Vortex Lattices
in Quantum Mechanics

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Abstract

Vortex lattices are constructed in terms of linear combinations of solutions for Schrödinger equation with a constant potential. The vortex lattices are mapped on the spaces with two-dimensional rotationally symmetric potentials by using conformal mappings and the differences of the mapped vortex-patterns are examined. The existence of vortex dipole and quadrupole is also pointed out.

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Vortices play an interesting roles in various aspects of present-day physics such as vortex matters (vortex lattices) in condensed matters [1,2], quantum Hall effects [3-7], various vortex patterns of non-neutral plasma [8-11] and Bose-Einstein gases [12-16]. The vortices are well known objects in hydrodynamics and has been investigated in many aspects [17-21]. In quantum mechanics hydrodynamical approach was vigorously investigated in the early stage of the development of quantum mechanics [22-29]. The fundamental properties of vortices in quantum mechanics were extensively examined by Hirschfelder and others [30-34] and the motions of vortex lines were also studied [35,36]. Recently an analysis of vortices has been carried out on the basis of non-linear Schrödinger equations [37,38]. The vortex dynamics will hold an important position in quantum mechanical problems. In vortex dynamics, however, the construction of various vortex patterns from fundamental dynamics is still not very clear. Recently Kobayashi and Shimbori has proposed a way to investigate vortex patterns from the special solutions of Schrödinger equations [39]. By using the conformal mappings the article has shown that the special solutions with zero energy eigenvalue in the two dimensional parabolic potential barriers (2D PPB) expressed by $V = -m\gamma^2(x^2 + y^2)/2$ [40] can be extended to all rotationally symmetric potentials and the solutions are infinitely degenerate as same as those in the 2D PPB [39]. It has also been shown that the infinite degeneracy can be an origin of variety of vortex patterns which appear at nodal points of wave functions and this idea can be applied in three dimensions. In the article a few examples of simple vortex patterns are presented. In this letter some patterns of vortex lattices that are constructed from simple linear combinations of the infinitely degenerate states will be studied.

Let us start from the arguments in two dimensions [39]. In two dimensions the eigenvalue problems with the energy eigenvalue $\mathcal{E}$ are explicitly written as

$$[-\frac{\hbar^2}{2m} \Delta + V_a(\rho)] \psi(x, y) = \mathcal{E} \psi(x, y),$$

where $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$, the rotationally symmetric potentials are generally given by $V_a(\rho) = -a^2 g_a \rho^{2(a-1)}$, with $\rho = \sqrt{x^2 + y^2}$, $a \in \mathbb{R}$ ($a \neq 0$), and $m$ and $g_a$ are, respectively, the mass of the particle and the coupling constant. Note here that the eigenvalues $\mathcal{E}$ should generally be taken as complex numbers that are allowed only in conjugate spaces of Gel’fand triplets [41]. Note also that $V_a$ represents repulsive potentials for $(g_a > 0, \ a > 1)$ and $(g_a < 0, \ a < 1)$ and attractive potentials for $(g_a > 0, \ a < 1)$ and $(g_a < 0, \ a > 1)$.

Let us consider the conformal mappings

$$\zeta_a = z^a, \quad \text{with} \quad z = x + iy.$$  \hspace{1cm} (2)

We use the notations $u_a$ and $v_a$ defined by $\zeta_a = u_a + iv_a$ that are written as

$$u_a = \rho^a \cos a\varphi, \quad v_a = \rho^a \sin a\varphi,$$  \hspace{1cm} (3)
where $\varphi = \arctan(y/x)$. In the $(u_a, v_a)$ plane the equations (1) are written down as

$$a^2 \rho_a^{2(1/a)} \left[ -\frac{\hbar^2}{2m} \Delta_a - g_a \right] \psi(u_a, v_a) = \mathcal{E} \psi(u_a, v_a),$$

(4)

where $\Delta_a = \partial^2/\partial u_a^2 + \partial^2/\partial v_a^2$. We can rewrite the equations as

$$\left[ -\frac{\hbar^2}{2m} \Delta_a - g_a \right] \psi(u_a, v_a) = a^{-2} \mathcal{E} \rho_a^{2(1-a)/a} \psi(u_a, v_a).$$

(5)

It is surprising that the equations become same for all $a \neq 0$ when the energy eigenvalue $\mathcal{E}$ have zero value. That is to say, for $\mathcal{E} = 0$ the equations have the same form as that for the free particle with the constant potentials $g_a$ as

$$\left[ -\frac{\hbar^2}{2m} \Delta_a - g_a \right] \psi(u_a, v_a) = 0.$$

(6)

It should be noticed that in the case of $a = 1$ where the original potential is a constant $g_1$ the energy does not need to be zero but can take arbitrary real numbers, because the right-hand side of (5) has no $\rho$ dependence. In the $a = 1$ case, therefore, we should take $g_1 + \mathcal{E}$ instead of $g_1$. Here let us briefly comment on the conformal mappings $\zeta_a = z^a$. We see that the transformation maps the part of the $(x, y)$ plane described by $0 \leq \rho < \infty$, $0 \leq \varphi < \pi/a$ on the upper half-plane of the $(u_a, v_a)$ plane for $a > 1$ and the lower half-plane for $a < -1$. Note here that the maps on the part of the $(u_a, v_a)$ plane with the angle $\varphi_a = \varphi + \alpha$ can be carried out by using the conformal mappings

$$\zeta_a(\alpha) = z^a e^{i\alpha}.$$

(7)

In the maps the variables are given by

$$u_a(\alpha) = u_a \cos \alpha - v_a \sin \alpha, \quad v_a(\alpha) = v_a \cos \alpha + u_a \sin \alpha.$$

(8)

The relations $u_a(0) = u_a$ and $v_a(0) = v_a$ are obvious.

It is trivial that the equations for all $a$ have the particular solutions

$$\psi_a^{\pm}(u_a) = N_a e^{\pm ika u_a}$$

(9)

$$\psi_a^{\pm}(v_a) = N_a e^{\pm ika v_a}, \text{ for } g_a > 0$$

(10)

and

$$\phi_a^{\pm}(u_a) = M_a e^{\pm ika u_a}$$

(11)

$$\phi_a^{\pm}(v_a) = M_a e^{\pm ika v_a}, \text{ for } g_a < 0$$

(12)

where $k_a = \sqrt{2m|g_a|}/\hbar$, and $N_a$ and $M_a$ are in general complex numbers. General solutions should be written by the linear combinations of (9) and (10) for $g_a > 0$ and those of (11) and (12) for $g_a < 0$. Examples for the 2D PPB with $g_a > 0$ are presented in ref.
In the following investigations we shall concentrate our attention on the solutions of (9) and (10) that are expressed in terms of plane waves. Hereafter we use the notations \( u_a(\alpha) \) and \( v_a(\alpha) \) with \(-\pi \leq \alpha < \pi\), which are used in the conformal mappings of (7). We see that
\[
e^{\pm ik_a u_a(\alpha)} \quad \text{and} \quad e^{\pm ik_a v_a(\alpha)}
\]
are the solutions of (6). (For details, see ref. [39].)

We have to comment on the degeneracy of the solutions. The origin of the infinite degeneracy can easily understand in the case of the 2D PPB [40]. It is known that energy eigenvalues of 1D PPB are given by pure imaginary values \( \mp i(n + 1/2)\hbar\gamma \) with \( n = 0, 1, 2, \ldots \) [42-47]. From this result we see that the energy eigenvalues of the 2D PPB, which are composed of the sum of the 1D-PPB eigenvalues with the opposite signs such as \( i(n_x - n_y)\hbar\gamma \) with \( n_x \) and \( n_y = 0, 1, 2, \ldots \), include the zero energy for \( n_x = n_y \).

It is apparent that all the states with the energy eigenvalues \( i(n_x - n_y)\hbar\gamma \) are infinitely degenerate [40]. This means that all the rotationally symmetric potentials have the same degeneracy for the zero energy states. By putting the wave function \( f^\pm(u_a; v_a) \) into (6) where \( f^\pm(u_a; v_a) \) is a polynomial function of \( u_a \) and \( v_a \), we obtain the equation
\[
[\Delta_a \pm 2ik_a \frac{\partial}{\partial u_a}] f^\pm(u_a; v_a) = 0. \tag{13}
\]

A few examples of the functions \( f \) are given by [40]
\[
\begin{align*}
f^0_0(u_a; v_a) & = 1, \\
f^1_1(u_a; v_a) & = 4k_a v_a, \\
f^2_2(u_a; v_a) & = 4(4k_a^2 v_a^2 + 1 \pm 4ik_a u_a).
\end{align*} \tag{14}
\]

We can obtain the general forms of the polynomials in the 2D PPB, which are generally written by the multiple of the polynomials of degree \( n \), \( H_n^\pm(\sqrt{2k_2 x}) \), such that
\[
f^\pm_n(u_2; v_2) = H_n^\pm(\sqrt{2k_2 x}) \cdot H_n^\mp(\sqrt{2k_2 y}), \tag{15}
\]
where \( x \) and \( y \) in the right-hand side should be considered as the functions of \( u_2 \) and \( v_2 \) [40]. Since the form of the equations (6) is the same for all \( a \), the solutions can be written by the same polynomial functions that are given in (15) for the PPB. That is to say, we can obtain the polynomials for arbitrary \( a \) by replacing \( u_2 \) and \( v_2 \) with \( u_a \) and \( v_a \) in (15).

Note that the polynomials \( H_n^\pm(\xi) \) with \( \xi = \sqrt{m\gamma/\hbar x} \) are defined by the solutions for the eigenstates with \( E = \mp i(n + 1/2)\hbar\gamma \) in 1D PPB of the type \( V(x) = -m\gamma^2 x^2/2 \) and they are written in terms of the Hermite polynomials \( H_n(\xi) \) as
\[
H_n^\pm(\xi) = e^{\mp in\pi/4} H_n(e^{\mp i\pi/4} \xi). \tag{16}
\]
(For details, see ref. [45,46].) The states expressed by these wave functions belong to the conjugate spaces of Gel’fand triplets of which nuclear space is given by Schwarz space.
Actually we easily see that the wave functions cannot be normalized in terms of Dirac’s delta functions except the lowest polynomial solutions \[39\].

The extension to three dimensions can easily be carried out in the cases with potentials that are separable into the \((x, y)\) plane and the \(z\) direction such that

\[
V(x, y, z) = V_0(\rho) + V(z).
\]

When the energy eigenvalues of the \(z\) direction is given by \(E_z\), we obtain the same equation as (6) for \(E - E_z = 0\). (For the \(a = 1\) case \(g_1 + E - E_z > 0\) should be taken.) If we take the free motion with the momentum \(p_z\) for the \(z\) direction, \(E_z = p_z^2/2m\) should be taken. It is important that the total energy \(E\) is in general not equal to zero in the three dimensions. Note that wave functions for the separable potentials are written by the product such as \(\psi(x, y, z) = \psi(x, y)\psi(z)\). Hereafter we shall not explicitly write \(\psi(z)\) in the wave functions.

Let us study vortices that appear in the linear combinations composed of the solutions with the polynomials of (15). Before going into the details we briefly describe vortices in quantum mechanical hydrodynamics. The probability density \(\rho(t, x, y)\) and the probability current \(j(t, x, y)\) of a wave function \(\psi(t, x, y)\) in non-relativistic quantum mechanics are defined by

\[
\rho(t, x, y) \equiv |\psi(t, x, y)|^2,
\]
\[
j(t, x, y) \equiv \text{Re} \left[ \psi(t, x, y)^* \left(-i\hbar \nabla\right) \psi(t, x, y) \right]/m.
\]

They satisfy the equation of continuity \(\partial \rho/\partial t + \nabla \cdot j = 0\). Following the analogue of the hydrodynamical approach [17-21], the fluid can be represented by the density \(\rho\) and the fluid velocity \(v\). They satisfy Euler’s equation of continuity

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0.
\]

Comparing this equation with the continuity equation, we are thus led to the following definition for the quantum velocity of the state \(\psi(t, x, y)\),

\[
v \equiv \frac{j(t, x, y)}{|\psi(t, x, y)|^2},
\]

in which \(j(t, x, y)\) is given by (17). Notice that \(\rho\) and \(j\) in the present cases do not depend on time \(t\). Now it is obvious that vortices appear at the zero points of the density, that is, the nodal points of the wave function, where the current \(j\) does not vanish \[39\]. We should here remember that the solutions of (6) degenerate infinitely. This fact indicates that we can construct wave functions having the nodal points at arbitrary positions in terms of linear combinations of the infinitely degenerate solutions \[39,40\].

The strength of vortex is characterized by the circulation \(\Gamma\) that is represented by the integral round a closed contour \(C\) encircling the vortex such that

\[
\Gamma = \oint_C v \cdot ds
\]
and it is quantized as
\[ \Gamma = 2\pi l \hbar / m, \]
where the circulation number \( l \) is an integer [31,33,36]. It should be stressed that we can perform the investigation of vortices for all the cases with \( a \neq 0 \) in the \((u_a, v_a)\) plane, because fundamental properties of vortices such as the numbers of vortices in the original plane and the mapped plane and the strengths of vortices do not change by the conformal mappings. Vortex patterns in the \((x, y)\) plane can be obtained by the inverse transformations of the conformal mappings. Let us here show that vortex lines and vortex lattices can be constructed from simple linear combinations of the low lying polynomial solutions. And also the mapped patterns of those lines and lattices are investigated by the conformal mappings for the \( a = 2 \) (PPB case; \( V_a \propto \rho^2 \)) and for the \( a = 1/2 \) (Coulomb type; \( V_a \propto \rho^{-1} \)). In the following discussions the suffixes \( a \) of \( u_a, v_a \) and \( k_a \) are omitted.

(I) Vortex lines

Let us consider the linear combination of two degree 1 solutions such that
\[ \Psi(u, v) = v e^{iku} - u e^{-ikv}, \]
where the complex constant corresponding to the overall factor of the wave function is ignored, because the wave function belongs to the conjugate space of Gel'fand triplet and it is not normalizable. This means that the wave function represents a stationary flow [40]. The nodal points of the probability density
\[ |\Psi(u, v)|^2 = u^2 + v^2 - 2uv \cos k(u + v) \]
appear at points satisfying the conditions
\[ u = \pm v, \quad \cos k(u + v) = \pm 1. \]
We have the nodal points at
\[ u = v = n\pi / k, \quad \text{for } n = \text{integers}. \]

In the \((u, v)\) plane the positions of vortices can be on a line of \( u = v \). After some elementary but tedious calculations we see that the circulation numbers of vortex strengths are given by \( l = -1 \) for \( n = \text{positive integers} \) and \( l = 1 \) for \( n = \text{negative ones} \). Note that the origin at \( u = v = 0 \) has no vortex. We can directly see the result by showing the fact that the strength of vortex \( \Gamma \) becomes zero for the closed circle around the origin. We can interpret this result as follows; at the origin there exist a pair of vortices having the opposite circulation numbers, that is, they, respectively, belong to the vortex line with \( l = -1 \) and that with \( l = 1 \). We may say that it is a vortex dipole.

For the case of \( a = 1 \) (constant potential) we can take as \( u = x \) and \( v = y \).
In the case of $a = 2$ (PPB) we have $u = x^2 - y^2$ and $v = 2xy$. The relations for the nodal points are written down as

$$y = \frac{1}{\sqrt{2} + 1} x, \quad y = \pm \frac{1}{2} \sqrt{\frac{n\pi}{(\sqrt{2} + 1)k}}, \quad \text{for } n = \text{positive integers}$$

$$y = -\frac{1}{\sqrt{2} - 1} x, \quad y = \pm \frac{1}{2} \sqrt{\frac{|n|\pi}{(\sqrt{2} - 1)k}}, \quad \text{for } n = \text{negative integers}.\quad (26)$$

We see that a vortex quadrupole composed of two vortex dipole appears at the origin.

In the case of $a = 1/2$ (Coulomb type), by using the relations $u^2 - v^2 = x$ and $2uv = y$, we obtain the conditions for the nodal points as follows;

$$x = 0, \quad y = 2\frac{n^2\pi^2}{k^2}, \quad \text{for } n = \text{non zero integers}.\quad (27)$$

Note that the origin is a singular point, where the source of the potential exists.

Figures for $a = 1, 2$ and $1/2$ are presented in figs.1, 2 and 3, which, respectively, represent the vortex pattern for the constant potential, that for the PPB, and that for the Coulomb type one. Note here that the differences of the potentials clearly appear not only in the vortex patterns but the properties of the singularities at the origin as well. In hydrodynamics it is known that vortex lines for the constant potential ($a = 1$) are stationary but generally unstable for perturbations [17-21]. Note that the parallel vortex lines are constructed from the linear combinations of the lowest and the degree 1 polynomials [39].

(II) Vortex lattices

Let us consider the linear combination of a stationary wave and a plane wave such that

$$\Psi(u, v) = \cos ku - e^{-ikv}.\quad (28)$$

The nodal points of the probability density

$$|\Psi(u, v)|^2 = 1 + \cos^2 ku - 2\cos ku \cos kv$$

appear at positions satisfying

$$u = m\pi/k, \quad v = n\pi/k,\quad (29)$$

where both of $m$ and $n$ must be even or odd, that is, $(-1)^m = (-1)^n$ must be fulfilled. These conditions produce a vortex lattice presented in fig.4, which was suggested in ref.[48].
In the cases of the PPB ($a = 2$) and the Coulomb type ($a = 1/2$) vortices appear at the cross points of the following two functions;

$$
x^2 - y^2 = m\pi/k, \quad xy = n\pi/2k, \quad \text{for the PPB},
$$

$$
x^2 + y^2 = (m^2 + n^2)^2\pi^4/k^4, \quad y = 2mn\pi^2/k^2, \quad \text{for the Coulomb type}. \quad (30)
$$

In these case we obtain the circulation number $l = -1$ for the all vortices. Figures for $a = 2$ and $1/2$ are given in figs.5 and 6, respectively.

In these arguments we see the following points:

(1) The construction of vortex lattices in experiments seems to be not very difficult. In fact the vortex lattice of (II) can be produced from a stationary wave and a plane wave perpendicular to the stationary wave.

(2) The differences of potentials can be clearly seen from the vortex patterns. Especially the distances between two neighbouring vortices are a good object to identify the type of the potentials. That is to say, the vortices appear in an equal distance $\pi/k$ in the case of the constant potential ($a = 1$), whereas the distances become smaller in the regions far from the origin for $a > 1$ and larger for $a < 1$ in comparison with those near the origin.

(3) The property of the singularity at the origin is also a good object to identify the potentials.

Though it is at this moment difficult to categorize the present experimental vortex patterns [8-16], we shall be able to understand fundamental dynamics of vortex phenomena from vortex patterns.

It is also noticed that the present results can be applicable not only to quantum phenomena but also those in classical fluids by changing the parameters $m$, $\hbar$ and $g_a$ in the original equation.

Up to now we have not discussed on the stability of the vortex lattices. In order to investigate the time development of the patterns we have to take account of the fact that the solutions used here belong to the conjugate spaces of Gel'fand triplets. In the spaces the eigenstates generally have complex energy eigenvalues such as resonances and the eigenvalues are expressed by pairs of complex conjugates such that $E = \epsilon \pm i\gamma$, where $\epsilon, \gamma \in \mathbb{R}$ [41]. It is known that the + and - signs of the imaginary part in the eigenvalues, respectively, represent resonance-formation and resonance-decay processes. We see that this pairing property is the origin of the infinite degeneracy of the solutions and the infinite degeneracy stems from the balance between the resonance-formation and resonance-decay processes. This fact seems to indicate that many vortex systems are possibly unstable for perturbations. Actually the existence of vortex lattices has already been pointed out and it has also been noticed that those systems will decay from their edges, where the balance between the formation and decay processes is broken [48]. In such processes we have to discuss on dynamics in many vortex systems and statistical mechanics should be extended into the conjugate spaces of Gel'fand triplets, where the freedom of the imaginary energy eigenvalues must be introduced [48-50]. In the theory the infinite degeneracy plays another important role, that is, it becomes the origin of new entropy. The new entropy brings essentially new processes in thermal non-equilibrium [49,50]. At present we still have
many fundamental questions in the investigation of dynamics in the conjugate spaces of Gel’fand triplets. The study of vortex dynamics will bring us a new prospect and open a new dynamics for the systems essentially described by the states in Gel’fand triplets.

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Fig. 1: Positions of vortices for $n = \pm 1, \pm 2, \pm 3$ in the constant potential $(a = 1)$, which are denoted by $\bullet$, and $\circ$ stands for the vortex dipole at the origin.

Fig. 2: Positions of vortices for $n = \pm 1, \pm 2, \pm 3$ in the PPB $(a = 2)$, which are denoted by $\bullet$, and $\circ$ stands for the vortex quadrupole at the origin.
Fig. 3: Positions of vortices for $n = 1, 2, 3$ in the Coulomb type potential ($a = 1/2$), which are denoted by $\bullet$, and $\circ$ stands for the source at the origin.
Fig. 4: Positions of vortices for $m, n = 0, \pm 1, \pm 2, \pm 3, \pm 4$ in a constant potential ($a = 1$), which are denoted by $\bullet$ and the distance between the neighbouring lines are taken by $\pi/k$. 