Maximum principles and Bôcher type theorems

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Edited by Sun-Yung Alice Chang, Princeton University, Princeton, NJ, and approved May 25, 2018 (received for review March 9, 2018)

We establish maximum principles and Bôcher-type theorems for superharmonic and fractional superharmonic nonnegative functions on a punctured ball. Connecting maximum principles with Bôcher-type theorems is a crucial observation.

fractional Laplacian | singular solution | Bôcher theorem

Maximum principles for superharmonic and fractional superharmonic functions on a punctured ball are very useful in studying singular solutions. Singular solutions appear naturally in many physical and geometry phenomena. It also appears when using the Kelvin transform.

The following is a “maximum principle for superharmonic functions on a punctured ball”:

Assume that \( v(x) \in L^\infty(B_1(0)) \) and \( v(x) \geq 0, \ x \in \mathbb{R}^n \) (\( n \geq 2 \)), and satisfies

\[
\begin{aligned}
-\Delta v(x) &\geq 0 \quad \text{on } B_1(0) \setminus \{0\}, \\
v(x) &> m > 0 \quad \text{on } B_1(0) \setminus B_2(0),
\end{aligned}
\]

then

\[ v(x) \geq m, \ x \in B_1(0) \setminus \{0\}. \]  

Here and hereafter, inequalities and identities are generally regarded in the sense of distributions. For example, the differential inequality \(-\Delta v(x) \geq 0 \) on \( B_1(0) \setminus \{0\} \) holds in the sense of distributions if for \( \phi(x) \geq 0 \) that

\[
\int_{B_1(0)} v(x)(-\Delta)\phi(x)dx \geq 0, \ \forall \phi \in C_0^\infty(B_1(0)).
\]

Notice that when \( n = 1 \), for \( v(x) = |x| \), it holds that \(-\Delta v(x) \geq 0 \), and \( v(x) > \frac{1}{2} \) in \( B_1(0) \setminus B_\frac{1}{2}(0) \); however, \( \inf_{B_1(0)} v(x) = 0 \). This maximum principle plays a key role when one makes a Kelvin transform and uses the method of moving planes to consider the symmetry of the solutions to some elliptic equations (1–6).

Inspired by the classical work of Berestycki–Nirenberg–Varadhan (7) and Caffarelli–Gidas–Spruck (1), we derive the following maximum principles for fractional superharmonic functions on a punctured ball. Also, our version of the above well-known maximum principle assumes very basic regularity of \( v(x) \in L^\infty(B_1(0) \setminus \{0\}) \).

Theorem 1 (Fractional Maximum Principle on a Punctured Ball)

Assume that

\[
\begin{aligned}
(-\Delta)_s^a v(x) + a(x)v(x) &\geq 0, \ \text{on } B_r(0) \setminus \{0\}, \ r > 0, \\
v(x) &> m > 0, \ \text{on } B_r(0) \setminus B_2(0), \\
v(x) &\geq 0, \ \text{on } \mathbb{R}^n, \\
v(x) &\in L^2, \ n \geq 2,
\end{aligned}
\]

and \( a(x) \leq D \) for some constant \( D \), then there exists a positive constant \( c = c(n, s, D) < 1 \) depending on \( n, s, D \) and only such that

\[ v(x) \geq c m, \ x \in B_r(0) \setminus \{0\}, \ \text{for all } r \leq 1. \]

Here \( L^2 = \{ w(x) : \mathbb{R}^n \to \mathbb{R} | \int_{\mathbb{R}^n} \frac{|w(x)|^2}{1 + |x|^{2s}} dx < +\infty \} \). For any \( v(x) \in L^2, \ (-\Delta)_s^a v(x) \) as a distribution is well-defined:

\[ (-\Delta)_s^a v(x) = \int_{\mathbb{R}^n} v(x)(-\Delta)_s^a \phi(x)dx, \ \forall \phi \in C_0^\infty(\mathbb{R}^n) \] since

\[ |(-\Delta)_s^a \phi(x)| \leq C \frac{1}{1 + |x|^{2s-2}}. \]

Remark 1

Comparing with ref. 8 for the Laplacian case, it is necessary to have a constant \( c < 1 \) due to the nonlocality of the fractional Laplacian even for the special case \( a(x) \equiv 0 \) in ref. 9. Following is an example indicating this:

\[ v(x) = \begin{cases} 1 - \epsilon \rho(x), & |x| \leq 1, \\ 1 - \eta(x), & |x| > 1, \end{cases} \]

where \( \rho(x), \eta(x) \in C_0^\infty(\mathbb{R}^n) \), \( \rho(x) = 0, \forall |x| \geq \frac{1}{2}, \rho(0) = 1, \) and \( \eta(x) = 0, \forall |x| \leq 1; \eta(x) = 1, \forall |x| \geq 2. \)

The maximum principle for anti-symmetric functions is also useful for the method of moving planes. It can be proved in a similar way, see ref. 10.

Theorem 2 (Fractional Maximum Principle on a Punctured Ball for Antisymmetric Functions)

Let \( H = \{ (x_1, x') \in \mathbb{R}^n | x_1 < 0 \} \), and assume that \( w(-x_1, x') = -w(x_1, x'), \forall x \in H, B_r(x_1) \subset H, \)

\[
\begin{aligned}
(-\Delta)_s^a w(x) + a(x)w(x) &\geq 0, \text{on } B_r(x_1 \setminus \{x_1\}), \\
v(x) &> m > 0, \text{on } B_r(x_1 \setminus \{x_1\}), \\
v(x) &\geq 0, \text{on } H, \\
w(x) &\in L^2, \ n \geq 2,
\end{aligned}
\]

and \( a(x) \leq D \) for some constant \( D \). Then there exists a positive constant \( c = c(n, s, D) < 1 \) depending on \( n, s, D \) and only such that

\[ w(x) \geq c m, \ x \in B_r(x_1 \setminus \{x_1\}), \ \text{for all } r \leq 1. \]

In proving these maximum principles, we further develop some new Bôcher-type theorems. These Bôcher-type theorems, like the classical Bôcher theorem, are also useful and interesting

Significance

The Bôcher theorem for fractional Laplacian extends the classical Bôcher theorem with a unified proof that can be adapted in other situations. Our distributional approach reduces the regularity requirement and connects the Bôcher theorem directly with the corresponding maximum principles. These maximum principles derived are simple and basic with many potential applications.

Author contributions: C.L. designed research; C.L., Z.W., and H.X. performed research; C.L. contributed new reagents/analytic tools; and C.L. wrote the paper.

The authors declare no conflict of interest.

This article is a PNAS Direct Submission.

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Published online June 20, 2018.
on their own. Connecting the maximum principles with these Bôcher-type theorems is a key ingredient of this paper.

The following classical Bôcher theorem deals with nonnegative harmonic functions on a punctured ball.

**Bôcher Theorem (8, 11)**

If \( v(x) \) is nonnegative and harmonic on \( B_1(0) \setminus \{0\} \), then there is a constant \( a \geq 0 \) such that for all \( x \in B_1(0) \setminus \{0\} \subset \mathbb{R}^n \) with \( n \geq 2 \) that

1. \( v(x) \in L^1(B_1(0)) \),
2. \( -\Delta v(x) = a\delta_0 \). \[9\]

Here and hereafter \( \delta_0 \) is the Delta distribution concentrated at the origin.

Later and hereafter, Brédiz–Lions (9) obtained another Bôcher theorem:

Let \( v(x) \in L^1_{\text{loc}}(B_1(0)) \) and \( v(x) \geq 0, \) a.e. in \( B_1(0) \) be such that \( \Delta v(x) \in L^1_{\text{loc}}(B_1(0)) \) in the sense of distributions on \( B_1(0) \setminus \{0\} \), and

\[-\Delta v(x) \geq -Dv(x) - f(x), \ D > 0, \ a.e. \ in \ B_1(0) \setminus \{0\}, \] \[10\]

with \( f \in L^1_{\text{loc}}(B_1(0)) \). Then \( v(x) \in L^1_{\text{loc}}(B_1(0)) \) and there exists \( \phi(x) \in L^1_{\text{loc}}(B_1(0)) \) and \( a \geq 0 \) such that

\[-\Delta v(x) = \phi(x) + a\delta_0, \ in \ \mathcal{D}'(B_1(0)). \] \[11\]

In ref. 9, they rely heavily on the assumption \( f(x) \in L^1_{\text{loc}}(B_1(0)) \) and the sphere average method.

However, due to the lack of sphere average for the fractional Laplacian, we need to find some new methods to deal with Bôcher theorems for superharmonic functions. In particular, we give a uniform proof for both the Laplacian and fractional Laplacian cases. The following are Bôcher theorems for superharmonic functions (Theorem 3) and fractional superharmonic functions (Theorem 4).

**Theorem 3**

Let \( v(x) \in L^1_{\text{loc}}(B_1(0) \setminus \{0\}) \) be a nonnegative solution in \( \mathbb{R}^n \) \((n \geq 2)\) to

\[-\Delta v(x) + c(x)v(x) = f(x) \geq 0 \ \text{on} \ B_1(0) \setminus \{0\} \] \[12\]

for some \( f(x) \in L^1_{\text{loc}}(B_1(0) \setminus \{0\}) \) and \( c(x) \) is bounded from above, then

1. \( v(x), f(x) \in L^1_{\text{loc}}(B_1(0)) \),
2. \( -\Delta v(x) + c(x)v(x) = f(x) + a\delta_0 \ \text{on} \ B_1(0), \) \[13\]

for some constant \( a \geq 0 \).

It is worth mentioning that under the additional assumption \( v(x) \in C^2(B_1(0) \setminus \{0\}) \), Ghergu and Taliaferro in their book (12) gave another proof for the above theorem, where they also rely on the method of sphere average. We emphasize that \( a \geq 0 \) in (ii) implies that \( v(x) \) is superharmonic on the whole ball in the sense of distributions when \( c(x) = 0 \).

**Remark 2**

The following are counterexamples when \( n = 1 \).

1. \( v(x) = |x|, \) and \( -v''(x) = -2\delta_0, \ a = -2 < 0; \)
2. \( v(x) = \begin{cases} 0, & x > 0, \\ 1, & x < 0, \end{cases} \) and \( -v''(x) = \delta_0; \)
3. \( v(x) = |x|^\theta, \ 0 < \theta < 1, \) then \( v''(x) = \theta(1 - \theta)|x|^{\theta-2} \) is not integrable in the interval \((0, 1)\).

When \( n \geq 3 \), the special case that \( c(x) = 0 \) in Theorem 3 has been proved in ref. 6. The outline of the complete proof for \( n \geq 2 \) is as follows:

First, we derive that \( v(x) \) and \( f(x) \) are integrable in \( B_1(0) \).

Second, we prove that \( -\Delta v(x) + c(x)v(x) = f(x) + a\delta_0 + \vec{b} \cdot \nabla \delta_0. \)

Third, we obtain \( \vec{b} = 0 \). Finally, we show that the constant \( a \geq 0 \). The details can be found in ref. 10; we just give the main steps in the proof here.

**Theorem 4**

Let \( v(x) \in L^2_{\text{loc}}(\mathbb{R}^n) \) with \( n > 2s \) be a nonnegative solution to

\[ (-\Delta)^sv(x) + c(x)v(x) = f(x) \geq 0 \ \text{on} \ B_1(0) \setminus \{0\}, \] \[14\]

for some \( f(x) \in L^1_{\text{loc}}(B_1(0) \setminus \{0\}) \) and \( c(x) \) is bounded from above, then there exists some constant \( a \geq 0 \) such that

1. \( v(x), f(x) \in L^1_{\text{loc}}(B_1(0)) \),
2. \( -\Delta v(x) + c(x)v(x) = f(x) + a\delta_0 \ \text{on} \ B_1(0). \) \[15\]

**Proof.** We only need to consider the special case \( c(x) \equiv D \). The general case is a simple consequence. The integrability of \( v(x) \) in \( B_1(0) \) is from the fact \( v(x) \in L_{2s} \). Thus, we first prove that \( f(x) \) is locally integrable in \( B_1(0) \).

Define \( w_0(x) := \max \{0, -\frac{c}{1/2s} - \frac{\rho(x)}{1/2s} \} \) and let \( \eta(x) \) be a nonnegative, nonincreasing, smooth function supported in \( |x| < \frac{1}{4} \) with \( \eta(0) = 1 \) for \( |x| \leq \frac{1}{2} \). The mollified function \( w_k(x) = w_{0 \ast \rho_k(x)} \) is also useful to our proof, where \( \rho_k(x) \) is a mollifier. By integration by parts for the fractional case and the fact \( (\Delta)^s w_k(x) \leq 0 \) from Lemma 1 and formula \( [25] \) in Lemma 2 below, we have

\[ \int_{B_2(0)} f(x)w_k(x) dx = \int_{B_{\frac{1}{2}}(0)} f(x)(1 - \eta(x))w_k(x) dx + \int_{\mathbb{R}^n} f(x)\eta(x)w_k(x) dx \]

\[ \leq C_1 + \int_{\mathbb{R}^n} v(x) dx \int_{\mathbb{R}^n} \left( \eta(x) - \eta(y) \right) \left( w_k(x) - w_k(y) \right) dy \]

\[ \leq C, \] \[16\]

where \( C \) is independent of \( \delta \) and \( \epsilon \).

Taking \( \delta \to 0 \) and then \( \epsilon \to 0 \), we obtain \( f(x) \) is integrable in \( B_2(0) \), which together with the assumption \( f \in L^1_{\text{loc}}(B_1(0) \setminus \{0\}) \) implies that \( f(x) \) is integrable in \( B_1(0) \).

Next, we prove \( [15(ii)] \) in three steps.

**Step 1.** We claim that the following identity holds in the sense of distributions

\[ (-\Delta)^s v(x) + Dv(x) = f(x) + a\delta_0 + \vec{b} \cdot \nabla \delta_0 \ \text{in} \ B_1(0). \]

**Given a test function \( \phi(x) \in C_0^\infty(B_1(0)) \), we linearize it as \( \phi(x) = \phi(0) + \nabla \phi(0) \cdot x \eta(x) + H(x) \), where \( H(x) \) is a smooth function with quadratic or higher order of \( x \) in \( B_1(0) \) and \( \eta(x) \) is a positive smooth function supported in \( B_1(0) \). Define \( \rho(x) \) as a smooth nonnegative function compactly supported in \( B_2(0) \), \( \rho(x) = 1, \forall x \in B_1(0), \) and \( \rho(x) = \rho(\frac{x}{2}) \). Then, we obtain:

\[ \int_{\mathbb{R}^n} v(x)(-\Delta)^sD \phi(x) dx = \int_{\mathbb{R}^n} v(x)(-\Delta)^sD(\phi(0) + \nabla \phi(0) \cdot x \eta(x)) dx \]

\[ + \int_{\mathbb{R}^n} v(x)(-\Delta)^sD H(x) dx \]

\[ = \phi(0) a + \nabla \phi(0) \cdot \vec{b} + \int_{\mathbb{R}^n} \phi(x)f(x) dx \]

\[ + \lim_{\epsilon \to 0} \int_{\mathbb{R}^n} v(x)(-\Delta)^sD \rho(x) H(x) dx \]

\[ + D \lim_{\epsilon \to 0} \int_{\mathbb{R}^n} v(x)\rho(x) H(x) dx \]

\[ = \phi(0) a + \nabla \phi(0) \cdot \vec{b} + \int_{\mathbb{R}^n} \phi(x)f(x) dx. \]
The first limit is zero follows from the fact \(|\rho_\epsilon(x)H(x)| \leq \epsilon^2\), and the second limit is zero is a consequence of the facts that \(|\rho_\epsilon(x)H(x)| \leq \epsilon^2\) and \(|2\rho_\epsilon(x)H(x) - \rho_\epsilon(x + y)H(x + y) - \rho_\epsilon(x - y)H(x - y)| \leq C \min(\epsilon^2, |y|^2)\).

**Step 2.** We derive that the vector \(\vec{b}\) in [16] is zero. Since \(v(x)\) satisfies [16], \(v(x)\) can be represented as

\[
v(x) = \omega(x) + \frac{ac_n}{|x|^{n-2}} + \frac{\vec{b} \cdot x}{|x|^{n-2}} + h(x), \tag{18}
\]

in the sense of distributions. Here \(c_n > 0\), \(\omega(x)\) is the Newtonian potential of the integrable function \(f(x) - \nabla v\), and \(h(x)\) is \(\sigma\)-harmonic and bounded.

Next, we show that \(\vec{b} = 0\). First, it is easy to see that \(\omega(x), h(x), \) and \(\frac{\vec{b} \cdot x}{|x|^{n-2}}\) are in \(L^{\frac{n-2}{n-1}}(\Omega)\) for any \(\Omega \subset B_1(0)\). If \(\vec{b} \neq 0\), then choose a conic domain \(\Omega = \{ x \in B_1(0) \mid \frac{\vec{b} \cdot x}{|x|^{n-2}} < 0 \} \subset B_1(0)\). We calculate that \(\|(\frac{\vec{b} \cdot x}{|x|^{n-2}})\|_{L^{\frac{n-2}{n-1}}(\Omega)} = \infty\).

These estimates imply that the negative part of \(v(x)\) is nonzero, which contradicts with \(v(x) \geq 0\) in \(\Omega\). As a result, we derive that \(\vec{b} = 0\).

**Step 3.** We prove that \(a \geq 0\) in [18]. First, a conclusion of [18] is

\[
v(x) = w_0(x) + \frac{a}{|x|^{n-2}} + h(x), \tag{19}
\]

where \(h(x)\) is a bounded \(\sigma\)-harmonic function.

If \(a < 0\), we can prove that the average of \(w_0(x) = \frac{w(x)}{|x|^{n-2}}\) is dominated by \(\frac{a}{|x|^{n-2}}\) in a sufficiently small ball. This contradicts with the nonnegativity of \(v\) and thus completes the proof of Theorem 4.

The following two lemmas are used to prove Theorem 4. They are also interesting in their own. A similar result for Laplacian operator is well-known and widely used.

**Lemma 1**

If \(w(x)\) is nonnegative and fractional superharmonic in the domain \(\Omega \subset \mathbb{R}^n\), then the mollification \(w_\epsilon(x) = w \ast \rho_\epsilon(x)\) is also fractional superharmonic in the domain \(\Omega_\epsilon = \{ x \in B_1(x) \subset \Omega \}\). Here \(\rho_\epsilon(x) = \epsilon^{-n}\rho(x/\epsilon), \rho(x) \in C_0^\infty(B_1(0)), \rho(x) \geq 0, \) and \(\int_{B_1(0)}(\rho(x)) dx = 1\).

**Lemma 2**

If \(u(x), v(x) \in L_{2s}, f(x), g(x) \in L_{\infty}(\mathbb{R})\) with \(n > 2s\), and satisfy

\[
(\Delta)^s u(x) \leq f(x), \quad (\Delta)^s v(x) \leq g(x),
\]

then for \(w(x) = \max\{u(x), v(x)\}\), we have in the sense of distributions that

\[
(\Delta)^s w(x) \leq \max\{f(x), g(x)\}. \tag{20}
\]

On the other hand, if

\[
(\Delta)^s u(x) \geq f(x), \quad (\Delta)^s v(x) \geq g(x),
\]

then for \(w(x) = \min\{u(x), v(x)\}\), we have in the sense of distributions that

\[
(\Delta)^s w(x) \geq \min\{f(x), g(x)\}. \tag{21}
\]

**Proof.** In the first step, we consider the case that the functions \(u\) is smooth in a neighborhood of \(\{ x \mid u(x) \geq v(x) \}\) and \(v\) is smooth in a neighborhood of \(\{ x \mid v(x) \geq u(x) \}\). Without loss of generality, we set \(w(x) = u(x)\) in \(\Omega\) and \(w(x) = v(x)\) in \(\Omega^c\). For simplicity, we denote \(h(x) = f(x)\chi_{u(x) > v(x)} + g(x)\chi_{u(x) \leq v(x)}\).

For any nonnegative test function \(\phi(x)\) and fixed \(\delta > 0\), we want to prove that

\[
\int_{\Omega^c} w(x)(-\Delta)^s \phi(x) dx \leq \int_{\Omega^c} h(x)\phi(x) dx \tag{22}
\]

To this end, by using method of difference, we first prove that for a fixed \(\delta > 0\)

\[
\int_{\Omega^c} \int_{|x-y| \leq \delta} \frac{w(x)(\phi(x) - \phi(y))}{|x-y|^{n+2s}} dy dx
\]

\[
= \int_{\Omega} \int_{|x-y| \leq \delta} \frac{\phi(x)(u(x) - u(y))}{|x-y|^{n+2s}} dy dx + \int_{\Omega^c} \int_{|x-y| \leq \delta} \frac{\phi(x)(v(x) - v(y))}{|x-y|^{n+2s}} dy dx
\]

\[
+ \int_{\Omega^c} \int_{|x-y| \leq \delta} \frac{\phi(x)(u(y) - v(y)) + \phi(y)(v(x) - u(x))}{|x-y|^{n+2s}} dy dx \leq 0, \quad \text{since } u(x) \geq v(x) \text{ in } \Omega \text{ and } u(x) \text{ in } \Omega^c
\]

\[
\int_{\Omega} \int_{|x-y| \leq \delta} \frac{\phi(x)(u(x) - u(y))}{|x-y|^{n+2s}} dy dx + \int_{\Omega^c} \int_{|x-y| \leq \delta} \frac{\phi(x)(v(x) - v(y))}{|x-y|^{n+2s}} dy dx
\]

\[
\leq \int_{\Omega} \int_{|x-y| \leq \delta} \frac{\phi(x)(u(x) - u(y))}{|x-y|^{n+2s}} dy dx + \int_{\Omega^c} \int_{|x-y| \leq \delta} \frac{\phi(x)(v(x) - v(y))}{|x-y|^{n+2s}} dy dx
\]

\[
\leq \int_{\Omega} (-\Delta)^s u(x)\phi(x) dx + \int_{\Omega^c} (-\Delta)^s v(x)\phi(x) dx, \tag{24}
\]

which implies that it holds in the sense of distributions

\[
(-\Delta)^s w(x) \leq h(x) = f(x)\chi_{u(x) > v(x)} + g(x)\chi_{u(x) \leq v(x)}. \tag{25}
\]

Second, we consider the case that \(u(x)\) and \(v(x)\) are not smooth. From Lemma 1 and the assumptions \((-\Delta)^s u(x) \leq f(x)\) and \((-\Delta)^s v(x) \leq g(x)\), we have \((-\Delta)^s u_\epsilon(x) \leq f_\epsilon(x)\) and \((-\Delta)^s v_\epsilon(x) \leq g_\epsilon(x)\), where \(u_\epsilon(x), v_\epsilon(x)\) are the mollifications of \(u(x)\) and \(v(x)\) respectively. Define \(w_\epsilon = \max\{u_\epsilon(x), v_\epsilon(x)\}\). The first step implies that \((-\Delta)^s w_\epsilon \leq f_\epsilon(x)\chi_{u_\epsilon(x) > v_\epsilon(x)} + g_\epsilon(x)\chi_{u_\epsilon(x) \leq v_\epsilon(x)}\). From the facts that \((-\Delta)^s w_\epsilon \rightarrow (-\Delta)^s w(x)\) in distributions, \(f_\epsilon(x) \rightarrow f(x)\) and \(g_\epsilon(x) \rightarrow g(x)\) in \(L_{\infty}(\mathbb{R}^n)\), we derive \((-\Delta)^s w(x) \leq \max\{f(x), g(x)\}\) by letting \(\epsilon \rightarrow 0\). This completes the proof of Lemma 2.

The special but essential case of Lemma 2 when \(f(x) = g(x) = 0\) has been proved in ref. 13 with the additional assumption that \(u(x)\) and \(v(x)\) are both lower semicontinuous.

**Outline of the Proof for Theorem 1**

The proof holds for general \(r\) with suitable scaling. In addition, it is obvious that we only need to consider the special case that \(a(x) \equiv D\) where \(D\) is the upper bound of \(a(x)\). Thus, in the following, we give the proof under the assumption that \((-\Delta)^s v(x) + Dv(x) \geq 0\) in \(B_0(0)\).

There are two basic steps.
First, we prove Theorem 1 when \( v(x) \) is a smooth function on \( B_{1}(0) \). We prove this by contradiction. Letting \( C_{1}(x_{0}) = \int_{B_{1}(0) \setminus B_{1/2}(0)} \frac{C_{n,s}(x_{0})}{|x_{0} - y|^{n+2s}} dy, \), \( C_{2}(x_{0}) = \int_{B_{1}(0) \setminus B_{1/2}(0)} \frac{C_{n,s}(x_{0})}{|x_{0} - y|^{n+2s}} dy, \) and \( c = \inf_{x_{0} \in B_{1}(0)} C_{2}(x_{0}) > 0, \) then we show that \( v(x) \geq cm, \) for \( x \in B_{1}(0). \) Assume the contrary that \( v(x) < cm \) for some \( x \in B_{1/2}(0) \), then there exists \( x_{0} \in B_{1/2}(0) \) with \( v(x_{0}) = \min_{x \in B_{1/2}(0)} v(x) < cm. \) A direct calculation yields

\[
0 \leq (-\Delta)^{s} v(x_{0}) + Dv(x_{0}) = C_{n,s} \left( \int_{B_{1}(0)} \frac{v(x_{0}) - v(y)}{|x_{0} - y|^{n+2s}} dy + \int_{B_{1}(0) \setminus B_{1/2}(0)} \frac{v(x_{0}) - v(y)}{|x_{0} - y|^{n+2s}} dy \right) + P.V. \int_{B_{1/2}(0)} \frac{v(x_{0}) - v(y)}{|x_{0} - y|^{n+2s}} dy + Dv(x_{0}) + \int_{B_{1}(0) \setminus B_{1/2}(0)} \frac{cm C_{n,s}}{|x_{0} - y|^{n+2s}} - \int_{B_{1}(0) \setminus B_{1/2}(0)} \frac{(1 - c)m C_{n,s}}{|x_{0} - y|^{n+2s}} + Dcm \leq (C_{1}(x_{0}) + C_{2}(x_{0}) + D) cm - C_{2}(x_{0}) m = (C_{1}(x_{0}) + C_{2}(x_{0}) + D) m \left( c - \frac{C_{2}(x_{0})}{C_{1}(x_{0}) + C_{2}(x_{0}) + D} \right) \leq 0.
\]

Second, we apply Theorem 4 and Lemma 1 to prove that \( v_{\epsilon}(x) = \rho_{\epsilon} * v(x) \) is fractional superharmonic in \( B_{1-\epsilon}(0) \). Then, from the result of the first step we can show that \( v_{\epsilon}(x) \geq cm \) in \( B_{1-\epsilon}(0) \). Finally, taking \( \epsilon \to 0 \), we arrive at \( v \geq cm \) in \( B_{1}(0) \setminus \{0\} \). This proves Theorem 1.

We emphasize here the importance of the Böcher-type Theorem 4: the nonnegative fractional superharmonic function on the punctured ball \( B_{1}(0) \setminus \{0\} \) is actually a fractional superharmonic function on the whole ball \( B_{1}(0) \) in the sense of distributions.

In fact, connecting Theorem 4 and Theorem 1 is the crucial observation.

ACKNOWLEDGMENTS. C.L. and H.X. were partially supported by National Natural Science Foundation of China Grant 11571233 and NSF Grant DMS-1405175. Z.W. was supported by National Science Foundation of Shanghai 16ZR1402100 and China Scholarship Council.

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