DISTANCE-SPARSITY TRANSFERENCE FOR VERTICES OF CORNER POLYHEDRA

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Abstract.
We obtain a transference bound for vertices of corner polyhedra that connects two well-established areas of research: proximity and sparsity of solutions to integer programs. In the knapsack scenario, it implies that for any vertex \( x^* \) of an integer feasible knapsack polytope \( P(a, b) = \{ x \in \mathbb{R}_+^n : a^T x = b \} \), \( a \in \mathbb{Z}_+^n \), there exists an integer point \( z^* \in P(a, b) \) such that, denoting by \( s \) the size of support of \( z^* \) and assuming \( s > 0 \),
\[
\| x^* - z^* \|_\infty \frac{2^{s-1}}{s} < \| a \|_\infty,
\]
where \( \| \cdot \|_\infty \) stands for the \( \ell_\infty \)-norm. The bound (1) gives an exponential in \( s \) improvement on previously known proximity estimates. In addition, for general integer linear programs we obtain a resembling result that connects the minimum absolute nonzero entry of an optimal solution with the size of its support.

1. Introduction and Statement of Results
The main contribution of this paper shows a surprising relation that holds between two well-established areas of research, proximity and sparsity of solutions to integer programs, in the case of Gomory’s corner polyhedra.

The proximity-type results provide estimates for the distance between optimal vertex solutions of linear programming relaxations and feasible integer points, with seminal works by Cook et al. [11] and, more recently, by Eisenbrand and Weismantel [13]. The sparsity-type results, in their turn, provide bounds for the size of support of feasible integer points and solutions to integer programs. Bounds of this type are dated back to the classical integer Carathéodory theorems of Cook, Fonlupt and Schriver [10] and Sebő [17]. More recent contributions include results of Eisenbrand and Shmonin [12] and Aliev et al. [3, 2, 1]. Further, in a very recent work Lee, Paat, Stallknecht and Xu [15] apply new sparsity-type bounds to refine the bounds for proximity.

To state the main results of this paper, we will need the following notation. Let \( A \in \mathbb{Z}^{m \times n} \), \( m < n \), and let \( \tau = \{ i_1, \ldots, i_k \} \subseteq \{ 1, \ldots, n \} \) with \( i_1 < i_2 < \cdots < i_k \). We will use the notation \( A_\tau \) for the \( m \times k \) submatrix of \( A \) with columns indexed by \( \tau \). In the same manner, given \( x \in \mathbb{R}^n \), we will denote by \( x_\tau \) the vector \(( x_{i_1}, \ldots, x_{i_k} )\). The complement of \( \tau \) in \( \{ 1, \ldots, n \} \) will be denoted as \( \bar{\tau} \). We will say that \( \tau \) is a basis of \( A \) if \( |\tau| = m \) and the submatrix \( A_\tau \) is nonsingular. By \( \Sigma(A) \) we will denote the maximum absolute \( m \times m \) subdeterminant of \( A \):
\[
\Sigma(A) = \max \{| \det(A_\tau) | : \tau \subset \{ 1, \ldots, n \} \text{ with } |\tau| = m \}.
\]
When \( \Sigma(A) \) is positive, \( \gcd(A) \) will denote the greatest common divisor of all \( m \times m \) subdeterminants of \( A \).

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For $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$, the $\ell_\infty$-norm of $x$ will be denoted as $\|x\|_\infty$. We will denote by supp$(x) = \{i : x_i \neq 0\}$ the support of $x$. Further, $\|x\|_0 := |\text{supp}(x)|$ will denote the 0-"norm", widely used in the theory of compressed sensing [8, 6], which counts the cardinality of the support of $x$.

Let $A \in \mathbb{Z}^{m \times n}$ with $m < n$ and $b \in \mathbb{Z}^m$. We will consider the polyhedron
$$P(A, b) = \{x \in \mathbb{R}^n_{\geq 0} : Ax = b\}$$
and take any vertex $x^*$ of $P(A, b)$. Without loss of generality, we may assume that
$$A = (A_\gamma, A_\bar{\gamma}) \in \mathbb{Z}^{m \times n}$$
where $A_\gamma$ is nonsingular. Let
$$x_\gamma^* = A_\gamma^{-1}b$$
and $x_\bar{\gamma}^* = 0$.

In general, there can be many choices for the basis $\gamma$. However, if $x^*$ is nondegenerate; that is, if the size of the support of $x^*$ is $m$, then there is a unique choice for $\gamma$, namely $\gamma = \text{supp}(x)$. For convenience, throughout this paper we will assume, given a choice for $\gamma$, that
$$\gamma = \{1, \ldots, m\}.$$ 

For a set $S \subset \mathbb{R}^n$ we will denote by conv$(S)$ the convex hull of $S$. Gomory [14] introduced the corner polyhedron $C_\gamma(A, b)$ associated with $\gamma$ as
$$C_\gamma(A, b) = \text{conv}((x \in \mathbb{Z}^n : Ax = b, x_\gamma \geq 0)).$$

**Theorem 1.** Suppose that $A$ satisfies (3), $b \in \mathbb{Z}^m$, $x^*$ is given by (3) and the corner polyhedron $C_\gamma(A, b)$ is nonempty. Let $z^*$ be a vertex of $C_\gamma(A, b)$ and let $r = \|z_\gamma^*\|_0$. Then
$$x^* = z^* \text{ if } r = 0,$$
$$\|x^* - z^*\|_\infty \leq \frac{\Sigma(A)}{\gcd(A)} - 1 \text{ if } r = 1 \text{ and}$$
$$\|x^* - z^*\|_\infty \leq \frac{2r}{r} \frac{\Sigma(A)}{\gcd(A)} \text{ if } r \geq 2.$$

The bounds (4) and (6) are optimal. Specifically, (6) is attained already in the knapsack scenario (with the choice of parameters (22)). The bound (4), in its turn, is attained for

$$A = \begin{pmatrix} 2 & 0 & 5 & 5 \\ 0 & 4 & 2 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 20 \\ 3 \end{pmatrix}.$$

The vertex $x^* = (10, 3/4, 0, 0)^T$ of $P(A, b)$. For this choice of parameters the corner polyhedron $C_\gamma(A, b)$ has the unique vertex $z^* = (0, 1, 1, 3)^T$.

Theorem 1 shows that for the corner polyhedron associated with a vertex $x^*$ of $P(A, b)$ a strong proximity-sparsity transference holds: the distance from $x^*$ to any vertex $z^*$ of the corner polyhedron exponentially drops with the size of support of $z^*$ and, vice versa, the size of support of $z^*$ reduces with the growth of its distance to $x^*$.

Suppose next that the polyhedron $P(A, b)$ is integer feasible and consider its integer hull $P_I = \text{conv}(P(A, b) \cap \mathbb{Z}^n)$. A natural direction for a further research would be to derive a distance-sparsity transference bound for the vertices of $P_I$. Notice that the set $P(A, b) \cap \mathbb{Z}^n$ is obtained from $C_\gamma(A, b) \cap \mathbb{Z}^n$ by enforcing back the nonnegativity constraints $x_\gamma \geq 0$ and this may potentially result in cutting off all vertices of the corner polyhedron. In Section 2, we show that in the knapsack scenario at least one vertex of $C_\gamma(A, b)$ avoids the cut and
Theorem \[1\] implies an optimal distance-sparsity transference bound for lattice points in the knapsack polytope.

Although it remains an open problem to extend Theorem \[1\] to vertices of \( P_I \) in the general setting, the next result of this paper allows enforcing back the constraints \( x_\tau \geq 0 \) that are tight at \( x^* \) when \( \tau := \text{supp}(x^*) \) has size strictly less than \( m \). In this situation, following \[18\], we may define the polyhedron

\[
C_\tau(A, b) = \text{conv}(\{ x \in \mathbb{Z}^n : Ax = b, x_\tau \geq 0 \}).
\]

In this setting the choice of basis \( \gamma \) in (\[2\]) is typically not unique. However, we show is that there exists at least one basis \( \gamma \) for which the conclusions of Theorem \[1\] remain valid for this polyhedron, up to a factor which depends on the number of zero coordinates of \( x^*_\tau \).

**Theorem 2.** Suppose that \( A \) satisfies (\[2\]) and \( b \in \mathbb{Z}^m \). Let \( x^* \) be given by (\[3\]) and let \( \tau = \text{supp}(x^*) \). Let \( z^* \) be a vertex of the polyhedron \( C_\tau(A, b) \). Then the basis \( \gamma \) in (\[2\]) may be chosen so that, letting \( r = \|z^*_\tau\|_0 \) and \( d = m - |\tau| \), we have

\[
\|x^* - z^*\|_\infty = 1 \quad \text{if} \quad r = 0,
\]

\[
\|x^* - z^*\|_\infty \leq \frac{\Sigma(A)}{\gcd(A)} - 1 \quad \text{if} \quad r = 1 \quad \text{and}
\]

\[
\|x^* - z^*\|_\infty \leq \frac{2r}{r^d + 1} \leq \frac{\Sigma(A)}{\gcd(A)} \quad \text{if} \quad r \geq 2.
\]

We remark that integer programs of the form \( Ax = b, x_\tau \geq 0, x \in \mathbb{Z}^n \), where \( \tau \) is the support of a vertex of \( P(A, b) \), have been investigated in \[18\]. Such an integer program is called a Gomory relaxation with respect to \( \tau \). See in particular \[18\] §2 for more details.

### 1.1. Distance-sparsity transference for knapsacks.

We will now separately consider the case \( A \in \mathbb{Z}_{>0}^{1 \times n} \), known as knapsack scenario. We will follow a traditional vector notation and replace \( A \) and \( b \) with a positive integer vector \( a = (a_1, \ldots, a_n)^\top \in \mathbb{Z}_{>0}^n \) and integer \( b \in \mathbb{Z} \). In this setting \( P(A, b) \) is referred to as the knapsack polytope

\[
P(a, b) = \{ x \in \mathbb{R}_{\geq 0}^n : a^\top x = b \}.
\]

In what follows, we will exclude the trivial case \( n = 1 \) and assume that \( n \geq 2 \). We also assume that the polytope \( P(a, b) \) contains integer points. Equivalently, \( b \) belongs to the semigroup

\[
Sg(a) = \{ a^\top z : z \in \mathbb{Z}_{\geq 0}^n \}
\]

generated by the entries of the vector \( a \). Note that any element of the semigroup \( Sg(a) \) must be divisible by the greatest common divisor \( \gcd(a_1, \ldots, a_n) \) of \( a_1, \ldots, a_n \). Hence, we may assume without loss of generality that \( a \) satisfies the following conditions:

\[
a = (a_1, \ldots, a_n)^\top \in \mathbb{Z}_{\geq 0}^n, n \geq 2, \quad \text{and} \quad \gcd(a_1, \ldots, a_n) = 1.
\]

Aliev et al. \[4\] Theorem 1] proved that for any vertex \( x^* \) of the polytope \( P(a, b) \) there exists an integer point \( z \in P(a, b) \) such that

\[
\|x^* - z\|_\infty \leq \|a\|_\infty - 1
\]

and that the bound (\[11\]) is sharp in the following sense. For any positive integer \( k \) and any dimension \( n \) there exist \( a \) satisfying (\[10\]) with \( \|a\|_\infty = k \) and \( b \in \mathbb{Z} \) such that the knapsack polytope \( P(a, b) \) contains exactly one integer point \( z \) and \( \|x^* - z\|_\infty = \|a\|_\infty - 1 \).
Theorem 3. Let \( z \in \{0, 1\}^n \) satisfy (10), if integer points are sufficiently far from a vertex of the knapsack polytope. Theorem 3 can be viewed as a transference result that allows strengthening the distance bound (12) if feasible integer points are sufficiently far from a vertex of the knapsack polytope. The next result will combine and refine the bounds (11) and (12) as follows. The knapsack polytope \( P(a, b) \) is an \((n-1)\)-dimensional simplex in \( \mathbb{R}^n \) with vertices \( (b/a_1)e_1, \ldots, (b/a_n)e_n \), where \( e_i \) denotes the \( i \)-th standard basis vector. Hence, without loss of generality, we may assume that the vertex \( x^* \) has the form

\[
x^* = \frac{b}{a_1} e_1.
\]

The corner polyhedron associated with the vertex \( x^* \) (with \( \gamma = \{1\} \)) can be written as

\[
C_\gamma(a, b) = \text{conv} \{ x \in \mathbb{Z}^n : a^\top x = b, x_2 \geq 0, \ldots, x_n \geq 0 \}.
\]

Theorem 3. Let \( a \) satisfy (11), \( b \in Sg(a) \) and \( x^* \) is given by (13). Then \( P(a, b) \) contains a vertex \( z^* \) of \( C_\gamma(a, b) \) with \( r = \| (z^*_2, \ldots, z^*_n)^\top \|_0 \) such that

\[
x^* = z^* \text{ if } r = 0,
\]

\[
\| x^* - z^* \|_\infty \leq \| a \|_\infty - 1 \text{ if } r = 1 \text{ and }
\]

\[
\| x^* - z^* \|_\infty \leq \frac{2r}{r} \| a \|_\infty \text{ if } r \geq 2.
\]

Theorem 3 can be viewed as a transference result that allows strengthening the distance bound (11) if integer points in the knapsack polytope are not sparse and, vice versa, strengthening the sparsity bound (12) if feasible integer points are sufficiently far from a vertex of the knapsack polytope.

Given a cost vector \( c \in \mathbb{Z}^n \), we will now consider the integer knapsack problem

\[
\min \{ c^\top x : x \in P(a, b) \cap \mathbb{Z}^n \}.
\]

Note that (17) is feasible since \( b \in Sg(a) \).

Let \( IP(c, a, b) \) and \( LP(c, a, b) \) denote the optimal values of (17) and its linear programming relaxation

\[
\min \{ c^\top x : x \in P(a, b) \},
\]

respectively. The \textit{integrality gap} \( IG(c, a, b) \) of (17) is defined as

\[
IG(c, a, b) = IP(c, a, b) - LP(c, a, b).
\]

As a corollary of Theorem 3, we obtain the following bound for the integrality gap.

\textbf{Corollary 4.} Let \( a \) satisfy (10), \( b \in Sg(a) \) and \( c \in \mathbb{Z}^n \). Suppose that \( x^* \) given by (13) is an optimal vertex solution to (18). Let further \( z^* \) be any vertex of \( C_\gamma(a, b) \) such that \( z^* \in P(a, b) \). Then for \( r = \| (z^*_2, \ldots, z^*_n)^\top \|_0 \) we have

\[
IG(c, a, b) = 0 \text{ if } r = 0,
\]

\[
IG(c, a, b) \leq 2 \| a \|_\infty - 1 \| c \|_\infty \text{ if } r = 1 \text{ and }
\]

\[
IG(c, a, b) < \frac{r(r + 1)}{2r} \| a \|_\infty \| c \|_\infty \text{ if } r \geq 2.
\]
It follows from the proof of Theorem 1(ii) in [4] that the bound (15) (and hence (5) and (8) for \(m = 1\)) corresponding to the case \(r = 1\) is optimal. For completeness, we recall that it is sufficient to choose a positive integer \(k\) and set

\[
a = (k, \ldots, k, 1)^\top, b = k - 1 \quad \text{and} \quad x^* = \frac{k - 1}{k} \cdot e_1.
\]

Then the knapsack polytope \(P(a, b)\) contains precisely one integer point, \(z^* = (k - 1) \cdot e_n\) and we obtain \(\|x^* - z^*\|_\infty = k - 1 = \|a\|_\infty - 1\). The next result of this paper shows that the bounds in Theorems 1 - 3 are optimal in the knapsack scenario for \(r \geq 2\).

**Theorem 5.** Fix integer \(s \geq 3\). For any \(\epsilon > 0\) there exists an integer vector \(a \in \mathbb{Z}^s\) satisfying (14) and \(b \in Sg(a)\) such that for \(x^* = (b/a_1)e_1\) the knapsack polytope \(P(a, b)\) contains a vertex \(z^*\) of \(C_\gamma(a, b)\) with \(\|z^*_\|_0 = s - 1\) and

\[
\|x^* - z^*\|_\infty \geq \frac{2^{s-1}}{s - 1} > (1 - \epsilon)\|a\|_\infty.
\]

### 1.2. A refined sparsity-type bound for solutions to integer programs.

The next result of this paper aims to refine the general sparsity-type bound obtained in [2, Theorem 1]. Let

\[
\rho(x) = \min\{|x_i| : i \in \text{supp}(x)\}
\]

denote the minimum absolute nonzero entry of \(x\). The notation \(\log(\cdot)\) will be used for logarithm with base two. Let \(A \in \mathbb{Z}^{m \times n}, b \in \mathbb{Z}^m\) and \(c \in \mathbb{Z}^n\). We will consider the general integer linear problem in standard form

\[
\max \{c^\top x : x \in P(A, b) \cap \mathbb{Z}^n\}.
\]

We assume that \(P(A, b)\) contains integer points, so that (24) is feasible. We will also assume without loss of generality that the matrix \(A\) has full row rank, i.e., \(\text{rank}(A) = m\).

It was shown in [2, Theorem 1] that there exists an optimal solution \(z^*\) for (24) satisfying the bound

\[
\|z^*_\|_0 \leq m + \log\left(\frac{\sqrt{\det(AA^\top)}}{\gcd(A)}\right).
\]

Note that any vertex solution for (24) has the size of support \(\leq m\). Any non-vertex solution \(z^*\), in its turn, belongs to the interior of the face \(F = P(A, b) \cap \{x \in \mathbb{R}^n : x_i = 0\} \forall i \notin \text{supp}(z^*)\) of the polyhedron \(P(A, b)\). Then the minimum absolute nonzero entry \(\rho(z^*)\) is the \(\ell_\infty\)-distance from \(z^*\) to the boundary of \(F\). To obtain a refinement of the bound (25) we will link the minimum absolute nonzero entry and the size of support of solutions to (24).

**Theorem 6.** Let \(A \in \mathbb{Z}^{m \times n}, b \in \mathbb{Z}^m, c \in \mathbb{Z}^n\) and suppose that (24) is feasible. Then there is an optimal solution \(z^*\) to (24) such that, letting \(s = \|z^*_\|_0\),

\[
(\rho(z^*) + 1)^{s - m} \leq \frac{\sqrt{\det(AA^\top)}}{\gcd(A)}.
\]
2. LATTICES AND CORNER POLYHEDRA

For linearly independent \( b_1, \ldots, b_l \) in \( \mathbb{R}^d \), the set \( \Lambda = \{ \sum_{i=1}^{l} x_i b_i, x_i \in \mathbb{Z} \} \) is a \( l \)-dimensional lattice with basis \( b_1, \ldots, b_l \) and determinant \( \det(\Lambda) = (\det(b_i \cdot b_j)_{1 \leq i, j \leq l}^{1/2}) \), where \( b_i \cdot b_j \) is the standard inner product of the basis vectors \( b_i \) and \( b_j \). Recall that the Minkowski sum \( X + Y \) of the sets \( X, Y \subseteq \mathbb{R}^d \) consists of all points \( x + y \) with \( x \in X \) and \( y \in Y \). The difference set \( X - X \) is the Minkowski sum of \( X \) and \( -X \). For a lattice \( \Lambda \subseteq \mathbb{R}^d \) and \( y \in \mathbb{R}^d \), the set \( y + \Lambda \) is an affine lattice with determinant \( \det(\Lambda) \).

Let \( \Lambda \subseteq \mathbb{Z}^d \) be a \( d \)-dimensional integer lattice. The point \( x \in \mathbb{Z}^d_{\geq 0} \) is called irreducible (with respect to \( \Lambda \)) if for any two points \( y, y' \in \mathbb{Z}^d_{\geq 0} \) with \( 0 \leq y_i \leq x_i, 0 \leq y'_i \leq x_i, i \in \{1, \ldots, d\} \) the inclusion \( y - y' \subseteq \Lambda \) implies \( y = y' \).

Lemma 7 (Theorem 1 in [14]). If \( x \in \mathbb{Z}^d_{\geq 0} \) is irreducible with respect to the lattice \( \Lambda \) then

\[
\prod_{i=1}^{d} (x_i + 1) \leq \det(\Lambda).
\]

Proof. The lattice \( \Lambda \) can be viewed as a subgroup of the additive group \( \mathbb{Z}^d \). The number of points \( y \in \mathbb{Z}^d_{\geq 0} \) with \( 0 \leq y_i \leq x_i, i \in \{1, \ldots, d\} \) is equal to \( \prod_{i=1}^{d} (x_i + 1) \). Since \( x \) is irreducible, each such \( y \) corresponds to a unique coset (affine lattice) \( y + \Lambda \) of \( \Lambda \). Finally notice that there are only \( \det(\Lambda) \) different cosets. \( \square \)

Let \( r \in \mathbb{Z}^d \) and consider the affine lattice \( \Gamma = r + \Lambda \). We will call the set \( E(\Gamma) = \text{conv}(\Gamma \cap \mathbb{R}^d_{\geq 0}) \) the sail associated with \( \Gamma \).

Lemma 8. Every vertex of the sail \( E(\Gamma) \) is irreducible.

Proof. Let \( x \) be a vertex of \( E(\Gamma) \). Suppose, to derive a contradiction, that \( x \) is reducible. Then there are distinct points \( y, y' \in \mathbb{Z}^d_{\geq 0} \) with \( 0 \leq y_i \leq x_i, 0 \leq y'_i \leq x_i, i \in \{1, \ldots, d\} \) such that \( y - y' \subseteq \Lambda \).

Since \( x - y \in \mathbb{Z}^d_{\geq 0} \) and \( x - y' \in \mathbb{Z}^d_{\geq 0} \), the vectors \( v_1 = x - y + y' \) and \( v_2 = x - y' + y \) have nonnegative integer entries. Further, \( v_1, v_2 \in \Gamma \) and \( x = (v_1 + v_2)/2 \). Therefore \( x \) is not a vertex of \( E(\Gamma) \). \( \square \)

Lemma 9. For \( d \geq 2 \) and \( x_1, \ldots, x_d \geq 1 \) the inequality

\[
(28) \quad x_1 + \cdots + x_d \leq \frac{d(x_1 + 1) \cdots (x_d + 1)}{2^d}
\]

holds.

Proof. Suppose that (28) is satisfied for \( x_1 = y_1, \ldots, x_d = y_d \). We will first show that for any \( \epsilon > 0 \) and any \( i \in \{1, \ldots, d\} \) the inequality (28) is satisfied for \( x_1 = y_1, \ldots, x_{i-1} = y_{i-1}, x_i = y_i + \epsilon, x_{i+1} = y_{i+1}, \ldots, x_d = y_d \). After possible renumbering, it is sufficient to consider the case \( i = 1 \). We have

\[
(y_1 + \epsilon) + y_2 + \cdots + y_d \leq \frac{d(y_1 + 1) \cdots (y_d + 1)}{2^d} + \epsilon
\]

\[
\leq \frac{d(y_1 + 1) \cdots (y_d + 1)}{2^d} + \epsilon \frac{d(y_2 + 1) \cdots (y_d + 1)}{2^d} = \frac{d(y_1 + 1 + \epsilon) \cdots (y_d + 1)}{2^d}.
\]

To complete the proof it is sufficient to observe that (28) holds for \( y_1 = \cdots = y_d = 1 \). \( \square \)
Given \( A \in \mathbb{Z}^{m \times n} \) and \( b \in \mathbb{Z}^m \), we will denote by \( \Gamma(A, b) \) the set of integer points in the affine subspace

\[ H(A, b) = \{ x \in \mathbb{R}^n : Ax = b \}, \]

that is

\[ \Gamma(A, b) = H(A, b) \cap \mathbb{Z}^n. \]

The set \( \Gamma(A, b) \) is an affine lattice of the form \( \Gamma(A, b) = r + \Gamma(A) \), where \( r \) is any integer vector with \( Ar = b \) and \( \Gamma(A) = \Gamma(A, 0) \) is the lattice formed by all integer points in the kernel of the matrix \( A \).

Let \( \pi_\gamma \) denote the projection map from \( \mathbb{R}^n \) to \( \mathbb{R}^{n-m} \) that forgets the first \( m \) coordinates, \( \pi_\gamma : u \mapsto u_{\bar{\gamma}} \). Recall that \( A_\gamma \) is nonsingular. It follows that the restricted map \( \pi_\gamma|_{H(A, b)} : H(A, b) \rightarrow \mathbb{R}^{n-m} \) is bijective. Specifically, for any \( u_{\bar{\gamma}} \in \mathbb{R}^{n-m} \) we have

\[ \pi_\gamma|_{H(A, b)}^{-1}(u_{\bar{\gamma}}) = \left( \begin{array}{c} u_\gamma \\ u_{\bar{\gamma}} \end{array} \right) \quad \text{with} \quad u_\gamma = A_\gamma^{-1}(b - A_{\bar{\gamma}}u_{\bar{\gamma}}). \]

For technical reasons, it is convenient to consider the projected affine lattice \( \Lambda(A, b) = \pi_\gamma(\Gamma(A, b)) \) and the projected lattice \( \Lambda(A) = \pi_\gamma(\Gamma(A)) \). Let \( g_1, \ldots, g_{n-m} \) be a basis of \( \Gamma(A) \). Since the map \( \pi_\gamma|_{H(A, b)} \) is bijective, the vectors \( b_1 = \pi_\gamma(g_1), \ldots, b_{n-m} = \pi_\gamma(g_{n-m}) \) form a basis of the lattice \( \Lambda(A) \). Let \( G \in \mathbb{Z}^{n \times (n-m)} \) be the matrix with columns \( g_1, \ldots, g_{n-m} \).

We will denote by \( F \) the \( (n-m) \times (n-m) \)-submatrix of \( G \) consisting of the last \( n-m \) rows; hence, the columns of \( F \) are \( b_1, \ldots, b_{n-m} \). Then \( \det(\Lambda(A)) = |\det(F)| \). The rows of the matrix \( A \) span the \( m \)-dimensional rational subspace of \( \mathbb{R}^n \) orthogonal to the \( (n-m) \)-dimensional rational subspace spanned by the columns of \( G \). Therefore, by Lemma 5G and Corollary 5I in [16], we have \( |\det(F)| = |\det(A_\gamma)|/\gcd(A) \) and, consequently,

\[ \det(\Lambda(A)) = \frac{|\det(A_\gamma)|}{\gcd(A)}. \]

**Theorem 10.** Suppose that \( A \) satisfies (2), \( b \in \mathbb{Z}^m \) and \( x^* \) is given by (3). For any vertex \( z^* \) of the corner polyhedron \( C_\gamma(A, b) \) the bound

\[ \prod_{j \in \gamma} (z_j^* + 1) \leq \frac{|\det(A_\gamma)|}{\gcd(A)} \]

holds.

**Proof.** Since \( \pi_\gamma|_{H(A, b)} \) is a bijection, the point \( y^* = \pi_\gamma(z^*) \) is a vertex of the sail \( E(\Lambda(A, b)) \). The result now follows by Lemma 8 and (20). \( \square \)

### 3. Proof of Theorem 11

Theorem 11 is an immediate consequence of Theorem 10 and the following lemma:

**Lemma 11.** Suppose that \( A \) satisfies (2), and let \( b \in \mathbb{Z}^m \). Let \( x^* \) be a vertex of \( P(A, b) \), and let \( \gamma \) be any basis of \( A \) containing \( \text{supp}(x^*) \). Let \( z^* \) be an integral vector satisfying \( Az^* = b \), with \( z_j^* \geq 0 \), and let \( r = \|z^*_\|_0 \). Then

\[ x^* = z^* \text{ if } r = 0, \]

\[ \|x^* - z^*\|_\infty \leq \frac{\sum(A)}{|\det(A_\gamma)|} \prod_{j \in \gamma} (z_j^* + 1) - 1 \text{ if } r = 1 \text{ and } \]

\[ \sum(A) \]
(34) \[ \| x^* - z^* \|_{\infty} \leq \frac{2^r}{r} \left( \frac{\Sigma(A)}{|\det(A_{\gamma})|} \prod_{j \in \bar{\gamma}} (z^*_j + 1) \right) \text{ if } r \geq 2. \]

Proof. If \( r = \| z^*_i \|_0 = 0 \) the vector \( z^*_i \) is the unique solution to the system \( A_{\gamma} x_{\gamma} = b \). Therefore (32) holds.

In the rest of the proof we assume that \( r \geq 1 \). We will set \( \delta = \| x^* - z^* \|_{\infty} \) and consider the following two cases. First suppose that there exists an index \( j \in \bar{\gamma} \) such that \( \delta = |x^*_j - z^*_j| = z^*_j \). Observe that \( r \) of the numbers \( z^*_m + 1, \ldots, z^*_n \) are nonzero. Hence,

(35) \[ (\delta + 1)2^{r-1} \leq \prod_{j \in \bar{\gamma}} (z^*_j + 1) \]

and so

(36) \[ \delta \frac{2^r}{r} \leq \frac{2}{r} \prod_{j \in \bar{\gamma}} (z^*_j + 1) - \frac{2^r}{r}. \]

Since \( \Sigma(A) \geq |\det(A_{\gamma})| \), inequality (36) justifies both (33) and (34).

Now suppose that \( \delta = x^*_j - z^*_j \) for \( j \in \gamma \). We can write

\[ A_{\gamma} z^*_\gamma + A_{\bar{\gamma}} z^*_\bar{\gamma} = b \]

and

\[ A_{\gamma} x^*_\gamma = b. \]

Therefore

(37) \[ A_{\gamma}(x^*_\gamma - z^*_\gamma) = A_{\bar{\gamma}} z^*_\bar{\gamma}. \]

Given a vector \( v \in \mathbb{R}^m \), we will denote by \( A^j_\gamma(v) \) the matrix obtained from \( A_\gamma \) by replacing its \( j \)-th column with \( v \). Let \( A_1, \ldots, A_n \) be the columns of the matrix \( A \). Solving (37) by Cramer’s rule, we have

(38) \[ \delta = x^*_j - z^*_j = \frac{\det(A^j_\gamma(A_{\gamma} z^*_\gamma))}{\det(A_{\gamma})}. \]

If \( r = 1 \), then for some \( i \in \bar{\gamma} \) we can write

(39) \[ \delta = \frac{z^*_i \det(A^i_\gamma(A_{\gamma}))}{\det(A_{\gamma})} = (z^*_i + 1) \frac{\det(A^i_\gamma(A_{\gamma}))}{\det(A_{\gamma})} - \frac{\det(A^i_\gamma(A_{\gamma}))}{\det(A_{\gamma})}. \]

Equation (39) plus the integrality of \( \delta \) imply (33).

To settle the case \( r \geq 2 \), observe that (38) implies

(40) \[ \delta \leq (z^*_{m+1} + \cdots + z^*_n) \frac{\Sigma(A)}{|\det(A_{\gamma})|}. \]

Without loss of generality, assume that \( z^*_i \neq 0 \) for \( i \in \{m+1, \ldots, m+r\} \) and \( z^*_i = 0 \) for \( m + r < i \leq n \). Then, by (40) and Lemma 9 we have

(41) \[ \delta \leq \frac{r(z^*_{m+1} + 1) \cdots (z^*_{m+r} + 1) \Sigma(A)}{2^r |\det(A_{\gamma})|}. \]

This establishes inequality (34). \( \square \)
4. Proof of Theorem 2

As in the proof of Theorem 1, we have that Theorem 2 is an immediate consequence of Lemma 11 and the generalization of Theorem 10 given below. Recall that $\tau$ denotes the support of $x^*$, and $C_\tau(A, b)$ denotes the polyhedron $C_\tau(A, b) = \text{conv} \left( \{ x \in \mathbb{Z}^n : Ax = b, \ x_\tau \geq 0 \} \right)$.

**Theorem 12.** Let $z^*$ be a vertex of $C_\tau(A, b)$. Then there exists a basis $\gamma$ of $A$ containing $\tau$ such that

$$
\prod_{j \in \gamma} (z^*_j + 1) \leq r d \left| \frac{\det(A_\gamma)}{\gcd(A)} \right|
$$

where $r = \|z^*_\|_0$ and $d = m - |\tau|$.

Theorem 12 is proved over the remainder of this section.

4.1. Convex geometry lemmas. For an affine subspace $F \subseteq \mathbb{R}^d$, let $\text{vol}_F(\cdot)$ denote the standard Lebesgue measure with respect to $F$. We denote $\text{vol}_{\mathbb{R}^d}(\cdot)$ simply by $\text{vol}_d(\cdot)$.

**Lemma 13** (Blichfeldt’s lemma [9, Chapter III, Theorem I]). Let $K \subseteq \mathbb{R}^d$ be bounded, nonempty, Lebesgue measurable and let $\Lambda$ be a full-dimensional lattice in $\mathbb{R}^d$. Suppose that the difference set $K - K$ contains no nonzero lattice points from $\Lambda$. Then $\text{vol}_d(K) \leq \det(\Lambda)$.

**Theorem 14** (Brunn’s concavity principle [5, Theorem 1.2.1]). Let $K$ be a convex body, and let $F$ be a $k$-dimensional subspace of $\mathbb{R}^d$. Then the function $g : F^\perp \rightarrow \mathbb{R}$ defined by

$$
g(x) = \text{vol}_{F^\perp + x}(K \cap (F + x))^{1/k}
$$

is concave on its support.

By a slab we mean the nonempty intersection of two halfspaces with antiparallel normals. Let $q \in \mathbb{R}^d$ be nonzero. The *width* of a set $K \subseteq \mathbb{R}^d$ along $q$ is defined to be

$$
w_q(K) := \left( \sup_{x \in K} q^\top x \right) - \left( \inf_{x \in K} q^\top x \right).
$$

**Proposition 15.** Let $K$ be a centrally symmetric convex body with centre $c$. Let $S$ be a slab centred at $c$ with a facet normal $q$. If $S$ does not contain $K$, then

$$
\text{vol}_d(K \cap S) \geq \frac{w_q(S)}{w_q(K)} \cdot \text{vol}_d(K).
$$

*Proof.* Without loss of generality, we may assume $c$ is the origin. For $\lambda \in [-1, 1]$, define the affine hyperplane

$$
L_\lambda := \{ x \in \mathbb{R}^d : q^\top x = \lambda \cdot w_q(K)/2 \}.
$$

Let $K_\lambda := K \cap L_\lambda$, and define the cross-sectional volume

$$
f(\lambda) := \text{vol}_{L_\lambda}(K_\lambda).
$$

By symmetry, we have $K_\lambda = -K_{-\lambda}$. Hence, $f$ is an even function on $[-1, 1]$, which means that $g(\lambda) := (f(\lambda))^{1/(d-1)}$ is an even function as well. Since $g$ is concave on $[-1, 1]$ by Brunn’s concavity principle, we have that $g$, and therefore $f$, is a decreasing function on $[0, 1]$. 


Now let \( \delta := w_q(S)/w_q(K) \). By Fubini’s theorem, symmetry, and monotonicity on \([0,1]\), we conclude
\[
\text{vol}_d(K \cap S) = \int_{-\delta}^{\delta} f(\lambda) d\lambda = 2 \int_{0}^{\delta} f(\lambda) d\lambda \geq 2\delta \int_{0}^{1} f(\lambda) d\lambda = \frac{w_q(S)}{w_q(K)} \cdot \text{vol}_d(K).
\]

The notion of irreducibility from Lemma 8 can be mildly generalized as follows. Let \( C \) be a pointed cone. Let \( \Lambda \subset \mathbb{Z}^d \) be a \( d \)-dimensional integer lattice. The point \( \mathbf{x} \in C \cap \mathbb{Z}^d \) is called \textit{irreducible} \((\text{with respect to } \Lambda \text{ and } C)\) if
\[
(-\mathbf{x} + C) \cap (\mathbf{x} - C) \cap \Lambda = \{\mathbf{0}\}.
\]

Let \( \mathbf{r} \in \mathbb{Z}^d \) and consider the affine lattice \( \Gamma = \mathbf{r} + \Lambda \). We will call the set \( E(\Gamma, C) = \text{conv}(\Gamma \cap C) \) the \textit{sail} associated with \( \Gamma \) and \( C \).

**Lemma 16.** Every vertex of the sail \( E(\Gamma, C) \) is irreducible.

\[\text{Proof.}\]
Let \( \mathbf{x} \) be a vertex of \( E(\Gamma, C) \). Suppose, to derive a contradiction, that \( \mathbf{x} \) is reducible. Then there exists nonzero \( \mathbf{\lambda} \in \Lambda \) and vectors \( \mathbf{y}, \mathbf{y}' \in C \) such that \( \mathbf{\lambda} = -\mathbf{x} + \mathbf{y} = -\mathbf{x} + \mathbf{y}' \).

The fact that \( \mathbf{x} \) is a vertex of \( E(\Gamma, C) \) implies \( \mathbf{x} \in \Gamma \), and therefore both \( \mathbf{y} = \mathbf{\lambda} + \mathbf{x} \) and \( \mathbf{y}' = -\mathbf{\lambda} + \mathbf{x} \) are contained in \( \Gamma \cap C \), and hence in \( E(\Gamma, C) \). Since \( \mathbf{\lambda} \) is nonzero, we conclude that \( \mathbf{x} = (\mathbf{y} + \mathbf{y}')/2 \) is not a vertex of \( E(\Gamma, C) \). \(\square\)

4.2. **Lemmas for Theorem 12.** In this subsection we further assume the condition that \( \overline{\tau} \subseteq \text{supp} (z^*) \). We fix a basis \( \gamma \) of \( A \) containing \( \tau \) such that for all \( j \in \gamma \setminus \tau \),
\[
(42) \quad z_i^* + 1 \geq \frac{1}{r} \sum_{j \in \gamma} |(A_{\gamma}^{-1} A_\gamma)_i,j(z_j^* + 1)|.
\]

The existence of such a basis \( \gamma \) is justified in Proposition 17. Without loss of generality, we continue with our notational assumption that \( \gamma = \{1,2,\ldots,m\} \) and we further assume \( \gamma \setminus \tau = \{1,2,\ldots,d\} \). We denote the rows of the matrix \(-A_\gamma^{-1} A_\gamma\) by \( q_1^T, q_2^T, \ldots, q_m^T \). Note that the equality \( A \mathbf{x}^* = \mathbf{b} \) implies \( q_i^T \mathbf{x}^*_\gamma = z_i^* - x_i^* \) for all \( i \in \gamma \).

**Proposition 17.** Assume \( \overline{\tau} \subseteq \text{supp} (z^*) \). Then there exists a basis \( \gamma \) of \( A \) containing \( \tau \) satisfying inequality (42).

\[\text{Proof.}\]
Among all bases of \( A \) containing \( \tau \), choose a basis \( \gamma \) so that the quantity \( |\det A_\gamma| \cdot \prod_{i \in \gamma} (z_i^* + 1) \) is as large as possible. If \( i \in \gamma \) and \( j \in \gamma \), then by Cramer’s rule we have
\[q_{i,j} = -\frac{\det(A_\gamma^i(A_j))}{\det(A_\gamma)},\]
where \( A_\gamma^i(A_j) \) denotes the matrix obtained by replacing column \( i \) of \( A_\gamma \) with column \( j \) of \( A \). The choice of \( \gamma \) implies that if \( i \in \gamma \setminus \tau \) and \( j \in \gamma \), then
\[z_i^* + 1 \geq \frac{\det(A_\gamma^i(A_j))}{\det(A_\gamma)} (z_j^* + 1) = |q_{i,j} (z_j^* + 1)| .\]

The condition \( \overline{\tau} \subseteq \text{supp} (z^*) \) implies \( r = |\gamma| \). Hence, for all \( i \in \gamma \setminus \tau \), we have
\[z_i^* + 1 \geq \frac{1}{r} \sum_{j \in \gamma} |q_{i,j} (z_j^* + 1)| .\]
\(\square\)
Let $1_{n-m} \in \mathbb{R}^{n-m}$ be the vector of all ones, and define, for each $i \in \gamma \setminus \tau$,
$$S_i := \{ x \in \mathbb{R}^{n-m} : -\frac{1}{2} < q_i^\top x < q_i^\top z_\gamma^* + \frac{1}{2} \}.$$  
Also define
$$B := \{ x \in \mathbb{R}^{n-m} : -\frac{1}{2} 1_{n-m} < x < z_\gamma^* + \frac{1}{2} 1_{n-m} \},$$
and for each $i \in \gamma \setminus \tau$, let $P_i := P_{i-1} \cap S_i$ with $P_0 = B$. Let $P := P_d$.

**Lemma 18.** Assume $\bar{\tau} \subseteq \text{supp}(z^*)$. Then
$$\text{vol}_{n-m}(P) \geq \frac{1}{r^d} \prod_{j \in \gamma} (z_j^* + 1).$$

**Proof.** Suppose $i \in \gamma \setminus \tau$. If $S_i$ contains $P_{i-1}$ then $P_i = P_{i-1}$, and hence
$$\text{vol}_{n-m}(P_i) = \text{vol}_{n-m}(P_{i-1}).$$

Otherwise, define
$$\lambda_i := \frac{w_{q_i}(S_i)}{w_{q_i}(P_{i-1})}.$$

The fact that $\bar{\tau} \subseteq \text{supp}(z^*)$ implies $z_j^* \geq 1$ for all $i \in \gamma \setminus \tau$. Applying Proposition 17, we get
$$\lambda_i \geq \frac{w_{q_i}(S_i)}{w_{q_i}(B)} = \frac{q_i^\top z_i^* + 1}{\sum_{j \in \gamma} |q_{i,j}(z_j^* + 1)|} = \frac{z_i^* + 1}{\sum_{j \in \gamma} |q_{i,j}(z_j^* + 1)|} \geq \frac{z_i^* + 1}{r(z_i^* + 1)} = \frac{1}{r}.$$

Since $S_i$ does not contain $P_{i-1}$, Proposition 15 applies, and so
$$\text{vol}_{n-m}(P_i) \geq \frac{w_{q_i}(S_i)}{w_{q_i}(P_{i-1})} \text{vol}_{n-m}(P_{i-1}) = \lambda_i \text{vol}_{n-m}(P_{i-1}) \geq \frac{1}{r} \text{vol}_{n-m}(P_{i-1}).$$

Applying induction to the sequence of polytopes $P = P_d, \ldots, P_1, P_0 = B$, we get
$$\text{vol}_{n-m}(P) \geq \frac{1}{r^d} \text{vol}_{n-m}(B) = \frac{1}{r^d} \prod_{j \in \gamma} (z_j^* + 1).$$

**Lemma 19.** Assume $\bar{\tau} \subseteq \text{supp}(z^*)$. Then
$$\text{vol}_{n-m}(P) \leq \frac{|\text{det}(A_\gamma)|}{\text{gcd}(A)}.$$

**Proof.** Recall we defined the lattice $\Lambda(A) = \pi_\gamma(\ker(A) \cap \mathbb{Z}^n)$, whose determinant is given by $|\text{det}(A_\gamma)| / \text{gcd}(A)$ by (30). We show that $(P - P) \cap \Lambda(A) = \{0\}$. The conclusion then follows from Lemma 13.

Suppose that $u, v \in P$ and $u - v \in \Lambda(A)$. Since $P$ is symmetric, $P - P$ is the origin-symmetric translate of $2P$, and therefore
$$-z_i^* - 1_{n-m} < u - v < z_i^* + 1_{n-m}$$
$$-q_i^\top z_i^* - 1 < q_i^\top (u - v) < q_i^\top z_i^* + 1 \text{ for all } i \in \gamma \setminus \tau.$$

The lattice $\Lambda(A)$ can be characterized as the set of points $x \in \mathbb{Z}^{n-m}$ such that $q_i^\top x \in \mathbb{Z}$ for each $i \in \gamma$. Hence, the inequalities from (43) imply
$$-z_i^* \leq u - v \leq z_i^*$$
$$-q_i^\top z_i^* \leq q_i^\top (u - v) \leq q_i^\top z_i^* \text{ for all } i \in \gamma \setminus \tau.$$
In particular, \( u - v \) lies in the polyhedron \((-z_i^\gamma + C) \cap (z_i^\gamma - C)\), where
\[
C := \{ x \in \mathbb{R}^{n-m} : x \geq 0, q_i^\top x \geq 0 \text{ for all } i \in \gamma \setminus \tau \}.
\]

By assumption, \( z_i^\gamma \) is a vertex of the sail \( E(\Lambda(A, b), C) \). Hence, \( z_i^\gamma \) is irreducible by Lemma 10 and therefore \( u = v \).

4.3. **Proof of Theorem 12** Let \( \mu = \tau \cup \text{supp}(z^\star) \), which we may assume without loss of generality is given by \( \mu = \{1, 2, \ldots, |\mu|\} \), and let \( A'_{\mu} \) be any full row rank integer matrix with the same rowspace as \( A_{\mu} \). We have that \( x^\star_{\mu} \) is a basic feasible solution of the system
\[
A'_{\mu} x_{\mu} = b', \ x_{\mu} \geq 0,
\]
where \( b' := A'_{\mu} x^\star_{\mu} \). Moreover, letting
\[
C_{\tau}(A'_{\mu}, b') := \text{conv}(\{ x_{\mu} \in \mathbb{Z}^{|\mu|} : A'_{\mu} x_{\mu} = b', x_{\mu\setminus\tau} \geq 0 \},
\]
we have that \( C_{\tau}(A'_{\mu}, b') \times \{0\}^{|\mu|} \) is the face of \( C_{\tau}(A, b) \) for which the constraints \( x_{\mu} \geq 0 \) are tight, and this face contains \( z^\star = (z^\star_{\mu}, 0) \). Hence, \( z^\star_{\mu} \) is a vertex of \( C_{\tau}(A'_{\mu}, b') \). We may therefore apply the above results to \( x^\star_{\mu}, z^\star_{\mu} \), and the system (44). Let \( \sigma \subseteq \mu \) be a basis of \( A'_{\mu} \) containing \( \tau \) satisfying (42), whose existence is guaranteed by (42). Then Lemmas 18 and 19 imply
\[
\prod_{j \notin \sigma}(z^\star_j + 1) \leq r^{|\sigma|-|\tau|} \frac{\det(A'_{\sigma})}{\gcd(A'_{\mu})}.
\]

Now let \( \gamma \) be a basis of \( A \) containing \( \sigma \). Then \( \mu \) and \( \gamma \setminus \sigma \) partition \( \mu \cup \gamma \). Up to invertible row operations, we can write
\[
A_{\mu \cup \gamma} = \begin{pmatrix} A'_{\mu} & A'_{\gamma \setminus \sigma} \\ 0 & A''_{\gamma \setminus \sigma} \end{pmatrix} = \begin{pmatrix} A'_{\mu \setminus \sigma} & A'_{\gamma \setminus \sigma} \\ 0 & A''_{\gamma \setminus \sigma} \end{pmatrix},
\]
where both \( A'_{\sigma} \) and \( A''_{\gamma \setminus \sigma} \) are both invertible. Now, every nonzero maximal subdeterminant of \( A_{\mu \cup \gamma} \) is the product of \( \det(A''_{\gamma \setminus \sigma}) \) with a maximal subdeterminant of \( A'_{\mu} \). It follows that
\[
\gcd(A_{\mu \cup \gamma}) = |\det(A''_{\gamma \setminus \sigma})| \cdot \gcd(A'_{\mu}),
\]
and hence
\[
\frac{|\det(A'_{\mu})|}{\gcd(A'_{\mu})} = \frac{|\det(A'_{\sigma})|}{\gcd(A_{\mu \setminus \gamma})} \cdot \frac{|\det(A''_{\gamma \setminus \sigma})|}{\gcd(A'_{\mu})} = \frac{|\det(A_{\gamma})|}{\gcd(A_{\mu \cup \gamma})}.
\]
We conclude
\[
\prod_{j \notin \gamma}(z^\star_j + 1) \leq \prod_{j \notin \sigma}(z^\star_j + 1) \leq r^{|\sigma|-|\tau|} \frac{\det(A_{\gamma})}{\gcd(A_{\mu \cup \gamma})} \leq r^{d} \frac{\det(A_{\gamma})}{\gcd(A)}.
\]

5. **Proof of Theorem 3**

First we will show that the knapsack polytope \( P(a, b) \) contains a vertex of the corner polyhedron \( C_{\gamma}(a, b) \). Let \( z^\star \) be a vertex of \( C_{\gamma}(a, b) \) that gives an optimal solution to the linear program
\[
\max \{ x_1 : x = (x_1, \ldots, x_n) \top \in C_{\gamma}(a, b) \}.
\]
By definition of \( C_{\gamma}(a, b) \) the vertex \( z^\star \) is in \( P(a, b) \) if and only if \( z^\star_i \geq 0 \). Since \( P(a, b) \subset C_{\gamma}(a, b) \), it is now sufficient to choose any integer point \( z = (z_1, \ldots, z_n) \top \in P(a, b) \) and observe that \( z_i^\star \geq z_i \geq 0 \).
Applying Theorem 1 with the vertex $z^* \in P(a, b)$ we immediately obtain (14) and (15). Further, the bound (19) implies for $r \geq 2$ the non-strict inequality

$$\|x^* - z^*\|_\infty \leq \frac{2r}{r} \leq \|a\|_\infty.$$  

To show that (45) is strict (and hence that (16) holds), it is sufficient to prove that the bound (40) in the proof of Theorem 1 is strict in the knapsack scenario. Specifically, we need to prove that for the vertex $z^*$

$$\delta = |x_1^* - z_1^*| < \frac{(z_2^* + \cdots + z_n^*)\|a\|_\infty}{a_1},$$  

(46)

Set $A = (a_1, \ldots, a_n) \in \mathbb{Z}^{1 \times n}$ and consider the affine lattice $\Lambda(a, b) := \Lambda(A, b)$. We can write

$$\Lambda(a, b) = \{(\lambda_2, \ldots, \lambda_n)^T \in \mathbb{Z}^{n-1} : \lambda_2a_2 + \cdots + \lambda_{n-1}a_{n-1} \equiv b \pmod{a_1}\}.$$  

(47)

Following (29), the map $\pi_{|\gamma|}(\Lambda(a, b))$, with $\gamma = \{1\}$ in the knapsack scenario, is a bijection. It follows that the point $y^* = \pi_{\gamma}(z^*)$ is a vertex of the sail $E(\Lambda(a, b))$.

Suppose, to derive a contradiction, that the equality

$$\delta = (z_2^* + \cdots + z_n^*)\|a\|_\infty$$  

(48)

holds. By (48) we have

$$\delta = \frac{z_2^*a_2 + \cdots + z_n^*a_n}{a_1}$$

and, consequently, (48) implies $a_2 = \cdots = a_n = \|a\|_\infty$. Therefore, using (17), the affine lattice $\Lambda(a, b)$ contains the points

$$(z_2^* + \cdots + z_n^*)e_j, \ j \in \{1, \ldots, n-1\}.$$  

(49)

The point $y = (z_2^*, \ldots, z_n^*)^T$, in its turn, belongs to the simplex with vertices (49) and has $\|y\|_0 = r \geq 2$. Therefore $y$ cannot be a vertex of the sail $E(\Lambda(a, b))$. The derived contradiction completes the proof of Theorem 3.

6. Proof of Corollary 4

By Theorem 3 the knapsack polytope $P(a, b)$ contains a vertex $z^*$ of $C_\gamma(a, b)$. Therefore

$$IG(c, a, b) \leq \|x^* - z^*\|_\infty \sum_{i \in supp(x^* - z^*)} |c_i|.$$  

(50)

If $r = 0$ we have $x^* = z^*$ that justifies (19). Further, (50) implies the bound

$$IG(c, a, b) \leq (r + 1)\|x^* - z^*\|_\infty \|c\|_\infty$$

that immediately gives (20) and (21).
7. Proof of Theorem 5

For $s \geq 2$ set $a^{(s)} = (2s^{-1}, 2s^{-2}, \ldots, 1)^\top$ and $b^{(s)} = 1^\top a^{(s)} = 2s - 1$. Let $P_I(a^{(s)}, b^{(s)}) = \text{conv}(P(a^{(s)}, b^{(s)}) \cap \mathbb{Z}^s)$ be the integer hull of the knapsack polytope $P(a^{(s)}, b^{(s)})$.

We will need the following observations.

Lemma 20. The point $1_s$ is a vertex of the polytope $P_I(a^{(s)}, b^{(s)})$.

Proof. We will use induction on $s$. The basis step $s = 2$ holds as there are only two integer points $1_2$ and $(0, 3)^\top$ in the polytope $P(a^{(s)}, b^{(s)})$. To verify the inductive step, suppose that the result does not hold for some $s \geq 3$. Observe that any integer point $z = (z_1, \ldots, z_s)^\top \in P(a^{(s)}, b^{(s)})$ has $z_1 \leq 1$. Consequently, $1_s$ belongs to the face $P_I(a^{(s)}, b^{(s)}) \cap \{x \in \mathbb{R}^s : x_1 = 1\}$ of the polyhedron $P_I(a^{(s)}, b^{(s)})$. Hence $1_s$ is a convex combination of some integer points in $P(a^{(s)}, b^{(s)})$ that have the first entry 1. Therefore, removing the first entry we obtain a convex combination of integer points from $P(a^{(s-1)}, b^{(s-1)})$ equal to $1_{s-1}$. The obtained contradiction completes the proof.

For the rest of the proof we assume $s \geq 3$.

Lemma 21. The point $1_{s-1}$ is a vertex of the sail $E(\Lambda(a^{(s)}, b^{(s)}))$.

Proof. Using (47), the affine lattice $\Lambda(a^{(s)}, b^{(s)})$ can be written as

\[ \Lambda(a^{(s)}, b^{(s)}) = \left\{ x \in \mathbb{Z}^{s-1} : 2s^{-2}x_2 + \cdots + x_s \equiv -1 \pmod{2s^{-1}} \right\}. \]

Therefore

\[ \mathcal{H} = \left\{ x \in \mathbb{R}^{s-1} : 2s^{-2}x_2 + \cdots + x_s = 2s^{-1} - 1 \right\} \]

is a supporting hyperplane of $E(\Lambda(a^{(s)}, b^{(s)}))$. Consequently,

\[ P_I(a^{(s-1)}, b^{(s-1)}) = \mathcal{H} \cap E(\Lambda(a^{(s)}, b^{(s)})) \]

is a face of $E(\Lambda(a^{(s)}, b^{(s)}))$. The result now follows by Lemma 20.

For a positive integer $t$ set

\[ a^{(s)}(t) = (a_1^{(s)}(t), \ldots, a_s^{(s)}(t))^\top = (2s^{-1}, 2s^{-2} + t2s^{-1}, \ldots, 1 + t2s^{-1})^\top \]

and $b^{(s)}(t) = 1^\top a^{(s)}(t) = 2s + (s-1)t2s^{-1} - 1$. Consider the vertex $a^{(s)}(t) = (b^{(s)}(t)/a_1^{(s)}(t))e_1$ of the knapsack polytope $P(a^{(s)}(t), b^{(s)}(t))$.

In view of (47), we have $\Lambda(a^{(s)}, b^{(s)}) = \Lambda(a^{(s)}(t), b^{(s)}(t))$. Therefore, by Lemma 21 the point $1_{s-1}$ is a vertex of the sail $E(\Lambda(a^{(s)}(t), b^{(s)}(t)))$. Observe that the sail $E(\Lambda(a^{(s)}(t), b^{(s)}(t)))$ is the image of the corner polyhedron $C_{v^{(s)}(t)}(a^{(s)}, b^{(s)})$ under the bijective linear map $\pi_\gamma|_{H(a^{(s)}(t), b^{(s)}(t))}$.

Using (29), the point

\[ 1_s = \pi_\gamma^{-1}|_{H(a^{(s)}(t), b^{(s)}(t))}(1_{s-1}) \]

is a feasible vertex of $C_{v^{(s)}(t)}(a^{(s)}, b^{(s)})$. Note also that $1_s \in P(a^{(s)}(t), b^{(s)}(t))$.

It is now sufficient to show that for any $\epsilon > 0$

\[ \|v^{(s)}(t) - 1_s\|_{\infty} \geq \frac{2s^{-1}}{8 - 1} > (1 - \epsilon)\|a^{(s)}(t)\|_{\infty} \]

(51)
for sufficiently large $t$. We have

$$\|v^{(s)}(t) - 1_s\|_\infty = \left\lfloor \frac{b^{(s)}(t)}{a_1^{(s)}(t)} - 1 \right\rfloor = (s - 1)t + 1 - \frac{1}{2^{s-1}}.$$ 

Finally,

$$\left\| \frac{u^{(s)}(t) - 1_s}{a^{(s)}(t)} \right\|_\infty = \frac{(s - 1)t + 1 - 2^{-(s-1)}}{2^{s-2} + t2^{s-1}} \to \frac{s - 1}{2^{s-1}}$$

as $t \to \infty$, that implies (51).

8. Proof of Theorem 6

We will apply the following result by Bombieri and Vaaler [7].

**Theorem 22 ([7, Theorem 2]).** Let $A \in \mathbb{Z}^{m \times n}$, $m < n$, be a matrix of rank $m$. There exist $n - m$ linearly independent integral vectors $y_1, \ldots, y_{n-m} \in \Gamma(A)$ satisfying

$$\prod_{i=1}^{n-m} \|y_i\|_\infty \leq \sqrt{\det(AA^\top) \gcd(A)}.$$ 

Let $z^*$ be a vertex of the integer hull $P_I(A, b)$ that gives an optimal solution to (24). We will show that $z^*$ satisfies (26). First, we argue that it suffices to consider the case $\|z^*\|_0 = n$. Suppose that $\|z^*\|_0 < n$. For $\tau = \text{supp}(z^*)$ set $\bar{A} = A_\tau$, $\bar{b} = b$, $\bar{c} = c_\tau$, and $\bar{z}^* = z^*_\tau$. By removing linearly dependent rows, we may assume that $\bar{A}$ has full row rank. Let $\bar{m} = \text{rank}(\bar{A}) \leq m$. Observe that $\bar{z}^*$ is an optimal solution for the corresponding problem (24) with minimal support. Furthermore, note that $\bar{z}^*$ has full support. Now, if (26) holds true for $\bar{z}^*$, then

$$\rho(\bar{z}^*) + 1)^{s-m} \leq \frac{\sqrt{\det(AA^\top)}}{\gcd(A)}.$$ 

Further, using [3, Lemma 2.3] we have

$$\frac{\sqrt{\det(\bar{A}\bar{A}^\top)}}{\gcd(\bar{A})} \leq \sqrt{\det(AA^\top) \gcd(A)}.$$ 

Combining (52) and (53), we obtain (26).

From now on, we may assume that $\|z^*\|_0 = n$. Suppose, to derive a contradiction, that (26) does not hold, that is $(\rho(z^*) + 1)^{n-m} > (\gcd(A))^{-1} \sqrt{\det(AA^\top)}$. By Theorem 22 there exists a vector $y \in \mathbb{Z}^{n} \setminus \{0\}$ such that

$$Ay = 0 \quad \text{and} \quad \|y\|_\infty \leq \left( \frac{\sqrt{\det(AA^\top)}}{\gcd(A)} \right)^{\frac{1}{n-m}} < \rho(z^*) + 1.$$ 

It follows that both $z^* + y$ and $z^* - y$ are in the knapsack polytope $P_I(A, b)$. Therefore $z^*$ is not a vertex of $P_I(A, b)$. The obtained contradiction completes the proof.


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