Applying Set Optimization to Weak Efficiency

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Abstract

Since the seminal papers by Giannessi, an interesting topic in vector optimization has been the characterization of (weak) efficiency thorough Minty and Stampacchia type variational inequalities. Several results have been proved to extend those known for the scalar case. However, in order to introduce a proper definition of variational inequality, some assumptions are usually made that may eventually be questioned.

We find two major drawbacks in the papers we considered, that arise when defining generalized derivatives for vector–valued functions. First, some authors introduce set–valued derivatives for single–valued problems, thus completely changing the setting of the problem. Second, when dealing with Dini–type derivatives, infinite elements may occurs. The approach to handle this problem is not yet uniquely defined in the literature, therefore, when considered, the definition proposed may seem arbitrary.

Indeed these problems are strictly related with the lack of a complete order in the image space of a vector–valued function. We propose an alternative approach to study vector optimization, by considering an set–valued counterpart defined with values in a conlinear space. The structure of this space allows to overcome the previous difficulties and to obtain variational inequality characterization of weak efficiency as a straightforward application of scalar arguments.

1 Introduction

Vector optimization has been extensively studied in the literature. Since the seminal papers by Giannessi [11, 12] one of the issues within this field has been the use of differentiable variational inequalities to characterize weak efficient solutions of a primitive optimization problem, see e.g. [5, 10]. In this paper we consider real vector spaces $X$ and $Z$, where $Z$ is locally convex and Hausdorff, with topological dual $Z^*$, and $\mathcal{P}(Z)$ the power set of $Z$, including $\emptyset$ and $Z$ as elements. The vector optimization problem is

$$\min \psi(x), \ x \in S \quad \text{(VOP)}$$

where $\psi : S \subseteq X \rightarrow Z$ is a vector-valued function and $S$ is a non empty subset of $X$. Throughout the paper we denote by $\mathcal{U}$ the set of all closed, convex and balanced $0$ neighborhoods in $Z$, a $0$–neighborhood base of $Z$ and by $\text{cl} A$, $\text{co} A$ and $\text{int} A$, the closed hull, the

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convex hull and the topological interior of of a set $A \subseteq Z$, respectively. The conical hull of a set $A$ is $\text{cone} \ A = \{ta \mid a \in A, \ 0 < t\}$. To define a solution concept to (VOP) we introduce a preorder on $Z$ by a closed convex cone $C \neq Z$ with nonempty topological interior, $\text{int} \ C \neq \emptyset$. As usual, by $z_1 \leq_C z_2$ we mean $z_2 \in \{z_1\} + C$. The (negative) dual cone of $C$ is the set $C^- = \{z^* \in Z^* \mid \forall z \in C : z^*(z) \leq 0\}$. Since $\text{int} \ C \neq \emptyset$ is assumed, there exists a weak$^*$ compact base $W^*$ of $C^-$, i.e. a convex subset with $C^- \setminus \{0\} = \text{cone} W^*$ with $z^*, tz^* \in W^*$ implying $t = 1$ and any net in $W^*$ has a weak$^*$ convergent subnet, compare [14, Lemma 2.2.17].

Most of the results attained in the field of vector variational inequalities refer to weak solutions of (VOP), that is an element $x^* \in S$ such that $\psi(x)$ is a weakly efficient element in the image set $\psi[X] = \{\psi(x) \mid x \in S\}$, i.e.

$$\forall z \in \psi[X] : \psi(x^*) \not\in \{z\} + \text{int} \ C.$$ 

Variational inequalities are usually interpreted as directional derivatives of some kind of the vector-valued function $\psi(\cdot)$. Whether the derivative is computed at $x^*$ or at some other $S \ni x \neq x^*$, the inequality is in the form of Stampacchia or Minty, respectively. Relations between the set of weak efficient solutions of (VOP) and those of the associated inequalities have been proved in various papers compare e.g. [2, 4, 28].

However, besides the differentiable case studied by Giannessi, further generalizations of the directional derivatives classically need to use upper or lower limits of differential quotients. Since the ordering in vector spaces generally lacks completeness, the definitions proposed become somewhat awkward. Indeed we see two major drawbacks in the papers we considered. First, some approaches provide a set valued derivative for single valued problems (see e.g. [10]), thus completely changing the image space of (VOP). Second, when considering generalized directional derivatives, infinite values come easily. To the best of our knowledge, this problem is either solved by introducing arbitrary notions of infinite elements for vector spaces (e.g. [10]), or avoided (e.g. [2]).

Following the lines of [20] we propose a 'fresh look' to the problem, replacing (VOP) by an equivalent set optimization problem. This approach allows to overcome the ambiguity of infinite elements, dealing with a fully set valued problem, and gaining a deeper insight on the original vector valued problem. We introduce a set valued extension, $\psi^C$, of any vector valued map $\psi$, mapping $X$ into an order complete space $G^\Delta$, as studied in [7, 8, 9, 16, 17, 24]. Inspired by this, we study the general set-optimization problem for this class of function, defining directional derivatives, mapping onto $G^\Delta$, by means of upper or lower limits of differential quotients in the image space. We obtain necessary and sufficient conditions in terms of Stampacchia and Minty variational inequalities, that allow us to characterize the set of weak solutions to (VOP). Eventually, results proved in [6, 12] follow, as a special cases, overcoming the necessity to introduce infinite elements in $Z$ and to study the topology of the extended vector space $\tilde{Z}$, that appears in the cited paper.

The remainder of the paper is organized as follows. In Section 2, we introduce the general setting and the basic notation. A definition of inf–residuated conlinear structure is given and some results for the special case of $G^\Delta$ are proven for subsequent reference. Section 3 is devoted to the concept of upper and lower Dini directional derivatives for functions mapping onto an order complete, inf–residuated conlinear space. We show that these concepts generalize the original definition for proper scalar functions, compare e.g. [13]. The final Section 4 collects our main results, applying the general scheme to (VOP). In this final section, we restrict
ourselves to the case of convex functions in order to achieve a greater simplicity of the arguments, rather than greatest possible generality, leaving the more general case for further research.

2 Setting

In the sequel, given any vector valued function \( \psi : S \subseteq X \rightarrow Z \), we define its set valued extension \( \psi^C : X \rightarrow \mathcal{P}(Z) \) as the function mapping \( x \) to the upper Dedekind cut of \( \psi(x) \) with respect to \( C \), namely

\[
\psi^C(x) = \begin{cases} 
\{\psi(x)\} + C & \text{if } x \in X \\
\emptyset & \text{elsewhere}
\end{cases}
\]

Obviously, \( \text{dom } \psi^C = S \).

Images of \( \psi^C : X \rightarrow \mathcal{P}(Z) \) are closed convex sets, closed under the addition with the ordering cone \( C \), that is \( \psi^C(x) = \text{cl co} (\psi^C(x) + C) \). Therefore we restrict to the set

\[
\mathcal{G}(Z,C) = \{A \in \mathcal{P}(Z) \mid A = \text{cl co} (A + C)\}
\]

as natural image space for the set-valued functions along this paper. Properties of \( \mathcal{G}(Z,C) \) have been extensively studied in recent years. First we recall that the ordering in \( Z \) can be extended to the power set of \( Z \) (compare [15] and the references therein) by setting

\[
A_1 \preceq A_2 \iff A_2 + C \subseteq A_1 + C
\]

for all \( A_1, A_2 \subseteq Z \). Especially, the relation \( \preceq \) coincides with \( \supseteq \) on the subset \( \mathcal{G}(Z,C) \) and \( (\mathcal{G}(Z,C), \preceq) \) is a complete lattice, see e.g. [17]. For any subset \( A \subseteq \mathcal{G}(Z,C) \), supremum and infimum of \( A \) in \( \mathcal{G}(Z,C) \) are given by

\[
\inf A = \text{cl co} \bigcup_{A \in A} A; \quad \sup A = \bigcap_{A \in A} A.
\]

When \( A = \emptyset \), we agree that \( \inf A = \emptyset \) and \( \sup A = Z \). Hence \( \mathcal{G}(Z,C) \) possesses a greatest and smallest element \( \inf \mathcal{G}(Z,C) = Z \) and \( \sup \mathcal{G}(Z,C) = \emptyset \). The Minkowsky sum and multiplication with non-negative reals need to be slightly adjusted to provide operations on \( \mathcal{G}(Z,C) \). We define

\[
\forall A, B \in \mathcal{G}(Z,C) : \quad A \oplus B = \text{cl} \{a + b \in Z \mid a \in A, b \in B\};
\]

\[
\forall A \in \mathcal{G}(Z,C), \forall 0 < t : \quad t \cdot A = \{ta \in Z \mid a \in A\}; \quad 0 \cdot A = C.
\]

Note that \( 0 \cdot \emptyset = 0 \cdot Z = C \) and \( \emptyset \) dominates the addition in the sense that \( A \oplus \emptyset = \emptyset \) is true for all \( A \in \mathcal{G}(Z,C) \). Moreover, \( A \oplus C = A \) is satisfied for all \( A \in \mathcal{G}(Z,C) \), thus \( C \) is the neutral element with respect to the sum.

As a consequence,

\[
\forall A \subseteq \mathcal{G}(Z,C), \forall B \in \mathcal{G}(Z,C) : \quad B \oplus \inf A = \inf \{B \oplus A \mid A \in A\},
\]
or, equivalently, the inf–residual \( A \uparrow B = \inf \{ M \in G(Z,C) \mid A \preceq B \uplus M \} \) exists for all \( A, B \in G(Z,C) \). It holds (compare [17, Theorem 2.1])

\[
A \uparrow B = \{ z \in Z \mid B + \{ z \} \subseteq A \};
A \preceq B \uplus (A \uparrow B).
\]

Overall, the structure of \( G^\Delta = (G(Z,C), \oplus, \cdot, C, \preceq) \) is that of an order complete inf–residuated conlinear space, compare also [7, 8, 16] for a more detailed study of this structure.

The recession cone of a nonempty closed convex set \( A \subseteq Z \) is the closed convex cone \( 0^+ A = \{ z \in Z \mid A + \{ z \} \subseteq A \} \), compare [29, p.6]. By definition, \( 0^+ \emptyset = \emptyset \) is assumed. If \( A \in G^\Delta \setminus \{ \emptyset \} \), then \( 0^+ A = A \uparrow A \) and \( C \subseteq 0^+ A \) are satisfied. Especially, \( \text{int} (0^+ A) \neq \emptyset \) and \( (0^+ A)^- \subseteq C^\ominus \), hence \( W^* \cap (0^+ A)^- \) is a weak* compact base of \( (0^+ A)^- \).

We first recall the notion of conlinear spaces as introduced in [16]. References and details on structural properties of conlinear spaces and inf–residuation can be found in [16, 17].

**Definition 2.1** A nonempty set \( Y \) together with two algebraic operations \( +, \cdot : Y \times Y \to Y \) and \( \cdot : \mathbb{R}_+ \times Y \to Y \) is called a conlinear space with neutral element \( \theta \) provided that

(C1) \( (Y, +, \theta) \) is a commutative monoid with neutral element \( \theta \),

(C2) The operations are compatible: (i) \( \forall w_1, w_2 \in Y, \forall r \in \mathbb{R}_+ : r \cdot (w_1 + w_2) = r \cdot w_1 + r \cdot w_2 \), (ii) \( \forall w \in Y, \forall r, s \in \mathbb{R}_+: s \cdot (r \cdot w) = (rs) \cdot w \), (iii) \( \forall w \in Y : 1 \cdot w = w \), (iv) \( \forall w \in Y : 0 \cdot w = \theta \).

Subsequently, these operations are referred to as addition and multiplication, respectively.

A conlinear space \( (Y, +, \cdot, \theta) \) together with an order relation \( \preceq \) on \( Y \) is called partially ordered, lattice ordered or order complete conlinear space provided that \( (Y, \preceq) \) has the respective structure and the order is compatible with addition and multiplication, that is

(C3) (i) \( \forall w, w_1, w_2 \in Y, w_1 \preceq w_2 \) implies \( w_1 + w \preceq w_2 + w \), and (ii) \( \forall w, w_1, w_2 \in Y, w_1 \preceq w_2, r \in \mathbb{R}_+ \) implies \( r \cdot w_1 \preceq r \cdot w_2 \).

A partially ordered conlinear space \( (Y, +, \cdot, \theta, \preceq) \) is called inf–residuated, when for all \( v, w \in Y \) the element \( w \downarrow v = \inf \{ u \in Y \mid w \preceq v + u \} \) exists. In this case, \( w \downarrow v \) is called the inf–residual of \( w \) and \( v \).

**Example 2.2** Let us consider \( Z = \mathbb{R}, C = \mathbb{R}_+ \). Then \( G(Z,C) = \{ [r, +\infty) \mid r \in \mathbb{R} \} \cup \{ \mathbb{R} \} \cup \{ \emptyset \} \), and \( G^\Delta \) can be identified (with respect to the algebraic and order structures which turn \( G(\mathbb{R}, \mathbb{R}_+) \) into an ordered conlinear space and a complete lattice admitting an inf-residuation) with \( \mathbb{R} = \mathbb{R} \cup \{ \pm \infty \} \) using the 'inf-addition' \( + \) (see [17, 26]). The inf-residuation on \( \mathbb{R} \) is given by

\[
r \downarrow s = \inf \{ t \in \mathbb{R} \mid r \leq s + t \}
\]

for all \( r, s \in \mathbb{R} \), compare [17] for further details.

A partially ordered conlinear space is inf–residuated, if and only if for all \( w \in Y \) and all \( A \subseteq Y \) such that \( \inf A \) exists, it holds \( (w + \inf A) = \inf \{ w + a \mid a \in A \} \) (compare [17, Theorem 2.1]).

**Lemma 2.3** Let \( Y \) be an order complete inf-residuated conlinear space, \( a, b, c, d \in Y \) and \( 0 < t, s \in \mathbb{R} \). Then

\[
(ta + sb) \downarrow (tc + sd) \preceq t(a \downarrow c) + s(b \downarrow d).
\]
PROOF. Since the ordering in $Y$ is compatible with the algebraic operations and $t(a - b) = ta - tb$ is true for all $0 < t$, without loss of generality we can assume $t = s = 1$. As $Y$ is inf–residuated,

\[(a - c) + (b - d) = \inf \{ u \in Y \mid a \not\leq c + u \} + \inf \{ v \in Y \mid b \not\leq d + v \} \]

\[= \inf \{ u + v \mid v \in Y \mid b \not\leq d + v \} \in Y \mid a \not\leq c + u \} \]

\[= \inf \{ u + v \in Y \mid a \not\leq c + u, b \not\leq d + v \} \]

but $a \not\leq c + u$ and $b \not\leq d + v$ together imply

\[a + b \not\leq (c + u) + (d + v) = (c + d) + (u + v), \]

hence

\[a + b \not\leq (c + d) + ((a - c) + (b - d)), \]

and equivalently

\[(a + b) - (c + d) \not\leq (a - c) + (b - d). \]

Since throughout the paper we only use the conlinear spaces $\mathbb{R}$ and $\mathcal{G}^\Delta$, we focus our attention on them. Especially $\mathcal{G}^\Delta$ enjoys some additional properties. Indeed, since each element of $\mathcal{G}^\Delta$ is closed and convex and $A = A + C$, by a separation argument we have

\[
\forall A \in \mathcal{G}^\Delta : \quad A = \bigcap_{z^* \in W^*} \{ z \in Z \mid -\sigma(z^*|A) \leq -z^*(z) \}, \tag{2.1}
\]

where $\sigma(z^*|A) = \sup \{ z^*(z) \mid z \in A \}$ is the support function of $A$ at $z^*$.

**Remark 2.4** $A = \emptyset$ if and only if there exists $z^* \in W^*$ such that $-\sigma(z^*|A) = +\infty$, or equivalently if the same holds true for all $z^* \in W^*$.

The right hand side in (2.1) provides a scalarization of elements $A \in \mathcal{G}^\Delta$. The following equivalent formulation holds as well

\[
\forall A \in \mathcal{G}^\Delta \setminus \{ \emptyset \} : \quad A = \bigcap_{z^* \in W^*, -\sigma(z^*|A) \in \mathbb{R}} \{ z \in Z \mid -\sigma(z^*|A) \leq -z^*(z) \}, \tag{2.2}
\]

Applying these characterizations, scalarized counterparts of infimum and supremum of a subset of elements in $\mathcal{G}^\Delta$ are provided.

**Lemma 2.5** [27, Proposition 3.5] Let $A \subseteq \mathcal{G}^\Delta$ be a set, then

\[\inf A = \bigcap_{z^* \in W^*} \{ z \in Z \mid \inf \{ -\sigma(z^*|A) \mid A \in A \} \leq -z^*(z) \} \]

\[\forall z^* \in W^* : \quad -\sigma(z^*|\inf A) = \inf \{ -\sigma(z^*|A) \mid A \in A \}. \]

**Lemma 2.6** Let $A \subseteq \mathcal{G}^\Delta$ be a set, then

\[\sup A = \bigcap_{z^* \in W^*} \{ z \in Z \mid \sup \{ -\sigma(z^*|A) \mid A \in A \} \leq -z^*(z) \} \]

\[\forall z^* \in W^* : \quad -\sigma(z^*|\sup A) \geq \sup \{ -\sigma(z^*|A) \mid A \in A \}. \]
Proof. For any set $A \subseteq \mathcal{G}^\triangle$, the supremum of $A$ is given by $\sup A = \bigcap_{A \in A} A$, and by the scalarization formula (2.1), it holds

$$\sup A = \bigcap_{A \in A} \bigcap_{z^* \in W^*} \{ z \in Z \mid -\sigma(z^*|A) \leq -z^*(z) \}$$

$$= \bigcap_{z^* \in W^*} \bigcap_{A \in A} \{ z \in Z \mid -\sigma(z^*|A) \leq -z^*(z) \}$$

$$= \bigcap_{z^* \in W^*} \{ z \in Z \mid \sup \{-\sigma(z^*|A) \mid A \in A \} \leq -z^*(z) \}. $$

This implies

$$\{ z \in Z \mid \sup \{-\sigma(z^*|A) \mid A \in A \} \leq -z^*(z) \} \subseteq \sup A$$

for all $z^* \in W^*$, hence $-\sigma(z^*|\sup A) \supseteq \sup \{-\sigma(z^*|A) \mid A \in A \}$ is proven. □

The following result has been proved in [17].

Lemma 2.7 [17, Proposition 5.20] Let $A, B \in \mathcal{G}^\triangle$, then

$$A \prec B = \bigcap_{z^* \in W^*} \{ z \in Z \mid (-\sigma(z^*|A)) \prec (-\sigma(z^*|B)) \leq -z^*(z) \};$$

$$\forall z^* \in W^*: -\sigma(z^*|A) \prec B \geq (-\sigma(z^*|A)) \prec (-\sigma(z^*|B)).$$

Remark 2.8 Let $A, B \in \mathcal{G}^\triangle$ be given with $A = \{a\} + C$, $a \in Z$. Then $0^+ A = C$ and $-\sigma(z^*|A) = -z^*(a)$ is satisfied for all $z^* \in W^*$. Moreover, $B \prec A = B + \{-a\}$ is true, hence

$$\forall z^* \in W^*: -\sigma(z^*|B \prec A) = (-\sigma(z^*|B)) \prec (-\sigma(z^*|A)).$$

The recession cone $0^+ A$ of any element $A \in \mathcal{G}^\triangle$ is related to the values of the support function of $A$ as the following two lemmas show.

Lemma 2.9 Let $A \in \mathcal{G}^\triangle$ be a nonempty set, then

$$0^+ A = \{ z \in Z \mid \forall z^* \in W^*: -\sigma(z^*|A) = -\infty \vee 0 \leq -z^*(z) \}. \quad (2.3)$$

Especially, for all $A \subseteq \mathcal{G}^\triangle$, either $A = \emptyset$, or

$$0^+ A = \bigcap_{z^* \in W^*} \{ z \in Z \mid 0 \leq -z^*(z) \}. \quad (2.4)$$

Proof. Assume $z \notin 0^+ A$, then either $A = \emptyset$ or there exists a $z^* \in Z^*$ such that $\sigma(z^*|A) < z^*(a + z)$ is satisfied for some $a \in A$. As $z^*(a + z) \leq \sigma(z^*|A) + z^*(z)$, this implies $-z^*(z) < 0$ and $-\sigma(z^*|A) \neq -\infty$. But as $C \subseteq 0^+ A$, $-\sigma(z^*|A) \neq -\infty$ implies $z^* \in C^- \setminus \{0\}$. Especially, $z$ is not an element of the left hand side of either (2.3) or (2.4), which obviously describe the same set.

On the other hand, assume $z \in 0^+ A$, then $A$ is nonempty and $A + \{z\} \subseteq A$, hence for all $z^* \in Z^*$ it holds $\sigma(z^*(A + \{z\}) \leq \sigma(z^*|A)$, hence $\sigma(z^*|A) + z^*(z) \leq \sigma(z^*|A)$. This implies that either $-\sigma(z^*|A) = -\infty$ or $0 \leq -z^*(z)$ is true for all $z^* \in Z^*$ and thus especially for $z^* \in C^- \setminus \{0\}$. □
Lemma 2.10 Let $A \in \mathcal{G}^\triangle$ be a nonempty set, then

\[ \{ z^* \in C^- \setminus \{0\} \mid -\sigma(z^*|A) \in \mathbb{R} \} \subseteq (0^+ A)^- \subseteq C^-. \]

Proof. Since $C \subseteq 0^+ A$ is always satisfied, the last inclusion is trivial. Now take $z^* \in C^- \setminus \{0\}$ such that $-\sigma(z^*|A) \in \mathbb{R}$ and $z \in 0^+ A$, i.e. $A + z \subseteq A$. Then

\[ -\sigma(z^*|A) \leq -\sigma(z^*|A + z) = -\sigma(z^*|A) - z^*(z) \]

implies $0 \leq -z^*(z)$, in other words $z^* \in (0^+ A)^-$. \qed

In the following proposition, we state some implications that are used in the main proofs.

Proposition 2.11 Let $A, B \in \mathcal{G}^\triangle$ be two sets, then

(a) $A \nsubseteq \text{int } B$ implies

(b) $\exists z^* \in W^*$: $-\sigma(z^*|A) \leq -\sigma(z^*|B) \neq -\infty$ which in turn implies

(c) $\forall U \in \mathcal{U}$: $A \oplus U \nsubseteq B$.

Proof. As $\text{int } C \neq \emptyset$, $B = \emptyset$ is equivalent to $\text{int } B = \emptyset$. In this case, $A \nsubseteq \text{int } B$ implies $A \neq \emptyset$, hence the inequality $-\sigma(z^*|A) \leq -\sigma(z^*|B) \neq -\infty$ is satisfied for all $z^* \in W^*$. Otherwise, $\text{int } B \neq \emptyset$ is a convex set and by a separation argument there exists a $a \in A$ such that $-z^*(a) \leq -\sigma(z^*|B)$ is true for some $z^* \in Z^* \setminus \{0\}$. But this implies $-\sigma(z^*|B) \neq -\infty$ and without loss $z^* \in W^*$.

For the second implication, consider that for any $U \in \mathcal{U}$ and any $z^* \in W^*$, $-\sigma(z^*|A + U) = (-\sigma(z^*|A)) + (-\sigma(z^*|U))$. Especially, if $-\sigma(z^*|A) \leq -\sigma(z^*|B) \neq -\infty$, then

\[ -\sigma(z^*|A + U) < -\sigma(z^*|A) \leq -\sigma(z^*|B), \]

implying $A + U \nsubseteq B$. \qed

The reverse implications do not hold in general, as the following example shows.

Example 2.12 (a) Let $Z = \mathbb{R}^2$ and $C = B = \mathbb{R}_+^2$. Setting $A = \{(x, y) \in Z \mid \frac{1}{2} \leq y, 0 < x \}$, then $A \subseteq \text{int } B$ but $-\infty = -\sigma((0, 1)^T|A) < -\sigma((0, 1)^T|B)$ is true.

(b) Let $Z = \mathbb{R}^2$ and $C = \text{cl } \text{cone } \{(0, 1)^T\}$. Set $A = \{(x, y) \in Z \mid x^2 \leq y \}$ and

\[ \forall n \in \mathbb{N}: \quad B_n = \left\{ (x, y) \in Z \mid \max \left\{ 2nx - n^2 - \frac{1}{n}, -2nx - n^2 - \frac{1}{n} \right\} \leq y \right\}, \]

and $B = \bigcap_{n \in \mathbb{N}} B_n$. Then for all $U \in \mathcal{U}$, $A + U \nsubseteq B$ is satisfied, but $-\sigma(z^*|B) = -\sigma(z^*|A) = -\infty$ is satisfied for $z^* \in \text{cone } \{(-1, 0)^T, (0, -1)^T\}$ while for all other $z^* \in W^*$ it holds $-\sigma(z^*|B) < -\sigma(z^*|A)$.  

7
Given a conlinear, inf–residuated space $Y$ and a function $f : X \to Y$ we denote the 
(effective) domain by the set $\text{dom } f = \{ x \in X \mid f(x) \neq \text{sup } Y \}$. The image set of a subset $A \subseteq X$ through $f$ is denoted by $f[A] = \{ f(x) \in Y \mid x \in A \} \subseteq Y$. A function $f : X \to Y$ is called proper, if $\text{dom } f \neq \emptyset$ and $\text{inf } Y \notin f[X]$.

The setting of conlinear spaces allows for intuitive definitions of properties of set–valued 
maps, such as convexity and homogeneity. A function $f : X \to Y$ is called convex when
\[
\forall x_1, x_2 \in X, \forall t \in (0, 1) : f(t x_1 + (1-t)x_2) \preceq t f(x_1) + (1-t) f(x_2).
\]
Moreover, $f$ is positively homogeneous (see e.g. [15]) when
\[
\forall 0 < t, \forall x \in X : f(tx) = tf(x),
\]
and it is called sublinear if it is positively homogeneous and convex. In Section 3, we motivate 
the choice of $t \neq 0$.

Using the setting of the paper, $Y = \mathcal{G}^\oplus$ we prove some more properties of set–valued 
functions $f : X \to \mathcal{G}^\oplus$. We introduce the family of extended real-valued functions $\varphi_{f,z^*} : X \to \mathbb{R} \cup \{\pm \infty\}$ defined by
\[
\varphi_{f,z^*}(x) = \inf \{ -z^*(z) \mid z \in f(x) \}, \quad z^* \in C^- \setminus \{0\}
\]
as the family of (linear) scalarizations for $f$. Some properties of $f$ are inherited by its 
scalarizations and vice versa. For instance, $f$ is convex if and only if $\varphi_{f,z^*}$ is convex for each $z^* \in W^*$. Moreover, (2.1) admits the following representation
\[
\forall x \in X : f(x) = \bigcap_{z^* \in W^*} \{ z \in Z : \varphi_{f,z^*}(x) \leq -z^*(z) \}.
\]

To state our main results, we need a notion of lower semicontinuity of set valued functions 
$f : X \to \mathcal{G}^\oplus$. The following definition recall some notion previously used in the literature, 
compare [21, 23, 25].

**Definition 2.13** (a) Let $\varphi : X \to \mathbb{R}$ be a function, $x_0 \in X$. Then $\varphi$ is said to be lower semicontinuous (l.s.c.) at $x_0$, iff
\[
\forall r \in \mathbb{R} : \quad r < \varphi_{f,z^*}(x_0) \Rightarrow \exists U \in \mathcal{U} : \forall u \in U : r < \varphi_{f,z^*}(x_0 + u).
\]

(b) Let $f : X \to \mathcal{G}^\oplus$ be a function, $M^* \subseteq C^- \setminus \{0\}$. Then $f$ is said $M^*$–lower semicontinuous ($M^*$–l.s.c.) at $x_0$, iff $\varphi_{f,z^*}$ is l.s.c. at $x_0$ for all $z^* \in M^*$.

(c) Let $f : X \to \mathcal{G}^\oplus$ be a function. If
\[
f(x) \preceq \liminf_{u \to 0} f(x + u) = \bigcap_{U \in \mathcal{U}} \text{cl co} \bigcup_{u \in U} f(x + u)
\]
is satisfied, then $f$ is lattice lower semicontinuous (lattice l.s.c.) at $x$. A function $f : X \to \mathcal{G}^\oplus$ is lattice l.s.c. if and only if it is lattice l.s.c. everywhere.
In [21], it has been proven that if \( f \) is \( C^-\{0\} \)-l.s.c. at \( x \), then it is also lattice l.s.c. at \( x \). Since we assume \( \text{int} \ C \neq \emptyset, f \) is \( C^-\{0\} \)-l.s.c. at \( x \) if and only if \( f \) is \( W^* \)-l.s.c. at \( x \). One can show that if \( f \) is convex, then \( f \) is lattice l.s.c. if and only if graph \( f = \{(x, z) \mid z \in f(x)\} \subseteq X \times Z \) is a closed set with respect to the product topology, see [18].

In [21], a detailed study of continuity concepts for set valued functions is proposed. Indeed it is also shown that none of the concepts in Definition 2.13 coincides with those used in some literature (see e.g. [1, 3, 14]).

Finally, we come back to weak efficiency. Obviously \( x \in S \) is a weak solution to (VOP) if and only if one of the following equivalent assumptions is satisfied.

(a) \( \forall u \in X : \psi_C(x) \notin \text{int} \psi_C(x + u) \);
(b) \( \forall u \in X, \exists z^* \in W^* : -\sup \{z^*(z) \mid z \in \psi_C(x)\} \leq -\sup \{z^*(z) \mid z \in \psi_C(x + u)\} \neq -\infty \);
(c) \( \forall u \in X, \forall U \in \mathcal{U} : \psi_C(x) + U \notin \psi_C(x + u) \).

**Remark 2.14** We note that, although 

\[-\sup \{z^*(z) \mid z \in \psi_C(x)\} = \begin{cases} -z^*(x) & \text{if } x \in S \\ +\infty & \text{elsewhere} \end{cases} \]

considering a more general set valued function \( f : X \to \mathcal{P}(Z) \), it may happen that the value \( f(x) = Z \) or \( -\sup \{z^*(z) \mid z \in f(x)\} = -\infty \) may occur. Neither of the previous happens when \( f = \psi_C \).

Therefore, when considering any set-valued function \( f : X \to \mathcal{G}^\Delta \) and the related (weak) optimization problem

\[
\min f(x), \ x \in X. \tag{P}
\]

a point \( x_0 \in \text{dom} \ f \) is called a weak minimizer of \( f \) when

\[
f(x) = Z \lor \forall x \in X, \forall U \in \mathcal{U} : f(x_0) \oplus U \notin f(x). \tag{W-Min}
\]

This notion of solution can be related to others known in the literature. In [19, Definition 2.4 (1)], weak \( l \)-minimal elements of a set \( \mathcal{A} \subseteq \mathcal{P}(Z) \setminus \{\emptyset\} \) are those elements \( A \in \mathcal{A} \), such that for all \( B \in \mathcal{A}, A \subseteq B + \text{int} \ C \) implies \( B \subseteq A + \text{int} \ C \). If \( \mathcal{A} \subseteq \mathcal{G}^\Delta \), \( \text{int} \ C \neq \emptyset \) implies \( \text{int} \ B = B + \text{int} \ C \) for all \( B \in \mathcal{G}^\Delta \). Hence \( A \) is a weakly \( l \)-minimal element of \( \mathcal{A} \subseteq \mathcal{G}^\Delta \), if for all \( B \in \mathcal{A} \), \( A \subseteq \text{int} \ B \) implies \( B \subseteq \text{int} \ A \). Thus, either \( A = Z \), or \( A \) is weak \( l \)-minimal in \( \mathcal{A} \subseteq \mathcal{G}^\Delta \), if and only if there exists no \( B \in \mathcal{A} \) such that \( A \subseteq \text{int} \ B \). Therefore for any \( x_0 \in \text{dom} \ f \), \( f(x_0) \) is a weak \( l \)-minimal element of \( f[X] \) if and only if

\[
f(x) = Z \lor \forall x \in X : f(x_0) \notin \text{int} f(x). \tag{W-l-Min}
\]

Applying Proposition 2.11 it easily follows that (W-l-Min) implies

\[
f(x_0) = Z \lor \forall x \in X \exists z^* \in W^* : \varphi_{f,z^*}(x_0) \leq \varphi_{f,z^*}(x) \neq -\infty, \tag{Sc-W-Min}
\]

which in turn implies (W-Min). While, in general, none of these implications can be reverted, we have some advantages when \( f = \psi_C \).
Proposition 2.15 Let \( \psi : S \subseteq X \rightarrow Z \) be a vector valued function. For \( f = \psi^C : X \rightarrow G^\triangle \), \( x_0 \in S \) the properties (W-Min), (W-l-Min) and (Sc-W-Min) are equivalent and satisfied if and only if \( x_0 \) is a weakly efficient solution of \( (VOP) \).

**Proof.** When \( f = \psi^C \), if \( x^0 \) is a weak minimizer, \( \forall x \in X \) and \( \forall U \in \mathcal{U} \) it holds

\[
\psi(x_0) + C + U \nsubseteq \psi(x) + C
\]

Hence \( \psi(x_0) \nsubseteq \text{int}(\psi(x) + C) \) and \( \psi(x_0) + C \nsubseteq \text{int}(\psi(x) + C) \).

**Remark 2.16** For notational simplicity we set the restriction of a set valued function \( f : X \rightarrow G^\triangle \) to a segment with end points \( x_0, x \in X \) as \( f_{x_0,x} : \mathbb{R} \rightarrow G^\triangle \), given by

\[
f_{x_0,x}(t) = \begin{cases} f(x_0 + t(x - x_0)), & \text{if } t \in [0, 1]; \\
\emptyset, & \text{elsewhere.} \end{cases}
\]

This is equivalent to the restriction of a scalar valued function \( \varphi : X \rightarrow \mathbb{R} \) to the same segment, defined by

\[
\varphi_{x_0,x}(t) = \begin{cases} \varphi(x_t), & \text{if } t \in [0, 1]; \\
+\infty, & \text{elsewhere.} \end{cases}
\]

Setting \( x_t = x_0 + t(x - x_0) \) for all \( t \in \mathbb{R} \), the scalarization of the restricted function \( f_{x_0,x} \) is equal to the restriction of the scalarization of \( f \) for all \( z^* \in C^- \setminus \{0\} \).

If \( f \) is convex, \( x_0, x_t \in \text{dom } f \) for some \( t \in (0, 1) \), then \( (\varphi_{f,z^*})_{x_0,x} \) is lower semicontinuous on \( (0, t) \), hence \( f_{x_0,x} \) is lattice l.s.c. on \( (0, t) \).

Notice that in general, if \( f \) is \( C^- \setminus \{0\} \)-l.s.c. in \( x_0 \), then \( f_{x_0,x} \) is \( C^- \setminus \{0\} \)-l.s.c. in \( 0 \) for all \( x \in X \), while the implication is not revertible.

### 3 Dini directional derivatives

As we anticipated in Section 2, inf–residuated and order complete structure allows for an immediate extension of the definitions of both the differential quotient and upper and lower limits. Thus we have the basic ingredients to define the notion of upper and lower Dini directional derivatives.

**Definition 3.1** Let \( Y \) be a inf–residuated order complete conlinear space, \( f : X \rightarrow Y \) and \( x, u \in X \). The upper and lower Dini directional derivative of \( f \) at \( x \) in direction \( u \) are given by

\[
f^\uparrow(x, u) = \limsup_{t \downarrow 0} \frac{1}{t} \left( f(x + tu) - f(x) \right) = \inf_{0 < s \leq t \leq s} \sup_{0 < s \leq t \leq s} \frac{1}{t} \left( f(x + tu) - f(x) \right); \\
f^\downarrow(x, u) = \liminf_{t \uparrow 0} \frac{1}{t} \left( f(x + tu) - f(x) \right) = \sup_{0 < s \leq t \leq s} \inf_{0 < s \leq t \leq s} \frac{1}{t} \left( f(x + tu) - f(x) \right).
\]

If both derivatives coincide, then \( f'(x, u) = f^\uparrow(x, u) = f^\downarrow(x, u) \) is the Dini directional derivative of \( f \) at \( x \) in direction \( u \).
The previous definition does not require \( f \) to be proper or \( x \in \text{dom} \, f \). Clearly, if \( f(x) = \sup Y \), then \( f'(x, u) = \inf Y \) is satisfied for all \( u \in X \).

For notational simplicity, we agree that in the sequel we refer simultaneously to upper and lower Dini derivatives by \( f^\pm(x, u) \), even if the two values can be different.

If \( 0 < s \) is given, then \( f^\pm(x, su) = sf^\pm(x, u) \), that is, both derivatives are positively homogeneous. We remark that, when \( u = 0 \), \( \frac{1}{h} (f(x + t0) - f(x)) \preceq \frac{1}{h} \cdot \theta \), so both derivatives in direction 0 in general are less than \( \theta \), the neutral element in \( Y \), motivating our choice for the definition of positive homogeneity not including \( f(0) = \theta \).

**Remark 3.2** When \( Y = \mathbb{R} \), Definition 3.1 provides an extension to the classical notion of Dini derivatives for scalar functions (see [13] and the references therein), without requiring neither \( x \in \text{dom} \, f \) nor \( f \) to be proper. However, since a vector space needs not to be order complete, the same definition may not apply to vector valued functions \( \psi : S \subseteq X \rightarrow Z \). For this setting, the limiting process for the differential quotient has been defined in different manners in order to define a Dini derivative for vector valued functions, compare e.g. [2, 10].

**Example 3.3** Let \( \varphi : X \rightarrow \mathbb{R} \) be an extended scalar function. If \( \varphi(x + tu) \in \mathbb{R} \) is satisfied for all \( t \in [0, t_0] \) for a given \( 0 < t_0 \), then the differential quotient is real

\[
\frac{1}{t} (\varphi(x + tu) - \varphi(x)) \in \mathbb{R}
\]

Hence in this case the derivatives coincide with the standard definition in the literature, compare [13].

If \( x \notin \text{dom} \, \varphi \), then \( \varphi(x + tu) - \varphi(x) = -\infty \) for all \( t > 0 \), so \( \varphi'(x, u) = -\infty \). On the other hand, if \( \varphi(x) = -\infty \), then \( \varphi(x + tu) - \varphi(x) = -\infty \), whenever \( \varphi(x + tu) = -\infty \) and \( \varphi(x + tu) - \varphi(x) = +\infty \), else. The value of the derivatives in this case depends on the behavior of \( \varphi \) in a proximity of \( x \).

It is easy to see that \( f^\pm(x, u) \preceq f^\pm(x, u) \) is always satisfied, hence \( f'(x, u) \) exists if and only if \( f^\pm(x, u) \preceq f^\pm(x, u) \) is satisfied.

**Proposition 3.4** Let \( Y \) be a in\(\text{-}\)residuated order complete conlinear space, \( f : X \rightarrow Y \). If \( f \) is convex and \( tf(x) + (1 - t)f(x) \preceq f(x) \) is satisfied for all \( t \in (0, 1) \) and all \( x \in X \), then the Dini derivative exists for all \( u \in X \) and it holds

\[
f'(x, u) = \inf_{t \in [0, 1]} \frac{1}{t} \left( f(x + tu) - f(x) \right).
\]

Moreover, \( f' : X \times X \rightarrow Y \) is sublinear in its second component.

**Proof.** Let \( 0 < s \) be given, then for all \( 0 < t \leq s \), there exists a \( 0 < h \leq 1 \) such that \( hs + (1 - h)t = t \) and by convexity of \( f \), \( f(x + tu) = f(h(x + su) + (1 - h)x) \preceq hf(x + su) + (1 - h)f(x) \). By assumption, \( hf(x) + (1 - h)f(x) \preceq f(x) \) is satisfied for all
\( h \in [0,1] \). Applying Lemma 2.3 we can prove
\[
\frac{1}{t} \left( f(x + tu) - f(x) \right) \leq \frac{1}{hs} \left( (hf(x + su) + (1 - h)f(x) - hf(x)) - (hf(x) + (1 - h)f(x)) \right)
\]
\[
\leq \frac{1}{hs} \left( (hf(x + su) - hf(x)) + ( (1 - h)f(x) - (1 - h)f(x)) \right)
\]
\[
= \frac{1}{hs} \left( h (f(x + su) - f(x)) + (1 - h) (f(x) - f(x)) \right)
\]
\[
\leq \frac{1}{hs} \left( h (f(x + su) - f(x)) + \theta \right)
\]
\[
= \frac{1}{s} \left( f(x + su) - f(x) \right).
\]
Hence especially
\[
f^1(x, u) \leq \inf_{0 < s, t \leq s} \left( f(x + su) - f(x) \right) \leq f^1(x, u)
\]
is proven.

Finally, let \( s \in (0,1) \) and \( 0 < t \leq r \) be given, \( u_1, u_2 \in X \). Then
\[
f'(x, su_1 + (1 - s)u_2) \leq \frac{1}{t} \left( f(x + t(su_1 + (1 - s)u_2)) - f(x) \right)
\]
\[
\leq \frac{1}{t} \left( s \left( f(x + tu_1) - f(x) \right) + (1 - s) \left( f(x + tu_2) - f(x) \right) \right)
\]
\[
\leq s \frac{1}{t} \left( f(x + tu_1) - f(x) \right) + (1 - s) \frac{1}{r} \left( f(x + ru_2) - f(x) \right)
\]
But as this holds for all \( 0 < t \leq r \),
\[
f'(x, su_1 + (1 - s)u_2) \leq sf'(x, u_1) + (1 - s) \frac{1}{r} \left( f(x + ru_2) - f(x) \right)
\]
and ultimately
\[
f'(x, su_1 + (1 - s)u_2) \leq sf'(x, u_1) + (1 - s)f'(x, u_2).
\]
As \( f'(x, \cdot) : X \to Y \) is convex and positively homogeneous, it is sublinear.

In the proof of Proposition 3.4, it is shown also that if \( f : X \to Y \) is convex and \( tf(x) + (1 - t)f(x) \) is satisfied whenever \( t \in (0,1) \) and \( x \in X \), then the differential quotient is decreasing.

If \( Y = G^\triangle \), the assumption \( tf(x) + (1 - t)f(x) \leq f(x) \) is always satisfied, because of the ordering relation given by \( \geq \). Thus, if \( f : X \to G^\triangle \) is convex, then \( tf(x) + (1 - t)f(x) = f(x) \) is satisfied for all \( x \in X \) and all \( t \in (0,1) \).

**Lemma 3.5** Let \( f : X \to G^\triangle \) be a convex function, \( x, u \in X \). Then
\[
f'(x, u) = \text{cl} \bigcup_{0 < t \leq s} \frac{1}{t} \left( f(x + tu) - f(x) \right)
\]
is true for all \( s > 0 \). Moreover it holds
\[
\text{int } f'(x, u) = \bigcup_{0 < t \leq s} \text{int } \frac{1}{t} \left( f(x + tu) - f(x) \right).
\]
Let Proposition 3.6 prove \( f'(x, u) = \inf_{0 < t \leq s} \frac{1}{t} (f(x + tu) - f(x)) \). Moreover, since the differential quotient is decreasing as \( t \) converges to 0, \( \bigcup_{0 < t \leq s} \frac{1}{t} (f(x + tu) - f(x)) \) is convex for all \( 0 < s \), the first statement is true.

It is left to show that \( \inf_{0 < t \leq s} \int \frac{1}{t} (f(x + tu) - f(x)) \) is satisfied. Let then \( z \in \int f'(x, u) \) be given, hence there exists \( \bar{z} \in \int C \) and \( U \in \mathcal{U} \) such that \( z - \bar{z} \in \int f'(x_0, x) \) and \( \bar{z} + U \subseteq \int C \). Therefore, there exists \( 0 < t \) such that \( (z - \bar{z}) \in \frac{1}{t} (f(x + tu) - f(x)) \) and

\[
(z - \bar{z} + \bar{z} + U) \subseteq \frac{1}{t} (f(x + tu) - f(x)) + \int C \subseteq \frac{1}{t} (f(x + tu) - f(x)),
\]

implying \( z + U \subseteq \frac{1}{t} (f(x + tu) - f(x)) \), or equivalently \( z \in \int \frac{1}{t} (f(x + tu) - f(x)) \). \( \square \)

**Proposition 3.6** Let \( f : X \to G^\delta \) be given, \( x, u \in X \). If \( f^\dagger(x, u) \neq \emptyset \), then \( 0^+ f(x) \subseteq 0^+ f^\dagger(x, u) \) is true. If additionally \( x \in \text{dom } f \), then \( f'(x, 0) = 0^+ f(x) \).

**Proof.** First we consider the case \( x \in \text{dom } f \) and \( u = 0 \), then

\[
f'(x, 0) = \liminf_{t \downarrow 0} \frac{1}{t} (f(x) - f(x)) = f(x) - f(x) = 0^+ f(x).
\]

Let \( z \in f^\dagger(x, u) \) be satisfied. This is equivalent to stating that it exists an \( s > 0 \) such that for all \( 0 < t \leq s \) it holds \( f(x) + \{tz\} \subseteq f(x + tu) \). But, as \( f(x) = \text{can be re-written as } f(x) + t \cdot 0^+ f(x) \) for all \( 0 < t \), the previous inclusion is equivalent to

\[
\forall 0 < t \leq s : \quad \{z\} + 0^+ f(x) \subseteq \frac{1}{t} (f(x + tu) - f(x)),
\]

hence \( 0^+ f(x) \subseteq 0^+ f^\dagger(x, u) \). Again, the argument for the lower derivative goes along the same lines. \( \square \)

Especially, if \( f \) is convex, \( x \in \text{dom } f \), then \( f'(x, \cdot) : X \to G^\delta(Z, 0^+ f(x)) \) is a sublinear function with \( f'(x, 0) = 0^+ f(x) \), the neutral element in a subspace of the image space. However, \( 0^+ f(x) \ni C \) and in general, the inequality will be strict.

In the special case of \( Y = G^\delta \), we are also interested in comparing the derivative of a given function with the set of the derivatives of its scalarization. The following inequalities holds true.

**Proposition 3.7** Let \( f : X \to G^\delta \) be given, \( x, u \in X \) and \( z^* \in W^* \). Then

\[
f^\dagger(x, u) \subseteq \bigcap_{z^* \in B} \left\{ z \in Z \mid \varphi_{f, z^*}^\dagger(x, u) \leq -z^*(z) \right\};
\]

\[
\varphi_{f, z^*}^\dagger(x, u) \leq -\sigma(z^*|f^\dagger(x, u)).
\]
Example 3.9 Let the scalarization of a function do not commute. That is, the operations of taking the derivative and taking the infimum of expressions do not commute. The following counterexample shows. That is, the operations of taking the derivative and taking the infimum do not commute.

Proof. Combining the scalarization formula (2.1) with Lemmas 2.5 to 2.7, it holds
\[
\begin{align*}
  f^\uparrow(x, u) &= \text{cl co} \bigcup_{0<s, 0<t\leq s} \frac{1}{t} \bigcap_{z\in W^*} (z \in Z | \varphi_{f, z^*}(x + tu) - \varphi_{f, z^*}(x) \leq -z^*(z)) \\
  &\subseteq \bigcap_{z\in W^*} \text{cl co} \bigcup_{0<s, 0<t\leq s} \frac{1}{t} (z \in Z | \varphi_{f, z^*}(x + tu) - \varphi_{f, z^*}(x) \leq -z^*(z)) \\
  &= \bigcap_{z\in W^*} \left( z \in Z | \inf_{0<s, 0<t\leq s} \sup_{0<s, 0<t\leq s} \frac{1}{t} (\varphi_{f, z^*}(x + tu) - \varphi_{f, z^*}(x)) \leq -z^*(z) \right)
\end{align*}
\]
and likewise
\[
\begin{align*}
  \varphi_{f, z^*}^\uparrow(x, u) &= \inf_{0<s, 0<t\leq s} \sup_{0<s, 0<t\leq s} \frac{1}{t} (\varphi_{f, z^*}(x + tu) - \varphi_{f, z^*}(x)) \\
  &\leq \inf_{0<s, 0<t\leq s} -\sigma(z^*) \frac{1}{t} (f(x + tu) - f(x)) \\
  &\leq \inf_{0<s} -\sigma(z^*) \bigcap_{0<t\leq s} \frac{1}{t} (f(x + tu) - f(x)) \\
  &= -\sigma(z^*) \text{cl co} \bigcup_{0<s, 0<t\leq s} \frac{1}{t} (f(x + tu) - f(x)).
\end{align*}
\]
The same chain of arguments proves both inequalities for the lower derivative as well. \(\square\)

In general, neither of the inequalities in Proposition 3.7 is satisfied with equality, as the following counterexample shows. That is, the operations of taking the derivative and taking the scalarization of a function do not commute.

Example 3.8 Let \(f : \mathbb{R} \to \mathcal{G}^\circ\mathbb{R} \to \{\emptyset\}\) be defined as \(f(x) = [-\sqrt{1-x^2}, \sqrt{1-x^2}]\), whenever \(x \in [-1, 1]\) and \(f(x) = \emptyset\), else. Then \(f'(0) + z \not\subseteq f(t)\) for any \(t \neq 0\), so \(f'(0, u) = \emptyset\). On the other hand, \(\varphi_{f, s}(x) = -|s| \cdot \sqrt{1-x^2}\) for all \(s \neq 0\) and thus \(\varphi_{f, s}(x, u) = -|s| \cdot \frac{\sqrt{1-x^2}}{\sqrt{1-x^2}} \cdot u\) for all \(x \in (-1, 1)\), especially \(\varphi_{f, s}(0, u) = 0\) for all \(s \neq 0\). Hence,
\[
\emptyset = f'(0, u) \not\subseteq \bigcap_{z^* \in \{\emptyset\} \setminus \{0\}} f_{z^*}'(0, u) = \{\emptyset\}
\]

Example 3.9 Let \(\psi : S \subseteq X \to Z\) be a \(C\)-convex function with set valued extension \(f = \psi_C : X \to \mathcal{G}^\circ\mathbb{R}\), then for all \(x, x + u \in S\) and all \(t \in (0, 1)\) it holds
\[
\frac{1}{t} (f(x + tu) - f(x)) = \frac{1}{t} (\psi(x + tu) - \psi(x)) + C.
\]
If \(x \not\in S\), then \(f'(x, u) = Z\) while if \(x \in S\) and \(x + tu \not\in S\) is satisfied for all \(0 < t\), then \(f'(x, u) = \emptyset\). Thus especially for \(x \in S\),
\[
f'(x, u) = \text{cl co} \bigcup_{0<t} \left( \frac{1}{t} (\psi(x + tu) - \psi(x)) + C \right).
\]
is satisfied. However the infimum of \(\frac{1}{t} (\psi(x + tu) - \psi(x))\) needs not exist, even if \(Z\) is lattice ordered.
Proposition 3.7 and the previous examples motivate to consider as a special case when equality is satisfied in either of the inequalities stated in Proposition 3.7. In the sequel we refer to

\[ \forall z^* \in B : \; \varphi^\dagger_{f,z^*}(x, u) = -\sigma(z^*)f^\dagger(x, u) \]  

as strong regularity assumption later on, in contrast to the weak regularity assumption,

\[ f^\dagger(x, u) = \bigcap_{z^* \in B} \left\{ z \in Z \mid \varphi^\dagger_{f,z^*}(x, u) \leq -z^*(z) \right\}. \]  

The following proposition states that if \( f = \psi^C \), then it satisfies (WR). Additional convexity assumption allows to prove (SR). It is left as an open question to identify necessary and sufficient conditions for either regularity assumption to be satisfied by a general set-valued function \( f : X \to \mathcal{G}^\Delta \).

**Proposition 3.10** Let \( \psi : S \subseteq X \to Z \) be given, \( x, u \in X \) and \( f = \psi^C : X \to \mathcal{G}^\Delta \) its set valued extension. Then property (WR) is satisfied for the lower derivative of \( f \),

\[ f^\dagger(x, u) = \bigcap_{z^* \in B} \left\{ z \in Z \mid \varphi^\dagger_{f,z^*}(x, u) \leq -z^*(z) \right\}. \]

If additionally \( \psi \) is \( C \)-convex, then property (SR) is true.

**Proof.** Recall that for all \( z^* \in W^* \) it holds \( \varphi_{f,z^*}(x) = -z^*\psi(x) \) for all \( x \in S \) and \( \varphi_{f,z^*}(x) = +\infty \), elsewhere. Hence especially

\[ -\sigma(z^*) \frac{1}{t} \left( \frac{f(x + tu)^\dagger - f(x)}{t} \right) = \frac{1}{t} \left( \varphi_{f,z^*}(x + tu)^\dagger - \varphi_{f,z^*}(x) \right) \]

is satisfied for all \( 0 < t \), in contrast to the inequality in the case of general set valued functions. Applying Lemma 2.5, then

\[ -\sigma(z^*) \inf_{0 < t} \frac{1}{t} \left( \frac{f(x + tu)^\dagger - f(x)}{t} \right) = \inf_{0 < t} \frac{1}{t} \left( \varphi_{f,z^*}(x + tu)^\dagger - \varphi_{f,z^*}(x) \right) \]

proving the equality in the convex case. Also, additionally applying Lemma 2.6, it holds

\[ f^\dagger(x, u) = \bigcap_{0 < s} \bigcap_{z^* \in W^*} \left\{ z \in Z \mid \inf_{0 < t \leq s} \frac{1}{t} \left( \varphi_{f,z^*}(x + tu)^\dagger - \varphi_{f,z^*}(x) \right) \leq -z^* \right\}. \]

\[ = \bigcap_{z^* \in W^*} \left\{ z \in Z \mid \sup_{0 < s} \inf_{0 < t \leq s} \frac{1}{t} \left( \varphi_{f,z^*}(x + tu)^\dagger - \varphi_{f,z^*}(x) \right) \leq -z^* \right\}. \]

\[ \square \]

**Example 3.11** (a) Let \( Z = \mathbb{R}^3 \) be ordered by the ordering cone \( C \), the closed conical hull of \( \text{co} \{(-1, 1, 1)^T, (1, 1, -1)^T, (1, 1, 1)^T\} \). Let a vector function \( \psi : S \subseteq X \to Z \) be given with \( \psi(x) = (0, 0, 0)^T \) and

\[ \psi(t) = \begin{cases} (-t, 0, 0)^T, & \text{if } \exists n \in \mathbb{N} : \frac{1}{2n} \leq t < \frac{1}{2n-1}; \\ (t, 0, 0)^T, & \text{if } \exists n \in \mathbb{N} : \frac{1}{2n+1} \leq t < \frac{1}{2n}. \end{cases} \]

Then \( (\psi^C)^\dagger(0, 1) = \text{co} \{(0, 1, 1)^T, (0, 1, -1)^T\} \uplus C \). For \( z^* = (0, -1, 0) \in C^- \setminus \{0\} \), it holds \( \varphi_{(\psi^C)^\dagger(0, 1)}(0, 1) = 0 < -\sigma(z^*)((\psi^C)^\dagger(0, 1)) = 1. \)
(b) Let \( Z = \mathbb{R}^2 \) be ordered by the natural ordering cone \( C = \mathbb{R}^2_+ \) and let a vector function 
\( \psi : S \subseteq X \to Z \) be given such that \( \psi(x) = (0,0)^T \) and \( \psi(t) = (1,0)^T \) is satisfied for all \( t > 0 \). Then \( (\psi^C)'(0,1) = 0 \), hence \(-\sigma(z^*)((\psi^C)'(0,1)) = \emptyset \), while for \( z^* = (0, -1)^T \in C^- \setminus \{0\} \) it holds \( \varphi'_{(\psi^C),z^*}(0,1) = 0 \).

In Proposition 3.10, we basically apply set valued arguments to obtain a definition of Dini derivatives for vector valued functions. In [2, 10], similar derivatives are introduced using vector valued arguments. Although a careful comparison among the different types of derivatives is beyond the limits of the paper, we conclude this section with a sneak view of some results that easily hold. First, we stress once more that Definition 3.1 allows to overcome two major drawbacks of other approaches. Indeed, to introduce a Dini type derivative we have no need to arbitrary define infinite elements in a vector space, while defining a difference quotient among elements of the image \( f(x) \).

To compare our approach to that in [2], let \( C \) be a polyhedral cone, \( M^* \subseteq W^* \) a finite set such that \( \text{co} M^* = W^* \). If \( \psi : S \subseteq X \to Z \) is a \( C \)-convex function, then 
\[
\exists \exists z^* \in W^* \text{ such that } z^* \psi(x) \leq \psi(x + tu) - \psi(x) \text{ for all } t > 0 \text{ and } u \in M^*.
\]
As any \( z^* \in W^* \) can be represented as a convex combination of elements of \( M^* \), and \( \varphi_{f,z^*}(x) = -z^* \psi(x) \) for all \( x \in S = \text{dom} \varphi_{f,z^*} \), this implies
\[
\forall z^* \in W^* \exists m^* \in M^* \text{ such that } \frac{1}{t} \left( \varphi_{f,m^*}(x + tu) - \varphi_{f,m^*}(x) \right) \leq -m^*(\bar{z}) + \varepsilon_t.
\]
\[
\exists \bar{z} \in M^* \exists \varepsilon_t > 0 \text{ such that } \frac{1}{t} \left( \varphi_{f,z^*}(x + tu) - \varphi_{f,z^*}(x) \right) \leq -z^*(\bar{z}) + \varepsilon_t,
\]
\[
(\psi^C)'(x,u) = \bigcap_{m^* \in M^*} \{ z \in Z | \varphi_{f,m^*}'(x,u) \leq z^* \},
\]
is satisfied, as \( (\psi^C)'(x,u) \subseteq \bigcap_{m^* \in M^*} \{ z \in Z | \varphi_{f,m^*}'(x,u) \} \) is always true. Especially, if \( Z = \mathbb{R}^n \) is ordered by the Pareto ordering cone, then the derivative of the set valued extension of a \( C \)-convex function \( \psi : S \subseteq X \to Z \) is characterized by the derivatives of the finite number of scalarizations with respect to the negative unit vectors in \( Z^* \). This approach has been chosen in [2], where the upper and lower Dini derivative of a function \( \psi : S \subseteq X \to \mathbb{R}^n \) is defined through the vector \( (\varphi^+_{\psi C_i},e_i^*(x,u), \ldots, \varphi^+_{\psi C_i},e_i^*(x,u)) \in \mathbb{R}^n, \text{ where } e_i^* \text{ denotes the i-th unit vectors in } \mathbb{R}^n \).

In [10], a set valued Dini derivative for vector valued functions \( \psi : S \subseteq X \to Z \) has been defined, using the Painlevé Kuratowski limit of the differential quotient. The original image space is extended by infinite elements \( \zeta_{\infty} = \lim t z \) for all \( z \in Z \setminus \{0\} \). Roughly speaking, \( \zeta_{\infty} \) is an element of \( \psi'(x,u) \), if for any \( U \in U \) and any \( s > 0 \), for any \( t_0 > 0 \) there exists a \( t \in (0, t_0) \) such that \( \frac{1}{t} \left( \psi(x + tu) - \psi(x) \right) \in sz + \text{cone}(\{z\} + U) \) and \( z \in \psi'(x,u) \), if \( z \) is a
cluster point of the net of differential quotients. It can be proven that if \( z \in \psi'(x, u) \), then 
\( z \in (\psi^C)^t(x, u) \), while the situation is somewhat more complicated for infinite elements. If 
\( z_\infty \in \psi'(x, u) \) and \( z \in -\text{int } C \), then \( (\psi^C)^t(x, u) = Z \). With Stampacchia type variational 
inequalities in mind, the following chain of implications can be proven.

If \( \psi(x, u) \cap (-C \cup \{z_\infty \mid z \in -C \setminus \{0\}\}) = \emptyset \), then \( 0 \notin (\psi^C)^t(x, u) \), which in turn implies 
\( \psi'(x, u) \cap (-\text{int } C \cup \{z_\infty \mid z \in \text{int } C\}) = \emptyset \).

However, consider the function \( \psi : \mathbb{R} \to \mathbb{R}^2 \)

\[
\psi(t) = \begin{cases} 
(-1, -\sqrt{t}) & t > 0 \\
(0, 0) & t = 0 \\
+\infty & \text{elsewhere}
\end{cases}
\]

and the image space \( Z = \mathbb{R}^2 \) being ordered by \( \mathbb{R}^2_+ \). Then the differential quotient of \( \psi \) is 
\( \frac{1}{t} (\psi(0 + t) - \psi(0)) = (-\frac{1}{t}, -\frac{1}{\sqrt{t}}) \), so \( (\psi^C)'(0, 1) = Z \), while \( \psi'(0, 1) = \{(-1, 0)_\infty\} \).

It turns out, by direct calculations, that also the function

\[
\Psi(t) = \begin{cases} 
(-1, \sqrt{t}) & t > 0 \\
(0, 0) & t = 0 \\
+\infty & \text{elsewhere}
\end{cases}
\]

then \( (\Psi^C)'(0, 1) = \emptyset \), while \( \Psi'(0, 1) = \{(-1, 0)_\infty\} \).

\section{Main results}

To characterize weak minimizers of (VOP) as solutions to (weak) variational inequalities of 
Stampacchia or Minty type, we first provide extensions of such inequalities for a general, 
convex, set valued function \( f : X \to \mathcal{G}^\delta \) and study their relations with solutions of (P).

We begin by considering the following variational inequality of Stampacchia type.

\textbf{Definition 4.1} Let \( f : X \to \mathcal{G}^\delta \) be a convex function and \( f' : X \times X \to \mathcal{G}^\delta \) its directional 
derivative. Then \( x_0 \) is a solution to the weak Stampacchia Variational Inequality, iff

\[ f(x_0) = Z \quad \forall x \in X : \quad 0 \notin \text{int } f'(x_0, x - x_0). \]  

\textbf{(W-SVI)}

\textbf{Remark 4.2} An element \( x_0 \in \text{dom } f \) solves (W-SVI) if and only if either

\[ f(x_0) = Z \quad \forall x \in X : \quad 0^+ f(x_0) \nsubseteq \text{int } f'(x_0, x - x_0) \]  

\text{or}

\[ f(x_0) = Z \quad \forall x \in X \forall U \in \mathcal{U} : \quad U \nsubseteq f'(x_0, x - x_0) \]

is satisfied.

Indeed, \( 0 \notin \text{int } f'(x_0, x - x_0) \) is equivalent to \( f'(x_0, x - x_0) = \emptyset \) or \( 0^+ f(x_0) \nsubseteq \text{int } f'(x_0, x - x_0) \). Obviously, \( 0 \notin \text{int } f'(x_0, x - x_0) \) implies \( 0^+ f(x_0) \nsubseteq \text{int } f'(x_0, x - x_0) \), as \( x_0 \in \text{dom } f \) and hence \( 0 \notin 0^+ f(x_0) \) is satisfied. On the other hand \( 0 \in \text{int } f'(x_0, x - x_0) \) implies \( 0^+ f(x_0) \subseteq 0^+ f'(x_0, x - x_0) \) (compare Proposition 3.6) and thus \( 0^+ f(x_0) \subseteq \text{int } f'(x_0, x - x_0) \).
According to the ordering relation introduced in $G^\phi$, (4.1) can be easily read as an inequality in the conlinear space that perfectly matches the form of scalar variational inequalities.

Applying scalarization, we can prove relations between the set–valued inequality (W-SVI) and a suitable family of scalar variational inequalities.

**Lemma 4.3** If $x_0 \in \text{dom } f$ satisfies

\[
f(x_0) = Z \quad \forall x \in X \exists z^* \in W^* : 0 \leq \varphi'_{f,z^*}(x_0, x - x_0)
\]

then it solves (W-SVI). If additionally the regularity assumption (SR) is satisfied, the reverse implication is true, too.

**Proof.** By a separation argument, $0 \not\in \text{int } f'(x_0, x - x_0)$ is satisfied if and only if there exists a $z^* \in W^*$ such that $0 \leq -\sigma(z^*) f'(x_0, x - x_0))$. But as by Proposition 3.7 the inequality $\varphi'_{f,z^*}(x_0, x - x_0) \leq -\sigma(z^*) f'(x_0, x - x_0))$ is always satisfied, the first implication is proven. On the other hand if (SR) is satisfied, then $\varphi'_{f,z^*}(x_0, x - x_0) = -\sigma(z^*) f'(x_0, x - x_0))$ is true for all $z^* \in W^*$ and thus the reverse implication holds true. \hspace{1cm} \Box

Under convexity assumption, inequality (W-SVI) is a necessary and sufficient condition for (W-Min) to hold.

**Theorem 4.4** Let $f : X \to G^\phi$ be a convex function, $x_0 \in \text{dom } f$. Then $x_0$ is a weak minimizer of $f$ if and only if it solves the Stampacchia variational inequality.

**Proof.** An element $x_0$ is a weak minimizer of $f$, iff $f(x_0) \subseteq U \not\subseteq f(x)$ is satisfied for all $U \in \mathcal{U}$ and all $x \in X$. With other words, iff $0 \not\in \text{int } (f(x) - f(x_0))$ is satisfied. Obviously, if this is not satisfied, then there exists $x \in X$ such that $0 \in \text{int } (f(x) - f(x_0)) \subseteq \text{int } f'(x_0, x - x_0)$. Hence, if $x_0$ solves the variational inequality, then $x_0$ is a weak minimizer of $f$. On the other hand, if $x_0$ is a weak minimizer of $f$, then especially for all $x \in X$ and all $t > 0$ it holds $0 \not\in \text{int } \frac{1}{t} (f(x_0 + t(x - x_0)) - f(x_0))$, hence by Lemma 3.5

\[
0 \not\in \bigcup_{t > 0} \text{int } \frac{1}{t} (f(x_0 + t(x - x_0)) - f(x_0)) = \text{int } f'(x_0, x - x_0).
\]

\hspace{1cm} \Box

In Section 2 we introduced also a scalarization of (W-Min), thorough condition (Sc-W-Min). The following results proves that, under some regularity condition, we have also equivalence between scalarized optimization and variational inequalities.

**Proposition 4.5** Let $f : X \to G^\phi$ be a convex function, $x_0 \in \text{dom } f$. If $x_0$ solves the scalarized Stampacchia variational inequality (Sc-W-SVI), then it satisfies (Sc-W-Min).

**Proof.** Since each scalarization $\varphi_{f,z^*} : X \to \mathbb{R}$ is convex, $0 \leq \varphi'_{f,z^*}(x_0, x - x_0)$ implies $\varphi_{f,z^*}(x_0) \leq \varphi_{f,z^*}(x) \neq -\infty$. Hence if $x_0$ solves the Stampacchia variational inequality (Sc-W-SVI), then for all $x \in X$ there exists a $z^* \in W^*$ such that $\varphi_{f,z^*}(x_0) \leq \varphi_{f,z^*}(x) \neq -\infty$ is satisfied and therefore $x_0$ satisfies (Sc-W-Min). \hspace{1cm} \Box

The reverse implication needs further assumptions to hold.
Proposition 4.6 Let $f : X \to G^\triangledown$ be a convex function, $x_0 \in \text{dom } f$. If $x_0$ satisfies \((\text{Sc-W-Min})\) and any of the following conditions is satisfied:

(a) The regularity assumption \((\text{SR})\) is satisfied;

(b) It exists a finite subset $M^* \subseteq W^*$ such that

$$\forall x \in X \exists z^* \in M^* : \; \varphi_{f,z^*}(x_0) \leq \varphi_{f,z^*}(x) \neq -\infty;$$

then $x_0$ solves \((\text{Sc-W-SVI})\)

**Proof.**

(a) If $x_0$ satisfies \((\text{Sc-W-Min})\) and $f(x_0) \neq Z$, then it satisfies \((\text{W-Min})\) and by theorem 4.4, this implies that $x_0$ solves the Stampacchia variational inequality \((\text{W-SVI})\). If additionally the regularity assumption \((\text{SR})\) is satisfied, then by Lemma 4.3 this implies that $x_0$ solves \((\text{Sc-W-SVI})\).

(b) Let $x \in X$ be given. Then for all $t \in (0,1)$ there exists a $z^* \in M^*$ such that $\varphi_{f,z^*}(x_0) \leq \varphi_{f,z^*}(x_0 + t(x - x_0)) \neq -\infty$. As $M^*$ is finite, there exists a $z_0^* \in M^*$ and a sequence $t_n \downarrow 0$ in $(0,1)$ such that $\varphi_{f,z_0^*}(x_0) \leq \varphi_{f,z_0^*}(x_0 + t_n(x - x_0)) \neq -\infty$, hence by convexity either $\{x_0, x\} \cap \text{dom } f = \{x_0\}$ and $\varphi'_{f,z^*}(x_0, x - x_0) = +\infty$, or $\varphi_{f,z_0^*}(x_0) \neq -\infty$, and

$$0 \leq \inf_{n \in \mathbb{N}} \frac{1}{t_n} (\varphi_{f,z^*}(x_0 + t_n(x - x_0)) - \varphi_{f,z^*}(x_0))$$

and as $t_n$ converges to $0$, this implies $0 \leq \varphi'_{f,z^*}(x_0, x - x_0)$, hence $x_0$ solves \((\text{Sc-W-SVI})\). \(\square\)

The study of variational inequalities related to optimization problems is classically divided into two parts. The first one relates to Stampacchia-type inequalities (see eg. [11]) and the second to Minty-type (see eg. [12]). Indeed, the differentiable Minty–type variational inequality, roughly speaking, evaluates the directional derivatives at some point $x$ along the direction $u = x_0 - x$. This motivates the following definition.

**Definition 4.7** Let $f : X \to G^\triangledown$ be a convex function and $f' : X \times X \to G^\triangledown$ its directional derivative. Then $x_0$ is said to be a solution to the Minty Variational Inequality, iff $x_0 \in \text{dom } f$ and

$$f(x_0) = Z \lor \forall x \in X : \; f'(x, x_0 - x) \notin \text{int } 0^+ f(x). \quad (\text{W-MVI})$$

As for Definition 4.1, we can provide a scalarization of \((\text{W-MVI})\) in the following Lemma. However, a complete equivalence holds only for set-valued extensions of vector-valued function.

**Lemma 4.8** If $x_0 \in \text{dom } f$ satisfies property \((\text{W-MVI})\), then it also satisfies

$$f(x_0) = Z \lor \forall x \in X \exists z^* \in W^* : \; \varphi_{f,z^*}(x) \neq -\infty \land \varphi'_{f,z^*}(x, x_0 - x) \leq 0. \quad (\text{Sc-W-MVI})$$

Moreover, if $f(x) = \psi^C(x)$, for some $\psi : X \to Z$ is satisfied for all $x \in X$, then equivalence holds true.
Proposition 4.9 Let $f(x, x_0 - x) \notin \text{int} 0^+ f(x)$ is satisfied then there exist $z^* \in W^*$ and $z \in f'(x, x_0 - x)$ such that $\varphi_{f, z^*}(x) \neq -\infty$ and $-\sigma(z^* | f'(x, x_0 - x)) \leq -z^*(z) \leq 0$, compare Lemma 2.9. By Proposition 3.7 the inequality $\varphi_{f, z^*}(x, x_0 - x) \leq -\sigma(z^* | f'(x, x_0 - x))$ is always satisfied, hence (W-MVI) implies (Sc-W-MVI).

On the other hand if $f$ is the set valued extension of a vector valued function, then $\varphi_{f, z^*}(x, x_0 - x) = -\sigma(z^* | f'(x, x_0 - x))$ is true for all $z^* \in W^*$ and applying Lemma 2.9 and Proposition 2.11 proves that (Sc-W-MVI) implies

$$f(x_0) = Z \lor \forall x \in X \forall U \in U : f'(x, x_0 - x) \oplus U \not\subset 0^+ f(x) \quad (4.2)$$

and it is left to prove that this implies (W-MVI). If $x \notin \text{dom } f$ or $0^+ f(x, x_0 - x) \neq C$, then there is nothing to prove. Hence, let $x \in S$ and $0^+ f'(x, x_0 - x) = C$. If $f'(x, x_0 - x) \subset \text{int } C$, then $\frac{1}{t} (\varphi(x + t(x_0 - x)) - \varphi(x))$ is a monotonely decreasing net in $\text{int } C$ and bounded from below by $0 \in Z$. If the set of differential quotients possesses a convergent subnet with limit $z_0$, then $z_0$ is a lower bound of the set $f'(x, x_0 - x)$ and $z_0 \in \text{int } C$, hence there exists a neighborhood $U \in U$ such that $f'(x, x_0 - x) \oplus U \subset \text{cl } (\{z_0\} + U) \subset C$.

Hence let $f'(x, x_0 - x) \subset \text{int } C$ be assumed and the set $\left\{ \frac{1}{t} (\varphi(x + t(x_0 - x)) - \varphi(x)) \mid t > 0 \right\}$ does not possess any convergent subnets.

For all $t > 0$, define

$$U_t = \left( C \cap \left\{ \frac{1}{t} (\psi(x + t(x_0 - x)) - \psi(x)) + (-C) \right\} \right) + \left\{ -\frac{1}{2t} (\psi(x + t(x_0 - x)) - \psi(x)) \right\},$$

a neighborhood of 0. It holds $U_s \subset U_t$ whenever $0 < s < t$ and $0 \in \bigcap_{0 < t} U_t = U_0$. First, assume $\text{int } U_0 = \emptyset$. Then for all $U \in U$ there is a $t > 0$ such that $\left\{ \frac{1}{2t} (\psi(x + t(x_0 - x)) - \psi(x)) \right\} + U \not\subset C$, hence the set $\left\{ \frac{1}{2t} (\psi(x + t(x_0 - x)) - \psi(x)) \mid t > 0 \right\}$ possess a convergent subnet with limit outside of $\text{int } C$, a contradiction. Therefore, $U_0 \in U$ is proven and

$$\left\{ \frac{1}{t} (\varphi(x + t(x_0 - x)) - \varphi(x)) \right\} + U_0 \subset C$$

for all $0 < t$. Hence,

$$f'(x, x_0 - x) \oplus U_0 = \text{cl} \bigcup_{0 < t} \left\{ \left\{ \frac{1}{t} (\varphi(x + t(x_0 - x)) - \varphi(x)) \right\} + C + U_0 \right\} \subset C.$$

\hfill \Box

In the general setting of problem (P), to prove the variational inequality characterization of weak minimizers, we need to apply scalarization argument. Therefore we begin to study the scalarized version of the Minty inequality. Indeed, the next Propositions show that the solution set to (Sc-W-Min) is always a subset of the solutions of (Sc-W-MVI), while equality is satisfied under additional regularity assumptions.

Proposition 4.9 Let $f : X \to G^\Delta$ be a convex function, $x_0 \in \text{dom } f$. If $x_0$ satisfies (Sc-W-Min) then it solves (Sc-W-MVI).
Proof. If $x_0$ satisfies (Sc-W-Min) then either $f(x_0) = Z$, or for all $x \in X$ there exists a $z^* \in W^*$ such that $\varphi_{f,z^*}(x) \neq -\infty$ and

$$\varphi'_{f,z^*}(x, x_0 - x) \leq \varphi_{f,z^*}(x_0) - \varphi_{f,z^*}(x) \leq 0.$$ 

$\square$

Proposition 4.10 Let $f : X \to G^\mathcal{G}$ be a convex function and $x_0 \in \text{dom } f$ solves (Sc-W-MVI). If it exists a finite subset $M^* \subseteq W^*$ such that $f_{x_0,x}$ is $M^*$-l.s.c. in $0 \in \text{dom } f_{x_0,x}$ for all $x \in X$ and

$$\forall x \in X \exists z^* \in M^* : \varphi_{f,z^*}(x) \neq -\infty \land \varphi'_{f,z^*}(x, x_0 - x) \leq 0;$$

then $x_0$ satisfies (Sc-W-Min).

Proof. Let $x \in X$ be given and $x_t = x_0 + t(x - x_0)$. By convexity of $f$, if $\varphi_{f,z^*}(x_t) \neq -\infty$ and $\varphi'_{f,z^*}(x_t, x_0 - x_t) \leq 0$, then $\varphi_{f,z^*}(x) \neq -\infty$ and $\varphi'_{f,z^*}(x, x_0 - x) \leq 0$ is satisfied and $\varphi_{f,z^*}(x_t) \leq \varphi_{f,z^*}(x)$. As by assumption the set $M^*$ is finite, for any $x \in X$ there exists a $z^* \in M^*$ such that for all $t > 0$ it holds $\varphi_{f,z^*}(x_t) \neq -\infty$ and $\varphi'_{f,z^*}(x_t, x_0 - x_t) \leq 0$. As $(\varphi_{f,z^*})_{x_0,x}$ is convex and l.s.c. in 0, this implies $\varphi_{f,z^*}(x_0) = \inf_{t \in [0,1]} \varphi_{f,z^*}(x_t) \leq \varphi_{f,z^*}(x).

$\square$

Based on the previous results we can prove equivalence between solutions of Minty type inequality and weak minimizers at least when $f = \psi^C$.

Corollary 4.11 Let $f : X \to G^\mathcal{G}$ be a convex function.

(a) If $f$ satisfies the regularity assumption given in proposition 4.10 for $x_0 \in \text{dom } f$ and $x_0$ solves (W-MVI), then it also satisfies (W-Min).

(b) If $f(x) = \psi^C(x)$ for all $x \in X$ and $x_0 \in \text{dom } f$ satisfies (W-Min), then $x_0$ also solves (W-MVI).

Proof. The implication in (a) is proven in lemma 4.8, proposition 4.10 and proposition 2.11, while the implication in (b) is a corollary of remark 2.15, proposition 4.9 and lemma 4.8.

The following example shows that we cannot obtain a result similar to Theorem 4.4 for Minty type variational inequality.

Example 4.12 Consider $Z = \mathbb{R}^2$, ordered by the natural ordering cone $C = \mathbb{R}^2_+$ and $X = \mathbb{R}$. The function $f : X \to G^\mathcal{G}$ given by

$$f(t) = \begin{cases} \{(z_1, z_2)^T \in Z | -t \leq z_1, z_2, t \leq z_1 + z_2\}, & \text{if } t \in (0,1); \\ \emptyset, & \text{elsewhere} \end{cases}$$

is convex and $C^\mathcal{G} \setminus \{0\}$-l.s.c. everywhere. Then $f'(1, -1) = (1, 1)^T + C$, hence there exists $t \in \text{dom } f$ and $U \in \mathcal{U}$ such that $f'(t, 0 - t) + U \subseteq 0^+ f(t) = C$ and obviously $f'(t, 0 - t)$
int $0^+ f(t)$. However, $f(0) \nsubseteq \text{int} f(t)$ for all $t \in \mathbb{R}$, hence $f(0)$ is a weak-$t$-minimal element of $f[X]$ and thus especially satisfies (W-Min) and (Sc-W-Min), but the Minty variational inequality (W-MVI) is not satisfied.

To summarize, we have proved the following chain of characterization of weak minimizer of problem (P) through set-valued variational inequalities.

Finally, when $f = \psi_C$ we can simplify the previous results to gain a better characterization of weak efficiency in vector optimization. First, scalarized variational inequalities (Sc-W-SVI) and (Sc-W-MVI) are equivalent to their set–valued counterparts, without further assumptions.

**Proposition 4.13** Let $\psi : S \subseteq X \to Z$ be a $C$–convex function, $x_0 \in S$ and $f(x) = \psi_C(x)$ for all $x \in X$. Then

(a) the Stampacchia variational inequalities of type (W-SVI) and (Sc-W-SVI) are equivalent;

(b) the Minty variational inequalities of type (W-MVI) and (Sc-W-MVI) are equivalent.

**Proof.**

(a) Assuming $f(x) = \psi_C(x)$ for all $x \in X$ is true, the regularity assumption (SR) is satisfied and equivalence follows from Lemma 4.3.

(b) This is Lemma 4.8. □

Next, we finally get the classical chain of relations for weak efficiency (compare e.g. [12, 6]) as corollaries of the results proved in the general case.
The following corollaries state the implications in the scheme.

**Corollary 4.14** Let \( \psi : S \subseteq X \to Z \) be a \( C \)-convex function and \( f(x) = \psi^C(x) \) for all \( x \in X \). Then \( x_0 \in S \) solves \((W-SVI)\) if and only if \( x_0 \) is a weakly efficient solution of the vector optimization problem \((VOP)\).

**Corollary 4.15** Let \( \psi : S \subseteq X \to Z \) be a \( C \)-convex function and \( f(x) = \psi^C(x) \) for all \( x \in X \).

(a) If \( x_0 \in S \) is a weakly efficient solution of the vector optimization problem \((VOP)\), then \( x_0 \) solves \((W-MVI)\).

(b) If additionally \( f_{x_0,x} \) is \( C^- \setminus \{0\} \)-l.s.c in 0 for all \( x \in X \) and \( C \) is polyhedral, then \( x_0 \in S \) solves \((W-MVI)\) if and only if \( x_0 \) is a weakly efficient solution of the vector optimization problem \((VOP)\).

**Proof.**

(a) If \( x_0 \in S \) is a weakly efficient solution of the vector optimization problem \((VOP)\), then \( x_0 \) is a weak minimizer of \( f \) this implies \((W-MVI)\).

(b) The reverse implication follows from proposition 4.10, as \( \varphi_{f,z^*} (x_0) = -z^* (\psi(x_0)) \in \mathbb{R} \) is true. \( \square \)

The main advantage of these results, compared with those in [5, 10] is that \( \psi(x_0) \in \text{wEff}_\psi[X] \) is characterized using a Minty or Stampacchia type variational inequality for the epigraphical extension of \( \psi \) saving us the effort of introducing “infinite elements” of \( Z \) to cope with possible unboundedness of the differential quotient \( \frac{1}{t} (\psi(x_0 + tu) - \psi(x_0)) \).

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