EXTREMAL DISCREPANCY BEHAVIOR OF LACUNARY SEQUENCES

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ABSTRACT. In 1975 Walter Philipp proved the law of the iterated logarithm (LIL) for
the discrepancy of lacunary sequences; that is, he showed that for any sequence \((n_k)_{k \geq 1}\)
satisfying the Hadamard gap condition \(n_{k+1}/n_k \geq q > 1, k \geq 1\), we have
\[
\frac{1}{4\sqrt{2}} \leq \limsup_{N \to \infty} \frac{ND_N(\{n_1x\}, \ldots, \{n_Nx\})}{\sqrt{2N \log \log N}} \leq C_q
\]
for almost all \(x\). In recent years there has been significant progress concerning the precise
value of the limsup in this LIL for special sequences \((n_k)_{k \geq 1}\) having a “simple” number-
theoretic structure. However, since the publication of Philipp’s paper there has been no
progress concerning the lower bound in this LIL for generic lacunary sequences \((n_k)_{k \geq 1}\).
The purpose of the present paper is to collect known results concerning this problem, to
investigate what the optimal value in the lower bound could be, and for which special
sequences \((n_k)_{k \geq 1}\) a small value of the limsup in the LIL for the discrepancy can be
obtained. We formulate three open problems, which could serve as the main targets for
future research.

1. INTRODUCTION AND STATEMENT OF RESULTS

It is a well-known fact that for quickly increasing \((n_k)_{k \geq 1}\) the systems \((\cos 2\pi n_k x)_{k \geq 1}\) and
\((\sin 2\pi n_k x)_{k \geq 1}\) show properties which are typical for systems of independent, identically
distributed (i.i.d.) random variables. For example, if \((n_k)_{k \geq 1}\) is an increasing sequence of
positive integers satisfying the Hadamard gap condition
\[
\frac{n_{k+1}}{n_k} \geq q > 1, \ k \geq 1,
\]
then we have the following:

- The series \(\sum_{k=1}^\infty a_k \cos 2\pi n_k x\) is almost everywhere (a.e.) convergent if and only if
  \[\sum_{k=1}^N a_k^2 < \infty.\]

- For all \(t \in \mathbb{R}\) we have
  \[
  \lambda \left\{ x \in (0, 1) : \sqrt{2} \sum_{k=1}^N \cos 2\pi n_k x < t\sqrt{N} \right\} \to \Phi(t).
  \]

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We have

\[
\limsup_{N \to \infty} \frac{\left| \sum_{k=1}^{N} \cos 2\pi n_k x \right|}{\sqrt{2N \log \log N}} = \frac{1}{\sqrt{2}} \quad \text{a.e.}
\]

The first result, which is due to Kolmogorov \[20\] and Zygmund \[29\], is a counterpart of Kolmogorov’s three-series theorem for series of i.i.d. random variables. The second result, where \(\lambda\) denotes the Lebesgue measure and \(\Phi\) the standard normal distribution function, clearly is a counterpart of the central limit theorem (CLT); it is due to Salem and Zygmund \[25\]. The third result is a counterpart of the law of the iterated logarithm, and has been proved by Erdős and Gál \[13\]. Actually, they proved a similar result for \(e^{2\pi ix}\) rather than \(\cos 2\pi x\), but the proof is solely based on orthogonality arguments and can be used for \(\cos 2\pi x\) as well. All the results in this list remain valid if the function \(\cos 2\pi x\) is replaced by \(\sin 2\pi x\). The Hadamard gap condition for \((n_k)_{k \geq 1}\) is not optimal in any of these three theorems; the a.e. convergence problem for Fourier series was solved by Carleson \[9\], and for both the CLT and the LIL the so-called Erdős gap condition is sufficient. See \[7, 12, 27\]. However, generally speaking, the almost-independent behavior of \((\cos 2\pi n_k x)_{k \geq 1}\) and \((\sin 2\pi n_k x)_{k \geq 1}\) breaks down unless very strong growth conditions or number-theoretic conditions are imposed on \((n_k)_{k \geq 1}\). Classical survey papers concerning the almost-independent behavior of lacunary trigonometric sums are \[18, 19\]; more recent ones are \[5, 15\]. Recently also the multidimensional version of these problems has gained some attention, see for example the papers of Levin \[22\] and Moore and Zhang \[23\].

A sequence \((x_k)_{k \geq 1}\) of real numbers from the unit interval is called uniformly distributed modulo one (u.d. mod 1) if for any subinterval \(A = [a, b]\) of the unit interval we have

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} 1_A(x_k) = \lambda(A) = b - a.
\]

By the famous Weyl criterion, first obtained by Hermann Weyl in his seminal paper \[28\] of 1916, this is equivalent to the fact that for all \(h \in \mathbb{Z} \setminus 0\) we have

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} e^{2\pi i h x_k} = 0.
\]

Using this criterion, Weyl showed that for any sequence \((n_k)_{k \geq 1}\) of distinct positive integers the sequence \((\{n_k x\})_{k \geq 1}\), where \(\{\cdot\}\) denotes the fractional part function, is u.d. mod 1 for almost all \(x\). By (3) this is equivalent to the fact that for any sequence \((n_k)_{k \geq 1}\) of distinct positive integers we have

\[
\frac{1}{N} \sum_{k=1}^{N} \cos 2\pi n_k x \to 0 \quad \text{and} \quad \frac{1}{N} \sum_{k=1}^{N} \sin 2\pi n_k x \to 0 \quad \text{for almost all } x;
\]

in other words, Weyl’s theorem can be seen as a variant of the strong law of large numbers for \((\cos 2\pi n_k x)_{k \geq 1}\) and \((\sin 2\pi n_k x)_{k \geq 1}\). The quality of uniform distribution is measured
in terms of the so-called discrepancy. There exist two classical notions of discrepancies, denoted by $D_N$ and $D_N^*$, which are defined by

$$D_N(x_1, \ldots, x_N) = \sup_{0 \leq a \leq b \leq 1} \left| \frac{1}{N} \sum_{k=1}^{N} 1_{[a,b]}(x_k) - (b - a) \right|$$

and

$$D_N^*(x_1, \ldots, x_N) = \sup_{0 \leq a \leq 1} \left| \frac{1}{N} \sum_{k=1}^{N} 1_{[0,a]}(x_k) - a \right|,$$

for points $x_1, \ldots, x_N$ in $[0, 1]$. $D_N$ is called the extremal discrepancy or just discrepancy, and $D_N^*$ is called the star discrepancy. It is easily seen that for any points we have $D_N^* \leq D_N \leq 2D_N^*$, and that an infinite sequence $(x_k)_{k \geq 1}$ is u.d. mod 1 if and only if its discrepancy tends to zero as $N \to \infty$. Koksma’s inequality (see e.g. [21, Chapter 2, Theorem 5.1]) states that

$$\left| \int_0^1 f(x) \, dx - \frac{1}{N} \sum_{k=1}^{N} f(x_k) \right| \leq \left( \text{Var}_{[0,1]} f \right) D_N^*(x_1, \ldots, x_N)$$

for any function $f$ of bounded variation on $[0, 1]$; the multidimensional generalizations of this inequality are the reason why discrepancy theory plays an important role in multidimensional numerical integration, see e.g. [10, 11, 21].

The strongest qualitative version of Weyl’s metric theorem which is known to date is the following result of Baker [6]: for any strictly increasing sequence $(n_k)_{k \geq 1}$ of positive integers we have

$$D_N(\{n_1 x\}, \ldots, \{n_N x\}) = O\left( \frac{(\log N)^{3/2+\epsilon}}{\sqrt{N}} \right) \quad \text{as } N \to \infty, \text{ for almost all } x.$$

The optimal value of the exponent of the logarithmic term in (5) is an important open problem in metric number theory; it is just known that it cannot be smaller than $1/2$, see [8]. In general, determining the precise asymptotic order of the discrepancy of $(\{n_k x\})_{k \geq 1}$ for typical $x$ is a very difficult problem, which is only solved for very few sequences $(n_k)_{k \geq 1}$ (for example when $n_k = k$, $k \geq 1$; see [26]). However, due to the analogy between lacunary series and sequences of independent random variables described in the first paragraph, very precise metric results for the asymptotic order of the discrepancy of $(\{n_k x\})_{k \geq 1}$ can be obtained if $(n_k)_{k \geq 1}$ is quickly increasing, as we will show in the next paragraph.

One could expect that the almost-independence property of lacunary series remains valid if the functions cos $2\pi x$ and sin $2\pi x$ are replaced by an other “nice” 1-periodic function $f$. However, this is only the case in a significantly weakened form, even if $f$ is a trigonometric polynomial. This is most easily seen by considering

$$f(x) = \cos 2\pi x - \cos 4\pi x, \quad n_k = 2^k, \quad k \geq 1.$$
In this case the sum $\sum_{k=1}^{N} f(n_k x)$ is a telescoping sum, and it is obvious that the normalized partial sums will satisfy neither the CLT nor the LIL. An other example, independently due to Erdős and Fortet (see [18]) is the following: set 

$$f(x) = \cos 2\pi x + \cos 4\pi x, \quad n_k = 2^k - 1, k \geq 1.$$ 

In this case the CLT fails (the limit distribution of the normalized partial sums can be shown to be a mixture normal distribution), while the LIL holds in the modified form

$$\lim_{N \to \infty} \frac{\sum_{k=1}^{N} f(n_k x)}{\sqrt{2N \log \log N}} = \sqrt{2} |\cos \pi x| \quad \text{a.e.}$$

Thus the precise form of the LIL may fail for Hadamard lacunary $(n_k)_{k \geq 1}$ in the case of general $f$. However, in this case by a result of Takahashi [27] we still have the following upper-bound version of the LIL: for $(n_k)_{k \geq 1}$ satisfying (1) and $f$ satisfying

$$f(x+1) = f(x), \quad \int_{0}^{1} f(x) \, dx = 0, \quad \text{Var}_{[0,1]} f < \infty,$$

we have

$$\limsup_{N \to \infty} \frac{\sum_{k=1}^{N} f(n_k x)}{\sqrt{2N \log \log N}} \leq C \quad \text{a.e.,}$$

for some appropriate constant $C$ (depending on $f$ and the growth factor $q$). From a probabilistic point of view, the star-discrepancy and extremal discrepancy are a version of the (one-sided and two-sided, respectively) Kolmogorov-Smirnov statistic, adjusted to the uniform distribution on $[0,1]$ and applied to the empirical distribution induced by $x_1, \ldots, x_N$. The Chung-Smirnov law of the iterated logarithm for the Kolmogorov-Smirnov statistic states (as a special case) that for $X_1, X_2, \ldots$ being i.i.d. random variables having uniform distribution on $[0,1]$ we have

$$\lim_{N \to \infty} \frac{1}{\sqrt{2N \log \log N}} \frac{N D_N(X_1, \ldots, X_N)}{\sqrt{2N \log \log N}} = \frac{1}{2} \quad \text{a.s.,}$$

and the same result holds if $D_N$ is replaced by $D_N^*$. Following the heuristics on the almost-independent behavior of lacunary series, Erdős and Gál conjectured that an upper-bound version of the Chung–Smirnov LIL should hold if the sequence of i.i.d. random variables is replaced by the identically $[0,1]$-uniformly distributed and “almost independent” system $(\{n_k x\})_{k \geq 1}$. This was confirmed by Philipp [24] in 1975; he showed that for $(n_k)_{k \geq 1}$ satisfying (1) we have

$$\limsup_{N \to \infty} \frac{N D_N(\{n_1 x\}, \ldots, \{n_N x\})}{\sqrt{2N \log \log N}} \leq C_q \quad \text{a.e.,}$$

where $C_q$ is a constant depending on $q$. The same result holds if $D_N$ is replaced by $D_N^*$. Together with Koksma’s inequality, (2) implies (3). On the other hand, by (2), Koksma’s
inequality and the fact that $\text{Var}_{[0,1]} \cos 2\pi x = 4$ we have
\begin{equation}
\limsup_{N \to \infty} \frac{ND_N^* \{\{n_1x\}, \ldots, \{n_Nx\}\}}{\sqrt{2N \log \log N}} \geq 1 \quad \text{a.e.,}
\end{equation}
and the same lower bound holds for $D_N$.

In the sequel, for a given sequence $(n_k)_{k \geq 1}$ we set
\begin{align*}
\Lambda(x) &= \limsup_{N \to \infty} \frac{ND_N \{\{n_1x\}, \ldots, \{n_Nx\}\}}{\sqrt{2N \log \log N}}, \\
\Lambda^*(x) &= \limsup_{N \to \infty} \frac{ND_N^* \{\{n_1x\}, \ldots, \{n_Nx\}\}}{\sqrt{2N \log \log N}}.
\end{align*}

In 2008 Fukuyama [14] calculated the precise value of the limsup in the LIL for the discrepancy of $(\{n_kx\})_{k \geq 1}$ for special sequences of the form $n_k = \theta^k$, $k \geq 1$. Amongst other results, for such sequences he obtained the following:
\begin{equation}
\Lambda = \Lambda^* = \begin{cases}
\frac{\sqrt{2\theta}}{9} & \text{if } \theta = 2, \\
\frac{\sqrt{(\theta+1)(\theta-2)}}{2\sqrt{n+1}} & \text{if } \theta \geq 4 \text{ is even}, \\
\frac{2\sqrt{n+1}}{\theta-1} & \text{if } \theta \geq 3 \text{ is odd}, \\
\frac{1}{2} & \text{if } \theta^r \not\in \mathbb{Q} \text{ for all } r \in \mathbb{Q},
\end{cases}
\end{equation}
for almost all $x$. Such results indicate that there is a close connection between number-theoretic properties of $(n_k)_{k \geq 1}$ on the one hand and the precise asymptotic order of $\sum f(n_kx)$ and of the discrepancy of $(\{n_kx\})_{k \geq 1}$ on the other hand. It turns out that the number of solutions of certain Diophantine equations plays an important role in this connection. Following the notation from [11], for $j_1 \geq 1, j_2 \geq 1, \nu \in \mathbb{Z}$ and $N \geq 1$ we set
\begin{equation}
S(j_1, j_2, \nu, N) = \# \{ (k, l) : (j_1, k) \neq (j_2, l), 1 \leq k, l \leq N, j_1n_k - j_2n_l = \nu \}.
\end{equation}

For some $d \geq 1$ we say that $(n_k)_{k \geq 1}$ satisfies condition $D_d$ if for all $1 \leq j_1, j_2 \leq d$ there exist real numbers $\gamma_{j_1, j_2, \nu}$ such that
\begin{equation}
\left| \frac{S(j_1, j_2, \nu, N)}{N} - \gamma_{j_1, j_2, \nu} \right| = O \left( \frac{1}{(\log N)^{1+\varepsilon}} \right)
\end{equation}
for some fixed $\varepsilon > 0$, uniformly in $\nu \in \mathbb{Z}$. Furthermore we say that $(n_k)_{k \geq 1}$ satisfies condition $D$ if it satisfies $D_d$ for all $d \geq 1$. Assume that $(n_k)_{k \geq 1}$ satisfies (11) and condition $D$, and that $f$ is a function satisfying (7). Then, writing
\begin{equation}
f(x) = \sum_{j=1}^{\infty} (a_j \cos 2\pi jx + b_j \sin 2\pi jx)
\end{equation}
and setting
\begin{equation}
\sigma^2_f(x) = \|f\|^2 + \sum_{\nu = -\infty}^{\infty} \sum_{j_1, j_2 = 1}^{\infty} \frac{\gamma_{j_1, j_2, \nu}}{2} \left( (a_{j_1}a_{j_2} + b_{j_1}b_{j_2}) \cos 2\pi \nu x + (b_{j_1}a_{j_2} - a_{j_1}b_{j_2}) \sin 2\pi \nu x \right),
\end{equation}
where \( \| \cdot \| \) denotes the \( L^2(0, 1) \) norm, we have

\[
\limsup_{N \to \infty} \frac{1}{\sqrt{2N \log \log N}} \left| \sum_{k=1}^{N} f(n_k x) \right| = \sigma_f(x) \quad \text{a.e.}
\]

(see [4, Theorem 2]). If \( f \) is a trigonometric polynomial of degree \( d \), it is sufficient to assume condition \( D_d \) and the second sum in (14) can be restricted to \( 1 \leq j_1, j_2 \leq d \). The Erdős-Fortet example (6) is a special case of this result. For the discrepancy it implies (see [4, Theorem 3]) that

\[
\Lambda^*(x) = \sup_{0 \leq a \leq 1} \sigma_{[0,a]}(x) \quad \text{a.e. and} \quad \Lambda^*(x) = \sup_{0 \leq a \leq b \leq 1} \sigma_{[a,b]}(x) \quad \text{a.e.,}
\]

where \( \sigma_{[a,b]} \) is the function defined in (14) for the (centered, extended with period 1) indicator functions

\[
I_{[a,b]}(x) = \sum_{m \in \mathbb{Z}} I_{[a,b]}(x + m) - (b - a), \quad \text{where} \ 0 \leq b - a < 1.
\]

In particular if the Diophantine equations in (12) have a “small” number of solutions, that is if \( \gamma_{j_1,j_2,\nu} = 0 \) for all \( j_1, j_2, \nu \), then

\[ \Lambda^* = \Lambda = \frac{1}{2} \quad \text{a.e.;} \]

that is, in this case we have the same limsup as in the Chung–Smirnov LIL for i.i.d. random variables (see [3]). On the other hand, if the number of solutions of these Diophantine equations is large, then \( \Lambda \) and \( \Lambda^* \) are in general different from \( 1/2 \), and can even show the “irregular” behavior of not being equal to a constant for almost all \( x \) (see [1, 2, 16]). This is in accordance with (11), which gives values for \( \Lambda \) and \( \Lambda^* \) different from \( 1/2 \) for sequences \( (n_k)_{k \geq 1} \) for which Diophantine equations such as \( \theta n_k - n_l = 0 \) have many solutions.

Despite considerable efforts to construct lacunary sequences \( (n_k)_{k \geq 1} \) with extremal discrepancy behavior, so far no lacunary \( (n_k)_{k \geq 1} \) has been found for which

- \( \Lambda^* < \frac{1}{2} \) on a set of positive measure, or
- \( \| \Lambda^* \| < \frac{1}{2}, \) or
- \( \Lambda < \frac{1}{2} \) on a set of positive measure.

The purpose of the present paper is to construct a sequence \( (n_k)_{k \geq 1} \) for which the first of these three properties holds, and to demonstrate why conventional constructions cannot provide an example of a sequence for which either the second or third property holds. Furthermore, we will show that the lower bound in (10) can be improved if \( D_N^* \) is replaced by \( D_N \).

**Theorem 1.** Let \( (n_k)_{k \geq 1} \) be defined by

\[
n_k = \begin{cases} 
3k^2 & \text{if } k \text{ is odd} \\
3(k-1)^2 + 1 & \text{if } k \text{ is even}.
\end{cases}
\]
Then for almost all $x \in [0, 1]$ we have

$$
\Lambda^*(x) = \begin{cases} 
\sqrt{-3x^2-x+2} & \text{if } 0 \leq x \leq \frac{1}{6}, \\
\sqrt{-24x+25} \cdot \frac{1}{72} & \text{if } \frac{1}{6} \leq x \leq \frac{3}{8}, \\
\sqrt{\frac{2}{9}} & \text{if } \frac{3}{8} \leq x \leq \frac{1}{2}, \\
\Lambda^*(1-x) & \text{if } \frac{1}{2} < x \leq 1.
\end{cases}
$$

In particular we have $\Lambda^*(x) < 1/2$ for almost all $x \in [7/24, 17/24]$.

![Figure 1. The function $\Lambda^*(x)$ for the sequence $(n_k)_{k \geq 1}$ defined in Theorem 1.](image)

The construction of the sequence in Theorem 1 is very similar to the constructions in [1, 2], and makes use of formula (15) and the simple structure of the sequence $(n_k)_{k \geq 1}$ with respect to linear Diophantine equations as those in (12). Sequences constructed in this way are (relatively simple) examples of sequences satisfying condition $\mathbf{D}$; and sequences satisfying condition $\mathbf{D}$ are essentially the only sequences for which the values of $\Lambda$ and $\Lambda^*$ can be calculated with currently available tools. The next theorem shows, roughly speaking, that the other two open problems on the previous list cannot be solved using such a standard construction. If a sequence satisfying one of the last two of the three points above exists, then it cannot satisfy condition $\mathbf{D}$ and accordingly must have a somewhat irregular Diophantine structure.

**Theorem 2.** If $(n_k)_{k \geq 1}$ satisfies the Diophantine condition $\mathbf{D}$, then

$$
\|\Lambda^*\| \geq \frac{1}{2}
$$

and

$$
\Lambda \geq \frac{1}{2} \quad \text{a.e.}
$$
The next theorem is an improved version of the general lower bound \( (10) \), which follows from a version of Koksma’s inequality for symmetric functions (see Lemma 3 in the next section).

**Theorem 3.** For any sequence of positive integers \((n_k)_{k \geq 1}\) satisfying \( (1) \) we have

\[
\Lambda \geq \frac{1}{2\sqrt{2}} \quad \text{a.e.}
\]

Some important problems concerning lower bounds for \( \Lambda \) and \( \Lambda^* \) remain open. In all three questions below it is assumed that \((n_k)_{k \geq 1}\) satisfies the Hadamard gap condition \( (1) \).

- **Open problem 1:** Is it possible that \( \| \Lambda^* \| < \frac{1}{2} \)?
- **Open problem 2:** Is it possible that \( \Lambda^* < 1/2 \) almost everywhere?
- **Open problem 3:** Is it possible that \( \Lambda < \frac{1}{2} \) on a set of positive measure?

Theorem 2 suggests that the answer of all three problems could be negative, but this is by no means certain. Problem 1 and Problem 2 are related: if the answer of Problem 1 is “no”, then the answer of Problem 2 must also be “no”; if the answer of Problem 2 is “yes”, then the answer of Problem 1 must also be “yes”.

The second part of Theorem 2 is proved with an argument which involves taking the average \( L^2 \) norm over all intervals of fixed length. This argument is related to an interesting phenomenon, which apparently has not been observed before. Since this observation might be fruitful for further investigations (also in the context of \( (5) \)), we state it as a theorem below. Note that for this theorem no growth conditions whatsoever for the numbers \( n_k \) are necessary.

**Theorem 4.** For any \( N \geq 1 \) let \( n_1, \ldots, n_N \) be distinct positive integers. Let \( z \in (0,1) \) be fixed. Then for any \( x \in [0,1] \) we have

\[
\int_0^1 \int_0^1 \left( \sum_{k=1}^{N} I_{[a,a+z]} (n_k x) \right)^2 \, dx \, da = z(1-z)N.
\]

Note that \( z(1-z) \) is the square of the \( L^2(0,1) \) norm of the function \( I_{[a,a+z]} \) for some \( a \). Theorem 4 should be compared with the fact that for a sequence \( X_1, \ldots, X_N \) of i.i.d. \([0,1]\)-uniformly distributed random variables for any \( a, z \) we have

\[
\int_{\Omega} \left( \sum_{k=1}^{N} I_{[a,a+z]} (X_k (\omega)) \right)^2 \, d\omega = z(1-z)N.
\]

Such a result does not hold for the (dependent) random variables \( \{n_1 x\}, \ldots, \{n_N x\} \) for all individual values of \( a \); however, as Theorem 4 shows, such a result holds “on average” by integrating over all indicator functions of equal length.

**2. Auxiliary results**

In this section we first state several auxiliary results, and give the proofs for them afterward.
Lemma 1. For the sequence defined in Theorem 1 we have for any fixed $1 \leq j_2 \leq j_1$ that

$$S(j_1, j_2, \nu, N) = \begin{cases} \frac{N}{2} + O(1) & \text{if } j_1 = 3j_2 \text{ and } j_2 = \nu, \\ O(1) & \text{otherwise,} \end{cases}$$

uniformly in $\nu \in \mathbb{Z}$.

Lemma 2. Let $f$ be a function satisfying (7), and write

$$f(x) = \sum_{j=1}^{\infty} (a_j \cos 2\pi j x + b_j \sin 2\pi j x).$$

Let $(n_k)_{k \geq 1}$ be a sequence satisfying (11) and the Diophantine condition D. Then we have

$$\sum_{\nu=-\infty}^{\infty} \sum_{j_1, j_2 = 1}^{\infty} \frac{\gamma_j}{2} (|a_{j_1} a_{j_2}| + |b_{j_1} b_{j_2}| + |b_{j_1} a_{j_2}| + |a_{j_1} b_{j_2}|) \leq \frac{2(\text{Var}_{[0,1]} f)^2 q}{3(q-1)},$$

where $q$ is the growth factor in (11).

Lemma 2 provides an uniform upper bound for the function $\sigma_f(x)$ defined in (14), and will be needed for the proof of the second part of Theorem 2.

Lemma 3. Let $f(x)$ be a function having bounded variation $\text{Var}_{[0,1]} f$ on $[0,1]$, which is symmetric around 1/2; that is, it satisfies $f(1/2 - y) = f(1/2 + y)$, $y \in [0,1/2]$. Then for any points $x_1, \ldots, x_N \in [0,1]$ we have

$$\left| \int_0^1 f(x) \, dx - \frac{1}{N} \sum_{k=1}^{N} f(x_k) \right| \leq \frac{\text{Var}_{[0,1]} f}{2} D_N(x_1, \ldots, x_N).$$

This lemma is a version of Koksma’s inequality (4), adapted to the case of symmetric functions. Note the two differences between this Lemma and the original version of Koksma’s inequality: On the one hand, in the error estimate $\text{Var}_{[0,1]} f$ is replaced by $(\text{Var}_{[0,1]} f)^2 / 2$; on the other hand, the star-discrepancy $D_N^*$ is replaced by $D_N$.

Proof of Lemma 1. The construction of the sequence $(n_k)_{k \geq 1}$ in Theorem 1 is exactly the same as the construction of the sequence in [1], except that the base 2 has been replaced by 3. Thus Lemma 1 is a variant of [1, Lemma 2], and can be shown in exactly the same way.

Proof of Lemma 2. Let $K = \text{Var}_{[0,1]} f$. Then for the Fourier coefficients of $f$ we have (see for example [17, Theorem 1] or [30, p. 48]) that

$$|a_j| \leq \frac{K}{\pi j}, \quad |b_j| \leq \frac{K}{\pi j}, \quad j \geq 1.$$ 

Thus the left-hand side of (17) is bounded by

$$(18) \quad \sum_{\nu=-\infty}^{\infty} \sum_{j_1, j_2 = 1}^{\infty} \frac{\gamma_j}{2} \frac{4K^2}{\pi^2 j_1 j_2} = \frac{4}{\pi^2} \sum_{\nu=-\infty}^{\infty} \sum_{1 \leq j_1 < j_2 < \infty} \frac{\gamma_j}{j_1 j_2} K^2.$$
where the equality follows from the fact that for all \( j_1, j_2, \nu \) we have \( \gamma_{j_1,j_2,\nu} = \gamma_{j_2,j_1,-\nu} \), and that by \( \text{(1)} \) we necessarily have \( \gamma_{j_1,j_2,\nu} = 0 \) whenever \( j_1 = j_2 \). We can rewrite the right-hand side of \( \text{(18)} \) in the form

\[
\frac{4}{\pi^2} \sum_{j_1=1}^{\infty} \sum_{i=0}^{\infty} \sum_{j_1q^i < j_2 \leq j_1q^{i+1}} \sum_{\nu=-\infty}^{\infty} \frac{\gamma_{j_1,j_2,\nu} K^2}{j_1j_2},
\]

where \( q \) is the growth factor from \( \text{(1)} \). From the definition of the numbers \( \gamma_{j_1,j_2,\nu} \) for any fixed \( j_1 \) and \( i \) we necessarily have

\[
\sum_{j_1q^i < j_2 \leq j_1q^{i+1}} \sum_{\nu=-\infty}^{\infty} \gamma_{j_1,j_2,\nu} \leq 1
\]

(it is easily seen that assuming the opposite leads to a contradiction). Thus for \( \text{(19)} \) we get the upper bound

\[
\frac{4}{\pi^2} \sum_{j_1=1}^{\infty} \sum_{i=0}^{\infty} \frac{K^2}{j_1^2 q^i} = \frac{2K^2 q}{3(q-1)},
\]

which by our previous consideration is also an upper bound for the left-hand side of \( \text{(17)} \). This proves the Lemma.

**Proof of Lemma** Let \( f \) and \( x_1, \ldots, x_N \in [0, 1] \) be given. We define points \( \bar{x}_1, \ldots, \bar{x}_N \) by

\[
\bar{x}_k = \begin{cases} 
  x_k & \text{if } x_k \leq 1/2 \\
  1 - x_k & \text{if } x_k > 1/2,
\end{cases}
\]

for \( 1 \leq k \leq N \). Then clearly all points \( \bar{x}_1, \ldots, \bar{x}_N \) lie in the interval \( [0, 1/2] \), and by the symmetry of \( f \) we have \( f(x_k) = f(\bar{x}_k), \ 1 \leq k \leq N \). Also by the symmetry of \( f \) we have

\[
\left| \int_0^1 f(x) \, dx - \frac{1}{N} \sum_{k=1}^N f(x_k) \right| &= \left| \int_0^1 f(x/2) \, dx - \frac{1}{N} \sum_{k=1}^N f(2\bar{x}_k/2) \right| \\
&\leq \text{Var}_{[0,1]} f(x/2) \ D_N^*(2\bar{x}_1, \ldots, 2\bar{x}_N),
\]

where the last line follows from Koksma’s inequality for the function \( f(x/2) \). Note that by the symmetry of \( f \) we have

\[
\text{Var}_{[0,1]} f(x/2) = \left( \text{Var}_{[0,1]} f(x) \right) / 2.
\]

For any \( a \in [0, 1] \) and any \( k \) we have

\[
2\bar{x}_k \in [1-a, 1] \quad \text{if and only if} \quad x_k \in [1/2 - a/2, 1/2 + a/2].
\]

Thus we have

\[
D_N^*(2\bar{x}_1, \ldots, 2\bar{x}_N) = \sup_{0 \leq a \leq 1} \left| \frac{1}{N} \sum_{k=1}^N 1_{[1-a,1]}(2\bar{x}_k) - a \right|
\]
By Lemma 1 we have, for the sequence \((a_j)_{j=1}^{\infty}\):

\[
\frac{1}{N} \sum_{k=1}^{N} \mathbb{1}_{[1/2-a/2,1/2+a/2]}(x_k) - a \leq D_N(x_1, \ldots, x_N).
\]

Together with (20) and (21) this proves the lemma. \(\square\)

3. Proofs

Proof of Theorem 1: Let \([0, a] \subset [0, 1]\) be given. Then the Fourier series of the centered indicator function \(I_{[0,a]}(x)\) is given by

\[
I_{[0,a]}(x) \sim \sum_{j=1}^{\infty} \frac{\sin 2\pi ja}{2\pi j} \cos 2\pi jx + \left(1 - \cos 2\pi ja\right) \sin 2\pi jx.
\]

By Lemma 1, we have, for the sequence \((n_k)_{k \geq 1}\) defined in Theorem 1, that for \(1 \leq j_2 \leq j_1\)

\[
\gamma_{j_1,j_2} = \begin{cases} 1 & \text{if } j_1 = 3j_2 \text{ and } j_2 = \nu, \\ 0 & \text{otherwise}, \end{cases}
\]

Note that \(S(j_1, j_2, \nu, N) = S(j_2, j_1, -\nu, N)\). Using this relation Lemma 1 can also be used to calculate \(\gamma_{j_1,j_2,\nu}\) in the case \(j_2 \geq j_1\), and we have

\[
\gamma_{j_1,j_2,\nu} = \gamma_{j_2,j_1,-\nu}.
\]

Thus according to formula (14), for the function \(\sigma^2_{[0,a]}(x)\) for \(I_{[0,a]}(x)\) we have

\[
\sigma^2_{[0,a]}(x) = a(1-a) + \sum_{j=1}^{\infty} \frac{1}{4} \left( (a_{3j}a_j + b_{3j}b_j) \cos 2\pi jx + (b_{3j}a_j - a_{3j}b_j) \sin 2\pi jx \right)
\]

\[
+ \sum_{j=1}^{\infty} \frac{1}{4} \left( (a_ja_{3j} + b_jb_{3j}) \cos 2\pi (-j)x + (b_ja_{3j} - a_jb_{3j}) \sin 2\pi (-j)x \right)
\]

(22)

\[
= a(1-a) + \sum_{j=1}^{\infty} \frac{1}{2} \left( (a_{3j}a_j + b_{3j}b_j) \cos 2\pi jx + (b_{3j}a_j - a_{3j}b_j) \sin 2\pi jx \right).
\]

For the Fourier coefficients \(a_j, b_j\) we have the relation

(23) \(a_j(1-a) = -a_j(a)\) and \(b_j(1-a) = b_j(a)\).

The convolution theorem of Fourier analysis states in its real form that for two functions \(g, h\) given by

\[
g(x) \sim \sum_{j=1}^{\infty} c_j \cos 2\pi jx + d_j \sin 2\pi jx, \quad h(x) \sim \sum_{j=1}^{\infty} \tilde{c}_j \cos 2\pi jx + \tilde{d}_j \sin 2\pi jx,
\]

the function

\[
\frac{1}{2} \sum_{j=1}^{\infty} \left( c_j \tilde{c}_j - d_j \tilde{d}_j \right) \cos 2\pi jx + \left( c_j \tilde{d}_j + \tilde{c}_j d_j \right) \sin 2\pi jx
\]
is the Fourier series of
\[ \int_0^1 f(t) g(x - t) \, dt. \]
Let \( g(x) = -\mathbf{1}_{[0,1-a]} \) and \( h(x) = \mathbf{1}_{[0,\{3a\}]} \). Then by (23) the Fourier coefficients \( c_j, d_j \) of \( g \) satisfy
\[ c_j = a_j, \quad d_j = -b_j, \quad j \geq 1, \]
and the Fourier coefficients \( \tilde{c}_j, \tilde{d}_j \) of \( h \) satisfy
\[ \tilde{c}_j = 3a_j, \quad \tilde{d}_j = 3b_j, \quad j \geq 1. \]
Using the convolution theorem for the functions \( g \) and \( h \) defined in this way, and comparing the Fourier coefficients in (24), (25) with those in equation (22), we get the expression
\[ \sigma^2_{[0,a]}(x) = a(1-a) - \frac{1}{3} \int_0^1 \mathbf{1}_{[0,1-a]}(t) \mathbf{1}_{[0,\{3a\}]}(x-t) \, dt \]
for the function \( \sigma^2_{[0,a]}(x) \). Using this formula for \( a \in [2/3,1] \) we see that for all \( x \) we have
\[ \sigma^2_{[0,a]}(x) \leq a(1-a) + \frac{(1-a)\{3a\}}{3} = a(1-a) + \frac{(1-a)(3a-2)}{3} = -2a^2 + \frac{8a}{3} - \frac{2}{3}. \]
It is easily seen that
\[ \max_{a \in [2/3,1]} \left( -2a^2 + \frac{8a}{3} - \frac{2}{3} \right) = \frac{2}{9}, \]
and thus
\[ (26) \quad \max_{a \in [2/3,1]} \sigma^2_{[0,a]}(x) \leq \frac{2}{9} \quad \text{for all } x \in [0,1]. \]
Furthermore, if \( a \in [0,1/3] \), then the length of \([0,1-a]\) plus the length of \([0,\{3a\}] = [0,3a]\) is \( 1 + 2a \); consequently, even if the second interval is translated (mod 1), there must be an overlap of length at least \( 2a \). Thus we have
\[ (27) \quad \sigma^2_{[0,a]}(x) \leq a(1-a) + \frac{(1-a)3a}{3} - \frac{2a}{3} \leq \frac{2}{9} \quad \text{for all } a \in [0,1/3] \text{ and } x \in [0,1], \]
where the last inequality again follows by standard methods. The only complicated case is when \( a \in [1/3,2/3] \). In this case we have
\[ (28) \quad \sigma^2_{[0,a]}(x) = a(1-a) + \frac{(1-a)(3a-1)}{3} - \frac{1}{3} \int_0^1 \mathbf{1}_{[0,1-a]}(t) \mathbf{1}_{[0,3a-1]}(\{x-t\}) \, dt. \]
If we assume that \( 0 \leq x \leq 1-a \), then we have the following values for the integral \( I_1 \) in (28):
Thus we can explicitly describe the values of \( \sigma_{[0,a]}(x) \) for all \( a \in [1/3, 2/3] \) and \( x \in [0,1] \) satisfying \( 0 \leq x \leq 1 - a \). Now we note that the function \( \sigma_{[0,a]}(x) \) satisfies the relation \( \sigma_{[0,a]}(x) = \sigma_{[0,1-a]}(1-x) \); this can be seen for example from (22) and (23). Consequently the values of \( \sigma_{[0,a]}(x) \) under the additional assumption \( 1 - a \leq x \leq 1 \) can be obtained by transforming the corresponding cases of \( 0 \leq x \leq 1 - a \). Overall, we have a full explicit description of \( \sigma_{[0,a]}(x) \) for all values \( a \in [1/3, 2/3] \) and \( x \in [0,1] \). Consequently, to calculate \( \max_{a \in [1/3, 2/3]} \sigma^2_{[0,a]}(x) \) it remains to solve a (simple, but quite laborious) maximization problem with several different cases. Using standard methods, it can be shown that for given \( x \in [0,1] \) the largest possible value of \( \sigma^2_{[0,a]}(x) \) (for some appropriate \( a \), depending on \( x \)) is

\[
\max_{a \in [1/3, 2/3]} \sigma^2_{[0,a]}(x) = \begin{cases} 
-\frac{3x^2-x+2}{24x+25} & \text{if } 0 \leq x \leq \frac{1}{6}, \\
\frac{2}{9} & \text{if } \frac{1}{6} \leq x \leq \frac{1}{3}, \\
\max_{a \in [1/3, 2/3]} \sigma^2_{[0,a]}(1-x) & \text{if } \frac{1}{3} \leq x \leq 1.
\end{cases}
\]

Together with (16), (26) and (27) this proves the theorem. \( \square \)

**Proof of Theorem 2**: Suppose that \((n_k)_{k \geq 1}\) satisfies the Diophantine condition \( \text{D} \). Then by (16) we have

\[(29) \quad \Lambda^*(x) = \max_{a \in [0,1]} \sigma_{[0,a]}(x) \geq \sigma_{[0,1/2]}(x) \]

for almost all \( x \), where \( \sigma_{[0,1/2]}(x) \) is defined according to (14) for the function \( f(x) = I_{[0,1/2]}(x) \).

The Fourier series of \( f \) is given by

\[
f(x) \sim \sum_{j=1}^{\infty} \frac{\sin \pi j x}{2 \pi j} \cos 2\pi j x + \left( \frac{1 - \cos \pi j x}{2 \pi j} \right) \sin 2\pi j x.
\]

Obviously \( a_j = 0 \) for all \( j \), while \( b_j = 1/(\pi j) \) for odd \( j \) and \( b_j = 0 \) for even \( j \). Thus for this function \( f \) the formula (14) can be reduced to

\[
\sigma^2_f(x) = \|f\|^2 + \sum_{\nu=\infty}^{\infty} \sum_{j_1,j_2=1}^{\infty} \frac{\gamma_{j_1,j_2,\nu}}{2} b_{j_1} b_{j_2} \cos 2\pi \nu x.
\]

Since all coefficients \( b_j \) are non-negative, clearly also all products \( b_{j_1} b_{j_2} \) must be non-negative, as are the numbers \( \gamma_{j_1,j_2,\nu} \). Thus we have

\[
\int_0^1 \sigma^2_f(x) \, dx = \|f\|^2 + \sum_{j_1,j_2=1}^{\infty} \frac{\gamma_{j_1,j_2,0}}{2} b_{j_1} b_{j_2} \geq \|f\|^2 = \frac{1}{4}.
\]
Together with (29) this yields

\[ \|A^*\| \geq \frac{1}{2}, \]

which proves the first part of the theorem.

To prove the second part, we will show that for any fixed \( x \in [0, 1] \) we have

\[ (30) \int_0^{1/2} \sigma_{[a,a+1/2]}^2(x) \, da = \frac{1}{8}. \]

This implies that for any \( x \) there exists an interval \([a, a+1/2]\) such that

\[ \sigma_{[a,a+1/2]}^2(x) \geq \frac{1}{4}, \]

which together with (16) proves the second statement of Theorem 2.

To prove (30), we note that for the Fourier coefficients of the function \( I_{[a,a+1/2]}(x) \) for some \( a \in [0, 1/2] \), given by

\[ I_{[0,a]}(x) \sim \sum_{j=1}^{\infty} a_j \cos 2\pi j x + b_j \sin 2\pi j x, \]

we have

\[ a_j = a_j(a) = \frac{\sin 2\pi j(a + 1/2) - \sin 2\pi ja}{2\pi j} = \begin{cases} -\frac{\sin 2\pi ja}{j\pi} & \text{if } j \text{ is odd}, \\ 0 & \text{if } j \text{ is even} \end{cases} \]

and

\[ b_j = b_j(a) = \frac{-\cos 2\pi j(a + 1/2) + \cos 2\pi ja}{2\pi j} = \begin{cases} \frac{\cos 2\pi ja}{j\pi} & \text{if } j \text{ is odd}, \\ 0 & \text{if } j \text{ is even}. \end{cases} \]

Note that the growth condition (1) implies that \( \gamma_{j_1,j_2,\nu} = 0 \) whenever \( j_1 = j_2 \). Thus according to (14) we have

\[ (31) \sigma_{[a,a+1/2]}(x) = \frac{1}{4} + \sum_{\nu=-\infty}^{\infty} \sum_{j_1 \neq j_2, \text{ and } j_1 \text{ and } j_2 \text{ odd}} \frac{\gamma_{j_1,j_2,\nu}}{2} \left( (a_{j_1}a_{j_2} + b_{j_1}b_{j_2}) \cos 2\pi \nu x + (b_{j_1}a_{j_2} - a_{j_1}b_{j_2}) \sin 2\pi \nu x \right) \]

We have

\[ \int_0^{1/2} \sum_{\nu=-\infty}^{\infty} \sum_{j_1 \neq j_2, \text{ and } j_1 \text{ and } j_2 \text{ odd}} \frac{\gamma_{j_1,j_2,\nu}}{2} a_{j_1}a_{j_2} \cos 2\pi \nu x \, da \]

\[ = \int_0^{1/2} \sum_{\nu=-\infty}^{\infty} \sum_{j_1 \neq j_2, \text{ and } j_1 \text{ and } j_2 \text{ odd}} \frac{\gamma_{j_1,j_2,\nu}}{2} \frac{\sin 2\pi j_1 a \sin 2\pi j_2 a}{j_1\pi j_2\pi} \cos 2\pi \nu x \, da \]
\[
= \int_0^{1/2} \sum_{\nu=\infty}^{\infty} \sum_{j_1 \neq j_2, j_1 \text{ and } j_2 \text{ odd}} \frac{\gamma_{j_1,j_2,\nu}}{2} \frac{\cos(2\pi(j_1 - j_2)a) - \cos(2\pi(j_1 + j_2)a)}{j_1j_2\pi^2} \cos 2\pi\nu x \, da
\]
\[
= \sum_{\nu=\infty}^{\infty} \sum_{j_1 \neq j_2, j_1 \text{ and } j_2 \text{ odd}} \int_0^{1/2} \frac{\gamma_{j_1,j_2,\nu}}{2} \frac{\cos(2\pi(j_1 - j_2)a) - \cos(2\pi(j_1 + j_2)a)}{j_1j_2\pi^2} \cos 2\pi\nu x \, da
\]
\[
= 0.
\]

Here we used the fact that when \(j_1\) and \(j_2\) are both odd and \(j_1 \neq j_2\), then both \(j_1 + j_2\) and \(j_1 - j_2\) are even and nonzero, and consequently all the integrals \(\int_0^{1/2} \cos(2\pi(j_1 - j_2)a) \, da\) and \(\int_0^{1/2} \cos(2\pi(j_1 + j_2)a) \, da\) are zero; furthermore, we used Lemma 2 and the dominated convergence theorem to exchange the order of summation and integration. The other parts of the sum in (31) can be treated in a similar way, and it turns out that their integrals are also equal to zero. Overall we get

\[
\int_0^{1/2} \sigma_{[a,a+1/2]}(x) \, da = \int_0^{1/2} \frac{1}{4} \, da = \frac{1}{8},
\]

which is (30). This proves the theorem.

□

Proof of Theorem 5: Theorem 5 is a simple consequence of (2) and Lemma 3.

Proof of Theorem 4: To prove the theorem, it is sufficient to show that for any distinct positive integers \(m, n\) we have

\[
\int_0^{1} \int_0^{1} I_{[a,a+z]}(mx)I_{[a,a+z]}(nx) \, dx \, da = 0.
\]

For this purpose, we set

\[
g(a) = \int_0^{1} I_{[a,a+z]}(mx)I_{[a,a+z]}(nx) \, dx.
\]

It is easily seen that the function \(g\) has period 1, that is \(g(a + 1) = g(a)\). By noting that \(I_{[a,a+z]}(y) = I_{[0,z]}(y - a)\) and by changing the variable \(x\) to \(x + a/m\) we have

\[
g(a) = \int_0^{1} I_{[0,z]}(mx - a)I_{[0,z]}(nx - a) \, dx
\]
\[
= \int_0^{1} I_{[0,z]}(mx)I_{[0,z]}(nx + (n/m - 1)a) \, dx
\]
(since the integrand has period 1, the integration interval does not change). Since \(g\) has period 1, we have

\[
\int_0^{1} g(a) \, da = \frac{1}{m} \int_0^{m} g(a) \, da
\]
\[
= \frac{1}{m} \int_0^{1} I_{[0,z]}(mx) \int_0^{m} I_{[0,z]}(nx + (n/m - 1)a) \, da \, dx
\]
This proves the theorem. \(\square\)

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