Potential equations for plasmas round a rotating black hole

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The generalized Helmholtz equations of relativistic multifluid plasmas can be integrated for axisymmetric equilibria in close analogy to the magnetic flux conservation law in ideal magnetohydrodynamics (J. D. Bekenstein and E. Oron, Phys. Rev. D 18, 1809 (1978)). The results are, for each fluid component, two flux functions and a potential equation for the poloidal stream function. Ampère’s equation for the four-potential $A_\nu$ is reduced to two coupled equations for the time-like and the toroidal component. So we have altogether four potential equations for a two-component plasma; they can be derived from a variational principle.
I. INTRODUCTION

Observations of active galactic nuclei and other massive objects have increased the interest in equilibrium plasma models within a Schwarzschild or Kerr geometry. Even if the metric is given and all physical quantities are assumed to be independent of a toroidal angle \( \varphi \) and of time \( t \), there is still some effort needed to reduce the whole set of equations for the electromagnetic field and the fluid quantities. This has been done within ideal magnetohydrodynamics (MHD) by several authors with various degrees of completeness and sophistication. A characteristic feature of the MHD description is the conservation of magnetic flux in a system co-moving with the center-of-mass four-velocity \( u^\mu \) (of ions, essentially); this equation (“Ohm’s law”, Eq. (2) below) has been “integrated” for general stationary and axisymmetric systems by Bekenstein and Oron [1] who give also a basic discussion and some historical background of general-relativistic MHD. As a result, one can represent the electromagnetic field tensor \( F_{\mu\nu} \) completely by the particle flux \( nu^\mu \), the two Killing vectors associated with the translational symmetry in \( t \) and \( \varphi \), and two “flux functions” which are constant along the poloidal stream lines (Eq. (14) below). Further reductions of the whole set of MHD equations and discussions of the astrophysical background have been given by Camenzind [2], Mobarry and Lovelace [3], Nitta, Takahashi and Tomimatsu [4], and Beskin and Par’ev [5]. Thus one arrives, as in the non-relativistic case, at a single potential equation for the magnetic flux function \( \Psi \) (the covariant toroidal component \( A_\varphi \) of the vector potential \( A \)), together with some constraints.

One problem with these MHD models is the large number of arbitrary flux functions – no dissipation mechanism has been included to reduce this number –, and a fluid picture may be questionable if the collision frequencies are too low. This, however, is not our point here, and we admit an ideal fluid picture on reasons of simplicity. The question is, however, whether the usual MHD theory is consistent for a really rotating black hole [6]. In this case the metric is necessarily non-diagonal; the relevant element \( g_{t\varphi} \) is the invariant scalar product of the two Killing vectors mentioned above \( \mathcal{K} \), so it can by no means be transformed to zero. On the other hand, it couples the components \( A_t \) and \( A_\varphi \) in Ampère’s law which will, in general, contradict the strong coupling of \( A_t \) and \( A_\varphi \) in the MHD models (here they have to be functions of each other). So we are led to the question of how to reduce the set of multifluid plasma equations and Maxwell’s equations for stationary axisymmetric systems without the discrepancy mentioned above.

In Section II we give a short account of the work of Bekenstein and Oron, leading to the strong coupling of \( A_t \) and \( A_\varphi \). The same solution, however, can be used for any generalized Helmholtz equation in ideal fluid descriptions. Two examples of these Helmholtz equations are derived in Sections III and IV, respectively: One refers to the ordinary non-relativistic MHD equation (momentum balance), the other to the relativistic multifluid plasma, assuming a constant temperature of all species. The latter example is used for the reduction problem in Section V. Here it is shown that the multifluid equations are reduced to one single potential equation for each species, and the poloidal components of Ampère’s equation are integrated. A numerical solution of this full set of four potential equations would be facilitated by the existence of a variational principle; the solution could then be approximated with finite elements by minimizing the corresponding functional. So it may be interesting that such a functional exists, as is shown in Section VI. Finally we discuss the results in Section VII.

II. FLUX CONSERVATION IN AXISYMMETRIC PLASMAS

Let us assume a plasma configuration where all physical quantities, including the metric tensor components \( g_{\mu\nu} \), are independent of time \( t \) and of a toroidal angle \( \varphi \). We choose a coordinate system \((x^\mu)\) with \( x^0 = ct \) \((c = 1)\), \( x^1 = \varphi \), and \( x^2, x^3 \) some poloidal coordinates. Assuming in addition that all physical quantities are invariant to the simultaneous inversion of \( t \) and \( \varphi \) – which is reasonable for any rotating equilibrium – the most general line element \( ds \) can be represented as follows [3]:

\[
(ds)^2 = g_{rs} \, dx^r \, dx^s + g_{ab} \, dx^a \, dx^b,
\]

where the indices \( r, s \) run from 0 to 1, and \( a, b \) from 2 to 3. We want to determine an electromagnetic field tensor \( F_{\mu\nu} \) which is consistent with Eq. (1), and which, in addition, obeys the condition of magnetic flux conservation in a medium moving with the Eulerian four-velocity \( u^\mu \). The latter condition is familiar from ideal magnetohydrodynamics (MHD) and means that a certain four-vector \( E_\mu \) (“co-moving electric field”) should vanish:

\[
E_\mu \equiv F_{\mu\nu} u^\nu = 0.
\]

The field tensor \( F_{\mu\nu} \), of course, should solve the homogeneous Maxwell equations; therefore it can be written as the curl of a four-potential \( A_\mu \) in the usual manner:

\[
F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}.
\]
where \( \frac{\partial}{\partial x^\mu} \) is the partial derivative of the quantity in brackets with respect to \( x^\mu \). Finally we use the continuity equation
\[
(\sqrt{-g} u^\mu)_{,\mu} = 0,
\] (4)
with \( g \) the determinant of \( g_{\mu\nu} \) and \( n \) the particle number density in the local inertial rest frame.

To solve Eq. (2) we remember that any skew-symmetric tensor \( F_{\mu\nu} \) can be represented by two four-vectors, \( E_\mu \) and \( B_\mu \) (“co-moving magnetic field”): denoting all components in the local inertial rest frame by primes (with \( u^\mu = c = 1; u^\nu = 0 \) for \( i = 1, 2, 3 \)) we define
\[
E'_\mu = 0 \quad ; \quad E'_i = F'_{i0}, \quad i = 1, 2, 3 \\
B'_\mu = 0 \quad ; \quad B'_i = F'_{jk},
\]
where in the last equation \((i, j, k)\) is a cyclic permutation of \((1, 2, 3)\). In this coordinate system we use the Minkowski metric
\[
g'_{\mu\nu} = \text{diag}(1, -1, -1, -1),
\]
and \( E'_\mu = -E'_\nu, \quad B'_\mu = -B'_\nu \) are the local electric and magnetic fields, respectively. It is then easily seen that the covariant representation of \( E_\mu \) in the laboratory frame with any velocity field \( u^\mu(x) \) is given by the left part of Eq. (2), while \( B_\mu \) is given by
\[
B_\mu = -\frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} u^\nu F^\rho\sigma,
\] (5)
The covariant equation (5) can easily be proved by writing it in the local inertial rest frame. Eq. (5) is generally valid; in the case of flux conservation according to Eq. (2) it simplifies to
\[
F_{\mu\nu} = -\varepsilon_{\mu\nu\rho\sigma} u^\rho B^\sigma.
\] (6)
Obviously the field tensor \( F_{\mu\nu} \) is then orthogonal not only to \( u^\mu \), as required by Eq. (2), but also to \( B^\mu \):
\[
B^\mu F_{\mu\nu} = 0.
\] (7)
The remaining task is now to construct \( B^\mu \) for axisymmetric equilibria.

For this purpose we consider the two Killing vectors \( (\xi^\mu) \) associated with these symmetries, namely:
\[
(\xi^\mu) = (k^\mu) \equiv (1, 0, 0, 0), \\
(\xi^\mu) = (m^\mu) \equiv (0, 1, 0, 0).
\]
In both cases they lead to a vanishing partial derivative of any physical quantity \( A \) in the direction of \( \xi^\mu \):
\[
\xi^\mu A_{,\mu} = 0.
\] (9)
Let us assume that all components \( A_\nu \) of the vector potential share this property (though a gauge transformation could destroy it). Then Eq. (3) leads to
\[
F_{r\nu} \equiv \xi^\mu F_{\mu\nu} = -\xi^\mu A_{\mu,\nu} = -A_{r,\nu},
\] (10)
where \( r = 0 \) for \( \xi^\mu = k^\mu \) and \( r = 1 \) for \( \xi^\mu = m^\mu \). The right-hand side of this expression is obviously non-zero only for \( \nu = a = 2 \) or 3. Differentiating Eq. (10) with respect to \( x^b \), \( b = 2 \) or 3, but \( b \neq a \), we find
\[
F_{ra,b} = -A_{r,ab} = F_{rb,a}.
\] (11)
To obtain the same expressions from Eq. (7) we write the factor \( \sqrt{-g} \) of \( \varepsilon_{\mu\nu\rho\sigma} \) explicitely, with the remaining constant permutation symbol \( \tilde{\varepsilon}_{\mu\nu\rho\sigma} \):
\[ \varepsilon_{\mu\nu\rho\sigma} = \sqrt{-g} \tilde{\varepsilon}_{\mu\nu\rho\sigma}. \]

Then our integrability condition from Eqs. (7) and (11) reads as follows:

\[
0 = F_{ra,b} - F_{rb,a} \\
= \tilde{\varepsilon}_{rsab} \left[ (\sqrt{-g} u^s B^r)_b - (\sqrt{-g} u^b B^r)_b \\
+ (\sqrt{-g} u^s B^b)_a - (\sqrt{-g} u^b B^b)_a \right].
\]

The right-hand side of this equation is understood with fixed and mutually different values for \( r, s, a \) and \( b \). We can re-write it by restoring the summation convention with respect to the index \( a \):

\[
(\sqrt{-g} u^s B^a)_a - (\sqrt{-g} u^b B^a)_a = 0; \quad s = 0, 1.
\]

The general solution \( B^\mu \) of Eq. (12) with \( B^\mu u_\mu = 0 \) can be written as follows:

\[
B^\mu = \alpha u^\mu + b^\mu; \quad \alpha \equiv -b^\mu u_\mu,
\]

where \( b^\mu \) solves the same Eq. (12) as \( B^\mu \). Since it is linear in \( b^\mu \) we can solve it separately for the poloidal part \( b^a \) and for \( b^r \). Ignoring now \( b^r \) we can solve Eq. (12) for \( b^a \) if the poloidal flow \( u^a \) is not identically zero. It is useful to represent \( b^a \) by a linear combination of the Killing vectors, namely:

\[
b^a = -nC K^a; \quad K^a \equiv k^a + \beta n^a,
\]

where the coefficients \( C \) and \( \beta \) have to be determined suitably. The factor \( n \) has been included in order to take advantage of the mass conservation law, Eq. (4), where the replacement \( \mu \rightarrow a \) is allowed due to the symmetries. Then we have a solution of Eq. (12) if \( C \) and \( \beta \) are constant along the poloidal stream lines (“flux functions”):

\[
u^a C, a = 0 = u^a \beta, a, \quad (\text{15})
\]

\[
B^\mu = -nC \left[ K^\mu - (K^\lambda u_\lambda)u^\mu \right], \quad (\text{16})
\]

\[
F_{\mu\nu} = nC \varepsilon_{\mu\nu\rho\sigma} u^\rho K^\sigma. \quad (\text{17})
\]

The last equation has been obtained by inserting Eq. (14) into Eq. (7). While \( C \) is an arbitrary flux function, \( \beta \) is fixed by the condition that \( F_{\mu\nu} \) is perpendicular to both \( u^\mu, B^\mu \) or \( u^\rho, K^\rho \). Using Eqs. (2), (8), (10), (14) and (16) we find

\[
0 = K^\mu F_{\mu\nu} = -(A0, \nu + \beta A1, \nu). \quad (\text{18})
\]

This equation can only be fulfilled if \( A_0 \) and \( A_1 \) are functions of each other and constant along the poloidal stream lines. This is indeed the case as can now easily be shown from Eq. (2) for \( \mu = r \) and Eq. (10):

\[
0 = F_{rr} u^r = -A_r, a u^a. \quad (\text{19})
\]

This completes our particular solution for \( B^\mu \) and \( F_{\mu\nu} \) if \( b^a = 0 \). It coincides with the result of Ref. [1] (their \( A \) is our \( -\beta \)). Why is a poloidal part of \( b^a \) not possible? Our symmetry requires \( F_{rs} = 0 \) according to Eq. (10); from Eq. (7) we have

\[
F_{rs} = -\varepsilon_{rsab} u^b B^a = -\varepsilon_{rsab} u^b B^b.
\]

Here we see that for a non-zero poloidal flow \( u^a \) the poloidal part of \( B^\mu \) must be proportional to \( u^a \), otherwise \( F_{rs} \) would be non-zero. Therefore the solution as given in Eqs. (10) and (13) is unique, up to the specification of two flux functions, \( C \) and \( \beta \).

Sometimes it is useful to re-write the result of this section in particular poloidal coordinates as defined by the poloidal stream lines, \( \Psi = \text{const.} \), and an angle-like coordinate \( \theta \) varying along the poloidal stream lines:

\[
x^2 = \Psi; \quad x^3 = \theta.
\]

From Eq. (10) we know that \( A_r = A_r(\Psi) \), so we may identify one of both components with \( \Psi \) itself, while the other component defines \( \beta \) according to Eq. (13), e.g.:
\[ A_1 \equiv \Psi \quad ; \quad \beta = -dA_0(\Psi)/d\Psi. \]  \hfill (20)

In this coordinate system we have from Eq. (10):

\[ F_{13} = 0 \quad ; \quad F_{12} = -1. \]

Inserting here Eq. (17) we find

\[ u^2 = 0 \quad ; \quad C = 1/(\sqrt{-g} n u^3). \]  \hfill (21)

The remaining elements of \( F_{\mu\nu} \) in this coordinate system are then:

\[ F_{0a} = \beta F_{1a} \quad ; \quad F_{23} = (\beta u^0 - u^1)/u^3. \]

Then our vector \( B^\mu \) and field tensor elements \( F_{\mu\nu} \) are, for a given geometry and flow field, completely determined by one single flux function \( \beta \).

### III. VORTICITIES IN MHD PLASMAS

Flux conservation laws are generally expected if dissipative processes are negligible. Let us first discuss the non-relativistic case. The oldest example is the Kelvin/Helmholtz theorem for a neutral fluid with pressure \( p = p(\varrho) \), where \( \varrho \) is the mass density. Euler’s equation may then be written using the vorticity \( \omega \) and the Bernoulli function \( U \) as follows:

\[ \frac{\partial v}{\partial t} + \omega \times v + \nabla U = 0, \]  \hfill (22)

and we obtain immediately Helmholtz’s equation for the vorticity by taking the curl:

\[ \frac{\partial \omega}{\partial t} + \nabla \times (\omega \times v) = 0. \]  \hfill (23)

This equation is the prototype of any vector field \( \omega \to \Omega \) which is “frozen in”, moving with the fluid velocity \( v \), and which is the curl of another field, say \( V \):

\[ \frac{\partial \Omega}{\partial t} + \nabla \times (\Omega \times v) = 0; \quad \Omega \equiv \nabla \times V. \]  \hfill (24)

To get an equation for \( V \) we integrate Eq. (24), introducing a scalar potential \( \Phi \):

\[ \frac{\partial V}{\partial t} + \Omega \times v + \nabla \Phi = 0. \]  \hfill (25)

For \( V = A \) and \( \Omega = B \) we obtain

\[ E + v \times B = 0, \]  \hfill (26)

where the electric field \( E \) is the usual expression in terms of the potentials \( A, \Phi \) (with \( c = 1 \)). Eq. (26) is just the non-relativistic limit of Eq. (2). For other ideal fluid models one may also find a Kelvin/Helmholtz theorem, though the corresponding vectors \( V, \Omega \) are more complicated. The momentum balance of ideal MHD theory refers to a “center-of-mass” fluid with total pressure \( p \), total mass density \( \varrho \), center-of-mass velocity \( v \), and the Lorentz force due to the electric current density \( j \):

\[ \varrho \frac{dv}{dt} + \nabla p = j \times B. \]

The picture of an idealized fluid requires not only zero resistivity according to Eq. (26), but, more generally, zero entropy production. So we expect for ideal MHD theory a second Kelvin/Helmholtz theorem and associated vectors \( V, \Omega \); they seem to be unknown, but they can be constructed using appropriate Lagrangian and Lin variables [9], [10]. Here we use three Lin variables: the entropy per mass \( s \) and the two Euler potentials \( q_\lambda (\lambda = 1, 2) \) of the magnetic field:
The constraints of entropy and magnetic flux conservation are then expressed by these three material invariants:

\[
\frac{ds}{dt} = \frac{dq_1}{dt} = \frac{dq_2}{dt} = 0,
\]

and the Ansatz for \( V \) reads as follows:

\[
V = v - r \nabla s - \sum_{\lambda=1}^{2} v_\lambda \nabla q_\lambda. \tag{27}
\]

The coefficients \( r, v_\lambda \) are now determined in order to match Eq. (25) with an arbitrary potential \( \tilde{\Phi} \). From Eq. (27) and its curl, the equation of \( \Omega \), we find after some rearrangements the following purely kinematical relation:

\[
\frac{\partial V}{\partial t} + \Omega \times v = -\nabla \left( \frac{v^2}{2} + h \right) + T \nabla s + \frac{1}{\rho} \mathbf{j} \times \mathbf{B},
\]

where \( h \) is the enthalpy per mass, \( T \) the temperature, and \( \mathbf{j} \) the current density as determined from Ampère’s law. The right-hand side of Eq. (28) is then just \( -\nabla \tilde{\Phi} \), with

\[
\tilde{\Phi} = \frac{v^2}{2} + h + r \frac{\partial s}{\partial t} + \sum_{\lambda} v_\lambda \frac{\partial q_\lambda}{\partial t},
\]

provided the coefficients \( r, v_\lambda \) obey the following equations of motion:

\[
\frac{dr}{dt} = T; \quad \frac{dv_1}{dt} = \frac{1}{\rho} \mathbf{j} \cdot \nabla q_2; \quad \frac{dv_2}{dt} = -\frac{1}{\rho} \mathbf{j} \cdot \nabla q_1.
\]

So we find a second flux conservation law in ideal MHD; it refers to the following generalized vorticity:

\[
\Omega \equiv \nabla \times V = \omega - \nabla r \times \nabla s - \sum_{\lambda} \nabla v_\lambda \times \nabla q_\lambda. \tag{29}
\]

The generalized Helmholtz equation (24) allows, of course, the “trivial” solution \( \Omega = 0 \), corresponding to a potential flow for \( V, V = -\nabla S \); then Eq. (27) leads to the so-called Clebsch representation of \( v \):

\[
v = -\nabla S + r \nabla s + \sum_{\lambda=1}^{2} v_\lambda \nabla q_\lambda.
\]

(For \( \Omega \neq 0 \) we would need a further pair of Clebsch variables \( v_\lambda, q_\lambda \) with \( dv_\lambda/dt = dq_\lambda/dt = 0 \). In contrast to the first Kelvin/Helmholtz theorem associated with Eq. (24) we have no simple advantage from this second vorticity law; in particular, the equations of motion for \( r \) and \( v_\lambda \) have to be solved, in addition to the conservation laws for \( s \) and \( q_\lambda \) and (not shown here) the Bernoulli equation for \( S \). The situation, however, becomes more transparent if we leave the MHD description and treat the plasma as a fluid of different species \( j \) (\( j = e \) : electrons, \( j = i \) : ions of any kind) interacting only via the electromagnetic field, with the following momentum balance for each species:

\[
g_j \left( \frac{\partial}{\partial t} + v_j \cdot \nabla \right) v_j + \nabla p_j = e_j n_j (\mathbf{E} + v_j \times \mathbf{B}).
\]
There is an obvious formal bridge to the MHD description for a two-component plasma: Neglecting the electron mass \( m_e \sim \rho_e \to 0 \) we obtain \( \mathbf{v}_e \) as the center-of-mass velocity \( \mathbf{v} \); assuming, in addition, quasi-neutrality with simply charged ions \( n_e = n_i = n \) we can replace \( \mathbf{v}_e \) as follows:

\[
\mathbf{v}_e = \mathbf{v} - \frac{j}{|e|n}.
\]

Adding both momentum equations gives then the correct ideal MHD balance with \( p = p_e + p_i \), while the momentum equation of electrons leads to a generalized Ohm’s law replacing Eq. (26):

\[
E + \mathbf{v} \times \mathbf{B} = \frac{1}{|e|n} (j \times \mathbf{B} - \nabla p_e).
\]

This “modified” MHD theory is the bridge mentioned above, but a simpler form of the two Kelvin/Helmholtz theorems is obtained if we come back to the multifluid description, now within general relativity.

**IV. VORTICITIES IN ISOTHERMAL PLASMAS**

The flux conservation theorems associated with perfect fluid models depend crucially on the equation of state for the pressure, e.g., \( p = p(\rho, s) \). The non-relativistic case in standard textbooks on neutral fluids usually assumes constant entropy \( s \) throughout the fluid volume, leading to the usual Helmholtz equation, Eq. (24). The corresponding equation in general relativity has been derived by Taub [13]. If the entropy of a non-relativistic neutral fluid varies in space, the situation is more subtle; it has been discussed already by Eckart [14] and later in Ref. [9]. The result is that the vector \( \mathbf{V} \) whose vorticity flux is conserved differs from \( \mathbf{v} \) by \( (-r \nabla s) \), the second term on the right-hand side of Eq. (27). For a non-relativistic multifluid plasma Eq. (27) is replaced by

\[
\mathbf{V}_j = \mathbf{v}_j - r_j \nabla s_j + \frac{e_j}{m_j} \mathbf{A},
\]

leading to a “three-circulation theorem” corresponding to the three constituents of \( \mathbf{V} \) (\( \mathbf{A} \) is the vector potential). In general relativity we have, instead of Eq. (25) (from which the general Helmholtz equation (24) follows), the equation corresponding to Eqs. (2) and (3) for each species (suppressing now the species index \( j \)):

\[
u^\nu \Omega_{\mu\nu} = 0, \\
\Omega_{\mu\nu} \equiv V_{\nu,\mu} - V_{\mu,\nu}.
\]

The vector \( V_{\nu} \) for a multifluid plasma with \( s = \text{const.} \) has been given by Lichnerowicz [15] and Carter [16] (“single constituent perfect fluid”, his example (c) in §4), and for varying \( s \) by Ref. [11], namely:

\[
V_{\mu} = \sigma u_{\mu} + r s_{,\mu} + \frac{e}{m} A_{\mu},
\]

where \( \sigma \) is the relativistic enthalpy per volume, and \( u_{\mu} \) the covariant Eulerian four-velocity (with \( c = 1 \)). One of Carter’s results refers also to a neutral fluid with two constituents and varying entropy, but the resulting canonical momentum per volume \( (= \rho V_{\mu}) \) reads, in our notation, as follows (from Eqs. (4.25), (4.26), (4.32) and (4.33) of Ref. [16]):

\[
\rho V_{\mu} \equiv n\pi_{\mu} = \sigma u_{\mu},
\]

so the term \( r s_{,\mu} \) is missing.

Here we consider a different physical situation which may be of astrophysical interest: We assume that a radiation field acts like a heat reservoir for electrons and protons, maintaining a constant temperature of them. Then the term \( r s_{,\mu} \) is again absent, and we find the relativistic Helmholtz equation for the ordinary canonical vorticity of both species. In this case, \( \sigma \) will turn out to be the free relativistic enthalpy per volume. We start with the material energy-momentum-stress tensor \( T_{\mu \nu} \) of an ideal electron or ion fluid without denoting the species index explicitly. Since both fluids are only coupled via the electromagnetic field by the Coulomb/Lorentz force, we can write the energy-momentum balance for both species as follows:

\[
T_{\mu \nu} = e n F_{\mu \nu} u^\nu.
\]
The semi-colon indicates the covariant derivative. The coupling of both species by the right-hand side of Eq. (30) implies now that the magnetic flux is not conserved, neither for the electron nor for the ion fluid. The tensor $T_{\mu \nu}$ for an ideal fluid is well-known (see, e.g., Ref. [12]):

$$T_{\mu \nu} = \sigma u^\mu u^\nu - p \delta_{\mu \nu}.$$  

The last term in this equation is the scalar pressure $p$ in the local inertial rest frame times the Kronecker symbol, and the scalar $\sigma$ depends on the equation of state. Using also the mass conservation, Eq. (4), we obtain

$$T_{\mu \nu} ;_{\nu} = \rho u^\nu \left( \sigma u^\mu \right)_{,\mu} - p_{,\nu}.$$  

This four-vector must be orthogonal to $u^\mu$, as is also the right-hand side of Eq. (30). With the normalization of $u^\mu (u^\mu u^\mu = 1)$ we find then

$$0 = u^\nu T_{\mu \nu} ;_{\nu} = \rho u^\nu \left( \sigma u^\mu \right)_{,\mu} - u^\nu p_{,\nu},$$  

where the differential $d$ means variation along the path of a fluid element. In the co-moving inertial rest frame we use the Gibbs-Duhem relation for the free enthalpy $\mu$ per mass:

$$d\mu = -s dT + \frac{1}{\rho} dp.$$  

(31)

Ignoring temperature variations, and including the relativistic rest energy $\rho c^2$ (with $c \neq 1$ for the moment) the resulting expression for $\sigma$ is then as follows:

$$\sigma = \rho \left( 1 + \frac{\mu}{c^2} \right).$$  

(32)

To derive a flux conservation law we insert the vector potential for $F_{\mu \nu}$ in Eq. (30) and put all terms to the left-hand side:

$$u^\nu \left[ \left( \frac{\sigma}{\rho} u^\mu \right)_{,\nu} + \frac{e}{m} A_{\mu \nu} - \frac{e}{m} A_{\nu \mu} \right] - \frac{1}{\rho} p_{,\mu} = 0.$$  

To eliminate the pressure term we calculate the partial derivative from Eqs. (31) and (32) with $dT = 0$ (and this time $c = 1$):

$$\frac{1}{\rho} p_{,\mu} = \left( \frac{\sigma}{\rho} \right)_{,\mu} = u^\nu \left( \sigma u^\mu \right)_{,\mu},$$  

where in the last step we used again the normalization of $u^\mu$. This is now just the term leading to flux conservation for the canonical vorticity of each species; we define

$$V_\mu = \frac{\sigma}{\rho} u^\mu + \frac{e}{m} A_\mu,$$  

(33)

$$\Omega_{\mu \nu} = V_{\nu ;\mu} - V_{\mu ;\nu} = V_{\nu ,\mu} - V_{\mu ,\nu},$$  

(34)

and we find

$$\Omega_{\mu \nu} u^\nu = 0.$$  

(35)

The solution of Eq. (32) for axisymmetric equilibria is now simply obtained from the previous section: We replace there $F_{\mu \nu}$ by $\Omega_{\mu \nu}$ and $A_\mu$ by $V_\mu$. The mean velocities of electrons and ions, however, are usually different; the fluxes of their general vorticities $\Omega_{\mu \nu}$ are therefore conserved in different frames. It is interesting to rewrite Eq. (34) for the spatial components of the canonical velocity, $V^i$, in the special-relativistic case (no gravity), namely:

$$\frac{\partial V}{\partial t} + \Omega \times v + \nabla U = 0,$$

where $U(\equiv cV_0)$ is the relativistic Bernoulli function, and $\Omega^i (\equiv -\Omega_{jk})$ are the spatial vector components associated with $\Omega_{\mu \nu}$. This equation is now again of the same form as Eq. (25), and we recover Eq. (24) as the prototype of any special-relativistic generalized Helmholtz equation.
V. MULTIFLUID PLASMA EQUATIONS FOR AXISYMMETRIC EQUILIBRIA

Let us use a general poloidal coordinate system to solve the mass conservation law, Eq. (4), for each species separately by introducing an appropriate stream function \( \chi \), namely (\( \varepsilon^{1ab} \) is again the permutation symbol, here for spatial indices):

\[
u^a = \frac{1}{\sqrt{-g}} \varepsilon^{1ab} \chi_b.
\]

(36)

The poloidal stream lines are then given by \( \chi = \text{const.} \), and Helmholtz’ equation (35) in the symmetry plane, \( \mu = r = 0, 1 \), reads as follows:

\[
0 = -V_{r,\nu} u^\nu = -V_{r,a} u^a.
\]

Similarly as in Eq. (19) we conclude that the components \( V_{r} \) can be any flux functions with respect to \( \chi \):

\[
V_{r} = V_{r}(\chi).
\]

These two flux functions and the corresponding components of \( A_{\nu} \) fix two components of \( u_{\nu} \) for each species up to a factor \( \frac{\sigma}{\rho} \), namely:

\[
u_r = \frac{\sigma}{\rho} \left( V_r(\chi) - \frac{e}{m} A_r \right).
\]

(37)

Assuming that \( \chi \) and \( A_{\nu} \) are given elsewhere we may read the normalization condition for \( u_{\nu} \) as an equation for \( n \):

\[
1 = g^{rs} u_r u_s + g_{ab} u^a u^b.
\]

(38)

So all fluid quantities besides \( \chi \) are determined by Eqs. (36) - (38), and we find all elements of \( \Omega_{\mu \nu} \) except \( \Omega_{ab} \):

\[
\Omega_{rs} = 0 ; \quad \Omega_{ra} = -V_{r,a} = -V'_{r,\chi,a},
\]

(39)

where the prime means differentiation with respect to \( \chi \). The general solution for \( \Omega_{\mu \nu} \), however, can be obtained from Eqs. (17) and (18) with appropriate changes of notation, namely:

\[
\Omega_{\mu \nu} = nC(\chi) \varepsilon_{\mu \nu \rho \sigma} u^\rho K^\sigma,
\]

\beta = -V'_{\mu}(\chi)/V'_{1}(\chi).

(40)

(41)

Comparing this with the results above (Eqs. \( (36) \) and \( (39) \)) we find the flux function \( C(\chi) \):

\[
C(\chi) = -V'_{1}(\chi).
\]

(42)

The equation for the stream function \( \chi \) is then obtained from Eq. (31) with \( \mu = 2 \) and \( \nu = 3 \), where the left-hand side is determined according to the definitions (34) and (33); the result is then the following:

\[
\Omega_{23} = \sqrt{-g} n V'_{r}(\chi) u^r,
\]

\[
\Omega_{23} \equiv \left( \sigma u^3 \right)_{,2} - \left( \sigma u^2 \right)_{,3} + e F_{23}.
\]

(43)

(44)

The complete set of fluid equations (36) - (38) and (43) - (44) for the unknown variables \( u_r, n \) and \( \chi \) is written for any poloidal coordinate system, thus allowing any number of particle species. To solve finally Ampère’s equation in the poloidal plane we are then free to use particular coordinates and a particular gauge of \( A_{\nu} \). To be consistent with the usual notation we denote the projections of the \( j^a \)-lines onto the poloidal plane the lines \( \Psi = \text{const.} \), where the flux function \( \Psi \) may be identified with a “radial” coordinate \( x^2 \) as in Sec. II, and the stream function \( \sim I \) of \( j^a \) is denoted as a flux function, \( I = I(\Psi) \). The continuity equation for \( j^a \) in the poloidal plane is then solved as follows:

\[
j^2 = 0 ; \quad 4\pi \sqrt{-g} j^3 = I'(\Psi),
\]

(45)

where the prime of \( I \) means differentiation with respect to \( \Psi \). The flux function \( I(\Psi) \) is, of course, not independent from the stream functions \( \chi \) of the different species. From
\[ j^\mu = \sum_j e n u^\mu \]

and Eq. (45) we find the following relation:

\[ \sum_j e \chi = -\frac{I(\Psi)}{4\pi} + \text{const.}, \] (46)

where the sum over \( j \) is the sum with respect to the different particle species. A convenient gauge of \( A_\nu \) is, as in the non-relativistic case, the condition that \( A \) is tangential to the surfaces \( \Psi = \text{const.} \) of the current lines:

\[ A \cdot \nabla \Psi = 0. \]

This equation can usually be fulfilled by an appropriate gauge function since \( \Psi \) is a radial coordinate; for \( \Psi = x^2 \) it reads

\[ A_2 = 0 \quad \text{or} \quad F_{23} = A_{3,2}. \] (47)

Let us now start with Ampère’s equation for \( A_\nu \):

\[ \left[ \sqrt{-g} g^{\mu\sigma} g^{\nu\tau} (A_{\sigma,\tau} - A_{\tau,\sigma}) \right]_{,\nu} = -4\pi \sqrt{-g} j^\mu. \] (48)

In the symmetry plane, \( \mu = r = 0, 1 \), this equation is decoupled from the poloidal components of \( A_\nu \); inserting the fluid quantities for \( j^r \) we have then the following set of two equations for the components \( A_s, s = 0, 1 \):

\[ \left[ \sqrt{-g} g^{rs} g^{ab} A_{s,b} \right]_{,a} = 4\pi \sqrt{-g} g^{rs} \sum_j e n \frac{\sigma}{\sigma} \left( V_s(\chi) - e \frac{m}{m} A_s \right). \] (49)

In the poloidal plane, Eq. (48) can be re-written with indices \( a, b, c, d \) which run from 2 to 3 only:

\[ \left[ \sqrt{-g} g^{ab} g^{cd} F_{bd} \right]_{,c} = -4\pi \sqrt{-g} j^a. \]

The left-hand side is easily evaluated due to the antisymmetry of \( F_{bd} \); we introduce the determinants of \( g_{ab} \) and \( g_{rs} \) explicitly:

\[ g_{\text{pol}} = \text{det}(g_{ab}); \quad g_{\text{sym}} = \text{det}(g_{rs}), \]

then we obtain the following form of Ampère’s equation in the poloidal plane for \( a = 2 \):

\[ \left[ \sqrt{-g_{\text{sym}}} g_{\text{pol}} F_{23} \right]_{,3} = -4\pi \sqrt{-g} j^2, \] (50)

and for \( a = 3 \):

\[ -\left[ \sqrt{-g_{\text{sym}}} g_{\text{pol}} F_{23} \right]_{,2} = -4\pi \sqrt{-g} j^3. \] (51)

Inserting here Eq. (45) we identify the expression in brackets as a flux function, namely:

\[ \sqrt{-g_{\text{sym}}} g_{\text{pol}} F_{23} = I(\Psi). \] (52)

In the non-relativistic case this is simply the covariant toroidal component of the magnetic field. Finally, the component \( A_3 \) is determined from Eqs. (52) and (47); it is not a flux function, and it is not needed in the remaining set of equations.
VI. A VARIATIONAL PRINCIPLE

The numerical solution of our potential equations for \( \chi \), Eqs. (43) - (44), and \( A_r \), Eq. (49), is simplified by the existence of a functional of \( \chi \) and \( A_r \) which is stationary in equilibrium. We need, however, still another quantity whose variation leads to the normalization condition, Eq. (38). So we look for a functional \( W \) of three quantities, \( W = W(\sigma, \chi, A_r) \) say, whose independent variations lead to the equilibrium conditions. The particle density \( n \) (which is not varied), and \( \sigma \) of each species are then obtained afterwards from a combination of the normalization condition and an equation of state according to Eq. (32).

Let us start with the normalization condition, where \( u^a \) and \( u_r \) are given by Eqs. (36) and (37), respectively.

The latter equation gives

\[
\delta u_r = \frac{1}{\sigma} u_r \delta \sigma + \frac{\partial}{\partial \chi} V_r'(\chi) \delta \chi - \frac{en}{\sigma} \delta A_r, \tag{53}
\]

while \( \delta u^a \) depends on \( \delta \chi \) only:

\[
\delta u^a = \frac{1}{\sqrt{-g \ n}} \varepsilon^{lab}(\delta \chi), \tag{54}
\]

It is then easily realized that \( W \) could be of the following form:

\[
W(\sigma, \chi, A_r) = \int d^2x \sqrt{-g} \sum_j \frac{\sigma}{2} (g_{ab} u^a u^b - g^{rs} u_r u_s - 1) + \cdots,
\]

where the terms indicated by dots are independent of \( \sigma \), and the integration is done in a fixed region of the poloidal plane \((x^2, x^3)\). It is a remarkable effect that in the expression above the usual invariant \( u^a u^a \) is replaced by \( u^a u^a - u_r u_r \) which is only invariant under transformations in the poloidal plane. A similar replacement will be needed in the invariant which produces Maxwell’s equations in vacuum. Using our solution for \( F_{23} \), Eq. (52), and \( F_{rs} = 0, F_{ur} = A_{r,a} \), we find

\[
-\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{8\pi (-g_{sym})} + \frac{I^2(\Psi)}{8\pi} - \frac{1}{8\pi} g^{rs} g_{ab} A_{r,a} A_{s,b}.
\]

Changing now the sign of the last expression above, we are led to the following functional:

\[
W(\sigma, \chi, A_r) = \int d^2x \sqrt{-g} \left[ \sum_j \frac{\sigma}{2} (g_{ab} u^a u^b - g^{rs} u_r u_s - 1) \right.
\]
\[
- \frac{1}{8\pi (-g_{sym})} + \frac{1}{8\pi} g^{rs} g_{ab} A_{r,a} A_{s,b} \right] \tag{55}
\]

Variations with respect to \( \chi \) and \( A_r \) are now done by eliminating derivatives of \( \delta \chi \) and \( \delta A_r \) by partial integrations, assuming that \( \delta \chi \) and \( \delta A_r \) vanish at the boundary. Furthermore, we have to vary \( I(\Psi) \) according to Eq. (52), namely:

\[
\delta I(\Psi) = (-4\pi e) \delta \chi. \tag{56}
\]

The total variation of \( W(\sigma, \chi, A_r) \) is then obtained with the following result:

\[
\delta W = \int d^2x \sqrt{-g} \left\{ \sum_j \frac{1}{2} (g_{ab} u^a u^b + g^{rs} u_r u_s - 1) \delta \sigma \right.
\]
\[
+ \sum_j \frac{1}{(-g_{sym})} \left( \tilde{\Delta} \chi + eI(\Psi) + g_{sym} \partial u^r V_r'(\chi) \right) \delta \chi \right.
\]
\[
+ \left[ \sum_j enu^r \frac{1}{4\pi \sqrt{-g}} (\sqrt{-g} g^{rs} g_{ab} A_{s,b},a) \delta A_r \right\}, \tag{57}
\]
the vanishing factor of \( \delta \chi \)
The vanishing factor of \( \delta A \)

This is just the condition that \( \chi \) is stationary with respect to variation of \( \chi \), assuming that \( g_{sym} \) is finite at this point.

VII. DISCUSSION

A plasma equilibrium near a rotating black hole has been considered in the ideal fluid picture. The usual MHD equations imply two flux conservation laws. One is the well-known conservation law of magnetic flux (Ohm’s law with vanishing resistivity); it leads to two constants of motion for stationary axisymmetric systems: The covariant time-like and toroidal components of the vector potential are constant on the poloidal stream lines of the plasma bulk velocity. This well-known fact is in contradiction with the coupling of \( A_t \) and \( A_\phi \) for a metric with \( g_{t\phi} \neq 0 \), as is appropriate for a rotating black hole. Simple Grad-Shafranov type MHD equilibria (see, e.g., Ref. [17]) are then ruled out in this case. The second flux conservation law of MHD has been derived in Section III for the non-relativistic case (Eqs. (24) and (27) - (29)); it is, however, not as simple as the magnetic flux conservation law. A more reasonable description for \( g_{t\phi} \neq 0 \) is given by the multifluid equations: They can be cast into the form of Helmholtz equations for each fluid component \([11]\); they are particularly simple for isothermal plasmas, as is shown here (Eqs. (33) - (35)). Since the form of these equations is exactly the same as the magnetic flux conservation law, we obtain two constants of motion for each species of an axisymmetric equilibrium, the time-like and toroidal components of the canonical angular momentum per mass of a fluid particle of a particular species; this has to be expected for axisymmetric equilibria if the fluid components interact only through the electromagnetic field. The poloidal components of Ampère’s equation can be integrated, too, by adjusting the coordinates to the lines of the electric current, \( \Psi = \text{const.} \); the relevant flux function \( J(\Psi) \), Eq. (12), corresponding to the toroidal magnetic field, is now simply related to the stream functions \( \chi \) of the different plasma components according to Eq. (43). The whole set of potential equations can now be summarized in a functional \( W(\sigma, \chi, A_t) \), where \( \sigma \) is the free enthalpy density of a particular species, \( \chi \) the stream function of its poloidal velocity, and \( A_t \) stands for \( A_t \) and \( A_\phi \). The total variation of \( W \) produces then the normalization condition for the Eulerian four-velocity of each species, the potential equation for \( \chi \) (Eq. (64)), and Ampère’s equations for \( A_t \) and \( A_\phi \).

It is interesting to consider possible solutions of these equations in a given geometry with \( g_{t\phi} \neq 0 \) like Kerr’s metric. Equilibria with poloidal velocity fields are of interest from an observational point of view, because they are able to exchange mass, angular momentum etc. between inner and outer parts. Plasma models with pure poloidal velocity fields, however, are not possible; the reason is that the toroidal velocity \( u_\phi \) is proportional to \( (V_\phi - (e/m)A_\phi) \) (Eq. (27)), where \( V_\phi \) is constant on the stream lines of the particular species which is considered, but \( A_\phi \) generally not. Equilibria with pure toroidal velocities are possible for an electron-positron plasma. In this case Eq. (60) is solved trivially with \( \chi \equiv 0, V_\phi \equiv \text{const.}, V_t = 0 \), and the velocity components \( u_t, u_\phi \) become equal but opposite in sign, up to a constant \( V_\phi \); they are determined from Ampère’s equation, which is highly nonlinear due to the normalization condition. This solution, however, seems to be artificial, and an acceptable solution will exhibit inevitably also poloidal velocity components.
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