Local GW Invariants of Elliptic Multiple Fibers

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Abstract
We use simple geometric arguments to calculate the dimension zero local Gromov-Witten invariants of elliptic multiple fibers. This completes the calculation of all dimension zero GW invariants of elliptic surfaces with $p_g > 0$.

Let $X$ be a Kähler surface with $p_g > 0$. By the Enriques-Kodaira classification (cf. [BHPV]), its minimal model is a $K3$ or Abelian surface, a surface of general type or an elliptic surface. Each holomorphic 2-form $\alpha$ on $X$ defines an almost complex structure $J_\alpha = (Id + JK_\alpha) J (Id + JK_\alpha)$. (0.1)

Here, $J$ is the complex structure on $X$ and the endomorphism $K_\alpha$ of $TX$ is defined by the formula $\langle u, K_\alpha v \rangle = \alpha(u, v)$ where $\langle , \rangle$ is the Kähler metric. This $J_\alpha$ satisfies:

Lemma 0.1 ([L]). If $f$ is a $J_\alpha$-holomorphic map that represents a nontrivial $(1,1)$ class then its image lies in the support of the zero divisor $D_\alpha$ of $\alpha$ and $f$ is, in fact, holomorphic.

The Gromov-Witten invariant $GW_{g,n}(X, A)$ is a (virtual) count of holomorphic maps representing the class $A$. In particular, the invariant $GW_{g,n}(X, A)$ vanishes unless $A$ is a $(1,1)$ class since every holomorphic map represents $(1,1)$ class. Note that each canonical divisor $D$ of $X$ is a zero divisor of a holomorphic 2-form. Lemma 0.1 thus shows that the GW invariant is a sum

$$GW_{g,n}(X, A) = \sum GW_{g,n}^{loc}(D_k, A_k)$$

over the connected components $D_k$ of the canonical divisor $D$ of local invariants that counts the contribution of maps whose image lies in $D_k$ (cf. [LP], [KL]). It follows that the GW invariants of minimal $K3$ or Abelian surfaces are trivial except possibly for the trivial homology class because their canonical divisors are trivial.

The local GW invariants have a universal property. If $X$ is a minimal surface of general type with a smooth canonical divisor $D$ then the local invariants associated with $D$, and hence GW invariants, are determined by the normal bundle of $D$ — in fact, there exists a universal function of $c_1^2$ and $c_2$ that gives the GW invariants of $X$ (cf. Section 7 of [LP]).

If $\pi : X \to C$ is a minimal elliptic surface with $p_g > 0$, after suitable deformation, we can assume $X$ has a canonical divisor of the form

$$\sum_i n_i F_i + \sum_k (m_k - 1) F_{m_k}$$
where $F^i$ is a regular fiber and $F_{m_k}$ is a smooth multiple fiber of multiplicity $m_k$ (cf. Proposition 6.1 of [LP]). In this case, the GW invariants of $X$ are sums of universal functions, and are completely determined by the multiplicities $m_k$ and the number

$$c_\pi = \chi(O_X) - 2\chi(O_C)$$

(cf. Section 6 of [LP]). In particular, the generating function for the set of all dimension zero GW invariants of $X$ is given by

$$GW^0_X = c_\pi \sum_{d>0} GW_{1}^{loc}(F, d) t^d + \sum_{k} \sum_{d>0} GW_{1}^{loc}(F_{m_k}, d) t_{m_k}^d$$

(0.2)

where the formal variables $t$ and $t_{m_k}$ are for the fiber class $[F]$ and the multiple fiber classes $[F_{m_k}]$ respectively; these satisfy $t_{m_k}^m = t$. The local invariants in (0.2) are counts of multiple covers of elliptic curves together with signs determined by the GW theory of 4-manifolds.

Some of the generating functions in (0.2) are known. In cases of the regular fiber $F$ and the multiple fiber $F_2$, it was proved in Section 10 of [LP] that

$$GW_1^{loc}(F, d) = -\frac{1}{d} \sigma(d) \quad \text{and} \quad GW_1^{loc}(F_2, d) = \frac{1}{d} \left( \sigma(d) - 2 \sigma\left(\frac{d}{2}\right) \right)$$

(0.3)

where $\sigma(d) = \sum_{k|d} k$ if $d$ is a positive integer and $\sigma(d) = 0$ otherwise. In this note we use geometric arguments to obtain the terms in (0.2) associated with fibers of higher multiplicity. Our main theorem is the following formula for the local invariants $GW_1^{loc}(F_m, d)$ for $m > 2$. This completes the calculation of all dimension zero GW invariants of all minimal elliptic surfaces with $p_g > 0$.

**Main Theorem.** Let $m \geq 3$. Then

$$GW_1^{loc}(F_m, d) = \frac{1}{d} \left( \sigma(d) - m \sigma\left(\frac{d}{m}\right) \right).$$

The contribution of each degree $d$ cover $f$ of elliptic curve $F_m$ is, as a map into a 4-manifold, determined by the normal bundle $N_m$ of $F_m$. In cases of $F_1 = F$ and $F_2$, the almost complex structure $J_\alpha$ on $X$ is generic in the sense that the linearized operator $L_f$ (see (1.7) below) is invertible and hence the contribution of $f$ is $(-1)^{h^0(N_m)/|\text{Aut}(f)|}$ (cf. Section 10 of [LP]). When $m \geq 3$, $J_\alpha$ is, in general, no longer generic. We need to perturb $J_\alpha$ to generic $J$. In Section 2, using the universal property of local invariants (see (1.3) below), we choose a local model that is convenient for our calculation. In Section 3, when $L_f$ is not invertible, we use a lifting property of covering space to calculate the contribution of $f$ that proves the Main Theorem. The information for dimension zero GW invariants of elliptic surfaces with $p_g > 0$ is the same as for its Seiberg-Witten invariants. We spell out the specific connection in Remark 3.4.

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1 Dimension Zero Genus One Local GW Invariants

Let $X$ be a (not necessarily compact) elliptic Kähler surface with a holomorphic 2-form $\alpha$. The 2-form $\alpha$ defines an almost complex structure $J_\alpha$ on $X$ by the formula (0.1). Suppose that the zero divisor $D_\alpha$ of $\alpha$ has a smooth reduction $D_\alpha = (m-1)D$ where $D$ is a regular fiber or a multiple fiber of multiplicity $m$ for some integer $m > 1$. The adjunction formula then shows $c_1(TX)([D]) = 0$ and $c_1(N) = 0$ where $N$ is the normal bundle of $D$. The moduli space

$$\overline{M}_1^d(X, d[D])$$

(1.1)

of stable $J_\alpha$-holomorphic maps from curves of genus one representing the class $d[D]$ ($d \neq 0$) carries a (virtual) fundamental class

$$[\overline{M}_1^d(X, d[D])]^{vir}$$

(1.2)

that is defined by the GW theory of 4-manifolds (cf. Section 4 of [LP]). This (virtual) fundamental class (1.2) has dimension zero since $c_1(TX)([D]) = 0$. The dimension zero genus one local Gromov-Witten invariant of $X$ associated with the zero divisor $D_\alpha$ is then

$$GW^1_{loc}(X, D_\alpha, d) := [\overline{M}_1^d(X, d[D])]^{vir}.$$ 

This local GW invariant has the following universal property. Let $X'$ be another elliptic Kähler surface with a holomorphic 2-form $\alpha'$ whose zero divisor $D_{\alpha'} = (m'-1)D'$ where $D'$ is a regular fiber or a multiple fiber of multiplicity $m'$. Let $N'$ be the normal bundle of $D'$. If $m = m'$ and $h^0(N) = h^0(N')$ then

$$GW^1_{loc}(X, D_\alpha, d) = GW^1_{loc}(X', D_{\alpha'}, d)$$

(1.3)

(cf. Section 6 of [LP]). We set

$$GW^1_{loc}(X, D_\alpha, d) = \begin{cases} GW^1_{loc}((m-1)F, d) & \text{if } D \text{ is a regular fiber} \\ GW^1_{loc}(F_m, d) & \text{if } D \text{ is a } m\text{-multiple fiber} \end{cases}$$

(1.4)

It was proved in Example 4.4 of [LP] that

$$GW^1_{loc}(mF, d) = m GW^1_{loc}(F, d).$$

(1.5)

As given in (1.2), all dimension zero GW invariants of minimal elliptic surfaces with $p_g > 0$ are sums of local invariants in (1.4).

In the below, we will give a precise description on the (virtual) fundamental class (1.2) which will be used for our calculation in Section 3. The point in the moduli space (1.1) is an equivalence class $[f, C]$ of stable maps $(f, C)$ where two stable maps $(f, C)$ and $(f', C')$ are equivalent if there is a biholomorphic map $\sigma : C \to C'$ with $f' \circ \sigma = f$. By Lemma 0.1 if $d \neq 0$ every representative $(f, C)$ of $[f, C]$ is a holomorphic $d$-fold covering map from $C$ to $D$. Thus, if $D$ is given by a lattice $\Lambda$ in the complex plane then $[f, C]$ is determined by an index $d$ sublattice of $\Lambda$. In particular, the moduli space (1.1) consists of $\sigma(d)$ points.
On the other hand, since the (virtual) fundamental class \( [\mathcal{M}_1^i(X, d[D])]^{vir} \) is defined by the GW theory of 4-manifolds, as described in Section 3 of [IP], it is a finite sum

\[
\sum c([f, C])
\]

over \([f, C] \in \overline{\mathcal{M}}_1^i(X, d[D])\) of the contributions \(c([f, C])\) that are defined as follows. Choose a \(p \in D\) and a small disk \(B\) in \(X\) with \(B \cap D = \{p\}\) and, once and for all, fix a map \((f, C, x)\) with \(f(x) = p\) such that \((f, C)\) represents \([f, C]\). Then for a generic almost complex structure \(J\) on \(X\) that is sufficiently close to \(J_0\) and tamed by the Kähler form on \(X\), there are finitely many \(J\)-holomorphic maps \((f_i, C_i, x_i)\) from smooth genus one curves with one marked point such that

(i) \(f_i(x_i) \in B\) (ii) each \((f_i, C_i, x_i)\) is \(C^0\)-close to \((f, C, x)\) (in a suitable space of maps) and (iii) the index zero operator

\[
L_{f_i} : \Omega^0(f_i^*N_i) \to \Omega^{0,1}(f_i^*N_i)
\]

has trivial kernel (or equivalently \(L_{f_i}\) is invertible) where the operator \(L_{f_i}\) is obtained by linearizing \(J\)-holomorphic map equation (see Remark 1.1 below) and restricting to the normal bundle \(N_i\) of the image of \(f_i\). Denote by

\[\mathcal{M}_{(f_i, C, x), B, J}\]

the set of such \(J\)-holomorphic maps \((f_i, C_i, x_i)\). Notice that for each \((f_i, C_i, x_i)\) the preimage \(f_i^{-1}(B)\) consists of \(d = |\text{Aut}(f)|\) distinct points \(x_{ij}\). Since the automorphism group of \(C_i\) acts transitively, for each \(x_{ij}\) there exists an automorphism \(\sigma_j\) of \(C_i\) with \(\sigma_j(x_i) = x_{ij}\) such that \((f_i \circ \sigma_j, C_i, x_i)\) is also contained in the set \(\mathcal{M}_{(f_i, C, x), B, J}\). The contribution \(c([f, C])\) is thus the (weighted) sum

\[
c([f, C]) = \frac{1}{d} \sum (-1)^{SF(L_{f_i})}
\]

over \(f_i\) in \(\mathcal{M}_{(f_i, C, x), B, J}\) where the sign of each \(f_i\) is given by the mod 2 spectral flow \(SF(L_{f_i})\) of the invertible operator \(L_{f_i}\). In particular, \(SF(L_{f_i}) = 0\) if \(L_{f_i}\) is complex linear, namely \(J\)-linear.

**Remark 1.1.** The operator \(D_{f_i} : \Omega^0(f_i^*TX) \to \Omega^{0,1}(f_i^*TX)\) obtained by linearizing \(J\)-holomorphic map equation at \(f_i\) is given by

\[
D_{f_i}(\xi)(v) = \nabla_v \xi + J\nabla_{jv} \xi + \frac{1}{2} \left( \nabla_{\xi J}(df_i(jv)) - J(\nabla_{\xi J}(v)) \right)
\]

where \(\xi \in \Omega^0(f_i^*TX), v \in TC_i\) and \(j\) is the complex structure on \(C_i\). Here \(\nabla\) is the pull-back connection on \(f_i^*TX\) of the Levi-Civita connection of the metric on \(X\) that is defined by the Kähler form and \(J\) (cf. Lemma 6.3 of [RT]).

### 2 Local Model

Once and for all, fix an integer \(m \geq 2\) and let \(D\) denote the elliptic curve given as the complex plane (with coordinate \(z\)) modulo the lattice \(\mathbb{Z} + (mi)\mathbb{Z}\). Then \(S = D \times \mathbb{C}\) has an automorphism \(\varphi\) of order \(m\) defined by

\[
\varphi(z, w) = (z + i, e^{2\pi i/m} \cdot w)
\]
such that all powers \( \varphi^i \) are fixed-point free where \( w \) is a coordinate on \( C \). Let \( S_m \) be the quotient of \( S \) by the group \( \{ \varphi^i \} \) and \( q : S \to S_m \) the quotient map. The map \( S \to \mathbb{C} : (z, w) \to w^m \) then factors through \( S_m \) to give an elliptic fibration \( S_m \to \mathbb{C} \) whose central fiber is a \( m \)-multiple fiber \( D_m \) given by the lattice \( \mathbb{Z} + i\mathbb{Z} \) with torsion normal bundle \( N_m \) of order \( m \):

\[
S = D \times \mathbb{C} \quad \xrightarrow{q} \quad S_m \quad \xrightarrow{\mathbb{C}} \quad \mathbb{C}
\]

The following simple observation is a key fact for our subsequent discussions. Let \( f : C \to D_m \) be a holomorphic map of degree \( d \) from an elliptic curve \( C \) that is given by a sublattice of \( \mathbb{Z} + i\mathbb{Z} \) of the form

\[
a\mathbb{Z} + (bi + k)\mathbb{Z} \quad \text{with} \quad d = ab, \quad 0 \leq k \leq a - 1.
\]

Write \( D \times \{0\} \subset S \) simply as \( D \).

**Lemma 2.1.** Let \( D, N_m \) and \( f : C \to D_m \) be as above. Then,

\[
\text{\( f \) factors through \( D \) } \iff \text{\( a \mid \frac{d}{m} \) } \iff \text{\( f^*N_m = \mathcal{O}_C \)}
\]

**Proof.** \( f \) factors through \( D \) \( \iff \) \( a\mathbb{Z} + (bi + k)\mathbb{Z} \) is a sublattice of \( \mathbb{Z} + (mi)\mathbb{Z} \iff m|b \iff a|\frac{d}{m} \). This shows the first assertion. Observe that for the restriction map \( g_m = q|_D : D \to D_m \),

\[
g_m^* (N_m) = g_m^* ([D_m]|_D) = q^* ([D_m]|_D) = [q^* D_m]|_D = [D]|_D = \mathcal{O}_D \quad (2.1)
\]

where \( [D_m] \) is the line bundle associated to the divisor \( D_m \), \( N_m = [D_m]|_D \) by adjunction, the pullback divisor \( q^* D_m = D \) and again by adjunction \( [D]|_D \) is the normal bundle of \( D \) that is trivial. Write as \( N_m = \mathcal{O}_{D_m}(p - q) \) where

\[
\int_{q}^{p} dz = \frac{k_1}{m} + i \frac{k_2}{m} \quad \text{for some} \quad 0 \leq k_1, k_2 \leq m - 1.
\]

Then, by (2.1) and the Abel’s Theorem, \( g_m^* N_m = \mathcal{O}_D(\sum_j (p_j - q_j)) \) for some \( p_j, q_j \) such that

\[
\sum_j \int_{q_j}^{p_j} dz = k_1 + i k_2 \equiv 0 \mod \mathbb{Z} + (mi)\mathbb{Z}.
\]

Consequently, \( k_2 = 0 \) and \( \gcd(m, k_1) = 1 \) since \( N_m \) is torsion of order \( m \). Now, again by the Abel’s Theorem, \( f^* N_m = \mathcal{O}_C(\sum_\ell (t_\ell - s_\ell)) \) for some \( s_\ell, t_\ell \) such that

\[
\sum_\ell \int_{s_\ell}^{t_\ell} dz = \frac{dk_1}{m} \equiv 0 \mod a\mathbb{Z} + (bi + k)\mathbb{Z} \iff f^* N_m = \mathcal{O}_C
\]

Therefore, \( a|\frac{d}{m} \iff m|b \iff a|\frac{dk_1}{m} \iff f^* N_m = \mathcal{O}_C \). This shows the second assertion. \( \square \)

**Remark 2.2.** Since \( q : S \to S_m \) is a covering map, Lemma 2.1 shows \( f : C \to D_m \subset S_m \) lifts to \( \tilde{f} : C \to D \subset S \) if and only if \( f^* N_m = \mathcal{O}_C \). On the other hand, the Kähler form \(-\frac{i}{2}(dz \wedge d\bar{z} + dw \wedge d\bar{w}) \) on \( \mathbb{C}^2 \) descends to a Kähler form \( \tilde{\omega} \) on \( S \) that is \( \varphi \)-invariant, so \( \tilde{\omega} \) also descends to a Kähler form \( \omega \) on \( S_m \) such that \( q^* \omega = \tilde{\omega} \).
3 Calculation

Let \( q : (S, D) \to (S_m, D_m) \) be as in Section 2. Fix a holomorphic 2-form

\[
\alpha = u^{m-1}dw \wedge dz
\]
on \( S \) whose zero divisor is \((m-1)D\) and let \( J_{\alpha} \) denote the almost complex structure on \( S \) defined by the formula (0.1). The 2-form \( \alpha \) is \( \varphi \)-invariant, so it descends to a holomorphic 2-form \( \alpha_m \) on \( S_m \) whose zero divisor is \((m-1)D_m\). We denote by \( J_m = J_{\alpha_m} \) the almost complex structure on \( S_m \) defined by the 2-form \( \alpha_m \). Since \( D_m \) is a multiple fiber of multiplicity \( m \),

\[
GW_1^{loc}(F_m, d) = [\mathcal{M}^{\alpha_m}(S_m, d[D_m])]^{vir}
\]
where the right-hand side is given by the sum of contributions as in (1.6). In order to calculate them, we decompose the moduli space \( \mathcal{M}^{\alpha_m}(S_m, d[D_m]) \) as a disjoint union

\[
\mathcal{M}^{\alpha_m}(S_m, d[D_m]) = \bigsqcup M^{+}_{m,d} \cup M^{-}_{m,d}
\]
where

\[
\begin{cases}
[f, C] \in M^{+}_{m,d} & \text{if } h^0(f^*N_m) = 0 \\
[f, C] \in M^{-}_{m,d} & \text{if } h^0(f^*N_m) = 1
\end{cases}
\]

It then follows from Lemma 2.1 that

\[
\# M^{+}_{m,d} = \sigma(d) - \sigma\left(\frac{d}{m}\right) \quad \text{and} \quad \# M^{-}_{m,d} = \sigma\left(\frac{d}{m}\right)
\]

where \( \# A \) is the cardinality of a set \( A \).

We first calculate the contribution \( c([f, C]) \) of \([f, C]\) in \( M^{+}_{m,d} \). In the below, we always assume \( m \geq 3 \) and \( m|d \).

**Lemma 3.1.** If \([f, C] \in M^{+}_{m,d}\) then \( c([f, C]) = \frac{1}{d} \).

**Proof.** The linearized operator \( L_f \) has the form \( L_f = \overline{\partial}_f + R_m \) where \( \overline{\partial}_f \) is the usual \( \overline{\partial} \)-operator on \( f^*N_m \) and the zeroth order term \( R_m \) is given by

\[
R_m(\xi) = -\nabla_\xi K_{\alpha_m} \circ J_{\alpha_m} \circ df \quad \text{for} \quad \xi \in \Omega^0(f^*N_m)
\]
(cf. Section 8 of [LP]). But, \( R_m \equiv 0 \) since \( \alpha_m \) (and hence \( K_{\alpha_m} \)) vanishes of order \( m - 1 \geq 2 \) along \( D_m \). Consequently, \( \dim \ker L_f = 2h^0(f^*N_m) = 0 \), so \( L_f \) is invertible with \( SF(L_f) = 0 \). Now, the proof follows from the fact \( f : C \to D_m \) has degree \( d \). \( \Box \)

Let \([f, C] \in M^{-}_{m,d}\). The proof of Lemma 3.1 shows \( L_f = \overline{\partial}_f \) is not invertible. In this case, we will uses the \( m \)-fold covering map \( q : S \to S_m \) to calculate the contribution \( c([f, C]) \). Observe that by Lemma 2.1 the map

\[
M^{-}_{m,d} \to \mathcal{M}^{\alpha_m}(S, d[D]) \quad \text{defined by} \quad [f, C] \to [\tilde{f}, C]
\]
is one-to-one and onto where \( \tilde{f} \) is a lift of \( f \).

**Lemma 3.2.** If \([f, C] \in M^{-}_{m,d}\) then \( c([f, C]) = \frac{1}{m} c([\tilde{f}, C]) \).
Proof. Let $B = \{0\} \times \Delta \subset S$ where $\Delta$ is a small disk around 0 in $\mathbb{C}$ and $B_m = q(B)$ and fix a map $(f, C, x)$ with $f(x) \in B_m$ such that $(f, C)$ represents $[f, C]$. Since the restriction map $q_{|B} : B \to B_m$ is one-to-one, Lemma 3.1 shows that $(f, C, x)$ uniquely lifts to a $J_\alpha$-holomorphic map $(\tilde{f}, C, x)$ with $\tilde{f}(x) \in B$ such that $(\tilde{f}, C)$ represents $[\tilde{f}, C]$ in $\overline{M}_1^\alpha(S, \frac{d}{m}[D])$.

Let $\omega$ and $\tilde{\omega}$ be the Kähler forms as in Remark 2.2 and choose a generic $\omega$-tamed almost complex structure $J$ on $S_m$ that is close to $J_m$. Then, we have

- $J$ lifts to an $\tilde{\omega}$-tamed almost complex structure $\tilde{J}$ on $S$ close to $J_\alpha$ such that $dq \circ \tilde{J} = J \circ dq$,
- each $f_i$ in $\mathcal{M}(f,C,x),B_m,J$ is homotopic to $f$ since $f_i$ is $C^0$-close to $f$, so $(f_i, C_i, x_i)$ also uniquely lifts to $J$-holomorphic maps $(\tilde{f}_i, C_i, x_i)$ with $\tilde{f}(x_i) \in B$ such that $(\tilde{f}_i, C_i, x_i)$ is $C^0$-close to $(\tilde{f}, C, x)$.

The pair $(\omega, J)$ defines a metric $g$ on $S_m$ whose lift $\tilde{g} = q^*g$ is the same metric defined by the pair $(\tilde{\omega}, \tilde{J})$. Let $\nabla$ and $\tilde{\nabla}$ respectively denote the pull-back connections on $\tilde{f}_i^*TS_m$ and $\tilde{f}_i^*TS$ of the Levi-Civita connection of $g$ and $\tilde{g}$. The differential $dq$ then induces a bundle isomorphism $dq : \tilde{f}_i^*TS \to \tilde{f}_i^*q^*TS_m = f_i^*TS_m$ such that $dq \circ \tilde{\nabla} = \nabla \circ dq$ (see [W] page 138) and hence by the formula (1.8) we have

$$dq \circ D_{\tilde{f}_i} = D_{f_i} \circ dq \quad (3.2)$$

The differential $dq$ also induces a bundle isomorphism $dq_i : \tilde{f}_i^*\tilde{N}_i \to f_i^*N_i$ and restricting the equation (3.2) to $\tilde{f}_i^*\tilde{N}_i$ and $f_i^*N_i$ gives

$$dq_i \circ L_{\tilde{f}_i} = L_{f_i} \circ dq_i$$

where $\tilde{N}_i$ and $N_i$ are normal bundles of $\text{Im}(\tilde{f}_i)$ and $\text{Im}(f_i)$ respectively. Therefore, $L_{\tilde{f}_i}$ is also invertible and hence there is one-to-one correspondence

$$\mathcal{M}(f,C,x),B_m,J \to \mathcal{M}(f,C,x),B,\tilde{J} \quad \text{given by} \quad (f_i, C_i, x_i) \to (\tilde{f}_i, C_i, x_i).$$

Let $\tilde{L}_i$ be a path of first order elliptic operators from an invertible $\tilde{J}$-linear operator $\tilde{L}_0$ to $\tilde{L}_1 = L_{\tilde{f}_i}$ with all $\tilde{L}_i$ invertible except at finitely many $t_k$. Then, $dq_i \circ \tilde{L}_i \circ (dq_i)^{-1}$ is also a path from invertible $\tilde{J}$-linear operator to $L_{\tilde{f}_i}$ such that

$$SF(L_{\tilde{f}_i}) = \sum_k \dim \ker \tilde{L}_{t_k} = \sum_k \dim \ker dq_i \circ \tilde{L}_{t_k} \circ (dq_i)^{-1} = \sum_k (-1)^{SF(L_{\tilde{f}_i})} = \frac{1}{m} c([f, C]). \quad \Box$$

Now, noting $\text{deg}(f) = d$ and $\text{deg}(\tilde{f}) = \frac{d}{m}$, we have

$$c([f, C]) = \frac{1}{d} \sum_{f_i} (-1)^{SF(L_{f_i})} = \frac{1}{d} \sum_{f_i} (-1)^{SF(L_{f_i})} = \frac{1}{m} c([f, C]).$$

We are now ready to prove the Main Theorem in the introduction.

Proof of the Main Theorem : It follows from Lemma 3.1, Lemma 3.2 and 3.1 that

$$GW_1^{loc}(F_m, d) = \sum_{[f,C] \in \mathcal{M}_{m,d}^+} c([f, C]) + \sum_{[f,C] \in \mathcal{M}_{m,d}^-} c([f, C]) = \frac{1}{d} \left( \sigma(d) - \sigma(\frac{d}{m}) \right) + \frac{1}{m} \left[ \overline{M}_1^\alpha(S, \frac{d}{m}[D]) \right]^{\text{vir}}. \quad (3.3)$$
Since the 2-form $\alpha$ on $S$ has the zero divisor $(m-1)D$, so by (1.5) and (0.3) we have

$$[\mathcal{M}_1^d(S, \frac{d}{m}[D])]^{\text{vir}} = GW_1^{\text{loc}}((m-1)F, \frac{d}{m}) = -(m-1) \frac{m}{d} \sigma\left(\frac{d}{m}\right).$$

(3.4)

Now, the proof follows from (3.3) and (3.4). □

**Remark 3.3.** One can also use the above argument to compute $GW_1^{\text{loc}}(F_2, d)$, replacing the “Taubes type” argument used in [LP]. Specifically, for each $f \in \mathcal{M}_2^d(S_2, d[D_2])$ the linearized operator $L_f$ is invertible with $SF(L_f) \equiv h^0(f^*N_2) \pmod{2}$ (cf. Proposition 9.2 of [LP]). Thus, by (3.1) we have

$$GW_1^{\text{loc}}(F_2, d) = [\mathcal{M}_2^d(S_2, d[D_2])]^{\text{vir}} = \frac{1}{d} \left(\sigma(d) - \sigma\left(\frac{d}{2}\right)\right) - \frac{1}{d} \sigma\left(\frac{d}{2}\right) = \frac{1}{d} \left(\sigma(d) - 2\sigma\left(\frac{d}{2}\right)\right).$$

**Remark 3.4.** Ionel and Parker [IP] showed how GW invariants for the class $A$ of a symplectic 4-manifold $X$ are related with the Taubes’ Gromov invariants $Gr_X(A)$ [T] that count embedded (not necessarily connected) curves in $X$ representing the class $A$. They used a particular function $F(t)$ that satisfies

$$\prod_d F\left(t^d\right)^{-\frac{1}{2}\sigma(d)} = (1 - t)$$

to relate Taubes’ counting of multiple covers of embedded tori with the dimension zero genus one GW invariants. Let $X$ be a minimal elliptic surface with $p_g > 0$. In this case, any GW invariant constrained to pass through generic points vanishes (cf. Corollary 3.4 of [LP]). So, by (0.2), (0.3) and the Main Theorem, the relation between two set of invariants (Theorem 4.5 of [IP]) yields

$$\sum_A Gr_X(A) t_A = \prod_{d,k} F\left(t^d\right) c_F GW_1^{\text{loc}}(F,d) F\left(t^{d_k}\right)GW_1^{\text{loc}}(F_{m_k},d)$$

$$= (1 - t)^{c_F} \prod_k \left(1 + t_{m_k} + \cdots + t_{m_k}^{m_k-1}\right).$$

This also gives the well-known Seiberg-Witten invariants $SW$ of $X$ (cf. [FM], [B], [FS]) due to the famous Taubes’ theorem $SW = Gr$.

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