Logarithmic Gradient Transformation and Chaos Expansion of Itô Processes

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Abstract
Since the seminal work of Wiener (Am J Math 60:897–936, 1938), chaos expansion has evolved to a powerful methodology for studying a broad range of stochastic differential equations. Yet its complexity for systems subject to the white noise remains significant. The issue appears due to the fact that the random increments generated by the Brownian motion result in a growing set of random variables with respect to which the process could be measured. In order to cope with this high dimensionality, we present a novel transformation of stochastic processes driven by the white noise. In particular, we show that under suitable assumptions, the diffusion arising from white noise can be cast into a logarithmic gradient induced by the measure of the process. Through this transformation, the resulting equation describes a stochastic process whose randomness depends only on the initial condition. Therefore, the stochasticity of the transformed system lives in the initial condition and it can be treated conveniently with chaos expansion tools.

Keywords Itô process · Chaos expansion · Fokker–Planck equation

Mathematics Subject Classification 60H10 · 35Q84 · 60J60

1 Introduction
Often stochastic descriptions of natural or social phenomena lead to more realistic mathematical models. The introduced stochastic notion may either arise from the uncertainty in the model inputs or from the underlying governing law. In particular, the white noise manifests itself in both circumstances, for example, as a random force

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acting on a deterministic system in the Landau–Lifschitz fluctuating hydrodynamics
[12] or as a Markovian process describing rarefied gases [8] or polymers [18].

The Monte Carlo methods are typically a natural choice for computational studies of
the systems driven by the white noise. Yet the slow convergence rate of the brute force
Monte Carlo motivates a quest for improved approaches. There exists an immense
list of advanced Monte Carlo techniques, each of which may yield to a substantial
improvement over the conventional Monte Carlo, provided certain regularities. One
of the promising examples belongs to the multi-level Monte Carlo approach [7] (and
its variants [9]). In short, MLMC makes use of abundant samples on a coarse-scale
discretization in order to improve the convergence rate of the fine-scale one. This can
be achieved by enforcing correlations between successive approximations, usually
through employing common random numbers among them.

Instead of producing numerical samples of a random variable, however, one can
expand the solution with respect to a set of (orthogonal) random functions which
possess a known distribution [29]. The polynomial chaos and stochastic collocation
schemes are among the main approaches built around this idea [27,28]. In particular,
the polynomial chaos schemes transform the random differential equations to a set of
deterministic equations, by which the evolution of the coefficients introduced in the
polynomial expansion of the random solution is governed. Therefore, by knowing the
distribution of the resulting orthogonal functions, different statistics of the solution can
be computed deterministically. While this approach may lead to efficient computations
for equations pertaining a finite set of random variables, its application to the Brownian
motion remains with a significant computational challenge. The problem arises due to
the fact that the dimension of the expansion should grow in time in order to keep the
solution measurable with respect to the Brownian motion [10]. Hence, the cost of the
chaos expansion schemes grows here significantly, in comparison with the counterpart
scenario where the solution remains measurable with respect to a fixed set of random
variables.

This study addresses the problem of deterministic solution algorithms for systems
subject to the white noise, in an idealized Itô process setting. Here, we introduce a
novel transformation, where the randomness of the Brownian motion is described as
a propagation of an (artificial) uncertainty of the initial condition. We show that the
measure induced by the transformed system is consistent with the one resulting from
the Itô process, in the moment sense. The key ingredient is the fact that both the
transformed and the original process result in an identical Fokker–Planck equation
for their probability densities. Afterwards, since the transformed system describes
an ordinary differential equation (ODE) with an uncertain initial condition, a chaos
expansion can be applied in a straightforward manner.

The remainder of the manuscript is structured as follows. First, we present our set-
ing for the Itô process followed by a short review of its Wiener chaos expansion. In
Sect. 3, the gradient transformation of the white noise is motivated and introduced. In
the follow-up Sect. 4, some theoretical aspects of the transformation are justified. In
particular, the solution existence and uniqueness of the transformed process are dis-
cussed. Therefore, in Sect. 5, the Hermite chaos expansion of the transformed process
is devised. Some algorithmic considerations and numerical example are discussed in
Sect. 6. The paper concludes with final remarks and future outlooks.
2 Review of the Itô Process

To start, a set of assumptions on the coefficients of the Itô Stochastic Differential Equations (SDEs) necessary for our analysis is provided in Sect. 2.1. Next, the conventional chaos expansion of the Itô process is reviewed in Sect. 2.2.

2.1 General Setting

We focus on a simple prototype of stochastic processes driven by the white noise. Let \((\Omega, \mathcal{F}_t, \mathcal{P})\) be a complete probability space, where \(\mathcal{F}_t = \mathcal{F}_t \otimes \mathcal{F}_0\) denotes the \(\sigma\)-algebra on the subsets of \(\Omega = \Omega_1 \cup \Omega_2\). Here, \(\mathcal{F}_t\) is an increasing family of \(\sigma\)-algebras induced by the \(n\)-dimensional standard Brownian path \(W(., .) : \mathbb{R}_+ \times \Omega_1 \rightarrow \mathbb{R}^n\) and \(\mathcal{F}_0\) the \(\sigma\)-algebra generated by the initial condition \(U_0(.) : \Omega_2 \rightarrow \mathbb{R}^n\).

We consider an Itô diffusion SDE

\[
dU_i(t, \omega) = b_i(U)dt + \beta dW_i(t, \omega),
\]

governing the evolution of the \(\mathcal{F}_U\)-measurable random variable \(U(., .) : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}^n\), with the initial value \(U_0\) and the law \(\mathcal{P}\).

Throughout this manuscript, we need certain regularity assumptions on the drift \(b(.) : \mathbb{R}^n \rightarrow \mathbb{R}^n\), the diffusion coefficient \(\beta \in \mathbb{R}\) and the initial condition \(U_0\).

We require that \(\beta \neq 0\) and that the drift \(b(x) = -\nabla \Psi(x)\) with \(\Psi(.) \in C_b^\infty(\mathbb{R}^n)\), where \(C_b^\infty\) denotes the space of bounded functions with bounded derivative of all orders. Finally, without loss of generality, we consider the initial condition to be \(f_{U_0}(u) = \delta(u - U_0)\), where \(\delta(.)\) is the \(n\)-dimensional Dirac delta and \(U_0 \in \mathbb{R}^n\).

For the above-described setting, interesting properties can be shown for the Itô process, including the following.

Remark 2.1 It is a classic result in the theory of SDEs that since \(\Psi(.) \in C_b^\infty(\mathbb{R}^n)\) and \(\beta\) is assumed to be a constant, Eq. (2.1) has a solution with a bounded variance for all \(t \geq 0\), which is unique in the mean square sense. Furthermore, the process is well posed, resulting in smooth variation of an expectation of the solution with respect to the initial condition; see Theorem 5.2.1 of [17].

Remark 2.2 Based on different results in the Malliavin calculus, since coefficients \(b\) and \(\beta\) fulfill the Hörmander criterion, and furthermore, \(b\) has bounded derivatives, the Borel measure generated by the process \(\mu_U = \mathcal{P}(U^{-1})\) has a well-defined probability density \(f_U(u; t)du = d\mu_U(u; t)\) which is smooth \(f_U(.) ; t \in C(\mathbb{R}^n)\), provided \(t > 0\); see, for example, Theorem 2.7 in [25]. Here and henceforth \(U^{-1}\) denotes the preimage of \(U\), i.e., the smallest \(\sigma\)-algebra with respect to which \(U\) is measurable.

Remark 2.3 Due to Corollary 4.2.2. in [2], since \(\mu_U\) is three times differentiable, the Fisher information

\[
I(f) := \int_{\mathbb{R}^n} \frac{1}{f} \nabla_x f \cdot \nabla_x f dx
\]

associated with the density \(f_U\) is bounded at \(t > 0\).
Remark 2.4  The density \( f_U \) evolves according to the Fokker–Planck equation (forward Kolmogorov equation)
\[
\frac{\partial f_U(u; t)}{\partial t} = -\frac{\partial}{\partial u_i} \left( b_i(u) f_U(u; t) \right) + \frac{\beta^2}{2} \frac{\partial^2}{\partial u_i \partial u_i} f_U(u; t)
\]
(2.3)
and the measure \( \mu_U \) is governed by the transport equation
\[
\frac{\partial \mu_U(u; t)}{\partial t} = -b_i(u) \frac{\partial}{\partial u_i} \mu_U(u; t) + \frac{\beta^2}{2} \frac{\partial}{\partial u_i \partial u_i} \mu_U(u; t).
\]
(2.4)
Since \( \Psi(.) \in C^\infty_b(\mathbb{R}^n) \) and \( \beta \neq 0 \), both the above-mentioned equations have unique solutions (for uniqueness results see [3,5,14]). Notice that the Einstein index convention is employed here and henceforth, to economize the notation.

In comparison with the natural setting of Itô processes, we have introduced strong assumptions on \( \Psi \) and \( \beta \). Though not straightforward, the generalization of our analysis may become possible as long as the corresponding Itô SDE has a unique solution with bounded variance and its corresponding Fisher information is bounded (e.g. by using Lyapunov functionals [11]). But to keep the study focused on the main idea, we postpone the generalization to the follow-up studies.

In typical applications associated with scientific computations, one is interested in some moments of the solution \( U \), which are in the form of an expectation \( \mathbb{E}[g(U(t, \omega))] \) of some smooth function \( g(.) \in C^\infty(\mathbb{R}^n) \).

2.2 Wiener Chaos Expansion

Due to slow convergence rates of Monte Carlo methods, deterministic solution algorithms for stochastic processes can be attractive. Besides stochastic collocation methods [19], a Wiener chaos expansion of Eq. (2.1) is possible due to the Cameron–Martin theorem [4], as carried out, for example, in [10,16,29]. It is useful for our sequel analysis to provide an overview of this expansion. To simplify the notation we explain the chaos expansion of \( U \) in a one-dimensional setting \( n = 1 \). For a multidimensional case, the following can be applied to each component of the solution.

Random events with respect to which the solution \( U \) is measurable are due to the initial condition \( U_0 \) and the corresponding Brownian integral \( \beta \int_0^t dW(s, \omega_2) \).
Therefore, for a deterministic \( U_0 \), \( U \) can be expressed as a map
\[
M \left( U_0, \int_0^t dW(s, \omega) \right) := U(t, \omega).
\]
(2.5)
Integral of the Brownian path \( \mathcal{I}(\omega) := \int_{s=0}^t dW(s, \omega) \) can be expanded as
\[
\mathcal{I}(\omega) = \sum_{j=1}^{\infty} \xi_j(\omega) \int_0^t \phi_j(s) ds,
\]
(2.6)
where \( \{\phi_j(s)\} \) is a sequence of orthonormal functions in \( L^2([0, t]) \) and \( \xi_j \) are independent standard normal random variables.

Suppose \( P^{(l)} = \left\{ t_j^{(l)} = j t / m_l \mid j \in \{1, \ldots, m_l\} \right\} \) is a partition for the time interval \((0, t]\). Intuitively, the Brownian motion generates an independent normally distributed random variable at each \( t_j^{(l)} \in P^{(l)} \). Along this picture, let

\[
\hat{I}^{(l)} = \sum_{j=1}^{m_l} \xi_j \int_0^t \phi_j(s) \, ds \tag{2.7}
\]

be an approximation of the integral Eq. (2.6) corresponding to the partition \( P^{(l)} \). It can be shown that

\[
\mathbb{E} \left[ \left( I - \hat{I}^{(l)} \right)^2 \right] < C \frac{t}{m_l}, \tag{2.8}
\]

where \( C < \infty \) is some constant \([13]\).

Analogously, let \( \hat{U}^{(l)} \) be an approximation of \( M \), computed on the partition \( P^{(l)} \). Therefore, due to Eq. (2.7), the solution at time \( t \) can be approximated as a function \( \hat{U}^{(l)}(t, \xi_1, \ldots, \xi_{m_l}) \) with a mean square error of \( O(t/m_l) \) due to truncation introduced in Eq. (2.7). At this point, the Wiener chaos expansion can be applied to \( \hat{U}^{(l)} \), as explained in the following.

In order to expand \( \hat{U}^{(l)} \) with respect to the Hermite basis, suppose \( \xi = (\xi_1, \ldots, \xi_{m_l}) \) is an \( m_l \)-dimensional normally distributed random variable and let \( \alpha = (\alpha_1, \ldots, \alpha_{p}) \in \mathcal{J}_{m_l}^p \) denote an index from the set of multi-indices

\[
\mathcal{J}_{m_l}^p = \left\{ \alpha = (\alpha_i, 1 \leq i \leq m_l) \mid \alpha_i \in \{0, 1, 2, \ldots, p\}, |\alpha| = \sum_{i=1}^{m_l} \alpha_i \right\}. \tag{2.9}
\]

Let the \(|\alpha|\)-order multi-variate Hermite polynomial

\[
H_{\alpha}(\xi) = \prod_{i=1}^{m_l} \hat{H}_{\alpha_i}(\xi_i) \tag{2.10}
\]

be a tensor product of the normalized \( \alpha_i \)-order Hermite polynomials \( \hat{H}_{\alpha_i}(\xi_i) \). According to the Cameron–Martin theorem, \( \hat{U}^{(l)}(t, \xi) \) admits the following Hermite expansion

\[
\hat{U}^{(l)}(t, \xi) = \lim_{p \to \infty} \sum_{\alpha \in \mathcal{J}_{m_l}^p} \hat{u}_{\alpha}^{(l)}(t) H_{\alpha}(\xi), \tag{2.11}
\]

where \( \hat{u}_{\alpha}^{(l)}(t) = \mathbb{E}[\hat{U}^{(l)}(t, \xi) H_{\alpha}(\xi)] \).

In fact, the expansion (2.11) provides a means to project the randomness of the solution \( U(t, \omega) \) into the Hermite basis. As a result, the Itô process is transformed
to a set of deterministic ODEs for the coefficients $\hat{u}_a^{(l)}(t)$, and thus, the expectations $\mathbb{E}[g(U(t, \omega))] \approx \mathbb{E}[g(\hat{U}^{(l)}(t, \xi))]$ can be computed deterministically. However, in order to keep the order of the approximation introduced in the expansion (2.7) constant, $m_l$ should grow linearly with respect to $t$. So does the dimension of the expansion (2.11), as $m_l$ shows up in the order of the Hermite polynomials. Thus, unless a short time behaviour of the solution is of interest, complexity of the Wiener chaos expansion of the Itô process may become prohibitive, even though the number of Hermite polynomials can be reduced significantly through sparse tensor compressions [22].

A more general insight into the problem can be sought by considering the fact that a smooth function of an $n$-dimensional Brownian path $f(W(t, \omega))$ at time $t = T$ is measurable with respect to the Borel $\sigma$-algebra on $\Omega = (\mathbb{R}^n)^{[0, T]}$ [17]. Therefore, in order to devise a chaos expansion of $f$, the orthogonal functions should span a rather high-dimensional space $L^2((\mathbb{R}^n)^{[0, T]})$.

### 3 Main Result

The main idea of this work is to find an alternative SDE with a similar probability density as the one generated by the Itô process, which yet remains measurable with respect to the $\sigma$-algebra induced by its initial condition.

More precisely, consider again the partition $P^l = \{0 = t_1^l < t_2^l < \cdots < t_m^l = t\}$ for the time interval $[0, t]$ with $|P^l| \to 0$ as $l \to \infty$. Obviously the solution of the Itô SDE $U(t, \omega)$ is measurable with respect to the family of $\sigma$-algebras

$$\left\{ \mathcal{F}_{t_1^l}^{U_0}, \mathcal{F}_{t_2^l}^{U_0}, \ldots, \mathcal{F}_{t_m^l}^{U_0} \right\} \quad \text{as} \quad l \to \infty.$$  

However, if we are only interested in some expectation $\mathbb{E}[g(U(t, \omega))]$ at time $t$, the knowledge of the Borel measure $\mu_U(B; t) = \mathcal{P}\{U^{-1}(t, B)\}$ where $B \in \mathcal{B}^n$, is sufficient. Note that $\mathcal{B}^n$ is the Borel $\sigma$-algebra on $\mathbb{R}^n$. Let $f_U(u; t)$ be the corresponding probability density, i.e., $f_U(u; t)du = d\mu_U(u; t)$, therefore

$$\mathbb{E}[g(U(t, \omega))] = \int_{\mathbb{R}^n} f_U(u; t)g(u)du.$$  

Suppose the random variable $X(t, \omega) : \mathbb{R}^+ \times \Omega \to \mathbb{R}^n$ belongs to a complete probability space $(\Omega, \mathcal{G}, \mathbb{Q})$ and generates a Borel measure $\mu_X = \mathcal{Q}(X^{-1})$. Let the probability density be $f_X(x; t)dx = d\mu_X$. We propose that under suitable assumptions on $f_X(x; 0)$ (as explained in the following section), the solution of the transformed Itô SDE

$$\frac{d}{dt}X_t(t, \omega) = b_t(X) - \frac{1}{2} \beta^2 \left[ \nabla_{x_i} \log f_X(x; t) \right]_{x=X(t, \omega)}$$  \hspace{1cm} (3.1)

with the initial condition $X_0(\omega) : \Omega \to \mathbb{R}^n$ uniquely exists for all $t$. Furthermore, the solution is consistent with the Itô process in a sense that for an arbitrary smooth $g \in C^\infty(\mathbb{R}^n)$ we have

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where $U$ is solution of the Itô SDE with the initial condition $U_0 = X_0$.

Notice that the type of nonlinearity introduced by this transformation is similar to McKean SDEs, where the evolution of the process is some function of the probability density [15]. Let us first review the motivation behind this transformation. According to Remark 2.4, the probability density generated by the Itô process follows the Fokker–Planck equation

$$
\frac{\partial f_U(u; t)}{\partial t} + \frac{\partial}{\partial u_i} \left( b_i(u) f_U(u; t) \right) = \frac{1}{2} \frac{\partial^2}{\partial u_i \partial u_j} \left( \beta^2 f_U(u; t) \right) .
$$

By rearranging the diffusion term, one can see that

$$
\frac{\partial f_U(u; t)}{\partial t} + \frac{\partial}{\partial u_i} \left\{ \left( b_i(u) - \frac{1}{2} \frac{\beta^2}{\partial u_j} \frac{\partial}{\partial u_j} \log(f_U(u; t)) \right) f_U(u; t) \right\} = 0,
$$

resulting in a stochastic process similar to (3.1). Intuitively, we observe that the effect of the diffusion on the probability density is equivalent to an advection induced by the gradient $\nabla u \log f_U$. We refer to this transformation as logarithmic gradient transformation.

4 Theoretical Justifications

The following arguments establish a connection between solutions of the main Itô SDE (2.1) and the transformed one (3.1).

4.1 Regularity of the Itô Process

To start, note that in order to make sense of (3.1), $f_U$ should admit certain regularities. Let us introduce a class of admissible probability densities for a measurable $f(x)$ as

$$
K_1 := \left\{ f(x) : \mathbb{R}^n \to (0, \infty) \left| \nabla \log f \in C^\infty_l(\mathbb{R}^n), \ M(f) < \infty, I(f) < \infty \right. \right\},
$$

where

$$
M(f) = \int_{\mathbb{R}^n} f x^2 dx
$$

and $C^\infty_l$ is the space of infinite times differentiable functions, with at most linear growth.

Before establishing the properties of $f_U$, let us review Itô’s formula (also known as Itô’s lemma) in the following theorem.
Theorem 4.1 Let $U(t, \omega)$ be solution of the Itô SDE. Let $g(t, x) \in \mathbb{R}^m$ be a $C^2$ map. Then the process $Y := g(t, U)$ follows

$$dY_i(t, U) = \frac{\partial g_i}{\partial t}(t, U)dt + \frac{\partial g_i}{\partial x_j}(t, U)dU_j + \frac{1}{2} \frac{\partial^2 g_i}{\partial x_j \partial x_k}(t, U)dU_jdU_k, \quad (4.2)$$

where $dW_fdW_f = d\delta_{ij}$ and $dW_fdt = dt = dW = 0$.

Proof See Theorem 4.2.1 of [17]. $\square$

Corollary 4.2 Let $h: \mathbb{R}^n \to \mathbb{R}$ be $C^2$, therefore

$$h(W(t, \omega)) = h(W(0, \omega)) + \int_0^t \nabla h(W(s, \omega))dW(s, \omega) + \frac{1}{2} \int_0^t \Delta h(W(s, \omega))ds,$$

(4.3)

where $W$ denotes $n$-dimensional Brownian motion, $\nabla$ denotes gradient and $\Delta$ is the Laplace operator.

Proof It follows directly from Theorem 4.1 by setting $U = W$ and $g = h$. $\square$

The next lemma provides a link between $f_U$ and $K_1$.

Lemma 4.3 Consider $U^\varepsilon(t, \omega)$ to be solution of the Itô SDE (2.1) in the probability space $(\Omega, \mathcal{F}^U_0, P^\varepsilon)$ with the drift $b = -\nabla \Psi$, $\Psi(\cdot) \in C^\infty_c(\mathbb{R}^n)$ and the diffusion $\beta \neq 0$. Suppose the initial condition reads $U^\varepsilon_0 = U_0 + \varepsilon Z$, where $U_0 \in \mathbb{R}^n$ is deterministic, $Z(\omega) \in \mathbb{R}^n$ is a normally distributed random variable and $\varepsilon \in \mathbb{R}$ is a small, arbitrary chosen nonzero constant.

Let $f^\varepsilon_U (u; t) = dP^\varepsilon ((U^\varepsilon)^{-1})$ be the probability density of the process, therefore

$$f^\varepsilon_U (.; t) \in K_1,$$

(4.4)

for $t \in [0, \infty)$. 

Proof The initial condition $U^\varepsilon_0$ has a Gaussian probability density of the form

$$f^\varepsilon_U_0 (u) = \mathcal{M}_\varepsilon (|u - U_0|), \quad (4.5)$$

where

$$\mathcal{M}_\varepsilon (h) := \frac{1}{(\sqrt{2\pi |\varepsilon|})^n} \exp \left( -\frac{h^2}{2\varepsilon^2} \right). \quad (4.6)$$

It is straightforward to see that $\mathcal{M}_\varepsilon (|u - U_0|) \in K_1$, and thus, we only need to prove the claim (4.4) for $t > 0$. Notice that here and afterwards, $|\cdot|$ denotes the Euclidean norm.

First let us show that log $f^\varepsilon_U_0 (.; t > 0) \in C^\infty (\mathbb{R}^n)$. According to Remarks 2.1-2.3 at each $t > 0$, we have $f^\varepsilon_U_0 (.; t) \in C^\infty (\mathbb{R}^n)$, $I(f^\varepsilon_U_0) < \infty$ and $M(f^\varepsilon_U_0) < \infty$. Hence, 

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it is sufficient to prove $f_{U_0^\varepsilon}(u; t) > 0$, for $t > 0$. For that, we make use of the Girsanov transformation. But before proceed, to prevent unnecessary notational complications we set $\beta = 1$ for the followings.

Let $W^\varepsilon(t, \omega)$ be a standard n-dimensional Brownian process with the initial condition $U_0^\varepsilon$ and the law $\mathcal{W}^\varepsilon$. Then since $b(.) \in C^\infty_b(\mathbb{R}^n)$, we have

$$E \left[ \exp \left( \frac{1}{2} \int_0^T b_i(W^\varepsilon(t, \omega))b_i(W^\varepsilon(t, \omega)dt) \right) \right] < \infty,$$

(4.7)

for any finite $T$. Therefore, the process

$$Z(t, \omega) := \exp \left( - \int_0^t b_i(W^\varepsilon(s, \omega))dW^\varepsilon_i(s, \omega) - \frac{1}{2} \int_0^t b_i^2(W^\varepsilon(s, \omega))ds \right)$$

(4.8)

is a martingale for $t \in [0, T)$ [17]. It follows from the Girsanov theorem that

$$d\mathcal{P}^\varepsilon(t, \omega) = Z(t, \omega)d\mathcal{W}^\varepsilon(t, \omega).$$

(4.9)

Since $d\mathcal{W}^\varepsilon$ is a Gaussian measure, it is strictly positive for $t > 0$, and hence, $d\mathcal{P} > 0$. It is then straightforward to check that $f_{U_0^\varepsilon}(u; t) > 0$, for any $u \in \mathbb{R}^n$, provided $t > 0$.

Now the final piece is to prove

$$|\nabla_u \log f_{U_0^\varepsilon}(u; t)| \leq C(t, U_0) (|u| + 1)$$

(4.10)

for every $u \in \mathbb{R}^n$, $t > 0$ and some constant $C(t, U_0) < \infty$ which depends on $t$ and the initial condition $U_0$. First note that using the result of Corollary 4.2 to compute $\int_0^T b_i(W^\varepsilon) dW^\varepsilon_i$, the martingale $Z$ reads

$$Z(t, \omega) = \exp \left( \Psi(W^\varepsilon(0, \omega)) - \Psi(W^\varepsilon(t, \omega)) \right)$$

$$\exp \left( \frac{1}{2} \int_0^t \left( b'(W^\varepsilon(t, \omega)) - b^2(W^\varepsilon(t, \omega)) \right) dt \right),$$

(4.11)

where $b' = \text{div}(b)$ and $b^2 = b_i b_i$. Next, consider the partition

$$P^{(l)} = \left\{ t^{(l)}_j = \left( jt / m_l \right), j \in \{1, \ldots, m_l\} \right\}$$

(4.12)
for the interval \((0, t]\) and \(\Delta t^{(l)} = t/m_l\). Suppose \(Z^{(l)}\) is discretized version of the martingale \(Z(t, \omega)\) on the partition \(P^{(l)}\), thus

\[
Z^{(l)}(t, \omega) = \exp \left( \Psi(W^\varepsilon(0, \omega)) - \Psi(W^\varepsilon(t, \omega)) \right) \\
\exp \left( \frac{1}{2} \sum_{j=0}^{m_l-1} \left( b'(W^\varepsilon(t_j^{(l)}, \omega)) - b^2(W^\varepsilon(t_j^{(l)}, \omega)) \right) \Delta t^{(l)} \right).
\]

Notice that following properties of Itô integrals (see, for example, Theorem 7.1. \[6\]), we have \(Z\) as a martingale for the density \(f_U\). In particular, the Girsanov transformation yields

\[
f_{U^\varepsilon}(u_{m_l}; t) = e^{-\Psi(u_{m_l})} \int_{\mathbb{R}^n} \ldots \int_{\mathbb{R}^n} \left( e^{\Psi(u_0) + 1/2 \sum_{j=0}^{m_l-1} (b'(u_j) - b^2(u_j))} \right) \mathcal{M}_\varepsilon(|u_0 - U_0|) \prod_{i=0}^{m_l-1} \mathcal{M}_{\Delta t^{(l)}}(|u_{i+1} - u_i|) du_0 du_1 \ldots du_{m_l-1},
\]

as \(m_l \to \infty\), where \(\mathcal{M}\) is the Gaussian density defined by Eq. (4.6). Since \(\Psi \in C^\infty_b\), \(\exp(\Psi(u_0) + 1/2 \sum_{j=0}^{m_l-1} (b'(u_j) - b^2(u_j)))\) is bounded above and below by some \(S(t) < \infty\) and \(I(t) > 0\), respectively. Therefore, we have

\[
\left| \nabla u_{m_l} \log f_{U^\varepsilon}(u_{m_l}; t) \right| \leq |b(u_{m_l})| \\
+ \frac{S(t)}{I(t)} \int_{\mathbb{R}^n} \ldots \int_{\mathbb{R}^n} \mathcal{M}_\varepsilon(\mathcal{M}_{\Delta t^{(l)}}) du_0 \ldots du_{m_l-1},
\]

as \(m_l \to \infty\). However, the integral terms can be computed explicitly. In fact in the limit of \(m_l \to \infty\), we get

\[
\int_{\mathbb{R}^n} \ldots \int_{\mathbb{R}^n} \mathcal{M}_\varepsilon(\mathcal{M}_{\Delta t^{(l)}}) du_0 \ldots du_{m_l-1} \Rightarrow \mathcal{M}_{\Delta t^{(l)}}(\mathcal{M}_{\Delta t^{(l)}})
\]

(4.16)
and
\[
\int_{\mathbb{R}^n} \ldots \int_{\mathbb{R}^n} \mathcal{M}_{\epsilon^2}(|u_0 - U_0|) \prod_{i=0}^{m-1} \nabla u_m \mathcal{M}_{\Delta_i}(|u_{i+1} - u_i|) du_0 \ldots du_{m-1} = \nabla u_m \mathcal{M}_{\epsilon^2 + t}(|u_m - U_0|).
\]

(4.17)

Therefore, the upper bound reads
\[
\left| \nabla u_m \log f_{U^\epsilon}(u_m; t) \right| \leq |b(u_m)| + \frac{S(t)}{I(t)} \left( \frac{\nabla u_m \mathcal{M}_{\epsilon^2 + t}(|u_m - U_0|)}{\mathcal{M}_{\epsilon^2 + t}(|u_m - U_0|)} \right),
\]

(4.18)

for \( t > 0 \). Observe that \( \nabla u_m \mathcal{M}_{\epsilon^2 + t}(|u_m - U_0|)/\mathcal{M}_{\epsilon^2 + t}(|u_m - U_0|) \) and \( b(u_m) \) have at most linear growth rates with respect to \( u_m \). Hence, the upper bound takes the form
\[
\left| \nabla u_m \log f_{U^\epsilon}(u_m; t) \right| \leq C(t, U_0) \left( |u_m| + 1 \right)
\]

(4.19)

for some constant \( C \).

\( \square \)

**Corollary 4.4** The measure of the process \( \mu_{U^\epsilon} \) is solution of the following transport equation
\[
\frac{\partial \mu_{U^\epsilon}(u; t)}{\partial t} = \left( -b(u) + \frac{\beta^2}{2} \frac{\partial}{\partial u_i} \log f_{U^\epsilon}(u; t) \right) \frac{\partial \mu_{U^\epsilon}(u; t)}{\partial u_i}.
\]

(4.20)

**Proof** Proof is straightforward by using Remark 2.4 and the result of Lemma 4.3 that \( f_{U^\epsilon}(., t) \in K_1 \).

\( \square \)

### 4.2 Solution Existence–Uniqueness and Consistency

**Theorem 4.5** Let \( U(t, \omega), U^\epsilon(t, \omega) \in \mathbb{R}^n \) be solutions of the Itô SDE (2.1) for initial conditions \( U_0 \) and \( U_0^\epsilon \), respectively, where the drift \( b = -\nabla \Psi \) fulfills \( \Psi \in C_b^\infty \) and \( \beta \neq 0 \). Here, \( U_0 \in \mathbb{R}^n \) is deterministic, whereas \( U_0^\epsilon = U_0 + \epsilon Z, Z(\omega) \in \mathbb{R}^n \) is a normally distributed random variable and \( \epsilon \in \mathbb{R} \) is a nonzero arbitrary chosen parameter.

Suppose \( X^\epsilon(t, \omega) \in \mathbb{R}^n \) is a random variable in a space \( (\Omega, \mathcal{G}^\epsilon, \mathbb{Q}^\epsilon) \) and evolves according to
\[
\frac{d}{dt} X^\epsilon_i(t, \omega) = b_i(X^\epsilon) - \frac{1}{2} \beta^2 \left[ \nabla x_i \log f_{X^\epsilon}(x; t) \right]_{x=X^\epsilon(t, \omega)},
\]

(4.21)

subject to the initial condition \( U_0^\epsilon \). Here, \( f_{X^\epsilon}(x; t) = d\mathbb{Q}^\epsilon \left( X^{\epsilon - 1} \right) \) is the probability density of the SDE (4.21). Therefore,
1. The SDE (4.21) has a unique solution with $\mathbb{E}[(X^\varepsilon(t,\omega))^2] < \infty$ for $\forall t \in [0, \infty)$.

2. For an arbitrary $g(.) \in C^2(\mathbb{R}^m)$, we have

$$\mathbb{E}[g(X^\varepsilon(t,\omega))] = \mathbb{E}[g(U^\varepsilon(t,\omega))].$$

(4.22)

and

$$\lim_{\varepsilon \to 0} \mathbb{E}[g(X^\varepsilon(t,\omega))] = \mathbb{E}[g(U(t,\omega))].$$

(4.23)

Proof First let us show that the SDE

$$\frac{d}{dt} Y^\varepsilon_i(t,\omega) = b_i(Y^\varepsilon) - \frac{1}{2} \beta^2 \left[ \nabla_{Y_i} \log f_{U^\varepsilon}(y; t) \right]_{y=Y^\varepsilon(t,\omega)}$$

(4.24)

with the initial condition $U^\varepsilon_0$ has a unique solution with bounded variance for all $t > 0$. Let $F(t, Y^\varepsilon)$ denote the right-hand side of (4.24). For the existence–uniqueness proof of a bounded variance solution, since $f_{U^\varepsilon}(.; t) \in K_1$ according to Lemma 4.3 and $b(.) \in C^\infty(\mathbb{R}^n)$, we get $F(t,.) \in C^\infty(\mathbb{R}^n)$. Therefore, the existence–uniqueness follows directly from the Picard iterations and Groenwall’s inequality (see [1] for details). Furthermore, the boundedness of the variance comes from the Chebyshev lemma (see Theorem 1.8 in [11]).

Now let us turn to the measure induced by $Y^\varepsilon$, i.e., $\mu_{Y^\varepsilon}$. Let us define the map $\sigma_t(U^\varepsilon_0(\omega)) = Y^\varepsilon(t,\omega)$, and hence, $\mu_{Y^\varepsilon}(\sigma_t(u); t) = \mu_{U^\varepsilon_0}(u)$. Therefore, $\mu_{Y^\varepsilon}$ fulfills the following transport equation

$$\frac{\partial}{\partial t} \mu_{Y^\varepsilon}(y; t) = -F_i(t,y) \frac{\partial}{\partial y_i} \mu_{Y^\varepsilon}(y; t).$$

(4.25)

Note that since (4.24) has a unique solution, so does (4.25). However, due to Corollary 4.4, the measure induced by $U^\varepsilon$ also fulfills (4.25). Therefore, $\mu_{Y^\varepsilon}(y; t) = \mu_{U^\varepsilon}(y; t)$, resulting in equivalence of (4.24) and (4.21). Furthermore,

$$\mathbb{E}[g(X^\varepsilon(\omega, t))] = \mathbb{E}[g(U^\varepsilon(\omega, t))].$$

(4.26)

But due to well-posedness of the Itô SDE (Theorem 5.2.1 [17]), we have

$$\lim_{\varepsilon \to 0} \mathbb{E}[g(U^\varepsilon(\omega, t))] = \mathbb{E}[g(U(\omega, t))],$$

(4.27)

and hence,

$$\lim_{\varepsilon \to 0} \mathbb{E}[g(X^\varepsilon(\omega, t))] = \mathbb{E}[g(U(\omega, t))].$$

(4.28)

To summarize, let $U^\varepsilon$ and $U$ be solutions of the Itô SDE subject to the initial conditions $U^\varepsilon_0$ and $U_0$, respectively. As a consequence of the regularization and the introduced transformation, we can approximate the statistics of the true solution $U$ by statistic of
\[ U^\epsilon \text{ via } \mathbb{E}[g(U^\epsilon(\omega, t))] = \mathbb{E}[g(X^\epsilon(\omega, t))]. \] However, due to well-posedness of (2.1), we obtain a mean square error
\[ \mathbb{E} \left[ (U(\omega, t) - U^\epsilon(\omega, t))^2 \right] < C(t)\epsilon^2 \] bounded by \( \epsilon^2 \) and some constant \( C(t) \) independent of \( \epsilon \). Therefore, the regularization costs us an error of \( O(\epsilon^2) \) in the mean square sense.

### 5 Chaos Expansion

Computational advantages of the gradient formulation (3.1) over the original Itô SDE (2.1) can be exploited through its chaos expansion. Actually while the dimension of the space in which the Brownian path is measurable increases in time, its gradient transformation only propagates randomness originated from the initial condition. Therefore, the resulting logarithmic gradient transformation behaves like an ODE with an uncertain initial condition.

Let us consider an initial condition \( X_0(\omega) : \Omega \to \mathbb{R}^n \) with a probability density \( f_{X_0}(x) = \mathcal{M}_\epsilon(|x - U_0|) \), where \( |\epsilon| > 0 \) and \( U_0 \in \mathbb{R}^n \). In the following, we present the corresponding Hermite chaos expansion of the SDE (3.1) for \( X(\omega, t) : \Omega \times \mathbb{R}^+ \to \mathbb{R}^n \) subject to \( X_0 \). For more details on the Hermite chaos, in general polynomial chaos expansions, see [28]. The expansion is performed on the map \( \mathcal{M}(\xi(\omega), t) := X(\omega, t) \), where \( \xi \in \mathbb{R}^n \) is a normally distributed random variable. Note that both \( X \) and \( \xi \) should be evaluated for the same \( \sigma \)-algebra. Hence, by equality of measures we have
\[ |\nabla_q \mathcal{M}| f_X(M; t) = f_\Xi(q), \] where \( f_\Xi(q) = \mathcal{M}_1(q) \) and \( q \in \mathbb{R}^n \). In practice, (5.1) is only employed to find the initial condition of \( \mathcal{M} \) (which in our case of \( X_0 \) initially being Gaussian distributed, the map becomes trivial); afterwards, simply the coefficients of the expanded \( \mathcal{M} \) are propagated.

The map evolves according to \( X \), and thus,
\[ \frac{d}{dt} M_i(\xi(\omega), t) = b_i(M) - \frac{1}{2}\beta^2 \left[ \nabla_{x_i} \log f_X(x; t) \right]_M. \] Since \( \mathbb{E}[(M_i(\xi, t))^2] < \infty \), we conclude \( M \in L^2(d\mu_\Xi) \), where \( L^2(d\mu_\Xi) \) is the space of square integrable functions with the weight \( d\mu_\Xi(q) = f_\Xi(q)dq \). Furthermore, note that since \( b(\cdot) \) and the Fisher information are bounded, we have \( F(t, \cdot) \in L^2(d\mu_\Xi) \). Therefore, \( M \) admits an orthogonal expansion [21] such as
\[ M_i(\xi, t) = \lim_{p \to \infty} \sum_{\alpha \in \mathcal{J}_p^n} m_{i, \alpha}(t) H_\alpha(\xi) \]
for each component $i \in \{1, \ldots, n\}$, where $H_\alpha$ and $J$ are defined in (2.10) and (2.9), respectively. The coefficients follow

$$m_{i, \alpha}(t) = \langle M_i, H_\alpha \rangle_{\mu \mathbb{Z}}$$

(5.4)

with the inner product defined based on the Gaussian weight

$$\langle h, g \rangle_{\mu \mathbb{Z}} = \int_{\mathbb{R}^n} h(q) g(q) f_\mathbb{Z}(q) dq.$$  

(5.5)

Therefore,

$$\frac{dm_{i, \alpha}}{dt} = \langle b_i, H_\alpha \rangle_{\mu \mathbb{Z}} - \frac{1}{2} \beta^2 \int_{\mathbb{R}^n} H_\alpha(\xi) \left( \nabla_{x_i} \log f_X(x; t) \right)_{x=M} d\mu \mathbb{Z}$$

$$= \langle b_i, H_\alpha \rangle_{\mu \mathbb{Z}} + \frac{1}{2} \beta^2 \left( \frac{\partial M_i}{\partial \xi_i} \right)^{-1} \left( \frac{\partial H_\alpha}{\partial \xi_l} \right)_{\mu \mathbb{Z}},$$

(5.6)

and

$$\frac{\partial M_i}{\partial \xi_k} \left( \frac{\partial M_j}{\partial \xi_l} \right)^{-1} = \delta_{ij},$$

(5.7)

with $\delta$ being the Kronecker delta. Note that in deriving the last step of (5.6), the fact that $f_\mathbb{Z}$ vanishes at the boundaries together with (5.1) has been used. Moreover, since $f_X, f_\mathbb{Z} \in K_1$, the inverse of $\nabla_\xi M$ exists which can be seen again from (5.1). It is important to emphasize that the evolution of the coefficients $m_{i, \alpha}$ do not directly depend on $f_X$. By taking advantage of the measure transform (5.1), no explicit knowledge of the density $f_X$ is required.

The convergence rate of Hermite functions in the context of the probability density estimates as well as random variables has been analysed extensively; for example, see [23,24]. Provided sufficient regularities, Hermite coefficients decay by a rate faster than any algebraic one, and thus, the so-called exponential convergence would be expected.

### 6 Solution Algorithm

#### 6.1 Numerical Scheme

In practice, two types of integrations should be carried out in order to evaluate evolution of the coefficients $m_{i, \alpha}$. One is applied to the time integration, and the other owes to the inner product $\langle \cdot , \cdot \rangle_{\mu \mathbb{Z}}$. We employ a Runge–Kutta scheme for the former and the Gauss–Hermite quadrature for the latter [20]. Using these two schemes, since the inner products become explicit in terms of $m_{i, \alpha}$, Eq. (5.6) reduces to a linear update equation for each mode $m_{i, \alpha}$.
In the following, we explain the scheme in a one-dimensional setting

\[
\frac{dm_\alpha}{dt} = \langle b, \hat{H}_\alpha \rangle_{\mu_Z} + \frac{\beta^2}{2} \left( \left( \frac{\partial M}{\partial \xi} \right)^{-1}, \frac{\partial \hat{H}_\alpha}{\partial \xi} \right)_{\mu_Z}.
\]  

(6.1)

Let \( \tilde{m}^{(s)}_\alpha \) be our numerical approximation of \( m_\alpha \) at \( t = t_s \) and

\[
\tilde{M}_n^{(s)}(\xi) = \sum_{\alpha=0}^{n} \tilde{m}_\alpha^{(s)} \hat{H}_\alpha(\xi).
\]  

(6.2)

First, let us consider the transformed term, i.e., the white noise contribution. Notice that the contribution vanishes for \( \alpha = 0 \). Using the Gauss–Hermite quadrature, we get

\[
\left( \left( \frac{\partial M}{\partial \xi} \right)^{-1}, \frac{\partial \hat{H}_\alpha}{\partial \xi} \right)_{\mu_Z} \approx \sum_{i=0}^{N} Q_\alpha(t_i, N, n) \left\{ \left( \sum_{\gamma=1}^{n} \sqrt{\gamma} \tilde{m}_\gamma^{(s)} \hat{H}_\gamma^{-1}(\tilde{\xi}_i) \right) \right\}^{-1},
\]  

(6.3)

for \( \alpha > 0 \), where \( \{\tilde{\xi}_i\}_{i=0}^{N} \) are the roots of (physicists) Hermite polynomials and \( \{w_i\}_{i=0}^{N} \) the corresponding weights. Similarly, projection of the drift follows

\[
\langle b, \hat{H}_\alpha \rangle_{\mu_Z} \approx \sum_{i=0}^{N} w_i \frac{1}{\sqrt{\pi}} b(\tilde{\xi}_i) \hat{H}_\alpha(\tilde{\xi}_i).
\]  

(6.4)

Adopting the \( l \)-stage Runge–Kutta scheme, the update takes the form

\[
\tilde{m}_\alpha^{(s+1)} = \tilde{m}_\alpha^{(s)} + P_\alpha(N, n) + \frac{\beta^2}{2} \sum_{j=1}^{l} W_j Q_\alpha(t_{s+j}, N, n),
\]  

(6.5)

where \( t_{s+j} = t_s + c_j(t_{s+1} - t_s) \). The coefficients \( c_j \) and \( W_j \) are the nodes and weights of the \( l \)-stage Runge–Kutta.

### 6.2 Random Walk Example

As a numerical example, evaluation of the scheme is carried out for a random walk with \( \beta = \sqrt{2} \). The four-stage Runge–Kutta with \( \Delta t = 10^{-4} \) besides \( N = 20 \) number of quadrature points is utilized. The initial condition

\[
M_0(\xi) = (2\pi)^{1/4} \left( a_1 \hat{H}_1(\xi) + \sqrt{2a_2} \hat{H}_2(\xi) + \sqrt{6a_3} \hat{H}_3(\xi) \right)
\]  

(6.6)
is adopted with $a_1 = 1$, $a_2 = 0.05$ and $a_3 = 0.005$. Convergence of the second- and the third-order moment is shown with respect to the number of Hermite modes $n$, at $t = 0.1$. Note that since the density of the initial condition belongs to $K_1$ [see Eq. (4.1)], no regularization was employed. Relative errors are computed with respect to the exact solution. Convergence results are shown in Fig. 1 for relative errors of the second- and the third-order moment.

7 Conclusion

This study proposes a transformation of the diffusion arising from the white noise into a transport induced by logarithmic gradient of the probability density. The well-posedness of such a transformation for an Itô process with strong regularity assumptions was shown. As a result, the transformed Itô process behaves similar to an ODE with uncertain initial condition. Therefore, the process remains measurable with respect to its initial condition resulting in interesting computational advantages. The relevance of the transformation is discussed by employing the chaos expansion technique. In follow-up studies, besides analysing the computational performance of the resulting chaos expansion, generalization of the transformation to a broader class of stochastic processes driven by the white noise will be pursued.

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