Model Selection of Nested and Non-Nested Item Response Models using Vuong Tests

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Abstract

In this paper, we apply Vuong’s (1989) general approach of model selection to the comparison of both nested and non-nested unidimensional and multidimensional item response theory (IRT) models. This approach, which has not been fully applied to IRT models, is especially useful because it allows for formal statistical tests of non-nested models. Further, in the nested case, it offers statistics that are highly competitive with the traditional likelihood ratio test. After summarising the statistical theory underlying the tests, we study the performance of all three Vuong tests in the context of IRT, using simulation studies and real data. We find that, in the non-nested case, the tests can reliably distinguish between the graded response model and the generalized partial credit model. In the nested case, the tests often perform as well as or even better than the traditional likelihood ratio test.

Keywords: item response theory, model selection, Vuong test, likelihood ratio test, likelihood inference
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Item response theory (IRT) consists of a variety of mathematical and statistical models aimed at describing the interaction between unobserved (latent) psychological constructs (traits) and item characteristics. Most commonly, IRT is adopted to understand examinee response behavior to aptitude tests, psychological inventories, ratings scales, and other forms of (typically categorical) response stimuli at the item and composite test score level. As such, a plethora of related, and often competing, IRT models have appeared in the literature for dichotomous and polytomous item response data. For example, regarding polytomous response data, graded response models (Samejima, 1969), (generalized) partial credit models (Muraki, 1992), sequential response models (Tutz, 1990), and nominal response models (Bock, 1972), have been studied extensively, where each model may be theoretically suitable for a given empirical investigation.

Aside from selecting a suitable IRT model a priori, which in many applications may itself be difficult, the selection of IRT models often consists of comparing “best fitting” models among sets of competing models. Best fitting in this context refers to favoring response models based on statistical decision and information theoretic grounds. This is often achieved by either selecting models that provide more statistically likely fit to the data (i.e., that result in relatively small data-model residuals), or by choosing the most parsimonious of the competing models that also explain the data well.

Depending on the nature of the competing IRT models, various statistical tests can be applied to conduct model selection and comparisons. For instance, when models are nested, model selection can be investigated using the traditional likelihood ratio test approach (Neyman & Pearson, 1928, 1933), which is sometimes derived from the difference in $G^2$- or $\chi^2$-statistics (Baker & Kim, 2004; Bock & Aitkin, 1981; Reckase, 2009; Schilling & Bock, 2005; Thissen, Steinberg, & Wainer, 1988). In the case when models are not nested, model selection can be performed using information criteria, such as Akaike’s Information Criterion (AIC; Akaike, 1974) or Schwarz’s Bayesian
In this paper, we recapitulate Vuong’s (1989) general approach to model selection and apply it to the comparison of both nested and non-nested unidimensional and multidimensional IRT models. Briefly stated, Vuong’s theory consists of three distinct statistical tests related to the distinguishability and relative model-fit of nested and non-nested models. We study all three tests as applied to multiple types of IRT models.

Recently, Freeman (2016) applied one of these three tests (the test of non-nested models) to compare compensatory and non-compensatory multidimensional IRT models, concluding that the test proved useful for correctly identifying the data generating model, so long as the correlations between latent dimensions stay below 0.8. The test that Freeman (2016) studied is a subset of the Vuong theory. This paper also considers the two other tests, one of which allows to test the distinguishability of competing non-nested models before evaluating their relative fit to the data. Testing this assumption is in line with the original work of Vuong (1989), because the distribution of the test statistic under the null hypothesis relies on the assumption that the models are distinguishable. To our knowledge, neither Vuong’s test of distinguishability nor Vuong’s test of nested models have been investigated in the context of IRT, yet. Additionally, we introduce IRT software that provides functionality to conduct Vuong tests.

Vuong tests have been successfully applied in other psychometric contexts such as structural equation modeling (SEM; Levy & Hancock, 2007, 2011; Merkle, You, & Preacher, 2016), with Merkle et al. (2016) being the first to make full use of Vuong’s framework. They specifically allowed for the calculation of all three statistical tests, including those requiring non-standard model output, through software implementations via the R package nonnest2 (Merkle & You, 2018). Vuong’s framework has also been investigated when comparing mixture distribution models with different numbers of components (Greene, 1994; Lo, Mendell, & Rubin, 2001; Nylund, Asparouhov, & Muthén, 2007), although it has been noted that the Vuong tests may be problematic when parameters are on the boundary of the parameter space (Jeffries, 2003; Wilson,
Recent extensions of Vuong’s (1989) seminal work have also focused on deriving nonparametric test statistics (Clarke, 2001, 2003, 2007) and overlapping non-nested models. For instance, Shi (2015) proposed a simulation based procedure to achieve correct null rejection rates uniformly over all data generating processes, and Liao and Shi (2016) extended Vuong’s work by deriving a new statistical test for the comparison of semi/non-parametric models that retain optimal asymptotic properties.

In the following pages, we give a brief summary of a selection of popular IRT models, and describe Vuong’s (1989) theory and the three related statistical tests. We then present the results of several Monte Carlo simulation studies, illustrating the properties of these tests when comparing both nested and non-nested IRT models. Next, we apply the Vuong tests to empirical data consisting of an online questionnaire quantifying “nerdiness”. Finally, we conclude with a general discussion regarding the utility and future use of the Vuong tests in the context of IRT investigations. To facilitate future applications, we have extended the functionality of the R package nonnest2 (Merkle & You, 2018) to allow the Vuong tests to be easily conducted on IRT models fitted via the R package mirt (Chalmers, 2012).

### Theoretical Background

In this section, we provide background and notation on the Vuong (1989) test statistics. Related discussion of the test statistics can also be found in Levy and Hancock (2007) and Merkle et al. (2016).

#### Models and Estimation

Let \( X_{ij} \) be the response from person \( i \) \((i = 1, \ldots, N)\) on item \( j \) \((j = 1, \ldots, J)\), with item \( j \) having \( K_j \) categories. We consider \( M \)-dimensional IRT models of the form

\[
X_{ij} | \theta_i, \Psi \sim \text{Multinomial}(n = 1, p_{ij0}, p_{ij1}, \ldots, p_{ijK_j-1}),
\]

\[
\log \left( \frac{p_{ijk}}{1 - p_{ijk}} \right) = \beta_{jk} + \sum_{m=1}^{M} \alpha_{jm} \theta_{im} \quad k = 0, \ldots, K_j - 1,
\]

where \( \theta_i \) contains person parameters (i.e., factor or trait scores) for person \( i \); \( \Psi \) contains item parameters and person hyper-parameters (e.g., means, variances, covariances); and \( p_{ijk}^{*} \) is a function of the original category probabilities, \( p_{ij0}, p_{ij1}, \ldots, p_{ijK_j-1} \).
The above equations cover many popular IRT models. For example, the graded response model (GRM; Samejima, 1969) is obtained by setting
\[ p_{ijk}^* = P(X_{ij} \geq k), \]
and the generalized partial credit model (GPCM; Muraki, 1992) is obtained by setting
\[ p_{ijk}^* = P(X_{ij} = (k + 1)|X_{ij} \in \{k,(k + 1)\}). \]
Further, when \( K_j = 2 \) for all \( j \), the models both reduce to the \( M \)-dimensional two-parameter logistic models (Md-2PL). Note that, for multidimensional models, there is a distinction to be made with respect to between-item and within-item multidimensionality; in the former, one restricts each item to only load on one dimension, resulting in a so-called simple structure, while in the latter, one allows each item to load on each dimension, see, e.g., Adams, Wilson, and Wang (1997). The usual unidimensional 2PL results when \( M = 1 \). Finally, Rasch-like versions of the models can be obtained by setting \( M = 1 \), fixing \( \alpha_j = 1 \) for all \( j \), and freely estimating the latent variance hyper-parameter. Across all versions of this model, we assume that the \( \theta_i \) are random variables (typically from a multivariate normal distribution), leading to models estimated via marginal maximum likelihood (marginal ML). However, the test statistics described below are potentially applicable to models estimated via other ML methods, e.g., conditional ML (see Baker & Kim, 2004).

Focusing on marginal ML, models are estimated by choosing values of \( \Psi \) to maximize the log-likelihood
\[ \ell(\Psi; \mathbf{x}_1, \ldots, \mathbf{x}_N) = \sum_{i=1}^{N} \ell(\Psi; \mathbf{x}_i) = \sum_{i=1}^{N} \log f(\mathbf{x}_i; \Psi), \]
where the log-likelihood for person \( i \) is marginalized over \( \theta_i \), i.e.,
\[ \ell(\Psi; \mathbf{x}_i) = \log \int \prod_{j=1}^{J} f(x_{ij}; \Psi, \theta) g(\theta; \Psi) d\theta, \]
with \( g(\theta; \Psi) \) often following a \( N_M(0, I_M) \), though some person hyper-parameters may also be involved in the distribution.

Maximizing the log-likelihood function involves searching for values of \( \Psi \) such that the gradient of the log-likelihood is \( \mathbf{0} \), and therefore has reached a (locally) optimal
parameter set. The gradient can be represented as the sum of scores across individuals, i.e.,
\[ s(\hat{\Psi}; x_1, \ldots, x_N) = \sum_{i=1}^{N} s(\hat{\Psi}; x_i) = 0, \]  
(7)
where
\[ s(\hat{\Psi}; x_i) = \left( \frac{\partial \ell(\hat{\Psi}; x_i)}{\partial \hat{\Psi}_1}, \ldots, \frac{\partial \ell(\hat{\Psi}; x_i)}{\partial \hat{\Psi}_P} \right)', \]  
(8)
contains derivatives of person i’s log-likelihood across all P parameters in \( \hat{\Psi} \).

Computation of these derivatives is aided by an identity attributed to Louis (1982), which is particularly useful when the IRT models are estimated using the Expectation-Maximization (EM) algorithm; see Baker and Kim (2004) and Glas (1998) for further detail.

Following estimation, we can obtain standard errors of parameter estimates via computation of the model’s observed or expected parameter information matrix, \( I(\Psi) \). Unfortunately, these matrices are more complicated to compute for IRT models than for many other types of statistical models, particularly when the EM algorithm is adopted during estimation (Bock & Aitkin, 1981). Recently, however, Chalmers (2018a) demonstrated accurate and efficient numerical schemes to obtain the observed information matrix which capitalize on Oakes’ (1999) identity (see also Pritikin, 2017). Throughout this paper, we utilized the observed information matrix results obtained via Oakes identity approximation method described by Chalmers (2018a).

**Vuong Statistics**

The test statistics studied in this paper are generally used to compare two models, which we label Model A and Model B. Once the two models are estimated, we have two parameter vectors, \( \Psi_A \) and \( \Psi_B \), along with their respective information matrices, \( I(\Psi_A) \) and \( I(\Psi_B) \). Each individual also has a log-likelihood \( \ell() \) and a score vector \( s() \) under each model. These are the building blocks used to construct the Vuong test statistics.

**Nesting, Non-nesting, and Equivalence.** Before defining the test statistics, we define different types of relationships between models. Researchers are generally familiar with nested models, whereby one model (a “reduced model”) is a special case of another model (a “full model”); that is, the reduced model’s predictions are a subset of
the full model’s predictions. However, researchers are often less familiar with the concept of “overlapping” classification of non-nested models. If two non-nested models are overlapping, they make identical predictions in some populations, but not in others. Conversely, in the non-overlapping or strictly non-nested case, two non-nested models make unique predictions in all populations. The “overlapping” attribute is somewhat similar to model equivalence, which is often discussed in the context of SEM models (e.g., Bentler & Satorra, 2010; Hershberger & Marcoulides, 2013; MacCallum, Wegener, Uchino, & Fabrigar, 1993). However, equivalent models make identical predictions across all populations, whereas overlapping models make identical predictions in only some populations.

To build an intuition for the “nested”, “overlapping” and “strictly non-nested” definitions, consider a simple regression involving three covariates:

$$g(Y) = \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + \ldots,$$

(9)

where $g()$ is a link function and where $\ldots$ may involve other terms in the model. This type of regression may appear in a linear regression model, latent regression IRT model, or other generalized linear model. In the context of the above model, a simplified model involving only $X_1$ and $X_2$ is overlapping with a simplified model involving only $X_1$ and $X_3$: These simplified models are non-nested and generally make different predictions, but they cannot be distinguished in populations where both $\beta_2$ and $\beta_3$ equal 0. Strictly non-nested models, on the other hand, may utilize different link functions $g()$ or different data distributions. Finally, nested models comprise the familiar situation where we sequentially add covariates to the model. For example, we start with a simplified model involving only $X_1$, then add $X_2$ to the model, then add $X_3$. While the relationships between models are easily seen in this example, they are more difficult to discern in more complex models. This is why the Vuong tests (especially the test of distinguishability) are useful for comparing non-nested models.

For pairs of non-nested models, the overlapping concept potentially leads to two separate statistical tests. First, if models are overlapping (or if we are unsure about whether they are overlapping), we can test whether the model predictions are identical.
in the population of interest; this is a test of distinguishability. Stated differently, we examine the fit of two models to sample data (which generally will not be identical), and test whether the sample fit statistics could have arisen from models that provide identical fit in the population of interest. If the test indicates indistinguishable models, then we have no basis for choosing one model over the other. However, if the test indicates distinguishable models, we can further examine whether one model provides a “significantly better” fit than the other. This second test is akin to the traditional likelihood ratio test, except that the two candidate models are non-nested. This is also the test that Freeman (2016) studied.

For pairs of nested models, the distinguishability and likelihood ratio tests can still be carried out to test the same hypotheses as the traditional likelihood ratio tests. However, unlike the traditional likelihood ratio test (see, e.g., Chun & Shapiro, 2009; Steiger, Shapiro, & Browne, 1985), the Vuong test statistics make no assumptions related to the full model being “correctly specified”. This point is further discussed in the next section.

Statistics. The Vuong statistics’ derivations focus on the Kullback-Leibler (K-L) distance (Kullback & Leibler, 1951) between each model and the population generating model (PGM). A better-fitting model is one whose distance to the PGM is smaller, and two models fit equally well if their distances are equal. The statistics focus on the case-wise log-likelihoods of the fitted models; each observation in the data will have a log-likelihood value under both candidate models. If two overlapping, non-nested models are indistinguishable from one another, then each observation’s log-likelihood will be nearly identical under both models. This concept is tested by computing the variance of differences between log-likelihoods under the two models. Similarly, if two distinguishable, non-nested models have the same overall goodness of fit, then the mean log-likelihood across observations will be the same for both models. This concept is tested by computing the mean difference between log-likelihoods. We now formalize these ideas.
Test of distinguishability. Define a population variance in case-wise log-likelihoods as
\[
\omega^2 = \text{VAR} \left[ \log \frac{f_A(x_i; \Psi^*_A)}{f_B(x_i; \Psi^*_B)} \right],
\] (10)
where \(\Psi^*_A\) is the Model A parameter vector that is closest to the PGM in K-L distance across the entire population (i.e., where \(i\) includes all members of the population). The vector \(\Psi^*_B\) is defined similarly. We can formally test the hypothesis that non-nested models are indistinguishable via
\[
H_0: \omega^2 = 0
\] (11)
\[
H_1: \omega^2 > 0,
\] (12)
with the associated estimate of \(\omega^2\) being
\[
\hat{\omega}^2 = \frac{1}{N} \sum_{i=1}^{N} \left( \log \frac{f_A(x_i; \hat{\Psi}_A)}{f_B(x_i; \hat{\Psi}_B)} \right)^2 - \left[ \frac{1}{N} \sum_{i=1}^{N} \log \frac{f_A(x_i; \hat{\Psi}_A)}{f_B(x_i; \hat{\Psi}_B)} \right]^2.
\] (13)
Under (11), Vuong showed that \(N\hat{\omega}^2\) follows a weighted sum of \(\chi^2\) distributions, where the weights are computed by taking squared eigenvalues of a matrix that involves the two models’ scores and information matrices (see the appendix of Merkle et al., 2016, for technical detail). Computations involving weighted sums of \(\chi^2\) distributions are generally complicated, and the computations are facilitated herein via use of the R package \textit{CompQuadForm} (Duchesne & De Micheaux, 2010).

Goodness of fit. Assuming that the two non-nested models are distinguishable, we can proceed to test their fits by comparing the mean log-likelihood under each model. The hypotheses are specified via
\[
H_0: \mathbb{E}[\ell(\Psi^*_A; x_i)] = \mathbb{E}[\ell(\Psi^*_B; x_i)]
\] (14)
\[
H_1: \mathbb{E}[\ell(\Psi^*_A; x_i)] \neq \mathbb{E}[\ell(\Psi^*_B; x_i)],
\] (15)
where the direction of \(H_1\) is typically considered in drawing final conclusions (i.e., instead of concluding that the two models differ in fit, the researcher interprets one model as fitting better than the other).

The test statistic associated with these hypotheses is similar to a paired-samples \(t\)-test: An observation has a log-likelihood under each model, and the test statistic is
based on the mean and variance of differences between log-likelihoods across observations. Formally, the test statistic is

$$LR_{AB} = N^{-1/2} \sum_{i=1}^{N} \log \frac{f_A(x_i; \hat{\Psi}_A)}{f_B(x_i; \hat{\Psi}_B)},$$

(16)

which, under (14), converges in distribution to $N(0, \omega^2)$ when models are distinguishable (Vuong, 1989). Note that in nonnest2 (Merkle & You, 2018), this test statistic is rescaled, resulting in a $z$ test statistic following the standard normal distribution.

**Testing nested models.** In the case of nested models, the two statistics described above are alternative ways of testing the same hypothesis. Assuming that Model B is nested within Model A, these hypotheses can be written as

$$H_0: \quad \Psi_A \in h(\Psi_B)$$

$$H_1: \quad \Psi_A \notin h(\Psi_B),$$

(17) (18)

where $h()$ is a function translating the $M_B$ parameter vector to an equivalent $M_A$ parameter vector.

The limiting distribution of the test statistics depends on whether or not we assume that Model A is correctly specified. If we do make this assumption (which is commonly employed for traditional tests of nested models), then both statistics ($N\hat{\omega}^2$ and $LR = 2N^{1/2}LR_{AB}$) weakly converge to the usual $\chi^2$ distribution. If we do not make this assumption, then the statistics strongly converge to weighted sums of $\chi^2$ distributions, where the weights again involve eigenvalues of a matrix containing the models’ scores and information matrices.

The test statistics described above are implemented for many classes of models in the R package nonnest2 (Merkle & You, 2018). As part of the current paper, the package functionality was extended to IRT models estimated via the R package mirt (Chalmers, 2012), which is often used to obtain parameter estimates in IRT models using the marginal ML criteria. It should be noted that Vuong’s theory can also be used for the computation of AIC and BIC confidence intervals for non-nested models (Merkle et al., 2016), which we will not discuss here. We use these package extensions
throughout the paper to study and illustrate the Vuong test statistics’ applications to IRT.

Alternative Methods

We now briefly discuss some widely used methods that aim at similar goals. Later, we will compare these methods against Vuong’s tests through a selection of simulation studies.

**Nested Models and Likelihood Ratio Tests.** Several authors have discussed the use of likelihood ratio tests for comparing the relative model fit of two competing models, both within the framework of IRT (e.g., Reckase, 2009) and in factor analysis (e.g., Hayashi, Bentler, & Yuan, 2007). The traditional likelihood ratio test (Neyman & Pearson, 1928, 1933), and the related difference in $G^2$ or $\chi^2$, have been shown to follow an asymptotic $\chi^2$ distribution under the null hypothesis under a wide range of conditions and models; examples include log-linear models (Haberman, 1977) and factor analytic models (Amemiya & Anderson, 1990). Drton (2009) discusses some models for which the asymptotic distribution does not follow a $\chi^2$ distribution. A typical application is the assessment of the dimensionality of a dataset (e.g., Maydeu-Olivares & Cai, 2006; Schilling & Bock, 2005; Tollenaar & Mooijaart, 2003). As it has already been stated, a basic assumption of the use of likelihood ratio tests for evaluating the relative model fit is that the less restrictive of the two models is correctly specified. If this assumption is not met, $p$-values are in general no longer uniformly distributed under the null hypothesis. Simulation studies in IRT and factor analysis indicate that the asymptotic distribution is no longer $\chi^2$ if neither of the two models being compared is the true model (Maydeu-Olivares & Cai, 2006; Yuan & Bentler, 2004). In contrast, the asymptotic distribution of Vuong’s test statistics are not based on the assumption that one of the two competing models is the true model.

**Information Criteria.** Information criteria (e.g., AIC and BIC) are widely known and commonly used tools for model selection. Taking into account the fit and complexity of the competing models, information criteria aim at providing an index of model fit, where a lower index expresses a better model fit in terms of both data-model
fit as well as parsimony; for an extensive overview see Burnham and Anderson (2002). While the application of information criteria is not limited to nested models, explicitly choosing a “better fitting” model can be somewhat difficult. Popular approaches include “rules of thumbs”, e.g., an absolute difference in AIC larger than 10 indicates strong support for the model with the lower AIC (Burnham & Anderson, 2004), or the approach of simply selecting the model with the lowest index. Kang and Cohen (2007) and Kang, Cohen, and Sung (2009) studied the performance of the AIC and BIC in the context of model selection of both dichotomous and polytomous IRT models, among other Bayesian measures of fit and other test statistics, and found them to generally perform well in that they often indicated correct preference for the true data generating model.

Assessing Absolute Model Fit. In the analysis of categorical data, $G^2$, and the related $\chi^2$-statistic, are used to assess the absolute fit of specific models (Agresti, 2002). For testing this hypothesis in IRT, limited information fit statistics have been proposed as an alternative to account for the sparsity of the underlying contingency table. Two prominent examples in IRT are $M_2$ (Maydeu-Olivares & Joe, 2005) and $M_2^*$ (Cai & Hansen, 2013). These statistics aim at testing a different null hypothesis than Vuong’s test statistics in that they compare the fit of a given model against the first and second moments of the data.

In the following sections, we study the Vuong tests’ application to IRT using both simulations and real data. We also compare Vuong’s tests to the aforementioned model assessment approaches discussed in this section to evaluate how effective the Vuong tests are relative to these previously studied popular methods.

Simulation 1: Non-Nested Models

In this section of simulations, we first focus on non-nested IRT models for polytomous data; namely, the GRM and the GPCM. In Simulation 1.1, we compare the fit of the GRM to the GPCM when the data generating process is not following either of the competing two models, but rather an “uninformative” binomial distribution. The purpose of this simulation is to demonstrate the behavior of Vuong’s tests when the
data do not suggest any meaningful preference towards the competing models. Hence, the test results should not demonstrate any systematic preferences for or against a given model. As an example of this feature, we contrast the results suggested by information-based statistics, such as AIC, and model fit approaches such as the $M_2^*$ statistic. Next, in Simulation 1.2, we compare the fit of the GRM to the GPCM with the data generated under a hybrid model, where items follow either of the two competing models in varying numbers.

In both simulations, we study Vuong’s test of distinguishability, as well as Vuong’s test of non-nested models, and compare these to the AIC and $M_2^*$ statistic. All models in this section were estimated via marginal ML, assuming the standard normal distribution of the person parameters. The EM algorithm for marginal ML estimation was implemented using the default estimation criteria and defaults found in mirt (Chalmers, 2012), with the exception that up to 5000 EM cycles were allowed, if necessary, before the algorithm was terminated; otherwise, the algorithm was terminated early if all elements of the sets of estimates between two succeeding EM cycles fell below $|0.0001|$, in which case the model was deemed to have successfully “converged”. Simulation results are reported based on the replications in which both models converged. Test statistics were evaluated at an $\alpha$ of 0.05.

Simulation 1.1: Data Generated Under a Binomial Distribution

Method. Simulation conditions were defined by the number of persons, $N = 500, 1000$, or $2000$, and the length of the test (fixed at $J = 10$). In each condition, $1000 \times J$ datasets were generated from a binomial distribution with hyper-parameters $n = 3$ and $p = 0.5$. This model serves as a generalization of the data generating process investigated by Wood (1978) to polytomous data.

In each condition, and for each generated dataset we computed four statistics after fitting the models: Vuong’s test of distinguishability, Vuong’s test of non-nested models, and each model’s AIC and $M_2^*$ statistic. We checked whether the models could

\footnote{We do not include the BIC for further comparison in this study because the GRM and GPCM share the same number of estimated model parameters.}
be distinguished, and if this was the case, whether the non-nested test implied preference of one model over the other one. We further checked which model was to be preferred based on the lower AIC, and whether the $M_2^*$ statistic indicated bad model fit.

Table 1

*Simulation 1.1: Comparing the GRM and the GPCM When Data Follow a Binomial Distribution.*

| N  | Dist | LRT$_v$ | AIC | $M_2^*$ | LRT$_v$ | AIC | $M_2^*$ |
|----|------|---------|-----|---------|---------|-----|---------|
|    | all  | (Dist sgn.) |     |         | all  | (Dist sgn.) |     |         |
| 500 | 0.00 | 0.01     | (0.00) | 0.55    | 0.01   | (0.00) | 0.45    | 0.01   |
| 1000| 0.01 | 0.01     | (0.00) | 0.53    | 0.01   | (0.00) | 0.47    | 0.01   |
| 2000| 0.01 | 0.00     | (0.11) | 0.49    | 0.01   | (0.00) | 0.51    | 0.02   |

*Note.* all = using all replications for checking the preference of the non-nested Vuong test (LRT$_v$). Dist sgn. = using only the replications in which Vuong’s test of distinguishability (Dist) yielded significant results. $J = 10$ items. $N =$ number of persons.

**Results.** In the condition of $N = 500$, the EM algorithm converged for both models in 85% of the replications. For the condition of $N = 1000$, this was the case in 97% of the replications and in the condition of $N = 2000$, this was the case in 100% of the replications. In all conditions and all replications, second-order tests based on the condition number of the estimated information matrices of the models indicated that possible local maxima were found.

Table 1 summarizes the simulation results. Regardless of the number of persons, Vuong’s test of distinguishability (Dist) indicates at a rate of 1% that the GRM and GPCM can be distinguished. Recall that this statistical test is not designed to determine which of the two competing models provides the better fit to the data. At a rate of around 1%, Vuong’s test of non-nested models (LRT$_v$) prefers the GRM over the GPCM, and at a rate of around 1% the GPCM is to be preferred over the GRM. However, as outlined in the introduction, Vuong’s test of non-nested models can only be applied validly if the test of distinguishability yielded a significant result beforehand.
Looking only at these few replications, the GRM is to be preferred over the GPCM at a rate of 0% to 11%, and the GPCM is to be preferred over the GRM at a rate of 0%. As to be expected, performing model selection based on the lower AIC results in choosing either model at a rate of 50%. Finally, the $M_2^*$ statistic indicates bad model fit for both models at a maximum rate of 2%.

Figure 1 shows both a boxplot as well as a histogram of the absolute differences in AIC values for the condition of $N = 500$. While these absolute differences tend to be small ($Mean = 0.91$, $SD = 0.91$, $Q_{10\%} = 0.13$ and $Q_{90\%} = 1.86$), substantial differences do occur nevertheless, making model selection based on the lower AIC quite misleading in some scenarios. These are potentially misleading because a researcher may conclude that one model is notably more supported by the data than another, when in fact the fit to the data is based completely on noise variation.

**Discussion.** In Simulation 1.1, we showed that Vuong’s test of distinguishability is useful in the context of IRT modeling, when the data generating process is uninformative for both competing models, i.e., when the data follow a binomial distribution. In this scenario, there is no basis for asking the question whether the GRM or the GPCM provides the better fit to the data, and the results from this test of distinguishability tells us exactly this; i.e., that the models result in nearly identical likelihoods for all persons. In contrary, comparing these two competing models based on
their AIC can lead to misleading conclusions: In the scenarios investigated, selecting the model with the lower AIC results in falsely declaring one model as the “better fitting” one simply by chance. Relying on cut-off heuristic values, such as interpreting an absolute difference in AIC larger than 10 as “substantial”, can mitigate this problem to some extent; however, due to the arbitrariness being involved in declaring such a cut-off value and other factors, such as sample size variability of information criteria, this procedure still leaves plenty of room for false positive declarations of one model advertised as the “better fitting” one.

**Simulation 1.2: Data Generated Under a Hybrid Model**

**Method.** Simulation conditions were defined by the number of persons, $N = 500, 1000, \text{ or } 2000$, the length of the test (fixed at $J = 10$), and the number of items of the data generating hybrid model following a four categories GPCM, $D = 0, 1, \ldots, 9, 10$. If an item was not generated under the GPCM then it was generated according to a four category GRM. Under both models, slopes were drawn from a log-normal distribution with a mean of 0 and a variance of 0.25$^2$. Intercepts were generated based on the distance vector $(1, 0, -1)'$, where for each item this vector was shifted by a random deviance term drawn from the standard normal distribution.

In each condition, we generated 1000 datasets and for each generated dataset we computed four statistics after fitting the models: Vuong’s test of distinguishability, Vuong’s test of non-nested models, and each model’s AIC and $M^*_2$ statistic. We checked whether the models could be distinguished, and if this was the case, whether Vuong’s test of non-nested models implied preference of one model over the other one. We further checked which model was to be preferred based on the lower AIC, and whether the $M^*_2$ statistic indicated bad model fit.

**Results.** In all conditions and all replications, the EM algorithm converged for both models. Moreover, in all conditions and all replications, second-order tests based on the condition number of the estimated information matrices of the models indicated that possible local maxima were found.

Results are displayed in Figure 2, where the x-axis indicates the number of GPCM
Figure 2. Simulation 1.2: Empirical preference/rejection rates associated with statistics. \( J = 10 \) items. \( D \) = number of items of the data generating (hybrid) model following the GPCM. \( N \) = number of persons.

items \((D)\). The three panels split the results with respect to the number of persons \( N \). Within each panel, the lines represent the four statistics. For Vuong’s test of distinguishability \((\text{Dist})\), there is only one line representing the power. For both Vuong’s test of non-nested models \((\text{LRT}_v)\) and the AIC, there are two lines: one for each model representing the relative frequency of the model being preferred over the other. Note that this is symmetric for the AIC but not for the test of non-nested models due to the possibility of neither model being preferred over the other one. For the \( M_2^* \) statistic, there are also two lines, one for each model, representing the relative frequency of the test statistic indicating a bad model fit.

Looking at the results for Vuong’s test of distinguishability, we observe a power of one for all conditions, implying that the GRM and GPCM can be perfectly distinguished from each other in the scenarios examined. Remember that based on this test, we only conclude that the models can potentially be differentiated based on their fit. We do not, however, draw any conclusions about which model fits the data better.

While a perfect power of one may seem inordinate, note that this result was somewhat to be expected in the context of IRT modeling if and when the data generating process is informative (c.f. Simulation 1.1). Contrary to other more malleable statistical models, IRT models are often more limited in their mathematical structure; namely,
that their probabilistic “predictors” are predetermined by the model parameters themselves, resulting in predicted values that are unlikely to overlap. In regression analysis, as an example, it may happen that different predictor variables make identical predictions in some populations, thereby sharing substantial overlap, which ultimately leads to competing models being statistically indistinguishable.

Regarding Vuong’s test of non-nested models, when all data generating items follow the GRM ($D = 0$), the GRM is preferred over the GPCM at a rate near 67% for the condition of $N = 500$, and this rate increases up to 99% as the number of persons increases. Analogously, the same pattern holds for the GPCM when all data generating items follow the GPCM ($D = 10$). Moreover, with $D$ increasing, the relative preference of the GRM over the GPCM decreases, whereas the relative preference of the GPCM over the GRM increases.

A similar pattern can be observed when inspecting the results for the AIC, although model selection based on the lowest AIC results in higher “power” for extreme values of $D$ (e.g., 0 and 10). On the other hand, this procedure does not allow for the conclusion that neither model is to be preferred, or that both models fit equally well, resulting in a relative preference rate close to chance for both models at $D = 5$. To allow for a comparison of the absolute differences in AIC values with the results reported in Simulation 1.1, we again computed their mean and standard deviation, as well as their 10% and 90% quantiles for some selected conditions. For the condition of $N = 500$ and $D = 0$: $Mean = 18.88$, $SD = 9.73$, $Q_{10\%} = 6.40$ and $Q_{90\%} = 31.62$. For the condition of $N = 500$ and $D = 3$: $Mean = 9.87$, $SD = 7.46$, $Q_{10\%} = 1.42$ and $Q_{90\%} = 20.77$. For the condition of $N = 500$ and $D = 5$: $Mean = 8.81$, $SD = 6.91$, $Q_{10\%} = 1.43$ and $Q_{90\%} = 18.44$. Notice that with $D$ increasing, i.e., up to $D = 5$, the absolute differences in AIC values tend to become smaller, and their distribution tends to (partially) overlap with the distribution reported in Simulation 1.1’s Figure 1. While this is the expected behavior, this also highlights the problematic aspects of performing model selection based on differences in information criteria. Looking back at Simulation 1.1, we are tempted to conclude that an absolute difference in AIC values of around 2 should not
be regarded as an indication of preference of one model over the other one, as we were not able to formally distinguish them based on Vuong’s test of distinguishability. However, observing the results reported here, we are tempted to conclude that this difference of around 2 should, in fact, be regarded as an indication of preference.

Inspecting the $M_2^*$ statistic, we reject a “good fit” for both models at a rate close to 5%. We conclude that both models fit the data well, being rejected at the nominal Type I error rate independent of the data generating process. This shows that in this scenario, the $M_2^*$ statistic cannot be used for model selection as we are left with no indication of preference of one model over the other, even when all items follow either the GRM or GPCM. Admittedly, the $M_2^*$ statistic was not originally designed to be used for model selection, but rather as a statistic for evaluating the absolute fit of a model according to the first and second moment structures. Nevertheless, this simulation highlights why goodness-of-fit statistics based on a subset of the moments of the data are often insufficient for evaluating the true population generating models.

**Discussion.** In Simulation 1.2, we showed how the Vuong tests could be used to compare the fit of a GRM to the fit of a GPCM. To our knowledge, these are the first formal test statistics for comparing such models. We found that the two models could be reliably distinguished from one another, and, in the cases of $D = 0$ and $D = 10$, Vuong’s test of non-nested models was able to select the data generating model with near perfect accuracy. As seen in this simulation, applying Vuong’s test of non-nested models can result in the conclusion that both models fit equally well. We argue that this is a benefit rather than a drawback, and further discuss the implications of the test of non-nested models in the General Discussion section. In the next simulation section, we apply the Vuong tests to the comparison of nested models.

**Simulation 2: Nested Models**

While the Vuong tests’ application to non-nested IRT models is relatively novel, the statistics can also be used to test nested models. In this case, they serve as alternatives to the traditional tests based on the likelihood function, such as the likelihood ratio test, Wald test, or score test (Engle, 1984). As mentioned earlier,
however, the Vuong tests do not rely on the assumption that either of the competing models are correctly specified. Thus, it is reasonable to expect that the Vuong tests’ properties will differ from the traditional tests in some scenarios. We study this expectation, among others, in this simulation section, and focus on nested IRT models for dichotomous data.

In Simulation 2.1, we compare the fit of the Rasch model (RM) to the 2PL when the data are either generated under the RM, 2PL or a modified three-parameter logistic model (3PL), relying on the latter to investigate performance under misspecification of the 2PL. In Simulation 2.2, we compare the fit of the 2PL to the within-item 2d-2PL, when the data are either generated under the 2PL or the 2d-2PL, varying the correlation of the two latent dimensions. In both simulations, we study Vuong’s test of distinguishability and Vuong’s test of nested models as alternatives to the traditional likelihood ratio test, and compare these further to the AIC, BIC and $M^2$ statistic (note that for dichotomous response data, $M_2 \equiv M^*_2$). Estimation defaults and assumptions were the same as in Simulation 1, if not stated otherwise. Simulation results are reported based on the replications in which both models converged. Test statistics were evaluated at an $\alpha$ of 0.05.

Simulation 2.1: Comparing the Rasch Model and the 2PL

Methods. Simulation conditions were defined by the number of persons, $N = 500, 1000, \text{ or } 2000$, the test length, $J = 10, 20, 30, \text{ or } 40$, and the data generating model either being the RM, the 2PL or a modified 3PL with varying lower asymptote parameters, restricted to be the same for all items. As we did not cover IRT models with lower or upper asymptotes in the Introduction section, we briefly introduce the 3PL in this section. The 3PL extends the 2PL by introducing another item parameter $g_j$ for each item, a lower asymptote acting as a so-called “guessing parameter”, modeling the probability of person $i$ “solving” item $j$ as:

$$p_{ij1} = g_j + \frac{(1 - g_j)}{1 + \exp(-(\beta_j + a_j \theta_i))}$$

(19)

In this section, we consider a modified 3PL, restricting these guessing parameters
$g_j$ to be the same for all items ($g = 0.01, 0.05, \text{ or } 0.25$) while simultaneously restricting the slopes to 1 for all items. Analogous to Maydeu-Olivares and Cai (2006), this allows us to evaluate the tests statistics' performance under misspecification of the less restrictive model, as the 2PL is not correctly specified when the data are generated under this modified 3PL.

In the conditions of the RM or the modified 3PL being the data generating model, slopes were fixed at 1 for all items. In the condition of the 2PL being the data generating model, slopes were drawn from a log-normal distribution with a mean of 0 and a variance of $0.25^2$. Intercepts were drawn from the standard normal distribution.

Regarding the RM, slopes were fixed at 1 for all items and the latent variance $\sigma^2_\theta$ was freely estimated. In each condition we generated 1000 datasets, and for each generated dataset, we computed six statistics after fitting the models: Vuong's test of distinguishability, Vuong's test of nested models, the traditional likelihood ratio test, and each model's AIC, BIC, and $M_2$ statistic. In addition to evaluating the difference in AIC and BIC, we checked whether Vuong’s test of distinguishability, Vuong’s test of nested models and the traditional likelihood ratio test indicated preference of the 2PL over the RM, and whether the $M_2$ statistic indicated bad model fit.

Results.
| DGM | J | Dist | LRTₜ | LRTᵣ | AIC | BIC | RM | 2PL | Dist | LRTₜ | LRTᵣ | AIC | BIC | RM | 2PL | Dist | LRTₜ | LRTᵣ | AIC | BIC | RM | 2PL |
|-----|---|------|------|------|-----|-----|----|----|------|------|------|-----|-----|-----|----|----|----|------|------|------|-----|-----|----|----|----|
| RM  | 10 | 0.04 | 0.05 | 0.06 | 0.04 | 0.00 | 0.06 | 0.05 | 0.04 | 0.04 | 0.05 | 0.03 | 0.00 | 0.05 | 0.04 | 0.04 | 0.04 | 0.03 | 0.00 | 0.05 | 0.04 |
| RM  | 20 | 0.02 | 0.04 | 0.06 | 0.01 | 0.00 | 0.05 | 0.05 | 0.03 | 0.04 | 0.05 | 0.01 | 0.00 | 0.04 | 0.04 | 0.03 | 0.04 | 0.05 | 0.00 | 0.00 | 0.05 | 0.05 |
| RM  | 30 | 0.01 | 0.03 | 0.04 | 0.00 | 0.00 | 0.06 | 0.07 | 0.02 | 0.03 | 0.04 | 0.00 | 0.00 | 0.05 | 0.05 | 0.03 | 0.04 | 0.05 | 0.00 | 0.00 | 0.05 | 0.05 |
| RM  | 40 | 0.01 | 0.04 | 0.06 | 0.00 | 0.00 | 0.05 | 0.05 | 0.02 | 0.04 | 0.05 | 0.00 | 0.00 | 0.05 | 0.05 | 0.03 | 0.04 | 0.05 | 0.00 | 0.00 | 0.05 | 0.05 |
| 2PL | 10 | 0.69 | 0.75 | 0.73 | 0.03 | 0.49 | 0.04 | 0.92 | 0.93 | 0.94 | 0.92 | 0.20 | 0.80 | 0.05 | 0.99 | 0.99 | 0.99 | 0.99 | 0.58 | 0.95 | 0.05 |
| 2PL | 20 | 0.96 | 0.98 | 0.96 | 0.03 | 0.75 | 0.06 | 1.00 | 1.00 | 1.00 | 0.37 | 0.95 | 0.05 | 1.00 | 1.00 | 1.00 | 1.00 | 0.88 | 1.00 | 0.05 |
| 2PL | 30 | 1.00 | 1.00 | 1.00 | 0.99 | 0.43 | 0.05 | 1.00 | 1.00 | 1.00 | 0.51 | 0.99 | 0.05 | 1.00 | 1.00 | 1.00 | 1.00 | 0.97 | 1.00 | 0.05 |
| 2PL | 40 | 1.00 | 1.00 | 1.00 | 1.00 | 0.04 | 0.90 | 0.07 | 1.00 | 1.00 | 1.00 | 0.62 | 1.00 | 0.07 | 1.00 | 1.00 | 1.00 | 1.00 | 0.99 | 1.00 | 0.07 |
| 3PL₁₀.₀₁ | 10 | 0.03 | 0.05 | 0.05 | 0.04 | 0.00 | 0.06 | 0.06 | 0.04 | 0.05 | 0.05 | 0.04 | 0.00 | 0.07 | 0.07 | 0.06 | 0.07 | 0.08 | 0.06 | 0.00 | 0.07 | 0.06 |
| 3PL₁₀.₀₁ | 20 | 0.02 | 0.05 | 0.06 | 0.01 | 0.00 | 0.05 | 0.05 | 0.03 | 0.06 | 0.06 | 0.01 | 0.00 | 0.05 | 0.05 | 0.05 | 0.06 | 0.07 | 0.01 | 0.00 | 0.05 | 0.05 |
| 3PL₁₀.₀₁ | 30 | 0.01 | 0.04 | 0.06 | 0.00 | 0.00 | 0.08 | 0.08 | 0.03 | 0.06 | 0.08 | 0.00 | 0.00 | 0.06 | 0.06 | 0.06 | 0.06 | 0.07 | 0.08 | 0.00 | 0.04 | 0.04 |
| 3PL₁₀.₀₁ | 40 | 0.01 | 0.05 | 0.06 | 0.00 | 0.00 | 0.05 | 0.06 | 0.02 | 0.05 | 0.06 | 0.00 | 0.00 | 0.05 | 0.05 | 0.06 | 0.06 | 0.07 | 0.08 | 0.00 | 0.04 | 0.04 |
| 3PL₁₀.₀₅ | 10 | 0.06 | 0.07 | 0.09 | 0.06 | 0.00 | 0.07 | 0.06 | 0.10 | 0.11 | 0.12 | 0.10 | 0.00 | 0.08 | 0.05 | 0.20 | 0.20 | 0.22 | 0.18 | 0.00 | 0.11 | 0.05 |
| 3PL₁₀.₀₅ | 20 | 0.06 | 0.10 | 0.14 | 0.03 | 0.00 | 0.08 | 0.06 | 0.18 | 0.22 | 0.26 | 0.07 | 0.00 | 0.08 | 0.04 | 0.41 | 0.43 | 0.45 | 0.21 | 0.00 | 0.13 | 0.05 |
| 3PL₁₀.₀₅ | 30 | 0.06 | 0.14 | 0.20 | 0.02 | 0.00 | 0.08 | 0.06 | 0.24 | 0.31 | 0.36 | 0.06 | 0.00 | 0.09 | 0.05 | 0.62 | 0.66 | 0.68 | 0.28 | 0.00 | 0.14 | 0.05 |
| 3PL₁₀.₀₅ | 40 | 0.05 | 0.16 | 0.23 | 0.00 | 0.00 | 0.09 | 0.07 | 0.29 | 0.42 | 0.48 | 0.05 | 0.00 | 0.12 | 0.07 | 0.74 | 0.78 | 0.80 | 0.32 | 0.00 | 0.17 | 0.06 |
| 3PL₁₀.₂₅ | 10 | 0.19 | 0.26 | 0.31 | 0.27 | 0.00 | 0.15 | 0.05 | 0.46 | 0.53 | 0.55 | 0.50 | 0.00 | 0.30 | 0.05 | 0.77 | 0.78 | 0.80 | 0.76 | 0.02 | 0.54 | 0.06 |
| 3PL₁₀.₂₅ | 20 | 0.47 | 0.61 | 0.67 | 0.40 | 0.00 | 0.27 | 0.08 | 0.87 | 0.91 | 0.91 | 0.80 | 0.00 | 0.50 | 0.07 | 0.99 | 1.00 | 1.00 | 0.97 | 0.05 | 0.82 | 0.08 |
| 3PL₁₀.₂₅ | 30 | 0.67 | 0.85 | 0.88 | 0.55 | 0.00 | 0.34 | 0.08 | 0.98 | 0.98 | 0.98 | 0.92 | 0.00 | 0.62 | 0.10 | 1.00 | 1.00 | 1.00 | 1.00 | 0.09 | 0.93 | 0.08 |
| 3PL₁₀.₂₅ | 40 | 0.78 | 0.94 | 0.96 | 0.66 | 0.00 | 0.41 | 0.15 | 1.00 | 1.00 | 1.00 | 0.98 | 0.00 | 0.74 | 0.11 | 1.00 | 1.00 | 1.00 | 1.00 | 0.14 | 0.96 | 0.14 |

*Note.* DGM = data generating model. $3PL_q = 3PL$ with lower asymptotes restricted to $q$ and slopes restricted to 1 for all items. $J =$ number of items. $N =$ number of persons.
In all conditions and all replications, the EM algorithm converged for both models. Moreover, in all conditions and all replications, second order tests based on the condition number of the estimated information matrices of the models indicated that possible local maxima were found.

Results are displayed in Table 2. With the RM being the data generating model, all statistics imply preference of the 2PL over the RM or bad model fit for either model at around the nominal Type I error rate of 5%. However, Vuong’s test of distinguishability (Dist) and the AIC and BIC generally appear to be conservative in their error control rates. When data are generated under the 2PL, all test statistics imply preference of the 2PL over the RM with high power, increasing with the number of items and persons, and Vuong’s test of nested models (LRT\textsubscript{v}) and the traditional likelihood ratio test (LRT\textsubscript{t}) show almost equivalent performance. While this also holds for the AIC, the BIC lacks power in comparison. Finally, the $M_2$ statistic is also sensitive to the 2PL being the data generating model, indicating a bad model fit of the RM at high rates, while holding its nominal Type I error rate for the 2PL. Evaluating the scenarios including misspecification, the Vuong tests and the traditional likelihood ratio test appear to be robust under minor misspecification ($g = 0.01$). However, with increasing misspecification ($g = 0.05$, or $0.25$), all tests increasingly prefer the 2PL over the RM, a finding Maydeu-Olivares and Cai (2006) already reported for the traditional likelihood ratio test. Although this degree of preference of the 2PL over the RM is generally smaller under Vuong’ test of distinguishability, showing the best performance compared to all other statistics, this difference in performance can be considered negligible in most of the scenarios examined in that the tests’ performance is far from being ideal. The same conclusions hold for the traditional likelihood ratio test and the AIC and BIC as well.

In this simulation, we further investigated whether the empirical distributions of the Vuong test statistics’ match their theoretical distributions under the null hypothesis. As outlined in the Introduction section, the limiting distributions of Vuong’s test of distinguishability and test of nested models are given by weighted sums
Figure 3. Simulation 2.1: Histograms of $p$-values for Vuong’s test of distinguishability ($\text{Dist}$) and test of nested models ($\text{LRT}_v$) under the null hypothesis, i.e., the RM being the data generating model. $J = 10$ items. $N = 2000$ persons.

of $\chi^2$ distributions and the computation of these weights depends on eigenvalues of a matrix involving the two models’ scores and information matrices and for this reason depend on the concrete data. We therefore simply checked whether the $p$-values are distributed uniformly under the null hypothesis. Figure 3 shows two histograms of $p$-values, one for Vuong’s test of distinguishability, and one for Vuong’s test of nested models for the scenario of the RM being the data generating model and $J = 10$ and $N = 2000$. Looking at these histograms, $p$-values seem to be uniformly distributed under the null hypothesis.

Discussion. In Simulation 2.1, we showed that the Vuong tests, especially Vuong’s test of nested models, perform as well as the traditional likelihood ratio test when comparing nested models under ideal scenarios, i.e., the models are truly nested, the parameters to be tested lie in the interior of the parameter space, and the larger model is correctly specified. However, this actually comes as no surprise, as under these conditions the equivalence of the Vuong tests and the traditional likelihood ratio test has been proven (see Vuong, 1989, Corollary 7.3, Corollary 7.5). We have also seen that when the less restrictive model is severely misspecified, the Vuong tests do not necessarily perform substantially better than the traditional likelihood ratio test; at least, in the scenarios examined here.
Simulation 2.2: Comparing Nested Models of Different Dimensions

Method. Simulation conditions were defined by the number of persons (fixed at $N = 2000$), the test length, $J = 10, 20, 30,$ or $40$ and the data generating process either being the 2PL or the within-item 2d-2PL, varying the correlation of the two latent dimensions, $\rho = \frac{2}{3}, \frac{1}{3}, $ or $0$. In the conditions of the 2PL being the data generating model, person parameters were assumed to follow the standard normal distribution. In the conditions of the 2d-2PL being the data generating model, person parameters were assumed to follow a bivariate normal distribution with means 0 and a covariance matrix with variances of 1 and a covariance of $\rho$. Both vectors of slopes were drawn independently from a log-normal distribution with a mean of 0 and a variance of 0.25$^2$, resembling a within-item multidimensional structure with uncorrelated factor loadings, while intercepts were drawn from the standard normal distribution.

Both models were estimated via marginal ML, assuming the standard normal distribution of the person parameters for the 2PL and a bivariate normal distribution with means 0 and the identity matrix as the covariance matrix for the 2d-2PL. Regarding the 2d-2PL, the second slope of the last item was always fixed at 0, resolving the rotational indeterminancy of the model. In each condition we generated 1000 datasets, and for each generated dataset we computed six statistics after fitting the models: Vuong’s test of distinguishability, Vuong’s test of nested models, the traditional likelihood ratio test, and each model’s AIC, BIC and $M_2$ statistic. We checked whether Vuong’s test of distinguishability, Vuong’s test of nested models, the traditional likelihood ratio test and the difference in AIC and BIC implied preference of the 2d-2PL over the 2PL, and whether the $M_2$ statistic indicated bad model fit.

Results. We observed the lowest rate of convergence of both models in the condition of $N = 2000$, $M = 20$ and $\rho = \frac{2}{3}$, where 96% of the time the models successfully converged. In all conditions and all replications, second-order tests based on the condition number of the estimated information matrices of the models indicated that possible local maxima were found.

Results are displayed in Figure 4, where the x-axis shows values of $J$. The four
Figure 4. Simulation 2.2: Empirical preference/rejection rates associated with
statistics. $J =$ number of items. $\rho =$ correlation between the two latent dimensions
under the data generating 2d-2PL. $N = 2000$ persons.

panels split the results with respect to the data generating model being either the 2PL
or the 2d-2PL with varying correlation $\rho$. Within each panel, the lines represent the six
statistics. For Vuong’s test of distinguishability (Dist), Vuong’s test of nested models
($\text{LRT}_v$), the traditional likelihood ratio test ($\text{LRT}_t$), and the AIC and BIC, there is only
one line representing Type I error rate/power. For the $M_2$ statistic, there are two lines,
one for each model, representing the relative frequency of the statistic indicating bad
model fit.

Regarding Vuong’s test of distinguishability, Vuong’s test of nested models, and
the traditional likelihood ratio test, we notice that the latter test shows a highly
inflated Type I error rate, implying preference of the 2d-2PL over the 2PL at a rate of
around 25%, increasing with the number of items up to 98% — even though no second
dimension is present in the data (the 2PL being the data generating model). Somewhat
remarkably, however, both Vuong’s test of distinguishability and test of nested models
imply preference of the 2d-2PL over the 2PL at much more reasonable Type I error
rates, with the former one being slightly too conservative and the latter one being more
liberal. Moreover, both tests are sensitive to the correlation $\rho$ decreasing, implying
increasing preference of the 2d-2PL over the 2PL with a peak of power for the former at
around 78%, and 97% for the latter ($J = 40$, $\rho = 0$).
Looking at the AIC, there is also a less pronounced bias for the 2d-2PL to be selected, implying preference of the 2d-2PL over the 2PL at rates of around 12% to 18% given no second dimension (the 2PL being the data generating model). Analogous to Vuong’s test of distinguishability and test of nested models, the AIC increasingly prefers the 2d-2PL over the 2PL as $\rho$ decreases. In contrast, the BIC shows itself to be strictly conservative, almost always expressing preference for the less complex 2PL.

Lastly, the $M_2$ statistic implies bad model fit for the 2d-2PL at overall Type I error rates of 1% to 5%, being conservative when no second dimension is present (the 2PL being the data generating model). Regarding the 2PL, the $M_2$ statistic implies bad model fit at Type I error rates close to 5%, and increasingly implies bad model fit as $\rho$ decreases (up to a power of 88% for $J = 40$, $\rho = 0$).

Discussion. In Simulation 2.2, we found the Vuong tests to exhibit good behavior for testing the dimension of the 2PL. In contrast, the traditional likelihood ratio test performed quite poorly, exhibiting very large Type I error rates. Both the AIC and $M_2$ statistic exhibited reasonable performance for model selection, though we reiterate that these statistics do not provide formal tests of model comparison, though we reiterate that these statistics do not provide formal tests of model comparison. In the General Discussion section, we provide further thoughts on the poor performance of the traditional likelihood ratio test as well as the future developments of the Vuong tests. In the following section, we study the Vuong test’s application to IRT using real data.

Application: The Nerdy Personality Attributes Scale

Background

The Nerdy Personality Attributes Scale (NPAS; Open Source Psychometrics Project, 2016) was developed as an online questionnaire by the Open Source Psychometrics Project aiming at quantifying a “nerdiness” construct. The NPAS consists of 26 items in total, each rated on a five-point Likert scale, where a total of $N = 1445$ participants were collected over several months in 2015. For the purpose of this analysis we limit our demonstration to a subset of science-related items only; namely, items 1, 2, 6, 13, 22 and 23. The exact item wordings are presented in the appendix. As an example of this science-related content, item 1 states: “I am interested
in science”. We excluded 384 participants due to failing the additional validity check items or failing to answer any of these six items. Our final dataset therefore consists of \( N = 1061 \) participants responding to six items.

**Method**

For this analysis we only consider the GRM and GPCM as suitable models to be fit to the data. First, we explored their fit using the AIC statistic. Second, we followed up with Vuong’s test of distinguishability, and if we concluded that the models can be distinguished, we then tested which model provides the better fit using Vuong’s test of non-nested models.

After having selected one of these unidimensional models, we then further wanted to test whether a two-dimensional version of the selected model provides an even better fit. Again, we first explored their overall fit using the AIC, but we also tested whether the unidimensional model fits as well as its two-dimensional version, using the traditional likelihood ratio test. We then compared these results to Vuong’s test of distinguishability and Vuong’s test of nested models.

For all models, we also investigated absolute model fit, however, the \( M^*_2 \) statistic could not be computed due to too few degrees of freedom. We therefore computed the \( C_2 \) statistic (Cai & Monroe, 2014), where only the bivariate moments are collapsed.

**Results**

Looking at the \( C_2 \) statistic, both models fit the data well, \( C_2(\text{GRM})_{(9)} = 9.78, p = 0.369 \), \( C_2(\text{GPCM})_{(9)} = 10.63, p = 0.302 \). Examining the two models’ AICs demonstrated that the GRM is preferred to the GPCM (\( \text{AIC}_{\text{GRM}} = 17412.12 \), \( \text{AIC}_{\text{GPCM}} = 17466.53, \Delta_{\text{AIC}} = -54.41 \)). Next, we followed up with Vuong’s test of distinguishability and found that we could distinguish the GRM from the GPCM (\( \hat{\omega}_*^2 = 0.04, p < 0.001 \)). Finally, we used Vuong’s test of non-nested models to compare the respective model fits. We found indeed that the GRM does fit better than the GPCM (\( z = 4.41, p < 0.001 \)), and selected the GRM as the better fitting unidimensional model for these data.
Following these initial model comparisons, we were interested in whether a
two-dimensional GRM provides a significantly better fit than the unidimensional model.
Looking at the $C_2$ statistic, the two-dimensional GRM also fits the data well,
$C_2(2\text{-GRM}) = 3.73, p = 0.443$. Examining these two models AICs’, we were left with
no strong evidence in favor of one model over the other ($\text{AIC}_{\text{GRM}} = 17412.12,$
$\text{AIC}_{2\text{-GRM}} = 17401.29, \Delta_{\text{AIC}} = 10.83$). Based on the criteria of selecting the model with
the lower information index, we would have chosen the 2d-GRM. Looking at the
traditional likelihood ratio test, we were left with the same conclusion as well
($\chi^2_{(5)} = 20.83, p < 0.001$), and the same holds for Vuong’s test of nested models
($LR = 20.83, p = 0.022$). However, applying Vuong’s test of distinguishability yielded
different results: $\hat{\omega}^2 = 0.02, p = 0.175$. As we have seen in Simulation 2.2, model
selection based on the traditional likelihood ratio test can be misleading when
comparing nested models of different dimensions, and Vuong’s test of distinguishability
was the only test statistic exhibiting a reasonable Type I error rate. In this scenario,
Vuong’s test of distinguishability is likely more reliable than the other test statistics.
Therefore, based on these results we conclude that there is little reason to adopt the
more complex 2d-GRM, and consequently retain the GRM as the most reasonable
modeling representation for these data.

**General Discussion**

As described in this paper, Vuong’s (1989) statistical framework of model
selection provides applied researchers with a useful set of statistical tests that allow for
the comparison of both nested and non-nested IRT models. In our simulation studies,
the tests could reliably distinguish between the GRM and GPCM, which are non-nested
models whose fits are typically not formally compared. Further, Vuong’s tests of
distinguishability and nested models generally performed as well as or even
outperformed the traditional likelihood ratio test, with the latter performing poorly
when comparing nested models with different numbers of latent traits, where it yielded
highly inflated Type I error rates. In the discussion below, we provide some additional
thoughts on indistinguishable or equally well fitting non-nested models, as well as
nested models of different dimensions, and provide directions for future research. Moreover, we discuss the regularity conditions of Vuong’s test statistics and address IRT models with lower and upper asymptotes.

**Non-Nested Models Being Indistinguishable or Fitting Equally Well**

As we have seen in our simulations, comparing non-nested models can result in Vuong’s test of distinguishability concluding that two competing models are not distinguishable; i.e., results demonstrate almost identical likelihoods for nearly all persons. Moreover, Vuong’s test of non-nested models can imply that two competing models, although distinguishable, provide equal fit to the data; resulting, for instance, in the same mean log-likelihood. In Simulation 1.1 we have shown that indistinguishability of non-nested models could hint at the data generating process being “uninformative”, where neither of the competing models should be selected. In Simulation 1.2, we demonstrated that the GRM and GPCM can be distinguished when the data generating items follow either the GRM or GPCM.

For practitioners who ultimately have to choose one model, the scenario of indistinguishable or equally well fitting non-nested models is arguably harder than the nested case. If the two competing models are indistinguishable or fit equally well and differ in their number of parameters, practitioners can argue for the merits of the less complex model, following the principle of parsimony, as is common when comparing nested models. If the two competing models share the same number of parameters, e.g., the GRM and the GPCM, and the Vuong tests suggest that the models are either indistinguishable or fit equally well, we argue that based on statistical information alone, there is no justification for choosing either model. This situation suggests an “uninformative” data generating process, and requires additional data before support for either competing model can be reached. In our opinion, this is a benefit rather than a drawback of these tests, particularly when compared to model selection based on differences in information criteria whereby practitioners will often interpret even small differences, such as $|4|$ or $|0.4|$, as an indication in favor of one model over another (see, e.g., Stochl et al., 2013).
Nested Models of Different Dimension

In our simulations, using the traditional likelihood ratio test for testing nested models of different dimensions (e.g., testing the 2PL vs. 2d-2PL) resulted in highly inflated Type I error rates. To our knowledge, the magnitude of the severity for the traditional likelihood ratio test has not been strongly emphasized in the literature of IRT. Reckase (2009, Chapter 7.2.4) discusses the use of the difference in $\chi^2$ for determining the number of dimensions of IRT models, and states that this test overestimates the true number of dimensions, which generally agrees with our findings. In contrast, Tate (2003) found this procedure to generally work well when slopes were restricted to one (e.g., between-item multidimensional Rasch-like models), although problems similar to ours arose when more extreme parameter values were evaluated.

The problematic behavior of the traditional likelihood ratio test in the context of nested models of different dimensions does however appear better known in the literature of factor analysis. Hayashi et al. (2007) discusses that, when the number of factors exceeds the true number of factors in exploratory factor analysis, the likelihood ratio test may no longer follow a $\chi^2$ distribution due to rank deficiency and non-identifiability of model parameters. Drton (2009) discusses problems of the traditional likelihood ratio test in the case of singularities, and more specifically the scenario of factor analysis with one factor; in the case of testing the complete independence model against the one-factor model, the limiting distribution of the traditional likelihood ratio test can be proven to be no longer $\chi^2$.

When testing the 2PL vs. the 2d-2PL, a typical parametrization of the 2d-2PL consists of freely estimating the (item) slopes of both dimensions (except for the second slope of the last item, which is fixed at 0, resolving the rotational indeterminancy of the model) and assuming the two latent dimensions to have means of 0 and the identity matrix as the covariance matrix. This is also the parametrization we used. When the null hypothesis holds, i.e., data follow the 2PL, the 2d-2PL can be reduced to the 2PL by restricting the (item) slopes of the second dimensions to 0. Looking at the second latent dimension, this would also imply a latent variance of 0, however, this latent
variance is upwardly biased due to the parametrization. All in all, one could argue that this results in a boundary/misspecification scenario and the traditional likelihood ratio test should not be used in the first place and our simulation results indeed confirm this. To our knowledge, this has not been explicitly discussed in the literature of IRT, yet.

While a possible solution may be to implement a bootstrap methodology (Efron & Tibshirani, 1998) to better approximate the distribution of the traditional likelihood ratio test in the scenarios described in this paper, we demonstrated that the Vuong tests (especially Vuong’s test of distinguishability) are robust alternatives to the traditional likelihood ratio test, holding more reasonable Type I error rates, while also demonstrating reasonable power. Nevertheless, we encourage future research to systematically investigate the problem of the traditional likelihood ratio test in the context of nested models with different dimensions.

**Regularity Conditions and Models with Lower and Upper Asymptotes**

As stated by Vuong (1989), and also discussed in Merkle et al. (2016), the conditions under which the assumptions of the Vuong tests hold are quite general (e.g., existence of second-order derivatives of the log-likelihood, invertibility of the models’ information matrices, and i.i.d distributed data vectors). As discussed in Jeffries (2003) and Wilson (2015), applying Vuong’s tests to compare mixture models with different number of components can violate the invertibility requirement due to the lower dimensional model lying on the boundary of the parameter space of the higher dimensional model, which can result in inflated Type I error rates. In the context of IRT, researchers are familiar with models including lower and upper asymptotes, which may share similar limitations. Comparing the 2PL to the 3PL, for example, mimics the same problems as described above due to the 2PL lying on the boundary of the parameter space of the 3PL, restricting all guessing parameters to 0. Brown, Templin, and Cohen (2015) point out that in this scenario, the application of the traditional likelihood ratio test results in deflated Type I error rates, while Chalmers, Pek, and Liu (2017) suggest similar issues when computing likelihood-based confidence intervals for these types of models. As such, similar problems may arise when applying Vuong’s
tests. Therefore, although technically already possible, we do not wish to encourage practical researchers to compare models including lower and upper asymptotes until future research has systematically examined these scenarios both theoretically and by simulation studies.

Conclusion

Vuong’s (1989) tests provide researchers with effective methods for comparing the fits of both nested and non-nested IRT models. We have shown in this paper that the statistics generally exhibit desirable properties, especially compared to statistics that are traditionally used for model comparison in IRT. While computation and evaluation of these statistics is generally difficult, the implementations in the R packages `mirt` and `nonnest2` make the statistics generally accessible to applied researchers. We look forward to future extensions of the statistics to boundary scenarios (involving, e.g., the 3PL) and to non-traditional IRT models, such as the explanatory item response framework described by De Boeck and Wilson (2004).

Computational Details

All results were obtained using the R system for statistical computing (R Core Team, 2018) version 3.5.1, employing the add-on packages `MASS` (Venables & Ripley, 2002) version 7.3-51.1 for simulating person parameters from a bivariate normal distribution, `mirt` (Chalmers, 2012) version 1.29 for simulating data, fitting of the models and information matrix, log-likelihood deriviates, likelihood ratio test, AIC, BIC and $M_2/M_2^*/C_2$ computation, `nonnest2` (Merkle & You, 2018) version 0.5-2 for carrying out the Vuong tests, and `SimDesign` (Chalmers, 2018b) version 1.13 for carrying out the simulation studies. R and the packages `MASS`, `mirt`, `nonnest2` and `SimDesign` are freely available under the General Public License from the Comprehensive R Archive Network at [https://cran.r-project.org/](https://cran.r-project.org/). Numerical values were rounded based on the IEC 60559 standard. Code for replicating our results is available at [https://github.com/sumny/vuong_mirt_code](https://github.com/sumny/vuong_mirt_code).
References

Adams, R. J., Wilson, M., & Wang, W. C. (1997). The multidimensional random
coefficients multinomial logit model. *Applied Psychological Measurement, 21*(1),
1–23.

Agresti, A. (2002). *Categorical data analysis* (2nd ed.). New York, NY: John Wiley &
Sons.

Akaike, H. (1974). A new look at the statistical model identification. *IEEE
Transactions on Automatic Control, 19*(6), 716–723.

Amemiya, Y., & Anderson, T. W. (1990). Asymptotic chi-square tests for a large class
of factor analysis models. *The Annals of Statistics, 18*(3), 1453–1463.

Baker, F. B., & Kim, S.-H. (2004). *Item response theory: Parameter estimation
techniques* (2nd ed.). Boca Raton, FL: Chapman & Hall/CRC.

Bentler, P. M., & Satorra, A. (2010). Testing model nesting and equivalence.
*Psychological Methods, 15*(2), 111–123. doi: https://doi.org/10.1037/a0019625

Bock, R. D. (1972). Estimating item parameters and latent ability when the responses
are scored in two or more nominal categories. *Psychometrika, 37*(1), 29–51.

Bock, R. D., & Aitkin, M. (1981). Marginal maximum likelihood estimation of item
parameters: Application of an EM algorithm. *Psychometrika, 46*(4), 443–459.

Brown, C., Templin, J., & Cohen, A. (2015). Comparing the two-and three-parameter
logistic models via likelihood ratio tests: A commonly misunderstood problem.
*Applied Psychological Measurement, 39*(5), 335–348. doi:
https://doi.org/10.1177/0146621614563326

Burnham, K. P., & Anderson, D. R. (2002). *Model selection and multimodel inference:
A practical information-theoretic approach* (2nd ed.). New York, NY: Springer.

Burnham, K. P., & Anderson, D. R. (2004). Multimodel inference: Understanding AIC
and BIC in model selection. *Sociological Methods & Research, 33*(2), 261–304.

Cai, L., & Hansen, M. (2013). Limited-information goodness-of-fit testing of hierarchical
item factor models. *British Journal of Mathematical and Statistical Psychology,
66*(2), 245–276. doi: https://doi.org/10.1111/j.2044-8317.2012.02050.x
Cai, L., & Monroe, S. (2014). A new statistic for evaluating item response theory models for ordinal data. CRESST report 839. *National Center for Research on Evaluation, Standards, and Student Testing (CRESST)*.

Chalmers, R. P. (2012). *mirt*: A multidimensional item response theory package for the R environment. *Journal of Statistical Software, 48*(6), 1–29. doi: https://doi.org/10.18637/jss.v048.i06

Chalmers, R. P. (2018a). Numerical approximation of the observed information matrix with Oakes’ identity. *British Journal of Mathematical and Statistical Psychology, 71*(3), 415–436. doi: https://doi.org/10.1111/bmsp.12127

Chalmers, R. P. (2018b). *SimDesign*: Structure for organizing Monte Carlo simulation designs [Computer software manual]. Retrieved from https://CRAN.R-project.org/package=SimDesign (R package version 1.13)

Chalmers, R. P., Pek, J., & Liu, Y. (2017). Profile-likelihood confidence intervals in item response theory models. *Multivariate Behavioral Research, 52*(5), 533–550. doi: https://doi.org/10.1080/00273171.2017.1329082

Chun, S. Y., & Shapiro, A. (2009). Normal versus noncentral chi-square asymptotics of misspecified models. *Multivariate Behavioral Research, 44*(6), 803–827. doi: https://doi.org/10.1080/00273170903352186

Clarke, K. A. (2001). Testing nonnested models of international relations: Reevaluating realism. *American Journal of Political Science, 45*(3), 724–744. doi: https://doi.org/10.2307/2669248

Clarke, K. A. (2003). Nonparametric model discrimination in international relations. *Journal of Conflict Resolution, 47*(1), 72–93. doi: https://doi.org/10.1177/0022002702239512

Clarke, K. A. (2007). A simple distribution-free test for nonnested model selection. *Political Analysis, 15*(3), 347–363. doi: https://doi.org/10.1093/pan/mpm004

De Boeck, P., & Wilson, M. (2004). *Explanatory item response models: A generalized linear and nonlinear approach*. New York, NY: Springer.

Drton, M. (2009). Likelihood ratio tests and singularities. *The Annals of Statistics,*
VUONG TESTS OF ITEM RESPONSE MODELS

37(2), 979–1012. doi: https://doi.org/10.1214/07-AOS571

Duchesne, P., & De Micheaux, P. L. (2010). Computing the distribution of quadratic forms: Further comparisons between the Liu-Tang-Zhang approximation and exact methods. *Computational Statistics and Data Analysis, 54*(4), 858–862. doi: https://doi.org/10.1016/j.csda.2009.11.025

Efron, B., & Tibshirani, R. J. (1998). *An introduction to the bootstrap*. New York, NY: Chapman & Hall.

Engle, R. F. (1984). Wald, likelihood ratio, and Lagrange multiplier tests in econometrics. In Z. Griliches & M. D. Intriligator (Eds.), *Handbook of econometrics* (Vol. II, pp. 776–826). Amsterdam, Netherlands: Elsevier.

Freeman, L. (2016). *Assessing model-data fit for compensatory and non-compensatory multidimensional item response models using Vuong and Clarke statistics*. Retrieved from https://dc.uwm.edu/etd/1366

Glas, C. A. W. (1998). Detection of differential item functioning using Lagrange multiplier tests. *Statistica Sinica, 8*(3), 647–667.

Greene, W. H. (1994). *Accounting for excess zeros and sample selection in poisson and negative binomial regression models* (Working Papers). NYU Working Paper No. EC94-10. Retrieved from https://ssrn.com/abstract=1293115

Haberman, S. J. (1977). Maximum likelihood estimates in exponential response models. *The Annals of Statistics, 5*(5), 815–841.

Hayashi, K., Bentler, P. M., & Yuan, K.-H. (2007). On the likelihood ratio test for the number of factors in exploratory factor analysis. *Structural Equation Modeling: A Multidisciplinary Journal, 14*(3), 505–526. doi: https://doi.org/10.1080/10705510701301891

Hershberger, S. L., & Marcoulides, G. A. (2013). The problem of equivalent structural models. In G. R. Hancock & R. O. Mueller (Eds.), *Structural equation modeling: A second course* (pp. 3–39). Charlotte, NC: Information Age Publishing.

Jeffries, N. O. (2003). A note on “Testing the number of components in a normal mixture”. *Biometrika, 90*(4), 991–994. doi:
Kang, T., & Cohen, A. S. (2007). IRT model selection methods for dichotomous items. *Applied Psychological Measurement, 31*(4), 331–358. doi: https://doi.org/10.1177/0146621606292213

Kang, T., Cohen, A. S., & Sung, H.-J. (2009). Model selection indices for polytomous items. *Applied Psychological Measurement, 33*(7), 499–518. doi: https://doi.org/10.1177/0146621608327800

Kullback, S., & Leibler, R. A. (1951). On information and sufficiency. *Annals of Mathematical Statistics, 22*(1), 79–86.

Levy, R., & Hancock, G. R. (2007). A framework of statistical tests for comparing mean and covariance structure models. *Multivariate Behavioral Research, 42*(1), 33–66. doi: https://doi.org/10.1080/00273170701329112

Levy, R., & Hancock, G. R. (2011). An extended model comparison framework for covariance and mean structure models, accommodating multiple groups and latent mixtures. *Sociological Methods & Research, 40*(2), 256–278. doi: https://doi.org/10.1177/0049124111404819

Liao, Z., & Shi, X. (2016). A uniform Vuong test for semi/nonparametric models. Retrieved from http://www.econ.ucla.edu/liao/papers_pdf/NPVuong08062016.pdf

Lo, Y., Mendell, N. R., & Rubin, D. B. (2001). Testing the number of components in a normal mixture. *Biometrika, 88*(3), 767–778. doi: https://doi.org/10.1093/biomet/88.3.767

Louis, T. A. (1982). Finding the observed information matrix when using the EM algorithm. *Journal of the Royal Statistical Society B, 44*(2), 226–233.

MacCallum, R. C., Wegener, D. T., Uchino, B. N., & Fabrigar, L. R. (1993). The problem of equivalent models in applications of covariance structure analysis. *Psychological Bulletin, 114*(1), 185–199.

Maydeu-Olivares, A., & Cai, L. (2006). A cautionary note on using $G^2$(dif) to assess relative model fit in categorical data analysis. *Multivariate Behavioral Research,*
Maydeu-Olivares, A., & Joe, H. (2005). Limited- and full-information estimation and goodness-of-fit testing in 2^n contingency tables. *Journal of the American Statistical Association, 100*(471), 1009–1020. doi: https://doi.org/10.1198/016214504000002069

Merkle, E. C., & You, D. (2018). *nonnest2*: Tests of non-nested models [Computer software manual]. Retrieved from http://cran.r-project.org/package=nonnest2 (R package version 0.5-2)

Merkle, E. C., You, D., & Preacher, K. J. (2016). Testing non-nested structural equation models. *Psychological Methods, 21*(2), 151–163. doi: https://doi.org/10.1037/met0000038

Muraki, E. (1992). A generalized partial credit model: Application of an EM algorithm. *Applied Psychological Measurement, 16*(2), 159–176.

Neyman, J., & Pearson, E. S. (1928). On the use and interpretation of certain test criteria for purposes of statistical inference. *Biometrika, 20A*(1/2, 3/4), 175–240, 263–294.

Neyman, J., & Pearson, E. S. (1933). On the problem of the most efficient tests of statistical hypotheses. *Philosophical Transactions of the Royal Society A, 231*(694-706), 289–337.

Nylund, K. L., Asparouhov, T., & Muthén, B. O. (2007). Deciding on the number of classes in latent class analysis and growth mixture modeling: A Monte Carlo simulation study. *Structural Equation Modeling, 14*(4), 535–569. doi: https://doi.org/10.1080/10705510701575396

Oakes, D. (1999). Direct calculation of the information matrix via the EM algorithm. *Journal of the Royal Statistical Society B, 61*(2), 479–482.

Open Source Psychometrics Project. (2016). *Data From: The Nerdy Personality Attributes Scale [Dataset].* Retrieved from https://openpsychometrics.org/_rawdata/

Pritikin, J. N. (2017). A comparison of parameter covariance estimation methods for
item response models in an expectation-maximization framework. *Cogent Psychology*, 4(1), 1279435. doi: https://doi.org/10.1080/23311908.2017.1279435

R Core Team. (2018). R: A language and environment for statistical computing [Computer software manual]. Vienna, Austria. Retrieved from https://www.R-project.org/

Reckase, M. (2009). *Multidimensional item response theory*. New York, NY: Springer.

Samejima, F. (1969). Estimation of latent ability using a response pattern of graded scores. *Psychometrika Monograph Supplement*.

Schilling, S., & Bock, R. D. (2005). High-dimensional maximum marginal likelihood item factor analysis by adaptive quadrature. *Psychometrika*, 70(3), 533–555. doi: https://doi.org/10.1007/s11336-003-1141-x

Schwarz, G. (1978). Estimating the dimension of a model. *The Annals of Statistics*, 6(2), 461–464.

Shi, X. (2015). A nondegenerate Vuong test. *Quantitative Economics*, 6(1), 85–121. doi: https://doi.org/10.3982/QE382

Steiger, J. H., Shapiro, A., & Browne, M. W. (1985). On the multivariate asymptotic distribution of sequential chi-square statistics. *Psychometrika*, 50(3), 253–264.

Stochl, J., Croudace, T., Perez, J., Birchwood, M., Lester, H., Marshall, M., . . . Jones, P. B. (2013). Usefulness of EQ-5D for evaluation of health-related quality of life in young adults with first-episode psychosis. *Quality of Life Research*, 22(5), 1055–1063. doi: https://doi.org/10.1007/s11136-012-0222-7

Tate, R. (2003). A comparison of selected empirical methods for assessing the structure of responses to test items. *Applied Psychological Measurement*, 27(3), 159–203. doi: https://doi.org/10.1177/0146621603027003001

Thissen, D., Steinberg, L., & Wainer, H. (1988). Use of item response theory in the study of group differences in trace lines. In H. Wainer & H. I. Braun (Eds.), *Test validity* (pp. 147–172). Hillsdale, NJ: Lawrence Erlbaum Associates.

Tollenaar, N., & Mooijaart, A. (2003). Type I errors and power of the parametric bootstrap goodness-of-fit test: Full and limited information. *British Journal of
Tutz, G. (1990). Sequential item response models with ordered response. *British Journal of Mathematical and Statistical Psychology, 43*(1), 39–55.

Venables, W. N., & Ripley, B. D. (2002). *Modern applied statistics with S* (4th ed.). New York, NY: Springer.

Vuong, Q. H. (1989). Likelihood ratio tests for model selection and non-nested hypotheses. *Econometrica, 57*(2), 307–333.

Wilson, P. (2015). The misuse of the Vuong test for non-nested models to test for zero-inflation. *Economics Letters, 127*(C), 51–53. doi: https://doi.org/10.1016/j.econlet.2014.12.029

Wood, R. (1978). Fitting the Rasch model - a heady tale. *British Journal of Mathematical and Statistical Psychology, 31*(1), 27–32.

Yuan, K.-H., & Bentler, P. M. (2004). On chi-square difference and z tests in mean and covariance structure analysis when the base model is misspecified. *Educational and Psychological Measurement, 64*(5), 737–757. doi: https://doi.org/10.1177/0013164404264853
Appendix

The Nerdy Personality Attributes Scale (NPAS)

In the Application section, we study the Vuong test’s performance using six items of the NPAS (Open Source Psychometrics Project, 2016), rated on a five-point Likert scale (0 = Disagree, 2 = Neutral and 4 = Agree). In this appendix, we provide the wording of these six items:

Q1  I am interested in science.
Q2  I was in advanced classes.
Q6  I prefer academic success to social success.
Q13 I would describe my smarts as bookish.
Q22 I enjoy learning more than I need to.
Q23 I get excited about my ideas and research.