On the Semisimple Orbits of Restricted Cartan Type Lie Algebras $W$, $S$ and $H$

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Abstract
In this short note, we give a description of semisimple orbits in the restricted Cartan type Lie algebras $W$, $S$, $H$.

Keywords Weyl groups · Lie algebras of Cartan type · Semisimple orbits

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1 Introduction

Let $(g, [p])$ be a restricted Lie algebra with connected automorphism group $G_g := \text{Aut}_p(g)^0$. The algebraic group $G_g$ acts naturally on the constructible set $S_g$ of semisimple elements of $g$. A basic problem is to understand the set $S_g/G_g$ of semisimple $G_g$-orbits.

In the classical case, where $g := \text{Lie}(G)$ is the Lie algebra of a connected reductive group $G$, all maximal tori of $g$ are $G$-conjugate (hence $G_g$-conjugate) and there is a bijective correspondence $S_g/G_g \rightarrow t/W$, where $t$ is a maximal torus and $W$ its corresponding Weyl group (cf. [5, (7.12)]). If $g$ is not an algebraic Lie algebra, then maximal tori are not necessarily $G_g$-conjugate. In fact, for the non-classical simple Lie algebras (which, by the classification theorem of Block-Wilson-Premet-Strade (cf. [9]), are of Cartan type provided that the characteristic of $k$ is larger than 5), the maximal tori are not all conjugate under the action of the automorphism group [9, Chapter 7].
In this paper, we study Lie algebras of Cartan type $\mathfrak{g} := W, S, H$. In these cases, $\mathfrak{g}$ possesses finitely many $G_{\mathfrak{g}}$-conjugacy classes of maximal tori. These algebras have a natural filtration

$$\mathfrak{g} = \mathfrak{g}(-1) \supseteq \mathfrak{g}(0) \supseteq \cdots \supseteq \mathfrak{g}(s) \supseteq (0)$$

by $[p]$-stable subspaces. Let $\langle x \rangle_p$ denote the torus generated by a semisimple element $x \in S_\mathfrak{g}$. We define a function

$$\text{Ind}_\mathfrak{g} : S_\mathfrak{g} \to \mathbb{N}_0; \ x \mapsto \dim_k \langle x \rangle_p / (\langle x \rangle_p \cap \mathfrak{g}(0)),$$

(1.1)

whose fibers are $G_{\mathfrak{g}}$-stable. Given a maximal torus $t_\mathfrak{g}$ of $\mathfrak{g}$, we consider the Weyl group $W(\mathfrak{g}, t_\mathfrak{g})$ relative to $t_\mathfrak{g}$, defined via $W(\mathfrak{g}, t_\mathfrak{g}) := N_{G_{\mathfrak{g}}}(t_\mathfrak{g})/C_{G_{\mathfrak{g}}}(t_\mathfrak{g})$, where $N_{G_{\mathfrak{g}}}(t_\mathfrak{g})$ and $C_{G_{\mathfrak{g}}}(t_\mathfrak{g})$ are the normalizer and the centralizer of $t_\mathfrak{g}$ in $G_{\mathfrak{g}}$, respectively. Using basic results on tori, due to Demushkin [2, 3], every maximal torus $t_\mathfrak{g} \subseteq \mathfrak{g}$ has the same dimension $\mu(\mathfrak{g})$.

Moreover, up to conjugacy, every integer $0 \leq r \leq \mu(\mathfrak{g})$ gives rise to a unique maximal torus $t_{\mathfrak{g},r}$ such that $\text{Ind}_\mathfrak{g}(x) \leq r$ for all $x \in t_{\mathfrak{g},r}$ and $t_{\mathfrak{g},r} := \text{Ind}_\mathfrak{g}^{-1}(r) \cap t_{\mathfrak{g},r} \neq \emptyset$. The $W(\mathfrak{g}, t_{\mathfrak{g},r})$-orbits on $t_{\mathfrak{g},r}$ are distinguished by the values of invariant functions, and the invariants were determined by the second author in [7, Proposition 3.2]. Actually, in the case $r = \mu(\mathfrak{g})$, the isomorphism $W(\mathfrak{g}, t_{\mathfrak{g},\mu(\mathfrak{g})}) \cong \text{GL}_{\mu(\mathfrak{g})}(\mathbb{F}_p)$ was established in [8] and [1], and the invariant functions on $t_{\mathfrak{g},\mu(\mathfrak{g})}$ under $\text{GL}_{\mu(\mathfrak{g})}(\mathbb{F}_p)$ action were determined in a classical work of L. Dickson [4].

The main result reads:

**Theorem** Let $r \in \{0, 1, \ldots, \mu(\mathfrak{g})\}$. Then there is a bijective correspondence

$$\text{Ind}_\mathfrak{g}^{-1}(r)/G_{\mathfrak{g}} \to t_{\mathfrak{g},r}'/W(\mathfrak{g}, t_{\mathfrak{g},r}).$$

More details refer to Theorem 3.6. We will give description the quotients $t_{\mathfrak{g},r}'/W(\mathfrak{g}, t_{\mathfrak{g},r})$ by employing $p$-polynomials in Proposition 3.7 respectively.

*Throughout this paper, $k$ denotes an algebraically closed field of characteristic $\text{char}(k) =: p > 3$.*

## 2 Preliminaries

### 2.1 Cartan Type Lie Algebras Type $W, S, H$

Let $A(n) := k[X_1, \ldots, X_n]/(X_1^p, \ldots, X_n^p)$ be the truncated polynomial ring in $n$ variables. We write $x_i$ for the image of $X_i$ in $A(n)$. Note that $A(n)$ is a finite-dimensional local algebra, with maximal ideal $\mathfrak{m} := (x_1, \ldots, x_n)$. The Lie algebra $W(n) := \text{Der}(A(n))$ is called the $n$-th Jacobson-Witt algebra. It is an $(A(n))$-module in an obvious way, and has a standard basis $\{x_1^{\alpha_1} \cdots x_n^{\alpha_n} \partial_i; \ 0 \leq \alpha_j < p, 1 \leq i \leq n\}$ where $\partial_i$ denotes the partial derivative with respect to the variable $x_i$.

Define the linear map $\text{div} : W(n) \to A(n)$ by

$$\text{div}(\partial) = \sum_{i=1}^n \partial_i(\partial(x_i)).$$

The Lie algebra $S(n)$ is defined via $S(n) := \{\partial \in W(n); \ \text{div}(\partial) = 0\}$ and the derived algebra $S(n)^{(1)}$ is called special algebra. If $n \geq 3$, then $S(n)^{(1)}$ is restricted and simple.
Let us move on to the family $H(2m)$. For $i \in \{1, \ldots, 2m\}$, we put
\[
\sigma(i) := \begin{cases} 
1, & 1 \leq i \leq m, \\
-1, & m + 1 \leq i \leq 2m.
\end{cases}
\]
In addition, we define
\[
i' := \begin{cases} 
i + m, & 1 \leq i \leq m, \\
i - m, & m + 1 \leq i \leq 2m.
\end{cases}
\]
Let $H(2m) := \{ \sum_{i=1}^{2m} f_i \partial_i \in W(2m); \ \sigma(i) \partial_{i'}(f_i) = \sigma(j) \partial_j(f_j) \ 1 \leq i, j \leq 2m \}$. The Lie subalgebra $H(2m)^{(2)}$ of $H(2m)$ is simple and restricted, and we call it a Hamiltonian algebra.

From now on we will (by abuse of notation) write $W(n)$, $S(n)$ and $H(n)$ for the corresponding simple derived subalgebra, with the convention that $n = 2m$ for the Hamiltonian type.

Suppose that $g = X(n)$, where $X \in \{W, S, H\}$. By definition, it possesses a restricted $\mathbb{Z}$-grading
\[
g = \bigoplus_{i=-1}^{s} g_i, \quad [g_i, g_j] \subseteq g_{i+j}, \quad g_i^{[p]} \subseteq g_{pi}, \quad s \geq 1. \tag{2.1}
\]
Given such an algebra $g$, we consider the associated descending filtration $(g(i))_{i \geq -1}$, defined via
\[
g(i) := \sum_{j \geq i} g_j. \tag{2.2}
\]

### 2.2 Automorphism Groups

Let us gather some facts on automorphisms. It is well known that we have an isomorphism $\text{Aut}(A(n)) \cong \text{Aut}_p(W(n))$; $\varphi \mapsto \sigma_\varphi$, given by
\[
\sigma_\varphi(\partial) = \varphi \circ \partial \circ \varphi^{-1} \tag{2.3}
\]
for all $\partial \in W(n)$. If $g \in \{W, S, H\}$, then the group $\text{Aut}_p(g)$ is connected, i.e. $G_g = \text{Aut}_p(g)$, and we have
\[
G_g \cong \{ g \in \text{Aut}_p(W(n)); \ g(g) \subseteq g \}.
\]
Moreover, the group $G_g$ is a semidirect product $G_0 \rtimes U$, where $G_0$ consists of those automorphisms preserving the $\mathbb{Z}$-grading (2.1) of $g$ and $U$ is the unipotent radical [10]. It is a consequence of the semidirect product decomposition that
\[
g(g(i)) = g(i) \tag{2.4}
\]
for every $g \in G_g$ and $i \in \mathbb{Z}$.

Recall that the Poisson Lie algebra structure on $A(2m)$ is given by $\{f, g\} = D_H(f)(g)$ for all $f, g \in A(2m)$ (cf. [9, Section 4.2]), where the linear map $D_H$ is defined by
\[
D_H : A(2m) \to W(2m); \ f \mapsto \sum_{i=1}^{2m} \sigma(i) \partial_i(f) \partial_{i'}.
\]
For ease of reference we record the following well-known result:

**Lemma 2.1** [9, Theorem 7.3.4, 7.3.6] Let $\varphi \in \text{Aut}(A(n))$. Then $\sigma_\varphi$ induces an automorphism of $S(n)$ if and only if

$$\det(\partial_i \varphi(x_j)) \in k \setminus \{0\},$$

and $\sigma_\varphi$ induces an automorphism of $H(n)$ if and only if

$$\{\varphi(x_i), \varphi(x_j)\} = a\sigma(i)\delta_{i,j}$$

for some $a \in k \setminus \{0\}$ and all $i, j$.

### 2.3 Maximal Tori

Given a restricted Lie algebra $(g, [p])$, we let $\mu(g)$ denote the maximal dimension of all tori $t \subseteq g$ and let

$$\text{Tor}(g) := \{t \subseteq g; t \text{ torus, } \dim_k t = \mu(g)\}$$

be the set of tori of maximal dimension.

In the case of restricted Cartan type Lie algebras, every maximal torus has maximal dimension (cf. [9, Section 7.5]). Assume that $g \in \{W,S,H\}$. Since the natural filtration (2.2) is stable under the action of $G_g$, it follows that the function

$$\chi_0 : \text{Tor}(g) \to \mathbb{N}_0; t \mapsto \dim_k (t \cap g(0))$$

is constant on the $G_g$-orbits of $\text{Tor}(g)$. As shown by Demushkin in [2, 3], there are exactly $\mu(g) + 1$ orbits $O_0, \ldots, O_{\mu(g)}$ in $\text{Tor}(g)$ under the $G_g$-action, and these have the following description:

$$O_r = \{t \in \text{Tor}(g); \chi_0(t) = \mu(g) - r\}.$$

For each of the three Cartan types we have canonical orbit representatives $t_{g,r}$ of $O_r$ given by

$$t_{g,r} = t_{g,r}' \oplus t_{g,r}'' \quad (2.7)$$

where

$$t_{g,r}' = \sum_{i=1}^r k(1 + x_i)\partial_i \quad \text{and} \quad t_{g,r}'' = \sum_{i=r+1}^n kx_i\partial_i, \quad \text{if } g = W(n) \quad (2.8)$$

$$t_{g,r}' = \sum_{i=1}^r k((1 + x_i)\partial_i - x_n\partial_n) \quad \text{and} \quad t_{g,r}'' = \sum_{i=r+1}^{n-1} k(x_i \partial_i - x_n \partial_n), \quad \text{if } g = S(n) \quad (2.9)$$

$$t_{g,r}' = \sum_{i=1}^r k((1 + x_i)\partial_i - x_{i'}\partial_{i'}) \quad \text{and} \quad t_{g,r}'' = \sum_{i=r+1}^m k(x_i \partial_i - x_{i'} \partial_{i'}), \quad \text{if } g = H(2m) \quad (2.10)$$

### 2.4 Weyl Groups of $W, S, H$

Let $(g, [p])$ be a restricted Lie algebra with connected automorphism group $G_g$. $t \subseteq g$ be a torus and we let $N_{G_g}(t)$ and $C_{G_g}(t)$ be the normalizer and the centralizer of $t$ in $G_g$, respectively. The group

$$W(g, t) := N_{G_g}(t)/C_{G_g}(t)$$

is referred to as the Weyl group of $g$ relative to $t$. 
For the three Cartan types $W$, $S$ and $H$, we are interested in the Weyl group relative to maximal torus. Let $t \in \text{Tor}(g)$. Since $W(g, g, t) \cong W(g, t)$ for every $g \in G_g$, the Weyl group $W(g, t)$ only depends on the orbit $G_g t \subseteq \text{Tor}(g)$. We may hence assume that $t = t_{g, r}$ (2.8), (2.9), (2.10) for some $0 \leq r \leq \mu(g)$.

The following result was proved by Jensen in [6], Prop. 3.6:

**Proposition 2.2** Assume $g \in \{W, S, H\}$. Then $W(g, t_{g, r}) \cong (W_1 \times W_2) \rtimes W_3$, with

$$W_1 \cong \begin{cases} S_{n-r} & \text{if } g = W(n) \\ S_{n-r} & \text{if } g = S(n) \\ S_{m-r} \ltimes \mathbb{Z}^{m-r} & \text{if } g = H(n) \end{cases}$$ (2.11)

$$W_2 \cong \text{GL}_r(\mathbb{F}_p)$$ (2.12)

$$W_3 \cong \text{Mat}_{r, \mu(g)-r}(\mathbb{F}_p)$$ (2.13)

3 Semisimple Orbits in $W$, $S$, $H$

3.1 Semisimple Elements in the Standard Tori

Assume that $g \in \{W, S, H\}$. If $x \in S_g$ is a semisimple element, then $\langle x \rangle_p$ denotes the torus generated by $x$. We define a function $\text{Ind}_g: S_g \to \mathbb{N}_0; x \mapsto \dim_k \langle x \rangle_p / (\langle x \rangle_p \cap g(0))$. (3.1)

The index of an element $x$ is defined as $\text{Ind}_g(x)$. In view of Section 2.3, we have $\text{Ind}_g(S_g) = \{r \in \mathbb{N}_0; 0 \leq r \leq \mu(g)\}$. Given $r \in \{0, 1, \ldots, \mu(g)\}$, it follows from (2.4) that each fiber $\text{Ind}_g^{-1}(r)$ is $G_g$-stable. Clearly, $S_g$ is the disjoint union of all fibers, i.e.,

$$S_g = \bigcup_{r=0}^{\mu(g)} \text{Ind}_g^{-1}(r).$$

Indeed, $\dim_k t_{g, r} / (t_{g, r} \cap g(0)) = r$ implies that

$$\text{Ind}_g(x) \leq r \quad \text{for all } x \in t_{g, r}.$$ (3.2)

We denote by $t_{g, r}^0 := t_{g, r} \cap \text{Ind}_g^{-1}(r)$ the set of those elements of $t_{g, r}$ whose index is $r$. If $r = 0$, then Eq. 3.2 yields $t_{g, 0}^0 = t_{g, 0}$. Note that the dimension $\text{Ind}_g(x)$ does not change when $x$ is replaced by its $G_g$-conjugate. Observing Eqs. 2.8, 2.9 and 2.10, we conclude that $\text{Ind}_g^{-1}(0)$ is just the $G_g$-saturation $\hat{G}_g t_{g, 0}$.

We put $y_i := x_i + 1$. In order to describe the set $\text{Ind}_g^{-1}(r)$, we introduce the following notations:

$$t_{g, r} = (d_1^{g, r}, \ldots, d_{\mu(g)}^{g, r}), \quad 0 \leq r \leq \mu(g).$$

$$d_i^{g, r} := \begin{cases} y_i \partial_i, & 1 \leq i \leq r, \\ x_i \partial_i, & r + 1 \leq i \leq n. \end{cases}$$ if $g = W(n)$ (3.3)

$$d_i^{g, r} := \begin{cases} y_i \partial_i - x_n \partial_n, & 1 \leq i \leq r, \\ x_i \partial_i - x_n \partial_n, & r + 1 \leq i \leq n - 1. \end{cases}$$ if $g = S(n)$ (3.4)

$$\hat{G}_g t_{g, 0}$$
\[ d_{1}^{g,r} := \begin{cases} 
 y_{i} \partial_{t} - x_{i} \partial_{t'}, & 1 \leq i \leq r, \\
 x_{i} \partial_{t} - x_{i} \partial_{t'}, & r + 1 \leq i \leq m. 
 \end{cases} \] 

(3.5)

The following lemma is well-known. We provide a proof here for convenience.

**Lemma 3.1** Given a restricted Lie algebra \((l, [p])\), we let \(t \subseteq \mathbb{T}\) be a torus with basis \(\{t_{1}, \ldots, t_{n}\}\) such that \(t_{i}^{[p]} = t_{i}\) for \(1 \leq i \leq n\). If \(t = \sum_{i=1}^{n} \lambda_{i} t_{i} \in t\), then \(t = \langle t, t^{[p]}, \ldots, t^{[p]^{p-1}} \rangle\) if and only if \(\lambda_{1}, \ldots, \lambda_{n}\) are \(\mathbb{F}_{p}\)-linearly independent.

**Proof** It is clear that \(\langle t, t^{[p]}, \ldots, t^{[p]^{p-1}} \rangle\) is linearly independent if and only if \(\det((\lambda_{i}^{p^{j-1}})_{1 \leq i, j \leq n}) \neq 0\). Since \(\det((\lambda_{i}^{p^{j-1}})_{1 \leq i, j \leq n}) = \prod_{i=1}^{n} \prod_{a_{1}, \ldots, a_{j-1} \in \mathbb{F}_{p}} (a_{1}\lambda_{1} + \cdots + a_{j-1}\lambda_{j-1} + \lambda_{i})\) (see [4, Section 2] for example), we have \(\det((\lambda_{i}^{p^{j-1}})_{1 \leq i, j \leq n}) \neq 0\) if and only if \(\lambda_{1}, \ldots, \lambda_{n}\) are \(\mathbb{F}_{p}\)-linearly independent. \(\Box\)

**Lemma 3.2** Keep the notations as above. Let \(1 \leq r \leq \mu(g)\) and \(t_{g,r}\) be the standard maximal torus with basis \(\{d_{1}^{g,r}, \ldots, d_{\mu(g)}^{g,r}\}\). Suppose that \(d = \sum_{i=1}^{\mu(g)} \lambda_{i} d_{i}^{g,r} \in t_{g,r}\). Then \(d \in t_{g,r}\) if and only if \(\lambda_{1}, \ldots, \lambda_{r}\) are \(\mathbb{F}_{p}\)-linearly independent.

**Proof** If \(\lambda_{1}, \ldots, \lambda_{r}\) are linearly independent over the prime field \(\mathbb{F}_{p}\), then the coset of \(d\) modulo \(t_{g,r} \cap g(0)\) generates an \(r\) dimensional torus by Lemma 3.1, so that \(\text{Ind}_{g}(d) = \dim_{k}(d_{p})/(d_{p} \cap g(0)) = r\), i.e. \(d \in t_{g,r}\).

Conversely, if \(\lambda_{1}, \ldots, \lambda_{r}\) are \(\mathbb{F}_{p}\)-linearly dependent, then

\[ d = \alpha_{1} t_{1} + \cdots + \alpha_{m} t_{m} \pmod{t_{g,r} \cap g(0)} \]

where \(m < r\), \(\alpha_{i} \in k\) and each \(t_{i}\) is a linear combination of \(d_{1}^{g,r}, \ldots, d_{\mu(g)}^{g,r}\) with coefficients in \(\mathbb{F}_{p}\). In this case

\[ \langle d_{p} \rangle \subseteq k t_{1} + \cdots + k t_{m} + g(0), \]

and it is immediately clear that \(\text{Ind}_{g}(d) < r\). \(\Box\)

**Lemma 3.3** Suppose that \(d = \sum_{i=1}^{\mu(g)} \lambda_{i} d_{i}^{g,r} \in t_{g,r}\). If \(\lambda_{1}, \ldots, \lambda_{r}\) are \(\mathbb{F}_{p}\)-linearly dependent, then there exists \(\sigma_{\varphi} \in G_{g}\) such that \(\sigma_{\varphi}(d) \in t_{g,r-1}\).

**Proof** By assumption, there exist \(u_{i} \in \mathbb{F}_{p}\) such that \(\sum_{i=1}^{r} u_{i} \lambda_{i} = 0\). We may assume without loss of generality that \(u_{r} = -1\), so that \(\lambda_{r} = \sum_{i=1}^{r-1} u_{i} \lambda_{i}\). We put \((u) := (u_{1}, \ldots, u_{r-1})\) and define \(y(u) := y_{1}^{u_{1}} \cdots y_{r-1}^{u_{r-1}}\).

Assume first that \(g = W(n)\). We define an automorphism of \(A(n)\) by

\[ \varphi(x_{i}) = \begin{cases} 
 x_{i} & \text{if } i \neq r, \\
 x_{r} + y(u) - 1 & \text{if } i = r.
 \end{cases} \]

Using Eq. 2.3 (see also the formula in [2, p. 234])) one can show by direct computation that

\[ \sigma_{\varphi}(d) = \sum_{i=1}^{r-1} \lambda_{i} y_{i} \partial_{i} + \sum_{i=r}^{n} \lambda_{i} x_{i} \partial_{i} \in t_{W,r-1}. \]
Assume now \( g = S(n) \). Define \( \varphi \in \text{Aut}(A(n)) \) by

\[
\varphi(x_i) = \begin{cases} 
  x_i & \text{if } i \neq r, \\
  x_r + y^{(u)} & \text{if } i = r.
\end{cases}
\]

As \( \det(\partial_i(\varphi(x_j))) = 1 \), Lemma 2.1 ensures that \( \sigma(\varphi) \in G_g \). Then we have:

\[
\sigma(\varphi)(d) = \sum_{i=1}^{r-1} \lambda_i(y_i \partial_i - x_n \partial_n) + \sum_{i=r}^{n-1} \lambda_i(x_i \partial_i - x_n \partial_n) \in t_{S,r-1}.
\]

Finally the case \( g = H(n) \), with \( n = 2m \) for some \( m \geq 1 \). Using multi-index notation (see [9, Section 2.1]), we define

\[
\varphi(x_i) = \begin{cases} 
  x_i & \text{if } i \neq r \text{ and } 1 \leq i \leq m, \\
  x_r + y^{(u)} - 1 & \text{if } i = r, \\
  x_i - u_j y^{(u) - \epsilon_j} x_{m+r} & \text{if } m + 1 \leq i \leq m + r - 1, \\
  x_i & \text{if } m + r \leq i \leq 2m.
\end{cases}
\]

Observing \( \det(\partial_i(\varphi(x_j))) = 1 \), we thus obtain \( \varphi \in \text{Aut}(A(n)) \). Moreover, we claim that

\[
\{\varphi(x_i), \varphi(x_j)\} = \sigma(i)\delta_{i,j} \quad \text{for all } i, j.
\]

We just deal with \( i = r \), the other cases are similar. Suppose that \( 1 \leq j \leq m \). Since both \( \varphi(x_r) \) and \( \varphi(x_j) \) lie in the algebra \( A(m) \), it follows that \( \{\varphi(x_r), \varphi(x_j)\} = 0 \). If \( m + 1 \leq j \leq m + r - 1 \), then

\[
\{\varphi(x_r), \varphi(x_j)\} = \{x_r + y^{(u)} - 1, x_j - u_j y^{(u) - \epsilon_j} x_{m+r}\}
\]

\[
= \sigma(r)\partial_r(x_r)\partial_{m+r}(-u_j y^{(u) - \epsilon_j} x_{m+r}) + \sigma(j)\partial_j(y^{(u)})\partial_j(x_j) = 0.
\]

Now for \( m + r \leq j \leq 2m \), so that \( \{\varphi(x_r), \varphi(x_j)\} = \{x_r + y^{(u)} - 1, x_j\} = \delta_{r,j} \). This establishes our claim. Consequently, Lemma 2.1 implies that \( \sigma(\varphi) \in G_g \). By the same token, we have

\[
\sigma(\varphi)(d) = \sum_{i=1}^{r-1} \lambda_i(y_i \partial_i - x_r \partial_r) + \sum_{i=r}^{m} \lambda_i(x_i \partial_i - x_r \partial_r) \in t_{H,r-1}.
\]

Corollary 3.4 Assume \( g \in \{W, S, H\} \) and let \( x \in S_g \) be a semisimple element of \( g \). Then

\[
\text{Ind}_g(x) = \min\{r; \ G_g x \cap t_{g,r} \neq \emptyset, \ 0 \leq r \leq \mu(g)\}. \tag{3.6}
\]

Let \( r \in \{0, 1, \ldots, \mu(g)\} \). In particular, up to conjugacy, there exists a unique maximal torus \( t \) such that \( t \cap \text{Ind}_g^{-1}(r) \neq \emptyset \) and \( \text{Ind}_g(x) \leq r \) for all \( x \in t \).

Proof We put \( l := \min\{r; \ G_g x \cap t_{g,r} \neq \emptyset, \ 0 \leq r \leq \mu(g)\} \). We may assume that \( x \in t_{g,l} \). Now Lemma 3.2 and Lemma 3.3 in conjunction with the minimality of \( l \) yields \( x \in t_{g,l} \). Consequently, \( \text{Ind}_g^{-1}(x) = l \).

To verify the last assertion, we note that \( \text{Ind}_g(x) \leq r \) for all \( x \in t_{g,i} \) and \( i \leq r \) (3.2). In view of Eq. 3.6, we obtain \( t_{g,i} \cap \text{Ind}_g^{-1}(r) = \emptyset \) whenever \( i < r \). This proves the uniqueness.
3.2 Semisimple Orbits

In this section, we turn to the set \( \text{Ind}_{g}^{-1}(r)/G_{g} \) for the restricted Cartan type Lie algebra \( g \in \{ W, S, H \} \). We have seen in the foregoing section that the set \( \text{Ind}^{-1}(r) \) coincides with the \( G_{g} \)-saturation \( G_{g} \cdot tr_{g} \cdot r \). It will be necessary to consider the \( G_{g} \)-conjugacy relation on the set \( tr_{g} \cdot r \).

Lemma 3.5 Assume that \( g \in \{ W, S, H \} \) with connected automorphism group \( G_{g} \). Let \( d, t \in tr_{g} \cdot r \). If \( d \) and \( t \) are conjugate under \( G_{g} \), then there exists \( \sigma_{\psi} \in N_{G_{g}}(tr_{g} \cdot r) \) such that \( \sigma_{\psi}(d) = t \).

Proof Let \( \sigma_{\psi} \in G_{g} \) be such that \( \sigma_{\psi}(d) = t \). We write \( d = \sum_{i=1}^{\mu(g)} \beta_{i} d_{i}^{g} \cdot r \) as well as \( t = \sum_{i=1}^{\mu(g)} \alpha_{i} d_{i}^{g} \cdot r \). Define

\[
  z_{i} = \begin{cases} 
    y_{i} & \text{if } i = 1, \ldots, r, \\
    x_{i} & \text{otherwise.}
  \end{cases}
\] and \( f_{i} = \varphi(z_{i}) \).

Setting \( \beta = (\beta_{1}, \ldots, \beta_{\mu(g)}) \) and \( \alpha = (\alpha_{1}, \ldots, \alpha_{\mu(g)}) \), we apply Eq. 2.3 in conjunction with Eqs. 3.3–3.5 to see that

\[
  t \cdot (f_{i}) = \begin{cases} 
    \beta_{i} f_{i}, & 1 \leq i \leq n - 1, \\
    -\sum_{j=1}^{n-1} \beta_{j} f_{j}, & i = n.
  \end{cases} \quad \text{if } g = S(n). (3.8)
\]

As a result, \( f_{i} \) is a weight vector with respect to the canonical action of \( t \) on \( A(n) \). We claim that

\[
  \tag{†} \text{there exists matrices } A, B, \tau \text{ such that } \beta = \alpha \left( \begin{array}{cc} A & B \\ 0 & \tau \end{array} \right), \text{ where } A = (a_{ij}) \in GL_{r}(\mathbb{F}_{p}), B = (b_{ij}) \text{ and } \tau \in W_{1}. (2.11)
\]

Suppose that \( g = W(n) \). Since \( \varphi \in \text{Aut}(A(n)) \), the weight space decomposition ensures that

\[
  f_{j} \text{ has the term } z_{a_{1}}^{r} \cdots z_{r}^{r} \text{ with weight } \sum_{i=1}^{r} a_{ij} 1 \leq j \leq r \quad (3.10)
\]

\[
  f_{j} \text{ has the term } c_{j} x_{\tau(j)}^{r} z_{a_{1}}^{r} \cdots z_{r}^{r} \text{ with weight } \sum_{i=1}^{r} a_{ij} + \alpha_{\tau(j)} 1 \leq j \leq r \quad (3.11)
\]

where \( \tau \) is a permutation on \( \{ r + 1, \ldots, n \} \).

Assume now \( g = S(n) \). We put \( \alpha_{n} := -\sum_{j=1}^{n-1} \alpha_{j} \). The same argument shows that

\[
  f_{j} \text{ has the term } z_{a_{1}}^{r} \cdots z_{r}^{r} \text{ with weight } \sum_{i=1}^{r} a_{ij} 1 \leq j \leq r \quad (3.12)
\]

\[ \square \]
$f_j$ has the term $c_j x_\tau(j) z_1^{b_{1j}} \cdots z_r^{b_{rj}}$ with weight $\sum_{i=1}^{r} \alpha_i b_{ij} + \alpha_{\tau(j)} r + 1 \leq j \leq n$ (3.13)

where $\tau$ is a permutation on \{r + 1, \ldots, n\}.

Finally consider the case $\mathfrak{g} = H(2m)$. As $\sigma_\varphi \in G_\mathfrak{g}$, it follows that $f_j$ has the form (modulo the corresponding weight space) $z_1^{\alpha_{1j}} \cdots z_r^{\alpha_{rj}}$ with weight $\sum_{i=1}^{r} \alpha_i a_{i,j}$ for every $j \in \{1, \ldots, r\}$.

For $r + 1 \leq j \leq m$, combining Eqs. 2.6 with 3.9 one obtains that $f_j$ has the term $z_1^{b_{1j}} \cdots z_r^{b_{rj}} x_{\tau(j)}$, where $\tau$ is a permutation on \{r + 1, \ldots, m, m + r + 1, \ldots, 2m\}. In view of Eq. 2.6, $\{f_j, f_{j'}\}$ is a non-zero constant. Direct computation shows that $\tau(j') = \tau(j)$.

So that $\tau$ can be identified with an element of $S_{m-r} \ltimes \mathbb{Z}_2^{m-r}$, where the copies of $\mathbb{Z}_2$ act by $'$. Consequently, $f_j$ has weight $\sum_{i=1}^{r} \alpha_i b_{ij} + \sigma(\tau(j)) \alpha_{\omega(j)}$ with $(\omega, a) = \tau \in S_{m-r} \ltimes \mathbb{Z}_2^{m-r}$.

Thanks to Lemma 3.2, both $\{\alpha_1, \ldots, \alpha_r\}$ and $\{\beta_1, \ldots, \beta_r\}$ are $\mathbb{F}_p$-linearly independent. It follows that $A \in GL_r(\mathbb{F}_p)$. This proves (†). Now, Proposition 2.2 ensures the existence of $\sigma_\varphi$.

\begin{theorem}
Assume that $\mathfrak{g} \in \{W, S, H\}$. Let $r \in \{0, 1, \ldots, \mu(\mathfrak{g})\}$. Then there is a bijective correspondence

$$
\text{Ind}^{-1}_\mathfrak{g}(r)/G_\mathfrak{g} \rightarrow t^r_{\mathfrak{g},r}/W(\mathfrak{g}, t_{\mathfrak{g},r}).
$$

\end{theorem}

\begin{proof}
Let $x \in \text{Ind}^{-1}_\mathfrak{g}(r)$. Corollary 3.4 readily yields $G_\mathfrak{g} x \cap t_{\mathfrak{g},r} \neq \emptyset$. It follows that $\text{Ind}^{-1}_\mathfrak{g}(r)$ coincides with the $G_\mathfrak{g}$-saturation $G_\mathfrak{g} t^r_{\mathfrak{g},r}$. The assertion now follows from Lemma 3.5.

\end{proof}

\section{Quotients}
In this section, we would like to give a description of the quotients $t^r_{\mathfrak{g},r}/W(\mathfrak{g}, t_{\mathfrak{g},r})$ by employing $p$-polynomials. Recall that a $p$-polynomial is a polynomial of the form

$$
f(t) = a_0 t^p + a_{-1} t^{p-1} + \cdots + a_0 t \in k[t].
$$

For each $x \in \mathfrak{g}$, define $f(x) := a_0 x^{|p|} + a_{-1} x^{|p|-1} + \cdots + a_0 x \in \mathfrak{g}$. Let $t \subseteq \mathfrak{g}$ be a torus. Given a semisimple element $x \in t$, there exists a monic $p$-polynomial $f_x(t)$ of lowest degree such that $f_x(x) = 0$. It is unique and we call it the minimal $p$-polynomial of $x$. By general theory, the orbit of an element $x \in t$ with respect to the whole group $\text{Aut}_p(t)$ is completely determined by the minimal $p$-polynomial of $x$.

In the case $r = \mu(\mathfrak{g})$, according to Proposition 2.2 (see also [8, Theorem 1] and [1, Theorem 5.3]), there is an isomorphism $W(\mathfrak{g}, t_{\mathfrak{g},\mu(\mathfrak{g})}) \cong GL_{\mu(\mathfrak{g})}(\mathbb{F}_p)$. Let $x, y \in t_{\mathfrak{g}, \mu(\mathfrak{g})}$. The foregoing observation implies that $x$ and $y$ are in the same $W(\mathfrak{g}, t_{\mathfrak{g}, \mu(\mathfrak{g})})$-orbit if and only if $f_x = f_y$.

For general $r$, let $x \in t_{\mathfrak{g},r}$. We denote by $\bar{x}$ the image of $x$ under the canonical projection $\pi : t_{\mathfrak{g},r} \rightarrow t_{\mathfrak{g},r}/(t_{\mathfrak{g},r} \cap \mathfrak{g}(0))$. Let $f_{\bar{x}}(t)$ be the minimal $p$-polynomial of $\bar{x}$. It follows that $f_{\bar{x}}(t)$ is the monic $p$-polynomial of smallest degree such that $f_{\bar{x}}(x) \in \mathfrak{g}(0)$.

\begin{proposition}
Keep the notations as before. Let $x, y \in t_{\mathfrak{g},r}$. Then $x$ and $y$ are in the same $W(\mathfrak{g}, t_{\mathfrak{g},r})$-orbit if and only if $f_{\bar{x}} = f_{\bar{y}}$ and $f_{\bar{x}}(x), f_{\bar{y}}(y)$ lie in the same $W_1$-orbit (2.11).

\end{proposition}
Proof Let \( \sigma \in W(\mathfrak{g}, t_{\mathfrak{g}, r}) \) be such that \( \sigma(\mathfrak{g}) = y \). The Weyl group \( W(\mathfrak{g}, t_{\mathfrak{g}, r}) \) leaves invariant the subtorus \( t_{\mathfrak{g}, r} \cap \mathfrak{g}_0 \). It follows that \( \sigma \in \text{Aut}_p(t_{\mathfrak{g}, r} / (t_{\mathfrak{g}, r} \cap \mathfrak{g}_0)) \) and \( \sigma(\mathfrak{x}) = \bar{y} \).

Consequently, \( f_{\bar{y}} = f_{\mathfrak{y}} \). Now, let \( f_{\bar{y}}(t) = f_{\mathfrak{y}}(t) = t^{p^j} + a_{l-1}t^{p^{l-1}} + \cdots + a_0t \). We have

\[
\sigma(f_{\bar{y}}(x)) = \sigma(x^{p^j} + a_{l-1}x^{p^{l-1}} + \cdots + a_0x) = \sigma(x)^{p^j} + a_{l-1}\sigma(x)^{p^{l-1}} + \cdots + a_0\sigma(x) = f_{\mathfrak{y}}(y).
\]

Since \( W(\mathfrak{g}, t_{\mathfrak{g}, r}) \) acts on the subtorus \( t_{\mathfrak{g}, r} \cap \mathfrak{g}_0 \) via the classical Weyl group \( W_1 \), there exists an element \( w \in W_1 \) such that \( \sigma(f_{\bar{y}}(x)) = w(f_{\bar{y}}(x)) = f_{\bar{y}}(y) \).

Conversely, denote \( f := f_{\bar{y}} = f_{\mathfrak{y}} \). Let \( \tau \in W_1 \) be such that \( \tau(f(x)) = f(y) \).

We write \( x = (x', x'') \) and \( y = (y', y'') \) with \( x', y' \in t_{\mathfrak{g}, r} \) and \( x'', y'' \in t_{\mathfrak{g}, r}^0 \) (see Eq. 2.7). It is easy to see that \( \tau(x) = \tau(x') = x'' - \tau(x'') \). By the same token, there exists an invertible matrix \( g \) such that \( \tau(x) = \tau(x'') = \bar{y} \). Hence, \( f \) is the minimal \( p \)-polynomial of \( x' + y'' - \tau(x'') \). By the same token, there exists an invertible matrix \( \begin{pmatrix} B & C \\ D & E \end{pmatrix} \in \text{GL}_m(\mathbb{F}_p) \) such that \( (x', 0) \begin{pmatrix} B & C \\ D & E \end{pmatrix} = (x', y'' - \tau(x'')) \). Consequently, \( (x', x'') \begin{pmatrix} A & C \\ 0 & \tau \end{pmatrix} = (y', y'') \), and our assertion now follows directly from Proposition 2.2.

\[\square\]

4 Normalizers and Centralizers in \( W(n) \)

In this section, \( \mathfrak{g} \) always denotes the \( n \)-th Jacobson-Witt algebra. For convenience, we set \( t_r := t_{\mathfrak{g}, r}, 0 \leq r \leq n \) (see Eq. 2.8). We shall compute \( N_{G_{\mathfrak{g}}}(t) \) and \( C_{G_{\mathfrak{g}}}(t) \) for every \( t \in \text{Tor}(\mathfrak{g}) \). Up to conjugacy, we may thus assume that \( t = t_r \) for some \( r \in \{0, 1, \ldots, n\} \). It should be noted that the isomorphisms

\[ N_{G_{\mathfrak{g}}}(t_n) \cong \text{GL}_n(\mathbb{F}_p) \quad \text{and} \quad C_{G_{\mathfrak{g}}}(t_n) \cong [\text{id}_{t_n}] \tag{4.1} \]

were established by Premet (see [8, p. 139]).

Recall that \( t_r = \sum_{i=1}^n k_{z_i} \partial_i \), where \( z_i = y_i \) for \( 1 \leq i \leq r \) and \( z_i = x_i \) for \( r + 1 \leq i \leq n \) (3.3).

Proposition 4.1 Assume \( r \in \{0, \ldots, n\} \). Then

\[ N_{G_{\mathfrak{g}}}(t_r) = \{ \varphi \in \text{Aut}(A(n)) \mid \varphi \text{ has form } (*) \} \]

\[ (*) \text{ If } 1 \leq j \leq r, \varphi(z_j) = \prod_{i=1}^r z_i^{m_{ij}}, \text{ where } (m_{ij})_{1 \leq i, j \leq r} \in \text{GL}_r(\mathbb{F}_p). \text{ If } r + 1 \leq j \leq n, \varphi(z_j) = a_j z_{\tau(j)} \prod_{i=1}^r z_i^{m_{ij}}, \text{ where } a_j \in k^*, \tau \in S_{n-r}, m_{ij} \in \mathbb{F}_p. \]

Proof Let \( \varphi \in \text{Aut}(A(n)) \) be such that \( \sigma_{\varphi} \in N_{G_{\mathfrak{g}}}(t_r) \). There exists an invertible matrix \( (m_{ij}) \in \text{GL}_n(k) \) such that

\[ z_i \partial_i = \sum_{j=1}^n m_{ij} \varphi(z_j \partial_j) \quad 1 \leq i \leq n. \]

We put \( f_j := \varphi(z_j) \), then it is a simple exercise in linear algebra to show

\[ z_i \partial_i f_j = m_{ij} f_j, \quad 1 \leq i, j \leq n. \tag{4.2} \]
Note that \( \{z_1^{a_1} \cdots z_n^{a_n} ; \ 0 \leq a_1, \ldots, a_n < p \} \) is a \( k \)-basis of \( A(n) \) consisting of weight vectors. Our assertion follows from Eq. 4.2 in conjunction with the weight space decomposition.

**Corollary 4.2** \( C_{G_0}(t_r) \cong (k^*)^{n-r} \). In particular, \( C_G(t_r) = N_G(t_r)^0 \).

**Proof** As \( \sigma_\varphi \in C_{G_0}(t_r) \), direct computation shows that the weights \( z_i \) and \( \varphi(z_i) \) are the same. The assertion follows by applying the similar argument as in Proposition 4.1.

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