Abstract. We prove a gluing theorem which allows to construct an ample divisor on a rational surface from two given ample divisors on simpler surfaces. This theorem combined with the Cremona action on the ample cone gives rise to an algorithm for constructing new ample divisors. We then propose a conjecture relating continued fractions approximations and Seshadri-like constants of line bundles over rational surfaces. By applying our algorithm recursively we verify our conjecture in many cases and obtain new asymptotic estimates on these constants. Finally, we explain the intuition behind the gluing theorem in terms of symplectic geometry and propose generalizations.

CONSTRUCTING NEW AMPLE DIVISORS OUT OF OLD ONES

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1. Introduction

The main objective of this paper is to propose a method for constructing new ample divisors on rational surfaces by gluing two given ones.

Recall that a divisor $D$ on an algebraic variety $X$ is ample if the corresponding line bundle $O_X(D)$ is ample, and is called nef (numerically effective) if there exists an ample divisor $A$ such that $A + kD$ is ample for every $k > 0$. We refer the reader to [Dem, Ha 1] for excellent expositions on various aspects of the theory of ample and nef line bundles.

Of fundamental importance is the determination of those classes in $\text{Pic}(X)$ which are ample. Although this problem has a very simple solution for smooth curves, already in dimension two the problem becomes much harder. It turns out the even for relatively simple surfaces, such as rational, the complete answer is not known. Several conjectures in this direction exist, however at the present time only estimates on the ample cone -- the cone generated by the ample classes in $\text{Pic}(X)$ -- are known. For example, let $d, m > 0$ and consider the divisor class

$$D = \pi^* O_{\mathbb{C}P^2}(d) - m \sum_{j=1}^N E_j$$

on the blow-up $\pi : V_N \to \mathbb{C}P^2$ of $\mathbb{C}P^2$ at $N \geq 9$ generic points. Nagata conjectured in [Nag] that $D$ is ample iff $D \cdot D > 0$, but was able to prove it only for $N$’s which are squares. In [Xu 1] Xu proved that $D$ is ample provided that $\frac{m}{d} < \frac{\sqrt{N} - 1}{N}$. By making a more detailed analysis of the case $m = 1$, Xu proved in [Xu 2] that when $d \geq 3$ the divisor class $D = \pi^* O_{\mathbb{C}P^2}(d) - \sum_{j=1}^N E_j$ is ample iff $D \cdot D > 0$ (see also Küchle [Ku] for a generalization for arbitrary surfaces and [Ang] for an analogous result for $\mathbb{C}P^3$).

Closely related is the problem of computing Seshadri constants of ample line bundles, which measure their local positivity. The Seshadri constant $E(L, p)$ of the line bundle $L$ at the point $p \in X$ is defined to be the supremum of all those $\epsilon \geq 0$ for which the $\mathbb{R}$-divisor class $\pi^* L - \epsilon E$ is nef on the blow-up $\pi : \tilde{X}_p \to X$ of $X$ at the point $p$ with exceptional divisor $E$. 

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Preliminary Version.
Seshadri constant has been studied much by Demailly ([Dem]), Ein, K"uchle, Lazarsfeld ([E-L],[E-K-L],[Laz]), and Xu ([Xu 3]). A considerable part of these works is devoted to computations and estimates from below on the values of these constants.

The present paper is largely motivated by the problem of computing Seshadri constants and the determination of the ample cone of rational surfaces. Our main results provide an algorithmic method for constructing new ample divisors out of the knowledge of ample divisors on simpler rational surfaces. By applying the algorithm recursively we obtain in Section 4 new estimates on Seshadri-like constants and detect new ample divisors. We then propose in Section 5 a conjecture naturally arising from our method which relates continued fractions expansions of \( \sqrt{N} \) with the ample cone of \( \mathbb{CP}^2 \) blown-up at \( N \) points. Finally we interpret in Section 7 our main results in the language of Symplectic Geometry and explain the intuition behind them.

Our main tool is Shustin’s version of the Viro method for gluing curves with singularities.

2. Main results

Our main results deal with simple rational surfaces \( S \), which by definition are blow-ups \( \Theta : S \to \mathbb{CP}^2 \) of \( \mathbb{CP}^2 \) at \( n \) distinct points \( p_1, \ldots, p_n \in \mathbb{CP}^2 \). We denote by \( E_i^S = \Theta^{-1}(p_i) \) \( i = 1, \ldots, n \) the standard exceptional divisors of the blow-up and write \( \Sigma^S \) for the union \( \bigcup_{i=1}^n E_i^S \). Finally, we write \( L^S \) for be a divisor on \( S \), obtained by pulling back via \( \Theta \) a projective line in \( \mathbb{CP}^2 \) which does not pass through any of the points \( p_1, \ldots, p_n \).

A vector \( (d; \alpha_1, \ldots, \alpha_k) \in \mathbb{Z}_+ \times \mathbb{Z}_{\geq 0}^k \) is called ample (resp. nef) if there exists a simple rational surface \( V \), on which the divisor \( dL^S - \sum_{j=1}^k \alpha_j E_j^S \) is ample (resp. nef).

Our first result is the following gluing theorem:

**Theorem 2.A.** Let \( (d; m_1, \ldots, m_n, m) \) be an ample (resp. nef) vector and \( (m; \alpha_1, \ldots, \alpha_k) \in \mathbb{Z}_{+}^{k+1} \) a nef vector. Then \( v = (d; m_1, \ldots, m_n, \alpha_1, \ldots, \alpha_k) \) is ample (resp. nef). Moreover, \( v \) can be realized by an ample (resp. nef) divisor on a very general rational surface.

By a very general choice of points \( q_1, \ldots, q_r \) in an algebraic variety \( X \) we mean that \( (q_1, \ldots, q_r) \) is allowed to vary in a subset of the configuration space \( C_r(X) = \{ (x_1, \ldots, x_r) \in X^r \mid x_i \neq x_j \} \) whose complement is contained in a countable union of proper subvarieties of \( C_r(X) \). By a very general rational surface we mean one which is obtained by blowing-up points \( q_1, \ldots, q_r \in \mathbb{CP}^2 \) which may be chosen to be very general.

We shall actually prove a stronger result which allows us to keep the blown-up points corresponding to the first ample vector fixed, thus giving information also on ample divisors on non-generic rational surfaces. The precise statement is:

**Theorem 2.B.** Let \( D \) be a divisor on a simple rational surface \( S \). Suppose that there exists a point \( p \in S \setminus (\Sigma^S \cup \text{Supp } D) \) and \( m > 0 \) such that \( \pi_p^* D - mE \) is ample on the blow-up \( \pi_p : \tilde{S}_p \to S \) of \( S \) at \( p \) with exceptional divisor \( E \). Let \( (m; \alpha_1, \ldots, \alpha_k) \in \mathbb{Z}_{+}^{k+1} \) be a nef vector. Then for a very general choice of points \( q_1, \ldots, q_k \in S \setminus (\Sigma^S \cup \text{Supp } D) \) the divisor

\[
\pi^* D - \sum_{j=1}^k \alpha_j E_j
\]

is ample on the blow-up \( \pi : \tilde{S} \to S \) of \( S \) at \( q_1, \ldots, q_k \) with exceptional divisors \( E_j = \pi^{-1}(q_j) \).

\(^1\)Note that we regard \( \mathbb{CP}^2 \) itself as a simple rational surface too (this corresponds to \( n = 0 \)).
Proofs of Theorems 2.A and 2.B appear in Section 3.

Theorem 2.A in combination with the action of the Cremona group on the ample cone give rise to an algorithmic procedure for detecting new ample classes in the Picard group of rational surfaces. The algorithm will be explained in Section 4.2.

2.1. Applications to Seshadri constants. Given an ample line bundle $\mathcal{L} \to S$ on a surface, and a vector $w = (w_1, \ldots, w_N)$ of positive numbers we define the $w$-weighted remainder of $\mathcal{L}$ at the $N$ distinct points $p_1, \ldots, p_N \in S$ to be the quantity

$$R^w(\mathcal{L}, p_1, \ldots, p_N) = \frac{1}{\mathcal{L}} \inf_{0 \leq \epsilon \leq R} \left\{ \mathcal{L}_\epsilon \cdot \mathcal{L}_e \mid \mathcal{L}_\epsilon = \pi^* \mathcal{L} - \epsilon \sum_{j=1}^N w_j E_j \text{ is nef} \right\},$$

where $\pi : \tilde{S} \to S$ is the blow-up of $S$ at the points $p_1, \ldots, p_N$ with exceptional divisors $E_i = \pi^{-1}(p_i)$. It is obvious that $0 \leq R^w < 1$. Note that $R^w$ remains invariant under rescalings of $\mathcal{L}$ and of $w$, namely $R^{aw}(b\mathcal{L}, p_1, \ldots, p_N) = R^w(\mathcal{L}, p_1, \ldots, p_N)$ for every $a, b > 0$. It is convenient to define also a more global invariant, namely

$$R^w_N(\mathcal{L}) = \inf \{ R^w(\mathcal{L}, p_1, \ldots, p_N) \mid p_1, \ldots, p_N \in S \text{ are distinct points} \}.$$ 

Restricting to the case of homogeneous weights we obtain the homogeneous remainders

$$R(\mathcal{L}, p_1, \ldots, p_N) = R^{w_h}(\mathcal{L}, p_1, \ldots, p_N), \quad R_N(\mathcal{L}) = R^{w_h}(\mathcal{L}),$$

where $w_h = (1, \ldots, 1)$.

The constants $R^w(\mathcal{L}, p_1, \ldots, p_N)$ are obvious generalizations of the Seshadri constants $E(\mathcal{L}, p)$ from section 1 (see also [Xu 3] for similar Seshadri-like constants). Several theorems and conjectures related to the ample cone can be neatly formulated using the constants $R_N$. For example, Nagata’s conjecture from Section 1 can be reformulated as “$R_N(O_{\mathbb{C}P^2}(1)) = 0$ when $N \geq 9$”. Similarly, Xu’s result from Section 1 asserts that $R_N(O_{\mathbb{C}P^2}(1)) \leq \frac{1}{N}$. In Section 4.3 we shall prove the following asymptotic result:

**Theorem 2.1.A.**

1) For $N = a^2l^2 + 2l$, $a, l \in \mathbb{N}$ $R_N(O_{\mathbb{C}P^2}(1)) \leq \frac{1}{(a^2l^2+1)^2}$.

2) For $N = a^2l^2 - 2l$, $a, l \in \mathbb{N}$ $R_N(O_{\mathbb{C}P^2}(1)) \leq \frac{1}{(a^2l^2-1)^2}$.

3) If $N = a^2l^2 + l$ with $l < 1 \notin \mathbb{N}$ and suppose that $l > \frac{a}{2}$, where $k$ is the maximal non-negative integer for which $a \equiv 0 \mod 2^k$. Then $R_N(O_{\mathbb{C}P^2}(1)) \leq \frac{1}{(2a^2+l+1)^2}$.

In Section 5 we shall view this result in a more general context by proposing a conjecture which bounds $R_N(O_{\mathbb{C}P^2}(1))$ in terms of continued fractions approximations of $\sqrt{N}$.

Our methods also yield, as a corollary, the following generalization of a theorem of Xu [Xu 2] and Küchle [Ku]:

**Corollary 2.1.B.** Let $d > 0$. The divisor $D = \pi^* O_{\mathbb{C}P^2}(d) - 2 \sum_{j=1}^N E_j$ on the blow-up of $\mathbb{C}P^2$ at $N$ very general points is nef iff $D \cdot D \geq 0$.

The proof appears in Section 4.3. Let us conclude this section with the following, somewhat amusing, corollary of Theorem 2.A.

**Corollary 2.1.C.** If Nagata’s conjecture holds for $N_1$ and $N_2$ then it holds also for $N_1N_2$.

The proof is given in Section 4.4.
3. Gluing curves on rational surfaces

We shall derive Theorem 2.A as a corollary from Theorem 2.B. The proof of Theorem 2.B is based on a technique for "gluing" singular curves, which was developed by Shustin in [Sh 1]. This method generalizes Viro’s method (see [Vi]) for gluing curves to singular cases.

Suppose that $C_1, \ldots, C_n$ are plane curves with Newton Polygons $\Delta_1, \ldots, \Delta_n$ which have mutually disjoint interiors and match together to a bigger polygon $\Delta = \Delta_1 \cup \ldots \cup \Delta_n$. Shustin’s method allows, under some transversality conditions on the equisingular strata corresponding to $C_1, \ldots, C_n$, to construct a new curve $C$ with Newton polygon $\Delta$ and with singular points "inherited" from $C_1, \ldots, C_n$. We refer the reader to [Sh 1] for a detailed presentation of the general method and to [Sh 2] for interesting applications in other directions. Here, we shall make use only of a tip of the power of this method, by applying it to two curves with disjoint Newton polygons.

The application of Shustin’s technique to our problem is summed up in the following proposition which will be the main ingredient in the proof of Theorem 2.B. Most of the proof presented below is essentially an adjustment of the arguments appearing in the proof of Theorem 3.1 of [Sh 1] to our specific situation.

**Proposition 3.A.** Let $D$ be an effective divisor on a simple rational surface $S$, and $p \in S \setminus \Sigma^S$ a point with $\text{mult}_p D = m > 0$. Let $C$ be an effective divisor on another simple rational surface $V$, lying in the linear system $|m'L^V - \sum_{j=1}^k \alpha_j E_j^V|$. Suppose that $D, C$ satisfy the following conditions:

1) $0 < m' < m$.
2) $H^1(S_p, \mathcal{O}_{S_p}(\pi_p^* D - mE)) = 0$, where $\pi_p : \tilde{S} \to S$ is the blow-up of $S$ at the point $p$ with exceptional divisor $E$.
3) $H^1(V, \mathcal{O}_V(C)) = 0$.
4) Each of $C, D$ does not have any of the standard exceptional divisors $E_i^V, E_j^S$ as one of its components.
5) $D$ is an irreducible curve.

Then, there exist $k$ distinct points $q_1, \ldots, q_k \in S \setminus (\Sigma^S \cup \text{Supp} D)$ and a curve $\tilde{D}$ on the blow-up $\pi : \tilde{S} \to S$ of $S$ at $q_1, \ldots, q_k$ with exceptional divisors $E_j = \pi^{-1}(q_j)$, which has the following properties:

1) $\tilde{D} \in |\pi^* D - \sum_{j=1}^k \alpha_j E_j|$.
2) The curve $\tilde{D}$ is irreducible.

**Proof.** The idea of the proof is basically the following. By passing to the underlying projective planes of $\tilde{S}_p$ and $V$ we obtain from $D$ and $C$ two singular curves $C_1$ and $C_2$ and a point, still denoted by $p$, such that $\text{mult}_p C_1 > \deg C_2$. This inequality implies that the Newton polygons of $C_1$ and $C_2$ with respect to an affine chart centered at $p$ are disjoint. The next step is to construct two deformations $C_{1,t}$ and $C_{2,t}$ of $C_1$ and $C_2$ which are equisingular for $t > 0$ and such that each of them contains a deformations of the union of the singular points of $C_1$ and $C_2$ except of the one at the point $p$ which might disappear. These two deformations are then glued using the Viro polynomial. Shustin’s method requires the Newton polygons of each of $C_{1,t}$ and $C_{2,t}$ to be contained in the union, say $\Delta$, of the ones of $C_1$ and $C_2$. In order to construct deformations which satisfy this, one has to prove roughly speaking that the equisingular strata of $C_1$ and $C_2$ intersect transversally the space of curves with Newton polygons $\Delta$. This is precisely what the conditions of vanishing of the $H^1$’s is needed for.
Let us give now the precise details of the proof. Suppose that \( S \) is obtained by blowing up \( \Theta_S : S \to \mathbb{C}P^2 \) at \( p_1, \ldots, p_n \in \mathbb{C}P^2 \) and that \( V \) is obtained by blowing-up \( \Theta_V : V \to \mathbb{C}P^2 \) at \( q_1^0, \ldots, q_k^0 \in \mathbb{C}P^2 \). Put \( p_0 = \Theta_S(p) \), \( C_1 = \Theta_S(D) \subset \mathbb{C}P^2 \) and \( C_2 = \Theta_V(C) \subset \mathbb{C}P^2 \). Assuming that \( D \in |dE^S - \sum_{i=1}^m t_i E_i^S| \) we see that:

- \( C_1 \) is a plane curve of degree \( d \) and has singularities of orders \( m_1, \ldots, m_n \) at the points \( p_1, \ldots, p_n \) and a singular point of order \( m \) at \( p_0 \).
- \( C_2 \) is a plane curve of degree \( m' \) and has singularities of orders \( \alpha_1, \ldots, \alpha_k \) at the points \( q_1^0, \ldots, q_k^0 \).

In view of what we have to prove there is no loss of generality in assuming that \( q_j^0 \not\subset C_1 \) for every \( 1 \leq j \leq k \). Choose an affine chart \( \mathbb{C}^2 \subset \mathbb{C}P^2 \) with coordinates \( (x, y) \) such that \( p_0 = (0, 0) \) and such that \( p_1, \ldots, p_n, q_1^0, \ldots, q_k^0 \in (\mathbb{C}^*)^2 \subset \mathbb{C}^2 \subset \mathbb{C}P^2 \).

Let \( F_1(x, y), F_2(x, y) \) be polynomials of degrees \( d \) and \( m' \) respectively, such that \( C_1 \cap \mathbb{C}^2 = \{F_1 = 0\} \) and \( C_2 \cap \mathbb{C}^2 = \{F_2 = 0\} \). Set

\[
\Delta_1 = \{(i, j) \in \mathbb{Z}^2_{\geq 0} | m \leq i + j \leq d\}, \quad \Delta_2 = \{(i, j) \in \mathbb{Z}^2_{\geq 0} | 0 \leq i + j \leq m'\},
\]

and put \( \Delta = \Delta_1 \cup \Delta_2 \). With these notations, we may write

\[
F_1(x, y) = \sum_{(i,j) \in \Delta_1} a_{ij} x^i y^j, \quad F_2(x, y) = \sum_{(i,j) \in \Delta_2} a_{ij} x^i y^j.
\]

Let \( \overline{\Delta}_1 \supset \Delta_1 \) and \( \overline{\Delta}_2 \supset \Delta_2 \) be two slightly larger triangles with disjoint interiors. More precisely, let \( \delta > 0 \) be a small enough number such that \( m' + 2\delta < d - 2\delta \) and set

\[
\overline{\Delta}_1 = \{(i, j) \in \mathbb{R}^2_{\geq 0} | m - \delta \leq i + j \leq d\}, \quad \overline{\Delta}_1 = \{(i, j) \in \mathbb{R}^2_{\geq 0} | 0 \leq i + j \leq m' + \delta\}.
\]

Next, choose a strictly convex continuous piecewise linear function \( \nu : \mathbb{R}^2 \to \mathbb{R} \) such that the restrictions of \( \nu \) to each of \( \overline{\Delta}_1, \overline{\Delta}_2 \) coincides with some linear function \( \ell_1, \ell_2 : \mathbb{R}^2 \to \mathbb{R} \) with \( \ell_1 \neq \ell_2 \). We shall define the curve \( \tilde{D} \) as the zero locus of a polynomial lying in the following family:

\[
F_t(x, y) = \sum_{(i,j) \in \Delta} A_{ij}(t) x^i y^j t^{\nu(i,j)} \quad t > 0,
\]

with \( \lim_{t \to 0} A_{ij}(t) = a_{ij} \). The polynomial \( F_t \) is called the Viro polynomial.

More precisely, we claim that by a correct choice of of the coefficients \( A_{ij}(t) \), and of a homogeneous change of coordinates \( (x, y) \to T_t(x, y) \), the curve \( D_t = \{F_t(T_t(x, y)) = 0\} \) will have the following properties for \( t > 0 \) small enough:

1) \( \text{mult}_{p_i} D_t = m_i \) for every \( 1 \leq i \leq n \).
2) There exists \( k \) points \( q_{1i}, \ldots, q_{ki} \) depending smoothly on \( t > 0 \) such that \( q_{ji} \neq p_i \) for every \( i, j \) and \( \text{mult}_{q_{ji}} D_t = \alpha_j \) for every \( 1 \leq j \leq k \).
3) \( D_t \) is an irreducible curve of degree \( d \).

If we manage to prove this then the statement of the proposition will immediately follow. Indeed, let \( t_0 > 0 \) be small enough such that properties 1-3 above hold. Consider \( \overline{D}_{t_0} \subset \mathbb{C}P^2 \), the closure of \( D_{t_0} \) in \( \mathbb{C}P^2 \), and let \( \tilde{D}_{t_0} \) be the proper transform of \( \overline{D}_{t_0} \) in \( \tilde{S} \), the blow-up of \( \mathbb{C}P^2 \) at \( p_1, \ldots, p_n, q_{1t_0}, \ldots, q_{kt_0} \). Clearly \( \tilde{D}_t \in |\pi^* D - \sum_{j=1}^k \alpha_j E_j| \), where \( \pi : \tilde{S} \to S \) denotes the blow-up of \( S \) at \( q_1 = q_{1t_0}, \ldots, q_{kt_0} \).

Let us prove the existence of the coefficients \( A_{ij}(t) \) having the claimed properties. For this end, set \( \nu_1 = \nu - \ell_1 \), \( \nu_2 = \nu - \ell_2 \). Note that since \( \nu \) is strictly convex and \( \ell_1 \neq \ell_2 \) by construction,
we must have \( \nu_1, \nu_2 > 0 \). Consider the following deformations of \( F_1(x, y), F_2(x, y) \):

\[
F_{1,t}(x, y) = \sum_{(i,j) \in \Delta} A_{ij}(t)x^iy^jt^{\nu_{1}(i,j)},
\]

\[
F_{2,t}(x, y) = \sum_{(i,j) \in \Delta} A_{ij}(t)x^iy^jt^{\nu_{2}(i,j)}.
\]

An easy computation gives:\(^2\)

\[
F_{1,t}(x, y) = F_1(x, y) + \sum_{(i,j) \in \Delta_2} A_{ij}(t)x^iy^jt^{\nu_{1}(i,j)} + \sum_{(i,j) \in \Delta_1} (A_{ij}(t) - a_{ij})x^iy^j,
\]

\[
F_{2,t}(x, y) = F_2(x, y) + \sum_{(i,j) \in \Delta_1} A_{ij}(t)x^iy^jt^{\nu_{2}(i,j)} + \sum_{(i,j) \in \Delta_2} (A_{ij}(t) - a_{ij})x^iy^j,
\]

and

\[
F_t(x, y) = t^{c_1}F_{1,t}(t^{c_1}x, t^{c_2}y) + t^{c_2}F_{2,t}(t^{c_1}x, t^{c_2}y),
\]

where \( c_1, c_2 \) are the coefficients of the linear functions \( \ell_1, \ell_2 \), namely

\[
\ell_1(i, j) = c_0 + c^1_i + c^2_j, \quad \ell_2(i, j) = c_0 + c^1_i + c^2_j.
\]

Since \( A_{ij}(t) \to a_{ij} \) we see from (3) above that \( F_{1,t} \to F_1 \) as \( t \to 0 \). As \( F_1(x, y) \) is assumed to be irreducible and \( F_{1,t} \) is a deformation of \( F_1 \) (of the same degree) we see that \( F_{1,t} \) is irreducible for \( t > 0 \) small enough. In view of (5) we conclude that \( F_1(x, y) \) is irreducible for \( t > 0 \) small too. From (5) above we also see that the curve \( \{F_1(x, y) = 0\} \) will have the same (topological) types of singularities as each of the curves \( \{F_{1,t}(x, y) = 0\}, \{F_{2,t}(x, y) = 0\} \). Put

\[
T_t(x, y) = (t^{-c_1}x, t^{-c_2}y), \quad T'_t(x, y) = (t^{-c_1}x, t^{-c_2}y)
\]

and write

\[
D_t = \{F_1(T_t(x, y)) = 0\}, \quad q_{ij} = T_{i}^{-1} \circ T'_t(q^0_i) \text{ for every } 1 \leq j \leq k.
\]

Clearly, the maps \( (x, y) \to T_t(x, y) \) and \( (x, y) \to T'_t(x, y) \) extend to a family of biholomorphisms of \( CP^2 \) depending smoothly on \( t > 0 \). Note that from the definition of the points \( q_{ij} \) it easily follows that for a generic choice of \( t > 0 \) the points \( q_{ij} \) will be distinct from the \( p_i \)’s. In particular there exit arbitrarily small values \( t \) for which the points \( q_{ij} \) will not collide with the \( p_i \)’s.

Putting \( D_t = \{F_1(T_t(x, y)) = 0\} \), the problem is reduced to proving the following

**Lemma.** There exists a smooth deformation \( \{A_{ij}(t)\}_{0 \leq t \leq \epsilon} \) of the coefficients \( a_{ij}, \quad (i, j) \in \Delta \) with the following properties:

1. \( \lim_{t \to 0} A_{ij}(t) = a_{ij} \).
2. The curve \( \{F_1(x, y) = 0\} \) passes through \( p_1, \ldots, p_n \) with multiplicities \( m_1, \ldots, m_n \).
3. The curve \( \{F_1(x, y) = 0\} \) passes through \( q_{i_1}^0, \ldots, q_{i_k}^0 \) with multiplicities \( \alpha_1, \ldots, \alpha_k \).

**Proof of the Lemma.** Let \( \mathcal{P}(\Delta_1), \mathcal{P}(\Delta_2) \) be the spaces of polynomials in the variables \( (x, y) \) with Newton diagrams contained in \( \Delta_1, \Delta_2 \), respectively. For every point \( q \in \mathbb{C}^2 \), we denote by \( J_q^{(r)} \) the space of \( r \) jets of holomorphic functions at the point \( q \), viewed as a vector space and write \( j_q^{(r)}(F) \in J_q^{(r)} \) the \( r \)’th jet of \( F \) at the point \( q \).

Consider the linear maps

\[
R_1 : \mathcal{P}(\Delta_1) \to \bigoplus_{i=1}^n J_{p_i}^{(m_i)}, \quad R_2 : \mathcal{P}(\Delta_2) \to \bigoplus_{j=1}^k J_{q_j}^{(\alpha_j)}
\]

\(^2\)Here we use the convention that \( t^0 \equiv 1 \) and so the families \( F_{1,t}, F_{2,t} \) extend smoothly to \( t \geq 0 \).
defined by
\[ R_1(F) = \left( j_{p_1}^{(m_1)}(F), \ldots, j_{p_n}^{(m_n)}(F) \right), \quad R_2(F) = \left( j_{q_1}^{(\alpha_1)}(F), \ldots, j_{q_k}^{(\alpha_k)}(F) \right). \]

We claim that they are both surjective.

To see this let us denote for every \( q \in \mathbb{CP}^2 \) by \( \mathcal{I}_q \) the ideal sheaf corresponding to the point \( q \). Consider the ideal sheaf \( \mathcal{I}_{X_1} = \prod_{i=1}^n \mathcal{I}_{p_i}^{m_i} \cdot \mathcal{I}_p^m \) on \( \mathbb{CP}^2 \), and let \( X_1 \subset \mathbb{CP}^2 \) be the zero-dimensional subscheme defined by \( \mathcal{I}_{X_1} \), with structure sheaf \( \mathcal{O}_{X_1} = \mathcal{O}_{\mathbb{CP}^2} / \mathcal{I}_{X_1} \). Tensoring the structural exact sequence of \( X_1 \) by \( \mathcal{O}_{\mathbb{CP}^2}(d) \) we obtain the following exact sequence
\[ 0 \to \mathcal{I}_{X_1}(d) \to \mathcal{O}_{\mathbb{CP}^2}(d) \to \mathcal{O}_{X_1}(d) \to 0, \]
where for any sheaf \( \mathcal{F} \) we denote \( \mathcal{F}(d) = \mathcal{F} \otimes \mathcal{O}_{\mathbb{CP}^2}(d) \). Passing to cohomologies we obtain:
\[ 0 \to H^0(\mathcal{I}_{X_1}(d)) \to H^0(\mathcal{O}_{\mathbb{CP}^2}(d)) \xrightarrow{R_{X_1}} H^0(\mathcal{O}_{X_1}(d)) \to H^1(\mathcal{I}_{X_1}(d)) \to \ldots \]
where the map \( R_{X_1} \) is induced by the restriction \( \mathcal{R}_{X_1} : \mathcal{O}_{\mathbb{CP}^2} \to \mathcal{O}_{X_1} \). Since \( H^1(\mathbb{CP}^2, \mathcal{I}_{X_1}(d)) \cong H^1(S_p, \mathcal{O}_{S_p}(D - mE)) \) and the latter vanishes by assumption we see that the map \( R_{X_1} \) is surjective.

The choice of the affine chart \( \mathbb{C}^2 \subset \mathbb{CP}^2 \) induces an isomorphism \( i_1 : \mathcal{P}(d) \to H^0(\mathcal{O}_{\mathbb{CP}^2}(d)) \), where \( \mathcal{P}(d) \) denotes the space of polynomials in \( (x, y) \) of degree not more than \( d \). Similarly, we obtain an isomorphism \( i'_1 : \bigoplus_{i=1}^n J_{p_i}^{(m_i)} \oplus J_{p_0}^{(m)} \to H^0(\mathcal{O}_{X_1}) \). Denoting by
\[ \tilde{R}_1 : \mathcal{P}(d) \to \bigoplus_{i=1}^n J_{p_i}^{(m_i)} \oplus J_{p_0}^{(m)} \]
the linear map
\[ \tilde{R}_1(F) = \left( j_{p_1}^{(m_1)}(F), \ldots, j_{p_n}^{(m_n)}(F), j_{p_0}^{(m)}(F) \right), \]
we obtain the following commutative diagram:
\[ \begin{array}{c}
\mathcal{P}(d) \xrightarrow{R_1} \bigoplus_{i=1}^n J_{p_i}^{(m_i)} \oplus J_{p_0}^{(m)} \\
\downarrow \quad \downarrow i'_1 \\
H^0(\mathcal{O}_{\mathbb{CP}^2}(d)) \xrightarrow{R_{X_1}} H^0(\mathcal{O}_{X_1}(d))
\end{array} \]

As \( R_{X_1} \) is surjective so is \( \tilde{R}_1 \). But \( \tilde{R}_1^{-1}(\bigoplus_{i=1}^n J_{p_i}^{(m_i)}) = \mathcal{P}(\Delta_1) \subset \mathcal{P}(d) \) and \( \tilde{R}_1|_{\mathcal{P}(\Delta_1)} = R_1 \). This implies that \( R_1 \) is indeed surjective.

The case of \( R_2 \) is easier. Replacing \( d \) by \( m' \), \( X_1 \) by the subscheme \( X_2 \subset \mathbb{CP}^2 \) defined by the ideal sheaf \( \mathcal{I}_{X_2} = \prod_{j=1}^k \mathcal{I}_{q_j}^{(\alpha_j)} \), and \( R_{X_1} \) by the the restriction map \( R_{X_2} \), we obtain the commutative diagram:
\[ \begin{array}{c}
\mathcal{P}(\Delta_2) \xrightarrow{R_2} \bigoplus_{j=1}^k J_{q_j}^{(\alpha_j)} \\
\downarrow i_2 \\
H^0(\mathcal{O}_{\mathbb{CP}^2}(m')) \xrightarrow{R_{X_2}} H^0(\mathcal{O}_{X_2}(m'))
\end{array} \]

where \( i_2 \) and \( i'_2 \) are obvious isomorphisms induced by the choice of the affine chart \( \mathbb{C}^2 \subset \mathbb{CP}^2 \). The vanishing of \( H^1(V, \mathcal{O}_V(C)) \cong H^1(\mathbb{CP}^2, \mathcal{I}_{X_1}(m')) \) implies, as before, the surjectivity of \( R_{X_2} \) and consequently that of \( R_2 \).
To conclude the proof of the lemma, consider the smooth family of linear maps

\[ R(t) : \mathcal{P}(\Delta) \rightarrow \bigoplus_{i=1}^{n} J_{p(i)}^{(m_i)} \bigoplus_{j=1}^{k} J_{q(j)}^{(\alpha_j)}, \]

defined by:

\[ R(t)\left( \sum_{(i,j) \in \Delta} A_{ij} x^i y^j \right) = \left( R_1\left( \sum_{(i,j) \in \Delta} A_{ij} x^i y^j t^{m(i,j)} \right), \ R_2\left( \sum_{(i,j) \in \Delta} A_{ij} x^i y^j t^{\alpha(i,j)} \right) \right). \]

Substituting \( t = 0 \) we have, under the direct sum decomposition \( \mathcal{P}(\Delta) = \mathcal{P}(\Delta_1) \oplus \mathcal{P}(\Delta_2) \), that \( R^{(0)} = R_1 \oplus R_2 \), hence \( R^{(0)} \) is surjective. Since the family \( R^{(t)} \) depends smoothly on \( t \) we conclude that \( R^{(t)} \) remains surjective for \( t > 0 \) small enough. By the (linear) implicit function theorem there exists a smooth deformation \( \{ A_{ij}(t) \}_{0 \leq t \leq \epsilon} \) of \( A_{ij} \), such that \( R(t)\left( \sum_{(i,j) \in \Delta} A_{ij}(t) x^i y^j \right) = 0 \). This means that \( F_{1,i}(x,y) \) vanishes to order \( m_i \) at \( p_i \) for every \( 1 \leq i \leq n \) and \( F_{2,i}(x,y) \) vanishes to order \( \alpha_j \) at \( q_j \) for every \( 1 \leq j \leq k \). This concludes the proof of the lemma and thus of the whole proposition. \( \blacksquare \)

### 3.1. Passing from specific points to very general.

In what follows we shall detect several useful ample (resp. nef) vectors \((d, \alpha_1, \ldots, \alpha_k)\) by choosing \( k \) points \( q_1, \ldots, q_k \in \mathbb{C}P^2 \) to lie in a very specific convenient position which is not generic. The following lemma shows that this vectors remain ample (resp. nef) also for a very general choice of the points \( q_1, \ldots, q_k \).

**Lemma 3.1.A.** Let \( F \) be a divisor on a simple rational surface \( S \), and \( q_1^{(0)}, \ldots, q_k^{(0)} \in S \setminus (\Sigma^S \cup \text{Supp} F) \) distinct points. Let \( \pi_0 : S_0 \rightarrow S \) be the blow-up of \( S \) at \( q_1^{(0)}, \ldots, q_k^{(0)} \) with exceptional divisors \( E_i^0 = \pi_0^{-1}(q_i^{(0)}) i = 1, \ldots, k \).

Suppose that for some \( f_1, \ldots, f_k \geq 0 \) the divisor \( \pi_0^* F - \sum_{j=1}^{k} f_j E_j^0 \) is ample (resp. nef). Then, for a very general choice of points \( q_1, \ldots, q_k \in S \setminus (\Sigma^S \cup \text{Supp} F) \) the divisor

\[ \pi^* F - \sum_{j=1}^{k} f_j E_j \]

is ample (resp. nef) on the blow-up \( \pi : \tilde{S} \rightarrow S \) of \( S \) at \( q_1, \ldots, q_k \) with exceptional divisors \( E_j = \pi^{-1}(q_j) j = 1, \ldots, k \).

**Proof.** The idea of the proof is very simple. Since \( \tilde{F}_0 = \pi_0^* F - \sum_{j=1}^{k} f_j E_j^0 \) is assumed to be nef on the blow-up of \( S \) at \( q_1^{(0)}, \ldots, q_k^{(0)} \), all the divisor classes which intersect \( \tilde{F}_0 \) negatively do not admit any effective representatives. Now, the point is that if a divisor class on the blow-up of \( S \) at specific points has no effective representatives then the same will continue to hold also on the blow-up at generic points. The lemma now follows because \( \text{Pic}(\tilde{S}) \) is countable. Let us give now the precise details.

We prove the lemma for the “nef” case, the “ample” being very similar. Consider the following subset of \( \text{Pic}(S) \times \mathbb{Z}_{\geq 0}^k \):

\[ \mathcal{B} = \left\{ (A, a_1, \ldots, a_k) \in \text{Pic}(S) \times \mathbb{Z}_{\geq 0}^k \mid F \cdot A - \sum_{j=1}^{k} f_j a_j < 0 \right\}. \]

\(^3\)As before, using the convention that \( t^0 \equiv 1 \) the family \( R^{(t)} \) extends also for \( t = 0 \).
Notice that $\mathcal{B}$ is a countable set. We claim that for every $B = (A, a_1, \ldots, a_k) \in \mathcal{B}$ there exists a non-empty Zariski-open subset $U_B \subset C_k(S \setminus (\Sigma^S \cup \text{Supp } F))$ such that for every $(q_1, \ldots, q_k) \in U_B$, the surface $\tilde{S}$ obtained by blowing-up $S$, $\pi : \tilde{S} \to S$, at $q_1, \ldots, q_k$ does not admit any effective divisor in the class $\pi^*[A] - \sum_{j=1}^k a_j [E_j]$. Once this is proved, we take $\mathcal{V} = \cap_{B \in \mathcal{B}} U_B$. Obviously $\mathcal{V}$ is a very general subset of $C_k(S \setminus (\Sigma^S \cup \text{Supp } F))$ having the needed properties.

Let us prove the existence of the Zariski-open sets $U_B$ claimed above. For this end put $\mathcal{C} = C_k(S \setminus (\Sigma^S \cup \text{Supp } F))$, $X = C \times S$, and denote by $pr : X \to C$ the obvious projection.

Consider the subvarieties $Y_j \subset X$, $j = 1, \ldots, k$, defined by

$$Y_j = \{(x_1, \ldots, x_k) | x = x_j \}.$$

The $Y_j$'s are smooth disjoint subvarieties of $X$ each of which is mapped by $pr$ isomorphically onto $\mathcal{C}$. Let $\Theta : X \to X$ be the blow-up of $X$ along $Y = \cup_{j=1}^k Y_j$ and write $E_j = \Theta^{-1}(Y_j)$ for the exceptional divisors.

Given $B = (A, a_1, \ldots, a_k) \in \mathcal{B}$, we denote by $\mathcal{L}$ the line bundle

$$\mathcal{L} = \mathcal{O}_{\tilde{X}}(\tilde{A} - \sum_{j=1}^k a_j E_j) \in \text{Pic}(\tilde{X}),$$

where $\tilde{A} \in \text{Div}(\tilde{X})$ is the divisor $\Theta^*(C \times A)$. Finally, for every $q \in \mathcal{C}$ we write $\mathcal{L}_q$ for the restriction of $\mathcal{L}$ to the surface $\tilde{S}_q = \Theta^{-1}pr^{-1}(q)$.

Let $q = (q_1, \ldots, q_k) \in \mathcal{C}$. It is easy to see that the map $\pi_q : \tilde{S}_q \to S$ defined by the composition $\tilde{S}_q \xrightarrow{\theta} X \xrightarrow{pr} S$ is just the blow-up of $S$ at $q_1, \ldots, q_k$, and that

$$\mathcal{L}_q = \mathcal{O}_{\tilde{S}_q}(\pi_q^* A - \sum_{j=1}^k a_j E_j).$$

Now, for $q_0 = (q_1^0, \ldots, q_k^0)$ we know that $\dim H^0(\tilde{S}_{q_0}, \mathcal{L}_{q_0}) = 0$ because $\pi_0^* F - \sum_{j=1}^k f_j E_j$ is nef. It follows from the semicontinuity theorem (see [Ha 2]) that there exits a Zariski-open neighborhood of $q_0^0$, $U_B \subset C$ such that for every $q \in U_B$, $H^0(\tilde{S}_q, \mathcal{L}_q) = 0$.

3.2. Proof of the gluing Theorem. Now we are in position to prove Theorem 2.B.

**Proof.** We divide the proof into three steps. In the first step we prove that the resulting divisor $D = \pi^* D - \sum_{j=1}^k \alpha_j E_j$ is nef provided that $D_p = \pi_p^* D - mE$ and $v = (m; \alpha_1, \ldots, \alpha_k)$ are nef. In the second step we prove that the theorem holds under the assumption that both $D_p$ and $v$ are ample. Finally, in the third step we prove the theorem in its full generality by reducing the problem to the first two steps.

**Step 1.** Assuming that $D_p = \pi_p^* D - mE$ and $v = (m; \alpha_1, \ldots, \alpha_k)$ are nef we prove that $D = \pi^* D - \sum_{j=1}^k \alpha_j E_j$ is nef.

We claim that there exists $N_0 > 0$ and a divisor $A$ on $S$ such that for every $N > 0$ there exists a very general subset $G_N \subset C_k(S \setminus (\Sigma^S \cup \text{Supp } D))$ such that for every $(q_1, \ldots, q_k) \in G_N$ the divisor

$$D_N = \pi^* A + (N + N_0) \pi^* D - N \sum_{j=1}^k \alpha_j E_j$$

is nef on the blow-up $\pi : \tilde{S} \to S$ of $S$ at $q_1, \ldots, q_k$. 

Once this is proved step 1 of the proof will be concluded as follows: put \( G = \cap_{N=1}^\infty G_N \). Clearly \( G \subset C_k(S \setminus (\Sigma^S \cup \text{Supp } D)) \) is a very general subset. Let \((q_1, \ldots, q_k) \in G\) and let \( C \subset \tilde{S} \) be a curve. Since \( D_N \) is nef we have
\[
0 \leq D_N \cdot C = A \cdot C + (N + N_0)D \cdot C - N \sum_{j=1}^k C \cdot E_j.
\]
Dividing by \( N \) and letting \( N \to \infty \) we obtain that
\[
\tilde{D} \cdot C = (\pi^*D - \sum_{j=1}^k \alpha_j E_j) \cdot C \geq 0.
\]
As \( v \) is nef, we have \( m^2 \geq \sum_{j=1}^k \alpha_j^2 \) and so
\[
\tilde{D} \cdot \tilde{D} = D \cdot D - \sum_{j=1}^k \alpha_j^2 \geq D \cdot D - m^2 = D_p \cdot D_p \geq 0,
\]
the latter inequality following from the nefness of \( D_p \). Thus \( \tilde{D} \) is nef.

Let us prove now the existence of \( N_0, A, G_N \) claimed above. For the divisor \( A \) we choose any divisor on \( S \) such that \( \pi^*A - E \) is ample on \( \tilde{S}_p \). The nefness of \( v \) means by definition that there exist \( k \) distinct points \( q_1^0, \ldots, q_k^0 \in \mathbb{C}P^2 \) such that the divisor \( mL^v - \sum_{j=1}^k \alpha_j E_j^v \) is nef on \( V \) - the blow-up of \( \mathbb{C}P^2 \) at \( q_1^0, \ldots, q_k^0 \). For \( N_0 > 0 \) we choose any integer for which \( B = N_0L^v - \sum_{j=1}^k E_j^v \) is ample on \( V \).

For every \( N > 0 \) define \( L'_N \in \text{Div}(\tilde{S}_p), \ L''_N \in \text{Div}(V) \) to be:
\[
L'_N = (\pi_p^*A - E) + (N + N_0)D_p = \pi_p^*A + (N + N_0)\pi_p^*D - (mN + mN_0 + 1)E,
\]
\[
L''_N = B + N(mL^v - \sum_{j=1}^k \alpha_j E_j^v) = (Nm + N_0)L^v - N \sum_{j=1}^k E_j^v.
\]
It easily follows from our assumptions on \( D_p \) and on \( v \) that \( L'_N, L''_N \) are ample for every \( N > 0 \). Choose an integer \( r_N > 0 \) for which the following two conditions are satisfied:

1) \( r_NL'_N \) and \( r_NL''_N \) are very ample.
2) \( H^1(\tilde{S}_p, \mathcal{O}_{\tilde{S}_p}(r_NL'_N)) = 0 \), and \( H^1(V, \mathcal{O}_V(r_NL''_N)) = 0 \).

Choose irreducible curves \( \tilde{C}'_N \subset |r_NL'_N| \) and \( \tilde{C}''_N \subset |r_NL''_N| \) and put \( C' \subset \pi_p(\tilde{C}'_N) \subset S \). By Proposition 3.A there exist \( q_1, \ldots, q_k \in S \setminus (\Sigma^S \cup C' \cap C''_N) \) such that the surface \( \tilde{S} \) obtained by blowing-up \( \pi : \tilde{S} \to S \) at \( q_1, \ldots, q_k \) admits an irreducible curve \( C_N \) in the linear system
\[
C_N \in r_N\left(\pi^*A + (N + N_0)\pi^*D - N \sum_{j=1}^k \alpha_j E_j \right) = |r_ND_N|.
\]
Noting that
\[
D_N \cdot D_N \geq N^2(D \cdot D - \sum_{j=1}^k \alpha_j^2) \geq N^2(D \cdot D - m^2) = N^2D_p \cdot D_p \geq 0,
\]
we conclude that \( D_N \) intersects every curve non-negatively and so it is is nef on \( \tilde{S} \). By Lemma 3.1.A we may assume that \( (q_1, \ldots, q_k) \) vary in some very general subset \( G_N \subset \mathcal{C}(S \setminus (\Sigma^S \cup \text{Supp } D)) \). This completes the proof of step 1.
Lemma 4.1.A. which essentially appears in [Do-Or].

Here it is more convenient to work with \( \mathbb{Q} \)-divisors. First note that step 1 remains true if we take \( m \) and \( \alpha_j \) to be rational numbers. It follows from Seshadri’s criterion for ampleness (see [Ha 1]) that there exists a positive rational number \( \epsilon \) such that \( \pi^* D - (1 + \epsilon) m E \) is ample. Since \( ((1 + \epsilon)n; (1 + \epsilon) \alpha_1, \ldots, (1 + \epsilon) \alpha_k) \) is ample too we have from step 1 that \( \tilde{D} = \pi^* D - (1 + \epsilon) \sum_{j=1}^k \alpha_j E_j \) is nef.

Let us prove that \( \tilde{D} \) is ample by applying Nakai-Moishezon criterion. Indeed, let \( \bar{C} \subset \bar{S} \) be a curve. If \( \tilde{C} = E_j \) is one of the standard exceptional divisors then \( \tilde{D} \cdot \tilde{C} = \alpha_j > 0 \) (recall that \( \alpha_j > 0 \) because we assume that \( v \) is ample). Otherwise, let \( C = \pi(\bar{C}) \). If \( \bar{C} \) does not pass through any of the exceptional divisors then \( \tilde{D} \cdot \tilde{C} = D \cdot C > 0 \) because \( D \) itself is ample for \( D_p \) is. Suppose now that there exits a \( j_0 \) such that \( \tilde{C} \cdot E_{j_0} > 0 \). In this case \( \tilde{D} \cdot \tilde{C} = D \cdot C - \sum_{j=1}^k \alpha_j E_j > \tilde{D}_v \cdot \tilde{C} \geq 0 \). Finally note that \( \tilde{D} \cdot \tilde{D} > \tilde{D}_v \cdot \tilde{D}_v \geq 0 \).

Step 3. Consider the general case. The case of \( D_p \) nef has been treated in step 1 so we may assume that \( D_p \) is ample and \( v \) is nef. Similarly to step 2 we choose a positive rational number \( \epsilon \) such that both \( \pi_p^* D - (1 + \epsilon) E \) and \( ((1 + \epsilon)m; \alpha_1, \ldots, \alpha_k) \) are ample. By step 2 we have that \( \pi_p^* D - \sum_{j=1}^k \alpha_j E_j \) is ample.

Theorem 2.A follows immediately from Theorem 2.B by taking \( D = dL^S - \sum_{i=1}^n m_i E_i^S \) for a suitable simple rational surface \( S \).

4. Asymptotics on the remainders of \( \mathcal{O}_{\mathbb{C} P^2}(1) \)

In order to obtain estimates on \( \mathcal{R}_N(\mathcal{O}_{\mathbb{C} P^2}(1)) \) we shall extensively use Theorem 2.A in combination with the Cremona action. The point is, that the Cremona group acts on the set of ample (resp. nef) vectors. Let us briefly summarize the needed facts about the Cremona action. We refer the reader to [Do-Or] for more details.

4.1. The Cremona action on the ample cone. Denote by \( (H_k, \langle , \rangle) \) \((k \geq 3)\) the hyperbolic lattice \( H_k = \mathbb{Z} l \oplus \mathbb{Z} e_1 \oplus \cdots \oplus \mathbb{Z} e_k \) with the bilinear form \( \langle , \rangle \) defined by \( \langle l, l \rangle = 1, \langle l, e_j \rangle = 0, \langle e_i, e_j \rangle = -\delta_{ij} \). Consider the subgroup \( Cr_k \subset Aut(H_k, \langle , \rangle) \), generated by:

1. The symmetric group \( S_k \to Aut(H_k, \langle , \rangle) \) acting on the last \( k \) components.
2. The reflection \( R_{123} : (H_k, \langle , \rangle) \to (H_k, \langle , \rangle) \) defined by \( R_{123}(\eta) = \eta + \langle \eta, r_{123} \rangle r_{123} \), where \( r_{123} = l - e_1 - e_2 - e_3 \).

The group \( Cr_k \) is called the Cremona group.

It is easily seen that the reflection \( R_{ijk}(\eta) = \eta + \langle \eta, r_{ijk} \rangle r_{ijk} \), where \( r_{ijk} = l - e_i - e_j - e_k \), belong to \( Cr_k \). Let us mention one more useful transformation which we denote by \( SR \). The transformation \( SR \) takes a vector \( v = (d; m_1, \ldots, m_k) \in H_k \) and sorts it. In other words \( SR(v) = (d; m_{\tau(1)}, \ldots, m_{\tau(k)}) \), where \( \tau \) is a permutation of \( \{1, \ldots, k\} \) for which \( m_{\tau(1)} \geq \cdots \geq m_{\tau(k)} \). It is obvious that for every vector \( v \in H_k \) there exists \( \sigma \in Cr_k \) such that \( SR(v) = \sigma(v) \).

Given a simple rational surface obtained by blowing up \( \Theta : V \to \mathbb{C} P^2 \) of \( p_1, \ldots, p_n \in \mathbb{C} P^2 \), there is an isomorphism of lattices \( m_\Theta : (Pic(V), \cdot) \to (H_k, \langle , \rangle) \), where \( \cdot \) stands for the intersection form on \( Pic(V) \). The isomorphism \( m_\Theta \) sends \( L^V \) to \( l \) and \( E_i^V \) to \( e_i \).

To deduce that \( Cr_k \) acts on the set of ample (resp. nef) vectors we need the following lemma which essentially appears in [Do-Or].

Lemma 4.1.A. Let \( V \) be a simple rational surface obtained by blowing-up \( \Theta : V \to \mathbb{C} P^2 \) points \( p_1, \ldots, p_n \in \mathbb{C} P^2 \) in general position. Then for every \( \sigma \in Cr_k \) there exists a simple rational
surface $V_{\sigma}$ obtained by blowing-up $\Theta_\sigma : V_{\sigma} \to \mathbb{C}P^2$ points $q_1, \ldots, q_k$ in general position and a biholomorphism $f_\sigma : V_{\sigma} \to V$ making the following diagram commutative:

$$
\begin{array}{ccc}
\text{Pic}(V) & \xrightarrow{f_\sigma^*} & \text{Pic}(V_{\sigma}) \\
\downarrow m_\sigma & & \downarrow m_{\sigma,\sigma} \\
H_k & \xrightarrow{\sigma} & H_k
\end{array}
$$

Combining this with Lemma 3.1.A we immediately obtain the following

**Lemma 4.1.B.** When $k \geq 3$ the group $C\mathcal{R}_k$ acts on the set of ample (resp. nef) vectors viewed as a subset of $H_k$.

**Remark.** From Lemma 3.1.A it follows that there exists (at least) one simple rational surface $S$, obtained by blowing-up $k$ distinct points in $\mathbb{C}P^2$, $\Theta : S \to \mathbb{C}P^2$ such that $\mathcal{L} \in \text{Pic}(S)$ is ample (resp. nef) iff $m_\Theta(\mathcal{L})$ is ample (resp. nef). Hence, the set of ample (resp. nef) vectors is closed under addition and multiplication by positive (resp. non-negative) integers. Henceforth we shall denote by $\mathcal{K}_k \subset H_k \otimes \mathbb{R}$ (resp. $\overline{\mathcal{K}}_k$) the cone generated by all ample (resp. nef) vectors.

### 4.2. An algorithmic procedure for detecting ample classes.

Given two vectors $v_1 = (d; m_1, \ldots, m_n) \in H_n$ and $v_2 = (\delta; \alpha_1, \ldots, \alpha_k) \in H_k$ with $\delta = m_1$ for some $1 \leq i \leq n$, define a new vector $v_1 \# v_2 = (d; m_1, \ldots, m_{i-1}, \alpha_1, \ldots, \alpha_k, m_{i+1}, \ldots, m_n)$.

Theorem 2.A asserts that if $v_1$ is ample (resp. nef) and $v_2$ is nef, then $v_1 \# v_2$ is ample (resp. nef).

Given a vector $v_0 \in H_N$ the ampleness of which we want to prove we try to find a decomposition $v_0 = v_1 \#_i u_i$ where $u_i \in H_{k_i}$ is known to be nef and $v_1 \in H_{n_1}$, $(k_1 + n_1 - 1 = N)$. If $v_1$ turns to be ample then we are done in view of Theorem 2.A. To check the ampleness of $v_1$ we first "simplify" it by applying to it Cremona transformations. For example, we may try, using Cremona transformations to reduce the degree of $v_1$ (by the degree of $v = (d; \mu_1, \ldots, \mu_k)$ we mean $\text{deg } v = d$). Let $v'_1$ be a simpler vector in the same orbit of $v_1$ under the action of $C\mathcal{R}_{n_1}$. Having minimal degree in the orbit, or having some other convenient feature). By Lemma 4.1.B $v_1$ is ample iff $v'_1$ is. Now we apply the whole process to $v'_1$ and so on. In this way we obtain a sequence of vectors $v_1, u_1, v_1', v_r, u_r, v_r'$ where $v_r'$ is a Cremona simplification of $v_j \in H_{n_j}, u_j \in H_{k_j}$, is a nef vector and $v'_r = v_{r+1} \#_{i_{j+1}} u_{j+1}$ for some $i_{j+1}$.

Note that at each stage the number of points decreases, namely $n_{j+1} < n_j$ provided that $k_j > 1$. The process ends successfully as soon as we are able to prove that $v_r$ is ample for some $r$. We remark that if one of the $v_j$ turns out not to be ample then process fails to give any information because the converse of Theorem 2.A is not true. However, we may attempt to find other decomposition sequences $v_1, u_1, v_1', v_r, u_r, v_r'$ (see Section 6).

The same procedure can be applied for proving nefness of a vector $v_0$ by requiring that $v_r$ is nef instead of ample. In the next subsection we shall apply this process in order to prove Theorem 2.1.A and Corollary 2.1.B.

In order to make the preceding procedure applicable we must first endow ourselves with an initial large enough collection of ample and nef vectors which will play the role of the $u_j$'s and of $v_r$. To simplify notations let us agree that $(d; \alpha_1^{r_1}, \ldots, \alpha_k^{r_k})$ stands for

$$
(d; \underbrace{\alpha_1, \ldots, \alpha_1}_{r_1 \text{ times}}, \ldots, \underbrace{\alpha_k, \ldots, \alpha_k}_{r_k \text{ times}}) \in H_N,
$$

where $1 \leq r_1, \ldots, r_k$.
where $N = \sum_{j=1}^{h} r_j$.

The next lemma provides a modest initial collection of ample and nef vectors which is sufficient for our purposes.

**Lemma 4.2.A.** The following vectors are nef (resp. ample) on a very general rational surface:

1) $(d; 1^{xr})$, where $d^2 \geq r$ (resp. $d^2 > r$).
2) $(d; m_1, m_2, 1^{xr})$ where $d \geq m_1 + m_2$ and $d^2 \geq m_1^2 + m_2^2 + r$.

**Remark.** The “ample” case of statement 1 above has been proved by Xu in [Xu 2] and by Küchle in [Ku]. Below however, we present an alternative proof suggested by Ilya Tyomkin.

**Proof.** Notice first that in view of Lemma 3.1.A it is enough to prove that the above vectors are nef (resp. ample) on a specific simple rational surface.

1) Consider first the case $d^2 > r$. In [Nag] (consult also [Sh-Ty]) Nagata proved that if $N$ is a square, then for generic points $p_1, \ldots, p_N \in \mathbb{C}P^2$ and for every irreducible curve $C \subset \mathbb{C}P^2$ the following strict inequality holds:

$$\deg(C) > \frac{\sum_{j=1}^{N} \text{mult}_{p_j}(C)}{\sqrt{N}}. \quad (1)$$

Let $V_r$ be the blow-up of $\mathbb{C}P^2$ at $r$ generic points and denote by $\Theta : \tilde{V}_r \to V_r$ the blow-up of $V_r$ at $d^2 - r$ generic points. Thus $\tilde{V}_r$ is the blow-up of $\mathbb{C}P^2$ at $N = d^2$ generic points and it follows from inequality (1) that the divisor $\tilde{D} = \Theta^*(dL^r - \sum_{j=1}^{r} E_j^r) - \sum_{j=r+1}^{d^2} E_j$ intersects every curve positively. This immediately implies that $D = dL^r - \sum_{j=1}^{r} E_j^r$ intersects any curve in $V_r$ positively. As $D \cdot D > 0$ the statement follows from Nakai-Moishezon criterion (see [Ha 1]).

The proof for the nef case ($d^2 \geq r$) is much easier. Indeed, let $C \subset \mathbb{C}P^2$ be an irreducible curve of degree $d$, and let $p_1, \ldots, p_r$ be distinct points on $C$ at which $C$ is smooth. Let $V$ be the blow-up of $\mathbb{C}P^2$ at $p_1, \ldots, p_r$ and let $D$ be the proper transform of $C$ in $V$, $D \in |dL^r - \sum_{j=1}^{r} E_j|$. As $D$ is an irreducible curve of non-negative self intersection the vector $(d; 1^{xr})$ corresponding to the divisor class of $D$ is nef on $V$.

2) Set $D = dL - m_1 E_1 - m_2 E_2$. Consider the linear system $|D|$ on $V_2$ — the blow-up of $\mathbb{C}P^2$ at 2 points. As $D = m_1(L - E_1) + m_2(L - E_2) + (d - m_1 - m_2)L$ it is easy to see that $|D|$ is not empty and has no base-points, hence by Bertini theorem there exists an irreducible (smooth) curve $C \in |D|$. Choose $r$ distinct points $p_1, \ldots, p_r \in C \setminus (E_1 \cup E_2)$ and let $\tilde{V}$ be the blow-up of $V$ at $p_1, \ldots, p_r$. Finally denote by $\tilde{C}$ be the proper transform of $C$ in $\tilde{V}$.

We have $\tilde{C} \in |dL - m_1 E_1 - m_2 E_2 - \sum_{j=3}^{r+2} E_j|$. As $\tilde{C}$ is irreducible and $\tilde{C} \cdot \tilde{C} \geq 0$, the vector $(d; m_1, m_2, 1^{xr})$ is nef.

**Remark.** Note that the cones $K_n$ (resp. $\overline{K}_n$) can be explicitly computed when $n < 9$ (see [Dmz], [F-M]), and so can be joined to the initial collection of ample and nef vectors to be applied in the framework of the process mentioned above.

### 4.3. Proofs of Theorem 2.1.A and Corollary 2.1.B

We start with the proof of Theorem 2.1.A.

**Proof.** 1) Let $N = a^2l^2 + 2l$ and $v_0 = (a^2l + 1; a^{xN})$. As $\langle v_0, v_0 \rangle = 1$, nefness of $v_0$ will give the needed estimate for $R_N(O_{C\mathbb{P}^2}(1))$. 

The decomposition $N = (al - 1)^2 + n$, where $n = 2al + 2l - 1$, leads us to $v_0 = v_1 \#_i u_1$ where

$$v_1 = \left( a^2l + 1; a(al - 1), a^{x_n} \right) \in H_{n+1}, \quad u_1 = a\left( al - 1; 1^{x(al-1)^2} \right) \in H_{(al-1)^2}. $$

By Lemma 4.2.A $u_1$ is nef, hence in view of Theorem 2.A we are reduced to proving that $v_1$ is nef. This turns out to be easy by using Cremona transformations. Indeed let

$$v'_1 = R_{1,n-1,n} \circ R_{1,n-3,n-2} \circ \ldots \circ R_{123}(v_1),$$

where $R_{ijk} \in CR_{n+1}$ are defined in Section 4.1. A straightforward computation shows that $v'_1 = (a + l; l - 1, 1^{x_{n-1}}, a)$. This vector is nef by Lemma 4.2.A, and therefore $v_1$ too.

2) Let $N = a^2l^2 - 2l$ and $v_0 = (a^2l - 1; a^N)$. Again $\langle v_0, v_0 \rangle = 1$, hence in order to prove the needed estimate on $\mathcal{R}_N(O_{\mathbb{CP}^2}(1))$ we have to prove that $v_0$ is nef. Using the decomposition $N = (al - 2)^2 + n_1$, where $n_1 = 4al - 2l - 4$, we note that $v_0 = v_1 \#_i u_1$ where

$$v_1 = \left( a^2l - 1; a(al - 2), a^{x_{n_1}} \right) \in H_{n_1+1}, \quad u_1 = a\left( al - 2; 1^{x(al-2)^2} \right) \in H_{(al-2)^2}.$$ 

By Lemma 4.2.A $u_1$ is nef. We are thus reduced to proving nefness of $v_1$. By applying similar Cremona transformations as in 1, we obtain that $v'_1 = (a^2l-2al+l+1; (al-l-2)(a-1), (a-1)^{x_{n_1}})$ lies in the same orbit as $v_1$.

Let us apply now the same algorithm again on $v'_1$. For this, consider the decomposition $v'_1 = v_2 \#_i u_2$, where

$$v_2 = \left( a^2l - 2al + l + 1; (al - l - 1)(a - 1), (a - 1)^{x_{2al-1}} \right) \in H_{2al-1},$$

$$u_2 = (a - 1)\left( al - l - 1; al - l - 2, 1^{x_{2al-2l-2}} \right).$$

By Lemma 4.2.A $u_2$ is nef, thus we are reduced to proving that $v_2$ is nef. Using similar Cremona transformations as in 1 we obtain that $v'_2 = (a + l - 1; l - 1, 1^{x_{2al-2}}, a - 1)$ lies in the same orbit as $v_2$. But by Lemma 4.2.A $v'_2$ is nef.

3) Let $N = a^2l^2 + l$ and suppose that $a = 2^k b$ with $k \geq 0$ and $b$ odd. The assumption appearing in the statement of the Theorem is that $l > 2b$. Note that we may assume that $l$ is odd, since when $l$ is even we have $N = (2a)^2\left(\frac{1}{2}\right)^2 + 2\frac{1}{2}$ and this is already covered in 1 above.

In order to prove the needed estimate on $\mathcal{R}_N(O_{\mathbb{CP}^2}(1))$ we have to show that the vector $v = (2a^2l + 1; 2a^{x(a^2l^2+l)}, 1)$ is nef. Let us prove a slightly stronger statement, namely:

Claim. The vector $v_0 = (2a^2l + 1; 2a^{x(a^2l^2+l)}, 1)$ is nef.

We argue by induction on $k$. Consider first the case $k = 0$. We have $v_0 = w \#_i u$, where

$$w = \left( 2a^2l + 1; 2a(al-1), 2a^{x_{2al+l-1}}, 1 \right), \quad u = 2a\left( al - 1; 1^{x(al-1)^2} \right).$$

The latter being nef, we are reduced to proving nefness of $w$. Applying suitable Cremona transformation to $w$, we obtain the vector

$$w' = \left( \frac{l+1}{2} + a; \frac{l-1}{2} - a, 1^{x_{2al+l}} \right).$$

Since $l > 2b = 2a$ we have that $\frac{l+1}{2} - a \geq 0$ and so $w'$ is nef by Lemma 4.2.A. This completes the basis of the induction.
Let us turn now to the case $k > 0$. We have
\[ v_0 = \left( (v_1 \#_1 u_1) \#_2 u_1 \right) \#_1 u_1, \]
where
\[ v_1 = \left( 2a^2 l + 1; (a^2 l)^3, 2a \times (\frac{a}{2} l)^2 + l, 1 \right), \quad u_1 = 2a(\frac{a}{2} l; 1 \times (\frac{a}{2} l)^2). \]
Again, $u_1$ is nef. As for $v_1$, it lies in the same orbit under the Cremona action as the vector $v'_1 = \left( (a^2 l + 2; 2a \times (\frac{a}{2} l)^2 + l, 1 \times 4 \right)$. Consider now the decomposition $v'_1 = v_2 \# (2; 1 \times 4)$, where $v_2 = (a^2 l + 2; 2a \times (\frac{a}{2} l)^2 + l, 2)$ and $\#$ stands for gluing at the last coordinate of $v_2$. As $(2; 1 \times 4)$ is nef, it is enough to prove that $v_2$ is nef. Putting $c = \frac{a}{2} = 2k^{-1}b$, we have that
\[ v_2 = 2(2c^2 l + 1; 2c \times c^2 l + l, 1). \]
By the induction hypothesis $v_2$ is nef. This completes the proof of the claim. The Theorem now follows easily.

Let us turn now to the proof of Corollary 2.1.B.

**Proof.** Let $D = \pi^* \mathcal{O}_{CP^2}(d) - 2 \sum_{j=1}^{N} E_j$ and suppose that $D \cdot D \geq 0$.

**Step 1.** Consider first the case $N = k^2 + k$ for some $k$. By Theorem 2.1.A-3
\[ R_N(\mathcal{O}_{CP^2}(1)) \leq \frac{1}{(2k + 1)^2}. \]
Since $D \cdot D = 1$, this implies that $D$ is nef.

**Step 2.** Consider the general case. The condition $D \cdot D \geq 0$ reads $d^2 \geq 4N$. We may assume that $d$ is odd, for the case of $d$ even is precisely the contents of Xu’s theorem from Section 1 (see [Xu]). Writing $d = 2k + 1$, the condition $d^2 \geq 4N$ gives $k + k > N$. By step 1, $\pi^* \mathcal{O}_{CP^2}(d) - 2 \sum_{j=1}^{k^2 + k} E_j$ is nef, hence also $\pi^* \mathcal{O}_{CP^2}(d) - 2 \sum_{j=1}^{N} E_j$.

**Remark.** More careful considerations, in the spirit of the proof of Theorem 2.1.A actually show that when $d > 5$, the divisor $D = \pi^* \mathcal{O}_{CP^2}(d) - 2 \sum_{j=1}^{N} E_j$ is ample iff $D \cdot D > 0$.

To prove this one has to sharpen first the second statement of Lemma 4.2.A and prove that $(d; m_1, m_2, 1 \times r)$ is ample when $d > m_1 + m_2$ and $d^2 > m_1^2 + m_2^2 + r$. This can be done by similar, though more delicate, arguments to those used to prove nefness of these vectors. Then, using the “ample+nef ⇒ ample” case of Theorem 2.A one deduces as in the proof of Theorem 2.1.A that the divisor $D$ is ample for $N = k^2 + k$, when $k > 2$. The case of general $N$ can be easily reduced to $N = k^2 + k$ as in the preceding proof.

4.4. **Proof of Corollary 2.1.C.**

**Proof.** Let $N = N_1N_2$. Nagata’s conjecture for $N$ is equivalent to the nefness of vector $v = (d; m^N)$ for every $d, m > 0$ which satisfy $d^2 - Nm^2 > 0$.

Let $d, m$ be two such numbers. Choose a positive rational number $x$ such that $d^2 > x^2 N_2 > m_2^2 N$. The assumption of Nagata’s conjecture for $N_1$ and $N_2$ implies that the vectors $u = (x; m_1^N)$ \( \in \mathbb{Q}^{N_{1}+1} \) and $w = (d; x^N) \in \mathbb{Q}^{N_{2}+1}$ are nef.

We have $v = \ldots ((w \#_{u, u}) \#_{u, u}) \ldots \#_{u, u}$. Observing that Theorem 2.B remains valid also for vectors of rational numbers, we conclude that $v$ is also nef.
5. A conjecture relating continued fractions and remainders of a line bundle

The goal of this section is to propose a conjecture concerning estimates on the values of the homogeneous remainders of $\mathcal{O}_{\mathbb{P}^2}(1)$, defined in Section 2.1. It turns out that all the cases appearing in the statement of Theorem 2.1.A are particular cases of this conjecture.

Let us first recall some relevant facts from classical number theory. Given a square-free number $N$, consider the following Diophantine equation in the unknowns $d, m$

$$d^2 - Nm^2 = 1.$$ 

This equation had been attached to the name Pell’s equation in the ancient literature and has been extensively studied by many mathematicians in the 17th and 18th centuries including Leonard Euler (see [Niv, Ir-Ro, VndP]). The classical result about the solutions of this equation is that all solutions come from continued fractions expansions of $\sqrt{N}$. Let us write \(d,m\) as follows: if $n$ is even put

$$d = \langle a_0, a_1, \ldots, a_{n-1} \rangle,$$

while for $n$ odd

$$d = \langle a_0, a_1, \ldots, a_{n-1}, 2a_0, a_1, \ldots, a_{n-1} \rangle.$$ 

It is well known that $(d, m)$ provides the minimal solution of Pell’s equation, called the fundamental solution. Moreover, any other solution of Pell’s equation is obtained in a similar manner – by truncating the infinite continued fraction of $\sqrt{N}$ one term before the end of one of its periods. More precisely, $(d, m)$ solves Pell’s equation iff

$$d = \langle a_0, a_1, \ldots, a_{n-1}, 2a_0, a_1, \ldots, a_{n-1} \rangle,$$

where $r$ is odd if $n$ is odd.

This formula means that the periodic part $a_1, \ldots, a_{n-1}, 2a_0$ should be taken $r$ times and then once more without the last member $2a_0$. The number $r$ is allowed to be any non-negative integer in case $n$ is even, and $r$ must be odd if $n$ is odd.

Our conjecture is the following
Applying this process enough times we finally arrive to the vector \((1; 0)\) we obtain again the vector \((1; 0)\).

Consider the case that \(a \times u\) with \(u\) is defined in 4.1.

Thus, we are reduced to proving that \(v\) by Lemma 4.2.A the vector \(u\) is nef. By Theorem 2.1.A our conjecture holds in this case.

Theorem 2.1.A our conjecture holds.

2. Consider \(N\)’s of the form \(N = a^2l^2 - 2l\). It is not hard to see that \(d = a^2l - 1, m = a\) satisfy Pell’s equation \(d^2 - Nm^2 = 1\). By Theorem 2.1.A we have \(\mathcal{R}_N(O_{CP^2}(1)) \leq \frac{1}{d^2}\). Therefore, if \((d', m')\) is the fundamental solution of Pell’s equation then \(\mathcal{R}_N(O_{CP^2}(1)) \leq \frac{1}{d^2} \leq \frac{1}{d'^2}\), and so the conjecture holds. Note that in this case the expansion of \(\sqrt{N}\) will usually be longer than 2 (example: \(\sqrt{14} = (3, 1, 2, 1, 6))\).

3. Let us mention two other examples which do not fall into the above categories.

A) \(N = 19\). In this case \(\sqrt{19} = (4, 2, 1, 3, 1, 2, 8)\). The fundamental solution is \(d = 170, m = 39\). Thus, our conjecture suggests that \(R_{19}(O_{CP^2}(1)) \leq \frac{1}{170^2}\).

B) \(N = 22\). In this case \(\sqrt{22} = (4, 1, 2, 4, 2, 1, 8)\). The fundamental solution is \(d = 197, m = 42\). Thus, our conjecture suggests that \(R_{22} \leq \frac{1}{197^2}\).

Let us prove that the conjecture indeed holds in the cases 3.A and 3.B. We start with \(N = 19\). By Lemma 4.2.A the vector \(u = 39(2; 1^x) = (78; 39^x)\) is nef. We have

\[(170; 39^{19}) = (((170; 78^3, 39^7)\#_3 u)\#_2 u)\#_1 u.\]

Thus, we are reduced to proving that \(v = (170; 78^3, 39^7)\) is nef. To do this we apply the following Cremona transformations successively:

1) Replace \(v\) by \(R_{123}(v)\).

2) Sort the vector \(v\), that is, replace \(v\) by \(SR(v)\), where the transformation \(SR\) is the one defined in 4.1.

Applying this process enough times we finally arrive to the vector \((1; 0^{10})\) which is nef.

The case \(N = 22\) is similar. Here we use the decomposition

\[(197; 42^{22}) = (((197; 84^{4}, 42^6)\#_4 u)\#_3 u)\#_2 u)\#_1 u,\]

with \(u = 42(2; 1^x) = (84; 42^4)\). Applying the preceding process successively to \((197; 84^4, 42^6)\) we obtain again the vector \((1; 0^{10})\) which is nef.

\[\text{It is not hard to see that if } N = a^{2}l^2 - 2l \text{ then } \sqrt{N} \text{ has } 2\text{-periodic expansion with minus signs.}\]
6. The limits of the algorithm

In its present version, the algorithm described in Section 4.2 has the disadvantage that it does not tell which decomposition \( v' = v_{j+1} \# u_{j+1} \) one should choose at each stage in order the whole process to end successfully. We would like to emphasize that this decision is sometimes crucial as the following example shows:

Let \( v_0 = (10; 3 \times 11) \in H_{11} \). If one tries to decompose \( v_0 \) as \( v_0 = (10; 3 \times 2, 9) \# (9; 3 \times 9) \) the process will fail to give any information on \( v_0 \). The reason is that although \((9; 3 \times 9)\) is nef \((10; 3 \times 2, 9)\) is not, and so we cannot apply the gluing theorem. However, the decomposition \( v_0 = (10; 3 \times 7, 6) \# (6; 3^4) \) will eventually lead to a successful ending of the algorithm, thus proving that \( v_0 \) is nef. It would be useful of course to find a rule for choosing the “optimal” decomposition at each stage.

Finally, let us mention one simple example for which it seems that the algorithm fails to give information always. Consider the vector \( v_0 = (19, 6 \times 10) \) which by Nagata’s conjecture should be ample. However, it seems that the vector \( v_0 \) is indecomposable in the sense that it is impossible to find even nef vectors \( v_1 \in H_{n_1}, u_1 \in H_{k_1} \) with \( n_1, k_1 < 10 \), such that \( v_0 = v_1 \# v_2 \) lies in the same orbit as \( v_0 \) under the Cremona action. It would be interesting to find the precise conditions for an ample (resp. nef) vector \( v \) to be indecomposable.

7. Symplectic interpretations

The purpose of this section is to explain the intuition which give rise to the gluing Theorem 2.B. Interestingly enough this comes from symplectic geometry.

Symplectic geometry is the branch of geometry dealing with the structure of symplectic manifolds which are by definition pairs, \((M, \Omega)\), consisting of a smooth manifold \(M\) and a non-degenerated closed differential 2-form \(\Omega\). The reader is referred to [A-G] and [M-S] for the foundations.

Due to developments in this field of research in the last decade, many analogies has been discovered between symplectic and complex manifolds. These become especially striking in dimension 4, where symplectic 4-manifolds play the role of complex surfaces. In several cases it turned out that algebro-geometric considerations, remain true when properly translated into the symplectic category, and so gave rise to new theorems in the symplectic framework. This principle is reflected very well in the classification of rational and ruled symplectic manifolds of Lalonde and McDuff, in the symplectic packing theorems of McDuff and Polterovich, in Ruan’s symplectization of the extremal rays theory etc.

In this paper we have, in some sense, reversed this direction of reasoning. Our main theorem is in fact an algebro-geometric translation of a very simple symplectic fact arising from the theory of symplectic packing. We refer the reader to [M-P] for an excellent exposition on the symplectic packing problem.

Recall from [M-P] that a symplectic packing of \((M, \Omega)\) by \(N\) balls of radii \(\lambda_1, \ldots, \lambda_N\) is a symplectic embedding

\[
\varphi = \prod_{j=1}^{N} \varphi_j : \prod_{j=1}^{N} B(\lambda_j) \to (M, \Omega),
\]

where \(B(\lambda_j)\) stands for the standard Euclidean closed ball of radius \(\lambda_j\) of the same dimension as \(M\), endowed with its standard symplectic structure \(\omega_{std} = dx_1 \wedge dy_1 + \ldots + dx_n \wedge dy_n\).

It was discovered by McDuff that every symplectic packing gives rise to a symplectic form \(\tilde{\Omega}\) on the blow-up \(\Theta : \tilde{M} \to M\) of \(M\) at the points \(p_1 = \varphi_1(0), \ldots, p_N = \varphi_N(0)\). This form lies in
the cohomology class
\[ [\tilde{\Omega}] = [\Theta^* \Omega] - \pi \sum_{j=1}^{N} \lambda_j^2 e_j, \tag{1} \]
where \( e_j \) denotes the Poincaré dual to the homology class of the exceptional divisor \( E_j \) of the blow-up. This procedure is called \textit{symplectic blowing-up}.

Conversely, given a symplectic form \( \Omega \) on \( M \) which is non-degenerated on the exceptional divisors \( E_j \) and with cohomology class as in (1) above, one can perform \textit{symplectic blowing-down} at the exceptional divisors and obtain a symplectic form \( \tilde{\Omega} \) on \( \tilde{M} \) and a symplectic packing \( \varphi = \prod_{j=1}^{N} \varphi_j : \prod_{j=1}^{N} B(\lambda_j) \to (M, \Omega) \).

Consider the symplectic manifold \( (\mathbb{C}P^2, \sigma) \) where \( \sigma \) is the Fubini-Studi Kähler form normalized such that the area of a projective line is \( \pi \). Its cohomology class is \( \pi l \), where \( l \in H^2(\mathbb{C}P^2, \mathbb{Z}) \) is the standard positive generator.

Call a vector of positive numbers \( (d;m_1, \ldots, m_k) \) \textit{symplectic} if the cohomology class
\[ d\Theta^*_Vl - \sum_{j=1}^{k} m_je_j \]
can be represented by a symplectic form \( \tilde{\omega} \) on some blow-up \( \Theta_V : V \to \mathbb{C}P^2 \) of \( \mathbb{C}P^2 \) at some \( k \) distinct points, in such a way that \( \tilde{\omega} \) is non-degenerated on the exceptional divisors.

Now, let \( M \) be a complex surface and \( \Theta_p : \tilde{M}_p \to M \) its blow-up at the point \( p \in M \) with exceptional divisor \( E \). Denote by \( e \) the Poincaré dual to the homology class of \( E \).

**Proposition 7.A.** Let \( a \in H^2(M) \) and suppose that there exists a positive number \( m \) such that the cohomology class \( \Theta^*_p a - me \in H^2(\tilde{M}_p) \) can be represented by a symplectic form whose restriction to \( E \) is non-degenerated.\(^5\) Then, for every symplectic vector \( (m; \alpha_1, \ldots, \alpha_k) \), the cohomology class \( \Theta^*_p a - \sum_{j=1}^{k} \alpha_j e_j \) on the blow-up \( \Theta : \tilde{M} \to M \) at some \( k \) points can be represented by a symplectic form.

The proof is based on the following very simple observation. If \( \Theta^*_p a - me \) has a symplectic representative \( \tilde{\Omega} \), then by symplectic blowing down one obtains a symplectic form \( \Omega \) on \( M \) and an embedding \( \varphi \) of a standard 4-dimensional ball of radius \( \sqrt{\frac{m}{\pi}} \) into \((M, \Omega)\). The same argument with slight modifications, applied to the vector \((m; \alpha_1, \ldots, \alpha_k)\), implies that the standard ball of radius \( \sqrt{\frac{1}{\pi}} \) admits a symplectic packing, say \( \phi \), by \( k \) balls of radii \( \sqrt{\frac{\alpha_1}{\pi}}, \ldots, \sqrt{\frac{\alpha_k}{\pi}} \). Composing these two embeddings we conclude that \((M, \Omega)\) admits a symplectic packing \( \varphi \circ \phi \) of \( k \) balls of radii \( \sqrt{\frac{\alpha_1}{\pi}}, \ldots, \sqrt{\frac{\alpha_k}{\pi}} \). The proposition follows now from symplectic blowing-up. For completeness, here are the precise arguments of the proof.

**Proof.** Let \( \tilde{\Omega} \) be a symplectic form on \( \tilde{M}_p \) lying in the cohomology class \( \Theta^*_p a - me \) and suppose that the restriction of \( \tilde{\Omega} \) to \( E \) is non-degenerated. Applying symplectic blowing-down to \( \tilde{\Omega} \) we obtain a symplectic form \( \Omega \) on \( M \) lying in the cohomology class \( a \) and a symplectic embedding \( \varphi : B(\sqrt{\frac{1}{\pi}}) \to (M, \Omega) \).

Let \( \tilde{\omega} \) be a symplectic form on the blow-up \( \pi : V \to \mathbb{C}P^2 \) of \( \mathbb{C}P^2 \) lying in the cohomology class \( m\Theta^*_V l - \sum_{j=1}^{k} \alpha_j e_j \) and whose restriction to the exceptional divisors \( E_j \) is non-degenerated. Blowing-down symplectically we obtain a symplectic form \( \omega \) on \( \mathbb{C}P^2 \) lying in the cohomology class \( ml \) and a symplectic packing \( \psi : \prod_{j=1}^{k} B(\sqrt{\frac{\alpha_j}{\pi}}) \to (\mathbb{C}P^2, \omega) \). Since any two cohomologous

\(^5\)This means that \( E \) is a symplectic submanifold with respect to this form.
symplectic forms on $\mathbb{C}P^2$ are symplectomorphic we may assume that $\omega = \frac{m}{n} \sigma$. It can be proved by the methods of [M-P] that there exists a symplectic submanifold (with respect to $\omega$) $L \subset M$, homologous to a projective line, which is disjoint from Image $\psi$. It is well known that $(\mathbb{C}P^2 \setminus L, \frac{m}{n} \sigma) \approx B(\sqrt{\frac{m}{n}})$. We thus obtain a symplectic packing $\phi : \prod_{j=1}^k B(\sqrt{\frac{m}{n}}) \to B(\sqrt{\frac{m}{n}})$. The composition $\varphi \circ \phi$ is a symplectic packing of $(M, \Omega)$ by $k$ balls of radii $\sqrt{\frac{m}{n}}, \ldots, \sqrt{\frac{m}{n}}$. Blowing-up symplectically with respect to this embedding yields a symplectic form on the blow-up $\Theta : \tilde{M} \to M$ of $M$ at $k$ points, which lies in the cohomology class $\Theta^*a - \sum_{j=1}^k \alpha_j e_j$. 

Let us try to translate Proposition 7.A to the language of algebraic geometry. Keeping in mind that in the symplectic category the role of Kähler forms is played by symplectic forms and the role of complex submanifolds by symplectic submanifolds, the Kählerian translation should read: "If the cohomology class $\Theta^*a$ me has a Kähler representative than for every Kähler vector $(m; \alpha_1, \ldots, \alpha_k)$ the cohomology class $\Theta^*a - \sum_{j=1}^k \alpha_j e_j$ has a Kähler representative too".

Here, we call a vector $(m; \alpha_1, \ldots, \alpha_k)$ Kähler if the cohomology class

$$m\Theta_V^*l = \sum_{j=1}^k \alpha_j e_j$$

can be represented by a Kähler form $\tilde{\omega}$ on some simple rational surface $\Theta_V : V \to \mathbb{C}P^2$ obtained by blowing-up $\mathbb{C}P^2$ at some $k$ distinct points.

Due to Lefschetz theorem on $(1, 1)$ classes and Kodaira’s embedding theorem it follows that on a complex manifold there is a bijection – via Poincaré duality – between the set of homology classes of ample $\mathbb{Q}$-divisors and the set of rational cohomology classes which can be represented by Kähler forms. Poincaré dualizing the “Kählerian translation” we are naturally led to the following: “Let $D$ be a divisor on $M$ such that $\Theta_p^*D - mE$ is ample. Then for every ample vector $(m; \alpha_1, \ldots, \alpha_k)$ the divisor $\Theta^*D - \sum_{j=1}^k \alpha_j E_j$ is ample too”. This is precisely the contents of Theorem 2.B for the case that $M$ is a simple rational surface. The technical machinery which made the whole translation rigorous is Shustin’s curve gluing technique which we used in Section 3.

We would like to emphasize that the same “symplectic reasoning” suggests that if we replace the surface $S$ in the statement of Theorem 2.B by any projective surface, the Theorem should remain true. Similar symplectic arguments suggest that an appropriate version of Theorem 2.B should hold also for higher dimensions than 2. It would be of course interesting to know whether Theorem 2.B continues to hold for smooth algebraic surfaces over an arbitrary algebraically closed field. We leave these discussions to another opportunity.

7.1. Symplectic meaning of the remainders $\mathcal{R}_N(\mathcal{L})$. In section 2.1 we have defined the homogeneous remainders $\mathcal{R}_N(\mathcal{L})$ of an ample line bundle $\mathcal{L}$ over a surface. The definition naturally extends to $n$-dimensional smooth varieties $X$ in the following obvious way. Given $p_1, \ldots, p_N \in X$, set

$$\mathcal{R}(\mathcal{L}, p_1, \ldots, p_N) = \frac{1}{\mathcal{L}^N} \inf_{0 \leq \epsilon \leq \epsilon} \left\{ \mathcal{L}_{\epsilon}^N \mid \mathcal{L}_{\epsilon} = \pi^* \mathcal{L} - \epsilon \sum_{j=1}^N E_j \text{ is nef} \right\},$$

where $\pi : \tilde{X} \to X$ is the blow-up of $X$ at the points $p_1, \ldots, p_N$ with exceptional divisors $E_i = \pi^{-1}(p_i)$. To get a more global invariant, define

$$\mathcal{R}_N(\mathcal{L}) = \inf \{ \mathcal{R}(\mathcal{L}, p_1, \ldots, p_N) \mid p_1, \ldots, p_N \in X \text{ are distinct points} \}.$$
Let us explain now the symplectic meaning of these constants. Let \((M, \Omega)\) be a symplectic manifold. Following McDuff and Polterovich define the following quantity

\[
v_N(M, \Omega) = \sup_{\phi, \lambda} \frac{\text{Vol(Image } \phi)}{\text{Vol}(M, \Omega)},
\]

where \((\phi, \lambda)\) passes over all the possible symplectic packings \(\phi\) of \((M, \Omega)\) with \(N\) equal balls of varying radius \(\lambda\). The volume of the manifolds is defined to be \(\text{Vol}(M, \Omega) = \int_M \frac{1}{n!} \Omega^\wedge n\).

The constants \(v_N(M, \Omega)\) admit values between 0 and 1 and measure the maximal part of the volume of \((M, \Omega)\) which can be filled by symplectic packing with \(N\) equal balls. When \(v_N = 1\) we say that there exists a full packing of \((M, \Omega)\) by \(N\) equal balls, while in the case \(v_N < 1\) we say that there exists a packing obstruction.

In view of the preceding discussion it is easy to see that the homogeneous remainders \(R_N(L)\) of an ample line bundle over a complex manifold \(M\), play the algebro-geometric role of the quantity \(1 - v_N(M, \Omega)\), where \(\Omega\) is a Kähler form representing the first Chern class of \(L\), \(c_1(L)\). In fact, it is not hard to prove that the following inequality holds

\[
1 - v_N(M, \Omega) \leq R_N(L).
\]

(Note that there are cases in which one always has equality in (2). For example, it follows from the work of McDuff and Polterovich (see [M-P]) that this is the case for \(\mathbb{C}P^2\) when \(N < 9\). The point is that the symplectic cone and the Kähler cone of del Pezzo surfaces coincide. Note that in (real) dimension 4, more is known about the constants \(v_N\) than about \(R_N\) (see [Bi]). It would be interesting to know whether there exist cases in which a strict inequality occurs in (2).

Let us conclude by pointing out another interesting approach to bounding Seshadri constants via symplectic packing, due to Lazarsfeld (see [Laz]). The idea is that given a Kähler form \(\Omega\) on a complex manifold and a symplectic packing \(\varphi\) which is also holomorphic, the symplectic blow-up of \(\Omega\) associated to \(\varphi\) will be Kähler. This situation happens when the associated Kähler metric on the image of \(\varphi\) is flat. Applying this to the case of a principally polarized abelian variety, Lazarsfeld obtains non-trivial estimates on Seshadri constants of the corresponding ample divisor.

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