Fast decaying and slow decaying solutions of Lane-Emden equations involving nonhomogeneous potential

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Abstract
Our purpose of this paper is to study positive solutions of Lane-Emden equation

$$-\Delta u = Vu^p \quad \text{in} \quad \mathbb{R}^N \setminus \{0\}$$

disturbing by a non-homogeneous potential $V$ when $p \in \left(\frac{N}{N-2}, p_c\right)$, where $p_c$ is the Joseph- Léglérison exponent. We construct a sequence of fast decaying solutions and slow decaying solutions with appropriated restrictions for $V$.

1 Introduction
Our concern in this paper is to consider fast decaying solutions of weighted Lane-Emden equation in punctured domain

$$\begin{cases}
-\Delta u = Vu^p & \text{in} \quad \mathbb{R}^N \setminus \{0\}, \\
u > 0 & \text{in} \quad \mathbb{R}^N \setminus \{0\},
\end{cases} \quad (1.1)$$

where $p > 1$, $N \geq 3$ and the potential $V$ is a locally Hölder continuous function in $\mathbb{R}^N \setminus \{0\}$.

When $V = -1$, the nonlinear term is known as an absorption and problem (1.1) admits a positive solution $c_p|x|^{\frac{-2}{p}}$ for any $p \in (0, \frac{N}{N-2})$ and Brezis-Veron in [6] showed that it has no positive solution when $p \geq \frac{N}{N-2}$. This type of nonexistence in the super critical case could also done in [25]. While the isolated singularities of this elliptic problems in punctured domain subject to Dirichlet boundary condition are well studied in [24, 26–29] and a survey in [30].

When $V = 1$, equation (1.1) is well known as Lane-Emden-Fowler equation

$$-\Delta u = u^p \quad \text{in} \quad \mathbb{R}^N \setminus \{0\}, \quad (1.2)$$

which has been extensively studied in the last decades. When $p \leq \frac{N}{N-2}$, problem (1.2) has no positive solution, see the reference [5] and for $p > \frac{N}{N-2}$, problem (1.2) always has a singular solution $w_p(x) = c_p|x|^{\frac{-2}{p-1}}$ with

$$c_p = \left(\frac{2}{p-1}(N - 2 - \frac{2}{p - 1})\right)^{\frac{1}{p-1}}. \quad (1.3)$$

When $p \in \left(\frac{N}{N-2}, \frac{N+2}{N-2}\right)$, positive isolated singular solutions of the problem (1.2) have the following structure:

\[\text{[References and footnotes]}\]

\[\text{[End of document]}\]
(i) a sequence $k$-fast decaying solutions $w_k$ with $k > 0$ such that

$$\lim_{|x| \to 0^+} w_k(x)|x|^{\frac{2}{p-1}} = c_p \quad \text{and} \quad \lim_{|x| \to +\infty} w_k(x)|x|^{N-2} = k.$$ 

Here a solution is called $k$-fast decaying if

$$\lim_{|x| \to +\infty} w_k(x)|x|^{N-2} = k.$$ 

(ii) a slow decaying solution $w_p(x) = c_p|x|^{-\frac{2}{p-1}}$ and $w_p = \lim_{k \to +\infty} w_k$.

Furthermore, the fast decaying solution $w_k$ could be written by

$$w_k(x) = |x|^{-\frac{2}{p-1}} \bar{w}_p(-\ln |x| + b_p^{-1}(\ln k - \ln c)),$$

where $b_p = N - 2 - \frac{4}{p-1} > 0$, $c > 0$ is independent of $k$ and $\bar{w}_p(\cdot)$ is a positive and bounded function independently of $k$. Assume that $t = -\ln |x| + b_p^{-1}(\ln k - \ln c)$, then the function $\bar{w}_p$ satisfies

$$\begin{cases}
\bar{w}_p'' - \left(N - 2 - \frac{4}{p-1}\right) \bar{w}_p' - b_p^{-1} \bar{w}_p + b_p^{-2} \bar{w}_p = 0 \quad \text{in} \quad \mathbb{R}, \\
\bar{w}_p(-\infty) = 0 \quad \text{and} \quad \bar{w}_p(+\infty) = c_p. 
\end{cases} \tag{1.4}$$

To be convenient for the analyze, let us denote

$$p_c = 1 + \frac{4}{N - 4 + 2\sqrt{N} - 1} \in \left(\frac{N}{N-2}, \frac{N + 2}{N - 2}\right), \tag{1.5}$$

which is the Joseph-Ludgren exponent, note that $\bar{w}_p$ is increasing for $p \in \left(\frac{N}{N-2}, p_c\right]$ and for $p \in (p_c, \frac{N+2}{N-2})$, $\bar{w}_p$ is oscillating as $t \to +\infty$, more information could be seen in Section 2. For the supercritical case that $p \geq \frac{N+2}{N-2}$, problem (1.2) has been studied [13][14][21]. In particular, the authors in [13] obtained a sequence of fast decay solutions of (1.2) with $p > \frac{N+2}{N-2}$ in an exterior domain.

During the last years there has been a renewed and increasing interest in the study of the semilinear elliptic equations with potentials, motivated by great applications in mathematical fields and physical fields, e.g. the well known scalar curvature equation in the study of Riemannian geometry, the scalar field equation for standing wave of nonlinear Schrödinger and Klein-Gördan equations, the Matukuma equation, see a survey [18][22] and more references on decaying solutions of semilinear elliptic equations with potentials, motivated by great applications in mathematical fields.

For Lane-Emden equation (1.1) involving nonhomogeneous potential $V(x) = |x|^{\alpha}(1 + |x|)^{\beta - \alpha}$, the authors in [4][5] showed the nonexistence provided $\beta > -2$ and $p \leq \frac{N + \beta}{N - 2}$, also see [4] Theorem 3.1. In [10], the infinitely many positive solutions of problem (1.1) are constructed for $p \in (\frac{N + \beta}{N - 2}, \frac{N + 2}{N - 2}) \cap (0, +\infty)$ with $\alpha_0 \in (-N, +\infty)$ and $\beta \in (-\infty, 0)$, by dealing with the distributional solutions of

$$-\Delta u = Vu^p + \kappa \delta_0 \quad \text{in} \quad \mathbb{R}^N, \tag{1.6}$$

where $k > 0$, $\delta_0$ is a Dirac mass at the origin and $p = \frac{N + \alpha_0}{N - 2}$ is the critical exponent named Serrin exponent, the value for problem (1.6) with recoverable isolated singularities. Compared to the case $V \equiv 1$, problem (1.1) would have totally different isolated singular solution structure for the supercritical case $p \geq \frac{N+2}{N-2}$, due to the behavior of potential at infinity.

Our interest of this paper is to classify the fast decaying and slow decaying solutions of problem (1.1) for the supercritical case and involving general potential $V$. Here, we say that $u \in C^2(\mathbb{R}^N \setminus \{0\})$ is a $\nu$-fast decaying solution if $u$ pointwisely verifies (1.7) in $\mathbb{R}^N \setminus \{0\}$ and has the asymptotic behavior at infinity

$$\lim_{|x| \to +\infty} u(x)|x|^{N-2} = \nu \quad \text{for} \quad \nu > 0.$$ 

Assume that the potential function $V$ is Hölder continuous and satisfies the following conditions:
Theorem 1.1 Assume that 

\[ V(x) - 1 \leq c_0 |x|^\tau_0 \quad \text{for } x \in B_1(0), \]  

(1.7) 

for some \( c_0 > 0 \) and \( \tau_0 > 0 \); 

(ii) global control, 

\[ 0 \leq V(x) \leq c_\infty (1 + |x|)^\beta \quad \text{for } |x| > 0, \]  

(1.8) 

where \( c_\infty \geq 1 \) and \( \beta \in \mathbb{R} \). 

Our main result is the following, which states the existence of fast decaying solutions of (1.1).

**Theorem 1.1** Assume that \( p_c \) is given by (1.3), \( p \in \left( \frac{N}{N - 2}, p_c \right) \), the potential function \( V \) verifies (\( V_0 \)) with \( \tau_0 \) and \( \beta \) verifying 

\[ \tau_0 > \tau^*_p \quad \text{and} \quad \beta < (N - 2)p - N, \]  

(1.9) 

where 

\[ \tau^*_p = \left( \frac{2}{p - 1} - \frac{N - 2}{2} \right) - \sqrt{\left( \frac{2}{p - 1} - \frac{N - 2}{2} \right)^2 - 2 \left( N - 2 - \frac{2}{p - 1} \right)}. \]  

(1.10) 

Then there exists \( \nu_0 > 0 \) such that for any \( \nu \in (0, \nu_0] \), problem (1.1) has a \( \nu \)-fast decaying solution \( u_\nu \), which has singularity at the origin as 

\[ \lim_{|x| \to 0} u_\nu(x)|x|^{\frac{2}{p - 1}} = c_p, \]  

(1.11) 

where \( c_p \) is given in (1.3).

Furthermore, the mapping \( \nu \in (0, \nu_0] \mapsto u_\nu \) is increasing, continuous and satisfies that 

\[ \lim_{\nu \to 0} \|u_\nu\|_{L^\infty(B_1(0))} = 0. \]  

(1.12) 

We remark that \( \tau^*_p > 0 \) is well-defined due to \( \left( \frac{2}{p - 1} - \frac{N - 2}{2} \right)^2 > 2(2 - N - \frac{1}{p - 1}) \) for \( p \in \left( \frac{N}{N - 2}, p_c \right) \). Theorem 1.1 constructs a parameterized fast decaying solutions \( u_\nu \) of (1.1) with \( \nu \in I \) being an interval and more properties of the mapping \( \nu \mapsto u_\nu \) are founded. Here the main difficulty is that the potential \( V \) breaks the scaling invariance of the equation. Moreover, due to the potential \( V \), we can not restrict to search the symmetric solutions by ODE’s tools such as the phase analysis, the variational method fails to apply due to the singularity at origin. First step of our method is to use the Schauder fixed point theorem to construct a solution \( v_k \) of the problem 

\[ -\Delta v = V(w_k + v_+^p - w_k^p) \quad \text{in } \mathbb{R}^N \setminus \{0\}, \]  

(1.13) 

for \( k > 0 \) sufficiently small and \( w_k \) is the \( k \)-fast decaying solution of (1.2). And then a \( \tilde{v}_k \)-fast decaying solution \( \tilde{u}_{\nu_k} := v_k + w_k \) of (1.1) is derived. However, the method of the Schauder fixed point theorem fails to build the increasing mapping \( k \mapsto \tilde{v}_k \). When \( V \) is comparable to value 1, we note that \( v_k \) could be determined its sign, then motivated by this observation, this solution could be used as a barrier for the sequence \( v_n = \Gamma * (V v_{n-1}^p) \) with initial data \( v_0 = w_k \), and its limit is our desirable \( v_k \)-fast decaying solution of (1.1) and more properties of the mapping \( k \mapsto v_k \) could be built, even for general \( V \) by dividing it as \( V = (1 + (V - 1)_+)(1 - (V - 1)_-) \).

Our another interest of this paper is whether the parameter \( \nu_0 \) can be taken \( +\infty \) in Theorem 1.1 that is, whether (1.1) has \( \nu \)-fast decaying solution with \( \nu \in (0, +\infty) \). To this end, we propose the following assumptions on the potential \( V \).

(V1) \( I \) \( V \geq 1 \) in \( \mathbb{R}^N \setminus \{0\} \) and there exist \( \alpha_1 \geq 0, l_1 > 1 \) such that 

\[ V(l_{1}^{-1}x) \geq l_{1}^{-\alpha_1} V(x), \quad \forall x \in \mathbb{R}^N \setminus \{0\}; \]  

(1.14) 

(II) \( V \leq 1 \) in \( \mathbb{R}^N \setminus \{0\} \) (i.e. \( c_\infty = 1, \beta = 0 \) in (V0)) and there exist \( \alpha_2 \leq 0, l_2 > 1 \) such that 

\[ V(l_{2}^{-1}x) \leq l_{2}^{-\alpha_2} V(x), \quad \forall x \in \mathbb{R}^N \setminus \{0\}. \]  

(1.15)
Theorem 1.2 Assume that the potential function \( V \) verifies (\( V_0 \)) with \( \tau_0 > 0 \) and \( \beta \) verifying (1.9) and \( p \in \left( \frac{N}{N-2}, p_c \right) \), where \( p_c \) is given by (1.3).

If \( (V_1) \) part (I) or part (II) holds, then for any \( \nu \in (0, +\infty) \), problem (1.1) has a \( \nu \)-fast decaying solution \( u_\nu \), which has singularity at the origin verifying (1.11) and the mapping \( \nu \in (0, \infty) \rightarrow u_\nu \) is increasing, continuous and (1.12) holds true.

Finally, our interest is to study the limit of \( \{u_\nu\}_\nu \) as \( \nu \to +\infty \) and we propose the following conditions on potential \( V \).

(\( V_\infty \)) Assume that \( V \) is radially symmetric, decreasing with respect to \( |x| \) and

\[
\frac{1}{\gamma} |x|^\alpha \leq V(x) \leq \gamma |x|^\alpha \quad \text{for} \quad |x| > 1, \tag{1.16}
\]

where \( \gamma > 1 \) and

\[
(N-2)p_c - N - 2 < \alpha \leq 0. \tag{1.17}
\]

Theorem 1.3 Assume that \( p_c \) is given by (1.2), \( p \in \left( \frac{N}{N-2}, p_c \right) \), \( V \) verifies (\( V_0 \)) part (i) with \( \tau_0 > 0 \), (\( V_1 \)) part (II) and (\( V_\infty \)). Let \( u_\nu \) be a \( \nu \)-fast decaying solution of problem (1.1) with \( \nu \in (0, +\infty) \) derived by Theorem 1.2. Then the limit of \( \{u_\nu\}_\nu \) as \( \nu \to +\infty \) exists, denoting \( u_\infty = \lim_{\nu \to +\infty} u_\nu \), and \( u_\infty \) is a solution of (1.1) verifying (1.11) and

\[
\frac{1}{c_1} u_\infty(x)|x|^{\frac{2+\alpha}{p-1}} \leq c_1, \quad |x| \geq 1, \tag{1.18}
\]

where \( c_1 > 1 \).

We see that the solution \( u_\infty \) is no longer a fast decaying solution of (1.1) by the decay estimate (1.18). Here the solution \( u_\infty \) of (1.1) is called as slow decay solution. Observe that the limit of \( u_p = \lim_{k \to +\infty} w_k \), \( k \)-fast decaying solution of (1.2), behaviors as \( c_p |x|^{-\frac{2}{p-1}} \) at infinity, and the fast decay solution \( u_\nu \) behaves as (1.8), where \( -\frac{2}{p-1} < -\frac{2+\alpha}{p} \). This means \( u_\infty > u_p \) for \( |x| > r \) for some \( r > 0 \), although for any \( \nu > 0 \), \( u_\nu \) is derived by iterating the decreasing sequence \( v_n = G[V_{\nu_{n-1}}] \) with the initial data \( v_0 = w_k \) for some \( k \) and \( u_\nu \leq w_k \).

The rest of this paper is organized as follows. In section 2, we show qualitative properties of the solutions to elliptic problem with homogeneous potential and some basic estimates. Section 3 is devoted to build fast decaying solutions of (1.1) by combining Schauder fixed point theorem and iteration method. Section 4 is devoted to consider the slow decaying solution as the limit of fast decaying solutions.

2 Preliminary

2.1 ODE analysis of Lane-Emden equation

In this subsection, we recall some results of Lane-Emden equation by ODE analysis. Denote a new independent variable \( t = -\ln |x| \) and set \( u(x) = |x|^{-\frac{2}{p-1}} w_p(-\ln |x|) \), then the function \( \tilde{w}_p(t) \) verifies

\[
\tilde{w}_p'' + a\tilde{w}_p' - c_p^{-1} \tilde{w}_p + \tilde{w}_p^p = 0 \quad \text{in} \quad \mathbb{R}, \tag{2.1}
\]

where \( a = \frac{\tau_0}{p-1} - N + 2 \).

Let \( X(t) = \tilde{w}(t) \) and \( Y(t) = \tilde{w}'(t) \), (2.1) can be rewritten as dynamic system

\[
\begin{align*}
X' &= Y \\ Y' &= -aY + c_p^{-1}X - X^p \quad \text{in} \quad \mathbb{R}.
\end{align*} \tag{2.2}
\]
We see that system (2.2) has two equilibrium points (0, 0) and \((c_p, 0)\). Our aim is to find a trajectory \((X, Y)\) that starts from point \((0, 0)\) as \(t \to -\infty\) and ends at point \((c_p, 0)\) as \(t \to +\infty\).

In fact, the trajectory \((X, Y)\) is contained within the homocyclic orbit of the Hamiltonian system

\[
v'' - \frac{p-1}{p} v + v^p = 0 \quad \text{in} \quad \mathbb{R},
\]

then we conclude that \(\sup \bar{w}_p \leq \sup \nu\). The conservation of Hamiltonian energy could be given as following

\[
E(t) := \frac{1}{2} v'(t)^2 - \frac{p-1}{2} v(t)^2 + \frac{1}{p+1} v(t)^p + 1 = 0.
\]

Hence \(v\) attains its supremum when \(v' = 0\), we get the upper bound \(\nu^{p-1} < c_{p+1}^{p-1}\).

On the other hand, the eigenvalues of linearizing system at \((0, 0)\) are

\[
\lambda_1 := N - 2 - \frac{2}{p - 1} > 0 > -\frac{2}{p - 1} =: \lambda_2,
\]

for \(t \to -\infty\), then we have that curve \((X, Y)\) goes out from point \((0, 0)\) along the direction \(Y = \lambda_1 X\) and then

\[
\bar{w}_p(t) \sim c e^{\lambda_1 t} \quad \text{as} \quad t \to -\infty.
\]

Therefore, we have that

\[
\|\bar{w}_p\|_{L^\infty(\mathbb{R})} < \left(\frac{p+1}{2}\right)^{1/p} c_p.
\]

Observe that the eigenvalues at \((c_p, 0)\) are given by zero points of

\[
H(\mu) := \mu^2 + a\mu + (p - 1) c_p^{p-1}.
\]

When \(p \in \left(\frac{N}{N-2}, p_c\right)\), where \(p_c\) is given by (1.5), (2.4) has two negative zero points of \(H\):

\[
\mu_1 = \frac{1}{2} \left(-a + \sqrt{a^2 - 4(p - 1) c_p^{p-1}}\right), \quad \mu_2 = \frac{1}{2} \left(-a - \sqrt{a^2 - 4(p - 1) c_p^{p-1}}\right),
\]

then we have that

\[
c_p - \bar{w}_p(t) \sim c_1 e^{\mu_1 t} + c_2 e^{\mu_2 t} \quad \text{as} \quad t \to +\infty.
\]

For \(p = p_c\), \(H = 0\) has two same roots \(\mu_1 = \mu_2 = -\frac{a}{2}\), then \(c_p - \bar{w}_p(t) \sim (c_1 + c_2 t) e^{\mu_1 t}\) as \(t \to +\infty\).

Note that for \(p \in \left(\frac{N}{N-2}, p_c\right)\), \(\bar{w}_p\) is increasing and

\[
\sup_{t \in \mathbb{R}} \bar{w}_p^{p-1} = \frac{2}{p-1} (N - 2 - \frac{2}{p-1})
\]

and then there exists \(c > 0\) such that

\[
0 < \bar{w}_p' \leq c \bar{w}_p \quad \text{in} \quad \mathbb{R}.
\]

Hence, the fast decaying solutions \(\{u_k\}_k\) of (1.2) have following properties:

\[
0 < w_{k_1} < w_{k_2} < w_p \quad \text{if} \quad 0 < k_1 < k_2 < +\infty.
\]

When \(p \in \left(p_c, \frac{N+2}{N-2}\right)\), \(H = 0\) has two complex roots as

\[
\mu_1 = \frac{1}{2} \left(-a + i \sqrt{-a^2 - 4(p - 1) c_p^{p-1}}\right) \quad \text{and} \quad \mu_2 = \frac{1}{2} \left(-a - i \sqrt{-a^2 - 4(p - 1) c_p^{p-1}}\right),
\]

where \(i\) is the unit imaginary number. Then \(\bar{w}\) oscillates around \(c_p\) and converges to \(c_p\) with the rate

\[
\lim_{t \to +\infty} \sup_{t \in \mathbb{R}} |\bar{w}_p(t) - c_p e^{\frac{2}{p} t}| = c_0.
\]

A crucial tool for the analysis is following.
Proposition 2.1 (i) For $p \in \left( \frac{N}{N-2}, p_c \right]$, we have that

$$ p \cdot \sup_{t \in \mathbb{R}} \bar{w}_p^{p-1} \leq \frac{(N - 2)^2}{4}, $$

(2.8)

where $' = '$ holds only for $p = p_c$.

(ii) For $p \in (p_c, \frac{N+2}{N-2})$, we have that

$$ p \cdot \sup_{t \in \mathbb{R}} \bar{w}_p^{p-1} > \frac{(N - 2)^2}{4}. $$

(2.9)

Proof. One hand, when $p \in \left( \frac{N}{N-2}, \frac{N+2}{N-2} \right)$, we have that $\frac{2}{p-1} \in \left( \frac{N-2}{2}, N - 2 \right)$, then

$$ \frac{2p}{p-1}(N - 2 - \frac{2}{p-1}) \leq (\frac{N-2}{2})^2 \quad \text{holds if and only if} \quad \frac{N}{N-2} < p \leq p_c. $$

On the other hand for $p \in (p_c, \frac{N+2}{N-2})$, we have that

$$ \sup_{t \in \mathbb{R}} \bar{w}_p^{p-1} > \frac{2}{p-1}(N - 2 - \frac{2}{p-1}), $$

combining (2.6), the assertion holds and the proof is complete. \hfill \Box

Lemma 2.1 Let $p \in \left( \frac{N}{N-2}, p_c \right]$ and $b_p = N - 2 - \frac{2}{p-1}$, then

$$ w_k(x) = \left| x \right|^{-\frac{2}{p-1}}w_p(-\ln |x| + b_p^{-1}(\ln k - \ln d_0)), $$

(2.10)

and for any $r \in (0, 1]$, there exists $k_r = r^{b_p}$ such that for $0 < k \leq k_r$,

$$ w_k(x) \leq c_1 k r^{-\frac{2}{p-1}} (1 + |x|)^{2-N} \chi_{\mathbb{R}^N \setminus B_r(0)}(x) + c_p |x|^{-\frac{2}{p-1}} \chi_{B_r(0)}(x) \quad \forall x \in \mathbb{R}^N \setminus \{0\}, $$

(2.11)

where $c_1 > 0$ is independent of $k, r$.

Proof. By above phase plane analysis, we have (2.10) just taking $t = -\ln |x| + b_p^{-1}(\ln k - \ln d_0)$. From (2.8), there exists $c > 0$ such that $\bar{w}_p(t) \leq c_1 e^{b_p t}$ for any $t \leq 0$, then if $-\ln |x| + b_p^{-1}(\ln k - \ln d_0) \leq 0$, i.e. $|x| \geq (k/d_0)^{b_p^{-1}}$, we have that

$$ w_k(x) \leq c_1 k r^{-\frac{2}{p-1}} (1 + |x|)^{2-N} \quad \forall x \in \mathbb{R}^N \setminus B_r(0). $$

For $|x| < (k/d_0)^{b_p^{-1}}$, we have $\bar{w}_p \leq c_p$ and (2.11) follows. \hfill \Box

Remark 2.1 Let $p \in (p_c, \frac{N+2}{N-2})$ and $b_p = N - 2 - \frac{2}{p-1}$, then for any $r \in (0, 1]$, there exists $k_r = r^{b_p}$ such that for $0 < k \leq k_r$,

$$ w_k(x) \leq c_1 k r^{-\frac{2}{p-1}} (1 + |x|)^{2-N} \chi_{\mathbb{R}^N \setminus B_r(0)}(x) + ||\bar{w}_p||_{L^\infty} |x|^{-\frac{2}{p-1}} \chi_{B_r(0)}(x), \quad \forall x \in \mathbb{R}^N \setminus \{0\}, $$

where $||\bar{w}_p||_{L^\infty} > c_p$. 

2.2 Basic estimate

In this subsection, some estimates are introduced, which play important roles in our construction of fast-decaying solutions for problem (1.1). Denote

\[ \Gamma(x) = c_N|x|^{2-N}, \quad \forall \, x \in \mathbb{R}^N \setminus \{0\}, \]

which is the fundamental solution of \(-\Delta \Gamma = \delta_0 \) in \(\mathbb{R}^N\) and \(c_N > 0\) is a normalized constant.

Lemma 2.2 Let

\[ U_1(x) = |x|^{-2-\theta} \chi_{B_1(0)}(x), \quad U_2(x) = (1 + |x|)^{-\tau} \quad \text{and} \quad U_3(x) = |x|^{-\theta-2}(1 + |x|)^{-\tau+\theta+2}, \]

where \(r \in (0, 1/2)\) and \(\tau > N > 2 + \theta\). Then there is \(r^* > 0\) small such that for \(r \in (0, r^*)\),

\[ (\Gamma * U_1)(x) \leq \frac{1}{\theta(N-2-\theta)}|x|^{-\theta}(1 + |x|)^{2-N+\theta} \quad \text{for} \quad x \in \mathbb{R}^N \tag{2.12} \]

and there exists \(c > 0\) such that

\[ (\Gamma * U_2)(x) \leq c(1 + |x|)^{2-N} \quad \text{for} \quad x \in \mathbb{R}^N \tag{2.13} \]

and

\[ (\Gamma * U_3)(x) \leq c|x|^{-\theta}(1 + |x|)^{2-N+\theta} \quad \text{for} \quad x \in \mathbb{R}^N. \tag{2.14} \]

Proof. By direct computation, we have that

\[ (\Gamma * U_1)(x) = c_N \int_{B_1(0)} \frac{|y|^{-2-\theta}}{|x-y|^{N-2}} \, dy. \]

From the fact that \(-\Delta(|x|^{-\theta}) = \theta(N-2-\theta)|x|^{-\theta-2}\) in \(\mathbb{R}^N \setminus \{0\}\), we can deduce

\[ c_N \int_{\mathbb{R}^N} \frac{|y|^{-\theta-2}}{|x-y|^{N-2}} \, dy = \frac{1}{\theta(N-2-\theta)}|x|^{-\theta}. \]

For \(x \in B_1(0) \setminus \{0\}\), we have that

\[ (\Gamma * U_1)(x) \leq c_N \int_{\mathbb{R}^N} \frac{|y|^{-\theta-2}}{|x-y|^{N-2}} \, dz = \frac{1}{\theta(N-2-\theta)}|x|^{-\theta}. \]

When \(x \in \mathbb{R}^N \setminus B_1(0)\),

\[ (\Gamma * U_1)(x) \leq c_N(|x|^{-2-N}) \int_{B_1(0)} |y|^{-\theta-2} \, dy \leq c_N 2^{N-2} |x|^{2-N} \int_{B_1(0)} |y|^{-\theta-2} \, dy, \tag{2.15} \]

where

\[ \int_{B_1(0)} |y|^{-\theta-2} \, dy = |S^{N-1}|r^{N-2-\theta} \to 0 \quad \text{as} \quad r \to 0^+. \]

Thus, (2.12) holds thanks to \(c_N 2^{N-2} |S^{N-1}|r^{N-2-\theta} \leq \frac{1}{\theta(N-2-\theta)}\) when \(r \leq r^*\).

Next we show (2.13). Note that

\[ (\Gamma * U_2)(x) = c_N \int_{\mathbb{R}^N} \frac{(1 + |y|)^{-\tau}}{|x-y|^{N-2}} \, dy, \]

then \((\Gamma * U_2)(x)\) is bounded locally in \(\mathbb{R}^N\) and so we only need to show the case \(|x| \to +\infty\).
In fact, for $|x| > 4$ large enough, there holds
\[
\int_{\mathbb{R}^N} \frac{(1 + |y|)^{-\tau}}{|x-y|^{N-2}} dy \leq \int_{B_{1/2}(0)} (1 + |y|)^{-\tau} dy + \frac{1}{2} \int_{\mathbb{R}^N \setminus B_{1/2}(0)} \frac{|y|^{-\tau}}{|x-y|^{N-2}} dy
\]
\[
\leq \left( \frac{1}{2} \right)^{2-N} \int_{B_{1/2}(0)} (1 + |y|)^{-\tau} dy + |x|^{2-\tau} \int_{\mathbb{R}^N \setminus B_{1/2}(0)} |y|^{-\tau} e^{2-\tau} dy
\]
\[
\leq c|x|^{2-N},
\]
where the last inequality holds thanks to the facts that $\tau > N$,
\[
\int_{B_{1/2}(0)} (1 + |y|)^{-\tau} dy \leq \int_{\mathbb{R}^N} (1 + |y|)^{-\tau} dy \quad \text{and} \quad \int_{\mathbb{R}^N \setminus B_{1/2}(0)} |y|^{-\tau} dy < +\infty.
\]
Then (2.13) holds true.

Finally, we observe that
\[
U_3(x) \leq 2|x|^{-2-\theta} \chi_{B_1(0)}(x) + 2(1 + |x|)^{\theta-\tau}, \quad x \in \mathbb{R}^N \setminus \{0\},
\]
then (2.14) follows by (2.15) with $r = 1$ and (2.13) directly.

**Corollary 2.1** Assume that $\alpha \in (0, N)$, $f$ is a nonnegative function satisfying that
\[
|f(x)| \leq |x|^{-\theta}(1 + |x|)^{\theta-\tau} \quad \text{for} \quad |x| > 0
\]
with $\alpha < \theta < N$ and $\tau > N$. Then there exists $c > 0$ such that
\[
\int_{\mathbb{R}^N} \frac{f(y)}{|x-y|^{N-\alpha}} \leq c|x|^{-\theta+\alpha}(1 + |x|)^{-N+\tau}, \quad \forall x \in \mathbb{R}^N \setminus \{0\}.
\]  

**Proof.** The same as the proof of Lemma 2.2 we can obtain (2.16). \qed

**Lemma 2.3** Suppose that $f \in L^1(\mathbb{R}^N)$ is a nonnegative function satisfying $|f(x)| \leq c|x|^{-\tau}$ for $|x| > r$, with $\tau > N$ and some $r > 0$, $c > 0$. Then
\[
\lim_{x \to +\infty} (\Gamma * f)(x)|x|^{N-2} = c_N \int_{\mathbb{R}^N} f(x) dx.
\]  

**Proof.** By the decay condition of $f$, we have that for any $\epsilon > 0$, there exists $R > R_0$ such that for $R$ large,
\[
\int_{B_R(0)} f(x) dx \geq (1-\epsilon)\|f\|_{L^1(\mathbb{R}^N)}.
\]
For $|x| \gg R$, there holds $(1-\epsilon)|x|^{2-N} \leq |x-y|^{2-N} \leq (1+\epsilon)|x|^{2-N}$ for $y \in B_R(0)$ and
\[
(\Gamma * f)(x) = c_N \int_{B_R(0)} \frac{f(y)}{|x-y|^{N-\alpha}} dy + c_N \int_{\mathbb{R}^N \setminus B_R(0)} \frac{f(y)}{|x-y|^{N-\alpha}} dy,
\]
which yields that for $|x|$ large,
\[
(1-\epsilon)\|f\|_{L^1(\mathbb{R}^N)} \leq |x|^{N-2} \int_{B_R(0)} \frac{f(y)}{|x-y|^{N-2}} dy \leq (1+\epsilon)\|f\|_{L^1(\mathbb{R}^N)}
\]
and
\[
\int_{\mathbb{R}^N \setminus B_R(0)} \frac{f(y)}{|x-y|^{N-2}} dy \leq c \int_{\mathbb{R}^N \setminus B_R(0)} \frac{|y|^{-\tau}}{|x-y|^{N-2}} dy
\]
\[
= c \int_{|y|<2|x|} \frac{|y|^{-\tau}}{|x-y|^{N-2}} dy + c \int_{|y|>2|x|} \frac{|y|^{-\tau}}{|x-y|^{N-2}} dy
\]
\[
\leq c R^{2-N} \int_{|x|} 2|x| r^{N-1-\tau} dr + c \int_{2|x|}^{+\infty} r^{1-\tau} dr
\]
\[
\leq \frac{c}{N-\tau} R^{2-N} ((2|x|)^{N-\tau} - R^{N-\tau}) - \frac{c}{2-\tau}(2|x|)^{2-\tau}.
\]
Passing to the limit as $\epsilon \to 0$ and letting $R \to +\infty$, we see that $|x| \to +\infty$ and then (2.17) holds. \qed
Corollary 2.2 Let $w_k$ be $k$-fast decaying solution of \((1.2)\), then
\[
c_N \int_{\mathbb{R}^N} w_k(x)^p dx = k.
\]

Proof. Since \(\lim_{|x| \to 0^+} w_k(x)|x|^\frac{2}{p+2} = c_p\) and \(\lim_{|x| \to +\infty} w_k(x)|x|^{(N-2)} = k\), then Lemma 2.3 implies that for \(p(N - 2) > N\), which implies that
\[
k = \lim_{|x| \to +\infty} w_k(x)|x|^{N-2} = c_N \int_{\mathbb{R}^N} w_k(x)^p dx,
\]
which ends the proof. \(\square\)

Lemma 2.4 Assume that \(a > 0, b \in \mathbb{R}\), then for \(p \in (1, 2]\),
\[(a + b)^p_+ \leq a^p + pa^{p-1}|b| + |b|^p;\]
for \(p > 2\),
\[(a + b)^p_+ \leq a^p + pa^{p-1}|b| + 2^p(p-1)a^{p-2}b^2 + 2^p|b|^p.\]

These are basic inequalities, here we omit the proof.

Finally, we introduce a comparison principle for general Hardy operator.

Lemma 2.5 Let \(\Omega\) be a bounded \(C^2\) domain containing the origin, \(W\) be Hölder continuous locally in \(\Omega \setminus \{0\}\) such that \(\lim_{|x| \to 0} W(x)|x|^2 = \mu\) with \(\mu \in \left(0, \frac{(N-2)^2}{4}\right)\) and \(W(x) \leq \frac{(N-2)^2}{4}|x|^{-2}\) in \(\Omega\). Then the operator
\[
\mathcal{L}_W w := -\Delta w - W w \tag{2.18}
\]
verifies the following comparison principle in \(\Omega\):

Assume that \(f_1, f_2\) are two functions in \(C^\gamma(\Omega \setminus \{0\})\) with \(\gamma \in (0, 1)\), \(g_1, g_2\) are two continuous functions on \(\partial \Omega\),
\[f_1 \geq f_2 \quad \text{in} \quad \Omega \setminus \{0\} \quad \text{and} \quad g_1 \geq g_2 \quad \text{on} \quad \partial \Omega.\]

Let \(u_i (i = 1, 2)\) be the classical solutions of
\[
\begin{cases}
\mathcal{L}_W u = f_i & \text{in} \quad \Omega \setminus \{0\}, \\
u = g_i & \text{on} \quad \partial \Omega.
\end{cases}
\]

If \(\lim_{x \to 0} u_1(x)|x|^{-\tau_-(\mu)} \geq \limsup_{x \to 0} u_2(x)|x|^{-\tau_-(\mu)}\) holds, then \(u_1 \geq u_2\) in \(\Omega \setminus \{0\}\).

Proof. From [9] Theorem 1.1, we note that operator \(\mathcal{L}_W\) has a positive solution \(\Phi_W\) such that
\[
\lim_{|x| \to 0} \frac{\Phi_W(x)}{|x|^{-\tau_-(\mu)}} = 1.
\]

Let \(w = u_2 - u_1\) be a solution of
\[
\begin{cases}
\mathcal{L}_W u \leq 0 & \text{in} \quad \Omega \setminus \{0\}, \\
u \leq 0 & \text{on} \quad \partial \Omega, \\
\limsup_{x \to 0} u(x)|x|^{-\tau_-(\mu)} \leq 0,
\end{cases}
\]
then for any \(\epsilon > 0\), there exists \(r_\epsilon > 0\) converging to zero as \(\epsilon \to 0\) such that \(w \leq \epsilon \Phi_W\) on \(\partial B_{r_\epsilon}(0)\). Observe that \(w \leq 0 < \epsilon \Phi_W\) on \(\partial \Omega\), then from [11] Lemma 2.1, we have that \(w \leq \epsilon \Phi_W\) in \(\Omega \setminus \{0\}\). By the arbitrary of \(\epsilon > 0\), we have that \(w \leq 0\) in \(\Omega \setminus \{0\}\). \(\square\)
3 Fast decaying solutions

3.1 Existence by Fixed point theory

In this subsection, we give the proof of Theorem 1.1. We are looking for a $k$-fast decaying solution $u$ of (1.2), with the division

$$u = w_k + v,$$

where $w_k$ is the $k$ fast decaying solution of (1.2) and $v$ verifies that

$$-\Delta v = V(w_k + v)^p - w_k^p \quad \text{in} \quad \mathbb{R}^N \setminus \{0\}.$$  \hspace{1cm} (3.1)

We will employ the Schauder fixed point theorem to obtain a solution of (3.1). To this end, let us clarify the key value $\tau^*_p$. In fact, the essential point in our following construction of fast decaying solutions is to find a $\theta_0 \in \left[\frac{N-2}{2}, N-2\right)$ such that

$$\theta_0(N - 2 - \theta_0) > p c_p^{p-1} = \left(2 + \frac{2}{p-1}\right) \left(N - 2 - \frac{2}{p-1}\right),$$  \hspace{1cm} (3.2)

which is possible since \(\left(2 + \frac{2}{p-1}\right) \left(N - 2 - \frac{2}{p-1}\right) < \frac{(N-2)^2}{4}\) when $p \in \left(\frac{N}{N-2}, p_c\right)$, and $\frac{(N-2)^2}{4}$ is maximum of $\theta_0(N - 2 - \theta_0)$ which is achieved at $\theta_0 = \frac{N-2}{2}$. Now the point is to find the smallest $\tau > 0$ such that

$$\left(2 + \frac{2}{p-1}\right)^2 \left(N - 2 + \tau - \frac{2}{p-1}\right) = \left(2 + \frac{2}{p-1}\right) \left(N - 2 - \frac{2}{p-1}\right) + \tau.$$  \hspace{1cm} (3.3)

Direct computation shows that for $p \in \left(\frac{N}{N-2}, p_c\right)$, $\tau_p^* > 0$ defined in (1.10) is the smallest zero of $\theta_0(N - 2 - \theta_0) > p c_p^{p-1}$ and letting $\tau^#_p = \left(\frac{2}{p-1} - \frac{N-2}{2}\right) + \sqrt{(\frac{2}{p-1} - \frac{N-2}{2})^2 - 2(N - 2 - \frac{2}{p-1})}$, for any $\tau \in \left(\tau_p^*, \tau^#_p\right)$, we have that

$$\left(2 + \frac{2}{p-1}\right) \left(N - 2 + \tau - \frac{2}{p-1}\right) < \left(2 + \frac{2}{p-1}\right) \left(N - 2 - \frac{2}{p-1}\right) + \tau.$$  \hspace{1cm} (3.4)

Now let us fix

$$\tau_1 = \tau_p^* + \frac{1}{2} \min \left\{\tau_0 - \tau_p^*, \frac{2}{p-1} - \frac{N-2}{2}\right\} \quad \text{and} \quad \theta_0 = \frac{2}{p-1} - \tau_1,$$  \hspace{1cm} (3.5)

then $\tau_1 \in \left(\tau_p^*, \tau^#_p\right) \cap (0, \tau_0)$ and $\theta_0 \in \left[\frac{N-2}{2}, \frac{2}{p-1}\right)$ verifying

$$\theta_0(N - 2 - \theta_0) > \frac{2p}{p-1}(N - 2 - \frac{2}{p-1}).$$  \hspace{1cm} (3.6)

Finally, we denote $\tau_2 = \tau_0 - \tau_1 > 0$.

**Proposition 3.1** Assume that $p_c$ is given by (1.3), $p \in \left(\frac{N}{N-2}, p_c\right)$, the potential function $V$ verifies $(\mathcal{V}_0)$ with $\tau_0, \beta$ verifying (1.4). Then there exist $k^* > 0$ and $c > 0$ such that for any $k \in (0, k^*)$, problem (3.7) has a classical solution $v_k$ such that

$$|v_k(x)| \leq c k |x|^{-\theta_0(1 + |x|)^2-N+\theta_0}, \quad \forall x \in \mathbb{R}^N \setminus \{0\}.$$  \hspace{1cm} (3.7)

where $\theta_0$ is defined in (3.3).

**Proof.** Step 1: to show basic setting for applying the Schauder fixed point Theorem. Note that for $p \in \left(\frac{N}{N-2}, p_c\right)$, there holds

$$\theta_0(N - 2 - \theta_0) > \frac{2p}{p-1}(N - 2 - \frac{2}{p-1}).$$  \hspace{1cm} (3.7)
Let \( q_0 \in \left( \frac{N}{N-1}, \frac{N}{2} \right) \), we denote
\[
\mathcal{D}_\epsilon := \left\{ v \in L^{q_0}(\mathbb{R}^N) : |v(x)| \leq \epsilon |x|^{-\theta_0}(1 + |x|)^{2-N+\theta_0}, \forall x \in \mathbb{R}^N \setminus \{0\} \right\},
\]
and
\[
\mathcal{T} v := \Gamma * (V(w_k + v))_+ - w_k^p, \quad \forall v \in \mathcal{D}_\epsilon,
\]
where \( \Gamma \) is the fundamental solution of \(-\Delta \) in \( \mathbb{R}^N \).

Step 2: to prove \( \mathcal{T} \mathcal{D}_\epsilon \subset \mathcal{D}_\epsilon \) for \( \epsilon, k > 0 \) small suitably.

Case i: \( p \in \left( \frac{N}{N-2}, p_0 \right) \cap (1, 2) \) (this happens when \( N \geq 5 \)). For given \( \epsilon > 0 \) small and \( v \in \mathcal{D}_\epsilon \), we have that
\[
|\Gamma * (V(w_k + v))_+ - w_k^p| \leq \Gamma * \left( |V - 1|w_k^p + pVw_k^{p-1}|v| + |v|^p \right).
\]
As \( \frac{pc_p^{p-1}}{\theta_0(N-2-\theta_0)} < 1 \) by (3.5), then there exists \( r^* \in (0, 1] \) such that \( \frac{pc_p^{p-1}}{\theta_0(N-2-\theta_0)} \max_{B_*} V < 1 \) and we denote
\[
\rho_0 = 1 - \frac{pc_p^{p-1}}{\theta_0(N-2-\theta_0)} \max_{B_*} V > 0.
\]
By (2.11) and Lemma 2.2, direct computation shows that
\[
\Gamma * (|V - 1|w_k^p) \leq c_0 c_p^{p-1} \int_{B_*} \frac{|y|^{r_1} - 2 - 2}{|x - y|^{N-2}} dy + c_1 r_1^{-r} c_\infty \int_{\mathbb{R}^N} \frac{(1 + |y|)^{p(2-N) + \beta}}{|x - y|^{N-2}} dy
\leq c_2 r_1 |x|^{-\theta_0}(1 + |x|)^{2-N+\theta_0} + c_3 r_1^{-r} c_\infty|x|^{p(2-N) - \beta},
\]
where \( c_2, c_3 > 0 \) are independent of \( k, r \). Now we fix \( r \in (0, r^*) \) such that \( c_2 r_1^{r_2} \leq \frac{1}{8} \rho_0 \), then for that \( r \), there exists \( k_1^* > 0 \) such that for \( k \in (0, k_1^*) \),
\[
c_3 r_1^{r-1} c_\infty \leq \frac{\rho_0}{8} \epsilon.
\]
Therefore, we have that \( \Gamma * (|V - 1|w_k^p)(x) \leq \frac{\rho_0}{4} \epsilon |x|^{-\theta_0}(1 + |x|)^{2-N+\theta_0} \). Moreover, we observe that
\[
p \Gamma * (Vw_k^{p-1}|v|) \leq \epsilon \left( \max_{B_*} \frac{V c_p^{p-1}}{\theta_0(N-2-\theta_0)} \int_{B_*} \frac{|y|^{r_1} - 2 - 2}{|x - y|^{N-2}} dy + c_1 c_p^{p-1} k^{p-1} r_1^{-r} \int_{\mathbb{R}^N} \frac{|y|^{-\theta_0}(1 + |y|)^{p(2-N) + \beta + \theta_0}}{|x - y|^{N-2}} dy \right)
\leq (1 - \frac{\rho_0}{2}) \epsilon |x|^{-\theta_0}(1 + |x|)^{2-N+\theta_0},
\]
where \( c_\infty, c_1 > 0 \) are independent of \( r, k \). Then for fixed \( r \), there exists \( k_1^* > 0 \) such that for \( k \in (0, k_1^*) \),
\[
c_\infty c_1 c_p^{p-1} k^{p-1} r_1^{-r} \leq \frac{\delta_0}{2}.
\]
Furthermore, we note that
\[
\Gamma * (V|v|^p) \leq c_\infty c_p |x|^{-\theta_0}(1 + |x|)^{2-N+\theta_0} \leq \frac{\rho_0}{4} \epsilon |x|^{-\theta_0}(1 + |x|)^{2-N+\theta_0},
\]
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where \( c_4 > 0 \) is independent of \( k \), we choose
\[
c_4 \epsilon^{p-1} \leq \frac{\rho_0}{4}.
\]

As a consequence, for \( \epsilon, r, k \) verifying (3.12), (3.13), we have that
\[
|\Gamma \ast (V(w_k + v)^p - w_k^p)| \leq \epsilon |x|^{-\theta_0}(1 + |x|)^{2-N+\theta_0},
\]
that is to say, \( TD \epsilon \subset D \epsilon \).

**Case ii:** \( p \in (\frac{N}{N-2}, p_c) \cap (2, +\infty) \). In this case, (3.10) should be replaced by
\[
|\Gamma \ast (V(w_k + v)^p - w_k^p)| \leq \Gamma \ast \left( |V - 1|w_k^p + pVw_k^{p-1}|v| + 2^p p(p - 1)w_k^{p-2}v^2 + V|v|^p \right).
\]

Here we only need to do the estimate for \( \Gamma \ast (Vw_k^{p-2}v^2) \) in addition. Indeed, as \( w_k(x) \leq c_p |x|^{-\frac{2}{p-2}(1 + |x|)} \) for \( k \leq k_0 \), we have that
\[
\Gamma \ast (Vw_k^{p-2}v^2) \leq c_5 \epsilon^2 |x|^{-\theta_0}(1 + |x|)^{2-N+\theta_0} \leq \frac{\rho_0}{4} \epsilon |x|^{-\theta_0}(1 + |x|)^{2-N+\theta_0},
\]
where \( c_5 > 0 \) is independent of \( \epsilon \), \( -\frac{2(p-2)}{p-1} - 2\theta_0 > -\theta_0 - 2 \), and the last inequality holds if we choose
\[
c_5 \epsilon \leq \frac{\rho_0}{4}.
\]

By our choice of \( \epsilon, r \) and \( k \), we have that \( LD \epsilon \subset D \epsilon \) for \( p \in (\frac{N}{N-2}, p_c) \cap (2, +\infty) \).

**Step 3: Applying Schauder fixed point theorem.** Note that for \( x \in \mathbb{R}^N \setminus \{0\} \),
\[
h(x) := |V(w_k(x) + v(x))^p - w_k^p(x)| \leq c|x|^{-\theta_0}(1 + |x|)^{p(2-N)+\beta},
\]
and then by **Step 2** and Corollary 2.1 we have that
\[
|\mathcal{T}v(x)| \leq c_N \int_{\mathbb{R}^N} \frac{h(y)}{|x - y|^{N-2}} dy \leq \epsilon |x|^{-\theta_0}(1 + |x|)^{2-N+\theta_0}
\]
and
\[
|\nabla \mathcal{T}v(x)| = |\nabla G [v_k(V(w_k + v)^p - w_k^p)] (x)| \leq c_N(N - 2) \int_{\mathbb{R}^N} \frac{h(y)}{|x - y|^{N-1}} dy \leq c|x|^{-\theta_0 - 1}(1 + |x|)^{2-N+\theta_0},
\]
thus, \( |\mathcal{T}v| \in L^q(\mathbb{R}^N) \) for \( \frac{N}{N-2} < q < \frac{N}{\theta_0} \); and \( |\nabla \mathcal{T}v| \in L^q(\mathbb{R}^N) \) for \( \frac{N}{N-1} < q < \frac{N}{\theta_0+1} \), where \( \theta_0 + 1 < N - 1 \).

For \( \sigma \geq 1 \), denote \( W^{1,\sigma}(\mathbb{R}^N) \) the Sobolev space with the norm
\[
\|u\|_{W^{1,\sigma}} = \left( \int_{\mathbb{R}^N} (|u|^\sigma + |\nabla u|^\sigma) dx \right)^{\frac{1}{\sigma}}.
\]
Therefore, we see that \( TD \epsilon \subset W^{1,q_0}(\mathbb{R}^N) \cap D \epsilon \).

We next show that the operator \( \mathcal{T} \) is compact. To this end, we only have to prove that \( W^{1,q_0}(\mathbb{R}^N) \cap D \epsilon \) is compact in \( L^{q_0}(\mathbb{R}^N) \). Since the embedding \( W^{1,q_0}(\mathbb{R}^N) \hookrightarrow L^{q_0}(\mathbb{R}^N) \) is locally compact in \( \mathbb{R}^N \), letting \( \{\zeta_j\}_j \) be a bounded functions in \( W^{1,q_0}(\mathbb{R}^N) \cap D \epsilon \) with \( \epsilon > 0 \) and \( \zeta \in L^p(\mathbb{R}^N) \cap D \epsilon \),
then for any \( \eta > 0 \), there exist \( R > 0 \), \( j_0 \in \mathbb{N} \) and a subsequence, still denote \( \{ \zeta_j \}_{j} \), such that for \( j \geq j_0 \),

\[
\| \zeta_j - \zeta \|_{L^p(B_R(0))} \leq \frac{\eta}{2} \quad \text{and} \quad \| \zeta_j \|_{L^p(\mathbb{R}^N \setminus B_R(0))} + \| \zeta \|_{L^p(\mathbb{R}^N \setminus B_R(0))} \leq \frac{\eta}{2},
\]

therefore, we have that for \( j \geq j_0 \),

\[
\| \zeta_j - \zeta \|_{L^p(\mathbb{R}^N)} \leq \eta.
\]

By the arbitrarily of \( \eta \), \( W^{1,p_0}(\mathbb{R}^N) \cap D_\varepsilon \hookrightarrow L^{p_0}(\mathbb{R}^N) \) is compact and we derive that \( T \) is a compact operator.

Observing that \( D_k \) is a closed and convex set in \( L^{p_0}(\mathbb{R}^N) \), we now can apply Schauder fixed point theorem to derive that there exists \( \nu_k \in D_\varepsilon \) such that

\[
T \nu_k = \nu_k.
\]

Since \( |\nu_k(x)| \leq c|x|^{-\theta_0}(1 + |x|)^{2-N+\theta_0} \), so \( \nu_k \) is locally bounded in \( \mathbb{R}^N \setminus \{0\} \), then \( \nu_k \) satisfies by standard interior regularity results and \( \nu_k \) is a classical solution of (3.1).

**Corollary 3.1** (i) From the proof of Proposition 3.1, the parameters \( \epsilon, r \) could be fixed by

\[
\epsilon = c_{\delta_0}k^p \quad \text{and} \quad r = c_{\delta_0}k^{2p},
\]

where \( c_{\delta_0} > 0 \) is a constant depending on \( \delta_0 \).

(ii) Under the assumption of Proposition 3.1, if \( V \leq 1 \), we can refine the solution \( \nu_k \) of (3.1) to be nonpositive, derived in

\[
D_{\epsilon,-} := \left\{ w \in L^{p_0}(\mathbb{R}^N) : -\epsilon|x|^{-\theta_0}(1 + |x|)^{2-N+\theta_0} \leq w(x) \leq 0, \forall x \in \mathbb{R}^N \setminus \{0\} \right\}; \quad (3.16)
\]

if \( V \geq 1 \), we can refine the solution \( \nu_k \) of (3.1) to be nonnegative, derived in

\[
D_{\epsilon,+} := \left\{ w \in L^{p_0}(\mathbb{R}^N) : 0 \leq w(x) \leq c|x|^{-\theta_0}(1 + |x|)^{2-N+\theta_0}, \forall x \in \mathbb{R}^N \setminus \{0\} \right\}. \quad (3.17)
\]

**Remark 3.1** In the critical case \( p = p_c \), our construction of fast decaying solution fails, due to the estimate of \( p \Gamma * (V w^{p-1}_k|v|) \) as the proof of Proposition 3.1, where \( \frac{p_0\tau p_c p_c}{2N-2p_c} = 1 \) even with \( \theta_0 \) taking the optimal value \( \frac{N-p_c}{2-p_c} \) for \( \theta_0(N-2-p_c) \), so there is no space for perturbing \( V \) near origin.

### 3.2 Existence when \( V \) is comparable to 1

**Theorem 3.1** Under assumptions of Theorem 1.1, we let \( V \geq 1 \). Then there is \( \nu_0 \in (0, +\infty) \) such that for any \( \nu \in (0, \nu_0) \), problem (1.1) has a \( \nu \)-fast decaying solution \( u_{\nu} \), which has singularity at the origin as \( x = 0 \) and the mapping \( \nu \in (0, \nu_0) \rightarrow u_{\nu} \) is increasing, continuous and \( (1.14) \) holds.

Moreover, if \( (1.14) \) holds for some \( \alpha_1 \geq 0 \) and \( l_1 > 1 \), then \( \nu_0 = +\infty \).

**Proof.** From Corollary 3.1, we take \( \epsilon = c_{\delta_0}k^p \) and let \( k \in (0, k^*) \), then Proposition 3.1 implies that problem (3.1) has a nonnegative solution \( \nu_k \) verifying (3.6). We denote

\[
\tilde{u}_{\nu_k} = u_k + \nu_k \geq u_k \quad \text{and} \quad \tilde{\nu}_k = c_N \int_{\mathbb{R}^N} V \tilde{u}_{\nu_k}^p \, dx,
\]

then \( \tilde{u}_{\nu_k} \) is a nonnegative classical solution of (1.1) such that

\[
\lim_{|x| \to 0^+} \tilde{u}_{\nu_k}(x)|x|^{\frac{-2}{p_c}} = c_p \quad \text{and} \quad \lim_{|x| \to +\infty} \tilde{u}_{\nu_k}(x)|x|^{N-2} = \tilde{\nu}_k, \quad (3.18)
\]

where the second estimate is obtain by Lemma 2.3 and the fact that \( k \leq \tilde{\nu}_k \leq k + c_{\delta_0}k^p \).

To complete the proof, we divide into four steps.
Step 1. Existence by iteration method. We initiate from \( v_0 := w_k \), denote by \( v_n \) iteratively the unique solution of

\[
v_n = \Gamma \ast (V v_{n-1}^p) \quad \text{in} \quad \mathbb{R}^N \setminus \{0\},
\]

that is,

\[
\begin{cases}
-\Delta v_n = V v_{n-1}^p & \text{in} \quad \mathbb{R}^N \setminus \{0\}, \\
\lim_{|x| \to 0} v_n(x)|x|^{N-2} = 0.
\end{cases}
\]

As \(-\Delta v_0 = v_0^p \) in \( \mathbb{R}^N \setminus \{0\} \) and \( \lim_{|x| \to 0} v_0(x)|x|^{2-p} = c_p \), by the Comparison Principle, we have that

\[
v_1 \geq v_0 \quad \text{in} \quad \mathbb{R}^N \setminus \{0\}.
\]

Inductively, we can deduce that \( v_n \geq v_{n-1} \) in \( \mathbb{R}^N \setminus \{0\} \). Thus, the sequence \( \{v_n\}_n \) is increasing.

Now we show that \( \tilde{u}_\nu \) is an upper bound for \( \{v_n\}_n \) for \( k \in (0, k^*) \). We observe that \( \tilde{u}_\nu \) is a solution of (1.1) and

\[
V w_k^p \leq V \tilde{u}_\nu^p, \quad \text{in} \quad \mathbb{R}^N \setminus \{0\}.
\]

Then Comparison Principle implies that

\[
v_1 \leq \tilde{u}_\nu \quad \text{in} \quad \mathbb{R}^N \setminus \{0\}.
\]

Inductively, we see that for any \( n \in \mathbb{N} \), we have that

\[
v_n \leq \tilde{u}_\nu \quad \text{in} \quad \mathbb{R}^N \setminus \{0\},
\]

so \( \{v_n\}_n \) has an upper barrier \( \tilde{u}_\nu \). Therefore, the sequence \( \{v_n\}_n \) converges. Denote \( u_\nu := \lim_{n \to \infty} v_n \), then for any compact set \( K \) in \( \mathbb{R}^N \setminus \{0\} \), and then \( u_\nu \) verifies the equation

\[
-\Delta u = V u^p \quad \text{in} \quad K
\]

and then \( u_\nu \) is a classical solution of (1.1) verifying

\[
w_k \leq u_\nu \leq \tilde{u}_\nu \quad \text{in} \quad \mathbb{R}^N \setminus \{0\}.
\]

From the classification of isolated singularities of positive solutions to problem (1.1), we have that

\[
\lim_{|x| \to 0} u_\nu(x)|x|^{2-p} = c_p.
\]

Here we let

\[
\nu_k = c_N \int_{\mathbb{R}^N} V w_k^p \, dx.
\]

and then

\[
k \leq \nu_k \leq \tilde{\nu}_k \leq k + c_{d_k} k^p \quad \text{and} \quad \lim_{|x| \to +\infty} u_\nu(x)|x|^{N-2} = \nu_k
\]

hold by Lemma 2.3.

Thus,

\[
\lim_{k \to 0} \nu_k = 0.
\]

Since \( u_\nu \leq \tilde{u}_\nu \leq w_k + v_k \),

\[
\lim_{k \to 0^+} \|w_k\|_{L^\infty(\mathbb{R}^N \setminus \{0\})} = 0, \quad \lim_{k \to 0^+} \|v_k\|_{L^\infty(\mathbb{R}^N \setminus \{0\})} = 0,
\]

so \( u_\nu \) verifies (1.12). Here and in what follows, we always denote \( u_\nu \) the \( \nu_k \)-fast decaying solution of (1.1) derived by the sequence \( v_n \) defined in (3.19) with initial value \( w_k \).
Step 2: the mapping \( k \mapsto \nu_k \) is increasing. For \( 0 < k_1 < k_2 \), by the increasing monotonicity of \( w_k \), we have that \( w_{k_1} < w_{k_2} \). Let \( \{v_{n,k}\} \) be sequence of (3.19) with the initial data \( v_0 = w_k \), here \( i = 1, 2, \ldots \).

Let
\[
\nu_{n,i} = \lim_{|x| \to +\infty} v_{n,k_i}(x)|x|^{N-2}, \quad i = 1, 2, \quad n = 1, 2, 3, \ldots
\]

We see that
\[
\nu_{1,1} = c_N \int_{\mathbb{R}^N} Vw_{k_1}^p \, dx < c_N \int_{\mathbb{R}^N} Vw_{k_2}^p \, dx = \nu_{1,2}
\]
and
\[
\nu_{1,2} - \nu_{1,1} = c_N \int_{\mathbb{R}^N} V(w_{k_2}^p - w_{k_1}^p) \, dx \geq c_N \int_{\mathbb{R}^N} (w_{k_2}^p - w_{k_1}^p) \, dx = k_2 - k_1.
\]

Inductively, we have that for any \( n \in \mathbb{N} \),
\[
\nu_{n,2} - \nu_{n,1} \geq k_2 - k_1,
\]
which implies that the limit \( u_{\nu_{k_1}} \) of \( \{v_{n,k}\} \) and the limit \( u_{\nu_{k_2}} \) of \( \{v_{n,k}\} \) as \( n \to +\infty \) verifies that
\[
\lim_{|x| \to +\infty} u_{\nu_{k_1}}(x)|x|^{N-2} - \lim_{|x| \to +\infty} u_{\nu_{k_2}}(x)|x|^{N-2} \geq k_2 - k_1,
\]
that is,
\[
u_{k_2} - \nu_{k_1} \geq k_2 - k_1.
\]
As a conclusion, for any \( k \in (0, k^*) \), there exists a \( \nu_k > 0 \) such that problem (1.1) has a solution \( u_{\nu_k} \) such that
\[
\lim_{|x| \to +\infty} u_{\nu_k}(x)|x|^{N-2} = \nu_k.
\]

For \( 0 < k_2 \leq k_1 \leq k_0 \), then \( w_{k_1,\mu} \geq w_{k_2,\mu} \) and \( v_{n,k_1} \geq v_{n,k_2} \) in \( \mathbb{R}^N \setminus \{0\} \), so we have that \( u_{\nu_{k_1}} \geq u_{\nu_{k_2}} \) in \( \mathbb{R}^N \setminus \{0\} \). That is to say that the mapping \( k \mapsto \nu_k \) is increasing.

Step 3. we prove that the mapping \( k \in (0, k^*) \mapsto \nu_k \) is continuous. Fix \( \bar{k} \in (0, k^*) \) and \( \delta < \frac{1}{2} \min\{\bar{k}, k_0 - \bar{k}\} \), then for \( k \in (\bar{k} - \delta, \bar{k} + \delta) \) and \( x \in \mathbb{R}^N \setminus \{0\} \), we have that
\[
|\omega_0(- \ln |x| + b_0^{-1} \ln(\frac{k}{d_0})) - \omega_0(- \ln |x| + b_0^{-1} \ln(\frac{\bar{k}}{d_0}))| \\
\leq b_0^{-1}|\ln k - \ln \bar{k}| \omega_0(- \ln |x| + b_0^{-1} \ln(\frac{\bar{k} + \delta}{d_0})) \\
\leq cb_0^{-1}|\ln k - \ln \bar{k}| \omega_0(- \ln |x| + b_0^{-1} \ln(\frac{\bar{k} + \delta}{d_0})),
\]
where the last inequality used (2.7).

For \( |k - \bar{k}| < \delta \), we have that
\[
b_0^{-1}|\ln k - \ln \bar{k}| \leq \bar{c}|k - \bar{k}|,
\]
where \( \bar{c} = b_0^{-1} \ln 2 \). So we have that
\[
|v_{0,k} - v_{0,\bar{k}}|(x) \leq c|k - \bar{k}| w_{k + \delta}(x) \leq c|k - \bar{k}| u_{\nu_{k + \delta}}(x).
\]
Then by the increasing monotonicity of the mapping \( k \mapsto \nu_k \), we have that
\[
|v_{1,k} - v_{1,\bar{k}}| \leq G|V| w_{0,k}^p - u_{0,\bar{k}}^p | \leq pG|V| w_{0,k}^p |v_{0,k} - v_{0,\bar{k}}| \\
\leq c|k - \bar{k}| G|V| u_{\nu_{k + \delta}}^p = c|k - \bar{k}| u_{\nu_{k + \delta}}
\]

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and since \(v_{1,k}, v_{1,k} \leq u_{\nu_k + \delta}\), it holds
\[
|v_{2,k} - u_{2,k}| \leq G|V|v_{1,k}^p - v_{1,k}^p| \leq cG[V u_{\nu_k}^{-1} v_{1,k} - v_{1,k}] = c|k - \bar{k}| u_{\nu_k + \delta},
\]

inductively, we obtain that
\[
|v_{n,k} - v_{n,k}| \leq c|k - \bar{k}| u_{\nu_k + \delta},
\]
so, the following holds
\[
|u_{\nu_k} - u_{\nu_k}| \leq c_3|k - \bar{k}| u_{\nu_k + \delta} \quad \text{in } \mathbb{R}^N \setminus \{0\}.
\]
Therefore, we have that
\[
|\nu_k - \nu_{k}| \leq c_\nu u_{\nu_k + \delta}|k - \bar{k}|.
\]

Let \(\nu_0 = \lim_{k \to k^*} \nu_k\), then for any \(\nu \in (0, \nu_0)\), problem (1.1) has a \(\nu\)-fast decaying solution \(u_\nu\) verifying
\[
\lim_{|x| \to 0} u_\nu(x)|x|^{\frac{2}{p-1}} = c_\nu.
\]

Observe that for any \(\tilde{\nu} \geq \nu_0\), if there is an upper barrier for the sequence (3.19) with the initial data \(w_k\), all above properties could be extended into \((0, \tilde{\nu})\).

Motivated by the fact that \(\nu_k \geq k\) when \(V \geq 1\), let us denote
\[
\nu_\infty = \sup \{\nu > 0 \text{ such that } \text{there is } k \in (0, \nu_\infty) \text{ problem (1.1) has solution } u_\nu \geq w_k \text{ in } \mathbb{R}^N \setminus \{0\} \}
\]
Note that using the above arguments, we can show that the mapping \(k \in (0, k_\infty) \mapsto \nu_k\) is increasing and continuous, where \(k_\infty = \sup\{k \in (0, \nu_\infty)\} \).

Finally, we prove that \(\nu_\infty = +\infty\) if (1.17) holds for some \(l_1 > 1\) and \(\alpha_1 \geq 0\). By contradiction, we may assume that
\[
\nu_\infty < +\infty.
\]

Now fix \(\bar{\nu} \in (0, \nu_\infty)\) such that for \(l_1 > 1\) and \(\alpha_1 \geq 0\) in (1.14) and \(\bar{\nu} l_1^{N-2-\frac{2+\alpha_1}{p-1}} > \nu_\infty\), where \(N - 2 - \frac{2+\alpha_1}{p-1} > 0\) by our assumption (1.9). Denote
\[
\psi_1(x) = l_1^{\frac{2+\alpha_1}{p-1}} u_{\nu_\infty}(l_1^{-1} x), \quad \forall x \in \mathbb{R}^N \setminus \{0\}.
\]
Let \(\bar{k}\) be the number such that \(\nu_{\bar{k}} = \bar{\nu}\). By direct computation, we have that
\[
\lim_{|x| \to +\infty} \psi_1(x)|x|^{N-2} = \bar{\nu} l_1^{N-2-\frac{2+\alpha_1}{p-1}}
\]
and \(-\Delta \psi_1 = V_1 \psi_1^p\) in \(\mathbb{R}^N \setminus \{0\}\), where \(V_1(x) = l_1^{\alpha_1} V(l_1^{-1} x) \geq V(x)\) by (1.14).

Note that \(w_{l_1^{N-2-\frac{2+\alpha_1}{p-1}} \bar{k}}(x) = l_1^{2+\alpha_1 \bar{k}} w_{l_1^{-1}}(l_1^{-1} x)\) and then we may initiate the iteration (3.19) with \(v_0 = w_{l_1^{N-2-\frac{2+\alpha_1}{p-1}} \bar{k}}\) and \(\psi_0\) is an upper bound, so we have a solution \(u_\nu\)

that means
\[
\nu_{l_1^{N-2-\frac{2+\alpha_1}{p-1}} \bar{k}} \leq w_{l_1^{N-2-\frac{2+\alpha_1}{p-1}} \bar{k}} \psi_1
\]
which contradicts (3.24). So we have that \(\nu_\infty = +\infty\). The proof ends. \(\Box\)
Theorem 3.2 Under assumptions of Theorem 1.1, we let $V \leq 1$, i.e. $\beta = 0$ and $c_\infty = 1$ in the assumption $\mathcal{V}_0$ part (ii). Then there is $v_0 \in (0, +\infty)$ such that for any $\nu \in (0, v_0)$, problem (1.1) has a $\nu$-fast decaying solution $u_\nu$, which has singularity at the origin verifying (1.11) and the mapping $\nu \in (0, v_0) \mapsto u_\nu$ is increasing, continuous and (1.12) holds.

If (1.12) holds for some $\alpha_2 \leq 0$ and $l_2 > 1$, then $v_0 = +\infty$.

Proof. From Corollary 3.1 we take $\epsilon = c_{\delta_0} k^p$, and let $k \in (0, k^*)$, then Proposition 3.1 implies that problem (3.1) has a nonpositive solution $v_k$ verifying (3.6). Denote
\[
\tilde{u}_{v_k} = w_k + v_k \leq w_k \quad \text{and} \quad \tilde{v}_k = \int_{\mathbb{R}^N} V \tilde{u}_{v_k}^p \, dx,
\]
then $\tilde{u}_{v_k}$ is a nonnegative classical solution of (1.1) such that
\[
\lim_{|x| \to 0^+} \tilde{u}_{v_k}(x)|x|^{2-p-1} = c_p \quad \text{and} \quad \lim_{|x| \to +\infty} \tilde{u}_{v_k}(x)|x|^{N-2} = \tilde{v}_k,
\]
where the second estimate is obtain by Lemma 2.3 and $(k - c_{\delta_0} k^p)^+ < \tilde{v}_k \leq k$.

Let $v_0 := w_k$ and denote $v_n$ iteratively
\[
v_n = \Gamma * (V v_{n-1}^p) \quad \text{in} \quad \mathbb{R}^N \setminus \{0\},
\]
(3.26)
Since $V \leq 1$, we have that the sequence $\{v_n\}_n$ is a decreasing sequence of functions. When $k \in (0, k^*)$, we can show that $\tilde{u}_{v_n}$, obtained in Proposition 3.1, is a positive lower barrier for this sequence and we derive a $k$-fast decaying solution $u_{v_k}$ of problem (1.1), which verifies (1.12), where $u_k = c_N \int_{\mathbb{R}^N} V u_{v_k}^p \, dx$. When $k \geq k^*$, $\tilde{u}_{v_k/2}$ is always a lower bound for the sequence $\{v_n\}_n$ and then a solution $u_{v_k}$ of problem (1.1).

We show that the mapping $k \in (0, +\infty) \to \nu_k$ is increasing and continuous. For $0 < k_1 < k_2$, we have that $w_{k_1} < w_{k_2}$. Let $\{v_{n, k_i}\}$ be sequence of (3.26) with the initial data $v_{0, i} = w_{k_i}$, here $i = 1, 2$.

Let
\[
v_{n, i} = \lim_{|x| \to +\infty} v_{n, k_i}(x)|x|^{N-2}, \quad i = 1, 2, \quad n = 1, 2, 3, \ldots
\]
(3.26)
We see that
\[
v_{1, 1} = c_N \int_{\mathbb{R}^N} V w_{k_1}^p \, dx < c_N \int_{\mathbb{R}^N} V w_{k_2}^p \, dx = v_{1, 2}
\]
and
\[
0 < v_{1, 2} - v_{1, 1} = c_N \int_{\mathbb{R}^N} V (w_{k_2}^p - w_{k_1}^p) \, dx \leq c_N \int_{\mathbb{R}^N} (w_{k_2}^p - w_{k_1}^p) \, dx = k_2 - k_1.
\]
Inductively, we have that for any $n \in \mathbb{N}$,
\[
0 \leq v_{n, 2} - v_{n, 1} \leq k_2 - k_1,
\]
which implies that the limit $u_{v_{k_1}}$ of $\{v_{n, k_1}\}$ and the limit $u_{v_{k_2}}$ of $\{v_{n, k_2}\}$ as $n \to +\infty$ verifies that
\[
0 \leq \lim_{|x| \to +\infty} u_{v_{k_2}}(x)|x|^{N-2} - \lim_{|x| \to +\infty} u_{v_{k_1}}(x)|x|^{N-2} \leq k_2 - k_1,
\]
that is,
\[
0 \leq v_{k_2} - v_{k_1} \leq k_2 - k_1.
\]
As a conclusion, $k \to \nu_k$ is increasing and continuous.

Let
\[
\nu_\infty = \lim_{k \to +\infty} \nu_k,
\]
(3.26)
then we have that for any \( k \in (0, +\infty) \), there exists \( \nu_k \in (0, \nu_{\infty}) \) such that problem (1.1) has a solution \( u_{\nu_k} \) such that
\[
\lim_{|x| \to +\infty} u_{\nu_k}(x)|x|^{N-2} = \nu_k.
\]

Finally, we prove that \( \nu_{\infty} = +\infty \). By contradiction, we may assume that
\[
\nu_{\infty} < +\infty. \tag{3.27}
\]
Now fix \( \tilde{\nu} \in (0, \nu_{\infty}) \), then there exist \( \alpha_2 \leq 0 \) and \( l_2 > 1 \) such that
\[
\nu_{\infty} < l_2^{N-2-\frac{2+\alpha_2}{p-1}} > \nu_{\infty}
\]
and denote
\[
\psi_2(x) = l_2^{-\frac{2+\alpha_2}{p-1}} w_{\nu}(l_2^{-1}x), \quad \forall \ x \in \mathbb{R}^N \setminus \{0\}.
\]
Let \( \bar{\nu} \) be the number such that \( \nu_k = \tilde{\nu} \). By direct computation, we have that
\[
\psi_2(x) \leq l_2^{-\frac{2+\alpha_2}{p-1}} w_{\nu_\bar{k}}(l_2^{-1}x), \quad \forall \ x \in \mathbb{R}^N \setminus \{0\}
\]
and
\[-\Delta \psi_2 = V_{12} \psi_2^0 \text{ in } \mathbb{R}^N \setminus \{0\},
\]
where \( V_{12}(x) := l_2^{\nu_{\infty}} V(l_2^{-1}x) \leq V(x) \) by (1.15).

Note that \( w_{l_2^{-N-2-\frac{2+\alpha_2}{p-1}} k}(x) = l_2^{\frac{\alpha_2}{p-1}} w_{\nu}(l_2^{-1}x) \) and then we may initiate the iteration (3.19) with \( v_{0} = w_{l_2^{-N-2-\frac{2+\alpha_2}{p-1}} k} \) and \( \psi_0 \) is a lower bound, so we have a solution \( u_{\nu} \), of (1.1) such that
\[
\nu_{l_2^{-N-2-\frac{2+\alpha_2}{p-1}} k} \leq \nu_{\nu} \leq w_{l_2^{-N-2-\frac{2+\alpha_2}{p-1}} k}
\]
that means
\[
\nu_{l_2^{-N-2-\frac{2+\alpha_2}{p-1}} k} > \nu_{\nu} \nu_{\infty} > \nu_{\nu} \nu_{\infty},
\]
which contradicts (3.27). Thus, we have that \( \nu_{\infty} = +\infty. \)

### 3.3 Proof of main Theorems

**Proof of Theorem 1.1.** Step 1: Existence and the decay at infinity. Let
\[
V_1 = 1 - (V - 1)_- \quad \text{and} \quad V_2 = 1 + (V - 1)_+,
\]
where \( a_\pm = \max\{0, \pm a\} \). Then \( V_1, V_2 \) are H"older continuous and \( V = V_1 V_2 \in \mathbb{R}^N \setminus \{0\} \).

From Theorem 3.2 there is \( \nu_1 \in (0, +\infty) \) such that for any \( \nu \in (0, \nu_1) \), problem
\[
\begin{cases}
-\Delta u = V_1 u^p & \text{in } \mathbb{R}^N \setminus \{0\}, \\
u > 0 & \text{in } \mathbb{R}^N \setminus \{0\},
\end{cases} \tag{3.28}
\]
has a \( \nu \)-fast decaying solution \( u_{\nu,1} \), which has singularity at origin satisfying (1.11) and the mapping \( \nu \in (0, \nu_1) \mapsto u_{\nu} \) is increasing, continuous and (1.12) holds.

We remark that for any \( \nu \in (0, \nu_1) \), there exists a unique \( k \) such that \( \nu = \nu_k \) and the solution \( u_{\nu} \) is derived as the limit of the sequence
\[
v_n = \Gamma \ast (V_1 v_{n-1}^p) \quad \text{in } \mathbb{R}^N \setminus \{0\},
\]
where \( v_n = \Gamma \ast (V_1 v_{n-1}^p) \).
with the initial data \( v_0 = w_k \). The mapping \( k \mapsto \nu_k \) is increasing, continuous and \( \lim_{k \to 0^+} \nu_k = 0 \).

As the proof of Theorem 3.1 there is a \( \mu_k \in (0, +\infty) \) such that for any \( \mu \in (0, \mu_k) \), problem

\[
\begin{cases}
-\Delta u = V_2 V_1 v^p & \text{in } \mathbb{R}^N \setminus \{0\}, \\
u > 0 & \text{in } \mathbb{R}^N \setminus \{0\},
\end{cases}
\tag{3.29}
\]

has a \( \mu \)-fast decaying solution \( u_{\mu} \), which has singularity \((1.11)\) at the origin and the mapping \( \mu \in (0, \mu_k) \mapsto u_{\mu} \) is increasing, continuous and \((1.12)\) holds.

We remark that for any \( \mu \in (0, \mu_k) \), there exists a unique \( \nu \in (0, \nu_1) \) such that \( \mu = \nu \mu_k \) and the solution \( u_{\mu} \) is derived as the limit of the sequence

\[
v_n = \Gamma \ast (V_2 V_1 v_{n-1}^p) \quad \text{in } \mathbb{R}^N \setminus \{0\}
\]

with the initial data \( v_0 := u_{\nu_1} \). The map \( \nu \mapsto \mu_{\nu} \) is increasing and continuous and \( \lim_{k \to 0^+} \mu_{\nu} = 0 \).

As a consequence, for some \( k^* \in (0, +\infty) \), the map: \( k \mapsto \nu_k \mapsto \mu_{\nu_k} \), denoting \( \mu_k = \mu_{\nu_k} \), is continuous and increasing, problem \((1.1)\) has a \( \mu_{k}\)-fast decaying solution \( u_{\mu_k} \), which has singularity at the origin

\[
\lim_{{|x| \to 0}} u_{\nu}(x) |x|^{-\frac{2^*}{N}} = c_p,
\]

where \( c_p \) is given in \((1.3)\). From Proposition 3.1 and Corollary 3.1 we have that

\[
\mu_k \leq k + c_{k_0} k^p
\]

which deduces \((1.12)\).

\(\square\)

**Proof of Theorem 1.2.** Theorem 1.2 follows by Theorem 3.1 and Theorem 3.2 directly. \(\square\)

## 4 Existence of slow decay solution

Under the assumption \((V_1)\) part \((II)\), the mapping \( \nu \in (0, \infty) \mapsto u_{\nu} \) is increasing, where \( u_{\nu} \) is a \( \nu \)-fast decaying solution of problem \((1.1)\), so our interest is to show the limit of \( \{u_{\nu}\}_\nu \) as \( \nu \to +\infty \) exists, denoting \( u_{\infty} = \lim_{{\nu \to +\infty}} u_{\nu} \) if the limit exists, which is a very weak solution of \((1.1)\) in the distributional sense that \( u_{\infty} \in L^1_{loc}(\mathbb{R}^N) \cap L^p_{loc}(\mathbb{R}^N, V dx) \) satisfies the identity

\[
\int_{\mathbb{R}^N} u_{\infty}(-\Delta) \xi dx = \int_{\mathbb{R}^N} V u_{\infty}^p \xi dx, \quad \forall \xi \in C_c^\infty(\mathbb{R}^N).
\tag{4.1}
\]

**Lemma 4.1** Let the hypotheses of Theorem 1.3 hold, then \( u_{\nu} \) is radially symmetric and decreasing with respective to \( |x| \).

**Proof.** Since \( V \) is radially symmetric and decreasing with respect to \( |x| \) and \( u_{\nu} \) decays as \( \nu |x|^{2-N} \) at infinity, so it is available to apply the moving planes method, see [3][16] to obtain that \( u_{\nu} \) is the radially symmetric and decreasing with respect to \( |x| \). \(\square\)

**Lemma 4.2** Let the hypotheses of Theorem 1.3 hold, then there exists \( c > 0 \) independent of \( \nu \) such that

\[
\|u_{\nu}\|_{L^1_{loc}(\mathbb{R}^N)} \leq c \quad \text{and} \quad \|u_{\nu}\|_{L^p_{loc}(\mathbb{R}^N, V dx)} \leq c.
\]

**Proof.** Let

\[
U_0(x) = \frac{c_N}{(1 + |x|^2)^{\frac{2^*}{2}}},
\]

then

\[
-\Delta U_0 = U_0^{2^*-1} \quad \text{in } \mathbb{R}^N,
\]

where \( 2^* = \frac{2N}{N-2}, \ c_N = (N(N-2))^\frac{N-2}{N} \).

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For $\epsilon \in (0, \frac{1}{4})$, denote
\[ \eta_\epsilon(x) = \eta_0(\epsilon|x|), \quad x \in \mathbb{R}^N, \] (4.2)
where $\eta_0 : [0, +\infty) \to [0, 1]$ is a smooth increasing function such that
\[ \eta_0(t) = 0, \quad \forall t \geq 2 \quad \text{and} \quad \eta_0(t) = 1, \quad \forall t \in [0, 1]. \]

Take $U_0 \eta_\epsilon^2$ as a test function of (1.1), then by Hölder inequality, we have that
\[
\int_{\mathbb{R}^N} V u_\nu^p U_0 \eta_\epsilon^2 dx = \int_{\mathbb{R}^N} u_\nu(-\Delta)(U_0 \eta_\epsilon^2) dx \\
= \int_{\mathbb{R}^N} u_\nu(U_0^{2^* - 1} \eta_\epsilon^2 + 4\eta_\epsilon \nabla U_0 \cdot \nabla \eta_\epsilon + U_0(-\Delta)(\eta_\epsilon^2)) dx \\
\leq \left( \int_{\mathbb{R}^N} V u_\nu^p U_0 \eta_\epsilon^2 dx \right)^{\frac{1}{p}} \left( \int_{\mathbb{R}^N} V^{1 - p} U_0^{1 - p + (2^* - 1)\frac{p}{p-1}} dx \right)^{\frac{p-1}{p}} \\
+ \epsilon \|\eta_0\|_{C^1(\mathbb{R})} \int_{B^*_\epsilon(0) \setminus B_{\frac{1}{2}}(0)} V^{1 - p} U_0^{1 - p + (2^* - 1)\frac{p}{p-1}} |\nabla U_0|^{p-1} dx \\
\leq \epsilon^2 \|\eta_0\|_{C^2(\mathbb{R})} \left( \int_{B^*_\epsilon(0) \setminus B_{\frac{1}{2}}(0)} V^{1 - p} U_0^{p-1} dx \right)^{1 - \frac{1}{p}} \\
= c_0 \left( \int_{\mathbb{R}^N} V u_\nu^p U_0 \eta_\epsilon^2 dx \right)^{\frac{1}{p}},
\]
where $c_0 > 0$ is dependent on $\epsilon, V$ but it is independent of $\nu$.

So we have that
\[
c_N(1 + \frac{4}{c^2})^{-\frac{N-2}{2}} \int_{B_{\frac{1}{2}}(0)} V u_\nu^p dx \leq \int_{B_{\frac{1}{2}}(0)} V u_\nu^p U_0 dx \leq \int_{\mathbb{R}^N} V u_\nu^p U_0 \eta_\epsilon^2 dx \leq c_0^\frac{p}{p-1},
\]
that is,
\[
\|u_\nu\|_{L^p_{\text{loc}}(\mathbb{R}^N, V dx)} \leq c
\]
for some $c > 0$ independent of $\nu$.

Furthermore,
\[
\int_{\mathbb{R}^N} u_\nu U_0^{2^* - 1} \eta_\epsilon^2 dx \leq c_0 \left( \int_{\mathbb{R}^N} V u_\nu^p U_0 \eta_\epsilon^2 dx \right)^{\frac{1}{p}} \leq c_0^\frac{p}{p-1}
\]
and
\[
\|u_\nu\|_{L^1_{\text{loc}}(\mathbb{R}^N)} \leq c.
\]
The proof ends. \qed

**Proof of Theorem 1.3.** From Theorem 1.2 and Lemma 4.2 the mapping $\nu \in (0, \infty) \mapsto u_\nu$ is increasing and uniformly bounded in $L^1_{\text{loc}}(\mathbb{R}^N) \cap L^p_{\text{loc}}(\mathbb{R}^N, V dx)$, so there exists $u_\infty \in L^1_{\text{loc}}(\mathbb{R}^N) \cap L^p_{\text{loc}}(\mathbb{R}^N, V dx)$ such that
\[
u \mapsto u_\nu \quad \text{as} \quad \nu \to +\infty \quad \text{a.e. in} \quad \Omega \quad \text{and in} \quad L^1_{\text{loc}}(\mathbb{R}^N) \cap L^p_{\text{loc}}(\mathbb{R}^N, V dx).
\]

It is known in [19] that $u_\nu$ is also a weak solution of (1.1), i.e.
\[\int_{\mathbb{R}^N} u_\nu(-\Delta)\xi dx = \int_{\mathbb{R}^N} V u_\nu^p \xi dx, \quad \forall \xi \in C^\infty_c(\mathbb{R}^N). \quad (4.3)\]
Passing to the limit of (4.3), we obtain that $u_\infty$ is a weak solution of (1.1) in the sense of (4.1).
Furthermore, by Lemma 4.4 we have that \( u_p \) is radially symmetric and decreasing with respect to \( |x| \), so is \( u_\infty \). So we have that \( u_\infty \in L^\infty_{\text{loc}}(\mathbb{R}^N \setminus \{0\}) \), then \( V u_\infty^p \) is in \( L^\infty_{\text{loc}}(\mathbb{R}^N \setminus \{0\}) \). By standard regularity results, we have that \( u \) is a classical solution of (1.1).

Since \( u_p \) verifies \((1.11)\) at the origin for any \( \nu > 0 \) and \( u_\infty \) is the limit of an increasing sequence \( \{u_p\}_p \), then we have that

\[
\liminf_{|x| \to 0} u_\infty(x)|x|^{-2 + \frac{2 + \bar{\nu}}{p-1}} \geq c_p. \tag{4.4}
\]

Next we claim

\[
\limsup_{|x| \to 0} u_\infty(x)|x|^{-2 + \frac{2 + \bar{\nu}}{p-1}} \leq c_p, \tag{4.5}
\]

From [23, Theorem 2.1], there exists \( c > 0 \) such that

\[
u \in \mathbb{R} \quad \text{and} \quad u \in C^2_{\text{loc}}(\mathbb{R}^N \setminus \{0\})
\]

By direct computation, we have that

\[
\Delta v(x) = \frac{1}{|x|^{N+2}} \Delta u_\infty(x) = \frac{2(2-N)}{|x|^4} v(x) + \frac{2}{|x|^2} \nabla u_\infty \cdot x \frac{x}{|x|^2}.
\]

Let \( u^p(x) = |x|^{2-N} v(x) \), then for \( x \in \mathbb{R}^N \setminus \{0\} \), we get that

\[
-\Delta u^p(x) = -\Delta v(x) |x|^{2-N} - 2 \nabla v(x) \cdot (\nabla |x|^{2-N}) = |x|^{-2-N} (-\Delta) u \frac{x}{|x|^2} = V^p(x) u^p(x),
\]

where

\[
V^p(x) = |x|^{-2-N+p(N-2)} V \frac{x}{|x|^2}.
\]

We see that

\[
-\Delta u^p(x) = V^p(x) u^p(x) \quad \text{in} \quad \mathbb{R}^N \setminus \{0\},
\]

where

\[
V^p(x) \sim |x|^\varrho \quad \text{as} \quad |x| \to 0 \quad \text{with} \quad \varrho = -\alpha - 4 + (p-1)(N-2).
\]

Note that it implies by (1.17) that \( \varrho \in (-2,0) \).

Now claim that \( p \in \left( \frac{N+\bar{\nu}}{N-2}, \frac{N+2+\bar{\nu}}{N-2} \right) \). Note that \( p > \frac{N+\bar{\nu}}{N-2} \) follows by the fact \( p > \frac{N}{N-2} \) and \( p < \frac{N+2+\bar{\nu}}{N-2} \) is equivalent to \( (N-2)p - N - 2 < -4 + (p-1)(N-2) - \alpha \), which is true thanks to \( \alpha \leq 0 \).

Then by [23, Theorem 2.1] and [17, Theorem 3.3] we have that

\[
\frac{1}{c} |x|^{-2 + \frac{2+\bar{\nu}}{p-1}} \leq u^p(x) \leq c |x|^{-2 + \frac{2+\bar{\nu}}{p-1}}, \quad \forall x \in B_1(0) \setminus \{0\},
\]

where \( c > 1 \) and

\[
\frac{2 + \bar{\nu}}{p-1} = -(N-2) + \frac{2 + \alpha}{p-1}.
\]

By Kelvin transformation, we turn back that

\[
\frac{1}{c} |x|^{-2 + \frac{2+\nu}{p-1}} \leq u_\infty(x) \leq c |x|^{-2 + \frac{2+\nu}{p-1}}, \quad x \in \mathbb{R}^N \setminus B_1(0)
\]
for some $c > 1$.  

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