Delooping of the $K$-theory of Waldhausen categories with factorizations

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Introduction

In this article, we will provide new delooping methods of the $K$-theory of certain Waldhausen categories (for precise conditions, see §2) and abelian categories. A specific feature of our delooping is that a suspension of an abelian category is again an abelian category. By utilizing this delooping method, we will show that negative $K$-groups of an abelian category is trivial (see Corollary 3.3) which is conjectured by Marco Schlichting in [Sch06, Conjecture 9.7] and as its consequence, we will obtain a generalization of a theorem of Auslander and Sherman in [She89] which says that negative direct sum $K$-groups and negative $K$-groups are isomorphisms for any small exact categories. (see Corollary 3.4.)

Now we give a guide for the structure of this article. In section 1, we give one to one correspondence between the class of admissible classes and the class of Serre subcategories in a Waldhausen category with factorizations. (for more precise statement, see Proposition 1.8.) In section 2, we will define the delooping of $K$-theory for certain Waldhausen categories by using unbounded filtered objects. (see Definition 2.10.) In the final section, by combining with the results in previous sections, we will prove negative $K$-groups of an abelian category is trivial and we obtain a generalization of a theorem of Auslander and Sherman as mentioned above.

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1 Admissible classes of morphisms and $w$-Serre subcategories

In this section, let $\mathcal{C}$ be a category with cofibrations and assume that $\mathcal{C}$ is enriched over the category of abelian groups. Since $\mathcal{C}$ is closed under finite coproducts, in this case $\mathcal{C}$ is an additive category.

1.1. Definition (Admissible class of morphisms). We say that a class $w$ of morphisms in $\mathcal{C}$ is admissible if $w$ satisfies the following three conditions.
• $w$ contains all isomorphisms.

• $w$ satisfies the two out of three property. Namely for a pair of composable morphisms $x \xrightarrow{f} y \xrightarrow{g} z$, if two of $gf$, $f$ and $g$ are in $w$, then the third one is also in $w$.

• In the commutative diagram of cofibration sequences below

\[
\begin{array}{c}
\begin{array}{c}
x \\ a
\end{array} \xrightarrow{j} \begin{array}{c}
y \\ b
\end{array} \xrightarrow{q} \begin{array}{c}
y/x \\ c
\end{array}
\end{array}
\]

if two of $a$, $b$ and $c$ are in $w$, then the third one is also in $w$.

1.2. Lemma. If $w$ is an admissible class of morphisms in $C$, then $w$ satisfies the gluing axiom of [Wal85]. In particular the pair $(C,w)$ is a category with cofibrations and weak equivalences.

Proof. In the commutative diagram below, assume that $a$, $b$ and $c$ are in $w$.

\[
\begin{array}{c}
\begin{array}{c}
y \\ b
\end{array} \leftarrow \begin{array}{c}
x \\ a
\end{array} \rightarrow \begin{array}{c}
z \\ c
\end{array}
\end{array}
\]

Then there are commutative diagrams of cofibrations sequences.

\[
\begin{array}{c}
\begin{array}{c}
y \oplus z \\ b
\end{array} \xrightarrow{g} \begin{array}{c}
z \\ a
\end{array} \xrightarrow{c} \begin{array}{c}
y \sqcup z \\ a \sqcup c
\end{array}
\end{array}
\]

By the left diagram above, it turns out that $\begin{pmatrix} b & a \\ c & c \end{pmatrix}$ is in $w$ and therefore $b \sqcup a \ast c$ is also in $w$ by the right diagram above.

We fix an admissible class $w$ of morphisms in $C$.

1.3. Definition ($w$-Serre subcategory). We say that a full subcategory $S$ of $C$ is a $w$-Serre subcategory of $C$ if $S$ satisfies the following two conditions.

• For a cofibration sequence $x \to y \to y/x$ in $C$, if two of $x$, $y$ and $y/x$ are in $S$, then the third one is also in $S$.

• $S$ is $w$-closed. Namely for an object $x$ in $C$, if there exists an object $y$ in $S$ and if there exists a zig-zag sequence of morphisms in $w$ which connects $x$ and $y$, then $x$ is also in $S$. 

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For an admissible class $u$ of morphisms in $\mathcal{C}$ which contains $w$, we write $\mathcal{C}^u$ for the full subcategory of $\mathcal{C}$ consisting of those objects $x$ such that the canonical morphism $0 \to x$ is in $u$. Then by the lemma below, $\mathcal{C}^u$ is a $w$-Serre subcategory of $\mathcal{C}$.

1.4. Lemma. For an admissible class $u$ of morphisms in $\mathcal{C}$ which contains $w$, $\mathcal{C}^u$ is a $w$-Serre subcategory of $\mathcal{C}$.

Proof. For a cofibration sequence $x \to y \to y/x$, by applying the axiom of admissible classes to the commutative diagram below, it turns out that if two of $x$, $y$ and $y/x$ are in $\mathcal{C}^u$, then the third one is also in $\mathcal{C}^u$.

\[
\begin{array}{cccc}
0 & \to & 0 & \to 0 \\
\downarrow & & \downarrow & \\
x & \to & y & \to y/x.
\end{array}
\]

Next let $x \xrightarrow{f} y$ be a morphism in $w$ and assume that $x$ (resp. $y$) is in $\mathcal{C}^u$, then $y$ (resp. $x$) is also in $\mathcal{C}^u$ by the two out of three property of $u$.

\[
\begin{array}{ccc}
x & \xrightarrow{f} & y \\
\downarrow & & \downarrow \\
0 \\
\end{array}
\]

The rest of this section, we assume that the pair $(\mathcal{C}, w)$ satisfies the following factorization axiom in [Sch06, A.5].

1.5. (Factorization axiom). For any morphism $x \xrightarrow{f} y$ in $\mathcal{C}$, there exists a cofibration $i_f: x \to u_f$ and a morphism $p_f: u_f \to y$ in $w$ such that $f = p_f i_f$. In this case, we call a triple $(i_f, p_f, u_f)$ a factorization of $f$.

1.6. Definition ($\mathcal{S}$-weak equivalences). Let $\mathcal{S}$ be a $w$-Serre subcategory of $\mathcal{C}$. Then for a morphism $x \xrightarrow{f} y$ in $\mathcal{C}$, the following two conditions are equivalent by Lemma 1.7 below and in this case, we say that $f$ is an $\mathcal{S}$-weak equivalence.

- There exists a factorization $(i_f, p_f, u_f)$ of $f$ such that $u_f/x$ is in $\mathcal{S}$.
- For any factorization $(i_f, p_f, u_f)$ of $f$, $u_f/x$ is in $\mathcal{S}$.

We denote the class of all $\mathcal{S}$-weak equivalences in $\mathcal{C}$ by $w_{\mathcal{S}, \mathcal{C}}$ or shortly $w_{\mathcal{S}}$.

1.7. Lemma. Let $\mathcal{S}$ be a $w$-Serre subcategory of $\mathcal{C}$. Then

1. Let $x \xrightarrow{i} y \xrightarrow{j} z$ be a pair of composable cofibrations in $\mathcal{C}$ such that $z/y$ is in $\mathcal{S}$. Then $y/x$ is in $\mathcal{S}$ if and only if $z/x$ is in $\mathcal{S}$.
(2) If \( x \xrightarrow{i} y \) is a cofibration and a morphism in \( w \), then \( y/x \) is in \( \mathcal{S} \).

(3) In the commutative diagram below, assume that both \( i \) and \( i' \) are cofibrations and both \( p \) and \( p' \) are morphisms in \( w \),

\[
\begin{array}{ccc}
& z & \\
\downarrow & & \downarrow \\
\downarrow & i' & \downarrow \\
& y & \\
\end{array}
\]

\[
\begin{array}{ccc}
& p & \\
\downarrow & & \downarrow \\
\downarrow & p' & \downarrow \\
& y' & \\
\end{array}
\]

then \( z/x \) is in \( \mathcal{S} \) if and only if \( z'/x \) is in \( \mathcal{S} \).

Proof. (1) By considering a cofibrations sequence \( y/x \rightarrow z/x \rightarrow z/y \), we obtain the result.

(2) By the commutative diagram below, it turns out that the canonical morphism \( y/x \rightarrow 0 \) is in \( w \) by the gluing axiom.

\[
\begin{array}{ccc}
& z & \\
\downarrow & & \downarrow \\
\downarrow & i' & \downarrow \\
& y & \\
\end{array}
\]

\[
\begin{array}{ccc}
& p & \\
\downarrow & & \downarrow \\
\downarrow & p' & \downarrow \\
& y' & \\
\end{array}
\]

Then since \( \mathcal{S} \) is \( w \)-closed, it turns out that \( y/x \) is in \( \mathcal{S} \).

(3) By the factorization axiom, there exists a factorization \( (i_p, j_p', q_p, j_p', u) \) of \( p \sqcup p' : z \sqcup z' \rightarrow y \). Then by the two out of three property for \( w \), the composition \( z \rightarrow z \sqcup z' \xrightarrow{i_p \sqcup i_p'} u \) is a cofibration and a morphism in \( w \). Thus by (2), \( u/z \) is in \( \mathcal{S} \). Then by (1), \( z/x \) is in \( \mathcal{S} \) if and only if \( u/x \) is in \( \mathcal{S} \). By symmetry, it is also equivalent to the condition that \( z'/x \) is in \( \mathcal{S} \).

1.8. Proposition. Let \( \mathcal{S} \) be a \( w \)-Serre subcategory of \( \mathcal{C} \) and let \( u \) be an admissible class of morphisms in \( \mathcal{C} \) which contains \( w \). Then

(1) \( w_{\mathcal{S}} \) is an admissible class of morphisms in \( \mathcal{C} \) which contains \( w \).

(2) We have the equalities

\[
\mathcal{C}^{w_{\mathcal{S}}} = \mathcal{S},
\]

\[
w_{\mathcal{C}^{u}} = u.
\]

To show Proposition, we will utilize the following lemma and a proof is straightforward.

1.9. Lemma. Let \( \mathcal{S} \) be a \( w \)-Serre subcategory of \( \mathcal{C} \) and let \( x \xrightarrow{f} y \) and \( x' \xrightarrow{f'} y' \) be a pair of morphisms in \( \mathcal{C} \) and let \( (i_h, p_h, u_h) \) be a factorization of \( h \) for \( h \in \{ f, g, (f \circ g) \} \). Then the triple \( (f \circ i', p \circ i', u \oplus u \circ p) \) is a factorization of \( (f \circ i', p \circ i', u \oplus u \circ p) \). In particular, \( u \xrightarrow{(f \circ i', p \circ i', u \oplus u \circ p)} (x \oplus x') \) is in \( \mathcal{S} \) if and only if \( u_f/x \) and \( u_{f'/x'} \) are in \( \mathcal{S} \).
Proof of Proposition 1.8. (1) For a morphism \( f : x \to y \) in \( w \), the triple \((\text{id}_x, f, x)\) is a factorization of \( f \) such that \( x/x \to 0 \) is in \( S \). Thus \( f \) is in \( w_S \). In particular, \( w_S \) contains all isomorphisms.

We consider the commutative diagram of cofibration sequences \([\text{1}]\). Let \((i_a, p_a, u_a)\) be a factorization of \( a \) and let \((i', p_b, u_b)\) be a factorization of \( y'p_a \vdash_x \)
\( b : u_a \vdash_x y \to y' \). We denote the composition \( y \to u_a \vdash_x y \to u_b \) by \( i_b \) and we set \( u_c := u_b/u_a \) the quotient of \( u_b \) by \( u_a \vdash_a \vdash_x y \to u_b \). Then the induced morphism \( i_h/i_a : y/x \to u_b/u_a = u_c \) is a cofibration by \([Wal85, 1.1.2]\) and \( p_b/p_a : u_c \to y'/x' \) is in \( w \) by the gluing axiom.

Thus the triple \((i_b/i_a, p_b/p_a, u_c)\) is a factorization of \( c \) and the sequence \( u_a/x \to u_b/y \to u_c/(y/x) \) is a cofibration sequence by \([Wal85, 1.1.2]\) again. Hence if two of \( u_a/x, u_b/y \) and \( u_c/(y/x) \) are in \( S \), then the third one is also in \( S \). Namely if two of \( a, b \) and \( c \) are in \( w_S \), then the third one is also in \( w_S \).

Next let \( x \xrightarrow{f} y \xrightarrow{g} z \) be a pair of composable morphisms in \( C \). By applying the argument in the previous paragraph to the commutative diagram of cofibration sequences below

\[
\begin{array}{ccc}
  x & \xrightarrow{(id_x, f)} & x \oplus y \\
  \downarrow f & & \downarrow g \\
  y & \xrightarrow{(0, id_y)} & z
\end{array}
\]

we obtain a factorization \((i_h, p_h, u_h)\) of \( h \) for \( h \in \{f, g, gf, id_y, (0, id_y)\} \) and a cofibration sequence \( u_f/x \to u_f(y/0) = u_f/y \). Since \( id_y : y \to y \) is in \( w_S \), \( u_{id_y}/y \) is in \( S \). Thus by Lemma 1.9 \( u_{id_y}/y \) is in \( S \) if and only if \( u_{gf}/x \) is in \( S \). Hence if two of \( u_{gf}/x \), \( u_{gf}/x \) and \( u_{gf}/y \) are in \( S \), then the third one is also in \( S \). Namely \( w_S \) satisfies the two out of three property.

(2) We will show the equality \([2]\). Let \( x \) be an object in \( S \). Then \((0 \to x, i_d, x)\) is a factorization of the canonical morphism \( 0 \to x \) such that \( x/0 \to x \) is in \( S \). Thus \( 0 \to x \in w_S \). Namely \( x \) is in \( C^w_S \). Next let \( y \) be an object in \( C^w_S \). That is, the canonical morphism \( 0 \to y \) is in \( w_S \). Then there exists a factorization \((0 \to z, z \to y, z)\) such that \( z/0 \to z \) is in \( S \). Since \( p \) is in \( w \) and \( S \) is \( w \)-closed, \( y \) is in \( S \).

Next we will prove the equality \([3]\). Let \( x \xrightarrow{f} y \) be a morphism in \( C \) and let \((i_f, p_f, u_f)\) be a factorization of \( f \). Then \( f \) is in \( w_{C^n} \) if and only if \( u_f/x \) is in \( w_{C^n} \) if and only if \( u_f/y \) is in \( w_{C^n} \).
In the commutative diagram below,

\[
\begin{array}{ccc}
\pi f & \rightarrow & u_f/x \\
\downarrow f & & \downarrow \\
y & \rightarrow & 0, \\
\end{array}
\]

\(f\) is in \(u\) if and only if the canonical morphism \(u_f/x \rightarrow 0\) is in \(u\) and the last condition is equivalent to the condition that \(0 \rightarrow u_f/x\) is in \(u\) by the two out of three property of \(u\).

1.10. Corollary. Assume \(C\) is an abelian category and let \(S\) be a \(w\)-Serre subcategory of \(C\). Then the quotient functor \(C 
\rightarrow C/S\) induces a homotopy equivalence of spectra \(K(C; w) \rightarrow K(C/S)\) on \(K\)-theory.

**Proof.** Notice that since \(S\) is \(w\)-closed, we have an equality \(S^w = C^w\) and the pair \((S, w|S)\) is a Waldhausen category with factorization. Thus by fibration theorem in [Sch06, A.3], there exists a left commutative diagram of fibration sequences below

\[
\begin{array}{ccc}
K(S^w) & \rightarrow & K(C^w) \\
\downarrow & & \downarrow \\
K(S) & \rightarrow & K(C) \\
\downarrow & \rightarrow & \downarrow 1 \\
K(S; w) & \rightarrow & K(C; w) \\
\downarrow & \rightarrow & \downarrow \\
w_S S(C) & \rightarrow & w_S S(C/S) \\
\end{array}
\]

where the map \(1\) is a zig-zag sequence of morphisms which makes the right diagram above commutative. Thus by \(3 \times 3\)-lemma, the map \(1\) is a homotopy equivalence of spectra and it is an inverse map of the induced map \(K(C; w_S) \rightarrow K(C/S)\).

2 Unbounded filtered objects

In this section, let \(C\) be an essentially small category with cofibrations such that \(C\) is an additive category and let \(w\) be an admissible class of morphisms in \(C\) (see 1.1) such that the pair \(C := (C, w)\) satisfies the factorization axiom (see 1.5).

2.1. Definition (Filtered objects). Let \(\mathbb{N}\) be the linearly ordered set of all natural numbers with the usual linear order. As usual we regard \(\mathbb{N}\) as a category and we denote the category of functors and natural transformations from \(\mathbb{N}\) to \(C\) by \(FC\) and call an object in \(FC\) a filtered object (in \(C\)). For a filtered object \(x\) and for a natural number \(n\), we write \(x_n\) and \(i_{x_n}^x\) for an object \(x(n)\) in \(C\) and a morphism \(x(n \leq n + 1)\) in \(C\) respectively.
Let \( f: x \to y \) be a morphism in \( FC \). We say that \( f \) is a level cofibration (resp. level weak equivalence) if for each natural number \( n \), \( f_n \) is a cofibration (resp. \( f_n \) is in \( w \)). We denote the class of all level weak equivalences by \( lw_{FC} \) or simply \( lw \). We can make \( FC \) into a category of cofibrations by declaring the class of all level cofibrations to be the class of cofibrations in \( FC \).

2.2. Lemma.

(1) \( lw \) is an admissible class of morphisms in \( FC \).

(2) The pair \((FC, lw)\) satisfies the factorization axiom.

Proof. A proof of assertion (1) is straightforward. We will give a proof of assertion (2). Let \( n \) be a natural number and assume that \( \sigma_{\leq n} f: \sigma_{\leq n} x \to \sigma_{\leq n} y \) admits a factorization \((i_{\leq n}, p_{\leq n}, u_{\leq n})\). Then let a triple \((i', p_{f_n+1}, u_{f_n+1})\) be a factorization of a morphism \( f_{n+1} \cup x_n u_f: x_{n+1} \cup x_n u_f \to y_{n+1} \) and we denote the compositions \( x_{n+1} \Rightarrow x_{n+1} \cup x_n u_f \Rightarrow u_{f_n+1} \) and \( u_f \Rightarrow x_{n+1} \cup x_n u_f \Rightarrow u_{f_n+1} \) by \( i_{f_n+1} \) and \( i'_{n+1} \) respectively. Then the pair of triples \((i_{\leq n}, p_{\leq n}, u_{\leq n})\) and \((i', p_{f_n+1}, u_{f_n+1})\) give a factorization of \( \sigma_{\leq n+1} f: \sigma_{\leq n+1} x \to \sigma_{\leq n+1} y \). By proceeding induction on \( n \), we finally obtain a factorization of \( f \). \( \square \)

2.3. Definition (Bounded and weakly bounded filtered objects). Let \( a \leq b \) be a pair of natural numbers and let \( x \) be a filtered object in \( C \). We say that \( x \) has amplitude contained in \([a, b]\) if for any \( 0 \leq k < a \), \( x_k = 0 \) and for any \( b \leq k \), \( x_k = x_b \) and \( i^n_k = id_{x_k} \). In this case we write \( x_\infty \) for \( x_b = x_{b+1} = \cdots \). Similarly for any morphism \( f: x \to y \) in \( FC \) between objects which have amplitude contained in \([a, b]\), we denote \( f_b = f_{b+1} = \cdots \) by \( f_\infty \). We denote the full subcategory of \( FC \) consisting of those objects having amplitude contained in \([a, b]\) by \( F_{[a, b]} C \). We also set \( F_b C := \bigcup_{a<b} F_{[a,b]} C \) and \( F_{b \geq a} C := \bigcup_{a<b} F_{[a,b]} C \) and call an object in \( F_b C \) a bounded filtered object (in \( C \)). For a natural number \( n \), we write \( F_{\leq n} C \) and \( F_{\geq n} C \) for the category \( F_{[0,n]} C \) and the full subcategory of \( FC \) consisting of those objects \( x \) such that \( x_k = 0 \) for all \( 0 \leq k < n \) respectively.

Let \( f: x \to y \) be a morphism in \( C \). We write \( j(x) \) and \( j(f): j(x) \to j(y) \) for the object and the morphism in \( F_{[0,0]} C \) such that \( j(x)_\infty = x \) and \( j(f)_\infty = f \). We define \((-)_\infty: F_b C \to C \) and \( j: C \to F_b C \) to be functors by sending an object \( x \) in \( F_b C \) to \( x_\infty \) and sending an object \( x \) in \( C \) to \( j(x) \). We denote \( F_{[0,0]} C \) by \( j(C) \) and sometimes identify it with \( C \) via the functor \( j \).

We say that a filtered object \( x \) in \( C \) is weakly bounded if there exists a natural number \( N \) such that \( x^n = x \) is in \( w \) for all \( n \geq N \). We write \( F_{wb} C \) for the full subcategory of \( FC \) consisting of all weakly bounded filtered objects. For a full subcategory \( D \) of \( FC \), we denote the class of level weak equivalences in \( D \) by \( lw_D \) or \( lw_{FD} \) or simply \( lw \).

2.4. Definition (Stable weak equivalences). A morphism \( f: x \to y \) in \( FC \) is stably weak equivalence if there exists a natural number \( N \) such that \( f_n \) is in \( w \) for all \( n \geq N \). For a full subcategory \( D \) of \( FC \), we denote the class of all stably weak equivalences in \( D \) by \( w_{st} \) or simply \( w_{st} \).
2.5. Lemma.

(1) $w_{st}$ is an admissible class of morphisms in $FC$.

(2) The exact functor $j : C = (C, w) \to (F_b C, w_{st})$ induces a homotopy equivalence $K(C) \to K(F_b C, w_{st})$ of spectra on $K$-theory.

Proof. A proof of assertion (1) is straightforward. For (2), notice that we have the equality $(-)_{\infty} \cdot j = id_C$ and there exists a natural weak equivalence $id_{F_b C} \to j \cdot (-)_{\infty}$ with respect to $w_{st}$. Thus $j$ induces a homotopy equivalence of spectra on $K$-theory. □

We will define several operations on $FC$.

2.6. Definition (Truncation functor). For a natural number $n$, the inclusion functors $F_{\geq n} C \to FC$ and $F_{\leq n} C \to FC$ admit right adjoint functors $\sigma_{\geq n} : FC \to F_{\geq n} C$ and $\sigma_{\leq n} : FC \to F_{\leq n} C$ respectively. Explicitly, for an object $x$ and a morphism $f : x \to y$ in $FC$, we set

$$(\sigma_{\leq n} x)_k := \begin{cases} x_k & \text{if } k \leq n \\ x_n & \text{if } k \geq n \end{cases}, \quad i_{k}^{\leq n} := \begin{cases} i_k & \text{if } k \leq n - 1 \\ id_{x_n} & \text{if } k \geq n \end{cases}

(\sigma_{\geq n} x)_k := \begin{cases} x_k & \text{if } k \geq n \\ 0 & \text{if } k < n \end{cases}, \quad i_{k}^{\geq n} := \begin{cases} i_k & \text{if } k \geq n \\ 0 & \text{if } k \leq n - 1 \end{cases}

(\sigma_{\leq n} f)_k := \begin{cases} f_k & \text{if } k < n \\ f_n & \text{if } k \geq n \end{cases}, \quad (\sigma_{\geq n} f)_k := \begin{cases} f_k & \text{if } k \geq n \\ 0 & \text{if } k \leq n - 1 \end{cases}

There are adjunction morphisms $\sigma_{\geq n} x \to x$ and $\sigma_{\leq n} x \to x$.

2.7. Definition (Degree shift). Let $n$ be a natural number and let $f : x \to y$ be a morphism in $FC$. We define $x[n] : x[n] \to y[n]$ for each natural number $n$. The association $(-)[n] : FC \to FC$, $x \mapsto x[n]$ gives an exact functor.

There exists a natural transformation $\theta : id_{FC} \to id_{FC[1]}$. Namely for a filtered object $x$ in $C$, $\theta_x : x \to x[1]$ is defined by the formula $\theta_x := i_k^x$ for each natural number $k$. For a pair of natural numbers $a < b$, We write $\theta[a,b] : id_{FC[a]} \to id_{FC[b]}$ for the compositions $\theta[b-1][b-2] \cdots \theta[a+1][a]$. For a full subcategory $D$ of $FC$ which is closed under the operation $(-)[1] : D \to D$, we write $\Theta_D$ for the family $\{ \theta_x : x \to x[1] \}_{x \in \text{Ob} D}$ of morphisms in $D$ indexed by the class of objects in $D$.

2.8. Lemma.

(1) $w_{st}|_{F_{wb}C}$ is the smallest admissible class of morphisms in $F_{wb}C$ which contains $lw$ and $\Theta_{F_{wb}C}$.

(2) $F_{wb}C$ is the smallest $w_{st}$-Serre subcategory of $FC$ which contains $j(C)$. 

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(3) For a morphism \( f: x \rightarrow y \) in \( F_{wb}C \) with \( x \in \text{Ob} \ F_{wb}C \), there is a factorization \( (i_f, p_f, u_f) \) of \( f \) with respect to \( w_{at} \) the class of stable weak equivalences such that \( u_f \) is in \( F_{wb}C \).

Proof. (1) Let \( u \) be a class of morphisms in \( F_{wb}C \) which contains \( lw \) and \( \Theta_{F_{wb}C} \). Then we will show that \( u \) contains \( w_{at} \). First we will show that for any natural number \( N \) and for any object \( x \) in \( F_{wb}C \), the natural morphism \( \sigma_{\geq N} x \rightarrow x \) is in \( u \). Notice that in the left commutative diagram below \( \theta^{[0,N]}_{\sigma_{\geq N} x}: \sigma_{\geq N} x \rightarrow \sigma_{\geq N}[N] \), \( \theta^{[0,N]}_x: x \rightarrow x[N] \) and \( \text{id}_{x[N]}: \sigma_{\leq N} x[N] \rightarrow x[N] \) are in \( u \). Thus the natural morphism \( \sigma_{\geq N} x \rightarrow x \) also in \( u \) by the two out of three property.

\[
\begin{array}{ccc}
\sigma_{\geq N} x & \xrightarrow{\theta^{[0,N]}_{\sigma_{\geq N} x}} & x \\
\downarrow & & \downarrow \\
\sigma_{\geq N}[N] & \xrightarrow{\theta^{[0,N]}_x} & x[N].
\end{array}
\]

Next let \( f: x \rightarrow y \) be a morphism in \( w_{at} \) and let \( M \) be a natural number such that \( f_n: x_n \rightarrow y_n \) is in \( w \) for all \( n \geq M \). Then \( \sigma_{\geq M} f: \sigma_{\geq M} x \rightarrow \sigma_{\geq M} y \) is in \( lw \) and there is the right commutative diagram above with \( \sigma_{\geq M} x \rightarrow x, \sigma_{\geq M} y \rightarrow y \) and \( \sigma_{\geq M} f: \sigma_{\geq M} x \rightarrow \sigma_{\geq M} y \) are in \( u \). Thus by the two out of three property of \( u, f \) is also in \( u \).

(2) First we will show that \( F_{wb}C \) is a \( w_{at} \)-Serre subcategory of \( FC \). Let \( x \rightarrow y \rightarrow z \) be a cofibration sequence in \( FC \). Then for each natural number \( n \), since \( w \) is admissible, if two of \( i_{x}^{y}, i_{n}^{y} \) and \( i_{n}^{y/x} \) are in \( w \), then the third one is also in \( w \).

\[
\begin{array}{ccc}
\sigma_{\geq M} x & \xrightarrow{\sigma_{\geq M} f} & \sigma_{\geq M} y \\
\downarrow & & \downarrow \\
x & \xrightarrow{f} & y.
\end{array}
\]

Namely if two of \( x, y \) and \( y/x \) are in \( F_{wb}C \), then the third one is also in \( F_{wb}C \).

Next let \( f: x \rightarrow y \) be a morphism in \( w_{at} \) and assume that \( x \) (resp. \( y \)) is in \( F_{wb}C \). Then there exists a natural number \( N \) such that \( i_{x}^{y}, i_{n}^{y} \) and \( f_n \) are in \( w \) for all \( n \geq N \). Then by considering the commutative diagram below and the two out of three property, it turns out that \( i_{x}^{y} \) (resp. \( i_{n}^{y} \)) is also in \( w \).

\[
\begin{array}{ccc}
x & \xrightarrow{f_n} & y_n \\
\downarrow & & \downarrow \\
x_n & \xrightarrow{i_n} & y_n/x_n
\end{array}
\]

Thus \( y \) (resp. \( x \)) is also in \( F_{wb}C \).

Next let \( S \) be a \( w_{at} \)-Serre subcategory of \( FC \) which contains \( j(C) \). We will show that \( S \) contains \( F_{wb}C \). For an object \( x \) in \( F_{wb}C \), there is an integer \( N \) such that \( i_{x}^{y} \) is in \( w \) for all \( n \geq N \). Then there is a zig-zag sequence \( j(x_N) \leftarrow \sigma_{\geq N} j(x_N) \rightarrow x \) of morphisms in \( w_{at} \). Since \( S \) is \( w_{at} \)-closed, it contains \( x \).
(3) If $y$ is in $F_b \mathcal{C}$, then by the same proof of 2.2 (2), we can show that there exists a factorization $(i_f, p_f, u_f)$ of $f$ with $u_f \in \text{Ob} \ F_b \mathcal{C}$ and $p_f \in \text{lw}$. In general case, let $N$ be a natural number such that $i^n_x = \text{id}_{xN}$ and $i^n_y \in w$ for any $n \geq N$. Then there exists a factorization $x \xrightarrow{f'_i} \sigma_{\leq N}x \xrightarrow{p'_i} y$ of $f$ with $p' \in w_{\text{st}}$. Now by applying the argument in the previous paragraph to $f'$, we obtain the desired factorization.

2.9. Corollary. The inclusion functor $\mathcal{C} = (\mathcal{C}, w) \xrightarrow{\cdot} (F_b \mathcal{C}, w_{\text{st}}) \rightarrow (F_{wb} \mathcal{C}, w_{\text{st}})$ induces a homotopy equivalence $K(\mathcal{C}) \rightarrow K(F_{wb} \mathcal{C}; w_{\text{st}})$ of $K$-theory.

Proof. By approximation theorem in [Sch06, A.2] and 2.8 (3), the inclusion functor $(F_b \mathcal{C}, w_{\text{st}}) \rightarrow (F_{wb} \mathcal{C}, w_{\text{st}})$ induces a homotopy equivalence of spectra on $K$-theory. Thus combining with 2.5 (2), we obtain the result.

2.10. Definition. We denote the smallest admissible class of morphisms in $F \mathcal{C}$ which contains $\text{lw}$ and $\Theta_{F \mathcal{C}}$ by $w_{\text{st}}$ and call it the class of strongly stably weak equivalences in $F \mathcal{C}$. We write $w_{\text{sst}}$ for the admissible class $w_{F_{wb} \mathcal{C}}$ of morphisms in $F \mathcal{C}$ which corresponds to the $w_{\text{st}}$-Serre subcategory $F_{wb} \mathcal{C}$ of $F \mathcal{C}$ and call it the class of suspension weak equivalences in $F \mathcal{C}$. We write $F \mathcal{C}$ and $S \mathcal{C}$ for the pairs $(F \mathcal{C}, w_{\text{sst}})$ and $(F \mathcal{C}, w_{\text{st}})$ and call them flasque envelope of $\mathcal{C}$ and the suspension of $\mathcal{C}$ respectively. The naming of $F \mathcal{C}$ and $S \mathcal{C}$ are justified by the following results.

2.11. Proposition.

(1) (Eilenberg swindle). $K(F \mathcal{C})$ is trivial.

(2) (Suspension). There exists a natural homotopy equivalence of spectra $K(S \mathcal{C}) \rightarrow \Sigma K(\mathcal{C})$.

Proof. (1) We denote the functor which sends an object $x$ in $F \mathcal{C}$ to $\bigsqcup_{n \geq 0} x[n]$ by $F$. Then there exists an equality $F = F[1] \oplus \text{id}_{F \mathcal{C}}$ and there exists a natural weak equivalence $\theta_F : F \rightarrow F[1]$ with respect to $w_{\text{st}}$. Thus $K(F) = K(F[1])$ on $K(F \mathcal{C})$ and the identity morphism of $K(F \mathcal{C})$ is trivial. Hence $K(F \mathcal{C})$ is trivial.

(2) By fibration theorem in [Sch06, A.3] and 2.9 there exists a fibration sequence $K(\mathcal{C}) \rightarrow K(F \mathcal{C}) \rightarrow K(S \mathcal{C})$ of spectra. Combining with the result (1), we obtain the homotopy equivalence of spectra $K(S \mathcal{C}) \rightarrow \Sigma K(\mathcal{C})$.

3 Delooping of the $K$-theory of abelian categories

In this section, let $\mathcal{A}$ be an essentially small abelian category.

3.1. Definition. We let denote $\mathcal{C}$ the category of bounded chain complexes on $\mathcal{A}$ and let $w$ be a class of all quasi-isomorphisms in $\mathcal{C}$. Then the pair $\mathcal{C} = (\mathcal{C}, w)$ satisfies the condition in [2] Since $F \mathcal{C}$ and $F_{wb} \mathcal{C}$ are again abelian categories, we
can consider the quotient abelian category $F \mathcal{C}/F_{wb} \mathcal{C}$ and we denote it by $\Sigma \mathcal{A}$ and call it the suspension abelian category of $\mathcal{A}$. The naming of $\Sigma \mathcal{A}$ is justified by the following result.

3.2. Corollary. There exists a natural homotopy equivalence of spectra $K(\Sigma \mathcal{A}) \sim \Sigma K(\mathcal{A})$.

Proof. By 1.10 the quotient functor $\mathcal{S} \mathcal{C} \to \Sigma \mathcal{A}$ induces a homotopy equivalence of spectra $K(\mathcal{S} \mathcal{C}) \to K(\Sigma \mathcal{A})$ on $K$-theory. On the other hand, by Gillet-Waldhausen theorem [Gil81, Theorem 6.2], [Wal85, Theorem 1.7.1], the inclusion functor $\mathcal{A} \to \mathcal{C}$ induces a homotopy equivalence of spectra $K(\mathcal{A}) \to K(\mathcal{C})$ on $K$-theory. Thus we obtain the result by 2.11 (2). □

Marco Schlichting showed $K_{-1}(\mathcal{A})$ the $-1$th $K$-theory of an essentially small abelian category $\mathcal{A}$ is trivial in Theorem 9.1 in [Sch06]. By applying this theorem to an abelian category $\Sigma^n \mathcal{A}$ for any natural number $n$, we obtain the following corollary which is conjectured by Schlichting in [Sch06, Conjecture 9.7].

3.3. Corollary. For any essentially small abelian category $\mathcal{A}$, $K_n(\mathcal{A})$ the $n$-th $K$-theory of $\mathcal{A}$ is trivial for $n < 0$. □

As pointed out in §10 in [Sch06], by virtue of Proposition 10.1 in [Sch06], Corollary 3.3 implies the following result which is a generalization of a theorem of Auslander and Sherman in [She89].

3.4. Corollary. Let $\mathcal{E}$ be an essentially small exact category. We write $\mathcal{E}^\oplus$ for an exact category with split exact sequences whose underlying additive category is $\mathcal{E}$. Then the identity functor $\mathcal{E}^\oplus \to \mathcal{E}$ induces an isomorphism of $K$-groups $K_n(\mathcal{E}^\oplus) \to K_n(\mathcal{E})$ for any negative integer $n$. □

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