An arbitrary-order discrete de Rham complex on polyhedral meshes. Part II: Consistency

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Abstract

In this paper we prove a complete panel of consistency results for the discrete de Rham (DDR) complex introduced in the companion paper \cite{1}, including primal and adjoint consistency for the discrete vector calculus operators, and consistency of the corresponding potentials. The theoretical results are showcased by performing a full convergence analysis for a DDR approximation of a magnetostatics model. Numerical results on three-dimensional polyhedral meshes complete the exposition.

Key words. Discrete de Rham complex, compatible discretisations, polyhedral methods, mixed methods

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1 Introduction

We prove complete consistency results for the discrete de Rham (DDR) complex introduced in the companion paper \cite{12}. Specifically, the first set of results concerns primal consistency of the local discrete vector calculus operators introduced in \cite{12} Section 3.3 and of the corresponding potentials defined in Section 3.1 below. The second set of results concerns adjoint consistency, that relates to the ability to approximate formal adjoint operators, and therefore requires to estimate the residuals of global integration by parts formulas.

For specific space dimensions, polynomial degrees, and operators, consistency results that bear relations to ours can be found in the literature on polytopal methods.

Starting from low-order methods, consistency results for Compatible Discrete Operator approximations of the Poisson problem based on nodal unknowns can be found in \cite{5}; see in particular the proof of Theorem 3.3 therein, which contains an adjoint consistency result for a gradient reconstructed from vertex values. In the same framework, an adjoint consistency estimate for a discrete curl constructed from edge values can be found in \cite{6} Lemma 2.3. A rather complete set of consistency results for Mimetic Finite Difference operators can be found in \cite{4}, where they appear as intermediate steps in the error analyses of Chapters 5–7. A notable exception is provided by the adjoint consistency of the curl operator, which is not needed in the error estimate of \cite{4} Theorem 7.3 since the authors consider an approximation of the current density based on the knowledge of a vector potential.

Moving to arbitrary-order methods, error estimates that involve the adjoint consistency of a gradient and the consistency of the corresponding potential have been recently derived in \cite{7} in the framework of the $H^1$-conforming Virtual Element method. The same method is considered in \cite{9} Section 3.2, where
a different analysis is proposed based on the third Strang lemma. The estimate of the consistency error in [9, Theorem 19] involves, in particular, the adjoint consistency of a discrete gradient reconstructed as the gradient of a scalar polynomial rather than a vector-valued polynomial. We note, in passing, that the concept of adjoint consistency for (discrete) gradients is directly related to the notion of limit-conformity in the Gradient Discretisation Method [15], a generic framework which encompasses several polytopal methods. Primal and dual consistency estimates for a discrete divergence and the corresponding vector potential similar (but not identical) to the ones considered here have been established in [14] in the framework of Mixed High-Order methods. Note that these methods, the H1-conforming Virtual Element method, and the Mixed High-Order method, do not lead to a discrete de Rham complex.

In the framework of arbitrary-order compatible discretisations, on the other hand, primal consistency results for the curl appear as intermediate results in [3], where an error analysis for a Virtual Element approximation of magnetostatics is carried out assuming interpolation estimates for three-dimensional vector valued virtual spaces; see Remark 4.4 therein. However, [3] does not establish any adjoint consistency property of the discrete curl (the formulation of magnetostatics considered in this reference does not require this).

The results presented in this paper are, to the best of our knowledge, the first ones to span the full set of discrete vector calculus operators for an arbitrary-order discrete de Rham complex on polyhedral meshes. The key ingredients to establish primal consistency are the polynomial consistency of discrete vector calculus operators along with the corresponding potentials, and their boundedness when applied to the interpolates of smooth functions. The proofs of adjoint consistency, on the other hand, rely on operator-specific techniques, all grounded in discrete integration by parts formulas for the corresponding potential reconstructions (see (3.1) along with Remark 3 for the gradient, (3.6) for the curl, and (3.10) for the divergence). Specifically, the key point for the adjoint consistency of the gradient are estimates for local H1-like seminorms of the scalar potentials. The adjoint consistency of curl requires, on the other hand, the construction of liftings of the discrete face potentials that satisfy an orthogonality and a boundedness condition. These reconstructions are inspired by the minimal reconstruction operators of [4, Chapter 3], with a key novelty provided by a curl correction which ensures the well-posedness of the reconstruction inside mesh elements.

In order to showcase the theoretical results derived here and in the companion paper [12], we carry out a full convergence analysis for a DDR approximation of magnetostatics. This is, to the best of our knowledge, the first full theoretical result of this kind for arbitrary-order polytopal methods.

The rest of this paper is organised as follows. In Section 2 we briefly recall the key elements of the setting introduced in [12]. Section 3 contains the statement of the primal and adjoint consistency results, whose proofs are given in Section 4. The application of the theoretical tools to the error analysis of a DDR approximation of magnetostatics is considered in Section 5 where numerical evidence supporting the error estimates is also provided. Finally, Appendix A contains an in-depth and novel study of the div–curl problems defining the curl liftings on polytopal elements: well-posedness, orthogonality and boundedness properties.

2 Setting

We briefly recall here the setting introduced in the companion paper [12], to which we refer for a more detailed description of the following notions.

2.1 Mesh and orientation

Let \( \mathcal{H} \subset \mathbb{R}_+^* \) be a countable set with 0 as its unique accumulation point. Let \( \Omega \subset \mathbb{R}^3 \) denote an open connected polyhedral set and \( (\mathcal{M}_h)_{h \in \mathcal{H}} \) a family of meshes indexed by their size \( h \). We write \( \mathcal{M}_h := \mathcal{T}_h \cup \mathcal{F}_h \cup \mathcal{E}_h \cup \mathcal{V}_h \) with \( \mathcal{T}_h \) the set of elements \( T \), \( \mathcal{F}_h \) the set of faces \( F \), \( \mathcal{E}_h \) the set of edges \( E \), and \( \mathcal{V}_h \) the set of vertices \( V \). We additionally denote by \( \mathcal{F}_h^b \) the subset of \( \mathcal{F}_h \) collecting the faces that lie on the boundary \( \partial \Omega \) of \( \Omega \). It is assumed that \((\mathcal{T}_h, \mathcal{F}_h)_{h \in \mathcal{H}} \) matches the regularity conditions in [11].
Definition 1.9] (with $\rho \in (0, 1)$ denoting the mesh regularity parameter), and that elements and faces are simply connected with Lipschitz continuous boundary. For $T \in T_h$, we set $T_T := \{F \in T_h : F \subset \partial T\}$ and, for $Y \in T_h \cup T_f, E_Y := \{E \in E_h : E \subset \partial Y\}$, the real number $h_Y$ denotes the diameter of a mesh element, face, or edge $Y \in T_h \cup T_f \cup E_h$.

Each face $F \in T_h$ is equipped with a unit normal vector $n_F$, and each edge $E \in E_h$ with a unit tangent vector $t_E$. Given $F \in T_h$ and $E \in E_F$, we also denote by $n_{FE}$ the unit vector normal to $E$ lying in the plane of $F$. The families of numbers \( \{\omega_{TF} \in \{-1, 1\} : T \in T_h, F \in T_T\} \) and \( \{\omega_{TE} \in \{-1, 1\} : F \in T_h, E \in E_F\} \) collect relative orientations selected so that: for all $T \in T_h$ and all $F \in T_T$, $\omega_{TF} n_F$ points out of $T$ and, for all $F \in T_h$ and all $E \in E_F$, $\omega_{TE} n_E$ points out of $F$. Given $F \in T_h$, the tangent gradient, divergence, two-dimensional vector and scalar curl operators are denoted by $\nabla F, \text{div} F, \nabla F$ and $\nabla F$, respectively.

### 2.2 Polynomial spaces

Let $\ell \geq -1$ be an integer. For $Y \in T_h \cup T_f \cup E_h$, with $n$ the dimension of $Y$, we denote by $P^\ell(Y)$ the space of polynomial functions over $Y$ of total degree $\leq \ell$, and we set $P^\ell(Y) = P^\ell(Y)^n$. The $L^2$-orthogonal projector on $P^\ell(Y)$ is $\pi_{P,Y}$, and $\pi_{P,Y} : L^2(Y) \to P^\ell(Y)$ is its vector-valued counterpart. The set $P^\ell(E_h)$ is made of all continuous functions over the mesh skeleton $\bigcup_{E \in \partial E_h} E$ that are polynomial of total degree $\leq \ell$ on each $E \in E_h$.

For all $Y \in T_h \cup T_f$, denote by $x_Y$ a point inside $Y$ such that $Y$ contains a ball centered at $x_Y$ and of diameter $\rho_{hY}$. For any mesh face $F \in T_h$, any mesh element $T \in T_h$, and any integer $\ell \geq -1$, we define

\[
\begin{align*}
\mathcal{G}^\ell(F) &:= \nabla F P^{\ell+1}(F), & \mathcal{G}^{c,\ell}(F) &:= (x - x_F)^\perp P^{\ell-1}(F), \\
\mathcal{R}^\ell(F) &:= \text{rot} P^{\ell+1}(F), & \mathcal{R}^{c,\ell}(F) &:= (x - x_F) P^{\ell-1}(F),
\end{align*}
\]

(2.1)

where $(x - x_F)^\perp$ denotes the vector $x - x_F$ rotated by an angle $-\pi/2$ in the plane spanned by $F$ and oriented by $n_F$. If $Y = F$ or $Y = T$, the following direct (but not necessarily orthogonal) decompositions hold:

\[
P^\ell(Y) = \mathcal{G}^\ell(Y) \oplus \mathcal{G}^{c,\ell}(Y) = \mathcal{R}^\ell(Y) \oplus \mathcal{R}^{c,\ell}(Y).
\]

(2.2)

With obvious notations, the $L^2$-orthogonal projectors on the subspaces appearing in these decompositions are denoted by $\pi_{\mathcal{G},Y}, \pi_{\mathcal{G},c,\ell,Y}, \pi_{\mathcal{R},Y},$ and $\pi_{\mathcal{R},c,\ell,Y}$. The local Nédélec and Raviart–Thomas spaces over $Y$ are denoted by

\[
\begin{align*}
\mathcal{N}^\ell(Y) &:= \mathcal{G}^{\ell-1}(Y) \oplus \mathcal{G}^{c,\ell}(Y), & \mathcal{RT}^\ell(Y) &:= \mathcal{R}^{\ell-1}(Y) \oplus \mathcal{R}^{c,\ell}(Y).
\end{align*}
\]

(2.3)

As detailed in [12] Lemma 4, the knowledge of the $L^2$-projections of a polynomial $z \in P^\ell(Y)$ on each element of the space pairs $(\mathcal{G}^\ell(Y), \mathcal{G}^{c,\ell}(Y))$ or $(\mathcal{R}^\ell(Y), \mathcal{R}^{c,\ell}(Y))$ appearing in (2.2) enables the recovery of $z$. Specifically, for $Y \in T_h \cup T_f, X \in \{\mathcal{G}, \mathcal{R}\}$, and $(v, w) \in X^\ell(Y) \times X^{c,\ell}(Y)$, letting

\[
\mathcal{R}^\ell_{X,Y}(v, w) := (\text{Id} - \pi_{X,Y}^{\ell,\ell} \pi_{X,Y}^{\ell,c,\ell})^{-1} (v - \pi_{X,Y}^{\ell,\ell} w) + (\text{Id} - \pi_{X,Y}^{c,\ell} \pi_{X,Y}^{c,c,\ell})^{-1} (v - \pi_{X,Y}^{c,\ell} w)
\]

(2.4)

we have

\[
\pi_{X,Y}^{\ell,\ell} (\mathcal{R}^\ell_{X,Y}(v, w)) = v \quad \text{and} \quad \pi_{X,Y}^{c,\ell} (\mathcal{R}^\ell_{X,Y}(v, w)) = w \quad \forall (v, w) \in X^\ell(Y) \times X^{c,\ell}(Y),
\]

(2.5)

\[
z = \mathcal{R}^\ell_{X,Y}(v, w) \quad \forall z \in P^\ell(Y),
\]

(2.6)

and

\[
\|\mathcal{R}^\ell_{X,Y}(v, w)\|_{L^2(Y)} \approx \|v\|_{L^2(Y)} + \|w\|_{L^2(Y)} \quad \forall (v, w) \in X^\ell(Y) \times X^{c,\ell}(Y).
\]

(2.7)
Above, writing \( a \leq b \) in place of \( a \leq Cb \) with \( C \) depending only on \( \Omega \), the mesh regularity parameter \( \rho \) of Definition 1.9], and the considered polynomial degree, we have used \( a \simeq b \) with the meaning of “\( a \leq b \) and \( b \leq a \)”. Both shorthand notations \( \leq \) and \( \simeq \) will be used throughout the paper.

2.3 Discrete spaces

The discrete counterpart of the space \( H^1(\Omega) \) in the DDR sequence is

\[
X_{\text{grad}, h}^k := \left\{ q_h = ((q_T)_{T \in \mathcal{T}_h}, (q_F)_{F \in \mathcal{F}_h}, q_{E_h}) : q_T \in \mathcal{P}^{k-1}(T) \text{ for all } T \in \mathcal{T}_h, q_F \in \mathcal{P}^{k-1}(F) \text{ for all } F \in \mathcal{F}_h, \text{ and } q_{E_h} \in \mathcal{P}_c^{k+1}(E_h) \right\},
\]

and the corresponding interpolator \( I_{\text{grad}, h}^k : C^0(\Sigma) \rightarrow X_{\text{grad}, h}^k \) is such that, for all \( q \in C^0(\Sigma) \),

\[
I_{\text{grad}, h}^k q := \left( (\pi_{p, T}^{k-1} q|_T)_{T \in \mathcal{T}_h}, (\pi_{p, F}^{k-1} q|_F)_{F \in \mathcal{F}_h}, q_{E_h} \right) \in X_{\text{grad}, h}^k,
\]

where \( \pi_{p, T}^{k-1}(q, E_h)|_E = \pi_{p, F}^{k-1}(q, E)|_E \) for all \( E \in \mathcal{E}_h \) and \( q_{E_h}(x_V) = q(x_V) \) for all \( V \in \mathcal{V}_h \), with \( x_V \) denoting the coordinates vector of the vertex \( V \). The discrete \( \mathbf{H}(\text{curl}; \Omega) \) space is

\[
X_{\text{curl}, h}^k := \left\{ v_h = ((v_{R,T}, v_{G,T})_{T \in \mathcal{T}_h}, (v_{R,F}, v_{G,F})_{F \in \mathcal{F}_h}, (v_E)_{E \in \mathcal{E}_h}) : (v_{R,T}, v_{G,T}) \in \mathcal{R}^{k-1}(T) \times \mathcal{G}^{c,k}(T) \text{ for all } T \in \mathcal{T}_h, (v_{R,F}, v_{G,F}) \in \mathcal{R}^{k-1}(F) \times \mathcal{G}^{c,k}(F) \text{ for all } F \in \mathcal{F}_h, \text{ and } v_E \in \mathcal{P}^{k}(E) \text{ for all } E \in \mathcal{E}_h \right\},
\]

with interpolator \( I_{\text{curl}, h}^k : C^0(\Sigma) \rightarrow X_{\text{curl}, h}^k \) such that, for all \( v \in C^0(\Sigma) \),

\[
I_{\text{curl}, h}^k v := \left( (\pi_{p, T}^{k-1} v|_T, \pi_{G, T}^{c,k} v|_T)_{T \in \mathcal{T}_h}, (\pi_{R, F}^{k-1} v|_F, \pi_{G, F}^{c,k} v|_F)_{F \in \mathcal{F}_h}, (\pi_{p, E}^k (v|_E \cdot I_E))_{E \in \mathcal{E}_h} \right),
\]

where, for all \( F \in \mathcal{T}_h, v_{l,F} := n_F \times (v|_F \times n_F) \) denotes the orthogonal projection of \( v \) on the plane spanned by \( F \). The role of \( \mathbf{H}(\text{div}; \Omega) \) is played, at the discrete level, by

\[
X_{\text{div}, h}^k := \left\{ v_h = ((v_{G,T}, v_{G,T})_{T \in \mathcal{T}_h}, (v_{G,F})_{F \in \mathcal{F}_h}) : (v_{G,T}) \in \mathcal{G}^{k-1}(T) \times \mathcal{G}^{c,k}(T) \text{ for all } T \in \mathcal{T}_h \text{ and } v_F \in \mathcal{P}^{k}(F) \text{ for all } F \in \mathcal{T}_h \right\},
\]

with interpolator \( I_{\text{div}, h}^k : H^1(\Omega) \rightarrow X_{\text{div}, h}^k \) such that, for all \( v \in H^1(\Omega) \),

\[
I_{\text{div}, h}^k v := \left( (\pi_{p, T}^{k-1} v|_T, \pi_{G, T}^{c,k} v|_T)_{T \in \mathcal{T}_h}, (\pi_{p, F}^k (v|_F \cdot n_F))_{F \in \mathcal{T}_h} \right).
\]

Finally, the discrete counterpart of \( L^2(\Omega) \) in the DDR sequence is

\[
\mathcal{P}^k(\mathcal{T}_h) := \{ q_h \in L^2(\Omega) : (q_h)|_T \in \mathcal{P}^k(T) \text{ for all } T \in \mathcal{T}_h \},
\]

equipped with the global \( L^2 \)-orthogonal projector \( \pi_{p, h}^k : L^2(\Omega) \rightarrow \mathcal{P}^k(\mathcal{T}_h) \) such that, for all \( q \in L^2(\Omega) \),

\[
(\pi_{p, h}^k q)|_T := \pi_{G, T}^{k}(q|_T) \text{ for all } T \in \mathcal{T}_h.
\]

2.4 Local discrete vector calculus operators

Given \( \bullet \in \{ \text{grad}, \text{curl}, \text{div} \} \) and a mesh entity \( Y \) appearing in the definition of \( X_{\emptyset, h}^k \), we denote by \( X_{\emptyset, Y}^k \) the restriction of this space to \( Y \), gathering the polynomial components on \( Y \) and on the geometrical entities on \( \partial Y \). The corresponding local interpolator is denoted by \( I_{\emptyset, Y}^k \).
2.4.1 Gradients

Throughout the rest of the paper, for $E \in \mathcal{E}_h$ and $q_E \in X^k_{\text{grad},h}$ we set $q_E : (q_E)_E \in X^k_{\text{grad},E} = \mathcal{P}^{k+1}(E)$. For any $E \in \mathcal{E}_h$, the edge gradient $G^k_E : X^k_{\text{grad},E} \to \mathcal{P}^k(E)$ is such that, for all $q_E \in X^k_{\text{grad},E}$,

$$G^k_E q_E := q'_E,$$

where the derivative is taken along $E$ according to the orientation of $t_E$. For all $F \in \mathcal{T}_h$, the face gradient $G^k_F : X^k_{\text{grad},F} \to \mathcal{P}^k(F)$ is such that, for all $q_F \in X^k_{\text{grad},F}$,

$$\int_F G^k_F q_F \cdot w_F = -\int_F q_F \text{div}_F w_F + \sum_{E \in \partial F} \omega_{EF} \int_E q_E (w_F \cdot n_{FE}) \quad \forall w_F \in \mathcal{P}^k(F).$$

The scalar trace $\gamma^k_F : X^k_{\text{grad},F} \to \mathcal{P}^{k+1}(F)$ is such that, for all $q_F \in X^k_{\text{grad},F}$,

$$\int_F \gamma^k_F q_F \text{div}_F v_F = -\int_F G^k_F q_F \cdot v_F + \sum_{E \in \partial F} \omega_{EF} \int_E q_E (v_F \cdot n_{FE}) \quad \forall v_F \in \mathcal{P}^{k+1}(F).$$

**Remark 1** (Validity of (2.10)). Relation (2.10) also holds for all $v_F \in \mathcal{P}^{k+1}(F)$; see [12, Remark 9].

Finally, for all $T \in \mathcal{T}_h$, the element gradient $G^k_T : X^k_{\text{grad},T} \to \mathcal{P}^k(T)$ is such that, for all $q_T = (q_T, (q_E)_{E \in F}) \in X^k_{\text{grad},T}$,

$$\int_T G^k_T q_T \cdot w_T = -\int_T q_T \text{div}_T w_T + \sum_{F \in \partial T} \omega_{TF} \int_F \gamma^k_F (w_T \cdot n_F) \quad \forall w_T \in \mathcal{P}^k(T).$$

2.4.2 Curls

For all $F \in \mathcal{T}_h$, the face curl $C^k_F : X^k_{\text{curl},F} \to \mathcal{P}^k(F)$ is such that, for all $v_F = (v_R,F, v_{R,F}, (v_E)_{E \in F}) \in X^k_{\text{curl},F}$,

$$\int_F C^k_F v_F \cdot r_F = \int_F v_R, F \cdot \text{rot}_F r_F - \sum_{E \in \partial F} \omega_{EF} \int_E v_{EF} r_E \quad \forall r_F \in \mathcal{P}^k(F).$$

The tangential trace $\gamma^k_{\text{curl},F} : X^k_{\text{curl},F} \to \mathcal{P}^k(F)$ is such that, for all $v_F \in X^k_{\text{curl},F}$, recalling the definition (2.4) of the recovery operator with $(X, Y) = (\mathcal{R}, F)$,

$$\gamma^k_{\text{curl},F} v_F := \mathcal{R}^k_{\text{curl},F} (\gamma^k_{\text{curl},F} v_F, v_{R,F}^k),$$

where $\mathcal{R}^k_{\text{curl},F} v_F \in \mathcal{R}^k(F)$ is defined by

$$\int_F \gamma^k_{\text{curl},F} v_F \cdot \text{rot}_F r_F = \int_F C^k_F v_F \cdot r_F + \sum_{E \in \partial F} \omega_{EF} \int_E v_{EF} r_E \quad \forall r_F \in \mathcal{P}^{0,k+1}(F).$$

**Remark 2** (Validity of (2.13)). We note that this relation actually holds for all $r_F \in \mathcal{P}^{0,k+1}(F)$ and also with $\gamma^k_{\text{curl},F}$ instead of $\mathcal{R}^k_{\text{curl},F}$; see [12] Remark 14.

Finally, for all $T \in \mathcal{T}_h$, the element curl $C^k_T : X^k_{\text{curl},T} \to \mathcal{P}^k(T)$ is such that, for all $v_T = (v_R,T, v_{R,T}, (v_R,F, v_{R,F})_{F \in \partial T}, (v_E)_{E \in \partial T}) \in X^k_{\text{curl},T}$,

$$\int_T C^k_T v_T \cdot w_T = \int_T v_R,T \cdot \text{curl}_T w_T + \sum_{F \in \partial T} \omega_{TF} \int_F \gamma^k_{\text{curl},F} v_F \cdot (w_T \times n_F) \quad \forall w_T \in \mathcal{P}^k(T).$$
2.4.3 Divergence

For all $T \in T_h$, the element divergence $D^k_T : X^k_{\text{div},T} \to P^k(T)$ is defined by: For all $v_T = (v_T^T, v_T^F, (v_T^F)_F \in F_T) \in X^k_{\text{div},T}$,

$$
\int_T D^k_T v_T \cdot r_T = -\int_T v_T^T \cdot \nabla r_T + \sum_{F \in T} \omega_T F \int_F v_F^F r_T \quad \forall r_T \in P^k(T).
$$

(2.15)

2.5 Global sequence

The global discrete gradient $G^k_h : X^k_{\text{grad},h} \to X^k_{\text{curl},h}$, curl $G^k_h : X^k_{\text{curl},h} \to X^k_{\text{div},h}$, and divergence $D^k_h : X^k_{\text{div},h} \to P^k(T_h)$ are obtained by projecting the local operators onto the corresponding spaces:

For all $(q_h, v_h, w_h) \in X^k_{\text{grad},h} \times X^k_{\text{curl},h} \times X^k_{\text{div},h}$,

$$
\begin{align*}
G^k_h q_h &:= (\pi^{R,T}_{\text{grad}}(G^k_T q^E), \pi^{\text{curl}}_{\text{grad}}(G^k_T q^E))_{T \in T_h}, \pi^{k-1}_{\text{grad}}(G^k_E q_E)_{E \in E_h},
C^k_h v_h &:= (\pi^{R,F}_{\text{curl}}(C^k_T v^E), \pi^{\text{div}}_{\text{curl}}(C^k_T v^E))_{T \in T_h}, \pi^{k-1}_{\text{curl}}(C^k_E v_E)_{E \in E_h},
D^k_h w_h &:= D^k_T w_T \quad \forall T \in T_h.
\end{align*}
$$

Following our previous notation for local spaces and interpolator, we will use the following notations for the restrictions of these discrete gradients and curl operators to mesh elements and faces:

$$
\begin{align*}
G^k_T q_T &:= (\pi^{R,F}_{\text{grad}}(G^k_T q_T^E), \pi^{\text{curl}}_{\text{grad}}(G^k_T q_T^E), \pi^{k-1}_{\text{grad}}(G^k_E q_E)_{E \in E_T}), \\
C^k_T v_T &:= (\pi^{R,F}_{\text{curl}}(C^k_T v_T^E), \pi^{\text{div}}_{\text{curl}}(C^k_T v_T^E), \pi^{k-1}_{\text{curl}}(C^k_E v_E)_{E \in E_T}).
\end{align*}
$$

The global sequence reads:

$$
\begin{align*}
\mathbb{R} &\xrightarrow{L^k_{\text{grad},h}} X^k_{\text{grad},h} \xrightarrow{G^k_h} X^k_{\text{curl},h} \xrightarrow{C^k_h} X^k_{\text{div},h} \xrightarrow{D^k_h} P^k(T_h) \xrightarrow{0} \{0\}.
\end{align*}
$$

(2.16)

It is proved in [12] that this sequence has exactness properties (the specific nature of which depends on the topology of $\Omega$ as for the continuous de Rham sequence), and that the discrete operators satisfy Poincaré inequalities.

3 Consistency results

3.1 Potential reconstructions and $L^2$-products on discrete spaces

Let $T \in T_h$. In this section, we define polynomial potential reconstructions on the discrete spaces $X_{\bullet,T}$ with $\bullet \in \{\text{grad}, \text{curl}, \text{div}\}$. These potentials have polynomial consistency properties, and enable the construction of discrete L2-inner products on DDR spaces that are also polynomially consistent.

3.1.1 Scalar potential on $X^k_{\text{grad},T}$

The scalar potential reconstruction $p^{k+1}_{\text{grad},T} : X^k_{\text{grad},T} \to P^{k+1}(T)$ is such that, for all $q_T \in X^k_{\text{grad},T}$,

$$
\int_T p_{\text{grad},T}^{k+1} q_T \text{ div } v_T = -\int_T G^k_T q_T \cdot v_T + \sum_{F \in T} \omega_T F \int_F \gamma_F^{k+1} q_T (v_T \cdot n_F) \quad \forall v_T \in \mathcal{R}^{k+2}(T),
$$

(3.1)

with $\gamma_F^{k+1}$ defined by (2.10). This relation defines $p_{\text{grad},T}^{k+1} q_T$ uniquely since div : $\mathcal{R}^{k+2}(T) \to P^{k+1}(T)$ is an isomorphism by [11] Corollary 7.3].
Remark 3 (Validity of (3.1)). The definition (2.11) of $G^k_T$ and the identity $\text{div} \, \text{curl} = 0$ show that both sides of (3.1) vanish when $v_T \in \mathcal{R}^k(T)$. Hence, (3.1) actually holds for any $v_T \in \mathcal{R}^k(T) \oplus \mathcal{R}^{c,k+2}(T) = \mathcal{P}^k(T) + \mathcal{R}^{c,k+2}(T)$, the second equality following from $\mathcal{R}^c(T) \subset \mathcal{R}^{c,k+2}(T)$ and the decomposition (2.2).

Using the polynomial consistency properties $G^k_T(I^k_{\text{grad},T} q) = \text{grad} q$ and $\gamma^{k+1}_T(I^k_{\text{grad,F}} q|_F) = q|_F$, valid for all $q \in \mathcal{P}^{k+1}(T)$ (see [12] Eqs. (5.13) and (5.11)), the following polynomial consistency of $p^{k+1}_{\text{grad},T}$ is obtained:

$$p^{k+1}_{\text{grad},T}(I^k_{\text{grad},T} q) = q \quad \forall q \in \mathcal{P}^{k+1}(T).$$

Moreover, applying (3.1) to $v_T \in \mathcal{R}^{c,k}(T)$ (which is possible since $\mathcal{R}^{c,k}(T) \subset \mathcal{R}^{c,k+2}(T)$), using the definition (2.11) of $G^k_T$ with $w_T = v_T$, and recalling that $\text{div} : \mathcal{R}^{c,k}(T) \to \mathcal{P}^{k-1}(T)$ is onto, we obtain

$$\pi^{k-1}_{\mathcal{P},T}(p^{k+1}_{\text{grad},T} q_T) = q_T \quad \forall q_T \in X^k_{\text{grad},T}.$$  

3.1.2 Vector potential on $X^k_{\text{curl},T}$

The vector potential reconstruction $P^k_{\text{curl},T} : \mathcal{X}^k_{\text{curl},T} \to \mathcal{P}^k(T)$ is such that, for all $v_T \in \mathcal{X}^k_{\text{curl},T}$,

$$P^k_{\text{curl},T} v_T := R^k_{R,T}(P^k_{\text{curl,R},T} v_T, v^c_{R,T}),$$

where $P^k_{\text{curl,R},T} v_T \in \mathcal{R}^k(T)$ is defined, using the isomorphism $\text{curl} : \mathcal{G}^{c,k+1}(T) \to \mathcal{R}^k(T)$ (see [12], Eq. (2.10)), by

$$\int_T P^k_{\text{curl,R},T} v_T \cdot \text{curl} w_T = \int_T C^{k}_{T} v_T \cdot w_T - \sum_{F \in \mathcal{T}} \omega_{TF} \int_F \gamma^{k}_T \, v_T \cdot (w_T \times n_F) \quad \forall w_T \in \mathcal{G}^{c,k+1}(T).$$

(3.5)

Remark 4 (Discrete integration by parts formula for $P^k_{\text{curl},T}$). Formula (3.5) can be extended to test functions in the Nédélec space $\mathcal{N}^{k+1}(T)$ defined by (2.3). To check it, simply notice that both sides vanish whenever $w_T \in \mathcal{G}^k(T)$ (use $\text{curl} \, \text{grad} = 0$ and the definition (2.14) of $C^k_T$). Since $\pi^{k}_{R,T}(P^k_{\text{curl},T} v_T) = P^k_{\text{curl,R},T} v_T$ (see (3.4) and (2.5)), we infer that

$$\int_T P^k_{\text{curl},T} v_T \cdot \text{curl} z_T = \int_T C^{k}_{T} v_T \cdot z_T - \sum_{F \in \mathcal{T}} \omega_{TF} \int_F \gamma^{k}_T \, v_T \cdot (z_T \times n_F) \quad \forall z_T \in \mathcal{N}^{k+1}(T).$$

(3.6)

Applying (3.6) to $v_T = I^k_{\text{curl},T} v$ with $v \in \mathcal{P}^k(T)$, using the consistency properties $\gamma^{k}_T(I^k_{\text{curl,F}} v) = \pi^{k}_{\mathcal{P},F} v = v|_F$ and $C^k_T(I^k_{\text{curl},T} v) = \text{curl} v$ (see [12] Eqs. (3.22) and (3.26)), and integrating by parts, and since $\text{curl} : \mathcal{N}^{k+1}(T) \to \mathcal{R}^k(T)$ is onto (due to the isomorphism property [12], Eq. (2.10)), we see that $\pi^{k}_{R,T}[P^k_{\text{curl},T}(I^k_{\text{curl},T} v)] = \pi^{k}_{R,T} v$. The definition (3.4) and the property (2.5) of the recovery operator also yield $\pi^{c,k}_{R,T}[P^k_{\text{curl},T}(I^k_{\text{curl},T} v)] = \pi^{c,k}_{R,T} v$. As a consequence,

$$P^k_{\text{curl},T}(I^k_{\text{curl},T} v) = v \quad \forall v \in \mathcal{P}^k(T).$$

(3.7)

Following similar arguments as in the proof of [12] Proposition 15, we also have

$$\pi^{k-1}_{R,T}(P^k_{\text{curl},T} v_T) = v_{R,T} \quad \text{and} \quad \pi^{c,k}_{R,T}(P^k_{\text{curl},T} v_T) = v^c_{R,T} \quad \forall v_T \in X^k_{\text{curl},T}.$$  

(3.8)
3.1.3 Vector potential on $X^k_{\text{div},T}$

The vector potential reconstruction $P^k_{\text{div},T} : X^k_{\text{div},T} \to \mathcal{P}^k(T)$ is such that, for all $w_{rT} \in X^k_{\text{div},T}$, 

$$P^k_{\text{div},T} w_{rT} = \mathcal{R}^k_{G,T}(P^k_{\text{div},G,T} w_{rT}, w^c_{G,T}),$$

where $P^k_{\text{div},G,T} w_{rT} \in \mathcal{G}^k(T)$ is defined by

$$\int_T P^k_{\text{div},G,T} w_{rT} \cdot \text{grad} r_T = - \int_T D^k_{\text{grad},T} w_{rT} r_T + \sum_{F \in \mathcal{T}_T} \omega_F F \int_F w_{rT} r_T \quad \forall r_T \in \mathcal{P}^{k+1}(T). \quad (3.9)$$

**Remark 5** (Discrete integration by parts formula for $P^k_{\text{div},T}$). Observing that $P^k_{\text{div},G,T} = \pi^k_{G,T} P^k_{\text{div},T}$ (use (2.5)) and that (3.9) holds for any $r_T \in \mathcal{P}^{k+1}(T)$ (as can be proved taking $r_T$ constant in $T$ and observing that both sides of this equation vanish due to the definition (2.15) of $D^k_{\text{grad},T}$), we infer

$$\int_T P^k_{\text{div},T} w_{rT} \cdot \text{grad} r_T = - \int_T D^k_{\text{grad},T} w_{rT} r_T + \sum_{F \in \mathcal{T}_T} \omega_F F \int_F w_{rT} r_T \quad \forall r_T \in \mathcal{P}^{k+1}(T). \quad (3.10)$$

Writing (3.10) for $w_{rT} = I^k_{\text{div},T} w$ with $w \in \mathcal{R}^k(T)$, observing that $D^k_{\text{grad},T} (I^k_{\text{div},T} w) = \pi^k_{\text{div},T}(\text{div} w) =$ div $w$ by [12, Eq. (A.4)] and $\pi^k_{F,F}(w_{F,T} \cdot n_F) = w_{F,T} \cdot n_F$ for all $F \in \mathcal{T}_T$ by [12, Eq. (A.4)], and integrating by parts the right-hand side of the resulting expression, we infer $\pi^k_{G,T} \{P^k_{\text{div},T} (I^k_{\text{div},T} w)\} = \pi^k_{G,T} w$; since $\pi^c_{G,T} \{P^k_{\text{div},T} (I^k_{\text{div},T} w)\} = \pi^c_{G,T} w$ by definition of $P^k_{\text{div},T} : I^k_{\text{div},T}$ and (2.5), we deduce that

$$P^k_{\text{div},T} (I^k_{\text{div},T} w) = \pi^k_{G,T} w \quad \forall w \in \mathcal{R}^k(T). \quad (3.11)$$

Moreover, following similar arguments as in [12] Proposition [15] we get

$$\pi^k_{G,T} (P^k_{\text{div},T} w_{rT}) = w_{G,T} \quad \text{and} \quad \pi^c_{G,T} (P^k_{\text{div},T} w_{rT}) = w^c_{G,T} \quad \forall w_{rT} \in X^k_{\text{div},T}. \quad (3.12)$$

3.1.4 Discrete $L^2$-products

We now define discrete $L^2$-inner products on the DDR spaces. These products are all constructed in a similar way: by assembling local contributions composed of a consistent term based on the potential reconstructions and a stabilisation term that provides a control of the polynomial components on the lower dimensional geometrical objects. Specifically, each $L^2$-product $(\cdot, \cdot)_{\text{grad},h} : X^k_{\text{grad},h} \times X^k_{\text{grad},h} \to \mathbb{R}$, $(\cdot, \cdot)_{\text{curl},h} : X^k_{\text{curl},h} \times X^k_{\text{curl},h} \to \mathbb{R}$, and $(\cdot, \cdot)_{\text{div},h} : X^k_{\text{div},h} \times X^k_{\text{div},h} \to \mathbb{R}$ is the sum over $T \in \mathcal{T}_h$ of its local counterpart defined by:

$$(L_{\text{grad},T} \cdot \mathcal{T}_{\text{grad},T}) := \int_T P^k_{\text{grad},T} L_{\text{grad},T} \mathcal{T}_{\text{grad},T} + s_{\text{grad},T} (L_{\text{grad},T} \cdot \mathcal{T}_{\text{grad},T}) \quad \forall (L_{\text{grad},T} \cdot \mathcal{T}_{\text{grad},T}) \in X^k_{\text{grad},T} \times X^k_{\text{grad},T}, \quad (3.13a)$$

$$(L_{\text{curl},T} \cdot \mathcal{T}_{\text{curl},T}) := \int_T P^k_{\text{curl},T} L_{\text{curl},T} \mathcal{T}_{\text{curl},T} + s_{\text{curl},T} (L_{\text{curl},T} \cdot \mathcal{T}_{\text{curl},T}) \quad \forall (L_{\text{curl},T} \cdot \mathcal{T}_{\text{curl},T}) \in X^k_{\text{curl},T} \times X^k_{\text{curl},T}, \quad (3.13b)$$

$$(L_{\text{div},T} \cdot \mathcal{T}_{\text{div},T}) := \int_T P^k_{\text{div},T} L_{\text{div},T} \mathcal{T}_{\text{div},T} + s_{\text{div},T} (L_{\text{div},T} \cdot \mathcal{T}_{\text{div},T}) \quad \forall (L_{\text{div},T} \cdot \mathcal{T}_{\text{div},T}) \in X^k_{\text{div},T} \times X^k_{\text{div},T}, \quad (3.13c)$$

8
where the symmetric, positive semidefinite stabilisation bilinear forms \( s_{T,T} \), \( \bullet = \{ \text{grad}, \text{curl}, \text{div} \} \), are defined as follows:

\[
\begin{align*}
\text{s}_{\text{grad},T} (\ell_F, q_F) & := \sum_{F \in T} h_F^2 \int_F \left( p_{\text{grad},T}^{k+1} - \gamma_F^{k+1} \ell_F \right) \left( p_{\text{grad},T}^{k+1} q_F - \gamma_F^{k+1} q_F \right) \\
& \quad + \sum_{E \in \partial T} h_E^2 \int_E \left( p_{\text{grad},E}^{k+1} - r_E \right) \left( p_{\text{grad},E}^{k+1} q_E - q_E \right), \\
\text{s}_{\text{curl},T} (\varphi_F, \psi_F) & := \sum_{F \in T} h_F^2 \int_F \left( (P_{\text{curl},T}\varphi)_F, F - \gamma_{\text{curl},F}^{k}(\varphi) \right) \cdot \left( (P_{\text{curl},T}\psi)_F, F - \gamma_{\text{curl},F}^{k}(\psi) \right) \\
& \quad + \sum_{E \in \partial T} h_E^2 \int_E \left( p_{\text{curl},E}^{k} \varphi_E \cdot t_E - t_E \right) \left( p_{\text{curl},E}^{k} \psi_E \cdot t_E - t_E \right),
\end{align*}
\]

where we recall that the index \( t, F \) denotes the tangential trace on \( F \), and

\[
\text{s}_{\text{div},T} (\varphi_F, \psi_F) := \sum_{F \in T} h_F^2 \int_F \left( p_{\text{div},T}^{k} \varphi_F \cdot n_F - w_F \right) \left( p_{\text{div},T}^{k} \psi_F \cdot n_F - v_F \right).
\]

These local stabilisation bilinear forms \( s_{T,T} \) are polynomially consistent, i.e., they vanish whenever one of their arguments is the interpolant of a polynomial of total degree \( \leq k + 1 \) for \( \bullet = \text{grad} \), or \( \leq k \) for \( \bullet \in \{ \text{curl}, \text{div} \} \). Further consistency properties for interpolates of smooth functions are stated in Theorem 8.

For \( \bullet \in \{ \text{grad}, \text{curl}, \text{div} \} \), we denote by \( \| \cdot \|_{T,T} \) the norm on \( X^k_{T,T} \) induced by the corresponding local discrete \( L^2 \)-product \( (\cdot, \cdot)_{T,T} \), and by \( \| \cdot \|_{T,h} \) the norm on \( X^k_{T,h} \) corresponding to the global discrete \( L^2 \)-product \( (\cdot, \cdot)_{T,h} \).

### 3.2 Primal consistency

In this section we state consistency results for the discrete potentials, vector calculus operators, and stabilisation bilinear forms. Because of the nature of the interpolator on \( X^k_{\text{curl},T} \) (which requires higher regularity of functions), we introduce the following notation: For \( T \in T_h \) and \( v \in H^{\max(k+1,2)}(T) \),

\[
|v|_{H^{(k+1,2)}(T)} := \begin{cases} 
|v|_{H^1(T)} + h_T |v|_{H^2(T)} & \text{if } k = 0, \\
|v|_{H^{k+1}(T)} & \text{if } k \geq 1.
\end{cases}
\]

The corresponding global broken seminorm \( |v|_{H^{(k+1,2)}(\Omega_h)} \) is such that, for all \( v \in H^{(k+1,2)}(\Omega_h) \),

\[
|v|_{H^{(k+1,2)}(\Omega_h)} := \left( \sum_{T \in T_h} |v|^2_{H^{(k+1,2)}(T)} \right)^{1/2}.
\]

The proofs of the following theorems are postponed to Section 4.3.

**Theorem 6** (Consistency of the potential reconstructions). It holds, for all \( T \in T_h \),

\[
\begin{align*}
\| P_{\text{grad},T}^{k} (\ell_k^{\text{grad},T} q) - q \|_{L^2(T)} & \leq h_T^{k+2} |q|_{H^{k+2}(T)} & \forall q \in H^{k+2}(T), \\
\| P_{\text{curl},T}^{k} (\varphi) - \psi \|_{L^2(T)} & \leq h_T^{k+1} |v|_{H^{k+1,2}(T)} & \forall \varphi \in H^{\max(k+1,2)}(T), \\
\| P_{\text{div},T}^{k} (\varphi_t) - w \|_{L^2(T)} & \leq h_T^{k+1} |w|_{H^{k+1}(T)} & \forall \varphi_t \in H^{k+1}(T). 
\end{align*}
\]

**Theorem 7** (Primal consistency of the discrete vector calculus operators). It holds, for all \( T \in T_h \),

\[
\begin{align*}
\| G^k_T (\ell_k^{\text{grad},T} q) - \text{grad} q \|_{L^2(T)} & \leq h_T^{k+1} |q|_{H^{k+2}(T)}, & \forall q \in H^{k+2}(T), \\
\| \nabla^k_T (\varphi) - \text{curl} \varphi \|_{L^2(T)} & \leq h_T^{k+1} |\varphi|_{H^{k+1}(T)} & \forall \varphi \in H^{k+1}(T), \\
\| D^k_T (\varphi_t) - \text{div} w \|_{L^2(T)} & \leq h_T^{k+1} |\text{div} w|_{H^{k+1}(T)}, & \forall \varphi_t \in H^{k+1}(T), \\
\| J^k_T (\varphi_{tt}) - \nabla \cdot \psi \|_{L^2(T)} & \leq h_T^{k+1} |\nabla \cdot \psi|_{H^{k+1}(T)}, & \forall \psi \in H^1(T),
\end{align*}
\]

9
Then, it holds, for all $q \in H^{k+2}(T)$,
\begin{equation}
\|q\|_{H^{k+2}(T)} \leq h_T^{k+2} \|q\|_{H^{k+2}(T)} \quad \forall q \in H^{k+2}(T), \tag{3.24}
\end{equation}
\begin{equation}
\|v\|_{H^{k+1}(T)} \leq h_T^{k+1} \|v\|_{H^{k+1}(T)} \quad \forall v \in H^{\max(k+1,2)}(T), \tag{3.25}
\end{equation}
\begin{equation}
\|w\|_{H^{k+1}(T)} \leq h_T^{k+1} \|w\|_{H^{k+1}(T)} \quad \forall w \in H^{k+1}(T). \tag{3.26}
\end{equation}

### 3.3 Adjoint consistency

Whenever a (formal) integration by parts is used to write the weak formulation of a PDE problem underpinning its discretisation, a form of adjoint consistency is required in the convergence analysis. We state here the adjoint consistency of the operators in the DDR sequence (2.16). Since this sequence does not incorporate boundary conditions, the corresponding adjoint consistency will be based on essential boundary conditions. The regularity requirements will be expressed in terms of the broken Sobolev spaces and norms such that, for any $\ell \geq 1$,

$$H^\ell(T_h) := \{ g \in L^2(\Omega) : g|_T \in H^\ell(T) \text{ for all } T \in T_h \} \quad \text{and} \quad |g|_{H^\ell(T_h)} := \left( \sum_{T \in T_h} |g|_{H^\ell(T)}^2 \right)^{\frac{1}{2}}.$$ 

The corresponding seminorms for vector-valued functions is denoted, as usual, using boldface letters. We denote in what follows by $H^1_0(\Omega)$, $H_0(\text{div}; \Omega)$, and $H_0(\text{curl}; \Omega)$ the subspaces of $H^1(\Omega)$, $H(\text{div}; \Omega)$, and $H(\text{curl}; \Omega)$ spanned by functions whose trace, normal trace, and tangential trace vanish on $\partial \Omega$, respectively.

#### Theorem 9 (Adjoint consistency for the gradient).
Let $\tilde{E}_{\text{div},h} : (C^0(\Omega) \cap H_0(\text{div}; \Omega)) \times X^k_{\text{grad},h} \rightarrow \mathbb{R}$ be such that, for all $q_h \in X^k_{\text{grad},h}$,

$$\tilde{E}_{\text{div},h}(v, q_h) := \sum_{T \in T_h} \left[ (I^k_{\text{grad},T} v|_T, q_h|_T)_{\text{curl},T} + \int_T \text{div} v \cdot P^{k+1}_{\text{grad},T} q_h \right].$$

Then, it holds, for all $v \in C^0(\Omega) \cap H_0(\text{div}; \Omega)$ such that $v \in H^{\max(k+1,2)}(T_h)$ and all $q_h \in X^k_{\text{grad},h}$,

$$|\tilde{E}_{\text{div},h}(v, q_h)| \leq h^{k+1} \|v\|_{H^{k+1,2}(T_h)} \|q_h\|_{\text{grad},h}. \tag{3.27}$$

**Proof.** See Section 4.4.1. \qed

#### Theorem 10 (Adjoint consistency for the curl).
Let $\tilde{E}_{\text{curl},h} : (C^0(\Omega) \cap H_0(\text{curl}; \Omega)) \times X^k_{\text{curl},h} \rightarrow \mathbb{R}$ be such that, for all $(w, v_h) \in (C^0(\Omega) \cap H_0(\text{curl}; \Omega)) \times X^k_{\text{curl},h}$,

$$\tilde{E}_{\text{curl},h}(w, v_h) := \sum_{T \in T_h} \left[ (I^k_{\text{curl},T} w|_T, v_h|_T)_{\text{div},T} - \int_T \text{curl} w \cdot P^{k}_{\text{curl},T} v_h \right]. \tag{3.28}$$

Then, for all $w \in C^0(\Omega) \cap H_0(\text{curl}; \Omega)$ such that $w \in H^{k+2}(T_h)$ and all $v_h \in X^k_{\text{curl},h}$,

$$|\tilde{E}_{\text{curl},h}(w, v_h)| \leq h^{k+1} \left( |w|_{H^{k+2}(T_h)} + |w|_{H^{k+2}(T_h)} \right) \left( \|v_h\|_{\text{curl},h} + \|C^k_{\text{curl},h} v_h\|_{\text{div},h} \right). \tag{3.29}$$

**Proof.** See Section 4.4.2. \qed
Theorem 11 (Adjoint consistency for the divergence). Let \( \tilde{\mathbf{E}}_{\text{grad}, h} : (C^0(\Omega) \cap H^1_0(\Omega)) \times X^k_{\text{div}, h} \to \mathbb{R} \) be such that, for all \( (q, \mathbf{v}_h) \in (C^0(\Omega) \cap H^1_0(\Omega)) \times X^k_{\text{div}, h} \),

\[
\tilde{\mathbf{E}}_{\text{grad}, h}(q, \mathbf{v}_h) := \int_{\Omega} \nabla^T q \cdot \mathbf{v}_h + \sum_{T \in \mathcal{T}_h} \int_{\Omega} \mathbf{grad} q \cdot P^k_{\text{div}, T} \mathbf{v}_T.
\]  

(3.30)

Then, for all \( q \in C^0(\Omega) \cap H^1_0(\Omega) \) such that \( q \in H^{k+2}(\mathcal{T}_h) \) and all \( \mathbf{v}_h \in X^k_{\text{div}, h} \),

\[
|\tilde{\mathbf{E}}_{\text{grad}, h}(q, \mathbf{v}_h)| \leq \frac{k+1}{2} \| q \|_{H^{k+2}(\mathcal{T}_h)} \| \mathbf{v}_h \|_{\text{div}, h}.
\]

(3.31)

Proof. See Section 4.3.3.

\[\square\]

4 Proofs of the consistency results

In this section, after establishing some preliminary results, we prove the primal and adjoint consistency results stated in Section 3.

4.1 Component norms and bounds on potentials

We recall the definition of the component \( L^2 \)-norm on \( X^k_{\text{grad}, T}, X^k_{\text{curl}, T} \) and \( X^k_{\text{div}, T} \) introduced in Section 4.1, and which correspond to the \( L^2 \)-norms of the components of the vectors of polynomials, with scaling appropriate to the dimensions of the geometrical objects on which these components are defined:

\[
\| q_f \|_{\text{grad}, T} := \left( \| q_f \|_{L^2(T)}^2 + \sum_{F \in \mathcal{F}_T} h_F \| q_f \|_{L^2(F)}^2 \right)^{1/2} \quad \text{for all} \quad q \in X^k_{\text{grad}, T},
\]

where \( \| q_f \|_{\text{grad}, F} := \left( \| q_f \|_{L^2(F)}^2 + \sum_{E \in \mathcal{E}_F} h_E \| q_f \|_{L^2(E)}^2 \right)^{1/2} \quad \text{for all} \quad F \in \mathcal{T}_F, \)

\[
\| \mathbf{v}_f \|_{\text{curl}, T} := \left( \| \mathbf{v}_f \|_{L^2(T)}^2 + \sum_{F \in \mathcal{F}_T} h_F \| \mathbf{v}_f \|_{L^2(F)}^2 \right)^{1/2} \quad \text{for all} \quad \mathbf{v}_f \in X^k_{\text{curl}, T},
\]

where \( \| \mathbf{v}_f \|_{\text{curl}, F} := \left( \| \mathbf{v}_f \|_{L^2(F)}^2 + \sum_{E \in \mathcal{E}_F} h_E \| \mathbf{v}_f \|_{L^2(E)}^2 \right)^{1/2} \quad \text{for all} \quad F \in \mathcal{T}_F, \)

and

\[
\| \mathbf{w}_f \|_{\text{div}, T} := \left( \| \mathbf{w}_f \|_{L^2(T)}^2 + \sum_{F \in \mathcal{F}_T} h_F \| \mathbf{w}_f \|_{L^2(F)}^2 \right)^{1/2} \quad \text{for all} \quad \mathbf{w}_f \in X^k_{\text{div}, T}.
\]

The next proposition follows from (2.7) and [12, Lemma 31], in a similar way as in the proof of [10, Proposition 13].

Proposition 12 (Boundedness of local potentials). It holds, for all \( T \in \mathcal{T}_h \) and all \( F \in \mathcal{F}_T \),

\[
\gamma^{k+1}_{f,F} \| q_f \|_{L^2(F)} \leq \| q_f \|_{\text{grad}, F} \quad \text{and} \quad \| P^{k+1}_{\text{grad}, F} q_f \|_{L^2(F)} \leq \| q_f \|_{\text{grad}, T} \quad \forall q_f \in X^k_{\text{grad}, T},
\]

(4.2)

\[
\gamma^k_{f,F} \| v_f \|_{L^2(F)} \leq \| v_f \|_{\text{curl}, F} \quad \text{and} \quad \| P^k_{\text{curl}, F} v_f \|_{L^2(T)} \leq \| v_f \|_{\text{curl}, T} \quad \forall v_f \in X^k_{\text{curl}, T},
\]

(4.3)

\[
\| P^k_{\text{div}, T} w_f \|_{L^2(T)} \leq \| w_f \|_{\text{div}, T} \quad \forall w_f \in X^k_{\text{div}, T}.
\]

(4.4)

For \( \bullet \in \{ \text{grad}, \text{curl}, \text{div} \} \), using triangle inequalities as in [10, Proposition 14], invoking the bounds of Proposition [12] the projection properties (3.3), (3.8) (and similar for \( \gamma^{k}_{f,F} \), see [12, Proposition 15]) or (3.12), and recalling (2.7), we have the norm equivalence: For all \( T \in \mathcal{T}_h \)

\[
\| q_f \|_{T} \approx \| q_f \|_{\bullet,T} \quad \forall q_f \in X^k_{\bullet,T}.
\]

(4.5)
Lemma 13 (Boundedness of local interpolators). It holds, for all \( T \in \mathcal{T}_h \),
\[
\| p^k_{\text{grad},T} q \|_{\text{grad},T} \leq \| q \|_{L^2(T)} + h_T \| q \|_{H^{1}(T)} + h_T^2 \| q \|_{H^2(T)} \quad \forall q \in H^2(T), \quad (4.6)
\]
\[
\| p^k_{\text{curl},T} v \|_{\text{curl},T} \leq \| v \|_{L^2(T)} + h_T \| v \|_{H^{1}(T)} + h_T^2 \| v \|_{H^2(T)} \quad \forall v \in H^2(T), \quad (4.7)
\]
\[
\| L^k_{\text{div},T} w \|_{\text{div},T} \leq \| w \|_{L^2(T)} + h_T \| w \|_{H^{1}(T)} \quad \forall w \in H^1(T). \quad (4.8)
\]

Proof. The definition of \( L^k_{\text{grad},T} \) (see (2.3)) clearly shows that \( \| p^k_{\text{grad},T} q \|_{\text{grad},T} \leq |T|^{1/2} \max_T |q| \). By \cite{11} Eq. (5.110), it holds
\[
\max_T |q| \leq |T|^{-1/2} \sum_{r=0}^{2} h_T^r |q|_{W^r(T)},
\]
which concludes the proof of (4.6). The estimate (4.7) is obtained the same way. As for (4.8), by the continuous trace inequality of \cite{11} Lemma 1.31, we have
\[
\| p^k_{\text{div},T} (w \cdot n_F) \|_{L^2(F)} \leq \| w \|_{L^2(T)} + h_F^{-1} \| w \|_{L^2(T)} + h_F^2 \| w \|_{H^1(T)}.
\]
Using this bound in the definition (2.9) of \( L^k_{\text{div},T} \) yields (4.8). \( \square \)

4.2 Links between discrete vector potentials and vector calculus operators

Proposition 14 (Link between discrete vector potentials and vector calculus operators). For all \( T \in \mathcal{T}_h \), it holds
\[
P^k_{\text{curl},T} (G^k_{T} q_{T}) = G^k_{T} q_{T} \quad \forall q_{T} \in X^k_{\text{grad},T}, \quad (4.9)
\]
\[
P^k_{\text{div},T} (G^k_{T} v_{T}) = C^k_{T} v_{T} \quad \forall v_{T} \in X^k_{\text{curl},T}. \quad (4.10)
\]

Proof. 1. Proof of (4.9). By the second projection property in (3.12), we have \( \pi^k_{R,T} [P^k_{\text{curl},T} (G^k_{T} q_{T})] = \pi^k_{R,T} (G^k_{T} q_{T}) \). To infer the conclusion, it then suffices to prove that
\[
\pi^k_{R,T} [P^k_{\text{curl},T} (G^k_{T} q_{T})] = \pi^k_{R,T} (G^k_{T} q_{T}) \quad (4.11)
\]
and invoke (2.6). To prove (4.11), we take \( z_T \in N^{k+1}(T) \) and apply (3.6) with \( z_T = G^k_{T} q_{T} \). Using the inclusion \( \text{Im} G^k_{T} \subset \text{Ker} G^k_{T} \) (see \cite{12} Remark 21) and the relation \( \gamma^k_{T,F} (G^k_{T} q_{T}) = G^k_{T} q_{F} \) valid for all \( F \in T \) (see \cite{12} Proposition 15), we obtain
\[
\int_{T} P^k_{\text{curl},T} (G^k_{T} q_{T}) \cdot \text{curl} z_T = - \sum_{F \in T} \int_{F} G^k_{T} q_{F} \cdot (z_T \times n_F) = \int_{T} G^k_{T} q_{T} \cdot \text{curl} z_T,
\]
the conclusion following from the link between element and face gradient, see \cite{12} Proposition 11. By the isomorphism \( \text{curl} : G^{c,k+1}(T) \rightarrow R^{k}(T) \) of \cite{12} Eq. (2.10) and since \( G^{c,k+1}(T) \subset N^{k+1}(T) \), this establishes (4.11) and concludes the proof of (4.9).

2. Proof of (4.10). The second projection property in (3.12) ensures that \( \pi^k_{G,T} [P^k_{\text{div},T} (C^k_{T} v_{T})] = \pi^k_{G,T} (C^k_{T} v_{T}) \). As before, it therefore remains to analyse the projections on \( G^k_{T} \). Apply (3.10) to \( w_{T} = C^k_{T} v_{T} \) and a generic \( r_{T} \in P^{k+1}(T) \), and use the inclusion \( \text{Im} C^k_{T} \subset \text{Ker} D^k_{T} \) (see \cite{12} Proposition 17) to get
\[
\int_{T} P^k_{\text{div},T} (C^k_{T} v_{T}) \cdot \text{grad} r_{T} = \sum_{F \in T} \int_{F} C^k_{F} v_{F} r_{T} = \int_{T} C^k_{T} v_{T} \cdot \text{grad} r_{T},
\]
where the conclusion is obtained applying the link between element and face curls of \cite{12} Proposition 16. This establishes that \( \pi^k_{G,T} [P^k_{\text{div},T} (C^k_{T} v_{T})] = \pi^k_{G,T} (C^k_{T} v_{T}) \), proving (4.10). \( \square \)
Proof of Theorem 7.

and (3.19) follows using the approximation properties of $F$ or all boundedness. In the case established in a similar way.

The definitions of

Proof. The definitions of $|||\cdot|||$ show that the edge gradient contributions in the left-hand sides of (4.12) and (4.13) are bounded by the corresponding right-hand sides. To bound the face and element gradient contributions in the left-hand sides of (4.12) and (4.13), simply apply (4.3) to $v_T = G_F^k q_F$, using $\gamma_F^k \circ G_F^k = G_F^k$ (see [12, Proposition 15]) and (4.9). The estimate (4.14) is established in a similar way.

4.3 Primal consistency

Proof of Theorem [2] Let us start with (3.18). Since $H^2(T) \subset C^0(\overline{\mathcal{T}})$, the mapping $P_{\text{grad}, T}^{k+1} \circ L_{\text{grad}, T}^{k} : H^2(T) \to \mathcal{P}^{k+1}(T)$ is well-defined and, owing to (3.2), it is a projector. Moreover, combining (4.6) and (4.7), it satisfies the $L^2(T)$-boundedness

$$\|P_{\text{grad}, T}^{k+1}(L_{\text{grad}, T}^{k} q)|L^2(T) \leq \|q|L^2(T) + h_T|q|H^1(T) + h_T^2|q|H^2(T) \quad \forall q \in H^2(T).$$

The approximation property (3.18) is thus a direct consequence of [11] Lemma 1.43. The proofs of (3.19) (for $k \geq 1$) and (3.20) are similar, using the fact that the considered operators are projectors onto $\mathcal{P}^k(T)$ (see (3.7) and (3.11)) and invoking Proposition [2] and Lemma [13] to establish their $L^2$-boundedness. In the case $k = 0$, since $P_{\text{curv}, T}^{0} \circ L_{\text{curv}, T}^{0}$ requires the $H^2$-regularity of its argument, with $2 > k + 1$, (3.19) cannot be deduced directly from [11] Lemma 1.43. However, using the bounds (4.3) and (4.7), a direct proof can be done by introducing $\pi_{P,T}^{0} v = I_{\text{curv}, T}^{0} (L_{\text{curv}, T}^{0} \pi_{P,T}^{0} v)$:

$$\|P_{\text{curv}, T}^{0}(L_{\text{curv}, T}^{0} v) - v|L^2(T) \leq \|P_{\text{curv}, T}^{0}(L_{\text{curv}, T}^{0} (v - \pi_{P,T}^{0} v))|L^2(T) + \|\pi_{P,T}^{0} v - v|L^2(T)

\leq \|v - \pi_{P,T}^{0} v|L^2(T) + h_T|v - \pi_{P,T}^{0} v|H^1(T) + h_T^2|v - \pi_{P,T}^{0} v|H^2(T),$$

and (3.19) follows using the approximation properties of $\pi_{P,T}^{0}$, the fact that the $H^1(T)$- and $H^2(T)$-seminorms of $\pi_{P,T}^{0} v$ vanish, and the definition (3.17) of $H^{k+1,2}(T)$.

Proof of Theorem [2] Let us prove (3.21). For any $q_F \in X_{\text{grad}, T}$, taking $w_F = G_F^k q_F$ in (2.11) and using Cauchy–Schwarz inequalities along with discrete inverse and trace inequalities, it is inferred, after simplification,

$$\|G_F^k q_F|L^2(T) \leq h_T^{-1}||q_T|L^2(T) + \sum_{F \in \mathcal{T}} h_F^{-1/2} \gamma_F^{k+1} q_F|L^2(F) \leq h_T^{-1}||q||_{\text{grad}, T},$$

where the conclusion follows from the estimate on $\gamma_F^{k+1} q_F$ in (4.2) and from the definition of $|||\cdot|||_{\text{grad}, T}$. As a result, for any $r \in H^2(T)$, making $q_F = L_{\text{grad}, T}^{k} r$ and invoking (4.6), we infer

$$\|G_F^k (L_{\text{grad}, T}^{k} r)|L^2(T) \leq h_T^{-1}||r|L^2(T) + ||r|H^1(T) + h_T |r|H^2(T).$$

(4.15)
Letting now \( q \in H^{k+2}(T) \), we use the polynomial consistency \cite[Eq. (3.13)]{12} of \( G_T^k \) followed by a triangle inequality to write

\[
\|G_T^k (L_{\text{grad},T} q) - \text{grad} \, q\|_{L^2(T)} \leq \|G_T^k (L_{\text{grad},T} (q - \pi_{\text{grad},T}^k q))\|_{L^2(T)} + \|\text{grad} (\pi_{\text{grad},T}^k q - q)\|_{L^2(T)}
\]

and conclude using (4.15) with \( r = q - \pi_{\text{grad},T}^k q \) for the first term in the right-hand side followed by the approximation properties of \( \pi_{\text{grad},T}^k \) (see \cite[Theorem 1.45]{11}).

To prove (3.22), we notice that \( C_T^k (L_{\text{curl},T} v) = P_{\text{div},T}^k [C_T^k (L_{\text{curl},T} v)] = P_{\text{div},T}^k [L_{\text{div},T}^k (\text{curl} \, v)] \) owing to (4.10) along with the commutation property \cite[Eq. (3.35)]{12}, and conclude using the approximation properties (3.20) with \( w = \text{curl} \, v \).

Finally, (3.23) is a straightforward consequence of the commutation property \( D_T^k (L_{\text{div},T}^k w) = \pi_{\text{div},T}^k \) (div \( w \)) stated in \cite[Eq. (3.36)]{12} together with \cite[Theorem 1.45]{11}.

**Remark 16** (Alternative proof of (3.21)). When \( q \in C^1(\overline{T}) \) is such that \( \text{grad} \, q \in H^{\max(k+1,2)}(T) \), the proof of (3.21) can be done following similar arguments as for (3.22), i.e., we write \( G_T^k (L_{\text{grad},T} q) = P_{\text{curl},T}^k [G_T^k (L_{\text{grad},T} q)] = P_{\text{curl},T}^k [L_{\text{curl},T}^k (\text{grad} \, q)] \) using (4.9) followed by \cite[Eq. (3.34)]{12}, and conclude using the approximation properties (3.19) with \( v = \text{grad} \, q \). This argument, however, requires additional regularity on \( q \) with respect to the one used above.

**Proof of Theorem 8** We only prove (3.25), the other consistency properties being established in a similar way. Let \( v \in H^{\max(k+1,2)}(T) \). By the polynomial consistency \cite[Eq. (3.22)]{12} of \( \gamma_{1,T}^k \) and \( \gamma_{1,T}^k \) of \( P_{\text{curl},T}^k \), it is easily checked that, for all \( z_T \in \mathcal{P}_k(T) \) and all \( w_T \in X_{\text{curl},T}^k \), it holds \( s_{\text{curl},T} (L_{\text{curl},T}^k z_T, w_T) = 0 \). Applying this with \( z_T = \pi_{\text{curl},T}^k v \) we infer

\[
s_{\text{curl},T} (L_{\text{curl},T}^k v, L_{\text{curl},T}^k v)^2 = s_{\text{curl},T} (L_{\text{curl},T}^k (v - \pi_{\text{curl},T}^k v), L_{\text{curl},T}^k (v - \pi_{\text{curl},T}^k v)) \leq \|L_{\text{curl},T}^k (v - \pi_{\text{curl},T}^k v)\|_{\text{curl},T}^2,
\]

the conclusion following from the definition of \( \|\|_{\text{curl},T} \) and the norm equivalence (4.3). Invoking then (3.7) we infer

\[
s_{\text{curl},T} (L_{\text{curl},T}^k v, L_{\text{curl},T}^k v)^2 \leq \|v - \pi_{\text{curl},T}^k v\|_{L^2(T)}^2 + \|v - \pi_{\text{curl},T}^k v\|_{H^1(T)}^2 + h_T^2 \|v - \pi_{\text{curl},T}^k v\|_{H^1(T)}^2
\]

and the estimate (3.25) follows from the approximation properties of \( \pi_{\text{curl},T}^k \), see \cite[Theorem 1.45]{11}, and the definition (3.17) of \( \|\|_{H^{k+2,2}(T)} \), using in the case \( k = 0 \) the same arguments as in the proof of Theorem 6.

**4.4 Adjoint consistency**

**4.4.1 Adjoint consistency for the gradient**

**Lemma 17** (Estimates on local \( H^1 \)-seminorms of potentials). For all \( F \in \mathcal{T}_h \) and all \( q_F \in X_{\text{grad},F}^k \), it holds

\[
\|\text{grad} \, \gamma_{F}^k q_F\|_{L^2(F)}^2 + \sum_{E \in \mathcal{E}_F} h_E^{-1} \|\gamma_{E}^{k+1} q_F - q_F\|_{L^2(E)}^2 \leq \|G_F^k q_F\|_{\text{curl},F}^2.
\]

For all \( T \in \mathcal{T}_h \) and all \( q_T \in X_{\text{grad},T}^k \), it holds

\[
\|\text{grad} \, P_{\text{grad},T}^k q_T\|_{L^2(T)}^2 + \sum_{F \in \mathcal{T}_F} h_F^{-1} \|P_{\text{grad},T}^k q_F - \gamma_{F}^k q_F\|_{L^2(F)}^2 \leq \|G_T^k q_T\|_{\text{curl},T}^2.
\]
Let \( u_1 \) and \( u_2 \) be the ideas similar to those used to prove (4.16), provided we can establish a discrete trace inequality on \( \gamma^{k+1}(q_F - A_{q,\partial F}) \), we have

\[
\sum_{E \in \mathcal{E}_F} h_E^{-1} \| \gamma^{k+1}_F q_F - q_E \|_{L^2(E)}^2 \leq \sum_{E \in \mathcal{E}_F} h_E^{-1} \| q_E - A_{q,\partial F} \|_{L^2(E)}^2 + h_E^{-2} \| \gamma^{k+1}_F (q_F - A_{q,\partial F}) \|_{L^2(F)}^2.
\]

(4.18)

Since \( q_{E_F} \) is continuous, recalling that \( q_E = (q_{E_0})_E \) for all \( E \in \mathcal{E}_F \) and using a Poincaré–Wirtinger inequality along \( \partial F \) followed by the definition (4.1) of \( G^k \), we have

\[
\sum_{E \in \mathcal{E}_F} h_E^{-1} \| q_F - A_{q,\partial F} \|_{L^2(E)}^2 \leq h_F \sum_{E \in \mathcal{E}_F} \| G^k_F q_F \|_{L^2(E)}^2 \leq \| G^k_F q_F \|_{\text{curl}, F}^2.
\]

(4.19)

We now turn to the second term in (4.18). Select \( v_F \in \mathcal{R}^{k+1}_F(F) \) such that \( \text{div} \ v_F = \gamma^{k+1}_F (q_F - \frac{1}{2}\partial G^k_F A_{q,\partial F}) \).

By the \( L^2 \)-estimate on \( v_F \) coming from [12], Lemma [31], the discrete trace inequality of [11] Lemma 1.32, and the consistency property [12], Eq. (5.10) of \( \mathcal{G}^k \), we have

\[
\| v_F \|_{L^2(F)} + \left( \sum_{E \in \mathcal{E}_F} h_E^{-1} \| v_F \|_{L^2(E)}^2 \right)^{\frac{1}{2}} \leq h_F \| \gamma^{k+1}_F (q_F - \frac{1}{2}\partial G^k_F A_{q,\partial F}) \|_{L^2(F)},
\]

\[
\mathcal{G}^k_F (q_F - \frac{1}{2}\partial G^k_F A_{q,\partial F}) = \mathcal{G}^k_F q_F.
\]

Hence, applying the definition (2.10) of \( \gamma^{k+1}_F \) to \( \frac{1}{2}\partial G^k_F A_{q,\partial F} \), taking \( v_F \) above as a test function, and using Cauchy–Schwarz inequalities, we obtain

\[
\| \gamma^{k+1}_F (q_F - \frac{1}{2}\partial G^k_F A_{q,\partial F}) \|_{L^2(F)}^2 \leq h_F \| \mathcal{G}^k_F q_F \|_{L^2(F)} \| \gamma^{k+1}_F (q_F - \frac{1}{2}\partial G^k_F A_{q,\partial F}) \|_{L^2(F)}
\]

\[
+ \left( \sum_{E \in \mathcal{E}_F} h_E^{-1} \| q_E - A_{q,\partial F} \|_{L^2(E)}^2 \right)^{\frac{1}{2}} h_F \| \gamma^{k+1}_F (q_F - \frac{1}{2}\partial G^k_F A_{q,\partial F}) \|_{L^2(F)}.
\]

Simplifying and recalling (4.12) and (4.19), we infer \( \| \gamma^{k+1}_F (q_F - A_{q,\partial F}) \|_{L^2(F)} \leq h_F \| \mathcal{G}^k_F q_F \|_{\text{curl}, F} \) which, plugged together with (4.19) into (4.18), gives the following estimate on the second term in the left-hand side of (4.16):

\[
\sum_{E \in \mathcal{E}_F} h_E^{-1} \| \gamma^{k+1}_F q_F - q_E \|_{L^2(E)}^2 \leq \| \mathcal{G}^k_F q_F \|_{\text{curl}, F}^2.
\]

(4.20)

Integrating by parts the definition (2.10) of \( \gamma^{k+1}_F \) applied to \( v_F \in \mathcal{D}_F \) (see Remark [1]), we have

\[
\int_F \text{grad}_F \gamma^{k+1}_F q_F \cdot v_F = \int_F \mathcal{G}^k_F q_F \cdot v_F + \sum_{E \in \mathcal{E}_F} \omega_{FE} \int_E (\gamma^{k+1}_F q_F - q_E)(v_F \cdot n_{FE}).
\]

Making \( v_F = \text{grad}_F \gamma^{k+1}_F q_F \), using Cauchy–Schwarz inequalities, (4.12), a discrete trace inequality, and (4.20) then yields the bound on the first term in the left-hand side of (4.16).

2. Proof of (4.17). The ideas are similar to those used to prove (4.16), provided we can establish a Poincaré–Wirtinger inequality for face potentials (which is not straightforward given their discontinuity).

\[
A_{q,\partial F} := \frac{1}{|\partial F|} \sum_{F \in \mathcal{F}_F} |F| A_{q,F} \quad \text{with} \quad A_{q,F} := \frac{1}{|F|} \int_F \gamma^{k+1}_F q_F
\]

15
denote the average over $\partial T$ of the piecewise polynomial function defined by $(\gamma_k^{k+1} q_F)$, $F \in \mathcal{T}$. We write, using triangle and Cauchy–Schwarz inequalities,

$$\sum_{F \in \mathcal{T}} h_F^{-1} \Vert P^{k+1}_{\text{grad}, T} q_F - \gamma_k^{k+1} q_F \Vert_{L^2(F)}^2 \leq \sum_{F \in \mathcal{T}} h_F^{-1} \Vert \gamma_k^{k+1} q_F - A_{q,F} \Vert_{L^2(F)}^2 + \sum_{F \in \mathcal{T}} h_F^{-1} \Vert A_{q,F} - A_{q,T} \Vert_{L^2(F)}^2 + \sum_{F \in \mathcal{T}} h_F^{-1} \Vert P^{k+1}_{\text{grad}, T} q_F - A_{q,T} \Vert_{L^2(F)}^2 =: \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3. \quad (4.21)$$

The first term is estimated using a Poincaré–Wirtinger inequality on $\gamma_k^{k+1} q_F$ and invoking (4.16) together with the definition (3.1) of $\|\cdot\|_\text{curl}$ to get

$$\mathcal{I}_1 \leq \sum_{F \in \mathcal{T}} h_F^{-1} \left(h_F \| \text{grad} \gamma_k^{k+1} q_F \|_{L^2(F)} \right)^2 \leq \sum_{F \in \mathcal{T}} h_F \| \nabla \gamma_k^{k+1} q_F \|_{L^2(F)}^2 \leq \| \nabla \gamma_k^{k+1} q_F \|_{L^2(T)}^2. \quad (4.22)$$

For the second term in (4.21), we follow the same steps as in [12] Lemma 29, working from face to face through common edges and using (4.16) to get $\mathcal{I}_2 \leq \| G^k q_F \|_{L^2(T)}^2$. Finally, for $\mathcal{I}_3$, we apply the definition (3.1) of $P^{k+1}_{\text{grad}, T}(q_F - I^k_{\text{grad}, T} A_{q,T})$ with $v_T \in C^{k+1}(T)$ such that $\operatorname{div} v_T = P^{k+1}_{\text{grad}, T}(q_F - I^k_{\text{grad}, T} A_{q,T})$ and $\| v_T \|_{L^2(T)} \leq h_T \| P^{k+1}_{\text{grad}, T}(q_F - I^k_{\text{grad}, T} A_{q,T}) \|_{L^2(T)}$, see [12] Lemma 31. Using the consistency properties (3.2) of $P^{k+1}_{\text{grad}, T}$, [12] Eq. (3.13) of $G^k$ and [12] Eq. (3.11) of $\gamma_k^{k+1}$, and a discrete trace inequality, this gives

$$\| P^{k+1}_{\text{grad}, T} q_F - A_{q,T} \|_{L^2(T)} \leq h_T \| G^k q_F \|_{L^2(T)} + h_T \left( \mathcal{I}_1 + \mathcal{I}_2 \right), \quad (4.23)$$

where the second line follows from (4.13), and a triangle inequality to write

$$\sum_{F \in \mathcal{T}} h_F^{-1} \| \gamma_k^{k+1} q_F - A_{q,T} \|_{L^2(F)} \leq \sum_{F \in \mathcal{T}} h_F^{-1} \| \gamma_k^{k+1} q_F - A_{q,F} \|_{L^2(F)} + \sum_{F \in \mathcal{T}} h_F^{-1} \| A_{q,F} - A_{q,T} \|_{L^2(F)}.$$

Using discrete trace inequalities and the previous estimates on $\mathcal{I}_1$ and $\mathcal{I}_2$, (4.23) leads to

$$\mathcal{I}_3 \leq h_T^{-2} \| P^{k+1}_{\text{grad}, T} q_F - A_{q,T} \|_{L^2(T)}^2 \leq \| G^k q_F \|_{L^2(T)}^2.$$

Plugging this bound together with the estimates on $\mathcal{I}_1$ and $\mathcal{I}_2$ into (4.21) concludes the proof of the bound on the second term in the right-hand side of (4.17). To bound the first term in the left-hand side of (4.17), we proceed as for $\text{grad} \gamma_k^{k+1} q_F$ in Step 1, using an integration by parts in the definition (3.1) of $P^{k+1}_{\text{grad}, T} q_F$ and selecting the test function $v_T = \text{grad} P^{k+1}_{\text{grad}, T} q_F$ (see Remark 3). \hfill \Box

**Proof of Theorem 9** It holds, by definition (3.13) of the local discrete $L^2$-product in $X^k_{\text{curl}, h}$ and (4.9),

$$\tilde{E}_{\operatorname{div}, h}(v, q_h) = \sum_{T \in \mathcal{T}_h} \left[ \int_T P^k_{\text{curl}, T} (P^k_{\text{curl}, T} v) \cdot G^k q_h + s_{\text{curl}, T} (P^k_{\text{curl}, T} v) \cdot G^k q_h + \int_T \operatorname{div} v P^{k+1}_{\text{grad}, T} q_F \right]. \quad (4.24)$$

Using Remark 3 we have, for all $w_T \in \mathcal{P}^k(T)$,

$$\int_T P^{k+1}_{\text{grad}, T} q_F \cdot \operatorname{div} w_T + \int_T G^k q_F \cdot w_T - \sum_{F \in \mathcal{T}_h} \omega_{TF} \int_F \gamma_k^{k+1} q_F (w_T \cdot n_F) = 0.$$

16
Subtracting this quantity from (4.24), we obtain
\[
\tilde{E}_{\text{div},h}(v, q_h) = \sum_{T \in \mathcal{T}_h} \left[ \int_T \left( P_{\text{curl},T}^k \left( \mathcal{L}^k_{\text{curl},T} v \right) - w_T \right) \cdot \mathbf{G}_T^k q_T + s_{\text{curl},T} \left( \mathcal{L}^k_{\text{curl},T} v_{|T} \cdot \mathbf{G}_T^k q_T \right) \right] \\
+ \sum_{T \in \mathcal{T}_h} \left[ \int_T \text{div}(v - w_T) P_{\text{grad},T}^{k+1} q_T + \sum_F \omega_{TF} \int_F (w_T - v) \cdot n_F \gamma_F^{k+1} q_F \right],
\]
where \( v \) is introduced in the boundary term by single-valuedness of the discrete trace, and using \( v_{|F} \cdot n_F = 0 \) whenever \( F \in \mathcal{T}_h^b \). Integrating by parts the third term in the right-hand side of the above expression, we obtain
\[
\tilde{E}_{\text{div},h}(v, q_h) = \sum_{T \in \mathcal{T}_h} \left[ \int_T \left( P_{\text{curl},T}^k \left( \mathcal{L}^k_{\text{curl},T} v \right) - w_T \right) \cdot \mathbf{G}_T^k q_T + s_{\text{curl},T} \left( \mathcal{L}^k_{\text{curl},T} v_{|T} \cdot \mathbf{G}_T^k q_T \right) \right] \\
+ \sum_{T \in \mathcal{T}_h} \left[ - \int_T (v - w_T) \cdot \text{grad} \left( P_{\text{grad},T}^{k+1} q_T \right) + \sum_F \omega_{TF} \int_F (w_T - v) \cdot n_F \gamma_F^{k+1} q_F \right. \\
\left. - P_{\text{grad},T}^{k+1} q_T \right].
\]
(4.25)

We set \( w_T = \pi_{\text{rot},T}^k v \) and use (3.19) and the approximation properties of \( \pi_{\text{rot},T}^k \) stated in [11, Theorem 1.45] to see that
\[
\| P_{\text{curl},T}^k \left( \mathcal{L}^k_{\text{curl},T} v \right) - \pi_{\text{rot},T}^k v \|_{L^2(T)} + \| v - \pi_{\text{rot},T}^k v \|_{L^2(T)} + \sum_F h_F^{\frac{1}{2}} \| v - \pi_{\text{rot},T}^k v \|_{L^2(F)} \leq h_F^{\frac{3}{2}} \| v \|_{H^{k+1,2}(T)}.
\]

Using Cauchy–Schwarz inequalities on the integrals and on the stabilisation bilinear form in (4.25), the bound (4.13) together with the norm equivalence (4.5), and the consistency property (3.25) of the stabilisation term, we arrive at
\[
\left| \tilde{E}_{\text{div},h}(v, q_h) \right| \leq \sum_{T \in \mathcal{T}_h} h_F^{k+1} \| v \|_{H^{k+1,2}(T)} \| \mathbf{G}_T^k q_T \|_{\text{curl},T} + \sum_{T \in \mathcal{T}_h} h_F^{k+1} \| v \|_{H^{k+1,2}(T)} \| \text{grad} \left( P_{\text{grad},T}^{k+1} q_T \right) \|_{L^2(T)} \\
+ \sum_{T \in \mathcal{T}_h} \sum_{F \in \mathcal{T}_F} h_F^{k+1} \| v \|_{H^{k+1,2}(T)} h_F^{\frac{1}{2}} \| \gamma_F^{k+1} q_F \|_{L^2(F)} - P_{\text{grad},T}^{k+1} q_T \|_{L^2(F)}.
\]

The conclusion follows from the estimate (4.17), and Cauchy–Schwarz inequalities on the sums. \( \square \)

### 4.4.2 Adjoint consistency for the curl

The proof of the adjoint consistency for the curl hinges on liftings defined as solutions of local problems. For any \( F \in \mathcal{T}_h \), the face lifting \( R_{\text{curl},F} : X_{\text{curl},F}^k \rightarrow \mathbf{H}(\text{rot}; F) \cap \mathbf{H}(\text{div}; F) \) is such that, for all \( v_F \in X_{\text{curl},F}^k \), \( R_{\text{curl},F} v_F = \phi_{\psi_F} + \text{grad}_F \psi_F \) with \( \phi_{\psi_F} \in \mathbf{H}(\text{rot}; F) \cap \mathbf{H}(\text{div}; F) \) such that
\[
\text{rot}_F \phi_{\psi_F} = C_F^k v_F \quad \text{in} \ F, \tag{4.26a}
\]
\[
\text{div}_F \phi_{\psi_F} = 0 \quad \text{in} \ F, \tag{4.26b}
\]
\[
\phi_{\psi_F} \cdot t_E = v_E \quad \text{on all} \ E \in \mathcal{E}_F, \tag{4.26c}
\]
while \( \psi_F \in C_0^\infty(F) \) is such that
\[
- \int_F \psi_F \cdot \text{div}_F z_F = \int_F (\gamma_F^{k+1} v_F - \phi_{\psi_F}) \cdot z_F \quad \forall z_F \in \mathcal{R}^{k+1}(F). \tag{4.27}
\]
Let now $T \in \mathcal{T}_h$. The curl correction $\delta_T : X^k_{\text{curl},T} \to \textbf{H}(\text{curl}; T) \cap \textbf{H}(\text{div}; T)$ is such that, for all $\psi_T \in X^k_{\text{curl},T}$,

\[
\begin{align*}
\text{div} \delta_T \psi_T &= - \text{div} C^k_T \psi_T & \text{in } T, & (4.28a) \\
\text{curl} \delta_T \psi_T &= 0 & \text{in } T, & (4.28b) \\
\delta_T \psi_T \cdot n_F &= C^k_T \psi_T \cdot n_F & \text{on all } F \in \mathcal{T}_h. & (4.28c)
\end{align*}
\]

The curl correction lifts the difference between the face curl $C^k_T \psi_T$ and the normal component of the element curl $C^k_T \psi_T$ as a function defined over $T$. Its role is to ensure the well-posedness of the problem defining the element lifting $R_{\text{curl},T} : X^k_{\text{curl},T} \to \textbf{H}(\text{curl}; T) \cap \textbf{H}(\text{div}; T)$ such that, for all $\psi_T \in X^k_{\text{curl},T}$,

\[
\begin{align*}
\text{curl } R_{\text{curl},T} \psi_T &= C^k_T \psi_T + \delta_T \psi_T & \text{in } T, & (4.29a) \\
\text{div } R_{\text{curl},T} \psi_T &= 0 & \text{in } T, & (4.29b) \\
(R_{\text{curl},T} \psi_T)_n &= R_{\text{curl},T} \psi_T & \text{on all } F \in \mathcal{T}_h. & (4.29c)
\end{align*}
\]

In Appendix $\Delta$ we prove that these lifting operators are well-defined, and that they satisfy the following two key properties:

- Orthogonality of the face lifting: For all $F \in \mathcal{T}_h$,

\[
\int_F (\gamma^k_{L,F} \psi_T - R_{\text{curl},T} \psi_T) \cdot z_F = 0 \quad \forall (\psi_T, z_F) \in X^k_{\text{curl},F} \times \mathcal{RT}^{k+1}(F); \quad (4.30)
\]

- Boundedness of the element lifting: For all $T \in \mathcal{T}_h$,

\[
\| R_{\text{curl},T} \psi_T \|_{L^2(T)} + \| \text{curl } R_{\text{curl},T} \psi_T \|_{L^2(T)} \leq \| \psi_T \|_{\text{curl},T} + \| C^k_T \psi_T \|_{\text{div},T} \quad \forall \psi_T \in X^k_{\text{curl},T}. \quad (4.31)
\]

**Lemma 18** (Approximation properties of $\mathcal{N}^{k+1}(T)$ on polyhedral elements). For all $T \in \mathcal{T}_h$ and all $w \in \textbf{H}^{k+2}(T)$, there exists $z_T \in \mathcal{N}^{k+1}(T)$ such that

\[
\begin{align*}
\| w - z_T \|_{L^2(T)} &\leq h_T^{k+1} \left( |w|_{\mathcal{H}^{k+1}(T)} + |w|_{\mathbf{H}^{k+2}(T)} \right), & (4.32) \\
\| \text{curl } w - \text{curl } z_T \|_{L^2(T)} &\leq h_T^{k+1} |w|_{\mathbf{H}^{k+2}(T)} . & (4.33)
\end{align*}
\]

**Proof.** By the mesh regularity assumption, there is a simplex $S \subset T$ whose inradius is $\geq h_T$. Following the arguments in the proof of [11] Lemma 1.25, we infer the norm equivalence

\[
\| q \|_{L^2(S)} \approx \| q \|_{L^2(T)} \quad \forall q \in \mathcal{P}^{k+1}(T). \quad (4.34)
\]

Let us take $z_T$ as the Nédélec interpolant in $\mathcal{N}^{k+1}(T)$ of $w$; $z_T$ can be uniquely extended as an element of $\mathcal{N}^{k+1}(T)$. By the arguments in the proof of [16] Theorem 3.14 and Corollary 3.17, and since $S \subset T$, it holds

\[
\begin{align*}
\| w - z_T \|_{L^2(S)} &\leq h_T^{k+1} \left( |w|_{\mathcal{H}^{k+1}(T)} + |w|_{\mathbf{H}^{k+2}(T)} \right), & (4.35) \\
\| \text{curl } w - \text{curl } z_T \|_{L^2(S)} &\leq h_T^{k+1} |w|_{\mathbf{H}^{k+2}(T)} .
\end{align*}
\]

We then write, introducing $\pi^{k+1}_{\mathcal{P},T} w$ and using triangle inequalities,

\[
\begin{align*}
\| w - z_T \|_{L^2(T)} &\leq \| w - \pi^{k+1}_{\mathcal{P},T} w \|_{L^2(T)} + \| \pi^{k+1}_{\mathcal{P},T} w - z_T \|_{L^2(T)} \\
&\leq h_T^{k+1} |w|_{\mathbf{H}^{k+1}(T)} + \| \pi^{k+1}_{\mathcal{P},T} w - z_T \|_{L^2(S)} \\
&\leq h_T^{k+1} \left( |w|_{\mathcal{H}^{k+1}(T)} + |w|_{\mathbf{H}^{k+2}(T)} \right),
\end{align*}
\]
where we have used the approximation property of \( \pi_{p,T}^{k+1} \) together with the norm equivalence (4.34) in the second equality, and concluded by introducing \( w \) and invoking (4.35) to write

\[
\| \pi_{p,T}^{k+1} w - z_T \|_{L^2(S)} \leq \| \pi_{p,T}^{k+1} w - w \|_{L^2(S)} + \| w - z_T \|_{L^2(S)} \\
\leq h_T^{k+1} \| w \|_{H^{k+1}(T)} + h_T^{k+1} \left( \| w \|_{H^{k+1}(T)} + \| w \|_{H^{k+2}(T)} \right).
\]

This concludes the proof of (4.32). The proof of (4.33) is done in a similar way, introducing \( \text{curl}(\pi_{p,T}^{k+1} w) \) and using the approximation property \( \| \text{curl} w - \text{curl}(\pi_{p,T}^{k+1} w) \|_{L^2(S)} \leq h_T^{k+1} \| w \|_{H^{k+2}(T)}. \)

**Proof of Theorem 10.** For all \( T \in T_h \), select \( z_T \in N^{k+1}(T) \) given by Lemma [18]. Using (3.13c) to expand \((-\cdot)_\text{div},h\) together with (4.10), and recalling (3.6), we see that it holds, for all \( \psi_h \in X_{\text{curl},h}^k \),

\[
\mathcal{E}_{\text{curl},h}(w, \psi_h) = \sum_{T \in T_h} \int_T (P_{\text{div},T}^k (T_{\text{div},T}^k w^T - z_T) \cdot C_{\text{curl}}^k \psi_f + \sum_{T \in T_h} s_{\text{div},T} (L_{\text{div},T}^k w^T \cdot C_{\text{curl}}^k \psi_f) \\
+ \sum_{T \in T_h} \int_T \text{curl}(z_T - w) \cdot P_{\text{curl},T}^k \psi_f + \sum_{T \in T_h} \sum_{F \in T} \omega_{TF} \int_F (z_T \times n_F) \cdot \gamma_{TF}^k \psi_f) \\
=: \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4.
\]

Using Cauchy–Schwarz and triangle inequalities, it is readily inferred for the first term

\[
|\mathcal{I}_1| \leq \left( \sum_{T \in T_h} \left( \| P_{\text{div},T}^k (L_{\text{div},T}^k w^T \cdot C_{\text{curl}}^k \psi_f + \sum_{T \in T_h} s_{\text{div},T} (L_{\text{div},T}^k w^T \cdot C_{\text{curl}}^k \psi_f) \right)^2 \right)^{\frac{1}{2}} \\
\leq h^{k+1} \left( \| w \|_{H^{k+1}(T_h)} + \| w \|_{H^{k+2}(T_h)} \right) \| C_{\text{curl}}^k \psi_f \|_{\text{div},h},
\]

where the conclusion follows using the approximation properties (3.20) and (4.32) to bound the first factor, and (4.14) along with the norm equivalence (4.5) to bound the second.

For \( \mathcal{I}_2 \), combining the consistency property (3.26) of \( s_{\text{div},T} \) with discrete Cauchy–Schwarz inequalities and the definition of the \( \| \cdot \|_{\text{div},h} \)-norm readily gives

\[
|\mathcal{I}_2| \leq h^{k+1} \| w \|_{H^{k+1}(T_h)} \| C_{\text{curl}}^k \psi_f \|_{\text{div},h}.
\]

For \( \mathcal{I}_3 \), Cauchy–Schwarz inequalities, the approximation property (4.35), and the definition of the norm \( \| \cdot \|_{\text{curl},h} \) yield

\[
|\mathcal{I}_3| \leq \left( \sum_{T \in T_h} \| \text{curl}(z_T - w) \|_{L^2(T)}^2 \right)^{\frac{1}{2}} \left( \sum_{T \in T_h} \| P_{\text{curl},T}^k \psi_f \|_{L^2(T)}^2 \right)^{\frac{1}{2}} \leq h^{k+1} \| w \|_{H^{k+2}(T_h)} \| \psi_f \|_{\text{curl},h}.
\]

Let us now consider the last term in the right-hand side of (4.36). Since \( (z_T)_F \times n_F \in \mathcal{RT}^{k+1}(F) \) (see Proposition [30]), by (4.30) we can replace \( \gamma_{TF}^k \psi_f \) by \( R_{\text{curl},F} \psi_f \) in the boundary integral. Using the fact that both \( R_{\text{curl},F} \sum_F \psi_f \) and the (rotated) tangential component of \( w \) are continuous across interfaces, along with the fact that \( \omega_{TF} \times n_F = 0 \) for all \( F \in T_h \) between two elements \( T_1, T_2 \), and \( w|_F \times n_F = 0 \) for all \( F \in T_h^b \), we then have

\[
\mathcal{I}_4 = \sum_{T \in T_h} \sum_{F \in T} \omega_{TF} \int_F (z_T - w) \times n_F \cdot R_{\text{curl},F} \psi_f \\
= \sum_{T \in T_h} \left( \int_T (z_T - w) \cdot \text{curl} R_{\text{curl},T} \psi_f - \int_T \text{curl}(z_T - w) \cdot R_{\text{curl},T} \psi_f \right).
\]
where the conclusion follows recalling that, by definition (4.29), \( R_{\text{curl}}.F \psi_F = (R_{\text{curl}}.T \psi_T)_F \) for all \( T \in T_h \) and all \( F \in F_T \), and by integrating by parts. Using Cauchy–Schwarz inequalities, it is inferred

\[
|\mathcal{I}_4| \leq \left[ \sum_{T \in T_h} \left( \| z_T - w \|_{L^2(T)}^2 + \| \text{curl}(z_T - w) \|_{L^2(T)}^2 \right) \right]^{1/2} \\
\times \left[ \sum_{T \in T_h} \left( \| \text{curl} R_{\text{curl}}.T \psi_T \|_{L^2(T)}^2 + \| R_{\text{curl}}.T \psi_T \|_{L^2(T)}^2 \right) \right]^{1/2}.
\]

The approximation properties (4.35) of \( z_T \) along with the boundedness (4.31) of \( R_{\text{curl}}.T \psi_T \) yield

\[
|\mathcal{I}_4| \leq h^{k+1} \left( |w|_{H^{k+1}(\Omega)} + |w|_{H^{k+2}(\Omega)} \right) \left( \| \psi_h \|_{L^2} + \| \text{curl} \psi_h \|_{W^{1,2}_0} \right).
\]

Plugging (4.37)–(4.40) into (4.36), (3.29) follows. \( \square \)

4.4.3 Adjoint consistency for the divergence

\[ \text{Proof of Theorem 11.} \] Combining the definition (3.30) of the adjoint consistency error for the divergence with (3.10) summed over \( T \in T_h \), we infer that it holds, for all \( (q, \psi_h) \) as in the theorem and all \( q_h \in P^{k+1}(T_h) \) with \( q_T := (q_h)_T \) for all \( T \in T_h \),

\[
\hat{E}_{\text{div},h}(q, \psi_h) = \\
\sum_{T \in T_h} \left[ \int_T (\nabla_h \cdot q - q_T) D_{T,h}^k \psi_T + \int_T \nabla \cdot (q - q_T) \cdot \mathbf{P}_{\text{div},T}^k \psi_T + \sum_{F \in F_T} \omega_{TF} \int_F (q_T - q)v_F \right],
\]

where the cancellation of \( \mathbf{R}_{\text{div},T}^k \) is justified by its definition along with \( D_{T,h}^k \psi_T \in P^k(T) \), while the insertion of \( q \) into the boundary integral is possible thanks to its single-valuedness at interfaces along with the fact that it vanishes on \( \partial \Omega \). Taking absolute values and using Cauchy–Schwarz inequalities in the right-hand side along with \( h_F = h_T \) for all \( T \in T_h \) and all \( F \in F_T \), we infer

\[
|\hat{E}_{\text{div},h}(q, \psi_h)| \leq \sum_{T \in T_h} \left( h_T^{-1} \| q - q_T \|_{L^2(T)}^2 + \| \nabla \cdot (q - q_T) \|_{L^2(T)}^2 + h_T^{-1} \| q_T - q \|_{L^2(T)}^2 \right) \\
\times \left[ \sum_{T \in T_h} \left( h_T^2 \| D_{T,h}^k \psi_T \|_{L^2(T)}^2 + \| \mathbf{P}_{\text{div},T}^k \psi_T \|_{L^2(T)}^2 + \sum_{F \in F_T} \omega_{TF} \| v_F \|_{L^2(F)}^2 \right) \right]^{1/2}.
\]

(4.41)

Taking \( q_h \) such that \( q_T = \nabla_h^{-1}(q) \) for all \( T \in T_h \) and using the approximation properties of the \( L^2 \)-orthogonal projector [11 Theorem 1.45], it is inferred that the first factor in the right-hand side of (4.41) is \( \leq h^{k+1} \| q \|_{H^{k+2}(\Omega)} \). Moving to the second factor, we use, for all \( T \in T_h \), [14 Lemma 8] followed by the local seminorm equivalence (4.5) to write \( h_T \| D_{T,h}^k \psi_T \|_{L^2(T)} \leq \| \psi_T \|_{W^{1,2}_0} \leq \| \psi_T \|_{\text{div},T} \). The same norm equivalence and the definition of the \( \| \cdot \|_{\text{div},T} \)-norm also yields \( \| \mathbf{P}_{\text{div},T}^k \psi_T \|_{L^2(T)}^2 + \sum_{F \in F_T} h_F \| v_F \|_{L^2(F)}^2 \leq \| \psi_T \|_{\text{div},T} \). The second factor in the right-hand side of (4.41) is therefore \( \leq \| \psi_h \|_{\text{div},h} \), and the proof is complete. \( \square \)

5 Convergence analysis for a DDR discretisation of magnetostatics

We analyse in this section the DDR approximation of the following magnetostatics model, in which the unknowns are the magnetic field \( \mathbf{H} \in \mathbf{H}(\text{curl}; \Omega) \) and the vector potential \( \mathbf{A} \in \mathbf{H}(\text{div}; \Omega) \):

\[
\mu \mathbf{H} - \text{curl} \mathbf{A} = 0, \quad \text{curl} \mathbf{H} = \mathbf{J}, \quad \text{div} \mathbf{A} = 0 \quad \text{in } \Omega, \quad A \times n = 0 \quad \text{on } \partial \Omega.
\]

(5.1)
The free current $J$ belongs to $\text{curl} \mathbf{H} (\text{curl}; \Omega)$ and we assume, for the sake of simplicity, that the magnetic permeability $\mu$ is piecewise-constant on the considered meshes, with $\mu \in [\mu_-, \mu_+]$ for some constant numbers $0 < \mu_- \leq \mu_+$.

5.1 Scheme

As shown in [10], a scheme based on the discrete de Rham tools can be written by replacing, in the weak formulation of (5.1), the continuous $L^2$-products by discrete ones built on the local products. Denote by $\mu_T$ the constant value of $\mu$ over $T \in \mathcal{T}_h$ and define the bilinear forms $a_h : X^k_{\text{curl}, h} \times X^k_{\text{curl}, h} \rightarrow \mathbb{R}$, $b_h : X^k_{\text{curl}, h} \times X^k_{\text{div}, h} \rightarrow \mathbb{R}$, and $c_h : X^k_{\text{div}, h} \times X^k_{\text{div}, h} \rightarrow \mathbb{R}$ as follows: For all $\mathbf{u}_h, \mathbf{z}_h \in X^k_{\text{curl}, h}$ and all $\mathbf{w}_h, \mathbf{v}_h \in X^k_{\text{div}, h}$,

$$a_h(\mathbf{u}_h, \mathbf{z}_h) := \sum_{T \in \mathcal{T}_h} \mu_T (\mathbf{u}_h, \mathbf{z}_h)_{\text{curl}, T}, \quad b_h(\mathbf{z}_h, \mathbf{v}_h) := (\mathbf{C}_h^k \mathbf{z}_h, \mathbf{v}_h)_{\text{div}, h},$$

$$c_h(\mathbf{w}_h, \mathbf{v}_h) := \int_{\Omega} D_h^k \mathbf{w}_h \cdot D_h^k \mathbf{v}_h.$$

The discrete problem then reads: Find $\mathbf{H}_h \in X^k_{\text{curl}, h}$ and $\mathbf{A}_h \in X^k_{\text{div}, h}$ such that

$$a_h(\mathbf{H}_h, \mathbf{z}_h) - b_h(\mathbf{z}_h, \mathbf{A}_h) = 0 \quad \forall \mathbf{z}_h \in X^k_{\text{curl}, h},$$

$$b_h(\mathbf{H}_h, \mathbf{v}_h) + c_h(\mathbf{A}_h, \mathbf{v}_h) = \sum_{T \in \mathcal{T}_h} \int_T \mathbf{J} \cdot P^k_{\text{div}, T} \mathbf{v}_r \quad \forall \mathbf{v}_h \in X^k_{\text{div}, h}. \quad (5.2)$$

The equations of this problem can be recast in the standard variational form $\mathcal{A}_h((\mathbf{H}_h, \mathbf{A}_h), (\mathbf{z}_h, \mathbf{v}_h)) = \mathcal{L}_h(\mathbf{z}_h, \mathbf{v}_h)$, where $\mathcal{A}_h : (X^k_{\text{curl}, h} \times X^k_{\text{div}, h}) \rightarrow \mathbb{R}$ and $\mathcal{L}_h : X^k_{\text{curl}, h} \times X^k_{\text{div}, h} \rightarrow \mathbb{R}$ are the bilinear and linear forms, respectively, such that

$$\mathcal{A}_h((\mathbf{u}_h, \mathbf{w}_h), (\mathbf{z}_h, \mathbf{v}_h)) := a_h(\mathbf{u}_h, \mathbf{z}_h) - b_h(\mathbf{z}_h, \mathbf{w}_h) + b_h(\mathbf{u}_h, \mathbf{v}_h) + c_h(\mathbf{w}_h, \mathbf{v}_h),$$

$$\mathcal{L}_h(\mathbf{z}_h, \mathbf{v}_h) := \sum_{T \in \mathcal{T}_h} \int_T \mathbf{J} \cdot P^k_{\text{div}, T} \mathbf{v}_r.$$

5.2 Error estimate

We establish an error estimate using the stability results of the companion paper [12] and the consistency results presented in Section 3. To measure the error, we introduce the following $\mathbf{H}(\text{curl}; \Omega)$- and $\mathbf{H}(\text{div}; \Omega)$-like norms on $X^k_{\text{curl}, h}$ and $X^k_{\text{div}, h}$, respectively:

$$\| \mathbf{x} \|_{\mu, \text{curl}, 1,h} := \left( a_h(\mathbf{x}, \mathbf{x}) + \| \mathbf{C}_h^k \mathbf{x} \|_{\text{div}, h}^2 \right)^{\frac{1}{2}} \quad \forall \mathbf{x} \in X^k_{\text{curl}, h},$$

$$\| \mathbf{y} \|_{\text{div}, 1,h} := \left( \| \mathbf{y} \|_{\text{div}, h}^2 + \| D_h^k \mathbf{y} \|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}} \quad \forall \mathbf{y} \in X^k_{\text{div}, h}.$$

Theorem 19 (Error estimate for the magnetostatics problem). Assume that both the first and second Betti numbers of $\Omega$ are zero (i.e., $\Omega$ is not crossed by any tunnel and does not enclose any void). Then, there exists a unique solution $(\mathbf{H}_h, \mathbf{A}_h) \in X^k_{\text{curl}, h} \times X^k_{\text{div}, h}$ to (5.2). Moreover, letting $(\mathbf{H}, \mathbf{A}) \in \mathbf{H}(\text{curl}; \Omega) \times \mathbf{H}(\text{div}; \Omega)$ be the weak solution to (5.1) and assuming that $\mathbf{H} \in C^0(\overline{\Omega}) \cap H^{k+2}(\mathcal{T}_h)$ and $\mathbf{A} \in C^0(\overline{\Omega}) \times H^{k+2}(\mathcal{T}_h)$, we have

$$\| \mathbf{H}_h - \mathbf{H} \|_{\mu, \text{curl}, 1,h} \leq h^{k+1} \left( \| \text{curl} \mathbf{H} \|_{H^{k+1}(\mathcal{T}_h)} + \| \mathbf{H} \|_{H^{k+2}(\mathcal{T}_h)} + \| \mathbf{A} \|_{H^{k+1}(\mathcal{T}_h)} + \| \mathbf{A} \|_{H^{k+2}(\mathcal{T}_h)} \right), \quad (5.3)$$

where the hidden constant in $\lesssim$ only depends on $\Omega$, $k$, the mesh regularity parameter, and $\mu_- \leq \mu_+$. 

21
Proof. As shown in the proof of [10] Theorem 10, the exactness of the rightmost part of the sequence (2.16), which holds owing to [12] Eqs. (3.44) and (3.41), and the Poincaré inequalities for $C_k^h$ and $D_k^h$ (see [12] Theorems 26 and 27) enable a reproduction of the arguments of the continuous inf-sup condition (see, e.g., [13] Section 2) or [11] Theorem 4.9) to see that $A_h$ satisfies a uniform inf-sup condition with respect to the norm on $X_{\text{curl},h}^k \times X_{\text{div},h}^k$ induced by $||| \cdot |||_{\mu,\text{curl},1,h}$ and $||| \cdot |||_{\text{div},1,h}$.

Using the Third Strang Lemma [9], we therefore obtain (5.3) provided we can prove that the consistency error

$$\mathcal{E}_h((H, A); (\xi_h, \psi_h)) = \mathcal{L}_h(\xi_h, \psi_h) - \mathcal{A}_h((I_{\text{curl},h}^kH, I_{\text{div},h}^kA), (\xi_h, \psi_h))$$

satisfies, for all $(\xi_h, \psi_h) \in X_{\text{curl},h}^k \times X_{\text{div},h}^k$:

$$\mathcal{E}_h((H, A); (\xi_h, \psi_h)) \preceq h^{k+1} \left( |\text{curl } H|_{H^{k+1}((\gamma_h)} + |H|_{H^{k+1,2}((\gamma_h)} + |A|_{H^{k+2}((\gamma_h)} + |A|_{H^{k+1}((\gamma_h)} \right) \times \left( ||\xi_h||_{H^{1,1},h} + ||\psi_h||_{H^{1,1},h} \right). \quad (5.4)$$

Expanding according to the respective definitions $A_h$, $\mathcal{L}_h$, $a_h$, $b_h$, and $c_h$, we have

$$\mathcal{E}_h((H, A); (\xi_h, \psi_h)) = \mathcal{E}_{h,1}((H, A); (\xi_h, \psi_h)) + \mathcal{E}_{h,2}((H, A); (\xi_h, \psi_h)) + \mathcal{E}_{h,3}((H, A); (\xi_h, \psi_h)), \quad (5.5)$$

with

$$\mathcal{E}_{h,1}((H, A); (\xi_h, \psi_h)) := \sum_{T \in \tau_h} \left( \int_T J \cdot P_{\text{div},T}^k \psi_T - (C_k^h (I_{\text{curl},T}^kH), \psi_T)_{\text{div},T} \right),$$

$$\mathcal{E}_{h,2}((H, A); (\xi_h, \psi_h)) := - \sum_{T \in \tau_h} \int_T D_k^h (I_{\text{div},T}^kA) D_k^h \psi_T,$$

$$\mathcal{E}_{h,3}((H, A); (\xi_h, \psi_h)) := - \sum_{T \in \tau_h} \left( \mu_T (I_{\text{curl},T}^kH, \xi_T)_{\text{curl},T} - (C_k^h \xi_T, I_{\text{div},T}^kA)_{\text{div},T} \right).$$

Let us first estimate $\mathcal{E}_{h,1}$. Recalling that $J = \text{curl } H$ and expanding $(\cdot, \cdot)_{\text{div},T}$ according to its definition (3.13b), we have

$$\mathcal{E}_{h,1}((H, A); (\xi_h, \psi_h))$$

$$= \sum_{T \in \tau_h} \left( \int_T \left( \text{curl } H - P_{\text{div},T}^k \left[ C_k^h (I_{\text{curl},T}^kH) \right] \right) \cdot P_{\text{div},T}^k \psi_T - \sum_{T \in \tau_h} \left( \mu_T (I_{\text{curl},T}^kH, \xi_T)_{\text{curl},T} - (C_k^h \xi_T, I_{\text{div},T}^kA)_{\text{div},T} \right).$$

where the second line comes from the relation (4.10) and the commutation formula $C_k^h (I_{\text{curl},T}^kH) = I_{\text{div},T}^k (\text{curl } H)$, see [12] Eq. (3.35). Using Cauchy–Schwarz inequalities on the integrals, on $s_{\text{div},T}$, and on the sums over $T \in \tau_h$, and recalling the consistency properties (3.22) and (3.26), we infer

$$\mathcal{E}_{h,1}((H, A); (\xi_h, \psi_h)) \preceq h^{k+1} ||\text{curl } H||_{H^{k+1}((\gamma_h)} ||\psi_h||_{H^{1,1},h}. \quad (5.6)$$

To handle $\mathcal{E}_{h,2}$, we invoke the commutation formula [12] Eq. (3.36) to see that $D_k^h (I_{\text{div},T}^kA) = \pi^k_{\text{div},T} (\text{div } A) = 0$, and thus

$$\mathcal{E}_{h,2}((H, A); (\xi_h, \psi_h)) = 0. \quad (5.7)$$
Finally, we turn to $E_{h,3}$. Since $A \in H_0(\text{curl}; \Omega)$, the adjoint consistency. Theorem 10 enables us to replace, in this consistency error, the term $(C^k_{\text{curl},T} \zeta_T, \mathcal{P}^k_{\text{curl},T} A)_{\text{div},T}$ with $\int_T \mu_T H \cdot P^k_{\text{curl},T} \zeta_T = \int_T H \cdot P^k_{\text{curl},T} \zeta_T$ up to a term that is controlled, i.e.,

$$E_{h,3}((H, A); (\zeta_j, \mathcal{P}_j)) \leq \sum_{T \in \mathcal{T}_h} \left[ \mu_T (I^k_{\text{curl},T} H, \zeta_T)_{\text{curl},T} - \mu_T \int_T H \cdot P^k_{\text{curl},T} \zeta_T \right] + h^{k+1} \left( |A|_{H^{k+1}(\mathcal{T}_h)} + |A|_{H^{k+2}(\mathcal{T}_h)} \right) \|\zeta_T\|_{\mu,\text{curl},1,h}$$

$$\leq \sum_{T \in \mathcal{T}_h} \mu_T \int_T \left[ P^k_{\text{curl},T} (I^k_{\text{curl},T} H) - H \right] \cdot P^k_{\text{curl},T} \zeta_T + s_{\text{curl},T} (I^k_{\text{curl},T} H, \zeta_T) + h^{k+1} \left( |A|_{H^{k+1}(\mathcal{T}_h)} + |A|_{H^{k+2}(\mathcal{T}_h)} \right) \|\zeta_T\|_{\mu,\text{curl},1,h},$$

where we have used $\|\zeta_T\|_{\text{curl},h} + \|\zeta_T\|_{\mu,\text{curl},1,h} \leq \|\zeta_T\|_{\mu,\text{curl},1,h}$ and the second inequality comes from expanding $(\cdot, \cdot)_{\text{curl},T}$ according to its definition. Cauchy–Schwarz inequalities and the consistency properties (3.19) and (3.25) then lead to

$$E_{h,3}((H, A); (\zeta_j, \mathcal{P}_j)) \leq h^{k+1} |H|_{H^{k+1,2}(\mathcal{T}_h)} \|\zeta_T\|_{\text{curl},h} + h^{k+1} \left( |A|_{H^{k+1}(\mathcal{T}_h)} + |A|_{H^{k+2}(\mathcal{T}_h)} \right) \|\zeta_T\|_{\mu,\text{curl},1,h}.$$

Plugging this estimate together with (5.6) and (5.7) into (5.5), we infer that (5.4) holds, which concludes the proof.

5.3 Numerical tests

We present here the results of some numerical tests obtained with the DDR scheme (5.2) for the magnetostatics model (5.1), focusing on comparing outputs obtained using either the complements (2.1), hereafter denoted by (K), or the orthogonal complements of (13) [10], denoted by $(\perp)$. Both versions of the DDR complex, and related schemes, have been implemented in the HArDCore3D C++ framework (see https://github.com/jdroniou/HArDCore), using linear algebra facilities from the Eigen3 library (see http://eigen.tuxfamily.org) and the Intel MKL PARDISO library (see https://software.intel.com/en-us/mkl) for the resolution of the global sparse linear system. All tests were run on a 16-inch 2019 MacBook Pro equipped with an 8-core Intel Core i9 processor (19-9980HK) and 32Gb of RAM and running macOS Big Sur version 11.1. We consider a constant permeability $\mu = 1$, and the same exact smooth solution and mesh families as in [10] Section 4.4.

Figure 1 presents the errors, for various values of $k$, computed in the relative discrete $H(\text{curl}; \Omega) \times H(\text{div}; \Omega)$ norm:

$$\left( \|H_i - L^k_{\text{curl},h} H\|_{\mu,\text{curl},1,h}^2 + \|A_i - L^k_{\text{div},h} A\|_{\text{div},1,h}^2 \right)^{1/2}.$$

In the case of the Koszul complements, Theorem 19 states that this error should decrease as $O(h^{k+1})$ with the mesh size. No such estimate is known for the DDR scheme using orthogonal complements and, due to the lack of key properties of these complements (hierarchical inclusions, structure of traces), it is not clear whether the analysis carried out in the rest of this paper could be adapted to such complements. Nonetheless, the graphs in Figure 1 show that both schemes converge with an order $k + 1$. The errors between (K) and $(\perp)$ are essentially indistinguishable, except for $k \geq 1$ on tetrahedral meshes, where $(\perp)$ leads to slightly larger errors than (K) – about twice as large on the finest mesh with $k = 3$.

The assembly of the $(\perp)$-DDR scheme requires, for any $Y \in \mathcal{T}_h \cup \mathcal{F}_h$, to compute bases for the $L^2$-orthogonal complements in $\mathcal{P}^k(Y)$ of $\mathcal{G}^k(Y)$ and $\mathcal{R}^k(Y)$, which is done by computing the kernels of local matrices through a full pivot LU algorithm [10] Section 5.1. On the contrary, in the (K) version, explicit
Figure 1: Relative error estimates in discrete $\mathbf{H}(\text{curl}; \Omega) \times \mathbf{H}(\text{div}; \Omega)$ norm vs. $h$, for the Koszul complements of (2.1) [(K), continuous lines], and the orthogonal complements of [13] [⊥], dashed lines].
Table 1: Dimension of meshes and spaces considered for the evaluation of computational times in the numerical tests of Section 5.3.

| Mesh       | $\text{card}(\mathcal{F}_h)$ | $\text{card}(\mathcal{E}_h)$ | $\text{card}(\mathcal{E}_h)$ |
|------------|-----------------------------|-----------------------------|-----------------------------|
| Cubic_Cells| 4 096                       | 13 056                      | 13 872                      |
| Tetgen_Cube-0| 2 925                      | 6 228                       | 3 965                       |
| Voro-small-0| 2 197                      | 15 969                      | 27 546                      |
| Voro-small-1| 356                        | 2 376                       | 4 042                       |

(a) Number of relevant mesh entities

| Mesh       | $\dim(X^0_{\text{curl},h})$ | $\dim(X^1_{\text{curl},h})$ | $\dim(X^2_{\text{curl},h})$ | $\dim(X^3_{\text{curl},h})$ |
|------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|
| Cubic_Cells| 13 872                       | 83 296                      | 207 504                     | 398 784                     |
| Tetgen_Cube-0| 3 956                      | 38 314                      | 105 594                     | 214 580                     |
| Voro-small-0| 27 546                      | 111 787                     | 243 345                     | —                           |
| Voro-small-1| 4 042                       | 16 636                      | 36 474                      | 64 624                      |

(b) Dimension of the space $X^k_{\text{curl},h}$ for $k \in \{0, \ldots, 3\}$

| Mesh       | $\dim(X^0_{\text{div},h})$ | $\dim(X^1_{\text{div},h})$ | $\dim(X^2_{\text{div},h})$ | $\dim(X^3_{\text{div},h})$ |
|------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|
| Cubic_Cells| 13 056                       | 63 744                      | 160 256                     | 314 880                     |
| Tetgen_Cube-0| 6 228                      | 36 234                      | 95 868                      | 193 905                     |
| Voro-small-0| 15 969                      | 61 089                      | 139 754                     | —                           |
| Voro-small-1| 2 376                       | 9 264                       | 21 376                      | 39 780                      |

(c) Dimension of the space $X^k_{\text{div},h}$ for $k \in \{0, \ldots, 3\}$

bases for $G^{c,\ell}(Y)$ and $R^{c,\ell}(Y)$ can be devised; even though these bases are then orthonormalised to ensure a better numerical stability of the scheme (especially on non-isotropic elements, see the discussion in [11, Section B.1.1] on this topic), the computational cost of creating the polynomial bases in ($\perp$) is expected to be larger than in ($K$). Figure 2 compares the processor times for the two DDR schemes required for (a) the creation of the bases for local polynomial spaces and (b) the model construction (computation of the discrete operators, potentials, and $L^2$-products, and global system assembly). We do not compare the linear system resolution times as they are very close for both schemes. In all the cases, the finest mesh of each sequence is considered; see Table 1. In the left column of Figure 2 we report the total CPU time, which constitutes the most reliable measure to assess performance. Since our code relies on multi-threading, we also report, in the right column, wall-clock times, which are more representative of real-life performance on the selected architecture. Wall-clock times are subject to outside influences, such as the impact of other processes, and should therefore be regarded with caution.

As expected, ($K$) polynomial bases are faster to create than ($\perp$) polynomial bases, but not by a large factor. There is a more pronounced difference when comparing the time for model construction, which is mostly dedicated to the creation of the discrete vector calculus operators and potentials in $X^k_{\text{curl},h}$ and $X^k_{\text{div},h}$ (once these are created, assembling the global linear system itself takes only a small fraction of the total model construction time). One possible explanation for the largest model construction time noticed with ($\perp$) complements is that the local systems defining the operators and potentials, solved using the Eigen::LDLT direct solver, have a worse condition number than those with Koszul complements. Drawing more definitive conclusions is always difficult, as running times highly depend on specific implementation choices, and our implementation is designed for flexibility rather than for efficiency on
one given model. The results presented in this section seem to show, however, that the DDR complex using Koszul complements is not only theoretically better (as it allows for complete consistency analysis and error estimates), but also requires less computational resources. The comparison of CPU times and wall clock times also confirms that the assembly step strongly benefits from parallel implementations.

## A Curl lifting

We prove here that the face $R_{\text{curl}, F}$ and element $R_{\text{curl}, T}$ liftings, detailed in Section 4.4.2 are well-defined and satisfy the key properties (4.30) and (4.31).

### A.1 Face lifting $R_{\text{curl}, F}$

#### A.1.1 Existence of $\phi_{L_F}$

Owing to (4.26), we look for $\phi_{L_F} = \mathbf{rot}_F q_F$ for some $q_F \in H^1(F)$. Using the property $\mathbf{rot}_F (\mathbf{rot}_F) = -\text{div}_F (\mathbf{grad}_F) = -\Delta_F$ (which stems from [12, Eq. (2.1)]) and that $\mathbf{rot}_F q_F$ (resp. $t_E$) is $\mathbf{grad}_F q_F$ (resp. $n_{E_F}$) rotated by $-\pi/2$ in the plane spanned by $F$, we see that (4.26) reduces to the following Neumann problem on $q_F$:

$$-\Delta_F q_F = C^k_F \Sigma_F \quad \text{in } F,$$

$$\mathbf{grad}_F q_F \cdot (\omega_{F E} n_{E_F}) = \omega_{F E} \forall E \in \mathcal{E}_F. \quad (A.1)$$

Recalling that $\omega_{F E} n_{E_F}$ is the outer normal, in the plane spanned by $F$, to $F$ on $E$, we see that the compatibility condition of this Neumann problem simply amounts to the definition (2.12) of $C^k_F$ with $r_F = 1$. There exists therefore a unique $q_F \in H^1(F)$ solution of this problem with $\int_F q_F = 0$. Using $q_F$ as a test function in the weak formulation and applying Cauchy–Schwarz inequalities leads to

$$\|\mathbf{grad}_F q_F\|_{L^2(F)}^2 \leq \|C^k_F \Sigma_F\|_{L^2(F)} \|q_F\|_{L^2(F)} + \sum_{E \in \mathcal{E}_F} \|v_E\|_{L^2(E)} \|q_F\|_{L^2(E)}$$

$$\leq h_F \|C^k_F \Sigma_F\|_{L^2(F)} \|\mathbf{grad}_F q_F\|_{L^2(F)} + \left( \sum_{E \in \mathcal{E}_F} h_E \|v_E\|_{L^2(E)}^2 \right) \|\mathbf{grad}_F q_F\|_{L^2(F)},$$

where the second line follows from the Poincaré–Wirtinger inequality $\|q_F\|_{L^2(F)} \leq h_F \|\mathbf{grad}_F q_F\|_{L^2(F)}$ together with the continuous trace inequality $\|q_F\|_{L^2(F)} \leq h_F h_E^{-1/2} \|q_F\|_{L^2(E)} + h_E^{-1/2} \|\mathbf{grad}_F q_F\|_{L^2(F)}$, see [11] Remark 1.46 and Lemma 1.31. As a consequence,

$$\|\mathbf{grad}_F q_F\|_{L^2(F)} \leq \|C^k_F \Sigma_F\|_{L^2(F)} \|q_F\|_{L^2(F)} + \|v_F\|_{\text{curl}, F} \quad (A.2)$$

#### A.1.2 Existence of $\psi_{L_E}$

Fix $\sigma_F \in C^\infty_c(F)$ such that $\sigma_F = 1$ on a ball $B_F \subset F$ of radius $\approx h_F$ (the existence of such a ball follows from the mesh regularity assumption) and $0 \leq \sigma_F \leq 1$. We look for $\psi_{L_E}$ under the form $\sigma_F r_{F F}$ with $r_F \in \mathcal{P}^k(F)$. Since $\text{div}_F : \mathcal{R}^{c,k+1}(F) \to \mathcal{P}^k(F)$ is an isomorphism, denoting as in [12] Lemma 31 its inverse by $(\text{div}_F)^{-1}$, the relation (4.27) is equivalent to

$$\int_F \sigma_F r_{F F} w_F = \int_F (\gamma_{L_F}^k \psi_F - \phi_{L_F}) \cdot (\text{div}_F)^{-1} w_F \quad \forall w_F \in \mathcal{P}^k(F).$$

Since $\sigma_F \geq 0$ is strictly positive on a ball, the mapping $(r_F, w_F) \mapsto \int_F \sigma_F r_{F F} w_F$ is an inner product on $\mathcal{P}^k(F)$ and there exists therefore a unique $r_F \in \mathcal{P}^k(F)$ that satisfies this property. This establishes the existence of $\psi_{L_E}$.
Figure 2: Comparison of CPU (left column) and wall times (right column), both measured in seconds, for the computation of the DDR bases ("Polynomial bases") and of the model construction ("Model") for Koszul (solid fill) and orthogonal (pattern fill) complements on the finest mesh of each sequence; see Table I.
Moreover, since \( \sigma_F = 1 \) on \( B_F \) and \( \| \cdot \|_{L^2(B_F)} \) and \( \| \cdot \|_{L^2(F)} \) are uniformly equivalent on \( \mathcal{P}^k(F) \) (see the proof of [11, Lemma 1.25]), using \( w_F = r_F \) above leads to

\[
\| r_F \|_{L^2(F)}^2 \leq \int_F \sigma_F r_F^2 \leq \| \gamma^k_{1,F} \varphi_F - \phi_F \|_{L^2(F)} \| (\text{div})^{-1} r_F \|_{L^2(F)} \leq (\| \gamma^k_{1,F} \|_{\text{curl}, F} + \| C^k_F \varphi_F \|_{L^2(F)}) h_F \| r_F \|_{L^2(F)},
\]

where the conclusion follows from a triangle inequality along with the boundedness (4.3) of \( \gamma^k_{1,F} \) and the estimate (A.2) for the first factor, and [12, Lemma 31] for the second factor. Simplifying, we obtain

\[
\| r_F \|_{L^2(F)} \leq h_F (\| \varphi_F \|_{\text{curl}, F} + \| C^k_F \varphi_F \|_{L^2(F)}).
\]

(A.3)

**A.1.3 Orthogonality property of \( R_{\text{curl}, F} \)**

We prove here the relation (4.30). Notice first that, since \( \psi_{\varphi_F} \) vanishes on \( \partial F \) and \( \text{rot}_F \varphi_F = 0 \), by (4.26) it holds

\[
\text{rot}_F (R_{\text{curl}, F} \varphi_F) = C^k_F \varphi_F \quad \text{and} \quad (R_{\text{curl}, F} \varphi_F) \cdot t_F = \varphi_E \quad \forall E \in \mathcal{E}_F.
\]

Let \( z_F \in \mathcal{R}^k(F) \) and write \( z_F = \text{rot}_F R_{\text{curl}, F} \psi_{\varphi_F} \) with \( r_F \in \mathcal{P}^{0,k+1}(F) \). By (2.13) and Remark 2 we have

\[
\int_F \gamma^k_{1,F} \varphi_F \cdot z_F = \int_F C^k_F \varphi_F r_F + \sum_{E \in \mathcal{E}_F} \omega_F \int_E \varphi_E r_F
\]

\[
= \int_F \text{rot}_F (R_{\text{curl}, F} \varphi_F) r_F + \sum_{E \in \mathcal{E}_F} \omega_F \int_E (R_{\text{curl}, F} \varphi_F) \cdot t_F = \int_F R_{\text{curl}, F} \varphi_F \cdot z_F,
\]

where the second equality follows from (A.4), and the conclusion has been obtained using an integration by parts. This proves that (4.30) holds for \( z_F \in \mathcal{R}^k(F) \).

Let us now take \( z_F \in \mathcal{R}^{c,k+1}(F) \). Integrating the left-hand side of (4.27) by parts yields

\[
\int_F \text{grad} \psi_{\varphi_F} \cdot z_F = \int_F (\gamma^k_{1,F} \varphi_F - \phi_F) \cdot z_F.
\]

Since \( R_{\text{curl}, F} \varphi_F = \phi_F + \text{grad} \psi_{\varphi_F} \), this establishes that (4.30) also holds for \( z_F \in \mathcal{R}^{c,k+1}(F) \), which completes the proof of this orthogonality relation since \( \mathcal{R} T^{k+1}(F) = \mathcal{R}^k(F) \oplus \mathcal{R}^{c,k+1}(F) \).

**A.2 Element lifting \( R_{\text{curl}, T} \)**

**A.2.1 Existence of \( \delta_T \varphi_F \)**

Owing to (4.28b), we look for \( \delta_T \varphi_F \) under the form of a potential gradient \( \text{grad} q_T \) with \( q_T \in H^1(T) \). Equations (4.28a) and (4.28c) then show that \( q_T \) must solve the Neumann problem

\[
\Delta q_T = - \text{div} C^k_F \varphi_F \quad \text{in } T,
\]

\[
\text{grad} q_T \cdot (\omega_{TF} n_F) = \omega_{TF} (C^k_F \varphi_F - C^k_T \varphi_T \cdot n_F) \quad \forall F \in \mathcal{F}_T,
\]

(A.5)

where we recall that \( \omega_{TF} n_F \) is the outer normal to \( T \) on \( F \). The compatibility condition of this problem is

\[
\sum_{F \in \mathcal{F}_T} \omega_{TF} \int_F (C^k_F \varphi_F - C^k_T \varphi_T \cdot n_F) = - \int_T \text{div} C^k_T \varphi_T = - \sum_{T \in \mathcal{F}_T} \omega_{TF} \int_F C^k_T \varphi_T \cdot n_F,
\]

which holds true owing to [12, Eq. (3.27)] with \( r_T = 1 \). There exists therefore a unique \( q_T \in H^1(T) \) with \( \int_T q_T = 0 \) solution to (A.5). Using \( q_T \) as a test function in the weak formulation of (A.5) yields

\[
\| \text{grad} q_T \|_{L^2(T)}^2 \leq \| C^k_T \varphi_T \|_{L^2(T)} \| \text{grad} q_T \|_{L^2(T)} + \sum_{F \in \mathcal{F}_T} \| C^k_F \varphi_F \|_{L^2(F)} \| q_T \|_{L^2(F)}.
\]

28
Using the Poincaré–Wirtinger and continuous trace inequalities as we did to obtain (A.2), and recalling that $\delta_T \nu_T = \nabla q_T$, we infer

$$\|\delta_T \nu_T\|_{L^2(T)} \leq \|C^k_T \nu_T\|_{L^2(T)} + \left( \sum_{F \in \mathcal{T}_T} h_F \|C^k_T \nu_T\|_{L^2(F)}^2 \right)^{\frac{1}{2}} \leq \|C^k_T \nu_T\|_{\text{div},T}, \quad (A.6)$$

where the conclusion follows from (4.14).

### A.2.2 Existence of $R_{\text{curl},T} \nu_T$

The equation (4.29b) suggests to look for $R_{\text{curl},T} \nu_T = \text{curl} z_T$. Since adding a gradient to $z_T$ does not change its curl, we can look for $z_T$ in the space

$$z_T \in (\text{grad} H^1(T))^\perp \doteq \left\{ w \in H(\text{curl}; T) : \int_T w \cdot \nabla r = 0 \quad \forall r \in H^1(T) \right\}. \quad (A.7)$$

The equations (4.29a) and (4.29c) then lead to a curl-curl problem on $z_T$, whose variational form is: Find $z_T \in (\text{grad} H^1(T))^\perp$ such that

$$\int_T \text{curl} z_T \cdot \text{curl} w = \int_T (C^k_T \nu_T + \delta_T \nu_T) \cdot w - \langle \omega_T \partial_T R_{\text{curl},\partial_T \nu_T} \cdot w \times n_{\partial T} \rangle_{H^{1/2}_\#(\partial T), H^{1/2}_\#(\partial T)}$$

$$\quad \forall w \in (\text{grad} H^1(T))^\perp, \quad (A.8)$$

where $\omega_T, \partial_T R_{\text{curl},\partial_T \nu_T}$ and $w \times n_{\partial T}$ are the functions defined on $\partial T$ by setting $(\omega_T, \partial_T R_{\text{curl},\partial_T \nu_T})(F) \doteq (\omega_{TF} R_{\text{curl},F} \nu_F)_{L^2}$ and $(w \times n_{\partial T})(F) \doteq w|_F \times n_F$ for all $F \in \mathcal{T}_T$, $H^{1/2}_\#(\partial T)$ is the set of functions on $\partial T$ whose restriction to each face $F \in \mathcal{T}_T$ belongs to $H^{1/2}(F)$, and whose tangential traces on the edges are weakly continuous (see [2] Definition 3.1.2 for details), and $H^{1/2}_\#(\partial T)$ is its dual space. Since the solution to (A.1) belongs to $H^{1/2}(F)$ (see [8] Corollary 23.5)), the edge tangential trace property in (A.4) ensures that $\omega_T, \partial_T R_{\text{curl},\partial_T \nu_T}$ indeed belongs to $H^{1/2}_\#(\partial T)$.

Owing to the Poincaré inequality (A.15) and to the fact that $(\text{grad} H^1(T))^\perp$ is a closed subspace of $H(\text{curl}; T)$, there exists a unique solution to (A.8). We now prove that $z_T$ satisfies (A.8) for all $w \in H(\text{curl}; T) = \text{grad} H^1(T) \oplus (\text{grad} H^1(T))^\perp$, which amounts to showing that the right-hand side vanishes whenever $w$ is $\text{grad} r$ for some $r \in H^1(T)$. By density of smooth functions in $H^1(T)$, we only need to prove this result for $r \in C^\infty(\overline{T})$. Plugging $w = \text{grad} r$ in the right-hand side of (A.8), the duality product can be written as standard integrals (since $R_{\text{curl},F} \nu_F \in L^2(F)$ for all $F \in \mathcal{T}_T$) and, integrating by parts, we obtain

$$\int_T (C^k_T \nu_T + \delta_T \nu_T) \cdot \text{grad} r - \sum_{F \in \mathcal{T}_T} \omega_{TF} \int_F R_{\text{curl},F} \nu_F \cdot (\text{grad} r \times n_F)$$

$$= - \int_T \text{div}(C^k_T \nu_T + \delta_T \nu_T) r + \sum_{F \in \mathcal{T}_T} \omega_{TF} \int_F (C^k_T \nu_T + \delta_T \nu_T) \cdot n_F r$$

$$- \sum_{F \in \mathcal{T}_T} \omega_{TF} \int_F R_{\text{curl},F} \nu_F \cdot \text{rot}_F (r|_F)$$

$$= \sum_{F \in \mathcal{T}_T} \omega_{TF} \int_F C^k_T \nu_F r - \sum_{F \in \mathcal{T}_T} \omega_{TF} \int_F \text{rot}_F (R_{\text{curl},F} \nu_F) r|_F$$

$$- \sum_{F \in \mathcal{T}_T} \sum_{E \in \mathcal{T}_E} \omega_{TF} \omega_{FE} \int_E (R_{\text{curl},F} \nu_F \cdot t_E) r|_F,$$
where we have used (4.28a) to cancel the term in the first equality, and (4.28c) together with integrations by parts on each face in the second equality. Recalling (A.4) and that \(\omega_{TF,\omega_{F,E}} + \omega_{TF,\omega_{F,E}} = 0\) if \(F_1, F_2\) are the two faces of \(T\) that share the edge \(E\), the right-hand side above vanish, which shows that (A.8) indeed holds for \(w = \nabla r\), and thus for all \(w \in H(\text{curl}; T)\).

Since \(R_{\text{curl},T} v_T = \text{curl} z_T\), applying this relation to a generic \(w \in C_c^\infty(T)\) and integrating by parts yields (4.29a); using then a generic \(w \in C_c^\infty(T)\) and again integrating by parts, we infer (4.29c).

A.2.3 Bound on \(R_{\text{curl},T}\)

We prove here the estimate (4.31). The estimate on \(\text{curl} R_{\text{curl},T} v_T\) follows from (4.29a), (4.14) and (A.6). It remains to bound the \(L^2\)-norm of \(R_{\text{curl},T} v_T\). To do so, we use \(g_{v_T}\) provided by Lemma 20 below and an integration by parts \([2, \text{Eq. (2.27)}]\) to re-cast (A.8) as

\[
\int_T \text{curl} z_T \cdot \text{curl} w = \int_T (C_T^k v_T + \delta_F v_T) \cdot w + \int_T \text{curl} w \cdot \text{curl} g_{v_T} - \int_T w \cdot \text{curl} g_{v_T}.
\]

Making \(w = z_T\), we deduce

\[
\|\text{curl} z_T\|_{L^2(T)} \lesssim \|C_T^k v_T\|_{\text{div}, T} h_T \|\text{curl} z_T\|_{L^2(T)} + \|\text{curl} z_T\|_{L^2(T)} \left(\|v_T\|_{\text{curl}, T} + \|\sum F_{\text{div}, T} v_T\|_{\text{div}, T}\right),
\]

where we have invoked (4.14), (A.6), the Poincaré inequality (A.15), and (A.9) below. Simplifying, using the norm equivalences (4.5), and recalling that \(R_{\text{curl},T} v_T = \text{curl} z_T\) concludes the proof of the \(L^2\)-estimate on \(R_{\text{curl},T} v_T\) stated in (4.31).

Lemma 20 (Lifting in \(H^1(T)\)). There exists \(g_{\Sigma_T} \in H^1(T)\) such that the tangential trace of \(g_{v_T}\) on \(\partial T\) is \(R_{\text{curl},\partial T} v_T\), and

\[
\|g_{v_T}\|_{L^2(T)} + h_T \|\text{curl} g_{v_T}\|_{L^2(T)} \leq \|v_T\|_{\text{curl}, T} + \|\sum F_{\text{div}, T} v_T\|_{\text{div}, T}.
\]

Proof. Recalling that

\[
R_{\text{curl},\partial T} v_T = \phi_{\Sigma_T} + \nabla r_{\Sigma_T},
\]

with obvious notations (each of these functions, restricted to a face \(F \in \mathcal{F}_T\), corresponds to the function obtained replacing \(\partial T\) by \(F\)), we construct \(g_{\Sigma_T} = g_{\Sigma_T} \cdot \phi + g_{\Sigma_T} \cdot \phi\), each addend corresponding to the addends in the decomposition (A.9) of \(R_{\text{curl},\partial T} v_T\).

1. Construction of \(g_{\Sigma_T} \cdot \phi\). We assume, for the moment, that \(h_T = 1\). By [8, Corollary 23.5] and inverse inequalities on the polynomials \(C_T^k v_{\Sigma_T}\) and \((v_{E})_{E \in \mathcal{F}_T}\) (recalling that \(1/h_T \approx h_T \approx h_E\) for all \(F \in \mathcal{F}_T\) and \(E \in \mathcal{E}_T\), there exists \(e \in (0, 1/2)\) such that \(\nabla f_{\Sigma_T} q_F \in H^{1/2+e}(F)\) and

\[
\|\nabla f_{\Sigma_T} q_F\|_{H^{1/2+e}(F)} \leq \|C_T^k v_{\Sigma_T}\|_{L^2(F)} + \sum_{E \in \mathcal{E}_F} \|v_{E}\|_{L^2(E)} \leq \|C_T^k v_{\Sigma_T}\|_{L^2(F)} + \|v_{\Sigma_T}\|_{\text{curl}, F}.
\]

Above, when invoking [8, Corollary 23.5], we have used the fact that, since \(e < 1/2\), the \(H^s(\partial F)\)-norm is equivalent to the sum of the \(H^s(\partial F)\)-norms over \(E \in \mathcal{F}_E\). By construction, \(\phi_{\Sigma_T}\) has strongly continuous tangential traces on the edges of \(T\) so

\[
|\phi_{\Sigma_T}|^2_{H^{1/2}(\partial T)} \leq \sum_{F \in \mathcal{F}_T} |\phi_{\Sigma_T}|^2_{H^{1/2}(F)} = \sum_{F \in \mathcal{F}_T} \|\nabla f_{\Sigma_T} q_F\|^2_{H^{1/2}(F)}
\]

\[
\leq \sum_{F \in \mathcal{F}_T} \|\nabla f_{\Sigma_T} q_F\|^2_{H^{1/2+e}(F)} \leq \sum_{F \in \mathcal{F}_T} \left(\|C_T^k v_{\Sigma_T}\|_{L^2(F)} + \|v_{\Sigma_T}\|_{\text{curl}, F}\right)^2.
\]
Combined with \((A.2)\) and recalling that the local length scales are \(\equiv 1\), this leads to
\[
\|\phi_{\partial T}\|_{L^1(\partial T)} + |\phi_{\partial T}|_{H^{1/2}(\partial T)} \lesssim \|C_kv_T\|_{\text{div}, T} + \|v_T\|_{\text{curl}, T}.
\]
Since \(\phi_{\partial T}\) belongs to \(H^{1/2}(\partial T)\), by [2, Theorem 3.1.3] there exists \(g_{s^T, \phi}\in H^1(T)\) such that the tangential trace of \(g_{s^T, \phi}\) is \(\phi_{\partial T}\) and
\[
\|g_{s^T, \phi}\|_{L^2(T)} + \|\text{curl} g_{s^T, \phi}\|_{L^2(T)} \lesssim \|\phi_{\partial T}\|_{L^2(\partial T)} + |\phi_{\partial T}|_{H^{1/2}(\partial T)} \lesssim \|C_kv_T\|_{\text{div}, T} + \|v_T\|_{\text{curl}, T}.
\]
This was done under the assumption that \(h_T = 1\). Using a scaling argument, we infer from the estimate above that, for an element \(T\) of generic diameter \(h_T\),
\[
\|g_{s^T, \phi}\|_{L^2(T)} + h_T \|\text{curl} g_{s^T, \phi}\|_{L^2(T)} \lesssim \|C_kv_T\|_{\text{div}, T} + \|v_T\|_{\text{curl}, T}.
\]
(A.11)

2. Construction of \(g_{s^T, \psi}\). By definition, \(g_{s^T, \psi}\) is the lifting of \(\text{grad}_{\partial T}\psi_{\partial T}\). Recalling the construction of each \(\psi_{\partial T} = \sigma_{TF}r_F\), for \(F\in \mathcal{F}_T\), we can extend \(r_F\) into a polynomial \(r_{TF} \in P^k(T)\) (for example, by making \(r_{TF}\) independent of the coordinate perpendicular to \(F\)). We then have, by \((A.3)\),
\[
\|r_{TF}\|_{L^2(T)} \lesssim \|r_F\|_{L^2(F)} \lesssim h_T \left(h_T^{-1/2} \|v_F\|_{\text{curl}, F} + h_T^{1/2} \|C_kv_F\|_{L^2(F)}\right).
\]
(A.12)
The smooth, compactly supported function \(\sigma_{TF}\) can be extended in \(T\) into \(\sigma_{TF}\) such that \(0 \leq \sigma_{TF} \leq 1\), \(\sigma_{TF}\) has a compact support in a ball of radius \(\approx h_T\) that does not touch the faces in \(\mathcal{F}_T \setminus \{F\}\), and \(\|\text{grad} \sigma_{TF}\| \leq h_T^{-1}\). Then, for each \(F\in \mathcal{F}_T\), the chain rule yields
\[
\|\text{grad}(\sigma_{TF}r_{TF})\|_{L^2(T)} \lesssim \|\text{grad} r_{TF}\|_{L^2(T)} + h_T^{-1} \|r_{TF}\|_{L^2(T)} 
\lesssim h_T^{1/2} \|v_F\|_{\text{curl}, F} + h_T^{1/2} \|C_kv_F\|_{L^2(F)},
\]
(A.13)
where the second inequality follows from an inverse inequality and \((A.12)\). We then set
\[
\begin{align*}
g_{s^T, \psi} &= \sum_{F\in \mathcal{F}_T} \text{grad}(\sigma_{TF}r_{TF}) \in C^0(\overline{T}).
\end{align*}
\]
By choice of the supports of \((\sigma_{TF})_{F\in \mathcal{F}_T}\), the tangential trace of \(g_{s^T, \psi}\) on each face \(F\in \mathcal{F}_T\) is \(\text{grad}_F(\sigma_{TF}r_{TF})|_{F} = \text{grad}_F \psi_{s^T, \psi}\). Moreover, the estimate \((A.13)\) gives
\[
\|g_{s^T, \psi}\|_{L^2(T)} \lesssim \left[\sum_{F\in \mathcal{F}_T} \left(h_T^{1/2} \|v_F\|_{\text{curl}, F} + h_T^{1/2} \|C_kv_F\|_{L^2(F)}\right)^2\right]^{1/2} \lesssim \|v_T\|_{\text{curl}, T} + \|v_T\|_{\text{div}, T}.
\]
(A.14)
Since \(g_{s^T, \psi}\) is a gradient, we also have \(\text{curl} g_{s^T, \psi} = 0\) and thus, combining \((A.11)\) and \((A.14)\) yields the estimate \((A.9)\) on \(g_{s^T, \psi} = g_{s^T, \phi} + g_{s^T, \psi}\). \(\square\)

**Lemma 21** (Local Poincaré inequality for \(\text{curl}\)). With \((\text{grad} H^1(T))^\perp\) defined by \((A.7)\), it holds
\[
\|w\|_{L^2(T)} \lesssim h_T \|\text{curl} w\|_{L^2(T)} \quad \forall w \in (\text{grad} H^1(T))^\perp.
\]
(A.15)

**Proof.** By [2, Theorem 3.1.4], for all \(v\in H(\text{div}; T)\) such that \(\text{div} v = 0\) and \(\langle v \cdot n_T, 1 \rangle_{\partial T} = 0\) (where \(\langle \cdot, \cdot \rangle_{\partial T}\) is the \(H^{-1}(\partial T)\)-\(H^{1/2}(\partial T)\) duality product and \(n_T\) is the outer normal to \(T\)), there exists \(z\in H(\text{curl}; T)\) such that \(\int_T z = 0\) and \(v = \text{curl} z\). Moreover, \(\|z\|_{L^2(T)} \lesssim C_0\|v\|_{L^2(T)} = C_0\|\text{curl} z\|_{L^2(T)}\)
and an inspection of the proof shows that $C_0 \leq h_T$ (this estimate is obtained via a scaling argument, and noticing that, if $h_T = 1$, the constants appearing in the proof of \cite{2} Theorem 3.4.1 do not depend on $T$ under our mesh regularity assumptions).

Take $w \in (\nabla H^1(T))^\perp$ and let $(w_m)_{m \in \mathbb{N}}$ be a sequence in $C^\infty(\overline{T})$ which converges to $w$ in $H(\text{curl}; T)$, see \cite{2} Proposition 2.2.12. Apply the result above to $v = \text{curl} w_m$, which satisfies the requirements since, on each $F \in \mathcal{F}_T$, we have $\text{curl} w_m \cdot n_{TF} = \text{rot}_F ((w_m)_T, F)$ (where $n_{TF} = (n_T)|_F$ and, as before, $(w_m)_T, F$ is the tangential trace of $w_m$ on $F$, oriented here according to $n_T$), and $w_m$ is continuous on $\partial T$. This yields $z_m \in H(\text{curl}; T)$ such that $\text{curl}(w_m - z_m) = 0$ and $\|z_m\|_{L^2(T)} \leq h_T \|\text{curl} w_m\|_{L^2(T)}$. In particular, since the second Betti number of $T$ is zero, $w_m - z_m \in \nabla H^1(T)$, and thus $\int_T (w_m - z_m) \cdot w = 0$. Hence,

$$\int_T w_m \cdot w = \int_T z_m \cdot w \leq \|w\|_{L^2(T)} \|z_m\|_{L^2(T)} \leq \|w\|_{L^2(T)} h_T \|\text{curl} w_m\|_{L^2(T)}.$$ 

The conclusion follows by letting $m \to \infty$ and simplifying by $\|w\|_{L^2(T)}$. \hfill \qed

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