Monotonicity with volume of entropy and of mean entropy for translationally invariant systems as consequences of strong subadditivity

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Abstract

We consider some questions concerning the monotonicity properties of entropy and mean entropy of states on translationally invariant systems (classical lattice, quantum lattice and quantum continuous). By taking the property of strong subadditivity, which for quantum systems was proven rather late in the historical development, as one of four primary axioms (the other three being simply positivity, subadditivity and translational invariance) we are able to obtain results, some new, some proved in a new way, which appear to complement in an interesting way results proved around thirty years ago on limiting mean entropy and related questions. In particular, we prove that as the sizes of boxes in $\mathbb{Z}^\nu$ or $\mathbb{R}^\nu$ increase in the sense of set inclusion, (1) their mean entropy decreases monotonically and (2) their entropy increases monotonically. Our proof of (2) uses the notion of $m$-point correlation entropies which we introduce and which generalize the notion of index of correlation (see e.g. R. Horodecki, Phys. Lett. A 187 p145 1994). We mention a number of further results and questions concerning monotonicity of mean entropy for more general shapes than boxes and for more general translationally invariant (homogeneous) lattices and spaces than $\mathbb{Z}^\nu$ or $\mathbb{R}^\nu$.

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I INTRODUCTION

The mid 1960’s saw the beginning of an intense period of research into the mathematical properties of the entropy of translationally invariant states on translationally invariant (infinite Euclidean) systems, both classical and quantum. Amongst the questions which were of interest at that time was the question of the existence of what we shall call in this paper limiting mean entropy. A simple variant of this question (see Corollary 1 below) is the question whether the mean entropy of a box tends to a definite limit as the lengths of each of its sides tend to infinity. Here, by the mean entropy of a (finite) box, we simply mean its entropy divided by its volume. (The reader should be warned that in the literature referred to here, no particular phrase is attached to this concept, and the term ‘mean entropy’ is used instead to denote what we call here ‘limiting mean entropy’.) This variant had been proven to be true both for classical systems by Robinson and Ruelle in [1] and for quantum systems by Lanford and Robinson in [2]. However, there were important reasons for wanting to prove variants of this result which involved more general shapes than boxes, such as the variant known as ‘(limiting) mean entropy in the sense of van Hove’ [1]. This had been proven in the classical case in the Robinson-Ruelle paper [1] as a consequence of a general property called strong subadditivity (SSA). The Lanford-Robinson paper [2] put forward the conjecture that SSA held also in the quantum case but, in the absence of a proof of this, could not immediately establish limiting mean entropy in the sense of van Hove. (It was in fact first proven for quantum systems by Araki and Lieb [3].) In fact, six years were to pass before SSA was finally established for quantum systems by Lieb and Ruskai [4].

Here we recall that, if $\rho_{123}$ is a state on a Hilbert space which is given to us as a triple tensor product of three preferred Hilbert spaces, $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$, and if $\rho_2$, $\rho_{12}$, and $\rho_{23}$ denote its partial traces over $\mathcal{H}_1 \otimes \mathcal{H}_3$, $\mathcal{H}_3$, and $\mathcal{H}_1$ respectively, then the property of SSA can be written as

$$S(\rho_{123}) + S(\rho_2) \leq S(\rho_{12}) + S(\rho_{23})$$

where for any density matrix $\rho$, $S(\rho)$ denotes its (von Neumann) entropy $-\text{Tr}(\rho \log \rho)$.

Since then, it has become increasingly clear that SSA is a key property of states on composite quantum systems, and in particular of translationally invariant states on translationally invariant quantum systems [3, 4]. However, we feel that our understanding of the significance of SSA has remained incomplete, partly because of the historical accident that its discovery and first use were very much bound up with the specific technical problem of generalizing results on limiting mean entropy from the case of simple boxes –
where it was not needed – to the case of van Hove – where it was useful. In an attempt to partially remedy this situation, we have considered a number of questions relating to monotonicity properties of entropy and of mean entropy of boxes, bearing in mind the SSA property from the outset. We have found that, while SSA might not be needed to establish limiting mean entropy for the case of boxes, it can, in fact, be used with profit to throw new light on this result. Namely, we shall show in this paper that, for translationally invariant states on translationally invariant quantum systems, SSA implies the stronger result that the mean entropy of boxes decreases monotonically as the boxes increase in size in the sense of set inclusion. We shall also mention a number of results and (as far as we are aware) open questions concerning monotonicity of mean entropy, suggested by our approach, which concern more general shapes than boxes and more general translationally invariant (homogeneous) lattices and spaces than the usual infinite Euclidean lattices and spaces. Finally, we shall give a new proof of the known result that SSA implies that the entropy of boxes increases monotonically, again as the boxes increase in size in the sense of set inclusion.

We now explain our basic setting and list our results in detail. We begin with the following discrete and continuous versions of the standard definition of a translationally invariant quantum system (see for example [2, 7]). In the case of a lattice, \( \mathbb{Z}^\nu \), we define a region \( \Lambda \) to be a non-empty finite subset. In the case of a continuum, \( \mathbb{R}^\nu \), we define a region \( \Lambda \) to be a measurable set with finite (non-zero) volume. In either case, there is an assignment of a separable Hilbert space \( \mathcal{H}_\Lambda \) to each region, satisfying, in the continuum case, the additional condition that this assignment be the same for any two regions which differ by a region of zero volume. Further, this assignment satisfies the compatibility condition that if two regions \( \Lambda_1 \) and \( \Lambda_2 \) are disjoint, then \( \mathcal{H}_{\Lambda_1 \cup \Lambda_2} = \mathcal{H}_{\Lambda_1} \otimes \mathcal{H}_{\Lambda_2} \), where, in the continuum case, two regions are said to be disjoint if their intersection has zero volume. We define a state mathematically to consist of a family \( \{ \rho_\Lambda \} \) of density operators (positive trace-class operators with trace 1) on the Hilbert spaces \( \mathcal{H}_\Lambda \) which are compatible in the sense that, for disjoint \( \Lambda_1 \) and \( \Lambda_2 \),

\[
\rho_{\Lambda_1} = \text{Tr}_{\Lambda_2}(\rho_{\Lambda_1 \cup \Lambda_2})
\]

where for any region \( \Lambda \), \( \text{Tr}_\Lambda \) means the partial trace over \( \mathcal{H}_\Lambda \).

We remark that it is well known that classical lattice systems can be regarded as special cases of quantum lattice systems, where the density matrices representing the state are simultaneously diagonal, and so any result for a quantum lattice system will also be true for a classical lattice system. However, our results below are not applicable to classical continuous systems since property (A) below fails (see [4]) in this case.
In this paper we shall mainly confine our interest to situations where not only the quantum system, but also the state is translationally invariant. This means that for all regions $\Lambda$ and all translations $\tau$ from the relevant translation group ($\mathbb{Z}^\nu$ for lattice systems or $\mathbb{R}^\nu$ for continuous systems) there exists a unitary operator $U(\tau, \Lambda)$ from $\mathcal{H}_\Lambda$ to $\mathcal{H}_{\tau(\Lambda)}$ such that

$$\rho_{\tau(\Lambda)} = U(\tau, \Lambda)\rho_\Lambda U(\tau, \Lambda)^{-1}. \quad (3)$$

Given any state on a translationally invariant quantum system, we define the entropy of a region $\Lambda$ to be the von Neumann entropy of $\rho_\Lambda$, i.e.

$$S(\Lambda) \overset{\text{def}}{=} -\text{Tr}(\rho_\Lambda \log \rho_\Lambda). \quad (4)$$

The entropy of a region is known to satisfy many properties [5]. However, in the present paper we shall focus on:

(A) Positivity.

$$S(\Lambda) \geq 0 \quad \text{for all } \Lambda$$

(B) Subadditivity (SA).

If $\Lambda_1$ and $\Lambda_2$ are disjoint, then

$$S(\Lambda_1 \cup \Lambda_2) \leq S(\Lambda_1) + S(\Lambda_2)$$

(C) Strong subadditivity (SSA).

$$S(\Lambda_1 \cup \Lambda_2) + S(\Lambda_1 \cap \Lambda_2) \leq S(\Lambda_1) + S(\Lambda_2).$$

(A) follows immediately from (4). (C) follows immediately from (3), (2), and (4). (B) is just a special case of (C) but we prefer to view it as a separate property. Furthermore, if our state is translationally invariant, it follows immediately from (3) and (4) that

(D) Translational invariance.

For any element $\tau$ of the relevant translation group

$$S(\Lambda) = S(\tau(\Lambda)).$$

As we discussed above, Property (C) (or rather (3) from which it is an easy consequence) has the status of a difficult theorem [4], but in spite of this, the game we wish to play from now on is to regard (A), (B), (C) and (D) as axioms and to see what one can easily prove about the class of functions
Λ ↦→ S(Λ) from regions of \( \mathbb{Z}^\nu \) or \( \mathbb{R}^\nu \) to the real numbers which obey these axioms.

We begin by defining the mean entropy \( \bar{S} \) of a region \( \Lambda \) by

\[
\bar{S}(\Lambda) \overset{\text{def}}{=} \frac{S(\Lambda)}{|\Lambda|}
\]

where \(|\Lambda|\) denotes, in the lattice case, the number of lattice points contained in \( \Lambda \) and, in the continuum case, the volume of \( \Lambda \).

We also define the notion of box regions, \( \Lambda_a, a = (a_1, \ldots, a_\nu) \), where \( a_1, \ldots, a_\nu \) are positive integers (in the lattice case) or positive real numbers (in the continuum case) by

\[
\Lambda_a \overset{\text{def}}{=} \{ x \in \mathbb{Z}^\nu \text{ or } \mathbb{R}^\nu : 0 < x_i \leq a_i \text{ for } i = 1, \ldots, \nu \}
\]

These have \(|\Lambda_a| = \prod_{i=1}^\nu a_i\). With these two definitions, we shall prove in Sections [I] and [V] that, both in the lattice and continuum cases, and for arbitrary dimension \( \nu \), Axioms (A), (B), (C) and (D) imply:

**Theorem 1.** \( \Lambda_a \subset \Lambda_b \implies \bar{S}(\Lambda_a) \geq \bar{S}(\Lambda_b) \)

**Theorem 2.** \( \Lambda_a \subset \Lambda_b \implies S(\Lambda_a) \leq S(\Lambda_b) \)

By Axiom (A) and the elementary result from real analysis that any monotonic sequence which is bounded below has a limit, we immediately have from Theorem [I] the corollary:

**Corollary 1.** Given any infinite sequence of boxes \( \Lambda_i, i = 1, 2, \ldots \), which increase in size in the sense of set inclusion

\[
\lim_{i \to \infty} \bar{S}(\Lambda_i)
\]

exists.

The special case of this where every edge length of \( \Lambda_i \) tends to infinity as \( i \) tends to infinity, is the result of Lanford and Robinson [4].

We have found a number of intriguing hints that it should be possible to considerably generalize Theorem [I] both to settings which involve a class of shapes more general than boxes and to translationally (and rotationally etc.) invariant systems more general than \( \mathbb{Z}^\nu \) and \( \mathbb{R}^\nu \). In Section [III] we outline a number of partial results in this direction and pose a number of open questions.
Theorem 2 is not an entirely new result. Robinson and Ruelle [1] proved such a monotonicity result, for classical lattice systems, which was more general in that our boxes were replaced by general regions. Also, in an article by Wehrl [8], Theorem 2 is proven in the one-dimensional quantum case; this can then easily be extended to higher dimensions as in our proof below. However, we remark that Wehrl’s proof both relies on SSA and requires the existence (on the line) of limiting mean entropy to have been established first. Instead, our proof proceeds directly from Axioms (A), (B), (C) and (D) and involves the concept of \textit{m-point correlation entropies} which we introduce in Section IV and which are related to the \textit{index of correlation} (see e.g. [9]) in somewhat the same way that truncated correlation functions are related to full correlation functions in quantum field theory and statistical mechanics [10, 11].

\section{II PROOF OF THEOREM 1}

We shall treat in turn the four cases of the one-dimensional lattice, the $\nu$-dimensional lattice, the one-dimensional continuum, and the $\nu$-dimensional continuum.

\textbf{Case 1 (one-dimensional lattice).} In this case, a box region, $\Lambda(n)$, is simply a set consisting of the first $n$ natural numbers for some natural number $n$. Writing $S(n)$ instead of $S(\Lambda(n))$ for ease of notation, it follows from Axioms (B) and (D) that

\begin{equation}
S(q + r) \leq S(q) + S(r)
\end{equation}

and from Axioms (C) and (D) that

\begin{equation}
S(q + r + t) + S(r) \leq S(q + r) + S(r + t)
\end{equation}

where $q, r, t \in \mathbb{N}$.

The statement of our theorem in this case amounts to the statement that the mean entropy $S(n)/n$ is monotonically decreasing. We prove this by establishing the proposition

\begin{equation}
\frac{S(n)}{n} \geq \frac{S(n + 1)}{n + 1}
\end{equation}

with the following simple inductive argument. First notice that a special case of (5) is the statement that $S(2) \leq 2S(1)$. This establishes Proposition (6).
in the case \( n = 1 \). Next, on the assumption that Proposition (7) is true for \( n = p \), we have, by (6) in the case \( r = p \) and \( q = t = 1 \) that

\[
S(p + 2) \leq S(p + 1) + S(p + 1) - S(p) \\
\leq 2S(p + 1) - \frac{p}{p + 1}S(p + 1) \\
= \frac{p + 2}{p + 1}S(p + 1)
\]

which implies that (7) is true for \( n = p + 1 \). We conclude that (7) is true and hence that Theorem 1 is true in the case of a one-dimensional lattice.

Case 2 (\( \nu \)-dimensional lattice). With a similar change in notation to that used above, we now need to prove

\[
\frac{S(a_1, \ldots, a_\nu)}{a_1 \cdots a_\nu} \geq \frac{S(b_1, \ldots, b_\nu)}{b_1 \cdots b_\nu}
\]

(8)

where \( a_i, b_i \in \mathbb{N} \) and \( a_i \leq b_i \) for \( i = 1, \ldots, \nu \). We first notice that the function \( S_{a_1, \ldots, a_\nu}(\cdot) \overset{\text{def}}{=} S(\cdot, a_2, \ldots, a_\nu) \), from the natural numbers to \( \mathbb{R} \), clearly satisfies (3) and (4). Thus, by Case 1, we have

\[
\frac{S(a_1, a_2, \ldots, a_\nu)}{a_1 a_2 \cdots a_\nu} \geq \frac{S(b_1, a_2, \ldots, a_\nu)}{b_1 a_2 \cdots a_\nu}
\]

We next notice that, in a similar way to above, the function \( S_{b_1, a_3, \ldots, a_\nu}(\cdot) \overset{\text{def}}{=} S(b_1, \cdot, a_3, \ldots, a_\nu) \) also satisfies (3) and (4). Thus, by applying Case 1 again, we have

\[
\frac{S(b_1, a_2, a_3, \ldots, a_\nu)}{b_1 a_2 a_3 \cdots a_\nu} \geq \frac{S(b_1, b_2, a_3, \ldots, a_\nu)}{b_1 b_2 a_3 \cdots a_\nu}
\]

One may clearly continue in this way, arriving at (8) after a total of \( \nu \) such steps.

Case 3 (one-dimensional continuum). In this case, a box region, \( \Lambda_{(x)} \), is simply a real interval \((0, x]\). Writing \( S(x) \) instead of \( S(\Lambda_{(x)}) \) we now need to prove

\[
\frac{S(y)}{y} \geq \frac{S(x)}{x}
\]

(9)

for \( y \leq x \).
We first argue that (9) holds on the rationals. For any two rationals \(x\) and \(y\), let \(c\) be their common denominator and define the function \(S_c(\cdot)\), taking its argument from the natural numbers, by \(S_c(n) \overset{\text{def}}{=} S(n/c)\). This function satisfies (3) and (4) of Case 1 and thus \(S_c(n)/n\) and hence \((n/c)/(n/c)\) are monotonically decreasing by the argument given there, thus establishing (9) for these \(x\) and \(y\). To extend (9) to the reals, it then clearly suffices to prove that \(S(x)\) is continuous. This follows immediately from the following lemmas and Axiom (A).

**Lemma 1 (Lieb).** \(S(x)\) is weakly concave i.e. for positive real numbers \(x\) and \(y\), \(S((x + y)/2) \geq S(x)/2 + S(y)/2\).

**Lemma 2.** A function which is weakly concave and bounded below is necessarily continuous.

To prove Lemma 1, first note that if \(x = y\) the statement is trivially true. Otherwise, assume without loss of generality that \(y < x\). The result then follows from (3) in the case established above where \(q, r\) and \(t\) are real, by identifying \(q = t = (x - y)/2\) and \(r = y\). We remark that this is essentially the same as an argument given in [8], where it is attributed to E. Lieb. Lemma 2 (or rather the alternative statement with “convex” substituted for “concave” and “bounded above” substituted for “bounded below”) is proved in [12]. We remark that this is the only place where we use Axiom (A). In particular, Axiom (A) is unnecessary for Cases 1 and 2.

**Case 4 (\(\nu\)-dimensional continuum).** This case can be established from Case 3 by an argument similar to that used above to go from Case 1 to Case 2.

This completes the proof of Theorem 1. We remark that it can be helpful to visualize the steps in the above proof using a geometrical picture in which lattice points are identified with \(\nu\)-dimensional continuum cubes of side 1. In detail, one identifies the particular lattice point \((1, \ldots, 1)\) with the particular continuum cube \(\Lambda_{(1, \ldots, 1)}\) and extends this identification by identifying the general lattice point \((a_1, \ldots, a_\nu)\), \(a_1, \ldots, a_\nu \in \mathbb{Z}\), with the result of translating the cube \(\Lambda_{(1, \ldots, 1)}\) by the vector \((a_1 - 1, \ldots, a_\nu - 1)\). We also remark that, in the continuum case, Theorem 1 can trivially be extended from the case of nested box regions to nested parallelepiped regions with parallel faces (by simply “squashing” the boxes in the theorem).
III REMARKS ABOUT POSSIBLE GENERALIZATIONS OF THEOREM 1

We now discuss two different directions in which one can attempt to generalize Theorem 1.

Firstly, one can ask whether Theorem 1 generalizes to more general shapes than boxes (or parallelepipeds). Indeed, one can ask the very general question

Question 1. Is mean entropy monotonically decreasing on any sequence of regions in $\mathbb{Z}^\nu$ or $\mathbb{R}^\nu$ which increase in size in the sense of set inclusion?

In other words, is the mean entropy of any region in the system less than or equal to the mean entropy of any subregion of that region? We remark that this question is more likely to have a positive answer if we extend the translation group of Section I to the appropriate full symmetry group of $\mathbb{Z}^\nu$ or $\mathbb{R}^\nu$, i.e. if we also include rotations and reflections. From now on we shall assume this extension to be made. We have been unable to answer this question in anything like its full generality, but we have found no negative answers and some partial positive answers in the case of a few specific simple shapes which go beyond the box-shapes (and parallelepiped shapes – cf. the second remark at the end of Section II) of Theorem 1. For example, in $\mathbb{Z}^2$ we can prove inequalities such as

$$\frac{S(\square)}{3} \leq \frac{S(\square)}{2}$$

where we are now using an obvious notation suggested by the first remark at the end of Section II.

Equation (10) may easily be proven from the special cases

$$S(\square) \leq S(\omega) + S(\omega)$$

$$S(\square) + S(\omega) \leq S(\square) + S(\square)$$

of subadditivity and strong subadditivity in an entirely analogous way to the way we established Case 1 of Theorem 1 from equations (5) and (6) in Section II in the case that $q = r = t = 1$. However we have been unable, for example, to prove or disprove either of the candidate inequalities

$$\frac{S(\square)}{4} \leq \frac{S(\square)}{3}$$

$$\frac{S(\square)}{4} \leq \frac{S(\square)}{3}$$

(11)

(12)
(but see after Corollary 2 below for a partial answer to these questions).

In fact, many of the cases where we have been able to answer Question 1 positively turn out to refer to consecutive figures in a one-dimensional “chain” of figures. For example the case (10) illustrated above clearly easily extends to a more general inequality which refers to an arbitrary pair of successive figures in the chain shown in Figure 1.

![Figure 1: Chain of figures](image)

An interesting special case of Question 1 is

**Question 2.** *Is mean entropy monotonically decreasing on any sequence of similar regions in \( \mathbb{Z}^n \) or \( \mathbb{R}^n \) which increase in size in the sense of set inclusion?*

Of course, we know from Theorem 1 that we can answer Question 2 positively for the case of similar boxes and parallelepipeds. But consideration of more general shapes forces us to leave the realm of one-dimensional chains and, for this reason, we have found it difficult to find other shapes for which we can prove anything. In fact, we have not even been able to answer Question 2 in the case of discs in \( \mathbb{R}^2 \) with increasing radii. However, we have been able to answer Question 2 positively in the case of two regular hexagons in the plane (i.e. \( \mathbb{R}^2 \)) with diameters in the ratio two-to-one.

![Figure 2: Hexagon figure](image)

To treat this situation we consider Figure 2. Denoting the smaller hexagon made of 6 small triangles by \( H \) and the diamond region made of two small triangles by \( D \), we begin by noting that the mean entropy of \( H \) is less than or equal to the mean entropy of \( D \). This follows immediately once one notices that \( H \) can be viewed as the disjoint union of three copies of \( D \), since by
applying Axiom (B) (twice) we have $S(H) \leq S(D) + S(D) + S(D)$ which implies that

$$
\frac{S(H)}{6} \leq \frac{S(D)}{2}
$$

(13)

Next, imagine that the vertices of the central small hexagon in Figure 2 are numbered (say) clockwise, starting at some particular vertex, from 1 to 6 and regard the large hexagon as the union of 6 copies of $H$, which we shall call $H_1, \ldots, H_6$, centred respectively at each of these 6 vertices. Also define the sequence of figures $F_1 = H_1$, $F_2 = F_1 \cup H_2$, $F_3 = F_2 \cup H_3$, etc. so that $F_6$ is our large hexagon. We may then argue that each of these figures $F_n$, taken successively, has a mean entropy less than or equal to that of $H$. The first step in this argument proceeds by first noticing that $F_2$ consists of the union of two copies of $H$ whose intersection is a copy of $D$ and hence by Axioms (C) and (D) that $S(F_2) + S(D) \leq S(H) + S(H)$. This is easily combined with the inequality (13) to conclude that $S(F_2)/10 \leq S(H)/6$. The subsequent steps proceed along similar lines, each using the result of the previous step, along with inequality (13) and the facts that (a) $F_i$ consists of the union of the figure $F_{i-1}$ and a copy of the figure $H$ (b) the intersection of $F_{i-1}$ and the same $H$ is a copy of $D$. After the fourth step we have the result that $S(F_5)/22 \leq S(H)/6$. For the final step we note that $F_6$ is the union of the figure $F_5$ and a copy of the figure $H$, but this time the intersection of these figures is a new figure $G$ (composed of 4 small triangles). To derive the final result that $S(F_6)/24 \leq S(H)/6$ we now need, instead of (13), the result that $S(H)/6 \leq S(G)/4$. This can easily be shown by using Axioms (C) and (D) to establish that $S(H) + S(D) \leq 2S(G)$ and combining this with (13).

Besides the above specific examples, we can also prove (say for a lattice $\mathbb{Z}^\nu$, and continuing to interpret lattice points as cubes and to refer to collections of cubes as ‘figures’) the general result:

Theorem 3. The mean entropy of a figure $F(n)$ composed of $n$ cubes ($n \geq 2$) is less than or equal to the average of the mean entropies of all the (connected or disconnected) figures contained in $F(n)$ which are composed of $n-1$ cubes.

We remark that Theorem 3 and Corollary 2 below actually only assume Axioms (B) and (C). In particular, the symmetry-invariance axiom (D) is not required in any form.

Proof of Theorem 3. First we introduce some new notation. Labelling the cubes of $F(n)$ by the integers $1, \ldots, n$ we let $F(n; i, j, \ldots)$ denote the figure
that is formed from the figure $\mathcal{F}(n)$ by taking away its $i$th, $j$th, ... cubes. Then the statement of Theorem 3 amounts to

$$\frac{S(\mathcal{F}(n))}{n} \leq \frac{1}{n} \sum_{j} S(\mathcal{F}(n; j)) \quad (14)$$

We prove this inequality by induction on $n$. First, (14) is true for all figures $\mathcal{F}$ with $n = 2$ by Axiom (B). Next, we assume that (14) is true for all figures $\mathcal{F}$ with $n = p$ cubes. Taking any figure $\mathcal{F}(p + 1)$, we note that $\mathcal{F}(p + 1; i)$ consists of just $p$ cubes, so by our assumption

$$\frac{S(\mathcal{F}(p + 1; i))}{p} \leq \frac{1}{p} \sum_{j \neq i} S(\mathcal{F}(p + 1; i, j)) \quad (15)$$

Also, for $j \neq i$, Axiom (C) implies that

$$S(\mathcal{F}(p + 1)) \leq S(\mathcal{F}(p + 1; i)) + S(\mathcal{F}(p + 1; j)) - S(\mathcal{F}(p + 1; i, j)) \quad (16)$$

Summing (16) for $j = 1, \ldots, p + 1$, with $j \neq i$, leads to

$$pS(\mathcal{F}(p + 1)) \leq pS(\mathcal{F}(p + 1; i)) + \sum_{j \neq i} S(\mathcal{F}(p + 1; j)) - \sum_{j \neq i} S(\mathcal{F}(p + 1; i, j))$$

Combining this with (13) we have

$$pS(\mathcal{F}(p + 1)) \leq pS(\mathcal{F}(p + 1; i)) + \sum_{j \neq i} S(\mathcal{F}(p + 1; j)) - (p - 1)S(\mathcal{F}(p + 1; i))$$

$$= \sum_{j} S(\mathcal{F}(p + 1; j))$$

Dividing this last equation by $p(p+1)$ shows that (14) is true for $n = p + 1$. □

This theorem also leads to the natural corollary:

**Corollary 2.**

$$\frac{S(\mathcal{F}(n))}{n} \leq \frac{\max_j S(\mathcal{F}(n; j))}{n - 1}$$
Thus the mean entropy of a figure on a lattice is less than or equal to the mean entropy of at least one of its subfigures composed of one less cube. Returning to an example discussed above, we see that this remark implies that the mean entropy of the figure \( S_2 \) is less than or equal to the mean entropy of one of its four subfigures each composed of 3 cubes. In fact, we have been able to prove, by an alternative route, the stronger result that its mean entropy is less than or equal to the mean entropy of one of its subfigures composed of one less cube. I n fact, this result, in the continuum case, to the 2-sphere:

\[
S(\Omega) + S(\omega) \leq S(\Omega) + S(\Omega)
\]

Combining this with (\( \Pi \)) we have:

\[
S(\Omega) \leq S(\Omega) + \frac{1}{3}S(\Omega)
\]

But, we must have either \( S(\Omega) \leq S(\Omega) \) or \( S(\Omega) \leq S(\Omega) \). Thus, we conclude from (\( \Pi \)) that one of the inequalities (\( \Pi \)) and (\( \Pi \)) must be true.

A second direction in which one can attempt to generalize Theorem (\( \Pi \)) is suggested by the fact that the basic setting of Section (\( \Pi \)) clearly generalizes to more general lattices than \( \mathbb{Z}^n \) and to more general homogeneous spaces than \( \mathbb{R}^n \) such as discs in one-dimension and spheres and tori in higher dimensions. One can thus ask to what extent Theorem (\( \Pi \)) generalizes to such settings, where Axiom (D) is now replaced by invariance under the relevant symmetry group. As far as more general lattices are concerned, we remark that the hexagon example discussed above could be regarded as an example concerning a triangular lattice. For the case of the one-dimensional circle and higher dimensional tori, it is easy to see that the obvious analogue of Theorem (\( \Pi \)) still holds. For example, on both a one-dimensional “lattice unit-circle” (where the allowed angles are \( 2m\pi/n, m = 1, \ldots, n \)) and a “continuum unit-circle”, one easily shows by a close analogue to the arguments in Cases (\( \Pi \)) and (\( \Pi \)) of Section (\( \Pi \)) that

\[
\frac{S(\theta_1)}{\theta_1} \geq \frac{S(\theta_2)}{\theta_2} \quad \text{for} \quad \theta_1 \leq \theta_2
\]

It is natural to ask the following question (and the obvious counterparts to this question in higher dimensions) concerning a possible generalization of this result, in the continuum case, to the 2-sphere:

**Question 3.** Does the mean entropy of a disc drawn on the surface of a sphere decrease monotonically as the solid angle subtended at the centre increases?

But, just as for discs in \( \mathbb{R}^2 \), we have been unable to answer this question.
IV PROOF OF THEOREM 2

We shall find it useful to begin by introducing, in the case of a one-dimensional lattice, the notion of the \textit{m-point correlation entropies} of a translationally invariant state.

To motivate this definition, we first recall the notion of the \textit{index of correlation} (see for example [9] where it is discussed in an abstract setting concerning states on tensor products of Hilbert spaces). In the case of a one-dimensional quantum lattice system we can interpret this as the difference between the entropy of the union of \(n\) consecutive lattice points (or, in our alternative interpretation, cubes) and the sum of their individual entropies:

\begin{equation}
I_n \overset{\text{def}}{=} nS(1) - S(n) \tag{18}
\end{equation}

By using Axiom (B) \(n - 1\) times, it is easy to show that \(I_n\) is positive.

Our new notion of \textit{m-point correlation entropies} may be regarded as designed so as to provide a new way of writing the index of correlation \(I_n\) as a sum of positive terms, each of which concerns \(m \leq n\) lattice points. Namely, we define the \textit{m-point correlation entropies} by

\begin{equation}
S_m^c \overset{\text{def}}{=} \begin{cases} 
2S(1) - S(2) & m = 2 \\
2S(m - 1) - S(m - 2) - S(m) & m \geq 3 
\end{cases} \tag{19}
\end{equation}

Note that \(S_m^c\) is positive by Axiom (B) for \(m = 2\) and by Axiom (C) for \(m \geq 3\). An easy calculation then shows that

\begin{equation}
I_n = \sum_{m=2}^{n} (n + 1 - m)S_m^c \tag{20}
\end{equation}

By (18) and (20), we can write the entropy of \(n\) consecutive lattice points as

\begin{equation}
S(n) = nS(1) - \sum_{m=2}^{n} (n + 1 - m)S_m^c \tag{21}
\end{equation}

We note that by adding an extra lattice point onto a region of \(n\) consecutive lattice points, the entropy increases by \(S(1)\) i.e. the entropy of one lattice point, but decreases by \(S_i^c\) (for \(i = 2, \ldots, n + 1\)). Thus it is natural to think of \(S_i^c\) as a measure of the degree of correlation of a chain of lattice points of length \(i\) over and above the correlations involving subchains of length \(j\) where \(j < i\). Thus as we mentioned in the introduction, our \(S_n^c\) is related to the index of correlation \(I_n\) in somewhat the same way that truncated correlation functions (sometimes known as connected correlation functions) are related
to full correlation functions in quantum field theory and statistical mechanics \[10, 11\].

We now use this formalism to prove Theorem 2 for the case of the lattice \( \mathbb{Z} \). This can then be proven to extend to \( \mathbb{Z}^\nu \) and \( \mathbb{R}^\nu \) in a similar way to that in which we proved Cases 2, 3 and 4 from Case 1 in Section II.

Proving Theorem 2 in the case of \( \mathbb{Z} \) is equivalent to proving that

\[
0 \leq S(N) - S(N - 1) \quad \text{for } N \geq 2
\]  

To do this, we first note that all the terms in the sum in (21) are positive. Thus for any \( n > N \), removing the last \( n - N \) terms gives us the inequality

\[
S(n) \leq nS(1) - \sum_{m=2}^{N} (n + 1 - m)S_m^c
\]

from which we have

\[
0 \leq \frac{S(n)}{n} \leq S(1) - \frac{1}{n} \sum_{m=2}^{N} (n + 1 - m)S_m^c.
\]

Taking the limit \( n \to \infty \), we deduce that

\[
0 \leq S(1) - \sum_{m=2}^{N} S_m^c.
\]

Substituting the expression for \( S_m^c \) given in Equation (19) into the right hand side of this inequality, one finds that all but two of the \( 3(N - 2) + 3 \) terms cancel and one is left with (22).

We remark that actually the above proof clearly proves a stronger statement than our theorem, namely that \( S(N) - S(N - 1) \) is greater than or equal to the limiting mean entropy!

We also remark that it is essential for Theorem 2 that the full system be infinite. For example, if instead of the one dimensional system \( \mathbb{Z} \) one were to take a closed lattice unit-circle consisting of \( n \) lattice points, as discussed in Section II, then it is obviously easy to have states (‘pure total states’) for which \( S(n) = 0 \) while \( S(m) > 0 \) for some \( m < n \). An amusing example of this is provided by the case where each point around our circle corresponds to a quantum system with Hilbert space \( \mathcal{H} = \mathbb{C}^2 \) and the pure total state is the generalized GHZ [13] state on the \( n \)-fold tensor product of \( \mathcal{H} \) with itself

\[
\Psi = \frac{1}{\sqrt{2}} \left| \ldots \uparrow \uparrow \uparrow \ldots \right\rangle + \frac{1}{\sqrt{2}} \left| \ldots \downarrow \downarrow \downarrow \ldots \right\rangle
\]
where $|\uparrow\rangle$ and $|\downarrow\rangle$ are a choice of orthonormal basis for $\mathcal{H}$. Clearly, in this case, we would have $S(m) = \log 2$ whenever $m < n$, but $S(n) = 0$!

Note that if we were to attempt to consider an analogue of this example in the case of an infinite row of lattice points, then there would be no such difficulty because we never actually assign an entropy to an infinite row of lattice points. (Note though that, at least if we take the view that all observables are local observables, it would still be correct to assign an entropy of $\log 2$ even to the state which formally corresponds to the above generalized GHZ state in the case “$n = \infty$” notwithstanding the fact that this “looks like” a vector state on an infinite tensor product of $\mathbb{C}^2$.)

V EPILOGUE

One immediate consequence of Axioms (A) and (C) is that, if two regions each have zero entropy, then both their intersection and their union must also have zero entropy. This might be expressed by saying: “If a state is pure on each of two regions, it must be pure on both their union and intersection.”

Amongst other things, this remark further illuminates one of the heuristic remarks (concerning Theorem 6.4 of [14]) made in a paper [14] by Kay and Wald on quantum field theory in curved spacetime. (See pages 55, 99 and 105 of [14].) Namely, that it is impossible for a state to be pure on each of two ‘double-wedge regions’ [14] but mixed on their intersection. In fact, one of the motivations for the present research was a desire to elucidate that remark.

With an extension of the reasoning behind the above remark, another result which one can easily derive, now from our full set of axioms (A), (B), (C) and (D), is:

**Theorem 4.** In both lattice and continuum cases, and for arbitrary dimension $\nu$, if the entropy of any box is zero the entropy of all boxes is zero.

One may prove this either as an immediate consequence of Theorems [1] and [2], or as an easy direct consequence of the axioms.

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