Exact constant in Sobolev’s and Sobolev’s trace inequalities
for Grand Lebesgue Spaces

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Abstract. In this article we generalize the classical Sobolev’s and Sobolev’s trace inequalities on the Grand Lebesgue Spaces instead the classical Lebesgue Spaces.

We will distinguish the classical Sobolev’s inequality and the so-called trace Sobolev’s inequality.

We consider for simplicity only the case of whole space.

Key words: Sobolev’s and Poincare’s inequalities, derivative, gradient, norm, Lebesgue spaces, Talenti’s estimate, Bilateral Grand Lebesgue spaces, trace, counterexamples.

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1 Introduction. Notations. Statement of problem.

A. Ordinary Sobolev’s inequality.

The classical Sobolev’s inequality in the whole space $\mathbb{R}^m$, see, e.g. [15], chapter 11, section 5; [30], [31] etc. asserts that for all function $f, f : \mathbb{R}^m \to \mathbb{R}$, $m \geq 3$ from the Sobolev’s space $W^1_p(\mathbb{R}^m)$, which may be defined as a closure in the Sobolev’s norm

$$||f||_{W^1_p(\mathbb{R}^m)} = |f|_p + |Df|_p$$

of the set of all finite continuous differentiable functions $f, f : \mathbb{R}^m \to \mathbb{R}$, that

$$|f|_q \leq K_m(p) |Df|_p, \quad q = q(p) = mp/(m-p), \quad p \in [1, m), \quad q \in (m/(m-1), \infty). \quad (1)$$
Here $m = 3, 4, \ldots$;

\[ |f|_p = |f|_{p,m} = |f|_{p,R^m} = \left( \int_{R^m} |f(x)|^p \, dx \right)^{1/p}, \]

\[ Df = \{ \partial f/\partial x_1, \partial f/\partial x_2, \partial f/\partial x_3, \ldots, \partial f/\partial x_m \} = \text{grad } f, \]

\[ |Df|_p = \left( \frac{1}{2} \sum_{i=1}^{m} ( \partial f/\partial x_i )^2 \right)^{1/2} \cdot \]

The best possible constant in the inequality (1) belongs to G. Talenti [31]:

\[ K_m(p) = \pi^{-1/2} m^{-1/p} \left[ \frac{p - 1}{m - p} \right]^{1-1/p} \left[ \frac{\Gamma(1 + m/2) \Gamma(m)}{\Gamma(m/p) \Gamma(1 + m - m/p)} \right]^{1/m}. \]

**B. Trace Sobolev’s inequality.**

Let $m, n = 1, 2, \ldots, x \in R^m, y \in R^n, z = \{x, y\} \in R^{m+n}, u = u(x, y) = u(z)$ be any function from the space $W^1_p(R^{m+n})$.

We consider in this case only the so-called radial functions. In detail, we define as usually for the vectors $x = \vec{x} = \{x_1, x_2, \ldots, x_m\}$ and $y = \vec{y} = \{y_1, y_2, \ldots, y_n\}$

\[ |x| = \left( \sum_{i=1}^{m} (x_i)^2 \right)^{1/2}, \quad |y| = \left( \sum_{j=1}^{n} (y_j)^2 \right)^{1/2} \]

and correspondingly

\[ |z| = |(x, y)| = \left( |x|^2 + |y|^2 \right)^{1/2}. \]

We assume that the function $u = u(x, y)$ depended only on the variable $|z|$ and we will write for simplicity

\[ u(x, y) = u(|z|). \]

Let us denote $N = m + n, (N \geq 3); S[u](x) = u(x, 0), \nabla u = \{ \partial u/\partial x_1, \partial u/\partial x_2, \ldots, \partial u/\partial x_m, \partial u/\partial y_1, \partial u/\partial y_2, \ldots, \partial u/\partial y_n \} = \nabla u \cap \{ \text{grad } u, \text{ grad } u \}; |\nabla u|_p = (|\text{grad } u|_p + |\text{grad } u|_p^p)^{1/p}.

We will denote the class of all the radial functions $\text{Rad} = \text{Rad}(R^N); u(\cdot) \in \text{Rad}$. Notice that the operator $S[u]$ is correct and continuously defined in the $L_p(R^m)$ in the following sense:

\[ \lim_{|y| \to 0} |u(\cdot, y) - S[u]|_{L_p(R^m)} = 0, \]
see [3], chapter 5, section 24.

The following inequality is called the Sobolev’s trace inequality:

\[ |S[u](\cdot)|_{q,m} \leq K_{m,n}(p) \cdot |\nabla u|_{p,N}, \ q = q(p) = mp/(N - p), \ p \in [1, N). \quad (2) \]

We will understand further under the constant \( K_{m,n}(p) \) in the inequality (2) its minimal value, namely:

\[ K_{m,n}(p) = \sup \left\{ \left[ \frac{|S[u](\cdot)|_q}{|\nabla u|_p} \right], \ u \in W^1_p(R^{m+n}) \cap \text{Rad}(R^N), \nabla u \neq 0 \right\}. \quad (3) \]

It is evident \( K(m, 0) = K(m) \).

More information about the constant \( K_{m,n}(p) \) see, for instance, in the articles [1], [21], [32], [5], [7], [18], [33], [8], [9], [19] etc., see also reference therein.

Our aim is generalization of Sobolev’s-type inequalities (1), (3) on some popular classes of rearrangement invariant (r.i.) spaces, namely, on the so-called Grand Lebesgue Spaces \( G(\psi) \). We intend to show also the exactness of offered estimations.

Hereafter \( C, C_j \) will denote any non-essential finite positive constants. We define also for the values \((p_1, p_2)\), where \( 1 \leq p_1 < p_2 \leq \infty \)

\[ L(p_1, p_2) = \cap_{p \in (p_1, p_2)} L_p. \]

The paper is organized as follows. In the next section we recall the definition and some simple properties of the so-called Grand Lebesgue Spaces \( G(\psi) \). In the section 3 we formulate and prove the main result: the classical Sobolev’s inequality for \( G(\psi) \) spaces with the exact constant computation.

In the section 4 we investigate the trace Sobolev’s inequality for radial functions, also with the exact constant computation.

The last section contains some concluding remarks: a weight generalizations of ordinary and trace Sobolev’s inequality.

2 Auxiliary facts. Grand Lebesgue Spaces.

Definition.

Recently, see [16], [10], [11], [12], [13], [14], [22], [23], [24], [25], [26], [27] etc. appears the so-called Grand Lebesgue Spaces \( GLS = G(\psi) = G(\psi; A, B) \), \( A, B = \text{const}, A \geq 1, A < B \leq \infty \), spaces consisting on all the measurable functions \( f : T \rightarrow R \) with finite norms

\[ ||f||_{G(\psi)} \overset{\text{def}}{=} \sup_{p \in (A, B)} [||f||_{p}/\psi(p)]. \]
Here $\psi(\cdot)$ is some continuous positive on the open interval $(A, B)$ function such that

$$\inf_{p \in (A, B)} \psi(p) > 0, \quad \sup_{p \in (A, B)} \psi(p) = \infty.$$ 

We can suppose without loss of generality

$$\inf_{p \in (A, B)} \psi(p) = 1.$$ 

This spaces are rearrangement invariant, see [2], and are used, for example, in the theory of probability [17], [16], [22]; theory of Partial Differential Equations [11], [14]; functional analysis [25], [26]; theory of Fourier series [28], theory of martingales [23] etc.

3 Classical Sobolev’s inequality for Grand Lebesgue Spaces.

We recall $q = q(p) = mp/(m - p), \ p \in [1, m)$, or equally

$$p = p(q) = \frac{mq}{m + q}, \ q \in [m/(m - 1), \infty).$$

Let $\psi(\cdot)$ be arbitrary function from the set $G\Psi(1, m)$. We define the new function $\nu_\psi(q) = \nu(q)$ from the set $G\Psi(m/(m - 1), \infty)$ as follows:

$$\nu_\psi(q) = K_m \left( \frac{mq}{m + q} \right) \cdot \psi \left( \frac{mq}{m + q} \right), \ q \in [m/(m - 1), \infty). \quad (4)$$

**Theorem 1.** The following Sobolev’s type inequality holds:

$$||u||_{G(\nu_\psi)} \leq 1 \cdot ||\nabla u||_{G(\psi)}, \quad (5)$$

and the constant ”one” in the inequality (5) is the best possible.

**Remark 1.** It is presumed in the assertion (5) of theorem 1 that the right-hand side of the proposition of theorem 1 is finite.

**Proof.**

A. The upper bound. Let the function $u = u(x)$ be such that

$$\nabla u \in \cap_{p \in (1, m)} W^1_p(R^m)$$

and

$$||\nabla u||_{G(\psi)} \in (0, \infty);$$

we can assume without loss of generality

$$||\nabla u||_{G(\psi)} = 1.$$
It follows from the direct definition of norm for the $G(\psi)$ spaces that

$$|\nabla u|_p \leq \psi(p), \ p \in (1, m).$$

We use now the Talenti’s inequality (2):

$$|u|_{pm/(m-p)} \leq K_m(p) \cdot \psi(p), \ p \in (1, m). \quad (6)$$

The proposition of theorem 1 follows from (6) after the substitution

$$p = \frac{mq}{m + q}, \ q \in (m(m-1), \infty):$$

$$|u|_q \leq K_m \left( \frac{mq}{m + q} \right) \cdot \psi \left( \frac{mq}{m + q} \right) = \nu(q) = \nu(q) \cdot ||\nabla u||_{G(\psi)};$$

$$\frac{|u|_q}{\nu(q)} \leq ||\nabla u||_{G(\psi)};$$

therefore

$$||u||_{G(\nu_\psi)} = \sup_q \left[ \frac{|u|_q}{\nu(q)} \right] \leq ||\nabla u||_{G(\psi)}.$$

B. Proof of the low bound.

1. Let us introduce the following important functional: $V =$

$$\sup_{m=3,4,\ldots} \sup_{\psi \in G(\Psi(m(m-1),m))} \sup \left\{ \frac{||f||_{G(\nu)}}{||\nabla f||_{G(\psi)}} : f \in G(\psi) \cap W^1_p(R^m), \nabla f \neq 0 \right\}. \quad (7)$$

We proved that $V \leq 1$; it remains to ground the opposite inequality.

Let us consider the following example (more exactly, a family of examples) of a functions:

$$f_\Delta = f_\Delta(x) = |\log |x||^\Delta \cdot I(|x| \leq 1),$$

where $\Delta = \text{const} \geq 1$,

$$I(|x| \leq 1) = 1, \ |x| \leq 1; I(|x| \leq 1) = 0, \ |x| > 1.$$

So, $f_\Delta(x)$ is radial function and $f_\Delta(\cdot) \in G(\psi) \cap W^1_p(R^m), \nabla f_\Delta(\cdot) \neq 0$.

We calculate denoting

$$\omega(m) = \frac{2\pi^{m/2}}{\Gamma(m/2)};$$

$$|f_\Delta|_q = \frac{\omega^{1/q}(m) \Gamma^{1/q}(\Delta q + 1)}{m^{\Delta + 1/q}};$$
\[ |\nabla f_\Delta|_p = \Delta \cdot \omega^{1/p}(m) \cdot \frac{\Gamma^{1/p}(\Delta - 1) + 1}{(m - p)^{\Delta - 1 + 1/p}}. \]

We can rewrite the expression for the value \( V \) as follows:

\[ V = \sup_m \sup_{\psi \in G^\Psi(m/(m-1), m)} \left\{ \frac{\sup_p |f|_q / \psi(q)}{\sup_p |\nabla f|_p / \psi(p)} : f \in G(\psi) \cap W^1_p(R^m), \nabla f \neq 0 \right\}. \quad (8) \]

When we choose \( f = f_\Delta \) and \( \psi(p) = \psi_\Delta(p) := |\nabla f_\Delta|_p \), we obtain the following lower bond for the value \( V : V \geq V_0 \), where

\[ V_0 = \sup_m \sup_{p \in (1,m)} \left[ \frac{|f_\Delta|_{q(p)}}{K_m(p) |\nabla f_\Delta|_p} \right]. \quad (9) \]

Substituting the expressions for \( |f_\Delta|_q \), \( |\nabla f_\Delta|_p \) and for \( K_m(p) \) into the formula (9), we obtain after some calculations by means of Stirling’s formula for all the admissible values \( m, \Delta \):

\[ V_0 \geq \lim_{p \to m-0} \frac{|f_\Delta|_{q(p)}}{K_m(p) |\nabla f_\Delta|_p} = \frac{m^\Delta \Delta^{-1} e^{-\Delta} m^{1/m}}{(m - 1)^{1/m} \Gamma^{1/m}((\Delta - 1)m + 1)} =: V_{00}(m, \Delta); \]

\[ V_0 \geq \lim_{m \to \infty} V_{00}(m, \Delta) = e^{-1} \left[ \frac{\Delta}{\Delta - 1} \right]^{\Delta - 1} =: V_{000}(\Delta). \]

Finally,

\[ V_0 \geq \lim_{\Delta \to \infty} V_{000}(\Delta) = 1. \]

This completes the proof of theorem 1.

4 Trace Sobolev’s inequality for Grand Lebesgue Spaces. Radial case.

We recall that we consider in this section only radial functions \( u(z) = u(x,y) = u(|z|) \) and that here

\[ q = q(p) = mp/(N - p), \quad p \in [p_0, N], \quad p_0 \overset{def}{=} \max(N/(m + 1), 1), \]

or equally

\[ p = p(q) = \frac{mq}{N + q}, \quad q \in [1, \infty). \]

Let \( \psi(\cdot) \) be arbitrary function from the set \( G^\Psi(p_0, N) \). We define the new function \( \zeta_\psi(q) = \zeta(q) \) from the set \( G^\Psi(1, \infty) \) as follows:
\[ \zeta_\psi(q) = K_m \left( \frac{mq}{m+q} \right) \cdot \psi \left( \frac{mq}{m+q} \right), \quad q \in [1, \infty). \]  

(10)

**Theorem 2.** The following trace Sobolev’s type inequality holds:

\[ ||u|| G(\nu_\psi) \leq 1 \cdot ||\nabla u|| G(\psi), \]  

and the constant "one" in the inequality (11) is the best possible.

**Remark 2.** As in the remark 1, it is presumed in the assertion (11) of theorem 2 that the right-hand side of the proposition of theorem 2 is finite.

**Proof** is alike to the proof of the assertion of theorem 1.

**A. The upper bound.** Let the function \( u = u(z) = u(|z|) \) be radial function: \( u(\cdot) \in \text{Rad}(\mathbb{R}^N) \) and such that

\[ \nabla u \in \cap_{p \in (p_0, N)} W^1_p(\mathbb{R}^N), \quad \nabla u \neq 0. \]

We assume without loss of generality that

\[ ||\nabla u|| G(\psi) = 1. \]

It follows from the definition of norm for the \( G(\psi) \) spaces that

\[ |\nabla u|_p \leq \psi(p), \quad p \in [p_0, N). \]

It follows from the definition of the constants \( K_{m,n}(p) \) (inequality (3)):

\[ |S[u]|_{p,m/(N-p)} \leq K_{m,n}(p) \cdot \psi(p), \quad p \in (p_0, N). \]  

(12)

The proposition of theorem 2 follows from (12) after the substitution

\[ p = \frac{Nq}{m+q}, \quad q \in (1, \infty); \]

\[ |S[u]|_q \leq K_{m,n} \left( \frac{Nq}{m+q} \right) \cdot \psi \left( \frac{Nq}{m+q} \right) = \zeta(q) = \zeta(q) \cdot ||\nabla u|| G(\psi); \]

\[ \frac{|S[u]|_q}{\zeta(q)} \leq ||\nabla u|| G(\psi); \]

thus

\[ ||S[u]| G(\zeta_\psi) = \sup_q \left[ \frac{|S[u]|_q}{\zeta(q)} \right] \leq ||\nabla u|| G(\psi). \]

**B. Proof of the low bound.**

1. Let us introduce again the following important functional (with at the same notation as in the last section): \( V(m, n) = \)
\[
\sup_{N} \sup_{\psi \in G(\psi_{\psi_{0}})} \sup \left\{ \frac{||S[f]||_{G(\zeta)}}{||\nabla f||_{G(\psi)}} : \nabla f \in G(\psi) \cap W^{1}_{p}(\mathbb{R}^{N}), \nabla f \neq 0 \right\}.
\]

We proved that \( V(m, n) \leq 1 \); it remains to ground an opposite inequality.

2. Let us consider again the used examples of a functions:
\[
f_{\Delta} = f_{\Delta}(z) = ||\log|z| \cdot I(|z| \leq 1),
\]
where \( \Delta = \text{const} \geq 1 \).

So, \( f_{\Delta}(x) \) is radial function and such that \( \nabla f_{\Delta}(\cdot) \in G(\psi) \cap W^{1}_{p}(\mathbb{R}^{N}), \nabla f_{\Delta}(\cdot) \neq 0 \).

We calculate as before:
\[
|S[f_{\Delta}]|_{q} = \frac{\omega^{1/q(m)} \Gamma^{1/q}(\Delta q + 1)}{m^{\Delta+1/q}};
\]
\[
|\nabla f_{\Delta}|_{p} = \Delta \cdot \omega^{1/p}(N) \cdot \frac{\Gamma^{1/p}(p(\Delta - 1) + 1)}{(N - p)\Delta-1+1/p}.
\]

3. We do not know, in contradiction to the case of ordinary Sobolev’s inequality, the exact value of a constant \( K_{m,n}(p) \). In order to prove the proposition of theorem 2, we need to obtain the upper estimate for this constant.

Let us estimate the constant \( K_{m,n}(p) \). As long as the function \( f(\cdot) \) is radial, we can rewrite the inequality (2) using the multidimensional spherical coordinates as follows:
\[
\omega^{1/q(m)} \left[ \int_{0}^{\infty} s^{m-1} \left( \int_{s}^{\infty} g(t) \, dt \right)^{1/q} \right]^{1/q} \leq \omega^{1/p}(N) K_{m,n}(p) \times \left[ \int_{0}^{\infty} s^{N-1}|g(s)|^{p} \, ds \right]^{1/p}.
\]

Further we will use the result belonging to Bradley [6]; see also [19], which is a weight generalization of the classical Hardy-Littlewood inequality. It asserts that the following inequality is true:
\[
\left\{ \left[ \int_{0}^{\infty} u(x) \int_{x}^{\infty} f(t) \, dt \right]^{q} \, dx \right\}^{1/q} \leq C \times \left\{ \int_{0}^{\infty} [v(x)f(x)]^{p} \, dx \right\}^{1/p},
\]
where \( u(x), v(x) \geq 0, 1 < p \leq q < \infty, \)
\[
Q(p) := p^{1/q} \cdot (p/(p - 1))^{(p-1)/p},
\]
\[
B \leq C \leq B \cdot Q(p),
\]
\[
B = \sup_{w>0} J(w), \quad J(w) = J_{1}(w) \cdot J_{2}(w),
\]
\[ J_1(w) = \left( \int_0^w u^q(x) dx \right)^{1/q}; \quad J_2(w) = \left( \int_w^\infty (v(x))^{-(p-1)/p} dx \right)^{(p-1)/p}. \] (19)

Note that the case \( p \leq q \), i.e. \( p \geq n \) is sufficient in order to prove the second assertion of theorem 2, as long as we put further \( p \to N - 0 \).

We compute in the considered case:

\[ J_1(w) = m^{-1/q} w^{m/q}, \quad J_2(w) = \left[ \frac{p-1}{N-p} \right]^{1-1/p} \cdot w^{-(N-p)/p}. \]

Hence, the expression for the value of \( B \) is finite only in the case when

\[ \frac{m}{q} = \frac{N-p}{p}, \]

or equally

\[ q = \frac{mp}{N-p}, \]

i.e. as in the conditions of theorem 2.

We conclude in the considered case:

\[ B = m^{-1/q} \left[ \frac{p-1}{N-p} \right]^{1-1/p} \]

and following

\[ C \leq Q(p) \cdot m^{-1/q} \cdot \left[ \frac{p-1}{N-p} \right]^{1-1/p}, \]

\[ K_{m,n}(p) \leq K_{m,n}^+(p), \quad K_{m,n}^+(p) \overset{\text{def}}{=} \omega^{1/q}(m) \cdot \omega^{-1/p}(N) \cdot C = \]

\[ Q(p) \cdot m^{-1/q} \cdot \left[ \frac{p-1}{N-p} \right]^{1-1/p} \cdot \omega^{1/q}(m) \cdot \omega^{-1/p}(N). \] (20)

4. We can rewrite the expression for the value \( V(m, n) \) as follows:

\[ V(m, n) = \sup \sup \sup_{N, \psi \in G \Psi(m/(m-1), m)} \left\{ \sup_q \left[ \frac{|f(q)/\zeta(q)|}{\sup_p \left[ \frac{\nabla f(p)}{\psi(p)} \right]} : \nabla f \in G(\psi) \cap W_1^p(R^m), \nabla f \neq 0 \right] \right\}. \] (21)

When we choose \( f = f_\Delta \) and \( \psi(p) = \psi_\Delta(p) := |\nabla f_\Delta|_p \), we obtain the following lower bond for the value \( V(m, n) : V(m, n) \geq V_0(m, n) \), where

\[ V_0(m, n) = \sup \sup \sup_{\Delta \in (1, \infty)} \sup_{p \in [p_0, m]} \left[ \frac{|f_\Delta(q(p))|}{K_{m,n}(p) \cdot |\nabla f_\Delta|_p} \right]. \]

Substituting the expressions for \( |f_\Delta|_q, \ |\nabla f_\Delta|_p \) into the formula for the value \( V(m, n) \), tacking into account the inequality \( K_{m,n}(p) \leq K_{m,n}^+(p) \), we obtain after some calculations by means of Stirling’s formula for all the admissible values \( m, p, \Delta : \)
\[ V_0(m, n) \geq \lim_{p \to N-0} \left[ \frac{|f_\Delta|_{q(p)}}{K_{m,n}^1(p) \left| \nabla f_\Delta \right|_p} \right] = \frac{\Delta^{\Delta-1} e^{-\Delta N^{\Delta-1}}}{\Gamma^{1/N}((\Delta-1)N+1)} =: V_{00}(m, n; \Delta, N); \] 

Finally,

\[ V_0(m, n) \geq \lim_{N \to \infty} V_{00}(m, n; \Delta, N) = e^{-1} \left[ \frac{\Delta}{\Delta - 1} \right]^{\Delta-1} =: V_{000}(m, n; \Delta). \] 

This completes the proof of theorem 2.

Note that at the same result may be obtained from the estimation in [4], chapter 1, section 2:

\[ \omega_{1/q}(m) |S[u]|_q \leq \omega_{1/p}(N) \cdot \left[ \frac{p(m-1) + N}{m(N-p)} \right]^{1-(p-n)/(mp)} \cdot |\nabla u|_p, \] 

hence

\[ K_{m,n}(p) \leq K_{m,n}^{(1)}(p) \overset{def}{=} \omega_{1/p}(N) \cdot \omega_{1/q}(m) \cdot \left[ \frac{p(m-1) + N}{m(N-p)} \right]^{1-(p-n)/(mp)}, \] 

\[ V(m, n) \geq \lim_{\Delta \to \infty} \lim_{N \to \infty} \lim_{p \to N-0} \frac{|S[f_\Delta]|_{q(p)}}{K_{m,n}^{(1)}(p) \cdot |\nabla f_\Delta|_p} = 1. \]

5 Concluding remarks. Weight Sobolev’s inequalities.

A. We introduce the so-called weight trace operator by the formula

\[ S_\alpha[u](x) = |x|^{-\alpha} \cdot u(x, 0), \quad u(z) = u(x, y) = u(|z|), \]

i.e. \( u(\cdot) \in \text{Rad}(R^N). \)

Let us consider a weight Poincare-Sobolev trace inequality of a view:

\[ |S_\alpha[u]|_q \leq K_{m,n}^\alpha(p) \cdot |\nabla u|_p, \]

where

\[ \alpha = \text{const} \in [0, 1], \quad p \in (p_1, N/(1-\alpha)), \]

\[ p_1 \overset{def}{=} \max(N/(m+1-\alpha), 1) < N/(1-\alpha); \quad N/0 = +\infty; \]
\[ q = q(p) = \frac{mp}{N - p(1 - \alpha)}; \quad q \in (1, \infty) \Leftrightarrow p = \frac{qN}{m + q(1 - \alpha)}. \quad (29) \]

and we understood the value \( K_{m,n}^\alpha(p) \) as the its minimal value:

\[ K_{m,n}^\alpha(p) = \sup_{u : \nabla u \neq 0} \left[ \left| S_{\alpha, m}[u]_q \right| \right] < \infty, p \in (p_1, N/(1 - \alpha)). \]

Notice that in the case \( \alpha = 0 \) we obtain the Sobolev’s inequality and the case \( \alpha = 1 \) correspondent the so-called modified Poincare’s inequality.

Let \( \psi \in G\Psi(p_1, N/(1 - \alpha)) \); we introduce a new function

\[ \theta_\psi(q) = \psi \left( \frac{qN}{m + q(1 - \alpha)} \right) \cdot K_{m,n}^\alpha \left( \frac{qN}{m + q(1 - \alpha)} \right). \quad (30) \]

**Theorem 3.** The following generalized trace Sobolev-Poincare type inequality holds:

\[ \| S_\alpha[u]\| G(\theta_\psi) \leq 1 \cdot \| \nabla u\| G(\psi), \quad (31) \]

and the constant ”one” in the last inequality is the best possible.

**Proof** is at the same as in the theorem 2, with at the same ”counterexamples” and may be omitted.

**B.** Note that the low bound for Sobolev’s trace embedding constants, for instance, \( V(m, n) \geq 1 \), are true for arbitrary, i.e. not only for radial functions.

**C.** The exact value for the degree \( q \) in the generalized Poincare-Sobolev inequality, namely

\[ q = q(p) = \frac{mp}{N - p(1 - \alpha)} \]

may be obtained by means of the so-called dilation method, offered by Talenti [31]. In detail, let us define as usually the family of dilation operators \( T_\lambda[f], \lambda \in (0, \infty), f : R^k \to R, k = 1, 2, \ldots \), of a view:

\[ T_\lambda[f](x) = f(x/\lambda). \quad (32) \]

Suppose the inequality (27) is satisfied for some admissible radial function such that \( \nabla u \in W^1_p(R^N), \nabla u \neq 0 \). As long as \( u \in W^1_p(R^N) \Rightarrow T_\lambda u \in W^1_p(R^N) \), we have rewriting the inequality (27) for the function \( T_\lambda u : \)

\[ |S_\alpha[T_\lambda u]_q \leq K_{m,n}^\alpha(p) \cdot |\nabla T_\lambda u|_p, \quad (33) \]

taking into account the equalities:

\[ |T_\lambda u|_{p,N} = \lambda^{N/p}|u|_{p,N}, \]

11
\[ |S_\alpha[T_\lambda u]|_{q,m} = \lambda^{m/q-\alpha}|S[u]|_q, \]

\[ \nabla T_\lambda[u](z) = \lambda^{-1}u(z/\lambda), \]

\[ |\nabla T_\lambda[u]|_{p,N} = \lambda^{N/p-1}|u|_{p,N}: \]

\[ \lambda^{m/q-\alpha}|S_\alpha[u]|_{q,m} \leq K^{\alpha}_{m,n}(p) \lambda^{N/p-1} |\nabla u|_{p,N}. \]  

(34)

Since the value \( \lambda \) in the last inequality is arbitrary in the set \((0, \infty)\), we conclude

\[ m/q - \alpha = N/p - 1 \]

or equally

\[ q = \frac{p \cdot m}{N - p(1 - \alpha)}. \]

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