QUANTUM AFFINE $\mathfrak{gl}_n$ VIA HECKE ALGEBRAS

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Abstract. We use the Hecke algebras of affine symmetric groups and their associated Schur algebras to construct a new algebra through a basis, and a set of generators and explicit multiplication formulas of basis elements by generators. We prove that this algebra is isomorphic to the quantum enveloping algebra of the loop algebra of $\mathfrak{gl}_n$. Though this construction is motivated by the work [1] by Beilinson–Lusztig–MacPherson for quantum $\mathfrak{gl}_n$, our approach is purely algebraic and combinatorial, independent of the geometric method which seems to work only for quantum $\mathfrak{gl}_n$ and quantum affine $\mathfrak{sl}_n$. As an application, we discover a presentation of the Ringel–Hall algebra of a cyclic quiver by semisimple generators and their multiplications by the defining basis elements.

1. Introduction

The quantum enveloping algebra of the loop algebra of $\mathfrak{gl}_n$, or simply quantum affine $\mathfrak{gl}_n$, has two usual definitions, the $R$-matrix one and the Drinfeld one, known as Drinfeld’s new realisation. Both are presented by generators and relations (see, e.g., [9, §2.3] and the references therein). In [3, 2.3.1,2.5.3], a third presentation is given via the double Ringle–Hall algebra. In this presentation, the Ringel–Hall algebra of the cyclic quiver and its opposite algebra become the $\pm$-part of quantum affine $\mathfrak{gl}_n$. Thus, with this construction, one may consider semisimple or indecomposable generators for quantum affine $\mathfrak{gl}_n$, defined by the semisimple or indecomposable representations of the quiver; see [3, §1.4]. In particular, one sees easily the fact that the subalgebra generated by simple generators is a proper subalgebra. This subalgebra is isomorphic to the quantum affine $\mathfrak{sl}_n$.

The double Ringel–Hall algebra construction of quantum affine $\mathfrak{gl}_n$ is an affine generalisation of a similar construction for a quantum enveloping algebra $\mathcal{U}$ of a finite type quiver via a Ringel–Hall algebra which, as the positive or negative part of $\mathcal{U}$, is spanned by the basis of isoclasses of representations of the quiver and whose multiplication is defined by Hall polynomials, see [15, 16, 20]. However, there is another construction for quantum $\mathfrak{gl}_n$ by Beilinson, Lusztig and MacPherson [1, 5.7], which directly displays a basis for the entire quantum enveloping algebra $\mathcal{U}(\mathfrak{gl}_n)$ and displays the multiplication rules by explicit formulas of basis elements by generators.

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This construction is geometric in nature and has been partially generalised to the affine case (more precisely, to affine \(\mathfrak{sl}_n\)) in [12, 14, 19].

BLM’s geometric approach uses the definition of quantum Schur algebras as the convolution algebras of (partial) flag varieties over finite fields and then, by a process of “quantumization”, to get a construction over the polynomials ring. Progress on generalising BLM’s work to the affine case via an algebraic approach has been made in the works [8, 3, 10]. In particular, a realisation conjecture [8, 5.5(2)] for quantum affine \(\mathfrak{g}l_n\) was formulated and was proved in the classical \((v = 1)\) case in [3, Ch. 6]. We will prove this conjecture in this paper. We will use directly the double cosets associated with semisimple representations will play a key role in the establishment of multiplication rules of BLM type basis elements by semisimple generators.

It should be pointed out that a recent work by Bridgeland constructs quantum enveloping algebras via Hall algebras of complexes and a complete realisation [2, Th. 4.9] is obtained for simply-laced finite types; see [21] for the general (finite type) case. We obtain here a complete realisation for quantum affine \(\mathfrak{g}l_n\).

We now describe the main result of the paper. For a positive integer \(n\), let \(\widehat{\mathfrak{g}l}_n := M_{\delta,n}(\mathbb{C})\) be the loop algebra of \(\mathfrak{g}l_n(\mathbb{C})\) consisting of all matrices \(A = (a_{i,j})_{i,j \in \mathbb{Z}}\) with \(a_{i,j} \in \mathbb{C}\) such that

\[
\begin{align*}
(a) & \quad a_{i,j} = a_{i+n,j+n} \text{ for } i, j \in \mathbb{Z}; \\
(b) & \quad \text{for every } i \in \mathbb{Z}, \text{ both sets } \{j \in \mathbb{Z} \mid a_{i,j} \neq 0\} \text{ and } \{j \in \mathbb{Z} \mid a_{j,i} \neq 0\} \text{ are finite.}
\end{align*}
\]

A basis for \(\widehat{\mathfrak{g}l}_n\) can be described as \(\{E_{i,j}^\delta \mid i, j \in \mathbb{Z}\}\), where the matrix \(E_{i,j}^\delta = (e_{k,l}^{i,j})_{k,l \in \mathbb{Z}}\) is defined by

\[
e_{k,l}^{i,j} = \begin{cases} 1 & \text{if } k = i + sn, l = j + sn \text{ for some } s \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases}
\]

Let \(\Theta_{\delta}(n) = M_{\delta,n}(\mathbb{N})\) be the \(\mathbb{N}\)-span of the basis. Then \(\Theta_{\delta}(n)\) serves as the index set of the PBW basis of the universal enveloping algebra \(U(\widehat{\mathfrak{g}l}_n)\). Let \(U(\widehat{\mathfrak{g}l}_n)\) be the quantum enveloping algebra of \(\widehat{\mathfrak{g}l}_n\) over \(\mathbb{Q}(v)\) and let

\[
\Theta^\pm_{\delta}(n) = \{A \in \Theta_{\delta}(n) \mid a_{i,i} = 0 \text{ for all } i\} \text{ and } Z^n = \{(\lambda_i)_{i \in \mathbb{Z}} \mid \lambda_i \in \mathbb{Z}, \lambda_i = \lambda_{i-n} \text{ for } i \in \mathbb{Z}\}.
\]

Then \(\Theta^\pm_{\delta}(n) \times Z^n\) serves as an index set of a PBW type basis for \(U(\widehat{\mathfrak{g}l}_n)\) (see, e.g., [3, 1.4.6]). We will construct a new basis \(\{A(j) \mid A \in \Theta^\pm_{\delta}(n), j \in \mathbb{Z}^n\}\) and prove the following main result.

**Main Theorem 1.1.** The quantum enveloping algebra \(U(\widehat{\mathfrak{g}l}_n)\) is the \(\mathbb{Q}(v)\)-algebra which is spanned by the basis \(\{A(j) \mid A \in \Theta^\pm_{\delta}(n), j \in \mathbb{Z}^n\}\) and generated by \(0(j), S_\alpha(0)\) and \(tS_\alpha(0)\) for all \(j \in \mathbb{Z}^n\) and \(\alpha \in \mathbb{N}^n\), where \(S_\alpha = \sum_{1 \leq i \leq n} \alpha_i E_{i,i+1}^\delta + t^i S_\alpha\) is the transpose of \(S_\alpha\), and whose multiplication rules are given by the formulas in Proposition 4.2(1)–(3).

We organise this paper as follows. We first recall in §2 some preliminary results on the double Ringel–Hall algebra of a cyclic quiver and affine quantum Schur algebras. In particular,
we display a PBW type basis for the former and a basis defined by certain double cosets for the latter. In §3, we derive in the affine quantum Schur algebra some multiplication formulas (Theorem 3.6) of the basis elements by those associated with semisimple representations of the cyclic quiver. We then prove the main result via Theorem 4.4 in §4. As an application of the work, we obtain certain multiplication formulas in the Ringel–Hall algebra which are not directly seen from the Hall algebra multiplication. In the Appendix, we give a proof for the length formula of the shortest representative of a double coset defined by a matrix.

**Further Notations 1.2.** We need the following index sets for bases of the triangular parts of $\mathcal{U}(\hat{\mathfrak{g}}_n)$. Let

$$\Theta^+(n) := \{ A \in \Theta(n) \mid a_{i,j} = 0 \text{ for } i > j \} \text{ and } \Theta^-(n) := \{ A \in \Theta(n) \mid a_{i,j} = 0 \text{ for } i < j \}.$$  

For $A \in \Theta(n)$, we write

$$A = A^+ + A^0 = A^+ + A^0 + A^-$$  

(1.2.1)

where $A^+ \in \Theta^+(n)$, $A^+ \in \Theta^+(n)$, $A^- \in \Theta^-(n)$ and $A^0$ is a diagonal matrix.

Further, for $r \geq 0$, let $N_0^n = \{ (\lambda_i)_{i \in \mathbb{Z}} \in \mathbb{Z}_0^n \mid \lambda_i \geq 0 \text{ for } i \in \mathbb{Z} \}$ and let

$$\Theta_\lambda(n,r) = \{ A \in \Theta(n) \mid \sigma(A) = r \} \text{ and } \Lambda_\lambda(n,r) = \{ \lambda \in N_0^n \mid \sigma(\lambda) = r \}$$

where $\sigma(A) = \sum_{1 \leq i \leq n, j \in \mathbb{Z}} a_{i,j}$ and $\sigma(\lambda) = \sum_{i} \lambda_i$. Note that $\Theta_\lambda(n,r)$ is the index set of a basis for a certain quotient algebra of $\mathcal{U}(\hat{\mathfrak{g}}_n)$—the affine quantum Schur algebras.

Moreover, we will use the standard notation for Gaussian polynomials. Thus, let $\mathbb{Z} = \mathbb{Z}[\mathbf{v}, \mathbf{v}^{-1}]$, where $\mathbf{v}$ is an indeterminate, and let $\mathbb{Q}(\mathbf{v})$ be the fraction field of $\mathbb{Z}$. For integers $N, t$ with $t \geq 0$ and $\mu \in \mathbb{Z}_n$ and $\lambda \in \mathbb{N}_0^n$, let

$$\left[ \begin{array}{c} N \\ t \end{array} \right] = \frac{\mathbf{v}^{2(N-t+1)} - 1}{\mathbf{v}^{2t} - 1} \quad \text{and} \quad \left[ \begin{array}{c} \mu \\ \lambda \end{array} \right] = \prod_{1 \leq i < n} \left[ \begin{array}{c} \mu_i \\ \lambda_i \end{array} \right].$$

Then $\left[ \begin{array}{c} N \\ t \end{array} \right] = \frac{\mathbf{v}^{2(N-t+1)} - 1}{\mathbf{v}^{2t} - 1}$. We also need the symmetric Gaussian polynomials $\left[ \begin{array}{c} N \\ t \end{array} \right] = \mathbf{v}^{-t(N-t)} \left[ \begin{array}{c} N \\ t \end{array} \right]$. For $\lambda, \lambda^{(1)}, \ldots, \lambda^{(m)} \in \mathbb{N}_0^n$ with $\lambda = \lambda^{(1)} + \cdots + \lambda^{(m)}$, we also need the following notation in 2.1(2)(e)

$$\left[ \begin{array}{c} \lambda \\ \lambda^{(1)}, \ldots, \lambda^{(m)} \end{array} \right] = \prod_{1 \leq i < n} \frac{[\lambda_i]!}{[\lambda^{(1)}_i]! \cdots [\lambda^{(m)}_i]!}.$$  

**2. Preliminary results**

In this section, we briefly discuss the affine symmetric group and its associated Hecke algebra, the affine $\mathbf{v}$-Schur algebra, the double Hall algebra interpretation of affine $\mathfrak{g}_n$ and the connections between them.

Let $\mathcal{S}_{\lambda,r}$ be the **affine symmetric group** consisting of all permutations $w : \mathbb{Z} \to \mathbb{Z}$ satisfying $w(i+r) = w(i) + r$ for $i \in \mathbb{Z}$. Let $W_r$ be the subgroup of $\mathcal{S}_{\lambda,r}$, the **Weyl group** of affine type $A$, generated by $S = \{ s_i \}_{1 \leq i < r}$, where $s_i$ is defined by $s_i(j) = j$ for $j \neq i, i+1 \mod r$, $s_i(j) = j - 1$ for $j \equiv i + 1 \mod r$, and $s_i(j) = j + 1$ for $j \equiv i \mod r$. Let $\rho$ be the permutation of $\mathbb{Z}$ sending $j$
to \( j+1 \) for all \( j \in \mathbb{Z} \). We extend the length function \( \ell \) on \( W_r \) to \( \mathfrak{S}_{\lambda r} \) by setting \( \ell(\rho^m w) = \ell(w) \) for all \( m \in \mathbb{Z}, w \in W_r \).

The (extended) affine Hecke algebra \( \mathcal{H}_\lambda(r) \) over \( \mathbb{Z} \) associated to \( \mathfrak{S}_{\lambda r} \) is the \( \mathbb{Z} \)-algebra which is spanned by (basis) \( \{T_w\}_{w \in \mathfrak{S}_{\lambda r}} \) and generated by \( T_\rho, T_{\rho-1}, T_s, s \in S \), and whose multiplication rules are given by the formulas, for all \( s \in S \) and \( w \in \mathfrak{S}_{\lambda r} \),

\[
T_s T_w = \begin{cases} 
(v^2 - 1)T_w + v^2 T_{sw}, & \text{if } \ell(sw) < \ell(w); \\
T_{sw}, & \text{if } \ell(sw) = \ell(w) + 1,
\end{cases}
T_\rho T_w = T_{\rho w}.
\]

Let \( \mathcal{H}_\lambda(r) = \mathcal{H}_\lambda(r) \otimes_{\mathbb{Z}} \mathbb{Q}(v) \). We will discover a similar description for quantum affine \( \mathfrak{gl}_n \).

For \( \lambda \in \Lambda_\lambda(n, r) \), let \( \mathfrak{S}_\lambda := \mathfrak{S}_{\lambda_1, \ldots, \lambda_n} \) be the corresponding standard Young subgroup of the symmetric group \( \mathfrak{S}_r \). For a finite subset \( X \subseteq \mathfrak{S}_{\lambda r} \), let

\[
T_X = \sum_{x \in X} T_x \in \mathcal{H}_\lambda(r) \quad \text{and} \quad x_\lambda = T_{x_\lambda}.
\]

The endomorphism algebras over \( \mathbb{Z} \) or \( \mathbb{Q}(v) \)

\[
\mathcal{S}_\lambda(n, r) := \text{End}_{\mathcal{H}_\lambda(r)} \left( \bigoplus_{\lambda \in \Lambda_\lambda(n, r)} x_\lambda \mathcal{H}_\lambda(r) \right) \quad \text{and} \quad \mathcal{S}_\lambda(n, r) := \text{End}_{\mathcal{H}_\lambda(r)} \left( \bigoplus_{\lambda \in \Lambda_\lambda(n, r)} x_\lambda \mathcal{H}_\lambda(r) \right)
\]

are called affine quantum Schur algebras or, more specifically, affine \( v \)-Schur algebras (cf. \([12, 13, 14]\)). Note that \( \mathcal{S}_\lambda(n, r) \cong \mathcal{S}_\lambda(n, r) \otimes_{\mathbb{Z}} \mathbb{Q}(v) \).

For \( \lambda \in \Lambda_\lambda(n, r) \), denote the set of shortest representatives of right cosets of \( \mathfrak{S}_\lambda \) in \( \mathcal{S}_\lambda(n, r) \) by

\[
\mathcal{D}_\lambda = \left\{ d \mid d \in \mathfrak{S}_{\lambda r}, \ell(wd) = \ell(w) + \ell(d) \text{ for } w \in \mathfrak{S}_\lambda \right\}.
\]

Note that elements in \( \mathcal{D}_\lambda \) can be characterised as follows:

\[
d^{-1} \in \mathcal{D}_\lambda \iff d(\lambda_{0, i-1} + 1) < d(\lambda_{0, i-1} + 2) < \cdots < d(\lambda_{0, i-1} + \lambda_i), \quad \forall 1 \leq i \leq n,
\]

where \( \lambda_{0, i-1} := \sum_{1 \leq t \leq i-1} \lambda_t \). Moreover, \( \mathcal{D}_\lambda^{\mu} := \mathcal{D}_\lambda \cap \mathcal{D}_\mu^{-1} \) is the set of shortest representatives of \( (\mathfrak{S}_\lambda, \mathfrak{S}_\mu) \) double cosets.

For \( \lambda, \mu \in \Lambda_\lambda(n, r) \) and \( d \in \mathcal{D}_\lambda^{\mu} \), define \( \phi_{\lambda, \mu}^d \in \mathcal{S}_\lambda(n, r) \) by

\[
\phi_{\lambda, \mu}^d(x_\mu h) = \delta_{\mu, \nu} \sum_{w \in \mathfrak{S}_\lambda \mathfrak{S}_\mu} T_w h
\]

where \( \nu \in \Lambda_\lambda(n, r) \) and \( h \in \mathcal{H}_\lambda(r) \). Then by \([13]\) the set \( \{\phi_{\lambda, \mu}^d \mid \lambda, \mu \in \Lambda_\lambda(n, r), d \in \mathcal{D}_\lambda^{\mu} \} \) forms a \( \mathbb{Z} \)-basis for \( \mathcal{S}_\lambda(n, r) \).

For \( 1 \leq i \leq n, k \in \mathbb{Z} \) and \( \lambda \in \Lambda_\lambda(n, r) \) let \( \lambda_{k, i-1} := kr + \sum_{1 \leq t \leq i-1} \lambda_t \) and

\[
R_{i+kn}^\lambda = \{\lambda_{k, i-1} + 1, \lambda_{k, i-1} + 2, \ldots, \lambda_{k, i-1} + \lambda_i = \lambda_{k, i}\},
\]

By \([19, 7.4]\) (see also \([8, 9.2]\)), there is a bijective map

\[
\beta : \{(\lambda, d, \mu) \mid d \in \mathcal{D}_\lambda^{\mu}, \lambda, \mu \in \Lambda_\lambda(n, r)\} \to \Theta_\lambda(n, r)
\]
sending \((\lambda, w, \mu, e)\) to \(A = (a_{k,l})\), where \(a_{k,l} = |R^\lambda_k \cap wR^\mu_l|\) for all \(k, l \in \mathbb{Z}\). For \(A \in \Theta_\Delta(n, r)\) let \(e_A = \phi_{\lambda, \mu}^d\) where \(A = \delta_\Delta(\lambda, d, \mu)\). Furthermore, let

\[
[A] = v^{-d_A}e_A, \quad \text{where} \quad d_A = \sum_{1 \leq i < j \leq n} a_{i,j}a_{k,l}.
\]

Later, in 3.5 and 3.6, we will consider basis elements associated with matrices of the form \(M = A + T - \tilde{T}\) for some \(T, \tilde{T} \in \Theta_\Delta(n)\). We will automatically set \(e_M = 0 = [M]\) if one of the entries of \(M\) is zero.

For \(A \in \Theta_\Delta^+(n)\) and \(j \in \mathbb{Z}_\Delta^n\), define elements in \(S_\Delta(n, r)\):

\[
A(j, r) = \sum_{\mu \in \Lambda_\Delta(n, r - \sigma(A))} v^{\mu j}[A + \text{diag}(\mu)]; \quad \text{(cf. [1])} \quad \text{(2.0.4)}
\]

where \(\mu \cdot j = \sum_{1 \leq i \leq n} \mu_i j_i\). The set \(\{A(j, r)\}_{A \in \Theta_\Delta^+(n), j \in \mathbb{Z}_\Delta^n}\) spans \(S_\Delta(n, r)\).

Let \(\Delta(n) (n \geq 2)\) be the cyclic quiver with vertex set \(I = \mathbb{Z}/n\mathbb{Z} = \{1, 2, \ldots, n\}\) and arrow set \(\{i \rightarrow i + 1 \mid i \in I\}\). Let \(\mathbb{F}\) be a field. For \(i \in I\), let \(S_i\) be the irreducible nilpotent representation of \(\Delta(n)\) over \(\mathbb{F}\) with \((S_i)_i = \mathbb{F}\) and \((S_i)_j = 0\) for \(i \neq j\). For any \(A = (a_{i,j}) \in \Theta_\Delta^+(n)\), let

\[
M(A) = M_\mathbb{F}(A) = \bigoplus_{1 \leq i \leq n} a_{i,j}M^{i,j},
\]

where \(M^{i,j} = M(E_{i,j}^\Delta)\) is the unique indecomposable nilpotent representation for \(\Delta(n)\) of length \(j - i\) with top \(S_i\). Thus, the set \(\{M(A)\}_{A \in \Theta_\Delta^+(n)}\) is a complete set of representatives of isomorphism classes of finite dimensional nilpotent representations of \(\Delta(n)\).

The Euler form associated with the cyclic quiver \(\Delta(n)\) is the bilinear form \(\langle \cdot, \cdot \rangle\): \(\mathbb{Z}_\Delta^n \times \mathbb{Z}_\Delta^n \rightarrow \mathbb{Z}\) defined by \(\langle \lambda, \mu \rangle = \sum_{1 \leq i \leq n} \lambda_i \mu_i - \sum_{1 \leq i \leq n} \lambda_i \mu_{i+1}\) for \(\lambda, \mu \in \mathbb{Z}_\Delta^n\).

By [17], for \(A, B, C \in \Theta_\Delta^+(n)\), let \(\varphi^{C}_{A,B} \in \mathbb{Z}[u^2]\) be the Hall polynomials such that, for any finite field \(\mathbb{F}_q\), \(\varphi^{C}_{A,B}|_{u^2=0}\) is equal to the number of submodules \(N\) of \(M_{\mathbb{F}_q}(C)\) satisfying \(N \cong M_{\mathbb{F}_q}(B)\) and \(M_{\mathbb{F}_q}(C)/N \cong M_{\mathbb{F}_q}(A)\).

By definition, the (generic) twisted Ringel–Hall algebra \(\mathcal{H}_\Delta(n)\) of \(\Delta(n)\) is the \(\mathbb{Q}(v)\)-algebra spanned by basis \(\{u_A = u_{[M(A)]} \mid A \in \Theta_\Delta^+(n)\}\) whose multiplication is defined by, for all \(A, B \in \Theta_\Delta^+(n)\),

\[
u_{A}u_{B} = v^{(d(A), d(B))} \sum_{C \in \Theta_\Delta^+(n)} \varphi^{C}_{A,B}u_{C},
\]

where \(d(A) \in NI\) is the dimension vector of \(M(A)\).

By extending \(\mathcal{H}_\Delta(n)\) to Hopf algebras (see 2.1(2)(b) for multiplication)

\[
\mathcal{H}_\Delta(n)^{\geq 0} = \mathcal{H}_\Delta(n) \otimes \mathbb{Q}(v)[K_1^{\pm 1}, \ldots, K_n^{\pm 1}] \quad \text{and} \quad \mathcal{H}_\Delta(n)^{\leq 0} = \mathbb{Q}(v)[K_1^{\pm 1}, \ldots, K_n^{\pm 1}] \otimes \mathcal{H}_\Delta(n)^{op},
\]

we define the double Ringel–Hall algebra \(\mathcal{D}_\Delta(n)\) (cf. [20] and [3, (2.1.3.2)]) to be a quotient algebra of the free product \(\mathcal{H}_\Delta(n)^{\geq 0} * \mathcal{H}_\Delta(n)^{\leq 0}\) via a certain skew Hopf paring \(\psi: \mathcal{H}_\Delta(n)^{\geq 0} \times \mathcal{H}_\Delta(n)^{\leq 0} \rightarrow \mathbb{Q}(v)\). In particular, there is a triangular decomposition

\[
\mathcal{D}_\Delta(n) = \mathcal{D}_\Delta^+(n) \otimes \mathcal{D}_\Delta^0(n) \otimes \mathcal{D}_\Delta^-(n),
\]
where $\mathcal{D}_\Delta^+(n) = S_\Delta(n)$, $\mathcal{D}_\Delta^0(n) = \mathcal{Q}(v)[K_1^\pm 1, \ldots, K_n^\pm 1]$ and $\mathcal{D}_\Delta^-(n) = S_\Delta(n)^{op}$.

For $\alpha \in \mathbb{N}_0^n$, let
\[
S_\alpha = \sum_{1 \leq i \leq n} \alpha_i E_{i,i+1}^0 \in \Theta_\Delta^+(n).
\] (2.0.5)

Then $M(S_\alpha) = \oplus_{1 \leq i \leq n} \alpha_i S_i$ is a semisimple representation of $\Delta(n)$. Let $u_\alpha = u_{S_\alpha}$.

For $\alpha, \beta \in \mathbb{Z}_0^n$, define a partial order on $\mathbb{Z}_0^n$ by setting
\[
\alpha \leq \beta \iff \alpha_i \leq \beta_i \text{ for all } i \in \mathbb{Z}.
\] (2.0.6)

We now collects some of the results we need later, see [3, Th. 2.5.3] for part (1) and [3, 2.6.7] for part (2)(e).

Theorem 2.1. (1) Let $U(\hat{\mathfrak{g}}_n)$ be the quantum enveloping algebra of the loop algebra of $\mathfrak{gl}_n$ defined in [6] or [3, §2.5]. Then there is a Hopf algebra isomorphism $\mathcal{D}_\Delta(n) \cong U(\hat{\mathfrak{g}}_n)$.

(2) The algebra $\mathcal{D}_\Delta(n)$ is the algebra over $\mathcal{Q}(v)$ which is spanned by basis
\[
\{u_A^+ K^j u_A^- | A \in \Theta_\Delta^+(n), j \in \mathbb{Z}_0^n\},
\] where $K^j = K_1^{j_1} \cdots K_n^{j_n}$, and generated by $u_A^+$, $K_i^\pm 1$, $u_B^-$ ($\alpha, \beta \in \Theta_\Delta^+(n)$, $1 \leq i \leq n$), and whose multiplication is given by the following relations:

(a) $K_i K_j = K_j K_i$, $K_i K_j^{-1} = 1$;

(b) $K^j u_A^+ = v^{(d(A),j)} u_A^+ K^j$, $u_A^- K^j = v^{(d(A),j)} K^j u_A^-$;

(c) $u_A^+ u_B^- = \sum_{C \in \Theta_\Delta^+(n)} v^{(\alpha,d(A))} \varphi_{S_{\alpha}} u_C^+$;

(d) $u_B^- u_A^- = \sum_{C \in \Theta_\Delta^+(n)} v^{(d(A),\beta)} \varphi_{S_{\beta}} u_C^-$;

(e) For $\lambda, \mu \in \mathbb{N}_0^n$, $u_\lambda^+ u_\mu^+ - u_\lambda^- u_\mu^- = \sum_{\alpha, \gamma} x_{\alpha, \gamma} K^{2\gamma - \alpha} u_{\alpha - \gamma}^+ u_{\gamma - \alpha}^-$, where
\[
x_{\alpha, \gamma} = v^{(\alpha, \lambda - \alpha + (\mu, 2\gamma - \alpha) + 2(\gamma, \alpha - \gamma - \lambda) + 2\sigma(\alpha)} \left[ \frac{\lambda}{\alpha - \gamma - \lambda - \alpha} \right] \left[ \frac{\mu}{\alpha - \gamma - \mu - \alpha} \right] \frac{a_\gamma a_{\lambda - \alpha} a_{\mu - \alpha}}{a_\lambda a_\mu}.
\]

\[
x_{\alpha, \gamma} = \sum_{m \geq 1, \gamma^{(i)} \neq 0, \gamma^{(i)} = \gamma} (-1)^m v^2 \left( \sum_{i < j} a_{\gamma^{(i)}} a_{\gamma^{(j)}} \right) \left( \sum_{i = 1}^n a_{\gamma^{(i)}} \right) 2
\]

with $a_\beta = \prod_{i=1}^n b_i$ as defined in [3, Lem. 3.9.1] and $K^\nu := (K_1)^{\nu_1} \cdots (K_n)^{\nu_n}$ with $K_i = K_i K_{i+1}^{-1}$ for $\nu \in \mathbb{Z}_0^n$.

For $A \in \Theta_\Delta^+(n)$, let
\[
\overline{u_A^\pm} = v^{\dim \text{End}(M(A)) - \dim M(A)} u_A^\pm,
\]
and let $^t A$ be the transpose matrix of $A$. The relationship between $\mathcal{D}_\Delta(n)$ and $\mathcal{S}_\Delta(n, r)$ can be seen from the following (cf. [12, 14] and [19, Prop. 7.6]).

Theorem 2.2 ([3, 3.6.3, 3.8.1]). For $r \geq 1$, the map $\zeta_r : \mathcal{D}_\Delta(n) \to \mathcal{S}_\Delta(n, r)$ is a surjective algebra homomorphism such that, for all $j \in \mathbb{Z}_0^n$ and $A \in \Theta_\Delta^+(n)$,
\[
\zeta_r(K^j) = 0(j, r), \quad \zeta_r(\overline{u_A^+}) = A(0, r), \quad \text{and} \quad \zeta_r(\overline{u_A^-}) = (^t A)(0, r).
\]
3. SOME MULTIPLICATION FORMULAS IN THE AFFINE $v$-SCHUR ALGEBRA

We now derive certain useful multiplication formulas in the affine $v$-Schur algebra and, hence, in the quantum affine $\mathfrak{gl}_n$. These formulas will be given in 3.6 and 4.2. They are the key to the realization of quantum affine $\mathfrak{gl}_n$.

We need some preparation before proving 3.6 and 4.2. The following result is given in [3, 3.2.3].

Lemma 3.1. Let $\lambda, \mu \in \Delta(n, r)$ and $d \in \mathcal{D}_\lambda^\mu$. Assume $A = j_\lambda(\lambda, d, \mu)$. Then $d^{-1} \mathcal{S}_\lambda d \cap \mathcal{S}_\mu = \mathcal{S}_\nu$, where $\nu = (\nu^{(1)}, \ldots, \nu^{(n)})$ and $\nu^{(i)} = (a_{ki})_{k \in \mathbb{Z}} = (\ldots, a_{i1}, \ldots, a_{ni}, \ldots)$. In particular, we have

$$x_\lambda T d x_\mu = \prod_{1 \leq i < j \leq n} [a_{ij}^{-1}] T \mathcal{S}_\lambda d \mathcal{S}_\mu.$$

Given $A \in \Theta_\lambda(n, r)$ with $A = j_\lambda(\lambda, w, \mu)$, let $y_A = w$ be the shortest representative of the double coset $\mathcal{S}_\lambda w \mathcal{S}_\mu$.

Lemma 3.2. (1) For $\lambda \in \Lambda(2, r)$ and $w \in \mathcal{D}_\lambda \cap \mathcal{S}_r$, we have $\ell(w) = \sum_{1 \leq i \leq \lambda_1} (w^{-1}(i) - i)$.

(2) For any $A \in \Theta_\lambda(n, r)$, $\ell(y_A) = \sum_{1 \leq i \leq \lambda_1} a_{ij} a_{kl}$.

Proof. Since $w \in \mathcal{S}_r$, $w^{-1}(1) < \cdots < w^{-1}(\lambda_1)$ and $w^{-1}(\lambda_1 + 1) < \cdots < w^{-1}(r)$, it follows that

$$\ell(w^{-1}) = \left| \{(i, j) \mid 1 \leq i < j \leq r, w^{-1}(i) > w^{-1}(j)\} \right|$$

$$= \left| \{(i, j) \mid 1 \leq i \leq \lambda_1, \lambda_1 + 1 \leq j \leq r, w^{-1}(i) > w^{-1}(j)\} \right|.$$ 

On the other hand, for every $1 \leq i \leq \lambda_1$, $w^{-1}(i) - i$ of the numbers $1, 2, \ldots, w^{-1}(i)$ must lie in $\{w^{-1}(\lambda_1 + 1), \ldots, w^{-1}(r)\}$ which contribute $w^{-1}(i) - i$ inversions. Hence, $\ell(w) = \ell(w^{-1}) = (w^{-1}(1) - 1) + (w^{-1}(2) - 2) + \cdots + (w^{-1}(\lambda_1) - \lambda_1)$, proving part (1).

Part (2) is probably known. Since we couldn’t find a proof in the literature, a proof is given in the Appendix.

Lemma 3.3. For $a \geq 0$, $r \geq 1$ and $0 \leq t \leq r$ we have

$$\sum_{X \subseteq \{a+1, \ldots, a+r\}} v^2 \sum_{x \in X} x = v^{2at+t}(t+1) \left[ \begin{array}{c} r \\ t \end{array} \right].$$

Proof. We proceed by induction on $r$. The case $r = 1$ is trivial. Assume now that $r > 1$. Then, by induction hypothesis,

$$\sum_{X \subseteq \{a+1, \ldots, a+r\}} v^2 \sum_{x \in X} x = \sum_{X \subseteq \{a+1, \ldots, a+r-1\}} v^2 \sum_{x \in X} x + \sum_{Y \subseteq \{a+1, \ldots, a+r-1\}} v^2 \sum_{x \in Y} x$$

$$= v^{2at+t}(t+1) \left[ \begin{array}{c} r-1 \\ t \end{array} \right] + v^{2(a+r)} v^{2a(t-1)+t}(t-1) \left[ \begin{array}{c} r-1 \\ t-1 \end{array} \right]$$

$$= v^{2at+t}(t+1) \left[ \begin{array}{c} r-1 \\ t \end{array} \right] + v^{2(r-t)} v^{2(t-1)} \left[ \begin{array}{c} r-1 \\ t-1 \end{array} \right]$$

$$= v^{2at+t}(t+1) \left[ \begin{array}{c} r \\ t \end{array} \right].$$
as desired. □

For $i \in Z$ let $e_i^\beta \in N_\Theta$ be such that

$$(e_i^\beta)_j = \begin{cases} 1 & \text{if } j \equiv i \mod n \\ 0 & \text{otherwise.} \end{cases}$$

**Lemma 3.4** ([10, 5.2]). Let $\mu \in \Lambda_0(n, r)$, $\beta \in N_\Theta$ and assume $\mu \geq \beta$.

1. If $\alpha = \sum_{1 \leq i \leq n} (\mu_i - \beta_i) e_i^\beta$, $\delta = (\alpha_0, \alpha_1, \beta_2, \cdots, \alpha_{n-1}, \beta_n)$ and

   $$\gamma = \{(Y_0, Y_1, \cdots, Y_{n-1}) \mid Y_i \subseteq R_{i+1}^\mu, \lvert Y_i \rvert = \alpha_i, \text{ for } 0 \leq i \leq n-1\},$$

then there is a bijective map

$$g : \mathcal{P}_0 \cap \Theta_\mu \rightarrow \gamma$$

defined by sending $w$ to $(w^{-1}X_0, w^{-1}X_1, \cdots, w^{-1}X_{n-1})$ where $X_i = \{\mu_{0,i}+1, \mu_{0,i}+2, \cdots, \mu_{0,i}+\alpha_i\}$, with $\mu_{0,i} = \sum_{1 \leq s < i} \mu_s$ and $\mu_{0,0} = 0$.

2. If $\gamma = \mu - \beta$, $\theta = (\beta_1, \gamma_1, \beta_2, \gamma_2, \cdots, \beta_n, \gamma_n)$ and

   $$\gamma' = \{(Y'_1, Y'_2, \cdots, Y'_n) \mid Y'_i \subseteq R_i^\mu, \lvert Y'_i \rvert = \gamma_i, \text{ for } 1 \leq i \leq n\},$$

then there is a bijective map

$$g' : \mathcal{P}_0 \cap \Theta_\mu \rightarrow \gamma'$$

defined by sending $w$ to $(w^{-1}X'_1, w^{-1}X'_2, \cdots, w^{-1}X'_n)$ where $X'_i = \{\mu_{0,i-1} + \beta_i + 1, \mu_{0,i-1} + \beta_i + 2, \cdots, \mu_{0,i}\}$.

The injection can be seen easily by noting that

$$(\alpha_0 + \beta_1, \alpha_1 + \beta_2, \cdots, \alpha_{n-1} + \beta_n) = (\mu_1, \mu_2, \cdots, \mu_n) = (\beta_1 + \gamma_1, \beta_2 + \gamma_2, \cdots, \beta_n + \gamma_n)$$

and $X_i$ (reps., $X'_i$) consists of the first $\alpha_i$ (reps., the last $\gamma_i$) numbers in $R_{i+1}^\mu$ for all $0 \leq i < n$, while the subjection is to define $w = g^{-1}(Y_0, Y_1, \cdots, Y_{n-1})$ by $w^{-1}(\mu_{0,i} + s) = k_{i,s}$ for all $0 \leq i \leq n - 1$ and $1 \leq s \leq \mu_{i+1}$, where $Y_i = \{k_{i,1}, \vdots, k_{i,\alpha_i}\}$, $R_{i+1}^\mu \setminus Y_i = \{k_{i,\alpha_i+1}, k_{i,\alpha_i+2}, \cdots, k_{i,\mu_{i+1}}\}$, and both are strictly increasing.

For $A \in M_{\lambda,n}(Z)$ with $\sigma(A) = r$, we denote $e_A = [A] = 0 \in \Theta_\lambda(n, r)$ if $a_{i,j} < 0$ for some $i, j \in Z$. There is a natural map

$$\tilde{\gamma} : \Theta_\lambda(n) \rightarrow \Theta_\lambda(n) \quad A = (a_{i,j}) \mapsto \tilde{A} = (\tilde{a}_{i,j}),$$

where $\tilde{a}_{i,j} = a_{i,j}$ for all $i, j \in Z$.

We are now ready to establish multiplication formulas of an arbitrary basis elements $e_A$ by certain basis elements $e_B$ in the affine Schur algebra $\mathcal{S}_\lambda(n, r)$ over $Z$, where $B^+$ or $\cdot (B^-)$ defines a semisimple representation of the cyclic quiver. The significance of these formulas is the generalisation of [14, 3.5] (cf. [1, 3.1]) from real roots to all roots including all imaginary roots.

**Proposition 3.5.** Let $A \in \Theta_\lambda(n, r)$ and $\mu = \rho_0(A)$. Assume $\beta \in N_\lambda$ and $\beta \leq \mu$. Let $\alpha = \sum_{1 \leq i \leq n} (\mu_i - \beta_i) e_i^\beta$, and $\gamma = \sum_{1 \leq i \leq n} (\mu_i - \beta_i) e_i^\beta = \mu - \beta$, $B = \sum_{1 \leq i \leq n} \alpha_i E_{i+1}^\beta + \text{diag}(\beta)$, and $C = \sum_{1 \leq i \leq n} \gamma_i E_{i+1,i}^\beta + \text{diag}(\beta)$. Then the following identities hold in $\mathcal{S}_\lambda(n, r)$.
\( \text{(1) } e_{BEA} = \sum \sum v^2 \sum_{1 \leq i, j \leq n} \prod_{i \leq \ell \leq n} \prod_{j \notin \mathbb{Z}} [a_{i,j} + t_{i,j} - t_{i-1,j}] e_{A + T - \bar{T}}^i \)

\( \text{(2) } e_{CEA} = \sum \sum v^2 \sum_{1 \leq i, j \leq n} \prod_{i \leq \ell \leq n} \prod_{j \notin \mathbb{Z}} [a_{i,j} - t_{i,j} + t_{i-1,j}] e_{A + T + \bar{T}}^i \)

**Proof.** We only prove (1). The proof for (2) is entirely similar.

Let \( \lambda = \text{ro}(B) \) and \( \nu = \text{co}(A) \). Assume \( d_1 \in \mathcal{D}_{\lambda, \mu}^\delta \) and \( d_2 \in \mathcal{D}_{\mu, \nu}^\delta \) defined by \( \delta(\lambda, d_1, \mu) = B \) and \( \delta(\mu, d_2, \nu) = A \). Then \( \lambda_i = \alpha_i + \beta_i \) and \( \mu_i = \alpha_i - \beta_i \) for all \( 1 \leq i \leq n \). From 3.1 we see that

\[
e_{BEA}(x_{\nu}) = T_{\mathfrak{g}_\lambda}d_1 \mathfrak{g}_\nu \cdot T_{d_2} \cdot T_{\mathfrak{g}_\nu \cap \mathfrak{g}_\mu}
\]

\[
= \frac{1}{\sum_{w \in \mathfrak{g}_\mu} v^{2\ell(w)} T_{\mathfrak{g}_\lambda d_1 \mathfrak{g}_\nu} \cdot T_{\mathfrak{g}_\mu d_2 \mathfrak{g}_\nu}}
\]

\[
= \frac{1}{\sum_{w \in \mathfrak{g}_\mu} v^{2\ell(w)} \prod_{1 \leq i \leq n} \prod_{j \notin \mathbb{Z}} \frac{1}{a_{i,j}^\nu} T_{\mathfrak{g}_\lambda d_1 \mathfrak{g}_\mu} \cdot T_{\mathfrak{g}_\mu} \cdot T_{d_2} \cdot T_{\mathfrak{g}_\nu}}
\]

\[
= \prod_{1 \leq i \leq n} \prod_{j \notin \mathbb{Z}} \frac{1}{[a_{i,j}]} T_{\mathfrak{g}_\lambda} \cdot T_{d_1} \cdot T_{\mathfrak{g}_\nu \cap \mathfrak{g}_\mu} \cdot T_{d_2} \cdot T_{\mathfrak{g}_\nu}
\]

where \( \mathfrak{g}_\omega = d_2^{-1} \mathfrak{g}_\mu d_2 \cap \mathfrak{g}_\nu \), \( \mathfrak{g}_\delta = d_1^{-1} \mathfrak{g}_\lambda d_1 \cap \mathfrak{g}_\mu \) with \( \delta = (\alpha_0, \beta_1, \alpha_1, \beta_2, \cdots, \alpha_{n-1}, \beta_n) \). By (2.0.2), we have \( d_1 = \rho^{-\alpha_0} \) (so \( \ell(d_1) = 0 \)). This together with the fact that \( d_2 \in \mathcal{D}_{\mu}^\delta \) implies that \( \ell(d_1 d_2) = \ell(d_1) + \ell(w) + \ell(d_2) = \ell(w) + \ell(d_2) \) for \( w \in \mathcal{D}_{\delta}^\mu \). Thus, we have

\[
e_{BEA}(x_{\nu}) = \prod_{1 \leq i \leq n} \prod_{j \notin \mathbb{Z}} \frac{1}{[a_{i,j}]} \sum_{w \in \mathfrak{g}_\mu \cap \mathfrak{g}_\delta} T_{\mathfrak{g}_\lambda} T_{d_1} d_2 T_{\mathfrak{g}_\nu} \tag{3.5.1}
\]

For \( w \in \mathcal{D}_{\delta}^\mu \) let \( C(w) = (c^w_{i,j}) \in \Theta_\lambda(n, r) \), where \( c^w_{i,j} = |R^\lambda_i \cap d_1 w d_2 R^\nu_j| \), and let \( T(w) = (t^w_{i,j}) \in \Theta_\delta(n, r) \), where \( t^w_{i,j} = |w^{-1} X_i \cap d_2 R^\nu_j| \) with \( X_i = \{ \alpha_{i,0} + 1, \mu_{0,i} + 2, \cdots, \mu_{0,i} + \alpha_i \} \) for \( 1 \leq i \leq n \) and \( j \in \mathbb{Z} \). Then \( \text{ro}(T(w)) = \alpha \) and \( \text{co}(T(w)) \leq \nu \). Since \( d_1^{-1} R^\lambda_i = \alpha_0 + R^\lambda_i = (R^\mu_i \setminus X_{i-1} \cup X_i) \), we see that \( c^w_{i,j} = |R^\lambda_i \cap d_1 w d_2 R^\nu_j| = |w^{-1} d_1^{-1} R^\lambda_i \cap d_2 R^\nu_j| = a_{i,j} - t^w_{i-1,j} + t^w_{i,j} \) (see the proof of [10, 5.3]). In other words, for all \( w \in \mathcal{D}_{\delta}^\mu \cap \mathfrak{g}_\mu \),

\[
C(w) = A + T(w) - \bar{T}(w) \tag{3.5.2}
\]

In particular, \( y_{C(w)} \in \mathfrak{g}_\lambda d_1 w d_2 \mathfrak{g}_\nu \cap \mathcal{D}_{\lambda, \mu}^\delta \).
Putting $\mathcal{G}_{\alpha w} = y_{C(w)}^{-1} \mathcal{G}_\lambda y_{C(w)} \cap \mathcal{G}_\nu$, we have by 3.1

$$
\sum_{w \in \mathcal{G}_\mu \cap \mathcal{G}_\delta} T_{\mathcal{G}_\lambda} T_{d_1 w d_2} T_{\mathcal{G}_\nu} = \sum_{w' \in \mathcal{G}_\mu \cap \mathcal{G}_\delta, d_1 w d_2 = w' y_{C(w)} w''} \sum_{w'' \in \mathcal{G}_\lambda, w' \in \mathcal{G}_\nu \cap \mathcal{G}_\delta} T_{\mathcal{G}_\lambda} T_{w'} T_{y_{C(w)}} T_{w''} T_{\mathcal{G}_\nu} \\
= \sum_{w \in \mathcal{G}_\mu \cap \mathcal{G}_\delta, d_1 w d_2 = w' y_{C(w)} w''} v^2(\ell(w') + \ell(w'')) T_{\mathcal{G}_\lambda} T_{y_{C(w)}} T_{\mathcal{G}_\nu} \\
= \sum_{w \in \mathcal{G}_\mu \cap \mathcal{G}_\delta} v^2(\ell(w) + \ell(d_2) - \ell(y_{C(w)})) \prod_{1 \leq i \leq n, j \in \mathbb{Z}} \left[ c_{i,j}(w) \right]! e_{C(w)}(x_{\nu}).
$$

Now by (3.5.1) and (3.5.2) and noting $\text{ro}(T(w)) = \alpha$ for $w \in \mathcal{G}_\delta \cap \mathcal{G}_\mu$, we have

$$
e_{B \varepsilon A} = \sum_{w \in \mathcal{G}_\mu \cap \mathcal{G}_\delta} v^2(\ell(w) + \ell(d_2) - \ell(y_{C(w)})) \prod_{1 \leq i \leq n, j \in \mathbb{Z}} \left[ c_{i,j}(w) \right]! e_{C(w)}
$$

$$
= \sum_{T \in \Theta_{\alpha}(n), \text{ro}(T) = \alpha} \prod_{1 \leq i \leq n, j \in \mathbb{Z}} \left[ a_{i,j} - t_{i-1,j} - t_{i,j} \right]! v^2(\ell(d_2) - \ell(y_{A \tau - T})) \left( \sum_{w \in \mathcal{G}_\mu \cap \mathcal{G}_\delta} v^2(\ell(w)) e_{A \tau - T} \right).
$$

(3.5.3)

Given $T \in \Theta_{\alpha}(n)$ with $\text{ro}(T) = \alpha$ let

$$Z(T) = \{ Z = (Z_{i,j})_{0 \leq i \leq n-1, j \in \mathbb{Z}} \mid |Z_{i,j}| = t_{i,j}, Z_{i,j} \subseteq R^u_{i+1} \cap d_2 R^v_j, \text{ for } 0 \leq i \leq n-1, j \in \mathbb{Z} \}.$$

If $T = T(w)$ then the bijective map $g$ in 3.4 induces a bijective map

$$h_T : \{ w \in \mathcal{G}_\delta \cap \mathcal{G}_\mu \mid T(w) = T \} \rightarrow Z(T)$$

defined by sending $w$ to $(w^{-1}(X_s) \cap d_2 R^v_j)_{0 \leq i \leq n-1, j \in \mathbb{Z}}$. Since for $0 \leq i \leq n-1$ and $j \in \mathbb{Z}$

$$R^u_{i+1} \cap d_2 R^v_j = \left\{ \mu_{0,i} + \sum_{s \leq j-1} a_{i+1,s} + 1, \mu_{0,i} + \sum_{s \leq j-1} a_{i+1,s} + 2, \cdots, \mu_{0,i} + \sum_{s \leq j} a_{i+1,s} \right\},$$

it follows from 3.2 and the definition of $g^{-1}$ that, for $Z = (Z_{i,j})_{0 \leq i \leq n-1, j \in \mathbb{Z}} \in Z(T)$,

$$\ell(h_T^{-1}(Z)) = \sum_{0 \leq i \leq n-1} \left( \sum_{j \in \mathbb{Z}} k - \sum_{1 \leq j \leq \alpha_i} (\mu_{0,i} + j) \right) = \sum_{0 \leq i \leq n-1} k - \sum_{0 \leq i \leq n-1} \sum_{j \in \mathbb{Z}} (\alpha_i \mu_{0,i} + \frac{\alpha_i (\alpha_i + 1)}{2}).$$

This implies that

$$\sum_{w \in \mathcal{G}_\mu \cap \mathcal{G}_\delta \mid T(w) = T} v^2(\ell(w)) = v^2 \sum_{Z \in Z(T)} v^2(\kappa_T^{-1}(Z))
$$

$$= v^{-2} \sum_{0 \leq i \leq n-1} \left( \alpha_i \mu_{0,i} + \alpha_i (\alpha_i + 1)/2 \right) \prod_{0 \leq i \leq n-1} \left( \sum_{j \in \mathbb{Z}} v^2 \sum_{Z_{i,j} \subseteq R^u_{i+1} \cap d_2 R^v_j} k \right).$$
Consequently, by 3.3, we have
\[
\sum_{w \in \Theta_{\mu}(n, r)} v^{2\ell(w)} = v^{2\alpha_T} \prod_{0 \leq i \leq n-1} \left[ \frac{a_{i+1,j}}{t_{i,j}} \right] = v^{2\alpha_T} \prod_{1 \leq i \leq n} \left[ \frac{a_{i,j}}{t_{i-1,j}} \right]
\]
(3.5.4)
where
\[
a_T = \sum_{0 \leq i \leq n-1} \left( t_{i,j}(\mu_0,i + \sum_{s \leq j-1} a_{i+1,s}) + \frac{t_{i,j}(t_{i,j} + 1)}{2} \right) - \sum_{0 \leq i \leq n-1} \left( \alpha_i \mu_0,i + \frac{\alpha_i(\alpha_i + 1)}{2} \right).
\]
Since \(\text{ro}(T) = \alpha\) we have \(\alpha_i = \sum_{j \in \mathbb{Z}} t_{i,j}\) and \(\alpha_i^2 = \sum_{j \in \mathbb{Z}} t_{i,j}^2 + 2 \sum_{j \leq l} t_{i,j} t_{i,l}\) for \(0 \leq i \leq n - 1\). This implies that
\[
a_T = \sum_{0 \leq i \leq n-1} a_{i+1,j} t_{i,j} - \sum_{0 \leq i \leq n-1} t_{i,j} t_{i,l}.
\]
Since \(d_2 = y_A\) is the shortest representative in the double coset associated with \(A\), by 3.2(2),
\[
\ell(d_2) - \ell(y_{A+T-\tilde{T}}) = \sum_{1 \leq i \leq n} a_{i,j} t_{i,l} - \sum_{1 \leq i \leq n} a_{i+1,j} t_{i,l} + \sum_{1 \leq i \leq n} t_{i,j} t_{i,l} - \sum_{1 \leq i \leq n} t_{i-1,j} t_{i,l}.
\]
It follows that
\[
a_T + \ell(d_2) - \ell(y_{A+T-\tilde{T}}) = \sum_{1 \leq i \leq n} a_{i,j} t_{i,l} - \sum_{1 \leq i \leq n} t_{i-1,j} t_{i,l}.
\]
Consequently, by (3.5.3), (3.5.4) and noting \(\prod_{1 \leq i \leq n, j \in \mathbb{Z}} [t_{i,j}]^! = \prod_{1 \leq i \leq n, j \in \mathbb{Z}} [t_{i-1,j}]^!\) we have
\[
\begin{align*}
\tilde{e}_{BE_{A}} &= \sum_{T \in \Theta_{\mu}(n) \atop \text{ro}(T) = \alpha} v^{2(\alpha_T + \ell(d_2) - \ell(y_{A+T-\tilde{T}}))} \prod_{1 \leq i \leq n} \frac{[a_{i,j} - t_{i-1,j} + t_{i,j}]^!}{[t_{i-1,j}]^!} e_{A+T-\tilde{T}} \\
&= \sum_{T \in \Theta_{\mu}(n) \atop \text{ro}(T) = \alpha} v^{2(\alpha_T + \ell(d_2) - \ell(y_{A+T-\tilde{T}}))} \prod_{1 \leq i \leq n} \frac{[a_{i,j} + t_{i,j} - t_{i-1,j}]^!}{[t_{i-1,j}]^!} e_{A+T-\tilde{T}},
\end{align*}
\]
proving (1).
\[\square\]

Let \(\tilde{\zeta} : \mathcal{Z} \to \mathcal{Z}\) be the ring homomorphism defined by \(\tilde{v} = v^{-1}\). We now use 3.5 to derive the corresponding formulas for the normalised basis \(\{[A]\}_{A \in \Theta_{\mu}(n, r)}\) defined in (2.0.3).

**Proposition 3.6.** Let \(A \in \Theta_{\alpha}(n, r)\) and \(\alpha, \gamma \in \mathbb{N}^n\).

1. For \(B \in \Theta_{\alpha}(n, r)\), if \(B - \sum_{1 \leq i \leq n} \alpha_i E_{i,i+1}^\alpha\) is a diagonal matrix and \(\text{co}(B) = \text{ro}(A)\), then
\[
[B][A] = \sum_{T \in \Theta_{\mu}(n) \atop \text{ro}(T) = \alpha} v^{\beta(T,A)} \prod_{1 \leq i \leq n} \left[ \frac{a_{i,j} + t_{i,j} - t_{i-1,j}}{t_{i,j}} \right] [A + T - \tilde{T}],
\]
where \(\beta(T, A) = \sum_{1 \leq i \leq n, j \geq 1} (a_{i,j} - t_{i-1,j}) t_{i,t} - \sum_{1 \leq i \leq n, j \geq 1} (a_{i+1,j} - t_{i,j}) t_{i,t}\).

2. For \(C \in \Theta_{\alpha}(n, r)\), if \(C - \sum_{1 \leq i \leq n} \gamma_i E_{i,i+1}^\alpha\) is a diagonal matrix and \(\text{co}(C) = \text{ro}(A)\), then
\[
[C][A] = \sum_{T \in \Theta_{\mu}(n) \atop \text{ro}(T) = \alpha} v^{\beta'(T,A)} \prod_{1 \leq i \leq n} \left[ \frac{a_{i,j} - t_{i,j} + t_{i-1,j}}{t_{i-1,j}} \right] [A - T + \tilde{T}],
\]
where \(\beta'(T, A) = \sum_{1 \leq i \leq n, j \geq 1} (a_{i,j} - t_{i,j}) t_{i-1,t} - \sum_{1 \leq i \leq n, j \geq 1} (a_{i,j} - t_{i,j}) t_{i,t}\).
Proof. We only prove (1). The proof for (2) is entirely similar. Note that we have

\[ \prod_{1 \leq i \leq n, j \in \mathbb{Z}} \left[ \frac{a_{ij} + t_{ij} - t_{i-1,j}}{t_{ij}} \right] = v^2 \sum_{1 \leq i \leq n, j \in \mathbb{Z}} (a_{ij} - t_{i-1,j}) \prod_{1 \leq i \leq n, j \in \mathbb{Z}} \left[ \frac{a_{ij} + t_{ij} - t_{i-1,j}}{t_{ij}} \right] \]

Thus by 3.5(1) we have

\[ [B][A] = \sum_{T \in \Theta_0(n)} v^{\beta(T, A)} \prod_{1 \leq i \leq n, j \in \mathbb{Z}} \left[ \frac{a_{ij} + t_{ij} - t_{i-1,j}}{t_{ij}} \right] [A + T - \tilde{T}], \]

where

\[ \beta(T, A) = 2 \sum_{1 \leq i \leq n, j > l} (a_{ij} - t_{i-1,j}) t_{i,l} + d_{A+T-\tilde{T}} - d_A - d_B + 2 \sum_{1 \leq i \leq n, j \in \mathbb{Z}} (a_{ij} - t_{i-1,j}) t_{i,j} \]

Fix \( T \in \Theta_0(n) \) satisfying \( \text{ro}(T) = \alpha \). Then by definition we have \( d_B = \sum_{1 \leq i \leq n} b_{i,i} \alpha_i \) and

\[ d_{A+T-\tilde{T}} - d_A = \sum_{1 \leq i \leq n} a_{i,j}(t_{k,l} - t_{k-1,l}) + \sum_{1 \leq i \leq n, j < l} a_{k,l}(t_{i,j} - t_{i-1,j}) + \sum_{1 \leq i \leq n, j > l} (t_{i,j} - t_{i-1,j})(t_{k,l} - t_{k-1,l}) \]

\[ = \sum_{1 \leq i \leq n, j > l} a_{i,j} t_{i,l} - \sum_{1 \leq i \leq n, j < l} a_{i+1,l} t_{i,j} + \sum_{1 \leq i \leq n, j > l} (t_{i,j} - t_{i-1,j}) t_{i,l} \]

\[ = \sum_{1 \leq i \leq n, j > l} (a_{i,j} - t_{i-1,j}) t_{i,l} - \sum_{1 \leq i \leq n, j < l} (a_{i+1,j} - t_{i,j}) t_{i,l}. \]

Furthermore, since \( \text{ro}(T) = \alpha \) and \( \text{co}(B) = \text{ro}(A) \) we have \( b_{i,i} = \sum_{j \in \mathbb{Z}} (a_{ij} - t_{i-1,j}) \) and \( \alpha_i = \sum_{j \in \mathbb{Z}} t_{i,j} \) for each \( i \), and hence

\[ d_{A+T-\tilde{T}} - d_A - d_B = - \sum_{1 \leq i \leq n, j > l} (a_{i,j} - t_{i-1,j}) t_{i,l} - \sum_{1 \leq i \leq n, j > l} (a_{i+1,j} - t_{i,j}) t_{i,l}. \]

Consequently, \( \beta(T, A) = \sum_{1 \leq i \leq n, j > l} (a_{i,j} - t_{i-1,j}) t_{i,l} - \sum_{1 \leq i \leq n, j > l} (a_{i+1,j} - t_{i,j}) t_{i,l} \). The proof is completed. \( \square \)

4. Proof of the main theorem

We now construct explicitly a subalgebra of the algebra \( S_\Delta(n) := \prod_{r \geq 1} S_\Delta(n, r) \) and prove that this subalgebra is isomorphic to \( U(\hat{\mathfrak{g}}_n) \). Recall the elements \( A(j, r) \) defined in (2.0.4) and let

\[ A(j) = (A(j, r))_{r \geq 0} \in S_\Delta(n) \quad \text{and} \quad \mathcal{B} = \{ A(j) \ | \ A \in \Theta_\Delta^\pm(n), j \in \mathbb{Z}_n \} \].

(4.0.1)

Then \( \mathcal{B} \) is linearly independent by [8, Prop.4.1(2)]. Let \( \mathcal{V}_\Delta(n) \) be the \( \mathbb{Q}(v) \)-subspace of \( S_\Delta(n) \) spanned by \( \mathcal{B} \). We will prove that \( \mathcal{V}_\Delta(n) \) is a subalgebra of \( S_\Delta(n) \) isomorphic to \( U(\hat{\mathfrak{g}}_n) \) or \( \mathcal{D}_\Delta(n) \) by 2.1(a). For this purpose, we need a larger spanning set containing \( \mathcal{B} \):

\[ \mathcal{B} = \{ A(j, \lambda) \ | \ A \in \Theta_\Delta^\pm(n), j \in \mathbb{Z}_n^\Delta, \lambda \in \mathbb{N}_\Delta^\Delta \}. \]
where \( A(j, \lambda) = (A(j, \lambda, r))_{r \geq 0} \) with \( A(j, \lambda, r) \) defined by
\[
A(j, \lambda, r) = \sum_{\mu \in \Lambda_0(n, r - \sigma(A))} v^{\mu j} \left[ \frac{\mu}{\lambda} \right] [A + \operatorname{diag}(\mu)] \quad \text{(cf. [11, \S2])} \tag{4.0.2}
\]

Note that, for \( \sigma(\lambda) \leq r \), \( 0(j, \lambda, r) = \sum_{\mu \in \Lambda_0(n, r), \lambda \leq \mu} v^{\mu j} \left[ \frac{\mu}{\lambda} \right] [\operatorname{diag}(\mu)] \)

**Lemma 4.1.** The space \( \mathcal{V}_\triangle(n) \) is spanned by the set \( \mathfrak{B} \). In other words, every \( A(j, \lambda) \in \mathcal{V}_\triangle(n) \).

**Proof.** Let \( \mathcal{V}_\triangle^0(n) \) be the \( \mathbb{Q}(v) \)-subalgebra of \( \mathcal{S}_\triangle(n) \) generated by \( 0(\pm e_i^\lambda) \) for \( 1 \leq i \leq n \). Then the set \( \{0(j) \mid j \in \mathbb{Z}_0^n\} \) forms a \( \mathbb{Q}(v) \)-basis for \( \mathcal{V}_\triangle^0(n) \). Since
\[
0(j, \lambda, r) = \prod_{1 \leq i \leq n} \left( 0(e_i^\lambda, r)^j_i \prod_{1 \leq s \leq \lambda_i} \frac{0(e_i^\lambda, r)v^{-s+1} - 0(-e_i^\lambda, r)v^{s-1}}{v^{s} - v^{-s}} \right), \quad \text{where } \sigma(\lambda) \leq r,
\]
we have
\[
0(j, \lambda) = \prod_{1 \leq i \leq n} \left( 0(e_i^\lambda)^j_i \prod_{1 \leq s \leq \lambda_i} \frac{0(e_i^\lambda)v^{-s+1} - 0(-e_i^\lambda)v^{s-1}}{v^{s} - v^{-s}} \right) \in \mathcal{V}_\triangle(n). \tag{4.1.1}
\]

On the other hand, by the proof of [11, 3.4], we have
\[
0(j, \lambda)A(0) = v^{\rho(A) - (j + \lambda)}A(j, \lambda) + \sum_{\mu \in \mathbb{N}^n, 0 \leq \mu \leq \lambda} v^{\rho(A) - (j + \lambda) - \mu} \left[ \frac{\rho(A)}{\mu} \right] A(j - \mu, \lambda - \mu).
\]

By induction, we see that \( A(j, \lambda) \in \text{span}\{0(j, \lambda)A(0) \mid A \in \Theta_\triangle^\pm(n), j \in \mathbb{Z}_0^n, \lambda \in \mathbb{N}_0^n\} \). Thus, by (4.1.1), \( A(j, \lambda) \in \text{span}\{0(j)A(0) \mid A \in \Theta_\triangle^\pm(n), j \in \mathbb{Z}_0^n\} \). This span equals \( \mathcal{V}_\triangle(n) \) by [8, (4.2.1)] (i.e., 4.2(1) below).

For \( T = (t_{i,j}) \in \Theta_\triangle(n) \) let \( \delta_T \) be the diagonal of \( T \), i.e.,
\[
\delta_T = (t_{i,i})_{i \in \mathbb{N}} \in \mathbb{N}_0^n.
\]

We now use 3.6 to derive multiplication formulas of an arbitrary basis element by a “semisimple generators”, which is the key to solving the realisation problem. Recall the notation in (1.2.1).

**Proposition 4.2.** Let \( j \in \mathbb{Z}_0^n, A \in \Theta_\triangle^\pm(n), \alpha \in \mathbb{N}_0^n, \) and \( S_\alpha = \sum_{1 \leq i \leq n} \alpha_i E_{i,i+1}^\lambda \). The following identities holds in \( \mathcal{V}_\triangle(n) \):
\begin{enumerate}
  
  \item \( 0(j')A(j) = v^{j' - j}A(j' + j) \) and \( A(j)0(j') = v^{j' - j}A(j' + j) \) ([8, (4.2.1)]).
  
  \item \( S_\alpha A(j) = \sum_{T \in \Theta_\triangle(n) \cap \mathcal{T}(T) = \alpha} v^{f_{A,T}} \prod_{i \in \mathbb{N}, j \in \mathbb{Z}, \lambda \neq i} \left[ \frac{a_{i,j} + t_{i,j} + t_{i-1,j}}{t_{i,j}} \right] (A + T^\pm - \tilde{T}^\pm)(j_T, \delta_T), \)

where \( j_T = j + \sum_{1 \leq i \leq n} \left( \sum_{j' < (t_{i,j} - t_{i-1,j})} e_i^{j'} \right) \) and
\[
f_{A,T} = \sum_{1 \leq i \leq n, \lambda \neq j} a_{i,j} t_{i,j} - \sum_{1 \leq i \leq n, \lambda \neq j, \gamma \neq i+1} a_{i+1,j} t_{i+1,j} - \sum_{1 \leq i \leq n} t_{i-1,j} t_{i,j} + \sum_{1 \leq i \leq n, \lambda \neq j} t_{i,j} t_{i+1,i+1} + \sum_{1 \leq i \leq n} j_i(t_{i-1,i} - t_{i,i});
\]
\end{enumerate}
\[
(3) \quad \mathcal{S}_\alpha(0)A(j) = \sum_{T \in \Theta_0(n) \cap ro(T) = \alpha} v_j f_{A,T} \prod_{1 \leq i \leq n, j \in E, j \neq i} \begin{bmatrix} a_{i,j} - t_{i,j} + t_{i-1,j} \\ t_{i-1,j} \end{bmatrix},
\]

where \( j_T = j + \sum 1_{i \leq n} (\sum_{j > i} (t_{i-1,j} - t_{i,j})) \mathbf{e}_i^\delta \) and

\[
f_{A,T}' = \sum_{1 \leq i \leq n} a_{i,j} t_{i-1,l} - \sum_{1 \leq i \leq n} a_{i,j} t_{i,l} - \sum_{1 \leq i \leq n} t_{i-1,j} t_{i,l} + \sum_{1 \leq i \leq n} t_{i,j} t_{i,l}
+ \sum_{1 \leq i \leq n} t_{i,j} t_{i-1,l} + \sum_{1 \leq i \leq n} j_i (t_{i,i} - t_{i-1,i}).
\]

In particular, \( \mathcal{V}_\delta(n) \) is closed under the multiplication by the “generators” \( 0(0), \mathcal{S}_\alpha(0), \mathcal{S}_\alpha'(0) \) for all \( j \in \mathbb{N} \) and \( \alpha \in \mathbb{N}^n \).

**Proof.** We only prove (2). If \( r < \sigma(A) \), the \( r \)-th components of both sides are 0. Assume now \( r \geq \sigma(A) \). By 3.6 and noting the fact that, for \( X, Y \in \Theta_\delta(n, r) \), \( [X][Y] \neq 0 \implies \co(X) = \co(Y) \) the \( r \)-th component of \( S_\alpha(0)A(j) \) becomes

\[
S_\alpha(0, r)A(j, r) = \sum_{\gamma \in \Lambda_0(n, r - \sigma(A))} v_j^\gamma [S_\alpha + \text{diag}(\gamma + \co(A) - \sum_{1 \leq i \leq n} \alpha_i \mathbf{e}_i^\delta)] [A + \text{diag}(\gamma)]
\]

where

\[
x_T = \sum_{\gamma \in \Lambda_0(n, r - \sigma(A))} v_j^\gamma \beta(T, A + \text{diag}(\gamma)) \begin{bmatrix} \gamma + \delta_T - \delta_T \\
\delta_T \end{bmatrix} [A + \text{diag}(\gamma) + T - T].
\]

Let \( A + \text{diag}(\gamma) = (a^i_j) \). Then \( a^\gamma_{i,j} = a_{i,j} \) for \( i \neq j \) and \( a^\gamma_{i,i} = \gamma_i \). Let \( \nu = \gamma + \delta_T - \delta_T \). Then

\[
\beta(T, A + \text{diag}(\gamma)) = \sum_{1 \leq i \leq n} (a^\gamma_{i,j} - t_{i-1,j}) t_{i,l} - \sum_{1 \leq i \leq n} (a^\gamma_{i+1,j} - t_{i,j}) t_{i,l}
= \sum_{1 \leq i \leq n} (a_{i,j} - t_{i-1,j}) t_{i,l} + \sum_{1 \leq i \leq n} (\nu_i - t_{i,i}) t_{i,l} - \sum_{1 \leq i \leq n} (a_{i+1,j} - t_{i,j}) t_{i,l}
- \sum_{1 \leq i \leq n} (\nu_{i+1} - t_{i+1,i+1}) t_{i,l}
= \beta_{A,T} + \beta_{\nu,T},
\]

where \( \beta_{\nu,T} = \sum_{1 \leq i \leq n, i > l} \nu_i t_{i,l} - \sum_{1 \leq i \leq n, i+1 > l} \nu_{i+1} t_{i,l} \) and

\[
\beta_{A,T} = \sum_{1 \leq i \leq n} (a_{i,j} - t_{i-1,j}) t_{i,l} - \sum_{1 \leq i \leq n} a_{i+1,j} t_{i,l} + \sum_{1 \leq i \leq n} t_{i,j} t_{i,l}
- \sum_{1 \leq i \leq n} t_{i,j}^2 + \sum_{1 \leq i \leq n} t_{i+1,i+1} t_{i,l}.
\]
Theorem 4.4. We now prove the conjecture formulated in [8, 5.5(2)].

Clearly, we have \( \sum_{\nu} \nu = v^{\delta_T - \nu} [\nu]_{\delta_T} \), \( \beta_{A,T} + \delta_T \cdot \delta_T + j \cdot (\delta_T - \delta_T) = f_{A,T} \) and \( \beta_{\nu,T} + \nu \cdot (j - \delta_T) = \nu \cdot j_T \). This implies that

\[
x_T = v^{\beta_{A,T} + \delta_T \cdot \delta_T + j \cdot (\delta_T - \delta_T)} \sum_{\nu \in \Xi(n,r - \sigma(A + T^+ - \overline{T}^+))} v^{\beta_{\nu,T} + \nu \cdot (j - \delta_T)} [\nu]_{\delta_T} \\
\times [A + T^+ - \overline{T}^+ + \text{diag}(\nu)] \\
= v^{f_{A,T}} (A + T^+ - \overline{T}^+)(j_T, \delta_T, r),
\]

proving (2). \( \square \)

For \( A, B \in \Theta_\Delta^\pm (n) \), define the ordering on \( \Theta_\Delta^\pm (n) \) by setting

\[
A \preceq B \iff \sum_{s \leq i, t \geq j} a_{s,t} \leq \sum_{s \leq i, t \geq j} b_{s,t}, \forall i < j, \text{ and } \sum_{s \geq i, t \leq j} a_{s,t} \leq \sum_{s \geq i, t \leq j} a_{s,t}, \forall i > j. \quad (4.2.1)
\]

Proposition 4.3. With the notation in (1.2.1) we have, for any \( A \in \Theta_\Delta^\pm (n) \) and \( j \in \mathbb{N}^n \),

\[
A^+(0)0(j)A^-(0) = v^{\frac{1}{2}(\text{co}(A^+) + \text{ro}(A^-))} A(j) + \sum_{B \in \Theta_\Delta^\pm (n) \atop B \preceq A, j' \in \mathbb{N}^n} f_{A,j'}^{B,j'} B(j'),
\]

where \( f_{A,j'}^{B,j'} \in \mathbb{Q}(v) \).

Proof. Let \( \mathfrak{D}_\Delta^+ (n) \) be the subspace of \( \mathfrak{D}_\Delta (n) \) spanned by the elements \( u_A^+ \) for \( A \in \Theta_\Delta^+ (n) \). According to [5, 6.2], the algebra \( \mathfrak{D}_\Delta^+ (n) \) is generated by the elements \( \tilde{u}_{S_a}^+ \) for \( a \in \mathbb{N}^n \), where \( S_a \) is defined as in (2.0.5). This together with 2.2 implies that \( A^+(0) \) can be written as a linear combination of monomials in \( S_a(0) \). Thus, by 4.2 and 4.1, we conclude that there exist \( f_{A,j'}^{B,j'} \in \mathbb{Q}(v) \) (independent of \( r \)) such that

\[
A^+(0)0(j)A^-(0) = \sum_{B \in \Theta_\Delta^\pm (n) \atop j' \in \mathbb{N}^n} f_{A,j'}^{B,j'} B(j'). \quad (4.3.1)
\]

On the other hand, by the triangular relation given in [3, 3.7.3], we have

\[
A^+(0,r)0(j, r)A^-(0, r) = v^{\frac{1}{2}(\text{co}(A^+) + \text{ro}(A^-))} A(j, r) + f
\]

where \( f \) is a \( \mathbb{Q}(v) \)-combination of \( |B| \) with \( B \in \Theta_\Delta(n, r) \) and \( B \prec A \). Combining this with (4.3.1) proves the assertion. \( \square \)

The maps \( \zeta_r \) given in 2.2 induce an algebra homomorphism

\[
\zeta = \prod_{r \geq 1} \zeta_r : \mathfrak{D}_\Delta(n) \to \mathfrak{S}_\Delta(n) = \prod_{r \geq 1} \mathfrak{S}_\Delta(n, r).
\]

We now prove the conjecture formulated in [8, 5.5(2)].

Theorem 4.4. The \( \mathbb{Q}(v) \)-space \( \mathfrak{V}_\Delta(n) \) is a subalgebra of \( \mathfrak{S}_\Delta(n) \) with \( \mathbb{Q}(v) \)-basis \( \mathfrak{B} \). Moreover, the map \( \zeta \) is injective and induces a \( \mathbb{Q}(v) \)-algebra isomorphism \( \mathfrak{D}_\Delta(n) \cong \mathfrak{V}_\Delta(n) \).
Proof. According to [8, 4.1], the set \( \mathcal{B} \) forms a \( \mathbb{Q}(v) \)-basis for \( \mathcal{V}_\alpha(n) \). This together with 4.3 implies that the set \( \{ A^+(0)0(j)A^-(0) \mid A \in \Theta^+_\alpha(n), j \in \mathbb{Z}_n^0 \} \) forms another basis for \( \mathcal{V}_\alpha(n) \). Note that, by 2.1(2), the set \( \{ u_A^+K_ju_A^- \mid A \in \Theta^+_\alpha(n), j \in \mathbb{Z}_n^0 \} \) forms a \( \mathbb{Q}(v) \)-basis for \( \mathcal{D}_\alpha(n) \). Furthermore, by 2.2,

\[
\zeta(u_A^+K_ju_A^-) = A^+(0)0(j)A^-(0).
\]

Thus, \( \zeta \) takes a basis for \( \mathcal{D}_\alpha(n) \) onto the basis for \( \mathcal{V}_\alpha(n) \). It follows that \( \zeta \) is injective and \( \zeta(\mathcal{D}_\alpha(n)) = \mathcal{V}_\alpha(n) \). The proof is completed.

Now, the Main Theorem 1.1 follows immediately.

We end the paper with an application to the (untwisted) Ringel–Hall algebra of a cyclic quiver.

Let \( \mathcal{V}_\alpha^+(n) \) be the subalgebra of \( \mathcal{V}_\alpha(n) \) spanned by \( A(0) \) for all \( A \in \Theta^+_\alpha(n) \). Then, by 2.2, the map sending \( \tilde{u}_A \) to \( A(0) \) is an algebra isomorphism from \( \mathcal{D}_\alpha^+(n) = \mathcal{S}_\alpha(n) \) to \( \mathcal{V}_\alpha^+(n) \). In particular, the formula 4.2(2) gives the following multiplication formula in the Ringel–Hall algebra \( \mathcal{S}_\alpha(n) \) over \( \mathbb{Z} \):

\[
\tilde{u}_\alpha \tilde{u}_A = \sum_{T \in \Theta^+_\alpha(n) \atop \text{ro}(T) = \alpha} v^{f_{A,T}} \prod_{1 \leq i \leq n \atop j \neq \bar{i}} \begin{bmatrix} a_{i,j} + t_{i,j} - t_{i-1,j} \\ t_{i,j} \end{bmatrix} \tilde{u}_{A+T-\tilde{T}^+}
\]

where

\[
f_{A,T} = \sum_{1 \leq i \leq n \atop j \neq \bar{i}} a_{i,j}t_{i,j} - \sum_{1 \leq i \leq n \atop j > \bar{i}} a_{i,j}t_{i,j} - \sum_{1 \leq i \leq n \atop j \neq \bar{i}} t_{i,j}t_{i-1,j} + \sum_{1 \leq i \leq n \atop j > \bar{i}} t_{i,j}t_{i-1,j}.
\]

Untwisting the multiplication for \( \mathcal{S}_\alpha(n) \) yields the following.

Theorem 4.5. The Ringel–Hall algebra \( \mathcal{S}_\alpha(n)^{\diamond} \) is the algebra over the polynomial ring \( \mathbb{Z}[q] \) (\( q = v^2 \)) which is spanned by the basis \( \{ u_A \mid A \in \Theta^+_\alpha(n) \} \) and generated by \( \{ u_\alpha \mid \alpha \in \mathbb{N}_n^0 \} \) and whose (untwisted) multiplication is given by the formulas: for any \( A \in \Theta^+_\alpha(n) \) and \( \alpha \in \mathbb{N}_n^0 \),

\[
u_{\alpha} \diamond u_A = \sum_{T \in \Theta^+_\alpha(n) \atop \text{ro}(T) = \alpha} q^{\sum_{1 \leq i \leq n, i < j} (a_{i,j}t_{i,j} - t_{i,j}t_{i+1,j})} \prod_{1 \leq i \leq n \atop j \neq \bar{i}} \begin{bmatrix} a_{i,j} + t_{i,j} - t_{i-1,j} \\ t_{i,j} \end{bmatrix} u_{A+T-\tilde{T}^+}
\]

In other words, if the Hall polynomial \( \varphi^{B}_{\mathcal{S}_\alpha,A} \) is nonzero, then there exists \( T = (t_{i,j}) \in \Theta^+_\alpha(n) \) with \( \text{ro}(T) = \alpha \) such that \( B = A + T - \tilde{T}^+ \) and

\[
\varphi^{A+T-\tilde{T}^+}_{\mathcal{S}_\alpha,A} = q^{\sum_{1 \leq i \leq n, i < j} (a_{i,j}t_{i,j} - t_{i,j}t_{i+1,j})} \prod_{1 \leq i \leq n \atop j \neq \bar{i}} \begin{bmatrix} a_{i,j} + t_{i,j} - t_{i-1,j} \\ t_{i,j} \end{bmatrix}.
\]

Note that, when \( \alpha \) defines a simple module or \( A \) defines another semisimple module, the formula coincides with the formulas given in [3, Th. 5.4.1] (built on [8, Th. 4.2]) and [4, Cor. 1.5].
5. Appendix — Proof of Lemma 3.2(2)

For \( w \in \mathcal{G}_{\Delta, r} \) and \( t \in \mathbb{Z} \), let

\[
\text{Inv}(w, t) = \{(i, j) \in \mathbb{Z}^2 \mid 1 + t \leq i \leq r + t, \ i < j, \ w(i) > w(j)\}
\]

\[
\text{Inv}(w) = \text{Inv}(w, 0).
\]

Then the number of inversions \( \ell'(w) := |\text{Inv}(w, t)| \) is clearly independent of \( t \).

**Proposition 5.1.** If \( w = ys \) with \( y, w \in \mathcal{G}_{\Delta, r} \) and \( s \in S \) satisfies \( \ell(w) = \ell(y) + 1 \), then \( \ell'(w) = \ell'(y) + 1 \).

**Proof.** Suppose \( s = s_{i_0} \) for some \( 1 \leq i_0 \leq r \). By the hypothesis and [18, 4.2.3], we have \( y(i_0) < y(i_0 + 1) \). Fix \( t \in \{0, 1\} \) with \( i_0, i_0 + 1 \in [1 + t, r + t] \). Let \( \mathcal{W} = \text{Inv}(w, t) \) and \( \mathcal{Y} = \text{Inv}(y, t) \).

We want to prove that \( |\mathcal{W}| = |\mathcal{Y}| + 1 \). For \( j \in \mathbb{Z} \) and \( i \in [i_0 + 1, i_0 + r] \), let

\[
c(i_0, i, j) := |\{i_0, i_0 + 1\} \cap \{i, \tilde{j}\}|
\]

where \( \tilde{j} \) denote the unique integer in \( [1 + t, r + t] \) such that \( j \equiv \tilde{j} \mod r \). For each \( x \in \{0, 1\} \), let

\[
\mathcal{W}_x = \{(i, j) \in \mathcal{W} \mid c(i_0, i, j) = x\} \quad \text{and} \quad \mathcal{Y}_x = \{(i, j) \in \mathcal{Y} \mid c(i_0, i, j) = x\}.
\]

Then we have disjoint unions \( \mathcal{W} = \mathcal{W}_0 \cup \mathcal{W}_1 \cup \mathcal{W}_2 \) and \( \mathcal{Y} = \mathcal{Y}_0 \cup \mathcal{Y}_1 \cup \mathcal{Y}_2 \).

For \( j \in \mathbb{Z} \) and \( i \in [i_0 + 1, i_0 + r] \), if \( c(i_0, i, j) = 0 \), then \( w(i) = y(i) \) and \( w(j) = y(j) \). This implies \( \mathcal{W}_0 = \mathcal{Y}_0 \). Hence, \( |\mathcal{W}_0| = |\mathcal{Y}_0| \). Since \( y(i_0) < y(i_0 + 1) \), it follows that

\[
\mathcal{W}_2 = \{(i_0, i_0 + 1 + kr) \mid k \in \mathbb{Z}_{\geq 0}, y(i_0 + 1) > y(i_0) + kr\},
\]

while

\[
\mathcal{Y}_2 = \{(i_0 + 1, i_0 + kr) \mid k \in \mathbb{Z}_{\geq 0}, y(i_0 + 1) > y(i_0) + kr\}.
\]

Hence, \( |\mathcal{W}_2| = |\mathcal{Y}_2| + 1 \). It remains to prove that \( |\mathcal{W}_1| = |\mathcal{Y}_1| \). In this case, we have

\[
\mathcal{W}_1 = \mathcal{W}_{1, (i_0, \bullet)} \cup \mathcal{W}_{1, (i_0 + 1, \bullet)} \cup \mathcal{W}_{1, (\bullet, i_0)} \cup \mathcal{W}_{1, (\bullet, i_0 + 1)},
\]

where \( \mathcal{W}_{1, (i_0, \bullet)} = \{(i, j) \in \mathcal{W}_1 \mid i = i_0\} \), etc. Define \( \mathcal{Y}_{1, (\bullet, \bullet)} \) similarly to get a similar partition for \( \mathcal{Y}_1 \). Then the condition \( y(i_0) < y(i_0 + 1) \) implies the following

\[
\mathcal{Y}_{1, (i_0, \bullet)} \subseteq \mathcal{W}_{1, (i_0, \bullet)}, \quad \mathcal{Y}_{1, (i_0 + 1, \bullet)} \supseteq \mathcal{W}_{1, (i_0 + 1, \bullet)}, \quad \mathcal{Y}_{1, (\bullet, i_0)} \supseteq \mathcal{W}_{1, (\bullet, i_0)}, \quad \mathcal{Y}_{1, (\bullet, i_0 + 1)} \subseteq \mathcal{W}_{1, (\bullet, i_0 + 1)}.
\]

But, since

\[
\mathcal{W}_{1, (i_0, \bullet)} \setminus \mathcal{Y}_{1, (i_0, \bullet)} = \{(i_0, j) \mid i_0 < j, j \in \mathbb{Z}, j \notin \{i_0, i_0 + 1\}, y(i_0) < y(j) < y(i_0 + 1)\}
\]

and

\[
\mathcal{Y}_{1, (i_0 + 1, \bullet)} \setminus \mathcal{W}_{1, (i_0 + 1, \bullet)} = \{(i_0 + 1, j) \mid i_0 + 1 < j, j \in \mathbb{Z}, j \notin \{i_0, i_0 + 1\}, y(i_0) < y(j) < y(i_0 + 1)\},
\]

it follows that \( |\mathcal{W}_{1, (i_0, \bullet)}| + |\mathcal{W}_{1, (i_0 + 1, \bullet)}| = |\mathcal{Y}_{1, (i_0, \bullet)}| + |\mathcal{Y}_{1, (i_0 + 1, \bullet)}| \). Similarly, one proves that \( |\mathcal{W}_{1, (\bullet, i_0)}| + |\mathcal{W}_{1, (\bullet, i_0 + 1)}| = |\mathcal{Y}_{1, (\bullet, i_0)}| + |\mathcal{Y}_{1, (\bullet, i_0 + 1)}| \). This completes the proof. \( \square \)

The following result given in [3, (3.2.1.1)] without proof follows immediately.

**Corollary 5.2.** For \( w \in \mathcal{G}_{\Delta, r} \) we have \( \ell(w) = \ell'(w) \).
We now generalise the construction for the shortest representatives of double cosets of the symmetric group \([7, \S 3]\) to the affine case. For \(A \in \Theta_0(n, r)\) with \(\lambda = \text{ro}(A)\) and \(\mu = \text{co}(A)\), define a pseudo matrix \(A^-\) as follows: the entry \(a_{i,j}\) is replaced by the sequence
\[
\varrho_{i,j}(A) = \left(\lambda_{k_0,i_0-1} + \sum_{t=1}^{i-1} a_{i,t} + 1, \ldots, \lambda_{k_0,i_0-1} + \sum_{t=1}^{j-1} a_{i,t} + (a_{i,j} - 1), \lambda_{k_0,i_0-1} + \sum_{t=1}^{j-1} a_{i,t}\right)
\]
where \(i = i_0 + k_0 n, j \in \mathbb{Z}\) with \(1 \leq i_0 \leq n\) and \(k_0 \in \mathbb{Z}\). We define \(\tilde{y}_A \in \mathcal{G}_{\lambda\gamma}\) by
\[
\tilde{y}_A(i + kr) = a_i + kr, \text{ for all } 1 \leq i \leq r, k \in \mathbb{Z},
\]
where \((a_1, a_2, \cdots, a_r)\) is the sequence obtained by reading the numbers in column 1 inside the subsequences from left to right and from top to bottom, and then in column 2, etc., and then in column \(n\). In other words, it is the sequence obtained by ignoring 0’s from \(((\varrho_{k,1})_{k \in \mathbb{Z}}, (\varrho_{k,2})_{k \in \mathbb{Z}}, \cdots, (\varrho_{k,n})_{k \in \mathbb{Z}})\) with \((\varrho_{k,i})_{k \in \mathbb{Z}} = (\cdots, \varrho_{1,i}, \varrho_{2,i}, \cdots, \varrho_{n,i}, \cdots)\).

We are ready to prove Lemma 3.2(2).

**Proposition 5.3.** Let \(A \in \Theta_0(n, r)\). Then \(\tilde{y}_A\) is the shortest representative of the double coset defined by \(A\), i.e., \(y_A = \tilde{y}_A\), and
\[
\ell(y_A) = \sum_{1 \leq i < j \leq n} a_{ij} k_{ij}.
\]

**Proof.** Let \(\lambda = \text{ro}(A)\) and \(\mu = \text{co}(A)\). For \(w \in \mathcal{G}_{\lambda \gamma} \mathcal{G}_\mu\), let
\[
\mathcal{N} = \bigcup_{1 \leq i \leq n, i < k, j > l} \left((R^k_i \cap w(R^\mu_j)) \times (R^k_i \cap w(R^\mu_j))\right) \text{ and } N = |\mathcal{N}| = \sum_{1 \leq i < j \leq n} a_{ij} k_{ij}.
\]

Then there is a injective map \(\varphi_w\) defined as follows:
\[
\varphi_w : \mathcal{N} \rightarrow \text{Inv}(w), \quad (c, d) \mapsto (w^{-1}(d), w^{-1}(c)).
\]
Hence by 5.2 we have \(N \leq \ell(w)\). In particular, we have \(\ell(y_A) \geq N\).

For \(i, j \in \mathbb{Z}\) let \(C_{ij}(A)\) denote the set of members of the sequence \(\varrho_{ij}(A)\). By definition we have, for \(i \in \mathbb{Z}\) and \(1 \leq j \leq n\),
\[
R^k_i = \bigcup_{l \in \mathbb{Z}} C_{il}(A), \quad \text{ and } \quad \tilde{y}_A(R^\mu_j) = \bigcup_{k \in \mathbb{Z}} C_{kj}(A).
\]
It is easy to see that \(C_{ij}(A) + tr = C_{i+t n,j+t n}(A)\) for \(i, j, t \in \mathbb{Z}\). Hence, for \(j = j_0 + t n\) with \(1 \leq j_0 \leq n\) and \(t \in \mathbb{Z}\),
\[
\tilde{y}_A(R^\mu_j) = tr + w_A(R^\mu_{j_0}) = tr + \bigcup_{k \in \mathbb{Z}} C_{kj_0}(A) = \bigcup_{k \in \mathbb{Z}} C_{k+t n,j+t n}(A) = \bigcup_{k \in \mathbb{Z}} C_{kj}(A).
\]
Thus, we have \(C_{ij}(A) = R^k_i \cap \tilde{y}_A(R^\mu_j)\) for \(i, j \in \mathbb{Z}\) and so \(a_{ij} = |R^k_i \cap \tilde{y}_A(R^\mu_j)|\) for \(i, j \in \mathbb{Z}\). This implies that \(\tilde{y}_A \in \mathcal{G}_{\lambda \gamma} \mathcal{G}_\mu\) and, hence, \(\ell(\tilde{y}_A) \geq \ell(y_A) \geq N\). Observe that \(\tilde{y}_A(C_{ji}(A)) = C_{ij}(A)\) for \(i, j \in \mathbb{Z}\), where \(A^t\) is the transpose of \(A\).

We now prove that \(\ell(\tilde{y}_A) = N\) by showing that \(\varphi_{\tilde{y}_A}\) is surjective. Let \((a, b) \in \text{Inv}(\tilde{y}_A)\). Since \(\mathbb{Z} = \bigcup_{s, t \in \mathbb{Z}} C_{s,t}(A)\), there exist \(i, j, k, l \in \mathbb{Z}\) such that \(\tilde{y}_A(a) \in C_{kl}(A)\) and \(\tilde{y}_A(b) \in C_{ij}(A)\). Since \(1 \leq a \leq r\) we have \(1 \leq l \leq n\). Since \(\tilde{y}_A(a) > \tilde{y}_A(b)\) we have either \(i < k\) or \(i = k\) and \(j < l\). On
the other hand, the conditions $a \in \tilde{y}_A^{-1}(C_{kl}(A)) = C_{lk}(\{A\})$, $b \in \tilde{y}_A^{-1}(C_{ij}(A)) = C_{ji}(\{A\})$ and $a < b$ imply either $l < j$ or $l = j$ and $k < i$. Hence, we must have $i < k$ and $l < j$. Therefore, the map $\varphi_{\tilde{y}_A}$ is bijective, proving $\ell(\tilde{y}_A) = N$. □

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