Hexagonal Warping Induced Nonlinear Planar Nernst Effect in Nonmagnetic Topological Insulators

Xiao-Qin Yu, Zhen-Gang Zhu, and Gang Su

1 School of Physics and Electronics, Hunan University, Changsha 410082, China.
2 School of Electronic, Electrical and Communication Engineering, University of Chinese Academy of Sciences, Beijing 100049, China.
3 Kavli Institute of Theoretical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China.
4 CAS Center for Excellence in Topological Quantum Computation, University of Chinese Academy of Sciences, Beijing 100190, China.
5 Theoretical Condensed Matter Physics and Computational Materials Physics Laboratory, College of Physical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China.

We propose theoretically a new effect, i.e. nonlinear planar Nernst effect (NPNE), in nonmagnetic topological insulator (TI) Bi$_2$Te$_3$ in the presence of an in-plane magnetic field. We find that the Nernst current scales quadratically with temperature gradient but linearly with magnetic field and exhibits a cosine dependence of the orientation of the magnetic field with respect to the direction of the temperature gradient. The NPNE has a quantum origin arising from the conversion of a nonlinear transverse spin current to a charge current due to a joint result of hexagonal warping effect, spin-momentum locking, and the time-reversal symmetry breaking induced by the magnetic field.

I. INTRODUCTION

The three-dimensional (3D) topological insulators (TI) represent a new class of 3D materials, owning an insulating bulk and conductive surface states. The surface Dirac electrons have their spin locked perpendicularly to their momenta, namely, spin-momentum locking, giving rise to highly efficient spin-to-charge conversion and magnetic switching and great potential application in spintronics and quantum computation.

Owing to the spin-momentum-locked surface states, a series of novel magneto-transport properties are identified in nonmagnetic TI film or bilayer structures composed of a ferromagnetic layer and a nonmagnetic TI layer, including both novel linear and nonlinear magnetoelectric effects, such as the non-saturating linear magnetoresistance, the anisotropic magnetoresistance, negative longitudinal magnetoresistance, bilinear magnetoresistance, unidirectional magnetoresistance, planar Hall effect, and nonlinear planar Hall effect, etc. The nonlinear planar Hall effect has recently been observed in nonmagnetic TI Bi$_2$Se$_3$, which describes the Hall resistance linear dependence on both the applied electric field and in-plane magnetic field and is shown to originate from concerted actions of spin-momentum locking and time-reversal symmetry breaking.

Unlike the extensive exploration on the magnetoelectric transport in TIs, only few works have recently focused on the magnetothermal transport. Unidirectional Seebeck effect, an nonlinear magnetothermal effect, owing to the asymmetry magnon scattering was discovered in magnetic TIs, which describes the thermoelectric voltage from Seebeck effect depending on the relative orientations of in-plane magnetization with respect to the temperature gradient.

In this paper, we report another type of nonlinear magnetothermal effect: nonlinear planar Nernst effect (NPNE) in a 3D nonmagnetic TI, i.e. Bi$_2$Te$_3$, in which the Nernst current is quadratically proportional to temperature gradient and linearly proportional to the in-plane magnetic field. NPNE manifests itself when the applied temperature gradient, magnetic field, and the induced transverse voltage are all coplanar, where the conventional Nernst effect vanishes. Unlike the recently reported topological nonlinear anomalous Nernst effect in strained MoS$_2$ and in bilayer WTe$_2$ that origins from Berry curvature in the absence of magnetic field, this nonlinear planar Nernst effect in nonmagnetic TIs is found to originate from the generation of a transverse nonlinear spin current via second-order response to temperature gradient, which can be converted into a transverse nonlinear planar Nernst current via in-plane magnetic field collinear with a temperature gradient in the presence of hexagonal warping effect of 2D Fermi contour. We believe that the pro-
posed effect is very useful in magnetotransport and spin caloritronics, which is an extension and combination of spintronics and the conventional thermoelectrics, investigating the interplay between a temperature gradient, spin and charge degrees of freedom and aiming at increasing the efficiency and versatility of spin-involved thermoelectric devices.

The paper is organized as follows. We derive the formula of the transverse nonlinear spin current \( j_y \) driven by a temperature gradient \( \nabla_x T \) up to the second order based on the Boltzmann theory in Sec. II. The expression of NPNE for TI is derived and determined in Sec. III. The behavior of NPNE is discussed in Sec. IV. Finally, we give a conclusion in Sec. V.

II. NONLINEAR SPIN NERNST CURRENT IN TOPOLOGICAL INSULATOR

With the relaxation time approximation, the Boltzmann equation for the distribution of electrons in the absence of electric field can be written as

\[
f - f_0 = -\tau \frac{\partial f}{\partial r_a} \cdot v_a. \tag{1}
\]

where \( \tau \) denotes the relaxation time, and \( r_a \) and \( v_a \) represent the \( a \) component of coordinate position and velocity of electrons, respectively. \( f_0 = 1/\left( \exp \left[ \frac{\epsilon(k) - E_f}{k_B T} \right] + 1 \right) \) is the equilibrium Fermi distribution, where \( \epsilon(k) \) is energy dispersion, \( E_f \) indicates the Fermi energy and \( k_B \) represents Boltzmann constant. The nonequilibrium distribution function response to the second order in temperature gradient can be expanded as \( f \approx f_0 + \delta f_1 + \delta f_2 \) with the term \( \delta f_a \) vanishing as \( (\partial T/\partial r_a)^n \). After detail derivation (see Appendix D), the formulas of \( \delta f_1 \) and \( \delta f_2 \) can be determined by Eq. (A11).

In the absence of a magnetic field \( B \), the effective Hamiltonian for the surface state of topological insulator is \( H_0(k) = E_0(k) + \sigma \cdot h(k) \),

\[
h(k) = v_F \hbar k \times z + \lambda k \times y \left( k_x^2 - 3k_y^2 \right), \tag{3}
\]

where \( \hbar \) is the Plank constant, \( v_F \) denotes the Fermi velocity, \( \sigma \) indicates the Pauli matrices for the two basis functions of the energy bands, and \( \lambda \) represents the energy warping parameter. The spin independent term \( E_0(k) = \hbar^2 k^2 / 2m^* \) generates the particle-hole asymmetry. Unlike the contribution to the nonlinear Hall planar effect, the signal of nonlinear planar Nernst effect arising from the particle-hole asymmetry is insignificant (the details can be found in Appendix D2). For simplicity and to emphasize the hexagonal warping effect, we will neglect the particle-hole asymmetry \( E_0(k) \) in main text. The second term is the hexagonal warping term which is invariant under threefold rotation \( C_{3v} \). \( H_0(k) \) is invariant under the following two operators: 1) mirror reflection \( M_z \) about the \( yz \) plane, and 2) threefold rotation \( C_3 \) about the \( z \)-axis. The energy eigenvalues

\[
c_n^0(k) = n \sqrt{(v_F \hbar k)^2 + \lambda^2 k^6 \cos^2 3\phi_k}, \tag{4}
\]

where \( c_n^0(k) \) denotes the energy dispersion of upper (lower) surface bands, respectively, and \( \phi_k \) is the azimuthal angle of wavevector \( k \) with respect to the \( k_y \)-axis. In the absence of a magnetic field, the time-reversal symmetry is guaranteed, which requires that the energy dispersion respects \( c_n^0(k) = c_n^0(-k) \) and the mirror symme-

![FIG. 1](image-url)
try $M_x$ imposes the constraint $\epsilon_0^1(k_x, k_y) = \epsilon_0^1(-k_x, k_y)$. Both constraints on the energy dispersion also imply the relation $\epsilon_n^0(k_x, k_y) = \epsilon_n^0(k_x, -k_y)$. In the following, the upper surface band, namely, $n = 1$ will be considered and $\epsilon_n^0(k)$ is written as $\epsilon_1^0$ for simplicity. The lower surface bands can be analysed in the similar way.

The spin current $j^{s,b}_{a,b}$ in $a$-direction with spin pointing to the $b$-direction is given by

$$ j^{s,b}_{a,b} = \frac{\hbar}{2} \int \frac{dk}{(2\pi)^2} \langle \sigma^b \rangle v_a(k) f(k), \quad (5) $$

where $\int dk$ is shorthand for $\int dk/(2\pi)^2$, the average $\langle \cdots \rangle$ is carried out over the surface state of the upper (lower) band and can be replaced by $\langle \sigma^b \rangle = n \hbar b(k)/\hbar$ with $h(k)$ defined by Eq. (3).

In the absence of a magnetic field, the time reversal symmetry guarantees that the energy dispersion is even in $k$, i.e., $\epsilon_0(k) = \epsilon_0(-k)$, which hints that the nonequilibrium electron distribution $\delta f_1 \sim (\epsilon_\mathbf{k} - E_f)\partial f_0/\partial \epsilon_\mathbf{k}(T)$ [Eq. (11)] in the first order of temperature gradient $\partial T$ is odd in $k$, i.e., $\delta f_1(-k) = -\delta f_1(k)$, as shown in Fig. (a). In other words, if the nonequilibrium surface states in $(\mathbf{k}, \sigma)$ excises/deplete due to the first-order variation of temperature gradient, then, the surface states with opposite momentum and spin will deplete/excess, which has no contribution to the spin Nernst current.

On the contrary, the second-order nonequilibrium electron distribution $\delta f_2(k)$ is even in $k$. Hence, the nonequilibrium surface states response to the second order of temperature gradient with opposite momentum and opposite spins (due to the spin-momentum locking) are equally populated as shown in Fig. (b), which leads to a nonzero nonlinear spin current $j^{s,n}_{s,b}$ with spin orientation in $a_\perp$ direction due to the spin-momentum locking, namely, the spins of topological surface states are locked perpendicular to their momenta. Therefore, when applying the temperature gradient in $x$-direction, only nonlinear spin Nernst current $j^{s,n}_{y}$ (where the subscript “nl” and superscript “$n$” refer to nonlinear and spin, respectively) with spin pointing to $x$-direction gives rise to a transverse spin current in $y$-direction and is found to be

$$ j^{s,n}_{y} = \frac{\hbar^2}{2} \int \frac{dk}{(2\pi)^2} \frac{(\alpha h k_y)}{\epsilon_0^0} \left[ \frac{\epsilon_0^0 - \mu}{T} v_x \frac{\partial f_0}{\partial k_x} + \left( \frac{\epsilon_0^0 - \mu}{h T^2} \right) v_y \left( \frac{\partial^2 f_0}{\partial k_y^2} \right)(\partial_x T)^2 \right], \quad (6) $$

where $y/x$ in $j^{s,n}_{y}$ indicates the movement direction of carrier/spin orientation, respectively. This nonlinear spin Nernst current originated from the topological surface states could be a source of spin injection and spin current generation in future applications of spin caloritronics.

A set of constant energy contours of $H_0(k)$ are obtained, as plotted in Fig. (c), where we have taken $\lambda = 250 \text{ eV}^3$ and $v_F h = 2.25 \text{ eV for Bi}_2\text{Te}_3$. When the Fermi energy gets close to the Dirac point ($E = 0 \text{ eV}$), the Fermi surface manifests itself as a circle and the warping effect is inapparent. The Fermi surface starts to deviate considerably from a circle and becomes more hexagonal-like around $E = 0.2 \text{ eV}$.

Figs. (a) and (b) illustrate the dependence of nonlinear spin Nernst current $(\text{NSNC})$ $j^{s,n}_{y}$ on the Fermi energy and the hexagonal warping effect. A larger $j^{s,n}_{y}$ can be generated by increasing the hexagonal warping parameter and the absolute value of the Fermi energy in which the hexagonal warping effect will be enhanced. An interesting finding is that in addition to the contribution of hexagonal warping term, the linear-$k$ Dirac dispersion ($\lambda = 0$) can also give rise to the signal of nonlinear spin Nernst current, which is distinguished from the electric-field-induced nonlinear spin Hall current. This can explain why the NSNC is nonzero when the energy is in the range of $[0 \text{ eV}, 0.2 \text{ eV}]$ [Fig. (a)], in which the trigonal warping effect is insignificant [Fig. (c)]. However, the signal of NSNC originated from the linear-$k$ Dirac dispersion cannot be converted into the nonlinear planar Nernst current when the Fermi energy is away from the...
III. NONLINEAR PLANAR NERNST EFFECT IN TOPOLOGICAL INSULATOR

In the absence of magnetic field, the carriers with opposite spins are equally populated and move in opposite directions in transverse direction (y-direction) [Fig. I(f)]. Hence, there is no charge current flux vertical to temperature gradient. However, when applying an in-plane magnetic field to the topological insulator, because of the spin-momentum locking, the Fermi surface will be distorted in the direction perpendicular to magnetic field (NPNC) [Fig. II(g)]. Hence, there is no charge current flux vertical to temperature gradient (NPNC) [Fig. II(g)].

It should be emphasized that the successful conversion from the spin current into NPNC is ensured by the hexagonal warping effect. If there is no hexagonal warping term, i.e., $\lambda = 0$, the energy dispersion will shift in the linear-$k$ Dirac dispersion and the Fermi surface returns to a circle instead of being distorted, as shown in Fig. III(c) and (d) due to the hexagonal warping term, which leads to the imbalance between the two spin fluxes from the linear dispersion when the Fermi energy is located near Dirac point within $10k_BT$. This weak signal can be attributed to the temperature broadening effect (see Appendix II for a detailed discussion).

In the presence of a magnetic field $\textbf{B}$, the effective Hamiltonian for the surface state of topological insulator Bi$_2$Te$_3$ is given by

$$H (\textbf{k}) = \sigma \cdot [\textbf{h} (\textbf{k}) + g \mu_B \textbf{B}],$$

(7)

where $g$ and $\mu_B$ represent the $g$-factor and Bohr magnetron, respectively. The energy eigenvalues are

$$\epsilon_n^M (\textbf{k}) = n |\textbf{h} (\textbf{k}) + g \mu_B \textbf{B}|,$$

(8)

In the following, we shall consider the upper surface bands, namely $n = 1$, and write $\epsilon_1^M (\textbf{k})$ as $\epsilon_1^M$ for simplicity. The lower surface bands can be analysed in the similar way.

The charge current $j_a$ in $a$-direction is $j_a = -e \int [\text{d}k] v_a f (r, k)$. After tedious derivation in Appendix III, the current $j_a^{(1)}$ and $j_a^{(2)}$ as the first-order and second-order responses to the temperature gradient in the first-order approximation of magnetic field are found, respectively, to be

$$j_a^{(1)} = \sum_b G_{ab} \partial_b T + \sum_{bc} K_{abc} \partial_b T \partial_c B_c,$$

$$j_a^{(2)} = \sum_{bc} W_{abc} \partial_b T \partial_c T + \sum_{bcd} Q_{abcd} \partial_b T \partial_c T \partial_d B_d,$$

(9)

where the relation $\partial \epsilon_1^M / \partial B_d = g \mu_B \partial \epsilon_1^M / \partial h_B$ has been applied. Explicit expressions for the linear current response ($G_{ab}$, $K_{abc}$) and nonlinear response function ($W_{abc}$, $Q_{abcd}$) are given in Eqs. (10) and (11).

Through exploiting the parity in Table II one can find the following tensor elements are zero, i.e.,

$$G_{xy} = G_{yx} = 0,$$

$$K_{abc} = 0, W_{abc} = 0, \ a, b, c = x, y$$

(10)

which suggest that when applying an in-plane magnetic field $\textbf{B} = B (\cos \theta, \sin \theta)$ and temperature gradient $\partial_x T$ along $x$-direction (i.e., $b = c = x$), the planar Nernst effect $j_y^{(1)}$ disappears in Bi$_2$Te$_3$ and has no contribution to the transverse thermal voltage signal. And the current density $j_y^{(2)}$ flowing along the $y$-direction (i.e., $d = y$) as the response to the second order in temperature gradient stems from the nonlinear planar Nernst current density $j_{yl}^{(2)}$ (where the subscript “nl” and superscript “p” denote nonlinear and planar, respectively) and is found to be

$$j_y^{(2)} = j_{yl}^{(2)} = \left(Q_{yxx} \cos \theta + Q_{yxy} \sin \theta \right) (\partial_x T)^2 B,$$

(11)

where the nonlinear planar coefficient $Q_{yxx}$ is given as

$$Q_{yxx} = - \frac{e^2 v_B}{\alpha T^2 h^2} \int [\text{d}k] \left[ \frac{\partial f_0}{\partial \epsilon_k^M} \hbar^2 \Upsilon_1 (\epsilon_0^M - \mu) \right]$$

$$\times \hbar \Upsilon_2 + \frac{3}{4} \frac{\partial^2 f_0}{\partial (\epsilon_k^M)^2} \hbar^2 \Upsilon_1 + (\epsilon_0^M - \mu)^2 \left[ \frac{\partial f_0}{\partial \epsilon_k^M} \Upsilon_3 + l \right]$$

$$+ \frac{\partial^2 f_0}{\partial (\epsilon_k^M)^2} \hbar \Upsilon_4 + \frac{3}{4} \frac{\partial^3 f_0}{\partial (\epsilon_k^M)^3} \hbar^2 \Upsilon_1 \right],$$

(12)

where the coefficients $\Upsilon_1, \Upsilon_2, \Upsilon_3$ and $\Upsilon_4$ are given in Eq. III.
It’s observed that a very weak signal appears near the Dirac point with a few $k_B T$, and $Q_{xxx}$ is almost zero when the energy is in the range of $[0, 0.2eV]$, as expected, since the trigonal warping effect is insignificant and the Fermi surface almost displays like a circle [Fig. 2(c)] in this range. The appearance of faint signal at the Dirac point can attributes to the thermal broadening effect of nonequilibrium Fermi distribution near Dirac point for the linear-$k$ Dirac dispersion [the details can be found in Appendix D3]. Besides, one might notice that the signal of $Q_{xxx}$ is still quite weak when the Fermi energy is in the range of $[0.2, 0.4]eV$, a regime where a warped Fermi surface is present [Fig. 2(c)]. This can be attributed to the low conversion efficiency from the nonlinear spin to charge current [Fig. 24]. However, when the absolute value of Fermi energy $|E_f|$ is increased sufficiently, the trigonal warping effect will become profound and lead to a large enhancement of nonlinear planar Nernst effect. It is interesting to point out that the impact of varying temperature is negligible [Fig. 25(a) (b)] when Fermi energy is away from Dirac point. Figure 25(c) and (d) present the Fermi energy and hexagonal warping dependence of $Q_{xxx}$. The magnitude of $Q_{xxx}$ increases monotonously with the enhanced energy warping parameter $\lambda$. As expected, when $\lambda$ tends to be zero, the nonlinear planar Nernst effect will disappear.

To numerically estimate the proposed effect, we take $Q_{xxx} \approx 0.8 nA \cdot \mu m / T K^2$ [Fig. 25(b)] for $T = 30K$ and $E_f = 0.5 eV$. In experiment, the temperature gradient can already reach $1 \text{K} \cdot \text{m}^{-1}$. Therefore, when applying the magnetic field $B = 3T$ parallel to temperature gradient, the nonlinear planar Nernst current $J_{nl}^p \times l$ [Eq. (11)] of Bi$_2$Te$_3$ is estimated to be order of $0.16 \mu A$ with the length of sample $l = 50 \mu m$, which is measurable.

A Rashba-split surface states in two-dimensional electron gas (2DEG) [26, 27] might coexists with topological surfaces states (TSS) due to the surface band bending in topological insulators, which may also have a significant contribution to the Planar Nernst effect. However, it is found that only when Fermi energy locates near the Lifshitz point within a few $k_B T$, a very weak signal (100 times smaller than the signal arising from TSS) can be generated [see the details in Appendix D3]. Therefore, the contribution of Rashba 2DEG to NPNE can be neglected.

V. CONCLUSION

In summary, we propose a new effect, i.e. the nonlinear planar Nernst effect (NPNE) in this work. It is found that a nonlinear spin-Nernst current, originated from the hexagonal warping effect and the nonequilibrium carrier distribution, flows transversely to temperature gradient direction and can be partially converted into the nonlinear-planar-Nernst current $J_{nl}^p$ when an in-plane magnetic field is applied to TI. The quantity of $J_{nl}^p$ is strongly dependent on the orientation of the magnetic vector. As shown in Fig. 1(f) and discussion in Sec. II, the spin orientation in the nonlinear spin Nernst current generated by temperature gradient $\nabla_x T$ is along $x$-direction. Therefore, only the $x$-component of magnetic field can lead to a transition between the two spin currents and induces the imbalance of two spin carriers [Fig. II(c)]. As a result the nonlinear spin Nernst current will be partially converted to nonlinear planar Nernst current [Figs. II(f) and (g)].

We use the following parameters for Bi$_2$Te$_3$: the Fermi velocity $v_F$ = 2.25 $eV\cdot \AA$, $g = 2$, and the scattering relaxation time $\tau \approx 5.864 \times 10^{-13}s$ is estimated by $\tau = \mu m / e$. The mobility of surface states in Bi$_2$Te$_3$ can range from $9 \times 10^3$ to $10^4$ cm$^2 V^{-1} s^{-1}$ [28] $\mu = 9000$ cm$^2 V^{-1} s^{-1}$ is used for an estimation.

FIG. 3. The nonlinear planar coefficient (NPC) $Q_{xxx}$ [(a) and (c)] as a function of Fermi energy for different energy warping parameter $\lambda$. (b) $Q_{xxx}(T)/Q^0_{xxx}$ versus temperature. $Q^0_{xxx}$ is the NPC for $T = 300K$. (d) $Q_{xxx}$ as a function of $\lambda$ at different Fermi energy. The energy warping parameter $\lambda$ is taken 250 eV$\cdot$Å$^3$ in (a) and (b). $T = 30K$ is fixed in (c) and (d). Parameters used: $v_F$ = 2.25$eV\cdot\AA$, $g = 2$ and $T = 38.64 \times 10^{-15}s$. Here, all parameters are taken from topological insulator Bi$_2$Te$_3$.

IV. RESULTS AND DISCUSSION

Eq. (11) indicates the nonlinear planar current $J_{nl}^p$ exhibits $\cos \theta$ dependence on the orientation of magnetic field and is proportional to the $x$-component of the magnetic field $B_x \propto B \cos \theta$. Thus, when the magnetic field is collinear with the temperature gradient (i.e., $\theta = 0, \pi, 2\pi$), the magnitude of $| J_{nl}^p |$ will reach its maximum. However, the nonlinear planar Nernst effect will disappear when the magnetic field $B$ is vertical to the temperature gradient. These features of the nonlinear planar Nernst current depending on the orientation of magnetic field can be ascribed to the spin-momentum locking. As shown in Fig. II(f) and discussion in Sec. II, the spin orientation in the nonlinear spin Nernst current generated by temperature gradient $\nabla_x T$ is along $x$-direction. Therefore, only the $x$-component of magnetic field can lead to a transition between the two spin currents and induces the imbalance of two spin carriers [Fig. II(c)]. As a result the nonlinear spin Nernst current will be partially converted to nonlinear planar Nernst current [Figs. II(f) and (g)].

We use the following parameters for Bi$_2$Te$_3$: the Fermi velocity $v_F$ = 2.25 $eV\cdot\AA$, $g = 2$, and the scattering relaxation time $\tau \approx 5.864 \times 10^{-13}s$ is estimated by $\tau = \mu m / e$. The mobility of surface states in Bi$_2$Te$_3$ can range from $9 \times 10^3$ to $10^4$ cm$^2 V^{-1} s^{-1}$ [28] $\mu = 9000$ cm$^2 V^{-1} s^{-1}$ is used for an estimation.
The magnitude of NPNE is strongly affected by the hexagonal warping term and the Fermi energy. Except for a faint signal of NPNE appearing near the Dirac point within a few $k_B T$ due to the temperature broadening effect, when the Fermi level is close to the Dirac point, the signal of the NPNE mostly disappears due to the weak hexagonal warping effect. Therefore, our findings have great potential application in magneto-thermal transport and spin caloritronics, and might pave a new way to the emerging field of nonlinear spin caloritronics.

This work is supported by the Fundamental Research Funds for the Central Universities and the NSFC (Grant No.12004107). G.S. and Z.G.Z. are supported in part by the National Key R&D Program of China (Grant No. 2018FYA0305800), the Strategic Priority Research Program of CAS (Grant Nos. XDB28000000), the NSFC (Grant No. 11834014), and Beijing Municipal Science and Technology Commission (Grant No. Z118100004218001). Z.G.Z. is also supported in part by the NSFC (Grant Nos. 11674317 and 11974348).

Appendix A: The non-equilibrium distribution function in the presence of temperature gradient

With the relaxation time approximation, the Boltzmann equation for the distribution of electrons in the absence of an electric field can be written as

$$\frac{\partial f}{\partial r_a} \cdot v_a + \frac{e}{\hbar} (\vec{v} \times \vec{B}) \cdot \frac{\partial f}{\partial k} = -\frac{f - f_0}{\tau}. \quad (A1)$$

In two-dimensional (2D) transport, the Lorentz force has no contribution to the electron dynamics for the in-plane magnetic field because of $(\vec{v} \times \vec{B}) \cdot \frac{\partial f}{\partial k} = 0$. Thus, in the presence of an in-plane magnetic field the Boltzmann equation in Eq. (A1) for 2D transport can be further simplified as

$$f - f_0 = -\tau \frac{\partial f}{\partial r_a} \cdot v_a. \quad (A2)$$

To the response up to the second order in temperature gradient $\nabla T$, the local distribution function $f(r, k)$ can be expanded as

$$f(k, r) = f_0(k, r) + A_0 \frac{\partial T}{\partial r_a} + B_{a\beta} \frac{\partial T}{\partial r_a} \frac{\partial T}{\partial r_\beta} + O[3]$$

$$\approx f_0(k, r) + \delta f_1(\partial_a T) + \delta f_2(\partial_a T \partial_b T), \quad (A3)$$

with

$$\begin{align*}
\delta f_1(\partial_a T) &= A_0 \partial_a T, \\
\delta f_2(\partial_a T \partial_b T) &= B_{a\beta} \partial_a T \partial_b T, \\
\partial_a &\rightarrow \frac{\partial}{\partial r_a}.
\end{align*} \quad (A4)$$

where $f_0(k, r)$ is the local equilibrium distribution, which is itself fixed by the temperature at $\tau \tilde{A}$, giving

$$\frac{\partial f_0}{\partial r_a} = \frac{\partial f_0}{\partial T} \frac{\partial T}{\partial r_a} = -\frac{(\epsilon_k - \mu_e) \delta f_0}{\tau} \frac{\partial T}{\tau}. \quad (A5)$$

Substituting the formula of $f$ in Eq. (A3) into Eq. (A2) and comparing the expansion coefficients in the first-order of $\partial_a T$, one obtains

$$\delta f_1(\partial_a T) = -\frac{\partial f_0}{\partial r_a} \cdot v_a + O[\partial_a T \partial_b T]. \quad (A6)$$

Thus, we can have

$$\delta f_1(\partial_a T) = -\tau \frac{\partial f_0}{\partial T} \cdot v_a. \quad (A7)$$

By iteration, then, we can have

$$\begin{align*}
\delta f_2(\partial_a T \partial_b T) &= -\tau \frac{\partial \delta f_1}{\partial r_a} \cdot v_a \\
\delta f_2(\partial_a T \partial_b T) &= \tau^2 \left( \frac{\partial^2 f_0}{\partial T^2} \partial_a T \partial_b T + \frac{\partial f_0}{\partial T} \partial_{ab} T \right) v_b v_a.
\end{align*} \quad (A8)$$

Here, we introduce a trick to transform $\frac{\partial f_0}{\partial T}$ into $\frac{\partial f_0}{\partial k}$ through a partial differential treatment,

$$\frac{\partial f_0}{\partial k} = \frac{\partial f_0}{\partial \epsilon_k} \frac{\partial \epsilon_k}{\partial k} = -\frac{\partial f_0}{\partial T} \hbar v T. \quad (A9)$$

In the above, we have used the relation: $\frac{\partial f_0}{\partial T} = \frac{\epsilon_k - \mu_e}{\tau} \frac{\partial f_0}{\partial \epsilon_k}$ and $\frac{\partial \epsilon_k}{\partial k} = \hbar v$.

From Eq. (A9), it is easily to obtain the following identities:

$$\begin{align*}
\frac{\partial^2 f_0}{\partial T^2} v_a v_b &= \frac{E_k - \mu_e}{\hbar T^2} \partial_k v_a v_b + \left( \frac{E_k - \mu_e}{\hbar T} \right)^2 \partial^2 f_0 \partial_k v_a v_b \\
\partial^2 f_0 &= \frac{E_k - \mu_e}{\hbar T} \partial_k v_a v_b + \left( \frac{E_k - \mu_e}{\hbar T} \right)^2 \partial^2 f_0 \partial_k v_a v_b.
\end{align*} \quad (A10)$$

Taking these identities into the formulas of $\delta f_1$ [Eq. (A7)] and $\delta f_2$ [Eq. (A8)] and assuming the uniform temperature gradient in the system, i.e., $\partial_a T = 0$, one obtains

$$\begin{align*}
\delta f_1 &= \frac{\tau}{T} (\epsilon_k - \mu_e) \frac{\partial f_0}{\partial k} \partial_k T \\
\delta f_2 &= \frac{\tau^2}{T^2 \hbar} \left( \hbar v_T \frac{\partial f_0}{\partial k} + (\epsilon_k - \mu_e) \frac{\partial^2 f_0}{\partial k^2} \right) \partial_k T.
\end{align*} \quad (A11)$$
Appendix B: The formula of nonlinear planar current for topological insulator

Based on Eq. [11], one can determine the charge current \( j_a = -e \int [dk] v_a f(r, k) \) in \( a \)-direction as the first-order and second-order responses to the temperature gradient, respectively, as

\[
j_a^{(1)} = -\tau e \int [dk] \frac{e k - \mu}{T h} v_a \frac{\partial f_0}{\partial k_b} \partial T,
\]

\[
j_a^{(2)} = -\tau^2 e \int [dk] \left[ \frac{e k - \mu}{T^2 h} v_a v_b \frac{\partial f_0}{\partial k_c} \right] \partial T \partial T + \left. \left( \frac{\epsilon_k - \mu}{T h} \right)^2 v_a \frac{\partial^2 f_0}{\partial k_b \partial k_c} \right] \partial T \partial T.
\]

(B1)

In presence of a magnetic field, the energy dispersion \( \epsilon_n^M(k) \) for nonmagnetic topological insulator Bi\(_2\)Te\(_3\) is given in Eq. [3]. We only consider the upper surface band, and write \( \epsilon_n^M(k) \) as \( \epsilon^M \). One can find \( \partial \epsilon^M / \partial B_d = g_u B \partial \epsilon^M / \partial h_d \), which hints

\[
\frac{\partial F(\epsilon^M)}{\partial B_d} = g_u \frac{\partial F(\epsilon^M)}{\partial h_d}.
\]

(B2)

To obtain Eqs. [14] and [15], we have used the relation

\[
\frac{\partial F(\epsilon^M)}{\partial h_d} = g_u B \frac{\partial F(\epsilon^M)}{\partial \epsilon^M} / \partial h_d
\]

with \( h_d = (d = x, y \text{ or } z) \). \( \epsilon^M = |h(k)| \) is the eigenvalues for the effective Hamiltonian \( H^{(0)} = -\sigma \cdot h(k) \). According to the formulas \( h(k) \) in Eq. [3], one can obtain

\[
\frac{\partial}{\partial h_x} = \frac{\partial}{\alpha h k_y}, \quad \frac{\partial}{\partial h_y} = \frac{\partial}{\alpha h k_x}.
\]

(B6)

When applying an in-plane magnetic field \( B = B(\cos \theta, \sin \theta) \) and temperature gradient \( \partial_x T \) along \( x \)-direction (i.e., \( b = c = x \)), the planar Nernst current density \( j_y^{(1)} \) in \( y \)-direction (i.e., \( d = y \)), as the response to the first order and the second order in temperature gradient, are found to be, respectively,

\[
j_y^{(1)} = [K_{yx} \partial_x T + (K_{yxx} \cos \theta + K_{yxy} \sin \theta) B] \partial_x T = 0,
\]

\[
j_y^{(2)} = [W_{yxx} + (Q_{yxx} \cos \theta + Q_{yxy} \sin \theta) B] (\partial_x T)^2 = Q_{yxx} \cos \theta (\partial_x T)^2 B.
\]

(B7)

To obtain Eq. [17], we have used the equations in Eq. [10]. Taking \( a = y, b = c = d = x \) into Eq. [15] and, therefore, to the first order of magnetic field, the current \( j_a^{(1)} \) and \( j_a^{(2)} \) in Eq. [11] is found to be

\[
j_a^{(1)} = \sum_b G_{ab} \partial h_b T + \sum_{bc} K_{abc} \partial h_b T B_c,
\]

\[
j_a^{(2)} = \sum_{bc} W_{abc} \partial h_b T \partial_b T + \sum_{bcd} Q_{abcd} \partial h_b T \partial_b T B_d,
\]

(B3)

with

\[
G_{ab} = -\tau e \int [dk] (\epsilon^M - \mu) v_a \frac{\partial f_0}{\partial k_b},
\]

\[
K_{abc} = -\tau^2 e g_u B \int [dk] \left[ v_a \frac{\partial \epsilon^M}{\partial k_c} \frac{\partial f_0}{\partial k_b} + (\epsilon^M - \mu) \right] \times \left( \frac{\partial v_a}{\partial k_b} \frac{\partial f_0}{\partial k_c} + v_a \frac{\partial^2 f_0}{\partial k_b \partial k_c} \right),
\]

\[
W_{abc} = -\tau^2 e g_u B \int [dk] v_a \left[ (\epsilon^M - \mu) \hbar v_b \frac{\partial f_0}{\partial k_c} + (\epsilon^M - \mu)^2 \frac{\partial^2 f_0}{\partial k_b \partial k_c} \right],
\]

(B4)

Meanwhile, using the relation in Eq. [16], the quantity \( Q_{xwxw} \) can be determined and is given in Eq. [12]. And the coefficients ( \( \Upsilon_1, \Upsilon_2, \Upsilon_3 \) and \( \Upsilon_4 \) ) in Eq. [12] are found to be

\[
\Upsilon_1 = v_x^2 v_y^2,
\]

\[
\Upsilon_2 = 2v_x v_y v_x v_y + 2v_x^2 v_y^2 + v_x^4 v_y^4,
\]

\[
\Upsilon_3 = v_x v_y v_x v_y + v_x^2 v_y^2,
\]

\[
\Upsilon_4 = (v_x^2 v_y^2 + 2v_x v_y v_x v_y + v_y^2 v_x^2),
\]

(B8)

To obtain Eq. [18], we have used the equations in Eq. [10]. Taking \( a = y, b = c = d = x \) into Eq. [15] and, where \( \epsilon_1 = \nu F \hbar k \) and \( \eta = \lambda (\nu F \hbar k)^2 \). For \( v_x \)

\[
v_x = \frac{v_F \epsilon_1 [\cos \phi_k + 1.5 \eta \epsilon_1 (\cos \phi_k + \cos 5 \phi_k)]}{\epsilon_1},
\]

(B10)
Appendix C: The conversion rate from nonlinear spin to charge current

Figure 1 illustrates the conversion rate from the nonlinear spin current (NSC) \( j_{nl}^{\text{NL}} \) to nonlinear planar Nernst current \( j_{nl}^{\text{NL}} \) against the Fermi energy \( E_F \). The conversion of \( j_{nl}^{\text{NL}} \) to nonlinear planar Nernst current \( j_{nl}^{\text{NL}} \) involves both contributions from hexagonal warping effect and linear Dirac dispersion. \( j_{nl}^{\text{NL}} \) represents the NSC stemmed from the linear-\( k \) Dirac dispersion, namely the nonlinear spin current for \( \lambda = 0 \). Thus, \( j_{nl}^{\text{NL}} - j_{nl}^{\text{NL}} \) denotes the hexagonal-warping-effect-induced NSC.

For \( v_y \)

\[
v_y = v_F \epsilon_1 \left[ \sin \phi_k + 1.5 \eta^2 \epsilon_1' \left( \sin \phi_k - \sin 5 \phi_k \right) \right].
\]

For \( v_{xx} \), \( v_{xy} \), \( v_{yy} \) and \( v_{xy} \)

\[
v_{xx} = \xi_k \left[ 6 \epsilon^2 \cos^4 \phi_k (2 \cos 2 \phi_k - 1)^3 + \sin^2 \phi_k \right. + \xi \left( 3.5 + 1.5 \cos 2 \phi_k + 6 \cos 4 \phi_k - 6 \cos 6 \phi_k \right),
\]

\[
v_{xy} = \xi_k \left[ -6 \epsilon^2 \cos \phi_k \sin \phi_k - \frac{1}{2} \sin 2 \phi_k \right. + 1.5 \left( \sin 2 \phi_k - 4 \sin 4 \phi_k \right) \xi,
\]

\[
v_{yy} = \xi_k \left[ -6 \epsilon^2 \cos^2 \phi_k - 2 \cos 2 \phi_k - 1 \right] + \xi \left( 15 - 6 \cos 2 \phi_k - 4 \cos 4 \phi_k \right),
\]

\[
v_{xyxy} = \frac{e_1 \epsilon_1}{(\epsilon_1')^2} \left[ 48 \epsilon \cos^6 \phi_k \sin \phi_k \cos^4 \phi_k - 9 \sin^4 \phi_k \right.
\]

\[
+ \frac{\epsilon}{3} \left( 35 \sin \phi_k - 162 \sin 3 \phi_k + 26 \sin 5 \phi_k \right)
\]

\[
+ 7 \sin 7 \phi_k \right) + \frac{1}{4} \left( 3 \sin 3 \phi_k - \sin 5 \phi_k \right) \right].
\]

with \( \xi_k = v_F^2 \hbar \epsilon_1' / (\epsilon_1')^3 \) and \( \zeta = \eta^2 \epsilon_1' \).

Appendix D: The other possible contributions to NPNE

1. The contribution of linear dispersion near the Dirac point with a few \( k_B T \)

In this section, the faint signal [Fig. D1(a)] arising from the linear dispersion near the Dirac point within a few \( k_B T \) will be analysed. Letting the involved hexagonal warping term to be zero (i.e., \( \lambda = 0 \)) in quantities [ \( Y_1, Y_2, Y_3, \text{ and } Y_4 \] ] and combining with a tedious derivation, the nonlinear planar coefficient \( Q_{yxxx} \) [Eq. D2] origin-
inated from the linear dispersion can be determined as
\[ Q_{yxxx} = \frac{\pi e^2 \mu_B V F}{4T^2 h^2} \int d\epsilon \left[ Q_1(\epsilon) + E_f Q_2(\epsilon) \right], \]
where
\[ Q_1(\epsilon) = (\epsilon - E_f) \left( \frac{\partial f_0}{\partial \epsilon} + Q_2(\epsilon) \right) + 4(\epsilon - E_f)^2 \frac{\partial^2 f}{\partial \epsilon^2}, \]
\[ Q_2(\epsilon) = \frac{\partial f_0}{\partial \epsilon} + 3(\epsilon - E_f) \frac{\partial^2 f}{\partial \epsilon^2} + (\epsilon - E_f) \frac{\partial^3 f}{\partial \epsilon^3}. \] \( \text{(D2)} \)

The quantities \( Q_1(\epsilon) \) and \( Q_2(\epsilon) \) are essentially zero when the energy is beyond the range of \( [E_f - 10k_B T, E_f + 10k_B T] \). When the Fermi level is larger than \( 10k_B T \) [Fig. 11 (d)], the term \( Q_1(\epsilon) \) will have no contribution to the nonlinear planar Nernst effect owing to the anti-symmetry property, namely \( Q_1(\epsilon + E_f) = -Q_1(\epsilon - E_f) \) [Fig. 11 (d)]. For \( Q''(\epsilon) \) term, although it is an even function of \( \epsilon \), it satisfies
\[ \int_{E_f - 10k_B T}^{E_f + 10k_B T} d\epsilon Q_2(\epsilon) \approx 0. \] \( \text{(D3)} \)

Thus, when Fermi energy is larger than \( 10k_B T \), \( Q_2(\epsilon) \) also has no contribution to \( Q_{yxxx} \). This is consistent with the result in the main text that the signal of nonlinear spin current originated from the linear-\( k \) Dirac dispersion will not be converted into the nonlinear planar Nernst current.

Next, let us analyse the appearance of the weak signal near the Dirac point within a few \( k_B T \), namely \( E_f < 10k_B T \). In this regime, the contribution of \( E_f Q_2(\epsilon) \) term in Eq. (1) to nonlinear planar Nernst coefficient \( Q_{yxxx} \) can be neglected since \( E_f \) can be viewed as a small quantity \( (k_B T \approx 2.5meV) \) for \( T = 30K \). The contribution to the nonlinear planar Nernst effect mainly come from \( Q'(\epsilon) \). Figure 11 (a) shows the variation of \( Q'(\epsilon) \) towards energy \( \epsilon \). When the Fermi energy is located in the range of \([0, 10k_B T] \), there is no states in the range of \([E_f - 10k_B T, 0eV] \) for upper band. Therefore, the depleted or excessive carriers below the Fermi energy due to the second-order variation of temperature gradient and magnetic field are no longer equal to the excessive or depleted carriers above the Fermi energy. As a result, the carriers are no longer in balance and lead to a weak signal of the nonlinear planar Nernst coefficient. Thus, the appearance of the weak signal from the linear-\( k \) Dirac dispersion could, physically, be attributed to the temperature broadening effect of nonequilibrium Fermi distribution near the Dirac point.

2. Contribution of the particle-hole asymmetry

In this section, we will discuss the contribution of the particle-hole asymmetry, namely \( E_0(k) = \frac{\hbar^2 k^2}{2m^*} \) term, to the nonlinear planar Nernst effect (NPNE). Unlike the contribution of particle-hole asymmetry to the nonlinear planar Hall effect (NPHE) in which the contributions related to particle-hole asymmetry and hexagonal warping are the same order of magnitude, we shall show below that the independent contribution of the particle-hole asymmetry to NPNE is insignificant. Explicitly, we start with the following model Hamiltonian without hexagonal warping effect for topological insulator in the presence of the in-plane magnetic field \( \mathbf{H} \)
\[ H' = \frac{\hbar^2 k^2}{2m^*} + v_F \hbar \sigma \cdot (k \times \hat{z}) + g u_B \sigma \cdot \mathbf{H}. \] \( \text{(D4)} \)
The energy eigenvalues are
\[ \epsilon_n^M = \frac{\hbar^2 k^2}{2m^*} + n \sqrt{(v_F \hbar k)^2 + 2v_F \hbar g u_B (k \times \hat{z}) + (g u_B H)^2}, \] \( \text{(D5)} \)
and the corresponding energy \( \epsilon_k^0 \) in Eq. (12) without the perturbation of magnetic field for upper band is
\[ \epsilon_k^0 = \frac{\eta_2}{2} \epsilon_1 + \epsilon_1, \] \( \text{(D6)} \)
where \( \eta_2 = \left( \frac{\hbar^2}{m^*} \right)/ (v_F \hbar)^2 \) and \( \epsilon_1 = v_F \hbar k \). Thus, the corresponding quantities \( [v_x, v_y, v_{xx}, v_{yy}, v_{xy}, v_{xxy}] \) in Eq. (13) are found to be
\[ v_x = v_F \cos \phi_k \left( 1 + \eta_2 \epsilon_1 \right), \]
\[ v_y = v_F \sin \phi_k \left( 1 + \eta_2 \epsilon_1 \right), \]
\[ v_{xx} = v_F^2 \hbar \left( \frac{\sin^2 \phi_k}{\epsilon_1} + \eta_2 \right), \]
\[ v_{yy} = v_F^2 \hbar \left( \frac{\cos^2 \phi_k}{\epsilon_1} + \eta_2 \right), \]
\[ v_{xy} = v_F^2 \hbar \left( \frac{\sin 2\phi_k}{2\epsilon_1} + \eta_2 \right), \]
\[ v_{xxy} = v_F^2 \hbar \left( 3 \sin 3\phi_k - \sin \phi_k \right). \]
With Eqs. D7, the nonlinear planar Nernst coefficient (NPNC) \( Q_{yx|xx} \) quantizing the NPNE can be determined. Figure 12(a) shows that the variation of \( Q_{yx|xx} \) as a function of Fermi energy with or without particle-hole asymmetry (PHA) \( \hbar^2k^2/2m^* \). It is found that with or without the particle-hole asymmetry makes no difference to the magnitude of \( Q_{yx|xx} \) [Fig. 12(a)] in Bi\(_2\)Te\(_3\), which means that the particle-hole asymmetry can not independently give rise to NPNE that is distinguish from the nonlinear planar Hall effect. In fact, the weak signal appeared near the Dirac point within 25 meV \((\sim 10k_BT \text{ for } T = 30K)\) is originated from the linear-

\[ k \text{ Dirac dispersion and induced by the thermal broadening effect [see detail in Sec. D1]. Parameters are used for } Bi\(_2\)Te\(_3\): the Fermi velocity } v_F \hbar = 2.25 \text{ eVÅ and } m^* = 0.09m_e^{10,15} \text{ where } m_e \text{ is free electron mass.} \]

**FIG. D3.** The nonlinear planar coefficient \( Q_{yx|xx} \) [(a)] and the nonlinear spin Nernst current \( j_{n|x}\sigma_y^n \) [(b)] from the Rashba-split surface states in two-dimensional electron gas (2DEG) as a function of Fermi energy. The red dashed line and the blue short-dot line are the contributions from upper \( n = +1 \) and down \( n = -1 \) subbands, respectively. The black solid-line is the sum of the two contributions. (c) \( Q_{yx|xx} \) vs \( E_F \) for different temperature. (d) Schematic depiction of the band structures for the surface states of topological insulators, where the shaded part indicates the Rashba 2DEG due to the surface band bending. \( T = 30K \) is fixed in (a) and (b). Parameters are used: \( m^*_e = 0.2m_e \) with bare mass of an electron \( m_e \), \( \alpha_R = 0.5 \text{ eVÅ} \) and \( g = 2 \).

3. **Contribution of the Rashba 2DEG**

Due to the surface band bending in topological insulator, a Rashba-split surface states in two-dimensional electron gas (2DEG) [Fig. D3(d)] might coexist with topological surfaces states (TSS). To theoretically investigate the contribution of a Rashba to nonlinear planar Nernst effect, we begin with the following model Hamiltonian

\[
H' = u_0 + \frac{\hbar^2k^2}{2m_R} + \alpha_R \hbar \sigma \cdot (k \times \hat{z}) + gu_B \sigma \cdot H, \tag{D8}
\]

where \( m^*_R \) represents the effective mass, \( u_0 \) is chemical potential, and \( \alpha_R \) denotes the strength of the Rashba spin-orbit coupling. The nonlinear planar Nernst coefficient \( Q_{yx|xx} \) [Eq. (12)] with an in-plane magnetic field and spin Nernst current \( j_{n|x}\sigma_y^n \) without a magnetic field as the second-order response to temperature gradient can be obtained accordingly based on the Hamiltonian Eq. D8 in the same manner as Sec. D2.

Figure D3(a) shows the contributions of the \( n = +1 \) and \( n = -1 \) subbands (namely, the inner and outer Fermi contours of Rashba 2DEG) to nonlinear planar Nernst effect. It is found that only when the Fermi energy located near the Lifshitz point [Fig. D3(d)] within a few \( k_BT \) [Fig. D3(a)], a nonzero \( Q_{yx|xx} \) can be generated and is almost 100 times smaller than the signal from TSS. In fact, the appearance of this faint signal might be attributed to the temperature broadening effect like the signal stemmed from the linear-k Dirac dispersion [see details in Sec. D1]. When modulating Fermi energy away from the Lifshitz point, the nonlinear planar Nernst effect disappears since there is no nonlinear spin current converted into charge current for both subbands.

---

* yuxiaoqin@hnu.edu.cn
  † zgzhu@ucas.ac.cn
  ‡ gst@ucas.ac.cn

1. M. Z. Hasan and C. L. Kane, Rev. Mod. Phys. 82, 3045 (2010).
2. X.-L. Qi and S. C. Zhang, Rev. Mod. Phys. 83, 1057 (2011).
3. Y. Shiomi, K. Nomura, Y. Kajiwara, K. Eto, M. Novak,
K. Segawa, Y. Ando, and E. Saitoh, Phys. Rev. Lett. 113, 196601 (2014).
4. Z. Jian, C.-Z. Chang, M. R. Masir, C. Tang, Y. Xu, J. S. Moodera, A. H. MacDonald, and J. Shi, Nat. Commun. 7, 11458 (2016).
5. H. Wang, J. Kally, J. Sue Lee, T. Liu, H. Chang, D. R. Hickey, K. A. Mkhyon, M. Wu, A. Richardella, and N. Smarth, Phys. Rev. Lett. 117, 076601 (2016).
6. J. Han, A. Richardella, S. A. Siddiqui, J. Finley, N. Samarth, and L. Liu, Phys. Rev. Lett. 119, 077702 (2017).
7. Y. Wang, D. Zhu, Y. Wu, Y. Yang, J. Yu, R. Ramaswamy, R. Mishra, S. Shi, M. Elyasi, K.-L. Teo, Y. Wu, and H. Yang Nat. Commun. 8, 1364 (2017).
8. M. De, R. Grassi, J.-Y. Chen, M. Jamali, D. R. Hickey, D. Zhang, Z. Zhao, H. Li, P. Quarterman, Y. Lv, M. Li, A. Manchon, K. A. Mkhyon, T. Low, and J.-P. Wang, Nat. Mater. 17, 800 (2018).
9. C.-F. Pai, Nat. Mater. 17, 755 (2018).
10. X. Wang, D. Yu, S. Don and C. Zhang, Phys. Rev. Lett. 108, 266806 (2012).
11. J. Wang, H. Li, C. Chang, K. He, J. S. Lee, H. Lu, Y. Sun, X. Ma, N. Samarth, S. Shen, Q. Xue, M. Xie, and M. H. Chan, Nano Res. 5, 739-746 (2012).
12. A. Sulaev, M. Zeng, S.-Q. Shen, S. K. Cho, W. G. Zhu, Y. P. Feng, S. V. Ereminov, Y. Kawazoe, L. Shen, and L. Wang, Nano lett. 15, 2061-2066 (2015).
13. S. Wiedmann, A. Jost, B. Fauqué, J. van Dijk, M. J. Meijer, T. Khouri, S. Pezzini, S. Grauer, S. Schreyer, C. Brue, H. Buhmann, L. M. Wollenkamp, and N. E. Hussey Phys. Rev. B 94, 081302 (R) (2016).
14. A. A. Taskin, H. F. Legg, F. Yang, S. Sasaki, Y. Kanai, K. Matsumoto, A. Rosch, and Y. Ando, Nat. Commun. 8, 1340 (2017).
15. P. He, S. S. L. Zhang, D. Zhu, Y. Liu, Y. Wang, J. Yu, G. Vignale, and H. Yang, Nat. Phys. 14, 495 (2018).
16. A. Dyrdal, J. Barnas, and A. Fert, Phys. Rev. Lett. 124, 046802 (2020).
17. C. O. Avci, K. Garello, A. Ghosh, M. Gabureac, S. F. Alvarado, and P. Gambardella, Nat. Phys. 11, 570 (2015).
18. C. O. Avci, K. Garello, J. Mendil, A. Ghosh, N. Bilsak, M. Gabureac, M. Trassin, M. Fiebig, and P. Gambardella, App. Phys. Lett. 107, 192405 (2015).
19. S. Langanfeld, V. Tshitoyan, Z. Fang, A. Wells, T. A. Moore, and A. J. Ferguson, App. Phys. Lett. 108, 192402 (2016).
20. K. Yasuda, A. Tsukazaki, R. Yoshimi, K. S. Takahashi, M. Kawasaki, and Y. Tokura, Phys. Rev. Lett. 117, 127202 (2016).
21. Y. Lv, J. Kally, D. Zhang, J. S. Lee, M. Jamali, N. Samarth, and J.-P. Wang, Nat. Commun. 9, 111 (2018).
22. A. A. Taskin, S. Sasaki, K. Segawa, and Y. Ando, Phys. Rev. Lett. 109, 066803 (2012).
23. B. Wu, X.-C. Pan, W. Wu, F. Fei, B. Chen, Q. Liu, H. Bu, L. Cao, F. Song, and B. Wang, Appl. Phys. Lett. 113, 011902(2018).
24. D. Rakhmilevich, F. Wang, W. Zhao, M. H. W. Chan, J. S. Moodera, C. Liu, and C.-Z. Chang, Phys. Rev. B 98, 094404 (2018).
25. S.-H. Zhang, H.-J. Duan, J.-K. Wang, J.-Y. Li, M.-X. Deng, and R.-Q. Wang, Phys. Rev. B 101, 041408(R) (2020).
26. P. He, S. S.-L. Zhang, D. Zhu, S. Shi, and O. G. Heinenon, Phys. Rev. Lett. 123, 016801 (2019).
27. X.-Q. Yu, Z.-G. Zhu, and G. Su, Phys. Rev. B 100, 195418 (2019).
28. X.-Q. Yu, Z.-G. Zhu, J.-S. You, T. Low, and G. Su, Phys. Rev. B 99, 201410(R) (2019).
29. C. Zeng, S. Nandy, A. Taraphder, and S. Tewari, Phys. Rev. B 100, 245102 (2019).
30. G. E. W. Bauer, E. Saitoh, and B. J. van Wees, Nat. Mater. 11, 391 (2012).
31. A. D. Avery, M. R. Pufall, and B. L. Zink, Phys. Rev. Lett. 109, 196602 (2012).
32. S. Y. Huang, W. G. Wang, S. F. Lee, J. Kwo, and C. L. Chien, Phys. Rev. Lett. 107, 216604 (2011).
33. S. R. Boona, R. C. Myerscbe and J. P. Heremans, Energy Environ. Sci. 7, 885-910 (2014).
34. X.-Q. Yu, Z.-G. Zhu, G. Su, and A.-P. Jauho, Phys. Rev. Lett. 115, 246601 (2015).
35. V. Baltz, A. Manchon, M. Tsioi, T. Moriyama, T. Ono, and Y. Tserkovnyak, Rev. Mod. Phys. 90, 015005 (2018).
36. L. Fu, Phys. Rev. Lett. 103, 266801 (2009).
37. C. M. Wang and X. L. Lei, Phys. Rev. B 89, 045415 (2014).
38. Dong-Xia Qu, Y. S. Hor, Jun Xiong, R. J. Cava, and N. P. Ong, Science 329, 5993 (2010).
39. J. Xu, W. A. Phelan, and C.-L. Chien, Nano Lett. 19, 8250-8254 (2019).
40. Y. Wang, P. Deorani, K. Banerjee, N. Koirala, M. Brahlek, S. Oh, and H. Yang, Phys. Rev. Lett. 114, 257202 (2015).
41. A. R. Mellnik et al., Nature (London) 511, 449 (2014).
42. A. Dankert, J. Geurs, M. V. Kamalakar, S. Charpentier, and S. P. Dash, Nano Lett. 15, 7976 (2015).
43. J. M. Ziman, Electrons and Phonons: The Theory of Transport phenomena in Solid (Oxford University Press, New York, 1960).
44. H.-J. Zhang, C.-X. Liu, X.-L. Qi, X. Dai, Z. Fang, and S.-C. Zhang, Nat. Phys. 5, 438 (2009).
45. C.-X. Liu, X.-L. Qi, H. J. Zhang, X. Dai, Z. Fang, and S.-C. Zhang, Phys. Rev. B 82, 045122 (2010).