PERIODIC POINTS OF ALGEBRAIC ACTIONS OF DISCRETE GROUPS

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Abstract. Let $\Gamma$ be a countable group. A $\Gamma$-action on a compact abelian group $X$ by continuous automorphisms of $X$ is called Noetherian if the dual of $X$ is Noetherian as a $\mathbb{Z}(\Gamma)$-module. We prove that any Noetherian action of a finitely generated virtually nilpotent group has a dense set of periodic points.

1. Introduction

Let $\Gamma$ be a countable group. An algebraic $\Gamma$-action is an action $\alpha$ of $\Gamma$ on a compact metrizable abelian group $X$ by continuous automorphisms of $X$. By duality theory, any such action induces a $\Gamma$-action on $\hat{X}$, the dual of $X$, by automorphisms of $\hat{X}$. Hence $\hat{X}$ can be viewed as a $\mathbb{Z}[\Gamma]$-module, where $\mathbb{Z}[\Gamma]$ is the integral group ring of $\Gamma$. The action $\alpha$ is called Noetherian if $\hat{X}$ is Noetherian as a $\mathbb{Z}[\Gamma]$-module. Equivalently, $\alpha$ is Noetherian if any decreasing sequence

$$X = X_0 \supset X_1 \supset X_2 \supset \cdots$$

of closed $\Gamma$-invariant subgroups stabilizes.

For $\Gamma = \mathbb{Z}^d$, the study of such actions was initiated in [5], and since then these systems have been extensively studied (see [9] for a comprehensive account). For general $\Gamma$, several important results about entropy and homoclinic points of Noetherian $\Gamma$-actions have been proved in recent years ([1], [2], [3], [6]). However, unlike the commutative case, many basic dynamical properties are only partly understood in the general situation.

In this paper, we study periodic points of Noetherian $\Gamma$-actions. Recall that a point $x \in X$ is periodic if the $\Gamma$-orbit of $x$ is finite. It is

2010 Mathematics Subject Classification. 37B05, 37B20.

Key words and phrases. Nilpotent groups, automorphisms, periodic points.
known that when $\Gamma = \mathbb{Z}^d$, for any Noetherian $\Gamma$-action $(X, \alpha)$ the space $X$ contains a dense set of periodic points ([9], Theorem 5.7). However, the proof uses tools from commutative algebra and does not generalize to actions of non-abelian groups. The question whether Noetherian actions of all residually finite groups admit a dense set of periodic points was raised in [7, Problem 8.5].

In this paper we show that for algebraic actions of countable groups, the density of periodic points is related to vanishing of certain cohomology groups. As a consequence, we obtain the following:

**Theorem 1.1.** Let $\Gamma$ be a finitely generated virtually nilpotent group and let $\alpha$ be a Noetherian action of $\Gamma$ on a compact metrizable abelian group $X$. Then $X$ contains a dense set of $\alpha$-periodic points.

As another application we show that there exist Noetherian actions of finitely generated residually finite groups that do not have a dense set of periodic points, thus giving a negative answer to the above question in the most general case.

## 2. Virtual first cohomology groups

Let $\alpha$ be an action of a countable group $\Gamma$ on a compact metrizable group $X$ by continuous automorphisms of $X$. A *virtual 1-cocycle* of $\alpha$ is a map $c$ from a finite index subgroup $\Lambda \subset \Gamma$ to $X$ that satisfies the equation

$$c(\gamma \gamma') = c(\gamma) + \alpha(\gamma)(c(\gamma'))$$

for all $\gamma, \gamma' \in \Lambda$. Two virtual 1-cocycles $c_1 : \Lambda_1 \to X$ and $c_2 : \Lambda_2 \to X$ are *equivalent* if there exists a finite index subgroup $\Lambda \subset \Lambda_1 \cap \Lambda_2$ such that $c_1(\gamma) = c_2(\gamma)$ for all $\gamma \in \Lambda$. For any $x \in X$, the map $c_x : \Gamma \to X$ defined by $c_x(\gamma) = \alpha(\gamma)(x) - x$ is a 1-cocycle. A virtual 1-cocycle $c : \Lambda \to X$ is said to be a *virtual coboundary* if it is equivalent to $c_x$ for some $x \in X$. We will call two virtual 1-cocycles $c_1 : \Lambda_1 \to X$ and $c_2 : \Lambda_2 \to X$ *cohomologous* if there exists a finite index subgroup $\Lambda \subset \Lambda_1 \cap \Lambda_2$, and a virtual coboundary $c : \Lambda \to X$ such that $c_1(\gamma) - c_2(\gamma) = c(\gamma)$ for all $\gamma \in \Lambda$. It is easy to see that the equivalence classes of virtual 1-cocycles is a group with respect to the pointwise addition, and the equivalence classes of virtual coboundaries form a subgroup.
We will denote the quotient group by $H^1_v(\alpha)$ and call it the virtual first cohomology group of $\alpha$.

Let $(Y, \beta)$ be an algebraic $\Gamma$-action, and let $X \subset Y$ be a closed $\beta$-invariant subgroup such that $Y/X$ is finite. Let $\alpha$ denote the restriction of $\beta$ to $X$ and let $\Lambda \subset \Gamma$ be a finite index subgroup that acts trivially on $Y/X$. For any $y \in Y$ we define a map $c_y : \Lambda \to X$ by $c_y(\gamma) = \beta(\gamma)(y) - y$. It is easy to see that $c_y$ is a virtual 1-cocycle of $\alpha$.

**Proposition 2.1.** Let $(Y, \beta)$, $(X, \alpha)$ and $y \in Y$ be as above. Then the virtual 1-cocycle $c_y$ is a virtual coboundary if and only if the coset $y + X$ contains a $\beta$-periodic point.

*Proof.* Suppose $y_1 \in y + X$ is a $\beta$-periodic point, and $\Lambda_1 \subset \Lambda$ is a finite index subgroup such that $\beta(\gamma)(y_1) = y_1$ for all $\gamma \in \Lambda_1$. Then $x = y - y_1 \in X$ and

$$c_y(\gamma) = \beta(\gamma)(y) - y = \beta(\gamma)(x) - x$$

for all $\gamma \in \Lambda_1$. Hence $c_y$ is a virtual coboundary of $\alpha$. Conversely, if $c_y$ is a virtual coboundary of $\alpha$, then there exists a finite index subgroup $\Lambda_1$ and $x \in X$ such that $c_y(\gamma) = \beta(\gamma)(x) - x$ for all $\gamma \in \Lambda_1$. If $p = y - x$ then $p \in y + X$ and it is fixed by $\beta(\Lambda_1)$. Hence it is a periodic point. \qed

A compact abelian group $X$ is zero-dimensional if the connected component containing $0_X$ is trivial. From the duality theory it follows that $X$ is zero-dimensional if and only if every element of $\hat{X}$ has finite order.

**Proposition 2.2.** Let $\Gamma$ be a countable group, and let $\alpha$ be a Noetherian action of $\Gamma$ on a zero-dimensional compact abelian group $X$. Then there exists $k \geq 1$ such that $kx = 0$ for all $x \in X$.

*Proof.* Let $\{\chi_1, \ldots, \chi_n\} \subset \hat{X}$ be a finite set that generates $\hat{X}$ as a $\mathbb{Z}(\Gamma)$-module. As every element of $\hat{X}$ has finite order, we can find $k \geq 1$ such that $k\chi_i = 0$ for $i = 1, \ldots, n$. Since for any $m$ the set $N_m = \{\chi : m\chi = 0\}$ is a $\mathbb{Z}(\Gamma)$-submodule of $\hat{X}$, we deduce that $N_k = \hat{X}$, i.e., $k\chi = 0$ for all $\chi \in \hat{X}$. If $x \in X$ then for all $\chi \in \hat{X}$, $\chi(kx) = k\chi(x) = 0$. Hence $kx = 0$. \qed
Our next proposition shows that all virtual 1-cocycles of Noetherian \( \Gamma \)-actions on zero-dimensional groups arise in the manner described in Proposition 2.1.

**Proposition 2.3.** Let \( \Gamma \) be a countable group and let \((X, \alpha)\) be a Noetherian action of \( \Gamma \) on a zero-dimensional group \( X \). Then for any virtual 1-cocycle \( c : \Lambda \to X \) there exists \( k \geq 1 \) and an algebraic \( \Lambda \)-action \( \beta \) on \( Y = (\mathbb{Z}/k\mathbb{Z}) \times X \) such that

1. For all \( \gamma \in \Lambda \), \( \beta(\gamma) \) fixes the first co-ordinate.
2. \( \beta(\gamma)(0, x) = (0, \alpha(\gamma)(x)) \) for all \( \gamma \in \Lambda \) and \( x \in X \).
3. \( c(1, 0) = c \).

**Proof.** Since \( \alpha \) is Noetherian and \( X \) is zero-dimensional, by the previous proposition there exists \( k \geq 1 \) such that \( kx = 0 \) for all \( x \in X \). For any \( z \in Y \) and a continuous automorphism \( \tau \) of \( X \) there exists a unique homomorphism \( \theta : Y \to Y \) such that \( \theta(1, 0) = z \) and \( \theta(0, x) = (0, \tau(x)) \) for all \( x \in X \). It is easy to see that \( \theta \) is continuous. For \( \gamma \in \Lambda \) let \( \beta(\gamma) \) denote the unique continuous endomorphism of \( Y \) satisfying the following two conditions:

1. \( \beta(\gamma)(0, x) = (0, \alpha(\gamma)(x)) \) for all \( x \in X \).
2. \( \beta(\gamma)(1, 0) = (1, c(\gamma)) \).

Similarly we define \( \beta'(\gamma) \) to be the unique continuous endomorphism of \( Y \) with the property that \( \beta'(\gamma)(0, x) = (0, \alpha(\gamma)^{-1}(x)) \) and \( \beta'(\gamma)(1, 0) = (1, -\alpha(\gamma)^{-1}c(\gamma)) \). Then

\[
\beta(\gamma)\beta'(\gamma)(1, 0) = \beta(\gamma)(1, 0) + \beta(\gamma)(0, -\alpha(\gamma)^{-1}c(\gamma)) = (1, 0).
\]

As \( \beta(\gamma)\beta'(\gamma) \) fixes both \( \{0\} \times X \) and the point \( (1, 0) \), we conclude that \( \beta(\gamma)\beta'(\gamma) = I \). A similar computation shows that \( \beta'(\gamma)\beta(\gamma) = I \). Hence \( \beta(\gamma) \) is a continuous automorphism of \( Y \) for each \( \gamma \in \Lambda \). As \( c(\gamma_1\gamma_2) = c(\gamma_1) + \alpha(\gamma_1)c(\gamma_2) \), it follows that

\[
\beta(\gamma_1\gamma_2)(1, 0) = (1, c(\gamma_1\gamma_2)) = \beta(\gamma_1)\beta(\gamma_2)(1, 0).
\]

Since \( \beta(\gamma_1\gamma_2)(0, x) = \beta(\gamma_1)\beta(\gamma_2)(0, x) \) for all \( x \in X \), this implies that \( \beta(\gamma_1\gamma_2) = \beta(\gamma_1)\beta(\gamma_2) \). Hence \( \beta \) defines an algebraic \( \Lambda \)-action on \( Y \). It is easy to see that \( \beta \) satisfies all three conditions stated in the proposition.
\( \square \)
**Lemma 2.4.** Let $\Gamma$ be a countable group and let $(X, \alpha)$ be a Noetherian $\Gamma$-action such that $X$ admits non-zero periodic points. Then $X$ also contains non-trivial $\alpha$-invariant finite subgroups.

**Proof.** Let $x \in X$ be a non-zero $\alpha$-periodic point and let $\Lambda \subset \Gamma$ be the stabilizer of $x$. Since the orbit of $x$ is finite, $\Lambda$ is a finite index subgroup of $\Gamma$. We define a subgroup $\Lambda_0 \subset \Gamma$ by

$$\Lambda_0 = \bigcap_{\gamma \in \Gamma} \gamma \Lambda \gamma^{-1}.$$ 

It is easy to see that $\Lambda_0 \subset \Gamma$ is normal and has finite index. Let $Y$ denote the set of points of $X$ that are fixed by all elements of $\Lambda_0$. Then $Y$ is a non-trivial $\alpha$-invariant closed subgroup of $X$. Since $\Lambda_0 \subset \Gamma$ has finite index, the action $\alpha|_{\Lambda_0}$ is also Noetherian. As $\Lambda_0$ acts trivially on $Y$, it follows that any collection of closed subgroups of $Y$ has a minimal element. Let $Y_0$ be a minimal element of the collection of all non-trivial subgroups of $Y$. Since closed subgroups of $Y_0$ are in one to one correspondence with subgroups of $\hat{Y}_0$, we deduce that $\hat{Y}_0$ does not admit non-trivial subgroups. Therefore $\hat{Y}_0$, and hence $Y_0$, are isomorphic with $\mathbb{Z}/p\mathbb{Z}$ for some prime $p$. In particular, $Y_0$ is finite. We define $H \subset X$ by

$$H = \sum_{\gamma \in \Gamma} \alpha(\gamma)(Y_0).$$

As $\alpha(\gamma)(Y_0) = Y_0$ for all $\gamma \in \Lambda_0$, and $\Lambda_0 \subset \Gamma$ has finite index, the above sum is finite and $H$ is a well defined closed subgroup of $X$. It is easy to see that $H$ is finite and $\alpha$-invariant. $\square$

The above lemma is not true if $\alpha$ is not assumed to be Noetherian. For example, suppose $\Gamma$ is an arbitrary countable group and $\alpha$ is the trivial action of $\Gamma$ on $X = \hat{\mathbb{Q}}$. Then every point of $X$ is $\alpha$-periodic, but since the group $\mathbb{Q}$ does not have non-trivial finite quotients, $X$ does not admit non-trivial finite subgroups.

**Theorem 2.5.** Let $\Gamma$ be a countable group satisfying :

1. Every Noetherian $\Gamma$-action $(X, \alpha)$ with $X \neq \{0\}$, has a non-zero $\alpha$-periodic point.
2. The group $H_v^1(\alpha)$ is trivial for every Noetherian $\Gamma$-action $(X, \alpha)$ on a zero-dimensional group $X$. 
Then every Noetherian action of \( \Gamma \) on a compact abelian group (not necessarily zero-dimensional) admits a dense set of periodic points. Conversely, if every Noetherian \( \Gamma \)-action admits a dense set of periodic points, then both the conditions are satisfied.

**Proof.** Suppose every Noetherian action of \( \Gamma \) admits a dense set of periodic orbits. Then the first condition is automatically satisfied. Let \( \alpha \) be a Noetherian \( \Gamma \)-action on a zero-dimensional group \( X \), and let \( c : \Lambda \to X \) be a virtual 1-cocycle of \( \alpha \). By Proposition 2.3, there exists \( k \geq 1 \) and an algebraic \( \Lambda \)-action \( \beta \) on \( Y = (\mathbb{Z}/k\mathbb{Z}) \times X \) such that \( c = c_{(1,0)} \) and \( \beta(\gamma)|_X = \alpha(\gamma) \) for all \( \gamma \in \Lambda \). Since the coset containing \((1,0)\) is an open subset of \( Y \), it contains a \( \beta \)-periodic orbit.

From Proposition 2.4, we deduce that \( c \) is a virtual coboundary. Hence \( H^1_v(\alpha) = \{ 0 \} \).

Now suppose \( \Gamma \) satisfies both the conditions stated above. Let \((Y, \beta)\) be a Noetherian action of \( \Gamma \). Let \( X \subset Y \) denote the closure of the set of all \( \beta \)-periodic points. Since the sum of two periodic points is again a periodic point it follows that \( X \) is a closed \( \beta \)-invariant subgroup. Suppose \( X \neq Y \). Let \( \bar{\beta} \) denote the induced action of \( \Gamma \) on the quotient \( Y/X \). Since \( \Gamma \) satisfies the first condition, by Lemma 2.4 there exists a finite \( \bar{\beta} \)-invariant non-zero subgroup \( F \subset Y/X \). We define \( Y' = \pi^{-1}(F) \), where \( \pi \) is the projection map from \( Y \) to \( Y/X \). We choose a finite index subgroup \( \Lambda \subset \Gamma \) that acts trivially on \( F \) under \( \bar{\beta} \).

Let \( y \notin X \) be a point in \( Y' \), and let \( c_y : \Lambda \to X \) denote the virtual 1-cocycle of \((X, \beta)\) defined by \( c_y(\gamma) = \beta(\gamma)(y) - y \).

Let \( X_0 \) denote the connected component of \( X \). It is easy to see that \( X_0 \) is a \( \beta \)-invariant closed subgroup of \( X \). Let \( \beta_1 \) denote the induced action of \( \Lambda \) on \( X/X_0 \). As \( X/X_0 \) is zero-dimensional and \( \beta_1 \) is Noetherian, applying Proposition 2.2 we deduce that there exists \( l \geq 2 \) such that \( lp \in X_0 \) for all \( p \in X \). Let \( k \geq 2 \) be a positive integer such that \( kx = 0 \) for all \( x \in F \). Then \( lky \in X_0 \). Since \( X_0 \) is connected, the map \( x \mapsto lkx \) is a surjective endomorphism of \( X_0 \). We find \( q \in X_0 \) such that \( lky = lkw \), and define a virtual 1-cocycle \( c_1 : \Lambda \to X \) by \( c_1 = c_y - c_q \). Then for any \( \gamma \in \Lambda \),

\[
lkc_1(\gamma) = lk(\beta(\gamma)(y-q) - (y-q)) = 0.
\]
Hence the image of $c_1$ is contained in $H = \{ x \in X : lkx = 0 \}$. Since $H$ is closed, $\beta$-invariant, and zero-dimensional; from the second condition we conclude that $c_1$ is a virtual coboundary. As $c_1$ is cohomologous to $c_y$, this implies that $c_y$ is also a virtual coboundary. Applying Proposition 2.1 we deduce that the coset $y + X$ contains a periodic point $y_1$. This contradicts the fact that $y \notin X$, and shows that $Y = X$, i.e., the set of $\beta$-periodic points is dense in $Y$. □

We now construct an example of a Noetherian action of a finitely generated residually finite group that does not have a dense set of periodic points. Let $H$ denote the group of all functions from $\mathbb{Z}$ to $\mathbb{Z}$ with finite support, equipped with pointwise addition. We define an action of $\mathbb{Z}$ on $H$ by $n \cdot f(i) = f(i + n)$. Let $\Gamma$ denote the semi-direct product of $\mathbb{Z}$ and $H$ defined by $(f_1, n_1) \cdot (f_2, n_2) = (f_1 + n_1 \cdot f_2, n_1 + n_2)$. It is easy to see that $\Gamma$ is torsion free. For $k \in \mathbb{Z}$, we define $f^k \in H$ by $f^k(i) = 0$ if $k \neq i$ and $f^k(i) = 1$ if $k = i$. Then for any $k \in \mathbb{Z}$, $(0, 1)(f^{k+1}, 0) = (f^k, 0)(0, 1)$, i.e., $(0, 1)(f^{k+1}, 0)((0, 1)^{-1} = (f^k, 0)$. As $\{(0, f^k) : k \in \mathbb{Z}\}$ generates $H$ as a $\mathbb{Z}$-module, this shows that $\Gamma$ is generated by $(0, 1)$ and $(f^0, 0)$. In particular, $\Gamma$ is finitely generated. For $k \geq 2$ let $\Gamma_k \subset \Gamma$ denote the set of all elements $(f, n)$ such that $n = 0 \pmod{k}$ and

$$\sum_{i=-\infty}^{\infty} f(ki + j) = 0 \pmod{k} \ \forall j = 0, 1, \ldots, k - 1.$$ 

It is easy to see that for each $k$, $\Gamma_k$ is a normal subgroup of $\Gamma$ and $\Gamma/\Gamma_k$ is finite. Moreover for each non-zero $(f, n) \in \Gamma$ there exists $k$ such that $(f, n) \notin \Gamma_k$. Hence $\Gamma$ is residually finite.

Let $X$ denote the compact abelian group $(\mathbb{Z}/2\mathbb{Z})^\mathbb{Z}$, equipped with the product topology and pointwise addition, and let $S : X \to X$ denote the shift map defined by $S(x)(i) = x(i + 1)$. We define an algebraic $\Gamma$-action $\alpha$ on $X$ by $\alpha(f, n)(x) = S^n(x)$. It is easy to see that the shift action of $\mathbb{Z}$ on $X$ is Noetherian. Since $\alpha|_H$ is trivial and the action of $\Gamma/H \cong \mathbb{Z}$ induced by $\alpha$ is the shift action on $X$, we deduce that $\alpha$ is also Noetherian.
Let $\pi : H \to X$ denote the homomorphism defined by $\pi(f)(i) = f(i) \pmod{2}$. We define $c : \Gamma \to X$ by $c(f, n) = \pi(f)$. Then $c((f_1, n_1)(f_2, n_2)) = \pi(f_1 + n_1 : f_2) = c(f_1, n_1) + S^{n_1}(\pi(f_2))$. Since

$$S^{n_1}(\pi(f_2)) = S^{n_1}(c(f_2, n_2)) = \alpha(f_1, n_1)(c(f_2, n_2)),$$

this shows that $c$ is a 1-cocycle of $\alpha$. We pick any $x \in X$ and define $c' : \Gamma \to X$ by $c'(\gamma) = \alpha(\gamma)(x) - x$. Let $\Lambda$ be an arbitrary finite index subgroup of $\Gamma$. Since $\alpha(\gamma) = I$ for all $\gamma \in H$, it follows that $c'(H \cap \Lambda) = \{0\}$. On the other hand, the restriction of $c$ to $H$ is a homomorphism with infinite image. As $H \cap \Lambda$ is a finite index subgroup of $H$, we deduce that $c(H \cap \Lambda)$ is also infinite. Hence $c|_\Lambda \neq c'|_\Lambda$. As $\Lambda$ and $x$ are arbitrary, we conclude that $c$ is not a virtual coboundary.

We now apply Proposition 2.3. Let $\beta$ denote the action corresponding to the cocycle $c$. Since $\alpha$ is Noetherian, so is $\beta$. Applying Proposition 2.1 we conclude that $\beta$ does not have a dense set of periodic points.

### 3. Density of periodic orbits

In this section we concentrate on the case when $\Gamma$ is polycyclic-by-finite. A countable group $\Gamma$ is *polycyclic* if there exists a decreasing sequence of subgroups

$$\Gamma = \Gamma_n \supset \Gamma_{n-1} \supset \cdots \supset \Gamma_0 = \{0\}$$

such that for each $i$, $\Gamma_i$ is a normal subgroup of $\Gamma_{i+1}$ and $\Gamma_{i+1}/\Gamma_i$ is cyclic. Any such series is called a *polycyclic series* of $\Gamma$. A group $\Gamma$ is called *polycyclic-by-finite* if it contains a finite index subgroup that is polycyclic. If $\Gamma$ is polycyclic-by-finite then every subgroup of $\Gamma$ is finitely generated, and $\mathbb{Z}(\Gamma)$ is a Noetherian ring.

Let $\Gamma$ be a polycyclic-by-finite group, $\Gamma_0 \subset \Gamma$ be a polycyclic subgroup of finite index and $\Gamma_0 = \Gamma_n \supset \Gamma_{n-1} \supset \cdots \supset \Gamma_0 = \{0\}$ be a polycyclic series of $\Gamma_0$. Then the number of $i$'s such that $\Gamma_i/\Gamma_i$ is infinite cyclic is independent of $\Gamma_0$ and the polycyclic series. This number is known as the *Hirsch number* of $\Gamma$. The following proposition summarizes some basic properties of this invariant (see [10]).

**Proposition 3.1.** Let $\Gamma$ be a polycyclic-by-finite group.

1. $h(\Gamma) = 0$ if and only if $\Gamma$ is finite.
(2) If $\Gamma_1$ is a subgroup of $\Gamma$ then $h(\Gamma_1) \leq h(\Gamma)$.
(3) If $\Gamma_1 \subset \Gamma$ is normal then $h(\Gamma) = h(\Gamma_1) + h(\Gamma/\Gamma_1)$.

We note that the previous proposition applies to finitely generated virtually nilpotent groups since they are polycyclic-by-finite. In the proof of Theorem 1.1 we will also use the following result about polycyclic-by-finite groups (see [8]):

**Proposition 3.2.** Let $\Gamma$ be a polycyclic-by-finite group and let $M$ be a simple $\mathbb{Z}[\Gamma]$-module. Then $M$ is finite.

Our next lemma is a direct consequence of this result.

**Lemma 3.3.** Let $\Gamma$ be a polycyclic-by-finite group and let $(X, \alpha)$ be a Noetherian action of $\Gamma$ on a non-trivial compact abelian group $X$. Then $X$ admits a non-zero periodic point.

*Proof.* Let $\mathcal{A}$ be the collection of all proper $\mathbb{Z}(\Gamma)$-submodules of $\hat{X}$. Since $\alpha$ is Noetherian, $\mathcal{A}$ contains a maximal element $M$. It is easy to see that $N = \hat{X}/M$ is a simple $\mathbb{Z}(\Gamma)$-module. By the previous proposition $N$ is finite. Let $i : \hat{N} \to X$ denote the dual of the projection map $\pi : \hat{X} \to N$. Since $\pi$ is surjective, $i$ is injective. Hence $i(\hat{N})$ is a non-trivial $\alpha$-invariant finite subgroup of $X$. Clearly every non-zero point of $i(\hat{N})$ is periodic. □

**Lemma 3.4.** Let $\Gamma$ be a polycyclic-by-finite group and let $\alpha$ be a Noetherian action of $\Gamma$ on a zero-dimensional group $X$ such that $H^1_v(X, \alpha) \neq \{0\}$. If $\text{Ker}(\alpha) = \{\gamma : \alpha(\gamma) = I\}$ is infinite then there exists a polycyclic-by-finite group $\Gamma'$ with $h(\Gamma') < h(\Gamma)$, and a Noetherian $\Gamma'$-action $\alpha'$ on $X$ such that $H^1_v(X, \alpha') \neq \{0\}$.

*Proof.* Let $\Lambda \subset \Gamma$ be a finite index subgroup, and let $c : \Lambda \to X$ be a virtual 1-cocycle of $\alpha$ that is not a virtual coboundary. Replacing $\Gamma$ by $\Lambda$ if necessary, we may assume that $\Lambda = \Gamma$. Since $\text{Ker}(\alpha)$ acts trivially on $X$, it follows that the restriction of $c$ to $\text{Ker}(\alpha)$ is a homomorphism. Since $\Gamma$ is polycyclic-by-finite, the subgroup $\text{Ker}(\alpha)$ is finitely generated. This implies that $c(\text{Ker}(\alpha))$ is also finitely generated. By Proposition 2.2 there exists $k \geq 2$ such that $kx = 0$ for
all \( x \in X \). Hence \( c(\text{Ker}(\alpha)) \) is a finite subgroup of \( X \). We define \( M = \{ \gamma \in \text{Ker}(\alpha) : c(\gamma) = 0 \} \). Since \( \text{Ker}(\alpha) \) is finitely generated and \( M \subset \text{Ker}(\alpha) \) has finite index, \( M \) contains a finite index characteristic subgroup of \( \text{Ker}(\alpha) \), i.e., there exists a finite index subgroup \( N \subset M \) such that \( \theta(N) = N \) for all automorphism \( \theta \) of \( \text{Ker}(\alpha) \). Since for any \( g \in \Gamma \) the map \( \gamma \mapsto g\gamma g^{-1} \) is an automorphism of \( \text{Ker}(\alpha) \), we deduce that \( N \) is a normal subgroup of \( \Gamma \). We define \( \Gamma' = \Gamma/N \). As \( N \) is a finite index subgroup of \( \text{Ker}(\alpha) \) and \( \text{Ker}(\alpha) \) is infinite, \( N \) is also infinite. In particular \( h(N) > 0 \). Hence 

\[
h(\Gamma/N) = h(\Gamma) - h(N) < h(\Gamma).
\]

As \( \alpha(\gamma) = I \) for all \( \gamma \in N \), \( \alpha \) induces a Noetherian \( \Gamma' \)-action \( \alpha' \) on \( X \). We define \( c' : \Gamma' \to X \) by \( c'(\gamma N) = c(\gamma) \). We note that for any \( \gamma \in \Gamma \) and \( n \in N \), \( c(\gamma n) = c(\gamma) + \alpha(\gamma)(c(n)) = c(\gamma) \). Hence \( c' \) is a well defined virtual 1-cocycle of \( \alpha' \). Suppose there exist a finite index subgroup \( \Lambda' \subset \Gamma' \) and \( x \in X \) such that \( c'(a) = \alpha'(a)(x) - x \) for all \( a \in \Lambda' \). We define \( \Lambda_1 = \pi^{-1}(\Lambda') \), where \( \pi : \Gamma \to \Gamma' \) is the projection map. Then for any \( \gamma \in \Lambda_1 \),

\[
c(\gamma) = c'(\pi(\gamma)) = \alpha'(\pi(\gamma))(x) - x = \alpha(\gamma)(x) - x.
\]

Since this contradicts the fact that \( c \) is not a virtual coboundary, we conclude that \( c' \) is also not a virtual coboundary. Hence \( H_1^v(X, \alpha') \neq 0 \).

Our next result shows that for Noetherian actions of virtually nilpotent groups on zero-dimensional compact abelian groups, the virtual first cohomology group vanishes. In view of Theorem 2.5 and Lemma 3.3 this completes the proof of Theorem 1.1.

**Theorem 3.5.** Let \( \Gamma \) be a finitely generated virtually nilpotent group, and let \( (X, \alpha) \) be a Noetherian action of \( \Gamma \) on a zero-dimensional group \( X \). Then \( H_1^v(\alpha) = \{0\} \)

**Proof.** Let \( \Lambda \subset \Gamma \) be a finite index subgroup, and let \( c : \Lambda \to X \) be a virtual 1-cocycle of \( \alpha \). Suppose \( \Gamma_0 \subset \Gamma \) is a nilpotent subgroup of finite index. Then \( c \) is equivalent to \( c|_{\Gamma_0 \cap \Lambda} \). Hence, to show that \( c \) is
a virtual coboundary, without loss of generality we may assume that $\Lambda \subset \Gamma_0$.

We will use induction on $h(\Lambda)$, the Hirsch number of $\Lambda$. If $h(\Lambda) = 0$ then $\Lambda$ is finite, and hence $c$ is a virtual coboundary. Suppose $h(\Lambda) \geq 1$. Then $\Lambda$ is a finitely generated infinite nilpotent group. Hence the center of $\Lambda$ is also finitely generated and infinite ([H], Proposition 2.8). We choose an element $\gamma_0$ in the center of $\Lambda$ that has infinite order. For $m \geq 1$, we define $K_m = (\alpha(\gamma_0^m) - I)(X)$. Since $\alpha(\gamma_0)$ commutes with $\alpha(\gamma)$ for all $\gamma \in \Gamma$, it follows that each $K_m$ is a closed $\alpha$-invariant subgroup. As $\alpha$ is Noetherian, the collection $\{K_m : m \geq 1\}$ has a minimal element $K$. Clearly $K$ is of the form $(\alpha(\gamma_1) - I)(X)$, where $\gamma_1 = \gamma_0^n$ for some $n \geq 1$. Let $\beta$ denote the induced algebraic $\Gamma$-action on the quotient $X/K$. If $P$ denotes the projection map from $X$ to $X/K$ then $P \circ c$ is virtual 1-cocycle of the action $\beta$. We note that the $\beta(\gamma) = I$ for all $\gamma$ in the infinite cyclic subgroup generated by $\gamma_1$. By Lemma 3.4 and the induction hypothesis we deduce that $P \circ c$ is a virtual coboundary of $\beta$. We find a finite index subgroup $\Lambda_1 \subset \Lambda$ and $p \in X/K$ such that

$$P \circ c(\gamma) = \beta(\gamma)(p) - p \quad \forall \gamma \in \Lambda_1.$$ 

We choose $q \in X$ such that $P(q) = p$. Let $c_1 : \Lambda_1 \to X$ denote the virtual 1-cocycle defined by $c_1 = c - c_q$. Since $P$ is an equivariant map it follows that $P \circ c_1 = 0$, i.e., the image of $c_1$ is contained in $K = (\alpha(\gamma_1) - I)(X)$. We choose $l \geq 1$ such that $\gamma_1^l \in \Lambda_1$. From the minimality of $K$ we deduce that $K = (\alpha(\gamma_2) - I)(X)$, where $\gamma_2 = \gamma_1^l$. We find $x \in X$ such that $\alpha(\gamma_2)(x) - x = c_1(\gamma_2)$. Let $c_2 : \Lambda_1 \to X$ be the virtual 1-cocycle defined by $c_2(\gamma) = c_1(\gamma) - c_x(\gamma)$. Then $c_2$ is cohomologous to $c_1$ and $c_2(\gamma_2) = 0$. Let $F \subset X$ denote the set of points that are fixed by $\alpha(\gamma_2)$. Since $\alpha(\gamma_2)$ commutes with $\alpha(\gamma)$ for all $\gamma$, $F$ is a $\alpha$-invariant closed subgroup. We note that for any $\gamma \in \Lambda_1$, $c_2(\gamma_2 \gamma) = c_2(\gamma_2) + \alpha(\gamma_2)(c_2(\gamma)) = \alpha(\gamma_2)(c_2(\gamma))$. As $\gamma_2$ lies in the center of $\Gamma$ we also have,

$$c_2(\gamma_2 \gamma) = c_2(\gamma_2 \gamma) = c_2(\gamma) + \alpha(\gamma)(c_2(\gamma_2)) = c_2(\gamma).$$
This shows that for all $\gamma \in \Lambda_1$ the element $c_2(\gamma)$ lies in $F$. Let $\alpha_F$ denote the restriction of $\alpha$ to $F$. Then $c_2$ can be viewed as a virtual 1-cocycle of $\alpha_F$. Since $\alpha(\gamma)|_F = I$ for all $\gamma$ in the infinite cyclic group generated by $\gamma_2$, from the previous lemma and the induction hypothesis we conclude that $c_2$ is a virtual coboundary. Since $c$ is cohomologous to $c_2$ this completes the proof. □

We conclude this paper with the following question:

**Question 3.6.** Does Theorem 1.1 hold if $\Gamma$ is polycyclic-by-finite?

In view of Theorem 2.5 and Lemma 3.3 this is true if and only if Theorem 3.5 holds for polycyclic-by-finite groups.

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