Analysis of Student’s Understanding of Exponential Concept: A Perspective of Pirie-Kieren Theory

N Syafiqoh, S M Amin, and T Y E Siswono

1 Mathematics Education Program, Postgraduate Program, Universitas Negeri Surabaya, Jl. Ketintang, Surabaya 60231, Indonesia
2 Mathematics Education Program, Postgraduate Program, Universitas Negeri Surabaya, Jl. Ketintang Surabaya 60231, Indonesia

*Corresponding Author: nursyafiqoh16070785028@mhs.unesa.ac.id

Abstract. Solving mathematical problem is frequently used by employing exponential concept, such as logarithm. The aim of the research is to analyze students’ understanding of exponential concept based on Pirie-Kieren theory. The participant of the research is one female student and one male student at the 11th grade. The data collection was conducted by giving two tasks and interviewing with semi-structure interview. The result showed that both students can solve the first task systematically with different process. The processes are correct although the last expression is not simple. For primitive understanding level, they had exponential basic knowledge. They could explain exponential definition by their own statements. At image making and image having level, they had different ideas to solve problem and they were able to deliver their solutions. At property noticing level, they had competences to identify and differentiate the exponential properties for multiplication and division. At formalizing level, they are able to express these properties to mathematics symbol. However they had no algebraic ability for proving exponential properties. At the second task, they couldn’t reach observing level. They answered the problem correctly. Therefore, it can be concluded that the female student and male student’s understanding have already reached formalizing level in the exponential concept.

1. Introduction
Mathematics is one of science which has important role in daily life. Therefore, mathematics has been taught since elementary school. In learning mathematics, students need understanding. National Council of Teachers of Mathematics (NCTM) stated that the students must learn mathematics with understanding, actively building new knowledge from the experience and the prior knowledge [1]. According to Hiebert and Carpenter’s explanation, the goal of mathematics education is to ensure learning with understanding [2]. Similarly, Kastberg explained that there are four important points about understanding. First, the goal of mathematics teaching is students’ understanding. Second, students’ understanding of mathematical concept exists in their mind. Third, what is in the mind of students cannot be known certainly but the workings of their minds can be concluded through working that is showed by evidence. Fourth, when students try to solve mathematical problems, they try to understand it by their own way [3]. In fact, there are still found many difficulties of students in solving mathematics problems, such as geometric problems [4], fraction problems [5], exponential problem [6], and so on. In addition, Confrey and Smith’s research showed that student’s understanding of exponential concept is quite limited [6]. It is caused by a lack of student’s understanding about
exponential concept. A common mistake in this concept is that the student thinks that the exponent is two values of the factors, for example $5^3 = 5 \times 3$ [7]. Exponents can provide a convenient shorthand notation. The symbol $a^m$ represents the product of $m$ factors of $a$ [8]. Moreover, the students also often forget the exponential properties shortly after they have studied and rarely be able to explain why these properties are true [6]. How such understanding occurs can be identified to know whether students understand or not [9]. A growth understanding theory from Pirie & Kieren explains that mathematical understanding process is illustrated like embedded rings (figure 1) [10]. There are eight levels in this theory, namely primitive knowing, image making, image having, property noticing, formalizing, observing, structuring, and inventising.

Figure 1. Growth of mathematical understanding on Pirie-Kieren model

1.1. Pirie Kieren’s Understanding Model
Firstly, Pirie & Kieren defined mathematical understanding by developing a definition of understanding by Von Glasersfeld which stated that the construction of individual knowledge is needed during understanding process [11]. Pirie & Kieren revealed that the growth of mathematical understanding is as a whole, dynamic, levelled but non-linear, transcendently recursive process [10]. There are eight understanding levels within the growth of understanding. The first is primitive knowing. Primitive knowing is a level of students’ understanding in using their preliminary knowledge to understand a new concept. The second level is image making. In this level, students try to make an image of the concept by using their prior knowledge with physical activities. Wright explained that the images can be pictures, symbols, words, or memories of events [12]. The next level is image having. In this level, students use a mental construction about a topic without having to perform certain activities that lead to the topic. It means that students have had ideas or images in their mind. The fourth level is property noticing. In this level, students use the ideas that had been obtained to establish the relevant properties about a topic. Formalizing is the next level. Students generalize what he has gained in the previous level into formal definition or algorithm about the topic. The sixth level is observing. Observing is a level where students coordinate formal activities on the formalizing level so they will be able to use it on the related problems faced. The next level is known as structuring. Structuring refers to the student's ability in proving theorem with a logical argument. The last level is inventising. Inventising is a level of students understanding which had gained a full understanding of a concept and they can ask a question which can generate a new concept. The four inner levels are described as informal levels, whereas the four outer levels are described as formal understanding of student [11]. Based on this theory, the growth of understanding does not always work in the same direction. A person understands often returns to the previous level to advance to the next level of understanding. The process is called by folding back.

1.2. Purpose of the study
The study was carried out to address the following question: how is the student’s mathematical understanding level based on Pirie-Kieren theory in the exponential concept? Thus, based on the research question, the aim of this research is to analyze 11th grade students’ understanding of
exponential concept in female and male students based on Pirie-Kieren theory. The researcher aimed to observe the students’ constructional development of the exponential concept restricted in exponential properties for multiplication and division.

2. Method
The descriptive method was used in the study. This method aims to describe students’ understanding level based on Pirie-Kieren Theory about exponential concept. The participant of the research is one female student and one male student at 11th grade who had middle mathematical ability. The participants were selected based on the mathematical score of the report and the score should be less than five points. The researcher selected these students because they had learnt about exponential in 9th grade and 10th grade. The students were given a test focused on exponential concept especially exponential properties for multiplication and division and then they were given semi-structured task based interviews. As a result, this research is delivered descriptively based on test result and interviews of participants.

In this study, the research data were collected by giving a test about exponential concept to participants. The test consists of two tasks about exponential properties for multiplication and division. They were asked to simplify the exponential form \(3^2x^4y^{-2}\) and were asked to determine the values \(a\) and \(b\) in the equation of \(a^b = 2^{22} - 2^{21}\). After they completed the test, they were given semi-structured task based interviews. The interviews were constructed to identify student’s mathematical understanding based on Pirie-Kieren theory. The process of the observation was equipped with the digital audio recorder. The recorder captures all of information which is given by participants during interviews, so that there is no left information.

The test of students’ results and the interview results were used as data sources. All data collected were analysed using descriptive method of Miles and Huberman. The stages for analyzing qualitative research data are data condensation, data display and conclusion drawing/verification. In the condensation stage, the data has been obtained from interview result which then selected, focused and simplified so that they will be appropriate with the purpose of research. After that, the next step is presenting the data in the form of narrative text and concerning understanding levels indicators described previously. Then, the data were interpreted for making conclusion.

3. Result and Discussion
Firstly, the participants were given a test consist of two tasks about exponential concept. The researcher supervised the participants during completing the test. After completing, they were given interviews based on test result. Students’ understanding level was identified based on test result and task based interviews and then analyzed according to the Pirie-Kieren Theory. Participants of this research are Kevin (K) as a male student and Nina (N) as female student. Both participants are pseudonym.

3.1. Female student’s understanding level
Student’ primitive knowing of exponential concept was identified based on her test result and task based interviews. Nina (N) had basic knowledge about exponential such as multiplication and she could express what the meaning of exponential. The following dialogue which occurred between Nina (N) and Researcher (R) supports this finding:
   R: Nina, can you tell me what is asked in the question number 1?
   N: Simplify the exponential form (Reading the question)
   R: What is your idea to solve this?
   N: I solve this by using exponential properties (Explained her answer step by step systematically) based on exponential properties.
   R: Okay, before we continue to the next problem, I want to know, what do you know about exponential definition?
N: The number which is multiplied by the number itself, for example $3^2 = 3 \times 3$
R: If $a^n$, what it means?
N: $a$ is multiplied as much as $n$.

Based on the short conversation above, she understood about what is the meaning of exponential by using her own statement. Nina could solve the given test systematically. The final result is correct; however it is not simple expression. The figure below depicts Nina’s answers. Image making level was showed based on students’ test result are as in the following:

![Figure 2. Nina’s answer for the first task](image)

In the figure 2, Nina spell out 6 became 2 times 3. She did it because there are numerators $3^2$ so that she could use the exponential property of the division to simplify the exponential form. In this level, student is able to acquire an idea to solve the given problem. It is important because the idea helps them early to solve the problem and make them to think continually during the working process. So, this level is very important because it is the key for mathematical understanding [13]. Similar with this, Borgen revealed that image making is an important and active process for the growth of understanding [14]. Image having level can be showed based on conversation between student and researcher above. The dialogue above indicated that Nina knew what the meaning of the question is. She knew what she should do for solving the problem. She also could explain her answer step by step systematically and clearly. She solved the given problem by using exponential property of division. She had has images in her mind for solving the problem by using exponential properties. In the property noticing level, she could distinguish about multiplication and division properties of exponential form. She told that the multiplication property of exponential form is by adding the exponents with notes of both bases which are actually the same, whereas for division of exponential form property is by subtracting the exponents with notes of both bases which are the same. For the formalizing level, she could generalize the exponential properties of multiplication and division into the mathematical symbols. She expressed the exponential properties for multiplication as $a^m \times a^n = a^{m+n}$ and for division as $a^m \div a^n = a^{m-n}$. According to Skemp, an ability to connect mathematical symbolism and notation with relevant mathematical ideas and to combine these ideas into chains of logical reasoning is called by formal understanding or logical understanding [15]. For checking the ability for proving the properties, the researcher asked Nina to prove it. She didn’t prove it algebraically, but she just showed that the value of equation in the left side is equal to the right side. The second problem was given to see students’ understanding on the observing. Based on her answer below, it showed that she couldn’t solve it correctly. She also didn’t reach the highest level because she was incapable to create new questions that lead to new concept.

![Figure 3. Nina’s answer for second task](image)

3.2. Male student’s understanding
Like Nina, Kevin had basic knowledge in completing the test. He could state the meaning of exponential in different statement compared to Nina. He gave an example about the meaning of
He expressed that $2^3 = 2 \times 2 \times 2$. He said that if $a^n$, it means $a$ times $a$ times $a$ until $n$. He solved the problem in different way than Nina. He solved the first task systematically but the final result is not simplest expression. The answer can be seen as follows.

$$\frac{9}{216} x^2 y^3$$

He should simplify $\frac{9}{216}$, but he did not.

For the second problem, he couldn’t solve it correctly like the figure below.

**Figure 4.** Kevin’s answer for first task

In the figure 4, Kevin separated the coefficients, variables of $x$ and variables of $y$ in each bracket. He used exponential definition for outlining the coefficient and used exponential properties for simplified $x$ and $y$ variables. But, the final result is not the simplest expression. His final answer is $\frac{9}{216} x^2 y^3$. He should simplify $\frac{9}{216}$, but he did not.

For image having, it is showed in the excerpt of dialogue between Researcher (R) and Kevin (K) as follows:

R: What is asked in the question?
K: Simplify the exponent
R: What is your idea to solve this?
K: I think it is must be separated between coefficient, $x$ variables and $y$ variables. If the denominator is changed its position upwards, so the sign of the power is also changes. Positive becomes negative. Negative becomes positive.

If we consider Kevin’s answer, he solved the problem by using exponential property of division. Actually, according to the conversation between Researcher and Kevin above, the image in his mind is not about exponential properties in solving the problem. But, rather than sign the power. If denominator is changed become numerator, the sign of power is also converted into the opponent’s sign, positive becomes negative and vice versa. However, his answer was correct and he could explain his answer step by step systematically and clearly.

4. Conclusion

Based on data analysis, the female student and male student could solve the first task systematically. The procedure of the solution is correct but the final expression is not simple. They had basic knowledge about exponential concept by explaining exponential definition in their own statements. Although they explained in different ways, their intentions were the same. They understood what the meaning of exponent is. For image making level, both students were able to acquire an idea to solve the given problem with different process and they could explain their solutions clearly. So they had indicator for image having.

They also could distinguish the properties of multiplication and division in the exponential form. For the formalizing level, they could generalize the exponential properties of multiplication and division into the mathematical symbols. When the researcher asked both students
They had no ability to prove it by providing a logical argument. For the second task, both students couldn’t solve it. Therefore, it can be concluded that the female student and male student’s understanding have reached formalizing level in the exponential concept. This study is useful for researchers who are interested on student’s mathematical understanding based on Pirie-Kieren theory. In the next study, it is suggested that the next research should more communicative and more detail in exploring students’ understanding level. It is caused by student’s understanding which is not written. In addition, this study result is useful for teachers for planning their exponential lesson.

5. References

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