Cosmological solutions with dilaton 
and maximally symmetric space in string theory

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We study time-dependent solutions of the leading order string effective equations for a non-zero central charge deficit and curved maximally symmetric space. Some regular solutions are found for the case of non-trivial antisymmetric tensor and vector backgrounds (in various dimensions) and negative spatial curvature. It remains an open question which conformal theories are exact generalisations of these solutions. We discuss the analogy between the string cosmological solutions and the solutions of the standard first order renormalisation group equations interpolating between “static” conformal theories.

To appear in Int. J. Mod. Phys.:D

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1. Introduction

The standard approach to string cosmology is based on the analysis of time dependent solutions of the effective field theory equations which may contain some “matter” contributions (e.g. density and pressure of a string gas) but only the leading terms in the derivative ($\alpha'$) expansion (see, for example, refs. [1-20]). This approach is justified under the assumption that the evolution is “adiabatic” (i.e. that gradients are small). This assumption may be valid at late enough times (e.g. after the supersymmetry is broken and dilaton got a mass). However, in order to describe an early (“string”) phase of evolution and, in particular, to understand the issue of cosmological singularity it is necessary to go beyond the derivative expansion. There is no much sense in including some of higher derivative terms (e.g. $R^2$ ones) in the effective action since if $\alpha'$ corrections are important one should account for all of them on an equal footing.

The only known alternative to the $\alpha'$- expansion is to find an exact 2d conformal theory which has an appropriate “cosmological” interpretation. While the first attempt in this direction used a direct product of the “time” and “3-space” conformal theories [5] some non-trivial time-dependent conformal theories based non-compact coset models associated with gauged Wess-Zumino-Witten theories were recently considered in [21–27] (see also [28]). This approach has its own obvious limitations. A conformal theory corresponds only to a perturbative (classical) solution of a superstring (Bose string) theory. It is not known whether non-perturbative solutions (e.g. extremals of an effective action which contains non-perturbative corrections like a dilaton potential) can be described in terms of 2d conformal theories. One may hope that the two approaches – the “phenomenological” one (based on a low energy effective action containing only leading terms in the $\alpha'$ expansion but including non-perturbative corrections and “matter” terms) and the “conformal” one may be complementary in the sense that they apply at different scales (times). For example, if at early enough times the dilaton potential is not yet generated, i.e. the dilaton is massless, one may hope to describe the cosmological evolution by an exact conformal field theory.
The usual method of giving a space-time interpretation to a conformal theory is based on relating it (in some approximation) to a solution of the leading order string effective equations (Weyl invariance conditions for the corresponding $\sigma$ model). While in $D = 2$ the most general solution [9,29] has the known conformal field theory counterpart [22] (see also [17]) the situation in higher dimensions $D \geq 3$ is very different: for most of the solutions of the leading order effective equations their conformal field theory generalizations are unknown. Conformal theories based on gauged WZW models seem to correspond to a rather small and special subclass of solutions of the $D \geq 3$ effective equations. In particular, they often have singularities and very few if any symmetries [23–28]. For example, only a subset of the simplest “toroidal” cosmological solutions with $N = D - 1$ commuting isometries [9] has known conformal field theory realisation [27].

The standard cosmological backgrounds have their spatial parts represented by maximally symmetric $N$-dimensional manifolds (or, in the Kaluza-Klein context, products of maximally symmetric manifolds), i.e. a sphere, a flat space or a pseudosphere. Since there should be such regular solutions of the effective string equations one is naturally led to the question about their possible conformal field theory generalisations. It is striking that the only example of a solution with maximally symmetric space which has known conformal field theory interpretation is the “static” $N = 3$ solution [5] corresponding to the direct product of the $D = 1$ (“time”) conformal theory with linear dilaton [4] and $SU(2)$ WZW model [30]. The $N = 3$ (pseudo)sphere is special being equivalent to a group space. $N \neq 3$ spheres and $D \neq 3$ de Sitter spaces do not directly correspond to conformal theories. For example, the “(anti) de Sitter string” of ref.[21] based on gauged $SO(D,1)/SO(D - 1,1)$ ($SO(D,1)/SO(D - 1,1)$) WZW model with $D > 26$ ($D < 26$) has a space-time interpretation not in terms of the de Sitter space but a background which does not have a maximally symmetric space [24–26].

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1 One should keep in mind, of course, that the assumption of maximal symmetry may not be a valid one at early times (e.g. before an inflationary stage).
One may hope to grasp some features of hypothetic “maximally symmetric” conformal theories by studying properties of particular solutions of the leading order string effective equations. This is one of the motivations behind the analysis of cosmological solutions in the present paper. Some of the solutions with maximal spatial symmetry may play a fundamental role, e.g. as building blocks for more complex cosmological backgrounds. In particular, it seems important to generalize the solution of ref.[9] (its isotropic limit) which has flat space sections to the case when the space has a non-zero curvature. That is why we shall not restrict the number of spatial dimensions $D$ and the value of the central charge deficit $c$.

Another motivation for a study of the time-dependent solutions is a possible application to a description of a “late time” string cosmology within the “phenomenological” approach mentioned above. Since the precise form of non-perturbative corrections to the effective action (e.g. to the dilaton potential) is unknown one would like to understand the behaviour of the solutions in various special cases in order to extract their universal features, in particular, some generalities in the evolution of the dilaton (which is non-trivial except in the radiation dominated phase where a $\phi = \text{const}$ solution may exist [17,13,20]).

Most of the previous discussions of the string theory cosmological solutions (see e.g. [6,7,19,20]) used the “Einstein frame”, i.e. the parametrisation of the metric in which there is no dilaton factor in front of the Einstein term in the effective action. Though the Einstein frame [31] and the string frame [32] should be physically equivalent, the string frame metric (i.e. the “$\sigma$ model metric” [32]) is the one strings directly interact with. That is why it is natural to use the string frame in order to try to understand a string theory interpretation of the solutions (see also [10,11]) and, in particular, to trace a possible relation to exact conformal field theories.

In Sect.2 we shall review the general structure of the cosmological evolution equations in arbitrary dimension $D$. The use of the string frame and “shifted” dilaton field [14,15] simplifies their form and makes a qualitative analysis of the solutions particularly transparent.
In Sect. 3 we shall consider the cosmological solutions in the case of non-trivial antisymmetric tensor and vector backgrounds. We shall see that for $c < 0$ the decreasing dilaton produces a damping force which stabilizes the solutions. We shall find a new analytic solution in the case of a rank $N$ antisymmetric tensor background and zero spatial curvature. The asymptotics of this solution is the same as that of the isotropic case of the vacuum solution of ref. [9], i.e. for $t \rightarrow \infty$ the scale factor approaches a constant while the dilaton is linearly decreasing. This does not necessarily imply that conformal theories which should correspond to these solutions are also the same and are trivial, i.e. equivalent to the direct product of time and flat $N$-space theories.

In Sect. 4 we shall study solutions with negative spatial curvature and $c < 0$. Solutions with $c > 0$ (or positive curvature in the absence of “matter”) appear to be singular. In particular, we shall find a regular vacuum solution (for arbitrary $D$) which is probably the “closest” string theory analog of the open de Sitter space. Though at small times it resembles the $D$-dimensional de Sitter space [17] its large $t$ behaviour is different: the scale factor is expanding as a power of time and not exponentially. The difference from the de Sitter space can be attributed to the non-trivial time dependence of the dilaton field. If the spatial curvature is non-zero there are no solutions with a constant dilaton even for $c = 0$.

The large $t$ asymptotics of the solutions of Sect. 4 in the string frame is different from that of the solutions of Sect. 3 and of refs. [4, 5, 9]: the scale factor is not approaching a finite

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2 The presence of a non-zero central charge deficit $c$ corresponds in the Einstein frame to the exponential dilaton potential. Similar systems were considered in the context of Kaluza-Klein cosmology based on higher dimensional supergravity models and in connection with $D = 4$ superstring theory (see e.g. [33–35, 3] and [7, 19, 20]).

3 For example, the solutions in refs. [22–28] and their asymptotics correspond to different conformal theories.

4 Our solution is different from the $D = 4$ solution of ref. [19] (even though they have the same large time asymptotics in the Einstein frame). The latter is a special “power law” solution which exists only in the presence of the $R^2$ term in the effective action.
constant, i.e. asymptotically the space–time is not a trivial product of time and space factors. This suggests that their conformal field theory generalisations should be different as well. Let us note that being transformed to the Einstein frame all the solutions (with $c \neq 0$) we have discussed have the same large time behaviour: the linear growth of the scale factor and the logarithmic decrease of the dilaton. This illustrates the point that it is not sufficient to find an asymptotic behaviour of a solution in the Einstein frame in order to identify the corresponding conformal theory.

The relation between cosmological solutions in the string frame and the Einstein frame will be studied in Appendix A. We shall find how the asymptotics of our solutions look like in the Einstein frame.

In Appendix B we shall demonstrate (using the analytic continuation which interchanges space and time coordinates) the correspondence between the $N = 1$ case of the solution found in Sect.3 and the charged black hole solution of $D = 2$ heterotic string theory [36].

In Appendix C we shall consider time–dependent solutions in the presence of an extra scalar field (e.g. a modulus of an extra compact dimension or a coupling corresponding to a nearly marginal perturbation of a conformal theory). We shall discuss a relation between string equations (which are of second order in $\frac{d}{dt}$) and the standard renormalisation group equation (which is of first order $\frac{d}{dt}$) providing two alternative pictures of an interpolation between different conformal points (cf. [39–41]). This illustrates the point that asymptotics of cosmological solutions may be related to different conformal theories (as well as being different from a conformal theory which generalizes the time–dependent solution itself). While solutions of the standard RG equations interpolating between “static” (or “$N$–dimensional”) conformal theories are not related to any “$N + 1$–dimensional” conformal theory, solutions of string equations should be interpretable as “$N + 1$–dimensional” conformal theories.
2. Cosmological equations for the scale factor and the dilaton

Our basic assumptions will be the following: (i) “adiabaticity” of cosmological evolution (in particular, a possibility to ignore higher derivative terms in the effective action); (ii) weak coupling (we shall ignore string loop corrections and impose the condition that the dilaton or the effective string coupling should not increase with time); (iii) maximal spatial symmetry. We shall study the following FRW-type cosmological background

\[ ds^2 = -dt^2 + a^2(t) \, d\Omega^2, \quad d\Omega^2 = g_{bc} dx^b dx^c, \]

\[ a(t) = e^{\lambda(t)}, \quad \phi = \phi(t), \quad b, c = 1, \ldots, N, \quad D = 1 + N, \]

where \( g_{bc} \) is the metric of a maximally symmetric \( N \) dimensional space with the radius of curvature \( k^{-1} \) (\( k = -1, 0, 1 \)), i.e. \( R_{bc} = k(N - 1)g_{bc} \).

The metric we shall consider corresponds to the “string frame” in which the leading order terms in the low energy expansion of the effective action of a closed string theory have the following form (we absorb the gravitational coupling constant into \( \phi \) and set \( \alpha' = 1 \))

\[ S = \int d^D x \sqrt{-G} \, e^{-2\phi} \left[ c + R + 4(\partial \phi)^2 - V(\phi) - L_m \right]. \]

Compared to the corresponding equations in the Einstein frame the resulting cosmological equations have simpler structure and more natural interpretation from string theory point of view. We shall discuss the relation between cosmological solutions in the string frame and the Einstein frame in Appendix A. \( V \) in (2.2) is a dilaton potential and

\[ c = -\frac{2}{3\alpha'}(D_{\text{eff}} - D_{\text{crit}}) \]

depends on details of particular string theory (\( D_{\text{eff}} = D, \quad D_{\text{crit}} = 26 \) in the Bose string theory, \( D_{\text{eff}} = \frac{3}{2}D, \quad D_{\text{crit}} = 15 \) in the superstring theory, etc). We consider \( c \) as an

\[ \text{One may, of course, consider a Kaluza–Klein–type cosmological ansatz with the space being a product (with different time dependent radii) of several maximally symmetric factors (see e.g. [33-35] and eqs. (2.24)–(2.31) below).} \]
arbitrary parameter and prefer not to include it into \(V\) (for discussions of “non-critical” string cosmology see e.g. [4,5,9,17,18]). The “matter” Lagrangian contains contributions of other “light” degrees of freedom (for example, vectors and the antisymmetric tensor \(B_{\mu\nu}\)). We can represent the sum of each of the matter field kinetic terms in the following generic way

\[
L_m = \sum_{n=1,2,3} \frac{1}{2n!} H_{\lambda_1...\lambda_n}^2 + ... , \quad H_{\lambda_1...\lambda_n} = n \partial_{[\lambda_1} B_{\lambda_2...\lambda_n]} . \tag{2.3}
\]

The combinations of the metric and dilaton equations which follow from (2.2) are

\[
R_{\mu\nu} + 2 D_\mu D_\nu \phi = - \frac{1}{4} \frac{\partial V}{\partial \phi} G_{\mu\nu} + \frac{e^{2\phi}}{\sqrt{-G}} \left( \frac{\delta S_m}{\delta \mu\nu} - \frac{1}{4} G_{\mu\nu} \frac{\delta S_m}{\delta \phi} \right) , \tag{2.4}
\]

\[
c + 2D^2 \phi - 4(\partial \phi)^2 = V + \frac{1}{4} (D - 2) \frac{\partial V}{\partial \phi} + \frac{e^{2\phi}}{\sqrt{-G}} \left[ \frac{1}{4} (D - 2) \frac{\delta S_m}{\delta \phi} - G_{\mu\nu} \frac{\delta S_m}{\delta G_{\mu\nu}} \right] . \tag{2.5}
\]

In the case of the metric (2.1) one finds from (2.4), (2.5)

\[
\ddot{\lambda} + N \dot{\lambda}^2 - 2 \dot{\phi} \dot{\lambda} + k(N - 1)e^{-2\lambda} = - \frac{1}{4} \frac{\partial V}{\partial \phi} - \frac{1}{2N} e^{2\phi - N\lambda} \left( \frac{\delta S_m}{\delta \lambda} + \frac{1}{2} N \frac{\delta S_m}{\delta \phi} \right) , \tag{2.6}
\]

\[
2 \ddot{\phi} - N \ddot{\lambda} - N \dot{\lambda}^2 = \frac{1}{4} \frac{\partial V}{\partial \phi} - e^{2\phi - N\lambda} \left( \frac{\delta S_m}{\delta \lambda} + \frac{1}{4} N \frac{\delta S_m}{\delta \phi} \right) , \tag{2.7}
\]

\[
c + N(N - 1) \dot{\lambda}^2 + 4 \dot{\phi}^2 - 4 \dot{\phi} \dot{\lambda} + kN(N - 1)e^{-2\lambda} = V - 2e^{2\phi - N\lambda} \frac{\delta S_m}{\delta G_{00}} , \tag{2.8}
\]

where \(G_{00} = -1\) after the variation. As is well known, eq. (2.8) is the “zero energy” constraint which is conserved as a consequence of (2.6) and (2.7) and hence gives only a restriction on the initial values of \(\dot{\phi}, \dot{\lambda}, \phi, \lambda\). The structure of the system (2.6)–(2.8) is made more transparent by introducing the “shifted” dilaton field \(\varphi\) [14,15,17]

\[
\varphi \equiv 2\phi - N\lambda , \quad \sqrt{-G} e^{-2\phi} = \sqrt{g} e^{-\varphi} . \tag{2.9}
\]

Then (2.2),(2.6)–(2.8) take the following form (we absorb the constant factor of the volume of \(N\)-space into \(\varphi\) )

\[
S = \int dt \ e^{-\varphi} \sqrt{-G_{00}} \left[ c - G^{00} N \dot{\lambda}^2 + G^{00} \dot{\varphi}^2 \right] - S_m'[G_{00}, \lambda, \varphi] , \tag{2.10}
\]
\[ c - N \dot{\lambda}^2 + \dot{\phi}^2 = 2U, \quad (2.11) \]
\[ \ddot{\lambda} - \dot{\phi} \dot{\lambda} = W_1, \quad \ddot{\phi} - N \dot{\lambda}^2 = W_2, \quad (2.12) \]

where

\[ U = -\frac{1}{2} k N (N - 1) e^{-2\lambda} + \frac{1}{2} V - e^\phi \frac{\delta S_m}{\delta G_{00}}, \quad (2.13) \]
\[ W_1 = -k (N - 1) e^{-2\lambda} - \frac{1}{2} \frac{\partial V}{\partial \phi} - \frac{1}{2N} e^\phi \frac{\delta S_m'}{\delta \lambda}, \quad (2.14) \]
\[ W_2 = \frac{1}{2} \frac{\partial V}{\partial \phi} - e^\phi \left( \frac{\delta S_m'}{\delta G_{00}} - \frac{1}{2} \frac{\delta S_m'}{\delta \phi} \right), \quad (2.15) \]

and

\[ V(\lambda, \phi) = V(\phi), \quad S'_m[G_{00}, \lambda, \phi] = S_m[G_{00}, \lambda, \phi], \quad \phi = \frac{1}{2}(\phi + N\lambda). \quad (2.16) \]

The conservation of (2.10) implies (we are assuming of course that \( S_m \) is covariant under reparametrisations of time)

\[ \frac{\partial U}{\partial \lambda} = -NW_1, \quad \frac{\partial U}{\partial \phi} = W_2. \quad (2.17) \]

The resulting system

\[ c - N \dot{\lambda}^2 + \dot{\phi}^2 = 2U, \quad (2.18) \]
\[ \ddot{\lambda} - \dot{\phi} \dot{\lambda} = -\frac{1}{N} \frac{\partial U}{\partial \lambda}, \quad (2.19) \]
\[ \ddot{\phi} - N \dot{\lambda}^2 = \frac{\partial U}{\partial \phi}, \quad (2.20) \]

has an obvious mechanical interpretation and is very useful for a qualitative analysis of the solutions [17,18]. Note that eq. (2.19) and a linear combination of (2.20) and (2.18) can be derived from the action (cf. (2.10))

\[ S = \int dt \; e^{-\phi} \left[ c + N \dot{\lambda}^2 - \dot{\phi}^2 - 2U(\lambda, \phi) \right]. \quad (2.21) \]

Assuming that \( \dot{\lambda} > 0 \) (expansion) \( \dot{\phi} < 0 \) is a necessary condition for a decreasing of the dilaton, i.e. \( \dot{\phi} < 0 \). Suppose \( \frac{\partial U}{\partial \phi} \geq 0 \). Then (2.20) implies that \( \ddot{\phi} > 0 \). If also \( 2U - c \geq 0 \)
we get $\dot{\phi} \neq 0$ from (2.18). As a result, if the initial value $\dot{\phi}(0)$ is negative (as we shall always assume) $\dot{\phi}$ will never change sign and so $\phi$ will always decrease. Then eq. (2.19) describes a motion of a particle in a potential (in general time–dependent through $\varphi(t)$) with a damping term $\sim \dot{\phi}$. Because of the dilaton damping effect the “energy”

$$E = \frac{1}{2} N \dot{\lambda}^2 + U(\lambda, \varphi) = \frac{1}{2}(c + \dot{\phi}^2)$$  \hspace{1cm} (2.22)

is decreasing with time

$$\frac{dE}{dt} = \dot{\phi}(N \dot{\lambda}^2 + \frac{\partial U}{\partial \varphi}) < 0 \ .$$  \hspace{1cm} (2.23)

One can generalize the above cosmological equations to the case of a Kaluza–Klein – type cosmological ansatz taking the space to be a product of several ($i = 1, \ldots, p$) maximally symmetric factors of dimensions $N_i$

$$ds^2 = -dt^2 + \sum_{i=1}^{p} a_i^2(t) g_{b_i c_i} dx_i^{b_i} dx_i^{c_i} ,$$  \hspace{1cm} (2.24)

$$a_i = e^{\lambda_i(t)} , \quad \phi = \phi(t) , \quad b_i, c_i = 1, \ldots, N_i , \quad N = \sum_{i=1}^{p} N_i , \quad D = 1 + N \ ,$$  \hspace{1cm} (2.25)

$$R_{b_i c_i}^{(i)} = (N_i - 1) k_i g_{b_i c_i} .$$  \hspace{1cm} (2.26)

Introducing the “shifted” dilaton

$$\varphi = 2\phi - \sum_{i=1}^{p} N_i \lambda_i \ ,$$  \hspace{1cm} (2.27)

one finds that the system (2.18)–(2.20) is replaced by (cf. (2.13)–(2.16))

$$c - \sum_{i=1}^{p} N_i \dot{\lambda}_i^2 + \dot{\varphi}^2 = 2U \ ,$$  \hspace{1cm} (2.28)

$$\ddot{\lambda}_i - \dot{\phi} \dot{\lambda}_i = -\frac{1}{N_i} \frac{\partial U}{\partial \lambda_i} ,$$  \hspace{1cm} (2.29)

$$\ddot{\varphi}_i - \sum_{i=1}^{p} N_i \dot{\lambda}_i^2 = \frac{\partial U}{\partial \varphi} ,$$  \hspace{1cm} (2.30)

$$U = -\frac{1}{2} \sum_{i=1}^{p} N_i(N_i - 1) k_i e^{-2\lambda_i} + \frac{1}{2} V - e^{\varphi} \frac{\delta S_m}{\delta G_{00}} .$$  \hspace{1cm} (2.31)
3. Solutions with antisymmetric tensor backgrounds

We would like to understand the behaviour of the solutions of the system (2.18)–(2.20) for various choices of the “potential” $U(\lambda, \varphi)$. The simplest possibility when the “matter” and dilaton potential are absent but the space has a non-zero curvature will be discussed in the next section. Here we consider the case when the “matter” part of the action (2.2) contains only the antisymmetric tensor terms (2.3).\footnote{Cosmological solutions with antisymmetric tensor backgrounds were studied in Kaluza–Klein context [33-35]. For a discussion of cosmological solutions of a similar system with dilaton in the Einstein frame see also [3,7].} The equation for the antisymmetric tensor of rank $n - 1$ is

$$D_{\lambda_1}(e^{-2\phi}H^{\lambda_1\ldots\lambda_n}) = 0.$$  \hspace{1cm} (3.1)

A non-trivial solution consistent with symmetries of the ansatz (2.1) is found if the number of space dimensions $N$ is equal to the rank $n$ of the antisymmetric tensor field strength [33]

$$H_{0a_1\ldots a_{N-1}} = 0, \quad H_{a_1\ldots a_N} = h\epsilon_{a_1\ldots a_N}, \quad h = \text{const}, \hspace{1cm} (3.2)$$

where $\epsilon_{a_1\ldots a_N}$ is the standard covariantly constant antisymmetric tensor. Then $U$ in (2.13) takes the form

$$U = -\frac{1}{2}kN(N-1)e^{-2\lambda} + \frac{1}{4N!}H_{\lambda_1\ldots\lambda_N}^2 + \frac{1}{2(N-1)!}H_{a_1\ldots a_{N-1}}^0 H_{0a_1\ldots a_{N-1}}^0$$

$$= -\frac{1}{2}kN(N-1)e^{-2\lambda} + \frac{1}{4}h^2 e^{-2N\lambda}. \hspace{1cm} (3.3)$$

The two particular well known cases correspond to $N = 2$ (“monopole” background of a vector field [35]) and $N = 3$ [33,34]

$$N = n = 2, \quad F_{ab} = h\epsilon_{ab}, \quad U = -ke^{-2\lambda} + \frac{1}{4}h^2 e^{-4\lambda}, \hspace{1cm} (3.4)$$

$$N = n = 3, \quad H_{abc} = h\epsilon_{abc}, \quad U = -3ke^{-2\lambda} + \frac{1}{4}h^2 e^{-6\lambda}. \hspace{1cm} (3.5)$$
If the space curvature is positive $k > 0$ (i.e. the space is a $N$-sphere) the potential $U$ has the minimum at $\lambda = \lambda_0$

$$h^2 = 2k(N - 1)e^{2(N-1)\lambda_0} , \quad U(\lambda_0) = -\frac{1}{4}(N - 1)h^2e^{-2N\lambda_0} < 0 . \quad (3.6)$$

If $c < 0$ the system (2.18)–(2.20) has the following “static” solution

$$\lambda = \lambda_0 , \quad \varphi = \varphi_0 - 2bt , \quad \phi = \phi_0 - bt , \quad (3.7)$$

$$4b^2 = -(c - 2U(\lambda_0)) = |c| - \frac{1}{2}(N - 1)h^2e^{-2N\lambda_0} . \quad (3.8)$$

We have assumed that $\lambda_0$ (or $h$) and $c$ are such that $b^2 \geq 0$. The corresponding solution in the Einstein frame is given in (A.16). For $N = 3$ this is the solution found in [5]. It has the obvious conformal field theory generalisation [5] represented by the direct product of the $D = 1$ “time” theory with linear dilaton [4] and the $SU(2)$ WZW theory (i.e. $S^3$ parallelised by the antisymmetric tensor background). The $N = 2$ case (with $b = 0$) was considered in [35].

As we discussed in sect.2, if $c < 0$ $\dot{\varphi}$ remains negative if it was negative at $t = 0$. That means the dilaton term in (2.19) plays the role of a damping force. As a result, the solution (3.7) is stable so that for a broad range of initial conditions a time-dependent solution with decreasing dilaton approaches (3.7), i.e. the space-time is asymptotically $R \times S^N$ (since the “energy” (2.22) is decreasing with time the trajectory on the $(E, \lambda)$ plane is approaching the minimum of $U$, reflecting from the walls of $U$).

The solution (3.7) exists only if

$$c_{\text{eff}} \equiv c - 2U(\lambda_0) \leq 0 . \quad (3.9)$$

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7 In this case the constant $F_{ab}$–flux “compensates” for the curvature of $S^2$. It is possible to interpret the $SU(2)$ WZW model as an exact conformal field theory which generalizes this solution to all orders in $\alpha'$ expansion [37].

8 In Kaluza–Klein context these solutions provide a model of “confinement” of internal dimensions (because of the damping by dilaton there are no oscillations of internal radii, so that the effective couplings also do not fluctuate).
If $c_{\text{eff}} > 0$ the sign of $\dot{\phi}$ changes at the point where $2U = c$. After this happens the dilaton term in (2.19) provides an accelerating force and the solution goes to infinity in a finite time. In general, it appears that all solutions with $c_{\text{eff}} > 0$ (or $c > 0$ if $U$ is non-negative and approaches zero at large $\lambda$) are singular. Since according to (2.22) $E \geq \frac{1}{2}c$ the trajectory $E(\lambda)$ with the initial condition $\dot{\phi} < 0$ first goes down, reflects off the line $E = \frac{1}{2}c$ and then goes up to the infinity in a finite time.

If $k \leq 0$, i.e. the curvature of the space is negative or zero, $U$ in (3.3) is positive and has no local minima. A qualitative behaviour of solutions does not significantly depend on the presence or absence of the antisymmetric tensor $O(h^2)$ term in $U$. We shall study the case of $k < 0$ in detail in the next section.

2. Let us now consider another class of cosmological solutions generated by the following solution of (3.1) [33] which exists if $n = N + 1$ \[10\]

$$H_{\lambda_1 \ldots \lambda_{N+1}} = h \ e^{2\phi} \epsilon_{\lambda_1 \ldots \lambda_{N+1}} \ , \ h = \text{const} \ . \quad (3.10)$$

Then

$$H^2_{\lambda_1 \ldots \lambda_{N+1}} = -(N + 1)! \ h^2 e^{4\phi} \ , \quad (3.11)$$

so that the potential $U$ in (2.13), (3.3) takes the form

$$U = -\frac{1}{2} k N (N - 1) e^{-2\lambda} + \frac{1}{4} h^2 e^{4\phi} = -\frac{1}{2} k N (N - 1) e^{-2\lambda} + \frac{1}{4} h^2 e^{2\phi + 2N\lambda} \ . \quad (3.12)$$

The antisymmetric tensor contribution to $U$ in this case is the same as that of the “two-loop” term in the dilaton potential (cf. (2.13))

$$V = \frac{1}{2} h^2 e^{4\phi} \ . \quad (3.13)$$

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9. In the bosonic string $c > 0$ corresponds to $D < 26$. Singularity of solutions with $c > 0$ was already observed in [9] in the simplest case of a flat space ($k = 0$) and the absence of matter.

10. Since in a closed string theory context the rank $n \leq 3$ this solution is possible only in $D = 2, 3$. However, one can consider a product of the corresponding $D = 2$ or $D = 3$ space-time and additional space factors thus enlarging the total dimension.
Let us first ignore the effect of the space curvature and set \( k = 0 \) (we shall consider solutions with non-zero \( k \) in the next section). Remarkably enough, the system (2.18)–(2.20) corresponding to the case of \( k = 0 \) and \( V \) given by (3.13), i.e.

\[
c - N\dot{\lambda}^2 + \dot{\varphi}^2 = 2U ,
\]

\[
\ddot{\lambda} - \dot{\varphi}\dot{\lambda} = -2U ,
\]

\[
\ddot{\varphi} - N\dot{\lambda}^2 = 2U
\]

has a simple analytic solution. In fact, subtracting (3.14) from (3.15) we get

\[
\ddot{y} + cy = 0 , \quad y \equiv e^{-\varphi} .
\]

To have a regular solution with \( \varphi \) decreasing with time we need to assume that \( c \leq 0 \). Then (for \( y(0) = 0 \))

\[
\varphi = \varphi_0 - \ln \sinh 2bt , \quad 4b^2 = -c .
\]

This is the same \( \varphi \) as in the solution of (3.14)–(3.15) in the case of \( U = 0 \) (the isotropic case of the solution of ref.[9])

\[
\varphi = \varphi_0 - \ln \sinh 2bt , \quad \lambda = \lambda_0 \pm \frac{1}{\sqrt{N}} \ln \tanh bt .
\]

The solution for \( \lambda \) is found by substituting (3.18) into (3.14) and integrating the resulting equation for \( e^{-N\lambda} \)

\[
\lambda = \lambda_0 - \frac{1}{N} \ln \left[ A^{-1}( \tanh bt )^{-\sqrt{N}} + A ( \tanh bt )^{\sqrt{N}} \right] ,
\]

\[
\phi = \phi_0 - \frac{1}{2} \ln \left[ \sinh bt \cosh bt \left[ A^{-1}( \tanh bt )^{-\sqrt{N}} + A ( \tanh bt )^{\sqrt{N}} \right] \right] ,
\]

\[
A^2 = \frac{h^2}{32b^2} e^{2\varphi_0 + N\lambda_0} = \frac{h^2}{8|c|} e^{4\phi_0} .
\]

The large \( t \) behaviour of the solution (3.20), (3.18) is

\[
\lambda \to \lambda_0 - \frac{1}{N} \ln (A + A^{-1}) = \text{const} , \quad \varphi = \varphi_0 - 2bt , \quad \phi = \phi_0 - bt .
\]
In the special case of $c = 0$ one finds

$$\varphi = \varphi_0 - \ln t , \quad \lambda = \lambda_0 - \frac{1}{N} \ln \left( A^{-1}t^{-\sqrt{N}} + At^{\sqrt{N}} \right) ,$$  

(3.22)

so that the scale factor $a$ grows at small $t$ until it reaches its maximum at $t_* = A^{-1/\sqrt{N}}$ and then asymptotically contracts to zero. The dilaton $\phi$ first grows and then starts decreasing (see also Appendix A).

If $h = 0$ (i.e. if $U = 0$) (3.20) reduces to (3.19). $\pm$ in (3.19) indicates the two solutions related by the duality transformation [14,15]. The solution (3.18),(3.20) is invariant under the shift $bt \to bt + \frac{1}{2}i\pi$, i.e. $\tanh bt \to (\tanh bt)^{-1}$, etc (which relates the two solutions in (3.19)) combined with $A \to A^{-1}$. Thus the system (3.14)–(3.16) with $h^2/|c| \to (h^2/|c|)^{-1}, \phi_0 \to -\phi_0$ (i.e. $A \to A^{-1}$) will have the “dual” solution (3.20) with $\tanh bt \to (\tanh bt)^{-1}$. This transformation is reminiscent of a “weak coupling–strong coupling” duality.

Since $e^\varphi$ is decreasing with time the same is true for the second term in the potential $U$ in (3.12). Therefore it is not surprising that the behaviour of (3.20) is similar to that of (3.19): the two solutions look the same for small $t$, then $\lambda$ grows (and, if $A > 1$, reaches a maximum at $t_* , A^2 ( \tanh bt_*)^{\sqrt{N}} = 1$) and finally approaches the constant value (3.21) at $t \to \infty$. We shall show in Appendix A that in the Einstein frame such behaviour corresponds to a linear expansion of the scale factor (note that asymptotically the dilaton (3.18) is linearly decreasing with time, $\varphi \to \varphi_0 - 2bt$). As was noted in [14] the analytic continuation ($t \to ir$ and $c \to -c$ or $b \to -ib$) of the $N = 1$ vacuum solution (3.19) coincides with the (euclidean) “black hole” solution of $D = 2$ Bose string theory [29,22]. In Appendix B we shall demonstrate that the analytic continuation of the $N = 1$ case of (3.20) (which interchanges the space and time coordinates and $c \to -c$, i.e. $b \to -ib , A \to -A$) is equivalent to the “charged black hole” solution of $D = 2$ heterotic string theory found in [36].
4. Solutions with negative spatial curvature

In the absence of “matter” and dilaton potential the system (2.18)–(2.20) takes the following form

\[ c - N \dot{\lambda}^2 + \varphi^2 = -N(N-1)k \, e^{-2\lambda} , \]  
(4.1)

\[ \ddot{\lambda} - \dot{\varphi} \dot{\lambda} = -(N-1)k \, e^{-2\lambda} , \]  
(4.2)

\[ \ddot{\varphi} - N \dot{\lambda}^2 = 0 . \]  
(4.3)

One may ask how “close” can string solutions resemble the maximally symmetric \( D \)-dimensional de Sitter space. Naively, one may expect that the role of \( c \) in (4.1) is similar to that of the cosmological constant in the corresponding Einstein equation. In fact, rewriting (4.1) in terms of the original dilaton \( \phi \) we get

\[ N(N-1)\dot{\lambda}^2 + 4\dot{\phi}^2 - 4N\dot{\phi}\dot{\lambda} = \Lambda - N(N-1)ke^{-2\lambda} , \quad \Lambda \equiv -c . \]  
(4.1\textsuperscript{′})

If \( \phi = \text{const} \) (4.1\textsuperscript{′}) has the following solutions:

\[ c < 0 \text{ (de Sitter)} : e^{\lambda} = H^{-1} \cosh Ht \quad (k = +1) ; \]
\[ e^{\lambda} = H^{-1} \sinh Ht \quad (k = -1) ; \quad \lambda = \lambda_0 + Ht \quad (k = 0) , \quad H^2 = -\frac{c}{N(N-1)} , \]  
(4.4)

\[ c > 0 \text{ (anti de Sitter)} : e^{\lambda} = H^{-1} \sin Ht \quad (k = -1) , \quad H^2 = \frac{c}{N(N-1)} . \]

The point, however, is that while \( \phi = \text{const} \) and (anti) de Sitter metric solve (4.1) and (4.2) they do not satisfy the remaining dilaton equation (4.3). That is why the dilaton should necessarily change with time and that produces a “deformation” of the de Sitter metric. In fact, it turns out that in the asymptotic region of large \( t \) it is the time variation

\[ 11 \text{ The question about possible relation between solutions of (4.1)–(4.3) and “de Sitter” coset conformal field theories [21] was raised in [17] and also studied in [38]. As we have mentioned in the introduction, no direct connection seems to exist.} \]

\[ 12 \text{ For a similar observation in the Einstein frame see Boulware and Deser [1] and also [6].} \]
of the dilaton and not that of the scale factor that “compensates” for the presence of the “cosmological constant” \(-c\) in (4.1’).

Let us consider the most interesting case when \(c \leq 0\) and the space has a negative curvature \(k < 0\) (if \(c\) or \(k\) are positive \(\dot{\varphi}\) may change sign at some point and the solution is singular, see also below). Then \(c_{\text{eff}} = c + N(N-1)ke^{-2\lambda}\) in (3.9) is negative and hence if \(\dot{\varphi}(0) < 0\) the dilaton \(\varphi\) is always decreasing with time (note that (4.1) is a constraint on initial values of \(\dot{\lambda}, \dot{\varphi}, \lambda\) so that \(\dot{\varphi}(0) \leq -\sqrt{|c_{\text{eff}}|}\)). The potential

\[
U = -\frac{1}{2}N(N-1)k e^{-2\lambda}
\]

grows at negative \(\lambda\) and thus prevents penetration into the region of small scales \(a = e^\lambda\). If \(a\) is contracting at the initial moment it reflects from the potential wall and eventually expands to infinity.\(^{13}\) The trajectory of the system on the energy plane \(E(\lambda)\) is going down because of the damping effect of the dilaton (see (2.22), (2.23)).

Let us assume that \(\lambda(0)\) is very large and negative (i.e. the scale factor \(a\) is very small). To study solutions at small times we may drop \(c\) in (4.1) since the potential term dominates. Then the system (4.1)–(4.3) has the following special solution\(^{14}\)

\[
\lambda = \lambda_0 + \ln t\ ,\ \varphi = \varphi_0 - N \ln t\ ,\ \phi = \phi_0 = \text{const}\ ,\ -k e^{-2\lambda_0} = 1\ . \tag{4.5}
\]

If \(N > 1\) such a solution exists only for the negative spatial curvature \(k < 0\) [17]. The corresponding energy \(E\) is infinite at \(t = 0\) so that the expansion starts from \(\lambda = -\infty\). This

\(^{13}\) The existence of a minimal radius of contraction in the case of the negative spatial curvature is similar to what was found in [17,18] in the case of cosmological evolution in the presence of the classical fields corresponding to the “momentum” modes of a string compactified on a torus. In fact, the (mass)\(^2\) of momentum modes is proportional to \(a^{-2}\) and hence the resulting cosmological system is similar to (4.1)–(4.3). The conclusion was [17] that the contribution of the classical momentum modes prevents the scale factor from contracting to zero.

\(^{14}\) For \(c = 0\) one can solve (4.1)–(4.3) analytically. Introducing \(f = \varphi\) one finds: \(f' = \frac{df}{d\lambda} = \sqrt{N} \sqrt{f^2 - m^2 e^{-2\lambda}}\ ,\ m^2 = N(N-1)|k|\). This gives \(w' = w + \sqrt{N} \sqrt{w^2 - m^2}\ ,\ w = e^\lambda f\) which is easily integrated.
solution describes a flat space (the scalar curvature is $R = -2(\ddot{\lambda} + \dot{\lambda}^2) - N(\dot{\lambda}^2 + ke^{-2\lambda}) = 0$) and may be interpreted (at small $t$) as a $D = N + 1$ dimensional open de Sitter space (4.4) “born” at $t = 0$ (note that the dilaton $\phi$ is approximately constant at early times).

Solutions with regular initial conditions ($\varphi(0), \dot{\lambda}(0)$) correspond to an expansion from some minimal non-vanishing $a$. One could expect that the large $t$ asymptotics of the solution should coincide with that of the $k = 0$ solution (3.19) because the effect of the potential should be negligible at large positive $\lambda$. This expectation, however, is wrong. According to (3.19) $\lambda$ is approaching a finite value as $t \to \infty$. Once we ignore the contribution of the potential the damping effect of the dilaton stops the expansion at a finite $\lambda$ but for each finite value of $\lambda$ the contribution of the potential is still non-vanishing. The resolution of this contradiction is that while for $k = 0$ the expansion stops at a finite $\lambda$, for $k \neq 0$ it actually continues towards $\lambda = \infty$. In fact, if $\lambda$ was approaching a finite value the asymptotic solution would be a direct product of the time line and the $N$-dimensional maximally symmetric space. The latter, however, is not a solution (i.e. does not correspond to a conformal theory) if $k$ is non-vanishing (and “matter” contributions are absent, cf. (3.3)–(3.7)).

It is easy to see that the correct large $t$ asymptotics are given by

$$\lambda \simeq \lambda_1 + \frac{1}{2} \ln t , \quad (4.6)$$

$$\varphi \simeq \varphi_1 - 2bt - \frac{1}{4} N \ln t , \quad \phi \simeq \phi_1 - bt + \frac{1}{4} N \ln t , \quad (4.7)$$

i.e. the scale factor is slowly growing while the dilaton is linearly decreasing as in (3.19) to compensate for the non-vanishing $c$. We have confirmed this behaviour by the numerical solution of the system (4.1)–(4.3). As expected, this solution is very different from the de Sitter space. We shall discuss how its asymptotics looks like in the Einstein frame in Appendix A. Let us comment on the case when $c > 0$ and (or) $k > 0$. If $\dot{\varphi}(0) < 15$ One can find analogous solutions also in the Euclidean case. The system (4.1)–(4.3) and its solutions are transformed into their Euclidean counterparts by the substitution: $t \to i\tau , \quad c \to -c , \quad k \to -k$. 

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0, $\lambda(0) > 0$ the expansion is first slowed down by the dilaton until $\dot{\varphi}$ changes sign to a positive one and accelerates the expansion so that $\lambda$ becomes infinite in a finite time. As follows from (4.1), (4.3) (cf.(3.17))

$$\ddot{y} + c_{\text{eff}}(t) y = 0, \quad y = e^{-\varphi}, \quad c_{\text{eff}} = c + N(N - 1)k e^{-2\lambda}.$$  (4.8)

For large $\lambda$ $c_{\text{eff}} \simeq c > 0$ so that

$$\varphi \simeq \varphi_0 - \ln (\sin \sqrt{c} t),$$  (4.9)

and the solution develops a singularity in a finite time. If $k > 0$ and $c < 0$ there exist solutions which correspond to an expansion to a maximal radius and subsequent contraction to zero in a finite time (since $c_{\text{eff}}$ in (3.9) is positive for sufficiently large negative $\lambda$ $\dot{\varphi}$ changes sign and the solution becomes singular in a finite time).

Now it is easy to understand a qualitative behaviour of the solutions when both the spatial curvature and the antisymmetric tensor (or dilaton potential) contributions are taken into account. If $k < 0$ the two terms in $U$ in (3.3) and in (3.12) have the same sign so that $c_{\text{eff}}$ (3.9) is negative if $c < 0$. In the case of the first antisymmetric tensor background (3.2) the $O(h^2)$ term in $U$ (3.3) is irrelevant at large distances but determines the behaviour at small $t$ (assuming $\lambda(0)$ is large negative). If $a$ starts contracting it eventually reflects from the potential wall and expands to infinity.

In the region $\lambda < 0$ the potential (3.12) can be approximated by the first term. One could expect that the large $t$ (or $\lambda > 0$) asymptotics of the corresponding solutions are determined by the second “matter” term in $U$. However, if $\varphi$ is decreasing rapidly enough it may dominate over the growth of $\lambda$ in the $O(h^2)$ term in (3.12). This is actually what happens. If one ignores the spatial curvature term in $U$ the large $t$ limit of the solution (3.20) is $\lambda \to \text{const}$ but as we have discussed above that means that the neglect of the first term in (3.12) is not justified. The large $t$ limit of the solution is, in fact, dominated by the curvature term in (3.12), i.e. is again given by (4.6). This conclusion seems to be
valid in the general case of \( c \leq 0 \), \( k < 0 \) and the dilaton potential \( V(\phi) \) given by a sum of exponentials \( e^{p\phi} \), \( p > 0 \) with positive coefficients. In fact, a slow growth of \( \lambda \) and a rapid decrease of the dilaton \( \phi \) with time implies that the dilaton potential term in \( U \) (2.13) will be negligible at late times.

I would like to acknowledge useful discussions of related issues with G. Gibbons and V. Linetsky. I am also grateful to Trinity College, Cambridge for a financial support through a visiting fellowship.
Appendix A. Relation between cosmological solutions in the string frame and the Einstein frame

We have discussed cosmological solutions using the “string frame” in which the action has the form (2.2). Though the string frame and the Einstein frame should be physically equivalent, the metric corresponding to (2.2) is the one which directly appears in the string action so it is natural to use it in order to try to understand a string theory interpretation of the solutions. Also, the form of the solutions in the string frame is often simpler than that of their counterparts in the Einstein frame. Below we shall discuss the relation between cosmological solutions in the two frames.\footnote{A similar relation is known in the context of the Brans-Dicke theory as a relation between Jordan frame and Einstein frame. It should be emphasized however that the string effective action does not correspond to $\omega = -1$ BD theory because of different couplings of the dilaton to “matter” (see also [11]).}

If $D > 2$ one can transform the string frame action (2.2)

$$S = \int d^Dx \sqrt{-G} \ e^{-2\phi} \left[ R + 4(\partial \phi)^2 - \hat{V}(\phi) - \frac{1}{12} H_{\lambda \mu \nu}^2 - \frac{1}{4} F_{\mu \nu}^2 + \cdots \right], \quad (A.1)$$

$$\hat{V} = V(\phi) - c,$$

into its Einstein frame form

$$S_E = \int d^Dx \sqrt{-g} \left[ R - 2p(\partial \phi)^2 - e^{2p\phi} \hat{V}(\phi) - \frac{1}{12} e^{-4p\phi} H_{\lambda \mu \nu}^2 - \frac{1}{4} e^{-2p\phi} F_{\mu \nu}^2 + \cdots \right], \quad (A.2)$$

which depends on the new metric $g_{\mu \nu}$

$$g_{\mu \nu} = e^{-2p\phi} G_{\mu \nu}, \quad p \equiv \frac{2}{D-2}. \quad (A.3)$$

Given a cosmological solution (2.1) in the string frame theory (A.1)

$$ds^2 = G_{\mu \nu} dx^\mu dx^\nu = -dt^2 + e^{2\lambda(t)} d\Omega^2, \quad \phi = \phi(t), \quad (A.4)$$
we can find the corresponding solution in the Einstein frame theory (A.2)

\[ ds_E^2 = g_{\mu\nu}dx^\mu dx^\nu = e^{-2p\phi(t)}(-dt^2 + e^{2\lambda(t)}d\Omega^2) , \quad (A.5) \]

i.e.

\[ ds_E^2 = -d\tau^2 + e^{2\lambdaE(\tau)}d\Omega^2, \quad d\tau = dt\,e^{-p\phi(t)} , \quad (A.6) \]

\[ \lambdaE \equiv \lambda - p\phi . \quad (A.7) \]

Using (2.9), i.e.

\[ \phi = \frac{1}{2}(\varphi + N\lambda) , \quad N = D - 1 , \quad (A.8) \]

we can rewrite (A.7) in the form

\[ \lambdaE = \lambda - \frac{1}{4}p(\varphi + N\lambda) = -\frac{1}{N - 1}(\lambda + \varphi) . \quad (A.9) \]

If \( \phi = \text{const} \) the solutions look of course the same in the two frames. Once \( \phi \neq \text{const} \) the relations (A.6)–(A.8) suggest that a correspondence between the behaviour of the functions \( \lambda(t), \phi(t) \) and \( \lambdaE(t), \phi(\tau) \) may be non-trivial. Suppose that \( \lambda(t) \) is growing while \( \varphi(t) \) is decreasing rapidly enough so that \( \phi(t) \) is also decreasing and is monotonic.\(^{17}\) Then according to (A.6) the Einstein frame time \( \tau \) is a monotonically growing function of \( t \). As a consequence, \( \phi(\tau) \) is still decreasing while \( \lambda(\tau) \) is increasing so that (A.7) implies that \( \lambdaE(\tau) \) is also increasing. In this case an expansion in the string frame corresponds to an expansion in the Einstein frame. However, the rates of expansions may be quite different.

As follows from (A.7), (A.9)

\[ \lambda'_{E} = e^{p\phi}(\dot{\lambda} - p\dot{\phi}) = \frac{1}{N - 1}e^{p\phi}(-\dot{\varphi} - \dot{\lambda}) , \quad \lambda'_{E} \equiv \frac{d\lambdaE}{d\tau} , \quad (A.10) \]

so that \( \lambda'_{E} \) depends strongly on the behaviour of the dilaton.

\(^{17}\) Note that our assumption that \( \dot{\varphi} < 0 \) is only a necessary but not sufficient condition for a decrease of the dilaton \( \phi(t) \).
To illustrate the above general remarks let us consider the following example

$$\lambda = \lambda_0 + q \ln t \, , \, \phi = \phi_0 - bt \, , \, \varphi \simeq \varphi_0 - 2bt \, , \, b > 0 \ . \quad (A.11)$$

Then

$$\tau = \tau_0 + m^{-1} e^{\frac{2bt}{N-1}} \, , \, m = \frac{2b}{N-1} e^{\frac{2\phi_0}{N-1}} \, ,$$

$$\phi = \phi_0 - \frac{1}{2} (N - 1) \ln m(\tau - \tau_0) \ , \quad (A.12)$$

$$\lambda_E = \lambda_{E0} + \ln m(\tau - \tau_0) + q \ln \ln m(\tau - \tau_0) \ . \quad (A.13)$$

While $\phi(\tau)$ is decreasing much slower than $\phi(t)$, $\lambda_E(\tau)$ is still growing logarithmically with the coefficient of the leading logarithm being universal, i.e. independent of $b$ in the dilaton $\varphi(t)$ or $q$ in $\lambda(t)$. As a result, the static metric $(q = 0)$ in the string frame corresponds to the linear expansion $a_E(\tau) \sim \tau$ in the Einstein frame [5,10,11]. The large $\tau$ asymptotics in $D = 4$ is

$$\lambda_E \simeq + \ln \tau \ , \ \phi \simeq - \ln \tau \ .$$

Solutions with such asymptotics were discussed in [5,6,19,20].

Another useful example is

$$\lambda = \lambda_0 + q \ln t \, , \, \phi = \phi_0 - s \ln t \, , \, \varphi = \varphi_0 - (2s + Nq) \ln t \ . \quad (A.14)$$

Assuming $sp \neq 1$ we get

$$\tau = \tau_0 + m^{-1} t^{sp-1} \, , \, m = e^{\rho\phi_0} (sp - 1) \ ,$$

$$\phi = \phi_0 - r \ln |m(\tau - \tau_0)| \ , \ \lambda_E = \lambda_{E0} + l \ln |m(\tau - \tau_0)| \ , \quad (A.15)$$

$$r = \frac{s}{sp - 1} \ , \ l = \frac{q + sp}{sp - 1} .$$

If $sp = 1$

$$\tau = \tau_0 + m^{-1} \ln \tau \ , \ m = e^{\rho\phi_0} \ , \phi = \phi_0 - sm(\tau - \tau_0) \ , \ \lambda_E = \lambda_{E0} + qm(\tau - \tau_0) \ ,$$

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i.e. we get the exponential inflation of $a_E = e^{\lambda_E}$. If $sp < 1$ we obtain (A.15) with negative $m, r, l$ and $\tau$ asymptotically approaching $\tau_0$ from below as $t \to \infty$.

We are now able to discuss the Einstein frame form of the solutions found in sections 3 and 4. The singular solutions with $c_{\text{eff}} > 0$ remain singular in the Einstein frame so we shall consider the case of $c_{\text{eff}} \leq 0$. The $k > 0$ solutions corresponding to the potential (3.3) asymptotically approach the “static” solution (3.7). Using (A.11)–(A.13) we conclude that their large $\tau$ behaviour in the Einstein frame is represented by

$$\lambda_E \simeq \lambda_{E1} + \ln \tau, \quad \phi \simeq \phi_1 - \frac{1}{2} (N - 1) \ln \tau, \quad (A.16)$$

i.e. is the linear expansion with a logarithmically decreasing dilaton.

The exact $k = 0$ solution (3.20), (3.18) found in the case of the potential (3.12) looks very complicated in the Einstein frame but its large $\tau$ asymptotics is easily found from (3.21). It is again given by (A.16). The small $t$ limit of the $c = 0$ solution (3.22)

$$\lambda = \lambda_0 + \frac{1}{\sqrt{N}} \ln t, \quad \varphi = \varphi_0 - \ln t, \quad \phi = \phi_0 - (1 - \sqrt{N}) \ln t \quad (A.17)$$

can be translated into the Einstein frame using (A.14), (A.15) (note that here $s < 0$). We find (A.15) with $l > 0$, $r > 0$, i.e. the scale factor is growing while the dilaton is decreasing with $\tau$.

Let us now turn to the solutions in the case of the “pure curvature” potential (4.1)–(4.3). The special solution (4.5) (which is valid in the small $t$ region) looks the same in the Einstein frame since $\phi = \text{const}$. The large $t$ asymptotics (4.6), (4.7) is of the type (A.11) with $q = \frac{1}{2}$ so that the corresponding $\phi(\tau), \lambda_E(\tau)$ are given by (A.12), (A.13), i.e. (A.16).

For completeness let us record the Einstein frame analog of the system of cosmological equations (2.18)–(2.20)

$$N(N - 1)\lambda_E^2 - \frac{4}{N - 1} \phi'^2 = 2 \mathcal{U}, \quad (A.18)$$

$$\frac{1}{N - 1} \phi'' + \frac{4}{N - 1} \phi'^2 = \frac{1}{N} \frac{\partial \mathcal{U}}{\partial \lambda_E}, \quad (A.19)$$

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\[ \phi'' + N \lambda'_E \phi' = -\frac{1}{4} (N - 1) \frac{\partial U}{\partial \phi} , \quad (A.20) \]

\[ U(\lambda_E, \phi) \equiv e^{\frac{4\phi}{N-1}} \left[ U(\lambda, \varphi) - \frac{1}{2} c \right] . \quad (A.21) \]

Here primes denote derivatives over \( \tau \) and \( U \) is given by (2.13). The expression for \( U \) can be derived directly from (A.2). As a simple example let us consider the case when \( U \) contains only the curvature term and \( c = 0 \). Then

\[ U = -\frac{1}{2} N(N - 1)e^{-2\lambda_E} , \quad \phi'' + N \lambda'_E \phi' = 0 , \quad (A.22) \]

i.e.

\[ \phi' = -q e^{-N\lambda_E} , \quad \lambda'_E^2 = -k e^{-2\lambda_E} + \frac{4q^2}{N(N - 1)^2} e^{-2N\lambda_E} . \quad (A.23) \]

If \( k > 0 \) there exists a maximal radius of expansion while for \( k < 0 \) the expansion continues to \( \lambda_E = \infty \). The case of \( c \neq 0 \) is obviously easier to analyse in the string frame.

**Appendix B. Correspondence between the \( N = 1 \) solution (3.18), (3.20) and the \( D = 2 \) charged black hole background**

If \( N = 1 \) the metric, the dilaton \( \phi \) and the gauge field corresponding to (3.20), (3.18) are given by

\[ ds^2 = -dt^2 + e^{2\lambda(t)} dx^2 , \]

\[ \lambda = \lambda_0 - \ln[ A^{-1} \left( \tanh bt \right)^{-1} + A \tanh bt ] , \quad (B.1) \]

\[ \phi = \frac{1}{2}(\varphi + \lambda) = \phi_0 - \frac{1}{2} \ln(A^{-1} \cosh^2 bt + A \sinh^2 bt) , \quad (B.2) \]

\[ F_{\mu\nu} = h e^{2\phi} \epsilon_{\mu\nu} , \quad A = \frac{h}{4\sqrt{2b}} e^{2\phi_0} . \quad (B.3) \]

This is to be compared with the \( D = 2 \) charged black hole metric of ref. [36]

\[ ds^2 = -g dy^2 + g^{-1} dr^2 , \quad \phi = \phi_0 - \frac{1}{2} Q r , \quad c = Q^2 , \quad (B.4) \]

\[ g = 1 - 2m e^{-Q r} + q^2 e^{-2Q r} . \quad (B.5) \]
The metric (B.5) has two horizons and a singularity at \( r = -\infty \). Introducing the new time parameter \( z \) one can represent (B.1), (B.2) in the form

\[
d s^2 = -f^{-1}dz^2 + f dx^2, \quad \phi = \phi_0 - bz, \quad c = -4b^2, \quad (B.6)
\]

\[
f = (1 + A^2 e^{-2bz})(1 - e^{-2bz}). \quad (B.7)
\]

The relation between (B.4), (B.5) and (B.6), (B.7) is established by the following identification

\[
r = iz + r_0, \quad y = ix, \quad g = f, \quad Q = -2ib, \quad 2me^{-Qr_0} = 1 - A^2, \quad qe^{-Qr_0} = iA. \quad (B.8)
\]

As in the case of the zero charge black hole [22,17] the cosmological solution corresponds to the region between the horizons and singularity.

**Appendix C. Time-dependent solutions interpolating between “static” conformal backgrounds**

The system of cosmological equations (2.18)–(2.19) has another interesting interpretation. Suppose the potential \( U(\lambda) \) has a maximum and a minimum. The two extremal points are solutions of (2.18)–(2.19) (for appropriate signs of \( c \) and \( U \) and a linear dilaton). If we start from the maximum we shall end the evolution at the minimum, i.e. \( \lambda(t) \) will be interpolating between the two “static” solutions. This is similar to the situation one encounters in the case of a conformal theory perturbed by a nearly marginal operator. In this context \( \lambda \) will be interpreted as a coupling associated with a nearly marginal operator. There are two conformal points: the original one (\( \lambda = 0 \)) and the nearby one. Embedding into string theory, i.e. introducing time-dependent dilaton and demanding the vanishing of the total central charge we shall get a system of equations for \( \varphi(t) \) and \( \lambda(t) \) (see also [39,41]). Below we shall discuss a relation between this system and the standard renormalisation group equation.
A simple way of deriving this system is the following. Consider the effective action for the metric, dilaton and tachyon couplings (cf. (2.2); $\alpha' = 1$)

$$S = \int d^D x \sqrt{-G} \ e^{-2\phi} \left[ c + R + 4(\partial \phi)^2 - \frac{1}{4}(\partial T)^2 + T^2 - \frac{1}{6}T^3 + ... \right]. \tag{C.1}$$

Suppose now that the dependence of $T$ on the space coordinates is such that it is nearly “on-shell” in $N$ space dimensions, i.e. an effective mass of a “reduced” field is small. Assuming that the metric (2.1) is spatially flat and isotropic and that the dilaton depends only on $t$ let us replace $T$ by a function of time only $T(t, x) \rightarrow 2f(t)^{18}$ As a result, we find the following “dimensionally reduced” action (cf. (2.10), (2.21))

$$S = \int dt \ e^{-\phi} \left[ c + N\dot{\lambda}^2 - \dot{\varphi}^2 + f^2 - 2U(f) \right]. \tag{C.2}$$

$$U = -2\mu f^2 + \frac{2}{3}gf^3 + ... , \quad 0 < \mu \ll 1 , \tag{C.3}$$

where $g$ is an effective coupling. The corresponding system of evolution equations is similar to (2.18)–(2.19) (except that now $U$ does not depend on $\varphi$ and $\lambda$)

$$c - N\dot{\lambda}^2 - \dot{f}^2 + \dot{\varphi}^2 = 2U , \tag{C.4}$$

$$\ddot{f} - \dot{\varphi}\dot{f} = -U' , \quad U' \equiv \frac{\partial U}{\partial f} , \tag{C.5}$$

$$\ddot{\varphi} - N\dot{\lambda}^2 - \dot{f}^2 = 0 , \tag{C.6}$$

$$\ddot{\lambda} - \dot{\varphi}\dot{\lambda} = 0 . \tag{C.7}$$

$f$ appears in the l.h.s. of equations of this system in the same way as $\lambda$ does (or as one of the “moduli” $\lambda_i$ in (2.28)–(2.30)). $\lambda$ is easily eliminated by solving (C.7)\(^{19}\) but

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\(^{18}\) For example, in $N = 1$ case this can be done by setting $T \sim f(t) \cos px$ (with a properly chosen $p$) and integrating over $x$ (more generally, $T = \sum_n f_n(t) e^{ip_n \cdot x}$ and integration over $x^a$ imposes “momentum conservation”; cf. [42,43]).

\(^{19}\) The result of solving (C.7) and substituting $\lambda$ (i.e. $\dot{\lambda} = he^\varphi$) in the remaining equations is equivalent to adding the term $\frac{1}{2}Nh^2e^{2\varphi}$ to the potential $U$ in (C.4) and including its derivative over $\varphi$ in (C.6) (cf. (2.20)). Since $\varphi$ is decreasing with time the effect of this term is not important. In general $\lambda$ will be slowly changing with time asymptotically approaching a constant value.
let us consider for simplicity the trivial solution \( \lambda = \text{const} \) (flat metric). In this case there is no difference in the behaviour of the original dilaton \( \phi \) and the “shifted” one \( \varphi \) (2.9). The resulting system of equations for \( \varphi(t) \) and \( f(t) \) is identical to (2.18)–(2.20) with \( N = 1, \quad U = U(\lambda), \quad \lambda = f_c - \dot{f}^2 + \varphi^2 = 2U \), \( (C.8) \)

\[
\ddot{f} - \dot{\varphi}\dot{f} = -U' \quad ,
\]

\( (C.9) \)

\[
\ddot{\varphi} - \dot{f}^2 = 0 \quad .
\]

\( (C.10) \)

For the potential (C.3) and \( c \leq 0 \) this system has two “static” solutions corresponding to the extremal points (the maximum and the minimum) of \( U \)

\[
f = f_0 = 0 \quad , \quad \varphi = \varphi_0 - q_0 t \quad , \quad q_0 > 0 \quad , \quad c + q_0^2 = 0 \quad , \quad (C.11)\]

\[
f = f_1 = \frac{2\mu}{g} \quad , \quad \varphi = \varphi_1 - q_1 t \quad , \quad (C.12)\]

\[
c + q_1^2 = 2U(f_1) = -\frac{16\mu^3}{3g} \quad , \quad q_1^2 - q_0^2 = -\frac{16\mu^3}{3g} \quad . \quad (C.13)\]

Following the discussion in Sect. 2 it is easy to analyse the behaviour of time-dependent solutions of (C.8)–(C.10). Since the maximum of the potential is an unstable point the solution with (C.11) and \( \dot{f} = \epsilon > 0 \) as the initial conditions will asymptotically approach the (stable) minimum point (C.10). The dilaton is decreasing with time (with \( \dot{\varphi} \) increasing from \(-q_0\) to \(-q_1\)) thus providing a damping force which eliminates oscillations near the minimum. Note that the effective central charge of the theory \( \sim -c - \varphi^2 \) is decreasing with time (in our notation \( c \) has the sign opposite to that of the standard central charge \( C \), \( c = -\frac{2}{3}c \)).

This is of course reminiscent of the well-known picture of the RG interpolation between nearby conformal theories [44]. Let \( \gamma^i \) be a set of couplings which parametrize a perturbation of a conformal theory by local operators \( O_i \) of dimensions \( 2 - w_i \). Then the leading terms in the corresponding renormalisation group \( \beta \) - function are expressed in
terms of \( w_i \) and the operator product coefficients \( C_{ijk} \) (in normal coordinates Zamolodchikov’s metric is trivial to the leading order)

\[
\beta^i = \frac{d\gamma^i}{d\tau} = w_i \gamma^i - \pi \sum_{j,k} C_{ijk} \gamma^j \gamma^k + O(\gamma^3) ,
\]

where \( \tau \) is the RG “time” parameter. If one of the operators (say \( O_1 \)) is nearly marginal (i.e. \( w_1 \ll 1 \)) it drives the system to another conformal point corresponding to the non-trivial zero of (C.14)

\[
\gamma^1 = \frac{w_1}{\pi C_{111}} + O(w_1^2) , \quad \gamma^i = O(w_1^2) , \quad i = 2, 3, ... ,
\]

with the difference of central charges being [44]

\[
\mathcal{C}(1) - \mathcal{C}(0) = -\frac{w_1^3}{\pi C_{111}} + O(w_1^4) .
\]

These relations are in direct correspondence with (C.11)–(C.13) with \( f = \gamma^1 , \mu \sim w_1 , g \sim C_{111} \).

Ignoring the effects of other couplings the RG equation for \( f(t) = \gamma^1(\tau) \) where \( t = q\tau \) is a rescaled “time” can be written in the form

\[
q \dot{f} = -U' , \quad \dot{f} = \frac{df}{dt} \quad (C.17)
\]

\[
U = -2\mu f^2 + \frac{2}{3} gf^3 + ... .
\]

Let us compare (C.8)–(C.10), (C.3) with (C.17), (C.18). Their static solutions (\( f = 0, f_1 \)) are in correspondence– the only difference is that (C.8)–(C.10) contains also the dilaton which ensures that the total central charge is zero. The time-dependent solutions \( f(t) \) of (C.8)–(C.10) and of (C.17) thus interpolate between the same asymptotic conformal points. However, they are obviously different as functions of \( t \).

The system (C.8)–(C.10) can be considered as a “string generalisation” of the standard RG equation (C.17): to satisfy the Weyl invariance condition in “\( N + 1 \) dimensions” (i.e.
the vanishing of the “N+1- dimensional” β-functions and of the total central charge) we need to introduce a time-dependent dilaton with its own equation of motion and to add the second derivative term to (C.17). One can try to establish a correspondence between (C.8)–(C.10) and (C.17) in a “semiclassical” limit of large negative \( c \) [40]: setting \( \dot{\phi} = -q + O(q^{-1}) \), \( q \gg 1 \) and \( t = q\tau \) we get from (C.8)–(C.10)

\[
\begin{align*}
c + q^2 & = 2U + O(q^{-2}) \quad , \\
\frac{df}{d\tau} & = -U' + O(q^{-2}) \quad , \\
q^{-2}(\frac{df}{d\tau})^2 & = O(q^{-2}) \quad .
\end{align*}
\]

(C.20)

(C.21)

Eq.(C.19) reduces to (C.17) but the zero central charge equation (C.19) is not satisfied in general. It is only if \( U \) is very small on the solution (which is true only for particular potentials \( U \), e.g. \( U = -b^2 f^n \)) that one can claim an agreement between (C.17) and (C.19)–(C.21) (compare [40]).

It is interesting to note that there exists a potential \( U \) for which any solution of (C.17) is also a solution of (C.8)–(C.10). Substituting (C.17) into (C.9), (C.10) one finds that \( U \) should satisfy

\[
U''' = U' , \quad \dot{\phi} = -q - q^{-1}U'' .
\]

(C.22)

As a result,

\[
U = U_0 + A \cosh f + B \sinh f ,
\]

(C.23)

\[
c + q^2 + A^2 - B^2 = 2U_0 ,
\]

(C.24)

where (C.24) follows from (C.8). It is because of the “dissipative” nature of the dilaton coupling that solutions of the second order system (C.8)–(C.10) can in principle be in correspondence with the solutions of the first order RG equation (C.17).
References

[1] D. Bailin, A. Love and D. Wong, Phys. Lett. B165(1986)409 ;
D. Boulware and S. Deser, Phys. Lett. B175(1986)409 ;
Y.S. Wu and Z. Wang, Phys. Rev. Lett. 57(1986)1978 ;
J.T. Wheeler, Nucl. Phys. B268(1986)737 ;
A. Henriques, Nucl. Phys. B277(1986)621 ;
A. Henriques and R. Moorhouse, Phys. Lett. B197(1987)353 ;
D. Lorentz–Petzold, Phys. Lett. B197(1987)71 ;
K. Maeda, Mod. Phys. Lett. A3(1988)243 .

[2] I. Antoniadis and C. Kounnas, Nucl. Phys. B284(1987)71 ;
S. Kalara, C. Kounnas and K.A. Olive, Phys. Lett. B215(1988)265 .

[3] G. Gibbons and P. Townsend, Nucl. Phys. B282(1987)610 ;
G. Gibbons and K. Maeda, Nucl. Phys. B298(1988)741 .

[4] R. Myers, Phys. Lett. B199(1987)371 .

[5] I. Antoniadis, C. Bachas, J. Ellis and D.V. Nanopoulos,
Phys. Lett. B211(1988)393; Nucl. Phys. B328(1989)115 .

[6] S. Kalara and K.A. Olive, Phys. Lett. B218(1989)148 .

[7] A. Liddle, R. Moorhouse and A. Henriques, Nucl. Phys. B311(1988)719 ;
N. Stewart, Class. Quant. Grav. 8(1988)1701 ;

[8] R. Brandenberger and C. Vafa, Nucl. Phys. B316(1988)391 ;
H. Nishimura and M. Tabuse, Mod. Phys. Lett. A2(1987)299 ;
J. Kripfganz and H. Perlt, Class. Quant. Grav. 5(1988)453 ;
M. Hellmund and J. Kripfganz, Phys. Lett. B241(1990)211 .

[9] M. Mueller, Nucl. Phys. B337(1990)37 .

[10] N. Sanchez and G. Veneziano, Nucl. Phys. B333(1990)253 .

[11] B.A. Campbell, A. Linde and K.A. Olive, Nucl. Phys. B355(1991)146 .

[12] I. Antoniadis, C. Bachas, J. Ellis and D.V. Nanopoulos,
Phys. Lett. B257(1991)278 .

[13] J.A. Kasas, J. Garcia–Bellido and M. Quiros, Nucl. Phys. B361(1991)713 ;
J. Garcia–Bellido and M. Quiros, Nucl. Phys. B368(1992)463 ;
M.C. Bento, O. Bertolami and P.M. Sa, Phys. Lett. B262(1991)11 .

[14] A.A. Tseytlin, Mod. Phys. Lett. A6(1991)1721 .

[15] G. Veneziano, Phys. Lett. B265(1991)287 .

[16] A.A. Tseytlin, in: Proc. of the First International A.D. Sakharov Conference on
Physics, Moscow 27 - 30 May 1991, ed. L.V. Keldysh et al., Nova Science Publ.,
Commack, N.Y., 1991 .

[17] A.A. Tseytlin and C. Vafa, Nucl. Phys. B372(1992)443 .

[18] A.A. Tseytlin, preprint DAMTP-37-1991; Class. Quant. Grav. 9(1992)1 .

[19] B.A. Campbell, N. Kaloper and K.A. Olive, Univ. of Minnesota preprint UMN-TH-
1006/91.

[20] N. Kaloper and K.A. Olive, Univ. of Minnesota preprint UMN-TH-1011/91.

[21] I.Bars and D. Nemeschanski, Nucl. Phys. B348(1991)89 ;
I. Bars, U. Southern California preprint USC-91/HEP-B4 ;
E.S. Fradkin and V.Ya. Linetsky, Phys. Lett. B261(1991)26 .

[22] E. Witten, Phys. Rev. D44(1991)314 ;
R. Dijgraaf, H. Verlinde and E. Verlinde, Princeton preprint PUPT-1252/91.

[23] M. Crescimanno, Berkeley preprint LBL-30947(1991) .

[24] I. Bars and K. Sfetsos, U. Southern California preprints USC-91/HEP-B5 ; USC-
91/HEP-B6 .

[25] E.S. Fradkin and V.Ya. Linetsky, Harvard preprint HUTP-91/A044 (1991) .

[26] A.H. Chamseddine, Zurich U. preprint ZU-TH-31-1991 .

[27] P. Ginsparg and F. Quevedo, Los Alamos preprint LA-UR-92-640 .

[28] P. Horava, Chicago U. preprint EFI-91-57 ;
D. Gershon, Tel-Aviv U. preprint TAUP-1937-91 .
[29] S. Elitzur, A. Forge and E. Rabinovici, Nucl. Phys. B359 (1991) 581;
    G. Mandal, A. Sengupta and S. Wadia, Mod. Phys. Lett. A6(1991)1685.
[30] E. Witten, Commun. Math. Phys. 92(1984)455.
[31] J. Scherk and J.H. Schwarz, Nucl. Phys. B81(1974)118.
[32] E.S. Fradkin and A.A. Tseytlin, Phys. Lett. B158(1985)316;
    C.G. Callan, D. Friedan, E. Martinec and M.J. Perry, Nucl. Phys. B262(1985)593.
[33] P.G.O. Freund, Nucl. Phys. B209(1982)146;
    P.G.O. Freund and M. Rubin, Phys. Lett. B97(1980)233.
[34] M. Gleiser, S. Rajpoot and J.G. Taylor, Phys. Rev. D30(1984)756; Ann. of Phys. 160(1985)299;
    D. Bailin, A. Love and J. Stein–Schabes, Nucl. Phys. B253(1985)387;
    F.S. Acceta, M. Gleiser, R. Holman and E.W. Kolb, Nucl. Phys. B276(1986)501;
    R. Holman, E.W. Kolb, S.L. Vadas and Y. Wang, Phys. Rev. D43(1991)995.
[35] S. Randjbar-Daemi, A.Salam and J. Strathdee, Nucl. Phys. B214(1983)491;
    E. Sezgin and A. Salam, Phys. Lett. B147(1984)47;
    K. Maeda and H. Nishino, Phys. Lett. B158(1985)381;
    Y. Okada, Nucl. Phys. B264(1986)197;
    J. Halliwell, Nucl. Phys. B286(1987)729;
    A. Linde and M. Zelnikov, Phys. Lett. B215(1988)59.
[36] M.D. McGuigan, C.R. Nappi and S.A. Yost, Princeton preprint IASSNS-HEP-91/57.
[37] I. Antoniadis, C. Bachas and A. Sagnotti, Phys. Lett. B235(1990)255.
[38] V.Ya. Linetsky, unpublished.
[39] S.Das, A.Dhar and S. Wadia, Mod. Phys. Lett. A5(1990)799;
    A. Sen, Phys. Lett. B252(1990)566.
[40] A. Cooper, L. Susskind and L. Thorlacius, Nucl. Phys. B363(1991)132;
    A. Polyakov, Princeton U. preprint PUPT-1289 (1991).
[41] S. Mukherji, Tata Inst. preprint TIFR/TH/92-11;
[42] M.T. Grisaru, A. Lerda, S. Penati and D. Zanon, Phys. Lett. B234(1990)88; Nucl. Phys. B346(1990)264.

[43] A.A. Tseytlin, Phys. Lett. B241(1990)233; Phys. Lett. B243(1990)465(E).

[44] A.B. Zamolodchikov, Sov. J. Nucl. Phys. 46(1987)1090; JETP Lett. 43(1986)731;
J.Cardy and C.Ludwig, Nucl. Phys. B285(1987)687.