A NOTE ON CARLEMAN’S INEQUALITY

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ABSTRACT. We study a weighted version of Carleman’s inequality via Carleman’s original approach. As an application of our result, we prove a conjecture of Bennett.

1. Introduction

The well-known Carleman’s inequality asserts that for convergent infinite series $\sum a_n$ with non-negative terms, one has

$$\sum_{n=1}^{\infty} \left( \prod_{k=1}^{n} a_k \right)^{\frac{1}{n}} \leq e \sum_{n=1}^{\infty} a_n,$$

with the constant $e$ best possible.

There is a rich literature on many different proofs of Carleman’s inequality as well as its generalizations and extensions. We shall refer the readers to the survey articles [7] and [5] as well as the references therein for an account of Carleman’s inequality.

From now on we will assume $a_n \geq 0$ for $n \geq 1$ and any infinite sum converges. Our goal in this paper is to study the following weighted Carleman’s inequality:

$$\sum_{n=1}^{\infty} G_n \leq C \sum_{n=1}^{\infty} a_n,$$

where

$$G_n = \prod_{k=1}^{n} a_k^{\lambda_k/\Lambda_n}, \quad \Lambda_n = \sum_{k=1}^{n} \lambda_k, \quad \lambda_k \geq 0, \quad \lambda_1 > 0.$$

The task here is to determine the best constant $C$ so that inequality (1.1) holds for any non-negative sequence $\{a_n\}_{n=1}^{\infty}$.

One approach to our problem here is to deduce inequality (1.1) via $l^p$ operator norm of the corresponding weighted mean matrix. We recall here that a matrix $A = (a_{j,k})$ is said to be a weighted mean matrix if its entries satisfy:

$$a_{j,k} = \lambda_k/\Lambda_j, \quad 1 \leq k \leq j; \quad a_{j,k} = 0, \quad k > j,$$

where the notations are as in (1.2). For $p > 1$, let $l^p$ be the Banach space of all complex sequences $b = (b_n)_{n \geq 1}$ with norm

$$\|b\| := \left( \sum_{n=1}^{\infty} |b_n|^p \right)^{1/p} < \infty.$$

The $l^p$ operator norm $\|A\|_{p,p}$ of $A$ for $A$ as defined in (1.3) is then defined as the $p$-th root of the smallest value of the constant $U$ so that the following inequality holds for any $b \in l^p$:

$$\sum_{n=1}^{\infty} \left| \sum_{k=1}^{\infty} \lambda_k b_k/\Lambda_n \right|^p \leq U \sum_{n=1}^{\infty} |b_n|^p.$$
In an unpublished dissertation \[4\], Cartlidge studied weighted mean matrices as operators on \(l^p\) and obtained the following result (see also \[1, p. 416, Theorem C\]).

**Theorem 1.1.** Let \(1 < p < \infty\) be fixed. Let \(A = (a_{j,k})\) be a weighted mean matrix given by \[(1.3)\]. If
\[
L = \sup_n \frac{\Lambda_{n+1}}{\lambda_{n+1}} \frac{\Lambda_n}{\lambda_n} < p,
\]
then \(||A||_{p,p} \leq \frac{p}{(p - L)}\).

The above theorem implies that one can take \(U = (p/(p - L))^p\) in inequality \[(1.4)\] for any weighted mean matrix \(A\) satisfying \[(1.5)\]. We note here by a change of variables \(b_k \to a_1^{1/p}k\) in \[(1.4)\] and on letting \(p \to +\infty\), one obtains inequality \[(1.1)\] with \(C = e^L\) as long as \[(1.5)\] is satisfied with \(p\) replaced by \(+\infty\) there.

In this note, we will study inequality \[(1.1)\] via Carleman’s original approach and we shall prove in the next section the following:

**Theorem 1.2.** Suppose that
\[
M = \sup_n \frac{\Lambda_n}{\lambda_n} \log \left( \frac{\Lambda_{n+1}/\lambda_{n+1}}{\Lambda_n/\lambda_n} \right) < +\infty,
\]
then inequality \[(1.1)\] holds with \(C = e^M\).

We point out here that the result of Theorem 1.2 is better than what one can deduce from Cartlidge’s result as discussed above. This can be seen by noting that \[(1.6)\] is equivalent to
\[
\frac{\Lambda_{n+1}}{\lambda_{n+1}} \leq e^{\lambda_n M/\Lambda_n},
\]
for any integer \(n \geq 1\). Suppose now \[(1.5)\] is satisfied, then the case \(n = 1\) of \[(1.6)\] implies \(L > 0\) and it is easy to check that
\[
\frac{\Lambda_{n+1}}{\lambda_{n+1}} \frac{\Lambda_n}{\lambda_n} = 1 + \frac{\Lambda_n}{\lambda_n} - \frac{\Lambda_{n+1}}{\lambda_{n+1}} - \frac{\Lambda_n}{\lambda_n} - \frac{\Lambda_n}{\lambda_n} L \leq e^{\lambda_n L/\Lambda_n},
\]
from which we deduce that \(M \leq L\).

Bennett \[2, p. 829\] conjectured that inequality \[(1.1)\] holds for \(\lambda_k = k^\alpha\) for \(\alpha > -1\) with \(C = 1/(\alpha + 1)\). As the cases \(-1 < \alpha \leq 0\) or \(\alpha \geq 1\) follow directly from Cartlidge’s result above (Theorem 1.1), the only case left unknown is when \(0 < \alpha < 1\). As an application of Theorem 1.2, we shall prove Bennett’s conjecture in Section 3.

2. **Proof of Theorem 1.2**

It suffices to establish our assertion with the infinite summation in \[(1.1)\] replaced by any finite summation, say from 1 to \(N \geq 1\) here. We now follow Carleman’s approach by determining the maximum value \(\mu_N\) of \(\sum_{n=1}^N G_n\) subject to the constraint \(\sum_{n=1}^N a_n = 1\) using Lagrange multipliers. It is easy to see that we may assume \(a_n > 0\) for all \(1 \leq n \leq N\) when the maximum is reached. We now define
\[
F(a, \mu) = \sum_{n=1}^N G_n - \mu \left( \sum_{n=1}^N a_n - 1 \right),
\]
where \(a = (a_n)_{1 \leq n \leq N}\). By the Lagrange method, we have to solve \(\nabla F = 0\), or the following system of equations:
\[
\mu a_k = \sum_{n=k}^N \frac{\lambda_k G_n}{\lambda_n}, \quad 1 \leq k \leq N; \quad \sum_{n=1}^N a_n = 1.
\]
We note that on summing over $1 \leq k \leq N$ of the first $N$ equations above, we get
\[ \sum_{n=1}^{N} G_n = \mu. \]
Hence we have $\mu = \mu_N$ in this case which allows us to recast the equations (2.1) as:
\[ \mu_N \frac{a_k}{\lambda_k} = \sum_{n=k}^{N} \frac{G_n}{\lambda_n}, \quad 1 \leq k \leq N; \quad \sum_{n=1}^{N} a_n = 1. \]
On subtracting consecutive equations, we can rewrite the above system of equations as:
\[ \mu_N \left( \frac{a_k}{\lambda_k} - \frac{a_{k+1}}{\lambda_{k+1}} \right) = \frac{G_k}{\lambda_k}, \quad 1 \leq k \leq N - 1; \quad \mu_N a_N \frac{a_N}{\lambda_N} = \frac{G_N}{\lambda_N}; \quad \sum_{n=1}^{N} a_n = 1. \]

Now we define for $1 \leq k \leq N - 1$,
\[ \omega_k = \frac{\Lambda_k}{\lambda_k} - \frac{\Lambda_k a_{k+1}}{\lambda_{k+1} a_k}, \]
so that we can further rewrite our system of equations as:
\[ \mu_N a_k \omega_k = G_k, \quad 1 \leq k \leq N - 1; \quad \mu_N a_N \frac{a_N}{\lambda_N} = \frac{G_N}{\lambda_N}; \quad \sum_{n=1}^{N} a_n = 1. \]

It is easy to check that for $1 \leq k \leq N - 2$,
\[ \omega_{k+1}^{\Lambda_{k+1}} = \frac{1}{\mu_N^{\Lambda_{k+1}}} \left( \frac{\omega_k}{\lambda_k} \right)^{\Lambda_k}. \]

We now define a sequence of real functions $\Omega_k(\mu)$ inductively by setting $\Omega_1(\mu) = 1/\mu$ and
(2.2) \[ \Omega_{k+1}^{\Lambda_{k+1}}(\mu) = \frac{1}{\mu^{\Lambda_{k+1}}} \left( \frac{\Omega_k}{\mu^a_{\Lambda_k}} \right)^{\Lambda_k}. \]

We note that $\Omega_k(\mu_N) = \omega_k$ for $1 \leq k \leq N - 1$ and
\[ \Omega_{N}^{\Lambda_N}(\mu_N) = \frac{1}{\mu^\Lambda_N} \left( \frac{\omega_{N-1}}{\Lambda_{N-1}/\Lambda_k - \omega_{N-1}} \right)^{\Lambda_{N-1}} = \frac{1}{\mu^\Lambda_N} \left( \frac{G_{N-1}/a_N}{\Lambda_N/\Lambda_k} \right)^{\Lambda_N} = \left( \frac{\Lambda_N/\Lambda_k}{\Lambda_{k+1}/\Lambda_{k+1}} \right)^{\Lambda_N}. \]

We now show by induction that if $\mu > e^M$, then for any $k \geq 1$,
(2.3) \[ \Omega_k(\mu) < \frac{\Lambda_k/\Lambda_k}{\Lambda_{k+1}/\Lambda_{k+1}}. \]
As we have seen above that $\Omega_N(\mu_N) = \Lambda_N/\Lambda_N$, this forces $\mu_N \leq e^M$ and hence our assertion for Theorem 1.2 will follow.

Now, to establish (2.3), we note first the case $k = 1$ follows directly from our assumption (1.6) on considering the case $n = 1$ there. Suppose now (2.3) holds for $k \geq 1$, then by the relation (2.2), we have
\[ \Omega_{k+1}^{\Lambda_{k+1}}(\mu) = \frac{1}{\mu^{\Lambda_{k+1}}} \left( \frac{\Omega_k}{\mu^a_{\Lambda_k}} \right)^{\Lambda_k} < \frac{1}{\mu^{\Lambda_{k+1}}} \left( \frac{\Lambda_k/\Lambda_k}{\Lambda_{k+1}/\Lambda_{k+1}} \right)^{\Lambda_{k+1}} = \frac{1}{\mu^{\Lambda_{k+1}}}. \]
This implies that
\[ \Omega_{k+1}(\mu) < \frac{1}{{\mu}^{\lambda_{k+1}/\lambda_{k+1}}} < \frac{\Lambda_{k+1}/\lambda_{k+1}}{\Lambda_{k+2}/\lambda_{k+2}}. \]

The last inequality follows from the case \( n = k + 1 \) of our assumption (1.6) and this completes the proof.

### 3. An Application of Theorem 1.2

Our goal in this section is to establish the following:

**Theorem 3.1.** Inequality (1.1) holds for \( \lambda_k = k^\alpha \) for \( 0 < \alpha < 1 \) with \( C = 1/(\alpha + 1) \).

We need a lemma first:

**Lemma 3.1.** [6, Lemma 1, 2, p.18] For an integer \( n \geq 1 \) and \( 0 \leq r \leq 1 \),
\[
\frac{1}{r+1} n(n+1)^r \leq \sum_{i=1}^{n} i^r \leq \frac{r}{r+1} n^r(n+1)^r - n^r.
\]

Now we return to the proof of Theorem 3.1. It suffices to check that condition (1.6) is satisfied with \( M = 1/(\alpha + 1) \) there. Explicitly, we need to show that for any integer \( n \geq 1 \),
\[
(3.1) \quad \frac{\sum_{k=1}^{n} k^\alpha}{n^\alpha} \log \left( \left( 1 + \frac{(n+1)^\alpha}{\sum_{k=1}^{n} k^\alpha} \frac{n^\alpha}{(n+1)^\alpha} \right) \right) \leq \frac{1}{\alpha + 1}.
\]

Now we apply Lemma 3.1 to obtain:
\[
1 + \frac{(n+1)^\alpha}{\sum_{k=1}^{n} k^\alpha} \leq 1 + \frac{\alpha + 1}{n}.
\]

We use this together with the upper bound in Lemma 3.1 to see that inequality (3.1) is a consequence of the following inequality:
\[
(3.2) \quad \alpha \left( \log(1 + \frac{\alpha + 1}{n}) - \log(1 + 1/n)^\alpha \right) \leq 1 - \frac{1}{(1 + 1/n)^\alpha}.
\]

We now define
\[
f(x) = 1 - (1 + x)^{-\alpha} - \alpha \left( \log(1 + (\alpha + 1)x) - \alpha \log(1 + x) \right).
\]

Note that inequality (3.2) is equivalent to \( f(1/n) \geq 0 \). Hence it suffices to show that \( f(x) > 0 \) for \( 0 < x \leq 1 \). Calculation shows that
\[
f'(x) = \frac{\alpha g(x)}{(1 + x)^{1+\alpha}(1 + (1 + \alpha)x)},
\]
where
\[
g(x) = 1 + (\alpha + 1)x - (\alpha + (1 - \alpha^2)x)(1 + x)^\alpha.
\]

Note that when \( 0 < \alpha < 1 \),
\[
(1 + x)^\alpha \leq 1 + \alpha x.
\]

It follows that
\[
g(x) \geq 1 + (\alpha + 1)x - (\alpha + (1 - \alpha^2)x)(1 + \alpha x)
\]
\[= 1 - \alpha + \alpha x - \alpha(1 - \alpha^2)x^2 := h(x). \]

It is easy to see that \( h(x) \) is concave for \( 0 \leq x \leq 1 \) and \( h(0) = 1 - \alpha > 0, h(1) = 1 - \alpha(1 - \alpha^2) > 0 \). It follows that \( h(x) > 0 \) for \( 0 < x < 1 \) so that \( g(x) > 0 \) and hence \( f'(x) > 0 \) for \( 0 < x < 1 \). As \( f(0) = 0 \), this implies \( f(x) \geq 0 \) for \( 0 < x \leq 1 \) and this completes the proof of Theorem 3.1.
References

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