Remarks on the Debye Length and the Topological Susceptibility in Non-Abelian Gauge Theory

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Abstract

We study the Debye mass, $m_D$, and the topological susceptibility, $\chi$, at high temperatures in non-abelian gauge theory. Both exhibit, at some order in the perturbation expansion, infrared sensitivity. As a result, a perturbative analysis can at best provide an estimate of these quantities, subject to some uncertainty. The size of these uncertainties, particularly in the case of $\chi$, has been the subject of some debate. For the perturbative free energy, reframing an analysis of Braaten and Pisarski, the estimate and the associated error, can be understood in terms of a Wilsonian effective action for the low energy effective three dimensional theory. This action can be obtained completely from a perturbative calculation, which terminates at a finite order. This action provides the desired estimate. The size of the error follows from dimensional analysis in the low energy theory. The Debye length computation and its error can be obtained from a similar study of a non-relativistic effective theory for an adjoint scalar in three dimensions. $\chi$ requires a four dimensional analysis involving finite temperature instantons, but again the dominant sources of uncertainty are three dimensional, and we provide a procedure to estimate and an associated error. This uncertainty is of order $g^4$ relative to the leading semiclassical result, and in situations of interest is small.
1 Introduction

At high temperatures, non-abelian gauge theories undergo a phase transition to an unconstrained phase. The high temperature theory exhibits two mass or length scales. The first of these is the Debye mass, $m_D \sim gT$, loosely speaking a scale beyond which static electric charges are screened. The second arises because the theory at high temperatures and long distances becomes a three (Euclidean) dimensional Yang-Mills theory with coupling

$$g_3^2 = g^2 T$$

without matter fields. The second scale is the mass gap of this theory, on dimensional grounds, $m_{\text{mag}} = g_3^2 T$.

But the high temperature theory is not exactly a weakly coupled theory. If one attempts to formulate a perturbation theory, quantities like the free energy, and even short distance Green’s functions, suffer severe infrared divergences, and can be calculated, at best at low orders. This is usually described by saying that these quantities are logarithmically divergent at some order, diverging with an additional power of momentum at each further order. Assuming an infrared cutoff

$$m_{\text{mag}}^2 = ag^4 T^2 = ag^4 T^2,$$

each additional order makes a comparable contribution. For the free energy, divergences first arise at order $g^6$ (four loop order). Gauge invariant Greens functions, like $\langle F^2(x)F^2(0) \rangle$ similarly exhibit such divergences at high enough order.

All of this arises because the theory, at high temperatures, is a three dimensional theory, with a dimensionful coupling $g_3^2 = g^2 T$. At best, one can hope for a perturbation expansion valid for short distances or high momenta, $g_3^2 r \ll 1; g_3^2 / |p| \ll 1$. But loop corrections, even in these limits, are dominated, at sufficiently high order, by low momenta, leading to a breakdown of weak coupling.

On the other hand, if one computes a Wilsonian effective action for the three dimensional theory, integrating out momenta between scales

$$\frac{1}{\epsilon} g_3^2 < k < \Lambda \sim T$$

one should have a valid expansion in powers of $\epsilon$. The remaining contributions to physical quantities, Greens functions, and the like must be obtained from a fully non-peturbative analysis of the strongly coupled three dimensional theory. This suggests that quantities such as the free energy can be calculated as a sum of two parts: the perturbative, Wilsonian contribution, which can be obtained reliably, and the non-perturbative contribution. This latter is typically, on dimensional grounds, a power of $g_3^2$ times an unknown, dimensionless number. Assuming that the dimensionless number is of order one, this means that, with a straightforward (if possibly challenging) perturbative computation, one can

\[^{1}\text{Our discussion can be viewed as a reframing of an analysis of Braaten and Pisarski.}\]
obtain an estimate of such quantities, accompanied by an error estimate, of irreducible size.

Applied to the free energy, as we will explain in section 2, this means that one can reliably compute through order $g^4$. At order $g^6$, there is a contribution which, again, can be reliably extracted proportional to the log of the ultraviolet cutoff, and a contribution without a log which cannot be obtained perturbatively. This non-perturbative contribution represents the irreducible uncertainty.

For the Debye length, we will see that there is a similar story. The existence of a mass – or correlation length – for $A_4$ is well known. At finite temperature, one does not have the full $O(4)$ (in Euclidean space) symmetry. At one loop, if one calculates the vacuum polarization tensor, gauge invariance and the remaining $O(3)$ symmetry are enough to insure vanishing of $\Pi_{ij}$ as $\vec{q} \to 0$. However, this is not the case for $\Pi_{44}$ at one loop. If $q_0$ is the discrete frequency of the finite temperature theory, one finds that for $q_0 = 0$, as $\vec{q} \to 0$,

$$\Pi^{44}(0, \vec{q}) \to m_D^2 \equiv g^2 T^2 (N + 3N_f) \quad (1.4)$$

In coordinate space, this mass for the $A_4$ field translates into a characteristic length scale. The $A_4$ Greens function, in leading order and at large distances, is given by

$$D^{44}(0, \vec{y}) = \frac{1}{4\pi|\vec{y}|} e^{-m_D y} \quad (1.5)$$

As a result of these considerations, there is a scale, of order $\sqrt{g_3 T} \ll T$ (for small $g_3$), at which one has a three dimensional gauge theory with an adjoint scalar, $\phi$, of mass $\mu = m_D$. Corrections to the Debye length have been considered in the literature\[2, 3\]. We will consider them from two points of view. We’ll first examine the direct computation of $D^{44}(\vec{y})$ in perturbation theory. We’ll see that one can obtain a reliable estimate of the Greens function at distances parameterically large compared to $\mu^{-1}$ by a factor $(g_3^2 \log(\mu/g_3^2))^{-1}$. Beyond this scale, the computation of the Greens function, order by order in the perturbation expansion, is not under control.

But as explained in \[3\] and we elaborate further here, it is possible to define a gauge-invariant, non-perturbative Debye mass which controls the very large $r$ behavior of the Greens function. As we explain, viewing the three dimensional theory as a Minkowski theory with an adjoint scalar, the theory has a $Z_2$ symmetry.\[2\] The mass of the lightest $Z_2$ odd state controls the Euclidean large distance behavior of Green’s functions of $Z_2$-odd operators, and is naturally defied as the Debye mass. One can obtain an estimate of

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\[2\]These are statements about QCD, in the absence of weak interactions. As noted in \[3\], in theories which are not vector-like, the low energy theory may not respect the $Z_2$. In this case, strictly speaking, there is no sharp definition of the Debye mass. However the $Z_2$ breaking is often highly suppressed. For example, in non-vector-like theories without scalars, there is still a $Z_2$, related to $CP$ in the four dimensional theory. As a result, there is often a range of (large) distances where correlators invariant under the approximate $Z_2$ exhibit a rapid exponential falloff, even if, at extremely large distances, the falloff is power law.
this mass by studying the divergence structure of the perturbation theory, noting that there is logarithmic infrared sensitivity at one loop in the computation of this mass. But the structure of the perturbation expansion is complicated, with a variety of infrared singularities at higher orders. Instead, one can also use the Wilsonian language, as for the free energy, applied to a suitable non-relativistic effective action. Here there is, again, an ultraviolet divergence (the cutoff for the low energy theory is now $\mu = m_D$). Again, one obtains an estimate of the Debye length, as well as an irreducible, perturbative uncertainty.

But while providing, perhaps, a different language, for the free energy and the Debye length, this serves simply to confirm longstanding results. But our particular interest is in the nature of the semiclassical estimate of $\theta$-dependent effects at finite temperature, interesting in themselves and relevant to the problem of axion cosmology. Here the object of interest is the free energy as a function of $\theta$ and $T$, $F(\theta, T)$. Much of the literature, particularly the lattice literature, focuses on the topological susceptibility

$$\chi(T) \equiv \frac{\partial^2 F}{\partial \theta^2}. \quad (1.6)$$

The leading term as a function of $g^2$ can be computed in the dilute gas approximation, and is known[4]. At some order, one expects infrared divergences to arise as in the computation of the free energy in ordinary perturbation theory. One of the goals of this paper is to determine the order of the corrections to the leading semiclassical approximation at which these divergences arise. We will seek to determine the nature of these divergences, in order to assess the uncertainties in the standard computation. We will examine the large distance behavior of Feynman diagrams in an instanton background, determining the order of the expansion in $g^2$ about the classical solution at which first logarithmic divergences and then power law divergences arise. We will interpret the log divergence, as for the perturbative free energy, as a term in a Wilsonian action involving the log of the cutoff ($\Lambda = T$). This will, again, permit us to make an estimate of $\chi$ and to determine an irreducible uncertainty in the semiclassical computation.

Apart from the intrinsic interest of finite temperature gauge theory, our work has been motivated, in part, by arguments in the literature for large corrections to the semiclassical computation of $\chi (F(\theta, T))$. In particular, it has been asserted that at one loop there is a correction to the instanton action which is (fractionally) of order order $g \log g^2$ rather than $g^2$ and that at temperatures of interest it is numerically of order one. This effect is exponentiated in $\chi$, and as such could lead to an uncertainty of orders of magnitude. This claim has been used, in turn, to argue that lattice computations are essential to determine the behavior of hypothetical axions in the early universe. Such computations have yielded values of $\chi$ varying by orders of magnitude at relevant temperatures, both from each other and from the leading semiclassical result. [5, 6, 7, 8, 9].

The basis for these concerns is the assertion that the corrections to the Debye length which we have described above are large[2, 3] and that the Debye length acts as an infrared cutoff on the instanton size in the dilute gas approximation. As explained in [4], the Debye mass term in the effective lagrangian does provide an infrared cutoff, but
as noted in [10], the actual cutoff in the instanton scale size in the computation of $\chi$ is $T^{-1}$. As a result, as we will explain further in this paper, uncertainties associated with the cutoff on the $\rho$ integration are small.

Still, the $\theta$-dependence of the free energy will, at some point, exhibit infrared sensitivity. We will demonstrate that the expansion for the topological susceptibility, $\chi$, exhibits infrared divergences at lower order than that for the perturbative free energy:

$$\chi(T) = aT^4(1 + bg^2 + cg^4\log(g^2) + C). \quad (1.7)$$

This follows from the explicit form of the instanton, and in particular its large distance behavior.

This paper is organized as follows. In section 2, we review the high temperature behavior of the free energy in QCD. In particular, we note that the “infrared” log is also an ultraviolet logarithm from the point of view of the three dimensional effective theory, and that the coefficient of this logarithm can be reliably calculated. As a result, there is a well-justified computation of the free energy, to a fixed order in $g^2$, whose error can be reliably estimated.

In section 3, we consider the perturbative calculation of the $\phi(A_4)$ Green’s function, with a focus on the large distance, coordinate space behavior. After reviewing the leading computation, we consider broad classes of Feynman diagrams. These become progressively more singular at large distances, and large, infinite sets are actually infrared divergent. Assuming a cutoff $m_{mag}$, one can establish that the leading correction dominates, up to some maximum distance.

In section 4, we explain that, with the assumption the three dimensional theory is gapped, there is a well-defined notion of the Debye mass, and we provide a definition. Then, in section 5, we turn to the question of to what extent we can estimate this mass, i.e. to what order in the perturbation expansion, and with what level of uncertainty. Using the language of Non-Relativistic Effective Theory (NRET), we will argue that this computation is robust.

In section 6, we turn to the question of the calculation of the topological susceptibility. We review some features of the finite temperature instanton computation, explaining that, at low orders in the semiclassical computation, the dominant instanton scales, $\rho$, are of order $T^{-1}$. Then we ask about the appearance of infrared divergences in this computation. We work in coordinate space; noting that at large distances, the instanton solution falls off rapidly, so that for purposes of isolating the infrared divergence, the modifications of the relevant Greens functions from their tree level forms are small. Essentially, we are able to treat the background instanton as a perturbation. We note that individual Feynman diagrams are actually divergent already at two loops, but gauge invariance requires that these divergences cancel, and the leading infrared divergence occurs at three loops (order $g^4$). Again, we argue that the computation of the logarithmic term at three loops is robust, and gives us both an estimate of the size of $\chi$ and of irreducible uncertainties.

Finally, in section 7, we consider the implications of these observations for some physical problems. We focus on the calculation of the finite temperature axion mass,
which is proportional to $\chi$. We note, again, that the cutoff on the instanton scale size integration is of order $T^{-1}$, but stress that this distance is not only parameterically small compared to the Debye length, but it is even smaller than the scale at which the large corrections to the Debye come into play. We note that the infrared divergence at order $g^4$ implies an error in the computation of the susceptibility at the 1% level or better, which is not significant for the calculation of the axion energy density.

2 $g^2$ Expansion of the Free Energy

In this section, we consider the expansion of the free energy in perturbation theory in powers of $g^2$. For the perturbative free energy, ignoring, at first, the adjoint scalar (i.e. $A_4$), there is formally an expansion of the form:

$$F(T) = T^4 \sum_{n=0}^{\infty} a_n g^{2n}$$

(2.1)

but this expansion breaks down due to infrared divergences at a certain order. This can be understood in terms of the behavior of the three dimensional theory, with effective coupling $g^2 = g^2 T$, and an ultraviolet cutoff $\Lambda \sim T$. Let’s first ignore the adjoint scalar $\phi$ and consider the computation of a Wilsonian effective action in the pure gauge theory, integrating out physics between scales $\Lambda$ and $\epsilon \Lambda$, where $\frac{g^2}{\epsilon \Lambda} \ll 1$. Then in the Wilsonian action, one obtains for the vacuum energy (coefficient of the unit operator) a series:

$$E_0 = a \Lambda^3 (1 + \mathcal{O}(\epsilon)) + b g^2 \Lambda^2 (1 + \mathcal{O}(\epsilon)) + c g^4 \Lambda (1 + \mathcal{O}(\epsilon)) + d g^6 \log(\epsilon).$$

(2.2)

In individual Feynman diagrams, the logarithmic behavior is readily identified by power counting. Higher order terms in the expansion are suppressed by powers of $g^3/\Lambda$. This is characteristic of a superrenormalizable theory. The computation of the Wilsonian action terminates at some power of the coupling.

From the requirement $\frac{g^2}{\epsilon \Lambda} \ll 1$, we have $1 \gg \epsilon \gg g^2(T)$. So thinking of $\epsilon$ as several times $g^2$, say $\epsilon = A g^2$, where $1 \ll A \ll \frac{1}{g^2}$, we can reliably say that the Wilsonian action includes a contribution to the coefficient of the unit operator (ground state energy) of the form of equation 2.2. In the low energy, cutoff theory, there will be a contribution to the energy of size $g^6 \log(A)$, which requires non-perturbative evaluation.

Including the adjoint field in the analysis, we might expect an expansion in $\mu^2$ and $g_3^2$. However, already at one loop, the expansion is actually an expansion in $\mu$; at one loop order, there is a contribution behaving as $\mu^3 T \sim g^2 T^3 [11]$. Higher orders yield more complicated dependence on $\mu$.

Returning to the infrared perspective we discussed in the introduction, we saw that, beyond the $g^6 \log g$ term, there are an infinite number of perturbative contributions which are nominally of order $g^6$. From the perspective of the three dimensional effective theory, assuming that the vacuum energy is well-defined, after integrating out modes well above
Figure 1: The leading contribution to the $\phi$ self-energy, $\Sigma(p)$.

$g_3^2$, any further contribution necessarily scales as $g_3^6$. Moreover, these contributions are not perturbatively accessible. Formally, we can attempt to tame the infrared divergences by various strategies. Apart from introducing a magnetic mass or simply cutting off momentum integrals at that scale, we can resum the contributions to the propagator, yielding $1/(g_3^2 k)$ behavior at small $k$. But it is easy to check that all diagrams beyond four loop order are of order $g_3^6$.

3 Perturbative Computation of the Greens function at Large Distances and its Limitations

In the next section we will see that from knowledge of the three dimensional Minkowski theory we can obtain information about the large distance behavior of the Euclidean theory. We first consider the direct computation of the Green’s function in coordinate space, and then move on to non-perturbative considerations.

At tree level, the Euclidean Greens function in coordinate space can be evaluated by Fourier transforming the momentum space expression. Performing the angular integrals, the remaining momentum integral (integral over $p$) can be treated as an integral in the complex plane. Deforming into, say, the upper half plane one picks up the pole at $p = i\mu$. This corresponds to the on shell point in the Minkowski description.

As one works to higher order in perturbation theory, the self energy, $\Sigma$, as we will shortly see, has a branch cut starting at $p = i\mu$. So now for the Greens function, deforming the contour, one encircles the branch cut. Calling the new integration variable $\delta$, the integral involves a factor $e^{-(m+\delta)r}$. For large $r$, it is dominated by $\delta \sim r^{-1}$. In the Minkowski language, this corresponds to $\phi$ being off shell by an amount of order $\mu \delta \sim \mu/r$. So in momentum space, we are interested in $\Sigma(p)$ for $p^2 = \mu^2 - 2\mu\delta$, with $\delta$ small. We want to ask: how small can $\delta$ be and still yield a reliable estimate.

Consider, first, the one-loop contribution to the self-energy, $\Sigma(p)$ (figure 1). Work slightly off shell:
Figure 2: One class of diagrams singular in the limit $\delta \to 0$.

\[ p^2 = \mu^2 - 2\mu\delta. \]  \hfill (3.1)

Then

\[ \Sigma = Ng^2 \int \frac{d^3k}{(2\pi)^3} \frac{4\mu^2}{k^2 + i\epsilon} \frac{1}{2\mu(-\delta + k^0) + i\epsilon} \]  \hfill (3.2)

\[ = \frac{Ng^2}{\pi} \mu \log(\mu/\delta) \]

Adding $\alpha k^\mu k^\nu/\mu$ to the propagator, it is easy to check that the logarithmic term in this expression is gauge invariant.

As we will now show, successive orders in the expansion of $\Sigma$ are progressively more singular in $\delta$. Moreover, for fixed $\delta$, one encounters actual infrared divergences at two loop order and beyond.

We can consider several classes of higher order perturbative corrections to $\Sigma(p)$ to illustrate the behavior of the perturbation expansion for small $\delta$ and to determine where it breaks down. There are three issues:

1. Singular behavior for small $\delta$

2. Actual infrared divergences

3. Divergent series for some value of $\delta$ and plausible infrared cutoff.

One class of diagrams involves “rainbows” of gluons emitted by $\phi$ (figure 2). For these, a typical contribution is of the form:

\[ g^{2n} \int \frac{d^3k_1 \cdots d^3k_n}{(2\pi)^{3n}k_1^2 \cdots k_n^2} \frac{1}{2\mu(k_1 + \delta)} \frac{1}{2\mu(k_1 + k_2 + \delta)} \cdots \frac{1}{2\mu(k_1 + k_2 + \delta)} \frac{1}{2\mu(k_1 + \delta)} \]  \hfill (3.3)

\[ \sim g^{2n} \delta^{-(n-1)}. \]
Figure 3: A second class of diagrams singular in the limit $\delta \to 0$, with actual infrared divergences.

Figure 4: One class of diagrams singular in the limit $\delta \to 0$.

If we restrict

$$\delta \geq \frac{Ng^2}{\pi^2}\log(\mu/\delta),$$  \hspace{1cm} (3.4)

then the perturbation series would appear to be an expansion in powers of $1/\log(\delta)$.

But other classes of diagrams leads to a stricter requirement. The first consists of “rainbows” as above, but each with a one loop vacuum polarization correction on the gluon line (figure 3). Such diagrams, by power counting, have an actual infrared divergences, and are also more singular as $g^4/\delta^2$ for each additional loop. In other words, assuming an infrared cutoff of order $g^2$, each additional loop comes with a factor:

$$\frac{g^4\mu \log(m_{mag}/\mu)}{\delta^2}. \hspace{1cm} (3.5)$$

So, for $\delta$ satisfying the condition $3.4$, these contributions are all nominally of similar size, each suppressed by $\log(\delta) \sim \log(\mu/g^2)$ relative to the leading contribution.

A third class of diagrams involve propagator corrections to the gluon line of the one loop contribution (figure 4). With $n$ one-loop corrections to the gluon line, these are infrared divergent, behaving as

$$\delta \Sigma^{(n)} \sim (g^2)^{n+1} \int \frac{d^3 k}{k^2 k^n} \sim \frac{g^2\mu (g^2)^n}{n! \delta^{\overline{n}} m_{mag}}. \hspace{1cm} (3.6)$$
Figure 5: One class of diagrams singular in the limit $\delta \to 0$.

These diagrams are individually suppressed at large $\delta$, only by a single power of $\delta$; the series $\sum_n \frac{1}{n}$ is log divergent. Assuming that this sum is of order $\log(\mu/m_{mag})$, this requires, again, that $\delta \gg g_3^2 \log(\mu/m_{mag})$.

A similar requirement arises from the class of diagrams in figure 5. These behave, with $n$ one-loop corrections to the $\phi$ propagator, as

$$
\delta \Sigma^{(n)} \sim g_3^2 \int \frac{d^3k}{k^2(k+\delta)^n} (g_3^2 \log(\delta/\mu))^n \sim \frac{g_3^2}{n} (g_3^2 \log(\delta/\mu))^n.
$$

So at each order in $n$, we have a comparable contribution, suppressed by a logarithm relative to our leading contribution. The sum would appear poorly behaved. But if we sum before integration, we obtain a correction to the leading contribution down by a logarithm, in other words, of order $g_3^2$.

Of course, there are many other diagrams – mixtures of these various types and other topological classes altogether. The main lesson is that, at best, one can calculate for $\delta \gg \delta_0 = \frac{g_3^2}{2\pi} \log(\mu/g_3^2)$. With this understanding of the behavior of $\Sigma$, let’s consider the fourier transform of the propagator in Euclidean space,

$$
G(r) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{p} \cdot \vec{x}} \frac{1}{k^2 + \mu^2 - \Sigma(p)}
$$

Performing the angular integrals leaves:

$$
G(r) = \int_{-\infty}^{\infty} \frac{dp}{(2\pi)^2} \frac{1}{i r^2 + \mu^2 - \Sigma(p)}. \quad (3.8)
$$

The integrand has branch cuts starting at $p = \pm i\mu^2$. Treating as a contour integral and deforming so as to encircle the branch cut in the upper half plane, the integral becomes:

$$
G(r) = \int_{\mu}^{\infty} \frac{d\delta p}{(2\pi)^2} \frac{1}{e^{-\delta r} \text{Disc} \frac{1}{2\delta + \Sigma(\delta)}}.
$$

The main noteworthy feature here is that for large $r$, $\delta \sim \frac{1}{r}$. So for sufficiently large $r$,

$$
r \approx (\log(\frac{\mu}{g_3^2}) g_3^2)^{-1} \quad (3.11)
$$
one has lost control of the expansion of $\Sigma(p)$, and the computation of the propagator has broken down. Still, parameterically in $g_3$, there is a large range of distance where the propagator can be reliably estimated.

In the following sections, we will see that general arguments establish that the falloff of the propagator is that of a massive scalar field. The corrections to the mass can be estimated with a definite error. The results are consistent with our estimates above, in that the propagator is exponentially modified from its perturbative form at distances $r > \delta_0$, but the error on the exponent is controlled.

4 Defining the Debye Mass

At high temperatures, as is well known, four dimensional field theories with massless fields behave as three dimensional systems. In the case of Abelian gauge theories with light matter, and general non-abelian theories, the field $A_4$ of the four dimensional theory behaves like a massive field, with mass parameterically less than $T$ by a single power of $g$, the gauge coupling. In this section, we’ll consider a theory with a massive adjoint field, $\phi$, with lagrangian mass parameter $\mu^2$, small compared to some ultraviolet cutoff, and attempt to determine the behavior of the charged field propagator at large distances.

In QCD, the resulting effective theory has a $Z_2$ symmetry, arising from the four dimensional symmetry of the Euclidean theory $A_4 \rightarrow -A_4, \ t \rightarrow -t$. Calling $A_4 \equiv \phi$ (with corresponding transformations on the fermions), the symmetry takes $\phi \rightarrow -\phi$ in the three dimensional theory.

To extract properties of the theory at large Euclidean distances, it is helpful to consider the theory continued to three dimensional Minkowski space (the argument which follows is familiar in lattice gauge theory and other contexts). We expect in the three dimensional system, there will be states odd under the $Z_2$, which are bound states of $\phi$ and gluons. Suitable interpolating fields for such states would be:

$$\Phi = \text{Tr}(\phi F_{\mu\nu}^2),$$

and operators with more $F_{\mu\nu}$ and/or $D_\mu$ factors.

If we continue the three dimensional Euclidean theory to Minkowski space, we can write a spectral representation of the Green’s function:

$$G_\Phi(x) = \langle \Phi(x)\Phi(0) \rangle = \int dM \rho(M) D_F(x, M).$$

In $\int \rho(M)$ is the free propagator for a field of mass $M$. If the spectrum is gapped, we can write

$$\rho(M) = Z\delta(M - M_{\text{phys}}) + \theta(M - M_0)f(M)$$

where we will refer to $M_{\text{phys}}$ as the physical mass of the lightest $Z_2$-odd particle, and $M_0 > M_{\text{phys}}$. Then we can continue eqn. $\int \rho(M)$ back to Euclidean space, and (in three
dimensions) show that the asymptotic behavior of the propagator is:

\[ G_{\phi}(x) = \frac{Z}{8\pi r} e^{-M_{\text{phys}} r} \]  

(4.4)

for \( r = |\vec{x}| \gg M_{\text{phys}}^{-1} \). So the spatial falloff of the propagator is determined in terms of \( M_{\text{phys}} \). This quantity is gauge invariant. If we can estimate \( M_{\text{phys}} \) in some regime where the perturbation expansion is reliable, then, using unitarity, we can continue to the regime of arbitrarily large distance, with an error in the estimate which we can hope to control. \( M_{\text{phys}} \) we define to be the Debye mass.

5 Calculating the Debye Mass

In this section, we determine how well one can estimate the Debye mass, and compare with the analysis of section 3.

5.1 Non-Relativistic Effective Theory (NRET)

In our discussion of the perturbative free energy, we were able to argue that the infrared logarithm at four loop order was robust; it could be understood as a renormalization of the unit operator of the effective three dimensional theory at low energies. Given that our focus, for the Debye mass, is also on infrared (long distance) issues, we can ask whether we can isolate a similar ultraviolet logarithm.

We can consider the problem from the point of view of non-relativistic effective field theory. For scalars (see, for example, [12]), one conventionally defines:

\[ \phi = \frac{1}{\sqrt{2m}} e^{-imv \cdot x} \chi; \quad v^2 = 1. \]  

(5.1)

The action for \( \chi \) is then:

\[ i\chi^* v_\mu D^\mu \chi. \]  

(5.2)

At one loop, there is an ultraviolet and infrared divergent correction to the \( \chi \) propagator, similar to the one loop correction to \( \Sigma \) which has been our focus:

\[ \frac{g_3^2 N}{2\pi} \log(\Lambda/m_{\text{mag}}). \]  

(5.3)

For the non-relativistic theory, the ultraviolet cutoff is the mass, \( \mu \). This divergence is cured by a counterterm for the operator \( \chi^\dagger \chi \); it corresponds to a mass shift in the non-relativistic theory. Because the operator has dimension 2, and \( g_3^2 \) has dimension one, there is only an ultraviolet divergent correction at one loop. We can again think of the logarithm as a term arising in a Wilsonian effective action from integrating out
high energy modes of the gauge field, in this case, between the ultraviolet cutoff, $\mu$, and a scale

$$\lambda = \frac{1}{\epsilon} g_3^2.$$  \hfill (5.4)

In principle $\epsilon$ is a small number, $1 \gg \epsilon >> \frac{g_3^2}{\mu}$.

This counterterm eliminates all $\mu$-dependence in the perturbation expansion. The theory does contain the parameter $\log \epsilon$, which can be thought of as an order one number. Thus the effective theory has only the dimensionful parameter, $g_3^2$. So necessarily any further correction to the mass, beyond the counterterm, is a constant times $g_3^2$. In other words, this argument establishes that

$$m^2 = \mu^2 + \frac{g_3^2 N}{2\pi} (\log(\mu/g_3^2) + A).$$ \hfill (5.5)

The results of the NRET analysis can be understood in a more conventional Feynman diagram analysis. Dividing up the integration over the gluon momentum into two regions, one with momentum $k > k_0 = \frac{1}{\epsilon m_{mag}}$, and one with $k < k_0$, we also take the external momentum to satisfy

$$p^2 = \mu^2 + \delta_0 + \delta'; \quad \delta_0 = \frac{N g^2}{2\pi} \log(\mu/k_0).$$ \hfill (5.6)

where $\delta_0$ represents the lowest order mass shift. We take the propagator to be the resummed propagator, with the one loop contribution to $\Sigma$. With this choice, all higher loop contributions to $\Sigma$ are dominated by the infrared, and are thus insensitive to $\mu$, apart from an overall factor. They behave as $(g_3^2 \mu)(g_3^2 / \delta')^n$. The mass shell condition takes the form

$$\Sigma(\delta') = \mu \delta'.$$ \hfill (5.7)

So assuming $\Sigma(\delta') = g_3^2 \mu A(\frac{g_3^2}{\delta'})$ we have that $\delta' = ag_3^2$ for some constant $a$.

5.2 Comparison of NRET Analysis to Conventional Perturbation Theory

The argument based on non relativistic effective theory gives a sharp, principled argument that one can calculate, perturbatively, a correction of order $g_3^2 \mu \log(\mu/g_3^2)$ to the mass of $\phi$, while the remaining correction is of the form $A g_3^2$. Note that this is in accord with our discussion of the previous section. We an compute, in conventional perturbation theory, the Green’s function out to distances of order $r \ll \frac{1}{g_3^2 \log(\mu/g_3)}$. Beyond this distance scale, the corrections quickly become large compared to one. The non-perturbative analysis gives us a reliable estimate of the Green’s function to scales of order $r \sim g_3^{-2}$. Note that both of these scales are parameterically large compared to $\mu^{-1}$. As we have remarked, for the question of $\theta$ dependence of the free energy, one is interested in much shorter distance scales.
5.3 Implications of the Debye Mass Calculation for the Topological Susceptibility

As we have remarked, one situation where it has been suggested that large corrections to the Debye mass might be important is in instanton computations of the free energy (topological susceptibility) at high temperatures \([8, 9, 13]\). It has been argued that the instanton computation is proportional to a large power of an infrared cutoff, and that this cutoff is the Debye length. But as noted in \([10]\) the actual cutoff is \(T\) (or \(T^{-1}\) in coordinate space)\([4]\). As we have now seen, this is a regime where perturbation theory is valid, at least until one encounters magnetic divergences at high orders. These corrections are not likely to be numerically large, or terribly important for estimating, for example, the axion dark matter density\([10]\).

6 Infrared Sensitivity in the Instanton Computation

While we have argued that the Debye mass is not the relevant cutoff for the instanton computation, we do expect actual infrared divergences to appear at some order; in other words, we do not expect to be able to perform a semiclassical computation of the topological susceptibility to arbitrary accuracy. In this section, after first considering the question of what does play the role of the cutoff on the \(\rho\) integration at large \(\rho\), we turn to a determination of the order in perturbation theory infrared divergences actually arise in the computation of the topological susceptibility. In the spirit of our earlier Wilsonian analyses, we use this result to determine the irreducible uncertainty of the semiclassical computation.

6.1 Instantons as a Perturbation at Large Distances

At finite temperatures, for \(r = |\vec{x}| \gg \beta\), the instanton takes the form:

\[
A^{ia} = \frac{\epsilon_{aij} x^j}{(r^2 + r^3/\rho^2 T)} \quad A^{4a} = -\frac{x^a}{(r^2 + r^3/\rho^2 T)}
\]

From a three dimensional perspective, the instanton solution is well behaved at large distances, with \(E\) and \(B\) fields falling off as \(1/r^2\), but singular at short distances. The temperature and the scale size (which is of order \(T\)) act as short distance cutoffs, yielding a finite action. This is complicated to describe from our Wilsonian perspective. In addition to generating contributions to local operators, the short distance physics yields boundary conditions for the three dimensional classical solutions as well as an integration measure. Rather than describe the process of integrating out short distance physics in this way, we will content ourselves with an examination of corrections to the leading semiclassical approximation, isolating contributions which behave as \(\log(T/m_{mag})\) and \(\log(\rho m_{mag})\). From this analysis, we will infer the extent to which one can estimate \(\chi\), and the irreducible uncertainty.
If we want to investigate actual infrared divergences, we need to study loop corrections to the instanton computation in this background. Because we are interested in effects at large distances, we are interested in integration regions where the fields of the instanton are small, and Greens functions of the fluctuating fields are close to their free field expressions. In particular, we can attempt to treat the instanton as a perturbation.

By this we mean that we break up the fields as

\[ A_4 = A_{\text{inst}}^4 + a_4; \quad A^i = A_{\text{inst}}^i + a^i. \]  

(6.1)

Then there are interactions involving two fluctuating fields, \( a^i \) proportional to one or two powers of the background field, and three powers of the fluctuating fields and one power of the background field. For the instanton fields, we will take the large \( r \) limits. We will need to integrate over collective coordinates for translations, dilatations and rotations. The integral over \( \rho \) will be controlled by the same exponential terms as in the leading approximation, up to small corrections. As a result, the dominant \( \rho \) is of order \( T^{-1} \). The rotational collective coordinate is simple to deal with as \( \chi \) is itself rotational and gauge invariant.

The translational collective coordinate, \( \vec{x}_0 \), requires closer attention. In perturbation theory at order \( n \), we will have vertices labeled by \( x_i, \ i = 1, \ldots n \). At vertices with insertions of the instanton field, \( A_{\text{inst}} = A_{\text{inst}}(\vec{x}_i - \vec{x}_0) \). We will also have products of free Greens functions (and derivatives), \( \Delta(x_i - x_j) \). So if we shift \( \vec{x}_i \to \vec{x}_i + \vec{x}_0 \), the integral over \( \vec{x}_0 \) factors out, yielding the factor of volume appropriate to the three dimensional vacuum energy. The free propagators, in coordinate space, are simply

\[ \Delta(\vec{x}_i - \vec{x}_j) = \frac{g_3^2}{4\pi |\vec{x}_i - \vec{x}_j|}. \]  

(6.2)

The asymptotic behavior of the instanton is:

\[ (A_4)_{\text{inst}}^a = \frac{\rho^2 T x^a}{x^3}; \quad (A^i)_{\text{inst}}^a = \frac{\rho^2 T \epsilon_{aij} x^j}{x^3}. \]  

(6.3)

So formally, in the large distance regime we have an expansion in powers of \( g_3^2 \) and \( \rho' = \rho^2 T \). We can power count on diagrams at \( n + 1 \) loops. These will have \( n \) factors of \( g_3^2 \). Then if there are \( m \) insertions of the background field we have \( m \) factors of \( \rho' = \rho^2 T \). Schematically, the graph has the structure

\[ (g_3^2)^n \rho'^m \prod \int d^3 x_{i}^{m+2n} \partial_i^{m+2n} \frac{1}{[\vec{x}_i - \vec{x}_j]^{3n+m}} \]  

(6.4)

where the partial derivatives are meant to indicate vertices with derivatives and the factors of \( \frac{1}{|\vec{x}_i - \vec{x}_j|} \) to indicate the number of propagators. Then we can assign a ”superficial degree of divergence”, \( n - m \) to each graph. Then if

1. \( n < m \), the graph has power law divergence in the ultraviolet, corresponding to domination of the contribution to the Wilsonian action by high mo’menta. Such diagrams will yield powers of \( T \) relative to the leading contribution.
2. $n = m$, the graph is logarithmically divergent in the ultraviolet and infrared, similar to the $g^6$ contributions to the perturbative free energy.

3. $n > m$, the diagram exhibits a power law divergence in the infrared, and should be thought of as a contribution from the low energy, three dimensional theory.

For $n = m = 1$, however, the relevant Feynman diagrams vanish. So the action indicates infrared sensitivity first at order $g^4 \rho^2$ (figure 6).

6.2 Subtleties at Two Loops

At two loop order, there are diagrams which, individually, are ultraviolet and infrared divergent. These arise if, for example, we consider an insertion of two instanton fields at a point, and integrate over the instanton pair. Graphically, this (and similar contributions) correspond to examining the effects of the instanton on the large distance behavior of the $A_i$ two-point function. This effect can be summarized in terms of the insertion of a local operator. Here it is necessary to consider

$$\langle A_i(\vec{z}_1)A_j(\vec{z}_2) \rangle$$

The corrections which arise from insertions of two powers of the background instanton field have the form:

$$\delta \langle A_i(\vec{z}_1)A_j(\vec{z}_2) \rangle = g_3^2 \int \frac{d\rho}{\rho^4} \int d^3 x_0 f(\rho) \int d^3 z_3 \frac{1}{|\vec{z}_1 - \vec{z}_3|} \frac{1}{|\vec{z}_2 - \vec{z}_3|} A_{\text{inst}}^2(\vec{z}_3 - \vec{x}_0).$$

Shifting $z_3 \to z_3 + x_0$, if one estimates the integral over $z_3$ by ignoring the $z_3$ dependence of the first two propagators, one has:

$$\delta \langle A_i(\vec{z}_1)A_j(\vec{z}_2) \rangle = g_3^2 \int \frac{d\rho}{\rho^4} \int d^3 x_0 f(\rho) \frac{1}{|\vec{z}_1 - \vec{x}_0|} \frac{1}{|\vec{z}_2 - \vec{x}_0|} \int d^3 z_3 A_{\text{inst}}^2(\vec{z}_3).$$

The $z_3$ integral is convergent and dominated by scales of order $\rho$. Since $\rho \ll |\vec{z}_1|, |\vec{z}_2|$, this is a short distance effect. This is the result one would obtain from a term in the
effective lagrangian

$$\delta \mathcal{L} = \mu^2 (A^i)^2, \quad (6.8)$$

with $\mu^2$ the result of the $z_3$ integral above. But such a term is not gauge invariant, so the leading short distance contributions must cancel. The lowest dimension gauge invariant operator is

$$\delta \mathcal{L} = F_{ij}^2. \quad (6.9)$$

This insertion of this operator at two loops does not lead to an expression which is infrared divergent.

So the leading divergence arises at three loops. The form of the susceptibility is:

$$\chi(T) = \chi_0(T)(1 + a g_3^2/T + b g_3^4 \log(T) + c g_3^4) \quad (6.10)$$

where $a$ and $b$ can be computed in the semiclassical approximation, but $c$ requires a lattice computation. This last term represents, again, the irreducible uncertainty in the semiclassical analysis.

### 7 Conclusions

We have studied two physical quantities in high temperature QCD: the Debye mass and the topological susceptibility. For $\chi$, we have seen that uncontrolled infrared divergences first arise at order $g^4$. The leading semiclassical approximation would appear to be reliable at the fraction of a percent level.

As we have reviewed, the Debye length – which determines the exponential falloff of the $A_4$ Green’s function (or the gauge invariant Green’s function which we have discussed) at very large distances – is not critical to understanding the behavior of the susceptibility. But it is interesting in its own right. We have explained why, even though fundamentally a strong coupling problem, one can obtain a reliable estimate for this length. This involves carefully considering the fact that the system is gapped and the structure of the perturbation series. We have seen that, as a Minkowski theory, one can calculate the leading order contribution to the position of the pole, which is larger by a logarithm then the expected uncertainties. Infrared divergences appear in this computation at precisely the order where one expects confinement effects to be important. We have stressed, as in [10], that these large corrections to the Debye mass are not important for the calculation of the susceptibility.

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