Observational constraints on running vacuum model

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Abstract

We investigate the power spectra of the CMB temperature and matter density in the running vacuum model (RVM) with the time-dependent cosmological constant of $\Lambda = 3\nu H^2 + \Lambda_0$, where $H$ is the Hubble parameter. In this model, dark energy decreases in time and decays to both matter and radiation. By using the Markov chain Monte Carlo method, we constrain the model parameter $\nu$ as well as the cosmological observables. Explicitly, we obtain $\nu \leq 1.54 \times 10^{-4}$ (68% confidence level) in the RVM with the best-fit $\chi^2_{\text{RVM}} = 13968.8$, which is slightly smaller than $\chi^2_{\Lambda\text{CDM}} = 13969.8$ in the $\Lambda$CDM model of $\nu = 0$.

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I. INTRODUCTION

One of the most important recent cosmological observations is that our universe is undergoing a late-time accelerating expansion phase, realized by introducing dark energy [1]. Among the many possible dark energy scenarios, the simplest one is the ΛCDM model, in which a cosmological constant Λ is added to the gravitational theory, predicting a constant energy density. Although this simplest model fits current cosmological observations very well, it faces several difficulties, such as the “fine-tuning” [2, 3] and “coincidence” [4, 5] problems.

The running vacuum model (RVM) is one of the popular attempts to solve the latter problem [6–17]. In the RVM, instead of a constant, Λ is defined to be a function of the Hubble parameter $H$, and decays to matter (non-relativistic fluid) and radiation (relativistic fluid) in the evolution of the universe, leading to the same order of magnitude for the energy densities of dark energy and dark matter [18–33]. Unlike the scalar tensor dark energy theory, such as the simplest realistic scalar field dynamical one [34, 35], the RVM has no Lagrangian formula, indicating that this model is an effective theory from some other fundamental gravity theories. One possible origin of the RVM is from quantum effects induced by the cosmological renormalization group, resulting in $\Lambda = 3\nu H^2 + \Lambda_0$ [36–45] with $\nu$ and $\Lambda_0$ constants. It has been shown in Ref. [46] that the RVM with $\nu = 0$, i.e., the ΛCDM limit, is not favored by the observational data, whereas the best-fit for the model occurs at $\nu = 4.8 \times 10^{-3}$, implying that the RVM with $\nu > 0$ could well describe the evolution of our universe. Results with similar constraints on $\nu$ have also been given in other types of the RVMs [47–51]. Clearly, it is interesting to investigate the matter power spectrum and cosmic microwave background radiation (CMB) temperature fluctuation in this scenario to see if the model is indeed better than the ΛCDM one.

In this paper, we will derive the growth equations of matter and radiation density fluctuations with the linear perturbation theory and illustrate the matter and CMB temperature power spectra, which can significantly deviate from the ΛCDM prediction. We will show that the parameter $\nu$ will be further constrained when the observational data from Planck 2015 is taken into account. Furthermore, one can also find at the end of this paper that the
constraint on the RVM is about the same order of magnitude as that given in Ref. [52], indicating that the CMB photon power spectrum provides a strong constraint on the RVM.

In principle, dark energy has to be dynamical and its density fluctuation should be taken into account when the dark energy decay model is considered. In order to investigate the dynamics of dark energy, the running vacuum energy should be rewritten as a Lorentz scalar at the field equation level. For example, in Refs. [53–56], the cosmological constant is rewritten as $\Lambda = \Lambda(H)$ with $H = \nabla_{\mu}U^{\mu}/3$. In this work, we follow the perturbation method in Refs. [56–59], with which dark energy simultaneously decays to relativistic and non-relativistic matter and dilutes the density fluctuation. We will also use the Markov chain Monte Carlo (MCMC) method to perform the global fit with the current observational data to further constrain the model.

This paper is organized as follows: In Sec. II, we introduce the RVM and review the background evolutions of matter, radiation and dark energy. In Sec. III, we calculate the linear perturbation theory and show the power spectra of the matter density distribution and CMB temperature by the CAMB program [60]. In Sec. IV, we use the CosmoMC package [61] to fit the model from the observational data. Our conclusions are presented in Sec. V.

II. RUNNING VACUUM MODEL

The Einstein equation of the running vacuum model (RVM) is given by,

$$R_{\mu\nu} - \frac{g_{\mu\nu}}{2}R + \Lambda g_{\mu\nu} = \kappa^2 T^M_{\mu\nu},$$

where $\kappa^2 = 8\pi G$, $R = g^{\mu\nu}R_{\mu\nu}$ is the Ricci scalar, $\Lambda = \Lambda(H)$ is the time-dependent cosmological constant, and $T^M_{\mu\nu}$ is the energy-momentum tensor of matter and radiation. In the Friedmann-Lemaître-Robertson-Walker (FLRW) metric of $ds^2 = a^2(\tau)[-d\tau^2 + \delta_{ij}dx^i dx^j]$, the Friedmann equations are derived to be,

$$H^2 = \frac{a^2\kappa^2}{3}(\rho_M + \rho_\Lambda),$$

$$\dot{H} = -\frac{a^2\kappa^2}{6}(\rho_M + 3P_M + \rho_\Lambda + 3P_\Lambda),$$

In Ref. [52], a large number of data from $f(z)\sigma_8(z)$ and $H(z)$ observations as well as the CMB photon power spectrum have been included in the calculation.
where $\tau$ is the conformal time, $H = da/(ad\tau)$ represents the Hubble parameter, $\rho_M = \rho_m + \rho_r$ ($P_M = P_m + P_r = P_r$) corresponds to the energy density (pressure) of matter and radiation, and $\rho_\Lambda$ ($P_\Lambda$) is the energy density (pressure) of the cosmological constant. From Eq. (1), we have

$$\rho_\Lambda = -P_\Lambda = \kappa^{-2}\Lambda(t),$$

which leads to the equation of state (EoS) of $\Lambda$, given by

$$w_\Lambda \equiv \frac{P_\Lambda}{\rho_\Lambda} = -1.$$  (5)

In Eq. (1), we consider $\Lambda$ to be a function of the Hubble parameter, given by $[36-45]$

$$\Lambda = 3\nu H^2 + \Lambda_0,$$  (6)

where $\nu$ and $\Lambda_0$ are two free parameters. In order to avoid the negative dark energy density in the early universe, we will concentrate on the RVM with $\nu \geq 0$ in our investigation.

Substituting Eq. (6) into the conservation equation, $\nabla^\mu(T^M_{\mu\nu} + T^\Lambda_{\mu\nu}) = 0$, we have

$$\dot{\rho}_\Lambda + 3H(1 + w_\Lambda)\rho_\Lambda = 6\nu H \dot{H} \neq 0,$$  (7)

implying that dark energy unavoidably couples to matter and radiation, given by

$$\dot{\rho}_l + 3H(1 + w_l)\rho_a = Q_l,$$  (8)

where $l$ represents matter ($m$) or radiation ($r$), $Q_l$ is the decay rate of the cosmological constant to $l = m$ or $r$, taken to be

$$Q_l = -\frac{\dot{\rho}_\Lambda (\rho_a + P_l)}{\rho_M} = 3\nu H (1 + w_l)\rho_l,$$  (9)

and $w_{m(r)} = 0$ ($1/3$) is the EoS of matter (radiation). Subsequently, we derive

$$\rho_a = \rho_a^{(0)} a^{-3(1 + w_a)\xi},$$  (10)

where $\xi = 1 - \nu$ and $\rho_a^{(0)}$ is the energy density of $a$ (matter or radiation) at $z = 0$.

III. LINEAR PERTURBATION THEORY

Since the RVM with the strong coupling $Q_l$, corresponding to $\nu \sim O(1)$, is unable to describe the evolution of the universe $[47, 49]$, we only focus on the case of $\nu \ll 1$. Note that
\( \nu \) has taken to be non-negative, i.e., \( \nu \geq 0 \), in order to avoid \( \rho_\Lambda < 0 \) in the early universe. The calculation follows the standard linear perturbation theory with the synchronous gauge [62]. The metric is given by,

\[ ds^2 = a^2(\tau) \left[ -d\tau^2 + (\delta_{ij} + h_{ij})dx^i dx^j \right], \tag{11} \]

where

\[ h_{ij} = \int d^3k e^{i\bar{k} \cdot \vec{x}} \left[ \hat{k}_i \hat{k}_j h(\bar{k}, \tau) + 6 \left( \hat{k}_i \hat{k}_j - \frac{1}{3} \delta_{ij} \right) \eta(\bar{k}, \tau) \right], \tag{12} \]

\( i, j = 1, 2, 3 \), \( h \) and \( \eta \) are two scalar perturbations in the synchronous gauge, and \( \hat{k} = \bar{k} / k \) is the k-space unit vector. From \( \nabla^\mu(T^M_{\mu v} + T^\Lambda_{\mu v}) = 0 \) with \( \delta T^0_0 = \delta \rho_M, \delta T^0_i = -T^i_0 = (\rho_M + P_M)v^i_M \) and \( \delta T^i_\gamma = \delta P_M \delta^i_\gamma \), one can obtain the matter and radiation density perturbations, given by [56–59],

\[ \dot{\delta}_l = -(1 + w_l) \left( \theta_l + \frac{\dot{h}}{2} \right) - 3H \left( \frac{\delta P_l}{\delta \rho_l} - w_l \right) \delta_l - \frac{a Q_l}{\rho_l} \delta_l, \tag{13} \]

\[ \dot{\theta}_l = -H \left( 1 - 3w_l \right) \theta_l - \frac{\dot{\omega}_l}{1 + w_l} \theta_l + \frac{\delta P_l / \delta \rho_l \bar{k}^2}{1 + w_l} \delta_l - \frac{a Q_l}{\rho_l} \theta_l, \tag{14} \]

where \( \delta_l \equiv \delta \rho_l / \rho_l \) and \( \theta_l = ik_i v^i_l \) are the density fluctuation and the divergence of fluid velocity, respectively. Note that Eqs. (13) and (14) describe the evolutions of density fluctuations of the perfect fluids without interactions between them. If the interactions between any two fluids are taken into account, these equations should be further modified. Taking the photon-proton interaction to be the example, one has to add the additional term, \( a n_e \sigma_T (\theta_b - \theta_\gamma) \), at the RHS of Eq. (14) when \( l = \gamma \), and the details of the equations can be found in Ref. [62]. In Eqs. (13) and (14), one can observe that the last terms in the two equation slow down the growths of \( \delta_l \) and \( \theta_l \) if \( \nu \) in Eq. (9) is positive.

To show how the running vacuum scenario in Eq. (6) influences the physical observables, we use the open-source program CAMB [60], in which we modify the background density evolutions and the evolution equations of \( \delta \) and \( \theta \) in terms of Eqs. (10), (13) and (14). By taking \( 1 \gg \nu \geq 0 \), most of particles are created at the end of inflation, whereas the energy density from the dark energy decay is tiny, hinting that the RVM shares the same initial condition as that in the ΛCDM model. In addition, the matter-radiation equality \( z_{eq} \) slightly changes, given by

\[ \frac{\rho_m(z)}{\rho_r(z)} \bigg|_{z = z_{eq}} = 1. \tag{15} \]
FIG. 1: The matter power spectrum $P(k)$ as a function of the wavelength $k$ with $\nu = 0$ (solid line), $10^{-3}$ (dashed line), $5 \times 10^{-3}$ (dotted line) and $10^{-2}$ (dash-dotted line), where the boundary conditions are taken to be $\Omega_b h^2 = 2.23 \times 10^{-2}$, $\Omega_c h^2 = 0.118$, $h = 0.68$, $A_s = 2.15 \times 10^{-9}$, $n_s = 0.97$, $\tau = 0.07$ and $\Sigma m_\nu = 0.06$ eV, respectively.

FIG. 2: The CMB temperature power spectra of (a) $l(l+1)C_l/2\pi$ and (b) $\Delta C_l/C_l = (C_l^{\text{RVM}} - C_l^{\Lambda \text{CDM}})/C_l^{\Lambda \text{CDM}}$ with $T = 2.73$ K, where legend is the same as Fig. 1 and the grey points are the unbinned TT mode data from the Planck 2015.

In Fig. 1, we present the matter power spectrum $P(k) \sim \langle \delta_m^2(k) \rangle$ as a function of the wavenumber $k$ with $\nu = 0$ (solid line), $10^{-3}$ (dashed line), $5 \times 10^{-3}$ (dotted line) and $10^{-2}$ (dash-dotted line). As discussed earlier in this section, the matter density fluctuation is diluted by the creation of particles, so that the results of $P(k)$ at large $k$ and $\nu$ in the RVM significantly deviate from that in the $\Lambda$CDM model (solid line).
Fig. 2 shows the CMB temperature spectra of (a) \(l(l + 1)C_l/2\pi\) and (b) \(\Delta C_l/C_l = (C_l^{\text{RVM}} - C_l^{\Lambda \text{CDM}})/C_l^{\Lambda \text{CDM}}\) in the RVM with \(\nu = 0\) (solid line), \(10^{-3}\) (dashed line), \(5 \times 10^{-3}\) (dotted line) and \(10^{-2}\) (dash-dotted line), where the grey points are the unbinned TT mode data from the Planck 2015. We see that the CMB temperature spectra are significantly suppressed in the RVM. The maximum deviations of \(C_l\) from that in the \(\Lambda\)CDM model can be 13.8%, 48.6% and 64.5% with \(\nu = 10^{-3}\), \(5 \times 10^{-2}\) and \(10^{-2}\), respectively. Due to the accurate measurement from the Planck 2015, we can estimate that the allowed range of \(\nu\) should be at the same order of or less than \(O(10^{-3})\). We note that it is important to note there is a degeneracy with spatial curvature when studying dynamical dark energy models. Clearly, it might not be reasonable to retain flat geometry if one wants to get a realistic set of observational constraints. As shown in the literature \([63, 64]\), a positive spatial curvature shifts the CMB temperature spectra to the smaller \(l\) and rises \(C_l\) in the small \(l\) region. The former phenomenon degenerate with our RVM, whereas the later case is not, i.e., the RVM moves the high-\(l\) and keeps the low-\(l\) spectra in Fig. 2a. Moreover, as pointed out in Ref. \([65]\), when curvature is allowed to be a free parameter, the constraints on dark energy dynamics weaken considerably. In this work, we are interested in the curvature-free case and leave the discussion of the spatial curvature to the future work.

IV. OBSERVATIONAL CONSTRAINTS

| Parameter                          | Prior                      |
|------------------------------------|-----------------------------|
| Model parameter \(\nu\)            | \(0 \leq \nu \leq 3 \times 10^{-4}\) |
| Baryon density                     | \(0.5 \leq 100\Omega_bh^2 \leq 10\) |
| CDM density                        | \(10^{-3} \leq \Omega_c h^2 \leq 0.99\) |
| Optical depth                      | \(0.01 \leq \tau \leq 0.8\) |
| Neutrino mass sum                  | \(0 \leq \Sigma m_{\nu} \leq 2 \text{ eV}\) |
| Sound horizon                      | \(0.5 \leq 100\theta_{MC} \leq 10\) |
| Angular diameter distance          | \(2 \leq \ln (10^{10}A_s) \leq 4\) |
| Spectral index                     | \(0.8 \leq n_s \leq 1.2\) |
FIG. 3: One and two-dimensional distributions of $\Omega_b h^2$, $\Omega_c h^2$, $\tau$, $\Sigma m_\nu$, $\nu$ and $\sigma_8$, where the contour lines represent 68% and 95% confidence levels, respectively.

We now perform the open-source **CosmoMC** program [61] with the MCMC method to explore a more precise range for the model parameter $\nu$. The dataset includes the cosmic microwave background radiation (CMBR), combined with the CMB lensing, from *Planck 2015 TT, TE, EE, low-l polarization* [66–68]; baryon acoustic oscillation (BAO) data from 6dF Galaxy Survey [69], SDSS DR7 [70] and BOSS [71]; matter power spectrum data from SDSS DR4 and WiggleZ [72–74], and weak lensing data from CFHTLenS [75]. The priors of the various parameters are listed in Table. I.

In Fig. 3, we show the global fit from the observational data. In Table. II, we list the allowed ranges for various cosmological parameters at 95% confidence level ($\nu$ at 68% one). We find that the best-fit occurs at $\nu = 1.19 \times 10^{-4}$ with $\chi^2_{\text{RVM}} = 13968.8$, which is smaller than $\chi^2_{\text{LCDM}} = 13969.8$ in the $\Lambda$CDM model. This result demonstrates that the RVM with $\Lambda = 3\nu H^2 + \Lambda_0$ is preferred by the cosmological observations, in which $\nu \lesssim 1.54 \times 10^{-4}$.
TABLE II: Fitting results for the RVM with $\Lambda = 3\nu H^2 + \Lambda_0$ and $\Lambda$CDM, where $\chi^2_{\text{Best-fit}} = \chi^2_{\text{CMB}} + \chi^2_{\text{BAO}} + \chi^2_{\text{MPK}} + \chi^2_{\text{lensing}}$, and limits are given at 95% confidence level ($\nu$ is calculated within 68% C.L.).

| Parameter                                      | RVM                  | $\Lambda$CDM          |
|------------------------------------------------|----------------------|------------------------|
| Model parameter ($10^4\nu$)                    | $1.19^{+0.35}_{-1.19}$ (68% C.L.) | --                     |
| Baryon density ($100\Omega_b h^2$)             | $2.23^{+0.02}_{-0.03}$ | $2.23 \pm 0.03$        |
| CDM density ($\Omega_c h^2$)                   | $0.118 \pm 0.002$    | $0.118 \pm 0.002$      |
| Matter density ($\Omega_m$)                    | $0.308^{+0.015}_{-0.013}$ | $0.306 \pm 0.014$     |
| Hubble parameter ($H_0$) ($\text{km/s} \cdot \text{Mpc}$) | $67.58^{+1.14}_{-1.23}$ | $67.87^{+1.07}_{-1.22}$ |
| Optical depth ($\tau$)                         | $6.66^{+2.82}_{-2.68} \times 10^{-2}$ | $6.99^{+2.83}_{-2.77} \times 10^{-2}$ |
| Neutrino mass sum ($\Sigma m_\nu$)             | $< 0.186$ eV         | $< 0.200$ eV           |
| $100\theta_{\text{MC}}$                       | $1.0411 \pm 0.0006$  | $1.0409 \pm 0.0006$    |
| $\ln \left(10^{10} A_s \right)$               | $3.06^{+0.06}_{-0.05}$ | $3.07 \pm 0.05$        |
| $n_s$                                          | $0.970^{+0.007}_{-0.008}$ | $0.970^{+0.007}_{-0.008}$ |
| $\sigma_8$                                     | $0.805^{+0.023}_{-0.027}$ | $0.808^{+0.025}_{-0.026}$ |
| $z_{eq}$                                       | $3345^{+46}_{-44}$    | $3348^{+45}_{-46}$      |
| $\chi^2_{\text{Best-fit}}$                    | 13968.8              | 13969.8                |

is constrained at 68% confidence level. However, the model is unable to be distinguished from the $\Lambda$CDM model within 1$\sigma$ confidence level. In addition, although our result of $\chi^2$ is smaller than that in $\Lambda$CDM, it is clearly not significant due to the large overall values of $\chi^2$ for both models. Comparing to the best fitted value of $\nu = 4.8 \times 10^{-3}$ in Ref. [46], our simulation further lowers the model parameter $\nu$ more than one order of magnitude.

V. CONCLUSIONS

We have studied the RVM with $\Lambda = 3\nu H^2 + \Lambda_0$, in which dark energy decays to both matter and radiation. We have calculated the evolution equations of the matter density fluctuation $\delta(k, a)$ and the divergence of the fluid velocity $\theta(k, a)$ with the linear perturbation theory. We have shown that the decaying dark energy suppresses both $\delta$ and $\theta$, while
the power spectra of the matter density distribution and CMB temperature fluctuation significantly deviate from those in the ΛCDM model. By performing the global fit from the cosmological observations, we have obtained that $\nu \leq 1.54 \times 10^{-4}$ with the best-fit $\chi^2_{\text{RVM}} - \chi^2_{\Lambda\text{CDM}} = -1.0$ at $\nu = 1.19 \times 10^{-4}$. Such a strong constraint on $\nu$ with a small $\chi^2$ difference is due to the TT mode of the CMB measurement, which only allows the physical observables of the modified gravity models slightly deviating from that of the ΛCDM model. This situation can be seen in not only the RVM but also some other dark energy models, such as XCDM and $\phi$CDM [76–78]. In summary, although the RVM perfectly describes the evolution of our universe, the accurate measurement from the Planck 2015 strongly constrains this scenario, and the allowed window of the model parameter $\nu$ is an order of magnitude smaller than those results in Refs. [47–51]. It is clear that the ΛCDM model is hardly to be ruled out by the current cosmological observations yet. Finally, it is interesting to mention that the results from the spatially-flat XCDM and $\phi$CDM models [76–78] also do not agree with those in Refs. [47–51].

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