Lévy Processes:
Hitting Time, Overshoot and Undershoot
I - Functional Equations

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Manuscript no. P660 submitted to SPA, October 2004

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Abstract

Let \((X_t, t \geq 0)\) be a Lévy process started at 0, with Lévy measure \(\nu\), and \(T_x\) the first hitting time of level \(x > 0\): 
\[
T_x := \inf \{ t \geq 0; X_t > x \}.
\]
Let \(F(\theta, \mu, \rho, .)\) be the joint Laplace transform of \((T_x, K_x, L_x)\):
\[
F(\theta, \mu, \rho, x) := E(e^{-\theta T_x - \mu K_x - \rho L_x} 1_{\{T_x < +\infty\}}),
\]
where \(\theta \geq 0, \mu \geq 0, \rho \geq 0, x \geq 0, K_x := X_{T_x} - x\) and \(L_x := x - X_{T_x-}\).

If \(\nu(\mathbb{R}) < +\infty\) and \(\int_1^{+\infty} e^{sy} \nu(dy) < +\infty\) for some \(s > 0\), then we prove that \(F(\theta, \mu, \rho, .)\) is the unique solution of an integral equation and has a subexponential decay at infinity when \(\theta > 0\) or \(\theta = 0\) and \(E(X_1) < 0\). If \(\nu\) is not necessarily a finite measure but verifies \(\int_{-\infty}^{-1} e^{-sy} \nu(dy) < +\infty\) for any \(s > 0\), then the \(x\)-Laplace transform of \(F(\theta, \mu, \rho, .)\) satisfies some kind of integral equation. This allows us to prove that \(F(\theta, \mu, \rho, .)\) is a solution to a second integral equation.

Keywords: Lévy processes, ruin problem, hitting time, overshoot, undershoot, asymptotic estimates, functional equation.

AMS 2000 Subject classification: 60E10, 60F05, 60G17, 60G40, 60G51, 60J65, 60J75, 60J80, 60K05.
Introduction

1. Let \((X_t, \ t \geq 0)\) be a Lévy process, right continuous with left limits, started at 0, with Lévy measure \(\nu\). We suppose that \((X_t, \ t \geq 0)\) may be decomposed as follows:
   \[
   X_t = \sigma B_t - c_0 t + J_t \quad t \geq 0, \tag{0.1}
   \]
   where \(c_0 \in \mathbb{R}, \sigma > 0, (B_t, \ t \geq 0)\) is an one-dimensional Brownian motion started at 0, \((J_t, \ t \geq 0)\) is a pure jump Lévy process, independent of \((B_t, \ t \geq 0)\) and \(J_0 = 0\). We will suppose that \(\sigma = 1\).

   Let us introduce the function \(\varphi\) which will play a central role in our study:
   \[
   \varphi(q) := \psi(-q) \quad \text{where} \quad \psi \text{ is the characteristic exponent of } (X_t, \ t \geq 0), \ i.e. \ E(e^{qX_t}) = e^{t\psi(q)}. \tag{0.2}
   \]
   By Lévy-Khintchine formula, we have:
   \[
   \varphi(q) = \frac{q^2}{2} + cq + \int_{\mathbb{R}} (e^{-qy} - 1 + qy1_{\{|y| < 1\}}) \nu(dy). \tag{0.2}
   \]

   Remark if:
   \[
   \int_{\mathbb{R}} |y|1_{\{|y| > 1\}} \nu(dy) < +\infty, \tag{0.3}
   \]
   then \(X_1\) has a finite expectation and:
   \[
   \mathbb{E}(X_1) = -c + \int_{\mathbb{R}} y1_{\{|y| \geq 1\}} \nu(dy). \tag{0.4}
   \]

2. In this paper we are interested in the first hitting time of level \(x > 0\):
   \[
   T_x := \inf\{t \geq 0; X_t > x\}. \tag{0.5}
   \]
   Setting \(Z_t := x - X_t\), then \(T_x := \inf\{t \geq 0; Z_t < 0\}\) is the ruin time to a company whose fortune is modelled by \((Z_t; \ t \geq 0)\).

   We also consider the overshoot \(K_x\), respectively the undershoot \(L_x\):
   \[
   K_x := X_{T_x} - x, \tag{0.6}
   
   L_x := x - X_{T_x^-}. \tag{0.7}
   \]

   The aim of this paper is the study of the joint distribution of \((T_x, K_x, L_x)\).

   Our approach makes appeal to the joint Laplace transform of \((T_x, K_x, L_x)\), namely, for all \(\theta \geq 0, \mu \geq 0, \rho \geq 0, x \geq 0\):
   \[
   F(\theta, \mu, \rho, x) = \mathbb{E}\left(e^{-\theta T_x - \mu K_x - \rho L_x 1_{\{T_x < +\infty\}}}\right). \tag{0.8}
   \]

   If \(\theta = \mu = \rho = 0\),
   \[
   F(0, 0, 0, x) = \mathbb{P}(T_x < +\infty) \tag{0.9}
   \]
   is the well-known ruin probability.
3. In section 1 we study $F(\theta, \mu, \rho, \cdot)$ when $\nu(\mathbb{R}) < +\infty$ (i.e. $(J_t, t \geq 0)$ is a compound Poisson process). Since $(X_t, t \geq 0)$ has a first jump time $\tau_1$, and $(X_{t+\tau_1} - X_{\tau_1}, t \geq 0)$ is distributed as $(X_t, t \geq 0)$, we show in Theorem 1.1 that $F(\theta, \mu, \rho, \cdot)$ verifies the integral equation (1.3). To go further we suppose moreover:

$$\int_{-\infty}^{+\infty} e^{sy} \nu(dy) < +\infty, \quad \text{for some} \ s > 0. \quad (0.10)$$

a) Introducing adapted functional Banach spaces, we establish (cf. Theorem 1.5 and Proposition 1.7) that $F(\theta, \mu, \rho, \cdot)$ is the unique solution of (1.3). Moreover if $\theta > 0$ or $\theta = 0$ and $E(X_1) < 0$, then $F(\theta, \mu, \rho, \cdot)$ has a sub-exponential decay:

$$F(\theta, \mu, \rho, x) \leq Ce^{-\gamma x}, \quad \forall x \geq 0, \quad \text{for some} \ C > 0, \gamma > 0. \quad (0.11)$$

The optimal value of $\gamma$ will be given in (2.5).

Note that if $\theta = 0$ and $E(X_1) \geq 0$, then $F(0,0,0,x) = 1$, hence there is no hope to obtain a sub-exponential decay.

b) Unfortunately the equation (1.3) does not permit to determine $F(\theta, \mu, \rho, \cdot)$ explicitly, but allows to obtain an approximation scheme. Suppose $\theta > 0$ or $\theta = 0$ and $E(X_1) < 0$. We define by induction a sequence of functions $\alpha_n(\theta, \mu, \rho, \cdot)$ verifying (0.11) and strongly approximating $F(\theta, \mu, \rho, \cdot)$:

$$\lim_{n \to +\infty} \left( \sup_{x \geq 0} |(F - \alpha_n)(\theta, \mu, \rho, x)| e^{\gamma x} \right) \leq \xi_n K, \quad (0.12)$$

where $K$ is a constant, and $\xi \in ]0,1[$ and depends on $\theta$ and $\gamma$.

4. It is worth pointing out that the previous analysis is only valid if $\nu(\mathbb{R}) < +\infty$. To remove this assumption, we introduce the Laplace transform $\hat{F}$ of $F$ with respect to the $x$ variable:

$$\hat{F}(\theta, \mu, \rho, q) := \int_{0}^{+\infty} e^{-qx} F(\theta, \mu, \rho, x) dx. \quad (0.13)$$

Since $|F| \leq 1$, $\hat{F}(\theta, \mu, \rho, q)$ is well defined for any $q \in \mathbb{C}$, Re $q > 0$.

Suppose $\nu(\mathbb{R}) < +\infty$ and (0.10). Starting with the integral equation (1.3) satisfied by $F(\theta, \mu, \rho, \cdot)$, we prove that if moreover:

$$\int_{-\infty}^{-1} e^{-qy} \nu(dy) < +\infty, \quad \forall q > 0, \quad (0.14)$$

then $\hat{F}(\theta, \mu, \rho, \cdot)$ verifies some kind of integral equation (identity (2.10) in Theorem 2.1). We observe that (2.10) is still valid when $\nu$ is a Lévy measure.
satisfying (0.10) and (0.14). Using an approximation scheme, it is easy to prove that \( \hat{F}(\theta, \mu, \rho, \cdot) \) verifies (2.10), under (0.10) and (0.14). Moreover the equation (2.10) gives an equation satisfied by the factors of the Wiener-Hopf decomposition of \( \theta + \varphi(-q) \) (see for detail Remark 2.2, 7.).

In the particular case of the support of \( \nu \) is included in \([0, +\infty[ \) (i.e. \((X_t, t \geq 0)\) has only positive jumps) then \( \hat{F}(\theta, \mu, \rho, \cdot) \) is explicit.

5. In section 3 we draw a first important consequence of Theorem 2.1. We show that we can go back to \( F(\theta, \mu, \rho, \cdot) \). A simple modification in (2.10) allows to prove that \( F(\theta, \mu, \rho, \cdot) \) verifies an integro-differential equation (cf. Theorem 3.1) which is new and different from the equation verified by \( F(\theta, \mu, \rho, \cdot) \) when \( \nu(\mathbb{R}) < +\infty \).

6. From equation (2.10), we deduce in [23] two main consequences:

a) If \( \nu \) has finite exponential moments, \( F(\theta, \mu, \rho, \cdot) \) has the following expansion:

\[
F(\theta, \mu, \rho, x) = C_0(\theta, \mu, \rho) e^{-\gamma_0(\theta)x} + \sum_{i=1}^{p} a_i \left( C_i(\theta, \mu, \rho, x) e^{-\gamma_i(\theta)x} + \overline{C_i}(\theta, \mu, \rho, x) e^{-\gamma_i(\theta)x} \right) + O(e^{-Bx}),
\]

where \( C_0(\theta, \mu, \rho) \) is a positive real number, \( C_1(\theta, \mu, \rho, x), \cdots, C_p(\theta, \mu, \rho, x) \) are \( x \)-polynomial functions with values in \( \mathbb{C} \), \((\gamma_0(\theta), \gamma_1(\theta), \cdots, \gamma_p(\theta), \overline{\gamma_1(\theta)}, \cdots, \overline{\gamma_p(\theta)}) \) are zeros of \( \varphi - \theta \) (where \( \varphi \) is the function defined by (1.23)) and \( a_i = \frac{1}{2} \) (resp. 1) if \( \gamma_i(\theta) \in \mathbb{R} \) (otherwise).

This result is an extension of the one of J. Bertoin and R.A. Doney [2].

b) The asymptotic behaviour of the law of the triplet \((T_x, K_x, L_x)\) when \( x \to +\infty \).

7. Let \((X_t, t \geq 0)\) be a Lévy process. It is well known that there exists a family of probability measures \((P^\lambda, 0 \leq \lambda \leq \gamma)\) such that, under \( P^\lambda \), \((X_t, t \geq 0)\) is still a Lévy process and:

\[
P^\lambda(X_t \in dx) = e^{\lambda x} e^{-t\varphi(\lambda)} P(X_t \in dx).
\]

Consequently \( \varphi(\lambda)(q) = \varphi(q-\lambda) - \varphi(-\lambda) \), where \( \varphi^\lambda \) is associated with \((X_t, t \geq 0)\) under \( P^\lambda \). Suppose \( \nu \) verifies the assumption of Theorem 2.1 then there exists \( \lambda \) such that \( \varphi(-\lambda) = \theta \) and \( \varphi'(0) \varphi'(-\lambda) < 0 \). Since \( \mathbb{E}(X_1) = -\varphi'(0) \), and \( \mathbb{E}^\lambda(X_1) = -\varphi^\lambda(0) = -\varphi'(-\lambda) \), then \( \mathbb{E}(X_1) \mathbb{E}^\lambda(X_1) < 0 \). This trick allows to only consider the case \( \mathbb{E}(X_1) > 0 \) (or \( \mathbb{E}(X_1) < 0 \)), and then simplify the proofs of Theorems 2.4, 4.1 of [23].
8. There is a hudge litterature concerning the so-called ruin problem. A good reference for the reader interested in this topic is the book written by T. Rolski, H. Schmidli and J. Teugels [22]. Historically, the first model (called classical model or the Cramér-Lundberg model) was initiated by F. Lundberg [19] and H. Cramér [3], [4]. It corresponds to the case: 
\[ X_t = -ct + J_t, \quad (J_t, \quad t \geq 0) \]
being a compound Poisson process. There is no Brownian component (i.e. \( \sigma = 0 \)). The authors proved that the Laplace transform \( \hat{F}(0,0,0,) \) of the ruin probability verifies a relation, and computed explicitly \( P(T_x < +\infty) \) when the jumps are exponentially distributed. Among the authors working with the classical model we we may mention Gerber [13], F. Delbaen, J. Haezendonck [6], A. Dassios, P. Embrecht [5], G.C. Taylor [24] and W. Feller [11]). The perturbed model was introduced by H.U. Gerber [12] and corresponds to our underlying process \( (X_t, \quad t \geq 0) \) with \( \sigma > 0 \).

In some specific cases, the ruin probability, the law of \( T_x \) or the distribution of the overshoot have been determined, more or less explicitely, see for instance [9], [18], [8], [16], [10], [7], [21], [14], [20].

1 A functional relation satisfied by \( F \), when \( \nu(\mathbb{R}) < +\infty \)

1.1 Functional equation satisfied by \( F \)

We keep the notations given in the Introduction. In this section it is assumed that :
\[ \lambda := \nu(\mathbb{R}) = \int_{-\infty}^{+\infty} \nu(dy) < +\infty. \tag{1.1} \]

Then \( (J_t, \quad t \geq 0) \) is a compound Poisson process. Hence it admits a first jump time \( \tau_1 \), exponentially distributed with parameter \( \nu(\mathbb{R}) \) and the process \( (X_{t+\tau_1} - X_{\tau_1}; \quad t \geq 0) \) is again a Lévy process distributed as \( (X_t, \quad t \geq 0) \). This property is the key of our approach that we briefly describe. We distinguish three cases :

- \( T_x := \inf \{ t \geq 0 ; \quad B_t - c_0 t > x \} < \tau_1 \) if \( \sup_{0 \leq t \leq \tau_1} (B_t - c_0 t) > x \),
- \( T_x = \tau_1 \) if \( \sup_{0 \leq t \leq \tau_1} (B_t - c_0 t) < x \) and \( J_{\tau_1} + B_{\tau_1} - c_0 \tau_1 > x \),
- \( T_x > \tau_1 \) otherwise. However, conditionally to \( \{ T_x > \tau_1 \} \), \( T_x - \tau_1 \) is distributed as \( \hat{T}_{x-X_{\tau_1}} \) where \( (\hat{T}_x; \quad x > 0) \) is an independent copy of \( (T_x; \quad x > 0) \), independent of \( (X_t, \quad t \geq 0) \). This renewal part gives rise to the integral kernel \( \Lambda_\theta \) defined by (1.7) below.
This leads us to decompose $F(\theta, \mu, \rho, \cdot)$ defined in (0.8), as follows:

$$
F(\theta, \mu, \rho, x) = \mathbb{E}\left( e^{-\theta T_x - \mu K_x - \rho L_x} 1_{\{T_x < \tau_1\}} \right) + \mathbb{E}\left( e^{-\theta T_x - \mu K_x - \rho L_x} 1_{\{T_x = \tau_1\}} \right) + \mathbb{E}\left( e^{-\theta T_x - \mu K_x - \rho L_x} 1_{\{T_x < +\infty\}} \right).
$$

(1.2)

Finally the main result of this subsection is the following.

**Theorem 1.1** Assume $\lambda = \nu(\mathbb{R}) < +\infty$. For any $\theta \geq 0$, $\mu \geq 0$ and $\rho \geq 0$, the function $F(\theta, \mu, \rho, \cdot)$ is solution of the following integral equation:

$$
G(x) = F_0(\theta, \mu, \rho, x) + F_1(\theta, \mu, \rho, x) + \Lambda_\theta G(x) \quad \forall x \geq 0
$$

(1.3)

where

$$
\alpha_\theta = \sqrt{c_0^2 + 2(\lambda + \theta)},
$$

(1.4)

$$
F_0(\theta, \mu, \rho, x) = e^{-(c_0+\alpha_\theta)x},
$$

(1.5)

$$
F_1(\theta, \mu, \rho, x) = \frac{e^{-(c_0+\alpha_\theta)x}}{\alpha_\theta(\mu - \rho + c_0 + \alpha_\theta)} \int_{[0,x]} \left( e^{-(\rho+\alpha_\theta)y - \mu y} - e^{-\mu y} \right) \nu(dy)
+ \frac{e^{-\rho x}}{\alpha_\theta(\mu - \rho + c_0 + \alpha_\theta)} \int_{[x,+,\infty[} \left( e^{-(\rho+\alpha_\theta)(y-x) - \mu(y-x)} - e^{-\mu(y-x)} \right) \nu(dy)
+ \frac{e^{(\mu-\rho)x} - e^{-(c_0+\alpha_\theta)x}}{\alpha_\theta(\mu - \rho + c_0 + \alpha_\theta)} \int_{[x,+,\infty[} e^{-\mu y} \nu(dy)
- \frac{e^{-(c_0+\alpha_\theta)x}}{\alpha_\theta(\mu - \rho + c_0 + \alpha_\theta)} \int_{0}^{+\infty} \left( e^{-(\rho+\alpha_\theta)y - \mu y} - e^{-\mu y} \right) \nu(dy),
$$

(1.6)

and $\Lambda_\theta$ is the operator:

$$
\Lambda_\theta G(x) = \frac{1}{\alpha_\theta} \int_{-\infty}^{+\infty} \nu(dy) \int_{-\infty}^{x-y} e^{-c_0a} \left( e^{-\alpha_\theta|a|} - e^{-(2x-a)\alpha_\theta} \right) G(x-a-y) da.
$$

(1.7)

**Proof of Theorem 1.1**

We compute the two first terms in (1.2) in Lemmas 1.2, 1.3 and the last one in Lemma 1.4.

**Lemma 1.2** Let $\alpha_\theta$ be the real number, defined by (1.4), then:

$$
\mathbb{E}\left( e^{-\theta T_x - \mu K_x - \rho L_x} 1_{\{T_x < \tau_1\}} \right) = e^{-(c_0+\alpha_\theta)x}.
$$

(1.8)
Proof of Lemma 1.3
Let \((\tilde{B}_t, \ t \geq 0)\) be the Brownian motion with drift \(-c_0\):
\[
\tilde{B}_t = B_t - c_0 t \quad \forall t \geq 0.
\] (1.9)
We set \(\tilde{T}_x := \inf \{ t \geq 0 ; \ \tilde{B}_t > x \}, \ x \geq 0\).
Then \(\{T_x < \tau_1\} = \{\tilde{T}_x < \tau_1\}\) and on \(\{T_x < \tau_1\}\), we have \(K_x = L_x = 0\).
Since \(\tau_1\) is exponentially distributed with parameter \(\lambda\) and independent of \(\tilde{T}_x\), we have :
\[
E \left( e^{-\theta T_x - \mu K_x - \rho L_x} \mathbb{1}_{\{T_x < \tau_1\}} \right) = E \left( e^{-(\lambda + \theta) \tilde{T}_x} \right).
\] (1.10)
By (17), exercise 5.10 page 197 we can conclude that (1.9) holds. \(\square\)

Lemma 1.3 We have :
\[
E \left( e^{-\theta T_x - \mu K_x - \rho L_x} \mathbb{1}_{\{T_x = \tau_1\}} \right) = \frac{e^{-(c_0 + \alpha \theta)x}}{\alpha (\mu - \rho + c_0 + \alpha \theta)} \int_{\mathbb{R}} \left( e^{-(\rho + c_0 + \alpha \theta)y} - e^{-\mu y} \right) \nu(dy)
\]
\[
\quad + \frac{e^{-\rho x}}{\alpha (\mu - \rho + c_0 + \alpha \theta)} \int_{\mathbb{R}} \left( e^{-(\rho + \alpha \theta - c_0)(y-x)} - e^{-\mu(y-x)} \right) \nu(dy)
\]
\[
\quad + \frac{e^{(\rho - \mu)x}}{\alpha (\mu - \rho + c_0 + \alpha \theta)} \int_{\mathbb{R}} e^{-\mu y} \nu(dy)
\]
\[
\quad - \frac{e^{(c_0 + \alpha \theta)x}}{\alpha (\mu - \rho + c_0 + \alpha \theta)} \int_{\mathbb{R}} \left( e^{-(\rho + \alpha \theta - c_0)y} - e^{-\mu y} \right) \nu(dy).
\] (1.11)

Proof of Lemma 1.3
Write \(Y_1 := J_{\tau_1}\). We observe that on \(\{T_x = \tau_1\}\), \(Y_1 > 0\). Moreover :
\[
\{T_x = \tau_1\} = \{\sup_{t \leq \tau_1} \tilde{B}_t < x, \tilde{B}_{\tau_1} + Y_1 > x\},
\] (1.12)
and
\[
K_x = \tilde{B}_{\tau_1} + Y_1 - x, \quad L_x = x - \tilde{B}_{\tau_1},
\] (1.13)
where \((\tilde{B}_t, \ t \geq 0)\) is defined by the relation (1.9). Since the distribution of \(Y_1\) is \(\frac{1}{\lambda} \nu\), conditioning by \(\tau_1\) and \(Y_1\), we have :
\[
\Delta := E \left( e^{-\theta T_x - \mu K_x - \rho L_x} \mathbb{1}_{\{T_x = \tau_1\}} \right)
\]
\[
= e^{-\rho x} \int_{\mathbb{R}} dt \ e^{-(\lambda + \theta)t} \int_{\mathbb{R}} \nu(dy) E \left( e^{-(\mu - \rho)\tilde{B}_t - \mu(y-x)} \mathbb{1}_{\{\sup_{u \leq t} \tilde{B}_u < x; \ x-y \leq \tilde{B}_t\}} \right)
\] (1.14)
The density function of \((\sup_{u \leq t} B_u, B_t)\) is given by \((\ref{eq:9})\) page 95, i.e.:

\[
P\left( B_t \in da; \sup_{u \leq t} B_u \in db \right) = \frac{2(2b-a)}{\sqrt{2\pi} t^3} e^{-\frac{(2b-a)^2}{2t}} 1_{\{a<b, b>0\}} db.
\] (1.15)

Applying Girsanov’s formula, we get:

\[
P\left( \tilde{B}_t \in da; \sup_{u \leq t} \tilde{B}_u \in db \right) = \frac{2(2b-a)}{\sqrt{2\pi} t^3} e^{-\frac{c_0a-a^2}{2t}} e^{-\frac{(2b-a)^2}{2t}} 1_{\{a<b, b>0\}} db.
\] (1.16)

Combining (1.16) and (1.14) leads to:

\[
\Delta = e^{(\mu-\rho)x} \int_0^\infty \nu(dy) e^{-\mu y} \int_{x-y}^x da e^{-(c_0+\mu-\rho)a} \int_{a\vee 0}^x db (2b-a) \\
\int_0^\infty \frac{2}{\sqrt{2\pi} t^3} e^{-\frac{1}{2}(2(\lambda+\theta) + c_0^2) y + \frac{(2b-a)^2}{2t}} dt.
\] (1.17)

Recall the classical identities (cf. \(\cite{[15]}\) sections 8.432 6 page 959, and 8.469 3 page 967):

\[
K_\frac{1}{2}(\delta) := \frac{1}{2} \int_0^{+\infty} \frac{1}{\sqrt{t}} e^{-\frac{\delta}{2}(t+\frac{1}{t})} dt = \sqrt{\frac{\pi}{2\delta}} e^{-\frac{\delta}{2}} \quad \forall \delta > 0,
\] (1.18)

and

\[
\int_0^{+\infty} \frac{1}{\sqrt{t^3}} e^{-\frac{1}{2}(\beta t + \frac{1}{t})} dt = \sqrt{\frac{2\pi}{\gamma}} e^{-\sqrt{\beta \gamma}} \quad \forall \beta > 0, \forall \gamma > 0.
\] (1.19)

obtained by derivation and the changing variable \(t \rightarrow \sqrt{\frac{2}{\gamma}} t\).

This allows to first compute explicitly the integral with respect to \(dt\) in (1.17):

\[
\Delta = 2 e^{(\mu-\rho)x} \int_0^{+\infty} \nu(dy) e^{-\mu y} \int_{x-y}^x da e^{-(c_0+\mu-\rho)a} \int_{a\vee 0}^x e^{-\alpha a(2b-a)} db
\] (1.20)

In a second step we evaluate the integral with respect to \(db\):

\[
\Delta = \frac{e^{(\mu-\rho)x}}{\alpha_\theta} \int_0^{+\infty} \nu(dy) e^{-\mu y} \int_{x-y}^x e^{-(c_0+\mu-\rho+\alpha_\theta)a} da \\
+ \frac{e^{(\mu-\rho)x}}{\alpha_\theta} \int_{[x, +\infty]} \nu(dy) e^{-\mu y} \left[ \int_{x-y}^0 e^{-(c_0+\mu-\rho-\alpha_\theta)a} da + \int_0^x e^{-(c_0+\mu-\rho-\alpha_\theta)a} da \right] \\
- \frac{e^{(\mu-\rho)x}}{\alpha_\theta} \int_0^{+\infty} \nu(dy) e^{-\mu y} \int_{x-y}^x e^{-(c_0+\mu-\rho-\alpha_\theta)a} da.
\] (1.21)

To drop \(|a|\), we introduce two cases \(x-y \geq 0\) and \(x-y < 0\):

\[
\Delta = \frac{e^{(\mu-\rho)x}}{\alpha_\theta} \int_{[0, x]} \nu(dy) e^{-\mu y} \int_{x-y}^x e^{-(c_0+\mu-\rho+\alpha_\theta)a} da \\
+ \frac{e^{(\mu-\rho)x}}{\alpha_\theta} \int_{[x, +\infty]} \nu(dy) e^{-\mu y} \left[ \int_{x-y}^0 e^{-(c_0+\mu-\rho-\alpha_\theta)a} da + \int_0^x e^{-(c_0+\mu-\rho-\alpha_\theta)a} da \right] \\
- \frac{e^{(\mu-\rho)x}}{\alpha_\theta} \int_0^{+\infty} \nu(dy) e^{-\mu y} \int_{x-y}^x e^{-(c_0+\mu-\rho-\alpha_\theta)a} da.
\] (1.22)
Computing the integral with respect to $da$ we easily obtain (1.11).

**Lemma 1.4** In (1.2), the third expectation is equal to:

$$E \left( e^{-\theta T_x - \mu K_x - \rho L_x} \mathbb{1}_{\{\tau_1 < T_x < +\infty\}} \right) = \frac{1}{\alpha \theta} \int_{-\infty}^{+\infty} \nu(dy) \int_{-\infty}^{(x-y)/\lambda x} e^{-c_0 a} \left( e^{-\alpha |a|} - e^{(2x-a)\alpha} \right) F(\theta, \mu, \rho, x - a - y) da. \quad (1.23)$$

Moreover:

$$e^{-\alpha |a|} - e^{(2x-a)\alpha} \geq 0 \quad \text{if} \quad a \leq (x-y) \land x, \quad (1.24)$$

so $\Lambda_{\theta}$ defined by (1.7) is an non-negative operator.

**Proof of Lemma 1.4**

Formula (1.23) may be proved proceeding analogously to the proof of previous Lemma.

---

**1.2 Study of $\Lambda_{\theta}$**

To investigate uniqueness in (1.3), we prove that $\Lambda_{\theta}$ is a contraction in some functional Banach spaces. Recall we have in mind to prove that $F$ has a sub-exponential decay at infinity, therefore it seems natural to introduce the Banach space:

$$B_\gamma := \{ f : \mathbb{R}_+ \to \mathbb{R}; \sup_{x \in \mathbb{R}_+} e^{\gamma x} |f(x)| < +\infty \} \quad \gamma \geq 0. \quad (1.25)$$

$B_\gamma$ is equipped with the norm:

$$\|f\|_\gamma = \sup_{x \in \mathbb{R}_+} e^{\gamma x} |f(x)|. \quad (1.26)$$

The values of $\gamma$ such that $\Lambda_{\theta}$ is a contraction in $B_\gamma$ are linked to the zeros of the function $\varphi_{\theta}$:

$$\varphi_{\theta}(q) := \varphi(q) - \theta. \quad (1.27)$$

Before stating our main result, we fix some notations. Let:

$$r_\nu := \sup \left\{ s \geq 0; \int_{1}^{+\infty} e^{sy} \nu(dy) < +\infty \right\}, \quad (1.28)$$

with the convention $\sup \emptyset = 0$.

$\hat{\nu}$ (resp. $\hat{\nu}^+$) denotes the Laplace transform of $\nu$ (resp. $\nu_{|[0, +\infty[}$), if they exist :

$$\hat{\nu}(q) := \int_{-\infty}^{+\infty} e^{-qy} \nu(dy) \quad (1.29)$$

$$\hat{\nu}^+(q) := \int_{0}^{+\infty} e^{-qy} \nu(dy). \quad (1.30)$$

---
Theorem 1.5  Suppose that \( \nu(R) < +\infty \).

(i) For any \( \theta \geq 0 \), the operator \( \Lambda_\theta \) defined by (1.7) is a linear and non-negative operator, with norm equals to \( \frac{\lambda}{\lambda + \theta} \) in \( L^\infty(R_+) \).

(ii) Assume \( r_\nu > 0 \). Let \( \gamma \in [0, r_\nu] \) and \( \theta > 0 \) or \( \theta = 0 \) and \( \mathbb{E}(X_1) < 0 \). Then :

a) \( \Lambda_\theta \) is a bounded operator from \( B_\gamma \) to \( B_\gamma \). More precisely :

\[
\|\Lambda_\theta f\|_\gamma \leq c_{\theta,\gamma} \|f\|_\gamma \quad \forall f \in B_\gamma ,
\]

(1.31)

with

\[
c_{\theta,\gamma} = \frac{\hat{\nu}(-\gamma)}{\hat{\nu}(-\gamma) - \varphi(-\gamma) + \theta} .
\]

(1.32)

b) There exists \( \gamma \in ]0, r_\nu[ \) such that \( \varphi(-\gamma) < \theta \). Therefore \( \Lambda_\theta \) is a contraction in \( B_\gamma \) since :

\[ 0 < c_{\theta,\gamma} < 1 . \]

(1.33)

Remark 1.6  1. It is clear that \( r_\nu > 0 \) is equivalent to \( (1.10) \) and if \( r_\nu \in ]0; +\infty[ \), then \( \varphi \) given formally by (0.2), is actually well-defined on \( ]-r_\nu, 0[ \).

2. If \( r_\nu > 0 \), it is easy to check (cf. Annex A) that \( \{q \in ]-r_\nu, 0[ / \varphi(q) < \theta \} \) is non empty if \( \theta > 0 \) or if \( \theta = 0 \) and \( \mathbb{E}(X_1) < 0 \). Observe that \( X_1 \) has a finite expectation if :

\[
\int_R |y| \nu(dy) < +\infty .
\]

(1.34)

In this case :

\[
\mathbb{E}(X_1) = -c_0 + \int_R y \nu(dy) = -c + \int_R y 1_{\{|y|>1\}} \nu(dy) ,
\]

(1.35)

where \( c \) comes from Lévy-Khintchine formula (1.2).

3. The assumption \( (1.10) \) means that the positive jumps are not too big. It corresponds to the intuition, since more the positive jumps are small, more time is needed to reach a positive level \( x \). Hence more \( F(\theta, \mu, \rho, ..) \) decreases.

Proof of Theorem 1.5

(i) Relation (1.21) implies that \( \Lambda_\theta \) is a non-negative operator.

It is easy to check that the function \( \ell : \)

\[
\ell(x) = 1_{\{a+y \leq x; a \leq x\}} e^{-c_0 a} \left( e^{-\alpha_B |a|} - e^{-(2x-a)\alpha_B} \right)
\]

(1.36)

is increasing, then :

\[
\ell(x) < \ell(+\infty) = e^{-c_0 a} e^{-\alpha_B |a|} \quad \forall x \in \mathbb{R} .
\]

(1.37)
A straightforward calculation shows that $|\Lambda_\theta h(x)| \leq \frac{\lambda}{\lambda + \theta} \| h \|_\infty$, for any $x \geq 0$.

Taking $h : x \to 1$, we have $\|\Lambda_\theta h\|_\infty = \frac{\lambda}{\lambda + \theta}$, then $\|\Lambda_\theta\|_{L^\infty(\mathbb{R})} = \frac{\lambda}{\lambda + \theta}$.

(ii) Let $f$ be an element of $B_\gamma$, then $|f(x)| \leq \| f \|_\gamma e^{-\gamma x}$, $\forall x \geq 0$. Consequently:

$$|\Lambda_\theta f(x)| \leq \frac{1}{\alpha_\theta} \| f \|_\gamma e^{-\gamma x} \int_{-\infty}^{+\infty} \nu(dy) \int_{-\infty}^{-(x-y)\wedge x} e^{-\alpha_\theta a} \left( e^{-\alpha_\theta |a|} - e^{-(2x-a)\alpha_\theta} \right) e^{\gamma(a+y)} da,$$

for any $\gamma \in [0, r_\nu[$.

Making use of (1.37), we get:

$$|\Lambda_\theta f(x)| \leq \frac{1}{\alpha_\theta} \| f \|_\gamma e^{-\gamma x} \int_{-\infty}^{+\infty} \nu(dy) e^{\gamma y} \left[ \int_{-\infty}^{0} e^{-(c_0-\alpha_\theta)\alpha_\theta} da + \int_{0}^{+\infty} e^{-(c_0+\alpha_\theta)\gamma} da \right].$$

(1.38)

Computing the integral with respect to $da$, yields directly to (1.31).

Proposition 1.7 Assume $\nu(\mathbb{R}) < +\infty$, $r_\nu > 0$, $\mu \geq 0$, $\theta > 0$ or $\theta = 0$ if $\mathbb{E}(X_1) < 0$. Let $\gamma$ be in $[0, r_\nu[$, such that $\varphi(-\gamma) < \theta$. Then the function $F(\theta, \mu, \rho, )$ belongs to $B_\gamma$ and the equation (1.3) has an unique solution in $B_\gamma$.

To prove Proposition 1.7, we need the following preliminary.

Lemma 1.8 Suppose $\theta > 0$ or $\theta = 0$ if $\mathbb{E}(X_1) < 0$, then for any $x > 0$,

$$\lim_{n \to +\infty} \Lambda_\theta^n F(\theta, \mu, \rho, )(x) = 0.$$  

(1.39)

Proof of Lemma 1.8

1) Suppose $\theta > 0$. Since $F$ is bounded by 1, and the norm of $\Lambda_\theta$ is $\frac{\lambda}{\lambda + \theta}$ (cf. Theorem 1.5) : $\|\Lambda_\theta^n F(\theta, \mu, \rho, )\|_\infty \leq \left( \frac{\lambda}{\lambda + \theta} \right)^n$. This proves (1.39).

2) We now turn to the case $\theta = 0$ and $\mathbb{E}(X_1) < 0$. Iterating the functional equation (1.3), we obtain:

$$F(\theta, \mu, \rho, x) = \sum_{p=0}^{n-1} \Lambda_\theta^n [F_0 + F_1](\theta, \mu, \rho, ) \text{ (x)} + \Lambda_\theta^n F(\theta, \mu, \rho, )(x).$$

(1.40)

The norm of $\Lambda_\theta$ in $B_\gamma$ is strictly less than 1, then the series in (1.40) converges. Consequently the remaining term $\Lambda_\theta^n F(\theta, \mu, \rho, )(x)$ converges in $B_\gamma$ to some function $G(\theta, \mu, \rho, x)$. It is easy to check the following:
a) $G(0, \mu, \rho, \cdot)$ is a bounded and non negative function ,
b) $G(0, \mu, \rho, \cdot)$ is a continuous function on $[0, +\infty[$ ,
c) $\lim_{x \to +\infty} G(0, \mu, \rho, x) = 0$ ,
d) $\Lambda_0 G(0, \mu, \rho, \cdot) = G(0, \mu, \rho, \cdot)$.

Hence by point (i) of Theorem 1.5 :

$$G(0, \mu, \rho, x) = \Lambda_0 G(0, \mu, \rho, \cdot)(x) \leq \|G(0, \mu, \rho, \cdot)\|_{\infty}, \quad x \geq 0. \quad (1.41)$$

As (1.37) is a strict inequality then (1.41) is strict if $\|G(0, \mu, \rho, \cdot)\|_{\infty} \neq 0$.

According to b) and c), there exists $x_0 \geq 0$ such that :

$$G(0, \mu, \rho, x_0) = \|G(0, \mu, \rho, \cdot)\|_{\infty}. \quad \text{This implies} \quad \|G(0, \mu, \rho, \cdot)\|_{\infty} = 0. \quad \square$$

**Proof of Proposition 1.7**

Using the explicit expression of $F_0$ and $F_1$ (cf. (1.5) and (1.6)), by a straightforward calculation, it may be concluded that $F_0(\theta, \mu, \rho, \cdot)$ and $F_1(\theta, \mu, \rho, \cdot)$ belong to $B_\gamma$ (for a detailed proof, cf. [25]).

By Lemma 1.8 and (1.40),

$$F(\theta, \mu, \rho, x) = \sum_{n=0}^{+\infty} \Lambda_\theta^n (F_0 + F_1)(\theta, \mu, \rho, \cdot)(x), \quad (1.42)$$

Because $F_0 + F_1 \in B_\gamma$ and $\Lambda_\theta$ is a contraction in $B_\gamma$, the serie converges in $B_\gamma$. This directly implies the result. \square

**Remark 1.9**

1. Under the conditions stated in Proposition 1.7 we have actually proved that $F(\theta, \mu, \rho, x)$ can be approximated by $\sum_{n=0}^{p} \Lambda_\theta^n (F_0 + F_1)(\theta, \mu, \rho, \cdot)(x)$. More precisely :

$$\left| F(\theta, \mu, \rho, x) - \sum_{n=0}^{p} \Lambda_\theta^n (F_0 + F_1)(\theta, \mu, \rho, \cdot)(x) \right| < c_{\theta, \gamma} K e^{-\gamma x}, \quad (1.43)$$

where $K = \left\| \sum_{n=0}^{+\infty} \Lambda_\theta^n (F_0 + F_1)(\theta, \mu, \rho, \cdot) \right\|_{\gamma} < +\infty$ and $c_{\theta, \gamma}$ is defined by (1.32).

2. Let us consider the case where the support of $\nu$ is included in $]-\infty, 0]$. Then $\varphi$ is well defined on $]-\infty, 0]$ and $r_{\nu} = +\infty$. Moreover $K_x = L_x = 0$ and $F_1(\theta, \mu, \rho, x) = 0$ for any $x \geq 0$. As a result (1.3) reduces to :

$$F(\theta, \mu, \rho, x) = e^{-(\alpha + \omega)x} + \Lambda_\theta F(\theta, \mu, \rho, \cdot)(x). \quad (1.44)$$
If \( \theta > 0 \) or \( \theta = 0 \) and \( \mathbb{E}(X_1) < 0 \), we prove in Annex, Properties \( \text{A.1} \) the existence of an unique real number \( \gamma_0(\theta) \) such that:

\[
-\gamma_0(\theta) < 0 \quad \text{et} \quad \varphi(-\gamma_0(\theta)) = \theta .
\]  

(1.45)

A direct (but fastidious !) calculation shows that \( x \to e^{-\gamma_0(\theta)x} \) is a solution of \( \text{(1.3)} \). For more details we refer the reader to \[25\]. Hence \( F(\theta, \mu, \rho, x) = F(\theta, 0, 0, x) = e^{-\gamma_0(\theta)x} \).

## 2 The Laplace transform of \( F(\theta, \mu, \rho, .) \)

### 2.1 The Laplace transform expression \( \hat{F}(\theta, \mu, \rho, .) \)

In the previous section we have proved that \( F(\theta, \mu, \rho, .) \) verifies the integral equation \[133\] when \( \nu \) is a probability measure. If moreover \( r_\nu < +\infty \), then \( F(\theta, \mu, \rho, .) \) is the unique solution of \[133\]. Unfortunately we cannot define the operator \( \Lambda_\theta \) if \( \nu \) is not a probability measure. We would like to consider Lévy processes that do not reduce to a Brownian motion with drift plus a compound Lévy process.

Our approach is based on the use of the Laplace transform of \( F(\theta, \mu, \rho, .) \). Since \( F(\theta, \mu, \rho, .) \) is a bounded function on \([0, +\infty[\), its Laplace transform :

\[
\hat{F}(\theta, \mu, \rho, q) := \int_0^{+\infty} e^{-qy} F(\theta, \mu, \rho, y) dy ,
\]

(2.1)

is well defined for any \( q \) such that \( \text{Re} (q) > 0 \).

We first suppose that \( \nu \) is a finite measure. Taking the Laplace transform in \[133\], it is proved (cf. Theorem 2.1) that under some additional assumption, \( \hat{F}(\theta, \mu, \rho, .) \) verifies some kind of integral equation. In the calculations, cancellations occur so that in the final identity and \( \nu(R) < +\infty \) may be removed.

Before stating the main result of this sub-section (i.e. Theorem 2.1), we introduce :

\[
D_0 = \{ q \in \mathbb{C} ; \text{Re} q > 0 \} .
\]

(2.2)

We suppose :

\[
\int_{-\infty}^{-1} e^{-qy} \nu(dy) < \infty \quad \forall q > 0 .
\]

(2.3)

Let \( R \) the operator :

\[
R h(q) := \int_{-\infty}^{0} \nu(dy) \int_{0}^{-y} \left( e^{-q(b+y)} - 1 \right) h(b) db ,
\]

(2.4)

where \( q \in D_0 \) and \( h \in L^\infty(\mathbb{R}_+) \).

Property \[2.3\] implies that \( R h(q) \) exists.
We suppose moreover that \( r_\nu \in ]0; +\infty[ \), with \( r_\nu \) defined in (1.28), i.e., there exist some \( s > 0 \) such that \( \int_1^{+\infty} e^{sy} \nu(dy) < +\infty \). Recall that from Remark 1.6 \( \varphi \) is well defined on \( ]-r_\nu, 0[ \).

We only consider the cases given by Figures 1a, 2a and 3 (cf Annex). As a result, there exists \( \kappa > 0 \) such that, for all \( \theta \in [0, \kappa] \):

\[ \exists \ -\gamma_0(\theta) \in ]-r_\nu, 0[ \text{ such that } \varphi(-\gamma_0(\theta)) = \theta . \] (2.5)

More precisely:

(i) if \( \theta > 0 \),

\[ -\gamma_0(\theta) < 0 , \] (2.6)

(ii) if \( \theta = 0 \) et \( \mathbb{E}(X_1) < 0 \),

\[ -\gamma_0(0) < 0 , \] (2.7)

(iii) if \( \theta = 0 \) et \( \mathbb{E}(X_1) \geq 0 \),

\[ -\gamma_0(0) = 0 . \] (2.8)

These hypotheses are in force in the whole paragraphs 2 and 3.

**Theorem 2.1** We suppose \( r_\nu < +\infty \), and (2.3), (2.5) hold.

1. There exist \( \gamma_0^*(\theta) \) such that

- If \( \theta > 0 \) or \( \theta = 0 \) and \( \mathbb{E}(X_1) > 0 \), \( \gamma_0^*(\theta) \) is the unique positive real number such that:

\[ \gamma_0^*(\theta) > 0 \text{ et } \varphi(\gamma_0^*(\theta)) = \theta . \] (2.9)

- If \( \theta = 0 \) and \( \mathbb{E}(X_1) \leq 0 \) then \( \gamma_0^*(0) = 0 \).

2. Let \( \theta, \mu, \rho \geq 0 \), \( q \in D_0 \). We have:

\[
\hat{F}(\theta, \mu, \rho, q) = \frac{1}{\varphi(q) - \theta} \left( \frac{q - \gamma_0^*(\theta)}{2} + \int_0^{+\infty} \left[ \frac{e^{-(\gamma_0^*(\theta) + \rho)y} - e^{-\mu y}}{q + \rho - \mu} \right] \nu(dy) \right. \\
\left. + RF(\theta, \mu, \cdot, \cdot)(q) - RF(\theta, \mu, \cdot, \cdot)(\gamma_0^*(\theta)) \right) \] (2.10)

where \( \theta, \mu, \rho \geq 0 \), \( q \in D_0 \).

**Remark 2.2**

1. In (2.10), \( \frac{e^a - e^b}{a - b} \) stands for \( e^a \) when \( a = b \).
2. Assumptions (2.3) and (2.5) are needed to obtain the existence of \( \gamma_0^*(\theta) \), for any \( \theta \geq 0 \) (cf. Annex A, subsection A.2).

3. The function \( \hat{F}(\theta, \mu, \rho, \cdot) \) being defined on \( D_0 \), then \( q = \mu \) and \( q = \gamma_0^*(\theta) \) are false singularities of the right-hand side of (2.10).

4. If \( \nu([-\infty,0]) = 0 \), then \( RF(\theta, \mu, \rho, \cdot) \) cancels, and \( \hat{F}(\theta, \mu, \rho, q) \) is given by the following explicit formula:

\[
\hat{F}(\theta, \mu, \rho, q) = 1 \frac{1}{\varphi(q)} - \theta \left( \frac{q-\gamma_0^*(\theta)}{2} + \int_0^{+\infty} \left[ \frac{e^{-qy} - e^{-\mu y}}{q + \rho - \mu} \right] \nu(dy) \right)
\]

(2.11)

5. If \( \nu([0, +\infty[) = 0 \), then (2.10) is equivalent to:

\[
\hat{F}(\theta, \mu, \rho, q) = 1 \frac{1}{\varphi(q)} - \theta \left( \frac{q-\gamma_0^*(\theta)}{2} + RF(\theta, \mu, \rho, \cdot)(q) - RF(\theta, \mu, \rho, \cdot)(\gamma_0^*(\theta)) \right)
\]

(2.12)

6. Let us detailed the case \( \theta = \mu = \rho = 0 \) (i.e. \( F(0,0,0,x) \) is the ruin probability). If \( \mathbb{E}(X_1) \geq 0 \), it is easy to check that \( f : x \rightarrow 1 \) verifies (2.10).

Let us concentrate on the more interesting case : \( \mathbb{E}(X_1) < 0 \).

Relation (2.10) becomes:

\[
\hat{F}(0,0,0,q) = 1 \frac{1}{\varphi(q)} \left( \frac{q-\gamma_0^*(\theta)}{2} + RF(0,0,\cdot)(q) - RF(0,0,\cdot)(\gamma_0^*(\theta)) \right)
\]

(2.13)

Suppose morever that \( \nu([-\infty,0]) = 0 \), then (2.13) reduces to:

\[
\hat{F}(0,0,0,q) = 1 \frac{1}{q} + \frac{\mathbb{E}(X_1)}{\varphi(q)}.
\]

(2.14)

It can be proved (see [25], for details) that (2.14) generalizes identity (2.9) of [9].

However if \( \nu([0, +\infty[) = 0 \), then \( F(0,0,0,x) = e^{-\gamma_0(0)x} \).

7. Recall the Wiener-Hopf decomposition (cf. [1], page 165) : for any \( \theta > 0 \), we have:

\[
\frac{\theta}{\theta + \varphi(-q)} = \psi^+_\theta(q)\psi^-_\theta(q),
\]

(2.15)
where
\[ \psi^+(q) = \mathbb{E}\left(e^{iqS_{\tau_\theta}}\right), \quad \psi^-(q) = \mathbb{E}\left(e^{iq(S_{\tau_\theta} - X_{\tau_\theta})}\right), \tag{2.16} \]
and \( \tau_\theta \) is an exponential r. v. with parameter \( \theta \), independent from process \((X_t, t \geq 0)\) and \( S_t := \sup_{s \leq t} X_s \).

Since :
\[ P(S_{\tau_\theta} > a) = P(T_a < \tau_\theta) = \mathbb{E}(e^{-\theta T_a}) = F(\theta, 0, 0, a), \tag{2.17} \]
it is easy to deduce the following identity :
\[ \psi^+_\theta(q) = 1 + iq\widehat{F}(\theta, 0, 0, iq). \tag{2.18} \]

Equation (2.10) implies that the Wiener-Hopf factor \( \psi^+_\theta \) verifies a functional equation. In particular, if \( \nu([-\infty, 0]) = 0 \), combining equations (2.11) and (2.18) an explicit form of \( \psi^+_\theta(q) \) may be obtained. Due to (2.15), \( \psi^-_\theta(q) \) is also explicit.

**Proof of Theorem 2.1**

For simplicity, we prove (2.10) in the particular case \( \rho = 0 \), and we write \( F(\theta, \mu, x) \) instead of \( F(\theta, \mu, \rho, x) \). The proof will be divided into two steps. We first prove (2.10) when \( \nu \) satisfied the assumptions given in Theorem 2.1 and \( \nu(\mathbb{R}) < +\infty \).

In a second step, we approximate \( \nu \) by a sequence of finite measures \((\nu_n)\) and we take the limit in (2.10).

**Step 1** We suppose \( \nu(\mathbb{R}) < +\infty \), \( r_\nu < +\infty \), and (2.3), (2.5), and (0.3) hold.

a) Taking the Laplace transform in functional equation (1.3) leads to :
\[ \widehat{F}(\theta, \mu, q) = \widehat{F}_0(\theta, \mu, q) + \widehat{F}_1(\theta, \mu, q) + \Lambda_\theta F(\theta, \mu, .)(q) \quad q \in D_0. \tag{2.19} \]
Relation (1.6) implies :
\[ \widehat{F}_0(\theta, \mu, q) = \frac{1}{c + \alpha_\theta + q}. \tag{2.20} \]
As for \( \widehat{F}_1(\theta, \mu, x) \), starting from (1.6), we split the integral in four parts :
\[ \widehat{F}_1(\theta, \mu, q) = \int_0^{+\infty} e^{-qx} F_1(\theta, \mu, x) dx \\
= I_1(\theta, \mu, q) + I_2(\theta, \mu, q) + I_3(\theta, \mu, q) + I_4(\theta, \mu, q) \tag{2.21} \]
where
\[I_1(\theta, \mu, q) = \int_0^{+\infty} \frac{e^{-(q+c+\alpha\theta)x}}{\alpha\theta(\mu + c + \alpha\theta)} \left( \int_{[0, x]} \left( e^{(c+\alpha\theta)y} - e^{-\mu y} \right) \nu(dy) \right) dx\]
\[= \frac{1}{\alpha\theta(q + c + \alpha\theta)(\mu + c + \alpha\theta)} \int_0^{+\infty} \left( e^{-qy} - e^{-(q+\mu+c+\alpha\theta)y} \right) \nu(dy),\] (2.22)

\[I_2(\theta, \mu, q) = \int_0^{+\infty} \frac{e^{-qx}}{\alpha\theta(\mu + c - \alpha\theta)} \left( \int_{[x, +\infty[} \left( e^{-(\alpha\theta-c)(y-x)} - e^{-(\mu-y)x} \right) \nu(dy) \right) dx\]
\[= \frac{1}{\alpha\theta(\mu + c - \alpha\theta)} \int_0^{+\infty} \left( e^{-(\alpha\theta-c)y} - e^{-\mu y} + \frac{e^{-qy} - e^{-\mu y}}{q - \mu} \right) \nu(dy),\] (2.23)

\[I_3(\theta, \mu, q) = \int_0^{+\infty} \frac{e^{-(q-\mu)x} - e^{-(q+c+\alpha\theta)x}}{\alpha\theta(\mu + c + \alpha\theta)} \left( \int_{[x, +\infty[} e^{-\mu y} \nu(dy) \right) dx\]
\[= -\frac{1}{\alpha\theta(\mu + c + \alpha\theta)} \int_0^{+\infty} \nu(dy) \left( e^{-qy} - e^{-\mu y} - \frac{e^{-(q+\mu+c+\alpha\theta)y} - e^{-\mu y}}{q + c + \alpha\theta} \right),\] (2.24)

\[I_4(\theta, \mu, q) = -\int_0^{+\infty} \frac{e^{-(q+c+\alpha\theta)x}}{\alpha\theta(\mu + c - \alpha\theta)} dx \left( \hat{\nu}^+(\alpha\theta - c) - \hat{\nu}^+(\mu) \right)\]
\[= \frac{\hat{\nu}^+(\mu) - \hat{\nu}^+(\alpha\theta - c)}{\alpha\theta(q + c + \alpha\theta)(\mu + c - \alpha\theta)},\] (2.25)

with:
\[\hat{\nu}^+(q) := \hat{\nu}_{[0, +\infty]}(q) = \int_0^{+\infty} e^{-qy} \nu(dy).\]

Consequently:
\[\hat{F}_1(\theta, \mu, q) = -\frac{\hat{\nu}^+(q)}{\alpha\theta(q + c + \alpha\theta)(\mu - \mu)} + \frac{\hat{\nu}^+(q)}{\alpha\theta(q + c - \alpha\theta)(\mu - \mu)} \]
\[+ \frac{2 \hat{\nu}^+(\alpha\theta - c)}{(\mu + c - \alpha\theta)(q + c - \alpha\theta)(q + c + \alpha\theta)} \]
\[- \frac{2 \hat{\nu}^+(\mu)}{(\mu + c - \alpha\theta)(\mu - \mu)(q + c + \alpha\theta)}.\] (2.26)
Let us introduce:

\[ C_\theta(q) := (q + c + \alpha_\theta)(q + c - \alpha_\theta) = q^2 + 2cq - 2(\lambda + \theta). \quad (2.27) \]

By a direct calculation we obtain:

\[ \hat{F}_1(\theta, \mu, q) = \frac{2}{C_\theta(q)} \left( \frac{\nu^+(q) - \nu^+(\mu)}{q - \mu} - \frac{\nu^+(\mu) - \nu^+(\alpha_\theta - c)}{\mu + c - \alpha_\theta} \right). \quad (2.28) \]

b) Let us now compute \( \hat{\Lambda}_\theta F(\theta, \mu, .)(q) \).

Setting \( b = x - a - y \) in (1.7) yields to:

\[
\begin{align*}
\hat{\Lambda}_\theta F(\theta, \mu, .)(x) & = \frac{1}{\alpha_\theta} \int_{-\infty}^{+\infty} \nu(dy) \int_{0}^{+\infty} e^{-c(x-b-y)} \left( e^{-\alpha_\theta |x-y-b|} - e^{-(x+y+b)\alpha_\theta} \right) F(\theta, \mu, b) db \\
& \quad - \frac{1}{\alpha_\theta} \int_{-\infty}^{0} \nu(dy) \int_{y}^{+\infty} e^{-c(x-b-y)} \left( e^{-\alpha_\theta (x-y-b)} - e^{-(x+y+b)\alpha_\theta} \right) F(\theta, \mu, b) db \\
& = H_1 F(\theta, \mu, .)(x) + I(\theta, \mu, x) \\
& \quad + \frac{e^{-(c+\alpha_\theta)x}}{\alpha_\theta} \left( RF(\theta, \mu, .)(\alpha_\theta - c) - RF(\theta, \mu, .)(-\alpha_\theta - c) \right). \tag{2.29}
\end{align*}
\]

with \( R \) operator defined by (2.4) and

\[
\begin{align*}
H_1 F(\theta, \mu, .)(x) & = \frac{1}{\alpha_\theta} \int_{-\infty}^{+\infty} \nu(dy) \int_{0}^{+\infty} e^{-c(x-b-y)} e^{-\alpha_\theta |x-y-b|} F(\theta, \mu, b) db, \tag{2.30} \\
I(\theta, \mu, x) & = \frac{1}{\alpha_\theta} \int_{-\infty}^{+\infty} \nu(dy) \int_{0}^{+\infty} e^{-c(x-b-y)} e^{-(x+y+b)\alpha_\theta} F(\theta, \mu, b) db \\
& = \frac{1}{\alpha_\theta} e^{-(c+\alpha_\theta)x} \nu(\alpha_\theta - c) \hat{F}(\theta, \mu, \alpha_\theta - c). \tag{2.31}
\end{align*}
\]

Hence

\[
\begin{align*}
\hat{\Lambda}_\theta F(\theta, \mu, .)(q) & = \hat{H}_1 F(\theta, \mu, .)(q) - \frac{\nu(\alpha_\theta - c) \hat{F}(\theta, \mu, \alpha_\theta - c)}{\alpha_\theta (q + c + \alpha_\theta)} \\
& \quad + \frac{RF(\theta, \mu, .)(\alpha_\theta - c) - RF(\theta, \mu, .)(-\alpha_\theta - c)}{\alpha_\theta (q + c + \alpha_\theta)}. \tag{2.32}
\end{align*}
\]
By definition of $H_1 F$, we have:

$$
\widehat{H}_1 F(\theta, \mu, \cdot)(q) = \frac{1}{\alpha_\theta} \int_{-\infty}^{0} e^{y} \nu(dy) \int_{0}^{+\infty} e^{c b} F(\theta, \mu, b) db \int_{0}^{+\infty} e^{-(q+c)x} e^{-\alpha_\theta(x-y)} dx
$$

$$= \frac{1}{\alpha_\theta} \int_{-\infty}^{0} e^{y} \nu(dy) \int_{0}^{-y} e^{c b} F(\theta, \mu, b) db \int_{0}^{+\infty} e^{-(q+c)x} e^{-\alpha_\theta(x-y)} dx
$$

$$+ \frac{1}{\alpha_\theta} \int_{-\infty}^{+\infty} e^{c y} \nu(dy) \int_{0}^{+\infty} e^{c b} F(\theta, \mu, b) db \int_{b+y}^{b} e^{-(q+c)x} e^{-\alpha_\theta(x-y)} dx
$$

The $x$-integrals can be computed:

$$
\widehat{H}_1 F(\theta, \mu, \cdot)(q) = \frac{1}{\alpha_\theta} \int_{-\infty}^{0} e^{(c+\alpha_\theta)y} \nu(dy) \int_{0}^{-y} e^{(c+\alpha_\theta)b} F(\theta, \mu, b) db \frac{1}{q+c+\alpha_\theta}
$$

$$+ \frac{1}{\alpha_\theta} \int_{-\infty}^{+\infty} e^{c(y-\alpha_\theta)} \nu(dy) \int_{0}^{+\infty} e^{(c-\alpha_\theta)b} F(\theta, \mu, b) \frac{1}{q+c-\alpha_\theta} db
$$

$$+ \frac{1}{\alpha_\theta} \int_{-\infty}^{+\infty} e^{c(y+\alpha_\theta)} \nu(dy) \int_{0}^{+\infty} e^{(c+\alpha_\theta)b} F(\theta, \mu, b) \frac{1}{q+c+\alpha_\theta} db.
$$

By (2.33) we obtain:

$$
\widehat{H}_1 F(\theta, \mu, \cdot)(q) = \frac{1}{\alpha_\theta(q+c+\alpha_\theta)} \left[ RF(\theta, \mu, \cdot)(-\alpha_\theta-c) + \int_{-\infty}^{0} \nu(dy) \int_{0}^{-y} F(\theta, \mu, b) db \right]
$$

$$+ \frac{1}{\alpha_\theta(q+c-\alpha_\theta)} \int_{-\infty}^{+\infty} e^{(c-\alpha_\theta)y} \nu(dy) \int_{0}^{+\infty} e^{(c-\alpha_\theta)b} F(\theta, \mu, b) db
$$

$$- \frac{1}{\alpha_\theta(q+c-\alpha_\theta)} \int_{-\infty}^{+\infty} e^{-qy} \nu(dy) \int_{0}^{+\infty} e^{-qb} F(\theta, \mu, b) db
$$

$$+ \frac{1}{\alpha_\theta(q+c+\alpha_\theta)} \int_{-\infty}^{+\infty} e^{-qy} \nu(dy) \int_{0}^{+\infty} e^{-qb} F(\theta, \mu, b) db.
$$

Since

$$
\frac{1}{\alpha_\theta(q+c+\alpha_\theta)} - \frac{1}{\alpha_\theta(q+c-\alpha_\theta)} = -\frac{2}{C_\theta(q)}, \quad (2.33)
$$

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we get:

\[
\hat{H}_1 F(\theta, \mu, .)(q) = \frac{RF(\theta, \mu, .)(-\alpha_\theta - c)}{\alpha_\theta(q + c + \alpha_\theta)}
+ \frac{\hat{\nu}(\alpha_\theta - c) \hat{F}(\theta, \mu, \alpha_\theta - c) - RF(\theta, \mu, .)(\alpha_\theta - c)}{\alpha_\theta(q + c + \alpha_\theta)}
- \frac{2}{C_\theta(q)} \left( \hat{\nu}(q) \hat{F}(\theta, \mu, q) - RF(\theta, \mu, .)(q) \right). 
\] (2.34)

Using (2.19), (2.32) gives:

\[
\hat{F}(\theta, \mu, q) \left( 1 + \frac{2\hat{\nu}(q)}{C_\theta(q)} \right) = \hat{F}_0(\theta, \mu, q) + \hat{F}_1(\theta, \mu, q) + \frac{2}{C_\theta(q)} RF(\theta, \mu, .)(q)
+ \frac{2}{C_\theta(q)} \left( \hat{\nu}(\alpha_\theta - c) \hat{F}(\theta, \mu, \alpha_\theta - c) - RF(\theta, \mu, .)(\alpha_\theta - c) \right).
\] (2.35)

As \( C_\theta(q) + 2\hat{\nu}(q) = 2(\varphi(q) - \theta) \), it is easy to check:

\[
(\varphi(q) - \theta) \hat{F}(\theta, \mu, q) = \frac{1}{2} C_\theta(q) \left[ \hat{F}_0(\theta, \mu, q) + \hat{F}_1(\theta, \mu, q) \right] + RF(\theta, \mu, .)(q)
+ \hat{\nu}(\alpha_\theta - c) \hat{F}(\theta, \mu, \alpha_\theta - c) - RF(\theta, \mu, .)(\alpha_\theta - c). 
\] (2.36)

Assumptions (2.3) and (2.5) imply the existence of \( \gamma^*_\theta(\theta) \) in \([0, r_\nu]\). Therefore taking \( q = \gamma^*_\theta(\theta) \) in (2.36) brings to:

\[
\hat{\nu}(\alpha_\theta - c) \hat{F}(\theta, \mu, \alpha_\theta - c) - RF(\theta, \mu, .)(\alpha_\theta - c)
= -\frac{1}{2} C_\theta(\gamma^*_\theta(\theta)) \left( \hat{F}_0(\theta, \mu, \gamma^*_\theta(\theta)) + \hat{F}_1(\theta, \mu, \gamma^*_\theta(\theta)) \right) - RF(\theta, \mu, .)(\gamma^*_\theta(\theta)). 
\] (2.37)

Determining \( \hat{F}_0(\theta, \mu, \gamma^*_\theta(\theta)) \) and \( \hat{F}_1(\theta, \mu, \gamma^*_\theta(\theta)) \) by (2.20) et (2.28), relation (2.37) and (2.36) imply directly (2.10).

**Step 2**  Let \( \nu_n \) be the finite measure on \( \mathbb{R} \):

\[
\nu_n(dy) := \nu_{[\mathbb{R} \setminus \mathbb{H}]}(dy) \quad \forall n \geq 1.
\] (2.38)

We set \( \lambda_n := \nu_n(\mathbb{R}) \). We consider \( (J^n_t, t \geq 0) \) a compound Poisson process with Lévy measure \( \nu_n \), and for any \( n \geq 1, x \geq 0 \) and \( t \geq 0 \):

\[
X^n_t = B_t - c_0 t + J^n_t 
\] (2.39)

\[
T^n_x = \inf \{ t \geq 0, X^n_t > x \} 
\] (2.40)

\[
K^n_x = X^n_{T^n_x} - x. 
\] (2.41)
Let $F_n$ be the Laplace transform of $(T^n_x, K^n_x)$:

$$F_n(\theta, \mu, x) = \mathbb{E} \left( e^{-\theta T^n_x - \mu K^n_x} 1_{\{T^n_x < +\infty\}} \right). \quad (2.42)$$

By (2.10), the Laplace transform $\hat{F}_n(\theta, \mu, \cdot)$ of $F_n(\theta, \mu, \cdot)$ verifies, for any $q \in D_0$:

$$\hat{F}_n(\theta, \mu, q) = 1 + \varphi_n(q) - \theta \left[ q - \gamma_n^*(\theta) \right] + \int_0^{+\infty} \left[ \frac{e^{-\gamma_n(\theta) y} - e^{-\mu y}}{y - \mu} - \frac{e^{-\gamma_n(\theta) y} - e^{-\mu y}}{\gamma_n(\theta) - \mu} \right] \nu_n(dy) \quad (2.43)$$

where $\varphi_n, R_n$ and $\gamma_n^*(\theta)$ are associated with $\nu_n$. It is well known:

$$\lim_{n \to +\infty} T^n_x = T_x \quad \text{p.s.} ; \quad \lim_{n \to +\infty} K^n_x = K_x \quad \text{p.s.}$$

Consequently:

$$\lim_{n \to +\infty} F_n(\theta, \mu, x) = F(\theta, \mu, x) \quad \forall \theta, \mu, x \geq 0.$$ 

It is easy to check that $\lim_{n \to +\infty} \varphi_n(q) = \varphi(q)$ and $\lim_{n \to +\infty} \gamma_n^*(\theta) = \gamma_0^*(\theta)$, the proof is left to the reader. Taking the limit $n \to +\infty$ in (2.43), we obtain (2.10). \hfill \Box

2.2 The particular cases $\nu([\cdot, 0]) = 0$ and $\nu([0, +\infty[) = 0$

a) Let us start with the case $\nu([\cdot, 0]) = 0$.

**Proposition 2.3** Assume $\nu([\cdot, 0]) = 0$. Under (2.3) and (2.5), then for any $\mu \geq 0$, $\theta \in [0, \kappa]$,

$$F(\theta, \mu, \rho, q - \gamma_0(\theta)) \sim_{q \to 0} \frac{C_0(\theta, \mu, \rho)}{q} \quad (2.44)$$

where

1. If $\theta > 0$ or if $\theta = 0$ and $\mathbb{E}(X_1) \neq 0$, 

$$C_0(\theta, \mu, \rho) = \frac{1}{\varphi'(-\gamma_0(\theta))} \left[ -\gamma_0(\theta) - \gamma_0^*(\theta) \right] + \int_0^{+\infty} \left[ \frac{e^{(\gamma_0(\theta) - \rho) y} - e^{-\mu y}}{\gamma_0(\theta) + \rho - \mu} - \frac{e^{-(\gamma_0(\theta) + \rho) y} - e^{-\mu y}}{\gamma_0(\theta) + \rho - \mu} \right] \nu(dy). \quad (2.45)$$
2. If $\theta = 0$ et $E(X_1) = 0$, 
\[
C_0(0, \mu, \rho) = \frac{1}{\varphi'(0)} \left( 1 - \frac{2}{(\rho - \mu)^2} \int_0^{+\infty} (1 - e^{(\rho - \mu)y} + (\rho - \mu)y \nu(dy)) \right).
\]
(2.46)

**Remark 2.4**

1. In the companion paper [23] we prove:
\[
\lim_{x \to +\infty} e^{\gamma_0(x)} F(\theta, \mu, \rho, x) = C_0(\theta, \mu, \rho).
\]
(2.47)

2. Since $\varphi'(0) = 0$ if $\theta = 0$ and $E(X_1) = 0$, and $\varphi'(-\gamma_0(\theta)) < 0$ otherwise, this explains why $C_0(\theta, \mu, \rho)$ is given by two different expressions, (2.45) and (2.46).

3. The constant $C_0(\theta, \mu, \rho)$ is positive because $\mu \to \hat{F}(\theta, \mu, q - \gamma_0(\theta))$ is decreasing and 
   \[
   \lim_{\mu \to +\infty} C_0(0, \mu, \rho) = \frac{1}{\varphi''(0)} > 0 \quad \text{if } \theta = 0 \text{ and } E(X_1) = 0,
   \]
   and 
   \[
   \lim_{\mu \to +\infty} C_0(\theta, \mu, \rho) = -\frac{\gamma_0(\theta) + \gamma^*_0(\theta)}{2\varphi'(-\gamma_0(\theta))} > 0 \quad \text{otherwise}.
   \]

4. The constant $C_0(0, 0, 0)$ can be computed explicitly:
\[
C_0(0, 0, 0) = -\frac{\varphi'(0)}{\varphi'(-\gamma_0(0))}
\]
(2.48)

when $E(X_1) < 0$, and $C_0(0, 0, 0) = 1$ otherwise.

**Proof of Proposition 2.3**

Once more we only deal with $\rho = 0$, and $F(\theta, \mu, \cdot)$ stands for $F(\theta, \mu, \rho, \cdot)$.

1) We first suppose $\theta > 0$ or $\theta = 0$ and $E(X_1) \neq 0$.

Recall (cf. Remark 2.4 point 2) $\varphi'(-\gamma_0(\theta)) \neq 0$, then:
\[
\varphi(q - \gamma_0(\theta)) - \theta = \varphi(q - \gamma_0(\theta)) - \varphi(-\gamma_0(\theta)) \sim_{q \to 0} q \varphi'(-\gamma_0(\theta)).
\]
(2.49)

Replacing in (2.11) $q$ by $q - \gamma_0(\theta)$ and taking the limit as $q \to 0$, we conclude immediately that (2.44) holds.

2) If $\theta = 0$ and $E(X_1) = 0$, then $\gamma_0(0) = 0$, $\varphi'(0) = 0$ and 
\[
\varphi(q) \sim_{q \to 0} \frac{q^2}{2} \varphi''(0).
\]
(2.50)

(2.44) follows easily.  \[\square\]

b) We now briefly investigate the case $\nu([0, +\infty]) = 0$.

We observe that $K_x = L_x = 0$, then $F(\theta, \mu, \rho, \cdot) = F(\theta, 0, 0, \cdot)$. We can check that the function $G_\theta : x \to e^{\gamma_0(\theta)x}$ verifies (2.10) and so is the unique solution of the functional equation (1.3) (For a proof, see [25]).
3 A new functional equation verified by $F$

Suppose that $\nu$ satisfied the assumption given in Theorem 2.1. If $\nu(\mathbb{R}) < +\infty$, since the Laplace transformation is one-to-one, $F$ is equivalent to (2.10). But relation (2.10) remains valid when $\nu(\mathbb{R}) = +\infty$. This brings us to ask what is the relation involving $F$ induced by (2.10)? In other words is it possible to inverse (2.10)? That strengthen the role of equation (2.10) and also the approach we have developed previously via the Laplace transform of $F(\theta, \mu, \rho, \cdot)$. Let $\tilde{L}$ be the operator:

$$\tilde{L}f(x) = \frac{1}{2}f''(x) + cf'(x) + \int_{-\infty}^{+\infty} (f(x-y) - f(x))\nu(dy). \quad (3.1)$$

We notice that $\tilde{L}$ is the formal adjoint of the infinitesimal generator $L$ of $(X_t, t \geq 0)$.

**Theorem 3.1** Suppose that $\nu$ satisfies the hypotheses given in Theorem 2.1 and moreover $\int_{-1}^{1} |y|\nu(dy) < \infty$. Then

$$\tilde{L}F(\theta, \mu, \rho, x) - \theta F(\theta, \mu, \rho, x) = g(\mu, \rho, x); \quad x > 0. \quad (3.2)$$

where

$$g(\mu, \rho, x) := -\left( \int_{[x, +\infty[} e^{-\mu(y-x) - \rho x} \nu(dy) \right) \mathbb{1}_{\{x > 0\}} , \quad (3.3)$$

with the boundary conditions:

$$F(\theta, \mu, \rho, 0) = 1, \quad (3.4)$$

$$F'(\theta, \mu, \rho, 0_{+}) = -2 \left[ c + \frac{\gamma_0^* (\theta)}{2} + \frac{1}{\gamma_0^* (\theta)} \mu \int_{0}^{+\infty} (e^{-(\gamma_0^* (\theta) + \rho) y} - e^{-\mu y}) \nu(dy) \right]$$

$$+ \int_{-\infty}^{0} \nu(dy) \left[ \int_{0}^{y} e^{-\gamma_0^* (\theta)(y+b)} F(\theta, \mu, \rho, b) db \right]. \quad (3.5)$$

**Remark 3.2**

1. In (3.2) and (3.5) the derivatives are $x$-derivatives.
2. Suppose $\nu([-\infty, 0]) = 0$. Then the last term in the right hand-side of (3.5) cancels and $F'(\theta, \mu, \rho, 0_{+})$ only depends on $\nu$. Consequently (3.2) is a classical integro-differential linear equation. If moreover $\mu = \rho = 0$, then (3.5) reduces to

$$F'(\theta, 0, 0, 0_{+}) = -\frac{2\theta}{\gamma_0^* (\theta)}. \quad \text{If additionally } \theta = \mu = \rho = 0, \text{ the ruin probability } F(0, 0, 0, .) \text{ solves :}$$

$$\tilde{L}F(0, 0, 0, x) = -\nu([x, +\infty[), \quad x > 0. \quad (3.6)$$

with $F(0, 0, 0, 0) = 1$ and $F'(0, 0, 0, 0_{+}) = 0$ if $\mathbb{E}(X_1) \geq 0$ and...
\( F'(0,0,0,0) = -2\varphi'(0) = 2E(X_1) \) if \( E(X_1) < 0 \). It is easy to check that \( x \to 1 \) is the unique solution of (3.6) with the boundary conditions (3.4) and (3.5) when \( E(X_1) \geq 0 \).

3. Suppose \( \nu([0, +\infty[) = 0 \). Then \( g(\mu, \rho, \cdot) \) cancels and (3.2) reduces to:

\[
\ddot{L}F(\theta, \mu, \rho, x) - \theta F(\theta, \mu, \rho, x) = 0; \quad x > 0.
\]

(3.7)

It is easy to verify that \( x \to e^{-\gamma_0(0)x} \) is the unique solution of (3.7), (3.4) and (3.5).

4. Obviously (3.2) may be written as:

\[
\frac{1}{2} F''(\theta, \mu, \rho, x) + cF'(\theta, \mu, \rho, x) - \theta F(\theta, \mu, \rho, x) = h(x),
\]

where

\[
h(x) = g(\mu, \rho, x) - \int_{-\infty}^{x} F(\theta, \mu, \rho, x - y) - F(\theta, \mu, \rho, x) \nu(dy)
\]

(3.9)

Considering (3.8) as a linear differential equation with given data \( h \), and integrating with the method of variation of parameter, we obtain:

\[
F(\theta, \mu, \rho, x) = e^{\alpha_1 x} \left[ (\alpha_2 - F'(\theta, \mu, \rho, 0_+)) - \int_{0}^{x} e^{-\alpha_1 y} h(y) dy \right]
+ e^{\alpha_2 x} \left[ (F'(\theta, \mu, \rho, 0_+)) - \int_{0}^{x} e^{-\alpha_2 y} h(y) dy \right]
\]

(3.10)

where \( \alpha_1 = -c + \sqrt{c^2 + 2\theta} \) and \( \alpha_2 = -c - \sqrt{c^2 + 2\theta} \).

5. We point out that (3.10) can be written as:

\[
F(\theta, \mu, \rho, x) = F_2(\theta, \mu, \rho, x) + \overline{L}_\theta F(\theta, \mu, \rho, \cdot)(x),
\]

(3.11)

where

\[
F_2(\theta, \mu, \rho, x) := e^{\alpha_1 x} \left[ (\alpha_2 - \int_{0}^{x} e^{-\alpha_1 y} g(\mu, \rho, y) dy) \right]
+ e^{\alpha_2 x} \left[ -\alpha_1 - \int_{0}^{x} e^{-\alpha_2 y} g(\mu, \rho, y) dy \right]
+ \frac{2(e^{\alpha_1 x} - e^{\alpha_2 x})}{\sqrt{c^2 + 2\theta} (\gamma_0(\theta) + \rho - \mu)} \int_{0}^{+\infty} (e^{-(\gamma_0(\theta)+\rho) y} - e^{-\mu y}) \nu(dy)
\]

(3.12)
and $Λ_θ$ is the linear operator:

$$Λ_θ f := 2 \left( e^{\alpha_1 x} - e^{\alpha_2 x} \right) \sqrt{c^2 + 2\theta} \int_{-\infty}^{0} \nu(dy) \int_{-\infty}^{-y} e^{-\gamma_0^*(\theta)(y+b)} F(\theta, \mu, \rho, b) db $$

$$+ \frac{e^{\alpha_1 x}}{\sqrt{c^2 + 2\theta}} \int_{0}^{x} e^{-\alpha_1 y} \left( \int_{-\infty}^{+\infty} (F(\theta, \mu, \rho, y - z) - F(\theta, \mu, \rho, y)) \nu(dz) \right) dy $$

$$+ \frac{e^{\alpha_2 x}}{\sqrt{c^2 + 2\theta}} \int_{0}^{x} e^{-\alpha_2 y} \left( \int_{-\infty}^{+\infty} (F(\theta, \mu, \rho, y - z) - F(\theta, \mu, \rho, y)) \nu(dz) \right) dy \tag{3.13}$$

**Proof of Theorem 3.1**

Multiplying both sides of (2.10) by $\varphi(q) - \theta$, we obtain:

$$(\varphi(q) - \theta) \hat{F}(\theta, \mu, \rho, q) - \frac{q}{2} + \gamma_0^*(\theta) =$$

$$\int_{0}^{+\infty} \left[ \frac{e^{-(q+\rho)y} - e^{-\mu y}}{q + \rho - \mu} - \frac{(e^{-(\gamma_0^*(\theta)+\rho)y} - e^{-\mu y})}{\gamma_0^*(\theta) + \rho - \mu} \right] \nu(dy)$$

$$+ RF(\theta, \mu, \rho, .)(q) - RF(\theta, \mu, \rho, .)(\gamma_0^*(\theta)) \tag{3.14}$$

We observe that the left hand-side and the right hand-side of (3.14) are Laplace transforms. This leads to (3.2), the details are left to the reader. \qed

### A Annex : Properties of $\varphi$ and $\varphi_\theta$

#### A.1 Properties of $\varphi$

Recall that:

$$\varphi(q) = \frac{q^2}{2} + cq + \int_{-\infty}^{+\infty} (e^{-qy} - 1 + qy\mathbb{1}_{\{|y| \leq 1\}}) \nu(dy). \tag{A.1}$$

(A.1) implies that $\varphi(q)$ exists if:

$$\int_{\mathbb{R}} \mathbb{1}_{\{|y| \geq 1\}} |e^{-qy}| \nu(dy) < +\infty, \quad q \in \mathbb{C}. \tag{A.2}$$

Recall that $r_\nu$ is defined by (1.28). Let

$$r_\nu^* := \sup \left\{ s \geq 0; \int_{-\infty}^{-1} e^{-sy}\nu(dy) < +\infty \right\}, \tag{A.3}$$

From now on, we suppose:

$$r_\nu > 0 \quad \text{et} \quad r_\nu^* > 0 \quad (r_\nu \text{ or } r_\nu^* \text{ may be infinite}). \tag{A.4}$$

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By definition $\varphi$ is a convex function defined on $]-r_\nu, r_\nu^*[$.

Moreover

$$\varphi'(0) = \mathbb{E}(X_1).$$  \hfill (A.5)

If $r_\nu < +\infty$ or $r_\nu^* < +\infty$, we extend $\varphi$ as follows :

$$\varphi(-r_\nu) = \lim_{q \to -r_\nu} \varphi(q) \quad \text{et} \quad \varphi(r_\nu^*) = \lim_{q \to r_\nu^*} \varphi(q).$$ \hfill (A.6)

We plot below (see figures 1, 2 and 3) the graph of $\varphi$, distinguishing three cases : $\mathbb{E}(X_1) < 0$, $\mathbb{E}(X_1) > 0$ and $\mathbb{E}(X_1) = 0$.

![Figure 1: Graph of $\varphi$, $\mathbb{E}(X_1) < 0$](image1)

![Figure 2: Graph of $\varphi$, $\mathbb{E}(X_1) > 0$](image2)

When $\varphi$ has two zeros (may be a double zero) in $]-r_\nu, r_\nu^*[$, $-\gamma_0(0)$ (resp. $\gamma_0^*(0)$) will denote the smallest (resp. biggest) one.

**Proposition A.1**

1. In particular in cases Fig 1 a, Fig 2 a and Fig 3, we have :

$$-r_\nu < -\gamma_0(0) \leq 0 \leq \gamma_0^*(0) < r_\nu^*. \hfill (A.7)$$

2. The set $\{s \in [-r_\nu, 0] / \varphi(s) < 0\}$ is an interval, being non empty as soon as $\mathbb{E}(X_1) < 0$. 

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A.2 Properties of $\varphi_\theta : q \to \varphi(q) - \theta$

Let us briefly mention properties of $\varphi_\theta = \varphi - \theta$, $\theta > 0$. For simplicity we restrict ourselves to Fig 1 a, Fig 2 a and Fig 3. The graph of $\varphi_\theta$ is given by Fig 4.

![Graph of $\varphi_\theta = \varphi - \theta$](image)

Figure 4: Graph of $\varphi_\theta = \varphi - \theta$

Then there exists $\kappa > 0$, such that for any $\theta \in ]0, \kappa]$, $\varphi_\theta$ has an unique positive (resp. negative) zero denoted $\gamma_0^*(\theta)$ (resp. $-\gamma_0(\theta)$) and:

$$-r_\nu < -\gamma_0(\theta) < -\gamma_0(0) \leq 0 \leq \gamma_0(0) < \gamma_0^*(\theta) < r_\nu^*. \quad (A.8)$$

A.3 The zeros of $C_\theta(q)$

Assume that $\lambda = \nu(\mathbb{R}) < +\infty$.

We notice that for any $q \in ]-r_\nu, r_\nu^[$, we have:

$$\varphi_\theta(q) = \frac{1}{2}C_\theta(q) + \tilde{\nu}(q), \quad (A.9)$$

where $\tilde{\nu}$ is the Laplace transform of $\nu$, i.e.:

$$\tilde{\nu}(q) := \int_{-\infty}^{+\infty} e^{-qy}\nu(dy), \quad (A.10)$$

and $C_\theta(q)$ is the polynomial function:

$$C_\theta(q) = q^2 + 2c_0q - 2(\lambda + \theta), \quad (A.11)$$
\[ y = C_\theta(q) \]

\[ y = \varphi_\theta(q) \]

\[ \theta \]

Figure 5: Comparison of the zeros of \( C_\theta(q) \) and those of \( \varphi_\theta \)

with \( c_0 = c + \int_R y \mathbb{1}_{\{|y| \leq 1\}} \nu(dy) \). Then

\[ \varphi_\theta(q) > \frac{1}{2} C_\theta(q), \quad q \in ]-r_\nu, r_\nu^*] . \] (A.12)

Recall

\[ C_\theta(q) = (q + c_0 + \alpha_\theta)(q + c_0 - \alpha_\theta), \] (A.13)

where \( \alpha_\theta = \sqrt{c_0^2 + 2(\lambda + \theta)} \).

Let us summarize the results in the following.

**Proposition A.2**  
Suppose \( \theta \geq 0 \). Then:

(i) the two real zeros of \( C_\theta(q) \) are \(-\alpha_\theta - c_0 < 0 \) and \( \alpha_\theta - c_0 > 0 \),

(ii) the set \( \{s \in ]-r_\nu, r_\nu^*] / \varphi_\theta(s) < 0\} \) is an interval included in \( ]-c_0 - \alpha_\theta, \alpha_\theta - c_0[, \)

(iii) if \( \varphi_\theta \) has two zeros \(-\gamma_0(\theta) \) et \( \gamma_0^*(\theta) \) in \([r_\nu, r_\nu^*]\) then:

\[ -c_0 - \alpha_\theta < -\gamma_0(\theta) \leq 0 \leq \gamma_0^*(\theta) < \alpha_\theta - c_0 . \] (A.14)
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