Fields of invariants for unipotent radicals of parabolic subgroups

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ABSTRACT
The paper is devoted to the problem of finding free generators in the fields of invariants for actions of unipotent groups on affine varieties. We consider the case when the unipotent group is the unipotent radical in an arbitrary parabolic subgroup in the reductive group of classical type $GL(n), SL(n), O(n)$ or $Sp(2n)$. In the explicit form, we present a system of free generators in the field of invariants for the action of the unipotent radical on the reductive group by conjugation.

1. Introduction
Let $G$ be a split reductive group defined over a field $K$ of characteristic zero. Let $U$ be the unipotent radical of an arbitrary parabolic subgroup $P$ in $G$. The group $U$ acts on $G$ by conjugation. There is a representation of the group $U$ on the space $A = K[G]$ by the formula $\rho(g)f(x) = f(g^{-1}xg), g \in U, x \in G$. This representation extends to the action of $U$ on the field $F = K(G)$ of rational functions on $G$. A function $f \in A$ (respectively $f \in F$) is called an $U$-invariant if $\rho(g)f = f$ for any $g \in U$.

In the present paper we consider the cases when $G$ is one of the following groups $GL(n), SL(n), O(n)$ or $Sp(2n)$. We aim to construct a system of free generators of the field of $U$-invariants $F^U$ in an explicit form as polynomials in matrix entries.

In the second section of this paper, we solve this problem for the case of the general linear group $GL(n)$. In Theorem 2.9, we construct the system of free generators $\{J_{ij} : (i, j) \in S\}$ for the field $F^U$, where $J_{ij}$ are determinants of special form. We obtain the similar system of generators for $SL(n)$ (see Corollary 2.10). The case when $U$ is the unitriangular group $UT(n)$ (i.e. the group of upper triangular matrices with ones on the diagonal) is treated by the author in the paper [1].
In the third section, we consider the case of $G = \text{O}(n)$ or $G = \text{Sp}(2n)$. In Theorem 3.5, using the restrictions $\{J_{ij}^c\}$ of polynomials $\{J_{ij}\}$ to $G$, we construct the system of free generators for the field of $U$-invariants on the group $G$.

Since the group $U$ is unipotent, the field of invariants $\mathcal{F}^U$ is a field of fractions of the algebra of invariant regular functions $\mathcal{A}^U$ [2, Theorem 3.3]. The algebra $\mathcal{A}^U$ is finitely generated [2, Theorem 3.13]. The problem of description of generators for the algebra $\mathcal{A}^U$ is still an unsolved problem.

In conclusion of the Introduction we emphasize a few papers which are devoted to the theory of invariants of unipotent groups [3–7].

## 2. Fields of invariants for radicals of parabolic subgroups in $\text{GL}(n)$

Let $K$ be a field of characteristic zero and $P$ be a standard upper-block parabolic subgroup of the general linear group $G = \text{GL}(n)$. There is a decomposition $P = LU$ of the group $P$ into a semidirect product of its unipotent radical $U$ and a Levi subgroup $L$. Our aim is to construct a system of free generators of the field $U$-invariants $\mathcal{F}^U$ explicitly.

The parabolic subgroup $P$ is defined by a partition of the integer segment $[1, n]$ into a system of consecutive segments

$$[1, n] = I_1 \sqcup I_2 \sqcup \cdots \sqcup I_\ell.$$ 

Put $n_i = |I_i|$ for each $1 \leq i \leq \ell$. The linear space $\mathcal{M} = \text{Mat}(n)$ of all $(n \times n)$ matrices is a direct sum

$$\mathcal{M} = \bigoplus_{1 \leq k, m \leq \ell} \mathcal{M}_{km},$$

where $\mathcal{M}_{km}$ is spanned by the matrix units $E_{ij}$, $(i, j) \in I_k \times I_m$. The Levi subgroup $L$ is isomorphic to the direct product

$$\text{GL}(n_1) \times \cdots \times \text{GL}(n_\ell).$$

The unipotent subgroup $U$ consists of matrices $E + A$, where $E$ is the identity matrix and $A$ belongs to the sum of subspaces $\mathcal{M}_{km}$, $k < m$.

For each $i \in [1, n]$, let $i' = n + 1 - i$ be the symmetric number to $i$ with respect to the centre of the segment. Respectively, for any $T \subseteq [1, n]$, we define $T' = \{i' : i \in T\}$.

We consider the set $\mathcal{S}$ of pairs $(i, j)$ such that

$$i \in I_k, \quad j \in I'_m, \quad k \geq m.$$ 

Let $S$ be the intersection of the group $\text{GL}(n)$ with the subspace in $\mathcal{M}$ spanned by $E_{ij}$, $(i, j) \in \mathcal{S}$.

**Proposition 2.1:** The subset

$$\bigcup_{g \in U} g S g^{-1}$$

is dense in $\text{GL}(n)$. 

**Proof:** Let \( \mathcal{N} = U T(n), \mathcal{N}_L = \mathcal{N} \cap L \) and \( B \) be the subgroup of upper triangular matrices in \( GL(n) \). The Bruhat cell \( \mathcal{N}_L w_0 B \), where \( w_0 \) is the element of greatest length in the Weyl group, is dense in \( GL(n) \). The subgroup \( \mathcal{N} \) decomposes into a product \( \mathcal{N} = U : \mathcal{N}_L \). Then \( \mathcal{N}_L w_0 B = U(\mathcal{N}_L w_0 B) \). By direct calculations one can verify \( \mathcal{N}_L w_0 B \subset S \) (see Example 2.11). Then \( \mathcal{N}_L w_0 B \subset US \). Since \( S \) is stable with respect to the right multiplication by \( U \), for each \( g \in U \), we have \( gSg^{-1} = gS \). The subset \( US \) coincides with the subset (1). Therefore (1) is dense in \( GL(n) \).

Denote by \( \pi \) the restriction map \( \mathcal{A} \rightarrow K[S] \). The map \( \pi \) establishes an embedding \( \mathcal{A}^U \hookrightarrow K[S] \) and it is extended to an embedding \( \pi : \mathcal{F}^U \hookrightarrow K(S) \).

Let \( (x_{ij}) \) be the system of standard coordinate functions on \( M \). Define the matrix \( X = (x_{ij})_{i,j=1}^n \). Consider the adjugate matrix

\[
X^* = (x_{ij}^*)_{i,j=1}^n.
\]

We have \( XX^* = X^*X = \det(X)E \).

Define the linear order on the set of pairs \( S \) as follows:

\[
(i_1, j_1) < (i, j) \quad \text{if} \quad j_1 < j, \text{ or } j_1 = j \text{ and } i_1 > i.
\]

Denote \( s_{ij} = \pi(x_{ij}) \) for \((i, j) \in S \).

Our aim is to attach a \( U \)-invariant \( J_{ij} \) in \( K[M] \) to each pair \((i, j) \in S \). Decompose \( S \) into two subsets \( S = S_0 \sqcup S_1 \), where \( S_0 \) consists of all \((i, j) \in S \), which is lying on or above the anti-diagonal, and \( S_1 \) consists of all \((i, j) \in S \), which is lying strictly below the anti-diagonal.

Let \((i, j) \in S_0 \). Then \((i, j)\) is lying on or above the anti-diagonal. In this case \( i + j \leq n + 1 \) (equivalent to \( i \leq j' \)). Since \( i \leq j' \), the number \( i \) is less than any number from the segment \([j' + 1, n] \). Observe that \([1, j - 1]' = [j' + 1, n] \). Consider the system of \( j \) rows \( \{i\} \sqcup [j' + 1, n] \).

For each \((i, j) \in S_0 \), we define the polynomial \( J_{ij} \) as the minor of the matrix \( X \) defined by the systems of columns \([1, j] \) and rows \( \{i\} \sqcup [j' + 1, n] \).

Let \((i, j) \in S_1 \). Then \((i, j)\) is lying below the anti-diagonal. In this case \( i + j > n + 1 \) (equivalent to \( j > i' \)). Then \( j = i' + (j - i') \). Consider the \((j \times j)\)-matrix

\[
Y_{ij} = \begin{pmatrix}
X_{i,j} & \cdots & \cdots \\
\cdots & \cdots & \cdots \\
X_{j^{*}i,j} & \cdots & \cdots
\end{pmatrix},
\]

where \( X_{i,j} \) is the submatrix \( X \) associated with the columns \([1, j] \) and the last \( i' \) rows, \( X_{j^{*}i,j} \) is the submatrix of \( X^* \) associated with the columns \([1, j] \) and the last \( j - i' \) rows. For each \((i, j) \in S_1 \), we define

\[
J_{ij} = \det Y_{ij}.
\]

**Proposition 2.2:** The polynomials \( \{ J_{ij} : (i, j) \in S \} \) are \( U \)-invariants.

**Proof:** Analogously to [1, Proposition 1].

**Proposition 2.3:** \( J_{ij} \neq 0 \) for each \((i, j) \in S \).
Proof: The statement is obvious for \((i, j) \in S_0\). Let \((i, j) \in S_1\). Consider the matrix \(A = (a_{st})\), where \(a_{st} = 1\) if \((s, t)\) lies on the anti-diagonal or \((s, t) = (i + k, j - k)\) for \(k \in [0, n - i]\), otherwise \(a_{st} = 0\).

Let us show that \(J_{ij}(A) \neq 0\). By definition, \(J_{ij}(A) = \det Y_{ij}(A)\). The matrix \(A\) is lower triangular with respect to anti-diagonal, and the submatrix \(X_{i', j}'(A)\) has the form

\[
X_{i', j}'(A) = \begin{pmatrix}
\times & \cdots & \times & \times & \cdots & 1 \\
\vdots & & \ddots & & \vdots \\
\times & \cdots & \times & 1 & \cdots & 0
\end{pmatrix}.
\]

The adjugate matrix \(A^*\) is upper triangular with respect to the anti-diagonal, and the submatrix \(X_{j-i, j}^*(A)\) has the form

\[
X_{j-i, j}^*(A) = \begin{pmatrix}
\times & \cdots & 1 & 0 & \cdots & 0 \\
\vdots & & \ddots & \vdots & \vdots \\
1 & \cdots & 0 & 0 & \cdots & 0
\end{pmatrix}.
\]

Then

\[
J_{ij}(A) = \begin{vmatrix}
\times & \cdots & \times & \times & \cdots & 1 \\
\vdots & & \ddots & & \vdots \\
\times & \cdots & \times & 1 & \cdots & 0 \\
\times & \cdots & 1 & 0 & \cdots & 0 \\
\vdots & & \ddots & \vdots & \vdots \\
1 & \cdots & 0 & 0 & \cdots & 0
\end{vmatrix} = \pm 1 \neq 0.
\]

For instance, in Example 2.11, for \(i = j = 4\) we have

\[
A = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 0
\end{pmatrix}, \quad A^* = \begin{pmatrix}
\times & \times & \times & \times & 1 \\
\times & \times & \times & 1 & 0 \\
\times & \times & 1 & 0 & 0 \\
\times & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0
\end{pmatrix},
\]

\[
J_{44}(A) = \begin{vmatrix}
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 \\
\times & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{vmatrix} = 1 \neq 0.
\]

Corollary 2.4: \(\pi(J_{ij}) \neq 0\) for each \((i, j) \in S\).

The proof follows from the fact that \(\pi\) is an embedding of \(\mathcal{F}^U\) to \(K(S)\).

For any \((n \times n)\)-matrix \(Y\), let \(M_{I, J}(Y)\) denote the minor of \(Y\) with systems of rows \(I\) and columns \(J\).

Lemma 2.5: If \(I\) is a segment and it ends in \(n\), then \(M_{I, J}(X^*)\) is invariant under the right multiplication of \(X^*\) by \(UT(n)\).
**Proof:** If $I$ is a segment and it ends in $n$, then for any $(n \times n)$-matrix $\mathbf{Y}$ and $g \in \text{UT}(n)$, we have $M_{I,J}(g^{*}\mathbf{Y}) = M_{I,J}((\mathbf{X})^{*})$. Therefore

$$M_{I,J}((\mathbf{X})^{*}) = M_{I,J}(g^{*}\mathbf{X}^{*}) = M_{I,J}(\mathbf{X}^{*}).$$

The subset $S$ is stable under the right multiplication by $\text{UT}(n)$. Denote by $K(S)^{\text{UT}(n)}$ the subfield of $\text{UT}(n)$-invariants in $K(S)$.

**Corollary 2.6:** If $I$ is a segment and it ends in $n$, then $\pi(M_{I,J}(\mathbf{X}^{*}))$ belongs to $K(S)^{\text{UT}(n)}$.

**Definition 2.7:** Let $\{\xi_{\alpha} : \alpha \in \mathcal{A}\}$ and $\{\eta_{\alpha} : \alpha \in \mathcal{A}\}$ be two finite systems of free generators of an extension $F$ of the field $K$. Let $\prec$ be a linear order on $\mathcal{A}$. We say that the second system of generators is obtained from the first one by a triangular transformation if each $\eta_{\alpha}$ can be presented in the form

$$\eta_{\alpha} = \phi_{\alpha} \xi_{\alpha} + \psi_{\alpha},$$

where $\phi_{\alpha} \neq 0$ and $\phi_{\alpha}, \psi_{\alpha}$ belong to the subfield generated by $\{\xi_{\beta} : \beta \prec \alpha\}$.

**Remark:** Using the induction method, it is easy to prove that if $\{\xi_{\alpha} : \alpha \in \mathcal{A}\}$ is a system of free generators of a field $F$ and the other system $\{\eta_{\alpha} : \alpha \in \mathcal{A}\}$ is linked with the first one by formulas (3), then it also freely generates $F$.

Denote by $P_{0}$ the subfield of $\mathcal{F}^{U}$ generated by $\{J_{ij} : (i,j) \in S_{0}\}$.

**Proposition 2.8:**

1. The subfield $K(S)^{\text{UT}(n)}$ is freely generated by the elements

$$\{\pi(J_{ij}) : (i,j) \in S_{0}\}.$$

2. The map $\pi$ establishes an isomorphism of the subfield $P_{0}$ of $\mathcal{F}^{U}$ to $K(S)^{\text{UT}(n)}$.

3. The subfield $P_{0}$ is freely generated by the elements

$$\{J_{ij} : (i,j) \in S_{0}\}.$$

**Proof:** Recall that $\pi : \mathcal{F}^{U} \hookrightarrow K(S)$ is an embedding. Then $\pi(P_{0})$ is a subfield of $K(S)$ generated by $\pi(J_{ij})$, $(i,j) \in S_{0}$.

First let us show that $\pi(J_{ij})$, $(i,j) \in S_{0}$, belongs to $K(S)^{\text{UT}(n)}$. Observe that $J_{ij}$ is a minor $M_{I,J}(\mathbf{X})$ where the system of columns $j$ is a segment beginning in 1. This implies that $J_{ij}$ is invariant under right-multiplication of $\mathbf{X}$ by $\text{UT}(n)$. Since $S \cdot \text{UT}(n) \subseteq S$, the polynomial $\pi(J_{ij})$, $(i,j) \in S_{0}$, is invariant under right-multiplication of $S$ by $\text{UT}(n)$. This proves the statement. We have $\pi(P_{0})$ is a subfield of $K(S)^{\text{UT}(n)}$.

Denote by $S_{0}$ the subset of $S$ that consists of matrices $(s_{ij})$ with $s_{ij} = 0$ for all $(i,j) \in S_{1}$. The subset $S_{0} \cdot \text{UT}(n)$ (that equals to $\mathcal{N}_{I,w_{0}}B$ from the proof of Proposition 2.1) is dense in $S$. Consider the restriction map $\pi_{0}$ from $K[S]$ to $K[S_{0}]$. Since $S_{0} \cdot \text{UT}(n)$ is dense in $S$, the map $\pi_{0}$ establishes the embedding $\pi_{0} : K(S)^{\text{UT}(n)} \hookrightarrow K(S_{0})$. 

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**LINEAR AND MULTILINEAR ALGEBRA**
By direct calculations, \( \pi_0 \pi (\mathcal{I}_{ij}) \) equals to
\[
\pm s_{n1} s_{n-1,2} \ldots s_{j+1,j-1} s_{ij}.
\]

For instance, in Example 2.11
\[
S = \begin{pmatrix}
0 & 0 & 0 & s_{15} \\
0 & 0 & s_{23} & s_{24} & s_{25} \\
0 & 0 & s_{33} & s_{34} & s_{35} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
s_{41} & s_{42} & s_{43} & s_{44} & s_{45} \\
s_{51} & s_{52} & s_{53} & s_{54} & s_{55}
\end{pmatrix}, \quad S_0 = \begin{pmatrix}
0 & 0 & 0 & s_{15} \\
0 & 0 & s_{23} & s_{24} & 0 \\
0 & 0 & s_{33} & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
s_{41} & s_{42} & 0 & 0 & 0 \\
s_{51} & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

In the case \( i = 2, j = 3 \), we obtain
\[
\pi_0 \pi (\mathcal{I}_{23}) = \begin{vmatrix}
0 & 0 & s_{23} \\
s_{41} & s_{42} & 0 \\
s_{51} & 0 & 0
\end{vmatrix} = -s_{51} s_{42} s_{23}.
\]

The system of elements \( \{\pi_0 \pi (\mathcal{I}_{ij}) : (i,j) \in S_0\} \) is obtained by a triangular transformation with respect to the linear order \( \prec \) (see (2)) from the system \( \{s_{ij} : (i,j) \in S_0\} \) of generators of the field \( K(S_0) \). Then the system \( \{\pi_0 \pi (\mathcal{I}_{ij}) : (i,j) \in S_0\} \) freely generates \( K(S_0) \). It follows that the embedding \( \pi_0 : K(S)^{\text{UT}(n)} \hookrightarrow K(S_0) \) is an isomorphism, the system \( \{\pi (\mathcal{I}_{ij}) : (i,j) \in S_0\} \) freely generates \( K(S)^{\text{UT}(n)} \), and \( \pi (\mathcal{P}_0) = K(S)^{\text{UT}(n)} \). This proves statements (1) and (2). The statement (3) follows from (1) and (2).

**Theorem 2.9:** The system of polynomials \( \{\mathcal{I}_{ij} : (i,j) \in S\} \) freely generates over \( K \) the field of invariants \( \mathcal{F}_U \) for the group \( \text{GL}(n) \).

**Proof:** *Item 1.* Let us show that the field \( K(S) \) is freely generated over \( K(S)^{\text{UT}(n)} \) by the system of elements \( \{s_{ij} : (i,j) \in S_1\} \).

For each \( (i,j) \in S_0 \), we expand the minor \( \mathcal{I}_{ij} \) along its first row. We get \( \mathcal{I}_{ij} = A_{ij} \cdot x_{ij} + \Psi_{ij} \). The cofactor \( A_{ij} \) and \( \Psi_{ij} \), which is the sum of the other cofactors multiplied by the respective entries, both are expressions in the \( x_{st}, (s,t) \prec (i,j) \). Therefore, the system of elements
\[
\{\pi (\mathcal{I}_{ij}) : (i,j) \in S_0\} \cup \{s_{ij} : (i,j) \in S_1\}
\]
is obtained by a triangular transformation from the system of free generators \( \{s_{ij} : (i,j) \in S\} \) of the field \( K(S) \). This proves the statement of Item 1.

*Item 2.* Let us show that the field \( \mathcal{F}_U \) is freely generated over the subfield \( \mathcal{P}_0 \) by the system of elements \( \{\mathcal{I}_{ij} : (i,j) \in S_1\} \). By Proposition 2.8, it is sufficient to prove that the system of elements \( \{\pi (\mathcal{I}_{ij}) : (i,j) \in S_1\} \) is obtained from \( \{s_{ij} : (i,j) \in S_1\} \) by a triangular transformation over the field \( K(S)^{\text{UT}(n)} \). We prove using the induction method that the formula of type (3) is valid for the linear order (2).

For the smallest element \( (i,j) = (n,1) \), we have \( \mathcal{I}_{ij} = x_{n1} \), and \( \pi (\mathcal{I}_{ij}) = s_{n1} \) belongs to \( K(S)^{\text{UT}(n)} \).

Assume that the statement is true for all elements \( \prec (i,j) \). Let us prove for \( (i,j) \in S_1 \). Denote by \( K(S)' \) the subfield of \( K(S) \) generated over \( K(S)^{\text{UT}(n)} \) by \( s_{km}, (k,m) \in S_1, (k,m) \prec (i,j) \).
(i, j). By the induction assumption, \(K(S)\)' is generated over \(K(S)^{\text{UT}(n)}\) by \(\pi(J_{km})\), \((k, m) \in S_1, (k, m) < (i, j)\).

It is sufficient to show that \(\pi(J_{ij})\) can be presented in the form

\[
\pi(J_{ij}) = A_1 s_{ij} + A_2,
\]

where \(A_1, A_2 \in K(S)'\) and \(A_1 \neq 0\).

Expand the determinant \(J_{ij}\) along its first row. We get

\[
J_{ij} = \pm M_{ij} \cdot x_{ij} + \Psi_{ij}, \quad \text{where} \quad \Psi_{ij} = \sum_{r=1}^{j-1} \pm x_{ir} M_{ir}.
\]

Here \(x_{i1}, \ldots, x_{ij}\) are the elements of the first row of the matrix \(\Upsilon_{ij}\), and \(M_{i1}, \ldots M_{ij}\) are their complementary minors. For instance, in Example 2.11 for \(i = 5, j = 3\), we have

\[
J_{44} = \begin{vmatrix}
    x_{41} & x_{42} & x_{43} & x_{44} \\
    x_{51} & x_{52} & x_{53} & x_{54} \\
    x_{41}^* & x_{42}^* & x_{43}^* & x_{44}^* \\
    x_{51}^* & x_{52}^* & x_{53}^* & x_{54}^*
\end{vmatrix} = -x_{44} \begin{vmatrix}
    x_{51} & x_{52} & x_{53} \\
    x_{41}^* & x_{42}^* & x_{43}^* \\
    x_{51}^* & x_{52}^* & x_{53}^*
\end{vmatrix} + \Psi_{44},
\]

where \(\Psi_{44} = x_{41} \begin{vmatrix}
    x_{52} & x_{53} & x_{54} \\
    x_{42}^* & x_{43}^* & x_{44}^* \\
    x_{52}^* & x_{53}^* & x_{54}^*
\end{vmatrix} - x_{42} \begin{vmatrix}
    x_{51} & x_{52} & x_{53} \\
    x_{41}^* & x_{42}^* & x_{43}^* \\
    x_{51}^* & x_{52}^* & x_{53}^*
\end{vmatrix} + x_{43} \begin{vmatrix}
    x_{51} & x_{52} & x_{54} \\
    x_{41}^* & x_{42}^* & x_{44}^* \\
    x_{51}^* & x_{52}^* & x_{54}^*
\end{vmatrix}.
\]

Restricting the above equality to \(S\), we obtain

\[
\pi(J_{ij}) = \pm \pi(M_{ij}) \cdot s_{ij} + \pi(\Psi_{ij}). \quad (4)
\]

If \(i < n\), then \(M_{ij} = J_{i+1,j-1}\) and \(M_{ij} \neq 0\) (see Proposition 2.3). According to induction assumption, \(\pi(J_{i+1,j-1})\) belongs to the subfield \(K(S)'\).

In the case \(i = n\), the polynomial \(M\) coincides with the minor \(M_{I,J}(X^*)\) for \(I = [1, j-1]\) and \(J = [j+1, n]\). The segment \(I\) ends in \(n\), therefore \(\pi(M_{I,J}(X^*)) \in K(S)^{\text{UT}(n)}\) (see Corollary 2.6).

In both cases \(i < n\) and \(i = n\), the element \(A_1\), that is equal to \(\pi(M_{ij})\), is a non-zero element of \(K(S)'\).

Denote

\[
A_2 = \pi(\Psi_{ij}) = \sum_{r=1}^{j-1} \pm s_{ir} \pi(M_{ir}).
\]

The elements \(s_{i1}, \ldots, s_{ij-1}\) are in \(K(S)'\). For each \(1 \leq r \leq j - 1\), the complementary minor \(M_{ir}\) is the determinate of the matrix \(\Upsilon_{ij}^{(r)}\) which is obtained from \(\Upsilon_{ij}\) by deleting the first row \(x_{i1}, \ldots, x_{ij}\) and the \(r\)th column. Applying the Laplace expansion along the first \(i' - 1\) rows of \(\Upsilon_{ij}^{(r)}\), we get

\[
M_{ir} = \det \Upsilon_{ij}^{(r)} = \sum \pm M_{r1,j1}(X) M_{r2,j2}(X^*),
\]

where \(M_{r1,j1}(X)\) (respectively \(M_{r2,j2}(X^*)\) are minors with systems of rows \(I_1\) (respectively \(I_2\)) and columns \(J_1\) (respectively \(J_2\)). The entries of the first \(i' - 1\) rows of \(\Upsilon_{ij}^{(r)}\) have the form \(x_{km}, (k, m) < (i, j)\). This implies the elements \(\pi(M_{I_1,J_1}(X))\) belong to \(K(S)'\).
The system of rows $I_2$ form a segment ending in $n$. By Corollary 2.6, the elements $\pi(M_{I_2}J_2(X^*))$ belong to $K(S)^{\mathrm{UT}(n)}$. We obtain $A_2 \in K(S')$. This proves the statement of Item 2.

Item 3. The field $\mathcal{F}^U$ is freely generated over the subfield $\mathcal{F}_0$ by the system of elements $\{J_{ij} : (i, j) \in S_1 \}$ (see Item 2). By Proposition 2.8, $\mathcal{F}_0$ is freely generated by $\{J_{ij} : (i, j) \in S_0 \}$. We conclude that $\mathcal{F}^U$ is freely generated over $K$ by the system of elements $\{J_{ij} : (i, j) \in S \}$. □

**Corollary 2.10:** The system of polynomials $\{J_{ij} : (i, j) \in S, (i, j) \neq (1, n) \}$ freely generates over $K$ the field of invariants $\mathcal{F}^U$ for the group $\text{SL}(n)$.

**Example 2.11:** Let $n = 5$ and the parabolic subgroup $P$ be defined by the partition $\{1, 5\} = \{1\} \sqcup \{2, 3\} \sqcup \{4, 5\}$. In the $(5 \times 5)$-table $\mathbb{P}$, by the symbol ‘$\times$’ (respectively, by the symbol ‘$\ast$’), we mark the cells that correspond to the subgroup $L$ (respectively, to the subgroup $U$).

$$
\mathbb{P} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad S = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

$$
S^* = \begin{pmatrix}
\ast_1 & \ast_2 & \ast_3 & \ast_4 & \ast_5 \\
\ast_1 & \ast_2 & \ast_3 & \ast_4 & \ast_5 \\
\ast_1 & \ast_2 & \ast_3 & \ast_4 & \ast_5 \\
\ast_1 & \ast_2 & \ast_3 & \ast_4 & \ast_5 \\
\ast_1 & \ast_2 & \ast_3 & \ast_4 & \ast_5
\end{pmatrix}
$$

$$
\mathcal{N}_L = \text{UT}(5) \cap L, \quad \mathcal{N}_L = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

$$
\mathcal{N}_L \mathcal{W}_0 B = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
0 & 0 & 0 & 0 & \ast \\
0 & 0 & 0 & 0 & \ast \\
0 & 0 & 0 & 0 & \ast \\
0 & 0 & 0 & 0 & \ast \\
0 & 0 & 0 & 0 & \ast
\end{pmatrix} \in S.
$$

The generators $J_{ij}, (i, j) \in S$, and their restrictions $\pi(J_{ij})$ to $S$ have the following form:

$$
J_{51} = x_{51}, \quad \pi(J_{51}) = s_{51}, \quad J_{41} = x_{41}, \quad \pi(J_{41}) = s_{41},
$$

$$
J_{52} = \begin{pmatrix}
x_{51} & x_{52} \\
x_{51} & x_{52}
\end{pmatrix}, \quad \pi(J_{52}) = \begin{pmatrix}s_{51} & s_{52} \\
s_{51} & 0
\end{pmatrix}, \quad J_{42} = \begin{pmatrix}x_{41} & x_{42} \\
x_{41} & x_{42}
\end{pmatrix}, \quad \pi(J_{42}) = \begin{pmatrix}s_{41} & s_{42} \\
s_{51} & s_{52}
\end{pmatrix},
$$

$$
J_{53} = \begin{pmatrix}
x_{51} & x_{52} & x_{53} \\
x_{41} & x_{42} & x_{43} \\
x_{51} & x_{52} & x_{53}
\end{pmatrix}, \quad \pi(J_{53}) = \begin{pmatrix}s_{51} & s_{52} & s_{53} \\
s_{41} & s_{42} & s_{43} \\
s_{51} & 0 & 0
\end{pmatrix}.
$$
\[ \mathcal{J}_{43} = \begin{bmatrix} x_{41} & x_{42} & x_{43} \\ x_{51} & x_{52} & x_{53} \\ \ast_{41} & \ast_{42} & \ast_{43} \end{bmatrix}, \quad \pi(\mathcal{J}_{43}) = \begin{bmatrix} s_{41} & s_{42} & s_{43} \\ s_{51} & s_{52} & s_{53} \\ s_{\ast 41} & s_{\ast 42} & s_{\ast 43} \end{bmatrix} ; \quad \mathcal{J}_{33} = \begin{bmatrix} x_{31} & x_{32} & x_{33} \\ x_{51} & x_{52} & x_{53} \end{bmatrix}, \quad \pi(\mathcal{J}_{33}) = \begin{bmatrix} s_{41} & s_{42} & s_{43} \\ s_{51} & s_{52} & s_{53} \end{bmatrix}, \quad \mathcal{J}_{23} = \begin{bmatrix} x_{21} & x_{22} & x_{23} \\ x_{51} & x_{52} & x_{53} \end{bmatrix}, \quad \pi(\mathcal{J}_{23}) = \begin{bmatrix} s_{41} & s_{42} & s_{43} \\ s_{51} & s_{52} & s_{53} \end{bmatrix} \]

\[ \mathcal{J}_{44} = \begin{bmatrix} x_{41} & x_{42} & x_{43} & x_{44} \\ x_{51} & x_{52} & x_{53} & x_{54} \\ \ast_{41} & \ast_{42} & \ast_{43} & \ast_{44} \\ \ast_{51} & \ast_{52} & \ast_{53} & \ast_{54} \end{bmatrix}, \quad \pi(\mathcal{J}_{44}) = \begin{bmatrix} s_{41} & s_{42} & s_{43} & s_{44} \\ s_{51} & s_{52} & s_{53} & s_{54} \\ s_{\ast 41} & s_{\ast 42} & s_{\ast 43} & s_{\ast 44} \\ s_{\ast 51} & s_{\ast 52} & s_{\ast 53} & s_{\ast 54} \end{bmatrix} \]

\[ \mathcal{J}_{34} = \begin{bmatrix} x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \\ x_{51} & x_{52} & x_{53} & x_{54} \\ \ast_{31} & \ast_{32} & \ast_{33} & \ast_{34} \end{bmatrix}, \quad \pi(\mathcal{J}_{34}) = \begin{bmatrix} s_{31} & s_{32} & s_{33} & s_{34} \\ s_{41} & s_{42} & s_{43} & s_{44} \\ s_{51} & s_{52} & s_{53} & s_{54} \\ s_{\ast 31} & s_{\ast 32} & s_{\ast 33} & s_{\ast 34} \end{bmatrix} \]

\[ \mathcal{J}_{24} = \begin{bmatrix} x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \\ x_{51} & x_{52} & x_{53} & x_{54} \end{bmatrix}, \quad \pi(\mathcal{J}_{24}) = \begin{bmatrix} s_{21} & s_{22} & s_{23} & s_{24} \\ s_{31} & s_{32} & s_{33} & s_{34} \\ s_{41} & s_{42} & s_{43} & s_{44} \\ s_{51} & s_{52} & s_{53} & s_{54} \end{bmatrix} \]
\[ J_{25} = \begin{vmatrix} \chi_{45} \\ \vdots \\ \chi_{1,5} \end{vmatrix}, \quad \pi(J_{25}) = \begin{vmatrix} 0 & 0 & s_{23} & s_{24} & s_{25} \\ 0 & 0 & s_{33} & s_{34} & s_{35} \\ s_{41} & s_{42} & s_{43} & s_{44} & s_{45} \\ s_{51} & s_{52} & s_{53} & s_{54} & s_{55} \\ s_{s1} & 0 & 0 & 0 & 0 \end{vmatrix}, \]

\[ J_{15} = \det X, \quad \pi(J_{15}) = \det S. \]

3. Fields of invariants for radicals of parabolic subgroups in the orthogonal and symplectic subgroups

Let \( K \) be a field of characteristic zero, \( G \) be the orthogonal or symplectic group over \( K \), and \( P \) be a parabolic subgroup. There is a decomposition of the subgroup \( P \) into a product \( P = LU \) of a Levi subgroup \( L \) and the unipotent radical \( U \). Let \( A = K[G] \) be the ring of regular functions on \( G \), and \( F = K(G) \) be the field of rational functions on \( G \). In this section, we construct a system of free generators in the field of invariants \( F^U \) with respect to the action of \( U \) on \( G \) by conjugation.

We introduce necessary notation. For each matrix \( A \), we denote by \( A^t \) (respectively, \( A^\sigma \)) its transpose (respectively, its transpose with respect to the anti-diagonal). Let \( \mathbb{I}_N \) be the matrix of size \( N \times N \) with ones on the anti-diagonal and zeros on other places. By definition, the orthogonal group \( O(N) \) is the group of all \( g \in \text{GL}(N) \) such that \( g^t \mathbb{I}_N g = \mathbb{I}_N \).

The symplectic group \( \text{Sp}(N) \), where \( N = 2n \), is defined analogously by the matrix

\[ \mathbb{J}_{2n} = \begin{pmatrix} 0 & -\mathbb{I}_n \\ \mathbb{I}_n & 0 \end{pmatrix}. \]

Consider a decomposition of the integer segment \([1, N]\) into a system of consecutive segments \([1, n] = I_1 \sqcup I_2 \sqcup \cdots \sqcup I_\ell \). Denote \( n_k = |I_k| \). Suppose that this decomposition is symmetric with respect to the centre of the segment, i.e. \( I_k^t = I_{\ell-k+1} \).

We introduce the notations \( \ell_0 = \lfloor \ell/2 \rfloor \), \( N_0 = n_1 + \cdots + n_{\ell_0} \). If \( \ell = 2\ell_0 \), then \( N = 2n \) and \( N_0 = n \). In the odd case \( \ell = 2\ell_0 + 1 \), denote \( I_0 = I_{\ell_0+1}, n_0 = n_{\ell_0+1} \). Let \( G_0 \) stand for the group \( O(n_0) \) in the orthogonal case or the group \( \text{Sp}(n_0) \) in the symplectic case.

By this decomposition, we construct the parabolic subgroup \( P = LU \) with the Levi subgroup \( L \) of block-symmetric matrices \( \text{diag}(A_1, \ldots, A_\ell) \), \( A_k \in \text{GL}(n_k) \), \( A_{\ell-k+1} = (A_k^\sigma)^{-1} \) for each \( 1 \leq k \leq \ell_0 \) and \( A_{\ell_0+1} \in G_0 \) (for \( \ell = 2\ell_0 + 1 \)).

If \( \ell = 2\ell_0 \), then each \( g \in P \) has the form

\[ g = \begin{pmatrix} A & 0 \\ 0 & (A^\sigma)^{-1} \end{pmatrix} \begin{pmatrix} E & B \\ 0 & E \end{pmatrix} = \begin{pmatrix} A & AB \\ 0 & (A^\sigma)^{-1} \end{pmatrix}, \]

where \( A \) is a matrix from the parabolic subgroup in \( \text{GL}(N_0) \), which relates to the decomposition \( I_1 \sqcup \cdots \sqcup I_{\ell_0} \) of the segment \([1, N_0] \), and \( B \) is a skew-symmetric (respectively, symmetric) matrix with respect to the anti-diagonal in the orthogonal case (respectively, in the symplectic case).

Consider the case \( \ell = 2\ell_0 + 1 \). Denote \( \mathbb{I}_0 = \mathbb{I}_{N_0} \),

\[ \mathbb{J}_0 = \begin{cases} \mathbb{I}_{n_0}, & \text{in the case } \text{O}(N), \\ \mathbb{J}_{n_0}, & \text{in the case } \text{Sp}(N). \end{cases} \]
Each element \( g \in P \) is presented as a product \( g = g_1g_2 \), where \( g_1 \in L \) and \( g_2 \in U \). Here
\[
g_1 = \begin{pmatrix} A & 0 & 0 \\ 0 & A_0 & 0 \\ 0 & 0 & (A^\sigma)^{-1} \end{pmatrix} \quad \text{and} \quad g_2 = \exp \begin{pmatrix} 0 & V & 0 \\ 0 & 0 & W \\ 0 & 0 & 0 \end{pmatrix} \cdot \exp \begin{pmatrix} 0 & 0 & B \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]
where \( A \) and \( B \) as above, \( V \) is a matrix of size \( N_0 \times n_0 \), \( W = -J_0V^tI_0 \), and \( A_0 \in G_0 \).

It follows
\[
g = \begin{pmatrix} A & 0 & 0 \\ 0 & A_0 & 0 \\ 0 & 0 & (A^\sigma)^{-1} \end{pmatrix} \begin{pmatrix} E & V & \frac{1}{2}VW \\ 0 & E & W \\ 0 & 0 & E \end{pmatrix} \begin{pmatrix} E & 0 & B \\ 0 & E & 0 \\ 0 & 0 & E \end{pmatrix}.
\]

Denote \( S^\circ = S \cap G \), where \( S \) is the subset of \( \text{GL}(N) \) defined in the previous section. Arguing analogously Proposition 2.1, we obtain that the subset
\[
\bigcup_{g \in U} gS^\circ g^{-1}
\]
is dense in \( G \).

The restriction map
\[
\pi : \mathcal{A} \to K[S^\circ]
\]
establishes the embedding of the field of \( U \)-invariants \( \mathcal{F}^U \) into \( K(S^\circ) \).

If \( \ell = 2\ell_0 \), then \( S^\circ \) consists of all matrices of the form
\[
\begin{pmatrix} 0 & S_{12} \\ S_{21} & S_{22} \end{pmatrix} = \begin{pmatrix} 0 & \pm I_0(A^\sigma)^{-1} \\ I_0A & I_0AB \end{pmatrix}.
\]

If \( \ell = 2\ell_0 + 1 \), then \( S^\circ \) consists of all matrices of the form
\[
\begin{pmatrix} 0 & 0 & S_{13} \\ 0 & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \pm I_0(A^\sigma)^{-1} \\ 0 & I_0A & I_0A_0W \\ I_0A & I_0AV & I_0A \left( B + \frac{1}{2}VW \right) \end{pmatrix}.
\]

Consider the set \( S^\circ \) of pairs \((i,j)\) such that

1. \( i > N_0 \) (for \( \ell = 2\ell_0 \)) or \( i > N_0 + n_0 \) (for \( \ell = 2\ell_0 + 1 \)),
2. if \( i \in I_k \) and \( j \in I_m \), then \( k \geq m \),
3. \( i > j \) in the orthogonal case or \( i \geq j \) in the symplectic case.

Similarly to the previous section, let \( S^\circ_0 \) consist of all pairs \((i,j)\) lying on or above the anti-diagonal, respectively, \( S^\circ_1 \) consist of all pairs \((i,j)\) lying below the anti-diagonal. Denote by \( G_0 \) the set \( I_0 \times I_0 \).

**Example 3.1:** Let \( G = \text{Sp}(8) \) and the parabolic subgroup is defined by the decomposition \([1,8] = \{1\} \sqcup \{2,3\} \sqcup \{4,5\} \sqcup \{6,7\} \sqcup \{8\} \). On the \((8 \times 8)\)-tables below, we mark by
the symbol ‘×’ the cells \((i,j)\) belonging to \(S^\circ\) (respectively, belonging to \(S_0^\circ, S_1^\circ, G_0\)). We have

\[
S^\circ = \begin{pmatrix}
\ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
\times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times \\
\end{pmatrix}, \quad G_0 = \begin{pmatrix}
\ldots & \times & \times & \ldots \\
\ldots & \times & \times & \ldots \\
\times & \times & \times & \times \\
\times & \times & \times & \times \\
\times & \times & \times & \times \\
\end{pmatrix},
\]

\[
S_0^\circ = \begin{pmatrix}
\ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
\times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times \\
\end{pmatrix}, \quad S_1^\circ = \begin{pmatrix}
\ldots & \times & \times & \times & \ldots \\
\ldots & \times & \times & \times & \ldots \\
\times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times \\
\end{pmatrix}.
\]

We attach to each \((i,j) \in G_0\) the rational function \(P_{ij}\) on \(G\) by the formula

\[P_{ij} = \frac{M_{ij}}{M_0},\]

where \(M_0\) is the minor of order \(N_0\) with the systems of rows \(I_{\ell_0+2} \sqcup \cdots \sqcup I_\ell\) and columns \(I_1 \sqcup \cdots \sqcup I_{\ell_0}\), and \(M_{ij}\) is the minor obtained from \(M_0\) by adding \(i\)th row and \(j\)th column.

For instance, in Example 3.1, if \((i,j) = (4,5)\), the minor \(M_0\) has rows \(\{6,7,8\}\) and columns \(\{1,2,3\}\), and the minor \(M_{45}\) has rows \(\{4,6,7,8\}\) and columns \(\{1,2,3,5\}\).

It is easy to see that the minors \(M_{ij}\) and \(M_0\) are invariants with respect to the action of \(U\) on \(G\) by conjugation. Therefore, \(P_{ij}\) is also an invariant.

Consider the subfield \(\mathcal{F}_0\) in \(\mathcal{F}^U\) generated by the system \(\{P_{ij} : (i,j) \in G_0\}\). One can treat \(G_0\) as a factor of the parabolic subgroup \(P\). This makes it possible to consider the field of rational functions \(K(G_0)\) as a subfield in the field \(K(G)\) and also as a subfield in \(K(S)\). The restriction map \(\pi\) establishes an isomorphism of the subfield \(\mathcal{F}_0\) in \(\mathcal{F}^U\) onto the subfield \(K(G_0)\) in \(K(S)\).

For each pair \((i,j) \in S^\circ\), we denote by \(s_{ij}\) the restriction of the matrix element \(x_{ij}\) to \(S^\circ\).

**Lemma 3.2:** The field \(K(S^\circ)\) is freely generated over the subfield \(K(G_0)\) by the system of matrix elements \(\{s_{ij} : (i,j) \in S^\circ\}\).

**Proof:** Let us prove for \(\ell = 2\ell_0 + 1\). The case \(\ell = 2\ell_0\) is treated similarly. We consider the system of matrix elements

\[\{a_{ij}, v_{ij}, b_{ij}\}\]

from (5), where \(A = (a_{ij})\), \(V = (v_{ij})\) and \(\{b_{ij}\}\) are entries of \(B\) lying above the anti-diagonal in the case of \(O(N)\) (respectively, on or above the anti-diagonal in the case of \(Sp(N)\)).
We treat these matrix elements as rational functions on \( S^\circ \). The formula (5) implies that the field \( K(S^\circ) \) is freely generated over the subfield \( K(G_0) \) by the system \( \{a_{ij}, v_{ij}, b_{ij}\} \). The subfield generated by the matrix elements from \( S_{31} \) and \( S_{32} \) coincides with the subfield generated by \( \{a_{ij}, v_{ij}\} \). Recall that \( S_{33} = \mathbb{I}_0A(B + \frac{1}{2}VW) \), where \( A \) is a matrix from the parabolic subgroup in \( \text{GL}(N_0) \) defined by the decomposition \( I_1 \sqcup \cdots \sqcup I_{\ell_0} \) of the segment \([1, N_0]\), \( B \) is the skew-symmetric (respectively, symmetric) matrix with respect to the anti-diagonal in the orthogonal case (respectively, in the symplectic case), and \( W = -\mathbb{I}_0 V^t \mathbb{I}_0 \). Rewrite \( S_{33} \) in the form

\[
S_{33} = \mathbb{I}_0A\mathbb{I}_0 \cdot C + \frac{1}{2}\mathbb{I}_0AVW,
\]

where \( C = \mathbb{I}_0B \) is a skew-symmetric matrix in the orthogonal case and symmetric in the symplectic case. Consider the system of matrix elements \( \{c_{ij}\} \) from \( C \) lying below the diagonal in the case \( \text{O}(N) \) (respectively, on or below the diagonal in the case of \( \text{Sp}(N) \)). The system \( \{c_{ij}\} \) coincides with the system of \( \{b_{ij}\} \) defined above. Observe that \( \mathbb{I}_0A\mathbb{I}_0 \) is a lower block-triangular matrix defined by the partition of the segment \([1, N_0]\) into consecutive segments of sizes \( n_{\ell_0}, \ldots, n_1 \). Present the matrix \( \mathbb{I}_0A\mathbb{I}_0 \) as a product of two matrices with rational entries \( \mathbb{I}_0A\mathbb{I}_0 = RQ \), where \( Q \) is a lower triangular matrix and \( R \) is an upper unitriangular matrix. Then the elements \( s_{ij}, (i, j) \in S^\circ \) from \( S_{33} \) are obtained from \( \{c_{ij}\} \) by a composition of two triangular transformations:

- \( C \rightarrow QC \) is a triangular transformation with respect to the linear order: \( (i_1, j_1) \) is less than \( (i, j) \) if \( j_1 < j \) or \( j_1 = j, i_1 < i \),
- \( QC \rightarrow RQC + \frac{1}{2}\mathbb{I}_0AVW \) is a triangular transformation with respect to the linear order \( < \) from (2).

The system \( \{s_{ij} : (i, j) \in S^\circ\} \) is obtained from \( \{a_{ij}, v_{ij}, b_{ij}\} \) by the composition of two triangular transformations. \hfill \blacksquare

Denote by \( \mathcal{J}_{ij}^\circ \) the restriction of the determinant \( \mathcal{J}_{ij} \) to the orthogonal or symplectic group \( G \).

**Proposition 3.3:** \( \mathcal{J}_{ij}^\circ \neq 0 \) for each \( (i, j) \in S^\circ \).

**Proof:** For \( (i, j) \in \mathbb{S}_0^\circ \) the statement is obvious. Consider the case \( (i, j) \in \mathbb{S}_1^\circ \). Define the lower triangular with respect to the anti-diagonal matrix \( A = (a_{st}) \in S^\circ \) that has nonzero entries on the anti-diagonal, has non-degenerate \( (i', i') \)-block defined by the system of last \( i' \) rows and the columns \([j - i' + 1, j]\), and \( a_{st} = 0 \) on other places. Then the determinant \( \mathcal{J}_{ij}^\circ (A) \) has the form

\[
\begin{vmatrix}
A_{11} & A_{12} \\
A_{21} & 0
\end{vmatrix},
\]

where \( A_{12} \) is the defined above non-degenerate \( (i', i') \)-block, and \( A_{21} \) is an upper triangular block with non-zeros on the anti-diagonal. Therefore \( \mathcal{J}_{ij}^\circ (A) \neq 0 \). \hfill \blacksquare

**Corollary 3.4:** \( \pi (\mathcal{J}_{ij}^\circ) \neq 0 \) for each \( (i, j) \in S^\circ \).
Recall that the subfield $F_0$ in $F^U$ is generated by $\{P_{ij} : (i,j) \in G_0\}$, and it is isomorphic to $K(G_0)$.

**Theorem 3.5:** For the orthogonal or symplectic group $G$, the system of polynomials $\{J_{ij}^\circ : (i,j) \in S^\circ\}$ freely generates over $F_0$ the field of invariants $F^U$.

**Proof:** Consider the extension $P_0$ generated over $F_0$ by

$$\{\pi(J_{ij}) : (i,j) \in S^\circ_0\}.$$ 

Let $Q$ be the parabolic subgroup in $G$ defined by the partition of the segment $[1,N]$ into the system of consecutive segments of sizes $(1, \ldots, 1, n_0, 1, \ldots, 1)$, and $U(Q)$ be its unipotent radical. The group $U(Q)$ acts on $S^\circ$ by right multiplication. Similar to the proof of Proposition 2.8, one can show that $\pi$ established isomorphism of the subfield $P_0$ onto the subfield of invariants in $K(S^\circ)$ under the right multiplication by $U(Q)$. It follows from the proof of Theorem 2.9 that the system of polynomials $\{\pi(J_{ij}^\circ) : (i,j) \in S^\circ\}$ is obtained by a triangular transformation from the system $\{s_{ij} : (i,j) \in S^\circ\}$. Applying Lemma 3.2 we conclude the proof. ■

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