Ultrahigh dimensional instrument detection using graph learning: an application to high dimensional GIS-census data for house pricing

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Abstract: The exogeneity bias and instrument validation have always been critical topics in statistics, machine learning and biostatistics. In the era of big data, such issues typically come with dimensionality issue and, hence, require even more attention than ever. In this paper we ensemble two well-known tools from machine learning and biostatistics — stable variable selection and random graph — and apply them to estimating the house pricing mechanics and the follow-up socio-economic effect on the 2010 Sydney house data. The estimation is conducted on an over-200-gigabyte ultrahigh dimensional database consisting of local education data, GIS information, census data, house transaction and other socio-economic records. The technique ensemble carefully improves the variable selection sparisty, stability and robustness to high dimensionality, complicated causal structures and the consequent multicollinearity, which is ultimately helpful on the data-driven recovery of a sparse and intuitive causal structure. The new ensemble also reveals its efficiency and effectiveness on endogeneity detection, instrument validation, weak instruments pruning and selection of proper instruments. From the perspective of machine learning, the estimation result both aligns with and confirms the facts of Sydney house market, the classical economic theories and the previous findings of simultaneous equations modeling. Moreover, the estimation result is totally consistent with and supported by the classical econometric tool like two-stage least square regression and different instrument tests (the code can be found at https://github.com/isaac2math/solar_graph_learning).

Keywords and phrases: instrument selection, endogeneity detection, subsample-ordered least-angle regression, lasso regression, elastic net regression, variable selection, random graph, grouping effect.

1. Introduction

The endogeneity bias has long been an important problem in causal analysis and has for decades aroused a universal awareness of statisticians, econometricians and biostatisticians. With the increase of the dimensionality, such topic requires much more attention than ever. On one hand, we seem to have more information and potential to observe and rectify the endogeneity bias by finding a valid instrument variable (referred to as instrument for short); on the other hand, the problem got mixed with the curse of high dimensionality and the consequential complication of dependence structure. As a result,
it is critical and urgent to investigate how to well utilise a high-dimensional database for endogeneity detection and instrument selection without be affected by the dimensionality curse. In this paper, we combine two theoretically well-founded machine learning and biostatistics tools — stable variable selection and random graph estimation — and demonstrates that such ensemble can well handle the endogeneity detection and instrument selection even in ultrahigh dimensional data.

In causal analysis, Pearl (2009, p246) shows that there are three definitions of a valid instrument: graphical criteria, error-based criteria and counterfactual criteria, where the graphical criteria implies the error-based criteria. The classical regression analysis relies mostly on the error-based criteria. In econometrics, the instrument \( z \) is defined by the data-generating process (equation (1.1)), where \( \{u, v\} \) are noise terms, \( \{z, v\} \) cause \( x \), \( \{x, u\} \) cause \( Y \) and \( x \) is endogenous.\(^1\)

\[
\begin{align*}
  x &= \alpha_0 + \alpha_1 z + v \\
  Y &= \beta_0 + \beta_1 x + u
\end{align*}
\]

For the validity of an instrument, we typically require \( \text{corr}(z, u) = 0 \) (denoted as \((C1)\)) and \( \text{corr}(z, x) \neq 0 \) (denoted as \((C2)\)) in the population. \((C1)\) implies that the change of \( z \) cannot affect \( u \), further implying that change of \( z \) cannot affect \( Y \) via \( u \). \((C2)\) implies the change of \( z \) can affect \( x \), further implying that change of \( z \) can affect \( Y \) via \( x \). Two conditions altogether mean that \( z \) can only impact \( Y \) via \( x \).

The idea about instruments can be generalized using probabilistic graph models (also referred to as Bayes net or causal network). In probabilistic graph models, the causal structure that equation (1.1) represents can be expressed equivalently as a directed acyclic graph (referred to as graph for short) as figure 1.\(^2\) In many causal inference monographs (see, e.g., Spirtes et al. (2000, p44)), the causal structure is directly defined as a graph. Using graphs, we can clearly identify the relation of each pair of variables and, hence, visually investigate their causal relations.

\(^1\)Unfortunately the causation assumption cannot be dropped; otherwise, endogeneity will inevitably arise. See Appendix 5.

\(^2\)There are more than one notation system for graphs. Throughout the paper we follow consistently the Koller and Friedman (2009) notation.
Fig 1: The graph representing equation (1.1), where the absence of arrows between z and u means corr (z, u) = 0.

A graph can be analogically considered as a family tree of variables and arrows in a graph represent causation. In Figure 1, arrows from \{z, v\} to x mean that z and v directly cause Y, further implying that corr (z, x) ≠ 0. We also use the terminology that \{z, v\} are the ‘parents’ of x, x is the ‘child’ of \{z, v\} and z is a ‘spouse’ of v. In Figure 1, v directly causes x and, hence, indirectly causes Y. The variables that directly or indirectly cause Y are the ‘ancestors’ of Y. Hence, Y and x are the ‘descendants’ of z. Lastly, two variables are ‘siblings’ if they share the same parents.³

In biostatistics and machine learning, graph learning is typically taken as a critical step of causal inference. Failing to learn an accurate graph may cause a series of causal inference problems, such as endogeneity, multicollinearity and misinterpretation. Such point can be demonstrated using the following three examples.

Motivating examples

In example 1, we demonstrate the importance of identifying a correct parent-child relation of a variable for causal inference.

Example 1. In this example, we demonstrate the possibility of endogeneity caused by wrong causation presumptions, which may be difficult to rectify. In both numerical and theoretical analysis of linear regression, typically the parents of Y are on the right-hand side of the equation and Y on the left. Assume the causal structure as figure 2.

³See Koller and Friedman (2009, Section 2.2) for further detail on the terminology.
and the data-generating process as equation (1.2),
\[ x_1 = \beta_0 + \beta_1 Y + e, \]  
(1.2)
where \( x_1, Y \) and \( e \) are \( n \times 1 \) vectors; both \( Y \) and \( e \) cause \( x_1 \); \( Y \) is independent from \( e \). 
Equation (1.2) implies that \( \text{corr}(x_1, e) \neq 0 \). If we take \( Y \) as the response variable for some empirical reason, the empirical model does not represent the correct causation and can be expressed as
\[ Y = \alpha_0 + \alpha_1 x_1 + u \]  
(1.3)
where \( \alpha_0 = -\beta_0/\beta_1, \alpha_1 = 1/\beta_1 \) and \( u = -e/\beta_1 \). This implies that \( \text{corr}(u, x_1) = \text{corr}(-e/\beta_1, x_1) \neq 0 \), which means absolute endogeneity. What’s worse, due to mistaking a parent as a child, the causal structure that equation (1.4) represents is totally wrong. In such case, it would be very difficult to find an instrument for remedy. Probably the only fix is put parents of \( Y \) back to the right-hand side. As a result, from the example above we can see that either the model represents the correct causal structure, or it will be contaminated by endogeneity; there is hardly a middle ground inbetween.  

Example 1 demonstrates the potential problem if we ignore the correct parent-child relation in the graph. In example 2, we demonstrate that an accurate graph can be greatly helpful on choosing the correct control variables. Failing to do so may cause a series of issues.

**Example 2.** In regression analysis, it is obvious and widely noted (e.g., Fox (1984)) that the regression coefficient estimate will be biased and inconsistent if some important covariate is omitted. To avoid that, it is often recommended (Pratt and Schlaifer, 1988) that investigators enlarge the set of potential covariates. In this example, we demonstrate that this may also cause problems in causal inference.

In this example, we assume that (i) the data is generated by the causal structure in figure 1; (ii) the data generating process does not suffer any endogeneity issue. The
data-generating process is still represented as equation (1.1). Suppose that we want to investigate the causal effect from \( z \) to \( Y \). Unfortunately, if we also control \( x \) in the regression equation like equation (1.4),

\[
Y = b_0 + b_1x + b_2z + e, \tag{1.4}
\]

we may never have an accurate inference on the causal effect from \( z \) to \( Y \). As shown in figure 1, \( z \to x \to Y \), implying there is an indirect causal relation from \( z \) to \( Y \) via \( x \). However, if we control the value of \( x \) in our regression, the value of \( x \) will be constant, implying that any change of \( z \) cannot affect the value of \( x \) and, hence, \( Y \). As a result, \( z \) will not be significant in regression equation, implying that we may wrongly conclude that \( z \) has no causal effect on \( Y \).

What is worse, in empirical analysis, this problem is highly likely to be mistaken as multicollinearity since the correlation between \( x \) and \( z \) is not low. Building on that, regression with robust standard error is often proposed. However, the source of the problem is the ‘overcontrol’ of variables. Hence, this issue will never be solved no matter how the sample size increases. We can get an accurate inference only if we remove \( x \) from the equation. □

Example 2 clearly reveals the issue of ‘overcontrolling’ variable in causal inference if we wrongly assume the causal structure, illustrating that a careful selection of variables is necessary in causal analysis. However, without careful thoughts on the graph, variable selection algorithms may also be misled, which may provide a counterintuitive and inaccurate result. For example, finding the correct parents of a variable is critical for both prediction and causal inference, especially when \( p \) is large. In the following example, we demonstrate numerically the difficulty of correctly identifying parents of \( Y \) under strong confounding effect and how it causes model misinterpretation. For precision and conciseness, we follow Zhao and Yu (2006); Tibshirani et al. (2012) and quantify the difficulty with a well-known condition called irrepresentable condition (IRC).
Example 3. (Zhao and Yu, 2006) Assume the data-generating process is

\[
\begin{align*}
    x_3 &= \omega_1 x_1 + \omega_2 x_2 + \sqrt{1 - \omega_1^2 - \omega_2^2} u, \\
    Y &= \beta_1 x_1 + \beta_2 x_2 + \sqrt{1 - \beta_1^2 - \beta_2^2} e,
\end{align*}
\] (1.5)

where all variables are Gaussian; \(x_1, x_2, x_3, Y, e\) and \(u\) are standardized \(n \times 1\) vectors; \(u\) and \(e\) are independent from \(\{u, e\}\). The causal structure shows that \(\{x_1, x_2\}\) are the common parents of \(\{x_3, Y\}\), which are siblings. IRC shows that, for variable selection accuracy of lasso (e.g., in this case selecting \(\{x_1, x_2\}\) while dumping \(x_3\)), \(\sum_i |\omega_i| < 1\); otherwise, with a large probability the lasso-type estimators will take the sibling of \(Y\) as a parent (see the last simulation in Xu et al. (2019) for detail). What’s worse, if a group of variables are highly correlated with each other, Zou and Hastie (2005) shows that lasso may randomly drop variable(s) from the group (referred to as the grouping effect), making the variable selection result extremely sensitive to sampling randomness. This may bring difficulty to model interpretation and causal inference like instrument selection, leading to selecting variables weakly correlated to \(Y\) (e.g., the weak instruments of \(Y\)).

As demonstrated above, the traditional lasso estimator has its caveats for empirical applications with severe multicollinearity and complicated causal structure. To improve the variable selection accuracy and robustness, we follow Xu et al. (2019) and apply the novel subsample-ordered least-angle regression algorithm (solar) instead, which is modified from least-angle regression (Efron et al., 2004) and significantly outperforms lasso (from the perspective of sparsity and variable-selection accuracy) on data with severe multicollinearity and complicated causal structure. Particularly Xu et al. (2019) shows that, with IRC violated, solar can still maintain the robustness while lasso completely loses its accuracy.
1.1. Literature review on graph learning and instruments

Learning a data-driven causal structure has been one of the central topics of machine learning and biostatistics for decades. Since 1980s Pearl (2009) starts the pioneer work of analyzing the causal structure problem with graphs. Spirtes et al. (2000, p197) carefully inspect causal inference from the perspectives of both regression analysis and graph learning. The classical machine learning and statistics researches show that, with joint distributions alone, it is not possible to validate whether some observable variable is an instrument (Brito and Pearl, 2002; Kuroki and Cai, 2005; Chu et al., 2013; Silva and Shimizu, 2017). They show that instrument assumptions can nevertheless be falsified by exploiting constraints in the joint distribution of multiple observable variables. They also illustrate that, as a special case of causal analysis, (i) the classical OLS modelling typically assumes an oversimplified causal structure; (ii) the regression methodology can easily be misled and problematic in causal inference due to the complication of causal structure in real-world data. Hence, they recommend careful graph learning for causal structure recovery. There are two basic methods for graph learning: constraint-based learning and score-based learning (see, for example, (Scutari and Denis, 2014)). Constraint-based learning carries out conditional and marginal correlation tests among all possible pairs of variables. Score-based learning assumes distribution for all variables and computes the AIC/BIC/BGE score for each possible dependence structure, selecting the structure with minimal BIC score, which can be done using different packages (for example, the R package bnlearn and Python package pgmpy).

As a major issue of linear graph learning, multicollinearity can cause problems on classical techniques of linear modelling from different perspectives. Firstly, since linear modelling can be considered as the error minimization in a linear space, the multicollinearity issue will reduce the magnitude of the minimal eigenvalue in the linear space, causing different issues on numerical convergence (e.g., the Cholesky decomposition or the gradient descent) and model estimation. Moreover, a severe multicollinearity will amplify the instability of the parameter estimate across samples. For example, the more severe the multicollinearity issue is, the more dramatically the sample regression coefficients will change across samples, implying that it is improbable to interpret the sample regression coefficients reliably and accurately. Furthermore, the multicollinearity issue also causes problems on statistical tests. A severe multicollinearity issue will unnecessarily overam-
plify the volume of the standard error of regression coefficients. As a result, the finite-sample performance of all the statistical tests that rely on the sample covariance (e.g., the post-OLS t-test or the covariance test of lasso (Lockhart et al., 2014), the conditional correlation tests of dependence structure estimation (Scutari and Denis, 2014)) will be weakened (Farrar and Glauber, 1967). Last but not least, the multicollinearity may also reduce the algorithmic stability of the model (Elisseeff et al., 2003), which reduce the generalization ability and the prediction ability of the estimated model.

Multicollinearity also affects the reliability of the variable selection algorithms in linear modelling. For example, the lasso regression (Tibshirani, 1996) will be unstable if a group of variables are highly correlated to each other (Zou and Hastie, 2005; Jia and Yu, 2010). Lasso will randomly select one from the group and drop the other out of the regression model, which is referred to as the **grouping effect**. For all linear modelling techniques, the variable selection decision is based on the conditional correlation between a covariate $x_j$ and the response $Y$ while controlling the other covariate. As a result, the grouping effect may well apply to other variable selection methods like the best subset method (including AIC, BIC and Mallow’s $C_p$), reducing the stability and accuracy of the variable selection in linear modelling.

The consequence of grouping effect and multicollinearity has gone beyond the field of variable selection in linear modelling. Since (i) it is NP-hard to estimate the dependence structure (also referred to as probabilistic graph learning) on data with large $p$ (Heckerman et al., 1995; Chickering et al., 2004); (ii) the dependence structure estimation algorithms typically work on data with large $n$ and very sparse $p$, variable selection methods in linear modelling (e.g., SCAD (Fan and Li, 2001), ISIS (Fan and Lv, 2008) and different lasso-type estimators (Fan et al., 2009)) are frequently used to filter out the redundant variables before estimating the linear dependence structures in biostatistics and machine learning. However, due to the complicated linear structure and, hence, the grouping effect, lasso or other classical variable selection methods may randomly drop some of the highly correlated variables, resulting in the omissions of important variables in the linear structure.

Different attempts have been made to reduce the effect of multicollinearity. For a more stable regression coefficients estimate, Hoerl and Kennard (1970) apply the Tikhonov regularization to OLS, resulting in the Ridge regression. However, since Ridge sacrifices
its unbiasedness for the smaller regression coefficient variance (a.k.a a James-stein-type estimator), extra difficulty is brought to the statistical tests and the post-estimation inference of Ridge. To reduce the grouping effect and obtain a stable variable-selection result, cross-validated group lasso and cross-validated elastic net (CV-en) are introduced Zou and Hastie (2005); Friedman et al. (2010). By grouping the highly correlated variables together (i.e. they will be dropped out or included as a group), group lasso improves the robustness of lasso to the grouping effect. However, group lasso relies on manual grouping of variables, which heavily relies on the accuracy of the field knowledge. On the other hand, even though Zou and Hastie (2005) and Jia and Yu (2010) show that in some cases CV-en improves the stability and accuracy of lasso variable selection, Jia and Yu (2010) also show that the improvement is marginal and “when the lasso does not select the true model, it is more likely that the elastic net does not select the true model either.”

1.2. Main results

In this paper we combine two well-known machine learning and biostatistics tools — stable variable selection and graph learning — and apply them to estimate the house pricing mechanics and the follow-up socio-economic effect on the 2010 Sydney house database, an ultrahigh dimensional database consisting of local education data, GIS information, census data, house transaction and other socio-economic records. The estimated graph of house pricing produces intuitive interpretations and matches the facts of the Sydney house market, economic theories and the previous findings of econometrics on house pricing. The estimated graph also returns an accurate and sparse house pricing model, which outmatch other methods on the bias-variance trade-off.

The estimated graph also visually demonstrates the causal structure of the house pricing dynamics. Based on the graph, we successfully detect the endogeneity on house price data-driven, which is also confirmed by simultaneous equations modelling. Further more, with the help of graph estimation, we are able to accomplish instrument validation and instrument selection effectively and efficiently, which is also confirmed significant and intuitive by the traditional instrument tests like Durbin, Wooldridge and Hausman. Moreover, using the graph-recommended instrument, we significantly rectify the endogeneity bias on house price, which is also confirmed in two-stage least square. Last but not least, the graph estimation method also helps in identifying weak instrument, which is totally
consisted with the traditional econometric tools and economic intuition.

The paper is written in the following order. In section 2, we introduce variable selection and instruments from the perspective of random graph. In section 3, we introduce the details of the 2011 Sydney house data and shows the detailed procedure of variable selection and graph estimation on 2011 Sydney house data. In section 4, we utilise the previous estimation result for endogeneity detection and instrument selection; we also show that the graph-based result is totally consistent with the classical

2. Graphs learning and instrument selection

2.1. Graphical criteria of exogeneity and instruments

To properly introduce an instrument using graphs, we need to first define how the change in a variable can affect another variable in a graph, which is summarized as the concept of ‘trail’ (also referred to as ‘path’ in other graph learning literature).

Definition 2.1 (Trail of a graph).

- for any pair of variables \((x_i, x_j)\) in a graph, we say that they are connected \((x_i \Leftarrow x_j)\) if either \(x_i \rightarrow x_j\) or \(x_j \rightarrow x_i\) (\(x_i\) and \(x_j\) have a parent-child relation).
- for variables \(x_1, \ldots, x_k\) in a graph, we say that they form a trail if \(\forall 1 \leq i \leq k-1, x_i \Leftarrow x_{i+1}\).

Intuitively, a trail is a sequence of variables that are sequentially connected by arrows. A change in \(x_1\) can affect \(x_k\) only if there is a trail between the two variables. In figure 1, for example, \(z \rightarrow x \rightarrow Y\) is a trail, meaning a change in \(z\) can be passed to \(Y\) if \(x\) is not conditioned on. In figure 1 and equation (1.1), \(z \rightarrow x \rightarrow Y \leftarrow u\) is also a trail, meaning a change in \(z\) can pass to \(u\) only if (i) \(Y\) is held constant and (ii) \(x\) is not fixed.\(^4\) In these two trails, \(x\) plays a key role. If \(x\) is held constant, any change in a variable at one end of the trail cannot affect the variable on the other end. To describe the role of variables like \(x\), we say the variables at both ends of the trail are \(d\)-separated by \(x\) (aka ‘blocked’ by \(x\)’ in other graph learning literature), defined as definition 2.2.

Definition 2.2 (d-separation). Let \(P\) be a trail from the variable \(u\) to the variable \(v\). We say that \(u\) and \(v\) to be \(d\)-separated by a set of variables \(Z\) (denoted as \(u \perp \perp v \mid Z\)) if \(u\) and \(v\) are independent after conditioning on all variables in \(Z\). For example,

\[^4\text{Taking } Y = x + u \text{ as an example. } x \text{ and } u \text{ will be negatively correlated after } Y \text{ is held constant.}\]
• $P$ contains a directed chain ($u \leftarrow \cdots \leftarrow m \leftarrow \cdots \leftarrow v$ or $u \rightarrow \cdots \rightarrow m \rightarrow \cdots \rightarrow v$) such that the middle variable $m \in Z$;

• $P$ contains a fork ($u \leftarrow \cdots \leftarrow m \rightarrow \cdots \rightarrow v$) such that the middle variable $m \in Z$;

• $P$ contains a collider ($u \rightarrow \cdots \rightarrow m \leftarrow \cdots \leftarrow v$) such that the middle variable $m \notin Z$ and no descendant of $m$ is in $Z$.

We also introduce two useful remarks for d-separation. Firstly, if $A$ directly causes $B$ (i.e., $A \rightarrow B$) via no intermediate variables, $A$ and $B$ will never be independent whatever variable you condition on (except $A$ and $B$). In that case, we say that no variable can d-separate $A$ and $B$ (some literature denote it as $A \perp \perp B$). Secondly, as illustrated in figure 4, if $A$ and $B$ have no causal relation of any possible sort,

\begin{center}
\begin{tikzpicture}
  \node (A) at (0,0) {A};
  \node (B) at (1,0) {B};
  \node (C) at (2,0) {C};

  \draw[->] (A) -- (B);
  \draw[->] (B) -- (C);
  \draw[->] (A) -- (C);
\end{tikzpicture}
\end{center}

Fig 4: $A$ and $B$ are d-separated by any possible variable, for example $C$.

we say any variable (for example, $C$) can d-separate $A$ and $B$ (some literature denote it as $A \not\perp \perp B$). Using the concept of d-separation, the graphical definition of an instrument can be precisely defined as definition 2.3 by Brito and Pearl (2002), Pearl (2009, p.247-248) and Silva and Shimizu (2017) and illustrated as figure 5.\(^5\)

**Definition 2.3** (Graphical criteria of instruments). Let $x$, $z$ and $Y$ be variables in graph $G$ and $x$ directly causes $Y$. $z$ is an instrument for $x$ if

- (G1): $z$ and $Y$ can be d-separated by any variable in $G_x$, where $G_x$ is the graph in which the effect from $x$ to $Y$ is cut off. Some literature denote this as $(z \perp \perp Y)_{G_x}$

- (G2): $z$ and $Y$ cannot be d-separated by any variable in $G$. Some literature denote this as $(z \not\perp \perp x)_G$

Consistent with (C1) and (C2), definition 2.3 can be interpreted intuitively. Graphically, condition (G1) means that, if we remove all the causal effects from $x$ to $Y$, $z$ cannot affect $Y$ any more.\(^6\) In a similar vein, condition (G2) means that the effect from $z$ to $x$ cannot

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\(^5\)As shown by Brito and Pearl (2002) and Pearl (2009, p.247-248), the complete graphical criteria of an instrument is somehow more complicated than definition 2.3. It incorporates the idea of ‘conditional instruments’ in a graph. To avoid being sidetracked, we also leave these discussions into Appendix 5.

\(^6\)In some graph learning literature, (C1) is modified as $(z \perp Y)_{G_x}$, where $G_x$ is obtained by removing all arrows entering $x$ from the graph $G$. We stick to the mainstream definition of instrument (definition 2.3). Nonetheless, both versions mean that the effect from $z$ to $Y$ must only go through $x$.
be broken by holding any variable constant. Both (G1) and (G2) mean that the effect from $z$ to $Y$ must only go through $x$. Put in another way, holding $x$ constant, $z$ cannot affect $Y$ by any mean. In graph learning, the effect from $z$ to $Y$ via the (endogenous) variable $x$ is also referred to analogically as the indirect effect (Figure 5a). Moreover, Definition 2.3 is a generalized version of the definition of an instrument in regression analysis. Assuming that $x$ causes $Y$ in equation (1.1), $\text{corr}(z, x) \neq 0$ means the existence of the indirect effect; likewise, $\text{corr}(z, u) = 0$ means there does not exist any effect from $z$ to $Y$ that does not go through $x$ (referred to analogically as no indirect effect).

Definition 2.3 can also be used to graphically identify variables that are instruments. Take Figure 6 as an example. As a classical econometric case, Figure 6 contains an arrow from $z$ to $u$. As a result, $\text{corr}(z, u) \neq 0$ in equation (1.1), implying $z$ is not a valid instrument. Equivalently, the arrow from $z$ to $u$ allows $z$ affect $Y$ not through the endogenous variable $x$, which induces the indirect effect. As a result, figure 6 violates condition (G1) in Definition 2.3 since $z$ and $Y$ are not independent even though $x$ is held constant. For more detailed analysis and examples, see Appendix 5.
2.2. Variable selection for graph learning

Figure 5 and Appendix 5 illustrate that a graph can be used for instrument selection. If we can successfully estimate the graph (or at least estimate the role of each variable to $Y$), we can reduce the severity of endogeneity and improve the reliability and robustness of instrument regression. To achieve graph estimation accuracy, we need a large number of variables to avoid possible variable omission to the maximum extent. Unfortunately, large variable number may also bring up the dimensionality issue. To avoid the dimensionality issue in graph learning, it is mandatory to accompany graph learning with variable selection. To reduce the computational load and show that graph and variable selection are consistent with the classical causal analysis and regression analysis, in this paper we follow the classical endogeneity analysis and assume that

**A1.** the data generating process of each variable can be somehow represented as a linear regression equation;

**A2.** the dependence of variables can be somehow represented as correlation (e.g., equation (1.1)).

It is worth noting that we do not assume that the linear representation is perfectly accurate. In fact, it is quite common to suffer misspecification when applying linear models, especially when we are not sure about the linearity of the data-generating process. As a crucial part of application study, we will discuss this in detail when analyzing the reliability of the graph learning result. Under these assumptions, all graphs in this paper are linear graphs and graph learning can be comprehended from the perspective of high-dimensional regression analysis.

In classical regression analysis, significance test and variable selection algorithm are applied to finding the variables with non-zero population coefficients. A regression coefficient represents the level of conditional correlation between the corresponding covariate and the response variable. As a result, variable selection algorithms are to find the variables that are conditionally correlated to $Y$ in the population, holding all other variables constants.\(^7\) In graph learning, the set of those variable are called **Markov blanket of $Y$ (MB(Y))**, which includes the correct parent(s), children and spouse(s) of $Y$. Hence,\(^7\)After standardizing the response variables and all covariates, the regression coefficients of $x_i$ is the conditional correlation between $x_i$ and $Y$, holding all other covariates constant.
in linear graphs, recovering the MB of $Y$ is equivalent to variable selection in the linear regression of $Y$ to all other variables, which can be illustrated graphically as figure 7.8

![Fig 7: Illustration: recovering the Markov blanket for $Y$.](image)

In Figure 7, $u$ and $v$ are independent latent noise; $\{x_1, x_2, u\}$ are the parents of $Y$; $\{x_3, v\}$ are the spouses of $Y$; $\{Y, x_3, v\}$ together cause $x_4$. Based on A1 and A2, The a data-generating process in figure 7 is the following linear regression system,

$$
\begin{align*}
Y &= \alpha_0 + \alpha_1 x_1 + \alpha_2 x_2 + u, \\
X_4 &= \beta_0 + \beta_1 Y + \beta_1 x_3 + v.
\end{align*}
$$

(2.1)

Holding $\{x_1, x_2\}$ constant, equation (2.1) shows that all the variation in $Y$ is only caused by $u$ (mathematically, $Y|\{x_1, x_2\} = u$ since $u$ is independent to all other parents). Put it another way, after partial out $\{x_1, x_2\}$ from $Y$, the variation of $u$ can be explained by the children of $Y$. As a result, the independence of $v$ and equation 2.1 implies that

$$
u = Y|\{x_1, x_2\} = - \frac{\beta_0}{\beta_1} + \frac{1}{\beta_1} x_4|\{x_1, x_2\} - \frac{\beta_2}{\beta_1} x_3|\{x_1, x_2\} - \frac{v}{\beta_1}.
$$

(2.2)

As a result, after replacing $u$ in equation (2.1) with the right-hand side of equation (2.2), the population linear regression system will reduce to the following population regression equation of $Y$ to its MB memebers,

$$
Y = \gamma_0 + \gamma_1 x_1 + \gamma_2 x_2 + \gamma_3 x_3 + \gamma_4 x_4 + e.
$$

(2.3)

8For more general explanations and examples, see Pearl (2009), Koller and Friedman (2009) or Scutari and Denis (2014) for detail.
where $e$ is a linear function of $v$. Equation (2.3) is the reduced form of the linear system (equation (2.2)), where only MB variables are the informative variables (also called ‘true variables’). As Zhao and Yu (2006) shows, if $p/n$ is reasonably small (say less than $1/1000$), IRC almost sure guarantees that only MB members of $Y$ are selected in the variable selection algorithm for the linear regression of $Y$ (e.g., least-angle regression, variable screening or lasso regression of $Y$), referred to as variable selection consistency, $L_0$ consistency of $\|\gamma\|$ and sometimes implies sign consistency). As a result, variable selection for the regression of $Y$ is equivalent to finding the MB of $Y$. However, due to multicollinearity among all covariates, variable selection algorithms typically will not perform well especially when $p/n$ is close or larger than 1. Moreover, since a complicated linear regression system probably induces endogeneity bias, it is also difficult to acquire the $L_2$ consistency of $\|\gamma\|$. For example, as equation 2.2 shows, $v$ is a spouse of $Y$ at the data-generating procedure, implying that $\text{corr}(v, x_4) \neq 0$. Since $e$ in equation 2.3 is a linear function of $v$, $\text{corr}(e, x_4)$ will also not be zero. As a result, children of $Y$ (e.g., $x_4$ in equation 2.3) will definitely suffer endogeneity bias. However, by combining variable selection with graph learning, we can demonstrate that such issue can be at least effectively spotted, giving us a fair chance to reduce the corresponding bias.

3. Graph estimation on Sydney house pricing

To demonstrate the power of graph estimation on the issue of endogeneity and instruments, we choose the high-dimensional house price database at Sydney, 2010. The house price database are ensembled from more than 10 different datasets, including 2010 Sydney house transaction data (including every transaction on the 2010 second-hand house market in City, Mid, North and East Sydney, Australia), 2010 Sydney house feature data (the features of on-market houses before sales), 2010 and 2011 Sydney crime data by suburb, 2010 GIS data (extracted and complied from Sydney geospatial topology data, climate data and Google Map), 2011 census and demographic data, 2009 local school quality and catchment data, 2010 Sydney traffic data and so on. With the ensemble of different high-dimensional datasets, we can fully observe the procedure of second-hand house pricing/transaction at 2010 and comprehensively estimate the 2011 socio-economic impact of each 2010 house transaction.

Graph estimation and MB selection typically works well on datasets with small $p/n$. 
However, many datasets (e.g., census data and GIS data) are above 5 GB and have over one million variables, the size of which is well above the computation limit of graph estimation. Also, with such high dimensionality, there must be a number of irrelevant variables. As a result, based on the suggestion of Fan and Lv (2008), we conducted an ensemble variable selection. First, we conduct a feature pre-elimination using repeated SIS (Fan and Lv, 2008) and rule out variables whose conditional correlation to 2010 house price are, ceteris paribus, approximately 0. Based on variables that survive pre-elimination, we conduct the proper variable selection and use its result for MB selection and graph estimation. The variables that survive the pre-elimination are listed in Table 1.

As shown in Table 1, 57 survived variables are in 4 categories: the features of the house, distances to key locations (public transport, shopping, etc), local school quality and localized socio-economic data. Some house features are reported in real-estate advertising and others are scrapped from Google search engine; the distance of each house to the nearest key locations are computed in QGIS—a open-source geographical information system—using the GPS location of each house and geo-data collected from Google Map database and Department of Land and Natural Resources, New South Wales. The 2009 ICSEA score—an measure of the socio-educational background of students at every school—is collected from the Australian Curriculum, Assessment and Reporting Authority (ACARA). The variables on local school quality (2010 average NAPLAN examination scores) are also collected from ACARA and Department of Education. The 2009 and 2010 crime data are collected from Australian Bureau of Statistics and Department of Justice. The 2011 census data are acquired from Australian Bureau of Statistics. It is also worth noting that all the socio-economic data are observed by SA1, the smallest statistic area in 2011 census. Each SA1 in our data contains typically around 200 local residents. After pre-eliminating tons of irrelevant variables, we begin the proper variable selection and MB estimation based on table 1.

It is worth noting that multicollinearity and grouping effect will be the concern in the Sydney house price data, which is supported by the pairwise correlation among all 57 covariates. Due to the size of the table, we report it in supplementary files, which shows that multiple covariates in the Sydney house price data are highly correlated to one another. As a result, variable selection need to be act with cautions.

Before conducting variable selection and graph learning, it is also worth noting that this
Table 1

Variable selection by CV-en, CV-lasso (lars and cd) and Solar for linear and log models in Sydney house price data

| Variable         | Description                                      | CV-en linear | CV-en log | CV-lasso linear (lars, cd) | CV-lasso linear (cd) | Solar linear | Solar log |
|------------------|--------------------------------------------------|--------------|-----------|----------------------------|----------------------|--------------|-----------|
| Bedrooms         | property, number of bedrooms                     | y            | y         | y                          | y                    | y            | y         |
| Baths            | property, number of bathrooms                    | y            | y         | y                          | y                    | y            | y         |
| Parking          | property, number of parking spaces               | y            | y         | y                          | y                    | y            | y         |
| Size             | property, land size                              | y            | y         | y                          | y                    | y            | y         |
| Airport          | distance, nearest airport                       | y            | y         | y                          | y                    | y            | y         |
| Beach            | distance, nearest beach                          | y            | y         | y                          | y                    | y            | y         |
| Boundary         | distance, nearest suburb boundary                | y            | y         | y                          | y                    | y            | y         |
| Cemetery         | distance, nearest cemetery                       | y            | y         | y                          | y                    | y            | y         |
| Child care       | distance, nearest child-care centre              | y            | y         | y                          | y                    | y            | y         |
| Club             | distance, nearest club                           | y            | y         | y                          | y                    | y            | y         |
| Community facility| distance, nearest community facility            | y            | y         | y                          | y                    | y            | y         |
| Gaol             | distance, nearest gaol                           | y            | y         | y                          | y                    | y            | y         |
| Golf course      | distance, nearest golf course                    | y            | y         | y                          | y                    | y            | y         |
| High             | distance, nearest high school                    | y            | y         | y                          | y                    | y            | y         |
| Hospital         | distance, nearest general hospital               | y            | y         | y                          | y                    | y            | y         |
| Library          | distance, nearest library                        | y            | y         | y                          | y                    | y            | y         |
| Medical          | distance, nearest medical centre                 | y            | y         | y                          | y                    | y            | y         |
| Museum           | distance, nearest museum                         | y            | y         | y                          | y                    | y            | y         |
| Park             | distance, nearest park                           | y            | y         | y                          | y                    | y            | y         |
| PO               | distance, nearest post office                    | y            | y         | y                          | y                    | y            | y         |
| Police           | distance, nearest police station                 | y            | y         | y                          | y                    | y            | y         |
| Pre-school       | distance, nearest preschool                      | y            | y         | y                          | y                    | y            | y         |
| Primary          | distance, nearest primary school                 | y            | y         | y                          | y                    | y            | y         |
| Primary High     | distance, nearest primary-high school            | y            | y         | y                          | y                    | y            | y         |
| Rubbish          | distance, nearest rubbish incinerator            | y            | y         | y                          | y                    | y            | y         |
| Sewage           | distance, nearest sewage treatment               | y            | y         | y                          | y                    | y            | y         |
| SportsCenter     | distance, nearest sports centre                  | y            | y         | y                          | y                    | y            | y         |
| SportsCourtField| distance, nearest sports court/field             | y            | y         | y                          | y                    | y            | y         |
| Station          | distance, nearest train station                  | y            | y         | y                          | y                    | y            | y         |
| Swimming         | distance, nearest swimming pool                  | y            | y         | y                          | y                    | y            | y         |
| Tertiary         | distance, nearest tertiary school                 | y            | y         | y                          | y                    | y            | y         |
| Mortgage         | SA1, mean mortgage repayment (log)               | y            | y         | y                          | y                    | y            | y         |
| Rent             | SA1, mean rent (log)                             | y            | y         | y                          | y                    | y            | y         |
| Income           | SA1, mean family income (log)                    | y            | y         | y                          | y                    | y            | y         |
| Income (personal)| SA1, mean personal income (log)                  | y            | y         | y                          | y                    | y            | y         |
| Household size   | SA1, mean household size                         | y            | y         | y                          | y                    | y            | y         |
| Household density| SA1, mean persons to bedroom ratio               | y            | y         | y                          | y                    | y            | y         |
| Age              | SA1, mean age                                    | y            | y         | y                          | y                    | y            | y         |
| English spoken   | SA1, percent English at home                     | y            | y         | y                          | y                    | y            | y         |
| Australian born  | SA1, percent Australian-born                    | y            | y         | y                          | y                    | y            | y         |
| Suburb area      | suburb, area                                     | y            | y         | y                          | y                    | y            | y         |
| Population       | suburb, population                               | y            | y         | y                          | y                    | y            | y         |
| TVO2010          | suburb, total violent offences, 2010              | y            | y         | y                          | y                    | y            | y         |
| TPO2010          | suburb, total property offences, 2010             | y            | y         | y                          | y                    | y            | y         |
| TVO2009          | suburb, total violent offences, 2009              | y            | y         | y                          | y                    | y            | y         |
| TPO2009          | suburb, total property offences, 2009             | y            | y         | y                          | y                    | y            | y         |
| ICSEA            | local school, ICSEA                              | y            | y         | y                          | y                    | y            | y         |
| ReadingY3        | local school, year 3 mean reading score          | y            | y         | y                          | y                    | y            | y         |
| WritingY3        | local school, year 3 mean writing score          | y            | y         | y                          | y                    | y            | y         |
| SpellingY3       | local school, year 3 mean spelling score         | y            | y         | y                          | y                    | y            | y         |
| GrammarY3        | local school, year 3 mean grammar score          | y            | y         | y                          | y                    | y            | y         |
| NumeracyY3       | local school, year 3 mean numeracy score         | y            | y         | y                          | y                    | y            | y         |
| ReadingY5        | local school, year 5 mean reading score          | y            | y         | y                          | y                    | y            | y         |
| WritingY5        | local school, year 5 mean writing score          | y            | y         | y                          | y                    | y            | y         |
| SpellingY5       | local school, year 5 mean spelling score         | y            | y         | y                          | y                    | y            | y         |
| GrammarY5        | local school, year 5 mean grammar score          | y            | y         | y                          | y                    | y            | y         |
| NumeracyY5       | local school, year 5 mean numeracy score         | y            | y         | y                          | y                    | y            | y         |

Number of variables selected | 57 | 53 | 44 | 36 | 9 | 11
database is ideal for the linear graphing learning technique. Firstly, we have over 200GB data and millions of variables from all sorts of databases, which confidently reduces the concerns of the possibility of variable omission. Secondly, as to be shown in the following sections, the regression $R^2$ is high in house pricing database (e.g., in next section, a linear regression of log (price) with only 11 variable has at least 73% $R^2$), which give us the confidence to say that the majority of the patterns in our data is linear. Necessary sanity checks are advised if the same trick is to be applied to other datasets.

3.1. Robust variable selection and prediction accuracy

Pearl (2009) points out that dependence structures and causations should not be affected by the forms of a variable. For example, if $Y$ is a parent of $x$, log ($Y$) → log ($x$) must be also true and vice versa. Thus, to avoid being misled by the form of each variable, we conduct variable selection in both linear and log terms, only selecting variables that are simultaneously selected in both scenarios. We also implement lasso and cross-validated elastic net (CV-en) for comparison, the latter of which is suppose to outperform lasso from the perspective of variable selection sparsity and accuracy on data with serious grouping effect. We optimize lasso using both cross-validated coordinate descent (CV-cd) and cross-validated lars (CV-lars), both of which returns the same variables selection result due to $p/n \leq 1/200$.

With all variables in linear form, Table 1 shows the selection results from solar, lasso and CV-en. Consistent with Friedman et al. (2010) and Jia and Yu (2010), both lasso and CV-en lose sparsity of variable selection due to the complicated dependence structures and severe multicollinearity in the data. Lasso only manage to drop 7 variables and CV-en selects all 57 variables. It is not recommended to heuristically increase the value of $\lambda$ in lasso-type estimators (e.g., the one- sd rule) since it may lead to the random dropping of variables and trigger even worse grouping effects. On the other hand, CV-en is designed to tolerate multicollinearity and grouping effects and is expected to return sparse and stable regression models. However, due to the complicated dependence structure in the house price setting, CV-en completely fails to accomplish any variable selection. Conclusively, due to the complicated multicollinearity, $L_1$ and composite $L_1$ shrinkage methods fail to maintain sparsity in data with $p/n$ less than 1/200. By contrast, the solar algorithm returns a very sparse regression model, with only 9 variables selected from 57.
Table 1 also shows the selection results when all variables measured in dollars (e.g., Rent, FamInc, Inc, Mortgage) are transformed by logs and the response variable is log (Price). The log transform decision are implement only on dollar-measured variables with both statistical and empirical reasons. Statistically, it is because that the remaining variables in the data are distributed almost symmetrically without heavy tails (subgaussianly). As illustrated with Gaol and Beach in figure 8, log transforms will induce left skewness and a long/heavy/fat left tail. As shown in figure 8f and 8c, such issue cannot be rectified by changing the unit of the variable before the log transform. Empirically, log transform may cause interpretation difficulties. For example, typically we are interested in by the Price response with one more bedroom instead of 1% more. As a result, we do not take the log transform on other variables.

Due to the sensitivity to variable form changes, lasso selects 35 variables and CV-en selects 54. Some of the lasso and CV-en selections seem odd. For example, lasso drops all Year 5 test scores, SpellingY3 and GrammarY3 but selects all the other examination scores in Year 3. CV-en selects all other scores, dropping only ReadingY5. These selections seem to suggest that only some primary school examination scores particularly matter in house pricing. By contrast, solar returns a very sparse regression model, with only 11 variables selected, 9 of which are the linear selection result. Since dependence structure and causations should not be affected by variable forms, we select variables chosen by...
solar simultaneously in Tables 1: {Bedrooms, Baths, Parking, Beach, ChildCare, Gaol, ICSEA, logMortgage, logRent, logFamInc}.

| Variable     | elas net | lasso | rec solar | solar |
|--------------|----------|-------|-----------|-------|
| constant     | 8.81***  | 8.76*** | 7.99***   | 7.21*** |
| (0.17)       | (0.15)   | (0.11) | (0.11)    |        |
| Bedrooms     | 0.21***  | 0.21*** | 0.23***   | 0.23*** |
| (0.00)       | (0.00)   | (0.00) | (0.00)    |        |
| Baths        | 0.09***  | 0.10*** | 0.09***   | 0.09*** |
| (0.00)       | (0.00)   | (0.00) | (0.00)    |        |
| Parking      | 0.08***  | 0.08*** | 0.08***   | 0.08*** |
| (0.00)       | (0.00)   | (0.00) | (0.00)    |        |
| Airport      | 3.67***  | 2.53*** | 2.88***   |        |
| (0.39)       | (0.20)   | (0.25) |          |        |
| Beach        | −1.78*** | −2.21*** | −1.34***  | −2.45*** |
| (0.31)       | (0.11)   | (0.14) | (0.14)    |        |
| ChildCare    | −4.39*** | −4.49*** | −3.63***  | −2.45*** |
| (0.20)       | (0.16)   | (0.12) | (0.11)    |        |
| Gaol         | −0.80**  | −1.01*** | 0.36***   |        |
| (0.32)       | (0.15)   | (0.14) |          |        |
| Rubbish      | −0.4     | 0.54*** |          |        |
| (0.35)       | (0.21)   |        |          |        |
| Mortgage     | 0.16***  | 0.16*** | 0.24***   | 0.26*** |
| (0.01)       | (0.01)   | (0.01) | (0.01)    |        |
| Rent         | 0.02***  | 0.04*** | 0.07***   | 0.07*** |
| (0.01)       | (0.01)   | (0.01) | (0.01)    |        |
| Income       | 0.19***  | 0.19*** | 0.17***   | 0.24*** |
| (0.02)       | (0.01)   | (0.01) | (0.01)    |        |
| Age          | 0.01***  | 0.01*** | 0.01***   | 0.01*** |
| (0.00)       | (0.00)   | (0.00) | (0.00)    |        |
| ICSEA        | 0.00***  | 0.00*** | 0.00***   | 0.00*** |
| (0.00)       | (0.00)   | (0.00) | (0.00)    |        |
| +            |          |        |          |        |
| \(k\)        | 54       | 36     | 13        | 11     |
| \(R^2\)     | 0.77     | 0.76   | 0.74      | 0.73   |
| \(\bar{R}^2\) | 0.77    | 0.76   | 0.74      | 0.73   |
| \(N\)        | 11,796   | 11,796 | 11,796    | 11,796 |

Last but not least, solar variable selection outperforms lasso-type estimators at the balance between complexity and prediction accuracy, which is shown in table 2 and 3. Table 3 shows the post-selection OLS results on CV-en, lasso and solar in log models. As we shown previously, solar only select 9 variable while lasso does 44 and CV-en does 57. Surprisingly, pruning 35/48 variables only reduce \(R^2\) by 5%. This confirms that, from the perspective of prediction, solar successfully identifies the most important variables in our database. A very similar result can also be found in table 2. Solar only selects 11 out of 57, which surprisingly explains 73% of the variation of logPrice. The extra variables selected
by lasso or CV-en only improve $R^2$ by two to three percent. Since Price and other dollar-measured variables typically has a long/heavy/fat right tail and looks asymmetrically distributed, the results of log post-selection regressions seem more reliable.

The high explanation power of solar variables reassures the reliability of MB selection. It is pretty common that dollar-measured variables typically have asymmetric and long/fat/heavy left tails. As a result, even though log and linear model should represent the same causal structure, without log transform the linear model estimation typically performs poorly. Hence, we focus on the result of the log regression. As explained previously, MB includes all variables that are conditionally correlated to Price in population, implying that MB variables should be able to explain all "non-noise" variation of Price.

**Table 3**

*Post-selection linear model OLS coefficients for variables selected by rectified solar.*

| Variable   | elas net  | lasso   | rec solar | solar   |
|------------|-----------|---------|-----------|---------|
| constant   | $-886234.40***$ | $-827387.49***$ | $-1445430.40***$ | $-2486422.15***$ |
| (188680.91) | (174136.59) | (112390.24) | (98422.98) |
| Bedrooms   | $165639.00***$ | $166225.82***$ | $183893.59***$ | $169510.52***$ |
| (6433.57)  | (6404.82) | (6015.93) | (6015.37) |
| Baths      | $210101.84***$ | $210600.55***$ | $203674.28***$ | $209626.52***$ |
| (8115.07)  | (8048.92) | (8147.93) | (8297.24) |
| Parking    | $97790.57***$ | $96883.13***$ | $104050.40***$ | $97623.23***$ |
| (6689.35)  | (6688.13) | (6861.16) | (6985.67) |
| Airport    | $2865246.35***$ | $3122719.74***$ | $1849108.51***$ | $1445430.25***$ |
| (735335.86) | (625604.38) | (454344.97) | (107204.64) |
| Beach      | $-5029681.74***$ | $-4051671.57***$ | $-1509612.24***$ | $-796281.77***$ |
| (600061.28) | (262369.00) | (260370.04) | (153431.50) |
| Child care | $-4802905.18***$ | $-4163486.91***$ | $-3961629.37***$ | $-190369.80***$ |
| (393577.54) | (316107.89) | (220752.57) | (107204.64) |
| Gaol       | $1614215.27***$ | $1137143.99***$ | $1909369.80***$ | $107204.64$ |
| (646392.84) | (267597.21) | (107204.64) | (107204.64) |
| Rubbish    | $-45084.93$ | $780180.22$ | $3136997.85***$ | $372355.34$ |
| (672532.42) | (594084.58) | (372355.34) | (372355.34) |
| Mortgage   | $133.67***$ | $134.18***$ | $174.55***$ | $185.99***$ |
| (7.98)     | (7.95)    | (7.82)    | (7.96)    |
| Rent       | $264.35***$ | $265.80***$ | $312.85***$ | $370.76***$ |
| (36.80)    | (35.27)   | (34.78)   | (35.42)   |
| Income     | $59.04***$ | $69.22***$ | $-8.39$    | $65.57***$ |
| (19.23)    | (14.97)   | (12.18)   | (11.48)   |
| Age        | $3673.46***$ | $4106.93***$ | $960.68***$ | $1756.92***$ |
| (103.69)   | (958.98)  | (104.07)  | (95.67)   |
| ICSEA      | $838.54***$ | $862.92***$ | $960.68***$ | $1756.92***$ |
| (172.42)   | (163.42)  | (104.07)  | (95.67)   |

$P$, $R^2$, $R^2$, $N$.
Since we do not know the population variance of noise, we cannot know with absolute certainty the magnitude of noise variation. However, with $R^2$ at 73%, we can be very confident to say that the majority of the Price variation is linear and explained by MB variables. The remaining 27% may come from other perspectives, like the noise term, possible functional form error (e.g., we should use polynomial or trigonometric equations instead of the first-order linear equation) and spatial clustering issue in geographical data. Even though we cannot rule out the possibility of nonlinearity, the severity of those problems is definitely under controlled.

### 3.2. Grouping effects in variable selection

Before moving on to the graph estimation, we need to check whether multicollinearity and grouping effects potentially cause solar dropping out any variable of Price MB by mistake. As shown in the introduction, the accuracy and robustness of variable selection may be reduced with multicollinearity and grouping effects embedded in the data, especially when the potential dependence structure and causations are complicated. As shown in the previous correlation table, the geographical features of a house (e.g., distances to different locations) are highly spatially correlated with one another. To investigate whether solar variable selection is affected by such multicollinearity, table 4 focuses on the group of variables highly correlated to ‘Gaol’, including ‘Airport’, ‘Rubbish’ and ‘ChildCare’ (pairwise correlations larger than 0.5). Such high correlations may potentially violate IRC and trigger variable selection grouping effects.

**Table 4**

| Table 4 | Marginal correlations to Gaol (absolute value > 0.5) |
|---------|---------------------------------|
|         | ChildCare | Airport | Rubbish | Beach |
| corr(·,Gaol) | 0.756     | 0.715   | 0.671   | 0.528 |

Based on table 4, we standardize all variables and estimate regression equation (3.1),

$$Gaol = \gamma_0 + \gamma_1 \cdot Airport + \gamma_2 \cdot ChildCare + \gamma_3 \cdot Rubbish + \gamma_4 \cdot Beach + e.$$  (3.1)

The result from equation (3.1) are reported in Table 5.
As Table 5 shows, the collinearity between Gaol and \{ChildCare, Airport, Rubbish, Beach\} is severe. Almost 90% of the variation of Gaol can be explained by \{ChildCare, Airport, Rubbish, Beach\} and $\sum_{i \neq 0} |\gamma_i| = 1.35$ in (3.1). This is very likely to lead to a IRC breach and serve grouping effect, further implying that variable selection algorithm may randomly drop any of them in one realization. This also implies that, even though we know that some variable(s) in \{ChildCare, Airport, Rubbish, Beach, Gaol\} is (are) the MB memeber of Price, variable-selection algorithm may have difficulty to pinpoint which one is (are). Hence, to avoid being misled by the grouping effect, it is statistically reasonable to consider the inclusion of \{Gaol, ChildCare, Beach\} as a placeholder for the group \{ChildCare, Airport, Rubbish, Beach, Gaol\} (short as the ‘gaol’ group) in the variable-selection results. We also refer to the union of solar variable and \{ChildCare, Airport, Rubbish, Beach, Gaol\} as the ‘rectified solar selection’.

As it turns out, there is an empirical reason why \{Gaol, ChildCare, Airport, Rubbish, Beach\} are highly correlated. The observations in the house price data are for a roughly 10km square area in eastern Sydney. The gaol (Long Bay correctional complex), childcare centers (e.g., Blue Gum Cottage Child Care, Alouette Child Care, etc.), the airport (Kingsford-Smith Airport) and rubbish incinerators (e.g., Malabar Wastewater Treatment Plant, Sydney Desalination Plant, Cronulla Wastewater Treatment Plant, Bondi Wastewater Treatment Plant) are all located in the southeast corner of the 10km square area, explaining the collinearity among the variables.

For completeness, we compare the OLS results in both linear and log forms with the selection results from lasso, CV-en, solar (equation (3.2) and (3.4)) and ‘rectified solar selection’ (equations (3.3) and (3.5)).
Price = $\beta_0 + \beta_1 \cdot \text{Mortgage} + \beta_2 \cdot \text{Rent} + \beta_3 \cdot \text{FamInc} + \beta_4 \cdot \text{Bedrooms}$ (3.2) \\
+ $\beta_5 \cdot \text{Baths} + \beta_6 \cdot \text{Parking} + \beta_7 \cdot \text{Beach} + \beta_8 \cdot \text{Gaol} + \beta_9 \cdot \text{ICSEA} + u;$

Price = $\beta_0 + \beta_1 \cdot \text{Mortgage} + \beta_2 \cdot \text{Rent} + \beta_3 \cdot \text{FamInc} + \beta_4 \cdot \text{Bedrooms}$ (3.3) \\
+ $\beta_5 \cdot \text{Baths} + \beta_6 \cdot \text{Parking} + \beta_7 \cdot \text{Beach} + \beta_8 \cdot \text{Airport} + \beta_9 \cdot \text{ChildCare}$
+ $\beta_{10} \cdot \text{Rubbish} + \beta_{11} \cdot \text{ICSEA} + u;$

$logPrice = \beta_0 + \beta_1 \cdot \text{logMortgage} + \beta_2 \cdot \text{logRent} + \beta_3 \cdot \text{logFamInc} + \beta_4 \cdot \text{Bedrooms}$ (3.4) \\
+ $\beta_5 \cdot \text{Baths} + \beta_6 \cdot \text{Parking} + \beta_7 \cdot \text{Beach} + \beta_8 \cdot \text{Gaol} + \beta_9 \cdot \text{ICSEA} + u;$

$logPrice = \beta_0 + \beta_1 \cdot \text{logMortgage} + \beta_2 \cdot \text{logRent} + \beta_3 \cdot \text{logFamInc} + \beta_4 \cdot \text{Bedrooms}$ (3.5) \\
+ $\beta_5 \cdot \text{Baths} + \beta_6 \cdot \text{Parking} + \beta_7 \cdot \text{Beach} + \beta_8 \cdot \text{Airport} + \beta_9 \cdot \text{ChildCare}$
+ $\beta_{10} \cdot \text{Rubbish} + \beta_{11} \cdot \text{ICSEA} + u.$

The comparisons are also summarized in Table 3. The most interesting thing is the value of $R^2$ of rectified solar. The difference between solar $R^2$ and CV-en $R^2$ tells us that the drop-out variables of solar (48 totally) only explain five extra percent of the Price variation; however, the difference between solar $R^2$ and rectified solar $R^2$ shows that $\{\text{Airport, Rubbish}\}$ – the ‘Gaol’ group member dropped out by solar – explains two extra percent of the Price variation. A very similar result can be found in the $R^2$ comparison of log models. As a result, among all 48 drop-outs, $\{\text{Airport, Rubbish}\}$ seems to be the most important variables. This, from another perspective, justifies our previous doubt on grouping effect.

3.3. **Graph estimation based on solar variable selection**

Based on the result of variable selection and grouping effect examination, we have a clear clue on the possible member of Price MB. Ceteris paribus, each variable selected by solar is highly likely to be conditionally related to Price in the population, implying that they are very likely to be the MB of Price. However, it is possible that these variables have different roles: some may serve as the parents of Price while others may serve as children or spouses. In order to accomplish the endogeneity detection and instrument selection,
we need to estimate the role of each MB member and all the arrows/ causations in the MB. As a result, we rely on the score-based learning method for graph estimation.

**Temporal ordering of MB variables and Markov equivalence**

Markov equivalence is a common problem in graph learning and causal inference. In a nutshell, markov equivalence says that, without exact time stamp (when a variable is generated or sampled), we cannot learn the exact population graph from the data. Instead, we can only learn the skeleton of the population graph (a graph without arrows, aka undirected graph), which represent a class of equivalent graphs as figure 9.

![Fig 9: Illustration of Markov Equivalence](image)

Figure 9a shows a simple population graph as $z \rightarrow x \rightarrow Y$. However, without knowing the time stamp of each variable (which variable is born first), it is impossible to find the correct parent-child relations. What we can find is only a skeleton (figure 9b), where we know that (i) $\text{corr}(z, x) \neq 0$; (ii) $\text{corr}(z, Y) \neq 0$ and (iii) $\text{corr}(Y, x) \neq 0$. Hence, any graph that fits three conditions above is included in the Markov equivalence class represented by figure 9b. All the mebers are figure 9a (population graph), figure 9c and figure 9d (aka the confounding/fork structure). However, figure 9e (aka the collider structure) is not included since the correlation between $z$ and $Y$ is zero unless $x$ is conditioned on.

Put in another way, without specific time stamps, in this example graph learning can only identify whether the population graph is a collider or not, which is not particularly useful. However, with huge amount of variables and the corresponding time stamps, we can clearly break the Markov equivalence and narrow down the possible candidates of population graph.
Based on the temporal order of variable generation, the selected variables can be ordered vertically as Figure 10. In Figure 10, the red nodes (including house features, distances and 2009 ICSEA score) are variables generated before the house transaction at 2010; the green node is the variable generated within the house transaction at 2010 and the blue nodes (the demographic variables) are the variables generated at 2011 census, which come after 2010 house transactions.\footnote{Multiple green nodes are included in our database, including the transaction method of a house (e.g., public bidding, private sale and so on), the history of bidding and other 30 variables. We center on the Price variable in this graph estimation.} Due to the probable IRC violation shown above, \{Gaol, ChildCare, Beach\} are grouped manually, which may be considered as the placeholder for \{Airport, Rubbish\}. The temporal order helps us identify the role of each variable in MB. Assuming that time can only flow forward, variables generated at 2011 cannot cause the change of those generated at or before 2010, implying that the red nodes cannot be the descendants of green and blue nodes. Hence, the red nodes are the parents of Price and Rent and Mortgage are the children of Price.

The role of Income is worth some extra discussion. 2009 ICSEA is computed partially based on household income at 2009 and Income is the household income at 2011. Since (i) there exists strong serial correlation of household income; (ii) we do not have the high dimensional database on family income and wealth, we cannot determine the correlation between ICSEA and Income is purely serial correlation or contains some kind of causation (e.g., the Matthew effect of accumulated advantage). Assuming there exists a causation, we cannot identify it statistically without detailed labour economics database. Hence, we
connect ICSEA and Income with an undirected dash edge. Since solar variable selection confirms that Income and the "Gaol" group also belong to both MB of Rent and MB of Mortgage (the detailed statistical output can be found in supplementary files), we connect them to Rent and Mortgage directly, which gives us the estimated graph as figure 11.

![Figure 11: The estimated graph based on solar variable selection.](image)

**Indirect effect estimation**

After estimating the role of each variable in figure 11, it clearly shows that a parent of Price can indirectly cause the change of Rent or Mortgage through Price. The last step of graph estimation is to determine whether the parents of Price can cause the change of Rent or Mortgage not via Price. Such causal effect is also referred to as the indirect effect and illustrated as black arrows in figure 12.
Fig 12: Illustration of indirect effect estimation between Baths and Mortgage.

Figure 12 is an illustration of indirect effect estimation based on figure 11. From the previous learning result we know that the indirect effect (Bath → Price → Mortgage) exists with large probability. Now we need to determine whether Baths can affect Mortgage not via Price (either cause Mortgage directly as figure 12b or through variable other than Price as figure 12c), also referred to as the indirect effect from Mortgage to the children of Price. The difference between figure 12b and 12c is that, in figure 12b, Baths can cause the change of Mortgage even after we controlling all other variables; by contrast, after controlling variable(s) ‘?’ and Price in figure 12c, Bath can no longer cause any change of Mortgage. As a result, we can estimate which causal structure fits the data better in the following method. First, we find out which of figure 12a and 12b fits data better without controlling any other variable. If figure 12a is chosen, it implies that there is no indirect effect; if figure 12b is chosen, there exists a indirect effect and we need to combinatorially determine whether the variable ‘?’ exists.\footnote{If figure 12b is chosen, it means that Bath can still cause Rent when controlling Price. As a result, we need to determine whether such conditional causation is established via some other variable(s). By putting each potential candidate of variable ‘?’ in the figure 12c, we can obtain a potential causal structure among Baths, Price and Mortgage. We need to combinatorially find the optimal causal structure among the potential structures above and figure 12b.}

Due to the robustness and accuracy, we rely on the score-based learning method to find the optimal causal structure. To be specific, we are going to estimate the AIC, BIC and
BGE scores of figure 12a (listed as column ‘no BE’ of table 6) and figure 12b (listed as column ‘BE’ of table 6) on the house pricing data and choose the one with lower score. If Bath can only cause Mortgage via Price, the conditional correlation between Baths and Mortgage shall be very close to zero after holding Price constant; hence, due to overfitting the causal structure in figure 12b, its AIC/BIC/BGE score will be higher than figure 12a, implying that the no-BE graph fits the data better. By replacing Baths with Parking or Bedrooms in figure 12, we can instead check whether a indirect effect exists between Mortgage and other parent of Price. In a similar vein, by replacing Mortgage with Rent, we can also check whether a indirect effect exists between Rent and any parent of Price. The results of indirect effect estimation are summarized in table 6.

|          | logRent   | logMortgage |
|----------|-----------|-------------|
|          | IE        | No IE       | IE        | No IE       |
| Baths    | AIC       | 23726.55    | -23731.28 | -19437.42   | 19482.37   |
|          | BIC       | 23759.74    | -23760.78 | -19470.61   | -19511.87  |
|          | BGE       | 23758.70    | -23760.01 | -19470.54   | -19512.35  |
| Bedrooms | AIC       | -27197.09   | -27500.08 | -23060.65   | -23251.17  |
|          | BIC       | -27230.27   | -27529.58 | -23093.84   | -23280.67  |
|          | BGE       | -27230.75   | -27529.87 | -23095.31   | -23282.22  |
| Parking  | AIC       | -25002.25   | -25064.54 | -20762.16   | -20815.63  |
|          | BIC       | -25035.44   | -25094.04 | -20795.35   | -20845.13  |
|          | BGE       | -25034.62   | -25093.35 | -20795.46   | -20845.70  |

In table 6, instead of Rent and Mortgage, we use logRent and logMortgage for the indirect effect estimation. It is because that, although the score-based learning method works on many subgaussian variables with small enough $p/n$ value, Rent and Mortgage typically come as asymmetric distributions with long/heavy/fat right tails, possibly biasing the AIC/BIC/BGE score of a graph. As a result, taking a log transform on these variable can significantly reduce such issue and improve the accuracy of AIC/BIC/BGE estimation. Table 6 clearly shows that the graph without indirect effect clearly have lower score. As a result, figure 11 is the final estimated graph as the MB of Price and we don’t need to add any indirect effect to it.

It is worth noting that we skip ICSEA, Incom and the "Gaol" group in table 6 due to statistical reasons. Due to the IRC violation in the "Gaol" group, it is difficult to accurately estimate the indirect effect from one or some variables in the group to the
children of Price. ICSEA is manually synthesized from a number of variables like household income, family wealth and other factors representing the household socio-economic status. As a result, there may exists a complicated causal relation among each part of ICSEA and Price, which we cannot investigate due to the synthesis.

Graph interpretations and further remarks on AreaSize

Figure 11 shows that the graph estimation and MB selection offers interpretations consistent with our intuition on the dynamics of the house market. (i) houses with better features and better locations are purchased by wealthier households at higher prices, which further indirectly causes higher mortgage payment for the house; (ii) higher price and mortgage also drives up the rent payment, which is typically also considered as a method to pay monthly mortgage. (iii) typically houses with higher rent are leased to wealthier people; from the demographic perspective, after wealthier house owners/tenants move into the newly purchased/leased houses, the average Inc and FamInc in the local SA1 also increase, which is reflected on the graph as Price causes Income. Moreover, the causation structure revealed in figure 11 can be interpreted from supply and demand. All the parents of Price can be categorized as the supply side and demand side of the second-hand house market. Supply side factors includes Bedrooms, Baths, Parking and geo-location (the ‘Gaol’ group) of the house, which represents different kinds of construction inputs of the house. ICSEA score, which is computed from the income and wealth the household, represents a demand side factor. A similar structure can also be found for the MB of Rent (see supplementary files for details), where variables like Income, FamInc and Household Size represent the demand side of house leasing and variables like the ‘gaol’ group represent the supply side.

\textsuperscript{11}Statistically, the minimal eigenvalue of the covariance matrix may be infinitely close to 0 and, hence, bring oversensitivity to the AIC/BIC/BGE score of a graph.
Among all the parents of Price, it is worth noting that AreaSize is not selected into the MB of Price, which appears somehow counterintuitive. However, this can be naturally explained using figure 13. Firstly, area size of houses at different locations are not comparable. A downtown terrace house with small area size can be much more pricy than one with large area size somewhere else. Hence, ceteris paribus, AreaSize has much more explanatory power on Price if we compare houses within a same/similar local area or spatial neighbourhood, which requires more spatial statistics technique (e.g., lasso variable selection within a spatial Gaussian kernel) than only controlling the distance to geo-locations. Secondly, in the procedure of house construction, AreaSize is determined when purchasing land; hence, the design and construction of baths, parking and bedrooms are typically under the constraint of AreaSize. As a result, AreaSize is typically considered as a parent of Parking and Bedrooms and may be simultaneously determined with the ”Gaol” group (e.g., a certain level of AreaSize may not be available at close-CBD suburbs) during the procedure of land purchase, implying that AreaSize certainly does not belong to the MB of Price. It is also interesting to note that (i) Bath is not taken as a child of AreaSize and (ii) there is a undirected edge between Bath and Bedrooms. These decisions are made data-driven since in our data corr(Bath, AreaSize) \approx 0 and corr(Bath, Bedrooms) is significantly nonzero. This phenomenon is quite expected. During the house design and construction, the bathroom number are typically determined alongside with the bedroom number (essentially the expected household size). Given the
household size and the number of bedrooms, there seems to be little incentive to build more bathrooms with more AreaSize. Due to the lack of data on the first hand house market and the decision making procedure of house construction, we don’t have enough variables to infer the graph of the procedure of house construction and land purchase; hence, we will not pursue this topic further.

4. Application of graph estimation: endogeneity detection and instrument selection

With estimated graph at hand, we can begin the instrument selection procedure based on definition 2.3. However, before selecting a valid instrument, we need to make sure that endogeneity does exist in the graph; otherwise it will be purely waste of degrees of freedom. Hence, we first verifies the existence of endogeneity using Figure 11.

4.1. Endogeneity detection using graphs

Price is endogenous both statistically and empirically. Figure 11 represents a graph that is considered as a statistical dynamic system. The input of this system is the ”Gaol” group, house features and ICSEA; Rent and Mortgage are two outputs and Price is internally determined within the system. Put it in the language of random graph, Rent, Mortgage and Price are in the MB of each other. As a result, Price is highly likely to be endogenous. The endogeneity of Price is also supported statistically by variable selection. For example, by running solar regressions of Rent on all other variables in both linear and log terms, we have the following estimated regression models,

\[
\begin{align*}
\text{logRent} &= \alpha_0 + \alpha_1 \cdot \text{TotPop} + \alpha_2 \cdot \text{Household size} + \alpha_3 \cdot \text{Beach} + \alpha_4 \cdot \text{ChildCare} \\
&\quad + \alpha_5 \cdot \text{Gaol} + \alpha_6 \cdot \text{PrimaryHigh} + \alpha_7 \cdot \text{ICSEA} + \alpha_8 \cdot \log\text{PersonInc} \\
&\quad + \alpha_9 \cdot \log\text{FamInc} + \alpha_{10} \cdot \log\text{Price} + u, \\
\text{Rent} &= \gamma_0 + \gamma_1 \cdot \text{Household size} + \gamma_2 \cdot \text{Beach} + \gamma_3 \cdot \text{ChildCare} + \gamma_4 \cdot \text{Gaol} \\
&\quad + \gamma_5 \cdot \text{PrimaryHigh} + \gamma_6 \cdot \text{Mortgage} + \gamma_7 \cdot \text{ICSEA} + \gamma_8 \cdot \text{FamInc} \\
&\quad + \gamma_9 \cdot \text{PersonInc} + \gamma_{10} \cdot \text{Price} + u.
\end{align*}
\]

Since the \(p/n\) ratio is almost 1/200, the solar variable selection result is statistically very robust and accurate. Solar includes logPrice and Price into eqn (4.1) and (4.2) respectively, implying that (i) Price is an very important covariate in Rent/logRent regression;
(ii) highly likely that Rent and Price are simultaneously determined. As a result, alongside with the solar variable selection result in eqn (3.2) and (3.4), we can establish a simultaneous equations models in log terms

$$\begin{align*}
\log\text{Rent} &= \alpha_0 + \alpha_1 \cdot \text{TotPop} + \alpha_2 \cdot \text{Household size} + \alpha_3 \cdot \text{Beach} + \alpha_4 \cdot \text{ChildCare} \\
&\quad + \alpha_5 \cdot \text{Gaol} + \alpha_6 \cdot \text{PrimaryHigh} + \alpha_7 \cdot \text{ICSEA} + \alpha_8 \cdot \log\text{PersonInc} \\
&\quad + \alpha_9 \cdot \log\text{FamInc} + \alpha_{10} \cdot \log\text{Price} + u_1,
\end{align*}$$

$$\log\text{Price} = \beta_0 + \beta_1 \cdot \log\text{Mortgage} + \beta_2 \cdot \log\text{Rent} + \beta_3 \cdot \log\text{FamInc} + \beta_4 \cdot \text{Bedrooms} \\
&\quad + \beta_5 \cdot \text{Baths} + \beta_6 \cdot \text{Parking} + \beta_7 \cdot \text{Beach} + \beta_8 \cdot \text{Gaol} + \beta_9 \cdot \text{ICSEA} + u_2; \quad (4.3)$$

or in linear terms

$$\begin{align*}
\text{Rent} &= \gamma_0 + \gamma_1 \cdot \text{Household size} + \gamma_2 \cdot \text{Beach} + \gamma_3 \cdot \text{ChildCare} + \gamma_4 \cdot \text{Gaol} \\
&\quad + \gamma_5 \cdot \text{PrimaryHigh} + \gamma_6 \cdot \text{Mortgage} + \gamma_7 \cdot \text{ICSEA} + \gamma_8 \cdot \text{FamInc} \\
&\quad + \gamma_9 \cdot \text{PersonInc} + \gamma_{10} \cdot \text{Price} + u_1,
\end{align*}$$

$$\begin{align*}
\text{Price} &= \delta_0 + \delta_1 \cdot \text{Mortgage} + \delta_2 \cdot \text{Rent} + \delta_3 \cdot \text{FamInc} + \delta_4 \cdot \text{Bedrooms} \\
&\quad + \delta_5 \cdot \text{Baths} + \delta_6 \cdot \text{Parking} + \delta_7 \cdot \text{Beach} + \delta_8 \cdot \text{Gaol} + \delta_9 \cdot \text{ICSEA} + u. \quad (4.4)
\end{align*}$$

The simultaneous determination of Rent, Price and Mortgage is also intuitive empirically. Before someone try to bid in house transaction, he needs to first approximately estimate the upper bound of mortgage that a bank will grant him; he also needs to estimate how much rent he can collect to pay the monthly mortgage. Before a bank makes decision on mortgage application, it typically first investigate the house price and the expected rent in case of mortgage default. In a similar vein, the rent of a house is typically determined by the price of house and monthly mortgage amount. Hence, in the decision making procedure of rent, price or mortgage, the other two factors are apparently critical, implying the simultaneous equations model is empirically solid.

### 4.2. Instrument selection using graphs

After confirming that the chance of Price/logPrice endogeneity is fair, we need to find a proper instrument for it in the regression anlaysis of Rent or Mortgage. Due to the similarity of two regression analysis and limit of the paper length, we focus on the regression
analysis of Rent/logRent in this paper. As shown before, a valid instrument needs to satisfied definition 2.3 and relevant discussions, which graphically implies the existence of a indirect effect and non-existence of the indirect effect (shown in figure 5 and 6). Based on the estimated graph (figure 11), we can directly find out 3 instrument variables: Baths, Bedrooms and Parking. None of them violates definition 2.3. All 3 variables are parents of Price/logPrice, implying no violation of condition (ii) in definition 2.3; in last subsection we confirms that all 3 variable can only affect Rent/logRent through Price/logPrice (illustrated in figure 12 and confirmed in table 6), which means no violation of condition (ii) in definition 2.3. However, these two conditions are for a general statistical dynamic system. The score-based learning method we use only requires the subgaussian distribution of each variable, which allows the causation among variable can be any nonlinear form. However, the classic endogeneity analysis in regression analysis requires the linearity among all variables. To make sure these 3 variables fit a system of multiple linear regression equations, we need to check the correlation between logRent/Rent and the instrument variable of logPrice/Price, which is stated in table 7.

| Table 7  |
|----------|
| **the correlation among instruments, the endogenous variable and Rent/logRent** |
|          | Baths | Parking | Bedroom | Price | logPrice |
| Rent     | 0.19  | 0.069   | 0.061   | 0.34  |          |
| logRent  | 0.16  | 0.043   | 0.023   | 0.32  |          |
| Price    | 0.52  | 0.34    | 0.46    |       |          |
| logPrice | 0.57  | 0.41    | 0.60    |       |          |

Table 7 clearly shows that Bedrooms and Parking does not fit the linear system as instruments of Price/logPrice. corr (logRent, Parking) and corr (logRent, Bedrooms) are too weak. As a result, even though the correlations between theses variables and logPrice are decent, the predicted value of logPrice by these two variables cannot explain enough variation of logRent in 2LSL or other IV regressions, which suggests the possibility of weak instruments and may lead to the wrong signs, values of the regression coefficients and wrong interpretations. The concerns are confirmed in table 8 and 9.

12For the nonlinear graph estimation and endogeneity bias correction, we need dependence measures like mutual information and Hilbert-Schmidt independence criterion, both of which requires huge computation loads in our high-dimensional database. Hence, we skip this topic in this paper.
Table 8 shows the 2SLS outputs in log equations. The covariates included in this table are determined data-driven by variable selection algorithms. In table 8, we use Baths, Bedrooms and Parking as a logPrice instrument respectively. Due to the endogeneity, OLS clearly underestimates the volume of the marginal effect of logPrice and takes it insignificant. The logPrice coefficient in Baths 2SLS is 7 times larger than the corresponding OLS coefficient; moreover, the t-value of logPrice in Baths 2SLS is 3 times larger then the one in OLS, which turns Price significant in 2SLS regression. These results clearly show the bias correction effect of 2SLS using Baths. By contrast, due to the weak correlation between Parking/Bedrooms and logRent, both Parking 2SLS and Bedrooms 2SLS alter the OLS coefficient of logPrice towards the wrong direction, which gives the wrong interpretation as ‘a higher house price is associated with a lower rent’. Moreover, logPrice in Parking 2SLS is even less significant than it in OLS.
Table 9 shows the 2SLS outputs in linear equations, which is similar to table 8. OLS still underestimates the volume of the marginal effect of Price. The Price coefficient in Baths 2SLS is around 50% larger than the corresponding OLS coefficient. The t-value of Price in Baths 2SLS is 40% larger than the one in OLS. Similar to table 8, Parking 2SLS and Bedrooms 2SLS again alter the OLS coefficient of Price negatively and reduce the volume of the corresponding marginal effect, which is even smaller than the corresponding marginal effect in the endogenous OLS.

Finally, to double check the validity of each instrument in 2SLS, we also implement 4 traditional instrument tests and report results in table 10.
With p-value less than 1%, all 4 tests confirm Baths significantly corrects the endogeneity bias on the logPrice marginal effect in 2SLS. Consistent with our intuition, graph estimation and MB selection successfully accomplish the endogeneity detection, instrument validation and selection. In linear equations, the p-values of Baths increases marginally, which still stay below 5%. This is reasonably expected because we does not log-transform dollar-measured variables, which leaves them with long/fat/heavy tails and highly likely non-subgaussian. Consistent with our previous concern, the p-values of all tests in Parking 2SLS are well above 5% (linear and log) due to the weak correlation between the instrument and the response variable. This shows that graph estimation and MB selection does work well on instrument validity in our data. Due to the similar reason, Bedrooms also alters the logPrice/Price marignal effect greatly towards an wrong direction, making Price insignificant and logPrice wrongly interpreted. As a result, low p-value of Bedrooms is only due to the volume of miscorrection, which itself does not implies the validity of the instrument.

### 4.3. Sanity check of the graph learning and interpretation on the validity of instruments

The previous subsection shows the instrument validation result. We also need sanity-check the reliability of the learning result and investigate its empirical appropriateness. Unlike the house pricing regression, the rent regression only return $R^2$ at around 40%. $R^2$ at such level suggests that some variation of rent is not specified as a linear model. As a result, we need to carefully check the model.

Firstly, the exclusion of house feature variable from mortgage equation is quite expected. In the mortgage market, banks directly focus on the possibility of mortgage
default, which can be efficiently analysed via direct factors like the potential ability of repayment (including income, rent, family wealth etc) and the house price (typically proportional to the loan amount). In such sense, Parking, Bedroom and Baths are not the most direct factor for mortgage determination.

In the determination of house price, it is intuitive that Bedroom directly causes the change of Price since the construction of bedrooms will increase the cost of house construction. However, someone may find it counterintuitive that Bedroom does not cause Rent directly. This is due to the consumers’ behavior and the condition of the Sydney leasing market. Based on 2011 census and the house leasing data, the majority of leasing demand comes from college and international students, young professionals and couples without children. Due to the short house supply and great leasing demand, a great number of landlords lease their houses via room-sharing. A house can accommodate more tenants via partitioning one bedroom into two or turning living spaces (e.g., lounges, living rooms, garages, storage rooms and dining rooms) into bedrooms. Being illegal and hard to check, this scheme can greatly increases the number of tenants in a house and the rent income, which makes room-sharing quite popular in Sydney leasing market. As a result, Bedroom does not accurately reflect the leasing capacity and, hence, does not cause Rent directly. This explains why the correlation between Bedroom and logRent is low. A similar reason can be found for Parking. Constructing garages will increase the construction cost of a house. However, since the local city council of Sydney issues permits to all local residents/tenants and allow them to park on the street without time limit, Parking does not accurately measures the parking spaces a house can offer.

Quite similar to Bedrooms, Bath also causes Price directly and Rent indirectly, both of which seems quite natural. However, in the second hand house market Sydney, Baths uniquely represents whether a house is newly refurbished or constructed. In our 2010 second-hand house database, we find out that more than 60% of the transacted houses (many of them are terrace house and town houses in or close to CBD) are constructed more than 30 years ago, many of which are even more than 50 year old. The houses designed at that time typically includes no more than two bathrooms (typically only one), no matter how many bedrooms associated. This is especially the case for town

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13See, for example, the house releasing report of Rent.com.au at https://www.rent.com.au/blog/room-sharing-overcrowding.
houses and terrace houses. This implies that, in Sydney second hand house market, a house with more than two bathrooms is quite possibly either a newly constructed house or an old but recently refurbished/reconstructed house. For those old town/terrace houses in or close to CBD, refurbishment or reconstruction is rare due to extra cost and restraints of city council. Hence, in this case Baths is a strong indicator of the house quality. That explains why the correlation between Baths and logRent is much larger than \( \text{corr} (\text{Baths}, \text{logRent}) \), which further explains why Bath can be applied as a valid instrument for logPrice in 2SLS.

5. Conclusion

In this paper we demonstrate the performance of solar variable selection with empirical data that have severe multicollinearity and, hence, severe grouping effects. As a competitor to solar, lasso is more sensitive to the grouping effect and returns unreliable variable-selection results. While more robust to the grouping effect than lasso, CV-en loses all sparsity in variable selection. By contrast, solar returns a stable and sparse variable selection and illustrates superior robustness to the grouping effect.
Appendix A: Graphical criteria of instruments and examples

*Conditional and unconditional instruments*

Invented by Pearl (2009), machine learning researchers and biostatisticians (e.g., Spirtes et al. (2000), Brito and Pearl (2002) and Silva and Shimizu (2017)) specify the concept of instruments from the perspective of the conditional distribution as definition A.1.

**Definition A.1.** Given a graph $G$ that includes the causal effect from $X$ to $Y$, a variable $z$ is a **conditional instrument** for $x \rightarrow Y$ (conditional on a set of variables $W$) if and only if

- $W$ does not d-separate $x$ and $z$ in $G$;
- $W$ does d-separate $Y$ and $z$ in $G_x$;
- $W$ are not the descendents of $Y$ and $x$ in $G$.

The idea of definition A.1 is very similar to definition 2.3. The first condition in definition A.1 means that, after conditioning on $W$, the causal effect between $x$ and $z$ still exists. Alongside with the assumption that graph $G$ includes the causal effect from $x$ to $Y$, the first condition implies that, after conditioning on $W$, $z$ can affect $Y$ via $x$. The second condition means that, after (i) holding $W$ constant and (ii) removing $x \rightarrow Y$, $z$ can no longer affect $Y$. This means that, after holding $W$ constant, $z$ can affect $Y$ only via $x$. As a result, definition A.1 is a special version that does not condition on any variable, which is very popular in econometrics and sociology. This kind of instruments is called **unconditional instruments**.\(^{15}\)

![Conditional Instrument Diagram](image.png)

**Fig 14:** (Brito and Pearl, 2002) An example with no possible unconditional instruments.

The reason that Pearl (2009) define instruments from the conditional distribution is...

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\(^{14}\)Since the terminology of graph sometimes could be different, we modify the wording of the definition in Silva and Shimizu (2017) to maintain the consistency with our framework.

\(^{15}\)Some literature also call it **marginal instruments**.
to increase the probability of finding a valid instrument, especially when unconditional instruments are hard to find. Take figure 14 as an example. In figure 14, due to existence of $w$, $z$ can still cause $Y$ even though we remove $x \rightarrow Y$, violating (G2) of definition 2.3. In this graph there is no unconditional instrument for $x$. However, if we hold $w$ constant, the graph will degenerate to figure 15.

![Diagram](image1)

Fig 15: (Brito and Pearl, 2002) with $w$ in figure 15 controlled, $z$ is a valid conditional instrument.

and $z$ does not violate (G1) (no variable can d-separate $z$ and $x$) and (G2) (any variable can d-separate $z$ and $Y$ after removing $x \rightarrow Y$) anymore, meaning $z$ is a proper conditional instrument for $x$.

**Graphical examples of instruments**

In this subsection we illustrate that, given an accurately estimated graph, graphic criteria can conveniently identify the invalid instruments.

**descendent of $Y$ are invalid instruments of $x$**

![Diagram](image2)

Fig 16: $z$ (descendant of $Y$) cannot be a valid instrument.

Figure 16 explains why descendants of $Y$ cannot be instruments. In all 4 figures, $z$ is the child of $Y$, meaning that $z \perp \perp Y$ by any variable in $G_{X}$. This implies that the effect from $z$ to $Y$ does not go through $x$ at all. As a result, condition (i) in Definition 2.3 is violated in all 4 figures.
Equivalently, such violation can be explained using the instrument definition in econometrics. No matter how the other arrows is set, there are only two possible relations among $z$, $Y$ and $u$: either $u \to Y \to z$ or $u \leftarrow Y \to z$.\(^{16}\) If $u \to Y \to z$ (as illustrated in figure 16a), $\text{corr} (z, u) \neq 0$. If $u \leftarrow Y \to z$ (as illustrated in figure 16b to 16d), $Y$ confounds $z$ and $u$, implying the unconditional correlation $\text{corr} (z, u)$ also nonzero. As a result, $\text{corr} (z, u)$ can be 0 only if both $z$ and $u$ are ancestors of $Y$.

\[\text{descendent of } x \text{ are invalid instruments of } x\]

Figure 17 explains why descendants of $x$ cannot be an instrument. Using descendants of $x$ as instruments will inevitably violates the instrument definition in econometrics. No matter how the other arrows is set, there are only two possible relations among $z$, $x$ and $u$: either $u \to x \to z$ or $u \leftarrow x \to z$.\(^{17}\) If $u \to x \to z$ (as illustrated in figure 17a and 17b), $\text{corr} (z, u) \neq 0$. If $u \leftarrow x \to z$ (as illustrated in figure 17c to 17d), $x$ confounds $z$ and $u$, implying the marginal correlation $\text{corr} (z, u)$ also nonzero. As a result, it is possible to have $\text{corr} (z, u) = 0$ only when both $u$ and $z$ are the ancestors of $x$.

Equivalently, Using descendants of $x$ as instruments will inevitably violates the definition 2.3. In the case $u \to x \to z$, if $Y$ is also the ancestor of $x$ (as shown in figure 17a and 17c), definition 2.3 will be violated since $x \to Y$ does not exist at all; if $Y$ is also a descendant of $x$ (as shown in figure 17b and 17d), $x$ confounds both $z$ and $Y$, meaning that the unconditional correlation between $z$ and $Y$ is not zero in $G_{X}$. Hence, $z \not \perp \perp Y$ by $u$ in $G_{X}$, which violates condition (i) in definition 2.3.

\(^{16}\)We assume $z$ as descendant of $Y$, hence it’s always $Y \to z$.

\(^{17}\)We assume $z$ as descendant of $x$, hence it’s always $x \to z$. 
remote ancestors of $x$ and $Y$ are highly like to be weak instruments of $x$

As explained in previous subsubsections, ancestors of $x$ and $Y$ are more likely to fit both graphical and error-based criteria. However, close ancestors are preferred when choosing the instruments. The farther the ancestor of $x$ and $Y$, the weaker correlation to $x$ and $Y$. Assuming all variables standardized, in two stage least square

$$b_{IV} = \frac{\text{corr}(z, Y)}{\text{corr}(z, x)}$$

The remote ancestors are more likely to be weak IVs and cause bias.

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