DYNAMICAL SYMMETRY BREAKING IN
FOUR-FERMIonic MODELS UNDER THE INFLUENCE OF
EXTERNAL ELECTROMAGNETIC FIELD IN CURVED
SPACETIME

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An investigation of the Nambu-Jona-Lasinio model with external constant electromagnetic and weak gravitational fields is carried out in three- and four-dimensional spacetimes. The effective potential of the composite bifermionic fields is calculated keeping terms linear in the curvature, while the electromagnetic field effect is treated exactly by means of the proper-time formalism. Numerical simulations of the dynamical symmetry breaking phenomenon accompanied by some phase transitions are presented.

1. Spontaneous symmetry breaking phenomenon is the absolutely necessary part of modern quantum field theory. It has very strong experimental basis and is usually incorporated in any new model.

2. One of the most attractive opportunities to involve the spontaneous generation of dimensional values is the dynamical symmetry breaking mechanism when masses, effective coupling constants and so on are expressed through the vacuum expectation value of bifermionic composite field \( \langle \psi \psi \rangle \) instead of some hypothetical Higgs scalar field \( \langle \phi \rangle \).

3. Dynamical symmetry breaking has been applied successfully to describe the overcritical behaviour of quantum electrodynamics, top quark condensate mechanism of mass generation in Weinberg-Salam model, technicolor models and especially to investigate nonpertubatively the composite fields generation in four-fermionic models that of Nambu-Jona-Lasinio one.

4. Recently this model has been studied in external electromagnetical field. The essential role of this field in the dynamical symmetry breaking realisation has been proved. Furthermore it has been paid a great attention to the dynamical symmetry breaking phenomenon in curved spacetime.

5. It seems to be important to build up the realistic scenario of early Universe. However it has been shown early Universe should contain a large
primodial magnetic field and have a huge electrical conductivity.

It makes us consider the dynamical symmetry breaking and dynamical fermion mass generation with the presence of both electromagnetic and gravitational fields.

We consider Nambu – Jona – Lasinio model in external constant electromagnetic field treated nonpertubatively in Schwinger propertim e formalism. The linear – curvature corrections for the effective potential of composite bifermionic field are calculated and the phase structure of the model is investigated. Both 4D and 3D cases are disussed. The phase transitions accompanied dynamical symmetry breaking on the spacetime curvature and electromagnetic field strength values are described numerically. Let us discuss now the Nambu-Jona–Lasinio model in the arbitrary dimension curved space-time with the following action:

$$S = \int d^d x \sqrt{-g} (i \bar{\psi} \gamma^\mu(x) D_\mu \psi + \frac{\lambda}{2N} \left[ (\bar{\psi} \psi)^2 + (\bar{\psi} i \gamma_5 \psi)^2 \right]), \quad (1)$$

where the covariant derivative $D_\mu$ includes the electromagnetic potential $A_\mu$, local Dirac matrices $\gamma_\mu(x)$ are expressed through the usual flat one $\gamma_a$ and tetrads $e^a_\mu$, $\sigma_{ab} = \frac{1}{4} [\gamma_a, \gamma_b]$, $\omega^{ab}_\mu$ is a spin- connection and $N$ is the number of bispinor fields $\psi_a$. Spinor representation dimension is supposed to be four. Greek and Latin indices correspond to the curved and flat tangent spacetimes.

Introducing the auxiliary fields $\sigma = -\frac{\lambda}{N} \bar{\psi} \psi$, $\pi = -\frac{\lambda}{N} \bar{\psi} i \gamma_5 \psi$, we can rewrite the action as:

$$S = \int d^d x \sqrt{-g} \{ i \bar{\psi} \gamma^\mu(x) D_\mu \psi - \frac{N}{2\lambda} (\sigma^2 + \pi^2) - \bar{\psi} (\sigma + i\pi \gamma_5) \psi \}, \quad (2)$$

Then the effective potential is given by:

$$V_{eff} = \frac{\sigma^2}{2\lambda} + i\text{Sp} \ln \langle \left[ i \gamma^\mu(x) D_\mu - \sigma \right] \rangle \quad (3)$$

By means of the usual Green function, which obeys the equation

$$(i \gamma^\mu D_\mu - \sigma) G(x, x', \sigma) = \delta(x - x'), \quad (4)$$

we obtain the following formula

$$V'_{eff}(\sigma) = \frac{\sigma}{\lambda} - i\text{Sp} G(x, x, \sigma) \quad (5)$$
To calculate the linear curvature corrections the local momentum expansion formalism is the most convenient one. Then in the special Riemannian normal coordinate framework

\[ g_{\mu\nu}(x) = \eta_{\mu\nu} - \frac{1}{3} R_{\mu\rho\alpha\nu} y^\rho y^\sigma \]  

and corresponding formulae for the others values with \( y = x - x' \). Then we suppose that:

\[ G(x, x', \sigma) = \Phi(x, x') \left[ \tilde{G}_0(x - x', \sigma) + \tilde{G}_1(x - x', \sigma) \ldots \right], \]

where \( \tilde{G}_n \sim R^n \), \( \Phi(x, x') = \exp\left[ ie \int_{x'}^x A^\mu(x') dx'' \right] \) and therefore our basic expression for the Green function \( \tilde{G}_1 \) in a constant curvature space-time of arbitrary dimension \( d \) is the following:

\[ \tilde{G}_1(0, \sigma) = -iR \frac{12(d-1)}{d} \int dz G_{00}(-z, \sigma) \left[ 2 \not z \partial_\mu \tilde{G}_0(z, \sigma) - 2 z^2 \gamma^\mu \partial_\mu \tilde{G}_0(z, \sigma) + 3(d-1) \not z \tilde{G}_0(z, \sigma) \right], \]

where \( G_{00}(z, \sigma) \) is the free Green function. This expression can be substituted into Eq. (5) directly, because \( \Phi(x, x) = 1 \). 

Sustituting an exact flat spacetime Green function \( G_0(z, \sigma) \) of fermions in external electromagnetic field into this formula after some algebra we have evident expression for the effective potential with the linear-curvature accuracy in the constant curvature spacetime.

For 3D spacetime effective potential in the constant magnetic field case is given by:

\[ V_{\text{eff}}(\sigma) = \frac{\sigma^2}{2\lambda} + \frac{1}{4\pi^{3/2}} \int_{1/\Lambda^2}^{\infty} \frac{ds}{s^{3/2}} \exp(-s\sigma^2) \tau \coth \tau \]

\[ + \frac{R}{144\pi^{3/2}} \int_{1/\Lambda^2}^{\infty} \int_{1/\Lambda^2}^{\infty} \frac{ds dt}{(t + s)^{5/2}(1 + \kappa \coth \tau)^2} \exp(-(t + s)\sigma^2) \times \]

\[ \left[ 2\kappa(\kappa + \tau) + (9\tau + 5\kappa) \coth \tau + \kappa(\tau - 3\kappa) \coth^2 \tau \right] \]

and for 4D case:

\[ V_{\text{eff}}(\sigma) = \frac{\sigma^2}{2\lambda} + \frac{1}{8\pi^2} \int_{1/\Lambda^2}^{\infty} \frac{ds}{s^3} \exp(-s\sigma^2) \tau \coth \tau \]
\[
\frac{R}{192\pi^2} \int_{1/\Lambda^2}^{\infty} \int_{1/\Lambda^2}^{\infty} \frac{dsdt}{(t+s)^3(1+\kappa \coth\tau)^2} \exp[-(t+s)\sigma^2] \times \]
\[
[\kappa(\kappa + \tau) + 2(\kappa + 3\tau) \coth\tau + 2\kappa(\tau - \kappa) \coth^2\tau],
\]
where \(\tau = eBs, \kappa = eBt\). Typical behaviour of effective potential in this case is shown on the Fig.1.

A proper-time representation of fermion Green function in external constant electrical field contains after Wick rotation \(\cot(eEs)\) in the contrast of magnetic field case with nonsingular \(\coth(eBs)\). Therefore imaginary part of effective potential caused by residue contribution appears. It means that particle creation takes place and our vacuum is unstable.

The solution of this problem lies out of our present investigation but the simplest possibility seems to consider a comparably small electrical field strength values which provide an exponentially depressed particle creation velocity.

3D four-fermions models have been shown to be renormalizable in the leading large-N order. Furthemore the external electromagnetic and gravitational field don’t influence with the renormalization procedure It caused by the fact that the local feature of renormalizability can’t be spoiled by the weak curvature of global spacetime or external electrical field. Formally it can be proved by the integrand leading terms calculations for the \(s \to 0, t \to 0\) limit which is the only essential to determine the UV-divergences of effective potential.

So after coupling constant renormalization
\[
1_{\Lambda_R} = \frac{1}{\lambda} - \frac{\Lambda}{3\pi^2}
\]
we have for renormalized effective potential the following expression:
\[
V_{eff,R}^{(3D)}(\sigma) = \frac{\sigma^2}{2\lambda_R} - \frac{(2ieE)^{3/2}}{4\pi} \left[ 2\zeta(-\frac{1}{2}, \frac{\sigma^2}{2ieE}) - \left(\frac{\sigma^2}{2ieE}\right)^{1/2} \right] +
\]
\[
\frac{R\sigma}{24\pi} + \frac{iR(eE)^{1/6}}{2\pi^23^{3/3}} \exp(-\tau \frac{\sigma^2}{eE}) \Gamma(\frac{2}{3})\sigma^{2/3}.
\]
However 4D four-fermionic models are not renormalizable. Thus we have to make a trick and consider "almost" 4D situation with \(d = 4 - 2\epsilon\) where renormalizability feature exists. Then effective potential is given by:
\[
V_{eff}^{(4D)}(\sigma) = \frac{\sigma^2}{2\lambda} - \frac{(eE)^2}{8\pi^2} \Gamma(-1 + \epsilon) \left[ 4\zeta(-1 + \epsilon, -i\frac{\sigma^2}{2eE}) + i\frac{\sigma^2}{eE} \right] -
\]
\[
\frac{R\sigma^2}{96\pi^2} \Gamma(-1 + \epsilon) + \frac{iR(\epsilon E)^{2/3}}{48\pi^3 3^{1/3}} \exp(-\pi \frac{\sigma^2}{\epsilon E}) \Gamma\left(\frac{2}{3}\right) \sigma^{2/3}
\]  

(13)

The results of numerical simulation are presented on the Fig.2- Fig.3.

1. We have studied the phase structure of Nambu-Jona-Lasinio model in curved spacetime with external constant electrical and magnetic fields.

2. In three dimensions first-order phase transition takes place both on electrical field strength and on spacetime curvature. Magnetic field induced second-order phase transition.

3. In four-dimensional case the typical second-order one occurs on both these external parameters.

4. Our approximation seems to be correct because all of the critical values are very small.

5. Positive spacetime curvature tries to restore chiral symmetry as well as external electrical field meanwhile magnetic field breaks symmetry for any finite strength value.

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Figure 1: The behavior of the effective potential $V_{eff}$ is shown with the varying $B$ or $R$ for fixed $\lambda (= 1/2.5)$ and fixed $\Lambda (= 10\mu)$ for 4D spacetime. Second-order phase transition takes place on both magnetic field strength and spacetime curvature.
Figure 2: Behaviour in 3D of $\text{Re}V_{\epsilon R}/\mu^3$ as a function of $\sigma/\mu$ for fixed $\epsilon E/\mu^2 = 0.00005$ and $\lambda\mu = -100$. From above to below, the curves in the plot correspond to the following values of $R/\mu^2 = 0.00; 0.005; 0.004; 0.0032; 0$, respectively. The critical values, defined as usual, are given by: $R_{c1}/\mu^2 = 0.005; R_{c2}/\mu^2 = 0.0032; R_{c2}/\mu^2 = 0$. First-order phase transition takes place.
Figure 3: Behaviour of $\text{Re}V_f/f/\mu^{(4-2\epsilon)}$ in a $(4-2\epsilon)$-dimensional spacetime, as a function of $\sigma/\mu$, depicted for $R/\mu^2 = 0.002$, $\epsilon = 0.005$ and fixed $\lambda\mu = -1000$. From above, the curves in the plot correspond to the following values of the electric field strength: $eE/\mu^2 = 0.004; 0.0028; 0.02; 0.015; 0.001$, respectively. The critical value is reached at $eE_c/\mu^2 = 0.0028$. Second-order phase transition occurs.