T-DUALITY AND EXCEPTIONAL GENERALIZED GEOMETRY THROUGH SYMMETRIES OF DG-MANIFOLDS.

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Abstract. We study dg-manifolds which are $\mathbb{R}^2$-bundles over $\mathbb{R}^1$-bundles over manifolds, we calculate its symmetries, its derived symmetries and we introduce the concept of T-dual dg-manifolds. Within this framework we construct the T-duality map as a degree -1 map between the cohomologies of the T-dual dg-manifolds and we show an explicit isomorphism between the differential graded algebra of the symmetries of the T-dual dg-manifolds. We furthermore show how the algebraic structure underlying $B_n$ generalized geometry could be recovered as derived dg-Leibniz algebra of the fixed points of the T-dual automorphism acting on the symmetries of a self T-dual dg-manifold, and we show how other types of algebraic structures underlying exceptional generalized geometry could be obtained as derived symmetries of certain dg-manifolds.

1. Introduction

Generalized complex manifolds and Dirac structures are geometric structures defined in [10, 9] and [6] respectively, which among other things, provide a unified framework on which symplectic manifolds, Poisson manifolds, complex manifold and certain types of foliations can be treated in a uniform way.

The algebraic structures underlying these geometries are the so called the Exact Courant algebroids. These algebroids are split extensions of the tangent bundle by the cotangent bundle, and are endowed with a Leibniz bracket known as the Courant-Dorfman bracket. Dorfman [7] in its original construction, obtained the Courant-Dorfman bracket by applying the derived bracket procedure to a 4-tiered differential graded Lie algebra associated to vector fields, functions, 1-forms and 2-forms. It was noted by Ševera that the construction of Dorfman could be recovered and generalized to the twisted case by performing the derived bracket procedure to the differential graded Lie algebra of symmetries of an $\mathbb{R}^2$-bundle over $T^1M$ in the language of dg-manifolds (cf. [16]), and it was in this framework of dg-manifolds that the third author [18] explored the conditions under which a Lie group action on a manifold could be lifted to an action on an exact Courant algebroid.

In recent works in string theory [11, 14] it has been proposed to consider a geometric structure called exceptional generalized geometry, structure similar in
nature to the geometry of generalized complex manifolds in the sense that incorporates vector fields and differential forms of higher degree, to study backgrounds on eleven-dimensional supergravity. The underlying algebraic structure of this exceptional generalized geometry was further studied in [1] where it is described how this structure can be obtained from a certain simple Lie algebra, and at the same time describes other related types of algebraic structures coming from other simple Lie algebras. Motivated by the relation developed by Ševera between Exact Courant algebroids and the derived symmetries of $\mathbb{R}[2]$-bundles over $T[1]M$, in this work we investigate how to obtain the algebraic structures defined in [1] as derived constructions from the symmetries of particular dg-manifolds.

One of the exceptional Generalized geometries introduced in [1] under the name of $B_n$ generalized geometry lives on the bundle $TM \oplus \mathbb{R} \oplus T^*M$ and can be twisted by a pair of a closed 2-form $F$ and a 3-form $H$ satisfying $dH + F \wedge F = 0$. Now, since the structural constants which define a $\mathbb{R}[2]$-bundle over a $\mathbb{R}[1]$-bundle over $T[1]M$ are closed 2-forms $F$ and $\bar{F}$ and a 3-form $H$ satisfying $dH + F \wedge \bar{F} = 0$, we note that the “$B_n$ generalized geometry” should be related to the symmetries of “self dual bundles”, namely the ones on which $F = \bar{F}$, this basic observation is the basis of this investigation.

Let us describe what we accomplish in this work. We start by reviewing the basics on dg-manifolds and on $\mathbb{R}[n]$-bundles over dg-manifolds. Then we study $\mathbb{R}[2]$-bundles over $\mathbb{R}[1]$-bundles over $T[1]M$ and we find that the structural constants of these bundles are closed 2-forms $F$ and $\bar{F}$ and a 3-form $H$ satisfying $dH + F \wedge \bar{F} = 0$ making the vector field $d + F \partial_q + (H + qF)\partial_t$ homological. Then we note that since the equations on $F$ and $\bar{F}$ are symmetric, we see that the homological vector field $d + \bar{F} \partial_q + (H + qF)\partial_t$ defines another $\mathbb{R}[2]$-bundle over another $\mathbb{R}[1]$-bundle and we remark that this construction is completely analogous to the original construction of topological T-duality for circle bundles; hence it is natural to call the two resulting dg-manifolds T-dual. We follow the construction of the T-duality isomorphism in twisted cohomologies done in [2] and we put it in the framework of dg-manifolds; we obtain a degree -1 map between the complexes that calculate the cohomology of the T-dual dg-manifolds, and we show that this map fits on the right hand side of a short exact sequence of complexes where the left hand side is given by the inclusion of the De Rham complex of $M$ into the complex of the first dg-manifold. This degree -1 map, in cohomology of degrees greater than the cohomology of $M$, produces an isomorphism and in particular it induces the known isomorphism in twisted cohomology for the T-dual pair, this is Theorem 3.3.

Then we proceed to explicitly calculate the symmetries and its derived symmetries of $\mathbb{R}[2]$-bundles over $\mathbb{R}[1]$-bundles and we show that the symmetries and the derived symmetries of T-dual pairs are isomorphic through a simple map that switches tangent information with cotangent information, this is Theorem 4.4. We note in Corollary 4.5 that this isomorphism provides an alternative proof for the T-duality isomorphism al the level of the Courant algebroids for dual pairs carried out in [3].

Then we find that the algebraic structure underlying $B_n$ generalized geometry in the sense of Baraglia [1] can be obtained from a self T-dual dg-manifold by looking at the derived symmetries fixed by the T-duality automorphism, this is Theorem 4.6. This in particular implies that the $B_n$ generalized geometry embeds into the generalized geometry of the principal $\mathbb{R}[1]$-bundle that the curvature form
F defines. We finish by giving an alternative description of the underlying algebraic structure of $E_6$ generalized geometry as the derived symmetries of a $\mathbb{R}[6]$-bundle over a $\mathbb{R}[3]$-bundle over $T[1]M$.

The results of this paper seem to support the general intuition (observed by many people, including Nigel Hitchin) that generalized geometry seems to be a way to encode a sort of poor man’s super-geometry, as dg-manifolds are enriched form of super-manifolds.

2. $\mathbb{R}[n]$-Bundles on dg-manifolds

2.1. dg-Manifolds. Let us start with some notational conventions. Let $\mathcal{N}$ be a differentiable (super)manifold and let us denote its sheaf of smooth functions by $\mathcal{O}_\mathcal{N}$. For $P = \{P_k\}_{k \in \mathbb{Z}}$ a graded vector bundle over $\mathcal{N}$, $S(P)$ will denote the the sheaf of graded commutative $\mathcal{O}_\mathcal{N}$-algebras freely generated by $P$; the locally ringed space $(\mathcal{N}, S(P^*))$ will also be denoted by $P$ where $P^*$ is the dual vector bundle. For an integer $k$, $P[k]$ denotes the shifted vector bundle with $P[k]_l := P_{k+l}$. To keep the notation simple, we will usually denote a vector bundle and its $\mathcal{O}$-module of sections with the same symbol.

**Definition 2.1.** A (non-negatively) graded manifold is a locally ringed space $\mathcal{P} = (\mathcal{N}, \mathcal{O}_\mathcal{P})$, which is locally isomorphic to $(U, \mathcal{O}_U \otimes S(P^*))$, where $U \subset \mathbb{R}^{m|r}$ is an open domain of $\mathcal{N}$ and $P = \{P_i\}_{-n \leq i \leq 0}$ is a finite dimensional negatively graded (super)vector space. The number $n$ is called the degree of the graded manifold $\mathcal{P}$.

The global sections of $\mathcal{P}$ will be called the functions on $\mathcal{P}$ and they will be denoted by $C^*(\mathcal{P})$, and the derivations of $C^*(\mathcal{P})$ will be the vector fields of $\mathcal{P}$ and they will be denoted $\text{Vect}^*(\mathcal{P})$.

**Definition 2.2.** A differential graded manifold (dg-manifold) is a graded manifold $\mathcal{P}$ equipped with a degree 1 vector field $Q$ of $\text{Vect}^1(\mathcal{P})$ satisfying $[Q, Q]/2 = Q^2 = 0$ (a homological vector field).

Morphisms of dg-manifolds are morphisms of locally ringed spaces respecting the homological vector field. We recommend [16, 12] for an introduction to the theory of differential graded manifolds.

A dg-manifold over a point of degree $n$ is the same as an $L^\infty$-algebra of degree $n$, also called Lie $n$-algebra. A dg-manifold of degree $n$ is what is known as a “Lie $n$ algebroid”.

2.1.1. Manifolds. For $M$ a manifold let us consider the graded manifold $T[1]M$, whose algebra of functions is the algebra of differential forms

$$C^*(T[1]M) = \Omega^\bullet M,$$

and endow it with homological vector field

$$d = dx^i \frac{\partial}{\partial x^i},$$

the degree 1 derivation which is the De Rham differential. Let us denote by

$$\mathcal{M} = (T[1]M, d)$$

the image of $M$ in the category of dg-manifolds.
2.2. \( \mathbb{R}[n] \)-bundles. A \( \mathbb{R}[n] \)-bundle over the dg-manifold \( \mathcal{N} = (\mathcal{O}_N, Q_N) \) consists of the graded manifold \( \mathcal{O}_N \oplus \mathbb{R}[n] \) together with a choice of derivation of degree 1
\[ Q_N + \Theta \partial_t \]
of the algebra of functions
\[ C^*(\mathcal{O}_N) \otimes S[t] \]
that satisfies the Maurer-Cartan equation, i.e. \( Q_N \Theta = 0 \). Here \( \Theta \) is a function of degree \( n + 1 \) of \( N \) and \( S[t] \) denotes the symmetric graded algebra generated by the parameter \( t \) whose degree is \( n \).

2.2.1. \( \mathbb{R}[1] \)-bundles over \( T[1]M \). A \( \mathbb{R}[1] \)-bundle over the dg-manifold \( M \) is given by the graded manifold \( T[1]M \oplus \mathbb{R}[1] \) together with a choice of homological vector field that lifts \( d \). The algebra of functions of \( T[1]M \oplus \mathbb{R}[1] \) becomes
\[ C^*(T[1]M \oplus \mathbb{R}[1]) = \Omega^*M \otimes \Lambda[q] \]
where \( q \) is a variable of degree 1. A generic choice of derivation of degree 1 which lifts \( d \) is given by
\[ d + F \partial_q \]
where \( F \) is a 2-form on \( M \), and this derivation becomes homological if it satisfies the Maurer-Cartan equation
\[ [d + F \partial_q, d + F \partial_q] = 0; \]
which is equivalent to saying that \( dF = 0 \). So, any choice of closed 2-form will define a \( \mathbb{R}[1] \)-bundle over \( M \). Let us denote this dg-manifold by
\[ \mathcal{R} := (T[1]M \oplus \mathbb{R}[1], d + F \partial_q). \]

2.2.2. Relation to principal \( S^1 \)-bundles. Let \( S^1 \to R \xrightarrow{\pi} M \) be a principal \( S^1 \) bundle and denote by \( \theta \in \Omega^1 R \) any connection 1-form and by \( F \in \Omega^2 M \) the associated curvature form; we have that \( d_R \theta = \pi^* F \), where \( d_R \) denotes the De Rham differential in \( R \). Consider also the dg-manifold \( \mathcal{R} := (T[1]M \oplus \mathbb{R}[1], d + F \partial_q) \) where \( F \) is the curvature 2-form of the \( S^1 \)-bundle.

We claim that

**Proposition 2.3.** Consider the principal bundle \( S^1 \to R \xrightarrow{\pi} M \), then the map of graded algebras
\[ j : \Omega^*M \otimes \Lambda[q] \to \Omega^*R \]
\[ \omega_0 + q \omega_1 \mapsto \pi^* \omega_0 + \theta \pi^* \omega_1 \]
induces a quasi-isomorphism
\[ j : (\Omega^*M \otimes \Lambda[q], d_M + F \partial_q) \to (\Omega^*R, d_R) \]

**Proof.** Let us prove first that the map \( j \) is a morphism of complexes, on the left hand side we have that:
\[ d_R(j(\omega_0 + q \omega_1)) = d_R(\pi^* \omega_0 + \theta \pi^* \omega_1) \]
\[ = d_R(\pi^* \omega_0) + \pi^* F \pi^* \omega_1 - \theta d_R(\pi^* \omega_1) \]
\[ = \pi^*(d_M \omega_0) + \pi^* F \pi^* \omega_1 - \theta \pi^*(d_M \omega_1), \]
while on the right hand side we can write:

\[ j(d_M + F \partial_q (\omega_0 + q \omega_1)) = j(d_M \omega_0 + F \omega_1 - q d_M \omega_1) = \pi^*(d_M \omega_0) + \pi^*F \pi^* \omega_1 - \theta \pi^*(d_M \omega_1), \]

hence \( d_R \circ j = j \circ (d_M + F \partial_q) \).

Let’s consider the filtration via a good cover on \( M \) that induces the Serre spectral sequence of the fibration \( S^1 \to R \to M \) and which converges to \( H^*(R) \), let us denote this spectral sequence by \( E_r^{p,q} \). Let’s consider a corresponding filtration associated to \( \Omega^* M \otimes \Lambda[q] \) via the same open sets, which converges to \( H^*(\Omega^* M \otimes \Lambda[q], d_M + F \partial_q) \); denote this spectral sequence by \( E'_r^{p,q} \). Since the filtrations are obtained via the same open sets, we have that \( j \) induces a map of spectral sequences

\[ j : E_r^{p,q} \to E'_r^{p,q}. \]

The second page of \( E \) is equal to

\[ E_2^{p,q} = H^p(M) \otimes \Lambda[q] \]

and since the fundamental group of \( M \) acts trivially on \( H^*(S^1) \), we have that

\[ E_2^{p,q} = H^p(M) \otimes H^q(S^1). \]

The map \( j \) induces an isomorphism of the second pages of the spectral sequences, and therefore it induces an isomorphism of the infinity pages. Since the complexes are bounded, the map \( j \) induces an isomorphism of the cohomologies

\[ j : H^*(\Omega^* M \otimes \Lambda[q], d_M + F \partial_q) \cong H^*(R). \]

\[ \square \]

2.2.3. \( \mathbb{R}[n] \)-bundles on \( T[1]M \). A \( \mathbb{R}[n] \)-bundle over \( T[1]M \) is determined by a closed \( n+1 \) form \( \Theta \in \Omega^{n+1}_{cl} M \) making the derivation

\[ d + \Theta \partial_t \]

of the algebra of functions \( \Omega^* M \otimes S[t] \) homological.

In view of the result of Proposition 2.3 we should think of the dg-manifold

\[ (T[1]M \oplus \mathbb{R}[n], d + \Theta \partial_t) \]

as the analogue of a principal \( K(\mathbb{Z}, n) \)-bundle \( K(\mathbb{Z}, n) \to P \to M \) over the manifold \( M \). Since we do not have a reasonable model for the differential forms for \( K(\mathbb{Z}, n) \) for \( n \geq 2 \), we cannot proceed as in said proposition. Instead, we could think of the dg-manifold \( (T[1]M \oplus \mathbb{R}[n], d + \Theta \partial_t) \) as a rigid model for the differential forms on \( P \), provided that the differential form \( \Theta \) has integral periods. In particular, we should think that the cohomology

\[ H^*(\Omega^* M \otimes S[t], d + \Theta \partial_t) \]

calculates the cohomology of the total space \( P \).
2.2.4. $\mathbb{R}[2]$-bundles on $T[1]M$ and twisted cohomology. A $\mathbb{R}[2]$-bundle over $T[1]M$ is the dg-manifold

$$\mathcal{P} = (T[1]M \oplus \mathbb{R}[2], Q = d + H \partial_t)$$

for $H$ a closed 3-form on $M$. The cohomology of $\mathcal{P}$ is the cohomology of $\Omega^* M \otimes \mathbb{R}[t]$ with respect to the differential $d + H \partial_t$ where $t$ is a variable of degree 2, denote this cohomology by

$$H^*(\mathcal{P}) = H^*(\Omega^* M \otimes \mathbb{R}[t], d + H \partial_t).$$

A fiber-wise gauge transformation is given by a the degree 0 vector field $B \partial_t$ where $B$ is any 2-form on $M$, transforming the homological vector field by

$$d + H \partial_t \mapsto (d + H \partial_t + [d + H \partial_t, B \partial_t]) = d + (H + dB) \partial_t.$$

The map

$$e^{B \partial_t} : \Omega^* M \otimes \mathbb{R}[t] \to \Omega^* M \otimes \mathbb{R}[t]$$

induces an isomorphism between the dg-manifolds

$$e^{B \partial_t} : \mathcal{P}' \cong \mathcal{P}$$

where $\mathcal{P}' = (T[1]M \oplus \mathbb{R}[2], d + (H + dB) \partial_t)$, and therefore it induces an isomorphism in cohomologies

$$e^{B \partial_t} : H^*(\mathcal{P}') \cong H^*(\mathcal{P}).$$

For a closed 3-form $H$, the $H$-twisted cohomology of $M$ is defined as the cohomology of the $\mathbb{Z}/2$-graded complex $\Omega^{ev,od} M$, with respect to the differential $d + H \wedge$, and this cohomology is defined by

$$H^{ev,od}(M; H) = H^*(\Omega^{ev,od} M, d + H \wedge).$$

Similarly, the map $e^{B \wedge} : \Omega^{ev,od} M \to \Omega^{ev,od} M$ produces an isomorphism of twisted complexes

$$e^{B \wedge} : (\Omega^{ev,od} M, d + (H + dB) \wedge) \cong (\Omega^{ev,od} M, d + H \wedge)$$

inducing an isomorphism

$$H^{ev,od}(M; H + dB) \cong H^{ev,od}(M; H).$$

We claim

**Proposition 2.4.** Let $d$ be the dimension of $M$ and consider a closed 3-form $H$ on $M$. Let

$$\mathcal{P} = (T[1]M \oplus \mathbb{R}[2], d + H \partial_t)$$

be the associated dg-manifold and let

$$H^{ev,od}(M; H) = H^*(\Omega^{ev,od} M, d + H \wedge)$$

be the twisted cohomology of $M$ twisted by $H$. Then there are isomorphisms of groups

$$H^{ev}(M; H) = H^{2k}(\mathcal{P}), \quad H^{od}(M; H) = H^{2k+1}(\mathcal{P})$$

for any $k$ provided that the degree $2k$ or $2k + 1$ respectively, is greater than $d$. 
Proof. Consider the homomorphism of vector spaces given by the rescaling
\[ \phi : \Omega^\bullet M \otimes \mathbb{R}[t] \to \Omega^{ev,od} M \]
\[ \omega t^k \mapsto k! \cdot \omega, \]
and note that \( \phi \) preserves the \( \mathbb{Z}/2 \) grading.

Since
\[ \phi((d + H \partial_t)\omega t^k) = \phi(d\omega t^k + kH\omega t^{k-1}) = k!(d\omega + H\omega) \]
we see that
\[ ((d + H \wedge)\phi(\omega t^k)) = \phi((d + H \partial_t)\omega t^k), \]
provided that \( k \geq 1 \).

If we denote by \( C^j(\mathcal{P}) \) the homogeneous forms on \( \Omega^\bullet M \otimes \mathbb{R}[t] \) of degree \( j \), we see that the map \( \phi \) induces a map of complexes
\[ \phi : (C^* > d(\mathcal{P}), d + H \partial_t) \to (\Omega^{ev,od} M, d + H \wedge). \]
and moreover it induces an isomorphism of vector spaces
\[ \phi : C^{2k}(\mathcal{P}) \xrightarrow{\cong} \Omega^{ev} M \quad C^{2k+1}(\mathcal{P}) \xrightarrow{\cong} \Omega^{od} M \]
provided that \( 2k \geq d \).

We conclude that the induced maps on cohomologies
\[ \phi : H^{2k}(\mathcal{P}) \xrightarrow{\cong} H^{ev}(M, H), \quad \phi : H^{2k+1}(\mathcal{P}) \xrightarrow{\cong} H^{od}(M, H) \]
are isomorphisms whenever \( 2k > d \) for the left hand side and \( 2k + 1 > d \) for the right hand side. \( \square \)

As a corollary we have that the cohomology of \( \mathcal{P} \) becomes periodic for degrees greater than the degree of the manifold \( M \), that is
\[ H^j(\mathcal{P}) \cong H^{j+2l}(\mathcal{P}) \quad \text{for } l \geq 0 \text{ and } j > \dim(M). \]

Remark 2.5. The \( H \)-twisted cohomology
\[ H^*(M, H) \]
could be understood as the cohomology of the dg-manifold
\[ \mathcal{P} = (T[1]M \oplus \mathbb{R}[2], d + H \partial_t) \]
for degrees greater than the dimension of the manifold \( M \). This simple remark explains the somewhat puzzling fact that one sometimes encounters in which some twisted cohomologies are zero in even \textit{and} in odd degrees. This fact is explained away by noting that the cohomology of the associated dg-manifold \( \mathcal{P} \) is zero for degrees greater than the degree of the manifold, nevertheless this does not say anything with respect to the cohomology of \( \mathcal{P} \) in lower degrees.

For example, when \( M = S^3 \) and \( H \) is a volume form for the sphere, we have that \( H^*(S^3, H) = 0 \) in both degrees. This means that the cohomology of \( \mathcal{P} = (T[1]S^3 \oplus \mathbb{R}[2], d + H \partial_t) \) is zero for degrees greater than 3; and this can be easily checked since the cohomology of \( \mathcal{P} \) is
\[ H^0(\mathcal{P}) = \mathbb{R} \quad \text{and} \quad H^{*>0}(\mathcal{P}) = 0. \]

With this setup in mind, most of the properties that twisted cohomology satisfy follow from the properties that ordinary cohomology satisfies. Hence the twisted cohomology becomes a \( \mathbb{Z}/2 \)-graded cohomology theory satisfying the axioms of a generalized cohomology theory.
2.2.5. Integration on the fibers on $\mathbb{R}[n]$-bundles for $n$ odd. Let $n$ an odd positive integer and consider the $\mathbb{R}[n]$-bundle

$$\mathcal{R} = (\mathcal{N} \oplus \mathbb{R}[n], Q_{\mathcal{R}} = Q_{\mathcal{N}} + \Theta \partial q)$$

over the dg-manifold $(\mathcal{N}, Q_{\mathcal{N}})$ where $q$ is an odd variable of degree $n$, and $\Theta$ is a function of $\mathcal{N}$ of degree $n + 1$ such that $Q_{\mathcal{N}} \Theta = 0$; denote the bundle map $p : \mathcal{R} \to \mathcal{N}$.

Denote by $p^*$ the pushforward map

$$p : C^*(\mathcal{N}) \otimes \Lambda[q] \to C^*(\mathcal{N})$$

$$\omega q \mapsto \omega$$

$$\eta \mapsto 0,$$

where $\omega$ and $\eta$ are functions on $\mathcal{N}$ and note that it is a degree $-n$ map from $C^*(\mathcal{R})$ to $C^*(\mathcal{N})$ and moreover that $p^*$ is a map of complexes, i.e.

$$p^* \circ Q_{\mathcal{R}} = Q_{\mathcal{N}} \circ p^*.$$

Therefore we have that the pushforward

$$p^* : (\mathcal{N} \oplus \mathbb{R}[n], Q_{\mathcal{R}} = Q_{\mathcal{N}} + \Theta \partial q) \to (\mathcal{N}, Q_{\mathcal{N}})$$

is a map of dg-manifolds of degree $-n$ and therefore it induces a map of degree $-n$ on the cohomologies

$$p^* : H^* (\mathcal{R}, Q_{\mathcal{N}} + \Theta \partial q) \to H^{*-n} (\mathcal{N}, Q_{\mathcal{N}})$$

associated to the dg-manifolds.

3. $\mathbb{R}[2]$-bundles over $\mathbb{R}[1]$-bundles over $T[1]M$ and T-duality

Let $\mathcal{R} = (T[1]M \oplus \mathbb{R}[1], Q_{\mathcal{R}} = d + F \partial q)$ be a $\mathbb{R}[1]$-bundle over $T[1]M$.

A $\mathbb{R}[2]$-bundle over the dg-manifold $\mathcal{R}$ is given the graded manifold $(T[1]M \oplus \mathbb{R}[1]) \oplus \mathbb{R}[2]$ together with a choice of homological vector field that lifts $d + F \partial q$.

The algebra of functions of $(T[1]M \oplus \mathbb{R}[1]) \oplus \mathbb{R}[2]$ becomes

$$C^* (T[1]M \oplus \mathbb{R}[1]) \oplus \mathbb{R}[2]) = \Omega^* M \otimes \Lambda[q] \otimes \mathbb{R}[t]$$

where $t$ is a variable of degree 2. A generic choice of derivation of degree 1 which lifts $d + F \partial q$ is given by

$$d + F \partial q + (H + q \bar{F}) \partial t$$

where $H$ is a 3-form on $M$ and $\bar{F}$ is a 2-form on $M$, and this derivation becomes homological if it satisfies the Maurer-Cartan equation

$$[d + F \partial q + (H + q \bar{F}) \partial t, d + F \partial q + (H + q \bar{F}) \partial t] = 0.$$

This last equation is equivalent to the equations

$$dF = 0$$

$$d\bar{F} = 0$$

$$dH + F \wedge \bar{F} = 0.$$
will define a $\mathbb{R}[2]$-bundle over $\mathcal{R}$. Let us denote this dg-manifold by

$$\mathcal{P} := ((T[1]M \oplus \mathbb{R}[1]) \oplus \mathbb{R}[2], d + F\partial_q + (H + q\bar{F})\partial_t).$$

**Remark 3.1.** Note that the 3-form $H + q\bar{F}$ is automatically closed under the differential $d + F\partial_q$ since this differential is the one that defines the structure on $\mathcal{R}$:

$$(d + F\partial_q)(H + q\bar{F}) = dH + F \wedge \bar{F} + qd\bar{F} = 0.$$ 

3.1. **Gauge transformations.** In order to find when two homological vector fields on $(T[1]M \oplus \mathbb{R}[1]) \oplus \mathbb{R}[2]$ are homologous, we calculate the bracket

$$[d + F\partial_q + (H + q\bar{F})\partial_t, \mathcal{L}_X + A\partial_q + (B + q\bar{A})\partial_t]$$

for $A$ and $\bar{A}$ 1-forms, $B$ a 2-form and $\mathcal{L}_X$ the Lie derivative of a vector field $X$ of $M$; the element

$$\mathcal{L}_X + A\partial_q + (B + q\bar{A})\partial_t$$

is a generic derivation of degree 0 of $\Omega^\bullet M \otimes \Lambda[q] \otimes \mathbb{R}[t]$. The bracket becomes

$$[d + F\partial_q + (H + q\bar{F})\partial_t, \mathcal{L}_X + A\partial_q + (B + q\bar{A})\partial_t] =$$

$$(dA - \mathcal{L}_X F)\partial_q$$

$$+(dB - \mathcal{L}_X H + F \wedge \bar{A} - A \wedge \bar{F})\partial_t$$

$$-q(d\bar{A} + \mathcal{L}_X \bar{F})\partial_t$$

which implies that the homological vector fields

$$d + F\partial_q + (H + q\bar{F})\partial_t$$

and

$$d + (F + dA - \mathcal{L}_X F)\partial_q + (H + dB - \mathcal{L}_X H + F \wedge \bar{A} - A \wedge \bar{F} + q(\bar{F} - d\bar{A} - \mathcal{L}_X \bar{F}))\partial_t$$

are homologous.

3.1.1. **Fiber-wise gauge transformations.** In the particular case on which $X = 0$ we get that the gauge transformation defined by $A, \bar{A}$ and $B$, via the degree 0 derivation $A\partial_q + (B + q\bar{A})\partial_t$, transforms $F, \bar{F}$ and $H$ in the following way:

$$F \mapsto F + dA$$

$$H \mapsto H + dB + F \wedge \bar{A} - A \wedge \bar{F}$$

$$\bar{F} \mapsto \bar{F} - d\bar{A}.$$ 

3.2. **T-duality.** In this section we will write the T-duality construction in the language of dg-manifolds. We recover the isomorphism of twisted cohomologies of the dual pairs by producing a degree -1 morphism between the cohomologies of the dual dg-manifolds. We believe that this alternative point of view might be of interest.
3.2.1. Consider the $\mathbb{R}[1]$-bundle over $\mathcal{M} = (T[1]M, d)$

$$\mathcal{E} = (T[1]M \oplus \mathbb{R}[1], Q_\mathcal{E} = d + F \partial_q)$$

where $F$ is a closed 2-form on $M$ and $q$ is a variable of degree 1. Recall from Proposition 2.3 that if $F$ is the curvature form of a circle bundle $E \to M$ then there is a canonical isomorphism $H^*(\mathcal{E}) \cong H^*(E)$.

Consider the $\mathbb{R}[2]$-bundle over $\mathcal{E}$

$$\mathcal{P} = (T[1]M \oplus \mathbb{R}[1] \oplus \mathbb{R}[2], Q_\mathcal{P} = d + F \partial_q + (H + q\bar{F}) \partial_t)$$

where $H + q\bar{F}$ is a closed three form on $\mathcal{E}$; hence we have that $\bar{F}$ is closed and $dH + F \bar{F} = 0$.

Since the equations $dF = 0 = d\bar{F}$ and $dH + F \bar{F} = 0$ are symmetric on $F$ and $\bar{F}$, we can consider the $\mathbb{R}[1]$-bundle

$$\bar{\mathcal{E}} = (T[1]M \oplus \mathbb{R}[1], Q_{\bar{\mathcal{E}}} = d + \bar{F} \partial_q)$$

together with the $\mathbb{R}[2]$-bundle over it

$$\bar{\mathcal{P}} = (T[1]M \oplus \mathbb{R}[1] \oplus \mathbb{R}[2], Q_{\bar{\mathcal{P}}} = d + \bar{F} \partial_q + (H + \bar{q}F) \partial_t).$$

Denote by $\mathcal{E} \times_{\mathcal{M}} \bar{\mathcal{E}}$ the dg-manifold

$$\mathcal{E} \times_{\mathcal{M}} \bar{\mathcal{E}} = (T[1]M \oplus \mathbb{R}[1] \oplus \mathbb{R}[1], Q_{\mathcal{E} \times_{\mathcal{M}} \bar{\mathcal{E}}} = d + F \partial_q + \bar{F} \partial_{\bar{q}}),$$

note that this dg-manifold fits in the pullback of the diagram:

![Diagram](image)

and consider the pullbacks of the $\mathbb{R}[2]$-bundles $\mathcal{P}, \bar{\mathcal{P}}$ to $\mathcal{E} \times_{\mathcal{M}} \bar{\mathcal{E}}$ which become the dg-manifolds

$$p^*\mathcal{P} = ((T[1]M \oplus \mathbb{R}[1] \oplus \mathbb{R}[1]) \oplus \mathbb{R}[2], Q_{p^*\mathcal{P}} = d + F \partial_q + \bar{F} \partial_{\bar{q}} + (H + q\bar{F}) \partial_t)$$

$$\bar{p}^*\bar{\mathcal{P}} = ((T[1]M \oplus \mathbb{R}[1] \oplus \mathbb{R}[1]) \oplus \mathbb{R}[2], Q_{\bar{p}^*\bar{\mathcal{P}}} = d + F \partial_q + \bar{F} \partial_{\bar{q}} + (H + \bar{q}F) \partial_t).$$

Since the 3-forms $(H + q\bar{F})$ and $(H + \bar{q}F)$ are cohomologous in $\mathcal{E} \times_{\mathcal{M}} \bar{\mathcal{E}}$ via the 2-form $\bar{q}q$, i.e.

$$(H + q\bar{F}) - (d + F \partial_q + \bar{F} \partial_{\bar{q}})(\bar{q}q) = (H + \bar{q}F),$$

we have that the degree 0 map

$$e^{\bar{q}q \partial_q} : \Omega^\bullet M \otimes \Lambda[q, \bar{q}] \otimes \mathbb{R}[t] \xrightarrow{\cong} \Omega^\bullet M \otimes \Lambda[q, \bar{q}] \otimes \mathbb{R}[t]$$

induces an isomorphism of dg-manifolds

$$e^{\bar{q}q \partial_q} : p^*\mathcal{P} \cong \bar{p}^*\bar{\mathcal{P}}.$$
Schematically we have

\[ \text{Diagram} \]

where \( p \) and \( \bar{p} \) denote the corresponding maps of dg-manifolds, and the dotted arrow corresponds to the vertical gauge transformation defined previously.

Consider the map of complexes à la Fourier-Mukai, which is the composition of the maps

\[ \text{Diagram} \]

where the pushforward map \( \bar{p}^* \) is the one that was defined in section 2.2.5, and \( C^*(\bar{P})[1]^k = C^{k-1}(P) \). Denoting the composition by

\[ T := \bar{p}^* \circ \bar{\varrho}^g \circ p^* \]

we get that the map

\[ T : (C^*(P), Q_P) \rightarrow (C^*(\bar{P})[1], Q_{\bar{P}}) \]

is a map of complexes.

**Definition 3.2.** The pair of dg-manifolds \((E, P)\) and \((\bar{E}, \bar{P})\) over \(M\) are said to be **T-dual** and they satisfy the usual relations

\[ \pi_*(H + q\bar{F}) = \bar{F} \]
\[ \bar{\pi}_*(H + qF) = F. \]

The map of complexes of degree -1

\[ T : (C^*(P), Q_P) \rightarrow (C^*(\bar{P})[1], Q_{\bar{P}}) \]

is called the **T-duality map**.

**Theorem 3.3.** Let \((E, P)\) and \((\bar{E}, \bar{P})\) be a T-dual pair of \(\mathbb{R}[2]\)-bundles over \(\mathbb{R}[1]\)-bundles over \(M\). Then the T-dual map

\[ T : (C^*(P), Q_P) \rightarrow (C^*(\bar{P})[1], Q_{\bar{P}}) \]

fits in the short exact sequence of complexes

\[ 0 \rightarrow (\Omega^* M, d) \rightarrow (C^*(P), Q_P) \xrightarrow{T} (C^*(\bar{P})[1], Q_{\bar{P}}) \rightarrow 0 \]

and therefore it induces a long exact sequence in cohomologies

\[ \rightarrow H^k(P, Q_P) \xrightarrow{T} H^{k-1}(\bar{P}, Q_{\bar{P}}) \rightarrow H^{k+1}M \rightarrow H^{k+1}(P, Q_P) \xrightarrow{T} H^k(P, Q_P) \rightarrow \]

In particular when \( k \geq \dim(M) \) the T-dual map induces an isomorphism in cohomologies

\[ T : H^{k+1}(P, Q_P) \xrightarrow{\cong} H^k(P, Q_P). \]
Proof. Let us calculate explicitly the map $T$ for generic elements of the form $\omega t^j$ and $q\eta t^l$ for $\omega$ and $\eta$ differential forms on $M$. We have that

$$T(\omega t^j) = p_* (e^{ij\theta_j} (\omega t^j)) = p_* (\omega t^j + jq\omega t^{j-1}) = (-1)^{|\omega|} jq\omega t^{j-1}$$

$$T(q\eta t^l) = p_* (e^{ij\theta_j} (q\eta t^l)) = p_* (q\eta t^l) = (-1)^{|\eta|} q\eta t^l$$

and therefore $T(\omega) = 0$ for $\omega$ a differential form on $M$. Therefore

$$\ker(T) = \Omega^* M$$

and since $Q_P$ restricted to $\Omega^* M$ is precisely the De Rham derivation, we have the desired short exact sequence of complexes

$$0 \to (\Omega^* M, d) \to (C^*(\mathcal{P}), Q_P) \xrightarrow{T} (C^*(\mathcal{P})[1], Q_P) \to 0.$$  

The connection homomorphism

$$\beta : H^{k-1}(\mathcal{P}, Q_P) \to H^{k+1} M$$

is defined as follows: take a cocycle representing the cohomology class in $H^{k-1}(\mathcal{P}, Q_P)$ and write it as

$$\sum_{i \geq 0} (\omega_i + q\eta_i) t^i.$$  

Then

$$\beta(\sum_{i \geq 0} (\omega_i + q\eta_i) t^i) := (-1)^{|\omega|} F\omega_0$$

since we have that $T((-1)^{|\omega|} q\omega_0) = \omega_0$ and the restriction of

$$(d + F\partial_q + (H + q\check{F})\partial_t)((-1)^{|\omega|} q\omega_0)$$

to $\Omega^* M$ is precisely $(-1)^{|\omega|} F\omega_0$.

The connection homomorphism is trivial whenever $|\omega| + 2$ is greater than the dimension of the manifold. Therefore we have that for $k \geq \dim(M)$ the T-dual map induces an isomorphism in cohomologies

$$T : H^{k+1}(\mathcal{P}, Q_P) \xrightarrow{\simeq} H^k(\mathcal{P}, Q_P).$$

Applying Proposition 2.3 to the dg-manifolds $\mathcal{E}$ and $\mathcal{E}$ we obtain the T-duality isomorphism in twisted cohomologies that was proven in [3, 2].

**Corollary 3.4.** Let $(\mathcal{E}, \mathcal{P})$ and $(\mathcal{E}, \mathcal{P})$ be a T-dual pair of $\mathbb{R}[2]$-bundles over $\mathbb{R}[1]$-bundles over $M$. Then the T-dual map induces an isomorphism of degree -1 of the twisted cohomologies

$$H^{\text{ev,od}}(\mathcal{E}, d + F\partial_q + (H + q\check{F})\wedge) \xrightarrow{\simeq} H^{\text{od,ev}}(\mathcal{E}, d + F\partial_q + (H + q\check{F})\wedge)$$

Here we should recall that Proposition 2.3 tells us that in the case that $F$ is the curvature form of a $S^1$-principal bundle $E \to M$, then the complex $C^*(\mathcal{E})$ is quasi-isomorphic to $\Omega^* E$, and therefore the twisted complex

$$(C^*(\mathcal{E}), d + F\partial_q + (H + q\check{F})\wedge)$$

is quasi-isomorphic to the twisted complex

$$(\Omega^* E, d_E + (H + \theta\check{F})\wedge)$$
where $\theta$ is a choice of connection 1-form of the $S^1$-bundle with curvature $F$. This previous argument has as a corollary the usual T-dual isomorphism for the twisted cohomology of T-dual pairs.

3.3. **Topological interpretation.** Recall that the rational homotopy of the Eilenberg-Maclane spaces $K(\mathbb{Z}, n)$ is equivalent to the graded symmetric algebras $S[z]$ generated by one variable of degree $n$ i.e. $S[z]$ is a polynomial algebra if $n$ is even and is an exterior algebra if $n$ is odd.

Then we should think that the analogue in the category of topological spaces of a $\mathbb{R}[n]$-bundle over a dg-manifold is a $K(\mathbb{Z}, n)$-bundle over a space. Recall that $K(\mathbb{Z}, n) = B^{n-1}S^1$ and that if $A$ is an abelian group, then $BA$ can be endowed with the structure of an abelian group. This means that the model $B^{n-1}S^1$ endows the Eilenberg-Maclane spaces with an abelian group structure and therefore it is plausible to work with principal $K(\mathbb{Z}, n)$-bundles.

The analogue of the construction done in the previous section should be understood as a principal $K(\mathbb{Z}, 2)$-bundle $P$ over the total space of a principal $S^1$-bundle $E$ over the manifold $M$. The closed 2-form $F$ on $M$ should be thought as the curvature of the principal bundle $E$, the variable $q$ should be thought as the connection 1-form on $E$, and the 3-form $H + \bar{q}F$ on $E$ should be thought as the curvature of the gerbe that classifies the $K(\mathbb{Z}, 2)$-bundle. Again, note that the form $H + \bar{q}F$ is closed under the differential $d + F\partial_q$ if and only if $dH + F\bar{F} = 0$ and $d\bar{F} = 0$.

4. **Symmetries of $\mathbb{R}[n]$-bundles**

4.1. **Symmetries of dg-manifolds.** Recall that a homological vector field $Q$ on the graded manifold $P$ is the same as a Maurer-Cartan element on the graded Lie algebra $\text{Vect}^*(P)$. Then any vector field $\alpha \in \text{Vect}^0(P)$ of degree 0 may define another Maurer-Cartan element by taking the action on $Q$ of the exponential of the adjoint action of $\alpha$

$$Q \mapsto e^{(\text{ad}_\alpha)}Q := Q + [\alpha, Q] + \frac{1}{2}[\alpha, [\alpha, Q]] + \cdots$$

whenever we know that the series above converge. The infinitesimal version of this action is given by the adjoint action of $\alpha$ on $Q$ and therefore the action is trivial whenever $[\alpha, Q] = 0$. We say then that the infinitesimal symmetries of the Maurer-Cartan element are given by vector fields $\alpha$ of degree 0 such that the adjoint action of $\alpha$ on $Q$ vanishes, i.e. $[\alpha, Q] = 0$. Note that these infinitesimal symmetries of $Q$ become a Lie algebra with respect to the brackets of $\text{Vect}^0(P)$, as we have that for $\alpha_1$ and $\alpha_2$ commuting with $Q$, the equality $[[\alpha_1, \alpha_2], Q] = 0$ follows from the Jacobi identity and the fact that $Q$ is a homological vector field.

Furthermore note that for any vector field $\beta$ of degree -1, the degree 0 vector field $[\beta, Q]$ commutes with $Q$ (again because of the Jacobi identity) and therefore it gives an infinitesimal symmetry of $Q$. This means that we have to see the symmetries of the dg-manifold $P$ as a differential graded Lie algebra, where the differential is defined by the operator $[Q, -]$ and the bracket is the one for vector fields.

**Definition 4.1.** Let $P$ be a dg-manifold with homological vector field $Q$. The (infinitesimal) symmetries of the dg-manifold $P$ with homology vector field $Q$ is
the differential graded Lie algebra $\mathfrak{sym}^*(P, Q)$ with
\[
\mathfrak{sym}^q(P, Q) = \begin{cases} 
\text{Vect}^q(P) & \text{for } q < 0 \\
\{\alpha \in \text{Vect}^0(P) | [\alpha, Q] = 0\} & \text{for } q = 0 \\
0 & \text{for } q > 0 
\end{cases}
\]
whose differential is $[Q, \cdot]$ and the bracket is the bracket of vector fields.

4.1.1. The derived algebraic structure. To any negatively graded differential graded Lie algebra there is an associated dg-Leibniz algebra (see [17] for the explicit construction and also [13, 12, 20, 21, 8]).

**Definition 4.2.** The derived dg-Leibniz algebra $D\mathfrak{sym}^*(P)$ of $\mathfrak{sym}^*(P, Q)$ is the complex
\[
D\mathfrak{sym}^*(P, Q) := \mathfrak{sym}^{*-0}(P, Q)[1]
\]
together with the differential $\delta := \lfloor Q_P, \cdot \rfloor$ and the derived bracket
\[
[a, b] := (-1)^{|a||b|} [[Q_P, a], b].
\]

It is a fact that $D\mathfrak{sym}^*(P, Q)$ becomes a dg-Leibniz algebra; namely that $\delta$ and $[,]$ satisfy the properties
\[
\delta [a, b] = [\delta a, b] + (-1)^{|a||b|} [a, \delta b]
\]

\[
[a, [b, c]] = [[a, b], c] + (-1)^{|a||b|} [b, [a, c]]
\]

where $|a|$ denotes the degree of $a$ in $D\mathfrak{sym}^*(P, Q)$, and therefore $|a| = |a| + 1$ where $|a|$ is the degree of $a$ in $\mathfrak{sym}^*(P, Q)$.

Note that there is a canonical graded action
\[
\mathfrak{sym}^*(P, Q) \times D\mathfrak{sym}^*(P, Q) \to D\mathfrak{sym}^*(P, Q)
\]
\[
a \cdot b \mapsto [a, b]
\]
whose properties follow from the fact that $[Q, \cdot]$ and $[,]$ form a differential graded Lie algebra. In particular note that since the elements $a \in \mathfrak{sym}^0(P, Q)$ satisfy $[Q, a] = 0$, then $\mathfrak{sym}^0(P, Q)$ acts by derivations on the Leibniz algebra $(D\mathfrak{sym}^0(P, Q), [,])$.

4.2. Symmetries of $\mathbb{R}[n]$-bundles over $T[1]M$. Let $\mathcal{R} = (T[1]M \oplus \mathbb{R}[n], Q_{\mathcal{R}} = d + \Theta \partial_t)$ be a $\mathbb{R}[n]$-bundle over $\mathcal{M}$ where $\Theta \in \Omega^{n+1}_{cl} M$ and $t$ is a variable of degree $n$. Since the bracket of $Q_{\mathcal{R}}$ with a generic degree 0 vector field gives
\[
[d + \Theta \partial_t, \mathcal{L}_X + B \partial_t] = (dB - \mathcal{L}_X \Theta) \partial_t
\]
then the degree 0 symmetries of $\mathcal{R}$ is the Lie algebra
\[
\mathfrak{sym}^0(\mathcal{R}, Q_{\mathcal{R}}) = \{ \mathcal{L}_X + B \partial_t | X \in \mathcal{X} M, B \in \Omega^n M \text{ and } (dB - \mathcal{L}_X \Theta) = 0 \},
\]
and the negatively graded symmetries are
\[
\mathfrak{sym}^{-1}(\mathcal{R}, Q_{\mathcal{R}}) = \{ \nu + A \partial_t | Y \in \mathcal{X} M \text{ and } A \in \Omega^{n-1} M \}
\]
\[
\mathfrak{sym}^{-k}(\mathcal{R}, Q_{\mathcal{R}}) = \{ \eta \partial_t | \eta \in \Omega^{n-k} M \} \text{ for } k > 1.
\]

The differential in $\mathfrak{sym}^*(\mathcal{R}, Q_{\mathcal{R}})$ becomes
\[
[Q_{\mathcal{R}}, \mathcal{L}_X + B \partial_t] = 0
\]
\[
[Q_{\mathcal{R}}, \iota_X + A \partial_t] = \mathcal{L}_X + (dA + \iota_X \Theta) \partial_t
\]
\[
[Q_{\mathcal{R}}, \eta \partial_t] = (d\eta) \partial_t,
\]
and the brackets become

\[
\begin{align*}
[L_{X_0} + B_0 \partial_t, L_{X_1} + B_1 \partial_t] &= L_{[X_0, X_1]} + (L_{X_0}B_1 - L_{X_1}B_0) \partial_t \\
[L_{X} + B \partial_t, \iota_Y + A \partial_t] &= \iota_{[X,Y]} + (L_X A - \iota_Y B) \partial_t \\
[L_{X} + B \partial_t, \eta \partial_t] &= (L_X \eta) \partial_t \\
[\iota_Y + A_0 \partial_t, \iota_Y + A_1 \partial_t] &= (\iota_Y A_1 + \iota_Y A_0) \partial_t \\
[\iota_X + \alpha \partial_t, \eta \partial_t] &= (\iota_X \eta) \partial_t.
\end{align*}
\]

Note that when the \(n+1\) form \(\Theta = 0\), the differential graded Lie algebra structure defined above is the same one that was defined by Dorfman in [7].

4.3. Derived symmetries of \(\mathbb{R}[n]\)-bundles over \(T[1]M\). The derived dg-Leibniz algebra of the \(\mathbb{R}[1]\)-bundle \(\mathcal{R}\) over \(\mathcal{M}\) is

\[
\text{Dom}^k(\mathcal{R}, Q_{\mathcal{R}}) \cong \begin{cases} 
\mathfrak{X}M \oplus \Omega^{n-1}M & \text{if } k = 0 \\
\Omega^{n-1-k}M & \text{if } k < 0
\end{cases}
\]

where the differential \([Q_{\mathcal{R}}, \cdot]\) becomes the De Rham differential

\[
\Omega^0M \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{n-2}M \xrightarrow{d} \mathfrak{X}M \oplus \Omega^{n-1}M,
\]

and the brackets are given by the formulas

\[
\begin{align*}
[\iota_{X_0} + A_0 \partial_t, \iota_{X_1} + A_1 \partial_t] &= \iota_{[X_0, X_1]} + (L_{X_0}A_1 - \iota_{X_1}dA_0 - \iota_{X_1}\iota_{X_0} \Theta) \partial_t \\
[\iota_X + \alpha \partial_t, \eta \partial_t] &= L_X \eta \partial_t \\
[\eta \partial_t, \iota_X + A \partial_t] &= -(\iota_X d\eta) \partial_t \\
[\mu \partial_t, \eta \partial_t] &= 0.
\end{align*}
\]

4.3.1. Derived symmetries of \(\mathbb{R}[1]\)-bundles. The derived symmetries of \((T[1]M \oplus \mathbb{R}[1], d + F \partial_q)\) is simply the Lie algebra \((\mathfrak{X}M \oplus C^\infty M, [\cdot, \cdot])\) where the bracket is

\[
[X \oplus f, Y \oplus g] = [X, Y] \oplus (X(g) - Y(f) - \iota_X \iota_Y F).
\]

Whenever \(F\) is the curvature 2-form of a principal \(S^1\)-bundle \(E \to M\), this Lie algebra is isomorphic to the Lie algebra \(\mathfrak{X}E \oplus S^1\) of invariant vector fields on \(E\).

4.3.2. Derived symmetries of \(\mathbb{R}[2]\)-bundles. In the case that \(n = 2\) and whenever \(\Theta = H\) is a closed three form, the derived algebra of \((T[1]M \oplus \mathbb{R}[2], d + H \partial_t)\) becomes the complex

\[
\Omega^0M \xrightarrow{d} \mathfrak{X}M \oplus \Omega^1M
\]

where the bracket \([\cdot, \cdot]\) is precisely the \(H\)-twisted Courant-Dorfman bracket of the exact Courant algebroid \(TM \oplus T^*M\) (cf. [4, 15]).

4.4. Symmetries of \(\mathbb{R}[2]\)-bundles over \(\mathbb{R}[1]\)-bundles. For the \(\mathbb{R}[2]\)-bundle over the \(\mathbb{R}[1]\)-bundle

\[
P := ((T[1]M \oplus \mathbb{R}[1]) \oplus \mathbb{R}[2], Q_P = d + F \partial_q + (H + qF) \partial_t)
\]
as in chapter 3 we have that its differential graded Lie algebra of symmetries $\mathfrak{sym}^*(\mathcal{P}, Q_P)$ is defined by

\[
\mathfrak{sym}^{*-2}(\mathcal{P}, Q_P) = 0 \\
\mathfrak{sym}^{-2}(\mathcal{P}, Q_P) \cong (C^\infty M)\partial_t \\
\mathfrak{sym}^{-1}(\mathcal{P}, Q_P) \cong \mathfrak{X} M \oplus (C^\infty M)\partial_q \oplus (\Omega^1 M \oplus C^\infty M)\partial_t \\
\mathfrak{sym}^0(\mathcal{P}, Q_P) = \{ \mathfrak{X} \in \text{Vect}^0(\mathcal{P}) | [Q_P, \mathfrak{X}] = 0 \}
\]

with differential $[Q_P, -]$

\[
0 \to \mathfrak{sym}^{-2}(\mathcal{P}, Q_P) \xrightarrow{[Q_P, -]} \mathfrak{sym}^{-1}(\mathcal{P}, Q_P) \xrightarrow{[Q_P, -]} \mathfrak{sym}^0(\mathcal{P}, Q_P)
\]

and with bracket $[\cdot, \cdot]$

The elements in $\mathfrak{sym}^0(\mathcal{P}, Q_P)$, which are the degree zero derivations that commute with $Q_P$, are the elements

$\mathfrak{X} = \mathcal{L}_X + A\partial_q + (B + q\bar{A})\partial_t \in \text{Vect}^0(\mathcal{P})$

such that $[Q_P, \mathfrak{X}] = 0$, and this is equivalent to the equations

\[
dA - \mathcal{L}_X F = 0 \\
dB - \mathcal{L}_X H + F \wedge \bar{A} - A \wedge \bar{F} = 0 \\
d\bar{A} + \mathcal{L}_X \bar{F} = 0.
\]

4.4.1. The differential structure. Vector fields of degree $-2$ will be denoted by $h\partial_t$ for $h$ in $C^\infty M$ and vector fields of degree $-1$ will be denoted by

$\iota_Y + f\partial_q + (C + q\bar{f})\partial_t$

with $\iota_Y$ the contraction on the vector field $Y$ of $M, f, \bar{f}$ in $C^\infty M$ and $C$ in $\Omega^1 M$. Then $[Q_P, -]$ acts as follows:

\[
[Q_P, h\partial_t] = dh\partial_t \\
[Q_P, \iota_Y + f\partial_q + (C + q\bar{f})\partial_t] = \mathcal{L}_Y + (df + \iota_Y F)\partial_q + (dC + \iota_Y H + F \bar{f} + f\bar{F})\partial_t - q(df + \iota_Y \bar{F})\partial_t \\
[Q_P, \mathcal{L}_X + A\partial_q + (B + q\bar{A})\partial_t] = 0
\]

4.4.2. The graded Lie algebra structure. The graded Lie algebra structure of $\mathfrak{sym}^*(\mathcal{P}, Q_P)$ is given by the following explicit formulas.

For two degree zero derivations, the bracket is:

\[
[\mathcal{L}_{X_0} + A_0\partial_q + (B_0 + q\bar{A}_0)\partial_t, \mathcal{L}_{X_1} + A_1\partial_q + (B_1 + q\bar{A}_1)\partial_t] = \\
\mathcal{L}_{[X_0, X_1]} + (\mathcal{L}_{X_0}A_1 - \mathcal{L}_{X_1}A_0)\partial_q + (\mathcal{L}_{X_0}B_1 - \mathcal{L}_{X_1}B_0 + A_0\bar{A}_1 - A_1\bar{A}_0)\partial_t + q(\mathcal{L}_{X_0}\bar{A}_1 - \mathcal{L}_{X_1}\bar{A}_0)\partial_t
\]

and note that when $X_0 = 0 = X_1$ the bracket simplifies to

\[
[A_0\partial_q + (B_0 + q\bar{A}_0)\partial_t, A_1\partial_q + (B_1 + q\bar{A}_1)\partial_t] = (A_0\bar{A}_1 - A_1\bar{A}_0)\partial_t.
\]
For two degree -1 derivations, the bracket is:

\[ [\iota_X + f_0 \partial_q + (C_0 + q f_0) \partial_t, \iota_Y + f_1 \partial_q + (C_1 + q f_1) \partial_t] = \]

\[ (\iota_X + \iota_Y ) C_1 + \iota_X C_0 + f_0 f_1 + f_1 f_0 \partial_t \]

and the remaining two brackets are

\[ [\mathcal{L}_X + \mathcal{A} \partial_q + (B + q \dot{A}) \partial_t, \iota_Y + f \partial_q + (C + q \dot{f}) \partial_t] = \]

\[ \iota_{[X,Y]} + (\mathcal{L}_X f - \iota_Y A) \partial_q \]

\[ + (\mathcal{L}_X C - \iota_Y B + A \dot{f} - f \dot{A}) \partial_t \]

\[ q(\mathcal{L}_X \dot{f} + \iota_Y \dot{A}) \partial_t \]

\[ [\mathcal{L}_X + \mathcal{A} \partial_q + (B + q \dot{A}) \partial_t, h \partial_t] = \mathcal{L}_X h \partial_t \]

4.4.3. The derived dg-Leibniz algebra. The derived algebra is then

\[ D\text{sym}^k(\mathcal{P}, Q_P) \cong \begin{cases} 
X M \oplus (C^\infty M) \partial_q \oplus (\Omega^1 M \oplus q(C^\infty M)) \partial_t & \text{if } k = 0 \\
(C^\infty M) \partial_t & \text{if } k = -1 
\end{cases} \]

where the differential \([Q_P, \_]\) becomes the De Rham differential, and the derived bracket is given by the formula

\[ (4.1) \]

\[ [\iota_X + f_0 \partial_q + (C_0 + q f_0) \partial_t, \iota_Y + f_1 \partial_q + (C_1 + q f_1) \partial_t] = \]

\[ \iota_{[X,Y]} + (\mathcal{L}_X f_1 - \iota_Y d f_0 - \iota_Y \iota_X F) \partial_q \]

\[ + (\mathcal{L}_X C_1 - \iota_Y d C_0 - \iota_Y \iota_X H - \iota_Y (F f_0) - \iota_Y (f_0 F) + (d f_0) \dot{f}_1 + (f_1 d f_0 + f_1 \iota_X F) \partial_t \]

\[ q(\mathcal{L}_X \dot{f}_1 - \iota_Y \dot{f}_0 - \iota_Y \iota_X \dot{F}) \partial_t \]

and the extra action is given by the bracket

\[ [\iota_X + f \partial_q + (C + q \dot{f}) \partial_t, h \partial_t] = \mathcal{L}_X h \partial_t. \]

Remark 4.3. Since \((H + q \dot{F})\) is a closed 3-form on the dg-manifold \(E = (T[1]M \oplus \mathbb{R}[1], Q_E = d + F \partial_q)\), the previous derived algebraic structure is equivalent to the derived algebraic structure associated to \((E \oplus \mathbb{R}[2], Q_E = (H + q \dot{F}) \partial_t)\) where the Courant-Dorfman bracket has the usual structure

\[ [(f_0 \partial_q) + (C_0 + q f_0) \partial_t, (f_1 \partial_q) + (C_1 + q f_1) \partial_t] = \]

\[ \iota_{[X,Y]} + (X(f_1) - Y(f_0) - \iota_Y \iota_X F) \partial_q \]

\[ + [Q_E, \iota_X + f_0 \partial_q](C_1 + q f_1) \partial_t \]

\[ - (\iota_Y + f_1 \partial_q)(Q_E(C_0 + q f_0)) \partial_t \]

\[ - (\iota_Y + f_1 \partial_q)(\iota_X + f_0 \partial_q)(H + q \dot{F}) \partial_t, \]

where we should think of \(Q_E\) as the De Rham differential and of \([Q_E, \iota_X + f_0 \partial_q]\) as the Lie derivative.
4.5. **Isomorphism of symmetries for T-dual pairs.** Consider a T-dual pair of \( \mathbb{R}^2 \)-bundles \((\mathcal{E}, \mathcal{P})\) and \((\bar{\mathcal{E}}, \bar{\mathcal{P}})\) with

\[
\mathcal{P} = ((T[1]\mathbb{M} \oplus \mathbb{R}[1]) \oplus \mathbb{R}[2], Q_\mathcal{P} = d + F\partial_\theta + (H + qF)\partial_t)
\]

\[
\bar{\mathcal{P}} = ((T[1]\bar{\mathbb{M}} \oplus \bar{\mathbb{R}}[1]) \oplus \mathbb{R}[2], Q_{\bar{\mathcal{P}}} = d + \bar{F}\partial_\theta + (\bar{H} + q\bar{F})\partial_t)
\]

as in section 3.2. Consider the differential graded Lie algebras of symmetries \( \mathfrak{sym}^*(\mathcal{P}, Q_\mathcal{P}) \) and \( \mathfrak{sym}^*(\bar{\mathcal{P}}, Q_{\bar{\mathcal{P}}}) \) of both dg-manifolds and define the isomorphism of graded vector spaces

\[
\Phi : \mathfrak{sym}^*(\mathcal{P}, Q_\mathcal{P}) \to \mathfrak{sym}^*(\bar{\mathcal{P}}, Q_{\bar{\mathcal{P}}})
\]

\[
\mathcal{L}_X + A\partial_\theta + (B + q\bar{A})\partial_t \mapsto \bar{\mathcal{L}}_X - \bar{A}\partial_\theta + (\bar{B} - \bar{q}A)\partial_t
\]

\[
\iota_Y + f\partial_\theta + (C + \bar{q}\bar{f})\partial_t \mapsto \bar{\iota}_Y + \bar{f}\partial_\theta + (C + \bar{q}\bar{f})\partial_t
\]

\[
h\partial_t \mapsto h\partial_t.
\]

A straightforward calculation shows that indeed the map \( \Phi \) is an isomorphism of differential graded Lie algebras.

**Theorem 4.4.** Let \((\mathcal{E}, \mathcal{P})\) and \((\bar{\mathcal{E}}, \bar{\mathcal{P}})\) be T-dual dg-manifolds over \( \mathcal{M} \). Then the map

\[
\Phi : \mathfrak{sym}^*(\mathcal{P}, Q_\mathcal{P}) \to \mathfrak{sym}^*(\bar{\mathcal{P}}, Q_{\bar{\mathcal{P}}})
\]

is an isomorphism of differential graded Lie algebras. In particular, the induced map

\[
D\Phi : D\mathfrak{sym}^*(\mathcal{P}, Q_\mathcal{P}) \to D\mathfrak{sym}^*(\bar{\mathcal{P}}, Q_{\bar{\mathcal{P}}})
\]

on the derived dg-Leibniz algebras is also an isomorphism.

The previous theorem implies the isomorphism of exact algebroids associated to twisted T-dual \( S^1 \)-principal bundles that was proved in [2]. Let us spell this out.

4.5.1. **T-duality isomorphism for exact Courant algebroids.** Consider the \( S^1 \)-principal bundles \( E \xrightharpoonup{\pi} M \) and \( \bar{E} \xrightharpoonup{\bar{\pi}} M \) with associated curvature closed 2-forms \( F \) and \( \bar{F} \) respectively, and consider \( S^1 \)-invariant and closed 3-forms \( \eta \in (\Omega^3_{cl}E)^{S^1} \) and \( \bar{\eta} \in (\Omega^3_{cl}\bar{E})^{S^1} \) such that \( \eta = \pi^*H + \theta \wedge \pi^*\bar{F} \) and \( \bar{\eta} = \bar{\pi}^*\bar{H} + \bar{\theta} \wedge \bar{\pi}^*F \) where \( \theta \) and \( \bar{\theta} \) are connection 1-forms on \( E \) and \( \bar{E} \) respectively.

The choice of connections provide us with isomorphisms \((TE)/S^1 \cong TM \oplus \langle \partial_\theta \rangle\), \((T\bar{E})/S^1 \cong \bar{TM} \oplus \langle \partial_{\bar{\theta}} \rangle\), \((T^*E)/S^1 \cong T^*M \oplus \langle \theta \rangle\) and \((T^*\bar{E})/S^1 \cong T^*M \oplus \langle \bar{\theta} \rangle\) where \( \partial_\theta \) and \( \partial_{\bar{\theta}} \) denote the vector fields generated by the circle action on \( E \) and \( \bar{E} \) respectively with period 1.

A section of \((TE \oplus T^*E)/S^1\) can be written as

\[
X + f\partial_\theta + C + \bar{f}\bar{\theta}
\]

where \( X \) is a vector field on \( M \), \( f, \bar{f} \) are functions on \( M \) and \( C \) is a 1-form on \( M \), and therefore we can write the isomorphism

\[
\Psi : \Gamma(TE \oplus T^*E)^{S^1} \to \mathfrak{sym}^0(\mathcal{P}, Q_\mathcal{P})
\]

\[
X + f\partial_\theta + C + \bar{f}\bar{\theta} \mapsto \iota_X + f\partial_\theta + (C + \bar{q}\bar{f})\partial_t
\]

which in view of Remark 3.3 it induces an isomorphism of Leibniz algebras

\[
\Psi : (\Gamma(TE \oplus T^*E)^{S^1}, [\cdot, \cdot]) \cong (\mathfrak{sym}^0(\mathcal{P}, Q_\mathcal{P}), [\cdot, \cdot])
\]
where $[,]_\eta$ is the $\eta$-twisted Courant-Dorfman bracket. Obtaining the equivalent isomorphism for the dual

$$\tilde{\Psi} : (\Gamma(T\tilde{E} \oplus T^*\tilde{E})^{S^1}, [ , ]_{\tilde{\eta}}) \xrightarrow{\cong} (\mathfrak{sym}^0(\mathcal{P}, Q_{\mathcal{P}}), [ , ])$$

$$X + f\partial_\theta + C + \bar{f}\tilde{\theta} \mapsto \iota_X + f\partial_\theta + (C + \bar{q}\tilde{f})\partial_t$$

we conclude that

**Corollary 4.5.** For the $T$-dual pair of principal $S^1$-bundles $(E, \eta)$ and $(\tilde{E}, \tilde{\eta})$ over $M$ defined above, the composition

$$\tilde{\Psi}^{-1} \circ \Phi \circ \Psi : (\Gamma(T E \oplus T^* E)^{S^1}) \rightarrow \Gamma(T\tilde{E} \oplus T^*\tilde{E})^{S^1}$$

$$X + f\partial_\theta + C + g\theta \mapsto X + g\partial_\theta + C + f\tilde{\theta}$$

induces an isomorphism of Courant algebroids

$$(\Gamma(T E \oplus T^* E)^{S^1}, [ , ]_\eta) \cong (\Gamma(T\tilde{E} \oplus T^*\tilde{E})^{S^1}, [ , ]_{\tilde{\eta}}).$$

### 4.6. Relation with exceptional generalized geometry

Exceptional generalized geometry is an algebraic framework suited to the study solutions of M-theory with fluxes \[11, 14\]. In \[1\] it is shown how the framework of the exceptional generalized geometry can be obtained from certain properties associated to simple Lie algebras. Rather than reproducing what has been done in the above cited references, we will show how the symmetries of $\mathbb{R}[n]$-bundles over dg-manifolds provide an alternative way to obtain some of the algebraic structures that appear in \[1\].

#### 4.6.1. $B_n$ generalized geometry

Consider a $\mathbb{R}[2]$-bundle over a $\mathbb{R}[1]$-bundle over $\mathcal{M}$ which is T-dual to itself, that is

$$\mathcal{P} = ((T[1]M \oplus \mathbb{R}[1]) \oplus \mathbb{R}[2], d + F\partial_q + (H + qF)\partial_t)$$

satisfying the equations $dF = 0$ and $dH + F \wedge F = 0$.

Consider the sub-differential graded Lie algebra $\mathcal{B}^*$ of $\mathfrak{sym}^*(\mathcal{P}, Q_{\mathcal{P}})$ which consist of elements of the form

$$\mathcal{L}_X + A\partial_q + (B - qA)\partial_t \in \mathcal{B}^0$$

$$\iota_Y + f\partial_q + (C + qf)\partial_t \in \mathcal{B}^{-1}$$

$$h\partial_t \in \mathcal{B}^{-2}$$

and note that the restrictions on $A$ and $B$ become the equations

$$dA - \mathcal{L}_X F = 0 \quad \text{and} \quad dB - \mathcal{L}_X H = 0.$$
then the derived bracket on $DB^0$ becomes
\[
[[X, f, C], (Y, g, D)] = ([X, Y], (\mathcal{L}_X g - \mathcal{L}_Y f - \iota_Y \iota_X F), \quad (\mathcal{L}_X D - \iota_Y dC - \iota_Y \iota_X H - 2f \iota_Y F + 2g \iota_X F + 2g F))
\]
and the pairing comes from the bracket of degree -1 derivations, i.e.
\[
\langle (X, f, C), (Y, g, D) \rangle = \iota_X D + \iota_Y C + fg.
\]

The Lie algebra $B^0$ act on $DB^0$ by the bracket $[,]$ and we see that closed 1-forms $A$ and closed 2-forms $B$ act as follows
\[
A(X, f, C) = (0, -\iota_X A, 2Af),
\]
\[
B(X, f, C) = (0, 0, -\iota_X B).
\]

Note furthermore that since the $\mathbb{R}[2]$-bundle $\mathcal{P}$ is T-dual to itself, the automorphism of differential graded Lie algebras $\Phi : \mathfrak{sym}^*(\mathcal{P}, Q_{\mathcal{P}}) \rightarrow \mathfrak{sym}^*(\mathcal{P}, Q_{\mathcal{P}})$ defined in section 4.5 induces an automorphism $D\Phi : \mathfrak{sym}^*(\mathcal{P}, Q_{\mathcal{P}}) \rightarrow \mathfrak{sym}^*(\mathcal{P}, Q_{\mathcal{P}})$ on the derived dg-Leibniz algebras. It is easy to see that the dg-Leibniz algebra $DB^*$ is precisely the fixed points of the automorphism $D\Phi$. Finally, noting that the previous algebraic structure on $\mathfrak{sym}^*(\mathcal{P}, Q_{\mathcal{P}})$ defined in section 2.4 of [1], we conclude

**Theorem 4.6.** The $B_n$ generalized geometry structure on $\mathfrak{sym}^*(\mathcal{P}, Q_{\mathcal{P}})$ is isomorphic to the fixed sub dg-Leibniz algebra of fixed points of the $\Phi$ automorphism of Courant-Dorfman algebroids

\[
D\Phi : \mathfrak{sym}^*(\mathcal{P}, Q_{\mathcal{P}}) \rightarrow \mathfrak{sym}^*(\mathcal{P}, Q_{\mathcal{P}})
\]
on the derived dg-Leibniz algebra of the symmetries of the self T-dual dg-manifold
\[
\mathcal{P} = ((T[1]M \oplus \mathbb{R}[1]) \oplus \mathbb{R}[2], d + F\partial_q + (H + q F)\partial_t).
\]

In particular we note that the algebraic structure underlying the $B_n$ generalized geometry can be obtained as a sub-algebra of the algebraic structure underlying the $L_n$ generalized geometry.

### 4.6.2. $E_6$ generalized geometry.
Consider a $\mathbb{R}[6]$-bundle over a $\mathbb{R}[3]$-bundle over $\mathcal{M}$ which is given by the data
\[
\mathcal{P} = ((T[1]M \oplus \mathbb{R}[3]) \oplus \mathbb{R}[6], d + F_3\partial_q + (F_7 + q F_1)\partial_t)
\]
for $q$ a variable of degree 3, $t$ a variable of degree 6, $F_4$ a closed 4-form and $F_7$ a 7-form satisfying the equation $dF_7 + \frac{1}{2}F_4 \wedge F_4 = 0$.

Consider the sub-differential graded Lie algebra $C^*$ of $\mathfrak{sym}^*(\mathcal{P}, Q_{\mathcal{P}})$ which consist of elements of the form
\[
\mathcal{L}_X + A_3\partial_q + (B_6 - q A_3)\partial_t \in C^0
\]
\[
\iota_Y + \sigma_2\partial_q + (\sigma_5 + q \frac{\sigma_2}{2})\partial_t \in C^{-1}
\]
\[
\eta_1\partial_q + (C_4 - q \frac{\eta_1}{2})\partial_t \in C^{-2}
\]
\[
f\partial_q + (D_3 + q f)\partial_t \in C^{-3}
\]
\[
h\partial_t \in C^{<-3}
\]
and note that the restrictions on $A_3$ and $B_6$ become the equations

$$dA_3 - \mathcal{L}_X F_4 = 0 \quad \text{and} \quad dB_6 - \mathcal{L}_X F_7 = 0.$$ 

If we denote

$$(X, \sigma_2, \sigma_5) := \iota_X + \sigma_2 \partial_q + (\sigma_5 + q \frac{\sigma_2}{2}) \partial_t$$

then the derived bracket on $DC^0$ becomes

$$\lfloor (X, \sigma_2, \sigma_5), (Y, \tau_2, \tau_5) \rfloor = (\mathcal{L}_X \tau_2 - \iota_Y \sigma_2 - \iota_Y \iota_X F_4),$$ 

$$(\mathcal{L}_X \tau_5 - \iota_Y d\sigma_5 - \iota_Y \iota_X F_7 - \iota_Y (F_4 \wedge \sigma_2) + (\iota_X F_4) \wedge \tau_2 + d\sigma_2 \wedge \tau_2)$$

and the pairing becomes

$$\langle (X, \sigma_2, \sigma_5), (Y, \tau_2, \tau_5) \rangle = (0, \iota_X \tau_2 + \iota_Y \sigma_2, \iota_X \tau_5 + \iota_Y \sigma_5 + \sigma_2 \wedge \tau_2).$$

The Lie algebra $C^0$ act on $DC^0$ by the bracket $[,]$ and we see that closed 3-forms $A_3$ and closed 6-forms $B_6$ act as follows

$$A_3(X, \sigma_2, \sigma_5) = (0, -\iota_X A_3, A_3 \wedge \sigma_2)$$

$$B_6(X, \sigma_2, \sigma_5) = (0, 0, -\iota_X B_6).$$

The dg-Leibniz algebra $DC^*$ just defined is one piece of the algebraic structure defined on the exceptional generalized geometry framework [11, 14]. The previous algebraic structure on $\mathfrak{X}M \oplus \Omega^2 M \oplus \Omega^5 M$ is the underlying algebraic structure of the $E_6$ generalized geometry outlined in [1, Chapter 11].

4.6.3. We conclude this section by noting that the category of dg-manifolds provides a natural framework for generalized geometry and for its exceptional relative. In particular we note that the derived algebraic structure obtained in the previous two sections satisfy the axioms of a dg-Leibniz algebra for they are constructed as derived differential graded Lie algebras. The derived brackets in section 4.6.2 may appear complicated at first sight, nevertheless they are the brackets that satisfy the desired properties (cf. [1, Chapter 11]).

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