A new improved class of ratio-product type exponential estimators of the population variance

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Abstract

Several auxiliary information-based estimators of the population variance are available in the existing literature of survey sampling. Mostly, these estimators are based on conventional dispersion measures of the auxiliary variable. In this study, a generalized class of ratio-product type exponential estimators of the population variance is proposed which integrates the auxiliary information on non-conventional dispersion measures under simple random sampling in the ratio-type exponential class of estimators. The performance of the proposed estimators is compared, theoretically and numerically, with the several existing estimators of the population variance. It is established that the proposed class of estimators outperforms the existing estimators in terms of the lower mean square and relative root mean square errors. Moreover, the percentage relative efficiency of the proposed estimators is much higher as compared to their counterparts.

Keywords: auxiliary variable, mean square error, percentage relative efficiency, relative root mean square error, simple random sampling.

1. Introduction

In survey sampling the auxiliary information, if available or easily obtainable without involving much cost, can be advantageously used in selection of appropriate sampling design, selection of sampling units for inquiry or measurement process, and the estimation of the characteristic of interest. For example, to study the sugar cane production, the auxiliary information about area under cultivation, the market price of sugar, the incentive, in terms of support price, given to the farmers and the production of sugar cane in previous year etc., can play a vital role for efficient estimation of the expected sugar cane production. The ratio, product, regression, exponential and their different combinations are a popular choice, in practice, to enhance the efficiency of the estimators of population mean and variance in the presence of auxiliary information correlated with the study variable. The use of these estimators is expanding to a variety of fields such as yield estimation in agriculture, demographic studies, environmental

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studies, statistical process monitoring in industry, medical and biological sciences, and many other related fields; see for example [1-7].

Along with population mean, the estimation of variance is of great interest to make certain policy decisions in many practical situations such as agriculture, business, stock investments, production planning in manufacturing industry, services industry, ecology, seismology, and medical sciences are few to mention[8-10]. Therefore, efficient estimation of the mean and variance are equally important for effective decision making. The estimation of variance in the context of ratio-type methods of estimation, using auxiliary information, has been considered by various researchers. Usually, conventional auxiliary measures such as mean, median, quartiles, variance, coefficient of kurtosis, variation, skewness, and the correlation between the study and auxiliary variables are employed under ratio and regression type estimation structures to improve the efficiency of the estimators of variance. For example, see [10-27] as well as their cited references for details on this subject. The auxiliary measures used in most of the existing ratio-type estimators of variance are non-resistant to the presence of outliers. The use of such measures can undermine the efficiency of the ratio-type estimators of variance if some outliers are present in the data. Thus, there is need for incorporation of some outlier resistant auxiliary measures to develop more stable ratio-type estimators.

Recently, ratio-type estimators for estimation of population mean have been developed which incorporate auxiliary information on nonconventional measures [28-32]. These non-conventional measures are somewhat robust and outlier resistant which aids in stabilizing the mean square error of the estimators in presence of outliers [8, 33, 34]. Use of auxiliary information on non-conventional or robust measures for estimation of population variance is still a neglected area. Efficient estimation of variance in the presence of outliers is of paramount interest in several practical situation, such as in agriculture, business, production processes, and so forth. Therefore, the problem of estimating the finite population variance is dealt in this study by incorporating auxiliary information on some non-conventional and robust measures of dispersion, detailed in Section 3, to develop more stable and outlier resistant ratio-product type exponential estimators. It is assumed that the auxiliary information on these non-conventional measures is readily available or it can be obtained economically. Suppose a finite population \( \Omega = \{ \Omega_1, \Omega_2, \ldots, \Omega_N \} \) consists of \( N \) different and identifiable units. Let \((y_i, x_i)\) be the measurable study and the auxiliary variables, respectively, with their values \((y_i, x_i)\) being ascertained on \( \Omega_i \) \( (i = 1, 2, \ldots, N) \). The purpose of the measurement process is to efficiently estimate the population variance of the variable of interest, 
\[ S_y^2 = (N - 1)^{-1} \sum_{i=1}^{N} (y_i - \bar{Y})^2, \]
by drawing a random sample of size \( n \) from \( \Omega \) using simple random sampling (SRS) without replacement scheme. Let \( s_y^2 = (n - 1)^{-1} \sum_{i=1}^{n} (y_i - \bar{y})^2 \) and 
\[ s_x^2 = (n - 1)^{-1} \sum_{i=1}^{n} (x_i - \bar{x})^2 \]
be the sample variance of the study and the auxiliary variable, respectively. Furthermore, let \( \rho_{yx} \) be the population coefficient of correlation between the study and the auxiliary variable, \( C_y = S_y / \bar{Y} \) and \( C_x = S_x / \bar{X} \) be population coefficient of variations of \( y \) and \( x \), respectively.
To determine the bias and mean square error (MSE) of the existing and the proposed estimators, the following preliminaries regarding the relative error terms are considered:

Let $\xi_0 = (s_y^2 - S_y^2) / S_y^2$ and $\xi_1 = (s_x^2 - S_x^2) / S_x^2$, so that $E(\xi_0) = E(\xi_1) = 0$; and $E(\xi_0) = \eta(\beta_{2(y)} - 1) = \beta_{2(y)}^*$, $E(\xi_1) = \eta(\beta_{2(x)} - 1) = \beta_{2(x)}^*$, $E(\xi_0 \xi_1) = \eta(\lambda_{22} - 1) = \lambda_{22}^*$; where $\eta = \left(1 - \frac{1}{n}\right)$, $\beta_{2(y)}$ and $\beta_{2(x)}$ are the population coefficient of kurtosis of the study variable $y$ and auxiliary variable $x$, respectively, and $\lambda_{22} = \frac{\mu_{22}}{\mu_{20} \mu_{02}}$ with

$$\mu_{rs} = \frac{1}{N-1} \sum_{i=1}^{N} (y_i - \bar{Y}) (x_i - \bar{X})^r.$$  

2. Some existing estimators of variance under SRS

Numerous estimators of finite population variance are available in literature. In this section, we briefly describe the structure of some of the existing estimators of finite population variance based on SRS.

The usual unbiased estimator of variance under SRS as defined in Cochran [35] is given as

$$S_y^2 = \frac{1}{n-1} \sum_{i=1}^{n} (y_i - \bar{y})^2,$$

The variance of $S_y^2$ is given as

$$\text{Var}(S_y^2) \equiv S_y^4 \beta_{2(y)}^*.$$  \hspace{1cm} (1)

Isaki [13] proposed a ratio type estimator of $S_y^2$, which is given as

$$S_R^2 = s_y^2 \left( \frac{s_x^2}{S_x^2} \right),$$

The MSE of $S_R^2$, to first degree of approximation, is given as

$$\text{MSE}(S_R^2) \equiv S_y^4 \left[ \beta_{2(y)}^* + \beta_{2(x)}^* - 2\lambda_{22}^* \right].$$  \hspace{1cm} (2)

The conventional regression type estimator due to Isaki [13] is given as

$$S_{\text{Reg}}^2 = s_y^2 + b_{(s_y^*, s_x^*)} (S_x^2 - s_x^2),$$

Where $b_{(s_y^*, s_x^*)}$ represents the regression coefficient to be estimated from the sample.

The MSE of $S_{\text{Reg}}^2$, up to the first degree of approximation, is given as
\[
\text{MSE}\left(S_{beg}^2\right) \approx S_y^4 \beta_{2(y)}^* \left[1 - \rho^2(s_i^*, s_i^*)\right] \tag{3}
\]

where \(\rho(s_i^*, s_i^*) = \lambda_{22}^* \sqrt{\beta_{2(y)}^* \beta_{2(x)}^*}\) denotes the population correlation coefficient between \(y\) and \(x\).

The difference type estimator of Singh, Upadhyaya and Namjoshi [19] is given as
\[
S_d^2 = c_1 s_y^2 + c_2 \left(S_x^2 - s_i^2\right),
\]
where \(c_1\) and \(c_2\) are unknown constants and their optimal values are determined in such a manner that the MSE of \(S_d^2\) is minimized.

The minimum MSE of \(S_d^2\) at optimum values \(c_{1(opt)} = \beta_{2(x)}^* \left(\beta_{2(y)}^* + \beta_{2(y)}^* \beta_{2(x)}^* - \lambda_{22}^2\right)\) and \(c_{2(opt)} = S_x^2 \lambda_{22}^* \left(S_y^2 \left(\beta_{2(x)}^* + \beta_{2(y)}^* \beta_{2(x)}^* - \lambda_{22}^2\right)\right)\), up to the first degree of approximation, is given,
\[
\text{MSE}\left(S_d^2\right)_{\text{min}} \approx \frac{S_y^4 \beta_{2(y)}^* \left[1 - \rho^2(s_i^*, s_i^*)\right]}{1 + \beta_{2(y)}^* \left[1 - \rho^2(s_i^*, s_i^*)\right]} \tag{4}
\]

The ratio-type exponential estimator proposed by Bahl and Tuteja [11] is given as
\[
S_{BT}^2 = S_y^2 \exp\left(\frac{S_y^2 - s_i^2}{S_x^2 + s_i^2}\right).
\]

The minimum MSE of \(S_{BT}^2\), up to first degree of approximation, is given as
\[
\text{MSE}\left(S_{BT}^2\right) \approx S_y^4 \left[\beta_{2(y)}^* + \frac{1}{4} \beta_{2(y)}^* - \lambda_{22}^2\right] \tag{5}
\]

Upadhyaya and Singh [25] used coefficient of kurtosis of the auxiliary variable to propose a modified ratio-type estimator of population variance, which is given as
\[
S_{US}^2 = S_y^2 \left(\frac{S_x^2 + \beta_{2(x)}^*}{S_x^2 + \beta_{2(x)}^*}\right)
\]

The MSE of \(S_{US}^2\), up to first degree of approximation, is given as,
\[
\text{MSE}\left(S_{US}^2\right) \approx S_y^4 \left[\beta_{2(y)}^* + \gamma_{US} \beta_{2(x)}^* - 2\gamma_{US} \lambda_{22}^2\right], \tag{6}
\]
where \(\gamma_{US} = \frac{S_y^2}{S_x^2 + \beta_{2(x)}^*}\).
Kadilar and Cingi [14] utilized population coefficient of variation and the population coefficient of kurtosis of the auxiliary variable to suggest some modified estimators of population variance as

\[ S_{KC1}^2 = \left( \frac{S_x^2 + C_x}{s_x^2 + C_x} \right); \quad S_{KC2}^2 = \left( \frac{C_xS_x^2 + \beta_{2(s)}}{C_xS_x^2 + \beta_{2(s)}} \right); \quad S_{KC3}^2 = \left( \frac{\beta_{2(s)}S_x^2 + C_x}{\beta_{2(s)}S_x^2 + C_x} \right) \]

The respective MSEs of \( S_{KC1}^2, S_{KC2}^2, \) and \( S_{KC3}^2, \) up to first degree of approximation, are given as,

\[ MSE\left( S_{KC1}^2 \right) \equiv S_y^4 \left[ \beta_{2(s)}^* + \gamma_{KC1}\beta_{2(s)}^* - 2\gamma_{KC1}\lambda_{22}^* \right], \quad (7) \]

\[ MSE\left( S_{KC2}^2 \right) \equiv S_y^4 \left[ \beta_{2(s)}^* + \gamma_{KC2}\beta_{2(s)}^* - 2\gamma_{KC2}\lambda_{22}^* \right], \quad (8) \]

\[ MSE\left( S_{KC3}^2 \right) \equiv S_y^4 \left[ \beta_{2(s)}^* + \gamma_{KC3}\beta_{2(s)}^* - 2\gamma_{KC3}\lambda_{22}^* \right], \quad (9) \]

where \( \gamma_{KC1} = \frac{S_x^2}{S_x^2 + C_x}, \quad \gamma_{KC2} = \frac{C_xS_x^2}{C_xS_x^2 + \beta_{2(s)}}, \) and \( \gamma_{KC3} = \frac{\beta_{2(s)}S_x^2}{\beta_{2(s)}S_x^2 + C_x}. \)

The estimator of population variance \( S_y^2, \) given by Shabbir and Gupta [17] is

\[ S_{SG} = c_3S_y^2 + c_4 \left( S_y^2 - S_x^2 \right) \exp \left( \frac{s_x^2 - s_y^2}{S_x^2 + s_x^2} \right) \]

where \( c_3 \) and \( c_4 \) are unknown quantities to be determined in a manner to minimize the MSE of \( S_{SG}^2. \)

The optimum values of \( c_3 \) and \( c_4 \) that minimizes the MSE of \( S_{SG}^2 \) are given as

\[ c_3(\text{opt}) = \frac{\beta_{2(s)}^*}{8} \left( \frac{8 - \beta_{2(s)}^*}{\beta_{2(s)}^* + \beta_{2(s)}^* - \lambda_{22}^*} \right) \]

\[ c_4(\text{opt}) = \frac{S_x^2}{8S_y^2} \left( \frac{-4\beta_{2(s)}^* + \beta_{2(s)}^* + 8\lambda_{22}^* - \lambda_{22}^* - 4\lambda_{22}^*}{\beta_{2(s)}^* + \beta_{2(s)}^* + \beta_{2(s)}^* - \lambda_{22}^*} \right), \]

whereas the minimized MSE of \( S_{SG}^2 \) is

\[ MSE\left( S_{SG}^2 \right)_{\text{min}} \equiv S_y^4 \left( \frac{-\beta_{2(s)}^* - 16\beta_{2(s)}^*}{1 + \rho_{(s_i,s_j)}^2} \left( \frac{\beta_{2(s)}^*}{1 - \rho_{(s_i,s_j)}^2} - 4 \right) \right). \]

Subramani and Kumarapandiyan [36] used the median of the auxiliary variable to propose an estimator of the population variance which is defined as.
\[ S_{SK1}^2 = s_y^2 \left( \frac{S_x^2 + M_x}{s_x^2 + M_x} \right) \]

The MSE of \( S_{SK1}^2 \), up to first degree of approximation, is given as,

\[
MSE \left( S_{SK1}^2 \right) \equiv S_y^4 \left[ \beta_{2(1)}^* + \gamma_{SK1}^* \beta_{2(1)}^* - 2 \gamma_{SK1}^* \lambda_{22}^* \right], \quad (11)
\]

where \( \gamma_{SK1} = \frac{S_y^2}{s_x^2 + M_x} \).

Taking motivation from Kadilar and Cingi [14] and Subramani and Kumarapandiyan [36] a new ratio-type estimator of the population variance was introduced by Subramani and Kumarapandiyan [22] that utilizes the population information on the coefficient of variation and the median of the auxiliary variable and is given as

\[
S_{SK2}^2 = \left( \frac{C_x S_x^2 + M_x}{C_x s_x^2 + M_x} \right)
\]

The MSE of \( S_{SK2}^2 \), to first order of approximation, is given as,

\[
MSE \left( S_{SK2}^2 \right) \equiv S_y^4 \left[ \beta_{2(2)}^* + \gamma_{SK2}^* \beta_{2(2)}^* - 2 \gamma_{SK2}^* \lambda_{22}^* \right], \quad (12)
\]

where \( \gamma_{SK2} = \frac{C_x S_x^2}{C_x s_x^2 + M_x} \).

Khan and Shabbir [15] used upper quartile and the population correlation coefficient to suggest an improved ratio estimator of population variance as

\[
S_{KS}^2 = \left( \frac{\rho_{xs} S_x^2 + Q_{x(s)}}{\rho_{xs} s_x^2 + Q_{x(s)}} \right)
\]

The MSE of \( S_{KS}^2 \), up to first degree of approximation, is given as,

\[
MSE \left( S_{KS}^2 \right) \equiv S_y^4 \left[ \beta_{2(3)}^* + \gamma_{KS}^* \beta_{2(3)}^* - 2 \gamma_{KS}^* \lambda_{22}^* \right], \quad (13)
\]

where \( \gamma_{KS} = \frac{\rho_{xs} S_x^2}{\rho_{xs} s_x^2 + Q_{x(s)}} \).

The generalized estimator of population variance proposed by Swain [24] is given below,

\[
S_{SW}^2 = S_y^2 \left[ k \left( \frac{s_x^2}{s_y^2} \right)^h + (1-k) \left( \frac{s_x^2}{s_y^2} \right)^{h-d} \right]
\]
where $k$, $q$, $h$ are suitably chosen constant and $\delta = (1, -1)$. The minimum MSE of $S_{SW}^2$, up to first degree of approximation, at optimum value $k = \left( \delta h + (Q_{22}^* \ell \beta_{2(\omega)}) \right) / (\delta (g + h))$, is given by

$$MSE\left( S_{SW}^2 \right)_{min} \cong S_{\gamma}^4 \beta_{2(\omega)}^* \left( 1 - \rho_{(s_i^*, s_j^*)}^2 \right). \quad (14)$$

It is to be noted that the $MSE\left( S_{SW}^2 \right)_{min}$ is equal to $MSE\left( S_{reg}^2 \right)$.

The general class of estimator for population variance proposed by Yadav, Kadilar, Shabbir and Gupta [26], given by

$$S_{YG}^2 = \left[ c_s s_i^2 + c_6 (s_i^2 - s_i^2) \right] \left\{ \lambda \left( \frac{a s_i^2 + b}{a s_i^2 + b} \right) + (1 - \lambda) \exp \left( \frac{a (s_i^2 - s_i^2)}{a (s_i^2 + s_i^2) + 2b} \right) \right\},$$

where $c_s$ and $c_6$ are suitably chosen constants that minimizes the MSE of $S_{reg}^2$, while $\lambda$ can take values 0 or 1 and $a$, $b$ be the known values of the auxiliary variable parameters. The minimum MSE of $S_{YG}^2$, up to first degree of approximation, at optimum values,

$$c_{s(ope)} = \left\{ \frac{1 - \frac{1}{8} g^2 (1 + 3 \lambda + 4 \lambda^2) \beta_{2(\omega)}^*}{1 - \frac{1}{4} g^2 \lambda (1 + 3 \lambda) \beta_{2(\omega)} \beta_{2(\omega)}^* + \beta_{2(\omega)}^* \left( 1 - \rho_{(s_i^*, s_j^*)}^2 \right)} \right\} \text{ and}$$

$$c_{6(ope)} = \frac{S_{Y}^2}{S_{x}^2} \left\{ \frac{1}{2} g (1 + \lambda) + c_{s(ope)} \left( \frac{\lambda_{22}^*}{\beta_{2(\omega)}^*} - g (1 + \lambda) \right) \right\} \text{ is given as}$$

$$MSE\left( S_{YG}^2 \right)_{min} \cong S_{\gamma}^4 \left\{ 1 - \frac{1}{4} g^2 (1 + \lambda)^2 \beta_{2(\omega)}^* \right\} - \frac{\left( 1 - \frac{1}{8} g^2 (1 + 3 \lambda + 4 \lambda^2) \beta_{2(\omega)}^* \right)^2}{1 - \frac{1}{4} g^2 \lambda (1 + 3 \lambda) \beta_{2(\omega)}^2 + \beta_{2(\omega)}^* \left( 1 - \rho_{(s_i^*, s_j^*)}^2 \right)},$$

where $g = \frac{a S_{x}^2}{a S_{x}^2 + b}$.

The minimum MSE of $S_{YG}^2$, up to degree order of approximation at $(\lambda, a, b) = (1, 1, 0)$, is given below,

$$MSE\left( S_{YG}^2 \right)_{min} \cong S_{\gamma}^4 \left\{ S_{\gamma}^4 MSE\left( S_{reg}^2 \right) \left( 1 - \beta_{2(\omega)}^* \right) \right\}$$

$$- \frac{1 - \beta_{2(\omega)}^* + S_{\gamma}^4 MSE\left( S_{reg}^2 \right)}{1 - \beta_{2(\omega)}^*} \left( 1 - \beta_{2(\omega)}^* \right)$$

$$\left( 1 - \rho_{(s_i^*, s_j^*)}^2 \right), \quad (15)$$

$$\left( 1 - \beta_{2(\omega)}^* \right)$$

$$\left( 1 - \rho_{(s_i^*, s_j^*)}^2 \right), \quad (16)$$
Yadav and Kadilar [37] proposed a ratio-product-ratio type estimator of population variance which is given as
\[
S_{1K}^2 = S_s^2 \left[ \alpha_1 \left( \frac{(1 - \beta_1) s_y^2 + \beta_1 S_s^2}{\beta_1^2 s_y^2 + (1 - \beta_1) S_s^2} \right) + (1 - \alpha_1) \left( \frac{\beta_2 s_y^2 + (1 - \beta_2) S_s^2}{(1 - \beta_2) s_y^2 + \beta_2 S_s^2} \right) \right]
\]

where \(\alpha_1\) and \(\beta_1\) are constants.

The minimum MSE of \(S_{1K}^2\), up to first degree of approximation, is given below,
\[
MSE\left( S_{1K}^2 \right)_{\text{min}} \approx S_s^4 \left[ \left( \beta_{2(1)}^* + \beta_{2(2)}^* - 2 \lambda_{22}^* \right) + 16 \alpha \beta \beta_{2(2)}^* \left( 1 - \alpha - \beta + \alpha \beta \right) + 4 \lambda_{22}^* \left( \alpha - \beta \right)^2 + 4 \beta_{2(1)}^* \left( -\alpha - \beta + \alpha \beta + \beta \right) \right]
\]

(17)

The minimum MSE of \(S_{1K}^2\), up to first degree of approximation at \((\alpha_{opt}, \beta_{opt}) = \left( \frac{1}{2}, \frac{1}{2} \right)\) is
\[
MSE\left( S_{1K}^2 \right)_{\text{min}} \approx S_s^4 \beta_{2(1)}^* \left( 1 - \rho^2 \left( s_y^2, s_s^2 \right) \right)
\]

(18)

And when \((\alpha_1, \beta_1) = \left( \left( \left( \beta_{2(1)}^*, 2 \beta_{2(2)}^* \right)/2, 0 \right), \right)\), the minimum MSE of \(S_{1K}^2\) is given as
\[
MSE\left( S_{1K}^2 \right)_{\text{min}} \approx S_s^4 \beta_{2(2)}^* \left( 1 - \rho^2 \left( s_y^2, s_s^2 \right) \right)
\]

(19)

Recently Yaqub and Shabbir [27] proposed an improved class of estimators for population variance given as,
\[
S_{1S}^2 = s_s^2 \left[ c_7 + c_8 \left( s_y^2 - s_s^2 \right) \right] \left\{ \frac{a S_y^2 + b}{a s_y^2 + b} \right\} \left[ \frac{1}{2} \exp \left( \frac{a \left( S_y^2 - s_s^2 \right)}{a (s_y^2 + s_s^2) + 2b} \right) + \frac{1}{2} \exp \left( \frac{a \left( s_y^2 - S_s^2 \right)}{a (s_y^2 + s_s^2) + 2b} \right) \right]
\]

where \(c_7\) and \(c_8\) are suitably chosen constants and \(a\) and \(b\) be the known population parameters of the auxiliary variable. Assuming \(a = 1\) and \(b = 0\); the minimum MSE of \(S_{1S}^2\), up to first degree of approximation, based on the optimum values,
\[
c_{7(opt)} = \frac{\beta_{2(2)}^*}{2} \left( \frac{1 + 7 (1 - \beta_{2(1)}^*)}{\beta_{2(1)}^2 + 4 \beta_{2(2)}^* (1 - \beta_{2(1)}^*) + 4 \beta_{2(1)}^* \beta_{2(2)}^* - 4 \lambda_{22}^2} \right), \text{and}
\]
\[
c_{8(opt)} = \frac{S_s^2}{2 S_y^2} \left( \frac{\lambda_{22}^2 + 7 \lambda_{22}^* (1 - \beta_{2(1)}^*) - 8 \beta_{2(2)}^* (1 - \beta_{2(1)}^*) + 8 \beta_{2(2)}^* \beta_{2(1)}^* - 8 \lambda_{22}^2}{\beta_{2(1)}^2 + 4 \beta_{2(2)}^* (1 - \beta_{2(1)}^*) + 4 \beta_{2(2)}^* \beta_{2(2)}^* - 4 \lambda_{22}^2} \right)
\]

is given below,
\[
MSE\left( S_{1S}^2 \right)_{\text{min}} \approx \frac{S_s^4}{16} \left[ 64 \left( 1 - \beta_{2(2)}^* \right) S_y^{-4} \text{MSE}\left( S_{\text{Reg}}^2 \right) - \beta_{2(2)}^2 \right]
\]

(20)
3. The proposed generalized estimator of variance

This section presents a generalized ratio product type exponential estimator of population variance which incorporates the information on some outlier resistant non-conventional measures of dispersion of the auxiliary variable. The non-conventional measures are used in a linear combination within the structure of the proposed estimator to stabilize it against possible outliers in the data. The non-conventional somewhat robust measures of the auxiliary variable considered in this study includes:

(i) **The interquartile range**: The interquartile range (IQR) is the difference between the upper quartile \(Q_{3(s)}\) and lower quartile \(Q_{1(s)}\). Symbolically, it is given as

\[ IQR_s = Q_{3(s)} - Q_{1(s)} \]

It is the most known, somewhat, robust measure of dispersion with a breakdown point of 25%.

(ii) **The Gini’s mean difference estimator**: Gini [38] suggested an estimator of dispersion which is also known as Gini’s mean difference estimator. It is given as

\[ GIN_s = \frac{4}{N(N-1)} \sum_{i=1}^{N} \left( \frac{2i-N-1}{2N} \right) x_{(i)} \]

where \(x_{(i)}\) denotes the \(i^{th}\) order statistics. It is robust to outliers and more efficient estimator as compared to the estimators based on range and standard deviation (cf. David [39]).

(iii) **The Downton’s estimator**: Like \(GIN_s\), Downton [40] suggested a robust and highly efficient estimator of dispersion. It is defined as

\[ DOW_s = \frac{2\sqrt{\pi}}{N(N-1)} \sum_{i=1}^{N} \left( \frac{i-N+1}{2} \right) x_{(i)} \]

where \(x_{(i)}\) denotes the \(i^{th}\) order statistics. The asymptotic efficiency of \(DOW_s\) is 97.8% (cf. David [39]).

(iv) **The probability weighted moment estimator**: Another similar estimator to \(GIN_s\) and \(DOW_s\) is the probability weighted moment estimator given in [41]. It is defined as

\[ SPW_s = \frac{\sqrt{\pi}}{N^2} \sum_{i=1}^{N} (2i-N-1) x_{(i)} \]

where \(x_{(i)}\) denotes the \(i^{th}\) order statistics. Its properties are like \(GIN_s\) and \(DOW_s\), as all these three estimators are proportional to each other.

(v) **The median absolute deviation from median**: Hampel [42] suggested an estimator based on median of the absolute deviations taken from median which is given as

\[ MADM_s = m \left[ \text{median} |x_i - X| \right] \text{ for } i = 1, 2, \ldots, N, \]

where \(m\) is the consistency coefficient, and \(X\) denotes the median of the observations. The \(MADM_s\) is robust against outliers with a breakdown point of 50% but under normality its efficiency is relatively low i.e. 37%. To make \(MADM_s\) a consistent estimator of \(\sigma\) under the normal distribution the value of \(m\) is set equal to 1.4826.
(vi) **The median of pairwise distances:** Shamos [43] and Bickel and Lehmann [44] suggested an estimator of dispersion based on median of pairwise distances as
\[ \text{median} \{ |x_i - x_j|; i < l \} \]. Rousseeuw and Croux [45] suggested to pre-multiply it with 1.0483 to achieve consistency under the Gaussian distribution and the resultant estimator can be defined as
\[ B_{x_i} = 1.0483 \text{median} \{ |x_i - x_j|; i < l \} \]
The \( B_{x_i} \) is, somewhat, robust to outliers with a breakdown point of 29% and has a relatively high efficiency (about 86%) under normality.

(vii) **The ordered statistic of sub-ranges:** Croux and Rousseeuw [46] proposed a class of location-free robust estimators of dispersion based on ordered statistics of subranges defined as
\[ S_{\alpha} = C_{\alpha} \left\{ x_{(i+\alpha N)\mod N} - x_{(i\mod N)} \right\} \]
where \( 0 < \alpha < 0.5 \) and \( x_{(1)\leq x_{(2)}\leq\ldots\leq x_{(N)}} \) are the \( N \) order statistics, respectively (here, the symbol \( \lceil \cdot \rceil \) represents the integer part). The value of \( S_{\alpha} \) is determined by first sorting the observations \( x \) and then we calculate the absolute differences
\[ |x_{(i+\alpha N)\mod N} - x_{(i\mod N)}| \]
for \( i = 1,2,\ldots,N - \lceil \alpha N \rceil - 1 \). From these calculated quantities, the \( \lceil \frac{N}{2} - \alpha N \rceil \)th order statistics yields the desired estimator. The constant \( C_{\alpha} \) is chosen in such a way that \( S_{\alpha} \) becomes a consistent estimator for a given value of \( \alpha \). In the present study we have used \( \alpha = 0.25 \) which corresponds to \( C_{\alpha} = 1.4826 \) under normality. The \( S_{\alpha} \) has a 50% breakdown point and it is more efficient than \( MADM_x \) for small samples.

(viii) **The trimmed mean of median deviations:** Rousseeuw and Croux [47] proposed an estimator with a high breakdown point of 50% and efficiency of 52% under normality which is relatively higher as compare to \( MADM_x \). It is defined as
\[ T_{x_i} = \frac{1.38}{h} \sum_{k=1}^{h} \text{median} \{ |x_i - x_l|; i \neq l \} \]
where for each \( i = 1,2,\ldots,N \), we compute median of \( |x_i - x_l|, l = 1,2,3,\ldots,N \) that yields \( N \) values, the average of first \( h \) order statistics gives the desired estimator (here, \( h = \lceil \frac{N}{2} \rceil + 1 \) which is roughly half of the number of observations).

(ix) **The 0.25-quantile of pairwise distances:** Similar to \( B_{x_i} \), Rousseeuw and Croux [45] suggested a robust estimator of dispersion based on the 0.25-quantile of pairwise distances between the observations. It is given as
\[ Q_{x_i} = d \{ \text{median} |x_i - x_l|; i < l \} \]
where, \( p = \left( \frac{h}{2} \right) \approx \left( \frac{N}{2} \right)/4 \) and \( h = \left\lfloor \frac{N}{2} \right\rfloor + 1 \). Hence, the \( p^{th} \) order statistic of the \( \left( \frac{N}{2} \right) \) interpoint distances yields the desired estimator. The value of \( d \) is set equal to 2.2219 for \( Qn_x \) to be a consistent estimator under normality. The estimator \( Qn_x \) has a 50% breakdown point and high Gaussian asymptotic efficiency of 82%.

(x) **The median of the median of distances:** Rousseeuw and Croux [45] suggested another robust estimator which has a high breakdown point of 50%. It is defined as

\[
S_{n_x} = q \left[ \text{median} \left\{ \text{median} \left| x_i - x_j \right| ; i \neq j \right\} \right],
\]

where \( q \) is the consistency factor with a default value of 1.1926 under the normal population. To compute \( S_{n_x} \), first we determine median of \( \left| x_i - x_j \right| ; j = 1, 2, \ldots, N \) for each \( i \) which results in \( N \) values. Finally, the median of these \( N \) values yields \( S_{n_x} \).

For detail properties of these non-conventional measures of dispersion the readers may see [38-40, 43-47] and the references cited therein.

Taking motivation from Shabbir and Gupta [17] and Naz, Abid, Nawaz and Pang [34] we have integrated the above mentioned non-conventional robust measures of dispersion to design a stable ratio product type exponential estimator of population variance defined as:

\[
S_{\text{Prop}}^2 = s_y^2 \left\{ p_1 \left( \frac{\varphi S^2_x + \theta}{\varphi S^2_x + \theta} \right)^{\frac{\varphi S^2_x}{\varphi S^2_x + \theta}} + p_2 \left( \frac{\varphi S^2_x + \theta}{\varphi S^2_x + \theta} \right)^{\frac{\varphi S^2_x}{\varphi S^2_x + \theta}} \right\} \exp \left\{ \frac{\varphi \left( S^2_x - s^2_x \right)}{\varphi \left( S^2_x + s^2_x \right) + 2\theta} \right\}
\]

(21)

where \( p_1 \) and \( p_2 \) are suitably chosen constants and their values are to be determined later in such a manner that MSE of \( S_{\text{Prop}}^2 \) is minimized. \( \varphi \) can either be some known real value or function of the known conventional population parameter – such as \( \rho \) of \( \rho \) or any other value – of the auxiliary variable, whereas \( \theta \) can be one the above mentioned non-conventional measure.

Setting \( \omega = \frac{\varphi S^2_x}{\varphi S^2_x + \theta} \) and expressing \( S_{\text{Prop}}^2 \) in terms of \( \xi \)'s, we have

\[
S_{\text{Prop}}^2 = s_y^2 \left( 1 + \xi_0 \right) \left\{ p_1 \left( 1 + \omega \xi_1 \right)^{-\alpha_0} + p_2 \left( 1 + \omega \xi_1 \right)^{\alpha_0} \right\} \exp \left\{ -\frac{\omega \xi_1}{2} \left( 1 + \frac{\omega \xi_1}{2} \right)^{-1} \right\}
\]

(22)

For Simplification, expanding Eqn. (22) and retaining terms only up to 2\(^{nd}\) order in \( \xi \)'s, we have

\[
S_{\text{Prop}}^2 - S^2_x \approx s_y^2 \left\{ p_1 \left( 1 + \xi_0 \left( \omega^2 + \frac{5}{4} \right) \xi_1 + \frac{\omega^2}{2} \xi_1^2 \right) + p_2 \left( 1 + \xi_0 \left( \omega^2 + \frac{5}{4} \right) \xi_1 + \frac{\omega^2}{2} \xi_1^2 \right) \right\} \]

For detail properties of these non-conventional measures of dispersion the readers may see [38-40, 43-47] and the references cited therein.
By applying expectation on both sides of Eqn. (23), we get the bias of $S^2_{PR}$ as below,

$$\text{Bias}\left(S^2_{PR} \right) \geq S^4 \left[ (p_1 + p_2 - 1) + p_1 \left\{ \left( \frac{\omega^4}{2} + \omega^3 + \frac{3}{8} \omega^2 \right) \beta^2_{2\omega} \right\} \beta^2_{2\omega} + \left( \frac{\omega^4}{2} + \omega^3 + \frac{3}{8} \omega^2 \right) \beta^2_{2\omega} + \left( \frac{\omega^4}{2} + \omega^3 + \frac{3}{8} \omega^2 \right) \beta^2_{2\omega} - 1 \right]$$

(23)

Squaring both sides of Eqn. (23) and then applying expectation, we get the MSE of $S^2_{PR}$ as below,

$$\text{MSE}\left(S^2_{PR} \right) = S^4 + p_1 S^4 \left[ 1 + \beta^2_{2\omega} + \left( 2 \omega^3 + 3 \omega^2 + \omega \right) \beta^2_{2\omega} - 4 \left( \omega^2 + \frac{\omega^4}{2} \right) \beta^2_{2\omega} + 1 + \beta^2_{2\omega} + \left( 2 \omega^3 + 3 \omega^2 + \omega \right) \beta^2_{2\omega} + 4 \left( \omega^2 + \frac{\omega^4}{2} \right) \beta^2_{2\omega} \right] + 2 p_1 S^4 \left[ 1 + \beta^2_{2\omega} + \omega^2 \beta^2_{2\omega} - 2 \omega \beta^2_{2\omega} \right] - 2 p_1 S^4 \left[ 1 + \left( \frac{\omega^4}{2} + \omega^3 + \frac{3}{8} \omega^2 \right) \beta^2_{2\omega} - \left( \omega^2 + \frac{\omega^4}{2} \right) \beta^2_{2\omega} \right] - 2 p_1 S^4 \left[ 1 + \left( \frac{\omega^4}{2} + \omega^3 + \frac{3}{8} \omega^2 \right) \beta^2_{2\omega} + \left( \omega^2 + \frac{\omega^4}{2} \right) \beta^2_{2\omega} \right]

(24)

To get the optimal values of $p_1$ and $p_2$, we minimize Eqn. (25) with respect to $p_1$ and $p_2$ which gives,

$$p_{1(\text{opt})} = \frac{1}{8} \left[ \frac{\omega \left( -1 + 2 \omega \right) \left( \omega^2 - 7 + 4 \omega \left( 1 + \omega \right) \right) \beta_{2\omega}^2 - 16 \beta_{2\omega}^2 \beta_{2\omega}^2 + 8 \beta_{2\omega}^2 \left( 1 + 3 \omega \beta_{2\omega}^2 \right) \right]$$

and

$$p_{2(\text{opt})} = \frac{1}{8} \left[ \frac{\omega \left( 1 + 2 \omega \right) \left( \omega^2 - 7 + 4 \omega \left( -1 + \omega \right) \right) \beta_{2\omega}^2 - 16 \beta_{2\omega}^2 \beta_{2\omega}^2 + 8 \beta_{2\omega}^2 \left( 1 + 3 \omega \beta_{2\omega}^2 \right) \right]$$

The optimal values of $p_1$ and $p_2$ are then substituted in Eqn. (25), which gives the minimum MSE of $S^2_{PR}$ as below:

$$\text{MSE}\left(S^2_{PR} \right)_{\text{min}} \approx S^4 \left[ \frac{16 \omega \beta_{2\omega}^2 \beta_{2\omega}^2 \left( \omega^4 - 4 \omega^3 \right) \beta_{2\omega}^2 - \omega^6 (1 - 4 \omega^2) \beta_{2\omega}^2 - 64 \omega \beta_{2\omega}^2 \left( 1 + \beta_{2\omega}^2 \right) + 16 \beta_{2\omega}^2 \left( 4 \omega^2 - 1 \right) \beta_{2\omega}^2 + 4 \left( 1 + 2 \omega \beta_{2\omega}^2 \right) \beta_{2\omega}^2 \right]$$

(26)
Many estimators of the population variance can be generated from class of estimators given in Eqn. (21) by setting different values of \( \varphi \) and \( \theta \). A few selected estimators, which are members of proposed class, are given in Table-1.

4. **Theoretical and numerical efficiency comparisons**

In this section, theoretical and numerical efficiency comparison of the proposed generalized class of ratio-product type exponential estimators of population variance is made with the existing estimators discussed in Section 2.

4.1. **Theoretical comparison**

For theoretical efficiency comparison, let the MSE of the proposed class of estimators \( S_{PR}^2 \) be written as

\[
MSE\left(S_{PR}^2\right) = \frac{S^2}{16} \left( \frac{A}{B} \right), \tag{27}
\]

where

\[
A = 16 \omega^2 \beta_{2(s)}^2 \left( (\omega - 4 \omega^1) \lambda_{22}^* - 4 \beta_{2(1)}^* \right) - \omega^2 (1 - 4 \omega^2)^2 \beta_{2(s)}^3 - 64 \lambda_{22}^2 \left( 1 + \beta_{2(1)}^* \right), \quad \text{and}
\]

\[
B = \omega^2 \left( 4 \omega^2 - 5 \right) \beta_{2(s)}^2 - 16 \lambda_{22}^2 + 4 \beta_{2(1)}^* \left( 1 + 4 \omega \lambda_{22} + \beta_{2(1)}^* \right) \lambda_{22}^2 \beta_{2(s)}^3
\]

(i) The proposed class of estimators has superior efficiency as compare to \( S_y^2 \) if \( Var\left(S_y^2\right) - MSE\left(S_{PR}^2\right)_{\min} > 0 \). Thus, by Eqns. (1) and (27), the efficiency condition is given as

\[
\frac{S^4}{16} \left( \frac{16B \beta_{2(1)}^* - A}{B} \right) > 0.
\]

(ii) Similarly, \( S_{PR}^2 \) is more efficient as compare to \( S_R^2 \) if \( MSE\left(S_R^2\right) - MSE\left(S_{PR}^2\right)_{\min} > 0 \). Thus, by Eqns. (2) and (27), the efficiency condition is given as

\[
\frac{S^4}{16} \left( \frac{16B \left( \beta_{2(1)}^* + 2 \lambda_{22}^* \right) - A}{B} \right) > 0.
\]

(iii) The estimators envisaged in the class \( S_{PR}^2 \) attains higher efficiency as compare to \( S_{Reg}^2 \), \( S_{SW}^2 \) and \( S_{YK}^2 \) if \( MSE\left(S_{Reg}^2\right) - MSE\left(S_{PR}^2\right)_{\min} > 0 \),

\[
MSE\left(S_{SW}^2\right)_{\min} - MSE\left(S_{PR}^2\right)_{\min} > 0 \text{ and } MSE\left(S_{YK}^2\right)_{\min} - MSE\left(S_{PR}^2\right)_{\min} > 0.
\]

Thus, by Eqns. (3), (14), (19) and (27), the efficiency condition is given as
\[
\frac{S_i^4}{16} \left[ \frac{16B\beta^*_{2(y)} \left(1 - \rho^2_{(s_i^1, s_i^2)}\right) - A}{B} \right] > 0.
\]

(iv) The efficiency of the proposed class \(S_{PR}^2\) is higher than \(S_d^2\) if
\[
MSE\left(S_d^2\right)_{\min} - MSE\left(S_{PR}^2\right)_{\min} > 0.
\]
Thus, by Eqns. (4) and (27), the efficiency condition is given as
\[
\frac{S_i^4}{16} \left[ 16B \left( \beta^*_{2(y)} \left(1 - \rho^2_{(s_i^1, s_i^2)}\right) - A \right) \left(1 + \beta^*_{2(y)} \left(1 - \rho^2_{(s_i^1, s_i^2)}\right)\right) \right] > 0.
\]

(v) \(S_{PR}^2\) shows better efficiency as compare to \(S_{BT}^2\) if
\[
MSE\left(S_{BT}^2\right) - MSE\left(S_{PR}^2\right)_{\min} > 0.
\]
Thus, by Eqns. (5) and (27), the efficiency condition is given as
\[
\frac{S_i^4}{16} \left[ 4B \left(4\beta^*_{2(y)} + \beta^*_{2(i)} - 4\lambda^*_{22}\right) - A \right] > 0.
\]

(vi) The proposed class of estimators \(S_{PR}^2\) display superior efficiency as compare to \(S_{US}^2, S_{KC1}^2, S_{KC2}^2, S_{SK1, SK2}^2\) and \(S_{KS}^2\) if their MSEs are greater than \(MSE\left(S_{PR}^2\right)_{\min}\).
Thus, by Eqns. (6), (7), (8), (9), (11), (12), (13) and (27), the efficiency condition is given as
\[
\frac{S_i^4}{16} \left[ 16B \left(\beta^*_{2(y)} + \gamma_i^2 \beta^*_{2(i)} - 2\gamma_i^2 \lambda^*_{22}\right) - A \right] > 0,
\]
where \(\gamma_i = \gamma_{US}, \gamma_{KC1}, \gamma_{KC2}, \gamma_{SK1, SK2}\) and \(\gamma_{KS}\) respectively.

(vii) The efficiency of \(S_{PR}^2\) is higher than \(S_{SG}^2\) if \(MSE\left(S_{SG}^2\right)_{\min} - MSE\left(S_{PR}^2\right)_{\min} > 0\). Thus, by Eqns. (10) and (27), the efficiency condition is given as
\[
\frac{S_i^4}{64} \left[ \left(-\beta^*_{2(i)} - 16\beta^*_{2(y)} \left(1 - \rho^2_{(s_i^1, s_i^2)}\right)\right) \left(\beta^*_{2(y)} - 4\right) \right] > 0.
\]

(viii) The estimators envisaged in \(S_{PR}^2\) achieve higher efficiency as compare to \(S_{YG}^2\) if \(MSE\left(S_{YG}^2\right)_{\min} - MSE\left(S_{PR}^2\right)_{\min} > 0\). Thus, by Eqns. (16) and (27), the efficiency condition is given as
\[
\frac{S_i^4}{16} \left[ \frac{16\beta^*_{2(i)} \left(1 - \rho^2_{(s_i^1, s_i^2)}\right) \left(1 - \beta^*_{2(i)}\right)}{1 - \beta^*_{2(i)} + \beta^*_{2(i)} \left(1 - \rho^2_{(s_i^1, s_i^2)}\right)} - A \right] > 0.
\]
(ix) The proposed estimators of class $S^2_{PR}$ are superior to $S^2_{YS}$ in terms of efficiency if
$$MSE\left(S^2_{YS}\right)_{min} - MSE\left(S^2_{PR}\right)_{min} > 0.$$ Thus, by (20) and (27). The efficiency condition is given as
$$S^4 \left[\frac{64\beta^*_2(y)\left(1 - \rho^2(s^2_x, s^2_y)\right)}{16} - \frac{A}{B}\right] > 0.$$

4.2. Numerical comparison in presence of outliers

As it has been pointed out in Section 3 that the proposed class of estimators incorporates the non-conventional measures which are somewhat robust to outliers, therefore, the numerical comparison of the proposed class of estimators of population variance with the existing estimators is made by using three population datasets which contains some outliers. The boxplots given in Figures 1-3 clearly shows that both the study and auxiliary variables in Population-I and III are crippled with outliers, while population-II has only one outlier observation in its auxiliary variable. Thus, these datasets are a good realization of the both with and without outlier observation cases. Moreover, these population datasets are frequently used in many studies to compare the performance of various estimators of population mean and variance, see for example, [28, 29, 33, 48, 49]). The description and various population characteristics are detailed as under:

[Place Figures 1-3 Here]

**Population 1:** This dataset is taken from Cochran [35], where $y$ represents the number of inhabitants (in 1000's) in United States cities in 1930 and $x$ is the number of inhabitants (in 1000's) in 1920.

$$N = 49, \ n = 20, \ \bar{Y} = 127.7959, \ \bar{X} = 103.1429, \ S^2_x = 15158.8299, \ S^2_y = 10900.4249, \ 
\beta_{2(y)} = 4.9245, \ \beta_{2(x)} = 5.9878, \ C_x = 1.0435, \ \lambda_{22} = 4.6977, \ \eta = 0.02959, \ \rho_{(s^2_x, s^2_y)} = 0.83577, \ 
Q_{1(x)} = 43.0, \ Q_{3(x)} = 120.0, \ IQR_x = 77, \ GIN_x = 97.7553, \ DOW_x = 86.6508, \ SPW_x = 84.8456, \ 
MADM_x = 39.2889, \ Bn_x = 52.415, \ Sr^x = 34.0998, \ Tn_x = 35.5488, \ Qn_x = 46.6599, \ 
Sn_x = 40.5484, \ \rho_{yx} = 0.9817, \ M_x = 64.0.$$

**Population 2:** This dataset is obtained from Murthy [50], where $y$ denotes the output (in 100,000 rupees) of factories in a region and $x$ is fixed capital (in 100,000 rupees).

$$N = 80, \ n = 20, \ \bar{Y} = 51.8264, \ \bar{X} = 11.2646, \ S^2_y = 336.9757, \ S^2_x = 70.6634, \ \beta_{2(y)} = 2.2667, \ 
\beta_{2(x)} = 2.8664, \ C_x = 0.751, \ \lambda_{22} = 2.2209, \ \eta = 0.0375, \ \rho_{(s^2_x, s^2_y)} = 0.79311, \ Q_{1(x)} = 5.1500, \ 
Q_{3(x)} = 16.975, \ IQR_x = 11.825, \ GIN_x = 10.3613, \ DOW_x = 9.1844, \ SPW_x = 9.0681, \ 
MADM_x = 4.8925, \ Bn_x = 7.7060, \ Sr^x = 4.0032, \ Tn_x = 4.3265, \ Qn_x = 5.1770, \ Sn_x = 4.6869, \ 
\rho_{yx} = 0.941, \ M_x = 7.575.$$

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Population 3: This dataset is obtained from Italian bureau of environment protection-IBEP (2004) [source: http://www.osservatorionazionalerifuti.it (2004)], where $y$ denotes the amount of recyclable waste (in tons) collected in different cities of Italy in 2003 and $x$ is the number of inhabitants living in those cities.

$$N = 103, \quad n = 40, \quad \bar{Y} = 62.6212, \quad \bar{X} = 556.5541, \quad S_y^2 = 8345.7177, \quad S_x^2 = 8345.7177, \quad \beta_{2(x)} = 17.8738, \quad C_x = 1.0963, \quad \lambda_{22} = 17.2220, \quad \eta = 0.01529, \quad \rho_{[S_y^2,S_x^2]} = 0.6570, \quad Q_{(x)} = 259.3830, \quad Q_{(2x)} = 628.0235, \quad IQR_x = 373.82, \quad GIN_x = 457.666, \quad DOW_x = 405.678, \quad SPW_x = 401.701, \quad MADM_x = 223.169, \quad Bn_x = 241.697, \quad Sr_x^a = 191.317, \quad Tn_x = 201.547, \quad Qn_x = 223.029, \quad Sn_x = 221.654, \quad \rho_{yS} = 0.7298, \quad M_x = 373.82.$$

For numerical comparison, we have computed the MSEs, Percentage relative efficiencies (PREs) and relative root mean square errors (RRMSEs) based on the above-mentioned datasets. The PREs of the proposed estimators and the existing estimators relative to usual SRS estimator of population variance $(S_y^2)$ are obtained by using the following expression:

$$PRE(Y^2) = \frac{\text{Var}(S_y^2)}{\text{MSE}(Y^2) or \text{MSE}(Y^2)_{\text{min}}} \times 100,$$

where $\text{MSE}(Y^2)$ or $\text{MSE}(Y^2)_{\text{min}}$ denotes the MSEs of the existing and proposed estimators of population variance considered in this study. An estimator with a higher value of PRE is considered superior to its counterparts. The RRMSE is obtained by using the following expression:

$$RRMSE = \sqrt{\frac{\text{MSE}(\theta_i)}{\theta}}, \quad \theta_i = S_y^2, S_x^2, S_{yS}, \ldots, S_{yS}, S_{PR-j}^2.$$

where $\theta$ is the true population variance, i.e. $S_y^2$, and $S_{PR-j}^2$ ($j = 1, 2, \ldots, 20$) denote the proposed estimators given in Table 1. An estimator with lowest RRMSE is usually declared as most efficient among the competing estimators.

The numerical results for MSEs, RRMSEs and PREs of the existing estimators and proposed estimators are given in Tables 2 and 3, respectively. A comparison of these results clearly establish that all the proposed estimators envisaged as member of the class $S_{PR}^2$ have smaller MSEs and RRMSEs as compare to the existing estimators of population variance in all the populations considered in this study. Moreover, the PREs of the proposed estimators are much higher as compare to their existing counterparts. It is also observed that in most cases the estimators using the auxiliary information on $C_x$. 

[Place Tables 2 and 3 Here]
and the non-conventional measures in tandem, that is \( S_{PR-1}^2 \) to \( S_{PR-10}^2 \), perform slightly better as compare to other estimators of the proposed class \( S_{PR}^2 \).

5. Conclusion

In this study, we have proposed a new generalized class of ratio-product type exponential estimators of population variance under SRS which incorporates both the conventional and somewhat robust non-conventional auxiliary information. Some theoretical results such as the bias, MSE and efficiency conditions under which the proposed estimators are better than the existing estimators are derived. Using three different datasets which contains outlier observations, the numerical efficiency comparison with the existing estimators is made based on MSEs, RRMSEs and PREs. It is established that the proposed estimators have superior efficiency as compare to their counterparts.

Acknowledgments

The authors are thankful to the reviewers and the editor for their valuable comments and suggestions that led to improving the article.

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**Figures Captions:**

**Figure 1:** Boxplots for Population-I
- Figure 1 (a): Study variable
- Figure 1 (b): Auxiliary Variable

**Figure 2:** Boxplots for Population-II
- Figure 2 (a): Study variable
- Figure 2 (b): Auxiliary Variable

**Figure 3:** Boxplots for Population-III
- Figure 3 (a): Study variable
- Figure 3 (b): Auxiliary Variable

**Tables Captions:**

**Table 1:** Some new members of proposed class-I estimators

**Table 2:** Estimated numerical results of the MSEs, RRMSEs, and PREs with respect to $S_y^2$ of the existing estimators.

**Table 3:** Estimated numerical results of the MSEs, RRMSEs, and PREs with respect to $S_y^2$ of the proposed estimators.
| Estimator | Value of Constant |
|-----------|------------------|
| $S_{PR-1} = s_y^2 \left\{ p_1 \left( \frac{\rho_{(\hat{\sigma}^2, \hat{\sigma}^2)}}{\rho_{(\hat{\sigma}^2, \hat{\sigma}^2)}} \frac{S_x^2 + IQR_x}{S_x^2 + IQR_x} \right) + p_2 \left( \frac{\rho_{(\hat{\sigma}^2, \hat{\sigma}^2)}}{\rho_{(\hat{\sigma}^2, \hat{\sigma}^2)}} \frac{S_x^2 + IQR_x}{S_x^2 + IQR_x} \right) \right\} \exp \left\{ \frac{\rho_{(\hat{\sigma}^2, \hat{\sigma}^2)}}{\rho_{(\hat{\sigma}^2, \hat{\sigma}^2)}} (S_y^2 - S_x^2) \right\} $ | $\rho_{(\hat{\sigma}^2, \hat{\sigma}^2)}$ $IQR_x$ |
| $S_{PR-2} = s_y^2 \left\{ p_1 \left( \frac{\rho_{(\hat{\sigma}^2, \hat{\sigma}^2)}}{\rho_{(\hat{\sigma}^2, \hat{\sigma}^2)}} \frac{S_x^2 + GIN_x}{S_x^2 + GIN_x} \right) + p_2 \left( \frac{\rho_{(\hat{\sigma}^2, \hat{\sigma}^2)}}{\rho_{(\hat{\sigma}^2, \hat{\sigma}^2)}} \frac{S_x^2 + GIN_x}{S_x^2 + GIN_x} \right) \right\} \exp \left\{ \frac{\rho_{(\hat{\sigma}^2, \hat{\sigma}^2)}}{\rho_{(\hat{\sigma}^2, \hat{\sigma}^2)}} (S_y^2 - S_x^2) \right\} $ | $\rho_{(\hat{\sigma}^2, \hat{\sigma}^2)}$ $GIN_x$ |
| $S_{PR-3} = s_y^2 \left\{ p_1 \left( \frac{\rho_{(\hat{\sigma}^2, \hat{\sigma}^2)}}{\rho_{(\hat{\sigma}^2, \hat{\sigma}^2)}} \frac{S_x^2 + DOW_x}{S_x^2 + DOW_x} \right) + p_2 \left( \frac{\rho_{(\hat{\sigma}^2, \hat{\sigma}^2)}}{\rho_{(\hat{\sigma}^2, \hat{\sigma}^2)}} \frac{S_x^2 + DOW_x}{S_x^2 + DOW_x} \right) \right\} \exp \left\{ \frac{\rho_{(\hat{\sigma}^2, \hat{\sigma}^2)}}{\rho_{(\hat{\sigma}^2, \hat{\sigma}^2)}} (S_y^2 - S_x^2) \right\} $ | $\rho_{(\hat{\sigma}^2, \hat{\sigma}^2)}$ $DOW_x$ |
| $S_{PR-4} = s_y^2 \left\{ p_1 \left( \frac{\rho_{(\hat{\sigma}^2, \hat{\sigma}^2)}}{\rho_{(\hat{\sigma}^2, \hat{\sigma}^2)}} \frac{S_x^2 + SPW_x}{S_x^2 + SPW_x} \right) + p_2 \left( \frac{\rho_{(\hat{\sigma}^2, \hat{\sigma}^2)}}{\rho_{(\hat{\sigma}^2, \hat{\sigma}^2)}} \frac{S_x^2 + SPW_x}{S_x^2 + SPW_x} \right) \right\} \exp \left\{ \frac{\rho_{(\hat{\sigma}^2, \hat{\sigma}^2)}}{\rho_{(\hat{\sigma}^2, \hat{\sigma}^2)}} (S_y^2 - S_x^2) \right\} $ | $\rho_{(\hat{\sigma}^2, \hat{\sigma}^2)}$ $SPW_x$ |
| $S_{PR-5}$ | $S_{PR-6}$ | $S_{PR-7}$ | $S_{PR-8}$ | $S_{PR-9}$ |
|-----------|-----------|-----------|-----------|-----------|
| $s_y^2$   | $s_y^2$   | $s_y^2$   | $s_y^2$   | $s_y^2$   |

\[
S_{PR-5} = s_y^2 \left\{ p_1 \left( \frac{\rho_{(i',i)^2} S_{x}^2 + MADM_s}{\rho_{(i',i)^2} S_{x}^2 + MADM_s} \right)^{\rho_{(i',i)^2} S_{x}^2 + MADM_s} \right\} + p_2 \left( \frac{\rho_{(i',i)^2} S_{x}^2 + MADM_s}{\rho_{(i',i)^2} S_{x}^2 + MADM_s} \right)^{\rho_{(i',i)^2} S_{x}^2 + MADM_s} \right\} \exp \left( \frac{\rho_{(i',i)^2} \left( S_{x}^2 - s_x^2 \right)}{\rho_{(i',i)^2} \left( S_{x}^2 + s_x^2 \right) + 2MADM_s} \right) \right\} \rho_{(i',i)^2} \ \text{MADM}_s

\[
S_{PR-6} = s_y^2 \left\{ p_1 \left( \frac{\rho_{(i',i)^2} S_{x}^2 + Bn_x}{\rho_{(i',i)^2} S_{x}^2 + Bn_x} \right)^{\rho_{(i',i)^2} S_{x}^2 + Bn_x} \right\} + p_2 \left( \frac{\rho_{(i',i)^2} S_{x}^2 + Bn_x}{\rho_{(i',i)^2} S_{x}^2 + Bn_x} \right)^{\rho_{(i',i)^2} S_{x}^2 + Bn_x} \right\} \exp \left( \frac{\rho_{(i',i)^2} \left( S_{x}^2 - s_x^2 \right)}{\rho_{(i',i)^2} \left( S_{x}^2 + s_x^2 \right) + 2Bn_x} \right) \right\} \rho_{(i',i)^2} \ \text{Bn}_x

\[
S_{PR-7} = s_y^2 \left\{ p_1 \left( \frac{\rho_{(i',i)^2} S_{x}^2 + Sr^a_x}{\rho_{(i',i)^2} S_{x}^2 + Sr^a_x} \right)^{\rho_{(i',i)^2} S_{x}^2 + Sr^a_x} \right\} + p_2 \left( \frac{\rho_{(i',i)^2} S_{x}^2 + Sr^a_x}{\rho_{(i',i)^2} S_{x}^2 + Sr^a_x} \right)^{\rho_{(i',i)^2} S_{x}^2 + Sr^a_x} \right\} \exp \left( \frac{\rho_{(i',i)^2} \left( S_{x}^2 - s_x^2 \right)}{\rho_{(i',i)^2} \left( S_{x}^2 + s_x^2 \right) + 2Sr^a_x} \right) \right\} \rho_{(i',i)^2} \ \text{Sr}^a_x

\[
S_{PR-8} = s_y^2 \left\{ p_1 \left( \frac{\rho_{(i',i)^2} S_{x}^2 + Tn_x}{\rho_{(i',i)^2} S_{x}^2 + Tn_x} \right)^{\rho_{(i',i)^2} S_{x}^2 + Tn_x} \right\} + p_2 \left( \frac{\rho_{(i',i)^2} S_{x}^2 + Tn_x}{\rho_{(i',i)^2} S_{x}^2 + Tn_x} \right)^{\rho_{(i',i)^2} S_{x}^2 + Tn_x} \right\} \exp \left( \frac{\rho_{(i',i)^2} \left( S_{x}^2 - s_x^2 \right)}{\rho_{(i',i)^2} \left( S_{x}^2 + s_x^2 \right) + 2Tn_x} \right) \right\} \rho_{(i',i)^2} \ \text{Tn}_x

\[
S_{PR-9} = s_y^2 \left\{ p_1 \left( \frac{\rho_{(i',i)^2} S_{x}^2 + Qn_x}{\rho_{(i',i)^2} S_{x}^2 + Qn_x} \right)^{\rho_{(i',i)^2} S_{x}^2 + Qn_x} \right\} + p_2 \left( \frac{\rho_{(i',i)^2} S_{x}^2 + Qn_x}{\rho_{(i',i)^2} S_{x}^2 + Qn_x} \right)^{\rho_{(i',i)^2} S_{x}^2 + Qn_x} \right\} \exp \left( \frac{\rho_{(i',i)^2} \left( S_{x}^2 - s_x^2 \right)}{\rho_{(i',i)^2} \left( S_{x}^2 + s_x^2 \right) + 2Qn_x} \right) \right\} \rho_{(i',i)^2} \ \text{Qn}_x

23
\[ S_{PR-10} = s_y^2 \left\{ p_1 \left( \frac{\rho_{(x_1, x_2)} (S_{x_1}^2 + S_{x_2}^2)}{\rho_{(x_1, x_2)} (S_{x_1}^2 + S_{x_2}^2)} \right) \right\} \]

\[ + p_2 \left( \frac{\rho_{(x_1, x_2)} (S_{x_1}^2 + S_{x_2}^2)}{\rho_{(x_1, x_2)} (S_{x_1}^2 + S_{x_2}^2)} \right) \exp \left\{ \frac{\rho_{(x_1, x_2)} (S_{x_1}^2 - S_{x_2}^2)}{\rho_{(x_1, x_2)} (S_{x_1}^2 + S_{x_2}^2) + 2S_{x_{12}}} \right\} \]

\[ \rho_{(x_1, x_2)} \quad Sn_x \]

\[ S_{PR-11} = s_y^2 \left\{ p_1 \left( \frac{C_x (S_{x_1}^2 + IQR_x)}{C_x (S_{x_1}^2 + IQR_x)} \right) \right\} \]

\[ + p_2 \left( \frac{C_x (S_{x_1}^2 + IQR_x)}{C_x (S_{x_1}^2 + IQR_x)} \right) \exp \left\{ \frac{C_x (S_{x_1}^2 - S_{x_2}^2)}{C_x (S_{x_1}^2 + S_{x_2}^2) + 2IQR_x} \right\} \]

\[ C_x \quad IQR_x \]

\[ S_{PR-12} = s_y^2 \left\{ p_1 \left( \frac{C_x (S_{x_1}^2 + GIN_x)}{C_x (S_{x_1}^2 + GIN_x)} \right) \right\} \]

\[ + p_2 \left( \frac{C_x (S_{x_1}^2 + GIN_x)}{C_x (S_{x_1}^2 + GIN_x)} \right) \exp \left\{ \frac{C_x (S_{x_1}^2 - S_{x_2}^2)}{C_x (S_{x_1}^2 + S_{x_2}^2) + 2GIN_x} \right\} \]

\[ C_x \quad GIN_x \]

\[ S_{PR-13} = s_y^2 \left\{ p_1 \left( \frac{C_x (S_{x_1}^2 + DOW_x)}{C_x (S_{x_1}^2 + DOW_x)} \right) \right\} \]

\[ + p_2 \left( \frac{C_x (S_{x_1}^2 + DOW_x)}{C_x (S_{x_1}^2 + DOW_x)} \right) \exp \left\{ \frac{C_x (S_{x_1}^2 - S_{x_2}^2)}{C_x (S_{x_1}^2 + S_{x_2}^2) + 2DOW_x} \right\} \]

\[ C_x \quad DOW_x \]

\[ S_{PR-14} = s_y^2 \left\{ p_1 \left( \frac{C_x (S_{x_1}^2 + SPW_x)}{C_x (S_{x_1}^2 + SPW_x)} \right) \right\} \]

\[ + p_2 \left( \frac{C_x (S_{x_1}^2 + SPW_x)}{C_x (S_{x_1}^2 + SPW_x)} \right) \exp \left\{ \frac{C_x (S_{x_1}^2 - S_{x_2}^2)}{C_x (S_{x_1}^2 + S_{x_2}^2) + 2SPW_x} \right\} \]

\[ C_x \quad SPW_x \]

\[ S_{PR-15} = s_y^2 \left\{ p_1 \left( \frac{C_x (S_{x_1}^2 + MADM_x)}{C_x (S_{x_1}^2 + MADM_x)} \right) \right\} \]

\[ + p_2 \left( \frac{C_x (S_{x_1}^2 + MADM_x)}{C_x (S_{x_1}^2 + MADM_x)} \right) \exp \left\{ \frac{C_x (S_{x_1}^2 - S_{x_2}^2)}{C_x (S_{x_1}^2 + S_{x_2}^2) + 2MADM_x} \right\} \]

\[ C_x \quad MADM_x \]
\[
S_{\text{PR-16}}^2 = s_y^2 \left\{ p_1 \left( \frac{C_s S_z^2 + Bn_x}{C_s s_y^2 + Bn_x} \right) \frac{C_s S_z^2}{C_s s_y^2 + Bn_x} + p_2 \left( \frac{C_s S_z^2 + Bn_x}{C_s S_z^2 + Bn_x} \right) \frac{C_s S_z^2}{C_s S_z^2 + Bn_x} \right\} \exp \left\{ \frac{C_s \left( S_z^2 - s_z^2 \right)}{C_s \left( S_z^2 + s_z^2 \right) + 2Bn_x} \right\}
\]

\[
S_{\text{PR-17}}^2 = s_y^2 \left\{ p_1 \left( \frac{C_s S_z^2 + Sr_x^\alpha}{C_s S_z^2 + Sr_x^\alpha} \right) \frac{C_s S_z^2}{C_s S_z^2 + Sr_x^\alpha} + p_2 \left( \frac{C_s S_z^2 + Sr_x^\alpha}{C_s S_z^2 + Sr_x^\alpha} \right) \frac{C_s S_z^2}{C_s S_z^2 + Sr_x^\alpha} \right\} \exp \left\{ \frac{C_s \left( S_z^2 - s_z^2 \right)}{C_s \left( S_z^2 + s_z^2 \right) + 2Sr_x^\alpha} \right\}
\]

\[
S_{\text{PR-18}}^2 = s_y^2 \left\{ p_1 \left( \frac{C_s S_z^2 + Tn_x}{C_s S_z^2 + Tn_x} \right) \frac{C_s S_z^2}{C_s S_z^2 + Tn_x} + p_2 \left( \frac{C_s S_z^2 + Tn_x}{C_s S_z^2 + Tn_x} \right) \frac{C_s S_z^2}{C_s S_z^2 + Tn_x} \right\} \exp \left\{ \frac{C_s \left( S_z^2 - s_z^2 \right)}{C_s \left( S_z^2 + s_z^2 \right) + 2Tn_x} \right\}
\]

\[
S_{\text{PR-19}}^2 = s_y^2 \left\{ p_1 \left( \frac{C_s S_z^2 + Qn_x}{C_s S_z^2 + Qn_x} \right) \frac{C_s S_z^2}{C_s S_z^2 + Qn_x} + p_2 \left( \frac{C_s S_z^2 + Qn_x}{C_s S_z^2 + Qn_x} \right) \frac{C_s S_z^2}{C_s S_z^2 + Qn_x} \right\} \exp \left\{ \frac{C_s \left( S_z^2 - s_z^2 \right)}{C_s \left( S_z^2 + s_z^2 \right) + 2Qn_x} \right\}
\]

\[
S_{\text{PR-20}}^2 = s_y^2 \left\{ p_1 \left( \frac{C_s S_z^2 + Sn_x}{C_s S_z^2 + Sn_x} \right) \frac{C_s S_z^2}{C_s S_z^2 + Sn_x} + p_2 \left( \frac{C_s S_z^2 + Sn_x}{C_s S_z^2 + Sn_x} \right) \frac{C_s S_z^2}{C_s S_z^2 + Sn_x} \right\} \exp \left\{ \frac{C_s \left( S_z^2 - s_z^2 \right)}{C_s \left( S_z^2 + s_z^2 \right) + 2Sn_x} \right\}
\]
Table 2: Estimated numerical results of the MSEs, RRMSEs, and PREs with respect to $S^2_y$ of the existing estimators.

| Estimator | Measure | POP-I | POP-II | POP-III |
|-----------|---------|-------|--------|---------|
| $S^2_y$   | MSE     | 26686254 | 5393.75 | 38482180 |
|           | RRMSE   | 0.3407832 | 0.2179449 | 0.7433034 |
|           | PRE     | 100    | 100    | 100     |
| $S^2_R$   | MSE     | 10314786 | 2952.368 | 21898276 |
|           | RRMSE   | 0.2118675 | 0.1612452 | 0.5607138 |
|           | PRE     | 258.7184 | 182.6923 | 175.7316 |
| $S^2_{Reg}$ | MSE     | 8045752 | 2000.995 | 21871290 |
|           | RRMSE   | 0.1871189 | 0.1327469 | 0.5603682 |
|           | PRE     | 331.6813 | 269.5533 | 175.9484 |
| $S^2_d$   | MSE     | 7773572 | 1966.345 | 16644658 |
|           | RRMSE   | 0.1839266 | 0.1315925 | 0.4888479 |
|           | PRE     | 343.2946 | 274.3033 | 231.1984 |
| $S^2_BT$  | MSE     | 10021370 | 2188.727 | 25695997 |
|           | RRMSE   | 0.2088324 | 0.1388349 | 0.607392 |
|           | PRE     | 266.2935 | 246.4332 | 149.7594 |
| $S^2_US$  | MSE     | 10305164 | 2750.183 | 21898209 |
|           | RRMSE   | 0.2117687 | 0.155626 | 0.560713 |
|           | PRE     | 343.9542 | 274.3033 | 231.1984 |
| $S^2_{KC-1}$ | MSE     | 10313107 | 2895.45 | 21898272 |
|           | RRMSE   | 0.2118503 | 0.1596833 | 0.5607138 |
|           | PRE     | 258.7184 | 182.6923 | 175.7316 |
| $S^2_{KC-2}$ | MSE     | 10305564 | 2691.57 | 21898215 |
|           | RRMSE   | 0.2117728 | 0.1539587 | 0.5607131 |
|           | PRE     | 258.95 | 200.3942 | 175.732 |
| $S^2_{KC-3}$ | MSE     | 10213530 | 2494.658 | 21896896 |
|           | RRMSE   | 0.2108251 | 0.14822 | 0.5606962 |
|           | PRE     | 261.2834 | 216.212 | 175.7426 |
| $S^2_{SG}$ | MSE     | 7411158 | 1923.464 | 15515489 |
|           | RRMSE   | 0.179588 | 0.1301498 | 0.471975 |
|           | PRE     | 360.8021 | 288.4185 | 248.0243 |
| $S^2_{SK-1}$ | MSE     | 10217681 | 2389.708 | 21897016 |
|           | RRMSE   | 0.2108679 | 0.1450688 | 0.560977 |
|           | PRE     | 261.1772 | 225.7074 | 175.7417 |
| $S^2_{SK-2}$ | MSE     | 10124389 | 2162.944 | 21895159 |
|           | RRMSE   | 0.209903 | 0.1380143 | 0.5606739 |
|           | PRE     | 263.5839 | 249.3707 | 175.7566 |
| $S^2_{KS}$ | MSE     | 8045752 | 2000.995 | 21871290 |
|           | RRMSE   | 0.1871189 | 0.1327469 | 0.5603682 |
|           | PRE     | 331.6813 | 269.5533 | 175.9484 |
| $S^2_{SW}$ | MSE     | 7728302 | 1963.789 | 15367096 |
|           | RRMSE   | 0.1833903 | 0.131507 | 0.4697126 |
| Estimator | Measure | POP-I       | POP-II       | POP-III      |
|-----------|---------|-------------|--------------|--------------|
|           | PRE     | 345.3055    | 274.6603     | 250.4193     |
| \( S_{YK}^2 \) | MSE     | 8045752     | 2000.995     | 21871290     |
|           | RRMSE   | 0.1871189   | 0.1327469    | 0.5603682    |
|           | PRE     | 331.6813    | 269.5533     | 175.9484     |
| \( S_{YS}^2 \) | MSE     | 7335157     | 1919.253     | 14417414     |
|           | RRMSE   | 0.1786648   | 0.1300072    | 0.4549671    |
|           | PRE     | 363.813     | 281.0338     | 266.9146     |
Table 3: Estimated numerical results of the MSEs, RRMSEs, and PREs with respect to $S^2_y$ of the proposed estimators.

| Estimator | Measure | POP-I | POP-II | POP-III |
|-----------|---------|-------|--------|---------|
| $S_{PR-1}$ | MSE     | 5458119 | 1862.839 | 13835683 |
|           | RRMSE   | 0.1541189 | 0.1280823 | 0.4456938 |
|           | PRE     | 488.9277 | 289.5445 | 278.1372 |
| $S_{PR-2}$ | MSE     | 5490879 | 1847.107 | 13842745 |
|           | RRMSE   | 0.1545807 | 0.1275403 | 0.4458075 |
|           | PRE     | 486.0106 | 292.0107 | 277.9953 |
| $S_{PR-3}$ | MSE     | 5473413 | 1832.747 | 13838622 |
|           | RRMSE   | 0.1543346 | 0.126936 | 0.4457412 |
|           | PRE     | 487.8158 | 294.5414 | 278.0845 |
| $S_{PR-4}$ | MSE     | 5470560 | 1831.236 | 13838307 |
|           | RRMSE   | 0.1542944 | 0.1269912 | 0.4457361 |
|           | PRE     | 489.9466 | 297.6411 | 278.3402 |
| $S_{PR-5}$ | MSE     | 5397326 | 1763.154 | 13824116 |
|           | RRMSE   | 0.1532582 | 0.1246082 | 0.4455075 |
|           | PRE     | 494.4348 | 305.9149 | 278.3699 |
| $S_{PR-6}$ | MSE     | 5418674 | 1812.166 | 13825591 |
|           | RRMSE   | 0.153561 | 0.1263282 | 0.4455312 |
|           | PRE     | 492.4868 | 297.6411 | 278.3402 |
| $S_{PR-7}$ | MSE     | 5388330 | 1744.32 | 13821579 |
|           | RRMSE   | 0.1531375 | 0.1239409 | 0.4454666 |
|           | PRE     | 495.2143 | 309.2179 | 278.4210 |
| $S_{PR-8}$ | MSE     | 5391205 | 1751.381 | 13822394 |
|           | RRMSE   | 0.1531713 | 0.1241915 | 0.4454797 |
|           | PRE     | 494.996 | 307.9712 | 278.4046 |
| $S_{PR-9}$ | MSE     | 5409339 | 1768.802 | 13824105 |
|           | RRMSE   | 0.1534286 | 0.1248076 | 0.4455073 |
|           | PRE     | 493.3367 | 304.938 | 278.3701 |
| $S_{PR-10}$ | MSE   | 5399383 | 1758.961 | 13823996 |
|           | RRMSE  | 0.1532874 | 0.1245999 | 0.4455055 |
|           | PRE    | 494.2464 | 306.644 | 278.3723 |
| $S_{PR-11}$ | MSE   | 5433607 | 1869.284 | 13823937 |
|           | RRMSE  | 0.1537724 | 0.1283037 | 0.4455046 |
|           | PRE    | 491.1333 | 288.5463 | 278.3735 |
| $S_{PR-12}$ | MSE   | 5460177 | 1853.613 | 13828183 |
|           | RRMSE  | 0.1541479 | 0.1277647 | 0.445573 |
|           | PRE    | 488.7434 | 290.9857 | 278.288 |
| $S_{PR-13}$ | MSE   | 5446001 | 1839.234 | 13825704 |
|           | RRMSE  | 0.1539477 | 0.1272682 | 0.445331 |
|           | PRE    | 490.0156 | 293.2607 | 278.3379 |
| $S_{PR-14}$ | MSE   | 5443688 | 1837.716 | 13825515 |
|           | RRMSE  | 0.153915 | 0.1272157 | 0.44553 |
|           | PRE    | 490.2238 | 293.5029 | 278.3418 |
| $S_{PR-15}$ | MSE   | 5384508 | 1768.603 | 13816990 |
|           | RRMSE  | 0.1530761 | 0.1248006 | 0.4453926 |
| Estimator | Measure | POP-I    | POP-II   | POP-III   |
|-----------|---------|----------|----------|-----------|
| $S_{PR-16}^2$ | MSE     | 5401720  | 1818.501 | 13817876  |
|           | RRMSE   | 0.1533205| 0.1265488| 0.4454069|
|           | PRE     | 494.0325 | 296.6042 | 278.4956  |
| $S_{PR-17}^2$ | MSE     | 5377668  | 1749.249 | 13815467  |
|           | RRMSE   | 0.1529788| 0.1241158| 0.4453681|
|           | PRE     | 496.2422 | 308.3467 | 278.5442  |
| $S_{PR-18}^2$ | MSE     | 5379580  | 1756.516 | 13815956  |
|           | RRMSE   | 0.153006 | 0.1243734| 0.445376  |
|           | PRE     | 496.0658 | 307.0709 | 278.5343  |
| $S_{PR-19}^2$ | MSE     | 5394190  | 1774.389 | 13816983  |
|           | RRMSE   | 0.1532136| 0.1250046| 0.4453925|
|           | PRE     | 494.7222 | 303.9779 | 278.5136  |
| $S_{PR-20}^2$ | MSE     | 5386165  | 1764.303 | 13816918  |
|           | RRMSE   | 0.1530996| 0.1246488| 0.4453915|
|           | PRE     | 495.4592 | 305.7156 | 278.5149  |
Figure 1: Boxplots for Population-I

(a) Study variable
(b) Auxiliary variable

Figure 2: Boxplots for Population-II

(a) Study variable
(b) Auxiliary variable
Figure 3: Boxplots for Population-III

(a) Study variable

(b) Auxiliary variable
Biographies

Farah Naz earned her MSc and MPhil degrees in Statistics from the Islamia University Bahawalpur, Pakistan, and Government College University Faisalabad, Pakistan, respectively. Currently, she is pursuing PhD in Statistics in the School of Mathematical Sciences, Institute of Statistics, Zhejiang University, Hangzhou, People’s Republic of China, under the Chinese Government Scholarship Program (2017). Her research interest is survey sampling and distribution theory.

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