DETERMINING WHEN THE UNIVERSAL ABELIAN COVER OF A GRAPH MANIFOLD IS A RATIONAL HOMOLOGY SPHERE.

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Abstract. It was shown in [Ped10a] that the splice diagram of a rational homology sphere graph manifold determines the manifolds universal abelian cover. In this article we use the proof given in [Ped10a] to give a condition on the splice diagram to determine when the universal abelian cover itself is a rational homology sphere.

1. Introduction

Graph manifolds is an interesting class of 3-manifolds. They are defined as the manifolds who only have Seifert fibered pieces in their JSJ-decomposition, or equivalently have no hyperbolic pieces in their geometric decomposition. They are also the 3-manifolds that are boundary of plumbed 4-manifolds, and therefore all links of isolated complex surface singularities are graph manifolds.

If we restrict to rational homology sphere graph manifolds, then there are interesting question involves the universal abelian cover. The first is of course when do two manifolds have the same universal abelian cover. An answer to this was given in [Ped10a] using an invariant called splice diagram, saying that if two graph manifolds have the same splice diagram, then their universal abelian covers are homeomorphic. There I gave a simple corollary:

Corollary 1.1. Let $M$ be a rational homology sphere graph manifold with splice diagram $\Gamma(M)$, such that around any node in $\Gamma(M)$ the edge weights are pairwise coprime. Then the universal abelian cover of $M$ is an integer homology sphere.

The present article will strengthen this result and answer when is the universal abelian cover a rational homology sphere. We are going to do this, by investigating the construction of the universal abelian cover from the splice diagram given in [Ped10a] see also [Ped10b] for this construction in more algorithmic form.

The splice diagram we use differs slightly from the original definition given in [Sie80] and [EN85] by only having non negative weights at edges and not demanding that the edges at a node are pairwise coprime, the last is of course because we are working with rational homology spheres and not only integer homology spheres. Our splice diagram also differ slightly from the once in [NW02, NW05b, and NW05a], by having signs at nodes, but for singularity links which is what concern Neumann and Wahl in those articles, our splice diagram are the same.

This article has two sections. In the first we introduce splice diagrams and and give some result about them need in the second section where we prove a condition for when the universal abelian cover is a rational homology sphere. This result was originally partly in my Ph.d. thesis, but I was at that time not able to prove what here is Proposition 3.4 and could therefore only show sufficiency of the condition in the case of singularity links, using that finite branched covers only branched over the singular point of singularity links are themselves singularity links. Because this

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2. Splice diagrams

A splice diagram is a weighted tree with no vertices of valence two, with signs on nodes, that is vertices of valence tree or higher, and with non negative integers on edges adjacent to nodes.

![splice_diagram](attachment:splice_diagram.png)

We want to assign a splice diagram $\Gamma(M)$ to any given QHS graph manifold $M$, this is done in the following way:

- Take a vertex for each of the Seifert fibered pieces of the JSJ-decomposition of $M$, these are the vertices there are going to be the nodes of $\Gamma(M)$ and we will hence forward not distinguish between a node and the Seifert fibered piece it represents.
- Connect two nodes by an edge if they are they are glued to make $M$ from the pieces.
- Add a vertex and connect it with an edge to a node for for each singular fiber of the node, we will call these vertices for leaf, and will in general not distinguish between a leaf and the edge leading to it.
- The sign added at a node is the linking number of two non singular fibers, see [Ped10a] for more details.
- If $e$ is an edge at a node $d$, the it corresponds to a torus $T_e \subset M$, either from the JSJ-decomposition or as the boundary of a tubular neighborhood of a singular fiber. Let $M'_{e_0}$ be the connected piece of $M - T_e$ not containing $v$, and let $M_{e_0} = M'_{e_0} \cup(D^2 \times S^1)$, where we identify a meridian of the solid torus with the image of a fiber of $v$. Then the edge weight $d_{e_0}$ is $|H_1(M_{e_0})|$, if $H_1(M_{e_0})$ is infinite $d_{e_0} = 0$.

The assumption that $M$ is a QHS insure that this construction gives a tree, since the decomposition structure (or decomposition graph see [Neu97]) of a QHS is a tree, a fact which will be used later when we find obstructions to the universal abelian cover being a QHS. If we consider $M$ as a plumbed manifold, then we can give the following characterization of being a QHS.

**Proposition 2.1.** Let $M$ be a plumbed 3-manifold, then $M$ is a QHS is and only if its plumbing diagram is a tree of spheres, and its intersection form is non degenerate.

One can construct the splice diagram of $M$ from a plumbing diagram of $M$, see [Ped10a] for details. This is in fact the way splice diagrams are defined in [NW02], [NW05a], and [NW05b].

From the splice diagram there are several numerical invariants that plays important roles. The first is the edge determinant which is a number associated to an edge $e$ between nodes $v_0$ and $v_1$, if the nodes look like

![edge_determinant](attachment:edge_determinant.png)
then it is defined as
\[ r_0 r_1 - \varepsilon_0 \varepsilon_1 \left( \prod_{i=1}^{k_0} n_{i0} \right) \left( \prod_{j=1}^{k_1} n_{1j} \right), \]
where \( \varepsilon_i \) is the sign on the \( i \)’th node.

The edge determinant is important in the following two result from [Ped10a].

**Proposition 2.2 (Edge Determinant Equation).** Let \( e \) be an edge between two nodes \( v \) and \( w \), let \( D(e) \) be its edge determinant. Then the fiber intersection number \( p \) in the corresponding torus is given by
\[ p = \frac{|D|}{|H_1(M)|}. \]

By the fiber intersection number one means, the intersection number in the torus of a fiber from each of the two Seifert fibered pieces \( v \) and \( w \) with appropriate chosen orientations, for more details see [Neu97] or [Ped10a].

**Theorem 2.3.** A QHS graph manifold is the link or an isolated complex surfaces singularity if and only if there are no negative signs in its splice diagram and all edge determinants are positive.

The other numbers derived from the splice diagram we need is called ideal generators and is associated to a vertex \( v \) and adjacent edge \( e \). To define it we first need following construction. Let \( v \) and \( w \) be two vertices in \( \Gamma \), then their linking number \( l_{vw} \) is the product of all edge weights adjacent two but not on the shortest path from \( v \) to \( w \). We define \( l'_{vw} \) similar except we do not include the weights adjacent to \( v \) and \( w \). Then if \( v \) is a node and \( e \) and adjacent edge we define an ideal of \( CI \) by
\[ I_{ve} = \langle l'_{vw} | w \text{ is a leaf of } \Gamma_{ve} \rangle \]
where \( \Gamma_{ve} \) is the connected component of \( \Gamma - e \) not including \( v \). We define the ideal generator \( d_{ve} \) as the positive generator of \( I_{ve} \).

The ideal generator is important because of the following ideal condition.

**Definition 2.4.** A splice diagram is said to satisfy the ideal condition if for any node \( v \) and adjacent edge \( e \), then the ideal generator \( d_{ve} \) divides the edge weight \( d_{ve} \).

Every splice diagram coming from a graph manifold satisfy the ideal condition. This follows from the following topological description of the ideal generator from [NW05a].

**Theorem 2.5.** Let \( M \) be a QHS and \( \Gamma(M) \) its splice diagram, let \( v \) be a node and \( e \) an adjacent edge, then \( d_{ve} = |H_1(M_{ve}, K)| \). Where \( K \) is the core of the solid torus one glue to \( M_{ve}' \) to construct \( M_{ve} \).

Since our proofs in the next section relies on the combinatorics of splice diagram, we will introduce some helpful notation.

**Definition 2.6.** We say that an edge weight \( r \) of a splice diagram sees a vertex \( v \) (or edge \( e \)) of the splice diagram if, when we delete the node which \( r \) is adjacent to, the vertex \( v \) (or the edge \( e \)) and the edge which \( r \) is on are in the same connected component.

**Remark 2.7.** Given a vertex \( v \) on any edge \( e \) between nodes, one of the edge weights at \( e \) sees \( v \) and the other does not see \( v \). Let us introduce the following notation. Let \( v \) be a vertex of \( \Gamma \) and let \( v' \) be a node of \( \Gamma \), where \( v \neq v' \). Then let \( r_{v}(v) \) be the unique edge weight at an edge adjacent to \( v \) which sees \( v \). Likewise let \( d_{v'}(v) \) be the unique ideal generator associated to \( v' \), which sees \( v \).
Definition 2.8. We say that an edge weight \( r_v \) sees an edge weight \( r_{v'} \) if \( r_v \) sees \( v' \) and \( r_v \neq r_{v'}(v) \). Likewise for ideal generators.

Proposition 2.9. Let \( v \) be a node of a splice diagram \( \Gamma \) of a manifold \( M \). Let \( r_v \) be a edge weight adjacent to \( v \) and let \( d_v \) be the corresponding ideal generator. Then \( r_v \) and \( d_v \) are divisible by every ideal generator they see. Moreover if \( r_v \) and \( d_v \) see a node \( v' \) and \( n, n' \) are edge weights at \( v' \) and \( n, n' \neq r_{v'}(v) \), then \( \gcd(n, n') | r_v, d_v \).

Proof. We first observe that it is enough to show the proposition only for \( d_v \), since \( d_v | r_v \) by \( \ref{2.5} \).

We will show this by induction on the number of edges between \( v \) and \( v' \). If \( e \) is adjacent to \( v' \), then \( d_v \) is the generator of an ideal \( I \), which can be generated by elements, each of which is divisible by the product of all but one of the edge weights at \( v' \) not on \( e \). But this implies that each of the elements in the generating set is divisible by either \( n \) or \( n' \), and hence each the elements is divisible by \( \gcd(n, n') \), and therefore \( d_v \) is divisible by \( \gcd(n, n') \).

Assume by induction that if there are \( k \) edges between \( v'' \) and \( v' \) then \( d_{v''}(v') \) are divisible by \( \gcd(n, n') \). Assume that there are \( k+1 \) edges between \( v \) and \( v' \). Let \( d_i \) for \( i \in 1, \ldots, k \) be \( d_e(v') \) on the vertex \( \tilde{v} \) on \( i \)th edge between \( v \) and \( v' \), then by induction \( \gcd(n, n') | d_i \) for all \( i \). Remember that \( d_v \) is the generator of the ideal \( \langle l'_{vw} \rangle \) is a leaf of \( \Gamma_{\langle v \rangle} \).

where \( l'_{vw} \) is the product of the edge weights adjacent to but not on the path from \( v \) to \( w \). Now there are two types of leaves \( w \): the leaves \( w \) where the path to \( w \) goes through \( v' \), and the one where the path does not go through \( v' \). In the first case, \( n | l'_{vw} \) or \( n' | l'_{vw} \) or both, so in this case \( \gcd(n, n') | l'_{vw} \). In the second case one of the \( d_i \)'s will divide \( l'_{vw} \). This implies that \( \gcd(n, n') | l'_{vw} \) for all \( w \), and hence \( \gcd(n, n') \) divides the generator of the ideal \( d_v \). The following illustrates how \( \Gamma \) looks in the first and the second case of the induction. In the first case, the path to \( w \) can also pass through the edges with \( n \) or \( n' \).

The statement about \( d_v \) being divisible by ideal generators it sees follows from a similar argument as above. \( \square \)

3. Main Theorems

To determine conditions on the splice diagram for the universal abelian cover to be a rational homology sphere, we investigate the construction of the universal abelian cover in Theorem 6.3 of [Ped10a]. Since we construct the universal abelian cover by induction, there are two places where obstructions to being a rational homology sphere can arise: in the inductive step, and in the base case.

We start by looking at the base case, that is a splice diagram with one node. We distinguish between diagrams with an edge weight of 0 and those without. In the case of an edge weight of 0, we never get rational homology sphere universal abelian
covers. The universal abelian cover $X$ of $L(p,q) \# L(p',q')$ is $p$ copies of $S^3$ with $p'$ balls removed, glued to $p'$ copies of $S^3$ with $p$ balls removed, where the former pieces are glued to the latter pieces exactly once each. Then a Meyer-Vietoris argument shows that the rank of the first homology group is $(p-1)(p'-1)$. Since the universal abelian covers of iterated connected sums of lens spaces will contain several copies of $X$ as connected summands, it is clear that a connected sum of lens space can not have rational homology sphere universal abelian covers.

This leaves the second case, determining which Seifert fibered, or more precisely, which $S^1$ orbifold bundles have rational homology sphere universal abelian covers. By the results of [Neu83a] and [Neu83b], which also works for graph orbifolds, this is the same as determining which links of Brieskorn complete intersections are rational homology spheres.

**Proposition 3.1.** $\Sigma(\alpha_1, \ldots, \alpha_n)$ is a rational homology sphere if and only if one of the following conditions holds.

1. $\gcd(\alpha_i, \alpha_j) = 1$ for all $i \neq j$.
2. There exist a single pair $k, l$, such that $\gcd(\alpha_k, \alpha_l) \neq 1$.
3. There exist a single triple $k, l, m$ such that $\gcd(\alpha_k, \alpha_l) = \gcd(\alpha_l, \alpha_m) = \gcd(\alpha_m, \alpha_k) = 2$; for all other indices $\gcd(\alpha_i, \alpha_j) = 1$.

The first condition is of course the case where $\Sigma(\alpha_1, \ldots, \alpha_n)$ is a rational homology sphere, as we saw earlier.

**Proof.** The if direction follows from [Ham72], where Hamm proves a sufficient condition for the link of Brieskorn complete intersections of any dimension to be rational homology spheres. He could only prove the other direction if the number of variables was at most twice the dimension plus two. We will give a different proof in the case of surfaces, using the description of the Seifert invariants given in Theorem 2.1 in [NR78].

A Seifert fibered manifold is a rational homology sphere if and only if the rational euler number $e$ is nonzero, and the genus $g$ is zero. From the formulas of Theorem 2.1 in [NR78] we see that $e(\Sigma(\alpha_1, \ldots, \alpha_n)) \neq 0$, so we need only show that the conditions above are equivalent to the genus being 0. In other words it is enough to show that the following equations hold if and only if one of the three conditions does:

$$0 = 2 + (n - 2) \sum_{i=1}^{n} \frac{\alpha_i}{\lcm_i(\alpha_i)} - \frac{\prod_{i=1}^{n} \alpha_i}{\lcm(\alpha_1, \ldots, \alpha_n)}.$$  

Let $A = \frac{\prod_{i=1}^{n} \alpha_i}{\lcm(\alpha_i)}$ and $A_i = \frac{\prod_{x \in X} \alpha_i}{\lcm_{x \in X}(\alpha_i)}$.

We start by proving the “if” direction. Assume condition 1 holds, then $A = 1$ and $A_i = 1$ for all $i \in 1, 2, \ldots, n$, and we get

$$g = 2 + (n - 2)A - \sum_{i=1}^{n} A_j = 2 + (n - 2) + \sum_{i=1}^{n} 1 = 2 + (n - 2) - n = 0$$

Assume that condition 2 holds, and let $\gcd(\alpha_k, \alpha_l) = B$. Then $A = B$, $A_k = A_l = 1$ and $A_i = B$ if $i \neq k, l$. We get

$$g = 2 + (n - 2)B - 1 - 1 - \sum_{i \neq k, l} B = 2 + (n - 2)B - 2 - (n - 2)B = 0$$

Finally for condition 3, $A = 4$, $A_k = A_l = A_m = 2$ and $A_j = 4$ if $j \neq k, l, m$. The genus is

$$g = 2 + (n - 2)4 - 2 - 2 - \sum_{i \neq k, l, m} 4 = (n - 2)4 - 4 - (n - 3)4 = 0.$$
This conclude the “if” direction.

For the “only if” direction we start by assuming the equation (2) holds. Suppose we have \( \alpha_j, \alpha_k, \alpha_s, \alpha_m \), such that \( \gcd(\alpha_j, \alpha_k) = B \) and \( \gcd(\alpha_s, \alpha_m) = C \). Notice that \( BC \mid A, B \mid A_j, A_m, C \mid A_j, A_k \) and \( BC \mid A_i \), for \( i \neq j, k, l, m \). Let \( A' = B_i \), \( A' = B_i \), \( A' = B_i \), \( A' = B_i \), \( A' = B_i \), \( A' = B_i \), and \( A' = B_i \) if \( i \neq j, k, l, m \). \( A \geq A_i \) for all \( i \) so clearly \( A' \geq A_i' \). Hence \( C \mid A_i' \) so we also get \( A' \geq A_i' \).

Hence

\[
0 = 2 + (n - 2)BCA' - CA_j' - CA_k' - BA_i' - BA_m' - \sum_{s \neq k, l, m} BCA'_s
\]

(6)

\[
\geq 2 + (n - 2)BCA' - 2CA' - 2BA' - \sum_{s \neq k, l, m} BCA' = 2 + 2A'(BC - C - B).
\]

Since \( A' \geq 1 \) this implies that \( BC - C - B < 0 \) and hence either \( B = 1 \) or \( C = 1 \).

We have now proved that \( \gcd(\alpha_i, \alpha_j) = 1 \) except that there might be \( \alpha_k, \alpha_s, \alpha_m \) such that \( \gcd(\alpha_k, \alpha_s) = B \) and \( \gcd(\alpha_s, \alpha_m) = C \) and \( \gcd(\alpha_i, \alpha_m) = D \). Notice that

\[
A = \frac{\alpha_j}{\text{lcm}(\alpha_j, \alpha_i, \alpha_u)}, \quad A_k = \frac{\alpha_k}{\text{lcm}(\alpha_k, \alpha_i, \alpha_u)}, \quad A_i = \frac{\alpha_i}{\text{lcm}(\alpha_i, \alpha_u)}, \quad A_m = \frac{\alpha_i}{\text{lcm}(\alpha_i, \alpha_u)}, \quad A_i = A
\]

for \( i \neq k, l, m \). The equation becomes

\[
0 = 2 + (n - 2)A - A_k - A_l - A_m - \sum_{s \neq k, l, m} A_j
\]

(7)

\[
= 2 + (n - 2)A - A_k - A_l - A_m - \sum_{s \neq k, l, m} A = 2 + A - A_k - A_l - A_m,
\]

which is exactly the same equation as if \( n = 3 \). But it is known in this case that either \( B = C = D = 2 \) or two of \( B, C, D \) is 1, from the article of Hamm [Ham72].

One can also see this directly, if \( \alpha_1 = ds_1s_2t_2, \alpha_2 = ds_1s_3t_2, \) and \( \alpha_3 = dl_1s_2s_3 - d(s_1 + s_2 + s_3) \). It is clear that \( d = 1 \) or \( 2 \). If \( d = 2 \) then the only solution is \( s_1 = s_2 = s_3 = 1 \) since the right hand side is increasing in \( s_i \). If \( d = 1 \) then the only solution is if two of the \( s_i \)'s are one, since the right hand side increases if we increase two of the \( s_i \)'s.

\[\square\]

Combining this result with an investigation of the inductive step yields a necessary condition on the splice diagram for the universal abelian cover to be a rational homology sphere. We remember how we defined the notation \( r_{v'}(v) \) in 2.7.

**Corollary 3.2.** Let \( \Gamma \) be the the splice diagram of a manifold \( M \), where the universal abelian cover of \( M \) is a rational homology sphere. Then all edge weights are nonzero, and there is a special node \( v \in \Gamma \), with the following properties. For all other nodes \( v' \in \Gamma \), the weights other than \( r_{v'}(v) \) are pairwise coprime, and at most one of these edge weights is not coprime with \( r_{v'}(v) / d_{v'}(v) \). At \( v \) all the edge weights satisfy one of the conditions from Proposition 2.7.

**Proof.** What we are going to show is that the condition on the splice diagram given above is equivalent to the absence of cycles in the decomposition graph (or a plumbing graph) of the universal abelian cover \( \tilde{M} \), and all the pieces of the decomposition having a base of genus 0. The corollary then follows by Proposition 2.7. That the decomposition graph must also have no cycles and bases of genus 0 follows from the relation between plumbing graphs and decomposition graph given in [Neu97].
We saw that, when we cut along an edge $e$ between nodes $v_0$ and $v_1$ in the inductive construction of $\tilde{M}$ given in the proof of Theorem 6.3 in [Ped10a], we took $d_0$ pieces above $v_0$ and glued to $d_1$ pieces above $v_1$, where $d_i$ is the ideal generator at $e$ associated to $v_i$. Each piece on the one side is glued exactly once to each piece on the other side. Each of these pieces has a Seifert fibered piece sitting above the corresponding $M_{v_i}$. If $d_0, d_1 > 1$ then a piece $v_{00}$ over $M_{v_0}$ is glued to a piece $v_{10}$ sitting over $M_{v_1}$, then $v_{10}$ is glued to a piece $v_{01}$ sitting over $M_{v_0}$, and $v_{01}$ is glued to a piece $v_{11}$ sitting over $M_{v_1}$. Finally $v_{11}$ is glued to $v_{00}$. We have now constructed a cycle in the decomposition graph of $\Delta(\tilde{M})$ since each of the $v_{ij}$ represent a vertex of $\Delta(\tilde{M})$. If one of the $d_i$’s is 1, then we do not get cycles, since we will have only one piece above the appropriate end of $e$.

\[
\begin{array}{c}
v_0 & \overset{v_{01}}{\searrow} & v_1 \\
\downarrow & & \downarrow \\
v_0 & \underset{v_{10}}{\nearrow} & v_1
\end{array}
\]

So we now proved that a cycle in the decomposition graph for $\tilde{M}$ occurs if an edge $e$ in the splice diagram has ideal generators $d_0$ and $d_1$ (associated to each end), such that both $d_0$ and $d_1$ are not equal to one.

Let $M_0$ and $M_1$ be graph manifolds with universal abelian covers $\tilde{M}_0$ and $\tilde{M}_1$, and assume that there are no cycles in $\tilde{M}_i$. Let $M_{01}$ be the universal abelian cover of $M_{01}$ which is $M_0$ glued to $M_1$ after removing a solid torus from each. Assume that $\tilde{M}_{01}$ has cycles in its decomposition graph. $\tilde{M}_{01}$ is a number of $\tilde{M}_0$ with $n_0$ solid tori removed glued to $\tilde{M}_1$ with $n_1$ solid tori removed, such that each of the first type is glued to each of the second type. If one of the $n_i$ is 1, then $\tilde{M}_{01}$ has no cycles, so $n_0, n_1 > 1$. But $n_i = d_i$ so we are in the situation above.

So there are cycles in the decomposition graph of $\tilde{M}$ if and only if there is an edge which has both associated ideal generators different from 1.

We need to show that the conditions we stated on $\Gamma$ are equivalent to the statement that for each edge one of the ideal generators associated to an end of it is 1.

Suppose there were two nodes $v$ and $w$ of $\Gamma$, such that the edge weights at $v$ that do not see $w$ are not pairwise coprime, and the same with $v$ and $w$ exchanged. On any edge $e$ on the string between $v$ and $w$, the ideal generator associated to either end of $e$ is then greater than 1 by Proposition 2.9, so we have a cycle in the decomposition graph. This implies that there can be at most be one node $v$, such that there are all other nodes, edge weights that do not see $v$ are pairwise coprime. On the other hand, if $\Gamma$ satisfies this, then it is not hard to see that all ideal generators that do not see $v$ are 1, since all the edge weight they see at a node are pairwise coprime.

We have so far shown that there are no cycles in the decomposition graph of $\tilde{M}$ if and only if there is a special node $v$ such that at all other nodes the edge weights that do not see $v$ are pairwise coprime. Next we have to see that our condition on $\Gamma$ also gives that all the pieces of the decomposition have genus 0.

Remember that when we do the induction in the proof of Theorem 6.3 in [Ped10a] and cut along an edge $e$ between $v_0$ and $v_1$, for be any node $v'$ in $\Gamma$ not equal to $v_0$ or $v_1$, the weight $r_{v'}(v_i)$ gets replaced by $r_{v'}(v_i)/d_{v'}(v_i)$, where $v_i$ ($i = 0$ or 1) is the node not in the same piece as $v'$ after cutting. When we cut $\Gamma$ along its edges,
we do it in the following way. Always choose an edge $e$ to an end node $w$, that is not the special node $v$ to cut along. Then after the cutting we get two new pieces. The first corresponds to the end node $w$ and has a one node splice diagram with as many edges as $w$ had in $\Gamma$, and the edges have the same weights, except $r_w(v)$ is divided by $d_w(v)$. The splice diagram of the other piece $\Gamma_e$ looks like $\Gamma$ with the node $w$ replaced by a leaf, and no edge weight is changed since all the $d_{v'}(w) = 1$ for any node $v'$. We then find an end node of $\Gamma_e$ which is not $v$ to cut along, and repeat until we have cut along all the edges between nodes.

We have now cut $\Gamma$ into a collection of one-node splice diagrams. Each of these will contribute at least one Seifert fibered piece to $\tilde{M}$, (the same one-node splice diagram may of course contribute with the same Seifert fibered piece of $\tilde{M}$ more than once). We distinguish the piece corresponding to our special node $v$. The pieces not corresponding to $v$ have splice diagrams with the same weights as in $\Gamma$, except $r_w(v)$ is replaced by $r_w(v)/d_w(v)$. Our assumptions on the $\Gamma$ then imply that all the weights are pairwise coprime, except possibly two weights who are pairwise coprime with the rest, but might have a common divisor. Since the Seifert fibered pieces corresponding to each of the nodes are the Brieskorn complete intersections defined by the edge weights, so condition one or two of Proposition 3.1 holds. Then the Seifert fibered pieces of the decomposition of $\tilde{M}$ corresponding to these nodes are rational homology spheres.

The special piece of the decomposition of $\tilde{M}$ (corresponding to $v$, there will in fact only be one), has genus 0, since the assumption on $\Gamma$ are equivalent to the Brieskorn complete intersection being genus 0, by proposition 3.1

Hence the assumptions on $\Gamma$ are equivalent to the decomposition graph of $\tilde{M}$ having no cycles, and all the pieces of the decomposition having a base of genus 0. □

The converse to the corollary does not immediately follow, since having no cycles and having genus 0 pieces are only two of the three conditions for a graph manifold to be a rational homology sphere. The last one (as we saw in proposition 2.1) is that the intersection matrix $I$ must have non zero determinant. Proving that $\det(I) \neq 0$, reduces to a simpler problem since Neumann showed in [Neu97] that, by doing row and column additions, $I$ becomes the direct sum of the decomposition matrix and a number of $1 \times 1$ matrices with non zero entries. Hence it is enough to show that the determinant of the decomposition matrix is non zero. To do this we need the following lemma describing the fiber intersection numbers in the universal abelian cover from the splice diagram.

**Proposition 3.3.** Let $v_0$ and $v_1$ be two nodes of $\Gamma(M)$ connected by an edge $e$, decorated as below. If there are no edge weights of 0 adjacent to any of the $v_i$’s, then the fiber intersection number $\tilde{p}$ in any torus in the universal abelian cover sitting above $T_e$ is

$$\tilde{p} = \frac{|D(e)|}{d_0 b_0 b_1},$$

where $\overline{d}_i$ are ideal generator corresponding to $r_i$ and

$$b_i = \frac{r_i/\overline{d}_i \text{lcm}(n_{i1}/\overline{d}_{i1}, \ldots, n_{ik}/\overline{d}_{ik})}{\text{lcm}(n_{i1}/d_{i1}, \ldots, n_{ik}/d_{ik}, r_i/d_i)},$$

again the $\overline{d}_{ij}$ are the ideal generators.
Proof. Let \( f_0 \) and \( f_1 \) be fibers from each of the sides in \( T_e \), and let \( p \) be the fiber intersection number in \( T_e \), i.e. \( p = f_0 \cdot f_1 \). It follows from the Edge Determinant Equation \((\ref{edge_det})\) that \( p = |D(e)|/|H_1(M)| \). Let \( \pi : M \to M \) be the universal abelian cover, and let \( \tilde{T} \tilde{M} \) be a connected component of \( \pi^{-1}(T_e) \). Then the intersection number of the preimage of \( p \) restricted to \( \tilde{T} \) is the intersection number before multiplied by the degree of the map restricted to \( \tilde{T} \), i.e. \( \pi|_\tilde{T}\cdot\pi|_\tilde{T}^{-1}(f_0) = p \deg(\pi|_\tilde{T}) \). Since \( \pi \) is the universal abelian cover its degree is \( |H_1(M)| \) and hence \( \deg(\pi|_\tilde{T}) = |H_1(M)|/t \) where \( t \) is the number of components of \( \pi^{-1}(T_e) \), and using the edge determinant equation we get that \( \pi\tilde{T}^{-1}(f_0) \cdot \pi\tilde{T}^{-1}(f_1) = |D(e)|/t \). Notice that \( t = \tilde{d}_0\tilde{d}_1 \), this follows from the proof of Theorem 6.3 in [Ped10a] and was also used in the proof Corollary 3.2.

Now \( \pi\tilde{T}^{-1}(f_i) \) consist of a collection of fibers \( \tilde{f}_i \), and hence using the biliniarity of the intersection product we get that \( |D(e)|/t = (\#\pi\tilde{T}^{-1}(f_0))(\#\pi\tilde{T}^{-1}(f_1))\tilde{f}_0 \cdot \tilde{f}_1 \). Since \( \tilde{f}_0 \cdot \tilde{f}_1 = \tilde{p} \) we just need to calculate \( \#\pi\tilde{T}^{-1}(f_i) \).

Let \( \tilde{M}_i \subset \tilde{M} - \pi^{-1}(T_e) \) be a connected component sitting above \( v_i \). Hence need to determine how many copies of \( \tilde{f}_i \) sits in each of the boundaries of \( \tilde{M}_i \). Remember that the Seifert fibered piece of \( \tilde{M}_i \), sitting above \( v_i \), is the Brieskorn complete intersection \( \Sigma = \Sigma(n_{i1}/\tilde{d}_{i1}, \ldots, n_{ik}/\tilde{d}_{ik}, r_i/\tilde{d}_i) \) where a tubular neighborhood around all the singular fibers \( a_i \) corresponding to \( r_i/\tilde{d}_i \) are removed. \( \tilde{f}_i \) is a non singular fiber of \( \Sigma \), and hence by the proof of Theorem 8.2 in [JN83] \( \pi|_{\tilde{f}_i} : \tilde{f}_i \to f_i \) has degree \( a_i |e| \), where \( e \) is the rational euler number of \( \Sigma \) and \( a_i = \text{lcm}(n_{i1}/\tilde{d}_{i1}, \ldots, n_{ik}/\tilde{d}_{ik}, r_i/\tilde{d}_i) \). Since \( \pi \) restricted to the Seifert fibered piece above \( v_i \) is the same as the restriction of the universal abelian of \( \Sigma \) its degree is \( \vert e \vert r_i/\tilde{d}_i \prod_{j} n_{ij}/\tilde{d}_{ij} \), and hence there are \( r_i/\tilde{d}_i \prod_{j} n_{ij}/\tilde{d}_{ij} \) copies of \( \tilde{f}_i \) in \( \Sigma \). These \( \tilde{f}_i \) all sit in the boundaries when we remove the tubular neighborhoods of the fibers sitting above \( a_i \), and by symmetry each of the boundary components of \( \tilde{M}_i \) has an equal number of copies. Since the number of fibers above \( a_i \) is \( \left( \prod_{j} n_{ij}/\tilde{d}_{ij} \right)/\text{lcm}(n_{ij}/\tilde{d}_{ij}) \), and we get that
\[
\#\pi\tilde{T}^{-1}(f_i) = \frac{r_i/\tilde{d}_i \text{lcm}(n_{i1}/\tilde{d}_{i1}, \ldots, n_{ik}/\tilde{d}_{ik})}{\text{lcm}(n_{i1}/\tilde{d}_{i1}, \ldots, n_{ik}/\tilde{d}_{ik}, r_i/\tilde{d}_i)}
\]
and the formula follows. \( \Box \)

**Proposition 3.4.** Let \( M \) be a graph orbifold whose splice diagram \( \Gamma(M) \) satisfies the conditions of Corollary 3.2 then the intersection form of the universal abelian cover \( \tilde{M} \) of \( M \) is non degenerate.

**Proof.** Remember from the earlier discussion that we only need to show that the decomposition matrix is non degenerate. The decomposition matrix has as diagonal entries the rational euler number of the pieces of the JSJ-decomposition, and on off diagonal entries is \( 1/p \) where \( p \) is the fiber intersection number if the corresponding pieces are connected by an edge.

It proof is going to be by induction by the number of nodes in \( \Gamma(M) \). If \( \Gamma(M) \) only has one node, then \( \tilde{M} \) is a Brieskorn complete intersection since we have no weights of value 0, and hence its intersection matrix is negative definite and therefore non degenerate.
So let $\Gamma(M)$ have $n$ nodes. Let $v$ be an end node other that the special node, that means that $v$ is only connected to one other node, call this node $w$ and let $v'$ be the special node, $v'$ can be equal to $w$. Assume we have named the weights in the following way

$$
\begin{array}{c}
\text{\hfill} \\
\begin{array}{c}
\text{\hfill}
\end{array}
\end{array}
$$

where the edges weighted with $m_i$ and $m$ leads to other nodes, and if $w \neq v'$ then the edge with $m$ one sees $v'$. Let $N = \prod_i n_i$, $M = \prod_i m_i$ and $R = \prod_i r_i$. The conditions on $\Gamma(M)$ implies that all the ideal generators except maybe $d_r$ and $\overline{d}_m$ are 1, and that $\gcd(n_i, n_j) = 1$ and $\gcd(r_i, r_j) = \gcd(m_i, m_j) = \gcd(s_i, s_j) = 1$. Let $\sigma = \gcd(r/d_r, n_{i_0}) = 1$ except maybe for one of the $n_i$ call this $n_{i_0}$, and assume $\gcd(r/d_r, n_{i_0}) = b_i$, likewise $\gcd(m_i/d_m, m_i) = \gcd(m_i/d_m, r_i) = 1$ except maybe for one of the $m_i$'s $r_i$'s or $s$. Let the value of the $\gcd$ not being 1 be $c$, and notice that if $\gcd(m_i/d_m, s) = c$ then $d_r = d_m$ els $d_r = cd_m$.

Above $v$ in $M$ sits $\overline{d}_v$ identical Seifert fibered pieces $\bar{v}$, and above $w$ sits $\overline{d}_m$ identical Seifert fibered pieces $\bar{w}$. Each of the $\bar{w}$ is connected two $\overline{d}_v/d_m$ of the $\bar{v}$'s, by an edge. This implies that in the decomposition matrix $A$ has $\overline{d}_m$ blocks looking like

$$
\begin{pmatrix}
\vdots & \vdots & \vdots \\
0 & 0 & \ldots & 0 \\
\vdots & 0 & e_{\bar{v}} & 0 & \ldots & \frac{1}{\sigma} & 0 & \ldots \\
\vdots & 0 & \ldots & \frac{1}{\sigma} & 0 & \ldots \\
\vdots & 0 & \ldots & e_{\bar{v}} & 0 & \ldots & \frac{1}{\sigma} & 0 & \ldots \\
\frac{1}{\sigma} & \frac{1}{\sigma} & \ldots & \frac{1}{\sigma} & e_{\bar{w}} & \ldots & \frac{1}{\sigma} & \frac{1}{\sigma} & \ldots \\
0 & 0 & \ldots & \frac{1}{\sigma} & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{pmatrix}
\overline{d}_v/d_m
$$

where $e_{\bar{v}}$ and $e_{\bar{w}}$ are the rational euler numbers of $\bar{v}$ and $\bar{w}$, and $\bar{p}$ is the fiber intersection number in the edges. We can calculate $\bar{p}$ using [Ped10a] and gets that $\bar{p} = |D(c)|/bcd_m$, the reason that it is $d_m$ and not $d_r$ in the formula, is that using $d_r$ gives two different formulas depending on whether $\gcd(m_i/d_m, s) = c$ or not, but using the relation ship between $d_m$ and $d_r$ to replace $d_m$ with $d_r$ makes the formulas the same. To calculate $e_{\bar{v}}$ and $e_{\bar{w}}$ we use the formula given in the end of the proof of 6.3 in [Ped10a], which gives that $e_{\bar{v}} = \frac{\lambda^2}{\sigma^2}e_v/|H_1(M)|$ and $e_{\bar{w}} = \frac{\lambda^2}{\sigma^2}e_w/|H_1(M)|$, where $\lambda = N r/d_m(n_1, \ldots, n_k, r/d_m) = bd_r$ and $\lambda_m = M R sm/\text{lcm}(m_1, \ldots, m_i, r_1, \ldots, r_i, s, m/d_m) = c d_m$. We find $e_{v_i}/|H_1(M)|$ and
$e_w/|H_1(M)|$ by using the formula of Proposition 3.4 in [Ped10a]. This gives that

$$e_v/|H_1(M)| = -\frac{\varepsilon_v s}{ND(e)}, \quad e_w/|H_1(M)| = -\frac{\varepsilon_w m'}{MD(m)} - \frac{\varepsilon_w N}{sD(e)} - E,$$

where $\varepsilon_v$ and $\varepsilon_w$ are the signs at the nodes, $D(m)$ is the edge determinant of the edge with $m$ on it $m'$ is the weight on the other end of that edge, and $E$ is a sum of contributions from the nodes seen be the $r_i$'s which dose not include any factors coming from $v$. This give the following values for $e_v$ and $e_w$

$$e_v = \frac{\varepsilon_v s \tilde{d}_v}{ND(e)} \quad e_w = -c^2\tilde{d}_m\left(\frac{\varepsilon_w m'}{MD(m)} + \frac{\varepsilon_w N}{sD(e)} + E\right).$$

We can clear all the $1/\tilde{p}$ in the row and column containing $e_w$ by using the rows and columns with the $e_v$ on the diagonal whit out changing anything other that the entry with $e_w$, hence our blocks will now look like

$$\begin{pmatrix}
\vdots & \vdots & \vdots \\
0 & 0 & \ldots & 0 \\
\ldots & 0 & e_v & 0 & \ldots & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & e_v & 0 & 0 & 0 & \ldots \\
\ldots & 0 & \vdots & \vdots & \vdots & \vdots & \vdots \\
\ldots & 0 & 0 & \ldots & e_v & 0 & 0 & \ldots \\
0 & 0 & \ldots & 0 & e_w = \frac{\varepsilon_v s}{ND(e)} & \frac{1}{\tilde{p}} & \frac{1}{\tilde{p}} & \ldots \\
0 & 0 & \ldots & 0 & \frac{1}{\tilde{p}} & \ldots & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots 
\end{pmatrix}$$

This implies that $A$ is row and column equivalent to $A' = \bigoplus \left(\frac{\varepsilon_v s}{ND(e)}\right)$, where $A'$ is equal to $A$, except the block has been replaced be a single entry of $e_w = \frac{\varepsilon_v s}{ND(e)}$. Since the $1 \times 1$ matrix $(e_v)$ has a non zero entry, $A$ is non degenerate if and only if $A'$ is non degenerate. So lets calculate the difference between $A$ and $A'$

$$e_w - \frac{\varepsilon_v s}{ND(e)} \frac{1}{\tilde{p}} e_v = -c^2\tilde{d}_m\left(\frac{\varepsilon_w m'}{MD(m)} + \frac{\varepsilon_w N}{sD(e)} + E\right) + \frac{\varepsilon_v s}{ND(e)} \frac{1}{\tilde{p}} c^2\tilde{d}_m N\varepsilon_v$$

$$= -c^2\tilde{d}_m\left(\frac{\varepsilon_w m'}{MD(m)} + \frac{\varepsilon_w N}{sD(e)} + E\right) + c^2\tilde{d}_m\frac{\varepsilon_w N}{sD(e)}$$

$$=-c^2\tilde{d}_m\left(\frac{\varepsilon_w m'}{MD(m)} + E\right).$$

But this is exactly the rational euler number of the seifert fibered pieces in the universal abelian cover of the manifold $M'$ with splice diagram $\Gamma(M')$ sitting above the node $w$, where

$$\Gamma(M') = \begin{tikzpicture}
  \node (w) at (0,0) {$w$};
  \node (m1) at (1,1) {$m_1$};
  \node (m) at (1,0) {$m$};
  \node (m2) at (1,-1) {$m_2$};
  \node (r1) at (-1,1) {$r_1$};
  \node (r) at (-1,0) {$r$};
  \node (r2) at (-1,-1) {$r_2$};
  \draw (w) -- (m1);
  \draw (w) -- (m);
  \draw (w) -- (m2);
  \draw (w) -- (r1);
  \draw (w) -- (r);
  \draw (w) -- (r2);
\end{tikzpicture}$$
the rest of $\Gamma(M')$ is identical to $\Gamma(M)$. It is not hard to see that $\Gamma(M')$ satisfy the conditions of Corollary 3.2. Since all the ideal generators in $\Gamma(M)$ that sees $v$ are $1$, all entries in the decomposition matrix of the universal abelian cover of $M'$ are the same as in the universal abelian cover of $M$ except the one above $v$ and $w$, and hence $A'$ is the decomposition matrix of the universal abelian cover of $M'$. This implies that $A'$ is non degenerate by the induction hypothesis, and hence $A$ is non degenerate and the intersection form of the universal abelian cover of $M$ is non degenerate.

We can now summarize the above proposition and Corollary 3.2 to the following result.

**Theorem 3.5.** Let $\Gamma$ be the the splice diagram of a manifold $M$, then the universal abelian cover of $M$ is a rational homology sphere if and only if all edge weights are nonzero, and there is a special node $v \in \Gamma$, with the following properties. For all other nodes $v' \in \Gamma$, the weights other than $r_{v'}(v)$ are pairwise coprime, and at most one of these edge weights is not coprime with $r_{v'}(v)/d_{v'}(v)$. At $v$ all the edge weights satisfy one of the conditions from Proposition 3.1.
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