Path and Path Deviation equations of Fractal Space-Times: A Brief Introduction

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The idea that the quantum space-time of microphysics may be fractal everywhere was intensively investigated recently, and several authors have presented the geodesic equations of different fractal space-times. In the present work we obtain the geodesic and the geodesic deviation equations in fractal space-times by using the Bazanski method. We also extend this approach to obtain the equations of motion for spinning and spinning charged particles in the above-mentioned spaces, in a similar way to their counterparts in Riemannian geometry.

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I. INTRODUCTION

The proposal that the space-time in which the evolution of the microscopic objects takes place may be a fractal has attracted a lot of attention recently [1, 2, 4, 7, 8, 9, 10, 11]. A fractal structure is a manifestation of the universality of self-organization processes, a result of a sequence of spontaneous symmetry breaking. In the fractal space-time model the typical trajectories of quantum particles are continuous, but non-differentiable, and can be characterized by a fractal dimension that jumps from $D = 1$ at large length scale to $D = 2$ at small length scale. The fractal dimension $D = 2$ is the fractal dimension of the Brownian motion, or, equivalently, of a Markov-Wiener process. As suggested by Nelson [6], quantum mechanics can also be interpreted as assuming that any particle is subjected to an underlying Brownian motion of unknown origin, which is described by two (forward and backward) Wiener processes: when combined together, they yield the complex nature of the wave function and they transform Newton’s equation of dynamics into the Schrödinger equation.

An alternative way for the description of the motion was introduced by Bazanski [3], an approach that has the advantage of providing both the equations of the geodesics as well as the geodesic deviation equation. It is the purpose of the present paper to generalize the Bazanski approach to the case of the fractal space-times.

II. FRACTAL SPACE-TIMES AND THE SCHRODINGER EQUATION

Let us consider a fractal curve $f(x)$ between two points $A$ and $B$ that is continuous, but nowhere differentiable. Such a curve has an infinite length. This is a direct consequence of the Lebesgue theorem, which states that a finite length curve is almost everywhere (i.e., except a set of points with null dimension) differentiable [7, 11]. Another important property of a fractal curve is that between any two points of the curve we can get a curve with the same properties as the initial curve, that is, a continuous, nowhere differentiable and infinite length curve. Therefore fractal curves are almost self-similar everywhere.

In the differentiable case the usual definition of the derivative of a given function are given by

$$\frac{df}{dt} = \lim_{\Delta t \to +0} \frac{f(t + \Delta t) - f(t)}{\Delta t} = \lim_{\Delta t \to -0} \frac{f(t) - f(t - \Delta t)}{\Delta t},$$

and one can pass from one definition to the other by the transformation $\Delta t \to -\Delta t$. The differentiable nature of the space-time implies the local differential (proper) time reflection invariance. In the non-differentiable case two
functions \( \frac{df_+}{dt} \) and \( \frac{df_-}{dt} \) are defined as explicit functions of \( t \) and \( dt \),

\[
\frac{df_+}{dt} = \lim_{\Delta t \to +0} \frac{f(t + \Delta t, \Delta t) - f(t, \Delta t)}{\Delta t}, \quad \frac{df_-}{dt} = \lim_{\Delta t \to -0} \frac{f(t, \Delta t) - f(t - \Delta t, \Delta t)}{\Delta t},
\]

with the plus sign corresponding to the forward process, while the minus sign corresponds to the backward process. In other words, the non-differentiable nature of the space-times implies the breaking of the local differential (proper) time reflection invariance. If we apply the definition of the derivatives to the coordinate functions, we obtain \( dX^i_+ = dx^i_+ + d\xi^i_+ \), where \( dx^i_+ \) are the usual classical variables, and \( d\xi^i_+ \) are the non-differentiable variables. By taking the average of these equations we obtain \( \langle dX^i_+ \rangle = \langle dx^i_+ \rangle \), since \( \langle d\xi^i_+ \rangle = 0 \). If we denote by \( d\bar{x}_+ / dt = \bar{v}_+ \) the forward speed and by \( d\bar{x}_- / dt = \bar{v}_- \) the backward speed, then \( (\bar{v}_+ + \bar{v}_-)/2 \) may be considered as the differentiable (classical) speed, while \((\bar{v}_+ - \bar{v}_-)/2\) is the non-differentiable speed. These two quantities can be combined in a single quantity if we introduce the complex speed \( \tilde{V} = \delta \bar{x} \), where \( \delta = (d_+ + d_-)/2dt - i (d_+ - d_-)/2dt \).

By considering that the continuous but non-differentiable curve of motion is immersed in a three-dimensional space, any function \( f(X^i, t) \) can be expanded into a Taylor series as

\[
f = f(X^i + dX^i, t + dt) - f(X^i, t) = [(\partial/\partial X^i) dX^i + (\partial/\partial t) dt] f(X^i, t).
\]

By assuming that the mean values of the function \( f \) and of its derivatives coincide with themselves we obtain \( d_+ f/dt = \partial f/\partial t + \bar{v}_+ \cdot \nabla f_+ \), while the operator \( \delta \) is given by \( \delta f/dt = \partial f/\partial t + \tilde{V} \cdot \nabla f \).

We can now apply the principle of the scale covariance, which postulates that the passage from classical (differentiable) mechanics to the non-differentiable mechanics can be realized by replacing the standard time derivative \( df/dt \) by the complex operator \( \delta f/dt \). Therefore in a covariant form the equation of geodesics of the fractal space-time can be written as

\[
\frac{\delta \tilde{V}}{dt} = \frac{\partial \tilde{V}}{\partial t} + \tilde{V} \cdot \nabla \tilde{V} = 0.
\]

By considering that the fluid is irrotational, \( \nabla \times \tilde{V} = 0 \), by introducing the complex speed potential \( \phi \) so that \( \tilde{V} = \nabla \phi \), and by assuming that \( \phi = -2tD \ln \psi \), where \( D \) is a constant, the equation of motion of the fluid takes the form of the Schrodinger equation,

\[
D^2 \Delta \psi + iD \frac{\partial \psi}{\partial t} - U \psi = 0,
\]

where \( U = D^2 \Delta \ln \psi \). \( D \) defines the differential-non-differential transition, that is, the transition from the explicit scale dependence to scale independence.

In order to study gravitational phenomena in fractal space-times it is necessary to extend the concept of metric by taking into account the fluctuating character of the paths. This corresponds to the passage from Special Scale Relativity to General Scale Relativity. The line element between two neighboring points in a fractal geometry can be described as

\[
d\tilde{s}^2 = \tilde{g}_{\mu \nu} dX^\mu dX^\nu = \tilde{g}_{\mu \nu} (dx^\mu + d\xi^\mu)(dx^\nu + d\xi^\nu),
\]

leading to a generalized metric in a curved fractal space-time of the form

\[
\tilde{g}_{\mu \nu}(x, t) = g_{\mu \nu}(x, t) + \gamma_{\mu \nu} \left( \frac{\lambda_c}{dx^\nu} \right) \left( \frac{\lambda_c}{dx^\nu} \right),
\]

where \( \gamma_{\mu \nu} \) is described as the first approximation in terms of stochastic variables \( \lambda_c \). Based on this fluctuating metric one can obtain the affine connection of the fractal space-time as \( \tilde{\Gamma}_{jk}^i = \Gamma_{jk}^i + \chi_{jk}^i \), where \( \Gamma_{jk}^i \) is the usual Christoffel connection, and \( \chi_{jk}^i \) is the fluctuating part. The mean values of the affine connection satisfy the conditions \( \langle \tilde{\Gamma}_{jk}^i \rangle = \Gamma_{jk}^i \) and \( \langle \chi_{jk}^i \rangle = 0 \). Similarly one can define the curvature tensor \( \tilde{R}_{jkl} = R_{jkl} + \Xi_{jkl} \), so that \( \langle \tilde{R}_{jkl} \rangle = R_{jkl} \) and \( \langle \Xi_{jkl} \rangle = 0 \).

III. THE BAZANSKI APPROACH IN FRACTAL SPACE-TIMES

Geodesic and geodesic deviation equations can be obtained simultaneously by applying the action principle on the Bazanski Lagrangian \( \Xi_{jkl} \):

\[
L = g_{\alpha \beta} U^\alpha \frac{D \Psi^\beta}{D\tilde{s}},
\]
where \( D/Ds \) is the covariant derivative. By taking the variation with respect to the deviation vector \( \Psi^\rho \) and with the unit tangent vector \( U^\rho \) one obtains the geodesic equation and the geodesic deviation equation respectively,

\[
\frac{dU^\alpha}{ds} + \left\{ \frac{\alpha}{\mu\nu} \right\} U^\mu U^\nu = 0, \quad \frac{D^2 \Psi^\alpha}{Ds^2} = R^\alpha_{\beta\gamma\delta} U^\beta U^\gamma \Psi^\delta,
\]

where \( \left\{ \alpha_{\mu\nu} \right\} \) is the Christoffel symbol of the second kind \( R^\alpha_{\beta\gamma\delta} \) is the Riemann-Christoffel curvature tensor. In the case of the fractal space-times we propose the following form of the Bazanski Lagrangian \( L \),

\[
L = g_{\mu\nu} \dot{V}^\mu \dot{\Psi}^\nu + f_\mu \dot{\Psi}^\nu,
\]

where \( f_\mu = (e/m) \tilde{F}_{\mu\nu} V^\nu + (1/2m) \tilde{R}_{\mu\nu\gamma\delta} S^{\gamma\delta} V^\nu \), and the covariant scale derivatives in the fractal space are defined as

\[
\tilde{D} \tilde{V}^\mu = \frac{D (V^\alpha + iU^\alpha)}{Ds} + (V^\mu + iU^\mu) \cdot D_\mu (V^\alpha + iU^\alpha) - i\mu (\partial_\mu \partial_\nu) (V^\alpha + iU^\alpha) \tag{10}
\]

and

\[
\tilde{D} \tilde{(V^\alpha + iU^\alpha)} = \frac{D (V^\alpha + iU^\alpha)}{Ds} + (V^\mu + iU^\mu) \cdot D_\mu (V^\alpha + iU^\alpha) - i\mu (\partial_\mu \partial_\nu) (V^\alpha + iU^\alpha) \tag{11}
\]

respectively.

By using the Bazanski approach we can immediately obtain the Lorentz force equation, the Papapetrou equation (corresponding to the motion of spinning particles) and the Dixon equation (describing the motion of spinning particles in electromagnetic fields as

\[
\tilde{D} \tilde{V}^\mu = \frac{e}{m} \tilde{F}_{\mu\nu} \tilde{V}^\nu, \quad \tilde{D} \tilde{V}^\mu = \frac{1}{2m} \tilde{R}_{\mu\nu\sigma\rho} \tilde{S}^{\sigma\rho} \tilde{V}^\nu, \quad \tilde{D} \tilde{V}^\mu = \frac{1}{2m} \tilde{R}_{\mu\sigma\rho} \tilde{S}^{\sigma\rho} \tilde{V}^\nu + \frac{e}{m} \tilde{F}_{\mu\nu} \tilde{V}^\nu. \tag{12}
\]

The quantum covariant derivative in fractal space-times can be generalized as

\[
\frac{\tilde{D}}{Ds} = \left[ \tilde{V}^0 D_0 - \frac{i\lambda}{2} (D^\rho D_\rho + \xi R) \right] = 0. \tag{13}
\]

The geodesic equation of motion is \( \tilde{D} \tilde{V}^\mu / \tilde{D}s = 0 \). By putting \( \tilde{V}_\mu = i\lambda D_\mu \ln \Psi \), we obtain the Klein-Gordon equation for a free particle in a curved space as

\[
\lambda^2 D^\mu \ln \Psi D_\mu D_\rho \ln \Psi + \frac{\lambda^2}{2} (D_\mu D_\mu D_\rho \ln \Psi + \xi R D_\rho \ln \Psi) = 0. \tag{14}
\]

The geodesic deviation equation is given by

\[
\frac{\tilde{D}^2 \tilde{\Psi}^\alpha}{Ds^2} = \left[ \frac{\tilde{D} \tilde{\Psi}^\alpha}{Ds} + \tilde{V}^\mu \cdot D_\mu \tilde{\Psi}^\alpha - i\mu (D^\alpha D_\mu + \xi R) \tilde{\Psi}^\alpha \right] = \tilde{R}^\alpha_{\beta\gamma\delta} \tilde{V}^\beta \tilde{V}^\gamma \tilde{\Psi}^\delta = (R^\alpha_{\beta\gamma\delta} + \Xi^\alpha_{\beta\gamma\delta}) \tilde{V}^\beta \tilde{V}^\gamma \tilde{\Psi}^\delta. \tag{15}
\]

Let us introduce now the covariant derivation operator with respect to the coordinates as \( D\Phi_\alpha / Ds = D_\alpha \ln \Phi (x, \Psi) \) and the covariant derivation operator with respect to the deviation vector \( \tilde{V}^\alpha = D\Phi_\alpha \ln \Phi (x, \Psi) \). Then the geodesic deviation equation in a fractal space-time is given by

\[
\frac{\tilde{D}^2 D_\alpha \ln \Phi}{Ds^2} = \left[ \frac{\tilde{D} (D_\alpha \ln \Phi)}{Ds} + D\Phi_\alpha \ln \Phi \cdot D_\mu D_\alpha \ln \Phi - i\mu (D^\alpha D_\mu + \xi R) D_\alpha \ln \Phi \right] = (R^\alpha_{\beta\gamma\delta} + \Xi^\alpha_{\beta\gamma\delta}) D\Phi^\beta \ln \Phi D\Phi^\gamma \ln \Phi D\Phi^\delta \ln \Phi. \tag{16}
\]
IV. DISCUSSIONS AND FINAL REMARKS

The theory of scale relativity extends Einstein’s principle of relativity to scale transformations of resolutions, and it gives up the concept of differentiability of space-time. Its main result is the reformulation of quantum mechanics from its first principle. In a fractal space time a small increment of displacement of the non differentiable four-coordinates along one of the geodesics can be generally decomposed into its mean and a fluctuating term. In the present paper we have obtained the path equations in general scale relativity, as described by a fractal Riemannian geometry, and we have combined the geodesic equations and the Schrodinger/ Klein-Gordon equations in a single equation, which can be reduced to each of them separately if and only if one uses the averaging procedure and solve the problem of the integrability of the affine connection due to the curvature of the space time. Also, we have obtained a quantum analog of the geodesic deviation equation as defined in a Fractal Space-Time.

[1] Agop, M. and Gottlieb, I. 2006, J. Math. Phys., 47, 053503
[2] Agop, M., Ioannou, P. D. and Nica, D. 2005, J. Math. Phys., 46, 062110
[3] Bazanski, S. L. 1989, J. Math. Phys., 30, 1018
[4] Gottlieb, I., Agop, M., Ciobanu, G. and Stroe, A. 2006, Chaos, Solitons & Fractals, 30, 380
[5] Kahil, M. E. 2006, J. Math. Phys., 47, 052501
[6] Nelson, E. 1966, Phys. Rev., 150, 1079
[7] Nottale, L. and Schneider, J. 1984, J. Math. Phys., 25, 1296
[8] Nottale, L. 1994, Chaos, Solitons & Fractals, 4, 361
[9] Nottale, L. 1998 Chaos, Solitons & Fractals, 9, 1051
[10] Nottale, L. 2005 Chaos, Solitons & Fractals, 25, 797
[11] Nottale, L., Celerier, M.-N. and Lehner, T. 2006, J. Math. Phys, 47, 032303
[12] Wanas, M. I., Melek, M. and Kahil, M. E. 1995 Astrophys. Space Sci., 228, 273
[13] Wanas, M. I. and Kahil, M. E. 1999, Gen. Rel. Grav., 31, 1921