1. Introduction and Statement of Results

The total free energy of two dimensional gravity, which is a generating function for certain intersection numbers on the compactified moduli spaces \( \overline{M}_{g,n} \) of stable \( n \)-pointed curves, was conjectured by Witten (and proved by Kontsevich) to satisfy certain KdV equations. This gave new insight in the geometry of those moduli spaces [WI, KO, DIJ].

The Mumford class \( \kappa_1 \) on \( \overline{M}_{g,0} \) was shown to be proportional to the cohomology class \( [\omega_{WP}] \) of the Weil-Petersson form by Wolpert in [WO]. Furthermore he showed that the restriction of this class to any component of the compactifying divisor coincides with the corresponding Weil-Petersson class. Arbarello and Cornalba introduced classes \( \kappa_1 \) on \( \overline{M}_{g,n} \), proved a similar restriction property for these and concluded proportionality on all \( \overline{M}_{g,n} \) [A-C2]. So Weil-Petersson volumes \( \text{vol}(\overline{M}_{g,n}) \) are up to a normalizing factor the intersection numbers

\[
V_{g,n} = \int_{\overline{M}_{g,n}} \kappa_1^{3g-3+n}.
\]

Recently, in papers by Kaufmann, Manin, Zagier, and Zograf [K-M-Z, M-Z, ZO3] a generating function was introduced for intersection numbers of Mumford’s tautological classes [MU2], and shown to be equal to the above generating function up to a change of variables. Previously, Zograf had computed the volumes for genus 0,1, and 2 explicitly [ZO1, ZO2]. Manin and Zograf also gave estimates of the volume for fixed genus and \( n \to \infty \).

The aim of this note is to study the Weil-Petersson volume of the moduli spaces \( M_{g,n} \) for fixed \( n \) and large \( g \). Introducing the decorated...
Teichmüller space, Penner [PE] gave a technique how to integrate top degree differential forms on $\mathcal{M}_{g,n}$, which led to an estimate of the volumes of $\mathcal{M}_{g,1}$ from below with respect to $g \to \infty$. With these methods, Grushevskii [GR] recently proved an upper bound for the volume of $\mathcal{M}_{g,n}$ for fixed $n > 0$ and large $g$. For $n = 1$ his upper estimate has the same order of growth as Penner’s lower estimate. However, for the classical moduli spaces $\mathcal{M}_{g,0}$ the asymptotics of the volume for $g \to \infty$ have not been treated.

Here, we give a different approach, which does not require the existence of punctures. On one hand we use the known push-pull type formulas in the spirit of Arbarello and Cornalba [A-C 1, A-C 2] to estimate the volume of $\mathcal{M}_{g,n+1}$ from below in terms of the volume of $\mathcal{M}_{g,n}$ for any given $g$ and $n \geq 0$. On the other hand, we base our estimates on the fact that $\kappa_1$ is ample and the above restriction property.

In this way it is possible to estimate the volume of the moduli space $\mathcal{M}_{g,0}$ from below in terms of the volume of moduli spaces of lower genus.

We set $V_{(0,3)} = 0$. The values $V_{(0,4)} = 1$, $V_{(0,5)} = 5$, $V_{(1,1)} = \frac{1}{24}$, and $V_{(1,2)} = \frac{1}{8}$ are known. We prove the following Theorems.

**Theorem 1.** Let $2g - 2 + n > 0$ and $(g, n) \neq (0, 4), (1, 1)$. Then

$$V_{g,n+1} \geq \frac{1}{2}(3g - 2 + n)(7g - 7 + 3n) \cdot V_{g,n} + \frac{1}{24g!}.$$  

**Theorem 2.** Let $g > 1$. Then

$$V_{g,0} \geq \frac{1}{28}V_{g-1,2} + \frac{1}{672}V_{g-1,1} + \frac{1}{14} \sum_{j=2}^{\lfloor g/2 \rfloor} V_{j,1}V_{g-j,1} - \frac{1}{28}(V_{g,1})^2,$$

with $V_{g,1} = 0$, if $g$ is odd.

Together with the results of Penner and Grushevsky these imply the existence of constants $0 < c < C$, independent of $n$ such that

$$c^g(2g)! \leq \frac{V_{g,n}}{(3g - 3 + n)!} \leq C^g(2g)!.$$
for all fixed $n \geq 0$ and large $g$.

In particular for all $n \geq 0$

$$\lim_{g \to \infty} \frac{\log \frac{V_{g,n}}{(3g-3+n)!}}{g \log g} = 2$$

2. Proof of the estimates

For any $g$ and $n$ with $2g - 2 + n > 0$ the map $\mathcal{M}_{g,n+1} \to \mathcal{M}_{g,n}$ forgetting the last puncture is known to extend holomorphically to a map

$$\pi_{n+1} : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}.$$  

For $n > 0$ it possesses natural sections $\sigma_j$: $j = 1, \ldots, n$ (cf. \cite[sect. 1]{AC2}) with corresponding divisors $D_j$. Denote the relative dualizing sheaf $\omega_{\overline{\mathcal{M}}_{g,n+1}/\overline{\mathcal{M}}_{g,n}}$ by $\omega_{n+1}$. Let

$$\psi_j := c_1(\sigma_j^*\omega_{n+1}),$$

and

$$K := c_1(\omega_{n+1}(D)) \in H^2(\overline{\mathcal{M}}_{g,n+1}, \mathbb{R}),$$

where $D = D_1 + \ldots + D_n$.

Finally

$$\kappa_j := \pi_{n+1*}(K^{j+1}) \in H^{2j}(\overline{\mathcal{M}}_{g,n+1}, \mathbb{R})$$

for $j = 0, \ldots, 3g - 3 + n$.

For $n = 0$ these are equal to the Mumford classes (denoted also by $\kappa_j$). Moreover $\kappa_1$ is ample on $\overline{\mathcal{M}}_{g,n}$.

According to Mumford \cite{MU2} the classes $\kappa_j$ are numerically effective on $\overline{\mathcal{M}}_{g,0}$ for $j = 1, \ldots, 3g - 3$ in the sense that for any complete subvariety $W \subset \overline{\mathcal{M}}_{g,0}$ of dimension $j$

$$\int_W \kappa_j \geq 0$$

holds. Also, for $j_1 + \ldots + j_k = 3g - 3$

$$\int_{\overline{\mathcal{M}}_{g,0}} \kappa_{j_1} \cdot \ldots \cdot \kappa_{j_k} \geq 0 \tag{5}$$
following [HA2].

We shall need the following formulas for cohomology classes to be found in [A-C 2, (1.7), (1.9), and (1.10)].

\[
\pi_n^*(\psi_1^{a_1} \cdot \ldots \cdot \psi_n^{a_n-1} \cdot \psi_n^{a_n+1}) = \psi_1^{a_1} \cdot \ldots \cdot \psi_n^{a_n-1} \cdot \kappa_n
\]

for \(a_j \geq 0\)

\[
\pi_n^*(\psi_1^{a_1} \cdot \ldots \cdot \psi_n^{a_n-1}) = 
\sum_{j; a_j > 0} \psi_1^{a_1} \cdot \ldots \cdot \psi_{j-1}^{a_{j-1}} \cdot \psi_j^{a_j-1} \cdot \psi_{j+1}^a \cdot \ldots \cdot \psi_n^{a_n-1}
\]

\[
\kappa_a = \pi_{n+1}^*(\kappa_a) + \psi_{n+1}^a \quad \text{on } \overline{M}_{g,n+1}
\]

\[
\kappa_0 = 2g - 2 + n.
\]

**Lemma 1.** Let \(m_j \geq 0\) be integers such that \(\sum_{j=1}^k j \cdot m_j = 3g - 3 + n\) with \(n \geq 0\). Then

\[
\int_{M_{g,n}} \kappa_1^{m_1} \cdot \ldots \cdot \kappa_k^{m_k} \geq 0.
\]

**Proof.** The above equation (11) is the statement for \(n = 0\). We proceed by induction on \(n\). Assume (11) for some \(n \geq 0\). Then

\[
\int_{M_{g,n+1}} \kappa_1^{m_1} \cdot \ldots \cdot \kappa_k^{m_k} = 
\int_{M_{g,n+1}} \pi_{n+1}^* \left( (\pi_{n+1}^* \kappa_1 + \psi_{n+1}^1)^{m_1} \cdot (\pi_{n+1}^* \kappa_2 + \psi_{n+1}^2)^{m_2} \cdot \ldots \cdot (\pi_{n+1}^* \kappa_k + \psi_{n+1}^k)^{m_k} \right)
\]

Since

\[
\int_{M_{g,n}} \pi_{n+1}^* \left( \pi_{n+1}^* (\kappa_1^{j_1} \cdot \ldots \cdot \kappa_k^{j_k}) \cdot \psi_{n+1}^{j+1} \right) = \int_{M_{g,n}} \kappa_1^{j_1} \cdot \ldots \cdot \kappa_k^{j_k} \cdot \kappa_\ell
\]

the above integral can be expressed as a sum of non-negative terms. \(\square\)

**Proof of Theorem 4.** We first note that for \(g > 0\)

\[
\int_{M_{g,n+1}} \psi_{n+1}^{3g-2+n} = \int_{M_{g,n+1}} \psi_1^{3g-2+n} = \int_{M_{g,n}} \psi_1^{3g-3+n} = \ldots = \int_{M_{g,1}} \psi_1^{3g-2}
\]
by (7). These integrals are known to be equal to \(1/(24^g \cdot g!)\) (cf. [P-P]).

For \(g = 0\)

\[
\int_{\mathcal{M}_{0,n+1}} \psi_{n+1}^{n-2} = \int_{\mathcal{M}_{0,4}} \psi_1 = \int_{\mathcal{M}_{0,3}} \kappa_0 = 1.
\]

By Lemma 1, for \(j < 3g - 2 + n\)

\[
\int_{\mathcal{M}_{g,n+1}} \pi_{n+1}^*(\kappa_1)^j \cdot \psi_n^{3g-2+n-j} = \int_{\mathcal{M}_{g,n}} \kappa_1^j \cdot \kappa_{3g-3+n-j} \geq 0,
\]

hence

\[
\int_{\mathcal{M}_{g,n+1}} \kappa_1^{3g-2+n} = \int_{\mathcal{M}_{g,n+1}} (\pi_{n+1}^*(\kappa_1) + \psi_n^{3g-2+n})^{3g-2+n} = \\
\sum_{j=0}^{3g-2+n} \binom{3g-2+n}{j} \int_{\mathcal{M}_{g,n+1}} (\pi_{n+1}^*(\kappa_1))^j \cdot \psi_n^{3g-2+n-j} \geq \\
(3g-2+n) \cdot \int_{\mathcal{M}_{g,n}} \kappa_1^{3g-3+n} \cdot \kappa_0 + \\
\frac{1}{2} (3g-2+n)(3g-3+n) \int_{\mathcal{M}_{g,n}} \kappa_1^{3g-4+n} \kappa_1 + \\
\int_{\mathcal{M}_{g,n+1}} \psi_n^{3g-2+n} = \\
\frac{1}{2} (3g-2+n)(2g-2+n) + \frac{1}{2} (3g-2+n)(3g-3+n)) \cdot \int_{\mathcal{M}_{g,n}} \kappa_1^{3g-2+n} + \\
\int_{\mathcal{M}_{g,n+1}} \psi_n^{3g-2+n} = \\
\frac{1}{2} (3g-2+n)(7g-7+3n) \int_{\mathcal{M}_{g,n}} \kappa_1^{3g-3+n} + \frac{1}{24^g g!}.
\]

For any family \(f : \mathcal{C} \to S\) of stable curves \(\operatorname{det}(f_*\omega_{\mathcal{C}/S})\) is a line bundle over \(S\), where \(\omega_{\mathcal{C}/S}\) denotes the relative dualizing sheaf. These determinant sheaves give rise to a \(\mathbb{Q}\)-divisor on \(\mathcal{M}_{g,0}\), which is usually denoted by \(\lambda\). In a similar way the singular fibers of the above family define devisors, which also give rise to \(\mathbb{Q}\)-divisors on \(\mathcal{M}_{g,0}\), usually denoted by \(\delta_i\). The irreducible components of the divisor at infinity on \(\mathcal{M}_{g,0}\) are denoted by \(\Delta_i, i = 0, \ldots, [g/2]\) with classes

\([\Delta_i] = \delta_i \text{ for } i \neq 1 \text{ and } [\Delta_1] = 2\delta_1\).
These are characterized as follows:

(i) The generic point of $\Delta_0$ corresponds to an irreducible, stable curve of genus $g - 1$ with one ordinary double point. In fact there is a generically 2:1 surjective holomorphic map $\overline{M}_{g-1,1} \to \Delta_0$.

(ii) For $i = 1, \ldots, [g/2]$ the generic points of $\Delta_i$ correspond to stable curves with one ordinary double point and two irreducible components of genus $i$ and $g - i$ resp. There exists a surjective holomorphic map

$$\overline{M}_{i,1} \times \overline{M}_{g-i,1} \to \Delta_i,$$

which is generically 1:1 for $i \neq g/2$ and 2:1 for $i = g/2$

(cf. [H-M]).

**Theorem 3.** Let

$$D = p \cdot \lambda - \sum_{j=0}^{[g/2]} q_j \cdot \delta_j; \quad p, q_j > 0$$

be an effective $\mathbb{Q}$-divisor on $\overline{M}_{g,0}$ such that

$$\mu_j = \frac{12 q_j - p}{p} > 0.$$

Then

$$V_{g,0} > \frac{\mu_0}{2} \cdot V_{g-1,2} + \frac{\mu_1}{48} \cdot V_{g-1,1} + \sum_{j=2}^{[g/2]} \mu_j \cdot V_{j,1} \cdot V_{g-j,1} - \mu_{g/2} \cdot (V_{g/2,1})^2$$

(where $\mu_{g/2} = V_{g/2,1} = 0$, if $g$ is odd).

**Proof.** According to [MU1]

$$\kappa_1 = 12 \lambda - \sum_{j=0}^{[g/2]} \delta_j.$$

We want to write the divisor in the form

$$D = \alpha \kappa_1 - \sum \beta_j \delta_j$$

for $\alpha, \beta_j > 0$. This gives

$$\alpha = \frac{p}{12}; \quad \beta_j = q_j - \frac{p}{12} > 0.$$
Now
\[ 0 < \kappa_1^{3g-4} \cdot D = \alpha \kappa_1^{3g-3} - \sum \beta_j \kappa_1^{3g-4} \cdot \delta_j. \]

We use the restriction property of \( \kappa_1 \) with respect to \( \Delta_j \).

On \( \overline{M}_{g-1,2} \) the class \( \kappa_1 \) is invariant under the action of \( \mathbb{Z}_2 \) exchanging the punctures. So under the natural map \( \overline{M}_{g-1,2} \to \Delta_0 \) it descends to the restriction of the class \( \kappa_1 \) on the ambient space \( \overline{M}_{g,0} \). Now
\[ \kappa_1^{3g-4} \cdot \delta_0 = \int_{\Delta_0} \kappa_1^{3g-4} = \frac{1}{2} V_{g-1,2}. \]

In a similar way, we get
\[ \kappa_1^{3g-4} \cdot \delta_1 = \frac{1}{2} V_{1,1} \cdot V_{g-1,1}, \]
and for \( i > 1 \)
\[ \kappa_1^{3g-4} \cdot \delta_i = V_{i,1} \cdot V_{g-i,1} \]
with an extra factor \( 1/2 \) for \( (V_{g,1})^2 \), if \( g \) is even. Also we use \( V_{1,1} = 1/24. \)

**Remark:** Combining push-pull formulas and computations of intersections of powers of \( \kappa_1 \) with various effective divisors, one can also estimate the intersection numbers \( V_{g,n} \) from above in terms of \( V_{g-1,n+2} \), and the numbers \( V_{j,\ell} \) with \( j = g, \ell < n \) or \( j < g, \ell \leq n + 1 \).

The above Theorem 2 follows, since for every rational \( \varepsilon > 0 \) the \( \mathbb{Q} \)-divisor \( D = (11.2 + \varepsilon) \cdot \lambda - \delta \) is ample \([\text{MU1}]\), where \( \delta = \sum \delta_j \).

A stronger estimate for \( g \geq 23 \) follows from the fact that \( \overline{M}_{g,0} \) has positive Kodaira dimension according to the theorems of Eisenbud, Harris, and Mumford \([\text{E-H}, \text{HA1}, \text{H-M}]\). The equality
\[ K_{\overline{M}_{g,0}} = 13 \lambda - 2 \delta_0 - 3 \delta_1 - 2 \sum_{j=2}^{\lfloor g/2 \rfloor} \delta_j \]
in \( \text{Pic}(\overline{M}_{g,0}) \otimes \mathbb{Q} \) was proved in \([\text{H-M}]\).

This implies for \( g \geq 23 \)
\[ V_{g,0} > \frac{11}{26} \cdot V_{g-1,2} + \frac{23}{624} \cdot V_{g-1,1} + \frac{11}{13} \sum_{j=2}^{\lfloor g/2 \rfloor} V_{j,1} \cdot V_{g-j,1} - \frac{11}{26} \cdot (V_{g/2,1})^2. \]
Further computations of effective divisors in terms of $\lambda$ and $\delta_j$ were provided in [H-M, E-H]. All of these divisors satisfy the hypothesis of Theorem 3 and can be used for new estimates of the Weil-Petersson volumes.

We finally discuss, how to arrive at the asymptotic estimates (3).

A rough estimate following from Theorem 1 is

$$V_{g,n} \geq V_{g,1},$$

which, together with Penner’s lower estimate for $V_{g,1}$, already implies the existence of a constant $c > 0$, independent of $n \geq 1$, such that for large $g$

$$\frac{V_{g,n}}{(3g - 3 + n)!} \geq c^g(2g)!. $$

The corresponding upper estimate is due to Grushevski, so that (3) follows for $n \geq 1$.

For $n = 0$ and $g > 1$ the above Theorem 1 and Theorem 2 give

$$\frac{1}{672} V_{g-1,1} \leq V_{g,0} < \frac{2}{(3g - 2)(7g - 7)} V_{g,1}.$$ 

Again, with [GR, PE] these inequalities yield the following corollary.

**Corollary 1.** There exist constants $0 < \tilde{c} < \tilde{C}$ such that for $g \gg 0$

$$\tilde{c}^g(2g)! \leq \frac{V_{g,0}}{(3g - 3)!} \leq \tilde{C}^g(2g)!.$$ 

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ESTIMATES OF WEIL-PETERSSON VOLUMES VIA EFFECTIVE DIVISORS

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