Conformal field theory: 
a case study

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Abstract

This is a set of introductory lecture notes devoted to the Wess-Zumino-Witten model of two-dimensional conformal field theory. We review the construction of the exact solution of the model from the functional integral point of view. The boundary version of the theory is also briefly discussed.

1 Introduction

Quantum field theory is a structure at the root of our understanding of physical world from the subnuclear scales to the astrophysical and the cosmological ones. The concept of a quantum field is very rich and still poorly understood although much progress have been achieved over some 70 years of its history. The main problem is that, among various formulations of quantum field theory it is still the original Lagrangian approach which is by far the most insightful, but it is also the least precise way to talk about quantum fields. The strong point of the Lagrangian approach is that it is rooted in the classical theory. As such, it permits a perturbative analysis of the field theory in powers of the Planck constant and also captures some semi-classical non-perturbative effects (solitons, instantons). On the other hand, however, it masks genuinely non-perturbative effects. In the quest for a deeper understanding of quantum field theory an important role has been played by two dimensional models. Much of what we have learned about nonperturbative phenomena in quantum field theory has its origin in such models. One could cite the Thirring model with its anomalous dimensions and the fermion-boson equivalence to the sine-Gordon model, the Schwinger model with the confinement of electric charge, the non-linear sigma model with the non-perturbative mass generation, and so on.

The two-dimensional models exhibiting conformal invariance have played a specially important role. On one side, they are not without direct physical importance, describing, in their Euclidean versions, the long-distance behavior of the two-dimensional
statistical-mechanical systems, like the Ising or the Potts models, at the second order phase transitions. On the other hand, the (quantum) conformal field theory (CFT) models constitute the essential building blocks of the classical vacua of string theory, a candidate “theory of everything”, including quantum gravity. The two-dimensional space-time plays simply the role of a string world sheet parametrizing the string evolution, similarly as the one-dimensional time axis plays the role of a world line of point particles. The recent developments seem to indicate that string theory, or what will eventually emerge from it, provides the appropriate language to talk about general quantum fields, whence the central place of two-dimensional CFT in the quantum field theory edifice.

Due to the infinite-dimensional nature of the conformal symmetry in two space-time dimensions, the two-dimensional models of CFT lend themselves to a genuinely non-perturbative approach based on the infinite symmetries and the concept of the operator product expansion \[1\]. In the present lectures, we shall discuss a specific model of two-dimensional CFT, the so called Wess-Zumino-(Novikov)-Witten model (WZW) \[2\][3][4]. It is an example of a non-linear sigma model with the classical fields on the space-time taking values in a manifold which for the WZW model is taken as a group manifold of a compact Lie group \(G\). We shall root our treatment in the Lagrangian approach and will work slowly our way towards a non-perturbative formulation. This will, hopefully, provide a better understanding of the emerging structure which, to a large extent, is common to all CFT models. In fact, the WZW theory is a prototype of general (rational) CFT models which may be obtained from the WZW one by different variants of the so called coset construction. In view of the stringy applications, where the perturbation expansion is built by considering two-dimensional conformal theories on surfaces of arbitrary topology, we shall define and study the WZW model on a general Riemann surface.

A word of warning is due to a more advanced audience. The purpose of these notes is not to present a complete up-to-date account of the WZW theory, even less of CFT. That would largely overpass the scope of a summer-school lecture notes. As a result, we limit ourselves to the simplest version of the model leaving completely aside the ramifications involving models with non-simply connected groups, orbifolds, etc, as well as applications to string theory. We profit, however, from this simple example to introduce on the way some of the main concepts of two-dimensional CFT. Much of the material presented is not new, even old, by the time-scale standard of the subject, with the possible exception of the last section devoted to the boundary WZW models. The author still hopes that the following exposition, which he failed to present at the 1998 Istanbul summer school, may be useful to a young reader starting in the field.

The notes are organized as follows. In Sect. 2, we discuss a simple quantum-mechanical version of the WZW model: the quantum particle on a group manifold. This simple model, exactly solvable by harmonic analysis on the group, permits to describe many structures similar to the ones present in the two-dimensional theory and to understand better the origin of those. Sect. 3 is devoted to the definition of the action functional of the WZW model. The action contains a topological term, which requires
a special treatment. We discuss separately the case of the surfaces without and with boundary, in the latter case postponing the discussion of local boundary conditions to the last section. In Sect. 4, we introduce the basic objects of the (Euclidean) quantum WZW theory: the quantum amplitudes taking values in the spaces of states of the theory and the correlation functions. We state the infinite-dimensional symmetry properties of the theory related to the chiral gauge transformations and to the conformal transformations. The symmetries give rise to the action of the two copies of the current and Virasoro algebras in the Hilbert space of states of the theory constructed with the use of the representation theory of those algebras. We discuss briefly the operator product expansions which encode the symmetry properties of the correlation functions. Sect. 5 is devoted to the relation between the WZW theory and the Schrödinger picture quantum states of the topological three-dimensional Chern-Simons theory. The relation is established via the Ward identities expressing the behavior of the WZW correlation functions, coupled to external gauge field, under the chiral gauge transformations. We discuss the structure of the spaces of the Chern-Simons states, the fusion ring giving rise to the Verlinde formula for their dimensions and their Hilbert-space scalar product, as well as the Knizhnik-Zamolodchikov connection which permits to compare the states for different complex structures. In particular, we explain how the knowledge of the scalar product of the Chern-Simons states permits to obtain exact expressions for the correlation functions of the WZW theory. In Sect. 6 we give a brief account of the coset construction of a large family of CFT models which may be solved exactly, given the exact solution of the WZW model. Finally, Sect. 7 is devoted to the WZW theory with local boundary conditions. Again, for the sake of simplicity, we restrict ourselves to a simple family of the conditions that do not break the infinite-dimensional symmetries of the theory. We discuss how to define the action functional of the model in the presence of such boundary conditions and what are the elementary properties of the corresponding spaces of states, quantum amplitudes and correlation functions.

2 Quantum mechanics of a particle on a group

2.1 The geodesic flow on a group

Non-linear sigma models describe field theories with fields taking values in manifolds. These lectures will be devoted to a special type of sigma models, known under the name of Wess-Zumino(-Novikov)-Witten (or WZW) models. They are prototypes of conformal field theories in two-dimensional space-time. As such they play a role in string theory whose classical solutions are built out of two-dimensional quantum conformal field theory models by a cohomological construction. Before we plunge, however, into the details of the WZW theory, we shall discuss a simpler but largely parallel model in one dimension, i.e. in the domain of mechanics rather than of field theory.

One-dimensional sigma models describe the geodesic flows on manifolds $M$ endowed with a Riemannian or a pseudo-Riemannian metric $\gamma$. The classical action for the
The trajectory \([0, T] \ni x(t) \in M\) of such a system is
\[
S(x) = \frac{1}{2} \int_0^T \gamma_{\mu\nu}(x) \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} \, dt
\]  
(2.1)

and the classical solutions \(\delta S = 0\) correspond to the geodesic curves in \(M\) parametrized by a rescaled length. If \(M = \mathbb{R}^n\), for example, and \(\gamma_{\mu\nu} = m \delta_{\mu\nu}\), we obtain the action of the free, non-relativistic particle of mass \(m\) undergoing linear classical motions \(x(t) = x_0 + \frac{p}{m} t\). The action (2.1) is not parametrization-invariant but it may be viewed as a gauged-fixed version of the action
\[
S_p(x) = \frac{1}{2} \int_0^T \gamma_{\mu\nu}(x) \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} \eta^{-\frac{1}{2}} \, dt + \frac{1}{2} \int_0^T \eta^{\frac{1}{2}} \, dt,
\]
where the reparametrization invariance is restored by coupling the system à la Polyakov to the metric \(\eta(t)(dt)^2\) on the word-line of the particle. The \(\eta = 1\) gauge reproduces then the action (2.1), whereas extremizing over \(\eta\), one obtains the relativistic action
\[
S_r(x) = \int_0^T \left(\frac{dx^\mu}{dt} \frac{dx^\nu}{dt}\right)^{\frac{1}{2}} \, dt
\]
given by the geodesic length of the trajectory.

The exact solvability of the geodesic equations can be achieved in sufficiently symmetric situations. In particular, we shall be interested in the case when \(M\) is a manifold of a compact Lie group \(G\) and when \(\gamma\) is a left-right invariant metric on \(G\) given by \(\frac{k}{2}\) times a positive bilinear \(ad\)-invariant form \(\text{tr}(XY)\) on the Lie algebra \(\mathfrak{g}\). For matrix algebras, as the algebra \(su(N)\) of the hermitian \(n \times n\) traceless matrices, the form is given by the matrix trace in the defining representation, hence the notation. The positive constant \(k\) will play the role of a coupling constant. The action (2.1) may be then rewritten as
\[
S(g) = -\frac{k}{4} \int_0^T \text{tr} \left( g^{-1} \frac{dg}{dt} g \right)^2 \, dt.
\]

The variation of the action under the infinitesimal change of \(g\) vanishing on the boundary is
\[
\delta S(g) = \frac{k}{2} \int_0^T \text{tr} \left( g^{-1} \delta g \right) \frac{d}{dt} \left( g^{-1} \frac{dg}{dt} g \right) \, dt.
\]

Consequently, the classical trajectories are solutions of the equations
\[
\frac{d}{dt} \left( g^{-1} \frac{dg}{dt} g \right) = 0.
\]  
(2.2)

The case \(G = \mathbb{T}^n\), where \(\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n\) is the \(n\)-dimensional torus, is the prototype of an integrable system whose trajectories are periodic or quasiperiodic motions with the condition
\[1\text{e we use the physicists' convention in which the exponential map between the Lie algebra and the group is } X \mapsto e^{iX}\]
angles evolving linearly in time. The case $G = SO(3)$ corresponds to the symmetric top whose positions are parametrized by rigid rotations. The classical trajectories solving Eq. (2.2) have a simple form:

$$g(t) = g_e e^{it\lambda/k} g_r^{-1},$$

where $g_e, r$ are fixed elements in $G$ and $\lambda$ may be taken in the Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}$. For the later convenience, we have introduced the factor $\frac{1}{k}$ in the exponential.

The space $\mathcal{P}$ of classical solutions forms the phase space of the system. It may be parametrized by the initial data $(g(0), p(0))$ where the momentum $p(t) = \frac{k}{2i} \frac{d}{dt} g$. As usually, the phase space $\mathcal{P}$ comes equipped with the symplectic form

$$\Omega = \frac{1}{i} \, d\ell \, (p g^{-1} dg),$$

where the right hand side may be calculated at any instance of time giving a result independent of time. The symplectic structure on $\mathcal{P}$ allows to associate the vector fields $\mathcal{X}_f$ to functions $f$ on $\mathcal{P}$ by the relation $-df = \iota_{\mathcal{X}_f} \Omega$, where $\iota_{\mathcal{X}_f}$ denotes the contraction of a vector field $\mathcal{X}$ with a differential form $\alpha$. These are the Hamiltonian vector fields that preserve the symplectic form: $\mathcal{L}_\mathcal{X} \Omega = 0$, where $\mathcal{L}_\mathcal{X}$ is the Lie derivative that acts on differential forms by $\mathcal{L}_\mathcal{X} \alpha = \iota_{\mathcal{X}} d\alpha + d\iota_{\mathcal{X}} \alpha$. The Poisson bracket of functions is defined by: $\{ f, g \} = \mathcal{X}_f(g)$. In particular, the time evolution is induced by the vector field associated with the classical Hamiltonian

$$h = \frac{1}{k} \, \text{tr} \, p^2 = - \frac{k}{4} \, \text{tr} \, (g^{-1} \frac{d}{dt} g)^2$$

which stays constant during the evolution. In the alternative way (2.3) to parametrize the solutions, $h = \frac{1}{4k} \text{tr} \lambda^2$ and the symplectic structure splits:

$$\Omega = \Omega_\ell - \Omega_r, \quad \text{where} \quad \Omega_\ell = \frac{i}{2} \, \text{tr} \left[ \lambda (g_\ell^{-1} dg_\ell) - d\lambda \, g_\ell^{-1} dg_\ell \right]$$

and $\Omega_r$ is given by the same formula with the subscript $\ell$ replaced by $r$.

There are two commuting actions of the group $G$ on $\mathcal{P}$: from the left $g(t) \mapsto g_0 g(t)$ and from the right $g(t) \mapsto g(t) g_0^{-1}$. Both preserve the symplectic structure and the Hamiltonian $h$. The vector fields corresponding to the left and right actions of the infinitesimal generators $t^a \in \mathfrak{g}$ are induced by the functions

$$j^a = \frac{k}{2} \, \text{tr} \left( t^a g \frac{d}{dt} g^{-1} \right) = \frac{1}{2} \, \text{tr} \left( t^a g_\ell \lambda g_\ell^{-1} \right)$$

$$\bar{j}^a = \frac{k}{2} \, \text{tr} \left( t^a g^{-1} \frac{d}{dt} g \right) = - \frac{1}{2} \, \text{tr} \left( t^a g_r \lambda g_r^{-1} \right),$$

respectively. Note that, if we normalize $t^a$’s so that $\text{tr} \left( t^a t^b \right) = \frac{1}{2} \delta^{ab}$, then

$$h = \frac{2}{k} \, j^a j_a = \frac{2}{k} \, \bar{j}^a \bar{j}_a$$

(summation convention!). The symplectic form $\Omega_\ell$ gives, for $\lambda$ fixed, the canonical symplectic form on the (co)adjoint orbit $\{ g_0 \lambda g_0^{-1} | \, g_0 \in G \}$ passing through $\lambda$. The left action of the group is $g_\ell \mapsto g_0 g_\ell$ so that it coincides with the (co)adjoint action on the orbit. As is well known, upon geometric quantization of the coadjoint orbits for appropriate $\lambda$, this action gives rise to irreducible representations of $G$.\footnote{we identify $\mathfrak{g}$ with its dual using the bilinear form $\text{tr}(XY)$}
2.2 The quantization

The geodesic motion on a group is easy to quantize. As the Hilbert space $\mathcal{H}$ one takes the space $L^2(G, dg)$ of functions on $G$ square integrable with respect to the normalized Haar measure $dg$. The two commuting actions of $G$ in $\mathcal{H}$:

$$f \mapsto h f = f(h^{-1} \cdot), \quad f \mapsto f^h = f(\cdot h),$$

give rise to the actions

$$J^a f = \frac{1}{i} \frac{d}{dt} \big|_{t=0} e^{i t a} f, \quad \tilde{J}^a f = \frac{1}{i} \frac{d}{dt} \big|_{t=0} f e^{i t a}$$

of the infinitesimal generators $t^a$ of $\mathfrak{g}$. The commutation relations

$$[J^a, J^b] = i f^{abc} J^c \quad [\tilde{J}^a, \tilde{J}^b] = i f^{abc} \tilde{J}^c,$$

reflect the relation $[t^a, t^b] = i f^{abc} t^c$ in the Lie algebra $\mathfrak{g}$. The quantum Hamiltonian

$$H = \frac{2}{k} J^a J^a = \frac{2}{k} \tilde{J}^a \tilde{J}^a$$

coincides with $-\frac{2}{k}$ times the Laplace-Beltrami operator on $G$ and is a positive self-adjoint operator.

The irreducible representations $R$ of the compact Lie group $G$ are finite dimensional and are necessarily unitarizable so that we may assume that they act in finite-dimensional vector spaces $V_R$ preserving their scalar product. We shall denote by $g_R$ and $X_R$ the endomorphisms of $V_R$ representing $g \in G$ and $X \in \mathfrak{g}$. Up to isomorphism, the irreducible representation of $G$ may be characterized by their highest weights. Let us recall what this means. The complexified Lie algebra may be decomposed into the eigenspaces of the adjoint action of its Cartan subalgebra $\mathfrak{t}$ as

$$\mathfrak{g}^C = \mathfrak{t}^C \oplus (\oplus \alpha \mathbb{C} e_\alpha) \quad (2.6)$$

where $[X, e_\alpha] = \text{tr} (\alpha X) e_\alpha$ for all $X \in \mathfrak{t}$. The set of the roots $\alpha \in \mathfrak{t}$ may be divided into the positive roots and their negatives. We shall normalize the invariant form $\text{tr}$ on $\mathfrak{g}$ so that the long roots have the length squared 2 (this agrees with the normalization of the matrix trace for $\mathfrak{g} = su(N)$). The “step generators” $e_{\pm \alpha}$ may be chosen so that $[e_\alpha, e_{-\alpha}]$ is equal to the coroot $\alpha^\vee \equiv \frac{2a}{\text{tr} \alpha^2}$ corresponding to $\alpha$. The elements $\lambda \in \mathfrak{t}$ such that $\text{tr} (\alpha^\vee \lambda)$ is integer for all roots are called weights. A non-zero vector $v \in V_R$ (unique up to normalization) is called a highest weight (HW) vector if it is an eigenvector of the action of the Cartan algebra: $X_R v = \text{tr} (\lambda_R X) v$ and if $(e_\alpha)_R v = 0$ for all positive roots $\alpha$. The element $\lambda_R$ of the Cartan algebra, a weight, is called the highest weight (HW) of the representation $R$ and it determines completely $R$. All weights $\lambda \in \mathfrak{t}$ such that $\text{tr} (\alpha^\vee \lambda)$ is a non-negative integer for each positive $\alpha$ appear as HW’s of irreducible representations of $G$. The representations $R$ may be obtained by the geometric quantization of the (co)adjoint orbit passing through $\lambda_R$. For the $su(2)$ Lie
algebra spanned by the Pauli matrices $\sigma_i$, one usually takes $\sigma_3$ as the positive root and the matrices $\sigma_{\pm} = \frac{1}{2} (\sigma_1 \pm i \sigma_2)$ as the corresponding step generators. The HW’s are of the form $j \sigma_3$ with $j = 0, \frac{1}{2}, 1, \ldots$ called the spin of the representation.

With respect to the left-right action of $G \times G$, the Hilbert space $L^2(G, dg)$ decomposes as

$$\mathcal{H} \cong \bigoplus_R V_R \otimes \overline{V_R}, \quad (2.7)$$

where the (infinite) sum is over the (equivalence classes of) irreducible representations of $G$ and $\overline{R}$ denotes the representation complex-conjugate to $R$, i.e. $V_{\overline{R}} = \overline{V_R}$ and $g_{\overline{R}} = \overline{g_R}$. Recall that the complex conjugate vector space $\overline{V}$ is composed of the vectors $v \in V$, denoted for distinction by $\overline{v}$, with the multiplication by scalars defined by $\mu \overline{v} = \overline{\mu v}$. A linear transformation $A$ of $V$, when viewed as a transformation $\overline{A}$ of $\overline{V}$, is still linear. The above factorization of the Hilbert space reflects the classical splitting (2.5). Let $(g^{ij}_R)$ be the (unitary) matrix of the endomorphism $g_R$ with respect to a fixed orthonormal bases $(e^i_R)$ in $V_R$. The decomposition (2.7) is given by the assignment

$$V_R \otimes \overline{V_R} \ni e^i_R \otimes e^j_{\overline{R}} \mapsto d^{\frac{j}{2}}_R g^{ij}_R \in L^2(G, dg). \quad (2.8)$$

The Schur orthogonality relations

$$\int_G \overline{g^{ij}_R} g^{rs}_R \, dg = \frac{1}{d_R} \delta^{ij} \delta^r_s$$

assure that this assignment preserves the scalar product. The matrix elements $g^{ij}_R$ span a dense subspace in $L^2(G, dg)$.

The function on $G$ invariant under the adjoint action $g \mapsto Ad_{g_0}(g) \equiv g_0 g g_0^{-1}$ are called class functions. They are constant on the conjugacy classes

$$C_\lambda = \{ g_0 e^{2\pi i \lambda/k} g_0^{-1} \mid g_0 \in G \} \quad (2.9)$$

with $\lambda$ in the Cartan algebra $t$. The characters $\chi_R(g) = \text{tr}_{V_R} g_R$ of the irreducible representations $R$ are class functions. The Schur relations imply that

$$\int_G \overline{\chi_{R'}(g)} \chi_R(g) \, dg = \delta_{R' R}.$$  

The class functions in $L^2(G, dg)$ form a closed subspace and the characters $\chi_R$ form an orthonormal bases of it. Note that under the isomorphism (2.7),

$$\overline{\chi_R} \cong d^{\frac{j}{2}}_R e^i_R \otimes e^j_{\overline{R}} \quad (2.10)$$

(sum over $i!$).

The Hamiltonian $H$ becomes diagonal in the decomposition (2.7) of the Hilbert space. It acts on $V_R \otimes \overline{V_R}$ as the multiplication by $\frac{2}{k} c_R$ where $c_R$ is the value of the quadratic Casimir $c = t^a t^a$ in the representation $R$. In terms of the HW’s, $c_R = \ldots$
\( \frac{1}{2} \text{tr} (\lambda_R (\lambda_R + 2\rho)) \), where \( \rho \), the \textbf{Weyl vector}, is equal to half the sum of the positive roots. The Hamiltonian generates a 1-parameter family of unitary transformation \( e^{itH} \) describing the time evolution of the quantum system. In the Euclidean spirit, we shall be more interested, however, in the semigroup of the thermal density matrices \( e^{-\beta H} \) obtained by the Wick rotation of time \( \beta = it \geq 0 \). Their (heat) kernels are given by:

\[
e^{-\beta H}(g_0, g_1) = \sum_R d_R e^{-\frac{2}{k} \beta c_R} \chi_R(g_0 g_1^{-1}).
\]

In particular, at \( \beta = 0 \), we obtain a representation for the delta-function concentrated at an element \( g_0 \in G \):

\[
\delta_{g_0}(g_1) = \sum_R d_R \chi_R(g_0 g_1^{-1}).
\]

We shall also need below the delta-functions concentrated on the conjugacy classes \( C_\lambda \). They may be obtained by smearing the delta-function \( \delta_g \) over \( C_\lambda \):

\[
\delta_{C_\lambda}(g_1) = \int_G \delta_{g_0} e^{2\pi i \lambda/k} \chi_R^{-1}(g_1) dg_0 = \sum_R d_R \int_G \chi_{R}(g_0 e^{2\pi i \lambda/k} g_0^{-1} g_1^{-1}) \delta_{g_0} = \sum_R \chi_{R}(e^{2\pi i \lambda/k}) \chi_{R}(g_1),
\]

where we have used the Schur relations. It follows from the correspondence (2.10) that in the language of the isomorphism (2.7),

\[
\delta_{C_\lambda} \approx \sum_R \chi_{R}(1)^{-\frac{1}{2}} \chi_{R}(e^{2\pi i \lambda/k}) e^i R \otimes \overline{e^J R}. \tag{2.11}
\]

More exactly, \( \delta_{C_\lambda} \) is not a normalizable state in \( H \) but it defines an antilinear functional on a dense subspace in \( H \), e.g. the one of vectors with a finite number of components in the decomposition (2.7).

The delta-functions \( \delta_{C_\lambda} \) may be used to disintegrate the Haar measure \( dg \) into the measures along the conjugacy classes and over the set of different conjugacy classes:

\[
dg = \frac{1}{|T|} \Pi(e^{2\pi i \lambda/k})^2 \delta_{C_\lambda}(g) d\lambda dg. \tag{2.12}
\]

We shall choose the measure \( d\lambda \) such that it corresponds to the normalized Haar measure on the Cartan group \( T \subset G \) under the exponential map \( \lambda \mapsto e^{2\pi i \lambda/k} \). Then

\[
\Pi(e^{2\pi i \lambda/k}) = \prod_{\alpha > 0} (e^{\pi i \text{tr} (\alpha \lambda)/k} - e^{-\pi i \text{tr} (\alpha \lambda)/k}) \tag{2.13}
\]

is the so called \textbf{Weyl denominator}. In particular, for class functions constant on the conjugacy classes one obtains the Weyl formula:

\[
\int_G f \, dg = \int f(e^{2\pi i \lambda/k}) |\Pi(e^{2\pi i \lambda/k})|^2 d\lambda, \tag{2.14}
\]

where each conjugacy class should be represented ones. We shall employ this representation of the integral of class functions later.
The Feynman-Kac formula allows to express the heat kernel on the group as a path integral:

\[ e^{-\beta H}(g_0, g_1) = \int_{g : [0, \beta] \rightarrow G \atop g(0) = g_0, g(\beta) = g_1} \ e^{-S(g)} \ Dg , \]

where \( Dg \) stands for the product of the Haar measures \( dg(t) \). The integral on the right hand side may be given a rigorous meaning as the one with respect to the Brownian bridge measure \( dW_{g_0,g_1}(g) \) supported by continuous paths in \( G \). The path integral may be also used to define the thermal correlation function

\[ \langle \prod_n g_{i_n j_n}^{i_n j_n}(t_n) \rangle_\beta \equiv \frac{\int \prod_{n=1}^N g_{R_n}^{i_n j_n}(t_n) \ e^{-S(g)} \ Dg}{\int \ e^{-S(g)} \ Dg} , \tag{2.15} \]

where on the right hand side one integrates over the periodic paths \( g : [0, \beta] \rightarrow G \).

Upon ordering the (Euclidean) times \( t_1 \leq \ldots \leq t_N \), the above path integral may be expressed in the operator language:

\[
\langle \prod_n g_{R_n}^{i_n j_n}(t_n) \rangle_\beta = \frac{\text{tr}_R \left( e^{-t_1 \mathcal{H}} g^{i_1 j_1}_{R_1} e^{-(t_2-t_1) \mathcal{H}} g^{i_2 j_2}_{R_2} \cdots e^{-(t_N-t_{N-1}) \mathcal{H}} g^{i_N j_N}_{R_N} e^{-(\beta-t_N) \mathcal{H}} \right) \text{tr}_R \ e^{-\beta H}}{\text{tr}_R \ e^{-\beta H}} , \tag{2.16}
\]

where the functions \( g^{ij}_{R} \) on \( G \) are viewed as the multiplication operators in \( L^2(G, dg) \).

The right hand side of Eq. (2.16) may be calculated using harmonic analysis on \( G \). Indeed, what is really needed for such a computation are the matrix elements

\[
\left( e^{t_1} \otimes e^{t_2} \right) : g^{i_1 j_1}_{R_1} g^{i_2 j_2}_{R_2} \otimes e^{R_2} = d_{R_1}^+ d_{R_2}^+ \int_G g^{i_1 j_1}_{R_1} g^{i_2 j_2}_{R_2} g^{i_3 j_3}_{R_3} g^{i_4 j_4}_{R_4} \ Dg \tag{2.17}
\]

encoding the decomposition of the tensor product of the irreducible representations

\[ V_R \otimes V_{R_1} \cong \bigoplus_{R_2} M_{R_2}^{R_1 R_2} \otimes V_{R_2} . \tag{2.18} \]

In particular, the dimensions \( N_{R_1 R_2}^{R_3} \) of the multiplicity spaces \( M_{R_1 R_2}^{R_2} \) may be obtained from the traces of the above matrix elements:

\[ N_{R_1 R_2}^{R_3} = \int_G \chi_{R_2}(g) \chi_{R_1}(g) \chi_{R}(g) \ Dg . \]

The finite combinations with integer coefficients \( \sum n_i \chi_{R_i} \) of characters of irreducible representations form a subring \( \mathcal{R}_G \) in the commutative ring of class functions. The identity

\[ \chi_R \chi_{R_1} = \sum_{R_2} N_{R_2}^{R_2 R_1} \chi_{R_2} \tag{2.19} \]

shows that the integers \( N_{R_1 R_2}^{R_3} = N_{R_2 R_1}^{R_3 R_2} \) play the role of structure constants of this ring.
One can define a version of the correlation functions by replacing the integral over the periodic paths \( g : [0, \beta] \to G \) in Eq. (2.13) by the one over the paths constraint to fixed conjugacy classes on the boundary of the interval:

\[
\int_{g : [0, \beta] \to G} \prod_{n=1}^{N} g_{R_n}^{i_n} (t_n) \delta_{c_{\lambda_1}} (g(0)) \delta_{c_{\lambda_2}} (g(\beta)) \ e^{-S(g)} \ Dg \equiv \langle \prod_{n} g_{R_n}^{i_n} (t_n) \rangle_{\beta, \lambda_1 \lambda_2} . \tag{2.20}
\]

For \( G = SU(2) = \{ x_0 + i x_i \sigma_i | x_0^2 + x_i^2 = 1 \} \cong S^3 \), the conjugacy classes are the 2-spheres with fixed \( x_0 \) so that one integrates over the paths as in Fig. 1.

![Fig. 1](image)

The above functional integral may be rewritten in the operator language as

\[
\langle \prod_{n} g_{R_n}^{i_n} (t_n) \rangle_{\beta, \lambda_1 \lambda_2} = \left( \delta_{c_{\lambda_1}} , e^{-t_1 H} g_{R_1}^{i_{1,1}} \cdots e^{-(t_N-t_{N-1}) H} g_{R_N}^{i_{N,1}} e^{-(\beta-t_N) H} \delta_{c_{\lambda_2}} \right) . \tag{2.21}
\]

Although \( \delta_{c_{\lambda}} \) are generalized functions rather than normalizable states in \( L^2(G, dg) \), the matrix elements on the right hand side are finite and may again be computed by harmonic analysis on \( G \).

We shall encounter field-theoretical generalization of the above quantum-mechanical constructions below.

### 3 The WZW action

#### 3.1 Two-dimensional sigma models

The two-dimensional sigma models describe field theories with fields mapping a surface \( \Sigma \) to a target manifold \( M \), both equipped with metric structures. Such field configurations represent evolution of a string in the target \( M \) with \( \Sigma \) being the string world
sheet. The (Euclidean) action functional of the field configuration \( X : \Sigma \to M \) is

\[
S^\gamma(X) = \frac{1}{4\pi} \int_\Sigma \gamma_{\mu\nu}(X) \partial_\alpha X^\mu \partial_\beta X^\nu \eta^{\alpha\beta} \sqrt{\eta}, \tag{3.1}
\]

where \( \gamma_{\mu\nu} \) is the Riemannian metric on \( M \), \( \eta_{\alpha\beta} \) the one on \( \Sigma \) and \( \sqrt{\eta} \equiv \sqrt{\text{det} \eta_{\alpha\beta}} \) is the Riemannian volume density on \( \Sigma \). In particular, if \( M = \mathbb{R}^n \) with the standard metric, we obtain the quadratic action of the free field on a two-dimensional surface leading to linear classical equations. The general case, however, results in a non-linear classical theory.

The term \( S^\gamma \) does not change under the local rescalings \( \eta_{\alpha\beta} \mapsto e^{2\sigma} \eta_{\alpha\beta} \) of the metric on \( \Sigma \) i.e. it possesses two-dimensional conformal invariance. For oriented \( \Sigma \), conformal classes of the metric are in one to one correspondence with complex structures on \( \Sigma \) such that \( \eta_{zz} = \eta_{\bar{z}\bar{z}} = 0 \) in the holomorphic coordinates and that the latter preserve the orientation. The action \( S^\gamma \) may be written using explicitly only the complex structure of \( \Sigma \):

\[
S^\gamma(X) = \frac{i}{2\pi} \int_\Sigma \gamma_{\mu\nu}(X) \partial X^\mu \bar{\partial} X^\nu, \tag{3.2}
\]

where \( \partial = dz \partial_z \) and \( \bar{\partial} = d\bar{z} \partial_{\bar{z}} \). One-dimensional complex manifolds are called Riemann surfaces. It follows that the action \( S^\gamma(X) \) may be defined on such surfaces.

To the \( S^\gamma \) term, one may add the expression

\[
S^\beta(X) = \frac{1}{4\pi i} \int_\Sigma \beta_{\mu\nu}(X) \partial_\alpha X^\mu \partial_\beta X^\nu \epsilon^{\alpha\beta} \tag{3.3}
\]

where \( \beta_{\mu\nu} = -\beta_{\nu\mu} \) are the coefficients of a 2-form \( \beta \) on \( M \). Geometrically, \( S^\beta \) is proportional to the integral of the pullback of \( \beta \) by \( X \):

\[
S^\beta(X) = \frac{1}{4\pi i} \int_\Sigma X^* \beta. \tag{3.4}
\]

The imaginary coefficient is required by the unitarity of the theory after the Wick rotation to the Minkowski signature. The term \( S^\beta \) does not use the metric on \( \Sigma \) but only the orientation and is often called a topological term. Hence the classical two-dimensional conformal invariance of the model with the action \( S = S^\gamma + S^\beta \).

On the quantum level, the sigma model requires a renormalization which often imposes the addition to the action of further terms

\[
S^{\text{tach}}(X) = \frac{1}{2\pi} \int_\Sigma \mathcal{T}(X) \sqrt{\eta} \quad \text{and} \quad S^{\text{dil}}(X) = \frac{1}{2\pi} \int_\Sigma \mathcal{D}(X) \sqrt{\eta}, \tag{3.5}
\]

where where \( \mathcal{T} \) and \( \mathcal{D} \) are functions on \( M \) called tachyonic and dilatonic potentials, respectively, and \( r \) is the scalar curvature of the metric \( \eta_{\alpha\beta} \). The renormalization breaks the conformal invariance (note that \( S^{\text{tach}} \) and \( S^{\text{dil}} \) are not conformal invariant). We shall be interested, however, in the case of the WZW model \([2]\), an example of a CFT, where the classical conformal invariance is (almost) not broken on the quantum level.
The WZW model is the two-dimensional counterpart of the particle on a group and may be thought of as describing the movement of a string on a group manifold \(M = G\) equipped with the invariant metric \(\gamma\) described before. We then have for \(g : \Sigma \to G\),

\[
S^\gamma(g) = \frac{k}{4\pi i} \int_\Sigma \text{tr} (g^{-1} \partial g)(g^{-1} \bar{\partial} g),
\]

where \(k\) is a positive constant. The quantization of a model with such an action leads, however, to a theory without conformal invariance. To restore the latter, one adds to the \(S^\gamma\) term, following Witten [2], a topological term, the so called Wess-Zumino (WZ) term \(S^{WZ}\). In the first approximation, \(S^{WZ} = kS^\beta\) where \(\beta\) is a 2-form on \(G\) such that \(d\beta\) is equal to the canonical 3-form \(\chi \equiv \frac{1}{3!} \text{tr} (g^{-1} dg)^3\) on \(G\). If the group \(G\) is abelian, such a description is indeed possible and the overall action is a simple version of the free field one. In the non-abelian case, however, the difficulty comes from the fact that the 3-form \(\chi\) is closed but not globally exact so that the forms \(\beta\) exist only locally and are defined only up to closed 2-forms. Hence the definition of the WZ term of the action requires a more refined discussion.

### 3.2 Particle in the field of a magnetic monopole

It may be useful to recall a simpler situation where one is confronted with a similar problem. Suppose that we want to define the contribution \(S^{Div}(x)\) to the action of a mechanical particle of the term

\[
e \int_0^T A_\nu(x) \frac{dx^\nu}{dt} dt = e \int x^* A
\]

describing the coupling to the abelian gauge field \(A = A_\nu dx^\nu\) with the field strength \(F_{\nu\lambda} = \frac{1}{2}(\partial_\nu A_\lambda - \partial_\lambda A_\nu)\), or in the language of differential forms, with \(F = dA\), where \(F = F_{\nu\lambda} dx^\nu \wedge dx^\lambda\). The constant \(e\) stands for the electric charge of the particle. For concreteness, suppose that \(F_{\mu\nu}\) corresponds to the magnetic field of a monopole of magnetic charge \(\mu\) placed at the origin of \(\mathbb{R}^3\):

\[
F_{\nu\lambda} = \frac{1}{2} \mu \epsilon_{\nu\lambda\kappa} \frac{x^\kappa}{|x|^3}.
\]

There is no global 1-form \(A\) on \(\mathbb{R}^3\) without the origin such that \(dA = F\). For a closed trajectory \(t \mapsto x(t)\), however, we may pose

\[
S^{Div}(x) = e \int_D \bar{x}^* F,
\]

where \(\bar{x}\) is a map of a disc \(D\) into \(\mathbb{R}^3 \setminus \{0\}\) coinciding on the boundary of the disc with \(x\). For two different extensions \(\bar{x}\), however, the above prescription may give different results. Their difference may be written as the integral

\[
e \int_{S^2} \bar{x}^* F
\]
over the 2-sphere $S^2$ obtained by gluing the two disc $D$, one with the inverted orientation, along the boundary and for the map $\tilde{x} : S^2 \to \mathbb{R}^3 \setminus \{0\}$ glued from the two extensions of $x$ to the respective discs, see Fig. 2.

![Fig. 2](image)

The ambiguities (3.8) are the periods of the closed form $F$ over the cycles of the 2nd integer homology $H_2(\mathbb{R}^3 \setminus \{0\}) = \mathbb{Z}$. They take discrete values which are multiples of $4\pi e\mu$. The latter value is obtained for the unit sphere in $\mathbb{R}^3$, a generator of $H_2(\mathbb{R}^3 \setminus \{0\})$. The discrete ambiguities are acceptable in classical mechanics where one studies the extrema of the action. In quantum mechanics, however, we have to give sense to the Feynman amplitudes $e^{iS_{\text{Dir}}(x)}$, hence only the ambiguities in the action with values in $2\pi Z$ are admissible. Demanding that the quantum-mechanical amplitudes be unambiguously defined reproduces this way the Dirac quantization condition $e\mu \in \frac{1}{2}Z$.

For open trajectories $[0, T] \ni t \mapsto x(t)$, the amplitudes $e^{ie\int x^*A}$ may not, in general, be unambiguously assigned numerical values. They may be only defined as maps between the fibers $\mathcal{L}_{x(0)}$ and $\mathcal{L}_{x(T)}$ of a line bundle $\mathcal{L}$. Geometrically, they give the parallel transport in the bundle corresponding to a $U(1)$-connection with the curvature form $F$. We shall recover the analogous situation below when discussing how to give meaning to the WZ term in the action of the WZW model.

### 3.3 Wess-Zumino action on surfaces without boundary

Let us first consider the case of compact Riemann surfaces without boundary.
Topologically, such surfaces are characterized by the genus $g_\Sigma$ equal to the number of handles of the surface. They may be viewed as world sheets of a closed string created from the vacuum, undergoing in the evolution $g_\Sigma$ splittings and recombinations and finally disappearing into the vacuum, see Fig. 3 where a surface of genus 2 was represented. At genus zero, there is only one (up to diffeomorphisms) Riemann surface, the Riemann sphere $\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$, see Fig. 4(a).

At genus one, there is a complex one-parameter family of Riemann surfaces: the complex tori $T_\tau = \mathbb{C}/(\mathbb{Z} + \tau \mathbb{Z})$ with $\tau$ in the upper half plane $H^+$ of the complex numbers such that $\text{Im}\tau > 0$, see Fig. 4(b). The tori $T_\tau$ and $T_{\tau'}$, where $\tau' = \frac{a\tau + b}{c\tau + d}$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in the modular group $SL(2, \mathbb{Z})$, may be identified by the map $z \mapsto z' = (c\tau + d)^{-1}z$. The space of the diffeomorphism classes (i.e. the moduli space) of genus one Riemann surfaces is equal to $H^+/SL(2, \mathbb{Z})$ and has complex dimension 1. For higher genera, the moduli spaces of Riemann surfaces have complex dimension $3(g_\Sigma - 1)$.

Let us return to the discussion of the action of the WZW model. Assume that $G$ is connected and simply connected and that $\Sigma$ is a compact Riemann surface without boundary. Following [2] and mimicking the trick used for a particle in a monopole field, one may extend the field $g: \Sigma \to G$ to a map $\tilde{g}: B \to G$ of a 3-manifold $B$ such that $\partial B = \Sigma$ and set:

$$S^{WZ}(g) = \frac{k}{4\pi i} \int_B \tilde{g}^* \chi.$$  \hspace{1cm} (3.9)

By the Stokes formula, this expression coincides with $kS^\beta(g)$ whenever the image of $\tilde{g}$ is contained in the domain of definition of a 2-form $\beta$ such that $d\beta = \chi$, but it makes sense in the general case. The price is that the result depends on the extension $\tilde{g}$ of the field $g$. The ambiguities have the form of the integrals

$$\frac{k}{4\pi i} \int_{\tilde{B}} \tilde{g}^* \chi$$  \hspace{1cm} (3.10)

over 3-manifolds $\tilde{B}$ without boundary with $\tilde{g}: \tilde{B} \to G$, see Fig. 5.
They are proportional to the periods of the 3-form $\chi$ over the integer homology $H_3(G)$. Such discrete contributions do not effect the classical equations of motion $\delta S = 0$. In quantum mechanics, however, where we deal with the Feynman amplitudes $e^{-S(g)}$, only ambiguities in $2\pi i \mathbb{Z}$ are allowed. Hence we have to find conditions under which the periods (3.10) lie in $2\pi i \mathbb{Z}$.

Recall that we normalized the invariant form $\text{tr}$ on the Lie algebra $g$ of $G$ so that the long roots in $t$ have length squared 2. When $G = SU(2) \cong \{ x \in \mathbb{R}^4 \mid ||x||^2 = 1 \}$, the 3-form $\chi$ equals then to 4 times the volume form of the unit 3-sphere. Since the volume of the latter is equal to $2\pi$, we infer that

$$\frac{1}{4\pi i} \int_{SU(2)} \chi = -2\pi i.$$  

(3.11)

For the other simple, simply connected groups, the roots $\alpha$ determine the sub-algebras $\text{su}(2)_\alpha \subset g$, obtained by identifying the corresponding coroot $\alpha^\vee = \frac{2\alpha}{\text{tr} \alpha^2}$ and the step generators $e_{\pm \alpha}$ with the Pauli matrices $\sigma_3$ and $\sigma_{\pm}$, respectively. By exponentiation, we obtain the $SU(2)_\alpha$ subgroups of $G$. Clearly,

$$\frac{1}{4\pi i} \int_{SU(2)_\alpha} \chi = -\frac{4\pi i}{\text{tr} \alpha^2}.$$  

(3.12)

The ratio $\frac{2}{\text{tr} \alpha^2}$ is equal to 1 for long roots and is a positive integer for the others. It appears that any of the subgroups $SU(2)_\alpha \cong S^3$ for $\alpha$ a long root generates $H_3(G) = \mathbb{Z}$. Thus the unambiguous definition of the amplitudes $e^{-S^{WZ}(g)}$ requires that the coupling constant $k$, called the level of the model, be a (positive) integer, in the analogy to the Dirac quantization of the magnetic charge.

It is easy to see that, although the action $S^{WZ}(g)$ cannot be expressed, in general, as a local integral over $\Sigma$, the variation of $S^{WZ}$ has such a form:

$$\delta S^{WZ}(g) = \frac{k}{4\pi i} \int_{\Sigma} \text{tr} (g^{-1} \delta g)(g^{-1} dg)^2.$$  

(3.13)

The above formula is a special case of the general, very useful, geometric identity: $\delta \int f^* \alpha = \int \mathcal{L}_f \alpha$, where $\mathcal{L}_X$ is the Lie derivative. Applied to $f = g$ and $\alpha = \chi$ it gives,

\footnote{there is no $i$ in front of $S$ since we work with the Euclidean action}
in conjunction with the Stokes formula, the above relation. It is also important to note the behavior of $S^{WZ}$ under the point-wise multiplication of fields, a basic property of the WZ term:

$$S^{WZ}(g_1 g_2) = S^{WZ}(g_1) + S^{WZ}(g_2) + W(g_1, g_2),$$

(3.14)

where

$$W(g_1, g_2) = \frac{k}{4\pi i} \int_{\Sigma} \text{tr} \left(\frac{g_1^{-1}}{d}\frac{g_1}{d}\frac{g_2^{-1}}{d}\frac{g_2}{d}\right).$$

(3.15)

The relation follows easily from the definition (3.9), again by applying the Stokes formula.

The complete action of the WZW model on a closed Riemann surface $\Sigma$ is the sum of the $\gamma$- and the WZ-terms with the same coupling constant $k$: $S(g) = S^\gamma(g) + S^{WZ}(g)$. Since $S^\gamma$ is unambiguous, it has the same ambiguities as $S^{WZ}$, requiring that $k$ be a (positive) integer. The relations (3.13) and (3.14) get also contributions from $S^\gamma$ and become:

$$\delta S(g) = -\frac{k}{2\pi i} \int_{\Sigma} \text{tr} \left(\frac{g^{-1}}{d}\frac{g}{d}\frac{\partial}{\partial g}\right),$$

(3.16)

$$S(g_1 g_2) = S(g_1) + S(g_2) + \frac{k}{2\pi i} \int_{\Sigma} \text{tr} \left(\frac{g_1^{-1}}{d}\frac{g_1}{d}\frac{g_2^{-1}}{d}\frac{g_2}{d}\right).$$

(3.17)

The last relation is often called the Polyakov-Wiegmann formula. From Eq. (3.16), we obtain the classical equations of motion

$$\partial(\frac{g^{-1}}{d}\frac{g}{d}) = 0 \quad \text{or, equivalently,} \quad \bar{\partial}(g\frac{d}{d}g^{-1}) = 0.$$  

(3.18)

They have few solutions with values in $G$ (this would not be the case if we considered $\Sigma$ with a Minkowski metric). In all the above formulae, however, we could have taken fields $g$ with values in the complexified group $G^C$. For such fields, the general local solutions of Eqs. (3.18) have the form

$$g(z, \bar{z}) = g_\ell(z) g_r(\bar{z})^{-1}$$

(3.19)

where $g_\ell$ ($g_r$) are local holomorphic (anti-holomorphic) maps with values in $G^C$. Thus Eqs. (3.18) constitute a non-linear generalization of the Laplace equation in two dimensions whose solutions are harmonic functions which are, locally, sums of holomorphic and anti-holomorphic ones. In particular, multiplying a solution (3.19) by a holomorphic map into $G^C$ on the left and by an anti-holomorphic one on the right we obtain another solution. Similarly, composing a solution with a local holomorphic map or inverting it in $G^C$ after composition with a local anti-holomorphic map of $\Sigma$ one produces new solutions. Hence a rich symmetry structure of the classical theory. This structure will be preserved by the quantization leading to the current and Virasoro algebra symmetries of the quantum WZW model.
3.4 Wess-Zumino action on surfaces with boundary

What if the surface $\Sigma$ has a boundary? Of course only the WZ term in the action causes problems due to its non-local character. The term $S^{\gamma}$ is defined unambiguously for any compact surface. It will be convenient to represent $\Sigma$ as $\Sigma' \setminus (\bigcup_n D_n)$ where $D_n$ are disjoint unit discs $\{ z | |z| \leq 1 \}$ embedded in a closed surface $\Sigma'$ without boundary, see Fig.6.

Note that the boundaries of $\Sigma$ are then naturally parametrized by the unit circles. One way to proceed in the presence of boundaries is to extend the field $g : \Sigma \to G$ to a map $g' : \Sigma' \to G$ and to consider the action $S^{WZ}_{\Sigma'}(g')$ pertaining to the surface $\Sigma'$, as stressed by a subscript. We are again confronted with the question as to how the action on $\Sigma'$ depends on the extension of the field. The answer is easy to work out. If $g'' = g'h$ is another extension of $g$ then, by Eq. (3.14),

$$S^{WZ}_{\Sigma'}(g'') = S^{WZ}_{\Sigma'}(g') + S^{WZ}_{D_n}(h) + W_{D_n}(g', h).$$  \hspace{1cm} (3.20)

It will be convenient to localize the changes in the discs $D_n$ by rewriting the last formula as

$$S^{WZ}_{\Sigma'}(g'') = S^{WZ}_{\Sigma'}(g') + \sum_n \left( S^{WZ}_{S^2}(h_n) + W_{D_n}(g', h) \right),$$  \hspace{1cm} (3.21)

where $h_n$, mapping spheres (compactified planes) $S^2_n$ to $G$, extend the maps $h|_{D_n}$ by unity and $W_{D_n}$ are as in Eq. (3.13) but with the integration restricted to $D_n$. To account for the change (3.21), we shall define the following equivalence relation between the pairs $(g', z)$ where $g' : D \to G$ and $z \in \mathbb{C}$:

$$(g', z) \sim (g' h, z e^{-S^{WZ}_{S^2}(h)-W_{D}(g', h)})$$

for $h : S^2 \to G$ equal to unity outside the unit disc $D \subset S^2$. The set of equivalence classes forms a complex line bundle $L$ over the loop group $LG$ of the boundary values of the maps $g'$. Comparing to Eq. (3.21), we infer that for $g : \Sigma \to G$ with $\Sigma$ as above, the amplitude $e^{-S^{WZ}_{\Sigma}(g)}$ makes sense as the element of a tensor product of the line bundles $\mathcal{L}$, one for each boundary component of $\Sigma$,

$$e^{-S^{WZ}_{\Sigma}(g)} \in \bigotimes_n \mathcal{L}_{g|_{\partial D_n}},$$
where \( \mathcal{L}_h \) denotes the fiber of \( \mathcal{L} \) over the loop \( h \in LG \). Hence the WZ amplitudes \( e^{-S_{\mathcal{WZ}}(g)} \) take values in line bundles instead of having numerical values, exactly as for the amplitudes giving the parallel transport in a \( U(1) \)-gauge field mentioned before.

The line bundle \( \mathcal{L} \) is an interesting object. It carries a hermitian structure given by the absolute value of \( z \). The fibers of \( \mathcal{L} \) over \( g \) and \( \tilde{g} \) where \( \tilde{g} \) is a reversed loop, \( \tilde{g}(e^{i\varphi}) = g(e^{-i\varphi}) \), may be naturally paired so that, for \( g : \tilde{\Sigma} \to G \), where \( \tilde{\Sigma} \) is obtained from two surfaces \( \Sigma \) and \( \Sigma' \) by gluing them along some boundary components, see Fig. 7,

\[
\langle e^{-S_{\mathcal{WZ}}(g|_{\Sigma})}, e^{-S_{\mathcal{WZ}}(g|_{\Sigma'})} \rangle = e^{-S_{\tilde{\Sigma}}(g)} .
\tag{3.22}
\]

\( \tilde{\Sigma} \)

Fig. 7

\( \mathcal{L} \) may be also equipped with a product structure such that

\[
e^{-S_{\Sigma}(g_1)} \cdot e^{-S_{\Sigma}(g_2)} = e^{-S_{\Sigma}(g_1 g_2)} e^{W_{\Sigma}(g_1, g_2)} ,
\tag{3.23}
\]

compare to Eq. (3.14). Under the product, the elements of unit length in \( \mathcal{L} \) form a group \( \hat{G} \) which is a central extension of the loop group \( LG \) by the circle group \( U(1) \):

\[
1 \to U(1) \to \hat{G} \to LG \to 1 .
\tag{3.24}
\]

The second arrow sends \( e^{i\varphi} \) to the equivalence class of \((1, e^{i\varphi})\). The extensions for \( k > 1 \) are powers of the universal one corresponding to \( k = 1 \). On the infinitesimal level, one obtains the central extensions of the loop algebra \( Lg \) of the maps of the circle \( S^1 \) to the Lie algebra \( g \) by the real line:

\[
0 \to \mathbb{R} \to \hat{g} \to Lg \to 0 .
\tag{3.25}
\]

The Lie algebra \( \hat{g} \), called the current or the affine Kac-Moody algebra, may be described explicitly in terms of the complexified generators \( t^a_n \) corresponding to the loops \( t^a e^{i\varphi} \) in \( Lg^C \) and the central element \( K \) satisfying the commutation relations:

\[
[t^a_n, t^b_m] = i f^{abc} t^c_{n+m} + \frac{1}{2} K n \delta^{ab} \delta_{n+m,0} .
\tag{3.26}
\]
The algebra $\hat{g}$ is the same for all levels $k$ but the central element $K \in \hat{g}$ is the image of $k \in \mathbb{R}$ under the second arrow in the exact sequences (3.25). Note that the generators $t^a_0$ span a subalgebra $g \subset \hat{g}$. As we shall see, the group $\hat{G}$ and the algebra $\hat{g}$ play in the WZW theory a similar role to that of $G$ and $g$ for the particle on the group.

### 3.5 Coupling to gauge field

We may couple the WZW model to the gauge field $iA \equiv i(A^{10} + A^{01})$, a 1-form with values in the Lie algebra $g$ (or, more generally, $g^C$), where $A^{10} = t^a A^a dz$ and $A^{01} = t^a A^a d\bar{z}$ are, respectively, a 1,0- and a 0,1-form (the chiral components of the gauge field). In most what follows, we shall treat the gauge field as external, i.e. non-dynamical. Nevertheless, the coupling will allow to test the variation of the quantum system under the changes of the gauge field background and, finally, will facilitate the exact solution of the model. For a surface without boundary, we define

$$S(g, A) = S(g) + \frac{ik}{2\pi} \int_\Sigma \text{tr} \left[ A^{10} (g^{-1} \partial g) + (g \partial g^{-1}) A^{01} + g A^{10} g^{-1} A^{01} \right].$$  \hspace{1cm} (3.27)

Under the local gauge transformations $h : \Sigma \to G$, the gauge fields transform in the standard way:

$$A^{10} \mapsto h A^{10} h^{-1} + h \partial h^{-1}, \quad A^{01} \mapsto h A^{01} = h A^{01} h^{-1} + h \partial h^{-1}.$$

The reaction of the action to the chiral changes of the gauge is encoded in the identity

$$S(g, A) = S(h_1 g h_2^{-1}, h_2 A^{10} + h_1 A^{01}) + S(h_2, A^{10}) + S(h_1^{-1}, A^{01}) \hspace{1cm} (3.28)$$

which follows in a direct manner from the Polyakov-Wiegmann formula (3.17). For the later convenience, we have chosen a modified way of coupling to the gauge field, as compared to the more standard way with the addition of the term $-A^{10} A^{01}$ in the brackets on the right hand side of Eq. (3.27). The latter way would render the action invariant with respect to the diagonal (i.e. non-chiral) gauge transformations with $h_1 = h_2$.

For surfaces with boundary, we define $\text{e}^{-S(g, A)}$ by the same prescription but on the level of the amplitudes with values in the product of line bundles. The definition (3.28) of the product implies then the transformation rule:

$$\text{e}^{-S(g, A)} = \text{e}^{-S(h_1^{-1}, A^{01})} \cdot \text{e}^{-S(h_1 g h_2^{-1}, h_2 A^{10} + h_1 A^{01})} \cdot \text{e}^{-S(h_2, A^{10})} \hspace{1cm} (3.29)$$

which extends the property (3.28) to the case with boundaries.

### 4 Quantization of the WZW model

#### 4.1 Quantum amplitudes

The Feynman quantization prescription instructs us that in the quantum WZW model we should sum the amplitudes of different classical configurations. This leads to formal
functional integrals such as, for example,

\[ \mathcal{A}_\Sigma = \int e^{-S_\Sigma(g)} \, Dg, \]

where one integrates over the maps \( g : \Sigma \to G \) and \( Dg \) stands for the local product \( \prod_\xi dg(\xi) \) of the Haar measures. If \( \Sigma \) is closed, then the above integral should take a numerical value \( Z_\Sigma \) called the **partition function** (because of the statistical physics analogy). For \( \Sigma \) with boundary, it should define, instead, a Hilbert space state. Let \( \Gamma(L) \) denote the space of sections of the line bundle \( L \) over the loop group \( LG \). \( \Gamma(L) \) plays the role of the space of states of the quantized theory. If \( \Sigma \) has a boundary, we should fix in the functional integration the boundary values \( g = (g_n) \) of fields \( g : \Sigma \to G \):

\[ \mathcal{A}_\Sigma(g) = \int_{g|_{\partial D_n} = g_n} e^{-S_\Sigma(g)} \, Dg. \]

The result, in its dependence on \( g \), should give an element of the tensor product \( \otimes \Gamma(L) \) of the state spaces: the **quantum amplitude** corresponding to the surface \( \Sigma \). More generally, we shall consider the quantum amplitudes in the presence of external gauge field:

\[ \mathcal{A}_\Sigma(g; A) = \int_{g|_{\partial D_n} = g_n} e^{-S_\Sigma(g; A)} \, Dg \quad (4.1) \]

again with the values in \( \otimes \Gamma(L) \). We would like to give a rigorous meaning to such objects. In general, the functional integrals require complicated renormalization procedures which, besides, work only in some cases (of the so called renormalizable theories) and even then, in most instances, have been implemented only on the level of formal perturbation series. The WZW models are perturbatively renormalizable. In this case, however, one may follow a shortcut by exploiting formal symmetry properties of the functional integrals and showing that they fix uniquely the quantum amplitudes. This will be the line of thought adopted below, although we shall only describe the essential points of the argument and make detours to introduce other important notions.

Let us start by discussing the formal structure of the space of states \( \Gamma(L) \). The scalar product and the bilinear form

\[ (\psi, \psi') = \int_{\mathcal{L}G} (\psi(g), \psi'(g)) \, Dg, \quad \langle \psi, \psi' \rangle = \int_{\mathcal{L}G} \langle \psi(g), \psi'(\hat{g}) \rangle \, Dg, \quad (4.2) \]

which employ the hermitian structure and the duality \([3.22]\) on the line bundle \( L \), should turn \( \Gamma(L) \) into a Hilbert space \( \mathcal{H} \) and should allow the identification of \( \mathcal{H} \) with its dual. The space \( \Gamma(L) \) carries also two commuting actions of the group \( \hat{G} : \psi \mapsto \hat{h} \psi \) and \( \psi \mapsto \psi \hat{h} \). They are defined by:

\[ \hat{h} \psi(g) = \hat{h} \cdot \psi(h^{-1}g) \quad \psi \hat{h} = \psi(gh) \cdot \hat{h}^{-1}, \quad (4.3) \]

where \( g \) and \( h \) are elements of the loop group \( LG \) and \( h \) is the projection of \( \hat{h} \in \hat{G} \). Formally, these actions preserve the scalar product and the bilinear form in \( \Gamma(L) \). On
infinitesimal level, they give rise to two commuting actions of the current algebra \( \hat{g} \) in \( \Gamma(\mathcal{L}) \). We shall denote by \( J^a_n \) and \( \tilde{J}^a_n \) the operators in \( \Gamma(\mathcal{L}) \) corresponding to the left and right action of the generators \( t^a_n \) of \( \hat{g} \). The central generator \( K \) acts in \( \Gamma(\mathcal{L}) \) as multiplication by \( k \). Of course, \( J^a_n \) and \( \tilde{J}^a_n \) satisfy the commutation relation (3.26).

As stressed by Segal in [5], there are two important properties of the quantum amplitudes \( \mathcal{A}_\Sigma \) which are crucial for their rigorous construction. The first one, is the gluing property

\[
\mathcal{A}_{\tilde{\Sigma}}(g_n, g_{n'}) = \int \langle \mathcal{A}_\Sigma(g_n, g_{n_0}), \mathcal{A}_{\Sigma'}(g_{n_0}, g_{n'}) \rangle \, Dg_{n_0}
\]

(4.4)

which states that for a surface \( \tilde{\Sigma} \) glued along boundary components of two pieces \( \Sigma \) and \( \Sigma' \), as in Fig. 7, the functional integral may be computed iteratively, by first keeping the values of \( g \) fixed on the gluing circle and integrating over them only after the integration over the fields on \( \Sigma \) and \( \Sigma' \), see Eq. (3.22). Using the bilinear form on \( \Gamma(\mathcal{L}) \) applied to the glued channel, we may write this relation as the identity

\[
\mathcal{A}_{\tilde{\Sigma}} = \langle \mathcal{A}_\Sigma, \mathcal{A}_{\Sigma'} \rangle.
\]

(4.5)

It is often more convenient to view the quantum amplitudes \( \mathcal{A}_\Sigma \) as operators from the tensor product of some of the boundary spaces \( \mathcal{H} \) into the others. This is always possible because of the linear isomorphism between \( \mathcal{H} \) and its dual. Then Eq. (4.5) may be simply rewritten with the use of the product of operators:

\[
\mathcal{A}_{\tilde{\Sigma}} = \mathcal{A}_\Sigma \mathcal{A}_{\Sigma'}.
\]

(4.6)

One may also glue two boundary components in a single connected surface \( \Sigma \). The amplitude for the resulting surface \( \tilde{\Sigma} \) is then obtained from that of \( \Sigma \) by pairing the two corresponding factors in the product of the Hilbert spaces or, in the operator interpretation, by the partial trace applied to the glued channel:

\[
\mathcal{A}_{\tilde{\Sigma}} = \text{tr}_\mathcal{H} \mathcal{A}_\Sigma.
\]

(4.7)

In fact, as pointed out in [5], the last relation encompasses also the previous one if one introduces the amplitudes for the disconnected Riemann surfaces defining them as the tensor product of the amplitudes of the components. Clearly, similar gluing relation should hold for the amplitudes in external gauge field.

The second important property of the quantum amplitudes follows formally from the transformation property (3.29) of the classical amplitudes under the chiral gauge transformations \( h_{1,2} : \Sigma \rightarrow G \). It reads:

\[
\hat{h}_1 \mathcal{A}_\Sigma(A) \hat{h}_2 = \mathcal{A}_\Sigma(h_2^1 A^{10} + h_1 A^{01})
\]

(4.8)

for \( \hat{h}_1^{-1} = e^{-S_\Sigma(h_1^1, A^{01})} \) and \( \hat{h}_2 = e^{-S_\Sigma(h_2 A^{10})} \). The identity (4.8) expresses the covariance of the quantum amplitudes under the chiral gauge transformations. It is at the basis of the rich symmetry structure of the quantized WZW theory.
4.2 The spectrum

To give a rigorous construction of the Hilbert space $\mathcal{H}$ of the WZW model, whose vectors represent quantum states of a closed string moving on the group manifold, one may resort to the representation theory of the current algebras. The algebra $\hat{g}$ possesses a distinguished family of irreducible unitary representations labeled by pairs $\hat{R} = (k, R)$, where $k$, a non-negative integer called the level, is the value taken in the representation by the central generator $K$ of $\hat{g}$ and where $R$ is an irreducible representation of $G$ (and of $\mathfrak{g}$). The irreducible unitary representations $\hat{R}$ act in spaces $V_{\hat{R}}$ possessing a (unique) subspace $V_{\hat{R}} \subset V_{\hat{R}}$ annihilated by all the generators $t_a^0$ with $n > 0$ and carrying the irreducible representation $R$ of the subalgebra $\mathfrak{g} \subset \hat{g}$ generated by $t_a^0$. They are characterized by this property. Not all irreducible representations $R$ of $G$ appear for the fixed level $k$ but only the ones corresponding to the so called integrable HW’s which satisfy the condition

$$\text{tr} (\phi^\vee \lambda_R) \leq k,$$

where $\phi = \phi^\vee$ is the highest root of $\mathfrak{g}$, i.e. such a root that $\phi + \alpha$ is not a root for any positive root $\alpha$. Given $k$, there is only a finite number of integrable HW’s. For $su(2)$, the integrable HW’s correspond to spins $j = 0, \frac{1}{2}, 1, \ldots, \frac{k}{2}$. If $\lambda_R$ satisfies the condition (4.9) then so does $\lambda_{\pi}$ and the space $V_{\hat{R}}$ is canonically isomorphic to $V_{\hat{R}}$. The scalar product on $V_{\hat{R}}$ may then be viewed as a bilinear pairing between $V_{\hat{R}}$ and $V_{\hat{R}}$.

The rigorous definition of the Hilbert space of states $\mathcal{H}$ for the WZW model of level $k$ makes the two notions of the level coincide:

$$\mathcal{H} = \bigoplus_{\hat{R} \text{ of level } k} \left( V_{\hat{R}} \otimes V_{\hat{R}} \right)^{-},$$

where the symbol $(\ldots)^-$ stands for the Hilbert space completion. This is the loop group analogue of the decomposition (2.7) of $L^2(G, dg)$. The operators $J^a_n$ and $\tilde{J}^a_n$ representing the action of the generators $t_a^n$ in, respectively, $V_{\hat{R}}$ and $V_{\hat{R}}$ satisfy the unitarity conditions $J^a_n \dagger = J^a_{-n}$, $\tilde{J}^a_n \dagger = \tilde{J}^a_{-n}$. It is not difficult to motivate the above choice of the Hilbert space. One may, indeed, realize the space (4.10) as a space of sections of $L$. Formally, this may be done by the assignment (compare to the relation (2.8))

$$V_{\hat{R}} \otimes V_{\hat{R}} \supset V_{\hat{R}} \otimes V_{\hat{R}} \ni e^j \otimes e^j \mapsto \zeta \int \frac{g_R^a(0)}{g_R^a} \ e^{-S_D(g)} \ Dg,$$

where the normalization constant $\zeta$ will be fixed later. The functional integral on the right hand side, as a function of $g|_{\partial D} = h$ is the corresponding section of $L$. One may argue that the above integral is given, up to normalization, by its semi-classical value,

$$\overline{(g_d)^\vee_R(0)} \ e^{-S_D(g_d)},$$

where $g_d : D \rightarrow G^C$ is the solution of the classical equations (3.18) with the boundary condition $g_d|_{\partial D} = h$. As shown in [3], the expression (4.12) defines a non-singular section of $L$ only if the HW of $R$ is integrable at level $k$ (recall that the action $S(g_d)$ is
proportional to \(k\). One obtains then a rigorous embedding of the space \(\oplus \hat{R}V_R \otimes V_{\pi}\) into \(\Gamma(\mathcal{L})\), and, identifying the actions of the current algebra \(\hat{g}\) in \(\Gamma(\mathcal{L})\) and in \(\oplus \hat{R}V_R \otimes V_{\pi}\), also of the latter space. The formal scalar product and the formal bilinear form \((4.12)\) on \(\Gamma(\mathcal{L})\) correspond to the scalar product and the bilinear form on \(\mathcal{H}\) induced by the scalar product on the representation spaces \(V_R\) and the bilinear pairing between \(V_R\) and \(V_{\pi}\) (the latter induces the pairing between the \(\hat{R}\) and \(\hat{R}\) summands in \(\mathcal{H}\)).

The action of the pair of the current algebras in \(\mathcal{H}\) leads to the (projective) action in \(\mathcal{H}\) of the algebra of conformal symmetries. Let us discuss how this occurs. The spaces \(V_R\) of the irreducible unitary representations \(\hat{R}\) of \(\hat{g}\) appear to carry also the unitary representations of the Virasoro algebra \(Vir\), the central extension of the algebra of the vector fields \(Vect(S^1)\) on the circle,

\[
0 \to \mathbb{R} \to Vir \to Vect(S^1) \to 0.
\]  (4.13)

The complex generators \(\ell_n\) of \(Vir\) corresponding to the vector fields \(ie^{int}\partial_x\) satisfy the commutation relations

\[
[\ell_n, \ell_m] = (n-m)\ell_{n+m} + \frac{1}{12} C(n^3-n)\delta_{n+m,0},
\]  (4.14)

where \(C\) is the central generator, the image of 1 under the second arrow in the exact sequence \((4.13)\). The action of the generators \(\ell_n\) in the spaces of the representations \(\hat{R}\) of \(\hat{g}\) gives rise to the set of operators \(L_n\) and \(\bar{L}_n\) acting in the space of states \(\oplus \hat{R}V_R \otimes V_{\pi}\). They implement a projective action of \(Vect(S^1) \oplus Vect(S^1)\), the Lie algebra of Minkowskian conformal transformations. \(Vect(S^1) \oplus Vect(S^1)\) is, indeed, the Lie algebra of the infinitesimal transformations preserving the conformal class of the Minkowski metric \(dx^2 - dt^2 = dx^+ dx^-\), where \(x^\pm = x \pm t\) are the light-cone coordinates on the cylinder with periodic space-coordinate \(x\).

Explicitly, the operators \(L_n\)'s and \(\bar{L}_n\)'s are given in terms of the operators \(J^a_n\) and \(\bar{J}^a_n\), generating the actions of \(\hat{g}\), by the so called Sugawara construction:

\[
L_n = \frac{1}{k+h^\vee} \sum_{m=-\infty}^{\infty} J^a_{n-m}J^a_m \quad \text{for} \quad n \neq 0, \quad L_0 = \frac{2}{k+h^\vee} \sum_{m=0}^{\infty} J^a_{-n}J^a_n
\]  (4.15)

and similarly for \(\bar{L}_n\). Above, \(h^\vee\) (the dual Coxeter number) stands for the value of the quadratic Casimir in the adjoint representation of \(g\) and is equal to \(N\) for the \(SU(N)\) group. The operators \(L_n\) and \(\bar{L}_n\) satisfy the relations \((4.14)\) with \(C\) acting as the multiplication by \(c = \frac{k\dim(G)}{k+h^\vee}\), the value of the Virasoro central charge of the WZW theory. Besides,

\[
[L_n, J^a_m] = -mJ^a_{n+m}
\]

and similarly for \([\bar{L}_n, \bar{J}^a_m]\). The operators \(L_n\) (and \(\bar{L}_n\)) satisfy the unitarity conditions \(L_n^* = L_{-n}\). In particular \(L_0\) is a self-adjoint operator, bounded below on \(V_R\) by the conformal dimensions \(\Delta_R = \frac{c_R}{k+h^\vee}\), the eigenvalue of \(L_0\) on the subspace \(V_R \subset V_{\pi}\).
The latter subspace is annihilated by all $L_n$ with $n > 0$. The Hamiltonian of the WZW theory is $H = L_0 + \tilde{L}_0 - \frac{c}{12}$ whereas $L_0 - \tilde{L}_0$ defines the momentum operator $P$. The tensor product of the HW vectors in the subspace $V_\hat{1} \otimes V_\hat{1} \subset H$ corresponding to the trivial representation $R = 1$ gives the vacuum state $\Omega$ of the theory annihilated by $L_0$ and $\tilde{L}_0$.

A certain role in what follows will be played by the characters of the representations $V_{\hat{R}}$ defined as traces of loop group operators acting in $V_{\hat{R}}$. To avoid domain problems, one often considers only the endomorphisms $g_{\hat{R}}$ of $V_{\hat{R}}$ representing the action of the elements $g \in G$ (or in $G^C$) and obtained by the integration of the action of the generators $t_0^a$. One then defines

$$\chi_{\hat{R}}(\tau, g) = \text{tr}_{V_{\hat{R}}} e^{2\pi i \tau (L_0 - \frac{c}{12})} g_{\hat{R}},$$

(4.16)

where $\tau$ is a complex number in the upper half plane: $\text{Im}\tau > 0$. The presence of the factor $e^{2\pi i \tau L_0}$ renders the trace finite. The characters $\chi_{\hat{R}}(\tau, g)$ are class functions of $g$ and may be explicitly computed. Their decomposition into characters of $G$ encodes the multiplicities of the eigenvalues of the Virasoro generator $L_0$ in the subspaces of $V_{\hat{R}}$ transforming according to a given representation of $G$.

The central charge $c$ entering the commutation relations of the Virasoro generators is an important characteristic of a conformal field theory. It appears also in the rigorous version of the quantum amplitudes $A_\Sigma$. It enters into them in a somewhat subtle way, measuring their change under the local rescalings $\eta \mapsto e^{2\sigma} \eta$ of the metric of $\Sigma$ (recall that the amplitudes of the classical configurations $e^{-S(g)}$ were invariant under such rescalings). Under the change $\eta \mapsto e^{2\sigma} \eta$ with $\sigma$ vanishing around the boundary,

$$A_\Sigma \mapsto e^{\frac{2\pi i}{12} \int_\Sigma \left[ \frac{1}{2} \sum_{\alpha, \beta} \partial_\alpha \sigma \partial_\beta \sigma \eta^{\alpha \beta} + \sigma r + \mu (e^{2\sigma} - 1) \right] \sqrt{\eta} A_\Sigma,$$

(4.17)

due to renormalization effects, with $\mu$ depending on the renormalization prescription. We shall use the prescription corresponding to $\mu = 0$. The same transformation rule is obeyed by the amplitudes $A_\Sigma(A)$ in external gauge field. Hence, the quantum amplitudes are only projectively invariant under the conformal rescalings of the metric $\eta$. This is an example of the standard effect leading to projective actions of symmetries in quantum theory. Due to this effect, some care will have to be taken when making sense out of formal properties of the quantum amplitudes, like the gluing property (4.4). We shall always assume that the metric $\eta$ of $\Sigma$, which, together with the orientation of $\Sigma$, defines its complex structure, is of the special form around the boundary. Namely, that, in terms of the complex coordinate of the unit discs $D_\alpha$ holomorphically embedded into the surface $\Sigma'$ without boundary such that $\Sigma = \Sigma' \setminus \bigcup_n \overline{D_n}$, it is equal to the cylindrical metric $|z|^{-2} |dz|^2$. Upon gluing of surfaces along boundary components, such metrics will automatically give smooth metrics on the resulting surfaces.

In particular, the metrics on the unit discs $D$ will have the form $e^{2\sigma} |dz|^2$ with $\sigma = -\ln |z|$ around the boundary of $D$. Unless otherwise stated, we shall also assume that $\sigma = 0$ around the center of $D$. Consider the Riemann sphere $CP^1 = \mathbb{C} \cup \{\infty\}$.
composed from the two copies of the disc $D$ glued along the boundary. The choice

$$\zeta = \mathcal{Z}_{\mathbb{C}P^1}^{-1}$$  \hspace{1cm} (4.18)

for the normalizing constant will make precise the assignment (4.11). This choice guarantees that the change of $\zeta$ under the rescalings of the metric on $D$ will cancel the change of the functional integral $\int g^{ij} e^{-S(g)} \, Dg$.

### 4.3 Correlation functions

The formalism of Green functions encoding the action of field operators constitutes a traditional tool in quantum field theory. In the Minkowski space, the Green functions allow to express easily the scattering matrix elements (at least for the massive theories, via the LSZ formalism) whereas in the Euclidean space they coincide with correlation functions of continuum statistical models, providing a bridge between quantum field theory and statistical mechanics. In the context of CFT, the correlation functions defined on a general Riemann surface $\Sigma$ without boundary constitute a somewhat easier objects to deal with than the quantum amplitudes $\mathcal{A}_\Sigma$ for surfaces with boundary. Besides, even considered on the simplest Riemann surface, the Riemann sphere $\mathbb{C}P^1$, they already contain the full information about the model. Formally, the correlation functions of the WZW model are given by the functional integrals

$$\langle \prod_n g^{i_n j_n}(\xi_n) \rangle_{\Sigma'}(A) = \mathcal{Z}_\Sigma(A)^{-1} \int \prod_n g^{i_n j_n} e^{-S_\Sigma(g,A)} \, Dg,$$

where $\xi_n$ are disjoint points in a Riemann surface $\Sigma$ without boundary. For the Riemann surface without boundary $\Sigma'$ obtained by gluing unit discs $D_n$ to a surface $\Sigma$ with boundary\footnote{The metric $\eta$ on $\Sigma'$ is assumed to come from metrics on $\Sigma$ and on the discs $D_n$ of the type described above}, see Fig. 6, and for the points $\xi_n$ placed at the centers of the discs $D_n$, the correlation functions without the gauge field may be expressed, with the use of the assignment (4.11) and of the gluing property (4.5), by the scalar products of the quantum amplitudes $\mathcal{A}_\Sigma$ with special vectors in the Hilbert space of states:

$$\langle \prod_n g^{i_n j_n}(\xi_n) \rangle_{\Sigma'} = \mathcal{Z}^{-1}_\Sigma(\otimes_n (e^i_n \otimes \bar{e}^i_n), \mathcal{A}_\Sigma).$$

The normalization factor is given by the partition function of the surface with boundary $\Sigma$ defined by

$$\mathcal{Z}_\Sigma = \mathcal{Z}_{\Sigma'} \prod_n \zeta_n$$  \hspace{1cm} (4.20)

with $\zeta_n$ as in Eq. (1.18). The combination of the partition functions on the right hand side does not change under local rescalings of the metric inside the discs $D_n$.

On the level of correlation functions, the symmetry properties of the theory are encoded in the so called Ward identities. For example, the behavior (3.28) of the action
under the chiral gauge transformations with for $h_1 = h$ and $h_2 = 1$ implies formally that

$$Z_\Sigma(A) < \otimes n g_{R_n}(\xi_n) >_{\Sigma} (A)$$

$$= e^{-S(h^{-1}, A^{01})} \otimes n h^{-1}_{R_n}(\xi_n) Z_\Sigma(A^{10} + h^{A^{01}}) < \otimes n g_{R_n}(\xi_n) >_{\Sigma} (A^{10} + h^{A^{01}}), \quad (4.21)$$

where we view $\otimes g_{R_n}$ as taking value in $\otimes \text{End}(V_{R_n})$ and collecting all the matrix elements $\prod g_{R_n}^{i_n j_n}$. Similarly, for $h_1 = 1$ and $h_2 = h$, we obtain the mirror relation:

$$Z_\Sigma(A) < \otimes n g_{R_n}(\xi_n) >_{\Sigma} (A)$$

$$= e^{-S(h, A^{10})} Z_\Sigma(h^{A^{10}} + A^{01}) < \otimes n g_{R_n}(\xi_n) >_{\Sigma} (h^{A^{10}} + A^{01}) \otimes n h_{R_n}(\xi_n). \quad (4.22)$$

These are the Ward identities expressing the symmetry of the correlation functions under the chiral gauge transformations.

It is useful and customary to introduce more general correlation functions with insertions of **currents** testing the reaction of the functions (4.19) to infinitesimal changes of the gauge fields. On the surface $\Sigma'$ they are defined by

$$Z_{\Sigma'} < J^a(z_m) \prod_n g_{R_n}^{i_n j_n}(\xi_n) >_{\Sigma'} = -\frac{\pi}{\delta J^a_\Sigma(z_m)} \bigg|_{A=0} Z_{\Sigma'}(A) < \prod_n g_{R_n}^{i_n j_n}(\xi_n) >_{\Sigma'} (A), \quad (4.23)$$

where $z_m$ is the complex coordinate of the disc $D_m$, or by

$$Z_{\Sigma'} < \bar{J}^\alpha(z_m) \prod_n g_{R_n}^{i_n j_n}(\xi_n) >_{\Sigma'} = -\frac{\pi}{\delta \bar{J}^\alpha_\Sigma(z_m)} \bigg|_{A=0} Z_{\Sigma'}(A) < \prod_n g_{R_n}^{i_n j_n}(\xi_n) >_{\Sigma'} (A). \quad (4.24)$$

It is not very difficult to show, expanding the Ward identities (4.21) and (4.22) to the first order in $h$ around 1, that the insertions of $J^a(z)$ ($J^\alpha(z)$) are analytic (anti-analytic) in $z \neq 0$ but that they have simple poles at $z = 0$, the location point of one of the insertions $g_{R}(\xi)$, with the behavior

$$J^a(z) g_{R}(\xi) = -\frac{1}{z} t^a_{R} g_{R}(\xi) + \cdots , \quad \bar{J}^\alpha(z) g_{R}(\xi) = \frac{1}{z} g_{R}(\xi) t^\alpha_{R} + \cdots , \quad (4.25)$$

where the dots denote non-singular terms. The latter are related to the action of the current algebra generators in the space of states by the following relations involving the contour integral$^6$:

$$\frac{1}{2\pi i} \int_{|z_m|=\rho} < J^a(z_m) \prod_n g_{R_n}^{i_n j_n}(\xi_n) >_{\Sigma'} z_m^p d z_m = -Z^{-1}_\Sigma (\otimes (J^a_p e^\alpha_{R_n} \otimes e^{\bar{\alpha}_{R_n}}), A_\Sigma), \quad (4.26)$$

$$-\frac{1}{2\pi i} \int_{|z_m|=\rho} < \bar{J}^\alpha(z_m) \prod_n g_{R_n}^{i_n j_n}(\xi_n) >_{\Sigma'} \bar{z}_m^p d \bar{z}_m = -Z^{-1}_\Sigma (\otimes (e^\alpha_{R_n} \otimes \bar{J}^\alpha_p \bar{e}^{\bar{\alpha}_{R_n}}), A_\Sigma) \quad (4.27)$$

$^6$oriented counter-clockwise
with $\rho < 1$ and the superscript $\#$ indicating that the operator appears only for $n = m$. Eqs. (4.23) are examples of the **operator product expansions**, in this case, the ones stating that $g_R(\xi)$ are **primary fields** of the current algebra, in the CFT jargon.

Multiple current insertions, see Fig. 8, integrated over contours of increasing radii lead to multiple insertions of the current algebra generators. For example:

$$
\frac{1}{2\pi i} \int \frac{1}{2\pi i} \int < J^a(z_m) J^b(w_m) \prod_n g^i_{Rn} (\xi_n) >_{\Sigma'} \ z_m^p w_m^q \ dz_m \ dw_m \nabla = \mathcal{Z}_{\Sigma}^{-1} ( \otimes (J^a_{\rho} J^b_{\rho} \epsilon_{Rn}^i \otimes \epsilon_{Rn}^i), \mathcal{A}_{\Sigma})
$$

for $\rho_1 > \rho_2$ whereas for $\rho_1 < \rho_2$ the order of $J^a_{\rho} J^b_{\rho}$ should be reversed. In particular, the commutator $[J^a_{\rho}, J^b_{\rho}]$ corresponds to the difference of the two double contour integrals. It follows, that general matrix elements of the quantum amplitudes $\mathcal{A}_{\Sigma}$ may be read of the correlation function in the external gauge. It is then enough to find the latter to describe the complete theory.

![Fig. 8](image)

The action of the Virasoro generators may be interpreted similarly in terms of the insertions of the **energy-momentum** tensor into the correlation functions which test their variation under the changes of the metric on the surface:

$$
\mathcal{Z}_{\Sigma'} < T(z_m) \prod_n g^i_{Rn} (\xi_n) >_{\Sigma'} = 4\pi \frac{\delta}{\delta \eta^{zz}(z_m)} \mathcal{Z}_{\Sigma'} < \prod_n g^i_{Rn} (\xi_n) >_{\Sigma'},
$$

$$
\mathcal{Z}_{\Sigma'} < \tilde{T}(\bar{z}_m) \prod_n g^i_{Rn} (\xi_n) >_{\Sigma'} = 4\pi \frac{\delta}{\delta \eta^{zz}(\bar{z}_m)} \mathcal{Z}_{\Sigma'} < \prod_n g^i_{Rn} (\xi_n) >_{\Sigma'}, \quad (4.28)
$$

Under the local rescaling of the metric $\eta \mapsto e^{2\sigma} \eta$ with $\sigma$ vanishing around the insertion points $\xi_n$, the correlation functions (4.19) are invariant. This is not any more the case for general $\sigma$ due to (the “wave function”) renormalization of the insertions. For general $\sigma$, the correlation functions pick up the product of local factors equal to $e^{-2\Delta_{Rn} \sigma(\xi_n)}$, where the **conformal dimension** $\Delta_R$ of the fields $g_R(\xi)$ coincide with the lowest eigenvalues of the Virasoro generator $L_0$ in the HW representations of the current algebra discussed.
The latter expansions state that generators in the space of states: insertions of the energy momentum tensor encode the action of the Virasoro algebra product expansions properties under the local rescalings of the metric and under diffeomorphisms of the surface Σ may be shown to imply that the insertions of $T(z_m)$ ($\bar{T}(\bar{z}_m)$) are analytic (anti-analytic)\(^7\) in $z_m$ for $z_m \neq 0$ with the singular part given by the operator product expansion

$$
T(z) g_R(\xi) = \frac{1}{z^2} \Delta_R g_R(\xi) + \frac{1}{z} \partial_z g_R(\xi) + \ldots,
$$

$$
\bar{T}(\bar{z}) g_R(\xi) = \frac{1}{\bar{z}^2} \Delta_R g_R(\xi) + \frac{1}{\bar{z}} \partial_{\bar{z}} g_R(\xi) + \ldots.
$$

The latter expansions hold when inserted into the correlation functions as above with the covariance of the whole scheme under infinitesimal diffeomorphisms of the surface Σ may be shown to imply that the insertions of $T(z_m)$ ($\bar{T}(\bar{z}_m)$) are analytic (anti-analytic)\(^7\) in $z_m$ for $z_m \neq 0$ with the singular part given by the operator product expansion

$$
\frac{1}{2\pi i} \int_{|z_m|=\rho} < T(z_m) \prod_n g_{\rho_n}^{i_n j_n}(\xi_n) > \frac{z^{n+1}}{|z_m|} dz_m = Z^{-1} (\otimes (L^\#_{\rho_n} e_{\rho_n}^{i_n} \otimes \bar{e}_{\rho_n}^{j_n}), A_w), \quad (4.29)
$$

$$
-\frac{1}{2\pi i} \int_{|z_m|=\rho} < \bar{T}(\bar{z}_m) \prod_n g_{\rho_n}^{i_n j_n}(\xi_n) > \frac{\bar{z}^{n+1}}{|z_m|} d\bar{z}_m = Z^{-1} (\otimes (\bar{e}_{\rho_n}^{i_n} \otimes \bar{L}_p \bar{e}_{\rho_n}^{j_n}), A_w). \quad (4.30)
$$

The Ward identities of the chiral gauge symmetry together with the transformation properties under the local rescalings of the metric and under diffeomorphisms of the surface, expanded to the second order in the symmetry generators, yield the operator product expansions

$$
J^a(z) J^b(w) = \frac{k \delta^{ab}}{2(z-w)^2} + \frac{i f^{abc}}{z-w} J^c(w) + \ldots,
$$

$$
T(z) T(w) = \frac{c}{2(z-w)^4} + \frac{2}{(z-w)^2} T(w) + \frac{1}{z-w} \partial_w T(w) + \ldots,
$$

$$
T(z) J^a(w) = \frac{1}{(z-w)^2} J^a(w) + \frac{1}{z-w} \partial_w J^a(w) + \ldots,
$$

$$
\bar{T}(\bar{z}) \bar{J}^b(\bar{w}) = \frac{k \delta^{ab}}{2(\bar{z}-\bar{w})^2} + \frac{i f^{abc}}{\bar{z}-\bar{w}} \bar{J}^c(\bar{w}) + \ldots,
$$

$$
\bar{T}(\bar{z}) \bar{T}(\bar{w}) = \frac{c}{2(\bar{z}-\bar{w})^4} + \frac{2}{(\bar{z}-\bar{w})^2} \bar{T}(\bar{w}) + \frac{1}{\bar{z}-\bar{w}} \partial_{\bar{w}} \bar{T}(\bar{w}) + \ldots,
$$

$$
\bar{T}(\bar{z}) \bar{J}^a(\bar{w}) = \frac{1}{(\bar{z}-\bar{w})^2} \bar{J}^a(\bar{w}) + \frac{1}{\bar{z}-\bar{w}} \partial_{\bar{w}} \bar{J}^a(\bar{w}) + \ldots,
$$

where the dots denote the non-singular terms analytic (anti-analytic) in $z$ around $w$. The above expansions hold when inserted into the correlation functions as above with $z$ and $w$ corresponding to the values of the same local coordinate for two different points\(^7\) in the standard metric $|dz|^2$ around the insertion point.
and in the standard metric. They encode through the relations (4.26), (4.27), (4.29) and (4.30) the commutation relations of the current and Virasoro generators $J^a_n, J^a_n, L_n, \tilde{L}_n$ obtained from the above expansions by the deformation of the integration contours

$$
\int_{|z|=\rho+\epsilon} \int_{|w|=\rho} dz \, dw - \int_{|z|=\rho-\epsilon} \int_{|w|=\rho} dz \, dw = \int_{|w|=\rho} dw \int_{|z-w|=\epsilon} dz,
$$

(4.32)

see Fig. 9, and the use of the residue theorem.

The operator expansion algebra of the insertions into the correlation functions substitutes then for the operator commutation relations but allows to encode also more complicated algebraic relations between the CFT operators, see the last section. It is the basic technique of two-dimensional CFT.

As we have discussed before, the action of the Virasoro generators in the space of states of the WZW model may be expressed in terms of the current algebra action, see Eq. (4.15). This relation may be translated into the language of the insertions into the correlation functions, giving rise to the Sugawara construction of the energy-momentum tensor:

$$
T(w) = \lim_{z \to w} \frac{1}{k+h^\ve} \left( J^a(z) J^a(w) - \frac{k \dim(G)}{2(z-w)^2} \right)
$$

and similarly for $\tilde{T}(\bar{w})$.

5 Chiral WZW theory and the Chern-Simons states

As we have seen, the whole information about the quantum amplitudes of the WZW theory resides in the correlation functions (4.19) in an external gauge field. We shall look now more closely into the gauge-field dependence of these functions. By (formal) analytic continuation, the chiral Ward identities (4.21) and (4.22) should also hold for the complexified gauge fields $A$ with values in $g^C$ and for the complexified gauge transformations $h$ with values in $G^C$. As we shall see, they give a powerful tool for analysis of the correlation functions.
Let us consider first the Ward identity (4.21). The holomorphic maps \( \Psi \) on the space \( \mathcal{A}^{01} \) of \( \mathfrak{g}^\mathbb{C} \)-valued 0,1-gauge fields \( A^{01} \) with values in \( \otimes_n V_{R_n} \) satisfying the equation

\[
\Psi(A^{01}) = e^{-S(h^{-1}, A^{01}) \otimes h_n^{-1}(\xi_n)} \Psi(h A^{01})
\]

(5.1)

for \( h \) in the group \( \mathcal{G}^\mathbb{C} \) of \( \mathcal{G}^\mathbb{C} \)-valued gauge transformations have an interesting geometric interpretation. On one side, they may be viewed as holomorphic sections of a vector bundle \( W \) with typical fiber \( \otimes_n V_{R_n} \) over the orbit space\( \mathcal{N} = \mathcal{A}^{01}/\mathcal{G}^\mathbb{C} \). Mathematically, the orbit space \( \mathcal{N} \) is the moduli space of the holomorphic \( \mathcal{G}^\mathbb{C} \)-bundles and the mathematicians like to view \( \Psi \)'s as non-abelian generalizations of the classical theta functions. Indeed, the latter are holomorphic sections of a line bundle over the moduli space (the Jacobian) of the holomorphic \( \mathbb{C}^\ast \)-bundles over a Riemann surface.

### 5.1 States of the Chern-Simons theory

Physically, the holomorphic maps \( \Psi \) satisfying the Ward identity (5.1) may be identified as the quantum states of the three-dimensional Chern-Simons (CS) gauge theory \([7]\). The classical phase space of the CS theory on the 3-manifold \( \Sigma \times \mathbb{R} \) is composed of the flat \( \mathfrak{g} \)-valued gauge fields \( iA \) on \( \Sigma \) modulo \( \mathcal{G} \)-valued gauge transformations. The flatness condition is

\[
F(A) \equiv dA + A^2 = 0.
\]

(5.2)

In the holomorphic quantization à la Bargmann, the quantum states of the theory are described as holomorphic functionals \( \Psi \) on the space \( \mathcal{A}^{01} \) with the condition (5.2) imposed as a quantum constraint:

\[
F(A) \Psi(A^{01}) = 0,
\]

(5.3)

with \( F(A) \) as in Eq. (5.2) but with \( A^{01} \) acting as the multiplication operator and \( A^{10} \) as the differentiation: \( A^a_\xi = -\frac{2\pi}{k} \frac{\delta}{\delta A^a_\xi} \). The constraint (5.3) is closely related to the infinitesimal Ward identity:

\[
(F(A) + \frac{4\pi i}{k} \sum_n \delta_{\xi_n} t^a \tau^a_{R_n}) \Psi(A) = 0
\]

(5.4)

obtained by expanding the global Ward identity (5.1) to the first order in \( h \) around 1. The infinitesimal identity (5.4) is equivalent to its global version (5.1). In the absence of insertions, it coincides with Eq. (5.3). The modifications involving the insertions correspond in the CS gauge theory language to the insertions of the Wilson lines \( \{\xi_n\} \times \mathbb{R} \) in representations \( R_n \).

It is a crucial fact that the \((k\text{-dependent})\) spaces \( \mathcal{W}_\Sigma(\mathcal{X}, \mathcal{R}) \) of the holomorphic maps \( \Psi \)'s satisfying the Ward identities (5.1) or (5.4) are finite-dimensional, with the dimension given by the celebrated Verlinde formula \([8]\). In particular, only representations

\[\text{this space requires a careful definition with a special treatment of bad orbits}\]
$R_n$ with HW’s $\lambda_{R_n}$ integrable at level $k$ may give rise to non-trivial spaces $W_{\xi, R}$. It is instructive to look more carefully into the case of the Riemann sphere $\mathbb{C}P^1$. On $\mathbb{C}P^1$ all gauge fields $A^{01}$ may be written in the form $h^{-1}\bar{\partial}h$ or may be approximated by the fields of this form. In other words, the gauge orbit of $A^{01} = 0$ is dense in $A^{01}$. But by Eq. (5.1),

$$\Psi(h^{-1}\bar{\partial}h) = e^{S(h)} \otimes h^{-1}_{R_n}(\xi_n) \Psi(0),$$

(5.5)

where $\Psi(0) \in \otimes_n V_{R_n}$ is an element of a finite-dimensional space. Hence $\Psi(0)$ determines $\Psi$ on a dense set of gauge fields $A^{01}$, so everywhere. In fact, $\Psi(0)$ belongs to the subspace $(\otimes_n V_{R_n})^G$ of tensors invariant under the diagonal action of $G$, as is easy to see by taking constant $h$ in Eq. (5.5). We obtain then a natural embedding

$$W_{\mathbb{C}P^1}(\xi, R) \subset (\otimes_n V_{R_n})^G.$$  

(5.6)

The images of $W_{\mathbb{C}P^1}(\xi, R)$ in the spaces of invariant tensors are, in general, proper subspaces of $(\otimes_n V_{R_n})^G$ depending on $k$. The reason is that the $\Psi$’s defined by Eq. (5.5) on $A^{01} = h^{-1}\bar{\partial}h$ do not extend holomorphically to all of $A^{01}$ for all invariant tensors $\Psi(0)$. In particular, the image of $W_{\mathbb{C}P^1}(\xi, R)$, which is zero if some of HW’s $\lambda_{R_n}$ are not integrable at level $k$, becomes the whole space of invariant tensors for sufficiently large $k$.

For genus one, i.e. on the complex tori $T_\tau = \mathbb{C}/(\mathbb{Z} + \tau \mathbb{Z})$, a dense set of gauge fields is formed by the gauge orbits of the fields

$$A^{01}_{u} = \frac{\pi}{\im \tau} u d\bar{z}$$

with $u$ in the complexified Cartan algebra $t^C \subset g^C$. It is then enough to know the CS states $\Psi$ only on the gauge fields $A^{01}_{u}$. In particular, in the case with no insertions, the holomorphic functions $\psi$ defined by

$$\psi(u) = e^{-\frac{\pi k}{\im \tau} \text{tr} u^2} \Psi(A^{01}_{u})$$

characterize completely the CS states $\Psi$.

It appears that the functions $\psi(u)$ are arbitrary combinations of the characters $\chi^R_{\tau}(\tau, e^{2\pi i u})$ of the HW representations of the current algebra $g$, see Eq. (4.16). This fact implies an important property of the latter. Recall that the tori $T_\tau$ and $T_{\tau'}$ for $\tau' = -\frac{1}{\tau}$ may be identified by the map $z \mapsto z' = -z/\tau$. Under this identification, $A^{01}_{u'} \mapsto A^{01}_{u}$ if $u' = -u/\tau$. It follows then that the characters $\chi^R_{\tau'}(\tau', e^{2\pi i u'})$ of the current algebra are combinations of the characters $\chi^R_{\tau}(\tau, e^{2\pi i u})$:

$$\chi^R_{\tau'}(\tau', e^{2\pi i u'}) = \sum_R S^R R_{R'}^\tau \chi^R_{\tau}(\tau, e^{2\pi i u}).$$

Hence the modular transformation $\tau \mapsto -1/\tau$ (and more generally, the transformations of $SL(2, \mathbb{Z})$) may be implemented on the characters of the current algebra. The symmetric unitary matrices $(S^R_{R'})$ representing the action of the transformation $\tau \mapsto -1/\tau$
may be expressed explicitly by the characters \(\chi_R\) of the group \(G\) and by the Weyl denominator of Eq. (2.13),

\[
S^R_{R'} = \frac{1}{|T|} |\chi_{R'}(e^{2\pi i \hat{\lambda}/\hat{k}}) \Pi(e^{2\pi i \hat{\lambda}/\hat{k}})| = S^R = S^R_{R'}
\]  

(5.7)
in the notation: \(\hat{\lambda} \equiv \lambda + \rho\) and \(\hat{k} \equiv k + h^{\vee}\) with \(\rho\) the Weyl vector and \(h^{\vee}\) the dual Coxeter number. The normalizing factor \(|\hat{T}|\) is the number of the Cartan group elements of the form \(e^{2\pi i \hat{\lambda}/\hat{k}}\) with \(\lambda\) a weight. \(\frac{1}{4}\) is a fourth root of unity. For the \(SU(2)\) group, the above formula reduces to

\[
S_j = \left(\frac{2}{k+2}\right)^{\frac{1}{4}} \sin \left(\frac{\pi(2j+1)(2j'+1)}{k+2}\right).
\]  

(5.8)

### 5.2 Verlinde dimensions and the fusion ring

The dimensions \(\hat{N}_\Sigma\) of the spaces \(W_\Sigma(\xi, R)\) are independent of the complex structure of \(\Sigma\) and the locations \(\xi\) of the insertion points (but dependent on the level \(k\) of the theory suppressed in the notation). They are given by the Verlinde formula which, in the present context, is a natural generalization of the classical formula for the dimensions \(N_R\) of the spaces \((\otimes V_{Rn})^G\) of group \(G\) invariant tensors. The dimensions \(N_R\) may be computed from the characters of the representations \(R_n\):

\[
N_R = \int_G \prod_n \chi_{Rn}(g) \, dg = \int \prod_n \chi_{Rn}(e^{2\pi i \lambda/k}) \, |\Pi(e^{2\pi i \lambda/k})|^2 \, d\lambda,
\]  

(5.9)

where we have used the relation (2.14). For simple, simply connected groups, the last integral may be taken over the symplex

\[
\Delta_k = \{ \lambda \in t \mid \text{tr}(\alpha^\lambda \gamma) \geq 0 \text{ for } \alpha > 0, \text{ tr}(\phi^\lambda) \leq k \}
\]  

(5.10)

whose elements label the conjugacy classes classes \(C_\lambda\) in a one to one way. Note that the weights in the symplex \(\Delta_k\) are exactly the HW’s integrable at level \(k\), see the definition (4.13). The numbers \(N_{R_n}^{R_1 R_2}\) coincide with the dimensions \(N_{R_1 R_2}\) of the multiplicity spaces in the decomposition (2.18) of the tensor product of representations, i.e. with the structure constants of the character ring \(\mathcal{R}_G\) of the group \(G\), see Eq. (2.19). For example for the \(SU(2)\) group, \(N_{j, j_1}^{j_2} = 1\) if \(|j - j_1| \leq j_2 \leq j + j_1\) and \(j + j_1 + j_2\) is an integer and \(N_{j, j_1}^{j_2} = 0\) otherwise. The ring \(\mathcal{R}_G\) comes with an additive \(Z\)-valued form \(\omega\) given by the integral over \(G\). \(\omega\) assigns to the combination \(\sum n_i \chi_{R_n}\) of characters the coefficient of the character \(\chi_1 = 1\) of the trivial representation \(R = 1\). The dimensions \(N_R\) are the values of \(\omega\) on the product of the characters \(\chi_{R_n}\) in \(\mathcal{R}_G\).

The dimensions \(\hat{N}_\Sigma\) of the spaces \(W_\Sigma(\xi, R)\) are given by the formula

\[
\hat{N}_\Sigma = \frac{1}{|T|} \sum_{\text{weights} \, \lambda \in \Delta_k} \prod_n \chi_{Rn}(e^{2\pi i \lambda/k}) \, |\Pi(e^{2\pi i \lambda/k})|^2, \Sigma
\]  

(5.11)
in the notations from the end of the last subsection and with \( g_\Sigma \) denoting the genus of the surface \( \Sigma \). The above equation is a rewrite of the original Verlinde formula:

\[
\hat{N}_{\Sigma, \xi}^R = \sum_{\bar{R}} \prod_n (S_{R_n}^R / S_1^R) (S_1^R)^{2-g_\Sigma} \tag{5.12}
\]

which may be easily obtained from Eq. (5.11) with the use of the explicit expression (5.7) for the modular matrix \( S_{R_R} \). For the particular case of \( \Sigma = \mathbb{C}P_1 \), Eq. (5.11) is clearly a deformation of Eq. (5.9). More exactly, the sum in Eq. (5.11) is a Riemann sum approximation of the integral in Eq. (5.9). The genus zero 3-point dimensions \( \hat{N}_{R_R R_2}^R \equiv \hat{N}_{R_R}^{R_2} \) give the structure constants of a commutative ring \( \hat{R}_G \) which is additively generated by the representations \( R \) with the HW’s integrable at level \( k \). The \((k\text{-dependent})\) ring \( \hat{R}_G \) is called the fusion ring of the WZW model. For the \( SU(2) \) group and all spins \( \leq k / 2 \), \( \hat{N}_{j,j_1}^{R_2} = 1 \) if \( |j-j_1| \leq j_2 \leq j + j_1 \) and \( j + j_1 + j_2 \) is an integer \( \leq k \) and \( \hat{N}_{j,j_1}^{R_2} = 0 \) otherwise. The fusion ring is a deformation of the character ring \( R_G \). More exactly,

\[
\hat{R}_G \cong R_G / \hat{I}, \tag{5.13}
\]

where \( \hat{I} \) is the \((k\text{-dependent})\) ideal in \( R_G \) composed of the functions vanishing on the Cartan group elements \( e^{2\pi i \lambda / k} \) for weights \( \lambda \in \Delta_k \). The isomorphism identifies the image of \( \chi_R \) in \( R_G / \hat{I} \) with the generator of \( \hat{R}_G \) corresponding to \( R \) for representations \( R \) with integrable HW’s. The coefficient at the generator corresponding to the trivial representation defines an additive \( \mathbb{Z} \)-valued form \( \hat{\omega} \) on \( \hat{R}_G \). For all \( R_n \) with integrable HW’s, the genus zero Verlinde dimensions \( \hat{N}_{\Sigma}^R \) are given by the values of \( \hat{\omega} \) on the image in the fusion ring of the product of the characters \( \chi_{R_n} \). For fixed representations, \( \hat{N}_{R_R R_1}^{R_2} = N_{R_R R_1}^{R_2} \) for sufficiently high \( k \). The fusion ring may be also identified as the character ring of the quantum deformation \( \hat{U}_q(g) \) of the enveloping algebra of \( g \) for \( q = e^{\pi i / (8 + h^\vee)} \), an example of the intricate relations between the WZW model and the quantum groups.

5.3 Holomorphic factorization

Consider now the Ward identity (4.22) for the mirror chiral gauge transformations. The anti-holomorphic maps \( \Phi \) of the space \( A^{10}_1 \) of the \( g^\mathbb{C} \)-valued 1,0-gauge fields \( A^{10}_1 \) with values in \( \bigotimes_n V_{\bar{R}_n} \) such that

\[
\Phi(A^{10}) = e^{-S(h,a^{10})} \bigotimes_n h^t_{\bar{R}_n} (\xi_n) \Phi(h^{10})
\]

are the complex conjugates of the holomorphic maps \( \Psi \) satisfying the relation (5.1):

\[
\Phi(A^{10}) = \overline{\Psi((-A^{10})^*)}, \tag{5.14}
\]

where the star denotes the anti-linear involution of the complexified Lie algebra \( g^\mathbb{C} \) leaving \( g \) invariant (it coincides with the hermitian conjugation for \( g = su(N) \)). It follows that the correlation functions, in their dependence of the external gauge field,
are sesqui-linear combinations of the elements of the space $W_{\Sigma}(\xi, R)$ of holomorphic solutions of Eq. (5.1):

$$Z_{\Sigma}(A) < \otimes_n g_{R_n}(\xi_n) >_{\Sigma}(A) = H^{\alpha\beta} \Psi_{\alpha}(A^{01}) \otimes \overline{\Psi_{\beta}(- (A^{01})^*)},$$

(5.15)

where the states $\Psi_{\alpha}$ form a basis of $W_{\Sigma}(\xi, R)$ and the right hand side should be summed over $\alpha$ and $\beta$. The equality involves the natural identification of the vector spaces $(\otimes_n V_{R_n}) \otimes (\otimes_n \overline{V_{R_n}}) \cong \otimes_n End(V_{R_n})$. The partition function $Z(A)$ has to be given by similar expressions pertaining to the case without insertions. In particular, on the complex torus $T_\tau$, and in the vanishing gauge field,

$$Z_{T_\tau} = \sum_{\hat{R}, \hat{R}'} H_{\hat{R}, \hat{R}'} \text{ch}_{\hat{R}}(\tau, 1) \overline{\text{ch}_{\hat{R}'}(\tau, 1)}. \quad (5.16)$$

The matrices $(H^{\alpha\beta})$ should be specified for a given choice of the basis $(\Psi_{\alpha})$ for each complex structure on the surface and for each configuration of the insertions points so that, if we did not have means to compute them, the above formulae would mean little progress towards the solution of the WZW theory. Fortunately, there exist effective ways to determine the coefficients $H^{\alpha\beta}$.

### 5.4 Scalar product of the Chern-Simons states

It was argued in [9], see also [10] for a formal functional integral argument, that the matrices $(H^{\alpha\beta})$ appearing in Eq. (5.15) are inverse to the matrices $(H_{\alpha\beta})$ with matrix elements

$$H_{\alpha\beta} = (\Psi_{\alpha}, \Psi_{\beta})$$

given by the scalar product of the CS states. According to the rules of holomorphic quantization, the latter is given by the functional integral

$$(\Psi, \Psi') = \int_{A^{01}} (\Psi(A^{01}), \Psi'(A^{01})) \otimes_{V_{R_n}} e^{-\frac{1}{2\pi} \|A\|^2} DA \quad (5.17)$$

over the $g$-valued gauge fields $iA = i(- (A^{01})^* + A^{01})$ with $\|A\|^2 \equiv i \int_{T_\tau} \text{tr}(A^{01})^* A^{01}$. This is again a formal expression. The point is, however, that the $DA$-integral may be calculated exactly by reducing it to doable Gaussian (i.e. free field) functional integrals. Ones this is done, the exact solution for the correlation functions follows by Eq. (5.15). Note that the above solution for coefficients $H^{\alpha\beta}$ guarantees that the right hand side of Eq. (5.13) is independent of the choice of a basis of the CS states. Let us briefly sketch how one achieves the reduction of the integral (5.17) to the free field ones.

In the first step, the integral (5.17) may be rewritten by a trick resembling the Faddeev-Popov treatment of gauge theory functional integrals. The reparametrization of the gauge fields

$$A^{01} = h^{-1} A^{01}(n) \quad (5.18)$$

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by the chiral $G^C$-valued gauge transforms of a (local) slice $n \mapsto A^{01}(n)$ in the space $\mathcal{A}^{01}$ cutting each gauge orbit in one point, see Fig. 10,

$$\mathcal{A}^{01}$$

$$\mathcal{A}^{01}/\mathcal{G}^C$$

$n$

$$\mathcal{G}^C$$

permits to rewrite the functional integral expression for the norm squared of a CS state in the new variables as

$$\|\Psi\|^2 = \int (\Psi(A^{01}(n)), \otimes (hh^*)^{-1} \Psi(A^{01}(n)))_{\mathcal{R}_n} e^{(k+2h^\gamma)(S(hh^*,A^{01})(n))} D(hh^*) \ d\mu(n). \ (5.19)$$

To obtain the expression on the right hand side, we have used Eq. (5.1). The term $2h^\gamma S(hh^*)$ in the action comes from the Jacobian of the change of variables (5.18) contributing also to the measure $d\mu(n)$ on the local slice in $\mathcal{A}^{01}$.

Unlike in the standard Faddeev-Popov setup, the integral over the group of gauge transformations did not drop out since the integrand in Eq. (5.17) is invariant only under the $G$-valued gauge transformations. Instead, we are left with a functional integral (5.19) similar to the one for the original correlation functions, see Eq. (4.19), except that it is over fields $hh^*$. These fields may be considered as taking values in the contractible hyperbolic space $G^C/G$. $D(hh^*)$ is the formal product of the $G^C$-invariant measures on $G^C/G$. The gain is that the functional integral (5.19) for the hyperbolic WZW model correlation functions is doable. For example for $G = SU(2)$ and for $\Sigma = CP^1$, where we may set $A^{01}(n) = 0$ (in this case the gauge orbit of $A^{01} = 0$ gives already a dense open subset $\mathcal{A}^{01}$),

$$S(hh^*) = -\frac{i}{2\pi} \int \partial \phi \wedge \bar{\partial} \phi - \frac{i}{2\pi} \int (\partial + \bar{\partial} \phi) \bar{v} \wedge (\bar{\partial} + \bar{\partial} \phi)v$$

in the Iwasawa parametrization $h = (e^{\phi/2} e^{-\phi/2}) (\begin{smallmatrix} 1 & v \\ 0 & 1 \end{smallmatrix}) u$ of the 3-dimensional hyperboloid $SL(2,\mathbb{C})/SU(2)$ by $\phi \in \mathbb{R}$ and $v \in \mathbb{C}$, with $u \in SU(2)$. Although the action is not quadratic in the fields, it is quadratic in the complex field $v$ so that the $v$ integral can be explicitly computed. Somewhat miraculously, the resulting integral appears to depend on the remaining field $\phi$ again in a Gaussian way so that the integration may be carried out further. The same happens for more complicated groups and on surfaces with handles, except that the procedure requires more steps. At the end, one obtains explicit finite-dimensional integrals. Hence, the integral (5.19) belongs to the class of functional integrals that may be explicitly evaluated. The Gaussian functional integrals encountered in its computation require mild renormalizations (the zeta-function...
or similar regularization of determinants, Wick ordering of insertions) but these are well understood. They are responsible for the mild non-invariance of the WZW correlation functions under the local rescalings of the metric, leading to the values of the Virasoro central charge and of the conformal dimensions discussed above.

On the complex tori $T_\tau$ with no insertions and in the constant metric $|dz|^2$, the scalar product of the CS states takes a particularly simple form: the current algebra characters $\chi^R(\tau, \cdot)$ appear to give an orthonormal basis of the space of states. It follows that the toroidal partition function in the constant metric is

$$Z(\tau) = \sum_R |\chi^R(\tau, 1)|^2,$$

see Eq. (5.16). The exact normalization of the constant metric on $T_\tau$ is not important since at genus one constant rescalings of the metric, exceptionally, do not effect the partition functions. The latter fact has an important consequence. It implies that the partition function $Z(\tau)$ has to be a modular invariant:

$$Z(\tau) = Z(\frac{a\tau + b}{c\tau + d})$$

for $(a, b, c, d) \in SL(2, \mathbb{Z})$. This is indeed the property of the right hand side of Eq. (5.20) since the matrices implementing the modular transformations on the characters of the current algebra are unitary.

Explicit finite-dimensional integral formulae for the scalar product (5.17) have been obtained for general groups at genus zero and one and, for $G = SU(2)$, for higher genera. It is clear that the case of general group and genus $>1$ could be treated along the same lines, but the explicit calculation has not been done. It should be also said that the general proof of the convergence of the resulting finite-dimensional integrals is also missing, although several special cases have been settled completely.

### 5.5 Case of $G = SU(2)$ at genus zero

To give a feeling about the form of the explicit expressions for the scalar product of the CS states, let us describe the result for $G = SU(2)$ and $\Sigma = \mathbb{C}P^1$ with insertions at points $z_n$ in the standard complex coordinate $z$. In this case, as we have discussed above, the CS states correspond to invariant tensors $v$ in $(\otimes^n V_j)^{SU(2)}$, where we label the irreducible representations of $SU(2)$ by spins. The spin $j$ representation acts in the space $V_j$ spanned by the vectors $(\sigma^-)^\ell v_j$ with $\ell = 0, 1, \ldots, 2j$, where $v_j$ is the normalized HW vector annihilated by $(\sigma_+)$, with $\sigma_i$ denoting the Pauli matrices. For the scalar product of the CS states, the procedure described in the previous subsection gives the following integral expression:

$$\|v\|^2 = f(\sigma, z, j, k) \int_{\mathbb{C}^j} \left| (v, \omega(z, y))_{\otimes V_j} e^{-i\pi U(z, y)} \right|^2 \prod_{a=1}^j d^2 y_a.$$
Let us explain the terms on the right hand side. First, 

\[ f(\sigma, z, j, k) = e^{-\frac{1}{2\pi i} \int \partial \sigma \bar{\partial} \sigma \left( \frac{\det'(-\Delta)}{\text{area}} \right)^{3/2} \prod_n e^{2j_n(j_n+1)/k+2} \sigma(z_n)} \]  

(5.22)

carries the dependence on the metric \( e^{2\sigma |dz|^2} \) on \( \mathbb{C}P^1 \), with \( \det'(-\Delta) \) denoting the zeta-function regularized determinant of the (minus) Laplacian on \( \mathbb{C}P^1 \) with omission of the zero eigenvalue. The \( \sigma \)-dependence of \( \left( \frac{\det'(-\Delta)}{\text{area}} \right)^{3/2} \) is given by a term \( e^{\frac{1}{4} \pi i \int \partial \sigma \bar{\partial} \sigma \left( \det'(-\Delta) \right)} \) leading altogether to the value \( \frac{3k}{k+2} \) of the Virasoro central charge of the WZW theory (recall that this is the inverse of the scalar product that enters the WZW correlation functions). Similarly, the conformal weight \( \Delta_j \) of the fields \( \phi_j(\xi) \) of the \( SU(2) \) WZW theory may be read from Eq. (5.22) to be \( j(j+1)/k+2 \). Next, \( \omega(\vec{z}, \vec{y}) \) is a meromorphic \( \otimes V_{jn} \)-valued function of \( \vec{z} \) and \( \vec{y} = (y_1, \ldots, y_J) \), where \( J = \sum_n j_n \):

\[ \omega(\vec{z}, \vec{y}) = \prod_{r=1}^J \left( \sum_n \frac{1}{y_r - z_n} (\sigma_+)_{j_n} \right) \otimes v_{jn}. \]  

(5.23)

Finally, \( U(\vec{z}, \vec{y}) \) is a multivalued function

\[ U(\vec{z}, \vec{y}) = \sum_{n<m} j_n j_m \ln(z_n - z_m) - \sum_{n,r} j_n \ln(z_n - y_r) + \sum_{r<s} \ln(y_r - y_s). \]  

(5.24)

The integral (5.21) is over a positive density with singularities at the coinciding \( y_r \)'s and the question arises as to whether it converges. A natural conjecture states that the integral is convergent if and only if the invariant tensor \( v \) is in the image of the space of states \( \mathcal{W}(\mathbb{C}P^1, \vec{z}, \vec{j}) \) explicitly described as the set of \( v \in \left( \otimes V_{jn} \right)^{SU(2)} \) such that

\[ (\otimes v_{jn}, \prod_n (\sigma_+)_{j_n} e^{z_n(\sigma_+)_{j_n}} v)_{\otimes v_{jn}} = 0 \quad \text{for} \quad \sum p_n \leq J - k - 1. \]  

(5.25)

In particular, for two or three points, the image does not depend on the location of the insertions and gives the whole space of invariant tensors if \( \sum j_n \leq k \) and zero otherwise. In this case, the integrals in Eq. (5.21) may indeed be computed in a closed form confirming the conjecture. Numerous other special cases have been checked. In general, the “only if” part of the conjecture is easy but the “if” part remains to be verified.

### 5.6 Knizhnik-Zamolodchikov connection

There is another way to construct the matrices \( (H^{\alpha\beta}) \) entering the formula (5.13) for the correlation functions. Let us describe it briefly.

The spaces \( \mathcal{W}_\Sigma(\xi, R) \) of the CS states depend on the complex structure of the surface \( \Sigma \) and on the insertion points. They form, in a natural way, a holomorphic vector bundle \( \mathcal{W} \) whose base is the space of complex structures on a given smooth surface \( \Sigma \) and of non-coinciding insertions \( \xi \) (modulo diffeomorphisms). The scalar product of the CS states equips this bundle with a hermitian structure. In turn, a hermitian structure on
a holomorphic vector bundle determines a unique connection such that the covariant derivatives of the structure and of the holomorphic sections vanish. Although the scalar product of the CS states has been rigorously defined in the general case only modulo the convergence of finite-dimensional integrals (see the end of the last subsection), the connection on the bundle of the states may be easily constructed with full mathematical rigor. It appears to be projectively flat (i.e. with a curvature that is a scalar 2-form on the base space), a crucial fact. In other words, the parallel transport of a CS state around a closed loop in the space of complex structures and insertions at most changes its normalization.

For the genus zero case, there is only one complex structure modulo diffeomorphisms. If we fix it as the standard complex structure on $\mathbb{C}P^1$, then we are only left with the freedom to move the insertion points $z$. The bundle $\mathcal{W}$ is in this case a subbundle of the trivial bundle with the fiber formed by the invariant group tensors $(\otimes^n V_{\mathfrak{h}})^G$ and the connection may be extended to this trivial bundle. The covariant derivatives of the sections of the latter are given explicitly by the equations:

$$\nabla_{z_n} v = \partial_{z_n} v, \quad \nabla_{\bar{z}_n} v = \partial_{z_n} v + \frac{2}{k+h} \sum_{m \neq n} \frac{t^a_{z_m} t^a_{z_n}}{z_m-z_n} v$$

which go back to the work $[3]$ of Knizhnik-Zamolodchikov on the WZW theory. In fact the above connection is flat as long as the insertion points stay away from infinity and the article $[3]$ studied their horizontal sections such that $\nabla v = 0$. The higher genus generalizations of these equations were first considered by Bernard $[11][12]$ and we shall call the connection on the bundle $\mathcal{W}$ the Knizhnik-Zamolodchikov (KZ) connection for genus zero or the Knizhnik-Zamolodchikov-Bernard (KZB) one for higher genera.

In general, the KZB connection can be made flat by some choices (as in the case of genus zero, where the curvature has been concentrated at infinity). For a flat connection, one may choose locally a basis $(\Psi_\alpha)$ of horizontal sections. The gain from such a choice of the basis of the CS states is that the coefficients $H^{\alpha\beta}$ in Eq. (5.13) become then independent of the complex structure or the positions of the insertions. Indeed, since $H^{\alpha\beta} = (\Psi_\alpha, \Psi_\beta)$ and the KZB connection preserves the scalar product of the states, the above scalar products are constant for horizontal $\Psi_\alpha$. Since the horizontal sections are, in particular, holomorphic, Eq. (5.13) gives then a holomorphic factorization of the correlation functions into sesqui-linear combinations of the conformal blocks holomorphic in their dependence on the complex structure and positions of insertions. Such a finite factorization is the characteristic feature of rational CFT’s. As we see, the conformal blocks of the WZW theory are given by the horizontal sections $\Psi_\alpha$ of the bundle $\mathcal{W}$ of the CS states. For example, at genus zero with no insertions, the conformal blocks are formed by the characters of the current algebra and Eq. (5.20) provides a particular realization of the holomorphic factorization.

Since the KZB connection, although flat, has nevertheless a non-trivial holonomy, the conformal blocks are, in general, multivalued. The coefficients $H^{\alpha\beta}$ in Eq. (5.13) may then be fixed, up to normalization, by demanding that the correlation functions be uni-valued. The overall normalization may be fixed, in turn, by considering the limits
when the insertion points coincide. This was the strategy used in the original work [3] to compute the 4-point correlation function of the spin $\frac{1}{2}$ field $g_{\frac{1}{2}}(\xi)$ of the $SU(2)$ WZW model on the Riemann sphere. The horizontality relations for the conformal blocks reduce in this case to the hypergeometric equation and the calculations of the conformal blocks and of their monodromy are easy to perform. For general genus-zero conformal blocks, one obtains generalizations of the hypergeometric equation whose solutions may be expressed by contour integrals [13][14]. The latter are, essentially, the holomorphic versions of the integrals (5.21) so that the two strategies to obtain explicit solutions for the correlation functions, one based on the study of the monodromy of the conformal blocks and the other one involving a calculation of the scalar product of the CS states, are closely related.

There appears to be a very rich structure behind the connection (5.26) and its generalizations. It is closely related to the integrable systems of mechanics and statistical mechanics, see e.g. [21][22]. The holonomy of the connection gives representations of the braid groups which played an important role in the construction of the Jones-Witten invariants of knots and 3-manifold invariants [1]. The perturbative solutions of the horizontality equations enter the Vasiliev invariants of knots [23]. The KZ connection is also closely connected to quantum groups [24] and to Drinfel’d’s quasi-Hopf algebras [25], to mention only some interrelated topics.

6 Coset theories

There is a rich family of CFT’s which may be obtained from the WZW models by a simple procedure known under the name of a coset construction [13][16]. On the functional integral level, the procedure consists of coupling the $G$-group WZW theory to a gauge field $iB = i(B_{10} + B_{01})$ with values in a subalgebra $h \subset g$. The field $B$ is then integrated over with gauge-invariant insertions [17][18][19][20]. Let $H \subset G$ be the subgroup of $G$ corresponding to $h$. Choose elements $t_n$ in the space $(\text{End}(V_{R_n}, V_{r_n}))^H$ of the intertwiners of the action of $H$ in the irreducible $G$- and $H$-representation spaces. The simplest correlation functions of the $G/H$ coset theory take the form

$$\langle \prod_{i=1}^{n} \text{tr}_{V_{r_n}}(t_n g_{R_n}(x_n) t_n^\dagger) \rangle_\Sigma = \frac{1}{Z_{G/H}^{\Sigma}} \int \prod_n \text{tr}_{V_{r_n}}(t_n g_{R_n}(x_n) t_n^\dagger) e^{-kS(g,B)-\frac{k}{\pi} \|B\|^2} Dg DB, \quad (6.1)$$

where $Z_{G/H}^{\Sigma} = \int e^{-kS(g,B)-\frac{k}{\pi} \|B\|^2} Dg DB$ is the partition function of the $G/H$ theory. Note that the $g$-field integrals are the ones of the WZW theory and are given by Eq. (5.15). Consequently, $Z_{G/H}^{\Sigma}$

$$Z_{G/H}^{\Sigma} = \prod_{n} \text{tr}_{V_{r_n}}(t_n g_{R_n}(x_n) t_n^\dagger) \rangle_\Sigma = H^{\alpha \beta} \int (\otimes t_n \Psi_\beta(B_{01}), \otimes t_n \Psi_\alpha(B_{01}))_{\otimes V_{r_n}} e^{-\frac{k}{\pi} \|B\|^2} DB. \quad (6.2)$$
The composition with $\otimes t_n$ defines a linear map $T$ between the spaces of the group $G$ and the group $H$ CS states, i.e. $T : W_\Sigma(\xi, R) \rightarrow W_\Sigma(\xi, \bar{R})$ with
\[
(T \Psi)(B^{01}) = \otimes t_n \Psi(B^{01}).
\] (6.3)

Indeed, it is straightforward to check that the right hand side satisfies the the group $H$ version of the Ward identity (5.1). Eq. (6.2) may be rewritten with the use of the map $T$ as
\[
Z_{G/H}^{\Sigma} < \prod_n \text{tr}_{V_{n+1}}(t_n g_{R_n}(x_n) t_n^\dagger) >_\Sigma = H^{\alpha\beta} (T \Psi_\beta, T \Psi_\alpha) = \text{tr}_{W_\Sigma(\xi, \bar{R})} T^\dagger T.
\] (6.4)

Let $(T^\mu_\alpha)$ denote the ("branching") matrix of the linear map $T$ in the bases $(\Psi_\alpha)$ and $(\psi_\mu)$ of, respectively, $W_\Sigma(\xi, R)$ and $W_\Sigma(\xi, \bar{R})$, i.e. $T \Psi_\alpha = T^\mu_\alpha \psi_\mu$. Then
\[
Z_{G/H}^{\Sigma} < \prod_n \text{tr}_{V_{n+1}}(t_n g_{R_n}(x_n) t_n^\dagger) >_\Sigma = H^{\alpha\beta} T^\mu_\beta h_{\mu\nu} T^\nu_\alpha,
\]
where $h_{\mu\nu} = (\psi_\mu, \psi_\nu)$. Since the above relations hold also for the partition function, it follows that the calculation of the coset theory correlation functions (5.11) reduces to that of the scalar products of group $G$ and group $H$ CS states, both given by explicit, finite-dimensional integrals.

Among the simplest examples of the coset theories is the case with the group $G = SU(2) \times SU(2)$ with level $(k, 1)$ (for product groups, the levels may be taken independently for each group) and with $H$ being the diagonal $SU(2)$ subgroup. The resulting theories coincide with the unitary minimal series of CFT’s with the Virasoro central charges $c = 1 - \frac{6}{(k+2)(k+3)}$, first considered by Belavin-Polyakov-Zamolodchikov [1]. The Hilbert spaces of these theories are built from irreducible unitary representations of the Virasoro algebras with $0 < c < 1$. The simplest one of them with $k = 1$ and $c = \frac{1}{2}$ describes the continuum limit of the Ising model at critical temperature or the scaling limit of the massless $\phi^4_2$ theory. In particular, in the continuum limit the spins in the critical Ising model are represented by fields $\text{tr} g_+(\xi)$ where $g$ takes values in the first $SU(2)$. The corresponding correlation functions may be computed as above. One obtains this way for the 4-point function on the complex plane (or the Riemann sphere) an explicit expression in terms of the hypergeometric function.

The $G/H$ coset theory with $H = G$ is a prototype of a two-dimensional topological field theory. As follows from Eq. (5.4), the correlation functions of the fields $\text{tr} g_+(\xi)$ are equal to the dimension of the spaces $W_\Sigma(\xi, R)$, normalized by the dimension of $W_\Sigma(\emptyset, \emptyset)$ and are given by the Verlinde formula (5.11). In particular, they do not depend on the position of the insertion points, a characteristic feature of the correlation functions in topological field theories.

7 Boundary conditions in the WZW theory

Discussing in Sects. 3.4 and 4.1 above the classical and the quantum amplitudes for the WZW model on surfaces with boundary, we have admitted an arbitrary behavior
of the fields on the boundary. In physical situations, one often has to constrain this behavior by imposing the boundary conditions (BC) on the fields. The simplest example is provided by the Dirichlet or Neumann BC’s for the free fields which fix to zero, respectively, the tangent or the normal derivative of the field (the absorbing versus the reflecting condition). Such conditions leave unbroken an infinite-dimensional set of symmetries of the free field theory. We shall be interested in BC’s in the WZW model with a similar property.

The theory of boundary CFT’s was pioneered by Cardy and Cardy-Lewellen. It found its applications e.g. in the theory of isolated impurities in condensed matter physics (in the so called Kondo problem, a traditional playground for theoretical ideas). In string theory, the use of the Neumann BC for free open strings has a long tradition. The realization that one should also consider free open strings with the Dirichlet BC came much later and gave rise to a theory of Dirichlet- or D-branes: the end of an open string, some of whose coordinates are restricted by the Dirichlet BC, moves on a surface (brane) of a lower dimension. The D-branes provide the basic tool in the analysis of the non-perturbative effects in string theory: of the stringy solitons and of the strong-weak coupling dualities. The general theory of boundary CFT’s is slowly becoming an important technique of string theory. Here, for the sake of illustration, we shall discuss a particular class of BC’s for the WZW theory. These conditions constrain the group $G$ valued field $g$ to stay over the boundary components in fixed conjugacy classes of $G$. Such BC’s were discussed in, see also. They clearly generalize the Dirichlet BC of the free fields, contrary to the claim in (based on the conventions of reference) associating them to the Neumann BC. Our presentation, along similar lines, clarifies, hopefully, some of the discussions of the above papers.

### 7.1 The action

As before, we shall represent a Riemann surface $\Sigma$ with boundary as $\Sigma' \setminus \bigcup_m \tilde{D}_m$, where $D_m$ are disjoint unit discs embedded in a closed surface $\Sigma'$ without boundary, see Fig. 6. We have seen in Sect. 3.4 that the classical amplitudes $e^{-S(g)}$ of the fields $g : \Sigma \to G$ of the WZW model take values in a line bundle $L$ rather than being numbers. The line bundle $L$ over the loop group $LG$ is not trivial but it may be trivialized over certain subsets of $LG$, for example the ones formed by the loops taking values in special conjugacy classes of $G$. Such BC’s were discussed in, see also. They clearly generalize the Dirichlet BC of the free fields, contrary to the claim in (based on the conventions of reference) associating them to the Neumann BC. Our presentation, along similar lines, clarifies, hopefully, some of the discussions of the above papers.

In order to achieve this goal, we shall fix the 2-forms

$$\omega_\lambda = \text{tr} \left( (g^{-1}dg)(1 - Ad_g)^{-1}(g^{-1}dg) \right) = \text{tr} \left( (g_0^{-1}d_{g_0})e^{2\pi i\lambda/k}(g_0^{-1}d_{g_0})e^{-2\pi i\lambda/k} \right)$$

on the conjugacy classes $\mathcal{C}_\lambda$ composed of the elements $g = g_0 e^{2\pi i\lambda/k}g_0^{-1}$ (the operator $(1 - Ad_g)$ is invertible on the vectors tangent to $\mathcal{C}_\lambda$). It is easy to check by a direct
calculation that $d\omega_\lambda$ coincides with the restriction of the 3-form $\chi = \frac{1}{3} \text{tr} (g^{-1}dg)^3$ to $C_\lambda$.

Since the conjugacy classes in a simply connected group $G$ are simply connected, any field $g : \Sigma \to G$ satisfying the BC’s (7.1) may be extended to a field $g' : \Sigma' \to G$ in such a way that $g'(D_m) \subset C_{\lambda m}$ and then to a field $\tilde{g}$ on a 3-manifold $B$ such that $\partial B = \Sigma'$, see Fig. 11.

![Fig. 11](image1)

Having done this, we define the Wess-Zumino part of the action as

$$S^W_{\Sigma}(g) = \frac{k}{4\pi i} \int_B \tilde{g}^* \chi - \frac{k}{4\pi} \sum_m \int_{D_m} \tilde{g}|_{D_m}^* \omega_{\lambda m}.$$  

The ambiguities in this definition are the values of the integrals

$$\frac{k}{4\pi i} \int_{\tilde{B}} \tilde{g}^* \chi - \frac{k}{4\pi} \sum_m \int_{S^2_m} \tilde{g}|_{S^2_m}^* \omega_{\lambda m} \quad (7.3)$$

for 3-manifolds $\tilde{B}$ with $\partial \tilde{B} = \bigcup_m S^2_m$ and for maps $\tilde{g} : \tilde{B} \to G$ such that $\tilde{g}(S^2_m) \subset C_{\lambda m}$, see Fig. 12.

![Fig. 12](image2)

In other words, they are proportional to the periods of $(\chi, \omega)$ over the cycles of the relative integer homology $H_3(G, \bigcup_m C_{\lambda m})$, as noticed in [32].
It is not difficult to get a hold on these ambiguities. Let us glue the unit 3-balls $B_m$ to $\tilde{B}$ along the boundary spheres $S^2_m$ to obtain a 3-manifold $\tilde{B}'$ without boundary and let us extend $\tilde{g}$ to a map $\tilde{g}' : \tilde{B}' \to G$. The expression (7.3) may be now rewritten as

$$\frac{k}{4\pi^2} \int_{\tilde{B}'} \tilde{g}'^* \chi - \frac{k}{4\pi^2} \sum_m \left( \int_{B_m} \tilde{g}'_m^* \chi - \int_{\partial B_m} \tilde{g}'_m^* \omega \right),$$

where $\tilde{g}'_m$ are the restrictions of $\tilde{g}'$ to $B_m$ and they satisfy $\tilde{g}'_m(\partial B_m) \subset C_m$. As we have discussed in Sect. 3.3, the first term, involving the integral over the 3-manifold without boundary $\tilde{B}'$, takes values in $2\pi i \mathbb{Z}$ as long as $k$ is an integer.

Let us consider the other terms. For $G = SU(2) = \{ x_0 + ix_i \sigma_i \mid x_0^2 + x_i^2 = 1 \} \cong S^3$, the conjugacy classes corresponding to $\lambda = j \sigma_3$, with $0 \leq 2j \leq k$, are the 2-spheres with $x_0 = \cos \frac{2\pi j}{k}$ fixed (except for $j = 0$ or $\frac{k}{2}$ corresponding to the center elements). They are boundaries of two 3-balls $B_j$ and $B'_j$ with $x_0 \geq \cos \frac{2\pi j}{k}$ and $x_0 \leq \cos \frac{2\pi j}{k}$, see Fig. 13.

A direct calculation shows that

$$\frac{k}{4\pi^2} \left( \int_{B_j} \chi - \int_{\partial B_j} \omega \right) = -4\pi i \bar{j}. \quad (7.4)$$

If we used $B'_j$ instead of $B_j$, the result would be $4\pi i \left( \frac{k}{2} - j \right)$. We infer that $\bar{j}$, between 0 and $\frac{k}{2}$, must be an integer or a half-integer for the ambiguity to belong to $2\pi i \mathbb{Z}$. This result has already been stated in [33].

For the other groups, the restrictions come from the the 2-spheres in $C_\lambda$ of the form

$$\{ g_0 e^{2\pi i \lambda/k} g_0^{-1} \mid g_0 \in SU(2)_\alpha \}, \quad (7.5)$$

where $SU(2)_\alpha$ is the $SU(2)$ subgroup of $G$ corresponding to a root $\alpha$, see Sect. 3.3. Decomposing $\lambda = (\lambda - \frac{1}{2} \alpha \vee \text{tr}(\alpha \lambda)) + \frac{1}{2} \alpha \vee \text{tr}(\alpha \lambda)$, we observe that the first term commutes with the generators $\alpha \vee$, $e_{\pm \alpha}$ of the Lie algebra of $SU(2)_\alpha$ and plays a spectator

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**Fig. 13**
role. The calculation of the ambiguity is now essentially the same as for $G = SU(2)$ with $j$ replaced by $\frac{1}{2} \text{tr} (\alpha \lambda)$ and an overall factor $\frac{2}{2 \text{tr} \alpha^2}$ due to the different normalization of $\text{tr} \alpha^2$. We infer the condition

$$\frac{2}{2 \text{tr} \alpha^2} \text{tr} (\alpha \lambda) = \text{tr} (\alpha^\gamma \lambda) \in \mathbb{Z}.$$ 

Since the conjugacy classes $C_\lambda$ are in one to one correspondence with $\lambda$ in the symplex (5.10), the admissible conjugacy classes are in one to one correspondence with the HW’s $\lambda$ integrable at level $k$, see the definition (4.9).

The full action $S_\Sigma(g)$ of the boundary WZW model is still obtained by adding to the WZ action $S^{WZ}$ the $S^\gamma$ term of Eq. (3.6). The coupling to the gauge field is given again by Eq. (3.27). The behavior of the complete action under the chiral gauge transformation may be shown to obey the following BC version of the Eq. (3.28):

$$S(g, A) = S(h_1 g h_2^{-1}, h^2 A^{01} + h^1 A^{01}) + S(h_1^{-1} h_2, A) - \frac{ik}{2\pi} \int_\Sigma (h^2 A^{10} h^1 A^{01}),$$

provided that $g$ satisfies the BC’s (7.1) and that $h_1|_{\partial D_m} = h_2|_{\partial D_m}$. Note that under this conditions, the field $h_1 g h_2^{-1}$ is constrained on the boundary to the same conjugacy classes as $g$ and $h_1 h_2^{-1}$ to the trivial one so that the actions on the right hand side make sense. The above relation will be employed below to infer the chiral gauge symmetry Ward identities for the boundary WZW theory. It may be used as the principle that selects the BC’s (7.1).

Summarizing: if the field $g : \Sigma \to G$ satisfies the BC’s (7.1) with integrable weights $\lambda_m$, then the amplitude $e^{-S^{WZ}_\Sigma(g)}$, and consequently also $e^{-S_\Sigma(g)}$ and $e^{-S_\Sigma(g, A)}$, may be well defined as complex numbers. Of course, the mixed case, where the BC’s (7.1) with integrable $\lambda_m$ are satisfied only on some boundary components of $\Sigma$ and no conditions are imposed on the other (“free”) boundary components can be treated in the same way. It results in the amplitudes

$$e^{-S_\Sigma(g, A)} \in \otimes^\infty_{n \text{ free}} \mathcal{L}_{g|_{\partial D_m}}.$$ 

### 7.2 Quantum amplitudes and correlation functions

The functional integral definition (11.1) of the quantum amplitudes of the WZW model may be naturally generalized to the case where on some boundary components of $\Sigma$ we impose the BC’s (7.1) with integrable weights $\lambda_m$. The resulting amplitudes $A_{\Sigma, \lambda}(A)$ will be now elements of $\otimes^\infty_{n \text{ free}} \Gamma(\mathcal{L})$. They may be represented as (partial) contractions of the amplitudes $A_{\Sigma}(A)$ with all boundaries free with appropriate states:

$$A_{\Sigma, \lambda}(A) = (\otimes_m \delta_{\lambda_m}, A_{\Sigma}(A)).$$ 

The non-normalizable states $\hat{\delta}_\lambda$ are given by Cardy’s formula (89):

$$\hat{\delta}_\lambda = \sum_R \langle S^R \rangle - \frac{1}{2} S^R \hat{e}^\Lambda \otimes \hat{e}^\Lambda_R.$$  

(7.7)
where $R_\lambda$ denotes the representation of $G$ with the HW $\lambda$, the vectors $\hat{e}_\lambda^R$ form an orthonormal basis of the space $V_\lambda$ and the sum over $\hat{\imath}$ is understood. Note the analogy with Eq. (2.11) for the delta-function supported by a conjugacy class $C_\lambda$. The matrix $(S^R_\lambda)$ has replaced the one with the elements $(\chi_\lambda(e^{2\pi i /k}))$ and the representation spaces $V_\lambda$ of the current algebras those of the finite-dimensional group. The state $\hat{\delta}_\lambda$ should be interpreted as a delta-function concentrated on the loops in $LG$ contained in the conjugacy class $C_\lambda$. The non-normalizable states $\hat{e}_\lambda^R \otimes \hat{e}_\lambda^R$ are called the Ishibashi states $[35]$ and generalize the (properly normalized) characters of the group, see Eq. (2.10).

The correlation functions $< \otimes g_{R_n}(\xi_n) >_{\Sigma \lambda}(A)$ in the presence of the boundaries with fields constrained by the the BC’s (7.1) may be again defined by the functional integral (4.19) taking numerical values. The transformation property (7.6) of the action implies now that

$$Z_{\Sigma, \lambda}(A) < \otimes g_{R_n}(\xi_n) >_{\Sigma \lambda}(A) = e^{-S(h_1^{-1}h_2, A)} + \frac{1}{\pi h_2} \int_{\Sigma} tr(h_2 A_{10} + h_1 A_{01}) Z_{\Sigma, \lambda}(h_2 A_{10} + h_1 A_{01})$$

$$\cdot \otimes \eta^{-1}(\xi_n) < \otimes g_{R_n}(\xi_n) >_{\Sigma \lambda}(h_2 A_{01} + h_1 A_{01}) \otimes (h_2)_{R_n}(\xi_n). \quad (7.8)$$

This is the chiral gauge symmetry Ward identity for the boundary WZW correlation functions, a variant of the identities (4.21) and (4.22) in presence of the BC’s. Note, however, that the identity (7.8) may be factorized as the latter ones only if we assume that $h_1$ and $h_2$ are equal to 1 on the boundary. The general case where on the boundary $h_1 = h_2$ requires the presence of both gauge transformations $h_1$ and $h_2$.

It is illuminating to rewrite the Ward identity (7.8) in a different way. To this end, let us define a “doubled” Riemann surface without boundary $\tilde{\Sigma}$ by gluing $\Sigma$ to its complex conjugate $\Sigma$ along the boundary components, see Fig. 14.

We shall denote by $\iota$ the anti-holomorphic involution of $\tilde{\Sigma}$ exchanging $\Sigma$ with its complex conjugate: $\iota(\xi) = \bar{\xi}$. Each chiral gauge field $\tilde{A}_{01}$ on the Riemann surface $\tilde{\Sigma}$...
defines a complexified gauge field \( A = (\iota^* A^01 + A^01)|_\Sigma \) on the surface \( \Sigma \). Let us define a holomorphic functional

\[
\Psi_\lambda : \tilde{A}^{01} \rightarrow (\otimes_n V_{R_n}) \otimes (\otimes_n \pi_{\bar{R}_n})
\]

of the chiral gauge fields on the doubled surface by

\[
\Psi_\lambda (\tilde{A}^{01}) = \mathcal{Z}_{\Sigma,\lambda}(A) < \otimes_n g_{R_n}(\xi_n) >_{\Sigma,\lambda}(A),
\]

where we identify the space \((\otimes V_{R_n}) \otimes (\otimes \pi_{\bar{R}_n})\) with \(\otimes \text{End}(V_{R_n})\). Let \(\tilde{\eta}\) be a \(G^C\)-valued gauge transformation on \(\tilde{\Sigma}\). We shall pose \(h_1 = \tilde{\eta}|_\Sigma\) and \(h_2 = \iota^* \tilde{\eta}|_\Sigma\). Note that \(h_1 = h_2\) on the boundary of \(\Sigma\). It is not difficult to prove that

\[
e^{-S_\Sigma(h_1^{-1} h_2, A)} + \frac{4\pi}{i} \int_{\Sigma} \text{tr}(h_2 A^{01} h_1 A^{01}) = e^{-S_\tilde{\Sigma}(\tilde{\eta}^{-1}, \tilde{A}^{01})}, \tag{7.9}
\]

The Ward identity (7.8) implies then that

\[
\Psi_\lambda (\tilde{A}^{01}) = e^{-S_\Sigma(h_1^{-1}, A^{01})} (\otimes_n h^{-1}_{R_n}(\xi_n)) \otimes (\otimes_n h^{-1}_{\bar{R}_n}(\bar{\xi}_n)) \Psi_\lambda (\tilde{A}^{01}),
\]

i.e. that \(\Psi_\lambda\) is a CS state on the doubled surface \(\tilde{\Sigma}\) with the doubled insertions at points \(\xi_n\) and at their complex conjugates \(\bar{\xi}_n\), associated, respectively, to the representations \(R_n\) and to the complex conjugate representations \(\bar{R}_n\).

We infer that the correlations functions of the boundary WZW theory on a surface \(\Sigma\) may be viewed, in their gauge field dependence, as the special states \(\Psi_\lambda\) belonging to the space \(W_{\tilde{\Sigma}}(\xi, \bar{\xi}, R, \bar{R})\) of the CS states on the doubled surface \(\tilde{\Sigma}\). The states \(\Psi_\lambda\) may be shown, moreover, to be preserved by the parallel transport with respect to the KZB connection on the restriction of the bundle \(\tilde{W}\) of the CS states on the doubled surface to the subspace of the “doubled” complex structures and insertions. These properties are often summarized by saying that the boundary CFT is chiral since its correlation functions are given by special conformal blocks of the WZW theory on \(\tilde{\Sigma}\).

It would be desirable to characterize geometrically the states \(\Psi_\lambda\) for different choices \(\lambda\) of the BC’s seems, however, still missing.

### 7.3 Piece-wise boundary conditions

Up to now, we have imposed the boundary conditions forcing the fields to take values in the special conjugacy classes uniformly on the component circles of \(\partial \Sigma\). Since
the conditions are local, it should be also possible to do this locally on the pieces of the boundary. Suppose that the boundary $\partial \Sigma$ is divided into intervals $I_r$ (the entire boundary circles are also admitted). We shall associate integrable weight labels $\lambda_r$ to some of these intervals in such a way that two labeled intervals in the same boundary component are separated by an unlabeled ("free") one, see Fig. 15.

![Fig. 15](image)

We shall now consider the fields $g$ on $\Sigma$ which on the labeled intervals take values in the corresponding conjugacy classes $C_{\lambda_r}$ and are not restricted on the free intervals. One may still define the classical amplitudes $e^{-S^{WZ}(g)}$ for such fields although this requires a more local (Čech cohomology type) technique than the one developed above \[36\]. Let us sketch how this is done.

![Fig. 16](image)

Recall from the end of Sect. 3.1 that $S^{WZ}(g)$ in the first approximation is equal to $\frac{k}{4\pi i} \int g^* \beta$ where $d\beta = \chi$ is the canonical closed 3-form on $G$. The problem stemmed
from the fact that such 2-forms $\beta$ exist only locally. However, on the sets of a sufficiently fine open covering ($O_\alpha$) of $G$, we may choose 2-forms $\beta_\alpha$ such that $d\beta_\alpha = \chi$. Choose a triangulation $T$ of $\Sigma$ with the triangles $t$, edges $e$ and vertices $v$. If $T$ is fine enough then each of the simplices $s$ of $T$ is mapped by $g$ into an open set, say $O_{\alpha_s}$, see Fig. 16. The main contribution to the amplitude $e^{-S^W_\Sigma (g)}$ will come from $\exp[-\frac{k}{4\pi i} \sum_t g^*\beta_\alpha_t]$. The above expression depends, however, on the choice of the forms $\beta_\alpha$ and of the triangulation. The idea is to compensate this dependence by contributions from simplices of lower dimension. Let $\eta_{\alpha_0\alpha_1} = -\eta_{\alpha_1\alpha_0}$ be 1-forms defined on the non-empty intersections $O_{\alpha_0\alpha_1} \equiv O_{\alpha_0} \cap O_{\alpha_1}$ such that

$$d\eta_{\alpha_0\alpha_1} = \beta_{\alpha_1} - \beta_{\alpha_0}$$

and let $f_{\alpha_0\alpha_1\alpha_2}$ be functions on the triple intersections $O_{\alpha_0\alpha_1\alpha_2}$, antisymmetric in the indices, satisfying

$$df_{\alpha_0\alpha_1\alpha_2} = \eta_{\alpha_1\alpha_2} - \eta_{\alpha_0\alpha_2} + \eta_{\alpha_0\alpha_1}$$

and such that on the four-fold intersections

$$f_{\alpha_1\alpha_2\alpha_3\alpha_4} - f_{\alpha_0\alpha_2\alpha_3\alpha_4} + f_{\alpha_0\alpha_1\alpha_3\alpha_4} - f_{\alpha_0\alpha_1\alpha_2\alpha_4} \in 8\pi^2 \mathbb{Z}.$$ 

Such data may, indeed, be chosen. We define then

$$e^{-S^W_\Sigma (g)} = \exp \left[ -\frac{k}{4\pi i} \left( \sum_t g^*\beta_\alpha_t - \sum_{e \in T} \int g^*\eta_{\alpha_e\alpha_t} + \sum_{v \in e \in T} (\pm) f_{\alpha_v\alpha_e\alpha_t}(g(v)) \right) \right], \quad (7.11)$$

where in the last sum the sign is taken according to the orientation of the vertices $v$ inherited from the triangles $t$ via the edges $e$. One may show that for the surface without boundary, the above expression does not depend on the choices involved and coincides with the definition given in Sect. 3.3.

In the presence of boundary circles with unconstrained fields, the above expression may be used to define the amplitudes with values in a line bundle and it provides an alternative construction of the bundle $L$ over the loop group $G$. In the presence of the boundary conditions on the intervals $I_r$ we shall still employ the same definition, but with some care. Namely, we include neighborhoods $O_\lambda$ of the conjugacy classes $C_\lambda$ into the open covering ($O_\alpha$) of $G$. We choose 2-forms $\beta_\lambda$ on $O_\lambda$ coinciding with $\omega_\lambda$ of Eq. (7.2) when restricted to $C_\lambda$. The triangulations used in Eq. (7.11) are required to be compatible with the splitting of the boundary. To the simplices in the labeled intervals $I_r$ we assign the open sets $O_{\lambda_r}$, see Fig. 16. The amplitudes resulting from Eq. (7.11) coincide then with those defined in the previous section for the special case when the labeled intervals fill entire circles. In the general case,

$$e^{-S^W_\Sigma (g)} \in \prod_{\text{free } I_r} (L_{I_r})_{g|_{I_r}} \quad (7.12)$$

where $L_{I_r}$ is a line bundle over the space of maps from an interval $I_r$ to $G$ taking on the boundary of $I_r$ the values in the conjugacy classes $C_{\lambda_r}$ and $C_{\lambda'_r}$ specified by the labels of the neighboring interval $I'_r$.\footnote{If $I_r$ is a full circle, $L_{I_r} = L$}
The space of sections $\Gamma(\mathcal{L}_{Ir})$ plays, as before, the role of the space of states of the WZW theory but, this time, on the interval and with boundary conditions specified by the conjugacy classes $\mathcal{C}_{\lambda r}$ and $\mathcal{C}_{\lambda' r}$. In the string language, these are states of the open string moving on the group with the ends on the branes $\mathcal{C}_{\lambda r}$ and $\mathcal{C}_{\lambda' r}$, see Fig. 17.

One may still define an action of the central extension of the loop group in the spaces $\Gamma(\mathcal{L}_{Ir})$ (a single one) and base on its analysis a rigorous construction of the open-string Hilbert spaces of states $\mathcal{H}_{\lambda \lambda'}$, as we did in Sect. 4.1 for the closed-string states, see Eq. (4.10). One obtains

$$\mathcal{H}_{\lambda \lambda'} = \bigoplus_{\hat{R}} M^{R}_{\lambda \lambda'} \otimes V_{\hat{R}}. \quad (7.13)$$

The multiplicity spaces may be naturally identified with the spaces of the genus zero CS states $\mathcal{W}(\mathbb{C}P^1, \xi, \hat{R})$ with three insertion points in representations $\mathbb{T}_{\lambda}, R_{\lambda'}$ and $R$. In particular, the dimension of the multiplicity spaces is given by the fusion ring structure constants $\hat{N}^{R}_{\lambda \lambda'}$. The spaces $\mathcal{H}_{\lambda \lambda'}$ carry the obvious action of the current algebra $\hat{g}$ and of the Virasoro algebra, the latter obtained by the Sugawara construction (4.15). The generator $L_0 - \frac{c}{24}$ gives the Hamiltonian of the open string sectors. The spaces $\mathcal{H}_{\lambda \lambda}$ with the same BC on both sides contain the vacua $\Omega_\lambda$, i.e. the states annihilated by $L_0$ (unique up to normalization).

### 7.4 Elementary quantum amplitudes

The quantum amplitudes with the general boundary conditions are given now by the formal functional integrals.

$$\mathcal{A}_{\Sigma, I_r}(A) = \int_{g(I_r) \subset \mathcal{C}_{\lambda r}} e^{-S_{\Sigma}(g.A)} \, Dg \quad (7.14)$$

and, should take values in the space $\bigotimes_{\text{free } I_r} \mathcal{H}_{\lambda \lambda'}$. They should possess a gluing property along free boundary intervals with opposite boundary weight assignment, generalizing the gluing properties (4.6) or (4.7). As discussed in detail by Segal in [5] for the closed string sector, the general amplitudes may be constructed by gluing from the elementary ones for the geometries listed on Fig. 18.
The elementary amplitudes (a) and (b) represent, respectively, the vacuum state $\Omega$ in the closed string space $\mathcal{H}$, and the vacua $\Omega_\lambda$ in the open string spaces $\mathcal{H}_\lambda$. The amplitudes (c) for arbitrary annuli encode the action of the pair of Virasoro algebras in $\mathcal{H}$. In particular, for a complex number $q \neq 0$ inside the unit disc one may consider the annular regions $A_q = \{ z \mid |q| \leq |z| \leq 1 \}$, see Fig. 19, obtained from $\mathbb{CP}^1$ by taking out the unit discs embedded by the maps $z \mapsto qz$ and $z \mapsto z^{-1}$.

Viewing the amplitude of $A_q$ as an operator from the space $\mathcal{H}$ associated to the first boundary component to $\mathcal{H}$ associated to the second boundary, one has:

$$Z_{A_q}^{-1} A_q = q^{L_0} \bar{q} \bar{L}_0.$$

The gluing of the two boundary circles of $A_q$ leads to the complex torus $T_\tau$ where
\[ q = e^{2\pi i \tau}. \]  According to the gluing relation (4.7), this produces the toroidal partition function

\[ \mathcal{Z}(T_\tau) = \mathcal{Z}_{Aq} \text{ tr}_\mathcal{H} (q^{L_0} \bar{q}^{\bar{L}_0}). \quad (7.15) \]

Upon choosing a flat metric on \( T_\tau \) and working out the partition function \( \mathcal{Z}_{Aq} \), one obtains finally

\[ \mathcal{Z}(\tau) = \text{ tr}_\mathcal{H} q^{L_0} \bar{q}^{\bar{L}_0} \bar{q}^{\bar{L}_0} \quad (7.16) \]

which is nothing else but Eq. (5.20).

The amplitude for a disc \( P_{w,q,q_1} \) with two round holes, as in Fig. 18(d), gives rise to a 3-linear form on \( \mathcal{H} \) which may be also viewed as an operator from the space \( \mathcal{H} \otimes \mathcal{H} \) associated to the inner discs to \( \mathcal{H} \) corresponding to the outer one. It is customary in CFT to rewrite this amplitude as an operator \( \Phi(e; w) \) in \( \mathcal{H} \) labeled by the vectors \( e \) in (a dense subspace of) \( \mathcal{H} \) and the point \( w \) inside the unit disc:

\[ \Phi(e; w) e' = \mathcal{Z}_{P_{w,q,q_1}} A_{P_{w,q,q_1}} (q^{-L_0} \bar{q}^{-\bar{L}_0} e') \otimes (q_1^{-L_0} \bar{q}_1^{-\bar{L}_0} e). \quad (7.17) \]

The combination with the powers of \( L_0 \) and \( \bar{L}_0 \) assures the independence of the expression of \( q \) and \( q_1 \). The vectors \( e \) can be recovered from the operators \( \Phi(e; w) \) by acting with them on the vacuum vector

\[ \lim_{w \to 0} \Phi(e; w) \Omega = e. \quad (7.18) \]

Pictorially, this corresponds to filling up the central whole of \( P_{w,q,q_1} \) by gluing a disc to its boundary. The operators \( \Phi(e; w) \) satisfy an important relation:

\[ \Phi(e; z) \Phi(e'; w) = \Phi(\Phi(e; z - w) e'; w). \quad (7.19) \]

The above identity holds for \( 0 < |w| < |z| \) and \( 0 < |z - w| < 1 \). It results from the two ways that one may obtain the disc with three holes by gluing two discs with two holes, see Fig. 20.

---

\( \mathcal{Z}_{Aq} \) is a ratio of two partition function on the Riemann sphere and may be easily found from the relation (4.17) to be equal to \( |q|^{-\bar{\tau}} \).
The relation (7.19) may be viewed as a global form of the operator product expansion. The local forms may be extracted from it by expanding the vector \( \Phi(e; z - w) \) into terms homogeneous in \( z - w \). In particular, for specially chosen vectors \( e \) and \( e' \) one obtains the operator versions of the relations (4.25) and (4.31), hence the name of the latter. The vector-operator correspondence together with the operator product expansion (7.19) are the cornerstones of the non-perturbative approach to CFT.

The amplitudes corresponding to the surfaces with boundary of Fig. 18(e) represent the action of the Virasoro algebra in the open string spaces \( \mathcal{H}_{\lambda\lambda'} \). The surfaces (f) give rise, in turn, to the amplitudes which, applied to vectors \( q_1^{-L_0}q_2^{-L_0}e, q_1^{-\bar{L}_0}q_2^{-\bar{L}_0}e \in \mathcal{H} \), define the action of the closed string sector fields \( \Phi(e, w) \) in the open string spaces \( \mathcal{H}_{\lambda\lambda'} \). Finally, the amplitudes of the disc (g) with three labeled and three free boundary intervals define 3-linear forms on the corresponding open string spaces. As before in the closed string sector, one may interpret them in terms of boundary operators labeled by vectors in, say, \( \mathcal{H}_{\lambda'\lambda''} \) and mapping from \( \mathcal{H}_{\lambda\lambda'} \) to \( \mathcal{H}_{\lambda'\lambda''} \).

The gluing properties give rise to non-trivial relations between various amplitudes. For example, gluing along the free sides a rectangle with a local BC imposed on the two other sides, see Fig. 21, one obtain a finite cylinder \( Z_L \) with the BC’s imposed on the boundary components.

![Fig. 21](image)

Its amplitude \( A_{z_L} \) (in the flat metric) may be computed in two ways. On one hand side, using the decomposition (7.13), we infer that
\[
A_{z_L} = \text{tr}_{H_{\lambda\lambda'}} q^{L_0 - \frac{i\tau}{2\pi}} = \sum_{\hat{R}} \tilde{N}_{\hat{R}} \text{tr}_{V_{\hat{R}}} q^{L_0 - \frac{i\tau}{2\pi}} = \sum_{\hat{R}} \tilde{N}_{\hat{R}} \chi_{\hat{R}}(\tau, 1) \quad (7.20)
\]
with \( \tau = \frac{L_i}{2\pi} \) and \( q = e^{2\pi i\tau} \). On the other hand, we may express this amplitude as a matrix element of the close string amplitude between the boundary states \( \hat{\delta}_{\lambda} \) and \( \hat{\delta}_{\lambda'} \). With \( q' = e^{2\pi i\tau'} \) and \( \tau' = -\frac{1}{\tau} = \frac{2\pi i}{L} \), we obtain
\[
A_{z_L} = (\hat{\delta}_{\lambda}, (q')^{\frac{1}{2}(L_0 - \tilde{\pi})}(q')^{\frac{1}{2}(\bar{L}_0 - \tilde{\pi})}\hat{\delta}_{\lambda'}) \quad (7.21)
\]
Upon the substitution of Cardy’s expression (c.4) for the boundary states \( \hat{\delta}_{\lambda} \), this becomes
\[
A_{z_L} = \sum_{\hat{R}, \hat{R}'} (S_{\hat{R}}^{-1})^{\frac{1}{2}} S_{\hat{R}'}^{\frac{1}{2}} (S_{\hat{R}'}^{-1})^{\frac{1}{2}} S_{\hat{R}}^{\frac{1}{2}} \left( e_{\hat{R}}^- \otimes \tilde{e}_{\hat{R}}^-, (q')^{\frac{1}{2}(L_0 - \tilde{\pi})}(q')^{\frac{1}{2}(\bar{L}_0 - \tilde{\pi})} e_{\hat{R}'}^\oplus \otimes \tilde{e}_{\hat{R}'}^\oplus \right)
\]
52
\[
\begin{align*}
&= \sum_{R'} (S_{R'})^{-1} S^\pi_{R'} S^{-R'}_{R'} (e_{R'}^\pi, (q')^{\frac{1}{2}(L_0 - \frac{c_2}{24})} e_{R'}^\pi) \left( (q')^{\frac{1}{2}(L_0 - \frac{c_2}{24})} e_{R'}^\pi \right) \\
&= \sum_{R'} (S_{R'})^{-1} S^\pi_{R'} S^{-R'}_{R'} \chi_{\hat{R}}^\pi (\tau, 1). 
\end{align*}
\]

(7.22)

With the use of the modular transformation property (5.7), we finally obtain:

\[
A_{z_L} = \sum_{R, R'} (S_{R'})^{-1} S^\pi_{R'} S^{-R'}_{R'} \chi_{\hat{R}}^\pi (\tau, 1).
\]

By virtue of the Verlinde formula (5.12), the last identity coincides with Eq. (7.20). We have, in fact, inverted here the logic of reference [26], where the consistency of the two ways of computing the amplitude \( A_{z_L} \) was used to obtain the expression (7.7) for the boundary states \( \hat{\delta}_\lambda \).

The whole system of elementary amplitudes represents an intriguing algebraic structure which is common to all (rational) boundary CFT’s. Already the case of boundary topological field theories, where the amplitudes depend only on the surface topology, leads to an interesting construction that remains to be fully understood. It entangles a commutative algebra structure on the closed string space of states and a non-commutative algebroid in the open string sector. An example of such a structure was inherent in the work of Kontsevich [37] on the deformation quantization of general Poisson manifolds, see [38]. Certainly, the two-dimensional CFT did not unveil yet all of its secrets.

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