ESTIMATES FOR THE LARGEST CRITICAL VALUE OF $T_n^{(k)}$

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Abstract. Here we study the quantity

$$\tau_{n,k} := \frac{|T_n^{(k)}(\omega_{n,k})|}{T_n^{(k)}(1)},$$

where $T_n$ is the $n$-th Chebyshev polynomial of the first kind and $\omega_{n,k}$ is the largest zero of $T_n^{(k+1)}$. Since the absolute values of the local extrema of $T_n^{(k)}$ increase monotonically towards the end-points of $[-1, 1]$, the value $\tau_{n,k}$ shows how small is the largest critical value of $T_n^{(k)}$ relative to its global maximum $T_n^{(k)}(1)$. This is a continuation of the recent paper [6], where upper bounds and asymptotic formulae for $\tau_{n,k}$ have been obtained on the basis of Alexei Shadrin’s explicit form of the Schaeffer–Duffin pointwise majorant for polynomials with absolute value not exceeding 1 in $[-1, 1]$.

We exploit a result of Knut Petras [9] about the weights of the Gaussian quadrature formulae associated with the ultraspherical weight function $w_\lambda(x) = (1 - x^2)^{\lambda-1/2}$ to find an explicit (modulo $\omega_{n,k}$) formula for $\tau_{2n,k}$. This enables us to prove a lower bound and to refine the upper bounds for $\tau_{n,k}$ obtained in [6]. The explicit formula admits also a new derivation of the asymptotic formula in [6] approximating $\tau_{n,k}$ for $n \to \infty$. The new approach is simpler, without using deep results about the ordinates of the Bessel function, and allows to better analyze the sharpness of the estimates.

Key Words and Phrases: Derivatives of Chebyshev polynomials, ultraspherical polynomials, hypergeometric functions.

Mathematics Subject Classification 2020: 41A17

1. Introduction and statement of the results

We study the quantity

$$\tau_{n,k} := \frac{|T_n^{(k)}(\omega)|}{T_n^{(k)}(1)},$$

where $T_n$ is the $n$-th Chebyshev polynomial of the first kind, and $\omega = \omega_{n,k}$ is the rightmost zero of $T_n^{(k+1)}$, $n \geq k + 2$. The value $\tau_{n,k}$ shows how small is the largest critical value of $T_n^{(k)}$ relative to its global maximum in $[-1, 1]$, $T_n^{(k)}(1)$. This quantity has found applications in studying some extremal problems such as Markov-type inequalities [4], [7], [11] and the Landau–Kolmogorov inequalities for intermediate derivatives [5], [12].

Some upper bounds for $\tau_{n,k}$ have been obtained in the recent paper [6]. The main ingredient for the results in [6] is the pointwise majorant $D_{n,k}(x)$ for polynomials of degree at most $n$ with absolute value less than or equal to one in $[-1, 1]$. This majorant was used by Schaeffer and Duffin [13] to obtain another proof of V.

The authors are partially supported by the Sofia University Research Fund through Contract No. 80-10-20/22.03.2021.
Markov’s inequality. An explicit formula for $D_{n,k}^2(x)$, $k \geq 2$, was found by Shadrin (see [11]), it reads

$$D_{n,k}^2(x) = \frac{n^2(n^2 - 1^2) \cdots (n^2 - (k-1)^2)}{(1 - x^2)^k} S_{n,k}(x),$$

where

$$S_{n,k}(x) = 1 + \sum_{m=1}^{k-1} \frac{(2m-1)!}{(2m)!} \frac{(k-m)_{2m}}{(1-x^2)^m} \prod_{j=1}^{m} \frac{1}{n^2 - j^2}$$

and $(a)_j := a(a+1) \cdots (a+j-1)$, $j \in \mathbb{N}$, is the Pochhammer function. This leads to the inequality

$$\tau_{n,k}^2 \leq \frac{D_{n,k}^2(\omega)}{[T_n^{(k)}(1)]^2} = \frac{(2k-1)!}{(2k)!} \frac{n+k}{n} \frac{1}{\omega^{n+k}} S_{n,k}(\omega)$$

which was the starting point in [6] for the derivation of upper bounds for $\tau_{n,k}^2$ which are uniform in $n$ and $k$.

Of course, one can obtain an explicit formula for $\tau_{n,k}$ from $T_n^{(k)} = 2^{k-1} k! n P_{n-k}^{(k)}$ and the representations of ultraspherical polynomials as hypergeometric functions, e.g., [14, eqn. 4.7.6] yields

$$\tau_{n,k} = -1 + \sum_{m=1}^{n-k} (-1)^{m+1} \left( \frac{n-k}{m} \right) \left( \frac{n+k}{n-k} \right) \omega^2 \left( \frac{1}{2} \right)^m.$$ 

However, this expression is difficult to estimate because of the sign changing summands. It turns out that an explicit formula for $\tau_{n,k}^2$ exists, which moreover admits easier estimation. We prove the following

**Theorem 1.1.** For all $n > k + 1$, the quantity $\tau_{n,k}^2$ admits the representation

$$\tau_{n,k}^2 = \frac{(2k-1)!}{(2k)!} \frac{n}{n-k} \frac{1}{\omega^{n-k}} \frac{1}{(1-\omega^2)^{n-k}} S_{n,k+1}(\omega),$$

where

$$S_{n,k+1}(x) = 1 + \sum_{m=1}^{k} \frac{(2m-1)!}{(2m)!} \frac{(k+1-m)_{2m}}{(1-x^2)^m} \prod_{j=1}^{m} \frac{1}{n^2 - j^2}.$$ 

We derive Theorem 1.1 from a result of Knut Petras [9], who has found explicit expressions for the coefficients of the Gaussian quadrature formulae associated with the ultraspherical weight function $w_{\lambda}(x) = (1 - x^2)^{\lambda-1/2}$ when $\lambda$ is a non-negative integer.

The similarity of formulae (1.2) and (1.3) is remarkable, as well as the opposite roles played by the quantities $S_{n,k}(\omega)$ and $S_{n,k+1}(\omega)$ therein. As (1.3) is the exact expression while in (1.2) we have a majorant for $\tau_{n,k}^2$, it is natural to expect that with (1.3) one could produce better upper bounds than the bounds obtained with (1.2). This however requires a lower estimate for $S_{n,k+1}(\omega)$, exactly as upper estimates for $S_{n,k}(\omega)$ are needed in (1.2). In fact, one can avoid estimation of $S_{n,k}(\omega)$ and $S_{n,k+1}(\omega)$ by simply combining (1.2) and (1.3) to obtain

$$\tau_{n,k}^2 \leq \sqrt{\frac{n+k}{n-k} \frac{(2k-1)!}{(2k)!} \frac{1}{\omega^{n-k}} \frac{1}{(1-\omega^2)^k} \frac{S_{n,k}(\omega)}{S_{n,k+1}(\omega)}},$$
and then use the inequality $S_{n,k}(\omega)/S_{n,k+1}(\omega) < 1$, which is easily verified by a termwise comparison. By elaborating further this idea, we prove the following sharper result.

**Theorem 1.2.** For all $n > k + 1$, there holds

$$
\tau_{n,k} \leq \frac{(2k - 1)!!}{(n + k - 1)(n + k - 3) \ldots (n - k + 1)} \frac{1}{(1 - \omega^2)^{\frac{k}{2}}}
$$

Since $1 - \omega^2 \geq \left(\frac{k + 2}{n}\right)^2$ (see, e.g., [7, Lemma 3.5] for an estimate for the largest zeros of ultraspherical polynomials), Theorem 1.2 implies:

**Corollary 1.3.** For all $n > k + 1$, there holds

$$
\tau_{n,k} \leq \frac{(2k - 1)!!}{(n + k - 1)(n + k - 3) \ldots (n - k + 1)} \left(\frac{n}{k + 2}\right)^k.
$$

We proved in [6, Theorem 1.1]) that the sequence $\{\tau_{n,k}\}_{n > k + 1}$ is monotonically decreasing, thus showing the existence of $\tau_k^* = \lim_{n \to \infty} \tau_{n,k}$. Corollary 1.3 implies immediately an upper bound for $\tau_k^*$.

**Corollary 1.4.** For all $k \in \mathbb{N}$, the quantity $\tau_k^*$ satisfies the inequality

$$
\tau_k^* \leq \frac{(2k - 1)!!}{(k + 2)^k}.
$$

The following counterpart of the inequality $\tau_{n,k} \leq \tau_{n+1,k}$ holds true:

**Theorem 1.5.** For all $n > k + 1$,

$$
\frac{\tau_{n,k}}{\tau_{n+1,k}} \leq \frac{(n + k)(n + k - 2) \ldots (n - k + 2)}{(n + k - 1)(n + k - 3) \ldots (n - k + 1)}.
$$

By iterating (1.6) and using that $\tau_{k+2,k} = 1/(2k + 1)$ (see [6, Theorem 1.2]), we obtain that the right-hand side of (1.5) without the factor $(1 - \omega^2)^{-k/2}$ is a lower bound for $\tau_{n,k}$.

**Corollary 1.6.** For all $n > k + 1$, there holds

$$
\tau_{n,k} \geq \frac{(2k - 1)!!}{(n + k - 1)(n + k - 3) \ldots (n - k + 1)}.
$$

We observe that the bounds for $\tau_{n,k}$ given by Corollaries 1.3 and 1.6 present correctly the magnitude of $\tau_{n,k}$ whenever $\left(\frac{n}{k + 2}\right)^k$ remains bounded. This is the case, e.g., when $m = n - k$ is fixed.

Theorem 1.1 admits also a new derivation of the asymptotic formula for $\tau_k^*$, obtained in [6, Theorem 1.8]. The new approach is simpler, without using deep results about the ordinates of the Bessel function, and allows to better analyze the sharpness of our estimates.

**Theorem 1.7.** The quantity $\tau_k^*$ admits the representation

$$
\tau_k^* = A \left(\frac{2}{e}\right)^{k+1/2} e^{-ak^{1/3} k^{-1/6}} \left(1 + O(k^{-1/6})\right),
$$

where $A = \left(\int_{1}^{\infty} e^{-\frac{x^3}{2} + 2ax} \frac{dx}{\sqrt{x}}\right)^{-1/2} \approx 1.3951$ and $a = 2^{-1/3}|i_1| \approx 1.8558$ with $i_1$ the first zero of the Airy function.
The rest of the paper is organized as follows. In Section 2 we prove Theorem 1.1. Theorems 1.2 and 1.5 are proven in Section 3, and the proof of Theorem 1.7 is given in Section 4. Section 5 contains comments and concluding remarks.

2. PROOF OF THEOREM 1.1

The derivatives of $T_n$ are expressed by ultraspherical polynomials, namely,

\begin{equation}
T_n^{(k)} = n 2^{k-1} (k-1)! P_{n-k}^{(k)}, \quad k = 1, \ldots, n.
\end{equation}

Here, $P_m^{(\lambda)}$ is the usual notation for the $m$-th ultraspherical polynomial, which is orthogonal in $[-1, 1]$ with respect to the weight function $w_\lambda(x) = (1 - x^2)^{\lambda-1/2}$, $\lambda > -1/2$. Well known properties of ultraspherical polynomials are

\begin{equation}
(1 - x^2)y'' - (2\lambda + 1)xy' + m(m + 2\lambda)y = 0, \quad y = P_m^{(\lambda)},
\end{equation}

\begin{equation}
\frac{d}{dx}\{P_m^{(\lambda)}(x)\} = 2\lambda P_{m-1}^{(\lambda + 1)}(x),
\end{equation}

\begin{equation}
P_m^{(\lambda)}(1) = \left(\frac{m + 2\lambda - 1}{m}\right).
\end{equation}

From (2.1) and (2.4) it follows that

\begin{equation}
\tau_{n,k}^2 = \frac{(P_{n-k}^{(\lambda)}(\omega))^2}{(P_{n-k}^{(\lambda)}(1))^2} = \frac{\Gamma^2(2k) \Gamma^2(n + 1 - k)}{\Gamma^2(n + k)} \frac{(P_{n-k}^{(\lambda)}(\omega))^2}{(P_{n-k}^{(\lambda)}(1))^2},
\end{equation}

where $\omega$ is the largest zero of $P_{n-k-1}^{(\lambda)}$.

An explicit representation of $[P_{n-k}^{(\lambda)}(\omega)]^2$ follows from a result of Knut Petras [9], where the author has found an asymptotic expansion for the coefficients of the Gaussian quadrature formulae associated with the ultraspherical weight function $w_\lambda(x) = (1 - x^2)^{\lambda-1/2}$. In the case $\lambda \in \mathbb{N}_0$ Petras has proved that the coefficients $a^{(\lambda)}_{\nu,n}$ of the $n$-point Gaussian quadrature formula $Q_n^{(\lambda)G}$,

\begin{equation}
Q_n^{(\lambda)G}[f] = \sum_{\nu=1}^{n} a^{(\lambda)}_{\nu,n} f(x^{(\lambda)}_{\nu,n}),
\end{equation}

where $x^{(\lambda)}_{\nu,n} = x^{(\lambda)}_{\nu,n}$ are the zeros of $P_n^{(\lambda)}$, admit the representation

\begin{equation}
a^{(\lambda)}_{\nu,n} = \frac{\pi}{n + \lambda} (1 - x^2)\lambda \left(1 + \sum_{m=1}^{\lambda-1} \frac{\alpha_{m}(\lambda)}{(1 - x^2)^m} \prod_{j=1}^{m} \frac{1}{(n + \lambda - j)^2 - j^2}\right),
\end{equation}

where

\begin{equation}
\alpha_m(\lambda) = \left(\frac{(2m)!}{2^{2m}m!}\right)^2 \left(\frac{m + \lambda - 1}{2m}\right).
\end{equation}

On the other hand, the weights $a^{(\lambda)}_{\nu,n}$ obey the representation (cf. [14, eqn. (15.3.2)])

\begin{equation}
a^{(\lambda)}_{\nu,n} = \frac{2^{2-2\lambda} \pi \Gamma(n + 2\lambda)}{\Gamma^2(\lambda) \Gamma(n + 1)} \frac{1}{(1 - x^2)_{\nu}^{(\lambda)'2}}
\end{equation},

where $x^2_{\nu} = x^{(\lambda)}_{\nu,n}$.
If $\lambda > 1$, then, by (2.3), $P_n^{(\lambda)}(x) = y''(x)/(2\lambda - 2)$, where $y = P_n^{(\lambda-1)}$. By (2.2), $(1-x^2)y''(x) = -(n+1)(n+2\lambda-1)y(x)$ at the zeros of $y' = 2(\lambda-1)P_n^{(\lambda)}$, therefore the above formula can be rewritten as

\begin{equation}
\alpha_{\nu,n}^{(\lambda)} = \frac{2^{2-2\lambda} \pi \Gamma(n+2\lambda-1)}{(n+1)(n+2\lambda-1) \Gamma^2(\lambda-1) \Gamma(n+2)} \frac{1-x^2_{\nu}}{[P_{n+1}^{(\lambda-1)}(x_{\nu})]^2}, \quad \lambda > 1.
\end{equation}

By equating the right-hand sides of (2.6) and (2.7) and then substituting (3.2) into (2.6), we obtain

\begin{equation}
\frac{\pi}{n}(1-x^2_{\nu})^{k+1} S_{n,k+1}(x_{\nu}) = \frac{2^{2-2k} \pi \Gamma(n+k)}{(n^2-k^2) \Gamma^2(k) \Gamma(n-k+1)} \frac{1-x^2_{\nu}}{[P_{n-k}^{(k)}(x_{\nu})]^2}.
\end{equation}

In particular, this last equality holds true when $x_{\nu}$ is the largest zero of $P_{n-k-1}^{(k)}(x)$, i.e., $x_{\nu} = \omega$. Therefore,

\[ [P_{n-k}^{(k)}(\omega)]^2 = \frac{2^{2-2k} \pi \Gamma(n+k)}{(n^2-k^2) \Gamma^2(k) \Gamma(n-k+1)} \frac{1}{(1-\omega^2)^k S_{n,k+1}(\omega)}. \]

By putting this expression in (2.5) we obtain (1.3). \hfill \Box

3. Proof of Theorems 1.2 and 1.5

Recall that the $\mathbf{3F}_2$ hypergeometric function is defined by the series

\[ \mathbf{3F}_2(a, b, c; d, e; x) = 1 + \sum_{m=1}^{\infty} \frac{(a)_m (b)_m (c)_m}{(d)_m (e)_m} \frac{x^m}{m!}. \]

Generally, it is assumed that $d, e$ are not negative integers or zero, but exceptions are allowed when some of parameters $a, b, c$ are a negative integer, in which case the series terminates. This is the situation with the finite sums $S_{m,k}(0)$ and $S_{m,k+1}(0)$, where $m \in \mathbb{N}$, $m > k + 1$, namely, we have

\[ S_{m,k}(0) = \mathbf{3F}_2(k, 1-k, \frac{1}{2}; 1 + m, 1-m; 1), \]

\[ S_{m,k+1}(0) = \mathbf{3F}_2(k + 1, -k, \frac{1}{2}; 1 + m, 1-m; 1). \]

A closed type formula for such $\mathbf{3F}_2$ expressions provides the Whipple identity (see [3, p. 189, eqn. (7)]).

\[ \mathbf{3F}_2(a, 1-a, c; d, e; 1+c-a; 1) = \frac{2^{1-2c} \pi \Gamma(d) \Gamma(2c + 1 - d)}{\Gamma(\frac{a+d}{2}) \Gamma(\frac{a+1+2c-a}{2}) \Gamma(\frac{1-a+d}{2}) \Gamma(\frac{2+2c-a-d}{2})} \frac{1}{\Gamma(\frac{1}{2})}. \]

By using Whipple’s identity and familiar properties of the Gamma function (considering separately the cases of even and odd $m-k$), we find that, under the assumption $m > k + 1$,

\begin{align}
S_{m,k}(0) &= m \frac{(m + k - 2)(m + k - 4) \cdots (m - k + 2)}{(m + k - 1)(m + k - 3) \cdots (m - k + 1)}, \\
S_{m,k+1}(0) &= m \frac{(m + k - 1)(m + k - 3) \cdots (m - k + 1)}{(m + k)(m + k - 2) \cdots (m - k)}.
\end{align}

For the proof of Theorems 1.2 and 1.5 we need the following simple lemma.
Lemma 3.1. Let the polynomials $P(x) = \sum_{m=0}^{k} a_m x^m$ and $Q(x) = \sum_{m=0}^{k} b_m x^m$ have positive coefficients (with $a_k$ allowed to be zero). If the sequence $\{a_m\}_{m=0}^{k}$ is monotonically increasing (resp. decreasing), then $R(z) = \frac{P(z)}{Q(z)}$ is strictly monotonically increasing (resp. decreasing) in $[0, \infty)$. 

Proof. A straightforward calculation shows that $R'(x) = r(x)/Q^2(x)$, where
\[
 r(x) = \sum_{s=1}^{k} \left( \left\lfloor \frac{s}{2} \right\rfloor \sum_{m=0}^{k} (s-2m)(a_{s-m} b_m - a_m b_{s-m}) \right) x^{s-1} 
 + \sum_{s=k+1}^{2k} \left( \left\lfloor \frac{s}{2} \right\rfloor \sum_{m=s-k}^{k} (s-2m)(a_{s-m} b_m - a_m b_{s-m}) \right) x^{s-1}.
\]
(Here, $\lfloor \cdot \rfloor$ stands for the integer part function.) We observe that if the sequence $a_m/b_m$, $(m = 0, 1, \ldots, k)$, is monotonically increasing (decreasing), then all the coefficients of the polynomial $r(x)$ are positive (negative), and hence $R'(x)$ is positive (negative) on $[0, \infty)$. \qed

Proof of Theorem 1.2. In the introduction we deduced inequality (1.4) by combining (1.2) and Theorem 1.1. Now we refine the trivial estimate $S_{n,k}(\omega)/S_{n,k+1}(\omega) < 1$. Let us set
\[
 (3.3) \quad z = \frac{1}{1-x^2}, \quad z \in [1, \infty)
\]
Consider the polynomials $P(z) = S_{n,k}(x)$ and $Q(z) = S_{n,k+1}(x)$, where $x \in [0, 1)$. The coefficients $\{a_m\}$ and $\{b_m\}$ of $P(z)$ and $Q(z)$, respectively, are
\[
 a_m = \frac{(2m-1)!!}{(2m)!!} (k-m)_{2m} \prod_{j=1}^{m} \frac{1}{n^2 - j^2},
 b_m = \frac{(2m-1)!!}{(2m)!!} (k+1-m)_{2m} \prod_{j=1}^{m} \frac{1}{n^2 - j^2},
\]
The sequence
\[
 \frac{a_m}{b_m} = \frac{k-m}{k+m}, \quad m = 0, 1, \ldots, k,
\]
is monotonically decreasing. By Lemma 3.1, $P(z)/Q(z)$ is monotonically decreasing in the interval $[0, \infty)$ and therefore in $[1, \infty)$. It follows from (3.3) and the definition of $P$ and $Q$ that $S_{n,k}(x)/S_{n,k+1}(x)$ is a monotonically decreasing function of $x$ in the interval $[0, 1)$, hence
\[
 \frac{S_{n,k}(\omega)}{S_{n,k+1}(\omega)} \leq \frac{S_{n,k}(0)}{S_{n,k+1}(0)}.
\]
By using (3.1) and (3.2) we find
\[
 \sqrt{n^2-k^2} (n+k-2)(n+k-4) \cdots (n-k+2)\quad (n+k-1)(n+k-3) \cdots (n-k+1).
\]
and putting the last expression in the right-hand side of (1.4), after some simplification we arrive at inequality (1.5). \qed
Proof of Theorem 1.5. From Theorem 1.1 we have

\[ \frac{r_{n,k}^2}{\tau_{n+1,k}} = \frac{n(n+1+k)}{(n+1)(n-k)} \frac{(1-\tilde{\omega})^k S_{n+1,k+1}(\tilde{\omega})}{(1-\omega^2)^k S_{n,k+1}(\omega)}, \]

where \( \tilde{\omega} \) is the largest zero of \( T_{n+1}^{(k+1)} \). Since \( (1-\tilde{\omega})^k S_{n+1,k+1}(\tilde{\omega}) \) is a polynomial in \( 1-\omega^2 \) with positive coefficients and \( 0 < 1-\tilde{\omega}^2 < 1-\omega^2 \), it follows that

\[ \frac{r_{n,k}^2}{\tau_{n+1,k}} \leq \frac{n(n+1+k)}{(n+1)(n-k)} \frac{(1-\omega^2)^k S_{n+1,k+1}(\omega)}{(1-\omega^2)^k S_{n+1,k+1}(\omega)}. \]

Let us consider the polynomials in \( z = 1-\omega^2 \), \( z \in (0,1) \),

\[ P(z) = z^k + \sum_{m=0}^{k-1} a_m z^m = (1-\omega^2)^k S_{n+1,k+1}(\omega), \]

\[ Q(z) = z^k + \sum_{m=0}^{k-1} b_m z^m = (1-\omega^2)^k S_{n,k+1}(\omega). \]

For \( m = 1, \ldots, k \) we have

\[ a_{k-m} = \frac{(2m-1)!!}{(2m)!!} (k-m+1)_{2m} \prod_{j=1}^{m} \frac{1}{(n+1)^2 - j^2}, \]

\[ b_{k-m} = \frac{(2m-1)!!}{(2m)!!} (k-m+1)_{2m} \prod_{j=1}^{m} \frac{1}{n^2 - j^2}, \]

therefore

\[ \frac{a_{k-m}}{b_{k-m}} = n^2 - m^2 \frac{a_{k+1-m}}{b_{k+1-m}} < \frac{a_{k+1-m}}{b_{k+1-m}}, \quad m = 1, \ldots, k. \]

Hence, the sequence \( \{a_m/b_m\} \), \( m = 0, 1, \ldots, k \), is monotonically increasing, and Lemma 3.1 implies that \( \frac{P(z)}{Q(z)} \) increases monotonically in \((0, \infty)\), in particular,

\[ \frac{(1-\omega^2)^k S_{n+1,k+1}(\omega)}{(1-\omega^2)^k S_{n,k+1}(\omega)} = \frac{P(z)}{Q(z)} \leq \frac{P(1)}{Q(1)} = \frac{S_{n+1,k+1}(0)}{S_{n,k+1}(0)}. \]

From (3.2) we find

\[ \frac{S_{n+1,k+1}(0)}{S_{n,k+1}(0)} = \frac{(n+1)(n-k)}{n(n+k+1)} \frac{(n+k+2)^2(n+k-2)}{(n+k+1)^2(n+k-3)^2 \cdots (n-k+2)^2}. \]

The claim of Theorem 1.5 now follows from (3.4), (3.5) and (3.6). \( \square \)

4. Proof of Theorem 1.7

In view of (1.3) we have \( (\tau_k^*)^2 = L_1/L_2 \), where

\[ L_1 = \lim_{n \to \infty} \frac{n(n-k-1)(2k-1)!!}{(n+k)!(1-\omega^2)^k} \quad \text{and} \quad L_2 = \lim_{n \to \infty} S_{n,k+1}(\omega). \]

We will use the following result from [14], (see §8.9 or Theorem 8.21.12).

Let \( \alpha > -1 \) and \( \beta \) be an arbitrary real number. Then, for the \( r \)-th zero of \( P_\alpha^{(\alpha,\beta)}(\cos \theta) \), where \( P_\alpha^{(\alpha,\beta)}(x) \) is the Jacobi polynomial, it holds the limit relation

\[ \theta_r = n^{-1}(j_{\alpha,r} + \epsilon_n), \quad \epsilon_n \to 0 \text{ for } n \to \infty, \]

where \( j_{\alpha,r} \) is the \( r \)-th positive zero of the Bessel function \( J_\alpha(x) \).
Since \( \omega \) is the largest zero of \( T_n^{(k+1)}(x) = C_n,k P_n^{(\nu,\nu)}_{n-k-1}(x) \) with \( \nu = k + 1/2 \), then
\[
\omega = \cos \theta_1 = \cos \frac{j_{\nu,1} + \epsilon_n - k - 1}{n - k - 1} = 1 - j_{\nu,1}^2 \frac{1 + \epsilon_n}{2n^2},
\]
where \( \epsilon_n = \epsilon'(n, k) \) tends to 0 as \( n \to \infty \) and \( k \) is fixed. Equivalently, we have
\[
1 - \omega^2 = \left( \frac{j_{\nu,1}}{n} \right)^2 (1 + \delta_n), \quad \delta_n \to 0 \quad \text{as} \quad n \to \infty.
\]
For \( L_1 \) we obtain
\[
L_1 = \lim_{n \to \infty} \frac{(2k - 1)!!^2}{(n^2 - 1^2)(n^2 - 2^2)(n^2 - k^2)(1 - \omega^2)^k} = \lim_{n \to \infty} \frac{(2k - 1)!!^2}{(1 - 1^2/n^2)(1 - 2^2/n^2)(1 - k^2/n^2)j_{\nu,1}^2(1 + \delta_n)^k},
\]
and hence
\[
L_1 = \frac{(2k - 1)!!^2}{j_{\nu,1}^{2k}}.
\]
For \( L_2 = \lim_{n \to \infty} S_{n,k}(\omega), \ k = k + 1 \), we have
\[
L_2 = \lim_{n \to \infty} \sum_{m=0}^{k} \frac{(2m - 1)!!}{(2m)!!} \frac{k}{\kappa + m} \left[ \frac{(k^2 - 1^2)(k^2 - 2^2) \cdots (k^2 - m^2)}{(n^2 - 1^2)(n^2 - 2^2) \cdots (n^2 - m^2)} \right]^{-m} \left( 1 - \omega^2 \right)^{-m} = \lim_{n \to \infty} \sum_{m=0}^{k} \frac{(2m - 1)!!}{(2m)!!} \frac{k}{\kappa + m} \left( \prod_{\ell=1}^{m} \frac{1 - \ell^2/k^2}{1 - \ell^2/n^2} \right) j_{\nu,1}^{2m}(1 + \delta_n)^{-m}
\]
therefore
\[
L_2 = \sum_{m=0}^{k} \frac{(2m - 1)!!}{(2m)!!} \frac{k}{\kappa + m} \left( \prod_{\ell=1}^{m} \frac{1 - \ell^2/k^2}{1 - \ell^2/n^2} \right) \left( \frac{j_{\nu,1}}{\kappa} \right)^{-2m},
\]
and hence
\[
L_2 = \sum_{m=0}^{k} a_m q^{2m} = S_{k+1}^*.
\]
Combining (4.2) and (4.3) we obtain
\[
\tau_k^* = \frac{(2k - 1)!!}{j_{\nu,1}^1 \sqrt{S_{k+1}^*}}, \quad \nu = k + 1/2.
\]
Notice that \( J_{k+\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi}} z^{k+\frac{1}{2}} \left( 1 - \frac{d}{z} \right)^k \sin z \), therefore \( j_{\nu,1} \) is a zero of an elementary function.

To estimate the factor \( \frac{(2k - 1)!!}{j_{\nu,1}^1} \) in (4.4) we use the Stirling approximation
\[
N! = \sqrt{2\pi N} \left( \frac{N}{e} \right)^N \left( 1 + O(N^{-1}) \right)
\]
and (see, e.g., [10])
\[
j_{\nu,1} = \nu + a\nu^{1/3} + \frac{3a^2}{10}\nu^{-1/3} + O(\nu^{-1}), \quad \nu > 0.
\]
Then,
\[
\frac{(2k - 1)!!}{j^k_{\nu,1}} = \frac{(2k)!}{(2k)!!} \nu^{-k} \left[ 1 + a \nu^{-2/3} + O(\nu^{-4/3}) \right]^{-k}
\]
\[
= 2^{-k} \frac{(2k)!}{k!} (k + \frac{1}{2})^{-k} \exp \left\{ -k \log \left[ 1 + ak^{-2/3} + O(k^{-4/3}) \right] \right\}
\]
\[
= 2^{-k} \sqrt{\frac{(2k/e)^{2k}}{(k/e)^k}} (1 + O(k^{-1})) k^{-k} \left( 1 + \frac{1}{2k} \right)^{-k} \exp \left\{ -ak^{1/3} + O(k^{-1/3}) \right\}
\]
\[
= \sqrt{2}(2/e)^k e^{-k/2} \left( 1 + O(k^{-1}) \right) e^{-ak^{1/3}} \left( 1 + O(k^{-1/3}) \right),
\]

hence
\[
(4.5) \quad \frac{(2k - 1)!!}{j^k_{\nu,1}} = \left( \frac{2}{e} \right)^{k+1/2} e^{-ak^{1/3}} \left( 1 + O(k^{-1/3}) \right).
\]

The approximation of \( S^*_{k+1} \) in the denominator needs more care. We start with the coefficients \( a_m \) in (4.3). We shall use Stirling’s formula in the form
\[
\log(N - 1)! = \left( N - \frac{1}{2} \right) \log N - N + \frac{1}{2} \log 2\pi + O(N^{-1}).
\]

With \( \kappa = k + 1 \), we have
\[
\frac{(2m)!! a_m}{(2m-1)!!} = \kappa^{-2m} \exp \left\{ \left[ \left( \kappa + m - \frac{1}{2} \right) \log(\kappa+m) - \kappa + m + \frac{1}{2} \log 2\pi + O\left( \frac{1}{\kappa+m} \right) \right] - \left[ \left( \kappa - m - \frac{1}{2} \right) \log(\kappa-m) - \kappa - m + \frac{1}{2} \log 2\pi + O\left( \frac{1}{\kappa-m} \right) \right] \right\}
\]
\[
= \kappa^{-2m} \exp \left\{ \log \kappa \left[ \left( \kappa + m - \frac{1}{2} \right) - \left( \kappa - m - \frac{1}{2} \right) \right] + \left( \kappa + m - \frac{1}{2} \right) \log \left( 1 + \frac{m}{\kappa} \right)
\]
\[
- \left( \kappa - m - \frac{1}{2} \right) \log \left( 1 - \frac{m}{\kappa} \right) - 2m + O\left( \frac{1}{\kappa-m} \right) \right\}
\]
\[
= \exp \left\{ \left( \kappa + m - \frac{1}{2} \right) \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j} \frac{m^j}{\kappa^j} + \left( \kappa - m - \frac{1}{2} \right) \sum_{j=1}^{\infty} \frac{1}{j} \frac{m^j}{\kappa^j} - 2m + O\left( \frac{1}{\kappa-m} \right) \right\}
\]
\[
= \exp \left\{ \sum_{\text{odd } j > 0} \frac{(2k-1)}{j} \frac{(m)}{\kappa^j} - \sum_{\text{even } j > 0} \frac{(2m)}{j} \frac{(m)}{\kappa^j} - 2m + O\left( \frac{1}{\kappa-m} \right) \right\}
\]
\[
= \sqrt{\frac{k-m}{k+m}} \exp \left\{ -2m \sum_{\text{even } j > 0} \frac{1}{j} \frac{1}{j+1} \frac{(m)}{\kappa^j} + O\left( \frac{1}{\kappa-m} \right) \right\}
\]
\[
= \sqrt{\frac{k-m}{k+m}} \exp \left\{ -\frac{m^3}{3k^2} + O\left( \frac{m^5}{k^4} \right) + O\left( \frac{1}{\kappa-m} \right) \right\}.
\]

It is important that the remainder denoted by \( -O\left( \frac{m^5}{k^4} \right) \) is negative. The same holds true for the other "O" term, but we will not use this fact.
Next, for the ratio \( q \) in (4.3) we have

\[
q = \left( \frac{\bar{b}_v}{\kappa} \right)^{-1} = \left( \frac{\nu + a \nu^{1/3} + O(\nu^{-1/3})}{\nu + 1/2} \right)^{-1} = \left( 1 + \frac{1}{2\nu} \right)^{-1} \left( 1 + a \nu^{-2/3} + O(\nu^{-4/3}) \right)^{-1} = 1 + \frac{1}{2\nu} - a \nu^{-2/3} + O(\nu^{-4/3})
\]

\[
= 1 - a \kappa^{-2/3} + O(\kappa^{-1}), \quad \nu \geq 1/2.
\]

From this it is clear that for sufficiently large \( k \) (respectively \( \kappa \) and \( \nu \)) we have \( q \in (0, 1) \). Moreover, the same holds for \( \nu \geq 1/2 \), as can be seen from the results in [10].

Now, we split the sum \( S^*_{k+1} = \sum_{m=0}^{k} a_m q^{2m} \) into three parts

\[
S^*_{k+1} = \sum_{m=0}^{m_1} (\cdot) + \sum_{m=m_1'}^{m_2} (\cdot) + \sum_{m=m_2'}^{k} (\cdot) =: S^{(1)}_\kappa + S^{(2)}_\kappa + S^{(3)}_\kappa,
\]

where \( m_1 = \lfloor \kappa^{1/3} \rfloor \), \( m_2 = \lfloor A_k \kappa^{2/3} \rfloor \) with \( A_k = \log \kappa \) and \( m_1' = m_1 + 1 \).

Note that without loss of generality we may assume that \( k \) is sufficiently large so that the three sums above are non-empty. For small \( k \) the assertion of the theorem is fulfilled on account of the choice of the constant in "O".

We estimate \( S^{(1)}_\kappa \) from above by using \( \frac{(2m-1)!!}{(2m)!!} = \frac{1}{\sqrt{m}} \left( 1 + O(m^{-1}) \right) \).

\[
S^{(1)}_\kappa < \sum_{m=0}^{m_1} a_{m,k} < 1 + \sum_{m=1}^{m_1} \frac{(2m-1)!!}{(2m)!!} \exp \left\{ O\left( \frac{1}{\kappa - m} \right) \right\} = 1 + \sum_{m=1}^{m_1} O\left( \frac{1}{\sqrt{m}} \right) \left( 1 + O\left( \frac{1}{\kappa} \right) \right) = O\left( \frac{1}{m^1} \right) = O(k^{1/6}).
\]

The third sum is also relatively small. Indeed,

\[
S^{(3)}_\kappa < \sum_{m=m_2'}^{k} a_{m,k} < \sum_{m=m_2'}^{k} \frac{(2m-1)!!}{(2m)!!} \exp \left\{ - \frac{m^3}{3\kappa^2} + O(1) \right\} < \sum_{m=m_2'}^{k} \frac{C}{\sqrt{m}} \exp \left\{ - \frac{m^3}{3\kappa^2} \right\},
\]

where \( C \) is an absolute constant independent of \( m \) and \( k \). Hence,

\[
S^{(3)}_\kappa < k \frac{C}{\sqrt{m_2}} e^{-m_2^2/(3\kappa^2)} < k \frac{C}{\sqrt{A_k \kappa^{1/3}}} e^{-A_k^1/3} < \frac{C k}{\kappa^{1/3}} e^{-2A_k/3} = C k / \kappa,
\]

provided \( A_k^2 > 2 \), i.e. for \( k \geq 4 \). As a consequence, \( S^{(3)}_\kappa = O(1) \) for \( k \in \mathbb{N} \).
For the main part of $S^*_\kappa$, we have
\[ S^{(2)}_\kappa = \sum_{m=m'\kappa}^{m\kappa} a_m q^{2m} = \sum_{m=m'\kappa}^{m\kappa} \frac{1+O(m^{-1})}{\sqrt{\pi m}} \left( 1+O\left( \frac{m}{\kappa} \right) \right) \times \exp \left\{ -\frac{m^3}{3\kappa^2} - O\left( \frac{m^5}{\kappa^4} \right) + O\left( \frac{1}{\kappa} \right) \right\} \left( 1 - a(\kappa)^{-2/3} + O(\kappa^{-1}) \right)^{2m} \]
\[ = \sum_{m=m'\kappa}^{m\kappa} \frac{1}{\sqrt{\pi m}} \left( 1 + O\left( \frac{A_k}{\kappa^{1/3}} \right) \right) \exp \left\{ -\frac{1}{3} \left( \frac{m}{\kappa^{2/3}} \right)^3 - 2a \left( \frac{m}{\kappa^{2/3}} \right) \right\} \]
\[ = \left( 1 + O\left( \frac{A_k}{\kappa^{1/3}} \right) \right) \frac{1}{\sqrt{\pi h}} \sum_{m=m'\kappa}^{m\kappa} \frac{h}{\sqrt{m h}} \exp \left\{ -\frac{(m h)^3}{3} - 2a (m h) \right\} \]
\[ = \frac{\kappa^{1/3}}{\sqrt{\pi}} \left( 1 + O\left( \frac{A_k}{\kappa^{1/3}} \right) \right) \bar{I}_k, \]
where $h = \kappa^{-2/3}$ and $\bar{I}_k$ is an integral sum of $I_k = \int_{m_1 h}^{m_2 h} e^{-x^3/3 - 2ax} \frac{dx}{\sqrt{x}}$.

Since the distance between an integral sum of a monotone function $f(x)$ on $[a, b]$ with uniform mesh $x_i = a + i h$ to the integral is less than $h|f(b) - f(a)|$, we have $|I_k - \bar{I}_k| < \frac{h}{\sqrt{m_1 h}} \sim \frac{1}{\kappa^{1/3}}$.

On the other hand,
\[ \left| I_k - \int_0^\infty e^{-x^3/3 - 2ax} \frac{dx}{\sqrt{x}} \right| < \int_0^{m_1 h} \frac{dx}{\sqrt{x}} + \int_{m_2 h}^\infty e^{-x^3/3} \frac{dx}{\sqrt{x}} \]
\[ = O(\sqrt{m_1 h} + e^{-(m_2 h)^3/3}) = O(k^{-1/6}), \]
which implies that $\bar{I}_k = \left( 1 + O(k^{-1/6}) \right) \int_0^\infty e^{-x^3/3 - 2ax} \frac{dx}{\sqrt{x}}$, and hence
\[ S^{(2)}_\kappa = (A^*)^{-2} \kappa^{1/3} \left( 1 + O(k^{-1/6}) \right), \]
Adding to this the estimates of $S^{(1)}_\kappa$ and $S^{(3)}_\kappa$, we conclude that the same magnitude has the whole sum $S^*_\kappa$, which finishes the proof of the theorem in view of (4.4) and (4.5).

\[ \square \]

5. Concluding remarks

(1) To obtain lower bounds for $\tau_{n,k}$ from Theorem 1.1, sharper than the one in Corollary 1.6, one needs upper estimates for $S_{n,k+1}(\omega)$ and $1 - \omega^2$. Regarding the first quantity, we point out that from considerations in [6, §4] it follows that
\[ S_{n,k}(x) \leq \frac{(2k)!!}{(2k - 1)!!} = \frac{\sqrt{\pi}}{2} \sqrt{k}, \quad x^2 \in [0, 1 - k^2/n^2]. \]
By a result of Driver and Jordaan [2] (see [8] for some improvements), the largest zero of $P_n^{(\lambda)}(x, n, n(\lambda))$, satisfies
\[ 1 - x_{n, n}(\lambda)^2 \leq \frac{(2\lambda + 1)(2\lambda + 3)}{n(n + 2\lambda) + 2(\lambda + 1)(2\lambda + 1)}. \]

This yields the following counterpart to the estimate $1 - \omega^2 \geq \left(\frac{k+2}{n}\right)^2$:
\[ 1 - \omega^2 \leq \frac{(2k + 3)(2k + 5)}{n^2 + 3k^2 + 12k + 11} \leq \left(\frac{k + 2}{n}\right)^2 \frac{4}{1 + 3\left(\frac{k + 2}{n}\right)^2}. \]

We observe that if $k$ is small relative to $n$, then the ratio of the upper and the lower bounds for $1 - \omega^2$ is nearly 4.

(2) From the proof of Theorem 1.7 it is clear that the second exponential term in the approximation of $\tau^*_k$, which is significant, but is missing in the estimate (1.11) in [6], comes from the upper estimate for $\omega$ chosen there. Actually, the method from [6] can cover this term and potentially it can overestimate $\tau_{n,k}$ only by a factor $ck^{5/12}$ (in the area $n > O(k^{3/2})$). Therefore, the upper estimate (1.5), obtained by combining the main result of [6] and the exact formula (1.3), overestimates $\tau_{n,k}$ by a factor $c k^{5/24}$ in the worst case ($n >> k$).

On the contrary, in the area $k \approx n$ both the estimates in [6] and those obtained here are sharp with respect to the order of $k$.

(3) Using the results in [1, §5], we get the asymptotic formula for the largest zero of $P_m^{(p, p)}(x)$, which holds uniformly according to the parameters in the domain $0 < p < Cm$:
\[ x_1 = \sqrt{1 - \tilde{p}^2} - \frac{a \tilde{p}^2}{\sqrt{1 - \tilde{p}^2}^{1/3} p^{-2/3} + O(p^{-4/3})}, \]
where $\tilde{p} = \frac{p}{m + p + 1/2}$ and $a$ is the same constant as in Theorem 1.7. Then, applying this to $\omega$, in the same manner as in the proof of Theorem 1.7 one can obtain the formula
\[ \tau_{n,k} = A \rho^{n/2}_\lambda e^{-a(1-\lambda^2)^{1/3}k^{1/3}} \left( \frac{1}{k^2} - \frac{1}{n^2} \right)^{1/2} \left( 1 + O(k^{-1/6}) \right), \]
where $\lambda = \tilde{p} = \frac{k + 1/2}{n}$ and $\rho_\lambda = \left( \frac{2}{1 + \lambda} \right)^{1 + \lambda} \left( \frac{1 - \lambda}{2} \right)^{1 - \lambda} < 1$ (cf. [6]). The constant for “$O$”-term in (5.1) does not depend on $n$ and $k$, provided $\lambda < 1 - \delta$, i.e. when $k$ is not close to $n$.

(4) As mentioned in the introduction, our interest in $\tau_{n,k}$ is motivated by the role it plays in certain inequalities of Markov- and Landau-type. However, Petras’ result yields an explicit representation for the local maxima of $[P_n^{(\lambda)}]_2^2$, $\lambda \in \mathbb{N}$, and therefore is applicable to the estimation of the largest critical values of the ultraspherical polynomials $P_n^{(\lambda)}$, $\lambda \in \mathbb{N}$.
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