Minima of quasisuperminimizers

Anders Björn
Department of Mathematics, Linköping University, SE-581 83 Linköping, Sweden; anders.bjorn@liu.se

Jana Björn
Department of Mathematics, Linköping University, SE-581 83 Linköping, Sweden; jana.bjorn@liu.se

Riikka Korte
Department of Mathematics, P.O. Box 68 (Gustaf Hällström min katu 2b), FI-00014 University of Helsinki, Finland; riikka.korte@helsinki.fi

Abstract. Let $u_i$ be a $Q_i$-quasisuperminimizer, $i = 1, 2$, and $u = \min\{u_1, u_2\}$, where $1 \leq Q_1 \leq Q_2$. Then $u$ is a quasisuperminimizer, and we improve upon the known upper bound (due to Kinnunen and Martio) for the optimal quasisuperminimizing constant $Q$ of $u$. We give the first examples with $Q > Q_2$, and show that in general $Q > Q_2$ whenever $Q_1 > 1$. We also study the blowup of the quasisuperminimizing constant in pasting lemmas.

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1. Introduction

Let $1 < p < \infty$ and let $\Omega \subset \mathbb{R}^n$ be a nonempty open set. A function $u \in W^{1,p}_{\text{loc}}(\Omega)$ is a $Q$-quasiminimizer, $Q \geq 1$, in $\Omega$ if

$$\int_{\varphi \neq 0} |\nabla u|^p \, dx \leq Q \int_{\varphi \neq 0} |(\nabla u + \varphi)|^p \, dx \quad (1.1)$$

for all $\varphi \in W^{1,p}_0(\Omega)$. A function $u$ is a $Q$-quasisuper(sub)minimizer if (1.1) holds for all nonnegative (nonpositive) $\varphi \in W^{1,p}_0(\Omega)$.

Quasiminimizers were introduced by Giaquinta and Giusti [15], [16] as a tool for a unified treatment of variational integrals, elliptic equations and quasiregular mappings on $\mathbb{R}^n$. They realized that De Giorgi’s method could be extended to quasiminimizers, obtaining, in particular, local Hölder continuity. DiBenedetto and Trudinger [14] proved the Harnack inequality for quasiminimizers, as well as weak Harnack inequalities for quasisub- and quasisuperminimizers. A little later, Ziemer [35] gave a Wiener-type criterion sufficient for boundary regularity for quasiminimizers, and Tolksdorf [33] obtained a Caccioppoli inequality and a convexity result for quasiminimizers. The results in [14]–[16] and [35] were extended to metric spaces by Kinnunen–Shanmugalingam [22] and J. Björn [10] in the beginning of this century, see also A. Björn–Marola [8]. Soon afterwards, Kinnunen–Martio [21]
showed that quasiminimizers have an interesting potential theory, in particular they introduced quasisuperharmonic functions, which are related to quasisuperminimizers in a similar way as superharmonic functions are related to supersolutions. The theory of quas(super)minimizers has been further studied in [1–5], [7], [9], [11–13], [17], [19], [20], [23–32] and [34].

It is well known that the minimum of two superharmonic functions is again superharmonic. This property is used extensively e.g. in balayage and in the Perron method for solving the Dirichlet problem. For quasisuperminimizers, Kinnunen–Martio [21] showed the following similar result. (We formulate it in $\mathbb{R}^n$, but it is valid also in metric measure spaces, see Section 2. The same holds for Theorems 1.2 and 1.4.)

**Theorem 1.1.** (Kinnunen–Martio [21]) Let $u_j$ be a $Q_j$-quasisuperminimizer, $j = 1, 2$. Then $\min\{u_1, u_2\} = \min\{Q_1 Q_2, Q_1 + Q_2\}$-quasisuperminimizer.

The blowup of the quasisuperminimizing constant in this result is the main focus of this paper. Our first result is the following better upper bound.

**Theorem 1.2.** Let $u_i$ be a $Q_i$-quasisuperminimizer in $\Omega$ for $i = 1, 2$. Then $u = \min\{u_1, u_2\}$ is a $Q$-quasisuperminimizer in $\Omega$, where

$$Q = \begin{cases} 1, & \text{if } Q_1 = Q_2 = 1, \\ \left(\frac{Q_1 + Q_2 - 2}{Q_1 Q_2 - 1}\right), & \text{otherwise.} \end{cases} \quad (1.2)$$

In particular, if $Q_1 = Q_2$, then $Q = 2 Q_j^2 / (Q_1 + 1)$.

Note that when $Q_1, Q_2 > 1$, we always have the following bounds for $Q$ in (1.2):

$$Q_1 + Q_2 - 2 < Q < Q_1 + Q_2 - 1 < \min\{Q_1 Q_2, Q_1 + Q_2\}.$$ 

This means that we obtain a better blowup constant than Kinnunen–Martio [21] whenever $Q_1, Q_2 > 1$.

In the converse direction it is clear that $u$ cannot (in general) have a better quasisuperminimizing constant than $\max\{Q_1, Q_2\}$ (and thus already Theorem 1.1 is optimal if $Q_1 = 1$ or $Q_2 = 1$). As far as we know, there have so far not been any examples showing that some blowup is indeed possible. We construct such examples in Section 3. In particular, we prove the following result.

**Theorem 1.3.** Let $p > 1$ and $1 < Q_1 \leq Q_2$. Then there exist functions $u_1$ and $u_2$ on $(0, 1) \subset \mathbb{R}$ such that $u_j$ is a $Q_j$-quasisuperminimizer in $(0, 1)$, $j = 1, 2$, but $\min\{u_1, u_2\}$ is not a $Q_2$-quasisuperminimizer in $(0, 1)$.

We also obtain estimates for the blowup in the quasisuperminimizing constant. In Section 4 we give an upper bound for the blowup when taking a minimum of three quasisuperminimizers, which is better than iterating Theorem 1.2.

Another result with a blowup in the quasisuperminimizing constant is the following pasting lemma.

**Theorem 1.4.** (Björn–Martio [9, Theorem 4.1]) Assume that $\Omega_1 \subset \Omega_2 \subset \mathbb{R}^n$ are open and that $u_j$ is a $Q_j$-quasisuperminimizer in $\Omega_j$, $j = 1, 2$. Let

$$u = \begin{cases} u_2, & \text{in } \Omega_2 \setminus \Omega_1, \\ \min\{u_1, u_2\}, & \text{in } \Omega_1. \end{cases}$$

If $u \in W^{1,p}_{\text{loc}}(\Omega_2)$, then $u$ is a $Q_1 Q_2$-quasisuperminimizer in $\Omega_2$. 
In Theorems 5.1 and 5.2 we show that the blowup constant $Q_1 Q_2$ is optimal in this result. There is also a similar pasting lemma for quasisuperharmonic functions in Björn–Martio [9, Theorem 5.1] and our optimality result applies also to this case, see Remark 5.3.

Yet another result with a blowup of the quasisuperminimizing constant is the reflection principle by Martio [26, Theorem 3.1]. In one dimension (i.e. on $\mathbb{R}$) he obtained a better result in Theorem 4.1 in [26]. The blowup constant in the latter result was subsequently improved upon by Uppman [34, Lemma 2.8], who also showed that his constant is the best possible.

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2. An upper bound for the blowup

In this section we are going to prove Theorem 1.2. Let us however first discuss some consequences and generalizations of it.

**Definition 2.1.** A function $u : \Omega \to (-\infty, \infty]$ is $Q$-quasisuperharmonic in $\Omega$ if $u$ is not identically $\infty$ in any component of $\Omega$, $\min\{u, k\}$ is a $Q$-quasisuperminimizer in $\Omega$ for every $k \in \mathbb{R}$, and $u$ is lower semicontinuously regularized, i.e.

$$u(x) = \text{ess lim inf}_{y \to x} u(y) \quad \text{for } x \in \Omega.$$ 

This definition is equivalent to Definition 7.1 in Kinnunen–Martio [21], see Theorem 7.10 in [21]. Using this definition we obtain the following corollary of Theorem 1.2.

**Corollary 2.2.** Let $u_i$ be a $Q_i$-quasisuperharmonic function in $\Omega$ for $i = 1, 2$. Then $u = \min\{u_1, u_2\}$ is $Q$-quasisuperharmonic in $\Omega$, where $Q$ is given by (1.2).

We have formulated Theorem 1.2 and Corollary 2.2 on (unweighted) $\mathbb{R}^n$, but they have direct counterparts valid in complete metric spaces equipped with doubling measures supporting a $p$-Poincaré inequality (and thus also on weighted $\mathbb{R}^n$ with a $p$-admissible weight), see Björn–Björn [6] for more on the metric space theory (note that Appendix C therein gives a short survey on quasiminimizers).

Below we have chosen to give an $\mathbb{R}^n$ proof of Theorem 1.2. However, it carries over verbatim to metric spaces, with the trivial modifications that $|\nabla u|$ is replaced by the minimal $p$-weak upper gradient $g_u$ (and similarly for the other gradients) and $dx$ is replaced by $du$. Note that $g_u = |\nabla u|$ on unweighted and weighted $\mathbb{R}^n$, see Appendices A.1 and A.2 in [6].

**Proof of Theorem 1.2.** Let $0 \leq \varphi \in W^{1,p}_0(\Omega)$ be arbitrary and set

$$A = \{x \in \Omega : \varphi(x) > 0\},$$

$$A_1 = \{x \in A : u_1(x) < u_2(x)\},$$

$$A_2 = \{x \in A : u_2(x) < u_1(x)\},$$

$$A_0 = \{x \in A : v(x) > \max\{u_1(x), u_2(x)\}\},$$

where $v = u + \varphi$. Note that $A = A_1 \cup A_2 \cup A_0$, though not pairwise disjointly.
We may assume that \( \int_A |\nabla u|^p \, dx < \infty \), as otherwise (1.1) holds trivially, since the triangle inequality together with the fact that \( \varphi \in W^{1,p}(\Omega) \) implies that
\[
\left( \int_A |\nabla (u + \varphi)|^p \, dx \right)^{1/p} \geq \left( \int_A |\nabla u|^p \, dx \right)^{1/p} - \left( \int_A |\nabla \varphi|^p \, dx \right)^{1/p} = \infty.
\]

Let \( \varphi_1 = (\min\{u_2, v\} - u_1) \), and note that \( 0 \leq \varphi_1 \leq \varphi \), which implies that \( \varphi_1 \in W^{1,p}_0(\Omega) \). The \( Q_1 \)-quasisuperminimizing property of \( u_1 \) yields
\[
\int_{\varphi_1 > 0} |\nabla u_1|^p \, dx \leq Q_1 \int_{\varphi_1 > 0} |\nabla (u_1 + \varphi_1)|^p \, dx. \tag{2.1}
\]

Note that \( \varphi_1(x) > 0 \) if and only if \( u_2(x) > u_1(x) \) and \( u_1(x) + \varphi(x) = v(x) > u_1(x) \), which in turn holds exactly when \( x \in A_1 \). Moreover,
\[
u_1 + \varphi_1 = \begin{cases} u_2, & \text{in } A_1 \cap A_0, \\ v, & \text{in } A_1 \setminus A_0. \end{cases}
\]

Multiplying (2.1) by \( (Q_2 - 1) \) then gives
\[
(Q_2 - 1) \int_{A_1} |\nabla u_1|^p \, dx \leq Q_1 (Q_2 - 1) \left( \int_{A_1 \cap A_0} |\nabla u_2|^p \, dx + \int_{A_1 \setminus A_0} |\nabla v|^p \, dx \right). \tag{2.2}
\]

Similarly, using \( \varphi_2 = (\min\{u_1, v\} - u_2) \), \( u_2 \in W^{1,p}_0(\Omega) \) and the \( Q_2 \)-quasisuperminimizing property of \( u_2 \) we obtain (after multiplication with \( (Q_1 - 1) \)),
\[
(Q_1 - 1) \int_{A_2} |\nabla u_2|^p \, dx \leq Q_2 (Q_1 - 1) \left( \int_{A_2 \cap A_0} |\nabla u_1|^p \, dx + \int_{A_2 \setminus A_0} |\nabla v|^p \, dx \right). \tag{2.3}
\]

Next, let \( \tilde{\varphi}_j = (v - u_j)_+ \), \( j = 1, 2 \). Since \( 0 \leq \tilde{\varphi}_j \leq \varphi \), we have \( \tilde{\varphi}_j \in W^{1,p}_0(\Omega) \) and (1.1) with \( u_j \) and \( \tilde{\varphi}_j \) gives
\[
\int_{\tilde{\varphi}_j > 0} |\nabla u_j|^p \, dx \leq Q_j \int_{\tilde{\varphi}_j > 0} |\nabla (u_j + \tilde{\varphi}_j)|^p \, dx. \tag{2.4}
\]

Now, \( \tilde{\varphi}_1(x) > 0 \) if and only if \( \min\{u_1(x), u_2(x)\} + \varphi(x) > u_1(x) \), which is equivalent to \( x \in A \) (i.e. \( \varphi(x) > 0 \)) and \( u_2(x) + \varphi(x) > u_1(x) \). This in turn holds exactly if \( x \in A_1 \) or \( u_2(x) \leq u_1(x) < u_2(x) + \varphi(x) = v(x) \), i.e. when \( x \in A_1 \cup A_0 \). Similarly, \( \tilde{\varphi}_2(x) > 0 \) if and only if \( x \in A_2 \cup A_0 \).

Since \( u_j + \tilde{\varphi}_j = v \) whenever \( \tilde{\varphi}_j > 0 \), the inequalities in (2.4) give
\[
Q_2 (Q_1 - 1) \int_{A_1 \cup A_0} |\nabla u_1|^p \, dx \leq Q_1 Q_2 (Q_1 - 1) \int_{A_1 \cup A_0} |\nabla v|^p \, dx \tag{2.5}
\]

and
\[
Q_1 (Q_2 - 1) \int_{A_2 \cup A_0} |\nabla u_2|^p \, dx \leq Q_1 Q_2 (Q_2 - 1) \int_{A_2 \cup A_0} |\nabla v|^p \, dx, \tag{2.6}
\]

where we have also multiplied by \( Q_2 (Q_1 - 1) \) and \( Q_1 (Q_2 - 1) \), respectively.

Next, we shall sum up the inequalities (2.2), (2.3), (2.5) and (2.6) as follows. The first term in the right-hand side of (2.2) can be subtracted from the left-hand side of (2.6), leaving
\[
Q_1 (Q_2 - 1) \int_{(A_2 \cup A_0) \setminus A_1} |\nabla u_2|^p \, dx
= Q_1 (Q_2 - 1) \left( \int_{A_2} |\nabla u_2|^p \, dx + \int_{A_0 \setminus (A_1 \cup A_2)} |\nabla u_2|^p \, dx \right)
\]
therein. Since \( u = u_2 \) in \( A \setminus A_1 \supset A_2 \), adding this to the left-hand side of (2.3) results in
\[
(Q_1(Q_2 - 1) + (Q_1 - 1)) \int_{A_2} |\nabla u|^p \, dx + Q_1(Q_2 - 1) \int_{A_0 \setminus (A_1 \cup A_2)} |\nabla u|^p \, dx
\]
\[
= (Q_1Q_2 - 1) \int_{A_2} |\nabla u|^p \, dx + Q_1(Q_2 - 1) \int_{A_0 \setminus (A_1 \cup A_2)} |\nabla u|^p \, dx \quad (2.7)
\]
as \(|\nabla u_2|\)'s contribution to the left-hand side of the final sum.

Similarly, subtracting the first term in the right-hand side of (2.3) from the left-hand side of (2.5), and adding the left-hand side of (2.2) contributes with
\[
(Q_1Q_2 - 1) \int_{A_1} |\nabla u|^p \, dx + Q_2(Q_1 - 1) \int_{A_0 \setminus (A_1 \cup A_2)} |\nabla u|^p \, dx \quad (2.8)
\]
to the left-hand side of the final sum. Since \( Q_1(Q_2 - 1) + Q_2(Q_1 - 1) \geq Q_1Q_2 - 1 \), summing up (2.7) and (2.8) shows that the left-hand side in the final sum will be
\[
(Q_1Q_2 - 1) \int_{A_1 \cup A_2} |\nabla u|^p \, dx + (Q_1(Q_2 - 1) + Q_2(Q_1 - 1)) \int_{A_0 \setminus (A_1 \cup A_2)} |\nabla u|^p \, dx
\]
\[
\geq (Q_1Q_2 - 1) \int_{A_1 \cup A_2 \cup A_0} |\nabla u|^p \, dx.
\]

We now turn to the right-hand side of the sum of (2.2), (2.3), (2.5) and (2.6). The remaining term in the right-hand side of (2.2) is
\[
Q_1(Q_2 - 1) \int_{A_1 \setminus A_0} |\nabla v|^p \, dx \leq Q_1Q_2(Q_2 - 1) \int_{A_1 \setminus A_0} |\nabla v|^p \, dx,
\]
which together with the right-hand side of (2.6) contributes with
\[
Q_1Q_2(Q_2 - 1) \int_{A_1 \cup A_2 \cup A_0} |\nabla v|^p \, dx
\]
to the right-hand side of the final sum. Similarly, the remaining term in the right-hand side of (2.3) together with the right-hand side of (2.5) gives
\[
Q_1Q_2(Q_1 - 1) \int_{A_1 \cup A_2 \cup A_0} |\nabla v|^p \, dx
\]
in the right-hand side of the final sum.

As \( A_1 \cup A_2 \cup A_0 = A = \{x \in \Omega : \varphi(x) > 0\} \), we have thus obtained
\[
(Q_1Q_2 - 1) \int_{\varphi > 0} |\nabla u|^p \, dx \leq Q_1Q_2(Q_1 + Q_2 - 2) \int_{\varphi > 0} |\nabla v|^p \, dx.
\]
Division by \( Q_1Q_2 - 1 \) concludes the proof of Theorem 1.2. (If \( Q_1 = Q_2 = 1 \), the result follows from Theorem 1.1.)

\[ \square \]

3. Lower bounds for the blowup

Consider two quasisuperminimizers defined on some open set \( \Omega \). More precisely let \( u_j \) be a \( Q_j \)-quasisuperminimizer in \( \Omega \), \( j = 1, 2 \). Also let \( u = \min\{u_1, u_2\} \) and assume that \( Q_1 \leq Q_2 \). Theorem 1.1 then shows that \( u \) is also a quasisuperminimizer, and it gives an upper bound on the optimal quasisuperminimizer constant \( Q \) for \( u \) (in terms of \( Q_1 \) and \( Q_2 \) only). In Theorem 1.2 we improved upon this upper bound.
As far as we know, there have not been any examples showing that the optimal $Q$ can be greater than $Q_2$. It is obvious that one cannot do any better than $Q_2$ in general (just consider the cases when $u_1 \geq u_2$ in $\Omega$). Note also that if $Q_1 = 1$, then Theorem 1.1 shows that $u$ is a $Q_2$-quasisuperminimizer, and hence the constants in Theorems 1.1 and 1.2 are sharp in this case.

In this section we will give several examples of pairs of quasisuperminimizers such that their minimum has a blowup of the quasisuperminimizer constant, i.e. in the notation above we get $Q > Q_2$. Even though the best (largest) bounds come just from one such example we feel that it can be of interest to mention several different examples as they may add a little to the knowledge on quasisuperminimizers.

Let us already now mention that in all our examples, the functions $u_1$ and $u_2$ will not only be quasisuperminimizers, but will in fact be quasiminimizers (with the same optimal constants) as well as subminimizers (i.e. 1-quasisubminimizers).

We will also prove Theorem 1.3, i.e. that whenever $Q_1 > 1$, then there are examples showing that one can have $Q > Q_2$ and thus that $\max\{Q_1, Q_2\}$ is an upper bound only when $Q_1 = 1$.

Our examples will all be on $\mathbb{R}$. The reason for this is that this is almost the only case when one can actually calculate optimal quasiminimizers and their constants. As far as we know, the only higher-dimensional quasi(super)minimizers for which their optimal quasi(super)minimizer constant has been determined, and is strictly larger than 1, are the power-type quasi(super)minimizers studied in Björn–Björn [5].

The easiest example of a blowup in the quasisuperminimizing constant is perhaps the following. (It was incidentally also the first example we discovered.)

**Example 3.1.** Let $p = 2$,

$$u_1(x) = \begin{cases} \frac{2}{3}x, & 0 \leq x \leq \frac{1}{5}, \\ \frac{4}{5}x - \frac{1}{3}, & \frac{1}{2} \leq x \leq 1, \end{cases} \quad \text{and} \quad u_2(x) = \begin{cases} \frac{2}{3}x, & 0 \leq x \leq \frac{1}{5}, \\ \frac{4}{5}x - \frac{2}{3}, & \frac{4}{5} \leq x \leq 1. \end{cases}$$

Then $u_1$ and $u_2$ are $\frac{2}{3}$-quasisuperminimizers (with $\frac{2}{3} = 1.125$ being the optimal constant), by Theorem 5.4 below. We will call functions such as $u_1$ and $u_2$ one-corner functions.

Let $u = \min\{u_1, u_2\}$. Note that $u_1(x) = u_2(x)$ for $x = 0, \frac{2}{5}, 1$. Then

$$\int_0^1 (u')^2 \, dx = \frac{1}{4} \left( \frac{2}{5} \right)^2 + \left( \frac{2}{5} - \frac{1}{3} \right) \left( \frac{2}{5} \right)^2 + \left( \frac{4}{5} - \frac{2}{3} \right) \left( \frac{2}{5} \right)^2 + \frac{4}{5} \left( \frac{2}{5} \right)^2 = \frac{1}{5}.$$

Comparison with $v(x) = x$ shows that $u$ is not a $Q$-quasisuperminimizer for any $Q < \frac{2}{5} = 1.1666 \ldots$. The upper bounds given by Theorems 1.1 and 1.2 are

$$\frac{81}{64} = 1.265625 \quad \text{and} \quad \frac{61}{68} = 1.191176 \ldots .$$

With $Q_1 = Q_2 = \frac{9}{5}$ and $p = 2$ this example has been optimized, i.e. $u_1$ and $u_2$ are one-corner functions with quotient between the slopes $\gamma = 2$ and the choices of their corner points have been optimized to get as large blowup as possible. For $p = 2$ it is a rather straightforward (although a bit lengthy) calculation to do this optimization by hand even for a general $Q = Q_1 = Q_2$, and it leads to the lower bound $\frac{Q}{Q - 1}$. We omit the details as we find better lower bounds below.

For other values of $p$ such optimization becomes more laborious, and we decided to do some such calculations using Maple 16. Some obtained values, correctly rounded to the nearest digit, are shown in Table 1. These calculations suggest that for a given $Q = Q_1 = Q_2$ the lower bounds increase with $p$, but the dependence on $p$ is very small (much smaller than we had expected).
Remark 3.2. Even though Example 3.1 has been optimized it should be possible, by considering more general piecewise linear functions and optimizing their parameters, to obtain better results, possibly even reaching the optimal constant for fixed $p$, $Q_1$ and $Q_2$, at least when the result is specialized to $\mathbb{R}$. Here, Theorem 4.1 in Martio [27] (which can also be found as Theorem C.2 in [6]) might be of help since it makes it possible to approximate quasiminimizers by other (e.g. piecewise linear) functions with almost the same quasiminimizing constant.

Another necessary ingredient would be a good control of the best quasiminimizing constant of such functions. Lemmas 2 and 8 in Martio–Sbordone [31] show that the quasiminimizing constant is at most $(\sup |u'| / \inf |u'|)^{p-1}$. In particular, all strictly increasing continuous piecewise linear functions (with finitely many corners) are quasiminimizers, but the best constant is not easy to obtain. Our Proposition 5.14 below is a partial step in that direction.

The above considerations open up for further numerical investigations of the blow up. We will not pursue this route as the following approach gives good lower bounds.

**Definition 3.3.** If $u$ is a $Q$-quasiminimizer in $\Omega \subset \mathbb{R}$ we say that $u$ has the maximal $p$-energy allowed by $Q$ on an interval $I \subset \Omega$ if

$$\int_I |u'|^p \, dx = Q \int_I |v'|^p \, dx = Q \frac{(u(b) - u(a))^p}{(b-a)^{p-1}},$$

where $v$ is the minimizer in $I$ with boundary values $v = u$ on $\partial I$, i.e.

$$v(x) = u(a) + \frac{u(b) - u(a)}{b-a}(x-a),$$

where $a < b$ are the end points of $I$.

**Example 3.4.** For $\alpha > 1 - 1/p$ and $x \in [0, 1]$ let $v_\alpha(x) = x^\alpha$. Theorem 6.2 in Björn–Björn [5] with $n = 1$ and $p > 1$ implies that $v_\alpha$ is a $Q_\alpha$-quasiminimizer in $(0, 1)$, where

$$Q_\alpha = \frac{\alpha^p}{p(\alpha - 1) + 1}$$

is optimal. In fact, if $1 - 1/p < \alpha \leq 1$, then $v_\alpha$ is a superminimizer and a $Q_\alpha$-quasisubminimizer, while for $\alpha \geq 1$, $v_\alpha$ is a subminimizer and a $Q_\alpha$-quasisuperminimizer in $(0, 1)$.

A simple calculation also shows that for every $x_0 \in (0, 1)$,

$$\int_0^{x_0} (v_\alpha')^p \, dx = \int_0^{x_0} \alpha^px^p(\alpha-1) \, dx = \frac{\alpha^px_0^{p(\alpha-1)+1}}{p(\alpha - 1) + 1} = Q_\alpha \int_0^{x_0} (x_0^{\alpha-1})^p \, dx,$$
where the latter integral is the $p$-energy of the linear segment from the origin to
the point $(x_0, v_\alpha(x_0))$. Thus, for every $x_0 \in (0, 1]$, $v_\alpha$ has the maximal $p$-energy in
$(0, x_0)$ allowed by $Q_\alpha$.

Note that, given $Q > 1$, there are exactly two exponents $1 - 1/p < \alpha < 1$ such that $Q = Q_\alpha = Q_{\alpha'}$. This is easily shown by differentiating (3.1) and noting
that the derivative is negative for $\alpha < 1$ and positive for $\alpha > 1$, and that $Q_\alpha \to \infty$ as \(\alpha \to 1 - 1/p\) and as \(\alpha \to \infty\). We let
\[
    u_Q(x) = x^\alpha \quad \text{and} \quad \bar{u}_Q(x) = 1 - (1-x)^{\alpha'}.
\] (3.2)

Then $u_Q(0) = \bar{u}_Q(0) = 0$ and $u_Q(1) = \bar{u}_Q(1) = 1$. Note that both $u_Q$ and $\bar{u}_Q$
are subminimizers and $Q$-quasisuperminimizers in $(0, 1)$. Moreover, $u_Q$ has the
maximal $p$-energy allowed by $Q$ on each interval $(0, x_0)$, while $\bar{u}_Q$ has the maximal
$p$-energy allowed by $Q$ on each interval $(x_0, 1)$.

We can now use the functions $u_Q$ and $\bar{u}_Q$ above to prove Theorem 1.3.

**Proof of Theorem 1.3.** By Theorem 1.2, the function $u := \min\{u_Q_1, \bar{u}_Q_2\}$ is a qua-
sisuperminimizer in $(0, 1)$ with a quasisuperminimizing constant given by (1.2). We
shall show that $u$ is not a $Q_2$-quasisuperminimizer. To do this, it suffices to show
that the $p$-energy
\[
    \int_0^1 (u')^p \, dx > Q_2.
\]

Since $\bar{u}_{Q_2}$ is a subminimizer in $(0, 1)$ (by Theorem 6.2 in [5]), we have that
\[
    \int_0^{x_0} (u_{Q_1}')^p \, dx > \int_0^{x_0} (\bar{u}_{Q_2}')^p \, dx,
\] where the strict inequality follows from the uniqueness of solutions to obstacle
problems (see e.g. Theorem 7.2 in [6]) and from the fact that $u_{Q_1} < \bar{u}_{Q_2}$ in a set of
positive measure. Hence
\[
    \int_0^1 (u')^p \, dx > \int_0^1 (\bar{u}_{Q_2}')^p \, dx = Q_2,
\] (3.3)
which finishes the proof. \(\square\)

Theorem 1.3 shows that in general there is a blow up in the quasisuperminimizing
constant when taking minimum of two quasisuperminimizers, but it does not give
any quantitative estimate of the blow up. Next, we shall give some lower bounds
for the blow up.

Given $Q_1, Q_2 > 1$, let $1 - 1/p < \alpha_2 < 1 < \alpha_1$ be such that $Q_1 = Q_{\alpha_1}$, $Q_2 = Q_{\alpha_2}$
and $u_{Q_1}$ and $\bar{u}_{Q_2}$ are the corresponding quasisuperminimizers. Let $x_0$ be the unique
number in $(0, 1)$ such that $u_{Q_1}(x_0) = \bar{u}_{Q_2}(x_0)$, i.e. the unique solution of the equation
\[
    x_0^{\alpha_1} + (1-x_0)^{\alpha_2} = 1.
\] (3.4)

(To see that there is a unique solution, consider $w = \bar{u}_{Q_2} - u_{Q_1}$, and note that
$w(0) = w(1) = 0$. Since $w'(0) > 0$ and $w'(1) = \infty$, there is at least one $x \in (0, 1)$
such that $w(x) = 0$. Next, a simple calculation shows that $w'(x) = 0$ if and only if
\[
    v(x) := x^\beta(1-x) = \left(\frac{\alpha_1}{\alpha_2}\right)^{1/(\alpha_2-1)}, \quad \text{where} \quad \beta = \frac{\alpha_1 - 1}{1 - \alpha_2} > 0.
\]
As $v(0) = v(1) = 0$ and $v(x)$ attains its maximum at (and only at) $x = \beta/(\beta + 1)$
we see that there are at most two solutions to $w'(x) = 0$, and thus there can be at
most one solution to (3.4), which must lie in between those two local extrema of $w$. 

\[

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Upper bound for $1.014821935$ have been obtained by letting Maple 16 evaluate $1.015024876$. Here we have used that both $u_{Q_1}$ and $u_{Q_2}$ have the maximal energies allowed by $Q_1$ and $Q_2$ in the respective intervals.

Note that $x_0$ is uniquely determined by $Q_1$ and $Q_2$ (through $\alpha_1$ and $\alpha_2$) and thus $\bar{Q}$ depends only on $Q_1$ and $Q_2$ (and on $p$). Comparing this $p$-energy with the $p$-energy of the linear function $u_l(x) = x$ shows that $\bar{Q}$ is a lower bound for the quasisuperminimizing constant of $u$. We would therefore like to estimate $\bar{Q}$.

The lower bounds in Table 2 have been obtained by letting Maple 16 evaluate $\bar{Q}$ for some values of $Q := Q_1 = Q_2$ and are compared with the upper bound obtained in Theorem 1.2. Note that these lower bounds are considerably larger, and much closer to the upper bounds, than those in Table 1.

Our next aim is to obtain more explicit estimates of $\bar{Q}$. Calculating $\bar{Q}$ in (3.5) involves first solving the equation (3.1) twice for $\alpha$, so that $Q_1 = Q_{\alpha_1}$ and $Q_2 = Q_{\alpha_2}$ as above, then finding $0 < x_0 < 1$ such that $x_0^{\alpha_1} + (1 - x_0)^{\alpha_2} = 1$, and finally evaluating $\bar{Q}$ for the obtained values of $\alpha_1$, $\alpha_2$ and $x_0$. This can be done numerically but not analytically (not even for $p = 2$).

A somewhat weaker, but more explicit, estimate for $\bar{Q}$ can be obtained in the following way. Let $x_1 \in (0, 1)$ be such that $u_{Q_1}(x_1) = \alpha_2 x_1$, i.e. $x_1 = \alpha_2^{1/(\alpha_1 - 1)}$. Since $u_{Q_1}(0) = 0$, $u_{Q_2}'(0) = \alpha_2$ and both $u_{Q_1}$ and $u_{Q_2}$ are convex, we have that

$$\bar{u}_{Q_2}(x) > \alpha_2 x > u_{Q_1}(x) \quad \text{for all } x \in (0, x_1).$$

In particular, $x_1 < x_0$.

As $\bar{u}_{Q_2}$ is a subminimizer in $(0, 1)$ and $\bar{u}_{Q_2} > \max\{u_{Q_1}, \alpha_2 x\}$ in $(0, x_0)$, we then obtain (using also that $u_{Q_1}(x_0) = u_{Q_2}(x_0)$)

$$\int_0^{x_0} (\bar{u}_{Q_2})^p \, dx < \int_0^{x_0} \alpha_2^p \, dx + \int_{x_1}^{x_0} (\bar{u}_{Q_1}')^p \, dx,$$

where the strict inequality follows as in (3.3) from the uniqueness of solutions to obstacle problems. From the fact that $u_{Q_1}$ has the maximal $p$-energy allowed by $Q_1$ on the interval $(0, x_1)$ we can conclude that

$$\int_0^{x_0} (\bar{u}_{Q_1})^p \, dx = \int_0^{x_1} (u_{Q_1}')^p \, dx + \int_{x_1}^{x_0} (u_{Q_1}')^p \, dx = Q_1 \int_0^{x_1} \alpha_2^p \, dx + \int_{x_1}^{x_0} (u_{Q_1}')^p \, dx.$$
Together with (3.6) this yields
\[ \tilde{Q} - Q_2 = \int_0^1 ((u')^p - (\tilde{u}_Q)^p) \, dx = \int_0^{x_0} (u_Q^p) \, dx - \int_0^{x_0} (\tilde{u}_Q)^p \, dx \]
\[ > (Q_1 - 1) \int_0^{x_1} \alpha_2^p \, dx = (Q_1 - 1)\alpha_2^p x_1 = (Q_1 - 1)\alpha_2^{p+1/\alpha_2-1}. \]
(3.7)

(This gives another proof of Theorem 1.3.) A similar argument shows that
\[ \tilde{Q} - Q_1 > (Q_2 - 1)\alpha_1^p (1 - x_2) = (Q_2 - 1)\alpha_1^{p+1/\alpha_1-1}, \]
(3.8)
where \(x_2 \in (0, 1)\) is the solution of \(\tilde{u}_Q(x_2) = 1 - \alpha_1 (1 - x_2)\), i.e. \(1 - x_2 = \alpha_1^{1/(\alpha_2-1)}\).
Note that \(x_2 > x_0\). Depending on the particular values of \(p, Q_1\) and \(Q_2\), one of (3.7) and (3.8) may be better than the other.

For \(p = 2\), when \(\alpha_1\) and \(\alpha_2\) can be explicitly calculated in terms of \(Q_1\) and \(Q_2\), we get after simplification (and for \(Q_1 \leq Q_2\)) that the blow up is at least the maximum of
\[ \tilde{Q} - Q_2 > (Q_2 - 1)(Q_2 + \sqrt{Q_2^2 - Q_2})^{1 - \sqrt{Q_2/(Q_1 - 1)}}, \]
\[ \tilde{Q} - Q_2 > (Q_2 - 1)(Q_2 - \sqrt{Q_2^2 - Q_2})^{1 + \sqrt{Q_2/(Q_1 - 1)}}. \]
(3.9)

For the values considered in Tables 1 and 2, the first estimate above is quite close to those in Table 2 and better than those in Table 1.

For \(p \neq 2\), we cannot obtain such explicit expressions. However, using Remark 5.10 and (5.6) below we can write
\[ \alpha_1 = \frac{p - 1}{p} \frac{\gamma_1^p - 1}{\gamma_1^{p-1} - 1} \quad \text{and} \quad \alpha_2 = \frac{p - 1}{p} \frac{\gamma_2^p - 1}{\gamma_2^{p-1} - 1}, \]
in terms of the quotients \(\gamma_1\) and \(\gamma_2\) associated with \(Q_1\) and \(Q_2\) as in (5.4) by means of Proposition 5.5 below. A direct calculation then gives
\[ \alpha_1^{p+1/(\alpha_2-1)} = \left( \frac{p}{p-1} \frac{\gamma_1^{p-1} - 1}{\gamma_1^{p-2} - 1} \right)^{\frac{p(p-1)(\gamma_1 - 1)}{p-2}} \]
\[ \alpha_2^{p+1/(\alpha_1-1)} = \left( \frac{p}{p-1} \frac{\gamma_2^{p-1} - 1}{\gamma_2^{p-2} - 1} \right)^{\frac{p(p-1)(\gamma_2 - 1)}{p-2}}. \]
In particular, for \(p = 2\) and \(Q_1 = Q_2 = Q\) (and thus \(\gamma_1 = \gamma_2 = \gamma\)), these formulas simplify to
\[ \alpha_1^{p+1/(\alpha_2-1)} = \left( \frac{2}{\gamma + 1} \right)^{2/(\gamma - 1)}, \]
which is increasing with respect to \(\gamma\) and has limit \(1/e\) as \(\gamma \to 1^+\), while
\[ \alpha_2^{p+1/(\alpha_1-1)} = \left( \frac{\gamma + 1}{2\gamma} \right)^{2\gamma/(\gamma - 1)} = \left( 1 - \frac{\gamma - 1}{2\gamma} \right)^{2\gamma/(\gamma - 1)} < \frac{1}{e} \]
for all \(\gamma > 1\). Thus \(\alpha_2^{p+1/(\alpha_1-1)} < 1/e < \alpha_1^{p+1/(\alpha_2-1)} < 1\) for all \(\gamma > 1\), and hence
\[ \tilde{Q} > Q + (Q - 1)/e \]
in this case, which is better than the estimate \(\frac{4}{3}Q - \frac{1}{3} = Q + \frac{1}{3}(Q - 1)\) in Example 3.1, but worse than (3.9).
4. An upper bound for three (or more) functions

It is possible to get estimates for the quasisuperminimizing constant for the minimum of several quasisuperminimizers by iteratively using the estimate for the minimum of two functions. The obtained estimate often depends on the order in which the minima are taken. This suggests that better estimates could be obtained, if we directly consider the minimum of all of the involved functions and as in the proof of Theorem 1.2 use all the information that is available from the fact that all the functions are quasisuperminimizers with the original constants.

To estimate the quasisuperminimizer constant for the minimum $u$ of $N$ quasisuperminimizers $u_i$, let $0 \leq \varphi \in W_0^{1,p}(\Omega)$ be arbitrary and set $v = u + \varphi$. For each $i = 1, \ldots, N$ and $S \subset \{1, \ldots, N\}$ with $i \notin S$ let

$$u_S = \min_{s \in S} \{u_s, v\} \quad \text{and} \quad \varphi_{i,S} = (u_S - u_i)_+.$$ 

Then $0 \leq \varphi_{i,S} \leq \varphi$ and hence $\varphi_{i,S} \in W_0^{1,p}(\Omega)$. (Note that $\varphi_{i,S} = 0$ if $i \in S$.)

Testing (1.1) for each $u_i$ with $\varphi_{i,S}$ provides us with $N^2 - 1$ inequalities of the form

$$\int_{u_i < u_S} |\nabla u_i|^p \, dx \leq Q_i \int_{u_i < u_S} |\nabla u_S|^p \, dx. \quad (4.1)$$

This leads to a linear programming problem, which is solvable in polynomial time with respect to the number of the conditions.

**Remark 4.1.** When formulating the linear programming problem one can without loss of generality assume that the sets \( \{x \in \Omega : u_i(x) = u_j(x)\} \), $1 \leq i < j \leq N$, all have measure zero; this follows from the fact that we can approximate each $u_i$ from below using $u_i - q_i$, with rational $q_i \geq 0$, and the corresponding minima increase to $u$, while preserving the quasisuperminimizing constant, by Theorem 6.1 in Kinnunen–Martio [21].

For example, when $N = 3$, we obtain 12 conditions. We used Mathematica to solve this linear programming problem and obtained the following result. Below we provide a direct proof without relying on Mathematica. However, the Mathematica calculation shows that the constant obtained here is the best possible using only the information above.

**Theorem 4.2.** Let $u_i$ be a $Q_i$-quasisuperminimizer for $i = 1, 2, 3$. Let

$$P = 2Q_1Q_2Q_3 - Q_1Q_2 - Q_2Q_3 - Q_3Q_1 + 1$$

and, with $\{j, k\} = \{1, 2, 3\} \setminus \{i\}$,

$$R_i = \begin{cases} 0, & \text{if } Q_j = Q_k = 1, \\ \frac{(Q_j - 1)(Q_k - 1)(Q_j - 1 + Q_k - 1)}{Q_jQ_k - 1}, & \text{otherwise}. \end{cases}$$

Then $\min\{u_1, u_2, u_3\}$ is a $\overline{Q}$-quasisuperminimizer with

$$\overline{Q} = \frac{Q_1Q_2Q_3}{P}(R_1 + R_2 + R_3)$$

unless at least two of the $Q_i$ equal 1, say $Q_2 = Q_3 = 1$, in which case $\overline{Q} = Q_1$.

It is easily verified that the choice $Q_3 = 1$ gives the expression in Theorem 1.2. When $Q_1 = Q_2 = Q_3$, it is also easy to verify that the constant gets the following simpler form.
Corollary 4.3. Let $u_1, u_2, u_3$ be $Q$-quasisuperminimizers. Then $\min\{u_1, u_2, u_3\}$ is a $6Q^3/(Q+1)(2Q+1)$-quasisuperminimizer.

This estimate is slightly better than what we would have obtained by iterating Theorem 1.2: First, the minimum of $u_1$ and $u_2$ is a $2Q^2/(Q+1)$-quasisuperminimizer, and then the minimum of $\min\{u_1, u_2\}$ and $u_3$, i.e. the minimum of a $2Q^2/(Q+1)$- and a $Q$-quasisuperminimizer, is a $2Q^3(3Q+2)/(Q+1)(2Q^2+2Q+1)$-quasisuperminimizer, by Theorem 1.2. However both of these estimates give values close to $3Q$ for large values of $Q$.

We now explain how Theorem 4.2 can be proved without the use of Mathematica.

Proof of Theorem 4.2. If $\min\{Q_1, Q_2, Q_3\} = 1$, we have already noticed that the result follows from Theorem 1.2, so we assume that $\min\{Q_1, Q_2, Q_3\} > 1$.

The proof is similar to the proof of Theorem 1.2, it just requires more book keeping. There are 12 inequalities of the form (4.1) at our disposal. More precisely, for $S = \emptyset$, there are three inequalities

\begin{equation}
\int_{u_i<0} |\nabla u_i|^p \, dx \leq Q_1 \int_{u_i<0} |\nabla v|^p \, dx,
\end{equation}

$E_i$

$i = 1, 2, 3$. For singleton $S = \{j\}$, $j \neq i$, we obtain six possible inequalities, namely

\begin{equation}
\int_{u_i<\min\{u_j, v\}} |\nabla u_i|^p \, dx \leq Q_1 \int_{u_i<\min\{u_j, v\}} |\nabla \min\{u_j, v\}|^p \, dx,
\end{equation}

$E_{ij}$

$i, j = 1, 2, 3$, $i \neq j$. Finally, for $S = \{j, k\}$, $i \notin S$, we have three inequalities

\begin{equation}
\int_{u_i<\min\{u_{j, k}, v\}} |\nabla u_i|^p \, dx \leq Q_1 \int_{u_i<\min\{u_{j, k}, v\}} |\nabla \min\{u_{j, k}, v\}|^p \, dx,
\end{equation}

$\hat{E}_i$

$i = 1, 2, 3$.

Depending on the choice of the set $S$ and on the sizes of the functions $u_1$, $u_2$, $u_3$ and $v$, the sets of integration in these equations split into three different sets, where also $u_{jk} = \min_{s \in S}\{u_s, v\}$ equals different $u_i$ or $v$.

Let $\pi = (ijk)$ be a fixed but arbitrary permutation of the set $\{1, 2, 3\}$. Then the following subsets of the set $A = \{x \in \Omega : \varphi(x) > 0 \text{ and } u_i(x) < u_j(x) < u_k(x)\}$ are of interest:

\begin{align*}
A_0 &= \{x \in A : u_i(x) < u_j(x) < u_k(x) < v(x)\}, \\
A_1 &= \{x \in A : u_i(x) < u_j(x) < v(x) < u_k(x)\}, \\
A_2 &= \{x \in A : u_i(x) < v(x) < u_j(x) < u_k(x)\}.
\end{align*}

(Note that by Remark 4.1, we can assume that all the sets $\{x \in \Omega : u_i(x) = u_j(x)\}$ have measure zero.) We shall now check in which of the above inequalities these sets appear as parts of the sets of integration. We shall also keep track of which function then appears in the left-hand side (LHS) and in the right-hand side (RHS).

It is immediate that none of $A_0$, $A_1$ and $A_2$ is present in the equations $(E_{ji})$, $(E_{ki})$, $(\hat{E}_j)$ or $(\hat{E}_k)$. The set $A_2$ appears only in $(E_i)$, $(E_{ij})$, $(E_{ik})$ and $(\hat{E}_i)$, and the function in the LHS is then always $u_i$, while the one in the RHS is always $v$. For the sets $A_0$ and $A_1$, the choices of funtions are more complicated and are summarized in Table 3.

We multiply the inequalities $(E_i)$, $(E_{ij})$ and $(\hat{E}_i)$ by $x_i$, $x_{ij}$ and $\hat{x}_i$, respectively, and sum up. We have $u = u_i$ everywhere in the set $A = A_0 \cup A_1 \cup A_2$, and hence to show that $u$ is a quasisuperminimizer, we need to keep track of $\int_A |\nabla u_i|^p \, dx$ in the LHSs and of $\int_A |\nabla v|^p \, dx$ in the RHSs. We also want to choose $x_i$, $x_{ij}$ and $\hat{x}_i$ so
that the integrals of $|\nabla u_j|^p$ and $|\nabla u_k|^p$ in the RHSs are compensated by the same integrals in the LHSs.

From Table 3, we see that $\nabla u_j$ cancels out in $A_0$ and $A_1$ if we have

$$x_j + x_{jk} - Q_i x_{ij} - Q_i \hat{x}_i = 0,$$

(4.2)

and that $\nabla u_k$ cancels out in $A_0$, if

$$x_k - Q_i x_{ik} - Q_j x_{jk} = 0.
\tag{4.3}$$

In addition, we want the coefficients in front of the terms containing $\nabla u_i = \nabla u$ in each of the sets $A_0$, $A_1$ and $A_2$ to sum up to 1, i.e.

$$x_i + x_{ij} + x_{ik} + \hat{x}_i = 1.
\tag{4.4}$$

Considering all permutations of $\{1, 2, 3\}$ we obtain a linear system of 12 equations with 12 unknowns. However, the system can be simplified, which we do now. From (4.4) we obtain $\hat{x}_i = 1 - (x_i + x_{ij} + x_{ik})$ and inserting this into (4.2) gives

$$x_j + x_{jk} + Q_i x_i + Q_i x_{ik} = Q_i.
\tag{4.5}$$

From (4.3) we have $Q_i x_{ik} = x_k - Q_j x_{jk}$, which together with (4.5) leads to

$$Q_i x_i + x_j + x_k + (1 - Q_j)x_{jk} = Q_i.$$
Now, note that this equation is for fixed $i$ symmetric in $j$ and $k$, except for the last term in the left-hand side, which thus must be symmetric in $j$ and $k$ as well. Hence, we see that

$$(1 - Q_j)x_{jk} = (1 - Q_k)x_{kj} =: y_i.$$  

Thus the above system transforms into the six equations

$$x_k + S_i y_j + S_j y_i = 0,$$

$$Q_i x_i + x_j + x_k + y_i = Q_i,$$

where $S_i = Q_i/(Q_i - 1)$. It can be written as

$$\begin{cases}
  x + Sy = 0, \\
  Rx + y = c,
\end{cases}$$

with

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix},$$

where

$$S = \begin{pmatrix} 0 & S_3 & S_2 \\ S_3 & 0 & S_1 \\ S_2 & S_1 & 0 \end{pmatrix}, \quad R = \begin{pmatrix} Q_1 & 1 & 1 \\ 1 & Q_2 & 1 \\ 1 & 1 & Q_3 \end{pmatrix} \quad \text{and} \quad c = \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix}.$$

From the second equation we have $y = c - Rx$, which transforms the first equation into $(SR - I)x = Sc$, whose solution is

$$x = (SR - I)^{-1}Sc,$$

where $I$ stands for the identity matrix.

Now, as we have chosen $x$ (and thus $y$), so that all extra terms in the equations $(E_i), (E_j), (E_k), i, j = 1, 2, 3$, cancel out and the remaining ones with $\nabla u$ always appear with coefficient 1 in the LHS, we need to check how large constants appear with $|\nabla v|^p$ in the RHS to determine $Q$. From Table 3, we see that $\int_{A_i} |\nabla v|^p \, dx$ appears in the right-hand side with a factor

$$Q_{A_0} = Q_i x_i + Q_j x_j + Q_k x_k.$$  

Similarly, the factors are

$$Q_{A_1} = Q_i x_i + Q_j x_j + Q_k x_k + Q_j x_{jk} = Q_i x_i + Q_j x_j + x_k \leq Q_{A_0} \quad \text{(by (4.3))},$$

$$Q_{A_2} = Q_i (x_i + x_j + x_k + \hat{x}_i) = Q_i \quad \text{(by (4.4))},$$

for $\int_{A_i} |\nabla v|^p \, dx$ and $\int_{A_2} |\nabla v|^p \, dx$, respectively. Since the quasiminimizing constant $Q$ of $u$ must be at least max \{$Q_1, Q_2, Q_3$\}, we conclude that $Q_{A_0}$ is the largest of the three and

$$Q = Q_{A_0} = c^T x = c^T (SR - I)^{-1} Sc = ((SR - I)^T)^{-1} c^T Sc,$$

where $^T$ denotes the matrix transpose. Observe that the value of $Q_{A_0}$ is symmetric in $i, j$ and $k$. An elementary calculation shows that

$$(SR - I) = \begin{pmatrix} L_1 & Q_2 L_1 \\ Q_1 L_2 & L_2 \\ Q_1 L_3 & Q_2 L_3 \end{pmatrix} \begin{pmatrix} Q_3 L_1 \\ Q_3 L_2 \\ Q_3 L_3 \end{pmatrix},$$

where $L_i = S_j + S_k - 1 = \frac{Q_j Q_k - 1}{(Q_j - 1)(Q_k - 1)}$, for $i \neq j \neq k \neq i$. Thus, $z := ((SR - I)^T)^{-1} c$ is the unique solution of the system $(SR - I)^T z = c$, which can be equivalently written as

$$\begin{pmatrix} L_1 \\ L_2 \end{pmatrix} \begin{pmatrix} L_1 & L_2 \\ L_1 & L_2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$
Denoting the matrix in the left-hand side by $L$, Cramer’s rule gives
\[ z_i = \frac{L_j L_k}{Q_j Q_k} \det L (Q_j - 1)(Q_k - 1), \quad i \neq j \neq k \neq i, \]
where
\[ \det L = \frac{L_1 L_2 L_3}{Q_1 Q_2 Q_3} (2Q_1 Q_2 Q_3 - Q_1 Q_2 - Q_2 Q_3 - Q_3 Q_1 + 1) =: \frac{L_1 L_2 L_3}{Q_1 Q_2 Q_3} P. \]
It follows that \( z_i = Q_j (Q_j - 1)(Q_k - 1)/L_i P \). We also have \( Sc = w \), where
\[ w_i = Q_j S_k + Q_k S_j = Q_j Q_k \left( \frac{1}{Q_j - 1} + \frac{1}{Q_k - 1} \right), \]
and hence,
\[ w_i z_i = Q_j Q_k \left( \frac{1}{Q_j - 1} + \frac{1}{Q_k - 1} \right) \frac{Q_1}{L_1 P} (Q_j - 1)(Q_j - 1) \]
\[ = \frac{Q_1 Q_2 Q_3 (Q_j - 1)(Q_k - 1)(Q_j - 1 + Q_k - 1)}{Q_j Q_k - 1} =: \frac{Q_1 Q_2 Q_3}{P} R_i. \]
Consequently, going back to (4.6) we obtain
\[ \tilde{Q} = Q_{A_0} = \sum_{i=1}^{3} w_i z_i = \frac{Q_1 Q_2 Q_3}{P} (R_1 + R_2 + R_3). \]

5. Blowup in pasting lemmas

In this section we shall show that the quasisuperminimizing constant \( Q_1 Q_2 \) in the pasting Theorem 1.4 is optimal. More precisely, we prove the following result.

**Theorem 5.1.** Let \( p, Q_1 \) and \( Q_2 \) be given. Then there are \( u_1, u_2 \) and open sets \( \Omega_1 \subset \Omega_2 \), such that \( u_j \) is a \( Q_j \)-quasiminimizer in \( \Omega_j \), \( j = 1, 2 \), and
\[ u = \begin{cases} u_2, & \text{in } \Omega_2 \setminus \Omega_1, \\ \min\{u_1, u_2\}, & \text{in } \Omega_1, \end{cases} \]
is a quasisuperminimizer in \( \Omega_2 \) with the optimal quasisuperminimizer constant \( Q_1 Q_2 \).

This is in sharp contrast to Theorem 1.2, where \( \min\{u_1, u_2\} \) is guaranteed to have a quasisuperminimizing constant \( \tilde{Q} < Q_1 Q_2 \), and moreover,
\[ Q_1 Q_2 - \tilde{Q} = Q_1 Q_2 \frac{(Q_1 - 1)(Q_2 - 1)}{Q_1 Q_2 - 1} > 0 \]
everywhere \( Q_1, Q_2 > 1 \).

A drawback of our proof of Theorem 5.1 is that \( \Omega_1 \) is not connected. However even when \( \Omega_1 \) is required to be connected we can show, by varying \( p \), the optimality of the blowup constant in Theorem 1.4 using the following result.

**Theorem 5.2.** Let \( Q_1, Q_2 \) and \( \varepsilon > 0 \) be given. Then there are \( p, u_1, u_2 \) and an interval \( I = (x_0, 1), 0 \leq x_0 < 1 \), such that \( u_1 \) is a \( Q_1 \)-quasiminimizer in \( I \), \( u_2 \) is a \( Q_2 \)-quasiminimizer in \( \Omega = (0, 1) \), and
\[ u = \begin{cases} u_2, & \text{in } \Omega \setminus I, \\ \min\{u_1, u_2\}, & \text{in } I, \end{cases} \quad (5.1) \]
is a \( Q \)-quasisuperminimizer in \( \Omega \) with optimal quasisuperminimizer constant
\[ Q \geq Q_1 Q_2 - \varepsilon. \quad (5.2) \]
Remark 5.3. The functions \( u_1, u_2 \) and \( u \) in the proofs below of Theorems 5.1 and 5.2 are continuous, and hence this also demonstrates the sharpness of the blowup in the pasting lemma for quasisuperharmonic functions (Theorem 5.1 in A. Björn–Martio [9]).

To prove these theorems we need to use some results on one-corner functions. In particular, we will use the following result which was obtained by Uppman [34, Section 2.2.3]. For \( p = 2 \) it is due to Judin [18, Example 4.0.25].

**Theorem 5.4.** Let \( 0 < \alpha < \beta < \infty \) and \( \gamma = \beta/\alpha \). The optimal quasiminimizer constant for

\[
u(x) = \begin{cases} 
\alpha x, & x \leq 0, \\
\beta x, & x \geq 0,
\end{cases}
\]

is

\[
Q = \frac{(\gamma^p + k)(1 + k)^{p-1}}{(\gamma + k)^p},
\]

where

\[
k = \frac{p\gamma^p(\gamma - 1) - \gamma(\gamma^p - 1)}{\gamma^p - 1 - p(\gamma - 1)}.
\]

Moreover \( u \) has the maximal \( p \)-energy allowed by \( Q \) on an interval of the form \([-a, b], a, b > 0\), if and only if \( a/b = k \).

The last part is a consequence of the proof by Uppman (or Judin in the case when \( p = 2 \)). Recall from Definition 3.3 that a quasiminimizer is said to have the maximal \( p \)-energy allowed by \( Q \) on an interval \( I \) if its \( p \)-energy therein is \( Q \)-times the \( p \)-energy of the linear function with the same boundary values on \( \partial I \). Note also that \( k = \gamma \) if \( p = 2 \).

We will say that \( u \) as in (5.3) is a one-corner function with corner 0 and quotient \( \gamma \). We will mainly be interested in convex one-corner functions as these are subminimizers and thus \( Q \) above is also the optimal quasisuperminimizer constant.

**Proposition 5.5.** The function \( Q(\gamma, p) \) is continuous, and moreover it is strictly increasing with respect to \( \gamma \).

**Proof.** The continuity follows directly from the expressions in Theorem 5.4.

Let \( \gamma' > \gamma \) and let \( I = [-a, 1] \) be an interval such that \( u \) has the maximal \( p \)-energy allowed by \( Q \) on \( I \), where \( u \) and \( Q \) are given by Theorem 5.4 with \( \gamma = \beta/\alpha \). Let \( \beta' = \gamma' \alpha > \beta \). Choose \( 0 < x_0 < 1 \) so that \( \alpha x_0 + \beta'(1 - x_0) = \beta \) and let

\[
w(x) = \begin{cases} 
\alpha x, & x \leq x_0, \\
\beta'(1 - x_0) + \beta, & x \geq x_0.
\end{cases}
\]

Then \( w \) is a \( Q' = Q(\gamma', p) \)-quasiminimizer in \( I \), \( w = u \) on \([-a, 0] \cup \{1\}\) and \( w < u \) in \((0, 1)\). Hence, if \( v \) is the linear function in \( I \) with boundary values \( v = w \) on \( \partial I \), then

\[
\int_I |w'|^p \, dx > \int_I |u'|^p \, dx = Q \int_I |v'|^p \, dx.
\]

This shows that \( Q' > Q \).

A direct consequence of Proposition 5.5 is that we can view \( \gamma \) as a function of \( Q \) and \( p \), and this function is strictly increasing with respect to \( Q \). We will also need the following estimate.

**Proposition 5.6.** It is always true that \( Q \leq \gamma^{p-1} \).
Proof. Let $c = k/\gamma$. Then

$$Q = \frac{(\gamma^p + k)(1 + k)^{p-1}}{(\gamma + k)^p} \leq \frac{\gamma^p + k}{\gamma + k} = \frac{\gamma^p + c\gamma}{(1 + c)\gamma} \leq \frac{\gamma^p + c\gamma^p}{(1 + c)\gamma} = \gamma^{p-1}. \quad \square$$

**Remark 5.7.** A direct calculation of $\gamma^p + k; 1+k$ and $\gamma + k$ yields after simplifications that

$$Q = \frac{(p - 1)^{p-1}(\gamma^p - 1)^p}{p^p(\gamma^p - \gamma)^p - 1)^p} = \frac{(p - 1)^{p-1}p^p}{p^p(\gamma^p - \gamma)^p - 1)^p} \gamma^p - 1)^p. \quad (5.4)$$

It is easily verified that $\gamma^p - 1 \geq (\gamma - 1)\gamma^{p-1}$, and inserting this into (5.4) gives, together with Proposition 5.6, the two-sided estimate

$$Q \leq \gamma^{p-1} \leq \frac{p^pQ}{(p - 1)^{p-1}}.$$

**Lemma 5.8.** Given $\gamma > 1$, let $Q$ be as in Theorem 5.4. Then the function

$$u(x) = \begin{cases} \alpha x, & 0 < x \leq x_0, \\ 1 + \alpha\gamma(x - 1), & x_0 \leq x < 1, \end{cases} \quad (5.5)$$

with

$$x_0 = \frac{p\gamma^p(\gamma - 1) - \gamma(\gamma^p - 1)}{(p - 1)(\gamma^p - 1)(\gamma - 1)} \quad \text{and} \quad \alpha = \frac{p - 1}{p} \gamma^p - 1)^p. \quad (5.6)$$

is the unique one-corner function with the boundary conditions $u(0) = 0$ and $u(1) = 1$ that is convex and has the maximal $p$-energy allowed by $Q$ on $(0, 1)$.

**Proof.** For $u$ to be continuous, it is required that $\alpha x_0 + \alpha\gamma(1 - x_0) = 1$, i.e. that

$$\alpha = \frac{1}{\gamma + x_0(1 - \gamma)}. \quad (5.7)$$

Theorem 5.4 with $a = x_0$ and $b = 1 - x_0$ gives $x_0/(1 - x_0) = k$, i.e. $x_0 = k/(k + 1)$. The formula for $k$ from Theorem 5.4 then yields after some simplification the formula for $x_0$ in (5.6). Inserting that into (5.7) then concludes the proof of the lemma, since uniqueness follows by construction. \square

**Remark 5.9.** Lemma 5.8 can also be proved without an appeal to Theorem 5.4 by maximizing the $p$-energy

$$E = \int_0^1 |u_2'|^p \, dx = \alpha^p x_0 + \alpha^p \gamma^p(1 - x_0) = \frac{x_0 + \gamma^p(1 - x_0)}{(\gamma + x_0(1 - \gamma))}.$$

with respect to $x_0$.

**Remark 5.10.** A straightforward calculation shows that for $Q$ and $\alpha$ from (5.4) and (5.7) it holds that $Q = Q_{\alpha} = Q_{\alpha\gamma}$, where $Q_{\alpha}$ and $Q_{\alpha\gamma}$ are related to $\alpha$ and $\alpha\gamma$ as in (3.1). Thus, the optimal one-corner function provided by Lemma 5.8 is tangent at the end points 1 and 0 to the power-like functions $u_Q$ and $u_{\gamma Q}$ from (3.2), respectively.

The proof below of Theorem 5.2 is based on varying $p$ and the fact that the constant in Theorem 1.4 is independent of $p$. For fixed $p$ we obtain the following somewhat weaker result.
Proposition 5.11. Let $p$, $Q_1$ and $Q_2$ be given. Then there are $u_1$, $u_2$ and an interval $I = (x_0, 1)$, $x_0 \geq 0$, such that $u_1$ is a $Q_1$-quasiminimizer in $I$, $u_2$ is a $Q_2$-quasiminimizer in $\Omega = (0, 1)$, and
$$u = \begin{cases} u_2, & \text{in } \Omega \setminus I, \\ \min\{u_1, u_2\}, & \text{in } I. \end{cases}$$
is a $Q$-quasisuperminimizer in $\Omega$ with optimal quasisuperminimizer constant
$$Q \geq Q_1(Q_2 - 1) + 1 = Q_1Q_2 - Q_1 + 1. \tag{5.8}$$

If moreover $Q_1 > 1$, then the inequality in (5.8) is strict, i.e. $Q > Q_1(Q_2 - 1) + 1$.

Proof. Using Proposition 5.5 and Lemma 5.8 we can find $0 \leq x_0 < 1$, $\gamma \geq 1$ and $0 < \alpha \leq 1$ such that the function $u_2$ given by (5.5) is a $Q_2$-quasiminimizer in $\Omega$ with the maximal $p$-energy allowed by $Q_2$ on $\Omega$. For $Q_2 = 1$ let $u_2(x) = x$ and $x_0 = 0$.

Another use of Lemma 5.8 provides us with a convex one-corner function $u_1$ which is a $Q_1$-quasiminimizer in $I = (x_0, 1)$ with boundary values $u_1 = u_2$ on $\partial I$ and maximal $p$-energy allowed by $Q_1$ on $I$. (If $Q_1 = 1$, we let $u_1 \equiv u_2$ on $I$, which is not a one-corner function.)

Since $\alpha \leq 1$ and $x_0 < 1$, we have
$$A := \int_0^{x_0} |u_2'|^p dx = \alpha^px_0 < 1.$$It then follows that
$$\int_0^1 |u'|^p dx = \int_0^{x_0} |u_2'|^p dx + \int_{x_0}^1 |u'|^p dx = A + Q_1 \int_{x_0}^1 |u_2'|^p dx \tag{5.9}$$
$$= A + Q_1(Q_2 - A) = Q_1Q_2 - A(Q_1 - 1) \geq Q_1Q_2 - (Q_1 - 1),$$where the inequality is strict if $Q_1 > 1$. As $v(x) = x$ is the minimizer with boundary values $v = u$ on $\partial \Omega$, and its $p$-energy on $\Omega$ is 1, this concludes the proof. \qed

We are now ready to prove Theorems 5.1 and 5.2.

Proof of Theorem 5.1. The argument is a modification of the proof of Proposition 5.11. Let $\Omega_2 = (0, 1) \subset \mathbb{R}$ and $u_2$ and $x_0$ be as in the proof of Proposition 5.11. Now let $\Omega_1 = (0, x_0) \cup (x_0, 1)$ and choose $u_1$ so that $u_1(x) = u_2(x)$ for $x = 0, x_0, 1$, and its restrictions to $(0, x_0)$ and to $(x_0, 1)$ are convex one-corner functions provided by Lemma 5.8, which are $Q_1$-quasiminimizers in the respective intervals and have the maximal energy therein allowed by $Q_1$. But then $u = u_1$ and
$$\int_0^1 |u'|^p dx = \int_0^{x_0} |u_2'|^p dx + \int_{x_0}^1 |u_1'|^p dx$$
$$= Q_1 \int_0^{x_0} |u_2'|^p dx + Q_1 \int_{x_0}^1 |u_1'|^p dx = Q_1 \int_0^1 |u_2'|^p dx = Q_1Q_2. \qed$$

Proof of Theorem 5.2. We proceed as in the proof of Proposition 5.11. By Proposition 5.6, $1 - \gamma^{1-p} \geq 1 - 1/Q$. It thus follows from Lemma 5.8 that
$$\alpha = \frac{p-1}{p} \frac{1-\gamma^{-p}}{1-\gamma^{1-p}} \leq \frac{p-1}{p} \frac{1}{1-1/Q} \to 0, \quad \text{as } p \to 1+. \quad \text{(Note that $u_1$, $u_2$ and $u$ depend on $p$.)}$$

Hence $A = \alpha^px_0 < \alpha^p \to 0$, as $p \to 1+$, so as in (5.9),
$$\int_0^1 |u'|^p dx = Q_1Q_2 - A(Q_1 - 1) \to Q_1Q_2, \quad \text{as } p \to 1+. \qed$$
Remark 5.12. The estimate (5.8) in Proposition 5.11 can be replaced by \( Q \geq Q_2(Q_1 + 1)/2 \), which gives a better lower bound when \( Q_2 < 2 \). Indeed, we always have
\[
\int_0^{x_0} |u_2'|^p \, dx \leq \frac{Q_2}{2} \quad \text{or} \quad \int_{x_0}^{1} |u_2'|^p \, dx \leq \frac{Q_2}{2}.
\]
In the former case, the proof goes through as before, in the latter case, replace \( u_2 \) and \( u_1 \) by decreasing convex one-corner functions in \( \Omega \) and \( I \), respectively, with the maximal \( p \)-energies allowed by \( Q_2 \) and \( Q_1 \) therein, so that \( u_2(0) = 1, u_1(x_0) = u_2(x_0) \) and \( u_1(1) = u_2(1) = 0 \). In both cases, a direct calculation gives
\[
Q \geq Q_1 Q_2 - \frac{Q_2}{2} (Q_1 - 1) = \frac{Q_2(Q_1 + 1)}{2}.
\]

Corollary 5.13. Let \( p, Q_1, Q_2 \) and open sets \( \Omega_1 \subseteq \Omega_2 = (0,1) \) be given. Then there are \( u_1 \) and \( u_2 \), which are \( Q_1 \)- and \( Q_2 \)-quasiminimizers in \( \Omega_1 \) and \( \Omega_2 \), respectively, such that
\[
u = \begin{cases} u_2, & \text{in } \Omega_2 \setminus \Omega_1, \\ \min\{u_1, u_2\}, & \text{in } \Omega_1, \end{cases}
\]
is a \( Q \)-quasiminimizer in \( \Omega \) with optimal quasiminimizer constant satisfying (5.8) and (5.10).

Proof. Since \( \Omega_1 \) is open, it can be written as a pairwise disjoint union of open intervals. Let \((x_1, x_2)\) be one of them and assume to begin with that \( x_1 > 0 \). We can then find \( \delta > 0 \) so that
\[
\Omega' := (x_1 - k\delta, x_1 + \delta) \subset (0, x_2),
\]
where \( k \) is the constant associated with \( Q_2 \) as in Theorem 5.4.

Rescale the functions in Proposition 5.11 or Remark 5.12 (depending on which gives a better estimate) so that they apply to the sets \( \Omega' \) and \( I' := (x_1, x_1 + \delta) \) in place of \( \Omega \) and \( I \). This provides us with one-corner functions \( v_1 \) and \( v_2 \), which are \( Q_1 \)- and \( Q_2 \)-quasiminimizers in \( I' \) and \( \Omega' \), respectively, and their pasted function is a \( Q \)-quasiminimizer in \( \Omega' \) with optimal quasiminimizer constant \( Q \) satisfying (5.8) and (5.10).

Now, let \( u_2 \) be the linear extension of \( v_2 \) which is a one-corner function on the whole of \((0,1)\). Also, let
\[
u_1 = \begin{cases} u_2, & \text{in } \Omega_1 \setminus (x_1, x_2), \\ v_1, & \text{in } (x_1, x_2), \end{cases}
\]
where \( v_1 \) is extended linearly as a one-corner functions on the whole of \((x_1, x_2)\).

Then the best quasiminimizing constants of \( u_1 \) and \( u_2 \) in \( \Omega_1 \) and \( \Omega_2 \) are still \( Q_1 \) and \( Q_2 \), but their pasted function \( u \) given by (5.11) will have its optimal quasiminimizing constant satisfying (5.8) and (5.10) in \( \Omega' \) and thus in \( \Omega_2 \).

If \( x_1 = 0 \) then necessarily \( x_2 < 1 \) and the above construction can be done for the interval \((1 - x_2, 1)\) instead, replacing \( u_1 \) and \( u_2 \) by the decreasing convex one-corner functions \( x \mapsto u_1(1 - x) \) and \( x \mapsto u_2(1 - x) \).

We conclude the paper with further examples of quasiminimizers with explicit optimal quasiminimizing constants.

Proposition 5.14. Every strictly increasing continuous piecewise linear function \( u \) in \((0,1)\) (having finitely many corners) with alternating slopes \( \alpha \) and \( \beta \), \( \alpha < \beta \), is a quasiminimizer in \((0,1)\) with the best quasiminimizing constant \( Q \) given by (5.4) with \( \gamma = \beta/\alpha \).

Moreover, if \( u \) has at least one convex (concave) corner, then \( Q \) is also the best quasiminimizing (quasisubminimizing) constant.
Clearly, replacing \( u \) with \( x \mapsto u(1-x) \) gives a strictly decreasing quasiminimizer with the same best quasiminimizing constant as \( u \). Note also that we do not require that the first segment defining \( u \) has slope \( \alpha \), nor that the last segment has slope \( \beta \). However, we do not allow \( u \) to be a linear function in the proposition, as then \( \gamma = 1 \) and \( Q \) cannot be defined using (5.4). Nevertheless, \( Q = 1 \) is trivially the best quasiminimizing constant for \( u \) in this case.

**Proof.** To show that \( Q \) will do, let \( 0 \leq a < b \leq 1 \) be arbitrary and consider the linear function \( h \) with \( h(a) = u(a) \) and \( h(b) = u(b) \). By splitting \((a, b)\) into several subintervals, whose energies can be estimated separately, we may assume that either \( h = u \) in \((a, b)\), \( h < u \) in \((a, b)\) or \( h > u \) in \((a, b)\).

If \( h > u \) in \((a, b)\), then moving from \( a \) to \( b \), we can successively eliminate the concave corners as follows: If

\[
u(x) = \max\{u(x') + \beta(x-x'), u(x'') + \alpha(x-x'')\}\]

in the interval \((x', x'')\), where \( x' \) and \( x'' \) are two convex corners, then replace \( u \) in that interval by

\[
\min\{u(x') + \alpha(x-x'), u(x'') + \beta(x-x'')\}.
\]

This will decrease the number of corners in \((a, b)\) by 2, while preserving the \( p \)-energy of \( u \) therein. In the end, this procedure leaves us with a function which in \((a, b)\) coincides with a one-corner function \( v \) with slopes \( \alpha \) and \( \beta \) and the same \( p \)-energy therein as \( u \). Theorem 5.4 shows that \( v \) is a \( Q \)-quasiminimizer in \((a, b)\) and hence

\[
\int_a^b |h'|^p \, dx \leq Q \int_a^b |v'|^p \, dx = Q \int_a^b |u'|^p \, dx.
\]

The argument is similar when \( h < u \) in \((a, b)\), while if \( h = u \) in \((a, b)\) we trivially have

\[
\int_a^b |h'|^p \, dx = \int_a^b |u'|^p \, dx < Q \int_a^b |u'|^p \, dx.
\]

As \( a \) and \( b \) were arbitrary, this shows that \( u \) is a \( Q \)-quasiminimizer.

Finally, if \( u \) has at least one convex (concave) corner, then considering intervals of type \((x_0 - k\delta, x_0 + \delta)\), where \( x_0 \) is one such corner, together with the last part of Theorem 5.4 shows that the quasisuperminimizing (quasisubminimizing) constant of \( u \) cannot be better than \( Q \). As every piecewise linear function with nonequal slopes has at least one corner, this concludes the proof. \( \square \)

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