Classification of spherical tilings by congruent quadrangles over pseudo-double wheels (I)—a special tiling by congruent concave quadrangles

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Abstract. Every simple quadrangulation of the sphere is generated by a graph called a pseudo-double wheel with two local expansions (Brinkmann et al. “Generation of simple quadrangulations of the sphere.” Discrete Math., Vol. 305, No. 1–3, pp. 33–54, 2005). So, toward a classification of the spherical tilings by congruent quadrangles, we propose to classify those with the tiles being convex and the graphs being pseudo-double wheels. In this paper, we verify that a certain series of assignments of edge-lengths to pseudo-double wheels does not admit a tiling by congruent convex quadrangles. Actually, we prove the series admits only one tiling by twelve congruent concave quadrangles such that the symmetry of the tiling has only three perpendicular 2-fold rotation axes, and the tiling seems to be new.

1. Introduction

A spherical tiling by congruent polygons is, by definition, a covering of the unit sphere by congruent spherical polygons such that (i) none of the polygons share their inner point, (ii) an edge of a spherical polygon matches an edge of another spherical polygon, and (iii) each vertex is incident to more than two edges. Each spherical polygon is called a tile.

If there is a spherical tiling by $p$-gons, then $p$ is either 3, 4 or 5 ([10]), because of Euler’s formula $V - E + F = 2$. In [11], Ueno-Agaoka classified all the spherical tilings by congruent triangles. By using their classification, we can complete the classification of all the spherical tilings by congruent quadrangles where the quadrangle can be divided into two congruent triangles [3]. Interestingly, there is a spherical tiling by 24 congruent kites with the graph being the same as deltoidal icositetrahedron, a Catalan-solid [6], while there is another spherical tiling by the congruent kites with different symmetry.

As for spherical tilings by congruent quadrangles, if the quadrangle cannot be divided into two congruent triangles, then the edge-lengths of the tile is

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necessarily one of two types of Figure 1, because an edge of a tile should match an edge of another tile with the same length and because Euler’s formula implies the existence of a 3-valent vertex [10]. Following [10], we call the two types type 2 and type 4. They gave a somehow unexpected, sporadic, 4-fold rotational symmetric spherical tiling by 16 congruent quadrangles of type 2 [10, Theorem 2]. As a necessary condition for a spherical quadrangle of type 2 to exist, they provided inequalities involving trigonometric functions, in terms of the inner angles of the tile ([10, Proposition 3]). In [10, Figure 11], they presented spherical tilings by congruent concave quadrangles of type 2, and suggested that the class of spherical tilings by congruent concave quadrangles is difficult to classify.

Among the spherical monohedral quadrangular tilings, we are concerned with the following spherical tilings consisting of \( F \geq 6 \) congruent quadrangles, having \( F/2 \)-fold rotational symmetry. See Figure 2 and the first three spherical tilings in [10, Figure 11], for example. The pair of the vertices and the

Fig. 1. A quadrangle of type 2 (left) and a quadrangle of type 4 (right). Designation of edge-lengths and angles. The variables \( a, b, c \) are for edge-lengths and they have different values. The variables \( \alpha, \beta, \gamma, \delta \) are for inner angles.

Fig. 2. The pictures in the upper row are, from left to right, a pseudo-double wheel of 6 faces, a pseudo-double wheel of 8 faces, and two expansions of spherical quadrangulations to increase the number of faces (Brinkmann et al. [5]). The graphs in this paper should be interpreted using Convention 1 stated at the end of Section 2. The lower pictures are spherical tilings by congruent quadrangles over pseudo-double wheel of \( F \) faces \((F = 6, 8, 10, \text{from left to right})\).
edges of the tiling forms a graph, which is obtained from the cycle consisting
of $F$ vertices, by adjoining the north and the south poles alternately to the
vertices of the cycle. Such a graph falls into the class of pseudo-double wheels.
According to Brinkmann et al. [5], every spherical quadrangulation is obtained
from a pseudo-double wheel of even or odd faces through finite applications of
two local expansions (see Figure 2 (upper right)).

In [2], we prove that for every spherical tiling by congruent convex
quadrangles of type 2 or type 4, the edge-lengths of the graph are none of
two graphs in Figure 3. As a direct consequence, we can prove the following
statement: If a spherical tiling by congruent convex quadrangles of type 2 or
type 4 has a pseudo-double wheel as a graph, then the length-assignment of the
graph must be $F/2$-fold or an $F/6$-fold “rotationally symmetric.”

The main theorem of this paper is: If a spherical tiling by congruent
quadrangles of type 2 or type 4 over a pseudo-double wheel has an $F/6$-fold
“rotationally symmetric” length-assignment, then the tiling necessarily consists
of twelve congruent concave quadrangles. Such a tiling actually exists uniquely
up to mirror image, and the symmetry of the tiling is low compared to the
number of tiles. By this theorem, we can complete the classification of all the
spherical tilings by congruent convex quadrangles over pseudo-double wheels
([2]).

This paper is organized as follows: In the next section, we present
basic relevant notions of spherical tilings by congruent polygons. In Section
3, we present our main theorem, the background and the organization of
the proof. The rest of the sections are devoted to the proof of the main
theorem.

2. Preliminaries

Throughout this paper, by the sphere we mean the sphere with the center
being the origin and the radius being 1, and say two figures on the sphere are
congruent if there is an orthogonal transformation between them.

By a spherical triangle (resp. spherical quadrangle), we mean a nonempty,
simply connected, closed subset $T$ of the sphere with the area less than $2\pi$ such
that the boundary is the union of three (resp. four) distinct geodesic lines but is
not the union of any two (resp. three) distinct geodesic lines.

To prove that a spherical triangle actually exists, we use the following:

**Proposition 1** ([4, p. 62]). If $0 < A, B, C < \pi$, $A + B + C > \pi$, $-A + B +
C < \pi$, $A - B + C < \pi$, $A + B - C < \pi$, then there exists uniquely up to orthog-
onal transformation a spherical triangle such that all the edges are geodesic lines
and the inner angles are $A$, $B$ and $C$. 
We formalize relevant combinatorial notions of spherical tilings by congruent polygons. Refer [8] for the terminology of the graph theory.

**Definition 1.** A map is a triple $M = (V, E, \{O_v \mid v \in V\})$ such that $(V, E)$ is a graph (see [8, Section 1.1]) and each $O_v$ is a cyclic order for the edges incident to the vertex $v$. For a vertex $v$ of $M$, the set $A_v$ of angles around $v$ is

$$A_v := \{(v_1, v, v_2), (v_2, v, v_3), \ldots, (v_{n-1}, v, v_n), (v_n, v, v_1)\}$$

where the list $v_1, v_2, \ldots, v_n$ is the enumeration of $\{vu \mid vu \in E\}$ without repetition by the cyclic order $O_v$. We write an inner angle $\langle u, v, w \rangle$ by $\langle uwv \rangle$.

The mirror of a map $M$ is the map $M^R$ with the cyclic orders reversed. Thus $\langle uwv \rangle$ is an inner angle of the original map, if and only if $\langle wuv \rangle$ is an inner angle of the mirror of the map.

We recall a pseudo-double wheel [5].

**Definition 2.** For an even number $F$ greater than or equal to 6, a pseudo-double wheel $\mathrm{pdw}_F$ with $F$ faces is a map such that

- the graph is obtained from a cycle $(v_0, v_1, v_2, \ldots, v_{F-1})$, by adjoining a new vertex $N$ to each $v_{2i}$ $(0 \leq i < F/2)$ and then by adjoining a new vertex $S$ to each $v_{2i+1}$ $(0 \leq i < F/2)$. We identify the suffix $i$ of the vertex $v_i$ modulo $F$.
- The cyclic order at the vertex $N$ is defined as follows: the edge $Nv_{2i+2}$ is next to the edge $Nv_{2i}$. The cyclic order at the vertex $v_{2i}$ $(0 \leq i < F/2)$ is: the edge $v_{2i}N$ is next to the edge $v_{2i}v_{2i+1}$, which is next to the edge $v_{2i}v_{2i-1}$. The cyclic order at the vertex $S$ is: the edge $Sv_{2i-1}$ is next to the edge $Sv_{2i+1}$. The cyclic order at the vertex $v_{2i+1}$ $(0 \leq i < F/2)$ is: the edge $v_{2i+1}S$ is next to the edge $v_{2i+1}v_{2i}$, which is next to the edge $v_{2i+1}v_{2i+2}$.

We call each edge $Nv_{2i}$ northern, each edge $Sv_{2i+1}$ southern, and the other edges non-meridian. The number of edges is $2F$. See Figure 2 (lower) for $\mathrm{pdw}_6$, $\mathrm{pdw}_8$ and $\mathrm{pdw}_{10}$.

We use frequently the following notion:

**Definition 3.** A chart is, by definition, a triple $\mathcal{A} = (M, L, K)$ such that

- $M$ is a map. Let $V$ be the vertex set of $M$;
- $L$ is a length-assignment, which is a function from the edge set $E$ of $M$ to $\{a, b, c\}$ with $a, b, c \in (0, 2\pi)$;
- $K$ is an angle-assignment, which is a function from the set $\bigcup_{e \in V} A_e$ of the angles to the set of affine combinations of the variables $\alpha, \beta, \gamma, \delta$.
over $\mathbb{R}$ subject to the equation

$$\sum_{\psi \in A_v} K(\psi) = 2\pi \quad \text{for each vertex } v \in V.$$ \hspace{1cm} (1)

Here $A_v$ is the set of angles around the vertex $v$, as in defined in Definition 1. For a vertex $v \in V$ and an angle $\psi \in A_v$, $K(\psi)$ is called the type of the angle $\psi$, and $\sum_{\psi \in A_v} K(\psi)$ is called the vertex type of the vertex $v$; and

- The pair of $K$ and $L$ satisfies the constraint described by Figure 1 (left) ("type 2") for each face or the constraint described by Figure 1 (right) ("type 4") for each face.

We say a spherical tiling $\mathcal{T}$ by polygons realizes a chart $\mathcal{A}$, provided that there is an embedding $G$ from $\mathcal{A}$ to the sphere such that

- $G$ is a bijection from the vertex set of the chart $\mathcal{A}$ to that of the tiling $\mathcal{T}$,
- $G$ is a bijection from the edge set of the chart $\mathcal{A}$ to that of the tiling $\mathcal{T}$, such that (i) if two edges $uv$ and $u'v'$ of $\mathcal{A}$ has intersection $\{w\}$, then the intersection of two edges $G(uv)$ and $G(u'v')$ is $\{G(w)\}$; and (ii) if $uv$ and $u'v'$ is disjoint, then $G(uv)$ and $G(u'v')$ do not intersect.
- $G$ preserves the cyclic order $O_v$ of each vertex $v$ of the chart $\mathcal{A}$ to the orientation of the sphere at $G(v)$. In other words, for any vertex $v$ of the chart, if we rotate a screw at the vertex $G(v)$ according to the cyclic order $O_v$, then the screw goes outbound from the center of the sphere.
- For any angle $\angle uvw$ of the chart $\mathcal{A}$, $\angle G(u)G(v)G(w)$ is an angle of the tiling $\mathcal{T}$ and is $K(\angle uvw)$.
- For any edge $uv$ of the chart $\mathcal{A}$, $G(u)G(v)$ is an edge of the tiling $\mathcal{T}$ and has length $L(uv)$.

The mirror $\mathcal{A}^R$ of a chart $\mathcal{A} = (M, L, K)$ is, by definition, a chart $(M^R, L, K^R)$ where $K(\angle uvw) = K^R(\angle uvw)$. Thus if a tiling $\mathcal{T}$ realizes $\mathcal{A}$, then the mirror image of $\mathcal{T}$ does the mirror $\mathcal{A}^R$.

The charts in Figure 2 and Figure 3 are subject to the following convention as in [5].

**Convention 1.** Each displayed vertex is distinct from the others; Edges that are completely drawn must occur in the cyclic order given in the picture; Half-edges indicate that an edge must occur at this position in the cyclic order around the vertex; A triangle indicates that one or more edges may occur at this position in the cyclic order around the vertex (but they need not); If neither a half-edge nor a triangle is present in the angle between two edges in the picture, then these two edges must follow each other directly in the cyclic order of edges around that vertex.
3. Main theorem and the background

In [9], Sakano classified the spherical tilings by six or eight congruent quadrangles of type 2 or of type 4. His argument is generalized to the case in which the number of the tiles is more than ten, as follows: Recall quadrangles of type 2 and those of type 4, by Figure 1.

**Theorem 1 ([2]).** Two charts in Figure 3 and their mirrors do not occur as the chart of a spherical tiling by ten or more congruent convex quadrangles of type \( t \neq 2, 4 \). Thick edges are of length \( b \) while the other edges are of length \( a \) or \( c \). The charts should be interpreted by Convention 1.

Let the leftmost vertex of the chart in Figure 3 be the vertex \( S \) of the pseudo-double wheel and let the central vertex incident to at least five edges be the vertex \( N \). In the left chart of Figure 3, an upper, designated, northern edge of length \( b \) is immediately followed by another northern edge of length \( b \). In the right chart, the designated northern edge of length \( b \) is followed immediately by two consecutive non-meridian edges of length \( b \). By the repeated applications of Theorem 1 to a slightly rotated pseudo-double wheel \( \text{pdw}_F \) around the vertices \( N \) and \( S \), we can observe that all the possibilities of the length-assignment of \( \text{pdw}_F \) (\( F \geq 10 \)) is as follows:

- \((p)\) the tile is a convex quadrangle, and no northern edges of \( \text{pdw}_F \) have length \( b \); or
- \((a)\) the tile is a convex quadrangle. The northern edges and non-meridian edges of \( \text{pdw}_F \) alternatingly have length \( b \). See Assumption (II) of Theorem 2, and Figure 4; or
- \((a^R)\) (a) with \( \text{pdw}_F \) replaced by the mirror \( (\text{pdw}_F)^R \).

As a consequence of the following Theorem 2, the possibilities \((a)\) and \((a^R)\) are impossible for any even number \( F \geq 10 \) of tiles. In other words, for any even number \( F \geq 10 \), there is no spherical tiling by \( F \) congruent convex quadrangles over \( \text{pdw}_F \) such that \((a)\) or \((a^R)\) holds. The only possibility \((p)\) of the length-
assignment of \( pdw_F \) is actually realized with a spherical tiling by \( F \) congruent quadrangles. Such a spherical tiling is obtained from a spherical tiling by congruent rhombic tiles, such as Figure 2 (lower), by deforming the tiles to quadrangles of type 2 or of type 4 with the tiling kept \( F/2 \)-fold rotational symmetric. For details, see [1, 2]. So Theorem 2, which is rather a long statement about tiles of type 2 or type 4, leads to a complete classification of the spherical tilings by congruent convex quadrangles of type 2 or of type 4 over pseudo-double wheels.

**Theorem 2.** Assume a chart \( \mathcal{A} \) satisfies the following assumptions:

(I) the map of the chart is \( pdw_F \) for some \( F \geq 10 \), and

(II) there is \( b > 0 \) such that all the edges \( v_{6i}, v_{6i+1} \) and \( v_{6i+3} v_{6i+4} \) have length \( b \) for each nonnegative integer \( i < F/6 \) while the other edges do lengths \( 0 \).

Then there exists a spherical tiling \( \mathcal{T} \) by congruent quadrangles, uniquely up to special orthogonal transformation. Moreover the tile is a concave quadrangle of type 2, and the following three properties hold:

1. The tiling \( \mathcal{T} \) realizes \( \mathcal{A} \), where the length-assignment and the angle-assignment are Figure 4 (\( F = 12 \) in particular) with the following equations:
   \[
   a = \arccos \frac{1}{3}, \quad (2)
   \]
   \[
   b = \arccos \frac{-5}{9}, \quad (3)
   \]

Fig. 4. The chart \( \mathcal{A} \) of the tiling. Thick (resp. thin) edges of the chart correspond to edges of length \( b \) (resp. \( a \)) of the tiling (see Theorem 2).
Define the spherical polar coordinate of a point \( p \) on the unit sphere to be the pair \((\theta, \phi)\) of the length \( \theta \) of the geodesic line \( Np \) and the longitude \( \phi \) of \( p \). The longitude is the angle from the geodesic line \( Nv_0 \) to \( Np \), defined consistently with the cyclic order of \( N \). Let

\[
\phi = \arccos \frac{13}{14}. \tag{8}
\]

Then the spherical polar coordinates of the vertices \( v_{i+6} \) is \((\theta, \rho + \pi)\) for \( v_i = (\theta, \rho) \) \((i = 0, 1, 2, 3, 4, 5)\). Moreover \( S = (\pi, 0), \ v_0 = (b, 0), \ v_1 = (\pi - b, \phi), \ v_2 = (a, \alpha), \ v_3 = (\pi - a, \phi + \delta), \ v_4 = (a, \alpha + \beta), \) and \( v_5 = (\pi - a, \phi + \delta + \beta) \).

The tiling \( T \) has only three perpendicular 2-fold rotation axes but no mirror planes. In other words, the Schönflies symbol of the tiling is \( D_2 \). See Figure 5 for the views of \( T \) from the three axes.

The proof of Theorem 2 is organized as follows: In Section 4, by assuming the existence of such a tiling \( T \), we prove the assertions (1) and (3) concerning the inner angle \( \alpha \) and the number of tiles \( F \), and prove that the tile

\[
\alpha = \arccos \frac{-1}{2\sqrt{7}}, \tag{4}
\]

\[
\beta = \frac{\pi}{3}, \tag{5}
\]

\[
\gamma = \frac{4\pi}{3}, \tag{6}
\]

\[
\delta = \arccos \frac{5}{2\sqrt{7}}. \tag{7}
\]

(2) The tiling \( T \) has only three perpendicular 2-fold rotation axes but no mirror planes. In other words, the Schönflies symbol of the tiling is \( D_2 \). See Figure 5 for the views of \( T \) from the three axes.

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Fig. 5. The tiling of twelve congruent concave quadrangles over a pseudo-double wheel and the three 2-fold rotation axes. The upper left figure is the view from a general position, and the upper right is from the antipodal of the first viewpoint. The thick edges are of length \( b \). The lower left figure is the view from a 2-fold rotation axis through the midpoint between the vertices \( v_0 \) and \( v_1 \). The lower middle figure is from a 2-fold rotation axis through the midpoint between the vertices \( v_3 \) and \( v_4 \). The lower right is from the other 2-fold rotation axis through the poles. In the figure (lower left), two great circles containing the two middle thick edges forms an angle \( \phi = \arccos(13/14) \) (see (8)).
is a concave quadrangle of type 2. In Section 5, we determine the absolute values of all inner angles, all edge-lengths of the tile, and $F$. In Section 6, by those values, we prove that the tile actually exists as a spherical quadrangle, by using Proposition 1. In Section 7, we prove the assertion (2). In Section 8, we complete the proof of Theorem 2 by establishing the assertion (3).

4. Which angles and which edges are equal?

We answer the question in this section by manipulations of systems of equations of the form (1) and by elementary geometry.

By an automorphism of a map $M$, we mean any automorphism [8, Section 1.1] $h$ of the graph that preserves the cyclic orders of the vertices. Here we let $h$ send any angle $uvw$ to $h(u)h(v)h(w)$. For a chart $\mathcal{A} = (M, L, K)$ and an automorphism $h$ of the map $M$, let $h(\mathcal{A})$ be a chart $(M, L \circ h^{-1}, K \circ h^{-1})$.

**Definition 4.** Let $\mathcal{A} = (M, L, K)$ be a chart satisfying the assumptions of Theorem 2, define two automorphisms $f$ and $g$ on the map $M$ of $\mathcal{A}$ as follows:

\[ f(N) = S, \quad f(S) = N, \quad f(v_i) = v_{1-i} \quad (0 \leq i \leq F - 1); \quad (9) \]

\[ g(N) = N, \quad g(S) = S, \quad g(v_i) = v_{i+6} \quad (0 \leq i \leq F - 1). \quad (10) \]

Here the suffix $i$ of the vertex $v_i$ is understood modulo $F$ as before. Then we can prove that both of $f$ and $g$ are indeed automorphisms.

**Lemma 1.** Suppose some tiling $\mathcal{T}$ realizes a chart $\mathcal{A} = (M, L, K)$ satisfying the assumptions of Theorem 2. Then,

1. The number $F$ of faces is a multiple of 6 greater than or equal to 12.
2. The tile is a quadrangle of type 2.
3. $\mathcal{A} = f(\mathcal{A}) = g(\mathcal{A})$.
4. The inner angles satisfy $\beta = \frac{4\pi}{F}$, $\gamma = 2\pi - \frac{8\pi}{F} > \pi$, and $\delta = \frac{8\pi}{F} - \alpha$.
5. $K$ has the same assignment as in Figure 4 on the angles $\angle_{v_3Nv_2}$, $\angle_{v_0Nv_2}$ and on all angles around the vertices $v_0, v_2, v_3$. Especially,

\[ \angle_{v_3v_2N} = 2\pi - \frac{8\pi}{F} > \pi, \quad \angle_{v_2Nv_4} = \frac{4\pi}{F}, \quad \angle_{Nv_4v_3} = \alpha. \quad (11) \]

To prove the assertion (1), assume $F$ is not a multiple of 6. Then the edge $Nv_2$ or $Nv_4$ should have length $b$, which contradicts against Assumption (II) of Theorem 2. By this and Assumption (I) of Theorem 2, the assertion (1) of Lemma 1 follows.

4.1. Basic general lemmas. To prove the other assertions of Lemma 1, we associate to the chart in question a system of equations and inequalities about
the inner angles $\alpha$, $\beta$, $\gamma$, $\delta$ of a tile. The system consists of the equations (1), inequalities $0 < \alpha, \beta, \gamma, \delta < 2\pi$, and an equation

$$\alpha + \beta + \gamma + \delta - 2\pi = \frac{4\pi}{F}. \quad (12)$$

The last equation holds because all $F$ tiles are congruent quadrangles and because the area of a spherical triangle is the sum of the inner angles subtracted by $\pi$ according to [4].

If the system of equations and inequalities associated to a chart $\mathcal{A}$ is unsolvable, no tiling realizes $\mathcal{A}$. To show that such systems are unsolvable, we use the following lemmas.

**Lemma 2.** In a quadrangle of type $2$, no inner angle is equal to the opposite inner angle (i.e. $\beta \neq \delta$ and $\alpha \neq \gamma$). In a quadrangle of type $4$, we have $\alpha \neq \gamma$.

**Proof.** Draw a diagonal line in the tiles, and then argue with the isosceles triangle(s).

**Lemma 3.** If a tiling by congruent quadrangle of type $4$ realizes a chart, then, for each $Z \in \{\alpha, \beta, \gamma, \delta\}$, the chart has no 3-valent vertex of type $Z + 2\delta$ or $Z + \beta + \delta$.

**Proof.** Recall $\delta$ is the type of an angle between an edge of length $b$ and that of length $c$. Because $b \neq c$, if a 3-valent vertex has vertex type $2\delta + Z$ for some type $Z$, then $Z$ is the type of an angle between two edges of length $b$, or is that of an angle between two edges of length $c$. But such an angle does not exists in any quadrangles of type $4$.

$\beta$ is the type of an angle between two edges of length $a$. $\delta$ is the type of an angle between an edge of length $b$ and that of length $c$. By $c \neq a \neq b$, no edge of the angle of type $\beta$ does not match an edge of the angle of type $\delta$. Thus an angle of type $\delta$ cannot be adjacent to an angle of type $\beta$ in a quadrangle of type $4$. Hence a vertex of type $Z + \beta + \delta$ is impossible.

**Lemma 4.** For any quadrangle of type $2$, $\alpha = \delta$ if and only if $\beta = \gamma$.

**Proof.** Assume $\alpha = \delta$. We identify the vertex of the quadrangle with the inner angle. Because the quadrangle $\alpha\beta\gamma\delta$ is of type $2$, the two edges $\alpha\beta$ and $\gamma\delta$ have length $a$, and then the triangle $\alpha\beta\delta$ is congruent to the triangle $\delta\gamma\alpha$. Thus $\angle\beta\delta\alpha = \angle\gamma\alpha\delta$. As $\alpha = \delta$, we have $\angle\beta\gamma\alpha = \angle\gamma\beta\alpha$. Hence $\angle\beta\gamma\delta = \angle\gamma\beta\delta = \angle\beta\alpha\gamma = \angle\beta\gamma\alpha$. Thus $\beta = \angle\alpha\beta\delta + \angle\gamma\beta\delta = \angle\delta\gamma\alpha + \angle\beta\gamma\alpha = \gamma$. Conversely, assume the inner angle $\beta$ is equal to the inner angle $\gamma$. As the triangle $\alpha\beta\gamma$ is congruent to the triangle $\delta\gamma\beta$, we have $\angle\beta\gamma\alpha = \angle\beta\gamma\delta = \angle\gamma\beta\delta = \angle\gamma\delta\beta$. Let the point $P$ be shared by the two diagonal segments $x\gamma$ and $\beta\delta$ of the quadrangle. As
\[ \angle PB\gamma = \angle P\gamma\beta, \] the length of the edge $\beta P$ is equal to that of the edge $\gamma P$. Since the length of the diagonal $x\gamma$ is equal to that of the other diagonal $\delta\beta$, we have $P\alpha = P\delta$, by which the triangle $zP\delta$ is an isosceles triangle. So $\angle \beta x\gamma = \angle \gamma \delta\beta$. Thus the inner angle $\alpha$ is equal to the inner angle $\delta$. \hfill \square

**Lemma 5.** Suppose a tiling by congruent quadrangles of type 2 realizes a chart such that (1) there is a vertex incident to only three edges of length $a$, and (2) there is a 3-valent vertex incident to two edges of length $a$ and to one edge of length $b$. Then $\alpha \neq \delta$ and $\beta \neq \gamma$.

**Proof.** Assume otherwise. By Lemma 4, we have $\alpha = \delta$ and $\beta = \gamma$. So the assumption (1) implies $\beta = \gamma = 2\pi/3$. By the assumption (2), there exists a vertex of type $X + X' + Y$ (where $X, X' \in \{\alpha, \delta\}$, $Y \in \{\beta, \gamma\}$). Thus $\alpha = \beta = \gamma = \delta = 2\pi/3$. This contradicts against Lemma 2. \hfill \square

### 4.2. The proof of the assertions (2), (3), (4) and (5) of Lemma 1.

**Lemma 6.** If a tiling by congruent quadrangles of type 2 realizes a chart satisfying the assumptions of Theorem 2, then $\alpha \neq \delta$ and $\beta \neq \gamma$.

**Proof.** Since the tile $Nv_0v_1v_2$ of the pseudo-double wheel $pdw_F$ in question is of type 2 and has an edge $Nv_0$ of length $b$, other edges $v_1v_2$ and $v_2N$ of the same tile have lengths $a$. The tile $Sv_1v_2v_3$ is of type 2, and has an edge $Sv_1$ of length $b$ because of Assumption (II) of Theorem 2. Thus the opposite edge $v_2v_3$ of the same tile $Sv_1v_2v_3$ has length $a$. Thus the vertex $v_2$ is incident to only three edges of length $a$. On the other hand, the vertex $v_1$ is incident to one edge of length $b$ and to two edges $v_0v_1$ and $v_0v_{F-1}$ of length $a$. By Lemma 5, we have $\alpha \neq \delta$ and $\beta \neq \gamma$. \hfill \square

We can easily verify the following Lemma for automorphisms $f$, $g$ defined in Definition 4.

**Lemma 7.** If $\mathcal{A}$ is any chart satisfying the assumptions of Theorem 2, then

1. $f(\mathcal{A})$ and $g(\mathcal{A})$ are charts satisfying the assumptions of Theorem 2.
2. Whenever we can prove a property $P(\mathcal{A})$, we can prove the properties $P(f(\mathcal{A}))$ and $P(g(\mathcal{A}))$.

**Definition 5.** Fix a chart. For any property $P(\alpha, \beta, \gamma, \delta)$ for the inner angles $\alpha, \beta, \gamma, \delta$ of the tile, the conjugate property $P^*(\alpha, \beta, \gamma, \delta)$ is, by definition, a property $P(\delta, \gamma, \beta, \alpha)$.

**Lemma 8.** Suppose a tiling by congruent quadrangles realizes a chart satisfying the assumptions of Theorem 2. Then the tile is of type 2, and the chart satisfies the following properties (A) and (B), or satisfies the conjugate properties (A)* and (B)*.
(A) For each integer $0 \leq i < F/6$, the types of $\angle Sv_{6i+1}v_{6i+2}$ and $\angle Nv_{6i}v_{6i-1}$ are $\delta$, those of $\angle v_{6i}v_{6i+1}N$ and $\angle v_{6i+1}v_{6i}N$ are $\alpha$, and those of $\angle v_{6i+2}v_{6i+1}v_{6i}$ and $\angle v_{6i-1}v_{6i}v_{6i+1}$ are $\beta$.

(B) For each integer $0 \leq i < F/6$, the types of $\angle v_{6i+2}Nv_{6i+4}$ and $\angle v_{6i+5}Sv_{6i+3}$ are all $\gamma$.

**Proof.** It is sufficient to prove the following properties and the conjugate properties Property 1* and Property 2*:

Property 1. It is not the case that: for some nonnegative integer $i < F/6$, both of $\angle v_{6i+1}v_{6i}N$ and $\angle Nv_{6i}v_{6i-1}$ have type $\delta$ and $\angle v_{6i-1}v_{6i}v_{6i+1}$ has type $\beta$.

Property 2. It is not the case that: for some nonnegative integer $i < F/6$, both of $\angle v_{6i+1}v_{6i}N$ and $\angle Nv_{6i}v_{6i-1}$ have type $\alpha$ and $\angle v_{6i-1}v_{6i}v_{6i+1}$ has type $\beta$.

We first verify the sufficiency. Assume we can prove Property 1, Property 2, Property 1* and Property 2*. Then for every nonnegative integer $i < F/6$, the type of the vertex $v_{6i}$ of $\text{pdw}_F$ is $\alpha + \beta + \delta$ or $\alpha + \gamma + \delta$, and thus the type of the vertex $v_{6i+1}$ of $\text{pdw}_F$ is so, because the automorphism $f$ satisfies $f(v_{6i}) = v_{6i+1}$ and by Lemma 7.

Because every edge $v_{6i+3}v_{6i+4}$ has length $b$, each angle mentioned in (B) is $\beta$ or $\gamma$.

First consider the case where all the vertices $v_{6i}$ and $v_{6i+1}$ have type $\alpha + \gamma + \delta$. Then all the angles $\angle v_{6i+2}Nv_{6i+4}$ and $\angle v_{6i+5}Sv_{6i+3}$ do $\beta$. Otherwise, $\alpha + \gamma + \delta < 2\pi$ because $\alpha$ and $\delta$ appear as types of angles around the poles and $F/2 \geq 5$. Hence we have (B)*.

Each $\angle v_{6i+1}v_{6i}N$ has type $\delta$. Otherwise $\angle v_{6i+1}v_{6i}N$ has type $\alpha$ and thus $\angle v_{6i+2}v_{6i+1}v_{6i}$ does $\beta$, which contradicts against the vertex type of $v_{6i+1}$ being $\alpha + \gamma + \delta$. Similarly we can prove $\angle v_{6i}v_{6i+1}S$ has type $\delta$. So we have (A)*.

Thus each $\angle v_{6i+4}Nv_{6i+6}$ has type $\delta$. If the tile is of type 4, then we have contradiction against (B)*, by considering the length of the edge $v_{6i}N$. So the tile is of type 2.

Next consider the case where all the vertices $v_{6i}$ and $v_{6i+1}$ have type $\alpha + \beta + \delta$. In this case we can prove (A) and (B) by the same argument as above but with $(\alpha, \beta) \leftrightarrow (\delta, \gamma)$ swapped.

In the remaining case, we have a vertex of type $\alpha + \beta + \delta = 2\pi$ and that of type $\alpha + \gamma + \delta = 2\pi$. The former vertex leads to a contradiction against Lemma 3, when the tile is of type 4. If the tile is of type 2, this case implies $\beta = \gamma$, which contradicts against Lemma 6.

Hence to verify Lemma 8, we prove Property 1, Property 2 and their conjugate properties below.

Property 1* holds when the tile is of type 4, by the following argument. If both of $\angle v_{6i+1}v_{6i}N$ and $\angle Nv_{6i}v_{6i-1}$ have type $\alpha$, then the length of the edge $v_{6i}v_{6i+1}$ and that of edge $v_{6i}v_{6i-1}$ are both $a$. So $\angle v_{6i-1}v_{6i}v_{6i+1}$ has type $\beta$. 
Property 1 and Property 2 hold when the tile is of type 4, by Lemma 3.

Property 1 holds when the tile is of type 2. Otherwise, \( \beta + 2\delta = 2\pi \).

See Figure 6 (left). The type of \( \angle v_{6i-1}v_{6i-2}N \) is \( \beta \). By Assumption (II) of Theorem 2, the edge \( v_{6i-2}v_{6i-3} \) has length \( b \), and the type of the vertex \( v_{6i-2} \) is one of \( \alpha + \beta + \delta \), \( \beta + 2\alpha \), and \( \beta + 2\delta \). For the first two cases, we have \( \alpha = \delta \) which contradicts against Lemma 6. Therefore among the vertex type of the vertex \( N \), the type \( \gamma \) occurs. However the vertex type of \( v_{6i-3} \) is \( 2\alpha + Y \) for some \( Y \in \{ \beta, \gamma \} \). In case \( Y = \beta \), we have \( \alpha = \delta \), which contradicts against Lemma 6. In the other case \( Y = \gamma \), the vertex type of \( v_{6i-3} \) is a proper subtype of \( N \), which is a contradiction.

We can prove Property 1 holds when the tile is of type 2, by swapping \( (\alpha, \beta) \leftrightarrow (\delta, \gamma) \) in the proof above.

**Claim 1.** Property 2 holds when the tile is of type 2 or of type 4.

**Proof.** Assume otherwise. See Figure 6 (right). Since the edge \( Nv_{6i} \) has length \( b \), the type of the angle \( \angle v_{6i+2}v_{6i+1}v_{6i} \) is \( \beta \). Since the edge \( Sv_{6i+1} \) has length \( b \), the type of \( \angle v_{6i}v_{6i+1}S \) is \( \alpha \). Assume the type of \( \angle Sv_{6i+1}v_{6i+2} \) is \( \delta \). When the tile is of type 4, we have a contradiction against Lemma 3. When the tile is of type 2, we have a contradiction against Lemma 6 because \( \alpha = \delta \) follows by comparing the vertex types of \( v_{6i} \) and \( v_{6i+1} \). Therefore the vertex types of \( v_{6i} \) and \( v_{6i+1} \) are both

\[
2\alpha + \beta = 2\pi,
\]

and thus the type of \( \angle v_{6i+1}v_{6i+2}v_{6i+3} \) is \( \beta \). If the vertex type of \( v_{6i+2} \) is \( \beta + 2\gamma \), then \( \alpha = \gamma \) by comparing the vertex types of \( v_{6i+1} \) and \( v_{6i+2} \). This contradicts against Lemma 2, whether the tile is of type 2 or of type 4. Thus the vertex type of \( v_{6i+2} \) is

\[
2\beta + \gamma = 2\pi.
\]
Because the type of $\angle v_{6i+2}v_{6i+3}S$ is $\gamma$, if the vertex type of $v_{6i+3}$ is $2\alpha + \gamma$, then (13) and (14) implies $\alpha = \beta = \gamma = 2\pi/3$, which contradicts against Lemma 2. Therefore the vertex type of $v_{6i+3}$ is

$$\alpha + \gamma + \delta = 2\pi. \quad (15)$$

Then the vertex type of $v_{6i+4}$ is (15), too. Otherwise $\alpha + \beta + \delta = 2\pi$. This contradicts against Lemma 6 and the equation (15). Hence the type of $\angle Nv_{6i+6}v_{6i+5}$ is $\alpha$. The type of $\angle S_{v_{6i+5}}v_{6i+6}$ is $\gamma$. Otherwise it is $\beta$. Then (14) implies $\beta = \gamma$, which contradicts against Lemma 6. Thus the type of $\angle v_{6i+5}v_{6i+6}v_{6i+7}$ is $\beta$.

Actually, the type of $\angle v_{6i+7}v_{6i+6}N$ is $\alpha$. Otherwise it is $\delta$, and the vertex type of $v_{6i+6}$ is $\alpha + \beta + \delta$. This contradicts against Lemma 6 and (15).

To sum up, from the negation of Property 2 we derived the negation of Property 2 with the indices increased by 6. This increase of the indices by 6 contributes $\gamma + 2\delta$ to the vertex type of the vertex $N$, and three tiles to the northern part of the tiling. Therefore $\gamma + 2\delta = 2\pi/(F/6) = 12\pi/F$. By solving the system of this equation, (14), (15), and (12), we have $0 < \delta = (10 - F)\pi/F$, which is, however, be negative from $F \geq 10$. This ends the proof of Claim 1.

We can prove Property 2* holds when the tile is of type 2, by the same argument as above with $(\alpha, \beta) \leftrightarrow (\delta, \gamma)$ swapped. We can prove Property 2* holds when the tile is of type 4 by Lemma 3. This completes the proof of Lemma 8.

We now resume the proof of Lemma 1. The readers are kindly advised to follow the argument with Figure 4.

**Proof.** By Lemma 8, the assertion (2) of Lemma 1 follows. Hence $L(e) = L(f(e)) = L(g(e))$ for every edge $e$ of the chart.

First consider the case where the properties (A)* and (B)* of Lemma 8 hold. By the property (A)*, for each nonnegative integer $i < F/6$,

the type of $\angle v_{6i}Nv_{6i+2}$ is $\alpha \quad (16)$

and the type of $\angle v_{6i+2}v_{6i+1}v_{6i}$ is $\gamma$. By Lemma 8,

the type of $\angle v_{6i-1}v_{6i}v_{6i+1}$ is $\gamma. \quad (17)$

By the property (B)*,

the type of $\angle v_{6i+2}Nv_{6i+4}$ is $\beta. \quad (18)$

By (16),

the type of $\angle Nv_{6i+2}v_{6i+1}$ is $\beta. \quad (19)$
By (17), the type of $\angle v_6 v_{6i+1} S$ is $\delta$. By Lemma 8, the type of $\angle S v_{6i+1} v_{6i+2}$ is $\alpha$. Thus $K(\psi) = K(f(\psi))$ if $\psi$ is an angle around $v_{6i}$ or $v_{6i+1}$. Observe

the type of $\angle v_{6i+1} v_{6i+2} h_{6i+3}$ is $\beta$, (20)

and thus

the type of $\angle v_{6i+2} v_{6i+3} S$ is $\gamma$. (21)

By (18),

the type of $\angle v_{6i+3} v_{6i+2} N$ is $\gamma$. (22)

Thus the vertex type of $v_{6i+2}$ is (14). Note the type of $\angle N v_{6i+4} v_{6i+3}$ is $\alpha$, and the type of $\angle v_{6i+4} v_{6i+3} v_{6i+2}$ is $\delta$. (23)

By the property (A)$^*$, we have the equation (15). By this, (21), and (23), if the type of $\angle S v_{6i+3} v_{6i+4}$ is $\delta$, then we have $\alpha = \delta$ which contradicts against Lemma 6. So

the type of $\angle S v_{6i+3} v_{6i+4}$ is $\alpha$, (24)

and

that of $\angle v_{6i+3} v_{6i+4} v_{6i+5}$ is $\delta$. (25)

By the property (A)$^*$, the type of $\angle v_{6i+3} v_{6i+4} N$ is $\gamma$. Hence $K(\psi) = K(f(\psi))$ if $\psi$ is an angle around $v_{6i+3}$ or $v_{6i+4}$. Therefore the vertex type of $v_{6i+3}$ is (15). The property (A)$^*$ furthermore implies that the type of $\angle v_{6i+1} v_{6i+2} v_{6i+4}$ is $\beta$. By (25), the type of $\angle v_{6i+4} v_{6i+5} S$ is $\gamma$. By (17), the type of $\angle S v_{6i+5} v_{6i+6}$ is $\beta$. Therefore $K(\psi) = K(f(\psi))$ if $\psi$ is an angle around $v_{6i+2}$ or $v_{6i+5}$. This completes the proof of the assertion (3) of Lemma 1. By the property (A)$^*$,

the type of $\angle N v_{6i} v_{6i-1}$ is $\alpha$ and the type of $\angle v_{6i-2} v_{6i}$ is $\delta$. (26)

Because of (16), (18) and (26), the vertex type of the north pole is

$$(\alpha + \beta + \delta) \frac{F}{6} = 2\pi.$$  

From this and the equations (12) and (15), we obtain the assertion (4) of Lemma 1. The claim $\gamma > \pi$ follows from Lemma 1 (1).

We prove the assertion (5) of Lemma 1. The assertion states the following four statements: (i) the three inner angles $\angle v_{-2} N v_0$, $\angle v_0 N v_2$, and $\angle v_2 N v_4$ are mapped by the angle-assignment $K$, to $\delta$, $\alpha$, and $\beta$; (ii) the inner angles around the vertex $v_0$ are $\angle N v_0 v_{-1}$, $\angle v_{-1} v_0 v_1$ and $\angle v_1 v_0 N$, which $K$ maps to $\alpha$, $\gamma$ and $\delta$; (iii) the inner angles around $v_2$ are $\angle N v_2 v_1$, $\angle v_1 v_2 v_3$ and $\angle v_3 v_2 N$, which $K$ maps to $\beta$, $\beta$ and $\gamma$; and (iv) the inner angles around $v_3$ are $\angle v_3 v_1 v_2$, $\angle v_2 v_3 S$ and $\angle S v_3 v_4$, which $K$ maps to $\delta$, $\gamma$ and $\alpha$. The statement (i) follows
from (26), (16), and (18); the statement (ii) does from the property (A)*; the statement (iii) does from (19), (20), and (22); and the statement (iv) does from (23), (21), and (24).

Next consider the case where the properties (A) and (B) of Lemma 8 hold. By the same argument as above but with \((\alpha, \beta) \leftrightarrow (\delta, \gamma)\) being swapped, we obtain a chart from Figure 4 with \((\alpha, \beta) \leftrightarrow (\delta, \gamma)\) swapped and obtain the assertion (4)*. But these data provide exactly the same angle-assignment for the map \(pdw_F\) which we have just obtained in the case the properties (A)* and (B)* hold. Hence the assertions (1), (2), (3), (4) and (5) of Lemma 1 hold also.

5. The absolute values of the angles, the edges and the number of tiles

By trigonometry argument, we finish the proof of Assertion (1) of Theorem 2. In the rest of the paper, we use the spherical polar coordinate system introduced in Assertion (2) of Theorem 2. The vertex \(S\) of the chart is located at the south pole \((\pi, 0)\). To see it, first we can compute the longitude \(\varphi\) of the vertex \(S\) by applying trigonometry arguments along the path \(Nv_0v_1S\). By rotating the path \(Nv_0v_1S\) by \((\alpha + \beta + \delta) = 12\pi/F\) radian, we obtain a path \(Nv_6v_7S\). By applying the same trigonometry arguments along the latter path, the longitude of \(S\) becomes \(\pi + \varphi\). Thus the vertex \(S\) should be the north pole or the south pole of the spherical polar coordinate system. If the vertex is the north pole, then some tile containing the vertex \(N\) and some tile containing the vertex \(S\) shares the interior, which contradicts against the definition of the tiling.

For the length \(a\) of an edge of the tile, \(\cos a\) is positive. To see it, assume \(a \geq \pi/2\). Then the two points \(v_2\) and \(v_4\) are located on the southern hemisphere while the point \(v_3\) on the northern hemisphere. So \(\gamma = \angle v_3v_2N\) is less than or equal to \(\pi\), which contradicts against the assertion (4) of Lemma 1.

Consider the triangle \(Nv_2v_3\). If we join the edge \(Nv_3\) of the triangle with the edge \(v_3S\) of the tiling, we obtain a geodesic line from the north pole to the south pole. Because the edge \(v_3S\) has length \(a\),

\[Nv_3 = \pi - a.\]

The first equation of (11) of Lemma 1 implies \(\angle v_3v_2N = \gamma > \pi\). So the angle between two edges \(Nv_2\) and \(v_2v_3\) is \(2\pi - \gamma = 8\pi/F\). By the spherical cosine theorem, we obtain a quadratic equation \(\cos(\pi - a) = \cos^2 a + \sin^2 a \cos(8\pi/F)\) of \(\cos a\). Since \(\cos a > 0\), the solution is

\[\cos a = -\frac{\cos(8\pi/F)}{1 - \cos(8\pi/F)} > 0.\]
Thus by the premise \( F \geq 10 \) of Theorem 2 and Lemma 1 (1), the number of tiles is \( F = 12 \). From this and (27), the length \( a \) is as in the equation (2) of Theorem 2.

By \( F = 12 \) and the assertion (4) of Lemma 1, the inner angles \( \beta \) and \( \gamma \) are as in the equations (5) and (6) of Theorem 2. By the spherical cosine theorem applied to the isosceles triangle \( v_2 N v_4 \), the equations (2) and (5) imply

\[
\cos v_4 v_2 = \cos^2 a + \sin^2 a \cos \beta = \frac{1}{3^2} + \left(1 - \frac{1}{3^2}\right) \cos \frac{\pi}{3} = \frac{5}{9}, \tag{28}
\]

The length \( b \) of an edge is less than \( \pi \), because otherwise two edges \( N v_0 \) and \( N v_6 \) of length \( b \) share more than two points. We compute the length \( b \) by considering a triangle \( N v_0 v_1 \). If we join the edge \( N v_1 \) of the triangle with the edge \( v_1 S \) of the tiling, we obtain a geodesic line from the north pole to the south pole. Moreover the edge \( v_1 S \) has length \( b \). Therefore the edge \( N v_1 \) of the triangle \( N v_0 v_1 \) has length \( \pi - b \). Because two tiles \( S v_11 v_0 v_1 \) and \( v_4 N v_2 v_3 \) are congruent, the segment \( v_2 v_4 \) has length \( \pi - b \). By (28), the length \( b \) is as in the equation (3) of Theorem 2.

Then we compute the other inner angles \( \alpha \) and \( \delta \) of the tile. To compute \( \alpha \), we apply the spherical cosine theorem to the triangle \( N v_3 v_4 \). By the equations (2), (3) and (11), we have \( \cos(\pi - a) = \cos b \cos a + \sin b \sin a \cos \alpha \). Here \( 0 < \alpha < \pi \) because \( \alpha \) occurs twice in the vertex type of the vertex \( N \). Hence the inner angles \( \alpha \) and \( \delta \) are as in the equations (4) and (7) of Theorem 2.

By the assertion (5) and the assertion (3) of Lemma 1, all angles of the chart \( \mathcal{A} \) are as in Assertion (1) of Theorem 2.

6. The existence of the tile

To prove that the tiling \( \mathcal{T} \) actually exists, we verify that the quadrangle \( N v_2 v_3 v_4 \) actually exists. It is sufficient to show the following three assertions:

(i) Let \( u \) be the value of \( \cos v_2 v_4 \) computed by applying the spherical cosine theorem to the triangle \( v_2 v_3 v_4 \). Then \( u \) is equal to the value \( 5/9 \) of \( \cos v_2 v_4 \) we have computed already in (28) by applying the spherical cosine theorem to the triangle \( v_2 N v_4 \).

(ii) The triangles \( N v_2 v_4 \) and \( v_2 v_3 v_4 \) actually exist.

(iii) The different edges of the tile \( N v_0 v_1 v_2 \) do not have a common inner point.

To prove the assertion (i), we observe that \( \sin a = 2\sqrt{2}/3 \) and \( \sin b = 2\sqrt{14}/9 \), because \( 0 < a, b < \pi \) and the equations (2) and (3) hold. Then \( u = \cos v_3 v_2 \cos v_1 v_4 + \sin v_3 v_2 \sin v_1 v_4 \cos v_4 v_3 v_2 = \cos a \cos b + \sin a \sin b \cos \delta = 5/9 \) as desired.
To prove the assertion (ii), we employ Proposition 1. First we verify the assumptions of Proposition 1 for the triangle $v_2v_3v_4$. Let $\phi = \angle v_0Nv_1$. By applying the spherical cosine theorem to the triangle $v_0Nv_1$, we have $\cos a = \cos b \cos(\pi - b) + \sin b \sin(\pi - b) \cos \phi$. By the equation (3), we have $\phi = \pm \arccos(13/14)$. We prove the equation (8), that is, $\phi = \arccos(13/14)$. If $\phi = -\arccos(13/14)$, then the spherical polar coordinate of $v_1$ and that of $v_2$ are $(\theta_1, \varphi_1) := (\pi - b, -\arccos(13/14))$ and $(\theta_2, \varphi_2) := (a, x)$ respectively. For the length $a$ of the geodesic line between two points $v_1$ and $v_2$, the value $\cos a = 1/3$ should be equal to the inner product of the cartesian coordinates of $v_1$ and $v_2$, that is, $\sin(\theta_1) \sin(\theta_2) \cos(\varphi_2 - \varphi_1) + \cos(\theta_1) \cos(\theta_2) = -5/21$. This is a contradiction. Hence we have (8). Thus

$$\angle v_2v_4v_3 = \angle v_0Nv_1 = \phi = \arccos \frac{13}{14} = 0.380251 \ldots$$ (29)

By the spherical cosine theorem applied to the triangle $v_2v_3v_4$, we have $\cos b = \cos a \cos(\pi - b) + \sin a \sin(\pi - b) \cos \angle v_3v_2v_4$. By the equations (2), (3), and (27), we have $\cos \angle v_3v_2v_4 = -5/(2\sqrt{7})$. If $\angle v_3v_2v_4$ is greater than $\pi$, then the edges $v_3v_4$ and $Nv_3$ of the tile $Nv_2v_3v_4$ has a common inner point, which is a contradiction. Hence

$$\angle v_3v_2v_4 = \arccos \frac{-5}{2\sqrt{7}} = 2.80812 \ldots$$ (30)

For a geodesic line from $N$ to $S$ through $v_1$, we have $\angle v_0v_1N = \angle v_3v_2v_4$. So

$$\angle v_4v_2v_3 = \delta = \pi - \angle v_3v_2v_4 = \arccos \frac{5}{2\sqrt{7}} = 0.3334373 \ldots$$ (31)

By (29), (30) and (31), the assumptions of Proposition 1 hold for the triangle $v_2v_3v_4$.

Next we verify the assumptions of Proposition 1 for the triangle $v_2Nv_4$. By the equations (6), (30), (5), (4), and (8), the inner angles of the triangle $v_2Nv_4$ are

$$\gamma - \angle v_3v_2v_4 = \frac{4\pi}{3} - \arccos \frac{-5}{2\sqrt{7}} = 1.38067 \ldots, \quad \beta = \frac{\pi}{3} = 1.0472 \ldots,$$

$$\alpha - \phi = \arccos \frac{-1}{2\sqrt{7}} - \arccos \frac{13}{14} = 1.38067 \ldots$$

These also satisfy the assumptions of Proposition 1. Thus the triangle $v_2Nv_4$ actually exists.

The assertion (iii) holds because the longitudes of the vertices $v_0$, $v_1$ and $v_2$ are, respectively, $0$, $\phi = \arccos(13/14)$, and $\alpha = \arccos(-1/(2\sqrt{7}))$, increasing between $0$ and $\pi$. To sum up, the quadrangle $Nv_2v_3v_4$ actually exists.
7. The uniqueness up to special orthogonal transformation

We prove Assertion (2) of Theorem 2. Observe that the spherical polar coordinate of each vertex $v_{2i}$ (resp. $v_{2i+1}$) of the tiling $\mathcal{T}$ is determined from Assertion (1) of Theorem 2 by considering the angle around the north pole $N$ (resp. around the south pole $S$ with the equation (8) of Theorem 2). The length of the geodesic line between $N$ and $v_{2i+1}$ is $\pi$ subtracted by the length of the edge $Sv_{2i+1}$.

We have a freedom to set a spherical polar coordinate system, so long that the north pole is the vertex $N$ and the “Greenwich meridian” of the spherical polar coordinate system contains the edge $Nv_0$. However, the spherical coordinate system must respect the cyclic orders of the vertices of the chart. So each vertex is unique up to special orthogonal transformation. In fact, the mirror image of the tiling cannot be transformed to the original tiling, by a special orthogonal transformation.

8. The symmetry

To complete the proof of Theorem 2, we prove Assertion (3) of Theorem 2: Schönflies symbol of the tiling $\mathcal{T}$ is $D_2$. The Schönflies symbol is determined according to [7, p. 55] and Step 1, Step 2, ..., of [7, Figure 3.10 (p. 56)]. As for Step 1 of Figure 3.10, the tiling $\mathcal{T}$ does not have continuous rotational symmetry, and so the Schönflies symbol is neither $C_{\infty v}$ nor $D_{\infty h}$. Moreover the tiling has none of 3-fold and 5-fold symmetry, so the Schönflies symbol is none of $T$, $T_h$, $T_d$, $O$, $O_h$, $I$, $I_h$.

Observe that the tiling $\mathcal{T}$ has the following three 2-fold rotation axes perpendicular to each other (cf. Figure 5):

- The axis through the both poles. The rotation around this axis by $\pi$-radian maps a point $(\theta, \varphi)$ on the sphere to a point $(\theta, \varphi + \pi)$. The action of the rotation on the vertices and the edges of the tiling $\mathcal{T}$ is the automorphism $g$ defined in (10).
- The axis through the ball’s center and the midpoint of $v_0$ and $v_1$. The rotation around this axis by $\pi$-radian maps a point $(\theta, \varphi)$ on the sphere to a point $(\pi - \theta, \phi - \varphi)$ where $\phi$ is defined in the equation (8). The action of the rotation on the vertices and the edges of the tiling $\mathcal{T}$ is the automorphism $f$ defined in (9).
- The axis through the ball’s center and the midpoint of $v_3$ and $v_4$. The 2-fold rotation around this axis corresponds to the automorphism $g \circ f$. Because of the three perpendicular 2-fold axes, we can skip the Step 2, and we see in Step 3 that the Schönflies symbol is none of $S_4, S_6, S_8, \ldots$. Hence, we go to Step 5 and see that the symbol is one of $D_2, D_{2d}, D_{2h}$. But by Figure 5,
the tiling has no mirror plane perpendicular to a 2-fold axis, the symbol is not \( D_{2h} \). Moreover the tiling has no mirror plane between two 2-fold axes, so the symbol is not \( D_{2d} \) either. Therefore the Schönflies symbol is \( D_2 \). This completes the proof of Theorem 2.

Three movies of the tiling \( T \) spinning around the rotation axes are available at the web-site http://www.math.tohoku.ac.jp/akama/stcq.

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