Abstract. The Teichmüller space $\text{Teich}(S)$ of a surface $S$ in genus $g > 1$ is a real sub-manifold of the quasifuchsian space $\text{QF}(S)$. We show that the determinant of the Laplacian $\det'(\Delta)$ on $\text{Teich}(S)$ has a unique holomorphic extension to $\text{QF}(S)$.

1. Introduction

Given a compact Riemannian manifold $M$, the Laplace-Beltrami operator $\Delta$ on functions on $M$ is an elliptic operator with discrete spectrum

$$\lambda_0 = 0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots \to \infty.$$ 

The determinant of the operator $\Delta$ may be defined formally as the product of the nonzero eigenvalues of $\Delta$. A regularization $\det'(\Delta)$ of this product was defined by Ray and Singer [RS1], [RS2], using the zeta function of $\Delta$.

The determinant $\det'(\Delta)$ has played an important role in such areas of mathematics as mathematical physics, differential geometry, algebraic geometry and number theory. In particular, it plays a central role in the theory of determinant line bundles, initiated by Quillen ([Q]) and further developed by Bismut and Freed [BF1], [BF2], and by Bismut, Gillet and Soulé [BGS1], [BGS2], [BGS3].

In a series of papers [OPS1], [OPS2] (see also [S2]), Osgood, Phillips and Sarnak studied $-\log \det'(\Delta)$ as a “height” function on the space of metrics on a compact orientable smooth surface $S$ of genus $g$. For $g > 1$, they showed that when restricted to a given conformal class of metrics on $S$, it attains its minimum at the unique hyperbolic metric in this conformal class, and has no other critical points. Thus, to find Riemannian metrics on $S$ which are extremal, in the sense that they minimize $-\log \det'(\Delta)$, it suffices to consider its restriction to the moduli space $\mathcal{M}_g$ of hyperbolic metrics on a Riemann surface $S$ of genus $g$. Osgood, Phillips and Sarnak showed that this restriction is a proper function.

The universal cover of the orbifold $\mathcal{M}_g$, with covering group the mapping class group $\Gamma_g$, is the Teichmüller space $\text{Teich}(S)$. The function $-\log \det'(\Delta)$ lifts to a function on the Teichmüller space $\text{Teich}(S)$ invariant under $\Gamma_g$. In this paper, we are interested in the function theoretic properties of $\log \det'(\Delta)$ on $\text{Teich}(S)$. Before stating the main theorem, consider the special case of genus 1.

The author thanks Ezra Getzler for his advice while this work was carried out. He is grateful to Curtis McMullen, Peter Sarnak and Jared Wunsch for helpful discussions. This research was partially supported by the NSF under grant DMS-9704320.
Example ([RS2] or [S1], p. 33, (A.1.7)). For \( z \in \mathbb{H} \), let \( T_z \) be the flat torus obtained by the lattice of \( \mathbb{C} \) generated by 1 and \( z \). Then the determinant of Laplacian of this flat torus is

\[
\log \det' (\Delta) (z) = \log (2 \pi y^{1/2} | \eta(z) |^2)
\]

where \( \eta(z) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \) for \( q = e^{2 \pi i z} \) is the eta function; this is a modular form of weight 1/2.

The manifold \( \mathbb{H} \) has a complexification \( \mathbb{H} \times \overline{\mathbb{H}} \), and the function \( \log (2 \pi y^{1/2} | \eta(z) |^2) \) on the diagonal \( \{ w = \overline{z} \} \) has a holomorphic extension to \( \mathbb{H} \times \overline{\mathbb{H}} \), namely,

\[
\log (\pi i (z - w)^{1/2} \eta(z) \eta(w)).
\]

In this paper, we show that even in higher genus \( g > 1 \), the function \( \log \det' (\Delta) \) has a unique holomorphic extension. In higher genus, the objects corresponding to \( \mathbb{H} \) and \( \mathbb{H} \times \overline{\mathbb{H}} \) are the Teichmüller space \( \text{Teich}(S) \) and the quasifuchsian space \( \text{QF}(S) = \text{Teich}(S) \times \text{Teich}(\overline{S}) \) respectively. The complex structure on quasifuchsian space \( \text{QF}(S) \) is induced by the complex structure of \( \text{Teich}(S) \), and the real analytic manifold \( \text{Teich}(S) \) imbeds as the diagonal in \( \text{QF}(S) \).

The following theorem is the main result of this paper. (For the precise statement, see Theorem 4.3.)

**Theorem 1.1.** The function \( \log \det' (\Delta) \) on \( \text{Teich}(S) \) has a unique holomorphic extension to the quasifuchsian space \( \text{QF}(S) \).

In the proof of Theorem 1.1, we use the Belavin-Knizhnik formula (see Theorem 2.7), proved by Wolpert [W3] and by Zograf and Takhtajan [ZT] and the holomorphic extension of the Weil-Petersson form constructed by I. Platis [P] (see Theorem 2.4).

The following is a key step in the proof of Theorem 1.1 and may be of independent interest.

**Theorem 1.2.** Let \( V \) and \( W \) be domains in the complex space \( \mathbb{C}^n \) diffeomorphic to the open unit ball. Consider \( V \times W \subset \mathbb{C}^n \times \mathbb{C}^n \), with holomorphic coordinates \( (z, w) \), and let \( \partial_z = dz^i \partial_z^i \) and \( \partial_w = dw^j \partial_w^j \). Suppose \( \Omega \) is a holomorphic closed 2-form on \( V \times W \) which is locally written as

\[
\Omega = \sum_{i,j} \Omega_{ij} dz^i \wedge dw^j.
\]

Then there is a holomorphic function \( q \) on \( V \times W \) such that \( \partial_z \partial_w q = \Omega \).

This theorem implies that there is a holomorphic function on \( \text{QF}(S) \) whose restriction on \( \text{Teich}(S) \) (the diagonal in \( \text{Teich}(S) \times \text{Teich}(\overline{S}) \)) is a Kähler potential for the Weil-Petersson form \( \omega_{WP} \). This suggests that quasifuchsian space is a useful tool in gaining a better understanding of function theory of the Teichmüller space. This idea is due to McMullen [M], who used quasifuchsian space to study the geometry of the Weil-Petersson metric on the Teichmüller space.

In a sequel to this paper, we will construct a holomorphic family of differential operators in a neighborhood of the diagonal in quasifuchsian space, which equal the Laplacian along
the diagonal, and such that the determinant of this family equals the holomorphic extension of \( \log \det'(\Delta) \) constructed in this paper.

The asymptotic behavior of \( \log \det'(\Delta) \) near the boundary of Teichmüller space is important in both geometry and physics and was studied in [W4] and [BB]. It would be interesting to understand the asymptotic behavior of the holomorphic extension of \( \log \det'(\Delta) \) near the boundary of the quasifuchsian space. We hope to address this in the future.

**Plan of the paper.** In Section 2, we review the facts that we need on Teichmüller spaces and quasifuchsian spaces, including the Belavin-Knizhnik formula and Platis’s theorem. We prove Theorem 3.1 in Section 3. In Section 4, we complete the proof of Theorem 1.1.

### 2. Preliminaries

**Determinants of Laplacians.** Let \( \Delta \) be the Laplace-Beltrami operator on functions on a compact Riemannian manifold \( M \). Let

\[
\zeta_\Delta(s) = \sum_{\lambda \in \text{Spec}(\Delta) \setminus \{0\}} \lambda^{-s}
\]

be the zeta-function of \( \Delta \). The determinant \( \det'(\Delta) \) is defined (see [RS1]) as

\[
- \log \det'(\Delta) = \frac{d \zeta_\Delta(0)}{ds}.
\]

Since \( M \) is compact and \( \Delta \) is elliptic and self-adjoint, the nonzero spectrum of \( \Delta \) is positive and discrete. Moreover, the sum in Example 2.1 is absolutely convergent for \( \Re s \) sufficiently large, and has a meromorphic extension to the whole complex plane, with possible poles only at \( \{-1, -2, -3, \ldots\} \). Thus, there is no difficulty in taking the derivative at \( s = 0 \) in (2.2).

**Teichmüller spaces.** A general reference for this section is [IT].

Let \( S \) be an oriented closed surface with genus \( g > 1 \). The Teichmüller space \( \text{Teich}(S) \) of \( S \) is the space of isotopy classes of hyperbolic Riemannian metrics on \( S \), that is, metrics with Gaussian curvature \(-1\). Two Riemannian metrics \( m_1 \) and \( m_2 \) on \( S \) are said to be in the same isotopy class if there is an isotopy \( \phi \) of \( S \), i.e. a diffeomorphism obtained by a flow of a vector field on \( S \), such that \( \phi^* m_1 = m_2 \). On a surface, there is one-to-one correspondence between complex structures and hyperbolic Riemannian metrics, i.e. each complex structure on \( S \) has unique hyperbolic metric which is a Kähler metric to the complex structure and each hyperbolic Riemannian metric on \( S \) has canonical complex structure such that the metric becomes Kähler. From this correspondence we see that \( \text{Teich}(S) \) is also the space of isotopy classes of complex structures on \( S \).

The set of equivalence classes of hyperbolic metrics (or equivalently complex structures) under orientation preserving diffeomorphisms on \( S \) forms the moduli space \( \mathcal{M}_g \) of compact Riemann surfaces of genus \( g \).

Denote the group of orientation preserving diffeomorphisms on \( S \) by \( \text{Diff}^+(S) \), and the group of isotopies by \( \text{Diff}_0(S) \). The mapping class group

\[
\Gamma_g = \frac{\text{Diff}^+(S)}{\text{Diff}_0(S)}
\]
is a discrete group which acts properly discontinuously on $\text{Teich}(S)$. Thus $\text{Teich}(S)$ is almost a covering space of $\mathcal{M}_g$, with covering transformation group $\Gamma_g$:

$$\Gamma_g \longrightarrow \text{Teich}(S)$$

$$\downarrow$$

$$\mathcal{M}_g = \Gamma_g \backslash \text{Teich}(S)$$

The only caveat is that the action of $\Gamma_g$ is not free, i.e. there are points in $\text{Teich}(S)$ which are fixed under some finite subgroups of $\Gamma_g$. These points descend to $\mathcal{M}_g$ as orbifold singularities.

Fixing a hyperbolic metric on $S$, we may decompose $S$ into $2g - 2$ pairs of pants, separated by closed geodesics $\gamma_1, \ldots, \gamma_{3g-3}$.

A hyperbolic pair of pants is determined up to isometry by the lengths of its boundary geodesics. Given the combinatorial pants decomposition of $S$, we get a hyperbolic metric by specifying the lengths $l_i$ ($l_i > 0$) of the geodesics $\gamma_i$ and the angle $\theta_i$ by which they are twisted along $\gamma_i$ before gluing. Let $\tau_i = l_i \theta_i / 2\pi$, $i = 1, \ldots, 3g - 3$. Then the system of variables

$$(l_1, \ldots, l_{3g-3}, \tau_1, \ldots, \tau_{3g-3})$$

is a real analytic coordinate system on $\text{Teich}(S)$, called the Fenchel-Nielsen coordinates of $\text{Teich}(S)$. This coordinate system gives a diffeomorphism

$$\text{Teich}(S) \approx \mathbb{R}_+^{3g-3} \times \mathbb{R}^{3g-3}.$$  

There is a natural symplectic form $\omega_{WP}$ on $\text{Teich}(S)$, called the Weil-Petersson form. By a theorem of Wolpert ([W1], [W2]; see also [IT]), this form is given in Fenchel-Nielsen coordinates by the formula

$$\omega_{WP} = \sum_{i=1}^{3g-3} dl_i \wedge d\tau_i.$$  

The Teichmüller space $\text{Teich}(S)$ has a natural complex structure, for which $\omega_{WP}$ is a Kähler form. The following theorem is well known. (See, for example, [A].)

**Theorem 2.1.** For a closed surface $S$ with genus $g > 1$, $\text{Teich}(S)$ is biholomorphic to a bounded open contractible domain in $\mathbb{C}^{3g-3}$.

**Corollary 2.2.** There are global holomorphic coordinates $z = (z^1, \ldots, z^{3g-3})$ on $\text{Teich}(S)$.

**Quasifuchsian spaces.** While Teichmüller space is a space of Riemann surfaces, the quasifuchsian space defined by Lipman Bers (See [B]) is a space of pairs of Riemann surfaces. The quasifuchsian space $QF(S)$ of the surface $S$ may simply be defined as

$$QF(S) = \text{Teich}(S) \times \text{Teich}(\overline{S}).$$

Here, $\overline{S}$ denotes the surface $S$ with the opposite orientation.
The complex conjugate $\overline{X}$ of a Riemann surface $X$ is defined by the following diagram:

$$
\begin{array}{c}
\mathbb{H} \\
\downarrow \\
X
\end{array} \quad \begin{array}{c}
\mathbb{H} \\
\downarrow \\
\overline{X}
\end{array}
$$

(2.4)

The upper arrow is complex conjugation, and the vertical arrows are the universal coverings given by the uniformization theorem for Riemann surfaces. There is a canonical map from $\text{Teich}(S)$ to $\overline{\text{Teich}(S)}$ defined by sending a Riemann surface $X \in \text{Teich}(S)$ to its complex conjugate $\overline{X} \in \overline{\text{Teich}(S)}$.

Let $\overline{\text{Teich}(S)}$ be the complex conjugate of $\text{Teich}(S)$.

**Proposition 2.3.** As complex manifolds, $\text{Teich}(S) \cong \overline{\text{Teich}(S)}$.

**Proof.** The complex structure of $\text{Teich}(S)$ is induced by the complex structure of the space of Beltrami differentials on $S$. Fix a complex structure on $S$. Beltrami differentials are $(-1, 1)$-forms on $S$ with $L_\infty$ norm less than 1; locally, $\mu(z) \overline{dz}/dz$ with $|\mu| < 1$. (See [A] or [AB] for details.)

If $z$ is a local coordinate on $S$, then $w = \overline{z}$ is a local coordinate on $\overline{S}$. Now Beltrami differentials on $\overline{S}$ are locally of the form $\mu(w) \overline{dw}/dw$. From this local expression it is clear that the complex structure on $\text{Teich}(\overline{S})$ is the same as the one on $\overline{\text{Teich}(S)}$. \qed

The diagonal map

$$
\text{Teich}(S) \hookrightarrow \text{Teich}(S) \times \overline{\text{Teich}(S)}
$$

sending $X \in \text{Teich}(S)$ to $(X, \overline{X})$ embeds $\text{Teich}(S)$ as a totally real submanifold into $QF(S)$. The action of $\Gamma_g$ on $\text{Teich}(S)$ extends to $QF(S)$ by the diagonal action: for $\rho \in \Gamma_g$ and $(X, Y) \in QF(S) = \text{Teich}(S) \times \overline{\text{Teich}(S)}$,

$$
\rho \cdot (X, Y) = (\rho \cdot X, \rho \cdot Y).
$$

By Corollary 2.2, $QF(S) = \text{Teich}(S) \times \overline{\text{Teich}(S)}$ has global holomorphic coordinates

$$(z^1, \ldots, z^{3g-3}, w^1, \ldots, w^{3g-3}).$$

We abbreviate this coordinate system to $(z, w)$.

**Holomorphic extension of Weil-Petersson form.** The following result is due to Platis ([P], Theorems 6 and 8).

**Theorem 2.4.** The differential form $i\omega_{WP}$ on the Teichmüller space $\text{Teich}(S)$ has an extension $\Omega$ to the quasifuchsian space $QF(S)$ which is a holomorphic non-degenerate closed $(2, 0)$-form whose restriction to the diagonal $\text{Teich}(S) \subset QF(S) \cong \text{Teich}(S) \times \overline{\text{Teich}(S)}$ is $i\omega_{WP}$.

**Lemma 2.5.** Let $U \subset \mathbb{C}^n$ be a connected complex domain, and let $\phi$ be a holomorphic function on $U \times \overline{U}$ whose restriction to the diagonal $U \subset U \times \overline{U}$ vanishes. Then $\phi$ vanishes on all of $U \times \overline{U}$. 

5
Proof. It suffices to show that \( \phi \) vanishes near a point \((z, \overline{z})\) on the diagonal. Choose holomorphic coordinates \(s\) on \(U\) which vanish at \(z\), and let \((s, t)\) be the corresponding holomorphic coordinates on \(U \times \overline{U}\). Consider the Taylor series expansion
\[
\phi(s, t) = \sum_{\alpha, \beta} a_{\alpha, \beta} s^\alpha t^\beta.
\]
Along the diagonal, where \(s = t\), we have
\[
0 = \phi(s, s) = \sum_{\alpha, \beta} a_{\alpha, \beta} s^\alpha s^\beta.
\]
It follows that \(a_{\alpha, \beta} = 0\) for all \(\alpha\) and \(\beta\), hence \(\phi(s, t)\) vanishes in a neighborhood of \((z, \overline{z})\). \(\square\)

**Proposition 2.6.** In terms of the holomorphic coordinate system
\[
(z, w) = (z^1, \ldots, z^{3g-3}, w^1, \ldots, w^{3g-3})
\]
on \(\text{Teich}(S) \times \overline{\text{Teich}(S)}\), the 2-form \(\Omega\) of Theorem 2.4 may be written locally as
\[
\Omega = \sum_{i,j} \Omega_{ij} dz^i \wedge dw^j.
\]

Proof. Since \(\Omega\) is \((2, 0)\) form, we may write
\[
\Omega = \sum_{i,j} (A_{ij} \, dz^i \wedge dz^j + B_{ij} \, dz^i \wedge dw^j + C_{ij} \, dw^i \wedge dw^j).
\]
Restricted to the diagonal \(\{w = \overline{z}\}\),
\[
\omega_{WP} = \Omega|_{w=\overline{z}} = \sum_{i,j} (A_{ij}|_{w=\overline{z}} \, dz^i \wedge dz^j + B_{ij}|_{w=\overline{z}} \, dz^i \wedge d\overline{z}^j + C_{ij}|_{w=\overline{z}} \, d\overline{z}^i \wedge d\overline{z}^j).
\]
Since \(\omega_{WP}\) is \((1, 1)\)-form on \(\text{Teich}(S)\), we see that \(A_{ij}\) and \(C_{ij}\) vanish on the diagonal \(\{w = \overline{z}\}\). Since \(\Omega\) is holomorphic, so are the functions \(A_{ij}\), \(B_{ij}\), and \(C_{ij}\). Applying Lemma 2.5, we see that \(A_{ij}\) and \(C_{ij}\) vanish. \(\square\)

**The Laplacian on hyperbolic surfaces and the Belavin-Knizhnik formula.** Let \(X\) be a compact hyperbolic surface of genus \(g > 1\), and let \(\Delta\) be the Laplacian on scalar functions on \(X\). On the universal cover \(\mathbb{H}\) of \(X\), the pull-back of \(\Delta\) by the covering map may be written as
\[
\Delta = (z - \overline{z})^2 \frac{\partial^2}{\partial z \partial \overline{z}}.
\]
The Siegel upper half space \(\mathcal{S}_g\) is the space of complex symmetric matrices in \(\mathbb{C}^{g \times g}\) with positive definite imaginary part. The period matrix \(\tau\) is a holomorphic map from \(\text{Teich}(S)\) to \(\mathcal{S}_g\).

We will use the Belavin-Knizhnik formula, proved by Wolpert and by Zograf and Takhtajan. (See [W3] and [ZT].) We only need the following special case of this theorem ([ZT], Theorem 2).
Theorem 2.7. In Teich(S),

$$\partial \overline{\partial} \log \left( \frac{\det'(\Delta)}{\det(\text{Im } \tau)} \right) = iC_g \omega_{WP},$$

where \( \text{Im } \tau \) is the imaginary part of the period matrix \( \tau \) and \( C_g \) is a constant depending only on the genus \( g \). The differential operator \( \partial \overline{\partial} \) comes from the complex structure on \( \text{Teich}(S) \).

This formula and the result of the next section together with the theorem of Platis are the key ingredients in the construction of the holomorphic extension of \( \log \det'(\Delta) \).

3. An extended Kähler potential

The proof of the following elementary theorem occupies the whole of this section.

Theorem 3.1. Let \( V \) and \( W \) be domains in the complex space \( \mathbb{C}^n \) which are diffeomorphic to the open unit ball, and let \((z, w) = (z^1, \ldots, z^n, w^1, \ldots, w^n)\) be holomorphic coordinates on the domain \( V \times W \subset \mathbb{C}^n \times \mathbb{C}^n \). Suppose \( \Omega \) is a holomorphic closed 2-form on \( V \times W \) of the form

$$\Omega = \sum_{i,j} \Omega_{ij} \, dz^i \wedge dw^j.$$

Then there exists a holomorphic function \( q \) on \( V \times W \) such that

$$\partial_z \partial_w q = \Omega,$$

where \( \partial_z = \sum_i \, dz^i \, p_{z^i} \) and \( \partial_w = \sum_j \, dw^j \, \partial_{w^j} \).

Proof. Choose smooth polar coordinates on \( V \) and \( W \), and denote the centers of these coordinate systems by \( z_0 \) and \( w_0 \) respectively. Denote the radial line in polar coordinates from \( z_0 \) to the point \( z \in V \) by \( v(z) \); similarly, denote the radial line in polar coordinates from \( w_0 \) to the point \( w \in W \) by \( w(w) \). More generally, if \( c \) is a smooth chain in \( V \), let \( v(c) \) denote the cone on \( c \) with vertex \( z_0 \), and similarly if \( c \) is a smooth chain in \( W \), let \( w(c) \) denote the cone on \( c \) with vertex \( w_0 \).

Define \( q(z, w) \) by the formula

$$q(z, w) = \int_{v(z) \times w(w)} \Omega.$$

Since the cycle \( v(z) \times w(w) \) varies smoothly as \((z, w)\) varies, the function \( q(z, w) \) is smooth. Observe that \( q \) is unchanged by isotopies of the coordinate systems on \( V \) and \( W \) which fix the centers \( z_0 \) and \( w_0 \), and that \( q \) vanishes on \( V \times \{w_0\} \) and on \( \{z_0\} \times W \).

If \( c \) is a differentiable curve in \( W \) parametrized by the interval \([0, t]\), we have by Stokes’s theorem

$$q(z, c(t)) - q(z, c(0)) = \int_{v(z) \times c} \Omega + \int_{\{z\} \times w(c)} \Omega - \int_{\{z_0\} \times w(c)} \Omega - \int_{v(z) \times w(c)} d\Omega.$$
The second and third terms on the right-hand side vanish, since $\Omega$ vanishes when restricted to the 2-simplex $\{z\} \times w(c)$, and the last term vanishes since $d\Omega = 0$. Taking the limit $t \to 0$, we see that

$$\iota(0, c'(0)) dq(z, c(0)) = - \int_{v(z) \times c(0)} \iota(0, c'(0)) \Omega. \quad (3.1)$$

Since $\Omega$ is holomorphic along $\{z\} \times W$, it follows that $q$ is holomorphic along $\{z\} \times W$ as well. A similar argument shows that $q$ is holomorphic along $V \times \{w\}$; combining these two calculations, we see that $q$ is holomorphic on $V \times W$.

We now calculate $\partial_w \partial_z q$. By (3.1),

$$\partial_w q(z, w) = - \sum_{i=1}^{n} dw^i \int_{v(z) \times \{w\}} \iota(\partial_w \iota) \Omega.$$ 

If $c$ is a differentiable curve in $V$, parametrized by the interval $[0,t]$, we have by Stokes’s theorem

$$(\partial_w q)(c(t), w) - (\partial_w q)(c(0), w) = \sum_{i=1}^{n} dw^i \left( - \int_{c \times \{w\}} \iota(\partial_w \iota) \Omega + \int_{v(c) \times \{w\}} \partial_w \iota \iota \Omega \right). \quad (3.2)$$

The second term on the right-hand side vanishes. Indeed,

$$\partial_w \iota(\partial_w \iota) \Omega = - \partial_w \Omega_{ji} dz^k \wedge dz^j - \partial_w \Omega_{ji} dw^k \wedge dz^j$$

$$= - \sum_{j<k} \left( \partial_w \Omega_{ji} - \partial_z \Omega_{ki} \right) dz^k \wedge dz^j - \partial_w \Omega_{ji} dw^k \wedge dz^j$$

$$= - \partial_w \Omega_{ji} dw^k \wedge dz^j.$$ 

Restricting to $v(c) \times \{w\}$, this differential form vanishes.

Taking $t \to 0$ in (3.2), we see that

$$\iota(c'(0), 0)(\partial_w q)(c(0), w) = - \sum_{i=1}^{n} dw^i \iota(c'(0), 0) \iota(\partial_w \iota) \Omega(c(0), w),$$

or in other words, $\partial_z \partial_w q = \Omega$. \hfill \qed

4. HOLOMORPHIC EXTENSION OF $\log \det' \Delta$

From Proposition 2.6, we know that the holomorphic 2-form $\Omega$ of Theorem 2.4 satisfies the hypotheses of Theorem 3.1. Restricted to the diagonal $Teich(S) = \{w = \overline{z}\} \subset QF(S)$, the differential equation in Theorem 3.1 for the holomorphic function $q$ on $QF(S)$ becomes

$$\bar{\partial} \bar{\partial} q = i \omega_{WP},$$

where $i \omega_{WP}$ is the restriction of $\Omega$ to the diagonal. Thus, Theorem 3.1 gives a method of constructing a Kähler potential for the Kähler form $i \omega_{WP}$ on the Teichmüller space, using the fact that it has a holomorphic extension to quasifuchsian space.
Example. (See p214 in [IT]) When $S$ has genus 1, the Teichmüller space $\text{Teich}(S)$ may be identified with the upper half plane $\mathbb{H}$, and

$$\omega_{WP} = -i(z - \overline{z})^{-2} \, dz \wedge d\overline{z}.$$ 

One easily finds the Kähler potential $q(z) = \log(z - \overline{z})$. The method used in the proof of Theorem 3.1, applied to the 2-form $\Omega = (z - w)^{-2} \, dz \wedge dw$, yields the holomorphic function

$$q(z, w) = \log(z - w) - \log(z_0 - w) - \log(z - \overline{w}_0) + \log(z_0 - \overline{w}_0)$$

on the quasifuchsian space $\mathbb{H} \times \mathbb{H}$.

**Lemma 4.1.** There is a holomorphic function $\tilde{q}(z, w)$ on the quasifuchsian space $QF(S) \cong \overline{\text{Teich}(S)} \times \overline{\text{Teich}(S)}$, whose restriction to the diagonal $\text{Teich}(S) = \{ w = \overline{z} \}$, $\tilde{q}$ is real, and such that

$$\partial \overline{\partial} \tilde{q} = i \omega_{WP}.$$

**Proof.** The function $q(w, z)$ is holomorphic, and

$$\partial_z \partial_w (q(z, w) + \overline{q(w, z)}) = \Omega(z, w) - \overline{\Omega(w, z)}.$$

Restricted on the diagonal $\{ w = \overline{z} \}$, we have

$$\partial \overline{\partial} (q(z, \overline{z}) + \overline{q(z, \overline{z})}) = 2i \omega_{WP}.$$

Thus, it suffices to take $\tilde{q}(z, w) = \frac{1}{2} (q(z, w) + \overline{q(w, z)})$. □

**Proposition 4.2.** If genus $g > 1$, there is a holomorphic function $f$ on $\text{Teich}(S)$ such that, in the notation of Theorem 2.7,

$$(4.1) \quad \log \det'(\Delta) = C_g \tilde{q} + \log \det(\text{Im} \, \tau) + f + \overline{f}.$$ 

**Proof.** Let $v$ be the real function

$$v = -C_g \tilde{q} + \log \left( \frac{\det'(\Delta)}{\det(\text{Im} \, \tau)} \right)$$

on $\text{Teich}(S)$. By Theorem 2.7 and Lemma 4.1,

$$d(\partial v) = 0.$$ 

Since $\text{Teich}(S)$ is diffeomorphic to an open ball, there is a smooth function $h$ on $\text{Teich}(S)$ such that

$$df = \partial v.$$ 

It follows that $f$ is holomorphic on $\text{Teich}(S)$, and $v = f + \overline{f}$, up to a constant which may be absorbed into the definition of $f$. □

This proposition gives rise to the holomorphic extension of $\log \det'(\Delta)$, since each of the terms on the right-hand side of (4.1) has a natural holomorphic extension to $QF(S)$. 

9
Theorem 4.3. There exists a unique holomorphic extension of \( \log \det'(\Delta) \) to the quasifuchsian space \( QF(S) \). In coordinates \((z, w)\) on \( QF(S) \approx \text{Teich}(S) \times \overline{\text{Teich}(S)} \), this extension has the form

\[
\log \det'(\Delta)(z, w) = C_0 q(z, w) + \log \det\left((\tau(z) - \overline{\tau(w)})/2i\right) + f(z) + \overline{f(w)}.
\]

Proof. The only term whose extension is not obvious is

\[
\log \det(\text{Im} \tau) = \log \det\left((\tau - \overline{\tau})/2i\right).
\]

This has the holomorphic extension

\[
\log \det\left((\tau(z) - \overline{\tau(w)})/2i\right);
\]

we need only to observe that the matrix \( \tau(z) - \overline{\tau(w)} \) is everywhere invertible on \( QF(S) \).

The uniqueness of the holomorphic extension of \( \log \det'(\Delta) \) follows from Lemma 2.5. □

Remark. We know that \( \log \det'(\Delta) \) on \( \text{Teich}(S) \) is invariant under the action of the mapping class group. It follows that its holomorphic extension to \( QF(S) \approx \text{Teich}(S) \times \overline{\text{Teich}(S)} \) is invariant under the diagonal action of the mapping class group, because this action is holomorphic.

Remark. Theorem 4.3 implies that \( \log \det'(\Delta) \) is a real analytic function on \( \text{Teich}(S) \), and in fact, may be used to give a lower bound for its radius of convergence.

References

[A] L. V. Ahlfors, *Lectures on quasiconformal mappings*, D. Van Nostrand, Princeton, New Jersey (1966)

[AB] L. V. Ahlfors and L. Bers, Riemann’s mapping theorem for variable metrics, *Ann. of Math.* 72 (1960), 385–404.

[B] L. Bers, Simultaneous uniformization, *Bull. Amer. Math. Soc.* 66 (1960), 94-97.

[BB] J.-M. Bismut and J.-B. Bost, Fibrés déterminants, métriques de Quillern et dégénérescence des courbes, *Acta Math.* 165 (1990), 1-103.

[BF1] J.-M. Bismut and D. Freed, The analysis of elliptic families. I. Metrics and connections on determinant bundles, *Comm. Math. Phys.* 106 (1986), 159–176.

[BF2] J.-M. Bismut and D. Freed, The analysis of elliptic families. II. Dirac operators, eta invariants, and the holonomy theorem, *Comm. Math. Phys.* 109 (1986), 103–163.

[BGS1] J.-M. Bismut, H. Gillet, and C. Soulé, Analytic torsion and holomorphic determinant bundles. I. Bott-Chern forms and analytic torsion, *Comm. Math. Phys.* 115 no. 1 (1988),49–78.

[BGS2] J.-M. Bismut, H. Gillet, and C. Soulé, Analytic torsion and holomorphic determinant bundles. II. Direct images and Bott-Chern forms, *Comm. Math. Phys.* 115 no. 1 (1988),79–126.

[BGS3] J.-M. Bismut, H. Gillet, and C. Soulé, Analytic torsion and holomorphic determinant bundles. III. Quillen metrics on holomorphic determinants, *Comm. Math. Phys.* 115 no. 2 (1988),301–351.

[IT] Y. Imayoshi and M. Taniguchi, *An introduction to Teichmüller Spaces*, Springer-Verlag Tokyo 1992.

[M] C. T. McMullen, The moduli space of Riemann surfaces is Kähler hyperbolic, *Ann. of Math.* 151 no. 1 (2000), 327–357.

[OPS1] B. Osgood, R. Phillips, and P. Sarnak, Extremals of determinants of Laplacians, *J. Funct. Anal.* 80 (1988), 148-211.
[OPS2] B. Osgood, R. Phillips, and P. Sarnak, Moduli space, Heights and Isospectral Sets of Plane domains, *Ann. of Math.* **129** (1989) 293-362.

[P] I. Platis, Complex symplectic geometry of quasi-fuchsian space, *Geometriae Dedicata* **87** (2001), 17-34.

[Q] D. Quillen, Determinants of Cauchy-Riemann operators on Riemann Surfaces, *Funct. Anal. Appl.* **19** (1985), 31–34.

[RS1] D.B. Ray and I.M. Singer, R-torsion and the Laplacian on Riemannian manifolds, *Adv. in Math.* **7** (1971), 145–210.

[RS2] D.B. Ray and I.M. Singer, Analytic torsion for complex manifolds, *Ann. of Math.* **98** (1973), 154–177.

[S1] P. Sarnak, *Some Applications of Modular Forms*, Cambridge Tracts in Math. vol 99, Cambridge University Press 1990

[S2] P. Sarnak, Extremal Geometries, *Contemp. Math.* **201**, 1-7, (1997).

[W1] S. Wolpert, The Fenchel-Nielsen deformation, *Ann. of Math.* **115** (1982), 501-528.

[W2] S. Wolpert, On the Weil-Petersson geometry of the moduli space of curves, *Amer. J. Math.* **107** (1985), 969-997.

[W3] S. Wolpert, Chern forms and the Riemann tensor for the moduli space of curves, *Invent. Math.* **85** no. 1 (1986), 119–145.

[W4] S. Wolpert, Asymptotics of the spectrum and the Selberg zeta function of Riemann surfaces, *Comm. Math. Phys.* **112** (1987), 283-315.

[ZT] P.G. Zograf and L.A. Takhtadzhyan, A local index theorem for families of $\overline{\partial}$-operators on Riemann surfaces, *Russian Math. Surveys* **42** no. 6 (1987), 169–190.

Department of Mathematics, Northwestern University, Evanston, IL 60208, USA

E-mail address: huns@math.northwestern.edu