Exact Traveling Wave Solutions of One-Dimensional Parabolic–Parabolic Models of Chemotaxis

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Abstract. In this paper, we consider three different one-dimensional parabolic-parabolic systems of chemotaxis. For these systems, we obtain the exact analytical solutions in terms of traveling wave variables. Not all solutions obtained possess an adequate physical and biological interpretation. However, some solutions seem interesting, and a more detailed analysis is possible in a future study.

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1. INTRODUCTION

Chemotaxis, or the directed cell movement (of bacteria or other organisms) up or down a chemical concentration gradient plays an important role in many biological and medical fields such as embryogenesis, immunology, cancer growth and invasion. The macroscopic classical model of chemotaxis was proposed by Patlak in 1953 [1] and by Keller and Segel in the 1970s [2–4]. Since then, the mathematical modelling of chemotaxis has been widely developed. This model is described by a system of coupled nonlinear partial differential equations. Proceeding from the study of the properties of these equations, it is concluded that the model demonstrates a deep mathematical structure. The survey of Horstmann [5] provides a detailed introduction to the mathematics of the Patlak–Keller–Segel model and summarizes different mathematical results; detailed reviews also can be found in the textbooks of Suzuki [6] and Perthame [7]. In the review of Hillen and Painter [8] a number of variations of the original Patlak–Keller–Segel model are explored in detail. The authors study their formulation from the biological perspective, summarise key results on their analytical properties and classify their solution form [8]. It should be noted that interest in the Patlak–Keller–Segel model does not weaken and new works appear devoted to the study of various properties of the equations and their solutions [9–12].

In this paper, we investigate a number of different models describing chemotaxis. The aim of this paper is to obtain exact analytical solutions of these models. For one-dimensional parabolic-parabolic systems under consideration, we present these solutions in explicit form in terms of traveling wave variables. Of course, not all of the solutions obtained can have appropriate biological interpretation, since the biological functions must be nonnegative in all domain of definition. However some of these solutions are positive and bounded and their analysis requires further investigation. Despite the large number of works devoted to the systems under consideration and their properties, as well as the properties of their solutions, it seems to us that the solutions obtained in this paper are new.

The Patlak–Keller–Segel model describes the space-time evolution of cell density \( u(t, \vec{r}) \) and the concentration of the chemical substance \( v(t, \vec{r}) \). The general form of this model is:

\[
\begin{align*}
    u_t - \nabla (\delta_1 \nabla u - \eta_1 u \nabla \phi(v)) &= 0 \\
    v_t - \delta_2 \nabla^2 v - f(u, v) &= 0,
\end{align*}
\]

where \( \delta_1 > 0 \) and \( \delta_2 \geq 0 \) are cell and chemical substance diffusion coefficients respectively, \( \eta_1 \) is the chemotaxis coefficient; when \( \eta_1 > 0 \) this is an attractive chemotaxis (“positive taxis”), and when \( \eta_1 < 0 \) this is a repulsive (“negative”) one [13, 14]. The function \( \phi(v) \) is the chemosensitivity function and \( f(u, v) \) characterizes the chemical growth and degradation. These functions are taken
in different forms that correspond to some variations of the original Patlak–Keller–Segel model. We follow the reviews of Hillen and Painter [8] and of Wang [15] and consider models presented therein.

This paper is concerned with one-dimensional simplified models, when the coefficients $\delta_1$, $\delta_2$ and $\eta_1$ are positive constants, $x \in \text{Re}, t \geq 0$, $u = u(x,t)$, $v = v(x,t)$.

2. SIGNAL-DEPENDENT SENSITIVITY MODEL

Let us start with a model that possesses nonnegative bounded solutions that may be of interest from the biological point of view. Now consider the “logistic” model, one of versions of signal-dependent sensitivity model [8] with the chemosensitivity function $\phi(v) = (1+b) \ln(v+b)$, $b = \text{const}$ and $f(u,v) = \hat{\sigma}u - \hat{v}v$. In the survey [5] there is a mathematical analysis of this model. When $b = 0$ and $\hat{\beta} = 0$, the existence of traveling waves were established in [16, 17]. The replacement $t \rightarrow \delta t$, $u \rightarrow \sigma u \hat{\sigma} / \delta_1$ gives $\delta_1 = 1$, $\alpha = \delta_2 / \delta_1$, $\hat{\beta} = \hat{\beta} / \delta_1$, $\sigma = \pm 1$. We also set $\eta = \eta_1(1 + b) / \delta_1$, $1 + b > 0$, as well as $\phi(v) = \ln |v+b|$. It should be noted that the sign of $\sigma$ may affect the mathematical properties of the system. So, $\sigma = 1$ corresponds to an increase of the chemical substance, proportional to cell density, whereas $\sigma = -1$ corresponds to its decrease. And as we shall see later, various solutions correspond to these two cases.

After the above replacements the model reads:

\[
\begin{aligned}
&u_t - u_{xx} + \eta(u \frac{v_x}{v+b})_x = 0 \\
&v_t - \alpha v_{xx} - \sigma u + \beta v = 0.
\end{aligned}
\]

If we introduce the function $v = v + b$ in terms of traveling wave variable $y = x - ct$, $c = \text{const}$, this system acquires the form:

\[
\begin{aligned}
&u_y + cu - \eta u (\ln(v))_y + \lambda = 0 \\
&\alpha v_{yy} + cv_y - \beta v + \beta b + \sigma u = 0,
\end{aligned}
\]

where $u = u(y)$, $v = v(y)$ and $\lambda$ is an integration constant.

In this paper, we will consider the case $\lambda = 0$. Then the first equation in (1*) gives

\[
u = C_u e^{-cy} v^n,
\]

where $C_u$ is a constant and we examine the following equation for $v$:

\[
\alpha v_{yy} + cv_y - \beta v + \beta b + \sigma C_u e^{-cy} v^n = 0.
\]

Since $\eta$ is a positive constant, we consider two cases: $\eta = 1$ when (3) is a linear nonhomogeneous equation, and $\eta \neq 1$.

2.1. $\eta = 1$

Let us begin with $\eta = 1$. We introduce the new variable $z$ and the new function $w$:

\[
z = \left( \frac{4\sigma C_u}{\alpha c^2} \right)^{\frac{1}{2}} e^{-\frac{cy}{\alpha c^2}}, \quad w = \left( \frac{4\sigma C_u}{\alpha c^2} \right)^{\frac{\alpha c^2}{2}} v e^{\frac{cy}{\alpha c^2}}
\]

and equation (3) becomes:

\[
z^2 w_{zz} + z w_z + w(z^2 - \nu^2) = \Lambda z^{-\frac{1}{2}},
\]

where

\[
\nu^2 = \frac{\alpha c^2}{2} \left( 1 + \frac{4\alpha \beta}{c^2} \right), \quad \Lambda = \frac{4 \beta b}{\alpha c^2} \left( \frac{4\sigma C_u}{\alpha c^2} \right)^{\frac{1}{4}}.
\]
Equation (5) is the Lommel differential equation [18, 19] with $\mu = -1 - \frac{1}{\alpha}$. For $\sigma C_u > 0$, its general solution has the form:

$$w(z) = C_J J_{\nu}(z) + C_Y Y_{\nu}(z) + \Delta S_{\mu,\nu}(z),$$

where $C_J, C_Y$ are constants, $J_{\nu}(z)$ and $Y_{\nu}(z)$ are Bessel functions and

$$S_{\mu,\nu}(z) = s_{\mu,\nu}(z) + 2^{\mu-1} \Gamma\left(\frac{\mu - \nu + 1}{2}\right) \Gamma\left(\frac{\mu + \nu + 1}{2}\right) \left[\sin\left(\frac{\pi}{2}(\mu - \nu)\right) J_{\nu}(z) - \cos\left(\frac{\pi}{2}(\mu - \nu)\right) Y_{\nu}(z)\right],$$

$$s_{\mu,\nu}(z) = \frac{z^{\mu+1}}{[(\mu + 1)^2 - \nu^2]} \text{I}_F(1; -\frac{\mu - \nu + 3}{2}, \frac{\mu + \nu + 3}{2}; -\frac{z^2}{4})$$

are Lommel functions, $\text{I}_F$ is generalized hypergeometric function [18, 19]. Further, substituting the initial variable $y$ and the function $v$ (see (4)) into (6) we obtain a formal solution.

We first consider the case $b = 0$. Then $\nu = \nu \geq 0$ and $C_u > 0$. Equation (5) becomes homogeneous and for $\sigma = 1$ its general solution is

$$w(z) = C_J J_{\nu}(z) + C_Y Y_{\nu}(z).$$

However one can check that the function $u = u(y)$ diverges as $cy \to -\infty$ for all $\nu$.

Consider now $\sigma = -1$. For $v(y)$ be real let $\alpha = 2$. Then (5) becomes the modified Bessel equation; the analysis of the behavior of solutions at $\pm \infty$ leads to suitable solutions for $v(y)$ and $u(y)$:

$$v(y) = e^{-cy} K_\nu\left(\sqrt{\frac{2Cy}{c^2}} e^{-\frac{cy}{c^2}}\right), \quad u(y) = C_u e^{-\frac{cy}{c^2}} K_\nu\left(\sqrt{\frac{2Cy}{c^2}} e^{-\frac{cy}{c^2}}\right)$$

with restrictions $\nu \leq 1/2$ and $\beta \leq 0$. So on can see that $v(y) \to 0$ as $cy \to -\infty$ for all $\nu \leq 1/2$; $v(y) \to 0$ for $\nu < 1/2$ and $v(y) \to (\pi^2 c^2 / (8 Cy))^1/4$ for $\nu = 1/2$ as $cy \to \infty$ and $u(y) \to 0$ as $y \to \pm \infty$ for all $\nu \leq 1/2$. The curves of these functions are presented in Figs. 1–2. Thus, the solution obtained may be considered as a biologically meaningful one, and it requires further investigation.

Consider the case $b > 0$. Let us return to equation (5) with $\Lambda \neq 0$. The analysis of the asymptotic forms of the solutions at $\pm \infty$ [18, 19] gives the following expressions for $v(y)$ and $u(y)$:

$$v(y) + b = -\frac{4\beta b}{ac^2} \left(\frac{4\sigma C_u}{ac^2}\right)^{\frac{1}{2}} e^{-cy} S_{\mu,\nu}\left(\sqrt{\frac{4\sigma C_u}{ac^2}} e^{-\frac{cy}{c^2}}\right)$$

$$u(y) = -C_u \frac{4\beta b}{ac^2} \left(\frac{4\sigma C_u}{ac^2}\right)^{\frac{1}{2}} e^{-cy(1+\beta)} S_{\mu,\nu}\left(\sqrt{\frac{4\sigma C_u}{ac^2}} e^{-\frac{cy}{c^2}}\right)$$
with $\sigma C_u > 0$ and for $\nu < 1/\alpha$. The latter condition leads to the requirement $-c^2/(4\alpha) \leq \beta < 0$. The $v(y) \to -b$ and $u(y) \to -\beta b/\sigma$ as $cy \to -\infty$ and $v(y) \to 0$, $u(y) \to 0$ as $cy \to \infty$. Thus, one can see that for $b > 0$, $\sigma = 1$ and $C_u > 0$ $u(y) \geq 0$ is satisfied but $v(y) < 0$. These functions are presented in Figs 3–4. It should be noted that $\nu \neq 1/\alpha$, or $\beta \neq 0$ because of the pole in the $\Gamma$-function.
Consider the case $b < 0$. Using the analysis of (10) one can see that the condition $b < 0$ along with $\sigma = -1$ and $C_u < 0$ ($\sigma C_u > 0$) leads to the fact that the function $u(y)$ has not changed, but $v(y)$ becomes positive on all domain of definition. This function is presented in Fig. 5.

2.2. $\eta \neq 1$

Let us return to equation (3) and rewrite it in terms of the variable $\xi = e^{-\alpha y}$:

$$
\xi^2 v_{\xi\xi} - \frac{\alpha\beta}{c^2} v + \frac{\sigma\alpha C_u}{c^2} \xi^\alpha v^\eta = -\frac{\alpha\beta b}{c^2}.
$$

(11)

To integrate this equation, we use the Lie group method of infinitesimal transformations \[20\]. We find a group invariant of a second prolongation of the one–parameter symmetry group vector of (11) and with its help we transform equation (11) into an equation of the first order. It turns out that nontrivial symmetry group requires some conditions:

$$
\frac{\alpha\beta b}{c^2} = 0, \quad \beta = \frac{(\alpha - 2)(\alpha + \eta + 1)c^2}{\alpha(\eta + 3)^2}
$$

(12)

and we consider the case $b = 0$. Thus, $v = v$ and for

$$
z = \frac{v^{1-\eta}}{v^\eta}, \quad w = v_y v^{-\frac{\alpha + \eta - 1}{\alpha}}
$$

(13)

we obtain the Abel equation of the second kind:

$$
w_z [(1 - \eta)w - \alpha z] + (\alpha + \eta - 1)z^{-1}w^2 + \alpha z(-\frac{\alpha\beta}{c^2} + \frac{\sigma\alpha C_u}{c^2}z^{-\alpha}) = 0.
$$

(14)

Then we find solutions of equation (14) in parametric form \[21\] with the parameter $t$. Now we consider the case $2\alpha + \eta \neq 1$. A combination of substitutions leads to:

$$
z = \left( -\frac{(\eta + 3)(\eta + 1) t^2 + 2\sigma\alpha C_u}{2(2\alpha + \eta - 1)} \frac{\partial_\xi(t)}{\partial(t)} \right)^{\frac{1}{2}}
$$

(15)

$$
w = z^{2-\alpha} \left( t + \frac{2(2\alpha + \eta + 1)}{(\eta - 1)(\eta + 3)}z^{\frac{\alpha}{2}} \right) + \frac{\alpha}{1 - \eta} z,
$$

where we take

$$
\vartheta(t) > 0 \quad \text{and} \quad (2\alpha + \eta - 1) \vartheta(t) < 0,
$$

(16)
where ˜t for \in order for the asymptotic representation of hypergeometric Gauss function as solutions of initial equations (2)–(3) in parametric form:

\[ \frac{\alpha C_u}{c^2} \] is done in [15]. An existence of a global solution is established in [24].

When ˜b \on different aspects of traveling waves solutions. When the constant ˜t = 0 and equation (14) becomes an equation for the function \( \eta u \alpha \) of \( f \) in Fig. 11.

The model with logarithmic chemosensitivity function \( \phi(v) \sim \ln v \) is also studied. For the case \( f(u,v) = -v^m u + \bar{b} v, \bar{b} = const \) an extensive analysis is carried out in [15]. This survey is focused on different aspects of traveling waves solutions. When \( m = 0 \), the model coincides with (1) for \( b = 0 \). When \( \bar{b} = 0 \) and \( m = 1 \), the system was studied in [22, 23]. The complete analysis for \( \bar{b} = 0 \) is done in [15]. An existence of a global solution is established in [24].

where \( \tilde{C}_\varphi, C_\varphi \) are constants and \( _2F_1 \) is the hypergeometric Gauss function. Further we obtain the solutions of initial equations (2)–(3) in parametric form:

\[ y(t) = -\frac{\alpha(\eta + 3)}{c(2\alpha + \eta - 1)} \ln \left( \tilde{\vartheta}(t) \right) \]

\[ v(t) = \left( -\frac{\tilde{C}_\varphi(\eta + 3)}{2(2\alpha + \eta - 1)} \right)^{\frac{\alpha}{\eta + 3}} \left( (\eta + 1)t^2 + \frac{2\sigma C_u}{c^2} \right)^{-\frac{1}{\eta + 1}} \left( \tilde{\vartheta}(t) \right)^{\frac{2-\alpha}{2\alpha + \eta - 1}} \]

\[ u(t) = C_u \left( -\frac{\tilde{C}_\varphi(\eta + 3)}{2(2\alpha + \eta - 1)} \right)^{\frac{\alpha}{\eta + 3}} \left( (\eta + 1)t^2 + \frac{2\sigma C_u}{c^2} \right)^{-\frac{1}{\eta + 1}} \left( \tilde{\vartheta}(t) \right)^{\frac{\alpha + 2\alpha + 2}{2\alpha + \eta - 1}} \]

where the constant \( \tilde{C}_\varphi \) is chosen so that \( (2\alpha + \eta - 1)\tilde{C}_\varphi < 0 \), what is consistent with (16). Using the asymptotic representation of hypergeometric Gauss function as \( t \to \pm \infty \) [18] we can take

\[ C_\varphi > |\tilde{C}_\varphi| \frac{\pi}{2\sqrt{\eta + 1}} \left( \frac{2\sigma C_u}{c^2} \right)^{-\frac{1}{\eta + 1}} \frac{\Gamma\left( \frac{1}{\eta + 1} \right)}{\Gamma\left( \frac{\eta + 3}{2(\eta + 1)} \right)} \]

in order for \( y, v, \) and \( u \) to be real. Then one can see that all functions (18) are continuous bounded for \( t \in \mathbb{R} \) and \( v, u \) are positive. Hence, one may try to biologically interpret the functions \( v(y) \) and \( u(y) \) and this requires further investigation. In Figure 6, one may see the different curves \( v(y) \) for \( \eta = 0.1 \) and different \( \alpha \). Figure 7 demonstrates \( v(y) \) and \( u(y) \) for \( \eta < 1 \). Further, for larger values of \( \alpha \) and \( \eta \), it seems more convenient to present curves \( y(t), v(t), \) and \( u(t) \) to analyze them, see Figs. 8–10. One can see from (12) that \( \beta \geq 0 \) when \( \alpha \geq 2 \), and the case of \( \beta = 0, \alpha = 2 \) is presented in Fig. 11.

### 3. LOGARITHMIC SENSITIVITY

The model with logarithmic chemosensitivity function \( \phi(v) \sim \ln v \) is also studied. For the case \( f(u,v) = -v^m u + \bar{b} v, \bar{b} = const \) an extensive analysis is carried out in [15]. This survey is focused on different aspects of traveling waves solutions. When \( m = 0 \), the model coincides with (1) for \( b = 0 \). When \( \bar{b} = 0 \) and \( m = 1 \), the system was studied in [22, 23]. The complete analysis for \( \bar{b} = 0 \) is done in [15]. An existence of a global solution is established in [24].
Now we consider the system with $\phi(v) = \ln v$ and $f(u, v) = \tilde{\sigma}vu - \tilde{\beta}v$. Similarly, a replacement $t \to \delta_1 t$, $u \to \sigma u \tilde{\sigma}/\delta_1$ gives $\delta_1 = 1$, $\eta = \eta_1/\delta_1$, $\alpha = \delta_2/\delta_1$, $\beta = \tilde{\beta}/\delta_1$, $\sigma = \pm 1$. Then the model has the form:

$$
\begin{align*}
\left\{
\begin{array}{c}
u_t - \nu_{xx} + \eta (u \frac{v_x}{v})_x = 0 \\
v_t - \alpha v_{xx} - \sigma vu + \beta v = 0.
\end{array}
\right.
\end{align*}
$$

(20)
Let us rewrite system (20) in terms of function $v(x, t) = \ln v(x, t)$:

$$
\begin{align*}
&\frac{\partial u}{\partial t} - u_{xx} + \eta(uv_x)_x = 0 \\
&\frac{\partial v}{\partial t} - \alpha v_{xx} - \alpha(v_x)^2 + \beta - \sigma u = 0,
\end{align*}
$$

(20')
we obtain an equation similar to (11) with zero right-hand side: $$\alpha v_{yy} + \alpha(v_y)^2 + cv_y - \beta + \sigma u = 0,$$ (20*) where $u = u(y)$, $v = v(y)$, and $\lambda$ is an integration constant. To integrate (20*), we tested this system on the Painlevé ODE test. One can show that for $\eta > 0$ it passes this test only if $\alpha = 2$ with the additional condition $\lambda = -\sigma c\beta (1 + \eta/2)$ [25]. If we express $u(y)$ as $v(y)$ from (20*), we obtain an equation only for $v(y)$; for $\alpha = 2$ it has the form:

$$2v_{yy} + 3cv_y + (e^2 + \eta \beta) v_y + 2(2 - \eta)v_y v_{yy} + 2(2 - \eta)(v_y)^2 - 2\eta (v_y)^3 - c\beta - \sigma \lambda = 0.$$ (21)

For $\lambda = -\sigma c\beta \left(1 + \frac{\eta}{2}\right)$, this equation can be linearized. It becomes equivalent to the following linear equation for $F$:

$$F_y + cF = 0, \quad \text{where} \quad F(y) = e^{2v} \left(2v_{yy} + cv_y - \eta (v_y)^2 + \frac{\eta \beta}{2}\right);$$ (22)

this gives the equation for $v(y)$:

$$2v_{yy} + cv_y - \eta (v_y)^2 + \frac{\eta \beta}{2} = C_F e^{-2v - cy},$$ (23)

where $C_F = \text{const.}$ If we rewrite (23) in terms of the variable $\xi = e^{-\frac{2v}{c}y}$ for the function $\Psi(\xi) = e^{-\frac{\eta}{2}v}$, we obtain an equation similar to (11) with zero right-hand side:

$$\xi^2 \Psi_{\xi\xi} - \frac{\eta^2 \beta}{2c^2} \Psi + \frac{\eta C_F}{c^2} \xi^2 \Psi^{\frac{1}{n} + 1} = 0.$$ (24)

Using the result of the symmetry group analysis of (11) we can write solution for $\beta = 0$ (see (18)):

$$y(t) = -\frac{2}{e} \ln \left(\vartheta(t)\right), \quad v(t) = \frac{C_\vartheta}{2} \left(\frac{2(\eta + 2)}{\eta} t^2 + \frac{2\eta C_F}{c^2}\right)^{-\frac{1}{2}},$$ (25)

where $\vartheta(t)$ is given in (17) and $u(y)$ may be expressed from (20*). However, one may see that $v \to \infty$ as $t \to \pm \infty$ and this solution is unacceptable as a biological function.

Another possibility to solve this equation exactly is to put $C_F$ equal to zero. When $C_F = 0$, which means $F(y) = 0$, and $\beta \neq 0$ equation (24) can be linearized by $\xi = e^\tau$ [21]. Its solution has three forms according to a sign of the expression $D = 1 + 2\eta^2 \beta/c^2$. Since $v$ should be a nonnegative and bounded function as $cy \to \pm \infty$, the only suitable solution is

$$v(y) = e^{\frac{\eta}{2}y} \left(C_- e^{-\frac{\eta}{2c}y} + C_+ e^{\frac{\eta}{2c}y}\right)^{-\frac{2}{\eta}}$$ (26)

where $C_\pm$ are positive constants and $\beta > 0$. Unfortunately, the corresponding solution for $u(y)$ is alternating and has the form:

$$u(y) = -\frac{\sigma c^2 (\eta + 2)}{2\eta^2} (C_-^2 (1 + \sqrt{D}) e^{-\frac{\eta}{2c}y} + C_+^2 (1 - \sqrt{D}) e^{\frac{\eta}{2c}y})$$

$$-\frac{4\eta^2 \beta}{c^2} C_- C_+ \left(C_- e^{-\frac{\eta}{2c}y} + C_+ e^{\frac{\eta}{2c}y}\right)^{-\frac{2}{\eta}}.$$ (27)
It is easy to see that
\[
\sigma u(y) \rightarrow \frac{c^2(\eta + 2)}{2\eta^2}(-1 \pm \sqrt{D}) \quad \text{as} \quad cy \rightarrow \pm \infty.
\]
These functions are presented in Figs. 12–13.

4. LINEAR SENSITIVITY

Let us consider the system with linear function \( \phi(v) \sim v \). When \( f(u, v) = u - v \), the system is called the minimal chemotaxis model, following the nomenclature of [26]. This model is often considered with \( f(u, v) = \tilde{\sigma}u - \tilde{\beta}v \) (\( \tilde{\sigma} \) and \( \tilde{\beta} \) are constants) and is studied in many papers. As was proved in [27, 28], the solutions of this system are global and bounded in time for space dimension one. The case of positive \( \tilde{\sigma} \) and nonnegative \( \tilde{\beta} \) is studied in [29–33]. As we noted earlier, the sign of \( \tilde{\sigma} \) may effect the mathematical properties of the system, which changes its solvability conditions [34].

Now we consider the linear chemosensitivity function \( \phi(v) = v \) and \( f(u, v) = \tilde{\sigma}u - \tilde{\beta}v \). The replacement \( t \rightarrow \delta_1 t, \; v \rightarrow \nu \eta_1 / \delta_1, \; u \rightarrow \sigma u \eta_1 / \delta_1^2 \) leads to \( \delta_1 = \eta_1 = 1, \; \alpha = \delta_2 / \delta_1, \; \beta = \tilde{\beta} / \delta_1, \; \sigma = \pm 1 \). Then the system has the form:

\[
\begin{align*}
\frac{u}{u} - u_{xx} + (uv_x)_x &= 0 \\
\frac{v}{v} - \alpha v_{xx} + \beta v - \sigma u &= 0.
\end{align*}
\]
One can see that for \( \nu > 0 \), the function \( u(y) \) has a point of discontinuity. One can say that when \( B < 0 \) we obtain the “blow up” solution in the sense that it goes to infinity for finite \( y \), and this is true for different \( \nu \). The functions (29) for \( \nu = 1/2 \) are presented in Figs. 14–15.
5. CONCLUSION

We have investigated three different one-dimensional parabolic-parabolic Patlak-Keller-Segel models. For each of them, we obtained the exact solutions in terms of traveling wave variables. Not all of these solutions are acceptable for biological interpretation, but there are solutions that require detailed analysis. It seems interesting to consider the latter for the experimental values of the parameters and see their correspondence with experiment. This question requires further investigations.

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