The Bayesian Estimation for The Shape Parameter of The Power Function Distribution (PFD-I) to Use Hyper Prior Functions

Jinan Abbas Naser Al-obedy
Technical College of Management-Baghdad, Baghdad, IRAQ
author@institute.xxx

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Abstract

The objective of this study is to examine the properties of Bayes estimators of the shape parameter of the Power Function Distribution (PFD-I), by using two different prior distributions for the parameter $\theta$ and different loss functions that were compared with the Maximum likelihood estimators. In many practical applications, we may have two different prior information about the prior distribution for the shape parameter of the Power Function Distribution, which influences to the parameter estimation. So, we used two different kind of the conjugate priors of shape parameter $\theta$ of the Power Function Distribution (PFD-I) to estimate it. The conjugate prior function of the shape parameter $\theta$ was considered as combination of two different prior distributions such gamma distribution with Erlang distribution and Erlang distribution with exponential distribution and Erlang distribution with non-informative distribution and exponential distribution with non-informative distribution. We derived Bayes estimators for shape parameter $\theta$ of the Power Function Distribution (PFD-I) according to different loss functions such as the squared error loss function (SELF), the weighted error loss function (WSELF) and modified linear exponential (MLINEX) loss function (MLF), with two different double priors. In addition to the classical estimation (maximum likelihood estimation). We used simulation to get results of this study, for different cases of the shape parameter ($\theta$) of the Power Function Distribution used to generate data for different samples sizes.

Paper type: Research paper.

Keywords: The power function distribution (PFD-I), MLE, Bayes Estimation, SELF, WSELF, MLINEX.
1. Introduction

The power function distribution (PFD-I) is a member of continuous probability distributions. The power function distribution used in a wide range of fields such as physics, earth science, economics, social science and the electrical component reliability [4]. The power function distribution used in the analysis of lifetime data and in problems related to the modeling of failure processes. Also, the power function distribution is a flexible life time distribution model that may offer a good fit to some sets of failure data. Theoretically, Power function distribution is a special case of Pareto distribution. The power function distribution is the best distribution to check the reliability of any electrical component. We mention some of studies in a brief manner:

Rahman et.al (2012) [5] estimated the shape parameter \( \theta \) of the power function distribution (PFD-I) using Bayes estimation under different loss functions such as squared error loss function, quadratic loss function, modified linear exponential (MLINEX) loss function and non-linear exponential (NLINEX) loss function and along with maximum likelihood to identify the best estimation among all methods. They concluded that except for few cases Bayes estimator under NLINEX loss function and squared error loss function are better than other estimators in their study.

Kifayat et.al (2012) [3] used Bayesian analysis of The power function distribution under different priors, which are informative (gamma and Rayleigh) priors and non-informative (Jeffreys and uniform) priors. They derived the posterior distribution for the unknown parameter \( \theta \) of the power distribution. Also they derived prior predictive distribution under informative priors, which is used for the elicitation of hyper parameters.

Zaka and Akhtar (2013) [10] used various methods to estimate the shape parameter \( \theta \) of the power function distribution, such as the least squares method and relative least squares method and ridge regression method. They obtain the results by using simulation. They used total deviation (T.D) and mean square error (M.S.E) to identify the best estimation among all methods.

Sultan et.al (2014) [7] derived the posterior distribution of power function distribution under three double priors (gamma-exponential distribution, chi-square-exponential distribution, gamma-chi-square distribution) and three type of single priors. Also they developed posterior predictive distributions under double priors. From the empirical results they determine the best method of estimation according to the smallest value of the posterior standard error and AIC and BIC values. They observed that in most cases, Bayesian estimator under the double prior gamma distribution with exponential distribution has the less posterior standard error and less AIC and BIC values.

Hanif et.al (2015) [2] estimated the shape parameter \( \theta \) of the power function distribution (PFD-I) using Bayes estimation assuming Weibull and Generalized Gamma distributions as priors for the unknown parameters. They derived posterior distribution for parameter \( \theta \) under different priors. Then they derived Bayes estimator of the shape parameter \( \theta \) under the squared error loss function. In addition to the classical estimation (maximum likelihood estimation) to identify the best estimation among all used methods. From the empirical results, they concluded for small sample sizes the Bayes estimator with weibull prior performed better as compared to other estimators.
Ronak and Achyut (2016) [6] estimated the shape parameter $\theta$ of the power function distribution (PFD-I) using Bayes estimation assuming gamma and uniform priors, gamma and jeffrey’s priors, gamma and priors and only gamma prior distributions as priors for the unknown parameters. They derived posterior distribution for parameter $\theta$ under different priors. Then they derived Bayes estimator of the shape parameter $\theta$ under the squared error loss function in addition to the reliability at time $t$, and they constructed of equal tail credible interval for future observation by using simulation to compare the performance of the estimators under different double priors. According to the type-II censored sample from the power function distribution.

A few studies have examined the Bayes estimator of the parameter by considering a combination of two prior distribution, so we try in this study to use Bayes estimator for the shape parameter $\theta$ of the power function distribution by using the conjugate prior of the parameter $\theta$ is considered as combination of two prior distribution and by classical estimation (Maximum Likelihood Estimation).

So the aim of this study is examine the properties of Bayes estimators of the shape parameter of the Power Function Distribution (PFD-I), by using two different prior distributions for the parameter $\theta$ according to each of the posterior distributions for the parameter $\theta$, and different loss functions, and compared these estimators with the Maximum likelihood estimators. Bayes estimation make under different double prior selection for continuous case and under different loss functions. We have assumed gamma with Erlang distribution and Erlang with exponential distribution and Erlang with non-information distribution and exponential with non-information distribution as double priors. And we derive Bayes estimator of shape parameter $\theta$ of the Power Function Distribution under different loss function such as the Squared Errors Loss Function (SELF) and Weighted Squared Errors Loss Function (WSELF) and Modified Linear Exponential (MLINEX) Loss Function.

2. The Power Function Distribution (PFD-I)

Let us consider $t_1, t_2, \ldots, t_n$ is a random sample of $n$ independent observations from a Power Function Distribution (PFD-I) having the probability density function (pdf) with the shape parameter $\theta$ as “Eqn (1)” [2]:

$$f(t; \theta) = \theta t^{(\theta-1)} , \ 0 < t < 1 , \ \theta > 0 \quad (1)$$

and the cumulative distribution function (cdf) is:

$$(t; \theta) = t^\theta , \ 0 \leq t \leq 1 , \ \theta > 0 \quad (2)$$

And the $r^{th}$ moment about origin is

$$E(t^r) = \left( \frac{\theta}{\theta + r} \right) \quad (3)$$

Also, the mean and the variance are as follow

$$\text{Mean} = E(t) = \left( \frac{\theta}{\theta + 1} \right) \quad (4)$$

$$\text{Variance} = \left( \frac{\theta}{(\theta + 2)(\theta + 1)^2} \right) \quad (5)$$
3. Estimation Methods

In this section, we used several methods to estimate of shape parameter \( \theta \) of the Power Function Distribution (PFD-I), such classical estimation (Maximum Likelihood Estimation) and Bayes Estimation Methods as shown below.

3.1 Maximum Likelihood Estimation (MLE)

Here we obtain the MLE for the \( \theta \) based on the density as given in “Eqn (1)” and we can define the likelihood function as follows[1]:

\[
L(\theta) = \prod_{i=1}^{n} f(t; \theta) = \prod_{i=1}^{n} \theta \left(\frac{t_i}{\theta}\right)^{(\theta-1)} = \theta^n \prod_{i=1}^{n} t_i^{(\theta-1)} \quad (6)
\]

We can rewrite it as follow:

\[
L(\theta) = \theta^n \exp \left\{ \sum_{i=1}^{n} \ln(t_i) \right\} \exp \left\{ -\sum_{i=1}^{n} \ln(t_i) \right\} \quad (7)
\]

By taking the log likelihood function on both side in “Eqn (7)” as follows:

\[
\log L(\theta) = n \log(\theta) + \sum_{i=1}^{n} \ln(t_i) - \sum_{i=1}^{n} \ln(t_i) \quad (8)
\]

The MLE for \( \theta \) is

\[
\hat{\theta}_{MLE} = -\frac{n}{\sum_{i=1}^{n} \ln(t_i)} \quad (9)
\]

3.2 Bayes Estimation Method

We used different estimation methods to estimate of shape parameter \( \theta \) of the Power Function Distribution (PFD-I). By assuming \( t_i, i = 1, 2, ..., n \) are (iid) from the Power Function Distribution (PFD-I) as in “Eqn (1)” and likelihood function in “Eqn (7)” from the Power Function Distribution pdf given in “Eqn (1)” can be written as follows[1]:

\[
L(\theta) = \theta^n \exp \left\{ \sum_{i=1}^{n} \ln(t_i) \right\} \exp \left\{ -\sum_{i=1}^{n} \ln(t_i) \right\} \quad (7)
\]

To derive the posterior distributions for the parameter \( \theta \), using a combination of two prior distributions such gamma distribution [8] with Erlang distribution [7] and Erlang distribution with exponential distribution [1], and Erlang distribution with non-informative distribution and exponential distribution with non-informative distribution.

3.2.1 The Conjugate Priors and posterior distributions

To derive the posterior distributions for the parameter \( \theta \), we need to determine the prior distributions for \( \theta \) with pdf , as given below: By assuming a gamma prior for \( \theta \) having pdf [9]

\[
h_1(\theta) = \frac{b^a}{\Gamma(a)} \theta^{a-1} \exp(-b \theta) \quad \text{with} \quad \theta, a, b \geq 0 \quad (10)
\]

And Erlang prior for \( \theta \) having pdf [7],

\[
h_2(\theta) = \lambda^2 \theta \exp(-\lambda \theta) \quad \text{with} \quad \theta, \lambda \geq 0 \quad (11)
\]

And an exponential prior for \( \theta \) having pdf [1],

\[
h_3(\theta) = \lambda_i \exp(-\lambda_i \theta) \quad \text{ith} \quad \theta, \lambda_i \geq 0 \quad (12)
\]

And a non-informative prior for \( \theta \) having pdf ,
h_{i}(\theta) \propto \frac{1}{\theta^{c_{i}}} \quad \text{with} \quad \theta, c_{i} > 0 \quad (13)

Here we define their conjugate prior of the parameter \( \theta \) by combining two priors as follows:

- If \( P_{i}(\theta) \propto \text{gamma}(a, b) \times \text{erlang}(\lambda) \), it means \( P_{i}(\theta) \propto h_{i}(\theta) h_{2}(\theta) \) then we have
  \[
P_{i}(\theta) \propto \left[ \frac{b^{a}}{\Gamma(a)} \right] \theta^{a} \exp(-\theta (b + \lambda)) \quad \text{for} \quad \theta \geq 0, \ a, b, \lambda > 0 \quad (14)
  \]

- If \( P_{2}(\theta) \propto \text{erlang}(\lambda) \times \text{exponential} \ (\lambda_{i}) \), it means \( P_{2}(\theta) \propto h_{2}(\theta) h_{3}(\theta) \) then we have
  \[
P_{2}(\theta) \propto \left[ \lambda^{2} \lambda_{i} \right] \theta \exp(-\theta (\lambda + \lambda_{i})) \quad \text{for} \quad \theta \geq 0, \ \lambda, \lambda_{i} > 0 \quad (15)
  \]

- If \( P_{3}(\theta) \propto \text{erlang}(\lambda) \times \text{non-informative} \ (c_{i}) \), it means \( P_{3}(\theta) \propto h_{3}(\theta) h_{4}(\theta) \) then we have
  \[
P_{3}(\theta) \propto \lambda^{2} \theta^{1-c_{i}} \exp(-\theta \lambda) \quad \text{for} \quad \theta \geq 0, \ \lambda, c_{i} > 0 \quad (16)
  \]

We obtain posterior distribution of the parameter \( \theta \) for the given random sample \( t \) is given by [1]:

\[
\pi(\theta \mid t) = \frac{L(\theta) P(\theta)}{\int_{\theta} L(\theta) P(\theta) d\theta} \quad (18)
\]

using “Eqn (7)” and for each \( P_{i}(\theta), i = 1,2,3,4 \) as shown above in “Eqn (14)” to “Eqn (15)” in “Eqn (18)”, after simplified steps, we get the posterior distributions for the parameter \( \theta \) as follows:

- \text{for} \( P_{i}(\theta) \propto \text{gamma}(a, b) \times \text{erlang}(\lambda) \), we have the posterior distribution is
  \[
  \pi_{i}(\theta \mid t) = \frac{(b + \lambda - \sum_{i=1}^{n} \ln t_{i})^{a+n+1}}{\Gamma(a+n+1)} \theta^{a+n+1-i} \exp(-\theta (b + \lambda - \sum_{i=1}^{n} \ln t_{i}))
  \quad \text{with} \quad \theta \geq 0, \ a, b, \lambda > 0 \quad (19)
  \]

\( \pi_{i}(\theta \mid t) \sim \text{gamma dist}^{a}. \ (a_{\text{new}} = a + n + 1, b_{\text{new}} = b + \lambda - \sum_{i=1}^{n} \ln t_{i}) \)

- \text{for} \( P_{2}(\theta) \propto \text{erlang}(\lambda) \times \text{exponential} \ (\lambda_{i}) \), we have the posterior distribution is
  \[
  \pi_{2}(\theta \mid t) = \frac{(\lambda + \lambda_{i} - \sum_{i=1}^{n} \ln t_{i})^{n+2}}{\Gamma(n+2)} \theta^{n+2-i} \exp(-\theta (\lambda + \lambda_{i} - \sum_{i=1}^{n} \ln t_{i}))
  \quad \text{for} \quad \theta \geq 0 \quad \text{and} \quad \lambda, \lambda_{i}, n > 0 \quad (20)
  \]

\( \pi_{2}(\theta \mid t) \sim \text{gamma dist}^{a}. \ (a_{\text{new}} = n + 2, b_{\text{new}} = (\lambda + \lambda_{i} - \sum_{i=1}^{n} \ln t_{i})) \)
• For $P_i(\theta) \alpha$ Erlang $(\lambda_i \times$ non-informative $(c_i)$, we have the posterior distribution

$$
\pi_i(\theta \mid t) = \frac{(\lambda_i - \sum_{i=1}^{\theta} \ln t_i)^{(n+2-c_i)}}{\Gamma(n+2-c_i)} \theta^{(n+2-c_i)-1} \exp(-\theta(\lambda_i - \sum_{i=1}^{\theta} \ln t_i))
$$

for $\theta > 0$, $\lambda_i, c_i > 0$ \hspace{1cm} (21)

\pi_i(\theta \mid t) \sim \text{gamma dist}^a. \ (a_{\text{new}} = (n + 2 - c_i), b_{\text{new}} = (\lambda_i - \sum_{i=1}^{\theta} \ln t_i))

• For $P_i(\theta) \alpha$ exponential $(\lambda_i \times$ non-informative $(c_i)$, we have the posterior distribution is

$$
\pi_i(\theta \mid t) = \frac{(\lambda_i - \sum_{i=1}^{\theta} \ln t_i)^{(n-c_i+1)}}{\Gamma(n-c_i+1)} \theta^{(n-c_i+1)-1} \exp(-\theta(\lambda_i - \sum_{i=1}^{\theta} \ln t_i))
$$

for $\theta > 0$, $n, \lambda_i, c_i$ \hspace{1cm} (22)

\pi_i(\theta \mid t) \sim \text{gamma dist}^a. \ (a_{\text{new}} = (n - c_i + 1), b_{\text{new}} = (\lambda_i - \sum_{i=1}^{\theta} \ln t_i))

3.2.2 Bayes' Estimators

Here we derive Bayes' estimators $(\hat{\theta})$ for the parameter $\theta$ according to different loss functions such as the squared error loss function (SELF), the weighted error loss function (WSELF) and modified linear exponential (MLINE) loss function (MLF), with two different double priors as follows:

we derive Bayes' estimators $(\hat{\theta})$ for the parameter $\theta$ according to the squared error loss function (SELF), by minimize the posterior expected loss, as follows:

$$
L_i(\hat{\theta}, \theta) = (\hat{\theta} - \theta)^2, \text{ the risk function is } R(\hat{\theta} - \theta) = \hat{\theta}^2 - 2\hat{\theta}E(\theta \mid t) + E(\theta^2 \mid t).
$$

Let $\frac{\partial}{\partial \hat{\theta}} R(\hat{\theta} - \theta) = 0$, we get Bayes estimator of $\theta$ denoted by $\hat{\theta}_{\text{Bayes}}$ for the above prior as follows

$$
\hat{\theta}_{\text{SE}} = E(\theta \mid t) = \int_0^\infty \theta \pi(\theta \mid t) d\theta \hspace{1cm} (23)
$$

So, we derive Bayes' estimators $(\hat{\theta})$ according to the squared error loss function (SELF) with different conjugate prior of the parameter as the mean of the posterior distribution as follows:

• For $\pi_1(\theta \mid t)$ when $P_i(\theta) \alpha$ Gamma $(a, b) \times$ Erlang $(\lambda)$, we have

$$
\hat{\theta}_{\text{SEI}} = \frac{(a + n + 1)}{n} (b + \lambda - \sum_{i=1}^{n} \ln t_i), \ a, n, b, \lambda > 0 \hspace{1cm} (24)
$$
For $\pi_2(\theta \mid t)$ when $P_2(\theta) \propto \text{erlang}(\lambda) \times \text{exponential} \left( \lambda_i \right)$, we have

$$\hat{\theta}_{\text{SE2}} = \frac{(n + 2)}{(n + 2 - \sum \ln t_i)} \quad n, \lambda, \lambda_i > 0 \quad (25)$$

For $\pi_3(\theta \mid t)$ when $P_3(\theta) \propto \text{erlang}(\lambda) \times \text{non-informative} \left( c_i \right)$, we have

$$\hat{\theta}_{\text{SE3}} = \frac{(n + 2 - c_i)}{n} \quad n, c_i, \lambda > 0 \quad (26)$$

For $\pi_4(\theta \mid t)$ when $P_4(\theta) \propto \text{exponential} \left( \lambda_i \right) \times \text{non-informative} \left( c_i \right)$, we have

$$\hat{\theta}_{\text{SE4}} = \frac{(n - c_i + 1)}{(n - c_i + 1 - \sum \ln t_i)} \quad n, c_i, \lambda_i > 0 \quad (27)$$

Also, we derive Bayes' estimators ($\hat{\theta}$) for $\theta$ according to the weighted squared error loss function (WSELF) by minimize the posterior expected loss, as follows:

$$L_2(\hat{\theta}, \theta) = \frac{(\hat{\theta} - \theta)^2}{\theta}, \text{ the risk function is}$$

$$R_2(\hat{\theta}) = \frac{(\hat{\theta} - \theta)^2}{\theta} = E\left\{\frac{1}{\theta} \mid t\right\} - 2\hat{\theta} + E\left\{\theta \mid t\right\}$$

Let $\frac{\partial}{\partial \hat{\theta}} R_2(\hat{\theta}) = 0$, we get Bayes estimator of $\theta$ denoted by $\hat{\theta}_{\text{WSE}}$ for the above prior as follows:

$$\hat{\theta}_{\text{WSE}} = \frac{1}{E\left\{\frac{1}{\theta} \mid t\right\}} = \frac{1}{\int_{0}^{\infty} \pi(\theta \mid t) d\theta} \quad (28)$$

So, we derive Bayes' estimators ($\hat{\theta}$) for $\theta$ according to the weighted squared error loss function (WSELF) with different conjugate prior of the parameter as follows:

For $\pi_1(\theta \mid t)$ when $P_1(\theta) \propto \text{gamma}(\alpha, b) \times \text{erlang}(\lambda)$, we have

$$\hat{\theta}_{\text{WSE1}} = \frac{(a + n)}{(b + \lambda - \sum \ln t_i)} \quad a, n, b, \lambda > 0 \quad (29)$$

For $\pi_2(\theta \mid t)$ when $P_2(\theta) \propto \text{erlang}(\lambda) \times \text{exponential} \left( \lambda_i \right)$, we have

$$\hat{\theta}_{\text{WSE2}} = \frac{(n + 1)}{(\lambda + \lambda_i - \sum \ln \lambda_i)} \quad n, \lambda, \lambda_i > 0 \quad (30)$$
• For $\pi_3(\theta \mid t)$ when $P_3(\theta) \propto \text{erlang}(\lambda) \times \text{non-informative} (c_i)$, we have

$$\hat{\theta}_{\text{WSE3}} = \frac{(n+1-c_i)}{(\lambda - \sum_{i=1}^{n} \ln t_i)} \quad n, c_i, \lambda > 0 \quad (31)$$

• For $\pi_4(\theta \mid t)$ when $P_4(\theta) \propto \text{exponential}(\lambda_1) \times \text{non-informative} (c_i)$, we have

$$\hat{\theta}_{\text{WSE4}} = \frac{(n-c_i)}{(\lambda_1 - \sum_{i=1}^{n} \ln t_i)} \quad n, c_i, \lambda_1 > 0 \quad (32)$$

And, we derive Bayes’ estimators ($\hat{\theta}$) for the parameter $\theta$ according to the Modified linear exponential (MLINEX) loss function (MLF), by minimize the posterior expected loss, as follows [5]:

$$L_3(\theta, \hat{\theta}) = w[\frac{\theta}{\hat{\theta}}]^c \log(\frac{\theta}{\hat{\theta}}) - 1] \quad w > 0, c \neq 0 \quad \text{the risk function is}$$

$$R_3(\theta, \hat{\theta}) = \hat{\theta}^c E(\theta^c \mid t) - c \log(\hat{\theta}) + c E(\log(\theta^c \mid t) - 1$$

Let $\frac{\partial}{\partial \hat{\theta}} R_3(\theta, \hat{\theta}) = 0 \Rightarrow \hat{\theta}^c E(\theta^c \mid t) - \frac{c}{\hat{\theta}} = 0$ Then we have the Bayes estimator of $\theta$ denoted by $\hat{\theta}_{\text{MLF}}$, for the above prior as follows

$$\hat{\theta}_{\text{MLF}} = \left[ E(\theta^c \mid t) \right]^{1/c} = \left[ \frac{\theta}{\Gamma(\alpha)} \Gamma(\alpha + \lambda_1) \right]^{1/c} \quad a, n, b, \lambda > 0, c \neq 0 \quad (33)$$

So, we derive Bayes’ estimators ($\hat{\theta}$) for $\theta$ according the MLINEX loss function (MLF) with different conjugate prior of the parameter as follows:

• For $\pi_1(\theta \mid t)$ when $P_1(\theta) \propto \text{gamma}(a, b) \times \text{erlang}(\lambda)$, we have

$$\hat{\theta}_{\text{MLF1}} = \left[ \frac{\Gamma(a+n-c+1)}{\Gamma(a+n+1)} \left( b + \lambda - \sum_{i=1}^{n} \ln t_i \right) \right]^{1/c} \quad a, n, b, \lambda > 0, c \neq 0 \quad (34)$$

• For $\pi_2(\theta \mid t)$ when $P_2(\theta) \propto \text{erlang}(\lambda) \times \text{exponential}(\lambda_1)$, we have

$$\hat{\theta}_{\text{MLF2}} = \left[ \frac{\Gamma(n-c+2)}{\Gamma(n+2)} \left( \lambda_1 + \lambda_1 - \sum_{i=1}^{n} \ln t_i \right) \right]^{1/c} \quad n, \lambda, \lambda_1 > 0, c \neq 0 \quad (35)$$

• For $\pi_3(\theta \mid t)$ when $P_3(\theta) \propto \text{erlang}(\lambda) \times \text{non-informative} (c_i)$, we have

$$\hat{\theta}_{\text{MLF3}} = \left[ \frac{\Gamma(n+2-c_i-c)}{\Gamma(n+2-c_i)} \right]^{1/c} \quad n, c_i, \lambda > 0, c \neq 0 \quad (36)$$

• For $\pi_4(\theta \mid t)$ when $P_4(\theta) \propto \text{exponential}(\lambda_1) \times \text{non-informative} (c_i)$, we have

$$\hat{\theta}_{\text{MLF4}} = \left[ \frac{\Gamma(n-c_i-c+1)}{\Gamma(n-c_i+1)} \right]^{1/c} \quad n, c_i, \lambda_1 > 0, c \neq 0 \quad (37)$$
4. Simulation Study

For compare the quality of the Bayes estimators with the Maximum likelihood estimator for \( \theta \) according to the assumed loss functions and the values for the parameters of each of the posterior distributions for the parameter \( \theta \), to determine the best estimation among all these estimators, according to smallest value our criterion which are listed below. We used empirical study. So we have considered several steps to perform simulation study by taking:

1. The samples of sizes \( n = 25, 50, 75 \) and \( 100 \) which represented small, moderate and large sample size.

2. Different values of the shape parameters were chosen as \( \theta = 0.5, 1, 1.5 \) for the Power Function Distribution (PFD-I).

3. The generated random samples for all sizes \( n \) (from uniform \( (0,1) \), then \( t_i = u_i^{(1/\theta)} \) is random sample from the Power Function Distribution (PFD-I).

4. assuming values of the parameters of each the posterior distributions of the parameter \( \theta \) as different combinations to be compare listed below:

5. The experimental results were repeated \( (r = 1000) \) times for each sample size \( n \).

We obtained Bayes’ estimators and the Maximum likelihood estimator (MLE) of shape parameter \( \theta \) of the Power Function Distribution (PFD-I), by using MATLAB-R2018a program. Then computed the Mean Square Errors (MSE) and Mean Weighted Square Errors (MWSE) and Mean Modified Linear Exponential (MLINEX) to determine the best estimation among all used methods, according to smallest value our criterion which are:

- The squared error loss function (SELF)

\[
\text{MSE}(\hat{\theta}) = \frac{1}{1000} \sum_{r=1}^{r=1000} (\hat{\theta}(r) - \theta)^2 \quad (38)
\]

- The weighted error loss function (WSELF)

\[
\text{MWSE}(\hat{\theta}) = \frac{1}{1000} \sum_{r=1}^{r=1000} \frac{(\hat{\theta}(r) - \theta)^2}{\theta} \quad (39)
\]

- Modified linear exponential (MLINEX) loss function (MLF) calculated with \( w=1 \) and two value for \( c=1,2 \).

\[
\text{MLINEX}(\hat{\theta}) = \frac{1}{1000} \sum_{r=1}^{r=1000} \left( w \left( \frac{\hat{\theta}(r)}{\theta} \right)^c - c \log \left( \frac{\hat{\theta}(r)}{\theta} \right) - 1 \right), \quad w > 0, c \neq 0 \quad (40)
\]

The experimental results for this study were summarized and presented in tables, according to our aim to the study, to determine the best estimation for of
of the parameter $\theta$, according to smallest value our criterion under different loss functions, under different conjugate prior of the parameter for all the true value of $\theta = 0.5, 1, 1.5$ that are assumed.

So the experimental results under the squared error loss function (SELF) are listed in tables (4-1) to (4-3). And the experimental results under the weighted error loss function (WSELF) are listed in tables (4-4) to (4-7). Also, the experimental results under Modified linear exponential (MLINEX) loss function (MLF) calculated with $w=1$ and two value for $c=1, 2$ are listed in tables (4-1) to (4-3).

Table (4-1): Estimated value ($\hat{\theta}_{SE}$) and $\text{MSE}(\hat{\theta})$ of PFD-I, under the SELF, under different conjugate prior of the parameter for $\theta = 0.5$.

| Method | parameter | Estimated value ($\hat{\theta}_{SE}$) | $\text{MSE}(\hat{\theta})$ |
|--------|-----------|------------------------------------|----------------------------|
|        |           | Sample Size(n) |                     | 25   | 50   | 75   | 100  |     | 25   | 50   | 75   | 100  |
| MLE    | -         | -                    |                          |      |      |      |      |     |      |      |      |      |
| Bayes  | $a$       | $b$                  | $\lambda$                | 3    | 2    | 4    | 0.53653 | 0.52049 | 0.51413 | 0.51149 | 0.01070 | 0.00512 | 0.00347 | 0.00257 |
| Bayes  | $\lambda$ | $\lambda_1$          | $\lambda_1$              | 3    | 2    | 6    | 0.51681 | 0.51048 | 0.50744 | 0.50647 | 0.00832 | 0.00445 | 0.00316 | 0.00238 |
| Bayes  | $c$       |                      | $c_1$                    | 1    | 2    | 6    | 0.50832 | 0.50589 | 0.5043  | 0.50407 | 0.00815 | 0.00438 | 0.00313 | 0.00236 |
| Bayes  | $\lambda_1$ | $c_1$           | $\lambda_1$              | -    | 4    | 1    | 0.50926 | 0.50617 | 0.50443 | 0.50416 | 0.00887 | 0.00457 | 0.00322 | 0.00241 |
| Bayes  | $\lambda_1$ | $c_1$           | $\lambda_1$              | -    | 4    | 2    | 0.49953 | 0.50121 | 0.50111 | 0.50166 | 0.00811 | 0.00436 | 0.00311 | 0.00235 |
| Bayes  | $\lambda_1$ | $c_1$           | $\lambda_1$              | -    | 4    | 4    | 0.48116 | 0.49157 | 0.49459 | 0.49673 | 0.00732 | 0.00409 | 0.00298 | 0.00226 |
| Bayes  | $\lambda_1$ | $c_1$           | $\lambda_1$              | -    | 6    | 4    | 0.46414 | 0.48230 | 0.48825 | 0.49189 | 0.00730 | 0.00403 | 0.00294 | 0.00223 |
| Bayes  | $\lambda_1$ | $c_1$           | $\lambda_1$              | -    | 4    | 1    | 0.50016 | 0.50141 | 0.50121 | 0.50172 | 0.00883 | 0.00454 | 0.00320 | 0.00240 |
| Bayes  | $\lambda_1$ | $c_1$           | $\lambda_1$              | -    | 4    | 2    | 0.48092 | 0.49157 | 0.49461 | 0.49675 | 0.00853 | 0.00444 | 0.00314 | 0.00226 |
| Bayes  | $\lambda_1$ | $c_1$           | $\lambda_1$              | -    | 3    | 3    | 0.47107 | 0.48662 | 0.49130 | 0.49427 | 0.00901 | 0.00455 | 0.00319 | 0.00238 |
| Bayes  | $\lambda_1$ | $c_1$           | $\lambda_1$              | -    | 2    | 3    | 0.48087 | 0.49159 | 0.49463 | 0.49677 | 0.00926 | 0.00463 | 0.00323 | 0.00241 |
### Table (4-2): Estimated value $\hat{\theta}_{SE}$ and MSE($\hat{\theta}$) of PFD-I, under the SELF, under different conjugate prior of the parameter for $\theta = 1$.

| Method | parameter | Estimated value $\hat{\theta}_{SE}$ | MSE($\hat{\theta}$) |
|--------|-----------|--------------------------------------|---------------------|
|        |           | Sample Size(n)                       |                     |
|        |           | 25        | 50        | 75        | 100       | 25        | 50        | 75        | 100       |
| MLE    | -         | -         | -         | -         | -         | 1.03580  | 1.01840  | 1.01550  | 1.01410  | 0.04866  | 0.02182  | 0.01446  | 0.01129  |
| Bayes  | $a$       | $b$       | $\lambda$ |           |           |           |           |           |           |           |           |           |           |           |
|        | 3         | 2         | 4         | 0.95577   | 0.97816   | 0.98381   | 0.99361   | 0.02667  | 0.01603  | 0.01161  | 0.00946  |           |           |           |
|        | 3         | 2         | 6         | 0.89527   | 0.94344   | 0.96302   | 0.97481   | 0.02990  | 0.01663  | 0.01108  | 0.00935  |           |           |           |
|        | 1         | 2         | 6         | 0.83535   | 0.90850   | 0.93952   | 0.95606   | 0.04412  | 0.02083  | 0.01351  | 0.01032  |           |           |           |
|        | 2         | 1         | 6         | 0.89263   | 0.94270   | 0.96361   | 0.97466   | 0.03164  | 0.01720  | 0.01196  | 0.00953  |           |           |           |
| Bayes  | $\lambda$ | $\lambda_1$ |           |           |           |           |           |           |           |           |           |           |           |           |
|        | -         | -         | -         |           |           |           |           |           |           |           |           |           |           |           |
|        | 4         | 1         | 0.92107   | 0.95960   | 0.97564   | 0.98400   | 0.03090  | 0.01718  | 0.01206  | 0.00968  |           |           |           |
|        | 4         | 2         | 0.89986   | 0.94193   | 0.96329   | 0.97450   | 0.03356  | 0.01779  | 0.01225  | 0.00971  |           |           |           |
|        | 4         | 3         | 0.83535   | 0.90850   | 0.93952   | 0.95606   | 0.04412  | 0.02083  | 0.01351  | 0.01032  |           |           |           |
|        | 4         | 6         | 0.78404   | 0.87740   | 0.91690   | 0.93832   | 0.05943  | 0.02585  | 0.01584  | 0.01158  |           |           |           |

### Table (4-3): Estimated value $\hat{\theta}_{SE}$ and MSE($\hat{\theta}$) of PFD-I, under the SELF, under different conjugate prior of the parameter for $\theta = 1.5$.

| Method | parameter | Estimated value $\hat{\theta}_{SE}$ | MSE($\hat{\theta}$) |
|--------|-----------|--------------------------------------|---------------------|
|        |           | Sample Size(n)                       |                     |
|        |           | 25        | 50        | 75        | 100       | 25        | 50        | 75        | 100       |
| MLE    | -         | -         | -         | -         | -         | 1.55380  | 1.52760  | 1.52320  | 1.52120  | 0.10950  | 0.04910  | 0.03254  | 0.02540  |
| Bayes  | $a$       | $b$       | $\lambda$ |           |           |           |           |           |           |           |           |           |           |           |
|        | 3         | 2         | 4         | 1.30180   | 1.39050   | 1.42830   | 1.44850   | 0.07683  | 0.04015  | 0.02737  | 0.02154  |           |           |           |
|        | 3         | 2         | 6         | 1.19260   | 1.32150   | 1.37790   | 1.40890   | 0.12085  | 0.05478  | 0.03414  | 0.02519  |           |           |           |
|        | 1         | 2         | 6         | 1.11030   | 1.27260   | 1.34300   | 1.38180   | 0.17468  | 0.07299  | 0.04292  | 0.03021  |           |           |           |
|        | 2         | 1         | 6         | 1.20180   | 1.33000   | 1.38490   | 1.41470   | 0.11810  | 0.05333  | 0.03340  | 0.02479  |           |           |           |
| Bayes  | $\lambda$ | $\lambda_1$ |           |           |           |           |           |           |           |           |           |           |           |           |
|        | -         | -         | -         |           |           |           |           |           |           |           |           |           |           |           |
|        | 4         | 1         | 1.27040   | 1.37490   | 1.41800   | 1.44090   | 0.09213  | 0.04471  | 0.02946  | 0.02273  |           |           |           |
|        | 4         | 2         | 1.21200   | 1.33900   | 1.39210   | 1.42070   | 0.11548  | 0.05203  | 0.03275  | 0.02447  |           |           |           |
|        | 4         | 4         | 1.11030   | 1.27260   | 1.34300   | 1.38180   | 0.17468  | 0.07299  | 0.04292  | 0.03021  |           |           |           |
|        | 6         | 4         | 1.02460   | 1.21250   | 1.2973    | 1.34510   | 0.24250  | 0.10016  | 0.05698  | 0.03857  |           |           |           |
| Bayes  | $\lambda$ | $c_1$    |           |           |           |           |           |           |           |           |           |           |           |           |
|        | -         | -         | -         |           |           |           |           |           |           |           |           |           |           |           |
|        | 4         | 1         | 1.28540   | 1.38570   | 1.42620   | 1.44740   | 0.09078  | 0.04428  | 0.02934  | 0.02276  |           |           |           |
|        | 4         | 2         | 1.23590   | 1.35860   | 1.40740   | 1.43310   | 0.11108  | 0.05002  | 0.03184  | 0.02408  |           |           |           |
| Method | Parameter | Estimated Value $\hat{\theta}_{\text{WSE}}$ | MWSE( $\theta$ ) |
|--------|-----------|---------------------------------|-----------------|
|        |           |                                 | Sample Size(n)  |
|        |           |                                 | MLE            |
|        |           |                                 | Bayes           |
| MLE    | - -       | 0.52269 0.51209 0.50820 0.50692 | 0.02045 0.01062 0.00710 0.00519 |
| Bayes  | a b $\lambda_i$ | $P_2(\theta) \sim \text{gamma}(a, b) \times \text{erlang}(\lambda_i)$ | $P_1(\theta) \sim \text{exponential}(\lambda_i)$ |
|        | 3 2 6 6  | 0.49898 0.50102 0.50160 0.46334 0.48212 0.48817 0.49186 | 0.01561 0.00838 0.00603 0.00455 |
|        | 1 2 6 6  | 0.46083 0.48156 0.48794 0.49175 | 0.01941 0.00943 0.00653 0.00483 |
|        | 3 2 1 1  | 0.90934 0.44200 0.29410 0.02289 |                                  |

**Table (4-4):** Estimated value $\hat{\theta}_{\text{WSE}}$ and MWSE($\theta$) of PFD-I, under the WSELF, under different conjugate prior of the parameter for $\theta = 0.5$.

| Method | Parameter | Estimated Value $\hat{\theta}_{\text{WSE}}$ | MWSE( $\theta$ ) |
|--------|-----------|---------------------------------|-----------------|
|        |           |                                 | Sample Size(n)  |
|        |           |                                 | MLE            |
|        |           |                                 | Bayes           |
| MLE    | - -       | 1.03580 1.01840 1.01550 1.01410 | 0.04866 0.02182 0.01446 0.01129 |
| Bayes  | a b $\lambda_i$ | $P_2(\theta) \sim \text{gamma}(a, b) \times \text{erlang}(\lambda_i)$ | $P_1(\theta) \sim \text{exponential}(\lambda_i)$ |
|        | 3 2 4 6  | 0.92282 0.96004 0.97580 0.98406 | 0.02900 0.01658 0.01177 0.00949 |
|        | 3 2 6 6  | 0.86440 0.92597 0.95172 0.96544 | 0.03603 0.01842 0.01244 0.00975 |
|        | 1 2 6 6  | 0.80266 0.89103 0.92731 0.94669 | 0.05416 0.02386 0.01488 0.01107 |
| Method | parameter | Estimated value (\(^\hat{\theta}_{WSE}\)) | MWSE(\(^\hat{\theta}\)) |
|--------|-----------|---------------------------------|----------------------|
|        |           | Sample Size(n) | 25 | 50 | 75 | 100 | Sample Size(n) | 25 | 50 | 75 | 100 |
| MLE    | -        | -                | 1.55380 | 1.52760 | 1.52320 | 1.52120 | 0.07300 | 0.03273 | 0.02169 | 0.01693 |
| Bayes  | a, b     | \(\lambda\) | 1.25690 | 1.36480 | 1.41020 | 1.43460 | 0.06273 | 0.03028 | 0.01982 | 0.01520 |
|        |          | \(c_1\) | 1.15140 | 1.29700 | 1.36050 | 1.39540 | 0.09736 | 0.04218 | 0.02548 | 0.01834 |
|        |          | \(\lambda_1\) | 1.06920 | 1.24810 | 1.32560 | 1.36830 | 0.13784 | 0.05594 | 0.03215 | 0.02219 |
|        |          | \(c_1\) | 1.15890 | 1.30490 | 1.36710 | 1.40100 | 0.09566 | 0.04105 | 0.02486 | 0.01799 |
| Bayes  | \(\lambda\) | \(c_1\) | 1.22330 | 1.34850 | 1.39960 | 1.42680 | 0.07539 | 0.03394 | 0.02149 | 0.01615 |
|        |          | \(\lambda_1\) | 1.16710 | 1.31330 | 1.37400 | 1.40670 | 0.09398 | 0.03999 | 0.02429 | 0.01768 |
|        |          | \(c_1\) | 1.06920 | 1.24810 | 1.32560 | 1.36830 | 0.13784 | 0.05594 | 0.03215 | 0.02219 |
|        |          | \(\lambda_1\) | 0.98666 | 1.18920 | 1.28050 | 1.3319 | 0.18588 | 0.07563 | 0.04245 | 0.02836 |

Table (4-6): Estimated value (\(^\hat{\theta}_{WSE}\)) and MWSE(\(^\hat{\theta}\)) of PFD-I, under the WSELF, under different conjugate prior of the parameter for \(\theta = 1.5\).

| Method | parameter | Estimated value (\(^\hat{\theta}_{MLE}\)) | MLINEX(\(^\hat{\theta}\)) |
|--------|-----------|---------------------------------|----------------------|
|        |           | Sample Size(n) | 25 | 50 | 75 | 100 | Sample Size(n) | 25 | 50 | 75 | 100 |
| MLE    | -        | -                | 1.23590 | 1.35860 | 1.40740 | 1.43310 | 0.07405 | 0.03334 | 0.02123 | 0.01605 |
|        |          | \(\lambda\) | 1.18650 | 1.33140 | 1.38860 | 1.41880 | 0.09092 | 0.03817 | 0.02337 | 0.01721 |
|        |          | \(c_1\) | 1.19800 | 1.34130 | 1.39640 | 1.42510 | 0.08970 | 0.03745 | 0.02303 | 0.01705 |
|        |          | \(\lambda_1\) | 1.26600 | 1.38050 | 1.42400 | 1.44630 | 0.07277 | 0.03274 | 0.02103 | 0.01605 |
|        |          | \(c_1\) | 1.34260 | 1.42220 | 1.45270 | 1.46820 | 0.06279 | 0.03024 | 0.02011 | 0.01569 |
|        |          | \(\lambda_1\) | 1.32110 | 1.40930 | 1.44350 | 1.46110 | 0.06084 | 0.02968 | 0.01978 | 0.01543 |
|        |          | \(c_1\) | 1.25000 | 1.36920 | 1.41550 | 1.43960 | 0.07310 | 0.03293 | 0.02107 | 0.01602 |

Table (4-7): Estimated value (\(^\hat{\theta}_{MLE}\)) and MLINEX(\(^\hat{\theta}\)) of PFD-I, under the MLINEX, under different conjugate prior of the parameter for \(\theta = 0.5\) and \(w=1\) and \(c=1\).
| Method | parameter | Estimated value (θ_MLF) | MLINEX (θ) |
|--------|-----------|-------------------------|------------|
|        |           | Sample Size(n) | Sample Size(n) | |
| MLE    | -         | 25 | 0.52269 | 0.02126 |
|        | -         | 50 | 0.51209 | 0.00985 |
|        | -         | 75 | 0.5082 | 0.00672 |
|        | -         | 100 | 0.50692 | 0.00497 |
| Bayes  | a         | 25 | 0.51803 | 0.01652 |
|        | b         | 50 | 0.51085 | 0.00869 |
|        | λ         | 75 | 0.50762 | 0.00612 |
|        |           | 100 | 0.50692 | 0.00467 |
|        |           |     |     |     |
| Bayes  | λ_1       | 25 | 0.49017 | 0.01550 |
|        |           | 50 | 0.49634 | 0.00832 |
|        |           | 75 | 0.49783 | 0.00600 |
|        |           | 100 | 0.49918 | 0.00454 |
| Bayes  | λ_1       | 25 | 0.49040 | 0.01669 |
|        |           | 50 | 0.49644 | 0.00865 |
|        |           | 75 | 0.49788 | 0.00616 |
|        |           | 100 | 0.49922 | 0.00470 |
| Bayes  | λ_1       | 25 | 0.48092 | 0.01791 |
|        |           | 50 | 0.49157 | 0.00948 |
|        |           | 75 | 0.49461 | 0.00630 |
|        |           | 100 | 0.49675 | 0.00470 |
| Bayes  | λ_1       | 25 | 0.48087 | 0.01936 |
|        |           | 50 | 0.49159 | 0.00931 |
|        |           | 75 | 0.49463 | 0.00680 |
|        |           | 100 | 0.49677 | 0.00496 |
| Bayes  | λ_1       | 25 | 0.47107 | 0.02185 |
|        |           | 50 | 0.48662 | 0.00986 |
|        |           | 75 | 0.49130 | 0.00671 |
|        |           | 100 | 0.49427 | 0.00491 |

Method parameters for MLF and MLINEX include:
- MLF: $\theta$ (MLE)
- MLINEX: $\theta$ (Bayes)
- Sample Size: n

Bayes parameter estimates for various distributions:
- $\Lambda_1$ for $P_\Lambda(\theta) \propto \text{Gamma}(a, b) \times \text{Erlang}(\lambda)$
- $\Lambda_2$ for $P_\Lambda(\theta) \propto \text{Gamma}(a, b) \times \text{Exponential}(\lambda)$
- $\Lambda_3$ for $P_\Lambda(\theta) \propto \text{Beta}(a, b)$
Table (4-8): Estimated value $\hat{\theta}_{\text{MLE}}$ and MLINEX $\hat{\theta}$ of PFD-I, under the MLINEX, under different conjugate prior of the parameter for $\theta = 1$ and $w=1$ and $c=1$.

| Method | Parameter | Estimated value $\hat{\theta}_{\text{MLE}}$ | MLINEX $\hat{\theta}$ |
|--------|-----------|-----------------------------------------------|----------------------|
|        |           | Estimate | Standard Error | Estimate | Standard Error |
|        |           | 25       | 50       | 75       | 100       | 25       | 50       | 75       | 100       |
| MLE    | - -       | 1.03580  | 1.01840  | 1.01550  | 1.01410  | 0.02155  | 0.01028  | 0.00694  | 0.00541  |
| Bayes  | a b $\lambda$ | for $\pi_1(\theta \mid t)$ when $P_2(\theta) \alpha \text{gamma}(a, b) \times \text{erlang}(\lambda)$ |
| 3 2 4 | 0.92282 | 0.96004 | 0.97580 | 0.98406 | 0.01636 | 0.00882 | 0.00613 | 0.00484 |
| 3 2 6 | 0.86440 | 0.92597 | 0.95172 | 0.96544 | 0.02173 | 0.01032 | 0.00675 | 0.00514 |
| 1 2 6 | 0.80266 | 0.89103 | 0.92731 | 0.94669 | 0.03409 | 0.01385 | 0.00832 | 0.00601 |
| 2 1 6 | 0.86075 | 0.92491 | 0.95125 | 0.96520 | 0.02307 | 0.01068 | 0.00691 | 0.00524 |
| Bayes  | - $\lambda$ $c_1$ | for $\pi_2(\theta \mid t)$ when $P_3(\theta) \alpha \text{erlang}(\lambda) \times \text{exponential}(\lambda_1)$ |
| - 4 1 | 0.88695 | 0.94115 | 0.96297 | 0.97435 | 0.02108 | 0.01010 | 0.00668 | 0.00514 |
| - 4 2 | 0.85690 | 0.92382 | 0.95078 | 0.96495 | 0.02455 | 0.01106 | 0.00708 | 0.00534 |
| - 4 4 | 0.80266 | 0.89103 | 0.92731 | 0.94669 | 0.03409 | 0.01385 | 0.00832 | 0.00601 |
| - 6 4 | 0.75501 | 0.86052 | 0.90499 | 0.92912 | 0.04631 | 0.01768 | 0.01011 | 0.00700 |
| Bayes  | - $\lambda$ $c_1$ | for $\pi_3(\theta \mid t)$ when $P_4(\theta) \alpha \text{erlang}(\lambda) \times \text{non-informative}(c_1)$ |
| - 4 1 | 0.88390 | 0.94035 | 0.96265 | 0.97419 | 0.02252 | 0.01047 | 0.00685 | 0.00524 |
| - 4 2 | 0.84855 | 0.92154 | 0.94981 | 0.96445 | 0.02799 | 0.01187 | 0.00744 | 0.00554 |
| - 3 3 | 0.84401 | 0.92035 | 0.94932 | 0.96420 | 0.02999 | 0.01231 | 0.00763 | 0.00565 |
| - 2 3 | 0.87734 | 0.93868 | 0.96199 | 0.97388 | 0.02591 | 0.01128 | 0.00721 | 0.00544 |

Table (4-9): Estimated value $\hat{\theta}_{\text{MLE}}$ and MLINEX $\hat{\theta}$ of PFD-I, under the MLINEX, under different conjugate prior of the parameter for $\theta = 1.5$ and $w=1$ and $c=1$.

| Method | Parameter | Estimated value $\hat{\theta}_{\text{MLE}}$ | MLINEX $\hat{\theta}$ |
|--------|-----------|-----------------------------------------------|----------------------|
|        |           | Estimate | Standard Error | Estimate | Standard Error |
|        |           | 25       | 50       | 75       | 100       | 25       | 50       | 75       | 100       |
| MLE    | - -       | 1.55350  | 1.52760  | 1.52320  | 1.52120  | 0.02155  | 0.01028  | 0.00694  | 0.00541  |
| Bayes  | a b $\lambda$ | for $\pi_1(\theta \mid t)$ when $P_2(\theta) \alpha \text{gamma}(a, b) \times \text{erlang}(\lambda)$ |
| 3 2 4 | 1.25690 | 1.36480 | 1.41020 | 1.43460 | 0.02566 | 0.01151 | 0.00728 | 0.00543 |
| 3 2 6 | 1.15140 | 1.29700 | 1.36050 | 1.39540 | 0.04125 | 0.01656 | 0.00966 | 0.00676 |
| 1 2 6 | 1.06920 | 1.24810 | 1.32560 | 1.36830 | 0.06053 | 0.02240 | 0.01237 | 0.00830 |
| 2 1 6 | 1.15890 | 1.30490 | 1.36710 | 1.40100 | 0.04058 | 0.01609 | 0.00939 | 0.00660 |
| Bayes  | - $\lambda$ $c_1$ | for $\pi_2(\theta \mid t)$ when $P_3(\theta) \alpha \text{erlang}(\lambda) \times \text{exponential}(\lambda_1)$ |
| - 4 1 | 1.22330 | 1.34850 | 1.39960 | 1.42680 | 0.03142 | 0.01305 | 0.00796 | 0.00581 |
| - 4 2 | 1.16710 | 1.31330 | 1.37400 | 1.40670 | 0.03992 | 0.01564 | 0.00915 | 0.00646 |
| - 4 4 | 1.06920 | 1.24810 | 1.32560 | 1.36830 | 0.06053 | 0.02240 | 0.01237 | 0.00830 |
| - 6 4 | 0.98666 | 1.18920 | 1.28050 | 1.33190 | 0.08449 | 0.03088 | 0.01659 | 0.01078 |
Table (4-10): Estimated value ($\hat{\theta}_{\text{MLF}}$) and $\text{MLINEX}(\hat{\theta})$ of PFD-I, under the $\text{MLINEX}$, under different conjugate prior of the parameter for $\theta = 0.5$ and $w=1$ and $c=2$.

| Method | parameter | Estimated value ($\hat{\theta}_{\text{MLF}}$) | $\text{MLINEX}(\hat{\theta})$ |
|--------|-----------|---------------------------------|--------------------------------|
|        |           | Sample Size(n) |                     | Sample Size(n) |                     |
| MLE    |           | 25       | 50       | 75       | 100                 | 25       | 50       | 75       | 100                 |
| Bayes  | $\lambda_1$ | 3 2 4  | 0.50869 | 0.50600 | 0.50435 | 0.50411 | 0.06606 | 0.03498 | 0.02493 | 0.01880               |
|        |           | 3 2 6  | 0.48999 | 0.49627 | 0.49780 | 0.49916 | 0.05922 | 0.03280 | 0.02385 | 0.01812               |
|        |           | 1 2 6  | 0.45345 | 0.47737 | 0.48495 | 0.48942 | 0.07158 | 0.03562 | 0.02503 | 0.01866               |
|        |           | 2 1 6  | 0.48100 | 0.49155 | 0.49459 | 0.49673 | 0.06248 | 0.03360 | 0.02421 | 0.01829               |
| Bayes  | $\lambda_1 c_1$ | - 4 1  | 0.48088 | 0.49155 | 0.49460 | 0.49674 | 0.06732 | 0.03495 | 0.02486 | 0.01866               |
|        |           | - 4 2  | 0.47168 | 0.48673 | 0.49134 | 0.49427 | 0.06728 | 0.03478 | 0.02474 | 0.01856               |
|        |           | - 4 4  | 0.45345 | 0.47737 | 0.48495 | 0.48942 | 0.07158 | 0.03562 | 0.02503 | 0.01866               |
|        |           | - 6 4  | 0.43827 | 0.46837 | 0.47873 | 0.48465 | 0.08081 | 0.03789 | 0.02598 | 0.01915               |
| Bayes  | $\lambda_1 c_1$ | - 4 1  | 0.47120 | 0.48663 | 0.49131 | 0.49426 | 0.07246 | 0.03618 | 0.02541 | 0.01894               |
|        |           | - 4 2  | 0.45196 | 0.47651 | 0.48471 | 0.48930 | 0.08322 | 0.03843 | 0.02636 | 0.01941               |
|        |           | - 3 3  | 0.44152 | 0.47169 | 0.48133 | 0.48677 | 0.09299 | 0.04094 | 0.02746 | 0.02000               |
|        |           | - 2 3  | 0.45070 | 0.47651 | 0.48459 | 0.48924 | 0.08863 | 0.03996 | 0.02707 | 0.01980               |
| Bayes  | $\lambda_1 c_1$ | - 1 2  | 0.46029 | 0.48144 | 0.48789 | 0.49173 | 0.08591 | 0.03939 | 0.02685 | 0.01971               |
|        |           | - 2 2  | 0.45070 | 0.47651 | 0.48459 | 0.48924 | 0.08863 | 0.03996 | 0.02707 | 0.01980               |
|        |           | - 2 1  | 0.47074 | 0.48655 | 0.49127 | 0.49425 | 0.07833 | 0.03767 | 0.02610 | 0.01933               |
|        |           | - 3 1  | 0.46116 | 0.48162 | 0.48797 | 0.49176 | 0.07948 | 0.03784 | 0.02614 | 0.01932               |
Table (4-11): Estimated value ($\hat{\theta}_{\text{MLF}}$) and MLINEX($\hat{\theta}$) of PFD-I, under the MLINEX, under different conjugate prior of the parameter for $\theta=1$ and w=1 and c=2.

| Method | parameter | $\theta_{\text{MLF}}$ | MLINEX($\theta$) |
|--------|-----------|------------------------|-----------------|
|        |           | Estimated value         | Sample Size(n)   | Sample Size(n)   |
|        |           |                        | 25   | 50   | 75   | 100  | 25   | 50   | 75   | 100  |
| MLE    | - -       | 1.03580                | 1.01840        | 1.01550        | 1.01410 | 0.09177 | 0.04239 | 0.02836 | 0.02211 |
| Bayes  | a b λ     | for $\alpha_1(\theta|t)$ when $P_2(\theta)\propto\text{gamma}(a,b)\times\text{erlang}(\lambda)$ |
| 3 2 4 | 0.90619 | 0.95094 | 0.96953 | 0.97927 | 0.06686 | 0.03560 | 0.02459 | 0.01944 |
| 3 2 6 | 0.84882 | 0.91720 | 0.94560 | 0.96074 | 0.08855 | 0.04170 | 0.02712 | 0.02067 |
| 1 2 6 | 0.78707 | 0.88225 | 0.92119 | 0.94199 | 0.13621 | 0.05556 | 0.03335 | 0.02408 |
| 2 1 6 | 0.88466 | 0.91597 | 0.94506 | 0.96045 | 0.09386 | 0.04311 | 0.02776 | 0.02105 |
| Bayes  | - λ 1     | for $\alpha_3(\theta|t)$ when $P_3(\theta)\propto\text{erlang}(\lambda)\times\text{exponential}(\lambda_1)$ |
| - 4 1 | 0.86973 | 0.93187 | 0.95662 | 0.96951 | 0.08591 | 0.04078 | 0.02682 | 0.02064 |
| - 4 2 | 0.84026 | 0.91471 | 0.94451 | 0.96016 | 0.09968 | 0.04459 | 0.02842 | 0.02144 |
| - 4 4 | 0.78707 | 0.88225 | 0.92119 | 0.94199 | 0.13621 | 0.05556 | 0.03335 | 0.02408 |
| - 6 4 | 0.74034 | 0.85205 | 0.89902 | 0.92451 | 0.18137 | 0.07030 | 0.04031 | 0.02799 |
| Bayes  | - λ c_1   | for $\alpha_5(\theta|t)$ when $P_5(\theta)\propto\text{erlang}(\lambda)\times\text{non-informative}(c_1)$ |
| - 4 1 | 0.86605 | 0.93089 | 0.95621 | 0.96931 | 0.09162 | 0.04225 | 0.02749 | 0.02104 |
| - 4 2 | 0.83068 | 0.91209 | 0.94337 | 0.95957 | 0.11312 | 0.04780 | 0.02983 | 0.02226 |
| - 3 3 | 0.82845 | 0.91071 | 0.94279 | 0.95926 | 0.12093 | 0.04954 | 0.03087 | 0.02269 |
| - 2 3 | 0.85806 | 0.92886 | 0.95537 | 0.96889 | 0.10502 | 0.04545 | 0.02890 | 0.02187 |
| Bayes  | - λ_1 c_1 | for $\alpha_7(\theta|t)$ when $P_7(\theta)\propto\text{exponential}(\lambda_1)\times\text{non-informative}(c_1)$ |
| - 1 2 | 0.89345 | 0.94775 | 0.96830 | 0.97872 | 0.09414 | 0.04281 | 0.02790 | 0.02144 |
| - 2 2 | 0.85806 | 0.92886 | 0.95537 | 0.96889 | 0.10502 | 0.04545 | 0.02890 | 0.02187 |
| - 2 1 | 0.89621 | 0.94841 | 0.96855 | 0.97883 | 0.08740 | 0.04119 | 0.02718 | 0.02101 |
| - 3 1 | 0.86216 | 0.92989 | 0.95558 | 0.96910 | 0.09796 | 0.04381 | 0.02818 | 0.02145 |

Table (4-12): Estimated value ($\hat{\theta}_{\text{MLF}}$) and MLINEX($\hat{\theta}$) of PFD-I, under the MLINEX, under different conjugate prior of the parameter for $\theta=1.5$ and w=1 and c=2.

| Method | parameter | $\theta_{\text{MLF}}$ | MLINEX($\theta$) |
|--------|-----------|------------------------|-----------------|
|        |           | Estimated value         | Sample Size(n)   | Sample Size(n)   |
|        |           |                        | 25   | 50   | 75   | 100  | 25   | 50   | 75   | 100  |
| MLE    | - -       | 1.55380                | 1.52760        | 1.52320        | 1.52120 | 0.09177 | 0.04239 | 0.02836 | 0.02211 |
| Bayes  | a b λ     | for $\alpha_1(\theta|t)$ when $P_2(\theta)\propto\text{gamma}(a,b)\times\text{erlang}(\lambda)$ |
| 3 2 4 | 1.23430 | 1.35180 | 1.40110 | 1.42760 | 0.10388 | 0.04641 | 0.02923 | 0.02179 |
| 3 2 6 | 1.13070 | 1.28470 | 1.35170 | 1.38860 | 0.16234 | 0.06601 | 0.03855 | 0.02703 |
| 1 2 6 | 1.04840 | 1.23580 | 1.31680 | 1.36150 | 0.23227 | 0.08814 | 0.04905 | 0.03305 |
| 2 1 6 | 1.13720 | 1.29230 | 1.35820 | 1.39410 | 0.16012 | 0.06422 | 0.03754 | 0.02643 |
| Bayes  | - λ 1     | for $\alpha_3(\theta|t)$ when $P_3(\theta)\propto\text{erlang}(\lambda)\times\text{exponential}(\lambda_1)$ |
| - 4 1 | 1.19960 | 1.33520 | 1.39040 | 1.41970 | 0.12613 | 0.05244 | 0.03191 | 0.02330 |
| - 4 2 | 1.14450 | 1.30300 | 1.36500 | 1.39980 | 0.15792 | 0.06251 | 0.03659 | 0.02588 |
| - 4 4 | 1.04840 | 1.23580 | 1.31680 | 1.36150 | 0.23227 | 0.08814 | 0.04905 | 0.03305 |
| - 6 4 | 0.96750 | 1.17740 | 1.27200 | 1.32530 | 0.31523 | 0.11953 | 0.06505 | 0.04255 |
| Bayes  | - λ c_1   | for $\alpha_5(\theta|t)$ when $P_5(\theta)\propto\text{erlang}(\lambda)\times\text{non-informative}(c_1)$ |

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### Table-A: The estimation of MLE and the best estimators of Bayes estimation for θ under the squared error loss function (SELF).

| Sample Size(a) | Method of estimation | Estimated value (LSE(θ)) | MSE(θ) |
|----------------|----------------------|--------------------------|--------|
|                |                      |                          |        |
| 25, 50, 75, 100|                      |                          |        |
| 4, 5, 6, 7, 100|                      |                          |        |

#### 6. Discussion

From empirical results in tables (4-1) to (4-3), corresponding to the smallest values of MSE, we listed the best estimators using Bayes estimation, under the squared error loss function (SELF), under different conjugate prior of the parameter for all the true value of $\theta = 0.5, 1, 1.5$. We see Bayes estimators under different double prior selection are too close to the true values $\theta = 0.5, 1, 1.5$, and the values of MSE for all sample size. Bayes estimation gave the best estimation according to the to the smallest values of MSE comparative with the values of MSE of MLE. As shown below in Table-A.
of the parameter for all the true value of \( \theta = 0.5, 1, 1.5 \). We see Bayes estimators under different double prior selection are too close the true values \( \theta = 0.5, 1, 1.5 \), and the values of MWSE for all sample size. Bayes estimation gave the best estimation according to the to the smallest values of MWSE comparative with the values of MWSE of MLE. As shown below in Table B.

Table-B: The estimation of MLE and the best estimators of Bayes estimation for under the weighted error loss function (WSELF).

| Method of estimation | Estimated value (\( \hat{\theta}_{\text{MLE}} \)) | MWSE (\( \hat{\theta} \)) |
|----------------------|---------------------------------|-----------------|
| Sample Size(n)       | 25 | 50 | 75 | 100 | 25 | 50 | 75 | 100 |
| MLE when the true value of \( \theta = 0.5 \) | 0.52269 | 0.51269 | 0.5002 | 0.50092 | 0.02405 | 0.01102 | 0.00710 | 0.00519 |

Bayes estimation when the true value of \( \theta = 0.5 \) when

\[ \frac{\theta}{\hat{\theta}_{\text{MLE}}} \] \( \text{MLE} \) with \( P(\theta) \) a gamma(a = 1, b = 2) x Erlang(\( \lambda = 0 \)) \[ 0.49998 \] \[ 0.30102 \] \[ ... \] \[ 0.01102 \] \[ 0.00519 \]

\[ \frac{\theta}{\hat{\theta}_{\text{MLE}}} \] \( \text{MLE} \) with \( P(\theta) \) a gamma(a = 1, b = 2) x Erlang(\( \lambda = 0 \)) \[ 0.40806 \] \[ 0.39086 \] \[ ... \] \[ 0.00602 \] \[ 0.00455 \]

\[ \frac{\theta}{\hat{\theta}_{\text{MLE}}} \] \( \text{MLE} \) with \( P(\theta) \) Erlang(\( \lambda = 2 \)) \[ 0.49212 \] \[ 0.48912 \] \[ ... \] \[ 0.00433 \] \[ 0.00333 \]

\[ \frac{\theta}{\hat{\theta}_{\text{MLE}}} \] \( \text{MLE} \) with \( P(\theta) \) Erlang(\( \lambda = 1 \)) \[ 0.49992 \] \[ 0.49063 \] \[ ... \] \[ 0.00433 \] \[ 0.00455 \]

\[ \frac{\theta}{\hat{\theta}_{\text{MLE}}} \] \( \text{MLE} \) with \( P(\theta) \) a normal-informative \( \text{c} = 1 \) \[ 0.49992 \] \[ 0.49063 \] \[ ... \] \[ 0.00433 \] \[ 0.00455 \]

\[ \frac{\theta}{\hat{\theta}_{\text{MLE}}} \] \( \text{MLE} \) with \( P(\theta) \) a normal-informative \( \text{c} = 1 \) \[ 0.49992 \] \[ 0.49063 \] \[ ... \] \[ 0.00433 \] \[ 0.00455 \]

Also for empirical results in tables (4-8) to (4-11), corresponding to the smallest values of MLINE(\( \theta \)), we listed the best estimators using Bayes estimation under modified linear exponential (MLINEX) loss function (MLF) with \( w=1 \) and \( c=1 \), under different conjugate prior of the parameter for all the true value of \( \theta = 0.5, 1, 1.5 \). We see Bayes estimators under different double prior selection are too close the true values \( \theta = 0.5, 1, 1.5 \), and the values of MLINEX(\( \theta \)) for all sample size. Bayes estimation gave the best estimation according to the to the smallest values of MLINEX(\( \theta \)) comparative with the values of MLINEX(\( \theta \)) of MLE. As shown below in Table-C.
Table-C: The estimation of MLE and the best estimators of Bayes estimation for under modified linear exponential (MLINEX) loss function (MLF) with $w=1$ and $c=1$.

| Method of estimation | Estimated value ($\hat{\theta}_{MLE}$) | MLINEX($\theta$) |
|----------------------|--------------------------------------|------------------|
|                      | 25        | 50        | 75        | 100       | 25        | 50        | 75        | 100       |
| MLE when the true value of $\theta = 0.5$ | 0.52159 | 0.51209 | 0.50392 | 0.50032 | 0.51216 | 0.00985 | 0.00762 | 0.00497 |
| Bayes estimation when the true value of $\theta = 0.5$ when $\pi_i(\theta \mid \lambda \alpha \gamma)_{\text{exponential}}(\lambda, \alpha)$ | 0.04891 | 0.05182 | 0.05102 | 0.05160 | 0.01785 | 0.08133 | 0.00892 | 0.00459 |
| $\pi_i(\theta \mid \lambda \alpha \gamma)_{\text{exponential}}(\lambda, \alpha)$ | 0.05102 | 0.05157 | 0.04849 | 0.04874 | 0.01660 | 0.08800 | 0.00612 | 0.00461 |
| $\pi_i(\theta \mid \lambda \alpha \gamma)_{\text{exponential}}(\lambda, \alpha)$ | 0.04902 | 0.05157 | 0.04846 | 0.04875 | 0.01784 | 0.08984 | 0.00650 | 0.00470 |
| $\pi_i(\theta \mid \lambda \alpha \gamma)_{\text{exponential}}(\lambda, \alpha)$ | 0.04897 | 0.04897 | 0.04846 | 0.04877 | 0.01784 | 0.08984 | 0.00650 | 0.00470 |
| Sample Size(s) | 25        | 50        | 75        | 100       | 25        | 50        | 75        | 100       |
| MLE when the true value of $\theta = 1$ | 1.03580 | 1.01840 | 1.01550 | 1.01410 | 0.02154 | 0.01028 | 0.00694 | 0.00543 |
| Bayes estimation when the true value of $\theta = 1$ when $\pi_i(\theta \mid \lambda \alpha \gamma)_{\text{exponential}}(\lambda, \alpha)$ | 0.92258 | 0.94064 | 0.97580 | 0.98460 | 0.01636 | 0.00882 | 0.06083 | 0.00484 |
| $\pi_i(\theta \mid \lambda \alpha \gamma)_{\text{exponential}}(\lambda, \alpha)$ | 0.92258 | 0.94064 | 0.97580 | 0.98460 | 0.01636 | 0.00882 | 0.06083 | 0.00484 |
| $\pi_i(\theta \mid \lambda \alpha \gamma)_{\text{exponential}}(\lambda, \alpha)$ | 0.92258 | 0.94064 | 0.97580 | 0.98460 | 0.01636 | 0.00882 | 0.06083 | 0.00484 |
| $\pi_i(\theta \mid \lambda \alpha \gamma)_{\text{exponential}}(\lambda, \alpha)$ | 0.92258 | 0.94064 | 0.97580 | 0.98460 | 0.01636 | 0.00882 | 0.06083 | 0.00484 |
| Sample Size(s) | 25        | 50        | 75        | 100       | 25        | 50        | 75        | 100       |
| MLE when the true value of $\theta = 1$ | 1.15770 | 1.12710 | 1.52120 | 1.52120 | 0.02155 | 0.01028 | 0.00694 | 0.00543 |
| Bayes estimation when the true value of $\theta = 1$ when $\pi_i(\theta \mid \lambda \alpha \gamma)_{\text{exponential}}(\lambda, \alpha)$ | 1.25690 | 1.34840 | 1.41020 | 1.43460 | 0.02566 | 0.01151 | 0.00728 | 0.00605 |
| $\pi_i(\theta \mid \lambda \alpha \gamma)_{\text{exponential}}(\lambda, \alpha)$ | 1.25690 | 1.34840 | 1.41020 | 1.43460 | 0.02566 | 0.01151 | 0.00728 | 0.00605 |
| $\pi_i(\theta \mid \lambda \alpha \gamma)_{\text{exponential}}(\lambda, \alpha)$ | 1.25690 | 1.34840 | 1.41020 | 1.43460 | 0.02566 | 0.01151 | 0.00728 | 0.00605 |
| $\pi_i(\theta \mid \lambda \alpha \gamma)_{\text{exponential}}(\lambda, \alpha)$ | 1.25690 | 1.34840 | 1.41020 | 1.43460 | 0.02566 | 0.01151 | 0.00728 | 0.00605 |

Finally, for empirical results in tables (7-1) to (7-3) , corresponding to the smallest values of $\text{MLINEX}(\theta)$ , we listed the best estimators using Bayes estimation , under modified linear exponential (MLINEX) loss function (MLF) with $w=1$ and $c=2$, under different conjugate prior of the parameter for all the true value of $\theta = 0.5, 1, 1.5$. We see Bayes estimators under different double prior selection are too close the true values $\theta = 0.5, 1, 1.5$, and the values of $\text{MLINEX}(\theta)$ for all sample size. Bayes estimation gave the best estimation according to the to the smallest values of $\text{MLINEX}(\theta)$ comparative with the values of $\text{MLINEX}(\theta)$ of MLE. As shown below in Table-D.
Table-D: The estimation of MLE and the best estimators of Bayes estimation for under modified linear exponential (MLINEX) loss function (MLF) with \( w=1 \) and \( c=2 \).

| Method of estimation | Estimated value (\( \hat{\text{MLE}} \)) | MLINEX (\( \hat{\theta} \)) |
|----------------------|--------------------------------------|--------------------------|
|                      | Sample Size (n)                      | 25 | 50 | 75 | 100 | 25 | 50 | 75 | 100 |
| MLE when the true value of \( \theta \) is 0.5 when \( \theta = \lambda \) | 0.9599 | 0.9599 | 0.9599 | 0.9599 | 0.9599 | 0.9599 | 0.9599 | 0.9599 | 0.9599 |
| \( \pi(\lambda; 0.5) \) with \( P(\lambda) \alpha \text{ gamma} (a = 3, b = 2) \times \text{exponential} (\lambda = 4) \) | 0.8769 | 0.8769 | 0.8769 | 0.8769 | 0.8769 | 0.8769 | 0.8769 | 0.8769 | 0.8769 |
| \( \pi(\lambda; 0.5) \) with \( P(\lambda) \alpha \text{ exponential} (\lambda = 4) \times \text{exponential} (\lambda = 2) \) | 0.8769 | 0.8769 | 0.8769 | 0.8769 | 0.8769 | 0.8769 | 0.8769 | 0.8769 | 0.8769 |
| \( \pi(\lambda; 0.5) \) with \( P(\lambda) \alpha \text{ exponential} (\lambda = 4) \times \text{non-informative} (c = 1) \) | 0.8769 | 0.8769 | 0.8769 | 0.8769 | 0.8769 | 0.8769 | 0.8769 | 0.8769 | 0.8769 |
| \( \pi(\lambda; 0.5) \) with \( P(\lambda) \alpha \text{ exponential} (\lambda = 4) \times \text{normal} (\lambda = 1) \) | 0.8769 | 0.8769 | 0.8769 | 0.8769 | 0.8769 | 0.8769 | 0.8769 | 0.8769 | 0.8769 |
| Sample Size (n) | 25 | 50 | 75 | 100 | 25 | 50 | 75 | 100 |
| MLE when the true value of \( \theta \) is \( \theta = 1 \) when \( \theta = \lambda \) | 1.0580 | 1.0580 | 1.0580 | 1.0580 | 1.0580 | 1.0580 | 1.0580 | 1.0580 | 1.0580 |
| \( \pi(\lambda; 1) \) with \( P(\lambda) \alpha \text{ gamma} (a = 3, b = 2) \times \text{exponential} (\lambda = 4) \) | 0.9599 | 0.9599 | 0.9599 | 0.9599 | 0.9599 | 0.9599 | 0.9599 | 0.9599 | 0.9599 |
| \( \pi(\lambda; 1) \) with \( P(\lambda) \alpha \text{ exponential} (\lambda = 4) \times \text{exponential} (\lambda = 2) \) | 0.8769 | 0.8769 | 0.8769 | 0.8769 | 0.8769 | 0.8769 | 0.8769 | 0.8769 | 0.8769 |
| \( \pi(\lambda; 1) \) with \( P(\lambda) \alpha \text{ exponential} (\lambda = 4) \times \text{non-informative} (c = 1) \) | 0.8769 | 0.8769 | 0.8769 | 0.8769 | 0.8769 | 0.8769 | 0.8769 | 0.8769 | 0.8769 |
| \( \pi(\lambda; 1) \) with \( P(\lambda) \alpha \text{ exponential} (\lambda = 4) \times \text{normal} (\lambda = 1) \) | 0.8769 | 0.8769 | 0.8769 | 0.8769 | 0.8769 | 0.8769 | 0.8769 | 0.8769 | 0.8769 |
| Sample Size (n) | 25 | 50 | 75 | 100 | 25 | 50 | 75 | 100 |
| MLE when the true value of \( \theta \) is \( \theta = 1.5 \) when \( \theta = \lambda \) | 1.5580 | 1.5580 | 1.5580 | 1.5580 | 1.5580 | 1.5580 | 1.5580 | 1.5580 | 1.5580 |
| \( \pi(\lambda; 1.5) \) with \( P(\lambda) \alpha \text{ gamma} (a = 3, b = 2) \times \text{exponential} (\lambda = 4) \) | 0.9599 | 0.9599 | 0.9599 | 0.9599 | 0.9599 | 0.9599 | 0.9599 | 0.9599 | 0.9599 |
| \( \pi(\lambda; 1.5) \) with \( P(\lambda) \alpha \text{ exponential} (\lambda = 4) \times \text{exponential} (\lambda = 2) \) | 0.8769 | 0.8769 | 0.8769 | 0.8769 | 0.8769 | 0.8769 | 0.8769 | 0.8769 | 0.8769 |
| \( \pi(\lambda; 1.5) \) with \( P(\lambda) \alpha \text{ exponential} (\lambda = 4) \times \text{non-informative} (c = 1) \) | 0.8769 | 0.8769 | 0.8769 | 0.8769 | 0.8769 | 0.8769 | 0.8769 | 0.8769 | 0.8769 |
| \( \pi(\lambda; 1.5) \) with \( P(\lambda) \alpha \text{ exponential} (\lambda = 4) \times \text{normal} (\lambda = 1) \) | 0.8769 | 0.8769 | 0.8769 | 0.8769 | 0.8769 | 0.8769 | 0.8769 | 0.8769 | 0.8769 |
| Sample Size (n) | 25 | 50 | 75 | 100 | 25 | 50 | 75 | 100 |

In general, the parameter estimates using Bayes estimation methods are close to the true values comparative to the other estimated values using maximum likelihood estimation, at certain values for the parameters of the conjugate prior of the parameter \( \theta \) was considered as combination for the two different prior distribution.

5. Conclusions

When we compared the estimated values for the shape parameter \( \theta \) of the Power Function Distribution (PFD-I) by using the methods in this study .We can conclude that Bayes' estimators for the unknown shape parameter \( \theta \) was considered , under various double priors at certain values for the parameters of the double prior distribution and under the squared error loss function and the weighted error loss function and modified linear exponential (MLINEX) loss function with \( w=1 \) and \( c=1,2 \), for all the true value of \( \theta \) and for all samples sizes \( n \) are better than other estimators by using maximum likelihood estimation.

We find that best estimation under the squared error loss function (SELF) for each of the value of \( \theta = 0.5,1,1.5 \), according to the smallest values of MSE, by using the conjugate prior of the parameter \( \theta \) was considered as

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• (erlang$(\lambda = 6) \times \text{exponential}(\lambda_1 = 4)$) distribution for all samples, when the true value $\theta = 0.5$.
• (gamma$(a = 3, b = 2) \times \text{erlang}(\lambda = 4)$) distribution and for all samples except $n=100$, when the true value $\theta = 1$ and $\theta = 1.5$.

And we find that best estimation under the weighted error loss function (WSELF) for each of the value of $\theta = 0.5, 1, 1.5$, according to the smallest values of MWSE, by using the conjugate prior of the parameter $\theta$ was considered as

• (gamma$(a = 3, b = 2) \times \text{erlang}(\lambda = 6)$) distribution for all samples, when the true value $\theta = 0.5$.
• (gamma$(a = 3, b = 2) \times \text{erlang}(\lambda = 4)$) distribution and for all samples except $n=100$, when the true value $\theta = 1$.
• (exponential $(\lambda_1 = 2) \times \text{non\text{-}informative}(c_1 = 1)$) distribution and for all samples except $n=100$, when the true value $\theta = 1.5$.

Also we find that best estimation under modified linear exponential (MLINEX) loss function (MLF) with $w=1$ and $(c=1, 2)$ , for each of the value of $\theta = 0.5, 1, 1.5$, according to the smallest values of MLINDEX$(\theta)$, by using the conjugate prior of the parameter $\theta$ was considered as

• (gamma$(a = 3, b = 2) \times \text{erlang}(\lambda = 6)$) distribution for all samples, when the true value $\theta = 0.5$.
• (gamma$(a = 3, b = 2) \times \text{erlang}(\lambda = 4)$) distribution and for all samples, when the true value $\theta = 1$.
• (exponential $(\lambda_1 = 2) \times \text{non\text{-}informative}(c_1 = 1)$) distribution and for all samples, when the true value $\theta = 1.5$. 
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التقدير البيزي لمعلمات الشكل لتوزيع (PFD-I)

لاستعمال دوال أولية فوقية

أ.م.د. جهان عباس ناصر العبيدي

الكلية التقنية الإدارية - بغداد، العراق

author@institute.xxx

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เมนستخص البحث
في هذا البحث، نختبر خصائص مقدرات بيز لمعلمات الشكل لتوزيع دالة القوى من النوع الأول باستخدام نوعين من التوزيعات أولية، واستعمال دوال خسارة مختلفة، ومقارنتها مع مقدرات الإمكان الأعظم. في العديد من التطبيقات العملية، ربما يكون لدينا معلومات مختلفة حول التوزيع الأولي لمعلمات الشكل لتوزيع دالة القوى من النوع الأول، الذي يكون له تأثير على تقدير معلمات الشكل. لذا، فقد استعملنا نوعين مختلفين من الدوال الأولى المرافقة لمعلمات الشكل (٠) لتوزيع دالة القوى من النوع الأول (PFD-I) لتقديرها. افترضت الدالة المرافقة الأولية لمعلمات الشكل (٠) كتوثيفة من توزيعين أولية مختلفة، وتوزيع كاما مع توزيع ارلنك وتوزيع ارلنك مع الأسي وتوزيع ارلنك مع توزيع غير معلوماتي وتوزيع الأسي مع توزيع غير معلوماتي. فقد تم إستغلال مقدرات بيز لمعلمات الشكل لتوزيع دالة القوى من النوع الأول باستخدام ثلاثة أنواع لدالة الخسارة كدالة الخسارة التريبية، ودالة الخسارة التريبية الموزونة، ودالة الخسارة الإسية الخطية المحورة. بالإضافة إلى المقدرات الكلاسيكية المتمثلة بطريقة تقدير الإمكان الأعظم، استعملنا أسهل المحاولات للحصول على نتائج هذا البحث لحالات مختلفة لمعلمات الشكل (٠) لتوزيع دالة القوى من النوع الأول (PFD-I) استعملت لتوليد البيانات واحجام مختلفة من العينات.

نوع البحث: ورقة بحثية.
المصطلحات الرئيسية للبحث: توزيع دالة القوى من النوع الأول، تقدير الإمكان الأعظم، تقدير بيز، دالة الخسارة التريبية، ودالة الخسارة التريبية الموزونة، ودالة الخسارة الإسية الخطية المحورة.