mu$ is the angle of a muon from the
beam pipe. We choose,

\[
\sqrt{s} = M_Z
\]  

Now for large $n$ the hard matrix element for process (4.1) can be very time consuming to
calculate, so we calculate up to 4 photons exactly – then for larger numbers of photons
we calculate the 4 most energetic photons exactly, and then use the soft approximation to
Fig. 1 The contributions from varying numbers of hard photons to the inclusive cross-section for the process $e^+e^- \rightarrow \mu^+\mu^-$ as a function of $E_{\text{cut}}$.

We choose $E_{\text{reg}} = 10$ GeV, $\sqrt{s} = M_Z$ and use the other cuts described in the text. On the right we expand the scale and just show the inclusive cross-section. We also show the Monte Carlo errors associated with each point.

calculate the remaining photons. In practice the contribution to the cross-section from 4 or more hard photons is always very small, and usually several of the photons are very soft; this means that we make only a small error by using this approximation, while speeding up the code considerably.

The first test of the program is the dependence upon $E_{\text{cut}}$ and $E_{\text{reg}}$. In Fig. 1 we show the contributions from the $n$ hard photon cross-sections to the total inclusive cross-section as a function of $E_{\text{cut}}$. Clearly these each individually have a large dependence upon $E_{\text{cut}}$. However when we sum the contributions from each number of photons to form the physical total inclusive cross-section where photon radiation satisfies the cut $\Omega_c$ we obtain the solid line which clearly has far less dependence upon $E_{\text{cut}}$. We also show the total inclusive cross-section with an expanded scale. It is clear that for small $E_{\text{cut}}$ the cross-section is independent of $E_{\text{cut}}$ within the Monte Carlo error; however there is also a clear rise in the total inclusive cross-section for values of $E_{\text{cut}} \gtrsim 1$ GeV. This can be understood as the soft approximation that we have used to derive the exponential factor (2.22) breaks down for photon energies larger than $O(1$ GeV).

As we take $E_{\text{cut}}$ smaller it takes a longer time to calculate the cross-section with the same accuracy as for smaller $E_{\text{cut}}$ values we have a larger contribution from larger numbers of hard photons; and the cross-sections for large numbers of photons are very
Fig. 2 The dependence of the inclusive \( e^+ e^- \rightarrow \mu^+ \mu^- \) exponentiated cross-section on \( E_{\text{reg}} \). We also show the LO result for the same cross-section.

time consuming to calculate due to the rapid increase in the number of Feynman diagrams. We have also calculated the \( E_{\text{cut}} \) dependence of the more exclusive 1 and 2 photon cross-sections, and for small \( E_{\text{cut}} \) values there is no visible dependence upon \( E_{\text{cut}} \). This gives us confidence that our computer code has no mistakes in it. In all the results that follows we will choose \( E_{\text{cut}} = 10^{-2} \text{GeV} \) which is safely in the region where our results are independent of \( E_{\text{cut}} \), also the value of \( E_{\text{cut}} \) is not so small that we spend large amounts of time calculating the cross-section for large numbers of hard photons.

If Fig. 2 we show the dependence of the inclusive cross-section upon \( E_{\text{reg}} \), and also the LO \( e^+ e^- \rightarrow \mu^+ \mu^- \) cross-section. The dependence of the cross-section on \( E_{\text{reg}} \) is clear to see. The two cross-sections are not directly comparable as the exponentiated result has dependence upon \( \Omega_{c} \); however it is clear that the cross-sections are comparable as we expect, and the difference is of \( \mathcal{O}(\alpha) \) for values of \( E_{\text{reg}} \simeq 10 \text{ GeV} \) as suggested by equation (2.12). In all following results we will choose \( E_{\text{reg}} = 10 \text{ GeV} \).

Moving onto the more exclusive 1 photon differential cross-section, if we consider events with 1 or more photons present then the energy of the most energetic photon is a physical quantity. This means that we can form the differential cross-section \( d\sigma/d\ln E_{1} \),
Fig. 3 The 1 photon differential cross-section $d\sigma/d\ln E_1$ plotted both at LO and exponentiated.

where $E_1$ is the energy of the most energetic photon, in a meaningful way. If we do this we find the results shown in Fig. 3; in this figure we have also plotted the LO prediction for $d\sigma/d\ln E_1$. As $E \to 0$ we see that the $d\sigma^{\text{LO}}/d\ln E_1 \to \text{constant}$; this is manifest from the soft photon approximation. Equation (2.10) tells us that,

$$
\frac{d\sigma^{\text{LO}}}{d\ln E_1} = \sigma_0 \frac{\alpha}{4\pi^2} g(\Omega_c) \approx \text{constant}
$$

That this breaks down for $E_1 \gtrsim 1$ GeV we can understand as being due to initial state radiation (ISR), the large cross-section comes from the process $e^+e^- \to Z \to \mu^+\mu^-$ with an extra photon radiated; however if the photon is radiated from the initial state with an energy greater that $\mathcal{O}(\Gamma_Z)$ this forces the exchanged $Z^0$ boson far off mass shell, and this suppresses the cross-section.

We can see that for large $E_1$ the LO 1 photon cross-section gives an answer similar to the exponentiated 1 photon cross-section; however for smaller $E_1$ the LO result over estimates the cross-section. The effect of resumming the logarithms associated with the soft singularity is to change the shape of the 1 photon cross-section. Notice that changing
the value of $E_{\text{reg}}$ largely just changes the overall normalisation by a factor of,

$$(E_{\text{reg}})^{-\alpha g(\Omega_c)/4\pi^2}$$

and does not change the overall shape of the 1 photon cross-section.

For $E_1 \simeq \sqrt{s}/2$ the cross-section has a small maximum, this is mainly due to $e^+ e^- \rightarrow \mu^+ \mu^-$ production followed by a muon fragmenting into a photon. It is clear to see why this produces a maxima for large photon energies, consider,

$$\mu^*(1) \rightarrow \mu(2) \gamma(3)$$

this fragmentation is proportional to the propagator of muon 1; i.e., it is inversely proportional to $p_1^2 = (p_2 + p_3)^2$, the final muon photon invariant mass. If we look at a fragmentation at a particular $p_1^2 = (p_2 + p_3)^2$ value, if the photon and muon 2 have similar energies then they are produced largely collinear and the cut (4.3) vetos this event. However if either the final muon, 2, or the photon is very soft then the angle $\theta_{\mu\gamma}$ is far larger and we pass the collinear cut (4.3). The configuration where the photon carries most of the energy of muon 1 gives rise to the small peak in Fig.3 at large $E_1$. 

\[ \text{Fig.4} \] The ratio of the exponentiated divided by the LO 1 photon differential cross-section as a function of the photon energy, $E_1$. 

\[ \text{Fig.3} \]
In Fig. 4 we plot the ratio,
\[ \frac{d\sigma^{\text{exp}}/d\ln E_1}{d\sigma^{\text{LO}}/d\ln E_1} \] (4.9)

We can see that there is no particular energy where exponentiation becomes important, rather that as we consider lower and lower energy photons exponentiation slowly becomes a more important factor. For our choice of \( E_{\text{reg}} = 10 \text{ GeV} \) the exponentiated result is comparable to the LO results for large \( E_1 \) (the dip close to \( \sqrt{s}/2 \) is due to the the process (4.8) mentioned before), however for photon energies of 1 GeV the exponentiated result is down to 90% of the LO result.

It is of interest to ask how the soft photon approximation compares to the exact matrix element result. However before we compare the two approaches there is a subtlety with the soft photon approximation that should be appreciated. The soft photon approximation tells us that the matrix element with a photon present is the matrix element without that photon multiplied by an eikonal factor. However what does the matrix element without the photon mean? Consider the process \( e^+ e^- \rightarrow \mu^+ \mu^- \gamma \), if we remove the photon then we no longer have 4 momenta conserved; in particular if we ask what is the \( Q^2 \) carried by the \( s \) channel \( Z^0 \) or \( \gamma \) we can form this in two ways as \( (p_{e^+} + p_{e^-})^2 \) or as \( (p_{\mu^+} + p_{\mu^-})^2 \) and these have different values. As such the soft photon approximation has ambiguities associated with what the matrix element without photons is. Clearly as the photon energy goes to zero this ambiguity disappears.

In the current case of \( e^+ e^- \rightarrow \mu^+ \mu^- \) with \( \sqrt{s} = M_Z \) this ambiguity can be of great importance for photon energies greater than \( \mathcal{O}(\Gamma_Z) \). If we have initial state radiation (ISR) with energy greater than \( \mathcal{O}(\Gamma_Z) \) then this forces the exchanged \( Z^0 \) boson off mass shell, and this strongly suppresses the cross-section; this means that the relevant \( Q^2 \) for the exchanged \( Z^0 \) is \( (p_{\mu^+} + p_{\mu^-})^2 \) which means that for photon energies greater than \( \mathcal{O}(\Gamma_Z) \) the \( Z^0 \) propagator is strongly suppressed. Similarly if we have final state radiation (FSR) with energy greater than \( \mathcal{O}(\Gamma_Z) \) then the relevant \( Q^2 \) is \( (p_{e^+} + p_{e^-})^2 \). With this in mind, and given that photon radiation is radiated off a pair of charged legs (at least for massless fermions) a good prescription is to boost the two legs that the photon can be radiated off in such a way that 4 momentum is conserved and that their momentum only changes by a small amount. In the present case this is messy as the soft eikonal factor (2.4) has 6 different combinations of legs that it can be radiated off, and so there are 6 different boosts that need to be carried out. This is beyond the scope of the current work; instead we consider the two simpler cases of pure ISR and pure FSR \((i.e., \text{the cases where})\).
we take the final and initial legs respectively to be chargeless). For pure ISR (FSR) it is clear which legs the photon is radiated off, and hence which legs should be boosted; for ISR we boost the electron legs, and for FSR we boost the muon legs. If we are to boost momenta $p_1$ and $p_2$ we use the boost defined by,

$$p_1' = a_{11} p_1 + a_{12} p_2 + p_T$$
$$p_2' = a_{21} p_1 + a_{22} p_2 + p_T$$

(4.10)

with $p_1'^2 = p_1^2$, $p_2'^2 = p_2^2$, $p_1 \cdot p_T = 0$ and $p_2 \cdot p_T = 0$; where we require $p_1'$ and $p_2'$ to satisfy 4 momentum in the system without photons.

We plot the ratio of the soft approximation to the hard matrix element in Fig.5. It is clear that the soft approximation is good for photon energies up to $\mathcal{O}(10 \text{ GeV})$ but underestimates the hard result for larger energies. For the case of ISR this makes little difference as radiation with energy larger than $\mathcal{O}(\Gamma_Z)$ is strongly suppressed; and so we only make mistakes where the contribution to the cross-section is small. The rate for hard FSR can be quite appreciable and for this hard radiation the soft approximation underestimates the cross-section by a large factor. It seems hard to improve the implementation of the soft photon approximation in such a way that it correctly describes these hard photons.

As a final numerical result we wish to consider the photon photon invariant mass distribution in the process,

$$e^+ e^- \rightarrow \mu^+ \mu^- \gamma \gamma$$

(4.11)
Fig. 6 The exponentiated differential cross-section $d\sigma/dM_{\gamma\gamma}$ both with hard matrix elements and in the case where the hardest photon is treated exactly and the remaining photons are described by the soft photon approximation.

This process has been the subject of much interest since the L3 collaboration at LEP announced 4 events with $M_{\gamma\gamma} = 60$ GeV [12] in apparent conflict with the Standard Model. Although the conflict with the Standard Model has since gone away with improved statistics this process serves as a good illustration of a process that contains both soft and hard photons and so both exponentiation of soft photons and exact matrix elements for hard photons are important.

We use cuts similar to the L3 experimental cuts:

$$
|\cos\theta_\gamma| < 0.9 \quad 36^\circ < \theta_\mu < 144^\circ \\
\theta_{\gamma\mu} > 5^\circ \quad \theta_{\mu\mu} > 20^\circ \\
E_\gamma > 1 \text{ GeV}
$$

(4.12)

We show the exponentiated hard differential cross-section we obtain in Fig. 6. Now we can consider various different approximations to this process. If we just use the LO hard matrix element result we would expect to describe large $M_{\gamma\gamma}$ well, but have larger difficulty with small $M_{\gamma\gamma}$; in practice with the cuts used there is no difference within the Monte Carlo errors from the exponentiated hard result for all values of $M_{\gamma\gamma}$.
We also compare the soft photon approximation, for the comparison we have boosted the final state muons in order to conserve 4 momenta within the soft approximation. This keeps the exchanged $Z^0$ on mass shell even when we have hard ISR, as such we would expect this to overestimate the cross-section. We show the exponentiated cross-section where we calculate the hardest photon using the exact matrix element and calculate additional photons using the soft approximation. It is clear that although this works well for small $M_{\gamma\gamma}$ this underestimates the cross-section by a large amount for large $M_{\gamma\gamma}$. Surprisingly the differential cross-section is almost unchanged when we calculate all photons within the soft approximation from the result when we treat one photon exactly.

For this process it is clear that describing the hard photons correctly is more important than exponentiating the soft photons.

5 Conclusions

In this paper we have presented a method for exponentiating the singularities associated with soft photon emission while still retaining exact matrix elements for hard photon emission. We have applied this method to the process, $e^+e^- \rightarrow \mu^+\mu^- + n\gamma$ (5.1) where the photons can be radiated from any of the charged legs, i.e., we have included ISR, FSR and all the interference terms. We have also retained the mass dependence of the muon.

Exponentiation is of importance for low energy photons where it suppresses the cross-section over the LO prediction, whereas exact matrix elements are important for hard photons when the soft approximation that is the basis of exponentiation breaks down. Experimentally an accurate description of both soft and hard photons is often simultaneously required as in a process where the hard photons are of primary interest the low energy photons are used to normalise the theoretical predictions for that process.

We have give a method by which one could also remove the collinear singularities, and resum the associated logarithms, in section 3. As we are not currently interested in these singularities we have made no attempt to include these effects in our Monte Carlo; but instead impose experimental angular cuts on all photons.
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Appendix

In this appendix we give the explicit form of the function \( g(\Omega_c) \). \( g \) is defined as,

\[
g(\Omega_c) = \int_{\Omega_c} f(\Omega) d\Omega \quad (A.1)
\]

with,

\[
f(\Omega) = E^2 \left( \frac{p_1}{p_1 \cdot k} - \frac{p_2}{p_2 \cdot k} + \frac{p_3}{p_3 \cdot k} - \frac{p_4}{p_4 \cdot k} \right)^2 \quad (A.2)
\]

Now \( f(\Omega) \) contains two types of term, \( E^2 p_1 p_2 / (p_1 \cdot k) \) and \( E^2 m^2 / (p_1 \cdot k)^2 \). Concentrating on the first of these we have to integrate,

\[
I_1 = \int 2E^2 \frac{p_1 \cdot p_2}{p_1 \cdot k p_2 \cdot k} d\Omega = \int \frac{2(E_1 E_2 - P_1 P_2 \cos \rho) \, d\cos \theta \, d\phi}{(E_1 - P_1 \cos \theta)(E_2 - P_2 \cos \rho \cos \theta - P_2 \sin \rho \sin \theta \cos \phi)} \quad (A.3)
\]

where we have written,

\[
p_1 = (E_1, P_1, 0, 0) \quad \quad p_2 = (E_2, P_2 \cos \rho, P_2 \sin \rho, 0) \quad \quad k = (E, E \cos \theta, E \sin \theta \cos \phi, E \sin \theta \sin \phi)
\]

Integrating over \( \phi \) gives,

\[
I_1 = \int \frac{2\pi \, d\cos \theta}{(E_1 - P_1 \cos \theta)((P_2 \cos \theta - E_2 \cos \rho)^2 + m_2^2 \sin^2 \rho)^{1/2}} \quad (A.5)
\]

and performing the indefinite integral over \( \cos \theta \),

\[
I_1(p_1, p_2, \cos \theta) = \frac{2\pi}{(z^2 + m_2^2 \sin^2 \rho P_1^2)^{1/2}} \times \ln \left( \frac{(y^2 + m_2^2 \sin^2 \rho)^{1/2} (z^2 + m_2^2 \sin^2 \rho P_1^2)^{1/2} + zy + m_2^2 \sin^2 \rho P_1}{E_1 - P_1 \cos \theta} \right) \quad (A.6)
\]

where \( y \) and \( z \) are defined as,

\[
y = -P_2 \cos \theta - E_2 \cos \rho \quad \quad z = P_2 E_1 - E_2 P_1 \cos \rho \quad (A.7)
\]

Now we want integrate \( f(\Omega) \) over the region defined by \( \Omega_c \), that is the whole of space with small angular regions removed about the charged particles. Two of those charged
particles are \( p_1 \) and \( p_2 \), now \( \theta \) is the angle between \( p_1 \) and the photon, and so we can form the integration over the region with \( \theta_1 \gamma > \theta_1 \) and \( \theta_2 \gamma > \theta_2 \) as,

\[
I = (I_1(p_1, p_2, \cos \theta_1) - I_1(p_1, p_2, -1)) - (I_1(p_2, p_1, 1) - I_1(p_2, p_1, \cos \theta_2)) \tag{A.8}
\]

Now as \( m_2 \to 0 \), \( I_1(p_1, p_2, -1) \) and \( I_1(p_2, p_1, 1) \) diverge, and these two divergences cancel on each other. In order to avoid numerical difficulties it is wise to extract the singular terms and cancel them by hand. This we can do by rewriting the logarithm in equation (A.6) as,

\[
\ln \left( \frac{y^2 + m_2^2 \sin^2 \rho}{m_1^2 + (1 - \cos \theta) P_1(1 - P_1)} \right) = \ln \left( \frac{E_1 + P_1}{E_1 - P_1 \cos \theta} \right) \left( \frac{\rho P_1^2}{(y^2 + m_2^2 \sin^2 \rho)^{1/2}(\rho P_1^2)^{1/2} + zy + m_2^2 \sin^2 \rho P_1} \right) \tag{A.9}
\]

In these forms we can explicitly see the \( \ln(m_2^2) \) term and as \( \cos \theta \to 1 \) the \( -\ln(m_1^2) \) term and cancel them by hand.

So far we have assumed that the angular cuts about charged particles do not overlap, in practice if the angular cuts overlap the charged particles are close in phase space and so \( p_1 \cdot p_2 \) is small and the eikonal factor as a whole is small. Numerically the problem that we encounter is that \( I \) goes negative, in the case that this happens we set \( I = 0 \) and as the eikonal factor is small this only makes a small error.

In addition to particles 1 and 2 we also have two additional particles 3 and 4 that the photon is kept separated from. However these particles are typically far away from the singularities of \( f \) and is slowly varying. This means we can approximate the effect of the cuts \( \theta_3 \gamma > \theta_3 \) and \( \theta_4 \gamma > \theta_4 \) by evaluating \( E^2 p_1 \cdot p_2 / p_1 \cdot k p_2 \cdot k \) in the direction of particles 3 and 4 and then multiplying this by the volume of space that the cut excludes,

\[
2\pi(1 - \cos \theta_3) \left. \left( \frac{E^2 p_1 \cdot p_2}{p_1 \cdot k p_2 \cdot k} \right) \right|_{k \parallel p_3} \quad \text{and} \quad 2\pi(1 - \cos \theta_4) \left. \left( \frac{E^2 p_1 \cdot p_2}{p_1 \cdot k p_2 \cdot k} \right) \right|_{k \parallel p_4} \tag{A.10}
\]

we then subtract these two contributions from (A.8).

It remains to calculate the integral of \( E^2 m^2 / (p_1 \cdot k)^2 \). This is given by

\[
I_2(\cos \theta) = \int \frac{E^2 m^2}{(p_1 \cdot k)^2} d\Omega = \int d\cos \theta \int_0^{2\pi} d\phi \frac{m^2}{(E_1 - P_1 \cos \theta)^2} = \frac{2\pi m^2}{P_1(E_1 - P_1 \cos \theta)} \tag{A.11}
\]

- 24 -
As before we impose the angular cut about particle 1 by integrating (A.11) between $-1 < \cos \theta < \cos \theta_1$, and impose the angular cuts about particles 2,3 and 4 by evaluating $E_2^2 m^2 / (p_1 \cdot k)^2$ in the directions of particles 2,3 and 4 and multiplying by the volume of the angular cut.
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