The connected domination polynomial of some graph constructions

Rafia Yoosuf¹, Preethi Kuttipulackal²
¹Department of Mathematics, MES Mampad College(Autonomous)
²Department of Mathematics, University of Calicut
E-mail: rafiafiroz@gmail.com, pretikut@rediffmail.com

Abstract. The connected domination polynomial of a connected graph $G$ of order $n$ is the polynomial

$$D_c(G, x) = \sum_{i=\gamma_c(G)}^{n} d_c(G, i)x^i,$$

where $d_c(G, i)$ is the number of connected dominating sets of $G$ of cardinality $i$ and $\gamma_c(G)$ is the connected domination number of $G$ [5]. In this paper we find the polynomial $D_c(G, x)$ for some constructive graphs.

1. Introduction

The domination polynomial of a graph is introduced by Saeid Alikhani and Yee-hock Peng in [3]. While extending the concept of domination polynomial in view of connected dominating set ($cd$-set), we came across with many interesting relations among the connected domination polynomials of different graphs [4].

Let $G = (V, E)$ be a simple graph. For any vertex $v \in V$, the open neighbourhood of $v$ is the set $N(v) = \{u \in V : uv \in E\}$ and the closed neighbourhood of $v$ is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the open neighbourhood of $S$ is $N(S) = \bigcup_{v \in S} N(v)$ and the closed neighbourhood of $S$ is $N[S] = N(S) \cup S$. A set $S \subseteq V$ is a dominating set of $G$, if $N[S] = V$, or equivalently every vertex in $V \setminus S$ is adjacent to atleast one vertex in $S$. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set in $G$ [3].

Let $G$ be a simple connected graph of order $n$. A connected dominating set ($cd$ - set) of $G$ is a set $S$ of vertices of $G$ such that every vertex in $V \setminus S$ is adjacent to some vertex in $S$ and the induced subgraph $<S>$ is connected. The connected domination number $\gamma_c(G)$ is the minimum cardinality of a connected dominating set in $G$ [4].

Saeid Alikhani and Yee-hock Peng introduced the concept of domination polynomial of a graph as the polynomial $D(G, x) = \sum_{i=\gamma(G)}^{n} d(G, i)x^i$, where $d(G, i)$ denotes the number of dominating sets of cardinality $i$ [3].

Considering the polynomial idea of Alikhani et.al., we studied the connected domination($cd$—polynomial) polynomial of a connected graph and the information about the graph that we can obtain from the polynomial. For the basic concepts in graph theory we refer mainly Bondy and Murthy [1]. The graphs considered here are all connected and simple of order $n$. 
2. Notation:

- $D_c[G, x]$: Connected domination polynomial of a graph $G$
- $d_c(G, i)$: number of connected dominating sets of $G$ of cardinality $i$
- $\gamma_c(G)$: Connected domination number of $G$
- $N(v)$: Open neighborhood of the vertex $v$ of a graph $G$
- $N[v]$: Closed neighborhood of the vertex $v$ of a graph $G$
- $D[G, x]$: Domination polynomial of a graph $G$
- $\gamma(G)$: The domination number of a graph $G$
- $cd - set$: Connected dominating set
- $cd - polynomial$: Connected domination polynomial

3. Connected domination polynomial

The minimum cardinality of a $cd-$ set is $\gamma_c(G)$ and the maximum cardinality is $n$.

**Definition 1.** [4]. Let $G$ be a connected graph of order $n$. The connected domination polynomial of $G$ is the polynomial

$$D_c(G, x) = \sum_{i = 1}^{\gamma_c(G)} d_c(G, i)x^i,$$

where $d_c(G, i)$ is the number of connected dominating sets of $G$ of cardinality $i$.

**Example 1.** Consider the path $P_3 = v_1v_2v_3$.

The connected dominating sets are $\{v_2\}$, $\{v_1, v_2\}$, $\{v_2, v_3\}$ and $\{v_1, v_2, v_3\}$. So that the polynomial is

$$x + 2x^2 + x^3.$$

**Observations**

For any connected graph with $n$ vertices, we have

(i) The coefficient of $x^n$, $d_c(G, n) = 1$.
(ii) $D_c(G, x)$ has no constant term.
(iii) Zero is a root of $D_c(G, x)$ with multiplicity $\gamma_c(G)$.

3.1. Main Theorems

**Theorem 1.** [5] The $cd -$ polynomial of the star graph $K_{1,n}$ is

$$D_c[K_{1,n}, x] = \sum_{i=1}^{n+1} \binom{n}{i-1}x^i.$$

The cartesian product of two graphs $G$ and $H$ is the graph $G \square H$ with vertex set $V(G) \times V(H)$ and the vertices $(u, v)$ and $(x, y)$ are adjacent if and only if $u = x$ and $vy \in E(H)$ or $ux \in E(G)$ and $v = y$.

An $n-$book graph is obtained as the cartesian product of the star graph $K_{1,n}$ and the path graph $P_2$.

**Remark 1.** For $n = 1$ $B_1 \cong C_4$. The $cd -$ polynomial for $B_1$ is $4x^2 + 4x^3 + x^4$.

**Theorem 2.** For $n > 1$, the $cd -$ polynomial of the $n-$book graph is given by

$$D_c[B_n, x] = xD_c[K_{1,2n}, x] + 2x^nD_c[K_{1,n}, x]$$
**Theorem 3.** The $cd-$ polynomial of the cycle switching of the cycle $C_n$, for $n > 6$ is

$$x^2D_c[K_{1,n-2},x] + x^{n-4}D_c[K_{1,3},x].$$

**Proof.** Let $\{u_1, u_2, ..., u_n\}$ be the vertices of the cycle $C_n$. Without loss of generality, we take the vertex $u_1$ as the switching vertex of $C_n$, then $u_1$ is adjacent to all vertices $u_i$ except $u_2$ and $u_n$ and denote the resulting graph by $G$. The $cd-$ number of the vertex switching graph cycle $G$ is 3. For every cycle $G$, $n > 6$ there exist only one $cd-$ set of cardinality 3, namely $\{u_1, u_{n-1}, u_3\}$. Any $cd-$ set must include the two vertices $u_{n-1}$ and $u_3$ to dominate $u_2$ and $u_n$ respectively and for connection.

A $cd-$ set that does not contain $u_1$ must include all the $n-3$ vertices $\{u_3, u_4, ..., u_{n-1}\}$. On the other hand any $cd-$ set of cardinality less than $n-3$, must include the vertex $u_1$. So that a $cd-$ set of cardinality $i < n-3$, must include the three vertices $u_1, u_3$ and $u_{n-1}$. The remaining $i-3$ vertices can be chosen arbitrarily from the $n-3$ vertices. Hence the number of such sets is $\binom{n-3}{i-3}$.

Consider $n-3 \leq i < n$. There are $cd-$ sets containing $u_1$ and not containing $u_1$. For a $cd-$ set containing $u_1$, we have $\binom{n-3}{i-3}$ choices as above. If a $cd-$ set does not contain $u_1$, the $cd-$ set
must contain the vertices \( \{u_3, u_4, ..., u_{n-1}\} \), for the remaining vertices we have \((\frac{n-2}{i-(n-3)})\) choices. For \( i = n \) we have one choice, for convenient of the proof we take it as \((\frac{n-2}{n-3})\) choices. Therefore the polynomial is

\[
D_c[G, x] = \sum_{i=3}^{n-2} \binom{n-3}{i-3} x^i + \sum_{i=n-3}^{n-1} \binom{n-3}{i-3} \left( \binom{2}{i-3} \right) x^i + \binom{n-3}{n-3} x^n
\]

\[
= \sum_{i=3}^{n} \binom{n-3}{i-3} x^i + \sum_{i=n-3}^{n-1} \binom{2}{i-3} x^i
\]

\[
= x^2 \sum_{i=1}^{n} \binom{n-3}{i-1} x^i + x^{n-4} \sum_{i=1}^{3} \binom{2}{i-1} x^i
\]

\[
= x^2 D_c[K_{1,n-2}, x] + x^{n-4} D_c[K_{1,2}, x]
\]

\[\square\]

**Theorem 4.** For an end vertex switching graph \( P_n, n \geq 5 \) the \( cd- \) polynomial is

\[
xD_c[K_{1,n-2}, x] + D_c[P_{n-1}, x].
\]

**Proof.** Let \( v_n \) be the switching end vertex of \( P_n \). If \( G \) denote the switching graph, we have \( \gamma_c(G) = 2 \), and \( \{v_n, v_{n-2}\} \) is the only \( cd- \) set of cardinality 2. Note that a connected dominating set of \( G \) that does not contain \( v_n \), must include all the \( n - 3 \) vertices \( \{v_2, v_3, ..., v_{n-2}\} \).

So that any \( cd- \) set of less than \( n - 3 \) elements must include \( v_n \) and \( v_{n-2} \); and since these two elements are always to dominate the entire graph, the remaining vertices in the \( cd- \) set can be chosen arbitrarily from the \( n - 2 \) vertices. Thus there are \((\frac{n-2}{2})\) \( cd- \) sets of cardinality \( i \), for \( 1 < i < n - 3 \).

Now for \( i = n - 3 \), we have one \( cd- \) set that does not contain \( v_n \), namely \( \{v_2, ..., v_{n-2}\} \) and \((\frac{n-2}{n-5})\) \( cd- \) sets containing \( v_n \) (and \( v_{n-2} \)).

Similarly there are two \( cd- \) sets of cardinality \( n - 2 \), that does not contain \( v_n \) and \((\frac{n-2}{n-4})\) \( cd- \) sets containing \( v_n \). Hence the polynomial is

\[
D_c[G, x] = \sum_{i=1}^{n-1} \binom{n-2}{i-1} x^i + x^{n-3} + \binom{2}{1} x^{n-2} + x^{n-1}
\]

\[
= x[x + \binom{n-2}{1} x^2 + ... + \binom{n-2}{n-2} x^{n-1}] + x^{n-3} + \binom{2}{1} x^{n-2} + x^{n-1}
\]

\[
= xD_c[K_{1,n-2}, x] + D_c[P_{n-1}, x].
\]

\[\square\]

**Theorem 5.** For the vertex switching of the path \( P_n, n \geq 5 \), with respect to a vertex which is neither an end vertex, nor a support vertex, the \( cd- \) polynomial is

\[
x^2 D_c[K_{1,n-3}, x]
\].
The first 2 types together would contain the switching vertex. Note that \( \{v_1, v_{i+2}, v_{i-1}\} \) is a \( \text{cd}^{-}\) set and every \( \text{cd}^{-}\) set must contain the vertices \( \{v_i, v_{i+2}, v_{i-2}\} \). So the number of \( \text{cd}^-\) sets of cardinality \( i \) have \( (\binom{n-3}{i-3}) \) choices. Therefore the polynomial is

\[
 x^3 + \left(\frac{n-3}{1}\right)x^4 + \left(\frac{n-3}{2}\right)x^5 + ... + \left(\frac{n-3}{n-3}\right)x^n = x^2[x + \left(\frac{n-3}{1}\right)x^3 + \left(\frac{n-3}{2}\right)x^4 + ... + \left(\frac{n-3}{n-3}\right)x^{n-1}]
 = x^2D_c[K_{1,n-3}, x].
\]

Now the vertex switching graph of \( K_{1,n} \) is disconnected if \( n = 1 \) and \( K_{1,2} \) itself if \( n = 2 \). For \( n > 2 \), we have the following result.

**Theorem 6.** The \( \text{cd}^-\) polynomial of the vertex switching of the star graph \( K_{1,n} \), \( n \geq 2 \), with respect to a vertex of degree 1 is

\[
 (2 + x)\{D_c[K_{1,n}, x] - x\}.
\]

**Proof.** The switching vertex of the star graph should be any vertex other than the centre vertex to retain the graph connected, and the graph is denoted by \( G \). The \( \text{cd}^-\) number of this vertex switching graph of \( K_{1,n} \) is 2. Let \( \{v, v_1, v_2, ..., v_n\} \) be the vertices of \( K_{1,n} \) with centre \( v \). Without loss of generality we assume that the vertex \( v_1 \) is the switching vertex. There are two types of \( \text{cd}^-\) sets of cardinality two, one which contains \( v \) but not \( v_1 \) and the other which contains \( v_1 \) but not \( v \). Both of these together have \( 2^{(n-1)} \) choices.

From \( i = 3 \) onwards the \( \text{cd}^-\) sets of cardinality \( i \) fall in 3 categories. One which contains \( v \) but not \( v_1 \), second which contains \( v_1 \) but not \( v \) and the third which contains both \( v \) and \( v_1 \). The first 2 types together would contain \( (\binom{n-1}{i-1}) \) \( \text{cd}^-\) sets in each as above and the third type has \( (\binom{n-1}{i-2}) \) choices. So that for \( i = 3 \) onwards we have

\[
 \sum_{i=3}^{n} 2^{(\binom{n-1}{i-1})} + \sum_{i=3}^{n+1} (\binom{n-1}{i-2})
\]

\( \text{cd}^-\) sets. Therefore the polynomial is

\[
 D_c[G, x] = \sum_{i=3}^{n} 2^{(\binom{n-1}{i-1})}x^i + \sum_{i=3}^{n+1} (\binom{n-1}{i-2})x^i + 2^{(\binom{n-1}{1})}x^2
 = 2\sum_{i=2}^{n} (\binom{n-1}{i-1})x^i + x\sum_{i=2}^{n} (\binom{n-1}{i-2})x^i
 = (2 + x)\sum_{i=2}^{n} (\binom{n-1}{i-1})x^i
 = (2 + x)\{D_c[K_{1,n}, x] - x\}
\]

A spider is a tree with atmost one vertex of degree more than two, called the center of spider.
Theorem 7. The \(cd\)- polynomial of the spider graph \(K_{1,n,n}\) is
\[
D_c[K_{1,n,n}, x] = x^n D_c[K_{1,n}, x]
\].

Proof. We have \(\gamma_c(K_{1,n,n}) = n+1\). Let \(v\) be the centre vertex of the graph \(K_{1,n,n}\), \(\{v_1, v_2, ..., v_n\}\) be the vertices adjacent to \(v\) and \(u_i\) be the vertices adjacent to \(v_i\) for each \(i = 1, 2, ..., n\). Any \(cd\)- set of cardinality \(i\), \(i \geq n+1\) must contain all the vertex \(\{v, v_1, v_2, ..., v_n\}\). First we consider \(cd\)- sets of cardinality \(n+1\), there is only one such set namely \(\{v, v_1, v_2, ..., v_n\}\). Any \(cd\)- sets of cardinality \(i\), \(i > n+1\) we have \(\binom{n}{i}\) choices. Therefore the polynomial is
\[
D_c[K_{1,n,n}, x] = x^{n+1} + \binom{n}{1} x^{n+2} + \binom{n}{2} x^{n+3} + ... + \binom{n}{n} x^{2n+1}
\]
\[
= x^n[x + \binom{n}{1} x^2 + ... + \binom{n}{n} x^{n+1}]
\]
\[
= x^n D_c[K_{1,n}, x]
\]

The bispider graph is a graph obtained by edge introducing between two star graphs and the introducing is the rooted vertices, which is denoted by \(S_{p_1,p_2}\) of order \(2p_1 + 2p_2 + 2\) [5].

Theorem 8. The \(cd\)- polynomial of the bispider graph \(S_{p_1,p_2}\) is
\[
D_c[S_{p_1,p_2}, x] = x^{p_1+p_2+1} D_c[K_{1,p_1+p_2}, x]
\]

Proof. Let \(u, v\) be the rooted vertices of the spider graph \(S_{p_1,p_2}\) and \(\gamma_{cd}(S_{p_1,p_2}) = p_1 + p_2 + 2\). All the \(cd\)- sets must consist of all the \(p_1 + p_2 + 2\) vertices of this graph. For \(p_1 + p_2 + 2 \leq i \leq 2p_1 + 2p_2 + 2\), the \(cd\)- set of cardinality \(i\) have \(\binom{i}{i-p_1-p_2-2}\) choices. Therefore the polynomial is
\[
D_c[S_{p_1,p_2}, x] = \sum_{i=p_1+p_2+2}^{2p_1+2p_2+2} x^i
\]
\[
= x^{p_1+p_2+1} \sum_{i=1}^{p_1+p_2+1} x^i
\]
\[
= x^{p_1+p_2+1} D_{cd}[K_{1,p_1+p_2}, x].
\]

Theorem 9. The \(cd\)- polynomial of the connected graph \(G \circ mK_p\) is
\[
D_c[G \circ mK_p, x] = x^{n-1} D_{cd}[K_{1,nmp}].
\]

Proof. We have \(\gamma_c(G \circ mK_p) = n\). All \(cd\)- sets must contain the \(n\) vertices of the graph \(G\), say \(\{v_1, v_2, ..., v_n\}\) otherwise it wont form a \(cd\)- set. The total number of vertices of this graph is \(nmp + n\). There are only one set of cardinality \(n\) namely the set \(\{v_1, v_2, ..., v_n\}\). For a \(cd\)- sets
of cardinality \(i\), \(i > n\) we have \(\binom{nmp}{i-n}\) choices. It follows that the polynomial is

\[
D_c[G \circ mK_p, x] = x^n + \binom{nmp}{1}x^{n+1} + \ldots + \binom{nmp}{nmp}x^{n+nmp}
\]

\[
= x^n[1 + \binom{nmp}{1}x + \ldots + \binom{nmp}{nmp}x^{nmp}] 
\]

\[
= x^{n-1}[x + \binom{nmp}{1}x^2 + \ldots + \binom{nmp}{nmp}x^{nmp+1}]
\]

\[
= x^{n-1}D_c[K_{1,nmp}].
\]

The friendship graph \(F_n\) can be constructed by joining \(n\) copies of the cycle graph \(C_3\) with a common vertex.

**Theorem 10.** The \(cd\)– polynomial of the friendship graph \(F_n\) \(n \geq 2\) is given by

\[
D_c[K_{1,2n}, x]
\]

**Proof.** The \(cd\)– number of \(F_n\) is one, there is only one such set namely the center vertex \(v\). Since the blocks of \(F_n\) are connected through the centre vertex only, the center vertex \(v\) must be an element of every \(cd\)– set. For \(i = 2\) onwards we have \(\binom{2n}{i-1}\) choices exists. Therefore the polynomial is

\[
D_c[F_n, x] = \sum_{i=1}^{2n+1} \binom{2n}{i-1}x^i 
\]

\[
= D_c[K_{1,2n}, x]
\]

**Theorem 11.** The \(cd\)– polynomial of the wheel graph \(W_n\), on \(n + 1\) vertices, where \(n \geq 4\) is given by

\[
D_c[K_{1,n}, x] + D_c[C_n, x]
\]

**Proof.** Let \(\{v, v_1, v_2, \ldots, v_n\}\) be the vertices of the the graph \(W_n\). The \(cd\)– number of \(W_n\) is 1, and the only \(cd\)– set of one element is \(\{v\}\), namely the center vertex \(v\). For \(1 < i < n-1\), every \(cd\)– sets of cardinality \(i\) must include the center vertex \(v\). Since all other vertices are adjacent to \(v\), it follows that there are \(\binom{n}{i-1}\) such \(cd\)– sets exists. For \(n-2 \leq i \leq n\) there are two types of \(cd\)– sets, one containing \(v\) for which there are \(\binom{n}{i-1}\) choices as above and the other which does not contain the vertex \(v\), for which there are \(n\) choices. Finally \(i = n + 1\) is the order of \(W_n\), and we have a unique \(cd\)– set of \(n + 1\) elements. Therefore the polynomial is

\[
D_c[W_n, x] = \sum_{i=1}^{n-3} \binom{n}{i-1}x^i + \sum_{i=n-2}^{n} \left[\binom{n}{i-1} + n\right]x^i + x^{n+1}
\]

\[
= \sum_{i=1}^{n+1} \binom{n}{i-1}x^i + \sum_{i=n-2}^{n} nx^i
\]

\[
= D_c[K_{1,n}, x] + D_c[C_n, x].
\]
Let \( (v_1, v_2, ..., v_n) \) be the vertices of the shellgraph \( S_n \) and \( \gamma_c(S_n) = 1 \). Without loss of generality we assume that \( v_1 \) is adjacent to all other vertices of the graph \( S_n \). For \( 1 \leq i \leq n-1 \), all the \( cd- \) sets must contain the vertex \( v_1 \), and number of \( cd- \) sets of cardinality \( i \) is \( \binom{n-1}{i-1} \). There are two types of \( cd- \) sets of cardinality \( n-1 \) and \( n-2 \) exists, the one which is which contain \( v_1 \) and the other type is which does not contain the vertex \( v_1 \). For \( i = n-2 \) the \( cd- \) sets which does not contain \( v_1 \) are \( \{v_2, v_3, ..., v_{n-2}\} \) and \( \{v_3, v_4, ..., v_{n-1}\} \) and the \( cd- \) sets which contain \( v_1 \) has \( \binom{n-1}{n-3} \) choices as above. For \( i = n-1 \), the set \( \{v_2, v_3, ..., v_{n-1}\} \) is the only \( cd- \) set which does not contain \( v_1 \), and also there are \( \binom{n-1}{n-2} \) \( cd- \) sets which contain \( v_1 \) exists. There is one \( cd- \) set of cardinality \( n \) exists. Therefore the polynomial is

\[
D_c[S_n, x] = \sum_{i=1}^{n-1} \binom{n-1}{i-1} x^i + \left[ \binom{n-1}{n-3} + 2 \right] x^{n-2} + \left[ \binom{n-1}{n-2} + 1 \right] x^{n-1} + x^n
\]

\[
= \sum_{i=1}^{n-1} \frac{(n-1)!}{(i-1)!} x^i + x^n + 2x^{n-2} + x^{n-1}
\]

\[
= D_c[K_{1,n-1}, x] + 2x^{n-2} + x^{n-1}.
\]
A bow graph is a double shell with same apex in which each shell has any order.

**Theorem 14.** If \( B_N \) is a bow graph where \( N > 5 \), the \( cd- \) polynomial is

\[
D_c[B_N, x] = D_c[K_{1,2N-1}, x].
\]

**Proof.** The bow graph \( B_N \) has \( 2N - 1 \) vertices and \( \gamma_c(B_N, x) = 1 \). Let \( v \) be the apex vertex of \( B_N \) and \( v \) must include in all the \( cd- \) sets of \( B_N \), otherwise that set won't form a \( cd- \) set. We can choose any vertices with \( v \) because it is adjacent to all other vertices. For \( 1 \leq i \leq 2N - 1 \), the number of \( cd- \) sets of cardinality \( i \) is \((\frac{2N-2}{i-1})\). Therefore the polynomial is

\[
D_c[B_N, x] = \sum_{i=1}^{2N-1} \left(\frac{2N-2}{i-1}\right) x^i
= D_c[K_{1,2N-1}, x].
\]

\( \square \)

4. References

[1] J. A. Bondy and U.S.R. Murty 2008 *Graph Theory*, Springer.
[2] Frank Harary 1969 *Graph Theory*, Addison-Wesley, Reading, MA 9.
[3] Saeid Alikhani and Yee-hock Peng 2009 *arXiv* 0905 225 V 1.
[4] E.Sampathkumar and H.B.Walikar 1979 *J.Math.phys.*
[5] Dhananjaya Murthy B.V. Deepak G. and N.D. Soner 2013 *Int.J.of Math.Ach.* 4(11) 90-96
[6] Dhananjaya Murthy B.V. Deepak G. and N.D. Soner *Amr.J.of Math.Sci and Appl.* 3.38
[7] Samir K. Vaidya and Udayan Prajapati 2012 *Open Journal of Discrete Mathematics* 2, 99-104.