THE HOMOLOGICAL DETERMINANT OF QUATUM GROUPS OF
TYPE A

PH `UNG H ` Ô H AÎ

INTRODUCTION

Let \( V \) be a vector space over a field \( k \) and \( G = GL(V) \), the general linear group. Let \( n = \dim_k V \). It is well known that elements of \( G \) acts on the \( n \)-th homogeneous component of the exterior over \( V \) by means of the determinant. More precisely, let \( x_1, x_2, \ldots, x_n \) be a basis of \( V \). Then \( \Lambda^n(V) \) is one-dimensional and a non-zero vector is \( x_1 \wedge x_2 \ldots \wedge x_n \). If \( g \in G \) has the matrix \( A \) with respect to this basis, then
\[
g \cdot (x_1 \wedge x_2 \ldots \wedge x_n) = \det A \cdot x_1 \wedge x_2 \ldots \wedge x_n.
\]

Let now \( V \) be a supervector space of dimension \((m|n)\). The (super)determinant of an endomorphism of \( V \) was introduced by Berezin. Fix a homogeneous basis of \( V \), \( x_1, x_2, \ldots, x_{m+n} \) where the first \( m \) elements are even and the rest are odd (such a basis is called distinguished). Then an endomorphism of \( V \), with respect to this basis, has the matrix of the following form
\[
Z = \begin{pmatrix} A & B \\ C & D \end{pmatrix}
\]
where \( A, D \) are square matrices of dimension \( m \times m \) and \( n \times n \), respectively, whose entries are even, and \( B, D \) are matrices of type \( m \times n \) and \( n \times m \), whose entries are odd elements. Assume that \( D \) is invertible in the usual sense, define the super determinant of \( Z \) to be
\[
\text{Ber}_q Z = \det T^{-1} \det(A - CD^{-1}B).
\]
It is shown that the the matrix \( Z \) is invertible iff its super determinant is and that the super determinant is multiplicative. It is however not clear why the definition of \( \text{Ber}_q \) is independent of the choice of bases (our basis is a distinguished basis).

In [19] Manin suggested the following construction to define the super determinant. Let \( V^* \) denote the vector space to dual \( V \) with the dual basis \( \xi^1, \xi^2, \ldots, \xi^n \), \( \xi^i(x_j) = \delta^i_j \). Manin introduced the following Koszul complex; its \((k,l)\)-term is given by \( K^{k,l} := \Lambda^k \otimes S^l \), where \( \Lambda^n \) and \( S^n \) are the \( n \)-th homogeneous components of the exterior and the symmetric tensor algebra over \( V \). The differential \( d^{k,l} : K^{k,l} \rightarrow K^{k+1,l+1} \) is given by
\[
d^{k,l}(h \otimes \phi) = \sum_i h x_i \otimes \xi^i \wedge \phi.
\]
It is easy to check that \( d_{k,l} \) is \( G \)-equivariant hence the homology groups of this complex are representations of \( G \). On the other hand, one can show that this complex is exact everywhere except at the term \((m,n)\), where the homology group is one-dimensional, thus, it defines a one-dimensional representation of \( G \). It turns out that elements of \( G \) acts on this representation by means of its super determinant, in other words, the definition of the super determinant is basis free.

The quantum semigroup of type \( A \) is the “spectrum” of the quadractic algebra
\[
E := k(\langle z_1^i \rangle)/(R_{iuv}^j z_i^u z_j^v = z_i^1 z_j^2 R_{12}^{i,j})
\]

2000 Mathematics Subject Classification. Primary 16W30, 17B37, Secondary 17A45, 17A70.
This work is supported by the National Program of Basic Sciences Research of Vietnam.
where $R$ is a Hecke symmetry (see [1]). The Hecke symmetry resembles the usual flipping operator $a \otimes b \mapsto b \otimes a$ or $a \otimes b \mapsto (-1)^{ab} b \otimes a \ (a, b \text{ are homogeneous})$ in super symmetry.

In [6, 15], a Koszul complex is defined for $R$. For that, one first has to define the quantum exterior and quantum symmetric tensor algebra by means of certain projectors on $V^{\otimes n}$. It is still an open question, whether this complex has the homology group concentrated at a certain term and its dimension is one. Some efforts have been made. Gurevich [6] showed this for even Hecke symmetries (i.e., those which induce finite-dimensional exterior algebra), Lyubashenko and Sudbery [15] showed this for Hecke sums of an odd and an even Hecke symmetries.

In this paper, assuming that $R$ depends algebraically on $q$, where $q$ runs in $\mathbb{C}$, we give the affirmative answer to this question for an algebraically dense set of value of $q$. Our tactic is first to use a new result of Deligne [1] to check the case $q = 1$. Then using standard arguments we show that for a dense set of values of $q$, the homology group of $K$ has the dimension less than that of the corresponding homology groups when $q = 1$. In other words, for an algebraically dense subset of $\mathbb{C}$, the homology has dimension at most 1. It remains to show the non-vanishing of the homology. Taking the tensor product of the complex with a suitably chosen comodule we obtain a new complex whose terms are all $E$-comodules. We decompose these comodule using the Littlewood-Richardson formula and derive the non-vanishing of the homology.

1. Hecke symmetries and the associated quantum groups

We work over an algebraically closed field $k$ of characteristic zero. Let $V$ be a vector space over $k$ of finite dimension $d$. Let $R : V \otimes V \rightarrow V \otimes V$ be an invertible operator. $R$ is called a Hecke symmetry if the following conditions are fulfilled:

- $R_1 R_2 R_1 = R_2 R_1 R_2$, where $R_1 := R \otimes \id_V, R_2 := \id_V \otimes R$,
- $(R + 1)(R - q) = 0$ for some $q \in k$,
- The half adjoint to $R$, $R^k : V^* \otimes V \rightarrow V \otimes V^*$; $\langle R^k (\xi \otimes v), w \rangle = \langle \xi, Rv \otimes w \rangle$, is invertible.

Through out this work we will assume that $q$ is not a root of unity other then the unity itself. If $q = 1$, $R$ is called vector symmetry. Vector symmetries were introduced by Lyubashenko [14] and generalized to Hecke symmetries by Gurevich [6].

Let us fix a basis $x_1, x_2, \ldots, x_d$ of $V$. Then $R$ can be given in terms of a matrix, also denoted by $R$, $R(x_i \otimes x_j) = x_k \otimes x_l R_{ij}^{kl}$, we adopt the convention of summing up after the indices that appear both in the lower and upper places. The matrix $R_{ij}^{kl}$ is given by $R_{ij}^{kl} = R_{jl}^{ki}$. Therefore, the invertibility of $R^k$ can be expressed as follows: there exists a matrix $P$ such that $P_{jm}^i R_{ml}^{nk} = \delta^i_j \delta^k_l$.

To a Hecke symmetry $R$, there associated the following quadratic algebras:

\begin{align*}
S &:= k\langle x_1, x_2, \ldots, x_d \rangle / \langle x_i x_j R_{ij}^{kl} = q x_i x_j \rangle \\
\Lambda &:= k\langle x_1, x_2, \ldots, x_d \rangle / \langle x_i x_j R_{ij}^{kl} = q x_i x_j \rangle \\
E &:= k\langle z_1^i, z_2^j, \ldots, z_d^l \rangle / \langle z_m^i z_n^j R_{mn}^{kl} = R_{pq}^{ij} R_{kl}^{pq} \rangle
\end{align*}

$S$ and $\Lambda$ are called respectively the function algebra and the exterior-algebra on the corresponding quantum space and $E$ is called the function algebra on the corresponding quantum endomorphism space or the matrix quantum semi-group.

$E_R$ is a coquasitriangular bialgebra [13, 15]. The coproduct on $E_R$ is given by $\Delta(Z) = Z \otimes Z$. The coquasitriangular structure is given by $r(z_j^i, z_k^l) = R_{jl}^{ki}$. $S$, $\Lambda$ and all their homogenous components are right $E$-comodules. In particular, $V$ is a comodule and the induced braiding on $V \otimes V$ is exactly $R$.

$E_R$ is naturally $\mathbb{N}$-graded, let $E_n$ be its $n$-th homogenous component. Then $E_n$ is a coalgebra and $V^{\otimes n}$ is its comodule, hence an $E_n^*$-module.
On the other hand, \( R \) induces a representation of the Hecke algebra \( \mathcal{H}_n \) (see, e.g., \[2\]) on \( V^\otimes n \), denoted by \( \rho_n : \mathcal{H}_n \to \text{End}(V^\otimes n) \), for any \( n \geq 2 \). Explicitly, \( \rho \) maps the generator \( T_i \) of \( \mathcal{H}_n \) to the operator \( R_i := \text{id}^{\otimes i-1} \otimes R \otimes \text{id}^{n-i-1} \). We have the following “Double centralizer theorem” \[4\]

1.1. The algebras \( \rho_n(\mathcal{H}_n) \) and \( E_n \) are centralizers of each other in \( \text{End}_k(V^\otimes n) \).

Let \( x_n \in \mathcal{H}_n \) be the central idempotent that induces the trivial representation, \( x_n = \sum_{w \in S_n} (-q)^{-l(w)} T_w/[n]_q \), where \( T_w \) are the generators of \( \mathcal{H}_n \) as a \( k \)-vector space, indexed by elements of the symmetric group \( S_n \). The operator \( X_n := \rho_n(x_n) \) is called the \( q \)-symmetrizer, it is a projection on \( V^\otimes n \). The projection \( V^\otimes n \to S^n \) restricted to \( \text{Im} X_n \) is an isomorphism. Since \( R \) is a morphism of \( E \)-comodules the above isomorphisms are isomorphism of \( E \)-comodules, too.

Analogously, let \( y_n \in \mathcal{H}_n \) be the central idempotent that induces the signature representation of \( \mathcal{H}_n \): \( y_n := [(n)_{1/q}^{-1} \sum_{w \in S_n} (-q)^{-l(w)} T_w \). The operator \( Y_n := \rho_n(y_n) \) is called the \( q \)-anti-symmetrizer, it is a projection on \( V^\otimes n \). The projection \( V^\otimes n \to \Lambda^n \) restricted to \( \text{Im} Y_n \) is an isomorphism.

A Hecke symmetry \( R \) is called even (resp. odd) Hecke symmetry of rank \( r \) iff \( Y_{r+1} = 0 \), and \( Y_r \neq 0 \) (resp. \( X_{r+1} = 0 \) and \( X_r \neq 0 \)). One can show that, in this case, \( Y_r \) (resp. \( X_r \)) has rank 1 \[3\].

By means of the above double centralizer theorem, \( E \) is cosemisimple (i.e. its comodules are semisimple) and simple \( E \)-comodules can be described by primitive idempotents of the Hecke algebras, thus, partitions. For partitions \( \lambda, \mu \) and \( \gamma \), the multiplicity of \( M_\lambda \) (the simple \( E_R \) comodule corresponding to \( \gamma \)) in \( M_\lambda \otimes M_\mu \) does not depend on \( R \); in fact, it is equal to the corresponding Littlewood-Richardson coefficient \( \ell^{(1)}_{\lambda \mu} = (s_\lambda s_\mu, s_\gamma) \), where \( s_\lambda \) are the Schur functions (cf. \[17\]). Note that however that not any partition defines a simple comodule, some of them may give zero-modules.

To have more precise information on the simple comodules of \( E \), we need the Poincaré series of \( S, \Lambda \). Using theory of symmetric functions \[17\] and a theorem of Edrei \[5\] we have the following \[3\]:

1.2. \( P_\lambda(t) \) is a rational function having negative roots and positive poles. Assume that \( P_\lambda \) has \( m \) roots and \( n \) poles, then simple \( E_R \)-comodules are parameterized by partitions \( \lambda \) for which \( \lambda_{m+1} \leq n \).

**Definition.** The pair \( (m, n) \) is called the birank of \( R \).

**Examples.** The following are so far examples of Hecke symmetries.

- The solutions of the Yang-Baxter equation of series \( A \), due to Drinfel’d and Jimbo \[12\] is an example of even Hecke symmetries. The associated quantum groups are called standard deformations of \( GL(n) \).
- Cramer and Gevais \[4\] found another series of solution which are also even Hecke symmetries.
- Hecke sums of odd and even Hecke symmetries are examples of non-even, non-odd Hecke symmetries \[13\].
- Takeuchi and Tambara found a Hecke symmetry which is neither even nor a Hecke sum of an odd and an even Hecke symmetries \[20\].
- Even Hecke symmetries of rank 2 was classified by Gurevich \[8\]. He also show that on each vector space of dimension \( \geq 2 \), there exists an even Hecke symmetries of rank 2.
- Hecke symmetry of birank \((1, 1)\) was classified by the author \[10\].

The quantum group of type \( A \) is define to be the “spectrum” of the subsequently defined Hopf algebra. Let \( T = (t^i_j) \) be a \( d \times d \) matrix of new variables. The Hopf algebra associated
to $R$ is a factor algebra of the free associative algebra over entries of $Z$ and $T$:

\[(1) \quad H_R := T(Z, T) / (RZ_1Z_2 = Z_1Z_2R, TZ = ZT = \text{id})\]

$H_R$ is a Hopf algebra, the antipode is given by $S(Z) = T$. The coquasitriangular structure on $E_R$ can be extended on to $H_R$ thanks to the closedness of $R$.

The structure of $H_R$-comodules is, in general, much more complicated than the one of $E_R$-comodules. The best handled case is when $R$ is an even Hecke symmetry, i.e., when $P_\lambda(t)$ is a polynomial. We have, however the following result [9].

**1.3. The natural map $E_R \rightarrow H_R$ is injective. Consequently, every simple $E_R$-comodule is a simple $H_R$-comodule.**

Among $H_R$-comodules which are not $E$-comodules, the super determinant plays an important role. The well-known tool for defining the quantum super determinant serves the Koszul complex (of second type) introduced by Manin [19]. This is a (bi-)complex, whose $(k, l)$ term is $\Lambda^k \otimes S^{*l}$. The differential is induced from the dual basis map. The homology group of this complex is an $H_R$-comodule, if it is one dimensional over $k$, it defines a group-like element in $H_R$ called homological determinant or quantum super determinant or, in some cases, quantum Berezinian.

### 2. The Koszul complex

We begin with the description of the Koszul complex. For convenience, we first fix the following notion of the dual comodule of a tensor product of two or more comodules. For two (rigid) comodules $V, W$, the dual to $V \otimes W$ is defined to be $W^* \otimes V^*$ with the evaluation map $ev_{V \otimes W} = ev_W \circ (W^* \otimes ev_V \otimes W)$. Analogously, one defines the dual to longer tensor products.

Fix a basis $x_1, x_2, \ldots, x_n$ of $V$ and let $\xi_1, \xi_2, \ldots, \xi_n$ be the dual basis in $V^*$, we define the dual basis map $db : k \rightarrow V \otimes V^*$, $db(1) = \sum_i x_i \otimes \xi_i$. This map does not depend on the choice of basis. The term $K^{k,l}$ of the Koszul complex associated to $R$ is $\Lambda^k \otimes S^{*l}$, the differential $d_{k,l}$ is given by:

$$\Lambda^k \otimes S^{*l} \rightarrow V^\otimes l \otimes V^* \otimes l \text{id} \otimes db \otimes \text{id} \otimes d_{k,l}$$

where $X_i, Y_k$ are the $q$-symmetrizer operators introduced in the previous section. One defines another differential $d'$ as follows:

$$\Lambda^k \otimes S^{*l} \rightarrow V^\otimes l \otimes V^* \otimes l \text{id} \otimes ev_{V^*} \otimes \tau_{V, V^*} \otimes id$$

where $\tau_{V, V^*}$ denotes the braiding on $V \otimes V^*$ induced from the coquasitriangular structure on $H_R$, its matrix is given by $P: R^m_n P_{nl} = \delta_{l}^{n} \delta_{l}^{n}$. Then $d$ and $d'$ satisfy [6]

$$(qdd' + d'd) |_{K^{k,l}} = q^k (\text{rank}_q R + [l-k]_q) \text{id}.$$ 

where $\text{rank}_q R := P^m_n$. Hence, if $\text{rank}_q R \neq [l-k]_q$, the cohomology group at the term $(k, l)$ vanishes. Thus, all complexes except at most one are acyclic.

**Theorem 1.** Let $R$ be a Hecke symmetry of birank $(m, n)$. Then $\text{rank}_q R = \lfloor n - m \rfloor_q$ and the homology of the Koszul complex at the term $(m, n)$ is non-vanishing. Consequently, on the Hopf algebra $H$, there exists a non-zero integral and the simple $H$-comodule $M_\lambda$ is injective and projective if and only if $\lambda_m \geq n$.

**Proof.** Since $R$ has birank $(m, n)$, simple $E$-comodules are parameterized by partions satisfying $\lambda_m + 1 \leq n$. Using this fact and the Littlewood-Richardson formula, we can easily
show that
\[ \text{Hom}^E(M_{((n+1)^m}) \otimes S^n, M_{(n+1)} \otimes \Lambda^m) = k \]
\[ \text{Hom}^E(M_{((n+1)^m}) \otimes S^{n-1}, M_{(n+1)} \otimes \Lambda^{m+1}) = 0 \]
\[ \text{Hom}^E(M_{((n+1)^m}) \otimes S^{n+1}, M_{(n+1)} \otimes \Lambda^{m+1}) = 0. \]

As a consequence, \( M_{(n+1)^m} \otimes \Lambda^m \otimes S^{n*} \) contains \( M_{((n+1)^m)} \) while the comodules \( M_{(n+1)} \otimes \Lambda^{m-1} \otimes S^{n*} \), \( M_{(n+1)} \otimes \Lambda^{m+1} \otimes S^{m+1*} \) do not.

Assume that \( \text{rank}_q R \neq [n - m]_q \). Then the complex is exact at \( K^{m,n} \) and \( dd' + d'd = q^m(\text{rank}_q R + [n - m]_q)\text{id} \neq 0. \)

On the other hand, since \( M_{(n+1)^m} \) cannot be a submodule of \( M_{(n+1)^m} \otimes \Lambda^{m+1} \otimes S^{n+1*} \), the restriction of \( \text{id}_{M_{((n+1)^m)}} \otimes d^{m,n} \) on it should be zero. Analogously, the restriction of \( \text{id}_{M_{((n+1)^m)}} \otimes d^{m,n} \) on \( M_{(n+1)^m} \) is 0. Thus, the restriction of \( dd' + d'd \) on \( M_{(n+1)^m} \) is zero, a contradiction. Therefore \( \text{rank}_q R = [n - m]_q \).

According to a result of \([11]\) if \( \text{rank}_q R = [n - m]_q \) then \( H \) possesses a non-zero integral and in this case, according to a result of \([10]\), \( M_\lambda \) is a splitting comodule (i.e., injective and projective in \( H\)-comod) iff \( \lambda_m \geq n \). Thus, \( M_{((n+1)^m)} \) is projective hence cannot be a subquotient of \( M_{((n+1)^m)} \otimes \Lambda^{m-1} \otimes S^{n-1*} \), in particular, it cannot be a submodule of \( M_{(n+1)^m} \otimes \text{Im}d^{m-1,n-1} \). Therefore
\[ M_{(n+1)^m} \otimes \text{Im}d^{m-1,n-1} \neq M_{(n+1)^m} \otimes \text{Ker}d^{m,n}. \]

Thus, the sequence
\[ M_{(n+1)^m} \otimes \Lambda^{m-1} \otimes S^{n*} \rightarrow M_{(n+1)^m} \otimes \Lambda^m \otimes S^{n*} \rightarrow M_{(n+1)^m} \otimes \Lambda^{m+1} \otimes S^{m+1*} \]
which is obtained by tensoring \( K^\sim \) with \( M_{(n+1)^m} \) is not exact at the term \((m,n)\), whence neither is \( K^\sim \). 

3. The case \( q = 1 \)

Assume in this section, \( q = 1 \), thus, \( R^2 = 1 \) and \( H\)-comod is a tensor category (i.e., symmetric rigid monoidal). By a theorem of Deligne, there exists a faithful and exact, tensor (i.e., symmetric monoidal) functor \( F \) from \( H\)-comod to the category of vector superspaces. Under this functor, \( V \) is mapped to a certain vector super space \( \overline{V} \) and \( R \) is mapped to the supersymmetry on \( \overline{V} \otimes \overline{V} \), denoted by \( T \).

We can therefore reconstruct a super bialgebra \( \overline{E} \) and a Hopf super algebra \( \overline{H} \) from \( V \) and \( T \). We will show that this Hopf superalgebra is isomorphic to the function algebra over the general linear supergroups \( GL(n) \), where \((m,n)\) is the birank of \( R \), or, in other words, the super dimension of \( V \) is \((m|n)\). Indeed, \( \overline{E} \) is the function algebra on \( \text{End}(\overline{V}) \) and the image of \( M_\lambda \) under the embedding \( F \) are simple \( \overline{E}\)-comodules. Since \( F \) is faithful and exact and since \( M_\lambda \neq 0 \Leftrightarrow \lambda_m + 1 < n \), we conclude that \( \overline{E} \) is isomorphic to the function algebra on \( M(m|n) \). Hence \( \overline{H} \) is isomorphic to the function algebra on \( GL(m|n) \), by virtue of \([13]\).

Let \( \overline{K} \) denote the image of the complex \( K^\sim \). Then the homology of \( \overline{K} \) is concentrated at the term \((m,n)\), and is one-dimensional; it defines the super determinant. As a consequence, the homology of \( K^\sim \) is also concentrated at the term \((m,n)\), for \( F \) is faithful and exact. Let \( D \) denote the homology of \( K^\sim \). Then \( \overline{D} \), the image of \( D \) under \( F \), is one-dimensional and hence invertible, consequently,
\[ F(D^* \otimes D) \cong F(D^*) \otimes F(D) \cong D^* \otimes D \cong k, \]
where the last isomorphism is given by the evaluation morphism, that is the image of ev\(_D\) under \( F \). Since \( F \) is faithful and exact, we conclude that \( D^* \otimes D \cong k \), that is \( D \) is invertible, hence one-dimensional. \( \overline{M}_\lambda \) denote the image where \( \lambda \) runs in the set of partitions for which
Theorem 2. Let $R$ be a vector symmetry of birank $(m,n)$. Then the associated Koszul complex is exact every where except at the term $(m,n)$ where it has a one-dimensional homology group which determines a group-like element called homological determinant.

4. The case $q$ generic

Using the result of the previous section we show in this section that given a Hecke symmetry of birank $(m,n)$ that depends algebraically on $q$, then, for a dense set of values $q$, the associated Koszul complex is exact every where except at the term $(m,n)$, where is has a one-dimensional homology group and thus determines a group-like element in $H$, called the homological determinant. In this section $k$ will be assumed to be the field $\mathbb{C}$ of complex numbers.

Thus let $R = R_q$ be a Hecke symmetry depending on a parameter $q \in \mathbb{C}$. We first observe that the dimension of $\Lambda^k_q$ does not depend on $q$, as far as $q$ is not a root of unity. Indeed, $\Lambda^k_q$ can be defined as the image of a projection, its dimension can be given as the trace of a matrix which depend algebraically on $q$, since $\mathbb{C}$ substracted the set of root of unity is still connected, we conclude that this trace, being always integral must be a constant. The same happens with $S^k_q$. Thus, the terms of $K^ -$ has the dimension not depending on $q$.

On the other hand, observe that the rank of the operator $d^k_q$, for almost any $q$ (that is except a finite number of values of $q$) is large then the rank of $d^1_q$ and for the kernel of $d^k_q$ we have the reversed inequality. Consequently, the dimension over $k$ of the homology group $H(K_q^K)$ for almost any $q$ is least then or equal to the dimension of $H(K_q^K)$. According to Theorems 1 and 2 we conclude that for an algebraically dense set of values of $q$, $H(K_q^K) = 0$, for all $(k,l) \neq (m,n)$ and $H(K_q^K) = k$.

Theorem 3. Let $R = R_q$ be a Hecke symmetry over $\mathbb{C}$, depending algebraically on $q$. Then there is an algebraically dense set of values of $q$ for which the homology of the Koszul complex is one-dimensional and concentrated at the term $(m,n)$, where $(m,n)$ is the birank of $R$.

References

[1] P. Deligne. Catégories tensorielles. Preprint, 2002.
[2] R. Dipper and G. James. Representations of Hecke Algebras of General Linear Groups. Proc. London Math. Soc., 52(3):20–52, 1986.
[3] R. Dipper and G. James. Block and Idempotents of Hecke Algebras of General Linear Groups. Proc. London Math. Soc., 54(3):57–82, 1987.
[4] E.Cremmer and J.-L.Gervais. The Quantum Groups Structure Associated With Non-linearly Extended Virasoro Algebras. Comm. Math. Phys., 134:619–632, 1990.
[5] A. Edrei. Proof of a Conjecture of Schoenberg on the Generating Function of a Totally Positive Sequence. Canad. J. of Math., 5:86–94, 1953.
[6] D.I. Gurevich. Algebraic Aspects of the Quantum Yang-Baxter Equation. Leningrad Math. Journal, 2(4):801–828, 1991.
[7] Phung Ho Hai. Koszul Property and Poincaré Series of Matrix Bialgebra of Type $A_n$. J. of Algebra, 192(2):734–748, 1997.
[8] Phung Ho Hai. Poincaré Series of Quantum Spaces Associated to Hecke Operators. Acta Math. Vietnam, 24(2):236–246, 1999.
[9] Phung Ho Hai. On Matrix Quantum Groups of Type $A_n$. Int. J. of Math., 11(9):1115–1146, 2000.
[10] Phung Ho Hai. Splitting comodules over Hopf algebras and application to representation theory of quantum groups of type $A_{(0)}$. J. of Algebra, 245(1):20–41, 2001.
[11] Phung Ho Hai. The integral on quantum super groups of type $A_{\ell|l}$. Asian J. of Math., 5(4):751–770, 2001.
[12] M. Jimbo. A q-analogue of $U(\mathfrak{g}(N + 1))$, Hecke algebra, and the Yang-Baxter equation. Lett. Math. Phys., 11:247–252, 1986.
[13] R. Larson and J. Towber. Two Dual Classes of Bialgebras Related To The Concepts of “Quantum Groups” and “Quantum Lie Algebra”. Comm. in Algebra, 19(12):3295–3345, 1991.
[14] V.V. Lyubashenko. Hopf Algebras and Vector Symmetries. Russian Math. Survey, 41(5):153–154, 1986.
[15] V.V. Lyubashenko. Superanalysis and Solutions to the Triangles Equation. PhD thesis, Kiev, 1987.
[16] V.V. Lyubashenko and A. Sudbery. Quantum Super Groups of $GL(n|m)$ Type: Differential Forms, Koszul Complexes and Berezinians. *Duke Math. Journal*, 90:1–62, 1997.

[17] I.G. Macdonald. *Symmetric functions and the Hall polynomials*. Oxford University Press, New York, 1979 (Second edition 1995).

[18] S. Majid and M. Markl. Glueing Operation for $R$-Matrices, Quantum Groups and Link-Invariants of Hecke Type. *Math. Proc. Camb. Philos. Soc.*, 119(1):139–166, 1996.

[19] Yu.I. Manin. *Gauge Field Theory and Complex Geometry*. Springer-Verlag, 1988.

[20] M. Takeuchi and D. Tambara. A new one-parameter family of $2 \times 2$ quantum matrices. *Hokkaido Math. Journal*, XXII(3):409–419, 1992. See also Proc. Japan. Acad., 67, no. 8, 267–269. 1991.

Institute of Mathematics, P.O.Box 631, 10000 BOHO, HANOI