Persistent fluctuations and scaling properties in galaxy number counts

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ABSTRACT

Counts of galaxies as a function of apparent magnitude are among the most time-honored observations in cosmology. In this Letter, we focus on some statistical properties of these counts which are fundamental in order to characterize the large scale correlations in the galaxy spatial distribution. There are, in fact, no longer doubts since two decades of the existence of very large scale structures. The still remaining problem concerns the correct characterization of their statistical properties. We propose to study two properties of galaxy counts data, in order to discriminate between a small scale ($\sim 5 \div 20 \, h^{-1} Mpc$) homogeneous distribution, and a fractal structure on large scales ($\sim 20 \div 300 \, h^{-1} Mpc$). Firstly, the average slope of the counts which can be associated to an eventual fractal dimension in real space by simple arguments. Note that in the magnitude range $11^m \lesssim B_J \lesssim 18^m$, the results are nearly independent from cosmological parameters, K-corrections and evolution effects, as the corresponding average redshift is $\langle z \rangle \ll 0.1$. Secondly, we propose to study fluctuations of counts around the average behavior as a function of apparent magnitude in the whole magnitude range $B_J \gtrsim 11^m$. These fluctuations can discriminate between a genuine fractal distribution and a homogeneous one. In fact, they are related to the very statistical properties of the spatial distribution, independently on cosmological corrections. More specifically, in a fractal distribution one expects to find persistent scale-invariant fluctuations around the average behavior, which do not decay with apparent magnitude. On the other hand, in an homogeneous distribution, on large enough scales, the relative variance of the counts should decrease exponentially with apparent magnitude. Smooth cosmological corrections cannot change such a behavior. Such a test can be applicable also at very faint magnitudes. We point out that
the application of these tests to the new generation of photometric (POSS-II) and spectroscopic surveys (SDSS, 2dF) will be crucial in order to characterize statistically the galaxy spatial distribution.

Subject headings: galaxies: general; galaxies: statistics; cosmology: large-scale structure of the universe
1. Introduction

In the counts of galaxies, large fluctuations from field to field and from author to author, both in faint and bright counts, and in different spectral bands, have been reported (e.g. Shanks et al 1989, Tyson 1988, Cowie et al. 1990, Maddox et al. 1990, Metcalfe et al., 1991, Picard 1991, Weir et al. 1995, Bertin & Dennefeld 1997, Arnout et al 1997). These fluctuations can be as large as a factor of two. There have been controversy as to whether these fluctuations are due to real clustering or to differences in the magnitude zero point of the various surveys. Hence, in order to avoid possible systematic errors, it is very important to understand the nature of fluctuations in a given field of a single survey, once the magnitude system and zero point have been carefully calibrated. It is, in fact, possible that discrepancies among these surveys are not due mostly to differences in photometric systems or in data reduction effects, but rather to real effects, i.e. large scale structures. In this Letter we propose a method to verify this latter possibility in the actual data. The slope and the amplitude of the counts are shown to be compatible with a fractal distribution of galaxies, and we point out that fundamental information about clustering can obtained by studying \textit{the fluctuations of counts as a function of apparent magnitude}.

2. Average number counts in a fractal distribution

As suggested in Baryshev (1981), and proposed in a series of papers (Sylos Labini et al. 1996, Montuori et al. 1997, Sylos Labini Montuori & Pietronero 1998) number counts versus apparent magnitude can be used to test whether the large scale distribution of galaxies can be \textit{compatible} with a fractal or with an homogeneous behavior. In this context, we discuss the case in which the joint space-luminosity distribution \( \nu(\vec{r}, L) \) can be factorized as the
product of the number spatial density \( n(\vec{r}) \) and the luminosity function \( \phi(L) \) (Binggeli, Tammann & Sandage 1988):

\[
\nu(\vec{r}, L)d^3r dL = n(\vec{r})\phi(L)d^3r dL .
\]  

(1)

This is known to be a good approximation in the case of small redshift \( z \ll 1 \). All the eventual corrections to Eq. (1) (space geometry, K-corrections, evolution, etc.) are in fact proportional to \( z \) (Yoshii & Takahara 1988, Sandage 1995).

In the case of a fractal distribution, the average density seen from a galaxy (averaged over enough many observing galaxies) can be written as \( \langle n(\vec{r}) \rangle \equiv \Gamma(r) = Br^{D-3} \) (Pietronero 1987, Sylos Labini et al. 1998) where \( D \) is the fractal dimension; then:

\[
\langle \nu(\vec{r}, L) \rangle = \Gamma(r)\phi(L) = Br^{D-3}AL^\delta e^{-L/L^*}.
\]  

(2)

In this case one ends up with a very simple relation for the integrated counts as a function of apparent flux \( (f = L/(4\pi r^2)) \), for unit of steradian:

\[
\langle N(> f) \rangle = N_0 f^{-\frac{D}{4}} .
\]  

(3)

By using the transformation between apparent flux and magnitude (Peebles 1993)

\[
f = \frac{L_*}{4\pi(10\text{pc})^2}10^{0.4(M_*-m)} ,
\]  

(4)

where \( M_* \) is the cut-off of the luminosity function \( L_* \) in terms of magnitude, one obtains

\[
\langle N(< m) \rangle = \tilde{N}_0 10^{-\frac{D}{5}m} ,
\]  

(5)

and hereafter we denote \( \alpha \equiv D/5 \).

\(^1\)which we take to be Schechter like (Schechter 1976) but its actual function does not change the final results
Note that, from what concerns the average behavior, the case of a homogeneous distribution is included in the fractal case with \( D = 3 \). Eq.2 has been tested (Sylos Labini, Montuori & Pietronero 1998; Joyce, Montuori & Sylos Labini 1999) to be a rather good approximation in local redshift surveys. Thus, the exponent of the average counts is simply related to the fractal dimension of galaxies in the three dimensional space (see also Sandage, Tammann & Hardy 1972, Peebles 1993). In Eq. 2 \( A \) is a normalizing constant such that

\[
A = \frac{1}{\int_{L_{\text{min}}}^{\infty} L^\delta e^{-L/L^*} dL},
\]

where \( L_{\text{min}} \) is the faintest object observed in current surveys. Such a lower cut-off, larger than zero is necessary to avoid divergences for \( \delta \leq -1 \). Therefore, Eq. 2 depends on a combination of five different parameters which can be independently measured. Three parameters are related with the luminosity function: the exponent \( \delta \), the luminosity cut-off \( L^* \) and the lower cut-off \( L_{\text{min}} \). These three quantities have been measured with good precision in different redshift surveys (Binggeli, Sandage & Tammann 1988, Efstathiou, Ellis & Peterson 1988). The fourth parameter is the fractal dimension \( D \) and the last one, \( B \), is the absolute normalization of the fractal distribution. This latter can be, for example, defined as the average number of galaxies of any luminosity as seen by an average observer in a ball of radius \( 1h^{-1}Mpc \) and can be measured in redshift surveys (see Sylos Labini, Montuori & Pietronero 1998, Joyce, Montuori & Sylos Labini 1999 for a more detailed discussion of the subject).

The amplitude \( N_0 \) in Eq.3 is given by

\[
N_0 = \frac{AB}{2(4\pi)^{D/2}} L^*^{\delta + \frac{D+2}{2}} \Gamma_e \left( \delta + \frac{D}{2} \right),
\]

where \( \Gamma_e \) is the Euler’s Gamma function.

In view of Eq.2 and Eq.4, one can compute the average redshift of a galaxy with
apparent magnitude \( m \). We obtain

\[
\langle z \rangle = \frac{h}{3 \cdot 10^8} \frac{\Gamma_e \left( \frac{D+3}{2} + \delta \right)}{\Gamma_e \left( \frac{D+2}{2} + \delta \right)} 10^{0.2(m - M*)},
\]

where \( h \) is the normalized Hubble’s constant.

From current data both the amplitude and the slope of counts can be estimated. In general in the standard \( BJ \) photometric system\(^3\), and in the range of magnitude from \( \sim 11^m \) to \( \sim 19^m \) (see Tab.1), one has \( \alpha = 0.50 \pm 0.04 \) corresponding to \( D = 2.5 \pm 0.2 \). The corresponding range of average redshift (Eq. 8) is \( 10^{-3} \lesssim \langle z \rangle \lesssim 10^{-1} \). Note that, in the faint end part of counts, where the cosmological corrections are known to be relevant, the slopes are consistent with the bright end (see Tab.2).

Such a value of \( \alpha \) (and hence of \( D \)) is slightly larger than the value of \( D \) found in nearby redshift surveys, which is \( D = 2.1 \pm 0.1 \) up to \( \sim 30 \div 50 h^{-1} Mpc \). Whether such a difference is due to an increase of fractal dimension with scale or it is related to some systematic effects in the counts will be discussed in forthcoming papers (e.g. Gabrielli & Sylos Labini 2000). It is worth to note that Teerikorpi et al. (1988) have found a dimension \( D = 2.35 \pm 0.05 \) up to \( \sim 100 h^{-1} Mpc \) by counting galaxies in real space and in volume limited samples.

Note that in the range of \( \langle z \rangle \ll 1 \) one expects eventual cosmological and luminosity evolution corrections to be negligible. However, we propose a further test to discriminate the importance of these effects.

More specifically, we propose to study in detail the fluctuations around the average behavior of number counts as a function of apparent magnitude. In fact, as shown below,

\(^3\)We adopt hereafter the standard Johnson-Cousins system following the choice of Arount et al (1997) and of Bertin & Dennefeld (1997).
this test can discriminate between the fractal or smooth cosmological nature of the deviation of the $\alpha$ exponent from Euclidean behavior ($\alpha = 0.6$). This study is motivated by the fact that through number counts we can analyze much larger space volumes than in redshift surveys. In fact, the deepest actual red-shift surveys where the fractal dimension has been estimated (Joyce et al., 1999) contains some thousand galaxies, whereas magnitude limited surveys can have as many as some millions of galaxies up to very faint magnitudes and deep scales (e.g. POSS-II).

3. Fluctuations

A very illustrative and simple case is a poissonian homogeneous distribution of galaxies. In this case the difference between the number of points in two equal non overlapping volumes is of the order of the square root of the average number. The variance of counts can be easily computed from the probabilistic definition of Poisson distribution $n(\vec{r})$ and by using again Eq.\ref{eq:9} obtaining

$$\sigma^2_m = \frac{\langle (N(<m) - \langle N(<m) \rangle)^2 \rangle}{\langle N(<m) \rangle^2} \sim 10^{-0.6m}, \quad (9)$$

where $N(<m)$ is the number of galaxies with apparent magnitude brighter than $m$. The average $\langle N(<m) \rangle$ is given by Eq.\ref{eq:5} with $D = 3$. A more rigorous derivation, considering three point correlation function, can be found in Gabrielli & Sylos Labini (2000). Thus, in the poissonian case, relative fluctuations decrease exponentially at faint magnitudes. The pre-factor in Eq.\ref{eq:9} is simply related to a combination of the parameters in Eq.\ref{eq:2}.

In a fractal distribution the typical fluctuation of the number of points $N(r)$ in a sphere of radius $r$, with respect to the average value over different observers $\langle N(r) \rangle$, is always of the same order of the average number (e.g. Mandelbrot 1977):

$$\delta N(r) = \sqrt{\langle (N(r) - \langle N(r) \rangle)^2 \rangle} \sim \langle N(r) \rangle . \quad (10)$$
This property is very important for counts, which are not averaged over different observers (Sylos Labini Montuori & Pietronero 1998). Eq. 10 means that, in a fractal, at any scale, one expects to find a void or a structure, the extension of which is of the same order of the scale itself: this is the source of geometrical self-similarity. This property implies that fluctuations in the number of points (differential or integral) should be, in absolute value, always proportional to the average number itself and never decreases with distance.

From Eq. 10 and Eq. 1 one obtains that the relative fluctuation in the counts as a function of apparent magnitude has a constant amplitude:

$$\sigma_m \sim \text{const.} > 0.$$  \hspace{1cm} (11)

Eq. 11 describes the “persistent” character of fluctuations in number counts induced by the fractal nature of the spatial distribution. The numerical value of $\sigma_m$ depends now on the same parameters in Eq.1, and on some other morphological characteristics of the specific studied fractal. In fact, fluctuations are characterized by higher order correlation functions (Blumenfeld & Ball 1993; Gabrielli, Sylos Labini & Pellegrini 1999) and the fractal dimension does not determine them univocally. Note that $\sigma_m$ can be also very small: its striking feature being in fact that it is constant as a function of $m$, and not its absolute amplitude. By using simple approximations, it is possible to relate the constant $\sigma_m$ to three point correlation function of the distribution (Gabrielli & Sylos Labini 2000). In a deterministic fractal, fluctuations have a nearly constant amplitude with a log-periodical modulation (Sornette, 1998) as a function of scale, because the algorithm generating such a structure is a deterministic one. In the more realistic case of stochastic fractals, the oscillations are in general a superposition of waves, which are periodic in log-space, but which have different frequencies and amplitudes.

The poissonian case describes also the situation in which one has a spatial distribution of galaxies with a small crossover scale $\lambda_0$ to homogeneity and a finite correlation length $r_c$
(Gaite et al. 1999, Gabrielli, Sylos Labini & Durrer 2000). A different situation occurs in the case of a spatial distribution with a finite homogeneity scale, but an infinite correlation length. This case can be thought as obtained by a superposition of a fractal distribution to a dominating flat constant density. In this case \( \langle N(< m) \rangle \) is again given by Eq. 7 with \( D = 3 \) (i.e. it is dominated by the flat constant distribution). On the other hand the absolute fluctuation \( \langle (N(< m) - \langle N(< m) \rangle)^2 \rangle \) is dominated by the fractal scale invariant correlations. Consequently, the normalized variance \( \sigma_m^2 \) is again an exponentially decreasing function of \( m \), even if with a slower behavior than a poissonian distribution

\[
\sigma_m^2 \sim 10^{(-0.2(3-D)m)},
\]

where \( D < 3 \) is the dimension of the fractal superimposed to the constant density.

Note that, in general one can have more complex situations, but the case described by Eq. 11 is an unambiguous indication of persistent and scale invariant real space fluctuations typical of statistically self similar irregular distributions.

4. Discussion and Conclusions

Up to \( B_J \sim 18^m \) (i.e. \( \langle z \rangle < 0.1 \)) cosmological models, in the framework of the Friedmann solutions, predict an exponent \( \alpha = 0.6 \). This is because, one assumes an homogeneous distribution starting at very small scale, i.e. \( 5 \div 20h^{-1}Mpc \) (Peebles 1993, Davis 1997, Wu, Lahav & Rees 1999). Evolutionary effects, as other cosmological corrections, become efficient at \( z \simeq 1 \) (Yoshii & Takahara 1989, Sandage 1995). Such a situation is nearly independent on the value of \( q_0 \), the amount of K-corrections, the possible evolution of galaxies with redshift, and the photometric band chosen. It is important to note that \( N(m, q_0) \) is degenerate to \( z \) in first order: i.e. it is independent on \( q_0 \) for small redshift. For instance, at \( \langle z \rangle = 0.1 \) (i.e. \( B_J \sim 18^m \)) the deviation from the Euclidean
$\alpha = 0.6$ slope is less than 10% for any value of $q_0$. Moreover, the slope at fainter magnitudes should be a rapidly varying function of the magnitude itself. Clearly, this is not the case for the data shown in Tab.1 and Tab.2.

From an experimental point of view, we propose to study the fluctuations with respect to the average in the integrated number counts $N(<m)$ instead of in the differential one $N(m)$, in order to avoid problems with shot noise in magnitude bins. In such a way it is clear that at bright magnitudes $\sigma_m$ shows an initial decay due to the paucity of bright galaxies in small solid angle fields. Then, after the integrated number of points has reached a large enough value, one should be able to detect only the effect of eventual intrinsic fluctuations.

However, calibration errors or other systematic field-to-field possible biases can affect the measurement. For this reason we suggest first to focus the study to a single sky field at time, instead of considering fluctuations in different sky fields. That is, after having determined the best fit as in Eq. 3 (or by adding eventual cosmological corrections) one can study fluctuations in a single well calibrated sky field. The intrinsic nature of fluctuations reveals as soon as the shot noise contribution becomes negligible. Clearly, there is a transient between the shot noise regime and the instrinsical fluctuations, where there is a combination of these two effects. The range of magnitudes of such a combination of shot noise and intrinsic fractal fluctuations, depends not only on the number of points, but also on the solid angle of the survey (see Sylos Labini, Montuori & Pietronero 1998 for a more detailed discussion about the shot noise effect on the galaxy counts).

It is important to note that the presence of eventual persistent and scale-invariant fluctuations, in the log $N(< m)$ vs. $m$ plot, cannot be due to any smooth correction to the data as cosmological and evolution effects, but they can be the outcome exclusively of strongly correlated fractal fluctuations. The reason being that smooth linear corrections are
not able to produce persistent scale-invariant fluctuations on $N(< m)$ of the same order of $N(< m)$ itself.

It is important to note that the exponent of the counts can be very sensible to the photometric band chosen, due to the different K-correction and kind of objects selected. However, it is important to stress that the nature of fluctuation structures must be the same in all different photometric bands. In other words, since these fluctuations are intrinsic, then they should not depend on the photometric band used. On the contrary other possible intervening effects, like galactic extinction fluctuations, strongly depend on the photometric band chosen.

From the present discussion an important challenge for the new generation of experiments is represented by the following questions: (i) Why the slope and the amplitude of the counts remain nearly constant beyond $B_J \approx 18^m$? (ii) Why there is no clear sign of change of slope due to galaxy evolution, space-time geometry effects and K-correction, even when the average redshift becomes to be of order unity? (iii) The last question concerns the detection of persistent and scale-invariant fluctuations in the counts. The new generation of redshift surveys like SSDS and 2dF, together with the new POSS-II (Djorgovski et al., 2000) photometric survey, will be able to answer to these fundamental questions.

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Table 1: Determination of the differential galaxy number counts in the bright end by various authors. In the first column it is reported the reference to the original paper, in the second the value of the slope and in the third the range of $B_J$ magnitudes (Standard Johnson system as in Arnouts et al., (1997)) in which the fit has been performed. For the transformation between different magnitude systems to the standard $B_J$ band see Arnouts et al. (1997). For the exponents of the CGCG (Zwicky et al. 1961-68) and the data of Bertin & Dennefeld (1997) see discussion in Gabrielli & Sylos Labini (2000)

| Author               | $\alpha$     | $\Delta B_J$       |
|----------------------|--------------|---------------------|
| Bertin & Dennefeld (1997) | 0.50 ± 0.01 | 15.0 ≤ $B_J$ ≤ 20.0 |
| Maddox et al. (1990)   | 0.52 ± 0.01 | 17.3 ≤ $B_J$ ≤ 20.8 |
| Weir et al. (1995)     | 0.49 ± 0.01 | 17.5 ≤ $B_J$ ≤ 20.5 |
| CGCG                  | 0.50 ± 0.01 | 11.0 ≤ $B_J$ ≤ 15.0 |
| Rousseau et al. (1994) | 0.49 ± 0.02 | 11.0 ≤ $B_J$ ≤ 15.5 |
Table 2: Determination of the slopes of differential galaxy counts in the faint end by various authors (see Tab.1). The data of Picard (1991) are in the $r$ band and the transformation is $B_J \approx r + 0.93$.

| Author            | $\alpha$  | $\Delta B_J$          |
|-------------------|-----------|------------------------|
| Tyson (1988)      | 0.45 ± 0.02 | $18.0 \leq B_J \leq 28.0$ |
| Lilly (1991)      | 0.38 ± 0.02 | $23.0 \leq B_J \leq 26.5$ |
| Metcalfe et al. (1991) | 0.49 ± 0.05 | $21.0 \leq B_J \leq 25.0$ |
| Metcalfe et al. (1995) | 0.49 ± 0.05 | $24.0 \leq B_J \leq 27.0$ |
| Arnouts et al. (1997) | 0.46 ± 0.02 | $20.5 \leq B_J \leq 24.5$ |
| Driver et al. (1994) | 0.44 ± 0.02 | $23.5 \leq B_J \leq 26.0$ |
| Picard (1991)     | 0.45 ± 0.01 | $16.0 \leq r \leq 19.0$ |