Accelerating and retarding anomalous diffusion

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Abstract
In this paper, Gaussian models of retarded and accelerated anomalous diffusion are considered. Stochastic differential equations of fractional order driven by single or multiple fractional Gaussian noise terms are introduced to describe retarding and accelerating subdiffusion and superdiffusion. Short- and long-time asymptotic limits of the mean-squared displacement of the stochastic processes associated with the solutions of these equations are studied. Specific cases of these equations are shown to provide possible descriptions of retarding or accelerating anomalous diffusion.

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(Some figures may appear in colour only in the online journal)

1. Introduction
Anomalous diffusion occurs in many physical, chemical and biological systems [1–3]. In normal diffusion, the mean-squared displacement of diffusion particles varies linearly with time $\langle \Delta x^2(t) \rangle \sim t$. In some complex disordered media, the diffusion becomes anomalous with $\langle \Delta x^2(t) \rangle \sim t^\alpha$, where the scaling exponent $\alpha \neq 1$ characterizes the anomalous diffusion. For $\alpha < 1$, the process is known as subdiffusion, and when $\alpha > 1$ it is called superdiffusion. The fractal dimension of the trajectory of anomalous diffusion was studied in 1982 [4], which can be considered the first attempt to link fractional derivative with anomalous diffusion. It is widely accepted now that differential equations of fractional order are well suited for describing fractal phenomena such as anomalous diffusion in complex disordered media. The constant memory and self-similar character of these phenomena can be taken into account by using the kinetic equations with fixed fractional order.

In certain locally heterogeneous media, diffusing processes do not satisfy the constant power-law-type scaling behavior like anomalous diffusion. Such processes include the
retarding and accelerating anomalous diffusion. Retardation of diffusion occurs in single-file diffusion, where particles are constrained to move in a single file due to confined one-dimensional geometry, such as diffusion in zeolites [5], or in the anomalous diffusion that occurs on the biological cell membrane [6, 7]. Possible causes for the presence of anomalous subdiffusion in biological systems include the presence of immobilized obstacles which hamper molecular motion by an excluded volume interaction and the cytoplasmic crowding in living cells. On the other hand, membrane-bound proteins exhibit transition from subdiffusion at short time to normal or superdiffusion at long times [8]; the diffusion of telomeres in the nucleus of mammalian cells shows accelerating subdiffusion [9]. Other examples are retarded and enhanced dopant diffusion in semiconductors [10–12], the accelerating superdiffusion of energetic charged particles across the magnetic field in astrophysical plasma physics [13, 14] and accelerated diffusion in the Josephson junction [15]. Such processes have memory and fractal dimension that may vary with position, temperature, density, or internal parameters such as elasticity and viscosity.

Anomalous diffusion having a scaling exponent which varies with position and time was studied by Glimm and co-workers in the early 1990s [16, 17] in the multifractal modeling of heterogeneous geological systems. More systematic studies of transport phenomena with variable scaling exponents were carried out in the late 1990s and early 2000s. Stochastic processes with variable fractional order such as multifractional Brownian motion was introduced to model phenomena with variable memory or variable fractal dimension [18–20]. In order to describe systems with variable scaling exponents, one may have to consider fractional differential equations of variable order. However, such variable order equations are in general mathematically intractable and cannot be solved without numerical approximations [21–26].

There exists a certain class of diffusion processes with a non-unique scaling exponent that can be described by fractional differential equations of distributed order. The notion of distributed-order differential operators was first introduced by Caputo in 1969 [27]. A distributed-order fractional diffusion equation has its fractional order derivatives integrated over the order of differentiation within a given range. Applications of distributed fractional order equations to fractional diffusion and fractional relaxation have been carried out by various authors [25–33].

This paper considers Gaussian models of retarded and accelerated anomalous diffusion. We introduce a class of multi-term fractional Langevin-type equations driven by single or multiple fractional Gaussian noise terms. These equations can be regarded as special cases of fractional Langevin equations of distributed order, and they can be used to describe retarded and accelerated anomalous diffusion. Detailed study of the short- and long-time asymptotic properties of the solutions to these equations is carried out.

2. Multi-fractional stochastic differential equations

In this section, we introduce a class of multi-fractional Langevin-like equations of the following form:

$$\sum_{i=1}^{m} a_i D^{\alpha_i} x(t) = \sum_{j=1}^{n} c_j \xi_j(t), \quad t \in \mathbb{R}, \ 0 < \alpha_i \leq 2,$$

(1)
where \( c_j > 0, \) \( D^\nu \) is the Riemann–Liouville or Caputo fractional derivative [34–38] which is defined for \( m - 1 \leq \alpha \leq m \) as

\[
D^\alpha f(t) = \begin{cases} 
\frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f(u) \, du}{(t-u)^{m-\alpha+1}}, & \text{Riemann–Liouville} \\
\frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{\frac{d^m}{du^m} f(u) \, du}{(t-u)^{m-\alpha+1}}, & \text{Caputo}.
\end{cases}
\]

(2)

The Gaussian noise \( \xi_j(t) \) is defined by

\[
\langle \xi_j(t) \rangle = 0,
\]

(3)

and

\[
\langle \xi_j(t) \xi_j(s) \rangle = \delta_{ij} d_j |t-s|^{-\gamma_j}, \quad 0 < \gamma_j, \gamma_j < 2,
\]

(4)

with

\[
d_j = \frac{1}{2 \sin(\pi \gamma_j /2) \Gamma(1 - \gamma_j)}.
\]

(5)

If we let \( \gamma_j = 2 - 2H_j \), where \( 0 < H_j < 1 \) is the Hurst index associated with fractional Brownian motion, and \( d_j = (2 \sin(\pi H_j) \Gamma(2H_j - 1))^{-1} \), then \( \xi_j(t) \) can be regarded as the derivative (in the sense of generalized functions) of the fractional Brownian motion indexed by \( H_j \). Note that the covariance of fractional Gaussian noise has the same algebraic sign as \( (2H_j - 1) \). For \( 1/2 \leq H_j < 1 \), the process exhibits long-range dependence with persistent positive covariance. On the other hand, when \( 0 < H_j < 1/2 \), \( d_j \) is negative and the process has an anti-persistent correlation structure. For \( H_j = 1/2 \), or \( \gamma_j = 1 \), it corresponds to white noise. Here, we remark that \( \lim_{H_j \to 1/2} d_j t^{2H_j - 2} = \delta(t) \) in the sense of generalized functions [39, 40]. We thus see that there is no need to include in the covariance of fractional Gaussian noise \( \xi(t) \) an extra term to cater for the white noise when \( H = 1/2 \) as given by [41]

\[
\langle \xi(t) \xi(s) \rangle = 4d_H H_j (2H_j - 1) |t-s|^{2H_j - 2} + 4d_H |t-s|^{2H_j - 1} \delta(t-s).
\]

(6)

Here, we would like to briefly discuss the covariance of fractional Gaussian noise [41–43] in terms of generalized functions. Just like for a proper description of fractional Gaussian noise, \( \xi(t) \) should not be defined pointwise for each \( t \). Instead, the process needs to be considered as a linear functional in some test function space such as Schwartz space \( \mathcal{S}(\mathbb{R}) \) of real-valued infinitely differentiable functions which decrease rapidly [39]. The generalized process \( \xi(t) \), \( f \in \mathcal{S}(\mathbb{R}) \), is a linear functional

\[
\xi(f) = \int_{-\infty}^{\infty} \xi(t) f(t) \, dt.
\]

(7)

\( \xi(f) \) is a generalized stationary Gaussian process with the covariance given by the bilinear functional

\[
C(f, g) = \langle \xi(f) \xi(g) \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) g(s) c(t-s) \, dt \, ds
\]

\[
= \int_{-\infty}^{\infty} f(t) \, dt \int_{-\infty}^{\infty} g(s) c(|t-s|) \, ds
\]

\[
= \int_{-\infty}^{\infty} f(t) \, dt \int_{-\infty}^{\infty} c(s) [g(t+s) + g(t-s)] \, ds,
\]

(8)

where \( c(t-s) \) is a generalized function or distribution. For example, for \( H = 1/2 \), \( \xi \) is white noise with \( c(t-s) = \delta(t-s) \). \( H \neq 1/2 \) corresponds to fractional Gaussian noise, with \( c(t) \), \( t > 0 \), given by

\[
c(t) = \frac{t^{2H-2}}{\sin(\pi H) \Gamma(2H-1)} = \begin{cases} 
\frac{1}{\sin(\pi H)} t^{2H-1} \delta(t), & 1/2 < H < 1 \\
\frac{1}{\sin(\pi H)} D^{1-2H} \delta(t), & 0 < H < 1/2.
\end{cases}
\]

(9)
When $1/2 < H < 1$, the covariance kernel in (8) can be regarded as the fractional integral of the delta function, $t^{2H-1}\delta(t)$, up to a multiplicative constant $(2\sin(\pi H))^{-1}$. In this case, $c(t - s)$ is locally integrable. It becomes the fractional derivative of the delta function for $(2\sin(\pi H))^{-1}D^{1-2H}\delta(t)$ for $0 < H < 1/2$ since $D^Hf = D^{-H}f$ [43].

Before we proceed further, a brief comment on the stochastic differential equation driven by fractional Gaussian noise will be given. Recall that stochastic calculus of Ito cannot be used to define the integrals with respect to a stochastic process which is not a semimartingale. Fractional Brownian motion (except for the case of Brownian motion case with $H = 1/2$) is not a semimartingale. Due to its widespread application, the question on how to obtain a well-defined stochastic integral with respect to fractional Brownian motion has become a long standing problem which has attracted considerable attention [44, 45]. Several methods which include Sokorohod–Stratonovich stochastic integrals, Malliavin calculus and pathwise stochastic calculus have been proposed to overcome this difficulty (see [45, 46] for details). However, for application purposes, theory based on abstract integrals may encounter difficulty in physical interpretations. As we shall restrict our discussion related to applications of fractional Gaussian noise involving only persistent case 1 in physical interpretations. For most of the examples considered subsequently, we shall restrict $0 < \gamma < 1$.

Here, we remark that the multi-term fractional-order Langevin-like equation (1) can also be regarded as a special class of the following distributed-order fractional time stochastic equation:

$$D_x^\alpha x(t) = \xi_\psi(t), \quad t \geq 0,$$

with the distributed fractional derivative

$$D_x^\alpha x(t) = \int_0^2 \varphi(\alpha)D_\alpha^\alpha x(t) \, d\alpha,$$

where $D_\alpha^\alpha$ is the fractional derivative as defined by (2), the weight function $\varphi(\alpha)$, $0 \leq \alpha \leq 2$, which satisfies $\varphi(\alpha) \geq 0$ is given by $\varphi(\alpha) = \sum_{i=1}^{m} a_i \delta(\alpha - \alpha_i)$. Note that in general the weight function is a positive generalized function. The Gaussian noise $\xi_\psi(t)$ is the distributed-order fractional Gaussian noise defined by

$$\xi_\psi(t) = \int_0^2 \xi_\gamma(t)\psi(\gamma) \, d\gamma,$$

with the weight function $\psi(\gamma)$, $0 \leq \gamma \leq 2$, which satisfies $\psi(\gamma) \geq 0$ and is given by $\psi(\gamma) = \sum_{j=1}^{N} c_j \delta(\gamma - \gamma_j)$. Here, we remark that in most of the examples considered subsequently, we shall restrict $0 \leq \gamma \leq 1$.

The solution of (1) can be solved formally by using the Laplace transform method which gives

$$A(p)\tilde{x}(p) - B(p) = \tilde{\xi}(p),$$

where $\tilde{x}(p)$ is the Laplace transform of $x(t)$ and

$$A(p) = \sum_{i=1}^{m} a_i p^{\alpha_i},$$

and

$$B(p) = \begin{cases}
\sum_{i=1}^{m} a_i \sum_{k=0}^{[\alpha_i]} \mathcal{L}[D_{RL}^{\alpha_i-k}x(t)_{t=0}^{\alpha_i}](p), & \text{(Riemann–Liouville)} \\
\sum_{i=1}^{m} a_i \sum_{k=0}^{[\alpha_i]} p^{\alpha_i-k}x(0), & \text{(Caputo)}.
\end{cases}$$
Here, \([\alpha_i]\) denotes the largest integer smaller or equal to \(\alpha_i\). Note that we have used \(p\) as a Laplace transform variable since its usual symbol \(s\) has been used to represent time earlier. For the Riemann–Liouville case, \([D_{RL}^{\alpha_i-k_i-1}x(t)]_{t=0}\) is the Riemann–Liouville derivative of order \(\alpha_i - k_i - 1\) evaluated at \(t = 0\). For the Caputo case, \(x^{(k_i)}(t)\) denotes the \(k_i\)th derivative of \(x(t)\). For simplicity, we assume the initial conditions \(x^{(k_i)}(0) = 0\) for all \(i = 1, \ldots, m\), and \([D_{RL}^{\alpha_i-k_i-1}x(t)]_{t=0} = 0\), such that \(B(p) = 0\) for both these cases. The Laplace transform of the Green function is then given by \(\tilde{G}(p) = 1/A(p)\). Therefore,

\[
\tilde{x}(p) = \tilde{G}(p)\tilde{\xi}(p) = \frac{\tilde{\xi}(p)}{A(p)}.
\]

The solution is then given by the inverse Laplace transform:

\[
x(t) = \int_0^t G(t-u)\tilde{\xi}(u)\,du.
\]

The covariance and variance of the process are given respectively by

\[
K(s, t) = \int_0^t du\int_0^s dvG(t-u)C(u-v)G(s-v)
\]

and

\[
\sigma^2(t) = 2\int_0^t G(u)\int_0^u C(u-v)G(v)\,du\,dv.
\]

Assuming \(t > s\), (18) becomes

\[
K(s, t) = \int_0^t dv[G(s-v)G_C(t-v) + G(t-v)G_C(s-v)],
\]

with

\[
G_C(t) = (G*C)(t) = \int_0^t G(t-u)C(u).
\]

Note that for the simplest case of (1) with \(m = 1, n = 1\) and \(\alpha_1 = \alpha\), if \(\gamma_1 = 1\), then for Riemann–Liouville (or Caputo) fractional derivative,

\[
D_{RL}^\alpha x(t) = \eta(t),
\]

where \(\xi_1(t) = \eta(t)\) is white noise. For \(D_{RL}^\alpha x(t)\big|_{t=0} = 0\) (or \(x(0) = 0\), (22) defines the Riemann–Liouville fractional Brownian motion or type II fractional Brownian motion with the Hurst index \(H\), \(\alpha = H + 1/2\) [20]. The solution of (22) with the above boundary condition is given by

\[
x(t) = \int_0^t (t-u)^{\alpha-1} \eta(u)\,du,
\]

with the variance

\[
\sigma^2(t) = \frac{t^{2\alpha-1}}{(2\alpha - 1)(\Gamma(\alpha))^2} = \frac{t^{2H}}{2H(\Gamma(H + 1/2))^2}.
\]

Note that the fractional Brownian motion of Riemann–Liouville type has a variance with the same time dependence as the standard fractional Brownian motion. In contrast to the latter, though, it is self-similar but its increment process is not stationary. However, it has
the advantage that the process begins at time $t = 0$, and the Hurst index can take any value $H > 0$ [20].

Another simple case is when $m = 1, n = 1$ and $\gamma_1 = \gamma \neq 1$, which leads to a Gaussian non-stationary mono-fractal process with the variance

$$\sigma^2(t) = \frac{t^{2\alpha - \gamma}}{\sin(\gamma \pi/2)(2\alpha - \gamma)\Gamma(\alpha)\Gamma(\alpha - \gamma + 1)}. \tag{25}$$

This process can be subdiffusion or superdiffusion, depending on whether $2\alpha - \gamma < 1$ or $2\alpha - \gamma > 1$.

In the subsequent sections, we shall consider various specific cases of (1) for modeling accelerating and retarding anomalous diffusion.

3. Accelerating anomalous diffusion

One of the simplest models for accelerating diffusion can be obtained by using a special case of (1):

$$D^\alpha x(t) = \sum_{j=1}^{n} c_j \xi_{\gamma_j}. \tag{26}$$

The Green function is given by $G(t) = T^{-1}(t)$ and

$$G_C(t) = \int_0^t du \frac{(t-u)^{\alpha-1}}{\Gamma(\alpha)} \left[ \sum_{j=1}^{n} c_j^2 \frac{u^{\gamma_j}}{2\sin(\pi \gamma_j/2)\Gamma(\alpha - \gamma_j + 1)} \right]$$

$$= \sum_{j=1}^{n} c_j^2 \frac{u^{\gamma_j}}{2\sin(\pi \gamma_j/2)\Gamma(\alpha - \gamma_j + 1)} \tag{27}.$$

The covariance is given by

$$K(s,t) = \sum_{j=1}^{n} K_j(s,t), \tag{28}$$

and from (20), one obtains

$$K_j(s,t) = \int_0^s \frac{c_j^2}{2\sin(\pi \gamma_j/2)} \left[ \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} \frac{(t-u)^{\alpha-\gamma_j}}{\Gamma(\alpha - \gamma_j + 1)} + \frac{(t-u)^{\alpha-1}}{\Gamma(\alpha)} \frac{(s-u)^{\alpha-\gamma_j}}{\Gamma(\alpha - \gamma_j + 1)} \right] \frac{d\gamma_j}{\Gamma(\alpha + 1)\Gamma(\alpha - \gamma_j + 1)} \tag{29a}$$

$$= \frac{c_j^2}{2\sin(\pi \gamma_j/2)} \left[ \frac{s^{\alpha - \gamma_j}}{\Gamma(\alpha)\Gamma(\alpha - \gamma_j + 1)} \int_0^1 du (1-u)^{\alpha-1} \left( 1 - \frac{s}{t} u^{\alpha-\gamma_j} \right)^{\alpha-\gamma_j} 
+ \frac{s^{\alpha - \gamma_j + 1}}{\Gamma(\alpha)\Gamma(\alpha - \gamma_j + 1)} \int_0^1 du (1-u)^{\alpha-1} \left( 1 - \frac{s}{t} u^{\alpha-\gamma_j} \right)^{\alpha-\gamma_j} \right]$$

$$= \frac{c_j^2}{2\sin(\pi \gamma_j/2)} \left[ \frac{s^{\alpha - \gamma_j}}{s^{\alpha - \gamma_j + 1}} \frac{F(\gamma_j - \alpha, 1, 1 + \alpha + s/t)}{\Gamma(\alpha + 1)\Gamma(\alpha - \gamma_j + 1)} 
+ \frac{s^{\alpha - \gamma_j + 1}}{s^{\alpha - \gamma_j + 1}} \frac{F(1 - \alpha, 1, 2 + \alpha - \gamma_j, s/t)}{\Gamma(\alpha)\Gamma(\alpha - \gamma_j + 2)} \right], \tag{29b}.$$
where we have used [47], #9.111, page 1005. By using the following identity:

\[ F(\alpha, \beta, \gamma, 1) = \frac{\Gamma(\gamma) \Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha) \Gamma(\gamma - \beta)}. \]  

one obtains the variance as

\[ \sigma^2(t) = \sum_{j=1}^{n} \sigma_j^2(t) = \sum_{j=1}^{n} K_j(t, t) = \sum_{j=1}^{n} \frac{c_j^2 t^{2\alpha - \gamma_j}}{(2\alpha - \gamma_j) \sin(\pi \gamma_j / 2) \Gamma(\alpha) \Gamma(\alpha - \gamma_j + 1)}. \]  

Note that the variance can also be obtained directly from (29a).

For the case with \( n = 2, \) and \( \gamma_1 > \gamma_2, \) the short-time and long-time limits of the mean-squared displacement (or variance) are given by

\[ \sigma^2(t) \sim \begin{cases} 
\frac{c_1^2 t^{2\alpha - \gamma_1}}{\sin(\pi \gamma_1 / 2)(2\alpha - \gamma_1) \Gamma(\alpha) \Gamma(\alpha - \gamma_1 + 1)} & \text{as } t \to 0 \\
\frac{c_2^2 t^{2\alpha - \gamma_2}}{\sin(\pi \gamma_2 / 2)(2\alpha - \gamma_2) \Gamma(\alpha) \Gamma(\alpha - \gamma_2 + 1)} & \text{as } t \to \infty.
\end{cases} \]  

Therefore, the process is accelerating subdiffusion if for \( i = 1, 2, \) \( 0 < (2\alpha - \gamma_i) < 1, \) and it becomes accelerating superdiffusion when \( 1 < (2\alpha - \gamma_i) < 2. \) For \( 0 < (2\alpha - \gamma_1) < 1 \) and \( 1 < (2\alpha - \gamma_2) < 3, \) the process begins as a subdiffusion and it accelerates to become a superdiffusion. Figure 1 shows the various possible types of anomalous diffusion for different values of \( \alpha \) and \( \gamma. \)

In order to have some idea about the type of the process \( x(t) \) represents, let us consider some specific examples corresponding to certain given values for \( \alpha \) and \( \gamma_i. \) For (26) with \( \alpha = 1, \) the \( i \)th component of the covariance of the process associated with the solution becomes
The variance of the process is given by
\[ K_i(s, t) = \frac{c_i^2}{2 \sin(\pi \gamma_i/2) \Gamma(2 - \gamma_i)} \int_0^t [(t - u)^{1-\gamma_i} + (t - u)^{1-\gamma_i}] \, du \]
\[ = \frac{c_i^2}{2 \sin(\pi \gamma_i/2) \Gamma(3 - \gamma_i)} [t^{2-\gamma} + s^{2-\gamma} - (t-s)^{2-\gamma}], \tag{33} \]
which is the covariance of the fractional Brownian motion (up to a multiplicative constant). Thus the covariance of the process \( K(s, t) = \sum_{i=1}^n K_i(s, t) \) is the covariance of a mixed fractional Brownian motion (also called the fractional mixed fractional Brownian motion by some authors) \([48–50]\) which is the sum of \( n \) independent fractional Brownian motion

\[ x(t) = \sum_{i=1}^n c_i B_{H_i}(t), \tag{34} \]

where \( H_i = (2 - \gamma_i)/2 \) is the Hurst index of the fractional Brownian motion \( B_{H_i}(t) \). Thus, in the mixed fractional Brownian motion model, anomalous diffusion begins with a lower diffusion rate, which can be represented by the fractional Brownian motion of lower Hurst index, and it is subsequently accelerated and is described by fractional Brownian motion of a higher Hurst index. It is interesting to note that if all \( H_i \) are not equal to 1/2 (that is when they are all not Brownian motion), then such a process is long-range dependent \([51, 52]\). The process satisfies a generalization of self-similar property called mixed self-similarity in the following sense:

\[ \sum_{i=1}^n B_{H_i}(rt) \cong \sum_{i=1}^n r^{H_i} B_{H_i}(t). \tag{35} \]

The variance of the process is given by

\[ \sigma^2(t) = \sum_{j=1}^n \frac{c_j^2 t^{2H_j}}{\sin(\pi H_j) \Gamma(2H_j + 1)}. \tag{36} \]

Just like the previous case, suffice to consider \( n = 2 \). For \( H_2 > H_1 \), the short- and long-time limits of the MSD are

\[ \sigma^2(t) \sim \begin{cases} 
\frac{c_1^2 t^{2H_1}}{\sin(\pi H_1) \Gamma(2H_1 + 1)} & \text{as } t \to 0, \\
\frac{c_2^2 t^{2H_2}}{\sin(\pi H_2) \Gamma(2H_2 + 1)} & \text{as } t \to \infty.
\end{cases} \tag{37} \]

Thus the process behaves as accelerating superdiffusion (or subdiffusion) for \( 1/2 < H_j < 1 \) (or \( 0 < H_j < 1/2 \)) with \( j = 1, 2 \).

Here, we would like to remark that there exists another process called the step fractional Brownian motion \([53–55]\) which can also be used to model both accelerating and retarding anomalous diffusion. Although the mixed fractional Brownian motion can only be used to describe accelerating anomalous diffusion, its mathematical structure is comparatively simpler than that of step fractional Brownian motion. It is also interesting to mention that the mixed fractional Brownian motion \( x(t) = B_{H/2} + B_H(t) \), \( t \) in a finite interval \([0, T]\), is a semimartingale if \( H \in (3/4, 1) \), and it is equivalent to Brownian motion in law \([48]\).

Next, we consider another special case with the exponents \( \alpha \) and \( \gamma_i \) satisfying the condition \( \alpha - \gamma_i = 0 \) for a particular \( \alpha \)th fractional Gaussian noise. Equation (29a) now becomes

\[ K_i(s, t) = \frac{c_i^2}{2 \sin(\pi \alpha/2) \Gamma(\alpha)} \int_0^t [(s - u)^{\alpha-1} + (t - u)^{\alpha-1}] \, du \]
\[ = \frac{c_i^2}{2 \sin(\pi \alpha/2) \Gamma(\alpha + 1)} [s^\alpha + t^\alpha - (t-s)^\alpha]. \tag{38} \]
Since $\gamma_i = 2 - 2H_i = \alpha$, the process with the above covariance is fractional Brownian motion indexed by $2 - 2H_i$. Note that the other components of covariance $\mathcal{K}_j(s, t), j \neq i$, are given by (29b); hence, they are not the fractional Brownian motion. The variance is given by

$$\sigma^2(t) = \sigma_i^2(t) + \sum_{j=1, j \neq i}^{n} \sigma_j^2(t)$$

$$= \frac{2\sin(\pi\alpha/2)}{\Gamma(\alpha + 1)} c_i^2 t^{2\alpha - \gamma_i} + \sum_{j=1, j \neq i}^{n} \frac{c_j^2 t^{2\alpha - \gamma_j}}{2\sin(\pi\gamma_j/2)(2\alpha - \gamma_j)\Gamma(\alpha)\Gamma(\alpha - \gamma_j + 1)}.$$  \hfill{(39)}

In this case, the type of anomalous diffusion depends on the values of $\alpha$ and $2\alpha - \gamma_j$. Assuming $0 < \gamma_j < 1$, then for $0 < \alpha < 1$ and $0 < 2\alpha - \gamma_j < 1$ one obtains accelerating subdiffusion; when $1 < \alpha < 2$ and $1 < 2\alpha - \gamma_j < 2$, the process is accelerating superdiffusion. The process accelerates (retards) from subdiffusion (superdiffusion) to become superdiffusion (subdiffusion) when $1 < \alpha < 2$ and $0 < 2\alpha - \gamma_j < 1$ (or when $0 < \alpha < 1$ and $1 < 2\alpha - \gamma_j < 2$).

Finally, we note that if $\xi_i$ is white noise, that is when $\gamma_i = 1$, then $\mathcal{K}_i(s, t)$ takes the following form:

$$K_i(s, t) = \frac{c_i^2 a^\alpha t^{\alpha - 1}}{\Gamma(\alpha + 1)\Gamma(\alpha)} F(1 - \alpha, 1 + \alpha, s/t),$$  \hfill{(40)}

which is just the covariance of ‘type II’ or the Riemann–Liouville fractional Brownian motion [20]. The variance for the process is given by

$$\sigma^2_i(t) = \frac{2c_i^2 a^{2\alpha - 1}}{(2\alpha - 1)(\Gamma(\alpha))^2}. \hfill{(41)}$$

For $n = 2$, $\gamma_1 = 1$ and $\gamma_2 < 1$, one obtains a process which is an accelerating superdiffusion if $1/2 < \alpha < 1$, an accelerating superdiffusion if $1 < \alpha < 3/2$ and finally it represents a process which accelerates from a subdiffusion to a superdiffusion if $1/2\alpha < 1$ and $\gamma_2 < 2\alpha - 1$.

From the above discussion, it is noted that a simple fractional Langevin equation driven by a single fractional Gaussian noise results in an anomalous diffusion. However, when the process is driven by more than single fractional Gaussian noise terms, the resulting process can be accelerating subdiffusion or superdiffusion, depending on the fractional order of the noise terms and the derivative term. Thus, the interplay between multiple driving fractional Gaussian noise terms in the fractional Langevin-like equation leads to a simple Gaussian model of accelerating anomalous diffusion.

### 4. Retarding anomalous diffusion

There exists another type of anomalous diffusion that slows down with time. In other words, for short times the mean-squared displacement of such a diffusion process varies as $t^{\alpha_1}$, and it varies as $t^{\alpha_2}$ with $\alpha_2 < \alpha_1$ for long times. Retarding anomalous diffusion occurs in various physical and biological systems. For example, in the single-file diffusion that occurs in cell membranes and narrow channels [56, 57] and the retarding anomalous diffusion in semiconductors [12, 58].
A simple Gaussian model for retarding anomalous diffusion can be obtained by considering a special case of (1) which takes the form of the following fractional stochastic differential equation:

\[ a_1 D^{\alpha_1} x(t) + a_2 D^{\alpha_2} x(t) = \sum_{j=1}^{n} c_j \xi_j(t), \quad 0 < \alpha_2 < \alpha_1 < 2. \] (42)

For both Riemann–Liouville and Caputo cases, the Laplace transform of (42) gives

\[ [a_1 p^{\alpha_1} + a_2 p^{\alpha_2}] \tilde{\chi}(p) - B(p) = \sum_{j=1}^{n} c_j \tilde{\xi}_j(p), \] (43)

where \( B(p) \) is given by (15). For simplicity, we again choose the initial conditions such that \( B(p) = 0 \). The Green function is then given by the inverse Laplace transform of

\[ \tilde{G}(p) = \frac{1}{a_1 p^{\alpha_1} + a_2 p^{\alpha_2} + (a_2/a_1)} \] (44)

such that its inverse Laplace transform gives

\[ G(t) = \frac{1}{a_1} t^{\alpha_1-1} E_{\alpha_1,\alpha_1} \left( -\frac{a_2}{a_1} t^{\alpha_1-\alpha_2} \right), \] (45)

where

\[ E_{\mu,\nu}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\mu j + \nu)}, \quad \mu > 0, \nu > 0, \] (46)

is the Mittag–Leffler function [59]. Using (20) and (21), and substituting \( C(t) = \sum_{j=1}^{n} c_j^2 \Gamma(\gamma_j) \), one obtains

\[ G_C(t) = \sum_{j=1}^{n} c_j^2 \int_0^t \frac{1}{a_1} (t-u)^{\alpha_1-1} E_{\alpha_1,\alpha_1} \left( -\frac{a_2}{a_1} (t-u)^{\alpha_1-\alpha_2} \right) \left( u^{-\gamma_j} / \Gamma(1-\gamma_j) \right) du, \] (47)

The covariance \( K(s, t) = \sum_{j=1}^{n} c_j^2 K_j(s, t) \) is given by (20) with

\[ K_j(s, t) = \int_0^t du \left[ \frac{(s-u)^{\alpha_1-1}}{a_1} E_{\alpha_1,\alpha_1} \left( -\frac{a_2}{a_1} (s-u)^{\alpha_1-\alpha_2} \right) \right] \]
\[ \left[ (t-u)^{\alpha_1-\gamma_j} / a_1 \right] E_{\alpha_1-\alpha_2,\alpha_1-\gamma_j} \left( -\frac{a_2}{a_1} (t-u)^{\alpha_1-\alpha_2} \right) \]
\[ + \int_0^s du \left[ (s-u)^{\alpha_1-\gamma_j} / a_1 \right] E_{\alpha_1-\alpha_2,\alpha_1-\gamma_j} \left( -\frac{a_2}{a_1} (s-u)^{\alpha_1-\alpha_2} \right) \]
\[ \left[ (s-u)^{\alpha_1-\gamma_j} / a_1 \right] E_{\alpha_1-\alpha_2,\alpha_1-\gamma_j} \left( -\frac{a_2}{a_1} (s-u)^{\alpha_1-\alpha_2} \right). \] (48)

The variance is given by \( \sigma^2(t) = \sum_{j=1}^{n} c_j^2 K_j(t, t) \),

\[ K_j(t, t) = 2 \int_0^t du \left[ \frac{(s-u)^{\alpha_1-\gamma_j-1}}{a_2} E_{\alpha_1-\alpha_2,\alpha_1} \left( -\frac{a_2}{a_1} (t-u)^{\alpha_1-\alpha_2} \right) \right] \]
\[ \times E_{\alpha_1-\alpha_2,\alpha_1-\gamma_j+1} \left( -\frac{a_2}{a_1} (t-u)^{\alpha_1-\alpha_2} \right). \] (49)
Note that the covariance and variance given above cannot be evaluated. However, by using the following asymptotic properties of Mittag–Leffler function [59],

\[
E_{\mu,v}(-z) \sim -\sum_{n=1}^{N} (-1)^{v-n}z^{-n} + O(|z|^{-1-N}), \quad |\arg(z)| < \left(1 - \frac{\mu}{2}\right)\pi, \quad z \to \infty, \quad (50)
\]

and

\[
E_{\mu,v}(-z) \sim \frac{1}{\Gamma(v)} + O(z), \quad z \to 0, \quad (51)
\]

it is possible to obtain the short- and long-time behaviors of the variance. In the case of single fractional Gaussian noise, one has for \(\gamma < 2\alpha_2\),

\[
\sigma^2(t) \sim \begin{cases} 2^{2\alpha_1-2}, & t \to 0 \\ 2^{2\alpha_2-2}, & t \to \infty. \end{cases} \quad (52)
\]

Since \(\alpha_1 > \alpha_2\), the process is a retarding subdiffusion (or superdiffusion) if for \(i = 1, 2\), \(0 < \alpha_i - \gamma/2 < 1/2\) (or \(1/2 < \alpha_i - \gamma/2 < 1\)). In the case when there are more than one noise, the dominant terms for the short time limit and long time limit for the variance are \(\sim t^{\min(2\alpha_i-\gamma)}\) and \(\sim t^{\max(2\alpha_i-\gamma)}\), respectively. Thus we see that the double-order fractional stochastic equation driven by single fractional Gaussian noise can be used to model retarding anomalous diffusion. The lower order fractional derivative term in (42) plays the role of a damping term which slows down the diffusion.

The main disadvantage of using (42) for modeling retarding anomalous diffusion is that in general the covariance and variance of the underlying process cannot be calculated explicitly. We would like to find a particular case of (42) such that its solution is a process with covariance and variance that can be completely determined. For this purpose, we consider the following double-order fractional Langevin-like equation:

\[
a_1D^{\alpha_1}x(t) + a_2D^{\alpha_2}x(t) = c_1\xi_1(t) + c_2\xi_2(t), \quad 0 < \alpha_2 < \alpha_1 < 2. \quad (53)
\]

The two independent fractional Gaussian noises \(\xi_1(t)\) and \(\xi_2(t)\) are chosen such that for \(t > s\) they have zero mean and the following covariance:

\[
\langle \xi_i(t)\xi_j(s) \rangle = \frac{(t-s)^{\gamma-\alpha_i}}{\Gamma(v-\alpha_i)}\delta_{ij} \equiv C_{ij}(t-s)\delta_{ij}, \quad i, j = 1, 2. \quad (54)
\]

We remark that the fractional Gaussian noise with covariance given by (54) is selected based on practical purposes as it gives the required results as well as provides a more manageable solution to (53). As a result (54), one can verify that the Laplace transforms of the covariance of the fractional Gaussian noise \(C(t) = c_1^2C_1(t) + c_2^2C_2(t)\) and the Green function \(G(t)\) satisfy the following relation:

\[
\tilde{G}(p)\tilde{C}(p) = p^{-\gamma}. \quad (55)
\]

The inverse Laplace transform of (55) is

\[
G(t) * C(t) = \frac{t^{\gamma-1}}{\Gamma(v)}. \quad (56)
\]

The covariance of the process associated with the solution of (53) can be calculated using (20) and (56):

\[
K(s, t) = \int_0^t da \left[ G(t-u)\frac{(s-u)^{\gamma-1}}{\Gamma(v)} + G(s-u)\frac{(t-u)^{\gamma-1}}{\Gamma(v)} \right]
\]

\[
= \int_0^t da \left[ \frac{(t-u)^{\alpha_1-2\alpha_1}}{a_1}E_{a_2, a_1}[\alpha_2, a_1] - \frac{a_2}{a_1}(t-u)^{\gamma-2\gamma} \right] \frac{(s-u)^{\gamma-1}}{\Gamma(v)}
\]

\[
+ \frac{(s-u)^{\alpha_1-2\alpha_1}}{a_1}E_{a_2, a_1}[\alpha_2, a_1] - \frac{a_2}{a_1}(s-u)^{\gamma-2\gamma} \right] \frac{(t-u)^{\gamma-1}}{\Gamma(v)}. \quad (57)
\]
Its variance is given by the following series expansion:

\[
\sigma^2(t) = 2 \int_0^t \frac{1}{a_1 \Gamma(v)} \left(t - u\right)^{a_1 + v - 2} E_{a_1 - a_2, a_1} \left(\frac{a_2}{a_1} (t - u)^{a_1 - a_2}\right) \, du
\]

\[
= \frac{2}{a_1 \Gamma(v)} \sum_{n=0}^{\infty} \frac{\left(\frac{a_2}{a_1}\right)^n}{n!} t^{n(a_1 - a_2) + a_1 + v - 1} \Gamma(n(a_1 - a_2) + a_1),
\]  

(58)

From (58), one gets the short and long time limits of the variance as

\[
\sigma^2(t) \sim \begin{cases} 
\frac{2}{a_1 \Gamma(v) \Gamma(a_1 + v - 1)} t^{a_1 + v - 1}, & t \to 0 \\
\frac{2}{a_1 \Gamma(v) \Gamma(a_2 + v - 1)} t^{a_2 + v - 1}, & t \to \infty 
\end{cases}
\]  

(59)

The process with covariance (58) and variance (59) is a retarding subdiffusion or superdiffusion depending on the on the values of \(a_i + v - 1, i = 1, 2\). Thus, one can regard the term \(a_2 D^{\alpha_2} x(t)\) in (53) as a damping term which slows down the anomalous diffusion.

A special case for which the covariance has a closed form is when \(v = 1\). For \(t > s\), the covariance (57) becomes

\[
K(t, s) = \int_0^t \frac{1}{a_1} \left(t - u\right)^{a_1 - 1} E_{a_1 - a_2, a_1} \left(-\frac{a_2}{a_1} (t - u)^{a_1 - a_2}\right) \\
+ \frac{1}{a_1} (s - u)^{a_1 - 1} E_{a_1 - a_2, a_1} \left(-\frac{a_2}{a_1} (s - u)^{a_1 - a_2}\right)
\]

\[
= \frac{t^{a_1}}{a_1} E_{a_1 - a_2, a_1 + 1} \left(-\frac{a_2}{a_1} \right) + \frac{s^{a_1}}{a_1} E_{a_1 - a_2, a_1 + 1} \left(-\frac{a_2}{a_1} \right) - \frac{(t - s)^{a_1}}{a_1} E_{a_1 - a_2, a_1 + 1} \left(-\frac{a_2}{a_1} \right).
\]  

(60)

It is interesting to note that the covariance consists of three terms of Mittag–Leffler functions of same order, and the time variables \(t\) and \(s\) enter the covariance expression (60) in a form similar to that of the fractional Brownian motion. Thus, it is not a coincidence that the asymptotic short and long time limits of the covariance are given by

\[
K(t, s) = \begin{cases} 
\frac{1}{a_1 \Gamma(a_1 + 1)} t^{a_1} + s^{a_1} - |t - s|^{a_1}, & t, s \to 0 \\
\frac{1}{a_2 \Gamma(a_2 + 1)} t^{a_2} + s^{a_2} - |t - s|^{a_2}, & t, s, |t - s| \to \infty.
\end{cases}
\]  

(61)

Note that these are just the covariance of fractional Brownian motion indexed respectively by \(a_1/2 = 1 - H_1\) and \(a_2/2 = 1 - H_2\), with \(0 < H_i < 1, i = 1, 2\). Since both the short and long time limits are the fractional Brownian motion, one can conclude that the stochastic process in this case is long-range dependent except when \(a_1 = a_2 = 1/2\) or when both the limiting processes are the Brownian motion [51, 52].

The variance is given by

\[
\sigma^2(t) = \frac{2}{a_1} t^{a_1} E_{a_1 - a_2, a_1 + 1} \left(-\frac{a_2}{a_1} t^{a_1 - a_2}\right).
\]  

(62)

The long and short time limits are given by

\[
\sigma^2(t) \sim \begin{cases} 
\frac{2}{a_1 \Gamma(a_1 + 1)} t^{a_1}, & t \to 0 \\
\frac{2}{a_2 \Gamma(a_2 + 1)} t^{a_2}, & t \to \infty
\end{cases}
\]  

(63)
From the above results, one has for $0 < \alpha_2 < \alpha_1 < 1$ a subdiffusion process which slows down with time, or a retarding subdiffusion. On the other hand, if $1 < \alpha_2 < \alpha_1 < 2$, the process is a retarding superdiffusion.

It would be interesting to see whether (53) can be used to describe accelerating anomalous diffusion as well. Suppose we replace condition (55) by the following:

$$C(t) = a_1 t^{\nu_2 - \nu_1} + a_2 t^{\nu_1 - \nu_2 - \nu} + a_3 t^{\nu_1 - \nu_2},$$

such that

$$C(p) = a_1 p^{\nu_2 - \nu_1} + a_1 p^{\nu_1 - \nu_2} + a_2 p^{\nu_1 - \nu_2 - \nu}.$$  

(65)

Inverse Laplace transform of (65) gives

$$C(t) = a_1 \left[ t^{\nu_2 - \nu_1} + t^{\nu_1 - \nu_2 - \nu} \right] + a_2 \left[ t^{\nu_1 - \nu_2 - \nu} \right].$$

(66)

The variance of the resulting process is

$$\sigma^2(t) = \sigma_1^2(t) + \sigma_2^2(t),$$

(67)

with $\sigma_1^2(t)$ given by (58) and similarly for $\sigma_2^2(t)$ with $\nu$ replaced by $\kappa$. Thus for $\nu < \kappa$, one obtains

$$\sigma^2(t) \sim \begin{cases} a_1 \Gamma(\nu) \Gamma(\alpha_1) / \Gamma(\alpha_1 + \nu - 1) t^{\alpha_1 + \nu - 1}, & t \to 0 \\ a_2 \Gamma(\kappa) \Gamma(\alpha_2) / \Gamma(\alpha_2 + \kappa - 1) t^{\alpha_2 + \kappa - 1}, & t \to \infty \end{cases}.$$  

(68)

It is interesting to note that the process with the above variance represents accelerating subdiffusion if $\alpha_1 + \nu < \alpha_2 + \kappa$, or $\kappa > \alpha_1 - \alpha_2 + \nu > 2\alpha_1 - \alpha_2$. However, to achieve such an accelerating subdiffusion it is necessary to consider four fractional Gaussian noise terms in (53), a situation that may be difficult to realize in practice.

On the other hand, we recall that for (42) with more than one fractional Gaussian noise, the dominant term of the variance for the associated process in the short and long time limits respectively varies as $t_{\min(2\alpha_i - \gamma_j)}^{\max(2\alpha_i - \gamma_j)}$ and $t_{\max(2\alpha_i - \gamma_j)}^{\min(2\alpha_i - \gamma_j)}$, respectively. If we consider the case with $\alpha_1 > \alpha_2$ and $\gamma_1 > \gamma_2$, one then has $2\alpha_1 - \gamma_2 > 2\alpha_1 - \gamma_1$ and $2\alpha_2 - \gamma_2 > 2\alpha_2 - \gamma_1$. As a result,

$$\sigma^2(t) \sim \begin{cases} t^{2\alpha_1 - \gamma_1}, & t \to 0 \\ t^{2\alpha_2 - \gamma_2}, & t \to \infty \end{cases}.$$  

(69)

Thus, it is possible to use (53) to model accelerating anomalous diffusion provided $2\alpha_2 - \gamma_2 > 2\alpha_1 - \gamma_1$ or $\alpha_1 - \alpha_2 < (\gamma_1 - \gamma_2)/2$. In other words, the fractional stochastic equation (53) can be used to model retarding and accelerating anomalous diffusion by an appropriate choice of the order of the fractional derivatives and fractional Gaussian noise terms. Note that retarding diffusion such as single-file diffusion can also be modeled by the fractional generalized Langevin equation [60].

5. Concluding remarks

We have shown that it is possible to model both accelerating and retarding anomalous diffusion by using fractional Langevin-like stochastic differential equations driven by one or more terms of fractional Gaussian noise. The solutions associated with some specific cases of these equations turn out to be some interesting processes in the short- and long-time limits. For example, two types of fractional Brownian motion, namely the usual standard fractional Brownian motion and the Riemann–Liouville fractional Brownian motion are the asymptotic
processes of special cases of the model. This model also includes another interesting process, namely the mixed fractional Brownian motion, which is a simple process for describing accelerating sub- and super-diffusion.

We note that the stochastic differential equations in our model can be regarded as the fractional Langevin-like equation of distributed order (10) with the weight function consisting of delta functions. One may want to consider cases with different types of weight functions in (10), such as uniform or power-law weight functions. However, the results of our previous study on fractional Langevin equations of distributed order with uniform and power-law type of weight functions indicate that such equations in general do not have closed solutions even for the case of simple fractional Langevin of distributed order driven by white noise [33]. One thus expects the situation to be even more complex when weight functions other than the delta functions are used for the multi-term fractional Langevin equation with more than one fractional noise terms.

One question of interest is that whether it is possible to model accelerating and retarding anomalous diffusion based on (26) and (42), using different Gaussian noise. If one is only interested in the asymptotic limits of the mean-squared displacement of the stochastic process, then instead of using fractional Gaussian noise in the stochastic differential equations (26) and (42), Gaussian noise with covariance which has the ‘correct’ asymptotic limits of power-law type can be used. For example, for Gaussian noise with covariance which varies as $A t^{d-2} (1 + At^{d-2} / B)$, $A$ and $B$ are positive constants, $0 < \mu, \nu < 2$. Such a covariance has respectively short and long time limits $A t^{d-2}$ and $B t^{d-2}$, respectively. Another example is the noise of Mittag–Leffler type with covariance that of the form $A t^{d-2} E_{\mu-\nu,\mu+1} (-At^{d-2} / B)$, $\mu > \nu$, which has short- and long-time asymptotic limits $A t^{d-2} / \Gamma(\mu + 1)$ and $B t^{d-2} / \Gamma(\nu + 1)$, respectively. These two examples give the possible alternatives to the fractional Gaussian noise $\xi_j$ as the driving noise. More discussion related to these cases will be given in a forthcoming paper [61].

The usual characterization of anomalous diffusion uses its mean-squared displacement (or variance in the context of this paper). However, it is well known that even for a Gaussian model mean-squared displacement does not determine completely the underlying stochastic process and hence the mechanism of the anomalous diffusion. Recent advances in particle tracking devices allow experiments to track the trajectories of a single molecule or nanoparticle in complex systems such as cells in a biological system. Mean-squared displacement obtained from the time series data gives the scaling exponent of the anomalous diffusion undertaken by such particles. Comparison of the experimental data so obtained with various models of anomalous diffusion allows one to distinguish the different possible subdiffusion mechanisms. In particular, information of single-particle trajectories allows one to test the validity of ergodic property of the associated diffusion process. A stochastic process is said to be ergodic if the ensemble average of certain physical quantity such as mean-squared displacement measured in bulk coincides with the time average of the same quantity over sufficiently long time from the single-molecule time series. The examples of the ergodic process are Brownian motion and the standard fractional Brownian motion. Another process of interest which is ergodic is the mixed fractional Brownian motion which is the sum of two independent fractional Brownian motions. On the other hand, the Riemann–Liouville fractional Brownian motion and heavy tailed continuous-time random walk are non-ergodic [62–65].

By using the single-molecule data, the comparisons of experimental data based on various models of anomalous diffusion such as the continuous-time random walk, fractional Brownian motion, fractional Levy stable motion, etc have been carried out by various authors recently [66–73]. For example, diffusion of beads in the entangled F-actin networks at shorter times exhibits continuous-time random walk behavior [66]. However, the analysis of the data of the
anomalous diffusion in crowded intracellular fluid such as cytoplasm of living cells rules out continuous-time random walk and favors the fractional Brownian motion [67]. The analysis of single-particle tracking data of lipid granules in yeast cells by Tejedor et al [68] seems to rule out continuous-time random walk and shows agreement with fractional Brownian motion, but a subsequent study [69] shows that at short times the granules perform continuous-time random walk subdiffusion, while at longer times the motion is consistent with the fractional Brownian motion. Various analyses of the biological data describing the motion of individual fluorescently labeled mRNA molecules inside live E. coli cells, a well-known experiment first conducted by Golding and Cox [70], do not lead to the consensus result. Magdziarz et al [71] showed the fractional Brownian motion as the underlying stochastic process, but subsequently Burnecki and Weron claimed that the data follow fractional Levy stable motion [72]. According to Kepten et al , the experiments on telomeres in the nucleus of the mammalian cell exhibit the fractional Brownian motion [73]. Recent study by Weigel et al [74] on the physical mechanism underlying Kv2.1 voltage gated potassium channel anomalous dynamics using single-molecule tracking showed that both ergodic (diffusion on a fractal) and non-ergodic (continuous-time random walk) processes co-exist in the plasma membrane. Although it is widely recognized that the diffusion pattern of membrane protein displays anomalous subdiffusion, there is still no agreement on the mechanisms responsible for this transport behavior. Currently, there is still no consensus on whether heavy-tailed continuous-time random walk, fractional Brownian motion, fractional Levy stable motion or some other stochastic processes can provide the correct description to anomalous diffusion in some biological systems. Thus, it is important to make use of data and information other than the mean-squared displacement or the anomalous diffusion exponent to determine the type of mechanism and the stochastic process describing the anomalous diffusion. We hope that some of the processes considered in this paper may be of relevance in describing the anomalous diffusion in biological systems.

Finally, we remark that it would be interesting to investigate the Fokker–Planck equations associated with the processes considered in this paper. The mean first passage time for some of the simpler cases can also be studied [61].

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