Abstract

We address the question of how a localized microscopic defect, especially if it is small with respect to certain dynamic parameters, affects the macroscopic behavior of a system. In particular we consider two classical exactly solvable models: Ulam’s problem of the maximal increasing sequence and the totally asymmetric simple exclusion process. For the first model, using its representation as a Poissonian version of directed last passage percolation on $\mathbb{R}^2$, we introduce the defect by placing a positive density of extra points along the diagonal line. For the latter, the defect is produced by decreasing the jump rate of each particle when it crosses the origin.

The powerful algebraic tools for studying these processes break down in the perturbed versions of the models. Taking a more geometric approach we show that in both cases the presence of an arbitrarily small defect affects the macroscopic behavior of the system: in Ulam’s problem the time constant increases, and for the exclusion process the flux of particles decreases. This, in particular, settles the longstanding “Slow Bond Problem”.

1 Introduction

One of the fundamental questions of equilibrium and non-equilibrium dynamics refers to the following problem: how can a localized defect, especially if it is small with respect to certain dynamic parameters, affect the macroscopic behavior of a system? Two canonical examples are directed last passage percolation (DLPP) with a diagonal defect line and the one dimensional totally asymmetric simple exclusion process (TASEP) with a slow bond at the origin. In their unmodified form, these models are exactly solvable and in the KPZ universality class. They have been the subject of intensive study yielding a rich and detailed picture including Tracey-Widom scaling limits [6, 14]. Under the addition of small modifications, however, the algebraic tools used to study these models break down. In this paper we bring a new more geometric approach to determine the effect of defects.

For TASEP with a slow bond one asks whether the flux of particles is affected at any arbitrarily small value of slowdown at the origin or if when the defect becomes too weak, the fluctuations in the bulk destroy the effect of the obstruction so that its presence becomes macroscopically undetectable. Originally posed by Janowsky and Lebowitz in 1992, this
question has proved controversial with various groups of physicists arriving at competing conclusions on the basis of empirical simulation studies and heuristic arguments (see [8] for a detailed background). In DLPP the question becomes whether the asymptotic speed is changed in the macroscopic neighborhood of such a defect at any value of its strength. Equivalently, one may ask if its asymptotic shape is changed and becomes faceted.

Such a vanishing presence of the macroscopic effect as a function of the strength of obstruction represents what sometimes is called, in physics literature, a dynamic phase transition. The existence of such a transition, its scaling properties and the behavior of the system near the obstruction are among the most important issues. In this work we prove that indeed an arbitrarily small defect affects the macroscopic behaviour of these models resolving the longstanding slow bond problem. We begin with a description of the models and our main results.

Maximal increasing subsequence. We consider the classical Ulam’s problem of the maximal increasing subsequence of a random permutation recast in the language of continuum last passage percolation: Let Π be a Poisson point process of intensity 1 on \( \mathbb{R}^2 \). We let \( L_n \) denote the maximum number of points in Π along any oriented path from \((0,0)\) to \((n,n)\) calling it the length of a maximal path. Conditional on the number of points in the square \([0,n]^2\) this is distributed as the length of the longest increasing subsequence of a random permutation. Using a correspondence with Young-Tableaus, Vershik and Kerov [26] and Logan and Shepp [18] established that

\[
\lim_{n \to \infty} \frac{E L_n}{n} = 2. \tag{1}
\]

(See also Aldous and Diaconis’s proof using interacting particle systems [1]). For \( \lambda > 0 \), let \( \Sigma_\lambda \) be a one dimensional poisson process of intensity \( \lambda \) on the line \( x = y \) independent of \( \Pi \) and let \( \Pi_\lambda \) be the point process obtained by the union of \( \Pi \) and \( \Sigma_\lambda \). We study the question of how the length of the maximal path is affected by this reinforcing of the diagonal.

Let \( L^\lambda_n \) denote the maximum number of points of \( \Pi_\lambda \) on an increasing path from \((0,0)\) to \((n,n)\). It is easy to observe that taking \( \lambda \) sufficiently large changes the law of large numbers for \( L^\lambda_n \) from that of \( L_n \), i.e., for \( \lambda \) sufficiently large

\[
\lim_{n \to \infty} \frac{E L^\lambda_n}{n} > 2. \tag{2}
\]

An important problem is whether there is a non-trivial phase transition in \( \lambda \), i.e., whether for any \( \lambda > 0 \) the law of large numbers for \( L^\lambda_n \) differs from that of \( L_n \), or there exists \( \lambda_c > 0 \), such that the law of large number for \( L^\lambda_n \) is same as that of \( L_n \) for \( \lambda < \lambda_c \). Our first main result settles this question:

**Theorem 1** For every \( \lambda > 0 \),

\[
\lim_{n \to \infty} \frac{E L^\lambda_n}{n} > 2. \tag{3}
\]

The slow bond problem. Consider DLPP on \( \mathbb{Z}_+^2 \), defined by associating with each vertex \( x \in \mathbb{Z}_+^2 \) an independent random variable \( \xi_x \sim \exp(1) \). The last passage time is defined as

\[
T^0_n = \max \sum_{i=0}^n \xi_{x_i};
\]
maximized over all oriented paths in $\mathbb{Z}_+^2$ from $(0,0)$ to $(n,n)$. It is well known \cite{22} that
\[ \lim_{n \to \infty} \frac{\mathbb{E}T_0^0}{n} = 4 \] (4)

By a well known mapping due to \cite{23}, $T_0^0$ also describes passage times of particles in the totally asymmetric exclusion process. In the continuous time TASEP $X(t) = (\eta_k(t))_{k=-\infty}^\infty \in \{0,1\}^\mathbb{Z}$ introduced and studied in \cite{22}. The dynamics of the particles is as follows, a particle at position $k$ (i.e. $\eta_k = 1$) jumps with exponential rate one to $k+1$ provided that position is vacant (i.e. $\eta_{k+1} = 0$). Started from the initial configuration is $\mathbb{1}_{(-\infty,0]}(k)$, the so called “step initial condition”, the time for the particle from position $-n$ to move to 1 is distributed as $T_0^0$. Indeed, it is exactly $T_0^0$ if we couple TASEP and DLPP so that the variable $\xi(i,j)$ represents the time which the particle starting at $-i$ has to wait to perform its $j$-th jump once that position is vacant. The inverse value of the expression in (4) corresponds to the asymptotic rate of particles crossing a bond.

Now let us modify the distribution of passage times, by taking
\[ \xi(x,y) \sim \begin{cases} 
\exp(1) & \text{if } x \neq y, \\
\exp(1 - \epsilon) & \text{if } x = y.
\end{cases} \] (5)

and ask the same question: does the law of large numbers for $T_0^\epsilon$ change for any $\epsilon > 0$ where $T_0^\epsilon$ denotes the last passage time in this setting.

In the TASEP representation this change corresponds to a local modification of the dynamics: the exponential clock governing particles jumping across the edge $(0,1)$ is decreased from rate 1 to rate $1 - \epsilon$ introducing a slow bond. This version of the process was proposed by Janowsky and Lebowitz \cite{12} (see also \cite{13}), as a model for understanding non-equilibrium stationary states.

The jump-rate decrease at the origin will increase the particle density to the immediate left of such a “slow bond” and decrease the density to its immediate right. The difficulty in analyzing this process comes from the fact that the effect of any local perturbation in non-equilibrium systems carrying fluxes of conserved quantities is felt at large scales. What was not obvious, was if this perturbation, in addition to local effects, may also have a global effect and in particular change the current in the system i.e. whether the LLN for $T_0^\epsilon$ changes for any value $\epsilon > 0$ or whether $\epsilon_c$ is strictly greater than 0.

This question generated considerable controversy in theoretical physics and mathematical community, which was supported from opposite sides by numerical analysis and some theoretical arguments (see §1.1), and became known in the literature as the “Slow Bond Problem” (\cite{12, 13, 24, 21, 10}), see \cite{8} for a detailed account. Our second result settles this problem:

\textbf{Theorem 2} In discrete last passage percolation model for every $\epsilon > 0$,
\[ \lim_{n \to \infty} \frac{\mathbb{E}T_0^\epsilon}{n} > 4. \] (6)
1.1 Background

Non-equilibrium interfaces with localized defect that display nontrivial scaling properties are common in physical, chemical and biological systems. The problem we are interested in can be cast in several different, but closely related forms: as a stochastic driven transport through narrow channels with obstructions \([10]\), as a growth model with defect line \([21]\), or as a polymer pinning problem of a one-dimensional interface \([11, 2]\). Most of these models in two dimensions (sometimes interpreted as 1+1 dimension) belong, in their unperturbed case, to the Kardar-Parisi-Zhang (KPZ) universality class. The question if arbitrarily small microscopic obstruction may change local macroscopic behavior of non-equilibrium systems became broadly discussed starting late eighties.

For the TASEP model with a slow bond Janowsky and Lebowitz in \([13]\) provided a mean field argument, showing that if the jump rate at the origin is \(1-\epsilon\), then the current becomes equal to \((1-\epsilon)/(2-\epsilon)^2\), thus supporting the conjecture that \(\epsilon_c = 0\). This conjecture was also supported by theoretical renormalization group arguments in the study of a directed polymer pinning transition at low temperatures \([11]\). An alternative heuristic argument based on “influence percolation” was discussed in \([7]\). In more recent work \([8]\), based on a theoretical argument and analysis of the first sixteen terms of formal power series expansion of the current, authors concluded that for small values of \(\epsilon > 0\) the current should behave as \(1/4 - \gamma \exp(-a/\epsilon)\) with \(a \approx 2\).

On the rigorous side, an first upper bound for the critical value of the slow-down rate was derived in \([9]\) by approximating the slow bond model with an exclusion process whose rates vary more regularly in space. An alternative bound for the critical slow-down was provided in \([17]\). Finally the most complete and general hydrodynamic limit results were obtained in \([24]\) for all values \(0 < \epsilon < 1\) of the slow-down. However the hydrodynamic limit can not make the distinction of whether the slow bond disturbs the hydrodynamic profile for all values of \(\epsilon > 0\). Letting \(\kappa_{1-\epsilon}\) denote the inverse maximal current in presence of a \(1-\epsilon\) slow bond \([24]\) obtained the following bound:

\[
\max \left\{ 4, \frac{3}{2} + \frac{(1-\epsilon)^2 + 2(2-\epsilon)}{2(1-\epsilon)(2-\epsilon)} \right\} \leq \kappa_{1-\epsilon} \leq 3 + \frac{1}{1-\epsilon}.
\]

At the same time, a competing set of theoretical arguments, mostly appearing in the theoretical physics literature, supported also by numerical data, pointed towards the possibility that \(\epsilon_c > 0\). In \([10]\) early numerical data for a related polynuclear growth model, involving parallel updating, was interpreted as suggesting that the critical delay value in TASEP with slow bond model should be \(\epsilon_c \approx 0.3\). In another study, based on a finite size scaling analysis of simulation data \([10]\) concluded that \(\epsilon_c \approx 0.2\).

An important rigorous step forward was made by Baik and Rains \([5]\) where, among several cases of interest, they also consider the so called “symmetrized” version of the maximal increasing sequence with the a defect line, for which they showed that \(\lambda_c = 1\). At first glance this may seem at odds with Theorem \([1]\) showing that \(\lambda_c = 0\) in the original model. It is shown in \([5, Theorem 3.2]\) that the constant in the LLN in the symmetrized system with \(0 < \lambda \leq 1\) reinforcement on the diagonal coincides with that of the LLN in the non-symmetrized system with no reinforcement on the diagonal, and is equal to 2. However, if we look in both processes at the picture of their level lines, sometimes also called Hammersley process trajectories, (see \([1]\)), we observe that in the non-symmetrized model with no perturbation the level lines are in equilibrium and in particular, their intersection
with the main diagonal forms a stationary point process of intensity 2 (see [25]). However, in the symmetrized case with no reinforcement the level lines are “out of equilibrium” in vicinity $n^{2/3}$ of the main diagonal. Adding an extra rate $\lambda$ Poisson point with $0 < \lambda < 1$ on the main diagonal brings this process closer to equilibrium as $\lambda$ increases from 0 to 1. When $\lambda$ reaches 1, at which point the level lines in symmetrized process become “equilibrated”. After that for any positive increase above the value 1 the LLN in the symmetrized (and now equilibrated) model changes. In the context of the original non-symmetrized model there is no need to pay this extra cost in order to equilibrate the system and this corresponds to a change of the LLN at any positive value of reinforcement.

1.2 Tracy-Widom Limit, Moderate Deviations and $n^{2/3}$ Fluctuations

The two models that we consider (i.e., the longest increasing subsequence and the exponential last passage percolation) are exactly solvable in absence of a defect and it is possible to obtain scaling limits and precise moderate deviation tail bounds for $L_n$ and $T_n$. We shall treat these results from the exactly solvable models as a “black box” in our arguments. Using these estimates the problems at hand can be treated as percolation type questions. Here we collect the results we need for the longest increasing subsequence model which is the model we shall primarily work with in this paper. Similar results are also available in the literature for the exponential directed last passage percolation model, and we shall quote them in §10 where we explain how to adapt our arguments to the discrete case.

1.2.1 Scaling limit

Baik, Deift and Johansonn [6] proved the following fundamental result about fluctuations of $L_n$. Let $\Pi$ be a homogeneous Poisson point process on $\mathbb{R}^2$ with rate 1. Let $u_t$ be a point on the first quadrant of $\mathbb{R}^2$ such that the area of the rectangle with bottom left corner $(0,0)$ and the top right corner $u_t$ is $t$. Let $X_{u_t}$ denote the maximum number of points on $\Pi$ on an increasing path from $(0,0)$ to $u_t$. By the scaling of Possion point process it is clear that the distribution of $X_{u_t} = X_{u_t}$ depends on $u_t$ only through $t$. The following Theorem is the main result from [6].

**Theorem 1.1** Let $F_{TW}$ be the Tracy-Widom distribution. As $t \to \infty$,

$$\frac{X_t - 2\sqrt{t}}{t^{1/6}} \xrightarrow{d} F_{TW}$$  

where $\xrightarrow{d}$ denotes convergence in distribution.

1.2.2 Moderate deviation estimates

We also require estimates from the tails of the distribution and quote the following moderate deviation estimates for upper and lower tails of longest increasing subsequence from [19] and [20] respectively. The following theorem is an immediate corollary of Theorem 1.3 of [19].

**Theorem 1.2** There exists absolute constants $C_1$, $s_0$ and $t_0 > 0$ such that for all $t > t_0$ and $s > s_0$, the following holds.

$$\mathbb{P}[X_t \geq 2\sqrt{t} + st^{1/6}] \leq e^{-C_1 s^{3/2}}.$$  

(9)
The corresponding estimate for the lower tail was proved in [20], the following theorem is an immediate corollary of Theorem 1.2 from that paper.

**Theorem 1.3** There exists absolute constants $C_1$, $s_0$ and $t_0 > 0$ such that for all $t > t_0$ and $s > s_0$, the following holds.

$$
\mathbb{P}\left[ X_t \leq 2\sqrt{t} - st^{1/6} \right] \leq e^{-C_1 s^{3/2}}. \quad (10)
$$

Observe that $t_0$, $s_0$ and $C_1$ can be taken to be same in Theorem 1.2 and Theorem 1.3.

It is also clear by the translation invariance of the Poisson process that the same bounds can be obtained for the the number of points on a maximal increasing path on any pair of points that determine a rectangle with area $t$.

### 1.2.3 Transversal Fluctuation

Consider all increasing paths $\gamma$ from $(0,0)$ to $(n,n)$ in $\Pi$ containing the maximum number of points. The maximum transversal fluctuation $F_n$ is defined as $\max_{x \in [0,n], \gamma} |\gamma(x) - x|$. The scaling exponent for the transversal fluctuation $\xi$ is defined by

$$
\xi = \inf\{\theta > 0 : \lim\inf_n \mathbb{P}[F_n \geq n^\theta] = 0\}.
$$

Johansson [15] proved the following theorem.

**Theorem 1.4** In the above set-up we have $\xi = \frac{2}{3}$.

This theorem bounds the maximal fluctuation of the maximal paths from the diagonal as order $n^{2/3+o(1)}$. This motivates a lot of our construction. However for our proof, we need a slightly sharper estimate which we establish using Theorem 1.3 and Theorem 1.2 (see Theorem 9.13).

### 1.3 Outline of the proof

In this subsection we present an outline of the proof. We do it here for the case of the continuum last passage percolation. The proof of Theorem 2 follows similarly and the necessary modifications are discussed in §10.

We start with the observation that since due to superadditivity of the passage times

$$
\mathbb{E}\left[ L_n^\lambda \right] \geq \mathbb{E}\left[ L_n^\lambda + E L_m^\lambda \right]
$$

for any $\lambda > 0$, it suffices to prove that for some $n$

$$
\mathbb{E}[L_n^\lambda] > 2n. \quad (11)
$$

Using the Tracy-Widom Limit Theorem 1.1 and the moderate deviation inequalities from Theorems 1.2 and 1.3 we have

$$
\mathbb{E}[L_n] = 2n - O(n^{1/3}). \quad (12)
$$

Thus, in order to obtain (11), it is enough to prove that for $n$ sufficiently large

in the environment with the diagonal reinforced by a one-dimensional Poisson point process of intensity $\lambda > 0$, the length of longest increasing path from $(0,0)$ to $(n,n)$ increases by at least $cn^{1/3}$ for some arbitrarily large positive constant $c > 0$. 

6
Most of the work is dedicated to showing \([\text{13}]\). We do it in several steps. First, we observe that the maximal path in unperturbed environment, \(i.e.\) with \(\lambda = 0\), is expected to spend \(O(n^{1/3})\) of time near the diagonal (within finite distance from it). If this happens, then essentially without an additional cost reinforcement results in an average increase of the length of the maximal path by \(\lambda O(n^{1/3})\). On the other hand, if the maximal path in the unperturbed environment deviates from the diagonal for substantially long time, then we may search for an alternative path in unperturbed environment which returns to the diagonal more frequently and does it in such a manner, that the loss of the length due to such maneuver in unperturbed configuration at the end is compensated, and actually improved upon once the reinforcement \(\lambda > 0\) is added to the diagonal. Thus we see that improvements of \(O(n^{1/3})\) are possible both on the shortest and longest length scales.

Next, we need to “bootstrap” small improvement \(\lambda n^{1/3}\) to \(cn^{1/3}\) for arbitrarily large constant \(c > 0\). However, the task of obtaining simultaneous local improvements to the optimal path at different locations (in the sense of previous paragraph) becomes complicated due to difficulties in tracing excursions of the maximal path from the diagonal. To proceed, we adopt the following strategy: consider translates of the diagonal \(\ell_m = \{y = x + m\},\) where \(m \in [-Kn^{2/3}, Kn^{2/3}]\), for some large \(K \in \mathbb{N}\). For each \(m \in [-Kn^{2/3}, Kn^{2/3}]\) and \(\lambda > 0\) consider the new reinforced environment, obtained from the original one, by adding Poisson point process of intensity \(\lambda\) on \(\ell_m\). Let \(L_n^{\lambda,m}\) be the length of the longest increasing path from \((0, 0)\) to \((n, n)\) in the environment with reinforced \(\ell_m\). Our main statement is that for any \(\lambda > 0\) and arbitrarily large \(c_1 > 0\), there exists \(m \in [-Kn^{2/3}, Kn^{2/3}]\), such that

\[
\mathbb{E}[L_n^{\lambda,m}] > 2n + c_1 n^{1/3},
\]

for any \(n\) large enough. Once this is obtained, a little extra work is needed to show that shifting the reinforcement back to the main diagonal still provides a sufficient increase and

\[
\mathbb{E}[L_n^\lambda] \equiv \mathbb{E}[L_n^{\lambda,0}] > 2n + c_2 n^{1/3},
\]

where \(c_2 > 0\) can be chosen arbitrarily large, which implies \([\text{13}]\).

To obtain \([\text{14}]\) we analyze the unperturbed environment at different scales simultaneously. Despite that we use multiple scales, analysis which we perform should not be confused with renormalization arguments, sometimes also called multi-scale analysis. For fixed length scale \(r\), and spatial location \(x = kr, k \in [n/r]\), we define a rectangular box \(B_{r,x} := [kr, (k+1)r) \times [0, n]\) of width \(r\) and height \(n\). We denote the unique top most maximal path from \((0, 0)\) to \((n, n)\) in the unperturbed environment as \(\Gamma\). Our argument relies on the fact that for a large fraction of boxes \(B_{r,x}\) at a fixed scale \(r\) there is a reasonable chance that \(\Gamma\) “behaves nicely” while crossing \(B_{r,x}\). By saying that we mean that with probability \(p_\lambda > 0\), bounded away from 0 and independent of the scale \(r\), the following event occurs:

there exists non-empty random set of indices \(I_{r,x} \subset [-Kn^{2/3}, Kn^{2/3}]\), depending on the chosen scale \(r\) and location \(x\), or, more precisely, depending on the shape and spatial localization of \(\Gamma\) within \(B_{r,x}\), such that if the line \(\ell_i\), for some fixed \(i \in I_{r,x}\), was reinforced by an independent one dimensional Poisson point process of intensity \(\lambda > 0\), then there exists a modification of \(\Gamma\) within \(B_{r,x}\), called a local modification at scale \(r\) and denoted by \(\Gamma_{r,x}\), which has the following properties:

- \(\Gamma_{x,r}^*\) coincides with \(\Gamma\) outside of \(B_{r,x}\).
• the restriction of the new path $\Gamma_{x,r}^*$ within the box $B_{r,x}$ has transversal fluctuations of order $O(r^{2/3})$;

• The increase of the length obtained from such local modifications at the scale $r$ within $B_{r,x}$, integrated over all lines $\ell_i$, $i \in [-Kn^{2/3}, Kn^{2/3}]$, and averaged over reinforcements by $\lambda > 0$ is at least $c(\lambda)r$.

Thus the total average improvement from doing local perturbations at fixed scale $r$ in all boxes of scale $r$ between (0,0) and (n,n) would result in an increase of $\tilde{c}(\lambda)n^{1/3}$.

By choosing scales properly we ensure that at any given location and given offset $m$ the improvement is obtained only at one scale. This allows us to sum up the integrated improvements over different scales, and by considering a large number of scales we obtain an average improvement over different $m$'s of $c_1n^{1/3}$ for any arbitrary constant $c_1$. In particular we can find at least one such offset $m$ establishing (14).

Organisation of the paper: The rest of this paper is organised as follows. As mentioned before we shall provide details only for the proof of Theorem 1 while pointing out the adaptations needed for the proof of Theorem 2. We start with setting up the notations and terminology in §2. In §3 we define for a fixed scale $r$, and a fixed location $x$, events $G_x$, $H_x$, $Q_x$, $R_x$ which are key to the construction of an alternative path as explained above, we also explain how we condition on $R_x$. Using estimates of probabilities of these events (Theorem 3.2 and Theorem 3.5 whose proofs are deferred until later), in §4 we show that with a probability bounded uniformly away from 0, an alternative path satisfying the necessary conditions exist which deviates from the topmost maximal path only near $x$. This is the heart of the argument. Using this, and adding extra points on different offset diagonals as explained above, we complete the proof of Theorem 1 in §5. In §6 we work out certain percolation-type estimates showing that the maximal path behaves sufficiently regularly at a typical location. Probability bounds on $G_x$ are proved in §7 and for the other events in §8 which ultimately finishes the proof of Theorem 3.2 and Theorem 3.5. Throughout these proofs we use a number of results, which are consequences of the moderate deviation estimates Theorem 1.3 and Theorem 1.2. For convenience, we have put together these results in §9, however they are quoted throughout the paper. Finally in §10 we briefly describe how to modify the arguments for the continuum last passage percolation case to prove Theorem 2.

2 Notations and Preliminaries

In this section, we introduce certain notations for the continuum last passage percolation model. The same notations with minor modifications can be used for the exponential discrete last passage percolation model also, see §10 for details of the discrete case.

2.1 Path, length and area

Define the partial order $<$ on $\mathbb{R}^2$ by $u = (x,y) < u' = (x',y')$ if $x < x'$, and $y < y'$. For $u < u' \in \mathbb{R}^2$, an increasing path $\gamma$ from $u$ to $u'$ is the piecewise linear path joining a finite sequence of points $u = u_0 < u_1 < \cdots < u_k = u'$. For $u_0 = (x_0,y_0) < u_k = (x_k,y_k)$ and an increasing path $\gamma$ from $u_0$ to $u_k$, and for $x_0 \leq x \leq x_k$, let $\gamma_x$ be such that $(x,\gamma_x) \in \gamma$. 

Notice that \( \gamma_x \) is uniquely defined. We shall sometimes identify the path with the sequence of points that define it.

We define the length of an increasing path with respect to a background point configuration on \( \mathbb{R}^2 \). Let \( \Omega \) be a point configuration on \( \mathbb{R}^2 \). Consider an increasing path \( \gamma \) from \( u \) to \( u' \) given by \( \gamma = \{ u = u_0 < u_1 < \cdots < u_k = u' \} \). Then length of \( \gamma \) in \( \Omega \), denoted \( \ell_\gamma^\Omega \) is defined by

\[
\ell_\gamma^\Omega = \#\{0 \leq j < k : u_j \in \Omega \}.
\]

Notice that, in the above definition, for definiteness, we count the starting point of the path, but not the end point.

For \( u < u' \) in \( \mathbb{R}^2 \), let \( A(u, u') \) denote the area of the rectangle \( \text{Box}(u, u') \) with bottom left corner \( u \) and top right corner \( u' \). For an increasing path \( \gamma \) containing \( u \) and \( u' \), let \( \gamma(u, u') \) denote the restriction of \( \gamma \) between \( u \) and \( u' \). Let \( \gamma(u, u') = \{ u = u_0 < u_1 < \cdots < u_k = u' \} \).

For a given environment \( \Omega \) let \( i_1 < i_2 < \cdots < i_\ell \in [k - 1] \) be such that \( u_{i_j} \) are all the points on \( \gamma \cap \Omega \) (ignoring the end points of \( \gamma \)). Set \( i_0 = 0 \) and \( i_{\ell+1} = k \). Then the region of \( \gamma(u, u') \) in the environment \( \Omega \), denoted \( O_\gamma(u, u') \), is defined to be the union of the rectangles \( \text{Box}(u_{i_j}, u_{i_{j+1}}) \) for \( j \in \{0, 1, \ldots, \ell \} \). The area of the path \( \gamma(u, u') \) in the environment \( \Omega \), denoted \( A_\gamma^\Omega \) is the area of the region \( O_\gamma(u, u') \), i.e.,

\[
A_\gamma^\Omega = \sum_{j=0}^{\ell} A(u_{i_j}, u_{i_{j+1}}).
\]

We shall drop the superscript \( \Omega \) if the environment is clear from the context.

### 2.2 Statistics of the Unperturbed Configuration

We let \( \Pi \) denote the a rate 1 Poisson process on \( \mathbb{R}^2 \) which we refer to as the unperturbed configuration, that is without reinforcements.

- For \( u, u' \in \mathbb{R}^2 \), let \( X_{u,u'} \) denote the length of longest increasing path in \( \Pi \) from \( u \) to \( u' \). While the longest increasing path need not be unique, \( X_{u,u'} \) is well defined.

For \( u = (x, y) < u' = (x', y') \) in \( \mathbb{R}^2 \), let \( d(u, u') = (x' - x) + (y' - y) \) be the \( \ell_1 \) distance between \( u \) and \( u' \). It will be useful for us to consider following centered versions of \( X_{u,u'} \).

- Let

\[
\tilde{X}_{u,u'} = X_{u,u'} - Ex_{u,u'}.
\]

- Let

\[
\hat{X}_{u,u'} = X_{u,u'} - d(u, u').
\]

Observe that, by Theorem \[1.1\] and superadditivity, it follows that \( \mathbb{E}[X_{u,u'}] < d(u, u') \). The reason behind the choice of centering by \( d(u, u') \) is the following. If the straight line joining \( u \) and \( u' \) has slope very close to 1, then \( d(u, u') \) gives the right centring up to first order. Also observe that for \( u_1 < u_2 < \cdots < u_k \) we have \( \sum_{i=1}^{k-1} X_{u_{i+1}, u_{i+1}} \leq X_{u_1, u_k} \).
2.2.1 Statistics of constrained paths

We make the following notations for paths subject to constraints certain constraints.

- For $u, u' \in \mathbb{R}^2$ with $u < u'$, and $S \subseteq \mathbb{R}^2$, we define $X^S_{u,u'}$ to be the length of the longest increasing path from $u$ to $u'$ that does not go through $S$. The centered length is denoted by $\tilde{X}^S_{u,u'}$, i.e., $\tilde{X}^S_{u,u'} = X^S_{u,u'} - Ex_{u,u'}$. Similarly we also define $\hat{X}^S_{u,u'}$.

- For $u, u' \in \mathbb{R}^2$ with $u < u'$, and $S \subseteq \mathbb{R}^2$, we define $S X^S_{u,u'}$ to be the length of the maximal increasing path from $u$ to $u'$ that intersects the set $S$. We define $S \tilde{X}^S_{u,u'}$ and $S \hat{X}^S_{u,u'}$ similarly.

2.3 Choice of Parameters

Throughout the proof we shall make use of a number of parameters which need to satisfy certain constraints among themselves. We record here the parameters used, the relationship between them, and the order in which we need to fix them. The precise values of the parameters will not be of importance to us.

Reinforcement parameter $\lambda$: $\lambda > 0$ will be kept fixed throughout the proof, this is the rate at which the diagonal $\{x = y\}$ (and its translates) are reinforced.

Scale $r$: As explained in the introduction, we shall work out estimates for functions of $\Pi$ at different length scales, the scale will be indexed by $r$. Let

$$\mathcal{R} = \left\{ 10^k \frac{n}{\log 10} : 1 \leq k \leq \frac{1}{100 \log \log n} \right\}.$$ 

We shall take $r$ to be one of the elements of $\mathcal{R}$.

Parameters: We choose the parameters in the following order. All these parameters are positive numbers and are independent of $r \in \mathcal{R}$, but they can depend on $\lambda$.

1. $\psi$ will be an absolute constant sufficiently large.
2. $\eta$ will be an absolute positive constant sufficiently small.
3. We choose $\tilde{C}$ sufficiently large depending on $\psi$.
4. We choose $M$ sufficiently large depending on other parameters chosen so far.
5. The parameter $C$ will be a sufficiently large constant depending on $M$.
6. $\alpha' < 1$ is chosen to be sufficiently small constant depending on $M$.
7. $\rho$ is chosen sufficiently small depending on $C$.
8. $\delta < 1$ is chosen to be sufficiently small depending all other constants chosen so far (and $\lambda$).
9. We choose $\varepsilon$ small enough depending on $C$ and $M$ and $\delta$ and $\rho$.
10. $L$ is chosen sufficiently large depending on $\delta$ and $\varepsilon$.

The functional form of the constraints that these parameters will need to satisfy will be specified later on.
3 Defining the Key Events

As explained in the introduction, we shall define some key events on which we shall be able to obtain local modifications of the longest path which will lead to improvements in the reinforced environment. These events will be defined for different locations in each scale \( r \).

For the rest of this section, let \( r \in \mathcal{R} \) be fixed. All of our events will be defined for this fixed \( r \).

3.1 Geometric Definitions: The \((x,y,r)\)-Butterfly

For a fixed \( r \), let

\[
\mathcal{X}_r = \left\{ \left( k + \frac{1}{2} \right) r : k \in \{ \frac{n}{3r}, \frac{n}{3r} + 1, \ldots, \frac{2n}{3r} - 1 \} \right\}.
\]

For a fixed \( x \in \mathcal{X}_r \) and for a fixed \( y \in r^{2/3} \mathbb{Z} \) we define a geometric object, which we shall call the \((x,y,r)\)-butterfly, denoted as \( \mathcal{B}(x,y,r) \), which will be a union of parallelograms as described below.

First we need the following notation. For \((x,y) \in \mathbb{R}^2\), \( \ell, h \geq 0 \), let \( \mathcal{P}(x,y,\ell,h) \) denote the parallelogram whose corners are given by \((x - \ell/2, x - \ell/2 + y), (x - \ell/2, x - \ell/2 + y + h), (x + \ell/2, x + \ell/2 + y), (x + \ell/2, x + \ell/2 + y + h)\).

The butterfly \( \mathcal{B}(x,y,r) \) consists of the following parallelograms.

- The **body** of the \( \mathcal{B}(x,y,r) \) is the rectangle
  \[
  T = T_{x,y,r} := \mathcal{P}(x,y - Lr^{2/3}, r, (L + M)r^{2/3}).
  \]

- The **left wing** \( W^1 \) and the **right wing** \( W^2 \) of \( \mathcal{B}(x,y,r) \) is defined as follows
  \[
  W^1 = W^1_{x,y,r} = \mathcal{P}\left(x - \frac{r}{2}(1 + L^{3/2}), y - L^{11/10}r^{2/3}, L^{3/2}r, 2L^{11/10}r^{2/3}\right)
  \]
  and
  \[
  W^2 = W^2_{x,y,r} = \mathcal{P}\left(x + \frac{r}{2}(1 + L^{3/2}), y - L^{11/10}r^{2/3}, L^{3/2}r, 2L^{11/10}r^{2/3}\right).
  \]

The \((x,y,r)\)-butterfly is defined as

\[
\mathcal{B} := T \cup W^1 \cup W^2.
\]

We further define some important subsets of the butterfly (omitting the subscript \((x,y,r)\)).

- Let \( \mathcal{C} = \mathcal{P}(x, y - Lr^{2/3}, \frac{4r}{5}, (L + M)r^{2/3}) \). We shall call \( \mathcal{C} \) the **central column** of \( \mathcal{B} \).
- Let \( D = \mathcal{P}(x, y - (M + \frac{1}{10})r^{2/3}, \frac{r}{10}, \frac{r^{2/3}}{10}) \).
- Let \( \Lambda = \mathcal{P}(x, y - 2Mr^{2/3}, \frac{4r}{5}, 3Mr^{2/3}) \).
Let
\[ B_1^* = \mathcal{P}(x - \frac{9r}{20}, y - Lr^{2/3}, \frac{r}{10}(L + M)r^{2/3}) \]
and
\[ B_2^* = \mathcal{P}(x + \frac{9r}{20}, y - Lr^{2/3}, \frac{r}{10}(L + M)r^{2/3}). \]
We shall call \( B_1^*, B_2^* \) barriers of the butterfly \( \mathbb{B} \).

- Let \( F = \mathcal{P}(x, y - Lr^{2/3}, r, 0) \) be called the floor of the butterfly \( \mathbb{B} \) and let \( F^+ \) denote the region in \( \mathbb{B} \) above \( F \).

Different parts of the anatomy of the butterfly is illustrated in Figure 1. This and other figures we use in this paper are drawn in the tilted coordinate \((x', y') = (x, y - x)\) in which the parallelograms one one pair of sides parallel to the line \( x = y \) and other pair of sides parallel to the \( y \)-axis (e.g. the parallelograms constituting a Butterfly) become rectangles with sides parallel to the axes. This is merely a convenience in drawing and does not have any other significance.

---

**Figure 1**: Anatomy of a butterfly \( \mathbb{B}(x, y, r) \)

### 3.2 Defining the event \( G_{x,y} \):

Now we are ready to define an event \( G_{x,y} \) for \( x \in \mathcal{X}_r \) and \( y \in r^{2/3}\mathbb{Z} \), which is one of the key events in our proof. We shall say \( G_{x,y} \) holds if a long list of conditions are satisfied. For convenience we have divided the conditions into 3 parts.
We shall use the following notation. For a region $U \subseteq \mathbb{R}^2$, we define $S(U) = S_\psi(U) \subseteq U^2$ as follows. For $u = (x, y) \in U$, $u' = (x', y') \in U$, $(u, u') \in S(U)$ iff $\frac{2}{\psi} < \frac{y' - y}{x' - x} \leq \frac{\psi}{2}$.

### 3.2.1 The local conditions: $G_{x,y}^{\text{loc}}$

We say $G_{x,y}^{\text{loc}}$ holds if the following conditions are satisfied.

1. Let $U = \mathcal{P}(x, y - Mr^2/3/10, r, 2Mr^2/3/10)$. For all $(u', u'') \in S(U)$ we have
   \[ |\tilde{X}_{u', u''}| \leq Cr^{1/3}. \tag{16} \]

2. We have $\forall (u, u') \in S(\mathcal{C})$
   \[ |\tilde{X}_{u, u'}| \leq CL^{1/2}r^{1/3}. \tag{17} \]

3. For all $u, u' \in S(\Lambda)$ we have
   \[ |\tilde{X}_{u, u'}| \leq C\tilde{r}^{1/3}. \tag{18} \]

   Also let $2D$ denote the dilation that doubles $D$ keeping the centre fixed. Then we have for all $u, u' \in S(\Lambda \setminus 2D)$, such that both of $u$ and $u'$ are not in $\{(x^*, y^*) : |x - x^*| \leq r/5\}$
   \[ |\tilde{X}_{u, u'}| \leq C\tilde{r}^{1/3}. \tag{19} \]

4. We have $\forall u \in \mathcal{P}(x, y - 2Mr^2/3, \frac{r}{5}, 2Mr^2/3)$, $u' \in B_1^* \cap \mathcal{C}$ (resp. $u' \in B_2^* \cap \mathcal{C}$) (i.e., $u'$ is in the boundary between the barriers and the central column)
   \[ \tilde{X}_{u', u} \leq C\tilde{r}^{1/3} \text{ (resp. } \tilde{X}_{u, u'} \leq C\tilde{r}^{1/3} \). \tag{20} \]

5. We have $\forall u, u' \in F$
   \[ |\tilde{X}_{u, u'}| \leq C\tilde{r}^{1/3}. \tag{21} \]

### 3.2.2 The Regularity condition: $G_{x,y}^{\text{reg}}$

The Regularity conditions consist of the following two conditions.

1. **Area condition** $G_{x,y}^{\text{a}}$: Consider the parallelogram $U = \mathcal{P}(x, y - \frac{3Mr^2/3}{2}, \frac{4r}{5}, Mr^2/3)$. We say that $G_{x,y}^{\text{a}}$ holds if for all paths $\gamma$ from $u = (x, y)$ to $u' = (x', y')$ with $u < u' \in U$, $|x - x'| \geq \alpha' r$ with $\alpha' r \leq \ell_{\Pi}^{\Pi} \leq 3r$, we have
   \[ A_{\gamma}^{\Pi} \geq \alpha' \eta r. \tag{22} \]

2. **No hole condition**: There does not exist a square of side length at least $r^{2/3}$ contained in $\Lambda$ which does not contain any point of $\Pi$. 

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3.2.3 Resampling Condition: $G_{x,y}^{rs}$

Taking equally spaced line segments parallel to its sides we divide the parallelogram $D$ into a grid of $\frac{1}{100}$ many parallelograms of size $\varepsilon r \times (\varepsilon r)^{2/3}$ each. We denote these by $D_1, D_2, \ldots$ so $D = \bigcup_i D_i$. Let $u_i = (x_i, y_i)$ denote the bottom left corner of $D_i$. Define the parallelogram $\tilde{D}_i$ whose corners are $(x_i, y_i - \varepsilon^{2/3} r^{2/3})$, $(x_i + \varepsilon r, y_i - \varepsilon^{2/3} r^{2/3} + \varepsilon r)$, $(x_i, y_i + 2\varepsilon^{2/3} r^{2/3})$ and $(x_i + \varepsilon r, y_i + 2\varepsilon^{2/3} r^{2/3} + \varepsilon r)$.

Now let $\Pi^*$ be another i.i.d. copy of $\Pi$. Let $\Pi^{(*,i)}$ denote the point process obtained by replacing the the point configuration of $\bigcup_{i=1}^h D_i$ in $\Pi$ by the corresponding point configuration in $\Pi^*$. From now on, whenever we have a statistics of a point configuration with a superscript $(i)$, this will denote the statistic for the point configuration $\Pi^{(*,i)}$.

Let

$$\Delta_i = \sup_{u, u' \in S(\tilde{D}_i)} |X_{u,u'}^{(i)} - X_{u,u'}^{(\tilde{i})}|.$$

We say $G_{x,y}^{rs}$ holds if the following condition is satisfied.

$$\mathbb{P} \left[ \max_i \Delta_i \leq \frac{\delta r^{1/3}}{2} \bigg| \Pi \right] \geq 1 - e^{-C/\varepsilon^{1/4}}. \quad (23)$$

Notice that all the above conditions can be checked by looking at the point configuration in $[x-r/2, x+r/2] \times \mathbb{R}$. i.e., these events will be independent for different values of $x$.

3.2.4 The Wing condition: $G_{x,y}^w$

We say $G_{x,y}^w$ holds if the following condition is satisfied. We have $\forall u, u' \in S(W^i)$ for $i = 1, 2$

$$|\tilde{X}_{u,u'}| \leq CL^{3/4} r^{1/3}. \quad (24)$$

Finally we define

$$G_{x,y} = G_{x,y}^{loc} \cap G_{x,y}^w \cap G_{x,y}^{rs} \cap G_{x,y}^{\text{reg}}.$$

3.3 Defining the event $G_x$:

Let $\Gamma$ be the topmost maximal path in $\Pi$ from $0 = (0,0)$ to $n = (n,n)$. For $x \in \mathcal{X}$, we define the event $G_x$ as follows. Let $y^\ast = y(x, \Gamma) = \inf_y \{y \in r^{2/3} \mathbb{Z} : x + y \geq \Gamma_x\}$. We shall denote $\mathbb{B}(x, r) = \mathbb{B}(x, y^\ast, r)$. Also let

$$B_1 = \{(x', y') \in B^\ast_1 : x - r/2 \leq x' \leq x - 2r/5, x' + y^\ast - Lr^{2/3} \leq y' < \Gamma_{x'}\};$$

$$B_2 = \{(x', y') \in B^\ast_2 : x + r/2 \geq x' \geq x + 2r/5, x' + y^\ast - Lr^{2/3} \leq y' < \Gamma_{x'}\};$$

For $x \in \mathcal{X}$, $i = 1, 2$, we shall call $B_i = B_i(x, r)$ walls in the column $x$. Also for an increasing path $\gamma$ from $(0, 0)$ to $(n,n)$, $B_i(x, \gamma, r)$ will be defined similarly, replacing $\Gamma$ by $\gamma$.

We say that $G_x$ holds if all of following conditions are satisfied.

- **The local conditions**: $G_{x}^{loc}$: We say that $G_{x}^{loc}$ holds if $G_{x,y}^{loc}$ holds with the following two modifications.
Instead of condition 1 in the definition of $G^\text{loc}_{x,y}$ above we have that for all $u' = (x', y'), u'' = (x'', y'') \in \Gamma$ with $x', x'' \in [x - r/2, x + r/2]$ and $|x' - x''| \geq r^{3/4}$ we have
\[ |\tilde{X}_{u', u''}| \leq C r^{1/3}. \] (25)

We replace $B_i^+(x, y^*, r)$ and $B_i^+(x, y^*, r)$ in condition 4 by $B_1(x, r)$ and $B_2(x, r)$ respectively.

- **The regularity condition** $G^\text{reg}_x := G^\text{reg}_{x,y^*}$.
- **The Wing condition** $G^w_x := G^w_{x,y^*}$.
- **The resampling condition** $G^\text{rs}_x := G^\text{rs}_{x,y^*}$.
- **The fluctuation condition:** $G^f_x$: We say $G^f_x$ holds if the following conditions are satisfied.

1. We have
   
   (a) $|\Gamma_x - x| \leq C n^{2/3}$, $|\Gamma_n - n/3| \leq C n^{2/3}$, $|\Gamma_{2n/3} - 2n/3| \leq C n^{2/3}$.
   
   (b) $\ell_r \geq 2n - C n^{1/3}$.
   
   (c) Let $L'_1$ (resp. $L'_2$) denote the line segment joining $(n/3, -3C n^{2/3})$ to $(n/3, 3C n^{2/3})$ (resp. the line segment joining $(2n/3, -3C n^{2/3})$ to $(2n/3, 3C n^{2/3})$). Then
   
   \[ \sup_{u \in L'_1} |\tilde{X}_{0,u}| \leq C^3 n^{3/2}, \sup_{u \in L'_2} |\tilde{X}_{u', u}| \leq C^3 n^{3/2}. \]

2. We have for $(x', y') \in \Gamma$

\[ \frac{|(y' - y) - (x' - x)|}{r^{2/3}} \leq \begin{cases} M/10 & \text{if } \frac{|x' - x|}{r} \leq 1, \\ L^{1/10} & \text{if } |x' - x| = (1/2 + L^{3/2})r. \end{cases} \] (26)

### 3.4 Defining $R_{x,\gamma}, R_{x,y}$ and $R_x$:

Let $\gamma$ be an increasing path from 0 to $n$. For $x \in \mathcal{X}_r$, define $y(x, \gamma) = \inf\{y' \in r^{2/3}\mathbb{Z} : x + y' \geq \gamma_x\}$. Set $B_i = B_i(x, \gamma_r, r)$. Also let $\partial^+(B_i)$ denote the union of $B_i \cap \mathcal{E}$ and the bottom boundary of $B_i$. We define $\partial^+ B_i^*$ similarly. We say $R_{x,\gamma}$ holds if the following conditions are satisfied.

- We have $\forall u = (x', y') \in B_1$ (resp. $B_2$) with $y' \geq x' + y(x, \gamma) - M r^{2/3}$ and $\forall u' \in \partial^+ B_1$ (resp. $B_2$)

\[ \tilde{X}_{u, u'}^{B^*_1(\cup \gamma)} \leq C r^{1/3} \] (resp. \[ \tilde{X}_{u, u'}^{B^*_2(\cup \gamma)} \leq C r^{1/3} \]. \] (27)

- We have $\forall u \in B_1 \cap W^1$ (resp. $\forall u \in B_2 \cap \mathcal{E}$) and $\forall u' \in B_1 \cap \mathcal{E}$ (resp. $\forall u' \in B_2 \cap W^2$)

\[ \tilde{X}_{u, u'}^{B^*_1(\cup \gamma)} \leq -C^* r^{1/3} \] (resp. \[ \tilde{X}_{u, u'}^{B^*_2(\cup \gamma)} \leq -C^* r^{1/3} \]. \] (28)

This condition means that any path that crosses the walls from left to right are much shorter than typical paths. Recall that $C^*$ is chosen sufficiently large depending on other parameters.

- For $x \in \mathcal{X}_r$ and $y \in r^{2/3}\mathbb{Z}$, let $R_{x,y}$ denote the event such that [27] and [28] holds in $\mathbb{B}(x, y, r)$ with $B_i$ replaced by $B_i^*(x, y, r)$.

- We define $R_x := R_{x,\Gamma}$ where $\Gamma$ is the topmost maximal path from 0 to $n$ in $\Pi$. 


3.5 Defining $H_{x,\gamma}$, $H_{x,y}$ and $H_x$:

Let $\gamma$ be an increasing path $\gamma$ from 0 to $n$. For $x \in \mathcal{X}_r$, define $y(x, \gamma)$ as before. We say $H_{x,\gamma}$ holds if the following conditions are satisfied in the butterfly $\mathbb{B}(x, y(x, \gamma), r)$.

- For all $u, u' \in F$, we have
  \[ \Lambda \tilde{X}_{u,u'} \leq -L r^{1/3}. \] (29)

- For all $u \in F, u' \in P(x, y - 2Mr^2/3, r, 3Mr^2/3)$ and $u'$ below $\gamma$ we have
  \[ \tilde{X}_{u,u'} \leq -L r^{1/3} \text{ or } \tilde{X}_{u',u} \leq -L r^{1/3} \] (30)
  depending on whether $u < u'$ or $u' < u$.

- Let $H_{x,y}$ denote the event such that in $\mathbb{B}(x, y, r)$ (29) holds without the requirement of avoiding $\gamma$ and (30) holds without the requirement of avoiding $\gamma$ or the requirement $u' \in \gamma$.

- We define $H_x := H_{x,\Gamma}$ where $\Gamma$ is the topmost maximal path from 0 to $n$ in $\Pi$.

3.6 Defining $Q_x$

Definition 3.1 (Steepness Condition) An increasing path $\gamma$ from 0 to $n$ is called steep if there exists $\frac{n}{10} < x_1 < x_2 < \frac{9n}{10}$ such that $(x_2 - x_1) \vee (\gamma x_2 - \gamma x_1) \geq \frac{n^{2/3}}{2 \log n}$ and $\frac{\gamma x_2 - \gamma x_1}{x_2 - x_1} \notin \left(\frac{20}{\psi}, \frac{\psi}{20}\right)$.

For a point configuration $\Pi$ we say that steepness condition holds if for every steep $\gamma$ from 0 to $n$ we have $\ell_\gamma \leq 2n - n^{2/5}$.

We say that the event $Q_x$ holds if the following conditions hold.

1. Steepness condition holds.

2. Let $\mathcal{G}_i$ denote the event that steepness condition holds for the point configuration $\Pi^{(i,0)}$. Then we have
  \[ \mathbb{P}[\bigcup_i \mathcal{G}_i^c \mid \Pi] \leq e^{-n^{1/100}}. \]

3.7 Conditioning on $R_x$

We want to show that for a fixed $r$, for a large fraction of $x \in \mathcal{X}_r$, $G_x \cap H_x \cap Q_x \cap R_x$ hold probability bounded away from 0 uniformly in $r$. It turns out that each of $G_x$, $H_x$ and $Q_x$ holds with probability close to 1, however $R_x$ only holds with a small probability (bounded away from 0). For this reason, in many of our probabilistic estimates we shall need to condition on $R_x$ for $x \in \mathcal{X}_r$ and deal with the conditional probability measures. For the sake of clarity we shall use the measure $\mu$ for the measure on configurations $\Pi$ distributed according to a homogeneous Poisson process of rate 1. The generic notation $\mathbb{P}$ will also refer to this measure unless specified otherwise.

The following theorem gives a lower bound on the probability of $R_x$. 

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Theorem 3.2 Let \( \gamma \) be an increasing path from \( 0 \) to \( n \). Let \( r \in \mathbb{R} \) be fixed. For \( x \in \mathcal{X}_r \) let \( A_\gamma^x := \mathbb{R}^2 \setminus (B_1(x, \gamma, r) \cup B_2(x, \gamma, r)) \). Let \( \Pi_A \) denote the point configuration \( \Pi \) restricted to \( A \). Then we have
\[
\mu(R_{x, \gamma} \mid \Pi_A^x, \Gamma = \gamma) \geq \mu(R_{x, \gamma}) \geq \min_{y \in r^2/3} \mu(R_{x, y}) \geq \beta > 0.
\]

Definition 3.3 (Conditional measure) Define the measure \( \mu_x^* \) on configurations in \( \mathbb{R}^2 \) by conditioning on the configuration in the walls of column \( x \) such that \( R_x \) holds, i.e. for an increasing path \( \gamma \) from \( 0 \) to \( n \) if \( A = A_\gamma^x = B_1(x, \gamma, r) \cup B_2(x, \gamma, r) \) then we have
\[
\mu_x^*(\Gamma = \gamma, \Pi_A, \Pi_B) := \mu_x(\Gamma = \gamma, \Pi_A) \mu_x(\Pi_B \mid \Gamma = \gamma, \Pi_A, R_x).
\]

We record the basic properties of \( \mu_x^* \) in the following lemma.

Lemma 3.4 The measure \( \mu_x^* \) satisfies the following two properties:

(i) We have \( \mu_x^* \preceq \mu \) where \( \preceq \) denotes stochastic domination.

(ii) We have
\[
\frac{d\mu_x^*}{d\mu} \leq \max_y \frac{1}{\mu(R_{x,y})} \leq \frac{1}{\beta}.
\] 

Proof. Notice that (i) follows from FKG inequality and it is clear from definition that the first inequality in (31) holds, the second inequality follow from Theorem 3.2.

Finally we have the following theorem.

Theorem 3.5 There exist \( \mathcal{X}_r^* \subseteq \mathcal{X}_r \) with \( |\mathcal{X}_r^*| \geq \frac{9}{10} |\mathcal{X}_r| \) such that for all \( x \in \mathcal{X}_r^* \) we have
\[
\mu_x^*(G_x \cap H_x \cap Q_x) \geq \frac{9}{10}.
\]

We shall prove Theorem 3.2 and Theorem 3.3 over §6, §7 and §8. Before that we show how using these two theorems we can prove Theorem 1.

4 Resampling in \( \mathcal{D} \): Getting an almost optimal alternative path

Let \( \Gamma \) be the topmost maximal path in \( \Pi \) from \( 0 \) to \( n \). The aim of this section is to prove for \( x \in \mathcal{X}_r \) such that \( G_x \cap H_x \cap Q_x \cap R_x \) holds, with probability bounded away from 0 independent of \( r \), there exists a sufficiently regularly behaving alternative path, which deviates from \( \Gamma \) only in \( (x - \frac{r}{2}, x + \frac{r}{2}) \) and is shorter than \( \Gamma \) by at most an amount of \( \delta r^{1/3} \), where \( \delta \) is a small constant depending on \( \lambda \). This is illustrated in Figure 2.

The strategy for showing the above is as follows. Consider the butterfly \( \mathbb{B}(x, r) = \mathbb{B}(x, y(x, \Gamma), r) \). Resample the rectangles \( D_i \) in \( \mathcal{D} = \mathcal{D}(\mathbb{B}(x, r)) \) one by one, conditioned on \( \Gamma \) and also the configuration outside \( D_i \). Since this process is reversible it gives us a way to estimate the probability of such a configuration. We shall show that by the end of this process with positive probability we get an alternative path satisfying the conditions, to be made precise later.

Before proving that our job is to ensure that the alternative path we get by the above procedure satisfies the required regularity conditions.
4.1 The alternative path deviates locally

Fix $r$ and $x \in X_r$. We first prove that if $G_x \cap H_x \cap Q_x \cap R_x$ holds, then any alternative path which passes through $D$ will deviate from $\Gamma$ only in the interval $[x - \frac{r}{2}, x + \frac{r}{2}]$.

Let $\gamma$ be another increasing path from 0 to $n$ such that $\gamma$ passes through $D = D(\mathbb{B}(x, r))$. Let $D$-entry of $\gamma$ be the point $u_1 = (x_1, y_1)$ where $\gamma$ intersects $D$ first, i.e., for each $x < x_1$, we have $(x, \gamma(x)) \notin D$. Similarly let $D$-exit of $\gamma$ be the point $u_2 = (x_2, y_2)$ where $\gamma$ intersects $D$ last. We define the split of $\gamma$ to be the point $u_3 = \{x' : \gamma(x') = \Gamma(x')\}$. Similarly the confluence of $\gamma$ is defined to be the point $u_4 = \{x_4 = \inf_{x' > x_2} \{x' : \gamma(x') = \Gamma(x')\}$. It will suffice to consider the paths that deviate from $\Gamma$ only between the split and the confluence. We have the following lemma.

**Lemma 4.1** Let $\Gamma$ be the topmost maximal increasing path from 0 to $n$ in $\Pi$. Let $x \in X_r$. Let $\gamma$ be another increasing path from 0 to $n$ passing through $D = D(\mathbb{B}(x, r))$ with $D$-entry $u_1 = (x_1, y_1)$, $D$-exit $u_2 = (x_2, y_2)$, split $u_3 = (x_3, y_3)$ and confluence $u_4 = (x_4, y_4)$. Suppose $\Gamma = \gamma$ except on $(x_3, x_4)$. Also suppose either $x_3 < x - \frac{r}{2}$ or $x_4 > x + \frac{r}{2}$. The on $G_x \cap H_x \cap Q_x \cap R_x$, we have

$$\ell_\gamma \leq \ell_\Gamma - \delta \varepsilon^{-2} r^{1/3}.$$

**Proof.** First let us make some notations. Recall that $T$ denotes the body of the butterfly
We define the \textit{T-entry} of $\gamma$ to be the point $u_5 = (x_5, y_5)$ such that
\[ x_5 = \sup\{x' < x_3 : \forall x_3 > x'' \geq x', (x'', \gamma_{x''}) \in T\}. \]

On $G_x \cap H_x \cap R_x \cap Q_x$, depending on $T$-entries we can classify $\gamma$ into following three categories. \textbf{Enter with $\Gamma$}: if $x_5 < x_3$. \textbf{Enter through $F$}: if $u_5 \in F$. \textbf{Enter through wall}: if $u_5$ is on the left boundary of $B_1$. Similarly we define the $T$-exit $u_6 = (x_6, y_6)$ of $\gamma$ and classify $\gamma$ as \textbf{exit with $\Gamma$}, \textbf{exit through $F$} and \textbf{exit through wall}.

The proof of the lemma is based on analysis of a few cases.

\textbf{Case 1. Enter with $\Gamma$}: We shall need to consider two subcases.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3.png}
\caption{Case 1.1: $\gamma$ denotes the path that does not deviate locally, $\gamma'$ is the path it is compared with.}
\end{figure}

\textbf{Case 1.1. Exit through $F$}: Let $u_7 = (x_7, y_7)$ be the point on $\Gamma$ such that $x_7 = x - \left(\frac{1}{2} + L^{3/2}\right)r$ and $u_8 = (x_8, y_8)$ be the point on $F$ with $x_8 = x - \frac{r}{2}$.

Notice that it suffices to prove that
\[ \hat{X}_{u_7,u_5} + \hat{X}_{u_5,u_2} + \hat{X}_{u_2,u_6} \leq \hat{X}_{u_7,u_8} + \hat{X}_{u_8,u_6} - \delta \varepsilon^{-2} r^{1/3}. \] (32)

See Figure 3. Observe that on $G_x \cap H_x \cap Q_x \cap R_x$ we have
\[ \hat{X}_{u_7,u_5} \leq 2CL^{3/4} r^{1/3}; \hat{X}_{u_7,u_8} \geq -4CL^{3/4} r^{1/3}; \]
\[ \hat{X}_{u_5,u_2} \leq 4Cr^{1/3}; \hat{X}_{u_2,u_6} \leq -Lr^{1/3}; \hat{X}_{u_8,u_6} \geq -2Cr^{1/3}. \]

It follows that (32) holds since $L$ is sufficiently large (recall that $L$ was chosen sufficiently large depending on $\varepsilon$).
Case 1.2. Exit through wall: In this case, let $u_9 = (x + \frac{2r}{5}, \gamma x + 2r/5)$, $u_{10} = (x_{10}, y_{10}) = (x + \frac{r}{2}, y_{10}) \in F$. Let $u_{11} = (x_{11}, y_{11})$ be the point where $\gamma$ last exits $W^2$. Observe that $u_9, u_6 \in B_2$. Also observe that if either $(u_6, u_{11}) \notin S(W^2)$ or $(u_{10}, u_{11}) \notin S(W^2)$, then $\gamma$ is steep and we are done by definition of $Q_x$. Hence assume otherwise. It suffices to show that

$$\hat{X}_{u_7,u_5} + \hat{X}_{u_5,u_9} + \hat{X}_{u_9,u_6} + \hat{X}_{u_6,u_{11}} \leq \hat{X}_{u_7,u_8} + \hat{X}_{u_8,u_{10}} + \hat{X}_{u_{10},u_{11}} - \delta \varepsilon^{-2} r^{1/3}. \quad (33)$$

Observe that on $G_x \cap H_x \cap Q_x \cap R_x$ we have

$$\hat{X}_{u_9,u_6} \leq -\frac{C^*}{2} r^{1/3}; \hat{X}_{u_10,u_{11}} \geq -4CL^{3/4} r^{1/3}; \hat{X}_{u_6,u_{11}} \leq 4CL^{3/4} r^{1/3}.$$  

Using these and arguments similar to Case 1.1. we see that (33) holds.

Case 2. Enter through $F$: We need to consider three subcases.

Case 2.1. Exit with $\Gamma$: This case is similar to Case 1.1 and we omit the details.

Case 2.2. Exit through wall: Define points $u_9, u_{10}, u_{11}$ as in Case 1.2. Clearly it suffices to show

$$\hat{X}_{u_5,u_9} + \hat{X}_{u_9,u_6} + \hat{X}_{u_6,u_{11}} \leq \hat{X}_{u_5,u_{10}} + \hat{X}_{u_{10},u_{11}} - \delta \varepsilon^{-2} r^{1/3}.$$  

This is proved in a similar manner to Case 1.2 and we omit the details.

![Figure 4: Case 2.3](https://example.com/figure4.png)

Figure 4: Case 2.3: $\gamma$ denotes the path that enters and exits through $F$, it is compared with $\gamma'$

Case 2.3. Exit through $F$: In this case it suffices to show

$$\hat{X}_{u_5,u_6} \leq \hat{X}_{u_5,u_6} - \delta \varepsilon^{-2} r^{1/3}$$
Figure 5: Case 2.3: $\gamma$ denotes the path that enters and exits through the wall, compared with $\gamma'$

which follows from the definition of $G_x$ and $H_x$ since $L$ is sufficiently large, see Figure 4.

**Case 3.** Enter through wall: Again we need to consider three subcases.

**Case 3.1.** Exit with $\Gamma$: This case is similar to Case 1.2, we omit the details.

**Case 3.2.** Exit through wall: Let $u_{12}$ denote the point where $\gamma$ first enters $W^1$. Let $u_{13} = (x_{13}, y_{13})$ be the point on $\gamma$ such that $x_{13} = x - \frac{2r}{5}$. Clearly $u_{13} \in B_1$ and also without loss of generality we can assume $(u_{12}, u_5), (u_{12}, u_8) \in S(W^1)$, see Figure 5. Clearly it suffices to show that

$$\dot{X}_{u_{12}, u_5} + \dot{X}_{u_5, u_{13}} + \dot{X}_{u_{13}, u_9} + \dot{X}_{u_9, u_{0}} + \dot{X}_{u_0, u_{11}} \leq \dot{X}_{u_{12}, u_8} + \dot{X}^{F+}_{u_8, u_{10}} + \dot{X}_{u_{10}, u_{11}} - \delta \varepsilon^{-2}.$$  

The proof can now be completed as in Case 1.2.

**Case 3.3.** Exit through $F$: This case is analogous to Case 2.2.

**4.2 The alternate path is not too steep**

The following lemma ensures that an alternative path through $\mathcal{B}(x, r)$ spends sufficiently long in in the region $(x + y(x, \Gamma) - \frac{3M}{2}r^{2/3}, x + y(x, \Gamma) - \frac{M}{2}r^{2/3})$.

**Lemma 4.2** Let $\Gamma$ be the topmost maximal increasing path from 0 to n in $\Pi$. Let $x \in X_r$. Let $\partial^+(2D)$ denote the union of top, left and bottom boundary of $2D$ in the butterfly $\mathcal{B}(x, r) = \mathcal{B}(x, y(x, \Gamma), r)$. Fix a point $u_0 = (x_0, y_0) \in \partial^+(2D)$. Let $\gamma$ be the path in $\Pi$ from 0 to $u$ of maximal length subject to the conditions

1. $\gamma$ does not intersect $D$, 


2. \( \{ x' : \Gamma x' \neq \gamma x' \} \subseteq [x - \frac{x}{2}, x_0] \).

Let \( u_2 = (x_2, y_2) \in \gamma \) be such that

\[
x_2 = \inf \left\{ x' : \gamma x'' \in \left[ x'' + y(x, \Gamma) - \frac{3M}{2} r^{2/3}, x'' + y(x, \Gamma) - \frac{M}{2} r^{2/3} \right] \forall x'' \in [x', x_0] \right\}.
\]

Then on the event \( G_x \cap H_x \cap Q_x \cap R_x \), we have that \( x_0 - x_2 \geq \alpha' r \).

**Proof.** We prove by contradiction. Let \( \gamma \) be a path given by the hypothesis of the Lemma. Suppose \( x_2 > x_0 - \alpha' r \). We shall prove that on \( G_x \cap H_x \cap Q_x \cap R_x \), there exists a path \( \gamma' \) satisfying the two conditions given in the lemma such that \( \ell_{\gamma'} > \ell_{\gamma} \).

Observe that without loss of generality we can assume that there exists \( u_1 = (x_1, y_1) \in \Gamma \) such that \( \Gamma = \gamma \) on \([0, x_1]\) and \( \Gamma_{x'} > \gamma_{x'} \) on \((x_1, x_0]\) with \( x - \frac{x}{2} < x_1 < x_2 \).

**Case 1:** \( x_1 \geq x - \frac{2r}{5} \). Set the point \( u_4 = (x_4, y_4) = (x - \frac{2r}{5}, \Gamma_{x - 2r/5}) \). It suffices to prove that \( \hat{X}_{u_4, u_0} \geq \hat{X}_{u_4, u_2} + \hat{X}_{u_2, u_0} \), which will contradict the maximality of \( \gamma \). This is what we prove next.

Notice that on \( G_x \) we have

\[
\hat{X}_{u_4, u_0} \geq -2Cr^{1/3}; \quad \hat{X}_{u_4, u_2} \leq 2Cr^{1/3}; \quad \hat{X}_{u_2, u_0} \leq -5Cr^{1/3}.
\]

To prove the third inequality, define \( u_3 = (x_0 - 2ar, x_0 - 2ar + y(x, \Gamma) - \frac{M}{2} r^{2/3}) \), use \( \hat{X}_{u_2, u_0} \leq \hat{X}_{u_3, u_0} - \hat{X}_{u_3, u_2} \). Notice that on \( G_x \), \( \hat{X}_{u_3, u_2} \geq -2Cr^{1/3} \) and \( \hat{X}_{u_3, u_0} \leq -10Cr^{1/3} \) since \( \alpha \) is sufficiently small using Lemma 9.2. This completes the proof in this case.

---

**Figure 6:** Case 1.2: \( \gamma' \) is a better path than \( \gamma \)
Case 1.2: \( x_1 < x - \frac{2r}{3} \). Let \( u_5 = (x_5, y_5) \in \gamma \) be the first point on \( \gamma \) which intersects the boundary of \( B_1 \), i.e., \( x_5 = \inf_{x' > x_1: (x', \gamma_x)} \in \partial B_1 \). Also let \( u_6 = (x - \frac{r}{3}, \Gamma_{x - \frac{r}{3}}) \), see Figure 6. As before it suffices to prove,
\[
\hat{X}_{u_1,u_6} + \hat{X}_{u_5,u_2} + \hat{X}_{u_2,u_0} \leq \hat{X}_{u_1,u_6} + \hat{X}_{u_6,u_0}.
\]
Notice that on \( G_x \) we have
\[
\hat{X}_{u_1,u_6} \geq -2Cr^{1/3}; \quad \hat{X}_{u_6,u_0} \geq -Cr^{1/3}.
\]
Also notice that as before on \( G_x \cap H_x \cap R_x \) we further have
\[
\hat{X}_{u_1,u_5} \leq Cr^{1/3}, \quad \hat{X}_{u_5,u_2} \leq Cr^{1/3}, \quad \hat{X}_{u_2,u_0} \leq -10Cr^{1/3}.
\]
In this case also we have a contradiction.

Case 2: \( y_2 = x_2 + y(x, \Gamma) - \frac{3}{2} \mu \gamma^{2/3} \). This can be dealt with in the same manner as above and we omit the details. \( \Box \)

4.3 Sequential Resampling

Recall our strategy of resampling to get a better path. As always, let \( \Gamma \) denote the topmost maximal path from \( 0 \) to \( n \) in \( \Pi \). For \( r \in R \), fix \( x \in X_r \) and consider the parallelogram \( D \) in the butterfly \( B(x, y(x, \Gamma), r) \). Our first lemma states that on resampling the configuration on \( D \) the length of the longest path increases with a chance bounded away from 0.

Lemma 4.3 Let \( \hat{\Pi} \) be the point configuration on \( \mathbb{R}^2 \) where \( \Pi |_D \) is replaced by \( \Pi^* |_D \). Let \( \Gamma' \) denote a longest increasing path in \( \hat{\Pi} \). Then
\[
\mathbb{P}[\ell_{\Gamma'} > \ell_{\Gamma} | \Pi, G_x, H_x] \geq \rho.
\]

Proof. Let \( u_1 \) and \( u_2 \) be the midpoint of the left boundary and the right boundary of \( D \) respectively. For \( u < u' \), let \( Y_{u,u'} \) (resp. \( \hat{Y}_{u,u'} \) etc.) denote the length of the longest increasing path from \( u \) to \( u' \) (resp. the other corresponding statistics) in the environment \( \Pi^* \). It follows from the definition of \( G_x \) and \( H_x \) and that it suffices to prove
\[
\hat{Y}^{D}_{u_1,u_2} \geq 10Cr^{1/3}.
\]
This can be established by using Theorem 1.1 arguing along the lines of the proof of Lemma 9.21 and we omit the details. \( \Box \)

Observe that this is not directly useful for us since we shall need to resample conditioning on the fact that \( \Gamma \) is the topmost maximal path. To this end we use the parallelograms \( D_i \) described in the definition of condition \( G_{x,y}^{rs} \).

For \( i = 1, 2, \ldots, \frac{1}{1000r^{5/7}} \), we define the measure \( \mu_{x,y}^{*,i} \) on point configurations on \( \mathbb{R}^2 \) inductively as follows. Let \( \mu_{x,y}^{*,0} = \mu_{x,y}^{*} \). For now, let \( \Pi \) denote a point configuration in \( \mathbb{R}^2 \) sampled from \( \mu_{x,y}^{*} \). Let \( \Gamma \) denote the topmost maximal path in \( \Pi \). Consider the parallelograms \( D_i \) in the butterfly \( B(x, r) \). For \( i \geq 0 \) sample a point configuration \( \hat{\Pi}^{*,i} \) sampled recursively as follows. Let \( \hat{\Pi}^{*,0} = \Pi \). Given \( \hat{\Pi}^{*,i-1} \), obtain \( \hat{\Pi}^{*,i} \) by resampling the point configuration on \( D_i \) with law \( \hat{\mu}_{D_i} := \mu_{D_i} (\cdot | \hat{\Pi}^{*,i-1}, \mathcal{H}_{x,i}) \) where \( \mu_{D_i} \) is the measure \( \mu \) restricted on \( D_i \) and \( \mathcal{H}_{x,i} \) denotes the event that \( \Gamma \) is the topmost maximal path in the new point configuration (after resampling \( D_i \)). Let \( \mu_{x,y}^{*,i} \) be the measure on the point configuration \( \Pi^{*,i} \) in \( \mathbb{R}^2 \) obtained as described above.
Lemma 4.4 Let \(x \in \mathcal{X}_r\) be fixed. Let \(\Pi\) and \(\tilde{\Pi}^{*,i}\) be defined as above. For \(i \geq 0\), let \(\mathcal{F}_{x,i}\) be the event that there is an increasing path \(\Gamma'\) from 0 to \(n\) in \(\tilde{\Pi}^{*,i}\) passing through \(D\) such that \(\ell_{\Gamma'} \geq \ell_{\Gamma} - \delta r^{1/3}\). Let \(\tau = \inf_{i \geq 0} \{\mathcal{F}_{x,i} \text{ holds}\}\). Then we have

\[
P[\tau \leq \frac{1}{100e^{5/3}} | \Pi, G_x, H_x, Q_x] \geq \frac{\rho}{2} > 0.
\]

**Proof.** For \(i \geq 1\), we sample \(\tilde{\Pi}^{*,i}\) in the following manner. Let \(\Pi^*\) be an independent sample from \(\mu\). For each \(i \geq 1\), we take the configuration \(\Pi^{*,i}_{D_i}\) (whose law is \(\mu_{D_i}\)). If replacing \(\Pi^{*,i}_{D_i}\) by \(\Pi^*_{D_i}\) does not violate the condition \(\mathcal{H}_{x,i}\), we take \(\Pi^*_{D_i}\) as a realization of \(\tilde{\mu}_{D_i}\), and use it to resample \(D_i\) and generate \(\Pi^{*,i}\), else generate a configuration according to \(\tilde{\mu}_{D_i}\) using some external randomness. Let \(\tau'\) be the first time this coupling fails.

Notice that on \(\{i < \tau \land \tau'\} \cap G_x \cap H_x \cap Q_x\), we have

\[
P[\tau' = i + 1] \leq e^{-\frac{C}{2x^{1/4}}}.
\]

To establish (35) observe the following. Let \(\tilde{\Pi}^{*,i}\) be the point configuration obtained from \(\tilde{\Pi}^{*,i-1}\) by replacing \(\tilde{\Pi}^{*,i-1}_{D_i}\) by \(\Pi^*_{D_i}\). Let \(\gamma\) be an increasing path from 0 to \(n\) passing through \(D\). Observe that either \(\gamma\) is steep or there are points \(u_1, u_2 \in \gamma\) such that \((u_1, u_2) \in S(D_i)\).

Now (35) follows from the resampling condition in \(G_x\) and the definition of \(Q_x\). It follows now from Lemma 4.3 that

\[
P[\tau \leq \frac{1}{100e^{5/3}}] \geq \rho - \varepsilon^{-2}e^{-\frac{C}{2x^{1/4}}} \geq \rho/2
\]

since \(\varepsilon\) is small enough. This completes the proof of the lemma. \(\square\)

4.4 Local Success

Let \(\Pi\) be a point configuration on \(\mathbb{R}^2\), not necessarily distributed according to \(\mu\). Let \(\Gamma\) be the topmost maximal increasing path in \(\Pi\) from 0 to \(n\). Fix \(r \in \mathcal{R}\). For \(x \in \mathcal{X}_r\) we define the event \(\textbf{"Success at } x \textbf{ in scale } r \textbf{ at cost } \delta^n\), denoted \(\mathcal{S}_{x,r,\delta}\) to be the event that the following conditions hold.

1. We have
   \begin{enumerate}
   \item \(|\Gamma'_{x'} - \Gamma_x| \leq \frac{Mr^{2/3}}{10}\) for all \(x' \in [x - \frac{r}{2}, x + \frac{r}{2}]\).
   \item \(|\Gamma_x - x| \leq Cn^{2/3}, |\Gamma_{n/3} - \frac{n}{3}| \leq Cn^{2/3}, |\Gamma_{2n/3}| \leq Cn^{2/3}.
   \item \(\sup_{u \in L'_1} |\hat{X}_{u,a}| \leq C^3n^{2/3}, \sup_{u' \in L'_2} |\hat{X}_{u',a}| \leq C^3n^{2/3}\) where \(L'_1\) and \(L'_2\) are as in the definition of \(G_x\).
   \end{enumerate}

2. There exists an increasing path \(\Gamma'\) in \(\Pi\) from 0 to \(n\) such that
   \begin{enumerate}
   \item \(\Gamma'\) passes through \(D\),
   \item \(\{x' : \Gamma'_{x'} \neq \Gamma_{x'}\} \subseteq [x - \frac{r}{2}, x + \frac{r}{2}]\),
   \item \(\ell_{\Gamma'} \geq \ell_{\Gamma} - \delta r^{1/3}\).
   \end{enumerate}
(d) There exists points \( u_1 = (x_1, \Gamma'_{x_1}) \) and \( u_2 = (x_2, \Gamma'_{x_2}) \) on \( \Gamma' \) such that \( x_1, x_2 \in [x - \frac{r}{2}, x + \frac{r}{2}] \) and \( O_{\Gamma'}(u_1, u_2) \) is contained in the region

\[
\{(x', y') \in \mathbb{R}^2 : (\Gamma_x - x) - \frac{2M}{5} x^{2/3} \leq y' - x' \leq (\Gamma_x - x) - \frac{8M}{5} x^{2/3}\}
\]

and \( A_{\Gamma'}(u_1, u_2) \geq \alpha' \eta r \).

Our goal is to prove the following theorem.

**Theorem 4.5** For \( \mathcal{X}^*_r \subseteq \mathcal{X}_r \) with \( |\mathcal{X}^*_r| \geq \frac{9}{10} |\mathcal{X}_r| \) given by Proposition 4.5 and for each \( x \in \mathcal{X}^*_r \), we have \( \mu(S_{x,r,\delta}) > p > 0 \) where \( p \) is a constant independent of \( r \).

First we prove the following lemma.

**Lemma 4.6** Let \( \mathcal{X}^*_r \) be as given by Proposition 4.5. For \( x \in \mathcal{X}^*_r \), there exists \( i \in \{0, 1, \ldots, \frac{1}{100 \varepsilon^{1/3}}\} \) such that

\[
\mu_x^*(S_{x,r,\delta}) \geq \frac{9 \varepsilon^2}{40}.
\]

**Proof.** Sample \( \Pi \) from the measure \( \mu_x^* \). Fix \( x \in \mathcal{X}_r \) such that \( G_x, H_x \) and \( Q_x \) holds. Consider the set-up of Lemma 4.4. It follows from Lemma 4.4 that there exists \( i \in \{0, 1, \ldots, \frac{1}{100 \varepsilon^{1/3}}\} \) such that \( \mathbb{P}[\tau = i \mid \Pi, G_x \cap H_x \cap Q_x] \geq \varepsilon^2 \rho / 2 \). Let \( \Pi^* \) be an independent sample of \( \mu \). Now generate a sample from \( \mu_x^* \) in the manner described in the proof of Lemma 4.4. Recall that \( \Pi^{(\ast,j)} \) denotes the point configuration obtained from \( \Pi \) by changing \( \Pi_{\cup_{j=1}^{i} D_j} \) to \( \Pi^*_{\cup_{j=1}^{i} D_j} \). Also recall the definition of \( \Delta_j \) from the resampling condition. Let \( A_x \) denote the event that \( \max_{j \leq i} \Delta_j \leq \frac{\delta^{1/3}}{2} \) and that Steepness condition holds for each \( \Pi^{(\ast,j)} \) for \( j \leq i \). It follows that

\[
\mathbb{P}[\tau' > i, \tau = i, A_x \mid \Pi, G_x \cap H_x \cap Q_x] \geq \frac{\varepsilon^2 \rho}{2} - \varepsilon^{-2} e^{-C \varepsilon^{1/3}} \geq \frac{\varepsilon^2 \rho}{4},
\]

since \( \varepsilon \) is sufficiently small. Now observe that on \( \{G_x \cap H_x \cap Q_x \cap A_x \cap \{\tau = i, \tau' > i\}\} \) there exists an increasing path \( \Gamma' \) in \( \Pi^{(\ast,i)} \) such that \( \Gamma' \) satisfies all the conditions in the definition of \( S_{x,r,\delta} \). To see this, observe that by construction \( \Pi \) and \( \Pi^{(\ast,i)} \) has the same topmost maximal path and hence Condition 1 holds by the definition of \( G_x \). Condition 2(a) and 2(c) holds by the definition of \( \tau \). That Condition 2(b) holds is a consequence of Lemma 4.1. Condition 2(d) is a consequence of Lemma 4.2 and the regularity conditions in the definition of \( G_x \). It follows that

\[
\mu_x^*(S_{x,r,\delta} \mid \Pi, G_x \cap H_x \cap Q_x) \geq \frac{\varepsilon^2 \rho}{4}.
\]

It follows from Proposition 3.5 that for \( x \in \mathcal{X}^*_r \), we have

\[
\mu_x^*(S_{x,r,\delta}) \geq \mu_x^*(G_x \cap H_x \cap Q_x) \frac{\varepsilon^2 \rho}{4} \geq \frac{9 \varepsilon^2 \rho}{40}.
\]

□

**Proof.** of Theorem 4.5 For \( x \in \mathcal{X}^*_r \), choose \( i \) as in the previous lemma. Notice that the resampling of \( D_i \) under the conditional measure is can be interpreted as step of the Glauber
dynamics and hence a smoothing operator which implies that $\sup \frac{d\mu^*_{x,i}}{d\mu}$ is decreasing in $j$, and hence

$$\sup \frac{d\mu^*_{x,i}}{d\mu} \leq \sup \frac{d\mu^*_{x}}{d\mu} \leq \frac{1}{\beta} < \infty$$

by Lemma 3.4 where $\beta$ is a constant independent of $r$. The result now follows from Lemma 4.6.

5 Global Success

Our goal in this section is to improve the length of the local almost optimal paths of the previous section using the extra points from the reinforced configuration, and then put together all these improvements to obtain a path longer than the optimal path in the unperturbed configuration.

5.1 Reinforcing on different lines

As explained in the introduction, our strategy is to consider, instead of only one reinforced configuration, a family of reinforced configurations, where the reinforcement is on different translates of the diagonal line $\{x = y\}$. Let $\lambda > 0$ be fixed. For each $m \in [-2Cn^{2/3}, 2Cn^{2/3}]$, let $\Sigma^{(m)}$ denote a one dimensional PPP with intensity $\lambda$ on the line $\mathbb{L}_m : \{y = x + m\}$. Let $\Pi^{(m)}$ be the point process obtained by superimposing $\Pi$ and $\Sigma^{(m)}$. Let $L_{(\lambda,m)}$ be the length of the maximal increasing path from $0$ to $n$ in $\Pi^{(m)}$. Hence to prove Theorem 1, it suffices to show that $\mathbb{E}L_{(\lambda,0)} > 2n$ for some $n$. It will be useful for us to consider a slightly weaker notion of maximal path in $\Pi^{(m)}$. For $m \in (-2Cn^{1/3}, 2Cn^{1/3})$, let $\Gamma^{(m)}$ denote the longest path in $\Pi^{(m)}$ from $(0,0)$ to $(n,n)$ which deviates from $\Gamma$ (the topmost maximal path in $\Pi$) only on $[n/3, 2n/3]$. We have the following lemma. For the next lemma we shall use the notation $\ell_{\Gamma^{(m)}} = \ell_{\Pi^{(m)}}$ and $\ell_{\Gamma} = \ell_{\Pi}$.

**Lemma 5.1** Let $L'_1$ and $L'_2$ be as in the definition of $G^f_r$. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ denote the following events.

$$\mathcal{E}_1 = \{ |\Gamma_{n/3}|, |\Gamma_{2n/3}| \leq Cn^{2/3} \};$$

$$\mathcal{E}_2 = \{ \sup_{u \in L'_1} |\tilde{X}_0,u| \leq C^3n^{1/3}, \sup_{u' \in L'_2} |\tilde{X}_{u',n}| \leq C^3n^{1/3} \}.$$

Let $\mathcal{E} = \mathcal{E}_1 \cap \mathcal{E}_2$. Then for some $n$ sufficiently large, there exists $m \in [-2Cn^{2/3}, 2Cn^{2/3}]$ such that we have $\mathbb{E}(\ell_{\Gamma^{(m)}} - \ell_{\Gamma})1_{\mathcal{E}} > 100C^3n^{1/3}$.

**Proof.** Let $r \in \mathcal{R}$ and $x \in \mathcal{X}_r$ be fixed. For a given $\Pi$ with the topmost maximal path $\Gamma$, if $S_{x,r,\delta}$ holds, let $\Gamma'(x,r,\delta)$ denote the alternative path given by the definition of $S_{x,r,\delta}$. Let $u_1$ and $u_2$ be as in the definition of $S_{x,r,\delta}$. Let $\Gamma^*(x,r,\delta,m)$ denote the increasing path in $\Pi^{(m)}$ which contains all points of $\Gamma'(x,r,\delta)$ that belong to $\Pi$ and also all points of $\Sigma^{(m)}$ that are contained in $O_{\Gamma^*}(u_1,u_2)$. Let us denote $\ell_{\Gamma^*} = \ell_{\Pi^{(m)}}$ and $\ell_{\Gamma} = \ell_{\Pi}$ $\Gamma^*(x,r,\delta)$. Also set

$$G^m_{x,r,\Gamma} = (\mathbb{E}[\ell_{\Gamma^*} | \Pi] - \ell_{\Gamma})1_{S_{x,r,\delta}}.$$
Now observe that on $S_{x,r,\delta}$, we have
\[
\int_{[(\Gamma_x-x)-\frac{8Mr^2/3}{5},(\Gamma_x-x)-\frac{2Mr^2/3}{5}]} (\mathbb{E}[\ell_{\Gamma} \mid \Pi] - \ell_{\Gamma'}) \, dm \geq \frac{\lambda}{10} A_{\Gamma}(u_1, u_2).
\]
It follows from the definition of $S_{x,r,\delta}$ and since $\delta$ is sufficiently small depending on $\lambda$ that
\[
\int_{[(\Gamma_x-x)-\frac{8Mr^2/3}{5},(\Gamma_x-x)-\frac{2Mr^2/3}{5}]} C_{x,r,\Gamma} \, dm \geq \left( \frac{a' \lambda r}{10} - \frac{6M \delta r}{5} \right) S_{x,r,\delta} \geq r c_{\lambda} S_{x,r,\delta}.
\]
Now notice that for a fixed $m$, and for a fixed $x \in \bigcup_{r \in \mathcal{R}} \mathcal{X}_r$, we have that
\[
1 \{ m \in (\Gamma_{x}-\frac{8Mr^2/3}{5},\Gamma_{x}-\frac{2Mr^2/3}{5}) \} = 1
\]
is nonzero for at most one value of $r \in \mathcal{R}$. Also notice that if
\[
1 \{ m \in (\Gamma_{x}-\frac{8Mr^2/3}{5},\Gamma_{x}-\frac{2Mr^2/3}{5}) \} = 1
\]
and $S_{x,r,\delta}$ holds for some value of $r \in \mathcal{R}$ and $x \in \mathcal{X}_r$, then for any $r_1 \in \mathcal{R}$ with $r_1 < r$, and for any $x' \in [x - \frac{r}{2}, x + \frac{r}{2}] \cap \mathcal{X}_{r_1}$, we have
\[
1 \{ m \in (\Gamma_{x'}-\frac{8Mr^2/3}{2},\Gamma_{x'}-\frac{4Mr^2/3}{2}) \} = 0.
\]
Finally notice that for a fixed $r$ and $x \in \mathcal{X}_r$, $\Gamma^*(x,r,\delta,m)$ and $\Gamma$ deviate from one another only in the interval $[x - \frac{r}{2}, x + \frac{r}{2}]$ on $S_{x,r,\delta}$. All these together imply
\[
(\mathbb{E}[\ell_{\Gamma(m)} \mid \Pi] - \ell_{\Gamma}) 1_{\mathcal{E}} \geq \sum_{r \in \mathcal{R}} \sum_{x \in \mathcal{X}_r} 1 \{ m \in (\Gamma_{x}-\frac{8Mr^2/3}{5},\Gamma_{x}-\frac{2Mr^2/3}{5}) \} C_{x,r,\Gamma}.
\]
By a series of interchanges of summation, expectation and integration and using Lemma 4.5 it follows that
\[
\int_{-2Cn^{2/3}}^{2Cn^{2/3}} \mathbb{E}[\ell_{\Gamma(m)} - \ell_{\Gamma}] 1_{\mathcal{E}} \, dm \geq \sum_{r} \sum_{x} r c_{\lambda} \mu(S_{x,r,\delta})
\geq \sum_{r} p c_{\lambda} \frac{n}{10r} \geq p c_{\lambda} |\mathcal{R}| n.
\]
Hence it follows that there exists $m \in [-2Cn^{2/3}, 2Cn^{2/3}]$ such that
\[
\mathbb{E}[\ell_{\Gamma(m)} - \ell_{\Gamma}] 1_{\mathcal{E}} \geq \frac{pc_{\lambda}}{4C} |\mathcal{R}| n^{1/3}.
\]
The lemma follows.

5.2 Proof of Theorem 1

Proof. [of Theorem 1] Take $m$ as given by the Lemma 5.1. For $u, u' \in \mathbb{R}^2$, let us denote the length of the longest path from $u$ to $u'$ in $\Pi_{(m)}^{(u)} \times \mathcal{X}_{(m)}^{(u',u)}$. Let $L_1$ denote the line segment joining $(n/3, n/3 - Cn^{2/3})$ to $(n/3, n/3 + Cn^{2/3})$ and let $L_2$ denote the line segment joining $(2n/3, 2n/3 - Cn^{2/3})$ to $(2n/3, 2n/3 + Cn^{2/3})$. Recall that $L_1'$ is the line segment joining
Lemma 6.1 Let \( \gamma \) be the counting lemma which gives a bound on the number of \( J \) of a path fluctuation. We omit the proof.

Lemma 6.2 Let \( E \) as in Lemma 5.1. Notice that on \( \delta \), \( \sup_{u_1 \in L_1, u_2 \in L_2} X_{u_1, u_2}^{(m)} \) is stochastically dominated by \( \sup_{u_1 \in L_1', u_2 \in L_2'} X_{u_1, u_2}^{(0)} \) and both of these variables are independent of \( E \). Hence it follows that

\[
E[(\ell_{\Gamma(m)} - \ell_{\Gamma(0)})1_{\mathcal{E}}] \leq E[(\sup_{u_1 \in L_1'} X_{0, u_1} - \inf_{u_1 \in L_1'} X_{0, u_1})1_{\mathcal{E}}] + E[(\sup_{u_2 \in L_2'} X_{u_2, n} - \inf_{u_2 \in L_2'} X_{u_2, n})1_{\mathcal{E}}].
\]

Using the definition of \( E \) we can bound the terms in the right hand side above and get

\[
E[(\ell_{\Gamma(m)} - \ell_{\Gamma(0)})1_{\mathcal{E}}] \leq 50Cn^{1/3}.
\]

This, in conjunction with Lemma 5.1 and the fact that \( E[\ell_{\Gamma}] = 2n - O(n^{1/3}) \) proves that \( E[\ell_{\Gamma(0)}] > 2n \) which completes the proof. \( \square \)

6 Maximal paths behave nicely most of the time

In this section we shall prove that the paths of maximal length with large probability, are not too steep for most values of \( x \).

For a fixed \( r \) and for \( i \in \{0, 1, \ldots, n/r\} \), let \( \mathcal{I}_i \) denote the set of all line segments of the form

\[ \{(ir, y') : \ell r^{2/3} \leq y' - ir \leq (\ell + 1)r^{2/3}\}, \]

which represents a discretization of the endpoints of the ith segment of the path. Let \( \mathcal{I} \) denote the set of all sequences of the form \( \mathcal{J} = \{J_i\}_{0 \leq i \leq n/r} \) where \( J_i \in \mathcal{I}_i \). Fix \( \mathcal{J} \in \mathcal{I} \) where

\[
J_i = \{(ir, y') : jir^{2/3} \leq y' - ir \leq (ji + 1)r^{2/3}\}. \tag{36}
\]

Define \( \Delta_i(\mathcal{J}) = j_{i+1} - j_i \).

Now let \( \gamma \) be an increasing path from \( 0 \) to \( n \). Define \( \mathcal{J} = \mathcal{J}(\gamma) \in \mathcal{I} \) as follows. Let \( j_i = \lceil \frac{2n-mi}{r^{2/3}} \rceil \). Define \( \mathcal{J}_i = \{J_i(\gamma)\} \) by (36). Set \( \Delta_i^\gamma = \Delta_i(\mathcal{J}(\gamma)) \). We define the total fluctuation of a path \( \gamma \) at scale \( r \) to be equal to \( \sum_i |\Delta_i^\gamma| \). We shall need the following easy counting lemma which gives a bound on the number of \( J(\gamma) \in \mathcal{I} \) that correspond to an increasing path \( \gamma \) of a given total fluctuation. We omit the proof.

Lemma 6.1 Let \( \mathcal{J}(T) = \{\mathcal{J} \in \mathcal{I} : \sum_i |\Delta_i(\mathcal{J})| \leq T\} \). Then \( |\mathcal{J}(T)| \leq 4^{n/r + T} \). Further if \( T \geq \ell n^{1/4} \), then \( |\mathcal{J}(T)| \leq e^{c(\ell)(n/r + T)} \) where \( c(\ell) \to 0 \) as \( \ell \to \infty \).

Lemma 6.2 Let \( \Gamma \) be the topmost maximal path from \( 0 \) to \( n \) in \( \Pi \). We have w.h.p.,

\[
\sum_i |\Delta_i^\Gamma| \leq \frac{Cn}{r}.
\]
Proof. Observe that by Theorem 9.13 it suffices to restrict our attention to the case \( \{ \sup |\Gamma_x - x| \leq n^{3/4} \} \). Let \( \mathcal{F}^* \subseteq \mathcal{F} \) be the set of all sequences \( \mathcal{J}(\gamma) \) corresponding to all increasing paths \( \gamma \) from 0 to \( n \) such that \( \{ \sup |\gamma_x - x| \leq n^{3/4} \} \). Denote the set of all such gamma by \( \mathcal{G}_* \).

For an increasing path \( \gamma \) in \( \mathcal{G}_* \) set \( \tilde{X}_\gamma = t_\gamma^{1/2} - 2n \). It is clear that for each increasing such \( \gamma \) we have

\[
\hat{X}_\gamma \leq \sum_{i=0}^{n/r} \sup_{u \in J_i(\gamma), u' \in J_{i+1}(\gamma)} \hat{X}_{u,u'}.
\]

First observe that for all \( \gamma \) in \( \mathcal{G}_* \) and all \( i \), the slope of the line segment joining any \( u \in J_i(\gamma) \) to any \( u' \in J_{i+1}(\gamma) \) is in \( (\frac{\gamma}{\delta}, \frac{\psi}{\delta}) \), by our choice of values taken by \( \gamma \). From Corollary 9.3 it follows that for \( u \in J_i(\gamma) \) and \( u' \in J_{i+1}(\gamma) \), we have since \( \tilde{C} \) is sufficiently large

\[
\hat{X}_{u,u'} - \hat{X}_{u,u'} \leq \frac{\tilde{C}}{10n^{1/3}} - ((|\Delta_i| - 1) \wedge 0)^2 \frac{r^{1/3}}{100\psi^{3/2}} \leq \frac{\tilde{C}}{50} r^{1/3} - 5|\Delta_i|r^{1/3}.
\]

Now let \( \mathcal{G}_T \subseteq \mathcal{G}_* \) be the set of increasing paths from 0 to \( n \) such that \( \sum_i |\Delta_i^\gamma| = T \). It follows that for each \( \gamma \in \mathcal{G}_T \), we have

\[
\hat{X}_\gamma \leq \left( \frac{\tilde{C}n}{50r} - 5T \right) r^{1/3} + \sum_{i=0}^{n/r} \sup_{u \in J_i(\gamma), u' \in J_{i+1}(\gamma)} \hat{X}_{u,u'}.
\]

Fix \( T \geq \frac{\tilde{C}n}{r} \). Using the exponential tails of \( \hat{X} \) established in Proposition 9.10, we conclude that for some absolute constant \( c > 0 \) we have for each \( \gamma \in \mathcal{G}_T \),

\[
P \left( \sum_{i=0}^{n/r} \sup_{u \in I_i^\gamma, u' \in I_{i+1}^\gamma} \hat{X}_{u,u'} \geq 4Tr^{1/3} \right) \leq K^{n/r} e^{-4cT} \leq e^{-2cT}
\]

for some constant \( K > 0 \) where the last inequality follows because \( \tilde{C} \) is sufficiently large (depending on \( K \)). Now using Lemma 6.1 and taking a union bound over all \( \gamma \in \mathcal{G}_T \), since \( \tilde{C} \) is sufficiently large, we get for \( T \geq \frac{\tilde{C}n}{r} \),

\[
P(\Gamma \in \mathcal{G}_T, \hat{X}_\Gamma \geq -\frac{T}{100} r^{1/3}) \leq e^{-cT}.
\]

Summing over all \( T \), and noticing that \( P(\hat{X}_\Gamma < -\ell r^{1/3}) \to 0 \) as \( \ell \to \infty \) completes the proof of the lemma. \( \square \)

Lemma 6.3 Let \( r \in \mathcal{R} \) be fixed. For \( x \in \mathcal{V}_r \), let \( A_x \) denote the event that for all \( x' \) with \( |x' - x| \leq \frac{r}{2} \), we have \( |\Gamma_x - \Gamma_{x'} - (x - x')| \leq M r^{2/3} \). Then we have

\[
P \left[ \sum_{x \in \mathcal{V}_r} 1_{A_x} \geq \frac{n}{10000r} \right] \leq e^{-cn/r} + o(1)
\]

for some constant \( c > 0 \).
We define \( \tilde{\Delta}_i \) with \( \tilde{\Delta}_i \) taking \( M \) sufficiently large so that \( M \geq 10^6 \tilde{C} \). From Markov’s inequality it follows that

\[
\# \{ i : |j_{i+1} - j_i| \geq \frac{M}{50} \} \leq \frac{n}{20000r}.
\]

For \( u \in J_i = (ir + ji)r^{2/3}, ir + (j_i + 1)r^{2/3} \) and \( u' \in J_{i+1} = (i + 1)r \times ((i + 1)r + j_{i+1}r^{2/3}, i(r + 1) + (j_{i+1} + 1)r^{2/3}) \) we say \( F_{u,u'} \) holds if all the maximal paths between \( u \) and \( u' \) are contained in \( \mathcal{P}((i + \frac{1}{2})r, r, (j_i - \frac{M}{50})r^{2/3}, (j_{i+1} + \frac{M}{50})r^{2/3}) \). Define

\[
A_{i, J} = \bigcap_{u \in J_i, u' \in J_{i+1}} F_{u,u'}.
\]

It follows from Corollary 9.20 that \( \mathbb{P}[A_{i, J}] \geq 1 - \epsilon(M) \) where \( \epsilon(M) \) can be made arbitrarily small by taking \( M \) sufficiently large. Since these are independent events for different values of \( i \) it follows that \( \mathbb{P}[\sum_i 1_{A_{i, J}} \geq \frac{n}{20000}] \leq 10^{-Cn/r} \) with \( M \) sufficiently large. Now notice that if \( \sum_i 1_{A_{i, J}} \leq \frac{n}{20000} \), then we have \( \sum_{x \in X_r} 1_{A_x} \leq \frac{n}{10000r} \). The lemma now follows by taking \( M \) sufficiently large, taking a union bound over \( J \) and using Lemma 6.2 and Lemma 6.1.

**Lemma 6.4** Let \( r \in \mathcal{R} \) be fixed. For \( x \in X_r \), let \( C_x \) denote the event that for all \( x' \) with \( |x' - x| \leq (1/2 + L^{3/2})r \), we have \( |\Gamma_x - \Gamma_{x'} - (x - x')| \leq \frac{1}{10} L^{11/10} r^{2/3} \). Then we have

\[
\mathbb{P} \left[ \sum_{x \in X_r} 1_{C_x} \geq \frac{n}{10000r} \right] \leq e^{-cn/L^{3/2}r} + o(1)
\]

for some constant \( c > 0 \).

We first need the following lemma.

**Lemma 6.5** Let \( \gamma \) be an increasing path from 0 to \( n \). For a fixed \( r \) and for \( i \in \{0, 1, \ldots, \frac{n}{L^{3/2}r} \} \), let us define \( Y^\gamma(i) = \left| \frac{\gamma_{iL^{3/2}r} - irL^{3/2}}{L^{3/2}} \right| \). Let \( J^\gamma_i \) denote the line segment

\[
J^\gamma_i = \{(irL^{3/2}, y') : Y^\gamma(i)L^{3/2} \leq y' - irL^{3/2} \leq (Y^\gamma(i) + 1)Lr^{2/3} \}.
\]

We define \( \hat{\Delta}^\gamma_i = Y^\gamma(i + 1) - Y^\gamma(i) \). Let \( \hat{\mathcal{J}}(T) \) denote the set of sequences of line segments \( \{\hat{J}_i\}_{0 \leq i \leq \frac{n}{L^{3/2}r}} \) such that \( \sum_i |\Delta^\gamma_i| \leq T \). Then \( |\hat{\mathcal{J}}(T)| \leq 4L^{3/2}rT^{-1} \). Also let \( \Gamma \) be the topmost maximal path from 0 to \( n \). Then with high probability, \( \sum_i |\Delta^\gamma_i| \leq \frac{Cn}{L^{3/2}r} \).

**Proof.** The proof of this lemma is identical to the proofs of Lemma 6.1 and Lemma 6.2 and we omit the proof.

**Lemma 6.6** Assume the set-up of Lemma 6.5. Fix \( \hat{\mathcal{J}} = \{\hat{J}_i\} \in \hat{\mathcal{J}}(\hat{C}) \) with \( \hat{J}_i = x_i \times (x_i + y_i, x_i + y_i + Lr^{2/3}) \). For a fixed \( i \), consider the parallelogram \( U_i \) whose corners are \((x_i, x_i + y_i - L^{21/20}r^{2/3}), (x_i, x_i + y_i + L^{21/20}r^{2/3}), (x_{i+3}, x_{i+3} + y_{i+3} - L^{21/20}r^{2/3}), (x_{i+3}, x_{i+3} + y_{i+3} + L^{21/20}r^{2/3}) \). Call \( i \) ‘bad’ if at least one of the following two conditions fail to hold.

}\]
(i) \( \tilde{\Delta}_i + \tilde{\Delta}_{i+1} + \tilde{\Delta}_{i+2} \leq 10^6 \tilde{C} \).

(ii) For all \( u \in \tilde{J}_i \) and for all \( u' \in \tilde{J}_{i+3} \), all the maximal paths from \( u \) to \( u' \) is contained in \( U_i \) (call this event \( D_i \)).

Then we have
\[
P[\# \{ i : i \text{ is bad} \} \geq n/20000L^{3/2}r] \leq 10^{-Cn/L^{3/2}r} e^{-cn/L^{3/2}r}
\]
for some constant \( c > 0 \).

**Proof.** Since \( \tilde{J} \in \tilde{\mathcal{I}}(\tilde{C}) \), by Markov’s inequality it follows that deterministically
\[
\# \{ i : i \text{ is bad for failing (i)} \} \leq n/50000L^{3/2}r.
\]
Also notice that it follows from 9.20 that
\[
P[\mathcal{D}_i] \geq 1 - \epsilon(L), \quad \epsilon(L) \text{ can be made arbitrarily small by taking } L \text{ sufficiently large.}
\]
Also notice that for each fixed \( k \in \mathbb{Z}/3\mathbb{Z} \), the family of events \( \{D_{3j+k}\} \) are independent. A large deviation bound followed by a union bound then shows that
\[
P[\# \{ i : i \text{ is bad for failing (ii)} \} \geq n/50000L^{3/2}r] \leq 10^{-Cn/L^{3/2}r} e^{-cn/L^{3/2}r}
\]
since \( L \) is sufficiently large, which completes the proof of the lemma. \( \square \)

**Proof.** [of Lemma 6.4] Assume the set up of Lemma 6.5 and Lemma 6.6. It is clear that if \( \{\tilde{J}_i\} = \{J_i^\Gamma\} \) and \( i \) is good (i.e., \( i \) is not bad), then we have the following. for each \( x \in X_r \) with \( x_{i+1} \leq x \leq x_{i+2} \), we have that \( C_x \) holds. The lemma now follows from Lemma 6.5, Lemma 6.6 and a union bound over \( \tilde{\mathcal{I}}(\tilde{C}) \). \( \square \)

### 7 Probability bounds for \( G_x \)

Let \( r \in \mathbb{R} \) be fixed. In this section, our task is to prove that for a large fraction of \( x \in X_r \), \( \mathbb{P}(G_x) \) is close to 1. We shall prove the following theorem.

**Theorem 7.1** For all \( n \) sufficiently large we have
\[
\mathbb{P}[\# \{ x \in X_r : G_x \text{ does not hold} \} \geq 1/1000 |X_r|] \leq 10^{-3}.
\]

We shall need the following corollary of Theorem 7.1.

**Corollary 7.2** There exist \( X^*_r \subseteq X_r \) with \( |X^*_r| \geq 9/10 |X_r| \) such that for all \( x \in X^*_r \) we have for all \( n \) sufficiently large
\[
\mathbb{P}(G_x) \geq 95/100.
\]

**Proof.** It follows from Theorem 7.1 that
\[
\sum_{x \in X_r} \mathbb{P}(G_x) \geq 995/1000 |X_r|.
\]
Corollary 7.2 follows immediately. \( \square \)

Since the condition \( G_x \) has many components we will need a few steps to prove Theorem 7.1. For the rest of this section \( \chi \) shall denote a small positive constant which can be made arbitrarily small by taking \( C \) sufficiently large.
7.1 Bounding Probabilities of $G_{x,y}$

First we need to prove that for a fixed $x \in \mathcal{X}_r$, and $y \in r^{2/3}\mathbb{Z}$, $G_{x,y}$ holds with large probability. We have the following lemma.

**Lemma 7.3** For $x \in \mathcal{X}_r$, $y \in r^{2/3}\mathbb{Z}$, we have

$$\mathbb{P}(G_{x,y}^{loc}) \geq 1 - \chi.$$  \hspace{1cm} (39)

**Proof.** It suffices to prove that for a fixed $x \in \mathcal{X}_r$, $y \in r^{2/3}\mathbb{Z}$, each of the 5 conditions defining $G_{x,y}^{loc}$ holds with probability at least $1 - \frac{\chi}{10}$. We analyse each of the conditions separately.

**Condition 1:** Let $U = \mathcal{P}(x, y - Mr^{2/3}/10, r, 2Mr^{2/3}/10)$. Let $A_{x,y,1}$ denote the event that for all $(u', u'') \in \mathcal{S}(U)$ we have $|\tilde{X}_{u', u''}| \leq Cr^{1/3}$. It follows from Proposition 9.10 and Proposition 9.22 that $\mathbb{P}[A_{x,y,1}^c] \leq \frac{C}{10}$ for $C$ sufficiently large.

**Condition 2:** Let $A_{x,y,2}$ denote the event that $\forall (u, u') \in \mathcal{S}(\mathfrak{c})$, we have $|\tilde{X}_{u, u'}| \leq CL^{1/2}r^{1/3}$. It follows from Corollary 9.9 and Corollary 9.12 that $\mathbb{P}[A_{x,y,2}^c] \leq \frac{C}{10}$ since $C$ sufficiently large.

**Condition 3:** Let $A_{x,y,3}$ denote the event that for all $(u, u') \in \mathcal{S}(\Lambda)$ we have $|\tilde{X}_{u, u'}| \leq Cr^{1/3}$. It follows from Proposition 9.10 and Proposition 9.6 that $\mathbb{P}[A_{x,y,3}^c] \leq \frac{C}{10}$ since $C$ sufficiently large.

Now define the following parallelograms. Let $U_1 = \mathcal{P}(x - \frac{9r}{20}, y - 2Mr^{2/3}, 7r, 4Mr^{2/3}/10)$, $U_2 = \mathcal{P}(x + \frac{9r}{20}, y - 2Mr^{2/3}, 7r, 4Mr^{2/3}/10)$, $U_3 = \mathcal{P}(x, y - 2Mr^{2/3}, 4r, (M - 1)r^{2/3})$, $U_4 = \mathcal{P}(x, y - Mr^{2/3}, 2r, Mr^{2/3})$. Let us define the following events. For $i = 1, 2$, let

$$B_{x,y,i} = \{\forall (u, u') \in \mathcal{S}(U_i) \mid |\tilde{X}_{u, u'}| \leq \frac{C}{10}r^{1/3}\}.$$  \hspace{1cm} (40)

For $i = 3, 4$, let

$$B_{x,y,i} = \{\forall (u, u') \in \mathcal{S}(U_i) \mid |\tilde{X}_{u, u'}| \leq \frac{C}{10}r^{1/3}\}.$$  \hspace{1cm} (41)

Since $C$ sufficiently large it follows from Proposition 9.6, Proposition 9.10 and Proposition 9.22 that $\mathbb{P}[\cap_{i=1}^2 B_{x,y,i}] \leq \frac{C}{100}$.  \hspace{1cm} (42)

Now we show that on $\cap_{i=1}^4 B_{x,y,i}$, condition (19) holds. To prove this let us fix $(u, u') \in \mathcal{S}(\Lambda \setminus 2D)$ satisfying the hypothesis of the condition. Without loss of generality let us assume $u \in U_1$. There are several cases depending on the position of $u$. If $u' \in U_1$, on $B_{x,y,1}$ it follows that $|\tilde{X}_{u,u'}| \leq Cr^{1/3}$. If $u' \in U_2$ (resp. $U_3$) and $u$ is also in $U_2$ (resp. $U_3$), then also it follows that on $\cap_{i=1}^4 B_{x,y,i}$, $|\tilde{X}_{u,u'}| \leq Cr^{1/3}/10$. Next let us consider the case where $u' \in U_2$. Clearly there exists $u_1^* \in U_1 \cap U_3, u_2^* \in U_2 \cap U_3$ such that $(u, u_1^*) \in \mathcal{S}(U_1), (u_1^*, u_2^*) \in \mathcal{S}(U_3)$ and $(u_2^*, u') \in \mathcal{S}(U_2)$ such that it follows using Lemma 9.5 that

$$|\tilde{X}_{u, u'} - \tilde{X}_{u_1^*, u_1} - \tilde{X}_{u_2^*, u_2} - \tilde{X}_{u_2^*, u_1}| \leq C/2r^{1/3}$$  \hspace{1cm} (43)

since $C$ is sufficiently large. It follows that on $\cap_{i=1}^4 B_{x,y,i}$, $|\tilde{X}_{u,u'}| \leq C r^{1/3}$. All other cases can be dealt with similarly and it follows that condition (19) holds with probability at least $1 - \frac{6}{100}$.

**Condition 4:** Let $A_{x,y,4}$ denote the event that $\forall u \in U = \mathcal{P}(x, y - 2Mr^{2/3}, r, 2Mr^{2/3})$, $u' \in B_2^t \cap \mathfrak{c}$ we have $\tilde{X}_{u,u'} \leq Cr^{1/3}$. Clearly it suffices to show that $\mathbb{P}[A_{x,y,4}^c] \leq 1 - \frac{C}{100}$.  \hspace{1cm} (44)
Let $C_{x,y}$ denote the event that for all $u \in U, u' \in \Lambda \cap B_2^+$ we have $\hat{X}_{u,u'} \leq \frac{C}{2} r^{1/3}$. Clearly since $C$ is sufficiently large we have $\mathbb{P}[C_{x,y}] \leq \frac{1}{1000}$. Now let us define the points $u_j = (x + 2r/5, x + 2r/5 + y - 2Mr^2/3 - jMr^2/3)$ for $\frac{k}{2} j \geq 0$. Let $C_{x,y,j}$ denote the event that for all $u \in 2D$ and for all $u'$ on the line segment $L_j$ joining $u_j$ and $u_{j+1}$, we have $\hat{X}_{u,u'} \leq \frac{C}{2} r^{1/3}$. Notice that it follows from Lemma 9.2 that since $M$ is sufficiently large we have that for all $u \in U$, and for all $u' \in L_j$, $\hat{X}_{u,u'} \leq \hat{X} - jr^{1/3}$. Hence it follows from Proposition 9.10 that for some constant $c > 0$, we have $\mathbb{P}[C_{x,y,j}] \leq e^{-c(C+j)}$ since $C$ is sufficiently large. Hence it follows that $\mathbb{P}[(\cap_j C_{x,y,j})^c] \leq \frac{1}{1000}$. It now follows that $\mathbb{P}[A_{x,y,4}] \leq \frac{1}{100}$. 

**Condition 5:** Let $A_{x,y,5}$ denote the event that for all $u, u' \in F$ we have $|\hat{X}_{u,u'}| \leq C r^{1/3}$. Using Proposition 9.22 it follows that $\mathbb{P}[A_{x,y,5}] \leq \frac{1}{10}$. 

Putting together all the steps above it follows that $\mathbb{P}(G_{x,y}^{\text{loc}}) \geq 1 - \chi$ which completes the proof of the lemma. \square

**Lemma 7.4** For $x \in X_r, y \in r^{2/3} \mathbb{Z}$, we have for all $n$ sufficiently large

$$\mathbb{P}(G_{x,y}^{\text{reg}}) \geq 1 - \chi. \tag{40}$$

We first need the following lemma.

**Lemma 7.5** Consider the rectangle $U_h$ whose opposite corners are $0,0$ and $(h,mh)$ where $0.99 \leq m \leq 1.01$. Let $A(\eta)$ denote the event that there exists an increasing path $\gamma$ from $(0,0)$ to $(h,mh)$ such that $\ell_\gamma \in [h,3h]$ and $A_\gamma \leq \eta h$. For a fixed absolute constant $\eta > 0$ and for $h$ sufficiently large we have $\mathbb{P}(A(\eta)) \leq e^{-h}$.

**Proof.** Notice that it suffices to take $\ell_\gamma = \ell$ fixed in the statement of the lemma, since then we can take a union bound over different $\ell \in [h,3h]$. Without loss of generality let us take $\ell_\gamma = h$ in the statement of the lemma. Let us first divide $U_h$ into the following subrectangles. For $i,j \in \{0,1,\ldots,h/10 - 1\}$, we define $D_{i,j}$ to be the rectangle whose opposite corners are given by $(10i,10mj)$ and $(10(i+1),10m(j+1))$.

Let $\mathbb{H}$ denote the set of all oriented paths in $\mathbb{Z}^2$ from $(0,0)$ to $(\frac{h}{10} - 1, \frac{h}{10} - 1)$. Clearly $|\mathbb{H}| \leq 2^h$. For $H \in \mathbb{H}$, let $N_H$ denote the set of all nonnegative integer valued sequences $\{N_{i,j}\}_{(i,j) \in H}$ with $\sum_{(i,j) \in H} N_{i,j} = h$. It is clear that $|\cup_{H \in \mathbb{H}} N_H| \leq 20^h$. Now fix $H \in \mathbb{H}$ and \{N_{i,j}\} $\in N_H$.

Let $A(\eta, \{N_{i,j}\})$ denote the event that there exists an increasing path $\gamma$ in $U_h$ from $(0,0)$ to $(h,mh)$ with $\ell_\gamma = h$ and $A_\gamma \leq \eta h$, and such that $\gamma$ contains exactly $N_{i,j}$ many points in $D_{i,j}$. Observe that, on $A(\eta, \{N_{i,j}\})$, there must exist $\sum_{(i,j) \in H} (N_{i,j} - 1) \geq \frac{h}{2}$ points on $\gamma$, such that the point (say $u$) and the next point on $\gamma$ (say $u'$) belong to the same subrectangle $D_{i,j}$ for some $(i,j) \in H$.

Now for $D_{i,j}$, let $U_{i,j}$ denote the number of points $u \in D_{i,j}$ such that there is a point $u' \neq u$ in $D_{i,j}$ such $A(u,u') \leq 10\eta$. It follows that on $A(\eta, \{N_{i,j}\})$

$$A(\gamma) \geq 10\eta \times \left(\frac{h}{2} - \sum_{(i,j) \in H} U_{i,j}\right).$$

Hence it suffices to show that

$$\mathbb{P}\left[\sum_{(i,j) \in H} U_{i,j} \geq \frac{2h}{5}\right] \leq (20e)^{-h}. \tag{41}$$
First observe that \( \{ U_{i,j} \}, (i,j) \in H \) is an independent sequence of random variables. Also observe that
\[
\mathbb{E}(e^{10U_{i,j}}) \to 1
\]
as \( \eta \to 0 \) by the DCT. Since \( \eta \) is chosen sufficiently small we have
\[
\mathbb{E}(e^{20U_{i,j}}) \leq 2.
\]
The independence of \( U_{i,j} \)'s and Markov's inequality then establishes (41). This completes the proof of the lemma.

**Proof.** [of Lemma 7.4] It follows from Lemma 7.5 and taking a union bound over different pairs of points \((u,u')\) that \( \mathbb{P}[G_{r,y}] \geq 1 - \frac{\chi}{2} \). It is also easy to see that the no hole condition fails with probability exponentially small in \( \frac{r^2}{3} \) and hence if \( n \) is sufficiently large (and so \( r \) is sufficiently large) we have that the no hole condition holds with probability at least \( 1 - \frac{\chi}{2} \). The lemma follows.

**Lemma 7.6** For each \( x \in \mathcal{X}_r \), \( y \in r^{2/3} \mathbb{Z} \), we have \( \mathbb{P}(G_{rs,x,y}) \geq 1 - \chi \).

**Proof.** For \( i = 0, 1, 2, \ldots \frac{1}{100} \varepsilon^{-5/3} \), let
\[
A_i = \{ \forall (u,u') \in S(\tilde{D}_i) \mid |\tilde{X}_{u,u'}^{(i)}| \leq \frac{\delta}{8} r^{1/3} \};
\]
\[
B_i = \{ \forall (u,u') \in S(\tilde{D}_i) \mid |\tilde{X}_{u,u'}^{(i-1)}| \leq \frac{\delta}{8} r^{1/3} \}.
\]
It is clear from Proposition 9.10 that by taking \( \varepsilon \) sufficiently small depending on \( \delta \) and \( C \), we have that \( \mathbb{P}[A_i \cup B_i] \leq e^{-2C/\varepsilon^{1/4}} \).

Clearly, on \( A_i \cap B_i \), we have
\[
\Delta_i \leq \sup_{u,u' \in S(\tilde{D}_i)} (|\tilde{X}_{u,u'}^{(i)}| + |\tilde{X}_{u,u'}^{(i-1)}|) \leq \frac{\delta}{4} r^{1/3}.
\]
It follows by taking a union bound over all \( i \) we get
\[
\mathbb{P}(\max_i \Delta_i \geq \frac{\delta}{2} r^{1/3}) \leq e^{-2} e^{-2C/\varepsilon^{1/4}} \leq e^{-C/\varepsilon^{1/4}} \chi^{-1}.
\]
by choosing \( \varepsilon \) small enough. It follows now from Markov’s inequality that
\[
\mathbb{P}\left[ \mathbb{P}[\max_i \Delta_i \geq \frac{\delta}{2} r^{1/3} \mid \Pi] \geq e^{-C/\varepsilon^{1/4}} \right] \leq \chi.
\]
This completes the proof of the lemma.

For \( x \in \mathcal{X}_r \), let \( y(\Gamma, x) = \inf_{y \in r^{2/3} \mathbb{Z}} \{ y + x \geq \Gamma_x \} \). We have the following proposition.

**Proposition 7.7** For all \( n \) sufficiently large we have,
\[
\mathbb{P}[\# \{ x \in \mathcal{X}_r : G_{loc,x,y(\Gamma, x)} \cap G_{reg,x,y(\Gamma, x)} \cap G_{rs,x,y(\Gamma, x)} \text{ does not hold} \} \geq \frac{1}{10000} |\mathcal{X}_r|] \leq 10^{-4}.
\]

**Proof.** The proposition follows from Lemma 7.3, Lemma 7.4, Lemma 7.6, the fact that \( G_{loc,x,y} \cap G_{reg,x,y} \cap G_{rs,x,y} \) are independent events for different values of \( x \), and a union bound using Lemma 6.2 and Lemma 6.1.
7.2 Proof of Theorem 7.1

To prove Theorem 7.1, we still need to estimate the probabilities of the wing condition and the fluctuation condition.

Proposition 7.8 Let \( \Gamma \) be the topmost maximal path in \( \Pi \) from \( 0 \) to \( n \). For \( x \in \mathcal{X}_r \), let \( y(\Gamma, x) = \inf_{y} \{ y \in r^{2/3}Z : y \geq \Gamma_x \} \). Then

\[
\mathbb{P}[\# \{ x \in \mathcal{X}_r : G_{x}^{w}(\Gamma, x) \text{ holds} \} < \frac{9999}{10000} |\mathcal{X}_r|] \leq 10^{-4}.
\]

The proof of Proposition 7.8 is similar to the proof of Proposition 7.7, but we need to work harder as the Wings are not disjoint for different values of \( x \in \mathcal{X}_r \). We first need the following lemma.

Lemma 7.9 Assume the set-up of Lemma 6.5. For \( \tilde{\mathcal{J}} = \{ \tilde{J}_j \} \in \tilde{\mathcal{J}} = \tilde{\mathcal{J}}(\infty) \) with \( \tilde{J}_j = x_j \times (x_j + y_j, x_j + y_j + L^{3/2}) \), let \( W_{1,j} = \mathcal{P}(x_j - \frac{3L^{3/2}}{2}, y_j - 10L^{11/10}r^{2/3}, 3L^{3/2}r, 20L^{11/10}r^{2/3}) \)
and \( W_{2,j} = \mathcal{P}(x_j + \frac{3L^{3/2}}{2}, y_j - 10L^{11/10}r^{2/3}, 3L^{3/2}r, 20L^{11/10}r^{2/3}) \). Let \( \mathcal{X}_{r,j} = \{ x \in \mathcal{X}_r : x_j - L^{3/2}r \leq x \leq x_j + L^{3/2}r \} \). Let \( \gamma \) be an increasing path from \( 0 \) to \( n \) such that \( \tilde{J}_j^\gamma = \tilde{J}_j \).

For \( x \in \mathcal{X}_{r,j}, \) let \( \mathcal{X}_r \) denote the event that \( W_1^1(\mathbb{B}(x, y(x, \gamma), r)) \cup W_2^1(\mathbb{B}(x, y(x, \gamma), r)) \subseteq W_{1,\gamma} \cup W_{2,\gamma} \). Call \( i \) ‘good for \( \gamma \)’ if \( \cap_{x \in \mathcal{X}_{r,i}} A_x^\gamma \) holds. Then for \( L \) sufficiently large,

\[
\mathbb{P}[\# \{ i \in \text{‘good’ for } \Gamma \} \leq (1 - 10^{-5}) \frac{n}{L^{3/2}r}] \leq e^{-cn/L^{3/2}r} + o(1).
\]

Proof. The proof is essentially similar to the proof of Lemma 6.4 and we omit the details. \( \square \)

Lemma 7.10 Fix \( \tilde{\mathcal{J}} = \{ \tilde{J}_j \} \in \tilde{\mathcal{J}} \) and define \( W_{1,i} \) and \( W_{2,i} \) as in Lemma 7.9. Let \( A_i \) denote the event that for all \( (u, u') \in S(W_{1,i} \cup W_{2,i}) \), we have \( |\mathcal{X}_{u, u'}| \leq C \frac{L^{3/4}r^{1/3}}{2} \). Then

\[
\mathbb{P}[\# \{ i : A_i \text{ does not hold} \} \geq 10^{-6} \frac{n}{L^{3/2}r}] \leq 10^{-Cn/L^{3/2}r} e^{-cn/L^{3/2}r}.
\]

Proof. Notice that \( A_{1,i} \) and \( A_{2,i} \) are independent if \( i_1 - i_2 \geq 6 \). More generally, we also have \( \{ A_{6i + k} \}_{i \geq 0} \) is independent for each \( k = 0, 1, 2, 3, 4, 5 \). By Corollary 9.12 and Corollary 9.9 it follows that for each \( i \), \( \mathbb{P}[A_i] \geq 1 - \chi \), where \( \chi \) can be made arbitrarily small by choosing \( C \) sufficiently large. It follows that for a fixed \( k \in \mathbb{Z}/6\mathbb{Z} \) we have with exponentially high probability, \( \# \{ i : A_{6i + k} \text{ does not hold} \} \leq \frac{1}{10^6} \frac{n}{L^{3/2}r^2} \). The lemma follows by taking a union bound over \( k \in \mathbb{Z}/6\mathbb{Z} \). \( \square \)

Proof. [of Proposition 7.8] Observe the following. If \( i \) is ‘good’ for \( \Gamma \) and \( A_i \) holds for \( \tilde{J} = \{ \tilde{J}_j^\gamma \} \) then \( G_{x}^{w}(\Gamma, x) \) holds for all \( x \in \mathcal{X}_{r,i} \), The proposition now follows similarly to Proposition 7.7 by taking a union bound over \( \tilde{J} \) and using Lemma 6.5, Lemma 7.9, and Lemma 7.10. \( \square \)

Proposition 7.11 We have for all \( n \) sufficiently large

\[
\mathbb{P}[\# \{ x \in \mathcal{X}_r : G_{x}^{f}(\Gamma, x) \text{ holds} \} < \frac{9998}{10000} |\mathcal{X}_r|] \leq 10^{-4}.
\]
Proof. Since $C$ is sufficiently large, it follows from Theorem 1.3, Proposition 9.10 and Theorem 9.13 that with probability at least $1 - 10^{-5}$ the first condition in the definition of $G^f_x$ holds for all $x$. It follows from Lemma 6.3 and Lemma 6.4 that

$$\mathbb{P}[\# \{x \in X_r : G^f_x \text{ does not hold for failing (26)} \} > \frac{2n}{10000r}] \leq 10^{-5}.$$ 

The lemma follows. \hfill \Box

Now we are ready to prove Theorem 7.1.

Proof. [of Theorem 7.1] Observe that for $x \in X_r$ if $G^{3}_{x,y}(x,\Gamma)$ holds and $G^f_x$ holds then $G^{3}_{x,y}(x,\Gamma)$ holds. The Theorem now follows from Proposition 7.7, Proposition 7.8 and Proposition 7.11. \hfill \Box

8 Probability bounds on $R_x$, $H_x$, $Q_x$ and the conditional measure

In this section, we work out estimates of probabilities of $R_x$ and prove Theorem 3.2 and also estimates for probabilities of $H_x$ and $Q_x$ conditional on $R_x$ and ultimately prove Theorem 3.5.

8.1 Bounds on $R_x$

First we prove Theorem 3.2. We start with the following proposition.

Proposition 8.1 For each $x \in X_r$ and for each $y \in r^{2/3}\mathbb{Z}$ we have $\mathbb{P}[R_{x,y}] > \beta > 0$ where $\beta$ is a constant independent of $r$.

This proposition will follow from the next two lemmas.

Lemma 8.2 Let $x \in X_r$ and $y \in r^{2/3}\mathbb{Z}$ be fixed. Let $R_{1,xy}$ denote the event that $\forall u = (x',y') \in B^*_1$ with $y' \geq y - Mr^{2/3}$ and $\forall u' \in \partial^+(B^*_1)$ we have $\hat{X}_{u,y'} \leq Cr^{1/3}$. Then we have $\mathbb{P}[R_{1,xy}] \geq 99/100$ for $C$ sufficiently large.

Proof. Let $U = \mathcal{P} = (x - \frac{9r}{20},y - Mr^{2/3}, \frac{r}{10},3Mr^{2/3})$. Let $A_{1,xy}$ denote the event that for all $(u,u') \in S(U)$ we have $|\hat{X}_{u,y'}| \leq \frac{C}{10}r^{1/3}$. Let $A_{2,xy}$ denote the event that for all $(u,u')$ in the line segment joining $(x - \frac{r}{2},x - \frac{r}{2} + y - Lr^{2/3})$ and $(x - \frac{4r}{5},x - \frac{4r}{5} + y - Lr^{2/3})$ (i.e., the bottom boundary of $B^*_1$) we have $|\hat{X}_{u,y'}| \leq \frac{C}{10}r^{1/3}$. It follows from Proposition 9.6 and Proposition 9.10 that $\mathbb{P}[A_{1,xy}] \geq 1 - 10^{-3}$ since $C$ is sufficiently large.

Let $L_U$ denote the left boundary of $U$. For $0 \leq \ell \leq L - M$ let $L_\ell$ denote the vertical line segment joining $(x - \frac{4r}{5},x - \frac{4r}{5} + y - Mr^{2/3} - \ell r^{2/3})$ and $(x - \frac{4r}{5},x - \frac{4r}{5} + y - Mr^{2/3} - (\ell + 1)r^{2/3})$. Let $A_\ell$ denote the event that for all $u \in L_U, u' \in L_\ell$ we have $\hat{X}_{u,u'} \leq \frac{C + \ell}{10}r^{1/3}$. It follows from Proposition 9.10 that $\mathbb{P}[A_\ell] \leq e^{-c(C + \ell)}$ for some absolute constant $c > 0$. It follows by taking a union bound over all $\ell$ that $\mathbb{P}[\bigcap_\ell A_\ell] \geq \frac{999}{1000}$ since $C$ is large enough.

It suffices now to show that

$$A_{1,xy} \cap A_{2,xy} \cap \bigcap_\ell A_\ell \subseteq R_{1,xy}.$$
To show this observe that if \( u \in L_U \) and \( u' \) is on the right boundary of \( B_1^t \), this follows from Lemma 9.2 since \( C \) is sufficiently large. Otherwise, set point \( u_1 \in L_U \) such that the line joining \( u_1 \) and \( u \) has slope 1. Set \( u_2 = u' \) if \( u' \) is on the right boundary of \( B_1^t \), otherwise set \( u_2 = (x - \frac{4t}{r}, x - \frac{4t}{r} + y - Lr^{2/3}) \). Then observe that

\[
\hat{X}_{u,u} \leq \hat{X}_{u_1,u_2} - \hat{X}_{u_1,u} - \hat{X}_{u',u_2}.
\]

The lemma now follows from the definition of \( A_{x,y}^1 \) and \( A_{x,y}^2 \) and Lemma 9.2.

**Lemma 8.3** Let \( x \in \mathcal{X}_r \) and \( y \in r^{2/3} \mathbb{Z} \) be fixed. Let \( R_{x,y} \) denote the event that for all \( u \in B_1^t \cap W^1 \), \( \forall u' \in B_1 \cap \mathcal{C} \) we have \( \hat{X}_{u,u}^{(B_1^t)^c} \leq -C^* r^{1/3} \). Then we have \( \mathbb{P}[R_{x,y}] \geq \beta' > 0 \) where \( \beta' \) is independent of \( r \).

**Proof.** For \( s, t = 0, 1, \ldots (L + 2M) \), define points \( u_s = (x - r/2, x - r/2 + y - Lr^{2/3} + tr/2^{3/3}) \) and \( u'_t = (x - 2r/3, x - 2r/3 + y - Lr^{2/3} + tr/2^{3/3}) \). Let \( L_1^s \) denote the line segment joining \( u_s \) and \( u_{s+1} \) and \( L_2^t \) denote the line segment joining \( u'_t \) and \( u'_{t+1} \). Let \( A_{s,t} \) denote the following event.

\[
A_{s,t} = \{ \sup_{u \in L_1^s, u' \in L_2^t} \hat{X}_{u,u'} \leq -C^* r^{1/3} \}.
\]

First we prove that \( \mathbb{P}[A_{s,t}] \) is bounded away from 0 uniformly in \( r \). Fix \( \theta > 0 \). Define the points \( v(s) = (x - r/2 - \theta r, x - r/2 - \theta r + y - Lr^{2/3} + (s + \frac{1}{2})r^{1/3}) \) and \( v(t) = (x - 2r/5 + \theta r, x - 2r/5 + \theta r + y - Lr^{2/3} + (t + \frac{1}{2})r^{1/3}) \). Observe that for \( C^* \) sufficiently large, for all \( u \in L_1^s \) and all \( u' \in L_2^t \) we have

\[
\hat{X}_{v(s),u} + \hat{X}_{u,u'} + \hat{X}_{u',v(t)} \leq \hat{X}_{v(s),v(t)} - \frac{C^* r^{1/3}}{10}.
\]

It clearly follows that

\[
\mathbb{P}[\sup_{u \in L_1^s, u' \in L_2^t} \hat{X}_{u,u'} \leq -C^* r^{1/3}] \geq \mathbb{P}[\hat{X}_{v(s),v(t)} \leq -2C^* r^{1/3}] - \mathbb{P}[\inf_{u \in L_1^s} \hat{X}_{v(s),u} \leq -C^* r^{1/3} / 5]
\]

\[
- \mathbb{P}[\inf_{u' \in L_2^t} \hat{X}_{u',v(t)} \leq -C^* r^{1/3} / 5].
\]

Now observe that by Theorem 1.1, it follows that there exists a constant \( \kappa \) (depending on \( C^* \)) such that for all sufficiently large \( r \), we have \( \mathbb{P}[\hat{X}_{v(s),v(t)} \leq -2C^* r^{1/3}] > 10\kappa \). By choosing \( \theta \) sufficiently small and using Proposition 9.6 we get that \( \mathbb{P}[A_{s,t}] \geq \kappa > 0 \). Now notice that since \( A_{s,t} \) is a decreasing event for all \( s \) and \( t \), by FKG inequality it follows that

\[
\mathbb{P}[R_{x,y}] = P[\cap_{s,t} A_{s,t}] \geq \kappa^{(L+2M)^2} \geq \beta' > 0
\]

where \( \beta' \) is independent of \( r \). This completes the proof of the lemma.

**Proof** [of Proposition 8.1] Observe that since \( R_{1,xy} \) and \( R_{2,xy} \) are both decreasing events, it follows by FKG inequality that that \( \mathbb{P}[R_{1,xy} \cap R_{2,xy}] \geq \frac{99\beta'}{100} \). By symmetry we establish the same bounds for the right barrier \( B_2^t \) and since the two barriers are independent it follows that \( \mathbb{P}[R_{xy}] \geq \frac{9\beta'}{10} \geq \beta > 0 \), which completes the proof of the Proposition.

Now we are ready to prove Theorem 3.2

**Proof** [of Theorem 3.2] The first inequality follows from FKG inequality by noting that both \( \{ \Gamma = \gamma \} \) and \( R_{x,\gamma} \) are decreasing events on the configuration on \( \mathbb{R}^2 \setminus \{ \gamma \} \). The second inequality is trivial and the last inequality was already proved in Proposition 8.1.
8.2 Probability bounds for $H_x$:

**Theorem 8.4** For each $x \in \mathcal{X}_r$, we have $\mu_x^*(H_x) \geq \frac{98}{100}$.

To prove Theorem 8.4 we need the following Proposition.

**Proposition 8.5** For $x \in \mathcal{X}_r$, $y \in r^{2/3}\mathbb{Z}$, we have

$$\mathbb{P}(H_{x,y}) \geq 98/100. \quad (42)$$

This proposition will follow from the next two lemmas.

**Lemma 8.6** For $x \in \mathcal{X}_r$, $y \in r^{2/3}\mathbb{Z}$ and $B(x,y,r)$, let $A_1$ denote the event that for all $u, u' \in F$, we have $\Lambda \hat{X}_{u,u'} \leq -Lr^{1/3}$. Then we have $\mathbb{P}[A_1] \geq 999/1000$.

**Proof.** Let $u_1 = (x - r/2, x - r/2 + y - Lr^{2/3})$ and $u_2 = (x + r/2, x + r/2 + y - Lr^{2/3})$. Let $B$ denote the event that for all $u, u' \in F$ (note that $F$ is the line segment joining $u_1$ and $u_2$), $\hat{X}_{u,u'} \geq -Cr^{1/3}$. Let $G$ denote the event that $\Lambda \hat{X}_{u_1,u_2} \leq -2Lr^{1/3}$. It is then clear that $A \supseteq B \cap G$ since $L$ is sufficiently large.

Now it follows from Proposition 9.6 that $\mathbb{P}[B] \geq 1 - 10^{-5}$ since $C$ is sufficiently large. Now let $G_1$ (resp. $G_2$) denote the event that for all $u \in \Lambda$, we have $\hat{X}_{u_1,u} \leq C\sqrt{L}r^{1/3}$ (resp. $\hat{X}_{u,u_2} \leq C\sqrt{L}r^{1/3}$). Observe that since $C$ and $L$ are sufficiently large we have the

$$\hat{X}_{u_1,u_2} \leq \hat{X}_{u_1,u} + \hat{X}_{u,u_2} - 10Lr^{2/3}.$$  

Using the above fact and Corollary 9.9 it follows that $\mathbb{P}[G] \geq \mathbb{P}[G_1 \cap G_2] \geq 1 - 10^{-5}$ which completes the proof of the lemma. \hspace{1cm} \square

**Lemma 8.7** For $x \in \mathcal{X}_r$, $y \in r^{2/3}\mathbb{Z}$ and $B(x,y,r)$, let $A_2$ denote the event that for all $u \in F$, $u' \in U = \mathbb{P}(x,y - 2Mr^{2/3}, r, 3Mr^{2/3})$ we have $\hat{X}_{u,u'} \geq -Lr^{1/3}$ if $u < u'$ and $\hat{X}_{u,u'}' \geq -Lr^{1/3}$ if $u' < u$. Then $\mathbb{P}[A_2] \geq 99/100$.

**Proof.** Let $A_2^*$ denote the event that for all $u \in F$, all $u' \in U$, we have $\hat{X}_{u,u'} \leq -Lr^{1/3}$. Let $u_1$ be the leftmost point on $F$ as in the proof of Lemma 8.6. Let $R_U$ be the right boundary of $U$. Consider the following events

$$C_1 = \{\forall u, u' \in F : \hat{X}_{u,u'} \geq -Cr^{1/3}\},$$  

$$C_2 = \{\forall (u, u') \in S(U) : |\hat{X}_{u,u'}| \leq Cr^{1/3}\},$$  

$$C_3 = \{\forall u \in R_U : \hat{X}_{u_1,u} \leq -2Lr^{1/3}\}.$$  

It can then be proved along the lines of Lemma 8.2 that $\mathbb{P}[A_2^*] \geq 1 - 10^{-3}$. The other cases can be dealt with similarly and the lemma follows. \hspace{1cm} \square

**Proof.** [of Theorem 8.4] Let $\Gamma$ be the topmost maximal path in $\Pi$ from $0$ to $n$ in $\Pi$. Fix $x \in \mathcal{X}_r$. Observe that, for an increasing path $\gamma$ from $0$ to $n$, the event $\{\Gamma = \gamma\}$, $R_{\gamma} H_{x,\gamma}$ are all decreasing in the configuration on $\mathbb{R}^2 \setminus \{\gamma\}$, Hence it follows from FKG inequality and Lemma 8.4 that

$$\mu_x^*(H_x | \Gamma = \gamma) \geq \mu(H_x | \Gamma = \gamma) \geq \mu[H_{x,\gamma}] \geq \min_y \mathbb{P}[H_{x,y}].$$

The theorem follows by averaging over $\gamma$ and using Proposition 8.5. \hspace{1cm} \square
8.3 Bound on $Q_x$

In this section we bound the probability of $Q_x$.

**Theorem 8.8** For each $r \in R$, for each $x \in X_r$, we have $\mu^*_x(Q_x) \geq \frac{999}{1000}$.

We start with the following lemma.

**Lemma 8.9** An increasing path $\gamma$ from 0 to $n$ is called to be steep at end if either $\frac{n - \gamma_{10}}{10} \notin \left(\frac{15}{\psi}, \frac{\psi}{15}\right)$ or $\frac{n - \gamma_{10}}{10} \notin \left(\frac{5}{\psi}, \frac{\psi}{5}\right)$. Let $A$ denote the event that there exists a steep at end path $\gamma$ from 0 to $n$ with $\ell_\gamma > 2n - n^{2/5}$. Then $P[A] \leq e^{-n^{0.1}}$.

**Proof.** This lemma is proved by showing that since $\psi$ is large enough the expected length of an increasing path which is steep at end is much smaller than the maximal increasing path and using Theorem 1.2. The proof is similar to Lemma 8.11 and we omit the details here.

**Lemma 8.10** Suppose $\gamma$ is a steep increasing path from 0 to $n$ that is not steep at end.

Then there exists $u_1 = (x_1, y_1)$ and $u_2 = (x_2, y_2)$ in $\mathbb{Z}^2 \cap [0, n]^2$ satisfying the following conditions.

1. $\frac{n}{10} \leq x_1 < x_2 \leq \frac{9n}{10}$.
2. Either $\frac{y_2 - y_1}{x_2 - x_1} \in \left(\frac{\psi}{10}, \frac{\psi}{2}\right)$ or $\frac{y_2 - y_1}{x_2 - x_1} \in \left(\frac{2}{\psi}, \frac{10}{\psi}\right)$.
3. $\frac{y_1}{x_1} + \frac{n - y_1}{n - x_1} \in \left(\frac{10}{\psi}, 0.99\psi\right)$
4. $(x_2 - x_1) \wedge (y_2 - y_1) \geq \frac{n^{2/3}}{\log^8 n}$.
5. $\gamma_{x_1} \in [y_1, y_1 + 1)$ and $\gamma_{x_2} \in (y_2 - 1, y_2]$.

**Proof.** This lemma follows from the definition of steep path and steepness at ends.

A pair of points $u_1$ and $u_2$ in $\mathbb{Z}^2 \cap [0, n]^2$ satisfying the first 4 conditions in Lemma 8.10 is called inadmissible. We have the following lemma.

**Lemma 8.11** For a pair of inadmissible points $u_1 = (x_1, y_1)$ and $u_2 = (x_2, y_2)$, let $u_1' = (x_1, y_1 + 1)$ and $u_2' = (x_2, y_2 - 1)$. Let $A$ denote the event that there exists a pair $(u_1, u_2)$ of inadmissible points such that

$$X_{0, u_1'} + X_{u_1, u_2} + X_{u_2', n} \geq 2n - n^{0.49}.$$ 

Then $P[A] \leq e^{-n^{0.1}}$.

**Proof.** Fix a pair $(u, u')$ of inadmissible points. Since $\psi$ is sufficiently large it follows from an elementary computation that

$$\mathbb{E}X_{0, u_1'} + \mathbb{E}X_{u_1, u_2} + \mathbb{E}X_{u_2', n} < 2n - n^{0.5}.$$ 

The lemma now follows from using Theorem 1.2 and taking a union bound over all pairs of inadmissible points.
Now we are ready to prove Theorem 8.8.

**Proof.** [of Theorem 8.8] From Lemma 8.9, Lemma 8.10 and Lemma 8.11 it follows that

\[ \mathbb{P}[\text{Steepness condition fails}] \leq e^{-n^{0.1}} \]  

(43)

Now fix \( x \in X_r \). Since resampling at a fixed location does not change the law of the point configuration it follows by using (43) and taking a union bound over \( y \in \mathbb{R}^{2/3} \mathbb{Z} \) and \( i \in \left[ \frac{1}{100e^{n^{0.1}}} \right] \) that for \( n \) sufficiently large

\[ \mathbb{P}[\bigcup_i S_i^c] \leq e^{-n^{0.05}}. \]

It follows from Markov’s inequality that

\[ \mathbb{P}[\mathbb{P}[\bigcup_i S_i^c] \geq e^{-n^{1/100}}] \leq 10^{-5}. \]

It now follows that for all \( x \in X_r \), \( \mu(Q_{x}) \geq \frac{999}{1000} \). Since \( Q_x \) is a decreasing event it follows from FKG inequality that for each \( x \in X_r \)

\[ \mu^*_x(Q_x | \Gamma = \gamma) \geq \mu(Q_x | \Gamma = \gamma). \]

The theorem now follows from averaging over \( \gamma \). \( \square \)

### 8.4 Proof of Theorem 3.5

Finally we are ready to prove Theorem 3.5.

**Proof.** [of Theorem 3.5] Notice that by definition of \( G_x \) does not depend on the configuration in the interior of the walls of \( B(x, r) \). Hence it follows that \( \mu^*_x(G_x) = \mu(G_x) \) The proposition now follows from Corollary 7.2, Theorem 8.4 and Theorem 8.8. \( \square \)

### 9 Consequences of moderate deviation estimates

In this section we establish several consequences of the moderate deviation estimates Theorem 1.3 and Theorem 1.2 which are useful for our purposes.

#### 9.1 First order and second order approximation of \( X_{u,u'} \)

We shall mostly have to deal with pairs of points \( u, u' \in \mathbb{R}^2 \) with \( u < u' \) such that the slope of the line joining \( u \) and \( u' \) is neither too large nor too small. We shall work with first and second order approximations of \( \mathbb{E}[X_{u,u'}] \) in this case.

First we have the following easy corollary of Theorem 1.2 and Theorem 1.3.

**Corollary 9.1** Let \( \psi > 0 \) be fixed. There exist constants \( C_1 = C_1(\psi), r_0 = r_0(\psi), \theta_0 = \theta_0(\psi) > 0 \) such that for points \( u = (x, y) \) and \( u' = (x', y') \) in \( \mathbb{R}^2 \) such that \( x' - x = r \geq r_0 \), and

\[ \frac{1}{\psi} \leq \frac{y' - y}{x' - x} \leq \psi, \]

we have that

\[ \mathbb{E}[X_{u,u'}] = 2\sqrt{r(y - y')} + O(r^{1/3}). \]

Further, for \( \theta > \theta_0 \) we have

\[ \mathbb{P}[|\hat{X}_{u,u'}| > \theta r^{1/3}] \leq e^{-C_1 \theta}. \]  

(44)
The following expression for $\mathbb{E}[X_{u,u'}]$ will be useful.

**Lemma 9.2** Let $u = (x, y) < u' = (x', y') \in \mathbb{R}^2$ be such that $|x' - x| = r$ and $\frac{y' - y}{x' - x} = m$ where $m \in \left(\frac{1}{\psi}, \psi\right)$. Suppose $u_0 = (x, y + h_0 r^{2/3})$ and $u_1 = (x', y' + h_1 r^{2/3})$ be such that the slope of the line joining $u_0$ and $u_1$ is in $\left(\frac{1}{\psi}, \psi\right)$, and $|h_1 - h_0| \leq r^{1/10}$. Then for $r$ sufficiently large

$$
\mathbb{E}[X_{u_0,u_1}] = 2\sqrt{mr} + \frac{h_1 - h_0}{\sqrt{m}} r^{2/3} + O(r^{1/3}) - \frac{(h_1 - h_0)^2}{4m^{1/2}} r^{1/3}.
$$

**Proof.** Follows from Corollary 9.1 and observing that for $x \in (-1, 1)$ we have $(1 + x)^{1/2} = 1 + \frac{x^2}{2} - \frac{x^4}{8} + O(x^3)$. \hfill \square

The following corollary is a special case of Lemma 9.2 which will be useful to us and hence we state it separately.

**Corollary 9.3** In the set-up of Lemma 9.2 with $m = 1$ we have

$$
\hat{X}_{u_0,u_1} \leq \hat{X}_{u,u'} + O(r^{1/3}) - \frac{(h_1 - h_0)^2}{8} r^{1/3}.
$$

The quadratic term above may be viewed as a penalty term which is incurred for deviating from the straight line path as illustrated in the next lemma.

**Lemma 9.4** Let $u = (x, y) < u' = (x', y') \in \mathbb{R}^2$ be such that $|x' - x| = r$ and $\frac{y' - y}{x' - x} = m$ where $m \in \left(\frac{2}{\psi}, \frac{\psi}{2}\right)$. Let $u_0 = (x_0, y_0) = (x + \frac{r}{2}, y + \frac{mr}{2} + h_1 r^{2/3})$ be such that slope of the lines joining $u_0$ to $u$ and $u'$ are in $\left(\frac{1}{\psi}, \psi\right)$. Then

$$
\mathbb{E}[X_{u,u}] + \mathbb{E}[X_{u_0,u}] - \mathbb{E}[X_{u,u'}] \leq O(r^{1/3}) - \frac{h^2}{8(m \vee 1)^{1/2}} r^{1/3}.
$$

**Proof.** Proof is similar to that of Lemma 9.2 and we omit the details. \hfill \square

We also need the following similar lemma.

**Lemma 9.5** Let $u = (x, y) < u' = (x', y') \in \mathbb{R}^2$ be such that $|x' - x| = r$ and $\frac{y' - y}{x' - x} = m$ where $m \in \left(\frac{2}{\psi}, \frac{\psi}{2}\right)$. Consider points $u = u_0 < u_1 < u_2 < \cdots < u_\ell = u'$ such that $u_i = (x_i, y_i)$, $y_i = (y_0 + m(x_i - x_0) + h_i)$. Where $|h_i| \leq h(|x_i - x_{i-1}|^{2/3} \wedge |x_{i+1} - x_i|^{2/3})$. Then there exists $r_0 = r_0(\psi, h) > 0$ and $\theta = \theta(\psi, h) > 0$ such that if $\min_i |x_{i+1} - x_i| \geq r_0$, then

$$
\left| \left( \sum_i \mathbb{E}[X_{u_i,u_{i+1}}] \right) - \mathbb{E}[X_{u,u'}] \right| \leq \theta(r^{1/3} + \sum_i |x_{i+1} - x_i|^{1/3}).
$$

**Proof.** Follows from Corollary 9.1 and Lemma 9.2. \hfill \square

### 9.2 Shorter paths are unlikely in a rectangle

For our purposes we shall need to control the fluctuation in the length of maximal paths between “most” pairs of points in a parallelogram of suitable length and height. We need the following notations to make a precise statement.

Consider the parallelogram $U = U_{h,m,\ell}$ whose four corners are $(0, -\ell h^{2/3})$, $(0, \ell h^{2/3})$, $(h, mh - \ell h^{2/3})$, $(h, mh + \ell h^{2/3})$. Recall the definition of $S(U) \subseteq U^2$. For $u = (x, y)$ and $u' = (x', y') \in U$, $(u, u') \in S(U)$ iff $\frac{2}{\psi} < \frac{y' - y}{x' - x} \leq \frac{\psi}{2}$. We have the following theorem.
Proposition 9.6 Consider the parallelogram $U = U_{h,1,1}$. There exists an absolute constant $c_1 > 0$, $h_0 = h_0(\psi) > 0$ and $\theta_0 = \theta_0(\psi) > 0$ such that for all $h > h_0$ and $\theta > \theta_0$

$$
P \left( \inf_{(u,u') \in S(U)} \hat{X}_{u,u'} \leq -\theta h^{1/3} \right) \leq e^{-c_1 \theta}. \tag{45}$$

We shall need the following lemma to prove Proposition 9.6.

Lemma 9.7 Consider the parallelogram $U = U_{h,m,1}$ where $m \in (\frac{4}{3^0}, \frac{4}{3^1})$. Define $L(U) = U \cap \{x \leq h/8\}$ and $R(U) = U \cap \{x \geq 7h/8\}$. There exists constants $h_1 > 0$, $\theta_1 > 0$, $c_2 > 0$ such that for all $h > h_1$ and $\theta > \theta_1$ we have

$$
P \left( \inf_{u \in L(U), u' \in R(U)} \hat{X}_{u,u'} \leq -\theta h^{1/3} \right) \leq e^{-c_2 \theta}. \tag{46}$$

Before proving Lemma 9.7, we first prove the following easier lemma.

Lemma 9.8 Consider the parallelogram $U = U_{h,m,2}$. Define $u_* = (h', h'm)$ and $L'(U) = U \cap \{x < h'/4\}$. Then there exist constants $h_0'$, $\theta_2$ and $c_3 > 0$ such that for all $h' > h_0'$ and $\theta > \theta_2$ we have

$$
P \left( \inf_{u \in L'(U)} \hat{X}_{u,u_*} \leq -\theta h^{1/3} \right) \leq e^{-c_3 \theta}. \tag{47}$$

Proof. For $h'$ sufficiently large fix $h' >> a >> r_0$ where $r_0$ is given by Corollary 9.1 chosen such that $\frac{h'}{a} = 8^K$ for some integer $K > 0$. For each $k \in \{1, \ldots, K\}$ we define the following sets of points. Let $S_k = \{a\ell 8^k : \ell = 0, 1, \ldots, 8^{K-k}\}$ and $T_k = \{2^{2/3}(4^k : \ell = -2 \times 4^{K-k}, \ldots, 2 \times 4^{K-k}\}$. Define $V_k$ to be the set of all points $(x,y) \in U$ such that $x \in S_k$ and $y - mx \in T_k$.

At level $k$, define a graph $T_k$ with the vertex set $V_k$ where $(x,y), (x',y') \in v_k$ is connected by an edge if $x \neq x'$, $|x - x'| \leq 20 \cdot 8^k a$ and $|(y - y') - m(x - x')| \leq 30.4^k a^{2/3}$.

Let $E_k$ denote the following event.

$$
E_k := \{ \hat{X}_{u,v'} \geq -\frac{\theta h^{1/3}}{100} (1.5)^{K-k} \forall v, v' \in T_k \}.
$$

Claim: We claim that for $h$ and $\theta$ sufficiently large we have

$$
\left\{ \inf_{u \in L'(U)} \hat{X}_{u,u_*} \leq -\theta h^{1/3} \right\} \supseteq \bigcap_{k=1}^K E_k.
$$

Proof of Claim.

Fix $u = u_{-1} = (x_{-1}, y_{-1}) \in L'(U)$. Let $\text{Int}(z) = \lfloor z \rfloor$ if $z > 0$ and $\lceil z \rceil$ if $z < 0$. Define points $u_i = (x_i, y_i)$ for $i \geq 0$ recursively as follows.

$$
x_i = a(\lfloor x_{i-1} - 8^{-i}a \rfloor + 1)8^i; y_i = mx_i + \text{Int} \left( (y_0 - mx_0) \left( \frac{h' - x_i}{h' - x_0} \right) a^{-2/3} 4^{-i} \right) a^{2/3} 4^i.
$$

Observe the following.
(i) \((x_i, y_i) \in V_i\) for each \(i \geq 0\). Also, there is \(i_0 \leq K\) such that \((x_{i_0}, y_{i_0}) = u_\ast\).

(ii) For \(h'\) sufficiently large there is an edge in \(T_i\) between \((x_i, y_i)\) and \((x_{i+1}, y_{i+1})\). To see this notice that \(0 < x_{i+1} - x_i < 8^{i+1}a\) and \(|(y_{i+1} - y_i) - m(x_{i+1} - x_i)| \leq 5.4^i a^{2/3} + 20 \frac{8^i a}{h'^{1/3}}\).

We can lower bound \(X_{u,u_\ast}\) by,

\[
X_{u,u_\ast} \geq \sum_{i=0}^{i_0} X_{u_{i-1},u_i}.
\]

Taking \(h' >> a\) ensures that \(\tilde{X}_{u,u_\ast} \geq -h'^{1/3}\) and Lemma 9.5 it follows that for \(\theta\) sufficiently large we have

\[
\tilde{X}_{u,u_\ast} - \sum_{i=0}^{i_0} \tilde{X}_{u_{i-1},u_i} \geq -\frac{\theta}{100} h'^{1/3} - \frac{\theta}{100} \sum_{i=0}^{i_0} 2i a^{1/3} \geq -\frac{\theta}{2} h'^{1/3}.
\]

which completes the proof of the claim.

To complete the proof of the lemma it remains to obtain a lower bound for \(\mathbb{P}[\cap_k \mathcal{E}_k]\). From Corollary 9.1 for \((v,v') \in T_k\) and for \(\theta\) sufficiently large

\[
\mathbb{P}[\tilde{X}_{v,v'} \leq - \frac{\theta h'^{1/3}}{100}] \leq e^{-c\theta(k^{4/3}K-k)}.
\]

Now the number of edges \((v,v') \in T_k\) is polynomial in \(8^{K-k}\), so taking a union bound over all \((v,v') \in T_k\) and over \(k\) we get that \(\mathbb{P}[\cup_k \mathcal{E}_k] \geq 1 - e^{-c_3\theta}\) provided that \(\theta > \theta_2\) sufficiently large and \(h'>h'_0\) sufficiently large completing the proof.

Now we are ready to prove Lemma 9.7.

**Proof.** [Proof of Lemma 9.7] Define \(u_\ast = \left(\frac{h}{2}, \frac{mk}{2}\right)\). Let \(E_L\) and \(E_R\) denote the events

\[
E_L = \{ \inf_{u \in L(U)} \tilde{X}_{u,u_\ast} \leq -\theta h^{1/3}/100\};
\]

\[
E_R = \{ \inf_{u' \in R(U)} \tilde{X}_{u_\ast,u'} \leq -\theta h^{1/3}/100\}.
\]

By Lemma 9.8 for \(h\) sufficiently large and for \(\theta\) sufficiently large we have that \(\mathbb{P}[E_L \cup E_R] \leq e^{-c_2 \theta}\) for some constant \(c_2 > 0\). Notice that it follows from Lemma 9.5 that for all \(u \in L(U)\) and \(u' \in R(U)\) we have for \(\theta\) sufficiently large

\[
\tilde{X}_{u,u'} \geq \tilde{X}_{u,u_\ast} - \tilde{X}_{u_\ast,u'} - \theta h^{1/3}/100.
\]

It hence follows that on \(E_L^c \cap E_R^c\) we have for \(\theta\) sufficiently large and \(h\) sufficiently large

\[
\inf_{u \in L(U), u' \in R(U)} \tilde{X}_{u,u'} \geq -\theta h^{1/3}.
\]

Now we are ready to give the proof of Proposition 9.6.

**Proof.** [Proof of Proposition 9.6] Pick \(h\) sufficiently large such that \(h^{1/3} >> h_1\) where \(h_1\) is given by Lemma 9.7. Let \(U = U_{h,1,1}\) be as in the statement of the theorem and let us define the following sets of points in \(U\). For \(k \geq 0\), define

\[
S_k = \{ \ell h 2^{-3} 2^{-k} : \ell = 0, 1, \ldots, 8 \times 2^k \}.
\]
and
\[ T_k = \{ \ell h^{2/3}2^{-2k/3} : \ell = -2^k, \ldots, 2^k \}. \]

Consider the parallelogram whose corners are \((x_1, y_1), (x_1, y_1 + h^{2/3}2^{-2k/3}), (x_1 + h2^{-k}, y_1 + h2^{-k} + (s_2 - s_1)h^{2/3}2^{-2k/3})\) and \((x_1 + h2^{-k}, y_1 + h2^{-k} + (s_2 - s_1 + 1)h^{2/3}2^{-2k/3})\) where \(x_1 = \ell h^{2/3}2^{-k} \in S_k \) and \(y_1 = x_1 + s_1 h^{2/3}2^{-2k/3} \in T_k\). Denote this parallelogram by \(U_{k,\ell,s_1,s_2}^*\). Let \(D_k\) denote the set of all parallelograms \(U_{k,\ell,s_1,s_2}^*\) such that
\[ \frac{4}{3\psi} < 1 + \frac{s_2 - s_1}{2^{-k/3}h^{1/3}} < \frac{3\psi}{4}. \]

Let \(E_k\) denote the following event.
\[ E_k = \{ \forall U^* \in D_k, \inf_{u \in L(U^*), u' \in R(U^*)} \tilde{X}_{u,u'} \geq -\theta h^{1/3} \}. \]

First we prove that
\[ \bigcap_{k=0}^{\lfloor 5 \log h/6 \log 2 \rfloor} E_k \leq \{ \inf_{(u,u') \in \mathcal{S}(U)} \tilde{X}_{u,u'} \geq -\theta h^{1/3} \}. \]

To prove this first observe the following. If \(u = (x,y) < u' = (x',y') \in U\) is such that \((u,u') \in \mathcal{S}(U)\) and \(|x' - x| > h^{1/6}\), then there exists \(k \leq 5 \log h/6 \log 2\) and \(U^* \in D_k\) such that \(u \in L(U^*), u' \in R(U^*)\). The assertion follows by observing that for \(u = (x,y)\) and \(u' = (x',y')\) with \((u,u') \in \mathcal{S}(U)\) and \(|x - x'| \leq h^{1/6}\), then \(X_{u,u'} \geq -\theta h^{1/3}\) for \(h\) sufficiently large.

It now remains to estimate \(P[E_k]\).

Notice that \(|D_k| \leq 8^{k+1}\). Using Lemma \[.7\] and union bound it follows that for \(h\) sufficiently large (with \(h^{1/6} >> h_1\)) and for all \(\theta\) sufficiently large we have for all \(k\)
\[ P[E_k] \leq 8^{k+1}e^{-c_2\theta 2^{k/3}}. \]

Taking a union bound over \(k \in \{0,1,\ldots,5 \log h/6 \log 2\}\) we get the assertion of the theorem.

Proposition \[.6\] has the following immediate corollary.

**Corollary 9.9** Consider \(U = U_{h,1,\ell}\) with \(\ell > 1\). There exists an absolute constant \(c_1 > 0\), \(h_0 > 0\) and \(\theta_0 = \theta_0(\psi) > 0\) such that we have for all \(h > h_0\) and \(\theta > \theta_0\)
\[ P \left( \inf_{(u,u') \in \mathcal{S}(U)} \tilde{X}_{u,u'} \leq -\theta \sqrt{\ell} h^{1/3} \right) \leq e^{-c_1\theta}. \] (48)

### 9.3 Longer paths are unlikely too

In this subsection we prove results analogous to the previous subsection concerning upper tails of \(\sup_{u,u'} \tilde{X}_{u,u'}\) where the supremum is taken over ‘most’ points in an appropriate rectangle. Recall the notation \(U_{h,m,\ell}\) from the previous subsection. We have the following theorem.
Proposition 9.10 Consider the parallelogram $U = U_{h,m,1}$ where $\frac{4}{\psi} < m < \frac{\psi}{4}$. There exists an absolute constant $c_1 > 0$, $h_0 > 0$ and $\theta_0 = \theta_0(\psi) > 0$ such that we have for all $h > h_0$ and $\theta > \theta_0$

$$\mathbb{P} \left( \sup_{(u,u') \in S(U)} \tilde{X}_{u,u'} \geq \theta h^{1/3} \right) \leq e^{-c_1 \theta}. \quad (49)$$

Observe that $h_0, \theta_0$ and $c_1$ in the above theorem can be taken to be the same as in Proposition 9.6. The proof of Proposition 9.10 follows from the following lemma in an identical manner to the proof of Proposition 9.6 using Lemma 9.7. We omit the proof.

Lemma 9.11 Consider the parallelogram $U = U_{h,m,1}$ where $m \in \left( \frac{4}{\psi}, \frac{3\psi}{4} \right)$. Define $L(U) = U \cap \{ x \leq h/8 \}$ and $R(U) = U \cap \{ x \geq 7h/8 \}$. There exists constants $h_1 > 0, \theta_1 > 0, c_2 > 0$ such that for all $h > h_1$ and $\theta > \theta_1$ we have

$$\mathbb{P} \left( \sup_{u \in L(U), u' \in R(U)} \tilde{X}_{u,u'} \leq -\theta h^{1/3} \right) \leq e^{-c_2 \theta}. \quad (50)$$

Proof. Consider the point $v = (-7h/8, -7mh/8)$ and $v' = (15h/8, 15mh/8)$. Now observe that it follows from Lemma 9.5 that for $h$ sufficiently large and for $\theta$ sufficiently large we have for some absolute constant $c > 0$

$$\mathbb{P} \left[ \inf_{u \in L(U)} \tilde{X}_{v,u} \leq -\theta h^{1/3}/50 \right] \leq e^{-c \theta}$$

and

$$\mathbb{P} \left[ \inf_{u' \in R(U)} \tilde{X}_{v',u'} \leq -\theta h^{1/3}/50 \right] \leq e^{-c \theta}.$$ Observe that it follows from Lemma 9.11 that for $\theta$ sufficiently large we have

$$\tilde{X}_{v,v'} \geq \inf_{u \in L(U)} \tilde{X}_{v,u} + \inf_{u' \in R(U)} \tilde{X}_{v',u'} + \sup_{u \in \partial L, u' \in \partial R} \tilde{X}_{u,u'} - \theta h^{1/3}/10. \quad (51)$$

Let $F_1, F_2, F_3$ denote the events

$$F_1 = \left\{ \inf_{u \in L(U)} \tilde{X}_{v,u} \geq -\theta h^{1/3}/50 \right\};$$

$$F_2 = \left\{ \inf_{u' \in R(U)} \tilde{X}_{v',u'} \geq -\theta h^{1/3}/50 \right\};$$

$$F_3 = \left\{ \sup_{u \in \partial L, u' \in \partial R} \tilde{X}_{u,u'} \geq \theta h^{1/3} \right\}.$$

Then it is clear that

$$\{ \tilde{X}_{v,v'} \geq \theta h^{1/3}/2 \} \supseteq F_1 \cap F_2 \cap F_3.$$ Observe that the events $F_1, F_2, F_3$ are increasing in point configurations and hence by FKG inequality we have

$$\mathbb{P} \left[ \tilde{X}_{v,v'} \geq \theta h^{1/3}/2 \right] \geq \mathbb{P}[F_1] \mathbb{P}[F_2] \mathbb{P}[F_3]. \quad (52)$$

Also observe that it follows from Corollary 9.1 that for $h$ sufficiently large and $\theta$ sufficiently large we have for some absolute constant $c > 0$ that

$$\mathbb{P} \left[ \tilde{X}_{v,v'} \geq \theta h^{1/3}/2 \right] \leq e^{-c \theta}.$$

The above equation, together with (52), completes the proof of the Lemma. Proposition 9.10 has the following immediate corollary.
Corollary 9.12 Consider the parallelogram $U = U_{h, m, \ell}$ where \( \frac{4}{\psi} < m < \frac{\psi}{4} \) and $\ell > 1$. There exists an absolute constant $c_1 > 0$, $h_0 > 0$ and $\theta_0 = \theta_0(\psi) > 0$ such that we have for all $h > h_0$ and $\theta > \theta_0$

\[
\mathbb{P} \left( \sup_{(u, u') \in S(U)} \hat{X}_{u, u'} \geq \theta \sqrt{\ell} r^{1/3} \right) \leq e^{-c_1 \theta}. \tag{53}
\]

9.4 Exponential tails in transversal fluctuation

It was proved by Johansson in [15] that the transversal fluctuations of the longest increasing subsequence from $(0, 0)$ to $(n, n)$ is of the order $n^{2/3+o(1)}$. Using the estimates proved in the previous subsections we prove the following sharper version of Johansson’s result.

Theorem 9.13 Let $\Gamma^r = \Gamma = \{(x, \Gamma_x) : x \in [0, r]\}$ be the topmost maximal increasing path from $(0, 0)$ to $(r, r)$. Let $\Gamma^*_r = \sup_{x \in [0, r]} |\Gamma_x - x|$. Then there exist absolute positive constants $r_0$ and $k_0$ and $c_4$ such that for all $r > r_0$, $k > k_0$, we have

\[
\mathbb{P}[\Gamma^*_r \geq kr^{2/3}] \leq e^{-c k}.
\]

We shall need a few lemmas to prove Theorem 9.13

Lemma 9.14 Consider the parallelogram $U = U_{r, m, 1}$ where \( \frac{100}{\psi} < m < \frac{\psi}{100} \). Let $\partial L$ and $\partial R$ denote the left boundary and the right boundary of $U$ respectively. For $u \in \partial L$, $u' \in \partial R$, let $\Gamma_{u, u'}$ denote the maximal increasing path from $u$ to $u'$. Then there exist constants $c = c(\psi) > 0$, $k_0 = k_0(\psi) > 0$ and $r_0 = r_0(\psi) > 0$ such that for all $k \geq 2$ and for all $r > r_0$ we have

\[
\mathbb{P} \left( \sup_{u \in \partial L, u' \in \partial R} |\Gamma_{u, u'}^{r/2} - \frac{mr}{2}| \geq kr^{2/3} \right) \leq e^{-ck}. \]

Proof. Let $A$ denote the event

\[
A = \left\{ \sup_{u \in \partial L, u' \in \partial R} |\Gamma_{u, u'}^{r/2} - \frac{mr}{2}| \leq kr^{2/3} \right\}.
\]

For $\ell \geq 1$, let $B_\ell$ denote the event $\{\sup_{u, u'} \Gamma_{r/2}^{u, u'} - \frac{mr}{2} \in [k \ell r^{2/3}, k(\ell + 1) r^{2/3}]\}$. Similarly let $B'_\ell$ denote the event $\{\inf_{u, u'} \Gamma_{r/2} - \frac{mr}{2} \in [-k(\ell + 1) r^{2/3}, -k \ell r^{2/3}]\}$. Let $G$ denote the event that $|\Gamma_{r/2} - mr/2| \geq \frac{4mr}{\psi}$. It is clear that

\[
A^c \subseteq \bigcup_{\ell=1}^{\left[\frac{3mr^{1/3}}{100}\right]-1} B_\ell \cup B'_\ell \cup C.
\]

Let $u_\ell = (\ell, \frac{mr}{2} + k \ell r^{2/3})$. Let $L_\ell$ denote the line segment joining $u_\ell$ and $u_{\ell+1}$. It is easy to see that

\[
B_\ell \subseteq \left\{ \sup_{u_0 \in L_\ell, u \in \partial L, u' \in \partial R} X_{u, u_0} + X_{u_0, u'} - X_{u, u'} \geq 0 \right\}.
\]
Now notice that for $r$ sufficiently large and $k$ sufficiently large we have from Corollary 9.9 that
\[ P\left[ \inf_{u \in \partial L, u' \in \partial R} \bar{X}_{u, u'} \leq -k^{3/2} r^{1/3} \right] \leq e^{-ck} \]
for some constant $c > 0$. Notice that by taking $\psi$ sufficiently large, by Corollary 9.12 we have for $r$ and $k$ sufficiently large and $\ell \leq \frac{3m r^{1/3}}{10k}$ and some constant $c > 0$
\[ P\left[ \sup_{u \in \partial L, u_0 \in L_\ell} \bar{X}_{u, u_0} \geq \ell k^{3/2} r^{1/3} \right] \leq e^{-c k \ell} \]
and similarly
\[ P\left[ \sup_{u_0 \in L_\ell, u' \in \partial R} \bar{X}_{u_0, u'} \geq \ell k^{3/2} r^{1/3} \right] \leq e^{-c k \ell}. \]

Using Lemma 9.4 we have for $k$ sufficiently large and for all $u_0 \in L_\ell, u \in \partial L, u' \in \partial R$, we have
\[ X_{u, u_0} + X_{u_0, u'} - X_{u, u'} \leq \bar{X}_{u_1, u} + \bar{X}_{u_2, u_2} - \bar{X}_{u_1, u_2} - k^{7/4} \ell^2 r^{1/3}. \]
Taking a union bound over all $\ell \geq 0$ we get that for some absolute constant $c > 0$, $P[\cup_{\ell} B_{\ell}] \leq e^{-ck}$. We can bound $P[\cup_{\ell} B'_{\ell}]$ in an identical manner.

To complete the proof of the lemma we still need to bound $P[G]$. To this effect notice that
\[ P[G] \leq 2P \left[ \sup_{u \in \partial L, u' \in \partial R} X_{u,(r/2, mr/2, r^{2/3})} + X_{(r/2, 4mr/5, u') \sup X_{u, u'} \right]. \]

An elementary computation as in Lemma 9.4 shows that
\[ \sup_{u \in \partial L, u' \in \partial R} \mathbb{E} X_{u,(r/2, mr/2, r^{2/3})} + \mathbb{E} X_{(r/2, 4mr/5, u') \sup X_{u, u'} \leq -cr \]
some constant $c > 0$. It follows from Proposition 9.10 that for $r$ sufficiently large $P[G] \leq e^{-cr^{2/3}}$ for some absolute constant $c > 0$. Since we only need to consider the values of $k \leq r^{1/2}$ this finishes the proof of the lemma. \( \square \)

Let $k$ now be fixed. Choose $j_0 = j_0(k) > 0$ such that $2^{-j_0} r = \frac{k}{10} r^{2/3}$. Without loss of generality we can assume that $j_0$ is an integer. For $j = 1, 2, \ldots, j_0$, define $S_j$ by
\[ S_j = \{ \ell r 2^{-j} : \ell = 0, 1, \ldots, 2^j \}. \]

Let $A_j$ denote the event that for all $x \in S_j$, we have
\[ |\Gamma_x - x| \leq \frac{k}{10^5} \prod_{\ell=1}^{j-1} (1 + 2^{-\ell/10}) r^{2/3}. \]

We have the following lemma.

**Lemma 9.15** If $A_j$ holds for each $j \leq j_0$, then we have $\Gamma_x' \leq k r^{2/3}$.

**Proof.** Let $x \in [x_1, x_2]$ where $x_1, x_2 \in S_{j_0}$. Clearly then $|\Gamma_x - x| \leq |\Gamma_{x_1} - x_1| \vee |\Gamma_{x_2} - x_2| + \frac{k}{10^5} r^{2/3}$. The result follows. \( \square \)

The following lemma is an immediate corollary of Lemma 9.14.
**Lemma 9.16** There exists absolute constants $r_0, k_0, c$ such that for $r > r_0$ and $k > k_0$, we have $\mathbb{P}[A_r^c] \leq e^{-ck}$.

**Lemma 9.17** There exists positive absolute constants $r_0, k_0, c$ such that for all $r > r_0$, $k > k_0$ and $j_0(k) \geq j > 1$, we have $\mathbb{P}[A_j^c \cap A_{j-1}] \leq 2^{-j}e^{-ck}$.

Lemma 9.17 will follow from the following lemma.

**Lemma 9.18** Fix $j < j_0$ and $0 \leq h \leq 2^j$. Let $k_j = \frac{k}{10^j} \prod_{i=0}^{j-1}(1 + 2^{-i/10})$. Consider the line segments $L_1 = h2^{-j}r \times [-k_jr^{2/3}, k_jr^{2/3}]$ and $L_2 = (h+1)2^{-j}r \times [-k_jr^{2/3}, k_jr^{2/3}]$. Let $A_{h,j}$ denote the topmost longest increasing path from $u$ to $u'$. Let

$$A_{h,j} = \left\{ \sup_{u \in L_1, u' \in L_2} |\Gamma_{u,u'}^{h,j} - (2h + 1)2^{-(j+1)r}| \leq k_{j+1}r^{2/3} \right\}.$$ 

Then $\mathbb{P}[A_{h,j}^c] \leq 4^{-j}e^{-ck}$ for some absolute constant $c > 0$ and $k$ sufficiently large (not depending on $j$).

**Proof.** For $s, t = 1, 2, \ldots, 2k_j2^{j/3}$ define line segments $L_1^s$ and $L_2^t$ to be the line segments joining $(h2^{-j}r, -k_jr^{2/3} + (s - 1)2^{-2/3}j^{2/3})$ and $(h2^{-j}r, k_jr^{2/3} + s2^{-2/3}j^{2/3})$ and joining $((h+1)2^{-j}r, -k_jr^{2/3} + (t-1)2^{-2/3}j^{2/3})$ and $((h+1)2^{-j}r, k_jr^{2/3} + t2^{-2/3}j^{2/3})$ respectively. Define

$$A_{h,s,t} = \left\{ \sup_{u \in L_1^s, u' \in L_2^t} |\Gamma_{u,u'}^{h,s,t} - (2h + 1)2^{-(j+1)r}| \leq k_{j+1}r^{2/3} \right\}.$$ 

Clearly to prove the lemma it suffices to prove that $\mathbb{P}[A_{h,s,t}^c] \leq 16^{-j}e^{-ck}$.

Fix $s$ and $t$. If $\psi$ is large enough it follows from Lemma 9.14 that for $r$ sufficiently some constant $c > 0$

$$\mathbb{P}[A_{h,s,t}^c] \leq e^{-c(k_j+1-k_j)2^{2j/3}} \leq e^{-ck2^{2j/3}} \leq 16^{-j}e^{-ck}$$

by taking $k$ sufficiently large.

Taking a union bound over $s$ and $t$, the lemma follows. □

**Proof.** [Proof of Lemma 9.17] Notice that taking a union bound we get that $\mathbb{P}[A_j^c \cap A_{j-1}] \leq \sum_{h=1}^{2^{j-1}} \mathbb{P}[A_{h,j-1}^c]$, and the lemma follows from Lemma 9.18 □

Finally we prove Theorem 9.13

**Proof.** [Proof of Theorem 9.13] Notice that if $\mathbb{P}[(\bigcap_j A_j)^c] \leq \sum_j \mathbb{P}[A_j^c \cap A_{j-1}]$. The proposition now follows from Lemma 9.15, Lemma 9.16 and Lemma 9.17 □

The proof of the following corollaries follow along the same lines of proof of Theorem 9.13 and hence we omit it.

**Corollary 9.19** Let $L_{1,k}$ be the line segment joining $(0, -\frac{kr^{2/3}}{100})$ and $(0, \frac{kr^{2/3}}{100})$. Let $L_2$ be the line segment joining $(r, r - \frac{kr^{2/3}}{100})$ and $(r, r + \frac{kr^{2/3}}{100})$. For $u \in L_{1,k}$, $u' \in L_{2,k}$, let $\Gamma_{u,u'}^{c}$ denote the topmost maximal path from $u$ to $u'$. Let $\Gamma_{u,u',s}^{c} = \sup_{x \in [0,r]} |\Gamma_{u,u'}^{c} - x|$. Then there exist absolute positive constants $r_0$ and $k_0$ and $c_4$ such that for all $r > r_0$, $k > k_0$, we have

$$\mathbb{P} \left[ \sup_{u \in L_{1,k}, u' \in L_{2,k}} |\Gamma_{u,u',s}^{c} | \geq kr^{2/3} \right] \leq e^{-c_4k}.$$
**Corollary 9.20** Let $L_{1,k}$ be the line segment joining $(0,-\frac{kr^{2/3}}{100})$ and $(0,\frac{kr^{2/3}}{100})$. Let $L_2$ be the line segment joining $(r, mr - \frac{kr^{2/3}}{100})$ and $(r, mr + \frac{kr^{2/3}}{100})$ where $m \in (\frac{1}{\psi}, \psi)$. For $u \in L_{1,k}$, $u' \in L_{2,k}$, let $\Gamma^u_{u'}$ denote the topmost maximal path from $u$ to $u'$. Let $\Gamma^{u,u',s} = \sup_{x \in [0,r]} |\Gamma^u_{x,u} - x|$. Then there exist positive constants $r_0(\psi)$ and $h_0(\psi)$ and $c_4(\psi)$ such that for all $r > r_0$, $k > k_0$, we have

$$
P \left( \sup_{u \in L_{1,k}, u' \in L_{2,k}} |\Gamma^{u,u',s}| \geq kr^{2/3} \right) \leq e^{-c_4 k}.
$$

**9.5 Paths staying within a parallelogram**

Our objective in this subsection is to obtain bounds on lengths of the longest increasing path between two points completely contained in certain parallelograms.

**Lemma 9.21** Let $k > 0$ be fixed. Consider the parallelogram $U = U_{h,m,k}$ where $m \in (\frac{1}{\psi}, \psi)$. Let $u = (0,0)$ and $u' = (r, mr + hr^{2/3})$ where $|h| \leq \frac{k}{2}$. Then there exist positive constants $r_0(\psi)$, $\theta_0(\psi)$ and a constant $c = c(k, \psi) > 0$ such that for all $r > r_0$ and $\theta > \theta_0$ we have

$$
P[X_{u,u'}^\partial \leq -\theta r^{1/3}] \leq e^{-c(k)\sqrt{\theta}}.
$$

**Proof.** Note that we only need to consider the case where $\theta \leq 10\psi r^{2/3}$. Let $\theta$ be sufficiently large and set $J = [\theta^{3/4}]$. For $0 \leq j \leq J$, define $u_j = (\frac{2j}{J}, \frac{j r + hr^{2/3}}{J})$. Notice that,

$$
P[X_{u_j,u_{j+1}}^\partial \leq -\frac{\theta}{J} r^{1/3}] \leq \P[X_{u_j,u_{j+1}} \leq -\frac{\theta}{J} r^{1/3}] + \P[\text{geodesic from } u_j \text{ to } u_{j+1} \text{ exits } U].
$$

Notice that for some absolute constant $c > 0$ and for $r$ sufficiently large, Proposition 9.6 implies that the first term is bounded by $e^{-c\frac{\theta}{J} r^{2/3}}$. For $\theta$ sufficiently large, by Corollary 9.19 the second term is bounded by $e^{-c\theta r^{2/3}}$. Taking a union bound over all $j$ the result follows.

In the previous lemma we considered paths from the midpoints of a parallelogram. In the following theorem we extend this to any points along the sides recalling the notation $U_{h,m,\ell}$ and $S(U)$ from the previous section.

**Proposition 9.22** Let $W$ be a fixed positive constant. Consider the parallelogram $U = U_{r,1,W}$. Then there is an absolute constant $\theta_0 > 0$ such that for all $\theta > \theta_0$ and a constant $c = c(W) > 0$ such that and for all sufficiently large $r \geq r_0(W)$ we have

$$
P[\inf_{(u,u') \in S^\theta(U)} X_{u,u'}^\partial \leq -\theta r^{1/3}] \leq e^{-c \theta^{1/3}}.
$$

The argument for proving this theorem is along the general lines of the proof of Proposition 9.6, where we use Lemma 9.21 instead of Theorem 1.3. We need the following Lemma.

**Lemma 9.23** Consider the parallelogram $U = U_{r,m,W}$ where $m \in (\frac{2}{\psi}, \frac{1}{2})$. Define $u_* = (r, rm)$ and $L'(U) = U \cup \{x < r/4\}$. Then there exist constants $r_0$, $\theta_2$ and $c_3 > 0$ such that for all $r > r_0$ and $\theta > \theta_2$ we have

$$
P\left( \inf_{u \in L'(U)} X_{u,u_*}^\partial \leq -\theta r^{1/3} \right) \leq e^{-c_3 \theta^{1/3}}. \quad (54)
$$
Proof. The proof of this lemma is similar to the proof of Lemma 9.8. For \( r \) sufficiently large fix \( r >> a >> r_0 \) where \( r_0 \) is given by Lemma 9.21 such that \( \frac{r}{a} = 8^K \) for some integer \( K > 0 \). For each \( k \in \{0,\ldots,K\} \) we define the following sets of points. Let \( S_k = \{a\ell 8^k : \ell = 0,1,\ldots,8^{K-k}\} \) and \( T_k = \{a^{2/3} \ell 8^k : \ell = -W \times 4^{K-k},\ldots,W \times 4^{K-k}\} \). Define \( V_k \) to be the set of all points \((x,y) \in U \) such that \( x \in S_k \) and \( y - mx \in T_k \).

At level \( k \), define a graph \( T_k^* \) with the vertex set \( V_k \) where \((x,y), (x',y') \in V_k \) is connected by an edge if \( x \neq x' \), \(|x - x'| \leq 20 \cdot 8^k a \), \(|(y - y') - m(x - x')| \leq 30.4 a^{2/3} \) and \(|y - mx| \vee |y' - mx'| \leq W r^{2/3} - r^{1/4} \).

Let \( E_k \) denote the following event.

\[
E_k := \{ \tilde{X}_{v,v'}^{\partial U} \geq -\frac{\theta r^{1/3}}{100} (1.5)^{k-K} \forall v,v' \in T_k^* \}.
\]

Claim: We claim that for \( r \) and \( \theta \) sufficiently large we have

\[
\{ \inf_{u \in L'(U)} \tilde{X}_{u,u_*}^{\partial U} \geq -\theta r^{1/3} \} \supseteq \bigcap_{k=1}^{K} E_k.
\]

Proof of Claim.

Fix \( u = (x,y) \in L'(U) \). Define \( g(u) = u_{-1} = (x_{-1},y_{-1}) \) as follows. If \(|y - mx| \leq W r^{2/3} - 10 r^{1/4} \), then set \( u_{-1} = u \). Otherwise, if \( y - mx \leq -W r^{2/3} - 10 r^{1/4} \), set \( u_{-1} = (x,mx - W r^{2/3} + 10 r^{1/4}) \). If \( y - mx \geq W r^{2/3} - 10 r^{1/4} \) then set \( u_{-1} = (x_{-1},mx_{-1} + W r^{2/3} - 10 r^{1/4}) \) where \( x_{-1} \) solves the equation \( mx_{-1} + W r^{2/3} - 10 r^{1/4} = y \). Let \( \text{Int}(z) = [z] \) if \( z > 0 \) and \( \lceil z \rceil \) if \( z < 0 \). It is clear that for \( \theta \) sufficiently large we have

\[
\{ \inf_{u \in L'(U)} \tilde{X}_{u,u_*}^{\partial U} \geq -\theta r^{1/3} \} \supseteq \{ \inf_{u \in L'(U)} \tilde{X}_{g(u),u_*}^{\partial U} \geq -\theta r^{1/3}/2 \}.
\]

Fix \( u \in L'(U) \). Set \( u_{-1} = g(u) \). Define points \( u_i = (x_i,y_i) \) for \( i \geq 0 \) recursively as follows.

\[
x_i = a \left( \frac{x_{i-1} 8^{-i}}{a} + 1 \right) S^i; y_i = mx_i + \text{Int} \left( (y_{i-1} - mx_{i-1}) \left( \frac{r - x_i}{r - x_{i-1}} \right) a^{-2/3} 4^{-i} \right) a^{2/3} 4^i.
\]

Observe the following.

(i) \((x_i,y_i) \in V_i \) for each \( i \geq 0 \). Also, there is \( i_0 \leq K \) such that \((x_{i_0},y_{i_0}) = u_* \).

(ii) For \( r \) sufficiently large there is an edge in \( T_i^* \) between \((x_i,y_i) \) and \((x_{i+1},y_{i+1}) \). To see this notice that \( 0 < x_{i+1} - x_i < 8^{i+1} a \) and \(|(y_{i+1} - y_i) - m(x_{i+1} - x_i)| \leq 5.4 a^{2/3} + 20 8^i a^{2/3} \).

Observe that trivially we have

\[
\tilde{X}_{g(u),u_*}^{\partial U} \geq \sum_{i=0}^{i_0} \tilde{X}_{u_{i-1},u_i}^{\partial U},
\]

Also by taking \( r >> a \) we can ensure that \( \tilde{X}_{u,u_0} \geq -r^{1/3} \) and from Lemma 9.5 it follows that for \( \theta \) sufficiently large we have

\[
\tilde{X}_{u,u_*}^{\partial U} \geq \sum_{i=0}^{i_0} \tilde{X}_{u_{i-1},u_i}^{\partial U} \geq -\frac{\theta}{100} r^{1/3} - \frac{\theta}{100} \sum_{i=0}^{i_0} 2^i a^{1/3} \geq -\frac{\theta}{10} r^{1/3}.
\]
This completes the proof of the claim.
To complete the proof of the lemma it remains to obtain a lower bound for \( \mathbb{P}[\cap_k \mathcal{E}_k] \).
Observe that for any pair of points \((u_{i-1}, u_i) \in T_i^*\), we have that \((W^{2/3} - |y_i - mx_i|) \land (W_i^{2/3} - |y_i - mx_{i-1}|) \geq \frac{1}{100}|x_{i+1} - x_i|^{2/3}\), and hence we know from Lemma 9.21 that there exists an absolute constant \(c > 0\) such that for \((v, v') \in T_k\) and for \(\theta\) sufficiently large
\[
\mathbb{P}[\hat{X}_{v, v'} \leq -\frac{\theta r^{1/3}}{100}(1.5)^{(k-K)}] \leq e^{-c\sqrt{\theta(1/3)^{K-k}}}.
\]
Now the number of edges \((v, v') \in T_k\) is polynomial in \(W^{8K-k}\), so taking a union bound over all \((v, v') \in T_k\) we get that for some \(c = c(W) > 0\)
\[
\mathbb{P}[^{\cap_k} \mathcal{E}_k] \geq 1 - e^{-c\sqrt{\theta(1/3)^{K-k}}}.
\]
Taking a union bound over \(k\), it follows that for some \(c_3 = c_3(W) > 0\) and for \(\theta > \theta_2\) sufficiently large and for \(r\) sufficiently large we have
\[
\mathbb{P}[\cap_k \mathcal{E}_k] \geq 1 - e^{-c_3 \theta^{1/3}}.
\]
This completes the proof of the lemma. □

Proof. [Proof of Proposition 9.22] Proposition 9.22 can be derived from Lemma 9.23 in exactly the same way as Proposition 9.6 is proved using Lemma 9.8 so we omit the details. □

10 Necessary Modifications for the Discrete Case

In this section we briefly describe how we can adapt these arguments to prove Theorem 2. The argument is in essence the same, though some minor modification is necessary. In this case we shall only consider increasing paths between points in \(\mathbb{Z}^2\). For \(u = (x, y), u' = (x', y') \in \mathbb{Z}^2\), we define
\[
X_{u, u'} = \max_{\pi} \sum_{v \in \pi \setminus \{u'\}} \xi_v
\]
where the maximum is taken over all increasing lattice paths \(\pi\) from \(u\) to \(u'\). The Tracy-Widom Fluctuation result in this case is due to Johansson [14].

Theorem 10.1 Let \(h > 0\) be fixed. Let \(v = (0, 0)\) and \(v_n = (n, [hn])\). Let \(T_n = X_{v, v_n}\). Then
\[
\frac{T_n - (1 + \sqrt{h})^2 n}{h^{-1/6}(1 + \sqrt{h})^{4/3} n^{1/3}} \overset{d}{\to} F_{TW}.
\] (55)

As before, we define \(\hat{X}_{u, u'} = X_{u, u'} - \mathbb{E}X_{u, u'}\) and \(\hat{X}_{u, u'} = X_{u, u'} - 2d(u, u')\) since now \(2d(u, u')\) is the first order term in \(\mathbb{E}X_{u, u'}\). We have the following moderate deviation estimates from [4] and [3].

Theorem 10.2 Let \(\psi > 1\) be fixed. Let \(Z_{h, n}\) denote the last passage time from \((0, 0)\) to \((n, [hn])\) where \(h \in (1/\psi, \psi)\). Then there exist constants \(N_0 = N_0(\psi), t_0 = t_0(\psi)\) and \(c = c(\psi)\) such that we have for all \(n > N_0, t > t_0\) and all \(h \in (1/\psi, \psi)\)
\[
\mathbb{P}[|Z_{h, n} - n(1 + \sqrt{h})^2| \geq tn^{1/3}] \leq e^{-ct}.
\]
Now the proof proceeds in exactly similar manner, we establish all consequences of moderate deviation estimates in §9 (with possibly changed constants) using Theorem 10.2 instead of Theorem 1.3 and Theorem 1.2.

The proof now proceeds in exactly similar manner. We define key events exactly as in §3. The results in §§6, 7, 8 and §4 follows in exactly the same manner.

To get an improved path in an analogous manner to the argument of §5, we do the following. Observe that

$$\zeta_{1-\epsilon} \overset{d}{=} \zeta_1 + B_\epsilon \zeta'_{1-\epsilon}$$

where $\zeta_{1-\epsilon}, \zeta'_{1-\epsilon}$ are exponential variables with rate $(1-\epsilon)$, $\zeta_1$ is an exponential variable with rate 1, $B_\epsilon$ is a Ber($\epsilon$) variable and all of these are independent. Introducing a defect on the diagonal is equivalent to reinforcing the diagonal (i.e., adding to the entries on the diagonal) with these independent variables with positive expectation. The rest of the arguments in §5 works in the same way as before, where we reinforce on discrete lines $y = x + m$ with $m \in \mathbb{Z}$, and add up the improvements instead of integrating. Notice that we can get rid of the regularity conditions in $G_x$ for the discrete case, as for any path $\gamma$ the expected increase in length in the reinforced environment is proportional to the number of points $\gamma$ hits the reinforced line.

Another thing one needs to take care of is the following. In the discrete set up to make sure that the length of two augmented paths is equal to the sum of their individual lengths, we do not add up the contribution of the very last vertex. To make sure, that this does not change any of our estimates (and also to make sure that the estimates about paths conditioned not to hit a certain paths work as before), we need to condition on the event that no passage time on $[0,n]^2$ is bigger than $\log^2 n$. This event holds with high probability and hence rest of the arguments will work as before.

**Remark:** Notice that the main difference between the discrete and the continuous case is that the continuous case, the moderate deviation estimates for the length of a path between two corners of a rectangle does not depend on the aspect ratio of the rectangle. In the discrete case, we can only get uniform moderate deviation estimates for rectangles with bounded aspect ratio, which forces us to work harder to avoid steep paths, or work out different estimates for steep paths. To deal with only the continuous case, one can get rid of all the conditions involving $\psi$ (e.g. the steepness condition) and also one can prove Proposition 9.10 and Proposition 9.6 without the assumptions that the slope between the pairs of points considered are bounded. However, we worked under these assumptions in order to have a proof which can be adapted to the discrete case with minimal changes.

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