Convergent Normal Form for Five Dimensional Totally Nondegenerate CR Manifolds in $\mathbb{C}^4$

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Abstract
Applying the equivariant moving frames method, we construct a convergent normal form for real-analytic 5-dimensional totally nondegenerate submanifolds of $\mathbb{C}^4$. We develop this construction by applying further normalizations, the possibility of which completely relies upon vanishing/non-vanishing of some specific coefficients of the normal form. This in turn divides the class of our CR manifolds into several biholomorphically inequivalent subclasses, each of them has its own specified normal form with no further possible normalization applicable on it. It also is shown that, biholomorphically, Beloshapka’s cubic model is the unique member of this class with the maximum possible dimension seven of the corresponding algebra of infinitesimal CR automorphisms. Our results are also useful in the study of biholomorphic equivalence problem between CR manifolds, in question.

Keywords CR manifolds · Equivariant moving frame · Normal form

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1 Introduction

One of the most elementary examples of moving frames is the well-known Frenet frame defined on curves. In the late nineteenth century, Darboux generalized the construction of moving frames on surfaces in Euclidean spaces. Later on, in the early twentieth century, Cartan [8] developed extensively the theory of moving frames by extending...
it to more general submanifolds of homogeneous spaces. In Cartan’s thought, moving frames were powerful tools to study geometric features of submanifolds under the action of transformation (pseudo-)groups. In 1990s, Peter Olver and his collaborators endeavored to develop a modern and far-reaching reformulation of Cartan’s classical approach [11,28]. This modernization, known by the equivariant moving frames theory, considerably expands on Cartan’s construction and provides new algorithmic tools for computing sought for differential invariants.

Let $M$ be an arbitrary smooth manifold acted on by a Lie (pseudo-)group $G$. Let $J^n(M, p)$ be the $n$th order jet space of $p$-dimensional submanifolds of $M$. In view of the ordinary chain rule (or total differentiation), the action of $G$ on $M$ induces a prolonged action $G^{(n)}$ on the jet space $J^n(M, p)$ (see Sect. 2 for the explicit formulae). An $n$th order moving frame is defined locally as a $G^{(n)}$-equivariant section of the trivial bundle $J^n(M, p) \times G^{(n)} \to J^n(M, p)$. Existence of such a map heavily relies upon providing a so-called cross-section which is a transverse submanifold $K$ of $J^n(M, p)$ intersecting each jet orbit at most once.

The constructive theory of equivariant moving frames has exhibited its large potential of applications in many other fields (see e.g. the bibliographies of [24,26] for a list of relevant works in the literature). As a nontrivial application concerned with the pseudo-group actions, one may consider the works [31,32] on the classification of the gauge differential invariants of the Laplace operators and the factorization of the linear partial differential operators. Of particular interest, the equivariant moving frames theory offers a modern reformulation of Cartan’s classical approach [23] to equivalence problems [34]. In this formalism, the cross-section associated with a moving frame plays the role of normalization in Cartan’s theory. In contrast to the classical approach, existence of the essential tool recurrence formula (see Sect. 2 for pertinent definition) enables one to normalize the corresponding Maurer–Cartan forms, if possible, without requiring explicit expressions of the appearing torsion coefficients and just by applying some linear algebra techniques. This method is applied so far to a wide range of standard equivalence problems including those of polynomials, differential equations, differential operators and variational problems (cf. [1,5,33,34]).

Another significant application of the equivariant moving frames is in construction of normal forms for real-analytic manifolds $M$ acted on by certain transformation (pseudo-)groups $G$ [26]. Roughly speaking, a normal form is made by employing the group transformations to simplify, as much as possible, the coefficients of the Taylor series expansion of $M$. This process runs quite parallel to find practical cross-sections underlying the construction of the corresponding moving frames (see Sect. 2.3). Serendipitously, normal forms also encode $G$-equivalence problems. Indeed, two manifolds $M$ and $M'$ with the same underlying group action $G$ are equivalent if and only if they have the identical normal forms at the matching points.

In the CR geometry setting, “biholomorphic equivalence” and “normal form” are two central issues of profound interest. Serious study of the former one was initiated by Cartan in [6,7] (see also [23]) while the latter was developed widely by Chern and Moser in [9] (see also [12]). Despite their notable overlapping, each of these theories has also its own merits. For example, Cartan’s approach not only enables one to solve certain equivalence problems but also provides the opportunity of reaching the structure equations of the corresponding coframes which is the key to unlock
the structure of the associated symmetry algebras. On the other hand, although the equivalence invariants find themselves among the coefficients of the corresponding normal forms but Cartan’s approach is unable to unveil explicit structure of such normal forms.

Surprisingly, Olver’s approach of equivariant moving frames provides one with the opportunity of reaching simultaneously almost all expected advantages of both theories of equivalence problems and normal forms. It builds a concrete and striking bridge between these two theories. Even more and slightly in contrast to the Chern–Moser [9]—which is the original impulse of most works in CR geometry relevant to normal forms—the method of moving frames is much systematic in the sense that one applies it order by order in an algorithmic way consisting of symbolic computations.

But, in spite of its applications, this modern approach has not gained yet its deserved attention in CR geometry, neither for solving biholomorpohic equivalence problems nor for constructing desired normal forms. That is while, historically, biholomorphic equivalence between 3-dimensional real hypersurfaces in $\mathbb{C}^2$ is one of the primary problems studied by Cartan in [6,7]. Anyway, the wide breadth of both theoretical and practical applications of the equivariant moving frames theory is encouraging and motivating enough to apply it in the rich geometry of CR manifolds.

A crucial drawback to the Chern-Moser construction of normal forms concerns the convergence. Indeed, the heuristic and intelligent method of Chern-Moser in constructing normal forms for real-analytic nondegenerate hypersurfaces entailed them to consider the formal transformations, instead of holomorphic ones. Although they eventually proved the convergence of their formal normal forms (see [9, Theorem 3.5]), but this key issue remained much challenging and occasionally unsolved in the subsequent studies inspired by Chern-Moser (see e.g. [4,10,13–17]). For instance, it turned out in [14] that the normal form introduced in [13] is generally divergent although it does not prevent its important application in solving biholomorphic equivalence problem between finite type degenerate hypersurfaces in $\mathbb{C}^2$. In contrast, the equivariant moving frames approach basically involves infinitesimal counterpart of the associated (pseudo-)groups. Thus, it does not restrict one to work with formal transformations and impressively results in constructing convergent normal forms, in essence.

Let $T^c M \subset TM$ be an even rank tangent distribution of a smooth real manifold $M$ equipped with a fiber preserving complex structure $J : T^c M \to T^c M$ enjoying $J \circ J = -\text{id}$. Define the holomorphic subdistribution of the complexification $\mathbb{C} \otimes T^c M$ as:

$$T^{1,0} M := \{ X - iJ(X) : X \in T^c M \}.$$  

By definition [2], $M$ is an (abstract) Cauchy–Riemann (CR for short) manifold with the CR structure $T^c M$ if $T^{1,0} M \cap \overline{T^{1,0} M} = \{0\}$ and $[T^{1,0} M, T^{1,0} M] \subset T^{1,0} M$. In this case, $n := \frac{1}{2} \dim T^c M$ and $k := \dim M - 2n$ are referred to as CR dimension and CR codimension of $M$.

Totally nondegenerate CR manifolds are investigated widely by Valerii Beloshapka following his researches in the last decades of twentieth century. Let $M$ be a CR manifold with the CR structure $T^c M$. For each $t \in \mathbb{N}$, let $g^{-t}$ be the vector space

\[ \mathbb{C} \otimes T^c M \]
spanned by the iterated Lie brackets between exactly \( t \) vector fields in \( T^c M \). By definition \([3, 30]\), \( M \) is totally nondegenerate of length \( \mu \) if: (a) \( T M \) is generated by \( g^{-1} + \cdots + g^{-\mu} \); (b) \( T^c M \) is a regular distribution meaning that for each \( t = 1, \ldots, \mu \), the rank of the subspace \( g^t \) is constant on \( M \) and (c) the truncation:

\[
(g^{-1} + \cdots + g^{-\mu})/g^{-\mu} \cong g^{-1} + \cdots + g^{-(\mu-1)}
\]

forms a depth \( \mu - 1 \) free Lie algebra generated by \( g^{-1} = T^c M \)—the latter is the total nondegeneracy condition. Beloshapka in [3] devised a certain machinery to construct in part the defining functions of these CR manifolds. Linear independency of the lower degree terms in the Taylor series of these functions around the origin ensures the total nondegeneracy of their corresponding CR manifolds (we refer the reader to [3] for more details). Beloshapka also extended Poincaré’s and Chern-Moser’s approach of introducing model hypersurfaces to higher codimensions. These models enjoy several interesting features; in particular in each fixed CR dimension \( n \) and codimension \( k \), the dimension of their associated Lie algebras of infinitesimal CR automorphisms is maximal. Moreover, as is proved in [30], if \( k > n^2 \) then all origin preserving CR automorphisms of these models are linear (see [3, Theorem 14] for further nice properties of the models).

Let us confine our discussion to the totally nondegenerate manifolds of CR dimension one, a case of which has received particular attention in the literature. In CR codimension one, such manifolds are actually real hypersurfaces in \( \mathbb{C}^2 \) investigated specifically by Chern and Moser in [9] (see also [35]). In CR codimension two, namely for 4-dimensional totally nondegenerate submanifolds of \( \mathbb{C}^3 \), the construction of normal form is done by Beloshapka, Ezhov and Schmalz in [4]. The main approach applied in this work was that of Chern-Moser.

As the next class in CR dimension one, our goal in this paper is to construct convergent normal forms for real-analytic 5-dimensional totally nondegenerate submanifolds of \( \mathbb{C}^4 \) that we denote them throughout this paper by \( M^5 \). For this goal, we apply the equivariant moving frames techniques as proposed in [26]. Biholomorphic equivalence between the elements of this class is completely investigated in [21], where the desired invariants of this problem are found. It is discovered in [21, Proposition 2.1] that, after certain elementary normalizations, each CR manifold \( M^5 \) can be represented in local coordinates \( z, w^1, w^2, w^3 \) of \( \mathbb{C}^4 \) with \( w^j := u_j + iv^j \) by:

\[
\begin{align*}
v^1 &= \Phi^1(z, \bar{z}, u) := z\bar{z} + O(3), \\
v^2 &= \Phi^1(z, \bar{z}, u) := z^2\bar{z} + z\bar{z}^2 + O(4), \\
v^3 &= \Phi^1(z, \bar{z}, u) := i(z^2\bar{z} - z\bar{z}^2) + O(4),
\end{align*}
\]

where, after assigning the weights \([z] = [\bar{z}] = 1, [u_1] = 2 \) and \([u_2] = [u_3] = 3 \) to the variables \( z, \bar{z}, u_j, j = 1, 2, 3 \), then \( O(t) \) denotes the sum taken over the monomials of weights \( \geq t \). In view of the proof of [21, Proposition 2.1], the linear independency of the lower weighted degree (real) terms \( z\bar{z}, z^2\bar{z} + z\bar{z}^2 \) and \( i(z^2\bar{z} - z\bar{z}^2) \) of the above defining functions \( \Phi_j, j = 1, 2, 3 \) ensures the total nondegeneracy of \( M^5 \).
Our results increase in interest if we realize that there are only two types of 5-dimensional Levi nondegenerate generic CR manifolds, namely those of CR dimension 1 and codimension 3 in $\mathbb{C}^4$ and those of CR dimension 2 and codimension 1 in $\mathbb{C}^3$. The construction of normal forms for the latter class is almost concluded specifically in the works of Ebenfelt and Loboda [10,19,20]. But still, there is no considerable work concerning the normal forms of the former class. In [22], this class is divided into two distinct subclasses $\text{III}_1$ and $\text{III}_2$. In this terminology, what we aim to consider here is the first class $\text{III}_1$. Thus, this work can be viewed as the first investigation of normal forms for 5-dimensional nondegenerate CR manifolds in $\mathbb{C}^4$.

The outline of this paper is as follows. In Sect. 2, we present a brief description of equivariant moving frames for Lie pseudo-group actions. We also present some fundamental formulae and results arising in this theory. In Sect. 3, we provide the requisite materials for constructing the desired equivariant moving frame and normal form associated with the pseudo-group action of holomorphic transformations on totally nondegenerate manifolds $M^5$. In particular, we produce the infinitesimal version of this action which is necessary for launching the construction in Sect. 4. Our main tool in this construction is the powerful recurrence formula. In Sect. 4, by producing a partial cross-section, we succeed to construct the desired normal form for totally nondegenerate manifolds $M^5$. It is achieved by normalizing the defining jet coefficients up to order four. Subsequently, we proceed with the method to construct a complete cross-section and moving frame. The results enable us to normalize further the appearing coefficients of our normal form. The possibility of such normalizations completely relies upon vanishing and non-vanishing of some specific higher order coefficients. This observation leads us in turn to divide, through three major subbranches 1, 2 and 3, the class of our CR manifolds $M^5$ into some inequivalent subclasses, each of which has its own specified normal form. This situation is analogues to that of Chern-Moser supplementary normalizations in part (d), page 246 of [9] for non-umbilical hypersurfaces in $\mathbb{C}^2$, or those applied by Webster in [35] for arbitrary non-umbilical hypersurfaces in $\mathbb{C}^{n+1}$. The three Branches 1, 2 and 3, appearing at the end of Sect. 4 are the first but not the last branches which emerge along the way and it turns out soon that several further subbranches are await for us in the next orders. We display the appearing branches via the following diagram (Fig. 1).
The size of computations in orders $\geq 5$ groves explosively and we perform them with the aid of MAPLE. In the next two Sects. 5 and 6 we apply all further possible normalizations on manifolds belonging to Branches 1 and 2. Such normalizations approximate the normal form to order five (cf. Theorems 5.1 and 6.1). Seeking for further possible normalizations in Branch 3, we encounter (unpleasantly!) in Sect. 7 several new subbranches as are visible in the diagram. We defer the study of Branch 3-2 to another investigation, but a complete list of possible normalizations in all other branches 3-1, 3-3-1-a, 3-3-1-b, 3-3-2 and 3-3-3 is provided. In Branch 3-1 such normalizations approximate the normal form to order 5 (cf. Theorem 7.1) while in the next branches, it is approximated to orders 6 and 7 (cf. Theorems 7.2, 7.3 and 7.4). We emphasize that—disregarding the branch 3-2 that we did not study it here—our convergent normal forms are complete in the sense that no further normalization is applicable on them.

Let us collect together and exhibit concisely the main results of this paper.

**Theorem 1.1** Every five-dimensional real-analytic totally nondegenerate CR manifold $M^5 \subset \mathbb{C}^4$ can be transformed holomorphically into the convergent normal form:

\[
\begin{align*}
v^1 &= z\overline{z} + \sum_{j+k+\ell \geq 5} \frac{1}{j!k!\ell!} V^1_{j,k,\ell} z^j \overline{z}^k u^\ell, \\
v^2 &= \frac{1}{2} (z^2 \overline{z} + z\overline{z}^2) + \sum_{j+k+\ell \geq 5} \frac{1}{j!k!\ell!} V^2_{j,k,\ell} z^j \overline{z}^k u^\ell, \\
v^3 &= -\frac{i}{2} (z^2 \overline{z} - z\overline{z}^2) + \frac{1}{6} V^3_{1,3,1} z \overline{z}^3 + \frac{1}{6} V^3_{3,1,3} z^3 \overline{z} + \frac{1}{2} V^3_{1,2,1} z^2 \overline{z} u_1 + \frac{1}{2} V^3_{1,3,2} z^3 u_1 + \\
&\quad + \sum_{j+k+\ell \geq 5} \frac{1}{j!k!\ell!} V^3_{j,k,\ell} z^j \overline{z}^k u^\ell,
\end{align*}
\]

(2)

where for $\ell = (l_1, l_2, l_3) \in \mathbb{N}^3$ we denote $\ell! := l_1! l_2! l_3!$ and $u^\ell := u_1^{l_1} u_2^{l_2} u_3^{l_3}$. Taking into account the conjugation relations $V^\ast z^k \overline{z}^\ell u^\ell = V_{z^k \overline{z}^\ell u^\ell}^\ast$, these coefficient functions enjoy the cross-section normalizations:

\[
\begin{align*}
0 &= V^1_{z^1 U^\ell} = V^1_{z^2 U^\ell} = V^1_{z^2 \overline{z} U^\ell} = V^1_{z^2 \overline{z}^2 U^\ell} = V^1_{z^3 \overline{z} U^\ell}, \\
0 &= V^2_{z^1 U^\ell} = V^2_{z^2 U^\ell} = V^2_{z^2 \overline{z} U^\ell} = V^2_{z^2 \overline{z}^2 U^\ell}, \\
0 &= V^3_{z^1 U^\ell} = V^3_{z^2 U^\ell} = V^3_{z^2 \overline{z} U^\ell} = V^3_{z^2 \overline{z}^2 U^\ell}
\end{align*}
\]

for $j, l \in \mathbb{N}$. Further normalizations on this normal form relies upon vanishing and non-vanishing of some of its specific coefficient functions which causes splitting the process into several branches. The results are displayed concisely in the following table (cf. [21, Theorem 1.3]):
where NNF stands for the term “Normalized Normal Form”, where \( \dim \) denotes the dimension of the infinitesimal CR automorphism algebra \( \mathfrak{aut}_{CR}(M^5) \), where \( A \) is the collection of four lifted differential invariants visible in (44) and where \( B, C \) and \( D \) are the assumptions made respectively in (51), (53) and (57). The origin-preserving holomorphic transformation which brings \( M^5 \) into its associated final normal form is essentially unique except when \( M^5 \) belongs to one of the two branches 3-3-1-b or 3-3-3. In the former branch, it is unique up to the action of the 1-dimensional isotropy group of the corresponding normal form and in the latter one it is unique up to the action of the 2-dimensional isotropy group of Beloshapka’s cubic model \( M^5_c \)—defined as (58)—at the origin. As a consequence, \( M^5_c \) is biholomorphically the unique totally nondegenerate manifold with the maximum possible dimension 7 of the corresponding algebra of infinitesimal CR automorphisms.

We notice that by [3, Proposition 12], it was actually a known fact that the dimension of the associated Lie algebra \( \mathfrak{aut}_{CR}(M^5_c) \) is maximal in the class of totally nondegenerate CR manifolds \( M^5 \). However, to the best of the author’s knowledge, the “uniqueness” feature mentioned at the end of the above theorem is new. Analogous to the Beloshapka–Ezhov–Schamlz normal form [4] for 4-dimensional totally nondegenerate manifolds in \( \mathbb{C}^3 \), our normal form (2) exhibits its lowest order invariants in order four. This order for the Chern–Moser normal form [9] of 3-dimensional real hypersurfaces in \( \mathbb{C}^2 \) is six. The equivariant moving frames method, applied in this work, seems vastly feasible to investigate also the convergence issue of the Beloshapka–Ezhov–Schmalz normal form and to further develop it through supplementary normalizations.

## 2 Equivariant Moving Frames for Lie Pseudo-Group Actions

The modern reformulation of the theory of moving frames is established first in the case of finite dimensional Lie group actions by Fels and Olver [11]. Later on, Olver and Pohjanpelto [27,28] extended it to the case of infinite dimensional pseudo-group actions. In this section, we present a brief description of equivariant moving frames theory in the latter case, as we aim to apply it in the next sections.

We fix \( M \) throughout this section as a \( m \)-dimensional smooth manifold with local coordinates \( x = (x^1, \ldots, x^m) \) and let \( \mathcal{D}(M) \) be the pseudo-group of local diffeomorphisms \( \varphi : M \rightarrow M \). We denote by \( \mathcal{D}^{(n)}(M) \) the bundle of \( n \)th order jets of \( \mathcal{D}(M) \).
We also denote by the capital letters $X := (X^1, \ldots, X^m)$ the target coordinates of $M$ under diffeomorphisms $\varphi \in \mathcal{D}(M)$, i.e. $X := \varphi(x)$. Assume that $\mathcal{G} \subset \mathcal{D}(M)$ is a Lie pseudo-subgroup acting on $M$. Naturally, $\mathcal{G}^{(n)} \subset \mathcal{D}^{(n)}(M)$ denotes the collection of all $n$th order jets of diffeomorphisms belonging to $\mathcal{G}$. The induced local coordinates of $\mathcal{G}^{(n)}$ are denoted by $(x, X^{(n)})$, where the components $X^j$ of the $n$th order jet coordinates $X^{(n)}$, for $\sharp J \leq n$, represent the partial derivatives $\partial^J \varphi/\partial x^J$ for $\varphi \in \mathcal{G}$. As is known (cf. [24]), $\mathcal{G}^{(n)}$ is identified as the solution of some system of partial differential determining equations:

$$F^{(n)}(x, X^{(n)}) = 0.$$  \(3\)

Every Lie pseudo-group $\mathcal{G} \subset \mathcal{D}(M)$ can be characterized by its corresponding algebra $\mathfrak{g}$ of locally defined vector fields on $M$ whose flows belong to $\mathcal{G}$. Such vector fields are called the infinitesimal generators of $\mathcal{G}$. Set up the lift of an arbitrary vector field:

$$v := \sum_{i=1}^m \xi^i(x) \frac{\partial}{\partial x^i}$$

on $M$ as the vector field:

$$\mathbf{V} := \sum_{i=1}^m \sum_{\sharp J \geq 0} \mathbb{D}_x^J \xi^i(X) \frac{\partial}{\partial x^i}$$

on the diffeomorphism jet bundle $\mathcal{D}^{(\infty)}(M)$, where the total derivative operators $\mathbb{D}_x^i$ are defined as:

$$\mathbb{D}_x^i := \frac{\partial}{\partial x^i} + \sum_{j=1}^m \sum_{\sharp J \geq 0} X^j_{,J,i} \frac{\partial}{\partial X^j_{,J,i}} \quad i = 1, \ldots, m$$  \(4\)

and for each multi-index $J$ with $\sharp J = k$, we set $\mathbb{D}_x^J := \mathbb{D}_x^{J,1} \ldots \mathbb{D}_x^{J,k}$. Then, the above vector field $v$ belongs to the infinitesimal Lie algebra $\mathfrak{g}$ if its vector components $\xi := (\xi^1, \ldots, \xi^m)$ satisfy the infinitesimal determining equations:

$$L^{(n)}(x, \xi^{(n)}) := \mathbf{V}[F^{(n)}(x, X^{(n)})]|_{1^{(n)}} = 0$$  \(5\)

where $F^{(n)}(x, X^{(n)})$ is the $n$th order determining equation (3) of $\mathcal{G}$ and $1^{(n)}$ is the $n$th order jet of the identity transformation (see [23,24] for more details).

In agreement with the notations introduced above and for each $i = 1, \ldots, m$, set:

$$\sigma^i := \sum_{j=1}^m X^j \, dx^j \quad \text{and} \quad \mu^i := dx^i - \sum_{j=1}^m X^j \, dx^j.$$  \(6\)

In this terminology, the 1-forms $\sigma^i$ and $\mu^i$ are called the invariant horizontal and zeroth order Maurer–Cartan forms, respectively. Dual to the forms $\sigma^1, \ldots, \sigma^m$, we have differential operators:
\[ D_{X^i} = \sum_{j=1}^{m} W^j_i D_{X^j} \quad i = 1, \ldots, m, \]

where \( W^j_i := W^j_i(x, X^{(1)}) \) is the \((j,i)\)th entry of the inverse matrix \((X^j_i)^{-1}\). For \( i = 1, \ldots, m \) and associated to each multi-index \( J \) with \( k = \#J \), we define the \( k \)th order Maurer–Cartan form:

\[ \mu^i_j := D^j_i \mu^i. \]

These right invariant 1-forms \( \mu^(\infty) := (\mu^1, \ldots, \mu^m, \ldots, \mu^j, \ldots) \) generate all Maurer–Cartan forms of the diffeomorphism pseudo-group \( \mathcal{G}^{(\infty)}(M) \). This collection together with the horizontal forms \( \sigma^1, \ldots, \sigma^m \), makes a right-invariant coframe on \( \mathcal{G}^{(\infty)}(M) \) with the associated structure equations (see [24, 33]):

\[
\begin{align*}
\mathrm{d}\sigma^i &= \sum_{j=1}^{m} \mu^i_j \wedge \sigma^j, \\
\mathrm{d}\mu^i_j &= \sum_{j=1}^{m} \sigma^j \wedge \mu^i_{j,j} + \sum_{j=1}^{m} \sum_{I+K=J \#K \geq 1} \binom{J}{I} \mu^i_{I,j} \wedge \mu^j_{K},
\end{align*}
\]

where \( I + K \) is the component-wise addition of multi-indices in \( \mathbb{N}^m \) and, when it is equal to \( J \), we define \( \binom{J}{I} = \frac{J!}{I!K!} \). Up to order two, the first few structure equations are:

\[
\begin{align*}
\mathrm{d}\mu^i &= \sum_{j=1}^{m} \sigma^j \wedge \mu^i_j, \quad \text{and} \quad \mathrm{d}\mu^i_k &= \sum_{j=1}^{m} \sigma^j \wedge \mu^i_{kj} + \sum_{j=1}^{m} \mu^i_j \wedge \mu^k_j. \quad (7)
\end{align*}
\]

When we restrict the Maurer–Cartan forms \( \mu^(\infty) \) to a pseudo-subgroup \( \mathcal{G} \) of \( \mathcal{G}^{\infty}(M) \), it expectedly appears among them some linear dependencies. By [24, Theorem 6.1] and in each order \( n \), these linear dependencies can be detected through the linear system (cf. (5)):

\[ L^{(n)}(X, \mu^{(n)}) = 0. \]

This provides one with the opportunity of finding a complete right-invariant coframe on the Lie pseudo-subgroup \( \mathcal{G}^{(\infty)} \), as well.

Now, let us view this fragment of the theory from a broader perspective by considering the \( n \)th order jet bundle \( J^n(M, p) \), \( 0 \leq n \leq \infty \), consisting of equivalence \( p \)-dimensional submanifolds \( S \) of \( M \) under the contact form equivalence. Splitting local coordinates \( x^1, \ldots, x^m \) of \( M \) into independent and dependent variables \( x^1, \ldots, x^p \) and \( u^1, \ldots, u^q \) with \( p + q = m \), every submanifold \( S \) can be represented locally as the graph of some \( q \) defining equations \( u^\alpha := u^\alpha(x^1, \ldots, x^p), \alpha = \ldots \)
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1, $\ldots$, $q$. Accordingly, the local coordinates of $J^n(M, p)$ are represented by independent variables $x^1, \ldots, x^p$ and dependent jet variables $u^{(n)} := u^a_j$ for $\alpha = 1, \ldots, q$ and for multi-indices $J$ with $\sharp J \leq n$. Consider again $\mathcal{G} \subset \mathcal{D}(M)$ as a Lie pseudo-group acting on $M$. In view of the ordinary chain rule, it induces as follows the so-called $n$th order prolonged action $\mathcal{G}^{(n)}$ on $J^n(M, p)$ (see [23] for more details). For a multi-index $J$ with $\sharp J \leq n$, denote by $U^\alpha_J := u^a_{\alpha J}$ for $\alpha = 1, \ldots, q$ and for multi-indices $J$ with $\sharp J \leq n$. Consider again $G \subset D(M)$ as a Lie pseudo-group acting on $M$. In view of the ordinary chain rule, it induces as follows the so-called $n$th order prolonged action $G^{(n)}$ on $J^n(M, p)$ (see [23] for more details). For a multi-index $J$ with $\sharp J \leq n$, denote by $U^\alpha_J := u^a_{\alpha J}$ the transformed jet coordinate under the action of $G^{(n)}$. Aiming for a concrete formula to compute the expression of $U^\alpha_J$, consider first the total derivations on all group and jet variables (cf. (4)):

$$D_{x^j} := \frac{\partial}{\partial x^j} + \sum_{\alpha=1}^{q} \left( u^a_j \frac{\partial}{\partial u^\alpha_j} + \sum_{\sharp J \geq 1} u^a_{j,J} \frac{\partial}{\partial u^\alpha_j} \right), \quad j = 1, \ldots, p.$$ 

Corresponding to each $D_{x^j}$, set up the total differential operator:

$$D_{X^i} = \sum_{j=1}^{p} W^j_i D_{x^j}$$

where $(W^j_i) = (D_{x^j} X^i)^{-1}$ is the inverse of the total Jacobian matrix. These differential operators serve to compute desired expression of the transformed coordinates as:

$$U^{\alpha}_J = D_{X^j} U^\alpha := D_{X^{j_1}} \ldots D_{X^{j_k}} U^\alpha.$$ 

We notice that by the way of defining the operators $D_{x^j}$ and $D_{u^\alpha}$ in (4), the prolonged group action on $J^n(M, p)$ is locally parameterized by the local coordinates of $\mathcal{G}^{(n)}$ introduced at the beginning of this section.

The above total differential operators $D_{x^j}$ make the lifted horizontal coframe consisting of the forms:

$$\omega^i := \sum_{j=1}^{p} D_{x^j} X^i \, dx^j \quad i = 1, \ldots, p.$$ 

Roughly, after providing an equivariant moving frame, this coframe plays the role of an invariant coframe on the submanifolds, in question. In the language of Cartan’s classical theory of equivalence [23], it has the role of lifted coframe associated with the $\mathcal{G}$-equivalence problem between the submanifolds $S$.

Let $\mathcal{H}^{(n)}$ be the groupoid obtained by pulling back the pseudo-group $\mathcal{G}^{(n)}$ via the canonical projection $\pi : J^n(M, p) \to M$. Also let $d_1$ denotes the differentiation with respect to only jet coordinates. Substituting each $x^i, u^\alpha, u^{\alpha_{x^i}}, u^{\alpha_{x^i x^j}}, \ldots$ and each standard form $dx^i, du^\alpha, du^{\alpha_{x^i}}, du^{\alpha_{x^i x^j}}, \ldots$ with its corresponding $X^i, U^\alpha, U^{\alpha_{x^i}}, U^{\alpha_{x^i x^j}}, \ldots$ and $d_1 X^i, d_1 U^\alpha, d_1 U^{\alpha_{x^i}}, d_1 U^{\alpha_{x^i x^j}}, \ldots$, every arbitrary jet form $\Omega$ defined on $J^{(n)}(M, p)$ lifts to a certain form $\lambda(\Omega)$ on $\mathcal{H}^{(n)}$. We call $\lambda$ the lifting operator.
2.1 Equivariant Moving Frame

Now, we are ready to define the equivariant moving frame associated with the action of a Lie pseudo-group $G$ on the manifold $M$.

**Definition 2.1** A moving frame of order $n$ is a map $\rho^{(n)} : J^n(M, p) \to \mathcal{H}^{(n)}$, provided for each jet point $z^{(n)} = (x, u^{(n)})$ with $z = \pi(z^{(n)})$ that the source point of $\rho^{(n)}(z^{(n)})$ is $z^{(n)}$ and moreover it enjoys the equivariant condition:

$$\rho^{(n)}(g^{(n)} \cdot z^{(n)}) = \rho^{(n)}(z^{(n)}) \cdot (g^{n})^{-1}$$

for every $g^{(n)} \in G^{(n)}|_z$ with the inverse $(g^{(n)})^{-1}$.

Local existence of such a moving frame map is guaranteed by regularity and freeness of the action. By definition [27], an action is regular when its corresponding orbits have the same dimension and moreover there are arbitrarily small neighborhoods whose intersection with each orbit is a connected subset thereof. The ($n$th order) freeness means that at every point $z^{(n)} \in J^n(M, p)$, the isotropy subgroup:

$$G^{(n)}_z := \{g^{(n)} \in G^{(n)}|_z : g^{(n)} \cdot z^{(n)} = z^{(n)}\}$$

is trivial. However, even in the case that the underlying action is not free—as is customary for example in the case of actions that underly many standard equivalence problems—it is still possible to construct a weaker version of moving frames, called by the partial moving frame (see [34] for more details).

Constructing an $n$th order moving frame completely relies upon choosing a cross-section to the pseudo-group orbits which is a transverse submanifold of the complementary dimension. By the regularity and freeness of the action and for every point $p \in M$, there exists a unique transformation $g \in \mathcal{G}$ which maps $p$ to the cross-section. Let us describe the most simple, but practical type of cross-sections which are named coordinate cross-section. Let $\mathcal{G}^{(n)}$ acts freely and regularly on the open subset $\mathcal{V}^n$ of $J^n(M, p)$. The regularity of this action permits one to assume that the dimension of its orbits at each jet point $z^{(n)} = (x, u^{(n)})$ is constant, say $r_n$. This means that $\mathcal{H}^{(n)}$ can be coordinatized locally by the jet coordinates $x, u^{(n)}$ together with $r_n$ group parameters $g^{(n)} = g_1, \ldots, g_{r_n}$. Now, one may pick some $r_n$ lifted jet coordinates $X, U^{(n)}$, renamed by $F_1, \ldots, F_{r_n}$, to define the normalization equations:

$$F_1(x, u^{n}, g^{(n)}) = c_1, \quad \ldots \quad F_{r_n}(x, u^{(n)}, g^{(n)}) = c_{r_n},$$

for some appropriate integers $c_1, \ldots, c_{r_n}$. Solving this system for the group parameters as unknowns, enables one to express each of them in terms of the jet coordinates, say $g_i = \gamma_i(x, u^{(n)})$ for $i = 1, \ldots, r_n$. Accordingly, one defines the desired $n$-order moving frame by:

$$\rho^{(n)}(x, u^{(n)}) := (x, u^{(n)}, \gamma(x, u^{(n)})),$$
If $f_1, \ldots, f_r$ are jet coordinates with the lifts $F_1, \ldots, F_r$, then $f_1 = c_1, \ldots, f_r = c_r$ is the cross-section associated to this moving frame.

**Definition 2.2** A differential function $I : J^n(M, p) \to \mathbb{R}$ is a differential invariant of order $n$ if it is unaffected by the action of $\mathcal{G}^{(n)}$, i.e.

$$I(g^{(n)} \cdot (x, u^{(n)})) = I(x, u^{(n)})$$

for each $g^{(n)} \in \mathcal{G}^{(n)}$ and $(x, u^{(n)}) \in J^n(M, p)$.

Once one succeeds to construct a complete moving frame $\rho^{(\infty)} : J^{\infty}(M, p) \to \mathcal{H}^{(\infty)}$, it appears the crucial invariantization operator $\iota$ which converts every differential function or form to its invariantized counterpart. More precisely, for each arbitrary form $\Omega$ of $J^{\infty}(M, p)$, the invariantization $\iota(\Omega)$ is defined as:

$$\iota(\Omega) := (\rho^{(\infty)})^*(\lambda(\Omega))$$

where $\lambda$ is the above defined lifting operator. In particular, if we invariantize those jet coordinates $x^i, u^\alpha_j$ which do not appear in the corresponding coordinate cross-section then, one receives a complete system of differential invariants. Such differential invariants are termed non-phantom (or basic) invariants. Roughly speaking and in agreement with the notations, introduced before Definition 2.2, they can be obtained by inserting $g_i = \gamma_i(x, u^{(n)})$, resulted by the cross-section, into the expressions of the lifted jet coordinates. Even more, the differential operators $\mathcal{D}_i, i = 1, \ldots, p$ dual to the invariantized 1-forms $\iota(\omega^i)$ (see (9)) transform each differential invariant $I$ to another differential invariant $\mathcal{D}_i(I)$.

### 2.2 Recurrence Formula

The recurrence formula is one of the fundamental tools in the equivariant moving frames theory which relates differential invariants and invariant forms to their normalized counterparts defined by the corresponding cross-section. It determines rigorously the structure of the associated algebra of differential invariants. The point that brings this formula to many applications is that it is established using purely infinitesimal information, requiring only linear algebra and differentiation. In particular, it does not need explicit expressions of neither Maurer–Cartan forms nor lifted differential invariants.

Consider the infinitesimal generators:

$$v := \sum_{i=1}^p \xi^i(x, u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^q \phi^\alpha(x, u) \frac{\partial}{\partial u^\alpha}$$
of the action of Lie pseudo-group $\mathcal{G}$ on $M$. As is known [23], the prolonged action of $\mathcal{G}(\infty)$ on $J^{\infty}(M, p)$ is specified by the prolonged infinitesimal generators:

$$v(\infty) = \sum_{i=1}^{p} \xi^i(x, u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \sum_{k: l, J \geq 0} \phi^{\alpha; J}(x, u^{(k)}) \frac{\partial}{\partial u^\alpha_J},$$

where the vector components $\phi^{\alpha; J}$ are defined recursively by the prolongation formula:

$$\phi^{\alpha; J, j} = D_x \phi^{\alpha; J} - \sum_{i=1}^{p} (D_x \xi^i) u^\alpha_J.$$

Reordering the vector components of $v$ as $\zeta^j := \xi^j$ for $j = 1, \ldots, p$ and $\zeta^j := \phi^{j-p}$ for $j = p + 1, \ldots, p + q$, the lifts of the jets $\xi^j_J$ and $\phi^\alpha_J$ are defined artificially [27] to be the corresponding Maurer–Cartan forms (cf. (6)):

$$\lambda(\xi^j_J) := \mu^j.$$

**Recurrence Formula** (cf. [28, Theorem 25]) If $\Omega$ is a differential form defined on $J^{\infty}(M, p)$ then:

$$d\iota(\Omega) = \iota[d\Omega + v(\infty)(\Omega)]$$

where $v(\infty)(\Omega)$ denotes the Lie derivative of $\Omega$ along $v(\infty)$.

This formula measures the distance between the invariantization of a jet form differentiation with the differential of its invariantization. In particular, by taking $\Omega$ to be one of the standard jet coordinates $x^i$ or $u^\alpha_J$, one receives modulo contact forms:

$$dX^i \equiv \omega^i + \mu^i,$$

$$dU^\alpha_J \equiv \sum_{j=1}^{p} U^\alpha_{J, j} \omega^i + \lambda(\phi^{\alpha; J}),$$

where, with the customary abuse of notations, the same symbols $U^\alpha_J$ and $\omega^i$ are used for their invariantizations $\iota(U^\alpha_J)$ and $\iota(\omega^i)$.

### 2.3 Normal Form Construction

Let $S \subset M$, acted on by the pseudo-group $\mathcal{G}$, be a real-analytic $p$-dimensional submanifold represented in local coordinates as the graph of some defining vector equations $u := u(x)$ for $x = x^1, \ldots, x^p$ and $u = u^1, \ldots, u^q$. At a reference point $z_0 := (x_0, u(x_0)) = (x_0, u_0)$, we identify the Taylor series expansion:

$$u(x) = \sum_{k=\sharp J \geq 0} \frac{u_J(x_0)}{J!} (x - x_0)^J$$
of the defining equations with the restriction of jet coordinates to the point $z_0$, i.e.:

$$u^{(\infty)}|_{z_0} = (x_0, u_0, u_J(x_0), \ldots).$$

Let $\rho$ be a complete moving frame associated with the prolonged action $G^{(\infty)}(M, p)$ equipped with a cross-section $z_0^{(\infty)} := (x = x_0, u = u_0, \ldots)$. As above and by convenient abuse of notations, we denote by the same $X$, $U_J$ the invariantization of the lifted coordinates. Then, through the already mentioned identification, the Taylor jet coordinate $u^{(\infty)}|_{z_0}$ is transformed to its corresponding invariantized jet. These new coefficients build the Taylor series expansion of the desired normal form of $S$ as:

$$\sum_{k=\#J\geq 0} \frac{U_J}{J!} (x - x_0)^J.$$

It follows from the way of defining moving frames that such transformation is done by a unique element $g^{(\infty)} \in G^{(\infty)}$ and it maps the reference point $z_0$ to itself. We refer the reader to [26] for further relevant information and details.

### 3 Preliminary Materials

Before applying the moving frame method, we need to provide some requisite materials. In particular, we need to find the infinitesimal counterpart of the Lie pseudo-group $G$ consisting of local biholomorphic maps $(\mathbb{C}^4, 0) \rightarrow (\mathbb{C}^4, 0)$ defined around the origin. This will help us to launch the crucial recurrence formula in the next sections which results in normalizing the Maurer–Cartan forms, as much as possible.

By definition [18], an invertible map $(z, w^1, w^2, w^3) \mapsto (Z, W^1, W^2, W^3)$ with $Z := Z(z, w)$ and $W^j := W^j(z, w)$, $j = 1, 2, 3$ belongs to $G$ providing:

$$Z_z = Z_{w^k} = W^j_z = W^j_{w^k} = 0 \quad (13)$$

for each $j, k = 1, 2, 3$. With the expansion $1) w^j := u_J + iv^j$, let us expand the target complex functions $W^j$ to their real and imaginary parts as:

$$W^j = W^j(z, u, v) := U_j(z, \bar{z}, u, v) + i V^j(z, \bar{z}, u, v).$$

Assuming $\overline{Z} = Z(z, u, v), \overline{W^j} = W^j(z, u, v)$ and thanks to the well-known relation $\frac{\partial}{\partial w} = \frac{1}{2}(\frac{\partial}{\partial u} + i \frac{\partial}{\partial v})$, Eq. (13) also implies that:

$$Z_z = W^j_z = 0, \quad Z_{u^j} = i Z_{u^j}, \quad Z_{v^j} = -i Z_{u^j}, \quad W^k_z = i W^k_{u^j}, \quad W^k_{v^j} = -i W^k_{u^j},$$

1 There is some technical reason for indicating a bit unnaturally the indices $j$ as $u^j$ and $v^j$. 

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for each \( j, k = 1, 2, 3 \). Keeping the two equations \( U_j = \frac{1}{2}(W^j + \overline{W}^j) \) and \( V^j = \frac{i}{2}(W^j - \overline{W}^j) \) in mind, it is easy to verify that:

\[
\frac{\partial U_j}{\partial z} = iV_j^z, \quad \frac{\partial U_j}{\partial \overline{z}} = -iV_j^{\overline{z}}, \quad \frac{\partial U_k}{\partial v^j} = -V_{u_j}^k, \quad V_{v_j}^k = \frac{\partial U_k}{\partial u_j}.
\]

Summing up, then an invertible map \((z, w) \mapsto (Z, W)\) belongs to the pseudo-group \( \mathcal{G} \) whenever it satisfies the system of determining equations (cf. (3)):

\[
Z_z = 
\begin{cases} 
\overline{Z}_z = 0, \\
Z_{v^j} = iZ_{u_j}, \\
\overline{Z}_{v^j} = -i\overline{Z}_{u_j}, \\
\frac{\partial U_k}{\partial z} = iV_z^k, \\
\frac{\partial U_k}{\partial \overline{z}} = -iV_{\overline{z}}^k, \\
\frac{\partial U_k}{\partial v^j} = -V_{v^j}^k, \\
V_{v_j}^k = \frac{\partial U_k}{\partial u_j}, \\
j, k = 1, 2, 3.
\end{cases}
\]

(14)

By [2, Proposition 12.4.22], for every real-analytic generic submanifold \( M \) of \( \mathbb{C}^n \) and each point \( p \in M \), the infinitesimal Lie algebra \( \mathfrak{hol}(M, p) \) of local holomorphic automorphisms of \( M \) is the collection of all real and real-analytic germs of vector fields at \( p \) which are tangent to it and, in addition, are the real parts of the germs of some holomorphic vector fields of \( \mathbb{C}^n \). Letting \( M, \mathbb{C}^n := \mathbb{C}^4 \) and \( p := 0 \), it follows that the Lie algebra \( \mathfrak{g} := \mathfrak{hol}(\mathbb{C}^4, 0) \) consists of the real parts of holomorphic vector fields in \( \mathbb{C}^4 \), real-analytic at the origin. It is actually the infinitesimal algebra associated with our Lie pseudo-group \( \mathcal{G} \) of local holomorphic automorphisms of \( \mathbb{C}^4 \). Accordingly, let us introduce the real vector field:

\[
v = \xi(z, u, v) \frac{\partial}{\partial z} + \overline{\xi}(z, u, v) \frac{\partial}{\partial \overline{z}} + \sum_{j=1}^{3} \eta^j(z, \overline{z}, u, v) \frac{\partial}{\partial u_j} + \sum_{j=1}^{3} \phi^j(z, \overline{z}, u, v) \frac{\partial}{\partial v^j},
\]

(15)

where \( \overline{\xi}(z, u, v) := \overline{\xi(z, u, v)} \), \( \eta^j := \frac{1}{2}(\xi^j + \overline{\xi}^j) \) and \( \phi^j := \frac{i}{2}(\overline{\xi}^j - \xi^j) \) for some holomorphic functions \( \xi^j(z, w), j = 1, 2, 3 \). By linearizing the determining Eqs. (14) at the identity transformation, one verifies that \( v \) belongs to the Lie algebra \( \mathfrak{g} \) if and only if its vector components satisfy the following infinitesimal determining equations for \( j, k = 1, 2, 3 \) (cf. (5)):

\[
\begin{align*}
\xi_z &= \overline{\xi} = 0, & \xi_u^j &= i \xi_{\overline{u}_j}, & \overline{\xi}_{v^j} = -i \overline{\xi}_{u_j}, & \xi_{v^j} = -i \overline{\xi}_{u_j}, & \phi_z^k &= -i \eta_z^k, & \phi_{\overline{z}}^k &= i \eta_{\overline{z}}^k, & \phi_u^k &= -\eta_u^k, & \phi_v^k &= \eta_v^k.
\end{align*}
\]

(16)

By differentiating these equations, we also obtain the following relations among the second order jets of the vector field coefficients:

\[
\begin{align*}
\xi_z,_{a} &= 0, & \xi_z,_{v^j} &= i \xi_{u^j}, & \xi_{u^j},_{v^k} &= i \xi_{u^j u^k}, & \xi_{v^j},_{v^k} &= -\xi_{u^j u^k}, \\
\overline{\xi}_z,_{a} &= 0, & \overline{\xi}_z,_{v^j} &= -i \overline{\xi}_{u^j}, & \overline{\xi}_{u^j},_{v^k} &= -i \overline{\xi}_{u^j u^k}, & \overline{\xi}_{v^j},_{v^k} &= -\overline{\xi}_{u^j u^k}, \\
\eta_z^k,_{a} &= 0, & \eta_z^k,_{v^j} &= -\eta_{u^j}^k, & \eta_{v^j}^k,_{v^j} &= -\eta_{u^j}^k, & \eta_{u^j}^k,_{v^j} &= \eta_{v^j}^k, & \eta_{v^j}^k,_{v^j} &= \eta_{u^j}^k, \\
\phi_z^k,_{a} &= -i \eta_z^k, & \phi_{\overline{z}}^k,_{a} &= i \eta_{\overline{z}}^k, & \phi_u^k,_{a} &= -\eta_u^k, & \phi_v^k,_{a} &= \eta_v^k.
\end{align*}
\]

(17)
for \( j, k, r = 1, 2, 3 \).

Now, let us turn our attention to the jet space \( J := J(\infty)(\mathbb{C}^4) = \mathbb{R}^8, 5 \) of 5-dimensional real submanifolds. We split the coordinates into \( z, \bar{z}, u_1, u_2, u_3 \) as independent and \( v^1, v^2, v^3 \) as dependent variables. The local jet coordinates are of the form \( v^a_j := v^{a\ell} \epsilon^j_{\ell} u_1^{r_1} u_2^{r_2} u_3^{r_3} \) for \( j, k, r_1, r_2, r_3 \in \mathbb{N} \), for \( \alpha = 1, 2, 3 \) and for multi-indices \( J \) with \( \# J \geq 0 \). It is convenient to verify that the conjugate relation \( \bar{v}^a_j = v^{a\ell} \epsilon^j_{\ell} \) holds also in higher order jets.

### 3.1 Maurer–Cartan Forms

As explained in Sect. 2 and corresponding to the variables \( z, \bar{z}, u, v, j = 1, 2, 3 \), we have zeroth order Maurer–Cartan forms \( \mu^z, \mu^\bar{z}, \mu^u, \mu^v \). As a consequence of Eq. (6), it is easy to check that \( \mu^z = \bar{\mu}^\bar{z} \) and also to check that \( \mu^u, \mu^v \) are real forms.

For simplicity, let us make some changes in the notations as:

\[
\mu := \mu^z, \quad \bar{\mu} := \mu^\bar{z}, \quad \alpha^j := \mu^u, \quad \gamma^j := \mu^v, \quad j = 1, 2, 3.
\]

Since here we deal only with the holomorphic Lie pseudo-subgroup \( \mathcal{G} \) of \( \mathcal{D}(\mathbb{C}^4) \) and as stated in Sect. 2, it is anticipated the appearing of some linear dependencies among these Maurer–Cartan forms. Such dependencies can be detected by lifting (cf. (11)):

\[
\lambda(\xi_j) = \mu_j, \quad \lambda(\bar{\xi}_j) = \bar{\mu}_j, \quad \lambda(\eta_j^1) = \alpha_j^1, \quad \lambda(\phi_j^1) = \gamma_j^1, \quad \# J \geq 0
\]

of the vector components of \( \mathbf{v} \) into the infinitesimal determining Eqs. (16) and (17).

This provides the relations:

\[
\begin{align*}
\mu_Z &= \bar{\mu}_Z = 0, & \mu_V & = i \mu_U, &\bar{\mu}_V &= -i \bar{\mu}_U, \\
\gamma^k_Z &= -i \alpha^k_Z, & \gamma^k_Z &= i \alpha^k_Z, & \gamma^k_U &= -\alpha^k_U, & \gamma^k_V &= \alpha^k_V. \\
\mu_{Z, a} &= 0, & \mu_{Z V} &= i \mu_{Z U}, & \mu_{V U} &= i \mu_{U U}, & \mu_{V V} &= -\mu_{U U}, & \bar{\mu}_{Z V} &= -i \bar{\mu}_{Z U}, & \bar{\mu}_{U U} &= -i \bar{\mu}_{U U}, & \bar{\mu}_{V V} &= -\bar{\mu}_{U U}, \\
\alpha^k_{Z Z} &= 0, & \alpha^k_{V V} &= -\alpha^k_{U U}, & \alpha^k_{Z V} &= i \alpha^k_{Z U}, & \alpha^k_{Z U} &= -i \alpha^k_{Z U}, & \alpha^k_{V V} &= i \alpha^k_{V V}, & \alpha^k_{V U} &= -i \alpha^k_{V U}. \\
\gamma^k_{Z, a} &= -i \alpha^k_{Z, a}, & \gamma^k_{Z, a} &= i \alpha^k_{Z, a}, & \gamma^k_{U, a} &= -\alpha^k_{U, a}, & \gamma^k_{V, a} &= \alpha^k_{V, a}. 
\end{align*}
\]

Thus, a basis of Maurer—Cartan forms is provided by the invariant group forms:

\[
\mu_{Z^j U^\ell}, \quad \bar{\mu}_{Z^j U^\ell}, \quad \alpha^j_{U^j V^r}, \quad \alpha^j_{Z^j U^\ell}, \quad \alpha^j_{Z^j U^\ell}, \quad \gamma^k.
\]

(19)

for \( j, r = 1, 2, 3 \), for \( k \in \mathbb{N} \) and for \( \ell = (l_1, l_2, l_3) \in \mathbb{N}^3 \) where \( U^\ell \) denotes \( U_1^{l_1} U_2^{l_2} U_3^{l_3} \).
3.2 Structure Equations

Henceforth and for brevity, we present our expressions *occasionally* by Einstein’s summation convention. Associated with the independent variables \( z, \overline{z}, u_1, u_2, u_3 \) of each 5-dimensional submanifold \( M \subset \mathbb{C}^4 \) represented as the graph of some three defining equations \( v^j = f^j(z, \overline{z}, u_1, u_2, u_3), \ j = 1, 2, 3 \), we have the *lifted horizontal coframe* consisting of the five 1-forms \( \omega^z, \omega^\overline{z}, \omega^k := \omega^{\mu_k}, k = 1, 2, 3 \) with the expressions (cf. (9)):

\[
\omega^z = D_z Z \, dz + D_{\overline{z}} Z \, d\overline{z} + D_{u_j} Z \, du_j \\
= (Z_z + Z_{v^j} f^j_z) \, dz + Z_{\overline{z}} f^j_{\overline{z}} \, d\overline{z} + (Z_{u_j} + Z_{v^j} f^j_{u_j}) \, du_j \\
= (Z_z + i Z_{u_j} f^j_z) \, dz + i Z_{u_j} f^j_{\overline{z}} \, d\overline{z} + (Z_{u_j} + i Z_{u_k} f^k_{u_j}) \, du_j,
\]

\[
\omega^\overline{z} = \overline{\omega^z},
\]

\[
\omega^k = D_k U_k \, dz + D_{\overline{z}} U_k \, d\overline{z} + D_{u_j} U_k \, du_j \\
= (U_{k_z} + U_{k v^j} f^j_{\overline{z}}) \, dz + (U_{k_{\overline{z}}} + U_{k v^j} f^j_z) \, d\overline{z} + (U_{k u_j} + U_{k v^j} f^j_{u_j}) \, du_j.
\]

In matrix form, if we put the coefficients of \( dz, d\overline{z}, du_j, j = 1, 2, 3 \) of the above equations into a \( 5 \times 5 \) matrix \( A \) so that:

\[
[w^z, \omega^\overline{z}, \omega^1, \omega^2, \omega^3] = A \cdot [dz, d\overline{z}, du_1, du_2, du_3]^t
\]

then, the *lifted total derivative operators* are given by (cf. (8)):

\[
\begin{bmatrix}
D_z \\
D_{\overline{z}} \\
D_{u_j}
\end{bmatrix}
= A^{-T}
\begin{bmatrix}
D_z \\
D_{\overline{z}} \\
D_{u_j}
\end{bmatrix}
.
\]

These operators serve to compute the expressions of the lifted jet coordinates \( V^\alpha_{Z, \overline{Z}, U^\ell} := D^j_Z D^k_{\overline{Z}} D^\ell_U V^\alpha \) for \( \alpha = 1, 2, 3 \), though unfortunately, the computations are complicated in practice, specifically in higher orders. We notice that our lifted jet coordinates also respect the conjugation, i.e. \( V^\alpha_{Z, \overline{Z}, U^\ell} = \overline{V^\alpha_{Z, \overline{Z}, U^\ell}} \).

Modulo contact forms,\(^2\) the lifted horizontal forms \( \omega^z, \omega^\overline{z}, \omega^j, j = 1, 2, 3 \), match up to their corresponding invariant horizontal forms \( \sigma^z, \sigma^\overline{z}, \sigma^j \) introduced in (6). Therefore, according to (2), they admit the *structure equations*:

\[
d\omega^z = \mu_Z \wedge \omega^z + \mu_{\overline{Z}} \wedge \omega^\overline{z} + \mu_{u_j} \wedge \omega^j + \mu_{v^j} \wedge (V^j_Z \omega^z + V^j_{\overline{Z}} \omega^\overline{z} + V^j_{U_k} \omega^k)
\]

\[
d\omega^\overline{z} = d\overline{\omega^z},
\]

---

\(^2\) Contact forms play no essential role in this study.
\[ \begin{align*}
\omega^k &= \alpha^k Z \wedge \omega + \alpha^k \bar{Z} \wedge \omega + \alpha^k U_j \wedge \omega + \alpha^k V_j \wedge (V_j Z \omega + V_j \bar{Z} \omega + V_j U_r \omega), \quad \text{for } k = 1, 2, 3. 
\end{align*}\]

### 3.3 Recurrence Relations

The prolongation of the infinitesimal generator (15) of \( G \) is given by:

\[ v(\infty) = \xi \frac{\partial}{\partial z} + \xi \frac{\partial}{\partial \bar{z}} + \sum_{j=1}^{3} \eta^j \frac{\partial}{\partial u_j} + \sum_{\alpha=1}^{3} \sum_{k=\sharp J \geq 0} \phi_{\alpha; J} \frac{\partial}{\partial v^\alpha_{J}}, \]

where the vector components \( \phi_{\alpha; J} \) are defined recursively by the prolongation formula:

\[ \phi_{\alpha; J, x^j} = D_{x^j} \phi_{\alpha; J} - (D_{x^j} \xi) v_{\alpha, J}^x - (D_{x^j} \bar{\xi}) v_{\alpha, J}^\bar{x} - \sum_{k=1}^{3} (D_{x^j} \eta^k) v_{\alpha, J}^{u_k}, \]

for \( x^j = z, \bar{z}, u_1, u_2, u_3 \). Accordingly, the recurrence relations in our case are represented as (cf. (12)):

\[ \begin{align*}
\omega &= \mu, \\
\omega &= \mu, \\
\omega &= \alpha^j, \\
\omega &= \alpha^j, \\
\omega &= \lambda(\phi_{\alpha; J}), \\
\end{align*}\]

for \( j, \alpha = 1, 2, 3 \) and \( \sharp J \geq 0 \), where \( \lambda(\phi_{\alpha; J}) \) is the lift of the vector coefficient \( \phi_{\alpha; J} \) (cf. (11)).

### 4 Normal Form Construction

After providing preliminary requirements in the last section, we are now ready to launch the construction of convergent normal forms for 5-dimensional totally non-degenerate CR manifolds \( M^5 \subset \mathbb{C}^4 \). As stated before, our strategy is to build an appropriate moving frame for the action of the biholomorphic Lie pseudo-group \( G \) on theses CR manifolds. This strategy runs quite parallel to construct the desired normal forms [26]. As is customary in moving frame theory [33, 34], we do this task by normalizing order by order the appearing differential invariants, based fundamentally on the above mentioned recurrence formula.

#### 4.1 Order Zero

According to (23), the recurrence formula in this order readily gives:
\[dZ = \omega^z + \mu, \quad d\bar{Z} = \omega^{\bar{z}} + \bar{\mu}, \quad dU_j = \omega^j + \alpha^j,\]
\[dV^j = V^j_Z \omega^z + V^j_Z \omega^{\bar{z}} + V^j_{U_j} \omega^j + \gamma^k\]

for \(j = 1, 2, 3\). By the discussion presented at the end of Sect. 2 and since we aim to construct our sought normal form at the origin, as the reference point, with some origin-preserving holomorphic transformation then, we choose the normalizations (order zero cross-section):

\[Z = \bar{Z} = U_j = V^j = 0.\]

This, according to the above recurrence relations, brings the zero order Maurer–Cartan forms as:

\[\mu = -\omega^z, \quad \bar{\mu} = -\omega^{\bar{z}}, \quad \alpha^k = -\omega^k, \quad \gamma^k = -V^k_Z \omega^z - V^k_Z \omega^{\bar{z}} - V^k_{U_j} \omega^j.\]

### 4.2 Order One

By choosing the order one cross-section normalizations:

\[V^k_Z = V^k_{\bar{Z}} = V^k_{U_j} = 0,\]

for \(j, k = 1, 2, 3\), one finds the corresponding recurrence relations (henceforth we do not present the recurrence relations of the conjugated lifted invariants due to the fact that \(dV^k = \bar{d}V^k\)):

\[0 = dV^k_Z = V^k_Z \omega^z + V^k_{\bar{Z}} \omega^{\bar{z}} + V^k_{U_j} \omega^j - i \alpha^k_Z,\]
\[0 = dV^k_{U_j} = V^k_{ZU_j} \omega^z + V^k_{\bar{Z}U_j} \omega^{\bar{z}} + V^k_{U_j U_r} \omega^r - \alpha^k_{V_j}.\]

This implies that:

\[\alpha^k_Z = -i V^k_Z \omega^z - i V^k_{\bar{Z}} \omega^{\bar{z}} - i V^k_{ZU_j} \omega^j, \quad \alpha^k_{\bar{Z}} = i V^k_Z \omega^z + i V^k_{\bar{Z}} \omega^{\bar{z}} + i V^k_{ZU_j} \omega^j,\]
\[\alpha^k_{V_j} = V^k_{ZU_j} \omega^z + V^k_{\bar{Z}U_j} \omega^{\bar{z}} + V^k_{U_j U_r} \omega^r,\]

for \(j, k = 1, 2, 3\).

Among performing the above symbolic computations and for each \(k = 1, 2, 3\), we observed that the first order prolonged vector coefficient \(\phi^{k; z}_\ell\) (see (21)) includes the term \(1 \eta^k_z\). This, according to the prolongation formula (22), implies that for each integer \(j \geq 0\) and each triple \(\ell \in \mathbb{N}^3\), we can see the term \(1 \eta^k_{z,j+1}u^\ell\) in the higher order prolonged vector coefficient \(\phi^{k; z}_\ell\). Since the lift of this term is \(\lambda(\eta^k_{z,j+1}u^\ell) = \alpha^k_{Z,j+1}U^\ell\), then we may find through the recurrence relation of \(dV^k_{Z,j+1}U^\ell\) a nonzero constant coefficient of the Maurer–Cartan form \(\alpha^k_{Z,j+1}U^\ell\). Thus, by the normalization \(V^k_{Z,j+1}U^\ell = 0\), one may specify this Maurer–Cartan form. Likewise, the term \(-1 \eta^k_{v,j}\) is
visible in the prolonged vector coefficient $\phi^{k, u_j}$ for $j = 1, 2, 3$. Then, similar argument shows that one can specify each Maurer–Cartan form $\alpha^{k}_{U^l V_j}$, $l \in \mathbb{N}^3$ by applying the normalization $V^k_{U^l U_j} = 0$. Thus, we have

**Lemma 4.1** For $k = 1, 2, 3$ and for each integer $j \geq 0$ and triple $\ell \in \mathbb{N}^3$, it is always possible to specify the Maurer–Cartan forms $\alpha^k_{Z_j + 1 U^\ell}$ by normalizing $V^k_{Z_j + 1 U^\ell} = 0$.

1. to specify the Maurer–Cartan forms $\alpha^k_{Z_j + 1 U^\ell}$ by normalizing $V^k_{Z_j + 1 U^\ell} = 0$.

2. to specify the Maurer–Cartan forms $\alpha^k_{U^\ell V_j}$ by normalizing $V^k_{U^\ell U_j} = 0$.

**Remark 4.2** Normalizations mentioned in the above lemma are well-known in CR geometry. Indeed, they correspond to the fact that the pluriharmonic terms are always removable from the defining equations of generic CR manifolds (see e.g. [9, Lemma 3.2]).

By Lemma 4.1, the three lifted differential invariants $V^k_{Z Z}$, $k = 1, 2, 3$ are the only ones, we shall consider in this order. We first compute the recurrence relation of $V^1_{Z Z}$:

$$
\frac{\partial^2 f}{\partial z_i \partial \bar{z}_j} \equiv 0.
$$

4.3 Order Two

By Lemma 4.1, the three lifted differential invariants $V^k_{Z Z}$, $k = 1, 2, 3$ are the only ones, we shall consider in this order. We first compute the recurrence relation of $V^1_{Z Z}$:

$$
dV^1_{Z Z} = V^1_{Z Z} \omega^\zeta + V^1_{Z Z} \omega^\bar{\zeta} + V^1_{Z Z U_j} \omega^j + V^1_{Z Z} (\alpha^1_{U_1} - \mu \zeta - \bar{\mu} \bar{\zeta})
+ V^2_{Z Z} \alpha^1_{U_2} + V^3_{Z Z} \alpha^1_{U_3}.
$$

This recurrence relation indicates that $V^1_{Z Z}$ is actually a relative differential invariant [23]. Thus, the possibility of normalizing some Maurer–Cartan form, say $\alpha^1_{U_1}$, by means of this relation entirely relies upon vanishing/non-vanishing of this lifted differential invariant. Fortunately, here we can ignore the case $V^1_{Z Z} \equiv 0$ since according to [21, Proposition 2.1], the total nondegeneracy of $M^5$ necessitates that the coefficient of $\zeta \bar{\zeta}$ in the defining equation $\Phi_1$ of (1) remains nonzero under biholomorphic transformations. Thus, we are permitted here to plainly set:

$$
V^1_{Z Z} = 1.
$$

This normalization, brings the recurrence relations of the remained two lifted differential invariants as:

$$
dV^2_{Z Z} = V^2_{Z Z} \omega^\zeta + V^2_{Z Z} \omega^\bar{\zeta} + V^2_{Z Z U_j} \omega^j + V^2_{Z Z} (\alpha^2_{U_2} - \mu \zeta - \bar{\mu} \bar{\zeta}) + V^3_{Z Z} \alpha^2_{U_3} + \alpha^1_{U_1},
$$

$$
dV^3_{Z Z} = V^3_{Z Z} \omega^\zeta + V^3_{Z Z} \omega^\bar{\zeta} + V^3_{Z Z U_j} \omega^j + V^3_{Z Z} \alpha^3_{U_3} + V^3_{Z Z} (\alpha^3_{U_3} - \mu \zeta - \bar{\mu} \bar{\zeta}) + \alpha^3_{U_1}.
$$
Accordingly, one can readily normalize the two Maurer–Cartan forms $\alpha_{U_1}^2$ and $\alpha_{U_1}^3$ by setting:

$$V_{Z\bar{Z}}^2 = V_{Z\bar{Z}}^3 = 0.$$ 

In view of our selected cross-section, the above three recurrence relations give the normalizations:

$$\begin{align*}
\alpha_{U_1}^1 &= -V_{Z^2\bar{Z}}^1 \omega^z - V_{Z^2\bar{Z}}^1 \omega^\bar{z} - V_{Z\bar{Z}U_j}^1 \omega^j + (\mu_Z + \bar{\mu}_Z), \\
\alpha_{U_1}^2 &= -V_{Z^2\bar{Z}}^2 \omega^z - V_{Z^2\bar{Z}}^2 \omega^\bar{z} - V_{Z\bar{Z}U_j}^2 \omega^j, \\
\alpha_{U_1}^3 &= -V_{Z^2\bar{Z}}^3 \omega^z - V_{Z^2\bar{Z}}^3 \omega^\bar{z} - V_{Z\bar{Z}U_j}^3 \omega^j.
\end{align*}$$

Before concluding this order, we notice that in the expression of the prolonged vector coefficient $\phi^k_{z\bar{z}}, k = 1, 2, 3$, one observes the term $v_{z\bar{z}}^1 \eta_{U_1}^k$. Thus, by an argument similar to that of the proof of Lemma 4.1, one finds

**Lemma 4.3** For $k = 1, 2, 3$ and for each triple $\ell \in \mathbb{N}^3$, it always is possible to specify the Maurer–Cartan forms $\alpha_{U_1 U_\ell}^k$ by normalizing to zero lifted differential invariants $V_{Z\bar{Z}U_\ell}^k$.

**4.4 Order Three**

Taking into account the general observations, found in the preceding orders, here we have to consider the recurrence relations of only three lifted differential invariants $V_{Z^2\bar{Z}}^j, j = 1, 2, 3$ which, after simplification, are of the form:

$$\begin{align*}
dV_{Z^2\bar{Z}}^1 &= V_{Z^2\bar{Z}}^1 \omega^z + V_{Z^2\bar{Z}}^1 \omega^\bar{z} + V_{Z^2\bar{Z}j}^1 \omega^j + V_{Z\bar{Z}}^1 (\alpha_{U_1}^1 - 2\mu_Z - \bar{\mu}_Z) \\
&\quad + V_{Z^2\bar{Z}}^2 \alpha_{U_2}^1 + V_{Z^2\bar{Z}}^3 \alpha_{U_3}^1 - \mu_Z Z - 4i \bar{\mu}_Z, \\
dV_{Z^2\bar{Z}}^2 &= V_{Z^2\bar{Z}}^2 \omega^z + V_{Z^2\bar{Z}}^2 \omega^\bar{z} + V_{Z^2\bar{Z}j}^2 \omega^j + V_{Z\bar{Z}}^2 (\alpha_{U_1}^2 - 2\mu_Z - \bar{\mu}_Z) \\
&\quad + V_{Z^2\bar{Z}}^3 \alpha_{U_2}^2 + V_{Z^2\bar{Z}}^3 \alpha_{U_3}^2, \\
dV_{Z^2\bar{Z}}^3 &= V_{Z^2\bar{Z}}^3 \omega^z + V_{Z^2\bar{Z}}^3 \omega^\bar{z} + V_{Z^2\bar{Z}j}^3 \omega^j + V_{Z\bar{Z}}^3 (\alpha_{U_1}^3 - 2\mu_Z - \bar{\mu}_Z) \\
&\quad + V_{Z^2\bar{Z}}^3 (\alpha_{U_3}^3 - 2\mu_Z - \bar{\mu}_Z).
\end{align*}$$

As is visible from the recurrence relations of $V_{Z^2\bar{Z}}^2$ and $V_{Z^2\bar{Z}}^3$—and in contrary to that of $V_{Z^2\bar{Z}}^1$—the possibility of normalizing some Maurer–Cartan forms by means of these two recurrence relations completely relies upon vanishing or non-vanishing of the corresponding differential invariants. Again, as in the former order, the total nondegeneracy of our CR manifold $M^3$ ensures that the coefficients of $z^{2 \bar{z}}$ and $z \bar{z}^{2}$ in the defining equations $\Phi_2$ and $\Phi_3$ in (1) remain nonzero under biholomorphic
transformations. Then, in consistent with these coefficients in (1), it permits us to set:

\[ V_1^{Z^2Z} = 0, \quad V_2^{Z^2Z} = 1, \quad V_3^{Z^2Z} = -i. \]

Accordingly, the above recurrence relations take the simple form:

\[
0 = dV_1^{Z^2Z} = V_1^{Z^2Z} \omega^z + V_1^{Z^2Z} \omega^z + V_1^{Z^2Z} \omega^j + \alpha_1^{U_2} - i \alpha_1^{U_3} - \mu_{ZZ} + 4i \bar{\mu}_{U_1},
\]

\[
0 = dV_2^{Z^2Z} = V_2^{Z^2Z} \omega^z + V_2^{Z^2Z} \omega^z + V_2^{Z^2Z} \omega^j + \alpha_2^{U_2} - 2 \mu_{Z - \bar{\mu}Z} - i \alpha_2^{U_3},
\]

\[
0 = dV_3^{Z^2Z} = V_3^{Z^2Z} \omega^z + V_3^{Z^2Z} \omega^z + V_3^{Z^2Z} \omega^j + \alpha_3^{U_2} - i \alpha_3^{U_3} + 2i \mu_{Z + i \bar{\mu}Z}.
\]

Then, our cross-section gives rise to the following expressions of the Maurer–Cartan forms:

\[
\mu_{U_1} = -\frac{i}{4} (V_1^{Z^2Z} \omega^z + V_1^{Z^2Z} \omega^z + V_1^{Z^2Z} \omega^j + \alpha_1^{U_2} + i \alpha_1^{U_3} - \bar{\mu}_Z Z),
\]

\[
\mu_Z = \frac{1}{6} (3 V_2^{Z^2Z} - 3 V_2^{Z^2Z} + i V_3^{Z^2Z} - i V_3^{Z^2Z}) \omega^z
\]
\[+ \frac{1}{6} (3 V_2^{Z^2Z} - 3 V_2^{Z^2Z} + i V_3^{Z^2Z} - i V_3^{Z^2Z}) \omega^z
\]
\[+ \frac{1}{6} (3 V_2^{Z^2Z} - 3 V_2^{Z^2Z} + i V_3^{Z^2Z} - i V_3^{Z^2Z}) \omega^j + \frac{1}{3} \alpha_3^{U_3} - i \alpha_2^{U_3},
\]

\[
\alpha_2^{U_2} = \frac{1}{2} (i V_3^{Z^2Z} - V_2^{Z^2Z} - V_2^{Z^2Z} - i V_3^{Z^2Z}) \omega^z
\]
\[+ \frac{1}{2} (i V_3^{Z^2Z} - V_2^{Z^2Z} - V_2^{Z^2Z} - i V_3^{Z^2Z}) \omega^z
\]
\[+ \frac{1}{2} (i V_3^{Z^2Z} - V_2^{Z^2Z} - V_2^{Z^2Z} - i V_3^{Z^2Z}) \omega^j + \alpha_3^{U_3}
\]

\[
\alpha_3^{U_2} = \frac{1}{2} (i V_2^{Z^2Z} - i V_2^{Z^2Z} - V_3^{Z^2Z} - V_3^{Z^2Z}) \omega^z
\]
\[+ \frac{1}{2} (i V_2^{Z^2Z} - i V_2^{Z^2Z} - V_3^{Z^2Z} - V_3^{Z^2Z}) \omega^z
\]
\[+ \frac{1}{2} (i V_2^{Z^2Z} - i V_2^{Z^2Z} - V_3^{Z^2Z} - V_3^{Z^2Z}) \omega^j - \alpha_3^{U_3}.
\]

Lemma 4.4 Let \( j, l \geq 0 \) and \( \ell \in \mathbb{N}^3 \). One can specify

1. the Maurer–Cartan form \( \mu_{U_1 U_2} \) by normalizing to zero the lifted differential invariant \( V_1^{Z^2Z U_1} \).
2. the Maurer–Cartan form \( \mu_{Z^{j+1} U_2} \) by normalizing to zero the lifted differential invariant \( V_2^{Z^{j+2} U_2} \).
3. the real Maurer–Cartan forms \( \alpha_2^{U_2 U_3} \) and \( \alpha_3^{U_2 U_3} \) by normalizing to zero the lifted differential invariant \( V_3^{Z^{2j+1} U_2} \).
In the expression of the third order prolonged vector coefficient \( \phi^1 z^2 \bar{z} \), one finds some nonzero constant coefficient of the term \((v^1_{z \bar{z}})^2 \bar{z} u_1\). Thus, according to the prolongation formula (22) and for each \( \ell \in \mathbb{N}^3 \), one expects the higher order prolonged vector coefficient \( \phi^1 z^2 \bar{z} u^\ell \) to include the monomial \((v^1_{z \bar{z}})^2 \bar{z} u_1 u^\ell\), as well. Due to the normalization \( V^1_{Z \bar{Z}} = 1 \), the lift of this monomial is nothing but the Maurer–Cartan form \( \overline{\mu}_{U_j U^\ell} \). Hence, one can specify this form—and also its conjugation—by normalizing \( V^1_{Z \bar{Z} U^\ell} = 0 \) in the recurrence relation of this lifted differential invariant. This completes the proof of the first assertion. The proof of the next two assertions is quite similar. Indeed, they hold since the monomials \(-v^2_{z \bar{z} z}(2\xi + \bar{\xi})\) and \( iv^2_{z \bar{z} z} u^2_{u_2} + i v^3_{z \bar{z} z} u^2_{u_3} \) appear respectively in the expressions of \( \phi^2 z^2 \bar{z} \) and \( i \phi^2 z^2 \bar{z} + \phi^3 z^2 \bar{z} \). 

It follows from Lemmas 4.1, 4.3 and 4.4 that the collection of linearly independent Maurer–Cartan forms (19) is now reduced to:

\[
\mu_{U^1_{j} U^1_{j}'}, \quad \overline{\mu}_{U^1_{j} U^1_{j}'} \quad \alpha^1_{U^2_{j} U^1_{j}'}, \quad \alpha^2_{U^2_{j} U^1_{j}+1}, \quad \alpha^3_{U^2_{j} U^1_{j}+1},
\]

for \((0, 0) \neq (k, l) \in \mathbb{N}^2\) and \(\ell \in \mathbb{N}^3\).

### 4.5 Order Four

We start this order with the normalizations \( V^1_{Z \bar{Z}} = V^j_{Z \bar{Z}^2} = 0 \) for \( j = 1, 2, 3 \). Thanks to Lemmas 4.1, 4.3 and 4.4, the recurrence relations of these lifted differential invariants simplify to:

\[
0 = dV^1_{Z \bar{Z}^2} = (4i v^2_{Z \bar{Z}^2 U_1} + V^1_{Z \bar{Z}^2}) \omega^z - (4i v^2_{Z \bar{Z}^2 U_1} - V^1_{z \bar{z}}) \omega^\bar{z} + V^1_{z \bar{z}} \omega^j
- 4i (v^2_{Z \bar{Z} U_1} - v^3_{Z \bar{Z}^2 U_1}) a^2_{U^3} - 4i (\mu_{U_2} - \overline{\mu}_{U_2}) - 12 (\mu_{U_3} + \overline{\mu}_{U_3}),
\]

\[
0 = dV^1_{Z \bar{Z}^2} = 5V^1_{Z \bar{Z}^2} \omega^z + \frac{1}{4} (4V^1_{Z \bar{Z}^2} + V^j_{Z \bar{Z}^2} - V^2_{Z \bar{Z}^2}) \omega^\bar{z} + V^1_{Z \bar{Z}^2} \omega^j
+ V^3_{Z \bar{Z}^2} a^3_{U^3} - \frac{1}{4} V^3_{Z \bar{Z}^2} a^2_{U^3} + 3i \overline{\mu}_{U_3} + 3 \overline{\mu}_{U_3},
\]

\[
0 = dV^2_{Z \bar{Z}^2} = V^2_{Z \bar{Z}^2} \omega^z + V^2_{Z \bar{Z}^2} \omega^\bar{z} + V^2_{Z \bar{Z}^2} \omega^j - 2a^1_{U^2},
\]

\[
0 = dV^3_{Z \bar{Z}^2} = (V^3_{Z \bar{Z}^2} - 2i V^3_{Z \bar{Z}^2 U_1}) \omega^z + (V^3_{Z \bar{Z}^2} + 2i V^3_{Z \bar{Z}^2 U_1}) \omega^\bar{z} + V^3_{Z \bar{Z}^2} \omega^j - 2a^1_{U^3}.
\]

These equations result in the following expressions for the Maurer–Cartan forms:

\[
\text{Im} \mu_{U_2} = -\frac{1}{8} \left( (4i v^2_{Z \bar{Z}^2 U_1} + V^1_{Z \bar{Z}^2}) \omega^z - (4i v^2_{Z \bar{Z}^2 U_1} - V^1_{z \bar{z}}) \omega^\bar{z} + V^1_{z \bar{z}} \omega^j
- 4i (v^2_{Z \bar{Z} U_1} - v^3_{Z \bar{Z}^2 U_1}) a^2_{U^3} - 12 (\mu_{U_3} + \overline{\mu}_{U_3}) \right),
\]

\[
\mu_{U_3} = \frac{1}{96} \left( 15 V^1_{Z \bar{Z}} - 12i V^3_{Z \bar{Z}^2} V^3_{Z \bar{Z}^2 U_1} + 20i V^3_{Z \bar{Z}^2} V^3_{Z \bar{Z}^2 U_1} + 12i V^2_{Z \bar{Z}^2 U_1} + 3 V^1_{Z \bar{Z}^2} \right).
\]
\[-20 V^1_{zz^2} - 5 V^1_{zz} + 5 V^2_{z^2z^2} - 10 V^3_{z^3z^2} V^3_{z^3z^2} + 6 V^3_{z^3z^2} V^3_{z^3z^2}\] \[\alpha^5 \] 
\[+ \frac{1}{96} \left( 12 V^1_{z^3z^2} - 25 V^1_{z^3z^2} - 10 V^3_{z^3z^2} - 20 i V^3_{z^3z^2} - 12 i V^2_{z^3z^2} \right) \omega^5 \] 
\[+ \frac{1}{96} \left( 3 V^1_{z^3z^2} - 3 V^2_{z^3z^2} - 10 V^3_{z^3z^2} + 6 V^3_{z^3z^2} V^3_{z^3z^2} \right) \omega^5 \] 
\[+ \frac{1}{96} \left( 3 V^1_{z^3z^2} - 10 V^3_{z^3z^2} + 12 i V^3_{z^3z^2} \right) \right) \omega^j \] 
\[\alpha^1_{U_3} = \left( \frac{1}{2} V^3_{z^3z^2} - i V^3_{z^3z^2} \right) \omega^5 + \left( \frac{1}{2} V^3_{z^3z^2} + i V^3_{z^3z^2} \right) \omega^5 + \frac{1}{2} V^3_{z^3z^2} \omega^j. \]

Following the same lines of the proofs of Lemmas 4.1, 4.3 and 4.4, a symbolic view to the computations of the recurrence relations (25) helps us to find some more general observations concerning possible normalizations in higher orders. For brevity, we omit the proof of these observations as they essentially resemble an slight repetition of the arguments presented in the proofs of the already mentioned lemmas.

**Lemma 4.5** For each \(j, l \geq 0\) it is always possible to

1. specify the Maurer–Cartan form \( \text{Im} \mu_{U_{j+1}} \) by normalizing to zero the lifted invariant \( V^1_{z^3z^2 U_{j}^i} \).
2. specify the Maurer–Cartan form \( \mu_{U_{j+1} U_{j}^i} \) by normalizing to zero the lifted invariant \( V^1_{z^3z^2 U_{j}^i U_{j}^i} \).
3. specify the Maurer–Cartan form \( \alpha_{U_{j+1} U_{j}^i} \) by normalizing to zero the lifted invariant \( V^2_{z^3z^2 U_{j}^i U_{j}^i} \).
4. specify the Maurer–Cartan form \( \alpha_{U_{j+1} U_{j}^i} \) by normalizing to zero the lifted invariant \( V^3_{z^3z^2 U_{j}^i} \).

Following this lemma, the collection of yet unnormalized Maurer–Cartan forms (24) reduces to:

\[\text{Re} \mu_{U_{j+1}}, \quad \alpha^2_{U_{j+1}}, \quad \alpha^3_{U_{j+1}}, \quad j, l \geq 0. \] (26)

By Lemmas 4.1, 4.3 and 4.4, it remains in this order to only consider the recurrence relations of the two lifted differential invariants \( V^3_{z^3z^2} \) and \( V^3_{z^3z^2 U_{j}} \). Our computations show that:

\[
dV^3_{z^3z^2} = \left( V^3_{z^3z^2} - \frac{2i}{3} V^3_{z^3z^2} V^3_{z^3z^2} \right) \omega^5 + \left( V^3_{z^3z^2} + i V^2_{z^3z^2} + \frac{2i}{3} V^3_{z^3z^2} V^3_{z^3z^2} \right) \omega^5 \] 
\[+ \left( \frac{2i}{3} V^3_{z^3z^2} V^3_{z^3z^2 U_{j}} - \frac{2i}{3} V^3_{z^3z^2} V^3_{z^3z^2 U_{j}} + V^3_{z^3z^2} \right) \omega^j + V^3_{z^3z^2} \left( 3i \alpha^2_{U_3} - \frac{2}{3} \alpha^3_{U_3} \right). \]
\[ \frac{dV^3}{z^3 \partial U_1} = \frac{1}{12} \left( 4i V^3 z^3 \partial^3 U_1 + 10 V^1 z^3 \partial^2 U_1 + 3i V^3 z^3 \partial^2 z^2 - 2i V^3 z^3 \partial^2 z^3 + V^3 z^3 \partial U_1 \right) \]
\[ - 7 V^3 z^3 \partial^3 z^2 \left( \partial^3 \right) \omega^z \]
\[ + \frac{1}{12} \left( 10i V^3 z^3 \partial^3 U_1 - 14i V^3 z^3 \partial^2 z^2 - 3i V^3 z^3 \partial^2 z^3 + V^3 z^3 \partial z^3 \right) \omega^z \]
\[ - 7 V^3 z^3 \partial^3 z^3 + i V^2 z^3 \partial^2 z^3 - 8 V^1 z^3 \partial z^3 - 2 V^3 z^3 + 2 V^2 z^3 \omega^z \]
\[ + \frac{1}{12} \left( 10i V^3 z^3 \partial^3 U_1 - 10i V^3 z^3 \partial^2 z^2 + V^3 z^3 \partial z^3 + 3i V^3 z^3 \partial z^3 \right) \omega^2 \]
\[ - 7 V^3 z^3 \partial^3 z^3 - 8 V^1 z^3 \partial U_1 \left( \partial^3 \right) \omega^j \]
\[ + \frac{1}{6} \left( V^3 z^3 \partial^3 z^3 - 1i V^3 z^3 \partial^2 z^3 + 12i V^3 z^3 \partial z^3 \right) \omega^2 \]
\[ - \left( \frac{2}{3} V^3 z^3 \partial^3 U_1 \alpha^3 \right) \]

These equations indicate that \( \frac{dV^3}{z^3 \partial U_1} \) and \( \frac{dV^3}{z^3 \partial U_1} \) are relative differential invariants. Therefore, vanishing or non-vanishing of the coefficients of the monomials \( z^3 \omega^z \) and \( z^2 \omega^z \) in the third defining equation of (1) is an invariant property under holomorphic transformations. In view of the above relations, the possibility of normalizing the two real Maurer–Cartan forms \( \alpha^3_U \) and \( \alpha^3_{U_3} \) relies upon the vanishing and non-vanishing of the two already mentioned differential invariants and thus, we have to distinguish three major branches in the future issues of constructing desired moving frame:

- **Branch 1.** \( \frac{dV^3}{z^3 \partial U_1} \neq 0 \).
- **Branch 2.** \( \frac{dV^3}{z^3 \partial U_1} = 0 \) and \( \frac{dV^3}{z^2 \partial U_1} \neq 0 \).
- **Branch 3.** \( \frac{dV^3}{z^3 \partial U_1} = \frac{dV^3}{z^2 \partial U_1} = 0 \).

Consequently, the current order four was the last order in which our desired normal form appears in a uniform fashion. Let us present in the following theorem this achieved universal normal form.

**Theorem 4.1** Every 5-dimensional real-analytic totally nondegenerate CR manifold \( M^5 \subset \mathbb{C}^4 \) can be transformed, through some origin-preserving holomorphic transformation, into the convergent normal form:

\[ v^1 = \tilde{z} \tilde{z} + \sum_{j+k+l \geq 5} \frac{1}{j! k! l!} V^1 z^j \tilde{z}^k u^l, \]
\[ v^2 = \frac{1}{2} (z^2 \tilde{z} + \tilde{z}^2) + \sum_{j+k+l \geq 5} \frac{1}{j! k! l!} V^2 z^j \tilde{z}^k u^l, \]
\[ v^3 = \frac{1}{2} (z^2 \tilde{z} - \tilde{z}^2) + \frac{1}{6} V^3 z^3 \tilde{z}^3 \omega^z + \frac{1}{6} V^3 z^3 \tilde{z}^3 + \frac{1}{2} V^3 z^3 \partial z^3 u_1 + \frac{1}{2} V^3 z^3 \partial z^3 u_1 \]
\[ + \sum_{j+k+l \geq 5} \frac{1}{j! k! l!} V^3 z^j \tilde{z}^k u^l, \]

(28)
where, taking into account the conjugation relation \( V_{\bar{z}z}^{\bullet} U^\ell = V_{\bar{z}z}^{\bullet} U^\ell \), we have the cross-section normalizations:

\[
\begin{align*}
0 &= V_{zI}^1 U^\ell = V_{z\overline{z}U}^1 \quad \text{for } j, l \in \mathbb{N}, \\
0 &= V_{zI}^2 U^\ell = V_{z\overline{z}U}^2 \quad \text{for } j, l \in \mathbb{N}, \\
0 &= V_{zI}^3 U^\ell = V_{z\overline{z}U}^3 \quad \text{for } j, l \in \mathbb{N},
\end{align*}
\]

(29)

for \( j, l \in \mathbb{N} \) and \( \ell \in \mathbb{N}^3 \).

In the rest of the paper our aim is to apply, as much as possible, supplementary normalizations on the already obtained normal form. For this aim, we pursue along the above mentioned three branches, the construction of a complete equivariant moving frame associated with our CR manifolds, in question. Along this way, we will find our normal form in the next sections without any further possible normalization applicable on it. It is analogous to the Chern-Moser supplementary normalizations in part (d), pages 246–7 of their paper [9] for non-umbilical hypersurfaces in \( \mathbb{C}^2 \) or those applied by Webster in [35] for arbitrary non-umbilical hypersurfaces in \( \mathbb{C}^{n+1} \).

5 Branch 1: \( V_{z\overline{z}2}^{3} \neq 0 \)

As the first branch, let us assume that the coefficient \( V_{z\overline{z}2}^{3} \) in the normal form (28) is nonzero. Hence, we are permitted to plainly normalize it to 1. This normalization brings the recurrence relation of this differential invariant in (27) into the simple form:

\[
0 = dV_{z\overline{z}2}^{3} = (V_{z\overline{z}2}^{3} - \frac{2i}{3}) \omega^z + (V_{z\overline{z}2}^{3} + i V_{z\overline{z}2}^{2} + \frac{2i}{3}) \omega^\overline{z} + (\frac{2i}{3} V_{z\overline{z}2}^{3} - 2i V_{z\overline{z}2}^{3} + V_{z\overline{z}2}^{3}) \omega^j + 3i \alpha_{U_2}^2 - \frac{1}{3} \alpha_{U_3}^3.
\]

Keeping in mind that \( \alpha_{U_2}^2 \) and \( \alpha_{U_3}^3 \) are real Maurer–Cartan forms, this equation leads us to specify them as:

\[
\begin{align*}
\alpha_{U_2}^2 &= \frac{i}{6} (i V_{z\overline{z}2}^{3} - V_{z\overline{z}2}^{3} + V_{z\overline{z}2}^{3} - V_{z\overline{z}2}^{3}) \omega^z + \frac{i}{6} (i V_{z\overline{z}2}^{3} + V_{z\overline{z}2}^{3} - V_{z\overline{z}2}^{3}) \omega^\overline{z} \\
&+ \frac{i}{6} (V_{z\overline{z}2}^{3} - V_{z\overline{z}2}^{3}) \omega^j, \\
\alpha_{U_3}^3 &= \frac{2}{3} (V_{z\overline{z}2}^{3} + V_{z\overline{z}2}^{3} - i V_{z\overline{z}2}^{3} - \frac{4i}{3}) \omega^z + \frac{2}{3} (V_{z\overline{z}2}^{3} + i V_{z\overline{z}2}^{3} + V_{z\overline{z}2}^{3} + \frac{4i}{3}) \omega^\overline{z} \\
&+ \frac{2}{3} (V_{z\overline{z}2}^{3} - 4i V_{z\overline{z}2}^{3} + V_{z\overline{z}2}^{3} + V_{z\overline{z}2}^{3}) \omega^j.
\end{align*}
\]

A close inspection of the computations concerned the recurrence relation of \( dV_{z\overline{z}2}^{3} \) reveals that

**Lemma 5.1** In Branch 1 and for any \( j \geq 0 \), it is possible to normalize Maurer–Cartan forms \( \alpha_{U_2}^2 \) and \( \alpha_{U_3}^3 \) by normalizing \( V_{z\overline{z}2}^{3} = 1 \) and \( V_{z\overline{z}2}^{3} U_{j}^l = 0 \), when \( j \neq 0 \).
At this stage, most of the Maurer–Cartan forms are normalized except those of the real form $\text{Re} \mu_{U_2^{i+i}}$ (cf. (26)). In order to check the possibility of their normalization, we have to proceed into the next order five.

### 5.1 Order 5—Branch 1

After examining several order five recurrence relations, we realized finally that the normalization $\text{Re} V^1_{Z^iZ^i} = 0$ may bring our desired result of normalizing the remained Maurer–Cartan forms. By this normalization, the corresponding recurrence formula gives the long expression:

\[
dV^1_{Z^iZ^i} = \left( \frac{i}{2} V^1_{Z^iZ^i} - \frac{11i}{12} V^3_{Z^iZ^i} V^3_{Z^iZ^i} U_1 + \frac{i}{48} V^3_{Z^iZ^i} V^2_{Z^iZ^i} - \frac{i}{12} V^3_{Z^iZ^i} V^3_{Z^iZ^i} U_1 \right) \frac{V^3_{Z^iZ^i}}{V^3_{Z^iZ^i}}
\]

\[
- \frac{i}{30} V^3_{Z^iZ^i} V^3_{Z^iZ^i} + \frac{i}{8} V^1_{Z^iZ^i}
\]

\[
+ \frac{i}{12} V^3_{Z^iZ^i} V^3_{Z^iZ^i} U_1 - \frac{5i}{144} V^3_{Z^iZ^i} V^2_{Z^iZ^i} + \frac{5i}{144} V^3_{Z^iZ^i} V^3_{Z^iZ^i} + \frac{1}{2} V^3_{Z^iZ^i} V^3_{Z^iZ^i}
\]

\[
+ \frac{1}{48} V^3_{Z^iZ^i} V^3_{Z^iZ^i}
\]

\[
- \frac{1}{48} V^3_{Z^iZ^i} V^3_{Z^iZ^i} + \frac{1}{12} V^3_{Z^iZ^i} V^2_{Z^iZ^i} - \frac{1}{12} V^3_{Z^iZ^i} U_1 V^2_{Z^iZ^i} + \frac{1}{30} V^3_{Z^iZ^i} V^2_{Z^iZ^i}
\]

\[
- \frac{i}{8} V^2_{Z^iZ^i} + \frac{6}{5} V^1_{Z^iZ^i}
\]

\[
+ \frac{i}{30} V^3_{Z^iZ^i} V^3_{Z^iZ^i} - \frac{1}{2} V^2_{Z^iZ^i} U_1 - \frac{1}{3} V^3_{Z^iZ^i} U_1 - \frac{5}{144} V^3_{Z^iZ^i} V^3_{Z^iZ^i}
\]

\[
- \frac{i}{12} V^2_{Z^iZ^i} U_1 V^3_{Z^iZ^i} - \frac{1}{6} V^3_{Z^iZ^i}
\]

\[
- \frac{7}{6} V^1_{Z^iZ^i} V^3_{Z^iZ^i} - \frac{1}{6} V^1_{Z^iZ^i} V^3_{Z^iZ^i} + \frac{i}{6} V^1_{Z^iZ^i} V^2_{Z^iZ^i} - \frac{7i}{9} V^1_{Z^iZ^i} \omega^2
\]

\[
+ \left( \frac{5i}{24} V^2_{Z^iZ^i} - \frac{5i}{6} V^1_{Z^iZ^i} - \frac{i}{6} V^1_{Z^iZ^i} + \frac{i}{8} V^1_{Z^iZ^i} - \frac{5i}{144} V^3_{Z^iZ^i} V^2_{Z^iZ^i}
\]

\[
- \frac{i}{30} V^3_{Z^iZ^i} V^3_{Z^iZ^i} + \frac{i}{30} V^3_{Z^iZ^i} V^2_{Z^iZ^i}
\]

\[
+ \frac{i}{12} V^3_{Z^iZ^i} U_1 V^3_{Z^iZ^i} + \frac{i}{48} V^3_{Z^iZ^i} V^2_{Z^iZ^i} - \frac{i}{12} V^3_{Z^iZ^i} U_1 V^3_{Z^iZ^i}
\]

\[
- \frac{i}{12} V^3_{Z^iZ^i} U_1 V^3_{Z^iZ^i} + \frac{i}{12} V^3_{Z^iZ^i} V^3_{Z^iZ^i} U_1 V^3_{Z^iZ^i}
\]

\[
+ \frac{i}{2} V^3_{Z^iZ^i} V^3_{Z^iZ^i} - \frac{5}{144} V^3_{Z^iZ^i} V^3_{Z^iZ^i} + \frac{5}{144} V^3_{Z^iZ^i} V^3_{Z^iZ^i}
\]

\[
+ \frac{1}{48} V^3_{Z^iZ^i} V^3_{Z^iZ^i} + \frac{1}{12} V^3_{Z^iZ^i} V^3_{Z^iZ^i} U_1 V^3_{Z^iZ^i}
\]

\[
- \frac{1}{12} V^3_{Z^iZ^i} U_1 V^3_{Z^iZ^i} + \frac{1}{30} V^3_{Z^iZ^i} V^3_{Z^iZ^i} + \frac{1}{2} V^2_{Z^iZ^i} U_1 + \frac{1}{3} V^3_{Z^iZ^i} U_1
\]

\[
+ \frac{1}{5} V^1_{Z^iZ^i} - \frac{1}{48} V^3_{Z^iZ^i} V^3_{Z^iZ^i}
\]

\[
+ V^1_{Z^iZ^i} - \frac{1}{5} V^2_{Z^iZ^i} + i V^3_{Z^iZ^i} U_1 V^3_{Z^iZ^i} - \frac{1}{6} V^1_{Z^iZ^i} V^3_{Z^iZ^i} + \frac{7}{6} V^1_{Z^iZ^i} V^3_{Z^iZ^i}
\]
In Branch Lemma 5.2 general we have
ining the monomials which appeared among the above performed computations shows that in
discussion presented at the end of Sect. 2, the holomorphic transformation which brings a CR
with no further applicable normalizations on it, is available in the current order five. By the
state that in this branch a complete equivariant moving frame and equivalently a normal form
At this stage, there is no further Maurer–Cartan form remained unnormalized. This amounts to
M
manifold enjoying the assumption V
3
Theorem 5.1
Let M
holomorphic transformation which brings it into the normal form:
This equation readily specifies the expression of the real Maurer–Cartan form Re μ
added by:

\[ 0 = \text{Re} V^1_{z^jzU_1} = V^3_{z^jzU_1} \]  

(31)
for $l \geq 0$. Moreover, in this branch, the infinitesimal CR automorphism algebra $\text{aut}_{CR}(M^5)$ is five dimensional and the biholomorphic equivalence problem to $M^5$ can be reduced to an absolute parallelism, namely $[e]$-structure, on itself with the structure equations of the lifted horizontal coframe $\omega^z, \omega^\tau, \omega^j, j = 1, 2, 3$ of $M^5$, obtained by (20) after applying the already mentioned normalizations and inserting the achieved expressions of the normalized Maurer–Cartan forms.

Following the Chern-Moser terminology in [9], we may represent the normal form (30) as:

$$v^k = \Phi^k(z, \bar{z}, u) := \sum_{i,j} \Phi^k_{ij}(u_1, u_2, u_3) z^i \bar{z}^j, \quad j = 1, 2, 3, \quad (32)$$

for some complex power series $\Phi^k_{ij} := \Phi^k_{ij}(u_1, u_2, u_3)$ with $\Phi^k_{ij} = \Phi^k_{ji}$. Via the following auxiliary schematic diagram we attempt to partly describe these functions in terms of the applied normalizations (29)–(31). In each diagram, a filled black circle standing at a point.
that it has as low an order as possible (see \cite[Definition 3.1]{25} for the explicit definition). It is easy to see that our cross-section (31) enjoys this property. By definition \cite[Definition 5.2]{25}, a differential invariant $V^k$ is an edge differential invariant of our cross-section if $V^k_j$ appears in (29)–(31). It is convenient to turn up the edge differential invariants of the cross-section, constructed in this branch.

**Corollary 5.3** The collection of edge differential invariants of the cross-section (29)–(31) forms a generating system of differential invariants for the biholomorphic equivalence problem to 5-dimensional totally nondegenerate CR manifolds $M^5$, belonging to Branch 1.

**Proof** It is a straightforward consequence of \cite[Theorem 7.2]{25}. \hfill $\square$

Notice that the above generating set is not necessarily minimal.

**6 Branch 2:** $V^3_{Z^3\overline{Z}} = 0$ and $V^3_{Z^2\overline{Z}U_1} \neq 0$

As the second branch, let us assume that in the normal form (28), we have $V^3_{Z^3\overline{Z}} = 0$ but, in contrary, the coefficient $V^3_{Z^2\overline{Z}U_1}$ does not vanish. By the former assumption, it follows from the first recurrence relation in (27) that we have:

\[
V^3_{Z^3\overline{Z}} = 0, \quad V^3_{Z^2\overline{Z}^2} = -iV^2_{Z^3\overline{Z}^2}, \quad V^3_{Z^3\overline{Z}U_j} = 0, \quad j = 1, 2, 3. \tag{33}
\]

Moreover, by normalizing $V^3_{Z^2\overline{Z}U_1} = 1$, the recurrence relation of this differential invariant in (27) converts into the form:

\[
0 = dV^3_{Z^3\overline{Z}U_1} = -\frac{5}{6} V^1_{Z^3\overline{Z}}\omega^\overline{z} + \frac{1}{12} \left( V^3_{Z^2\overline{Z}^2U_1} + iV^2_{Z^2\overline{Z}U_1} - 8V^1_{Z^3\overline{Z}^2} - 2V^1_{Z^1\overline{Z}^2} + 2V^2_{Z^1\overline{Z}^2} \right)\omega^\overline{z}
+ \frac{1}{12} \left( 10iV^3_{Z^2\overline{Z}U_j} - 10iV^2_{Z^2\overline{Z}U_1} + V^3_{Z^2\overline{Z}U_1U_j} - 8V^1_{Z^3\overline{Z}U_j} \right)\omega^{\overline{z}} + 2i\alpha^2_{U_3} - \frac{2}{3} \alpha^3_{U_3}.
\]

Taking into account that $\alpha^2_{U_3}$ and $\alpha^3_{U_3}$ are real Maurer–Cartan forms, this equation results in the expressions:

\[
\alpha^2_{U_3} = \frac{i}{48} \left( iV^2_{Z^2\overline{Z}U_1} + 8V^1_{Z^2\overline{Z}U_1} + 10V^1_{Z^3\overline{Z}} + 2V^1_{Z^1\overline{Z}^2} - 2V^2_{Z^2\overline{Z}^2} - V^3_{Z^3\overline{Z}U_1} \right)\omega^z
+ \frac{i}{48} \left( iV^2_{Z^2\overline{Z}U_1} - 8V^1_{Z^3\overline{Z}} - 2V^1_{Z^1\overline{Z}^2} + 10V^1_{Z^3\overline{Z}^2} + 2V^2_{Z^2\overline{Z}^2} + V^3_{Z^3\overline{Z}U_1} \right)\omega^\overline{z}
+ \frac{i}{48} \left( 8V^1_{Z^2\overline{Z}U_1} - 8V^1_{Z^3\overline{Z}U_1} + V^3_{Z^2\overline{Z}U_1U_j} - V^3_{Z^3\overline{Z}U_1U_j} \right)\omega^j,
\]

[Springer]
$\alpha_{U_3}^3 = \left( \frac{1}{16} V_3^3 z^3 z U_1 - \frac{3}{8} V_3^1 z^3 z - \frac{i}{16} V_3^2 z^3 U_1 - \frac{1}{2} V_3^1 z^3 z - \frac{1}{8} V_3^1 z z + \frac{1}{8} V_3^2 z^3 \right) \omega^z \right.
+ \left( \frac{i}{16} V_3^2 z^3 z U_1 - \frac{1}{2} V_3^1 z^3 z - \frac{1}{8} V_3^1 z^3 z + \frac{1}{8} V_3^2 z^3 U_1 + \frac{1}{16} V_3^2 z^3 U_1 + \frac{5}{8} V_3^1 z^3 \right) \omega^x
+ \left( \frac{5i}{4} V_3^3 z^3 z U_1 - \frac{5i}{4} V_3^3 z U_1 - \frac{1}{2} V_3^1 z^3 U_1 + \frac{1}{16} V_3^3 z^3 U_1 U_1 - \frac{1}{2} V_3^1 z^3 U_1 + \frac{1}{16} V_3^3 U_1 U_1 \right) \omega^j.

Generally, inspecting monomials appeared among the performed computations shows that

**Lemma 6.1** In Branch 2 and for each $j \geq 0$, it is possible to identically specify the Maurer–Cartan forms $\alpha_{U_3}^{j+1}$ and $\alpha_{U_4}^{j+1}$ by normalizing $V_3^3 z^3 U_1 = 1$ and $V_3^3 U_1 U_1 = 0$, when $j \neq 0$.

Similar to the case of Branch 1 and up to the end of order four, the only yet unnormalized Maurer–Cartan forms are those of the real form $\text{Re} \mu_{U_3}^{j+1}$. Let us proceed into the next order to check the possibility of normalizing these Maurer–Cartan forms.

**6.1 Order 5—Branch 2**

In this branch and as suggested by the performed computations, we normalize to zero the imaginary part of the lifted invariant $V_3^2 z^3$. We have the following long recurrence expression:

$$dV_3^2 z^3 = \left( \frac{5}{6} V_3^1 z^3 z - \frac{5}{24} V_3^2 z^3 z - \frac{5}{24} V_3^1 z z - \frac{23}{8} V_3^1 z^3 z + \frac{i}{3} V_3^2 z^3 V_3^3 z z^3 \right)
+ \left( \frac{1}{24} V_3^2 z^3 U_1 V_3^2 z^3 z\right.
- \frac{1}{12} V_3^2 z^3 z V_3^1 z^3 z + \frac{1}{3} V_3^2 z^3 z V_3^1 z^3 z - \frac{1}{24} V_3^3 z^3 U_1 V_3^1 z^3 z
+ \left. \frac{5}{12} V_3^2 z^3 z V_3^1 z^3 z + \frac{1}{12} V_3^2 z^3 z V_3^1 z^3 z \right)
- \frac{19}{24} V_3^1 z^3 z - \frac{1}{24} V_3^3 z^3 U_1 - V_3^2 z^3 z V_3^1 z^3 + V_3^2 z^3 z
+ \left. \frac{13i}{24} V_3^2 z^3 u_1 \right) \omega^z
+ \left( \frac{1}{24} V_3^1 z^3 z + \frac{25}{24} V_3^1 z z - \frac{11}{6} V_3^1 z^3 z - \frac{15}{8} V_3^1 z^3 z + \frac{i}{3} V_3^2 z^3 V_3^3 z z^3 \right)
- \frac{1}{24} V_3^2 z^3 U_1 V_3^2 z^3 z
+ \left. \frac{1}{12} V_3^2 z^3 z V_3^1 z^3 z + \frac{1}{3} V_3^2 z^3 z V_3^1 z^3 z + \frac{5}{12} V_3^2 z^3 z V_3^1 z^3 z - \frac{1}{12} V_3^2 z^3 z V_3^2 z^3 z + V_3^2 z^3 z
+ \left. \frac{1}{24} V_3^3 z^3 U_1 \right) \omega^z
- \frac{1}{24} V_3^2 z^3 z - V_3^2 z^3 z V_3^2 z^3 z - \frac{1}{24} V_3^3 z^3 U_1 V_3^2 z^3 z
+ \left. \frac{13i}{24} V_3^2 z^3 U_1 \right) \omega^z.
Lemma 6.2 In Branch 2 and for each \( j \geq 0 \), it is possible to specify the real Maurer–Cartan form \( \text{Re} \mu_{U_j} \) by normalizing \( \text{Im} V^2_{z^j z^j U_j} = 0 \).

As one sees and similar to Branch 1, all Maurer–Cartan forms have gained at this stage their normalized expressions. Thus, we have succeeded again in constructing a complete moving frame and equivalently a normal form without further possible normalizations. It follows from the possibility of constructing this moving frame that the holomorphic transformation which brings each CR manifold to the achieved normal form is unique. Summing up the results, we have

Theorem 6.1 Let \( M^5 \subset \mathbb{C}^4 \) be a five dimensional real-analytic totally nondegenerate CR manifold enjoying two assumptions \( V^3_{z^j z^j} = 0 \) and \( V^3_{z^j z^j U_j} \neq 0 \). Then, there exists a unique origin-preserving holomorphic transformation that brings \( M^5 \) into the convergent normal form:

\[
\begin{align*}
v^1 &= z \bar{z} + \sum_{j+k+l \geq 5} \frac{1}{j! k! l!} V^1_{z^j \bar{z}^k u^l} z^j \bar{z}^k u^l, \\
v^2 &= \frac{1}{2} (z^2 \bar{z} + z \bar{z}^2) + \sum_{j+k+\ell \geq 5} \frac{1}{j! k! \ell!} V^2_{z^j \bar{z}^k \ell u^\ell} z^j \bar{z}^k u^\ell, \\
v^3 &= -\frac{1}{2} (z^2 \bar{z} - z \bar{z}^2) + \frac{1}{2} (z^2 \bar{z} u_1 + z \bar{z}^2 u_1) + \sum_{j+k+\ell \geq 5} \frac{1}{j! k! \ell!} V^3_{z^j \bar{z}^k \ell u^\ell} z^j \bar{z}^k u^\ell,
\end{align*}
\]

(34)

where, in addition to the relations (33), we have the cross-section normalizations (29) together with:

\[
0 = \text{Im} V^2_{z^j z^j U_j} = V^3_{z^j z^j U_j^{j+1}}
\]

(35)

for \( j \geq 0 \). Moreover, in this branch, the Lie algebra \( \text{aut}_{CR}(M^5) \) is five dimensional and the biholomorphic equivalence problem to \( M^5 \) can be reduced to an absolute parallelism, namely \( \{e\}\)-structure, on itself with the structure equations of the horizontal coframe \( \omega^s, \omega^r, \omega^j, j = 1, 2, 3 \) of \( M^5 \), obtained by (20) after applying the already mentioned normalizations and inserting the achieved expressions of the normalized Maurer–Cartan forms.

Representing the achieved normal form (34) as (32), the following schematic diagrams describe partly the associated functions \( \Phi^k_{U_j}, k = 1, 2, 3 \), in terms of the applied normalizations (29)–(35) (Figs. 5, 6, 7):
The interpretation of the circles at each point of these diagrams is as before, but it may be necessary to state that the circle standing at the point \((2, 1)\) in the diagram of \(\Phi^3\) amounts to the fact that \(\Phi^3_{21}\) not only does not include the monomials \(u_j^1 u_l^1\) for \(j, l \geq 0\) but also monomials of the form \(u_1 u_j^1\) are absent in it.

Our cross-section \((29)–(35)\) associated to this branch is minimal, as well. Thus, we have:

**Corollary 6.3** *The collection of edge differential invariants of the cross-section \((29)–(35)\) forms a generating system of differential invariants for the biholomorphic equivalence problem to 5-dimensional totally nondegenerate CR manifolds \(M^5\), belonging to Branch 2.*

### 7 Branch 3: \(V^3_{z^3 \bar{z}} = V^3_{z^2 \bar{z} u_1} = 0\)

Now we arrive at Branch 3, where the coefficients of \(z^3 \bar{z}\) and \(z^2 \bar{z} u_1\) in the moving frame \((29)\) are identically zero. This branch includes Beloshapka’s cubic model \(M_C^5\):
v^1 = z\bar{z},
v^2 = z^2\bar{z} + z\bar{z}^2, \quad (36)
v^3 = i(z^2\bar{z} - \bar{z}^2).

The Lie algebra \( \text{aut}_{CR}(M_5^5) \) of infinitesimal CR automorphisms of \( M_5^5 \) is of the maximum possible dimension 7 [21, Proposition 3.2] and since none of the CR manifolds \( M_5 \) in the former branches exhibits such CR symmetry dimension, then submanifolds equivalent to this model may possibly emerge in this branch.

Losing the benefit of the lifted differential invariants \( V_3^3 Z_3 Z_2 \) and \( V_3^3 Z_2 Z U_1 \) for normalizing further Maurer–Cartan forms in order four may convince oneself to expect more complicated computations at this branch. Before proceeding into the next order, we point it out that by equating to zero the coefficients of the linearly independent lifted horizontal forms \( \omega_z, \omega_\bar{z}, \omega_j, j = 1, 2, 3 \), in the recurrence relations of \( V_3^3 Z_3 Z_2 \) and \( V_3^3 Z_2 Z U_1 \) in (27), we receive the following helpful equations:

\[
\begin{align*}
V_3^3 Z_4 Z_2 &= 0, \\
V_3^3 Z_3 Z_2 &= -i V_2^2 Z_3 Z_2, \\
V_3^3 Z U_j &= 0, \\
V_1^1 Z_4 Z_2 &= 0, \\
V_2^2 Z_4 Z_2 &= 4 V_1^1 Z_3 Z_2 - \frac{1}{2} V_3^3 Z_3 Z_2 U_1 - \frac{i}{2} V_2^2 Z_3 Z_2 U_1, \\
V_3^3 Z U_1 U_1 &= 8 V_1^1 Z_3 Z U_1, \\
V_3^3 Z U_1 U_2 &= 0, \\
V_3^3 Z U_1 U_3 &= 0. \\
\end{align*}
\] (37)

Recall that, up to the end of order four, the remained unnormalized Maurer–Cartan forms are those visible in (26).

### 7.1 Order 5—Branch 3

In comparison to the former orders, computations in order five grow explosively. Then, for the sake of brevity, we will present from now on the upcoming long expressions modulo the lifted horizontal coframe \( \omega_z, \omega_\bar{z}, \omega_j, j = 1, 2, 3 \). Nevertheless, the full expressions and intermediate computations are accessible in the MAPLE worksheet [29]. Let us start with the recurrence relation of \( V_1^1 Z_3 Z^2 \). After a large amount of simplification, we receive:

\[
\begin{align*}
dV_1^1 Z_3 Z^2 &= V_1^1 Z_3 Z^2 (2i \alpha_3^2 - \frac{2}{3} \alpha_1^3) - 8 \alpha_3^3 U_1 U_3 + 12i \alpha_1^2 U_3 U_3, \mod \omega^z, \omega^\bar{z}, \omega^j. \\
\end{align*}
\]

Thus, by the normalization \( V_1^1 Z_3 Z^2 = 0 \), one receives that:

\[
\alpha_3^2 U_3 U_3 \equiv 0, \quad \text{and} \quad \alpha_1^3 U_3 U_3 \equiv 0, \quad \mod \omega^z, \omega^\bar{z}, \omega^j.
\]

We continue with the recurrence formula of \( dV_2^2 Z_3 Z_3 \). Our computations simplify to:

\[
\begin{align*}
dV_2^2 Z_3 Z_3 &= -\frac{2}{3} V_2^2 Z_3 Z_3 \alpha_1^3 U_3 - 4i \text{Re} \mu U_2, \quad \mod \omega^z, \omega^\bar{z}, \omega^j. \\
\end{align*}
\]

Hence, by normalizing \( \text{Im} V_2^2 Z_3 Z_3 = 0 \), one may also specify:

\[
\text{Re} \mu U_2 \equiv 0, \quad \mod \omega^z, \omega^\bar{z}, \omega^j.
\]
A close inspection of the performed computations for the above two recurrence relations shows that

**Lemma 7.1** Let \( j \geq 0 \). It is always possible

1. to specify each two Maurer–Cartan forms \( \alpha^2_{U_{j+2}} \) and \( \alpha^3_{U_{j+2}} \) by normalizing to zero the lifted differential invariant \( V^1_{Z^jZ^jU_{j}} \).
2. to specify each real Maurer–Cartan form \( \text{Re} \mu_{Z^jU_{j+1}} \), by normalizing to zero the lifted differential invariant \( \text{Im} V^2_{Z^jZ^jU_{j}} \).

We notice that following the normalization \( \text{Im} V^2_{Z^jZ^j} = 0 \), the recurrence relation of this differential invariant converts into the form:

\[
d(\text{Re} V^2_{Z^jZ^j}) = -\frac{2}{3} (\text{Re} V^2_{Z^jZ^j}) \alpha^3_{U_{j}}, \mod \omega^z, \omega^\bar{z}, \omega^j.
\]

(38)

Thanks to Lemma 7.1, the collection of yet unnormalized Maurer–Cartan forms (26) is now extensively reduced to only two Maurer–Cartan forms:

\[
\alpha^2_{U_{j}} \quad \text{and} \quad \alpha^3_{U_{j}}.
\]

(39)

At this stage, let us probe the recurrence relations corresponding to some of the lifted differential invariants appearing in (37). Although these equations will not result in normalizing the remained Maurer–Cartan forms but they can reveal some key relations between lifted differential invariants. After a large amount of simplifications, we receive that:

\[
0 = dV^3_{Z^jZ} = V^3_{Z^jZ} \omega^z + (V^3_{Z^jZ} + i V^2_{Z^jZ}) \omega^\bar{z} + V^3_{Z^jZU_{j}} \omega^j,
\]

\[
0 = dV^3_{Z^jZU_{j}} = V^3_{Z^jZU_{j}} \omega^z + (V^3_{Z^jZU_{j}} + i V^2_{Z^jZ}) \omega^\bar{z} + V^3_{Z^jZU_{j}U_{j}} \omega^j,
\]

\[
0 = dV^3_{Z^jZU_{j}} = V^3_{Z^jZU_{j}} \omega^z + (V^3_{Z^jZU_{j}} + i V^2_{Z^jZ}) \omega^\bar{z} + V^3_{Z^jZU_{j}U_{j}} \omega^j,
\]

\[
0 = dV^3_{Z^jZ} = \frac{6}{5} V^1_{Z^jZ} \omega^z + (V^1_{Z^jZ} + \frac{1}{5} V^2_{Z^jZ} - \frac{1}{5} V^2_{Z^jZ}) \omega^\bar{z} + V^1_{Z^jZU_{j}} \omega^j - \frac{1}{5} V^3_{Z^jZ} \alpha^2_{U_{j}}.
\]

Equating to zero the coefficients of the horizontal lifted 1-forms implies the following helpful relations:

\[
0 = V^3_{Z^jZ} = V^3_{Z^jZU_{j}} = V^3_{Z^jZU_{j}U_{j}} = V^1_{Z^jZ} = V^1_{Z^jZU_{j}}
\]

\[
V^3_{Z^jZ} = -i V^2_{Z^jZ}, \quad V^3_{Z^jZU_{j}} = 2i V^1_{Z^jZU_{j}} - i V^2_{Z^jZU_{j}},
\]

\[
V^3_{Z^jZU_{j}} = -i V^2_{Z^jZU_{j}}, \quad V^3_{Z^jZU_{j}U_{j}} = -i V^2_{Z^jZU_{j}} - V^2_{Z^jZU_{j}} = 5 V^1_{Z^jZ}.
\]

(40)

for \( j, k = 1, 2, 3 \). Now, let us continue examining the yet unnormalized lifted invariants in the current order five. According to the observations realized in the former orders, we shall check the recurrence relations of the six lifted differential invariants:

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\[ dV^1_{Z^2U_1} = (2i V^1_{Z^2U_1} - \frac{3}{2} V^3_{Z^2U_2}) \alpha^2_{U_1} - \frac{4}{3} V^1_{Z^2U_1} \alpha^3_{U_3}, \]
\[ dV^1_{Z^2U_1} = (4 V^3_{Z^2U_2} - \frac{8i}{3} V^1_{Z^2U_1} + \frac{8i}{3} V^1_{Z^2U_1}) \alpha^2_{U_3} - \frac{4}{3} V^1_{Z^2U_1} \alpha^3_{U_3}, \]
\[ dV^1_{Z^2U_3} = (4 V^2_{Z^2U_2} + 4 V^2_{Z^2U_2} - 4i V^2_{Z^2U_3} + 4i V^2_{Z^2U_3}) \alpha^2_{U_3} \]
\[ - 4i V^3_{Z^2U_2} + 4i V^3_{Z^2U_2} \alpha^2_{U_3}, \]
\[ - \frac{5}{3} V^1_{Z^2U_3} \alpha^3_{U_3}, \]
\[ dV^2_{Z^2U_1} = \frac{V^3_{Z^2U_2}}{z^2Z^2U_1} \alpha^2_{U_3} - \frac{V^2_{Z^2U_1}}{z^2Z^2U_1} \alpha^3_{U_3}, \]
\[ dV^3_{Z^2U_1} = -\frac{V^2_{Z^2U_1}}{z^2Z^2U_1} \alpha^2_{U_3} - \frac{V^3_{Z^2U_1}}{z^2Z^2U_1} \alpha^3_{U_3}, \]
\[ dV^3_{Z^2U_2} = -\frac{4}{3} \frac{V^3_{Z^2U_2}}{z^2Z^2U_2} \alpha^3_{U_3} , \]

presented modulo the lifted horizontal forms. Together with (38), these expressions indicate the appearance of several new subbranches, relying upon the values of the involving lifted differential invariants.

**Theorem 7.1** Let $M^5 \subset C^4$ be a five dimensional real-analytic totally nondegenerate CR manifold belonging to Branch 3 which amounts to assume that it enjoys $V^3_{Z^2U_1} = V^3_{Z^2U_2} = 0$. Then, there exists some origin-preserving holomorphic transformation, bringing $M^5$ into the convergent normal form:

\[ v^1 = z^2 + \frac{1}{6} V^1_{Z^2U_1} z^2 \alpha U_1 + \frac{1}{6} V^3_{Z^2U_1} z^2 \alpha U_1 + \frac{1}{4} V^1_{Z^2U_1} z^2 \alpha U_1 + \frac{1}{4} V^3_{Z^2U_1} z^2 \alpha U_1 \]
\[ + \sum_{j+k+\ell \geq 6} \frac{1}{j!k!\ell!} V^1_{Z^2U_1} z^j \alpha u^k \alpha \ell, \]
\[ v^2 = \frac{1}{2} (z^2 - z^2) + \frac{1}{24} V^2_{Z^2U_1} z^3 \alpha U_1 + \frac{1}{4} V^2_{Z^2U_1} z^3 \alpha U_1 \]
\[ + \sum_{j+k+\ell \geq 6} \frac{1}{j!k!\ell!} V^2_{Z^2U_1} z^j \alpha u^k \alpha \ell, \]
\[ v^3 = \frac{1}{2} (z^2 - z^2) + \frac{1}{24} V^3_{Z^2U_1} z^3 \alpha U_1 + \frac{1}{4} V^3_{Z^2U_1} z^3 \alpha U_1 + \frac{1}{4} V^3_{Z^2U_1} z^3 \alpha U_1 \]
\[ + \frac{1}{2} V^3_{Z^2U_1} z^3 \alpha U_1 + \frac{1}{2} V^3_{Z^2U_1} z^3 \alpha U_1 + \sum_{j+k+\ell \geq 6} \frac{1}{j!k!\ell!} V^2_{Z^2U_1} z^j \alpha u^k \alpha \ell, \]

where in addition to the relations (37) and (40), we have the cross-section normalizations (29) together with:

\[ 0 = V^1_{Z^2U_1} = \text{Im} V^2_{Z^2U_2}, \]

for $j \geq 0$. Furthermore,

**Branch 3-1.** If at least one of the four lifted differential invariants:

\[ V^2_{Z^2U_1}, \quad V^3_{Z^2U_1}, \quad V^1_{Z^2U_1}, \quad V^3_{Z^2U_2}, \]

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is nonzero then, by normalizing it to 1, the corresponding recurrence relation brings \( 0 \equiv \alpha_2^2 U_1 = \alpha_3^3 U_3 \), modulo \( \omega^z, \omega^\bar{z}, \omega^j, j = 1, 2, 3 \). In this case, the CR automorphism algebra \( \text{aut}_{CR}(M^5) \) is five dimensional and the biholomorphic equivalence problem to \( M^5 \) can be reduced to an absolute parallelism, namely \( \{e\} \)-structure, on itself with the structure equations of the lifted horizontal coframe, obtained by (20) after applying the already mentioned normalizations and substituting the achieved expressions of the normalized Maurer–Cartan forms.

Branch 3-2. Otherwise, if \( V_2^2 Z^2 U_1 = V_3^3 Z^2 U_1 = V_1^1 Z^2 U_2 = 0 \) but at least one of the three real lifted differential invariants \( V_1^1 Z^2 U_1, V_2^2 Z^2 U_3, \text{Re} V_2^2 Z^2 U_3 \) is nonzero then, by normalizing it to 1, the real Maurer–Cartan form \( \alpha_3^3 U_3 \) can be specified.

Branch 3-3. Otherwise, if all of the already mentioned seven differential invariants vanish identically, then none of the remained Maurer–Cartan forms \( \alpha_2^2 U_3 \) and \( \alpha_3^3 U_3 \) is normalizable in the current order five.

Proof If at least one of the invariants \( V_1^1 Z^2 U_1, V_2^2 Z^2 U_1, V_3^3 Z^2 U_1 \) is nonzero then, by normalizing it to 1, the corresponding first, fourth or fifth equation in (41) yields \( 0 \equiv \alpha_2^2 U_3 = \alpha_3^3 U_3 \), modulo the lifted horizontal coframe (notice that \( V_3^3 Z^2 U_2 \) is real). Otherwise, if these invariants vanish but \( V_3^3 Z^2 U_2 \neq 0 \), then by normalizing it to 1, the first and last equations in (41) imply the same result. In both of these cases, all the appearing Maurer–Cartan forms are normalized and hence the Lie algebra \( \text{aut}_{CR}(M^5) \) has the same dimension as the CR manifold \( M^5 \).

Otherwise, if the above four mentioned differential invariants vanish identically but at least one of the real differential invariants \( \text{Re} V_2^2 Z^2 U_3, V_1^1 Z^2 U_1, V_2^2 Z^2 U_3 \) is nonzero, then its corresponding recurrence relation in (38) or (41) results in specifying the Maurer–Cartan form \( \alpha_3^3 U_3 \).

If all of these seven differential invariants vanish, then the recurrence relations (38) and (41) are of no use to make any normalization on the remained two Maurer–Cartan form.

Representing the achieved normal form (34) as (32) with the corresponding cross-section (29)–(43), the general schematic diagram of the normal forms in Branch 3 can be displayed as follows—disregarding the normalizations which may appear in the mentioned subbranches (Figs. 8, 9, 10):

As all the appeared Maurer–Cartan forms are normalized in Branch 3-1, we notice that the normal form achieved in this branch admits no further normalizations applicable on it. It also follows, from the possibility of constructing a moving frame in this branch, that the transformation which brings each CR manifold \( M^5 \) to this normal form is unique. For Branch 3-2, further normalizations on the normal form depends on the possibility of normalizing the
last remained Maurer–Cartan form $\alpha_2^{U_3}$ in the next orders, however, at least we know that the infinitesimal CR automorphism algebras in this branch are either of dimension 5 or 6. Similarly, to seek further normalizations in Branch 3-3 we need to proceed to the next orders. Since this last branch includes the model cubic $M^5_c$ (cf. (36)), let us skip Branch 3-2 and pursue with it.

7.2 Order 6—Branch 3-3

Although the order five recurrence relations (38) and (41) are of no help to normalize Maurer–Cartan forms in this branch, nevertheless, equating to zero the coefficients of the independent lifted horizontal 1-forms brings us some helpful relations. Because of their length, we did not present these coefficients in (41) but they are available in the Maple worksheet [29]. Taking into account the expressions (37) and (40) and also the normalizations (43), the arisen system results in the solution:

\begin{align}
0 &\equiv V_1^{z^2 \bar{z}^2} = V_1^{z^2 \bar{z}^2 U_1 U_1} = V_1^{z^2 \bar{z}^2 U_1 U_2} = V_1^{z^2 \bar{z}^2 U_1 U_3} \\
&= V_1^{z^3 \bar{z} U_1 U_1} = V_1^{z^3 \bar{z} U_1 U_2} = V_1^{z^3 \bar{z} U_1 U_3} = V_1^{z^3 \bar{z}^2 U_1} = V_1^{z^3 \bar{z}^2 U_2} \\
&= V_2^{z^2 \bar{z}^2 U_1 U_1} = V_2^{z^2 \bar{z}^2 U_1 U_2} = V_2^{z^2 \bar{z}^2 U_1 U_3} = V_2^{z^2 \bar{z}^2 U_1} = V_2^{z^2 \bar{z}^2 U_2} = V_2^{z^2 \bar{z}^2 U_3} \\
&= V_3^{z^3 \bar{z} U_1 U_1} = V_3^{z^3 \bar{z} U_1 U_2} = V_3^{z^3 \bar{z} U_1 U_3} = V_3^{z^3 \bar{z} U_1 U_2} = V_3^{z^3 \bar{z} U_1 U_3} = V_3^{z^3 \bar{z} U_1 U_2} = V_3^{z^3 \bar{z} U_1 U_3} = V_3^{z^3 \bar{z} U_1 U_2} \\
&= V_3^{z^3 \bar{z} U_1 U_2 U_3} = V_3^{z^3 \bar{z} U_1 U_2 U_3}.
\end{align}
Furthermore, the recurrence relation of order five differential invariant \( V^3 z^4 \), vanished in this subbranch, can be simplified extensively by means of the above equations together with (37) and (40) as:

\[
0 = dV^3 z^4 = -\frac{i}{2} V^2 z^4 \omega^s + (V^3 z^3 + \frac{i}{2} V^2 z^4) \omega^r.
\]

Equating to zero the coefficients of \( \omega^s \) and \( \omega^r \) of this equation and also inserting (45) in (37) and (40), give in addition:

\[
0 = V^1 z^4 = V^1 z^4 U_j = V^2 z^5 = V^2 z^5 \quad \text{for } j = 1, 2, 3.
\]

Accordingly and taking into accounts also the formerly provided normalizations (29)–(43), we shall consider in this order just the following 3 out of 130 possible recurrence relations:

\[
dV^1 z^4 = -\frac{4}{3} V^1 z^3 \alpha^3_{U_3},
\]

\[
dV^1 z^4 U_2 U_3 = -\frac{1}{3} V^1 z^4 U_3 \alpha^3_{U_2} - \frac{8}{9} V^1 z^2 U_2 U_3 \alpha^3_{U_3}, \quad \text{mod } \omega^s, \omega^r, \omega^j. \tag{47}
\]

Following these equations, we find ourselves forced to deal again with some new subbranches, depending upon vanishing/non-vanishing of the three lifted relative invariants \( V^1 z^4, V^1 z^4 U_2 U_3 \) and \( V^1 z^4 U_3 U_3 \). Thus, we have to imagine the following three subbranches at the heart of Branch 3-3:

- **Branch 3-3-1**: \( V^1 z^4 \neq 0 \) and \( V^1 z^4 U_2 U_3 = \frac{V^1 z^2 U_2 U_3}{V^1 z^2 U_3} = 0 \).
- **Branch 3-3-2**: \( V^1 z^4 = 0 \) and either \( V^1 z^4 U_2 U_3 \neq 0 \) or \( V^1 z^4 U_3 U_3 \neq 0 \).
- **Branch 3-3-3**: \( V^1 z^4 = V^1 z^4 U_2 U_3 = V^1 z^4 U_3 U_3 = 0 \).

### 7.2.1 Branch 3-3-1

In this case and by normalizing \( V^1 z^4 = 1 \), the first recurrence relation in (47) plainly specifies the Maurer–Cartan form \( \alpha^3_{U_3} \). Our computations show that, after simplification, we have:

\[
\alpha^3_{U_3} = \frac{3}{4} V^1 z^3 \omega^s + \frac{3}{4} V^1 z^4 \omega^r + \frac{3}{4} V^1 z^3 U_j \omega^j.
\]

Then, only one Maurer–Cartan form, namely \( \alpha^2_{U_3} \), is remained yet unnormalized here. In order to realize the possibility of its normalization, we have to proceed into order seven. Although along the same way of discovering the relations (45), we can find the value of many order seven lifted differential invariants but, unfortunately, it needs to perform enormous computations for both finding the corresponding recurrence relations and solving the arisen system. In order to
Thus, these two lifted differential invariants are essentially of no use to normalize the Maurer–Cartan form in (49). After a large amount of simplifications, we found that:

$$\mu Z = -i \alpha_{U_3}^2 + \frac{1}{3} \alpha_{U_3}^3, \quad \mu U_j = 0, \quad j = 1, 2, 3,$$

$$\alpha_{U_1}^1 = -(\omega^z + \omega^\tau), \quad \alpha_{U_2}^2 = \alpha_{U_3}^3, \quad \alpha_{U_1}^3 = \frac{2}{3} \alpha_{U_3}^3, \quad \alpha_{U_2}^1 = \alpha_{U_3}^1 = 0,$$

with $\alpha_{U_3}^3$ as determined above. Inserting these expressions into the structure equations (20) of the lifted horizontal coframe gives:

$$d\omega^z = -i \alpha_{U_3}^2 \wedge \omega^z - \frac{1}{4} V^1_{Z^3Z^3} \omega^z \wedge \omega^\tau - \frac{1}{4} V^1_{Z^3Z^3} U_j \omega^z \wedge \omega^j,$$

$$d\omega^\tau = \overline{d\omega^z},$$

$$d\omega^1 = 2i \omega^z \wedge \omega^\tau + \frac{1}{2} V^1_{Z^3Z^3} \omega^z \wedge \omega^\tau - \frac{1}{2} V^1_{Z^3Z^3} U_j \omega^\tau \wedge \omega^j,$$

$$d\omega^2 = \alpha_{U_3}^2 \wedge \omega^3 - \omega^z \wedge \omega^1 - \omega^\tau \wedge \omega^1$$

$$+ \frac{3}{4} \left( V^1_{Z^3Z^3} \omega^z + V^1_{Z^3Z^3} \frac{1}{4} \omega^\tau + V^1_{Z^3Z^3} U_j \omega^j \right) \wedge \omega^2,$$

$$d\omega^3 = -\alpha_{U_3}^2 \wedge \omega^3 + i \omega^z \wedge \omega^1 - i \omega^\tau \wedge \omega^1$$

$$+ \frac{3}{4} \left( V^1_{Z^3Z^3} \omega^z + V^1_{Z^3Z^3} \frac{1}{4} \omega^\tau + V^1_{Z^3Z^3} U_j \omega^j \right) \wedge \omega^3.$$

In view of these structure equations and by the principles of Cartan’s theory (cf. [6,7,21,23]), normalization of the last Maurer–Cartan form $\alpha_{U_3}^2$ relies on the expressions of the four lifted differential invariants $V^1_{Z^3Z^3}, V^1_{Z^3Z^3} U_j, j = 1, 2, 3$. Our much complicated computations show that two lifted differential invariants $V^1_{Z^3Z^3}$ and $V^1_{Z^3Z^3} U_1$ are now independent of the remained group parameters. Indeed we have:

$$dV^1_{Z^3Z^3} \equiv 0 \text{ and } dV^1_{Z^3Z^3} U_1 \equiv 0, \mod \omega^z, \omega^\tau, \omega^j.$$

Thus, these two lifted differential invariants are essentially of no use to normalize the Maurer–Cartan form $\alpha_{U_3}^2$. Moreover, checking the recurrence relations of $V^1_{Z^3Z^3} U_2$ and $V^1_{Z^3Z^3} U_3$ shows that:

$$V^1_{Z^3Z^3} U_2 = V^1_{Z^3Z^3} U_3.$$

Accordingly, $V^1_{Z^3Z^3} U_2$ is the only lifted differential invariant which may help one to normalized the remained Maurer–Cartan form in (49). After a large amount of simplifications, we found that:

$$dV^1_{Z^3Z^3} U_2 = V^1_{Z^3Z^3} U_2 \alpha_{U_3}^2 \mod \omega^z, \omega^\tau, \omega^j.$$
Consequently, the normalization of \( \alpha_{U_3}^2 \) depends upon vanishing/non-vanishing of the order seven lifted differential invariant \( V_1^{Z^3 U_2} \). Thus we shall consider the following two sub-branches of Branch 3-3-1:

- **Branch 3-3-1-a:** \( V_1^{Z^3 U_2} \neq 0 \),
- **Branch 3-3-1-b:** \( V_1^{Z^3 U_2} = 0 \).

In the first branch, the normalization \( V_1^{Z^3 U_2} = 1 \) plainly specifies the Maurer–Cartan form \( \alpha_{U_3}^2 \). In this case, the construction of a complete equivariant moving frame and hence a unique normal form, without freedom, is successfully finalized and the corresponding Lie algebras of infinitesimal CR automorphisms are 5-dimensional. Moreover, according to the structure equations (49), the equivalence problem to CR manifolds belonging to this branch is determined by the two lifted differential invariants \( V_1^{Z^3 U_1} \) and \( V_1^{Z^3 U_1} \).

But in Branch 3-3-1-b, the above recurrence relation \( dV_1^{Z^3 U_2} \) is of no use to specify \( \alpha_{U_3}^2 \).

More precisely, in this case the structure equations (49) show that this remained Maurer–Cartan form is *never normalizable*. One can verify from these structure equations that the horizontal lifted coframe \( \{ \omega^z, \omega^\tau, \omega^1, \omega^2, \omega^3 \} \) is *non-involutive* (see [23, Sect. 11] for definition). Hence by [23, Proposition 12.1], the solution of the biholomorphic equivalence problem to 5-dimensional CR manifolds \( M^5 \), belonging to this subbranch, completely relies upon the solution of the equivalence problem between 6-dimensional *prolonged spaces* \( M^{6 \times \mathcal{G}^{red}} \) equipped with the coframe \( \{ \omega^z, \omega^\tau, \omega^1, \omega^2, \omega^3, \alpha_{U_3}^2 \} \). Here, \( \mathcal{G}^{red} \) is the 1-dimensional subgroup obtained by the normalizations applied to the original pseudo-group \( \mathcal{G} \). According to the principles of Cartan’s classical method, we only need to find the structure equations of the only added 1-form \( \alpha_{U_3}^2 \). By the help of the formula (7), we have:

\[
d\alpha_{U_3}^2 = \omega^z \wedge \alpha_{Z U_3}^2 + \omega^\tau \wedge \alpha_{Z U_3}^2 + \omega^j \wedge \alpha_{U_j U_3}^2 + \alpha_{Z U_3}^2 \wedge \mu_{U_3} + \alpha_{U_3}^2 \wedge \mu_{U_3} \]

\[
+ \alpha_{U_j U_3}^2 \wedge \alpha_{U_3}^2 + \alpha_{V_j U_3}^2 \wedge \gamma_{U_3}^j.
\]

Our computations show that in this branch:

\[
0 = \alpha_{Z U_3}^2 = \alpha_{U_1 U_3}^2 = \alpha_{U_2 U_3}^2 = \mu_{U_3} = \alpha_{U_3}^1 = \alpha_{V_j U_3}^j, \quad \alpha_{U_2 U_3}^2 = \frac{1}{16} (\omega^1 + \omega^2),
\]

\[
\alpha_{U_2}^2 = \alpha_{U_3}^2 = \frac{3}{4} V_1^{Z^3 U_2} \omega^z + \frac{3}{4} V_1^{Z^3 U_1} \omega^\tau + \frac{3}{4} V_1^{Z^3 U_1} \omega^1
\]

and thus, we have:

\[
d\alpha_{U_3}^2 = -\frac{1}{16} (\omega^z + \omega^\tau) \wedge \omega^2. \tag{50}
\]

This structure equation together with those in (49)—modified by the branch assumptions

\[
0 = V_1^{Z^3 U_2} = V_1^{Z^3 U_1}
\]

determine the biholomorphic equivalence problem between CR manifolds \( M^5 \) of this subbranch. They also indicate that the Lie algebras of infinitesimal CR automorphisms corresponding to these manifolds are all of dimension six. Summing up the results, we have

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3. Indeed, the reason is that the *degree of indeterminancy* of the system (49) is zero (cf. [23, Definition 11.2]).
Theorem 7.2 Let $M^5 \subset C^4$ be a five dimensional real-analytic totally nondegenerate CR manifold of Branch 3-3-1, namely with the assumptions:

$$0 = V^3 z^i \bar{z} = V^3 z^i \bar{z} u_1,$$

$$0 = V^2 z^i \bar{z} ^2 u_1 = V^3 \bar{z} z^i u_1 = V^3 z^3 \bar{z} u_1 = V^1 z^2 \bar{z} u_2 = V^1 z^2 \bar{z} u_3 = V^1 z^2 \bar{z} u_3 = 0,$$

$$0 \neq V^1 z^3 \bar{z} ^3 and V^1 z^2 \bar{z} u_2 u_3 = V^1 z^2 \bar{z} u_3 = 0.$$  

Then, there exists some origin-preserving holomorphic transformation, bringing $M^5$ to the convergent normal form:

$$v^1 = z \bar{z} + \frac{1}{36} z^3 \bar{z}^3$$

$$+ \sum_{j+k+\ell \geq 7} \frac{1}{j! k! \ell !} V^1 z^j \bar{z} ^k u_\ell,$$

$$v^2 = \frac{1}{2} (z^2 \bar{z} + \bar{z}^2) + \sum_{j+k+\ell \geq 7} \frac{1}{j! k! \ell !} V^2 z^j \bar{z} ^k u_\ell,$$

$$v^3 = -\frac{1}{2} (z^2 \bar{z} - \bar{z}^2) + \sum_{j+k+\ell \geq 7} \frac{1}{j! k! \ell !} V^3 z^j \bar{z} ^k u_\ell,$$

(52)

with the cross-section normalizations (29)–(43). Moreover,

Branch 3-3-1-a. If in addition we have $V^1 z^i \bar{z} u_2 \neq 0$ then, by adding the normalization $V^1 z^3 \bar{z} u_2 = 1$ to the already mentioned cross-section, the lastly remained Maurer–Cartan form $\alpha^2_{U_2}$ will be normalized and one constructs a complete equivariant moving frame. In this case, the already mentioned origin-preserving transformation is unique and the Lie algebra $\text{aut}_{CR}(M^5)$ is 5-dimensional. Furthermore, the biholomorphic equivalence problem to $M^5$ is reducible to an absolute parallelism, namely $\{e\}$-structure, on itself with the structure equations (49), modified by $V^1 z^i \bar{z} u_2 = V^1 z^i \bar{z} u_3 = 1.$

Branch 3-3-1-b. Otherwise, if $V^1 z^i \bar{z} u_2 = 0$ then, the remained Maurer–Cartan form $\alpha^2_{U_3}$ is never normalizable. In this case, the Lie algebra $\text{aut}_{CR}(M^5)$ is 6-dimensional and the biholomorphic equivalence problem to $M^5$ is reducible to an absolute parallelism, namely $\{e\}$-structure, on the 6-dimensional prolonged space $M^5 \times G^{red}$ with the structure equations (49)–(50), modified by $V^1 z^i \bar{z} u_2 = V^1 z^i \bar{z} u_3 = 0.$

Remaining finally unnormalized the Maurer–Cartan form $\alpha^2_{U_3}$ demonstrates that the construction of a complete equivariant moving frame is impossible in Branch 3-3-1-b. In view of the results of [34], it shall be for this reason that the action of $G$ on CR manifolds $M^5$ belonging to this branch is not free and admits a 1-dimensional isotropy subgroup with the associated Maurer–Cartan coframe $\{\alpha^2_{U_3}\}.$ However, as suggested in [34], our normalizations lead us to construct a partial moving frame on $M^5.$ Fortunately, at this stage that all possible normalizations are applied in this branch and as stated at the page 18 of [34], none of the lifted differential invariants will depend on the remained unnormalized isotropy group parameter. Thus, the normal form $N$ associated with every CR manifold $M^5$ of Branch 3-3-1-b is unique. If $\varphi, \psi : M^5 \to N$ are two origin-preserving holomorphic transformations which bring $M^5$
into its normal form, then clearly the combination $\varphi \circ \psi^{-1}$ belongs to the isotropy group of $N$ at the origin (see also [34, Theorem 4.11]).

### 7.2.2 Branch 3-3-2

Now, let us assume that $V_1^1 z^3 \bar{z}^3 = 0$ but at least one of the lifted differential invariants $V_1^1 z^2 \bar{z}^2 u_2 u_2$ or $V_1^1 z^2 \bar{z}^2 u_3 u_3$ is nonzero. Disregarding the very specific case that $2V_1^1 z^2 \bar{z}^2 u_2 u_2 = 9V_1^1 z^2 \bar{z}^2 u_3 u_3$, an slightly careful glance on the second and third recurrence relations in (47) shows that by normalizing $V_1^1 z^2 \bar{z}^2 u_2 u_2, V_1^1 z^2 \bar{z}^2 u_3 u_3 = 1$ or 0, depending to their vanishing/non-vanishing, it is possible in this branch to specify both the remained Maurer–Cartan forms $\alpha_{U_2}^2$ and $\alpha_{U_3}^3$ as:

$$0 \equiv \alpha_{U_3}^2 \quad \text{and} \quad 0 \equiv \alpha_{U_3}^3, \mod \omega^z, \omega^\bar{z}, \omega^j.$$

Therefore, all the appearing Maurer–Cartan forms are normalizable here and we can construct successfully a complete moving frame and equivalently a normal form with no further applicable normalizations.

**Theorem 7.3** Let $M^5 \subset \mathbb{C}^4$ be a five dimensional real-analytic totally nondegenerate CR manifold enjoying the branch assumptions:

$$0 = V_3^1 z^3 \bar{z} = V_3^3 z^2 \bar{z} u_1,$$

$$0 = V_2^1 z^2 \bar{z} u_1 = V_3^3 z^2 \bar{z} u_1 = V_2^1 z^3 \bar{z} u_1 = V_3^1 z^2 \bar{z} u_3 = V_1^1 z^2 \bar{z} u_3 = \text{Re} V_2^2 z^2 \bar{z}, \quad (53)$$

$$0 = V_1^1 z^1 \bar{z} \quad \text{and} \quad (0, 0) \neq (V_1^1 z^2 \bar{z} u_2 u_3, V_1^1 z^2 \bar{z} u_3 u_3)$$

together with the minor assumption $2V_1^1 z^2 \bar{z} u_2 u_3 \neq 9V_1^1 z^2 \bar{z} u_3 u_3$. Then, there exists a unique origin-preserving holomorphic transformation which brings $M^5$ into the normal form:

$$v_1 = z\bar{z} + \frac{1}{4} V_1^1 z^2 \bar{z} u_2 u_3 z^2 \bar{z}^2 u_2 u_3 + \frac{1}{4} V_1^1 z^2 \bar{z} u_3 u_3 z^2 \bar{z}^2 u_3 u_3 + \sum_{j+k+\ell \geq 7} \frac{1}{j! k! \ell!} V_1^1 z^j \bar{z}^k U^\ell z^j \bar{z}^k u_\ell,$$

$$v_2 = \frac{1}{2} (z^2 \bar{z} + z \bar{z}^2) + \sum_{j+k+\ell \geq 7} \frac{1}{j! k! \ell!} V_2^2 z^j \bar{z}^k U^\ell z^j \bar{z}^k u_\ell,$$

$$v_3 = -\frac{i}{2} (z^2 \bar{z} - z \bar{z}^2) + \sum_{j+k+\ell \geq 7} \frac{1}{j! k! \ell!} V_3^3 z^j \bar{z}^k U^\ell z^j \bar{z}^k u_\ell,$$

(54)

with the cross-section normalizations (29)–(43) added by either $V_1^1 z^2 \bar{z} u_2 u_3, V_1^1 z^2 \bar{z} u_3 u_3 = 1$ or 0, depending upon their vanishing/non-vanishing. In this branch, the infinitesimal CR automorphism algebra $\mathfrak{aut}_{CR}(M^5)$ is five dimensional and the biholomorphic equivalence problem to $M^5$ is reducible to an absolute parallelism, namely $\{e\}$-structure, on itself with the structure equations of the lifted horizontal coframe, obtained by (20) after applying the already mentioned cross-section normalizations and inserting the achieved expressions of the normalized Maurer–Cartan forms.
7.2.3 Branch 3-3-3

By the assumptions of this branch, all order six differential invariants are vanished identically and none of the remained Maurer–Cartan forms $\alpha^2_{U_3}$ and $\alpha^3_{U_3}$ is normalizable in this order. In this case, the Maurer–Cartan forms appeared in (48) have the same expressions with the only difference that here $\alpha^3_{U_3}$ is not specified. Inserting these expressions into the structure equations (20) of the lifted horizontal coframe gives:

$$\begin{align*}
d\omega^z &= \left( -\frac{1}{3} \alpha^3_{U_3} - i \alpha^2_{U_3} \right) \wedge \omega^z, \\
d\omega^\tau &= \left( \frac{1}{3} \alpha^3_{U_3} + i \alpha^2_{U_3} \right) \wedge \omega^\tau, \\
d\omega^1 &= 2i \omega^z \wedge \omega^\tau + \frac{2}{3} \alpha^3_{U_3} \wedge \omega^\tau, \\
d\omega^2 &= -\omega^z \wedge \omega^1 - \omega^\tau \wedge \omega^1 + \alpha^3_{U_3} \wedge \omega^2 + \alpha^2_{U_3} \wedge \omega^3, \\
d\omega^3 &= \omega^z \wedge \omega^1 - i \omega^\tau \wedge \omega^1 - \alpha^3_{U_3} \wedge \omega^2 + \alpha^2_{U_3} \wedge \omega^3.
\end{align*}$$

(55)

These structure equations are of constant type and hence it will be impossible, even in higher orders, to normalize two remained Maurer–Cartan forms. Moreover, they plainly imply that the horizontal lifted coframe $\{\omega^z, \omega^\tau, \omega^1, \omega^2, \omega^3\}$ is non-involutive and hence the solution of biholomorphic equivalence problem to 5-dimensional CR manifolds $M^5$, belonging to this subbranch, completely relies upon the solution of equivalence problem between 7-dimensional prolonged spaces $M^{pr}$ := $M^5 \times \mathcal{G}^{red}$ equipped with the coframe $\{\omega^z, \omega^\tau, \omega^1, \omega^2, \omega^3, \alpha^3_{U_3}, \alpha^3_{U_3}\}$. Here, $\mathcal{G}^{red}$ is the 2-dimensional subgroup obtained by the normalizations applied to the original pseudo-group $\mathcal{G}$. Thus, we need only to find the structure equations of the two added 1-forms $\alpha^2_{U_3}$ and $\alpha^3_{U_3}$. In this branch, our computations show that:

$$0 = \alpha^k_{ZU_3} = \alpha^k_{U_3U_3} = \alpha^k_{UjU_3} = \mu_{U_3} = \alpha^1_{U_3} = \alpha^k_{Vj}, \quad \alpha^2_{U_3} = \alpha^3_{U_3}, \quad \alpha^3_{U_3} = -\alpha^2_{U_3}$$

and thus, by the help of the formula (7), one plainly receives that:

$$d\alpha^2_{U_3} = 0, \quad d\alpha^3_{U_3} = 0.$$  

(56)

The above two structure equations together with (55) provide the final desired $[e]$-structure of biholomorphic equivalence problem to 5-dimensional totally nondegenerate CR manifolds, belonging to this branch. Since these seven structure equations are of the constant type, all CR manifolds in this branch are biholomorphically equivalent. The cubic model $M^5_\mathbb{C}$ defined as (36) belongs to this branch and thus we may assert that

**Theorem 7.4** Let $M^5 \subset \mathbb{C}^4$ be a five dimensional real-analytic totally nondegenerate CR manifold enjoying the branch assumptions:

$$\begin{align*}
0 &= V^3_{ZU} = V^3_{ZU_1}, \\
0 &= V^2_{ZU_1} = V^2_{ZU_1} = V^1_{ZU_1} = V^3_{ZU_2} = V^1_{ZU_2} = V^1_{ZU_3} = V^1_{ZU_3} = \text{Re} V^2_{ZU} \quad (57)
\end{align*}$$

and

$$\begin{align*}
0 &= V^1_{ZU} = V^1_{ZU_2U_3} = V^1_{ZU_2U_3}. \\
\end{align*}$$

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Then, $M^5$ is biholomorphically equivalent to the model cubic $M^5_c$ and its defining equations can be converted into the simple normal form:

\begin{align}
v^1 &= z\overline{z}, \\
v^2 &= z^2\overline{z} + z\overline{z}^2, \\
v^3 &= i(z^2\overline{z} - \overline{z}^2).
\end{align}

(58)

In this case, the infinitesimal CR automorphism algebra $\text{aut}_{CR}(M^5)$ is of the maximum possible dimension 7 with the structure, displayed in the table on page 3231 of [21].

**Remark 7.2** The structure equations (55)–(56) obtained in this branch are fully equivalent to the structure equations [21, Eq. (36)]. The correspondence is given by:

\begin{align*}
\zeta &\leftrightarrow \omega, \\
\rho &\leftrightarrow \frac{1}{2}\omega^1, \\
\sigma &\leftrightarrow \frac{1}{4}(\omega^2 + i\omega^3), \\
\alpha &\leftrightarrow \frac{1}{3}\alpha^3_{U^3} - i\alpha^2_{U^3}.
\end{align*}

As the construction of a complete moving frame was here impossible, the origin-preserving holomorphic transformations which bring CR manifolds of this branch into the simple normal form (58) are no longer unique. For instance, applying each CR automorphism of the 2-dimensional isotropy group $\text{Aut}_0(M^5_c)$ of the cubic model at the origin to the normal form (58) keeps it still in normal form. This isotropy group is found in [21, Proposition 3.2] as the group of local flows of holomorphic vector fields generated by:

\begin{align*}
D &= z\frac{\partial}{\partial z} + 2w^1\frac{\partial}{\partial w^1} + 3w^2\frac{\partial}{\partial w^2} + 3w^3\frac{\partial}{\partial w^3}, \\
R &= iz\frac{\partial}{\partial z} - w^3\frac{\partial}{\partial w^2} + w^2\frac{\partial}{\partial w^3}.
\end{align*}

If $\varphi, \psi : M^5 \to M^5_c$ are two origin-preserving holomorphic transformations which bring a CR manifold $M^5$ to the normal form $M^5_c$, then clearly there exists a unique transformation $\gamma \in \text{Aut}_0(M^5_c)$ satisfying $\varphi = \gamma \circ \psi$. Thus, we have

**Corollary 7.3** The origin-preserving holomorphic transformation which brings a 5-dimensional totally nondegenerate manifold $M^5$ of Branch 3-3-3 into the normal form (58) is unique, up to the action of the isotropy subgroup $\text{Aut}_0(M^5_c)$, generated by the above holomorphic vector fields $D$ and $R$.

The above Theorem 7.4 has still another interesting consequence. Indeed, among all branches appeared in this paper, only the present branch 3-3-3 consists of totally nondegenerate CR manifolds $M^5$ with $\text{dim} \text{ aut}_{CR}(M^5) = 7$. This characterizes such CR manifolds plainly in terms of their infinitesimal CR automorphisms.

**Corollary 7.4** (cf. [21, Theorem 5.1]) A 5-dimensional real-analytic totally nondegenerate manifold $M^5$ is biholomorphically equivalent to the cubic model $M^5_c$ if and only if $\text{dim} \text{ aut}_{CR}(M^5) = 7$. In other words, up to the biholomorphic equivalence and in the class of totally nondegenerate CR manifolds considered in this paper, $M^5_c$ is the unique member with the maximal dimension of the associated algebra of infinitesimal CR automorphisms.

The results achieved in this branch may remind one of the Chern-Moser discussion of umbilical points. In [9], a point $p$ of a nondegenerate real hypersurface of $\mathbb{C}^2$ is called umbilical.
if the coefficient $c_{42}$ of the monomial $z^4 \overline{z}$ in the associated normal form [9, eq. (3.18)] vanishes. It is well-known that locally, a hypersurface is umbilical at each point if and only if it is spherical, that is: biholomorphically equivalent to the Heisenberg sphere $\mathbb{H}^3$, represented in local coordinates $z, w = u + iv$ as:

$$v = z \overline{z}.$$ 

Following this terminology and thanks to Corollary 7.4, we may regard the assumptions (57) as umbilical conditions in which satisfying them by the CR manifold $M^5$ guarantees its biholomorphic equivalence to the cubic model $M^5_c$.

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