The definition of Neveu-Schwarz superconformal fields and uncharged superconformal transformations

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ABSTRACT

The construction of Neveu-Schwarz superconformal field theories for any $N$ is given via a superfield formalism. We also review some results and definitions of superconformal manifolds and we generalise contour integration and Taylor expansion to superconformal spaces. For arbitrary $N$ we define (uncharged) primary fields and give their infinitesimal change under superconformal transformations. This leads us to the operator product expansion of the stress-energy tensor with itself and with primary fields. In this way we derive the well-known commutation relations of the Neveu-Schwarz superconformal algebras $K_N$. In this context we observe that the central extension term disappears for $N \geq 4$ for the Neveu-Schwarz theories. Finally, we give the global transformation rules of primary fields under the action of the algebra generators.

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1 Introduction

The interplay of symmetries and conservation laws is one of the most intriguing features of physics and it can be found in many different areas of physics. Very often, physical systems for which one would have a priori thought that they have nothing in common, share in fact the same symmetry properties and thus have common physical properties. A recent example that illustrates this is local conformal symmetry, a concept which concerns various sectors of physics. The conformal symmetry group includes both Poincaré symmetry and scale invariance. The group of globally defined conformal transformations on the Riemann sphere are the well-known Möbius transformations, the transformations keeping angles invariant. In two dimensions, local conformal transformations are simply the locally holomorphic functions.

One of the most fascinating facts about statistical systems is the existence of special critical points where the systems become scale invariant and thus locally conformally invariant. Sweeping to an entirely different part of physics, string theory, we again find local conformal symmetry, here in two dimensions; after fixing the local symmetry of the string we are left with a conformally invariant field theory in two flat dimensions. Watching out for conformal invariance through physics we come across percolation systems, random walk models among many more still to be discovered.

Conformal invariance in a conformal field theory of two dimensions turns out to be particularly interesting since the algebra of symmetry generators becomes infinite dimensional. The algebra of generators of conformal transformations in two dimensions is given by the Virasoro algebra. This is an infinite dimensional Lie algebra with the commutation relations

\[
[L_m, L_n] = (m - n)L_{m+n} + \frac{C}{12}(m^3 - m)\delta_{m+n,0}, \\
[L_m, C] = 0, \quad m, n \in \mathbb{Z}.
\]

For a classical theory the central extension \(C\) would be trivial and therefore Eq. (1) would represent the de Witt algebra. Starting with the paper of Belavin, Polyakov and Zamolodchikov, many statistical models at their critical points have been identified as conformally invariant theories. In the canonical quantisation scheme, \(L_0\) generates time translations and hence represents the energy of the system. Since the energy is bounded below, the space of states of the physical system is confined to be a sum over highest weight representations of the algebra Eq. (1). A highest weight representation of the Virasoro algebra is a representation containing a vector \(|h, c\rangle\) such that

\[
L_0 |h, c\rangle = h |h, c\rangle, \quad C |h, c\rangle = c |h, c\rangle, \quad L_n |h, c\rangle = 0, \quad \forall n \in \mathbb{N}.
\]

|h, c\rangle is called a highest weight vector with conformal weight \(h\). In an irreducible representation of the Virasoro algebra the central extension operator \(C\) has a fixed value \(c \in \mathbb{C}\) since it commutes with the whole Virasoro algebra. Therefore it is common practice to omit \(c\) in the highest weight vector \(|h, c\rangle\) and consider it as a fixed constant. We construct the freely generated module \(V_{h, c}\) on a highest weight vector \(|h, c\rangle\) which is called the Verma module of \(|h, c\rangle\). A basis for \(V_{h, c}\) is given by

\[
B_{h,c} = \left\{ L_{-n_1}L_{-n_2}L_{-n_3} \ldots L_{-n_i} |h, c\rangle : n_i \geq \ldots \geq n_1, \ n_j \in \mathbb{N}, \ i \in \mathbb{N}_0 \right\}.
\]

The action of the Virasoro algebra on \(V_{h, c}\) is simply given by its commutation relations and the action on the highest weight state \(|h, c\rangle\). By defining the triangular decomposition of the Virasoro
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algebra, which we shall call the algebra \(K_0\), we can write \(V_{h,c}\) using tensor product notation:

\[
K_0 = K_0^- \oplus \mathcal{H}_0 \oplus K_0^+ ,
\]

\[
K_0^\pm = \text{span}\{L_{\pm n} : n \in \mathbb{N}\} , \quad \mathcal{H}_0 = \text{span}\{L_0, C\} ,
\]

\[
V_{h,c} = U(K_0) \otimes \mathcal{H}_0 \otimes \mathcal{K}_0 + \{h, c\} ,
\]

where \(U(K_0)\) denotes the universal enveloping algebra of the Virasoro algebra \(K_0\). The Verma module \(V_{h,c}\) decomposes into a direct sum of \(L_0\) grade spaces \(V^n_{h,c}\) with the basis

\[
B^n_{h,c} = \left\{ L_{-n_1} L_{-n_2} \cdots L_{-n_i} L_{-n_j} |h, c\rangle : n_1 \geq \ldots \geq n_j, \quad n_i + \ldots + n_j = n, \quad n_j \in \mathbb{N} \right\} ,
\]

where \(B^n_{h,c}\) is defined to be \(\{|h, c\rangle\}\). If \(R_{h,c}\) is an irreducible highest weight representation of the Virasoro algebra with highest weight \(h\) and fixed value \(c\) for \(C\), then there exists a homomorphism \(\phi_{h,c}\) from \(V_{h,c}\) onto \(R_{h,c}\). If \(V_{h,c}\) is reducible, then the kernel \(K_{h,c}\) of \(\phi_{h,c}\) is non-trivial. It can be shown that in this case \(K_{h,c}\) can also be decomposed in \(L_0\) grade spaces \(K^n_{h,c}\). If a vector lies in \(K_{h,c}\), it is obvious that all its descendant vectors, obtained by acting with Virasoro operators of negative index on the vector and taking linear combinations, also lie in \(K_{h,c}\). Hence, if \(K^n_{h,c}\) is non-trivial we find that the homomorphism \(\phi_{h,c}\) is trivial and we obtain the trivial representation. If \(R_{h,c}\) is not the trivial representation we have \(K^n_{h,c} = \{0\}\). Thus there exists a smallest index \(j \in \mathbb{N}\) such that \(K^j_{h,c}\) is non-trivial. If we take \(\psi \in K^j_{h,c}\) and \(L_m\) with positive index, then \(\phi_{h,c}(L_m\psi) = L_m\phi_{h,c}(\psi) = 0\) and hence due to the minimality of \(j\) we find \(L_m\psi = 0\). The vector \(\psi\) is not proportional to the highest weight vector \(|h, c\rangle\) but satisfies highest weight vector conditions with highest weight \(h + j\).

We call such a vector a singular vector in \(V_{h,c}\) at level \(j\): a vector \(\psi_n \in V_{h,c}\) is called singular vector at level \(n\) if

\[
L_0\psi_n = (h + n)\psi_n , \quad L_m\psi_n = 0 , \quad \forall m \in \mathbb{N} .
\]

For the Virasoro algebra it can be shown\(^{21}\) that any vector in the kernel \(K_{h,c}\) is either a descendant of a singular vector or is singular itself. For all algebras, a highest weight representation is irreducible if and only if there are no singular vectors in the representation. This is fundamental to understand the significance of the singular vectors. Furthermore, as proven by Feigin and Fuchs\(^{21}\), if we know in the Virasoro case the singular vectors in \(V_{h,c}\) we can construct the irreducible representation \(R_{h,c}\) by acting on \(V_{h,c}\) with a homomorphism whose kernel consists of the sum of the submodules spanned by the singular vectors and their descendants\(^{b}\). The structure of the highest weight representations of Eq. (1) are by now very well understood thanks to the combined effort of several authors\(^{5, 6, 8, 9, 21, 22, 25, 31, 36, 37}\).

So far, we have been looking at a theory describing a conformally invariant physical model at algebraic level. The underlying quantum field theory contains the quantum fields \(\Phi_h(z)\) which generate the energy eigenstates \(|h, c\rangle\) from the vacuum: \(|h, c\rangle = \Phi_h(z) |0\rangle_c\). Here we fixed again the central extension term: \(C|0\rangle_c = c |0\rangle_c\). These fields are called the primary fields. The field generating the conformal transformations and hence having the Virasoro generators as modes, is the stress-energy tensor \(T(z)\):

\[
T(z) = \sum_{n \in \mathbb{Z}} (z - w)^{-n-2} L_n(w) .
\]

\(^{b}\)For other algebras, after acting with the homomorphism which puts all singular vectors and their descendants equal to zero in the Verma module, new singular vectors may appear which were initially not singular in the Verma module. Such vectors are the so-called subsingular vectors of the Verma module.
As we explain later in the context of superconformal field theory, performing two conformal transformations and using contour integration methods allow us to compute the commutator of algebra generators and primary fields:

\[ [L_m(w), \Phi_h(z)] = [h(m + 1)(z - w)^m + (z - w)^{m+1} \frac{d}{dz}] \Phi_h(z). \tag{8} \]

We can always shift any point \( z \) to the origin since the group of translations is contained in the group of conformal transformations. Therefore we denote \( L_m(0) \) simply by \( L_m \) and we take the vector \( |h,c) \) as generated at the origin: \( |h,c) = \Phi_h(0) |0) \). We can give a complete set of fields of the conformally invariant theory by acting with the modes of \( T(z) \) on primary fields \( \Phi_h(w) \) to obtain the descendant fields of \( \Phi_h(w) \).

Singular vectors vanish in the physical theory. Therefore, correlators with singular vector operators inserted have to vanish. Using Eq. (8) one can thus obtain differential equations for the correlators of the theory by inserting singular operators. Hence, singular vectors together with Eq. (8) describe the dynamics of the physical model. For this reason we need to know not only singular vectors and thus irreducible representations, but also the action of the algebra generators on the primary field as this describes the dynamics. In this paper we will focus on the definition of primary fields in superspace using a superfield formalism. These are exactly the theories which are known as the Neveu-Schwarz theories.

In a physical model for elementary particles which has a Lie algebra as symmetry generators, the statistics of the particles are left unchanged under the action of this algebra. However, there is a common belief that a theory of everything should have a symmetry, transforming particles of different statistics into one another and hence providing a geometrical framework in which fermions and bosons receive a common treatment. Such a symmetry can be realised using a symmetry algebra which is \( \mathbb{Z}_2 \)-graded in the sense that some of their elements satisfy anticommutation relations rather than commutation relations and the underlying geometry can be provided by supermanifolds. These \( \mathbb{Z}_2 \)-graded algebras form Lie superalgebras. Motivated not only by string theory but also by two-dimensional statistical critical phenomena, the Lie superalgebra extensions of the Virasoro algebra became very attractive, as first suggested by Ademollo et al.\(^1\). At the same time Kac\(^{33}\) independently constructed several series of simple infinite-dimensional Lie superalgebras, among them superextensions of the Virasoro algebra. Since then, many applications for superconformal field theory were found, not only of theoretical interest. The tricritical Ising model, which can be realised experimentally\(^{48}\), was identified by Friedan, Qiu, and Shenker\(^{23}\) as a \( N = 1 \) superconformal model (we will reveal later the significance of the parameter \( N \) in that context). Moreover, the \( N = 2 \) superconformal models find applications in critical phenomena since under certain circumstances \( O(2) \) Gaussian models are \( N = 2 \) superconformally invariant\(^{49}\). There has recently been great interest in superconformal field theories because of their applications in superstring theory. The \( N = 2 \) superstring seems to be particularly interesting because of its connexion to quantum gravity\(^{41,43,44}\) and two-dimensional black holes. Furthermore it has been conjectured that the \( N = 2 \) string should give us insight into integrable systems\(^4\). Just recently, Kac has proven a complete classification of superconformal algebras\(^{35}\).

After setting up the necessary supergeometric framework in Sec. 2, we define in Sec. 3 the notion of superconformal transformations. In Sec. 4 and Sec. 5 we construct the foundation of superintegration and super Taylor expansions. This enables us to define in Sec. 6 superconformal field theories and to derive the well-known examples of \( N = 1 \) and \( N = 2 \) superconformal field theories in Sec. 7 and Sec. 8. The Hilbert space of states of a conformal field theory is created by the action of superconformal primary fields on the vacuum state and furthermore the action
of the whole superconformal algebra on these highest weight vectors. Therefore the action of the
algebra generators on the superconformal primary fields is of particular interest. We investigate
the global transformation properties of uncharged superconformal primary fields in Sec. 9. Like a
conformal field theory, a superconformal field theory consists of two chiral sectors having equivalent
representation theories. For this reason we will restrict our definition to one chiral sector only: the
holomorphic part. We therefore leave the antiholomorphic coordinates always unchanged and omit
them in the notation.

2 Supergeometry

As pointed out earlier, the ideas of having symmetry algebras which transform particles of different
statistics into one another requires the extension of Lie algebras by anticommuting objects. This
can be done by extending a Lie algebra to a \(\mathbb{Z}_2\)-graded algebra which defines the notion of a Lie
superalgebra. The theory of Lie superalgebras is well established in the mathematical literature
and we certainly do not want to rederive this here. The interested reader will find a vast amount
of literature on this topic among which we want to point out the paper by Kac and the book
by Scheunert. For our purpose we shall give a more simplified definition of a Lie superalgebra
starting already from an associative algebra:

**Definition 2.A Lie superalgebra**

Consider a \(\mathbb{Z}_2\)-graded associative algebra \(A = A_0 \oplus A_1\) with \(a_p b_q \in A_{p+q\ (\text{mod} \ 2)}\) for \(a_p \in A_p\), \(b_q \in A_q\)
and \(p, q \in \{0, 1\}\). We define the bilinear supercommutator by defining it for \(a_p \in A_p\) and \(b_q \in A_q\),
\(p, q \in \{0, 1\}\):

\[
[a_p, b_q]_S = a_p b_q - (-1)^{pq} b_q a_p .
\]

\(A\) is called Lie superalgebra. The elements of \(A_0\) are qualified as even and the ones of \(A_1\) as odd.

We have now defined what in a supertheory will play the rôle of the symmetry algebra. However,
in order to define a quantum field theory, we need to define the underlying manifold. The
concept of supermanifolds, extensions of differential manifolds, is well understood. Among other
references we shall point out the book by Manin. However, we want to achieve the extension
of Riemann surfaces, the underlying manifolds of conformal field theories. A subclass of these
so-called superconformal manifolds, also known as super-Riemann surfaces, were first studied by
Friedan. This was later generalised by Cohn. The definition we give here claims by no means
to be exhaustive but should rather be understood as an incentive.

In order to extend an ordinary quantum field theory to a super quantum field theory with
underlying supermanifold one would construct a fibre bundle of anticommutative rings over the
manifold of the model. As far as coordinates are concerned we obtain the ones of the manifold plus
anticommuting Grassmann variables arising due to the attached anticommutative rings.

**Definition 2.B Anticommutative ring**

An algebra \(R\) over the complex numbers \(\mathbb{C}\) is called anticommutative ring if it is \(\mathbb{Z}_2\)-graded \(R = R_0 \oplus R_1\) such that \(a_p b_q \in R_{p+q\ (\text{mod} \ 2)}\) for \(a_p \in R_p\) and \(b_q \in R_q\) where \(p, q \in \{0, 1\}\). Moreover the
bilinear supercommutator is trivial:

\[
[a_p, b_q]_S = a_p b_q - (-1)^{pq} b_q a_p = 0 ,
\]
In a conformal field theory the derivatives transform covariantly obtaining a prefactor only:

\[ \theta_1 \theta_2 = -\theta_2 \theta_1 , \quad \theta_1, \theta_2 \in R_\theta , \]

and since they generate \( R \), we can write \( R \) as the ring of polynomials over \( \mathbb{C} \) generated by \( R_\theta \):

\[ R = \mathbb{C}[R_\theta] \]  

The number of anticommutative rings we tensor together in the fibres has to be the same for the whole manifold. It is called the classification parameter \( N \) of the supermanifold. The theories we aim to construct are based on the manifolds of conformal field theories, more precisely on Riemann surfaces having the complex coordinate \( z \). To construct the super extension we take \( N \) anticommutative rings in its fibres. We obtain the set of coordinates \((z, \theta_1, \ldots, \theta_N)\) and we then extend the complex differential structure by anticommuting derivatives \( \frac{\partial}{\partial \theta_i} \):

\[ \frac{\partial}{\partial \theta_i} \frac{\partial}{\partial \theta_j} = - \frac{\partial}{\partial \theta_j} \frac{\partial}{\partial \theta_i} , \quad \frac{\partial}{\partial \theta_i} \theta_j = \delta_{i,j} - \theta_j \frac{\partial}{\partial \theta_i} \]  

Finally, we define the superderivatives \( D_i = \frac{\partial}{\partial \theta_i} + \theta_i \frac{\partial}{\partial z} \) for \( i = 1, \ldots, N \). The superderivatives are the square roots of the complex derivative \( \bar{D}_i = D_i^2 \), and therefore they describe exactly the fermionic structure we expected. Performing a coordinate transformation from \((z, \theta_1, \ldots, \theta_N)\) to \((\bar{z}, \theta_1, \ldots, \theta_N)\) the superderivatives transform as follows\(^d\):

\[ D_i = (D_i \bar{\theta}_j) \bar{D}_j + (D_i \bar{z} - \bar{\theta}_j D_i \bar{\theta}_j) \bar{\theta}_j . \] 

In a conformal field theory the derivatives transform covariantly obtaining a prefactor only: \( \frac{\partial}{\partial \zeta} = \frac{\partial}{\partial z} \). We apply the analogue statement for the square roots \( D_i \) of \( \bar{D}_i \) to define a superconformal field theory. Among all supertransformations we pick those which transform the superderivatives by a scaling factor only and which are therefore conformal in both the even and the odd variables. Hence, we require the inhomogeneous part in Eq. (11) to vanish:

**Definition 2.C Superconformal transformations**

A transformation from \((z, \theta_1, \ldots, \theta_N)\) to \((\bar{z}, \bar{\theta}_1, \ldots, \bar{\theta}_N)\) is called superconformal if

\[ D_i = (D_i \bar{\theta}_j) \bar{D}_j , \quad 1 \leq i \leq N \]  

We conclude this section by defining the underlying manifold of a superconformal field theory:

**Definition 2.D Superconformal manifold**

A superconformal manifold \( S^N \) of classification parameter \( N \) is a fibre bundle of \( N \) anticommutative rings over a one-dimensional complex manifold where the transition functions are superconformal transformations. The coordinates shall be called superpoints \( Z = (z, \theta_1, \ldots, \theta_N) \).

The space of functions \( \mathcal{F}_N \) defined on a superconformal manifold consists of functions \( f(Z) = f^0(z) + \theta_1 f_1(z) + \ldots + \theta_N f_N(z) \) where the functions \( f^0(z), \ldots, f^N(z) \) are complex functions evolving according to superconformal transformations.

\(^d\)Including the zero-power product, i.e. the identity.

\(^d\)The usual summation convention applies.
3 Superconformal transformations

Following the original approach of Kac\textsuperscript{33} we define the differential form $\kappa = dz + \theta_i d\theta_i$. Superconformal transformations are the only transformations under which $\kappa$ will simply be scaled. We find the following equivalences:

Theorem 3.1

$$D_i = (D_i \bar{\theta}_j) \bar{D}_j \ , \forall i \iff D_j \bar{z} = \bar{\theta}_i D_j \bar{\theta}_i \ , \forall j \iff \bar{\kappa} = p \kappa \ ,$$

where the prefactor $p(z, \theta_1, \ldots, \theta_N)$ is given by

$$p = \frac{\partial \bar{z}}{\partial z} + \bar{\theta}_i \frac{\partial \bar{\theta}_i}{\partial z} .$$

Proof: The transformation of $\kappa$ is

$$\bar{\kappa} = \left( \frac{\partial \bar{z}}{\partial z} + \bar{\theta}_i \frac{\partial \bar{\theta}_i}{\partial z} \right) dz + \left( -\frac{\partial \bar{z}}{\partial \theta_j} + \bar{\theta}_i \frac{\partial \bar{\theta}_j}{\partial \theta_j} \right) d\theta_j .$$

$\bar{\kappa} = p \kappa$ implies $(\frac{\partial \bar{z}}{\partial z} + \bar{\theta}_i \frac{\partial \bar{\theta}_i}{\partial z}) \theta_j = \left( -\frac{\partial \bar{z}}{\partial \theta_j} + \bar{\theta}_i \frac{\partial \bar{\theta}_j}{\partial \theta_j} \right)$ which is consequently equivalent to $D_j \bar{z} = \bar{\theta}_i D_j \bar{\theta}_i$. Thus the scaling factor $p$ can be found in Eq. (14).

Hence, finding the generators of superconformal transformations is equivalent to finding the superderivatives acting on the space of differential forms $\mathcal{D} = \mathbb{C}[z, z^{-1}] \otimes \mathbb{C} \otimes \mathbb{C}[d z, \theta_1, \ldots, \theta_N]$ and leaving $\kappa \in \mathcal{D}$ invariant up to a scalar multiple, i.e. $\bar{\kappa} = p \kappa$ for some $p$ depending on the coordinates. For our further considerations we present the result of Kac\textsuperscript{33} in the form recently given by Bremner\textsuperscript{12}. One takes elements $i_1, \ldots, i_I \in \{1, \ldots, N\}$ which form the sequence $S = (i_1, \ldots, i_I)$. In addition one defines the complement of $S$ as a set $\bar{S} = \{1, \ldots, N\} \setminus S$ and finally constructs operators labeled by a sequence $S$ and an index $a$ which is taken from $\mathbb{Z}$ if the number of elements in $S$ is even or otherwise $a$ is taken from $\mathbb{Z}_+$.

$$X_a(i_1, \ldots, i_I) = (1 - \frac{I}{2}) z^{-\frac{I}{2} + 1} \theta_{i_1} \ldots \theta_{i_I} \partial z + \frac{1}{2} \sum_{p=1}^{I} (-1)^p z^{-\frac{I}{2} + 1} \theta_{i_1} \ldots \hat{\theta}_{i_p} \ldots \theta_{i_I} \partial \theta_{i_p}$$

$$+ \frac{1}{2} \left( a - \frac{I}{2} + 1 \right) \sum_{k \in \bar{S}} z^{-\frac{I}{2} + 1} \theta_{i_1} \ldots \theta_{i_I} \theta_k \partial \theta_k ,$$

where $\hat{\theta}_{i_p}$ signifies that $\theta_{i_p}$ is omitted in the product. $X_a(i_1, \ldots, i_I)$ is defined to be even if $a \in \mathbb{Z}$, otherwise it is qualified as odd. A basis for the space of operators leaving $\kappa$ invariant up to a scalar multiple is given by the set of $X_a(i_1, \ldots, i_I)$ with $1 \leq i_1 < \ldots < i_I \leq N$. The operators (15) satisfy the supercommutation relations

$$[X_a(i_1, \ldots, i_I), X_b(j_1, \ldots, j_J)]_S = \sum_{p=1}^{I} \sum_{q=1}^{J} \frac{(-1)^p q \delta_{i_p, j_q}}{2} X_{a+b}(i_1, \ldots, \hat{i}_p, \ldots, i_I, j_1, \ldots, \hat{j}_q, \ldots, j_J)$$

$$+ \left[ (1 - \frac{I}{2}) b - (1 - \frac{J}{2}) a \right] X_{a+b}(i_1, \ldots, i_I, j_1, \ldots, j_J) ,$$

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which are closed in the set of basis elements by reordering the union of the sequences \((i_1, \ldots, i_I)\) and \((j_1, \ldots, j_J)\). Moreover for the transformation of \(\kappa\) one can find the scaling factor:

\[
\bar{\kappa} = X_a(i_1, \ldots, i_I)\kappa = (a - \frac{I}{2} + 1)z^{a - \frac{I}{2}}\theta_{i_1} \ldots \theta_{i_I}\kappa. \tag{17}
\]

The result by Kac gives the symmetry generators of a classical superconformally invariant field theory. In particular for \(N = 0\) we obtain a conformally invariant classical model having the de Witt algebra as symmetry algebra. It contains the operators \(X_a = z^a \partial_z\) where \(a \in \mathbb{Z}\), satisfying the commutation relations

\[
[X_a, X_b] = (b - a)X_{a+b}. \tag{18}
\]

It is a feature of infinite dimensional Lie algebras to allow in the quantised theory central terms which extend the classical symmetry algebra but do not change the infinitesimal transformations of tensors. In the case of \(N = 0\) the allowed central term leads us to the Virasoro algebra. Later we shall find that for the superconformal theories we have as well at most one central term which, however, disappears completely if \(N\) is at least equal to 4.

4 Superconformal integration

4.1 Superdifferentials

We have found the superconformal analogues \(D_i\) of the conformal derivatives \(\frac{d}{dz}\). In order to find the analogue of the radial quantisation procedure used for two-dimensional conformal field theories, we have to develop the corresponding integration and contour integration techniques. In order to do so we need to define differentials \(dZ_j\) as duals of \(D_i\).

**Definition 4.A Superdifferentials**

We define the differentials \(dZ_i\), \(i = 1, \ldots, N\), as the dual elements of the superderivatives \(D_i\):

\[
D_i dZ_j = \delta_{i,j}, \quad i,j \in \{1, \ldots, N\}. \tag{19}
\]

If we are given a superconformal transformation \(\bar{\theta}_1(z, \theta_1, \ldots, \theta_N), \ldots, \bar{\theta}_N(z, \theta_1, \ldots, \theta_N)\) then the matrix of superderivatives will contain all the information of directional derivatives and hence be crucial to define integration.

**Definition 4.B Super Jacobi matrix**

We define the matrix of superderivatives

\[
D\bar{\theta} = \begin{pmatrix}
D_1\bar{\theta}_1 & \cdots & D_N\bar{\theta}_1 \\
\vdots & \ddots & \vdots \\
D_1\bar{\theta}_N & \cdots & D_N\bar{\theta}_N
\end{pmatrix}, \tag{20}
\]

which we can write in index notation as \((D\bar{\theta})_{i,j} = D_j\bar{\theta}_i.\)

\(^a\)Note that \(X_a(i_1, \ldots, i_I)\) is trivial if \((i_1, \ldots, i_I)\) contains the same element twice or otherwise it is proportional to the basis element \(X_a(i'_1, \ldots, i'_I)\) with \((i'_1, \ldots, i'_I)\) being \((i_1, \ldots, i_I)\) reordered appropriately.

\(^b\)Due to Eqs. (13) the derivatives of \(\bar{z}\) are determined by \(\bar{\theta}_1, \ldots, \bar{\theta}_N\) and hence contain no linearly independent information.
As the first fruitful result of these definitions we can check easily that the usual chain rule holds:

$$D\bar{\theta} = (\bar{D}\theta)(\bar{D}\theta) .$$

(21)

We thus obtain the transformation rule for the vector of differentials $dZ = (dZ_1, \ldots, dZ_N)^T$ under superconformal transformations:

$$d\bar{Z} = D\theta dZ .$$

(22)

### 4.2 Riemann superintegrals

We say $F(Z)$ is a $Z_i$-integral of $f(Z)$ if $D_i F(Z) = f(Z)$. In symbols we write

$$F(Z) = \int dZ_i f(Z) .$$

(23)

Obviously, $D_i F(Z) = 0$ with $F(Z) = F^0(z) + \theta_j F_j^1(z) + \ldots + \theta_1 \theta_2 \ldots \theta_N F^N(z)$ and fixed $i$ implies that the only possible non-trivial components of $F_i(Z)$ are those which are not a coefficient of the coordinate $\theta_i$. Furthermore, their derivative $\frac{\partial}{\partial z}$ has to vanish and they are therefore constant. Thus, $F(Z)$ is constant in $\theta_i$ direction. This implies automatically that due to the linearity of the differential operator $D_i$ two $Z_i$-integrals of $f(Z)$ differ at most by a factor which is constant if we fix $\theta_j$ for $j = 1, \ldots, N, j \neq i$. Defining integration over the superdifferentials $dZ_i$ using a generalisation of the fundamental theorem of calculus is therefore justified.

**Definition 4.3 Riemann superintegrals**

We define the Riemann superintegral of a function $f(Z)$ as

$$\int_{Z_1}^{Z_2} dZ_i f(Z) = F(Z_2) - F(Z_1) ,$$

(24)

where $F(Z)$ is a $Z_i$-integral of $f(Z)$, $i = 1, \ldots, N$, and the superpoints $Z_1$ and $Z_2$ coincide with the possible exceptions of their $z$ and $\theta_i$ coordinates.

In the view of further applications we define for two superpoints a $Z_i$-integral $Z_i = (z_i, \theta_{i,1}, \ldots, \theta_{i,N})$, $i \in \{1, 2\}$, the differences $Z_{12} = z_1 - z_2 - \theta_{1,j} \theta_{2,j}$ and $\theta_{12,j} = \theta_{1,j} - \theta_{2,j}$. The importance of these superdifferences lies in $D_{2,j} Z_{12}^n = n \theta_{12,j} Z_{12}^{n-1}$ and $D_{2,(i)} \theta_{12,(i)} Z_{12}^n = -Z_{12}^n$. This means that they are the successive $Z_i$-integrals of 1:

$$\int_{Z_1}^{Z_2} dZ_{3,i} Z_{13}^n = -\theta_{12,i} Z_{12}^n ,$$

(25)

$$\int_{Z_1}^{Z_2} dZ_{3,(i)} \theta_{13,(i)} Z_{13}^n = \frac{1}{n + 1} Z_{12}^{n+1} .$$

(26)

Moreover, we now define integrals over a volume of the superconformal space:

---

*aOne should not confuse the component index with the label index of superpoints, superderivatives and superdifferentials.

*b$(i)$ denotes no summation convention applies to the index $i$. 

Definition 4.D  Superconformal integral over a volume
We take $f(Z) \in \mathcal{F}_N$ and let $V$ be a volume in the superconformal space $\mathcal{S}_N$. We then define the superconformal integral over volume $V$:
\[
\int_V dZ f(Z) = \int_V dZ_1 \ldots dZ_N f(Z) .
\tag{27}
\]

The chain rule (22) leads us to the substitution rule for integrals of the form (27). We take a scalar function $f(Z)$ of the superpoint $Z$ and perform a superconformal transformation $Z \mapsto \bar{Z}$:
\[
\int_V dZ f(Z) = \int_{\bar{V}} d\bar{Z} f(\bar{Z}) \det D\bar{\theta} .
\tag{28}
\]

4.3 Supercontour integrals

Finally we want to define contour integrals on the superconformal manifold $\mathcal{S}_N$. Whilst for Riemann superintegrals we aimed to find a function which has a given function as derivative and satisfies certain boundary conditions, for contour integrals we are more interested in defining an extension which is translation invariant and linear just like ordinary contour integrals. These two properties fix the contour integrals already up to a scalar factor. We follow the standard approach to define contour integration over Grassmann variables.

Definition 4.E  We define contour integrals over $\theta_j$ as
\[
\oint_{\mathcal{C}_0} d\theta_i \theta_j = \delta_{i,j} ,
\tag{29}
\]
\[
\oint_{\mathcal{C}_0} d\theta_i 1 = 0 ,
\]
where $\mathcal{C}_0$ is a supercontour about the origin.

These simple integration rules have the effect that for a function $f(Z) \in \mathcal{F}_N$ the only contributing term towards the contour integral is $f^N(z)$, due to:
\[
\oint_{\mathcal{C}_0} d\theta_1 \ldots d\theta_N \theta_1 \ldots \theta_N = (-1)^{N(N-1)} ,
\tag{30}
\]
and the integral vanishes whenever some of the $\theta_i$ are missing in the product $\theta_1 \ldots \theta_N$. This leads to the definition:

Definition 4.F  Supercontour integrals

For the function $f(Z) = f^0(z) + \theta_i f^1_i(z) + \ldots + \theta_1 \ldots \theta_N f^N(z) \in \mathcal{F}_N$ we define the integral along the supercontour $\mathcal{C}$ as
\[
\oint_{\mathcal{C}} dZ f(Z) = \oint_{\mathcal{C}} dz d\theta_1 \ldots d\theta_N f(Z)
= \epsilon^N \oint_{\mathcal{C}'} dz f^N(z) ,
\tag{31}
\]
where $\epsilon^N = (-1)^{N(N-1)}$, and $\mathcal{C}'$ is the projection of the supercontour $\mathcal{C}$ into the underlying Riemann surface.

\*Note that we do not consider any superdeterminants. $D\bar{\theta}$ is merely an even object.
5 Super Taylor expansion

We already know from the definition of a function on a superconformal space how it can be expanded in a power series about the origin. In this section we derive an expansion about non-trivial superpoints. We thus obtain an expansion in terms of the superdifferences $Z_{12}$ and $\theta_{12,j}$.

**Theorem 5.1 Super Taylor expansion**

The super Taylor expansion of $f(Z) \in \mathcal{F}_N$ is given by

$$f(Z_1) = \sum_{n=0}^{\infty} \frac{1}{n!} Z_{12}^n \partial_z^n \prod_{j=1}^{N} (1 + \theta_{12,j} D_{2,j}) f(Z_2) .$$

Proof: We first consider the case $j = 1$: $Z_i = (z_i, \theta_1, \ldots, \theta_{N})$, $i \in \{1, 2\}$.

$$f(Z_2) = f(Z_1) + \int_{Z_1}^{Z_2} dZ Df(Z)$$

$$= f(Z_1) + (\int_{Z_1}^{Z} dZ') Df(Z) \bigg|_{Z_1}^{Z_2} + \int_{Z_1}^{Z_2} dZ (\int_{Z_1}^{Z} dZ') D^2 f(Z)$$

$$= f(Z_1) + (\theta - \theta_1) Df(Z) \bigg|_{Z_1}^{Z_2} + \int_{Z_1}^{Z_2} dZ (\theta - \theta_1) D^2 f(Z)$$

$$= f(Z_1) - \theta_{12} D_{2} f(Z_2) + (\int_{Z_1}^{Z} dZ' (\theta' - \theta_1)) D^2 f(Z) \bigg|_{Z_1}^{Z_2}$$

$$- \int_{Z_1}^{Z_2} dZ (z - z_1 - \theta \theta_1) D^3 f(Z)$$

$$\vdots$$

$$= f(Z_1) - \theta_{12} D_{2} f(Z_2) - Z_{12} D_{2}^2 f(Z_2) - \frac{1}{2} \theta_{12} Z_{12} D_{2}^3 f(Z_2)$$

$$- \frac{1}{2} Z_{12}^2 D_{2}^4 f(Z_2) - \frac{1}{3!} Z_{12}^2 \theta_{12} D_{2}^5 f(Z_2) - \ldots .$$

Using Eqs. (25) and (26) we can easily prove by induction:

$$f(z_1, \theta_1) = \sum_{n=0}^{\infty} \frac{1}{n!} Z_{12}^n (1 + \theta_{12} D_{2}) D_{2}^n f(Z_2)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} Z_{12}^n \partial_{z_1}^n (1 + \theta_{12} D_{2}) f(Z_2) .$$

(33)

For the general case we define the sequence of superpoints:

$$Z_1^0 = (z_1, \theta_{1,1}, \theta_{1,2}, \ldots, \theta_{1,N}) ,$$

Note that for odd functions $f$ we can integrate by parts by altering the signs: $\int dZ [Df(Z)]g(Z) = f(Z)g(Z) + \int dZ f(Z) [Dg(Z)]$. If $f(Z)$ is even we can integrate by parts as usual.
We apply Eq. (33) which was found for \( N = 1 \):

\[
f(Z_1^0) = \sum_{n=0}^{\infty} \frac{1}{n!} (Z_{12}^1)^n \partial_{z_2}^n (1 + \theta_{12,1} D_{2,1}) f(Z_1^1)
\]

\[
= \sum_{n=0}^{\infty} \frac{1}{n!} (Z_{12}^1)^n \partial_{z_2}^n (1 + \theta_{12,1} D_{2,1}) \sum_{m=0}^{\infty} \frac{1}{m!} (Z_{12}^2)^m \partial_{z_2}^m (1 + \theta_{12,2} D_{2,2}) f(Z_1^2),
\]

where \( Z_{12}^1 = z_1 - z_2 - \theta_{1,1} \theta_{2,1}, \quad \theta_{12,i} = \theta_{1,i} - \theta_{2,i}, \quad Z_{12}^2 = -\theta_{1,2} \theta_{2,2}. \) Using this last expression we obtain \( (Z_{12}^2)^m = 0 \forall m \geq 2. \) This leads to:

\[
f(Z_1) = \sum_{n=0}^{\infty} \frac{1}{n!} (Z_{12}^1)^n \partial_{z_2}^n (1 + Z_{12}^2 \partial_{z_2}) (1 + \theta_{12,2} D_{2,2}) f(Z_1^2)
\]

\[
\Rightarrow f(Z_1) = \sum_{n=0}^{\infty} \frac{1}{n!} [(Z_{12}^1)^n \partial_{z_2}^n + (Z_{12}^1)^n Z_{12}^2 \partial_{z_2}^{n+1}] (1 + \theta_{12,2} D_{2,2}) f(Z_1^2)
\]

\[
= \sum_{n=0}^{\infty} \frac{1}{n!} [(Z_{12}^1)^n + n(Z_{12}^1)^{n-1} Z_{12}^2 \partial_{z_2}^n] (1 + \theta_{12,2} D_{2,2}) f(Z_1^2).
\]

Repeated application of this step until we reach \( Z_2 = Z_1^N \) completes the proof. 

We conclude this subsection with the main theorem of supercontour integration techniques: the Cauchy formulae. These formulae will be the essential tools to evaluate commutation relations in superconformal field theories.

**Theorem 5.8 Cauchy formulae**

\[
\frac{1}{2\pi i} \oint_{C_2} dZ_1 Z_{12}^{n-1} f(Z_1) = \frac{1}{n!} \partial_{z_2}^n D_{2,1} \cdots D_{2,N} f(Z_2),
\]

\[
\frac{1}{2\pi i} \oint_{C_2} dZ_1 \theta_{12,i_1} \cdots \theta_{12,i_k} Z_{12}^{n-1} f(Z_1) = \frac{(-1)^N n! (-1)^k}{n!} \epsilon_{i_1, \ldots, i_N} \epsilon_{j_1, \ldots, j_N} \partial_{z_2}^n D_{2,(j_1)} \cdots D_{2,(j_N)} f(Z_2),
\]

where the \( j \)'s are taken out of the complement of the \( i \)'s in \( 1, \ldots, N \) in increasing order\(^4\). \( \epsilon_{i_1, \ldots, i_N} \) denotes the totally antisymmetric tensor with \( \epsilon_{1, \ldots, N} = 1 \). Two particular cases of the last equation

\(^4\)Note that the function \( f(Z_1) \) has to be on the right of the integral.
are:
\[
\frac{1}{2\pi i} \oint_{C_2} dZ_1 \theta_{12,1} \ldots \theta_{12,i} \ldots \theta_{12,N} Z_{12}^{-n-1} f(Z_1) = \frac{1}{n!} (-1)^{N-l} \epsilon^N \partial_{z_2}^n D_{2,l} f(Z_2),
\]
\[
\frac{1}{2\pi i} \oint_{C_2} dZ_1 \theta_{12,1} \ldots \theta_{12,N} Z_{12}^{-n-1} f(Z_1) = \frac{1}{n!} \epsilon^N \partial_{\bar{z}_2}^n f(Z_2).
\]

Proof: Using the definition 4.4 it is easy to show that
\[
\frac{1}{2\pi i} \oint_{C_2} dZ_1 \theta_{12,1} \ldots \theta_{12,N} Z_{12}^{-n-1} = \epsilon^N \delta_{n,0},
\]
and the integral vanishes whenever the product $\theta_{12,1} \ldots \theta_{12,N}$ is not complete. We apply the super Taylor expansion [Eq. (32)] to the integral
\[
\frac{1}{2\pi i} \oint_{C_2} dZ_1 \theta_{12,i_1} \ldots \theta_{12,i_k} f(Z_1) Z_{12}^{-n-1},
\]
where we expand $f(Z_1)$ about $Z_2$. The only contributions can arise from the term leading to a complete product of $\theta_{12,1} \ldots \theta_{12,N}$. Hence, for each $\theta_{12,j}$ missing in Eq. (35) we introduce a derivative $D_{2,j}$ acting on $f(Z_2)$. Finally, we use the usual Cauchy formulae for contour integrals in the complex plane to obtain $\partial_{\bar{z}_2}^n f(Z_2)$.

\section{Superconformal field theory}

So far, we have defined the underlying geometry of the quantum field theory which we want to construct. The theory is meant to be superconformally invariant. Hence, we have a chiral stress-energy tensor $T(Z)$ generating the local symmetry group of superconformal transformations. The main objects in a conformal field theory are the primary fields, fields which play the role of the tensors. This means that they form conformally invariant differential forms. Exactly in the same way we define the (uncharged) superprimary fields.

**Definition 6.4** Uncharged superprimary fields

Fields $\Phi_h(Z)$ defined on a superconformal manifold $\mathcal{S}_N$ transforming under a superconformal transformation $Z \mapsto \tilde{Z}$ as

\[
\Phi_h(Z) = \Phi_h(\tilde{Z})(\text{det} D\tilde{\theta})^{\frac{2h}{N}}
\]

are called (uncharged) superprimary fields. The complex number $h$ is called the (super)conformal weight of $\Phi_h(Z)$. This definition is chosen in such a way that the differential form $[\Phi_h(Z)]^{\frac{N}{2h}} dZ$ is invariant.

We perform an infinitesimal superconformal transformation $z \mapsto \tilde{z} = z + \delta z$ and $\theta_i \mapsto \tilde{\theta}_i = \theta_i + \delta \theta_i$ for $i = 1, \ldots, N$. We define $\delta \Phi_h(Z)$ by $\Phi_h(Z) = \Phi_h(\tilde{Z}) + \delta \Phi_h$. Hence using Eq. (36) we obtain $\delta \Phi_h = \Phi_h(\tilde{Z}) - \Phi_h(Z)(\text{det} D\tilde{\theta})^{-\frac{2h}{N}}$. Since $(\text{det} D\tilde{\theta})^2 = \text{det} D\theta (D\bar{\theta})^T$ we calculate the variation $\delta D$ defined as $D\bar{\theta}(D\theta)^T = 1 + \delta D$. For the trivial variation we have $\delta D = 0$. Hence the variation of the determinant can be found as $\delta (\text{det} D\theta)^2 = \text{tr} (\delta D)$. For the $i$-th diagonal element $(D\tilde{\theta}_{(i)})(D\tilde{\theta}_{(i)})$ we obtain a variation of $2D_{(i)} \delta \theta_{(i)}$. Thus $\delta (\text{det} D\theta)^2 = 2D_i \delta \theta_i$. We use the Taylor
Neveu-Schwarz superconformal fields and uncharged superconformal transformations

...the infinitesimal version of the differential $\kappa$. Taking these results together we reach $\delta \Phi_h = [\delta \theta_i D_i + \nu(Z) \partial_z + \frac{2\pi i}{L} D_i \delta \theta_i] \Phi_h(Z)$. Finally, we want to write the variations $\delta \theta_i$ in terms of derivatives, for which we use the definition of superconformal transformations (12). Neglecting higher order terms in $D_j \bar{z} = \bar{\theta}_i D_j \bar{\theta}_i$ leads to $\delta \theta_i = D_j \delta z - \bar{\theta}_i D_j \delta \theta_i$ for $i = 1, \ldots, N$. We replace then $\delta z$ by $\nu(Z)$: $\delta \theta_i = \frac{i}{2} D_j \nu(Z)$ and $D_j \delta \theta_i = \frac{\pi i}{2} \partial_z \nu(Z)$. We can thus give the infinitesimal transformation of $\Phi_h(Z)$ under infinitesimal superconformal transformations:

**Theorem 6.B** Under an infinitesimal superconformal transformation $z \mapsto \bar{z} = z + \delta z$, $\theta_i \mapsto \bar{\theta}_i = \theta_i + \delta \theta_i$, the change of an (uncharged) superprimary field is given by

$$\delta \Phi_h(Z) = \frac{1}{2} (D_j \nu(Z)) D_j \Phi_h(Z) + \nu(Z) \partial_z \Phi_h(Z) + h(\partial_z \nu(Z)) \Phi_h(Z),$$

where $\nu(Z)$ is the infinitesimal version of $\kappa$: $\nu(Z) = \delta z + \theta_i \delta \theta_i$ and it corresponds to the superdifference: $\bar{z} - z - \bar{\theta}_i \theta_i = \nu(Z)$.

The field theory we constructed so far contains the stress-energy tensor, the superprimary fields and all the descendant fields obtained from the superprimary fields by applying superconformal transformations, that is acting with modes of $T(Z)$ on them. Altogether this forms the closed set $\Phi_N$ of fields contained in the theory. The radial quantisation procedure defines the meromorphic function $\phi_1(Z_1) \phi_2(Z_2)$ of two fields in $\Phi_N$, which is meant to be understood inside correlation functions such as $\langle 0 | \phi_1(Z_1) \phi_2(Z_2) | 0 \rangle$ for time-ordered points $Z_1$ and $Z_2$: $|z_1| > |z_2|$. For $|z_2| > |z_1|$ we define $\phi_1(Z_1) \phi_2(Z_2)$ to be its analytic continuation. $\phi_1(Z_1) \phi_2(Z_2)$ is called the operator product of $\phi_1(Z_1)$ and $\phi_2(Z_2)$. In these terms integrals over equal time commutators become contour integrals which we extended to supercontour integrals.

We assumed that the superprimary fields and its descendants form the complete set of fields $\Phi_N$ for the theory. Hence, the function $\Phi_{h_1}(Z_1) \Phi_{h_2}(Z_2)$ can be expanded about the superpoint $Z_j$ where the expansion coefficients are superprimary fields or descendants of superprimary fields. This expansion is called the operator product expansion (OPE) of the fields $\Phi_{h_1}(Z_1)$ and $\Phi_{h_2}(Z_2)$.

In a radially quantised theory the time-ordered Euclidean symmetry generator generating the infinitesimal transformation $z \mapsto z + \delta z$ and $\theta_i \mapsto \bar{\theta}_i + \delta \theta_i$ becomes the contour integral $\frac{1}{2\pi i} \oint dZ \nu(Z) T(Z)$. Here we have chosen $T(Z)$ in such a way that $\frac{1}{2\pi i} \oint dZ \theta_i T(Z)$ of $\theta_i$. We use the infinitesimal transformation (37) of the superprimary field $\Phi_h(Z)$ with conformal weight $h$ in order to determine the singular terms of the OPE $T(Z_1) \Phi_h(Z_2)$:

$$\delta \nu \Phi_h(W) = \frac{1}{2\pi i} \oint_{C_W} dZ \nu(Z) T(Z) \Phi_h(W)$$

$$\Rightarrow T(Z_1) \Phi_h(Z_2) = \frac{h \pi N}{Z_{12}^2} \Phi_h(Z_2) + \left( \frac{1}{2} \frac{\Delta N}{Z_{12}^2} D_{2,j} \Phi_h(Z_2) + \frac{\pi N}{Z_{12}^2} \partial_{z_2} \Phi_h(Z_2) + \ldots (\text{reg}) \right),$$

$$\pi N_{12} = \epsilon^N \theta_{12,1} \ldots \theta_{12,N}, \quad \Delta_{12,j} = \epsilon^N (-1)^{N-j} \bar{\theta}_{12,j} \ldots \bar{\theta}_{12,N}.\;$$

Here $\ldots (\text{reg})$ indicates non-singular terms.
We can determine most of the singular terms of the operator product $T(Z_1)T(Z_2)$ by performing two successive superconformal transformations $\delta_{i_{a2}}\delta_{b_{11}}\Phi_h$ and evaluating the double supercontour integral in two different ways using contour deformation. However, the singular terms are fixed except for a term of the form $\frac{1}{Z_{12}^{2-N}}$; and this is the only degree of freedom we can find. This term is called the \textit{central extension} term. It does not contribute towards the infinitesimal change of $\Phi_h(W)$ and may therefore be contained in the quantum field theory:

$$T(Z_1)T(Z_2) = \frac{\hat{C}}{Z_{12}^{2-N}} + \frac{4-N}{2} \frac{\pi_1^N Z_{12}^2 T(Z_2)}{Z_{12}^2} + \frac{\Delta_{i_{a2}}^N D_{2,j}^j T(Z_2)}{2 Z_{12}^2} + \frac{\pi_1^N}{Z_{12}^2} \partial_{z_2} T(Z_2) + \ldots (\text{reg}) \quad (40)$$

It is worth remarking that Eq. (40) does not allow a central extension for theories with $N \geq 4$ because the central extension term does not belong to a singularity any more and hence will not contribute to the commutation relations of the modes as we shall see. For $0 \leq N \leq 3$ we set $\hat{C} = e^{N(3-N)/12}C$ which will lead us to the central term of the Virasoro algebra.

We have now defined the main objects in a superconformal field theory. In order to look at the space of states of the physical model we expand $T(Z)$ in its modes. Therefore we take the sequences $(i_1, \ldots, i_I)$ where $i_j \in \{1, \ldots, N\}$ and define the operators:

$$J_{i_1^{\ldots, i_I}}(Z) = \frac{1}{2\pi i} \oint_{C_0} \oint_{C_2} dZ_1 \theta_{12, i_1} \ldots \theta_{12, i_I} Z_1^{a+1-\frac{\ell}{2}} T(Z_1) \quad (41)$$

The index $a$ is chosen out of $Z$ if $I$ is even in which case $J_{i_1^{\ldots, i_I}}(Z_2)$ is classified as even or otherwise $a$ is taken out of $Z_2$ and $J_{i_1^{\ldots, i_I}}(Z_2)$ is classified as odd. $J_{i_1^{\ldots, i_I}}(Z_2)$ is trivial if $(i_1, \ldots, i_I)$ contains the same number twice. Moreover if $(i_1', \ldots, i_I')$ is a reordering of $(i_1, \ldots, i_I)$, then the corresponding operators differ at most by a sign factor. The mode $J_{a_1^{\ldots, a_I}}(Z_2)$ corresponds to the expansion term $e^{N(\epsilon_{12, \ldots, i_I} + \delta_{12, \ldots, i_I})} T_{Z_2^{a_1^{\ldots, a_I}}}$ of $T(Z_1)$ where the $j_k$'s are taken from the complement of $(i_1, \ldots, i_I)$ in the sense of a set. Eq. (39) allows us to derive the commutation relations of the modes. It is common practice to consider the symmetry algebra generators taken at the origin. This is not a constraint since we can shift the generators to any other point by superconformal conjugation. Nevertheless, the commutation relations do not depend on the chosen base point. Instead of $J_{a_1^{\ldots, a_I}}(0)$ we shall just write $J_{a_1^{\ldots, a_I}}(Z)$ unless we explicitly want to base the generator on a different point than the origin in which case we write $J_{a_1^{\ldots, a_I}}(Z)$. Standard contour deformation techniques result in

$$[J_{a_1^{\ldots, a_I}}, J_{b_1^{\ldots, b_J}}]_S = \oint_{C_0} \oint_{C_1} \frac{dZ_2}{2\pi i} \frac{dZ_1}{2\pi i} \frac{\theta_{10, i_1} \ldots \theta_{10, i_I}}{\theta_{20, j_1} \ldots \theta_{20, j_J}} Z_{10}^{a+1-\frac{\ell}{2}} Z_{20}^{b+1-\frac{\ell}{2}} T(Z_1)T(Z_2) \quad (42)$$

Performing the supercontour integrals produces the following result:

$$[J_{a_1^{\ldots, a_I}}, J_{b_1^{\ldots, b_J}}]_S = (-1)^{N(I+J)}[a(1 - \frac{J}{2}) - b(1 - \frac{I}{2})] J_{a+b}^{i_1^{\ldots, i_I}, j_1^{\ldots, j_J}} + \frac{(-1)^{N(I+J)}}{2} \sum_{p=1}^{I} \sum_{q=1}^{J} (-1)^{I+p+q} \delta_{p, q} J_{a+b}^{i_1^{\ldots, i_I}, j_1^{\ldots, j_J}}$$

$$+ \frac{C(a + 1 + \frac{3}{2})^{3}}{12} \delta_{a+b, 0} (-1)^NI - \frac{(I+J)}{2} \epsilon(i_1) \ldots \epsilon(i_I) \epsilon(j_1) \ldots \epsilon(j_J) \delta_{N}^{0, 1, 2, 3}, \quad (43)$$

\footnote{This calculation is straightforward but not trivial. For the case of $N = 2$ see for instance Ref. 10.}

\footnote{By convention we moved $T(Z_1)T(Z_2)$ to the right of the integral.}

\footnote{The falling product $(x)_n$ is defined as $x(x-1) \ldots (x-n+1)$ for $n \in \mathbb{N}$ and $(x)_0 = 1$.}
In the following two sections we give the results for \( N \). To conclude this section we calculate the commutators of \( K \). The geometry would then be defined thanks to the generators.

The simplest superconformal extensions of conformal field theories are the \( N \) field theories. In particular we obtain for the Virasoro generators \( L_m \) with \( I = 0 \) we easily obtain the commutation relations

\[
[J_m, J_n] = (m - n)J_{m+n} + \frac{C}{12}(m^3 - m)\delta_{m+n,0}\epsilon_N^{0,1,2,3}.
\]

Hence\(^6\) we find the Virasoro algebra as a subalgebra of the symmetry algebra generated by the modes \( (41) \). This algebra is called the Neveu-Schwarz superconformal algebra with parameter \( N \) which mathematicians denote by \( K_N \). As expected Eq. \((16)\) is a representation\(^9\) of \( K_N \) with \( \hat{C} = 0 \). In the following two sections we give the results for \( N = 1 \) and \( N = 2 \) explicitly. We have derived the algebras \( K_N \) by using superfield formalism. Independently to this approach one could try to find superconformal extensions of the Virasoro algebra just on Lie superalgebra level. This could be done without constructing the underlying superconformal geometry in terms of extended differential manifolds. The geometry would then be defined thanks to the generators.

To conclude this section we calculate the commutators of \( K_N \) generators with (uncharged) superprimary fields:

\[
[J_a^{i_1 \ldots i_l}(Z_0), \Phi_h(Z_2)] = \oint_{C_2} \frac{1}{2\pi i} \theta_{10,i_1} \ldots \theta_{10,i_l} Z_{10}^{a+1-\frac{l}{2}} T(Z_1) \Phi_h(Z_2).
\]

We use Eq. \((39)\) to evaluate the contour integrals. Hence we can write the commutator as a differential operator acting on the superprimary field:

**Theorem 6C** The commutator of the algebra generator \( J_a^{i_1 \ldots i_l}(Z_0) \) with the superprimary field \( \Phi_h(Z_2) \) can be written as

\[
[J_a^{i_1 \ldots i_l}(Z_0), \Phi_h(Z_2)] = (-1)^N I \left[ h(a + 1 - \frac{l}{2}) \theta_{20,i_1} \ldots \theta_{20,i_l} Z_{20}^{a+\frac{l}{2}} + \theta_{20,i_1} \ldots \theta_{20,i_l} Z_{20}^{a+1-\frac{l}{2}} \partial_{z_2} \right. \\
- \frac{(-1)^l}{2} \sum_{p=1}^{l} (-1)^p \theta_{20,i_1} \ldots \theta_{20,i_p} \ldots \theta_{20,i_l} Z_{20}^{a+1-\frac{l}{2}} D_{2,i_p} \\
+ \left. \frac{(a + 1 - \frac{l}{2})(-1)^l}{2} \theta_{20,i_1} \ldots \theta_{20,i_l} Z_{20}^{a+\frac{l}{2}} D_{2,i_l} \right] \Phi_h(Z_2).
\]

In particular we obtain for the Virasoro generators \( L_m = J_m \):

\[
[L_m, \Phi_h(Z)] = \left[ h(m+1)z^m + z^m \partial_z + \frac{1}{2} (m+1) \theta_j z^m \partial_{\theta_j} \right] \Phi_h(Z).
\]

7 \( N = 1 \) Superconformal theories

The simplest superconformal extensions of conformal field theories are the \( N = 1 \) superconformal field theories\(^{23}\). The embedding structure of the corresponding highest weight representations has been analysed by Astashkevich\(^3\). Their superconformal structure is not large enough to show

\(^6\)If \( (i_1, \ldots, i_l) \) is the empty set we define \( \epsilon_0 = 1 \). However, \( \epsilon_0 \) should not be confused with \( \epsilon^N \).

\(^9\)We still have to scale the generators with factors of the complex unit \( i \) in order to obtain the notation of Eq. \((16)\).
significant differences to the representation theory of the Virasoro algebra. This makes the study of \( N = 1 \) superconformal representation theory not very spectacular. Maybe this is the reason why the same was suspected for the representation theory of superconformal theories with bigger \( N \) and hence literature did not treat the \( N = 2 \) representation theory as one could wish. However, we showed in reference 16 why especially \( N = 2 \) representations are much more appealing and their structures much different than already discussed by other authors\(^{14,39}\). Besides and as an exercise we want to use the definitions from the previous sections to define explicitly \( N = 1 \) superconformal theories. We define a quantum field theory containing a chiral stress-energy tensor: \( T(Z) \), which generates the local \( N = 1 \) superconformal transformations, and we have superprimary fields \( \Phi_h(Z) \) which have in our quantisation scheme according to Eq. (39) an operator product expansion with the stress-energy tensor of the form

\[
T(Z_1)\Phi_h(Z_2) = \frac{\hbar \theta_{Z_1^{12}}}{Z_1^{12}} \Phi_h(Z_2) + \frac{1}{2} Z_1^{12} D_2 \Phi_h(Z_2) + \frac{\theta_{Z_1^{12}}}{Z_1^{12}} \partial_{z_2} \Phi_h(Z_2) + \ldots (\text{reg}) .
\]  

(47)

We obtain the symmetry algebra of the theory by expanding the stress-energy tensor:

\[
T(Z_1) = \theta_{Z_1^{12}} \sum_{m \in \mathbb{Z}} Z_1^{12-m-2} L_m(Z_2) + \sum_{r \in \mathbb{Z}} \left( Z_1^{12} \right) ^{-\frac{r}{2}} G_r(Z_2) ,
\]

(48)

where \( C_2 \) is a \( N = 1 \) supercontour about \( Z_2 \). Finally we need to know the operator product of \( T(Z) \) with itself. According to Eq. (40) we obtain:

\[
T(Z_1)T(Z_2) = \frac{C}{6} Z_1^{12} + \frac{3 \theta_{Z_1^{12}}}{2} Z_2 T(Z_2) + \frac{1}{2} Z_1^{12} D_2 T(Z_2) + \frac{\theta_{Z_1^{12}}}{Z_1^{12}} \partial_{z_2} T(Z_2) + \ldots (\text{reg}) .
\]

(50)

This enables us to derive the commutators of the symmetry algebra generators [Eqs. (49)] using Eq. (50) and standard contour deformation methods. The well-known result contains the Virasoro algebra and the generators of one anticommuting field:

**Definition 7.A** The \( N = 1 \) Neveu-Schwarz superconformal algebra \( \mathfrak{K}_1 \) has the following commutation relations:

\[
[L_m, L_n] = (m - n) L_{m+n} + \frac{C}{12} (m^3 - m) \delta_{m+n,0} ,
\]

\[
[L_m, G_r] = \frac{m}{2} - r) G_{m+r} ,
\]

\[
\{G_r, G_s\} = 2 L_{r+s} + \frac{C}{3} (r^2 - \frac{1}{4}) \delta_{r+s,0} ,
\]

\[
[L_m, C] = [G_r, C] = 0 ,
\]

(51)

where \( m, n \in \mathbb{Z} \) and \( r, s \in \mathbb{Z} \).

We have found the algebra (51) by using superfield formalism. As mentioned earlier there may be other super extensions of the Virasoro algebra. In the case of \( N = 1 \) algebras one finds another \( N = 1 \) superconformal algebra. The only difference to (51) is that the operators \( G_r \) have integer indices rather than half integer indices. This algebra is commonly called the \( N = 1 \) Ramond algebra.
8 N = 2 Superconformal theories

The algebra of chiral superconformal transformations in N = 2 superconformal space\textsuperscript{[13]} is generated by the super stress-energy tensor $T(Z_1)$, where $Z_1$ denotes a superpoint $(z_1, \theta_{1,1}, \theta_{1,2})$ and $D_{2,1}$ the superderivative $\frac{\partial}{\partial z_{1,1}} + \theta_{1,1} \frac{\partial}{\partial \theta_{1,1}}$. Superconformal invariance of the theory determines the singular part of the operator product of $T(Z)$ with itself, according to (40). In this case the central non-fixed term is of the form $\frac{1}{Z_{12}^2}$:

$$T(Z_1)T(Z_2) = -\frac{C}{12Z_{12}^2} + \left(-\frac{\theta_{12,1}\theta_{12,2}}{Z_{12}^2} + \frac{\theta_{12,2}D_{2,1} - \theta_{12,1}D_{2,2}}{2Z_{12}} - \frac{\theta_{12,1}\theta_{12,2}}{Z_{12}^2} \partial_z^2 \right) T(Z_2)$$

$$+ \ldots \ (reg) .$$

Eq. (52) agrees with the result for the OPE derived by Blumenhagen\textsuperscript{[10]}. Expanding $T(Z)$ in its modes allows us to find the symmetry algebra generators\textsuperscript{[5]}:

$$T(Z_1) = -\theta_{12,1}\theta_{12,2} \sum_{m \in \mathbb{Z}} Z_{12}^{-m-2} L_m(Z_2) - \frac{1}{2} \sum_{r \in \mathbb{Z}_+} Z_{12}^{-r-\frac{3}{2}} \theta_{12,2} G_{r}^1(Z_2)$$

$$+ \frac{1}{2} \sum_{r \in \mathbb{Z}_+} Z_{12}^{-r-\frac{3}{2}} \theta_{12,1} G_{r}^2(Z_2) - \frac{1}{2} i \sum_{m \in \mathbb{Z}} Z_{12}^{-m-1} T_m(Z_2) ,$$

$$L_m(Z_2) = J_m(Z_2) = \oint_{C_2} \frac{dZ_{12}}{2\pi i} Z_{12}^{m+1} T(Z_1) ,$$

$$G_{r}^i(Z_2) = 2J_{r}^i(Z_2) = 2 \oint_{C_2} \frac{dZ_{12}}{2\pi i} \theta_{12,i} Z_{12}^{r+\frac{1}{2}} T(Z_1) ,$$

$$T_m(Z_2) = 2i J_{m}^{21}(Z_2) = -2i \oint_{C_2} \frac{dZ_{12}}{2\pi i} \theta_{12,1}\theta_{12,2} Z_{12}^{m} T(Z_1) .$$

If we use complex coordinates for the odd generators

$$\theta^\pm = \frac{1}{\sqrt{2}} (\theta_1 \pm i \theta_2) ,$$

$$G^\pm = \frac{1}{\sqrt{2}} (G^1 \pm i G^2) ,$$

$$D^\pm = \frac{1}{\sqrt{2}} (D_1 \pm i D_2) = \frac{\partial}{\partial \theta^+} + \theta^+ \frac{\partial}{\partial \theta^-} ,$$

we find for the symmetry algebra $k_2$ a decomposition in the Virasoro algebra, two anticommuting fields, and a U(1) Kac-Moody algebra with the commutation relations

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{C}{12} (m^3 - m) \delta_{m+n,0} ,$$

$$[L_m, G_r^\pm] = \left( \frac{1}{2} m - r \right) G_{m+r}^\pm ,$$

\textsuperscript{[r]}As mentioned earlier $L_m(0)$ is denoted by $L_m$ and respectively $T_m(0)$ by $T_m$ etc.
Neveu-Schwarz superconformal fields and uncharged superconformal transformations

\[ [L_m, T_n] = -nT_{m+n}, \]
\[ [T_m, T_n] = \frac{1}{3}Cm\delta_{m+n,0}, \]
\[ [T_m, G^\pm_r] = \pm G^\pm_{m+r}, \]
\[ \{G^+_r, G^-_s\} = 2L_{r+s} + (r-s)T_{r+s} + \frac{C}{3}(r^2 - \frac{1}{4})\delta_{r+s,0}, \]
\[ [L_m, C] = [T_m, C] = [G^+_r, C] = 0, \]
\[ \{G^+_r, G^-_s\} = \{G^-_r, G^+_s\} = 0, \quad m, n \in \mathbb{Z}; \quad r, s \in \mathbb{Z}_\pm. \]

Eq. (39) gives us the singular terms of the operator product of \( T(Z_1) \) and superprimary fields \( \Phi_h(Z) \). It turns out, that with respect to the adjoint representation, \( \Phi_h \) has the \( T_0 \) eigenvalue 0. This restricts the highest weight representations of (58) as we want \( T_0 \) in the Cartan subalgebra of \( K_2 \). We can extend the theory by introducing charged superprimary fields \( \Phi_{h,q}(Z) \). The action of superconformal transformations on \( \Phi_{h,q}(Z) \) is altered by a term \( \frac{1}{2Z_{12}} \) in the operator product expansion of \( T(Z_1)\Phi_{h,q}(Z_2) \). This corresponds to a standard \( U(1) \) charge term.

**Definition 8.A** We define the charged superprimary fields on the superconformal space \( S_2 \) by the operator product expansion

\[
T(Z_1)\Phi_{h,q}(Z_2) = -\frac{h\theta_{12,1}\theta_{12,2}}{Z_{12}^2}\Phi_{h,q}(Z_2) + \frac{1}{2}\frac{\theta_{12,2}D_{2,1} - \theta_{12,1}D_{2,2}}{Z_{12}}\Phi_{h,q}(Z_2)
-\frac{\theta_{12,1}\theta_{12,2}}{Z_{12}^2}\partial_z\Phi_{h,q}(Z_2) - \frac{q}{2Z_{12}^2}i\Phi_{h,q}(Z_2) \ldots .
\]

We call \( h \) the conformal weight of \( \Phi_{h,q} \) and \( q \) its conformal charge, corresponding to the scaling dimensions of \( L_0 \) and \( T_0 \) transformations respectively.

Using Eqs. (54) and performing the contour integrals we find the infinitesimal transformations for all generators:

\[
[L_m, \Phi_{h,q}(Z)] = \left[ h(m+1)z^m + \frac{1}{2}(m+1)z^m(\theta^+D^- + \theta^-D^+) + z^{m+1}\partial_z
+\frac{q}{2}\theta^+\theta^-z^{m-1}(m+1) \right]\Phi_{h,q}(Z),
\]
\[
[G^+_r, \Phi_{h,q}(Z)] = \left[ 2h(r + \frac{1}{2})\theta^+z^r - \frac{1}{2}z^rD^+ + \theta^+\theta^-(r + \frac{1}{2})z^{r+\frac{1}{2}}D^+
+2\theta^+z^r\partial_z \right] \Phi_{h,q}(Z),
\]
\[
[T_m, \Phi_{h,q}(Z)] = \left[ 2h\theta^+\theta^-mz^{m-1} + z^m(\theta^-D^+ - \theta^+D^-) + 2\theta^+\theta^-z^m\partial_z
+qz^m \right]\Phi_{h,q}(Z).
\]

Once more, we note that the \( N = 2 \) superconformal algebra \( K_2 \) which we consider is known as the \( N = 2 \) *Neveu-Schwarz* or antiperiodic algebra. Reference 47 has shown that it is isomorphic to the \( N = 2 \) *Ramond* or periodic algebra what makes a separate discussion redundant.

\(^*\)This term was not included in Eq. (39) since the number of charges increases with \( N \).
We can write \( K_2 \) in a triangular decomposition\(^4\) such that \( \mathcal{H}_2 \) contains the energy operator \( L_0 \): \( K_2 = K_2^- \oplus \mathcal{H}_2 \oplus K_2^+ \), where \( \mathcal{H}_2 = \text{span}\{L_0, T_0, C\} \) is the grading preserving Cartan subalgebra, and

\[
K_2^\pm = \text{span}\{L_{\pm n}, T_{\pm n}, G^+_{\pm r}, G^-_{\pm r} : n \in \mathbb{N}, r \in \mathbb{N}_1 \}.
\]

A simultaneous eigenvector \( |h, q, c\rangle \) of \( \mathcal{H}_2 \) with \( L_0, T_0 \) and \( C \) eigenvalues \( h, q \) and \( c \) respectively and vanishing \( K_2^+ \) action, \( K_2^+ |h, q, c\rangle = 0 \), is called a highest weight vector. It is easy to see that a primary field \( \Phi_{h,q} \) generates a highest weight vector \( |h, q, c\rangle \) on the vacuum: \( |h, q, c\rangle = \Phi_{h,q}(0) |0\rangle_c \). \( |0\rangle_c \) denotes the vacuum with fixed central extension \( C |0\rangle_c = c |0\rangle_c \). The Verma module \( \mathcal{V}_{h,q,c} \) is defined as the \( K_2 \) left-module \( U(K_2) \otimes \mathcal{H}_2 \otimes K_2^+ |h, q, c\rangle \), where \( U(K_2) \) denotes the universal enveloping algebra of \( K_2 \).

The representation theory of the \( N = 2 \) Neveu-Schwarz algebra contains many surprising features which have not appeared in any other conformal field theory so far. Singular vectors can be degenerated\(^6\), embedded modules may not be complete and subsingular vectors can appear\(^7,8,9\), rousing the curiosity about what one has to deal with for even higher \( N \).

### 9 Transformation of primary fields

As an application of the formalism developed above we compute the global transformation rules for (uncharged) primary fields under transformations generated by the algebra generators. Restricted to the Virasoro case our results represent the inverse problem of the transformation formula found by Gaberdiel\(^26\). There the author derived for the Virasoro case the algebra element that belongs to a given holomorphic coordinate transformation, whilst we are interested in the holomorphic coordinate transformation corresponding to the given algebra generators. Furthermore, in our superfield framework we can very easily obtain transformation rules also for all the superconformal cases as we shall demonstrate in some specific examples.

The only globally defined conformal transformations on the Riemann sphere are the Möbius transformations. They are generated\(^30\) by \( L_{-1}, L_0 \) and \( L_1 \). \( L_{-1} \) and \( L_0 \) respectively correspond to translations and to scaling transformations on the Riemann surface\(^6\):

\[
e^{\lambda L_{-1}} \Phi_h(z) e^{-\lambda L_{-1}} = \Phi_h(z + \lambda),
\]
\[
\lambda^{L_0} \Phi_h(z) e^{-L_0} = \lambda^h \Phi_h(\lambda z).
\]

\( \lambda L_1 \) corresponds to the coordinate transformation \( z \mapsto \frac{z}{1 - \lambda z} \). Since the central extension does not contribute towards the infinitesimal change of \( \Phi_h \), we can obtain the transformation rules by using the action of the de Witt algebra, which forms a representation of the Virasoro algebra with \( C = 0 \). In this section we want to find general transformation formulae for the primary fields under transformations generated by the Virasoro generators. We shall then extend this result to the \( N = 1 \) and \( N = 2 \) superconformal algebra. The extension to superconformal algebras with \( N \geq 3 \) follows the same rules as we shall see.

\(^4\)There are in fact triangular decompositions having a four-dimensional space \( \mathcal{H}_2 \). However, these decompositions are not consistent with our \( \mathbb{Z}_2 \) grading.

\(^6\)Note that this is after mapping the system from the cylinder to the complex plane for radial quantisation.
9.1 Primary fields in the Virasoro case

The action of the Möbius generators $L_{-1}, L_0$ and $L_1$ on a primary field $\Phi_h(z)$ is well-known in conformal field theory\(^{30}\). In a first step we generalise these results to the Virasoro generators $L_n$:

$$e^{\lambda L_n} \Phi_h(z) e^{-\lambda L_n}.$$  \hspace{1cm} (62)

We use the identity\(^w\)

$$e^A B e^{-A} = \sum_{j=0}^{\infty} \frac{1}{j!} [A, B]_j ,$$  \hspace{1cm} (63)

to rewrite Eq. (62):

$$e^{\lambda L_n} \Phi_h(z) e^{-\lambda L_n} = \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} [L_n, \Phi_h(z)]_j .$$  \hspace{1cm} (64)

Substituting Eq. (8) in the successive commutator of Eq. (64) leads us to:

$$e^{\lambda L_n} \Phi_h(z) e^{-\lambda L_n} = \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} [z^{n+1} \partial_z + h(n+1) z^n j] \Phi_h(z)$$  
$$= e^{\lambda [z^{n+1} \partial_z + h(n+1) z^n]} \Phi_h(z) .$$  \hspace{1cm} (65)

Before we continue, we prove the following differential operator identity which will constitute to be the main tool for the rest of this section:

Theorem 9.A

$$e^{\lambda [z^{n+1} \partial_z + h(n+1) z^n]} = \left[ \frac{\partial (e^{\lambda w^{n+1} \partial_w w})}{\partial w} \right]_{w=z}^h e^{\lambda z^{n+1} \partial_z}$$  \hspace{1cm} (66)

Proof: To prove the first part\(^w\) of Eq. (66) it is sufficient to show it by acting on the monomials $z^m$ for any positive integer $m$. By straightforward calculation we obtain the following identities for integer $n \neq 0$:

$$e^{\lambda z^{n+1} \partial_z} z = \frac{z}{(1 - n \lambda z^n)^\frac{n}{n-1}} ,$$  \hspace{1cm} (67)

$$e^{\lambda \partial_z z^{n+1}} = \frac{1}{(1 - n \lambda z^n)^\frac{n+1}{n}} .$$  \hspace{1cm} (68)

For $n = 0$ we have the identities:

$$e^{\lambda z \partial_z} z = e^\lambda z ,$$  \hspace{1cm} (69)

$$e^{\lambda \partial_z z} = e^\lambda .$$  \hspace{1cm} (70)

\(^w\)Successive commutators are defined as $[A, B]_i = [A, [A, B]_{i-1}]$ and $[A, B] = [\ldots [A, B], B]$ with $[A, B]_0 = B$ and $0[A, B] = A$.

\(^w\)We are grateful to Adrian Kent for this proof.
We show Eq. (68) and acting on the monomial $z^m$ with the first of the Eqs. (66) leads to:

$$
\sum_{j=0}^{\infty} \frac{\lambda^j}{j!} z^{m+jn} [m + h(n+1)] [m + h(n+1) + n] \ldots [m + h(n+1) + (j-1)n] = \frac{1}{(1 - n\lambda z^n)^{1/n}} \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} z^{m+jn} m [m + n] \ldots [m + (j-1)n].
$$

We expand both sides in power series in $\lambda$ about 0 and compare the coefficients. We thus compare the derivatives $\frac{\partial^q}{\partial \lambda^q} \bigg|_{\lambda=0}$ on both sides for any positive integer $q$. The left-hand side turns into:

$$z^{m+qn} [m + h(n+1)] [m + h(n+1) + n] \ldots [m + h(n+1) + (q-1)n].$$

In order to take the derivative of the right-hand side we use the differentiation rule:

$$\frac{\partial^q}{\partial \lambda^q} A(\lambda) B(\lambda) = \sum_{r=0}^{q} \binom{q}{r} \left( \frac{\partial^r A(\lambda)}{\partial \lambda^r} \right) \left( \frac{\partial^{q-r} B(\lambda)}{\partial \lambda^{q-r}} \right).$$

Hence, we obtain for the right-hand side:

$$\sum_{r=0}^{q} \binom{q}{r} z^{m+qn} h(n+1) [h(n+1) + n] \ldots [h(n+1) + (r-1)n] m [m + n] \ldots [m + (q-r-1)n].$$

Therefore, we only need to prove the following identity:

$$[m + h(n+1)] [m + h(n+1) + 1] \ldots [m + h(n+1) + (q-1)n] = \sum_{r=0}^{q} \binom{q}{r} m [m + n] \ldots [m + (q-r-1)n] [h(n+1)] \ldots [h(n+1) + (r-1)n]. \quad (71)$$

We show Eq. (71) by induction on $q$: Eq. (71) is obviously valid for $q = 1$. We assume (71) is true for $q$ and we show that it is true for $q+1:

$$[m + h(n+1)] \ldots [m + h(n+1) + qn] = \sum_{r=0}^{q} \binom{q}{r} m \ldots [m + (q-r-1)n] [h(n+1)] \ldots [h(n+1) + (r-1)n] [m + h(n+1) + qn]$$

$$= \sum_{r=0}^{q} \binom{q}{r} m \ldots [m + (q-r-1)n] [h(n+1)] \ldots [h(n+1) + (r-1)n] [h(n+1) + rn + m + (q-r)n]$$

$$= \sum_{r=1}^{q+1} \binom{q+1}{r-1} m \ldots [m + (q-r)m] [h(n+1)] \ldots [h(n+1) + (r-1)n]$$

$^x$For $q = 0$ the expression should be $z^m$.\
\[ + \sum_{r=0}^{q} \binom{q}{r} m \ldots [m + (q-r)m][h(n+1)] \ldots [h(n+1) + (r-1)n] \]
\[ = \sum_{r=0}^{q+1} \left[ \left( \begin{array}{c} q \\ r \end{array} \right) + \left( \begin{array}{c} q \\ r - 1 \end{array} \right) \right] m \ldots [m + (q-r)m][h(n+1)] \ldots [h(n+1) + (r-1)n] \]
\[ = \sum_{r=0}^{q+1} \left( \begin{array}{c} q + 1 \\ r \end{array} \right) m \ldots [m + (q-r)m][h(n+1)] \ldots [h(n+1) + (r-1)n] . \]

This completes the induction. The second equality of identity (66) follows from
\[ e^{\lambda \partial_w w^{n+1}} \bigg|_{w=z} = \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} \prod_{k=0}^{j} (kn + 1) z^{2n} = \frac{\partial (e^{\lambda w^{n+1} \partial_w w})}{\partial w} \bigg|_{w=z} . \tag{72} \]

We have herewith completed the proof of theorem 9.A.

Hence Eq. (62) simplifies to\(^g\):
\[ e^{\lambda L_n} \Phi_h(z) e^{-\lambda L_n} = \left( e^{\lambda \partial_w w^{n+1}} \bigg|_{w=z} \right)^h e^{\lambda z^{n+1} \partial_z \Phi_h(z)} . \tag{73} \]

We can provide the final step with the following theorem:

**Theorem 9.B** Let \( A_z \) be a linear differential operator acting on \( z \), then
\[ e^{A_z} f(z) = f(e^{A_w} w \bigg|_{w=z}) \]
for any function \( f(z) \) which can be expanded in a power series.

Proof: It is sufficient to show the theorem for functions \( f(z) = z^m \) for any \( m \in \mathbb{N}_0 \). Since \( A_z \) is a linear differential operator, we find:
\[ A_z^n z^m = \sum_{n_1 + n_2 + \ldots + n_m = n} \frac{n!}{n_1! \ldots n_m!} (A_w^{n_1} w \bigg|_{w=z}) \ldots (A_w^{n_m} w \bigg|_{w=z}) \]
\[ \Rightarrow e^{A_z} z^m = \sum_{n \geq 0} \frac{1}{n!} A_z^n z^m = \sum_{n_1, \ldots, n_m} \frac{1}{n_1! \ldots n_m!} (A_w^{n_1} w \bigg|_{w=z}) \ldots (A_w^{n_m} w \bigg|_{w=z}) = (e^{A_w} w \bigg|_{w=z})^m . \]

Using theorem 9.B we obtain for Eq. (62):
\[ e^{\lambda L_n} \Phi_h(z) e^{-\lambda L_n} = \sum_{j=0}^{\infty} \frac{1}{j!} [\lambda L_n, \Phi_h(z)]_j = \left( e^{\lambda \partial_w w^{n+1}} \bigg|_{w=z} \right)^h \Phi_h \left( e^{\lambda w^{n+1} \partial_w w} \bigg|_{w=z} \right) . \tag{74} \]

The main result of this subsection is now at hand after applying Eqs. (67)-(70):

**Theorem 9.C** A primary field \( \Phi_h(z) \) transforms for \( n \in \mathbb{Z} \) according to
\[ e^{\lambda L_n} \Phi_h(z) e^{-\lambda L_n} = \frac{1}{(1-n\lambda z)^{n+h}} \Phi_h \left( \frac{z}{(1-n\lambda z)^n} \right) , \quad n \neq 0 , \]
\[ e^{\lambda L_0} \Phi_h(z) e^{-\lambda L_0} = e^{\lambda h} \Phi_h \left( e^{\lambda z} \right) . \]

\(^g\)Note that \( \left( e^{\lambda \partial_w w^{n+1}} \bigg|_{w=z} \right)^h \neq e^{\lambda h \partial_w w^{n+1}} \bigg|_{w=z} . \)
9.2 Superprimary fields in the $N = 1$ case

The previous section has to be slightly altered in order to apply theorem 9.C for $K_1$. The operator product of the stress-energy tensor with superprimary fields leads to the following commutation relations where the superprimary fields are taken at the superpoint $Z = (z, \theta)$:

$$[L_n, \Phi_h(z, \theta)] = \left(z^{n+1}\partial_z + \frac{1}{2}(n + 1)z^n\partial_z + h(n + 1)z^n\right)\Phi_h(z, \theta), \quad n \in \mathbb{Z},$$

$$[G_r, \Phi_h(z, \theta)] = \left(z^{r+\frac{1}{2}}(\partial_\theta - \partial_z) - h(r + \frac{1}{2})z^{r-\frac{1}{2}}\right)\Phi_h(z, \theta), \quad r \in \mathbb{Z}_\frac{1}{2}.$$

In exactly the same manner as in the Virasoro case we can find:

$$e^{\lambda L_n} \Phi_h(z, \theta) e^{-\lambda L_n} = e^{\lambda (z^{n+1}\partial_z + \frac{1}{2}z^n(n+1)\partial_\theta + h(n+1)z^n)} \Phi_h(z, \theta). \quad (75)$$

If we then split $\Phi_h(z, \theta)$ into even and odd parts $\Phi_h(z, \theta) = \varphi_h(z) + \psi_h(z)$ and act on the two parts separately we notice that both cases come back to Eq. (66).

$$e^{\lambda (z^{n+1}\partial_z + \frac{1}{2}z^n(n+1)\partial_\theta + h(n+1)z^n)} \varphi_h(z) = e^{\lambda (z^{n+1}\partial_z + h(n+1)z^n)} \varphi_h(z),$$

$$e^{\lambda (z^{n+1}\partial_z + \frac{1}{2}z^n(n+1)\partial_\theta + h(n+1)z^n)} \theta \psi_h(z) = \theta e^{\lambda (z^{n+1}\partial_z + (h+\frac{1}{2})(n+1)z^n)} \psi_h(z).$$

By using theorem 9.A we thus obtain:

$$e^{\lambda (z^{n+1}\partial_z + \frac{1}{2}z^n(n+1)\partial_\theta + h(n+1)z^n)} \varphi_h(z) = \left( e^{\lambda \partial_w w^{n+1}} \right) \varphi_h \left( e^{\lambda w^{n+1} \partial_w w} \right),$$

$$e^{\lambda (z^{n+1}\partial_z + \frac{1}{2}z^n(n+1)\partial_\theta + h(n+1)z^n)} \theta \psi_h(z) = \theta \left( e^{\lambda \partial_w w^{n+1}} \right) \psi_h \left( e^{\lambda w^{n+1} \partial_w w} \right).$$

Taking both results together leads to

$$e^{\lambda L_n} \Phi_h(z, \theta) e^{-\lambda L_n} = \left( e^{\lambda \partial_w w^{n+1}} \right)^h \Phi_h \left( e^{\lambda w^{n+1} \partial_w w} \right), \quad n \neq 0,$$

$$e^{\lambda L_0} \Phi_h(z, \theta) e^{-\lambda L_0} = e^{\lambda h} \Phi_h \left( e^{\lambda z, \frac{1}{2} \theta} \right). \quad (76)$$

Using Eqs. (67)-(70) we obtain the transformation rules for the even generators.

**Theorem 9.D** In the even sector the transformation rules are ($n \in \mathbb{Z}$):

$$e^{\lambda L_n} \Phi_h(z, \theta) e^{-\lambda L_n} = \frac{1}{(1 - n\lambda z^n)^{\frac{n+1}{2}} h} \Phi_h \left( \frac{z}{(1 - n\lambda z^n)^{\frac{1}{2}}} \right) \left( \frac{\theta}{(1 - n\lambda z^n)^{\frac{n+1}{2}} h} \right), \quad n \neq 0,$$

$$e^{\lambda L_0} \Phi_h(z, \theta) e^{-\lambda L_0} = e^{\lambda h} \Phi_h \left( e^{\lambda z, \frac{1}{2} \theta} \right).$$

For the odd sector the calculation is less spectacular since the Taylor expansion comes to an end after the very first order.

**Theorem 9.E** In the odd sector the transformation rules are ($r \in \mathbb{Z}_\frac{1}{2}$):

$$e^{\epsilon G_r} \Phi_h(z, \theta) e^{-\epsilon G_r} = \left[ 1 - \epsilon \theta (r + \frac{1}{2}) z^{r-\frac{1}{2}} \right] h \Phi_h \left( z - \epsilon z^{r+\frac{1}{2}} \theta, \theta + \frac{1}{2} \epsilon z^{r+\frac{1}{2}} \right),$$

where $\epsilon$ is an anticommuting quantity.

We point out the fact that both results are consistent with the definition of superprimary fields [Eq. (36)].
9.3 Superprimary fields in the $N = 2$ case

Thanks to the complex coordinates (55)-(57) we can write the charged superprimary fields in the $N = 2$ case as $\Phi_{h,q}(z, \theta^+, \theta^-) = \varphi_{h,q}(z) + \theta^+\psi_{h,q}(z) + \theta^-\psi_{h,q}^+(z) + \theta^+\theta^-\chi_{h,q}(z)$. As in the previous subsection the commutators of $K_2$ elements with superprimary fields can be written as differential operators acting on the superprimary fields. These relations are given in the Eqs. (60). We then try to find the action of the exponential of these differential operators on the fields $\varphi_{h,c}(z)$, $\theta^+\psi_{h,q}(z)$, $\theta^-\psi_{h,q}^+(z)$ and $\chi_{h,q}(z)$. Once again theorem 9.A is the main tool in our calculation. After analysing all possible cases we obtain:

$$e^{\lambda L_m} \Phi_{h,q}(z, \theta^+, \theta^-) e^{-\lambda L_m} = \left[ \frac{1}{(1 - n\lambda z^n)^\frac{1}{2m}} \right]^{h + \frac{nq}{2}} \frac{\theta^+ \theta^-}{z} \Phi_{h,q} \left( \frac{z}{1 - \theta^+ z^{r - \frac{1}{2}}}, \frac{\theta^+}{1 - \theta^+ (r + \frac{1}{2}) z^{r - \frac{1}{2}}}, \frac{\theta^- - \epsilon z^{r + \frac{1}{2}}}{1 - \theta^- z^{r - \frac{1}{2}}} \right),$$

$$e^{\lambda L_0} \Phi_{h,q}(z, \theta^+, \theta^-) e^{-\lambda L_0} = e^{\lambda h} \Phi_{h,q} \left( e^{\lambda' z}, e^{\frac{1}{2} \theta^+}, e^{\frac{1}{2} \theta^-} \right),$$

$$e^{G^+} \Phi_{h,q}(z, \theta^+, \theta^-) e^{-G^+} = \Phi_{h,q}(z + \epsilon \theta^+ + \theta^+, \theta^- - \epsilon),$$

$$e^{G^-} \Phi_{h,q}(z, \theta^+, \theta^-) e^{-G^-} = \Phi_{h,q}(z - \epsilon \theta^- + \theta^+, \theta^-),$$

$$e^{\eta T_m} \Phi_{h,q}(z, \theta^+, \theta^-) e^{-\eta T_m} = e^{\eta q z^m} \Phi_{h,q} \left( \frac{z, e^{-\lambda z^m} \theta^+, e^{\lambda z^m} \theta^-} {1 - \lambda m \theta^+ \theta^- z^{m-1}} \right),$$

where $m \in \mathbb{Z}$, $n \in \mathbb{Z} \setminus \{0\}$ and $r \in \mathbb{Z}_4 \setminus \{-\frac{1}{2}\}$. It is easy to check that for $q = 0$ we obtain a transformation according to the definition of (uncharged) superprimary fields [Eq. (36)] where the transformed super Jakobi determinant takes the form

$$D \tilde{\theta} = \begin{pmatrix} D^+ \tilde{\theta}^- & D^+ \tilde{\theta}^+ \\ D^- \tilde{\theta}^- & D^- \tilde{\theta}^+ \end{pmatrix}.$$
to the evaluation of the action of the differential operator

\[ T^{a_1 \ldots a_l} = (-1)^{N_l} \left\{ h(a + 1 - \frac{I}{2}) \theta_{i_1} \ldots \theta_{i_l} z^{a_{i_1} - \frac{1}{2}} + \theta_{i_1} \ldots \theta_{i_l} z^{a_{i_1} + 1 - \frac{1}{2}} \partial_{z^2} \right\} \]

\[ - \frac{(-1)^f}{2} \sum_{p=1}^{I} (-1)^p \theta_{i_1} \ldots \theta_{i_p} \ldots \theta_{i_l} z^{a_{i_1} + 1 - \frac{1}{2}} \partial_{\theta_{i_p}} \]

\[ + \frac{(a + 1 - \frac{I}{2})(-1)^f}{2} \theta_{i_1} \ldots \theta_{i_l} z^{a_{i_1} - \frac{1}{2}} \partial_{\theta_{i_l}} \right\} , \]

on the superprimary field \( \Phi_h(Z) \). The most interesting cases are the Virasoro generators \( L_m = J_m \) and the generators \( G^k_r = J^k_r \). Again theorem 9.A is the main device:

\[ e^{\lambda L_0} \Phi_h(z, \theta_1, \ldots, \theta_N) e^{-\lambda L_0} = e^{\lambda h} \Phi_h \left( e^{\lambda h} z, e^{\frac{\lambda}{2} \theta_1}, \ldots, e^{\frac{\lambda}{2} \theta_N} \right) , \quad n \in \mathbb{Z} \setminus \{0\} , \]

\[ e^{G^k_r} \Phi_h(z, \theta_1, \ldots, \theta_N) e^{-G^k_r} = \left( 1 + (-1)^N \epsilon \theta_k (r + \frac{1}{2}) z^{r - \frac{1}{2}} \right)^h \]

\[ \Phi_h \left( z + (-1)^N \epsilon \theta_k z^{r + \frac{1}{2}}, \theta_1 - \frac{(-1)^N}{2} \epsilon \theta_1 \theta_k (r + \frac{1}{2}) z^{r - \frac{1}{2}}, \ldots, \right) \]

\[ \theta_k - \frac{(-1)^N}{2} \epsilon z^{r + \frac{1}{2}}, \ldots, \theta_N - \frac{(-1)^N}{2} \epsilon \theta_1 \theta_N (r + \frac{1}{2}) z^{r - \frac{1}{2}} \right) , \quad r \in \mathbb{Z}_{\frac{1}{2}} . \]

In particular \( G^k_{-\frac{1}{2}} \) is the supertranslation in \( \theta_k \) direction:

\[ e^{G^k_{-\frac{1}{2}}} \Phi_h(z, \theta_1, \ldots, \theta_N) e^{-G^k_{-\frac{1}{2}}} = \Phi_h \left( z, \theta_1, \ldots, \theta_k - \frac{(-1)^N}{2} \epsilon, \ldots, \theta_N \right) , \]

and \( L_{-1} \) translates in \( z \) direction:

\[ e^{\lambda L_{-1}} \Phi_h(z, \theta_1, \ldots, \theta_N) e^{-\lambda L_{-1}} = \Phi_h \left( z + \lambda, \theta_1, \ldots, \theta_N \right) . \]

10 Conclusions and Prospects

We presented a superfield framework based on superconformal manifolds to derive all superconformal Neveu-Schwarz theories. This enabled us to derive the most general OPE for the stress-energy tensor with itself and with (uncharged) primary fields for any \( N \). The resulting commutation relations agree for the classical case \( C = 0 \) with the commutation relations of superderivatives on the space of differential forms over a superconformal manifold leaving the differential form \( \kappa \) invariant. This is the exact analogue to the de Witt algebra being a representation of the Virasoro algebra for the classical case \( C = 0 \). The general expressions given for the generators of the Neveu-Schwarz \( N \) algebra and their commutators can be extremely helpful in many applications of conformal field theory. As an example we computed the transformation properties of (uncharged) primary fields under superconformal transformations generated by algebra generators.
Neveu-Schwarz superconformal fields and uncharged superconformal transformations

The representation theories of Neveu-Schwarz algebras are known for the Virasoro case $N = 0$ and for $N = 1$. However, already for $N = 2$ one finds new structures like degenerated singular vectors$^{15, 16, 29}$ and subsingular vectors$^{28, 29, 18}$. Besides, the embedding structure of singular vectors of the $N = 2$ Neveu-Schwarz algebra is only known up to subsingular vectors$^{17}$. The derivation of the Neveu-Schwarz algebras presented in this paper is valid for arbitrary $N$ and therefore serves as a framework for the study of the representation theory of superconformal field theories for any $N$. This, however, first requires that the charged primary fields for arbitrary $N$ would be defined.

Finally, we will present in a forthcoming publication$^{19}$ how to derive the Ramond superconformal field theories using a similar superfield formalism.

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