Tree Amplitudes in Gauge and Gravity Theories

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Abstract

Gauge theory amplitudes in a non-helicity format are generated at all $n$-point and at tree level. These amplitudes inherit structure from $\phi^3$ classical scattering, and the string inspired formalism is used to find the tensor algebra. All of the classical gravity amplitudes are also given. The classical effective action can also be constructed. Generalizations to amplitudes with non spin-1 or 2 is possible.
Introduction

The tree amplitudes of gauge theories have recently been under much scrutiny, in view of the simplified derivation using the weak-weak duality of the gauge theory with a twistor formulation. The string inspired formulation of perturbative amplitudes, in addition to the techniques based on factorization and unitarity, have prompted further interest in their computation. Helicity tree amplitudes up to 10 point are presented in the literature, due to these methods.

The gauge and gravity tree amplitudes are presented here at \( n \)-point. String inspired methods and scalar scattering are required to find their form. The form can be used to find the full classical effective action in gauge theory and in gravity, in practical applications such as jet physics, and in further studies of duality. An automation of the \( n \)-point tree amplitudes in gauge theory is useful for Monte Carlo simulations.

The massless \( \phi^3 \) diagrams and their specification are required in order to find the gauge theory and gravity theory tree-level amplitudes. The string inspired representation of the latter utilizing the Koba-Nielsen amplitude, and its field theory limit, require the momentum routing of the propagators in a diagram and its correlated tensor algebra. The form of the \( \phi^3 \) classical scattering presented in [1] has the required momentum parameterization. The tensor algebra is computed with the scalar form.

The gauge and gravity amplitudes are obtained in a no helicity basis, but in closed form via the known string-inspired tree-level rules. Spinor helicity is used in conjunction with the gauge invariance of the amplitudes to shorten the expressions. The choice of reference momenta in the gauge invariant sets of diagrams that generates the simplest amplitude expression is not known in the literature; this algebraic question discussed in [2], and its answer, is relevant for both formal and applied uses. Different forms of the amplitudes lend to different interpretations and make manifest different properties, such as a twistor representation, a self-dual field form including a WZW model, or a potential iterative number basis form as in [2]. The compact nature is important for practical applications.

These classical amplitudes, both in scalar and gauge field theories, are required to bootstrap to higher orders. The derivative expansion has been developed in [3]-[11] and the classical amplitudes are the initial conditions.

Gauge Amplitudes
The general scalar $\phi^3$ momentum routing of the propagators is presented. Their form is required with the string scattering expression and the string inspired formulation to find the amplitudes.

A general scalar field theory diagram at tree-level is parameterized by the set of propagators at the momenta labeling them. In a color ordered form, consider the ordering of the legs as in $(1, 2, \ldots, n)$. The graphs are labeled by

$$D_{\sigma} = g^{n-2} \prod \frac{1}{t_{\sigma(i,p)} - m^2},$$

and the Lorentz invariants $t_{\sigma(i,p)}$ are defined by $t_i^{[p]}$,

$$t_i^{[p]} = (k_i + \ldots + k_{i+p-1})^2.$$  

Factors of $i$ in the propagator and vertices are placed into the prefactor of the amplitude. The sets of permutations $\sigma$ are what are required in order to specify the individual diagrams. The full sets of $\sigma(i, p)$ form all of the diagrams, at any $n$-point order.

The expansions in mass of a massive $\phi^3$ diagram follows from expanding the propagators,

$$\mathcal{A}_{\sigma, \tilde{\sigma}}^n = \sum_{\sigma, \tilde{\sigma}} C_{\sigma \tilde{\sigma}} g^{n-2} \prod \frac{t^{\tilde{\sigma}(i,p)}}{m^{2\tilde{\sigma}(i,p)}},$$

with the coefficient $C_{\sigma}$ determined from the momentum routing of the tree graphs; the $C$ coefficients take on non-empty values when there is a diagram. An additional set of permutations $\tilde{\sigma}$ is required in order to specify the expansion of the propagators as in $(m^2 - p^2)^{-1} = m^{-2} \sum (p^2/m^2)^l$.

The massless diagrams are,

$$\mathcal{A}_{\sigma}^n = \sum_{\sigma} C_{\sigma} g^{n-2} \prod t_{\sigma(i,p)}^{-1},$$

with the $C_{\sigma}$ spanning the set of all $(i, p)$ values at a given $n$-point. The form and permutation set of $C_{\sigma}$ and $C_{\sigma \tilde{\sigma}}$ is given in [1].
The vertices of the ordered $\phi^3$ diagram are labeled so that the outer numbers from a two-particle tree are carried into the tree diagram in a manner so that $j > i$ is always chosen from these two numbers. The numbers are carried in from the $n$ most external lines.

The labeling of the vertices is such that in a current or on-shell diagram the unordered set of numbers are sufficient to reconstruct the current; the set of numbers on the vertices are collected in a set $\phi_m(j)$. For an $m$-point current there are $m-1$ vertices and hence $m-1$ numbers contained in $\phi_m(j)$. These $m-1$ numbers are such that the greatest number may occur $m-1$ times, and must occur at least once, the next largest number may occur at most $m-2$ times (and may or may not appear in the set, as well as the subsequent ones), and so on. The smallest number can not occur in the set contained in $\phi_m(j)$. Amplitudes are treated in the same manner as currents. Examples and a more thorough analysis is presented in \[1\].

Two example permutation sets pertaining to 4- and 5-point currents are:

\[
\begin{pmatrix}
444 \\
443 \\
442 \\
433 \\
432
\end{pmatrix}
\]  

\[
\begin{pmatrix}
5555 \\
5554 \\
5553 \\
5552 \\
5544 \\
5543 \\
5542 \\
5(3)
\end{pmatrix}
\]

with the 5(3) representing the (3)-permutation set attached to the 5 in the total count. There are 5 and 15 in the counts. The set of numbers in $\phi(j)$ is ordered from largest to least.

The numbers $\kappa(i)$ and $\phi(j)$ are used to find the propagators in the labeled diagram. The procedure to determine the set of $l_i^{[p]}$, or the $\sigma(i,p)$, is as follows. First, label all momenta as $l_i = k_i$. Then, the invariants are found with the procedure,
1) \( i = \phi(m - 1), \quad p = 2, \) then \( l_{m-1} + l_m \to l_{m-1} \)

2) \( i = \phi(m - 2), \quad p = 2, \) then \( l_{m-2} + l_{m-1} \to l_{m-2} \)

\[ \ldots \]

\(-1) \quad i = 1, \quad p = m \]

The labeling of the kinematics, i.e. \( t_i^{[q]} \), is direct from the definition of the vertices. \hfill (7)

The numbers \( \phi_n(i) \) can be arranged into the numbers \( (p_i, [p_i]) \), in which \( p_i \) is the repetition of the value of \([p_i]\). Also, if the number \( p_i \) equals zero, then \([p_i]\) is not present in \( \phi_n \). These numbers can be used to obtain the \( t_i^{[q]} \) invariants without intermediate steps with the momenta. The branch rules are recognizable as, for a single \( t_i^{[q]} \),

0) \( l_{\text{initial}} = [p_m] - 1 \)

1)

\[ \begin{align*}
& r = 1 \text{ to } r = p_m \\
& \text{if } r + \sum_{j=1}^{m-1} p_j = [p_m] - l_{\text{initial}} \text{ then } i = [p_m] \quad q = [p_m] - l_{\text{initial}} + 1 \\
& \text{beginning conditions has no sum in } p_j \\
& 2) \quad \text{else } l_{\text{initial}} \to l_{\text{initial}} - 1 : \text{ decrement the line number} \\
& \quad l_{\text{initial}} > [p_i] \text{ else } l \to l - 1 : \text{ decrement the } p \text{ sum} \\
& 3) \text{ goto 1)}
\] \hfill (8)

The branch rule has to be iterated to obtain all of the poles. This rule checks the number of vertices and matches to compare if there is a tree on it in a clockwise manner. If not, then the external line number \( l_{\text{initial}} \) is changed to \( l_{\text{initial}} \) and the tree is checked again. The \( i \) and \( q \) are labels to \( t_i^{[q]} \).

The previous recipe pertains to currents and also on-shell amplitudes. There are \( m - 1 \) poles in an \( m \)-point current \( J_\mu \) or \( m - 3 \) in an \( m \)-point amplitude. The comparison between amplitudes and currents is as follows: the three-point vertex
is attached to the current (in $\phi^3$ theory), and then the counting is clear when the attached vertex has two external lines with numbers less than the smallest external line number of the current (permutations to other sets of $\phi_n$ does not change the formalism). There are $n - 3$ poles are accounted for in the amplitude with $\phi_n$ and the branch rules.

**Gauge Amplitudes: Tensor Algebra**

The gauge theory amplitudes are computed with the string inspired formulation. The amplitudes are projected onto a color basis, so that only the amplitude with leg ordering $1, \ldots, n$ is analyzed. The color prefactor is $\text{Tr} T_{a_1} \cdots T_{a_n}$; permutations are used to obtain different orderings. The momentum routing of the propagators, $C_\sigma$ is required in order to specify the combinations of products $\epsilon_i \cdot k_j$ and $\epsilon_i \cdot \epsilon_j$ appearing in a $\phi^3$ diagram. The string-inspired formalism based on pinching the Koba-Nielsen formula generates a symmetric set of graph rules [13] which is well adapted to the general expressions of scalar tree amplitudes.

The multi-linear string scattering expression is,

$$
\prod_{i \neq j} \exp \left( k_i \cdot \varepsilon_j \hat{G}_B(i, j) - \frac{1}{2} \varepsilon_i \cdot \varepsilon_j \hat{G}_B(i, j) \right),
$$

in which the exponential is expanded to contain products of the polarization variables (i.e. multi-linear). The world-sheet propagator term is

$$
\prod_{i \neq j} \exp \frac{1}{2} k_i \cdot k_j G_B,
$$

which is useful in the field limit when the integration by parts is carried out on the $\hat{G}_B$.

The expansion of the polarizations in (9) generate the terms,

$$
\prod_{j=1}^n \left( \sum_{i \neq j} k_j \cdot \varepsilon_i \hat{G}_B(i, j) \right)
$$

and

$$
\sum_\rho \prod_m \left( -\frac{1}{2} \right) \varepsilon_{\rho(m,1;i)} \cdot \varepsilon_{\rho(m,2;i)} \hat{G}_B(\rho(m,1;i), \rho(m,2;i))
$$
\[ \times \prod_{i \neq \rho(m, 1; i), \rho(m, 2; 1)} \left( \sum_{j \neq i} k_j \cdot \varepsilon_i \hat{G}_B(i, j) \right), \]  

which contains all possible products of the polarization vectors.

The integration by parts on all of the \( \dot{G}_B(i, j) \) terms using the factor in (10) generates the string-inspired form,

\[
\sum_p \prod_m \left(-\frac{1}{2}\right) \varepsilon_{\rho(m, 1; a)} \cdot \varepsilon_{\rho(m, 2; a)} \dot{G}_B(\rho(m, 1; a), \rho(m, 2; a))
\]

\[
\left( \frac{1}{2} \times \sum_{c \neq b, d \neq \rho(m, 2; a)} k_b \cdot k_c \hat{G}_B(b, c) \right) \prod_{i \neq \rho(m, 1; i), \rho(m, 2; 1)} \left( \sum_{j \neq i} k_j \cdot \varepsilon_i \hat{G}_B(i, j) \right). \]

The products in (15) are used in conjunction with the individual \( \phi^3 \) diagrams to determine the gauge theory amplitudes. The string-inspired rules, found in the field theory limit of the KN tree formula, are applied to the formula in (15).

The permutation set \( \rho(m, 1; i) \) and \( \rho(m, 2; i) \) are derived via picking \( p \) numbers from the collection 1, \ldots, \( n \) for each of the sets. These \( p \) numbers are non-overlapping in both \( \rho(m, 1; ) \) and \( \rho(m, 2; ) \) as there is no duplication of the \( \varepsilon \) vectors. All possible combinations are required; there are \( 2p! \) permutations of the sets’ elements, and \( n! / p!(n - p)! \) choices of the \( p \) elements from the total number of \( n \) polarizations. The overcounting is \( 2^p \) as each of the pairs of polarizations is unordered.

The string inspired rules require that, from the expansion in (15), there are a certain number of \( \dot{G}_B \) terms appearing in coordination with the labeling of the color ordered \( \phi^3 \) diagram. The presence of the \( \dot{G}_B \) terms is accompanied by the kinematic factors multiplying them, i.e. the tensor algebra.

The vertices of the \( \phi^3 \) are numbered as in the previous section. The labeling of the numbers in the vertices is correlated with the set of numbers contained in \( \sigma(i, p) \); physically, the specification of the multi-particle poles generates the labeling. Each vertex number in a tree is labeled by taking the clockwise number in the two outer nodes \( i \) and \( j \) of the same tree. The numbers are found by starting with the outermost external leg numbers, which range from 1 to \( n \) in a cyclic fashion for the color ordering 1, \ldots, \( n \). The numbers are such that \( i > j \), with \( i \) and \( j \) the two outer points on the two-particle tree, is always chosen from the choice of the two numbers.

The vertices are associated with \( \dot{G}_B(i, j) \) worldsheet bosonic Greens functions. Each vertex requires one of these \( \dot{G}_B \) with the two indices as: one of them labeled by
Figure 1: The ordering and labeling of a sample $\phi^3$ diagram.
the node, the other with one of the numbers so as to be in the outer tree. The choice
of the latter to encompass the outer tree generates a ‘primary’ choice of the \( \hat{G}_B \). Each
vertex is associated with precisely one \( \hat{G}_B \) factor; these factors are set equal to unity
and the kinematics associated with the combination generates the tensor algebra.

The set of numbers \( \phi_n(i) \) discussed in the previous section gives a route to finding
the poles \( t_i^{[p]} \). These numbers, as discussed in [1], are representative of a discrete
symmetry in the classical scattering. The same numbers are used to find the tensor
algebra on the scalar graphs of the gauge theory.

The primary set of vertex labels and their \( \hat{G}_B \) factors are obtained from the
momentum routing of the scalar diagram given in [1]. The collection of indices in \( \sigma(i,p) \), which label \( t_i^{[p]} \), generate the \( \hat{G}_B \) factors with the indices \( i, i + p - 1 \). Denote
this set of numbers as \( \kappa(a, b) \) with \( a \) and \( b \) numbered by the propagator indices \( i + p - 1 \)
and \( i \). Beyond the primary set of indices, the remaining sets of indices have numbers
with \( i \) and \( i + 1, \ldots, i + p - 1 \), or \( a \) to \( a + 1, \ldots, b \). The numbers \( i + p - 1 \) are identical
to the \( n - 2 \) vertex numbers in \( \phi_n(i) \) (there are \( n - 2 \) vertices and \( n - 3 \) propagators
and an overcount is on the last of the largest number in \( \phi_n \)). The two sets of numbers
are placed in the correlated \( \kappa_1(i) \) and \( \kappa_2(i) \)

There are two more entries in \( \kappa(a, i) \) beyond the \( n - 2 \) vertices; these entries are
orthogonal numbers to the set contained in \( \phi_n(i) \). The two entries in \( \kappa(i, i) \) beyond
the \( n - 2 \) primary and descendent numbers (\( i \) to \( i + 1, \ldots, i + p - 1 \)) and comprise an
orthogonal set of \( \hat{G}_B(i, j) \).

The vertex labels are used to extract \( n - 2 \) non-identical \( \hat{G}_B(i, j) \) factors. These
factors are pulled from the kinematic expression in (15). The remaining factors must
have non-overlapping \( \hat{G}_B(i, j) \); all of the bosonic Greens functions are then set to 1
or \(-1\) for \( i > j \) or \( j < i \).

The \( \kappa(a; 1) \) and \( \kappa(b; 2) \) set of primary numbers used on (15) produces a contribution,

\[
(-\frac{1}{2})^{a_1}(\frac{1}{2})^{n-a_2}\prod_{i=1}^{a_1} \varepsilon(\kappa(i; 1)) \cdot \varepsilon(\kappa(i; 1)) \times \prod_{j=a_1+1}^{a_2} \varepsilon(\kappa(j; 1)) \cdot k_{\kappa(j; 2)} \times \prod_{p=a_2+1}^{n} k_{\kappa(p; 1)} \cdot k_{\kappa(p; 2)}, \tag{16}
\]

together with the permutations of \( 1, \ldots, n \). The permutations extract all possible
combinations from the (15), after distributing the numbers into the three categories.

The form of the amplitudes are expressed as,

\[
\mathcal{A}_g^n = \sum_{\sigma} C_{\sigma} g^{n-2} T_\sigma \prod t_{\sigma(i,p)}^{-1}, \tag{17}
\]

9
with $T_\sigma$ in (16) derived from the tensor set of $\kappa$, e.g. found from $\phi_n$ or the momentum routing of the propagators with $\sigma(i,p)$. The normalization is $i(-1)^n$. The numbers $a_1$ and $a_2$ are summed so that $a_1$ ranges from 1 to $n/2$, with the boundary condition $a_2 \geq a_1 + 1$. Tree amplitudes in gauge theory must possess at least one $\varepsilon_i \cdot \varepsilon_j$.

All $\phi^3$ diagrams are summed at $n$-point, which is represented by the sum in $\sigma$ in (17). The color structure is $\text{Tr} (T_{a_1} \ldots T_{a_n})$, and the complete amplitude involves summing the permutations of $1, \ldots, n$.

The first $n-2$ numbers in $\kappa_2$ are summed beyond those of the primary numbers in accord with the set $i$ to $i + p - 1$ for a given vertex label $i + p - 1$, which labels the vertex in $\phi_n$.

**Gravity Amplitudes**

Graviton scattering is straightforward given the gauge theory results. The holomorphic gauge theory string derivation is squared, i.e. the tensor algebra must include an identical non-holomorphic piece. The multi-linear string scattering expression is,

$$\prod_{i \neq j} \left| \exp \left( k_i \cdot \varepsilon_j \hat{G}_B(i,j) - \frac{1}{2} \varepsilon_i \cdot \varepsilon_j \hat{G}_B(i,j) \right) \right|^2 ,$$

(18)

and contains the holomorphic square of the function in (9).

The world-sheet propagator term is squared

$$\prod_{i \neq j} \exp \frac{1}{2} k_i \cdot k_j (G_B + \hat{G}_B) .$$

(19)

The integration by parts on all of the $\hat{G}_B(i,j)$ terms using the factor in (10) includes the product of the non-holomorphic half of the string-inspired form, i.e. the barred piece of (15).

The gravitational amplitudes are, via the holomorphic splitting,

$$A^n_\sigma = \sum_\sigma C_\sigma g^{n-2} T_\sigma \hat{T}_\sigma \prod t_{\sigma(i,p)}^{-1} ,$$

(20)

with a holomorphically squared $T_\sigma$, the same as in gauge theory,

$$T_\sigma = \sum_{\mu,\nu,\gamma} \hat{T}_{\mu\nu\gamma} \prod_{i,k_{\mu(i,1)} \cdot \varepsilon_{\mu(i,2)} \prod_{j,\varepsilon_{\nu(j,1)} \cdot \varepsilon_{\nu(j,1)} \prod_{s,k_{\gamma(s,1)} \cdot k_{\gamma(s,2)} .}$$

(21)
This form is the complete gravitational S-matrix, after summing the orderings of the external legs.

Both the gauge and gravitational scatterings may be gauge covariantized using the field strengths $F_{\mu\nu}$ and $R_{\mu\nu}$, to write the classical effective action. The classical effective action is relevant to the DBI work, soliton effects including black hole dynamics, and the anti-de Sitter correspondence with gauge theory.

**Concluding remarks**

Gauge and gravity amplitudes are found at tree level and with any number of legs. The analogous scalar field theory amplitudes have appeared in [1]. The amplitudes are generated without specifying the helicity content. Two sets of numbers are required to delimit the contributions, $\phi_n$ of the vertices and $t_i^{[p]}$ of the poles, and they are equivalent.

Gravity interactions, by varying $\int d^d x R \sqrt{g}$, possess an infinite number of vertices and makes the calculation of an all $n$-point formula tedious. The amplitudes are found by utilizing the KLT factorization of string tree scattering.

These gauge and gravity amplitudes use the number parameterization of $\phi^3$ diagrams. The latter have 'symmetries' accorded to them via the collection of numbers $\phi_n(j)$ used to construct the individual graphs (see [1]). These sets may also be used to classically quantize gauge and gravity, which should generalize to the quantum level.

The string form of the scattering amplitudes together with the scalar field results should allow for a generalization of the all $n$ amplitudes to contain fermionic modes, $(p, q)$ tensor modes, mixed spins in the asymptotic states, and including the electroweak sector.

These amplitudes are required in order to bootstrap to the quantum level. Also, a closed form of the classical amplitudes is useful for Monte Carlo simulations for particle beam simulations.
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