A Variance-Reduced Stochastic Gradient Tracking Algorithm for Decentralized Optimization with Orthogonality Constraints

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Abstract

Decentralized optimization with orthogonality constraints is found widely in scientific computing and data science. Since the orthogonality constraints are nonconvex, it is quite challenging to design efficient algorithms. Existing approaches leverage the geometric tools from Riemannian optimization to solve this problem at the cost of high sample and communication complexities. To relieve this difficulty, based on two novel techniques that can waive the orthogonality constraints, we propose a variance-reduced stochastic gradient tracking (VRSGT) algorithm with the convergence rate of $O(1/k)$ to a stationary point. To the best of our knowledge, VRSGT is the first algorithm for decentralized optimization with orthogonality constraints that reduces both sampling and communication complexities simultaneously. In the numerical experiments, VRSGT has a promising performance in a real-world autonomous driving application.

1 Introduction

This paper focuses on the multi-agent optimization problem (1.1) with orthogonality constraints that is defined on a connected network $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with $\mathcal{V} = [d] := \{1, 2, \ldots, d\}$ being composed of all the agents and $\mathcal{E} = \{(i, j) \mid i < j, i$ and $j$ are connected\} being the set of all the communication links.

$$\min_{X \in \mathbb{R}^{n \times p}} f(X) := \frac{1}{d} \sum_{i=1}^{d} f_i(X)$$

s.t. $X^\top X = I_p$. (1.1)

Here, $I_p$ stands for the $p \times p$ identity matrix, and $f_i : \mathbb{R}^{n \times p} \to \mathbb{R}$ is a local loss function privately known by the $i$-th agent for $i \in [d]$. The set of all $n \times p$ orthogonal matrices is referred to as Stiefel manifold $\mathcal{S}_{n,p} = \{X \in \mathbb{R}^{n \times p} \mid X^\top X = I_p\}$, which is a special case of Riemannian manifold $\mathcal{M}$ embedded in $\mathbb{R}^{n \times p}$.

Problems of the form (1.1) have profound applications in various scientific and engineering areas, such as spectral analysis [17, 15], dictionary learning [28], eigenvalue estimation of the covariance matrix [24], and deep neural networks with orthogonal constraints [2, 33, 14]. It is noteworthy that, in these applications, each local function $f_i$ is an average cost of different samples. Without loss of generality, we assume that each agent owns $l$ samples and $f_i$ can be represented as follows.

$$f_i(X) = \frac{1}{l} \sum_{j=1}^{l} f_{ij}^j(X),$$

where $f_{ij}^j$ denotes the cost for the $j$-th data sample at the $i$-th agent.
1.1 Decentralized Formulation

In this paper, we consider the scenario that only local communications are permissible, namely, the agents can only exchange information with their immediate neighbors through the network. Thus, the global information is not available, and the agent $i$ needs to maintain a local copy $X_i$ of the common variable $X$. Each agent only uses the local information to update its local variable and then communicates with neighbors to reach a consensus. This type of problem is usually called decentralized optimization.

For convenience, we use the notation $X = [X_1^T, \ldots, X_d^T]^T \in \mathbb{R}^{dn \times p}$ to stack all the local variables. Now the concerned problem (1.1) can be recast as the following decentralized formulation.

\[
\min_X \frac{1}{d} \sum_{i=1}^{d} f_i(X_i) \\
\text{s.t. } X_i = X_j, \ (i, j) \in E, \\
X_i^T X_i = I_p, \ i \in [d].
\]

Since the network $G$ is assumed to be connected, the constraints (1.3b) enforce the consensus condition $X_1 = X_2 = \cdots = X_d$. Therefore, our goal is to seek a consensus such that each local variable $X_i$ is a first-order stationary point of (1.1) whose definition can be stated as follows.

**Definition 1.1 ([34]).** A point $X \in \mathbb{R}^{n \times p}$ is called a first-order stationary point of (1.1) if and only if

\[
0 = \text{grad } f(X) := \nabla f(X) - X \text{sym } (X^T \nabla f(X)) \quad \text{and } X^T X = I_p.
\]

Here, $\text{grad } f(X)$ denotes the Riemannian gradient [1] of $f$ at $X$, and $\text{sym}(B) = (B + B^T)/2$ represents the symmetric part of the given matrix $B$.

In the decentralized optimization, the structure of the communication network $G$ is described by a mixing matrix $W = [W(i, j)] \in \mathbb{R}^{d \times d}$. We assume that $W$ satisfies the following properties, which are standard in the literature. There are a few common choices for the mixing matrix $W$. Interested readers are referred to [35, 30] for more details.

**Assumption 1.** The mixing matrix $W$ is symmetric, nonnegative, and doubly stochastic (i.e., $W 1_d = W^T 1_d = 1_d$). Moreover, $W(i, j) = 0$ if $i \neq j$ and $(i, j) \notin E$.

1.2 Existing Works

In this subsection, we briefly review some existing approaches closely related to the present work.

**Riemannian SVRG methods.** Recent years have witnessed the rapid development of optimization algorithms on the Riemannian manifolds. Interested readers are referred to a comprehensive survey [12] for more details. In the following, we briefly review several Riemannian SVRG algorithms. Based on the geometric tools developed in [1], a large amount of efforts has been devoted to extending the SVRG-type algorithm from the Euclidean space to the Riemannian manifolds. Existing SVRG algorithms for manifold optimization can be divided into two categories. The first category constructs stochastic variance reduced Riemannian gradients by invoking parallel transports or vector transports [1], including [42, 29]. As illustrated in [26], the evaluation of parallel transports or vector transports is computationally expensive. Therefore, the second category of Riemannian SVRG algorithms, such as [16, 21], is proposed to waive the computation of these two operations. However, the aforementioned methods can hardly be extended to the decentralized setting.

**Decentralized algorithms in the Euclidean space.** Decentralized optimization has been adequately investigated in the Euclidean space with different types of decentralized algorithms emerging in large numbers, such as decentralized gradient descent (DGD) [23, 40, 41], decentralized gradient tracking (DGT) [30, 39, 27], decentralized stochastic gradient descent (DSGD) [19], decentralized SVRG
method [32, 37], and decentralized primal-dual algorithm [4, 20, 11, 9]. It is worthy of mentioning that none of the above algorithms are able to deal with nonconvex constraints, and thus not applicable to the problem (1.3). We refer to three recent survey papers [22, 38, 3] for a complete review on the decentralized algorithms in the Euclidean space.

Decentralized Riemannian gradient descent methods. To solve decentralized problems over the Riemannian manifold, such as problem (1.3), Chen et al. [5] propose two algorithms DRGTA and DRSGD, which are the Riemannian extensions of DGT and DSGD, respectively. Although the exact convergence is guaranteed with fixed stepsizes, full batch gradient computation is involved in each iteration of DRGTA, giving rise to high sample complexity. On the contrary, DRSGD requires diminishing stepsizes to find an exact solution, which significantly slows down the convergence. This incurs high communication complexity, since it is necessary for DRSGD to take numerous rounds of communications to reach certain accuracy. Finally, we note that both DRGTA and DRSGD require $t \geq \lceil O(-\log_\lambda(2\sqrt{d})) \rceil$ rounds of communications per iteration to guarantee the convergence. For sparse networks, $t$ is in the order of $O(d^2 \log d)$ [5]. This prerequisite further imposes an enormous burden on the communications.

1.3 Our Contributions

The main contributions of this paper are three folds.

(i) We propose a variance-reduced stochastic gradient tracking (VRSGT) algorithm for solving problem (1.3) to reduce both sampling and communication complexities at the same time. Please see Table 1 for detailed comparisons with existing peer methods.

(ii) VRSGT is based on two novel techniques, augmented Lagrangian estimation and gradient approximation, to deal with orthogonality constraints, which are of independent values. Basically, one can utilize any unconstrained algorithms to solve the problem (1.3) by resorting to these two techniques. The convergence of VRSGT is established in Theorem 3.1.

(iii) We apply VRSGT to solve dual principal component pursuit (DPCP) problems that play an important role in autonomous driving. VRSGT has the ability to recognize the objects in the road quickly and precisely.

Table 1: A comparison of sample and communication complexities to find a first-order $\epsilon$-stationary point (to be defined in Definition 2.2) on decentralized algorithms for the problem (1.3).

| Algorithm       | Sample complexity | Communication complexity |
|-----------------|-------------------|--------------------------|
| DRGTA [5]       | $O(dl/\epsilon)$  | $O(t/\epsilon)^1$        |
| DRSGD [5]       | $O(d/\epsilon^2)$| $O(t/\epsilon^2)^1$      |
| VRSGT (this work)| $O(d\sqrt{\ell}/\epsilon)$ | $O(1/\epsilon)$ |

$^1 t \geq \lceil O(-\log_\lambda(2\sqrt{d})) \rceil$. For sparse networks, $t$ can be $O(d^2 \log(d))$ [5].

1.4 Notations

The Euclidean inner product of two matrices $Y_1, Y_2$ with the same size is defined as $\langle Y_1, Y_2 \rangle = \text{tr}(Y_1^T Y_2)$, where $\text{tr}(B)$ stands for the trace of a square matrix $B$. And the notation $\text{sym}(B) = (B + B^T)/2$ represents the symmetric part of $B$. The Frobenius norm and 2-norm of a given matrix $C$ is denoted by $\|C\|_F$ and $\|C\|_2$, respectively. The $(i, j)$-th entry of a matrix $C$ is represented by $C(i, j)$. The notation $\mathbf{1}_d$ stands for the $d$-dimensional vector of all ones. The Kronecker product is denoted by $\otimes$. Given a differentiable function $g(X) : \mathbb{R}^{n \times p} \to \mathbb{R}$, the gradient of $g$ with respect to $X$ is represented by $\nabla g(X)$. Further notation will be introduced wherever it occurs.
2 Algorithm Design

In this section, we develop two novel techniques to deal with the nonconvex orthogonality constraints. Based on these techniques, we propose a variance-reduced stochastic gradient tracking algorithm.

2.1 Augmented Lagrangian Estimation

The augmented Lagrangian function of (1.3) can be represented as follows.

$$L(X, \Lambda) := \frac{1}{d} \sum_{i=1}^{d} L_i(X_i, \Lambda_i),$$

where

$$L_i(X_i, \Lambda_i) := f_i(X_i) - \frac{1}{2} \langle \Lambda_i, X_i^T X_i - I_p \rangle + \frac{\beta}{4} \|X_i^T X_i - I_p\|_F^2.$$

In the above augmented Lagrangian function, we only penalize the orthogonality constraints (1.3c). Here, $\beta > 0$ is a penalty parameter, $\Lambda_i \in \mathbb{R}^{p \times p}$ denotes the local Lagrangian multiplier corresponding to the orthogonality constraint of $X_i$, and $\Lambda = [\Lambda_1^T, \ldots, \Lambda_d^T]^T \in \mathbb{R}^{dp \times p}$ stacks all the local multipliers.

According to the discussions in [7, 36, 34], the multiplier associated with orthogonality constraints admits a closed-form expression at any first-order stationary point. Therefore, it is straightforward to verify that each local multiplier admits the following explicit expression.

$$\Lambda_i = \text{sym} \left( X_i^T \nabla f_i(X_i) \right), i \in [d].$$

The second term of $L_i(X_i, \Lambda_i)$ with the above expression of $\Lambda_i$ can be rewritten as the following form.

$$\langle \Lambda_i, X_i^T X_i - I_p \rangle = \langle \nabla f_i(X_i), X_i X_i^T X_i - X_i \rangle.$$

Then in light of the definition of directional derivative, we have

$$\lim_{\sigma \to 0} \frac{f_i(X_i + \sigma (X_i X_i^T X_i - X_i)) - f_i(X_i)}{\sigma} = \langle \nabla f_i(X_i), X_i X_i^T X_i - X_i \rangle. \quad (2.1)$$

For convenience, we denote

$$A(X_i) := X_i + \sigma (X_i X_i^T X_i - X_i) = (1 - \sigma)X_i + \sigma X_i X_i^T X_i. \quad (2.2)$$

This motivates us to replace the second term of $L_i(X_i, \Lambda_i)$ with $(f_i(A(X_i)) - f_i(X_i))/\sigma$ for a small constant $\sigma \neq 0$, and the following function is obtained.

$$h_i(X_i) := g_i(X_i) + \beta b(X_i),$$

where

$$g_i(X) := \left(1 + \frac{1}{2\sigma}\right) f_i(X_i) - \frac{1}{2\sigma} f_i(A(X_i)), \text{ and } b(X_i) := \frac{1}{4} \|X_i^T X_i - I_p\|_F^2.$$

Finally, the augmented Lagrangian function $L(X, \Lambda)$ can be estimated by the following function.

$$h(X) = \frac{1}{d} \sum_{i=1}^{d} h_i(X_i), \quad (2.3)$$

which is called augmented Lagrangian estimation.
2.2 Gradient Approximation

Now we discuss the computation related to the gradient of the augmented Lagrangian estimation \( h(X) \). After straightforward calculations, we have

\[
\nabla h(X) = \left[ \nabla h_1(X_1)^T, \ldots, \nabla h_d(X_d)^T \right]^T,
\]

where

\[
\nabla h_i(X_i) = \left( 1 + \frac{1}{2\sigma} \right) \nabla f_i(Z)|_{Z=X_i} - \nabla f_i(Z)|_{Z=A(X_i)} \left( \frac{1-\alpha}{2\sigma} I_p + \frac{1}{2} X_i^T X_i \right) - X_i \text{sym} \left( X_i^T \nabla f_i(Z)|_{Z=A(X_i)} \right) + \beta X_i (X_i^T X_i - I_p).
\]  

(2.4)

The gradient of \( f_i \) is evaluated twice at different points \( X_i \) and \( A(X_i) \) in \( \nabla h_i(X_i) \), which incurs high computational costs. We find that these two points are very close as long as the orthogonality violation of \( X_i \) is small since \( \|A(X_i) - X_i\|_F \leq \sigma \|X_i\|_2 \|X_i^T X_i - I_p\|_F \). Therefore, we propose to approximate \( \nabla f_i(Z)|_{Z=A(X_i)} \) by \( \nabla f_i(Z)|_{Z=X_i} \) to reduce the computational costs. This yields the following direction \( H_i(X_i) \in \mathbb{R}^{n \times p} \) as an approximation of \( \nabla h_i(X_i) \).

\[
H_i(X_i) := G_i(X_i) + \beta E(X_i),
\]

where \( E(X_i) := X_i (X_i^T X_i - I_p) \) and

\[
G_i(X_i) := \nabla f_i(Z)|_{Z=X_i} \left( \frac{3}{2} I_p - \frac{1}{2} X_i^T X_i \right) - X_i \text{sym} \left( X_i^T \nabla f_i(Z)|_{Z=X_i} \right).
\]  

(2.5)

Let \( H(X) := \sum_{i=1}^d H_i(X)/d \) and \( G(X) := \sum_{i=1}^d G_i(X)/d \). We have the following lemma for \( H(X) \) and \( G(X) \).

**Lemma 2.1.** Let \( \mathcal{R} := \{ X \in \mathbb{R}^{n \times p} \mid \|X^T X - I_p\|_F \leq 1/6 \} \) be a bounded region and \( M := \sup\{\|\nabla f(X)\|_F \mid X \in \mathcal{R} \} \) be a positive constant. Then if \( \beta \geq (6 + 21M)/5 \), we have

\[
\|H(X)\|_F^2 \geq \|G(X)\|_F^2 + \beta \|X^T X - I_p\|_F^2,
\]

for any \( X \in \mathcal{R} \).

The proof of Lemma 2.1 is arranged to Appendix A. It is noteworthy that \( G(X) = \text{grad} \ f(X) \) for any \( X \in \mathcal{S}_{n,p} \). Hence, \( G(\cdot) \) can be interpreted as the generalized Riemannian gradient of \( f \) in the Euclidean space. Furthermore, Lemma 2.1 shows that both the orthogonality violation of \( X \) and the norm of \( G(X) \) is controlled by the norm of \( H(X) \) as long as \( X \) is restricted in the bounded region \( \Omega \). Consequently, under the decentralized setting, \( H_i(\cdot) \) can be employed as a local descent direction since the substationarity and orthogonality violation can be reduced simultaneously. In addition, for the finite sum setting (1.2) considered in this paper, we have

\[
H_i(X_i) = \frac{1}{l} \sum_{j=1}^l H_i^{[j]}(X_i),
\]

where \( H_i^{[j]}(X_i) = \nabla f_i^{[j]}(Z)|_{Z=X_i} \left( 3 I_p - X_i^T X_i \right)/2 - X_i \text{sym}(X_i^T \nabla f_i^{[j]}(Z)|_{Z=X_i}) + \beta E(X_i) \).

Finally, under the decentralized setting, we introduce the following stationarity gap \( \text{StaGap}(X) \) for (1.3).

\[
\text{StaGap}(X) := \left\| \frac{1}{d} \sum_{i=1}^d G_i(\bar{X}) \right\|_F^2 + \left( \frac{1}{d} \sum_{i=1}^d \|X_i - \bar{X}\|_F^2 + \left\| \bar{X}^T \bar{X} - I_p \right\|_F^2, \right.
\]

(2.7)

\[
\]  

where \( \bar{X} := \sum_{i=1}^d X_i/d \).

Based on the stationarity gap, we define the following first-order \( \epsilon \)-stationary point of (1.3).

**Definition 2.2.** For a sufficiently small constant \( \epsilon > 0 \), a point \( X = [X_1^T, \ldots, X_d^T]^T \) is called a first-order \( \epsilon \)-stationary point of (1.3) if and only if \( \text{StaGap}(X) \leq \epsilon \).
2.3 Algorithmic Development

Based on the augmented Lagrangian estimation (2.3) developed in Subsection 2.1, (1.3) can be converted to the following unconstrained problem.

\[
\begin{aligned}
\min_X \quad & b(X) \\
\text{s.t.} \quad & X_i = X_j, \quad (i,j) \in E,
\end{aligned}
\]

(2.8)

Now we can design decentralized algorithms to solve (2.8) without consideration of orthogonality constraints. The gradient approximation techniques developed in Subsection 2.2 can be applied to reduce the computational costs.

To introduce our algorithm, we first define two auxiliary local variables \( S_i \) and \( D_i \). Specifically, \( S_i \) aims at approximating the local full batch descent direction \( \frac{\sum_{j=1}^{t} H_{ij}^{[j]}(X_i)}{\lambda} \) by only using a subset of samples, while \( D_i \) is designed to track the global descent direction \( \sum_{r=1}^{d} H_r(X_r) / d \) based on \( S_i \). After the local and global descent direction estimates are obtained, our algorithm performs a local update along the direction of \(-D_i\).

In the following discussions, we use \( X_i^{(k,t)} \) to denote the \( t \)-th inner iterate of \( X_i \) at the \( k \)-th outer iteration. Other notations \( S_i^{(k,t)} \) and \( D_i^{(k,t)} \) are similar. Please see the main steps of our algorithm below.

**Step 1: X-update.** At the \( k \)-th outer iteration, each agent \( i \) first performs a descent in the direction of the global descent direction estimate \( D_i^{(k,0)} \), and then combines the results with its neighbors.

\[
X_i^{(k+1,1)} = \sum_{r=1}^{d} W(i,r) \left( X_r^{(k,0)} - \eta D_r^{(k,0)} \right),
\]

where \( \eta > 0 \) is the stepsize. The inner iteration is similar.

**Step 2: S-update.** At the \( k \)-th outer iteration, each agent \( i \) directly computes the local full batch descent direction as follows.

\[
S_i^{(k+1,1)} = H_i(X_i^{(k+1,1)}).
\]

In contrast, the \( t \)-th inner iteration estimates it by using a subset \( C^{(k+1,t)} \) of \( \tau \) samples.

\[
S_i^{(k+1,t+1)} = \frac{1}{\tau} \sum_{j \in C^{(k+1,t)}} \left( H_{ij}^{[j]}(X_i^{(k+1,t+1)}) - H_{ij}^{[j]}(X_i^{(k+1,t)}) \right) + S_i^{(k+1,t)},
\]

where \( \tau \in [1, \bar{t}] \) is the batch size.

**Step 3: D-update.** At the \( k \)-th outer iteration, each agent \( i \) combines the previous estimate \( D_i^{(k,0)} \) with its neighbors, and then make a new estimation based on the fresh information \( S_i^{(k+1,1)} \).

\[
D_i^{(k+1,1)} = \sum_{r=1}^{d} W(i,r) D_r^{(k,0)} + S_i^{(k+1,1)} - S_i^{(k,0)}.
\]

The inner iteration is similar.

We summarize the proposed algorithm as a compact form in Algorithm 1, which is called variance-reduced stochastic gradient tracking and abbreviated to VRSGT. In Algorithm 1, we use \( \mathbf{X}^{(k,t)} \in \mathbb{R}^{dn \times p}, \mathbf{S}^{(k,t)} \in \mathbb{R}^{dn \times p}, \mathbf{D}^{(k,t)} \in \mathbb{R}^{dn \times p}, \mathbf{H}^{(k,t)} \in \mathbb{R}^{dn \times p}, \) and \( \mathbf{H}^{(j)}(\mathbf{X}^{(k,t)}) \in \mathbb{R}^{dn \times p} \) to denote the concatenation of \( X_i^{(k,t)} \in \mathbb{R}^{n \times p} \), \( S_i^{(k,t)} \in \mathbb{R}^{n \times p} \), \( D_i^{(k,t)} \in \mathbb{R}^{n \times p} \), \( H_i(X_i^{(k,t)}) \in \mathbb{R}^{n \times p} \), and \( H_i^{[j]}(X_i^{(k,t)}) \in \mathbb{R}^{n \times p} \) across all the agents, respectively. For convenience, we denote \( \mathbf{W} = \mathbf{W} \otimes \mathbf{I}_n \in \mathbb{R}^{dn \times dn} \).

The total communication rounds required by VRSGT is in the same order as the total number of iterations (both inner and outer iterations), since only two rounds of communications are performed per iteration, via broadcasting the local variables \( X_i \) and \( D_i \) to the neighbors. And the total number of samples used per iteration is either \( d \tau \) (inner iteration) or \( dl \) (outer iteration).
Algorithm 1: Variance-Reduced Stochastic Gradient Tracking (VRSGT).

1. **Input:** initial guess $X_{\text{init}}$, stepsize $\eta > 0$, batch size $\tau \in [1, l]$, inner iteration interval $q > 0$, and penalty parameter $\beta > 0$.
2. Set $k := 0$.
3. Initialize $X^{(0,0)} = (1_d \otimes I_n)X_{\text{init}}$, $S^{(0,0)} := H(X^{(0,0)})$ and $D^{(0,0)} := H(X^{(0,0)})$.
4. while “not converged” do
   5. $X^{(k+1,1)} := W(X^{(k,0)} - \eta D^{(k,0)})$.
   6. $S^{(k+1,1)} := H(X^{(k+1,1)})$.
   7. $D^{(k+1,1)} := WD^{(k,0)} + S^{(k+1,1)} - S^{(k,0)}$.
   8. for $t = 1, 2, \ldots, q$ do
      9. Each node draws a subset $C^{(k+1,t)}$ of $\tau$ samples from $[l]$ with replacement.
     10. $S^{(k+1,t+1)} := \sum_{j \in C^{(k+1,t)}} (H_j[X^{(k+1,t+1)}] - H_j[X^{(k+1,t)}]) / \tau + S^{(k+1,t)}$.
     11. $D^{(k+1,t+1)} := WD^{(k+1,t)} + S^{(k+1,t+1)} - S^{(k+1,t)}$.
   12. Set $X^{(k+1,0)} := X^{(k+1,q+1)}$, $S^{(k+1,0)} := S^{(k+1,q+1)}$, and $D^{(k+1,0)} := D^{(k+1,q+1)}$.
   13. Set $k := k + 1$.
14. Output: $X^{(k,0)}$.

### 3 Convergence Analysis

Throughout this paper, we make the following blanket assumptions about local cost functions.

**Assumption 2.** Each local cost function $f_i^{[j]}$ is first-order differentiable. And $\{f_i^{[j]}\}_{i=1}^l$ satisfies the following mean-squared local Lipschitz smoothness property with a constant $L > 0$.

$$
\sqrt{\frac{1}{l} \sum_{j=1}^l \|\nabla f_i^{[j]}(X) - \nabla f_i^{[j]}(Z)\|_F^2} \leq L \|X - Z\|_F, \quad \text{for any } i \in [d] \text{ and } X, Z \in \tilde{S}, \tag{3.1}
$$

where $\tilde{S}$ is an arbitrary bounded set containing $S_{n,p}$.

Assumption 2 is standard in the decentralized stochastic optimization literature, which is also used in [32, 37]. It should be noted that, in these works, the condition (3.1) is assumed to hold for any $X, Z \in \mathbb{R}^{n \times p}$. Since the Stiefel manifold is compact, this global Lipschitz smoothness property is too stringent for (1.3). Therefore, in Assumption 2, we only impose the local version of this property.

Now we present the global convergence of VRSGT as follows, whose proof is given in Appendix B.

**Theorem 3.1.** Suppose Assumptions 1 and 2 hold. Let $\{X^{(k,t)}\}$ be the iterate sequence generated by Algorithm 1 with $X_{\text{init}} \in \mathbb{R}$, $\eta \in (0, \bar{\eta})$, $\bar{\beta} > \underline{\beta}$, and $\tau = q = \sqrt{l}$. The constants $\bar{\eta}$ and $\underline{\beta}$ are given in Appendix B. Then it holds that

$$
\min_{k=0,1,\ldots,K} \mathbb{E}\left[\text{StaGap}(X^{(k,t)})\right] \leq \frac{C}{\sqrt{lk}},
$$

where $C > 0$ is a constant defined in Appendix B and the expectation is taken over the randomness from the sampling step.

Theorem 3.1 demonstrate that, in order to achieve a first-order $\epsilon$-stationary point of (1.3) by VRSGT, the total number of communication rounds is in the order of $O(1/\epsilon)$, and the total number of samples evaluated across the whole network is in the order of $O(d\sqrt{l}/\epsilon)$.
4 Numerical Experiments

In this section, the numerical performance of VRSGT is evaluated through comprehensive numerical experiments in comparison with state-of-the-art algorithms. The corresponding experiments are performed on a workstation with two Intel Xeon Gold 6242R CPU processors (at 3.10GHz×20×2) and 510GB of RAM under Ubuntu 20.04. All the tested algorithms are implemented in the Python language, and the communication is realized via the package mpi4py. The code of DRSGD [5] is downloaded from GitHub.

4.1 Test Problems

In the numerical experiments, we test two different types of problems to evaluate the performances of compared algorithms, including decentralized (principal component analysis) PCA problem and dual principal component pursuit (DPCP) problem.

Decentralized PCA Problem. As a fundamental tool for dimensionality reduction, PCA is usually a critical procedure or preprocessing step in a large number of numerous statistical and machine learning tasks. under the decentralized setting, PCA amounts to solving the following optimization problem.

\[
\min_{X \in \mathbb{R}^{n \times p}} - \frac{1}{2d} \sum_{i=1}^{d} \left( \frac{1}{l} \sum_{j=1}^{l} \text{tr}\left( X^\top a_{i(j)} a_{i(j)}^\top X \right) \right) \\
\text{s.t. } X \in S_{n,p},
\]

where \(a_{i(j)} \in \mathbb{R}^{n}\) is the \(j\)-th sample owned by the \(i\)-th agent. For convenience, we denote \(A_i = [a_{i(1)}, a_{i(2)}, \ldots, a_{i(l)}] \in \mathbb{R}^{n \times l}\) as the local data matrix and \(A = [A_1, A_2, \ldots, A_d] \in \mathbb{R}^{n \times m}\) as the global data matrix with \(m = dl\).

Decentralized DPCP Problem. DPCP [43, 6] is a recently proposed model for learning subspaces of high relative dimension from datasets contaminated by outliers, which has wide applications in the computer vision, such as detecting planar structures in 3D point clouds in KITTI dataset [8] and estimating relative poses in multiple-view geometry [10]. Specifically, DPCP aims to recover an \(n-1\) dimensional hyperplane in \(\mathbb{R}^{n}\) by a normal vector from \(m\) samples stored in the matrix \(B \in \mathbb{R}^{n \times m}\) to distinguish inliers and outliers. Under the decentralized setting, we assume that \(B = [B_1, B_2, \ldots, B_d]\) with \(B_i = [b_{i(1)}, b_{i(2)}, \ldots, b_{i(l)}]\), where \(B_i \in \mathbb{R}^{n \times l}\) contains all the \(l\) samples owned by the \(i\)-th agent, and \(b_{i(j)} \in \mathbb{R}^{n}\) is the \(j\)-th sample of \(B_i\). Recently, the following optimization problem with orthogonality constraints is proposed in [13] for DPCP.

\[
\min_{X \in \mathbb{R}^{n \times p}} - \frac{1}{3d} \sum_{i=1}^{d} \left( \frac{1}{l} \sum_{j=1}^{l} \|X^\top b_{i(j)}\|_3^3 \right) \\
\text{s.t. } X \in S_{n,p},
\]

where the \(\ell_3\)-norm is defined as \(\|Y\|_3^3 = \sum_{i,j} Y(i,j)^3\).

4.2 Robustness to Penalty Parameter

In this subsection, we show that VRSGT is not sensitive to the choices of penalty parameter \(\beta\). The following experiments are performed on the Erdos-Renyi network with a fixed probability \(\text{prob} = 0.5\), and the batch size is set to 10.

In the following experiments, the global data matrix \(A \in \mathbb{R}^{n \times m}\) (assuming \(n \leq m\) without loss of generality) is constructed by its (economy-form) singular value decomposition as follows.

\[
A = U \Sigma V^\top,
\]
where both $U \in \mathbb{R}^{n \times n}$ and $V \in \mathbb{R}^{m \times n}$ are orthogonal matrices orthonormalized from randomly generated matrices, and $\Sigma \in \mathbb{R}^{n \times n}$ is a diagonal matrix with diagonal entries

$$
\Sigma_{ii} = \xi^{1/2}, \quad i \in [n],
$$

(4.4)

for a parameter $\xi \in (0, 1)$ that determines the decay rate of the singular values of $A$. In general, smaller decay rates (with $\xi$ closer to 1) correspond to more difficult cases. Finally, the global data matrix $A$ is uniformly distributed into $d$ agents.

Figure 1 depicts the performances of VRSGT with different values of $\beta$, which are distinguished by different colors. A synthetic matrix $A$, generated by (4.3) with $n = 200$, $m = 64000$, and $\xi = 0.9$, is tested with $p = 10$, $d = 32$, and $\eta = 0.0001$. We can observe that the curves in Figure 1 almost coincide with each other, which indicates that VRSGT has almost the same performance in a wide range of penalty parameters. Therefore, in the following experiments, we fix $\beta = 1$ by default.

### 4.3 Comparison on PCA Problems

We first compare the performance of VRSGT with the existing algorithm DRSGD [5] in solving the decentralized PCA problem on the MNIST dataset [18]. The corresponding numerical experiments are performed on three different networks with $d = 16$ and $p = 5$, including ring network, star network, and Erdos-Renyi network. In the Erdos-Renyi network, two agents are connected with a fixed probability $\text{prob} = 0.5$. And all the networks are associated with the Metropolis constant matrix [30] as the mixing matrix $W$. The batch size is set to 10 for both VRSGT and DRSGD. Figure 2 depicts the decay of the stationarity gap defined in (2.7) versus the round of communications. We can observe that VRSGT outperforms DRSGD in all the three networks.

![Figure 2: Comparison between VRSGT and DRSGD in solving the decentralized PCA problem.](image)

### 4.4 Comparison on DPCP Problems

Our next experiment is to test VRSGT and DRSGD in solving the decentralized DPCP problem on the KITTI dataset [8], which plays an important role in autonomous driving applications. DPCP is
employed to distinguish the 3D points lying on the road plane (inliers) and other objects off that plane (outliers). Each point cloud in the KITTI dataset contains around $10^5$ points with approximately 50% outliers. And the data is homogenized and normalized to unit $\ell_2$-norm. Specifically, given a 3D point cloud of a road scene, DPCP reconstructs an affine plane \( \{ x \in \mathbb{R}^3 \mid a^\top x - b = 0 \} \) as a representation of the road. This task can be converted to a linear subspace learning problem by embedding the affine plane from $\mathbb{R}^3$ into a hyperplane from $\mathbb{R}^4$ with a normal vector $w = [a, -b]$ through the mapping $x \to [x^\top, 1]^\top$ \[6\].

![Frame 1 of KITTI-CITY-5](image1)

![Frame 45 of KITTI-CITY-5](image2)

![Frame 0 of KITTI-CITY-48](image3)

![Frame 328 of KITTI-CITY-71](image4)

Figure 3: Recovery results of four frames from the KITTI dataset with inliers in blue and outliers in red. Both inliers and outliers are detected by using a threshold on the distance to the hyperplane recovered by each tested algorithm. The results are represented by projecting 3D point clouds onto the image.

In this test, VRSGT and DRSGD are set to run 100 and 200 epochs, respectively, such that they perform the same rounds of communications. And the batch size is set to 16 for both VRSGT and DRSGD. This experiment is conducted on an Erdos-Renyi network with $\text{prob} = 0.5$ and $d = 16$. The Metropolis constant matrix is also associated with the network as the mixing matrix $W$. We
select four frames from the KITTI dataset to test. Table 2 tabulates the corresponding recovery error \( \sqrt{1 - \langle \hat{w}, w^* \rangle^2} \) for each frame, where \( \hat{w} \) is the normal vector recovered by the tested algorithm and \( w^* \) denotes the ground truth. Moreover, we present the recovery results of these four frames in Figure 3. Both Table 2 and Figure 3 illustrate that VRSGT produces better classification accuracy than DRSGD.

Table 2: Recovery errors of VRSGT and DRSGD on selected frames from the KITTI dataset.

| Dataset        | KITTI-CITY-5 | KITTI-CITY-48 | KITTI-CITY-71 |
|----------------|--------------|---------------|---------------|
| Frame          | 1            | 45            | 0             | 328           |
| VRSGT          | 1.074e-02    | 7.997e-03     | 3.399e-02     | 4.088e-02     |
| DRSGD          | 1.063e-01    | 9.618e-02     | 1.646e-01     | 2.144e-01     |

5 Concluding Remarks

In this paper, we propose the algorithm VRSGT to minimize a finite sum of smooth functions with orthogonality constraints, which are distributed over a decentralized network of multiple agents. With appropriate algorithmic parameters, VRSGT achieves significantly improved sampling and communication complexities compared with existing decentralized Riemannian methods [5]. In the future work, we will focus on how to accelerate the training process of deep neural networks with orthogonality constraints [2, 33, 14]. Here, the main difficulty is to tackle the nonsmoothness and possible non-regularity of the objective function.

References

[1] P.-A. Absil, R. Mahony, and R. Sepulchre, *Optimization algorithms on matrix manifolds*, Princeton University Press, 2008.

[2] M. Arjovsky, A. Shah, and Y. Bengio, *Unitary evolution recurrent neural networks*, in International Conference on Machine Learning, PMLR, 2016, pp. 1120–1128.

[3] T.-H. Chang, M. Hong, H.-T. Wai, X. Zhang, and S. Lu, *Distributed learning in the nonconvex world: From batch data to streaming and beyond*, IEEE Signal Processing Magazine, 37 (2020), pp. 26–38.

[4] T.-H. Chang, M. Hong, and X. Wang, *Multi-agent distributed optimization via inexact consensus ADMM*, IEEE Transactions on Signal Processing, 63 (2015), pp. 482–497.

[5] S. Chen, A. Garcia, M. Hong, and S. Shahrampour, *Decentralized Riemannian gradient descent on the Stiefel manifold*, in Proceedings of the 38th International Conference on Machine Learning, vol. 139, PMLR, 2021, pp. 1594–1605.

[6] T. Ding, Z. Zhu, T. Ding, Y. Yang, D. P. Robinson, M. C. Tsakiris, and R. Vidal, *Noisy dual principal component pursuit*, in Proceedings of the 36th International Conference on Machine Learning, 2019, pp. 1617–1625.

[7] B. Gao, X. Liu, and Y.-X. Yuan, *Parallelizable algorithms for optimization problems with orthogonality constraints*, SIAM Journal on Scientific Computing, 41 (2019), pp. A1949–A1983.

[8] A. Geiger, P. Lenz, C. Stiller, and R. Urtasun, *Vision meets robotics: The KITTI dataset*, The International Journal of Robotics Research, 32 (2013), pp. 1231–1237.

[9] D. Hajinezhad and M. Hong, *Perturbed proximal primal-dual algorithm for nonconvex nonsmooth optimization*, Mathematical Programming, 176 (2019), pp. 207–245.
[10] R. Hartley and A. Zisserman, *Multiple view geometry in computer vision*, Cambridge University Press, 2003.

[11] M. Hong, D. Hajinezhad, and M.-M. Zhao, *Prox-PDA: The proximal primal-dual algorithm for fast distributed nonconvex optimization and learning over networks*, in International Conference on Machine Learning, PMLR, 2017, pp. 1529–1538.

[12] J. Hu, X. Liu, Z.-W. Wen, and Y.-X. Yuan, *A brief introduction to manifold optimization*, Journal of the Operations Research Society of China, 8 (2020), pp. 199–248.

[13] X. Hu and X. Liu, *An efficient orthonormalization-free approach for sparse dictionary learning and dual principal component pursuit*, Sensors, 20 (2020), p. 3041.

[14] L. Huang, X. Liu, B. Lang, A. W. Yu, Y. Wang, and B. Li, *Orthogonal weight normalization: Solution to optimization over multiple dependent Stiefel manifolds in deep neural networks*, in Thirty-Second AAAI Conference on Artificial Intelligence, 2018.

[15] L.-K. Huang and S. Pan, *Communication-efficient distributed PCA by Riemannian optimization*, in International Conference on Machine Learning, PMLR, 2020, pp. 4465–4474.

[16] B. Jiang, S. Ma, A. M.-C. So, and S. Zhang, *Vector transport-free SVRG with general retraction for Riemannian optimization: Complexity analysis and practical implementation*, arXiv:1705.09059, (2017).

[17] D. Kempe and F. McSherry, *A decentralized algorithm for spectral analysis*, Journal of Computer and System Sciences, 74 (2008), pp. 70–83.

[18] Y. LeCun, L. Bottou, Y. Bengio, and P. Haffner, *Gradient-based learning applied to document recognition*, Proceedings of the IEEE, 86 (1998), pp. 2278–2324.

[19] X. Lian, C. Zhang, H. Zhang, C.-J. Hsieh, W. Zhang, and J. Liu, *Can decentralized algorithms outperform centralized algorithms? A case study for decentralized parallel stochastic gradient descent*, Advances in Neural Information Processing Systems, 30 (2017).

[20] Q. Ling, W. Shi, G. Wu, and A. Ribeiro, *DLM: Decentralized linearized alternating direction method of multipliers*, IEEE Transactions on Signal Processing, 63 (2015), pp. 4051–4064.

[21] H. Liu, A. M.-C. So, and W. Wu, *Quadratic optimization with orthogonality constraint: explicit Łojasiewicz exponent and linear convergence of retraction-based line-search and stochastic variance-reduced gradient methods*, Mathematical Programming, 178 (2019), pp. 215–262.

[22] A. Nedić, A. Olshevsky, and M. G. Rabbat, *Network topology and communication-computation tradeoffs in decentralized optimization*, Proceedings of the IEEE, 106 (2018), pp. 953–976.

[23] A. Nedić and A. Ozdaglar, *Distributed subgradient methods for multi-agent optimization*, IEEE Transactions on Automatic Control, 54 (2009), pp. 48–61.

[24] F. Penna and S. Stańczak, *Decentralized eigenvalue algorithms for distributed signal detection in wireless networks*, IEEE Transactions on Signal Processing, 63 (2014), pp. 427–440.

[25] S. U. Pillai, T. Suel, and S. Cha, *The Perron-Frobenius theorem: some of its applications*, IEEE Signal Processing Magazine, 22 (2005), pp. 62–75.

[26] C. Qi, K. A. Gallivan, and P.-A. Absil, *Riemannian BFGS algorithm with applications*, in Recent advances in optimization and its applications in engineering, Springer, 2010, pp. 183–192.

[27] G. Qu and N. Li, *Harnessing smoothness to accelerate distributed optimization*, IEEE Transactions on Control of Network Systems, 5 (2017), pp. 1245–1260.
[28] H. Raja and W. U. Bajwa, *Cloud K-SVD: A collaborative dictionary learning algorithm for big, distributed data*, IEEE Transactions on Signal Processing, 64 (2015), pp. 173–188.

[29] H. Sato, H. Kasai, and B. Mishra, *Riemannian stochastic variance reduced gradient algorithm with retraction and vector transport*, SIAM Journal on Optimization, 29 (2019), pp. 1444–1472.

[30] W. Shi, Q. Ling, G. Wu, and W. Yin, *EXTRA: An exact first-order algorithm for decentralized consensus optimization*, SIAM Journal on Optimization, 25 (2015), pp. 944–966.

[31] E. Stiefel, *Richtungsfelder und Fernparallelismus in n-dimensionalen Mannigfaltigkeiten*, Commentarii Mathematici Helvetici, 8 (1935), pp. 305–353.

[32] H. Sun, S. Lu, and M. Hong, *Improving the sample and communication complexity for decentralized non-convex optimization: Joint gradient estimation and tracking*, in Proceedings of the 37th International conference on machine learning, PMLR, 2020, pp. 9217–9228.

[33] E. Vorontsov, C. Trabelsi, S. Kadoury, and C. Pal, *On orthogonality and learning recurrent networks with long term dependencies*, in International Conference on Machine Learning, PMLR, 2017, pp. 3570–3578.

[34] L. Wang, B. Gao, and X. Liu, *Multipliers correction methods for optimization problems over the Stiefel manifold*, CSIAM Transactions on Applied Mathematics, 2 (2021), pp. 508–531.

[35] L. Xiao and S. Boyd, *Fast linear iterations for distributed averaging*, Systems & Control Letters, 53 (2004), pp. 65–78.

[36] N. Xiao, X. Liu, and Y.-X. Yuan, *A class of smooth exact penalty function methods for optimization problems with orthogonality constraints*, Optimization Methods and Software, (2020), pp. 1–37.

[37] R. Xin, U. A. Khan, and S. Kar, *Fast decentralized nonconvex finite-sum optimization with recursive variance reduction*, SIAM Journal on Optimization, 32 (2022), pp. 1–28.

[38] R. Xin, S. Pu, A. Nedić, and U. A. Khan, *A general framework for decentralized optimization with first-order methods*, Proceedings of the IEEE, 108 (2020), pp. 1869–1889.

[39] J. Xu, S. Zhu, Y. C. Soh, and L. Xie, *Augmented distributed gradient methods for multi-agent optimization under uncoordinated constant stepsizes*, in 54th IEEE Conference on Decision and Control, IEEE, 2015, pp. 2055–2060.

[40] K. Yuan, Q. Ling, and W. Yin, *On the convergence of decentralized gradient descent*, SIAM Journal on Optimization, 26 (2016), pp. 1835–1854.

[41] J. Zeng and W. Yin, *On nonconvex decentralized gradient descent*, IEEE Transactions on Signal Processing, 66 (2018), pp. 2834–2848.

[42] H. Zhang, S. J Reddi, and S. Sra, *Riemannian SVRG: Fast stochastic optimization on Riemannian manifolds*, Advances in Neural Information Processing Systems, 29 (2016).

[43] Z. Zhu, Y. Wang, D. Robinson, D. Naiman, R. Vidal, and M. Tsakiris, *Dual principal component pursuit: Improved analysis and efficient algorithms*, Advances in Neural Information Processing Systems, 31 (2018).
Appendix A  Proof of Lemma 2.1.

**Proof of Lemma 2.1.** Suppose $\sigma_{\text{min}}$ is the smallest singular value of $X$. Then for any $X \in \mathcal{R}$, we have $\|X\|_F^2 \leq 7/6$ and $\sigma_{\text{min}}^2 \geq 5/6$, and hence,

$$\|X(X^TX - I_p)\|_F^2 \geq \sigma_{\text{min}}^2 \|X^TX - I_p\|_F^2 \geq \frac{5}{6} \|X^TX - I_p\|_F^2.$$ 

Moreover, simple algebraic manipulations give us

$$\langle G(X), X(X^TX - I_p) \rangle = \left\langle \nabla f(Z)|_{Z=X} \left(\frac{3}{2} I_p - \frac{1}{2} X^TX\right), X(X^TX - I_p) \right\rangle$$

$$- \langle X \text{sym} (X^T \nabla f(Z)|_{Z=X}), X(X^TX - I_p) \rangle$$

$$= \langle \text{sym} (X^T \nabla f_i(Z)|_{Z=X}), (X^TX - I_p) \left(\frac{3}{2} I_p - \frac{1}{2} X^TX\right) - X^TX(X^TX - I_p) \rangle$$

$$= -\frac{3}{2} \langle \text{sym} (X^T \nabla f(Z)|_{Z=X}), (X^TX - I_p)^2 \rangle,$$

which yields that

$$\|\langle G(X), X(X^TX - I_p) \rangle\|_F \leq \frac{3}{2} \|\text{sym} (X^T \nabla f(Z)|_{Z=X})\|_F \|X^TX - I_p\|_F^2 \leq \frac{7}{4} M \|X^TX - I_p\|_F^2.$$

Now it can be readily verifies that

$$\|H(X)\|_F^2 = \|G(X)\|_F^2 + 2\beta \langle G(X), X(X^TX - I_p) \rangle + \beta^2 \|X(X^TX - I_p)\|_F^2$$

$$\geq \|G(X)\|_F^2 + \frac{1}{6} \beta (5\beta - 21M) \|X^TX - I_p\|_F^2,$$

which together with $\beta \geq (6 + 21M)/5$ completes the proof. \hfill \Box

Appendix B  Convergence Analysis

Existing convergence guarantees of decentralized SVRG algorithms, such as [32, 37], are constructed for globally Lipschitz smooth functions, which are restrictive for (1.3) since $\mathcal{S}_{n,p}$ is compact. In this section, the global convergence of Algorithm 1 is rigorously established under rather mild conditions. The objective function is only assumed to be locally Lipschitz smooth (see Assumption 2).

To begin with, according to the Perron-Frobenius Theorem [25], the second largest eigenvalue in magnitude of the mixing matrix $W$, denoted by $\lambda$, is strictly less than 1. Recall that $\mathcal{R}$ is a bounded set defined in Lemma 2.1. And we define $\mathcal{B} := \{X \in \mathbb{R}^{n \times p} \mid \|X\|_F \leq \sqrt{\frac{dp}{6} + \sqrt{d}}\}$.

All the special constants to be used in this section are listed below. We divide these constants into four categories.

- **Category I:**

$$M_G = \sup \{\|G_i^j(X)\|_F \mid X \in \mathcal{B}, i \in [d], j \in [l]\}; \quad M_E = \sup \{\|bE(X)\|_F \mid X \in \mathcal{B}\};$$

$$L_G = \sup \left\{ \frac{\|G_i^j(X) - G_i^j(Y)\|_F}{\|X - Y\|_F} \mid X \neq Y, X \in \mathcal{B}, Y \in \mathcal{B}, i \in [d], j \in [l] \right\};$$

$$L_E = \sup \left\{ \frac{\|E(X) - E(Y)\|_F}{\|X - Y\|_F} \mid X \neq Y, X \in \mathcal{B}, Y \in \mathcal{B} \right\};$$

$$M_D = \left\| \mathbf{D}^{(0,0)} - \mathbf{D}^{(0,0)} \right\|_F; \quad \gamma = \frac{1 - \lambda^2}{2\lambda^2}; \quad \underline{f} = \inf \{f(X) \mid X \in \mathcal{B}\}. $$
• Category II:
  \[
  \hat{L} = L_G + \beta L_E; \quad C_D = M_D + \sqrt{d}((2q + 1)M_G + \beta M_E);
  \]
  \[
  C_1 = \frac{\eta - \eta^2 \hat{L}}{2} - 4\eta^3 \hat{L}^2 - 24 \left(1 + \frac{1}{\gamma}\right) \eta^3 \hat{L}^2;
  \]
  \[
  C_2 = \frac{1 - \lambda^2}{2} - 48 \left(1 + \frac{1}{\gamma}\right) \eta \hat{L}^2 - 9\eta \hat{L}^2;
  \]
  \[
  C_3 = \frac{\eta(1 - \lambda^2)}{2} - \left(1 + \frac{1}{\gamma}\right) \eta^2 - 24 \left(1 + \frac{1}{\gamma}\right) \eta^3 \hat{L}^2 - 4\eta \hat{L}^2;
  \]
  \[
  C_4 = \left(f(X_{\text{init}}) - f\right) \left(\frac{8\eta^2 \hat{L}^2 + 2}{C_1} + \frac{16\hat{L}^2 + 1}{dC_2} + \frac{8\eta^2 \hat{L}^2}{dC_3}\right);
  \]
  \[
  C = 2(\hat{L}^2 + 1)C_4.
  \]

• Category III:
  \[
  \eta_1 = \frac{216}{49\beta}; \quad \eta_2 = \frac{12}{7\beta}; \quad \eta_3 = \frac{1}{18(4 + 3(2q + 1)M_G)};
  \]
  \[
  \eta_4 = \frac{5\beta/72 - 4 - 3(2q + 1)M_G}{(1 + (2q + 1)M_G)^2}; \quad \eta_5 = \frac{\sqrt{d}(1 - \lambda)}{\beta^2 L_E \lambda C_D}; \quad \eta_6 = \frac{\sqrt{d}}{\beta^2 L_E C_D};
  \]
  \[
  \eta_7 = \frac{-\hat{L}/2 + \sqrt{L^2/4 + 48(1 + 1/\gamma)\hat{L}^2 + 8L^2}}{48(1 + 1/\gamma)\hat{L}^2 + 8L^2}; \quad \eta_8 = \frac{1 - \lambda^2}{96(1 + 1/\gamma)\hat{L}^2 + 18L^2};
  \]
  \[
  \eta_9 = \frac{-(1 + 1/\gamma) + \sqrt{(1 + 1/\gamma)^2 + 8(1 - \lambda^2)(6(1 + 1/\gamma) + \hat{L})}}{48(1 + 1/\gamma)\hat{L}^2 + 8L^2};
  \]
  \[
  \eta = \min\{\eta_1, \eta_2, \eta_3, \eta_4, \eta_5, \eta_6, \eta_7, \eta_8, \eta_9\}.
  \]

• Category IV:
  \[
  \beta_1 = \frac{6 + 21M}{5}; \quad \beta_2 = \frac{72(4 + 3(2q + 1)M_G)}{5};
  \]
  \[
  \beta_3 = \sqrt{\frac{1}{L_E}}; \quad \beta_4 = \frac{L_G}{L_E}; \quad \beta_5 = \frac{6\sqrt{d}}{(1 - \lambda)M_D};
  \]
  \[
  \beta = \max\{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5\}.
  \]

Category I consists of the constants defined in (B.1) that is independent of \( \beta \), while the constants in Category II (B.2) depend on \( \beta \). Categories III and IV are composed of the constants related to the stepsize \( \eta \) and penalty parameter \( \beta \), respectively.

In the following, we define an auxiliary sequence \( \{T_i^{(k,t)}\} \) for VRSGT. At the \( k \)-th outer iteration, \( T_i^{(k+1,1)} \) is first updated by

\[
T_i^{(k+1,1)} = G_i(X_i^{(k+1,1)}).
\]

Then at the \( t \)-th inner iteration, \( T_i^{(k+1,t+1)} \) is updated by

\[
T_i^{(k+1,t+1)} = \frac{1}{\tau} \sum_{j \in \mathcal{C}(k+1,t)} \left(G_i^{[j]}(X_i^{(k+1,t+1)}) - G_i^{[j]}(X_i^{(k+1,t)})\right) + T_i^{(k+1,t)}.
\]

Now we prove that \( S_i^{(k,t)} = T_i^{(k,t)} + \beta E(X_i^{(k,t)}) \) in the following lemma.

**Lemma B.1.** Suppose \( \{X_i^{(k,t)}\} \) is the iterate sequence generated by Algorithm 1. Then for any \( k \in \mathbb{N} \) and \( t \in [q+1] \), it holds that \( S_i^{(k,t)} = T_i^{(k,t)} + \beta E(X_i^{(k,t)}) \).
Proof. We use mathematical induction on $t$ to prove this lemma. At first, for $t = 1$, we have

$$S_i^{(k,1)} = H_i(X_i^{(k,1)}) = G_i(X_i^{(k,1)}) + \beta E(X_i^{(k,1)}) = T_i^{(k,1)} + \beta E(X_i^{(k,1)}).$$

Now, we assume that the assertion of this lemma holds for $S_i^{(k,t)}$ and $T_i^{(k,t)}$, and investigate the situation at for $S_i^{(k,t+1)}$ and $T_i^{(k,t+1)}$. In fact, it can be readily verified that

$$S_i^{(k,t+1)} = \frac{1}{\tau} \sum_{j \in C^{(k,t)}} \left( H_i^{[j]}(X_i^{(k,t+1)}) - H_i^{[j]}(X_i^{(k,t)}) \right) + S_i^{(k,t)}$$

$$= \frac{1}{\tau} \sum_{j \in C^{(k,t)}} \left( G_i^{[j]}(X_i^{(k,t+1)}) + \beta E(X_i^{(k,t+1)}) - G_i^{[j]}(X_i^{(k,t)}) - \beta E(X_i^{(k,t)}) \right)$$

$$+ T_i^{(k,t)} + \beta E(X_i^{(k,t)})$$

$$= \frac{1}{\tau} \sum_{j \in C^{(k,t)}} \left( G_i^{[j]}(X_i^{(k,t+1)}) - G_i^{[j]}(X_i^{(k,t)}) \right) + T_i^{(k,t)} + \beta E(X_i^{(k,t+1)})$$

$$= T_i^{(k,t+1)} + \beta E(X_i^{(k,t+1)}).$$

The proof is completed. \hfill \Box

Next, we prove that $T_i^{(k,t)}$ is bounded if $X_i^{(k,s)} \in B$ for $s \in [t]$.

Lemma B.2. Suppose $X_i^{(k,s)} \in B$ for $s \in [t]$. Then it holds that $\|T_i^{(k,t)}\|_F \leq (2q + 1)M_g$.

Proof. By straightforward manipulations, we have

$$T_i^{(k,t)} = \sum_{s=2}^{t} \left( T_i^{(k,s-1)} - T_i^{(k,s-1)} \right) + T_i^{(k,1)}$$

$$= \frac{1}{\tau} \sum_{s=2}^{t} \sum_{j \in C^{(k,s)}} \left( G_i^{[j]}(X_i^{(k,s)}) - G_i^{[j]}(X_i^{(k,s-1)}) \right) + G_i(X_i^{(k,1)}),$$

which is followed by

$$\|T_i^{(k,t)}\|_F \leq \frac{1}{\tau} \sum_{s=2}^{t} \sum_{j \in C^{(k,s)}} \left\| G_i^{[j]}(X_i^{(k,s)}) - G_i^{[j]}(X_i^{(k,s-1)}) \right\|_F + \left\| G_i(X_i^{(k,1)}) \right\|_F$$

$$\leq 2(t-1)M_g + M_g \leq (2q + 1)M_g.$$

We complete the proof. \hfill \Box

For the sake of convenience, we now define the following averages of local variables.

- $\hat{X}^{(k,t)} = \frac{1}{d} \sum_{i=1}^{d} X_i^{(k,t)} \in \mathbb{R}^{n \times p}$, $\bar{X}^{(k,t)} = (1_d \otimes I_n) \hat{X}^{(k,t)} \in \mathbb{R}^{dn \times p}$.

- $\bar{D}^{(k,t)} = \frac{1}{d} \sum_{i=1}^{d} D_i^{(k,t)} \in \mathbb{R}^{n \times p}$, $\bar{D}^{(k,t)} = (1_d \otimes I_n) \bar{D}^{(k,t)} \in \mathbb{R}^{dn \times p}$.

- $\bar{S}^{(k,t)} = \frac{1}{d} \sum_{i=1}^{d} S_i^{(k,t)} \in \mathbb{R}^{n \times p}$, $\bar{S}^{(k,t)} = (1_d \otimes I_n) \bar{S}^{(k,t)} \in \mathbb{R}^{dn \times p}$.

- $\bar{T}^{(k,t)} = \frac{1}{d} \sum_{i=1}^{d} T_i^{(k,t)} \in \mathbb{R}^{n \times p}$, $\bar{T}^{(k,t)} = (1_d \otimes I_n) \bar{T}^{(k,t)} \in \mathbb{R}^{dn \times p}$.

- $\bar{E}^{(k,t)} = \frac{1}{d} \sum_{i=1}^{d} E(X_i^{(k,t)}) \in \mathbb{R}^{n \times p}$, $\bar{E}^{(k,t)} = (1_d \otimes I_n) \bar{E}^{(k,t)} \in \mathbb{R}^{dn \times p}$.
Then it is not difficult to check that the relationship $\tilde{D}(k,t) = \tilde{S}(k,t) = \tilde{T}(k,t) + \beta \tilde{E}(k,t)$ holds for any $k \in \mathbb{N}$ and $t \in [q+1]$. Moreover, we denote $J = 1_d 1_d^\top /d \in \mathbb{R}^{d \times d}$, and $\mathbf{J} = J \otimes I_n \in \mathbb{R}^{dn \times dn}$. It holds that $(\mathbf{W} - \mathbf{J}) \mathbf{X}^{(k,t)} = (\mathbf{W} - \mathbf{J}) \mathbf{D}^{(k,t)} = 0$.

Based on Lemmas B.1 and B.2, we can prove the following technical lemma.

**Lemma B.3.** Let the conditions in Assumptions 1 and 2 hold and the stepsize $\eta$ and penalty parameter $\beta$ of Algorithm 1 satisfy

$$0 < \eta < \min \{\bar{\eta}_1, \bar{\eta}_2, \bar{\eta}_3, \bar{\eta}_4\}, \quad \text{and} \quad \beta > \max \left\{1, \frac{1}{\eta_2^2}\right\},$$

respectively. Suppose $\bar{X}^{(k,t)} \in \mathcal{R}$, $\|T_i^{(k,t)}\|_F \leq (2q + 1)M_G$, $\|X_i^{(k,t)}\|_F \leq \sqrt{7dp/6} + \sqrt{d}$, and $\|\bar{X}^{(k,t)} - X^{(k,t)}\|_F \leq \sqrt{d} / (\beta^2 L_E)$. Then we have $\bar{X}^{(k,t+1)} \in \mathcal{R}$.

**Proof.** Since $\|X^{(k,t)}\|_F \leq \sqrt{7dp/6} + \sqrt{d}$, we have $X_i^{(k,t)} \in \mathcal{B}$. Then it can be readily verified that

$$\left\| E(\bar{X}^{(k,t)}) - \bar{E}^{(k,t)} \right\|_F \leq \frac{d}{\eta} \sum_{i=1}^d \left\| E(\bar{X}^{(k,t)}) - E(X_i^{(k,t)}) \right\|_F \leq \frac{L_E}{d} \sum_{i=1}^d \left\| \bar{X}^{(k,t)} - X_i^{(k,t)} \right\|_F \leq \frac{L_E}{\sqrt{d}} \left\| \bar{X}^{(k,t)} - X^{(k,t)} \right\|_F \leq \frac{1}{\beta}.$$
This further implies that
\[
\left\| (\bar{X}(k,t+1)\top \bar{X}(k,t+1) - I_p) \right\|_F \\
\leq \left\| (I_n - \eta \beta (\bar{X}(k,t)\top \bar{X}(k,t))^2 (\bar{X}(k,t)\top \bar{X}(k,t) - I_p) \right\|_F \\
+ \eta^2 \beta^2 \left\| (\bar{X}(k,t)\top \bar{X}(k,t) (\bar{X}(k,t)\top \bar{X}(k,t) - I_p)\right\|_F + 2\eta^2 \left\| (\bar{X}(k,t)\top Y_k \right\|_F \\
+ 2\eta^2 \beta \left\| (\bar{X}(k,t) - I_p) (\bar{X}(k,t)\top Y_k \right\|_F + \eta^2 \left\| Y_k\top Y_k \right\|_F \\
\leq \left( 1 - \frac{5}{6}\eta^2 \beta^2 \right) \left\| (\bar{X}(k,t)\top \bar{X}(k,t) - I_p) \right\|_F + \eta \left( 4 + 3(2q + 1)M_G \right) + \eta^2 \left( 1 + (2q + 1)M_G \right)^2,
\]
where the last equality follows from the condition \( \eta < \min \{ \bar{\eta}_1, \bar{\eta}_2 \} \). Now we consider the above relationship in the following two cases.

**Case I:** \( \left\| (\bar{X}(k,t)\top \bar{X}(k,t) - I_p) \right\|_F \leq 1/12 \). Since \( \eta < \bar{\eta}_3 \), we have
\[
\left\| (\bar{X}(k,t+1)\top \bar{X}(k,t+1) - I_p) \right\|_F \leq \left\| (\bar{X}(k,t)\top \bar{X}(k,t) - I_p) \right\|_F + \frac{1}{12} = \frac{1}{6}.
\]

**Case II:** \( \left\| (\bar{X}(k,t)\top \bar{X}(k,t) - I_p) \right\|_F > 1/12 \). It can be readily verified that
\[
\left\| (\bar{X}(k,t+1)\top \bar{X}(k,t+1) - I_p) \right\|_F - \left\| (\bar{X}(k,t)\top \bar{X}(k,t) - I_p) \right\|_F \\
\leq \left( 1 - \frac{5}{6}\eta^2 \beta^2 - 1 \right) \left\| (\bar{X}(k,t)\top \bar{X}(k,t) - I_p) \right\|_F \\
+ \eta \left( 4 + 3(2q + 1)M_G \right) + \eta^2 \left( 1 + (2q + 1)M_G \right)^2 \\
\leq - \frac{5}{72}\eta^2 \beta + \eta \left( 4 + 3(2q + 1)M_G \right) + \eta^2 \left( 1 + (2q + 1)M_G \right)^2,
\]
which together with \( \beta > \beta_2 \) and \( \eta < \bar{\eta}_4 \) yields that
\[
\left\| (\bar{X}(k,t+1)\top \bar{X}(k,t+1) - I_p) \right\|_F - \left\| (\bar{X}(k,t)\top \bar{X}(k,t) - I_p) \right\|_F \leq 0.
\]
Hence, we arrive at \( \left\| (\bar{X}(k,t+1)\top \bar{X}(k,t+1) - I_p) \right\|_F \leq \left\| (\bar{X}(k,t)\top \bar{X}(k,t) - I_p) \right\|_F \leq 1/6 \). Combing the above two cases, we complete the proof.

**Lemma B.3** illustrates that, in the inner iteration of VRSST, the iterates are restricted in the bounded region \( \mathcal{R} \) if some conditions hold. In fact, the same is true for the outer iteration of VRSST.

**Lemma B.4.** Let all the conditions in Lemma B.3 hold. Suppose \( \bar{X}^{(k,0)} \in \mathcal{R}, T_i^{(k,0)} \right\|_F \leq (2q + 1)M_G, \|
X^{(k,0)} \right\|_F \leq \sqrt{7dp/6 + \sqrt{d}}, \|\bar{X}^{(k,0)} - \bar{X}^{(k,0)} \|_F \leq \sqrt{d}/(\beta^2 \ell) \). Then we have \( \bar{X}^{(k+1,1)} \in \mathcal{R} \).

**Proof.** The proof is similar to that of Lemma B.3, which is omitted here.

**Proposition B.5.** Suppose the conditions in Assumptions 1 and 2 hold. Let \( \{X^{(k,t)}\} \) be the iterate sequence generated by Algorithm 1 with \( X_{init} \in \mathcal{R} \) and the stepsize \( \eta \) and penalty parameter \( \beta \) satisfy
\[
0 < \eta < \min \{ \bar{\eta}_1, \bar{\eta}_2, \bar{\eta}_3, \bar{\eta}_4, \bar{\eta}_5, \bar{\eta}_6 \}, \text{ and } \beta > \max \left\{ 1, \frac{\beta_2}{\beta_3}, \frac{\beta_4}{\beta_5}, \frac{\beta_6}{\beta_5} \right\},
\]
0 < \eta < \min \{ \bar{\eta}_1, \bar{\eta}_2, \bar{\eta}_3, \bar{\eta}_4, \bar{\eta}_5, \bar{\eta}_6 \}, \text{ and } \beta > \max \left\{ 1, \frac{\beta_2}{\beta_3}, \frac{\beta_4}{\beta_5}, \frac{\beta_6}{\beta_5} \right\},
respectively. Then for any \( k \in \mathbb{N} \) and \( t \in [q+1] \), it holds that

\[
\hat{X}^{(k, t)} \in \mathcal{R}, \quad \|\hat{X}^{(k, t)} - X^{(k, t)}\|_F \leq \frac{\sqrt{d}}{\beta^2 L_E}, \quad \|X^{(k, t)}\|_F \leq \sqrt{\frac{7dp}{6}} + \sqrt{d},
\]

(B.5)

**Proof.** We use mathematical induction to prove this proposition. The argument (B.5) directly holds at \((X^{(0,0)}, D^{(0,0)})\) resulting from the initialization. To complete the proof, we consider the following two scenarios.

**Case I: Inner iterations.** We assume that the argument (B.5) holds at \((X^{(k,t)}, D^{(k,t)})\), and investigate the situation at \((X^{(k,t+1)}, D^{(k,t+1)})\).

Our first purpose is to show that \(\|X^{(k,t+1)} - X^{(k,t)}\|_F \leq \sqrt{d}/(\beta^2 L_E)\). By straightforward calculations, we can attain that

\[
\|\hat{X}^{(k,t+1)} - X^{(k,t+1)}\|_F = \|W - J\| \left(\hat{X}^{(k,t)} - X^{(k,t)}\right) + \eta \|W - J\| D^{(k,t)}
\leq \|W - J\| \left(\hat{X}^{(k,t)} - X^{(k,t)}\right) + \eta \|W - J\| D^{(k,t)}
\leq \lambda \|\hat{X}^{(k,t)} - X^{(k,t)}\|_F + \eta \lambda \|D^{(k,t)}\|_F
\leq \lambda \|\hat{X}^{(k,t)} - X^{(k,t)}\|_F + \eta \lambda C_D,
\]

which together with \(\eta < \bar{\eta}_3\) implies that

\[
\|\hat{X}^{(k,t)} - X^{(k,t)}\|_F \leq \frac{\sqrt{d} \lambda}{\beta^2 L_E} + \frac{\sqrt{d} (1 - \lambda)}{\beta^2 L_E} \leq \frac{\sqrt{d}}{\beta^2 L_E}.
\]

Then we aim to prove that \(\hat{X}^{(k,t+1)} \in \mathcal{R}\) and \(\|\hat{X}^{(k,t+1)}\|_F \leq \sqrt{7dp/6} + \sqrt{d}\). According to Lemmas B.2 and B.3, we have \(\hat{X}^{(k+1)} \in \mathcal{R}\). And it follows that

\[
\|X^{(k,t+1)}\|_F \leq \|\hat{X}^{(k,t+1)}\|_F + \|\hat{X}^{(k,t+1)} - X^{(k,t+1)}\|_F \leq \sqrt{\frac{7dp}{6}} + \sqrt{\lambda} \leq \sqrt{\frac{7dp}{6}} + \sqrt{\lambda},
\]

as a result of the condition \(\beta > \bar{\beta}_3\).

In order to finish the proof, we still have to show that \(\|\hat{D}^{(k,t+1)} - D^{(k,t+1)}\|_F \leq M_D\) and \(\|\hat{D}^{(k,t+1)}\|_F \leq C_D\). In fact, we have

\[
\|S^{(k,t+1)} - S^{(k,t)}\|_F \leq \frac{1}{\tau} \sum_{i \in [q+1]} \left\|H_i^{[j]}(X^{(k,t+1)}_i) - H_i^{[j]}(X^{(k,t)}_i)\right\|_F
\leq (L_G + \beta L_E) \|X^{(k,t+1)} - X^{(k,t)}\|_F, \quad i \in [d],
\]

which together with \(\beta > \bar{\beta}_4\) implies that

\[
\|S^{(k,t+1)} - S^{(k,t)}\|_F \leq 2 \beta L_E \|X^{(k,t+1)} - X^{(k,t)}\|_F.
\]

Moreover, it can be readily verified that

\[
\|X^{(k,t+1)} - X^{(k,t)}\|_F = \|W (X^{(k,t)} - \eta D^{(k,t)}) - X^{(k,t)}\|_F
\leq \left\|(I_{dn} - W) (\hat{X}^{(k,t)} - X^{(k,t)}) - \eta WD^{(k,t)}\right\|_F
\leq 2 \|\hat{X}^{(k,t)} - X^{(k,t)}\|_F + \eta \|D^{(k,t)}\|_F
\leq \frac{2 \sqrt{d}}{\beta^2 L_E} + \eta C_D \leq \frac{3 \sqrt{d}}{\beta^2 L_E}.
\]

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where the last inequality follows from \( \eta < \bar{\eta}_6 \). Combing the above two relationships, we can obtain that

\[
\| \mathbf{D}^{(k,t+1)} - \mathbf{D}^{(k,t+1)} \|_F \\
= \| (\mathbf{W} - \mathbf{J}) \left( \mathbf{D}^{(k,t)} - \mathbf{D}^{(k,t)} \right) - (\mathbf{I}_d - \mathbf{J}) \left( \mathbf{S}^{(k,t+1)} - \mathbf{S}^{(k,t)} \right) \|_F \\
\leq \| (\mathbf{W} - \mathbf{J}) \left( \mathbf{D}^{(k,t)} - \mathbf{D}^{(k,t)} \right) \|_F + \| (\mathbf{I}_d - \mathbf{J}) \left( \mathbf{S}^{(k,t+1)} - \mathbf{S}^{(k,t)} \right) \|_F \\
\leq \lambda \| \mathbf{D}^{(k,t)} - \mathbf{D}^{(k,t)} \|_F + \| \mathbf{S}^{(k,t+1)} - \mathbf{S}^{(k,t)} \|_F \\
\leq \lambda \mathbf{M}_D + \frac{6\sqrt{d}}{\beta} \leq \mathbf{M}_D,
\]

where the last inequality follows from \( \beta > \beta_5 \). Therefore, we can obtain that

\[
\| \mathbf{D}^{(k,t+1)} \|_F \leq \| \mathbf{D}^{(k,t+1)} - \mathbf{D}^{(k,t+1)} \|_F + \| \mathbf{D}^{(k,t+1)} \|_F \\
= \| \mathbf{D}^{(k,t+1)} - \mathbf{D}^{(k,t+1)} \|_F + \| \mathbf{S}^{(k,t+1)} \|_F \\
\leq \mathbf{M}_D + \sqrt{d}((2q + 1)\mathbf{M}_G + \beta\mathbf{M}_E) = \mathbf{C}_D.
\]

**Case II: Outer iterations.** Using the similar techniques, we can prove that the argument \((B.5)\) holds at \((X^{(k+1,1)}, D^{(k+1,1)})\) provided that it holds at \((X^{(k,0)}, D^{(k,0)})\). The corresponding proof is omitted here.

Combining the above two cases, we complete the proof.

Proposition B.5 demonstrates that the sequence \(X^{(k,t)}\) is bounded and the average of iterates \(\bar{X}^{(k,t)}\) is restricted in the region \(\mathcal{R}\). Therefore, we can directly apply the existing convergence results of decentralized SVRG algorithms established in [32, 37], although they impose the global Lipschitz smoothness property. The following proposition is an implication of Theorem 1 in [32].

**Proposition B.6.** Let the conditions in Assumptions 1 and 2 hold and \(\{X^{(k,t)}\}\) be the iterate sequence generated by Algorithm 1 with \(X_{init} \in \mathcal{R}\) and \(t = q = \sqrt{I}\). Suppose the stepsize \(\eta\) and penalty parameter \(\beta\) satisfy

\[
0 < \eta < \min \{ \bar{\eta}_1, \bar{\eta}_2, \bar{\eta}_3, \bar{\eta}_4, \bar{\eta}_5, \bar{\eta}_6, \bar{\eta}_7, \bar{\eta}_8, \bar{\eta}_9 \}, \quad \text{and} \quad \beta > \max \left\{ 1, \beta_2, \beta_3, \beta_4, \beta_5 \right\},
\]

respectively. Then it holds that

\[
\frac{1}{\sqrt{I_K}} \sum_{k=0}^{K} \sum_{t=0}^{q} \mathbb{E} \left[ \left\| \frac{1}{d} \sum_{i=1}^{d} H_i(X_{i}^{(k,t)}) \right\|_F^2 + \frac{1}{d} \sum_{i=1}^{d} \left\| X_{i}^{(k,t)} - \bar{X}^{(k,t)} \right\|_F^2 \right] \leq \frac{C_4}{\sqrt{I_K}},
\]

where \(C_4 > 0\) is a constant defined in \((B.2)\), and the expectation is taken over the randomness from the sampling step.
Proof of Theorem 3.1. By straightforward calculations, we have

$$\|H(\bar{X}^{(k,t)})\|_F^2 = \left\| \frac{1}{d} \sum_{i=1}^{d} H_i(\bar{X}^{(k,t)}) \right\|_F^2$$

$$= \left\| \frac{1}{d} \sum_{i=1}^{d} H_i(\bar{X}^{(k,t)}) - \frac{1}{d} \sum_{i=1}^{d} H_i(X_i^{(k,t)}) + \frac{1}{d} \sum_{i=1}^{d} H_i(X_i^{(k,t)}) \right\|_F^2$$

$$\leq 2 \left\| \frac{1}{d} \sum_{i=1}^{d} H_i(\bar{X}^{(k,t)}) - \frac{1}{d} \sum_{i=1}^{d} H_i(X_i^{(k,t)}) \right\|_F^2 + 2 \left\| \frac{1}{d} \sum_{i=1}^{d} H_i(X_i^{(k,t)}) \right\|_F^2$$

$$\leq \frac{2\bar{L}^2}{d} \sum_{i=1}^{d} \|X_i^{(k,t)} - \bar{X}^{(k,t)}\|_F^2 + 2 \left\| \frac{1}{d} \sum_{i=1}^{d} H_i(X_i^{(k,t)}) \right\|_F^2,$$

which implies that

$$\|H(\bar{X}^{(k,t)})\|_F^2 + \frac{1}{d} \sum_{i=1}^{d} \|X_i^{(k,t)} - \bar{X}^{(k,t)}\|_F^2$$

$$\leq 2 \left( \bar{L}^2 + 1 \right) \left( \left\| \frac{1}{d} \sum_{i=1}^{d} H_i(X_i^{(k,t)}) \right\|_F^2 + \frac{1}{d} \sum_{i=1}^{d} \|X_i^{(k,t)} - \bar{X}^{(k,t)}\|_F^2 \right).$$

According to Lemma 2.1, it follows that

$$\|H(\bar{X}^{(k,t)})\|_F^2 \geq \|G(\bar{X}^{(k,t)})\|_F^2 + \beta \| (\bar{X}^{(k,t)})^\top - I_p \|_F^2$$

$$\geq \|G(\bar{X}^{(k,t)})\|_F^2 + \| (\bar{X}^{(k,t)})^\top - I_p \|_F^2,$$

Combining the above two relationships, we can obtain that

$$\left\| \frac{1}{d} \sum_{i=1}^{d} G_i(\bar{X}^{(k,t)}) \right\|_F^2 \geq \|G(\bar{X}^{(k,t)})\|_F^2 + \| (\bar{X}^{(k,t)})^\top - I_p \|_F^2$$

$$\leq 2 \left( \bar{L}^2 + 1 \right) \left( \left\| \frac{1}{d} \sum_{i=1}^{d} H_i(X_i^{(k,t)}) \right\|_F^2 + \frac{1}{d} \sum_{i=1}^{d} \|X_i^{(k,t)} - \bar{X}^{(k,t)}\|_F^2 \right).$$

This together with Proposition B.6 yields that

$$\frac{1}{\sqrt{K}} \sum_{k=0}^{K} \sum_{t=0}^{q} \mathbb{E} \left[ \text{StaGap}(X^{(k,t)}) \right] \leq \frac{2(\bar{L}^2 + 1)C_4}{\sqrt{K}} = \frac{C}{\sqrt{K}}.$$

Since

$$\frac{1}{\sqrt{K}} \sum_{k=0}^{K} \sum_{t=0}^{q} \mathbb{E} \left[ \text{StaGap}(X^{(k,t)}) \right] \geq \min_{k=0,1,...,K} \mathbb{E} \left[ \text{StaGap}(X^{(k,t)}) \right],$$

we complete the proof. □