Studying thermodynamics in heavy ion collisions

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Abstract

We discuss the possibility to measure entropy of the system created in heavy ion collisions using the Ma coincidence method.

1 Introduction

The assumption of thermodynamic equilibrium is one of the most commonly used when discussing the system created in central collisions of two relativistic nuclei. It is by no means obvious, however, that the equilibration actually can be achieved, since it is recognized as a process which may take longer time than the life time of the system in question. Be it or not, it is certainly important to verify if the created system is indeed in thermal equilibrium. To test this, one needs to check if the various measured quantities do satisfy the relations following from thermodynamics. In the present note we discuss the possibility of testing the relations [1]

\[
\left. \frac{\partial S(E, n)}{\partial E} \right|_n = \frac{1}{T},
\]

(1)

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and, with \( \mu \) denoting the chemical potential,

\[
\frac{\partial S(E,n)}{\partial n} \bigg|_E = -\frac{\mu}{T},
\]

which should be valid in any system at thermal equilibrium.

Testing (1) requires measurement of the temperature \( T \), the energy \( E \), the number of particles \( n \) and the entropy \( S \) of the system in question. It is clear that by measuring the energies of the particles created in the collision we can measure the energy of the system. It is also generally accepted that by measuring the slope in the transverse momentum distribution we can measure the temperature\(^1\). The real difficulty is the measurement of entropy. In the present note we propose to adapt to this end the coincidence method proposed some time ago by Ma \(^2\). We also present Monte Carlo estimates of the feasibility of the method, based on a simple model. We conclude that the method has a large potential, as it requires much smaller number of events \( \sim \sqrt{\text{no. of states}} \), then the conventional approach. As a consequence its errors are significantly reduced compared to the simple-minded estimates. It is certainly worth to employ it in the present and future high energy experiments.

## 2 Measurement of entropy by the coincidence method

Below we give a summary of the idea presented in \(^2\).

Ma proposes to count the pairs of configurations of the investigated system. Call \( N_c \) the number of pairs of ”identical” configurations. Call \( N_t \) the total number of pairs of configurations. If all configurations considered are ”equivalent” (i.e. if they correspond to the same conditions), then entropy is given by the formula

\[
S = \log \left( \frac{N_t}{N_c} \right).
\]

The reason is that \( \frac{N_t}{N_c} \) is the volume in the phase-space occupied by the system. This can be seen as follows.

\(^1\)This requires correction to the effects of the hydrodynamic flow which seem to be under a reasonable control.
Suppose that the phase-space is divided in cells. Suppose furthermore that our system occupies $\Gamma$ cells (with uniform probability). Each cell represents a different state of the system (each cell has as many dimensions as is the number of variables describing the system). Our problem is to calculate $\Gamma$: $S = \log(\Gamma)$. Let us select randomly $N$ configurations of the system (in general $N \ll \Gamma$, that’s the main point). These configurations occupy some cells. The average occupation number of a cell is $N/\Gamma \ll 1$. Under this condition, the average number of pairs in the same cell is

$$\left(\frac{N}{\Gamma}\right)^2 \approx \frac{N_t}{\Gamma^2}, \quad (4)$$

where $N_t \approx N^2$ is the total number of pairs selected. The total number of coincidences is the sum of (4) over all cells

$$N_c = \Gamma \left(\frac{N}{\Gamma}\right)^2, \quad (5)$$

hence

$$\Gamma = \frac{N_t}{N_c}, \quad (6)$$

and thus (3).

If the configurations are not equivalent, one has to divide them into classes: within each class they are now equivalent. If the probability distribution of classes is $P(\lambda)$, then

$$S = \sum_{\lambda} P(\lambda) \log \left( \frac{N_t(\lambda)}{N_c(\lambda)P(\lambda)} \right). \quad (7)$$

The derivation is given in [2] but it can be easily understood as a sum of the ”average over classes” $= \sum_{\lambda} P(\lambda) \log \left( \frac{N_t(\lambda)}{N_c(\lambda)} \right)$ and of the ”entropy of the distribution of classes” $= - \sum_{\lambda} P(\lambda) \log[P(\lambda)]$.

The classes cannot be too small, so that number of configurations in each class is sufficient to make a reasonable statistics.

This is what we retain from Ma. In the next section we present a suggestion how to apply this method to measure the entropy of a system of particles produced in high energy interactions.

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2The formula (4) is only approximate. The exact formula is $\frac{N_t}{\Gamma^2}$ which leads again to (6).
3 Application of coincidence method to multiparticle production

A natural possibility to apply the coincidence method to multiparticle production is to identify the configurations of the statistical system with the events observed in experiment. Once this is accepted, one can proceed as follows.

(a) Select \(\mathcal{N}\) events and split them into classes according to the total transverse energy \(E\) and multiplicity \(n\) recorded. The number of events in each class is denoted by \(\mathcal{N}(E, n)\).

(b) Define a "lattice" in momentum space (e.g., rapidity, azimuth and \(\log p_t\), or \(\log E_t\)) of individual particles.

Within each class:

(i) Call the two configurations "identical" if they have the same occupation numbers within the accuracy of the grid. The number of such pairs is denoted by \(N_c(E, n)\).

(ii) Calculate the ratio (8), i.e.,

\[
S(E, n) = \log \left( \frac{N_t(E, n)}{N_c(E, n)} \right) = \log \left( \frac{\mathcal{N}(E, n)(\mathcal{N}(E, n) - 1)}{N_c(E, n)} \right)
\]  

where \(\mathcal{N}(E, n)\) is the number of events in a given class.

Actually, the condition that the events in one "equivalence class" must have strictly the same multiplicity (otherwise they could never be really identical) could be relaxed, e.g. by accepting that in the definition of the "identity" of the two configurations, the occupation numbers may differ by a small amount.

If this procedure is going to have any sense, the results should depend on the lattice spacing \(a\) in a trivial way: when \(a \to a'\), \(S \to S' = S + \log(a/a')\) (once the spacing is small enough). This must be checked, of course.

Also question of size of the energy bins must be analysed. Again, the result should depend on the chosen size of the energy bins in a trivial way: if the size of the energy bin \(\Delta\) changes into \(\Delta'\) the entropy \(S\) changes into \(S' = S + \log(\Delta/\Delta')\). But this is delicate: the bins should be small enough so that within a bin the energy may be considered constant but large enough

\[3\] Probably the better method is to transform the momenta into variables which give uniform distributions (see e.g. [3]).
so that there is a reasonable statistics within each bin. Same applies to the multiplicity.

One sees from these arguments that in this way one can measure the entropy only up to an additive constant. Therefore the interesting thing is not to measure the absolute value of entropy but rather its dependence on energy or multiplicity.

As we already noted in the Introduction, the measurement described by (8) allows one to perform a simple test of thermalization. When thermodynamics is valid, then the Eqs. (1,2) should be satisfied. Clearly the additive constant is irrelevant. One needs, however, a rather precise measurements because otherwise the numerical estimates of the derivatives \( \frac{\partial S(E,n)}{\partial E} \) and \( \frac{\partial S(E,n)}{\partial n} \) are not reliable. In the next section we show the results of a simple Monte Carlo estimate of the accuracies one can achieve.

4 The classical gas of identical particles

Of course the main problem is for how big systems the coincidence method works in practice. The number of states grows exponentially with the number of particles and the number of subdivisions. Therefore obviously there is a limit to what one can achieve with the finite computer. We will show here however that, the onset of the thermodynamic behaviour occurs still for the sizes where the coincidence method is feasible practically.

Second, we will also show what must be the minimal size of the system for the continuum behaviour to set in. For very small number of subdivisions the coincidence method and our Monte Carlo are quickly convergent, however they converge to the correct description of the discrete problem, which is far from the continuum, hence not interesting.

We consider the classical gas of noninteracting, nonrelativistic particles in \( d \) dimensions. Since, as mentioned earlier, the method may be applied to the transverse degrees of freedom only we prefer to retain the discussion in arbitrary dimensions. For the same reason we will use the number of degrees of freedom \( N \) to characterise the size of a system. Of course \( N = d n \) in \( d \) dimensions \(^4\).

Since for noninteracting particles momentum and space degrees of freedom factorize, we consider for simplicity only the momentum states. The

\(^4\)Provided only the momentum degrees of freedom are considered.
discretized expression for the number of states of \(N\) degrees of freedom with the total energy \(E\) reads

\[
\Gamma(M, N) = \sum_{n_1, \ldots, n_N, n_1^2 + \ldots + n_N^2 = M} 1, \tag{9}
\]

where the momentum \(p_i = an_i\) with some discretization scale \(a\). Accordingly \(2mE = a^2M\), where the integer \(M\) labels the energy of the system and \(m\) denotes the mass of a particle.

The generating function

\[
Z_N(\tau) = \sum_{M=0}^{\infty} \tau^M \Gamma(M, N) = \left( \sum_{n=-\infty}^{\infty} \tau^{n^2} \right)^N = \exp \left( N \log c(\tau) \right), \tag{10}
\]

factorizes and is expressed by a single sum \(c(\tau)\). The coefficients \(\Gamma\) can now be simply obtained by calculating recursively expansion of the \(f(\tau) = \log c(\tau)\) from that of \(c(\tau)\), and subsequently expansion of \(Z_N(\tau)\) from that of \(f(\tau)\). This procedure provided us with the exact numbers for the density of states \(\Gamma(M, N)\), which were used to benchmark the performance of our Monte Carlo.

For large \(M\) and \(N\) the density of states reaches its continuum limit

\[
\Gamma(M, N) \cong \frac{\pi^{N/2} M^{(N/2-1)}}{(N/2-1)!}, \quad M, N \text{ large.} \tag{11}
\]

We will see later that this relation is rather well satisfied even for moderate values of \(M\) and \(N\).

On the other hand, the thermodynamic limit, \(M, N \to \infty, M/N\)-fixed, is reached rather slowly. In this limit the entropy density \(^5\) scales depending only on the energy density \(\epsilon = M/N\).

\[
\frac{1}{N} \log \Gamma(M, N) \cong \frac{1}{2} (\log(\epsilon) + \log(2\pi) + 1), \quad M, N \to \infty, \epsilon = M/N - \text{const.} \tag{12}
\]

The purpose of this exercise is to see if the coincidence method can detect this behaviour.

\(^5\)Here and in the following, we will refer to the entropy per one degree of freedom as the entropy density.
4.1 Monte Carlo simulations

In principle one should generate a sample of $\mathcal{N}$ configurations $\{n_1, n_2, \ldots, n_N\}_k$, $k = 1, \ldots, \mathcal{N}$ of, integer-valued, one-dimensional momenta $n_1, \ldots, n_N$, which satisfy the energy conservation. For our purpose, however, the details of particle kinematics, although practically cumbersome, are not relevant. In order to measure the coincidences, it is enough to label uniquely all multiparticle states and compare the labels. In this way the problem simplifies considerably, yet the essential question of the onset of the thermodynamic behaviour can be addressed.

Consequently each Monte Carlo run consisted of a generation of a sample of $\mathcal{N}$ configurations, represented by integer indices, $(I_1, I_2, \ldots, I_N), 1 \leq I_k \leq \Gamma(M, N), k = 1, \ldots, \mathcal{N}$, uniformly distributed in the whole space of available states. Then we counted the number of coincidences $\hat{N}_c$, i.e., the number of pairs $(I_j, I_k)$ such that $I_j = I_k$. The estimate for the number of all states is then

$$\hat{\Gamma} = \mathcal{N}(\mathcal{N} - 1)/\hat{N}_c. \quad (13)$$

Moreover assuming the multinomial distribution of $\mathcal{N}$ integers among $\Gamma$ bins we have calculated also the higher moments of the distribution of the number of coincidencies $N_c$. In particular, the dispersion of $N_c$ reads

$$\sigma^2[N_c] = 2 < N_c >= \frac{2\mathcal{N}^2}{\Gamma}, \quad (14)$$

which gives for the relative error of the determination of $\Gamma$ after $\mathcal{N}$ trials

$$\frac{\sqrt{\sigma^2[\Gamma]}}{\Gamma} = \frac{\sqrt{2\Gamma}}{\mathcal{N}}. \quad (15)$$

Therefore the estimate of the error, based on the MC data only, is

$$\hat{\sigma}[\Gamma]/\hat{\Gamma} = 2/\sqrt{\hat{N}_c}. \quad (16)$$

Eqs. (13,15) show directly another advantage of the coincidence method. Namely, it works for much smaller number of trials ($\sim \sqrt{\Gamma}$) than the standard approach which measures average occupation of a single state.

A sample of runs is summarized in Table 1. Exact results for $\Gamma(M, N)$ are also quoted. The last column gives the relative deviation of the current

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6 After some approximations valid for $1 \ll \mathcal{N} \ll \Gamma$
estimate (col. 5) from the exact value. It should be compared with the estimate of the error based only on the Monte Carlo data, Eq.(16), given in column 6. The estimated error is steadily decreasing like $1/N$ and actual deviation follows the suit albeit with some fluctuations. In all runs we have made (about 20 times more than shown in the Table) approximalety 30% of actual deviations were bigger than the MC estimate, as they should. Of course the formula (15) is essential for planning future Monte Carlo simulations. It is interesting to note that the errors decrease as a number of trials and not as $1/\sqrt{N}$. This is because the true random variable in this problem is the number of pairs, i.e. $N^2$. In particular the computing effort (counting pairs) grows like $N^2$, and consequently the square root of the computational effort determines decrease of errors as it should. Altogether the Monte Carlo results are well under control and show that the method is quite reliable. It is however practical only if the total number of states is less than several hundred milions. The last run shown in Table 1 lasted few hours on a 200 MHz PC. This translates into $N, M \leq \sim 25$. We will discuss now if this is sufficient to see the onset of themodynamic properties.

4.2 Results

Figure 1 shows the entropy density as a function of a scaling variable $\epsilon = M/N$. Statistical errors of MC results (and the deviation from the exact discrete values given by $\Gamma(M, N)$) are much smaller than the size of symbols. The data follow nicely the curves obtained from the classical formula in the continuum, Eq.(11). Considered as a function of $\epsilon$ and $N$ they obviously show a substantial $N$-dependence. The $N$ varies from 8 (lowest curve) to 24 in this plot. On the other hand, the deviation from the ultimate scaling limit, (Eq.(12), the uppermost curve), is around 30% in the worst case ($N=8, M=30$). With $N$ starting from 12, deviations from the infinite system are smaller than 20%. Note that $N$ denotes the number of degrees of freedom, which in $d$ space dimensions corresponds to $N/d$ particles.

As a second test we have checked a differential form of Eq.(12)

$$\frac{\partial \log \Gamma}{\partial E} = \frac{N}{2E},$$

(17)

which, together with the equipartition of energy, is the basis of the equilib-
rium thermodynamics. Changing the variable $2mE = a^2 M$ gives
\[
\frac{\partial \log \Gamma}{\partial E} = \frac{\partial \log \Gamma}{\partial M} \frac{dM}{dE} = \frac{\partial \log \Gamma}{\partial M} \frac{M}{E} = \frac{N}{2E},
\]
or
\[
\frac{\partial \log \Gamma}{\partial M} = \frac{N}{2M},
\]
Finally after discretization of the derivative we obtain
\[
\log \left( \frac{\Gamma(M+1, N)}{\Gamma(M, N)} \right) = \frac{N}{2M+1}.
\]
This equation is tested in Fig.2, where a half of the inverse of the left hand side, as obtained from simulations, is plotted as a function of $\epsilon$. Solid line represents the right hand side. Similarly to the previous case agreement is very good for $N \geq 12$. It was necessary to reduce MC errors to the level of 1%-3% to achieve this agreement. Of course this test is much more sensitive than the previous one since it requires precise measurement of the derivatives.

To conclude, the coincidence method is satisfactory in practice for the number of degrees of freedom below $\sim 25$. This turns out to be sufficient to see the signatures of the thermal equilibrium. For more than 12 degrees of freedom the scaling of the entropy density is confirmed with the accuracy better than 20%. The saddle point relation $\partial S/\partial E = 1/T$ is also very well reproduced.

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\[\text{Of course } \epsilon = (M + 1/2)/N \text{ in this case.}\]
| N  | M  | N | N_c | Γ   | δ/Γ  | δ/Γ^ | 
|----|----|---|-----|-----|------|------|
| 4 000 | 218 | 73 376. | 0.096 | 0.140 |
| 8 000 | 1000 | 63 299. | 0.045 | 0.007 |
| 16 000 | 3866 | 66 214. | 0.023 | 0.028 |
| 32 000 | 15884 | 64 465. | 0.011 | 0.001 |
| 6 8 000 | 30 | 2 133 067. | 0.260 | 0.100 |
| 16 000 | 124 | 2 064 387. | 0.127 | 0.065 |
| 32 000 | 516 | 1 984 434. | 0.062 | 0.024 |
| 64 000 | 2 110 | 1 941 201. | 0.031 | 0.001 |
| 128 000 | 8 358 | 1 960 262. | 0.015 | 0.011 |
| 12 20 000 | 4 | 99 995 000. | 0.707 | 0.615 |
| 40 000 | 16 | 99 997 504. | 0.354 | 0.615 |
| 80 000 | 106 | 60 376 604. | 0.137 | 0.025 |
| 160 000 | 422 | 60 663 128. | 0.069 | 0.020 |
| 320 000 | 1596 | 64 160 200. | 0.035 | 0.036 |
| 16 6 8 000 | 8 | 7 999 000. | 0.500 | 0.077 |
| 32 000 | 132 | 7 757 334. | 0.123 | 0.105 |
| 64 000 | 442 | 9 266 824. | 0.067 | 0.070 |
| 128 000 | 1842 | 8 894 610. | 0.033 | 0.027 |

Table 1: Monte Carlo results for \( \Gamma(M, N) \) (col. 5) for different \( N \) and \( M \). The third and fourth column give the number of generated configurations \( N \), and the number of observed coincidences \( N_c \). In the last two columns we quote the Monte Carlo estimate of the relative error, c.f. Eq. (16), and the actual relative deviation \( \delta/\Gamma = |\hat{\Gamma} - \Gamma|/\Gamma \) from the exact value \( \Gamma \) also quoted in the Table.
Figure 1: Entropy density $s = \frac{1}{N} \log W(M, N)$ vs. the energy density $\epsilon = M/N$. Black symbols represent our Monte Carlo results for $N=8$ (diamonds), 12 (circles), 16 (boxes) and 24 (a triangle). Lower solid lines correspond to the continuum approximation, Eq.(11), for each $N$. The uppermost solid line represents the scaling, thermodynamical limit, Eq.(12).
Figure 2: Testing the relation \((20)\). Half of the inverse of the finite difference (with respect to \(M\)) of the entropy \(\log \Gamma(M, N)\), as a function of \(\epsilon = M/N\) for \(N = 8\) (diamonds), 12 (circles), 16 (boxes) and 24 (a triangle). Solid line corresponds to the thermodynamical limit.