Research Article

How to Contract a Vertex Transitive 5-Connected Graph

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Received 18 April 2020; Accepted 17 June 2020; Published 9 July 2020

1. Introduction

All graphs considered here are supposed to be simple, finite, and undirected graphs. For a connected graph $G$, a subset $T \subseteq V(G)$ is called a smallest separator if $|T| = \kappa(G)$ and $G - T$ has at least two components. Let $G$ be a $k$-connected graph, and let $H$ be a subgraph of $G$. Let $G/H$ stand for the graph obtained from $G$ by contracting every component of $H$ to a single vertex and replacing each resulting double edges by a single edge. A subgraph $H$ of $G$ is said to be $k$-contractible if $G/H$ is still $k$-connected. An edge $e$ is a $k$-contractible edge if $G/e$ is $k$-connected; otherwise, we call it a noncontractible edge. Clearly, two end-vertices of a noncontractible edge are contained in some smallest separator. A $k$-connected graph without a $k$-contractible edge is said to be a contraction-critical $k$-connected graph.

Tutte’s [1] wheel theorem showed that every 3-connected graph on more than four vertices contains a 3-contractible edge. For $k \geq 4$, Thomassen and Toft [2] showed that there were infinitely many contraction-critical $k$-regular $k$-connected graphs. On the other hand, one can find that every 4-connected graph can be reduced to a smaller 4-connected graph by contracting at most two edges. Therefore, Kriesell [3] posted the following conjecture.

Conjecture 1 (see [3]). There exists $b(k), h(k)$ such that every $k$-connected graph $G$ with at least $b(k)$ vertices can be contracted to a $k$-connected graph $G_0$ such that $0 < |V(G)| - |V(G_0)| < h(k)$.

Clearly, Conjecture 1 is true for $k \leq 4$. By Kriesell’s examples [3], Conjecture 1 fails for $k \geq 6$. Hence, it is still open for $k = 5$.

A smallest separator $T$ of a $k$-connected graph is said to be trivial if $G - T$ has exactly two components and one of them has exactly one vertex. A 5-connected graph is essentially a 6-connected graph if every smallest separator of $G$ is trivial. In ([3]), Kriesell proved the following results.

Theorem 1 (see [3]). Every essentially 6-connected graph $G$ with at least 13 vertices can be contracted to a 5-connected graph $H$ such that $0 < |V(G)| - |V(H)| < 5$.

In this paper, we will show that Conjecture 1 is true for vertex transitive 5-connected graphs. Clearly, Conjecture 1 holds for 5-connected graphs which contain a contractible edge. Hence, in order to show that Conjecture 1 holds for vertex transitive 5-connected graphs, we have to show that all vertex transitive contraction-critical 5-connected graphs have a small contractible subgraph. So, the key point of this paper is to characterize the local structure of a vertex transitive contraction-critical 5-connected graph and, then, to find the contractible subgraph of it. In the following, for convenience, a vertex transitive contraction-critical 5-connected graph will be called a TCC-5-connected graph. For a
of (see Figures 1(a)–1(d)).

To state our results, we need to introduce some further definitions. Let \( G \) be a 5-connected graph which is 5-regular. For any \( x \in V(G) \), we say that \( x \) has one of the following four types according the graph induced by the neighborhood of \( x \) (see Figure 1(e)). One can check that \( G^* \) is vertex transitive, and \( G^* \) can be reduced to \( K_5 \) by contracting \( yx_1 \) and \( xx_2 \).

Moreover, for \( i \in \{1, 2, 3, 4\} \), \( G \) has type \( i \) if every vertex of \( G \) has type \( i \).

Furthermore, we need to introduce the graph \( G^* \) (see Figure 1(e)). One can check that \( G^* \) is vertex transitive, and \( G^* \) can be reduced to \( K_5 \) by contracting \( yx_1 \) and \( xx_2 \).

First, we have the following results on the local structure of TCC-5-connected graphs.

**Theorem 2.** Let \( G \) be a TCC-5-connected graph. If \(|V(G)| \leq 9\), then either \( G \) is isomorphic to \( K_5 \) or \( G \cong G^* \).

**Theorem 3.** Let \( G \) be a TCC-5-connected graph. If \(|V(G)| \geq 10\), then \( G \) has type 1, type 2, type 3, or type 4.

**Theorem 4.** Let \( G \) be a TCC-5-connected graph. If \( G \) has type 2, then \( G \) is isomorphic to icosahedron.

Then, we will prove the following main result of the paper.

**Theorem 5.** Let \( G \) be a 5-connected vertex transitive graph which is neither \( K_5 \) nor icosahedron, and then, \( G \) can be contracted to a 5-connected \( G' \) such that \( 0 < |V(G)| - |V(G')| < 3\).

The organization of the paper is as follows. Section 2 contains some preliminary results. In Section 3, we will characterize the local structure of 5-connected TCC-graphs. In Section 4, we will prove Theorem 5.

### 2. Terminology and Lemma

For terms not defined here, we refer the reader to [11]. Let \( G = (V(G), E(G)) \) be a graph, where \( V(G) \) denotes the vertex set of \( G \) and \( E(G) \) denotes the edge set of \( G \). Let \( \text{Aut}(G) \) denote the automorphism group of \( G \), and let \( \kappa(G) \) denote the vertex connectivity of \( G \). Let \( P_n \) denote a path on \( n \) vertices. An edge joining vertices \( x \) and \( y \) will be written as \( xy \). Let \( \{xy\} \) stand for the new vertex obtained by contracting the edge \( xy \). For \( x \in V(G) \), we define \( N_G(x) = \{x | xy \in E(G)\} \). For \( F \subseteq V(G) \), we define \( N_G(F) = \cup x \in F N_G(x) - F \). Furthermore, let \( G[F] \) denote the subgraph induced by \( F \), and let \( G - F \) denote the graph obtained from \( G \) by deleting all the vertices of \( F \) together with the edges incident with them. Let \( \partial(F) \) stands for the set of edge with one end in \( F \) and the other end in \( G - F \).

Let \( T \) be a smallest separator of a noncomplete connected \( G \), and the union of at least one but not of all components of \( G - T \) is called a \( T \)-fragment. A fragment of \( G \) is a \( T \)-fragment for some smallest separator \( T \). Let \( F \) be a \( T \)-fragment, and let \( \overline{F} = V(G) - (F \cup T) \). Clearly, \( \overline{F} \neq \emptyset \), and \( T \) is also a \( T \)-fragment such that \( N_G(F) = T = N_G(\overline{F}) \). A fragment with least cardinality is called an atom. For \( N_G(x), d_G(x), \) and \( N_G(F) \), we often omit the index \( G \) if it is clear from the context.

Furthermore, we need some special terminologies for 5-connected graphs. Let \( A \) be a fragment of \( G \), and let \( S = N(A) \). Let \( x \in S \), and \( y \in N(x) \cap A \). A vertex \( z \) is said to be an admissible vertex of \( (x, y; A) \) if both of the following two conditions hold:

\[
z \in N(x) \cap N(y) \cap S \cap N_z(G), |N(z) \cap A| \geq 2.
\]

A vertex \( z \) is said to be an admissible vertex of \( (x; A) \), if \( z \) is an admissible vertex of \( (x, y; A) \) for some \( y \in N(x) \cap A \). Let \( \text{Ad}(x, y; A) \) (resp. \( \text{Ad}(x; A) \)) stand for the set of admissible vertices of \( (x, y; A) \) (resp. \( (x; A) \)). Let \( e \) be an edge of \( G \), and a fragment \( A \) is said to be a fragment with respect to \( e \) if \( V(e) \subseteq N(A) \).

The following properties of fragment are well known (for the proof, see [12]), and we will use them without any further reference.

**Lemma 1** ([12]). Let \( F \) and \( F' \) be two distinct fragments of \( G; T = N(F), T' = N(F') \). Then, the following statements hold.

1. If \( F \cap T' \neq \emptyset \) and \( |F \cap T'| \geq |F' \cap T'| \), then \( |F \cap T'| \geq |F' \cap T'| \).
2. If \( F \cap T' \neq \emptyset \) and \( F \cap T' \) is not a fragment of \( G \), then \( F \cap T' = \emptyset \) and \( |F \cap T'| > |F' \cap T'| \).
3. If \( F \cap T' \neq \emptyset \) and \( F \cap T' \) are fragments of \( G \), and \( N(F \cap T') = (T' \cap T) \cup (T' \cap T) \), then both \( F \cap T' \) and \( F \cap T' \) are fragments of \( G \).

**Lemma 2** ([4]). Let \( G \) be a k-connected graph, and \( A \) is a fragment of \( G \). Let \( B \subseteq N(A) \). If \( |N(B) \cap A| < |B| \), then \( A = N(B) \cap A \).

**Lemma 3** ([5]). Let \( G \) be a contraction-critical 5-connected graph, and then, \( G \) contains a vertex \( x \) such that every edge incident with \( x \) is contained in some triangle.

**Lemma 4** ([6]). Let \( G \) be a contraction-critical 5-connected graph. Let \( x \in V(G) \), and \( A \) be a fragment such that \( x \in N(A) \), \( |A| \geq 3 \), and \( |A| \geq 2 \). If \( |N(x) \cap A| = 1 \), then \( \text{Ad}(x; A) \neq \emptyset \).

**Lemma 5** ([7]). Let \( A \) be a fragment of a contraction-critical 5-connected graph such that \( |A| = 2 \), and let \( t_1, t_2 \) be two vertices of \( N(A) \) such that \( |N(t_1) \cap A| = |N(t_2) \cap A| = 1 \). Then, either \( \text{Ad}(t_1; A) \neq \emptyset \) or \( \text{Ad}(t_2; A) \neq \emptyset \).
Lemma 6. Let $G$ be a vertex transitive connected graph, and then, for any two vertices $x$ and $y$, $G[N(x)] \cong G[N(y)]$.

Proof. Since $G$ is a vertex transitive graph, there exist $g \in \text{Aut}(G)$ such that $x^g = y$. It follows that $(N(x))^g = N(y)$. Hence, $g|_{N(x)}$ is isomorphic to $G[N(x)]$ and $G[N(y)]$, where $g|_{N(x)}$ is the restriction of $g$ on $N(x)$. □

Lemma 7. Let $p \geq 2$ be a prime integer, and let $G$ be a vertex transitive graph with $\kappa(G) = p$; then, $G$ is a $p$-regular graph.

Proof. To the contrary, we may assume that $\delta(G) > p$ since $\delta(G) \geq \kappa(G)$. It follows that every atom of $G$ has at least two vertices. Since $G$ is a vertex transitive graph, then every vertex of $G$ is contained in some atom.

First, we show that any two atoms of $G$ are disjoint. Otherwise, let $A$ and $B$ be two distinguished atoms of $G$ such that $A \cap B \neq \emptyset$. By the definition of atom, $A \cap B$ is not a fragment. Lemma 1 assures us that $\overline{A} \cap B = \emptyset$ and $|A \cap B| = |B \cap N(A)|$. This implies that $|A| > |B|$, a contradiction. Thus, any two atoms of $G$ are disjoint.

Let $A$ and $C$ be two atoms of $G$ such that $C \cap N(A) \neq \emptyset$. It follows that $A \cap C = \emptyset$. We show that $C \subseteq N(A)$. Otherwise, suppose $C \cap \overline{A} \neq \emptyset$. If $C \cap \overline{A}$ is a fragment of $G$, then we see that $|C \cap \overline{A}| < |C|$, since $|C \cap N(A) \neq \emptyset$. This contradicts the definition of atom. So, $C \cap \overline{A}$ is not a fragment of $G$. Lemma 1 assures us that $A \cap C = \emptyset$ and $|C \cap N(A)| = |A \cap N(C)| = |A|$. It follows that $|C| > |A|$, a contradiction. Hence, $C \subseteq N(A)$. Therefore, $N(A)$ is the disjoint union of some atom, since any two atoms of $G$ are disjoint and every vertex of $G$ is contained in some atom. This means that $|A|$ is a subdivision of $|N(A)|$, and hence, $|A| = p$. It follows that $N[A] = C$. By symmetry, we see that $N(C) = A$, which implies that $\overline{A} = \emptyset$, a contradiction. □

Lemma 8. Let $G$ be a TCC-5-connected graph. If $G$ does not contain $K_4$ as subgraph, then for any $x \in V(G)$, $\Delta(G[N(x)]) \leq 2$.

Proof. Clearly, Lemma 7 assures us that $G$ is 5-regular, which implies that $G$ has an even order. Suppose that $x \in V(G)$ with $N(x) = \{x_1, \ldots, x_5\}$ such that $x_1$ adjacent to at least three vertices of $N(x)$. Let $A = \{x\}$, and it follows that $N(A) = N(x)$. If $G - A - N(A) = \emptyset$, then $G$ has six vertices. It follows that $G \cong K_6$, which implies $G$ contains $K_4$, a contradiction. Hence, we may assume that $G - A - N(A) \neq \emptyset$. It follows that $A$ is a fragment of $G$. By symmetry, we assume that $\{x_2, x_3, x_4\} \subseteq N(x)$. Now, we observe that $N(x_1) \cap N(A) = \{x_2, x_3, x_4\}$ and $|N(x_1) \cap N(A)| = 1$. Let $N(x_1) \cap N(A) = \{t_1\}$. Let $B = \{x_1, x_4\}$, and then $N(B) = \{x_2, x_3, x_4\}$. Now, the fact that $|V(G)|$ is even assures us that $G - B - N(B) \neq \emptyset$. It follows that $B$ is a fragment of $G$. Furthermore, we see that $N(t_1) \cap B = \{x_1\}$. Now, Lemma 5 assures us that either $Ad(t_1; B) \neq \emptyset$ or $Ad(x_2; B) \neq \emptyset$. Without loss of generality, assume $x_2 \in Ad(x_1; B)$. Therefore, $G[N(x)]$ is a connected graph. If $Ad(t_1; B) \neq \emptyset$, then, similarly, we have that $G[N(x)]$ is a connected graph. Now, since $G$ is vertex transitive, the following claims hold. □

Claim 1. For any $t \in V(G)$, $G[N(t)]$ is a connected graph.

Claim 2. For any $t \in V(G)$, $G[N(t)]$ contains a cycle of length 4.

Proof. Since $G$ is a vertex transitive graph, we only show that $N(x_1)$ has a cycle of length 4. By Claim 1, we see that for $y \in N(B)$, $N(y) \cap N(B) \neq \emptyset$. On the other hand, we observe that $G[\{x_2, x_3, x_4\}] \cong K_3$, since $G$ does not contain $K_4$. This implies that every member of $\{x_2, x_3, x_4\}$ is either adjacent to $t_1$ or $x_5$. It follows that $|N(t_1) \cap \{x_2, x_3, x_4\}| \geq 2$ or
implies that 4, then (a path. Suppose x \in N(x) \cap N(y) \cap N(z), we see that x, y, z has an even degree of at most 4, a contradiction. Hence, we may assume that |B| = 3. Now, Lemma 4 shows that Ad(x_2; B) \neq \emptyset, which implies \(|N(z) \cap N(x) \cap N(y)| \geq 2 for some z \in N(B), a contradiction. □

**Lemma 9.** Let G be a TCC-5-connected graph. If G has type 4, then G is essentially 6-connected.

**Proof.** Since G has type 4, we see that for any x \in V(G), \Delta(G[N(x)]) \leq 2. □

**Claim 3.** If A is a fragment of G, then |A| \neq 2.

**Proof.** Suppose A = [x, y]. If xy \in E(G), then x has three neighbors in G[N(y)], a contradiction. So, we may assume xy \notin E(G). It follows that G[N(A)] \cong P_5. Let x_1, x_2, x_3, x_4 be the path of G[N(A)]. Then \(|N(x_3) \cap A| = |N(y) \cap A| = |N(x) \cap A| = 1. If |A| = 3, then Lemma 4 implies that Ad(x_3, A) \neq \emptyset. Hence, either x_3 \in Ad(x_1, A) or x_1 \in Ad(x_3, A). This is a contradiction, since \(|N(x_3) \cap A| = |N(x) \cap A| = 1. Hence, we may assume that |A| \leq 2. If |A| = 1, then we see that d(x_1) < 5, a contradiction. So, we may assume that |A| = 2. It follows that \(|V(G)| = 9, which contradicts the fact that G has an even order. Hence, Claim 3 holds. □

**Claim 4.** If A is a fragment of G, then |\overline{A}| \neq 3.

**Proof.** We first show that G[A] is a connected graph. Otherwise, let A_1 be a component of G[A] such that A_1 has exactly one vertex. It follows that A_2 = A - A_1 is a fragment of cardinality 2, a contradiction. Next, we show that G[A] is a path. Suppose G[A] is a cycle, then a simple calculation shows that |\partial(A)| = 9. This implies that one vertex of N(A), say w, has exactly one neighbor in A. Now, we find that A - N(w) is a fragment of cardinality 2, a contradiction. Let xy be the path of G[A], and let N(y) = \{x_1, x_2, x_3, x_4\}. Without the loss of generality, let N(y) = \{x_1, x_2, x_3, x_4\}. Subclaim 1. |N(x) \cap \{x_1, x_2, x_3\}| = 2 and |N(z) \cap \{x_1, x_2, x_3\}| = 2. Proof. Notice that G has type 4; we find that |N(x) \cap \{x_1, x_2, x_3\}| \leq 2. If |N(x) \cap \{x_1, x_2, x_3\}| \leq 1, then we find that d(x) \leq 4, a contradiction. Hence, |N(x) \cap \{x_1, x_2, x_3\}| = 2. By symmetry, |N(z) \cap \{x_1, x_2, x_3\}| = 2. Without the loss of generality, we may assume that \{x_1, x_2\} \subseteq N(x). Now, if \{x_1, x_2\} \subseteq N(z), then \{x_1, x_2\} is a cycle of G[N(y)], a contradiction. Therefore, \{x_2, x_3\} \subseteq N(z), which implies that \{x_1, x_3\} \subseteq N(x) \cap N(z).

Subclaim 2. G[\{x_1, x_2, x_3\}] \cong K_3.

Proof. If x_1, x_2, x_3 \in E(G), then N(y) has a triangle, a contradiction. It follows that x_1, x_2 \notin E(G). Similarly, we have x_2, x_3 \notin E(G). Furthermore, if x_1, x_3 \notin E(G), then we find that there is a cycle of length four in N(y), a contradiction. Thus, x_1, x_3 \notin E(G). It follows that G[\{x_1, x_2, x_3\}] \cong K_3.

Now, we are ready to complete the proof of Claim 4. Focusing on x_2, we find that N(x_2) \cap N(A) \neq \emptyset since G[N(x_2)] is connected. By Subclaim 2, we may assume that x_4 \in N(x_2). Now, we find that there is a cycle of length four in N(x_2), a contradiction. □

**Claim 5.** For every smallest separator T, G - T has exactly two components.

**Proof.** Otherwise, assume that G - T has at least three components. Let A_1, A_2, and A_3 be three connected components of G - T.

Subclaim 3. For any y \in T, |N(y) \cap T| = 2, and |N(y) \cap A_i| = 1, i \in [1, 2, 3].

Proof. Let y \in T, let N(y) = \{y_1, \ldots, y_5\}. Without the loss of generality, we may assume that y_i \in A_i, i \in [1, 2, 3]. Now, we find that N(y) \cap A_i \neq \emptyset, since G has type 4. Suppose y_i \in N(y) \cap T. If N(y) \cap T = \{y_1\}, then the fact that G[N(y)] is connected shows that y_1 has three neighbors in G[N(y)], which contradicts the fact that G has type 4. So, we have N(y) \cap T = \{y_1, y_2\}. It follows that |N(y) \cap A_i| = 1, i \in [1, 2, 3] and |N(y) \cap T| = 2.

By Subclaim 3, \delta(G[T]) = 2, which implies that G[T] is a cycle of length 5. Hence, we see that |A_i| \neq 1, i \in [1, 2, 3]. Furthermore, by Subclaims 3 and 4, |A_i| \neq 3 for each i = 1, 2, 3. Focusing on A_1, we find that \overline{A_1} = A_2 \cup A_3, which implies that |\overline{A_1}| \geq 6. Recall that |N(x) \cap A_1| = 1, and Lemma 4 shows that Ad(x; A_1 \cup A_2) \neq \emptyset. Without the loss of generality, assume that y \in Ad(x; A_1). This implies that |N(y) \cap A_1| \geq 2, which contradicts Subclaim 3. Hence, Claim 5 holds.

Next, we assume that G is not essentially 6-connected. It follows that there is a fragment B such that |B| \geq 2 and |\overline{B}| \geq 2. Let \mathcal{B} = B \cup \overline{B} is a fragment such that |B| \geq 2 and |\overline{B}| \geq 2, and let t = min{|B|, |\overline{B}|}. By Claims 3 and 4, we see that t \geq 4. Let \mathcal{B}_1 = B \cup \overline{B} and \mathcal{B}_2 = t. Let A = \overline{B_1}, and let y \in N(A). Now, since G is vertex transitive, every vertex of G is contained in some member of \mathcal{B}_1. Therefore, there exist \mathcal{B} \in \mathcal{B}_1 such that y \in B. Next, we will analyse the local structure of A and B. □

**Claim 6.** If A \cap B \neq \emptyset, then \overline{A} \cap \overline{B} \neq \emptyset.
Proof. Suppose \( A \cap B \neq \emptyset \) and \( \overline{A} \cap \overline{B} = \emptyset \). Now, Lemma 1 assures us that \( |A \cap N(B)| \geq |B \cap N(A)| \). It follows that

\[
\begin{align*}
|A| & = |A \cap B| + |A \cap N(B)| + |A \cap \overline{B}| \\
& \geq |A \cap B| + [B \cap N(A)] + |A \cap \overline{B}| \\
& > |B \cap N(A)| + |A \cap \overline{B}| = |B|.
\end{align*}
\]

This contradicts the choice of \( A \). □

Claim 7. \( A \cap B \neq \emptyset \) if and only if \( \overline{A} \cap B \neq \emptyset \).

Proof. Suppose \( A \cap B \neq \emptyset \). Now, Lemma 1 assures us that \( |A \cap N(B)| \geq |B \cap N(A)| \). If \( \overline{A} \cap B = \emptyset \), then we see that

\[
\begin{align*}
|A| & = |A \cap B| + |A \cap N(B)| + |A \cap \overline{B}| \\
& \geq |A \cap B| + |B \cap N(A)| + |A \cap \overline{B}| \\
& > |A \cap B| + |B \cap N(A)| = |B|.
\end{align*}
\]

This contradicts the choice of \( A \). Hence, we see that \( \overline{A} \cap B \neq \emptyset \). By symmetry, we see that \( \overline{A} \cap B \neq \emptyset \) implies \( A \cap \overline{B} \neq \emptyset \). □

Claim 8. \( A \cap B = \emptyset \).

Proof. Suppose \( A \cap B = \emptyset \). By Claim 6, we know that \( \overline{A} \cap B = \emptyset \). Hence, \( A \cap B \) is a fragment of \( G \). By the choice of \( A \), we know that \( |A \cap B| = 1 \). Furthermore, since \( \overline{A} \cap \overline{B} = \emptyset \), Lemma 1 assures us that \( |B \cap N(A)| \geq |A \cap N(B)| \).

If \( A \cap B = \emptyset \), then Claim 7 assures us that \( A \cap B = \emptyset \). Furthermore, \( |A| = |B| \geq 4 \) implies that \( |A \cap N(B)| = |B \cap N(A)| \geq 3 \). Hence, we find that \( |N(A)| \geq |B \cap N(A)| + |B \cap N(A)| \geq |B \cap N(A)| + |A \cap B| \geq 6 \), a contradiction.

So, we may assume \( A \cap B \neq \emptyset \). Then, Claim 7 assures us that \( A \cap B \neq \emptyset \). Hence, \( A \cap B \neq \emptyset \). By the choice of \( A \) and \( B \), we know that \( |A \cap B| = |B \cap N(A)| = 1 \). It follows that \( |A \cap N(B)| = |B \cap N(A)| \geq |A \cap N(B)| + |A \cap B| \geq 6 \), a contradiction.

Therefore, let \( |A \cap N(B)| = 2 \). It follows that \( |A \cap N(B)| = |B \cap N(A)| = 2 \). Since \( \overline{A} \cap B \neq \emptyset \) and \( A \cap B \neq \emptyset \), Lemma 1 assures us that \( |A \cap N(B)| = |B \cap N(A)| = 2 \). This implies that \( |N(B) \cap N(A)| = 1 \). Let \( N(B) \cap N(A) = \{t\} \). Now, \( N(t) \cap A \cap B \neq \emptyset \), since \( A \cap B \) is a fragment. Similarly, we find that \( N(t) \cap A \cap \overline{B} \neq \emptyset \), \( N(t) \cap \overline{A} \cap B \neq \emptyset \), and \( N(t) \cap \overline{A} \cap \overline{B} \neq \emptyset \). Now, we find that \( G[N(t)] \) has at least two components, a contradiction. □

Claim 9. \( A \cap B = \emptyset \) and \( B \cap \overline{A} = \emptyset \).

Proof. Suppose \( A \cap \overline{B} \neq \emptyset \). By Claim 7, we see that \( \overline{A} \cap B \neq \emptyset \). Hence, both \( A \cap B \) and \( A \cap B \) are fragments of \( G \). By the choice of \( A \), we see that \( |A \cap \overline{B}| = |A \cap B| = 1 \). It follows that \( |A \cap N(B)| = |B \cap N(A)| \geq 3 \).

If \( \overline{A} \cap B \neq \emptyset \), then \( |N(A)| \geq |B \cap N(A)| + |B \cap N(A)| \geq 6 \), a contradiction.

Hence, we may assume that \( \overline{A} \cap B = \emptyset \). Then, by the choice of \( B \), we know that \( |B| \geq |B| \). It follows that \( |B \cap N(A)| \geq 3 \). Hence, we find that \( |N(A)| \geq |B \cap N(A)| + |B \cap N(A)| \geq 6 \), a contradiction.

Now, we are ready to complete the proof of the Lemma. By Claims 8 and 9, we find that \( A = A \cap N(B) \) and \( B = B \cap N(A) \). Now, we find that \( |B \cap N(A)| \leq |N(A)| - |B \cap N(A)| \leq 4 \), since \( |B \cap N(A)| = |B| \geq 4 \). It follows that \( |B \cap N(A)| < |A \cap N(B)| \). Now, Lemma 1 implies that \( \overline{A} \cap B = \emptyset \). It follows that \( |B| = |B \cap N(A)| \geq 1 \), a contradiction. □

3. The Local Structure of TCC-5-Connected Graphs

In this section, since Lemma 7 holds, all TCC-5-connected graphs were supposed to be 5-regular and have an even order.

Theorem 6. Let \( G \) be a TCC-5-connected graph. If \( |V(G)| \leq 9 \), then either \( G \cong K_6 \) or \( G \cong G^* \).

Proof. Recall that \( G \) has an even order. It follows that either \( |V(G)| = 6 \) or \( |V(G)| = 8 \). If \( |V(G)| = 6 \), then \( G \cong K_6 \). So, we may assume \( |V(G)| = 8 \). It follows that \( G \) has a fragment of cardinality 2. Let \( A = \{x, y\} \) be a fragment of \( G \). Let \( N(A) = \{x_1, x_2, x_3, x_4, x_5\} \), and let \( \overline{A} = \{z\} \).

Claim 10. \( x, y \in E(G) \).

Proof. Otherwise, we find that \( G[N(x)] \cong C_5 \). It follows that \( G[N(x)] \cong C_5 \). On the other hand, \( G[N(x)] \cong K_5 \). It follows that \( G[N(x)] \cong G[N(x)] \), a contradiction.

Let \( N(x) = \{x_1, x_2, x_3, x_4, y\} \). By symmetry, we may assume that \( N(y) = \{x_1, x_2, x_3, x_4, y\} \). We find that \( y \) has at least three neighbors in \( G[N(x)] \). Hence, Lemma 8 implies that \( G \) contains \( K_4 \). It follows that \( G[N(x)] \) contains a triangle. □

Claim 11. \( x_i x_j \in E(G) \).

Proof. Suppose \( x_i x_j \notin E(G) \). Notice that for \( i \in \{4, 5\} \), \( |N(x_i) \cap (A \cup \overline{A})| = 2 \), we see that \( x_4, x_5 \subseteq N(x_i) \cap N(x_j) \cap N(x_1) \cap N(x_3) \). Therefore, \( G[N(x)] \cong K_5 \), which implies that \( G[N(x)] \) does not contain a triangle, a contradiction.

Now, we observe that \( G[N(x)] \cap x_4 x_5 \) is 2-regular. Hence, \( G[N(x)] \cap x_4 x_5 \) is a cycle of length 5. Now, by symmetry, we may assume that \( x_2, x_3 \subseteq N(x_4) \) and \( x_1, x_3 \subseteq N(x_5) \). It follows that \( x_1 x_3 \in E(G) \) since \( G[N(x)] \cap x_4 x_5 \) is a cycle of length 5. Therefore, we have \( G \cong G^* \). □

Lemma 10. Let \( G \) be a TCC-5-connected graph with \( |V(G)| \geq 10 \). If \( G \) contains \( K_4 \) as a subgraph, then \( G \) has type 1.
Proof. Since $G$ is a vertex transitive graph, we know that every vertex of $G$ is contained in a $K_4$. Let $x$ be a vertex of $G$, and let $N(x) = \{x_1, \ldots, x_5\}$. Without the loss of generality, let $G[\{x, x_1, x_2, x_3\}] \cong K_4$.

Claim 12. $|N(x_i) \cap \{x_1, x_2, x_3\}| \leq 1, i \in \{4, 5\}$.

Proof. We only show that $|N(x_4) \cap \{x_1, x_2, x_3\}| \leq 1$, and the other one can be handled similarly. Otherwise, by symmetry, we may assume $\{x_1, x_2\} \subseteq N(x_4)$. Let $N(x_4) = \{x_1, x_2, x_3, x_4, x_5\}$. Let $A = \{x, x_1, x_2\}$, and it follows that $N(A) = \{x_2, x_3, x_4, x_5, t_1\}$. Furthermore, recall that $|V(G)| \geq 10$, $G - A - N(A) \neq \emptyset$. If $t_1 = x_5$, then $N(A)$ is a separator of order $4$, a contradiction. Thus, $t_1 \neq x_5$. Therefore, $A$ is a fragment of $G$. Furthermore, since $|V(G)| \geq 10$, we see that $|A| \geq 3$. Let $B = \{x, x_2\}$. Clearly, $B \cap N(x) = \{x_2, x_3, x_4, t_2\}$, and therefore, we have $[A] = [B]$.

Notice that $t_2 \in A$ and $t_4 \in N(x), x \not\in t_2$. Now, we see that $A \cap B = \{x_1\}, A \cap N(B) = \{x_1\}, B \cap N(A) = \{x_2\}, N(A) \cap N(B) = \{x_3, x_4, x_5\}, \overline{A} \cap B = \{x, x_2\}$, and $\overline{B} \cap N(A) = \{t_4\}$. Now, since $|A| = |B| \geq 3$, we find that $|\overline{A}| \geq 2$. Let $C = A \cup B$. Clearly, $C$ is a fragment with $A \cup B$. Notice that $N(t_1) \cap C = \{x\}, N(t_2) \cap C = \{x_2\}$, and $N(x) \cap C = \{x\}$. Now, Lemma 4 implies that $A \cap N(t_1) = \emptyset$, $B \cap N(t_2) = \emptyset$, $C \subseteq \{x, x_1, x_2\}$. It follows that either $x_3$ or $x_4$ has two neighbors in $N(C)$. By symmetry, let $x_3$ have two neighbors in $N(C)$. It follows that $d(x_3) \geq 6$, a contradiction. Hence, Claim 12 holds.

Claim 13. $N(x_i) \cap \{x_1, x_2, x_3\} = \emptyset, i \in \{4, 5\}$.

Proof. We only show that $N(x_4) \cap \{x_1, x_2, x_3\} = \emptyset$, and the other one can be handled similarly. Otherwise, by symmetry, we may assume that $x_4 \in E(G)$. Now, by Claim 12, $N(x_4) \cap \{x_1, x_2\} = \emptyset$. Let $A = \{x, x_1, x_2\}$ and $N(x_4) = \{x_1, x_2, x_3, x_4, x_5, t_1\}$. Clearly, $N(A) = \{x_2, x_3, x_4, x_5, t_1\}$ and $G - A - N(A) \neq \emptyset$. Now, since $G$ is 5-connected, we observe that $t_1 \neq x_5$. Therefore, $A$ is a fragment of $G$.

Subclaim 4. $N(x_i) \cap \{x_1, x_2, x_3\} = \emptyset$.

Proof. Suppose $x_4 \in E(G)$, and then, $\{x, x_3\}$ is fragments of $G$. Furthermore, we see that $G[N(x)]$ is a connected graph, and this implies that for any $t \in V(G)$, $G[N(t)]$ is a connected graph. Let $B = \{x, x_2\}$, and it follows $N(B) = \{x_1, x_3, x_4, x_5, t_2\}$, where $t_2 \in N(x_2) - \{x_2, x_3, x_5\}$. Now, since $G$ is 5-connected, we see that $t_2 \neq x_4$. We observe that $A \cap B = \emptyset$, $A \cap N(B) = \{x_1\}$, $B \cap N(A) = \{x_2\}$, $N(A) \cap N(B) = \{x_3, x_4, x_5\}$, $\overline{A} \cap B = \{x, x_2\}$, and $\overline{B} \cap N(A) = \{t_1\}$. Furthermore, we see that $A \cap B = \emptyset$ and $A \cap N(A) = \emptyset$. Notice that $G[N(x)]$ is connected, and we see that for every vertex $t$ of $G$, $G[N(t)]$ is connected.

Now, since $A \cap \overline{B} = \emptyset$ and $\overline{B} \cap N(A) = \{t_1\}$, the fact $|\overline{A}| = |\overline{B}| \geq 3$ shows that $|A \cap \overline{B}| \geq 2$. Furthermore, $\overline{A} \cap \overline{B}$ is a fragment.

If $|N(x) \cap \overline{A} \cap \overline{B}| \geq 2$, then $G[N(x) \cap \overline{A} \cap \overline{B}]$ has at least two components, a contradiction. Therefore, $|N(x) \cap \overline{A} \cap \overline{B}| = 1$ and $N(x) \cap (\overline{A} \cap \overline{B}) \neq \emptyset$.

On the other hand, by Claim 12, $N(x_4) \cap \overline{A} \cap \overline{B} \neq \emptyset$, we see that $G[N(x_4)]$ has only one vertex of degree $3$. On the other hand, we know that $G[N(x)]$ has two vertex of degree $3$, and this implies that $G[N(x_3)] \not\cong G[N(x)]$, a contradiction. This contradiction shows that $x_4 \in E(G)$. By symmetry, $x_5 \in E(G)$. Hence, Subclaim 4 holds.

Subclaim 5. $x_4 x_5 \notin E(G)$. Proof. Suppose $x_4 x_5 \notin E(G)$. Let $P^r$ be a graph which is got from the path $x_1 x_2 x_3 x_4 x_5$ by adding the edge $x_1 x_2$. Clearly, $G[N(x)] \cong P^r$. Now, since $G[N(x)] \not\cong G[N(x_4)]$, we find that $x_4 \notin E(G)$. This implies that $|N(x_4) \cap \overline{A}| = 1$. Let $N(x_4) \cap \overline{A} = \{t_1\}$. Furthermore, $G[N(x)]$ has a triangle, since $G[N(x)]$ has a triangle. Therefore, $G[t_1, t_4, x_5] = \triangle$. It follows that $G[N(x)]$ has a Hamilton cycle. This implies that $G[N(x)] \not\cong G[N(x_4)]$, a contradiction. Thus, Subclaim 5 holds.

By Subclaims 4 and 5, $G[N(x)]$ has two components, and one of them has exactly one vertex. If $|N(x_4) \cap \overline{A}| = 3$, then $G[N(x)] \not\cong G[N(x_3)]$, a contradiction. So, assume that $|N(x_4) \cap \overline{A}| \leq 2$, which implies that $N(x_4) \cap N(A) \neq \emptyset$. By Claim 12 and Subclaim 5, $N(x_4) \cap N(A) = \{t_1\}$. Now, we see that $K_4$, which contains $x_4$, is contained in $N(A) \cup \overline{A}$.

Hence, $G[N(x)]$ is a connected graph, a contradiction. Thus, Claim 13 holds.

By Claim 13 and Lemma 3, $x_4 x_5 \in E(G)$, and hence, $x$ has type $1$. Therefore, $G$ has type $1$.

Theorem 7. Let $G$ be a TCC-5-connected graph. If $|V(G)| \geq 10$, then $G$ has type $1$, type $2$, type $3$, or type $4$.

Proof. If $G$ contains $K_4$, then Lemma 10 assures us that $G$ has type $1$. So, we may assume that $G$ does not contain $K_4$. Hence, Lemma 8 assures us that for any $x \in V(G), \Delta(G[N(x)]) \leq 2$. Now, Lemma 3 assures us that $G$ has either type 2 or type 3 or type 4.

Theorem 8. Let $G$ be a TCC-5-connected graph. If $G$ has type 2, then $G$ is isomorphic to icosahedron.

Proof. Let $N(x) = \{x_1, \ldots, x_5\}$, and let $x_1 x_3 \cdots x_5 x_1$ be the cycle of $G[N(x)]$. Furthermore, let $N(x) = \{x_1, x, x_4, y_1, y_2\}$. Since $G$ has type 2, we may assume that $x_1 x_3 x_2 y_1 y_2$ is a cycle of $G[N(x)]$. Let $N(x) = \{x_1, x, x_3, y_1, y_2\}$, and then $y_1 \neq y_2$ and $y_2 \neq \{x_1, \ldots, x_5\}$. Now, $x_1 x_3 y_1 y_2 x_1$ is a cycle of $G[N(x)]$. Let $N(x) = \{x_1, x, x_5, y_1, y_2\}$. If $y_1 = y_2$, then, since $G$ has type
Since G is 5-regular, we see that \(|V(G)|\) is even. If G has a contractible edge, then we are done. Therefore, in the rest of the paper, we may assume that G is a contraction-critical 5-\(|\Delta|\)-connected graph. Hence, by Theorem 1, we can assume that \(|V(G)| \geq 10\). By Theorem 2, we see that G has type 1, type 2, type 3, or type 4. Next, we complete the proof of Theorem 5 by showing that the following lemmas are true.

**Lemma 11.** Let G be a TCC-5-connected graph. Let \(x \in V(G)\), \(abc\) be a path of \(G[N(x)]\), and \(G_0 = G[\{xa, bc\}]\). If \(G[N(x) - \{a, b, c\}] \cong K_2\), then \(\kappa(G_0) \geq 4\).

**Proof.** Suppose \(\kappa(G_1) \leq 3\). Let \(T_1\) be a smallest separator of \(G_1\), and let \(A_1\) be a \((T_1)\)-fragment. Clearly, \(|T_1| = 3\) and \(|\{xa\}, \{bc\}\| \subseteq T_1\). Let \(T = T_1 \cup \{x, a, b, c\} - \{xa, bc\}\). Clearly, \(|T| = 5\) and \(|x, a, b, c\| \subseteq T\). Furthermore, \(A = A_1\) is a fragment of G such that \(|N(A) = T|\). Since \(G[N(x)] - \{a, b, c\}\) is a complete graph, either \(N(x) \cap A = \emptyset\) or \(N(x) \cap A = \emptyset\), a contradiction. Hence, the lemma holds.

**Lemma 12.** Let G be a TCC-5-connected graph such that \(|V(G)| \geq 10\). If G has type 1, then G can be contracted to a 5-\(|\Delta|\)-connected H such that \(0 < |V(G)| - |V(H)| < 3\).

**Proof.** By the definition of type 1, we know that G contains \(K_4\) as a subgraph. Since G is vertex transitive graph, every vertex of G is contained in some \(K_4\). Let x be a vertex of G, and let \(N(x) = \{x, 1, \ldots, 3\}\). Furthermore, without the loss of generality, suppose \(G[\{x, x_1, x_2, x_3\}] \cong K_4\).

Since G has type 1, we may let \(N(x_1) = \{x, x_2, x_3, y_1, w_1\}, N(x_2) = \{x, x_1, x_3, y_2, w_2\}\), and \(N(x_3) = \{x, x_1, x_2, y_3, w_3\}\). Clearly, \(x, x_1, x_2, x_3, y_1, w_1, y_2, w_2, y_3, w_3\) are all different to each other since G has type 1.

Let \(G_1 = G[\{x, x_1, x_2, x_3\}]\), and let \(G_2 = G[\{x_1, x_2, x_3\}]\). Now, we see that \(\delta(G_1) \geq 5\) and \(\delta(G_2) \geq 5\), since \(x_1, x_2, x_3, y_1, w_1, y_2, w_2, y_3, w_3\) are all different to each other.

If either \(\kappa(G_1) = 5\) or \(\kappa(G_2) = 5\), then we are done. So, by Lemma 11, we may assume that \(\kappa(G_1) = 4\) and \(\kappa(G_2) = 4\).

Let \(T_1\) be a smallest separator of \(G_1\), and let \(A_1\) be a \((T_1)\)-fragment. Since \(\delta(G_1) \geq 5\), we see that \(|A_1| \geq 2\) and \(|A_1| \geq 2\). Furthermore, we can observe that \(T_1 \cap \{x, x_1, x_2, x_3\} \neq \emptyset\).

**Claim 14.** \(\{x, x_1, x_2, x_3\} - T_1 \neq \emptyset\).

**Proof.** Suppose \(\{x, x_1, x_2, x_3\} \subseteq T_1\). Let \(T = T_1 \cup \{x, x_1, x_2, x_3\} - \{x, x_1, x_2, x_3\}\). It follows that \(|T| = 6\) and \(\{x, x_1, x_2, x_3\} \subseteq T\). Furthermore, \(G[T] = A_1 \cup \{x, x_1, x_2, x_3\}\). Let \(x_2, x_3, y_2, w_2\) be a path of \(G[N(x)]\). Then, \(\delta(G) \geq 5\), hence, either \(N(x) \cap A_1 = \emptyset\) or \(N(x) \cap A_1 = \emptyset\). Without loss of generality, we may assume that \(N(x) \cap A_1 = \emptyset\). Then, \(A_1\) is a fragment of G such that \(\delta(G) \geq 5\).

Similarly, we may assume G has a fragment B such that \(|x, x_2, x_3| \subseteq B|\) and \(|x, x_1, x_2, x_3| \subseteq B|\). Furthermore, we may assume that \(|B| \geq 3\) and \(|B| \geq 2\).

Focusing on A and B, we see that \(x \in A \cap B\), \(x_1 \in N(A) \cap B\), \(x_2 \in N(A) \cap B\), and \(x_3 \in N(A) \cap N(B)\). If \(N(x_1) \cap N(B) \neq \emptyset\), then, since \(y_1, w_1 \in E(G)\), we see that \(N(x_1) - B = \emptyset\), a contradiction. Hence, we may assume \(N(x_1) - B = \emptyset\). By symmetry, let \(N(x_1) \cap (A \cap B) = \emptyset\).

**Claim 15.** \(B \cap A = \emptyset\).

**Proof.** Suppose \(B \cap A \neq \emptyset\). Since \(N(x_1) \cap (B \cap A) = \emptyset\), Lemma 1 assures us that \(B \cap A = \emptyset\) and \(|N(A) \cap B| > |N(B) \cap A|\). If \(B \cap A \neq \emptyset\), then \(A \cap B\) is a fragment. On the other hand, we find that \(N(x_1) \cap A \cap B = N(x_1) - A \cap B = \emptyset\). Now, Lemma 2 assures us that \(A \cap B \neq \emptyset\) and \(A \cap B \neq \emptyset\). Thus, \(|N(A) \cap B| \geq |N(B) \cap A| \geq 1 \geq 2 + 1 = 3\). It follows that \(|N(A) \cap B| \leq 1\). Now, Lemma 1 assures us that \(B \cap A = \emptyset\), a contradiction. Hence, \(B \cap A = \emptyset\). Now, we find that \(B \cap A = \emptyset\). This implies \(|B \cap N(A)| = |B| > 2\). Since \(A \cap B \neq \emptyset\) and \(B \cap A \neq \emptyset\), Lemma 1 implies that \(|N(B) \cap A| \geq 2\) and \(|N(B) \cap N(A)| \geq 2\). Thus, \(|N(B) \cap A| = |N(A) \cap B| = |N(B) \cap N(A)| = 2\), which implies that \(B \cap A\) is a fragment of G. It follows that \(N(x_1) \cap (B \cap A) \neq \emptyset\), a contradiction. Hence, we have \(B \cap A = \emptyset\), and similarly, \(B \cap A = \emptyset\).

If \(\bar{B} \cap \bar{A} \neq \emptyset\), then Lemma 1 assures us that \(A \cap B\) is a fragment of G. Since every vertex of G has type 1, we see that \(N(x_1) \cap A \cap B = N(x_1) \cap A \cap B = \emptyset\). Now, Lemma 2 assures us that \(A \cap B\) is a fragment. This implies \(|N(B) \cap A| \geq 2\) and \(|N(A) \cap B| \geq 2\). Now, Lemma 1 assures us that \(N(B) \cap A = \emptyset\) and \(|N(B) \cap N(A)| = 2\). Thus, we may assume that \(N(B) \cap A = |N(A) \cap B| = |N(B) \cap N(A)| = 2\). Furthermore, \(|B \cap N(A)| = |B| = 2\) and \(|A \cap N(B)| = |A| = 2\). Now, Lemma 1 assures us that \(|N(B) \cap A| = |N(A) \cap B| = |N(B) \cap N(A)| = 2\). Hence, we see that \(A \cap B\) is a fragment of G.
we see that $d(x_i) = 4$, a contradiction. Hence, we see that either $\kappa(G_i) \geq 5$ or $\kappa(G_j) \geq 5$.

**Lemma 13.** Let $G$ be a TCC-5-connected graph such that $|V(G)| \geq 10$. If $G$ has type 3, then $G$ can be contracted to a 5-connected $H$ such that $0 < |V(G) - |V(H)| < 3$.

**Proof.** Clearly, $G$ does not contain $K_5$. Suppose $G$ has a fragment of cardinality two, say $A = \{x, y\}$. Since $G$ is 5-regular, we see that $|N(x) \cap N(y)| = 3$. Hence, we see that $\Delta(G[N(x)]) \geq 3$. This contradicts Lemma 8. Hence, every fragment of $G$ contains either one vertex or at least three vertices. Let $x$ be a vertex of $G$ such that $N(x) = \{x_1, \ldots, x_5\}$. Let $x_1x_2x_3$ be a path of $G[N(x)]$. Furthermore, let $N(x_1) = \{x_2, x_3, y_1, y_2, y_3\}$, $N(x_2) = \{x_1, x_3, u_1, u_2\}$, and $N(x_3) = \{x_1, x_2, z_1, z_2, z_3\}$. Since $G$ has type 3, we see that $\{y_1, y_2, y_3\} \cap \{x_4, x_5, u_1, u_2\} = \emptyset$ and $\{z_1, z_2, z_3\} \cap \{x_4, x_5, u_1, u_2\} = \emptyset$.

Let $G_1 = G[\{x_1, x_2, x_3\}]$ and $G_2 = G[\{x_4, x_5, x_1\}]$. By Lemma 11, we have $\kappa(G_1) \geq 4$ and $\kappa(G_2) \geq 4$. If either $G_1$ or $G_2$ is 5-connected, then we are done. So we may assume $\kappa(G_1) = 4$ and $\kappa(G_2) = 4$.

Clearly, $\delta(G_1) \geq 5$ and $\delta(G_2) \geq 5$. For $i \in \{1, 2\}$, let $T_i$ be a smallest separator of $G_i$ and $A_i$ be a $T_i$-fragment. Since $\delta(G_1) \geq 5$ and $\delta(G_2) \geq 5$, we see that every component of $G - T_i$ has at least two vertices, where $i \in \{1, 2\}$. Furthermore, $T_1 \cap \{x, x_1, x_2, x_3\} \neq \emptyset$ and $T_2 \cap \{x, x_1, x_2, x_3\} \neq \emptyset$. Let $T_i^* = (T_i \cup \{a, b\} \setminus \{a, b\}, T_i \setminus \{a, b\})$ where $i \in \{1, 2\}$. It follows that $T_i^* \cap \{x, x_1, x_2, x_3\} \neq \emptyset$, $i \in \{1, 2\}$. Clearly, either $|T_i^* \cap \{x, x_1, x_2, x_3\}| = 3$, or $|T_i^* \cap \{x, x_1, x_2, x_3\}| = 4$, $i \in \{1, 2\}$.

**Claim 16.** For a smallest separator $T$ of $G$, the following holds.

1. $\{x, x_1, x_2\} \not\subseteq T$ and $\{x, x_2, x_3\} \not\subseteq T$.
2. If either $\{x_2, x_2, x_3\} \subseteq T$ or $\{x, x_1, x_2\} \subseteq T$, then one component of $G - T$ has exactly one vertex.

**Proof.** By symmetry, we only show that $\{x, x_1, x_2\} \not\subseteq T$, and the other one can be handled similarly. Suppose $\{x, x_1, x_2\} \not\subseteq T$, which implies that $\{x, x_1, x_2\} \not\subseteq T$. Let $A$ be a $T$-fragment. Since $G[N(x)] = \{x, x_1, x_2, x_3, x_4\}$, we see that $x \not\in T$. Hence, without the loss of generality, let $x_3 \in A$. Notice the fact that $x_3 \notin E(G)$, and we see that $A$ has at least two vertices. Now, the fact that $G[N(x)] = \{x, x_1, x_2, x_3\}$, $\kappa(G[N(x)]) \geq 5$, implies that $\kappa(A) = \{x\}$. Hence, $\kappa(A) = \{x\}$. We see that $A = \{x\}$, a contradiction.

**Claim 17.** $\{x_1, x_2, x_3\} \not\subseteq T_i^* \not\subseteq \emptyset$, $i \in \{1, 2\}$.

**Proof.** We only show that $\{x_1, x_2, x_3\} \not\subseteq T_i^*$. The other one can be handled similarly. Suppose $\{x_1, x_2, x_3\} \subseteq T_i$. It follows that $|T_i| = 3\cdot \delta(G_i) \geq 6$. Let $A'$ be a component of $G - T_i$, $A' = G - T_i - A$. As $x \notin E(G)$, it follows that $\kappa(A') = \emptyset$. Without the loss of generality, let $N(x) \cap A = \emptyset$. It follows that $N(x) \cap A' = \emptyset$, a contradiction. Hence, we have $\kappa(A') = 3$. Notice that $A' = A \cup \{x\}$, and we see that $A' \not\subseteq T$. It follows that $|A' \setminus T| \geq 3$.

Hence, $T$ is a smallest separator of $G$ such that $T \cap \{x, x_2, x_3\} = \{x_1, x_2, x_3\}$, but both $A$ and $A'$ have cardinality at least two, which contradicts Claim 16.

By Claim 17, without the loss of generality, we may assume that $T_i \cap \{x, x_2, x_3\} = \{x, x_2\}$. It follows that $|T_i| = 5$. Let $B_i$ be a $T_i$-fragment. Without the loss of generality, let $|T_i| \geq 5$. On the other hand, by Claim 17, either $T_i^* \cap \{x, x_2, x_3\} = \{x_1, x_2\}$ or $T_i^* \cap \{x, x_2, x_3\} = \{x, x_3\}$. Furthermore, since every component of $G - T_i$ has at least two vertices, we see that every component of $G - T_i$ has at least two vertices, where $i \in \{1, 2\}$. This implies that every component of $G - T_i$ has at least three vertices for each $i = 1, 2$. We will complete the proof of the lemma according the following two cases.

**Case 1.** $T_i^* \cap \{x, x_2, x_3\} = \{x_1, x_2\}$. It follows that $|T_i^*| = 5$. Let $B_2$ be a $T_i^*$-fragment. Without the loss of generality, suppose $\{x_1, x_2\} \subseteq B_2$. Now, we see that $x_2 \in B_1 \cap B_2$, $x_1 \in T_i^* \cap B_2$, $x_1 \in T_i^* \cap B_1$, and $x \in T_i^* \cap T_i^*$.
let \( x_4 \in B_2 \cap T_1 \). Furthermore, we see that \( N(x) \cap B_1 \cap B_2 = \emptyset \) and \( N(x) \cap B_2 \cap B_1 = \emptyset \). If \( B_1 \cap B_2 = \emptyset \), then Lemma 1 assures us that \( B_1 \cap B_2 = \emptyset \). It follows that \( \lvert B_1 \cap T_2 \rvert \geq \lvert B_1 \rvert \geq 3 \). Now, Lemma 1 assures us that \( \lvert B_2 \cap T_1 \rvert \geq \lvert B_1 \cap T_1 \rvert \geq 3 \). Hence, \( \lvert T_1 \rvert \geq 6 \), a contradiction.

Thus, we may assume that \( B_1 \cap B_2 = \emptyset \) and, similarly, \( B_1 \cap B_2 = \emptyset \). Similar to the argument of the last paragraph, we have \( \lvert B_1 \cap T_2 \rvert \geq \lvert B_1 \rvert \geq 3 \). It follows that \( \lvert B_2 \cap T_1 \rvert \geq \lvert B_1 \cap T_1 \rvert \geq 3 \). Hence, \( \lvert T_1 \rvert \geq 6 \), a contradiction. Hence, Claim 18 holds.

By Claim 18 and Lemma 1, we see that \( B_1 \cap B_2 = \emptyset \) is a fragment and \( N(B_1 \cup B_2) \cap \{ x_1, x_2, x_3, x_4 \} = \{ x_1, x_3 \} \). Now, Claim 16 implies that \( B_1 \cap B_2 = \{ x_3 \} \). Therefore, \( \{ w_1, w_2 \} \subseteq B_1 \cup T_1 \). If \( \lvert B_1 \cup T_1 \rvert \leq 8 \), it follows that \( \lvert N(x) \cap B_1 \cap T_1 \rvert \leq 8 \). \( \lvert N(x) \rvert \leq \min \{ 2, 8 \} \). Hence, \( \lvert T_1 \rvert \geq 6 \), a contradiction. Therefore, \( \lvert B_1 \cup T_1 \rvert \geq 9 \). Similarly, we may assume that \( \lvert B_2 \cup T_2 \rvert \geq 9 \).

Claim 19. \( B_1 \cap B_2 = \emptyset \) and \( B_2 \cap B_1 = \emptyset \).

Proof. Clearly, \( B_1 \cap B_2 = \emptyset \) is a fragment. It follows that \( N(x) \cap B_1 \cap B_2 = \emptyset \). Hence, \( N(x) \cap B_1 \cap B_2 = \emptyset \) and \( N(x) \cap B_1 \cap B_2 = \emptyset \), since \( G[N(x)] \cap \{ x_1, x_2, x_3 \} = K_2 \).

Now, if \( B_1 \cap B_2 = \emptyset \), then we may assume that \( B_1 \cap B_2 = \emptyset \). Notice that \( \lvert B_1 \cup T_1 \rvert \geq 9 \) and \( \lvert B_1 \cup T_2 \rvert \geq 9 \). And we have \( \lvert B_1 \cup T_1 \rvert \geq 9 \). Then, \( B_1 \cap B_2 = \emptyset \). Similarly, \( B_1 \cap B_2 = \emptyset \).

Now, we are ready to complete the proof of Case 1. Notice that \( \lvert B_1 \cup T_1 \rvert \geq 9 \) and \( B_1 \cup B_2 = \{ x_2 \} \). We may see that \( \lvert B_1 \cup B_2 \rvert \geq 3 \). Similarly, \( \lvert B_1 \cup T_2 \rvert \geq 3 \). Now, let \( B_1 \cup B_2 = \emptyset \) and \( B_1 \cap B_2 = \emptyset \). \( \lvert B_1 \cup T_1 \rvert \geq 6 \). Hence, \( \lvert T_1 \rvert \geq 6 \), a contradiction.

Case 2. \( T_1 = \{ x_1, x_2, x_3, x_4 \} = \{ x_1, x_3 \} \).

It follows that \( \lvert T_1 \rvert = 5 \). Let \( B_1 \) be a \( T_1 \)-fragment. Without the loss of generality, suppose \( \{ x_1, x_2 \} \subseteq B_1 \). Now, we see that \( x_3 \in B_1 \cap B_2, x_4 \in T_1 \cap B_2, x_2 \in T_2 \cap B_1, \) and \( x_3 \in T_1 \cap T_2 \).

Claim 20. \( B_1 \cap B_2 = \emptyset \).

Proof. Suppose \( B_1 \cap B_2 = \emptyset \). If \( B_1 \cap B_2 = \emptyset \), then \( \lvert B_1 \cup T_2 \rvert = \lvert B_1 \rvert \geq 3 \). Now, Lemma 1 assures us that \( \lvert B_1 \cap T_2 \rvert \geq \lvert B_1 \rvert \geq 3 \). It follows that \( \lvert B_1 \cap T_2 \rvert \geq \lvert B_1 \cap T_1 \rvert \geq 3 \). Hence, \( \lvert T_1 \rvert \geq 6 \), a contradiction. Thus, we may assume that \( B_1 \cap B_2 = \emptyset \). Similarly, \( B_1 \cap B_2 = \emptyset \) and \( B_1 \cap B_2 = \emptyset \) are fragments of \( G \). Without the loss of generality, we may assume \( x_3 \in B_1 \cap B_2 \).

Therefore, \( y_1 \in B_1 \cap B_2 \) and \( y_2 \in \lvert T_1 \rvert \). Notice that \( G[N(x)] \cap \{ x_1, x_2, y_1, y_2 \} = \emptyset \). Without the loss of generality, we may assume \( \{ x_1, x_2 \} \subseteq B_1 \cap B_2 \). Now, we see that \( \lvert N(x) \cap B_1 \cap B_2 \rvert = 1 \).
$B \subseteq \{x_4, x_5\}$. Hence, $x_1$ is adjacent to either $x_4$ or $x_5$, a contradiction. Hence, $[x_2, x_3] \in T'$.

Let $T_0 = T' \cup \{x, x_2, x_3\} \setminus \{x_2, x_3\}$. Let $A$ be a component of $G - T_0$, $A' = G - T_0 - A$. As $x_4x_5 \in E(G)$, $N(x) \cap A = \emptyset$ or $N(x) \cap A' = \emptyset$. Similarly, $N(x_4) \cap A = \emptyset$ or $N(x_4) \cap A' = \emptyset$. Without the loss of generality, let $N(x) \cap A = \emptyset$. Then, $N(x) \cap A' = \emptyset$ and $N(x_4) \cap A \neq \emptyset$, $N(x_4) \cap A' \neq \emptyset$.

Hence, $T = T_0 - \{x\}$ is a smallest separator of $G$. Let $A$ be a $T$–fragment. Clearly, $A = A' \cup \{x\}$. This implies that $|A| \geq 2$. Now,Lemma 9 shows that $|A| = 1$. So, $A \subseteq N(x_2) \cap N(x_3)$. Recall that $x \in A$, and we find that $|N(x_2) \cap N(x_3)| \geq 2$. This contradicts the fact that $N(x_2) \cap N(x_3) = \{x\}$.

Data Availability

No data were used to support this study

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This work was supported by NSFC (no. 11961051) and Natural Sciences Foundation of Guangxi Province (no. 2018GXNSFAA050117).

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