New Hermite–Hadamard type inequalities for $n$-polynomial harmonically convex functions

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Abstract
In the article, we introduce a class of $n$-polynomial harmonically convex functions, establish their several new Hermite–Hadamard type inequalities which are the generalizations and variants of the previously known results for harmonically convex functions.

MSC: 26A51; 26D10; 26D15

Keywords: $n$-polynomial; Harmonic convex function; Hermite–Hadamard inequality

1 Introduction and preliminaries
Let $I \subseteq \mathbb{R}$ be an interval. Then a real-valued function $f : I \rightarrow \mathbb{R}$ is said to be convex (concave) if the inequality

$$f(\lambda x + (1-\lambda)y) \leq (\geq) \lambda f(x) + (1-\lambda)f(y)$$

holds for all $x, y \in I$ and $\lambda \in [0, 1]$.

It is well known that the convexity (concavity) has wide applications in many branches of pure and applied mathematics [1–30], many inequalities can be derived via the convexity or concavity theory [31–56]. Recently, the generalizations, extensions, and variants for the convexity have attracted the attention of many researchers [57–65].

The classical Hermite–Hadamard inequality [66] is the most famous one in convex functions theory, which states that the double inequality

$$f\left(\frac{x+y}{2}\right) \leq (\geq) \frac{1}{y-x} \int_x^y f(t) dt \leq (\geq) \frac{f(x) + f(y)}{2}$$

holds for all $x, y \in I$ with $x \neq y$ if $f : I \rightarrow \mathbb{R}$ is a convex (concave) function.

In the past few decades, many generalizations, improvements, refinements, and variants for the Hermite–Hadamard inequality have been made by several researchers, we recommend the literature [67–72] to interested readers.
Now, we recall the definition of harmonically convex function [73] as follows.

**Definition 1.1** (see [73]) Let $H \subseteq (0, \infty)$ be an interval. Then a real-valued function $f : H \to \mathbb{R}$ is said to be harmonically convex if

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq tf(y) + (1-t)f(x)$$

for all $x, y \in H$ and $t \in [0,1]$.

İşcan [73] provided the Hermite–Hadamard type inequality for the harmonically convex function.

**Theorem 1.2** (see [73]) Let $H \subseteq (0, \infty)$ be an interval and $f : H \to \mathbb{R}$ be a harmonically convex function. Then the Hermite–Hadamard type inequality

$$f\left(\frac{2ab}{a + b}\right) \leq \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} \, dx \leq \frac{f(a) + f(b)}{2}$$

(1.1)

holds for all $a, b \in H$ with $a < b$ if $f \in L[a, b]$.

Very recently, Toplu et al. [74] introduced and investigated a new class of $n$-polynomial convex functions and established several new Hermite–Hadamard type inequalities for this class of functions.

The main purpose of the article is to introduce the notion of $n$-polynomial harmonically convex functions, derive the variants of the classical Hermite–Hadamard inequality by use of the class of $n$-polynomial harmonically convex functions. We also discuss several new special cases for the obtained results which show that our obtained results are the generalizations and extensions of some previously known results.

### 2 Results and discussions

In this section, we first introduce the definition of $n$-polynomial harmonically convex function.

**Definition 2.1** Let $n \in \mathbb{N}$ and $H \subseteq (0, \infty)$ be an interval. Then a nonnegative real-valued function $f : H \to [0, \infty)$ is said to be an $n$-polynomial harmonically convex function if

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq \frac{1}{n} \sum_{s=1}^{n} (1-(1-t)^s)f(y) + \frac{1}{n} \sum_{s=1}^{n} (1-t^s)f(x)$$

for all $x, y \in H$ and $t \in [0,1]$.

From Definitions 1.1 and 2.1, we clearly see that the class of $n$-polynomial harmonically convex functions reduces to the class of harmonically convex functions if $n = 1$ and the 2-polynomial harmonically convex function $f$ satisfies the inequality

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq \frac{3t - t^2}{2} f(y) + \frac{2 - t - t^2}{2} f(x).$$
Theorem 2.2 Let $b > a > 0, f_α : [a, b] \to [0, \infty)$ be a family of $n$-polynomial harmonically convex functions and $f(u) = \sup_α f_α(u)$. Then $f$ is an $n$-polynomial harmonically convex function if $J = \{x \in [a, b] : f(x) < \infty\}$ is an interval.

Proof Let $t \in [0, 1]$ and $x, y \in J$. Then we clearly see that
\[
f\left(\frac{xy}{tx + (1-t)y}\right) = \sup_α f_α\left(\frac{xy}{tx + (1-t)y}\right)
\leq \frac{1}{n} \sum_{s=1}^{n} (1 - (1-t)^s) \sup_α f_α(y) + \frac{1}{n} \sum_{s=1}^{n} (1 - t^s) \sup_α f_α(x)
= \frac{1}{n} \sum_{s=1}^{n} (1 - (1-t)^s) f(y) + \frac{1}{n} \sum_{s=1}^{n} (1 - t^s) f(x) < \infty,
\]
which completes the proof. □

Theorem 2.3 Let $f : [a, b] \subseteq (0, \infty) \to [0, \infty)$ be an $n$-polynomial harmonically convex function. Then one has
\[
\frac{1}{2} \left(\frac{n}{n + 2^{-n} - 1}\right) f\left(\frac{2ab}{a + b}\right) \leq \frac{ab}{b - a} \int_a^b \frac{f(x)}{x^2} dx \leq \left(\frac{f(a) + f(b)}{n}\right) \sum_{s=1}^{n} \frac{s}{s + 1}
\]
if $f \in L[a, b]$.

Proof It follows from the $n$-polynomial harmonic convexity of $f$ that
\[
f\left(\frac{xy}{tx + (1-t)y}\right) \leq \frac{1}{n} \sum_{s=1}^{n} (1 - (1-t)^s) f(y) + \frac{1}{n} \sum_{s=1}^{n} (1 - t^s) f(x),
\]
which leads to
\[
f\left(\frac{2xy}{x + y}\right) \leq \frac{1}{n} \sum_{s=1}^{n} \left[1 - \left(\frac{1}{2}\right)^s\right] f(y) + \frac{1}{n} \sum_{s=1}^{n} \left[1 - \left(\frac{1}{2}\right)^s\right] f(x).
\]
Using the change of variables, we have
\[
f\left(\frac{2ab}{a + b}\right) \leq \frac{1}{n} \sum_{s=1}^{n} \left[1 - \left(\frac{1}{2}\right)^s\right] \left[f\left(\frac{ab}{ta + (1-t)b}\right) + f\left(\frac{ab}{tb + (1-t)a}\right)\right].
\]

Integrating both sides of the above inequality with respect to $t$ on $[0, 1]$, we get
\[
\frac{1}{2} \left(\frac{n}{n + 2^{-n} - 1}\right) f\left(\frac{2ab}{a + b}\right) \leq \frac{ab}{b - a} \int_a^b \frac{f(x)}{x^2} dx. \quad (2.1)
\]

Note that
\[
\frac{ab}{b - a} \int_a^b \frac{f(x)}{x^2} dx = \int_0^1 f\left(\frac{ab}{ta + (1-t)b}\right) dt
\]
\[
\leq \int_0^1 \frac{1}{n} \sum_{s=1}^n \left[ \frac{1}{n} \sum_{s=1}^n [1 - (1-t)^s]f(b) + \frac{1}{n} \sum_{s=1}^n [1-t^s]f(a) \right] dt \\
= \frac{f(b)}{n} \sum_{s=1}^n \int_0^1 [1 - (1-t)^s] dt + \frac{f(a)}{n} \sum_{s=1}^n \int_0^1 [1-t^s] dt \\
= \left[ \frac{f(a)+f(b)}{n} \right] \sum_{s=1}^n \frac{s}{s+1}. \\
\]

Therefore, Theorem 2.3 follows from (2.1) and (2.2). \qed

Remark 2.4 Let \( n = 1 \). Then Theorem 2.3 leads to the Hermite–Hadamard inequality for harmonically convex functions of [73].

Lemma 2.5 Let \( f: [a, b] \subseteq (0, \infty) \to \mathbb{R} \) be a differentiable function. Then the identity

\[
\frac{\lambda f(a) + \mu f(b)}{2} + \frac{2 - \lambda - \mu}{2} f\left(\frac{2ab}{a+b}\right) - \frac{ab}{b-a} \int_a^b f(x) \frac{\lambda}{x^2} dx \\
= \frac{4(b-a)}{4} \int_0^1 \left[ \frac{4(1-\lambda-t)}{(1-t)a + (1+t)b)^2} f\left(\frac{2ab}{(1-t)a + (1+t)b}\right) \\
+ \frac{4(\mu-t)}{(2-t)a + tb)^2} f\left(\frac{2ab}{(2-t)a + tb}\right) \right] dt
\]

holds for \( \lambda, \mu \in [0, 1] \) if \( f'' \in L[a, b] \).

Proof Integrating by parts and changing variable of the definite integral give

\[
I_1 = \int_0^1 \frac{4(1-\lambda-t)}{(1-t)a + (1+t)b)^2} f\left(\frac{2ab}{(1-t)a + (1+t)b}\right) dt \\
= - \frac{2}{ab(b-a)} \left[ (1-\lambda-t) f\left(\frac{2ab}{(1-t)a + (1+t)b}\right) \right]_0^1 \\
+ \int_0^1 f\left(\frac{2ab}{(1-t)a + (1+t)b}\right) dt \\
= \frac{2}{ab(b-a)} \left[ \lambda f(a) + (1-\lambda)f\left(\frac{2ab}{a+b}\right) \right] - \frac{4}{(b-a)^2} \int_a^b f(x) \frac{\lambda}{x^2} dx.
\]

Similarly, we have

\[
I_2 = \int_0^1 \frac{4(\mu-t)}{(2-t)a + tb)^2} f\left(\frac{2ab}{(2-t)a + tb}\right) dt \\
= - \frac{2}{ab(b-a)} \left[ (\mu-t) f\left(\frac{2ab}{(2-t)a + tb}\right) \right]_0^1 + \int_0^1 f\left(\frac{2ab}{(2-t)a + tb}\right) dt \\
= \frac{2}{ab(b-a)} \left[ (1-\mu)f\left(\frac{2ab}{a+b}\right) + \mu f(b) \right] - \frac{4}{(b-a)^2} \int_a^b f(x) \frac{\mu}{x^2} dx.
\]

Multiplying \( I_1 \) and \( I_2 \) by \( \frac{ab(b-a)}{4} \) and combining both equalities, we get the desired result. \qed
For the sake of simplicity, in what follows we denote

\[ A_a = (1 - t)a + (1 + t)b \]

and

\[ A_b = (1 - t)a + tb. \]

Before we give our next result, let us recall the definitions of the gamma function \( \Gamma(\cdot) \), beta function \( B(\cdot, \cdot) \), and hypergeometric function \( _2F_1(\cdot, \cdot; \cdot; \cdot) \) as follows:

\[
\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} \, dt,
\]

\[
B(x, y) = \int_0^1 t^{x-1}(1 - t)^{y-1} \, dt,
\]

\[
B(x, y) = \frac{\Gamma(x) \Gamma(y)}{\Gamma(x + y)},
\]

\[
_2F_1(x, y; c; z) = \frac{1}{B(y, c - y)} \int_0^1 t^{x-1}(1 - t)^{c-1}(1 - zt)^{-y} \, dt.
\]

Theorem 2.6 Let \( p, q > 1 \) with \( 1/p + 1/q = 1 \), \( \lambda, \mu \in [0, 1] \), and \( f : [a, b] \subseteq (0, \infty) \to \mathbb{R} \) be a differentiable function such that \( f' \in L[a, b] \) and \( |f'|^q \) is an \( n \)-polynomial harmonically convex function. Then we have

\[
\left| \frac{2f(a) + \mu f(b)}{2} + \frac{2 - \lambda - \mu}{2} f\left( \frac{2ab}{a + b} \right) - \frac{ab}{b - a} \int_a^b \frac{f(x)}{x^2} \, dx \right| \leq \frac{ab(b - a)}{4} \left\{ \psi_1 \left( \frac{1}{\psi_1} C_1 |f'(a)|^q + C_2 |f'(b)|^q \right)^{\lambda} \right. \\
+ \left. \psi_2 \left( \frac{1}{\psi_2} C_3 |f'(a)|^q + C_4 |f'(b)|^q \right)^{\lambda} \right\},
\]

where

\[
\psi_1 = 4 \int_0^1 (1 - \lambda - t)^p \, dt = 4 \left( \frac{(1 - \lambda)^{p+1} + (\lambda)^{p+1}}{p + 1} \right), \tag{2.3}
\]

\[
\psi_2 = 4 \int_0^1 (1 - \mu - t)^p \, dt = 4 \left( \frac{(1 - \mu)^{p+1} + (\mu)^{p+1}}{p + 1} \right), \tag{2.4}
\]

\[
C_1 = \frac{1}{2n} \sum_{s=1}^n \left( (a + b)^{-2q} 2F_1\left( 2q, 1; 2; \frac{a - b}{a + b} \right) \\
- \frac{(a + b)^{-2q}}{s + 1} 2F_1\left( 2q, 1; 2; \frac{a - b}{a + b} \right) \right), \tag{2.5}
\]

\[
C_2 = \frac{1}{2n} \sum_{s=1}^n \left( (a + b)^{-2q} 2F_1\left( 2q, 1; 2; \frac{a - b}{a + b} \right) \\
- \frac{(a + b)^{-2q}}{s + 1} 2F_1\left( 2q, 1; 2; \frac{a - b}{a + b} \right) \right), \tag{2.6}
\]
\[ C_3 = \frac{1}{2^n} \sum_{s=1}^{n} \binom{2a}{s} F_1 \left( \binom{2q}{s}; 2q, 1; \frac{a-b}{2a} \right) \]
\[ - \binom{2a}{s+1} F_1 \left( \binom{2q}{s+1}; 2q, 1; \frac{a-b}{2a} \right), \quad (2.7) \]
\[ C_4 = \frac{1}{2^n} \sum_{s=1}^{n} \binom{2(2a)}{s} F_1 \left( \binom{2q}{s}; 2q, 1; \frac{a-b}{2a} \right) \]
\[ - \binom{2a}{s+1} F_1 \left( \binom{2q}{s+1}; 2q, 1; \frac{a-b}{2a} \right). \quad (2.8) \]

**Proof** It follows from Lemma 2.5 and Hölder’s integral inequality together with the \( n \)-polynomial harmonic convexity of \( |f'|^q \) that

\[
\left| \frac{f(a) + \mu f(b)}{2} + \frac{(2 - \lambda - \mu)}{2} f \left( \frac{2ab}{a+b} \right) - \frac{ab}{b-a} \int_a^b f(x) \frac{dx}{x^2} \right| \\
\leq \frac{ab(b-a)}{4} \left[ \int_0^1 \left( \frac{4(1-\lambda-t)}{(1-t)a+(1+t)b)^2} \right) \left| f' \left( \frac{2ab}{(1-t)a+(1+t)b} \right) \right| dt \right]^\frac{1}{p} \\
\times \left[ \int_0^1 \frac{1}{A^2} \left( \frac{1}{2n} \sum_{s=1}^{n} \left[ 2 - (1-t)^s \right] \left| f'(a) \right|^q + \frac{1}{2n} \sum_{s=1}^{n} \left[ 1 - t^s \right] \left| f'(b) \right|^q \right) dt \right]^\frac{1}{q} \\
+ \left( \int_0^1 4(\mu-t)^p dt \right)^\frac{1}{p} \\
\times \left[ \int_0^1 \frac{1}{A^2} \left( \frac{1}{2n} \sum_{s=1}^{n} \left[ 1 - (1-t)^s \right] \left| f'(a) \right|^q + \frac{1}{2n} \sum_{s=1}^{n} \left[ 2 - t^s \right] \left| f'(b) \right|^q \right) dt \right]^\frac{1}{q} \right\} \\
= \frac{ab(b-a)}{4} \left\{ \left( \psi_1^\frac{1}{p} \left( C_1 \left| f'(a) \right|^q + C_2 \left| f'(b) \right|^q \right)^\frac{1}{q} \right) + \left( \psi_2^\frac{1}{p} \left( C_3 \left| f'(a) \right|^q + C_4 \left| f'(b) \right|^q \right)^\frac{1}{q} \right) \right\}. 
\]

This completes the proof. \( \square \)

From Theorem 2.6 we get the following Corollaries 2.7 and 2.8 immediately.

**Corollary 2.7** Let \( \lambda = \mu \). Then Theorem 2.6 leads to the conclusion that

\[
\left| \frac{\lambda f(a) + \lambda f(b)}{2} + (1-\lambda) f \left( \frac{2ab}{a+b} \right) - \frac{ab}{b-a} \int_a^b f(x) \frac{dx}{x^2} \right| \\
\leq \frac{ab(b-a)}{4} \psi_1^\frac{1}{p} \left[ \left( C_1 \left| f'(a) \right|^q + C_2 \left| f'(b) \right|^q \right)^\frac{1}{q} + \left( C_3 \left| f'(a) \right|^q + C_4 \left| f'(b) \right|^q \right)^\frac{1}{q} \right]. 
\]

where \( \psi_1, C_1, C_2, C_3, \) and \( C_4 \) are given by (2.3), (2.5), (2.6), (2.7), and (2.8), respectively.
Corollary 2.8 Let $\lambda = \mu = 1/2$ and $\lambda = \mu = 1/3$. Then Theorem 2.6 leads to

$$
\left| \frac{f(a) + f(b)}{4} + \frac{1}{2} f\left( \frac{2ab}{a+b} \right) - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} \, dx \right| \\
\leq \frac{ab(b-a)}{4} \left( \frac{8}{2^{1+\frac{1}{p}}(p+1)^{\frac{1}{p}}} \right)^{\frac{1}{2}} \left[ \{C_1 |f'(a)|^q + C_2 |f'(b)|^q \} \right]^{\frac{1}{q}} \\
+ \left[ C_3 |f'(a)|^q + C_4 |f'(b)|^q \right]^{\frac{1}{q}}
$$

and

$$
\left| \frac{f(a) + f(b)}{6} + \frac{2}{3} f\left( \frac{2ab}{a+b} \right) - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} \, dx \right| \\
\leq \frac{ab(b-a)}{4} \left( \frac{4}{(2p+1;2]} \left[ (\frac{a-b}{a+b}) \right]^{\frac{1}{p}} \frac{1}{q} \right) \left[ \{C_5 |f'(a)|^q + C_6 |f'(b)|^q \} \right]^{\frac{1}{q}} \\
+ \left[ C_7 |f'(a)|^q + C_8 |f'(b)|^q \right]^{\frac{1}{q}},
$$

where $C_1, C_2, C_3, \text{and } C_4$ are given by (2.5), (2.6), (2.7), and (2.8), respectively.

Theorem 2.9 Let $p, q > 1$ with $1/p + 1/q = 1$, $\lambda, \mu \in [0,1]$, and $f : [a,b] \subseteq (0,\infty) \rightarrow \mathbb{R}$ be a differentiable function such that $f' \in L[a,b]$ and $|f'|^q$ is an $n$-polynomial harmonically convex function. Then

$$
\left| \frac{\lambda f(a) + \mu f(b)}{2} \right| - \frac{2 - \lambda - \mu}{2} f\left( \frac{2ab}{a+b} \right) - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} \, dx \\
\leq \frac{ab(b-a)}{4} \left[ \frac{4}{(a+b)^2} \left( 2F_1 \left( 2p+1;2;\frac{a-b}{a+b} \right) \right) \left( C_5 |f'(a)|^q + C_6 |f'(b)|^q \right) \right]^{\frac{1}{q}} \\
+ \frac{1}{a^2} \left( 2F_1 \left( 2p+1;2;\frac{a-b}{2a} \right) \right) \left( C_7 |f'(a)|^q + C_8 |f'(b)|^q \right) \left[ \frac{1}{a^2} \right]^{\frac{1}{q}},
$$

where

$$
C_5 = \frac{1}{2n} \sum_{s=1}^n \left( \frac{2(1-\lambda)^q + \lambda}{q+1} \right) - \int_0^1 |1-\lambda - t|^q (1-t)^q \, dt, \\
C_6 = \frac{1}{2n} \sum_{s=1}^n \left( \frac{(1-\lambda)^q + \lambda}{q+1} \right) - \int_0^1 |1-\lambda - t|^q t^q \, dt, \\
C_7 = \frac{1}{2n} \sum_{s=1}^n \left( \frac{(1-\mu)^q + \mu}{q+1} \right) - \int_0^1 |\mu - t|^q (1-t)^q \, dt, \\
C_8 = \frac{1}{2n} \sum_{s=1}^n \left( \frac{(1-\mu)^q + \mu}{q+1} \right) - \int_0^1 |\mu - t|^q t^q \, dt.
$$
Proof. Using Lemma 2.5, Hölder’s integral inequality, and the \( n \)-polynomial harmonic convexity of \( |f'|^q \), we have

\[
\begin{align*}
\frac{2f(a) + \mu f(b)}{2} + \frac{(2 - \lambda - \mu) f \left( \frac{2ab}{a + b} \right)}{2} - \frac{ab}{b - a} \int_a^b f(x) \frac{dx}{x^2} & \\
\leq \frac{ab(b - a)}{4} \left[ \int_0^1 \left| \frac{4(1 - \lambda - t)}{((1 - t)a + (1 + t)b)^2} \right| f \left( \frac{2ab}{(1 - t)a + (1 + t)b} \right) dt \right. \\
& + \left. \int_0^1 \left| \frac{4(\mu - t)}{(2 - t)a + tb)^2} \right| f \left( \frac{2ab}{(2 - t)a + tb} \right) dt \right] \\
\leq \frac{ab(b - a)}{4} \left\{ 4 \left( \int_0^1 \frac{1}{A_n^p} dt \right)^{\frac{1}{q}} \\
& \times \left[ \int_0^1 |1 - \lambda - t|^q \left( \frac{1}{2n} \sum_{s=1}^n [2 - (1 - t)^s] |f'(a)|^q + \frac{1}{2n} \sum_{s=1}^n [1 - t^s] |f'(b)|^q \right) dt \right]^{\frac{1}{q}} \\
& + 4 \left( \int_0^1 \frac{1}{A_n^p} dt \right)^{\frac{1}{q}} \\
& \times \left[ \int_0^1 |\mu - t|^q dt \left( \frac{1}{2n} \sum_{s=1}^n [1 - (1 - t)^s] |f'(a)|^q + \frac{1}{2n} \sum_{s=1}^n [2 - t^s] |f'(b)|^q \right) \right]^{\frac{1}{q}} \right\} \\
= \frac{ab(b - a)}{4} \left[ \frac{4}{(a + b)^2} \left( \frac{1}{2n} \sum_{s=1}^n [2 - (1 - t)^s] C_5 \left| f'(a) \right|^q + C_6 \left| f'(b) \right|^q \right) \right]^{\frac{1}{q}} \\
& + \frac{1}{a^2} \left( \frac{1}{2n} \sum_{s=1}^n [2 - t^s] C_7 \left| f'(a) \right|^q + C_8 \left| f'(b) \right|^q \right)^{\frac{1}{q}} \right].
\end{align*}
\]

This completes the proof. \(\square\)

Remark 2.10. Let \( n = 1 \). Then inequality (2.9) reduces to the following inequality for harmonically convex function:

\[
\begin{align*}
\frac{2f(a) + \mu f(b)}{2} + \frac{2 - \lambda - \mu} {2} f \left( \frac{2ab}{a + b} \right) - \frac{ab}{b - a} \int_a^b f(x) \frac{dx}{x^2} & \\
\leq \frac{ab(b - a)}{4} \left[ \frac{4}{(a + b)^2} \left( \frac{1}{2n} \sum_{s=1}^n [2 - (1 - t)^s] C_5 \left| f'(a) \right|^q + C_6 \left| f'(b) \right|^q \right) \right]^{\frac{1}{q}} \\
& + \frac{1}{a^2} \left( \frac{1}{2n} \sum_{s=1}^n [2 - t^s] C_7 \left| f'(a) \right|^q + C_8 \left| f'(b) \right|^q \right)^{\frac{1}{q}} \right],
\end{align*}
\]

where

\[
C_5^+ = \frac{1}{2} \int_0^1 |1 - \lambda - t|^q (1 + t) dt
\]
\[
= \frac{1}{2} \frac{(1 - \lambda) q^q + (\lambda) q^{q+1} + (1 - \lambda) q^{q+2} + (q + 2 - \lambda) (\lambda) q^{q+1}}{q + 1 (q + 2)}
\]
\[
= \frac{1}{2 (q + 1) (q + 2)} ((q + 3 - \lambda) (1 - \lambda) q^{q+1} + (2q + 4 - \lambda) \lambda q^{q+1}),
\]
\[ C_6^* = \frac{1}{2} \int_0^1 |1 - \lambda - t|^q (1 - t) \, dt \\
= \frac{1}{2} \left( (1 - \lambda)^{q+1} + (\lambda)^{q+1} \right) \frac{q + 1}{q + 1} - (1 - \lambda)^{q+2} + (q + 2 - \lambda)(\lambda)^{q+1} \right) \frac{q + 1}{(q + 1)(q + 2)} \\
= \frac{1}{2(q + 1)(q + 2)}((q + 1 + \lambda)(1 - \lambda)^{q+1} + \lambda^{q+2}), \]

\[ C_7^* = \frac{1}{2} \sum_{i=1}^n \int_0^1 |\mu - t|^q (2 - t) \, dt \\
= \frac{1}{2} \left( (\mu)^{q+1} + (1 - \mu)^{q+1} \right) \frac{q + 1}{q + 1} - (\mu)^{q+2} + (q + 1 + \mu)(1 - \mu)^{q+1} \right) \frac{q + 1}{(q + 1)(q + 2)} \\
= \frac{1}{2(q + 1)(q + 2)}((q + 3 - \mu)(1 - \mu)^{q+1} + (2q + 4 - \mu)\mu^{q+1}). \]

**Remark 2.11** If \( q = 1 \) and \( n = 1 \), then Theorem 2.9 reduces to

\[
\left| \frac{2f(a) + \mu f(b)}{2} + (1 - \lambda - \mu) f \left( \frac{2ab}{a + b} \right) - \frac{ab}{b - a} \int_a^b f(x) \, dx \right| \\
\leq \frac{ab(b - a)}{4} \left[ \frac{4}{(a + b)^2} \left( \beta \left(2F_1 \left( 2p, 1; 2; \frac{a - b}{a + b} \right) \right) \left( C_5^{**} |f''(a)| + C_6^{**} |f''(b)| \right) \right) + \frac{1}{a^2} \left( \beta \left(2F_1 \left( 2p, 1; 2; \frac{a - b}{2a} \right) \right) \left( C_7^{**} |f''(a)| + C_8^{**} |f''(b)| \right) \right) \right],
\]

where

\[ C_5^{**} = \frac{1}{12} \left[ (4 - \lambda)(1 - \lambda)^2 + (6 - \lambda)\lambda \right], \]

\[ C_6^{**} = \frac{1}{12} \left[ (2 + \lambda)(1 - \lambda)^2 + \lambda^3 \right], \]

\[ C_7^{**} = \frac{1}{12} \left[ (2 + \mu)(1 - \mu)^2 + \mu^3 \right], \]

\[ C_8^{**} = \frac{1}{12} \left[ (4 - \mu)(1 - \mu)^2 + (6 - \mu)\mu \right]. \]

**Remark 2.12** If \( q = 1 \), \( n = 1 \), and \( \lambda = \mu \), then inequality (2.9) becomes

\[
\left| \frac{2f(a) + \lambda f(b)}{2} + (1 - \lambda) f \left( \frac{2ab}{a + b} \right) - \frac{ab}{b - a} \int_a^b f(x) \, dx \right| \\
\leq \frac{ab(b - a)}{4} \left[ \frac{4}{(a + b)^2} \left( \beta \left(2F_1 \left( 2p, 1; 2; \frac{a - b}{a + b} \right) \right) \left( C_5^{**} |f''(a)| + C_6^{**} |f''(b)| \right) \right) + \frac{1}{a^2} \left( \beta \left(2F_1 \left( 2p, 1; 2; \frac{a - b}{2a} \right) \right) \left( C_7^{**} |f''(a)| + C_8^{**} |f''(b)| \right) \right) \right].
\]
Remark 2.13 Let $q = 1$, $n = 1$, and $\lambda = \mu = 1/2$. Then Theorem 2.9 leads to the conclusion that
\[
\left| \frac{f(a) + f(b)}{4} - \frac{1}{2} f \left( \frac{2ab}{a+b} \right) - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} \, dx \right|
\leq \frac{ab(b-a)}{384} \left[ \frac{4}{(a+b)^2} \left( {}_2F_1 \left( \frac{2}{2p}, 1; 2; \frac{a-b}{a+b} \right) \right)^{\frac{1}{p}} (18|f'(a)| + 6|f'(b)|) \right.
+ \frac{1}{a^2} \left( {}_2F_1 \left( 2p, 1; 2; \frac{a-b}{2a} \right) \right)^{\frac{1}{p}} (6|f'(a)| + 18|f'(b)|) \right].
\]

3 Conclusion
In this paper, we have introduced a new class of harmonically convex functions, which are called $n$-polynomial harmonically convex functions, derived several new versions of the Hermite–Hadamard inequality using the class of $n$-polynomial harmonically convex functions and a new integral identity for the differentiable function. We have also discussed some special cases of the obtained results which show that our results are the generalizations and extensions of some previously known results for the harmonically convex functions. Our ideas and approach may lead to a lot of follow-up research.

Acknowledgements
The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Funding
The work was supported by the Natural Science Foundation of China (Grant Nos. 61673169, 11301127, 11701176, 11626101, 11601485).

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors’ contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 18 February 2020 Accepted: 23 April 2020 Published online: 06 May 2020

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