Alternative Decomposition of Two-Qutrit Pure States and Its Relation with Entanglement Invariants

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Based on maximally entangled states in the full- and sub-spaces of two qutrits, we present an alternative decomposition of two-qutrit pure states in a form $|\Psi\rangle = \frac{p_1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle) + \frac{p_2}{\sqrt{2}}(|01\rangle + |12\rangle) + p_3e^{i\theta}|02\rangle$. Similar to the Schmidt decomposition, all two-qutrit pure states can be transformed into the alternative decomposition under local unitary transformations, and the parameter $p_1$ is shown to be an entanglement invariant.

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I. INTRODUCTION

Decomposition of quantum states is an interesting topic in quantum information theory [1–3]. Given an arbitrary bipartite state, it is well-known that the Schmidt decomposition is always applicable [4]. For instance, under local unitary transformations any two-qubit state $|\Psi\rangle = \sum_{i,j=0}^{1} a_{ij} |i\rangle_A |j\rangle_B$ can be transformed into its Schmidt-form as $|\Psi\rangle = U_A \otimes U_B |\Psi\rangle = \kappa_1 |00\rangle + \kappa_2 |11\rangle$.

Besides the Schmidt decomposition, other decompositions are possible. For example, in 2001, Abouraddy et al. have proposed an alternative decomposition for two-qubit pure states based on the maximally entangled state [5]:

$$|\Psi\rangle = p_1 |\psi\rangle_e + p_2 e^{i\varphi} |\phi\rangle_f,$$  \hspace{1cm} (1)

where $p_1 \geq 0$, $p_2 = \sqrt{1 - p_1^2}$, $|\psi\rangle_e$ is the two-qubit maximally entangled state, and $|\phi\rangle_f$ is a factorizable state orthogonal to $|\psi\rangle_e$. They showed that such a decomposition always exists and is not unique, but the parameter $p_1$ is unique. In comparison to the Schmidt decomposition, the merit of the new kind of decomposition is that the parameter $p_1$ has a definite physical significance as the degree of entanglement of two qubits. In this work, we would like to generalize the alternative decomposition to a two-qutrit system based on the maximally entangled states in the full- and sub-spaces. To our knowledge, such a generalization has not been reported in the literature.

This paper is organized as follows: In section II, we make a brief review for the previous result of Abouraddy et al., but from a different viewpoint of entanglement invariants. In section III, we present a Theorem on the alternative decomposition of two-qutrit pure states, and also show its relation with the entanglement invariants. Conclusion and discussion are made in the last section.

II. BRIEF REVIEW OF ENTANGLEMENT INVARIANTS AND PREVIOUS RESULT OF ABOURADDY ET AL.

Let us consider a general pure state of two $d$-dimensional quantum systems (two qudits), which takes of the following form:

$$|\Psi\rangle_{AB} = \sum_{i,j=0}^{d-1} a_{ij} |i\rangle_A |j\rangle_B,$$ \hspace{1cm} (2)

where $|i\rangle_A$ and $|j\rangle_B$ are the orthonormal bases of the Hilbert spaces $A$ and $B$ respectively, and $a_{ij}$’s are complex numbers satisfying the normalization condition $\sum_{i,j=0}^{d-1} |a_{ij}|^2 = 1$.

Let $A$ denote the matrix whose matrix elements are given by $\langle A \rangle_{ij} = a_{ij}$. It has been shown that the following quantities are entanglement invariants under local unitary transformations [6]:

$$I_n = \text{Tr}[|A A^\dagger|^n], \hspace{0.5cm} n = 0, 1, ..., d - 1. \hspace{1cm} (3)$$

Denote $\rho_{AB} = |\Psi\rangle_{AB} \langle \Psi|$, since the reduced density matrices $\rho_A = \text{Tr}_B[\rho_{AB}] = A A^\dagger$, $\rho_B = \text{Tr}_A[\rho_{AB}] = A^\dagger A$, thus Eq. (3) can be also expressed as

$$I_n = \text{Tr}[\rho_A^n], \hspace{0.5cm} n = 0, 1, ..., d - 1. \hspace{1cm} (4)$$

For $n = 0$, one easily has $I_0 = 1$, which is nothing but the normalization condition of the reduced density matrix of $\rho_A$ or $\rho_B$. Therefore, for a two-qudit system, there are only $(d - 1)$ nontrivial entanglement invariants.

After performing an appropriate local unitary transformation, one may transform the general state $|\Psi\rangle_{AB}$
into its Schmidt-form as
\[ |\Psi\rangle_{2\text{-qudit}} = \kappa_1|00\rangle + \kappa_2|11\rangle + \cdots + \kappa_d|d-1, d-1\rangle, \] (5)
where \( \kappa_j \)'s \((j = 1, 2, \ldots, d)\) are the Schmidt coefficients, which satisfy the normalization condition: \( \sum_{j=1}^{d} |\kappa_j|^2 = 1. \) In the Schmidt representation, it is easy to obtain the entanglement invariants as
\[ I_n = \text{Tr}[\rho^n] = \text{Tr}[\rho^{n+1}] = \sum_{j=1}^{d} |\kappa_j|^{2(n+1)}. \] (6)

Now, the previous result of Abouraddy et al. can be re-expressed as the following theorem:

**Theorem 1.** Under local unitary transformations any two-qubit state can be always transformed into an alternative decomposition as
\[ |\Psi\rangle_{AB} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) + p_2|01\rangle, \] (7)
where \( p_1 = 2(I_0 - I_1) = 4\kappa_1^2\kappa_2^2 = 4\text{Det}[\rho_A] \) is unique and is an entanglement invariant under the local unitary transformations.

By comparing Eq. (1) and Eq. (7), one notes that we have chosen the maximally entangled state of two-qubit as \( |\Psi\rangle_e = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \) and the factorizable state as \( |\Psi\rangle_f = |01\rangle \). Moreover, the phase factor \( e^{i\varphi} \) in Eq. (1) can be eliminated further by a suitable \( U(1) \otimes U(1) \) transformation. Therefore the decomposition in Eq. (7) is unique for the pure states of a two-qubit system.

The standard way to prove Theorem 1 is owing to the local unitary transformations, which has been actually given in Ref. \([5]\), namely, by acting the appropriate local unitary transformations \( U_A \otimes U_B \) on an arbitrary two-qubit pure state \( |\Psi\rangle = \sum_{i,j=0}^{1} a_{ij}|iangle_A |j\rangle_B \), then one obtains the decomposition (7). However, there is another equivalent way to prove Theorem 1, which is due to the entanglement invariants. Now we use the new approach to prove Theorem 1, the same approach will be used to prove the corresponding Theorem for the two-qutrit case.

**Proof.** On one hand, for the two-qubit state in the Schmidt-form
\[ |\Psi\rangle_{2\text{-qudit}} = \kappa_1|00\rangle + \kappa_2|11\rangle, \] (8)
one has the entanglement invariants as
\[ I_0[\vec{\kappa}] = |\kappa_1|^2 + |\kappa_2|^2 = 1, \]
\[ I_1[\vec{\kappa}] = |\kappa_1|^4 + |\kappa_2|^4, \] (9)
here \( \vec{\kappa} = (\kappa_1, \kappa_2) \), \( I_n[\vec{\kappa}] \) means that \( I_n \) is expressed by the parameters \( \kappa_1 \) and \( \kappa_2 \).

On the other hand, for the two-qubit state in the alternative decomposition as in Eq. (7), one has the matrices
\[ \mathcal{A} = \begin{pmatrix} p_1 & p_2 \\ p_2 & p_1 \end{pmatrix}, \quad \mathcal{A}^\dagger = \begin{pmatrix} \frac{p_1}{\sqrt{2}} & \frac{p_2}{\sqrt{2}} \\ \frac{p_2}{\sqrt{2}} & \frac{p_1}{\sqrt{2}} \end{pmatrix}. \] (10)
Thus the corresponding entanglement invariants reads
\[ I_0[p] = p_1 + p_2^2 = 1, \]
\[ I_1[p] = 1 - \frac{p_1^4}{2}, \] (11)
here \( p = (p_1, p_2) \), \( I_n[p] \) means that \( I_n \) is expressed by the parameters \( p_1 \) and \( p_2 \).

Because an arbitrary two-qubit state can be transformed into the Schmidt decomposition under the local unitary transformation, if one can prove that for any given \( \kappa_1 \) and \( \kappa_2 \), there always exists \( \vec{p} \) satisfying \( I_n[\vec{p}] = I_n[\vec{\kappa}] \), \((n = 0, 1)\), then it implies that an arbitrary two-qubit state can be transformed into the alternative decomposition as shown in Eq. (7) under the local unitary transformation. Since \( I_0[p] = I_0[\vec{\kappa}] = 1 \) is the normalization condition, one only need to study \( I_1[p] = I_1[\vec{\kappa}] \), this yields the following solution:
\[ p_1^2 = 2(I_0 - I_1) = 4\kappa_1^2\kappa_2^2 \in [0, 1], \] (12)
which means that an arbitrary two-qubit state can be transformed into the alternative decomposition (7) under the local unitary transformation if relation (12) is satisfied. This ends the proof.

By the way, it is easy to show that the determinants of matrices \( \mathcal{A} \) and \( \mathcal{A}^\dagger \) are
\[ \text{Det}[\mathcal{A}] = \text{Det}[\mathcal{A}^\dagger] = \frac{p_1^4}{2}, \] (13)
therefore one has
\[ p_1^4 = 4\text{Det}[\mathcal{A}] \text{Det}[\mathcal{A}^\dagger] = 4\text{Det}[\mathcal{A}\mathcal{A}^\dagger] = 4\text{Det}[\rho_A]. \] (14)
One will find later that such a similar relation holds for the any two-qudit system.

### III. Entanglement Invariants of Two-Qutrit and the Alternative Decomposition

Under local unitary transformations an arbitrary two-qutrit pure state can be transformed into its Schmidt-form as
\[ |\Psi\rangle_{2\text{-qutrit}} = \kappa_1|00\rangle + \kappa_2|11\rangle + \kappa_3|22\rangle, \] (15)
one has the entanglement invariants as
\[ I_0[\vec{\kappa}] = |\kappa_1|^2 + |\kappa_2|^2 + |\kappa_3|^2 = 1, \]
\[ I_1[\vec{\kappa}] = |\kappa_1|^4 + |\kappa_2|^4 + |\kappa_3|^4, \]
\[ I_2[\vec{\kappa}] = |\kappa_1|^6 + |\kappa_2|^6 + |\kappa_3|^6, \] (16)
Here \( \kappa = (\kappa_1, \kappa_2, \kappa_3) \), and \( I_0[\kappa] = 1 \) is trivial as the normalization condition of a quantum state.

By expanding \((I_0[\kappa])^3 = (|\kappa_1|^2 + |\kappa_2|^2 + |\kappa_3|^2)^3\), one may get an interesting and useful relation:

\[
I_2[\kappa] - \frac{3}{2}I_1[\kappa] = -\frac{1}{2}I_0[\kappa] + 3\mathcal{K},
\]

with

\[
\mathcal{K} = \kappa_1^2\kappa_2^2\kappa_3^3.
\]

Since \( I_1[\kappa] \) and \( I_2[\kappa] \) are entanglement invariants, thus \( \mathcal{K} \) is an entanglement invariant under local unitary transformation. \( \mathcal{K} \in [0, \frac{3}{2}] \), \( \mathcal{K} \) reaches its maximum value \( \frac{3}{2} \) when \( \kappa_1^2 = \kappa_2^3 = \kappa_3^3 = \frac{1}{3} \). We shall use such a useful relation to prove the Theorem 2 in this section.

Actually, the entanglement property of a two-qutrit system is completely characterized by two entanglement invariants \( I_1[\kappa] \) and \( I_2[\kappa] \), or equivalently,

\[
I_1'[\kappa] = \frac{3}{2}(1 - I_1[\kappa]),
\]

\[
I_2'[\kappa] = \frac{9}{8}(1 - I_2[\kappa]),
\]

where the normalized entanglement invariants \( I_1', I_2' \in [0, 1] \).

In Fig.1, we have plots points \((I_1', I_2')\) for the two-qutrit state |ψ⟩\textsubscript{2-qutrit} = κ₁|00⟩ + κ₂|11⟩ + κ₃|22⟩ by randomly taking 10⁷ values of \( \kappa_1, \kappa_2, \) and \( \kappa_3 \), see the red region of figure, whose contour lines form a curved triangle \( \Delta OBG \). In the \( I_1 - I_2 \) coordinate, one may observe that there are three special points: the first point is the origin \( O = (0, 0) \), which corresponds to the factorizable states, such as |00⟩; the second is the point \( G = (1, 1) \), which corresponds to the maximally entangled state (or say the GHZ state) in the full-space of two-qutrit, such as \( |\psi⟩\textsubscript{GHZ} = \frac{1}{\sqrt{3}}(|00⟩ + |11⟩ + |22⟩) \); and the third is the point \( B = (\frac{3}{4}, \frac{27}{25}) \), which corresponds to the entangled state in the sub-space of two-qutrit, such as \( \frac{1}{\sqrt{2}}(|00⟩ + |11⟩) \).

Inspired by the success of Theorem 1, we suggest the following decomposition for two-qutrit pure states:

\[
|\Psi⟩\textsubscript{AB} = p_1|\psi_1⟩ + p_2|\psi_2⟩ + p_3|\psi_3⟩,
\]

and

\[
|\psi_1⟩ = \frac{1}{\sqrt{3}}(e^{i\theta_1}|00⟩ + e^{i\theta_2}|11⟩ + e^{i\theta_3}|22⟩),
\]

\[
|\psi_2⟩ = \frac{1}{\sqrt{2}}(e^{i\theta_4}|01⟩ + e^{i\theta_5}|12⟩),
\]

\[
|\psi_3⟩ = e^{i\theta_6}|02⟩.
\]

Here |ψ₁⟩ is the maximally entangled state (or say the GHZ state) in the full-space of two-qutrit spanned by \{00⟩, |11⟩, |22⟩\}, |ψ₂⟩ is the maximally entangled state in the sub-space of two-qutrit spanned by \{01⟩, 12⟩\}, and |ψ₃⟩ is the factorizable state, they are mutually orthogonal, i.e., ⟨ψᵢ|ψⱼ⟩ = δᵢⱼ, \( \theta_j \)'s \((j = 1, 2, \ldots, 6)\) are some phase factors. However, five phases can be eliminated by the transformation \( U_a \otimes U_b \), with \( U_a = \sum_{i=0}^{2} e^{i\theta_i}|i⟩⟨i| \) and \( U_b = \sum_{j=0}^{2} e^{i\phi_j}|j⟩⟨j| \), thus there is only one phase factor is survival. In general, one may select the phase factor involved in |ψ₃⟩ is not zero. Consequently, one arrives at the alternative decomposition of two-qutrit pure states as follows: \( |\Psi⟩\textsubscript{AB} = p_1\frac{1}{\sqrt{3}}(|00⟩ + |11⟩ + |22⟩) + p_2\frac{1}{\sqrt{2}}(|01⟩ + |12⟩) + p_3 e^{i\theta_6}|02⟩ \).

Our main result is the following Theorem.

**Theorem 2.** Under local unitary transformations any two-qutrit state can be always transformed into an alternative decomposition as

\[
|\Psi⟩\textsubscript{AB} = p_1 \frac{1}{\sqrt{3}}(|00⟩ + |11⟩ + |22⟩) + p_2 \frac{1}{\sqrt{2}}(|01⟩ + |12⟩) + p_3 e^{i\theta_6}|02⟩,
\]

\[
p_1 \geq 0, \quad p_2 \geq 0, \quad p_3 = \sqrt{1 - p_1^2 - p_2^2},
\]

where \( p_1^6 = 9(I_2 - \frac{3}{2}I_1 + \frac{1}{2}I_0) = 27\det[p_A] \) is unique and is an entanglement invariant under the local unitary transformations.

**Proof.** Similarly, for the 2-qutrit pure quantum state in form (22), one can write the related matrices as

\[
A = \begin{pmatrix}
\frac{p_1}{\sqrt{3}} & \frac{p_2}{\sqrt{2}} & p_3 e^{i\theta_6} \\
0 & \frac{p_1}{\sqrt{3}} & \frac{p_2}{\sqrt{2}} \\
0 & 0 & \frac{p_3 e^{-i\theta_6}}{\sqrt{3}}
\end{pmatrix},
\]

\[
A^\dagger = \begin{pmatrix}
\frac{p_1}{\sqrt{3}} & 0 & 0 \\
\frac{p_2}{\sqrt{2}} & \frac{p_1}{\sqrt{3}} & 0 \\
\frac{p_3 e^{-i\theta_6}}{\sqrt{3}} & \frac{p_2}{\sqrt{2}} & \frac{p_3 e^{i\theta_6}}{\sqrt{3}}
\end{pmatrix}
\]

(23)
Its entanglement invariants are obtained immediately
\[
I_0[p] = p_1^2 + p_2^2 + p_3^2 = 1, \\
I_1[p] = 1 - \frac{2}{3}p_1^3 - \frac{1}{2}p_2^3 + \frac{2}{3\sqrt{3}}p_1p_2^2p_3\cos \theta, \\
I_2[p] = 1 - p_1^2 + \frac{p_1^6}{9} - \frac{3}{4}p_2^4 + \sqrt{3}p_1p_2^2p_3\cos \theta. \quad (24)
\]
From them, one can find the relation
\[
I_2[p] - \frac{3}{2}I_1[p] = -\frac{1}{2}I_0[p] + \frac{p_1^6}{9}. \quad (25)
\]
On condition that the state in Eq. (22) is equivalent to the one in Eq. (15) under local unitary (LU) transformations, the parameter \(p_1\) should satisfies
\[
p_1^6 = 27\text{Det}[\rho_A] = 27\kappa_1^2\kappa_2^2\kappa_3^2, \quad (26)
\]
which always has a root in the interval \(p_1 \in [0,1]\) for any value of \(K \in [0, 1/27]\). Then, the two nontrivial entanglement invariants can be replaced by
\[
\bar{I}_1 = I_1, \\
\bar{I}_2 = I_2 - \frac{3}{2}I_1. \quad (27)
\]
For a fixed value of \(\bar{I}_2[p] = \bar{I}_2[\kappa] = \bar{I}_2\), if the range of \(\bar{I}_1[\kappa] = I_1[\kappa]\) in Eq. (24) is the same as the one of \(\bar{I}_1[\kappa] = I_1[\kappa]\) in Eq. (16), one can conclude there exists a pure state in the form (22) equivalent the one (15) with any value of \(\kappa\) under LU transformations. Let us denote the minimum and maximum of \(\bar{I}_1\) as \(\bar{I}_1^\nu\) and \(\bar{I}_1^\sigma\), based on the fact that the values of \(\bar{I}_1[\kappa]\) and \(\bar{I}_1[p]\) vary continuously from their minimums to maximums, it is only to prove
\[
\frac{\bar{I}_1^\nu}{\bar{I}_1}[\kappa] = \frac{\bar{I}_1^\nu}{\bar{I}_1}[p], \\
\frac{\bar{I}_1^\sigma}{\bar{I}_1}[\kappa] = \frac{\bar{I}_1^\sigma}{\bar{I}_1}[p]. \quad (28)
\]
for a given value of \(\bar{I}_2\). In Appendix A, we show the two relations come into existence. Since an arbitrary two-qutrit pure state can be transformed into the form (16) under LU operation, it can always be decomposed as Eq. (22). This ends the proof.

**IV. CONCLUSION AND DISCUSSION**

In conclusion, we show that all 2-qutrit pure states can be rewritten as \(|\Psi\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle) + \frac{1}{\sqrt{2}}(|01\rangle + |12\rangle) + \rho_2 e^{i\theta}|02\rangle\). The method we have used is to verify the invariant space is as same as achieved by expression of Schmidt-form. The parameter \(p_1 \in [0,1]\) is unique and it is an entanglement invariant under LU operations. The values of \(p_2\) and \(\theta\) can be derived from the relations in Eq. (22).

In this paper, we concerns us in the pure states of two-qutrit system. There are two natural extensions of this issue: (i) to decompose the pure states in a bipartite arbitrary-dimensional system, (ii) to decompose the pure states in a multipartite system. For the case (i), we can foretell a two-qudit state can be transformed as
\[
|\Psi\rangle_{2-qudit} = \sum_{j=0}^{d} p_{d-M+1} \sum_{m=0}^{M-1} e^{i\theta_m}\sqrt{M} |m, d - M + m\rangle, \quad (29)
\]
where the parameters \(\theta_m^M \in [0, 2\pi]\), \(p_{d-M+1} \in [0,1]\) and \(\sum_{M=1}^{d} p_{d-M+1}^2 = 1\). And, here \(\sum_{m=0}^{M-1} e^{i\theta_m}\sqrt{M} |m, d - M + m\rangle\) is a maximally entangled state in the sub-space \(|m, d - M + m\rangle\) whose spacial case is shown in Eq. (21) for \(d = 3\). Under locally phase transformations \(U_a \otimes U_b = \sum_{j=0}^{d} e^{i\phi_j} \langle j | \otimes \sum_{k=0}^{d-1} e^{i\phi_k} | k \rangle\), the phases \(\theta_m^M\) and \(\theta_m^{-1}\) can be eliminated. We have numerically verified that the entanglement invariants of the states (29) cover the the ones of Schmidt-form states (2) perfectly for \(d = 4\). For (ii), the quantum correlation or entanglement in a multipartite state carry more nonclassical characteristics of quantum mechanics [7, 8]. Many perspectives have been presented to attempt an understanding of the problem in recent studies [7–13]. In our subsequent investigation, we hope to give a decomposition of a multipartite pure state, dividing it into subspaces which reflect the entanglement in different levels.

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Appendix A: Equivalence of the Ranges of $I_1[\vec{\kappa}]$ and $\overline{I}_1[\vec{\kappa}]$

a. $I_1[\vec{\kappa}]$ and $\overline{I}_1[\vec{\kappa}]$. Firstly, for the Schmidt-decomposed state (15), we consider the extremal values of $I_1[\vec{\kappa}]$, when $I_2[\vec{\kappa}]$ (or say $\kappa$) is fixed. From the relations (16), one can obtain

$$I_1[\vec{\kappa}] = I_1[\kappa_1^1] = 2(-\kappa_1^2 + \kappa_1^4 - \frac{\kappa}{\kappa_1^2}) + 1,$$  

(A1)

$$\kappa_1^4 - (1 - \kappa_2^2)\kappa_2^2 + \frac{\kappa}{\kappa_1^2} = 0.$$  

(A2)

Then the problem is transformed to derive extremal values of $I_1[\vec{\kappa}]$ in Eq. (A1) in the range $\kappa_1 \in [0, 1]$, under the constraint that the values of $\kappa_2$ and $\kappa_3$ should be legitimate. Solving the Eq. (A2), we find

$$\kappa_{2,3}^2 = \frac{1}{2}(1 - \kappa_1^2) \pm \frac{1}{2} \sqrt{(1 - \kappa_1^2)^2 - \frac{4\kappa}{\kappa_1^2}},$$  

(A3)

or permutation. Therefore the constraint can be explicitly expressed as the discriminant

$$(1 - \kappa_1^2)^2 - \frac{4\kappa}{\kappa_1^2} \geq 0.$$  

(A4)

This leads to $\kappa_1^2 \in [t_-, t_+]$, where $t_{\pm}$ are two of the roots of the cubic equation $t^3 - 2t^2 + t - 4K = 0$. They are given by

$$t_{\pm} = \frac{2}{3}(1 + \cos \frac{\phi_1 + 2\pi}{3}),$$  

(A5)

where the angle satisfies $\cos \phi_1 = 54K - 1 \in [-1, 1]$. The minimal value of $I_1[\vec{\kappa}]$ occurs when $\kappa_1^2 = t_-$ and $\kappa_2^2 = \kappa_3^2 = (1 - t_-)/2$ or

$$\frac{\partial I_1[\kappa_1^2]}{\partial (\kappa_1^2)} = 0, \quad \frac{\partial^2 I_1[\kappa_1^2]}{\partial (\kappa_1^2)^2} > 0.$$  

(A6)

Substituting the solutions of Eq. (A6) into Eq. (A2), one can find the result is only a permutation of the former case, e.g. $\kappa_2^2 = t_-$ and $\kappa_1^2 = \kappa_3^2 = (1 - t_-)/2$. In the same way, one can conclude that the maximal value of $I_1[\vec{\kappa}]$ occurs when $\kappa_2^2 = t_+$ and $\kappa_1^2 = \kappa_3^2 = (1 - t_+)/2$.

Uniformly, we write the minimum and maximum of $I_1[\vec{\kappa}]$ as $I_1[\vec{\kappa}] = I_1[t_-]$ and $\overline{I}_1[\vec{\kappa}] = I_1[t_+]$ with

$$I_1[t_{\pm}] - \overline{I}_1[\vec{\kappa}] = \frac{4}{9}(1 + \cos \frac{\phi_1 \pm 2\pi}{3})^2 - \frac{2}{3}(1 + \cos \frac{\phi_1 \mp 2\pi}{3})$$

$$- \frac{3\kappa}{2(1 + \cos \frac{\phi_1 \mp 2\pi}{3})}.$$  

(A7)

b. $\overline{I}_1[\vec{\kappa}]$ and $\overline{I}_1[\vec{\kappa}]$. For the pure states (22), when the parameter $p_2$ or say the entanglement invariant $I_2[p_2]$ is fixed, $\overline{I}_1[\vec{\kappa}]$ can be expressed as the function of $p_3$ and $\theta$

$$I_3[\vec{\kappa}] = 1 - \frac{2}{3}p_1^2 - \frac{1}{2}(1 - p_1^2 - p_3^2)^2 + \frac{2}{\sqrt{3}}p_1p_3(1 - p_1^2 - p_3^2)\cos \theta.$$  

(A8)

Because $p_1p_3(1 - p_1^2 - p_3^2) \geq 0$, the maximum value of $I_1[\vec{\kappa}]$ happens at $\cos \theta = 1$ and the minimum one at $\cos \theta = -1$.

When $\cos \theta = -1$, the derivative on Eq. (A8) $\partial I_1[\vec{\kappa}] / \partial p_3 = 0$ leads to

$$p_3^3 - \sqrt{3}p_1p_3^2 + (p_1^2 - 1)p_3 + \frac{p_1(1 - p_1^2)}{\sqrt{3}} = 0.$$  

(A9)

One of its three roots lies in $[0, \sqrt{1 - p_1^2}]$ being

$$p_3 = \frac{1}{\sqrt{3}}(p_1 + 2\cos \frac{\phi_2 - 2\pi}{3}),$$  

(A10)

where $\cos \phi_2 = p_1^3$. It corresponds to the minimal value of $I_1[\vec{\kappa}]$ as

$$I_1[\vec{\kappa}] = 1 - \frac{2}{9}p_1^2 + \frac{4}{3}\cos^2 \phi_2 - \frac{2}{3}.$$  

(A11)

When $\cos \theta = 1$, by completely the same analysis, we obtain the maximum

$$\overline{I}_1[\vec{\kappa}] = \frac{1}{3} - \frac{8}{9}p_1^3 \cos \phi_3 - \frac{4}{3}\cos^2 \phi_3 + \frac{8}{9}\cos^4 \phi_3.$$  

(A12)

(A12)

where $\cos \phi_3 = -p_1^3$.

c. Comprising the Ranges. The relation $I_1[\vec{\kappa}] = I_2[\vec{\phi}]$ leads to $2K = p_1^2$ and consequently $\phi_1 = 2\phi_2$. Therefore we have

$$\cos \phi_1 + 2\pi = 2\cos \frac{\phi_2 - 2\pi}{3} - 1.$$  

(A13)
Let \( x = \cos \frac{\phi_2 - 2\pi}{3} \), one can obtain
\[
\begin{align*}
p_1^3 &= \cos \phi_2 = \cos(\phi_2 - 2\pi) = 4x^3 - 3x \\
\mathcal{K} &= \frac{1}{27}p_1^6 = \frac{1}{27}(4x^3 - 3x)^2. \quad (A14)
\end{align*}
\]
Substituting them and Eq. (A13) into Eqs. (A7) and (A11), we get the first relation in Eq. (28), \( \mathcal{I}_1[\bar{r}] = \mathcal{I}_1[\bar{p}] \).

In the same process, the angle \( \phi_3 = \pi - \phi = \pi - \phi_1/2 \).

Setting \( y = \cos(\phi_3/3) \), we obtain
\[
\begin{align*}
p_1^3 &= -4y^3 + 3y, \\
\mathcal{K} &= \frac{1}{27}(4y^3 - 3y)^2. \quad (A15)
\end{align*}
\]
These relations in company with Eqs. (A7) and (A12) lead to \( \mathcal{I}_1[\bar{r}] = \mathcal{I}_1[\bar{p}] \), which is the second relation in Eq. (28).