Sparse constrained projection approximation subspace tracking

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Abstract

In this paper we revisit the well-known constrained projection approximation subspace tracking algorithm (CPAST) and derive, for the first time, non-asymptotic error bounds. Furthermore, we introduce a novel sparse modification of CPAST which is able to exploit sparsity in the underlying covariance structure. We present a non-asymptotic analysis of the proposed algorithm and study its empirical performance on simulated and real data.

1 Introduction

Subspace tracking methods are intensively used in statistical and signal processing community. Given observations of a multidimensional signal, one is interested in estimating or tracking a subspace spanning the eigenvectors corresponding to the first largest eigenvalues of the signal covariance matrix. Over the past few decades many variations of the original projection approximation subspace tracking (PAST) method [1] were developed which found

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applications in data compression, filtering, speech enhancement, etc. (see [2] and references therein). Despite popularity of the subspace tracking methods, only partial results are known about their convergence. The asymptotic convergence of the PAST algorithm was first established in [3, 4] using a general theory of stability for ordinary differential equations. However, no finite sample error bounds are available in the literature. Furthermore, in the case of a high-dimensional signal the empirical covariance matrix estimator performs poorly if the number of observations is small. A common way to improve the estimation quality in this case is to impose some kind of sparsity assumptions on the signal itself or on the eigensubspace of the underlying covariance matrix. In [5] a sparse modification of the orthogonal iteration scheme for a fixed number of observations was proposed. A thorough analysis in [5] shows that under appropriate sparsity assumptions on the leading eigenvectors, the orthogonal iteration scheme combined with thresholding allows to perform dimension reduction in high-dimensional setting. Our main goal is to propose a novel modification of constraint projection approximation subspace tracking method (CPAST) [6], called sparse projection approximation subspace tracking method (SCPAST), which can be used for efficient subspace tracking in the case of high-dimensional sparse signal and small number of available observations. Another contribution of our paper is a non-asymptotic convergence analysis of CPAST and SCPAST algorithms showing the advantage of SCPAST algorithm in the case of sparse covariance structure. Last but not the least, we analyse numerical performance of SCPAST algorithm on simulated and real data. In particular, the problem of tracking the leading subspace of a music signal is considered.

The structure of the paper is as follows. In Section 2 we introduce our observational model and formulate main assumptions. Section 3 first reviews the CPAST algorithm and then provides the non-asymptotic error bounds for CPAST in a "stationary" case. In Section 4 we introduce our sparse constraint approximation subspace tracking method and prove the non-asymptotic upper bounds for the estimation error. A numerical study of the proposed algorithm is presented in Section 5. Finally the proofs are collected in Section 6.
2 Main setup

One important problem in signal processing is adaptive estimation of a dominant subspace given incoming noisy observations. Specifically one considers a model

\[ x(t) = s(t) + \sigma(t)\xi(t), \quad t = 1, \ldots, T, \]

where the observations \( x(t) \in \mathbb{R}^n \) contain the signal \( s(t) \in \mathbb{R}^n \) corrupted by a vector \( \xi(t) \in \mathbb{R}^n \) with independent standard Gaussian components. The signal \( s(t) \) is usually modelled as

\[ s(t) = A(t)\eta(t), \]

where \( A(t) \) is a deterministic \( n \times d \) matrix of rank \( d \) with \( d \ll n \) and \( \eta(t) \) is a random vector in \( \mathbb{R}^d \) independent of \( \xi(t) \), such that \( \mathbb{E}[\eta(t)] = 0 \) and \( \mathbb{E}[\eta_i^2(t)] = 1, i = 1, \ldots, d \). Under these assumptions, the process \( x(t) \) has a covariance matrix \( \Sigma(t) \) which may be decomposed in the following way

\[ \Sigma(t) = \mathbb{E}[x(t)x^\top(t)] = A(t)A^\top(t) + \sigma^2(t)I_n, \]

where \( I_n \) stands for the unit matrix in \( \mathbb{R}^n \). Note that the matrix \( A(t)A^\top(t) \) has the rank \( d \) and by the singular value decomposition (SVD)

\[ A(t)A^\top(t) = \sum_{i=1}^d \lambda_i(t)v_i(t)v_i^\top(t), \]

where \( v_i(t) \in \mathbb{R}^n, i = 1, \ldots, d, \) are the eigenvectors of \( A(t)A^\top(t) \) corresponding to the eigenvalues \( \lambda_1(t) \geq \lambda_2(t) \geq \cdots \geq \lambda_d(t) > 0 \). It follows from (2) that the first \( d \) eigenvalues of \( \Sigma(t) \) are \( \lambda_1(t) + \sigma^2(t), \ldots, \lambda_d(t) + \sigma^2(t) \), whereas the remaining \( n - d \) eigenvalues are equal to \( \sigma^2(t) \). Since \( \lambda_d(t) > 0 \), the subspace corresponding to the first \( d \) eigenvectors of \( A(t)A^\top(t) \) is identifiable. The subspace tracking methods aim to estimate the subspace \( \text{span}(v_1(t), \ldots, v_d(t)) \) based on the observations \( (x(k))_{k=1}^T \). The overall number of observations \( T \) is assumed to be fixed and known.

Relying on a heuristic assumption of slow (in time) varying \( \Sigma(t) \), the subspace tracking methods use the following estimator of the covariance matrix (up to scaling)

\[ \hat{\Sigma}(t) = \sum_{i=0}^t \gamma^{t-i}x(i)x^\top(i), \]

where \( \gamma \) is a forgetting factor.
where $0 < \gamma \leq 1$ is the so-called forgetting factor. The estimator $\hat{\Sigma}_\gamma(t)$ can adapt to the change in $\Sigma(t)$ by discounting the past observations. In the stationary regime, that is, if $\Sigma(t)$ is a constant matrix, one would use $\gamma = 1$. It is well known, that in the case of Gaussian independent noise the estimator $\hat{\Sigma}_1(t)$ is consistent.

3 CPAST

For the general model (1) and non-stationary case, constrained projection approximation subspace tracking (CPAST) method allows to iteratively compute a matrix $\hat{V}_\gamma(t)$, $t = 1, \ldots, T$, containing the first $d$ leading eigenvectors of the matrix $\hat{\Sigma}_\gamma(t)$ (see (3)) based on sequentially arriving observations $x(j)$, $j = 1, \ldots, t$. The procedure starts with some initial approximation $\hat{V}_\gamma(0) = \hat{V}^0$ and consists of the following two steps

- **multiplication**: compute the $n \times d$ matrix
  \[
  \hat{\Sigma}_{\gamma,V}(t) = \hat{\Sigma}_\gamma(t)\hat{V}_\gamma(t - 1);
  \]

- **orthogonalization**: compute an estimator $\hat{V}_\gamma(t)$ of the matrix $V(t)$ containing $d$ leading eigenvectors via
  \[
  \hat{V}_\gamma(t) = \hat{\Sigma}_{\gamma,V}(t)[\hat{\Sigma}_{\gamma,V}(t)\hat{\Sigma}_{\gamma,V}(t)^\top]^{-1/2}.
  \]

In the ”stationary” case ($\gamma = 1$) the method may be regarded as the ”online”-version of the orthogonal iterations scheme (see [7]) for computing the eigen-subspace of the non-negatively definite matrix. With the use of the Sherman-Morrison-Woodbury formula for the inversion at each time $t$, one has to perform $O(nd^2)$ operations to compute the updated matrix $\hat{V}_\gamma(t)$ given $\hat{V}_\gamma(t - 1), \hat{\Sigma}_\gamma(t - 1)$ and $x(t)$.

3.1 Convergence of CPAST

Throughout this section we consider the stationary case where $\Sigma(t) = \Sigma$, $A(t) = A$, $v_i(t) = v_i$, $\lambda_i(t) = \lambda_i$, $i = 1, \ldots, d$, $\sigma^2(t) = \sigma^2$. In this situation one would like to keep all the available information to estimate $V$, that is, to use the estimator (3) for $\Sigma$ with $\gamma = 1$. For notational simplicity, from now
on we skip the dependence on $\gamma$ and use the notation $\hat{\Sigma}(t)$ for the empirical covariance matrix $\hat{\Sigma}(t) = \frac{1}{t} \sum_{i=1}^{t} x(i)x^\top(i)$ and $\hat{V}(t)$ for CPAST estimator. Thus in the stationary case the CPAST estimator takes the form

$$\hat{V}(t) = [\hat{\Sigma}(t)\hat{V}(t-1)][\hat{V}^\top(t-1)\hat{\Sigma}(t)\hat{V}(t-1)]^{-1/2}. \quad (4)$$

We assume that the random vectors $\eta(t)$ and $\xi(t)$ have independent $\mathcal{N}(0,1)$ components for $t = 1, \ldots, T$. Under these assumptions the covariance matrix $\Sigma$ becomes

$$\Sigma = \sum_{i=1}^{d} \lambda_i v_i v_i^\top + \sigma^2 I_n = V\Lambda_d V^\top + \sigma^2 I_n, \quad (5)$$

where $V$ is $n \times d$ matrix with columns $\{v_i\}_{i=1}^{d}$, $\Lambda_d$ is $d \times d$ diagonal matrix with $\{\lambda_i\}_{i=1}^{d}$ on the diagonal. Note that the observational model (1) in stationary case can be alternatively written as the so-called spike model

$$x(t) = \sum_{i=1}^{d} \sqrt{\lambda_i} u_i(t) v_i + \sigma \xi(t), \quad (6)$$

where $u_i(t)$ are i.i.d. standard Gaussian random variables independent from $\xi(t)$.

For our non-asymptotic error analysis of CPAST, we assume that $d$, $\sigma^2$ and $\lambda_i$, $i = 1, \ldots, d$ are known. With the known $\sigma^2$ we can always normalize the data and therefore without loss of generality we can assume that $\sigma^2 = 1$.

The typical condition while analyzing the quality of the eigenvectors estimation is the so-called spectral gap condition, which says that the adjacent eigenvectors explain distinguishably different portion of the variance in the data, namely there exists $\tau \geq 1$, such that for all $j = 1, \ldots, d$,

$$\tau(\lambda_j - \lambda_{j+1}) \geq \lambda_1, \quad \text{where } \lambda_{d+1} = 0 \text{ by definition.}$$

Since our goal is the estimation of the $d$-dimensional subspace of the first eigenvectors, and we are not interested in the estimation of each particular eigenvector, we need only the condition for the separation of this $d$-dimensional subspace, namely that the gap between $\lambda_d$ and $\lambda_{d+1}$ is sufficiently large:

$$\tau \lambda_d \geq \lambda_1. \quad (7)$$

5
Define a distance $l$ between two subspaces $W$ and $Q$ spanning orthonormal columns $w_1, \ldots, w_d$ and $q_1, \ldots, q_d$ correspondingly via

$$l(W, Q) = l(W, Q) = \|WW^T - QQ^T\|^2,$$  

where the nuclear norm $\|A\|$ of a matrix $A \in \mathbb{R}^{n \times d}$ is defined as $\|A\| = \sup_{x \in \mathbb{R}^d} \frac{\|Ax\|_2}{\|x\|_2}$, and $W = \{w_1, \ldots, w_d\}$ and $Q = \{q_1, \ldots, q_d\}$ are the matrices in $\mathbb{R}^{n \times d}$ with orthonormal columns.

The next result shows that with high probability the subspace which spans the CPAST estimator $\hat{V}(t)$ is close, in terms of $l$, to the subspace spanning $V$ when the number of observation is large enough. We assume that the initial estimator $\hat{V}(t_0) = \hat{V}_0$ is constructed from $t_0$ first observations by means of the singular value decomposition of $\hat{\Sigma}(t_0)$.

**Theorem 1.** Suppose that the spectral gap condition (7) holds and

$$\sqrt{t_0} \geq 4\sqrt{2}R_{\text{max}} \frac{\lambda_1 + 1}{\lambda_d},$$

where $R_{\text{max}} = 5\sqrt{n - d} + 5\sqrt{6\ln(n \vee T)}$. Then after $t - t_0$ iterations we get with probability at least $1 - C_0(n \vee t)^{-2}$,

$$l(V, \hat{V}(t)) \leq C_1 \frac{\lambda_d + 1}{\lambda_d^2} \frac{n - d}{t} + C_2 \frac{\lambda_1 + 1}{\lambda_d} \frac{\log(n \vee t)}{t},$$

where $C_0, C_2$ are absolute constants and $C_1$ depends on $\tau$.

**Remark 1.** The second term on the right-hand side of (9) corresponds to the error of separating the first $d$ eigenvectors from the rest. The first term is an average error of estimating all components of $d$ leading eigenvectors. It originates from the interaction of the noise terms with the different coordinates, see [8].

### 4 Sparse CPAST

#### 4.1 Sparsity assumptions on leading eigenvectors

We assume that in the stationary case [5] the first $d$ leading eigenvectors $v_i, i = 1, \ldots, d$, of $\Sigma$ have most of their entries close to zero. Namely, we
suppose that each $v_i$ fulfills the so-called weak-$l_r$ ball condition [9, 10], that is, for some $r \in (0, 2)$,

$$|v_i|_{(k)} \leq s_i k^{-1/r}, \quad k = 1, \ldots, n.$$  

where $|v_i|_{(k)}$ is the $k$-th largest coordinate of $v_i$. The weak-$l_r$ ball condition is known to be more general than $l_r$ ball condition (which is $\|q\|_r \leq s$ for $q \in \mathbb{R}^n$, $r \in (0, 2)$, $s \geq 1$), as it combines different definitions of sparsity used in statistics, see [11].

Define a thresholding function $g(x, \beta)$ with a thresholding parameter $\beta > 0$ and $x \in \mathbb{R}$ via

$$x - \beta \leq g(x, \beta) \leq x + \beta, \quad g(x, \beta)1_{|x| \leq \beta} = 0. \quad (10)$$

For example, the so-called hard-thresholding function $g_H(x, \beta)$ given by

$$g_H(x, \beta) = x 1_{|x| \leq \beta}$$  

and the so-called soft-thresholding function defined as

$$g_S(x, \beta) = (x - \beta)_+ \cdot \text{sign}(x)$$

fulfill the conditions [10]. When $\beta$ is a vector with components $\beta_i$, $i = 1, \ldots, d$, and $V$ is a matrix with columns $v_i \in \mathbb{R}^n$, $i = 1, \ldots, d$, we denote by $g(V, \beta)$ a $n \times d$ matrix with the elements $\{g(v_{ij}, \beta_i)\}, i = 1, \ldots, d, j = 1, \ldots, n$.

Our primal goal is to propose a subspace tracking method for estimating a $d$-dimensional subspace of the process under weak-$l_r$ ball assumption on the leading eigenvectors of the covariance matrix $\Sigma$ and to analyze it’s convergence.

### 4.2 Initialization and main steps

Our sparse modification of CPAST relies on the orthogonal iteration scheme with an additional thresholding step (cf. [5, 10]). From now on, by a slight abuse of notation, we will denote by $\hat{V}(t)$ an iterative estimator obtained with the help of the modified CPAST, given $t$ observations. To get the initial approximation $\hat{V}(t_0)$, we use the following modification of a standard SPCA scheme, see [5, 10].

1. First compute the empirical covariance $\hat{\Sigma}(t_0)$ based on $t_0$ observations:

$$\hat{\Sigma}(t_0) = \frac{1}{t_0} \sum_{i=1}^{t_0} x(i)x^\top(i).$$
2. Define a set of indices $G$, corresponding to large enough diagonal elements of $\hat{\Sigma}(t_0)$:

$$
G = \left\{ k : \hat{\Sigma}_{kk}(t_0) > 1 + \gamma_0 \frac{\log(n \vee t_0)}{t_0} \right\}
$$

for $\gamma_0 \geq 3\sqrt{2 \log(n \vee T) / \log(n \vee t_0)}$.

3. Let $\hat{\Sigma}_0(t_0)$ be a submatrix of $\hat{\Sigma}(t_0)$ corresponding to the row and column indices in $G \times G$.

4. As an estimator at step zero, we take the first $d$ eigenvectors of $\hat{\Sigma}_0(t_0)$ completed with zeros in the coordinates $\{1, \ldots, n\} \setminus G$ to the vectors of length $n$.

Now we describe a sparse modification of CPAST, which we called SCPAST. We start with $\hat{V}(t_0)$ obtained by the above procedure. Then for $t = t_0 + 1, \ldots, T$, we perform the following steps

1. multiplication: $\hat{\Upsilon}(t) = \hat{\Sigma}(t)\hat{V}(t-1)$,

2. thresholding: define a matrix $\hat{\Upsilon}^\beta(t) = g(\hat{\Upsilon}(t), \beta(t))$,

   where $g$ is a thresholding function satisfying [10] and $\beta(t)$ is the corresponding thresholding vector;

3. orthogonalization:

   $$
   \hat{V}(t) = \hat{\Upsilon}^\beta(t)[\hat{\Upsilon}^\beta(t)^\top \hat{\Upsilon}^\beta(t)]^{-1/2}.
   $$

4.3 Convergence of SCPAST

First we define the thresholding parameter $\beta(t)$ as follows. For $t = t_0 + 1, \ldots, T$ and

$$
a \geq 3\sqrt{2 \log(n \vee T) / \log(n \vee t_0)}
$$

the components $\beta_i(t), i = 1, \ldots, d$ of the vector $\beta(t)$ are given by

$$
\beta_i(t) = a \sqrt{(\lambda_i + 1) \frac{\log(n \vee t)}{t}}.
$$

(12)
The motivation for thresholding of the column vectors of \( \hat{\Upsilon}(t) \) comes from the following connection between sparsity of the leading eigenvectors \( v_j, j = 1, \ldots, d \), and the vector \( \zeta_v \) with the components \( \sqrt{\sum_{j=1}^{d} \lambda_j v_{jk}^2}, k = 1, \ldots, n \). (\( \zeta_{vk}^2 \) is the variance of the \( k \)-th coordinate of the signal part [12]): the weak-\( l_r \) sparsity of the vector \( \zeta_v \) implies the weak-\( l_r \) sparsity of \( v_j, j = 1, \ldots, d \).

Suppose that \( d \) and the eigenvalues \( \lambda_1, \ldots, \lambda_d \) are known. In the case of unknown \( d \) and \( \lambda_1, \ldots, \lambda_d \) one might first estimate the eigenvalues of \( \hat{\Sigma}_0(t_0) \) defined in the previous section and then select the largest set of eigenvalues satisfying the spectral gap condition (7) with some parameter \( \tau \) (see [5] for more details).

Denote by \( S(t) \) the set of indices of “large” eigenvectors components (\( S \) stands for ”signal”), that is, for a fixed \( t \),

\[
S(t) = \left\{ j : |v_{ij}| \geq bh_i \sqrt{\frac{\log(n \lor t)}{t}}, \text{ for some } i = 1, \ldots, d \right\},
\]

where \( h_i = \frac{\sqrt{\lambda_i+1}}{\lambda_i} \) and \( b = \frac{0.1r}{\sqrt{\tau \sqrt{d}}} \). In fact, the quantity \( h_i^2/t \) is an estimate of the noise variance in the entries of the \( i \)-th leading eigenvector [8]. The number of “large” entries of the first \( d \) leading eigenvectors to estimate thus might be estimated by the cardinality of \( S(t) \), which we denote by \( \text{card}(S(t)) \). One can bound \( \text{card}(S(t)) \) as

\[
\text{card}(S(t)) \leq \sum_{i=1}^{d} \text{card}(S_j(t)),
\]

where \( S_j(t) = \left\{ j : |v_{ij}| \geq bh_i \sqrt{\frac{\log(n \lor t)}{t}} \right\} \). From Lemma 14 (see Appendix B) we see that \( d \leq \text{card}(S(t)) \leq CM(t) \), where \( C \) depends on \( b, r \) and

\[
M(t) = n \land \left[ \sum_{j=1}^{d} \frac{s_j^r}{h_j^r} \left( \frac{\log(n \lor t)}{t} \right)^{-r/2} \right].
\]

(13)

Note that in the sparse case, the number of non-zero components \( \text{card}(S(t)) \) is much smaller than \( n \). For example, if \( \|v_j\|_r \leq s, j = 1, \ldots, d \), then

\[
M(t) \leq n \land d \frac{s_j^r}{h_d^r} \left( \frac{\log(n \lor t)}{t} \right)^{-r/2}.
\]
The value $\frac{c}{h_j}$ is often referred to as an effective dimension of the vector $v_j$. Thus $M(t)$ is the number of effective coordinates of $v_j$, $j = 1, \ldots, d$ in the case of disjoint $S_j(t)$.

Since $h_d^2/t$ is an upper-bound for the estimation error for the components of the first $d$ leading eigenvectors, the right hand side of the above inequality gives, up to a logarithmic term, the overall number of components of the $d$ leading eigenvectors to estimate. The next theorem gives non-asymptotic bounds for a distance between $V$ and $\hat{V}(t)$.

**Theorem 2.** Let

$$\sqrt{t_0} \geq \left(C_1 h_d M^{1/2}(T) + C_2 \right) \frac{\lambda_1 + 1}{\lambda_d} \sqrt{\log(n \lor T)},$$

where $C_1$ depends on $\tau$ in (7), $r$, $a$, $C_2$ depends on $\tau$. After $t$ iterations one has with probability at least $1 - C_0(n \lor t)^{-2}$,

$$l(V, \hat{V}(t)) \leq C_1 h_d^2 M(t) \frac{\log(n \lor t)}{t} + C_2 \frac{\lambda_1 + 1}{\lambda_d} \frac{\log(n \lor t)}{t}.$$

(15)

with some absolute constant $C_0 > 0$.

**Remark 2.** The second term in (15) is the same as in the non-sparse case, see Theorem 1. This term is always present as an error of separating the first $d$ eigenvectors from the rest eigenvectors regardless how sparse they are. The first term in (15) and (9) is responsible for the interaction of the noise with different coordinates of the signal. The average error of estimating one entry of the first $d$ leading eigenvectors based on $t$ observation can be bounded by $\frac{1}{t} \frac{\lambda_1 + 1}{\lambda_d}$, see [13]. The number of components to be estimated in SCPAST for each vector is bounded by $M(t)$ (see (13)), which is small compared to $n$ in the sparse case. Thus, the first term in (15) can be significantly smaller than the first one in (9), provided the first $d$ leading eigenvectors are sparse. Note also that the computational complexity of SCPAST at each step $t = t_0 + 1, \ldots, T$ is $O(d \text{card}(S(t)))$ with probability given by Theorem 2.

5 Numerical results

5.1 Single spike

To illustrate the advantage of using SCPAST for the sparse case, we generate $T = 2000$ observations from (6) for the case of a single spike, that is, $d = 1$
Figure 1: The components of the leading eigenvector to recover (a) step function, (b)–(d) contain the results for the error $l(v_1, \hat{v}_1)$ for $\lambda_1 = \{5, 30, 100\}$ and $n = 1024$. Our aim is to estimate the leading eigenvector $v_1$. We shall use three functions depicted in subplots (a) of Fig. 1–3 with different sparsity levels in the wavelet domain.

The observations are generated for the noise variance 1 and following cases of maximal eigenvalue $\lambda_1 \in \{5, 30, 100\}$. We used the Symmlet 8 basis from the Matlab package SPCALab to transform the initial data into the wavelet domain. We applied CPAST and SCPAST for the recovery of wavelet coefficients of the vector $v_1$ and then transformed the estimates to the initial domain and computed the error $l(v_1, \hat{v}_1)$ depending on the number of observations. The results for the hard thresholding (11) with the $a = 1.5$ are shown in Fig. 1–3 in subplots (b)-(d). Note that one peak function has sparser wavelet coefficients than those of three peak functions and the error of the recovery with SCPAST is significantly smaller for the case of one peak.
Figure 2: The components of the leading eigenvector to recover (a) three peaks function, (b)–(d) contain the results for the error $l(v_1, \hat{v}_1)$ for $\lambda_1 = \{5, 30, 100\}$
Figure 3: The components of the leading eigenvector to recover (a) one peak function, (b)–(d) contain the results for the error $l(v_1, \tilde{v}_1)$ for $\lambda_1 = \{5, 30, 100\}$
5.2 Real data example

Natural acoustic signals like the musical ones exhibit a highly varying temporal structure, therefore there is a need in adaptive unsupervised methods for signal processing which reduce the complexity of the signal. In [14] a method was proposed which reduces the spectral complexity of music signals using the adaptive segmentation of the signal in the spectral domain for the principal component analysis for listeners with cochlear hearing loss. In the following we apply CPAST and SCPAST as an alternative method for the complexity reduction of music signals. To illustrate the use of SCPAST and CPAST we set the memory parameter \( \gamma = 0.9 \) to be able to adapt to the changes in the spectral domain of the signal. We focus on the first leading eigenvector recovery. As an example we consider a piece from Bach Siciliano for Oboe and Piano. A wavelet-kind CQT-transform [15] is computed for the signal (see a spectrogram of the transform in Fig. 4). The warmer colors correspond to the higher values of the amplitudes of the harmonics present in the signal at a particular time frame. It is clear that the signal has some regions of “stationarity” (e.g. approximately in time frame interval \([1200, 2600]\)). We regard the corresponding spectrogram as a matrix with 4500 observations of 168-dimensional signal modeled by (16) and apply SCPAST and CPAST methods to recover the leading eigenvector \( v_1 \).

Fig. 5 contains the results of the recovery of the leading eigenvalue with 168 components. The results show that SCPAST method allows to obtain sparse representation of the leading eigenvectors and seems to be promising for construction of the structure-preserving compressed representations of the signals.

6 Sketch of the proofs

Denote by \( \tilde{V} \) a matrix with \( n - d \) column vectors \( v_i \), \( i = d + 1, \ldots, n \), which complete the orthonormal columns \( \{v_i\}_{i=1}^d \) of the matrix \( V \) to the orthonormal basis in \( \mathbb{R}^n \). Denote by \( X(t) \) a matrix with the columns \( \{x(i)\}_{i=1}^t \). From (16) one gets a representation

\[
X(t) = V \Lambda_d^{1/2} U^\top(t) + \sigma \Xi(t), \quad t = 1, \ldots, T,
\]
Figure 4: CQT-Spectrogram of Bach Siciliano for Oboe and Piano. The deep blue color corresponds to the zero values, the red color corresponds to the higher values.

where \( U(t) \in \mathbb{R}^{t \times d} \), \( \Xi(t) \in \mathbb{R}^{n \times t} \) are matrices with independent \( \mathcal{N}(0,1) \) entries, \( V \) is the orthonormal matrix with columns \( \{v_i\}_{i=1}^n \), \( \Lambda_d \) is a diagonal matrix with \( \lambda_i, i = 1, \ldots, d \) on the diagonal. Denote a set of indices to the small components of leading eigenvectors as \( N(t) = \{1, \ldots, n\} \setminus S(t) \) (where \( N \) here stands for “noise”).

From (16) the empirical covariance matrix can be decomposed as

\[
\hat{\Sigma}(t) = \frac{1}{t} V \Lambda_d^{1/2} U^\top(t) U(t) \Lambda_d^{1/2} V^\top + \frac{1}{t} \Xi(t) \Xi^\top(t) \\
+ \frac{1}{t} V \Lambda_d^{1/2} U^\top(t) \Xi^\top(t) + \frac{1}{t} \Xi(t) U(t) \Lambda_d^{1/2} V^\top.
\]  

(17)

It is well known \( \bigcirc \) that the distance (8) between subspaces \( \mathcal{W} \) and \( \mathcal{Q} \), spanning \( n \times d \) matrices with orthonormal columns \( W \) and \( Q \) correspondingly, is related to \( d \)-th principal angle between subspaces \( \mathcal{W} \) and \( \mathcal{Q} \) as \( l(\mathcal{W}, \mathcal{Q}) = \sin^2 \phi_d(\mathcal{W}, \mathcal{Q}) \), where the principal angles \( 0 \leq \phi_1 \leq \cdots \leq \phi_d \).
between subspaces $W$ and $Q$ are recursively defined as \[16\]

$$\phi_i(W, Q) = \arccos \frac{\langle x_i, y_i \rangle}{\|x_i\|_2 \|y_i\|_2}, \text{ where}$$

$$\{x_i, y_i\} = \arg\min_{x \in W, y \in Q, x \perp x_j, y \perp y_j, j < i} \left\{ \arccos \frac{\langle x, y \rangle}{\|x\|_2 \|y\|_2} \right\}.$$  

From the variational characterization of the singular values and the above definition of the principal angles, the $d$-th principal angle between subspaces spanning the columns of $W$ and $Q$ has the following non-recursive definition

$$\cos \phi_d(W, Q) = \min_{\|x\|_2 = 1, x \in \mathbb{R}^d} \frac{\|W^T Q x\|}{\|Q x\|}, \quad (18)$$

$$\tan \phi_d(W, Q) = \max_{\|x\|_2 = 1, x \in \mathbb{R}^d} \frac{\|W^T Q x\|}{\|W^T Q x\|}. \quad (19)$$

In the next sections we derive the error bounds for CPAST and SCPAST by looking at the change of the $d$-th principal angle between the eigensubspace spanning the columns of $V$ and its estimators based on $t$ observation, where $t = t_0 + 1, \ldots, T.$
6.1 Bound for CPAST

The aim of this section is to show that with the high probability the subspace which spans CPAST estimator \( \hat{V}(t) \) (6.1) is close to the subspace, which spans \( V \) when the number of observations is large enough. We assume that the initial estimator \( \hat{V}^0 \) is constructed from first \( t_0 \) observations with the help of SVD of \( \hat{\Sigma}(t_0) \). Let us first state the bound for the error \( l(\hat{V}(t), V) \) which depends on the error on the previous iteration \( l(\hat{V}(t - 1), V) \) for the fixed \( t = t_0 + 1, \ldots, T \). Denote \( r(t) = l^{1/2}(\hat{V}(t), V) = \sin \phi_d(\hat{V}(u), V) \).

**Lemma 1.** For CPAST (4) with probability \( 1 - C_0(n \lor t)^{-3} \)

\[
0 \leq \frac{(\lambda_{d+1} + 1) \tan \phi_d(\hat{V}(t - 1), V)}{\lambda_d + 1 - (\lambda_1 + 1)E(t) \sec \phi_d(\hat{V}(t - 1), V)} + \frac{(\lambda_1 + 1)^{1/2}E(t) \sec \phi_d(\hat{V}(t - 1), V)}{\lambda_d + 1 - (\lambda_1 + 1)E(t) \sec \phi_d(\hat{V}(t - 1), V)},
\]

where

\[
E(t) = 5\sqrt{\frac{n - d}{t}} + 5\sqrt{6}\sqrt{\log(n \lor t)}.
\]

The following lemma gives the bound for the error \( l(\hat{V}(u), V) \) depending on the error of the previous iteration \( l(\hat{V}(u - 1), V) \) for all \( u \in \{t_0 + 1, \ldots, t\} \).

**Lemma 2.** With probability greater than \( 1 - C_0(n \lor t)^{-3} \)

\[
0 \leq \frac{\alpha_0 r(u - 1) + \alpha_1 \frac{R(t)}{\sqrt{u}}}{\sqrt{1 - r^2(u - 1) - \alpha_2 \frac{R(t)}{\sqrt{u}}}},
\]

where \( r(u) = \sin \phi_d(\hat{V}(u), V) \),

\[
R(t) = 5\sqrt{n - d} + 5\sqrt{6}\sqrt{\log(n \lor t)},
\]

\[
\alpha_0 = \frac{1}{\lambda_d + 1}, \quad \alpha_1 = \frac{\sqrt{\lambda_1 + 1}}{\lambda_d + 1}, \quad \alpha_2 = \frac{\lambda_1 + 1}{\lambda_d + 1}.
\]
Given Lemma 2 it is possible to derive a bound for \( l(\hat{V}(t), V) \). First let us state the result which allows to bound the error of the initial estimate \( \hat{V}(t_0) \).

**Lemma 3.** Let \( \hat{V}(t_0) \) be a matrix containing first \( d \) leading eigenvectors of the matrix \( \hat{\Sigma}(t_0) \). Then with probability \( 1 - C_0(n \lor T)^{-2} \)

\[
r^2(t_0) = l(\hat{V}(t_0), V) \leq \alpha^2 \frac{1}{t_0},
\]

where \( \alpha = R_{\text{max}} \frac{\lambda_1 + 1}{\lambda_d} \), with \( R_{\text{max}} = R(T) \).

The following Lemma gives the error of CPAST after observing \( K \) vectors \( x_i, i = t_0 + 1, \ldots, t_0 + K \) based on the recursive bound (21). Note that the proof of Lemma 4 also insures that the denominator in (21) is bounded away from zero.

**Lemma 4.** Suppose that \( r(t_0) \leq \alpha \frac{1}{\sqrt{t_0}} \), where

\[
\sqrt{t_0} \geq 8R_{\text{max}} \frac{\alpha_2}{(1 - \alpha_0)^{3/2}}.
\]

Then for \( K \geq K(\rho, t_0) \)

\[
r(t_0 + K) \leq 2 \frac{\alpha_1}{\alpha_0} \frac{1}{1 - \alpha_0} \frac{R(t_0 + K)}{\sqrt{K + t_0}}.
\]

The statement of the Theorem 1 follows from Lemma 2 applied to the inequality (21) with (22), which holds with probability \( 1 - C_0(n \lor t)^{-2} \) with the initial conditions given by Lemma 3.

### 6.2 Bound for SCPAST

Define \( \hat{\Sigma}^\circ(t) \), the oracle version of \( \hat{\Sigma}(t) \) and the corresponding expectation \( \Sigma^\circ(t) = E[\hat{\Sigma}^\circ(t)] \) as follows

\[
\hat{\Sigma}^\circ(t) = \begin{bmatrix} \hat{\Sigma}_{S(t)} & 0 \\ 0 & I_{N(t)} \end{bmatrix}, \quad \Sigma^\circ(t) = \begin{bmatrix} \Sigma_{S(t)} & 0 \\ 0 & I_{N(t)} \end{bmatrix},
\]

where \( \hat{\Sigma}_{S(t)} \) and \( \Sigma_{S(t)} \) are the sub-matrices of the size \( \text{card}(S(t)) \times \text{card}(S(t)) \) with column and row indices from \( S(t) \). The identity matrix \( I_{N(t)} \) has the
size $\text{card}(N(t)) \times \text{card}(N(t))$. Here we assumed without loss of generality that indices in $S(t)$ are always smaller than ones in $N(t)$.

First we obtain the oracle sequence $\hat{V}^o(t)$ of the solutions by iterating SCPAST with matrices $\hat{\Sigma}^o(t)$ instead of $\hat{\Sigma}(t)$. We define the initial estimate $\hat{V}^o(t_0)$ with the steps (a)-(d) in the section 6.1 applied to the matrix $\hat{\Sigma}^o(t_0)$. And then bound $\sin \phi_d(\hat{V}^o(t), V)$. Denote the result of the thresholding the columns of the matrix $\hat{\Upsilon}^o(t) = \hat{\Sigma}^o(t) \hat{V}(t-1)$ with the thresholding parameters given by the vector $\beta(t)$ as

$$
\hat{\Upsilon}^{o,\beta}(t) = g(\hat{V}^o(t), \beta(t)).
$$

Denote the submatrix $V_S(t)$ obtained by selecting the rows of $V$ with indices in $S(t)$. Denote by $V_k$, $k = 1, \ldots, n$ the rows of $V$ (recall that the columns are $v_j$, $j = 1, \ldots, m$). For the estimators of $V$ we omit the dependence of $S(t)$ on $t$ as the estimator itself depends on $t$, that is, $\hat{V}_S(t)$ is a matrix of the rows of $\hat{V}(t)$ with indices from $S(t)$.

The following bound for the oracle error $r^2(t) = l(\hat{V}^o(t), V)$ of SCPAST method is analogous to Lemma 2.

**Lemma 5.** For $u = t_0 + 1, \ldots, t$ with probability greater than $1 - C_0(n \vee t)^{-3}$ the following bound holds true

$$
r(u) \leq \frac{\alpha_0 r(u - 1) + \alpha_1 \frac{R^o(t)}{\sqrt{u}}}{\sqrt{1 - r^2(u - 1) - \alpha_2 \frac{R^o(t)}{\sqrt{u}}}}, \quad \text{where}
$$

$$
\alpha_0 = (\lambda_d + 1)^{-1}, \quad \alpha_1 = \alpha_2 = (\lambda_1 + 1)/(\lambda_d + 1), \quad \alpha = \left(\frac{1}{\lambda_d} C_1 \lambda_1 h_d M^{1/2}(t_0) + C_2 \lambda_1 \lambda + 1 \right) \frac{1}{\lambda_d} \sqrt{\log(n \vee t)},
$$

$$
R^o(t) = C_1 h_d M^{1/2}(t) \sqrt{\log(n \vee t)} + C_2 \sqrt{\log(n \vee t)}, \quad \text{where } C_0 \text{ and } C_2 \text{ are constants and } C_1 \text{ depends on } r, d, a, \tau.
$$

In the sparse case the similar result to Lemma 3 holds true giving a bound on the error of the initial oracle estimator.

**Lemma 6.** The error of initial oracle estimation is bounded as follows $r(t_0) \leq \frac{\alpha}{\sqrt{t_0}}$, with probability $1 - C_0(n \vee T)^{-2}$, where

$$
\alpha = \left(\frac{1}{\lambda_d} C_1 \lambda_1 h_d M^{1/2}(t_0) + C_2 \lambda_1 \lambda + 1 \right) \frac{1}{\lambda_d} \sqrt{\log(n \vee T)},
$$

where $C_0$ and $C_2$ are constants and $C_1$ depends on $r$. 

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Lemma 7. Thus after $t$ iterations, $t = t_0 + 1, \ldots, T$, with the probability $1 - C_0(n \lor t)^{-2}$ one has

$$l(V, \hat{V}^\circ(t)) \leq C_1 h_d^2 M(t) \frac{\log(n \lor t)}{t} + C_2 \frac{\lambda_1 + 1 \log(n \lor t)}{\lambda_d^2 t},$$

where $C_1$ depends on $d$, $r$, $\tau$, $a$, and $C_2$ depends on $\tau$.

The convergence of the oracle scheme doesn’t immediately imply the convergence of the SCPAST estimators. The following two lemmas state that with the high probability $\hat{V}^\circ(t) = \hat{V}(t)$. Thus the bound in Lemma 7 holds for SCPAST and the Theorem 2 is justified.

Lemma 8. For $\gamma_0 \geq 3\sqrt{2} \frac{\log(n \lor T)}{\log(n \lor t_0)}$ with probability $1 - C_0(n \lor T)^{-2}$ the initial oracle estimate coincide with the initial SPCA estimate, that is, $\hat{V}^\circ(t_0) = \hat{V}(t_0)$.

Lemma 9. With probability $1 - C_0(n \lor t)^{-2}$ for $u = t_0 + 1, \ldots, t$ the oracle SCPAST and SCPAST solutions coincide $\hat{V}^\circ(t) = \hat{V}(t)$.

Conclusions

We developed a new method SCPAST based on constraint projection approximation subspace tracking method for subspace tracking in the sparsity assumptions on the underlying signal eigen subspace. The thresholding step was introduced in order to ensure the sparsity of the solution. We presented the non-asymptotical bounds for the errors of subspace recovery with SCPAST and CPAST as well as the empirical studies of the methods. The results of experiments show that SCPAST method allows to obtain sparse representation of the leading eigenvector of music signals and might be used for adaptive compression of the musical signal in the spectral domain.
Appendix A. Proofs of Lemmas

Proof of Lemma 1

From the definition (19) \( \tan \phi_d(\hat{\Sigma}(t)\hat{V}(t-1), V) = \max_{\|x\|_2=1} \frac{\|V^T\hat{\Sigma}(t)\hat{V}(t-1)x\|}{\|V^T\hat{\Sigma}(t)\hat{V}(t-1)x\|} \).

Using \( \hat{\Sigma}(t) = \hat{\Sigma}(t) - \Sigma + \Sigma \) the latter maximum can be bounded by

\[
\max_{\|x\|_2=1} \frac{\|V^T\hat{V}(t-1)x\| + \|V^T(\hat{\Sigma}(t) - \Sigma)\hat{V}(t-1)x\|}{(\lambda_d + 1)\|V^T\hat{V}(t-1)x\| - \|V^T(\hat{\Sigma}(t) - \Sigma)\hat{V}(t-1)x\|}.
\]

Using (18)

\[
\max_{\|x\|_2=1} \frac{\|V^T(\hat{\Sigma}(t) - \Sigma)\hat{V}(t-1)x\|}{\|V^T\hat{V}(t-1)x\|} \leq \frac{\|V^T(\hat{\Sigma}(t) - \Sigma)\|}{\cos \phi_d(\hat{V}(t-1), V)},
\]

\[
\max_{\|x\|_2=1} \frac{\|V^T(\hat{\Sigma}(t) - \Sigma)\hat{V}(t-1)x\|}{\|V^T\hat{V}(t-1)x\|} \leq \frac{\|V^T(\hat{\Sigma}(t) - \Sigma)\|}{\cos \phi_d(\hat{V}(t-1), V)}.
\]

From (17)

\[
\hat{\Sigma}(t) = \Sigma + \frac{1}{t}\Xi(t)\Xi^T(t) - I_n \\
+ \frac{1}{t}VA_d^{1/2}U(t)^T\Xi(t) + \frac{1}{t}\Xi(t)U(t)A_d^{1/2}V^T \\
+ VA_d^{1/2} \left( \frac{1}{t}U^T(t)U(t) - I_d \right) A_d^{1/2}V^T. \tag{27}
\]
Therefore using $VV^T + VV^T = I_n$

\[
\|\tilde{V}^T(\tilde{\Sigma}(t) - \Sigma)\| \leq \left\| \frac{1}{t}V^T\Xi(t)[V^T\Xi(t)]^T - I_{n-d} \right\| \\
+ \frac{1}{t}\|V^T\Xi(t)[V^T\Xi(t)]^T\| \\
+ \sqrt{\lambda_1} \frac{1}{t}\|V^T\Xi(t)U(t)\|,
\]

\[
\|V^T(\tilde{\Sigma}(t) - \Sigma)\| \leq \lambda_1 \left\| \frac{1}{t}U^T(t)U(t) - I_d \right\| \\
+ \left\| \frac{1}{t}V^T\Xi(t)[V^T\Xi(t)]^T \right\| \\
+ \left\| \frac{1}{t}V^T\Xi(t) \left[ V^T\Xi(t) \right]^T - I_d \right\| \\
+ 2\sqrt{\lambda_1} \left\| \frac{1}{t}\Xi(t)U(t) \right\|.
\]

Using $\sqrt{\lambda_1} \leq \sqrt{\lambda_1 + 1} \leq \lambda_1 + 1$, Lemma 10 and Lemma 11 with $p = \sqrt{6}$ we bound the terms to the right of the above two inequalities with the probability $1 - C_0(n \vee t)^{-3}$, for big enough $t$

\[
\|\tilde{V}^T(\tilde{\Sigma}(t) - \Sigma)\| \leq \sqrt{\lambda_1} E(t), \\
\|V^T(\tilde{\Sigma}(t) - \Sigma)\| \leq (\lambda_1 + 1) E(t),
\]

where $E(t) = 5\sqrt{\frac{n-d}{t}} + 5\sqrt{6\frac{\log(n \vee t)}{t}}$, with probability $1 - C_0(n \vee t)^{-3}$, where $C_0$ is a constant. The statement of the lemma follows from the observation that

\[
\tan \phi_d(V(t), V) = \tan \phi_d(\tilde{\Sigma}(t) \tilde{V}(t-1), V).
\]

**Proof of Lemma 2**

Lemma 1 gives a probabilistic bound on the error of subspace estimation $l(\tilde{V}(t), V)$ based on the previous iteration $l(\tilde{V}(t-1), V)$. Our goal is to bound $l(\tilde{V}(t), V)$ for $t \in \{t_0 + 1, \ldots, T\}$. Due to Lemmas 10 and 11 we get for $p > 1$, $u = t_0 + 1, \ldots, t$ the term $\sqrt{u} \left\| \frac{1}{u} V^T \Xi(u) [V^T \Xi(u)]^T - I_d \right\|$ is bounded from above by $3(\sqrt{d} + p \sqrt{\log(n \vee t)})$, $\sqrt{u} \left\| \frac{1}{u} V^T \Xi(u) [V^T \Xi(u)]^T - I_{n-d} \right\|$ by
3(\sqrt{n - d} + p\sqrt{\log(n \lor t)}), \sqrt{u} \left\| \frac{1}{u}V^\top \Xi(u)[\bar{V}^\top \Xi(u)]^\top \right\| \leq \sqrt{1 + 2p \frac{\log(n \lor t)}{\sqrt{u}}} (\sqrt{n - d} + \sqrt{d} + p\sqrt{\log(n \lor t)}). \text{ Finally, } \sqrt{u} \left\| \frac{1}{u}V^\top \Xi(u)U(u) \right\| \text{ is bounded by } \sqrt{1 + 2p \frac{\log(n \lor t)}{\sqrt{u}}} (\sqrt{n - d} + \sqrt{d} + p\sqrt{\log(n \lor t)}).

Each of the bounds holds with the probability \( 1 - (n \lor t)^{-3} \) for \( p = \sqrt{6} \). Using the union bound we get the statement of the Lemma for the intersection of events with the probability \( 1 - C_0(t - t_0)(n \lor t)^{-3} \).

**Proof of Lemma 3**

The proof is based on Davis sin \( \theta \) Theorem [4], Lemma [10] (see Appendix B), and Weyl's theorem [17]. From Davis sin \( \theta \) Theorem

\[
I(V, \hat{V}(t_0)) \leq \left\| (\hat{\Sigma}(t_0) - \Sigma)V \right\|^2 \frac{1}{\left( \lambda_d + 1 - \lambda_{d+1}(\hat{\Sigma}(t_0)) \right)^2}, \tag{29}
\]

where \( \lambda_{d+1}(A) \) is a \((d + 1)\)-th singular value of the matrix \( A^\top A \). Weil's theorem gives for \( j = 1, \ldots, n \)

\[
|\lambda_j + 1 - \lambda_j(\hat{\Sigma}(t_0))| \leq \left\| \hat{\Sigma}(t_0) - \Sigma \right\|.
\]

Therefore the denominator in (29) may be bounded as

\[
|\lambda_d + 1 - \lambda_{d+1}(\hat{\Sigma}(t_0))| \geq \lambda_d - 2\left\| \hat{\Sigma}(t_0) - \Sigma \right\|.
\]

From (27)

\[
\left\| \hat{\Sigma}(t_0) - \Sigma \right\| \leq \lambda_1 \left\| \frac{1}{t_0}U(t_0)^\top U(t_0) - I_d \right\| + \left\| \frac{1}{t} \Xi(t_0) \Xi^\top(t_0) - I_n \right\| + 2\sqrt{\lambda_1} \left\| \frac{1}{t_0}U^\top(t_0) \Xi^\top(t_0) \right\|,
\]

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by Lemma 10 and 11

$$|\lambda_d + 1 - \lambda_{d+1}(\hat{\Sigma}(t_0))| \geq (1 + o(1))\lambda_d.$$  

From (28) and Lemma 10 one has that with probability 1

$$1 - C_0(n \lor t)^{-2}$$

$$\|\hat{\Sigma}(t_0) - \Sigma\| \leq (\lambda_1 + 1) \left( \sqrt{\frac{n}{t_0}} + \sqrt{\frac{\log(n \lor t)}{t_0}} \right).$$

Combining the last two inequalities we get the statement of Lemma.

**Proof of Lemma 4**

First we prove (23) for the pair \((t_0, r(t_0))\) which satisfies by induction for all \(k = 1, \ldots, K\), and some \(\rho \in (\alpha_0, 1)\)

$$\sqrt{1 - \left( r(t_0) + R \left( \frac{\alpha_1}{\alpha_0} \right) \sqrt{\frac{1}{1 - \rho \frac{1}{\sqrt{t_0}}} \right)^2} - \frac{\alpha_2 R}{\sqrt{t_0}} > \frac{\alpha_0}{\rho}. \quad (30)$$

We have for \(K = 1\)

$$r(t_0 + 1) \leq \frac{\alpha_0 r(t_0) + \alpha_1 \frac{R}{\sqrt{t_0 + 1}}}{\sqrt{1 - r^2(t_0)}} - \frac{\alpha_2 \frac{R}{\sqrt{t_0 + 1}}}{\alpha_0 \sqrt{t_0 + 1}} \leq \rho r(t_0) + \frac{\rho \alpha_1}{\alpha_0} \frac{R}{\sqrt{t_0 + 1}}.$$  

Furthermore suppose that (30) holds for \(K = L\), then

$$r^2(t + L) \leq \left( \rho^L r(t_0) + R \left( \frac{\alpha_1}{\alpha_0} \right) \sum_{k=1}^{L} \rho^{L+1-k} \right)^2$$

and

$$r(t_0 + L + 1) \leq \frac{\alpha_0 r(t + L) + \alpha_1 \frac{R}{\sqrt{t + L + 1}}}{\sqrt{1 - r^2(t + L)}} - \frac{\alpha_2 \frac{R}{\sqrt{t + L + 1}}}{\alpha_0 \sqrt{t + L + 1}}$$

$$\leq \rho^L r(t_0) + R \left( \frac{\alpha_1}{\alpha_0} \right) \sum_{k=1}^{L+1} \rho^{L+1-k} \frac{1}{\sqrt{t_0 + k}}.$$
A sufficient condition for the above formula to hold reads as

\[ \sqrt{1 - \left( \rho^{k-1} r(t_0) + \frac{\alpha_1 R}{\alpha_0} \sum_{j=1}^{k-1} \rho^{k-j} \frac{\rho^j}{\sqrt{t_0+j}} \right)^2 - \frac{\alpha_2 R}{\sqrt{t_0+k}}} > \frac{\alpha_0}{\rho}. \]

Note that \( \sum_{j=1}^{k-1} \frac{\rho^{k-j}}{\sqrt{t_0+j}} \leq \frac{\rho}{1-\rho} \frac{1}{\sqrt{t_0}} \), therefore the above condition is fulfilled given (30). Furthermore

\[ r(t_0 + K + 1) \leq \rho^K r(t_0) + R \left( \frac{\alpha_1}{\alpha_0} \sum_{k=1}^{K+1} \rho^{1+K-k} \right) \sum_{j=1}^{K} \frac{\rho^j}{\sqrt{t_0+k}}, \]

where for \( K > K_0(\rho), \ t_0 > 1 \) and \( j_{K,\rho} = \log(K)/(2 \log(1/\rho)) \)

\[ \sum_{k=1}^{K} \frac{\rho^{K-k}}{\sqrt{t_0+k}} \leq \sum_{j=0}^{j_{K,\rho}} \frac{\rho^j}{\sqrt{t_0+K-j}} + \sum_{j=j_{K,\rho}+1}^{K-1} \frac{\rho^j}{\sqrt{t_0+K-j}} \]

\[ \leq \frac{1}{1-\rho} \frac{1}{\sqrt{t_0+K-j_{K,\rho}}} + \frac{1}{1-\rho} \frac{1}{\sqrt{K+t_0}} \]

and

\[ r(t_0 + K + 1) \leq \frac{\alpha_1}{\alpha_0} \frac{\rho}{1-\rho} \frac{R}{\sqrt{K+t_0+1}}. \]

From (30) the condition on the starting value \( r(t_0) \) is

\[ r(t_0) \leq \sqrt{1 - \left( \frac{\alpha_2 R}{\sqrt{t_0}} + \frac{\alpha_0}{\rho} \right)^2 - \frac{R\alpha_1}{\alpha_0} \frac{\rho}{1-\rho} \frac{1}{\sqrt{t_0}}}. \] \( (31) \)

Thus the number of initial observations \( t_0 \) for (31) to be satisfied given \( r(t_0) \leq \frac{\alpha_2}{\sqrt{t_0}} \) reads as

\[ \frac{\alpha}{\sqrt{t_0}} \leq \sqrt{1 - \left( \frac{\alpha_2 R}{\sqrt{t_0}} + \frac{\alpha_0}{\rho} \right)^2 - \frac{R\alpha_1}{\alpha_0} \frac{\rho}{1-\rho} \frac{1}{\sqrt{t_0}}}. \]

Therefore, taking into account (26) and \( \rho > \alpha_0 \), the sufficient condition on \( \sqrt{t_0} \) is

\[ \sqrt{t_0} \geq \frac{2\alpha_2 R}{\left(1 - \frac{\alpha_0^2}{\rho^2}\right)} + \frac{2R \left( \frac{\alpha_1}{\alpha_0} \frac{\rho}{1-\rho} \right)}{\sqrt{1 - \frac{\alpha_0^2}{\rho^2}}}. \]
From Lemma 3 and (26) \( \alpha = R_{\text{max}}^{\alpha_2} \). Set \( \rho = \rho(\epsilon) = 1 - \epsilon(1 - \alpha_0) \). It is easy to check that \( \alpha_0 < \rho(\epsilon) < 1 \) for \( \epsilon \in (0, 1/2] \). Recall \( R_{\text{max}} = R(T) \), therefore

\[
\sqrt{t_0} \geq \frac{2R_{\text{max}}\alpha_2}{\epsilon(1 - \alpha_0)^{3/2}} \frac{1}{\sqrt{1 - \epsilon}}.
\]

The value \( K_0(\rho) \) might be defined by \( \rho^K \leq \frac{\alpha_1}{\alpha_0 + \sqrt{t_0}} \), thus (using \( |\ln(1 - x)| \geq x \) for \( x \in (0, 1) \)) it is sufficient to set \( K \geq \frac{1}{\epsilon(1 - \alpha_0)} \ln \left( \frac{\alpha_0}{\alpha_1} \sqrt{T} \right) \). Put \( \epsilon = 1/2 \) to get the result.

**Proof of Lemma 5**

Using the triangle inequality

\[
\sqrt{l(\hat{V}_o(t), V)} \leq \sqrt{l(\hat{Y}_o(t), V)} + \sqrt{l(\hat{Y}_o(t), \hat{V}_o(t))}. \tag{32}
\]

We bound the first term as

\[
l^{1/2}(\hat{Y}_o(t), V) \leq \tan \phi_d(\hat{Y}_o(t)\hat{V}_o(t - 1), V).
\]

Using the variational definition of \( \tan \phi_d \)

\[
l^{1/2}(\hat{Y}_o(t), V) \leq \max_{\|x\|_2 = 1} \frac{\|V^T\hat{Y}_o(t)\hat{V}_o(t - 1)x\|}{\|V^T\hat{Y}_o(t)\hat{V}_o(t - 1)x\|}
\]

The right hand side may be bounded with

\[
\max_{\|x\|_2 = 1} \frac{\|V^T\Sigma\hat{V}_o(t - 1)x\| + \|\hat{V}_o(t) - \Sigma\hat{V}_o(t - 1)x\|}{\|V^T\Sigma\hat{V}_o(t - 1)x\| - \|\hat{V}_o(t) - \Sigma\hat{V}_o(t - 1)x\|}
\]

Triangle inequality gives

\[
\|\hat{X}_o(t) - \Sigma\hat{V}\| \leq \|\hat{X}_o(t) - \Sigma\hat{V}\| + \|\hat{X}_o(t) - \Sigma\hat{V}\|.
\]

Note that

\[
\hat{X}_o(t) - \Sigma = \begin{bmatrix} \hat{X}_S(t) - \Sigma_S(t) & 0 \\ 0 & 0 \end{bmatrix}, \tag{33}
\]

\[
\Sigma_0(t) - \Sigma = \begin{bmatrix} 0 & -V_S(t)\Lambda_dV_S^T(t) \\ -V_N(t)\Lambda_dV_N^T(t) & -V_N(t)\Lambda_dV_N^T(t) \end{bmatrix}. \tag{34}
\]
where $V_S(t)$ is a submatrix of $V$ with the row indices in $S(t)$. Decompose $\hat{\Sigma}_S(t) - \Sigma_S(t)$ using (27) and (33)

$$
\| (\hat{\Sigma}^o(t) - \Sigma^o(t)) \bar{V} \| \leq \lambda_1 \| \bar{V}_S(t) \| \left\| \frac{1}{t} U(t)^\top U(t) - I_d \right\|
$$

$$
+ \left\| \frac{1}{t} \Xi_S(t) \Xi_S^\top(t) - I_S(t) \right\| + 2\sqrt{\lambda_1} \left\| \frac{1}{t} U(t)^\top \Xi_S(t) \right\|,
$$

where $\Xi_S(t)$ is $t \times \text{card}(S(t))$ matrix, $U(t)$ is $t \times d$ matrix. The elements of both matrices are i.i.d. $\mathcal{N}(0,1)$. Using $\bar{V}^\top V = \bar{V}_S(t)^\top V_S(t) + \bar{V}_N(t)^\top V_N(t) = 0$ we may bound

$$
\| \bar{V}_S(t)^\top V_N(t) \| \leq \| \bar{V}_N(t)^\top V_N(t) \| \leq \| V_N(t) \|_F,
$$

where $\| \cdot \|_F$ is Frobenius norm, i.e. $\| A \|_F = \sqrt{\text{tr}(A^\top A)}$ for any matrix $A$, is small since it depends only on the components of the eigenvectors below the corresponding thresholds (see Lemma [13] and definition [13])

$$
\| V_N(t) \|_F^2 = \sum_{i=1}^d \| v_{i,N}(t) \|^2 
\leq \sum_{j=1}^d \left[ \frac{2}{2 - r [\log(n \lor t)]^2} \right] b_j^2 h_j^2 \frac{\log(n \lor t)}{t}
\leq C M(t) h_d^2 \frac{\log(n \lor t)}{t},
$$

where $C$ depends on $d, r$.

From Lemma [10], [11] and [12] (see Appendix B) with the probability $1 - C\_0 (n \lor t)^{-3}$ one can bound

$$
\| (\hat{\Sigma}^o(t) - \Sigma^o(t)) \bar{V} \| \leq C_1 \lambda_1 h_d M^{1/2}(t) \frac{\sqrt{\log(n \lor t)}}{\sqrt{t}}
$$

$$
+ C_2 (\sqrt{\lambda_1} \lor 1) \frac{\sqrt{\log(n \lor t)}}{\sqrt{t}}.
$$

From (34) $\| V^\top (\Sigma^o(t) - \Sigma) \| \leq \lambda_1 \| V_N(t) \|$. Thus

$$
\| (\hat{\Sigma}^o(t) - \Sigma^o(t)) \bar{V} \| \leq C_1 \lambda_1 h_d M^{1/2}(t) \sqrt{\frac{\log(n \lor t)}{t}}
$$

$$
+ C_2 (\sqrt{\lambda_1} \lor 1) \sqrt{\frac{\log(n \lor t)}{t}}. \tag{35}
$$
where $C_1$ depends on $r$, $d$ and $C_2$ is a constant. Similarly, from (34) $\| V^\top (\Sigma^o(t) - \Sigma) \| \leq |V_N(t)|(1 + o(1))$ and

$$\| V^\top (\widehat{\Sigma}^o(t) - \Sigma) \| \leq C_1 \lambda_1 h_d M^{1/2}(t) \sqrt{\frac{\log(n \vee t)}{t}}$$

$$+ C_2 (\sqrt{\lambda_1} \vee 1) \sqrt{\frac{\log(n \vee t)}{t}},$$

where $C_1$ depends on $r$, $d$ and $C_2$ is a constant.

The bound on $l(\widehat{\Omega}^o(t), \widehat{\Omega}^o(t)) = l(\widehat{\Omega}^o(t), \widehat{\Omega}^o(t))$ relies on Wedin’s sin $\theta$ Theorem $\mathbf{B}$ (see Appendix B)

$$l(\widehat{\Omega}^o(t), \widehat{\Omega}^o(t)) \leq \frac{\| \widehat{\Sigma}^o(t) \widehat{\Omega}^o(t - 1) - \widehat{\Sigma}^o(t) \widehat{\Omega}^o(t - 1) \|}{\lambda_d (\widehat{\Sigma}^o(t) \widehat{\Omega}^o(t - 1))}.$$ (37)

Note that $\| \widehat{\Sigma}^o(t) \widehat{\Omega}^o(t - 1) - \widehat{\Omega}^o(t - 1) \| \leq \| Z(t) \|_F$, where $Z_{ij}(t)$ is a matrix with the entries $Z_{ij}(t) = \beta_j(t)$ if $i \in S(t)$ and $Z_{ij}(t) = 0$ if $i \in N(t)$. Thus $\| \widehat{\Sigma}^o(t) \widehat{\Omega}^o(t - 1) - \widehat{\Omega}^o(t - 1) \|^2 \leq CM(t) \sum_{i=1}^d \beta_i^2(t)$ and from (12)

$$\sum_{i=1}^d \beta_i^2(t) \leq a^2 \frac{\log(n \vee t)}{t} \sum_{i=1}^d (\lambda_i + 1) \leq da^2 \frac{\lambda_1}{t} \frac{\log(n \vee t)}{t} h_d^2$$

That is

$$\| \widehat{\Sigma}^o(t) \widehat{\Omega}^o(t - 1) - \widehat{\Omega}^o(t - 1) \|^2 \leq C M(t) \frac{2 \lambda_1 \log(n \vee t)}{t} h_d^2,$$

where $C'$ depends on $d$, $a$ and $r$.

To bound the denominator of (37), note that one may decompose $\| \Sigma \widehat{\Omega}^o(t - 1)x \|^2 = \| \Sigma z_1 \|^2 + \| \Sigma z_2 \|^2$, where $\widehat{\Omega}^o(t - 1)x = z_1 + z_2$ and $z_1 \in \text{ran}(V)$ and $z_2 \in \text{ran}(\widehat{V})$. Thus $\| \Sigma \widehat{\Omega}^o(t - 1)x \|^2 \geq \| \Sigma z_1 \|^2$. Using $z_1 \in \text{ran}(V)$ one has

$$\| \Sigma z_1 \| \geq (\lambda_d + 1) \| z_1 \| \geq (\lambda_d + 1) \cos(V, \widehat{\Omega}^o(t - 1))$$

and taking into account (34) we get

$$\lambda_d^{1/2} \left( \Sigma^o(t) \widehat{\Omega}^o(t - 1) \right) \geq (\lambda_d + 1) \cos(V, \widehat{\Omega}^o(t - 1))$$

$$- \| \widehat{\Sigma}^o(t) - \Sigma^o(t) \| - \lambda_1 \| V_N(t) \|.$$
Thus using (33) and Lemmas 10, 11 and summarizing the bounds for denominator and nominator in (37) we get
\[
1^{1/2}(\hat{\Upsilon}(t), \hat{V}(t)) \leq \frac{\lambda_d CM^{1/2}(t) \sqrt{\log(n\lor t)} h_d}{(\lambda_d + 1) \cos(V, \hat{V}(t - 1)) - E^o(t)},
\]
where \(E^o(t) = (C_1\lambda_d h_d M^{1/2}(t) + C_2(\sqrt{\lambda_1} \lor 1)) \sqrt{\log(n\lor t)} \).

Combining the above inequality, (35), (36), (32) and the spectral gap condition (7) we get the result in the flavour of (1), that is with probability \(1 - C_0(n\lor t)^{-3}\) for one step of SCPAST algorithm. To get the bounds for \(u = t_0 + 1, \ldots, t\) simultaneously, similarly to Lemma 2 define the events, each of which occurs with probability \(1 - C_0(n\lor t)^{-3}\), namely that \(\sqrt{u} \left\| \frac{1}{u} U(u)^\top U(u) - I_d \right\| \) is bounded from above by \(2(\sqrt{d} + p\sqrt{\log(n\lor t)}), \sqrt{u} \left\| \frac{1}{u} \Xi(u) \Xi^\top(u) - I_S(u) \right\| \)
by \(2(\sqrt{\text{card}(S(t))} + p\sqrt{\log(n\lor t)}), \sqrt{u} \left\| \frac{1}{u} U(u)^\top \Xi_S(u) \right\| \) by \(1 + 2p\frac{\log(n\lor t)}{\sqrt{u}}(\sqrt{\text{card}(S(t))} + \sqrt{d} + p\sqrt{\log(n\lor t)})\). Taking the intersection of the above events for \(u = t_0 + 1, \ldots, t\) and using Lemma 12 we get the statement of the Lemma.

**Proof of Lemma 6**

Using Wedin sin \(\theta\) Theorem 3 (Appendix B)
\[
l(V, \hat{V}(t_0)) \leq \frac{\left\| V^\top (\Sigma - \hat{\Sigma}(t_0)) \right\|^2}{(\lambda_d - \lambda_{d+1}(\hat{\Sigma}(t_0))^2),}
\]
(38)

Using Weyl theorem 17 it may be shown that \(\lambda_{d+1}(\hat{\Sigma}(t_0)) = \lambda_{d+1} + o(\lambda_1)\) and thus \(|\lambda_d - \lambda_{d+1}(\hat{\Sigma}(t_0))| \geq \lambda_d(1 + o(1)).\) From (36) with probability \(1 - (n\lor T)^{-2}\)
\[
\left\| (\hat{\Sigma}(t_0) - \Sigma)V \right\| \leq C_1\lambda_1 h_d M^{1/2}(t_0) \sqrt{\frac{\log(n\lor t)}{t_0}}
+ C_2(\sqrt{\lambda_1} \lor 1) \sqrt{\frac{\log(n\lor T)}{t_0}}.
\]

Thus \(r(t_0) \leq \alpha \frac{\alpha}{\sqrt{t_0}}\) holds with probability \(1 - (n\lor T)^{-2}\), where
\[
\alpha = \left( \frac{1}{\lambda_d} C_1\lambda_1 h_d M^{1/2}(t_0) + C_2 \frac{\sqrt{\lambda_1 + 1}}{\lambda_d} \right) \sqrt{\log(n\lor T)}.
\]
Proof of Lemma 7
The proof follows from Lemma 4 applied to (25) with \( \alpha_0 = \frac{1}{\lambda_{d+1}} \), \( \alpha_1 = \alpha_2 = \frac{\lambda_{d+1}}{\lambda_{d+1}} \), and initial conditions given by Lemma 6.

Proof of Lemma 8
Following [12] define \( \eta_j = \sum_{i=1}^{d} \lambda_i v_{ji}^2 \), \( j = 1, \ldots, n \) and for \( 0 < a_- < 1 \) define \( G^+ = \left\{ j : \eta_j > a_- \gamma_0 \sqrt{\frac{\log(n \vee t_0)}{t_0}} \right\} \). To show that \( \hat{V}^o(t_0) = \hat{V}(t_0) \) one has to prove that for the proper choice of \( \gamma_0 \) and \( a_- \) it holds \( G \subseteq G^+ \subseteq S(t_0) \) with probability \( 1 - C_0 (n \vee T)^{-2} \). To show that we first note that \( \hat{\Sigma}_{jj}(t_0) \sim (1 + \sum_{i=1}^{d} \lambda_i v_{ji}^2) \xi/t_0 \), where \( \xi \) is \( \chi^2 \) r.v. Therefore

\[
P(G \not\subset G^+) = \mathbb{P} \left\{ \bigcup_{j \notin G^+} \left( \hat{\Sigma}_{jj}(t_0) > 1 + \gamma_0 \sqrt{\frac{\log(n \vee t_0)}{t_0}} \right) \right\}
\]

\[
\leq \sum_{j \notin G^+} \mathbb{P} \left( \hat{\Sigma}_{jj}(t_0) > 1 + \gamma_0 \sqrt{\frac{\log(n \vee t_0)}{t_0}} \right)
\]

\[
\leq n \mathbb{P} \left( \xi > 1 + \frac{\gamma_0 (1 - a_-) \sqrt{\log(n \vee t_0) / t_0}}{1 + a_- \gamma_0 \sqrt{\log(n \vee t_0) / t_0}} \right)
\]

\[
\leq \sqrt{n} \exp \left\{ - \frac{\gamma_0^2 (1 - a_-^2) \log(n \vee t_0)}{4 \left( 1 + a_- \gamma_0 \sqrt{\log(n \vee t_0) / t_0} \right)^2} \right\}
\]

\[
\leq \frac{\sqrt{n}}{\gamma_0} \exp \left\{ - \frac{\gamma_0^2 (1 - a_-^2)}{4 (1 + a_- \gamma_0 \sqrt{\log(n \vee t_0) / t_0})} \right\}
\]

Thus \( G \subset G^+ \) holds with probability \( 1 - C_0 (n \vee T)^{-2} \), e.g. for \( a_- = 1 - \sqrt{2}/\sqrt{3} \), \( \gamma_0 \geq 3 \sqrt{2} \sqrt{\frac{\log(n \vee T)}{\log(n \vee t_0)}} \). Note that for any \( j \in G^+ \) there exists \( i \in \{1, \ldots, d\} \), \( \lambda_i v_{ji}^2 \geq \frac{a_- \gamma_0}{d} \sqrt{\frac{\log(n \vee t_0)}{t_0}} \), thus for \( G^+ \subset S(t_0) \) to hold it is sufficient that

\[
\frac{a_-}{d \lambda_i} \sqrt{\frac{\log(n \vee T)}{t_0}} > b \lambda_i + 1 \sqrt{\frac{\log(n \vee t_0)}{t_0}}.
\]

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Thus for sufficiently big $T$, $G \cap S(t_0) = G$, that is $\hat{V}^o(t_0) = \hat{V}(t_0)$ with probability $1 - C_0(n \vee T)^{-2}$.

**Proof of Lemma 9**

From Lemma 8 with probability $1 - (n \vee T)^{-2}$ the results of the original and oracle version of the zero-step estimation procedure coincide, that is $V^o(t_0) = \hat{V}(t_0)$. First let us show that the similar statement holds for $\hat{V}(t_0 + 1)$ and $\hat{V}(t_0)$. Denote $t_1 = t_0 + 1$. On the event for which $\hat{V}^o(t_0) = \hat{V}(t_0)$ holds it is true that $\hat{Y}(t_1) = \hat{\Sigma}(t_1)\hat{V}(t_0) = \hat{\Sigma}(t_1)\hat{V}^o(t_0)$. From the construction of $\hat{V}^o(t_0)$, the submatrix $\hat{V}_N^o(t_0)$ has zero entries. Note that $S(t_0) \subseteq S(t_1)$ and $N(t_1) \subseteq N(t_0)$. Thus

$$\hat{Y}(t_1) = \hat{\Sigma}_{k,S}(t_1)\hat{v}_{k,S}(t_0), \quad (39)$$

where $\hat{v}_{i,S}(t_0)$ is a vector of size $\text{card}(S(t_1))$ containing the components of $\hat{v}_i(t_0)$ indexed by $S(t_1)$, $\hat{\Sigma}_{k,S}(t_1)$ is a row containing the components of $k$-th row of $\hat{\Sigma}(t_1)$ indexed by $S(t_0)$.

Let us show that for $k \in N(t_1)$ with high probability, which is equivalent to

$$\hat{Y}(t_1) \leq \beta(t_1) = a\sqrt{(\lambda t + 1) \frac{\log(n \vee t_1)}{t_1}},$$

that is during the thresholding step the components from $N(t_1)$ would be set to zero with high probability. From $[16]$

$$t_1\hat{\Sigma}_{k,S}(t_1) = V_k\Lambda_d^{1/2}U(t_1)^T U(t_1)\Lambda_d^{1/2}V_S^\top(t_1)$$

$$+ \Xi_k(t_1)\mathbb{X}_S^\top(t_1)$$

$$+ V_k\Lambda_d^{1/2}U(t_1)^T \Sigma_S^\top(t_1)$$

$$+ \Xi_k(t_1)U(t_1)\Lambda_d^{1/2}V_S^\top(t_1),$$

(40)

where $\Xi_k(t_1)$ is $k$-th row of $\Xi(t_1)$. Denote by $V_S^o(t_1)$ a matrix containing the first $d$ eigenvalues of $\Sigma_S(t_1)$ as columns (recall (24)) and by $V_S^o(t_1)$ a matrix with $\text{card}(S(t_1)) - d$ columns which complete columns of $V_S^o(t_1)$ to the orthonormal basis in $\mathbb{R}^{	ext{card}(S(t_1))}$. Note that

$$\hat{V}_S^o(t_0)\hat{V}_S^o(t_0)^\top + V_S^o(t_0)\Sigma_S^\top V_S^o(t_0) = I_S(t_0).$$

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Plugging in above equality in (39) (before $V^\top_S (t_1)$) in the view of (40) one gets

$$\tilde{\Upsilon}(t_1)_{k,l} = q_{11} + q_{12} + q_{14} + q_{21} + q_{22} + q_{24},$$

where $q$-s with the listed below with the first index 1 depend on $V^\top_S (t_0)$ and with the first index 2 depend on $V^\top_S (t_0)$. Let us first bound the terms $q_{11}$ and $q_{21}$. To this end we add and subtract $V_k \Lambda_d V^\top_S (t_1)$ in the first term in (40) and use that $\| U(t_1)^\top U(t_1) - I \| = o(1)$, thus

$$|q_{11}| \leq (1 + o(1)) \| V_k \Lambda_d \| \| V^\top_S (t_0) \tilde{v}^o_{i,S}(t_0) \|$$

(41)

where it was also used that $\| V^\top_S (t_1) \tilde{v}^o_S(t_0) \| \leq 1$.

Consider $k \in N(t_1)$, that is, $|V_k| = |v_{kl}| \leq b \sqrt{h_{i1}^2 \log(n \vee t_1) / t_1}$. Using the definition of $\beta_k(t_1)$ (recall (12))

$$\| V_k \Lambda_d \| \leq \frac{b}{a} \beta_k(t_1) \left( \sum_{i=1}^d \frac{\lambda_i + 1}{\lambda_k + 1} \right),$$

(42)

$$\| V_k \Lambda_d^{1/2} \| \leq \frac{b}{a} \beta_k(t_1) \left( \sum_{i=1}^d \frac{\lambda_i + 1}{(\lambda_k + 1) \lambda_i} \right).$$

(43)

Thus using (42) and (43) the term (41) may be bounded as

$$|q_{11}| \leq (1 + o(1)) \frac{b}{a} \beta_k(t_1) \| V^\top_S (t_0) \tilde{v}^o_{i,S}(t_0) \| \left( \sum_{i=1}^d \frac{\lambda_i + 1}{\lambda_i + 1} \right)$$

and in the same way it can be shown that

$$|q_{21}| \leq (1 + o(1)) \frac{b}{a} \beta_k(t_1) \| V^\top_S (t_0) \tilde{v}^o_{i,S}(t_0) \| \left( \sum_{i=1}^d \frac{\lambda_i + 1}{\lambda_i + 1} \right).$$

Next

$$|q_{12}| = \frac{1}{t_1} |\Xi_k(t_1) \Xi^\top_S (t_1) V^\top_S (t_0) \tilde{v}^o_{i,S}(t_0) |$$

$$\leq \frac{1}{t_1} \zeta(k, S(t_1)) \| \Xi^\top_S (t_1) \| \| V^\top_S (t_0) \tilde{v}^o_{i,S}(t_0) \|.$$
Thus for big enough \( t_1 \), \( k \in N(t_1) \), and since \( N(t_1) \subseteq N(t_0) \), \( \Xi_k(t_1) \) is independent from \( \Xi_S(t_1) \), thus \( \zeta(k, l, S(t_1)) \) has \( N(0, 1) \) distribution. Define the events

\[
|\zeta(k, l, S(t_1))| \leq \sqrt{c_1 \log(n \lor t)}
\]

and

\[
|\Xi_S(t_1)| \leq \sqrt{t_1 + \sqrt{\text{card}(S(t_1))}} + 2\sqrt{\log(n \lor t)}.
\]

For big enough \( t_1 \) (guaranteed by (14)) \( t_1 \) dominates \( \text{card}(S(t_1)) \) and \( \log(n \lor t) \). Thus

\[
|q_{12}| \leq \frac{1}{t_1} |\zeta(k, S(t_1))| |\Xi_S^\top(t_1)||V_{\theta_S^0}(t_1)\hat{v}_{t,S}^0(t_0)|
\]

\[
\leq \frac{1}{a} \left( \frac{c_1}{\lambda_i + 1} \right)^{1/2} \frac{\beta_i(t_1)}{\log(n \lor t_1)} |\log(n \lor t)| |V_{\theta_S^0}(t_1)\hat{v}_{t,S}^0(t_0)|.
\]

On the events defined in the end of the proof of Lemma 5 the bound for the term \( q_{13} \) is as follows

\[
|q_{13}| \leq \frac{1}{t_1} |V_k\Lambda_1^{1/2}||U(t_1)^\top \Xi_S(t_1)||V_{\theta_S^0}(t_1)\hat{v}_{t,S}^0(t_0)|
\]

\[
\leq \frac{b \text{card}(S(t_1))}{a} \frac{\log(n \lor t)}{\log(n \lor t_1)} \frac{\beta_i(t_1)}{\text{card}(S(t_1))} |V_{\theta_S^0}(t_1)\hat{v}_{t,S}^0(t_0)|.
\]

From \( \frac{1}{\lambda_i} \frac{\text{card}(S(t_1))}{t_1} = o(1) \) (see Supplementary materials for 5 p.16) it follows that \( |q_{13}| = o(\beta_i(t_1)) \). To bound the term \( q_{14} \) one may utilize the same argument as for \( q_{12} \)

\[
|q_{14}| \leq \frac{1}{t_1} |g(k, S(t_1))| |U(t_1)^\top \Lambda_1^{1/2}||V_{\theta_S^0}(t_1)\hat{v}_{t,S}^0(t_0)|,
\]

where

\[
g(k, l, S(t_1)) = \frac{t_1 q_{14}}{|U(t_1)^\top \Lambda_d^{1/2} V_S^\top(t_1) V_{\theta_S^0}(t_1) V_{\theta_S^0}(t_1) \hat{v}_{t,S}^0(t_0)|}
\]

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Similarly to the case of $\tilde{v}_{l,S}(t_0)$, furthermore $g(k,l,S(t_1))$ has $\mathcal{N}(0,1)$ distribution. Define the following two events \{\|U(t_1)\| \leq \sqrt{1+d+2\log(n \lor t)}\} and \{\|g(k,l,S(t_1))\| \leq \sqrt{c_1 \log(n \lor t)}\}. On these events with probability $1 - C_0(n \lor t)^{-3}$

$$|q_{14}| \leq \frac{1}{t_1} g(k,l,S(t_1)) \|U(t_1)\| \|V_{1/2}^\top (t_1) \tilde{v}_{l,S}(t_0)\|$$

$$\leq \frac{1}{a} \left( \frac{c_1 \lambda_1}{\lambda_1 + 1} \right) \|V_{1/2}^\top (t_1) \tilde{v}_{l,S}(t_0)\| \frac{\log(n \lor t)}{\log(n \lor t_1)}.$$

In the similar way term $q_{22}$ may be bounded as follows

$$q_{22} \leq \left\| \frac{1}{t_1} \sum_k \Xi_k \Xi_{S(t_1)}^\top V_{S}^\top (t_1) \right\| \|V_{S}^\top (t_1) \tilde{v}_{l,S}(t_0)\|$$

$$\leq \frac{b}{a} (1 + o(1)) \sqrt{\frac{c_2 \lambda_1}{\lambda_1 + 1} \beta_l(t_1) \frac{\log(n \lor t)}{\log(n \lor t_1)} \|V_{S}^\top (t_1) \tilde{v}_{l,S}(t_0)\|}.$$

The bound on the term $q_{23}$ due to (43) reads as

$$|q_{23}| = \frac{1}{t_1} \|V_{k} \Lambda_{1/2}^\top \| \|U(t_1)^\top \Xi_{S(t_1)} V_{S}^\top (t_1) \| \|V_{S}^\top (t_1) \tilde{v}_{l,S}(t_0)\|$$

$$\leq 2 \sqrt{\frac{\log(n \lor t_1)}{t_1}} \|V_{k} \Lambda_{1/2}^\top \| \|V_{S}^\top (t_1) \tilde{v}_{l,S}(t_0)\|$$

$$= o(\beta_l(t_1)).$$

Similarly to the case of $q_{14}$ one can show that

$$|q_{24}| \leq \frac{1}{t_1} \|\Xi_k U(t_1)\| \|V_{k} \Lambda_{1/2}^\top V_{S}^\top (t_1) V_{S}^\top (t_1) \| \|V_{S}^\top (t_1) \tilde{v}_{l,S}(t_0)\|$$

$$\leq \sqrt{\frac{c_2 \lambda_1}{\lambda_1 + 1} \frac{1}{a} \beta_l(t_1) \frac{\log(n \lor t)}{\log(n \lor t_1)} \|V_{S}^\top (t_1) \tilde{v}_{l,S}(t_0)\|}.$$

Note that (see [3]) $\|V_{S}^\top (t_1) \tilde{v}_{l,S}(t_0)\| = 1 + o(1)$ and $\|V_{S}^\top (t_1) \tilde{v}_{l,S}(t_0)\| = o(1)$, that is from above bounds $\sum_{i=1}^4 |q_i| = o \left( \sum_{i=1}^4 |q_{2i}| \right)$. Therefore

$$\hat{\gamma}(t_1)_{k,l} \leq \frac{b}{a} \beta_l(t_1) \sqrt{\sum_{j=1}^d \frac{\lambda_j + 1}{\lambda_j + 1} \beta_l(t_1) \sqrt{2c_1 \frac{\log(n \lor t_1)}{\log(n \lor t_1)} \lambda_j + 1}}.$$
Observe that \( \sqrt{\sum_{j=1}^{d} \frac{\lambda_j+1}{\lambda_j+1}} \leq \sqrt{\tau} \sqrt{d} \) and \( \lambda_1/\lambda_j \leq \tau \). Let us bound \( \log(n \lor t) \) by \( \log(n \lor T) \) thus

\[
a \geq \sqrt{2c_1} \frac{\log(n \lor T)}{\log(n \lor t_0)}, \quad b = \frac{0.9a - \sqrt{2c_1} \frac{\log(n \lor T)}{\log(n \lor t_0)}}{\sqrt{\tau} \sqrt{d}}.
\]

Therefore one gets for all \( k \in N(t_1) \)

\[
|\hat{\Upsilon}(t_1)_{k,l}| \leq \beta_l(t_1)
\]

and \( \hat{V}^*_N(t_0+1) = 0 \), and so, \( \hat{V}_N^*(t_0+1) = \hat{V}_N(t_0+1) \).

To show that

\[
\hat{V}^*_N(u) = \hat{V}_N(u), \quad u = t_0 + 2, \ldots, t
\]

we consider the events defined for the standard normal random variables \( z(k, l, S(u)) \) and \( g(k, l, S(u)) \)

\[
\{z(k, l, S(u)) \leq \sqrt{c_1 \log(n \lor t)}\},
\]

\[
\{|g(k, l, S(u))| \leq \sqrt{c_1 \log(n \lor t)}\}.
\]

Using the union bound

\[
P \left\{ \bigcup_{l \in N(u), k=1,\ldots,d, \atop u=t_0+1,\ldots,t} \{z(k, l, S(u)) \leq \sqrt{c_1 \log(n \lor t)}\} \right\}
\]

\[
\leq 1 - \sum_{k=1}^{d} \sum_{u=t_0+1}^{t} \sum_{l \in N(t)} P\{z(k, l, S(u)) \leq \sqrt{c_1 \log(n \lor t)}\}
\]

\[
\leq 1 - nd(t - t_0)P\{z(k, l, S(t)) \leq \sqrt{c_1 \log(n \lor t)}\}
\]

\[
\leq 1 - C_0n(t - t_0)\log(n \lor t)^{-1}(n \lor t)^{-c_1/2}.
\]

Take \( c_1 \geq 9 \) to obtain the statement of the lemma.
Appendix B. Concentration of the spectral norm of the perturbation

Denote $\delta_{n,t} = \log(n \lor t)$.

**Lemma 10.** [18] Let $X$ be a $t \times n$ matrix with i.i.d. $\mathcal{N}(0,1)$ entries. The following result holds true

$$
P \left( \left\| \frac{1}{t}X^\top X - I_n \right\| \geq E_1(t, n, p) \right) \leq 2(n \lor t)^{-p^2/2},$$

where

$$E_1(t, n, p) = 3 \max \left( \sqrt{\frac{n}{t}} + p \frac{\sqrt{\delta_{n,t}}}{\sqrt{t}}, \left[ \sqrt{\frac{n}{t}} + \frac{p \sqrt{\delta_{n,t}}}{\sqrt{t}} \right]^2 \right).$$

**Lemma 11.** [19] Let $X$ and $Y$ be $t \times q$ and $t \times m$ matrices, $q > m$, with i.i.d. $\mathcal{N}(0,1)$ entries then for any $0 < x < 1/2$ and $c > 0$

$$
P \left( \|XY\| \geq t E_2(t, q, m, x, c) \right) \leq e^{-\frac{x^2 \delta_{n,t}}{2}} + q e^{-\frac{3x^2 \delta_{n,t}}{16}},$$

where

$$E_2(t, q, m, x, c) = \sqrt{1 + x \frac{\delta_{n,t}}{t}} \left( \sqrt{\frac{q}{t}} + \sqrt{\frac{m}{t}} + c \frac{\sqrt{\delta_{n,t}}}{\sqrt{t}} \right).$$

**Lemma 12.** [5] There exist constants $\tilde{C}_1$ and $\overline{C}_1$ depending on $r$ and $\tilde{C}_2$ and $\overline{C}_2$ such that

$$E_1(t, \text{card}(S(t)), p) \leq \tilde{C}_1 \frac{\lambda_1 M^{1/2}(t)}{\sqrt{t}} h_d + \tilde{C}_2 \frac{\delta_{n,t}}{t},$$

$$E_2(t, \text{card}(S(t)), d, 4, p) \leq \overline{C}_1 \frac{M^{1/2}(t) h_d}{\sqrt{t}} + \overline{C}_2 \frac{\delta_{n,t}}{t}.$$

**Theorem 3.** *(Wedin sin $\theta$)* Let $A$ and $B$ be $n \times k$, $n \geq k$, full-column rank matrices. Let the columns of a $n \times (n - k + 1)$ matrix $U$ be the orthogonal matrices spanning the orthogonal complement of range of $B$. If the $\lambda_{\min}(A) \geq \epsilon \geq 0$ then

$$l(A, B) \leq \frac{\|A^\top U\|^2}{\epsilon^2} \leq \frac{\|B - A\|^2}{\epsilon^2}.$$
Theorem 4. (Davis sin θ) [20] Let A and B be the symmetric matrices with the decomposition $A = W_1 \Lambda_1 W_1^\top + W_2 \Lambda_2 W_2^\top$ and $B = U_1 \Delta_1 U_1^\top + U_2 \Delta_2 U_2^\top$, with conditions $[U_1, U_2]$ is orthogonal, $W_2$ is orthonormal and $W_1^\top W_1 = 0$, the eigenvalues of $\Lambda_1 W_1^\top W_1$ are contained in the interval $(a_1, a_2)$ and the eigenvalues of $\Delta_1$ are laying outside of the interval $(a_1 - \epsilon, a_2 - \epsilon)$ for some $\epsilon > 0$ then

$$l(W_1, U_1) \leq \frac{\|U_2^\top (B - A) W_1\|^2}{\epsilon \lambda_{\min}^2(W_1)}.$$ 

Lemma 13. The norms of the subvectors $v_{j,N(t)}$ of $v_j$ satisfy

$$\|v_{j,N(t)}\|^2 \leq \left[ \frac{2}{2 - r \left(bh_j(t) \right)^r} \left( \frac{\delta_{n,t}}{t} \right)^{-r/2} \wedge n \right] b^2 h_j^2(t) \frac{\delta_{n,t}}{t}.$$ 

Lemma 14. Bound on the effective dimension $\text{card} (S(t))$ is given by

$$d \leq \text{card} (S(t)) \leq CM(t) = C \left[ n \wedge \sum_{j=1}^{d} s_j^r h_j^{-r} \left( \frac{\delta_{n,t}}{t} \right)^{-r/2} \right].$$ 

Lemma 15. For a $\chi^2_t$ random variable $\zeta_t$ the following bounds hold [21]

$$\text{P}(\zeta_t > t(1 + \epsilon)) \leq e^{-\frac{3t \epsilon^2}{16}}, \quad 0 < \epsilon < 1/2,$$

$$\text{P}(\zeta_t < t(1 - \epsilon)) \leq e^{-\frac{t \epsilon^2}{4}}, \quad 0 < \epsilon < 1,$$

$$\text{P}(\zeta_t > t(1 + \epsilon)) \leq \frac{\sqrt{2}}{\epsilon \sqrt{t}} e^{-\frac{t \epsilon^2}{4}}, \quad 0 < \epsilon < t^{1/16}, \quad t > 16.$$ 

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