GEOMETRIC AND COMBINATORIAL ASPECTS OF SUBMONOIDS OF A FINITE-RANK FREE COMMUTATIVE MONOID

FELIX GOTTI

Abstract. Every torsion-free atomic monoid $M$ can be embedded into a real vector space via the inclusion $M \hookrightarrow \text{gp}(M) \hookrightarrow \mathbb{R} \otimes_{\mathbb{Z}} \text{gp}(M)$, where $\text{gp}(M)$ is the Grothendieck group of $M$. Let $C$ be the class consisting of all submonoids (up to isomorphism) that can be embedded in a finite-rank free commutative monoid. Here we investigate how the atomic structure and factorization properties of members of $C$ reflect in the combinatorics and geometry of their conic hulls $\text{cone}(M) \subseteq \mathbb{R} \otimes_{\mathbb{Z}} \text{gp}(M)$. First, we establish geometric characterizations in terms of $\text{cone}(M)$ for a monoid $M$ in $C$ to be factorial, half-factorial, and other-half-factorial. Then we show that the submonoids of $M$ determined by the faces of $\text{cone}(M)$ amount for all divisor-closed submonoids of $M$. Finally, we investigate the cones of finitary, primary, finitely primary, and strongly primary monoids in $C$ (monoids in these classes have been relevant in the development of factorization theory). Along the way, we study the cones that can be realized by monoids in $C$ and by finitary monoids in $C$.

Contents

1. Introduction 2
2. Atomic Monoids and Convex Cones 4
   2.1. General Notation 4
   2.2. Atomic Monoids 4
   2.3. Convex Cones 6
3. Monoids in $C$ 7
   3.1. The Class $C$ 7
   3.2. The Cones of Monoids in $C$ 10
   3.3. Cones Realized by Monoids in $C$ 12
4. Faces and Divisor-Closed Submonoids 14
   4.1. Face Submonoids 14
   4.2. Characterization of Face Submonoids 15
5. Geometry and Factoriality 16
   5.1. Unique Factorization Monoids 16
   5.2. Half-Factorial Monoids 17

Date: July 2, 2019.

2010 Mathematics Subject Classification. Primary: 20M14; Secondary: 51M20, 20M13.

Key words and phrases. free abelian monoid, positive convex cone, finitary monoid, weakly finitary monoid, primary monoid, strongly primary monoid.

arXiv:1907.00744v1 [math.AC] 27 Jun 2019
1. Introduction

Let $\mathcal{C}$ denote the class containing, up to isomorphism, all monoids that can be embedded into finite-rank free commutative monoids. If $F$ is one of the fields $\mathbb{Q}$ or $\mathbb{R}$ and $M$ is a monoid in $\mathcal{C}$, then the chain of natural inclusions

$$M \hookrightarrow \text{gp}(M) \hookrightarrow F \otimes_{\mathbb{Z}} \text{gp}(M)$$

yields an embedding of $M$ into the finite-dimensional vector space $F \otimes_{\mathbb{Z}} \text{gp}(M)$, where $\text{gp}(M)$ is the Grothendieck group of $M$. Here we provide a systematic study on the connection between atomic and factorization aspects of monoids $M$ in $\mathcal{C}$ and both the geometry of the conic hull $\text{cone}(M)$ and the combinatorics of the face lattice of $\text{cone}(M)$.

A commutative cancellative monoid is called atomic if any non-invertible element can be expressed as a product of irreducible elements. All monoids in $\mathcal{C}$ are atomic. After settling down the necessary terminology and recalling a few standard concepts in factorization theory and convex geometry, we begin the main core of this paper giving some characterizations of monoids in $\mathcal{C}$. Right after this, we will exhibit some motivating examples of monoids in $\mathcal{C}$, and then we show that the geometric and combinatorial aspects of the conic hulls of monoids in $\mathcal{C}$ do not depend on the vector space such monoids are embedded into.

As for integral domains, an atomic monoid is called a unique factorization monoid (or a UFM) if every non-invertible element has an essentially unique factorization into irreducibles. UFMs are the simplest monoids in the realm of factorization theory, as the main goal of this field is to study the deviation of an atomic monoid (resp., integral domain) from being a UFM (resp., a UFD). A huge variety of atomic conditions between being an atomic monoid (resp., an atomic integral domain) and being a UFM (resp., a UFD) have been considered in the literature during the last four decades, including half-factoriality, other-half-factoriality, being finitary, and being strongly primary. In this paper, we investigate some of these intermediate atomic conditions for monoids in the class $\mathcal{C}$. 

1. Introduction
An atomic monoid $M$ is half-factorial (or an HFM) provided that for all non-invertible $x \in M$, any two factorizations of $x$ have the same number of irreducibles (counting repetitions). In addition, an integral domain is half-factorial (or an HFD) if its multiplicative monoid is an HFM. The concept of half-factoriality was first investigated by L. Carlitz in the context of algebraic number fields; he proved that an algebraic number field is an HFD if and only if its class group has size at most two [6]. However, the term “half-factorial domain” is due to A. Zaks [45]. In [46], Zaks studied Krull domains that are HFDs in terms of their divisor class groups. Parallel to this, L. Skula [42] and J. Śliwa [43], motivated by some questions of W. Narkiewicz on algebraic number theory [39, Chapter 9], carried out systematic studies of HFDs. Since then HFMs and HFDs have been actively studied (see [7] and references therein).

Other-half-factoriality, on the other hand, is a dual version of half-factoriality, and it was introduced by J. Coykendall and W. Smith [12]. An atomic monoid $M$ is called other-half-factorial (or an OHFM) provided that for all non-invertible $x \in M$ no two distinct factorizations of $x$ in $M$ contain the same number of irreducibles (counting repetitions). Although an integral domain is a UFD if and only if its multiplicative monoid is an OHFM [12, Corollary 2.11], OHFMs are not always factorial or half-factorial, even in the class $\mathcal{C}$. In the second part of this paper, we offer geometric and combinatorial characterizations for the HFMs in $\mathcal{C}$ and for the OHFMs in $\mathcal{C}$.

The study of primary monoids was initiated by T. Tamura [44] and M. Petrich [40] in the 1970s and has received a great deal of attention since then [34, 35, 20]. Primary monoids naturally appear in commutative algebra: an integral domain is 1-dimensional and local if and only if its multiplicative monoid is primary. One of the most useful subclasses of primary monoids in factorization theory is that one consisting of finitely primary monoids. The initial interest in this subclass also comes from commutative algebra: the multiplicative monoid of a 1-dimensional local Mori domain with nonempty conductor is finitely primary [23, Proposition 2.10.7.6]. Finitely primary monoids were introduced in [20] by A. Geroldinger. Definitions of primary and finitely primary monoids will be given in Section 6.

Motivated by the non-unique factorization phenomenon of certain Noetherian domains, Geroldinger et al. introduced in [24] the class of finitary monoids. More precisely, the multiplicative monoid of a Noetherian domain $R$ is finitary if and only if $R$ is 1-dimensional and semilocal [24, Proposition 4.14]. In addition, finitary monoids conveniently capture certain aspects of the arithmetic and factorization structure of more sophisticated monoids, including $v$-Noetherian $G$-monoids [21] and congruence monoids [22]. Strongly primary monoids are those that are simultaneously primary and finitary. Numerical monoids and $v$-Noetherian primary monoids are strongly primary. On the other hand, the multiplicative monoid of a 1-dimensional local Mori domain is strongly primary. In the last section, we study the conic hull of monoids in $\mathcal{C}$ that are either primary or finitary. We conclude this paper with a few words about strongly primary monoids in $\mathcal{C}$. 
2. Atomic Monoids and Convex Cones

In this section we introduce most of the relevant concepts on commutative monoids, factorization theory, and convex geometry required to follow the results presented later. General references for any undefined terminology or notation can be found in [32] for commutative monoids, in [23] for atomic monoids and factorization theory, and in [41] for convex geometry.

2.1. General Notation. Recall that \( \mathbb{N} := \{0, 1, 2, \ldots\} \). If \( x, y \in \mathbb{Z} \) and \( x \leq y \), then we let \( [x, y] \) denote the interval of integers between \( x \) and \( y \), i.e.,

\[
[x, y] := \{ z \in \mathbb{Z} \mid x \leq z \leq y \}.
\]

In addition, for \( X \subseteq \mathbb{R} \) and \( r \in \mathbb{R} \), we set \( X \geq r := \{ x \in X \mid x \geq r \} \) and we use the notation \( X > r \) in a similar way. Lastly, if \( Y \subseteq \mathbb{R}^d \) for some \( d \in \mathbb{N} \setminus \{0\} \), then we set \( Y \cdot := Y \setminus \{0\} \).

2.2. Atomic Monoids. A monoid is commonly defined in the literature as a semigroup along with an identity element. However, in what follows all monoids are also assumed to be commutative and cancellative, and we omit these two attributes accordingly. As we only consider commutative monoids, unless otherwise specified we will use additive notation. In particular, the identity element of a monoid \( M \) will be denoted by 0, and we let \( M^\bullet \) denote the set \( M \setminus \{0\} \). A monoid is called reduced if its only invertible element is the identity element. Unless we specify otherwise, monoids here are also assumed to be reduced. For \( x, y \in M \), we say that \( x \) divides \( y \) in \( M \) and write \( x \mid_M y \) provided that \( y = x + x' \) for some \( x' \in M \). A submonoid \( N \) of \( M \) is called divisor-closed if for all \( y \in N \) and \( x \in M \) the condition \( x \mid_M y \) implies that \( x \in N \).

We write \( M = \langle S \rangle \) when \( M \) is generated by a set \( S \). If \( M \) can be generated by a finite set, we say that \( M \) is finitely generated. An element \( a \in M^\bullet \) is called an atom if for each pair of elements \( x, y \in M \) such that \( a = x + y \) either \( x = 0 \) or \( y = 0 \). The set consisting of all atoms of \( M \) is denoted by \( A(M) \), that is,

\[
A(M) := M^\bullet \setminus (M^\bullet + M^\bullet).
\]

Since \( M \) is reduced, it follows that \( A(M) \) will be contained in each generating set of \( M \). If \( M = \langle A(M) \rangle \), then \( M \) is said to be atomic. All monoids addressed in this paper are atomic. We say that \( p \in M^\bullet \) is prime if whenever \( p \mid_M x + y \) for some \( x, y \in M \) either \( p \mid_M x \) or \( p \mid_M y \). The monoid \( M \) is called a UFM (or a unique factorization monoid) if every nonzero element can be written as a sum of primes (up to permutation). Clearly, every prime element of \( M \) is an atom. Thus, if \( M \) is a UFM, then it is, in particular, an atomic monoid.

A subset \( I \) of \( M \) is an ideal of \( M \) if \( I + M \subseteq I \). An ideal \( I \) is principal if \( I = x + M \) for some \( x \in M \). Furthermore, \( M \) satisfies the ascending chain condition on principal
ideals (or the \( \text{ACCP} \)) provided that every increasing sequence of principal ideals of \( M \) eventually stabilizes. It is well known that every monoid satisfying the ACCP is atomic [23, Proposition 1.1.4]. The Gram’s monoid, introduced in [31], is an atomic monoid that does not satisfy the ACCP.

For any monoid \( M \) there exist an abelian group \( \text{gp}(M) \) and a monoid homomorphism \( \iota : M \hookrightarrow \text{gp}(M) \) such that any monoid homomorphism \( \phi : M \to G \) (where \( G \) is a group) uniquely factors through \( \iota \). The group \( \text{gp}(M) \), which is unique up to isomorphism, is called the difference group (or Grothendieck group) of \( M \). If \( M \) is a monoid in \( \mathcal{C} \), then the rank of \( M \), denoted by \( \text{rank}(M) \), is the rank of the abelian group \( \text{gp}(M) \), that is, the dimension of the \( \mathbb{Q} \)-space \( \mathbb{Q} \otimes_\mathbb{Z} \text{gp}(M) \). The monoid \( M \) is torsion-free if and only if its difference group is torsion-free (see [5, Section 2.A]).

A multiplicative commutative monoid \( F \) is free on a subset \( A \) of \( F \) if every element \( x \in F \) can be written uniquely in the form

\[
x = \prod_{a \in A} a^{v_a(x)},
\]

where \( v_a(x) \in \mathbb{N} \) and \( v_a(x) > 0 \) only for finitely many \( a \in A \). It is well known that for each set \( A \), there exists a unique (up to isomorphism) monoid \( F \) such that \( F \) is a free commutative monoid on \( A \). For a monoid \( M \), the free commutative monoid on \( \mathcal{A}(M) \), denoted by \( \mathcal{Z}(M) \), is called the factorization monoid of \( M \), and the elements of \( \mathcal{Z}(M) \) are called factorizations. If \( z = a_1 \ldots a_n \) is a factorization in \( \mathcal{Z}(M) \) for some \( n \in \mathbb{N} \) and \( a_1, \ldots, a_n \in \mathcal{A}(M) \), then \( n \) is called the length of \( z \) and is denoted by \( |z| \).

In addition, the unique monoid homomorphism \( \phi : \mathcal{Z}(M) \to M \) satisfying \( \phi(a) = a \) for all \( a \in \mathcal{A}(M) \) is called the factorization homomorphism of \( M \). For each \( x \in M \) the set

\[
\mathcal{Z}(x) := \mathcal{Z}_M(x) := \phi^{-1}(x) \subseteq \mathcal{Z}(M)
\]

is called the set of factorizations of \( x \). In addition, for \( k \in \mathbb{N} \) we set

\[
\mathcal{Z}_k(x) := \{ z \in \mathcal{Z}(x) : |z| = k \} \subseteq \mathcal{Z}(M).
\]

Observe that \( M \) is atomic if and only if \( \mathcal{Z}(x) \) is nonempty for all \( x \in M \) (notice that \( \mathcal{Z}(0) = \{ \emptyset \} \)). The monoid \( M \) is called a finite factorization monoid (or an FFM) provided that \( |\mathcal{Z}(x)| < \infty \) for all \( x \in M \). For each \( x \in M \), the set of lengths of \( x \) is defined by

\[
\mathcal{L}(x) := \mathcal{L}_M(x) := \{ |z| : z \in \mathcal{Z}(x) \}.
\]

The system of sets of lengths of monoids in \( \mathcal{C} \) has been considered in [27]. If \( |\mathcal{L}(x)| < \infty \) for all \( x \in M \), then \( M \) is called a bounded factorization monoid (or a BFM). Clearly, every FFM is a BFM.

A very special class of atomic monoids is that of all numerical monoids, i.e., cofinite additive submonoids of \( \mathbb{N} \). Each numerical monoid \( M \) has a unique minimal set of generators, which is finite; such a unique minimal generating set is precisely \( \mathcal{A}(M) \). As a result, every numerical monoid is atomic and contains only finitely many atoms. A
friendly introduction to numerical monoids can be found in [19]. The class of finitely generated submonoids of \((\mathbb{N}^d, +)\) naturally generalizes that one of numerical monoids. Although members of the former class are finitely generated and, therefore, finitary, numerical monoids are the only primary monoids in this class (Proposition 6.1(2)). However, we shall see later that there are many non-finitely generated submonoids of \((\mathbb{N}^d, +)\) that are primary. In addition, we will provide necessary and sufficient conditions for a submonoid of \((\mathbb{N}^d, +)\) to be finitary.

2.3. Convex Cones. We let \(e_1, \ldots, e_d\) denote the canonical basic vectors of \(\mathbb{R}^d\). In addition, we denote the standard inner product of \(\mathbb{R}^d\) by \(\langle \cdot, \cdot \rangle\), that is, \(\langle x, y \rangle = \sum_{i=1}^{d} x_i y_i\) for all \(x = (x_1, \ldots, x_d)\) and \(y = (y_1, \ldots, y_d)\) in \(\mathbb{R}^d\). As usual, for \(x \in \mathbb{R}^d\) we let \(\|x\|\) denote the Euclidean norm of \(x\). We always consider the space \(\mathbb{R}^d\) endowed with the topology induced by the Euclidean norm. Finally, we let the \(\mathbb{Q}\)-space \(\mathbb{Q}^d\) inherit the inner product and the topology of \(\mathbb{R}^d\). For a subset \(S\) of \(\mathbb{R}^d\), we let \(\text{int} S\), \(\overline{S}\), and \(\text{bd}\ S\) denote the interior, closure, and boundary of \(S\), respectively.

Let \(V\) be a vector space over an ordered field. A nonempty convex subset \(C\) of \(V\) is called a cone provided that \(C\) is closed under linear combinations with nonnegative coefficients. A cone \(C\) is called pointed if \(C \cap -C = \{0\}\). Unless otherwise stated, we assume that the cones we consider here are pointed. If \(X\) is a nonempty subset of \(V\), the conic hull of \(X\), denoted by \(\text{cone}(X)\), is defined as

\[
\text{cone}(X) := \{c_1 x_1 + \cdots + c_n x_n \mid x_i \in X \text{ and } c_i \geq 0 \text{ for all } i \in [1, n]\},
\]

i.e., \(\text{cone}(X)\) is the smallest cone in \(V\) containing \(X\). A cone in \(V\) is called simplicial, if it is the conic hull of a linearly independent set of vectors. In addition, a cone in \(\mathbb{R}^d\) is called rational if it is the conic hull of vectors with integer coordinates.

A face of \(C\) is a cone \(F\) contained in \(C\) satisfying the following condition: for all \(x, y \in C\) the fact that the open line segment \(\{tx + (1-t)y \mid 0 < t < 1\}\) intersects \(F\) implies that both \(x\) and \(y\) belong to \(F\). If \(F\) is a face of \(C\) and \(F'\) is a face of \(F\), then it is clear that \(F'\) must be a face of \(C\). Now suppose that \(F\) is either \(\mathbb{Q}\) or \(\mathbb{R}\). For a nonzero vector \(u \in \mathbb{R}^d\), consider the hyperplane \(H := \{x \in \mathbb{R}^d \mid \langle x, u \rangle = 0\}\), and denote the closed half-spaces \(\{x \in \mathbb{R}^d \mid \langle x, u \rangle \leq 0\}\) and \(\{x \in \mathbb{R}^d \mid \langle x, u \rangle \geq 0\}\) by \(H^-\) and \(H^+,\) respectively. If a cone \(C\) satisfies that \(C \subseteq H^-\) (resp., \(C \subseteq H^+\)), then \(H\) is called a supporting hyperplane of \(C\) and \(H^-\) (resp., \(H^+\)) is called a supporting half-space of \(C\). A face \(F\) of \(C\) is called exposed if there exists a supporting hyperplane \(H\) of \(C\) such that \(F = C \cap H\). The cone \(C\) is called polyhedral provided that it can be expressed as the intersection of finitely many half-spaces. The Farkas-Minkowski-Weyl Theorem states that a convex cone is polyhedral if and only if it is the conic hull of a finite set. On the other hand, Gordan’s Lemma states that if \(C\) is a rational polyhedral cone in \(\mathbb{R}^d\) and \(G\) is an additive subgroup of \(\mathbb{Q}^d\), then \(C \cap G\) is finitely generated (see [5, Lemma 2.9]).
A subset $S$ of $\mathbb{R}^n$ is called an affine set (or an affine subspace) provided that for all $x, y \in S$ with $x \neq y$, the line determined by $x$ and $y$ is contained in $S$. Affine sets are translations of subspaces, and an $(n - 1)$-dimensional affine set is called an affine hyperplane. The affine hull of $S$, denoted by $\text{aff}(S)$, is the smallest affine set containing $S$. The relative interior of $S$, denoted by $\text{relin}(S)$, is the Euclidean interior of $S$ when considered as a subset of $\text{aff}(S)$. If $C$ is a cone, then $C$ is the disjoint union of all the relative interiors of its nonempty faces [41, Theorem 18.2].

3. Monoids in $C$

3.1. The Class $C$. In this section we introduce the class of monoids we shall be concerned with in this paper. We also introduce the cones associated to such monoids.

**Theorem 3.1.** For a monoid $M$, the following conditions are equivalent.

1. $M$ can be embedded into a finite-rank free commutative monoid.
2. $M$ has finite rank and can be embedded into a free commutative monoid.
3. There exists $d \in \mathbb{N}$ such that $M$ can be embedded in $\left( \mathbb{N}^d, + \right)$ as a maximal-rank submonoid.

**Proof.** Let us verify first that (1) implies (2). Suppose that $F$ is a finite-rank commutative monoid containing $M$. Assuming that $\text{gp}(M) \subset \text{gp}(F)$, one can consider $\text{gp}(M)$ as a $\mathbb{Z}$-submodule of $\text{gp}(F)$. Since $\text{gp}(F)$ is a finite-rank $\mathbb{Z}$-module, so is $\text{gp}(M)$. Hence $M$ has finite rank, which yields (2).

Now we argue that (2) implies (1), consider a set $X$ such that $M$ is embedded into the free commutative monoid $\bigoplus_{x \in X} \mathbb{N}x$. After identifying $M$ with its image, we can assume that $M \subseteq \bigoplus_{x \in X} \mathbb{N}x$ and also that $\text{gp}(M)$ is a subgroup of $\bigoplus_{x \in X} \mathbb{Z}x$. Since $M$ has finite rank, the dimension of the subspace $W$ of $V := \bigoplus_{x \in X} \mathbb{Q}x$ generated by $\text{gp}(M)$ is finite. Let $\{b_1, \ldots, b_k\}$ be a basis for $W$. For each $i \in [1, k]$ there exists a finite subset $Y_i$ of $X$ such that $b_i \in \bigoplus_{x \in Y_i} \mathbb{Q}x$. As a result, $W \subseteq \bigoplus_{y \in Y} \mathbb{Q}y$, where $Y = \bigcup_{i=1}^k Y_i$. Then

$$M \subseteq \left( \bigoplus_{y \in Y} \mathbb{Q}y \right) \bigcap \left( \bigoplus_{x \in X} \mathbb{N}x \right) = \bigoplus_{y \in Y} \mathbb{N}y.$$

Since $Y$ is a finite set, $\bigoplus_{y \in Y} \mathbb{N}y$ is a finite-rank free commutative monoid, and so (1) holds.

Clearly, (3) implies (1). So it suffices to prove that (1) implies (3). To do this, let $M$ be a monoid of rank $d$, and suppose that $M$ is a submonoid of a free commutative monoid of rank $r$ for some $r \in \mathbb{N}$ with $r \geq d$. There is no loss of generality in assuming that $M$ is a submonoid of $\left( \mathbb{N}^r, + \right)$. Let $V$ be the subspace of the $\mathbb{Q}$-space $\mathbb{Q}^r$ generated by $M$. Since $M$ has rank $d$, the subspace $V$ has dimension $d$. Now consider the submonoid $M' := \mathbb{N}^r \cap V$ of $\left( \mathbb{N}^r, + \right)$. As $M'$ is the intersection of the rational cone $\text{cone}(\mathbb{N}^r \cap V)$ and the lattice $\mathbb{Z}^r \cap V \cong \mathbb{Z}^d$, it follows by Gordan’s Lemma that $M'$ is finitely generated. On the other hand, $M \subseteq M' \subseteq V$ guarantees that
rank($M'$) = $d$. Since $M'$ is a finitely generated additive submonoid of $\mathbb{N}^r$ of rank $d$, it follows by [5, Proposition 2.17] that $M'$ is isomorphic to a submonoid of $(\mathbb{N}^d, +)$. This, in turn, implies that $M$ is isomorphic to a submonoid of $(\mathbb{N}^d, +)$, which concludes our argument. □

As we are interested in studying monoids satisfying the equivalent conditions of Theorem 3.1, we introduce the following notation.

**Notation:** Let $\mathcal{C}$ denote the class consisting of all monoids (up to isomorphism) satisfying the conditions in Theorem 3.1. In addition, for every $d \in \mathbb{N}^\ast$, we set

$$C_d := \{ M \in \mathcal{C} \mid \text{rank}(M) = d \}.$$

A monoid is *affine* if it is isomorphic to a finitely generated submonoid of the free abelian group $\mathbb{Z}^d$ for some $d \in \mathbb{N}$. The interested reader may find a self-contained treatment of affine monoids in [5, Part 2]. Clearly, the class $\mathcal{C}$ contains all affine monoids. Computational aspects of affine monoids and factorization invariants of half-factorial affine monoids have been studied in [18] and [17], respectively. Diophantine monoids form a special subclass of that one consisting of affine monoids and has been studied in [9]. Monoids in $\mathcal{C}$ of small rank have been recently studied in [11]. Some other special subclasses of $\mathcal{C}$ have been previously considered in the literature as they naturally arise in the study of algebraic curves, toric geometry, and homological algebra. Here we offer a few examples.

**Example 3.2.** If $M$ is finitely primary, then $M$ is primary and satisfies that $\hat{M} \cong \mathbb{N}^d$ [23, Theorem 2.9.2]. Hence $\mathcal{C}$ contains all finitary primary monoids.

**Example 3.3.** Good semigroups, which also form a subclass of $\mathcal{C}$, were introduced in [3] in the context of algebraic curves. Good semigroups are submonoids of $(\mathbb{N}^d, +)$ that naturally generalize value semigroups of an algebraic curve in the sense that monoids on both classes satisfy certain common “good” properties. For instance, the value semigroup $S$ of the ring

$$R := \mathbb{C}[[x, y]]/(x^7 - x^6 + 4x^5y + 2x^3y^2 - y^4)(x^3 - y^3)$$

is represented in Figure 1. As $\{(x, y) \in S \mid x < 13\}$ is finite, the affine line $x = 13$ of $\mathbb{R}^2$ contains infinitely many atoms of $S$. Hence the good semigroup $S$ is not finitely generated (for more details on this example, see [4, page 8]). In addition, it has been verified in [3, Example 2.16] that the good semigroup

$$\{(x, y) \in \mathbb{N}^2 \mid x \geq 25 \text{ and } y \geq 27\}$$

is not the semigroup value of any algebraic curve. Good semigroups have received substantial attention since they were introduced; see for example [3, 4, 14] and see [15, 37] for more recent studies.
Example 3.4. From the structure theorem for modules over a PID, we have that if $R$ is a 1-dimensional integrally-closed local domain and $M$ is a finitely generated torsion-free $R$-module, then $M$ is free if and only if $M \otimes_R \text{Hom}(M, R)$ is torsion-free. It has been conjectured by C. Huneke and R. Wiegand that this property also holds for any 1-dimensional Gorenstein domain. Given a numerical monoid $\Gamma$ and $s \in \mathbb{N} \setminus \Gamma$, consider the collection

$$M^s_\Gamma := \{(0, 0)\} \cup \{(x, n) \mid \{x, x + s, x + 2s, \ldots, x + ns\} \subseteq \Gamma\}$$

consisting of all arithmetic sequences of step size $s$ contained in $\Gamma$. It is clear that $M^s_\Gamma$ is a monoid; it is called a Leamer monoid. The atomic structure of Leamer monoids is connected to the Huneke-Wiegand conjecture via [16, Corollary 7]. Notice that Leamer monoids are non-finitely generated rank-2 monoids contained in the class $\mathcal{C}$. Factorization properties of Leamer monoids have been considered in [33] and, more recently, in [10].

The following example has been kindly provided by Roger Wiegand, and will appear in [2].

Example 3.5. Let $\alpha$ and $\beta$ be two positive irrational numbers such that $\alpha < \beta$, and consider the monoid $M_{\alpha, \beta}$ defined as follows:

$$M_{\alpha, \beta} := \{(0, 0)\} \cup \{(m, n) \in \mathbb{N}^2 \mid \alpha < \frac{n}{m} < \beta\}.$$ 

It follows from Farkas-Minkowski-Weyl Theorem that $M_{\alpha, \beta}$ is not finitely generated and, therefore, $|A(M_{\alpha, \beta})| = \infty$. In addition, $M_{\alpha, \beta}$ is a primary FFM (see Proposition 6.1 and Proposition 5.7). For every $n \geq 3$, the sequence of monoids $\{M_n\}$ obtained by setting

$$\alpha = \frac{2}{n + \sqrt{n^2 - 4}} \quad \text{and} \quad \beta = \frac{n + \sqrt{n^2 - 4}}{2}.$$
shows up in the study of Betti tables of short Gorenstein algebras. In an ongoing project, Avramov, Gibbons, and Wiegand have proved that
\[ A(M_n) = \{ \omega^{1-a}(1, b) \mid (a, b) \in \Gamma \}, \]
where \( \omega: (p, q) \mapsto (np - q, p) \) is an automorphism of \( M_n \) and \( \Gamma := \mathbb{Z} \times \left[ 1, n - 2 \right] \). This suggests the following question.

**Question 3.6.** For any irrational (or algebraic) numbers \( \alpha \) and \( \beta \) with \( \alpha < \beta \), can we generalize Example 3.5 to describe the set of atoms of \( M_{\alpha, \beta} \)?

### 3.2. The Cones of Monoids in \( \mathcal{C} \)

A **lattice** is a partially ordered set \( L \), in which every two elements have a unique **join** (i.e., least upper bound) and a unique **meet** (i.e., greatest lower bound). The lattice \( L \) is **complete** if each \( S \subseteq L \) has both a join and a meet. Two complete lattices are isomorphic if there is a bijection between them that preserves joints and meets. For background information on (complete) lattices and lattice homomorphisms, see [13, Chapter 2]. For a cone \( C \), the collection of all its faces, denoted by \( F(C) \), is a complete lattice (under inclusion) [41, page 164], where the meet is given by intersection and the join of a given set of faces is the smallest face in \( F(C) \) containing all the given faces. The lattice \( F(C) \) is called the **face lattice** of \( C \). Two cones \( C \) and \( C' \) are **combinatorially equivalent** provided that their face lattices are isomorphic.

Let \( \mathbb{F} \) denote either \( \mathbb{Q} \) or \( \mathbb{R} \). As mentioned in the introduction, a monoid \( M \) in \( \mathcal{C} \) of rank \( d \) can be embedded in a \( d \)-dimensional vector space over \( \mathbb{F} \) via
\[ M \hookrightarrow \text{gp}(M) \hookrightarrow \mathbb{F} \otimes_{\mathbb{Z}} \text{gp}(M) =: V, \]
where the flatness of \( \mathbb{F} \) as a \( \mathbb{Z} \)-module ensures the injectivity of the second map. Then we can consider the conic hull \( \text{cone}_V(M) \) of \( M \) in \( V \). It turns out that the combinatorial and geometric structures of \( \text{cone}_V(M) \) do not depend on the proposed embedding \( M \hookrightarrow V \), as we proceed to show.

**Proposition 3.7.** Let \( \mathbb{F} \in \{ \mathbb{Q}, \mathbb{R} \} \). Let \( M \) and \( M' \) be two monoids in \( \mathcal{C} \), and let \( V \) and \( V' \) be two finite-dimensional vector spaces over \( \mathbb{F} \) containing \( M \) and \( M' \), respectively. If the monoids \( M \) and \( M' \) are isomorphic, then
1. \( \text{cone}_V(M) \) is homeomorphic to \( \text{cone}_{V'}(M') \);
2. \( \text{cone}_V(M) \) is combinatorially equivalent to \( \text{cone}_{V'}(M') \).

**Proof.** Let \( d \) be the rank of \( M \). By Theorem 3.1 the monoid \( M \) can be embedded in \( (\mathbb{N}^d, +) \). After identifying \( M \) with its image, we can assume that \( M \subseteq \mathbb{N}^d \) and \( \text{gp}(M) \) is a subgroup of \( \mathbb{Z}^d \). Let \( \varphi: M \rightarrow M' \) be a monoid isomorphism. Then \( \varphi \) extends to an injective group homomorphism \( \text{gp}(M) \rightarrow V' \) with image \( \text{gp}(M') \). By tensoring
gp(M) and V' with the flat Z-module F, such a group homomorphism extends to a linear transformation
\[ \varphi: V := F \otimes \text{gp}(M) \to F \otimes_{\mathbb{Z}} V' = V'. \]
Since F is flat, ker \(\varphi\) is trivial and, therefore, \(\varphi\) is a linear embedding. Hence \(\varphi\) is a homeomorphism onto its image. As
\[ \varphi(\text{cone}_V(M)) = \text{cone}_{\varphi(V)}(M') = \text{cone}_{V'}(M'), \]
the cones \(\text{cone}_V(M)\) and \(\text{cone}_{V'}(M')\) are homeomorphic. Notice that we have chosen the vector space V but not \(V'\). This, along the fact that being homeomorphic is a transitive relation, yields (1).

To argue (2), it suffices to observe that the fact that \(\varphi\) is a linear bijection taking \(\text{cone}_{F^d}(M)\) onto \(\text{cone}_{V'}(M')\) guarantees that the map given by the assignment \(F \mapsto \varphi(F)\) is an order-preserving bijection from \(F(\text{cone}_{F^d}(M))\) to \(F(\text{cone}_{V'}(M'))\) and, therefore, a lattice isomorphism.

From now on we shall tacitly assume Proposition 3.7 when referring to the cone of a monoid \(M\) in \(\mathcal{C}\) over a field \(F \in \{\mathbb{Q}, \mathbb{R}\}\), and feel free to choose (or let unspecified) the finite-dimensional \(F\)-vector space in which \(M\) is embedded into.

**Corollary 3.8.** If \(M\) is a monoid in \(\mathcal{C}\), then \(\dim \text{cone}(M) = \text{rank}(M)\).

**Proof.** Set \(d = \text{rank}(M)\). By Theorem 3.1, we can assume that \(M \subseteq \mathbb{N}^d\). Then we have that \(\text{cone}(M) \subseteq \mathbb{R}^d\) and, therefore, \(\dim \text{cone}(M) \leq d\). On the other hand, \(\text{rank} \text{gp}(M) = d\), along with the fact that \(\text{gp}(M)\) is contained in the subspace of \(\mathbb{R}^d\) generated by \(M\), implies that \(M\) contains \(d\) linearly independent vectors. Hence \(\dim \text{cone}_{F^d}(M) \geq d\), which concludes the proof. \(\square\)

**Proposition 3.9.** Let \(M\) be a cone in \(\mathcal{C}\). Then \(\overline{\text{cone}(M)}\) is a pointed cone.

**Proof.** Set \(k = \text{rank}(M)\), and for \(F \in \{\mathbb{Q}, \mathbb{R}\}\) set \(V := \mathbb{R} \otimes_{\mathbb{Z}} \text{gp}(M)\). Suppose, by way of contradiction, that \(\overline{\text{cone}_V(M)}\) is not pointed. Using Theorem 3.1, one can assume that \(M\) can be embedded into \((\mathbb{N}^k, +)\). Let \(\iota: M \to (\mathbb{N}^k, +)\) be an injective monoid homomorphism. After tensoring both \(\text{gp}(M)\) and \(\text{gp}(\mathbb{N}^k) = \mathbb{Z}^k\) with the flat \(\mathbb{Z}\)-module \(F\), the homomorphism \(\iota\) extends to a linear transformation \(\overline{\iota}: \mathbb{R} \otimes_{\mathbb{Z}} \text{gp}(M) \to \mathbb{F}^k\). Since \(\overline{\text{cone}_V(M)}\) is not pointed, it contains a 1-dimensional subspace \(L\). As \(\overline{\iota}\) is linear, it must be continuous and, therefore,
\[ \overline{\iota}(L) \subseteq \overline{\iota(\text{cone}_V(M))} \subseteq \overline{\iota(\text{cone}_V(M))} = \overline{\iota(\text{cone}_V(M))}. \]
This, along with the fact that \(\iota(\text{cone}_V(M)) \subseteq \mathbb{R}_{\geq 0}^k\), implies that \(\overline{\iota}(L) \subseteq \mathbb{F}_{\geq 0}^k\). Since \(\overline{\iota}(L)\) is a subspace of \(\mathbb{F}^k\), it must be trivial, which contradicts the injectivity of \(\iota\). Thus, \(\overline{\text{cone}_V(M)}\) must be pointed, which completes our argument. \(\square\)
Members of $C$ are finite-rank torsion-free monoids. However, not every finite-rank torsion-free monoid is in $C$. The next two examples shed some light upon this observation.

**Example 3.10.** A nontrivial submonoid $M$ of $(\mathbb{Q}_{\geq 0}, +)$ is obviously a rank 1 torsion-free monoid. It follows by Theorem 3.1 that $M$ belongs to $C$ if and only if $M$ is isomorphic to a numerical monoid. Hence [26, Proposition 3.2] guarantees that $M$ is in $C$ if and only if $M$ is finitely generated. As a result, non-finitely generated submonoids of $(\mathbb{Q}_{\geq 0}, +)$ such as $\langle 1/p \mid p \text{ is prime} \rangle$ are finite-rank torsion-free monoids that do not belong to the class $C$. The atomic and factorization structures of submonoids of $(\mathbb{Q}_{\geq 0}, +)$ have been fairly considered lately; see, for instance, [28, 29, 30]. Clearly, the Grothendieck group of a non-finitely generated submonoid of $(\mathbb{Q}_{\geq 0}, +)$ cannot be free.

The following example, courtesy of Winfried Bruns, shows that a finite-rank torsion-free monoid might not belong to $C$ even though its Grothendieck group is free.

**Example 3.11.** Consider the additive monoid
$$M := \{(0, 0)\} \cup \{(m, n) \in \mathbb{Z}^2 \mid n > 0\} \subseteq \mathbb{Z}^2.$$ It is clear that $M$ is an additive submonoid of $\mathbb{Z}^2$ and, therefore, it has finite rank. In addition, it is clear that $M$ is torsion-free. On the other hand, $\text{cone}_{\mathbb{R}^2}(M) = \{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}$, which is not a pointed cone. As a consequence, it follows from Proposition 3.9 that $M$ does not belong to the class $C$.

### 3.3. Cones Realized by Monoids in $C$.

We conclude this section characterizing the positive cones that can be realized by the monoids in $C$. First, let us argue the following lemma.

**Lemma 3.12.** For $d \in \mathbb{N}$, let $C$ be a $d$-dimension positive cone in $\mathbb{R}^d$ and let $x \in \text{int} C$. Then there exists a $d$-dimensional rational simplicial cone $C_p$ such that $\mathbb{R}_{>0}x \subset \text{int} C_p$ and $C_p \subseteq \{0\} \cup \text{int} C$.

**Proof.** For $d = 1$, take $C_p = \mathbb{R}_{>0}x$. Then suppose that $d \geq 2$ and write $x = (x_1, \ldots, x_d)$. As $C$ is positive and $x \in \text{int} C$, we have that $x_i > 0$ for $i \in [1, d]$. Let $\ell$ be the distance from $x$ to the complement of $\text{int} C$. Since the complement of $\text{int} C$ is closed and $\{x\}$ is compact, $\ell > 0$. Consider the $d$-dimensional regular simplex $\Delta_n := \text{conv}(e_1, \ldots, e_d)$, and choose $N \in \mathbb{N}$ large enough such that $\text{diam} (\Delta_n/N) = \sqrt{2}/N < \ell$. In addition, take $q = (q_1, \ldots, q_d) \in \mathbb{Q}_{>0}^d$ such that $q_i < x_i$ for $i \in [1, d]$ and $\sum_{i=1}^d (x_i - d_i) < 1/N$. Now set $\Delta := q + \Delta_n/N$. Clearly, $x - q$ is an interior point of $\Delta_n/N$ and, therefore, $x$ is an interior point of $\Delta$. This, along with the fact that $\text{diam}(\Delta) = \text{diam}(\Delta_n/N) < \ell$, ensures that $\Delta \subset \text{int} C$. Lastly take $C_p := \text{cone}(\Delta)$. It is clear that $C_p$ is a closed cone contained in $\{0\} \cup \text{int} C$. In addition, $x \in \text{int} \Delta$ implies that $\mathbb{R}_{>0}x \subset \text{int} C_p$. As
dim $\Delta = d$, we have that $\dim C_p = d$. Hence the set of 1-dimensional edges of the polyhedral $C_p$ has size at least $d$. On the other hand, the 1-dimensional faces of $C_p$ are determined by some of the vertices of $\Delta$. As $\mathbb{R}_{>0}q \subseteq \text{int} C_p$, the 1-dimensional faces of $C_p$ are precisely the $d$ nonnegative rays containing the points $q + e_i/N$ for $i \in [1, d]$. Thus, $C_p$ is a $d$-dimensional rational simplicial cone. \qed

**Notation:** We call a 1-dimensional subspace of $\mathbb{R}^d$ (resp., an infinite ray) a rational line (resp., a rational ray) if it contains a nonzero point with rational components.

**Theorem 3.13.** For $d \in \mathbb{N}$, let $C$ be a positive cone in $\mathbb{R}^d$. Then $C$ can be generated by a monoid in $\mathcal{C}$ if and only if each 1-dimensional face of $C$ is a rational ray.

**Proof.** For the direct implication, suppose that $C$ is generated by a monoid in $\mathcal{C}$. Then one can assume that $C = \text{cone}(M)$, where $M$ is a rank $d$ submonoid of $(\mathbb{N}^d, +)$. Let $L$ be a 1-dimensional face of $C$, and let $x$ be a nonzero point in $L$. Now take $c_1, \ldots, c_k \in \mathbb{R}_{>0}$ and $x_1, \ldots, x_k \in M^*$ such that $x = c_1 x_1 + \cdots + c_k x_k$. If $k = 1$, then $x_1 = x \in L$ and, therefore, $L$ is a rational ray. If $k > 1$, then $x' := c_2 x_2 + \cdots + c_k x_k \in \text{cone}(M)^*$ and

$$\frac{c_1}{1 + c_1} x_1 + \frac{1}{1 + c_1} x' = \frac{1}{1 + c_1} x \in L.$$

As $L$ is a face of $C$ and the segment line from $x_1$ to $x'$ intersects $L$, it follows that the whole segment is contained in $L$. In particular, $x_1 \in L$. Hence $L$ is a rational ray.

For the reverse implication, assume that all 1-dimensional faces of $C$ are rational rays. Consider the set $M := C \cap \mathbb{N}^d$. Clearly, $M$ is an additive submonoid of $\mathbb{N}^d$ and $\text{cone}(M) \subseteq C$. Take $x \in C^*$, and set $\ell := \mathbb{R}_{>0}x$. Since $C$ is the disjoint union of all the relative interiors of its nonempty faces, there exists a face $C'$ of $C$ such that $x \in \text{relin} C'$. Suppose that $C'$ is $d'$-dimensional. Then by Lemma 3.12 there exists a rational cone $C_x \subseteq \text{relin} C'$ with $d'$ 1-dimensional faces such that $\ell \subseteq \text{relin} C_x$. Now take $v_1, \ldots, v_{d'} \in \mathbb{N}^d \setminus \{0\}$ such that $\mathbb{R}_{>0}v_i$ for $i \in [1, d']$ are the 1-dimensional faces of $C_x$. As $x \in \text{relin} C_x$ we can write $x = c_1 v_1 + \cdots + c_{d'} v_{d'}$ for some $c_1, \ldots, c_{d'} \in \mathbb{R}_{>0}$. Because $v_i \in C \cap \mathbb{N}^d$ for $i \in [1, d']$, it follows that $x \in \text{cone}(M)$. \qed

Not every positive cone in $\mathbb{R}^d$ can be generated by a monoid in $\mathcal{C}$. The following example sheds some light upon this observation.

**Example 3.14.** Let $C$ be the cone in $\mathbb{R}^d$ generated by the set $\{e_1, \ldots, e_{d-1}, v_d\}$, where $v_d := \pi e_d + \sum_{i=1}^{d-1} e_i$. It is clear that $C$ is a positive cone. Note, in addition, that $\mathbb{R}_{>0}v_d$ is a 1-dimensional face of $C$. Finally, observe that $\mathbb{R}_{>0}v_d$ contains no point with rational components. Hence it follows by Theorem 3.13 that $C$ cannot be generated by any monoid in $\mathcal{C}$.
4. Faces and Divisor-Closed Submonoids

4.1. Face Submonoids. For \( M \) in \( \mathcal{C} \) we would like to understand the structure of the face lattice of \( \text{cone}(M) \) in connection with the divisibility aspects of \( M \). In particular, the submonoids of \( M \) obtained by intersecting \( M \) with the faces of \( \text{cone}(M) \) are very special as they inherit many divisibility and atomic properties from \( M \), as we shall see in the next three sections.

**Definition 4.1.** Let \( M \) be a nontrivial monoid in \( \mathcal{C} \). A submonoid \( N \) of \( M \) is called a **face submonoid** of \( M \) if \( N = M \cap F \) for some face \( F \) of \( \text{cone}(M) \subseteq \mathbb{R} \otimes_{\mathbb{Z}} \text{gp}(M) \).

It follows from Proposition 3.7 that the definition of a face submonoid only depends on \( M \).

**Proposition 4.2.** Let \( M \) be a monoid in \( \mathcal{C} \), and let \( N \) be the face submonoid of \( M \) determined by the face \( F \). Then the following conditions hold.

1. \( N \) is a monoid in \( \mathcal{C} \) satisfying that \( \mathcal{A}(N) = \mathcal{A}(M) \cap F \).
2. \( \text{cone}(N) = F \).

**Proof.** Let us argue (1) first. The fact that \( N \) is a monoid in \( \mathcal{C} \) is a direct implication of Theorem 3.1. Since \( \mathcal{A}(M) \cap F \subseteq N \), one has that \( \mathcal{A}(M) \cap F \subseteq \mathcal{A}(N) \). To verify the reverse inclusion, take \( a' \in \mathcal{A}(N) \), and let \( a \in \mathcal{A}(M) \) such that \( a \mid_{M} a' \). Take \( b \in M \) such that \( a' = a + b \). Then we have that \( a'/2 \) belongs to the intersection of \( F \) and \( \text{relin}\{ta + (1-t)b \mid 0 \leq t \leq 1\} \). This implies that both \( a \) and \( b \) belong to \( F \). As a result,

\[
a \in \mathcal{A}(M) \cap F \subseteq N \subseteq \mathcal{A}(N).
\]

Then \( a' = a \in \mathcal{A}(M) \cap F \), which yields the desired inclusion. Hence (1) holds.

To argue (2), it suffices to assume that \( M \) is a submonoid of \( (\mathbb{N}^d, +) \) of rank \( d \) for some \( d \in \mathbb{N}^* \) and \( F \) is a face of \( \text{cone}_{\mathbb{Q}^d}(M) \). Since \( N \subseteq F \) and \( F \) is a cone, \( \text{cone}_{\mathbb{Q}^d}(N) \subseteq F \).

To show the reverse inclusion, take \( x \in F^* \). Then \( x = \sum_{i=1}^{k} q_ia_i \) for \( a_1, \ldots, a_k \in \mathcal{A}(M) \) and \( q_1, \ldots, q_k \in \mathbb{Q}_{>0} \). Then \( mx \in M \cap F = N \), where \( m \) is the least common multiple of the denominators of the \( q_i's \). So \( x \in \text{cone}(N) \). As a result, \( F \subseteq \text{cone}(N) \), and then (2) follows. \( \square \)

**Remark 4.3.** With notation as in Proposition 4.2, the condition that the submonoid \( N \) is a face submonoid of \( M \) is needed to guarantee that \( \mathcal{A}(N) = \mathcal{A}(M) \cap F \). To see this consider, for instance, any submonoid \( N \) of \( M \) such that \( N \cap \mathcal{A}(M) \) is an empty set. It is clear in this case that \( \mathcal{A}(N) \neq \mathcal{A}(M) \cap F \).

For a monoid \( M \) in \( \mathcal{C} \), there might be submonoids of \( M \) obtained by intersecting \( M \) with certain non-supporting hyperplanes whose sets of atoms can be obtained as in Proposition 4.2.
Example 4.4. Consider the submonoid $M = \langle 2e_1, 2e_2, e_1 + e_2 \rangle$ of $(\mathbb{N}^2, +)$. It can be readily checked that $A(M) = \{2e_1, 2e_2, e_1 + e_2\}$. Now consider the hyperplane $H = \mathbb{R}(e_1 + e_2)$ of $\mathbb{R}^2$ and set $N = M \cap H$. It is clear that $N$ is a submonoid of $M$ satisfying that

$$A(N) = \{e_1 + e_2\} = A(M) \cap H.$$ 

However, notice that $N$ is not a face submonoid of $M$.

4.2. Characterization of Face Submonoids. Recall that a submonoid $N$ of a monoid $M$ is said to be divisor-closed provided that for all $y \in N$ and $x \in M$ the condition $x |_M y$ implies that $x \in N$. For any monoid $M$ in $\mathcal{C}$, the concepts of a face submonoid and a divisor-closed submonoid coincide.

Theorem 4.5. Let $M$ be a monoid in $\mathcal{C}$. Then a submonoid $N$ of $M$ is divisor-closed in $M$ if and only if $N$ is a face submonoid of $M$.

Proof. Suppose that $M \in \mathcal{C}_k$, and assume that $M \subseteq \mathbb{N}^k \subset \mathbb{R}^k$. We verify first that face submonoids of $M$ are divisor-closed. To do so, take a face $F$ of cone $(M)$ and set $N := M \cap F$. To argue that $N$ is a divisor-closed submonoid of $M$, take $x \in N$ and $y \in M \setminus \{x\}$ such that $y |_M x$. Then $x = y + y'$ for some $y' \in M$, which implies that

$$x/2 \in F \cap \text{relin} \left\{ty + (1 - t)y' \mid 0 \leq t \leq 1\right\}.$$ 

As $F$ is a face both $y$ and $y'$ belong to $F$, and so $y \in N$. Hence $N$ is divisor-closed.

Let us argue the reverse implication by induction. Notice that when $M$ has rank 1, it is isomorphic to a numerical monoid and the only submonoids of $M$ that are divisor-closed are the trivial and $M$ itself, which are the face submonoids of $M$ corresponding to the origin and to cone $(M)$, respectively. Fix now $k > 1$ and assume that the divisor-closed submonoids of any monoid in $\mathcal{C}$ with rank less than $k$ are face submonoids. Let $M$ be a maximal-rank submonoid of $(\mathbb{N}^k, +)$ and let $N$ be a submonoid of $M$ that is not a face submonoid.

CASE 1. rank$(N) = k$. Since $N$ is not a face submonoid of $M$, it follows that $N \neq M$. Take $v \in M \setminus N$ and a basis $v_1, \ldots, v_k \in N$ of $\mathbb{Q}^k$ such that $v = \sum_{i=1}^k q_i v_i$, where the rational coefficients satisfy that $q_1, \ldots, q_j \leq 0$ and $q_{j+1}, \ldots, q_k > 0$ (not all zeros) for some index $j \in \mathbb{N}^*$. Then

$$dv + \sum_{i=1}^j (dq_i) v_i = \sum_{i=j+1}^k (dq_i) v_i \in N,$$

where $d$ is the least common multiple of the denominators of all the nonzero $q_i$’s. Since $v \notin N$, the monoid $N$ cannot be divisor-closed.

CASE 2. rank$(N) < k$. Take $u \in \mathbb{Q}^k$ such that the hyperplane

$$H := \{h \in \mathbb{R}^k \mid \langle h, u \rangle = 0\}$$
of \( \mathbb{R}^k \) contains linearly independent vectors \( v_1, \ldots, v_{k-1} \in M \) such that \( v_1, \ldots, v_r \in N \), where \( r = \text{rank}(N) \). Consider the following two subcases.

CASE 2.1. \( H \) is a supporting hyperplane of \( \text{cone}(M) \). Because \( N \) is not a face submonoid of \( M \), the face \( F := H \cap \text{cone}(M) \) of \( \text{cone}(M) \) must contain an element of \( M \setminus N \). Since \( \text{cone}(M \cap F) = \text{cone}(M) \cap F = F \), we have that \( N \) is not a face submonoid of \( M \cap F \). As \( \text{rank}(M \cap F) < \text{rank}(M) \), it follows by induction that \( N \) is not a divisor-closed submonoid of \( M \cap F \). Therefore \( N \) cannot be a divisor-closed submonoid of \( M \).

CASE 2.2. \( H \) is not a supporting hyperplane of \( \text{cone}(M) \). In this case, there exist \( w_{r+1}, w'_{r+1} \in M \) such that \( \langle w_{r+1}, u \rangle > 0 \) and \( \langle w'_{r+1}, u \rangle < 0 \). As \( \{v_1, \ldots, v_{k-1}, w_{r+1}\} \) is a basis for \( \mathbb{R}^k \) there exists \( w_{r+2} \in M \cap \text{intcone}(v_1, \ldots, v_{k-1}, w'_{r+1}) \) satisfying that \( S := \{v_1, \ldots, v_r\} \cup \{w_{r+1}, w_{r+2}\} \) is linearly dependent. Clearly, \( \langle w_{r+2}, u \rangle < 0 \). After relabeling the vectors \( v_1, \ldots, v_r \) (if necessary),

\[
(4.1) \quad \sum_{i=1}^j q_i v_i = \left( \sum_{i=j+1}^r q_i v_i \right) + q_{r+1} w_{r+1} + q_{r+2} w_{r+2}
\]

for some \( j \in [1,r] \), and coefficients \( q_1, \ldots, q_{r+1} \in \mathbb{Q}_{\geq 0} \), and \( q_{r+2} \in \mathbb{Q} \) (not all zeros). Observe that both coefficients \( q_{r+1} \) and \( q_{r+2} \) are different from zero. After taking the scalar product with \( u \) in both sides of (4.1), one obtains that

\[
q_{r+2} \langle w_{r+2}, u \rangle = -q_{r+1} \langle w_{r+1}, u \rangle.
\]

Hence \( q_{r+1} \) and \( q_{r+2} \) are both positive. Now we can multiply (4.1) by the common denominator \( d \) of all nonzero \( q_i \), to obtain that \( w_{r+1} \mid_M \sum_{i=1}^j (dq_i) v_i \). Since \( w_{r+1} \not\in N \), we have that \( N \) is not divisor-closed, which concludes the proof. \( \square \)

5. Geometry and Factoriality

5.1. Unique Factorization Monoids. In this section we study the factoriality of members of \( \mathcal{C} \) in connection with the geometric properties of their corresponding cones. We shall provide geometric characterizations of the UFMs, HFMs, and OHFMs in \( \mathcal{C} \).

To begin with, let us characterize the UFMs in \( \mathcal{C} \).

**Proposition 5.1.** For a monoid in \( \mathcal{C} \), the following conditions are equivalent.

1. \( M \) is a UFM.
2. Each face submonoid of \( M \) is a UFM.
3. \( |\mathcal{A}(M)| = \text{dim} \text{cone}(M) \).

**Proof.** To prove that (1) implies (3), we will first verify that \( |\mathcal{A}(M)| \geq \text{dim} \text{cone}(M) \). Such inequality holds trivially if \( M \) contains infinitely many atoms. Then suppose that \( \mathcal{A}(M) \) is finite. By Farkas-Minkowski-Weyl Theorem, \( \text{cone}(M) \) is polyhedral. As a result, \( \text{cone}(M) \) contains at least \( \text{dim} \text{cone}(M) \) 1-dimensional edges. Since we
have that \( \text{cone}(M) = \text{cone}(\mathcal{A}(M)) \), any 1-dimensional edge of \( \text{cone}(M) \) must contain an atom of \( M \). Thus, \( |\mathcal{A}(M)| \geq \dim \text{cone}(M) \), as desired. Suppose now, by way of contradiction, that \( |\mathcal{A}(M)| > \dim \text{cone}(M) \). Let \( a_1, \ldots, a_{d+1} \in \mathcal{A}(M) \) be distinct atoms. Then

\[
\sum_{i=1}^{d+1} \beta_i a_i = 0 \quad \text{for some } \beta_1, \ldots, \beta_{d+1} \in \mathbb{Q} \text{ not all zeros.}
\]

There is no loss in assuming that there exists an index \( k \in [1, d] \) such that \( \beta_i < 0 \) for \( i \in [1, k] \) and \( \beta_i \geq 0 \) for \( i \in [k+1, d+1] \). Hence

\[
\sum_{i=1}^{k} \beta_i a_i \quad \text{and} \quad \sum_{i=k+1}^{d+1} (-\beta_i) a_i
\]

are two distinct factorizations of the same element of \( M \), contradicting that \( M \) is a UFM.

To show that (3) implies (2), set \( d := \dim \text{cone}(M) \) and suppose that \( |\mathcal{A}(M)| = d \). Let \( N \) be a face submonoid of \( M \). Since \( |\mathcal{A}(M)| = d \), Farkas-Minkowski-Weyl Theorem ensures that \( \text{cone}(M) \) is polyhedral. Then \( N = M \cap H \) for some supporting hyperplane \( H = \{ x \in \mathbb{R}^d \mid \langle x, u \rangle = 0 \} \) determined by \( u \in \mathbb{R}^d \). Suppose that \( \text{cone}(M) \subseteq H^\perp \). Then if \( x \in N \) and \( \sum_{i=1}^{t} a_i \in \mathbb{Z}_M(x) \), one has that \( \sum_{i=1}^{t} \langle a_i, u \rangle = 0 \) and, therefore, \( \langle a_i, u \rangle = 0 \) for \( i \in [1, t] \). This implies that \( \sum_{i=1}^{t} a_i \in \mathbb{Z}_N(x) \). As a consequence, \( \mathcal{A}(N) = \mathcal{A}(M) \cap N \). Thus, \( N \) is a UFM, and (2) follows.

As (2) trivially implies (1), our proof is complete.

**Corollary 5.2.** Let \( M \) be a UFM in \( \mathcal{C} \). Then \( \text{cone}(M) \) is rational and polyhedral.

**Proof.** By Proposition 5.1, the monoid \( M \) is finitely generated and so \( \text{cone}(M) \) is the conic hull of a finite set. Now the corollary follows by Farkas-Minkowski-Weyl Theorem. \( \square \)

### 5.2. Half-Factorial Monoids

The concept of half-factoriality is a weaker version of that one of factoriality (or being a UFD). We proceed to offer characterizations of half-factorial monoids in the class \( \mathcal{C} \) in terms of their face submonoids and in terms of the convex hull of their sets of atoms.

**Definition 5.3.** An atomic monoid \( M \) is called a half-factorial monoid (or an HFM) provided that for all \( x \in M^\bullet \) and \( z, z' \in \mathbb{Z}(x) \), we have that \( |z| = |z'| \).

HFMs in \( \mathcal{C} \) can be characterized as follows.

**Proposition 5.4.** For a monoid \( M \) in \( \mathcal{C} \) the next conditions are equivalent.

1. \( M \) is an HFM.
2. Each face submonoid of \( M \) is an HFM.
3. \( \dim \text{conv}(\mathcal{A}(M)) < \dim \text{cone}(M) \).
Proof. First, we show that (1) implies (3). To do this, suppose that $M$ is an HFM. Set $d := \dim \text{cone}(M)$. Since $\text{cone}(M) = \text{cone}(\mathcal{A}(M))$, one can take linearly independent vectors $a_1, \ldots, a_d$ in $\mathcal{A}(M)$. Take also $u \in \mathbb{Q}^d$ and $\alpha \in \mathbb{Q}$ such that the polytope $\text{conv}(a_1, \ldots, a_d)$ is contained in the affine hyperplane $H := \{q \in \mathbb{Q}^d \mid \langle q, u \rangle = \alpha \}$. In addition, fix $a \in \mathcal{A}(M)$, and write $a = \sum_{i=1}^d \beta_i a_i$ for some $\beta_1, \ldots, \beta_d \in \mathbb{Q}$. From the fact that $M$ is an HFM, we can deduce that $\sum_{i=1}^d \beta_i = 1$. As a result,

$$\langle a, u \rangle = \sum_{i=1}^d \beta_i \langle a_i, u \rangle = \alpha \sum_{i=1}^d \beta_i = \alpha,$$

which means that $a \in H$. Hence $\mathcal{A}(M) \subset H$, which implies that $\dim \text{conv}(\mathcal{A}(M))$ is at most $d - 1$. Then we have that $\dim \text{conv}(\mathcal{A}(M)) < \dim \text{cone}(M)$, as desired.

To argue that (3) implies (2), suppose that $\dim \text{conv}(\mathcal{A}(M)) < \dim \text{cone}(M)$. Then there exists an affine hyperplane $H$ containing $\text{conv}(\mathcal{A}(M))$. As in the previous paragraph, take $u \in \mathbb{Q}^d$ and $\alpha \in \mathbb{Q}$ such that $H = \{q \in \mathbb{Q}^d \mid \langle q, u \rangle = \alpha \}$. Now if $x \in M$ and

$$z := \sum_{a \in \mathcal{A}(M)} \beta_a a \in \mathbb{Z}(x),$$

then

$$|z| = \sum_{a \in \mathcal{A}(M)} \beta_a = \frac{1}{\alpha} \sum_{a \in \mathcal{A}(M)} \beta_a \langle a, u \rangle = \frac{1}{\alpha} \left\langle \sum_{a \in \mathcal{A}(M)} \beta_a a, u \right\rangle = \frac{1}{\alpha} \langle x, u \rangle.$$

Hence $L(x) = \{1/\alpha \langle x, u \rangle \}$ for all $x \in M^\bullet$, and so $M$ is an HFM.

That (2) implies (1) follows trivially. \qed

Corollary 5.5. A monoid $M$ in $\mathcal{C}_d$ is an HFM if and only if $\mathcal{A}(M)$ is contained in an affine hyperplane of $\mathbb{Q} \otimes_{\mathbb{Z}} \text{gp}(M)$.

Remark 5.6. Corollary 5.5 has been previously established by F. Kainrath and G. Lettl in [36]. Fairly similar versions of the same result were first given by Zaks [46] and Narkiewicz [38].

The chain of implications (5.1), where being a UFM, an HFM, and an atomic monoid are included, has received a great deal of attention since it was first studied (in the context of integral domains) by Anderson, Anderson, and Zafrullah [1]:

(5.1) UFM $\Rightarrow$ HFM $\Rightarrow$ FFM $\Rightarrow$ BFM $\Rightarrow$ ACCP monoid $\Rightarrow$ atomic monoid.

The first three implications above are obvious, while the last two implications follow from [23, Proposition 1.1.4] and [23, Corollary 1.3.3]. In addition, all the implications above are strict, and examples witnessing this observation (in the context of integral domains) can be found in [1]. We have already seen that not every monoid in $\mathcal{C}$ is an HFM. However, each monoid in $\mathcal{C}$ is an FFM, as the next proposition illustrates.

Proposition 5.7. Each monoid in $\mathcal{C}$ is an FFM.
Proof. By Theorem 3.1, it suffices to show that for every $d \in \mathbb{Z}_{\geq 1}$, any additive submonoid $M$ of $\mathbb{N}^d$ is an FFM. Fix $x \in M$. It is clear that $\langle x, y \rangle \geq 0$ for all $y \in M$. Thus, $y \mid_M x$ implies that $\|y\| \leq \|x\|$. As a result, the set
\[
\{ a \in A(M) : a \mid_M x \}
\]
is finite, which implies that $Z(x)$ is also finite. Hence $M$ is an FFM. □

As an immediate consequence of Proposition 5.7, every monoid in $C$ satisfies the last four conditions in the chain of implications (5.1).

5.3. Other-Half-Factorial Monoids. Other-half-factoriality is a dual version of half-factoriality and was introduced by Coykendall and Smith in [12].

Definition 5.8. An atomic monoid $M$ is called an other-half-factorial monoid (or an OHFM) provided that for all $x \in M^*$ and $z, z' \in Z(x)$ with $|z| = |z'|$, we have that $z = z'$.

Although an integral domain is a UFD if and only if its multiplicative monoid is an OHFM [12, Corollary 2.11], an OHFM is not, in general, a UFM or an HFM, as one can deduce from the next theorem.

A set of points in a $d$-dimensional $\mathbb{F}$-space $V$ (where $\mathbb{F}$ is either $\mathbb{Q}$ or $\mathbb{R}$) is said to be affinely independent provided that no $k$ of such points lie in a $(k - 2)$-dimensional affine subspace of $V$ for $k \in [2, d + 1]$. If a set is affinely independent, its points are said to be in general linear position.

Theorem 5.9. Let $M$ be a nontrivial monoid in $C$. Then the following statements are equivalent.

1. $M$ is an OHFM.
2. every face submonoid of $M$ is an OHFM.
3. The points in $A(M)$ are affinely independent.
4. $\text{conv}(A(M))$ is a simplex with dimension either $\text{rank}(M) - 1$ or $\text{rank}(M)$.

Proof. It is omitted to avoid duplications (it will appear in [8]). □

Corollary 5.10. Let $N$ be a numerical monoid. Then $N$ is an OHFM if and only if the embedding dimension of $N$ is at most 2.

Remark 5.11. The characterization proposed in Theorem 5.9 was indeed motivated by Corollary 5.10, which was first proved by Coykendall and Smith in [12].

The fact that every proper face submonoid of a monoid $M$ in $C$ is an OHFM does not guarantee that $M$ is an OHFM, as one can see in the following example.
Example 5.12. Consider the submonoid $M := \langle 2e_1, 3e_1, 2e_2, 3e_2 \rangle$ of $(\mathbb{N}^2, +)$. It is easy to argue that $\mathcal{A}(M) = \{2e_1, 3e_1, 2e_2, 3e_2\}$. Notice that the 1-dimensional faces of $\text{cone}_{\mathbb{Z}}(M)$ are $\mathbb{R}_{\geq 0}e_1$ and $\mathbb{R}_{\geq 0}e_2$. Then there are two face submonoids of $M$ corresponding to 1-dimensional faces of $\text{cone}(M)$, and they are both isomorphic to the numerical monoid $\langle 2, 3 \rangle$, which is an OHFM by Corollary 5.10. Hence every proper face submonoid of $M$ is an OHFM. However, $\text{conv}(\mathcal{A}(M))$ is not a simplex and, therefore, it follows by Theorem 5.9 that $M$ is not an OHFM.

We conclude this section with the following proposition.

Proposition 5.13. Let $M$ be an OHFM in $\mathcal{C}$. Then the faces of $\text{cone}(M)$ whose corresponding face submonoids are not UFMs form a (possibly empty) interval in the face lattice $\mathcal{F}(\text{cone}(M))$.

Proof. It is omitted to avoid duplications (it will appear in [8]). \qed

The reverse implication of Proposition 5.13 does not hold, as the next example illustrates.

Example 5.14. Consider the submonoid $M := \langle 3e_1, 3e_2, 2e_3, 3e_3 \rangle$ of $(\mathbb{N}^3, +)$. It can be readily verified that $\mathcal{A}(M) = \{3e_1, 3e_2, 2e_3, 3e_3\}$. Since $\{2e_3, 3e_3\}$ is an affinely dependent set, it follows by Theorem 5.9 that $M$ is not an OHFM. However, the non-UFM face submonoids of $M$ are precisely those determined by the faces of $\text{cone}(M)$ contained in the interval $[\mathbb{R}e_3, \text{cone}(M)]$.

6. Cones of Primary Monoids and Finitary Monoids

As mentioned at the beginning of this paper, primary monoids and finitary monoids have been crucial in the development of non-unique factorization theory as the factorization structure of members in these two classes abstracts certain properties of important classes of integral domains. In the first part of this section, we investigate some geometric aspects of primary monoids in $\mathcal{C}$. Then we shift our focus to the study of finitary monoids of $\mathcal{C}$.

6.1. Primary Monoids. A monoid $M$ is called primary provided that $M$ is nontrivial and for all $a, b \in M^\bullet$ there exists $n \in \mathbb{N}$ such that $nb \in a + M$. The primary monoids in $\mathcal{C}$ are precisely those minimizing the number of face submonoids.

Proposition 6.1. For a nontrivial monoid $M$ in $\mathcal{C}$, the following conditions are equivalent.

1. $M$ is primary.
2. The only face submonoids of $M$ are $\{0\}$ and $M$.
3. $\text{cone}(M)^\bullet$ is an open subset of $\mathbb{R} \otimes_{\mathbb{Z}} \text{gp}(M)$. 
Proof. It follows from [23, Lemma 2.7.7] that $M$ is primary if and only if the only divisor-closed submonoids of $M$ are $\{0\}$ and $M$. This, along with Theorem 4.5, implies that the conditions (1) and (2) are equivalent.

To argue that (2) implies (3), take $x \in \text{cone}(M)\bullet$. Since $\text{cone}(M)$ is the disjoint union of the relative interiors of all its faces, there exists a face $F$ of $\text{cone}(M)$ such that $x \in \text{relin} F$. As $x \neq 0$, the dimension of $F$ is at least 1 and, therefore, $M \cap F$ is a nontrivial face submonoid of $M$. It follows now by (2) that $M \cap F = M$ and, therefore, $x \in \text{relin cone}(M)$. Hence $\text{cone}(M)^\bullet$ is open.

Finally, let us verify that (3) implies (2). Since every proper face of $\text{cone}(M)$ is contained in the boundary of $\text{cone}(M)$, the fact that $\text{cone}(M)^\bullet$ is open implies that the only proper face of $\text{cone}(M)$ is the origin, from which (2) follows. □

Remark 6.2. We want to emphasize that the fact that (1) and (3) are equivalent conditions in Proposition 6.1 was first established by Geroldinger, Halter-Koch, and Lettl [25, Theorem 2.4]. However, we obtain such a result here from the poset structure of the face lattice of $\text{cone}(M)$.

Primary monoids in $C$ account for all primary submonoids of any (non-necessarily finite-rank) free commutative monoid, as the next proposition illustrates.

Proposition 6.3. Let $M$ be a primary submonoid of a free commutative monoid. Then $M$ has finite rank, and $M$ can be embedded into $(\mathbb{N}^r, +)$, where $r = \text{rank}(M)$.

Proof. Let $F_P$ be a free commutative monoid on an infinite set $P$ such that $M$ is a submonoid of $F_P$. For $s \in F_P$ and $S \subseteq F_P$, write

$$\text{Spec}(s) := \{p \in P \mid p \text{ divides } s \text{ in } F_P\} \quad \text{and} \quad \text{Spec}(S) := \bigcup_{s \in S} \text{Spec}(s).$$

Suppose, by way of contradiction, that $\text{Spec}(M)$ contains infinitely many elements. Fix $x \in M^\bullet$, and take $p \in P$ such that $p \in \text{Spec}(M) \setminus \text{Spec}(x)$. Since $p$ is a prime element of $F_P$, it is clear that the set

$$S := \{x \in M \mid p \text{ does not divide } x \text{ in } F_P\}$$

is a divisor-closed submonoid of $M$. The fact that $p \notin \text{Spec}(x)$ implies that $S$ is a nontrivial submonoid of $M$, and the fact that $p \in \text{Spec}(M)$ implies that $S \neq M$. Thus, $S$ is a proper nontrivial divisor-closed submonoid of $M$, which contradicts that $M$ is primary. Hence $\text{Spec}(M)$ is finite and, as a result, $M$ can be naturally embedded into $\mathbb{N}p_1 \oplus \cdots \oplus \mathbb{N}p_t$, where $p_1, \ldots, p_t$ are the prime elements $\text{Spec}(M)$. It follows now from Theorem 3.1 that $M$ can be embedded into $(\mathbb{N}^r, +)$. □
6.2. Finitely Primary Monoids. Now we restrict our attention to a special subclass of primary monoids, that one consisting of finitely primary monoids. The complete integral closure of a monoid $M$, denoted by $\hat{M}$, is defined as follows:

$$\hat{M} := \{ x \in \text{gp}(M) \mid \text{there exists } y \in M \text{ such that } nx + y \in M \text{ for every } n \in \mathbb{N} \}.$$ 

Clearly, $\hat{M}$ is a submonoid of $\text{gp}(M)$ containing $M$, and so $\text{rank}(\hat{M}) = \text{rank}(M)$. A monoid $M$ is called finitely primary if there exist $d \in \mathbb{N}$ and a UFM $F := \langle p_1, \ldots, p_d \rangle$, where $p_1, \ldots, p_d$ are pairwise distinct prime elements, such that

1. $M$ is a submonoid of $F$,
2. $M^* \subseteq p_1 + \cdots + p_d + F$, and
3. $\alpha(p_1 + \cdots + p_d) + F \subseteq M$ for some $\alpha \in \mathbb{N}^*$.

In this case, it follows by [23, Theorem 2.9.2] that $\hat{M} \cong (\mathbb{N}^d, +)$. Then $\text{rank}(M) = d$ and, moreover, any finitely primary monoid of rank $d$ is in $C_d$. On the other hand, it also follows from [23, Theorem 2.9.2] that finitely primary monoids are primary. Therefore, it follows from Proposition 6.1 that for any finitely primary monoid $M$ the set $\text{cone}(M)^*$ is open. As the next proposition reveals, the closure of the same set happens to be a simplicial cone.

**Proposition 6.4.** If $M$ is a finitely primary monoid, then $M$ is in $\mathcal{C}$ and $\overline{\text{cone}(M)}$ is a rational simplicial cone.

**Proof.** Let $d$ be the rank of $M$. We have already observed that $M$ is in the class $\mathcal{C}$. For the rest of the proof, assume that $M \subseteq \hat{M} \subseteq \mathbb{N}^d$. Because $\hat{M} \cong (\mathbb{N}^d, +)$, one can take distinct prime elements $p_1, \ldots, p_d$ of $\hat{M}$ such that $\hat{M} = \langle p_1, \ldots, p_d \rangle = \mathbb{N}p_1 \oplus \cdots \oplus \mathbb{N}p_d$. It follows from [23, Theorem 2.9.2] that

$$M^* \subseteq p_1 + \cdots + p_d + \hat{M} \quad \text{and} \quad \alpha(p_1 + \cdots + p_d) + \hat{M} \subseteq M,$$

for some $\alpha \in \mathbb{N}^*$. Let $C_p$ be the cone in $\mathbb{R}^d$ generated by $p_1, \ldots, p_d$. Clearly, $C_p$ is a rational simplicial cone of dimension $d$. We claim that $\overline{\text{cone}(M)} = C_p$. Since

$$M^* \subseteq p_1 + \cdots + p_d + \hat{M} \subseteq \text{int} C_p,$$

we have that $M \subseteq C_p$. Therefore $\overline{\text{cone}(M)} \subseteq C_p$ and, as the cone $C_p$ is closed, $\overline{\text{cone}(M)} \subseteq C_p$. Let us proceed to argue that $C_p \subseteq \overline{\text{cone}(M)}$. To do so, fix $\epsilon > 0$ and fix also an index $j \in [1, d]$. Let $L$ be the 1-dimensional face of $C_p$ in the direction of the vector $p_j$, and consider the conical open ball with central axis $L$ given by

$$B(p_j, \epsilon) := \left\{ w \in \mathbb{R}^d \setminus \{0\} \mid \frac{\|w - P_L(w)\|}{\|w\|} < \epsilon \right\},$$

where $P_L: \mathbb{R}^d \to \mathbb{R}p_j$ is the linear projection of $\mathbb{R}^d$ onto its subspace $\mathbb{R}p_j$. It is clear that the set

$$\{0\} \cup (B(p_j, \epsilon) \cap \text{int} C_p)$$
is a $d$-dimensional subcone of $C_p$ and, therefore, it must intersect $\widehat{M}$. Then one can take $y \in \widehat{M} \cap \text{int} C_p$ such that $\mathbb{R}_{>0}y \subset B(p_j, \epsilon)$. Because

$$\alpha(\widehat{M} \cap \text{int} C_p) \subseteq \alpha(p_1 + \cdots + p_d) + \widehat{M} \subseteq M,$$

we have that $\alpha y \in M$. As a result, $\mathbb{R}_{>0}y \subset \text{cone}(M)$. As $\text{cone}(M)$ and every open conical ball with central axis $L$ have an open ray in common, $p_j \in L \subset \text{cone}(M)$. As the index $j$ was arbitrarily taken, $p_j \in \text{cone}(M)$ for every $j \in [1, d]$, and so $C_p \subseteq \text{cone}(M)$. Hence $\text{cone}(M)$ is a rational simplicial cone. \qed

For a primary monoid $M$ in $\mathcal{C}$, the fact that $\overline{\text{cone}(M)}$ is rational and simplicial does not imply that $M$ is finitely primary. The following example sheds some light upon this observation.

**Example 6.5.** Consider the subset $M$ of $\mathbb{N}^2$ defined by

$$M := \{(0, 0)\} \cup \{(n, m) \in \mathbb{N}^2 \mid n, m \in \mathbb{N}^* \text{ and } m \leq 2^n\}.$$

From the fact that $f(x) = 2^x$ is a convex function, one can readily verify that $M$ is a submonoid of $(\mathbb{N}^2, +)$. Since $M$ contains $(n, 1)$ for every $n \in \mathbb{N}^*$ and $M$, the ray $\mathbb{R}_{\geq 0}e_1$ is contained in $\text{cone}_{\mathbb{R}^2}(M)$. On the other hand, the fact that $\{(n, 2^n) \mid n \in \mathbb{N}^*\} \subset M$, along with $\lim_{n \to \infty} 2^n/n = \infty$, guarantees that the ray $\mathbb{R}_{\geq 0}e_2$ is contained in $\text{cone}_{\mathbb{R}^2}(M)$. Thus,

$$\text{cone}_{\mathbb{R}^2}(M) = \{(0, 0)\} \cup \{(x, y) \in \mathbb{R}^2 \mid x, y > 0\} = \{(0, 0)\} \cup \mathbb{R}_{>0}^2.$$

As $\text{cone}(M)^*$ is open, Proposition 6.1 ensures that $M$ is a primary monoid. On the other hand, $\text{cone}(M) = \mathbb{R}_{\geq 0}^2$ is a rational simplicial cone.

To argue that $M$ is not finitely primary, it suffices to verify that $\widehat{M} \not\cong (\mathbb{N}^2, +)$. To do so, fix $m \in \mathbb{N}$, and then take $N \in \mathbb{N}$ large enough so that $nm \leq 2^n$ for every $n \geq N$. Note that $y := (N,Nm)$ belongs to $M$. Moreover,

$$n(1, m) + y = (n + N, (n + N)m) \in M$$

for every $n \in \mathbb{N}$. Therefore $(1, m) \in \widehat{M}$ for every $m \in \mathbb{N}$. On the other hand, for any $m \in \mathbb{N}^*$ and $(a, b) \in M^*$,

$$2^n(0, m) + (a, b) = (a, 2^n m + b) \notin M.$$

Hence $(n, m) \in \widehat{M}^*$ implies that $n > 0$. As a result,

$$\widehat{M} = \{(n, m) \in \mathbb{N}^2 \mid n > 0\}.$$

Since $\mathcal{A}(\widehat{M}) = \{(1, n) \mid n \in \mathbb{N}\}$ contains infinitely many elements, $\widehat{M} \not\cong (\mathbb{N}^2, +)$. Hence $M$ cannot be finitely primary.
6.3. Finitary Monoids. Let $M$ be a monoid. We say that $M$ is weakly finitary if there exist a finite subset $S$ of $M$ and $n \in \mathbb{N}^*$ such that $nx \in S + M$ for all $x \in M^*$. In addition, a BFM $M$ is called finitary if there exist a finite subset $S$ of $M$ and $n \in \mathbb{N}^*$ such that $nM^* \subseteq S + M$. Clearly, every finitary monoid is weakly finitary. In addition, every finitely generated monoid is finitary. Also, affine monoids are finitary.

The face submonoids of a monoid in $\mathcal{C}$ inherit the condition of being (weakly) finitary.

**Proposition 6.6.** Let $M$ be a monoid in $\mathcal{C}$. Then $M$ is finitary (resp., weakly finitary) if and only if each face submonoid of $M$ is finitary (resp., weakly finitary).

**Proof.** We will prove only the finitary version of the proposition as the weakly finitary version follows similarly. Suppose that $M$ is finitary, and let $d$ be the rank of $M$. Take $F$ to be a face of $\text{cone}(M)$, and consider the face submonoid $N := M \cap F$. Since $M$ is finitary, there exist $n \in \mathbb{N}$ and a finite subset $S$ of $M$ such that $nM^* \subseteq S + M$. We claim that $nN^* \subseteq SF + N$, where $SF := S \cap F$. Take $x_1, \ldots, x_n \in N^* = M^* \cap F$. As

$$n(M^* \cap F) \subseteq nM^* \subseteq S + M,$$

there exist $s \in S$ and $y \in M$ such that $x_1 + \cdots + x_n = s + y$. Since $M \cap F$ is a divisor-closed submonoid of $M$, we find that $s, y \in F$. Therefore $s \in SF$ and $y \in N$, which implies that $x_1 + \cdots + x_n \in SF + N$. Hence $N$ is a finitary monoid. The reverse implication follows trivially as $\text{cone}(M)$ is a face of itself. \hfill $\Box$

Our next goal is to give a sufficient geometric condition for a monoid in $\mathcal{C}$ to be finitary. First, let us recall the concept of triangulation. A *conical polyhedral complex* $P$ in $\mathbb{R}^d$ is a collection of polyhedral cones in $\mathbb{R}^d$ satisfying the following conditions:

1. Every face of a polyhedron in $P$ is also in $P$;
2. The intersection of any two polyhedral cones $C_1$ and $C_2$ in $P$ is a face of both $C_1$ and $C_2$.

Clearly, the underlying set of the face lattice of a given polyhedral cone is a conical polyhedral complex. For a conical polyhedral complex $P$ in $\mathbb{R}^d$, we set $|P| := \cup_{C \in P} C$. Let $P$ and $P'$ be two conical polyhedral complexes. We say that $P'$ is a *polyhedral subdivision* of $P$ provided that $|P| = |P'|$ and each face of $P$ is the union of faces of $P'$. A polyhedral subdivision $P'$ of $P$ is called a *triangulation* of $P$ if $P'$ consists of simplicial cones. Every conical polyhedral complex has certain special triangulations.

**Theorem 6.7.** [5, Theorem 1.54] Let $P$ be a conical polyhedral complex, and let $S \subseteq |P|$ be a finite set of nonzero vectors such that $S \cap C$ generates $C$ for each $C \in P$. Then there exists a triangulation $P'$ of $P$ such that $\{ \mathbb{R}_{\geq 0}v \mid v \in S \}$ is the set of 1-dimensional faces of $P'$.

We are in a position now to offer a sufficient geometric condition for a monoid in $\mathcal{C}$ to be finitary.

**Theorem 6.8.** Let $M$ be a monoid in $\mathcal{C}$. If $\text{cone}(M)$ is polyhedral, then $M$ is finitary.
Proof. Let $d$ be the rank of $M$, and assume that $M \subseteq \mathbb{N}^d$. Since $\text{cone}(M)$ is polyhedral, it follows by Farkas-Minkowski-Weyl Theorem that $\text{cone}(M)$ is the conic hull of a finite set of vectors. As the vectors in such a generating set are nonnegative rational linear combinations of vectors in $M$, there exists $S = \{v_1, \ldots, v_k\} \subseteq M$ with $k \geq d$ such that $\text{cone}(M) = \text{cone}(S)$. By Theorem 6.7, there exists a triangulation $\mathcal{T}$ of the face lattice of $\text{cone}(M)$ whose set of 1-dimensional faces is $\{\mathbb{R}_{\geq 0}v_i \mid i \in [1,k]\}$. Then for any $T \in \mathcal{T}$ there are unique indices $t_1, \ldots, t_d$ satisfying that

$$1 \leq t_1 < \cdots < t_d \leq k \quad \text{and} \quad T = \text{cone}(v_{t_1}, \ldots, v_{t_d}),$$

and we can use this to assign to $T$ the paralllelepiped

$$\Pi_T := \{\alpha_1v_{t_1} + \cdots + \alpha_dv_{t_d} \mid 0 \leq \alpha_i < 1 \text{ for every } i \in [1,d]\},$$

It is clear that

$$|\Pi_T \cap \mathbb{Z}^d| < \infty \quad \text{and} \quad \Pi_T \cap \mathbb{Z}^d \subseteq \mathbb{Q}_{\geq 0}v_{t_1} + \cdots + \mathbb{Q}_{\geq 0}v_{t_d}.$$  

Then we can choose $N_T \in \mathbb{N}$ large enough so that $N_Tv \in \mathbb{N}v_{t_1} + \cdots + \mathbb{N}v_{t_d}$ for every $v \in \Pi_T \cap \mathbb{Z}^d$. Now take

$$m := \max\{N_T |\Pi_T \cap \mathbb{Z}^d| : T \in \mathcal{T}\}$$

and set $n := m|\mathcal{T}|$. In order to show that $M$ is finitary, it suffices to verify that $nM^* \subseteq S + M$.

Take (possibly repeated) elements $x_1, \ldots, x_n \in M^*$. For every $x \in \{x_1, \ldots, x_n\}$, there exists $T \in \mathcal{T}$ with $x \in T$. Let $T = \text{cone}(v_{t_1}, \ldots, v_{t_d})$ for $t_1 < \cdots < t_d$ be a simplicial cone in $\mathcal{T}$. Observe that we can naturally partition $T$ into (translated) copies of the paralllelepiped $\Pi_T$, that is, $T$ equals the disjoint union of the sets $v + \Pi_T$ for $v \in \Pi_T \cap \mathbb{Z}^d$. As a result, there exist $z \in \Pi_T \cap \mathbb{Z}^d$ and coefficients $\alpha_1, \ldots, \alpha_d \in \mathbb{N}$ such that

$$(6.1) \quad x = z + \sum_{i=1}^d \alpha_iv_{t_i}.$$  

Hence for $i \in [1,n]$, we can write $x_i = z_i + m_i$ for some $z_i \in \bigcup_{T \in \mathcal{T}} \Pi_T \cap \mathbb{Z}^d$ and $m_i \in M$. Since $n = m|\mathcal{T}|$, there exists $T_0 \in \mathcal{T}$ such that

$$|\{i \in [1,n] \mid z_i \in \Pi_{T_0} \cap \mathbb{Z}^d\}| \geq m.$$  

Consider now the equivalence relation on the set of indices $\{i \in [1,n] \mid z_i \in T_0\}$ defined by $i \sim j$ whenever $z_i = z_j$. The fact that $m \geq N_{T_0}|\Pi_{T_0} \cap \mathbb{Z}^d|$ guarantees the existence of a class $I$ determined by the relation $\sim$ and containing at least $N_{T_0}$ distinct indices. Take $I_0 \subseteq I$ such that $|I_0| = N_{T_0}$. Setting $z := z_i$ for some $i \in I_0$, one has that

$$\sum_{i \in I_0} z_i = N_{T_0}z \in \mathbb{N}v_1 + \cdots + \mathbb{N}v_n \in S + M.$$
and, therefore, there exist \( v \in S \) and \( m \in M \) such that \( \sum_{i \in I_0} z_i = v + m \). As a result, one can set \( m' = \sum_{i=1}^n x_i - \sum_{i \in I_0} x_i \in M \) to obtain that
\[
\sum_{i=1}^n x_i = \left( \sum_{i \in I_0} x_i \right) + m' = \sum_{i \in I_0} z_i + m' + \sum_{i \in I_0} m_i = v + \left( m + m' + \sum_{i \in I_0} m_i \right) \in S + M.
\]
Since the elements \( x_1, \ldots, x_n \) were arbitrarily taken in \( M^* \), the inclusion \( nM^* \subseteq S + M \) holds. Hence the monoid \( M \) is finitary, as desired. \( \square \)

According to the characterization of cones generated by monoids in \( C \) we have provided in Theorem 3.13, every \( d \)-dimensional positive cone \( C \) of \( R^d \) with \( C^* \) open can be generated by a monoid in \( C \). Indeed, any such a cone can be generated by a finitary monoid in \( C \).

**Proposition 6.9.** For \( d \in \mathbb{N} \), let \( C \) be a positive cone in \( R^d \). If \( C^* \) is open in \( R^d \), then \( C \) can be generated by a finitary monoid in \( C \).

**Proof.** Assume that \( C^* \) is open in \( R^d \). Take \( M = \mathbb{N}^d \cap C \). It is clear that \( C = \text{cone}(M) \).

Now take \( v_0 \in M^* \), and consider the monoid \( M' := \{0\} \cup (v_0 + M) \). Let \( C' \) be the cone generated by \( M' \). Notice that \( Q^d \cap C \) and \( Q^d \cap C' \) are the cones generated by \( M \) and \( M' \) over \( Q \), respectively. So proving that \( C' = C \) amounts to showing that \( Q^d \cap C' = Q^d \cap C \) (see [5, Proposition 1.70]). Since \( M' \subseteq M \) it follows that \( Q^d \cap C' \subseteq Q^d \cap C \). Now let \( \ell_0 \) be the distance from \( \{v_0\} \) to \( R^d \setminus C \). As \( R^d \setminus C \) is closed and \( \{v_0\} \) is compact, \( \ell_0 > 0 \). Now take \( v \in Q^d \cap C^* \) such that \( \|v\| > 1 \), and let \( \ell \) be the distance from \( v \) to \( R^d \setminus C \).

By a similar argument, \( \ell > 0 \). Notice that the conical ball
\[
B(v, \ell) := \left\{ w \in Q^d \left| \frac{\|w - P_v(w)\|}{\|w\|} < \frac{\ell}{2} \right. \right\}
\]
is contained in \( Q^d \cap C \). Take \( N \in \mathbb{N} \) such that
\[
N > \max \left\{ \frac{\|v_0\|}{\|v\| - 1}, \frac{2 \|v_0\|}{\ell} \right\}
\]
and \( Nv \in Q^d \). Now set \( w_0 := Nv - v_0 \). Notice that \( \|w_0\| \geq N \|v\| - \|v_0\| > N \). Then we have that
\[
\frac{\|w_0 - P_v(w_0)\|}{\|w_0\|} < \frac{\|v_0 - P_v(v_0)\|}{N} \leq \frac{\|v_0\|}{N} \leq \frac{\ell}{2}.
\]

Hence \( w_0 \in Q^d \cap B(v, \ell) \subseteq Q^d \cap C \), and so there exist coefficients \( c_1, \ldots, c_k \in Q_{> 0} \) and elements \( v_1, \ldots, v_k \in M^* \) such that \( w_0 = \sum_{i=1}^k c_i v_i \). As a consequence, one has that \( nv \in M' \subseteq \text{cone}(M') \) for some \( n \in \mathbb{N} \), and, therefore, \( v \in Q^d \cap \text{cone}(M') \). Hence \( Q^d \cap C^* \subseteq Q^d \cap C \).

As \( M' \) generates \( C \), we only need to verify that \( M' \) is finitary. Take \( w_1, w_2 \in M^* \), and then \( v_1, v_2 \in M \) such that \( w_1 = v_0 + v_1 \) and \( w_2 = v_0 + v_2 \). Then
\[
w_1 + w_2 = v_0 + (v_0 + v_1 + v_2) \in v_0 + (v_0 + M) \subseteq v_0 + M'.
\]
As a result, $2M' \subseteq v_0 + M'$, which implies that $M'$ is a finitary monoid, as desired. \hfill \Box

Theorem 6.8 and Proposition 6.9 indicate that there is a huge variety of finitary monoids in $C$. We proceed to exhibit a monoid in $C_2$ that is not even weakly finitary. First, let us introduce the following notation.

**Notation:** For $x \in \mathbb{R}_{\geq 0}^2 \setminus \{0\}$, we let $\text{slope}(x) \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ denote the slope of the line $Rx$, and for $X \subset \mathbb{R}^2_{\geq 0}$ we set

$$\text{slope}(X) := \{\text{slope}(x) \mid x \in X^*\}.$$ 

**Example 6.10.** Construct a sequence $\{v_n\}$ of vectors in $\mathbb{N}^* \times \mathbb{N}^*$ as follows. Set $v_1 = (1, 1)$ and suppose that, for $n \in \mathbb{N}$, we have chosen vectors $v_i = (x_i, y_i) \in \mathbb{N}^* \times \mathbb{N}^*$ satisfying that $\text{slope}(v_i) < \text{slope}(v_{i+1})$ and $i \|v_i\| < \|v_{i+1}\|$ for every $i \in [1, n - 1]$. Then take $v_{n+1} = (x_{n+1}, y_{n+1}) \in \mathbb{N}^2$ such that $x_{n+1} > 0$, $\text{slope}(v_{n+1}) > \text{slope}(v_n)$, and $\|v_{n+1}\| > n \|v_n\|$. Now consider the submonoid $M := \langle v_n \mid n \in \mathbb{N}^* \rangle$ of $(\mathbb{N}^2, +)$. Clearly, $A(M) \subseteq \{v_n \mid n \in \mathbb{N}\}$. On the other hand, the fact that $\|v_m\| > \|v_n\|$ when $m > n$ implies that only atoms in $\{v_1, \ldots, v_{n-1}\}$ can divide $v_n$ in $M$. This, along with the fact that

$$\text{slope}(v_n) > \max\{\text{slope}(v_i) \mid i \in [1, n - 1]\}$$

for every $n \in \mathbb{N}$, ensures that

$$A(M) = \{v_n \mid n \in \mathbb{N}\}.$$ 

Finally, let us verify that $M$ is not weakly finitary. Assume for a contradiction that there exist $n \in \mathbb{N}$ and a finite subset $S$ of $M$ such that $nx \in S + M$ for all $x \in M^*$. We can assume without loss of generality that $S \subseteq A(M)$, so we let $S = \{v_{n_1}, \ldots, v_{n_k}\}$, where $n_1 < \cdots < n_k$. Take $N > \max\{n_i, n_k\}$. Then write $n'v_N = v_{n_i} + m$ for some $i \in [1, k]$ and $m \in M$ such that $n' \leq n$ and $v_N \not\mid_M m$. Since $\text{slope}(n'v_N) > \text{slope}(v_{n_i})$, there exists $j > N$ such that $v_j \mid_M m$. Therefore

$$\|Nv_N\| > \|n'v_N\| = \|v_{n_i} + m\| > \|m\| \geq \|v_j\| \geq \|v_{n+1}\|,$$

which is a contradiction. Hence $M$ is not weakly finitary.

**6.4. Strongly Primary Monoids.** We conclude this section with a few words about strongly primary monoids in $C$. A monoid is called strongly primary if it is simultaneously primary and finitary. The class of strongly primary monoids contains that of finitely primary monoids [23, Theorem 2.9.2]. Let $M$ be a monoid. For $x \in M^*$ the smallest $n \in \mathbb{N}$ satisfying that $nM^* \subseteq x + M$ is denoted by $M(x)$. When such $n$ does not exist, we set $M(x) = \infty$. If $M$ is strongly primary, then $M(x) < \infty$ for all $x \in M^*$ [23, Lemma 2.7.7]. In addition, set

$$M(M) := \sup\{M(a) \mid a \in A(M)\} \subseteq \mathbb{N} \cup \{\infty\}.$$
Example 6.11. Consider the monoid
\[ M := \{(0, 0)\} \cup \{(x, y) \in \mathbb{N}^2 \mid x, y > 0\}. \]
It is clear that
\[ A := \{(a, b) \in M \mid a = 1 \text{ or } b = 1\} \subseteq \mathcal{A}(M) \]
on the other hand, if \((x, y) \in M^* \setminus A\), then \(x, y \geq 2\) and, therefore,
\[ (x, y) = (1, 1) + (x - 1, y - 1) \in M^* + M^*. \]
Hence \(\mathcal{A}(M) = A\). In addition, the fact that \((1, 1) \mid_M (x, y)\) for all \((x, y) \in M^* \setminus A\) implies that \(\mathcal{M}((1, 1)) = 2\). The inclusion \(2M^* \subseteq (1, 1)+M\) implies that \(M\) is a finitary monoid. On the other hand, \(\text{cone}(M)^*\) is the open first quadrant, which implies via Proposition 6.1 that \(M\) is a primary monoid. As a result, \(M\) is strongly primary. Now fix \(n \in \mathbb{Z}_{\geq 2}\). Note that if \((n, 1) \mid_M m(1, 1)\) for some \(m \in \mathbb{N}\), then \(m \geq n + 1\). Thus, \(\mathcal{M}((n, 1)) \geq n + 1\). On the other hand, if \((x, y) \in (n + 1)M^*\), then \(x \geq n + 1\) and \(y \geq 2\), which implies that \((x, y) - (n, 1) \in M\). As a result, \(\mathcal{M}((n, 1)) = n + 1\) and, by a similar argument, \(\mathcal{M}((1, n)) = n + 1\). Hence \(\mathcal{M}((a, b)) = a + b\) for every \((a, b) \in A\) and, in particular, \(\mathcal{M}(M) = \infty\).

Unlike the computations shown in Example 6.11, an explicit computation of the set \(\{M(a) \mid a \in M\}\) for a monoid \(M\) in \(\mathcal{C}\) can be hard to carry out. However, for most monoids \(M\) in \(\mathcal{C}\) one can argue that \(\mathcal{M}(M) = \infty\) without performing such computations.

Proposition 6.12. Let \(M\) be a strongly primary monoid in \(\mathcal{C}\). Then the following conditions are equivalent.

1. \(\mathcal{M}(M) < \infty\).
2. \(\dim \text{cone}(M) = 1\).
3. \(M\) is isomorphic to a numerical monoid.

Proof. Conditions (2) and (3) are obviously equivalent. Therefore it suffices to verify that (1) and (2) are equivalent. To argue that (1) implies (2) suppose, by way of contradiction, that \(\dim \text{cone}(M) \neq 1\). Since \(M\) is strongly primary \(M^*\) is not empty and, thus, \(\dim \text{cone}(M) \geq 2\). As \(M\) is primary, \(\text{cone}(M)\) is open by Proposition 6.1. Therefore \(M\) cannot be finitely generated, which means that \(|\mathcal{A}(M)| = \infty\). Since
\[ \{a \in \mathcal{A}(M) \mid \|a\| < n\} \]
is a finite set for every \(n \in \mathbb{N}\), there exists a sequence \(\{a_n\}\) of atoms of \(M\) satisfying that \(\lim_{n \to \infty} \|a_n\| = \infty\). Now fix \(x \in M^*\). Because \(\mathcal{M}(a_n)x = a_n + b\) for some \(b \in M\), we have that
\[ \lim_{n \to \infty} \mathcal{M}(a_n) = \lim_{n \to \infty} \frac{\|a_n + b\|}{\|x\|} \geq \frac{1}{\|x\|} \lim_{n \to \infty} \|a_n\| = \infty. \]
Hence $\mathcal{M}(M) = \infty$, which is a contradiction. For the reverse implication, suppose that $\dim \text{cone}(M) = 1$. In this case, $M$ is isomorphic to a numerical monoid. Since numerical monoids are finitely generated, $\mathcal{M}(M) < \infty$, and the proof follows. $\square$

ACKNOWLEDGEMENTS

While working on this paper, the author was supported by the NSF AGEP and the UC Year Dissertation Fellowship.

REFERENCES

[1] D. D. Anderson, D. F. Anderson, and M. Zafrullah: *Factorizations in integral domains*, J. Pure Appl. Algebra 69 (1990) 1–19.

[2] L. L. Avramov, C. Gibbons, and R. Wiegand: *Monoids of Betti tables over short Gorenstein algebras*. Under preparation.

[3] V. Barucci, M. D’Anna, and R. Fröberg: *Analytically unramified one-dimensional semilocal rings and their value semigroups*, J. Pure Appl. Algebra 147 (2000) 215–254.

[4] V. Barucci, M. D’Anna, and R. Fröberg: *The Apery Algorithm for a Plane Singularity with Two Branches*, Beiträge zur Algebra und Geometrie 46 (2005) 1–18.

[5] W. Bruns and J. Gubeladze: *Polytopes, Rings and K-theory*, Springer Monographs in Mathematics, Springer, Dordrecht, 2009.

[6] L. Carlitz: *A characterization of algebraic number fields with class number two*, Proc. Amer. Math. Soc. 11 (1960) 391–392.

[7] S. T. Chapman and J. Coykendall: *Half-factorial domains, a survey*, Non-Noetherian Commutative Ring Theory, Mathematics and Its Applications, vol. 520, Kluwer Academic Publishers, 2000, pp. 97–115.

[8] S. T. Chapman, J. Coykendall, and F. Gotti: *Other-half-factorial monoids*. Under preparation.

[9] S. T. Chapman, U. Krause, and E. Oeljeklaus: *On Diophantine monoids and their class groups*, Pacific J. Math. 207 (2002) 125–147.

[10] S. T. Chapman and Z. Tripp: $\omega$-Primality in arithmetic Leamer monoids, Semigroup Forum (to appear). DOI: https://doi.org/10.1007/s00233-019-10036-x

[11] J. Coykendall and G. Oman: *Factorization theory of root closed monoids of small rank*, Comm. Algebra 45 (2017) 2795–2808.

[12] J. Coykendall and W. W. Smith: *On unique factorization domains*, J. Algebra 332 (2011) 62–70.

[13] B. A. Davey and H. A. Priestley: *Introduction to Lattices and Orders*, Cambridge University Press, Cambridge, 2002.

[14] M. D’Anna: *The canonical module of a one-dimensional reduced local ring*, Comm. Algebra 25 (1997) 2939–2965.

[15] M. D’Anna, P. A. García-Sánchez, V. Micale, and L. Tozzo: *Good subsemigroups of N^n*, Internat. J. Algebra Comput. 28 (2018) 179–206.

[16] P. A. García-Sánchez and M. J. Leamer: *Huneke-Wiegand Conjecture for complete intersection numerical semigroup*, J. Algebra 391 (2013) 114–124.

[17] P. A. García-Sánchez, I. Ojeda, and A. Sánchez-R.-Navarro: *Factorization invariants in half-factorial affine semigroups*, Internat. J. Algebra Comput. 23 (2013) 111–122.

[18] P. A. García-Sánchez, C. O’Neill, and G. Webb: *On the computation of factorization invariants for affine semigroups*, J. Algebra Appl. 18 (2019) 1950019.
[19] P. A. García-Sánchez and J. C. Rosales: *Numerical Semigroups*, Developments in Mathematics Vol. 20, Springer-Verlag, New York, 2009.
[20] A. Geroldinger: *On the structure and arithmetic of finitely primary monoids*, Czech. Math. J. 46 (1996) 677–695.
[21] A. Geroldinger: *The complete integral closure of monoids and domains*, PU.M.A. 4 (1993) 147–165.
[22] A. Geroldinger and F. Halter-Koch: *Congruence monoids*, Acta Arith. 112 (2004) 263–296.
[23] A. Geroldinger and F. Halter-Koch: *Non-unique Factorizations: Algebraic, Combinatorial and Analytic Theory*, Pure and Applied Mathematics Vol. 278, Chapman & Hall/CRC, Boca Raton, 2006.
[24] A. Geroldinger, F. Halter-Koch, W. Hassler, and F. Kainrath: *Finitary monoids*, Semigroup Forum 67 (2003) 1–21.
[25] A. Geroldinger, F. Halter-Koch, and G. Lettl: *The complete integral closure of monoids and domains II*, Rendiconti di Matematica Serie VII, 15 (1995) 281–292.
[26] F. Gotti: *On the atomic structure of Puiseux monoids*, J. Algebra Appl. 16 (2017) 1750126.
[27] F. Gotti: *On the system of sets of lengths and the elasticity of submonoids of a finite-rank free commutative monoid*, J. Algebra Appl. (to appear). Available on arXiv: https://arxiv.org/abs/1806.11273
[28] F. Gotti: *Puiseux monoids and transfer homomorphisms*, J. Algebra 516 (2018) 95–114.
[29] F. Gotti: *Systems of sets of lengths of Puiseux monoids*, J. Pure Appl. Algebra 223 (2019) 1856–1868.
[30] F. Gotti and C. O’Neil: *The elasticity of Puiseux monoids*, J. Commut. Algebra (to appear), DOI: https://projecteuclid.org/euclid.jca/1523433696. Available on arXiv: https://arxiv.org/abs/1703.04207
[31] A. Grams: *Atomic domains and the ascending chain condition for principal ideals*. Math. Proc. Cambridge Philos. Soc. 75 (1974) 321–329.
[32] P. A. Grillet: *Commutative Semigroups*, Advances in Mathematics Vol. 2, Kluwer Academic Publishers, Boston, 2001.
[33] J. Haarmann, A. Kalauli, A. Moran, C. O’Neill, R. Pelayo: *Factorization properties of Leamer monoids*, Semigroup Forum 89 (2014) 409–421.
[34] F. Halter-Koch: *Divisor theories with primary elements and weakly Krull domains*, Boll. Un. Mat. Ital. B 9 (1995) 417–441.
[35] F. Halter-Koch: *Elasticity of factorizations in atomic monoids and integral domains*, J. Théor. Nombres Bordeaux 7 (1995) 367–385.
[36] F. Kainrath and G. Lettl: *Geometric notes on monoids*, Semigroup Forum 61 (2000) 298–302.
[37] N. Maugeri and G. Zito: *Embedding dimension of a good semigroup*. Available on the arXiv: https://arxiv.org/pdf/1903.02057.pdf
[38] W. Narkiewicz: *Finite abelian groups and factorization problems*, Coll. Math. 42 (1979) 319–330.
[39] W. Narkiewicz: *Elementary and analytic theory of algebraic numbers*, Springer Monographs in Mathematics. Springer-Verlag, Berlin, third edition, 2004.
[40] M. Petrich: *Introduction to Semigroups*, Charles E. Merrill Publishing Co., 1973.
[41] R. T. Rockafellar: *Convex Analysis*, Princeton Landmarks in Mathematics, Vol. 28, Princeton Univ. Press., New Jersey, 1970.
[42] L. Skula: *On c-semigroups*, Acta Arith. 31 (1976) 247–257.
[43] J. Sliwa: *Factorizations of distinct lengths in algebraic number fields*, Acta Arith. 31 (1976) 399–417.
[44] T. Tamura: *Basic study of N-semigroups and their homomorphisms*, Semigroup Forum 8 (1974) 21–50.
[45] A. Zaks: *Half-factorial domains*, Bull. Amer. Math. Soc. **82** (1976) 721–723.

[46] A. Zaks: *Half-factorial domains*, Israel J. Math. **37** (1980) 281–302.

Department of Mathematics, UC Berkeley, Berkeley, CA 94720
Department of Mathematics, Harvard University, Cambridge, MA 02138

*E-mail address*: felixgotti@berkeley.edu

*E-mail address*: felixgotti@harvard.edu