Convergence of the rescaled Whittaker stochastic differential equations by independent sums*

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Abstract

We study some SDEs derived from the $q \to 1$ limit of a 2D surface growth model called the $q$-Whittaker process. The fluctuations are proven to “come down from infinity”: After renormalization, convergence to the time-inverted stationary additive stochastic heat equation holds. The point of view in this paper is a probabilistic representation of the SDEs by independent sums. With this connection, normal and Poisson approximations fully explain the convergence. The proof extends these approximations to some particular integrated forms.

Keywords: Stochastic heat equations; surface growth models; pure death processes; normal approximations; Poisson approximations.

Mathematics Subject Classification (2000): 60J27, 60K35, 60F99

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1 Introduction

Our goal in this paper is to study Gaussian fluctuations in the case of a discrete interacting particle system, called the $q$-Whittaker process, in the limit of $q$ tending to 1. The recent work by Borodin, Corwin and Ferrari [3] proves convergence to the additive stochastic heat equation by taking an iterated scaling limit of the particle systems. Our objective is to study the final stage of this limiting scheme. It shows a convergence of covariance functions of SDEs. We present a very different proof for this convergence and reinforce the convergence to the process level. Due to some explicit representations to be introduced below for the SDEs, the entire spectrum of limit theorems for sums of Bernoulli random variables forms the basis of this study.

Given $q \in (0,1)$, the $q$-Whittaker process $\Lambda^q$ is a model for finitely many particles in the plane. For any integer $N \geq 1$, the particles are arranged as an interlacing triangular array and are labelled by

$$\mathcal{T}_N = \{a = (a_1, a_2) \in \mathbb{N}^2; 1 \leq a_1 \leq a_2 \leq N\}. \tag{1.1}$$

The particles indexed by $\{(a_1, a_2); 1 \leq a_1 \leq a_2\}$ for any fixed $a_2$ live in the space $\mathbb{Z} \times \{a_2\}$. It is assumed that $\Lambda^q_0(a) = 0$ for all $a$ and the system obeys Markovian dynamics: Given a state $\lambda$ of the system and $a \in \mathcal{T}_N$, all the particles labelled by $a, a + (0, 1), \ldots, a + (0, \ell)$ jump to the right by one unit. Here, $\ell \in \mathbb{Z}_+$ is such that $\lambda(a + (0, 0)) = \cdots = \lambda(a + (0, \ell))$ and $\lambda(a + (0, \ell)) \neq \lambda(a + (0, \ell + 1))$. Besides, the jump rate for this update is given by

$$\frac{(1 - q^{\lambda(SW)} - \lambda(a))}{(1 - q^{\lambda(a) - \lambda(a_E) + 1})} \frac{(1 - q^{\lambda(a) - \lambda(a_S) + 1})},$$

where $a_{SW} = (a_1 - 1, a_2 - 1)$, $a_E = (a_1 + 1, a_2)$ and $a_S = (a_1, a_2 - 1)$ ($SW$, $E$ and $S$ refer to the standard compass points with $a$ at the center of the compass rose). More precisely, some extension on the foregoing jump rate defines the updates on the edges of the triangular lattice.

The study in [3, Sections 3 and 4] of the above Markovian dynamics obtains the following iterated limits in distribution:

$$\xi_t = \lim_{\varepsilon_2 \to 0^+} \lim_{\varepsilon_1 \to 0^+} (\varepsilon_2 \varepsilon_1)^{1/2} \left(\Lambda^{e_2 \varepsilon_1}_{(\varepsilon_2 \varepsilon_1)^{1/2}} - \varepsilon_1^{-1} \Lambda_{(\varepsilon_2 \varepsilon_1)^{1/2}}\right).$$

Here, $\Lambda_t = \lim_{\varepsilon_0 \to 0^+} \varepsilon_0 A^{e_0}_{\varepsilon_0^{-1}}$ is proven to be deterministic and so defines a strong law of large numbers limit. The limiting process $(\xi_t)$ above obeys a system of SDEs with a time-dependent drift coefficient $b(t, \xi) = t^{-1} A\xi$ and the noise given by the standard Brownian motion. For definiteness, these SDEs are called the Whittaker SDEs henceforth.

The main results of this paper obtain convergence of the Whittaker SDEs after rescaling as $N \to \infty$. We consider convergence in the covariance functions (Theorem 3.1) and as distribution-valued processes (Theorem 4.3). Whereas the convergence of the covariance functions is already obtained in [3], the present proof entirely circumvents the use of complex contour integrals and special functions, which is essential in its original derivation. Another feature of our results is that the proofs have very mild technical overlaps with those in our previous paper [6], although all these models are believed to fall in the anisotropic class of the Kardar–Parisi–Zhang equation for surface growth. See Wolf [19] for the conjecture behind these studies and [1, 17] for the broader background.

More specifically, the convergence of the Whittaker SDEs is obtained for the stochastic integral parts of the solutions. The rescaling applies the Edwards–Wilkinson diffusive scaling
in two spatial dimensions in the characteristic direction \((x, t) \mapsto (t + tx, t)\). Along the way, the limiting covariance function is for the stationary additive stochastic heat equation with the time inverted \((t \mapsto t^{-1})\). Therefore, the informal process-level picture reads that the limiting fluctuations “come down from infinity” along the clock of the additive stochastic heat equation.

The proofs stem from an initial step in [3]. It shows that the matrix \(A\) in the drift coefficient \(t^{-1}A\xi\) of the Whittaker SDEs is the generator of a linear transformation of two independent linear pure death processes. See Lemma 2.3 for a restatement. Since these Markov chains model the population size of individuals with i.i.d. exponential lifetimes, they can be identified as sums of independent Bernoulli random variables that keep track of the numbers of survivors. The central limit theorem thus comes into play for the convergence problem as \(N \to \infty\). It induces the rescaling mentioned above and relates the independent sums to the heat kernels. The additive stochastic heat equation arises since these kernels characterize the explicit solution. Moreover, this method is viable since we solve the SDEs by Gaussian stochastic integral of the semigroup generated by \(A\).

Given the crucial connection to independent sums, we are still faced with the basic question of whether normal approximations are enough to carry out the above sketch. The issue arises from the fact that parameters of the Bernoulli random variables are integrated out in the explicit forms of the covariance functions. Moreover, due to the choice of rescaling, we have to work with almost the full range of parameters of Bernoulli variables. See Section 3, especially (3.6) and (3.7). Hence, the sketch above actually leaves out the Poisson approximation, since it makes up the entire spectrum of limit theorems of sums of independent Bernoulli random variables. We apply sharp bounds for normal approximations under the \(L_1\)-Wasserstein distance and Poisson approximations under the total variation distance. The results are from [2, 4], and the sharpness is attributable to Stein’s method and the Stein–Chen method. Much of the present work is to show that the two approximation results continue to suffice in some particular integrated forms. Additionally, the proof handles an explosion in the rescaled covariance functions as in [3], but now it is due to the long transition from Poisson approximations to normal approximations in those integrated forms. An appropriate renormalization is applied accordingly.

Finally, the proof of tightness considers the weak formulation for the stochastic integral parts of the solutions of the Whittaker SDEs. It shows that the squared metrics induced by the Gaussian covariance functions for time are Lipschitz uniformly in \(N\). The bounds for the Lipschitz constants require, however, some calculations which are almost exact to cancel divergent constants. This difficulty is reminiscent of what we dealt with in the tightness proof in [6]. But the mathematical objects and methods we have to turn to in this part are also very different. The binomial integration by parts enters as a central tool to obtain the sharp cancellations.

**Organization.** In Section 2, we specify the Whittaker SDEs and present the key probabilistic representation. Section 3 begins with basic setups for the convergence and then analyzes the covariance function. In particular, Assumption 3.3 and the conditions in Definition 3.4 are used throughout this work. We identify the limiting process with the appropriate additive stochastic heat equation in Section 4 and postpone the proof of tightness until Section 5. Finally, we collect some basic properties of the stationary additive stochastic heat equation in Section 6.

**Convention for constants.** \(C(T) \in (0, \infty)\) is a constant depending only on \(T\) and can change from inequality to inequality. Other constants are defined analogously. We write \(A \lesssim B\) or \(A \gtrsim B\) if \(A \leq CB\) for a universal constant \(C \in (0, \infty)\). \(A \asymp B\) means both \(A \lesssim B\) and \(A \gtrsim B\).
2 The Whittaker SDEs

In this section, we introduce the Whittaker SDEs derived in [3, Proposition 5.5] and then show that the coefficients of these SDEs can be represented as sums of i.i.d. Bernoulli random variables. Due to this connection to independent sums, we turn to the related limit theorems in the next section.

2.1 Definition and well-posedness

Recall that $T_N$’s are the finite triangular lattices defined in (1.1). Now we define
\[ T_\infty = \{ a = (a_1, a_2) \in \mathbb{N}^2; 1 \leq a_1 \leq a_2 < \infty \}. \]  

(2.1)

Given an integer $N \geq 1$, the Whittaker SDEs are given by
\[ d\xi_t(a) = t^{-1}A_N\xi_t(a)dt + dW'_t(a), \quad a \in T_N, \quad 0 < t < \infty. \]  

(2.2)

Here, $W' = \{ W'(a); a \in T_\infty \}$ is a family of independent standard Brownian motions, and the coefficient matrix $A_N$ is the restriction to $T_N \times T_N$ of the infinite matrix $A = (A(a, b))_{a, b \in T_\infty}$ defined by
\[ \forall a \neq b \in T_\infty, \quad A(a, b) = \begin{cases} 
    a_1 - 1, & b = a - (1, 1); \\
    a_2 - a_1, & b = a - (0, 1); \\
    0, & \text{otherwise},
\end{cases} \]  

(2.3)

with
\[ \sum_{b \in T_\infty} A(a, b) = 0, \quad \forall a \in T_\infty. \]

Note that $A$ defines a generator so that the semigroup $(e^{tA}; t \geq 0)$ is Markovian. Since $A_N$ depends only on lattice points in $T_N$, it also holds that $A_N$ is a generator and $(e^{tA_N}; t \geq 0)$ is
Markovian, for every $N \geq 1$. As an example, see Figure 2.1 for the trajectories of the Markov chain with generator $A_5$.

The following proposition solves the Whittaker SDEs.

**Proposition 2.1.** Consider the Whittaker SDEs defined in (2.2) for an integer $N \geq 1$.

(1°) For any solution $(\xi_t)$, the following two properties hold almost surely:

$$\xi_0 \overset{\text{def}}{=} \lim_{t \to 0^+} e^{(-\log t)AN} \xi_t$$

(2.4)

exists and

$$\xi_t(a) = e^{(\log t)AN} \xi_0(a) + \sum_{b \in T_N} \int_0^t e^{[\log(t/r)]AN} (a, b) dW'_r(b)$$

$$=: \eta_t(a) + \zeta_t(a), \quad a \in T_N, \quad t \in (0, \infty).$$

(2°) For any $\xi_0 \in \mathbb{R}^{T_N}$, $(\xi_t)$ defined by (2.5) satisfies both (2.2) and (2.4).

**Proof.** The proofs of (1°) and (2°) both rely on the following almost-sure identity:

$$e^{-tAN} \xi_{e^0} = e^{-tAN} \xi_{e^t} + \int_0^t e^{-rAN} dW'_{e^r}, \quad -\infty < t \leq t_0 < \infty.$$  

(2.6)

To see this, a change of variables shows that any solution $\xi$ of the SDEs in (2.2) satisfies

$$\xi_{e^0} = \xi_{e^t} + \int_0^{t_0} A_N \xi_{e^r} dr + W'_{e^t_0} - W'_{e^t}$$

$$= \xi_{e^t} + \int_0^{t_0-t} A_N \xi_{e^{r+t}} dr + W'_{e^t_0} - W'_{e^t}, \quad -\infty < t \leq t_0 < \infty.$$  

(2.7)

Hence, for any fixed $t \in \mathbb{R}$, $(\xi_{e^{t+s}}; s \geq 0)$ obeys a linear SDE with initial condition $\xi_{t^0}$. The driving noise $(W_{e^{t+s}} - W_{e^t}; s \geq 0)$ is a continuous vector martingale that can be identified as $\int_0^s e^{\gamma/2} dB_s$, where $B_t = \int_0^t e^{-\gamma/2} d(W_{e^{t+s}} - W_{e^t})$ is a standard Brownian motion by Lévy’s characterization of Brownian motions. Pathwise uniqueness in this linear equation follows from the standard result for SDEs with Lipschitz coefficients [8, Theorem 2.5 in Chapter 5]. Moreover, the explicit solution in [8, (6.6) in Chapter 5] carries over with an obvious modification when the driving noise is generalized from Brownian motion to a continuous vector martingale. The almost-sure identity in (2.6) thus follows.

(1°) First, we prove that the limit in (2.4) exists almost surely. It suffices to show that the stochastic integral in (2.6) with $t_0 = 0$ converges almost surely as $t \to -\infty$.

Now, with probability one, the Riemann-sum approximations for stochastic integrals [8, Section 3.2.B] imply

$$\int_0^t e^{-rAN} dW'_{e^r} = -\int_0^{-t} e^{rAN} dM_r, \quad \forall \, t \in (-\infty, 0],$$  

(2.7)

where $(M_s = W'_{e^t} - W'_{e^{-s}}; s \geq 0)$ is a continuous vector martingale by the independence of increments of the Brownian motion $W'$. Since $d\langle M(b), M(b') \rangle_s = \delta_{b,b'} e^{-s} ds$, the stochastic
integrals on the right-hand side of (2.7) satisfy, for all \(a, a' \in \mathcal{T}_N\),

\[
\left\langle \sum_{b \in \mathcal{T}_N} \int_0^s e^{rA_N} (a, b) dM_r(b), \sum_{b \in \mathcal{T}_N} \int_0^s e^{rA_N} (a', b) dM_r(b) \right\rangle_s = \int_0^s \sum_{b \in \mathcal{T}_N} e^{rA_N} (a, b) e^{rA_N} (a', b) e^{-r} dr.
\]

Using the fact that \(e^{rA_N}\) for \(r \geq 0\) are stochastic matrices, we obtain convergence of the integral on the right-hand side as \(s \to \infty\). The existence of this limit and the martingale convergence theorem [8, Problem 3.19 in Section 1.3] give the almost sure convergence of the improper vector stochastic integral \(\int_0^\infty e^{-rA_N} dW'_{er}\), and hence, the limit in (2.4).

Given the two limits just obtained, we pass \(t \to -\infty\) in (2.6) and get

\[
e^{-t_0A_N} \xi_{e_0} = \xi_0 + \int_{-\infty}^{t_0} e^{-rA_N} dW'_{e_r}
\]

(2.8)

\[
\Rightarrow \xi_{e_0} = e^{t_0A_N} \xi_0 + \int_{-\infty}^{t_0} e^{(t_0 - r)A_N} dW'_{e_r} = \xi_0 + \int_{0}^{t_0} e^{[\log(e^{t_0}/r)]A_N} dW'_{e_r}.
\]

A change of variables for both sides of the last equality yields (2.5). We have proved (1°).

(2°) Given \(\xi_0 \in \mathbb{R}^{\mathcal{T}_N}\), reversing the above arguments for (2.8) and (2.6) proves that \((\xi_t)\) defined by (2.5) satisfies both (2.2) and (2.4). The proof is complete.

\[\square\]

Remark 2.2 (Spatial consistency). Let integers \(1 \leq N_0 < N_1 < \infty\) and \(\xi_0^{(N_j)} \in \mathbb{R}^{\mathcal{T}_{N_j}}\) for \(j = 0, 1\) be given such that \(e^{(N_j)}_0 (a) = e^{(N_0)}_0 (a)\) for all \(a \in \mathcal{T}_{N_0}\). Proposition 2.1 shows that the solutions of the Whittaker SDEs defined on \(\mathcal{T}_{N_0}\) coincide with the restriction to \(\mathcal{T}_{N_0}\) of the unique solutions of the Whittaker SDEs on \(\mathcal{T}_{N_1}\). This consistency together with Proposition 2.1 (2°) can be used to construct solutions of the Whittaker SDEs on \(\mathcal{T}_N\) for all \(N \in \mathbb{N}\) simultaneously from a given vector \(\xi \in \mathbb{R}^{\mathcal{T}_\infty}\).

Because of the spatial consistency in Remark 2.2 and a similar property mentioned below (2.3), we write \(A\) for \(A_N\) for the Whittaker SDEs in the sequel whenever the context is clear.

### 2.2 Representation by independent sums

In this subsection, we discuss a probabilistic representation of the Markovian semigroup \((e^{tA}; t \geq 0)\) in terms of sums of i.i.d. Bernoulli random variables. This representation uses two sets of ingredients defined as follows.

First, define a sum function \(\Sigma : \mathbb{Z}_+^2 \to \mathcal{T}_\infty\) and a difference function \(\Delta : \mathcal{T}_\infty \to \mathbb{Z}_+^2\) by

\[
\Sigma : (m_1, m_2) \mapsto \Sigma(m_1, m_2) = (\Sigma_1(m_1, m_2), \Sigma_2(m_1, m_2)) \overset{\text{def}}{=} (m_1 + 1, m_1 + m_2 + 1),
\]

\[
\Delta : (a_1, a_2) \mapsto \Delta(a_1, a_2) = (\Delta_1(a_1, a_2), \Delta_2(a_1, a_2)) \overset{\text{def}}{=} (a_1 - 1, a_2 - a_1).
\]

The linear map \(\Sigma\) is bijective with \(\Delta\) being its inverse. The following result is observed in [3, Lemma 5.7].

**Lemma 2.3.** If \(D^{(1)}\) and \(D^{(2)}\) are independent linear pure death chains on \(\mathbb{Z}_+\) such that \(k \to k - 1\) with rate \(k\) and \(D = (D^{(1)}, D^{(2)})\), then \(\Sigma D\) is a Markov chain with generator \(A\).
Proof. A jump in \( D^{(1)} \) leads to the change in \( \Sigma D \) from \((\Sigma_1 D, \Sigma_2 D)\) to \((\Sigma_1 D - 1, \Sigma_2 D - 1)\) by the definition of \( \Sigma \). The rate is \( D^{(1)} = \Sigma_1 D - 1 \), which recovers the rate \( a_1 - 1 \) for transition from \( a \) to \( a - (1,1) \) in the definition of \( A \). For the other rate in the definition of \( A \), note that a jump in \( D^{(2)} \) leads to the change in \( \Sigma D \) from \((\Sigma_1 D, \Sigma_2 D)\) to \((\Sigma_1 D, \Sigma_2 D - 1)\) with rate \( D^{(2)} = \Sigma_2 D - \Sigma_1 D \). The lemma is proved. \[ \blacksquare \]

As the second set of ingredients, we take a sequence of i.i.d. exponential variables \( \{e_n\} \) under \( P \) and define

\[ S_m(e^{-t}) \overset{\text{def}}{=} \sum_{n=1}^{m} \mathbf{1}_{\{t,\infty\}}(e_n), \quad t \in [0, \infty]. \tag{2.11} \]

We let \( e^{-t} \) parametrize \( S_m \) since the Bernoulli random variables \( \mathbf{1}_{\{t,\infty\}}(e_n) \) satisfy \( E[\mathbf{1}_{\{t,\infty\}}(e_n)] = e^{-t} \). The independent sum in (2.11) is applied in the form that, as processes with càdlàg paths,

\[ (D_t^{(j)}; t \geq 0) \overset{\text{(d)}}{=} (S_{D_0^{(j)}}(e^{-t}); t \geq 0) \tag{2.12} \]

for any deterministic initial condition \( D_0^{(j)} \in \mathbb{Z}_+ \). This probabilistic representation follows essentially because memorylessness of exponential random variables supplies the Markov property and the property that \( e_n \)'s are independent with \( E[e_n] = 1 \) gives the linear death rates. See [12, Section 6.2.1, pp.287–290].

The probabilistic representation of \( (e^{tA}) \) is now defined as follows: By Lemma 2.3, the identity \( \Delta \Sigma = \text{Id} \) and (2.12), we have

\[ \forall a, b \in \mathcal{T}_\infty, \quad e^{tA}(a, b) = P(\Sigma D_t = b|\Sigma D_0 = a) \]

\[ = P(D_t = \Delta b|D_0 = \Delta a) \]

\[ = P(S_{\Delta_t a}(e^{-t}) = \Delta_1 b) \quad P(S_{\Delta_2 a}(e^{-t}) = \Delta_2 b). \tag{2.13} \]

For the subsequent application of this representation, \( P \) and \( E \) continue to denote the probability and the expectation for the pure death processes \( D^j \) and the independent sums \( S_m(e^{-t}) \). They are distinguished from the probability \( \mathbb{P} \) and the expectation \( \mathbb{E} \) for the “random environment” defining the Brownian motions \( W''(a) \)'s.

We close this section with an application of (2.13), which is the starting point of the next section. From now on, write \( \xi \Sigma(m_1, m_2) \) for the value of \( \xi : \mathcal{T}_\infty \to \mathbb{R} \) at the lattice point \( \Sigma(m_1, m_2) \), \( S' \) for an independent copy of the process \( S \) defined by (2.11), and \( \text{Cov}[U; V] = \mathbb{E}[UV] - \mathbb{E}[U]\mathbb{E}[V] \) for complex-valued random variables \( U \) and \( V \).

**Proposition 2.4.** The stochastic integral \( \zeta \) defined in (2.5) is a Gaussian process with covariance function satisfying the following probabilistic representation:

\[ \text{Cov}[\zeta_s \Sigma(m_1, m_2); \zeta_t \Sigma(m'_1, m'_2)] = \int_0^s \prod_{j=1}^2 P \left( S_{m_{j}(\frac{r}{s})} \left( \frac{r}{t} \right) = S'_{m'_{j}(\frac{r}{s})} \left( \frac{r}{t} \right) \right) \text{dr} \tag{2.14} \]

for all \((m_1, m_2), (m'_1, m'_2) \in \mathbb{Z}_+^2 \) and \( 0 < s \leq t < \infty \).

**Proof.** By the definition of \( \zeta \) and Itô’s isometry, the covariance at points \( \Sigma(m_1, m_2) \) and \( \Sigma(m'_1, m'_2) \) satisfies

\[ \text{Cov}[\zeta_s \Sigma(m_1, m_2); \zeta_t \Sigma(m'_1, m'_2)] \]
\[ = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \int_0^s e^{[\log(s/r)]A} \left( \Sigma(m_1, m_2), \Sigma(n_1, n_2) \right) e^{[\log(t/r)]A} \left( \Sigma(m'_1, m'_2), \Sigma(n_1, n_2) \right) \, dr \]

\[ = \int_0^s \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \mathbb{P} \left( S_{m_1} \left( \frac{r}{s} \right) = n_1 \right) \mathbb{P} \left( S_{m_2} \left( \frac{r}{s} \right) = n_2 \right) \mathbb{P} \left( S'_{m_1'} \left( \frac{r}{t} \right) = n_1 \right) \mathbb{P} \left( S'_{m_2'} \left( \frac{r}{t} \right) = n_2 \right) \, dr, \]

where the last equality follows from (2.13) and the fact that $\Sigma \Delta = \text{Id}$ for the maps $\Sigma$ and $\Delta$. (Note that the above infinite series are only finite sums.) We obtain the required equality in (2.14) from the foregoing equality by summing over $n_1$ and $n_2$ and using the independent copy $S'$ of $S$. \[\blacksquare\]

3 Rescaled limit of the covariance function

In this section, we prove convergence of the covariance function in (2.14) after rescaling and specify the error bounds for the forthcoming applications.

First, the rescaling can be chosen from the central limit theorem if we consider the probabilities in (2.14). Write

\[ \mathbb{P} \left( S_{m_j} \left( \frac{r}{s} \right) = S'_{m'_j} \left( \frac{r}{t} \right) \right) = \mathbb{P} \left( \Sigma_{m_j} \left( \frac{r}{s} \right) - \Sigma_{m'_j} \left( \frac{r}{t} \right) = -m_j \left( \frac{r}{s} \right) + m'_j \left( \frac{r}{t} \right) \right), \quad (3.1) \]

with the shorthand

\[ \mathbf{W} = W - \mathbb{E}[W]. \]

Then a nontrivial limit of the random variable in (3.1) follows if we set $m_j$ and $m'_j$ to be

\[ M_j = M(x_j, s) \quad \text{and} \quad M'_j = M(y_j, t), \]

where

\[ M(x_j, s) = \left[ Ns + Ns \cdot \frac{x_j}{N^{1/2}} \right], \quad 1 \leq j \leq 2. \quad (3.2) \]

More precisely, under this setup, the central limit theorem applies in the form where

\[ \mu_j(r; N) = \mathbb{E} \left[ \frac{1}{N^{1/2}} \left( S_{M_j} \left( \frac{r}{s} \right) - S'_{M'_j} \left( \frac{r}{t} \right) \right) \right] \xrightarrow{N \to \infty} \mu_j(r), \]

\[ \sigma_j(r; N)^2 = \text{Var} \left[ \frac{1}{N^{1/2}} \left( S_{M_j} \left( \frac{r}{s} \right) - S'_{M'_j} \left( \frac{r}{t} \right) \right) \right] \xrightarrow{N \to \infty} \sigma_j(r)^2 \]

with $\sigma_j(r; N), \sigma_j(r) \geq 0$. We have

\[ \mu_j(r; N) = \frac{M_j \left( \frac{r}{s} \right) - M'_j \left( \frac{r}{t} \right)}{N^{1/2}}, \quad \sigma_j(r; N) = \left[ \frac{M_j}{N} \left( \frac{r}{s} \right) \left( 1 - \frac{r}{s} \right) + \frac{M'_j}{N} \left( \frac{r}{t} \right) \left( 1 - \frac{r}{t} \right) \right]^{1/2}, \quad (3.3) \]

\[ \mu_j(r) = (x_j - y_j) r, \quad \sigma_j(r) = \left[ r \left( 2 - \frac{r}{s} - \frac{r}{t} \right) \right]^{1/2}. \quad (3.4) \]

At the process level, we consider the rescaled version of $\zeta$ defined by

\[ \zeta_N(x, s) \overset{\text{def}}{=} \zeta_N \Sigma \left[ \left( Ns + Ns \cdot \frac{x_1}{N^{1/2}} \right), \left( Ns + Ns \cdot \frac{x_2}{N^{1/2}} \right) \right], \quad (x, s) \in \mathbb{R}^2 \times \mathbb{R}_+, \quad (3.5) \]
so that

\[
\text{Cov}[\zeta_N(x, s); \zeta_N(y, t)] = \int_0^{N_s} \prod_{j=1}^{N_s^2} P\left( S_{M_j} \left( \frac{r}{N_s} \right) = S'_{M'_j} \left( \frac{r}{N_t} \right) \right) dr
\]  

(3.6)

\[
= \int_0^s \prod_{j=1}^{N_t} N_j^{1/2} P\left( S_{M_j} \left( \frac{r}{s} \right) = S'_{M'_j} \left( \frac{r}{t} \right) \right) dr
\]  

(3.7)

for \(0 \leq s \leq t < \infty\). Notice that the integral representation in (3.7) corresponds to the “ideal case” discussed above for (3.1). If \(r, s, t\) are fixed such that \(\sigma_j(r) \neq 0\), the above view for the probability in (3.1) applies to the integrand in the form of the local central limit theorem. But due to the integral nature of the covariance function, we cannot neglect the contribution of \(r \approx 0\) as \(N \to \infty\). This is where the central limit theorem can break down. A similar issue arises if \(r \approx s = t\). Nevertheless, Poisson approximations apply over these two ranges of \(r\) so that the integrand in (3.6) is useful.

The first main result of this paper is the following theorem. It reproduces one of the main findings in [3]. Here and in what follows, we write \(V(\lambda)\) and \(V'(\lambda')\) for independent Poisson random variables with means \(\lambda\) and \(\lambda'\), respectively. Also, \((Q_t)\) stands for the probability semigroup of the two-dimensional standard Brownian motion.

**Theorem 3.1 (First main result).** For all \(0 < s \leq t < \infty\) and \(x, y \in \mathbb{R}^2\) such that either \(s < t\) or \(x \neq y\), it holds that

\[
\lim_{N \to \infty} \left( \text{Cov}[\zeta_N(x, s); \zeta_N(y, t)] - \mathcal{C}_N \right)
\]

\[
= \frac{1}{2\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \mathcal{Q}_{t-1}(x, y) \left( -\ln |y_1 - y_2| \right) Q_{s-1}(y_2, y) dy_1 dy_2
\]

(3.8)

\[
+ \int_0^{t-1} \int_{\mathbb{R}^2} \mathcal{Q}_{s-1-r'}(z, x) Q_{t-1-r'}(z, y) dz dr',
\]

where \(M_j\) and \(M'_j\) are defined before (3.2) and \(\mathcal{C}_N\), independent of \(x, s, y, t\), is defined by

\[
\mathcal{C}_N \overset{\text{def}}{=} \frac{\ln N}{4\pi} + \int_0^\infty \left( \prod_{j=1}^{N_t} P(V(r) = V'(r)) - \frac{1_{[1,\infty)}(r)}{4\pi r} \right) dr
\]

(3.9)

\[
+ \int_0^\infty \frac{1}{4\pi v} \left[ \exp \left\{-\frac{1}{4v}\right\} - 1_{[1,\infty)}(v) \right] dv.
\]

**Remark 3.2.** For \(\lambda, \lambda' \in (0, \infty)\), the independent difference \(V(\lambda) - V'(\lambda')\) is distributed explicitly as the **Skellam distribution** [15] given by

\[
P(V(\lambda) - V'(\lambda') = k) = e^{-(\lambda+\lambda')} \left( \frac{\lambda}{\lambda'} \right)^{k/2} I_k(2\sqrt{\lambda\lambda'}), \quad k \in \mathbb{Z},
\]

where \(I_k\) is the modified Bessel function of the first kind. ■
Theorem 3.1 combines the more detailed results, Theorems 3.11 and 3.17, to be proven in the rest of this section. For the proofs, we apply two schemes of integration which formalize the consideration below (3.7), and they are defined as follows. For $0 < \ell_N < r_N < 1$ and $0 < \tau_N < 1$, we subdivide $r \in [0, s]$ into the following three intervals:

$$r \in [0, s\ell_N], \ r \in [s\ell_N, sr_N], \ \text{and} \ r \in [sr_N, s] \ \text{if} \ 0 \leq t - s \leq \tau_N$$

(3.10)

or into the following two different intervals:

$$r \in [0, s\ell_N] \ \text{and} \ r \in [s\ell_N, s] \ \text{if} \ t - s > \tau_N.$$  

(3.11)

Then under (3.10), we work with the following decomposition:

$$\text{Cov}[\xi^N(x, s); \xi^N(y, t)] = \int_0^{Ns\ell_N} \prod_{j=1}^2 \mathbb{P} \left(S_{M_j} \left(\frac{r}{Ns}\right) = S'_{M_j} \left(\frac{r}{Nt}\right)\right) \, \text{dr}$$

$$+ \int_{s\ell_N}^{sr_N} \prod_{j=1}^2 N^{1/2} \mathbb{P} \left(S_{M_j} \left(\frac{r}{s}\right) = S'_{M_j} \left(\frac{r}{Nt}\right)\right) \, \text{dr}$$

(3.12)

$$+ \int_{NsrN}^{Ns} \prod_{j=1}^2 \mathbb{P} \left(S_{M_j} \left(\frac{r}{Ns}\right) = S'_{M_j} \left(\frac{r}{Nt}\right)\right) \, \text{dr.}$$

The decomposition corresponding to (3.11) is

$$\text{Cov}[\xi^N(x, s); \xi^N(y, t)] = \int_0^{Ns\ell_N} \prod_{j=1}^2 \mathbb{P} \left(S_{M_j} \left(\frac{r}{Ns}\right) = S'_{M_j} \left(\frac{r}{Nt}\right)\right) \, \text{dr}$$

(3.13)

$$+ \int_{s\ell_N}^* \prod_{j=1}^2 N^{1/2} \mathbb{P} \left(S_{M_j} \left(\frac{r}{s}\right) = S'_{M_j} \left(\frac{r}{Nt}\right)\right) \, \text{dr.}$$

**Assumption 3.3.** Fix $\eta \in (0, 1/2)$. For all integers $N \geq 16$, set $\ell_N = 1 - r_N = \tau_N = N^{-1/2+\eta}$ and choose $2 \leq \delta_N \leq N^{1/4}$ such that $\delta_N \to \infty$ and $\delta N/N \to 0$.  

This additionally specified sequence $(\delta_N)$ will be used in the normal approximations. Lastly, we introduce some convenient conditions of $x_1, x_2, y_1, y_2, s, t, N$ for the forthcoming proofs.

**Definition 3.4.** Fix $0 < T_0 < 1 < T_1 < \infty$ and let $\eta \in (0, 1/2)$ be the constant fixed in Assumption 3.3. **The primary condition (over $[T_0, T_1]$)** refers to the following condition for $x_1, x_2, y_1, y_2, s, t, N$:

$$x_1, x_2, y_1, y_2 \in [-\frac{1}{2}N^n, \frac{1}{2}N^n]; \ s, t \in [T_0, T_1]; \ N \geq 16 \ \text{such that} \ |\frac{1}{2}T_0 N^{1/2-\eta}| \geq 1. \ \ (3.14)$$

The **secondary condition (over $[T_0, T_1]$)** refers to the following condition for $x_1, x_2, y_1, y_2$:

$$|x_1 - y_1| \wedge |x_2 - y_2| \geq \frac{4}{T_0} N^{-1/2}.$$  

(3.15)

Under the primary condition, it holds that $[\frac{1}{2}T_0 N] \leq M_j, \ M'_j \leq [\frac{3}{2}T_1 N]$ so that $M_j, M'_j \geq 1$. The secondary condition will be used only in the proof of Proposition 3.10 ($4^\circ$).
3.1 Poisson approximations

In this subsection, we study the integrals

\[ \int_0^{Nst_N} \prod_{j=1}^2 \mathbb{P} \left( S_{M_j} \left( \frac{r}{N_s} \right) = S'_{M'_j} \left( \frac{r}{N_t} \right) \right) \, dr, \quad \int_N^{Ns} \prod_{j=1}^2 \mathbb{P} \left( S_{M_j} \left( \frac{r}{N_s} \right) = S'_{M'_j} \left( \frac{r}{N_t} \right) \right) \, dr \]

that appear in (3.12) and (3.13).

Lemma 3.5. Fix \( 0 < T_0 < 1 < T_1 < \infty \) and assume the primary condition (3.14). Then for any \( L \in (0, \infty) \) and \( R \in (0, s) \), we have

\[ \int_0^{NL} \prod_{j=1}^2 \mathbb{P} \left( S_{M_j} \left( \frac{r}{N_s} \right) = S'_{M'_j} \left( \frac{r}{N_t} \right) \right) \, dr \leq NL^2 \left( \frac{1}{s} + \frac{1}{t} \right) \tag{3.16} \]

and

\[ \int_N^{Ns} \prod_{j=1}^2 \mathbb{P} \left( S_{M_j} \left( \frac{r}{N_s} \right) = S'_{M'_j} \left( \frac{r}{N_t} \right) \right) \, dr \tag{3.17} \]

\[ - \prod_{j=1}^2 \mathbb{P} \left( V \left( \frac{M_j r}{Ns} \right) = V' \left( \frac{M'_j r}{Nt} \right) + M_j - M'_j \right) \, dr \]

\[ \leq N(s-R)^2 \left( \frac{1}{s} + \frac{1}{t} \right) + N(s-R) \left( \frac{t-s}{t} \right). \]

Proof. We state some preliminary results first. Write \( d_{TV} \) for the total variance distance of probability measures defined on the same space. The central tool of this proof is the following bound for Poisson approximations from [2, Theorem 1]: for independent Bernoulli random variables \( \beta_n \) with \( E[\beta_n] = p_n \), for independent Bernoulli random variables \( \beta_n \) with \( E[\beta_n] = p_n \),

\[ d_{TV} \left( \mathbb{P} \left( \sum_{n=1}^m \beta_n \in \cdot \right), \mathbb{P} \left( V \left( \sum_{n=1}^m p_n \right) \in \cdot \right) \right) \leq \frac{1 - \exp \{-\sum_{n=1}^m p_n \}}{\sum_{n=1}^m p_n} \sum_{n=1}^m p_n^2. \tag{3.18} \]

We only use the particular case that \( p_n = p \) for all \( n \), for which the above bound is reduced to \( (1 - e^{-mp})p \). Also, we recall that for probability distributions \( \mu_1, \mu_2, \nu_1, \nu_2 \) on \( \mathbb{Z} \), we have

\[ d_{TV}(\mu_1 \otimes \mu_2, \nu_1 \otimes \nu_2) \leq d_{TV}(\mu_1, \nu_1) + d_{TV}(\mu_2, \nu_2) \tag{3.19} \]

(e.g. [5, Proposition 2.3]).

We are ready to prove (3.16). By (3.18) and (3.19), for all \( m, m' \in \mathbb{N}, n \in \mathbb{Z} \), and \( p, p' \in (0, 1) \), it holds that

\[ \left| \mathbb{P}(S_m(p) = S'_{m'}(p') + n) - \mathbb{P}(V(mp) = V'(m'p') + n) \right| \leq \frac{p + p'}{2}. \tag{3.20} \]

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The foregoing inequality and the discrete product rule
\[ X Y - A B = (X - A)(Y - B) + (X - A)B + (Y - B)A \] (3.21)
imply that
\[ \int_0^{NL} \left| \prod_{j=1}^2 P \left( S_{M_j} \left( \frac{r}{N s} \right) = S'_{M'_j} \left( \frac{r}{N t} \right) \right) - \prod_{j=1}^2 P \left( V \left( \frac{M_j r}{N s} \right) = V' \left( \frac{M'_j r}{N t} \right) \right) \right| \, dr \]
\[ \lesssim \int_0^{NL} \frac{r}{N} \left( \frac{1}{s} + \frac{1}{t} \right) \, dr \]
since the \( X, Y, A, B \) in this application of (3.21) are all bounded by 1. The required bound in (3.16) follows. In the sequel, the discrete product rule in (3.21) will be used repeatedly without being mentioned.

The proof of (3.17) is similar. If \( X \) is binomial with parameters \((M, p)\), then \( M - X \) is binomial with parameters \((M, (1 - p))\). Hence,
\[ P(S_m(p) = S'_{m'}(p') + n) = P(S_m(1 - p) = S_{m'}(1 - p') + m - m' - n). \] (3.22)
By (3.20) and (3.21), the integral on the left-hand side of (3.17) can be \( \lesssim \)-bounded by
\[ \int_{NR}^{Ns} \left( \frac{Ns - r}{Ns} + \frac{Nt - r}{Nt} \right) \, dr = \int_0^{N(s-R)} \left( \frac{r}{Ns} + \frac{Nt - Ns + r}{Nt} \right) \, dr. \]
This is enough for the required bound in (3.17). The proof is complete. \( \blacksquare \)

As an immediate result of Lemma 3.5, we obtain the following integrated Poisson approximations.

**Proposition 3.6.** Fix \( 0 < T_0 < 1 < T_1 < \infty \). Under the primary condition (3.14),
\[ \int_0^{Nst_N} \left| \prod_{j=1}^2 P \left( S_{M_j} \left( \frac{r}{Ns} \right) = S'_{M'_j} \left( \frac{r}{Nt} \right) \right) - \prod_{j=1}^2 P \left( V \left( \frac{M_j r}{Ns} \right) = V' \left( \frac{M'_j r}{Nt} \right) \right) \right| \, dr \lesssim C(T_0, T_1)N\ell_N^2. \]

If, in addition, we assume \( 0 \leq t - s \leq \tau_N \), then it holds that
\[ \int_{NsN}^{Ns} \left| \prod_{j=1}^2 P \left( S_{M_j} \left( \frac{r}{Ns} \right) = S'_{M'_j} \left( \frac{r}{Nt} \right) \right) - \prod_{j=1}^2 P \left( V \left( \frac{M_j (Ns - r)}{Ns} \right) = V' \left( \frac{M'_j (Nt - r)}{Nt} \right) + M_j - M'_j \right) \right| \, dr \lesssim C(T_0, T_1)N(1 - r_N)^2. \]
Under Assumption 3.3, these integrals converge to zero.

Since \( Nst_N \to \infty \) for \( s > 0 \) by Assumption 3.3, we still need to compute the limit of
\[ \int_0^{N(sN)} P(V(M_j r/Ns) = V'(M'_j r/Nt)) \, dr \] by passing limit under the integral sign. The other integral from Proposition 3.6 can be written as
\[ \int_0^{N(s - sN)} \prod_{j=1}^2 P \left( V \left( \frac{M_j r}{Ns} \right) = V' \left( \frac{M'_j (Nt - Ns) + M'_j r}{Nt} \right) + M_j - M'_j \right) \, dr, \] (3.23)
where $N(s - s r_N) \to \infty$ for $s > 0$. Under the assumption $0 \leq t - s \leq \tau_N$, we have $M_j - M'_j \to \text{sgn}(x_j - y_j) \cdot \infty$ whenever $x_j \neq y_j$, and so in this case, the probability indexed by $j$ in (3.23) is zero in the limit. We prove the next three lemmas to pass these limits. From now on, write
\[
\vartheta(x) = \frac{1}{\sigma \sqrt{2\pi}} \exp \left\{ -\frac{x^2}{2\sigma^2} \right\}
\]
for $\sigma \in (0, \infty)$.

**Lemma 3.7.** For all $\lambda, \lambda' \in (0, \infty)$, it holds that
\[
\sup_{a \in \mathbb{Z}} \left| \mathbb{P} \left( V(\lambda) = V'(\lambda') + a \right) - \frac{1}{(\lambda + \lambda')^{1/2}} \vartheta \left( \frac{-\lambda + \lambda' + a}{(\lambda + \lambda')^{1/2}} \right) \right| \leq \frac{1}{(\lambda + \lambda')} + \frac{1}{(\lambda + \lambda')^{1/2}} \exp \left\{ -\frac{\pi^2(\lambda + \lambda')}{4} \right\}.
\]

**Proof.** We apply the argument for the local central limit theorem of lattice distributions. See [7, Chapters XV and XVI]. By Fourier inversions,
\[
f(a) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ia\theta} \sum_{b \in \mathbb{Z}} f(b) e^{i\theta b} d\theta, \quad a \in \mathbb{Z};
\]
\[
\vartheta(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\theta x} \exp \left\{ -\frac{\theta^2\sigma^2}{2} \right\} d\theta, \quad x \in \mathbb{R}.
\]
Hence, writing $\phi_{\lambda, \lambda'}(\theta)$ for $\mathbb{E} \exp \{i\theta [V(\lambda) - V'(\lambda')]\}$, it holds that, for all $a \in \mathbb{Z}$,
\[
(\lambda + \lambda')^{1/2} \mathbb{P} \left( V(\lambda) = V'(\lambda') + a \right) - \vartheta \left( \frac{-\lambda + \lambda' + a}{(\lambda + \lambda')^{1/2}} \right) = \frac{1}{2\pi} \int_{|\theta| \leq \pi(\lambda + \lambda')^{1/2}} e^{i\theta \frac{\lambda + \lambda'}{(\lambda + \lambda')^{1/2}}} \left[ \phi_{\lambda, \lambda'} \left( \frac{\theta}{(\lambda + \lambda')^{1/2}} \right) - \exp \left\{ -\frac{\theta^2}{2} \right\} \right] d\theta
\]
\[
+ \frac{1}{2\pi} \int_{|\theta| \geq \pi(\lambda + \lambda')^{1/2}} e^{i\theta \frac{\lambda + \lambda'}{(\lambda + \lambda')^{1/2}}} \exp \left\{ -\frac{\theta^2}{2} \right\} d\theta
\]
\[
= I_{3.27} + II_{3.27}.
\]

We show that the decomposition in (3.27) implies (3.24). To bound $I_{3.27}$, we consider
\[
\phi_{\lambda, \lambda'} \left( \frac{\theta}{(\lambda + \lambda')^{1/2}} \right) \exp \left\{ \frac{\theta^2}{2} \right\}
\]
\[
= \exp \left\{ \lambda \left( e^{i\theta/(\lambda + \lambda')^{1/2}} - 1 \right) - \frac{i\theta}{(\lambda + \lambda')^{1/2}} \right\} + \lambda' \left( e^{-i\theta/(\lambda + \lambda')^{1/2}} - 1 + \frac{i\theta}{(\lambda + \lambda')^{1/2}} \right) + \frac{\theta^2}{2} \]
\[
= \exp \left\{ (\lambda + \lambda') \left[ \cos \left( \frac{\theta}{(\lambda + \lambda')^{1/2}} \right) - 1 + \frac{\theta^2}{2(\lambda + \lambda')} \right] \right\}
\]
\[
+ i(\lambda - \lambda') \left[ \sin \left( \frac{\theta}{(\lambda + \lambda')^{1/2}} \right) - \frac{\theta}{(\lambda + \lambda')^{1/2}} \right] \right\}.
\]

(3.28)
By the inequality
\[ |e^{z_1} - e^{z_2}| \leq \max\{|e^{z_1}|, e^{|z_2|}\} \cdot |z_1 - z_2|, \quad \forall z_1, z_2 \in \mathbb{C}, \tag{3.29} \]
and Taylor’s theorem, (3.28) implies that
\[ |\phi_{\lambda, \lambda'} \left( \frac{\theta}{(\lambda + \lambda')^{1/2}} \right) \exp \left\{ \frac{\theta^2}{2} \right\} - 1 | \leq (\lambda + \lambda') \cdot \frac{|\theta|^3}{(\lambda + \lambda')^{3/2}}, \quad \forall |\theta| \leq \pi(\lambda + \lambda')^{1/2}, \tag{3.30} \]
and so
\[ |I_{3.27}| \leq \int_{|\theta| \leq \pi(\lambda + \lambda')^{1/2}} \exp \left\{ -\frac{\theta^2}{2} \right\} \frac{|\theta|^3}{(\lambda + \lambda')^{1/2}} d\theta \leq \frac{1}{(\lambda + \lambda')^{1/2}}. \tag{3.31} \]

To bound \( \Pi_{3.27} \), we use the simple bound
\[
\frac{1}{2\pi} \int_{|\theta| \geq a} \exp \left\{ -\frac{c\theta^2}{b} \right\} d\theta \leq \frac{1}{2\pi} \int_{|\theta| \geq a} \exp \left\{ -\frac{c\theta^2}{2b} \right\} d\theta \cdot \exp \left\{ -\frac{ca^2}{2b} \right\} \]
\[ \leq \left( \frac{b}{2\pi c} \right)^{1/2} \exp \left\{ -\frac{ca^2}{2b} \right\}, \quad \forall a, b, c \in (0, \infty). \tag{3.32} \]

With \( a = \pi(\lambda + \lambda')^{1/2}, b = 2 \) and \( c = 1 \), we get
\[ |\Pi_{3.27}| \leq \frac{1}{\pi^{1/2}} \exp \left\{ -\frac{\pi^2(\lambda + \lambda')}{4} \right\}. \tag{3.33} \]
The bound in (3.24) follows upon applying (3.31) and (3.33) to (3.27).

**Lemma 3.8.** For all \( 0 < a \leq b < \infty \) and \( x, y \in \mathbb{R} \), it holds that
\[ |g(a; x) - g(b; y)| \lesssim \frac{|b - a|}{a^{3/2}} + \frac{|x - y|}{b}. \]

**Proof.** We have the following bounds:
\[
\left| \frac{d}{da} \frac{1}{\sqrt{2\pi a}} \exp \left\{ -\frac{x^2}{2a} \right\} \right| \leq \frac{1}{2\sqrt{2\pi}} \left( \frac{1}{a^{3/2}} + \frac{1}{a^{3/2}} e^{-1} \right) \leq \frac{1}{\sqrt{2\pi a^3}}, \tag{3.34} \]
\[
\left| \frac{d}{dx} \frac{1}{\sqrt{2\pi a}} \exp \left\{ -\frac{x^2}{2a} \right\} \right| \leq \frac{\sqrt{2}}{\sqrt{2\pi a^3}}, \tag{3.35} \]
where the first inequality in (3.34) uses the fact that \( xe^{-x} \leq e^{-1} \) for all \( x \geq 0 \) and (3.35) uses \( xe^{-x^2} \leq 1 \) for all \( x \geq 0 \). The required inequality then follows by applying the mean value theorem to \( [g(a; x) - g(b; x)] + [g(b; x) - g(b; y)] \).

As the setup of the next lemma, let \( \lambda_j(r), \lambda'_j(r), \Lambda_j(r), \Lambda'_j(r) \) be increasing functions taking values in \((0, \infty)\) for all \( r \in (0, \infty) \) and \( a_j, \Lambda_j \in \mathbb{Z} \). We assume that \( \lambda_j(r) \) and \( \lambda'_j(r) \) are linear:
\[ \lambda_j(r) = \alpha_j + \beta_j r, \quad \lambda'_j(r) = \alpha'_j + \beta'_j r. \tag{3.36} \]
Next, define an auxiliary function \( g(r) = \prod_{j=1}^{2} g_j(r) \) by
\[ g_j(r) = g \left( \lambda_j(r) + \lambda'_j(r); -\lambda_j(r) + \lambda'_j(r) + a_j \right) \tag{3.37} \]
and \( G(r) = \prod_{j=1}^{2} G_j(r) \) for similarly defined \( G_j(r) \) using \( \Lambda_j, \Lambda'_j, A_j \) in place of \( \lambda_j, \lambda'_j, a_j \).
Lemma 3.9. Under the above setup, the following two inequalities hold: for all \( r \in [1, \infty) \)

\[
\left| \prod_{j=1}^{2} \mathbf{P} \left( V(\lambda_j(r)) = V' \left( \lambda'_j(r) \right) + a_j \right) - g(r) \right| 
\leq \frac{1}{\prod_{j=1}^{2}[\lambda_j(r) + \lambda'_j(r) + 1]} + \sum_{1 \leq i \neq j \leq 2} \frac{1}{[\lambda_i(r) + \lambda'_i(r)][\lambda_j(r) + \lambda'_j(r)]^{1/2}} \tag{3.38}
\]

\[
|g(r) - G(r)| 
\leq \prod_{j=1}^{2} K_j \frac{|\lambda_j(r) - \Lambda_j(r)| + |\lambda'_j(r) - \Lambda'_j(r)| + |a_j - A_j|}{[\lambda_j(r) + \lambda'_j(r)] \wedge [\Lambda_j(r) + \Lambda'_j(r)]} 
+ \sum_{1 \leq i \neq j \leq 2} K_i \frac{|\lambda_i(r) - \Lambda_i(r)| + |\lambda'_i(r) - \Lambda'_i(r)| + |a_i - A_i|}{[\lambda_i(r) + \lambda'_i(r)] \wedge [\Lambda_i(r) + \Lambda'_i(r)]} \tag{3.39}
\]

where \( K_j \overset{\text{def}}{=} 1/\{[\lambda_j(1) + \lambda'_j(1)]^{1/2} \wedge [\Lambda_j(1) + \Lambda'_j(1)]^{1/2} \wedge 1 \} \).

Proof. By Lemma 3.7, the left-hand side of (3.38) is \( \leq \)-bounded by

\[
\prod_{j=1}^{2} \left( \frac{1}{[\lambda_j(r) + \lambda'_j(r)]} + \frac{1}{[\lambda_j(r) + \lambda'_j(r)]^{1/2}} \exp \left\{ -\frac{\pi^2[\lambda_j(r) + \lambda'_j(r)]}{4} \right\} \right) 
+ \sum_{1 \leq i \neq j \leq 2} \left( \frac{1}{[\lambda_i(r) + \lambda'_i(r)]} + \frac{1}{[\lambda_i(r) + \lambda'_i(r)]^{1/2}} \exp \left\{ -\frac{\pi^2[\lambda_i(r) + \lambda'_i(r)]}{4} \right\} \right) g_j(r).
\]

We bound the exponentials in the foregoing display by \( e^{-x} \leq 1/\sqrt{x} \). Hence, (3.38) follows from the foregoing bound and the definition (3.37) of \( g_j \).

For the proof of (3.39), it is enough to note that by Lemma 3.8 and the assumed monotonicity of the functions, for all \( r \in [1, \infty) \),

\[
\left| g \left( \lambda_j(r) + \lambda'_j(r); -\lambda_j(r) + \lambda'_j(r) + a_j \right) - g \left( \Lambda_j(r) + \Lambda'_j(r); -\Lambda_j(r) + \Lambda'_j(r) + A_j \right) \right| 
\leq \frac{\left| \lambda_j(r) + \lambda'_j(r) - \Lambda_j(r) - \Lambda'_j(r) \right|}{[\lambda_j(r) + \lambda'_j(r)]^{3/2} \wedge [\Lambda_j(r) + \Lambda'_j(r)]^{3/2}} 
+ \frac{\left| \lambda_j(r) - \lambda'_j(r) - a_j - \Lambda_j(r) + \Lambda'_j(r) + A_j \right|}{[\lambda_j(r) + \lambda'_j(r)] \wedge [\Lambda_j(r) + \Lambda'_j(r)]} 
\leq \frac{\left| \lambda_j(r) - \Lambda_j(r) \right| + \left| \lambda'_j(r) - \Lambda'_j(r) \right| + \left| a_j - A_j \right|}{\{[\lambda_j(1) + \lambda'_j(1)]^{1/2} \wedge [\Lambda_j(1) + \Lambda'_j(1)]^{1/2} \wedge 1 \} \times \{[\lambda_j(r) + \lambda'_j(r)] \wedge [\Lambda_j(r) + \Lambda'_j(r)]\}}.
\]

The proof is complete. \( \blacksquare \)

The next proposition is the last step for the Poisson approximations.

Proposition 3.10. Fix \( 0 < T_0 < 1 < T_1 < \infty \) and let Assumption 3.3 be in force. 

(1°) For all \( 0 < s \leq t < \infty \) and \( x, y \in \mathbb{R}^2 \),

\[
\lim_{N \to \infty} \int_0^{Nt/N} \left( \prod_{j=1}^{2} \mathbf{P} \left( V \left( \frac{M_jr}{Ns} \right) = V' \left( \frac{M'_j r}{Nt} \right) \right) - \frac{1}{4\pi r} \mathbf{1}_{[1, \infty)}(r) \right) dr
\]

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\[
= \int_0^\infty \left( \prod_{j=1}^{2} P \left( V(r) = V'(r) \right) - \frac{1}{4\pi r} \right) dr.
\]

(2°) Under the primary condition \([T_0, T_1]\), it holds that
\[
\int_0^{N_{st}\cdot N} \left| \prod_{j=1}^{2} P \left( V \left( \frac{M_j r}{N_s} \right) = V' \left( \frac{M'_j r}{N_t} \right) \right) - \frac{1}{4\pi r} \right| dr \leq C(T_0, T_1) (1 + |x|^2 |y|^2).
\]

(3°) For all \(s, t \in (0, \infty)\) with \(s = t\) and \(x, y \in \mathbb{R}^2\) with \(x \neq y\), we have
\[
\lim_{N \to \infty} \int_{NsrN}^{Ns} \prod_{j=1}^{2} P \left( V \left( \frac{M_j r}{N_s} \right) = V' \left( \frac{M'_j r}{N_t} \right) \right) dr = 0.
\]

(4°) Under the primary and secondary conditions over \([T_0, T_1]\), it holds that
\[
\int_{NsrN}^{Ns} \left| \prod_{j=1}^{2} P \left( V \left( \frac{M_j r}{N_s} \right) = V' \left( \frac{M'_j r}{N_t} \right) \right) - \frac{1}{4\pi r} \right| dr \leq C(T_0, T_1) \left( 1 + |x|^2 |y|^2 + |\ln |x - y|| \right).
\]

**Proof.** We present the proofs of the first two statements and the last two separately. (1°) and (2°). We choose
\[
\lambda_j(r) = \frac{M_j r}{N_s}, \quad \lambda'_j(r) = \frac{M'_j r}{N_t}, \quad \Lambda_j(r) = \Lambda'_j(r) = r, \quad a_j = A_j = 0
\]
in the setup of Lemma 3.9. Now, (3.38) gives
\[
\left| \prod_{j=1}^{2} P \left( V \left( \frac{M_j r}{N_s} \right) = V' \left( \frac{M'_j r}{N_t} \right) \right) - g(r) \right| \leq \frac{C(T_0, T_1)}{r^{3/2}}, \quad \forall r \in [1, \infty),
\]
since the primary condition applies. The required limit in (1°) follows from the dominated convergence theorem.

Next, notice that \(G(r) = \prod_{j=1}^{2} g(2r; 0) = 1/(4\pi r)\). Hence, (3.39) and (3.40) give the following bound: for all \(r \in [1, \infty),\)
\[
\left| \prod_{j=1}^{2} P \left( V \left( \frac{M_j r}{N_s} \right) = V' \left( \frac{M'_j r}{N_t} \right) \right) - \frac{1}{4\pi r} \right|
\leq C(T_0, T_1) \left[ \frac{1}{r^{3/2}} + \sum_{j=1}^{2} \left( \left| \frac{M_j}{N_s} - 1 \right| + \left| \frac{M'_j}{N_t} - 1 \right| \right) + \frac{1}{r^{1/2}} \sum_{j=1}^{2} \left( \left| \frac{M_j}{N_s} - 1 \right| + \left| \frac{M'_j}{N_t} - 1 \right| \right) \right]
\leq C(T_0, T_1) \left( \frac{1}{r^{3/2}} + \frac{1}{N} (1 + |x| + |y|)^2 + \frac{1}{r^{1/2}} N^{1/2} (1 + |x| + |y|) \right)
\]
(3.41)
since
\[
|M_j/(N_s) - 1| \leq C(T_0, T_1) (N^{-1} + N^{-1/2} |x_j|)
\]
(3.42)
and a similar bound for $|M_j^/(Nt) - 1|$ holds. The bound in (2°) follows upon integrating both sides of (3.41) over $r \in [1, \bar{N}s\ell_N]$.

(3°) and (4°). Now, we work with the alternative expressions in (3.23) for the integrals under consideration. Choose

$$
\lambda_j(r) = \frac{M_j}{N_s}r, \quad \lambda'_j(r) = \frac{M_j'}{N_t}(Nt - Ns) + \frac{M_j'}{N_t}r,
$$

$$
\Lambda_j(r) = r, \quad \Lambda'_j(r) = \frac{Nt + N^{1/2}y_j}{N_t}(Nt - Ns) + r, \quad a_j = A_j = M_j - M'_j
$$

for the setup of Lemma 3.9. We proceed with the following steps.

**Step 1.** First, by (3.38), we have

$$
\left| \prod_{j=1}^{2} P \left( V \left( \frac{M_j r}{N_s} \right) = V' \left( \frac{M_j'}{N_t} \right) + \frac{M_j'}{N_t} (Nt - Ns) \right) + M_j - M'_j \right| - g(r)
$$

$$
\leq \frac{C(T_0, T_1)}{r^{3/2}}, \quad \forall \ r \geq 1.
$$

Hence,

$$
\int_1^{N(s-sr_N)} \prod_{j=1}^{2} P \left( V \left( \frac{M_j r}{N_s} \right) = V' \left( \frac{M_j'}{N_t} \right) + \frac{M_j'}{N_t} (Nt - Ns) \right) + M_j - M'_j \right| - g(r) \, dr
$$

is bounded by $C(T_0, T_1)$ and tends to zero by dominated convergence.

**Step 2.** We handle $\int_1^{N(s-sr_N)} |g(r) - G(r)| \, dr$ by using (3.39). To this end, note that under the setup in (3.43),

$$
|\lambda_j(r) - \Lambda_j(r)| + |\lambda'_j(r) - \Lambda'_j(r)| + |a_j - A_j|
$$

$$
\leq r \left( \frac{M_j}{N_s} - 1 \right) + \frac{1}{N_t} \cdot N(t - s) + r \left( \frac{M_j'}{N_t} - 1 \right)
$$

$$
\leq C(T_0, T_1) \left( \tau_N + r N^{-1} + r N^{-1/2} |x_j| + r N^{-1/2} |y_j| \right), \quad \forall \ r \geq 1,
$$

where the second inequality uses (3.42) and the last one uses $N^{1/2} \tau_N \leq 1$ by Assumption 3.3. Hence, (3.39) in the present case can be simplified to

$$
|g(r) - G(r)| \leq C(T_0, T_1) \prod_{j=1}^{2} (N^{-1/2} + N^{-1/2} |x_j| + N^{-1/2} |y_j|)
$$

$$
+ C(T_0, T_1) \sum_{j=1}^{2} \frac{N^{-1/2} + N^{-1/2} |x_j| + N^{-1/2} |y_j|}{r^{1/2}}
$$

so that

$$
\int_1^{N(s-sr_N)} |g(r) - G(r)| \, dr
$$
\begin{equation}
\leq C(T_0, T_1) \left( (1 - r_N) \prod_{j=1}^{2} (1 + |x_j| + |y_j|) + (1 - r_N)^{1/2} \sum_{j=1}^{2} (1 + |x_j| + |y_j|) \right)
\leq C(T_0, T_1)(1 - r_N)^{1/2}(1 + |x|^2|y|^2). \tag{3.44}
\end{equation}

**Step 3.** Recall the setup in (3.43) and the assumption $0 \leq t - s \leq \tau_N$. We consider

\begin{equation}
\int_{1}^{N(s-sr_N)} G(r) dr = \int_{1}^{N(s-sr_N)} \prod_{j=1}^{2} \left( 2r + \frac{Nt + N^{1/2}ty_j}{Nt} (Nt - Ns) \right) \frac{Nt + N^{1/2}ty_j}{Nt} (Nt - Ns) + M_j - M'_j \right) dr. \tag{3.45}
\end{equation}

This step is where we use the secondary condition (3.15).

To bound the Gaussian densities in (3.45), we first note that the variances therein satisfy the following bounds:

\begin{equation}
2r + C(T_0, T_1)N(t - s) \leq 2r + \frac{Nt + N^{1/2}ty_j}{Nt} (Nt - Ns) \leq 2r + C'(T_0, T_1)N(t - s), \tag{3.46}
\end{equation}

where the primary condition (3.14) is used. For the other arguments in the Gaussian densities, we consider

\begin{align}
\frac{Nt + N^{1/2}ty_j}{Nt} (Nt - Ns) + M_j - M'_j \\
\geq \frac{Nt + N^{1/2}ty_j}{Nt} (Nt - Ns) + Ns + Ns \cdot \frac{x_j}{N^{1/2}} - 1 - Nt - Nt \cdot \frac{y_j}{N^{1/2}} \\
\geq -|y_j|N^{1/2}N + N^{1/2}s(x_j - y_j) - |y_j|N^{1/2}\tau_N - 1 \\
\geq N^{1/2}s(x_j - y_j) - 2, \tag{3.47}
\end{align}

where the last inequality uses the primary condition and the definition of $\tau_N$ in Assumption 3.3. Similarly,

\begin{align}
-\frac{Nt + N^{1/2}ty_j}{Nt} (Nt - Ns) - M_j + M'_j \geq N^{1/2}t(y_j - x_j) - 2.
\end{align}

By (3.47) and the foregoing inequality, the secondary condition (3.15) implies that

\begin{equation}
\sum_{j=1}^{2} \left( \frac{Nt + N^{1/2}ty_j}{Nt} (Nt - Ns) + M_j - M'_j \right)^2 \geq \frac{1}{2} NT_0^2|x - y|^2. \tag{3.48}
\end{equation}

Recall that $1 - r_N = \tau_N$. Applying (3.46) and (3.48) to (3.45), we get

\begin{align}
\int_{1}^{N(s-sr_N)} G(r) dr &\leq \int_{1}^{NT_1\tau_N} C(T_0, T_1) \exp \left\{ - \frac{C'(T_0, T_1)|x - y|^2}{r + N(t - s)} \right\} dr \\
&= \int_{1/N}^{T_1\tau_N} C(T_0, T_1) \exp \left\{ - \frac{C'(T_0, T_1)|x - y|^2}{r + (t - s)} \right\} dr
\end{align}
Besides, since $N^{-1} \leq \tau_N \leq N^{-1/2} \leq C(T_0, T_1)|x - y|$ under the secondary condition (3.15), the last inequality implies that
\[
\int_1^{N(s - sr_N)} G(r)dr \leq C(T_0, T_1)(1 + |\ln |x - y||). \tag{3.50}
\]

**Step 4.** Finally, (3°) follows from the conclusion of Step 1, (3.44) and (3.49). (Notice that $\int_{0+} r^{-1}e^{-r^{-1}}dr < \infty$.) As for (4°), it is enough to prove the required inequality with the term $\mathbb{1}_{[1, \infty)}(r)/(4\pi r)$ removed. Then we use the conclusion of Step 1, (3.44) and (3.50).

The following theorem summarizes the asymptotic results proven in Propositions 3.6 and 3.10.

**Theorem 3.11.** (1°) For all $0 < s \leq t < \infty$ and $x, y \in \mathbb{R}^2$,
\[
\lim_{N \to \infty} \int_0^{Nsr_N} \left( \prod_{j=1}^2 \mathbb{P} \left( S_{M_j} \left( \frac{r}{Ns} \right) = S_{M_j'} \left( \frac{r}{Nt} \right) \right) - \frac{\mathbb{1}_{[1, \infty)}(r)}{4\pi r} \right)dr
= \int_0^\infty \left( \prod_{j=1}^2 \mathbb{P} \left( V(r) = V'(r) \right) - \frac{\mathbb{1}_{[1, \infty)}(r)}{4\pi r} \right)dr.
\]

(2°) If $s = t$ and $x \neq y$, then
\[
\lim_{N \to \infty} \int_0^{Ns} \prod_{j=1}^2 \mathbb{P} \left( S_{M_j} \left( \frac{r}{Ns} \right) = S_{M_j'} \left( \frac{r}{Nt} \right) \right)dr = 0.
\]

### 3.2 Normal approximations

We study the remaining integrals in (3.12) and (3.13):
\[
\int_{sr_N}^{s} \prod_{j=1}^2 N^{1/2} \mathbb{P} \left( S_{M_j} \left( \frac{r}{s} \right) = S_{M_j'} \left( \frac{r}{t} \right) \right) dr,
\int_{sr_N}^{s} \prod_{j=1}^2 N^{1/2} \mathbb{P} \left( S_{M_j} \left( \frac{r}{s} \right) = S_{M_j'} \left( \frac{r}{t} \right) \right) dr
\]
by normal approximations. Let us begin with an elementary result. For a Bernoulli random variable $\beta(p)$ with mean $p$, we set
\[
\psi_p(u) \overset{\text{def}}{=} \mathbb{E}[e^{iu\beta(p)}] = pe^{iu(1-p)} + (1 - p)e^{-iu}, \tag{3.51}
\]
where $\overline{\beta}(p) = \beta(p) - p$.

**Lemma 3.12.** It holds that
\[
|\psi_p(u)|^2 = 1 - 2p(1 - p)(1 - \cos u), \quad \forall \ p \in (0, 1), \ u \in \mathbb{R}; \tag{3.52}
\sup_{p \in (0, 1)} \sup_{u \in [-1, 1]} |\psi_p(u) - 1| < 1. \tag{3.53}
\]
Proof. To prove (3.52), notice that, for all \( p \in (0,1) \) and \( u \in \mathbb{R} \),
\[
|\psi_p(u)|^2 = |E[e^{iu\beta(p)}]|^2 = |p \cos u + ip \sin u + (1 - p)|^2
= (1 - p)^2 + 2p(1 - p) \cos u + p^2
= 1 - 2p(1 - p)(1 - \cos u).
\]
Also, to show (3.53), we note that
\[
|\psi_p(u) - 1| \leq (1 - p)|e^{-ip} - 1| + p|e^{ip(1 - p)} - 1|
\]
by (3.51). We have \(|e^{i\pi/3} - 1| = 1, \pi/3 > 1\), and the fact that \( u \mapsto |e^{iu} - 1|^2 = 2 - 2 \cos u \)
is an increasing function on \([0, \pi]\). Hence, the required inequality follows from the foregoing inequality. \(\blacksquare\)

The following proposition gives a counterpart of Lemma 3.5 in the context of normal approximations. Recall that we choose the auxiliary constants \( \delta_N \) in Assumption 3.3.

Lemma 3.13. Fix \( 0 < T_0 < 1 < T_1 < \infty \) and let Assumption 3.3 and the primary condition (3.14) be in force. Then for all \( 0 < r < s \) and \( 1 \leq j \leq 2 \), we have
\[
\left| N^{1/2} \mathbb{P} \left( S_{M_j} \left( \frac{r}{s} \right) = S'_{M'_j} \left( \frac{r}{t} \right) \right) - g(\sigma_j(r; N)^2, \mu_j(r; N)) \right|
\leq \frac{\delta_N^2}{N^{1/2}} + \frac{1}{N^{1/2} \sigma_j(r; N)^2} \int_{\delta_N \sigma_j(r; N)}^{N^{1/2} \sigma_j(r; N)} \theta^2 \exp \left\{ -\frac{\theta^2}{5} \right\} \, d\theta
+ \frac{1}{\sigma_j(r; N)} \int_{N^{1/2} \sigma_j(r; N)}^{\infty} \exp \left\{ -\frac{\theta^2}{5} \right\} \, d\theta.
\]

Proof. Throughout this proof, we use the following shorthand: \( q = r/s, q' = r/t, M = M_j, M' = M'_j, \mu = \mu_j(r; N), \) and \( \sigma = \sigma_j(r; N) \).

We reconsider the proof of Lemma 3.7. Applying (3.25) and (3.26), we obtain the following decomposition:
\[
N^{1/2} \mathbb{P} \left( S_{M}(q) = S'_{M'}(q') \right) - g(\sigma^2; \mu)
= \frac{1}{2\pi} \left( \int_{|\theta| \leq \delta_N} + \int_{\delta_N < |\theta| \leq N^{1/2}} + \int_{N^{1/2} < |\theta| \leq N^{1/2} \pi} \right)
\times e^{i\theta \mu} \left[ \psi_q \left( \frac{\theta}{N^{1/2}} \right)^M \psi_{q'} \left( -\frac{\theta}{N^{1/2}} \right)^{M'} - \exp \left\{ -\frac{\sigma^2 \theta^2}{2} \right\} \right] \, d\theta
+ \frac{1}{2\pi} \int_{|\theta| \geq N^{1/2} \pi} e^{i\theta \mu} \exp \left\{ -\frac{\sigma^2 \theta^2}{2} \right\} \, d\theta
= I_{\text{3.55}} + I_{\text{3.55}} + \text{III}_{\text{3.55}} + \text{IV}_{\text{3.55}}.
\]

The required bound in (3.54) follows from (3.56), (3.62) and (3.63) to be proven below.

Step 1. To bound \( I_{\text{3.55}} \), we recall the following bound from Stein’s method for normal approximations: if \( \beta_1, \ldots, \beta_n \) are independent mean zero random variables with variances \( \sigma_i^2 = \text{Var}(\beta_1) \) satisfying \( \sum_{i=1}^{n} \sigma_i^2 = 1 \), then the law of \( \sum_{i=1}^{n} \beta_i \) satisfies
\[
\mathcal{W}_1 \left( \mathcal{L} \left( \sum_{i=1}^{n} \beta_i \right), \mathcal{L}(Z) \right) \leq \sum_{i=1}^{n} E[|\beta_i|^3],
\]

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where $Z$ is a standard normal random variable and $W_1$ is the $L_1$-Wasserstein distance. The distance $W_1$ can be represented as

$$W_1(\nu_1, \nu_2) = \inf E[|X - Y|]$$

for $(X, Y)$ ranging over all couplings on a joint probability space such that their marginals are given by $\nu_1$ and $\nu_2$. See [4, p.64 and Corollary 4.2] for these properties.

Now, we obtain from the global Lipschitz continuity of $x \mapsto e^{10\sigma x}$ on the real line and the last two displays that

$$|E \left[ \exp \left\{ i\theta \overline{\Sigma}_M(q) - \overline{\Sigma}_{M'}(q') \right\} \right] - E[\exp(i\theta \sigma Z)]| \leq |\theta| |\sigma| \left( M \exp \left\{ \left| \overline{\beta}(q) \right|^{3} \right\} + M' \exp \left\{ \left| \overline{\beta}(q') \right|^{3} \right\} \right) \leq \frac{|\theta|}{N^{1/2}}$$

by the definition (3.3) of $\sigma = \sigma_j(r; N)$ and the fact that $\max\{(1 - p)^2 + p^2; p \in [0, 1]\} = 1$. It follows that

$$|I_{3.55}| \leq \frac{\delta_N^2}{2\pi N^{1/2}}. \quad (3.56)$$

**Step 2.** For $II_{3.55}$, we view

$$\left| \psi_q \left( \frac{\theta}{N^{1/2}} \right)^M \psi_{q'} \left( - \frac{\theta}{N^{1/2}} \right)^{M'} \exp \left\{ \frac{\sigma^2 \theta^2}{5} \right\} - \exp \left\{ - \frac{3\sigma^2 \theta^2}{10} \right\} \right|, \forall |\theta| \leq N^{1/2}, \quad (3.57)$$

as $|e^{z_1} - e^{z_2}|$ and bound this difference in the way of (3.29). This use of (3.29) is legitimate since we can take the logarithms of these $\psi_q(\theta/N^{1/2})$ and $\psi_{q'}(\theta/N^{1/2})$ by (3.53).

To bound the term corresponding to $\max\{|e^{z_1}, |e^{z_2}|\}$ in (3.29), we use (3.52) and the inequality $1 - x \leq e^{-x}$ for all $x \geq 0$. They give

$$\left| \psi_q \left( \frac{\theta}{N^{1/2}} \right)^M \exp \left\{ \frac{Mq(1 - q)\theta^2}{5N} \right\} \leq \exp \left\{ \frac{2Mq(1 - q)(\cos \frac{\theta}{N^{1/2}} - 1 + \frac{\theta^2}{10N})}{5N} \right\} \leq \exp \left\{ - \frac{Mq(1 - q)\theta^2}{5N} \right\}, \quad (3.58)$$

where the last inequality holds whenever $|\theta| \leq N^{1/2}$ since $\cos x - 1 + x^2/5 \leq 0$ on $x \in [-1, 1]$; the same bound with $(q, M)$ replaced by $(q', M')$ holds.

Next, we bound the term corresponding to $|z_1 - z_2|$ in (3.29). We expand $\kappa(\xi u) = \log \psi_p(u)$ for $u \in \mathbb{R}$ around 0, where $\kappa$ denotes the cumulant of $\overline{\beta}(p)$. Since $\log \psi_p(0) = 0$,

$$\log \psi_p(u) = \frac{d}{dr} \log \psi_p(r) \big|_{r=0} \cdot u + \frac{d^2}{dr^2} \log \psi_p(r) \big|_{r=0} \cdot \frac{u^2}{2} + \frac{1}{2!} \int_0^u (u - r)^2 \frac{d^3}{dr^3} \log \psi_p(r) dr.$$
By definition, $\bar{\beta}(p)$ has mean zero and variance $p(1-p)$, and so the first two derivatives are given by 0 and $(i)^2 p(1-p)$, respectively.

To bound the third-order derivative, we take a route not in the spirit of the central limit theorem: Observe that

$$\frac{d}{dr} \log \psi_p(r) = \frac{E[\bar{\beta}(p) e^{ir\bar{\beta}(p)}]}{E[e^{ir\bar{\beta}(p)}]},$$

and this implies that every higher-order derivative is given by a ratio where the denominator is a power of $\psi$ and the numerator is a sum of products of expectations of the form $\pm E[(\bar{\beta}(p))^\ell e^{ir\bar{\beta}(p)}]$. Each product carries at least one expectation such that $\ell \geq 1$ and so can be bounded by $E[|\bar{\beta}(p)|] = p(1-p)$. Hence, it follows from (3.53) that

$$\sup_{r \in [0,u]} \left| \frac{d^3}{dr^3} \log \psi_p(r) \right| \lesssim p(1-p), \quad \forall |u| \leq 1, \; p \in (0,1). \quad (3.59)$$

Up to this point in this paragraph, we have proved that

$$\left| \log \psi_p(u) + \frac{p(1-p)u^2}{2} \right| \lesssim p(1-p)|u|^3, \quad \forall |u| \leq 1, \; p \in (0,1). \quad (3.60)$$

We remark that the factor $p(1-p)$ in (3.60) will be useful in the argument below. An application of (3.60) gives

$$\begin{align*}
M \left[ \log \psi_q \left( \frac{\theta}{N^{1/2}} \right) + \frac{1}{2} q(1-q) \frac{\theta^2}{N} \right] + M' \left[ \log \psi_q' \left( -\frac{\theta}{N^{1/2}} \right) + \frac{1}{2} q'(1-q') \frac{\theta^2}{N} \right] \\
\lesssim \frac{\theta^3}{N^{1/2}} \left[ \frac{M}{N} q(1-q) + \frac{M'}{N} q'(1-q') \right] \\
\lesssim \frac{\theta^3}{N^{1/2}} \sigma^2, \quad \forall |\theta| \leq N^{1/2}, \quad (3.61)
\end{align*}$$

where the last equality uses the definition (3.3) of $\sigma$.

To close the proof in this step, we put together (3.58), the analogous bound with $(q, M)$ replaced by $(q', M')$, and (3.61). Then we deduce that

$$\begin{align*}
\left| \psi_q \left( \frac{\theta}{N^{1/2}} \right)^M \psi_q' \left( -\frac{\theta}{N^{1/2}} \right)^{M'} - \exp \left\{ -\frac{\sigma^2 \theta^2}{2} \right\} \right| \lesssim \exp \left\{ -\frac{\sigma^2 \theta^2}{5} \right\} \frac{\theta^3}{N^{1/2}} \sigma^2, \quad \forall |\theta| \leq N^{1/2}. \pi.
\end{align*}$$

We arrive at the following bound:

$$|\Pi_{3.55}| \lesssim \frac{\sigma^2}{N^{1/2}} \int_{\delta_N} \theta^{3} \exp \left\{ -\frac{\sigma^2 \theta^2}{5} \right\} d\theta. \quad (3.62)$$

**Step 3.** Finally, we bound $\Pi_{3.55} + IV_{3.55}$. Notice that

$$|\psi_p(u)|^2 \leq 1 - 2p(1-p) \cdot \frac{u^2}{\pi^2} \leq \exp \left\{ -\frac{2p(1-p)u^2}{\pi^2} \right\},$$

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where the first inequality follows from (3.52) and the fact that $1 - \cos x - x^2/\pi^2 \geq 0$ for all $|x| \leq \pi$ and the second inequality uses $1 - x \leq e^{-x}$ for all $x \geq 0$. It follows that

\[
|\text{III}\ 3.55 + \text{IV}\ 3.55| \leq \frac{1}{2\pi} \int_{N^{1/2} < |\theta| \leq N^{1/2}\pi} \exp \left\{ -\frac{2\theta^2}{\pi^2} \left[ \frac{M}{N} q(1-q) + \frac{M'}{N} q'(1-q') \right] \right\} d\theta + \frac{1}{2\pi} \int_{N^{1/2} < |\theta| \leq N^{1/2}\pi} \exp \left\{ -\frac{\sigma^2\theta^2}{2} \right\} d\theta + \frac{1}{2\pi} \int_{|\theta| \geq N^{1/2}\pi} \exp \left\{ -\frac{\sigma^2\theta^2}{2} \right\} d\theta 
\]

\[
\leq \int_{N^{1/2}}^{\infty} \exp \left\{ -\frac{\sigma^2\theta^2}{5} \right\} d\theta = \frac{1}{\sigma} \int_{N^{1/2}\sigma}^{\infty} \exp \left\{ \frac{\theta^2}{5} \right\} d\theta. \tag{3.63}
\]

Here, the definition (3.3) of $\sigma^2 = \sigma_j(r; N)^2$ is used. The proof is complete. 

The following proposition proves an integrated normal approximation for the covariance function in (2.14).

**Proposition 3.14.** Fix $0 < T_0 < 1 < T_1 < \infty$ and let Assumption 3.3 and the primary condition (3.14) be in force.

(1°) It holds that

\[
\sup_{s,t: T_{0i} \leq s \leq t \leq T_{1i}} \int_{s_{\ell_N}}^{s_{\ell_N}} \left| \prod_{j=1}^{2} N^{1/2} P \left( S_{M_j} \left( \frac{T}{s} \right) = S_{M'_j} \left( \frac{T}{t} \right) \right) - \prod_{j=1}^{2} \mathcal{g}(\sigma_j(r)^2; \mu_j(r)) \right| dr 
\]

\[
\leq C(T_0, T_1) \left( \frac{\delta_N^4}{N} + \frac{\delta_{N^2}}{N^{1/2}} + \frac{\delta_{N^2}}{N^{1/2}} + \frac{1}{N\ell_N} + \frac{1}{(N\ell_N)^{1/2}} + \int_{C(T_0, T_1)\ell_N}^{\infty} e^{-q/10\delta_q} \frac{d\delta_q}{q} \right). \tag{3.64}
\]

(2°) If we replace the condition $0 \leq t - s \leq \tau_N$ in the supremum in (3.64) with $t - s > \tau_N$ and the upper limits $s_{\ell_N}$ of the integrals there with $s$, then the same bound applies.

(3°) The suprema in (1°) and (2°) tend to zero as $N \to \infty$ for all $x, y \in \mathbb{R}^2$.

**Proof.** We write $I_j(r; N)$ for the right-hand side of (3.54) and

\[
\sigma(q) \overset{\text{def}}{=} \min \left\{ \sigma_j(sq; N), \sigma_j(sq); 1 \leq j \leq 2 \right\} = \min \left\{ \frac{M_j}{N} q(1-q) + \frac{M'_j}{N} q \left( \frac{s}{t} \right) \frac{(t-s) + s(1-q)}{t}, sq(1-q) + tq \left( \frac{s}{t} \right) \frac{(t-s) + s(1-q)}{t} \right\}.
\]

For the proof of the proposition, we consider the following integrals: for $0 < L < 1/2 < R \leq 1$,

\[
\int_{sL}^{sR} I_1(r; N)I_2(r; N) dr, \quad \int_{sL}^{sR} I_j(r; N) \mathcal{g}(\sigma_j'(r; N)^2; \mu_j'(r; N)) dr, \quad 1 \leq j, j' \leq 2, \ j \neq j', \tag{3.65}
\]

and

\[
\int_{sL}^{sR} \left| \prod_{j=1}^{2} \mathcal{g}(\sigma_j(r; N)^2; \mu_j(r; N)) - \prod_{j=1}^{2} \mathcal{g}(\sigma_j(r)^2; \mu_j(r)) \right| dr. \tag{3.66}
\]
If we set $L = \ell_N$ and $R = r_N$, then the sum of all of these integrals bounds the integral considered in (1°). The integral in (2°) can be handled in the same way if we set $R$ to be 1.

Let us simplify the task of bounding the sum of the integrals in (3.65) and (3.66) by making some observations. First, using the explicit forms of $I_j(r; N)$ and changing variable $r$ to $r/s$, we can bound the sum of all the integrals in (3.65) by the sum of the following integrals up to a multiplicative $C(T_1)$:

$$
\int_L^R P(q)Q(q)\,dq, \quad \int_L^R \frac{1}{\sigma(q)}P(q)\,dq,
$$

where $P$ and $Q$ range over the following three functions:

$$
\alpha(q) = \frac{\delta_N^2}{N^{1/2}}, \quad \beta(q) = \frac{1}{N^{1/2}\sigma(q)^2}, \quad \gamma(q) = \frac{e^{-N\sigma(q)^2/10}}{\sigma(q)}.
$$

There are nine different such integrals:

$$
\int_L^R \alpha(q)\alpha(q)\,dq \leq \frac{\delta_N^4}{N}, \quad (3.69)
$$

$$
\int_L^R \alpha(q)\beta(q)\,dq = \frac{\delta_N^2}{N} \int_L^R \frac{dq}{\sigma(q)^2}, \quad (3.70)
$$

$$
\int_L^R \alpha(q)\gamma(q)\,dq \leq \frac{\delta_N^2}{N^{1/2}} \int_L^R \frac{dq}{\sigma(q)}, \quad (3.71)
$$

$$
\int_L^R \beta(q)\beta(q)\,dq = \frac{1}{N} \int_L^R \frac{dq}{\sigma(q)^4}, \quad (3.72)
$$

$$
\int_L^R \beta(q)\gamma(q)\,dq \leq \frac{1}{N^{1/2}} \int_L^R \frac{dq}{\sigma(q)^3}, \quad (3.73)
$$

$$
\int_L^R \gamma(q)\gamma(q)\,dq = \int_{N_L}^{NR} \frac{N\sigma(q/N)^2}{\sigma(q)^2}\,dq, \quad (3.74)
$$

$$
\int_L^R \frac{1}{\sigma(q)}\alpha(q)\,dq = \frac{\delta_N^2}{N^{1/2}} \int_L^R \frac{dq}{\sigma(q)}, \quad (3.75)
$$

$$
\int_L^R \frac{1}{\sigma(q)}\beta(q)\,dq = \frac{1}{N^{1/2}} \int_L^R \frac{dq}{\sigma(q)^3}, \quad (3.76)
$$

$$
\int_L^R \frac{1}{\sigma(q)}\gamma(q)\,dq = \int_{N_L}^{NR} \frac{e^{-N\sigma(q/N)^2/10}}{N\sigma(q/N)^2}\,dq. \quad (3.77)
$$

Also, to bound the integral in (3.66), we use Lemma 3.8 and the fact that $|\sigma_j(sq; N)^2 - \sigma_j(sq)^2| \leq 2/N$ and $|\mu_j(sq; N) - \mu_j(sq)| \leq 2/N^{1/2}$, which gives

$$
\int_L^R \left| \prod_{j=1}^2 g(\sigma_j(sq; N)^2; \mu_j(sq; N)) - \prod_{j=1}^2 g(\sigma_j(sq)^2; \mu_j(sq)) \right| \,dq
\leq \int_L^R \left( \frac{1}{N\sigma(q)^{3/2}} + \frac{1}{N^{1/2}\sigma(q)^{1/2}} \right)^2 + \left( \frac{1}{N\sigma(q)^{3/2}} + \frac{1}{N^{1/2}\sigma(q)^{1/2}} \right) \cdot \frac{1}{\sigma(q)} \,dq. \quad (3.78)
$$
We only need to study the integrals in (3.70)–(3.77) and (3.78) accordingly in the rest of the proof, since the bound in (3.69) already gives the first term on the right-hand side of (3.64).

(1°). We give the proof according to the decomposition $\int_{s^N}^{s} = \int_{s^N}^{s/2} + \int_{s/2}^{s}$ of the integral under consideration. We first consider the case $L = \ell_N$ and $R = 1/2$ for the setup above. Then the following bound for $\sigma(q)$ holds:

$$\sigma(q)^2 \geq C(T_0, T_1)q, \quad \forall \ q \in (0, 1/2]. \quad (3.79)$$

Then by (3.70)–(3.77) and (3.78),

$$\int_{\ell_N}^{1/2} \alpha(q)\beta(q) dq \leq \frac{C(T_0, T_1)\delta_N^2|\ln \ell_N|}{N},$$

$$\int_{\ell_N}^{1/2} \alpha(q)\gamma(q) dq \leq \int_{\ell_N}^{1/2} \frac{1}{\sigma(q)}\alpha(q) dq \leq \frac{C(T_0, T_1)\delta_N^2}{N^{1/2}},$$

$$\int_{\ell_N}^{1/2} \beta(q)\beta(q) dq \leq \frac{C(T_0, T_1)}{N\ell_N},$$

$$\int_{\ell_N}^{1/2} \beta(q)\gamma(q) dq = \int_{\ell_N}^{1/2} \frac{1}{\sigma(q)}\beta(q) dq \leq \frac{C(T_0, T_1)}{(N\ell_N)^{1/2}},$$

$$\int_{\ell_N}^{1/2} \gamma(q)\gamma(q) dq = \int_{\ell_N}^{1/2} \frac{1}{\sigma(q)}\gamma(q) dq \leq C(T_0, T_1) \int_{C^T(T_0, T_1)N\ell_N}^{\infty} \frac{e^{-q/10} dq}{q},$$

$$\int_{\ell_N}^{1/2} \left| \prod_{j=1}^{2} g(\sigma_j(sq; N)^2; \mu_j(sq; N)) - \prod_{j=1}^{2} g(\sigma_j(sq)^2; \mu_j(sq)) \right| dq \leq \frac{C(T_0, T_1)|\ln \ell_N|}{N}.$$

These bounds give the terms after the first one on the right-hand side of (3.64).

Next, we take $0 \leq t - s \leq \tau_N$, $L = 1/2$ and $R = r_N = 1 - \ell_N$. The lower bound for $\sigma(q)$ is now taken to be

$$\sigma(q)^2 \geq C(T_0, T_1)(1 - q), \quad \forall \ q \in [1/2, 1). \quad (3.80)$$

A change of variables with $1 - q$ replaced shows that the bound in the above case applies. We have proved (1°).

(2°). Now according to the decomposition $\int_{s^N}^{s} = \int_{s^N}^{s/2} + \int_{s/2}^{s}$ of the integral under consideration. The result for $\int_{s^N}^{s/2}$ in Step 1 still applies. We change the argument for its second case by taking $t - s > \tau_N$, $L = 1/2$ and $R = 1$. The lower bound in (3.80) is replaced by

$$\sigma(q)^2 \geq C(T_0, T_1)[\tau_N + (1 - q)], \quad \forall \ q \in [1/2, 1). \quad (3.81)$$

Applying this lower bound to the integral bounds in (3.70)–(3.77) and (3.78) amounts to a translation of $(1 - q)$ by $\tau_N$. This implies that, up to a universal multiplicative positive constant, the sum of these integrals can be bounded as in (3.64) with $\ell_N$ replaced by $\tau_N$. Since $\tau_N = \ell_N$ by assumption, (2°) is proved.

(3°). Finally, $\delta_N^2|\ln \ell_N|/N \to 0$ is implied by $N\ell_N \to \infty$ and $\delta_N^2/N^{1/2} \to 0$. Hence, (3°) holds under Assumption 3.3. The proof is complete.

Finally, we pass limit under the integral sign by the following proposition. Note that the first integral in (3.83) converges absolutely by the inequality $1 - e^{-x} \leq x$ for all $x \geq 0$.  

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Proposition 3.15. Fix $0 < T_0 < 1 < T_1 < \infty$ and let Assumption 3.3 be in force.

(1°) Let $0 < s \leq t < \infty$ and $x, y \in \mathbb{R}^2$ be such that either $s < t$ or $x \neq y$. Then for $0 < \ell_N < 1/2 < \bar{r}_N \leq 1$ with $\bar{r}_N \to 1$, it holds that

$$\lim_{N \to \infty} \int_{s \bar{r}_N}^{s t N} \left( \prod_{j=1}^{2} g(\sigma_j(r)^2; \mu_j(r)) - \frac{1_{[1, \infty)}(N r)}{4 \pi r} \right) \, dr$$

$$= -\frac{\ln s}{4 \pi} + \int_0^\infty \frac{1}{4 \pi v} \left[ \exp \left\{ -\frac{1}{4 v} \right\} - 1_{[1, \infty)}(v) \right] \, dv$$

$$+ \int_0^{t-1} \int_{\mathbb{R}^2} Q_{s-1-r}(z, x) Q_{t-1-r}(z, y) dz \, dr'$$

$$+ \frac{1}{2 \pi} \int_0^T \int_{\mathbb{R}^2} Q_{t-1}(x, y_1) \left( -\ln |y_1 - y_2| \right) Q_{s-1}(y_2, y) dy_1 dy_2.$$  

(2°) Under the primary condition (3.14), it holds that, for all $x \neq y$,

$$\int_{s \bar{r}_N}^{s t N} \left| \prod_{j=1}^{2} g(\sigma_j(r)^2; \mu_j(r)) - \frac{1_{[1, \infty)}(N r)}{4 \pi r} \right| \, dr \leq C(T_0, T_1) \left( 1 + |x|^2 |y|^2 + \ln |x - y| \right).$$

The main argument of the proof is given in the following lemma.

Lemma 3.16. Fix $0 < T_0 < 1 < T_1 < \infty$. It holds that, for all $y_1, y_2 \in \mathbb{R}^2$ with $y_1 \neq y_2$,

$$\int_0^T Q_{2r}(y_1, y_2) \, dr' = \int_0^\infty \frac{1}{4 \pi v} \left[ \exp \left\{ -\frac{1}{4 v} \right\} - 1_{[1, \infty)}(v) \right] \, dv - \frac{1}{2 \pi} \ln |y_1 - y_2| + \frac{\ln T}{4 \pi}$$

$$+ \sum_{j=1}^{3} \varepsilon_j(y_1, y_2; T)$$

for functions

$$\varepsilon_1(y_1, y_2; T) = -\int_{|y_1 - y_2|^2}^{T} \frac{1}{4 \pi v} \left[ \exp \left\{ -\frac{1}{4 v} \right\} - 1_{[1, \infty)}(v) \right] \, dv,$$

$$\varepsilon_2(y_1, y_2; T) = 1_{(0, 1)} \left( \frac{T}{|y_1 - y_2|^2} \right) \cdot \frac{1}{4 \pi} \ln T,$$

$$\varepsilon_3(y_1, y_2; T) = -1_{(0, 1)} \left( \frac{T}{|y_1 - y_2|^2} \right) \cdot \frac{1}{2 \pi} \ln |y_1 - y_2|$$

satisfying

$$\lim_{T \to \infty} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} Q_{t-1}(x, y_1) \varepsilon_j(y_1, y_2; T) Q_{s-1}(y_2, y) dy_1 dy_2 = 0,$$

$$\forall 1 \leq j \leq 3, T_0 \leq s \leq t \leq T_1.$$  

Proof. Notice that (3.84) follows since by changing variables,

$$\int_0^T Q_{2r}(y_1, y_2) \, dr' = \int_{|y_1 - y_2|^2}^{T} \frac{1}{4 \pi v} \left[ \exp \left\{ -\frac{1}{4 v} \right\} - 1_{[1, \infty)}(v) \right] \, dv$$

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where we change variable by \( r \) by the Markov inequality. Passing \( T \to \infty \) for \( \tilde{\jmath} \), the limit is zero. For the limit with \( \jmath = 3 \), the dominated convergence theorem applies to evaluate it in the obvious way, since for any \( \alpha \in (0, 1) \),

\[
0 \leq \frac{\ln T}{4\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} Q_{t-1}(x, y_1) 1_{(0,1)} \left( \frac{T}{|y_1 - y_2|^2} \right) Q_{s-1}(y_2, y) dy_1 dy_2
\]

\[
= \frac{\ln T}{4\pi} P \left( |x + B_{s-1} - y - B_{t-1}'| > T \right)
\]

\[
= \frac{\ln T}{4\pi} P \left( |x - y + B_{s-1+t-1}| > T \right)
\]

\[
\leq \frac{\ln T}{4\pi} \cdot \frac{|x - y| + E[|B_{s-1+t-1}|]}{T}
\]

by the Markov inequality. Passing \( T \to \infty \) in the foregoing inequality, we see that the required limit is zero. For the limit with \( \jmath = 3 \), the dominated convergence theorem applies to evaluate it in the obvious way, since for any \( \alpha \in (0, 1) \),

\[
\frac{1}{2\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} Q_{t-1}(x, y_1) |\ln |y_1 - y_2|| Q_{s-1}(y_2, y) dy_1 dy_2
\]

\[
\leq \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} Q_{t-1}(x, y_1) \left( \frac{C(\alpha)}{|y_1 - y_2|^\alpha} + |y_1 - y_2| \right) Q_{s-1}(y_2, y) dy_1 dy_2
\]

\[
\leq C(\alpha) \int_{\mathbb{R}^2} Q_{t-1}(x, y_1)
\]

\[
\times \left( \int_{\mathbb{R}^2} Q_{s-1}(y_2, y) dy_2 + \|Q_{s-1}(y, \cdot)\|_\infty \int_{|y_1 - y_2| \leq 1} \frac{1}{|y_1 - y_2|^\alpha} dy_2 \right) dy_1
\]

\[
+ E \left[ |x + B_{t-1} - y - B_{s-1}'| \right] < \infty.
\]

This is enough for the lemma upon fixing a choice of \( \alpha \).

\[
\text{Proof of Proposition 3.15. (i)} \quad \text{Write}
\]

\[
\int_{s t \leq 2} \prod_{j=1}^{n} a(\sigma_j(r)^2, \mu_j(r)) dr = \int_{s t \leq 2} \prod_{j=1}^{n} a \left( \frac{2}{r} - \frac{1}{s} - \frac{1}{t} ; x_j - y_j \right) dr
\]

\[
= \int_{0}^{s^{-1}(t^{-1}-1)} Q_{2r'+s^{-1}-t^{-1}}(x, y) dr',
\]

where we change variable by \( r' = r^{-1} = s^{-1} \). The foregoing integral is obviously finite whenever \( s < t \) or \( x \neq y \) due to the integrability at \( r' = 0+ \). Hence, by dominated convergence, the proof for \( \tilde{\tau}_N = 1 \) suffices by dominated convergence and we consider this case in the rest of this proof.

By the Chapman–Kolmogorov equation, we can write

\[
\int_{0}^{s^{-1}(t^{-1}-1)} Q_{2r'+s^{-1}-t^{-1}}(x, y) dr'
\]

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whenever \( N \) is large enough such that \( s^{-1}(\ell_N^{-1} - 1) > t^{-1} \). Otherwise, the integral on the left-hand side is \( \ll \)-bounded by \( C(T_0, T_1) \). The integral in the parentheses is a particular case of the integral in Lemma 3.16 with \( T = s^{-1}(\ell_N^{-1} - 1) \). Hence, applying the lemma to \((3.89)\) via \((3.84)\) and using \( s^{-1}(\ell_N^{-1} - 1) - t^{-1} \sim s^{-1}\ell_N^{-1} \) as \( N \to \infty \), we get

\[
\lim_{N \to \infty} \int_{s/\ell_N}^{s} \left( \prod_{j=1}^{2} g(\sigma_j(r)^2; \mu_j(r)) - \frac{1_{[1, \infty)}(N r)}{4 \pi r} \right) \mathrm{d}r
\]

\[
= \lim_{N \to \infty} \int_{s/\ell_N}^{s} \prod_{j=1}^{2} g(\sigma_j(r)^2; \mu_j(r)) \mathrm{d}r - \frac{\ln s}{4 \pi} + \frac{\ln(s/\ell_N)}{4 \pi}
\]

\[
= -\frac{\ln s}{4 \pi} + \int_{0}^{t^{-1}} \int_{\mathbb{R}^2} Q_{s^{-1}}(z, x) Q_{t^{-1}}(z, y) \mathrm{d}z \mathrm{d}r' + \int_{0}^{\infty} \frac{1}{4 \pi v} \left[ \exp \left\{ -\frac{|y - y_2|^2}{2 r^2} \right\} - 1_{[1, \infty)}(v) \right] \mathrm{d}v
\]

\[
+ \frac{1}{2 \pi} \int_{0}^{\infty} \int_{\mathbb{R}^2} Q_{t^{-1}}(x, y) \left( -\ln |y_1 - y_2| \right) Q_{s^{-1}}(y_2, y) \mathrm{d}y_1 \mathrm{d}y_2.
\]

We obtain \((3.83)\) for the case \( r_N = 1 \) from \((3.88)\) and \((3.90)\).

\((2^o)\). Under the primary condition, \( NT_0 \ell_N \geq 1 \). We have

\[
\int_{s/\ell_N}^{s/2} \left| \prod_{j=1}^{2} g(\sigma_j(r)^2; \mu_j(r)) - \frac{1}{4 \pi r} \right| \mathrm{d}r
\]

\[
\leq \int_{s/\ell_N}^{s/2} \left| \frac{1}{2 \pi r (2 - r/s - r/t)} \exp \left\{ -\frac{|x - y|^2 r^2}{2 r (2 - r/s - r/t)} \right\} - \frac{1}{2 \pi r (2 - r/s - r/t)} \right| \mathrm{d}r + C(T_0, T_1)
\]

\[
\leq C(T_0, T_1)(1 + |x - y|^2) \leq C(T_0, T_1)|x|^2 |y|^2,
\]

where the third line uses the inequality \( 1 - e^{-x} \leq x \) which is valid for all \( x \geq 0 \). Also,

\[
\int_{s/2}^{s} \prod_{j=1}^{2} g(\sigma_j(r)^2; \mu_j(r)) \mathrm{d}r
\]

\[
\leq \int_{s/2}^{s} \frac{1}{2 \pi r^2 (2/r - 1/s - 1/t)} \exp \left\{ -\frac{|x - y|^2}{2 (2/r - 1/s - 1/t)} \right\} \mathrm{d}r
\]

\[
= \int_{s/2}^{s} \frac{1}{4 \pi v} e^{-\frac{|x - y|^2}{4 \pi v}} \mathrm{d}v
\]

\[
\lesssim C(T_0, T_1)(1 + | \ln |x - y||).
\]

The required inequality follows upon combining \((3.91)\) and \((3.92)\).

The following theorem summarizes Proposition 3.14 \((3^o)\) and Proposition 3.15 \((1^o)\).
Theorem 3.17. Let $0 < s \leq t < \infty$ and $x, y \in \mathbb{R}^2$ and let Assumption 3.3 be in force.

(1°) If $s < t$, then
\[
\lim_{N \to \infty} \int_s^t \left( \prod_{j=1}^2 N^{1/2} P \left( S_{M_j} \left( \frac{r}{s} \right) = S'_{M_j} \left( \frac{r}{t} \right) \right) - \frac{1_{[1,\infty)}(Nr)}{r} \right) dr
= -\frac{\ln s}{4\pi} + \int_0^\infty \frac{1}{4\pi v} \left[ \exp \left\{ -\frac{1}{4v} \right\} - 1_{[1,\infty)}(v) \right] dv
+ \int_0^{t-1} \int_{\mathbb{R}^2} Q_{s^{-1}-r'}(z, x) Q_{t^{-1}-r'}(z, y) dz dr'
+ \frac{1}{2\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} Q_{t^{-1}}(x, y_1) \left( -\ln |y_1 - y_2| \right) Q_{s^{-1}}(y_2, y) dy_1 dy_2.
\]

(2°) In the case of $s = t$ and $x \neq y$, the same limit holds if we change the upper limits $s$ of the integrals on the left-hand side to be $sr_N$.

4 Convergence to the additive stochastic heat equation

In this section, we relate the limiting covariance function in Theorem 3.1 to the covariance function of an additive stochastic heat equation. Whereas some of these connections are already pointed out in [3], we proceed with the weak formulation.

From now on, $S(\mathbb{R}^2)$ denotes the space of real-valued Schwartz functions on $\mathbb{R}^2$ and $S'(\mathbb{R}^2)$ denotes the space of bounded linear functionals on $S(\mathbb{R}^2)$ over $\mathbb{R}$. By convention, $S'(\mathbb{R}^2)$ is equipped with the weak topology.

4.1 Weak formulations

With respect to the process $\zeta_N(x, s)$ in (3.5), we define
\[
\zeta_s^N(\phi) = \int_{x \geq -N^{1/2}1} \phi(x) \zeta_N(x, s) dx, \quad \phi \in S(\mathbb{R}^2).
\]

Here, $1 = (1,1)$ and the constraint $x \geq -N^{1/2}1$ is maximal for using the Whittaker SDEs since for any $s > 0$, $M(x_j, s) \geq 0$ if and only if $x_j \geq -N^{1/2}$. (Recall (3.2) for the notation $M(x_j, s)$.) In this subsection, we show some basic growth properties of the process $\zeta^N(x, s)$ and then translate Theorem 3.1 to a convergence result under the weak formulation.

By Proposition 2.4, the metric induced by the covariance function of $\zeta_s \Sigma(m_1, m_2)$ can be represented as follows: for every $(m_1, m_2), (m'_1, m'_2) \in \mathbb{Z}_+^2$ and $0 \leq s \leq t < \infty$,
\[
\mathbb{E}[|\zeta_s \Sigma(m_1, m_2) - \zeta_t \Sigma(m'_1, m'_2)|^2]
\]
Lemma 4.1. (1°) Given \(0 < r < a\) and integers \(m, n \geq 0\), it holds that

\[
\frac{\partial}{\partial a} \mathbf{P}\left(S_m(\frac{r}{a}) = n\right) = \frac{1}{a} \left[ (n+1) \mathbf{P}\left(S_m(\frac{r}{a}) = n+1\right) - n \mathbf{P}\left(S_m(\frac{r}{a}) = n\right) \right].
\]

(2°) Given \(T \in (0, \infty)\), it holds that

\[
\mathbb{E}[\|\zeta_t \Sigma(m_1, m_2) - \zeta_t \Sigma(m_1, m_2)\|^2] \\
\lesssim C(T)(\|\Sigma(m_1, m_2)\|_\infty \vee 1) \times |t-s|, \quad \forall 0 \leq s, t \leq T, \ (m_1, m_2) \in \mathbb{Z}_+^2,
\]

where \(\|\Sigma(m_1, m_2)\|_\infty = \max\{|m_1|, |m_2|\}\).

Proof. To obtain (4.3), we may assume that \(0 \leq n \leq m\). In this case,

\[
\frac{\partial}{\partial a} \mathbf{P}\left(S_m(\frac{r}{a}) = n\right) = \left(\begin{array}{c} m \\ n \end{array}\right) \left(\frac{r}{a}\right)^{n-1} \left(1 - \frac{r}{a}\right)^{m-n} + \left(\begin{array}{c} m \\ n \end{array}\right) \left(\frac{r}{a}\right)^{n} \left(1 - \frac{r}{a}\right)^{m-n-1} \frac{r}{a^2}
\]

\[
= -\frac{n}{a} \mathbf{P}\left(S_m(\frac{r}{a}) = n\right) + \frac{n+1}{a} \mathbf{P}\left(S_m(\frac{r}{a}) = n+1\right),
\]

as required.

Next, we apply (4.3) to get (4.4). In the case \(s = 0\) or \(m_1 = m_2 = 0\), the required bound holds obviously since then the second and third terms on the right-hand side of (4.2) are zero.

For \(0 < s \leq t < \infty\) and nonzero \((m_1, m_2) \in \mathbb{Z}_+^2\), (4.3) shows the following bound for the second term in (4.2) with \((m'_1, m'_2) = (m_1, m_2)\):

\[
\int_0^s \left| \sum_{j=1}^{2} \mathbf{P}\left(S_{m_j}(\frac{r}{s}) = S'_{m_j}(\frac{r}{s})\right) - \sum_{j=1}^{2} \mathbf{P}\left(S_{m_j}(\frac{r}{s}) = S'_{m_j}(\frac{r}{t})\right) \right| dr
\]

\[
\lesssim \int_0^s \frac{t-s}{s} \sum_{j=1}^{2} \left( \mathbb{E}\left[S_{m_j}(\frac{r}{s})\right] + 1 \right) dr \lesssim \|\Sigma(m_1, m_2)\|_\infty \times |t-s|.
\]

The third term in (4.2) with \((m'_1, m'_2) = (m_1, m_2)\) can be bounded similarly. Hence, (4.4) holds whenever \(0 < s \leq t < \infty\) and \((m_1, m_2)\) is nonzero. We have proved (4.4). \[\square\]

As an application, we obtain the a.s. polynomial growth of \(\zeta\) in the following lemma.
Lemma 4.2. \(1^\circ\) For all \(T \in (0, \infty)\) and \(\alpha \in (1, \infty)\), we can find \(C(\alpha)\) such that
\[
\mathbb{E} \left[ \sup_{(m_1, m_2) \in \mathbb{Z}_+^2} \sup_{s \in [0, T]} \frac{|\zeta_s \Sigma(m_1, m_2)|^{2\alpha}}{1 + \|(m_1, m_2)\|_C^{\alpha(C)}} \right] < \infty.
\]

\(2^\circ\) For each \(N \geq 1\), the following statement holds with probability one: the integral in (4.1) converges absolutely for all \(s \in [0, \infty)\) and \(\phi \in \mathcal{S}(\mathbb{R}^2)\), and \(\zeta^N\) takes values in \(D(\mathbb{R}_+, \mathcal{S}'(\mathbb{R}^2))\).

Proof. \(1^\circ\). We modify the proof of [6, Proposition 4.1]. For any integer \(n \geq 1\), set \(E_n = \{(m_1, m_2) \in \mathbb{Z}_+^2; 2^{n-1} \leq \|(m_1, m_2)\|_\infty < 2^n\}\), which satisfies \(|E_n| \lesssim 2^n\). For any \(\beta \in (0, \infty)\) and \(\alpha \in (1, \infty)\),
\[
\mathbb{E} \left[ \sup_{(m_1, m_2) \in \mathbb{Z}_+^2} \sup_{s \in [0, T]} \frac{|\zeta_s \Sigma(m_1, m_2)|^{2\alpha}}{1 + \|(m_1, m_2)\|_C^{\beta r}} \right] \leq \mathbb{E} \left[ \sup_{s \in [0, T]} |\zeta_s \Sigma(0, 0)|^{2\alpha} \right] + \sum_{n=1}^{\infty} \frac{1}{1 + 2^{(n-1)\beta}} \sum_{(m_1, m_2) \in E_n} \mathbb{E} \left[ \sup_{s \in [0, T]} |\zeta_s \Sigma(m_1, m_2)|^{2\alpha} \right]. \tag{4.5}
\]
Since \(\zeta\) is a Gaussian process, Lemma 4.1 \(2^\circ\) is enough to apply Kolmogorov’s theorem for continuity [14, (2.1)Theorem in Chapter I]. Moreover, we can find \(C'(\alpha, T), C(\alpha)\) such that
\[
\mathbb{E} \left[ \sup_{s \in [0, T]} |\zeta_s \Sigma(m_1, m_2)|^{2\alpha} \right] \leq C'(\alpha, T)\|(m_1, m_2)\|_C^{\alpha(C)} \tag{4.6}
\]
Applying (4.6) to (4.5) with \(\beta = 3 + C(\alpha)\), we obtain the required result.

\(2^\circ\). For \(x_j \geq -N^{1/2}, s \mapsto sN + sN \cdot x_j/N^{1/2}\) is nondecreasing on \([0, \infty)\) and so \(s \mapsto M_j\) is càdlàg. For these \(x_j\)’s, we also have \(0 \leq M_j \leq Ns + Ns \cdot |x_j|/N^{1/2}\). Hence, by \(1^\circ\), \(\zeta^N_s(\phi)\) is absolutely convergent as an integral for any fixed \(s\) and is a càdlàg function in \(s\). The weak topology of \(\mathcal{S}'(\mathbb{R}^2)\) gives the required path property of \(\zeta^N\). \(\blacksquare\)

Now we extend Theorem 3.1 to a convergence under the weak formulation.

Theorem 4.3. Let \(\zeta^N\) be the \(\mathcal{S}'(\mathbb{R}^d)\)-valued processes defined by (4.1). Then it holds that
\[
\lim_{N \to \infty} \left[ \text{Cov} \{\zeta^N_s(\phi_1); \zeta^N_s(\phi_2)\} - \mathcal{E}_N \left( \int \phi_1 \right) \left( \int \phi_2 \right) \right] = \frac{1}{2\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} Q_{t-1} \phi_1(y_1) (-\ln |y_1 - y_2|) Q_{s-1} \phi_2(y_2) dy_1 dy_2 \tag{4.7}
\]
\[+ \int_0^{t-1} \int_{\mathbb{R}^2} Q_{s-1-r} \phi_1(z) Q_{t-1-r} \phi_2(z) dz dr, \quad \forall \phi_1, \phi_2 \in \mathcal{S}(\mathbb{R}^2),\]
where \(\mathcal{E}_N\) is defined in (3.9).

The next proposition summarizes Propositions 3.6, 3.10, 3.14 and 3.15.
Proposition 4.4. Fix $0 < T_0 < 1 < T_1 < \infty$. Let Assumption 3.3 be in force and $x, y \in \mathbb{R}^2$, $T_0 \leq s \leq t \leq T_1$ and $N \geq 16$ be subject to the primary condition (3.14). In the case that $0 \leq t - s \leq \tau_N$, we also require the secondary condition (3.15). Then it holds that

$$\int_0^{Ns} \left| \sum_{j=1}^{2} P \left( S_{M_j} \left( \frac{r}{Ns} \right) = S'_{M_j} \left( \frac{r}{Nt} \right) \right) \right| dr \leq C(T_0, T_1) \left( 1 + |x|^2 |y|^2 + |\ln |x - y|| \right).$$

Proof of Theorem 3.1. We turn to the integral representations of the covariance function in (3.6). Then the bound in Proposition 4.4 enables the use of the dominated convergence theorem and Theorem 3.1. It remains to show that the contribution from the complementary $x_1, x_2, y_1, y_2$ vanishes as $N \to \infty$.

When the primary condition fails, we use the property that $\int_{[-N^2/2, N^2/2]} |\phi(x)| dx_j$ decays polynomially in $N$ of any fixed order. Also, the secondary condition is only assumed to deal with the integrals in Proposition 3.10 (4°). In this case, if $F$ denotes the set of $x_1, x_2, y_1, y_2$ such that the secondary condition fails, then we still have

$$\int_F dx dy |\phi_1(x)\phi_2(y)| \int_{Nsr_N}^{Ns} dr \leq C(T_0, T_1, \phi)(N^{-1/2})^2 \cdot N(1 - r_N) \to 0 \quad \text{as } N \to \infty$$

by Assumption 3.3. This completes the proof.

4.2 Identification of the limit

Given $X_0 \in \mathcal{S}'(\mathbb{R}^2)$, the additive stochastic heat equation is defined by

$$X_t(\phi) = X_0(\phi) + \int_0^t X_s \left( \frac{\Delta}{2} \phi \right) \, ds + \int_0^t \int_{\mathbb{R}^2} \phi(x) W(dx, dr). \quad (4.8)$$

The solution is

$$X_t(\phi) = X_0[Q_t(\phi)] + \int_0^t \int_{\mathbb{R}^2} Q_{t-s}(\phi) W(dx, dr), \quad (4.9)$$

where $Q_t = e^{t\Delta/2}$ is the transition semigroup of the two-dimensional standard Brownian motion and $W$ is a space-time white noise. See [18, pp.339–343 in Chapter 5].

Theorem 4.3 and (4.9) suggest that the limiting process of the rescaled Whittaker SDEs is the solution $X$ of an additive stochastic heat equation: for $0 < s \leq t < \infty$,

$$\lim_{N \to \infty} \left[ \text{Cov}[\zeta_s^N(\phi_1); \zeta_t^N(\phi_2)] - \mathcal{C}_N \left( \int \phi_1 \right) \left( \int \phi_2 \right) \right] = \text{Cov}[X_{s-1}(\phi_1); X_{t-1}(\phi_2)] \quad (4.10)$$

if $X_0$ is independent of the space-time white noise and has a covariance function given by

$$\mathbb{E}[X_0(\phi_1)X_0(\phi_2)] = B_0(\phi_1, \phi_2) \overset{\text{def}}{=} \frac{1}{2\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \phi_1(y_1) \phi_2(y_2) \left( -\ln |y_1 - y_2| \right) dy_1 dy_2. \quad (4.11)$$

The foregoing equation well defines $X_0$ as a Gaussian random field indexed by

$$\mathcal{S}_0(\mathbb{R}^2) \overset{\text{def}}{=} \{ \phi \in \mathcal{S}(\mathbb{R}^2); \int \phi = 0 \}.$$
See [10] and the references therein for a reproducing kernel approach of the construction of $X_0$. Alternatively, $X_0$ can be defined as the stationary solution of the additive stochastic heat equation (Proposition 6.1).

Given the conditionally positive definiteness of $(x, y) \mapsto -\ln |x - y|$, the relation in (4.11) cannot apply on the full space $\mathcal{S}(\mathbb{R}^2)$. To circumvent technical issues from this restriction of domain, first we fix $\psi \in \mathcal{S}(\mathbb{R}^2)$ such that $\int \psi = 1$ and define a re-centering operator $R = R_\psi : \mathcal{S}(\mathbb{R}^2) \to \mathcal{S}_0(\mathbb{R}^2)$ by $R\phi = \phi - (\int \phi \psi$. Note that $R$ is a projection onto $\mathcal{S}_0(\mathbb{R}^2)$: $R^2\phi = R\phi$. Then with the Gaussian random field $X_0$ specified below (4.11), we modify the definition of $X$ in (4.10) to the following $\mathcal{S}'(\mathbb{R}^2)$-valued continuous process:

$$X_t(\phi) = X_0[Q_t R\phi] + \int_0^t \int_{\mathbb{R}^2} Q_t R\phi(x) W(dx, dr), \quad \phi \in \mathcal{S}(\mathbb{R}^2). \quad (4.12)$$

The first term in (4.12) is well-defined since the Lebesgue measure is an invariant measure of $(Q_t)$. Note that for $\phi \in \mathcal{S}_0(\mathbb{R}^2)$, $X_t(\phi)$ still satisfies (4.8). (For example, [16, 5° in the proof of Theorem 2.1 on page 430] allows for a straightforward extension beyond one dimension to this case.) Now, (4.10), (4.11), (4.12) can be made precise in the following form:

$$\lim_{N \to \infty} \text{Cov}[\zeta^N_t \circ R(\phi_1); \zeta^N_t \circ R(\phi_2)] = \text{Cov}[X_{s-1}(\phi_1); X_{t-1}(\phi_2)], \quad \forall \phi_1, \phi_2 \in \mathcal{S}(\mathbb{R}^2). \quad (4.13)$$

**Theorem 4.5 (Second main result).** As $N \to \infty$, the sequence of laws of $(\zeta^N_t \circ R; t > 0)$ converges weakly to the law of $(X_{t-1}; t > 0)$ as probability measures on $D((0, \infty), \mathcal{S}'(\mathbb{R}^2))$, where $X$ is defined by (4.12).

For the proof of this theorem, the convergence of finite-dimensional marginals follows readily from (4.13). To obtain the convergence at the process level, Mitoma’s theorem [11] requires the tightness of $\zeta^N(\phi)$ for all fixed $\phi \in \mathcal{S}_0(\mathbb{R}^2)$. The proof of this property is the subject of the next section.

## 5 Tightness of the rescaled Whittaker SDEs

Our goal in this section is to prove that the family of laws of the real-valued continuous processes $\zeta^N(\phi)$ defined by (4.1) is tight, for a fixed $\phi \in \mathcal{S}_0(\mathbb{R}^2)$. It is enough to prove that the induced metrics $(s, t) \mapsto \mathbb{E}[(\zeta^N_s(\phi) - \zeta^N_t(\phi))^2]^{1/2}$ are uniformly Hölder-1/2 continuous.

Consider the explicit expressions of these metrics by applying the rescaling under consideration. By (4.2), we have

$$\mathbb{E}[(\zeta^N_s(\phi) - \zeta^N_t(\phi))^2]$$

$$= \int_s^t dr \int_{x \geq -N^{1/2}} dx \phi(x) \int_{y \geq -N^{1/2}} dy \phi(y) \prod_{j=1}^2 N^{1/2} \mathbb{P} \left( S_{M(x,j)}(t) \left( \frac{r}{t} \right) = S_{M(y,j)}(t) \left( \frac{r}{t} \right) \right)$$

$$- \left[ \int_0^s dr \int_{x \geq -N^{1/2}} dx \phi(x) \int_{y \geq -N^{1/2}} dy \phi(y) \prod_{j=1}^2 N^{1/2} \mathbb{P} \left( S_{M(x,j)}(t) \left( \frac{r}{t} \right) = S_{M(y,j)}(t) \left( \frac{r}{t} \right) \right) \right.$$
where the notation \( M(x_j, s) \) is defined in (3.2). Then the job is to show that all of the three terms in (5.1) satisfy the following bound:

\[
\sup_{N \in \N} |\mathcal{L}_N(s, t)| \leq C(T_0, T_1, \phi)(t - s), \quad \forall \, s, t : T_0 \leq s \leq t \leq T_1, \; 0 < T_0 < 1 < T_1 < \infty.
\]

(5.2)

The results are presented as Propositions 5.1, 5.6 and 5.7.

**Proposition 5.1.** (5.2) is satisfied by \( \mathcal{L}_N = \mathcal{I}_N \).

**Proof.** We bound the integrand of \( \mathcal{I}_N(s, t) \), for \( r \in [s, t] \), according to \( r/t \geq 1 - N^{-1/2} \) and the complementary case. The point here is that since \( r \) is bounded away from zero, we can be saved from the need of dealing with divergent re-centering constants as in Section 3. In both Steps 1 and 2, we assume the primary condition (3.14).

**Step 1.** First, we consider \( r \) such that \( r/t \geq 1 - N^{-1/2} \). We obtain from (3.20) and (3.22) that

\[
\begin{align*}
&\mathcal{P} \left( S_{M(x_j, t)} \left( \frac{r}{t} \right) = S'_{M(y_j, t)} \left( \frac{r}{t} \right) \right) \\
&\quad - \mathcal{P} \left( V \left( M(x_j, t) \left( 1 - \frac{r}{t} \right) \right) = V' \left( M(y_j, t) \left( 1 - \frac{r}{t} \right) \right) + M(x_j, t) - M(y_j, t) \right) \leq \frac{2}{N^{1/2}}.
\end{align*}
\]

(5.3)

To bound the Poisson probabilities, notice that for all \( x_j, y_j \in \mathbb{R} \) and \( t \geq T_0 \),

\[
|M(x_j, t) - M(y_j, t)| \geq T_0 N^{1/2}|x_j - y_j| - 1.
\]

(5.4)

Hence, for \( r/t \geq 1 - N^{-1/2} \) and \( t, x_j, y_j \) satisfying the primary condition over \([T_0, T_1]\), we obtain from the elementary inequality \( \mathcal{P}(V \geq m) \leq e^{-m}E[e^V] \), (5.3) and (5.4) that the next three inequalities hold:

\[
\begin{align*}
N^{1/2} \mathcal{P} \left(V \left( M(x_j, t) \left( 1 - \frac{r}{t} \right) \right) = V' \left( M(y_j, t) \left( 1 - \frac{r}{t} \right) \right) + M(x_j, t) - M(y_j, t) \right) \\
&\leq 1_{\{M(x_j, t)-M(y_j, t) \geq 0\}} N^{1/2} e^{-\left[M(x_j, t)-M(y_j, t)\right]} e^{M(x_j, t) \left(1 - \frac{r}{t}\right) (e - 1)} \\
&\quad + 1_{\{M(x_j, t)-M(y_j, t) < 0\}} N^{1/2} e^{-\left|M(x_j, t)-M(y_j, t)\right|} e^{M(y_j, t) \left(1 - \frac{r}{t}\right) (e - 1)} \\
&\leq C(T_1) N^{1/2} e^{-\left|M(x_j, t)-M(y_j, t)\right|} \\
&\leq C(T_1) N^{1/2} e^{-T_0 N^{1/2}|x_j-y_j|}.
\end{align*}
\]

(5.5)

Combining (5.3) and (5.4) proves that for \( x_j, y_j, t, N \) satisfying the primary condition over \([T_0, T_1]\),

\[
N^{1/2} \mathcal{P} \left( \frac{r}{t} \right) = S'_{M(y_j, t)} \left( \frac{r}{t} \right) \leq 2 + C(T_1) N^{1/2} e^{-T_0 N^{1/2}|x_j-y_j|}.
\]

(5.6)
Step 2. Recall the definitions in (3.3) and (3.3). For $1 - r/t \geq N^{-1/2}$, the following bound holds:

$$C(T_0, T_1) \leq N^{1/2} \sigma_j(r; N)^2.$$  

Also, if the secondary condition (3.15) is in force, then $|M(x_j, t) - M(y_j, t)| \geq C(T_0, T_1)N^{1/2}|x_j - y_j|$. In this case, Lemma 3.13 gives

$$N^{1/2}P \left( S_{M(x_j,t)} \left( \frac{r}{t} \right) \right) = S_{M(y_j,t)}' \left( \frac{r}{t} \right) \leq \sigma_j(r; N)^2|x_j - y_j| + C(T_0, T_1). \quad (5.7)$$

Step 3. For all $s, t, N$ satisfying the primary condition, the bounds from Steps 1 and 2 contribute to the first three terms of the following inequality:

$$|I_N(s, t)| \leq \int_s^t dr \int_{\mathbb{R}^2} dx |\phi(x)| \int_{\mathbb{R}^2} dy |\phi(y)| \prod_{j=1}^2 \left[ 2 + C(T_1)N^{1/2}e^{-T_0N^{1/2}|x_j - y_j|} \right]$$

$$+ C(T_0, T_1) \int_s^t dr \int_{\mathbb{R}^2} dx |\phi(x)| \int_{\mathbb{R}^2} dy |\phi(y)| \prod_{j=1}^2 \left[ \sigma_j(r; N)^2|x_j - y_j| + C(T_0, T_1) \right]$$

$$+ C(T_0, T_1, \phi) \int_s^t dr \cdot N^{-1/2} \cdot N^{1/2}$$

$$+ N \int_s^t dr \int_{x \in [N^{1/2}N^2]} dx |\phi(x)| \int_{\mathbb{R}^2} dy |\phi(y)|$$

$$+ N \int_s^t dr \int_{\mathbb{R}^2} dx |\phi(x)| \int_{y \in [N^{1/2}N^2]} dy |\phi(y)|$$

$$\lesssim C(T_0, T_1, \phi)|t - s|. \quad (5.8)$$

Here, in the first inequality, the first and second integrals follow from (5.5) and (5.7), and the factor $N^{-1/2}$ in the third integral is contributed by $y_1, y_2$ which fail to satisfy the secondary condition (3.15) for any fixed $x_1, x_2$. Also, (5.8) uses the fast decay property of $\phi$. The last inequality proves the proposition.  

The main theme of this section is to bound $J_N$. We start with an interpolation to present the difference form of this term:

$$J_N(s, t) = \int_0^s dr \int_s^t da \int_{x \geq N^{1/2}1} dx \phi(x)$$

$$\times \frac{\partial}{\partial a} \int_{y \geq N^{1/2}1} dy \phi(y) \prod_{j=1}^2 N^{1/2}P \left( S_{M(x_j,s)} \left( \frac{r}{s} \right) = S_{M(y_j,a)}' \left( \frac{r}{a} \right) \right) \quad (5.9)$$

The foregoing derivative can be computed by changing variables with $N^{1/2}y' = aN^{1/2}(y + N^{1/2}1)$:

$$\frac{\partial}{\partial a} \int_{y \geq N^{1/2}1} dy \phi(y) \prod_{j=1}^2 N^{1/2}P \left( S_{M(x_j,s)} \left( \frac{r}{s} \right) = S_{M(y_j,a)}' \left( \frac{r}{a} \right) \right)$$

$$= \frac{\partial}{\partial a} \frac{1}{a^2} \int_{y' \geq 0} dy' \phi \left( \frac{y'}{a} - N^{1/2}1 \right) \prod_{j=1}^2 N^{1/2}P \left( S_{M(x_j,s)}' \left( \frac{r}{s} \right) = S_{M(y_j,a)}' \left( \frac{r}{a} \right) \right)$$

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\[ \begin{align*}
&\frac{2}{a^3} \int_{y \geq 0} dy' \phi \left( \frac{y'}{a} - N^{1/2} \right) \prod_{j=1}^{2} N^{1/2} P \left( S_{M(x_j,s)} \left( \frac{r}{s} \right) \right) = \frac{S'_r[N^{1/2}y'] \left( \frac{r}{a} \right)}{\prod_{j=1}^{2} N^{1/2}} \\
&- \frac{1}{a^4} \int_{y \geq 0} dy' y' \cdot \nabla \phi \left( \frac{y'}{a} - N^{1/2} \right) \prod_{j=1}^{2} N^{1/2} P \left( S_{M(x_j,s)} \left( \frac{r}{s} \right) \right) = \frac{S'_r[N^{1/2}y'] \left( \frac{r}{a} \right)}{\prod_{j=1}^{2} N^{1/2}} \\
&+ \frac{1}{a^2} \int_{y \geq 0} dy' \phi \left( \frac{y'}{a} - N^{1/2} \right) \frac{\partial}{\partial a} \prod_{j=1}^{2} N^{1/2} P \left( S_{M(x_j,s)} \left( \frac{r}{s} \right) \right) = \frac{S'_r[N^{1/2}y'] \left( \frac{r}{a} \right)}{\prod_{j=1}^{2} N^{1/2}}. 
\end{align*} \]

Apply \( \int_0^s dr \int_s^t da \int_{x \geq -N^{1/2} x_1} dx \phi(x) \) to both sides of the last equality, and then we obtain from (5.9) that \( \mathcal{J}_N \) can be decomposed into

\[ \mathcal{J}_N(s, t) = -2\mathcal{J}_{N,1}(s, t) - \mathcal{J}_{N,2}(s, t) + \mathcal{J}_{N,3}(s, t). \tag{5.10} \]

We handle \( \mathcal{J}_{N,1} \) and \( -\mathcal{J}_{N,2} + \mathcal{J}_{N,3} \) separately.

**Lemma 5.2.** (5.2) is satisfied by \( \mathcal{L}_N = \mathcal{J}_{N,1} \).

**Proof.** We undo the change of variables below (5.9):

\[ \mathcal{J}_{N,1}(s, t) \]

\[ = \int_0^s dr \int_s^t da \int_{x \geq -N^{1/2} x_1} dx \phi(x) \int_{y \geq -N^{1/2}} dy \phi(y) \prod_{j=1}^{2} N^{1/2} P \left( S_{M(x_j,s)} \left( \frac{r}{s} \right) = S'_r[M_{y_j,a}] \left( \frac{r}{a} \right) \right) \\
= \int_s^t da \int_{x \geq -N^{1/2} x_1} dx \phi(x) \int_{y \geq -N^{1/2}} dy \phi(y) \times \int_0^{N_s} dr \left[ \prod_{j=1}^{2} P \left( S_{M(x_j,s)} \left( \frac{r}{N_s} \right) = S'_r[M_{y_j,a}] \left( \frac{r}{Na} \right) \right) - \frac{1_{[1,\infty)}(r)}{4\pi r} \right] \\
+ \int_s^t da \int_{x \geq -N^{1/2} x_1} dx \phi(x) \int_{y \geq -N^{1/2}} dy \phi(y) \left( -\frac{\ln(Ns)}{4\pi} \right). 
\]

It follows from Proposition 4.4 that the first term in the foregoing equality can be bounded by \( C(T_0, T_1)|t - s| \). For the second term, the assumption \( \phi \in S_0(\mathbb{R}^2) \) enables the cancellation of \( \frac{-\ln(Ns)}{4\pi} \) up to an error term, so that this term can be bounded by \( C(T_0, T_1, \phi)|t - s| \). We have proved the proposition. \[ \blacksquare \]

For the remaining terms in (5.10) for \( \mathcal{J}_N \), we start with some preliminaries.

**Notation 5.3.** Write \( \{e_1, e_2\} \) for the standard ordered basis of \( \mathbb{R}^2 \) and \( j \mathcal{C} \) for the coordinate in \( \{1, 2\} \) different from \( j \in \{1, 2\} \).

**Lemma 5.4.** Let \( F : \mathbb{R} \to \mathbb{R} \) be bounded, \( p \in (0, 1) \) and \( M \in \mathbb{Z}_+ \).

(1°) The independent sums \( S_M = S_M(p) \)'s satisfy

\[ p \left( \mathbb{E} \left[ F(S_M) \right] - \mathbb{E} \left[ F(S_M + 1) \right] \right) = \mathbb{E} \left[ F(S_M) \right] - \mathbb{E} \left[ F(S_M + 1) \right]. \tag{5.11} \]

\[ \mathbb{E} \left[ F(S_M + 1) \right] = \frac{1}{p} \mathbb{E} \left[ F(S_{M+1}) \right] - \frac{1 - p}{p} \mathbb{E} \left[ F(S_M) \right]. \tag{5.12} \]

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\(2^o\) (Binomial integration by parts)

\[
\mathbf{E}[S_M F(S_M)] = \mathbf{E}[S_M]\mathbf{E}[F(S_{M-1})].
\]

\(3^o\) For all \(\phi \in \mathcal{S}(\mathbb{R}^2)\), \(L \in \mathbb{R}\), \(a \in (0, \infty)\) and \(\ell \in \mathbb{Z}\),

\[
\int_{L}^{\infty} dx_j \phi(x) F(M(x_j, a) + \ell) = \int_{L+ \frac{\ell}{aN^{1/2}}}^{\infty} dx_j \phi \left( x - \frac{\ell}{aN^{1/2}e_j} \right) F(M(x_j, a)).
\]

**Proof.** We omit the proof of \((1^o)\). For \((2^o)\), we write out the expectation on its left-hand side:

\[
\mathbf{E}[S_M F(S_M)] = \sum_{j=1}^{M} \binom{M}{j} p^j (1 - p)^{M-j} F(j)
\]

\[
= M p \sum_{j=1}^{M} \frac{(M - 1)!}{(j - 1)!(M - 1 - j + 1)!} p^{j-1} (1 - p)^{M-1-j+1} F(j - 1 + 1)
\]

\[
= M p \sum_{j=0}^{M-1} \binom{M - 1}{j} p^j (1 - p)^{M-1-j} F(j + 1)
\]

\[
= M p \mathbf{E}[F(S_{M-1} + 1)],
\]

as required. For \((3^o)\), note that since \(M(x_j, a) = [aN^{1/2}(x_j + N^{1/2})]\), \(M(x_j, a) + \ell = M(x_j + \ell/(aN^{1/2}), a)\) for all \(x_j \in \mathbb{R}\). \(\square\)

We are ready to complete the argument for \(J_N\).

**Lemma 5.5.** \((5.2)\) is satisfied by \(\mathcal{L}_N = -J_{N,2} + J_{N,3}\).

**Proof.** We divide the proof into six steps. Step 1 is for \(J_{N,2}\), whereas we need Steps 2, 2-1-2-2 to handle \(J_{N,3}\). A summary is given in Step 3 to complete the proof.

**Step 1.** For \(J_{N,2}\), we change variables back and get

\[
J_{N,2}(s, t) = \int_{s}^{t} \int_{s}^{t} da \int_{x \geq -N^{1/2}} dy \phi(x) \int_{y \geq -N^{1/2}} \frac{dy}{a} \cdot \nabla \phi(y)
\]

\[
= \prod_{j=1}^{2} N^{1/2} P \left( S_{M(x_j, s)} \left( \frac{r}{s} \right) = S'_{M(y_j, a)} \left( \frac{r}{a} \right) \right). \tag{5.13}
\]

According to \(\int \cdot \nabla \phi(y) = y \cdot \nabla \phi(y) + N^{1/2} \text{div} \phi(y)\), we can write the last integral as

\[
\int_{s}^{t} \int_{s}^{t} \frac{da}{a} \int_{x \geq -N^{1/2}} dx \phi(x) \int_{y \geq -N^{1/2}} dy \cdot \nabla \phi(y) \prod_{j=1}^{2} N^{1/2} P \left( S_{M(x_j, s)} \left( \frac{r}{s} \right) = S'_{M(y_j, a)} \left( \frac{r}{a} \right) \right)
\]

\[
+ N^{1/2} \int_{s}^{t} \int_{s}^{t} \frac{da}{a} \int_{x \geq -N^{1/2}} dx \phi(x) \int_{y \geq -N^{1/2}} dy \text{div} \phi(y)
\]

\[
= \prod_{j=1}^{2} N^{1/2} P \left( S_{M(x_j, s)} \left( \frac{r}{s} \right) = S'_{M(y_j, a)} \left( \frac{r}{a} \right) \right)
\]

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\[ J_{N, 2, 1}(s, t) + J_{N, 2, 2}(s, t). \]  

(5.14)

Since \( y \mapsto y \cdot \nabla \phi(y) \in S_0(\mathbb{R}^2) \), the proof in Lemma 5.2 shows that (5.2) for \( \mathcal{L}_N = J_{N, 2, 1} \) holds. We postpone the consideration of \( J_{N, 2, 2}(s, t) \) until Step 3.

**Step 2.** In the rest of this proof, we write \( \mathcal{R}_{N,j} = \mathcal{R}_{N,j}(a, r) \) for functions such that

\[
\sup_{N \in \mathbb{N}} \sup_{s, a \in [T_0, T_1]} \left| \int_0^s \, dr \int_{-N^{1/2}}^\infty \, dx \int_{-N^{1/2}}^\infty \, dy \right| j = N^{1/2} \mathbb{P} \left( S_{M(x, s)} \left( \frac{r}{s} \right) = S'_{M(y, a)} \left( \frac{r}{a} \right) \right) < \infty.
\]

(5.15)

(Recall Notation 5.3 for \( j \mathcal{L} \).) The functions \( \mathcal{R}_{N,j} \) can change from term to term unless otherwise specified.

To rewrite \( J_{N, 3} \), we first compute the derivative in its definition. By Lemma 4.1 \((^1)^o\), it holds that

\[
\frac{\partial}{\partial a} N^{1/2} \mathbb{P} \left( S_{M(x, s)} \left( \frac{r}{s} \right) = S'_{M'} \left( \frac{r}{a} \right) \right) \bigg|_{m' = M(y, a)} = N^{1/2} \mathbb{E} \left[ \frac{\partial}{\partial a} \mathbb{P} \left( S'_{m'} \left( \frac{r}{a} \right) = n \right) \bigg|_{n = S_{M(x, s)} \left( \frac{r}{s} \right)} \right] \bigg|_{m' = M(y, a)} = N^{1/2} \mathbb{E} \left[ \left( S_{M(x, s)} \left( \frac{r}{s} \right) + 1 \right) \mathbb{I} \left\{ S_{M(x, s)} \left( \frac{r}{s} \right) + 1 = S'_{M(y, a)} \left( \frac{r}{a} \right) \right\} \right] - N^{1/2} \mathbb{E} \left[ S_{M(x, s)} \left( \frac{r}{s} \right) \mathbb{I} \left\{ S_{M(x, s)} \left( \frac{r}{s} \right) = S'_{M(y, a)} \left( \frac{r}{a} \right) \right\} \right].
\]

(5.16)

By the binomial integration by parts [Lemma 5.4 \((^2)^o\)], the right-hand side of (5.16) is equal to

\[
\frac{N^{1/2} M(x, s) r}{a s} \mathbb{P} \left( S_{M(x, s) - 1} \left( \frac{r}{s} \right) + 2 = S'_{M(y, a)} \left( \frac{r}{a} \right) \right) - \frac{N^{1/2} M(x, s) r}{a s} \mathbb{P} \left( S_{M(x, s) - 1} \left( \frac{r}{s} \right) + 1 = S'_{M(y, a)} \left( \frac{r}{a} \right) \right) + \frac{N^{1/2}}{a} \mathbb{P} \left( S_{M(x, s)} \left( \frac{r}{s} \right) + 1 = S'_{M(y, a)} \left( \frac{r}{a} \right) \right).
\]

Next, we apply the first identity in Lemma 5.4 \((^1)^o\) for \( F(n) = \mathbb{P} \left( S_{M(x, s) - 1} \left( \frac{r}{s} \right) + 2 = n \right) \) and \( S_M = S'_{M(y, a)} \left( \frac{r}{a} \right) \) to the difference of the first two terms. Also, recall Theorem 3.1 and the fact that \( \lim_N M(x, s)/(Ns) = 1 \). Taking all of these into account, we write the foregoing expression as

\[
N \times \frac{M(x, s)}{Ns} \times N^{1/2} \mathbb{P} \left( S_{M(x, s) - 1} \left( \frac{r}{s} \right) + 2 = S'_{M(y, a)} \left( \frac{r}{a} \right) \right) - N \times \frac{M(x, s)}{Ns} \times N^{1/2} \mathbb{P} \left( S_{M(x, s) - 1} \left( \frac{r}{s} \right) + 2 = S'_{M(y, a) + 1} \left( \frac{r}{a} \right) \right) + \frac{1}{a} \times N^{1/2} \mathbb{P} \left( S_{M(x, s)} \left( \frac{r}{s} \right) + 1 = S'_{M(y, a)} \left( \frac{r}{a} \right) \right). 
\]

(5.17)
By the discussion between (5.16) and (5.17), we can write
\[
\int_{-N^{1/2}}^{\infty} dx_j \phi(x) \int_{-N^{1/2}}^{\infty} dy_j \phi(y) \frac{\partial}{\partial a} N^{1/2} \mathcal{P} \left( S_{M(x_j, s)} \left( \frac{r}{s} \right) = S'_{M(y_j, a)} \left( \frac{r}{a} \right) \right) \\
= \int_{-N^{1/2}}^{\infty} dx_j \phi(x) \int_{-N^{1/2}}^{\infty} dy_j \phi(y) \left\{ \text{the difference of the first two terms in (5.17)} \right\} + \mathcal{R}_{N,j},
\]
where \( \mathcal{R}_{N,j} \) is contributed by the last term in (5.17) since we can apply Proposition 4.4 as in the proof of Lemma 5.2. Notice that in the first two terms of (5.17), there are additional \( \pm 1 \) in the numbers of summands in the random sums. These unwanted integers can be removed by Lemma 5.4 (3°) and translating variables by \( \mp 1/(bN^{1/2}) \), for \( b \in \{s, a\} \). Hence,
\[
\int_{-N^{1/2}}^{\infty} dx_j \phi(x) \int_{-N^{1/2}}^{\infty} dy_j \phi(y) \left\{ \text{the difference of the first two terms in (5.17)} \right\}
= N \int_{-N^{1/2}}^{\infty} dx_j \frac{M(x_j, s) + 1}{Ns} \times \phi \left( x + \frac{1}{sN^{1/2}e_j} \right) \int_{-N^{1/2}}^{\infty} dy_j \phi(y) \\
\times N^{1/2} \mathcal{P} \left( S_{M(x_j, s)} \left( \frac{r}{s} \right) + 2 = S'_{M(y_j, a)} \left( \frac{r}{a} \right) \right) \\
- N \int_{-N^{1/2}}^{\infty} dx_j \frac{M(x_j, s) + 1}{Ns} \times \phi \left( x + \frac{1}{sN^{1/2}e_j} \right) \\
\times \int_{-N^{1/2}}^{\infty} dy_j \phi \left( y - \frac{1}{aN^{1/2}e_j} \right) \times N^{1/2} \mathcal{P} \left( S_{M(x_j, s)} \left( \frac{r}{s} \right) + 2 = S'_{M(y_j, a)} \left( \frac{r}{a} \right) \right).
\]
We formulate a discrete partial derivative \( [\phi(y) - \phi(y - \frac{1}{aN^{1/2}e_j})] \times aN^{1/2} \) from the foregoing difference. Hence, the foregoing expression shows that
\[
\int_{-N^{1/2}}^{\infty} dx_j \phi(x) \int_{-N^{1/2}}^{\infty} dy_j \phi(y) \left\{ \text{the difference of the first two terms in (5.17)} \right\}
= \frac{N}{a} \int_{-N^{1/2}}^{\infty} dx_j \frac{M(x_j, s) + 1}{Ns} \times \phi \left( x + \frac{1}{sN^{1/2}e_j} \right) \\
\times \int_{-N^{1/2}}^{\infty} dy_j \left[ \phi(y) - \phi(y - \frac{1}{aN^{1/2}e_j}) \right] \times aN^{1/2} \\
\times N^{1/2} \mathcal{P} \left( S_{M(x_j, s)} \left( \frac{r}{s} \right) + 2 = S'_{M(y_j, a)} \left( \frac{r}{a} \right) \right) + \mathcal{R}_{N,j}.
\]
This \( \mathcal{R}_{N,j} \) is given by
\[
N \int_{-N^{1/2}}^{\infty} dx_j \frac{M(x_j, s) + 1}{Ns} \times \phi \left( x + \frac{1}{sN^{1/2}e_j} \right) \int_{-N^{1/2}}^{\infty} dy_j \phi \left( y - \frac{1}{aN^{1/2}e_j} \right) \\
\times N^{1/2} \mathcal{P} \left( S_{M(x_j, s)} \left( \frac{r}{s} \right) + 2 = S'_{M(y_j, a)} \left( \frac{r}{a} \right) \right)
\]
and satisfies the definition of an \( \mathcal{R}_{N,j} \) by the fast decay property of \( \phi \). In the rest of this step, we show how to deal with the integral on the right-hand side of (5.20) so that it can be used to cancel \( \mathcal{J}_{N,2,2} \) left unsettled in Step 1.
The purpose of the next steps is to get probabilities of the form $P(S_M = S_{M'})$ which are present in $J_{N,2,2}$. In Steps 2-1–2-2, we calculate the integral in (5.20) with the following method: Iterating the second identity in (5.12), we get

$$E[F(S_M + 2)] = \frac{1}{p} \left( \frac{1}{p} E[F(S_{M+2})] - \frac{1-p}{p} E[F(S_{M+1})] \right)$$

$$- \frac{1-p}{p} \left( \frac{1}{p} E[F(S_{M+1})] - \frac{1-p}{p} E[F(S_M)] \right)$$

$$= \frac{1}{p^2} E[F(S_{M+2})] - \frac{2(1-p)}{p^2} E[F(S_{M+1})] + \frac{(1-p)^2}{p^2} E[F(S_M)].$$

(5.21)

Then we remove $+2$ and $+1$ in the numbers of summands by Lemma 5.4 (3°) in a way similar to how we obtain (5.19).

**Step 2-1.** In this step, we restrict our attention to $r$ such that $r/s \geq 1/2$. The integral in (5.20) is written as the following partial integral with respect to $x_j$ and then we apply the method outlined below (5.21):

$$\frac{N^{1/2}}{a} \int_{-N^{1/2}}^{N^{1/2}} dx_j \frac{M(x_j, s)}{N s} + \frac{1}{N s} \phi \left( x + \frac{1}{s N^{1/2} e_j} \right) E \left[ F(S(x_j, s) \left( \frac{r}{s} \right) + 2 \right]$$

$$= \frac{N^{1/2}}{a} \int_{-N^{1/2}}^{N^{1/2}} dx_j \frac{M(x_j, s) - 1}{N s} \times \phi \left( x - \frac{1}{s N^{1/2} e_j} \right) \frac{1}{(r/s)^2} E \left[ F(S(x_j, s) \left( \frac{r}{s} \right) \right]$$

$$- \frac{N^{1/2}}{a} \int_{-N^{1/2}}^{N^{1/2}} dx_j \frac{M(x_j, s)}{N s} \times \phi \left( x + \frac{1}{s N^{1/2} e_j} \right) (1-r/s)^2 (r/s)^2$$

$$\times E \left[ F(S(x_j, s) \left( \frac{r}{s} \right) \right]$$

(5.22)

$$= \frac{N^{1/2}}{a} \int_{-N^{1/2}}^{N^{1/2}} dx_j \overline{\phi}_N(x; r, s) E \left[ F(S(x_j, s) \left( \frac{r}{s} \right) \right]$$

$$- \frac{N^{1/2}}{a} \int_{-N^{1/2}}^{N^{1/2}} dx_j \left\{ 1\text{st integrand in (5.22)} \right\}$$

$$+ \frac{N^{1/2}}{a} \int_{-N^{1/2}}^{N^{1/2}} dx_j \left\{ 3\text{rd integrand in (5.22)} \right\},$$

(5.23)

where $\overline{\phi}_N$ gathers all the integrands in (5.22) and excludes the expectations:

$$\overline{\phi}_N(x; r, s) \overset{\text{def}}{=} \frac{M(x_j, s) - 1}{N s} \times \phi \left( x - \frac{1}{s N^{1/2} e_j} \right) \frac{1}{(r/s)^2} \frac{M(x_j, s)}{N s} \times \phi \left( x + \frac{1}{s N^{1/2} e_j} \right) (1-r/s)^2 (r/s)^2$$

(5.24)

Note that $\partial_{x_j} \phi \in \mathcal{S}(\mathbb{R}^2)$ and the sum of the coefficients of the expectations in (5.21) is 1. By these properties, (3.42) and the condition $r/s \geq 1/2$, we deduce that the equation of $\overline{\phi}_N(x; r, s)$
can be written in a simpler form:

\[
\overline{\phi}_N(x; r, s) = \overline{\phi}_N^0(x; r, s) + \phi(x),
\]

(5.25)

where \(\overline{\phi}_N^0(x; r, s)\) satisfies the following bound:

\[
|\overline{\phi}_N^0(x; r, s)| \leq \frac{C(T_0, T_1, \phi, n)}{N^{1/2}(1 + |x|^n)}, \quad \forall n \in \mathbb{N}.
\]

(5.26)

Now, it follows from (5.20) and (5.23) that

\[
1_{\{r/s \geq 1/2\}} \int_{-N^{1/2}}^{\infty} dx_j \phi(x) \int_{-N^{1/2}}^{\infty} dy_j \phi(y) \left\{ \text{the difference of the first two terms in (5.17)} \right\}
\]

\[
= 1_{\{r/s \geq 1/2\}} \frac{N^{1/2}}{a} \int_{-N^{1/2}}^{\infty} dx_j \overline{\phi}_N(x; r, s) \int_{-N^{1/2}}^{\infty} dy_j \left[ \phi(y) - \phi \left( y - \frac{1}{aN^{1/2}} e_j \right) \right] \times aN^{1/2}
\]

\[
\times N^{1/2} P \left( S_{M(x_j, s)} \left( \frac{r}{s} \right) = S'_{M(y_j, a)} \left( \frac{r}{a} \right) \right) + R_{N,j}.
\]

(5.27)

Here, \(\overline{\phi}_N(x; r, s)\) is defined by (5.24), and the new contribution for this \(R_{N,j}\) comes from the last two terms in (5.23). Besides, replacing \(\overline{\phi}_N(x; r, s)\) by \(\overline{\phi}_N^0(x; r, s)\) in the integral of (5.27) and applying (5.26) and Proposition 4.4 lead to another \(R_{N,j}\). In more detail, we use the integration with respect to \(dy\) and the property

\[
\int dy \left[ \phi(y) - \phi \left( y - \frac{1}{aN^{1/2}} e_j \right) \right] = 0
\]

to bring in the necessary renormalization function \(1_{\{1, \infty\}}(r)/(4\pi r)\) in applying Proposition 4.4.

Now, we see that (5.27) can be simplified to

\[
1_{\{r/s \geq 1/2\}} \int_{-N^{1/2}}^{\infty} dx_j \phi(x) \int_{-N^{1/2}}^{\infty} dy_j \phi(y) \left\{ \text{the difference of the first two terms in (5.17)} \right\}
\]

\[
= 1_{\{r/s \geq 1/2\}} \frac{N^{1/2}}{a} \int_{-N^{1/2}}^{\infty} dx_j \phi(x) \int_{-N^{1/2}}^{\infty} dy_j \mathcal{P} \left( S_{M(x_j, s)} \left( \frac{r}{s} \right) = S'_{M(y_j, a)} \left( \frac{r}{a} \right) \right) + R_{N,j}.
\]

(5.28)

**Step 2-2.** For the complementary case \(r/s < 1/2\), we turn to (3.22) and write

\[
P \left( S_{M(x_j, s)} \left( \frac{r}{s} \right) + 2 = S'_{M(y_j, a)} \left( \frac{r}{a} \right) \right)
\]

\[
= P \left( S'_{M(x_j, s)} \left( 1 - \frac{r}{s} \right) + M(y_j, a) - M(x_j, s) = S'_{M(y_j, a)} \left( 1 - \frac{r}{a} \right) + 2 \right).
\]

The above argument between (5.22) and (5.23) applies similarly if we use the random sums \(S'_{M}(1 - r/a)\) instead. This results in the replacement of the foregoing probability in (5.20) by

\[
P \left( S_{M(x_j, s)} \left( 1 - \frac{r}{s} \right) + M(y_j, a) - M(x_j, s) = S'_{M(y_j, a)} \left( 1 - \frac{r}{a} \right) \right)
\]

\[
= P \left( S_{M(x_j, s)} \left( \frac{r}{s} \right) = S'_{M(y_j, a)} \left( \frac{r}{a} \right) \right)
\]
with an additive $R_{N,j}$ as in (5.27).

Specifically, the integral in (5.20) is written as the following partial integral with respect to $y_j$: with $\phi_N(y) = \phi(y) - \phi(y - \frac{1}{aN^{1/2}}e_j)$,

$$
\frac{N^{1/2}}{a} \int_{-N^{1/2}}^{\infty} dy_j \phi_N(y) E \left[ F \left( S_{M(y_j,a)}' \left( 1 - \frac{r}{a} \right) + 2 \right) \right] = \frac{N^{1/2}}{a} \int_{-N^{1/2} + \frac{2}{aN^{1/2}}}^{\infty} dy_j \phi_N \left( y - \frac{2}{aN^{1/2}}e_j \right) \frac{1}{(1 - r/a)^2} E \left[ F \left( S_{M(y_j,a)}' \left( 1 - \frac{r}{a} \right) \right) \right] - \frac{N^{1/2}}{a} \int_{-N^{1/2} + \frac{2}{aN^{1/2}}}^{\infty} dy_j \phi_N \left( y - \frac{1}{aN^{1/2}}e_j \right) \frac{2[1 - (1 - r/a)]}{(1 - r/a)^2} E \left[ F \left( S_{M(y_j,a)}' \left( 1 - \frac{r}{a} \right) \right) \right] + \frac{N^{1/2}}{a} \int_{-N^{1/2}}^{\infty} dy_j \phi_N(y) \left[ 1 - (1 - r/a)^2 \right] \frac{2[1 - (1 - r/a)]}{(1 - r/a)^2} E \left[ F \left( S_{M(y_j,a)}' \left( 1 - \frac{r}{a} \right) \right) \right] \tag{5.29}
$$

$$
= \frac{N^{1/2}}{a} \int_{-N^{1/2}}^{\infty} dy_j \phi_N' \left( y; r, a \right) E \left[ F \left( S_{M(y_j,a)}' \left( 1 - \frac{r}{a} \right) \right) \right] - \frac{N^{1/2}}{a} \int_{-N^{1/2} + \frac{2}{aN^{1/2}}}^{\infty} dy_j \left\{ \text{1st integrand in (5.29)} \right\} \tag{5.30} + \frac{N^{1/2}}{a} \int_{-N^{1/2}}^{\infty} dy_j \left\{ \text{2nd integrand in (5.29)} \right\}.
$$

Here,

$$
\phi_N(y; r, a) \overset{\text{def}}{=} \phi_N \left( y - \frac{2}{aN^{1/2}}e_j \right) \frac{1}{(1 - r/a)^2} - \phi_N \left( y - \frac{1}{aN^{1/2}}e_j \right) \frac{2[1 - (1 - r/a)]}{(1 - r/a)^2} + \phi_N(y) \left[ 1 - (1 - r/a)^2 \right] \tag{5.31}
$$

Now, the simpler form of $\phi_N(y; r, a)$ is $\phi_N(y; r, a) = \phi^0_N(y; r, a) + \phi(y)$, where

$$
|\phi^0_N(y; r, a)| \leq \frac{C(T_0, T_1, \phi, n)}{N^{1/2}(1 + |y|^n)}, \quad \forall \, n \in \mathbb{N},
$$

since the condition $r/s < 1/2$ implies $1 - r/a \geq 1 - r/s \geq 1/2$. Therefore, arguing as before with what we do for (5.27) gives

$$
1_{\{r/s < 1/2\}} \int_{-N^{1/2}}^{\infty} dx_j \phi(x) \int_{-N^{1/2}}^{\infty} dy_j \phi(y) \left\{ \text{the difference of the first two terms in (5.17)} \right\} = 1_{\{r/s < 1/2\}} \frac{N^{1/2}}{a} \int_{-N^{1/2}}^{\infty} dx_j \phi(x) \int_{-N^{1/2}}^{\infty} dy_j \partial_y \phi(y) + N^{1/2} \mathcal{P} \left( S_{M(x_j,s)}' \left( \frac{r}{s} \right) = S_{M(y_j,a)}' \left( \frac{r}{a} \right) \right) + \mathcal{R}_{N,j}. \tag{5.32}
$$

**Step 3.** We recall the term $J_{N,2,2}$ in Step 1 and observe that, from (5.18), (5.28) and (5.32),

$$
J_{N,3}(s, t) = J_{N,2,2}(s, t) + \tilde{R}_N(s, t), \tag{5.33}
$$

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where
\[ \tilde{R}_N(s, t) = \sum_{j=1}^{2} \int_s^t da \int_0^s dr \int_{-N^{1/2}}^{N^{1/2}} dx_j \int_{-N^{1/2}}^{N^{1/2}} dy_j \mathcal{R}_{N,j}(a, r) N^{1/2} P \left( S_{M(x_j, s)} \left( \frac{r}{s} \right) = S'_{M(y_j, a)} \left( \frac{r}{a} \right) \right). \]

The definition (5.15) of \( \mathcal{R}_{N,j} \)'s show that (5.33) is enough for the proof of the lemma. ■

A summary of Lemmas 5.2 and 5.5 gives the following proposition upon recalling the decomposition in (5.10).

**Proposition 5.6.** (5.2) for \( \mathcal{L}_N = \mathcal{J}_N \) holds.

The Lipschitz continuity of \( \mathcal{K}_N \) can be obtained by an almost identical argument if we restart with (5.13) and replace \( P(S_{M(x_j, s)}(\xi) = S'_{M(y_j, a)}(\xi)) \) there by \( P(S_{M(x_j, a)}(\xi) = S'_{M(y_j, t)}(\xi)) \). One minor difference is that we replace the conditions \( r/s \leq 1/2 \) and \( r/s > 1/2 \) in Steps 2-1 and 2-2 by \( r/t \leq T_0/(2T_1) \) and \( r/t > T_0/(2T_1) \), since the latter gives us \( 1 - r/a \geq 1 - (T_1/T_0) \cdot T_0/(2T_1) \). Without giving further details, we state the result of this modification as the following proposition.

**Proposition 5.7.** (5.2) for \( \mathcal{L}_N = \mathcal{K}_N \) holds.

## 6 Stationary additive stochastic heat equation

In this section, we collect some well-known results for the stationary additive stochastic heat equation which are mentioned in Section 4 but seem difficult to find in the literature.

Let \( W \) be a two-sided space-time white noise. For any \( \phi \in L^2(\mathbb{R}^2, dx) \), \( W(\phi) \) is a two-sided Brownian motion with \( W_0(\phi) = 0 \) and \( \mathbb{E}[W_1(\phi)^2] = \mathbb{E}[W_{-1}(\phi)^2] = \|\phi\|_{L^2(\mathbb{R}^2, dx)}^2 \). An inspection of the proof of Proposition 3.15 shows that, without the renormalization, we would obtain the following divergent integral instead of the first integral in (3.8):

\[ \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} Q_{t-1}(x, y_1) \int_0^\infty Q_{2r}(y_1, y_2) dr Q_{s-1}(y_2, y) dy_1 dy_2. \]

Then at least formally, the initial condition \( X_0 \) below (4.11) needs to be replaced by

\[ \tilde{X}_0(\phi) = \int_{-\infty}^{0} \int_{\mathbb{R}^2} Q_{-r}(x) W(dx, dr) \]

so that the process corresponding to (4.9) is

\[ \tilde{X}_t(\phi) = \int_{-\infty}^{t} \int_{\mathbb{R}^2} Q_{t-r}(x) W(dx, dr). \]

The process in (6.2) is an analogue of the classical stationary solution of the Ornstein–Uhlenbeck process. The next proposition shows that it is well-defined whenever \( \phi \in \mathcal{S}_0(\mathbb{R}^2) \).

**Proposition 6.1.** For every \( \phi \in \mathcal{S}_0(\mathbb{R}^2) \), the improper stochastic integral in (6.2) converges a.s. for any fixed \( t \geq 0 \) and, as a process, has the same finite-dimensional marginals as \( X(\phi) \) defined by (4.12). In particular, \( (X_t; t \geq 0) \) as an \( \mathcal{S}'(\mathbb{R}^2) \)-valued process is stationary.
Proof. For the first assertion regarding $\widetilde{X}$, we notice that, for $-\infty < -S \leq -T \leq 0 \leq t < \infty$,

$$\Cov \left[ \int_{-S}^{-t} \int_{\mathbb{R}^2} Q_{t-r} \phi(x) W(dx, dr); \int_{-T}^{t} \int_{\mathbb{R}^2} Q_{t-r} \phi(x) W(dx, dr) \right]$$

$$= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \phi(x) \phi(y) \left( \int_{0}^{T+t} Q_{2r}(x, y) dr \right) dxdy$$

$$\overset{T \to \infty}{\longrightarrow} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \phi(x) \phi(y) \left[ \int_{-\infty}^{\infty} \left( Q_{2r}(x, y) - \mathbb{1}_{[1, \infty)}(\frac{r}{4\pi}) \right) dr \right] dxdy \quad (6.3)$$

by dominated convergence upon using the fast decay property of $\phi_1, \phi_2$ and the inequality $1 - e^{-x} \leq x$ for all $x \geq 0$. The improper stochastic integral in (6.2) converges in $L^2(\mathbb{P})$ by (6.3), and so, almost surely by the martingale convergence theorem as in the proof of Proposition 2.1.

To see that $X(\phi)$ and $\widetilde{X}(\phi)$ have the same finite-dimensional marginals, we use Lemma 3.16 to rewrite the log kernel in the definition (4.11) of $X_0$. For all $0 \leq s \leq t < \infty$,

$$\Cov[X_s(\phi); X_t(\phi)]$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \phi(x) \phi(y) \left( \int_{\mathbb{R}^2} Q_s(x, y_1) \left( -\ln|y_1 - y_2| \right) Q_t(y_2, y) dy_1 dy_2 \right) dxdy$$

$$+ \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \phi(x) \phi(y) \left( \int_{0}^{s} \int_{\mathbb{R}^2} Q_{s-r}(z, x) Q_{t-r}(z, y) dz dr \right) dxdy$$

$$= \lim_{T \to \infty} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \phi(x) \phi(y) \left( \int_{0}^{T} \int_{\mathbb{R}^2} Q_{s-r}(z, x) Q_{t-r}(z, y) dz dr \right) dxdy$$

$$+ \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \phi(x) \phi(y) \left( \int_{0}^{s} \int_{\mathbb{R}^2} Q_{r}(z, x) Q_{t-s+r}(z, y) dz dr \right) dxdy$$

$$= \lim_{T \to \infty} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \phi(x) \phi(y) \left( \int_{0}^{T+s} Q_{t-s+2r}(x, y) dr \right) dxdy$$

$$= \lim_{T \to \infty} \Cov \left[ \int_{-T}^{-s} \int_{\mathbb{R}^2} Q_{s-r} \phi(x) W(dx, dr); \int_{-T}^{t} \int_{\mathbb{R}^2} Q_{t-r} \phi(x) W(dx, dr) \right],$$

where we use the assumption $\phi \in \mathcal{S}_0(\mathbb{R}^2)$ in the second equality. The last equality shows that $\Cov[X_s(\phi); X_t(\phi)] = \Cov[\widetilde{X}_s(\phi); \widetilde{X}_t(\phi)]$, which is enough for the required identity in finite-dimensional marginals.

Finally, given $\phi_1, \phi_2 \in \mathcal{S}_0(\mathbb{R}^2)$, $\Cov[X_t(\phi_1); X_t(\phi_2)]$ is given by (6.3) with $\phi(x)\phi(y)$ replaced by $\phi_1(x)\phi_2(y)$. Hence, $\Cov[X_t(\phi_1); X_t(\phi_2)]$ does not depend on $t$. This proves the stationarity of $X$.

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