Sliding mode control of continuous-time switched systems with signal quantization and actuator nonlinearity

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Abstract
This article investigates sliding mode control for a class of continuous-time switched systems with signal quantization, actuator nonlinearity and persistent dwell-time switching that can guarantee the globally uniformly asymptotical stability of the closed-loop system. First, a sliding surface is devised for the switched system and sufficient conditions are proposed to ensure the globally uniformly asymptotical stability of the sliding motion equation by utilizing multiple Lyapunov function technique. Second, the sliding mode control laws, based on the parameters of quantizer, actuator nonlinearity and disturbance, are devised to stabilize the closed-loop systems. Moreover, sufficient conditions are given to guarantee the devised sliding surface’s reachability. Finally, the superiority and effectiveness of developed results is illustrated via a numerical simulation.

Keywords
Sliding mode control, actuator nonlinearity, quantized signal, multiple Lyapunov function, persistent dwell-time switching

Introduction
The past decades have witnessed a great advance in studies of switched systems which are widely applied in robot systems, networked systems, chemical process and so on. Switched systems, which comprised multiple subsystems and a signal deciding to activate one of them, can effectively model processes or multiple-mode...
systems. Moreover, the idea of controller switching is carried over to many intelligent control strategies in order to make up for the shortage of single controller.

For the switched system, the design of the switching signal yields a direct impact on system stability. Basic stability analysis for switched systems with various switching signals such as stochastic switching or average dwell-time (ADT) has been broadly addressed. ADT switching implies bounded switching times within bounded period, which suggests that there exists a lower bound for the average time between switchings. However, persistent dwell-time (PDT) switching is a class of switching signal consisting of infinitely many dispersed intervals in which the subsystem mode remains stationary. In the intermissions of such intervals, the subsystem mode can randomly switch. Compared with ADT switching, PDT switching, as a more general switching signal, has no limit of switching times. Although PDT switching is more complicated, it is essential to study the control problem under PDT switching.

Based on stability analysis for switched systems, diverse control methods have been studied including $H_\infty$ control, sliding mode control (SMC) and fuzzy control. Comparing with other control methods, the advantage of SMC method is to eliminate so-called matched plant parameters and external disturbances with insensitivity property. The SMC method offers a sliding surface for system trajectories to approach by utilizing intermittent control input such that the system can have required attributes, such as disturbance rejection capability, tolerance ability and stability. Since it is difficult to analyze the stability of the switched systems by constructing different sliding surfaces for each subsystem through the idea of common SMC method, numerous efforts about constructing sliding surface for switched systems have been devoted in existing literature. In these results, a nonswitched sliding surface with weighted parameter has been studied for the switched systems. However, most existing SMC methods in switched systems are considered under the condition of stochastic switching or ADT switching. Due to the complexity of PDT switching, the problem for SMC methods in switched systems with PDT switching remains open until now.

On another research forefront, the network technology has been widely used in modern engineering applications due to the superiority in remote operation capability and high installation flexibility. One distinctive feature of network technology is that signal transmission among components extremely depends on the performance of communication networks. However, the applicability of communication network can be affected by signal quantization. Different from the accurate used in conventional sliding mode controller, quantization error which caused by signal quantization may prevent the state trajectories of system from arriving on the pre-defined sliding surface, while it may make the closed-loop systems unstable if the quantization errors cannot be compensated. Hence, it is essential to solve this problem by studying the quantized SMC approach.

Motivated by the above discussions, this article is concerned with the quantized SMC design for a class of continuous-time switched systems with PDT switching and actuator nonlinearities. The contribution of this article is twofold: (1) the
improved logarithmic quantizer is applied to SMC approach instead of the traditional logarithmic quantizer to reduce the restriction of quantization density; (ii) the SMC laws for switched systems with signal quantization are developed to compensate the quantization effect.

**Problem formulation and preliminaries**

Consider the following switched system with actuator nonlinearities and bounded disturbance

\[ \dot{x}(t) = A_{\varepsilon(t)}x(t) + B_{\varepsilon(t)} \Psi(u(t)) + W_{\varepsilon(t)}w(t) \]  

(1)

where \( x(t) \in \mathbb{R}^n \), \( u(t) \in \mathbb{R}^{n_u} \) and \( w(t) \in \mathbb{R}^{n_w} \) represent the system state, the control input and the bounded disturbance, respectively. The switching signal \( \varepsilon(t) \) is a piecewise constant function taking the value from a finite set \( \mathcal{G} = \{1, \ldots, G\} \), where \( G \) represents the serial number of subsystems. The system matrices \( A_{\varepsilon(t)}, B_{\varepsilon(t)} \) and \( W_{\varepsilon(t)} \) denote the real known matrices with appropriate dimensions at the \( \varepsilon(t) \)th system, where \( A_{\varepsilon(t)} \in \mathbb{R}^{n \times n}, B_{\varepsilon(t)} \in \mathbb{R}^{n \times n_u} \) and \( W_{\varepsilon(t)} \in \mathbb{R}^{n \times n_w} \). It is assumed that \( \text{rank}(B_{\varepsilon(t)}) = n_u, W_{\varepsilon(t)} = B_{\varepsilon(t)} \hat{B} \), where \( \hat{B} \in \mathbb{R}^{n_u \times n_d} \) and \( \| w(t) \| \leq \hat{w} \). The dead-zone input \( \Psi(u(t)) \) is nonlinear vector with \( \Psi(u(t)) = [\psi_1(u_1(t)) \cdots \psi_i(u_i(t)) \cdots \psi_n(u_n(t))]' \) of which \( \psi_i(u_i(t)) \) can be described as

\[
\psi_i(u_i(t)) = \begin{cases} \gamma_i^+(t)(u_i - \xi_i^+), & u_i > \xi_i^+ \\ 0, & -\xi_i^- \leq u_i(t) \leq \xi_i^+ \\ \gamma_i^-(t)(u_i + \xi_i^-), & u_i < -\xi_i^- \end{cases}
\]

(2)

where \( \xi_i^+ \) and \( \xi_i^- \) are positive constants; \( \gamma_i^+(t) \) and \( \gamma_i^-(t) \) are the nonlinear functions of input \( u_i \). Let \( \gamma_{\min} = \min\{\gamma_i^+(t), \gamma_i^-(t)\} \) be positive constants. Therefore, one can obtain

\[
\begin{cases} 
\psi_i(u_i(t)) \geq \gamma_{\min}(u_i - \xi_i^+), & u_i > \xi_i^+ \\
\psi_i(u_i(t)) \leq \gamma_{\min}(u_i + \xi_i^-), & u_i < -\xi_i^- 
\end{cases}
\]

(3)

Some definitions should be introduced before proceeding further.

**Definition 1.** Consider the switching instants \( t_1, t_2, \ldots, t_s, \ldots \) with \( t_1 = 0 \). A positive constant \( \tau \) is the PDT if there exists an infinite number of disjoint intervals of length no smaller than \( \tau \) on which \( \varepsilon \) is constant at subsystem \( \Omega_s \), and consecutive intervals with this property are separated by no more than \( T \), where \( T \) is called the period of persistence.\(^{25}\)

**Remark 1.** According to the above definition, a PDT switching signal is composed of infinitely many consecutive switching stages. Each stage includes a period with length at least \( \tau \) and a period with length no greater than \( T \). The former period is called \( \tau \)-portion, in which subsystem switching is prohibited, and the latter period
is regarded as $T$-portion, in which no constrain is applied to the sequence and frequency of subsystem switching.

**Remark 2.** Some notations for PDT switching signal should be introduced for the sake of conciseness. As shown in Figure 1, $t_p$ indicates the instant entering $p$th stage and $t_p^i$ is the $i$th switching instant within $T$-portion. Let $T_{[t_p^i, t_p^{i+1}]}$ denote the actual running time from $t_p^i$ to $t_p^{i+1}$ in the $T$-portion of the $p$th stage, and $T^{(p)}$ denotes the duration of entire $T$-portion. It follows that $T^{(p)} = \sum_{i=1}^{S_{[t_p^i, t_p^{i+1}]}{}} T_{[t_p^i, t_p^{i+1}]} \leq T$ where $S_{[t_p^i, t_p^{i+1}]}$ denotes the switching times within $[t_p^i, t_p^{i+1}]$ and $S_{\max} \triangleq \max\{S_{[t_p, t_p^{+1}]}\}, \forall p \in N_+$.

Some notations for PDT switching signal should be introduced for the sake of conciseness. Let $t_p^i$ denote the actual running time of the $T$-portion of the $p$th stage, and $T^{(p)}$ denotes the duration of entire $T$-portion. It follows that $T^{(p)} = \sum_{i=1}^{S_{[t_p^i, t_p^{i+1}]}{}} T_{[t_p^i, t_p^{i+1}]} \leq T$ where $S_{[t_p^i, t_p^{i+1}]}$ denotes the switching times within $[t_p^i, t_p^{i+1}]$. In addition, $t_p$ indicates the instant entering $p$th stage and $t_p^i$ is the $i$th switching instant within $T$-portion.

**Definition 2.** The switched system (equation (1)) is globally uniformly asymptotically stable (GUAS) under certain switching signals $s(t)$ if for initial condition $x(t_0)$, there exists a class of $\kappa_\infty$ function $\kappa$ such that the solution of the system (equation (1)) satisfies

$$\|x(t)\| \leq \kappa\|x(t_0)\|, \forall t \geq t_0 \text{ and } \|x(t)\| \to 0 \text{ as } t \to \infty$$

As a consequence, the main objective of this article is to determine a set of laws based on SMC approach such that the closed-loop system (equation (1)) is GUAS under quantized signal.
Main results

Sliding surface design and stability analysis

First, a sliding surface is devised and the stability criterion for sliding motion with PDT switching is presented, upon which the parameter matrix of SMC law is obtained. The integral-type sliding surface is considered as follows

\[ s(t) = Hx(t) - \int_{0}^{t} H(A_r + B_r K_r)x(\tau)d\tau \]  

(4)

where \( e(\tau) = r \) and the parameter matrices \( K_r \in \mathbb{R}^{n_x \times n_x} \) are to be devised. The parameter matrix \( H \in \mathbb{R}^{n_x \times n_x} \) is selected such that \( HB_r \) is positive definite.

Remark 3. Note that the proposed sliding surface associated with the system mode will not be switched with the change of system mode, due to the fact that only the integral part of the mode surface depends on the system mode in this article. Considering the continuity of the sliding function at the switching instant \( t_p \), one can obtain

\[
\lim_{\Delta t \to 0} \left[ s(t_p + \Delta t) - s(t_p - \Delta t) \right] = Hx(t_p + \Delta t) - Hx(t_p - \Delta t) - \int_{t_p - \Delta t}^{t_p + \Delta t} H(A_{e(\tau)} + B_{e(\tau)} K_{e(\tau)})x(\tau)d\tau = 0
\]

According to continuity definition, one can conclude the continuity of the sliding function.

The derivative of sliding surface (equation (4)) is derived that

\[
\dot{s}(t) = H\dot{x}(t) - H(A_r + B_r K_r)x(t) = H[B_A \Psi(u(t)) + W_r w(t) - B_r K_r x(t)]
\]

(5)

So as to make the trajectories of the switched system state to approach the sliding hyperplane, we acquire \( s(t) = 0 \) and \( \dot{s}(t) = 0 \). As a consequence, it be derived from equation (5) that

\[
\Psi_{eq}(u(t)) = -(HB_r)^{-1}H W_r w(t) + K_r x(t)
\]

(6)

Substituting equation (6) into equation (1) yields the sliding motion equation

\[
\dot{x}(t) = A_r x(t) + B_r[K_r x(t) - (HB_r)^{-1} H W_r w(t)] + W_r w(t) = A_r x(t) - B_r (HB_r)^{-1} H B_r \dot{w}(t) + B_r \dot{w}(t) + B_r K_r x(t)
\]

(7)

\[
= A_r x(t) + B_r K_r x(t)
\]
The following lemmas present the stability criterion for continuous-time switched system and the sufficient condition for the GUAS for the sliding motion equation (7), respectively.

**Lemma 1.** Consider the sliding motion equation (7), and \( T, S_{\text{max}}, \alpha \) and \( \mu \) are known positive constants with \( \alpha > 0, \mu > 1 \). For \( \forall (a(t_p) \times a(t_{p+1})) = (r \times m) \in \mathcal{G} \times \mathcal{G}, r \neq m \), if there exist a family of function \( V_r \) and two class \( \kappa_\infty \) functions \( \vartheta_1 \) and \( \vartheta_2 \), such that

\[
\vartheta_1 \| x(t) \| \leq V_r(x(t), t) \leq \vartheta_2 \| x(t) \| \tag{8}
\]

Then, the sliding motion equation (7) is GUAS and PDT switching signal satisfies

\[
\alpha(\tau + T) \geq (S_{\text{max}} + 1) \ln \mu \tag{11}
\]

**Proof.** Suppose that \( a(t_p) = r \) is the mode of \( \tau_r \) portion and \( a(t_{p+1}) = m \) is the mode at \( t_p + T^{(p)} \) in the \( p \)th stage of switching, then it can be derived that

\[
V_m(x(t_{p+1}), t_{p+1}) \leq e^{-\alpha T_m} V_m(x(t_p + 1 - T_m), t_p + 1 - T_m)
\]

\[
\leq \mu e^{-\alpha T_m} V_1(x(t_p + 1 - T_m), t_p + 1 - T_m)
\]

\[
\leq \cdots
\]

\[
\leq \left( \prod_{i=1}^{r_{[p]} + 1} \mu e^{-\alpha T_{r_{[p]} + 1}} \right) V_r(x(t_p), t_p)
\]

\[
\leq \mu^{S_{\text{max}} + 1} e^{-\alpha (T + \tau)} V_r(x(t_p), t_p)
\]

From equation (12), it follows that

\[
V_{\ell(t_p)}(x(t_p), t_p) \leq \left( \mu^{S_{\text{max}} + 1} e^{-\alpha (T + \tau)} \right)^p V_{\ell(t_1)}(x(t_1), t_1)
\]

Combining equations (13) and (8), one can obtain that

\[
\| x(t_p) \| \leq \vartheta_1^{-1} \left( \mu^{S_{\text{max}} + 1} e^{-\alpha (T + \tau)} \right)^p \vartheta_2 \| x(t_1) \|
\]

Therefore, if the PDT switching signal satisfies (equation (11)), it can be derived that \( \mu^{S_{\text{max}} + 1} e^{-\alpha (T + \tau)} \leq 1 \). One can draw a conclusion that \( \| x(t_p) \| \to 0 \) when \( p \to \infty \), that is, \( \| x(t) \| \to 0 \) as \( t \to \infty \). The proof is completed.
Lemma 2. Consider the sliding motion equation (7), and $T$, $S_{\text{max}}$, $\alpha$ and $\mu$ are known positive constants with $\alpha > 0$, $\mu > 1$. For $\psi(\epsilon(t))^{{\times}}\psi(t^\text{p}) = (r \times m) \in \mathbb{G} \times \mathbb{G}$, $r \neq m$, if there exist a set of matrices $X_r \succ 0$ such that

\[ A_rX_r + X_rA_r^T + B_rY_r + Y_rB_r^T + \alpha X_r \preceq 0 \]  
\[ X_m - \mu X_r \preceq 0 \]  

Then, the sliding motion equation (7) is GUAS and PDT switching signal satisfies (equation (11)). Moreover, if equations (15) and (16) have a solution, the parameter matrix can be given by

\[ K_r = Y_rX_r^{-1} \]  

Proof. Consider the following Lyapunov functions $V_r = x(t)^{T}P_r x(t)$, where $P_r$ are positive and definite matrices and define $X_r = P_r^{-1}$, $Y_r = K_r X_r$

\[ \dot{V}_r(x(t),t) + \alpha V_r(x(t),t) = \dot{x}(t)P_r x(t) + x(t)P_r \dot{x}(t) \]
\[ = x(t)[P_r A_r + A_r^T P_r + \alpha P_r]x(t) \]
\[ = x(t)[P_r (A_r + B_rK_r) + (A_r + B_rK_r)^T P_r + \alpha P_r]x(t) \]

From equation (15), we can have

\[ A_rP_r^{-1} + (A_rP_r^{-1})^T + B_rK_pP_r^{-1} + (B_rK_pP_r^{-1})^T + \alpha P_r^{-1} \preceq 0 \]  

After equivalence transformation, one can conclude

\[ P_r(A_r + B_rK_r) + (A_r + B_rK_r)^T P_r + \alpha P_r \preceq 0 \]  

From equation (20), one can conclude that equation (9) holds. Similarly, equation (10) can be derived from equation (16). The proof is completed.

Remark 4. Distinguished from the gain matrix $K_r$ used for controller design directly in Liu and Wang, the parameter matrix $K_r$ is only a part of SMC law which is devised in later section.

SMC with improved logarithmic quantizer

In order to mitigate network congestion brought by limited communication network capacity, the signal has to be quantized before transmission. As a sketch of networked system layout is shown in Figure 2, system state $x(t)$, sliding surface variable $s(t)$ and controller output $u(t)$ should be quantized, respectively.

We are interested in a class of improved logarithmic quantized signals with following form: $\tilde{u}_i(t) = Q_u, i(\bar{u}_i(t))$, $\tilde{s}_i(t) = Q_s, i(\bar{s}_i(t))$ and $\tilde{x}_i(t) = Q_x, i(\bar{x}_i(t))$, where $\bar{u}_i(t)$, $\bar{s}_i(t)$ and $\bar{x}_i(t)$ are the quantized signals of $u_i(t)$, $s_i(t)$ and $x_i(t)$, respectively. The signal
\[ z(t) \in \{u(t), s(t), x(t)\} \text{ is vector with } z(t) = [z_1(t)z_2(t) \cdots z_n(t)]'. \]

The improved logarithmic quantizer \( Q_z, i(z_i(t)) \) which is proposed by Li Qiu et al.\(^{28}\) is defined as

\[
Q_z, i(z_i(t)) = \begin{cases} 
\mu_{z,i,q} & z_{i_{\min}} < z_i \leq z_{i_{\max}} \\
0 & z_i = 0 \\
-Q_z, i(-z_i(t)) & z_i < 0 
\end{cases}
\] (21)

where \( \mu_{z,i,q} \) denotes a quantization level for a corresponding subinterval \( (z_{i_{\min}}, z_{i_{\max}}) \) and \( \mu_{z,i,q} > 0 \)

\[
\delta_{z,i} = \frac{1 - \rho_{z,i}}{1 + \rho_{z,i}}
\]

\[
z_{i_{\min}} = \mu_{z,i,q}(1 + \delta_{z,i})
\]

\[
z_{i_{\max}} = \mu_{z,i,q}(1 - \delta_{z,i})
\]

where \( \delta_{z,i} \) is the sector bound of \( z_i \); \( \rho_{z,i} \) represents the quantization density of the quantizer \( Q_z, i(z_i(t)) \) and \( \rho_{z,i} \in (0, 1) \). It is assumed that the quantization density is invariant and the values in quantization subinterval \( (z_{i_{\min}}, z_{i_{\max}}) \) is mapped to the corresponding quantization level \( \mu_{z,i,q} \) which can be described as

\[
U_{z,i} = \left\{ \pm \mu_{z,i,q} : \mu_{z,i,q} = \rho_{z,i,0}^q, \quad q = 0, \pm 1, \pm 2, \cdots \right\} \cup \{0\}
\] (22)

The bound of quantization error is

\[
\frac{|Q_z, i(z_i(t)) - z_i(t)|}{|Q_z, i(z_i(t))|} \leq \delta_{z,i}
\] (23)

and define

\[
e_{x,i} = \bar{x}_i(t) - x_i(t)
\] (24)
\[ e_{x,i} = \tilde{s}_i(t) - s_i(t) \] (25)

The quantization errors \( e_x(t) \) and \( e_s(t) \) satisfy the following constraints

\[ |e_{x,i}(t)| \leq \delta_{x,i} |\tilde{x}_i(t)| \leq |\tilde{x}_i(t)| \] (26)

\[ |e_{s,i}(t)| \leq \delta_{s,i} |\tilde{s}_i(t)| \leq |\tilde{s}_i(t)| \] (27)

**Remark 5.** It is noted that the logarithmic quantizer and quantization error constraints distinguish from the ones in Chen et al.\(^{29}\). To be specific, the length of quantization level in improved logarithmic quantizer is different from traditional logarithmic quantizer, that is, \( z_{\text{min}} \) and \( z_{\text{max}} \) distinguish from the ones in traditional logarithmic quantizer. Moreover, the quantizer density is required to satisfy \( \rho_{z,i} > 1/3 \) in Chen et al.\(^{29}\); however, there are no additional constraints on quantizer density in this article. Therefore, the improved logarithmic quantizer applied in this article has wider application range.

For the convenience of later discussion, another form of quantization error constraints is given

\[ \|e_x(t)\| \leq \delta_x \|\tilde{x}(t)\| \leq \|\tilde{x}(t)\| \] (28)

\[ \|e_s(t)\| \leq \delta_s \|\tilde{s}(t)\| \leq \|\tilde{s}(t)\| \] (29)

where \( \delta_x = \max \delta_{x,i}, \ i = 1, \ldots, n_x \) and \( \delta_s = \max \delta_{s,i}, \ i = 1, \ldots, n_u \), respectively. From equation (26) and \( \delta_x \in (0, 1) \), one can obtain that

\[ \|e_x(t)\| \leq \sqrt{\sum_{i=1}^{n_x} (\delta_{x,i} |\tilde{x}_i(t)|)^2} \leq \sqrt{\sum_{i=1}^{n_x} (\tilde{x}_i(t))^2} \leq \delta_x \|\tilde{x}(t)\| \leq \|\tilde{x}(t)\| \] (30)

Similarly, equation (29) can be derived from equation (27).

The above results about the quantization errors \( e_x(t) \) and \( e_s(t) \) will be used for SMC design later. The following theorem is presented to design the SMC law via the improved logarithmic quantizer and guarantee the reachability of sliding surface.

**Theorem 1.** Considering the switched system (equation (7)) and the sliding surface (equation (4)), construct the SMC law \( u_{i,r}(t) \) via the improved logarithmic quantizer as the following form

\[
\begin{align*}
    u_{i,r}(t) &= \left\{ \begin{array}{ll}
    -g_r(t) \frac{\tilde{x}_i(t)}{|\tilde{x}(t)|} + \xi_i^+, & \tilde{s}_i(t) < 0 \\
    0, & \tilde{s}_i(t) = 0 \\
    -g_r(t) \frac{\tilde{x}_i(t)}{|\tilde{s}(t)|} - \xi_i^-, & \tilde{s}_i(t) > 0
    \end{array} \right.
\end{align*}
\] (31)

where

\[
g_r(t) = \frac{(1 + \delta_x)}{\gamma_{\min}(1 - \delta_s)} \left[ \tilde{w} \| \tilde{B} \| + (1 + \delta_x) \| K_r \| \| \tilde{x}(t) \| \right] \] (32)
\(\varepsilon(t) = r\), \(\dot{w}\) is the bound of disturbance \(w(t)\). Then, the trajectory can reach the sliding surface (equation (4)).

**Proof.** Considering the Lyapunov function \(V_s(t) = 0.5s'(t)(HB_r)^{-1}s(t)\) and sliding mode surface (equation (4)) as well as the switched system (equation (1)), one can obtain

\[
\dot{V}_s(t) = s'(t)(HB_r)^{-1}\dot{s}(t) = s'(t)\left[\Psi(u(t)) + \dot{B}w(t) - K_rx(t)\right]
\]

(33)

From the definition of quantization errors \(e_s(t)\) and \(e_r(t)\), equation (33) can be rewritten to

\[
\dot{V}_s(t) = \left[\tilde{s}(t) - e_s(s(t))\right]'\left[\Psi(u_r(t)) + \dot{B}w(t) - K_rx(t)\right] = \tilde{s}'(t)\left[\Psi(u_r(t)) + \dot{B}w(t) - K_rx(t)\right] - e_s'(s(t))\left[\Psi(u_r(t)) + \dot{B}w(t) - K_rx(t)\right]
\]

(34)

The related terms of equation (34) can be enlarged

\[
\tilde{s}'(t)\dot{B}w(t) \leq \dot{w}\|\tilde{B}\|\|\tilde{s}'(t)\|
\]

(35)

and

\[
-\tilde{s}'(t)K_rx(t) \leq \|\tilde{s}'(t)\|\|K_r\|\|\dot{x}(t) - e_r(t)\|
\leq \|\tilde{s}'(t)\|\|K_r\|\|\dot{x}(t)\| + \|e_r(t)\|
\leq (1 + \delta_x)\|\tilde{s}'(t)\|\|K_r\|\|\dot{x}(t)\|
\]

(36)

where \(\|w(t)\| \leq \dot{w}\) and \(\delta_x = \max \delta_{x,i}, i = 1, \ldots, n_x\). Combining equations (35) and (36), one can have

\[
\tilde{s}'(t)\left[\Psi(u_r(t)) + \dot{B}w(t) - K_rx(t)\right] \leq \tilde{s}'(t)\Psi(u_r(t)) + \dot{w}\|\tilde{s}'(t)\|\|\tilde{B}\| + \|\tilde{s}'(t)\|\|K_r\|\|\dot{x}(t)\|
\]

(37)

Considering another term of equation (34), one can obtain that

\[
-e_s'(s(t))\Psi(u(t)) \leq \sum_{i=1}^{n_u} |e_{s,i}(t)||\Psi_i(u_{i,r}(t))| \leq \sum_{i=1}^{n_u} \delta_{s,i} |\tilde{s}_i(t)||\Psi_i(u_{i,r}(t))|
\]

(38)

and

\[
-e_s'(s(t))\dot{B}w(t) \leq \dot{w}\|e_s'(s(t))\|\|\tilde{B}\| \leq \dot{w}\delta \|\tilde{s}'(t)\|\|\tilde{B}\|
\]

(39)

where \(\delta_s = \max \delta_{s,i}, i = 1, \ldots, n_u\)
Combining equations (38), (39) and (40), it follows that

\[
\begin{align*}
\epsilon'(s(t))K_r x(t) & \leq \|\epsilon'(s(t))\|_\infty \|K_r\|_\infty \|x(t) - e_x(t)\| \\
& \leq \delta_s \|\tilde{s}'(t)\|_\infty (\|\tilde{x}(t)\|_\infty + \|e_x(t)\|_\infty) \\
& \leq (1 + \delta_s)\|\tilde{s}'(t)\|_\infty \|\tilde{x}(t)\|_\infty
\end{align*}
\] (40)

To further simplify the related items in equation (42), the following proof is discussed in three cases:

Case 1. \( u_{i,r}(t) > \xi_i^+ \)

\[
\begin{align*}
\tilde{s}'(t)\psi(u_r(t)) + \sum_{i=1}^{n_u} \tilde{s}_i(t) \|\psi_i(u_{i,r}(t))\| & = \sum_{i=1}^{n_u} \tilde{s}_i(t) \gamma_i^+(u_{i,r} - \xi_i^+) + \sum_{i=1}^{n_u} \tilde{s}_i(t) \|\gamma_i^+(u_{i,r} - \xi_i^+)\| \\
& = -\sum_{i=1}^{n_u} \left[ \tilde{s}_i(t) \gamma_i^+(u_r - \xi_i^+) \tilde{s}_i(t) \|\tilde{s}_i(t)\| - \tilde{s}_i(t) \|\gamma_i^+(u_r - \xi_i^+)\| \tilde{s}_i(t) \|\tilde{s}_i(t)\| \right] \\
& = 0
\end{align*}
\] (43)

Case 2. \( u_{i,r}(t) < -\xi_i^- \)

\[
\begin{align*}
\tilde{s}'(t)\psi(u_r(t)) + \sum_{i=1}^{n_u} \tilde{s}_i(t) \|\psi_i(u_{i,r}(t))\| & = \sum_{i=1}^{n_u} \tilde{s}_i(t) \gamma_i^+(u_{i,r} - \xi_i^-) + \sum_{i=1}^{n_u} \tilde{s}_i(t) \|\gamma_i^+(u_{i,r} - \xi_i^-)\| \\
& = -\sum_{i=1}^{n_u} \left[ \tilde{s}_i(t) \gamma_i^+(u_r - \xi_i^-) \tilde{s}_i(t) \|\tilde{s}_i(t)\| - \tilde{s}_i(t) \|\gamma_i^+(u_r - \xi_i^-)\| \tilde{s}_i(t) \|\tilde{s}_i(t)\| \right] \\
& = 0
\end{align*}
\]
\[\tilde{s}'(t)\psi(u_r(t)) + \sum_{i=1}^{n_n} |\tilde{s}_i(t)||\psi_i(u_{i,r}(t))| = 0\] (45)

Therefore, equation (42) is simplified

\[
\dot{V}_s(t) \leq (1 - \delta_s)\tilde{s}'(t)\psi(u_r(t)) + \tilde{w}||\tilde{s}'(t)||||\dot{B}|| + (1 + \delta_s)||\tilde{s}'(t)||||K_r||||\tilde{x}(t)||
\]
\[
+ \delta_s \sum_{i=1}^{n_n} |\tilde{s}_i(t)||\psi_i(u_{i,r}(t))| + \tilde{w}\delta_s||\tilde{s}'(t)||||\dot{B}|| + (1 + \delta_s)\delta_s||\tilde{s}'(t)||||K_r||||\tilde{x}(t)||
\]
\[
\leq (1 - \delta_s)\tilde{s}'(t)\psi(u_r(t)) + \tilde{w}||\tilde{s}'(t)||||\dot{B}|| + (1 + \delta_s)||\tilde{s}'(t)||||K_r||||\tilde{x}(t)||
\]
\[
+ \tilde{w}\delta_s||\tilde{s}'(t)||||\dot{B}|| + \delta_s(1 + \delta_s)||\tilde{s}'(t)||||K_r||||\tilde{x}(t)||
\]
\[
\leq (1 - \delta_s)\tilde{s}'(t)\psi(u_r(t)) + \tilde{w}||\tilde{s}'(t)||||\dot{B}|| + (1 + \delta_s)||\tilde{s}'(t)||||K_r||||\tilde{x}(t)||
\]
\[
\leq (1 - \delta_s)\tilde{s}'(t)\psi(u_r(t)) + \tilde{w}||\tilde{s}'(t)||||\dot{B}|| + (1 + \delta_s)||K_r||||\tilde{x}(t)||||\tilde{s}'(t)||
\]
\[
\leq (1 - \delta_s)\tilde{s}'(t)\psi(u_r(t)) + \tilde{w}||\tilde{s}'(t)||||\dot{B}|| + (1 + \delta_s)||K_r||||\tilde{x}(t)||||\tilde{s}'(t)||
\]
\[
\leq (1 - \delta_s)\tilde{s}'(t)\psi(u_r(t)) + \tilde{w}||\tilde{s}'(t)||||\dot{B}|| + (1 + \delta_s)||K_r||||\tilde{x}(t)||||\tilde{s}'(t)||
\]
\[
\leq (1 - \delta_s)\tilde{s}'(t)\psi(u_r(t)) + \tilde{w}||\tilde{s}'(t)||||\dot{B}|| + (1 + \delta_s)||K_r||||\tilde{x}(t)||||\tilde{s}'(t)||
\]
\[
\leq (1 - \delta_s)\tilde{s}'(t)\psi(u_r(t)) + \tilde{w}||\tilde{s}'(t)||||\dot{B}|| + (1 + \delta_s)||K_r||||\tilde{x}(t)||||\tilde{s}'(t)||
\]
\[
\leq (1 - \delta_s)\tilde{s}'(t)\psi(u_r(t)) + \tilde{w}||\tilde{s}'(t)||||\dot{B}|| + (1 + \delta_s)||K_r||||\tilde{x}(t)||||\tilde{s}'(t)||
\]
\[
\leq (1 - \delta_s)\tilde{s}'(t)\psi(u_r(t)) + \tilde{w}||\tilde{s}'(t)||||\dot{B}|| + (1 + \delta_s)||K_r||||\tilde{x}(t)||||\tilde{s}'(t)||
\]
\[
\leq (1 - \delta_s)\tilde{s}'(t)\psi(u_r(t)) + \tilde{w}||\tilde{s}'(t)||||\dot{B}|| + (1 + \delta_s)||K_r||||\tilde{x}(t)||||\tilde{s}'(t)||
\]
\[
\leq (1 - \delta_s)\tilde{s}'(t)\psi(u_r(t)) + \tilde{w}||\tilde{s}'(t)||||\dot{B}|| + (1 + \delta_s)||K_r||||\tilde{x}(t)||||\tilde{s}'(t)||
\]
\[
\leq (1 - \delta_s)\tilde{s}'(t)\psi(u_r(t)) + \tilde{w}||\tilde{s}'(t)||||\dot{B}|| + (1 + \delta_s)||K_r||||\tilde{x}(t)||||\tilde{s}'(t)||
\]
\[
\leq (1 - \delta_s)\tilde{s}'(t)\psi(u_r(t)) + \tilde{w}||\tilde{s}'(t)||||\dot{B}|| + (1 + \delta_s)||K_r||||\tilde{x}(t)||||\tilde{s}'(t)||
\]

where \(g_r(t)\) satisfies equation (32).

Then, we will further prove that equation (47) holds in three cases.

Case 1. \(u_{i,r}(t) > \xi_i^+\), which means \(u_{i,r}(t) = -g_r(t)\times((\bar{s}_i(t))/|\tilde{s}_i(t)|) + \xi_i^+\) and \(\tilde{s}_i(t) < 0\), then
Referring to the discussion on the finite-time reachability, the trajectory can reach the sliding surface in finite time, if

\[ \tilde{s}'(t)\psi(u_r(t)) = \sum_{i=1}^{n_u} \tilde{s}_i(t)\gamma_i^+(t)(u_{i,r} - \xi_i^+) = -\sum_{i=1}^{n_u} \tilde{s}_i(t)\gamma_i^+(t)g_r(t)\frac{\tilde{s}_i(t)}{|\tilde{s}_i(t)|} \]

\[ = -\sum_{i=1}^{n_u} |\tilde{s}_i(t)|\gamma_i^+(t)g_r(t) \leq -\gamma_{\min}g_r(t)\sum_{i=1}^{n_u} |\tilde{s}_i(t)| \leq -\gamma_{\min}g_r(t)||\tilde{s}'(t)|| \]

(48)

Case 2. \( u_{i,r}(t) < -\xi_i^- \), which means \( u_{i,r}(t) = -g_r(t)\times((\tilde{s}_i(t))/|\tilde{s}_i(t)|) - \xi_i^- \) and \( \tilde{s}_i(t) > 0 \), then

\[ \tilde{s}'(t)\psi(u_r(t)) = \sum_{i=1}^{n_u} \tilde{s}_i(t)\gamma_i^-(t)(u_{i,r} + \xi_i^-) = -\sum_{i=1}^{n_u} \tilde{s}_i(t)\gamma_i^-(t)g_r(t)\frac{\tilde{s}_i(t)}{|\tilde{s}_i(t)|} \]

\[ = -\sum_{i=1}^{n_u} |\tilde{s}_i(t)|\gamma_i^-(t)g_r(t) \leq -\gamma_{\min}g_r(t)\sum_{i=1}^{n_u} |\tilde{s}_i(t)| \leq -\gamma_{\min}g_r(t)||\tilde{s}'(t)|| \]

(49)

Case 3. \( \xi_i^- < u_i(t) < \xi_i^+ \), which means \( u_i(t) = 0 \) and \( \tilde{s}_i(t) = 0 \), then

\[ \tilde{s}'(t)\psi(u_r(t)) = -\gamma_{\min}g_r(t)||\tilde{s}'(t)|| = 0 \]

(50)

According to the above discussion of three cases, it follows that

\[ \tilde{s}'(t)\psi(u(t)) \leq -\gamma_{\min}g_r(t)||\tilde{s}'(t)|| \]

(51)

from which one can conclude that equation (47) holds. The proof is completed. Therefore, the trajectory \( x(t) \) can reach the sliding surface (equation (4)).

Remark 6. Referring to the discussion on the finite-time reachability, the trajectory can reach the sliding surface in finite time, if \( g_r(t) \) is replaced by \( \tilde{g}_r(t) \)

\[ \tilde{g}_r(t) = \frac{(1 + \delta_s)[\tilde{w}||\tilde{B}|| + (1 + \delta_s)||K_r||||\tilde{x}(t)||] + \phi}{\gamma_{\min}(1 - \delta_s)} \]

where \( \phi \) is a positive constant. Then, \( \dot{V}_s(t) \leq 0 \) becomes \( \dot{V}_s(t) \leq -\phi \|s(t)\| \) and we can get

\[ \dot{V}_s(t) \leq -\phi\|s(t)\| \leq -\phi\sqrt{s^T(t)s(t)} \]

\[ \leq -\phi\sqrt{s^T(t)\lambda_{\min}\lambda_{\min}^{-1}s(t)} \leq -\phi\sqrt{\lambda_{\min}^{-1}\sqrt{s^T(t)\lambda_{\min}s(t)}} \]

Due to the fact \( s^T(t)(HB_r)^{-1}s(t) \geq s^T(t)\lambda_{\min}s(t) \), one can obtain that

\[ \dot{V}_s(t) \leq -\phi\sqrt{\lambda_{\min}^{-1}\sqrt{s^T(t)(HB_r)^{-1}s(t)}} \leq -\phi\sqrt{\lambda_{\min}^{-1}\sqrt{2V_s(t)}} \]  

(52)
where $\lambda_{\text{min}}$ is the minimum eigenvalue of $(HB_r)^{-1}$. According to equation (52), it concludes that the trajectory can reach the sliding surface in finite time.

**Numerical example**

A numerical example is provided to illustrate the effectiveness of the proposed result. Consider switched system (equation (1)) given by

$$A_1 = \begin{bmatrix} 1.3 & 0.7 \\ 0.6 & -0.1 \end{bmatrix}, A_2 = \begin{bmatrix} 0.9 & 0.62 \\ 0.5 & -0.4 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0.72 & 0.52 \\ 0.9 & 0.6 \end{bmatrix}, B_2 = \begin{bmatrix} 1.2 & 0.4 \\ 0.6 & 0.2 \end{bmatrix},$$

$$W_1 = \begin{bmatrix} 0.9 \end{bmatrix}, W_2 = \begin{bmatrix} 0.6 \end{bmatrix},$$

and initial state $x(0) = [2.76 \ -3.17]'.$

The related items of nonlinearity input (equation (2)) and the exogenous disturbance $w(t)$ are presented as follows

$$\gamma^+_i(t) = e^{\sin(u(t))}, \xi^+_i = 0.1$$

$$\gamma^-_i(t) = e^{\sin(u(t))}, \xi^-_i = 0.2$$

$$w(t) = 0.5 \cos(t)$$

where $\|w(t)\| \leq \bar{w}, \bar{w} = 0.5$ and $\gamma_{\text{min}} = \min e^{\sin(u(t))} = 1$. The parameters of the improved logarithmic quantizer in equation (21) are selected as follows

$$\rho_{z,i} = 0.9, q = 100, \mu_{z,i,0} = 8$$

where we can calculate that $\delta_x = \delta_y = 0.0526$.

Assigning associated parameters $\alpha = 0.6, \mu = 1.2, S_{\text{max}} = 20$ and period of persistence $T = 2s$, we can get the minimal PDT $\tau = 4.3813s$ from Lemma 1. Based on Lemma 2, the parameter matrices $K_r$ can be obtained as follows

$$K_1 = [-3.9885 \ 1.0136], K_2 = [-1.5060 \ -0.0638]$$

Therefore, the SMC law $u_r(t)$ can be constructed as follows

$$u_r(t) = \begin{cases} 
  g_r(t) + 0.1, & \ddot{s}(t) < 0 \\
  0, & \ddot{s}(t) = 0 \\
  -g_r(t) - 0.2, & \ddot{s}(t) > 0
\end{cases}$$

and

$$g_r(t) = 1.0555[0.5 + 1.0026||K_r||||\ddot{s}(t)||]$$

Compared with Figure 3 in which the uncontrolled system state diverges, Figure 4 demonstrates the performance of the closed-loop system via SMC law obtained
by Theorem 1. Although there exists slight chattering phenomenon, the state response of the closed-loop system with signal quantization and actuator nonlinearities converges in Figure 4. Therefore, the SMC method for switched systems with signal quantization and actuator nonlinearities can effectively guarantee the closed-loop system is GUAS. The comparisons of $x_1(t)$ and its quantized values are shown.
in Figures 5 and 6, from which can be seen that the trajectories of $x_1(t)$ and its quantize values almost coincide in steady state.

The state response $x_1(t)$ at different ranges of dead zone is shown in Figure 7 which indicates that the state response of switched system at wider range of dead zone has larger chattering phenomenon in steady state. $x_1(t)$ with different bounds of disturbance is shown in Figure 8, from which we can see that more powerful
disturbance will cause more larger chattering phenomenon in steady state. Although there exist different amplitudes of chattering, the closed-loop system is still stable. It indicates that the SMC method is robust to actuator dead zone and disturbance.

The state response of switched system in different quantization densities is shown in Figure 9. It indicates that there is little difference in chattering phenomenon under the different quantized errors caused by different quantization
densities. Figure 9 further strengthens the evidence that the quantization error of \( x(t) \) has been compensated, since the quantized effects have been considered in the controller design. Therefore, the SMC method which is against the signal quantization error in the networked channel and nonlinearities in the actuator is effective.

**Conclusion**

This article investigates the quantized SMC design method for switched systems with signal quantization, actuator nonlinearity and PDT switching. The improved logarithmic quantizer is applied to the signals transmitted in network instead of the traditional logarithmic quantizer to reduce the restriction of quantization density, and based on this quantizer the quantized SMC method is demonstrated to guarantee the globally uniformly asymptotical stability of the closed-loop system. A numerical simulation is given to illustrate the superiority and effectiveness of the developed results. Future work will be applied to practical system, for instance, attitude stability control of variable mass spacecraft to verify developed theoretical result.

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