Homogeneous involution on graded division algebras and their polynomial identities

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Abstract. In this short note, we describe the so-called homogeneous involution on finite-dimensional graded-division algebra over an algebraically closed field. We also compute their graded polynomial identities with involution. As pointed out by L. Fonseca and T. de Mello, a homogeneous involution naturally appears when dealing with graded polynomial identities and a compatible involution.

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1. Introduction

The main purpose of this short note is to investigate the homogeneous involution on finite-dimensional graded-division algebras over an algebraically closed field of characteristic zero and their graded polynomial identities with involution.

It is known that graded algebras play a fundamental role in several branches of Mathematics, being the main topic of research or being an important tool to understand an object. The monograph [8] compiles the state-of-art of the theory, where understanding the gradings is the central objective. It is worth mentioning that the classification of involutions that are compatible with a given grading, the so called degree-preserving involution or graded involution, is crucial in the approach.

On the other hand, an involution that inverts the degree appears naturally on several contexts, for instance, the Leavitt path algebras (see [11]) and Matrix algebras endowed with an elementary grading and the transpose involution (see [5] for details, and see also [9]).
Considering group gradings on the upper triangular matrices (a classification is obtained in [7, 13]), one sees that the natural transpose with respect to the secondary diagonal sends a homogeneous element to a homogeneous element. If the grading is adequate, the natural involution will map a whole homogeneous component in another homogeneous component. So, T. de Mello studied and described the so called homogeneous involution on upper triangular matrix algebra [4].

Following the paper by T. de Mello, we shall investigate the homogeneous involutions on finite-dimensional graded division algebras over an algebraically closed field.

Next, it is known the relevance of the description of the T-ideal of all polynomial identities satisfied by a given algebra, and the related main problems of the theory. We shall describe the graded polynomial identities with homogeneous involution on the graded division algebras endowed with a homogeneous involution. The graded version is done in [1], and in [10] the graded identities with an involution were computed for certain types of gradings.

Finally, it is worth mentioning the following phenomena (see [6]) when dealing with the free graded algebra with a homogeneous involution (we shall give the precise construction in the next section). Let \( X_G = \bigcup_{g \in G} X_g \), where \( X_g = \{ x_1^{(g)}, x_2^{(g)}, \ldots \} \), for each \( g \in G \). Let \( F\langle X, \iota \rangle \) be the free \( G \)-graded algebra with an involution. Then, since the unary operation \( \iota \) is an antiautomorphism, one obtains \( \iota(x^{(g)}x^{(h)}) = \iota(x^{(h)})\iota(x^{(g)}) \). If we require that \( \iota \) preserves the homogeneous degree, then this equation makes sense only if the grading group is abelian. Otherwise, there should be a antihomomorphism \( \tau : G \to G \) such that \( \text{deg}_G \iota(x^{(g)}) = \tau(g) \). Thus, in general, it seems to be natural to consider homogeneous involutions in the context of graded polynomial identities endowed with a compatible involution.

2. Preliminaries

2.1. Graded algebra

Let \( G \) be any group. We use the multiplicative notation for \( G \), and denote its neutral element by 1. We say that an algebra \( A \) is \( G \)-graded if there exists a vector-space decomposition \( A = \bigoplus_{g \in G} A_g \) such that \( A_gA_h \subseteq A_{gh} \), for all \( g, h \in G \). The choice of the decomposition is called a \( G \)-grading, and one usually denotes by \( \Gamma \). The subspace \( A_g \) is called homogeneous component of degree \( g \). A nonzero element \( x \in A_g \) is called a homogeneous element of degree \( g \). We denote \( \text{deg}_G x = g \). The support of the grading \( \Gamma \) is \( \text{Supp} \Gamma = \{ g \in G \mid A_g \neq 0 \} \). By abuse of language, we shall denote the support by \( \text{Supp} A \). A graded division algebra is an associative algebra \( D \) with 1, where each nonzero homogeneous element \( x \in D \) is invertible.

Finally, we provide a precise definition of the following:
Definition. Let $A = \bigoplus_{g \in G} A_g$ be a $G$-graded algebra, and let $\tau : G \to G$ be a map. An involution $\psi$ on $A$ is a homogeneous involution with respect to $\tau$ or a $\tau$-involution if $\psi(A_g) \subseteq A_{\tau(g)}$, for all $g \in G$.

In this paper, involution will mean an $\mathbb{F}$-linear map involution. Also, we are specially interested in the case where the map $\tau$ is an anti-automorphism of order 2 of the grading group.

Examples.

1. If $G$ is an abelian group, then every degree-preserving involution is a homogeneous involution with respect to the identity map of $G$.
2. A degree-inverting involution is a homogeneous involution with respect to the inversion of $G$. It is worth mentioning that the degree-inverting involution on matrix algebras and upper triangular matrices were described in [5, 9].
3. If $D = \bigoplus_{g \in G} D_g$ is a graded-division algebra (where $G = \text{Supp} \ D$) and $\iota$ is a $\tau$-involution on $D$, then $\tau$ is an involution of $G$. Indeed, for any $g, h \in G$, let $x_g$ and $x_h$ be nonzero homogeneous elements of $G$-degrees $g$ and $h$, respectively. Then $xy \neq 0$ and

$$\tau(gh) = \deg_G \iota(x_g x_h) = \deg_G (\iota(x_h) \iota(x_g)) = \deg_G \iota(x_h) \deg_G \iota(x_g) = \tau(h) \tau(g),$$

$$g = \deg_G x_g = \deg_G \iota(x_g) = \tau(\iota(g)).$$

Thus, $\tau$ is an anti-automorphism of order 2.

2.2. Factor sets

We let $T$ be a finite group, and $\mathbb{F}^\times$ denote the set of invertible elements of $\mathbb{F}$.

A map $\sigma : T \times T \to \mathbb{F}^\times$, is called a 2-cocycle or a factor set if

$$\sigma(u, v)\sigma(uv, w) = \sigma(u, vw)\sigma(v, w), \quad \forall u, v, w \in T.$$  

We denote the set of all 2-cocycles by $Z^2(T, \mathbb{F}^\times)$. Note that, using the usual multiplication, $Z^2(T, \mathbb{F}^\times)$ is an abelian group.

For each map $\mu : T \to \mathbb{F}^\times$, we define $\delta \mu : T \times T \to \mathbb{F}^\times$ by

$$\delta \mu(u, v) = \mu(u)\mu(v)\mu(uv)^{-1}, \quad u, v \in T.$$  

We define $B^2(T, \mathbb{F}^\times) = \{\delta \mu \mid \mu : T \to \mathbb{F}^\times\}$. An easy exercise shows that $B^2(T, \mathbb{F}^\times)$ is a subgroup of $Z^2(T, \mathbb{F}^\times)$. The 2nd cohomological group of $T$ is the quotient $H^2(T, \mathbb{F}^\times) = Z^2(T, \mathbb{F}^\times)/B^2(T, \mathbb{F}^\times)$.

We can construct algebras from factor sets. Given an arbitrary map $\sigma : T \times T \to \mathbb{F}^\times$ denote by $\mathbb{F}^\sigma T$ the following algebra: $\mathbb{F}^\sigma T$ has a basis $\{X_u \mid u \in T\}$, and the product is defined by $X_u X_v = \sigma(u, v)X_{uv}$. Note that $\mathbb{F}^\sigma T$ is associative if and only if $\sigma \in Z^2(T, \mathbb{F}^\times)$. For instance, if $\sigma = 1$ (the constant function), then $\mathbb{F}^\sigma T$ is the group algebra of $T$. Clearly such algebra have a natural $T$-grading. It is known that $\mathbb{F}^\sigma T \cong \mathbb{F}^{\sigma'} T$ if and only if $[\sigma] = [\sigma']$ (equality in $H^2(T, \mathbb{F}^\times)$.

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2.3. Graded division algebra

Assume that $\mathbb{F}$ is algebraically closed and let $\mathcal{D} = \bigoplus_{g \in G} \mathcal{D}_g$ be a finite-dimensional graded division algebra over $\mathbb{F}$. Let $T = \{ g \in G \mid \mathcal{D}_g \neq 0 \}$ be its support. Then it is easy to see that $T$ is a subgroup of $G$. We use multiplicative notation for the product of $T$, and denote by $1$ its neutral element.

Moreover, $\mathcal{D}_1 \supseteq \mathbb{F}$ is a division algebra. So $\mathcal{D}_1 = \mathbb{F}$, since $\mathbb{F}$ is algebraically closed and $\dim_{\mathbb{F}} \mathcal{D}_1 < \infty$. This also implies $\dim \mathcal{D}_g = 1$, for all $g \in T$. Let $\{ X_u \mid u \in T \}$ be a homogeneous basis of $\mathcal{D}$. Then $X_u X_v = \sigma(u, v) X_w$, for some $\sigma(u, v) \in \mathbb{F}^\times$. Since $\mathcal{D}$ is associative, from $(X_u X_v) X_w = X_u (X_v X_w)$, we derive that $\sigma$ is a $2$-cocycle. Hence, $\mathcal{D} \cong \mathbb{F}^\sigma T$, the twisted group algebra of $T$ by $\sigma$. Conversely, for any finite group $T$ and any $\sigma \in Z^2(T, \mathbb{F}^\times)$, the natural $T$-grading on $\mathbb{F}^\sigma T$ turns it into a graded division algebra.

2.4. Free graded algebra with homogeneous involution

We shall provide a construction of the free algebra endowed with a homogeneous involution. This is done using a particular case of the (relatively) free universal algebra in an adequate variety (see, for instance, [12] for a general discussion, and [2] for a particular graded version). Let $G$ be any group, and $X^G = \bigcup_{g \in G} X^g$, where $X^g = \{ x_1^{(g)}, x_2^{(g)}, \ldots \}$. Let $\tau : G \to G$ be an involution, that is, an anti-automorphism of order 2. We define the free $G$-graded associative algebra with a homogeneous involution with respect to $\tau$, $\mathbb{F}\langle X^G, \iota \rangle$, in the following way. First, we let $\mathbb{F}\{ X^G, \iota \}$ denote the absolute free $G$-graded binary algebra endowed with an unary operation (denote by $\iota$). We define the following polynomial identities

$$
\begin{align*}
&x_1^{(g_1)}(x_2^{(g_2)} x_3^{(g_3)}) - (x_1^{(g_1)} x_2^{(g_2)}) x_3^{(g_3)} = 0 \\
&\iota(x_1^{(g_1)} x_2^{(g_2)}) - \iota(x_2^{(g_2)}) \iota(x_1^{(g_1)}) = 0 \\
&\iota(\iota(x^{(g)})) - x^{(g)} = 0 \\
&\deg_G \iota(x^{(g)}) = \tau(\deg_G x^{(g)}).
\end{align*}
$$

The first relation is the associativity while the second and third indicate that $\iota$ acts as an involution. The last relation is also a polynomial identity, but to see this we need to describe the $G$-grading in terms of the projections (see, for instance, [2]). For each $g \in G$, let $\pi_g$ denote the unary operation on a $G$-graded algebra $\mathcal{A}$ given by the projection and inclusion $\pi_g : \mathcal{A} \to \mathcal{A}$. Then, the absolute free $G$-graded algebra $\mathbb{F}\{ X^G, \iota \}$ is a quotient of the absolute free $\Omega$-algebra, where $\Omega$ contains one binary operation and $|G| + 1$ unary operations (corresponding to each projection, and the involution). The quotient is given by the relations that define the $G$-grading, that is, $\pi_g(\pi_h(x)) = \delta_{gh} \pi_h(x)$ and $\pi_g(\pi_{g_1}(x) \pi_{g_2}(y)) = \delta_{g_1 g_2} \pi_g(xy)$. Hence, in the language of this $\Omega$-algebra, the last equation of (1) is equivalent to

$$
\iota(\pi_g(x)) - \pi_{\tau(g)}(\iota(x)) = 0.
$$
Using either the absolute free $G$-graded algebra or the (relatively) free $\Omega$-algebra, the free $G$-graded algebra with a $\tau$-involution $F\langle X^G, \iota \rangle$ is the quotient of $F\{X^G, \iota \}$ by the identities (1). As mentioned before, the forth identity is natural in the context of graded polynomial identities with an involution.

As discussed in [6], in the special case where $\iota$ is a degree-preserving involution, then we can define the new variables $x^g_+ := x^g + \iota(x^g)$ and $x^g_- = x^g - \iota(x^g)$ (the symmetric and skew symmetric variables). Then we get the classical construction of the free (graded) $*$-algebra. Since $\iota$ does not necessarily preserve the homogeneous degree, we cannot use such technique, since these variables would not be homogeneous.

Given a $G$-graded algebra $A$ with a homogeneous involution $\iota$, we denote by $\text{Id}_{G}(A)$ its ideal of graded polynomial identities, and by $\text{Id}_{G,\iota}(A)$ the set of all of its graded polynomial identities with involution $\iota$.

2.5. Codimension sequence

Let $T$ be a finite group and consider the natural $T$-grading on $F_{\sigma}T$. Given a sequence $s = (u_1, \ldots, u_m) \in T^m$, we let

$$P^s_m = \text{Span} \left\{ x^{(u_{\pi(1)})}_\pi \cdots x^{(u_{\pi(m)})}_\pi \mid \pi \in S_m \right\}, \quad P^s_m(\mathbb{F}^\sigma T) = P^s_m / P^s_m \cap \text{Id}_T(\mathbb{F}^\sigma T),$$

where $S_m$ is the symmetric group on the set of $m$ elements. The graded codimension sequence of $\mathbb{F}^\sigma T$ is

$$c^T_m(\mathbb{F}^\sigma T) = \dim \sum_{s \in T^m} P^s_m(\mathbb{F}^\sigma T).$$

Now, given $I = (j_1, \ldots, j_m) \in \{0, 1\}^m$, let

$$P^{T}_{s,I} = \text{Span} \left\{ \iota^{j_1} \left( x^{(u_{\pi(1)})}_\pi \right) \cdots \iota^{j_m} \left( x^{(u_{\pi(m)})}_\pi \right) \mid \pi \in S_m \right\}.$$ 

As before, we let

$$P^{T}_{s,I}(\mathbb{F}^\sigma T) = P^{T}_{s,I} / P^{T}_{s,I} \cap \text{Id}_{T,I}(\mathbb{F}^\sigma T), \quad c^{T,\iota}_m(\mathbb{F}^\sigma T) = \dim \sum_{s,I} P^{T}_{s,I}(\mathbb{F}^\sigma T).$$

The graded exponent and the graded-involution exponent (if exists) are respectively defined by

$$\exp^T(\mathbb{F}^\sigma T) = \lim_{m \to \infty} m \sqrt[c^{T}_m(\mathbb{F}^\sigma T)]{c^{T,\iota}_m(\mathbb{F}^\sigma T)}, \quad \exp^{T,\iota}(\mathbb{F}^\sigma T) = \lim_{m \to \infty} m \sqrt[c^{T,\iota}_m(\mathbb{F}^\sigma T)]{c^{T,\iota}_m(\mathbb{F}^\sigma T)}$$

3. Homogeneous involution on graded division algebra

We let $\mathbb{F}$ be a field of characteristic not 2. Let $T$ be a finite group. Given $\sigma \in Z^2(T, \mathbb{F}^*)$, we consider the natural $T$-grading on $\mathbb{F}^\sigma T$.

We shall denote by $\text{Aut}(T)$ the group of all automorphisms of $T$, and by $\overline{\text{Aut}}(T)$ the group of all automorphism and antiautomorphism of $T$. For each $\varphi \in \text{Aut}(T)$, let $\varphi \sigma : T \times T \to \mathbb{F}^*$ be defined by

$$\varphi \sigma (u, v) = \sigma(\varphi(u), \varphi(v)).$$
On the other hand, if \( \psi \in \overline{\text{Aut}}(T) \) is an antiautomorphism, then let \( \psi \sigma : T \times T \to \mathbb{F}^\times \) be defined by
\[
\psi \sigma(u, v) = \sigma(\psi(v), \psi(u)).
\]
The next lemma is an elementary exercise. We include the proof for completeness.

**Lemma 1.** For each \( \theta \in \overline{\text{Aut}}(T) \) and \( \sigma' \in Z^2(T, \mathbb{F}^\times) \), \( \theta \sigma' \in Z^2(T, \mathbb{F}^\times) \).

**Proof.** Assume that \( \theta \) is an antiautomorphism. Then, for any \( u, v, w \in T \), we have
\[
\theta \sigma'(u, v) \theta \sigma'(uv, w) = \sigma'(\theta(v), \theta(u)) \sigma'(\theta(w), \theta(uv))
= \sigma'(\theta(w), \theta(v)) \sigma'(\theta(u), \theta(u))
= \theta \sigma'(v, w) \theta \sigma'(u, vw).
\]
In an analogous way we show that \( \theta \sigma' \in Z^2(T, \mathbb{F}^\times) \) if \( \theta \) is an automorphism of \( T \). \( \square \)

Hence, it is easy to see that we have an action of \( \overline{\text{Aut}}(T) \) on \( Z^2(T, \mathbb{F}^\times) \). This action factors through \( B^2(T, \mathbb{F}^\times) \):

**Lemma 2.** If \( \sigma' \in B^2(T, \mathbb{F}^\times) \) and \( \theta \in \overline{\text{Aut}}(T) \), then \( \theta \sigma' \in B^2(T, \mathbb{F}^\times) \).

**Proof.** Let \( \sigma' = \delta \mu \). Then \( \theta \delta \mu = \delta(\mu \circ \theta) \in B^2(T, \mathbb{F}^\times) \). \( \square \)

Thus, we get:

**Corollary 3.** There is an action of \( \overline{\text{Aut}}(T) \) on \( H^2(T, \mathbb{F}^\times) \). \( \square \)

Now, the next result states the conditions so that we do have a homogeneous involution on \( \mathbb{F}^\sigma T \). For this, we need maps satisfying the following condition:

**Definition.** Let \( \sigma \in Z^2(T, \mathbb{F}^\times) \) and \( \tau \in \overline{\text{Aut}}(T) \). We say that the pair \( (\sigma, \tau) \) is compatible if there is a map \( \mu : T \to \mathbb{F}^\times \) such that \( \sigma = \delta \mu \cdot \tau \sigma \) (where \( \cdot \) is the product of \( Z^2(T, \mathbb{F}^\times) \)) and \( \mu(u \tau(u)) = 1 \), for all \( u \in T \).

For instance, let \( \tau : u \in T \mapsto u^{-1} \in T \) be the inversion. Then, for any \( \sigma \in Z^2(T, \mathbb{F}^\times) \) such that \([\sigma]^2 = 1\), the pair \( (\sigma, \tau) \) is compatible (see the proof of [9, Proposition 4]).

**Theorem 4.** Let \( \tau : T \to T \). Then there exists a \( \tau \)-homogeneous involution on \( \mathbb{F}^\sigma T \) if and only if \( \tau \) is an antiautomorphism of order 2 and \( (\sigma, \tau) \) is compatible.

**Proof.** Let \( \{X_u \mid u \in T\} \) be a homogeneous basis of \( \mathbb{F}^\sigma T \). First, assume that \( \iota \) is a \( \tau \)-homogeneous involution on \( \mathbb{F}^\sigma T \). Let \( \tau : T \to T \) and \( \mu : T \to \mathbb{F}^\times \) be the maps such that
\[
\iota(X_u) = \mu(u)X_{\tau(u)}, \quad u \in T.
\]
In this section, we describe the graded polynomial identities with involution of $F$. Polynomial identities of graded division algebra

$\alpha$

Then we can find a constant $z$ we obtain that $\mu$ we obtain that $\mu(\tau(u)) = \mu(\tau(u))X_{\tau(u)}$.

$$
\begin{align*}
\mu(\tau(u)) &= \mu(\tau(u))X_{\tau(u)} \\
&= \sigma(v, \tau(v))\mu(v)X_{\tau(v)\tau(u)}.
\end{align*}
$$

we obtain $\tau(uv) = \tau(v)\tau(u)$. Hence, $\tau$ is an antiautomorphism. Note that $\mu$ also shows that $\sigma = \delta \mu \cdot (\tau \sigma)$, thus $[\sigma] = [\tau \sigma]$. Finally, from $\mu(X_uX_{\tau(u)}) = \mu(X_{\tau(u)})\mu(X_u)$, we get

$$
\mu(u\tau(u))\sigma(u, \tau(u)) = \mu(u)\mu(\tau(u))\sigma(u, \tau(u)),
$$

thus $\mu(u\tau(u)) = \mu(u)\mu(\tau(u))$. Since $X_u = \mu(\tau(u)) = \mu(\tau(u))\mu(u)X_u$, we get that $\mu(u\tau(u)) = 1$, for all $u \in T$.

On the other hand, assume that $\tau$ is an antiautomorphism of order 2 such that $(\sigma, \tau)$ is compatible. So let $\mu : G \to F$ satisfy $\sigma = \delta \mu \cdot (\tau \sigma)$. We claim that

$$
\mu(X_u) := \mu(u)X_{\tau(u)}
$$

is a $\tau$-homogeneous involution on $F^\sigma T$. Indeed,

$$
\begin{align*}
\mu(X_uX_v) &= \sigma(u, v)\mu(uv)X_{\tau(uv)} \\
&= \sigma(u, v)\mu(uv)\sigma(v, \tau(u))^{-1}X_{\tau(v)\tau(u)} \\
&= \mu(u)\mu(v)X_{\tau(v)\tau(u)} = \mu(X_v)\mu(X_u).
\end{align*}
$$

Now,

$$
\mu(\tau(u))\mu(u)\mu(\tau(u)) = \mu(u)\mu(\tau(u))X_u.
$$

It remains to prove that $\mu(u)\mu(\tau(u)) = 1$. Since

$$
\mu(\tau(u))\mu(u)\mu(\tau(u))^{-1} = \sigma(u, \tau(u))\tau\sigma(u, \tau(u))^{-1} = 1,
$$

we obtain that $\mu(\tau(u)) = \mu(u\tau(u)) = 1$, since $(\sigma, \tau)$ is compatible.

Problem. Classify the compatible pairs $(\sigma, \tau)$.

4. Polynomial identities of graded division algebra

In this section, we describe the graded polynomial identities with involution of $F^\sigma T$. Let $F$ be a field of characteristic zero and $\iota$ a $\tau$-homogeneous involution on $F^\sigma T$. Let $U = (u_1, \ldots, u_m) \in T^m$, $I = (i_1, \ldots, i_m), I' = (i'_1, \ldots, i'_m) \in \{0, 1\}^m$ and $\theta, \theta' \in S_m$ be such that

$$
\tau^{i_\theta'(1)}(u_{\theta'(1)}) \cdots \tau^{i_\theta'(m)}(u_{\theta'(m)}) = \tau^{i_\theta(1)}(u_{\theta(1)}) \cdots \tau^{i_\theta(m)}(u_{\theta(m)}).
$$

Then we can find a constant $\alpha_{U, I', \theta, \theta'} \in F$ such that

$$
f^{i_\theta(1)}(X_{u_{\theta(1)}}) \cdots f^{i_\theta(m)}(X_{u_{\theta(m)}}) = \alpha_{U, I', \theta, \theta'}f^{i_\theta(1)}(X_{u_{\theta(1)}}) \cdots f^{i_\theta(m)}(X_{u_{\theta(m)}}).
$$

Hence, denoting $z_i = x^{(u_i)}_i$, $f_{U, I', \theta, \theta'} = f^{i_\theta'(1)}(z_{u_{\theta'(1)}}) \cdots f^{i_\theta'(m)}(z_{u_{\theta'(m)}}) = \alpha_{U, I', \theta, \theta'}f^{i_\theta(1)}(z_{\theta(1)}) \cdots f^{i_\theta(m)}(z_{\theta(m)})$
is a $G$-graded polynomial identity with involution of $\mathbb{F}^\sigma T$, which shall be called an elementary identity (following the graded case of \([1]\)). Let $\mathcal{T}_U$ be the set of all triples $(U, I, I', \theta, \theta')$ such that \([3]\) holds valid.

Theorem 5. $\mathrm{Id}_{T,\ell}(\mathbb{F}^\sigma T)$ is generated by \(\{f_{U, I, I', \theta, \theta'} \mid (I, I', \theta, \theta') \in \mathcal{T}_U, |U| \leq |T|\}\).

Proof. First, we shall prove that any multilinear polynomial identity is a consequence of the elementary ones. Then, we show that the elementary identities follow from the ones having length at most $|T|$. Let $\mathcal{I}$ be the $T_{T,\ell}$-ideal generated by \(\{f_{U, I, I', \theta, \theta'} \mid (I, I', \theta, \theta') \in \mathcal{T}_U, |U| \leq |T|\}\), and $\mathcal{J}$ be the $T_{T,\ell}$-ideal generated by \(\{f_{U, I, I', \theta, \theta'} \mid (I, I', \theta, \theta') \in \mathcal{T}_U\}\).

Let $f \in \mathrm{Id}_{T,\ell}(\mathbb{F}^\sigma T)$ be a $G$-homogeneous polynomial. Since $\mathrm{char} \mathbb{F} = 0$, we may assume that $f$ is multilinear. Write

\[ f = \sum_{\theta \in S_m} \alpha_{\theta, I} p_{\theta, I}, \quad (4) \]

where $p_{\theta, I} = \iota^\sigma_{s(1)}(z_{s(1)}) \cdots \iota^\sigma_{s(m)}(z_{s(m)})$. Since every monomial in \([4]\) satisfies $\tau^\sigma_{s(1)}(u_{s(1)}) \cdots \tau^\sigma_{s(m)}(u_{s(m)}) = \deg_T f$, we see that $(I, I', \theta, \theta') \in \mathcal{T}_U$ for every pair of $(I, \theta)$ and $(I', \theta')$ appearing with nonzero coefficient in \([4]\). Here, $U = (\deg_T z_1, \ldots, \deg_T z_m)$. Hence, modulo $\mathcal{J}$, $f$ is equal to a single monomial, up to a scalar. As $\mathbb{F}^\sigma T$ is a graded division algebra, a monomial cannot be a graded polynomial identity of it. Thus, $f = 0$ modulo $\mathcal{J}$, so $f \in \mathcal{J}$.

Finally, let $p = f_{U, I, I', \theta, \theta'}$ with $|U| > |T|$. We shall prove by induction on $|U|$ that $f \in \mathcal{I}$. For the sake of simplicity, we may replace $\theta$ by the identity, and then $\theta'$ becomes $\theta'' = \theta'\theta^{-1}$, and we may assume that $I = (0,0,\ldots,0)$ and $I'$ is replaced by $I'' = (i'_1 + i_1, \ldots, i'_m + i_m)$ (where the sum is taken modulo 2). Formally, we shall call $z_j = \iota^{ij}(x_{u_{ij}(j)})$, and then

\[ p = z_1 \cdots z_m - \iota^{i''(1)}(z_{\theta''(1)}) \cdots \iota^{i''(m)}(z_{\theta''(m)}). \]

To make notations cleaner, we may also suppress the double prime, so we shall write $p = f_{U,(0,\ldots,0),I,1,\theta}$. Denote also $v_i = \tau(u_{ij}(i))$.

Consider the elements \(\{v_1, v_1v_2, \ldots, v_1v_2\cdots v_m\}\), having exactly $|U| > |T|$ elements. By the pigeonhole principle, there should be two distinct sequences whose product coincides. Thus, there exists $1 < i < j \leq m$ such that $v_i v_{i+1} \cdots v_j = 1$. Since $(v_i \cdots v_{j-1})v_j = v_j(v_i \cdots v_{j-1}) = 1$, we may, modulo $\mathcal{I}$, cyclically permute the variables $\iota^i(z_i) \cdots \iota^j(z_j)$. As $(v_i \cdots v_j)v_k = v_k(v_i \cdots v_j)$ for any $k$, we may also move the string $\iota^i(z_i) \cdots \iota^j(z_j)$ anywhere in the last monomial, modulo $\mathcal{I}$. Finally, since $\tau(1) = 1$, we may apply $i$ to $\iota^i(z_i) \cdots \iota^j(z_j)$, modulo $\mathcal{I}$.

Hence, modulo $\mathcal{I}$, we may assume that there exists $\ell$ such that $z_\ell$ and $z_{\ell+1}$ appears consecutively (in this order) in the second monomial of $p$, and $i$ is applied in the product or $z_\ell z_{\ell+1}$ or not. More precisely, modulo $\mathcal{I}$, either

\[ p = w_1 z_\ell z_{\ell+1} w_2 - w_1' z_\ell z_{\ell+1} w_2' \]
or
\[ p = w_1 z_\ell z_{\ell+1} w_2 - w'_1 \ell (z_\ell + 1) \ell (z_\ell) w'_2. \]

Defining \( g = u_\ell u_{\ell+1} \) we have that \( p \) is a consequence of either one of the elementary identities \( w_1 x(g) w_2 - w'_1 x(g) w'_2 \) or \( w_1 x(g) w_2 - w'_1 \ell (x(g)) w'_2. \) In any case, \( p \) is a consequence of an elementary identity of total degree \( |U| - 1. \)

By induction, this implies that \( p \in \mathcal{Z} \), proving the result. \( \square \)

Finally, we shall obtain some estimates to the graded-involution codimension sequence of the \( \mathbb{F}^\sigma T \). We shall prove the following:

**Theorem 6.** Let \( T \) be a finite group, and \( \sigma \in \mathbb{Z}^2(T, \mathbb{F}^\times) \). Assume that \( \mathbb{F}^\sigma T \) admits a \( \tau \)-homogeneous involution \( \iota \). Then

1. \( c^T_m(\mathbb{F}^\sigma T) \leq c^{T,\iota}_m(\mathbb{F}^\sigma T) \leq |T| c^T_m(\mathbb{F}^\sigma T), \forall m \in \mathbb{N}, \)
2. For all \( m \in \mathbb{N}, \)
\[
|T|^m \leq c^{T,\iota}_m(\mathbb{F}^\sigma T) \leq |T'||T|^m+1,
\]

where \( T' = [T,T] \).

**Proof.** For each sequence \( s = (u_1, \ldots, u_m) \in T^m \), let
\[
B_s = \{ u_\pi(1) \cdots u_\pi(m) \mid \pi \in S_m \},
\]
and let (in \( \mathbb{F}(X)/\text{Id}^{T,\iota}(\mathbb{F}^\sigma T) \))
\[
P^T_s(\mathbb{F}^\sigma T) = \text{Span} \left\{ x^{(u_\pi(1))} \cdots x^{(u_\pi(m))} \mid \pi \in S_m \right\}.
\]

Since \( x^{(u_\pi(1))} \cdots x^{(u_\pi(m))} = x^{(u_\pi(1))} \cdots x^{(u_\pi(m))} \) if and only if \( u_\pi(1) \cdots u_\pi(m) = u_{\theta(1)} \cdots u_{\theta(m)} \) (see the proof of Theorem 5), we see that
\[
\dim P^T_s(\mathbb{F}^\sigma T) = |B_s|.
\]

Hence, we may conclude that
\[
c^T_m(\mathbb{F}^\sigma T) = \sum_{s \in T^m} \dim P^T_s(\mathbb{F}^\sigma T) = \sum_{s \in T^m} |B_s|. \tag{5}
\]

Now, for each sequence \( s \in T^m \), note that the elements of \( B_s \) are constant modulo \([T,T]\) (since \( T/T' \) is an abelian group). Thus, \( B_s \) assumes at most \(|T'|\) elements, that is, \( 1 \leq |B_s| \leq |T'|. \) From (5), this gives us
\[
|T|^m \leq c^T_m(\mathbb{F}^\sigma T) \leq |T'||T|^m.
\]

Finally, consider a basis of \( P^T_m(\mathbb{F}^\sigma T) \) consisting of monomials. Given \( x^{(u_1)}_i \cdots x^{(u_m)}_m \), by the proof of Theorem 5 the set
\[
\{ j_1(x^{(u_1)}_i) \cdots j_m(x^{(u_m)}_m) \mid j_1, \ldots, j_m \in \{0,1\} \}
\]
contains at most \(|T|\) distinct elements modulo \( \text{Id}^{T,\iota}(\mathbb{F}^\sigma T) \). Hence, \( c^{T,\iota}_m(\mathbb{F}^\sigma T) \leq |T| c^T_m(\mathbb{F}^\sigma T). \) \( \square \)

As a consequence, we obtain the graded-involution exponent of \( \mathbb{F}^\sigma T \):

**Corollary 7.** Let \( T \) be a finite group, \( \sigma \in \mathbb{Z}^2(T, \mathbb{F}^\times) \) and \( \iota \) a homogeneous involution on \( \mathbb{F}^\sigma T \). Then \( \exp^{T,\iota}(\mathbb{F}^\sigma T) = |T|. \) \( \square \)
Specializing Theorem 5 in the special case where $T$ is an abelian group, we obtain the exact codimension sequence:

**Corollary 8.** If $T$ is an abelian group and $\iota$ is a degree-preserving involution on $F^\sigma T$, then for each $m \in \mathbb{N}$,

$$c_m^{T,\iota}(F^\sigma T) = c_m^T(F^\sigma T) = |T|^m.$$

□

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