Some examples of log Fano structures on blow-ups along subvarieties in products of two projective spaces

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Abstract
We give a series of examples of log Fano manifolds in any dimension greater than or equal to three by using successive blow-ups along subvarieties in products of two projective spaces. More precisely, we first blow-up a product of two projective spaces along a smooth hypersurface contained in a fiber of one of the two projections and then along the strict transform of a fiber (intersecting the center of the first blow-up) of the other projection. By computing the nef cone and giving an explicit boundary divisor, we show that the resulting variety is always a log Fano manifold. Moreover, the effective cone and the extremal contractions are described. Note that this variety depends on three integral parameters: the dimensions of the two projective spaces and the degree of the hypersurface, which is the center of the first blow-up. Hence, we obtain a series of examples of log Fano manifolds. We also determine which ones are weak Fano.

Keywords Fano varieties · Nef cones · Extremal rays · Birational geometry

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1 Introduction

In the study of higher dimensional Fano varieties, it is interesting to find examples with special geometric structures. The papers [1] and [5] investigate log Fano manifolds obtained by blowing up points in projective spaces or products of projective spaces. It is also natural to consider blow-ups along subvarieties of dimension greater than or equal to one.

Let $Y$ be a smooth projective variety of dimension $n \geq 3$. Let $S$ and $C$ be smooth subvarieties of $Y$ of codimension greater than or equal to two with $\dim S + \dim C = n - 1$. Assume that $S$ and $C$ intersect transversally at one point. Let $X$ be the blow-up of $Y$ along $C$ and $\tilde{X}$ the blow-up of $X$ along the strict transform of $S$. Then, the non-trivial fiber in $\tilde{X}$ over the point $S \cap C$ has an irreducible component isomorphic to the projective space $\mathbb{P}^{\dim S}$.
with ample conormal bundle (cf. [2] Example 2.6). Such $\widetilde{X}$ is expected to be geometrically interesting.

The problem is to determine the triples $(Y, S, C)$ such that $\widetilde{X}$ is (weak) Fano or more generally log Fano. Recall that many examples of smooth Fano 3-folds are obtained by blowing up $\mathbb{P}^3$ (see [6]). Hence, the case $Y = \mathbb{P}^n$ is very important but it seems hard to treat the problem for $n \geq 4$. In this note, we consider the case in which $Y$ is a product of two projective spaces, $S$ is a fiber of a projection, and $C$ is a hypersurface in a fiber of the other projection (see [7] for a result about other types of $S$ and $C$ in $Y = \mathbb{P}^{n-1} \times \mathbb{P}^1$).

We work over the field of complex numbers.

**Theorem 1** Let $Y = \mathbb{P}^{n-k} \times \mathbb{P}^k$ with $n \geq 3$ and $k \in \{2, \ldots, n-1\}$. Let $S$ be a fiber of the projection $Y \to \mathbb{P}^k$. Let $C$ be a smooth hypersurface of degree $d \geq 1$ in a fiber of the other projection $Y \to \mathbb{P}^{n-k}$. Assume $S \cap C \neq \emptyset$. Let $\pi : X \to Y$ be the blow-up along $C$. Let $S'$ be the strict transform of $S$ by $\pi$. Let $\widetilde{X}$ be the blow-up of $X$ along $S'$. Then $\widetilde{X}$ is log Fano. If $n = 3$, $\widetilde{X}$ is weak Fano only for $d \in \{1, 2, 3\}$, and is not Fano for any $d \geq 1$. Assume $n \geq 4$. Then $\widetilde{X}$ is weak Fano if and only if one of the following holds.

1. $d = 1$, $n$ is even and $k = \frac{n}{2}$
2. $d = 1$, $n$ is odd and $k \in \{\frac{n-1}{2}, \frac{n+1}{2}\}$
3. $(n, k, d) = (5, 3, 2)$.

Moreover, $\widetilde{X}$ is Fano only in the case (1).

**Remark 1** The choice of the pair of centers $(S, C)$ is very special. However, the log Fano variety $\widetilde{X}$ has two small contractions if $2 \leq k < n-1$ (see the end of Sect. 4). Hence, it is interesting from the point of view of birational geometry. Note that the Picard number of $\widetilde{X}$ is four by construction. The nef cone of $\widetilde{X}$ is generated by four extremal rays, but the effective cone has five extremal rays if $d \geq 2$ (see Sect. 3). If $k = 1$, $C$ is considered as $d$ points on a fiber of the projection $Y = \mathbb{P}^{n-1} \times \mathbb{P}^1 \to \mathbb{P}^{n-1}$. This case is treated in [8].

Definition: Let $X$ be a smooth projective variety. We call $X$ Fano (resp. weak Fano) if the anti-canonical divisor $-K_X$ is ample (resp. nef and big). A log Fano pair $(X, \Delta)$ is a klt pair such that $-(K_X + \Delta)$ is ample. We call $X$ log Fano if there exists an effective $\mathbb{Q}$-divisor $\Delta$ such that $(X, \Delta)$ is a log Fano pair. The divisor $\Delta$ is called boundary divisor. We have the following implications:

$$\text{Fano} \Rightarrow \text{weak Fano} \Rightarrow \text{log Fano} \Rightarrow -(K_X) \text{ is big}.$$ 

The first implication is obvious. For the second one, see [1] Lemma 2.5. The last one is due to Kodaira’s lemma (see [4] Corollary 2.2.7).

This note is organized as follows. First, we find a nef divisor on a special prime divisor $\widetilde{D} \subset \widetilde{X}$ (Sect. 2). Then, we determine the structure of the nef cone of $\widetilde{X}$ by using the restriction map $\text{Nef}(\widetilde{X}) \to \text{Nef}(\widetilde{D})$. We also compute the effective cone of $\widetilde{X}$ (Sect. 3). Once the nef cone of $\widetilde{X}$ is described explicitly, it is easy to find conditions for $-(K_{\widetilde{X}} + \Delta)$ to be ample. We prove Theorem 1 by solving a system of linear inequalities in the coefficients of the divisor $\Delta$ and the values $n, k$ and $d$ (Sect. 4).

The numerical equivalence class of an $\mathbb{R}$-divisor (or an $\mathbb{R}$-1-cycle) $A$ is denoted by $[A]$. Let $\mathbb{R}^+$ be the set of non-negative real numbers. We put

$$\mathbb{R}^+[A_1, \cdots, A_r] : = \mathbb{R}^+[A_1] + \cdots + \mathbb{R}^+[A_r].$$
2 A nef divisor on a special prime divisor

We use the following lemmas to determine the nef cone of $\widetilde{X}$ in Theorem 1.

Lemma 1 ([7] Lemma 1.1) Let $X$ be a smooth projective variety of Picard number $r$. Assume that there exist $r$ nef divisors $N_1, \ldots, N_r$ and $r$ curves $\Gamma_1, \ldots, \Gamma_r$ on $X$ such that $N_i \cdot \Gamma_j = \delta_{ij}$ (Kronecker delta) for any $i, j \in \{1, \ldots, r\}$. Then we have

$$\text{Nef}(X) = \mathbb{R}^+ [N_1, \ldots, N_r] \quad \text{and} \quad \overline{\text{NE}}(X) = \mathbb{R}^+ [\Gamma_1, \ldots, \Gamma_r].$$

Lemma 2 ([7] Lemma 1.2) Let $X$ be a smooth projective variety and let $D$ be a smooth prime divisor on $X$. Let $N$ be a divisor on $X$ such that $[N - D] \in \text{Nef}(X)$. If $[N]_D \in \text{Nef}(D)$, then $[N] \in \text{Nef}(X)$.

Lemma 3 Let $X$ be a smooth projective variety and $N$ a divisor on $X$. Let $X = D_0 \supset D_1 \supset \cdots \supset D_k$ be a sequence of smooth subvarieties. Put $N_0 := N$, $N_i := N|_{D_i}$, $N_k := N|_{D_k}$. Assume that $\dim D_{i+1} = \dim D_i - 1$ and $[N_i - D_{i+1}] \in \text{Nef}(D_i)$ for any $i \in \{0, 1, \ldots, k-1\}$. If $[N_k] \in \text{Nef}(D_k)$, then $[N] \in \text{Nef}(X)$.

Proof. Let $i \in \{0, 1, \ldots, k-1\}$. Let $C$ be a curve on $D_i$. If $C \not\subset D_{i+1}$, then

$$N_i \cdot C = (N_i - D_{i+1}) \cdot C + D_{i+1} \cdot C \geq 0 + 0 = 0.$$

If $C \subset D_{i+1}$, we have

$$N_i \cdot C = N|_{D_i} \cdot C = N \cdot C = N|_{D_{i+1}} \cdot C = N_{i+1} \cdot C.$$

Hence, $[N_{i+1}] \in \text{Nef}(D_{i+1})$ implies $[N_i] \in \text{Nef}(D_i)$. Thus, if $[N_k] \in \text{Nef}(D_k)$, we conclude that $[N] \in \text{Nef}(D)$. \qed

Remark 2 Lemma 2 is a special case ($k = 1$) of Lemma 3.

We fix the following notation (valid only in this section):

Let $m$ and $k$ be natural numbers such that $m \geq 1$ and $k \geq 2$. Let $V$ be a smooth hypersurface of degree $d \geq 1$ in $\mathbb{P}^k$. Put $D := \mathbb{P}^m \times V$. Let $p : D \to \mathbb{P}^m$ and $q : D \to V$ be the projections. Let $C$ (resp. $S$) be a fiber of $p$ (resp. $q$). Note that $C$ (resp. $S$) is isomorphic to $V$ (resp. $\mathbb{P}^m$).

If $m = 1$ (resp. $k = 2$), then $C$ (resp. $S$) is a divisor on $D$. Let $H \in |p^*O_{\mathbb{P}^m}(1)|$ and $L \in |q^*O_V(1)|$ where $O_V(1) := O_{\mathbb{P}^k}(1)|_V$. We define the morphisms $\pi : D' \to D$ and $\beta : D \to D'$ as follows.

- If $C$ is not a divisor on $D$, let $\pi : D' \to D$ be the blow-up along $C$. We put $E := \text{Exc}(\pi)$, $H' := \pi^*H$, $L' := \pi^*L$ and $S' := \pi^{-1}_sS$. If $C$ is a divisor on $D$, we put $\pi := id$, $D' := D$, $E := C$, $H' := H$, $L' := L$ and $S' = S$.

- If $S'$ is not a divisor on $D'$, let $\beta : D \to D'$ be the blow-up along $S'$. We put $F := \text{Exc}(\beta)$, $\tilde{H} := \beta^*H'$, $\tilde{L} := \beta^*L'$ and $\tilde{E} := \beta^*E$. If $S'$ is a divisor on $D'$, we put $\beta := id$, $D' := D'$, $F := S'$, $\tilde{H} := H'$, $\tilde{L} := L'$ and $\tilde{E} := E$.

Proposition 1 We have $[\tilde{H} + d\tilde{L} - \tilde{E} - F] \in \text{Nef}(D)$.

Proof Note that $\tilde{L}$ is nef. Since $\tilde{H} + d\tilde{L} - \tilde{E} - F = (d - 1)\tilde{L} + (\tilde{H} - \tilde{E}) + (\tilde{L} - \tilde{F})$, the statement follows from the following two lemmas. \qed

Lemma 4 $\tilde{H} - \tilde{E}$ is nef.
Proof Case $m = 1$: $C$ is a divisor on $D = \mathbb{P}^1 \times V$ and $C \in |p^* \mathcal{O}_{\mathbb{P}^1}(1)|$. Hence, $H' = H \sim C = E$ and $\tilde{H} - \tilde{E} = \beta^*(H' - E) \sim 0$.

Case $m \geq 2$: Consider the blow-up $\varepsilon : \text{Bl}_c(\mathbb{P}^m) \to \mathbb{P}^m$ at the point $c := p(C)$. Let $p' : D' \simeq \text{Bl}_c(\mathbb{P}^m) \times V \to \text{Bl}_c(\mathbb{P}^m)$ be the projection. Then, we have the commutative diagram:

$$
\begin{array}{ccc}
D' & \xrightarrow{p'} & \text{Bl}_c(\mathbb{P}^m) \\
\pi & & \varepsilon \\
D & \xrightarrow{p} & \mathbb{P}^m.
\end{array}
$$

Note that $E = \text{Exc}(\pi) = (p')^*\text{Exc}(\varepsilon)$. Let $H_{\mathbb{P}^m}$ be a hyperplane on $\mathbb{P}^m$. We see that $H' - E \sim (p')^*(\varepsilon^* H_{\mathbb{P}^m} - \text{Exc}(\varepsilon))$ is nef. Hence so is $\tilde{H} - \tilde{E} = \beta^*(H' - E)$. \hfill $\Box$

Lemma 5 $\tilde{L} - F$ is nef.

Proof Let $\mathbb{P}^k \supset G_1 \supset \cdots \supset G_{k-2}$ be a sequence of linear subspaces such that $G_i \simeq \mathbb{P}^{k-i}$, $G_i \not\subseteq V$, and $V_i := V \cap G_i$ is smooth for any $i \in \{1, \cdots, k-2\}$. We put $V_0 := V$. Put $D_i := q^{-1}(V_i), D'_i := \pi^{-1}_* D_i, \tilde{D}_i := \beta^{-1}_* D'_i$ for $i \in \{1, \cdots, k-2\}$ and $D_0 := D, D'_0 := D', D_0 := D$. Note that $D_1 \simeq \mathbb{P}^m \times V_i$ and $\dim V_i = k - i - 1$. Assume that the point $s := q(S)$ is contained in $G_{k-2}$. Then we have $s \in V_i = V \cap G_i$ and the fiber $S = q^{-1}(s)$ of the projection $q : D \to V$ is contained in $D_i$ for any $i \in \{0, 1, \cdots, k-2\}$.

By the definition of the divisor $L$, there exists a hyperplane $H_{\mathbb{P}^k} \subset \mathbb{P}^k$ such that $L \sim q^*(H_{\mathbb{P}^k}|_V)$. Note that $V_{i+1}$ is linearly equivalent to $H_{\mathbb{P}^k}|_{V_i}$ as a divisor on $V_i$. Hence,

$$L|_{D_i} \sim (q|_{V_i})^*(H_{\mathbb{P}^k}|_{V_i}) \sim (q|_{V_i})^* V_{i+1} = D_{i+1}.$$  

Put $N := \tilde{L} - F, N_0 := N$ and $N_i := N|_{\tilde{D}_i}$ ($i = 1, \cdots, k-2$).

Recall that $V_{k-2}$ is a smooth curve of degree $d \geq 1$ on $G_{k-2} \simeq \mathbb{P}^2$ and $S$ is a fiber of the projection $q|_{D_{k-2}} : D_{k-2} \to V_{k-2}$. We have $L|_{D_{k-2}} \equiv dS$ because $L|_{D_{k-2}} \in |(q|_{D_{k-2}})^*(\mathcal{O}_{\mathbb{P}^k}(1)|_{V_{k-2}})|$. Since $S$ is a divisor on $D_{k-2}$ (hence so is $S'$ on $D'_{k-2}$), $\beta|_{D_{k-2}} : \tilde{D}_{k-2} \to D'_{k-2}$ is an isomorphism. Therefore,

$$N_{k-2} = (\tilde{L} - F)|_{\tilde{D}_{k-2}} = L'|_{D'_{k-2}} - S' = (\pi|_{D'_{k-2}})^*(L|_{D_{k-2}} - S).$$

Since $L|_{D_{k-2}} - S \equiv (d-1)S$ is nef, we have $[N_{k-2}] \in \text{Nef}(D_{k-2})$.

Let $i \in \{0, 1, \cdots, k-3\}$. Consider $D_{i+1}$ as a divisor on $D_i$. Note that $\pi|_{D'_i} : D'_i \to D_i$ is the blow-up along $C \cap D_i$ for $m \geq 2$ and is an isomorphism for $m = 1$, where $\pi|_{D'_0}$ means $\pi$ itself. In any case, we have $(\pi|_{D'_i})^* D_{i+1} = D_{i+1}$. Recall that $S \subset D_{i+1}$ hence $S' \subset D'_{i+1}$. We see that $\beta|_{\tilde{D}_i} : \tilde{D}_i \to D'_i$ is the blow-up along $S'$ whose exceptional divisor is $F|_{\tilde{D}_i}$, where $\beta|_{\tilde{D}_0}$ and $F|_{\tilde{D}_0}$ means $\beta$ and $F$ respectively. Therefore,

$$((\pi \circ \beta)|_{\tilde{D}_i})^* D_{i+1} = \tilde{D}_{i+1} + F|_{\tilde{D}_i}.$$  

On the other hand,

$$((\pi \circ \beta)|_{\tilde{D}_i})^* D_{i+1} \sim ((\pi \circ \beta)|_{\tilde{D}_i})^* (L|_{D_i}) = \tilde{L}|_{\tilde{D}_i}.$$  

We conclude that $\tilde{D}_{i+1} \sim \tilde{L}|_{\tilde{D}_i} - F|_{\tilde{D}_i} = N_i$. Thus $[N_i - \tilde{D}_{i+1}] = 0 \in \text{Nef}(\tilde{D}_i)$.

The proof completes by applying Lemma 3 to the sequence $\tilde{D} = \tilde{D}_0 \supset \tilde{D}_1 \supset \cdots \supset \tilde{D}_{k-2}$ and the divisor $N = \tilde{L} - F$ on $\tilde{D}$. \hfill $\Box$
Remark 3 The proofs of the lemmas above depend purely on the nefness of divisors and do not require any subtle argument of base point freeness. Hence, it is expected that our method in this section can be applied to investigate the nef cones of other types of projective varieties.

3 Structure of nef cone and effective cone

From now on, we fix the following.

Notation (*): Let $n$ and $k$ be natural numbers such that $n \geq 3$ and $2 \leq k \leq n - 1$. Put $Y := \mathbb{P}^{n-k} \times \mathbb{P}^k$. Let $p : Y \to \mathbb{P}^{n-k}$ and $q : Y \to \mathbb{P}^k$ be the two projections. Let $H \in |p^*O_{\mathbb{P}^{n-k}}(1)|$ and $L \in |q^*O_{\mathbb{P}^k}(1)|$. Let $P_0$ be a fiber of $p : Y \to \mathbb{P}^{n-k}$. Let $C$ be a smooth hypersurface of degree $d \geq 1$ in $P_0 \cong \mathbb{P}^k$. Let $S$ be a fiber of $q : Y \to \mathbb{P}^k$ such that $S \cap C \neq \emptyset$. Put $Y_0 := S \cap C$. Let $h$ be a line in a fiber of $p$ such that $h \cap S = \emptyset$ and $h \cap C = \emptyset$. Let $l$ be a line in a fiber of $q$ such that $l \cap S = \emptyset$ and $l \cap C = \emptyset$. Let $h_0$ be a line in $P_0 \cong \mathbb{P}^k$ such that $h_0 \not\subset C$ and $y_0 \neq h_0$. Let $l_0$ be a line in a fiber of $q$ such that $l_0 \not\subset S$ and $l_0 \cap C = \emptyset$. Put $D := q^{-1}(q(C))$. Note that we have $D \sim dL$.

Let $\pi : X \to Y$ be the blow-up along $C$. Let $E := \text{Exc}(\pi)$ and $E_0 := \pi^{-1}(y_0)$. Put $S' := \pi^{-1}_*S$ and $P_0' := \pi^{-1}_*P_0$. We have $S \cap P_0 = y_0$, but $S' \cap P_0' = \emptyset$. Let $e_0$ be a line in $E_0 \cong \mathbb{P}^{n-k}$ such that $e_0 \not\subset S'$. Let $e$ be a line in a fiber different from $E_0$ of the $\mathbb{P}^{n-k}$-bundle $\pi|_E : E \to C$ (if $k = n-1$, then $e$ and $e_0$ are fibers of the $\mathbb{P}^1$-bundle $\pi|_E$). Put $H' := \pi^*H$ and $L' := \pi^*L$. Let $D', h', l', h'_0$ and $l'_0$ be the strict transforms by $\pi$ of $D, h, l, h_0$ and $l_0$, respectively.

Let $\beta : \tilde{X} \to X$ be the blow-up along $S'$. Put $F := \text{Exc}(\beta)$. Let $f$ be a line in a fiber of the $\mathbb{P}^{k-1}$-bundle $\beta|_F : F \to S'$ (if $k = 2$ then $f$ is a fiber of the $\mathbb{P}^1$-bundle $\beta|_F$). Put $\tilde{H} := \beta^*H'$, $\tilde{L} := \beta^*L'$, $\tilde{E} := \beta^*E$, $\tilde{E}_0 := \beta^{-1}_*E_0$ and $P_0 := \beta^{-1}_*P_0$. Finally, let $\tilde{d}, \tilde{h}, \tilde{l}, \tilde{h}_0, \tilde{l}_0, \tilde{e}$ and $\tilde{e}_0$ be the strict transforms by $\beta$ of $D', h', l', h'_0, l'_0, e$ and $e_0$, respectively.

Lemma 6 We have the following table of intersection numbers.

| $\tilde{H}$ $\tilde{L}$ $\tilde{E}$ $-d\tilde{L}$ $-\tilde{E}$ $\tilde{H}$ $+d\tilde{L}$ $-\tilde{E}$ $-F$ |
|-----------------|----------------|----------------|----------------|----------------|
| $l_0$           | 1              | 1              | 0              | 0              |
| $\tilde{h}_0$   | 0              | 1              | 0              | 0              |
| $\tilde{e}_0$   | 0              | 0              | 1              | 0              |
| $f$             | 0              | 0              | 0              | 1              |

Proof By construction, we have the following intersection numbers:

| $\tilde{H}$ $\tilde{L}$ $\tilde{E}$ $F$ |
|-----------------|----------------|----------------|----------------|
| $l$             | 1              | 0              | 0              | 0              |
| $\tilde{h}$     | 0              | 0              | 0              | 0              |
| $\tilde{e}$     | 0              | 1              | 0              | 0              |
| $f$             | 0              | 0              | -1             | 0              |

We have $e \equiv e_0$ in $X$. Note that $e_0$ intersects $S'$ transversally at one point. Hence $\tilde{e} \equiv \tilde{e}_0 + f$ in $\tilde{X}$. We have $l \equiv l_0$ in $Y$. Since $l_0$ intersects $C$ transversally at one point, we have $l' \equiv l'_0 + e$ in $X$. This implies $\tilde{l} \equiv \tilde{l}_0 + \tilde{e}$ in $\tilde{X}$. Note that $h \equiv h_0$ and $C \cdot h_0 = d$ in $P_0 \cong \mathbb{P}^k$. Hence
\( h' \equiv h'_0 + d \varepsilon \) and \( \tilde{h} \equiv \tilde{h}_0 + d \tilde{\varepsilon} \) because \( S' \cap h'_0 = \emptyset \). Therefore, we have the following table.

| \( \tilde{h}_0 \equiv \tilde{h} - d \tilde{\varepsilon} \) | 1 | 0 | \( \tilde{L} \) | \( \tilde{E} \) | \( \tilde{B} \) | \( \tilde{L} \) | \( \tilde{E} \) | \( \tilde{h} + d \tilde{L} - \tilde{E} \) | \( \tilde{h} + d \tilde{L} - \tilde{E} - F \) |
|---|---|---|---|---|---|---|---|---|---|
| \( \tilde{h}_0 \equiv \tilde{h} - d \varepsilon \) | 0 | 1 | \( \tilde{E} \) | \( \tilde{E} \) | \( \tilde{B} \) | \( \tilde{L} \) | \( \tilde{E} \) | \( \tilde{h} + d \tilde{L} - \tilde{E} \) | \( \tilde{h} + d \tilde{L} - \tilde{E} - F \) |

\[ \square \]

**Proposition 2** We have

\[ \text{Nef}(\tilde{X}) = \mathbb{R}^+ [\tilde{H}, \tilde{L}, \tilde{H} + d \tilde{L} - \tilde{E}, \tilde{H} + d \tilde{L} - \tilde{E} - F]. \]

**Proof** We have \([\tilde{H}], [\tilde{L}] \in \text{Nef}(\tilde{X})\) and \([\tilde{h}_0], [\tilde{h}_0], [\tilde{e}_0], [\tilde{f}] \in \text{NE}(\tilde{X})\). Recall that \( \rho(\tilde{X}) = 4 \). By Lemmas 1 and 6, it is sufficient to show that \( \tilde{H} + d \tilde{L} - \tilde{E} \) and \( \tilde{H} + d \tilde{L} - \tilde{E} - F \) are nef.

First, we show that \([\tilde{H} + d \tilde{L} - \tilde{E} - F] \in \text{Nef}(\tilde{X})\). Put \( V := q(C) \). We have \( D \cong \mathbb{P}^{n-k} \times V \). Consider the projections \( p_D : D \to \mathbb{P}^{n-k} \) and \( q_D : D \to V \). Note that \( C \) is a fiber of \( p_D \) and \( S \) is a fiber of \( q_D \). If we put \( m := n - k \), then the divisors \( \tilde{H} \mid D, \tilde{L} \mid D, \tilde{E} \mid D \) and \( F \mid D \) correspond exactly to the divisors \( \tilde{H}, \tilde{L}, \tilde{E} \) and \( F \) in Sect. 2. Hence, by Proposition 1, we have \([\tilde{H} + d \tilde{L} - \tilde{E} - F] \in \text{Nef}(\tilde{D})\). Note that \( \tilde{D} \sim d \tilde{L} - \tilde{E} - F \). By Lemma 2, we conclude that \([\tilde{H} + d \tilde{L} - \tilde{E} - F] \in \text{Nef}(\tilde{X})\).

Now, we show that \([\tilde{H} + d \tilde{L} - \tilde{E}] \in \text{Nef}(\tilde{X})\). If \( n - k = 1 \), we have \( S' \simeq S \simeq \mathbb{P}^1 \), \( H' \mid S' \simeq O_{\mathbb{P}^1}(1) \) and \( E \mid S' \simeq O_{\mathbb{P}^1}(1) \). Hence, \( H' \mid S' - E \mid S' \sim 0 \). If \( n - k \geq 2 \), then \( \pi \mid S' : S' \to S \) is the blow-up at the point \( y_0 \). Let \( H_{\mathbb{P}^{n-k}} \) be a hyperplane on \( S \simeq \mathbb{P}^{n-k} \). We see that \( H' \mid S', E \mid S' = (\pi \mid S')^* (H_{\mathbb{P}^{n-k}} - \text{Exc}(\pi \mid S')) \) is nef. Since \( L \mid S \sim 0 \), we have \( L' \mid S' \sim 0 \). Hence \( [H' + dL' - E] \mid S' = [H' - E] \mid S' \in \text{Nef}(S') \). Thus,

\[ [\tilde{H} + d \tilde{L} - \tilde{E}] \mid F = (\beta \mid F)^* ([H' + dL' - E] \mid S') \in \text{Nef}(F). \]

Since \([\tilde{H} + d \tilde{L} - \tilde{E} - F] \in \text{Nef}(\tilde{X})\) as shown above, we have \([\tilde{H} + d \tilde{L} - \tilde{E}] \in \text{Nef}(\tilde{X})\) by Lemma 2.

\[ \square \]

**Remark 4** \( H' - E \) is not nef because \( (H' - E) \mid p_0^* \simeq O_{\mathbb{P}^d}(-d) \).

Let \( H_0 \in [H \otimes \mathcal{I}_B] \) and \( L_0 \in [L \otimes \mathcal{I}_S] \). If \( d = 1 \), we assume that \( L_0 \) does not contain \( C \). Let \( \tilde{H}_0 \) and \( \tilde{L}_0 \) be the strict transforms by \( \pi \circ \beta \) of \( H_0 \) and \( L_0 \) respectively.

**Proposition 3** We have

\[ \text{Eff}(\tilde{X}) = \mathbb{R}^+ [\tilde{H}_0, \tilde{L}_0, \tilde{E}, F, \tilde{D}]. \]

**Proof** The divisors \( \tilde{H}_0, \tilde{L}_0, \tilde{E}, F, \tilde{D} \) are all effective. Hence, the right-hand side is included in the left-hand side. Let \( A \subset \tilde{X} \) be a prime divisor. Assume \( A \neq \{ \tilde{E}, F, \tilde{D} \} \). Since \( B := (\pi \circ \beta)(A) \) is a prime divisor on \( Y = \mathbb{P}^{n-k} \times \mathbb{P}^d \), there exist \( a, b \in \mathbb{Z}_{\geq 0} \) such that \( B \sim aH + bL \). Put \( \mu := \text{mult}_C B \) and \( \nu := \text{mult}_S B \). Since \( A = (\pi \circ \beta)^{-1} B \), we have \( A \sim aH + bL - \mu \tilde{E} - \nu F \).

Remark that if \( q^{-1}(u) \subset B \) for any \( u \in q(C) \), we have \( D = q^{-1}(q(C)) \subset B \), hence \( B = D \) which contradicts the assumption \( A \neq \tilde{D} \). Let \( u \) be a point on \( q(C) \) such that \( q^{-1}(u) \not\subset B \).

Put \( Y_u := q^{-1}(u) \) and \( c_u := C \cap Y_u \). Note that there exists a line \( l_u \subset Y_u \simeq \mathbb{P}^{n-k} \) such that \( c_u \in l_u \) and \( l_u \not\subset B \). If \( n - k = 1 \), we put \( l_u := Y_u \simeq \mathbb{P}^1 \). Since \( B \sim aH + bL \) and \( l_u \equiv l \), we have \( B \cdot l_u = a \) if \( n - k = 1 \), we have \( B \mid Y_u \cdot l_u = \text{deg}(B \mid Y_u) \).

Therefore,

\[ a = B \cdot l_u = B \mid Y_u \cdot l_u \geq \text{mult}_{c_u} (B \mid Y_u) \geq \text{mult}_C B \cdot \mu. \]
Since $B$ is a divisor on $Y = \mathbb{P}^{n-k} \times \mathbb{P}^k$, there exists a point $v \in \mathbb{P}^{n-k}$ such that $p^{-1}(v) \not\subset B$. Put $P_v := p^{-1}(v)$ and $s_v := S \cap P_v$. Note that there exists a line $h_v \subset P_v \simeq \mathbb{P}^k$ such that $s_v \in h_v$ and $h_v \not\subset B$. Since $B \sim aH + bL$ and $h_v \equiv h$, we have $B \cdot h_v = b$. Hence,

$$b = B \cdot h_v = B|_{P_v} \cdot h_v \geq \mult_{s_v}(B|_{P_v}) \geq \mult_s B = v.$$

Note that $\tilde{H} \sim \tilde{H}_0 + \tilde{E}$ and $\tilde{L} \sim \tilde{L}_0 + \tilde{F}$. We conclude that

$$[A] = [a\tilde{H} + b\tilde{L} - \mu \tilde{E} - vF] = [a\tilde{H}_0 + b\tilde{L}_0 + (a - \mu)\tilde{E} + (b - v)\tilde{F}] \in \mathbb{R}^+[\tilde{H}_0, \tilde{L}_0, \tilde{E}, \tilde{F}].$$

\[\square\]

**Remark 5** Since $\tilde{D} \sim d\tilde{L} - \tilde{E} - F \sim d\tilde{L}_0 - \tilde{E} + (d-1)\tilde{F}$, we have $[\tilde{D}] \not\in \mathbb{R}^+[\tilde{H}_0, \tilde{L}_0, \tilde{E}, \tilde{F}]$ for any $d \geq 2$. If $d = 1$, we have $\tilde{L}_0 \sim \tilde{D} + \tilde{E}$ and

$$\mathbb{R}^+[\tilde{H}_0, \tilde{L}_0, \tilde{E}, \tilde{F}, \tilde{D}] = \mathbb{R}^+[\tilde{H}_0, \tilde{E}, \tilde{F}, \tilde{D}].$$

## 4 Proof of theorem

We continue to use Notation $(\ast)$ in Sect. 3. First, we find conditions for $-(K_X + \Delta)$ to be ample where $\Delta$ is a $\mathbb{Q}$-divisor on $X$.

**Lemma 7** Consider a divisor $\Delta = \alpha \tilde{H} + \beta \tilde{L} + \gamma \tilde{E} + \delta \tilde{F}$ with $\alpha, \beta, \gamma, \delta \in \mathbb{Q}$. Then, $-(K_X + \Delta)$ is ample if and only if

$$\begin{align*}
\alpha + \gamma &< 1 \\
\beta + d\gamma &< k + 1 - (n-k)d \\
-\gamma + \delta &< n - 2k + 1 \\
-\delta &< k - 1.
\end{align*}$$

**Proof** Since $Y = \mathbb{P}^{n-k} \times \mathbb{P}^k$, we have $-K_Y \sim (n-k+1)H + (k+1)L$. By the adjunction formula for the blow-ups $\pi$ and $\beta$,

$$-K_X \sim \beta^*(-K_X) - (k-1)F$$

$$\sim \beta^*(\pi^*(-K_Y) - (n-k)E) - (k-1)F$$

$$\sim (n-k+1)\tilde{H} + (k+1)\tilde{L} - (n-k)\tilde{E} - (k-1)F.$$

Hence, $-(K_X + \Delta)$ is linearly equivalent to

$$(n-k+1-\alpha)\tilde{H} + (k+1-\beta)\tilde{L} - (n-k+\gamma)\tilde{E} - (k-1+\delta)F$$

$$= (1-\alpha-\gamma)\tilde{H} + (k+1+d\gamma - dn - \beta - d\gamma)\tilde{L}$$

$$+(n-2k+1+\gamma-\delta)(\tilde{H} + d\tilde{L} - \tilde{F}) + (k-1+\delta)(\tilde{H} + d\tilde{L} - \tilde{E} - F).$$

By Proposition 2, we conclude that $-(K_X + \Delta)$ is ample (i.e. the numerical equivalence class is an interior point of the nef cone) if and only if

$$\begin{align*}
1 - \alpha - \gamma &> 0 \\
k + 1 + d\gamma - dn - \beta - d\gamma &> 0 \\
n - 2k + 1 + \gamma - \delta &> 0 \\
k - 1 + \delta &> 0.
\end{align*}$$

\[\square\]
Remark 6 We obtain the condition for $-(K_{\tilde{X}} + \Delta)$ to be nef if we replace “$$<$$” by “$$\leq$$”.

Lemma 8 $-K_{\tilde{X}}$ is ample (resp. nef) if and only if

\[
\begin{align*}
0 < k + 1 - (n - k)d \\
0 < n - 2k + 1
\end{align*}
\text{ (resp. } \begin{align*}
0 \leq k + 1 - (n - k)d \\
0 \leq n - 2k + 1
\end{align*} \).
\]

Proof We put $\Delta = 0$ in Lemma 7.

We prove Theorem 1, which is divided into the following Propositions.

Proposition 4 $\tilde{X}$ is log Fano.

Proof Let $H_1, \ldots, H_{n-k}$ be general members of $|H \otimes I_P|$\). Let $L_1, \ldots, L_{k-1}$ be general members of $|L \otimes I_S|$. Let $\tilde{H}_i$ and $\tilde{L}_j$ be the strict transforms in $\tilde{X}$ of $H_i$ and $L_j$ respectively. Put

\[
\lambda := n - k - \frac{1}{d}, \quad \mu := k - 1 - \frac{1}{2d}.
\]

Note that we have $\frac{\lambda}{n-k}, \frac{\mu}{k-1} \in [0, 1)$. Consider the $\mathbb{Q}$-divisor

\[
\Delta = \frac{\lambda}{n-k} \sum_{i=1}^{n-k} \tilde{H}_i + \frac{\mu}{k-1} \sum_{j=1}^{k-1} \tilde{L}_j.
\]

Since the divisors $H_1, \ldots, H_{n-k}$ and $L_1, \ldots, L_{k-1}$ are chosen to be general, $\Delta$ has simple normal crossing support. Hence $(\tilde{X}, \Delta)$ is a klt pair. Since $\tilde{H}_i \sim \tilde{H} - \tilde{E}$ and $\tilde{L}_j \sim \tilde{L} - F$ for any $i \in \{1, \ldots, n-k\}$ and $j \in \{1, \ldots, k-1\}$, we have

\[
\Delta \sim \lambda(\tilde{H} - \tilde{E}) + \mu(\tilde{L} - F) = \lambda \tilde{H} + \mu \tilde{L} - \tilde{E} - \mu F.
\]

Applying Lemma 7 to $\alpha = \lambda$, $\beta = \mu$, $\gamma = -\lambda$ and $\delta = -\mu$, we conclude that $-(K_{\tilde{X}} + \Delta)$ is ample.

Remark 7 In particular, $-K_{\tilde{X}}$ is big. Hence, $\tilde{X}$ is weak Fano if and only if $-K_{\tilde{X}}$ is nef.

Proposition 5 Assume $n = 3$ (hence $k = 2$). Then $\tilde{X}$ is weak Fano if and only if $d \in \{1, 2, 3\}$, and is not Fano for any $d \geq 1$.

Proof By Lemma 8, $-K_{\tilde{X}}$ is nef if and only if $d \in \{1, 2, 3\}$, and is not ample for any $d \geq 1$.

Proposition 6 Assume $n \geq 4$. Then $\tilde{X}$ is weak Fano if and only if

\[ (n, k, d) = (2k - 1, k, 1), (2k, k, 1), (2k + 1, k, 1) \text{ or } (5, 3, 2). \]

Moreover, $\tilde{X}$ is Fano if and only if $(n, k, d) = (2k, k, 1)$.

Proof By Lemma 8, $-K_{\tilde{X}}$ is nef if and only if

\[
\begin{align*}
\frac{dn}{2k - 1} &\leq (d + 1)k + 1 \\
2k - 1 &\leq n.
\end{align*}
\]
This yields $d(2k - 1) \leq dn \leq (d + 1)k + 1$. In particular, we have $d(2k - 1) \leq (d + 1)k + 1$, which implies

$$d \leq \frac{k + 1}{k - 1} = 1 + \frac{2}{k - 1} \leq 1 + 2 = 3.$$ 

If $d = 1$, we have $2k - 1 \leq n \leq 2k + 1$. If $d = 2$, we have $4k - 2 \leq 2n \leq 3k + 1$. Since $n \geq 4$, we have $8 \leq 3k + 1$, i.e. $k \geq 3$. On the other hand, we have $4k - 2 \leq 3k + 1$, i.e. $k \leq 3$. Thus, $k = 3$ and $n = 5$. If $d = 3$, we have $6k - 3 \leq 3n \leq 4k + 1$. Since $n \geq 4$, we have $12 \leq 4k + 1$, i.e. $k \geq 3$. On the other hand, we have $6k - 3 \leq 4k + 1$, i.e. $k \leq 2$, a contradiction. Therefore, we have

$$\frac{n}{k} \in \{2, 3, 4\}.$$ 

In any of these cases, $k + 1 - (n - k)d$ and $n - 2k + 1$ are non-negative, and strictly positive only for the case $(n, k, d) = (2k - 1, k, 1)$. \hfill \Box

**Remark 8** By Lemma 1, 6 and Proposition 2, we have

$$\text{NE}(\tilde{X}) = \mathbb{R}^+[\tilde{l}_0] + \mathbb{R}^+[\tilde{l}_0] + \mathbb{R}^+[[\tilde{z}_0]] + \mathbb{R}^+[f].$$

By Contraction Theorem ([3] Theorem 3.7 (3)), there exist contraction morphisms of these four extremal rays. The blow-up $\beta : \tilde{X} \to X$ corresponds to the extremal ray $\mathbb{R}^+[f]$. Let $\varphi_1, \varphi_2$ and $\varphi_3$ be the contraction associated to $\mathbb{R}^+[\tilde{l}_0], \mathbb{R}^+[[\tilde{z}_0]]$ and $\mathbb{R}^+[[\tilde{z}_0]]$ respectively.

- $\varphi_1$ is a divisorial contraction. The exceptional divisor is $\tilde{D}$. Put $W := \varphi_1(\tilde{D})$. Then the restriction morphism $\varphi_1|_{\tilde{D}} : \tilde{D} \to W$ coincides with the $\mathbb{P}^1$-bundle $\text{Bl}_{\text{pt}}(\mathbb{P}^{n-k}) \times C \to \mathbb{P}^{n-k} \times C$, where $\text{Bl}_{\text{pt}}(\mathbb{P}^{n-k})$ denotes the blow-up of $\mathbb{P}^{n-k}$ at a point.

- $\varphi_2$ contracts $\tilde{l}_0 \simeq \mathbb{P}^k$ to a point. Hence, $\varphi_2$ is divisorial for $k = n - 1$ and is small for $k < n - 1$.

- $\varphi_3$ contracts $\tilde{E}_0 \simeq \mathbb{P}^{n-k}$ to a point. Hence, $\varphi_3$ is a small contraction because $\text{codim}_{\tilde{X}} \tilde{E}_0 = n - (n - k) = k \geq 2$.

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**Conflict of interest** The author has no conflict of interest associated with this article.

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