On the Chow groups of Plücker hypersurfaces in Grassmannians

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Abstract. Motivated by the generalized Bloch conjecture, we formulate a conjecture about the Chow groups of Plücker hypersurfaces in Grassmannians. We prove weak versions of this conjecture.

Mathematics Subject Classification. Primary 14C15, 14C25, 14C30.

Keywords. Algebraic cycles, Chow groups, Motive, Bloch–Beilinson conjectures.

1. Introduction. Given a smooth projective variety $Y$ over $\mathbb{C}$, let $A_i(Y) := CH_i(Y)_{\mathbb{Q}}$ denote the Chow groups of $Y$ (i.e. the groups of $i$-dimensional algebraic cycles on $Y$ with $\mathbb{Q}$-coefficients, modulo rational equivalence). Let $A_i^{hom}(Y) \subset A_i(Y)$ denote the subgroup of homologically trivial cycles.

The “generalized Bloch conjecture” [26, Conjecture 1.10] predicts that the Hodge level of the cohomology of $Y$ should have an influence on the size of the Chow groups of $Y$. In case $Y$ is a surface, this is the notorious Bloch conjecture, which is still an open problem. In the case of hypersurfaces in projective space, the precise prediction is as follows:

Conjecture 1.1. Let $Y \subset \mathbb{P}^n$ be a smooth hypersurface of degree $d$. Then

$$A_i^{hom}(Y) = 0 \quad \forall \ i \leq \frac{n}{d} - 1.$$

Conjecture 1.1 is still open; partial results have been obtained in [8, 9, 19, 21, 23].

In the case of Plücker hypersurfaces in Grassmannians, we hazard the following prediction (cf. Subsection 2.1 below for motivation):

Conjecture 1.2. Let $\text{Gr}(k, n)$ denote the Grassmannian of $k$-dimensional subspaces of an $n$-dimensional vector space, and let

$$Y = \text{Gr}(k, n) \cap H \subset \mathbb{P}^{(n)}_{(i)} - 1$$
be a smooth hyperplane section (with respect to the Plücker embedding). Then
\[ A_i^{\text{hom}}(Y) = 0 \quad \forall i \leq n - 2. \]

The main result of this note is that a weak version of Conjecture 1.2 is true:

**Theorem** (=Theorem 3.1). Let
\[ Y = \text{Gr}(k, n) \cap H \subset \mathbb{P}^{k-1} \]
be a smooth hyperplane section (with respect to the Plücker embedding). Then
\[ A_i^{\text{hom}}(Y) = 0 \quad \forall i \leq n - k. \]

The argument proving Theorem 3.1 is very easy and straightforward; it combines the recent construction of jumps among subvarieties of Grassmannians [2] and a motivic version of the Cayley trick [11].

In some cases, we can do better and the conjecture is completely satisfied:

**Theorem** (=Theorem 3.2). Let
\[ Y = \text{Gr}(3, n) \cap H \subset \mathbb{P}^{3-1} \]
be a smooth hyperplane section (with respect to the Plücker embedding). Assume \( n \leq 13, n \neq 12 \). Then
\[ A_i^{\text{hom}}(Y) = 0 \quad \forall i \leq n - 2. \]

The \( n = 10 \) case of Theorem 3.2 was already proven by Voisin as an application of her technique of spread of algebraic cycles [25, Theorem 2.4]. Hyperplane sections \( Y \subset \text{Gr}(3, 10) \) are also known as Debarre–Voisin hypersurfaces because they give rise to the Debarre–Voisin hyperkähler fourfolds [7]. The new proof of [25, Theorem 2.4] provided by Theorem 3.2 does not rely on Voisin’s spread technique, nor on the relation with hyperkähler fourfolds.

As a consequence of Theorem 3.2, some instances of the generalized Hodge conjecture are verified:

**Corollary** (=Corollary 4.1). Let \( Y \) be as in Theorem 3.2. Then \( H^{\dim Y}(Y, \mathbb{Q}) \) is supported on a subvariety of codimension \( n - 1 \).

As another consequence, we find some new examples of varieties with finite-dimensional motive:

**Corollary** (=Corollary 4.2). Let
\[ Y = \text{Gr}(3, 9) \cap H \subset \mathbb{P}^{83} \]
be a smooth hyperplane section (with respect to the Plücker embedding). Then \( Y \) has finite-dimensional motive (in the sense of [13]).

Varieties \( Y \) as in Corollary 4.2 are studied in [2, Section 5.1], where they are related to Coble cubics and abelian surfaces.
Conventions. In this note, the word variety will refer to a reduced irreducible scheme of finite type over \( \mathbb{C} \). A subvariety is a (possibly reducible) reduced subscheme which is equidimensional.

All Chow groups will be with rational coefficients: we denote by \( A_j(Y) := CH_j(Y)_{\mathbb{Q}} \) the Chow group of \( j \)-dimensional cycles on \( Y \) with \( \mathbb{Q} \)-coefficients; for \( Y \) smooth of dimension \( n \), the notations \( A_j(Y) \) and \( A^{n-j}(Y) \) are used interchangeably. The notations \( A^j_{\text{hom}}(Y) \) and \( A^j_{\text{AJ}}(X) \) will be used to indicate the subgroup of homologically trivial (resp. Abel–Jacobi trivial) cycles.

For a vector bundle \( E \), we write \( \mathbb{P}(E) := \text{Proj}(\oplus_{m>0} \text{Sym}^m E) \).

2. Preliminaries.

2.1. Motivating the conjecture.

Theorem 2.1 (Bernardara–Fatighenti–Manivel [2], Kuznetsov [16]). Let \( Y = \text{Gr}(k,n) \cap H \subset \mathbb{P}^{(n^2-1)} \) be a smooth hyperplane section (with respect to the Plücker embedding). Assume either \( n > 3k > 6 \), or \( n \) and \( k \) are coprime. Then \( Y \) has Hodge coniveau \( n - 1 \). More precisely, the Hodge numbers satisfy

\[
h^{p,\dim Y-p}(Y) = \begin{cases} 1 & \text{if } p = n - 1, \\ 0 & \text{if } p < n - 1. \end{cases}
\]

Proof. The case \( n > 3k > 6 \) is [2, Theorem 3]. In case \( n \) and \( k \) are coprime, Kuznetsov [16, Corollary 4.4] has constructed an exceptional collection for the derived category of \( Y \) whose right orthogonal is a Calabi–Yau category of dimension \( k(n-k)+1-2n \). Taking Hochschild homology, one obtains the assertion about the Hodge numbers. \( \square \)

As mentioned in [2], the assumptions on \( n \) and \( k \) are probably not optimal. (And in view of the examples given in loc. cit., it seems likely that for any \( n, k \), the Hodge coniveau of \( Y \) is \( \geq n - 1 \), while one needs some condition on \( n, k \) to get an equality.)

The generalized Bloch conjecture [26, Conjecture 1.10] predicts that any variety \( Y \) with Hodge coniveau \( \geq c \) has

\[
A^i_{\text{hom}}(Y) = 0 \quad \forall \ i < c.
\]

This motivates Conjecture 1.2. Note that at least for \( n > 3k > 6 \) (or \( n \) and \( k \) coprime), the bound of Conjecture 1.2 is optimal: assuming \( A^i_{\text{hom}}(Y) = 0 \) for \( j \leq n - 1 \) and applying the Bloch–Srinivas argument [5], one would get the vanishing \( h^{n-1,k(n-k)-n}(Y) = 0 \), contradicting Theorem 2.1.

2.2. Jumps.

Proposition 2.2 (Bernardara–Fatighenti–Manivel [2]). Let \( Y = \text{Gr}(k,n) \cap H \subset \mathbb{P}^{(n^2-1)} \) be a general hyperplane section (with respect to the Plücker embedding).
There is a Cartesian diagram

\[
\begin{array}{cccc}
F & \rightarrow & q^*Y & \rightarrow & Y \\
\downarrow & & \downarrow p & & \\
T & \hookrightarrow & \text{Gr}(k-1,n).
\end{array}
\]

Here the morphism \( q \) is a \( \mathbb{P}^{k-1} \)-bundle, and the morphism \( p \) is a \( \mathbb{P}^{n-k-1} \)-bundle over \( \text{Gr}(k-1,n) \backslash T \) and a \( \mathbb{P}^{n-k} \)-bundle over \( T \). The subvariety \( T \subset \text{Gr}(k-1,n) \) is smooth of codimension \( n-k+1 \), given by a section of \( Q^*(1) \) (where \( Q \) denotes the universal quotient bundle on \( \text{Gr}(k-1,n) \)).

Proof. This is a special case of the construction of a jump in [2, Section 3.3]. The idea is to consider the flag variety \( F \ell(k-1,k,n) \) as a correspondence

\[
\begin{array}{ccc}
F \ell(k-1,k,n) & \rightarrow & \text{Gr}(k,n) \\
\downarrow & & \downarrow p \\
\text{Gr}(k-1,n)
\end{array}
\]

and look at what happens over \( Y \). The hyperplane \( Y \subset \text{Gr}(k,n) \) corresponds to a \( k \)-form \( \Omega \) on an \( n \)-dimensional vector space. The variety \( q^*Y \subset F \ell(k-1,k,n) \) is defined by \( q^*\Omega \); this is the inverse image of \( Y \) under \( q \). The flag variety \( F \ell(k-1,k,n) \) can be identified with the bundle of hyperplanes \( \mathbb{P}(Q^*(1)) \) on \( \text{Gr}(k-1,n) \), and the locus \( T \subset \text{Gr}(k-1,n) \) where the fiber dimension of \( p: q^*Y \rightarrow \text{Gr}(k-1,n) \) jumps is the zero locus of the section of \( Q^*(1) \) defined by \( \Omega \). For general \( Y \), the locus \( T \) will be smooth of codimension equal to \( \text{rank } Q^*(1) = n-k+1 \). (For the smoothness of \( T \), we note that \( Q^*(1) \) is globally generated and so the universal family \( T \) of zero loci of sections of \( Q^*(1) \) is a projective bundle over \( Gr(k,n) \) hence \( T \) is smooth; the smoothness of general \( T \) then follows from generic smoothness applied to \( T \rightarrow \mathbb{P}H^0(\text{Gr}(k,n),Q^*(1)) \).)

\[ \square \]

2.3. Cayley’s trick and Chow groups.

Theorem 2.3 (Jiang [11]). Let \( E \rightarrow X \) be a vector bundle of rank \( r \geq 2 \) over a smooth projective variety \( X \), and let \( T := s^{-1}(0) \subset X \) be the zero locus of a section \( s \in H^0(X,E) \) such that \( T \) is smooth of dimension \( \dim X - \text{rank } E \). Let \( Y := w^{-1}(0) \subset \mathbb{P}(E) \) be the zero locus of the section \( w \in H^0(\mathbb{P}(E),\mathcal{O}_{\mathbb{P}(E)}(1)) \) that corresponds to \( s \) under the natural isomorphism \( H^0(X,E) \cong H^0(\mathbb{P}(E),\mathcal{O}_{\mathbb{P}(E)}(1)) \). There are (correspondence-induced) isomorphisms of Chow groups

\[
A_i(Y) \cong A_{i+1-r}(T) \oplus \bigoplus_{j=0}^{r-2} A_{i-j}(X) \quad \forall \ i.
\]
In particular, there are isomorphisms

$$A^\text{hom}_i(Y) \cong A^\text{hom}_{i+1-r}(T) \oplus \bigoplus_{j=0}^{r-2} A^\text{hom}_{i-j}(X) \quad \forall \ i.$$ 

**Proof.** The first statement is a special case of [11, Theorem 3.1] (the statement is actually true with integer coefficients). Both the isomorphism and its inverse are explicitly described. The crucial point is that the projection $Y \to X$ is a $\mathbb{P}^{r-2}$-fibration over $X \setminus T$, and a $\mathbb{P}^{r-1}$-fibration over $T$.

As for the second statement, one observes that the first statement also holds on the level of Chow motives (this is [11, Corollary 3.8]). This implies that there is a commutative diagram (where vertical arrows are cycle class maps)

$$A_i(Y) \cong A_{i+1-r}(T) \oplus \bigoplus_{j=0}^{r-2} A_{i-j}(X)$$

$$\downarrow \quad \downarrow$$

$$H_{2i}(Y, \mathbb{Q}) \cong H_{2i+2-2r}(T, \mathbb{Q}) \oplus \bigoplus_{j=0}^{r-2} H_{2i-2j}(X, \mathbb{Q}).$$

This proves the second statement. \qed

**Remark 2.4.** In the set-up of Theorem 2.3, a cohomological relation between $Y$, $X$, and $T$ was established in [15, Prop. 4.3] (cf. also [10, Section 3.7], as well as [2, Proposition 46] for a generalization). A relation on the level of derived categories was established in [20, Theorem 2.10] (cf. also [12, Theorem 2.4] and [2, Proposition 47]).

### 2.4. A variant of the Cayley trick.

**Proposition 2.5.** Let

$$Y_T \hookrightarrow Y$$

$$\downarrow \quad \downarrow p$$

$$T \hookrightarrow X$$

be a Cartesian diagram of projective varieties, with $T \subset X$ of codimension $c$. Assume that $p$ is a proper morphism which is a $\mathbb{P}^n$-bundle over $X \setminus T$, and a $\mathbb{P}^m$-bundle over $T$. Assume also that there exists $h \in A^1(Y)$ such that $h|_{Y \setminus Y_T}$ is relatively ample for the $\mathbb{P}^n$-bundle and $h|_{Y_T}$ is relatively ample for the $\mathbb{P}^m$-bundle. Then there is an exact sequence

$$\bigoplus_{j=n+1}^m A_{i-j}(T) \to A_i(Y) \to \bigoplus_{j=0}^n A_{i-j}(X) \to 0.$$
Proof. We use Bloch’s higher Chow groups $A_i(-, j)$ [3, 4]. There is a commuta-
tive diagram with long exact rows

$$A_i(Y \setminus Y_T, 1) \to A_i(Y_T) \to A_i(Y) \to A_i(Y \setminus Y_T) \to 0$$

\[ \downarrow \cong \downarrow \Phi_T \downarrow \Phi \downarrow \cong \]

\[ \bigoplus_{j=0}^n A_{i-j}(X \setminus T, 1) \to \bigoplus_{j=0}^n A_{i-j}(T) \to \bigoplus_{j=0}^n A_{i-j}(X) \to \bigoplus_{j=0}^n A_{i-j}(X \setminus T) \to 0. \]

The vertical arrows in this diagram are defined as follows: the map $\Phi$ is $\sum_{j=0}^n p_* (h^j \cap -)$, and the map $\Phi_T$ is $\sum_{j=0}^n p_* ((h|Y_T)^j \cap -)$. Similarly, the left and right vertical maps are defined by restricting $h$ to $Y \setminus Y_T$. Commutativity of the diagram is proven as in [18, Diagram (10)] by looking at the level of the underlying complexes. The left and right vertical arrows are isomorphisms because of the projective bundle formula for higher Chow groups [3].

The projective bundle formula says that $\Phi_T$ is surjective with kernel $\ker \Phi_T \cong \bigoplus_{j=n+1}^m A_{i-j}(T)$.

A diagram chase now yields the desired exact sequence. \hfill \Box

Remark 2.6. The case $(m, n) = (r - 2, r - 1)$ of Proposition 2.5 gives back a weak version of Jiang’s result (Theorem 2.3). Versions of Proposition 2.5 on the level of cohomology and on the level of derived categories are given in [2, Appendix A], resp. [2, Appendix B].

At least when all varieties are smooth, it seems likely that a stronger version of Proposition 2.5 is true: we guess that in this case, there is an isomorphism of Chow groups

$$A_i(Y) \cong \bigoplus_{j=n+1}^m A_{i-j}(T) \oplus \bigoplus_{j=0}^n A_{i-j}(X).$$

Since this is not needed below, we have not pursued this guess.

2.5. Spreading out.

Proposition 2.7. Let $\mathcal{Y} \to B$ be a family of smooth projective varieties. Assume there is some $c \in \mathbb{N}$ that

$$A_i^{\text{hom}}(Y_b) = 0 \quad \forall i \leq c \quad (1)$$

for the very general fiber $Y_b$. Then

$$A_i^{\text{hom}}(Y_b) = 0 \quad \forall i \leq c$$

for every fiber $Y_b$.

Proof. Let $B^\circ \subset B$ denote the intersection of countably many Zariski open subsets such that the vanishing (1) holds for all $b \in B^\circ$. 

Doing the Bloch–Srinivas argument [5] (cf. also [17]), this implies that for each \( b \in B^\circ \), one has a decomposition of the diagonal
\[
\Delta_{Y_b} = \gamma_b + \delta_b \quad \text{in} \quad A^\dim_{Y_b}(Y_b \times Y_b)
\]
where \( \gamma_b \) is completely decomposed (i.e. \( \gamma_b \in A^*(Y_b) \otimes A^*(Y_b) \)) and \( \delta_b \) is supported on \( Y_b \times W_b \) with \( \operatorname{codim} W_b = c + 1 \).

Using Hilbert schemes as in the proofs of [27, Theorem 2.1(i)], [24, Proposition 3.7], the fiberwise data \((\gamma_b, \delta_b, W_b)\)
can be encoded by a countably infinite number of varieties, each carrying a universal object. By a Baire category argument, one of these varieties must dominate \( B \). Taking a linear section, this means that after a generically finite base change, the \( \gamma_b, \delta_b, W_b \) exist relatively. That is, there exist a generically finite morphism \( B' \to B \), a cycle \( \gamma \in (\gamma')^* A^*(\gamma') \cdot (p_2)^* A^*(\gamma') \), a subvariety \( \mathcal{W} \subset \gamma' \) of codimension \( c + 1 \), and a cycle \( \delta \) supported on \( \gamma' \times B' \mathcal{W} \) such that
\[
\Delta_{\gamma'}|_b = \gamma|_b + \delta|_b \quad \text{in} \quad A^\dim_{Y_b}(Y_b \times Y_b) \quad \forall \ b \in B^\circ.
\]
(Here \( \gamma' := \gamma \times_B B' \).)

Let \( \bar{\gamma}, \bar{\delta} \in A^\dim_{Y_b}(\gamma' \times B' \mathcal{W}) \) be cycles that restrict to \( \gamma \) resp. \( \delta \). The spread lemma [27, Proposition 2.4], [26, Lemma 3.2] then implies that
\[
\Delta_{\gamma'}|_b = \bar{\gamma}|_b + \bar{\delta}|_b \quad \text{in} \quad A^\dim_{Y_b}(Y_b \times Y_b) \quad \forall \ b \in B.
\]
Given any \( b_1 \in B \setminus B^\circ \), one can find representatives for \( \bar{\gamma} \) and \( \bar{\delta} \) in general position with respect to the fiber \( Y_{b_1} \times Y_{b_1} \). Restricting to the fiber, this implies that the diagonal of \( Y_{b_1} \) has a decomposition as in (2), and so (2) holds for all \( b \in B \). Letting the decomposition (2) act on Chow groups, this shows that
\[
A^i\hom(Y_b) = 0 \quad \forall \ i \leq c, \quad \forall \ b \in B.
\]
\( \square \)

3. Main results.

**Theorem 3.1.** Let
\[
Y = \Gr(k, n) \cap H \subset \mathbb{P}^{\binom{n}{k} - 1
\]
be a smooth hyperplane section (with respect to the Plücker embedding). Then
\[
A^i\hom(Y) = 0 \quad \forall i \leq n - k.
\]

**Proof.** In view of Proposition 2.7, it suffices to prove this for generic hyperplane sections, and so we may assume that \( Y \) is as in Proposition 2.2. The *jump* of Proposition 2.2 gives rise to a commutative diagram
\[
\begin{array}{ccc}
F & \rightarrow & q^*Y \\
\downarrow & & \downarrow p \\
T & \rightarrow & \Gr(k - 1, n).
\end{array}
\]
The morphism $q^*Y \to Y$ is a $\mathbb{P}^{k-1}$-bundle, and so the projective bundle formula implies that there are injections
\[ A_i^\hom(Y) \hookrightarrow A_i^\hom(q^*Y) \quad \forall i. \tag{3} \]
For $Y$ sufficiently general, the locus $T$ will be smooth of codimension $n-k+1$ (Proposition 2.2). The set-up is thus that of Cayley’s trick, with $X = \text{Gr}(k-1, n)$ and $E = Q^*(1)$. Applying Theorem 2.3 (with $r = \text{rank } Q^*(1) = n-k+1$), we find that there are isomorphisms
\[ A_i^\hom(q^*Y) \cong A_i^{\hom}(T). \tag{4} \]
But $T$ is a smooth Fano variety (indeed, using adjunction, one can compute that the canonical bundle of $T$ is $\mathcal{O}_T(1-k)$), hence $T$ is rationally connected \cite{6, 14} and so in particular $A_0(T) \cong \mathbb{Q}$. It follows that
\[ A_i^{\hom}(T) = 0 \quad \forall i \leq n-k. \tag{5} \]
Combining (3), (4), and (5), the theorem is proven. \hfill \qed

**Theorem 3.2.** Let
\[ Y = \text{Gr}(3, n) \cap H \subset \mathbb{P}(3)^{-1} \]
be a smooth hyperplane section (with respect to the Plücker embedding). Assume $n \leq 13$, $n \neq 12$. Then
\[ A_i^{\hom}(Y) = 0 \quad \forall i \leq n-2. \]

*Proof.* In view of Theorem 3.1, it only remains to treat the case $i = n-2$.

Applying Proposition 2.2, we may assume that $Y$ is sufficiently generic. Doing the jump as in the proof of Theorem 3.1 above, one finds a smooth variety $T \subset \text{Gr}(2, n)$ (of codimension $n-2$) and an injection of Chow groups
\[ A_{n-2}^{\hom}(Y) \hookrightarrow A_1^{\hom}(T). \tag{6} \]

In order to understand $A_1^{\hom}(T)$, we perform a second jump. This second jump (cf. \cite[Section 3.4]{2}) induces a diagram
\[
\begin{array}{ccc}
F & \hookrightarrow & q^*T \\
\downarrow p|_F & & \downarrow p \\
\dashrightarrow & & \dashrightarrow \\
P' & \hookrightarrow & P,
\end{array}
\]
Here $q^*T \subset \text{Fl}(1, 2, n)$, and the morphism $q^*T \to T$ is a $\mathbb{P}^1$-bundle. The projective bundle formula gives an injection
\[ A_1^{\hom}(T) \hookrightarrow A_1^{\hom}(q^*T). \tag{7} \]
The varieties $P$ and $P'$ depend on the parity of $n$:

- If $n$ is even, $P = \mathbb{P}^{n-1}$ and $P'$ is the $(n-4)$-dimensional Pfaffian variety called $P(1, n)$ in \cite{2}. For $n \leq 10$, the generic $P(1, n)$ is smooth and in this case, $p$ is the blow-up of $P = \mathbb{P}^{n-1}$ with center $P' = P(1, n)$. 

\[ \text{R. Laterveer} \quad \text{Arch. Math.} \]
• If $n$ is odd, $P$ is the Pfaffian hypersurface $P(1, n) \subset \mathbb{P}^{n-1}$ (in the notation of [2]). For $n \leq 15$, the generic $P(1, n)$ has singular locus $P' \subset P(1, n)$ of codimension 5 and $P'$ is smooth. In this case, the morphism $p$ is a $\mathbb{P}^1$-bundle over $P \setminus P'$, and a $\mathbb{P}^3$-bundle over $P'$.

Let us first treat the case $n$ even, $n \leq 10$. The blow-up formula gives an isomorphism

$$A_{\text{hom}}^1(q^*T) = A_{\text{hom}}^0(P').$$

Since $P' = P(1, n)$ is a smooth Fano variety, we have

$$A_{\text{hom}}^0(P') = 0,$$

and so (combining with (6) and (7)) the theorem follows for $n$ even and generic $Y$.

Next, let us treat the case $n$ odd, $n \leq 13$. In this case, we apply Proposition 2.5 (with $h \in A^1(q^*T)$ the restriction of the relatively ample class for $\text{Fl}(1, 2, n) \rightarrow \mathbb{P}^{n-1}$). This gives a (correspondence-induced) isomorphism

$$A_1(q^*T) = A_0(P) \oplus A_1(P),$$

and in particular an injection

$$A_{\text{hom}}^1(q^*T) \hookrightarrow A_{\text{hom}}^0(P) \oplus A_{\text{hom}}^1(P).$$

But $P = P(1, n) \subset \mathbb{P}^{n-1}$ is a (singular) hypersurface of degree $(n - 3)/2$. For $n \leq 13$, it is known that any hypersurface $P \subset \mathbb{P}^{n-1}$ of degree $\leq (n - 3)/2$ has

$$A_{\text{hom}}^0(P) = A_{\text{hom}}^1(P) = 0.$$

(For smooth $P$, this is proven in [21], the extension to singular $P$ is done in [9]. Note that, at least for smooth $P$, Conjecture 1.1 states that the restriction to $n \leq 13$ is not necessary.) Combined with (6) and (7), the theorem follows for $n$ odd and $Y$ generic.

4. Some consequences.

**Corollary 4.1.** Let $Y$ be as in Theorem 3.2. Then $H^\text{dim} Y(Y, \mathbb{Q})$ is supported on a subvariety of codimension $n - 1$.

**Proof.** This follows in standard fashion from the Bloch–Srinivas argument. The vanishing of Theorem 3.2 is equivalent to the decomposition

$$\Delta_Y = \gamma + \delta \quad \text{in } A^\text{dim} Y(Y \times Y),$$

where $\gamma$ is a completely decomposed cycle (i.e. $\gamma \in A^*(Y) \otimes A^*(Y)$), and $\delta$ has support on $Y \times W$ with $W \subset Y$ of codimension $n - 1$. Let $H^{\text{dim}}_{\text{tr}} Y(Y, \mathbb{Q})$ denote the transcendental cohomology (i.e. the complement of the algebraic part under the cup product pairing). The cycle $\gamma$ does not act on $H^{\text{dim}}_{\text{tr}} Y(Y, \mathbb{Q})$. The action of $\delta$ on $H^{\text{dim}}_{\text{tr}} Y(Y, \mathbb{Q})$ factors over $W$, and so

$$H^{\text{dim}}_{\text{tr}} Y(Y, \mathbb{Q}) \subset H^{\text{dim}}_{\text{tr}} Y(Y, \mathbb{Q}).$$

Since the algebraic part of $H^{\text{dim}} Y(Y, \mathbb{Q})$ is (by definition) supported in codimension $\text{dim} Y/2$, this settles the corollary. □
Corollary 4.2. Let
\[ Y = \text{Gr}(3, 9) \cap H \subset \mathbb{P}^{83} \]
be a smooth hyperplane section (with respect to the Plücker embedding). Then
\[ A^*_A(Y) = 0. \]
In particular, \( Y \) has finite-dimensional motive (in the sense of \[1, 13\]).

Proof. Theorem 3.2 implies that
\[ A^i_{\text{hom}}(Y) = 0 \quad \forall \ i \leq 7. \]
The Bloch–Srinivas argument \[5, 17\] then implies that
\[ A^j_{A,J}(Y) = 0 \quad \forall \ j \leq 9. \]
Since \( Y \) is 17-dimensional, these two facts taken together mean that
\[ A^*_A(Y) = 0, \]
as claimed. The fact that any variety \( Y \) with \( A^*_A(Y) = 0 \) is Kimura finite-dimensional is \[22, \text{Theorem } 4\]. \qed

Acknowledgements. Thanks to Kai and Len for making me listen to Sébastien Patoche. Thanks to the referee for many constructive comments that significantly improved the paper.

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Received: 24 June 2020

Revised: 25 September 2020

Accepted: 23 October 2020.