Classification of biharmonic $C$-parallel Legendrian submanifolds in 7-dimensional Sasakian space forms

Toru Sasahara

Abstract

In [5], D. Fetcu and C. Oniciuc presented the classification result for biharmonic $C$-parallel Legendrian submanifolds in 7-dimensional Sasakian space forms. However, it is incomplete. In this paper, all such submanifolds are explicitly determined.

1. Introduction

In [5, Theorem 5.1], Fetcu and Oniciuc presented the classification result for proper biharmonic $C$-parallel Legendrian submanifolds in 7-dimensional Sasakian space forms. The case (2) of the theorem is proved by applying Lemma 4.4 in [1]. However, the Lemma is wrong, and hence Fetcu and Oniciuc’s classification is incomplete. This paper corrects errors in [1], and moreover, completes the classification.

Our main result is the following, which determines explicitly all proper biharmonic $C$-parallel Legendrian submanifolds in 7-dimensional Sasakian space forms.

Theorem 1.1. Let $f : M^3 \to N^7(\epsilon)$ be a 3-dimensional $C$-parallel Legendrian submanifold in a 7-dimensional Sasakian space form of constant $\varphi$-sectional curvature $\epsilon$. Then $M^3$ is proper biharmonic if and only if either:

1. $M^3$ is flat, $N^7(\epsilon) = S^7(\epsilon)$ with $\epsilon > -1/3$, where $S^7(\epsilon)$ is a unit sphere in $\mathbb{C}^4$ equipped with its canonical and deformed Sasakian structures, and $f(M^3)$ is an open part of

\[
\begin{align*}
&f(u, v, w) = \left( \frac{\lambda}{\sqrt{\lambda^2 + \alpha^{-1}}} \exp \left( i \frac{1}{\alpha \lambda} u \right) \right), \\
&\frac{1}{\sqrt{\alpha(c - a)(2c - a)}} \exp(-i(\lambda u - (c - a)v)), \\
&\frac{1}{\sqrt{\alpha\rho_1(\rho_1 + \rho_2)}} \exp(-i(\lambda u + cv + \rho_1w)), \\
&\frac{1}{\sqrt{\alpha\rho_2(\rho_1 + \rho_2)}} \exp(-i(\lambda u + cv - \rho_2w)).
\end{align*}
\]

1

Tohoku Mathematical Journal 71 (2019), 157-169. A section is added at the end.

2010 Mathematics Subject Classification. Primary 53C42; Secondary 53B25.

Key words and phrases. Biharmonic submanifolds, $C$-parallel Legendrian submanifolds, Sasakian space forms.
where \( \alpha = 4/(\epsilon + 3) \), \( \rho_{1,2} = (\sqrt{4c(2c - a) + d^2} \pm d)/2 \) and \( \lambda, a, c, d \) are real constants given by

\[
\begin{aligned}
(3\lambda^2 - \alpha^{-1})(3\lambda^4 - 2(\epsilon + 1)\lambda^2 + \alpha^{-2}) + \lambda^4((a + c)^2 + d^2) &= 0, \\
(a + c)(5\lambda^2 + a^2 + c^2 - 7\alpha^{-1} + 4) + cd^2 &= 0, \\
d(5\lambda^2 + d^2 + 3c^2 + ac - 7\alpha^{-1} + 4) &= 0,
\end{aligned}
\]

such that \(-1/\sqrt{\alpha} < \lambda < 0 \), \( 0 < a \leq (\lambda^2 - \alpha^{-1})/\lambda \), \( a \geq d \geq 0 \), \( a > 2c \), \( \lambda^2 \neq 1/(3\alpha) \);

or

(2) \( M^3 \) is non-flat, \( N^7(\epsilon) = S^7(\epsilon) \) with \( \epsilon \geq (-7 + 8\sqrt{3})/13 \) and \( f(M^3) \) is an open part of

\[
f(x,y) = \left( \sqrt{\frac{\mu^2}{\mu^2 + 1}} e^{-\frac{x}{\mu^2}} \right), \left( \sqrt{\frac{1}{\mu^2 + 1}} e^{i\mu x} \right),
\]

where \( y = (y_1, y_2, y_3) \), \( ||y|| = 1 \) and

\[
\mu^2 = \begin{cases} 
1 & (\epsilon = 1) \\
\frac{4\epsilon + 4 \pm \sqrt{13\epsilon^2 + 14\epsilon - 11}}{3(3 + \epsilon)} & (\epsilon \neq 1).
\end{cases}
\]

**Remark 1.1.** The flat case (1) of Theorem 1.1 has been proved by Fetcu and Oniciuc in [5, Theorem 5.1]. However, they did not give the explicit representation of non-flat biharmonic \( C \)-parallel Legendrian submanifolds in \( S^7(\epsilon) \).

**Remark 1.2.** The immersion (1.1) can be rewritten as

\[
f(u, v, w) = (z_1(u), z_2(u)v(v, w)),
\]

where \((z_1(u), z_2(u))\) is a Legendre curve with constant curvature \((\lambda^2 - \alpha^{-1})/\lambda\) in \( S^5(\epsilon) \) given by

\[
(z_1(u), z_2(u)) = \left( \frac{\lambda}{\sqrt{\lambda^2 + \alpha^{-1}}} e^{\frac{\lambda}{\alpha}u}, \frac{1}{\sqrt{\lambda^2 + 1}} e^{-i\lambda u} \right)
\]

and \( y(u, v) \) is a Legendrian surface in \( S^5(\epsilon) \) given by

\[
y(v, w) = \left( \frac{\sqrt{\alpha\lambda^2 + 1}}{\sqrt{\alpha(c - a)(2c - a)}} e^{i(c-a)v}, \frac{\sqrt{\alpha\lambda^2 + 1}}{\sqrt{\alpha\rho_1(\rho_1 + \rho_2)}} e^{-i(cv + \rho_1 w)}, \frac{\sqrt{\alpha\lambda^2 + 1}}{\sqrt{\alpha\rho_2(\rho_1 + \rho_2)}} e^{-i(cv - \rho_2 w)} \right).
\]

**Remark 1.3.** (i) For each fixed \( x \) in (1.2) has constant Gauss curvature \((\mu^2 + 1)/\alpha\) with respect to the induced metric from \( S^7(\epsilon) \). We can check that the surface is an integral \( C \)-parallel surface in \( S^7(\epsilon) \).

(ii) The curve

\[
z(x) := \left( \sqrt{\frac{\mu^2}{\mu^2 + 1}} e^{-\frac{x}{\mu^2}}, \sqrt{\frac{1}{\mu^2 + 1}} e^{i\mu x} \right)
\]

given in (1.2) is a Legendre curve with constant curvature \((\mu^2 - 1)/(\mu\sqrt{\alpha})\) in \( S^3(\epsilon) \).
Remark 1.4. (i) In [5, Theorem 5.1], it is stated that when $\epsilon = 5/9$, $M^3$ is locally isometric to a product $\gamma \times \bar{M}^2$, where $\gamma$ is a curve of constant curvature $1/\sqrt{2}$ in $S^7(5/9)$ and $\bar{M}^2$ is a surface of constant Gauss curvature $4/3$. However, $1/\sqrt{2}$ should be replaced by $2/3$ because $\gamma$ coincides with $z(x)$ in Remark 1.3.

(ii) The function $\lambda$ in the case (2) of [5, Theorem 5.1] and the function $\mu$ in (1.3) are related by the equation $\mu^2 = \alpha \lambda^2$. Hence, in view of Remark 1.3, the case $\epsilon = 1$ and the case $\mu^2 = (4\epsilon + 4 + \sqrt{13}\epsilon^2 + 14\epsilon - 11)/(3(3 + \epsilon))$ with $\epsilon > 1$ in (2) of Theorem 1.1 are missing from [5, Theorem 5.1].

Applying Theorem 1.1, we have the following result which corrects [5, Corollary 5.2].

Corollary 1.1. Let $f : M^3 \to S^7(1)$ be a $C$-parallel Legendrian submanifold. Then $M^3$ is proper biharmonic if and only if either:

1. $M^3$ is flat, and $f(M^3)$ is an open part of

$$f(u, v, w) = \left(-\frac{1}{\sqrt{6}} \exp(-i\sqrt{5}u), \frac{1}{\sqrt{6}} \exp(i(\frac{1}{\sqrt{5}}u - \frac{4\sqrt{3}}{\sqrt{10}}v)), \frac{1}{\sqrt{6}} \exp(i(\frac{1}{\sqrt{5}}u + \frac{\sqrt{3}}{\sqrt{10}}v - \frac{3\sqrt{2}}{2}w)), \frac{1}{\sqrt{2}} \exp(i(\frac{1}{\sqrt{5}}u + \frac{\sqrt{3}}{\sqrt{10}}v + \frac{\sqrt{2}}{2}w))\right); \text{ or}$$

2. $M^3$ is non-flat, and $f(M^3)$ is an open part of

$$(1.4) \quad f(x, y) = \frac{1}{\sqrt{2}}(e^{ix}, e^{-ix}y),$$

where $y = (y_1, y_2, y_3)$ and $||y|| = 1$.

Remark 1.5. The flat case (1) of Corollary 1.1 has been proved in [5, Corollary 5.2]. However, the non-flat submanifold (1.4) is missing from [5, Corollary 5.2].

Remark 1.6. The author classified proper biharmonic Legendrian surfaces in 5-dimensional Sasakian space forms (see [10] and [12]). Those surfaces are flat and $C$-parallel.

In the last section, by the same argument as in the proof of Theorem 1.1, we determine explicitly all proper biharmonic parallel Lagrangian submanifolds in 3-dimensional complex projective space.

2. Preliminaries

2.1. Sasakian space forms

A $(2n + 1)$-dimensional manifold $N^{2n+1}$ is called an almost contact manifold if it admits a unit vector field $\xi$, a one-form $\eta$ and a $(1, 1)$-tensor field $\varphi$ satisfying

$$\eta(\xi) = 1, \quad \varphi^2 = -I + \eta \otimes \xi.$$

3
Every almost contact manifold admits a Riemannian metric $g$ satisfying
\[ g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y). \]

The quadruplet $(\varphi, \xi, \eta, g)$ is called an *almost contact metric structure*. An almost contact metric structure is said to be *normal* if the tensor field $S$ defined by
\[ S(X, Y) = \varphi^2[X, Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] + 2d\eta(X)\eta(Y) \]
vanishes identically. A normal almost contact structure is said to be *Sasakian* if it satisfies
\[ d\eta(X, Y) := (1/2) (X(\eta(Y)) - Y(\eta(X)) - \eta([X, Y])) = g(X, \varphi Y). \]

The tangent plane in $T_p N^{2n+1}$ which is invariant under $\varphi$ is called a *$\varphi$-section*. The sectional curvature of $\varphi$-section is called the *$\varphi$-sectional curvature*. Complete and connected Sasakian manifolds of constant $\varphi$-sectional curvature are called *Sasakian space forms*. Denote Sasakian space forms of constant $\varphi$-sectional curvature $\epsilon$ by $N^{2n+1}(\epsilon)$.

Let $S^{2n+1} \subset \mathbb{C}^{n+1}$ be the unit hypersphere centered at the origin. Denote by $z$ the position vector field of $S^{2n+1}$ in $\mathbb{C}^{n+1}$ and by $g_0$ the induced metric. Let $\xi_0 = -Jz$, where $J$ is the usual complex structure of $\mathbb{C}^{n+1}$ which is defined by $JX = iX$ for $X \in T\mathbb{C}^{n+1}$. Let $\eta_0$ be a 1-form defined by $\eta_0(X) = g_0(\xi_0, X)$ and $\varphi_0$ the tensor field defined by $\varphi_0 = s \circ J$, where $s : T_s \mathbb{C}^{n+1} \rightarrow T_s S^{2n+1}$ denotes the orthogonal projection. Then, $(S^{2n+1}, \varphi_0, \xi_0, \eta_0, g_0)$ is a Sasakian space form of constant $\varphi$-sectional curvature 1. If we put
\[ \eta = \alpha \eta_0, \quad \xi = \alpha^{-1} \xi_0, \quad \varphi = \varphi_0, \quad g = \alpha g_0 + \alpha (\alpha - 1) \eta_0 \otimes \eta_0 \]
for a positive constant $\alpha$, then $(S^{2n+1}, \varphi, \xi, \eta, g)$ is a Sasakian space form of constant $\phi$ sectional curvature $\epsilon = (4/\alpha) - 3 > -3$. We denote it by $S^{2n+1}(\epsilon)$. Tanno [13] showed that a simply connected Sasakian space form $N^{2n+1}(\epsilon)$ with $\epsilon > -3$ is isomorphic to $S^{2n+1}(\epsilon)$; i.e., there exists a $C^\infty$-diffeomorphism which maps the structure tensors into the corresponding structure tensors.

### 2.2. Legendrian submanifolds in Sasakian space forms

Let $M^m$ be an $m$-dimensional submanifold $M$ in a Sasakian space form $N^{2n+1}(\epsilon)$. If $\eta$ restricted to $M^m$ vanishes, then $M^m$ is called an *integral submanifold*, in particular if $m = n$, it is called a *Legendrian submanifold*. In particular a Legendrian submanifold in a 3-dimensional Sasakian space form is called a *Legendre curve*. One can see that a curve $z(s)$ in $S^3(\epsilon) \subset \mathbb{C}^2$ is a Legendre curve if and only if it satisfies $\text{Re}(z'(s), iz(s)) = 0$ identically in $\mathbb{C}^2$, where $(\cdot, \cdot)$ is the standard Hermitian inner product on $\mathbb{C}^2$.

We denote the second fundamental form, the shape operator and the normal connection of a submanifold by $h$, $A$ and $D$, respectively. The mean curvature vector field $H$ is defined by $H = (1/m)\text{Tr} h$. If it vanishes identically, then $M^m$ is called a *minimal submanifold*. In particular, if $h \equiv 0$, then $M^m$ is called a *totally geodesic submanifold*. A Legendrian submanifold in a Sasakian manifold is parallel, i.e., satisfies $\nabla h = 0$ if and only if it is totally geodesic. Here, $\nabla h$ is defined by
\[ (\nabla_X h)(Y, Z) = D_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z). \]
A Legendrian submanifold is called $C$-parallel if $\nabla h$ is parallel to $\xi$.

For a Legendrian submanifold $M$ in a Sasakian space form, we have (cf. [2])

$$(2.1) \quad A_\xi = 0, \quad \varphi h(X, Y) = -A_{\varphi Y} X, \quad \langle h(X, Y), \varphi Z \rangle = \langle h(X, Z), \varphi Y \rangle$$

for any vector fields $X, Y$ and $Z$ tangent to $M$, where $\langle \cdot, \cdot \rangle$ is the inner product. We denote by $K_{ij}$ the sectional curvature determined by an orthonormal pair $\{X_i, X_j\}$. Then from the equation of Gauss we have

$$(2.2) \quad K_{ij} = (\epsilon + 3)/4 + \langle h(X_i, X_i), h(X_j, X_j) \rangle - ||h(X_i, X_j)||^2.$$ 

The following Legendrian submanifolds can be regarded as the simplest Legendrian submanifolds next to totally geodesic ones in Sasakian space forms.

**Definition 2.1.** An $n$-dimensional Legendrian submanifold $M^n$ in a Sasakian space form is called $H$-umbilical if every point has a neighborhood $V$ on which there exists an orthonormal frame field $\{e_1, \ldots, e_n\}$ such that the second fundamental form takes the following form:

$$h(e_1, e_1) = \lambda \varphi e_1, \quad h(e_2, e_2) = \cdots = h(e_n, e_n) = \mu \varphi e_1,$$

$$h(e_i, e_j) = \mu \varphi e_j, \quad h(e_i, e_k) = 0, \quad j \neq k, \quad j, k = 2, \ldots, n,$$

where $\lambda$ and $\mu$ are some functions on $V$.

**Remark 2.1.** If in Definition 2.1 we assume that the mean curvature vector field is nowhere vanishing, then $e_1 = -\varphi H/||H||$ holds and hence it is a globally defined differentiable vector field, and $\lambda$ is also a globally defined differentiable function. Moreover, at each point $p$ of $M^n$, the shape operator $A_{jH}$ has only one eigenvalue $\mu(p)$ on $D(p) = \{X \in T_p M^n | \langle X, JH \rangle = 0\}$. Since $\mu = (n||H|| - \lambda)/(n - 1)$ holds, it is also a globally defined differentiable function.

**2.3. Biharmonic submanifolds**

Let $f : M^n \to N$ be a smooth map between two Riemannian manifolds. The tension field $\tau(f)$ of $f$ is a section of the vector bundle $f^*TN$ defined by

$$\tau(f) := \sum_{i=1}^n \left\{ \nabla_i^f df(e_i) - df(\nabla_{e_i} e_i) \right\},$$

where $\nabla^f$, $\nabla$ and $\{e_i\}$ denote the induced connection, the connection of $M^n$ and a local orthonormal basis of $M^n$, respectively.

A smooth map $f$ is called a harmonic map if it is a critical point of the energy functional

$$E(f) = \int_\Omega ||df||^2 dv$$

over every compact domain $\Omega$ of $M^n$, where $dv$ is the volume form of $M^n$. A smooth map $f$ is harmonic if and only if $\tau(f) = 0$ at each point on $M^n$ (cf. [1]).

The bienergy functional $E_2(f)$ of $f$ over compact domain $\Omega \subset M^n$ is defined by

$$E_2(f) = \int_\Omega ||\tau(f)||^2 dv.$$
Thus $E_2$ provides a measure for the extent to which $f$ fails to be harmonic. If $f$ is a critical point of $E_2$ over every compact domain $\Omega$, then $f$ is called a biharmonic map. In [6], Jiang proved that $f$ is biharmonic if and only if its bitension field defined by

$$
\tau_2(f) := \sum_{i=1}^{n} \left\{ (\nabla^f e_i \cdot \nabla^f e_i - \nabla^f e_i \cdot e_i) \tau(f) + R^N(\tau(f), df(e_i)) df(e_i) \right\}
$$

vanishes identically, where $R^N$ is the curvature tensor of $N$.

A submanifold is called a biharmonic submanifold if the isometric immersion that defines the submanifold is biharmonic map. Minimal submanifolds are biharmonic. A biharmonic submanifold is said to be a proper biharmonic submanifold if it is non-minimal.

Loubeau and Montaldo introduced a class which includes biharmonic submanifolds as follows.

**Definition 2.2** ([9]). An isometric immersion $f : M \to N$ is called biminimal if it is a critical point of the bienergy functional $E_2$ with respect to all normal variation with compact support. Here, a normal variation means a variation $f_t$ through $f = f_0$ such that the variational vector field $V = df_t / dt |_{t=0}$ is normal to $f(M)$. In this case, $M$ or $f(M)$ is called a biminimal submanifold in $N$.

An isometric immersion $f$ is biminimal if and only if the normal part of $\tau_2(f)$ vanishes identically. Clearly, biharmonic submanifolds are biminimal. Biminimal $H$-umbilical Legendrian submanifolds in Sasakian space forms have been classified by the author as follows.

**Theorem 2.3** ([12]). Let $f : M^n \to N^{2n+1}(\epsilon)$ be a non-minimal biminimal $H$-umbilical Legendrian submanifold, where $n \geq 3$. Then $N^{2n+1}(\epsilon) = S^{2n+1}(\epsilon)$ with

$$
\epsilon \geq \frac{-3n^2 - 2n + 5 + 32\sqrt{n}}{n^2 + 6n + 25} > -3
$$

and $f(M^n)$ is an open part of

$$
f(x,y) = \left( \sqrt{\frac{\mu^2}{\mu^2 + 1}} e^{-\frac{1}{\mu^2}}, \sqrt{\frac{1}{\mu^2 + 1}} e^{i\mu x} y \right),
$$

where $y = (y_1, \ldots, y_n), \|y\| = 1$ and

$$
\mu^2 = \begin{cases}
1 & (\epsilon = 1) \\
\frac{(n + 5) \epsilon + 3n - 1 \pm \sqrt{P(n, \epsilon)}}{2(3 + \epsilon)n} & (\epsilon \neq 1),
\end{cases}
$$

where $P(n, \epsilon) := (n^2 + 6n + 25) \epsilon^2 + (6n^2 + 4n - 10) \epsilon + 9n^2 - 42n + 1$.

**Remark 2.2.** Submanifolds given in Theorem 2.3 are in fact proper biharmonic.
3. \( C \)-parallel Legendrian submanifolds

3.1. A special orthonormal basis

We recall a special local orthonormal basis which is used in \([1]\) (see also \([5]\)). Let \( M \) be a non-minimal Legendrian submanifold of \( N^7(\epsilon) \). Let \( p \) be an arbitrary point of \( M \), and denote by \( U_pM \) the unit sphere in \( T_pM \). We consider the function \( f_p : U_pM \to \mathbb{R} \) given by

\[
f_p(u) = \langle h(u, u), \varphi u \rangle.
\]

A function \( f_p \) attains a critical value at \( X \) if and only if \( \langle h(X, X), \varphi Y \rangle = 0 \) for all \( Y \in U_pM \) with \( \langle X, Y \rangle = 0 \), i.e., \( X \) is an eigenvector of \( A_{\varphi X} \).

We take \( X_1 \) as a vector at which \( f_p \) attains its maximum. Then there exists a local orthonormal basis \( \{X_1, X_2, X_3\} \) of \( T_pM \) such that the shape operators take the following forms (cf. \([1]\)):

\[
A_{\varphi X_1} = \begin{pmatrix}
\lambda_1 & 0 & 0 \\
0 & \lambda_2 & 0 \\
0 & 0 & \lambda_3 \\
\end{pmatrix}, \quad A_{\varphi X_2} = \begin{pmatrix}
0 & \lambda_2 & 0 \\
\lambda_2 & a & b \\
0 & b & c \\
\end{pmatrix}, \quad A_{\varphi X_3} = \begin{pmatrix}
0 & 0 & \lambda_3 \\
0 & b & c \\
\lambda_3 & c & d \\
\end{pmatrix},
\]

where

\[
\lambda_1 > 0, \quad \lambda_1 \geq 2\lambda_2, \quad \lambda_1 \geq 2\lambda_3, \quad a \geq 0, \quad a^2 \geq d^2,
\]

and moreover, if \( \lambda_2 = \lambda_3 \), then \( b = 0 \) and \( a \geq 2c \).

**Lemma 3.1.** The vector \( X_1 \in T_pM \) can be differentiably extended to a vector field \( X_1(x) \) on a neighborhood \( V \) of \( p \) such that at every point \( x \) of \( V \), \( f_x \) attains a critical value at \( X_1(x) \), that is, \( X_1(x) \) is an eigenvector of \( A_{\varphi X_1(x)} \).

**Proof.** Let \( E_1(x), E_2(x), E_3(x) \) be an arbitrary local differentiable orthonormal frame field on a neighborhood \( V \) of \( p \), such that \( E_i(p) = X_i \). The purpose is to find a local differentiable vector field \( X_1(x) = \sum y^i(x)E_i(x) \) such that \( (y^1)^2 + (y^2)^2 + (y^3)^2 = 1 \) and at every point \( x \) of \( V \), \( f_x \) attains a critical value at \( X_1(x) \). As in the proof of Theorem A in \([7]\), we apply Lagrange’s multiplier method.

Consider the following function:

\[
F(x, y^1, y^2, y^3, \lambda) := \sum_{i,j,k} h_{ijk} y^i y^j y^k - \lambda \{ (y^1)^2 + (y^2)^2 + (y^3)^2 - 1 \},
\]

where \( h_{ijk} := \langle h(E_i(x), E_j(x)), \varphi E_k(x) \rangle \). We shall show that there exist differentiable functions \( y^1, y^2, y^3 \) defined a neighborhood of \( p \) satisfying the following system of equations:

\[
\begin{align*}
\frac{\partial F}{\partial y^i} &= 3 \sum_{j,k} h_{ijk}(x) y^j y^k - 2 y^i \lambda = 0, \quad i \in \{1, 2, 3\}, \\
\frac{\partial F}{\partial \lambda} &= (y^1)^2 + (y^2)^2 + (y^3)^2 - 1 = 0.
\end{align*}
\]

Define functions \( G_i \) by

\[
\begin{align*}
G_i(x, y^1, y^2, y^3, \lambda) &= 3 \sum_{j,k} h_{ijk}(x) y^j y^k - 2 y^i \lambda \quad \text{for } i = 1, 2, 3, \\
G_4(x, y^1, y^2, y^3) &= (y^1)^2 + (y^2)^2 + (y^3)^2 - 1.
\end{align*}
\]
Since $X_1(p) = X_1 = E_1(p)$, we have $(y^1, y^2, y^3) = (1, 0, 0)$ at $p$. It follows from (3.1) and (3.3) that $2\lambda(p) = 3\lambda_1$. We set $y^4 = \lambda$. A straightforward computation yields

$$\det\left(\frac{\partial G_\alpha}{\partial y^3}\right)(p) = 36(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1).$$

By (3.2), we have $\lambda_2 \neq \lambda_1 \neq \lambda_3$. Hence the RHS of (3.4) is not zero. The implicit function theorem shows that there exist local differentiable functions $y^1(x), y^2(x), y^3(x), \lambda(x)$ on a neighborhood of $p$ satisfying (3.3). The proof is finished.

**Remark 3.1.** In [1] and [2], the differentiability of $X_1(x)$ is not proved.

If the eigenvalues of $A_{\varphi X_1(x)}$ have constant multiplicities on a neighborhood $V$ of $p$, we can extend $X_2$ and $X_3$ differentiably to vector fields $X_2(x)$ and $X_3(x)$ on $V$. We work on the open dense set of $M$ defined by this property.

### 3.2. Correction to a paper by Biakoussis, Blair and Koufogiorgos

Let $M$ be a $C$-parallel Legendrian submanifold of $N^7(\epsilon)$. The condition that $M$ is $C$-parallel is equivalent to $\nabla \varphi h = 0$, where $\nabla$ is the Levi-Civita connection of $M$. Hence we have

$$R \cdot \varphi h = 0,$$

where $R$ is the curvature tensor of $M$.

By using (2.2), (3.1) and (3.5), Biakoussis et al. obtained a system of algebraic equations with respect to $\lambda_1, \lambda_2, \lambda_3, a, b, c, d, K_{12}, K_{13}$ and $K_{23}$ (see [1] pages 211-212).

However, the equation (3.19)-(iv) in [1], i.e., $c(a - 2c)(\lambda_2 - \lambda_3) = 0$ is incorrect. It should be replaced by

$$b(a - 2c)(\lambda_2 - \lambda_3) = 0,$$

which is obtained by $\langle (R(X_1, X_2) \cdot \varphi h)(X_2, X_2), X_3 \rangle = 0.$

In [1] Lemma 4.4, it is stated that if $\lambda_1 = 2\lambda_3 \neq 2\lambda_2$, then $\epsilon = -3$. However, the proof is based on the wrong equation (3.19)-(iv) (see page 214, line 11), and hence the statement is also wrong. The following is a counterexample to [1] Lemma 4.4: The submanifold (3.4) is a $H$-umbilical Legendrian submanifold such that, with respect to some orthonormal local frame field $e_1, e_2, e_3$ with $e_1 = \partial/\partial x$, the second fundamental form $h$ satisfies

$$h(e_1, e_1) = 0, \quad h(e_2, e_2) = h(e_3, e_3) = \varphi e_1,$$
$$h(e_1, e_2) = \varphi e_2, \quad h(e_1, e_3) = \varphi e_3, \quad h(e_2, e_3) = 0.$$

We put $X_1 = (e_1 - \sqrt{2}e_2)/\sqrt{3}$, $X_2 = (\sqrt{2}e_1 + e_2)/\sqrt{3}$ and $X_3 = e_3$. Then the shape operators take the forms (3.1) with $\lambda_1 = 2/\sqrt{3}, \lambda_2 = -1/\sqrt{3}, \lambda_3 = 1/\sqrt{3}$, $a = c = \sqrt{2}/\sqrt{3}$ and $b = d = 0$.

On the other hand, following the wrong statement of [1] Lemma 4.4, the non-flat case (2) of [5] Theorem 5.1] is investigated. Therefore, the classification presented in the theorem is incomplete.
3.3. Biharmonic C-parallel Legendrian submanifolds

We shall prove Theorem 1.1. First, we recall the following.

Proposition 3.1 ([5]). A C-parallel Legendrian submanifolds in a 7-dimensional Sasakian space form $N^7(\epsilon)$ is proper biharmonic if and only if $\epsilon > -1/3$ and

$$ (3.6) \quad \text{Tr} h(\cdot, A_H\cdot) = (3\epsilon + 1)/2. $$

By applying the proof of [1, Lemmas 4.2-4.6] and Proposition 3.1, we obtain the following.

Proposition 3.2. Let $M^3$ be a proper biharmonic C-parallel Legendrian submanifold in $N^7(\epsilon)$. If $M$ is non-flat, then it is $H$-umbilical.

Proof. By [1, Lemma 4.2], the case $\lambda_1 \neq 2\lambda_2 \neq 2\lambda_3 \neq \lambda_1$ cannot hold. According to the proof of [1, Lemma 4.6], the case $\lambda_1 = 2\lambda_2 = 2\lambda_3$ cannot hold for $\epsilon > -3$. Hence, by Proposition 3.1 the proof is divided into the following three cases.

Case (i). $\lambda_1 = 2\lambda_2 \neq 2\lambda_3$. In the proof of [1, Lemma 4.3], we have

$$ (3.7) \quad \lambda_1 = 2\lambda_2 = -\lambda_3 = \sqrt{2(\epsilon + 3)}/4, \quad a = c = d = 0, \quad b = \pm \sqrt{6(\epsilon + 3)}/8. $$

We choose a local orthonormal frame field $\{e_1, e_2, e_3\}$ as follows:

$$ e_1 = (X_1 \pm \sqrt{5}X_3)/2, \quad e_2 = X_2, \quad e_3 = (\mp \sqrt{5}X_1 + X_3)/2, $$

where the $\pm$ signs are determined by the sign of $b$. Then, by a straightforward computation using (3.7), we obtain

$$ (3.8) \quad h(e_1, e_1) = -(\sqrt{2(\epsilon + 3)}/4)\varphi e_1, \quad h(e_2, e_2) = h(e_3, e_3) = (\sqrt{2(\epsilon + 3)}/4)\varphi e_1, $$

$$ h(e_2, e_3) = 0, \quad h(e_1, e_i) = (\sqrt{2(\epsilon + 3)}/4)\varphi e_i, \quad i \in \{2, 3\}, $$

which implies that $M$ is $H$-umbilical. Moreover, from (3.6) and (3.8) we have $\epsilon = 5/9$ (see the subcase (a) of (2) in [5, Theorem 5.1]).

Case (ii). $\lambda_1 = 2\lambda_3 \neq 2\lambda_2$. Following the proof of [1, Lemma 4.4] (page 214, lines 7-10), we have

$$ (3.9) \quad K_{12} = 0, $$

$$ (3.10) \quad b = d = 0, \quad c \neq 0. $$

Moreover, in [1, (3.16)-(iv), (3.21)]) the following equations have been obtained:

$$ (3.11) \quad c(K_{23} + \lambda_3(\lambda_2 - \lambda_3)) = 0, $$

$$ (3.12) \quad (\lambda_2 - \lambda_3)(K_{23} - b^2 - c^2) = 0. $$

From (3.10), (3.11), (3.12) and $\lambda_2 \neq \lambda_3$, we have

$$ (3.13) \quad K_{23} + \lambda_3(\lambda_2 - \lambda_3) = 0, \quad K_{23} = c^2. $$

We note that (3.10) and (3.13) show $\lambda_3 \neq 0$. It follows from (2.2), (3.1), (3.9) and (3.13) that

$$ (3.14) \quad \left\{ \begin{array}{l} \lambda_2^2 = 4c^2 - 2ac - \beta, \\ \lambda_3^2 = 3c^2 - ac - \beta, \\ \lambda_2\lambda_3 = 2c^2 - ac - \beta, \end{array} \right. $$

9
where \( \beta = (\epsilon + 3)/4 \).

We choose a local orthonormal frame field \( \{e_1, e_2, e_3\} \) as follows:

\[
\begin{align*}
    e_1 &= (\lambda_3 X_1 + cX_2)/\sqrt{\lambda_3^2 + c^2}, \\
    e_2 &= (-cX_1 + \lambda_3 X_2)/\sqrt{\lambda_3^2 + c^2}, \\
    e_3 &= X_3.
\end{align*}
\]

We set

\[
k(a, c) := 8c^4 - 6ac^3 + (a^2 - 3\beta)c^2 + a\beta c.
\]

Then, by a straightforward computation using (3.14), we obtain

\[
\begin{align*}
    \langle h(e_1, e_1), \varphi e_2 \rangle &= \frac{ck(a, c)}{\lambda_3(\lambda_3^2 + c^2)^{3/2}}, \\
    \langle h(e_1, e_1), \varphi e_3 \rangle &= 0, \\
    \langle h(e_2, e_2) - h(e_3, e_3), \varphi e_1 \rangle &= -\frac{k(a, c)}{\lambda_3^2 + c^2}^{3/2}, \\
    h(e_2, e_2) &= 0, \\
    \langle h(e_2, e_2), \varphi e_2 \rangle &= -\frac{\lambda_3 k(a, c)}{c(\lambda_3^2 + c^2)^{3/2}}, \\
    \langle h(e_3, e_3), \varphi e_3 \rangle &= 0.
\end{align*}
\] (3.15)

On the other hand, substituting (3.14) into the identity \( \lambda_2 \lambda_3 - (\lambda_2 \lambda_3)^2 = 0 \) gives

\[
k(a, c) = 0.
\]

Hence, it follows from (2.1) and (3.15) that \( M \) is \( H \)-umbilical.

**Case (iii).** \( \lambda_1 \neq 2\lambda_2 = 2\lambda_3 \). By rotating the vector fields \( X_2 \) and \( X_3 \), if necessary, we may assume that \( b = 0 \). In [1, Lemma 4.5], it is proved that if \( M \) is non-flat, then \( a \neq 2c \) and \( a = c = d = 0 \). Thus, \( M \) is \( H \)-umbilical.

**Proof of Theorem 1.1** The flat case (1) has been proved in (1) of [5] Theorem 5.1. Applying Proposition 3.2 and Theorem 2.3 for \( n = 3 \), we can prove the non-flat case (2).

**Remark 3.2.** In [5], the case (ii) of Proposition 3.2 was not investigated.

### 4. Biharmonic parallel Lagrangian submanifolds

Let \( \mathbb{CP}^n(4) \) denote the complex projective space of complex dimension \( n \) and constant holomorphic sectional curvature 4. We denote by \( J \) the almost complex structure of \( \mathbb{CP}^n(4) \). An \( n \)-dimensional submanifold \( M^n \) of \( \mathbb{CP}^n(4) \) is said to be Lagrangian if \( J \) interchanges the tangent and the normal spaces at each point.

In [5], Theorem 6.3, Fetcu and Oniciuc presented the classification result of proper biharmonic parallel Lagrangian submanifolds in \( \mathbb{CP}^3(4) \). However, the theorem is proved by applying the wrong statement of [1] Lemma 4.4, and hence the classification is incomplete. This section completes it. First, we recall the following.

**Proposition 4.1** ([5]). Let \( L : M^3 \rightarrow \mathbb{CP}^3(4) \) be a proper biharmonic parallel Lagrangian immersion. Then \( L \) is locally given by \( \pi \circ f \), where \( \pi : S^{2n+1}(1) \rightarrow \mathbb{CP}^n(4) \) is the Hopf fibration and \( f : M^3 \rightarrow S^3(1) \) is a non-minimal \( C \)-parallel Legendrian immersion satisfying

\[
\text{Tr} h(\cdot, A_H \cdot) = 6H.
\]
The following theorem determines explicitly all proper biharmonic parallel Lagrangian submanifolds in $\mathbb{CP}^3(4)$.

**Theorem 4.1.** Let $L : M^3 \to \mathbb{CP}^3(4)$ a proper biharmonic parallel Lagrangian submanifold. Then $L$ is locally congruent to $\pi \circ f$, where $f : M^3 \to S^7(1)$ is one of the following:

1. $M^3$ is flat and
   \[
   f(u, v, w) = \left( \frac{\lambda}{\sqrt{\lambda^2 + 1}} \exp\left( i \left( \frac{1}{\lambda} u \right) \right), \right.
   \frac{1}{\sqrt{(c-a)(2c-a)}} \exp(-i(\lambda u - (c-a)v)),
   \frac{1}{\sqrt{\rho_1(\rho_1 + \rho_2)}} \exp(-i(\lambda u + cv + \rho_1 w)),
   \frac{1}{\sqrt{\rho_2(\rho_1 + \rho_2)}} \exp(-i(\lambda u + cv - \rho_2 w)) \right),
   \]
   where $\rho_{1,2} = (\sqrt{4c(2c-a) + d^2} \pm d)/2$ and the 4-tuple $(\lambda, a, c, d)$ is given by one of the following:
   \[
   \begin{pmatrix}
   - \sqrt{\frac{4 - \sqrt{13}}{3}}, & \sqrt{\frac{7 - \sqrt{13}}{6}}, & - \sqrt{\frac{7 - \sqrt{13}}{6}}, & 0 \\
   - \sqrt{\frac{5 + 2\sqrt{3}}{6}}, & \sqrt{\frac{45 + 21\sqrt{3}}{13}}, & - \sqrt{\frac{6}{21 + 11\sqrt{3}}}, & 0 \\
   - \sqrt{\frac{6 + \sqrt{13}}{6}}, & \sqrt{\frac{523 + 139\sqrt{13}}{138}}, & - \sqrt{\frac{79 - 17\sqrt{13}}{138}}, & \sqrt{\frac{14 + 2\sqrt{13}}{3}}
   \end{pmatrix};
   \]

2. $M^3$ is non-flat and
   \[
   f(x, y) = \left( \sqrt{\frac{\mu^2}{\mu^2 + 1}} e^{-\frac{i}{\mu} x}, \sqrt{\frac{1}{\mu^2 + 1}} e^{i\mu x} y \right),
   \]
   where $y = (y_1, y_2, y_3)$, $||y|| = 1$ and $\mu^2 = (4 \pm \sqrt{13})/3$.

**Proof.** The flat case (1) has been proved in [5, Corollary 6.4]. Applying Propositions 3.2 and 4.1 and modifying the second equation of [12, (5.33)] to $\lambda^2 + 2\mu^2 = 6$, we can prove the non-flat case (2). \qed

**Remark 4.1.** Fetcu and Oniciuc [5] did not give the explicit representation of non-flat proper biharmonic parallel Lagrangian submanifolds in $\mathbb{CP}^3(4)$.

**Remark 4.2.** The immersion (4.1) can be rewritten as the one with $\alpha = 1$ in Remark 1.2 (cf. [3], [8]).

**Remark 4.3.** The immersion (4.2) has the same properties as in Remark 1.3 where $\alpha = 1$. From this, we see that (4.2) with $\mu^2 = (4 + \sqrt{13})/3$ is missing from [5, Theorem 6.3].

**Remark 4.4.** The author classified proper biharmonic Lagrangian surfaces of constant mean curvature in $\mathbb{CP}^2(4)$ (see [11]). Those surfaces are flat and parallel.
5. Corrections to this paper (added on November 16, 2022)

5.1. Correction to Lemma 3.1

Equation (3.4) should be replaced by

\[
\det \left( \frac{\partial G_{\alpha}}{\partial y^{\beta}} \right) (p) = 36(2\lambda_2 - \lambda_1)(2\lambda_3 - \lambda_1).
\]

Therefore, Lemma 3.1 should be replaced by

Lemma 5.1. If \( \lambda_1 \neq 2\lambda_2 \) and \( \lambda_1 \neq 2\lambda_3 \), then the vector \( X_1 \in T_pM \) can be differentiably extended to a vector field \( X_1(x) \) on a neighborhood \( V \) of \( p \) such that at every point \( x \) of \( V \), \( f_x \) attains a critical value at \( X_1(x) \), that is, \( X_1(x) \) is an eigenvector of \( A_{\varphi X_1(x)} \).

5.2. Correction to the proof of Proposition 3.2

Proof of Proposition 3.2: Let \( p \) be an arbitrary point of \( M \), and we choose a local orthonormal basis \( \{X_1, X_2, X_3\} \) of \( T_pM \) such that the shape operators take the form (3.1) with (3.2).

By an argument given in the proof of [1, Lemma 4.2], the case \( \lambda_1 \neq 2\lambda_2 \neq 2\lambda_3 \neq \lambda_1 \) cannot hold. According to the proof of [1, Lemma 4.6], the case \( \lambda_1 = 2\lambda_2 = 2\lambda_3 \) cannot hold for \( \epsilon > -3 \). Note that these two assertions can be obtained without using (3.4)-(3.13) in [1]. By Proposition 3.1, \( \epsilon > -1/3 \) must be satisfied, and hence the proof is divided into the following three cases.

Case (i). \( \lambda_1 = 2\lambda_2 \neq 2\lambda_3 \). In the proof of [1, Lemma 4.3], we have

\[
(\ln e_1, e_1) = -(\sqrt{2(\epsilon + 3)/4})\varphi e_1, \quad h(e_2, e_2) = h(e_3, e_3) = (\sqrt{2(\epsilon + 3)/4})\varphi e_1, \\
h(e_2, e_3) = 0, \quad h(e_1, e_i) = (\sqrt{2(\epsilon + 3)/4})\varphi e_i, \quad i \in \{2, 3\}.
\]

Case (ii). \( \lambda_1 = 2\lambda_3 \neq 2\lambda_2 \). Following the proof of [1, Lemma 4.4] (page 214, lines 7-10), we have

\[
K_{12} = 0, \\
b = d = 0, \quad c \neq 0.
\]

Moreover, in [1] (3.16)-(iv), (3.21)) the following equations have been obtained:

\[
c(K_{23} + \lambda_3(\lambda_2 - \lambda_3)) = 0, \\
(\lambda_2 - \lambda_3)(K_{23} - b^2 - c^2) = 0.
\]
From (5.5), (5.6), (5.7) and \( \lambda_2 \neq \lambda_3 \), we have

\[
K_{23} + \lambda_3 (\lambda_2 - \lambda_3) = 0, \quad K_{23} = c^2.
\]

We note that (5.5) and (5.8) show \( \lambda_3 \neq 0 \). It follows from (2.2), (3.1), (5.4) and (5.8) that

\[
\begin{align*}
\lambda_2^2 &= 4c^2 - 2ac - \beta, \\
\lambda_3^2 &= 3c^2 - ac - \beta, \\
\lambda_2 \lambda_3 &= 2c^2 - ac - \beta,
\end{align*}
\]

where \( \beta = (\epsilon + 3)/4 \).

We choose a local orthonormal basis \( \{ e_1, e_2, e_3 \} \) of \( T_p M \) as follows:

\[
e_1 = (\lambda_3 X_1 + c X_2)/\sqrt{\lambda_3^2 + c^2}, \quad e_2 = (-c X_1 + \lambda_3 X_2)/\sqrt{\lambda_3^2 + c^2}, \quad e_3 = X_3.
\]

We set

\[
k(a, c) := 8c^4 - 6ac^3 + (a^2 - 3\beta)c^2 + a\beta c.
\]

Then, by a straightforward computation using (5.9), we obtain

\[
\langle h(e_1, e_1), \varphi e_2 \rangle = \frac{ck(a, c)}{\lambda_3 (\lambda_3^2 + c^2)^{3/2}}, \quad \langle h(e_1, e_1), \varphi e_3 \rangle = 0,
\]

\[
\langle h(e_2, e_2) - h(e_3, e_3), \varphi e_1 \rangle = -\frac{k(a, c)}{(\lambda_3^2 + c^2)^{3/2}}, \quad \langle h(e_2, e_3), \varphi e_3 \rangle = 0,
\]

\[
\langle h(e_2, e_2), \varphi e_2 \rangle = -\frac{\lambda_3 k(a, c)}{c(\lambda_3^2 + c^2)^{3/2}}, \quad \langle h(e_3, e_3), \varphi e_3 \rangle = 0.
\]

On the other hand, substituting (5.9) into the identity \( \lambda_2^2 \lambda_3^2 - (\lambda_2 \lambda_3)^2 = 0 \) gives

\[
k(a, c) = 0.
\]

Hence, it follows from (2.1) and (5.10) that the second fundamental form takes the form in Definition 2.1 at \( p \).

**Case (iii).** \( \lambda_1 \neq 2\lambda_2 = 2\lambda_3 \). By Lemma 5.1, the basis \( \{ X_1, X_2, X_3 \} \) of \( T_p M \) can be differentiably extended to an orthonormal frame field \( \{ X_1(x), X_2(x), X_3(x) \} \) on a neighborhood \( V \) of \( p \) such that at every point \( x \) of \( V \) the shape operators take the form (3.1) with \( \lambda_1 \neq 2\lambda_2 \) and \( \lambda_1 \neq 2\lambda_3 \). It follows from (3.4)-(3.10) in [4] that \( \lambda_1, \lambda_2, \lambda_3 \) and \( a \) are constant on \( V \). By rotating the vector fields \( X_2(x) \) and \( X_3(x) \), if necessary, we may assume that \( b = 0 \) on \( V \). In [4, Lemma 4.5], it is proved that if \( M \) is non-flat, then \( a = c = d = 0 \) on \( V \).

Consequently, \( M \) is \( H \)-umbilical.

**References**

[1] C. Baikoussis, D. E. Blair and T. Koufogiorgos, Integral submanifolds of Sasakian space forms \( M^7(k) \), Results Math. 27 (1995), 207–226.

[2] D. E. Blair, Riemannian geometry of contact and symplectic manifolds, Progress in Mathematics 203, Birkhäuser, Boston, 2010.
[3] F. Dillen, H. Li, L. Vrancken and X. Wang, *Lagrangian submanifolds in complex projective space with parallel second fundamental form*, Pacific J. Math. **255** (2012), 79–115.

[4] J. Eells and J. H. Sampson, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math. **86** (1964), 109–160.

[5] D. Fetcu and C. Oniciuc, *Biharmonic integral C-parallel submanifolds in 7-dimensional Sasakian space forms*, Tohoku Math. J. **64** (2012), 195–222.

[6] G. Y. Jiang, *2-harmonic maps and their first and second variational formulas* (Chinese), Chinese Ann. Math. A **7** (1986), 389–402.

[7] A.-M. Li and G. Zhao, *Totally real minimal submanifolds CP^n*, Arch. Math. **62** (1994), 562–568.

[8] H. Li and X. Wang, *Calabi product Lagrangian immersions in complex projective space and complex hyperbolic space*, Results Math. **59** (2011), 453–470.

[9] E. Loubeau and S. Montaldo, *Biminimal immersions*, Proc. Edinburgh Math. Soc. **51** (2008), 421–437.

[10] T. Sasahara, *Legendre surfaces in Sasakian space forms whose mean curvature vectors are eigenvectors*, Publ. Math. Debrecen **67** (2005), 285–303.

[11] T. Sasahara, *Biharmonic Lagrangian surfaces of constant mean curvature in complex space forms*, Glasgow Math. J. **49** (2007), 497–507.

[12] T. Sasahara, *A class of biminimal Legendrian submanifolds in Sasakian space forms*, Math. Nachr. **287** (2014), 79–90.

[13] S. Tanno, *Sasakian manifolds with constant φ-holomorphic sectional curvature*, Tohoku Math. J. **21** (1969), 501–507.

Center for Liberal Arts and Sciences
Hachinohe Institute of Technology
Hachinohe 031-8501
JAPAN

E-mail address: sasahara@hi-tech.ac.jp