General theory of cosmological perturbations in open and closed universes from the Horndeski action

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Our Universe is nearly spatially flat, but this does not mean that it is exactly spatial flat. In this paper we derive general quadratic actions for cosmological perturbations in non-flat models from the Horndeski theory. This allows us to study how the spatial curvature influences the behavior of cosmological perturbations in the early universe described by some general scalar-tensor theory. We show that a tiny spatial curvature at the onset of inflation is unlikely to yield large (or \( \mathcal{O}(1) \)) effects on the primordial spectra even if one modifies gravity. We also argue that non-singular cosmological solutions in the Horndeski theory are unstable in spatially open cases as well as in flat cases.

PACS numbers: 98.80.Cq, 04.50.Kd

I. INTRODUCTION

Different cosmological observations indicate that the Universe is nearly spatially flat. According to Planck’s observations [1, 2], Planck TT,TE,EE+lowE+lensing gives the constraint in terms of the curvature density parameter \( \Omega_K \) as

\[
\Omega_K = -0.011^{+0.013}_{-0.012} \quad (95\% \text{ CL}),
\]

and Planck TT,TE,EE+lowE+lensing+BAO gives

\[
\Omega_K = 0.0007 \pm 0.0037 \quad (95\% \text{ CL}).
\]

The observed flatness of the universe can be explained naturally by the inflationary expansion in the early time [3–5]. For this reason it is often assumed that the universe is exactly flat. However, the fact that \( \Omega_K \) is bounded to be small does not mean that \( \Omega_K \) is actually equal to zero. Some works even suggest that flat models are less favored by data [6–11]. Therefore, it is interesting to consider the possibility that there is a tiny spatial curvature \( K \neq 0 \) and

\[
-\frac{K}{a^2 H^2} \bigg|_{t=t_s} \sim \Omega_K = \mathcal{O}(10^{-3}),
\]

where \( H := d \ln a / dt \) is the Hubble parameter and \( t_s \) is the onset of inflation or the time at which the largest observable scales exited the horizon during inflation. If this is indeed the case, the dynamics of inflation and the evolution of cosmological perturbations are expected to be affected by non-zero \( K \).

So far the effects of the spatial curvature on the early universe have been investigated mostly in the context of conventional inflation driven by a canonical scalar field with a potential [12–19]. However, in light of recent developments in early universe models based on scalar-tensor theories or extended theories of gravity, it is important to study the impact of the spatial curvature on more general models of inflation (and possible alternative scenarios) than previously considered. This motivates us to formulate a general theory of cosmological perturbations in non-flat universes, and in this paper we do this by using the Horndeski theory, the most general second-order scalar-tensor theory with second-order field equations [20].

As applications of our general theory of cosmological perturbations, first we explore whether it is possible or not to yield significant effects on the primordial spectra from a tiny spatial curvature by modifying gravity. Second, we consider non-singular cosmology as an alternative to inflation (see, e.g., [21] for a review) and discuss the stability of such cosmology with a negative spatial curvature.

This paper is organized as follows. In the next section we present the background equations that govern the dynamics of a FLRW universe in the presence of the spatial curvature. We then derive the general quadratic actions for perturbations of a non-flat universe from the Horndeski theory in Sec. III. In Sec. IV we describe the first application of our general theory of cosmological perturbations: inflation in non-flat universes. In Sec. V we analyze the stability of non-singular open cosmologies as the second application. Section VI is devoted to conclusions. In Appendix a further extension of the quadratic actions for cosmological perturbations in non-flat models is given.

II. BACKGROUND EQUATIONS

Our discussion in the main body of the paper is based on the Horndeski theory [20, 22, 23], whose action is given by

\[
S = \int d^4 x \sqrt{-g} \mathcal{L}_{\text{Hor}},
\]
with
\[ L_{\text{Hor}} = G_2(\phi, X) - G_3(\phi, X) \Box \phi + G_4(\phi, X) R + G_{4X} \left[ (\Box \phi)^2 - (\nabla_\mu \phi \nabla_\nu \phi)^2 \right] + G_5(\phi, X) G_{\mu \nu} \nabla^\mu \nabla^\nu \phi - \frac{G_{5X}}{6} (\Box \phi)^3 - 3 \Box \phi (\nabla_\mu \nabla_\nu \phi)^2 + 2 (\nabla_\mu \nabla_\nu \phi)^3, \tag{5} \]
where \( X := -g^{\mu \nu} \nabla_\mu \phi \nabla_\nu \phi / 2 \) and we denote \( \partial G / \partial X \) by \( G_X \). This action leads to the most general second-order equations of motion for the scalar field \( \phi \) and the metric \( g_{\mu \nu} \), and hence is appropriate for describing general single-field cosmology.

Although a number of spatially flat cosmological models have been explored so far based on both canonical and non-canonical scalar-field theories, non-flat early universe models have been studied mostly only in the context of the canonical scalar field. In light of this situation, we study the effects of the spatial curvature (denoted by \( K \) in this paper) on the background dynamics and perturbations using the Horndeski theory.

We use the metric of the FLRW universe including the non-flat cases,
\[ ds^2 = -N^2(t) dt^2 + a^2(t) \gamma_{ij} dx^i dx^j, \tag{6} \]
where \( \gamma_{ij} \) is the metric of maximally symmetric spatial hypersurfaces. It can be written explicitly as \( \gamma_{ij} dx^i dx^j = d\chi^2 + S_K^2(\chi) d\Omega^2 \) with
\[ S_K(\chi) := \begin{cases} \sin(\sqrt{K} \chi) / \sqrt{K} & (\text{closed}: K > 0), \\ \chi & (\text{flat}: K = 0), \\ \sinh(\sqrt{-K} \chi) / \sqrt{-K} & (\text{open}: K < 0), \end{cases} \tag{7} \]
and \( d\Omega^2 := d\theta^2 + \sin^2 \theta d\phi^2 \). Hereafter we write \( (\chi, \theta, \varphi) =: \vec{x} \).

Substituting the metric to the action, varying it with respect to \( N(t) \) and \( a(t) \), and then taking \( N = 1 \), we obtain the background equations corresponding to the Friedmann and evolution equations in the following form,
\[ E_0 + \mathcal{E}_K = 0, \tag{8} \]
\[ P_0 + \mathcal{P}_K = 0, \tag{9} \]
where \( E_0 \) and \( P_0 \) are independent of \( K \) and \( \mathcal{E}_K \) and \( \mathcal{P}_K \) are proportional to \( K \). They are given explicitly by
\[ E_0 = 2X G_{2X} - G_2 + 6X \phi H G_{3X} - 2X G_{3\phi} - 6H^2 G_4 + 2H^2 X (G_{4X} + X G_{4XX}) - 12HX \phi G_{4\phi X} - 6H \phi G_{4\phi} + 2H^3 X \phi (5G_{5X} + 2X G_{5XX}) - 6H^2 X (3G_{5\phi} + 2X G_{5\phi X}), \tag{10} \]
\[ P_0 = G_2 - 2X (G_{3\phi} + \phi G_{3X}) + 2(3H^2 + 2H) G_4 - 12H^2 X G_{4X} - 4HX X G_{4XX} - 8HX G_{4XX} - 8HX X G_{4XX} + 2(\phi + 2H \phi) G_{4\phi} + 4X G_{4\phi}, \tag{11} \]
and
\[ \mathcal{E}_K = -3G T \frac{K}{a^2}, \quad \mathcal{P}_K = \mathcal{F}_T \frac{K}{a^2}, \tag{12} \]
with time-dependent coefficients
\[ \mathcal{F}_T = 2 \left[ G_4 - X(\phi G_{5X} + G_{5\phi}) \right], \tag{13} \]
\[ G_T = 2 \left[ G_4 - 2X G_{4X} - X(\phi G_{5X} - G_{5\phi}) \right]. \tag{14} \]
These general equations clarify how the spatial curvature comes into play in the background dynamics. In general, we have
\[ \mathcal{E}_K \mathcal{P}_K \sim \mathcal{E}_0 \mathcal{P}_0 \sim \frac{K}{a^2 H^2}. \tag{15} \]
We will see that \( \mathcal{F}_T \) and \( G_T \) also appear in the quadratic action for cosmological perturbations.

One can also derive the scalar-field equation by varying the action with respect to \( \phi(t) \):
\[ \frac{1}{a^3} \frac{d}{dt} \left[ a^3 (J_0 + J_K) \right] = P_{\phi \theta} + P_{\phi K}, \tag{16} \]
where
\[ J_0 = \phi G_{2X} + 6H X G_{4X} - 2\phi G_{3\phi} + 6H^2 \phi (G_{4X} + 2X G_{4XX}) - 12HX \phi G_{4\phi X} + 2H^3 X (3G_{5X} + 2X G_{5XX}) - 6H^2 \phi (G_{5\phi} + X G_{5\phi X}) + 6H \phi (\dot{X} + 2HX) G_{4\phi} - 6H^2 X G_{5\phi} - 8H^3 X \phi G_{5\phi}. \tag{17} \]
\[ J_K = 6 \left( \phi G_{4X} - \phi G_{3\phi} + H X G_{5X} \right) \frac{K}{a^2}, \tag{18} \]
\[ P_{\phi \theta} = G_{2\phi} - 2X (G_{3\phi} + \phi G_{3X}) + 6(2H^2 + \dot{H}) G_{4\phi} + 6H (\dot{X} + 2HX) G_{4\phi X} - 6H^2 X G_{5\phi} + 2H^3 X \phi G_{5\phi}, \tag{19} \]
\[ P_{\phi K} = 3 \frac{\partial \mathcal{F}_T}{\partial \phi} \frac{K}{a^2}. \tag{20} \]

In contrast to the case of a minimally coupled scalar field, the spatial curvature appears in the scalar-field equation of motion if the scalar field is non-minimally coupled to gravity (i.e., \( G_4 \neq \text{const}, G_5 \neq \text{const} \)). The above equations were derived earlier in Ref. [24] to study the effects of the spatial curvature in the context of the Galilean Genesis scenario.

### III. COSMOLOGICAL PERTURBATIONS

In this section, we derive for the first time the general quadratic actions for scalar and tensor perturbations in the presence of the spatial curvature.
Choosing the unitary gauge $[\delta \phi(t, \vec{x}) = 0]$, the perturbed metric can be written as
\[
ds^2 = -N^2 dt^2 + g_{ij} (dx^i + N^i dt) (dx^j + N^j dt),
\] (21)
with
\[
N = 1 + \delta n, \quad N_i = D_i \chi, \\
g_{ij} = a^2 e^{2\chi} \left( \gamma_{ij} + h_{ij} + \frac{1}{2} \mathcal{D} h_{ij} \right),
\] (22)
where Latin indices are raised and lowered with $\gamma_{ij}$, $D_i$ is the covariant derivative compatible with $\gamma_{ij}$, and the tensor perturbation $h_{ij}$ satisfies $h_{ij}^T = D_i h_{ij}^j = 0$. Substituting this metric to the action and expanding it to second order in perturbations, we will obtain the general quadratic actions for tensor and scalar perturbations in open and closed universes. Below we will do this based on the Horndeski theory, but the results can be extended straightforwardly to the beyond Horndeski theories. This extension is summarized in the Appendix.

### A. Tensor perturbations

With some manipulation the general quadratic action for tensor perturbations in open and closed universes is found to be
\[
S_T^{(2)} = \frac{1}{8} \int dt d^3 x \sqrt{\gamma} d^3 \left[ G_T \dddot{h}_{ij}^2 + \mathcal{F}_T \dddot{h}_{ij} (D^2 - 2\kappa) h_{ij} \right],
\] (23)
where $\mathcal{F}_T$ and $G_T$ were already defined in Eqs. (13) and (14), respectively, and $D^2 := \gamma_{ij} D_i D_j$. We may define the propagation speed of tensor perturbations as $c_T^2 := \mathcal{F}_T / G_T$. Recalling that $\mathcal{F}_T$ and $G_T$ do not depend explicitly on $\kappa$, the effects of the spatial curvature can be seen only in the spatial derivative operator $D^2 - 2\kappa$. In the case of $G_4 = M_4^4 / 2$, $G_5 = 0$, the action (23) reduces to the standard result known in general relativity (see, e.g., [25]).

From Eq. (23) we see that ghost instabilities are absent if $G_T > 0$. Gradient instabilities can be avoided by requiring that $\mathcal{F}_T / G_T > 0$. Therefore, the stability conditions in open and closed universes are given by
\[
\frac{\mathcal{F}_T}{G_T} > 0, \quad G_T > 0.
\] (24)

To canonically normalize the tensor perturbations, we follow Ref. [23] and introduce a new time coordinate
\[
dy_T = \frac{\mathcal{F}_T^{1/2}}{a G_T^{1/2}} dt,
\] (25)
and
\[
v_{ij}(y_T, \vec{x}) = z_T h_{ij}, \quad z_T := \frac{a}{2} (\mathcal{F}_T G_T)^{1/4}.
\] (26)

In spatially flat models, it is convenient to expand the tensor perturbations in terms of the eigenfunctions $e^{i\vec{k} \cdot \vec{x}}$ of the flat-space Laplacian and the polarization tensor. Similarly, in open and closed models we introduce the tensor harmonics $Q^{nlm(s)}_{ij}(\vec{x})$ satisfying
\[
\mathcal{D}^2 Q^{nlm(s)}_{ij} = -k^2 Q^{nlm(s)}_{ij},
\] (27)
\[
\mathcal{D}^i Q^{nlm(s)}_{ij} = \gamma^{ij} Q^{nlm(s)}_{ij} = 0.
\] (28)

Here $k^2 = \kappa (n^2 - 3)$ ($n = 3, 4, 5, \cdots$), $2 \leq l \leq n - 1$, $-l \leq m \leq l$ for $\kappa > 0$ and $k^2 = |\kappa| (n^2 + 3)$ ($n \geq 0$), $l \geq 2$, $-l \leq m \leq l$ for $\kappa < 0$. The index $(s)$ distinguishes between even- and odd-parity harmonics. See Refs. [26, 27] for an explicit form of $Q^{nlm(s)}_{ij}$.

We now quantize $v_{ij}$ by promoting it to the operator $\hat{v}_{ij}$. Using the tensor harmonics, one can expand $\hat{v}_{ij}$ as
\[
\hat{v}_{ij}(y_T, \vec{x}) = \sum_{n l m} v^{(s)}_{nlm}(y_T) \hat{a}^{(s)}_{nlm} Q^{nlm(s)}_{ij}(\vec{x}) + v^{(s)}_{nlm}(y_T) \hat{a}^{(s)*}_{nlm} Q^{nlm(s)*}_{ij}(\vec{x}),
\] (30)
where $\hat{a}^{(s)}_{nlm}$ and $\hat{a}^{(s)*}_{nlm}$ are the annihilation and creation operators satisfying the commutation relations
\[
\left[ a^{(s)}_{nlm}, a^{(s)*}_{n'l'm'} \right] = \delta_{nn'} \delta_{ll'} \delta_{mm'} \delta_{ss'},
\] (31)
where a prime denotes differentiation with respect to $y_T$. The normalization condition is
\[
v^{(s)}_{nlm} v^{(s)*}_{nlm} = \frac{\left( |\kappa| n^2 - \kappa - \frac{27}{2} \right)}{2 \pi^2} \left( v^{(s)}_{nlm} \right)^2.
\] (32)

The mode functions $v^{(s)}_{nlm}$ obey
\[
v^{(s)}_{nlm}'' + \left( |\kappa| n^2 - \kappa - \frac{27}{2} \right) v^{(s)}_{nlm} = 0,
\] (33)
where a prime denotes differentiation with respect to $y_T$. The normalization condition is
\[
v^{(s)}_{nlm} v^{(s)*}_{nlm} = \frac{\left( |\kappa| n^2 - \kappa - \frac{27}{2} \right)}{2 \pi^2} \left( v^{(s)}_{nlm} \right)^2.
\] (34)

Given the background solution, one can solve Eq. (32) with an appropriate initial condition, i.e., an appropriate choice of the positive frequency mode. The primordial power spectrum can then be obtained by evaluating
\[
P_T(n) := \sum_s \left[ \frac{|\kappa| n^2 - 3 |\kappa| n^2 - 3\kappa}{2 \pi^2} \right] \left| v^{(s)}_{nlm} \right|^2.
\] (35)

Here we have followed the definition of the power spectrum given in [28].

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1 The symbol of summation should be understood as
\[
\sum_{n l m} = \sum_{n=3}^{\infty} \sum_{l=2}^{n-1} \sum_{m=-l}^{l-1} \begin{cases} \infty & (K > 0), \\ 0 & (K < 0). \end{cases}
\] (29)

2 In the case of $K < 0$, the Kronecker delta $\delta_{n m'}$ should be understood as the Dirac delta function $\delta(n - n')$. 

B. Scalar perturbations

The quadratic action for scalar perturbations can be obtained as

$$S^{(2)}_S = \int dt d^3 x \sqrt{-g} a^3 \left\{ -3G_T \xi^2 - \frac{F_T}{a^2} \xi \partial^2 \xi + \Sigma \delta n^2 - 2 \Theta \delta n \frac{\partial^2 \xi}{a^2} + 2G_T \xi \frac{\partial^2 \xi}{a^2} + 6 \Theta \delta n \xi - 2G_T \delta n \frac{\partial^2 \xi}{a^2} - 3F_T \xi \frac{\partial^2 \xi}{a^2} \right\},$$

(35)

where

$$\Theta = \Theta_0 + \Theta_K, \quad \Sigma = \Sigma_0 + \Sigma_K,$$

(36)

with

$$\Theta_0 := -\dot{\phi} X G_{3,X} + 2HG_4 - 8HXG_{4X}$$

$$-8H^2X^2 G_{4XX} + \phi \hat{G}_{4,\phi} + 2X \phi G_{4,\phi}X$$

$$-H^2 \hat{G}(5X^2G_{5,X} + 2X^2G_{5XX})$$

$$+ 2HX (3G_{5,\phi} + 2G_{5,\phi}X),$$

(37)

$$\Theta_K := -\dot{\phi} X G_{5,X} + \frac{K}{a^2},$$

(38)

$$\Sigma_0 := X G_{2X} + 2X^2 G_{2XX} + 12H \dot{\phi} X G_{3X}$$

$$+ 6H \dot{\phi} X^2 G_{3XX} - 2X G_{3,\phi} - 2X^2 G_{3,\phi}X - 6H^2 G_4$$

$$+ 6[\hat{H}^2 (7X G_{4X} + 16X^2 G_{4XX} + 4X^3 G_{4XXX})]$$

$$- H^2 \hat{G}(G_{4,\phi} + 5X G_{4,\phi}X + 2X^2 G_{4,\phi}X)$$

$$+ 2H^2 \hat{G}(15X G_{5,X} + 13X^2 G_{5,XX} + 2X^3 G_{5,XXX})$$

$$- 6H^2 X (6G_{5,\phi} + 9X G_{5,\phi}X + 2X^2 G_{5,\phi}X),$$

(39)

$$\Sigma_K := 6(X G_{4X} + 2X^2 G_{4XX} - X G_{5,\phi} - X^2 G_{5,\phi}X$$

$$+ 2H \dot{\phi} X G_{5,X} + H \dot{\phi} X^2 G_{5,XX}) \frac{K}{a^2}.$$ 

(40)

We can also express these quantities as

$$\Theta_{0,K} = -\frac{1}{6} \frac{\partial \xi_{0,K}}{\partial H},$$

(41)

$$\Sigma_{0,K} = X \frac{\partial \xi_{0,K}}{\partial X} + 2 \frac{H}{6} \frac{\partial \xi_{0,K}}{\partial H},$$

(42)

Therefore, even in the presence of the spatial curvature the relation between the background equation and these coefficients remains the same as in the flat models [23].

In the case of general relativity with a canonical scalar field, $G_2 = X - V(\phi)$, $G_3 = M_{pl}^2/2$, $G_4 = G_5 = 0$, the action (35) reproduces Eq. (B.4) of Ref. [25] in the unitary gauge.

It can be seen that $\Theta$ and $\Sigma$ depend explicitly on $K$, while the other coefficients $F_T$ and $G_T$ do not. Similarly to the background equations, we roughly have

$$\frac{\Theta_K}{\Theta_0} \sim \frac{\Sigma_K}{\Sigma_0} \sim \frac{K}{a^2 H^2}.$$ 

(43)

Note that $\Theta_K$ and $\Sigma_K$ are non-vanishing if $\phi$ is non-minimally coupled to gravity except for the simplest case with $G_4 = G_4(\phi)$ and $G_5 = 0$. Without fine-tuning, $\Theta_K$ and $\Sigma_K$ cannot give rise to $O(1)$ effects.

Variation with respect to $\delta n$ and $\chi$ gives the constraint equations

$$\Sigma \delta n - \Theta \frac{\partial \xi^2}{a^2} + 3\Theta \xi - G_T \frac{\partial \xi^2}{a^2} - 3G_T \xi \frac{\partial \xi}{a^2} = 0,$$ 

(44)

$$\Omega \delta n - G_T \xi - G_T \xi \frac{\partial \xi}{a^2} = 0,$$ 

(45)

where we used $D^2 D_i \chi - D_i D^2 \chi = 2K D_i \chi$. Substituting these equations into Eq. (35), we obtain the quadratic action for the curvature perturbation

$$S^{(2)}_\zeta = \int dt d^3 x \sqrt{-g} a^3 \left[ \zeta \xi^2 + \frac{F_T}{a^2} \left( D^2 + 3K \xi \right) \right],$$ 

(46)

where

$$G_S := \frac{D^2 + 3K}{D^2 - (G_T \Sigma / \Theta^2)} \frac{G_T}{\Theta^2} + 3,$$ 

(47)

$$F_S := \frac{1}{a} \frac{d}{dt} \left[ \frac{D^2 + 3K}{D^2 - (G_T \Sigma / \Theta^2)} \frac{aG_T^2}{\Theta} \right],$$

(48)

This is the general quadratic action for $\zeta$ in open and closed universes. In contrast to the case of the tensor perturbations, the coefficients can depend non-trivially on $K$ through $\Theta$ and $\Sigma$. The squared sound speed may be defined as $c_s^2 = F_S/G_S$, which is also dependent on $K$ in general.

In the case of general relativity with a canonical scalar field, the reduced action (46) reproduces the standard result found in [29] written in terms of the different variable $Q := \phi/H(\xi)$ and, with a non-trivial transformation of the variables (see Appendix C of [29]), the result of [29] can be confirmed to reproduce the result of the $p(\phi, X)$ theory [30] when $p = X - V(\phi)$. However, we have not been able to reproduce the quadratic action of [30] for general $G_2 = p(\phi, X)$ directly from Eq. (46). (Note that the definition of our $\zeta$ is different from that of $\zeta$ used in [30].)

One can derive the conditions for avoiding gradient and ghost instabilities based on the action (46). These instabilities are dangerous particularly for short wavelength perturbations, because the growth rates of short modes are high. Therefore, we take $D^2 \gg K$ and impose

$$G_{S,\text{short}} := G_T \left( \frac{G_T \Sigma}{\Theta^2} + 3 \right) > 0,$$ 

(49)

$$F_{S,\text{short}} := \frac{1}{a} \frac{d}{dt} \left( \frac{aG_T^2}{\Theta} \right) - F_T + \frac{G_T^3 K}{a^2} > 0.$$ 

(50)

Let us move on to the quantum theory. The quadratic action for the curvature perturbation can be written in
a canonically normalized form by introducing
\[ dy_S = \frac{\mathcal{F}_S^{1/2}}{aG_S^2} dt \] (51)
and
\[ u(y_S, \vec{x}) = z_S \zeta, \quad z_S := \sqrt{2a(F_S G_S)^{1/4}}. \] (52)

We expand \( u \) in terms of the scalar harmonics \( Q_{nlm}(\vec{x}) \) satisfying
\[ D^2 Q_{nlm} = -k^2 Q_{nlm}, \] (53)
where the eigenvalues are given by \( k^2 = |\mathcal{K}|n^2 - \mathcal{K} \) with \( n = 3, 4, 5, \ldots, 0 \leq l \leq n - 1, -l \leq m \leq l \) \( (\mathcal{K} > 0) \) and \( n \geq 0, l \geq 0, -l \leq m \leq l \) \( (\mathcal{K} < 0) \). See Refs. [26, 27] for an explicit form of \( Q_{nlm} \). The operator \( \hat{u} \) can then be expanded as
\[ \hat{u}(y_S, \vec{x}) = \sum_{nlm} u_{nlm}(y_S) \hat{a}_{nlm} Q_{nlm}(\vec{x}), \] (54)
where the annihilation and creation operators \( \hat{a}_{nlm} \) and \( \hat{a}_{nlm}^\dagger \) satisfy the standard commutation relations. The mode functions \( u_{nlm} \) are subject to the same normalization condition as Eq. (33). The equation of motion for \( u_{nlm} \) is given by
\[ u''_{nlm} + \left( |\mathcal{K}|n^2 - 4\mathcal{K} - \frac{z_S'^2}{z_S} \right) u_{nlm} = 0, \] (55)
where a prime stands for differentiation with respect to \( y_S \). Following the definition given in [28], the power spectrum for \( \zeta \) is computed as
\[ \mathcal{P}_\zeta(n) = \frac{|\mathcal{K}|^{1/2} n \left( |\mathcal{K}|n^2 - \mathcal{K} \right)}{2\pi^2} \left| \frac{u_{nlm}}{z_S} \right|^2. \] (56)

IV. APPLICATION 1: INFLATION

As an application, let us study the case where the inflationary universe had a spatial curvature and its effects could be seen at the beginning of inflation.

To catch the flavor, we start with the simplest prototype with
\[ G_2 = -V_0 < 0, \quad G_4 = \frac{M^2}{2}, \quad G_3 = G_5 = 0, \] (57)
where \( V_0 \) and \( M \) are constants, namely, general relativity with a positive cosmological constant. The background equations are given by
\[ V_0 - 3M^2H^2 - \frac{3M^2\mathcal{K}}{a^2} = 0, \] (58)
\[ -V_0 + M^2(3H^2 + 2\dot{H}) + \frac{M^2\mathcal{K}}{a^2} = 0, \] (59)
which are solved by
\[ a = \begin{cases} a_0 \cosh(ht) & (\mathcal{K} > 0), \\ a_0 \sinh(ht) & (\mathcal{K} < 0), \end{cases} \] (60)
with
\[ h = \frac{1}{M} \sqrt{\frac{V_0}{3}}, \quad a_0h = \sqrt{|\mathcal{K}|}. \] (61)

This is nothing but the de Sitter spacetime in closed/open slicing. For \( ht \gg 1 \) we recover exponential expansion, \( a \propto e^{ht} \), which implies that \( h \) is essentially the inflationary Hubble parameter. However, for \( ht \lesssim 1 \) the evolution of the scale factor deviates from that of the usual flat case.

In [31], the power spectrum of a test scalar field has been evaluated for the background (60), without taking into account the mixing with gravity.

Below we will consider two cases within the Horndeski theory where the scale factor is given (approximately) by the hyperbolic functions, but with different expressions for \( a_0 \) and \( h \), depending on the concrete model. Note that if \( a_0h \neq \sqrt{|\mathcal{K}|} \), the spacetime is something different from de Sitter in closed/open slicing and in particular it is no longer de Sitter for \( ht \lesssim 1 \).

A. Potential-driven inflation

Let us consider the “slow-roll” version of the above prototype. We assume that \( \phi(t) \) moves very slowly, and expand the functions in terms of \( X \) as
\[ G_i = g_i(\phi) + h_i(\phi)X + \cdots, \quad g_i \gg h_iX, \] (62)
with \( g_2(\phi) = -V(\phi) < 0 \). Since \( g_3 \) and \( g_5 \) can be absorbed into the redefinition of \( h_2 \) and \( h_4 \), respectively, after integration by parts, we may set \( g_3 = g_5 = 0 \) without loss of generality. Then, the Friedmann and evolution equations reduce to
\[ V - 6g_4H^2 - 6g_1 \frac{\mathcal{K}}{a^2} \simeq 0, \] (63)
\[ -V + 2g_4 \left( 3H^2 + 2\dot{H} \right) + 2g_4 \frac{\mathcal{K}}{a^2} \simeq 0, \] (64)
where we ignored the terms that vanish in the \( \dot{\phi}, \ddot{\phi} \to 0 \) limit, assuming that \( \dot{\phi} \ll H\dot{\phi} \). Although in reality we should require that \( \mathcal{K}/a^2H^2 \lesssim O(10^{-3}) \) at the onset of inflation and this might be as small as or smaller than ignored terms, we keep the curvature terms because we want to see how in general \( \mathcal{K} \) appears in various equations. From these equations we obtain
\[ H^2 + \frac{\mathcal{K}}{a^2} \simeq \frac{V}{6g_4}, \] (65)
\[ \dot{H} - \frac{\mathcal{K}}{a^2} \simeq 0. \] (66)
Therefore, by assuming the “slow-variation” conditions, \( \dot{g}_i \ll H g_i \), the solution of the hyperbolic form (60) can be obtained, but now with

\[
h = \sqrt{\frac{V}{6g_4}} \simeq \text{const}, \quad a_0 h \simeq \sqrt{|K|}, \tag{67}
\]
as an approximate solution. This is indeed (approximately) de Sitter spacetime. Note that in a closed universe we have \( H = 0 \) at \( t = 0 \) (the bounce point), which implies that the slow-variation conditions are subtle there. However, one does not need care about this subtle situation as long as one focuses on the realistic case where \( K/a^2 H^2 \) is small enough at the onset of inflation.

The functions in the quadratic actions (23) and (46) can be evaluated as follows. It is easy to see that

\[
G_T \simeq F_T \simeq 2g_4 \simeq \text{const}. \tag{68}
\]

To leading order in small quantities (we assume that \( \dot{g}_i/H g_i \sim h_i X/g_i \ll 1 \)), we obtain

\[
\hat{G}_S \simeq \left[ h_2 + 6h_4 \left( H^2 + \frac{K}{a^2} \right) \right] \frac{X}{H^2} + 6 \left[ h_3 + h_5 \left( H^2 + \frac{K}{a^2} \right) \right] \frac{\dot{X}}{H}, \tag{69}
\]

\[
F_S \simeq \left[ h_2 + 6h_4 \left( H^2 + \frac{K}{a^2} \right) \right] \frac{X}{H^2} + 4 \left[ h_3 + h_5 \left( H^2 + \frac{K}{a^2} \right) \right] \frac{\dot{X}}{H}. \tag{70}
\]

One may use the background equation to replace \( H^2 + K/a^2 \) with \( h^2 \simeq \text{const} \). Therefore, the quantities inside the square brackets are approximately constant. However, \( H \) itself in the denominators can be regarded as a constant only when \( H^2 \gg K/a^2 \) and we actually have

\[
\frac{\dot{G}_S}{H g_5} \frac{\dot{F}_S}{H f_S} \sim - \frac{\dot{H}}{H^2} \sim - \frac{K}{a^2 H^2}. \tag{71}
\]

As already argued above, in reality we must consider the situation where this is indeed sufficiently small.

**B. Kinetically-driven inflation**

Kinetically-driven inflation is realized in the shift-symmetric theories which are invariant under \( \phi \rightarrow \phi + \text{const} \). The free functions in the Lagrangian are therefore functions of only \( X \). In shift-symmetric theories, the scalar-field equation reduces to

\[
\frac{1}{a^3} \frac{d}{dt} \left[ a^3 (J_0 + J_K) \right] = 0 \quad \Rightarrow \quad J_0 + J_K = \text{const}. \tag{72}
\]

Hence, \( J_0 + J_K = 0 \) is an attractor (as long as we focus on expanding solutions, which is the case in this paper).

Below we study non-flat inflation in (a certain subclass of) the shift-symmetric Horndeski theory.

The Lagrangian we consider is given by

\[
G_2 = G_2(X), \quad G_4 = G_4(X), \quad G_3 = G_5 = 0. \tag{73}
\]

We now seek for the solutions of the hyperbolic form (60) with \( X = \text{const} \). Substituting this ansatz to the Friedmann and evolution equations, we obtain

\[
2XG_{2X} - 6|K_0|(G_4 - 2XG_{4X}) - \left( K_0 \right)(G_4 - 2XG_{4X}) = 0, \tag{74}
\]

\[
G_2 + 2|K_0|G_4 + 4\dot{h}(G_4 - 2XG_{4X}) + 2f^2(t) \left[ h^2(G_4 - 2XG_{4X}) - |K_0|G_4 \right] = 0, \tag{75}
\]

where \( K_0 := K/a^2 \) and \( f(t) = \tanh(ht) (\mathcal{K} > 0), \coth(ht) (\mathcal{K} < 0) \). The shift current vanishes for the attractor, \( J_0 + J_K = 0 \), leading to

\[
G_{2X} + 6|K_0|XG_{4X} + 6f^2(t) \left[ h^2(G_4 + 2XG_{4X}) - 6|K_0|XG_{4X} \right] = 0. \tag{76}
\]

Equations (74)–(76) have time-dependent and time-independent pieces. Each piece vanishes if

\[
G_2 + 6|K_0|G_4 = 0, \quad G_{2X} + 6|K_0|G_{4X} = 0, \quad h^2(G_4 - 2XG_{4X}) - |K_0|G_4 = 0, \quad h^2(G_4 + 2XG_{4XX}) - |K_0|G_{4X} = 0. \tag{77} \tag{78} \tag{79} \tag{80}
\]

In order for these four equations to be consistent,

\[
G_2G_{4X} - G_{2X}G_4 = 0, \quad G_3G_{4XX} + G_{2X}^2 = 0, \tag{81} \tag{82}
\]

must be satisfied for the solution \( X = X_0(= \text{const}) \), and then one determines \( a_0 \) and \( h \) as

\[
h = \sqrt{\frac{G_2(X_0)}{6|G_4(X_0) - 2X_0G_{4X}(X_0)|}}, \tag{83}
\]

\[
a_0 h = \sqrt{|K|} \sqrt{\frac{G_4(X_0)}{G_4(X_0) - 2X_0G_{4X}(X_0)}}. \tag{84}
\]

This is not a de Sitter spacetime because \( a_0 h \neq \sqrt{|K|} \).

Note that if one included \( G_3(X) \) and \( G_5(X) \), then the background equations would contain terms proportional to \( H \) and \( H^3 \). In that case, constructing non-flat solutions would not be so simple as above.

As an example, let us consider a simple model with

\[
G_2 = -6\beta G_4, \quad G_4 = \frac{M_p^2}{2} - \frac{\alpha}{2} X^2, \tag{85}
\]
where $\alpha$ and $\beta$ are positive constants. Then, Eq. (81) is satisfied automatically. From Eq. (82) one can determine the solution $X_0$ as

$$X_0 = \frac{M_{\text{Pl}}}{\sqrt{3\alpha}},$$

(86)

and from Eqs. (83) and (84) we obtain

$$h = \sqrt{\frac{\beta}{3}}, \quad a_0 h = \sqrt{\frac{|K|}{3}}.$$  

(87)

Note, however, that our kinetically-driven inflation with the hyperbolic scale factor shows a problematic behavior at the level of perturbations. Indeed, in the above example we have

$$G_S = \frac{8M^2}{3} \frac{D^2 + 3K}{D^2 + 5K/3}, \quad F_S = \frac{2M^2}{3} \frac{4K/3}{D^2 + 5K/3}$$

and so $F_S \simeq 0$ for large $D^2$. More generically, one can check that $F_{S,\text{short}} = 0$ irrespective of the concrete model. This is not a surprise, because the quadratic action for scalar perturbations becomes singular in the (flat) de Sitter limit of usual k-inflation. To avoid this singular behavior, we suppose that the functions in the Lagrangian depend weakly on $\phi$ and inflation takes place slightly away from the exact hyperbolic-type expansion. Such a situation can be analyzed following Ref. [32]. Then, $F_{S,\text{short}}$ is expected to acquire a small (“slow-roll” order) correction.

As for tensor perturbations we have

$$G_T = 2M_{\text{Pl}}^2, \quad F_T = \frac{2M_{\text{Pl}}^2}{3}.$$  

(88)

C. Primordial perturbations

Let us evaluate the primordial power spectra under the assumptions that the background is given by the hyperbolic form (60) and $F_T$, $G_T$, $F_S$, and $G_S$ are approximately constant. These assumptions are analogous to those often made in usual flat inflationary cosmology: one assumes the de Sitter background $a \propto e^{Ht}$ and $F_S = G_S \propto -\dot{H}/H^2 = \dot{\phi}^2/2H^2 = \text{const} (= \text{slow-roll order} \ll 1)$ to evaluate the power spectrum analytically.

The “$y$” coordinate used in Sec. III is given by

$$dy = C_0 \frac{dt}{a},$$

(90)

where the constant $C_0$ depends on the concrete model of interest as well as on the type of the perturbations. Note that $C_0$ corresponds the propagation speed of the perturbations under consideration. In closed models, this gives

$$a(y) = \frac{a_0}{\sin(-Ay)}, \quad A := \frac{a_0 h}{C_0},$$

(91)

where the range of $y$ is $-\pi/(2A) < y < 0$ for $0 < t < \infty$. Since $a_0 h \approx \sqrt{|K|}$, $1/A$ roughly gives the effective curvature radius. In open models, we have

$$a(y) = \frac{a_0}{\sinh(-Ay)},$$

(92)

where the range of $y$ is $-\infty < y < 0$ for $0 < t < \infty$. The evolution of the mode functions depends on $z_S''/z_S$ and $z_T''/z_T$, and now these quantities can be expressed solely in terms of the scale factor as $z_S''/z_S \simeq a''/a$ and $z_T''/z_T \simeq a''/a$. Thus, we want to solve the equation of the form

$$\psi'' + (|K|n^2 - BK - \frac{a''}{a}) \psi_n = 0,$$

$$\frac{a''}{a} = \begin{cases} A^2 \left[-1 + \frac{2}{\sin^2(Ay)}\right] & (K > 0), \\ A^2 \left[1 + \frac{2}{\sin^2(Ay)}\right] & (K < 0), \end{cases}$$

(93)

where $B = 1$ for tensor perturbations and $B = 4$ for scalar perturbations. Even in the case of general relativity plus a canonical scalar field, the $K$-dependent part of the equation for the curvature perturbation is different from what is obtained for a test scalar field [31].

Equation (93) can be solved analytically. For $K > 0$ the general solution is given by

$$\psi_n = C_1 \left[-\frac{A}{\tan(Ay)} + i\kappa_n\right] e^{i\kappa_n y}$$

$$+ C_2 \left[-\frac{A}{\tan(Ay)} - i\kappa_n\right] e^{-i\kappa_n y},$$

(94)

where

$$\kappa_n := \sqrt{K(n^2 - B) + A^2}.$$  

(95)

This is a generalization of the solutions obtained in [28, 31]. We take $C_1 = 0$ so that it is a positive frequency solution. In the case of the de Sitter geometry, this corresponds to the Bunch-Davies vacuum. (See Refs. [33, 34] for a different choice of the initial state.) Then, it follows from the normalization condition (33) that

$$|C_2|^2 = \frac{1}{2\kappa_n(\kappa_n^2 - A^2)}.$$  

(96)

We will evaluate the power spectrum in the limit $y \to 0$, so the following result will be useful:

$$\lim_{y \to 0} \frac{|\psi_n|^2}{a^2} = \frac{A^2|C_2|^2}{a_0^2}. $$

(97)

The general solution for $K < 0$ is given by

$$\psi_n = C_1 \left[-\frac{A}{\tanh(Ay)} + i\kappa_n\right] e^{i\kappa_n y}$$

$$+ C_2 \left[-\frac{A}{\tanh(Ay)} - i\kappa_n\right] e^{-i\kappa_n y},$$

(98)
where
\[ \kappa_n := \sqrt{-\mathcal{K}(n^2 + B) - A^2}. \]  
(99)

We take \( C_1 = 0 \) in order for this to be a positive frequency solution. The normalization condition (33) yields
\[ |C_2|^2 = \frac{1}{2\kappa_n(\kappa_n^2 + A^2)}. \]  
(100)

and also in this case we have
\[ \lim_{n \to 0} \frac{|\psi_n|^2}{a^2} = \frac{A^2|C_2|^2}{a_0^2}. \]  
(101)

Using the above general formulas to evaluate the primordial spectrum of tensor perturbations, we obtain
\[ \mathcal{P}_T = \frac{2}{\pi^2} \frac{G_0^{1/2}}{F_T^{1/2}} h^2 f_T(K, n) \]
\[ = \frac{2}{\pi^2} \frac{G_0^{1/2}}{F_T^{1/2}} \left( H^2 + \frac{a_0^2 h^2}{a^2} \right) f_T(K, n), \]  
(102)

where
\[ f_T(K, n) := \frac{n(n^2 - 3\sigma_K)}{n^2 - \sigma_K + A^2/K)^{1/2}(n^2 - \sigma_K)}, \]  
(103)

with \( A^2 = a_0^2 h^2 G_T/F_T \) and \( \sigma_K := \text{sgn}(K) = \pm 1 \). Similarly, for scalar perturbations we have
\[ \mathcal{P}_\zeta = \frac{1}{8\pi^2} \frac{G_0^{1/2}}{F_S^{1/2}} h^2 f_S(K, n) \]
\[ = \frac{1}{8\pi^2} \frac{G_0^{1/2}}{F_S^{1/2}} \left( H^2 + \frac{a_0^2 h^2}{a^2} \right) f_S(K, n), \]  
(104)

where
\[ f_S(K, n) := \frac{n(n^2 - \sigma_K)}{(n^2 - 4\sigma_K + A^2/K)^{1/2}(n^2 - 4\sigma_K)}, \]  
(105)

with \( A^2 = a_0^2 h^2 G_S/F_S \).

Since the largest observable scales correspond to \( n \sim 1/\sqrt{\Omega_K} \), we generically expect that
\[ f_T, f_S = 1 + \mathcal{O}(|\Omega_K|), \]  
(106)

but more quantitatively the correction depends on \( A^2/K \).

Let us look at potential-driven inflation as an example. For tensor perturbations we have \( A^2/K = \sigma_K \), and for scalar perturbations we have \( A^2/K = \sigma_K \) or \( A^2/K = (3/2)\sigma_K \), depending on which of the \( h_i \) is dominant. Also for tensor perturbations in kinetically driven inflation we have \( A^2/K = \sigma_K \). In any case, we do not find a large enhancement factor for the \( \mathcal{O}(|\Omega_K|) \) corrections. We have thus clarified how \( \mathcal{O}(|\Omega_K|) \) corrections enter the expressions for primordial power spectra in the present analytic toy example.

The analytic results obtained in this subsection rely on the assumptions that the background is given by the exact hyperbolic form and \( F_T, G_T, F_S, \) and \( G_S \) are constant. More precise evaluation of the power spectra for a given model with non-vanishing \( K \) requires numerical calculations. However, we have already derived all the necessary basic equations and hence performing numerical calculations is straightforward.

### V. APPLICATION 2: STABILITY OF NON-SINGULAR UNIVERSES

Our quadratic actions for cosmological perturbations have been derived without assuming any specific background dynamics such as inflation. Therefore, one can use the quadratic actions for studying alternative scenarios as well. In this section, let us consider the stability of non-singular universes.

It has been proven that non-singular cosmological solutions in the Horndeski theory are generally plagued with gradient instabilities if we do not admit some pathology for tensor perturbations [35–37]. However, the proof assumes spatially flat models. While it is clear that stable non-singular cosmology is allowed in the case of closed universes,\(^3\) in open universes it is not obvious whether or not non-singular cosmology is possible in the Horndeski theory. However, by using the stability conditions derived in Sec. III one can extend the previous no-go argument to open models.

For open models of non-singular universes \((a \geq \text{const} > 0, K < 0)\), the stability conditions \( F_{S,\text{short}} > 0, F_T > 0, \) and \( G_T > 0 \) lead to
\[ \frac{d\xi}{dt} > a F_T - \frac{G_T^2}{\Theta^2} \frac{\kappa}{a^2} > 0, \]  
(107)

where
\[ \xi := \frac{aG_T^2}{\Theta}. \]  
(108)

From Eq. (107) it can be seen that \( \xi \) a monotonically increasing function of \( t \). Thus, the proof in [36] can be extended straightforwardly. Note here that \( K < 0 \) is the essential assumption.

We integrate Eq. (107) from \( t_i \) to \( t_f \) and have
\[ \xi(t_f) - \xi(t_i) > \int_{t_i}^{t_f} a F_T dt' + \int_{t_i}^{t_f} \frac{G_T^2}{\Theta^2} \frac{(-\kappa)}{a^2} dt'. \]  
(109)

\(^3\) The simplest example is the de Sitter solution in closed slicing in general relativity plus a cosmological constant. This implies that instabilities of closed models are not generic and one can avoid them by elaborating a model, suggesting that the instability of the closed models of non-singular cosmology in [38] can in principle be removed.
Note that both integrands in Eq. (109) are always positive. We may suppose that Θ is finite because it contains $H, \phi$ and $\phi$. Then, $\xi$ never crosses zero, and by taking $t_i \rightarrow -\infty$ or $t_f \rightarrow +\infty$ we see that $\xi(\infty) - \xi(t_i) < 0$ or $\xi(t_f) - \xi(-\infty) < 0$ is required. This means that both of the integrals in the right hand side of Eq. (109) must be convergent for $t_i \rightarrow -\infty$ or $t_f \rightarrow +\infty$. However, this then requires that $F_T$ must decay sufficiently rapidly as $t \rightarrow \infty$ or $t \rightarrow -\infty$, implying a kind of pathology in the tensor perturbations [36] (this is directly related to geodesic incompleteness for gravitons [37]). The proof can also be applied to the case where $\Theta$ crosses zero at some moment and hence $\xi$ has a discontinuity there.\(^4\) In this case the integrals must be convergent both for $t_i \rightarrow -\infty$ and $t_f \rightarrow +\infty$.

Thus, we have proven that all open models of nonsingular universes in the Horndeski theory are unstable if one requires geodesic completeness for gravitons.

VI. SUMMARY AND CONCLUSIONS

In this paper, we have formulated the non-flat inflationary dynamics and cosmological perturbations on a non-flat background in the Horndeski theory. We have obtained the general quadratic actions for tensor and scalar perturbations (see the Appendix for further generalization) and clarified their curvature dependence. Within the simple models we have investigated, the corrections to the power spectra received from the spatial curvature are of order $\Omega_K$ and cannot be enhanced by modifying gravity.

Using our general quadratic actions for cosmological perturbations, we have also studied the stability of nonsingular universes with the spatial curvature. It is obvious that one can have a stable bouncing solution in closed models. In contrast, we have generalized the previous no-go argument for non-singular cosmologies in the Horndeski theory to open models.

It would be interesting to extend the effective field theory of inflation [46, 47] to non-flat models and compare the results obtained from the two different approaches.

Acknowledgments

We thank Tomohiro Harada, Shuichiro Yokoyama, and Shin’ichi Hirano for helpful discussions. The work of SA was supported by the JSPS Research Fellowships for Young Scientists No. 18J22305. The work of TK was supported by MEXT KAKENHI Grant Nos. JP15H05888, JP17H06359, JP16K17707, JP18H04355, and MEXT-Supported Program for the Strategic Research Foundation at Private Universities, 2014-2018 (S1411024).

Appendix A: Cosmological perturbations in non-flat universes from the beyond Horndeski action

In this Appendix, we extend the theory of cosmological perturbations in non-flat models to beyond Horndeski theories [48, 49]. For this purpose it is convenient to use the Lagrangian expressed in terms of the ADM variables in the unitary gauge [$\phi(t, \vec{x}) = \phi(t)$]. We thus use the Lagrangian given by

$$
\mathcal{L} = A_2(t, N) + A_3(t, N)K + A_4(t, N)K^2 + A_5(t, N)K^2_{ij} + A_6(t, N)N^3 + A_7(t, N)KK^2_{ij} + A_8(t, N)K^3_{ij} + B_6(t, N)K_{ij}R^{ij} + B_6(t, N)K R, \tag{A1}
$$

where $K_{ij}$ and $R_{ij}$ are the extrinsic and intrinsic curvature tensors on $t = $ const hypersurfaces (on which $\phi$ is homogeneous), respectively. The above theory contains 10 free functions of $\phi(t)$ and $X = \dot{\phi}^2/(2N^2)$, which are expressed as functions of $t$ and $N$ in Eq. (A1). One may further add terms constructed from $K_{ij}, R_{ij},$ and $\partial_t N/N$ in such way that the theory preserves the three-dimensional spatial covariance [49]. However, in order to avoid unwanted complexity, we focus on the theory (A1) as a possible extension of the Horndeski theory.

When the specific relations

$$
A_5 = -A_4, \quad A_7 = -3A_6, \quad A_8 = 2A_6, \quad B_6 = -\frac{1}{2}B_5 \tag{A2}
$$

hold among the functions, the Lagrangian (A1) reduces to that of the GLPV theory [48]. When the following two conditions

$$
A_4 = -B_4 - N \frac{\partial B_4}{\partial N}, \quad A_6 = \frac{N}{6} \frac{\partial B_3}{\partial N} \tag{A3}
$$

are satisfied in addition to (A2), the Horndeski theory is recovered.

Starting from the metric (6), one can derive the background equations by varying the action with respect to $N$ and $a$:

$$
-\mathcal{E} := (N A_2)' + 3N A_3' H + 3N^2 (N^{-1} a_1)' H^2 + 3N^3 (N^{-2} a_2)' H^3 + \frac{6K}{a^2} (N B_4)' + \frac{6K}{a^2} N (B_5' + 3B_6')H = 0, \tag{A4}
$$

$$
\mathcal{P} := A_2 - 3a_1' H^2 - 6a_2 H^3 - \frac{1}{N} \frac{d}{dt} (A_3 + 2a_1' H + 3a_2 H^2)
$$

\(^4\) Note that there is some debate about zero-crossing of $\Theta$ [37, 39–44]. Ilijas claimed that the unitary gauge, which is used also in this paper, is ill-defined at $\Theta = 0$ [41], while Mironov, Rubakov, and Volkova argued that the unitary gauge is well-defined there and the no-go theorem in the Horndeski theory still holds even if zero-crossing of $\Theta$ occurs [43], as is also argued in [45].
\[ \begin{align*}
+ \frac{2K}{a^2} B_4 - \frac{2K}{a^2} \frac{d}{N \, dt} \left( B_5 + 3B_6 \right) = 0, \quad (A5)
\end{align*} \]

where the prime stands for differentiation with respect to \( N \) and we defined
\[ a_1 := 3A_4 + A_5, \quad a_2 := 9A_6 + 3A_7 + A_8. \quad (A6) \]

Equations \((A4)\) and \((A5)\) can be used to determine the background evolution of \( a(t) \) and \( N(t) \) in the presence of the spatial curvature.

Let us now move on to deriving the quadratic actions for cosmological perturbations. The perturbed metric is given by
\[ ds^2 = -N^2 dt^2 + g_{ij}(N^i dt + dx^i)(N^j dt + dx^j), \quad (A7) \]

where
\[ g_{ij} = a^2 e^{2\zeta} \left( \gamma_{ij} + h_{ij} + \frac{1}{2} \psi^{kl} h_{ik} h_{lj} + \cdots \right), \quad (A8) \]
\[ N = \overline{N}(1 + \delta n), \quad N_i = \overline{N} \overline{D}_i \chi, \quad (A9) \]

and \( \overline{N} \) is the background value of the lapse function. To keep generality, we do not take \( \overline{N} = 1 \). Hereafter we will simply write the background value as \( N \).

We expand the action to quadratic order in perturbations. The quadratic action for the tensor perturbations is obtained as
\[ S_T^{(2)} = \frac{1}{8} \int \! dt \! dx \sqrt{g} a^3 \left[ G_T h_{ij}^2 + \frac{F_T}{a^2} h^{ij} (D^2 - 2K) h_{ij} \right], \quad (A10) \]

where
\[ G_T := 2(5A + 3A_7 H + 3A_8 H), \quad (A11) \]
\[ F_T := 2B_4 + 3B_5 H + \frac{1}{N} \frac{d B_3}{dt} + 6B_6 H. \quad (A12) \]

Similarly to the case of the Horndeski theory, the explicit dependence on \( K \) appears only in the spatial derivative operator \( D^2 - 2K \).

The quadratic action for scalar perturbations is given by
\[ S_S^{(2)} = \int \! dt \! dx \sqrt{g} L_S, \quad (A13) \]

with
\[ \frac{L_S}{Na^3} = -3G_A \frac{\dot{\zeta}^2}{N^2} - \frac{F_A}{a^2} \zeta (D^2 + 3K) \zeta + \Sigma \delta n^2 - 2 \Theta \delta n \frac{D^2 \chi}{a^2} + 2G_A \frac{\zeta}{N} \frac{D^2 \chi}{a^2} + 6 \Theta \delta n \frac{\dot{\zeta}}{N} - 2 \frac{G_B}{a^2} \delta n (D^2 + 3K) \zeta + \frac{G_T}{a^2} \frac{D^2 \chi}{a^2} + 2 \frac{G_T}{a^2} (D^2 + 3K) \zeta - \frac{C_A}{a^2} \frac{(D^2 \chi)^2}{a^4}, \quad (A14) \]

where
\[ \Sigma := NA'_2 + \frac{1}{2} N^2 A''_2 + \frac{3H}{2} N^2 A''_3 + \frac{3}{2} \left( 2a_1 - 2Na'_1 + N^2 a''_1 \right) H^2 + \frac{3}{2} \left( 6a_2 - 4Na'_2 + N^2 a''_2 \right) H^3 + \frac{3K}{a^2} (2NB'_4 + N^2 B''_4) + \frac{3K}{a^2} N^2 (B''_5 + 3B''_6) H, \quad (A15) \]
\[ \Theta := \frac{1}{2} NA'_3 - (a_1 - Na'_1) H - \frac{3}{2} (2a_2 - Na'_2) H^2 + \frac{K}{a^2} N(B' + 3B'_6), \quad (A16) \]
\[ G_A := -a_1 - 3Ha_2, \quad (A17) \]
\[ F_A := 2B_4 - \frac{2B_5}{N} - \frac{6B_6}{N}, \quad (A18) \]
\[ G_B := 2(B_4 + NB'_4) + 2(NB'_5 + 3NB'_6) H, \quad (A19) \]
\[ C_A := -A_4 - A_5 - (9A_6 + 5A_7 + 3A_8) H, \quad (A20) \]
\[ C_F := B_5 + 2B_6, \quad (A21) \]

and the relation \( G_T = G_A - 3C_A \) holds.

The Euler-Lagrange equations for \( \delta n \) and \( \chi \) give the following constraint equations:
\[ \Sigma \delta n - \Theta \frac{D^2 \chi}{a^2} + 3\Theta \frac{\dot{\zeta}}{N} - \frac{G_B}{a^2} (D^2 + 3K) \zeta = 0, \quad (A22) \]
\[ -\Theta \delta n + 2G_A \frac{\dot{\zeta}}{N} + G_T \frac{\zeta}{a^2} - C_A \frac{D^2 \chi}{a^2} \]
\[ + \frac{C_F}{a^2} (D^2 + 3K) \zeta = 0. \quad (A23) \]

Solving the above equations for \( \delta n \) and \( \chi \) and substituting the results back to \((A22)\), we get the reduced Lagrangian for the curvature perturbation,
\[ \frac{L}{Na^3} = G_S \frac{\dot{\zeta}^2}{N^2} + \zeta F_S \left( \frac{D^2 + 3K}{a^2} \right) \zeta - \zeta H_S \left( \frac{D^2 + 3K}{a^4} \right)^2 \zeta, \quad (A24) \]

where
\[ G_S := \hat{O}_1 \left( 3 + \frac{G_T \Sigma}{\Theta^2 + C_A \Sigma} \right) G_T, \quad (A25) \]
\[ F_S := \frac{1}{Na} \frac{d}{dt} \left[ \hat{O}_1 a^2 G_B G_T \Sigma \right. \]
\[ - \left. \hat{O}_2 a C_F \left( 3 + \frac{G_T \Sigma}{\Theta^2 + C_A \Sigma} \right) \right], \quad (A26) \]
\[ H_S := \frac{\hat{O}_2}{\Theta^2 + C_A \Sigma} \left( G_B C_A + 2G_B C_F \Theta - C_A^2 \right), \quad (A27) \]
with
\[ \hat{\dot{\Omega}}_1 := \frac{D^2 + 3K}{D^2 - G_T^2 \Sigma K/(\Theta^2 + C_A \Sigma)} \]  
(A28)
\[ \hat{\dot{\Omega}}_2 := \frac{D^2}{D^2 - G_T^2 \Sigma K/(\Theta^2 + C_A \Sigma)} \]  
(A29)

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