On the trivial solutions for the rotating patch model

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Abstract. In this paper, we study the clockwise simply connected rotating patches for Euler equations. By using the moving plane method, we prove that Rankine vortices are the only solutions to this problem in the class of slightly convex domains. We discuss in the second part of the paper the case where the angular velocity $\Omega = \frac{1}{2}$, and we show without any geometric condition that the set of the V-states is trivial and reduced to the Rankine vortices.

1. Introduction

We shall study in this paper some aspects of the vortex motion for the two-dimensional incompressible Euler system which can be written with the vorticity–velocity formulation in the form,

$$\begin{cases}
\partial_t \omega + v \cdot \nabla \omega = 0, & x \in \mathbb{R}^2, \ t > 0, \\
v = \nabla^\perp \Delta^{-1} \omega, \\
\omega(0, x) = \omega_0(x).
\end{cases}$$

(1)

Here, $\nabla^\perp = (-\partial_2, \partial_1)$, $v = (v_1, v_2)$ is the velocity field, and the $\omega$ its vorticity given by the scalar $\omega = \partial_1 v_2 - \partial_2 v_1$. The classical theory dealing with the local\global well-posedness of smooth solutions is well developed, and we refer for instance to [1,3].

According to Yudovich result [10], the vorticity equation has a unique global solution in the weak sense provided the initial vorticity $\omega_0$ belongs to $L^1 \cap L^\infty$. This result allows to deal rigorously with the so-called vortex patches which are initial vortices uniformly distributed in a confined region $D$, that is, $\omega_0 = \chi_D$ the characteristic function of $D$. Since the vorticity is transported along trajectories, we conclude that the vorticity preserves the vortex patch structure for any positive time. This means that for any $t \geq 0$, $\omega(t) = \chi_{D_t}$, with $D_t = \psi(t, D)$ is the image of $D$ by the flow $\psi$ which satisfies the ordinary differential equation

$$\partial_t \psi(t, x) = v(t, \psi(t, x)), \quad \psi(0, x) = x.$$  

(2)

The dynamics of the boundary of $D_t$ is in general complex and very difficult to follow. By using the contour dynamics method, we may parametrize the boundary
by a function $\gamma_t : \mathbb{T} \to \partial D_t$ satisfying a nonlinear and non-local equation of the following type

$$\partial_t \gamma_t = -\frac{1}{2\pi} \int_{\partial D_t} \log |\gamma_t - \xi| \, d\xi.$$ 

There are few examples known in the literature with explicit dynamics. The first one is Rankine vortex where $D$ is a disc; in this case, the particle trajectories are circles centered at the origin, and therefore, $D_t = D, \forall t \geq 0$. The second example is a remarkable one and discovered by Kirchhoff [8] is the ellipses. In this case, the domain $D_t$ does not change its shape and undergoes a perpetual rotation around its barycenter with uniform angular velocity $\Omega$ related to the semi-axes $a$ and $b$ through the formula $\Omega = ab/(a + b)^2$. See, for instance, [1, p. 304].

It seems that the ellipses are till now the only explicit example with such properties but whether or not other non-trivial implicit rotating patches exist has been discussed in the last few decades from numerical and theoretical point of view. To be more precise about these structures, we say that $\omega_0 = \chi_D$ is a V-state or a rotating patch if there exists a real number $\Omega$ called the angular velocity such that the support of the vorticity $\omega(t) = \chi_{D_t}$ is described by

$$D_t = R_{x_0,\Omega_t} D, \quad \forall t \geq 0,$$

with $R_{x_0,\Omega_t}$ being the planar rotation with center $x_0$ and angle $\Omega t$. Deem and Zabusky [4] were the first to reveal numerically the existence of simply connected V-states with the $m$-fold symmetry for the integer $m = 3, 4, 5$. Recall that a domain $D$ is said to be $m$-fold symmetric if it is invariant by the dihedral group $D_m$ which is the symmetry group of a regular polygon of $m$ sides. A few years later, Burbea [2] gave an analytic proof by using the bifurcation theory showing the existence of a countable family of V-states with the $m$-fold symmetry for any $m \geq 2$. They can be identified to one-dimensional branches bifurcating from the Rankine vortex at the simple “eigenvalues” $\{\Omega = \frac{m-1}{2m}, m \geq 2\}$. See also [6], where the $C^\infty$ boundary regularity of the bifurcated V-states close to the disc was proven. It seems that close to the disc, the bifurcating branches rotate with bounded angular velocities, $\Omega \in [0, \frac{1}{2}].$ It is important to know whether all the V-states possess an angular velocity in this strip. From the implicit function theorem, we know that close to the disc there are no non-trivial V-states associated with $\Omega \notin [0, \frac{1}{2}]$.

In this paper, we give a partial answer to this problem. We shall first show that there is no clockwise rotating patches, that is, $\Omega \leq 0$, but with some geometric constraints. When $\Omega = 0$, this corresponds to stationary patches, and we know from a recent result of Fraenkel [5] in gravitational theory that the discs are the only stationary patches. He used the techniques of moving plane method which can be adapted to our framework only when $\Omega \leq 0$. More precisely, we obtain the following result.

**THEOREM 1.** Let $D$ be a $C^1$ bounded simply connected domain convex or more generally being in the class $\Sigma_{\arccos \frac{1}{\sqrt{3}}}$ introduced in the Definition 2. Assume that $\chi_D$