Functional integral method in quantum field theory of plasmons in graphene

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Abstract
In the present work we apply the functional integral method to the study of quantum field theory of collective excitations of spinless Dirac fermion in graphene at vanishing absolute temperature and at Fermi level $E_F = 0$. After introducing the Hermitian scalar field $\varphi(x)$ describing these collective excitations we establish the expression of the functional integral $Z^\varphi$ containing a functional series $I[\varphi]$. The explicit expressions of several terms of this functional series were derived. Then we consider the functional series $I[\varphi]$ in second order approximation and denote $I_0[\varphi]$ the corresponding approximate expression of $I[\varphi]$. We shall demonstrate that in this approximation the scalar field $\varphi(x)$ can be divided into two parts: a background field $\varphi_0(x)$ corresponding to the extremum of $I_0[\varphi]$ and another scalar field $\xi(x)$ describing the fluctuation of $\varphi(x)$ around the background $\varphi_0(x)$. We call $\xi(x)$ the fluctuation field. Then we establish the relationship between this fluctuation field $\xi(x)$ and the quantum field of plasmons in graphene. Considering some range of values of frequency (energy) and wave vector (momentum) of plasmons, when the analytical calculations can be performed, we derived the differential equation for the quantum field of graphene plasmons. From this field equation we establish the relation between frequency and wave vector of plasmons in the long wavelength limit.

Keywords: functional integral, collective excitation, fluctuation, plasmons, quantum field

Classification numbers: 2.01, 3.00, 3.02, 5.15

1. Introduction

Recently there was a significant attention to the experimental research on graphene plasmons. Basov \textit{et al} \cite{1} investigated electronic and plasmonic phenomena at graphene grain boundaries, and Hillenbrand \textit{et al} \cite{2} demonstrated the strong plasmon reflection at nanometer-size gap in monolayer graphene on SiC. Subsequently Kim, Ham \textit{et al} \cite{3} demonstrated the existence of plasmon in graphene by measuring its effective mass, Basov \textit{et al} \cite{4} investigated ultrafast and nanoscale plasmonic phenomena in exfoliated graphene revealed by infrared pump-probe nanoscopy, Avouris \textit{et al} \cite{5} performed graphene plasmon enhanced vibrational sensing of surface-adsorbed layers, Goldflam \textit{et al} \cite{6} investigated tuning and persistent switching of graphene plasmons on ferroelectric substrate, and Basov \textit{et al} \cite{7} studied plasmons in graphene moiré superlatties. Lately Constant \textit{et al} \cite{8} demonstrated all optical generation of surface plasmons in graphene, Hillenbrand \textit{et al} \cite{9} revealed acoustic terahertz graphene plasmons by using photocurrent nanoscopy and Koppens \textit{et al} \cite{10} performed thermoelectric detection and imaging of propagating graphene plasmons.

Theoretical research on graphene plasmons began more than a decade ago. Wunsch \textit{et al} \cite{11} studied dynamical polarization of graphene at finite doping and employed the dynamical polarization to calculate the decay rate of plasmons. Dielectric
function, screening and plasmons in graphene were studied by Hwang and Das Sarma [12]. In [13] Polini et al studied plasmons and spectral functions of graphene. Collective modes of doped graphene in a strong magnetic field, the magnetoplasmons, were studied by Roldán et al [14]. In [15] Pyatakovskiy studied plasmons in gapped graphene. The terahertz surface plasmons in optically pumped graphene were studied by Dubinov et al [16]. In [17] Xu et al studied terahertz plasmon and infrared coupled plasmon-phonon modes in graphene. In [18] Despoja et al studied plasmon spectra in pristine and doped graphene. The effects of screening on the propagation of graphene surface plasmons were studied by Sasaki et al [19]. In [20] Hess et al developed the theory of nonequilibrium plasmons with grain in graphene. The localized plasmons in graphene-coated nanospheres were studied by Mortensen et al [21]. In [22] Novko et al considered graphene plasmons as the consequence of changing character of electronic transitions in graphene from single-particle excitations to collective excitations. Plasmons in spin-polarized graphene were studied by Agarwal et al [23]. In [24] Iurov et al studied plasmon dissipation in gapped graphene open systems at finite temperature. Dynamical polarization and plasmons in graphene as a 2D system with merging Dirac points were investigated by Pyatakovskiy and Chakraborty [25]. The emergence of anisotropic plasmons when electrons in graphene are pumped to the M point in the Brillouin zone was demonstrated by Chaves et al [26]. In [27] Wenger et al studied optical signatures of nonlocal plasmons in graphene. In many above-mentioned theoretical works the authors have used explicit expressions of 2-component spinor wave functions of spinless Dirac fermions established in the comprehensive review of Castro Neto et al [28].

In our previous work [29] we have extended the method proposed by Nguyen and Nguyen [30, 31] and obtained following functional integral of the Hermitian scalar field \( \varphi(x) \) describing the collective excitations of the spinless Dirac fermion gas in graphene at vanishing absolute temperature

\[
Z^\varphi = Z_0 \int [D\varphi(x)] \exp \{iI[\varphi(x)]\},
\]

where \( Z_0 \) is an arbitrary constant and \( I[\varphi] \) has the form of a series

\[
I[\varphi] = \sum_{n=1}^{\infty} I^{(n)}[\varphi],
\]

where

\[
I^{(1)}[\varphi] = - \int dx \left\langle \psi^\dagger(x) \psi(x) + \psi^\dagger(x) \psi(x) \right\rangle_0 V(x - x') \varphi(x'),
\]

\[
I^{(2)}[\varphi] = \frac{1}{2} \int dx \int dx' \int dy \int dy' V(x - y) V(y - x') \varphi(y) \varphi(y') \left\langle \psi^\dagger(x) \psi(x) \psi^\dagger(x') \psi(x') \right\rangle_0,
\]

\[
\times \left[ S^{\alpha\beta}_{\alpha\beta}(x, y) S^{\alpha\beta}_{\alpha\beta}(y, x) + S^\alpha_{\alpha\beta}(x, y) S^{\alpha\beta}_{\alpha\beta}(y, x) \right],
\]

\[
I^{(3)}[\varphi] = \frac{1}{3} \int dx \int dx' \int dy \int dy' \int dz \int dz' V(x - y) V(x - x') V(z - z') \varphi(x') \varphi(y') \varphi(x') \varphi(y') \left\langle \psi^\dagger(x) \psi(x) \psi^\dagger(x') \psi(x') \right\rangle_0
\]

\[
\times \left[ S^{\alpha\beta}_{\alpha\beta}(x, y) S^{\alpha\beta}_{\alpha\beta}(y, z) S^{\alpha\beta}_{\alpha\beta}(z, x) + S^\alpha_{\alpha\beta}(x, y) S^{\alpha\beta}_{\alpha\beta}(y, z) S^{\alpha\beta}_{\alpha\beta}(z, x) \right],
\]

and so on,

\[
V(x - y) = \delta(x_0 - y_0) V(x - y),
\]

\[
V(x - y)\text{ being the potential energy of the Coulomb interaction between two Dirac fermions, } S^{E, K}_{\alpha\beta}(x, y) \text{ being matrix elements of } 2 \times 2 \text{ matrix functions } S^{E, K}(x, y) \text{ satisfying inhomogeneous Dirac equations for Dirac fermions, Explicit expressions of the } 2 \times 2 \text{ matrix functions } S^{E, K}(x, y) \text{ were established in our previous work [29].}
\]

In the present work we study the functional integral (1) in the special and simplest non-trivial case, when the polynomial functional (1) is limited at the second order approximation with respect to the scalar field \( \varphi(x) \). Obtained approximate expression of \( I[\varphi] \) will be denoted \( I_0[\varphi] \). We shall demonstrate that in this approximation the scalar field \( \varphi(x) \) can be divided into two parts: a background field \( \varphi_0(x) \) corresponding to the extremum of \( I_0[\varphi] \) and another scalar field \( \xi(x) \) describing the fluctuation of \( \varphi(x) \) around the background \( \varphi_0(x) \). We call \( \xi(x) \) the fluctuation field. Then we establish the relationship between the quantum field \( \xi(x) \) and plasmons in graphene.

2. Quantum theory of fluctuation field

Consider the second order approximation of functional integral (1) with respect to the Hermitian scalar field \( \varphi(x) \) describing the collective excitations of the spinless Dirac fermion gas at vanishing absolute temperature. In this approximation functional integral (1) of the system becomes

\[
Z^\varphi = Z_0 \int [D\varphi(x)] \exp \{iI_0[\varphi(x)]\},
\]

with following expression of the functional \( I_0[\varphi] \):

\[
I_0[\varphi] = - \int dx \left\langle \psi^\dagger(x) \psi(x) \psi^\dagger(x) \psi(x) \right\rangle_0 V(x - x') \varphi(x'),
\]

\[
+ \frac{1}{2} \int dx \int dx' \int dy \int dy' V(x - y) V(y - x') \varphi(y) \varphi(y') \left\langle \psi^\dagger(x) \psi(x) \psi^\dagger(x') \psi(x') \right\rangle_0
\]

\[
\times \left[ S^{\alpha\beta}_{\alpha\beta}(x, y) S^{\alpha\beta}_{\alpha\beta}(y, x) + S^\alpha_{\alpha\beta}(x, y) S^{\alpha\beta}_{\alpha\beta}(y, x) \right],
\]

\[
\frac{n^{E, K}(x)}{n^{E, K}(x)} \text{ being the Dirac fermion densities in the ground state of the Dirac fermion gas at vanishing absolute temperature}
\]

\[
n^{E, K}(x) = \left\langle \psi^\dagger(x) \psi(x) \right\rangle_0
\]

\[
A^{E, K}(x) = i \int dx' \int dy' S^{E, K}_{\alpha\beta}(x', y') \left\langle \psi^\dagger(x') \psi(x') \psi^\dagger(x') \psi(x') \right\rangle_0
\]

\[
\times S^{E, K}_{\alpha\beta}(x', y') V(x' - x) V(y' - y).
\]

Denote \( \varphi_0(x) \) the scalar field \( \varphi(x) \) corresponding to the extremum value of \( I_0[\varphi] \). It is determined by the variational equation

\[
\frac{\delta I_0[\varphi]}{\delta \varphi(x)} \bigg|_{\varphi(x)=\varphi_0(x)} = 0.
\]

From this equation it follows that
\( \omega \tilde{K}(x) + \phi \tilde{K}(x) = \int dy \left[ V(x - y) + A^K(x, y) + A^\tilde{K}(x, y) \right] \varphi_0(y). \)

The extremum field \( \varphi_0(x) \) plays the role of the background, and the fluctuation of \( \varphi(x) \) around \( \varphi_0(x) \) is described by the difference

\[ \xi(x) = \varphi(x) - \varphi_0(x). \]

In terms of the fluctuation field \( \xi(x) \) formula (8) becomes

\[ I_0 [\varphi_0 + \xi] = I_0 [\varphi_0] + \frac{1}{2} \int dx \int dy \xi(x) \left[ V(x - y) + A^K(x, y) + A^\tilde{K}(x, y) \right] \xi(y). \]

and the functional integral (7) has following form

\[ Z^\xi = \exp \left\{ \int I_0[\varphi_0] \right\} Z^\xi, \]

where

\[ I_2[\xi] = \int \frac{1}{2} dx \int dy \xi(x) \left[ V(x - y) + A^K(x, y) + A^\tilde{K}(x, y) \right] \xi(y). \]

According to formula (11) functions \( A^K(x, y) \) and \( A^\tilde{K}(x, y) \) are expressed in terms of the matrix elements of the \( 2 \times 2 \) matrix functions \( S^K_{\sigma}(x, y) \) and \( A^K_{\sigma}(x, y) \) the potential energy of the Coulomb interaction between two Dirac fermions in equation (6). Explicit formula for the \( 2 \times 2 \) matrix functions \( S^K_{\sigma}(x, y) \) was presented in our previous work [29]. We also use the same notations of space-time coordinates as those in [29]. In this work it was also shown that \( 2 \times 2 \) matrix functions \( S^K_{\sigma}(x, y) \) have following expression

\[ S^K_{\sigma}(x, y) = \frac{1}{(2\pi)^2} \int \frac{d\omega e^{-i\omega(x-y)}}{\omega} \int dk \frac{1}{1 + \frac{\omega^2 - \omega_0^2}{\omega_0^2}} \int d\phi \omega \left\{ \frac{C^{K,\sigma}_+(k)}{\omega - \omega_0 + i\delta} + \frac{C^{K,\sigma}_-(k)}{\omega - \omega_0 - i\delta} \right\}. \]

Explicit formulae for the 2-component spinor wave functions \( \tilde{U}^K_{\sigma\alpha}(x) \) of Dirac fermions were given in the comprehensive review [28]. Functions \( C^K_{\sigma}(k) \) are determined by the characteristics of the Dirac fermion gas in graphene at the vanishing absolute temperature. By means of the same reasonings as those presented in [30, 31] it can be shown that they equal to the occupation numbers \( n^K_{\sigma\alpha}(k) \) at the quantum states with 2-component spinor wave functions \( \tilde{U}^K_{\sigma\alpha}(x) \):

\[ C^K_{\sigma}(k) = n^K_{\sigma\alpha}(k). \]

Due to the symmetry of two Dirac points \( K \) and \( K' \) in the reciprocal lattice of graphene, the occupation numbers at both states with wave functions \( n^K_{\sigma\alpha}(x) \) and \( \tilde{U}^K_{\sigma\alpha}(x) \) are equal:

\[ n^K_{\sigma\alpha}(k) = n^{K'}_{\sigma\alpha}(k) = n_{\sigma\alpha}(k). \]

Thus we obtain following new form of the expression (19):

\[ S^{K',\sigma}(x, y) = \frac{1}{2\pi} \int d\omega e^{-i\omega(x-y)} \left( \int \frac{1}{(2\pi)^2} \int dk \frac{1}{1 + \frac{\omega^2 - \omega_0^2}{\omega_0^2}} \int d\phi \omega \left\{ \frac{C^{K',\sigma}_+(k)}{\omega - \omega_0 + i\delta} + \frac{C^{K',\sigma}_-(k)}{\omega - \omega_0 - i\delta} \right\}, \]

where

\[ \tilde{s}_{\sigma}(k, x) = \frac{1 - n_{\sigma}(k)}{\omega - E_\sigma(k) + i\delta} + \frac{n_{\sigma}(k)}{\omega - E_\sigma(k) - i\delta}. \]

The matrix elements \( S^{K',\sigma}_{\alpha\beta}(x, y) \) of \( 2 \times 2 \) matrix functions (22) are

\[ S^{K',\sigma}_{\alpha\beta}(x, y) = \frac{1}{(2\pi)^2} \int d\omega e^{-i\omega(x-y)} \int d\phi \omega \left\{ \tilde{s}_{\sigma}(k, x) \tilde{s}_{\sigma}(k, y) \right\}. \]

Functional \( I_2[\xi] \) determined by formula (18) contains functions \( A^K(x, y) \) and \( A^\tilde{K}(x, y) \) determined by formula (11). Using formula (24) for \( S^{K',\sigma}_{\alpha\beta}(x, y) \) and denoting \( \tilde{V}(p) \) the Fourier component of Coulomb potential energy \( V(x - y) \) in equation (6)

\[ \tilde{V}(p) = \frac{1}{(2\pi)^2} \int dpe^{-ip(x-y)} \tilde{V}(p), \]

after standard calculations we obtain following result

\[ A(x, y) = \frac{1}{\pi} \int d\omega e^{-i\omega(x-y)} \int d\phi \omega \left\{ \tilde{s}_{\sigma}(k, x) \tilde{s}_{\sigma}(k, y) \right\}. \]

with

\[ \tilde{s}_{\sigma}(k, x) = \frac{1 - n_{\sigma}(k)}{\omega - E_\sigma(k) + i\delta} + \frac{n_{\sigma}(k)}{\omega - E_\sigma(k) - i\delta}. \]

where

\[ \theta(k) = \arctg \frac{k_y}{k_x}, \]

\[ k_t \] and \( k_l \) being two components of vector \( k \).

Consider now expression (18) of the functional \( I_2[\xi] \) in the functional integral \( Z^\xi \) determined by formula (17). It comprises two parts

\[ I_2[\xi] = I_2^Y[\xi] + I_2^\xi[\xi], \]

where

\[ I_2^Y[\xi] = \int dx dy \xi(x) \left( V(x - y) \xi(y) \right), \]

and

\[ I_2^\xi[\xi] = \frac{1}{2} \int dx \int dy \xi(x) A(x, y) \xi(y). \]

Note that function \( V(x - y) \) in the expression of \( I_2^Y[\xi] \) depends only on the difference \( x - y \) of two space-time coordinates \( x \) and \( y \) while function \( A(x, y) \) in the expression of \( I_2^\xi[\xi] \) separately depends on both space-time coordinates \( x \) and \( y \). The appearance of the function \( A(x, y) \) separately depending on both space-time coordinates \( x \) and \( y \) is a significant peculiarity of the functional integral \( Z^\xi \) for the fluctuation field \( \xi \) in Dirac fermion gas in comparison to the corresponding functional.
integral in electron gas investigated in [30, 31]. Another significant peculiarity of the functional integral $Z^\xi$ for the fluctuation field $\xi$ in Dirac fermion gas is the special $k$-dependence
\[
E_\perp(k) = \pm \sqrt{v_F k}
\]
of the Dirac fermion energies in comparison with the non-relativistic $k$-dependence
\[
E(k) = \frac{\hbar^2}{2m} k^2
\]
of the electron energy. Therefore, in order to study plasmons in Dirac fermion gas of graphene it is necessary to extend the reasonings elaborated in [30, 31].

Let us now perform the Fourier expansion of $\xi(x)$ over the complete system of orthogonal and normalized plane waves. We have
\[
\xi(x) = \frac{1}{2\pi} \int \omega e^{-i\omega x} \frac{1}{(2\pi)^2} \int dk e^{ikx} \tilde{\xi}(\omega, k),
\]
therefore
\[
\xi(x)^+ = \frac{1}{2\pi} \int \omega e^{-i\omega x} \frac{1}{(2\pi)^2} \int dk e^{ikx} \tilde{\xi}(-\omega, -k)^+.
\]
From the Hermitian property $\xi(x) = \xi(x)^+$ of the fluctuation field $\xi(x)$ it follows that
\[
\tilde{\xi}(\omega, k) = \tilde{\xi}(-\omega, -k)^+.
\]
This Hermitian property of the fluctuation field $\xi(x)$ in term of its Fourier components $\xi(\omega, k)$, namely formula (34) will be used in the sequel.

Consider the functional $I^2_2[\xi]$ determined by formula (30). It can be shown that this functional has following explicit expression
\[
I^2_2[\xi] = \frac{1}{2} \frac{1}{(2\pi)^2} \int dk \tilde{\xi}(\omega, k) \tilde{V}(\omega, k) \tilde{\xi}(-\omega, -k).
\]
Concerning functional $I^2_2[\xi]$ determined by formula (31), by means of lengthy but standard calculations we obtain following result
\[
I^2_2[\xi] = \frac{1}{2} \frac{1}{(2\pi)^2} \int \omega \frac{1}{(2\pi)^2} \int dk \tilde{\xi}(\omega, k) \tilde{V}(\omega, k) \tilde{\xi}(-\omega, -k),
\]
where
\[
\tilde{V}(\omega, k) = \frac{1}{(2\pi)^2} \int \omega' \frac{1}{(2\pi)^2} \int dk' \frac{1}{(2\pi)^2} \int dk'' \frac{1}{(2\pi)^2} \int d\omega'' \times 2\pi \delta(\omega + \omega' - \omega'') (2\pi)^2 \delta(k + k' - k'')
\times i \{ \tilde{S}_+(\omega', k') + \tilde{S}_-(\omega', k') \} \tilde{S}_+((\omega', \omega''), k') + \tilde{S}_-((\omega', \omega'') k') \}
\times \cos(\theta(k') - \theta(\omega)).
\]
Thus the functional $I^2_2[\xi]$ in the expression (17) of the functional integral $Z^\xi$ of the fluctuation field $\xi(x)$ has following formula
\[
I^2_2[\xi] = \frac{1}{2} \frac{1}{(2\pi)^2} \int \omega \frac{1}{(2\pi)^2} \int dk \tilde{\xi}(\omega, k) \tilde{V}(\omega, k) \times [1 + \tilde{W}(\omega, k)] \tilde{\xi}(-\omega, -k).
\]
By means of lengthy but standard calculations it can be shown that
\[
\tilde{W}(\omega, k) = \tilde{V}(k) \frac{1}{(2\pi)^2} \int dk' \frac{1}{(2\pi)^2} \int d\omega' \times (2\pi)^2 \delta(k + k' - \omega') \times 2\pi \delta(\omega + \omega' - \omega) \{ 1 - \cos(\theta(k') - \theta(\omega')) \}. 
\]
(39)

3. Quantum field of plasmons in graphene

In the preceding Section we have demonstrated that the functional $I^2_2[\xi]$ of the fluctuation field $\xi(x)$ is expressed in terms of its Fourier components $\xi(\omega, k)$ by formula (38), in which function $\tilde{V}(\omega, k)$ has the explicit expression (39). It is easy to remark that $\tilde{V}(k)$ is a function of $k^2$, while $\tilde{W}(\omega, k)$ is a function of $\omega^2$ and $k^2$, so that we can set
\[
\tilde{V}(k) \left[ 1 + \tilde{W}(\omega, k) \right] = \tilde{R}(\omega^2, k^2),
\]
(40)

where
\[
\tilde{R}(\omega^2, k^2) = \tilde{V}(k) + \tilde{V}(k)^2 \frac{1}{(2\pi)^2} \int dk' \omega \frac{1}{(2\pi)^2} \int d\omega' (2\pi)^2 \delta(k + k' - \omega') \times \frac{2\pi \delta(\omega + \omega')}{2\pi \delta(\omega + \omega') \times \{ 1 - \cos(\theta(k') - \theta(\omega')) \}}.
\]
Then formula (38) becomes
\[
I^2_2[\xi] = \frac{1}{2} \frac{1}{(2\pi)^2} \int \omega \frac{1}{(2\pi)^2} \int dk \tilde{\xi}(\omega, k) \tilde{R}(\omega^2, k^2) \tilde{\xi}(-\omega, -k).
\]
(42)

Because of the complicated form of the function $\tilde{R}(\omega^2, k^2)$, there does not exist a simple analytical calculation method enabling to establish the differential equation for the quantum field of plasmons in graphene, in constrast to the case of plasmons in electron gas. Therefore let us consider a special case with $\omega^2 \gg \hbar^2 k^2$ and the long wavelength limit $k \to 0$, when we can elaborate the analytical calculation method by extending the reasonings presented in [30, 31]. In this case $\tilde{R}(\omega^2, k^2)$ has following approximate expression
\[
\tilde{R}(\omega^2, k^2) = \frac{\tilde{V}(k)}{\omega^2} \left[ \omega^2 - U(k) \right],
\]
(43)

where
\[
U(k) = \tilde{V}(k) \frac{1}{(2\pi)^2} \int dk'' \frac{1}{(2\pi)^2} \int d\omega'' (2\pi)^2 \delta(k + k' - \omega'') \times 2\pi \delta(\omega + \omega') \{ 1 - \cos(\theta(k') - \theta(\omega')) \}.
\]
(44)

Using expression (43) of $\tilde{R}(\omega^2, k^2)$, we rewrite formula (42) as follows
\[
I^2_2[\xi] = \frac{1}{2} \frac{1}{(2\pi)^2} \int \omega \frac{1}{(2\pi)^2} \int dk \tilde{\xi}(\omega, k) \tilde{V}(k)^{1/2} \frac{1}{\omega} \times [ U(k) - \omega^2 ] \tilde{\xi}(-\omega, -k). \tilde{V}(k)^{1/2} \frac{1}{\omega}.
\]
(45)

Introducing new functions
\[
\tilde{\lambda}(\omega, k) = i \tilde{\xi}(\omega, k) \tilde{V}(k)^{1/2} \frac{1}{\omega} \tilde{V}(k)^{1/2} \frac{1}{\omega},
\]
(46)
and
\[
\tilde{\lambda}(-\omega, -k) = i \tilde{\xi}(-\omega, -k) \tilde{V}(k)^{1/2} \frac{1}{\omega} \tilde{V}(k)^{1/2} \frac{1}{\omega},
\]
(47)
we obtain a functional containing $\tilde{\lambda}(\omega, \mathbf{k})$ and $\tilde{\lambda}(-\omega, -\mathbf{k})$. It is reasonable to use the new notation $I_2[\lambda]$ for this functional

$$I_2[\lambda] = \frac{1}{2} \int d\omega \frac{1}{(2\pi)^2} \int d\mathbf{k} \tilde{\lambda}(\omega, \mathbf{k}) \left[ \omega^2 - \tilde{U}(\mathbf{k}) \right] \tilde{\lambda}(-\omega, -\mathbf{k}).$$

Denote $\lambda(x)$ the quantum field with Fourier components $\tilde{\lambda}(\omega, \mathbf{k})$, namely

$$\lambda(x) = \frac{1}{2\pi} \int d\omega e^{-i\omega x} \int d\mathbf{k} e^{i\mathbf{k}x} \tilde{\lambda}(\omega, \mathbf{k}),$$

and therefore

$$\tilde{\lambda}(\omega, \mathbf{k}) = \int dx_0 e^{i\omega x_0} \int dx e^{-i\mathbf{k}x} \lambda(x).$$

Substituting expressions of the form (50) for $\tilde{\lambda}(\omega, \mathbf{k})$ and $\tilde{\lambda}(-\omega, -\mathbf{k})$ into r.h.s. of formula (48), after lengthy but standard calculations we obtain the expression of the functional (48) in terms of the quantum field $\lambda(x)$:

$$I_2[\lambda] = \frac{1}{2} \int dx_0 \int dx \left[ \frac{\partial \lambda(x, 0, x)}{\partial x_0} \right]^2 - \frac{1}{2} \int dx_0 \int dx \int dx' \lambda(x_0, x) U(x - x') \lambda(x_0, x'),$$

where

$$U(x - x') = \frac{1}{(2\pi)^2} \int d\mathbf{k} e^{-i\mathbf{k}(x - x')} \tilde{U}(\mathbf{k}).$$

From the definition (44) of $\tilde{U}(\mathbf{k})$ it follows that this function does not depend on the sign of vector $\mathbf{k}$:

$$\tilde{U}(-\mathbf{k}) = \tilde{U}(\mathbf{k}).$$

Therefore function $U(x - x')$ is also an even function of $x - x'$:

$$U(x' - x) = U(x - x').$$

The appearance of nonlocal term containing function $U(x - x')$ is the peculiarity of the quantum field $\lambda(x)$ of plasmons in graphene. The physical origin of this term is the extended structure of the quasiparticles which are the quanta of collective excitations in the system of interacting Dirac fermions in graphene.

Thus we have demonstrated that the collective excitations of Dirac fermions in graphene are described by the quantum field $\lambda(x)$. The quanta of this field are compounded from a large number of interacting Dirac fermions. Therefore the interpretation of these quasiparticles to be plasmons in graphene is quite reasonable.

Following many previous works, for example [30, 31], let us consider functional $I_2[\lambda]$ as the effective action of the graphene plasmon quantum field $\lambda(x)$ and study the physical consequence of the principle of extremum action. For this purpose we must calculate the functional derivative

$$\frac{\delta I_2[\lambda]}{\delta \lambda(x)} = I_2[\lambda + \delta \lambda] - I_2[\lambda]$$

where the variation $\delta \lambda(x)$ satisfies following conditions at $x_0 \to \pm \infty$:

$$\delta \lambda(x_0, x) |_{x_0=\pm \infty} = 0.$$  

Calculating functional derivative (55) of the effective action (51) we obtain

$$\frac{\delta I_2[\lambda]}{\delta \lambda(x_0, x)} = \frac{\partial^2 \lambda(x_0, x)}{\partial x_0^2} + \int dx' U(x - x') \lambda(x_0, x').$$

From the principle of extremum action

$$\frac{\delta I_2[\lambda]}{\delta \lambda(x_0, x)} = 0,$$

and the expression (57) of the functional derivative of the effective action we derive following differential equation for the quantum field $\lambda(x)$ of graphene plasmons in the region with $\omega^2 \gg v_F^2 k^2$:

$$\frac{\partial^2 \lambda(x_0, x)}{\partial x_0^2} + \int dx' U(x - x') \lambda(x_0, x') = 0.$$  

In terms of the Fourier components $\tilde{\lambda}(\omega, \mathbf{k})$ of $\lambda(x)$ and Fourier components $\tilde{U}(\mathbf{k})$ of $U(x - x')$ equation (58) becomes

$$\omega^2 = \tilde{U}(\mathbf{k}).$$

Note that according to the definition (44) function $\tilde{U}(\mathbf{k})$ is positive and therefore algebraic equation (59) has two real solutions with opposite signs. Let consider the positive solution

$$\omega = \tilde{U}(\mathbf{k})^{1/2}.$$  

In order to derive the analytical formula for the function $\tilde{U}(\mathbf{k})$ we rewrite formula (44) as follows:

$$\tilde{U}(\mathbf{k}) = 2v_F \frac{\tilde{V}(\mathbf{k})}{(2\pi)^2} \int d\mathbf{k}' (\mathbf{k}' + |\mathbf{k}' + \mathbf{k}|) \times \left[ 1 - \cos[\theta(\mathbf{k}') - \theta(\mathbf{k}' + \mathbf{k})] \right].$$

In the long wavelength limit with $k \to 0$, we have

$$\tilde{U}(\mathbf{k}) = 2e^2 v_F k^2.$$  

Finally we obtain the explicit formula of plasmon frequency as a function of its wave vector

$$\omega = (4\pi n)^{1/4} \left( e^2 v_F k \right)^{1/2},$$

where $n$ is the density of Dirac fermions in graphene. This formula is compatible with the result of [32] in which Sarma and Hwang studied the dispersion of graphene plasmons by applying the RPA method.

4. Conclusion and discussion

In the present work we have applied the functional integral method of plasmons in graphene at vanishing absolute temperature and at Fermi level $E_F = 0$. For this purpose we have introduced a Hermitian scalar field $\varphi(x)$ describing the collective excitations of the spinless Dirac fermion gas in graphene.
at vanishing absolute temperature and proposed the general form (1) of the functional integral \( Z^\phi \) of Hermitian scalar field \( \phi(x) \). This functional integral \( Z^\phi \) contains the series (2).

Then we studied functional integral (1) in the special and simplest non-trivial case, when the polynomial functional series (2) in limited at the second order approximation with respect to the scalar field \( \phi(x) \) which can be devided into two parts: a background field \( \phi_0(x) \) and another scalar field \( \xi(x) \) describing the fluctuation around background field \( \phi_0(x) \).

Subsequently we have investigated the relationship of fluctuation field \( \xi(x) \) with the quantum field of plasmons in graphene and have demonstrated that there exists a range of values of frequency (energy) and wave vector (momentum) of plasmons, when the analytical calculations can be performed. From the extremum action principle we derived the differential equation (58) for the quantum field of plasmons in graphene. Using this field equation we established the relation (63) between frequency and wave vector of plasmons in the long wavelength limit.

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