An Optimal Control Economy Model of Investing in Three Industries for a Region and Some Properties

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Abstract. In this article, we first construct an optimal control mathematics model about economy investing in three industries in a region; then give the existence of the fastest and valid method of investing in three industries for a region; at last, we show that the fastest way to invest in three industries to obtain some target is to input the largest total amount of the capital all through the investing process.

1. Introduction
The efficient investment for a region is very important for the economic development in a certain area. But how to invest efficiently is different when the target of the investment is different, such as how to invest to get to the largest output is different from how to invest to obtain some decided target as soon as possible. As we all know, the main economy in a society are three industries, so how to invest in the three industries to activate each of them to improve the quantity and quality of three industries economy is usually considered by local government in a region. Sometimes in a special history period, we need to invest to acquire some objective in the lest time, this is a time optimal control economy model. What’s more, any industry is just a part of the whole economy, so in fact we need to consider some industry which is just a part of the total economy body, and sometimes we should consider to only activate some industries of them even only one of them. Therefore, as we take account of any industry as one part of the unit, the investment in three industries problem becomes an local time optimal control problem if we set the investment to be a control for the economy development system of three industries.

This paper is organized as following in five sections. In section 2, we formulate a time optimal control (TOC) of investing in three industries problem with local control governed by a diffusion equation which is used to describe a regional economic development of three industries system. In section 3, we should give the existence of TOC problem which show that we are able to find a valid method to invest fastest when we input to the three industries for some decided goal. In section 5, by way of contradiction, we show the bang-bang property for the TOC problem of invest economic model about three industries, At last, the results on the model hint that the fastest and efficient investing in three industries is to keep the investment quantity to be the largest from the start.

2. The TOC problem of economy model on investment in three industries
To construct the TOC problem of economy model for investing on three industries in a region, we need some reasonable assumptions about the model. At first, we should omit the influence produced by the factors which owe to the nature and can not be controlled by human being. The second, we should neglect the element which is not due to economic relation. The third, we consider the economy
system of three industries abiding by the general economic growth law as in [1-5]. Let the vector variable $x = (x_1, x_2, x_3)$ denoted the three industries, $x_1$ the first industry, $x_2$ the second industry, and $x_3$ the third industry, since the total quantity of three industries in a region can not be infinite and the up-boundary is a certain quantity $M$ which is decided by the real situations of the economy development in the region, so it can denote by $x \in \Omega = \{x | x_1 + x_2 + x_3 \leq M\}$, it is clear that $\Omega$ is convex and closed. Assume $u(x, t)$ be the total quantity of three industries at time $t$ with $u(\cdot, \cdot) : \Omega \times [0, +\infty) \rightarrow R$. Suppose $f(x, t)$ be the investment in three industries by the government with $f(\cdot, \cdot) : \Omega \times [0, +\infty) \rightarrow R$, which is depend on the government plan, the capacity of the local government economy, and the requirement of the region in certain period, so $f \in U_{\text{ad}} = \{f \in L^2(0, +\infty; \Omega) | \|f(t)\|_{L^2(\Omega)} \leq Q\}$, where $Q$ is the maximum quantity of investment by local government at time $t$. For convenience, we set $u \in C(0, +\infty; L^2(\Omega))$.

2.1. The optimal control economic model of three industries

As we all know that the operating mechanism of economic development of three industries is similar with the one of heat diffusion, the distribution of the total quantity of regional economy is related to each quantity of three industries and the quantity invested in three industries. The difference between them is that the quantity keeps the conservation law during the course of heat diffusion, but not in the process of investing in three industries, the quantity of the total industries will be some ratio of $k(x)$ to the three industries and invest in three industries for which is activate by the invest, the three industries and their proportion in the total quantity. So we have the efficient invest economy model about three industries as following as in [6,7]:

$$
\begin{align*}
\left\{ 
\begin{array}{ll}
\frac{\partial u}{\partial t} - \nabla[k(x)\nabla u] = f, & x \in \Omega, \ t \in [0, +\infty); \\
u(\cdot, 0) = 0, & x \in \partial \Omega, t \in [0, +\infty); \\
u(\cdot, 0) = u_0, & x \in \Omega.
\end{array}
\right.
\end{align*}
$$

(1)

Here $f$ is the economic magnitude invested at time $t$ for part of three industries, $u_0$ is the initial quantity of three industries before the industries system are invested, as we all know in real economic life, we usually assume that $u$ is null on the boundary of the domain $\Omega$. So the model tell us about the quantity $u$ varies when the three industries are invested on a subdomain $\omega$ of $\Omega$.

2.2. The problem of the fastest achievement to some target

From the model of invest in three industries, we know the problem of fastest achievement to some target means that: find a suitable invest $f \in U_{\text{ad}}$ such that $u \in S$ as soon as possible, where $S$ is some target set of the three industries and $S \subset C(0, +\infty; L^2(\Omega))$. In fact, we know we can not invest all full the three industries, but just local of them, therefore the invest just act on subdomain $\omega$ of $\Omega$. So the fast efficient invest in three industries to obtain some decided target just is to invest the most suitable quantity in some part of three industries, to obtain a certain object as fast as possible, as the matter of fact, this is a local TOC problem of PDE as bellowing. Considering the controlled system

$$
\begin{align*}
\left\{ 
\begin{array}{ll}
\frac{\partial u}{\partial t} - \nabla[k(x)\nabla u] = \chi_\omega f, & x \in \Omega, t \in [0, +\infty), \\
u(\cdot, 0) = 0, & x \in \partial \Omega, t \in [0, +\infty), \\
u(\cdot, 0) = u_0, & x \in \Omega,
\end{array}
\right.
\end{align*}
$$

(2)

find a control $f \in U_{\text{ad}}$, such that $T^* = \inf_{f \in U_{\text{ad}}} \{T | u(x, T) \in S\}$, where $\omega \subset \Omega$ is local domain of three industries, $\chi_\omega$ is the characteristic function of subdomain $\omega$ of the whole quantity of three industries $\Omega$, $u$ is the solution of the above system.

3. The existence of TOC problem of investing in three industries

Since the problem of fastest and efficient investing in three industries is a TOC problem governed by a diffusion equation as the above system (2), to discuss any TOC problem, the first thing we have to do is to decide whether the TOC problem exists or not.

As we all know, if there is an optimal invest such that the state $u$ of system (2) can arrive at some point $u_T \in S$, i.e. $u(\cdot, T) = u_T$. From the property of $u_T$, and letting $z(\cdot, \cdot) = u(\cdot, \cdot) - u_T$, then we
know that the TOC of system (2) will exist if the following optimal control problem (P) is null controllable. Considering the system below
\[ \begin{align*}
    z_t - \nabla [k(x) \nabla z] &= \chi_{\omega} f, \quad z \in \Omega, \quad t \in [0, +\infty); \\
    z(x, 0) &= z_0, \quad x \in \partial \Omega; \\
    z(x, t) &= 0, \quad x \in \partial \Omega, \quad t \in [0, T], \\
    z(\cdot, T) &= 0, \quad x \in \Omega.
\end{align*} \] (3)

and find an optimal control \( f \in U_{ad} \), such that \( z(T) = 0 \), where \( z \) is the solution of system (3), the optimal time is defined by \( T^* = \inf_{f \in U_{ad}} \{ T | z(T) = 0 \} \), the corresponding investment \( f \) is named the optimal control, denoted by \( f^* \).

To show the state \( z \) of system (4) can reach null, we need the observability inequality, with which the null controllability of system (4) can be proved. From [8,9,10], we can get the similar observability estimation bellowing.

**Lemma 1.** Suppose \( \Omega \times [T_1, T_2] \) to be a non-empty subset of \( \Omega \times [0, +\infty) \) and open. Let \( E \) be a subset of \( (0, T) \) with positive measure, and \( T > 0 \). Then there exists a constant \( K(\Omega, \omega, E) \) such that for any \( z_0 \in L^2(\Omega) \), the solution \( z(x, t) \) of following system

\[ \begin{align*}
    z_t - \nabla [k(x) \nabla z] &= \chi_{\omega} f, \quad z \in \Omega, \quad t \in [0, T), \\
    z(x, t) &= 0, \quad x \in \partial \Omega, \quad t \in [0, T], \\
    z(\cdot, T) &= 0, \quad x \in \Omega.
\end{align*} \] (4)

satisfies
\[ \| z \|_{L^2(\Omega)} \leq K(\Omega, \omega, E) \int_{\omega \times E} |z(x, t)| dx dt. \] (5)
The proof is omitted here for its analogy to the result in [9,10]. Through the lemma 1, we find that the state \( z \) of system (4) can reach null under the control force \( f \).

**Theorem 2.** Suppose \( 0 \leq T_0 < T_1 < T_2 \) and \( E \subset [T_1, T_2] \) with \( |E| > 0 \), to any \( z_0 \in L^2(\Omega) \), there exists a control function \( f_0 \in U_{ad} \) such that the following system

\[ \begin{align*}
    z_t - \nabla [k(x) \nabla z] &= f_0, \quad z \in \Omega \times [T_0, T_2], \\
    z(x, t) &= 0, \quad (x, t) \in \partial \Omega \times [T_0, T_2], \\
    z(T_0) &= z_0, \quad x \in \Omega.
\end{align*} \] (6)

has a solution \( z(x, t) \), and satisfies
\[ z(T_2) = 0 \] (7)

and
\[ \| f_0 \|_{L^2(\Omega)} \leq K(\Omega, \omega, E) \| z_0 \|_{L^2(\Omega)} \]

here the constant \( K(\Omega, \omega, E) \) is not relative to \( T_0 \) and positive.

**Proof.** For convenience, the whole proof is listed to three parts.

First, we start to solve the following equation
\[ \begin{align*}
    z_t - \nabla [k(x) \nabla z] &= 0, \quad (x, t) \in \Omega \times [T_0, T_1], \\
    z(x, t) &= 0, \quad (x, t) \in \partial \Omega \times [T_0, T_1], \\
    z(T_0) &= z_0, \quad x \in \Omega.
\end{align*} \] (8)

Therefore, \( z(\cdot, T_1) \in L^2(\Omega) \) and we all known that
\[ \| z(\cdot, T_1) \|_{L^2(\Omega)} \leq C_1 \| z_0 \|_{L^2(\Omega)}, \] (9)

where \( C_1 \) depends on \( T_0, T_1 \) and \( \Omega \).

Second, we establish the existence of control function \( f \in L^2(0, \infty; \Omega) \) control the state \( \tilde{z} = \tilde{z}(x, t) \) of system (10)
\[ \begin{align*}
    \tilde{z}_t - \nabla [k(x) \nabla \tilde{z}] &= \chi_{\omega} f, \quad (x, t) \in \Omega \times [T_1, T_2], \\
    \tilde{z}(x, t) &= 0, \quad (x, t) \in \partial \Omega \times [T_1, T_2], \\
    \tilde{z}(T_1) &= z(T_1), \quad x \in \Omega.
\end{align*} \] (10)

satisfies
\[ \tilde{z}(T_2) = 0 \] (11)

and holds
\[ \| f \|_{L^2(\Omega)} \leq C_2 \| z(T) \|_{L^2(\Omega)} \]

here the constant \( C_2 \) depends on \( T_1, T_2 \) and \( \Omega \) but not on \( T_0 \).
Finally, we choose
\[ f_0 = \begin{cases} 0, & \text{if } t \in [T_0, T_1] \cup [T_2, +\infty), \\ f, & \text{if } t \in [T_1, T_2]. \end{cases} \]  
(12)

Since \( \|f_0\|_{L^2(T_0, T_2; \Omega)} = \|f\|_{L^2(T_1, T_2; \Omega)} \), let \( z = \begin{cases} 0, & t \in [T_0, T_1], \\ z, & t \in [T_1, T_2], \end{cases} \)

is still denoted by \( z \), therefore, there exists a control function \( f_0 \in L^2(0, +\infty; \Omega) \subset U_{ad} \), which leads \( z(\cdot, T_2) = 0 \) in \( L^2(\Omega) \), furthermore, \( \|f_0\|_{L^2(0, +\infty, \Omega)} \leq K(\Omega, \omega, E) \|z_0\|_{L^2(\Omega)} \), where \( K(\Omega, \omega, E) \) is a constant, and derive from \( C_1, C_2 \). This will lead to the theorem 2. and the result comes from a key observability estimate lemma in [11]. From above null controllability of system (6), the existence of TOC problem of invest in three industries is obtained.

**Theorem 3.** For any \( z_0 \in L^2(\Omega) \setminus \{0\} \) and \( Q > 0 \), there exists a TOC \( f^* \) such that
\[ \|f^*\|_{L^2(\Omega)} \leq Q \]
and \( z \) satisfies \( z(\cdot, T^*) = 0 \) in \( \Omega \), where \( z \) is the solution of system (10) corresponding to \( f^* \), and the optimal time is defined by \( T^* = \inf_{f \in U_{ad}} \{T|u(x, T) \in S\} \).

**Proof.** According to the Theorem 2. and definition of \( T^* \) by infimum to those \( T > 0 \), then \( 0 \leq T^* \leq T \), and there exists a positive real number sequence \( (T_m)_{m \geq 1} \) and corresponding control functions \( (f_m)_{m \geq 1} \) in \( L^2(0, +\infty; \Omega) \) so that
\[ T^* = \lim_{m \to \infty} T_m \],\[ \|f_m\|_{L^2(0, +\infty, \Omega)} \leq Q \] and the solutions \( z_m = z_m(x, t) \) which correspond to the control \( f_m \) satisfies
\[ t \in [0, T_m]; \]
\[ (z_m(x, t) - V[k(x)]z_m = \chi_0 f_m(x, t) \in \Omega \times [0, T_m], \]
\[ z_m(x, 0) = z_0, \quad x \in \Omega, \]
\[ z_m(T_m) = 0, \quad (x, t) \in \Omega \times [0, T_m]. \]
With the energy method, for the boundary \( Q \) of \( f_m \), we have
\[ \|z_m\|^2_{C(0, T_m; L^2(\Omega))} + \int_0^{T_m} \|z(x, t)\|^2_{L^2(\Omega)} \, dt \leq C_3, \]
(14)
where \( C_3 \) denote a generic constant independent of \( m \). Therefore, the sequence \( (z_m)_{m \geq 1} \) is bounded in \( C(0, +\infty; L^2(\Omega)) \). Therefore, through the embedding theorem of Sobolev Space, there exist a subsequence \( \{f_{m'}\} \) and a subsequence \( \{z_{m'}\} \) such that
\[ f_{m'} \to f^* \text{ weakly star in } L^2(0, +\infty; \Omega) \text{ with } \|f^*\|_{L^2(0, +\infty, \Omega)} \leq Q, \]
\[ z_{m'} \to z^* \text{ strongly in } C(0, +\infty; L^2(\Omega)). \]
Furthermore,
\[ \begin{cases} z_{m'}(x, t) = \chi_0 f^*, & x \in \Omega, t \in [0, T'], \\ z^*(x, t) = 0, & x \in \partial \Omega, t \in [0, T^*], \\ z^*(0) = z_0, & x \in \Omega. \end{cases} \]
(15)
and
\[ \|z^*(\cdot, T^*)\|_{L^2(\Omega)} \leq \|z^*(\cdot, T') - z^*(\cdot, T_m')\|_{L^2(\Omega)} + \|z^*(\cdot, T_m') - z_{m'}(\cdot, T_m')\|_{L^2(\Omega)} \to 0, \]
through \( z_{m'} \to z^* \) strongly converge in the Sobolev space \( C(0, +\infty; L^2(\Omega)) \), as \( m \to \infty \). This shows \( z^*(T^*) = 0 \). This proof completes.

From above theorem proved, we know that the fast invest in three industries is feasible, that is if we want to invest in three industries in some region for a certain objective, we can acquire the target as fast as possible under some condition of the invest.

**4. The property for TOC problem of the fast investing in three industries**

From above sections, we know it is possible for us to obtain some economic target through investing in part of three industries under the local economy condition in the fastest method, but how to get it is still a problem, in this section, we will give the detail way through the control property for TOC of system (2). i.e., the bang-bang property for the TOC: the following result.
From above three sections, we know that the development mechanism of the economy system of three industries invested by local government in a region can be constructed into a model of TOC governed control problem. Through the existence of the TOC of the invest model of three industries, we know the fast efficient invest can be realized in real economy life, what's more, the efficient and fastest invest is to put into a kind of diffusion equation, which is deduced from the law of interaction among three industries.

For convenience, the proof is divided into several steps.

**Step1.** Suppose $\delta_0 = \frac{|E|^c}{Z}$, since that $T^* > 0$, $0 < |E^c| \leq T^*$ are given, then let $E = E^c \cap (\delta_0, T^*)$, therefore $|E| > 0$. According to the Theorem 2, for $0 < T_0 < T_1 < T_2$, $E \subset (T_1, T_2)$ and $|E| > 0$, there exists a constant $K > 0$ and control function $f_t \in L^2(0, \infty; \Omega)$, which satisfies the system

$$\begin{align*}
\begin{cases}
\frac{d}{dt}z_t - \nabla[k(x)\nabla z_t] = f_t + \chi_{\delta_0} f_1, & x \in \Omega, \; t \in [T_0, T_1], \\
z_0(x, t), &= 0, \; (x, t) \in \partial \Omega \times [T_0, T_1], \\
z(x, T_0) = z_0, & x \in \Omega,
\end{cases}
\end{align*}$$

and keep the inequality

$$\|f_t\|_{L^2(\Omega)} \leq K\|z_{T_0}\|$$

to hold, where $K$ doesn't depend on $T_0$.

**Step2.** Applying step 1. with $T_0 = \delta, T_1 = \delta_0, T_2 = T^*$, $z_{T_0} = z_0 - z^*(\cdot, \delta)$, then $z = \bar{z} + z^*$ satisfies the following system

$$\begin{align*}
\begin{cases}
\frac{d}{dt}z_t - \nabla[k(x)\nabla z_t] = f^* + \chi_{\delta_0} f_1, & x \in \Omega, \; t \in [\delta, T^*], \\
z_0(x, t), &= 0, \; (x, t) \in \partial \Omega \times [\delta, T^*], \\
z(\cdot, \delta) = z_0, & x \in \Omega,
\end{cases}
\end{align*}$$

Denoted $f_2 = f^* + \chi_{\delta_0} f_1$, then

$$\|f_2\|_{L^2(\Omega)} \leq \begin{cases}
\|f^*\|_{L^2(\Omega)} \leq Q, & \text{if } t \in [0, T^*) \setminus E, \\
\|f^* + \chi_{\delta_0} f_1\|_{L^2(\Omega)} \leq Q - \varepsilon + K\|z^*(\cdot, 0) - z^*(\cdot, \delta)\|_{L^2(\Omega)}, & \text{if } t \in E.
\end{cases}$$

Since $z^* \subset C(0, +\infty; L^2(\Omega))$, letting $\delta$ approach to $0$ enough to have

$$\|z^*(\cdot, 0) - z^*(\cdot, \delta)\|_{L^2(\Omega)} \leq \frac{\varepsilon}{K}.$$

Consequently, $\|f_2\|_{L^2(\Omega)} \leq Q$, for almost everywhere $t \in [0, T^*)$.

**Step 3.** Set $f = f_2(\cdot, \delta)$, then $\|f\|_{L^2(\Omega)} \leq Q$, for almost everywhere $t \in [0, T^*)$, taking $u(\cdot, \cdot) = z(\cdot, \delta)$, then it satisfies

$$\begin{align*}
\begin{cases}
\frac{d}{dt}z_t - \nabla[k(x)\nabla z_t] = \chi_{\delta_0} f, & x \in \Omega \times [\delta, T^* - \delta], \\
z(x, t), &= 0, \; (x, t) \in \partial \Omega \times [\delta, T^* - \delta], \\
z(\cdot, \delta) = z_0, & x \in \Omega,
\end{cases}
\end{align*}$$

this means $T^* - \delta$ is the optimal time for the optimal control problem, so it is contrary to that $T^*$ is the optimal time, hence $\|f^*\|_{L^2(\Omega)} = Q$.

**5. Conclusion**

From above three sections, we know that the development mechanism of the economy system of three industries invested by local government in a region can be constructed into a model of TOC governed by a kind of diffusion equation, which is deduced from the law of interaction among three industries. Through the existence of the TOC of the invest model of three industries, we know the fast efficient invest can be realized in real economy life, what's more, the efficient and fastest invest is to put into...
the maximum quantity of what the local government can invest in some period and in the part of three industries planned all through the progress.

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References
[1] Thomas A. Weber, Optimal Control Theory with Applications in Economic, the MIT Press Cambridge, Massachusetts Lond, Englad, 2011.
[2] Acemoglu D, Introduction to Modern Economic Growth, Princeton, N.J.:Princeton University Press,2009.
[3] Arrow K.J., Applications of Control Theory to Economic Growth, Lectures in Applied Mathematics, Vol. 12, Providence,R.I.: American Mathematical Society,1968.
[4] Arrow K.J. and M. Kurz, Optimal Growth with Irreversible Investment in a Ramsey Model, Econometrica 38 (2):331-344(1970).
[5] Arrow K.J. and M. Kurz, Public Investment, the Rate of Return , and Optimal Fiscal Policy, Baltimore,Md: Johns Hopkins Press,1970.
[6] Cass D., Optimum Growth in an Aggregative Model of Capital Accumulation,Review of Economic Studies (1965), 32 (3):233-240.
[7] Duyckaerts,E., Zhang,X., Zuazua,E.,On the optimality of the observability inequalities for parabolic and hyperbolic systems with potentials.Ann.Inst.H.Poincare Anal. Non Linaire (2008) 25,1-41.
[8] Rousseau J L, Robbiano L. Carleman estimate for elliptic operators with coefficients with jumps at an interface in arbitrary dimension and application to the null controllability of linear parabolic equations.Archive for rational mechanics and analysis, 2010, 195(3): 953-990.
[9] K.D.Phung and G.Wang, Quantitative unique continuation for the semilinear heat equation in a convex domain, J.Funct. Anal.,259(2010),1230-1247.
[10] K.D.Phung, L.J.Wang and C.Zhang, Bang-bang property for time optimal control of semilinear heat equation, Ann. Inst.H.Poincare Anal.Non Linear, 31(2014),477-499.
[11] K.Kunisch and L.J.Wang, Time optimal controls of the linear Fitzhugh-Nagumo equation with pointwise control constraints,J.Math.Anal.Appl.,395(2012),114-130.