EXPLICIT FORMULAS FOR THE DIVERGENCE OPERATOR IN ISONORMAL GAUSSIAN SPACE

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ABSTRACT. In this paper, we first derive some explicit formulas for the computation of the \( n \)-th order divergence operator in Malliavin calculus in the one-dimensional case. We then extend these results to the case of isonormal Gaussian space. Our results generalize some of the known results for the divergence operator. Our approach in deriving the formulas is new and simple.

1. INTRODUCTION

Malliavin calculus is an infinitesimal differential calculus on the Weiner space. It deals with the random elements, which are functions of possible infinite dimensional Gaussian field. The Malliavin derivative and the divergence operator, the adjoint of the derivative operator, form some of the main tools of the Malliavin calculus. The divergence operator can be viewed also as a generalized stochastic integral or Skorohod integral which forms the basis of extending stochastic calculus from adaptive to anticipating one.

The main objective of the paper is to provide some explicit formulas for the divergence operator of the \( n \)-th order \( n \), in certain cases. First, we discuss the one-dimensional case, in the spirit of Chapter 1 of Nourdin and Peccati (2012), referred to hereafter as NP(2012). We then extend the results to the usual setup of isonormal Gaussian spaces, which are discussed in detail in several books; see for example, NP (2012) and Nualart (2009). For a recent introduction to this topic, see Nualart and Nualart (2018).

The divergence operator is closely connected with Hermite polynomials. Therefore, we start with some basic properties of Hermite polynomials that are used in the paper. Though most of these results are known, the proofs are rather different. For the one-dimensional setup, we derive an interesting and explicit formula for the divergence operator of the \( n \)-th order \( n \) that is acting on smooth functions. This new result appears in Theorem 3.1 which serves as a basis to proving all the results that follow. Another result, Theorem 3.2, presents an interesting alternative formula to that of Theorem 3.1. It seems that Theorem 3.2 formulation cannot be achieved without the help of Theorem 3.1. Also, using Theorem 3.1, we prove the analogous result for the usual setup of Malliavin calculus, the so-called isonormal Gaussian spaces. The outcome is Theorem 4.1 which generalizes a fundamental result in the literature, as illustrated in Example 4.1. Another concrete formula, that is new and concerns the Malliavin derivative and the divergence operator, is Corollary 4.1. A special case of Corollary 4.1 is similar to a well-known result in the literature.

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The paper is organized as follows: In Section 2, some basic properties of Hermite polynomials are presented with proofs that are based on our approach. In Section 3, we deal with the one-dimensional case with the high point being the explicit formula that is presented in Theorem 3.1. For the one-dimensional case, this result leads to some additional and interesting formulas for the divergence operator and also a result that connects Hermite polynomial with the standard normal variate. In Section 4, we deal with the extension of the main result of Section 3 to isonormal Gaussian spaces. This main result generalizes a result which is considered as an important result in the literature.

2. SOME BASIC RESULTS ON HERMITE POLYNOMIALS

In this section, we briefly discuss Hermite polynomials and their properties, as they are closely connected with the divergence operator. The Hermite polynomials, denoted by \( H_n(x), x \in \mathbb{R}, \) is defined as

\[
H_n(x) = (-1)^n e^{x^2/2} D^n e^{-x^2/2},
\]

(2.1)

where \( D = \frac{d}{dx}. \) Indeed, the above definition of Hermite polynomials is called Rodriques’ formula. We express \( H_n(x) \) using the standard normal density, as this approach is easier to study its properties. Let \( N(0, 1) \) denote the standard normal distribution. Then

\[
H_n(x) = (-1)^n \sqrt{2\pi e^{x^2/2}} \phi^{(n)}(x)
\]

\[
= (-1)^n \frac{\phi^{(n)}(x)}{\phi(x)},
\]

(2.2)

where \( \phi(x) \) denotes the density of the random variable \( N \sim N(0, 1). \) Here and henceforth, \( \phi^{(0)}(x) = \phi(x) \) and \( \phi^{(n)}(x) \) denotes the \( n \)-th derivative of \( \phi(x). \) Also, it can easily be checked that \( \phi^{(1)}(x) = \phi'(x) = -x\phi(x). \) This fact is used often in proving our results related to Hermite polynomials.

It follows easily from (2.2) that \( H_0(x) = 1, \) \( H_1(x) = x, \) \( H_2(x) = x^2 - 1, \) \( H_3(x) = x^3 - 3x, \) \( H_4(x) = x^4 - 6x^2 + 3, \) and etc.

First we state and prove some important properties of \( H_n(x), \) using our approach, which will be useful later. Though most of the the results are known, our proofs and approach are new and different. We have also used both the prime and the integer 1 in the superscript of a function to denote its first derivative. This is used as per the context and to avoid any further confusion.

**Lemma 2.1.** Let \( H_n(x) \) be \( n \)-th degree Hermite polynomial. Then the following holds: For \( n \geq 1, \)

(i) We have

\[
H_{n+1} = H_1 H_n - nH_{n-1}.
\]

(2.3)

(ii) The derivative \( H'_n(x) \) of \( H_n(x) \) satisfies

\[
H'_n = H_1 H_n - H_{n+1} = nH_{n-1}.
\]

(2.4)

(iii) The exponential generating function of \( H_n(x) \) is

\[
G_H(t, x) = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x) = \frac{\phi(t-x)}{\phi(x)}.
\]

(2.5)
Proof. (i) It can be seen that \( \phi \) satisfies the recurrence relation, for \( n \geq 1 \),
\[
\phi^{(n)} = - \left[ x\phi^{(n-1)} + (n-1)\phi^{(n-2)} \right].
\] (2.6)

Using the above relation, we get
\[
(-1)^n \phi^{(n)} = (-1)^{n+1} \left[ x\phi^{(n-1)} + (n-1)\phi^{(n-2)} \right]
\]
\[
= \left[ (-1)^2 (-1)^{n-1} x\phi^{(n-1)} + (-1)^3 (-1)^{n-2} (n-1)\phi^{(n-2)} \right]
\]
\[
= \left[ (-1)^{n-1} x\phi^{(n-1)} - (-1)^{n-2} (n-1)\phi^{(n-2)} \right]
\]
which when divided by \( \phi \) leads to, for \( n \geq 2 \),
\[
H_n = \left[ H_1 H_{n-1} - (n-1)H_{n-2} \right].
\] (2.7)

or equivalently, we have for \( n \geq 1 \),
\[
H_{n+1} = \left[ H_1 H_n - nH_{n-1} \right].
\] (2.8)

(ii). From (2.2),
\[
H_n'(x) = (-1)^n \left[ \frac{\phi^{(n+1)}(x)}{\phi(x)} - \frac{\phi^{(n)}(x)}{\phi^2(x)} \phi'(x) \right]
\]
\[
= (-1)^n \left[ \frac{\phi^{(n+1)}(x)}{\phi(x)} + x \frac{\phi^{(n)}(x)}{\phi(x)} \right]
\]
\[
= H_1(x)H_n(x) - H_{n+1}(x).
\]

Thus, we get
\[
H_n' = H_1 H_n - H_{n+1}.
\] (2.9)

Also, using (2.3) and (2.9),
\[
H_n' = H_1 H_n - \{ H_1 H_n - nH_{n-1} \}
\]
\[
= nH_{n-1}.
\] (2.10)

(iii) The Taylor series expansion of \( \phi(t) \) about \( x \) leads to
\[
\phi(t - x) = \phi(x) - t\phi^{(1)}(x) + \frac{t^2}{2!}\phi^{(2)}(x) + \cdots + (-1)^n \frac{t^n}{n!}\phi^{(n)}(x) + \cdots
\]
\[
= \sum_{n=0}^{\infty} (-1)^n \frac{t^n}{n!}\phi^{(n)}(x).
\]

Hence,
\[
\frac{\phi(t - x)}{\phi(x)} = \sum_{n=0}^{\infty} (-1)^n \frac{t^n}{n!}\phi^{(n)}(x)
\]
\[
= \sum_{n=0}^{\infty} t^n \frac{H_n(x)}{n!}
\]
\[
= G_H(t, x),
\]
as claimed. □

We start with a result that connects $H_n$ with the standard normal variate $N$.

**Lemma 2.2.** Let $N$ be a standard normal variate and $H_n(x)$ be the $n$-th degree Hermite polynomial. Then, for $n \geq 1$,

$$H_n = E(H_1 + iN)^n,$$

where $i = \sqrt{-1}$ and $E$ denotes the expectation operator.

**Proof.** Let $K_n(x) = E(H_1(x) + iN)^n = E(x + iN)^n$ and $\phi_X(t)$ denote the characteristic function of the rv $X$. Then the exponential generating function of $K_n(x)$ is, for $t > 0$,

$$G_K(t, x) = \sum_{n=0}^{\infty} \frac{t^n}{n!} K_n(x) = \sum_{n=0}^{\infty} \frac{t^n}{n!} E(x + iN)^n$$

$$= \sum_{n=0}^{\infty} \frac{t^n}{n!} \int_{-\infty}^{\infty} (x + iy)^n \phi(y) dy$$

$$= \int_{-\infty}^{\infty} \sum_{n=0}^{\infty} \frac{(tx + ity)^n}{n!} \phi(y) dy$$

$$= \int_{-\infty}^{\infty} e^{(tx + ity)} \phi(y) dy$$

$$= e^{tx} \phi_N(t)$$

$$= e^{tx - t^2/2}$$

$$= \phi(t - x) \phi(x),$$

which is the exponential generating function of $H_n(x)$ (see (2.5)).

As the generating functions of $H_n(x)$ and $K_n(x)$ agree for all $t > 0$, we have $H_n(x) = K_n(x)$ for all $x \in \mathbb{R}$. □

An application of Lemma 2.2 shows that

$$H_n(0) = E(iN)^n = (-1)^{\frac{n}{2}} E(N)^n = \begin{cases} 0, & \text{if } n \text{ is odd}, \\ \frac{(-1)^{\frac{n}{2}} (n)!}{2^{\frac{n}{2}} (\frac{n}{2})!}, & \text{if } n \text{ is even}. \end{cases}$$

(2.12)

3. **Divergence Operator in One-Dimensional Case**

First we introduce some notations and definitions (see NP (2012), Chapter 1). Let $L^2(\phi)$ denote the set of square integrable functions with respect to the standard normal density $\phi$. Let $S \subset C^\infty(\mathbb{R})$ denote the set of real-valued functions that have, together with all their derivatives, polynomial growth.

The $n$-th divergence operator $\delta^n : Dom(\delta^n) \to L^2(\phi)$ is such that for each $g \in Dom(\delta^n)$, the function $\delta^n g$ satisfies

$$\int f^{(n)}(x) g(x) \phi(x) dx = \int f(x) (\delta^n g)(x) \phi(x) dx, \text{ for all } f \in S,$$

(3.1)
where \( \text{Dom}(\delta^n) \subset L^2(\phi) \) is defined by

\[
\text{Dom}(\delta^n) = \{ g \in L^2(\phi) : \int f^n(x)g(x)\phi(x)dx \leq c\sqrt{\int f^2(x)\phi(x)dx} \},
\]

for all \( f \in S \) and some constant \( c = c(g) > 0 \).

**Remark 3.1. Notational Convention:** In the rest of the paper, the symbol \( H \) stands for the generic Hermite polynomial. Also, we will follow the following rule for the binomial theorem expansion of \((H - g)^n\). We denote \( H^0 = H_0 = 1, H^k = H_k, k \geq 1 \), and for the real-valued function \( g \), we denote \( g^0 = g, g^k = g^{(k)}, k \geq 1 \), the \( k \)-th derivative of \( g \). For example,

\[
(H - g)^1 = (H^1g^0 - H^0g^1) = (H_1g - H_0g^1) = (H_1g - g^1),
\]

and for \( x \in \mathbb{R} \),

\[
(H - g)^1(x) = (H_1g - g^1)(x) = H_1(x)g(x) - g^1(x).
\]

**Remark 3.2.** In fact, (3.1) holds for \( f \in D^{n,2} \), where \( D^{n,2} \supset S \) is a Sobolev space defined appropriately, see (1.1.6) in NP (2012). That is, \( D^{n,2} \) is the closure of \( S \) with respect to the norm

\[
\|f\|_{D^{n,2}} = \left( \int_{\mathbb{R}} |f(x)|^2\phi(x)dx + \int_{\mathbb{R}} |f'(x)|^2\phi(x)dx + \cdots + \int_{\mathbb{R}} |f^{(n)}(x)|^2\phi(x)dx \right)^{1/2}.
\]

It follows that \( \delta^0g = g, g \in \text{Dom}(\delta^0) = L^2(\phi) \). Also, it follows, from equation (1.2.4) of NP (2012), that if \( f \in \text{Dom}(\delta^{n+m}) \), then \( \delta^n f \in \text{Dom}(\delta^m) \) and

\[
\delta^{n+m}f = \delta^n(\delta^m f). \tag{3.2}
\]

It is known (see equation (1.2.5) of NP (2012)) that the operator \( \delta^1 \) satisfies

\[
\delta^1 g = \delta g = xg - g^1 = H_1g - g^1, \quad g \in D^{1,2}, \tag{3.3}
\]

since \( H_1(x) = x \). Also, we write, for example, \( \delta^1 g \) as

\[
(\delta^1 g)(x) = (H_1g - g^1)(x) = H_1(x)g(x) - g^1(x), \quad x \in \mathbb{R}. \tag{3.4}
\]

Using the result in (3.3), one could obtain the expression for \( \delta^2 g \) which would involve \( g, g^1 \) and \( g^2 \). However, proceeding this way and obtaining a general expression for \( \delta^n g \) will be very complicated. Further, an explicit formula for \( \delta^n \), as in (3.3) for the case \( n = 1 \), is not available in the literature. Our goal in this paper is to obtain some compact representations and explicit computational formulas for \( \delta^n \). First, we obtain it for the one-dimensional case, exploiting the connections to Hermite polynomials. We then extend some of these results to case of isonormal Gaussian space. In the process, we generalize some of the known results for the divergence operator in the isonormal Gaussian space.

First, we state and prove the main result of this section. This result provides a simple and compact representation formula for \( \delta^n \) and its proof is combinatorial in nature.

**Theorem 3.1.** Let \( \delta^n \) be the \( n \)-th divergence operator defined in (3.1) and \( g \in S \). Then for \( n \geq 1 \), we have

\[
\delta^n g = (H - g)^n, \tag{3.5}
\]

where the right-hand side is expanded using binomial theorem using the rule in Remark 3.1.
Proof. Note first when $n = 1$, we get from (3.5), $\delta^1 g = (H - g)^1 = H_1 g - H_0 g^1$, which coincides with (3.3). Consider the case $n = 2$. Using (3.2), we get

$$
\delta^2 g = \delta(\delta^1 g) = \delta(H_1 g - g^1)
$$

$$
= H_1 (H_1 g - g^1) - (H_1 g - g^1)^1 \quad \text{(using (3.3))}
$$

$$
= (H_1^2 - H_1') g - 2 H_1 g^1 + g^2 \quad \text{(using (2.9))}
$$

$$
= H_2 g - 2 H_1 g^1 + g^2.
$$

which coincides with (3.5).

Assume $\delta^k g = (H - g)^k$, for $1 \leq k \leq n$. We next show that the result in (3.5) is true for $n + 1$. Observe that

$$
\delta^{n+1} g = \delta(\delta^n g) = \delta(H - g)^n
$$

$$
= \delta \left( H_n g - \binom{n}{1} H_{n-1} g^1 + \cdots + (-1)^k \binom{n}{k} H_{n-k} g^k + \cdots \right)
$$

$$
+ (-1)^{n-1} \binom{n}{n-1} H_1 g^{n-1} + (-1)^n g^n
$$

$$
= H_1 \left( H_n g - \binom{n}{1} H_{n-1} g^1 + \cdots + (-1)^k \binom{n}{k} H_{n-k} g^k + \cdots \right)
$$

$$
+ (-1)^{n-1} \binom{n}{n-1} H_1 g^{n-1} + (-1)^n g^n
$$

$$
- \left( H_n g - \binom{n}{1} H_{n-1} g^1 + \cdots + (-1)^k \binom{n}{1} H_{n-k} g^k + \cdots \right)
$$

$$
+ (-1)^{n-1} \binom{n}{n-1} H_1 g^{n-1} + (-1)^n g^n
$$

$$
= (H_1 H_n - H_n') g - \left( \binom{n}{1} (H_1 H_{n-1} - H_{n-1}') + H_n \right) g^1 + \cdots
$$

$$
+ (-1)^k \left( \binom{n}{k} (H_1 H_{n-1} - H_{n-1}') + \binom{n}{k-1} H_{n-k+1} \right) g^k + \cdots
$$

$$
+ (-1)^n \left( H_1 + (-1)^{n-1} n H_1 \right) g^n + (-1)^{n+1} g^{n+1}
$$

$$
= H_{n+1} g - \left( \binom{n}{1} (H_1 H_{n-1} - H_{n-1}') + H_n \right) g^1 + \cdots
$$
\[ + (-1)^k \left( \binom{n}{k} (H_1 H_{n-k} - H'_{n-k}) + \binom{n}{k-1} H_{n-k+1} \right) g^k + \cdots \]
\[ + (-1)^n \left( H_1 + nH_1 \right) g^n + (-1)^{n+1} g^{n+1} \]
\[ = H_{n+1} g - \binom{n+1}{1} H_n g^1 + \cdots + \]
\[ + (-1)^k \left( \binom{n}{k} H_{n-k+1} + \binom{n}{k-1} H_{n-k+1} \right) g^k + \cdots \]
\[ + (-1)^n(n+1)H_1 g^n + (-1)^{n+1} g^{n+1} \]
\[ = H_{n+1} g - \binom{n+1}{1} H_n g^1 + \cdots + (-1)^k \binom{n+1}{k} H_{n+1-k} g^k \]
\[ + \cdots + (-1)^n \binom{n+1}{1} H_1 g^n + (-1)^{n+1} g^{n+1} \]
\[ = (H - g)^{n+1}. \]

Observe that we have used the following facts

(i) \( H_1 H_n - H'_n = H_{n+1}, \) (see (ii) of Lemma 2.1)

(ii) \( \binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k} \).

in the steps above.

Thus, we get, for \( n \geq 1, \)
\[ \delta^n g = (H - g)^n, \]

which proves the result.

When \( g \equiv 1, \) we have
\[ \delta^n 1 = (H - 1)^n = H_n 1 - 0 = H_n, \]

which is sometimes taken as the definition of \( H_n \) (see p. 13 of NP (2012)).

Next, we obtain another expression for \( \delta^n, \) which could be helpful for computational purposes.

**Lemma 3.1.** Let \( \delta^n \) be the \( n \)-th degree divergence operator defined in (3.1) and \( g \in S. \) Then for \( n \geq 1, \) we have

\[ \delta^{n+1} g = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \left( H_{k+1} g^{n-k} - H_k g^{n-k+1} \right). \]  

(3.6)

**Proof.** Using Theorem 3.1 and (3.2), we get
\[ \delta^{n+1} g = \delta(\delta^n g) \]
\[ = \delta(H - g)^n \]
\[ = \delta \left( \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} H_k g^{n-k} \right) \]
\[
\begin{align*}
&= \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \delta(H_k g^{n-k}) \\
&= \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} (H_1 H_k g^{n-k} - H_1' g^{n-k} - H_k g^{n-k+1}) \\
&= \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} (H_{k+1} g^{n-k} - H_k g^{n-k+1}),
\end{align*}
\]

using (2.4). This proves the result. \hfill \Box

Using Theorem 3.1, we get the interesting following result which generalizes Lemma 2.2 and also the result in (3.3). Further, it connects \( \delta^n \) with the moments of the standard normal distribution.

**Theorem 3.2.** Let \( N \) denote the standard normal variable, \( K = H_1 + iN \), \( g \in S \) and \( E \) denote the expectation is with respect to \( N \). Then, for \( n \geq 1 \),

\[
\delta^n g = E((K - g)^n),
\]

where the rhs is expanded using binomial theorem and the rule in Remark 3.1 is applied only for \( g \).

**Proof.** It follows, from Lemma 2.2, that \( H_n = E(K^n) \), for \( n \geq 1 \). Also, by Theorem 3.1 we have for \( n \geq 1 \),

\[
\begin{align*}
\delta^n g &= \sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r} H_r g^{n-r} \\
&= \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} E(K^r) g^{n-r} \\
&= E\left( \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} K^r g^{n-r} \right) \\
&= E((K - g)^n),
\end{align*}
\]

which completes the proof. \hfill \Box

**Remark 3.3.** (i) When \( g \equiv 1 \), we have \( \delta^n = E(K - 1)^n = E(K^n g^0 - g^1 + \cdots) = E(K^n) = H_n \), which is Lemma 2.2.

(ii) When \( n = 1 \), \( \delta^1 g = E(K - g)^1 = E(K^1 g^0 - g^1) = E ((H_1 + iN)g - g^1) = H_1 g - g^1 \), which coincides with (3.3).

Next, we give a different proof of the result in (1.3.5) of NP (2012, p. 12), using Theorem 3.1.

**Lemma 3.2.** Let \( D \) denote the usual derivative operator, \( \delta^n \) be the \( n \)-th divergence operator and \( g \in S \). Then

\[
D \delta^n g = n \delta^{n-1} g + \delta^n g^1, \quad (3.7)
\]

for \( n \geq 1 \).
Proof. Using Theorem 3.1, we get

\[ D^n \delta g = D(H - g)^n \]

\[ = D \left( \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} H_k g^{n-k} \right) \]

\[ = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} D \left( H_k g^{n-k} \right) \]

\[ = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \left( H_k' g^{n-k} + H_k g^{n-k+1} \right) \]

\[ = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} H_k' g^{n-k} + \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} H_k (g^1)^{n-k} \]

\[ = n \sum_{j=0}^{n-1} (-1)^{n-j-1} \binom{n-1}{j} H_j g^{n-1-j} + \delta^n g^1 \]

\[ = n \delta^{n-1} g + \delta^n g^1, \]

which proves the result. \[\Box\]

Remark 3.4. When \( n = 1 \), Lemma 3.2 reduces to \( D \delta g = g + \delta g^1 \), which is Proposition 1.3.8 of NP (2012). Also, the formula in (3.7) can also be written as

\[ (D^n \delta - \delta^n D) g = n \delta^{n-1} g \]

which is equation (1.3.5) of NP (2012, p. 12). Note the form in (3.7) is similar to the derivative of the product of two functions.

4. Divergence Operator in Isonormal Gaussian space

Let \( (\Omega, \mathcal{F}, P) \) be the underlying probability space, \( \mathcal{H} \) be a real separable Hilbert space and \( X = \{X(h) : h \in \mathcal{H}\} \) be the associated isonormal Gaussian process over \( \mathcal{H} \). That is, the \( X(h) \) are zero mean normal rvs defined on \( (\Omega, \mathcal{F}, P) \) such that \( E(X(h)X(g)) = \langle h, g \rangle \mathcal{H} \), for \( h, g \in \mathcal{H} \). Assume \( \mathcal{F} \) is generated by \( X \) and let \( L^2(\Omega) = L^2(\Omega, \mathcal{F}, P) \). It follows that \( X : \mathcal{H} \to L^2(\Omega) \) is a liner operator.

Let \( f : \mathbb{R}^m \to \mathbb{R} \) be a \( C^\infty \)-function such that its partial derivatives have polynomial growth. A random variable \( f(X(h_1), \cdots, X(h_m)), h_i \in \mathcal{H}, \) is called a smooth rv. Let \( S \) be a collection of all smooth rvs.

Let \( F \in S \) be a smooth rv. The \( n \)-th Malliavin derivative \( D^n \) of \( F \) is defined as

\[ D^n F = \sum_{j_1, \cdots, j_n=1}^{m} \frac{\partial^n}{\partial j_1, \cdots, \partial j_n} f(X(h_1), \cdots, X(h_m)) h_1 \otimes \cdots \otimes h_n \]
Note when $n = 1$,
\[ DF = \sum_{j=1}^{m} \frac{\partial}{\partial j} f(X(h_1), \ldots, X(h_m)) h_j. \] (4.2)

The $n$-th divergence operator $\delta^n : \text{Dom}(\delta^n) \to L^2(\Omega)$ satisfy, for each $u \in \text{Dom}(\delta^n)$,
\[ E\left(\langle D^n F, u \rangle_{H^{\otimes n}}\right) = E(F\delta^n(u)), \text{ for all } F \in \mathcal{S}, \]
where $\text{Dom}(\delta^n) \subset L^2(\Omega, H^{\otimes n})$ is defined by
\[ \text{Dom}(\delta^n) = \{ u \in L^2(\Omega, H^{\otimes n}) | \langle D^n F, u \rangle_{H^{\otimes n}} \leq c\sqrt{E(F^2)} \}, \]
for some $c = c(u) > 0$ and all $F \in \mathcal{S}$. For more details and properties of $D^n$ and $\delta^n$, see NP (2012).

In Section 3, the derivative and the divergence operators are acting on one dimensional deterministic functions, that is, for the case $\mathcal{H} = \mathcal{R}$, while in this section the domain of those operators is a collection of random elements. However, we will continue to use the same notations and this won’t create a difficulty.

The next result gives the divergence of a Hilbert space valued rv. If no confusion arises, we write sometimes $\delta^n g(x)$ for $(\delta^n g)(x)$.

**Lemma 4.1.** Let $\mathcal{H}$ be a real separable Hilbert space and $X : \mathcal{H} \to L^2(\Omega)$ be an isonormal process. Let $h \in \mathcal{H}$ be such that $\|h\| = 1$ and $g : \mathcal{R} \to \mathcal{R}$ be a function so that $g(X(h))$ is a smooth rv. Then
\[ \delta(g(X(h)) h) = (H - g)^1(X(h)), \] (4.3)
using the rule in Remark 3.1.

**Proof.** Let $D$ be the Malliavin derivative defined in (4.2). Recall that $H_0(x) = 1$ and $H_1(x) = x$. Using Proposition 2.5.4 of NP (2012), we have
\[
\delta(g(X(h)) h) = g(X(h)) \delta(h) - \langle Dg(X(h)), h \rangle_{\mathcal{H}} \\
= g(X(h)) X(h) - g^1(X(h)) h, h \rangle_{\mathcal{H}} \\
= g(X(h)) X(h) - g^1(X(h)) \langle h, h \rangle_{\mathcal{H}} \\
= H_1(X(h)) g(X(h)) - H_0(X(h)) g^1(X(h)) \\
= (H - g)^1(X(h)),
\]
which proves the result. \hfill \Box

**Remark 4.1.** It follows from Theorem 3.1 and for $n = 1$ that (4.3) is equivalent to
\[ \delta(g(X(h)) h) = (\delta^1 g)(X(h)) = (\delta g)(X(h)), \] (4.4)
where $\delta^1 = \delta$ on the rhs of (4.4) is an operator on functions in the sense of Section 3. Alternatively, $(\delta g)$ can also be viewed as an operator on the isonormal process $X = \{(X(h)|h \in \mathcal{H}\}$.

**Lemma 4.2.** Let, for $n \geq 1$, $h_1, \ldots, h_n \in \mathcal{H}$ and $g \in \mathcal{S}$ be a smooth random variable. Then,
\[ (\delta^{n+1}(g h_1 \otimes \cdots \otimes h_{n+1}) = \delta(\delta^n(g h_1 \otimes \cdots \otimes h_n) h_{n+1}). \] (4.5)
Proof. Let \( f : \mathbb{R}^m \to \mathbb{R} \) be a smooth function whose derivatives have polynomial growth and \( F = f(X(v_1), \cdots , X(v_m)), v_1, \cdots , v_m \in \mathcal{H} \) be a smooth random variable. For simplicity, let us denote \( h = h_1 \otimes \cdots \otimes h_n \in H^\otimes n \). Then by the definition of \( \delta \), we have
\[
E[F \cdot \delta(\delta^n(gh)h_{n+1})] = E\langle DF, \delta^n(gh)h_{n+1} \rangle_{\mathcal{H}}
\]
(definition of Malliavin derivative \( D \))
\[
= E\sum_{k=1}^m \langle \frac{\partial f}{\partial x_k}(X(v_1), \cdots , X(v_m))v_k, \delta^n(gh)h_{n+1} \rangle_{\mathcal{H}}
\]
(definition of Malliavin derivative \( D \))
\[
= \sum_{k=1}^m (v_k, h_{n+1})_{\mathcal{H}} E\langle D^n \frac{\partial f}{\partial x_k}(X(v_1), \cdots , X(v_m)), (gh) \rangle_{\mathcal{H}^\otimes n}
\]
(using the duality between \( \delta^n \) and \( D^n \))
\[
= E\sum_{k=1}^m D^n \frac{\partial f}{\partial x_k}(X(v_1), \cdots , X(v_m)) \otimes v_k, gh \otimes h_{n+1} \rangle_{\mathcal{H}^\otimes n+1}
\]
(inner product in \( \mathcal{H}^\otimes n+1 \))
\[
= E\langle D^{n+1}F, gh \otimes h_{n+1} \rangle_{\mathcal{H}^\otimes n+1}
\]
(definition of Malliavin derivative \( D^{n+1} \))
\[
= E[F \cdot \delta^{n+1}(gh \otimes h_{n+1})]
\]
(duality between \( D^{n+1} \) and \( \delta^{n+1} \))

Thus, for each \( F \in \mathcal{S} \), we obtain
\[
E[F \cdot \delta(\delta^n(gh_1 \otimes \cdots \otimes h_n)h_{n+1})] = E[F \cdot \delta^{n+1}(gh_1 \otimes \cdots \otimes h_{n+1})]
\]
and the result follows. \( \square \)

The following result generalizes Theorem 2.7.7 of NP (2012).

**Theorem 4.1.** Let \( \mathcal{H} \) be a real separable Hilbert space and \( X : \mathcal{H} \to L^2(\Omega) \) be an isonormal process. Let \( h \in \mathcal{H} \) be such that \( \|h\| = 1 \) and \( g : \mathbb{R} \to \mathbb{R} \) be a function so that \( g(X(h)) \) is a smooth rv. Then, for \( n \geq 1 \),
\[
\delta^n(g(X(h))h^\otimes n) = (H - g)^n(X(h)), \tag{4.6}
\]
using the rule in Remark 3.1.

**Proof.** The case \( n = 1 \) is proved in Lemma 4.1. Assume now (4.6) holds so that
\[
\delta^n(g(X(h))h^\otimes n) = (H - g)^n(X(h)).
\]
By Theorem 3.1 and Remark 4.4, the above assumption is equivalent to
\[
\delta^n(g(X(h))h^\otimes n) = (\delta^n g)(X(h)), \tag{4.7}
\]
where \( \delta^n \) on the rhs of (4.7) is an operator on the set of functions, in the sense of Section 3 Consider now
\[
\delta^{n+1}(g(X(h))h^\otimes (n+1)) = \delta \left( \delta^n(g(X(h))h^\otimes n)h \right) \quad \text{(using Lemma 4.2)}
\]
\[= \delta \left( (\delta^n g)(X(h))h \right) \quad \text{(by (4.7))} \]
\[= (\delta(\delta^n g))(X(h)) \quad \text{(using (4.4))} \]
\[= (\delta^{n+1} g)(X(h)) \quad \text{(using (3.2))} \]
\[= (H - g)^{n+1}(X(h)), \]
using Theorem 3.1. This proves the result. \(\square\)

**Remark 4.2.** Indeed, Theorem 4.1 implies, for \(n \geq 1\),
\[\delta^n (g(X(h))h^\otimes n) = (\delta^n g)(X(h)). \quad (4.8)\]

**Example 4.1.** (i) When \(g(x) = 1\), we get from (4.6),
\[\delta^n (h^\otimes n) = (H - 1)^n(X(h)) \]
\[= H_n(X(h)),\]
which is Theorem 2.7.7 of NP (2012).

(ii) Similarly, When \(g(x) = x\), we have
\[\delta^n (X(h)h^\otimes n) = (H - x)^n(X(h)) \]
\[= (H_1H_n - nH_{n-1})(X(h)) \]
\[= H_{n+1}(X(h)),\]
using (2.3).

(iii) When \(n = 2\), we obtain
\[\delta^2(g(X(h))h^\otimes 2) = (H - g)^2(X(h)) \]
\[= (H_2g - 2H_1g^1 + g^2)(X(h)) \]
\[= (X^2(h) - 1)g(X(h)) - 2X(h)g^1(X(h)) + g^2(X(h)),\]
using the properties of Hermite polynomials.

**Corollary 4.1.** Assume the conditions of Theorem 4.1 hold and let \(D\) be the Malliavin derivative. Then
\[D(\delta^n(g(X(h))h^\otimes n)) = \left( n(H - g)^{n-1} + (H - g^1)^n \right)(X(h))h. \quad (4.9)\]

**Proof.** Note first for a real-valued function \(f\) on \(R\), we have from (4.2) that
\[D(f(X(h))) = f'(X(h))h = (Df)(X(h))h. \quad (4.10)\]

Also, from Theorem 4.1
\[D(\delta^n(g(X(h))h^\otimes n)) = D((\delta^n g)(X(h))) \quad \text{(using (4.8))} \]
\[= (D(\delta^n g))(X(h))h \quad \text{(using (4.10))} \]
\[= [n\delta^{n-1}g + \delta^n g^1](X(h))h \quad \text{(using Lemma 3.2)} \]
\[= [n(H - g)^{n-1} + (H - g^1)^n](X(h))h, \]
using Theorem 3.1. This proves the corollary. \(\square\)
When \( g(x) = 1 \) and \( g^1(x) = 0 \) and \( (H - 0)^n = 0 \), using our rule in Remark 3.1. Also, from (4.9),
\[
D(\delta^n(h^\otimes n)) = (n(H - 1)^{n-1} + (H - 0)^n)(X(h))h
= nH_{n-1}(X(h))h,
\]
which is similar to Proposition 2.6.1 of NP (2012).
Similarly, when \( g(x) = x \), we get
\[
D(\delta^n(X(h)h^\otimes n)) = \left( n(H - x)^{n-1} + (H - 1)^n \right)(X(h))h
= \left( n(H_{n-1}x - (n - 1)H_{n-2}) + H_n \right)(X(h))h
= \left( H_n + nH_1H_{n-1} - n(n - 1)H_{n-2} \right)(X(h))h,
\]
using \( H_1 = x \).
Note that the formulas given above are explicit and can easily be computed.

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