Locally symmetric spaces and K-theory of number fields

Thilo Kuessner

Abstract

We describe an invariant of flat bundles over locally symmetric spaces with values in the K-theory of number fields and discuss the nontriviality and Q-independence of its values.

1 Introduction

While elements in topological K-theory $K^{-}(X)$ are, by definition, represented by (virtual) vector bundles over the space $X$, it is less evident what the topological meaning of elements in algebraic K-theory $K_{*}(A)$, for a commutative ring $A$, may be. An approach, which can be found e.g. in the appendix of [22], is to consider elements in $K_{d}(A)$ associated to a flat $GL(A)$-bundle over a $d$-dimensional homology sphere $M$. Namely, let

$$\rho : \pi_{1}M \to GL(A)$$

be the monodromy representation of the flat bundle, then its plusification

$$(B\rho)^{+} : M^{+} \to BGL^{+}(A)$$

can, in view of $M^{+} \simeq S^{d}$, be considered as an element in algebraic K-theory

$$K_{d}(A) := \pi_{d}BGL^{+}(A).$$

It was proved by Hausmann and Vogel (see [18]) that for a finitely generated, commutative, unital ring $A$ and $d \geq 5$ or $d = 3$, all elements in $K_{d}(A)$ arise from such a construction.

If the manifold $M$ is not a homology sphere, but still possesses a fundamental class $[M] \in H_{d}(M; \mathbb{Q})$, one can still consider

$$(B\rho)_{d}[M] \in H_{d}(BGL(A); \mathbb{Q})$$

and can use a suitably defined projection (see Section 2.4) to the primitive part of the homology to obtain

$$\gamma(M) \in P_{d}(BGL(A); \mathbb{Q}) \simeq K_{d}(A) \otimes \mathbb{Q}.$$
very natural way, an element in $B(\mathbb{C})$, the Bloch invariant. By [28], this element does not depend on the chosen ideal triangulation.

A generalization to higher-dimensional hyperbolic manifolds was provided by Goncharov. He associated to an odd-dimensional hyperbolic manifold and flat bundles coming from the half-spinor representations, an element $\gamma(M) \in K_* \left(\mathbb{Q}\right) \otimes \mathbb{Q}$, and proved its nontriviality by showing that application of the Borel regulator yields (a fixed multiple of) the volume.

It thus arises as a natural question, whether other locally symmetric spaces and different flat bundles give nontrivial elements in the K-theory of number fields (and eventually how much of algebraic K-theory in odd degrees can be represented by locally symmetric spaces and representations of their fundamental groups).

In section 2, we generalize the argument in [15] to the extent that, for a compact locally symmetric space $M^{2n-1} = \Gamma \backslash G/K$ of noncompact type and a representation $\rho : G \to GL(N, \mathbb{C})$, nontriviality of the associated element $\gamma(M) \in K_{2n-1} \left(\mathbb{Q}\right) \otimes \mathbb{Q}$ is (independently of $\Gamma$) equivalent to nontriviality of the Borel class $\rho^* b_{2n-1}$.

While it does, in general, not work to associate elements in algebraic K-theory to flat bundles over manifolds with boundary, we show in section 4 that an element $\rho$ for which $\rho^* b_{2n-1}$ gives nontrivial elements in the K-theory of number fields (and eventually assuming that $\partial M$ be connected.) The results of section 2 (for closed manifolds) and section 4 (for cusped manifolds) are subsumed in the following theorem.

**Theorem.** For each symmetric space $G/K$ of noncompact type and odd dimension $d = 2n - 1$, and to each representation $\rho : G \to GL(N, \mathbb{C})$ with $\rho^* b_{2n-1} \neq 0$, there exists a constant $c_\rho \neq 0$, such that the following holds. If $M = \Gamma \backslash G/K$ is a finite-volume, orientable, locally symmetric space and either $M$ is compact or $rk(\rho(G/K)) = 1$, then there is an element

$$\gamma(M) \in K_{2n-1} \left(\mathbb{Q}\right) \otimes \mathbb{Q}$$

such that the Borel regulator $r_{2n-1} : K_{2n-1} \left(\mathbb{Q}\right) \otimes \mathbb{Q} \to \mathbb{R}$ fulfills

$$r_{2n-1} (\gamma(M)) = c_\rho \cdot vol(M).$$

In particular, if $\rho^* b_{2n-1} \neq 0$, then locally symmetric spaces $\Gamma \backslash G/K$ of $\mathbb{Q}$-independent volume give $\mathbb{Q}$-independent elements in $K_{2n-1} \left(\mathbb{Q}\right) \otimes \mathbb{Q}$.

(In many cases one actually associates an element in $K_{2n-1} (\mathbb{F}) \otimes \mathbb{Q}$, for some number field $\mathbb{F}$, see Theorem 2 in section 2.5.)

In section 3, we work out the list of fundamental representations $\rho : G \to GL(N, \mathbb{C})$ for which $\rho^* b_{2n-1} \neq 0$ holds true. It is easy to prove that $\rho^* b_{2n-1} \neq 0$ is always true if $2n - 1 \equiv 3 \mod 4$. We work out, for which fundamental representations $\rho^* b_{2n-1} \neq 0$ holds if $2n - 1 \equiv 1 \mod 4$. (In [15] it was stated that the half-spinor representations would seem to be the only fundamental representations of Spin $(2n - 1, 1)$ that yield nontrivial invariants of odd-dimensional hyperbolic manifolds. This is however not the case. Indeed, if $2n - 1 = dim(M) \equiv 3 \mod 4$, then each irreducible representation of Spin $(2n - 1, 1)$ yields nontrivial invariants.)

The proof uses only standard Lie algebra and representation theory. The result reads as follows.
Theorem. The following is a complete list of irreducible symmetric spaces $G/K$ of noncompact type and fundamental representations $\rho : G \to GL(N, \mathbb{C})$ with $\rho^* b_{2n-1} \neq 0$ for $2n - 1 := \dim (G/K)$.

- $SL_2l(\mathbb{R})/SO_2l, l \equiv 0, 3, 4, 7 \mod 8$, any fundamental representation,
- $SL_2l(\mathbb{C})/SU_2l, l \equiv 0 \mod 2$, any fundamental representation,
- $SL_2l(\mathbb{H})/Sp_2l, l \equiv 0 \mod 2$, any fundamental representation,
- $Spin_{p,q}/(Spin_p \times Spin_q), p, q \equiv 1 \mod 2, p \neq q \mod 4$, any fundamental representation,
- $Spin_{p,q}/(Spin_p \times Spin_q), p, q \equiv 1 \mod 2, p \equiv q \mod 4$, positive and negative half-spinor representation,
- $Spin_4(\mathbb{C})/Spin_4, l \equiv 3 \mod 4$, the spinor representation and its conjugate,
- $Spin_4(\mathbb{C})/Spin_4, l \equiv 2 \mod 4$, any fundamental representation,
- $Spin_4(\mathbb{C})/Spin_4, l \equiv 3 \mod 4$, any fundamental representation,
- $Spin_4(\mathbb{C})/Spin_4, l \equiv 3 \mod 4$, any fundamental representation,
- $E_7(\mathbb{C})/E_7$, any fundamental representation.

For hyperbolic manifolds and half-spinor representations, the construction of $\gamma(M)$ is due to Goncharov. (Though the proof in [15] implicitly assumes that $\partial M$ be connected.) For hyperbolic 3-manifolds, another construction is due to Cisneros-Molina and Jones in [9]. (It was related in [9] to the construction of Neumann-Yang in [28].) The latter has the advantage that the number of boundary components does not impose technical problems, contrary to the group-homological approach in [15].

Our construction for closed locally symmetric spaces is a straightforward generalization of [15].

In the case of cusped locally symmetric spaces (with possibly more than one cusp) it would have seemed more natural to stick to the approach of Cisneros-Molina and Jones, and in fact this approach generalizes to locally symmetric spaces in a completely straightforward way (see section 4.1). However, we did not succeed to evaluate the Borel regulator (in order to discuss nontriviality and $\mathbb{Q}$-independence of the obtained invariants) in this approach. On the other hand, Goncharov’s approach, even in the case of only one cusp, uses very special properties of the spinor representation, which can not be generalized to other representations.

Therefore, our argument is sort of a mixture of both approaches. On the one hand it is closer in spirit to the arguments of [15] (but with the cuspidal completion in section 4.2 memorizing the geometry of distinct cusps), on the other hand the argument in section 4.3 uses arguments from [9] to circumvent the very special group-homological arguments that were applied in [15] in the special setting of the half-spinor representations.

Of course, it should be interesting to relate the different constructions in a more direct way.

2 Preparations

The results of this section are fairly straightforward generalizations of the results in [15] from hyperbolic manifolds to locally symmetric spaces of noncompact type. We will define a notion of representations with nontrivial Borel class and will, mimicking the arguments
in [15], show that representations with nontrivial Borel class give rise to nontrivial elements in algebraic K-theory of number fields. The problem of constructing representations with nontrivial Borel class will be tackled in the section 3.

2.1 Construction of elements in algebraic K-theory

In this paper, rings $A$ will always be commutative rings with unit. (To be explicit: we will always consider subrings of $\mathbb{C}$.)

Assume that $M$ is a closed, orientable $n$-manifold with $\Gamma := \pi_1 M$. Assume that we are given a ring $A$ and a representation $\rho : \Gamma \to \text{GL}(A)$, where $\text{GL}(A)$ denotes the increasing union of $\text{GL}(N,A)$ over all $N \in \mathbb{N}$.

We get an induced map $B\rho : M \to B\text{GL}(A)$.

Throughout this paper $B\text{GL}(A)$ (resp. $BG$ for any Lie group) will mean the classifying space for $G$, that is the group $G$ with the discrete topology. Thus $\pi_1 BG = G$.

Quillen’s plus construction (see [30]) provides us with a map $(B\rho)^+ : M^+ \to B\text{GL}^+(A)$.

If $M$ happens to be a $d$-dimensional homology sphere, then $M^+$ is homotopy equivalent to $S^d$ and one gets a map $S^d \simeq M^+ \to B\text{GL}^+(A)$

which may be considered as representative of an element

\[
\left[(B\rho)^+\right] \in K_d(A) := \pi_d \left(B\text{GL}^+(A)\right).
\]

It was actually shown by Hausmann and Vogel (cf. [18] or [17]) that, for $d \geq 5$ or $d = 3$, each element in $K_d(A)$ for a finitely generated commutative ring $A$ can be constructed by some homology sphere $M$ and some representation $\rho$.

If $M$ is not a homology sphere, but a closed and oriented $d$-manifold, and $A$ satisfies mild assumptions (see Section 2.4), e.g. for $A = \mathbb{Q}$, then we will construct an element in $K_d(A) \otimes \mathbb{Q}$, as follows. The continuous map $B\rho$ induces a homomorphism

\[
(B\rho)_* : H_d(M; \mathbb{Q}) \to H_d(B\text{GL}(A); \mathbb{Q})
\]

Since $M$ is a closed, oriented $d$-manifold, we have the fundamental class $[M] \in H_d(M; \mathbb{Q})$, which is the image of a generator of $H_d(M; \mathbb{Z})$ under the change-of-rings homomorphism associated to the inclusion $\mathbb{Z} \to \mathbb{Q}$. Let

\[
(B\rho)_* [M] \in H_d(B\text{GL}(A); \mathbb{Q}) \cong H_d \left(B\text{GL}(A)^+; \mathbb{Q}\right).
\]

By the Milnor-Moore Theorem, the Hurewicz homomorphism

\[
K_d(A) := \pi_d \left(B\text{GL}(A)^+\right) \to H_d \left(B\text{GL}(A)^+; \mathbb{Z}\right)
\]

gives, after tensoring with $\mathbb{Q}$, an injective homomorphism

\[
I_d : K_d(A) \otimes \mathbb{Q} = \pi_d \left(B\text{GL}(A)^+\right) \otimes \mathbb{Q} \to H_d \left(B\text{GL}(A)^+; \mathbb{Q}\right) \cong H_d(B\text{GL}(A); \mathbb{Q}).
\]
Again by the Milnor-Moore Theorem, its image consists of the subgroup of primitive elements, which we denote by $PH_d (BGL (A) ; \mathbb{Q})$.

(If $d$ is even and $A$ is a ring of integers in any number field, then $PH_d (BGL (A) ; \mathbb{Q}) = 0$. Therefore one is only interested in the case that $d$ is odd, $d = 2n - 1$.)

Whenever we fix a projection $pr_d : H_d (BGL (A) ; \mathbb{Q}) \to PH_d (BGL (A) ; \mathbb{Q})$, we can define an element $\gamma (M) \in K_d (A) \otimes \mathbb{Q}$ as

$$\gamma (M) := I_d^{-1} pr_d (B\rho)_* [M].$$

In Section 2.4 we are going to show e.g. for $A = \mathbb{Q}$ (and also for many other rings) the projection $pr_d$ can be chosen such that the Borel regulators of $h$ and $pr_d (h)$ agree for all $h \in H_d (BGL (\mathbb{Q}) ; \mathbb{Q})$. In particular, to check nontriviality of $\gamma (M)$ it will then suffice to apply the Borel regulator to $(B\rho)_* [M]$.

If $M$ is a (compact, orientable) manifold with nonempty boundary, then there is no general construction of an element of algebraic $K$-theory. However, we will show in section 4 that for finite-volume locally symmetric spaces one can generalize the above construction and again construct a canonical invariant $\gamma (M) \in K_d (\mathbb{Q}) \otimes \mathbb{Q}$.

### 2.2 The volume class in $H^d_c \left( \text{Isom} \left( \tilde{M} \right) \right)$

For a group $G$, its classifying space $BG$ (with respect to the discrete topology on $G$) is the geometric realisation of the simplicial complex $BG$ defined as follows:

- the $k$-simplices of $BG$ are the $n+1$-tuples $(g_1, \ldots, g_k)$ with $g_1, \ldots, g_k \in G$,
- the boundary operator is defined by

$$\partial (1, g_1, \ldots, g_k) = (1, g_2, \ldots, g_k) + \sum_{i=1}^{k-1} (1, g_1, g_{i+1}, \ldots, g_k) + (-1)^k (1, g_1, \ldots, g_{k-1}).$$

If $M$ is an aspherical space, $x_0 \in M$ and $\Gamma = \pi_1 (M, x_0)$, then Eilenberg and MacLane defined in [13] a chain homotopy equivalence

$$EM : C^*_{\text{simp}} (B\Gamma) \to C^*_{\text{sing}} (M)$$

such that, for $g_1, \ldots, g_k \in \Gamma$, the $k$-simplex $EM (1, g_1, \ldots, g_k)$ has all its vertices in $x_0$ and the edge connecting the $i$-th and $i+1$-th vertex represents $g_i$ for $i = 1$ resp. $g_i^{-1} g_{i-1} \in \Gamma$, for $i = 2, \ldots, k$.

**Volume class.** We recall that the continuous cohomology $H^*_c (G; \mathbb{R})$ of a Lie group $G$ is defined as the homology of the complex $\left( C^*_c (G^{*+1}, \mathbb{R})^G, \delta \right)$, where $C^*_c (G^{*+1}, \mathbb{R})^G$ stands for the continuous (with respect to the Lie group topology) $G$-invariant mappings from $G^{*+1}$ to $\mathbb{R}$ and $\delta$ is the usual coboundary operator. In particular, the group cohomology of $G$ is the continuous cohomology for $G$ with the discrete topology.

The comparison map $comp : H^*_c (G; \mathbb{R}) \to H^*_{\text{simp}} (BG; \mathbb{R})$ is defined by

$$comp (f) (1, g_1, \ldots, g_k) := f (1, g_1, g_1 g_2, \ldots, g_1 g_2 \ldots g_k)$$

for $f \in C^*_c (G)$. In particular, elements of $H^*_c (G; \mathbb{R})$ can be evaluated on $H_* (BG; \mathbb{R})$. 
Let $\tilde{M} = G/K$ be a symmetric space of noncompact type. It is well-known ([19], Ch.V, Thm.3.1) that $\tilde{M}$ has nonpositive sectional curvature. $G$ acts by isometries on $\tilde{M}$.

The volume class

$$v_d \in H^d_c (G; \mathbb{R})$$

is defined as follows. Fix an arbitrary point $\tilde{x} \in \tilde{M} = G/K$. Then we define a cocycle $cv_d \in C^d (BG)$ by

$$cv_d(1, g_1, \ldots, g_d) := \int_{\text{str}(\tilde{x}, g_1 \tilde{x}, \ldots, g_d \tilde{x})} \text{dvol},$$

that is the signed volume of the straight simplex with vertices $\tilde{x}_0, g_1 \tilde{x}_0, \ldots, g_d \tilde{x}_0$. (Note that in a simply connected space of nonpositive sectional curvature each ordered $k+1$-tuple of vertices $(p_0, \ldots, p_k)$ determines a unique straight $d$-simplex $\text{str}(p_0, \ldots, p_k)$.)

By Stokes' Theorem we have

$$\delta cv_d (1, g_1, \ldots, g_{d+1}) = cv_d (1, g_2, \ldots, g_d) + \sum_{i=1}^{d-1} (-1)^i (1, g_1, \ldots, g_i g_{i+1}, \ldots, g_d) + (-1)^d (1, g_1, \ldots, g_{d-1}) =$$

$$\sum_{i=0}^{d} \int_{\text{str}(\tilde{x}, g_0 \tilde{x}, \ldots, g_i \tilde{x}, \ldots, g_d \tilde{x})} \text{dvol} = \int_{\partial \text{str}(g_1 \tilde{x}, \ldots, g_d \tilde{x})} \text{dvol} = \int_{\text{str}(g_1 \tilde{x}, \ldots, g_d \tilde{x})} \text{d} (\text{dvol}) = 0.$$

Thus $cv_d$ is a simplicial cocycle on $BG$. (Its cohomology class does not depend on $\tilde{x}$, because any $\tilde{x} \in G/K$ can be mapped to any other $\tilde{x}_0 \in G/K$ by some $g \in G$ and the action of $G$ on $G/K$ is homotopic to the identity and therefore preserves cohomology classes.) The corresponding cocycle $\nu_d \in C^d_c (G; \mathbb{R})$ is given by the (clearly continuous) mapping $\nu_d (g_0, \ldots, g_d) = cv_d (1, g_0^{-1} g_1, \ldots, g_d^{-1} g_d) = \int_{\text{str}(\tilde{x}, g_0^{-1} \tilde{x}, g_1 \tilde{x}, \ldots, g_d \tilde{x})} \text{dvol}$. Thus we have defined a cohomology class $\nu_d := [\nu_d] \in H^d_c (G; \mathbb{R})$ such that $\text{comp} (\nu_d) \in H^d (BG; \mathbb{R})$ is represented by $cv_d$.

**Theorem 1.** Let $M = \Gamma \backslash G/K$ be a (closed, oriented, $d$-dimensional) locally symmetric space of noncompact type, and $j : \Gamma \to G$ the inclusion of $\Gamma = \pi_1 M$. Let $j_* : H_d (M; \mathbb{Z}) \cong H_d (\Gamma; \mathbb{Z}) \to H_d (G; \mathbb{Z})$ be the induced homomorphism, and denote $[M] \in H_d (M; \mathbb{Z})$ the fundamental class of $M$. Then

$$\text{vol} (M) = \langle v_d, j_* [M] \rangle.$$

**Proof:** Let $\pi : G/K = \tilde{M} \to M$ be the covering map. Fix a point $x_0 \in M$ and a lift $\tilde{x}_0 \in \tilde{M}$. Let $C^*_{\text{str}, x_0} (M)$ be the chain complex of straight simplices with all vertices in $x_0$. Let $w_0, \ldots, w_k$ be the vertices of the standard simplex $\Delta^K$. Each $\sigma \in C^k_{\text{str}, x_0} (M)$ is a continuous map $\sigma : \Delta^K \to M$ which (by asphericity of $M$) is uniquely determined by the homotopy classes of the sub-1-simplices $\gamma_j \subset \sigma$ with $\partial \gamma_j = \sigma (w_j) - \sigma (w_{j-1})$ for $j = 1, \ldots, k$. Thus

$$C^*_{\text{str}, x_0} (M) \cong C^\text{simp} (BG)$$

for $\Gamma = \pi_1 (M, x_0)$, where the bijection maps $\sigma$ to $(1, \gamma_1, \ldots, \gamma_d)$. It follows from the definition of the boundary operator on $BG$ that this bijection is a chain map, thus an isomorphism of chain complexes.
Moreover, inclusion $\gamma^{str,x_0}_* (M) \to C_\ast (M)$ induces an isomorphism in homology. Indeed, each cycle in $C_\ast (M)$ can first be homotoped such that all vertices are in $x_0$, and then be straightened (by induction on dimension of subsimplices, depending on a chosen order of vertices). Straightening a simplex $\sigma$ with straight boundary means to chose the unique geodesic simplex which is homotopic rel. $\partial$ to $\sigma$. (In particular, its edges represent the same elements of $\pi_1 (M,x_0)$ as the corresponding edges of $\sigma$.) It is well-known\(^1\) that straightening all simplices of a cycle yields a cycle in the same homology class\(^2\).

So, let $\sum_{i=1}^r a_i \sigma_i$ be a representative of the fundamental class. (One may choose e.g. a triangulation $\sigma_1 + \ldots + \sigma_r$.) Then $vol (M) = \sum_{i=1}^r a_i vol (\sigma_i)$. The cycle $\sum_{i=1}^r a_i \sigma_i$ is homologous to some $\sum_{i=1}^r a_i \tau_i \in C^{str,x_0}_\ast (M)$. (Possibly after straightening some simplices overlap, so we do not get a triangulation. However, it will be sufficient to have a fundamental cycle consisting of geodesic simplices.) By Stokes Theorem, $\int M \circ \nabla = \sum_{i=1}^r a_i vol (\tau_i)$.

The isomorphism $\gamma^{str,x_0}_* (M) \cong C^{simp}_d (BG)$ maps each $\tau_i$ to $(1, \gamma^i_1, \ldots, \gamma^i_d) \in \Gamma^{d+1}$, where $\gamma^i_j \in \Gamma$ is the class of the (closed) edge from $\tau_i (w_{j-1})$ to $\tau_i (w_j)$. Then $j_* \left[ M \right] \in H_d (BG; \mathbb{Z})$

is represented by

$$\sum_{i=1}^r \left[ a_i (1, \gamma^i_1, \ldots, \gamma^i_d) \right] \in C^{simp}_d (BG).$$

But

$$< cv_d, (1, \gamma^1_1, \ldots, \gamma^d_d) > = \int_{str(x_0,0,\ldots,0)} dvol = \int_{\tau_i} dvol = vol (\tau_i),$$

which implies

$$< cv_d, \sum_{i=1}^r a_i (1, \gamma^i_1, \ldots, \gamma^i_d) > = \sum_{i=1}^r a_i vol (\tau_i) = vol (M).$$

\(^{QED}\)

2.3 **Borel classes**

Let $\tilde{M} = G/K$ be a symmetric space of noncompact type. Then $G$ is a semisimple, noncompact Lie group and $K$ is a maximal compact subgroup.

\(^1\)This is proved in \[^2\] Lemma C.4.3, for hyperbolic manifolds. Word by word the same proof works if $M$ is any Riemannian manifold of nonpositive sectional curvature, in particular if $M$ is a locally symmetric space of noncompact type.

\(^2\)This construction uses that $M$ is a manifold of nonpositive sectional curvature. That assumption might actually be avoided, since it was shown in \[^3\] that, for each aspherical space $M$ with fundamental group $\Gamma$, there is a subcomplex $K (M) := EM \left( C^{simp}_\ast (BG) \right) \subset C^{sing}_\ast (M)$ such that:

- $EM : C^{simp}_\ast (BG) \to K (M)$ is a chain isomorphism and
- the inclusion $K (M) \subset C^{sing}_\ast (M)$ induces an isomorphism in homology.

If $M$ is a Riemannian manifold of nonpositive sectional curvature, then one can define $EM$ such that $K (M) = C^{str,x_0}_\ast (M)$. 

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Let $g$ be the Lie algebra of $G$, and $\mathfrak{k} \subset g$ be the Lie algebra of $K$. There is the Cartan decomposition $g = \mathfrak{k} \oplus p$ with $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{k}, p] \subset p$, $[p, p] \subset \mathfrak{k}$. It is a well-known fact that the Killing form $B(X, Y) = Tr(ad(X) \circ ad(Y))$ is negatively definite on $\mathfrak{k}$ and positively definite on $p$.

The dual symmetric space is $G_u/K$, where $G_u$ is the simply connected Lie group with Lie algebra $\mathfrak{g}_u = \mathfrak{k} \oplus ip \subset g \otimes \mathbb{C}$. The Killing form on $\mathfrak{g}_u$ is negatively definite, thus $G_u/K$ is a compact symmetric space.

The Lie algebra cohomology $H^*(g)$ is the cohomology of the complex $(\Lambda^* g, d)$ with $d\phi(X_0, \ldots, X_d) = \sum_{i<j} \phi(-1)^{i+j} \phi([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_n)$. The relative Lie algebra cohomology $H^*(g, k)$ is the cohomology of the subcomplex $(C^*(g, k), d)$ with $C^*(g, k) = \{ \phi \in \Lambda^* g \colon X_i \phi = 0, ad(X) \phi = 0 \ \forall X \in \mathfrak{k} \}$. If $G/K$ is a symmetric space of noncompact type, and $G_u/K$ its compact dual, then there is an obvious isomorphism $H^*(g, k) = H^*(g_u, k)$, dual to the obvious linear map $\mathfrak{k} \oplus ip \to \mathfrak{k} \oplus p$. Moreover, $H^*(g, k)$ is the cohomology of the complex of $G$-invariant differential forms on $G/K$. Since $G_u$ is compact and connected, there is an isomorphism (defined by averaging) $H^*(g_u, k) \simeq H^*(G_u/K; \mathbb{R})$. (Each closed form on $G_u/K$ is cohomologous to a $G_u$-invariant form.)

For example,

$$H^*(\text{spin}(d, 1), \text{spin}(d)) \cong H^*(\text{Spin}(d + 1)/\text{Spin}(d); \mathbb{R}) = H^*(S^d; \mathbb{R}).$$

**Dualizing representations.** Let $\rho : G \to GL(N, \mathbb{C})$ be a representation. $\rho$ can be conjugated such that $K$ is mapped to $U(N)$. We will henceforth always assume that $\rho$ has been fixed such that $\rho$ sends $K$ to $U(N)$.

**Definition 1.** Let $\widetilde{M} = G/K$ be a symmetric space of noncompact type. Let $\rho : (G, K) \to (GL(N, \mathbb{C}), U(N))$ be a smooth representation. We denote

$$D_r\rho : (\mathfrak{g}_u, k) \to (gl(N, \mathbb{C}), u(N))$$

the associated Lie-algebra homomorphism, and, with $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, $\mathfrak{g}_u := \mathfrak{k} \oplus ip$,

$$D_r\rho_u : (\mathfrak{g}_u, k) \to (u(N) \oplus u(N), u(N))$$

the induced homomorphism on $\mathfrak{k} \oplus ip$. The corresponding Lie group homomorphism

$$\rho_u : (G_u, K) \to (U(N) \times U(N), U(N))$$

will be called the dual homomorphism to $\rho$.

Here $\mathfrak{g}_u$, $k$ and $ip$ are to be understood as subsets of the complexification $g \otimes \mathbb{C}$, and $G_u$ is the simply connected Lie group with Lie algebra $\mathfrak{g}_u$. In particular, the complexification of $gl_N \mathbb{C}$ is isomorphic to $gl_N \mathbb{C} \oplus gl_N \mathbb{C}$, and $ip \simeq u(N)$ in this case. We emphasize that $\rho_u$ sends $K$ to the first factor of $U(N) \times U(N)$, and not to the diagonal subgroup as has been claimed in [13].
Van Est Theorem. The van Est theorem states that there is a natural isomorphism

\[ H^*_c(G; \mathbb{R}) = H^*(g, k) \].

If \( \rho : G \to GL(N, \mathbb{C}) \) is a representation, sending \( K \) to \( U(N) \), then we conclude that there exists the following commutative diagram, where all vertical arrows are isomorphisms

\[
\begin{array}{ccc}
H^*_c(GL(N, \mathbb{C}); \mathbb{R}) & \xrightarrow{\rho^*} & H^*_c(G; \mathbb{R}) \\
\cong & & \cong \\
H^*(gl(N, \mathbb{C}), u(N)) & \xrightarrow{D_\rho \rho^*} & H^*(g, k) \\
\cong & & \cong \\
H^*(u(N) \oplus u(N), u(N)) & \xrightarrow{D_\rho \rho^*} & H^*(g, k) \\
\cong & & \cong \\
H^*(U(N); \mathbb{R}) & \xrightarrow{\rho_u^*} & H^*(G_u/K; \mathbb{R})
\end{array}
\]

If \( \dim(G/K) = d \), then \( G_u/K \) is an \( d \)-dimensional, compact, orientable manifold and we have \( H^d_c(G; \mathbb{R}) \cong H^d(G_u/K, \mathbb{R}) \cong \mathbb{R} \). Thus the volume class \([v_d] \) is the (up to multiplication by real numbers unique) nontrivial continuous cohomology class in degree \( d \).

Corollary 1. The volume class \([v_d] \in H^d_c(G; \mathbb{R}) \) corresponds (under the van Est isomorphism) to the (de Rham) cohomology class of the volume form \([dvol] \in H^d(G_u/K; \mathbb{R}) \).

Proof: According to [10], Prop. 1.5, \( v_d \) corresponds to the class of the volume form in \( H^d(g, k) \). It is obvious that the isomorphism \( H^d(g, k) \cong H^d_i(g, k) \) maps the volume form of \( G/K \) to the volume form of \( G_u/K \). \( \square \)

Borel classes

Let \( G \) be a compact Lie group. Let \( I^S_k(G) \) resp. \( I^A_k(G) \) be the \( ad \)-invariant symmetric resp. antisymmetric multilinear \( k \)-forms on \( g \). We have the isomorphism \( I^S_k(G) \to H^k(G; \mathbb{R}) \). Moreover, we remind that there is the Chern-Weil isomorphism \( I^S_k(G) \to H^{2k}(BG; \mathbb{R}) \), where in this section (contrary to the remainder of the paper) \( BG \) means the classifying space for \( G \) with its Lie group topology. In particular, if \( G = U(N) \), then the invariant polynomial

\[
\frac{1}{(2\pi i)^n} Tr(A^{n+1})
\]

is mapped to

\[
C_n \in H^{2n}(BU(N); \mathbb{Z})
\]

the \( n \)-th component of the universal Chern character.
There is a fibration $G \to EG \to BG$ and an associated 'transgression map' $\tau$ which maps a subspace of $H^{2n-1}(G; \mathbb{Z})$ (the so-called transgressive elements) to a quotient of $H^{2n}(BG; \mathbb{Z})$, cf. [4], p.410. If $G = U(N)$, then, according to Borel ([4], p.412), one has

$$H^*(U(N); \mathbb{Z}) \cong \Lambda_2(b_1, b_3, b_5, \ldots, b_{2N-1}),$$

for transgressive elements $b_{2n-1} \in H^{2n-1}(U(N); \mathbb{Z})$ which satisfy

$$\tau(b_{2n-1}) = C_n.$$

The classes $b_{2n-1}$ are called Borel classes. By definition, Borel classes exist only in odd degrees. (We will not distinguish between $b_{2n-1}$ and its image in $H^{2n-1}(U(N); \mathbb{R})$.)

According to Cartan ([4]), there is a homomorphism

$$R : I^3(G) \to I^{2n-1}(G),$$

whose image corresponds (after the isomorphism $I^{2n-1}(G) \cong H^{2n-1}(G; \mathbb{R})$) precisely to the transgressive elements. It is explicitly given by

$$R(f)(X_1, \ldots, X_{2n-1}) = \sum_{\sigma \in S_{2n-1}} (-1)^\sigma f(X_{\sigma(1)}, [X_{\sigma(2)}, X_{\sigma(3)}], \ldots, [X_{\sigma(2n-2)}, X_{\sigma(2n-1)}]).$$

In particular, for $H^k(U(N); \mathbb{R}) \cong H^k(u(N))$, a representative of $b_{2n-1} \in H^{2n-1}(u(N))$ is given by the cocycle

$$b_{2n-1}(X_1, \ldots, X_{2n-1}) = \frac{1}{(2\pi i)^n (2n-1)!} \sum_{\sigma \in S_{2n-1}} (-1)^\sigma Tr(X_{\sigma(1)} [X_{\sigma(2)}, X_{\sigma(3)}] \cdots [X_{\sigma(2n-2)}, X_{\sigma(2n-1)}]).$$

It will be clear from the context whether we consider the Borel classes as elements of $H^k(u(N)) \cong H^k(U(N); \mathbb{R})$ or as the (under the van Est isomorphism) corresponding elements of $H^*_c(GL(N, \mathbb{C}); \mathbb{R})$.

Stabilization $H^*(U(N+1); \mathbb{R}) \to H^*(U(N); \mathbb{R})$ preserves $b_{2n-1}$, thus $b_{2n-1}$ may also be considered as an element of $H^{2n-1}(U; \mathbb{R}) \cong H^{2n-1}_c(GL(\mathbb{C}); \mathbb{R})$.

We consider $K_{2n-1}(\mathbb{C}) \otimes \mathbb{Q}$ as a subgroup of $H_{2n-1}(BGL(\mathbb{C}); \mathbb{Q})$, as in section 2.1. The Borel regulator

$$r_{2n-1} : K_{2n-1}(\mathbb{C}) \otimes \mathbb{Q} \to \mathbb{R}$$

is, for $x \in K_{2n-1}(\mathbb{C}) \otimes \mathbb{Q}$, defined as applying the Borel class

$$b_{2n-1} \in H^{2n-1}_c(GL(\mathbb{C}); \mathbb{R})$$

to $x \in K_{2n-1}(\mathbb{C}) \otimes \mathbb{Q} = PH_{2n-1}(BGL(\mathbb{C}); \mathbb{Q}) \subset H_{2n-1}(BGL(\mathbb{C}); \mathbb{Q})$.

(We will show in section 2.4 that there is a projection $pr_{2n-1} : H_{2m-1}(BGL(\mathbb{C}); \mathbb{Q}) \to K_{2m-1}(\mathbb{C}) \otimes \mathbb{Q}$ such that $b_{2+1}(pr_{2n-1}(x)) = b_{2n-1}(x)$ for all $x \in H_{2n-1}(BGL(\mathbb{C}); \mathbb{Q})$.)

If $A \subset \mathbb{C}$ is a subring, then inclusion induces a homomorphism $K_{2n-1}(A) \otimes \mathbb{Q} \to K_{2n-1}(\mathbb{C}) \otimes \mathbb{Q}$, thus the Borel regulator also defines a homomorphism

$$r_{2n-1} : K_{2n-1}(A) \otimes \mathbb{Q} \to \mathbb{R}.$$

Borel class of representations.
Definition 2. Let $\widetilde{M} = G/K$ be a symmetric space of noncompact type of odd dimension $d = 2n-1$. We say that a (continuous) representation $\rho : G \to \text{GL}(N, \mathbb{C})$ has nonvanishing Borel class if $\rho^* b_{2n-1} \neq 0 \in H^{2n-1}_c(G; \mathbb{R})$.

Lemma 1. Let $G/K$ be a symmetric space of noncompact type, of odd dimension $d = 2n-1$. A representation $\rho : G \to \text{GL}(N, \mathbb{C})$ has nonvanishing Borel class if and only if $\rho^* b_{2n-1} \neq 0 \in H^{2n-1}(G_u/K; \mathbb{R})$, and the latter holds if and only if

$$< b_{2n-1}, (\rho_u)_* [G_u/K] > \neq 0.$$

Proof: The first equivalence follows from naturality of the van Est isomorphism. The second equivalence follows from $H_d(G_u/K; \mathbb{R}) \simeq \mathbb{R}$. ($G_u/K$ is a closed, orientable $d$-manifold.)

QED

2.4 Projection $H_* (\text{BGL} (\mathbb{C}) ; \mathbb{Q}) \to K_* (\mathbb{Q}) \otimes \mathbb{Q}$

Let $A \subset \mathbb{C}$ be a subring and $G = \text{GL}(A)$. Let $I = H^{* \geq 1} (BG; \mathbb{Q})$ be the augmentation ideal of $H^* (BG; \mathbb{Q})$ and let $D = I^2$ be the subspace of decomposable cohomology classes.

Let $PH_* (BG; \mathbb{Q})$ be the subspace of primitive elements in homology. It is easy to check that $c(h) = 0$ for all $c \in D, h \in PH_* (BG; \mathbb{Q})$. By [27], Prop.3.10., $I/D$ is the dual of $PH_* (BG; \mathbb{Q})$, which implies

$$D = \{ c \in I : c(h) = 0 \ \forall \ h \in PH_* (BG; \mathbb{Q}) \} \text{ and } PH_* (BG; \mathbb{Q}) = \{ h \in H_* (BG; \mathbb{Q}) : c(h) = 0 \ \forall \ c \in D \}.$$

In section 2.3 we have defined the Borel classes as elements $b_{2n-1} \in H^{2n-1}_c(\text{GL} (\mathbb{C}) ; \mathbb{R})$.

If $A \subset \mathbb{C}$ is a subring, then by composition with the inclusion $\text{GL} (A) \to \text{GL} (\mathbb{C})$ we can also consider them as elements $b_{2n-1} \in H^{2n-1}(\text{BGL} (A) ; \mathbb{R})$.

Lemma 2. Let $A \subset \mathbb{C}$ be a subring. Assume that the Borel class $b_{2n-1} \in H^{2n-1} (\text{BGL} (A) ; \mathbb{R})$ is not decomposable: $b_{2n-1} \notin D$.

Then there exists a projection

$$pr_{2n-1} : H_{2n-1} (\text{BGL} (A) ; \mathbb{Q}) \to PH_{2n-1} (\text{BGL} (A) ; \mathbb{Q})$$

such that

$$b_{2n-1} (pr_{2n-1}(h)) = b_{2n-1}(h)$$

for all $h \in H_{2n-1} (\text{BGL} (A) ; \mathbb{Q})$.

Proof: Let $G = \text{GL}(A)$. Since $b_{2n-1} \notin D$ we can choose a basis $\{ e_i : i \in I \}$ of the $\mathbb{Q}$-vector space $H^{2n-1}(\text{BGL} (A) ; \mathbb{Q})$ such that the following two conditions are satisfied:

- There exists a subset $J \subset I$ such that $\{ e_j : j \in J \}$ is a basis of $D$.
- There exists some $i_0 \in I \setminus J$ such that $b_{2n-1} = e_{i_0}$.

Let $\{ f_i : i \in I \}$ be the dual basis of $H_{2n-1} (BG; \mathbb{Q})$, that is $e_i (f_k) = \delta_{ik}$ for all $i, k \in I$. We have

$$\text{PH}_{2n-1} (BG; \mathbb{Q}) = \{ h \in H_{2n-1} (BG; \mathbb{Q}) : c(h) = 0 \ \forall c \in D \} = < f_j : j \notin J >.$$

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We use this basis to define the projection $pr_{2n-1}$ by $pr_{2n-1}(f_i) = f_i$ if $i \notin J$ and $pr_{2n-1}(f_i) = 0$ if $i \in J$.

Since $i_0 \notin J$ we have $pr_{2n-1}(f_{i_0}) = f_{i_0}$, thus $b_{2n-1}(pr_{2n-1}(e_{i_0})) = b_{2n-1}(e_{i_0})$ and, of course, for $i \neq i_0$ we have $b_{2n-1}(pr_{2n-1}(e_{i_0})) = 0 = b_{2n-1}(e_{i_0})$. This implies $b_{2n-1}(pr_{2n-1}(h)) = b_{2n-1}(h)$ for all $h \in H_{2n-1}(BG; \mathbb{Q})$.

QED

To decide whether the Borel class is indecomposable we apply Borel’s computation of K-theory of integer rings in number fields in [5].

Let $O_F$ be the ring of integers in a number field $F$, which has $r_1$ real and $2r_2$ complex embeddings. Borel proves that the Borel regulator, applied to the different embeddings, is even resp. odd. Borel proves that the Borel regulator, applied to the different embeddings, is even resp. odd. Borel proves that the Borel regulator, applied to the different embeddings, is even resp. odd. Borel proves that the Borel regulator, applied to the different embeddings, is even resp. odd.

Since decomposable cohomology classes vanish on primitive homology classes, this implies in particular:

If $A = O_F$ for a number field $F$, then the Borel class $b_{2n-1}$ is not decomposable for even $n$.

If moreover $F$ is not totally real, then the Borel class $b_{2n-1}$ is not decomposable for all $n$.

In particular, we can apply Lemma 2 to $A = O_F$.

If $A_1 \subset A_2 \subset \mathbb{C}$ are subrings and the Borel class is not decomposable for $A_1$, then of course it is also not decomposable for $A_2$. Thus we can actually apply Lemma 2 to all rings $A$ with $O_F \subset A \subset \mathbb{C}$. In particular to $A = \mathbb{Q}$ or $A = \mathbb{C}$:

**Corollary 2.** There exists a projection

$$pr_{2n-1} : H_{2n-1}(BGL(\mathbb{Q}); \mathbb{Q}) \to PH_{2n-1}(BGL(\mathbb{Q}); \mathbb{Q}) = K_{2n-1}(\mathbb{Q}) \otimes \mathbb{Q}$$

such that

$$b_{2n-1}(pr_{2n+1}(h)) = b_{2n-1}(h)$$

for all $h \in H_{2n-1}(BGL(\mathbb{Q}); \mathbb{Q})$.

---

3We remark that in the already interesting case $A = \mathbb{C}$ one can prove indecomposability of the Borel class without using Borel’s K-theory computation.

First, $H^*_Q(GL(N, \mathbb{C}); \mathbb{Q}) = \Lambda_Q(b_1, b_3, b_5, \ldots, b_{2N-1})$ implies that $b_{2n-1}$ is not decomposable in $H^*_Q(GL(N, \mathbb{C}); \mathbb{Q})$ for any $N$. Next, by homology stability of the linear group ([7], p.77), inclusion induces an isomorphism $H^{2n-1}(BG; \mathbb{Q}) = H^{2n-1}(BGL(N, \mathbb{C}); \mathbb{Q})$ if $2n-1 \leq \frac{N-1}{2}$, that is if $N \geq 4n+3$.

By Borel’s Theorem (see [5], Thm.9.6.), for any arithmetic subgroup $\Gamma \subset SL(N, \mathbb{C})$ we have an isomorphism $j^* : H^{2n-1}(BG; \mathbb{Q}) \to H^{2n-1}(BSL(N, \mathbb{C}); \mathbb{Q})$ whenever $2n-1 \leq \frac{N}{4}$, that is if $N \geq 8n+4$. This isomorphism is constructed via the van Est isomorphism, that is by integration of forms over simplices. In particular, if $h \in H^*(BSL(N, \mathbb{C}); \mathbb{Q})$ and $i : \Gamma \to SL(N, \mathbb{C})$ is the inclusion, then $j^*i^*h$ maps to $h$ under the canonical homomorphism from continuous to ordinary group cohomology.

Now we prove $b_{2n-1} \neq D$ by contradiction. Assume $b_{2n-1}$ were decomposable, that is $b_{2n-1} = xy$, where $x, y \in I$ are cohomology classes of degree $\geq 1$. Fix some $N \geq 8n+4 > 4n+3$. Then $j^*i^*h = (j^*i^*) (j^*i^*)$ is decomposable in $H^*_Q(GL(N, \mathbb{C}); \mathbb{Q})$, giving a contradiction.
2.5 Compact locally symmetric spaces and K-theory

In this subsection, we finally show that to each representation of nontrivial Borel class, and each compact, oriented, locally symmetric space of noncompact type we can find a nontrivial element in $K_* (\mathbb{Q}) \otimes \mathbb{Q}$.

**Theorem 2.** For each symmetric space $G/K$ of noncompact type and odd dimension $d = 2n - 1$, and to each representation $\rho : G \to GL(N, \mathbb{C})$ with $\rho^* b_{2n-1} \neq 0$, there exists a constant $c_\rho \neq 0$, such that the following holds: to each compact, oriented, locally symmetric space $M = \Gamma \backslash G/K$, with $\rho (\Gamma) \subset GL(N, \mathbb{A})$ and $\rho (\Gamma) \subset GL(N, \mathbb{C})$ for a subring $\mathbb{A} \subset \mathbb{C}$ satisfying the conclusion of **Lemma 3**, there exists an element

$$ \gamma (M) \in K_{2n-1} (A) \otimes \mathbb{Q} $$

such that the Borel regulator $r_{2n-1} : K_{2n-1} (A) \otimes \mathbb{Q} \to \mathbb{R}$ fulfills

$$ r_{2n-1} (\gamma (M)) = c_\rho \text{vol} (M) . $$

**Proof:**

In **Theorem 1** we have produced an element $j_* [M] \in H_d (BG; \mathbb{Z})$. Applying $(B\rho)_*$ we get an element

$$(B\rho)_* j_* [M] \in H_{d}(BGL (N, \mathbb{C}); \mathbb{Z}) .$$

Since $(B\rho)_* j_*$ maps $B\Gamma$ to $BGL (N, \mathbb{A})$, we have

$$(B\rho)_* j_* [M] \in H_{d}(BGL (N, \mathbb{A}); \mathbb{Z}) .$$

By assumption $\rho^* b_{2n-1} \neq 0$. Since $H^2_{2n-1} (G)$ is one-dimensional, this implies $\rho^* b_{2n-1} = c_\rho v_{2n-1}$ for some real number $c_\rho \neq 0$.

By **Lemma 2** we have a projection $pr_{2n-1}$. Thus, as in section 2.1., we can consider

$$ \gamma (M) := I_{2n-1}^{-1} pr_{2n-1} (B\rho)_{2n-1} j_{2n-1} [M] \in K_{2n-1} (A) \otimes \mathbb{Q} . $$

Since the Borel regulator is defined by pairing with $b_{2n-1}$, and by Lemma 2, we get

$$ r_{2n-1} (\gamma (M)) = < b_{2n-1}, \gamma (M) > = < b_{2n-1}, (B\rho)_{2n-1} j_{2n-1} [M] > $$

$$ = < b_{2n-1}, (B\rho)_{2n-1} j_{2n-1} [M] > = < \rho^* b_{2n-1}, j_{2n-1} [M] > $$

$$ = < c_\rho v_{2n-1}, j_{2n-1} [M] > = c_\rho \text{vol} (M) $$

where the last equality is true by **Theorem 1**. $\quad \Box$

**Corollary 3.** For each symmetric space $G/K$ of noncompact type and odd dimension $d = 2n - 1$, and to each representation $\rho : G \to GL(N, \mathbb{C})$ with $\rho^* b_{2n-1} \neq 0$, there exists a constant $c_\rho \neq 0$, such that the following holds: to each compact, oriented, locally symmetric space $M = \Gamma \backslash G/K$ there exists an element

$$ \gamma (M) \in K_{2n-1} (\mathbb{Q}) \otimes \mathbb{Q} $$

such that the Borel regulator $r_{2n-1} : K_{2n-1} (\mathbb{Q}) \otimes \mathbb{Q} \to \mathbb{R}$ fulfills

$$ r_{2n-1} (\gamma (M)) = c_\rho \text{vol} (M) . $$
Proof: $G$ is a linear semisimple Lie group without compact factors. $dim\,(G/K) = 2n-1$ implies that $G$ is not locally isomorphic to $SL\,(2,\mathbb{R})$, because $dim\,(SL\,(2,\mathbb{R})/SO\,(2)) = 2$. Thus $G$ satisfies the assumptions of Weil’s rigidity theorem, which implies that there exists some $g \in G$ with $gGg^{-1} \in G(\mathbb{Q})$. Thus, upon replacing $\Gamma$ by $gGg^{-1}$, $M$ is of the form $M = \Gamma\backslash G/K$ with $\Gamma \subset G(\mathbb{Q})$.

Each irreducible representation $\rho : G \to GL\,(N,\mathbb{C})$ is isomorphic to a representation $\rho'$ such that $G(\mathbb{Q})$ is mapped to $GL\,(N,\mathbb{Q})$. This follows from the classification of irreducible representations of Lie groups, see [14].

Moreover, by Corollary 2 $A = \mathbb{Q}$ satisfies the conclusion of Lemma 2. Thus we can apply Theorem 2. QED

Corollary 4. Let $G/K$ be a symmetric space of noncompact type and $\rho : G \to GL\,(N,\mathbb{C})$ a representation with $\rho^* b_{2n-1} \neq 0$, for $2n-1 = dim\,(G/K)$. Then compact, oriented, locally symmetric spaces $\Gamma\backslash G/K$ of rationally independent volumes yield rationally independent elements in $K_{2n-1}(\mathbb{Q}) \otimes \mathbb{Q}$.

Remark: In [13] it was claimed that for $(2n-1)$-dimensional compact hyperbolic manifolds one can construct an element $\gamma\,(M) \in K_{2n-1}(\mathbb{Q}) \otimes \mathbb{Q}$ such that $\nu_{2n-1}\,(\gamma\,(M)) = vol\,(M)$. However, since $\rho^* b_{2n-1}$ is an integer cohomology class, $c_\rho$ is rational if and only if $\nu_{2n-1}$ is a rational cohomology class, and this is equivalent to $vol\,(M) = < v_{2n-1}, [M] > \in \mathbb{Q}$. Since, conjecturally, all hyperbolic manifolds have irrational volumes, one can probably not get rid of the factor $c_\rho$ in Theorem 3.

In conclusion, we are left with the problem of finding representations of nontrivial Borel class, which will be solved in section 3.

The Matthey-Pitsch-Scherer construction. The following construction gives a somewhat stronger invariant under the assumption that $M$ is stably parallelizable. Assume that $M^d \to \mathbb{R}^n$ is an embedding with trivial normal bundle $\nu M$. Let $U$ be a regular neighborhood. Then there is the composition

$$S^n \to U/\partial U \to U/\partial U \wedge M_+ = Th\,(\nu M) \wedge M_+ = \Sigma^{n-d} M_+ \wedge M_+ \to S^{n-d} \wedge M_+$$

giving an element $\gamma\,(M) \in \pi_0^d(M)$. By [20], if $M$ is a closed hyperbolic 3-manifold and $\rho : M \to BSL$ is the map given by the stable trivialization, then $\rho_*\,(\gamma\,(M))$ is the Bloch invariant.

An analogous construction works for locally symmetric spaces, as long as they are stably parallelizable.

It is known by a Theorem of Deligne and Sullivan that each hyperbolic manifold $M$ admits a finite covering $\hat{M}$ which is stably parallelizable. Let $k$ be the degree of this covering. Then, rationally, we can define $\gamma\,(M) := \frac{1}{k}\gamma\,(\hat{M}) \in \pi_0^d(M) \otimes \mathbb{Q}$, and thus get a finer invariant which gives back $\gamma\,(M) \in K_d(\mathbb{C}) \otimes \mathbb{Q}$. We will not pursue further that approach in this paper.

2.6 Examples

Compact examples can e.g. be obtained by Borel’s construction of arithmetic hyperbolic manifolds using quadratic forms.
Let $u$ be an algebraic integer such that all roots of its minimal polynomial have multiplicity 1 and are real and negative (except $u$). Assume moreover that $(0, \ldots, 0)$ is the only integer solution of $x_1^2 + \ldots + x_{2n-1}^2 - ux_{2n}^2 = 0$. Let $\hat{\Gamma} \subset GL(2n)$ be the group of maps preserving $x_1^2 + \ldots + x_{2n-1}^2 - ux_{2n}^2$. It is isomorphic to a discrete cocompact subgroup of $SO(2n-1,1)$. By Selberg’s lemma, it contains a torsionfree cocompact subgroup $\Gamma \subset SO(2n-1,1;\mathbb{Z}[u])$. With the computations in section 3 below one concludes: If $n$ is even, then the compact manifold $M := \Gamma \backslash \mathbb{H}^{2n-1}$ (and, for example, a half-spinor representation) gives a nontrivial element $\gamma(M) \in K_{2n-1}(\mathbb{Z}[u]) \otimes \mathbb{Q}$. If $n$ is odd, then there is no canonically defined projection to $K_{2n-1}(\mathbb{Z}[u]) \otimes \mathbb{Q}$ but one gets at least a nontrivial element $\gamma(M) \in K_{2n-1}(\mathbb{Q}) \otimes \mathbb{Q}$.

3 Existence of representations of nontrivial Borel class

3.1 Invariant polynomials

Lemma 3. Let $G/K$ be a symmetric space of noncompact type, of dimension $2n-1$. Let $\mathfrak{l}$ be a Cartan subalgebra of $\mathfrak{g}$.

Then a representation $\rho : G \to GL(N, \mathbb{C})$ has nonvanishing Borel class if and only if $\text{Tr}(D_t \rho(t)^n) \neq 0$ for some $t \in \mathfrak{l}$.

Proof: As in section 2.3, we consider the dual representation

$$\rho_u : G_u/K \to U(N) \times U(N)/U(N) \simeq U(N).$$

We know that $\rho$ has nonvanishing Borel class if and only if

$$\rho_u^* b_{2n-1} \neq 0 \in H^{2n-1} (G_u/K).$$

The projection $p : G_u \to G_u/K$ induces an injective map $p^* : H^* (G_u/K) \to H^* (G_u)$, because a left inverse to $p^*$ is given by averaging differential forms over the compact group $K$. Hence, $\rho_u^* b_{2n-1} \neq 0$ if and only if its image in $H^{2n-1} (G_u)$ does not vanish. The latter equals

$$(\pi_2 \rho_u)^* b_{2n-1},$$

where $\pi_2 : U(N) \times U(N) \to U(N)$ is projection to the second factor.

We identify $H^{2n-1} (G_u)$ with $I_{\mathfrak{A}}^{2n-1} (G_u)$ and $H^{2n} (BG_u)$ with $I_{\mathfrak{S}}^\mathfrak{S} (G_u)$. Let $R : I_{\mathfrak{S}}^\mathfrak{S} (G_u) \to I_{\mathfrak{A}}^{2n-1} (G_u)$ be Cartan’s homomorphism (see section 2.3). According to [8], the image of $R$ are the transgressive elements and one has

$$R \circ \tau = \text{id}$$

for the universal transgression map $\tau$. (In particular $b_{2n-1} = R(C_n)$.)

In view of naturality of the transgression map, $C_n = \tau (b_{2n-1})$ implies that

$$(\pi_2 \rho_u)^* C_n = \tau ((\pi_2 \rho_u)^* b_{2n-1}),$$

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thus \((\pi_2 \rho_u)^* b_{2n-1} \neq 0\) is implied by
\[
(\pi_2 \rho_u)^* C_n \neq 0 \in H^{2n}(BG_u).
\]
Moreover, \(R \circ \tau = id\) implies injectivity of \(\tau\), hence \((\pi_2 \rho_u)^* C_n \neq 0\) is also a necessary condition for \((\pi_2 \rho_u)^* b_{2n-1} \neq 0\).

Recall that \(C_n\) corresponds to
\[
\frac{1}{(2\pi i)^m} Tr(A^n)
\]
under the isomorphism \(H^{2n}(BU(N)) \simeq I_S^2(u(N))\). Hence it suffices to check that the invariant polynomial
\[
Tr(\pi_2 D \rho_u(A)^n)
\]
is not trivial on \(G\).

Let \(\mathfrak{t}_u\) be the Cartan subalgebra of \(g\), which corresponds to \(\mathfrak{t}\) under the canonical bijection \(\mathfrak{k} \oplus \mathfrak{p} \simeq \mathfrak{k} \oplus ip\). There is an action of the Weyl group \(W\) on \(\mathfrak{t}_u\), we denote its space of invariant polynomials by \(S^W_*(\mathfrak{t}_u)\). By a theorem of Chevalley (see \([3]\)), restriction induces an isomorphism
\[
I_S^*(\mathfrak{g}_u) \cong S^W_*(\mathfrak{t}_u).
\]
In particular, it suffices to check that \(Tr(\pi_2 D \rho_u(\cdot)^n)\) is not trivial on \(\mathfrak{t}_u\).

We note that the Cartan algebra \(\mathfrak{t}\) can be conjugated into a subspace of \(\mathfrak{p}\). Since the conclusion of Lemma 2 is invariant under conjugation, we can without loss of generality assume that \(\mathfrak{t} \subset \mathfrak{p}\) and thus \(\mathfrak{t}_u \subset ip\). This implies that, for \(t \in \mathfrak{t}_u\), \(D \rho_u(\pi u)\) belongs to the second factor of \(u(N) \oplus u(N)\), and thus \(\pi_2 D \rho_u(t) = D \rho_u(t)\) on \(\mathfrak{t}_u\). Finally we note that, for \(t \in \mathfrak{p}\), \(Tr(D \rho(\pi u t)^n)\) and \(Tr(D \rho_u(\pi u t)^n)\) coincide up to a power of \(i\). The claim follows. 

QED

**Example: Spinor representations.** We consider the half-spinor representations of \(so(m-1,1)\) because their Borel classes (for \(m = 2n\) even) have been computed in \([13]\) and the computation given there appears not to be correct. (The point of confusion seems to be that \([15]\) computes a supertrace rather than a trace. This seems to be related to the wrong assertion that \(K = U(N)\) embeds as a diagonal subgroup into \(G_u = U(N) \times U(N)\). However, \(K\) corresponds to the first factor of \(U(N) \times U(N)\), see section 2.3.)

For simplicity, we consider the complexified representation of \(so(m-1,1) \otimes \mathbb{C} \cong so(m,\mathbb{C})\). We use the description of the half-spinor representations as it can be found in \([13]\).

**m=2n even.** Let \(V = \mathbb{C}^{2n}\) with \(\mathbb{C}\)-basis \(e_1, \ldots, e_{2n}\), and \(Q\) the quadratic form given by \(Q(e_i,e_{n+i}) = Q(e_{n+i},e_i) = 1\) and \(Q(e_i,e_j) = 0\) else. Let \(so(Q)\) be the Lie algebra of skew-adjoint matrices with respect to \(Q\) and let \(Cl(Q) = Cl(Q)^{even} \oplus Cl(Q)^{odd} = TV/I(Q)\) be the Clifford algebra of \(Q\) with the grading induced from the grading of the tensor algebra \(TV = \bigoplus_{k=0}^{m} V^{\otimes k}\), where \(I(Q)\) is the generated by all \(v \otimes v + Q(v,v)1, v \in V\).

There is an injective homomorphism \(so(Q) \rightarrow Cl(Q)^{even}\) which maps, in particular, \(E_{i,i} - E_{n+i,n+i}\) to \(\frac{1}{2}(e_i e_{n+i} - 1)\) (see \([12]\), pp.303-305). Let \(W\) be the \(\mathbb{C}\)-subspace of \(V\) spanned by \(e_1, \ldots, e_n\), \(W'\) the subspace spanned by \(e_{n+1}, \ldots, e_{2n}\).
Let \( \{ \) resp. negative half-spinor representations. We will denote them by \( e_i \) for \( i = 1, \ldots, n \). (This follows from the proof of Lemma 20.9.) This action extends in the obvious way to an action of \( \text{Cl} \) on \( \Lambda^* W \). In particular \( \frac{1}{2} (e_i e_{n+i} - 1) \) acts by sending \( v \in W \) to \( e_i \wedge v - \frac{1}{2} v \). Thus it maps \( e_i \) to \( -\frac{1}{2} e_i \) and \( e_j \) to \( -\frac{1}{2} \delta_{ij} \) for \( j \neq i \).

This action gives rise to an isomorphism \( \text{Cl} (Q)^{\text{even}} \cong \text{End} (\Lambda^{\text{even}} W) \oplus \text{End} (\Lambda^{\text{odd}} W) \) (see p.305). The induced actions of \( \text{so}(Q) \) on \( \Lambda^{\text{even}} W \) resp. \( \Lambda^{\text{odd}} W \) are the positive resp. negative half-spinor representations. We will denote them by \( S^+ \) and \( S^- \).

Thus \( E_{i,j} - E_{n+i,n+j} \) acts on \( e_{i_k} \wedge \ldots \wedge e_{i_k} \) by multiplication with \( \frac{2k}{2} \) if \( i \in \{ i_1, \ldots, i_k \} \) and by multiplication with \( -\frac{k}{2} \) if \( i \not\in \{ i_1, \ldots, i_k \} \).

As a Cartan-algebra we choose the algebra of diagonal matrices

\[
\text{diag}(h_1, \ldots, h_n, -h_1, \ldots, -h_n).
\]

Let \( \{ A_i : i = 1, \ldots, n \} \) be a basis, where

\[
A_i = E_{i,i} - E_{n+i,n+i}.
\]

For the positive half-spinor representation (and any \( l \)) we have

\[
\text{Tr} \left( S^+ (A_i)^l \right) = \sum_{k \text{ even}} \left( \sum_{i \in \{ i_1, \ldots, i_k \}} \left( 1 - \frac{k}{2} \right)^l \right) + \sum_{i \not\in \{ i_1, \ldots, i_k \}} \left( -\frac{k}{2} \right)^l
\]

\[
= \sum_{1 \leq k \leq m, k \text{ even}} \left( \left( \frac{m-1}{k-1} \right) \left( 1 - \frac{k}{2} \right)^l \right) + \left( \frac{m-1}{k} \right) \left( -\frac{k}{2} \right)^l.
\]

In particular, we have \( \text{Tr} \left( S^+ (A_i)^l \right) < 0 \) for \( l \) odd and \( \text{Tr} \left( S^+ (A_i)^l \right) > 0 \) for \( l \) even.

For the negative half-spinor representation we have

\[
\text{Tr} \left( S^- (A_i)^l \right) = \sum_{1 \leq k \leq m} \left( \sum_{1 \leq m_1 \leq m, k \text{ even}} \left( 1 - \frac{k}{2} \right)^l \right) + \sum_{i \not\in \{ i_1, \ldots, i_k \}} \left( -\frac{k}{2} \right)^l
\]

\[
= \sum_{k \text{ odd}} \left( \frac{m-1}{k-1} \right) \left( 1 - \frac{k}{2} \right)^l + \left( \frac{m-1}{k} \right) \left( -\frac{k}{2} \right)^l.
\]

In particular, we have \( \text{Tr} \left( S^- (A_i)^l \right) < 0 \) for \( l \) odd and \( \text{Tr} \left( S^- (A_i)^l \right) > 0 \) for \( l \) even.

Since \( \text{dim} (\text{so}(m-1)/\text{so}(m)) = m-1 = 2n-1, \) with Lemma 3 and \( l := n, \) this implies nontriviality of the Borel class for odd-dimensional hyperbolic manifolds and the half-spinor representations.

**m=2n-1 odd.** Let \( V = \mathbb{C}^{2n-1} \) with \( \mathbb{C} \)-basis \( e_1, \ldots, e_{2n-1} \), and \( Q \) the quadratic form given by \( Q(e_{2n-1}, e_{2n-1}) = 1, Q(e_i, e_{n+i}) = Q(e_{n+i}, e_i) = 1 \) and \( Q(e_i, e_j) = 0 \) else. As in the case \( m \) even, we have \( \text{so}(Q) \to \text{Cl}(Q)^{\text{even}} \) which maps \( E_{i,j} - E_{n+i,n+j} \) to \( \frac{1}{2} (e_i e_{n+i} - 1) \). Let \( W \) be the \( \mathbb{C} \)-subspace of \( V \) spanned by \( e_1, \ldots, e_n \), \( W' \) the subspace spanned by \( e_{n+1}, \ldots, e_{2n} \).
It follows from the proof of [14], Lemma 20.16., that \( Cl(Q) \) acts on \( \Lambda^* W \) as follows: the action of \( e_i \) and \( e_{n+i} \), for \( i = 1, \ldots, n \) is defined as in the case \( n \) even, and \( e_{2n-1} \) acts as multiplication by 1 on \( \Lambda^{\text{even}} W \) and as multiplication by -1 on \( \Lambda^{\text{odd}} W \). In particular, we have again that \( \frac{1}{2} (e_i e_{n+i} - 1) \) acts by sending \( e_i \) to \( \frac{1}{2} e_i \) and \( e_j \) to \( -\frac{1}{2} e_j \) for \( j \neq i \).

This action gives rise to an isomorphism \( Cl(Q) \approx \text{End}(\Lambda^* W) \) (see [14], p. 306).

As a Cartan-algebra we choose the algebra of diagonal matrices

\[
diag(h_1, \ldots, h_n, -h_1, \ldots, -h_n, 0).
\]

Let \( \{A_i : i = 1, \ldots, n\} \) be a basis, where

\[
A_i = E_{i,i} - E_{n+i,n+i}.
\]

Thus we have (for any \( l \))

\[
\text{Tr} \left( S(A_i)^l \right) = \sum_k \left( \sum_{i \in \{i_1, \ldots, i_k\}} \left( 1 - \frac{k}{2} \right)^l + \sum_{i \notin \{i_1, \ldots, i_k\}} \left( -\frac{k}{2} \right)^l \right)
\]

\[
= \sum_k \left( \binom{m-1}{k-1} \left( 1 - \frac{k}{2} \right)^l + \binom{m-1}{k} \left( -\frac{k}{2} \right)^l \right).
\]

In particular, we have \( \text{Tr} \left( S(A_i)^l \right) < 0 \) for \( l \) odd and \( \text{Tr} \left( S(A_i)^l \right) > 0 \) for \( l \) even.

### 3.2 Borel class of Lie algebra representations

Let \( g \) be a semisimple Lie algebra and \( R(g) \) its (real) representation ring, with addition \( \oplus \) and multiplication \( \otimes \). Let \( \mathfrak{t} \) be a Cartan subalgebra of \( g \).

In this section we consider, for \( n \in \mathbb{N} \), the map

\[
b_{2n-1} : R(g) \rightarrow \mathbb{Z}[\mathfrak{t}]
\]

given by

\[
b_{2n-1} (\pi) (t) = \text{Tr} (\pi (t)^n).
\]

It is obvious that

\[
b_{2n-1} (\pi_1 \oplus \pi_2) = b_{2n-1} (\pi_1) + b_{2n-1} (\pi_2)
\]

holds for representations \( \pi_1, \pi_2 \). Therefore \( b_{2n-1} \) is uniquely determined by its values for irreducible representations. Moreover,

\[
b_{2n-1} (\pi_1 \otimes \pi_2) = b_{2n-1} (\pi_1) b_{2n-1} (\pi_2)
\]

for representations \( \pi_1, \pi_2 \).

**Complex-linear representations.** First we consider complex simple Lie algebras \( g \)
Let $R$ and the ring $R_C (g) \subset R (g)$ of their $C$-linear representations.

Let $V = \mathbb{C}^{l+1}$ be the standard representation, with basis $e_1, \ldots, e_{l+1}$. Then

$$R_C (g) = \mathbb{Z} [A_1, \ldots, A_l]$$

with $A_k$ the induced representation on $\Lambda^k V$, cf. [13]. In particular, irreducible representations occur as representations of dominant weight in tensor products of the fundamental representations $A_1, \ldots, A_l$. We compute $b_{2n-1}$ on the fundamental representations $A_k, k = 1, \ldots, l$.

A basis of $\Lambda^k V$ is given by

$$\{ e_{i_1} \wedge \ldots \wedge e_{i_k} : 1 \leq i_1 < \ldots < i_k \leq l+1 \}.$$ 

As Cartan-subalgebra we may choose the diagonal matrices

$$\text{diag} (h_1, \ldots, h_l, h_{l+1}) \text{ with } h_1 + \ldots + h_{l+1} = 0.$$ 

$\text{diag} (h_1, \ldots, h_l, h_{l+1})$ acts on $e_{i_1} \wedge \ldots \wedge e_{i_k}$ by multiplication with $h_{i_1} + \ldots + h_{i_k}$. Hence

$$b_{2n-1} (A_k) \left( \begin{array}{ccc} h_1 & 0 & \ldots & 0 \\ 0 & h_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & h_{l+1} \end{array} \right) = \sum_{1 \leq i_1 < \ldots < i_k \leq l+1} (h_{i_1} + \ldots + h_{i_k})^n.$$ 

For $n = 1$, the sum is a multiple of $h_1 + \ldots + h_{l+1} = 0$. For $l = k = 1$ and $n$ odd, we have that $b_n (A_1) = h_1^n + h_2^n$ is a multiple of $h_1 + h_2 = 0$. In all other cases, i.e. for $l \geq 2, n > 1$ or $l = k = 1, n$ even, the sum is not divisible by $h_1 + \ldots + h_{l+1}$ and thus not trivial. This is obvious for even $n$. In the case of odd $n > 1$ and $l \geq 2$, it follows for example from the computation $b_{2n-1} (A_k) \text{diag} (2, -1, -1, 0, \ldots, 0) = (2^n - 1) \left( \begin{array}{c} l-2 \\ k-1 \end{array} \right) - \left( \begin{array}{c} l-1 \\ k-1 \end{array} \right) \neq 0$. Thus, fundamental representations have nontrivial $b_{2n-1}$, for all $l \geq 2, n > 1$ or $l = k = 1, n$ even.

$g = \text{sp} (l, \mathbb{C})$.
Let $V = \mathbb{C}^{2l}$ be the standard representation. Then

$$R_C (g) = \mathbb{Z} [B_1, \ldots, B_l]$$

with $B_k$ the induced representation on $\Lambda^k V$. We compute $b_{2n-1}$ on the fundamental representations $B_k, k = 1, \ldots, l$.

$\text{sp} (l, \mathbb{C})$ consists of matrices $\left( \begin{array}{cc} A & B \\ C & D \end{array} \right)$, such that the $l \times l$-blocks $A, B, C, D$ satisfy $B^T = B, C^T = C, A^T = D$. As Cartan-subalgebra we may choose the diagonal matrices

$$\text{diag} (h_1, \ldots, h_l, -h_1, \ldots, -h_l).$$

Let $\{ e_1, \ldots, e_l, f_1, \ldots, f_l \}$ be a basis of $\mathbb{C}^{2l}$ for the standard representation.
A basis of $\Lambda^k V$ is given by
\[
\{e_{i_1} \wedge \ldots \wedge e_{i_p} \wedge f_{j_1} \wedge \ldots \wedge f_{j_{k-p}} : 0 \leq p \leq k, 1 \leq i_1 < \ldots < i_p \leq l, 1 \leq j_1 < \ldots < j_{k-p} \leq l\}. 
\]
\[
diag (h_1, \ldots, h_l, -h_1, \ldots, -h_l) \text{ acts on } e_{i_1} \wedge \ldots \wedge e_{i_p} \wedge f_{j_1} \wedge \ldots \wedge f_{j_{k-p}} \text{ by multiplication with } h_{i_1} + \ldots + h_{i_p} - h_{j_1} - \ldots - h_{j_{k-p}}. \text{ Hence}
\]
\[
b_{2n-1}(B_k) = \sum_{1 \leq i_1 < \ldots < i_p \leq l, 1 \leq j_1 < \ldots < j_{k-p} \leq l} (h_{i_1} + \ldots + h_{i_p} - h_{j_1} - \ldots - h_{j_{k-p}})^n.
\]
If $n$ is even, we clearly get a nonvanishing polynomial without cancellations. If $n$ is odd, then the permutation, which transposes $i_r$ and $j_r$ simultaneously for all $r$, multiplies the sum by $-1$, but on the other hand preserves the sum. Thus $b_{2n-1}(B_k) = 0$ if $n$ is odd.

$g = so(2l+1, \mathbb{C})$.

Let $V = \mathbb{C}^{2l+1}$ be the standard representation. Then
\[
R_C(g) = \mathbb{Z}[C_1, \ldots, C_{l-1}, S]
\]
with $C_k$ the induced representation on $\Lambda^k V$, and $S$ the spin representation.

As Cartan-subalgebra we may choose the diagonal matrices
\[
diag (h_1, \ldots, h_l, -h_1, \ldots, -h_l, 0).
\]
Then the computation of $b_{2n-1}$ on $C_k$ is exactly the same as for $sp(l, \mathbb{C})$, in particular $b_{2n-1}(C_k) \neq 0$ for $n$ even and $b_{2n-1}(C_k) = 0$ for $n$ odd.

Moreover, we have computed in section 3.1 that $b_{2n-1}(S) \neq 0$ for all $n$.

$g = so(2l, \mathbb{C})$.

Let $V = \mathbb{C}^{2l}$ be the standard representation. Then
\[
R_C(g) = \mathbb{Z}[D_1, \ldots, D_{l-2}, S^+, S^-]
\]
with $D_k$ the induced representation on $\Lambda^k V$, and $S^\pm$ the half-spinor representations.

As Cartan-subalgebra we may choose the diagonal matrices
\[
diag (h_1, \ldots, h_l, -h_1, \ldots, -h_l).
\]
Again the same computation as for $sp(l, \mathbb{C})$ shows that $b_{2n-1}(D_k) \neq 0$ for $n$ even and $b_{2n-1}(D_k) = 0$ for $n$ odd. Moreover, we have computed in section 3.1 that $b_{2n-1}(S^\pm) \neq 0$ for all $n$. 

20
Exceptional Lie groups.

We will see in the next section that we will only be interested in Lie groups which admit a symmetric space of odd dimension. The only exceptional Lie group admitting an odd-dimensional symmetric space is $E_7$ with $\dim (E_7/E_7(\mathbb{R})) = 163 = 2.82 - 1$. The fact that $163 \equiv 3 \mod 4$, i.e. $n = 82$ even, implies automatically that $\rho^* b_{163} \neq 0$ holds for each irreducible representation $\rho$.

For completeness we also show, at least for a specific representation, that $\rho^* b_{2n-1} \neq 0$ holds for each $n > 1$. Namely, we consider the representation $\rho : E_7 \to GL(56, \mathbb{C})$, which has been constructed in [1], Corollary 8.2, and we are going to show that this representation satisfies $\rho^* b_{2n-1} \neq 0$ for each $n > 1$, in particular for $n = 82$.

By [1], chapter 7/8, there is a monomorphism $Spin(12) \times SU(2)/\mathbb{Z}_2 \to E_7$ and the Cartan-subalgebra of the Lie algebra $\mathfrak{e}_7$ coincides with the Cartan-subalgebra $\mathfrak{t}$ of $spin(12) \oplus su(2)$. According to [1], Corollary 8.2, the restriction of $\rho$ to $Spin(12) \times SU(2)$ is $\lambda_{12}^{\mathbb{C}} \otimes \lambda_1 \otimes S^- \otimes 1$, where $\lambda_{12}$ resp. $\lambda_1$ are the standard representations and $S^-$ is the negative spinor representation.

For even $n$, we know that $\rho^* b_{2n-1} \neq 0$. If $n$ is odd then, for the derivative $\pi_1$ of the standard representation $\lambda_1$ of $SU(2)$ we have $Tr (\pi_1 (h)^n) = 0$, whenever $h \in \mathfrak{t} \cap su(2)$ belongs to the Cartan-subalgebra of $su(2)$, because the latter are the diagonal 2x2-matrices of trace 0. Thus the first direct summand $\lambda_{12}^{\mathbb{C}} \otimes \lambda_1$ does not contribute to $Tr (\pi (h)^n)$. Hence, for $h = (h_{spin}, h_{su}) \in \mathfrak{t} \subset spin(12) \oplus su(2)$, we have $Tr (\pi (h)^n) = Tr (S^- (h_{spin})^n)$. But the nontriviality of the latter has already been shown in section 3.1.

**Noncomplex Lie algebras.** Let $\pi : g \to gl(N, \mathbb{C})$ be an $\mathbb{R}$-linear representation of a simple Lie-algebra $g$ which is not a complex Lie algebra. Then $g \otimes \mathbb{C}$ is a simple complex Lie algebra and $\pi$ is the restriction of some $\mathbb{C}$-linear representation $g \otimes \mathbb{C} \to gl(N, \mathbb{C})$. Let $\mathfrak{t}$ be a Cartan subalgebra of $g$. Then it is obvious that an element $\mathfrak{t} \in \mathfrak{t} \otimes \mathbb{C}$ with

$$Tr (\pi (t)^n) \neq 0$$

exists if and only if such an element exists in $\mathfrak{t}$. Thus $\pi$ has nontrivial Borel class if and only if $\pi \otimes \mathbb{C}$ has nontrivial Borel class. Hence we can use the results for complex-linear representations.

**Real representations of complex Lie algebras.** If $g$ is a simple complex Lie algebra, then each $\mathbb{R}$-linear representation $\pi : g \to gl(N, \mathbb{C})$ is of the form $\pi = \pi_1 \otimes \pi_2$ for $\mathbb{C}$-linear representations $\pi_1, \pi_2$. We have

$$Tr (\pi (t)^n) = Tr (\pi_1 (t)^n) Tr (\pi_2 (t)^n).$$

In particular, real representations with nontrivial $b_{2n-1}$ can only exist if there are complex representations of nontrivial $b_{2n-1}$.

### 3.3 Conclusion

In this section, we discuss, for which symmetric spaces $G/K$ (irreducible, of noncompact type, of dimension $2n - 1$) and which representations $\rho : G \to GL(N, \mathbb{C})$ the inequality $\rho^* b_{2n-1} \neq 0$ holds.
Definition 3. We say that a Lie algebra representation \( \pi : g \rightarrow gl(N, \mathbb{C}) \) has nontrivial Borel class if \( b_{2n-1}(\pi) \neq 0 \), for \( b_{2n-1} : R(g) \rightarrow \mathbb{Z}[t] \) defined in section 3.2.

Proposition 1. Let \( \rho : G \rightarrow GL(N, \mathbb{C}) \) be a representation of a Lie group \( G \), and \( \pi : g \rightarrow gl(N, \mathbb{C}) \) the associated Lie algebra representation \( \pi = D_{e}\rho \). Then \( \rho \) has nontrivial Borel class if and only if \( \pi \) has nontrivial Borel class.

Proof: This is precisely the statement of Lemma 3. QED

We use the classification of symmetric spaces as it can be read off Table 4 in [29]. Of course, we are only interested in symmetric spaces of odd dimension. A simple inspection shows that all odd-dimensional irreducible symmetric spaces of noncompact type are given by the following list:

\[
\begin{align*}
SL_l(\mathbb{R})/SO_l, l \equiv 0, 3 \pmod{4}, \\
SL_{2l}(\mathbb{H})/Sp_l, l \equiv 0 \pmod{2}, \\
Spin_{p,q}/(Spin_p \times Spin_q), p, q \equiv 1 \pmod{2}, \\
SL_l(\mathbb{C})/SU_l, l \equiv 0 \pmod{2}, \\
SO_l(\mathbb{C})/SO_l, l \equiv 2, 3 \pmod{4}, \\
Sp_l(\mathbb{C})/Sp_l, l \equiv 1 \pmod{2}, \\
E_7(\mathbb{C})/E_7.
\end{align*}
\]

First we note that for \( n \) even all representations satisfy \( \rho^*b_{2n-1} \neq 0 \). This applies to locally symmetric spaces of dimension \( \equiv 3 \pmod{4} \). In the above list this are the following symmetric spaces:

\[
\begin{align*}
SL_l(\mathbb{R})/SO_l, l \equiv 0, 7 \pmod{8}, \\
SL_{2l}(\mathbb{H})/Sp_l, l \equiv 2 \pmod{4}, \\
SO_{p,q}/(SO_p \times SO_q), p, q \equiv 1 \pmod{2}, p \neq q \pmod{4}, \\
SL_l(\mathbb{C})/SU_l, l \equiv 0 \pmod{2}, \\
SO_l(\mathbb{C})/SO_l, l \equiv 2 \pmod{4}, \\
Sp_l(\mathbb{C})/Sp_l, l \equiv 3 \pmod{4}, \\
E_7(\mathbb{C})/E_7.
\end{align*}
\]

We now analyze the irreducible locally symmetric spaces of dimension \( \equiv 1 \pmod{4} \).

For those locally symmetric spaces, whose corresponding Lie algebra \( g \) is not a complex Lie algebra (this concerns the first 3 cases), we can, as observed at the end of section 3.2, directly apply the results for the complexifications. Thus we get:

- for \( SL_l(\mathbb{R})/SO_l, l \equiv 3, 4 \pmod{8} \), every fundamental representation yields a nontrivial element,
- for \( SL_{2l}(\mathbb{H})/Sp_l, l \equiv 0 \pmod{4} \), every fundamental representation yields a nontrivial element,
- for \( Spin_{p,q}/(Spin_p \times Spin_q), p, q \equiv 1 \pmod{2}, p \equiv q \pmod{4} \), the positive and negative
half-spinor representations are the only fundamental representations yielding nontrivial elements.

For those locally symmetric spaces whose corresponding Lie algebra \( g \) is a complex Lie algebra, we use the fact that each real representation is of the form \( \rho \otimes \bar{\rho} \). We get:

- for \( SO_l(\mathbb{C})/SO_l, l \equiv 3 \mod 4 \), the spinor representation and its conjugate are the only fundamental representations yielding nontrivial elements,

- for \( Sp_l(\mathbb{C})/Sp_l, l \equiv 1 \mod 4 \), no fundamental representation yields nontrivial elements.

**Example (Goncharov):** Hyperbolic space \( \mathbb{H}^n \) is the symmetric space
\[
\mathbb{H}^n = \text{Spin}_{n,1}/(\text{Spin}_n \times \text{Spin}_1).
\]

Let \( n \) be odd. It was shown in [15] that the positive and negative half-spinor representation have nontrivial Borel class. The question was raised ([15], p.587) whether these are the only fundamental representations of \( \text{Spin}_{n,1} \) with this property. As a special case of the above results we see that for \( n \equiv 3 \mod 4 \) each irreducible representation has nontrivial Borel class, but for \( n \equiv 1 \mod 4 \) the positive and half-negative spinor representation are the only fundamental representations with this property.

On the other hand, if \( n = 3 \), then the invariants coming from different irreducible representations, albeit distinct and nontrivial, all are rational multiples of each other. This will follow from the computation in Section 3.4.

### 3.4 Some clues on computation

So far we have only discussed how to decide whether \( \rho^*b_{2n-1} \neq 0 \), which is in view of Lemma 3 easier than computing \( \rho^*b_{2n-1} \). The aim of this subsection is only to give some clues to the computation of \( \rho^*b_{2n-1} \). Its results are not needed throughout the paper, except for the explicit values of the Borel regulator in section 5.

For each Lie-algebra-cocycle \( P \in C^n \left( g_u, k \right) \), we denote by \( \omega_P \in \Omega^n (G_u/K) \) the corresponding \( G_u \)-invariant differential form. Then we have the following obvious observation. (\([\omega_P]\) denotes the cohomology class of \( \omega_P \), and \([G_u/K]^\vee \in H^n (G_u/K, \mathbb{R})\) denotes the dual of the fundamental class \([G_u/K]\). The Riemannian metric is given by \(-B\), that is the negative of the Killing form.)

**Lemma 4.** Let \( X_1, \ldots, X_n \) be an orthonormal basis for \( ip \) with respect to \(-B\). Then, for each \( P \in I^n \left( g_u, k \right) \), we have
\[
[\omega_P] = [G_u/K]^\vee \operatorname{vol} (G_u/K) P (X_1, \ldots, X_n).
\]

**Corollary 5.** \([\omega_P] \neq 0 \) iff \( P (X_1, \ldots, X_n) \neq 0 \) for some (hence any) basis of \( ip \).

We will apply this to the Borel class \( b_{2n-1} \in H^{2n-1} (u (N) \oplus u (N), u (N)) \) which is given by the Lie-algebra-cocycle
\[
b_{2n-1} (X_1, \ldots, X_{2n-1}) = \frac{1}{(2\pi i)^m} \frac{1}{(2n-1)!} \sum_{\sigma \in S_{2n-1}} (-1)^\sigma \operatorname{Tr} (X_{\sigma(1)} [X_{\sigma(2)}, X_{\sigma(3)}] \cdots [X_{\sigma(2n-2)}, X_{\sigma(2n-1)}]).
\]
Example: Hyperbolic 3-manifolds.

Let $G = SL(2, \mathbb{C})$. Then

$$i\mathfrak{p} = \left\{ iA \in Mat(2, \mathbb{C}) : Tr(A) = 0, A = A^T \right\}.$$ 

An ON-basis of $i\mathfrak{p}$ (with respect to the Killing form) is given by $\frac{1}{2\sqrt{2}}H, \frac{1}{2\sqrt{2}}X, \frac{1}{2\sqrt{2}}Y$, with

$$H = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad X = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$ 

We have

$$[H, X] = -2Y, \quad [H, Y] = -2X, \quad [X, Y] = 2H.$$ 

Thus, for each representation $\rho : SL(2, \mathbb{C}) \to GL(m + 1, \mathbb{C})$ with associated Lie algebra representation $\Pi : sl(2, \mathbb{C}) \to Mat(m + 1, \mathbb{C})$ we have

$$\rho^* b_3(H, X, Y) = \frac{1}{(2\pi i)^2} \frac{1}{6\pi^2} \left\{ 2Tr(\Pi H [\Pi X, \Pi Y]) + 2Tr(\Pi X [\Pi Y, \Pi H]) + 2Tr(\Pi Y [\Pi H, \Pi X]) \right\}$$

$$= -\frac{1}{6\pi^2} Tr((\Pi H)^2) - \frac{1}{6\pi^2} Tr((\Pi X)^2) - \frac{1}{6\pi^2} Tr((\Pi Y)^2).$$

By the classification of irreducible representations of $sl(2, \mathbb{C})$, each $m + 1$-dimensional irreducible representation is equivalent to $\pi_m$ given by

$$\pi_m(H) = \begin{pmatrix} im & 0 & 0 & \ldots & 0 \\ 0 & i(m-2) & 0 & \ldots & 0 \\ 0 & 0 & i(m-4) & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & -im \end{pmatrix},$$

$$\pi_m(X) = \begin{pmatrix} 0 & -i & 0 & \ldots & 0 \\ -im & 0 & -2i & \ldots & 0 \\ 0 & -i(m-1) & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & -i \end{pmatrix},$$

$$\pi_m(Y) = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ -m & 0 & 2 & \ldots & 0 \\ 0 & -(m-1) & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & -1 \end{pmatrix}.$$ 

Therefore, the diagonal entries of $\pi_m(H)^2$ are

$$\left( -m^2, -(m-2)^2, \ldots, 0, \ldots, -(m-2)^2, -m^2 \right),$$

and the diagonal entries of $\pi_m(X)^2$ resp. $\pi_m(Y)^2$ are both equal to

$$\left( -m, -m - 2(m-1), -2(m-1) - 3(m-2), \ldots \right).$$
In particular, $Tr \left( \pi_m(X)^2 \right) = Tr \left( \pi_m(Y)^2 \right)$ and we conclude

$$\rho_m^* b_3 \left( \frac{1}{2\sqrt{2}} H, \frac{1}{2\sqrt{2}} X, \frac{1}{2\sqrt{2}} Y \right) = -\frac{1}{96\sqrt{2}\pi^2} Tr \left( (\pi_m H)^2 \right) - \frac{1}{48\sqrt{2}\pi^2} Tr \left( (\pi_m X)^2 \right)$$

$$= \frac{1}{96\sqrt{2}\pi^2} \sum_{k=0}^{m} (m - 2k)^2 + \frac{1}{48\sqrt{2}\pi^2} \sum_{k=0}^{m} k(m - k + 1)(k + 1)(m - k).$$

If $m = 1$, we get

$$\rho_1^* b_3 \left( \frac{1}{2\sqrt{2}} H, \frac{1}{2\sqrt{2}} X, \frac{1}{2\sqrt{2}} Y \right) = \frac{1}{16\sqrt{2}\pi^2}.$$

One should note that the hyperbolic metric is given by a half of the negative of the Killing form. Thus an orthonormal basis of $\mathfrak{p} = T_e \mathbb{H}^3$ with respect to the hyperbolic metric is given by $\{-\frac{1}{2}H, -\frac{1}{2}X, -\frac{1}{2}Y\}$. It follows that the Borel regulator is $\frac{1}{8\pi}$ times the hyperbolic volume.

(There seem to be different normalizations of the Borel regulator in the literature. \cite{11} computes the Borel regulator to be $\frac{1}{2\pi}$ times the hyperbolic volume, while \cite{28} defines the imaginary part of the Borel regulator to be $\frac{1}{4\pi}$ times the hyperbolic volume.)

In \cite{15} it was stated that the half-spinor representations seem to be the only fundamental representations of $Spin(d, 1)$ that yield nontrivial invariants of odd-dimensional hyperbolic manifolds. This is however not the case. Indeed, if $d = \dim(M) \equiv 3 \mod 4$, then each irreducible representation of $Spin(d, 1)$ yields nontrivial invariants. (But, as the computation above shows, the invariants of hyperbolic 3-manifolds for different representations all yield rationally dependent values of the Borel regulator. It would be interesting to know in general whether the invariants of a given d-dimensional locally symmetric space for different representations do or do not yield $\mathbb{Q}$-dependent elements of $K_d(\mathbb{Q}) \otimes \mathbb{Q}$.)

**Example: $SL(3, \mathbb{R})/SO(3)$.**

Let $\rho : SL(3, \mathbb{R}) \to GL(3, \mathbb{C})$ be the inclusion. Since $SL(3, \mathbb{R})/SO(3)$ is 5-dimensional, we wish to compute $\rho^* b_5$.

Let

$$H_1 = \begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix}, X_1 = \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, Y_1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We will use the convention that, for $A \in \{H, X, Y\}$ if $A_1$ is defined (in a given basis), then $A_2$ is obtained via the base change $e_1 \to e_2, e_2 \to e_3, e_3 \to e_1$ and $A_3$ is obtained via the base change $e_1 \to e_3, e_3 \to e_2, e_2 \to e_1$.

We have $[H_1, H_2] = 0, [H_1, X_1] = 2Y_1, [H_1, X_2] = -Y_2, [H_1, X_3] = -Y_3, [X_1, X_2] = iY_3$ and more relations are obtained out of these ones by base changes.

A basis of $i\mathfrak{p}$ is given by $H_1, H_2, X_1, X_2, X_3$. The formula for $\rho^* b_5(H_1, H_2, X_1, X_2, X_3)$ contains 120 summands. (24 of them contain $[H_1, H_2] = 0$ or $[H_2, H_1] = 0$.)
Each summand appears four times because, for example, $H_1[H_2, X_1] [X_2, X_3]$ also shows up as $-H_1[X_1, H_2] [X_2, X_3], -H_1[H_2, X_1] [X_3, X_2]$ and $H_1[X_1, H_2] [X_3, X_2]$. Thus one has to add 30 summands (6 of them zero), and multiply their sum by 4.

We note that all summands of the form $H_1[H_2, \ldots]$ give after base change corresponding elements of the form $H_2[H_1, \ldots]$, which are summed with the opposite sign. Thus these terms cancel each other. The same cancellation occurs between summands of the form $X_2[\ldots],[\ldots]$ and $X_3[\ldots],[\ldots]$. Thus we only have to sum up summands of the form $X_1[\ldots],[\ldots]$ and we get

$$
(2\pi i)^3 5! \rho^* b_5 (H_1, H_2, X_1, X_2, X_3) =
$$
$$
4Tr(X_1[H_1, H_2][X_2, X_3]) + 4Tr(X_1[X_2, X_3][H_1, H_2]) +
+4Tr(X_1[H_1, X_2][X_3, H_2]) + 4Tr(X_1[X_3, H_2][H_1, X_2]) +
+4Tr(X_1[H_1, X_3][H_2, X_2]) + 4Tr(X_1[H_2, X_2][H_1, X_3])
$$
$$= 0 + 0 + 4Tr(X_1Y_2Y_3) + 4Tr(X_1Y_3Y_2) + 4Tr(-2X_1Y_3Y_2) + 4Tr(-2X_1Y_2Y_3)
$$
$$= 0 + 0 + 4i + 4i - 8i - 8i = -8i.
$$

We note that $H_1, H_2, X_1, X_2, X_3$ are pairwise orthogonal and have norm $2\sqrt{3}$. Dividing each of them by $2\sqrt{3}$ gives an ON-basis, on which evaluation of $\rho^* b_5$ gives

$$
\rho^* b_5 \left( \frac{1}{2\sqrt{3}} H_1, \frac{1}{2\sqrt{3}} H_2, \frac{1}{2\sqrt{3}} X_1, \frac{1}{2\sqrt{3}} X_2, \frac{1}{2\sqrt{3}} X_3 \right) = \frac{1}{(2\sqrt{3})^5} \frac{1}{5!} (2\pi i)^3 (-8i) = \frac{1}{34560\sqrt{3}\pi^3}.
$$

4 The cusped case

4.1 Preparations

Let $G$ be a connected, semisimple Lie group with maximal compact subgroup $K$. Thus $G/K$ is a symmetric space of noncompact type. In this section we will assume that $G/K$ is a symmetric space of rank one.

Let

$$
\rho : G \to GL(N, \mathbb{C})
$$

be a representation. We assume that $\rho$ maps $K$ to $U(N)$, which can be achieved upon conjugation. We note that connected, semisimple Lie groups are perfect, hence $\rho$ has image in $SL(N, \mathbb{C})$ and maps $K$ to $SU(N)$.

Negative curvature and visibility manifolds

If $M = \Gamma \setminus G/K$ is a locally symmetric space of rank one, then its sectional curvature $sec$ is bounded between two negative constants, after scaling with a constant factor one has

$$-4 \leq sec \leq -1.$$ 

In particular, by [12], page 440, the universal covering $\tilde{M}$ is a 'visibility manifold' in the sense of [12]. By [12], Theorem 3.1, this implies that each end of $int(M) = M - \partial M$
has a neighborhood $E$ homeomorphic to $U_c/P_c$, where $c \in \partial_\infty \text{int}(M)$, $U_c$ is a horoball centered at $c$ and $P_c \subset \Gamma$ is a discrete group of parabolic isometries fixing $c$.

$\pi_1$-injective boundary

In the proof of Proposition 2 and Theorem 3 we will use that $\pi_1 \partial_i M \to \pi_1 M$ is injective for each path-component $\partial_i M$ of $\partial M$. We are going to explain how this fact follows from well-known properties of negatively curved manifolds.

**Observation 1.** Let $M$ be a compact manifold with boundary $\partial M$. If $\text{int}(M)$ admits a Riemannian metric of finite volume such that the universal covering $\tilde{M}$ with the pull-back metric is a visibility manifold of nonpositive sectional curvature, then $\pi_1 \partial_i M \to \pi_1 M$ is injective for each path-component $\partial_i M$ of $\partial M$.

**Proof:** $M = \tilde{M}/\Gamma$ for a group of isometries $\Gamma \subset \text{Isom}(\tilde{M})$ with $\Gamma \cong \pi_1 M$.

By the proof of [12], Theorem 3.1, we have that each end of $M$ is of the form $N \times (0, \infty)$ and that $N \times \{0\} = L_c/P_c$ for a simply connected horosphere $L_c$ and a subgroup $P_c \subset \Gamma$ that acts freely and properly discontinuously on $L_c$.

In particular, the boundary component $\partial_i M$ that corresponds to this end is homeomorphic to $N$ and its fundamental group is isomorphic to $P_c$. We have $N = L_c \subset \tilde{M}$, which implies that $\pi_1 N \to \pi_1 M$ is injective because, by covering theory, $N$ resp. $\tilde{M}$ are the set of homotopy classes (rel. $\{0, 1\}$) of paths $\gamma : [0, 1] \to N$ resp. $M$ with $\gamma(0) = x_0$, for a fixed point $x_0 \in N \subset M$.

QED

Moreover $M$ and all $\partial_i M$ are aspherical by the Cartan-Hadamard Theorem resp. by [12].

**Relative classifying spaces.**

First, we briefly discuss the approach via relative classifying spaces, which works exactly as in [9]. Let $M$ be a compact $d$-manifold with boundary such that $\text{int}(M) = M - \partial M$ admits a locally symmetric Riemannian metric of finite volume. Let $M_+$ be the quotient space obtained by identifying points in respectively each boundary component. In particular $H_d(M_+)$ has a fundamental class.

(Remark: In [9], $M_+$ is the one-point compactification. This is not homeomorphic to our $M_+$, but has the same homology in degrees $\geq 2$.)

Let $P \subset G$ and $B \subset SL(N; \mathbb{C})$ be maximal unipotent subgroups, such that $\rho : (G, K) \to (SL(N, \mathbb{C}), SU(N))$ sends $P$ to $B$. To a rank one locally symmetric space $\Gamma \backslash G/K$ of finite volume and dimension $n$, we can consider $G/K \cup C$, where $C$ denotes the set of parabolic fixed points in $\partial_\infty G/K$ and get as in [9], section 3, an (up to $\Gamma$-equivariant homotopy unique) $\Gamma$-equivariant map

$$G/K \cup C \to E(G, \mathcal{F}(P)).$$

The quotient of $G/K \cup C$ by $\Gamma$ is homeomorphic to $M_+$. In particular, $H_d(\Gamma \backslash (G/K \cup C))$ has a fundamental class, and we get as in [9] an element

$$\alpha(M) \in H_d(B(G, \mathcal{F}(P))).$$

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which by the representation \( \rho \) is pushed forward to an element

\[
\rho_*(\alpha(M)) \in H_d(B(SL(N,C), F(B))).
\]

In the case of hyperbolic 3-manifolds, Cisneros-Molina and Jones \([9]\) lifted this invariant to \( K(C) \otimes \mathbb{Q} \), and proved its nontriviality by relating it to the Bloch invariant. We describe here how to do a very similar construction for arbitrary locally symmetric spaces. (Unfortunately we did not succeed to compute the Borel regulator of this invariant. This is the reason why we will actually pursue another approach, using relative group homology and closer in spirit to \([15]\), in the remainder of this section. The construction is however included at this point because its main step, Lemma 4, will be crucial for the proof of Proposition 2.)

**Generalized Cisneros-Molina-Jones construction**

Let \( M \) be an aspherical \( d \)-manifold with aspherical boundary, \( F \subset C \) a subring and \( \rho : \pi_1 M \to SL(F) \) a representation\(^4\). To push forward the fundamental class \( [M_+] \in H_d(M_+; \mathbb{Q}) \) one would like to have a map \( R : M_+ \to BSL(N,F)^+ \) such that the following diagram commutes up to homotopy:

\[
\begin{array}{ccc}
M & \xrightarrow{B\rho} & M_+ \\
\downarrow & & \downarrow \\
BSL(F) & \xrightarrow{incl} & BSL(F)^+
\end{array}
\]

If this is the case, then one can use \( Q_* : H_d(BSL(N,F)^+; \mathbb{Q}) \to H_d(BSL(F); \mathbb{Q}) \) (the inverse of Quillen’s isomorphism) to define \( Q_* R_* [M_+] \in H_d(BSL(F); \mathbb{Q}) \) (and thus in the setting of Section 2.4 obtain an element in \( K_d(F) \otimes \mathbb{Q} \)).

**Lemma 5.** Let \( M \) be a manifold with boundary such that \( M \) and the path-components \( \partial_1 M, \ldots, \partial_s M \) of \( \partial M \) are aspherical. Let \( F \subset C \) a subring and \( \rho : \pi_1 M \to SL(N,F) \) be a representation such that \( \rho(\pi_1 \partial_i M) \) is unipotent for \( i = 1, \ldots, s \). Then there exists a continuous map \( R : M_+ \to BSL(N,F)^+ \) such that \( R \circ q = incl \circ B\rho \).

*Proof:* Let \( F \) be the homotopy fiber of \( BSL(N,F) \to BSL(N,F)^+ \). It is well-known (e.g. \([9]\), p.336) that \( \pi_1 F \) is isomorphic to the Steinberg group \( St(N;F) \). Let \( \Phi : St(N,F) \to SL(N,F) \) be the canonical homomorphism.

By assumption, \( \rho \) maps \( \pi_1 \partial_i M \) into some maximal unipotent subgroup \( B \subset SL(n,F) \) of parabolic elements. \( B \) is conjugate to \( B_0 \), the group of upper triangular matrices with all diagonal entries equal to 1. By \([30]\), Lemma 4.2.3, there exists a homomorphism \( \Pi : B_0 \to St(N,F) \) with \( \Phi \Pi = id \). Applying conjugations and composing with \( \rho \), we get a homomorphism \( \tau : \pi_1 \partial_i M \to St(N,F) \) such that \( \Phi \tau = \rho |_{\pi_1 \partial_i M} \).

\( \partial_1 M \) is aspherical, hence \( \tau \) is induced by some continuous mapping \( g_0 : \partial_1 M \to F \), and the diagram

\(^4\)Notation: We will denote by \( F \subset C \) an arbitrary subring, while \( A \subset C \) will denote a subring satisfying the assumptions of \([Lemma 2]\).
commutes up to some homotopy $H_t$.

This construction can be repeated for all connected components $\partial_1 M, \ldots, \partial_s M$ of $\partial M$. For each $r = 1, \ldots, s$ we get a continuous map $g_r : \partial_r M \to F$ such that $j g_r \sim (B \rho) i_r$.

Altogether, we get a continuous map $g : \partial M \to F$ such that $j g$ is homotopic to $(B \rho) i$.

By [9], Lemma 8.1. this implies the existence of the desired map $R$. QED

Hence one obtains an element $Q_* R_* [M_+] \in H_3(B SL(F); \mathbb{Q})$. Unfortunately we did not succeed to prove its nontriviality, i.e. to evaluate the Borel regulator. Therefore we will in the remainder of this section pursue a different approach, closer in spirit to [15], but surrounding the problem that $\partial M$ may be disconnected. We mention that another "basis-point independent" approach might use multicomplexes in the sense of Gromov, but also here we were not able to compute the Borel regulator in the case that there are 3 or more boundary components. Also, in the case of hyperbolic 3-manifolds, yet another approach is due to Neumann-Yang [28]. It should be interesting to generalize and compare the different constructions.

For hyperbolic 3-manifolds of finite volume, Zickert has given in [31] a direct construction of a fundamental class $[M, \partial M] \in H_3(SL(2, \mathbb{C}), B_0)$, even in the case of possibly disconnected boundary.

### 4.2 Cuspidal completion

We start with a notational remark. Let $X$ be a topological space and $A_1, \ldots, A_s \subset X$ a set of (not necessarily disjoint) subspaces. There is a (not necessarily injective) continuous mapping

$$i : \bigcup_{i=1}^{s} A_i \to X$$

from the disjoint union $A_1 \cup \ldots \cup A_s$ to $X$.

We define the disjoint cone

$$DCone(\bigcup_{i=1}^{s} A_i \to X)$$

to be the pushout of the diagram

$$\begin{array}{ccc}
A_1 \cup \ldots \cup A_s & \xrightarrow{i} & X \\
\downarrow & & \downarrow \\
Cone(A_1) \cup \ldots \cup Cone(A_s) & \xrightarrow{Dcone} & DCone(\bigcup_{i=1}^{s} A_i \to X)
\end{array}$$

If $X$ is a CW-complex and $A_1, \ldots, A_s$ are disjoint sub-CW-complexes, then clearly

$$H_*(DCone(\bigcup_{i=1}^{s} A_i \to X)) \cong H_*(Cone(\bigcup_{i=1}^{s} A_i \to X)) = H_*(X, \bigcup_{i=1}^{s} A_i)$$

in degrees $* \geq 2$. 29
A special case is that of a compact manifold $M$ with disconnected boundary $\partial M$, consisting of path-components $\partial_1 M \cup \ldots \cup \partial_s M$. Then $D\text{Cone} (\bigcup_{i=1}^s \partial_i M \to M)$ is the space $M_+$ from the Cisneros-Molina-Jones construction. (In this case, the union of components is a disjoint union. Nonetheless $D\text{Cone}$ is different from $\text{Cone}$.)

Another special case is the cuspidal completion of a classifying space, whose geometry will be described in this section. The point of the construction is that it may remember the geometry of the cusps of locally symmetric spaces. Thus it serves as a technical device to handle the cusped case.

**Construction of $BG^{\text{comp}}$:**

We recall from the beginning of section 2.2 that $BG$ is the simplicial complex realizing the bar construction. Thus its $n$-simplices are of the form $(1, g_1, \ldots, g_n)$ with $g_1, \ldots, g_n \in G$.

**Definition 4.** Let $G/K$ be a symmetric space of noncompact type. We define the cuspidal completion $BG^{\text{comp}}$ of $BG$ to be the pushout of the following diagram:

$$
\begin{array}{ccc}
\bigcup_{c \in \partial_\infty G/K} BG & \rightarrow & BG \\
\downarrow & & \downarrow \\
\bigcup_{c \in \partial_\infty G/K} \text{Cone}(BG) & \rightarrow & BG^{\text{comp}}
\end{array}
$$

In other words, $BG^{\text{comp}}$ is the mapping cone $\text{Cone} \left( \bigcup_{c \in \partial_\infty G/K} BG \to BG \right)$, but with the union $\bigcup_{c \in \partial_\infty G/K} BG$ to be understood as a disjoint union.

Notation: The cone point of $\text{Cone}(BG)$ corresponding to $c \in \partial_\infty G/K$ will also be denoted $c$.

For the remainder of this section we assume some $\tilde{x} \in G/K$ to be fixed.

A **simplexwise Riemannian metric** on $\text{Int} \left( BG^{\text{comp}} \right) := BG^{\text{comp}} - \{ \text{cone points} \}$ is defined by identifying a $k$-simplex

$$
(1, g_1, \ldots, g_k) \in BG
$$

isometrically with the straight simplex

$$
\text{str} \left( \tilde{x}, g_1 \tilde{x}, \ldots, g_k \tilde{x} \right) \subset G/K
$$

---

4The mapping cone is a simplicial complex in the natural way: k-simplices of $BG^{\text{comp}}$ are
- either k-simplices in the target $BG$,
- or cones (with cone point $c \in \partial_\infty G/K$) over k-1-simplices in $BG$.

More generally, if $X$ is a simplicial complex and $A_1, \ldots, A_s$ are simplicial subcomplexes, then $D\text{Cone} \left( \bigcup_{i=1}^s A_i \to X \right)$ is a simplicial complex whose k-simplices are either k-simplices in $X$ or cones over k-1-simplices in some $A_i$. 

---

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resp. identifying 
\[(1, p_1, \ldots, p_{k-1}, c) \in \text{Cone} (BG)\]

isometrically with the straight ideal simplex 
\[
\text{str} (\tilde{x}, p_1 \tilde{x}, \ldots, p_1 \ldots p_{k-1} \tilde{x}, c) \subset G/K.
\]

This is compatible with the boundary operator defined in section 2.2: 
indeed \(\partial_0 (1, g_1, \ldots, g_k) = (1, g_2, \ldots, g_k)\) corresponds to \(\text{str} (\tilde{x}, g_2 \tilde{x}, \ldots, g_2 \ldots g_k \tilde{x})\) which is isometric to \(\text{str} (g_1 \tilde{x}, g_1 g_2 \tilde{x}, \ldots, g_1 \ldots g_k \tilde{x})\), and, for \(i \geq 1\), \(\partial_i (1, g_1, \ldots, g_k)\) corresponds to \(\text{str} (\tilde{x}, \ldots, \hat{g}_i \tilde{x}, \ldots, g_1 \ldots g_k \tilde{x})\). Similarly for \(\partial (1, p_1, \ldots, p_{k-1}, c)\).

Thus we obtain a well-defined simplex-wise Riemannian metric on the push-out. In particular, we have a simplex-wise local isometry 
\[p : \text{Int} (BG^{\text{comp}}) \to G/K.\]

The local isometry \(p\) defines a volume form \(p^* \text{dvol}\) on \(\text{Int} (BG^{\text{comp}})\). (Recall that a differential form \(\omega\) on a simplicial complex consists of a smooth form \(\omega_\sigma\) on every simplex \(\sigma\) such that \(\omega_\sigma |_\tau = \omega_\tau\) whenever \(\tau \subset \sigma\) is a subsimplex.) By Thom’s analogue of the deRham Theorem one has \(H^k_{\text{dR}} (BG^{\text{comp}}) = H^k_{\text{simp}} (BG^{\text{comp}})\), where \(H^k_{\text{dR}}\) is the cohomology of the complex of differential forms. The isomorphism is given by integration of differential forms over simplices. In particular, \(p^* \text{dvol}\) gives a simplicial cohomology class \(v_d \in H^d_{\text{simp}} (BG^{\text{comp}})\) represented by the cocycle \(\Delta \to \int_\Delta p^* \text{dvol}\). (This is defined because ideal \(d\)-simplices in a \(d\)-dimensional symmetric space \(G/K\) of noncompact type have finite volume, see e.g. [24], which provides even a uniform bound.) By construction, \(v_d |_{BG} = v_d\).

Recall that in section 2.3 we defined, for \(d = 2n - 1\) odd, the Borel class \(b_d \in H^d (GL (\mathbb{C}); \mathbb{R})\) which, of course, can also be considered as a class \(b_d \in H^d (BSL (N, \mathbb{C}); \mathbb{R})\).

Lemma 6. Let \(d, N \in \mathbb{N}\) with \(d\) odd.
There exists a simplicial complex \(BSL (N, \mathbb{C})^{fb}\) with 
\[BSL (N, \mathbb{C}) \subset BSL (N, \mathbb{C})^{fb} \subset BSL (N, \mathbb{C})^{\text{comp}}\]

and a homomorphism 
\[\overline{b}_d : C^\text{simp}_d (BSL (N, \mathbb{C})^{fb}; \mathbb{R}) \to \mathbb{R},\]

such that 
\[i) \overline{b}_d |_{C^\text{simp}_d (BSL (N, \mathbb{C}); \mathbb{R})} \text{ is a cocycle representing } b_d,\]
\[ii) \text{if } G/K \text{ is an } n\text{-dimensional symmetric space of noncompact type and } \rho : G \to SL (N, \mathbb{C}) \text{ a representation, then }\]
\[\rho \left( C^\text{simp}_d (BG^{\text{comp}}; \mathbb{R}) \right) \subset C^\text{simp}_d \left( BSL (N, \mathbb{C})^{fb}; \mathbb{R} \right)\]

and \(\rho^* \overline{b}_d\) represents \(c_\rho \overline{v}_d\). (In particular, \(\overline{b}_d\) is well-defined on \((\rho)_* H_d (BG^{\text{comp}}; \mathbb{R})\).)
Proof: By the van Est Theorem (section 2.3) there is an isomorphism \( I : H^d_c (SL (N, \mathbb{C})) \cong H^d (sl (N, \mathbb{C}), su (N)) \), where \( H^\ast (sl (N, \mathbb{C}), su (N)) \) is the cohomology of the complex of \( SL (N, \mathbb{C}) \)-invariant differential forms on \( SL (N, \mathbb{C})/SU (N) \). Let \( dvol \) be a differential form representing \( I (b_d) \). This means that a representative \( \beta_d \) of \( b_d \) is given by

\[
\beta_d (1, g_1, \ldots, g_d) := \int_{\str (\bar{x}, g_1 \bar{x}, \ldots, g_d \bar{x})} dvol
\]

for each simplex \( (1, g_1, \ldots, g_d) \in BSL (N, \mathbb{C}) \).

Since the van Est isomorphism is functorial, and \( \rho^\ast b_d = c_d v_d \), we have that \( \rho^\ast dvol - c_d dvol \) is an exact differential form. Moreover, \( \rho^\ast dvol \) and \( dvol \) are \( G \)-invariant differential forms on \( G/K \). Hence they are harmonic and \( \rho^\ast dvol - c_d dvol \) is an exact harmonic form, thus zero and we conclude

\[ \rho^\ast dvol = c_d dvol. \]

Define

\[ BSL (N, \mathbb{C})^{fb} \coloneqq BSL (N, \mathbb{C})_d \cup \bigcup_{c \in \partial_c SL (N, \mathbb{C})/SU (N)} \left\{ (1, p_1, \ldots, p_{d-1}, c) \in \cone (BSL (N, \mathbb{C})): \int_{(1, p_1, \ldots, p_{d-1}, c)} p^\ast dvol < \infty \right\}. \]

(Recall that \( \int_{(1, p_1, \ldots, p_{d-1}, c)} p^\ast dvol = \int_{\str (\bar{x}, p_1 \bar{x}, \ldots, p_{d-1} \bar{x}, c)} dvol \).)

This defines the \( d \)-simplices of \( BSL (N, \mathbb{C})^{fb} \) and we define \( BSL (N, \mathbb{C})^{fb} \) to be the simplicial complex generated by \( BSL (N, \mathbb{C})^{fb}_d \) under face maps.

Define \( \overline{d} : BSL (N, \mathbb{C})^{fb}_d \to \mathbb{R} \) by

\[ \overline{d} (\sigma) := \int_{\sigma} p^\ast dvol. \]

By construction, \( \overline{d} |_{C^{simp}_d (BSL (N, \mathbb{C}); \mathbb{R})} \) is a cocycle representing \( b_d \).

To prove \( b \), let \( z \in C^{simp}_d (BG^{comp}; \mathbb{R}) \). Since \( \rho^\ast dvol = c_d dvol \), we have

\[ \int_{\rho(z)} dvol = c_d \int_{z} dvol. \]

In particular \( \rho \left( C^{simp}_d (BG^{comp}; \mathbb{R}) \right) \subset C^{simp}_d \left( BSL (N, \mathbb{C})^{fb}; \mathbb{R} \right) \) and \( \rho^\ast b_d \) represents \( c_d \overline{d} \).

QED

4.3 Construction of \( \pi (M) \)

For a manifold \( M \), with \( \pi_1 \)-injective boundary \( \partial M \) consisting of path-components \( \partial_1 M, \ldots, \partial_s M \), let \( \Gamma_i := \pi_1 (\partial_i M, x_i) \) for \( i = 1, \ldots, s \) and some \( x_i \in \partial_i M \), and fix (using some path from \( x_0 \) to \( x_i \)) isomorphisms of \( \Gamma_i \) with subgroups of \( \Gamma = \pi_1 (M, x_0) \), for \( i = 1, \ldots, s \).

Moreover, if \( M \) and all \( \partial_i M \) are aspherical, then the classifying map induces an isomorphism \( H_\ast (Dcone (\bigcup_{i=1}^s \partial_i M \to M)) \to H_\ast (Dcone (\bigcup_{i=1}^s \partial \Gamma_i \to \partial \Gamma)) \).
Proposition 2. Let $M$ be a compact, oriented manifold with boundary components $\partial_1 M, \ldots, \partial_s M$ such that $\text{Int}(M) = \Gamma \setminus G/K$ is a finite-volume locally symmetric space of noncompact type. Assume that, for some subring $\mathbb{F} \subset \mathbb{C}$, we have an inclusion

$$j : (\Gamma, \Gamma_i) \to (G(\mathbb{F}), \Gamma_i)$$

of $\Gamma = \pi_1 M, \Gamma_i = \pi_1 \partial_i M \subset \pi_1 M$. Let

$$\rho : G(\mathbb{F}) \to SL(N, \mathbb{F})$$

be a representation, such that $\Gamma_i' := \rho(\Gamma_i)$ is unipotent for $i = 1, \ldots, s$.

Denote $[M, \partial M] \in H_d(D\text{Cone}(\bigcup_{i=1}^s \partial_i M \to M) ; \mathbb{Q}) \cong H_d(D\text{Cone}(\bigcup_{i=1}^s B\Gamma_i \to B\Gamma) ; \mathbb{Q})$ the fundamental class of $M$. Then

$$(B\rho)_d j_d [M, \partial M] \in H_d(D\text{Cone}(\bigcup_{i=1}^s B\Gamma_i' \to BSL(\mathbb{F})) ; \mathbb{Q})$$

has a preimage

$$\pi(M) \in H_d(BSL(\mathbb{F}) ; \mathbb{Q}).$$

Proof: First we notice that it suffices to prove that $(B\rho)_{d-1} j_{d-1} [\partial_i M] = 0 \in H_{d-1}(B\Gamma_i ; \mathbb{Q})$ for $i = 1, \ldots, s$. Indeed, if $z$ is a relative cycle that represents $[M, \partial M]$, and for $i = 1, \ldots, s$ we have chains $z_i \in C_d(B\Gamma_i)$

$$\partial z_i = (B\rho)_{d-1} j_{d-1} (\partial z |_{\partial_i M}),$$

then

$$(B\rho)_d j_d z - \sum_{i=1}^s z_i \in C_d(BSL(\mathbb{F}))$$

is a cycle, whose image in $C_d(D\text{Cone}(\bigcup_{i=1}^s B\Gamma_i \to BSL(\mathbb{F})))$ again represents $(B\rho)_d j_d [M, \partial M]$. Thus $(B\rho)_d j_d z - \sum_{i=1}^s z_i$ represents the desired $\pi(M)$.

To prove $(B\rho)_{d-1} j_{d-1} [\partial_i M] = 0$, let $f : \partial M \to M$ be the inclusion, $g : M \to M_+$ the projection. Thus $gf$ is constant on each boundary component and the induced map $(gf)_*$ in homology is trivial. Since $\rho(\Gamma_i)$ is unipotent, we can apply Lemma 5 and obtain a continuous map

$$R : M_+ \to BSL^+(\mathbb{F})$$

such that

$$R \circ g \circ f = \text{incl} \circ B\rho \circ f.$$ 

Since $\text{incl}_* : H_d(BSL(\mathbb{F}) ; \mathbb{Q}) \to H_d(BSL(\mathbb{F})^+ ; \mathbb{Q})$ is an isomorphism and $(gf)_* = 0$, this implies

$$(B\rho)_* f_* = 0,$$

in particular

$$(B\rho)_{d-1} j_{d-1} [\partial_i M] = 0 \in H_{d-1}(B\Gamma_i ; \mathbb{Q})$$

for $i = 1, \ldots, s$. \hfill QED

Remark: For the special case of hyperbolic manifolds and half-spinor representations, Proposition 2 was proved in [15], Theorem 2.12. The proof in [15] uses very special properties of the half-spinor representations and seems not to generalize to other representations.
4.4 Straightening of interior and ideal simplices

Definition 5. For a manifold $M$ with boundary components $\partial_1 M, \ldots, \partial_s M$, we denote

$$\hat{C}_s (M) := C_+ (DCone (\cup_{i=1}^s \partial_i M \to M)).$$

A vertex in $DCone (\cup_{i=1}^s \partial_i M \to M)$ is an ideal vertex, if it is in one of the cone points, and an interior vertex else. For fixed $x_0 \in M$, $x_i \in \partial_i M$, and a fixed identification of $\pi_1 (\partial_i M, x_i)$ with a subgroup of $\pi_1 (M, x_0)$, let

$$\hat{C}_{s_0} (M) \subset \hat{C}_s (M)$$

be the subcomplex freely generated by those simplices for which
- either all vertices are in $x_0$,
- or the last vertex is an ideal vertex, corresponding to some boundary component $\partial_i M$, all other vertices are in $x_0$, and the homotopy classes of all edges between interior vertices belong to the image of $\pi_1 (\partial_i M, x_i)$ in $\pi_1 (M, x_0)$.

Remark : If the universal cover $\hat{int}(M)$ is a visibility manifold ([12]) of nonpositive sectional curvature and $int(M)$ has finite volume, then each end of $int(M) = M - \partial M$ has a neighborhood $E$ homeomorphic to $U_c / P_c$, where $c \in \partial_\infty int(M)$. $U_c$ is a horoball centered at $c$ and $P_c \subset \Gamma$ is a discrete group of parabolic isometries fixing $c$. (See [12], Theorem 3.1.)

In particular, since each boundary component $\partial_i M$ corresponds to an end $E_i$ of $int(M)$, we conclude that each boundary component $\partial_i M$ corresponds to a unique $\Gamma$-orbit $\Gamma c_i$ with $c_i \in \partial_\infty \hat{M}$, and that $E_i \cup \{ \Gamma c_i \}$ is homeomorphic to $Cone (\partial_i M)$. Since $int(M) - \cup_{i=1}^s E_i$ is homeomorphic to $M$, this means that we have a projection

$$\pi : \hat{int}(M) \bigcup_{\gamma \in \Gamma} \cup_{i=1}^s \{ \gamma c_i \} \to DCone (\cup_{i=1}^s \partial_i M \to M)$$

with $\pi \left( \hat{int}(M) - \cup_{i=1}^s \Gamma U_{c_i} \right) = int(M) - \cup_{i=1}^s E_i$ and $\pi \left( \Gamma U_{c_i} \right) = Cone (\partial_i M)$ for all $\gamma \in \Gamma$ and $i = 1, \ldots, s$.

A simplex in $DCone (\cup_{i=1}^s \partial_i M \to M)$ is said to be straight if some (hence any) lift to $\hat{M} \cup \partial_\infty \hat{M}$ is straight.

Definition 6. Let $M$ be a manifold with boundary $\partial M$ such that $int(M)$ is a visibility manifold. Let $x_0 \in \hat{M}$. Then we define

$$\hat{C}_{s, str, x_0} (M) := \mathbb{Z} \left[ \{ \sigma \in \hat{C}_{s_0} (M) : \sigma \text{ straight} \} \right]$$

to be the subcomplex generated by the straight simplices.

Recall that $DCone (\cup_{i=1}^s B\Gamma_i \to B\Gamma)$ is a simplicial complex as defined in the footnote in section 4.2.

Lemma 7. Let $M$ be a manifold with boundary $\partial M$ with a Riemannian metric such that $int(M)$ is a visibility manifold of nonpositive curvature. Let $x_0 \in M$, $\Gamma := \pi_1 (M, x_0)$
and \( \Gamma_i := \pi_1(\partial_i M, x_i) \) for the boundary components \( \partial_i M \) of \( \partial M \) and some \( x_i \in \partial_i M \). Fix (using some path from \( x_0 \) to \( x_i \)) isomorphisms of \( \Gamma_i \) with subgroups of \( \Gamma \). Then there is an isomorphism of chain complexes

\[
\hat{C}_*^{str,x_0}(M) \cong \mathcal{C}_*^{simp}(D\text{Cone}(\bigcup_{i=1}^n B\Gamma_i \to B\Gamma)).
\]

\textbf{Proof:} \( \text{int}(\hat{M}) \) is a simply connected manifold of nonpositive curvature. Hence a straight (possibly ideal) simplex is uniquely determined by its ordered set of vertices. Since each straight simplex in \( \hat{M} \) can be lifted to a straight simplex in \( \hat{M} \), this implies:

- each interior simplex \( \sigma \in \hat{C}_*^{str,x_0}(M) \) is uniquely determined by the homotopy classes of the \( k \) edges \( \gamma_j \) between its vertices \( \sigma(v_{j-1}) \) and \( \sigma(v_j) \) for \( j = 1, \ldots, k \). That is, the interior simplices generate a subcomplex in \( \hat{C}_*^{str,x_0}(M) \) that is isomorphic, as a chain complex, to \( \mathcal{C}_*^{simp}(B\Gamma) \). The bijection \( \Phi_1 \) is given by

\[
\sigma \mapsto (1, \gamma_1, \ldots, \gamma_k),
\]

where \( \gamma_i \in \Gamma = \pi_1(M, x_0) \) is the homotopy class of the edge from \( \sigma(v_{j-1}) \) to \( \sigma(v_j) \), for \( j = 1, \ldots, k \). It is easy to check that the bijection \( \Phi_1 \) is a chain map.

- each ideal simplex \( \sigma \in \hat{C}_*^{str,x_0}(M) \) is uniquely determined by the choice of a boundary component \( \partial_i M \) (which corresponds to the \( \Gamma \)-orbit \( \Gamma c_i \) of a parabolic fixed point \( c_i \)) and by the homotopy classes of interior edges from \( \sigma(v_{j-1}) \) to \( \sigma(v_j) \) for \( j = 1, \ldots, k-1 \). By definition, the interior edges belong to \( \Gamma_i \). Thus, for \( i = 1, \ldots, s \), ideal simplices with ideal vertex at \( \Gamma c_i \) generate a subcomplex in \( \hat{C}_*^{str,x_0}(M) \) that is isomorphic to \( \mathcal{C}_*^{simp}(\text{Cone}(B\Gamma_i)) \subset \mathcal{C}_*^{simp}(\text{Cone}(B\Gamma_i \to B\Gamma)) \). The bijection \( \Phi_2 \) is given by

\[
\sigma \mapsto (1, \gamma_1, \ldots, \gamma_{k-1}, c_i),
\]

where \( \gamma_j \in \Gamma_i \subset \Gamma = \pi_1(M, x_0) \) is the homotopy class of the edge from \( \sigma(v_{j-1}) \) to \( \sigma(v_j) \), for \( j = 1, \ldots, k-1 \). It is easy to check that the bijection \( \Phi_2 \) is a chain map.

The two chain isomorphisms \( \Phi_1 \) and \( \Phi_2 \) are compatible: if \( \sigma \) is an ideal \( k \)-simplex, then \( \partial_k \sigma \) is an interior simplex and we have by construction:

\[
\Phi_1(\partial_k \sigma) = \partial_k \Phi_2(\sigma).
\]

Thus \( \Phi_1 \) and \( \Phi_2 \) define a chain isomorphism

\[
\Phi : \hat{C}_*^{str,x_0}(M) \cong \mathcal{C}_*^{simp}(D\text{Cone}(\bigcup_{i=1}^n B\Gamma_i \to B\Gamma)).
\]

\( \text{QED} \)

\subsection{4.5 Evaluation of Borel regulator}

\textbf{Theorem 3.} a) Let \( M \) be a compact, oriented \( 2n-1 \)-manifold with boundary components \( \partial_1 M, \ldots, \partial_s M \) such that \( \text{Int}(M) \) is a finite-volume rank-one locally symmetric space of noncompact type: \( \text{Int}(M) = \Gamma \backslash G / K \). Let \( \rho : G \to \text{GL}(N, \mathbb{C}) \) be a representation and let \( c_\rho \) be defined by \textbf{Theorem 2}. Let

\[
\overline{\gamma}(M) \in H_{2n-1}(\text{BSL}(\overline{\mathbb{Q}}), \mathbb{Q})
\]
be defined by Proposition 2, let \( \gamma(M) \) be the image of \( \tau(M) \) in \( H_{2n-1} \left( BGL(\mathbb{Q}), \mathbb{Q} \right) \), and define 
\[
\gamma(M) := pr_{2n-1} \left( \gamma(M) \right) \in PH_{2n-1} \left( BGL(\mathbb{Q}), \mathbb{Q} \right) \cong K_{2n-1} (\mathbb{Q}) \otimes \mathbb{Q},
\]
where \( pr_{2n-1} \) is defined in Corollary 2. Then 
\[
< b_{2n-1}, \gamma(M) > = c_p \text{vol} (M).
\]

b) If \( A \subset \mathbb{C} \) satisfies the assumption of Lemma 2, and we have an inclusion \( j : \Gamma \to G(A) \), and if 
\[
\gamma(M) := pr_{2n-1} \left( \gamma(M) \right) \in PH_{2n-1} \left( BGL(A), \mathbb{Q} \right) \cong K_{2n-1} (A) \otimes \mathbb{Q},
\]
where \( \gamma(M) \) is the image of \( \tau(M) \) in \( H_{2n-1} \left( BSL(A), \mathbb{Q} \right) \) (defined by Proposition 2) in \( H_{2n-1} \left( BGL(A), \mathbb{Q} \right) \), and \( pr_{2n-1} \) is given by Lemma 2, then 
\[
< b_{2n-1}, \gamma(M) > = c_p \text{vol} (M).
\]

**Proof:** Denote \( d = 2n - 1 \).

\( G \) is a linear semisimple Lie group without compact factors, not locally isomorphic to \( SL(2, \mathbb{R}) \). By Weil rigidity we can assume (upon conjugation) that \( \Gamma \subset G(\mathbb{Q}) \). For a semisimple Lie group \( G \), each representation \( \rho : G \to GL(N, \mathbb{C}) \) is isomorphic to a representation which maps \( G(A) \) to \( SL(N, A) \). (This can be read off the classification of representations of semisimple Lie groups, see \[14\].) By Corollary 2, \( A = \mathbb{Q} \) satisfies the assumptions of Lemma 2. Thus a) is a consequence of b). We are going to prove b).

Let 
\[
\sum_{i=1}^{r} \tau_i
\]
be some triangulation of \( (M, \partial M) \). Let 
\[
\sum_{j=1}^{p} \kappa_j := DCone \left( \partial \left( \sum_{i=1}^{r} \tau_i \right) \right)
\]
be the cone over the induced triangulations of \( \partial_1 M, \ldots, \partial_s M \), with one cone point for each path-component of \( \partial M \). Thus 
\[
\sum_{i=1}^{r} \tau_i + \sum_{j=1}^{p} \kappa_j
\]
is a triangulation of \( DCone \left( \bigcup_{i=1}^{s} \partial_i M \to M \right) \).

Let \( \pi : G/K = \tilde{M} \to M \) be the covering map. Fix a point \( x_0 \in M \), an identification \( \Gamma \cong \pi_1(M, x_0) \), and a lift \( \tilde{x}_0 \in \tilde{M} \) of \( x_0 \in M \).

There is a homeomorphism 
\[
DCone \left( \bigcup_{i=1}^{s} \partial_i M \to M \right) \cong \Gamma \backslash G/K \cup \{C_1, \ldots, C_s\},
\]

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where \(C_1, \ldots, C_s\) are points ("cusp points") corresponding to the path-components \(\partial_1 M, \ldots, \partial_s M\) of \(\partial M\). For \(l = 1, \ldots, s\) fix some \(x_l \in \partial_l M\). Fix (using some path from \(x_0\) to \(x_l\)) isomorphisms of \(\pi_1(\partial_l M, x_l)\) with subgroups \(\Gamma_l \subset \pi_1(M, x_0)\).

\[
\sum_{i=1}^{r} \tau_i + \sum_{j=1}^{p} \kappa_j \text{ is an (ideal) triangulation of } \Gamma \setminus G/K \cup \{C_1, \ldots, C_s\}, \text{ therefore }
\]

\[
\text{vol}(M) = \text{vol}(\Gamma \setminus G/K) = \sum_{i=1}^{r} \text{vol}(\tau_i) + \sum_{j=1}^{p} \text{vol}(\kappa_j),
\]

where volume is computed by integration of the volume form (and defining the volume of \(\{C_1, \ldots, C_s\}\) to be zero).

We homotope the cycle \(\sum_{i=1}^{r} \tau_i + \sum_{j=1}^{p} \kappa_j\) into a cycle \(\sum_{i=1}^{r} \tau'_i + \sum_{j=1}^{p} \kappa'_j\) such that all interior vertices are homotoped into \(x_0\), all ideal vertices remain fixed during the homotopy, and all edges between interior vertices of \(\kappa'_j\) are homotoped into loops representing elements of \(\Gamma_l\). That is \(\tau'_1, \ldots, \tau'_r, \kappa'_1, \ldots, \kappa'_p \in C^x \omega_0(M)\).

\(G/K\) has negative sectional curvature. In particular it is a visibility manifold and we have the subcomplex \(\tilde{C}_{str,x_0}^{\ast}(M) \subset \tilde{C}_{x_0}^{\ast}(M)\) as defined by [Definition 6].

By \[12\], Theorem 3.1, some neighborhood of \(C_l\), for \(l \in \{1, \ldots, s\}\), is homeomorphic to \(U_{c_l}/P_{c_l}\), where \(c_l \in \partial_{\infty} G/K\), \(U_{c_l}\) is a horoball centered at \(c_l\) and \(P_{c_l} \subset \Gamma\) is a discrete group of parabolic isometries fixing \(c_l\). In particular, each ideal simplex with an ideal vertex at \(C_l\) lifts to an ideal simplex in \(G/K \cup \partial_{\infty} G/K\) with an ideal vertex at \(c_l\).

Literally the same proof as for \[2\], Lemma C.4.3 (where the case of hyperbolic manifolds is considered) shows that the cycle \(\sum_{i=1}^{r} \tau'_i + \sum_{j=1}^{p} \kappa'_j\) is homotopic (rel. vertices) to a cycle \(\sum_{i=1}^{r} \text{str}(\tau'_i) + \sum_{j=1}^{p} \text{str}(\kappa'_j)\), such that \(\text{str}(\tau'_1), \ldots, \text{str}(\tau'_r), \text{str}(\kappa'_1), \ldots, \text{str}(\kappa'_p)\) are straight simplices.

By the homotopy axiom, both homotopies do not change the homology class. By Stokes Theorem we have

\[
\text{vol}(M) = \sum_{i=1}^{r} \text{vol}(\text{str}(\tau'_i)) + \sum_{j=1}^{p} \text{vol}(\text{str}(\kappa'_j)).
\]

By section 4.2, we have a simplex-wise defined map

\[
p : BG^{\text{comp}} \to G/K \cup \partial_{\infty} G/K,
\]

whose restriction to \(BG^{\text{comp}} - \{\text{cone points}\}\) is a local isometry.

Let \(dvol \in \Omega^d(M)\) be the volume form of \(M\). Then \(\pi^*dvol\) is the volume form of \(G/K\) and, by section 4.2, the pull-back \(p^*(\pi^*dvol) \in \Omega^d(BG^{\text{comp}})\) is a differential form, which realizes the simplicial cocycle \(\overline{v}_d\).

Let \(w_0, \ldots, w_d\) be the vertices of the standard simplex \(\Delta^d\). By [Lemma 7] the isomorphism

\[
\tilde{C}_{str,x_0}^{\ast}(M) \cong C_{\sim}^{\text{simp}}(DCone(\cup_{i=1}^{s} B\Gamma_i \to B\Gamma))
\]

maps the interior simplex \(\text{str}(\tau'_i)\) to

\[
(1, \gamma^i_1, \ldots, \gamma^i_d) \in B\Gamma,
\]

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where $\gamma_i^j \in \Gamma$ is the homotopy class of the (closed) edge from $\tau_i^j(w_{k-1})$ to $\tau_i^j(w_k)$, and the ideal simplex $\text{str}(\kappa^j_i)$ to

$$\left(1, p_1^j, \dots, p_{d-1}^j, c_k^j\right) \in \text{Cone}(B\Gamma_k \to B\Gamma),$$

where $c_k^j \in \partial G/K$ and $p_l^j$ is the homotopy class of the (closed) edge from $\kappa^j_i(w_{l-1})$ to $\kappa^j_i(w_l)$. Thus

$$j_*[M, \partial M] \in H_d(BG^{\text{comp}}; \mathbb{Q})$$

is represented by

$$\sum_{i=1}^r (1, \gamma_1^i, \dots, \gamma_d^i) + \sum_{j=1}^p \left(1, p_1^j, \dots, p_{d-1}^j, c_j^i\right).$$

Let $\text{str}(\tilde{x}, \gamma_1^i \tilde{x}, \dots, \gamma_d^i \tilde{x}) \in C_\text{str}(G/K)$ be the unique straight simplex with vertices $\tilde{x}, \gamma_1^i \tilde{x}, \dots, \gamma_d^i \tilde{x}$, and $\text{str}(\tilde{x}, p_1^j \tilde{x}, \dots, p_{d-1}^j \tilde{x}, c_j^i)$ the unique ideal straight simplex with interior vertices $\tilde{x}, p_1^j \tilde{x}, \dots, p_{d-1}^j \tilde{x}$ and ideal vertex $c_j^i$.

By construction we have

$$\pi p \left(1, \gamma_1^i, \dots, \gamma_d^i\right) = \pi \left(\text{str}(\tilde{x}, \gamma_1^i \tilde{x}, \dots, \gamma_d^i \tilde{x})\right) = \text{str}(\tau_i^j),$$

$$\pi p \left(1, p_1^j, \dots, p_{d-1}^j, c_j^i\right) = \pi \left(\text{str}(\tilde{x}, p_1^j \tilde{x}, \dots, p_{d-1}^j \tilde{x}, c_j^i)\right) = \text{str}(\kappa_i^j).$$

Hence

$$\int_{(1, \gamma_1^i, \dots, \gamma_d^i)} p^* \text{dvol} = \int_{\text{str}(\tilde{x}, \gamma_1^i \tilde{x}, \dots, \gamma_d^i \tilde{x})} \pi^* \text{dvol} = \int_{\text{str}(\tau_i^j)} \text{dvol} = \text{vol}(\text{str}(\tau_i^j))$$

and

$$\int_{(1, p_1^j, \dots, p_{d-1}^j, c_j^i)} p^* \text{dvol} = \int_{\text{str}(\tilde{x}, p_1^j \tilde{x}, \dots, p_{d-1}^j \tilde{x}, c_j^i)} \pi^* \text{dvol} = \int_{\text{str}(\kappa_i^j)} \text{dvol} = \text{vol}(\text{str}(\kappa_i^j)).$$

Since $p^* \text{dvol}$ represents the cohomology class $\overline{\nu}_d$, this implies

$$<\overline{\nu}_d, \sum_{i=1}^r (1, \gamma_1^i, \dots, \gamma_d^i) + \sum_{j=1}^p \left(1, p_1^j, \dots, p_{d-1}^j, c_j^i\right)> = \sum_{i=1}^r \text{vol}(\text{str}(\tau_i^j)) + \sum_{j=1}^p \text{vol}(\text{str}(\kappa_i^j)) = \text{vol}(M).$$

$\Gamma_i \subset G$ consists of parabolic isometries with the same fixed point in $\partial_\infty G/K$ ([12], Theorem 3.1), thus $\rho(\Gamma_i) \subset SL(N, \mathbb{C})$ is unipotent. By Proposition 2 the image of

$$\overline{\nu}(M) \in H_d(BSL(N, A); \mathbb{Q}),$$

in $H_d(D\text{Cone}(\bigcup_{i=1}^s B\Gamma_i \rightarrow BSL(N, A); \mathbb{Q}))$ equals $B(\rho j)_d [M, \partial M]$.

By Lemma 6 there is $\overline{b}_d : C_d(BSL(N, \mathbb{C})^{\mathbb{R}}) \rightarrow \mathbb{R}$ such that $\overline{b}_d \mid_{C_d(BSL(N, \mathbb{C})^{\mathbb{R}})}$ represents $b_d$ and $\rho^* \overline{b}_d$ represents $c_p \overline{\nu}_d$. (In particular, $\overline{b}_d$ is well-defined on $(B\rho)_d H_d(BG^{\text{comp}}; \mathbb{R})$.)

Then we have

$$<\overline{b}_d, B(\rho j)_d [M, \partial M]>.$$
By Lemma 2 this implies $<\rho^*\bar{b}_d, j_d [M, \partial M] > = c_\rho < \nabla_d, j_d [M, \partial M] > = c_\rho \text{vol} (M)$.

Thus, with $i : BSL(N, \mathbb{C}) \to BSL(N, \mathbb{C})^{\text{comp}}$ the inclusion, and confusing $\gamma(M) \in H_d(\text{BSL}(A), \mathbb{Q})$ with its image in $H_d(\text{BSL}(\mathbb{C}), \mathbb{Q})$, we have

$$< b_d, \gamma(M) > = < i^* \overline{b}_d, \gamma(M) >$$

$$= < \overline{b}_d, i_d \gamma(M) >= < \overline{b}_d, B(\rho)_d [M, \partial M] > = c_\rho \text{vol} (M).$$

By Lemma 2 this implies $< b_d, \gamma(M) > = c_\rho \text{vol} (M)$. QED

Remark: The same argument would work for any relative fundamental cycle, not necessarily coming from a triangulation.

4.6 Examples

4.6.1 Examples from hyperbolic manifolds

The case of hyperbolic 3-manifolds has been discussed to some extent in [28].

If $M$ is any hyperbolic 3-manifold of finite volume, then $\pi_1 M$ can be conjugated to be contained in $SL(2, F)$, where $F$ is an at most quadratic extension of the trace field ([25]), thus one gets an element in $K_3(F) \otimes \mathbb{Q}$. In [28], section 9, some examples of this construction are given. (The discussion in [25] is about elements in $B(F) \otimes \mathbb{Q}$ for the Bloch group $B(F)$, but of course analogously one is getting elements in $K_3(F) \otimes \mathbb{Q}$ associated to the respective manifolds.)

For example (cf. [28], section 9.4) for any number field $F$ with just one complex place there exists a hyperbolic 3-manifold of finite volume, such that its invariant trace field equals $F$. The associated $\gamma(M)$ gives a nontrivial element, and actually a generator, in $K_3(F) \otimes \mathbb{Q}$.

4.6.2 Representation varieties

Let $M$ be a compact $d$-manifold with boundary. If $\psi : \pi_1 M \rightarrow G$ is a homomorphism, one can construct a $\psi$-equivariant map $f : M \rightarrow G/K$ which is unique up to homotopy. In particular, $\text{vol} (\psi) := \int_F f^* \text{vol}$, for a fundamental domain $F \subset \hat{M}$, is well-defined. If $\psi$ preserves parabolics, i.e. $\psi(\pi_1 \partial M)$ is unipotent for all components $\partial M \subset \partial M$, then literally the same argument as in the proof of Theorem 3 shows

$$< v_d, (B\psi)_d [M] >= \text{vol} (\psi).$$

Thus, if $\rho : G \rightarrow GL(N, \mathbb{C})$ is a representation with $\rho^* b_d \neq 0$, one does again get a nontrivial element $\gamma(\psi) := I_d^{-1} \rho (B\psi)_d [M, \partial M] \in K_d(\mathbb{C}) \otimes \mathbb{Q}$. Of course, continuous families of parabolics-preserving representations give us constant images in K-theory, because already $(B\psi)_d [M] \in H_d(D\text{Cone} (\hat{U}_i \rightarrow BG) ; \mathbb{Z})$ is constant. We note that the map is however not constant on the variety of parabolics-preserving representations. This follows, for example, from the volume rigidity theorem (which for hyperbolic manifolds has been proved by Thurston and Dunfield and in the higher rank case is a consequence of Margulis superrigidity theorem) which states that elements of the component of $\text{Rep}(\pi_1 M, G)$ that contains the discrete representation are the only representations of maximal volume.
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Thilo Kuessner
Mathematisches Institut, Universität Münster
Einsteinstraße 62
D-48149 Münster
Germany
e-mail: kuessner@math.uni-muenster.de