Generalized Lyapunov-Sylvester operators for Kuramoto-Sivashinsky Equation

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Abstract

A numerical method is developed leading to algebraic systems based on generalized Lyapunov-Sylvester operators to approximate the solution of two-dimensional Kuramoto-Sivashinsky equation. It consists of an order reduction method and a finite difference discretization which is proved to be uniquely solvable, stable and convergent by using Lyapunov criterion and manipulating generalized Lyapunov-Sylvester operators. Some numerical implementations are provided at the end to validate the theoretical results.

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1 Introduction

The present paper is devoted to the development of a computational method based on two-dimensional finite difference scheme to approximate the solution of the nonlinear Kuramoto-Sivashinsky equation

\[
\frac{\partial u}{\partial t} = q\Delta u - \kappa \Delta^2 u + \lambda |\nabla u|^2, \quad ((x, y), t) \in \Omega \times (t_0, +\infty),
\]

with initial conditions

\[
u(x, y, t_0) = \varphi(x, y) \quad (x, y) \in \Omega \]

and boundary conditions

\[
\frac{\partial u}{\partial \eta}(x, y, t) = 0 \quad ((x, y), t) \in \partial\Omega \times (t_0, +\infty),
\]

on a rectangular domain \( \Omega = [L_0, L_1] \times [L_0, L_1] \) in \( \mathbb{R}^2 \), \( t_0 \geq 0 \) is a real parameter fixed as the initial time. \( \frac{\partial}{\partial t} \) is the time derivative, \( \nabla \) is the space gradient operator and \( \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \) is the Laplace operator in \( \mathbb{R}^2 \), \( q, \kappa, \lambda \) are real parameters. \( \varphi \) and \( \psi \) are twice differentiable real valued functions on \( \Omega \).

We propose to apply an order reduction of the derivation and thus to solve a coupled system of equation involving second order differential operators. We set \( v = qu - \kappa \Delta u \) and thus we have
to solve the system

\[
\begin{cases}
\frac{\partial u}{\partial t} = \Delta v + \lambda |\nabla u|^2, \quad (x, y, t) \in \Omega \times (t_0, +\infty) \\
v = qu - \kappa \Delta u, \quad (x, y, t) \in \Omega \times (t_0, +\infty) \\
(u, v)(x, y, t_0) = (\varphi, \psi)(x, y), \quad (x, y) \in \overline{\Omega} \\
\nabla^2(u, v)(x, y, t) = 0, \quad (x, y, t) \in \partial\Omega \times (t_0, +\infty)
\end{cases}
\]

(4)

The Kuramoto-Sivashinsky equation (KS) is one of the most famous equations in math-physics for many decades. It has its origin in the work of Kuramoto since the 70-th decade of the 20-th century in his study of reaction-diffusion equation. The equation was then considered by Sivashinsky in modeling small thermal diffusion instabilities for laminar flames and modeling the reference flux of a film layer on an inclined plane. Since then the KS equation has experienced a growing development in theoretical mathematics, numerical as well as physical mechanics, nonlinear physics, hydrodynamics, chemistry, plasma physics, particle distributions advection, surface morphology, ...etc. See [1], [11], [13], [16], [20], [23], [25], [28], [29]. In [20], the symmetry problem of the model was studied based on the theory of Lie algebras. In [1], an anisotropic version of the KS equation has been proposed leading to global resolutions of the equation on rectangular domains. Sufficient conditions were given for the existence of global solution.

From the dimensional point of view, KS equation has been widely studied in one dimension, [8], [10], [14], [21], [22], [32]. However, this equation since its appearance is related to the modeling of flame spread which is a two-dimensional problem. In this context, an important result was developed in [30] where the authors showed by adapting the method developed in [24] the existence of a set of bounded solutions on a rectangular domain. This major importance of the two-dimensional model was a main motivation behind this work. A model representing a nonlinear dynamical system defined in a two-dimensional space is considered, where the solution \(u(x, y, t)\) satisfies a fourth order partial differential equation of the form (1), where \(u\) is the height of the interface, \(q\) is a pre-factor proportional to the coefficient of surface tension expressed by \(q\nabla^2 u\). The quantity \(\kappa \nabla^4 u\) represents the result of the diffusion surface due to the chemical potential gradient induced curvature. The pre-factor \(\kappa\) represents the surface diffusion term. The quantity \(\lambda |\nabla u|^2\) is due to the existence of overhangs and vacancies during the deposition process. Finally, the quantity \(\nu \nabla^2 u + \lambda |\nabla u|^2 u\) is referred to as modeling the effect of deposited atoms. cf. [13].

In the present work we propose to serve of algebraic operators to approximate the solutions of the Kuramoto-Sivashinsky(K-S) equation in the two partial and one temporal dimention without adapting classical developments based on separation of variables, radial solutions, tridiagonal operators, ... etc.

This was onefold of the present paper. A second crucial idea is to transform the continuous K-S equation into a generalized Lyapunov-Sylvester equation of the form

\[
\sum_i A_i X_n B_i = C_n
\]

(5)

where \(A_i\) and \(B_i\) are appropriate matrices depending on the discretization procedure and the problem parameters. \(X_n\) represents the numerical solution at time \(n\) and \(C_n\) is usually depending on the past values \(X_k, k \leq n - 1\) of the the solution. The equation (5) is known as generalized Lyapunov-Sylvester equation. Such equations have their origin in the work of Sylvester on classical matrices equations. In the particular case

\[
\sum_i A_i X A_i^T = C
\]

(6)
the equation is known restrictively as Lyapunov one [27]. Generally speaking, the equation

\[ \sum_i A_i X B_i = C \]  

(7)

is very difficult to be inverted and remains an open problem in algebra. Nevertheless, some works have been developed and proved that under suitable conditions on the coefficient matrices, one may get a unique solution, but it’s exact computation remains hard. It necessitates to compute eigenvalues and precisely bounds/estimates of eigenvalues or direct inverses of big matrices which remains a complicated problem and usually inappropriate.

In [19], a native method to solve (7) is investigated based on Kronecker product and equivalent matrix-vector equation. The Sylvester’s equation is transformed into a linear equivalent one on the form \( Gx = c \), with a matrix \( G \) obtained by tensor products issued from the \( A'_j \)s and the \( B'_j \)s. However, the general case remained already complicated. The authors have been thus restricted to special cases where the matrices \( A_j \) and \( B_j \) are scalar polynomials based on spacial and fixed matrices \( A \) and \( B \). Denote by \( \sigma (A) \) the spectrum of \( A \) and \( \sigma (B) \) the one of \( B \), the spectrum \( \sigma (G) \) may then be determined in terms these spectra. Indeed, with the assumptions the \( A'_j \)s and the \( B'_j \)s, the equation (7) can be written as

\[ \sum_{j,k} \alpha_{j,k} A'^j X B'^k = C, \]

where \( \alpha_{j,k} \) are complexe numbers. Hence, the tensor matrix \( G \) will be written on the form \( G = \phi (A, B) \), where \( \phi \) is the 2-variables polynomial \( \phi (x, y) = \sum_{j,k} \alpha_{j,k} x^j y^k \). Thus, \( G \) is singular if and only if \( \phi (\lambda, \mu) = 0 \) for some eigenvalues \( \lambda \) and \( \mu \) of \( A \) and \( B \) respectively.

In the case of square matrices an other criterion of existence of the solution of (7) was pointed out by Roth [31]. It was shown that the solution is unique if and only if the matrices

\[
\begin{bmatrix}
A & C \\
0 & B
\end{bmatrix}, \quad \text{and} \quad \begin{bmatrix}
A & 0 \\
0 & B
\end{bmatrix}
\]

are similar.

However, in numerical studies of PDE’s we may be confronted with matrices \( G \) where the computation of spectral properties are not easy and necessitate enormes calculus and sometimes, induces slow algorithms and bed convergence rates. Thus, one motivation here and issued from [6] consists to resolve such problem and prove the invertibility of the algebraic operator yielded in the numerical scheme by applying topological method instead of using classical ones such as tri-diagonal transformations. We thus aim to prove that generalized Lyapunov-Sylvester operators can be good candidates for investigating numerical solutions of PDEs in multi-dimensional spaces.

The paper is organized as follows. In next section the discretization of (4) is developed, the solvability of the scheme is analyzed. In section 4, the consistency of the method is shown and next, the stability and convergence are proved based on Lyapunov method and Lax-Richtmyer theorem. Finally, a numerical implementation is provided leading to the computation of the numerical solution and error estimates.
2 The numerical scheme

Let \( J \in \mathbb{N}^* \) and \( h = \frac{L_f}{J} \) be the space step. Denote for \((j,m) \in I^2 = \{0, 1, ..., J\}^2\), \( x_j = L_0 + jh \) and \( y_m = L_0 + mh \). Next, let \( l = \Delta t \) be the time step and \( t_n = t_0 + nl \), \( n \in \mathbb{N} \) be the time grid. We denote also \( \Omega = \{(x_j, y_m, t_n); (x_j, y_m, t_n) \in I^2 \times \mathbb{N}\} \) the associated discrete space. Finally, for \((j,m) \in I^2 \) and \( n \geq 0 \), \( u_{j,m}^n \) denotes the net function \( u(x_j, y_m, t_n) \) and \( U_{j,m}^n \) is the numerical solution.

The following discrete approximations will be applied for the different differential operators involved in the problem. For time derivatives, we set

\[
\frac{\partial u}{\partial t} \approx \frac{U_{j,m}^{n+1} - U_{j,m}^{n-1}}{2l}
\]

and for space derivatives, we shall use for the one order

\[
\frac{\partial u}{\partial x} \approx \frac{U_{j+1,m}^n - U_{j-1,m}^n}{2h}, \quad \text{and} \quad \frac{\partial u}{\partial y} \approx \frac{U_{j,m+1}^n - U_{j,m-1}^n}{2h}
\]

and for the second order ones, we set

\[
\frac{\partial^2 u}{\partial x^2} \approx \frac{U_{j+1,m}^{n,\alpha} - 2U_{j,m}^{n,\alpha} + U_{j-1,m}^{n,\alpha}}{h^2}, \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} \approx \frac{U_{j,m+1}^{n,\alpha} - 2U_{j,m}^{n,\alpha} + U_{j,m-1}^{n,\alpha}}{h^2}
\]

Finally, for \( n \in \mathbb{N}^* \) and \( \alpha \in \mathbb{R} \), we denote

\[
U_{j,m}^{n,\alpha} = \alpha U_{j,m}^{n-1} + (1 - 2\alpha) U_{j,m}^{n} + \alpha U_{j,m}^{n+1}
\]

to designate the estimation of \( U_{j,m}^{n} \) with an \( \alpha \)-extrapolation/interpolation barycenter method. By applying these discrete approximations, we obtain

\[
U_{j,m}^{n+1} - U_{j,m}^{n-1} = \frac{2l}{h^2} \left[ V_{j-1,m}^{n,\beta} - 2V_{j,m}^{n,\beta} + V_{j+1,m}^{n,\beta} + V_{j,m-1}^{n,\beta} - 2V_{j,m+1}^{n,\beta} + V_{j,m+1}^{n,\beta} \right]
\]

\[
+ \frac{2l}{h^2} \lambda \left[ \frac{1}{4} (U_{j+1,m}^{n} - U_{j-1,m}^{n})^2 + \frac{1}{4} (U_{j,m+1}^{n} - U_{j,m-1}^{n})^2 \right]
\]

In what follows, we set

\[
F_{j,m}^{n} = \frac{1}{4} \left[ (U_{j+1,m}^{n} - U_{j-1,m}^{n})^2 + (U_{j,m+1}^{n} - U_{j,m-1}^{n})^2 \right]
\]

We thus get

\[
U_{j,m}^{n+1} - U_{j,m}^{n-1} = \sigma [\beta V_{j-1,m}^{n+1} + (1 - 2\beta) V_{j-1,m}^{n} + \beta V_{j-1,m}^{n-1}
\]

\[
- 2\beta V_{j,m}^{n+1} + 2(1 - 2\beta) V_{j,m}^{n} - 2\beta V_{j,m}^{n-1}
\]

\[
+ \beta V_{j+1,m}^{n+1} + (1 - 2\beta) V_{j+1,m}^{n} + \beta V_{j+1,m}^{n-1}
\]

\[
+ \beta V_{j,m+1}^{n+1} + (1 - 2\beta) V_{j,m+1}^{n} + \beta V_{j,m+1}^{n-1}
\]

\[
- 2\beta V_{j,m+1}^{n+1} + 2(1 - 2\beta) V_{j,m+1}^{n} - 2\beta V_{j,m+1}^{n-1}
\]

\[
+ \beta V_{j,m+1}^{n+1} + (1 - 2\beta) V_{j,m+1}^{n} + \beta V_{j,m+1}^{n-1}] + \sigma \lambda F_{j,m}^{n}
\]
where $\sigma = \frac{2H}{\kappa}$. Otherwise, this can be written as

$$
U_{j,m}^{n+1} - \sigma \beta \left[ V_{j-1,m}^{n+1} - 2V_{j,m}^{n+1} + V_{j+1,m}^{n+1} + V_{j,m-1}^{n+1} - 2V_{j,m}^{n+1} + V_{j,m+1}^{n+1} \right] \\
= U_{j,m}^{n+1} + \sigma (1 - 2\beta) \left[ V_{j-1,m}^{n} - 2V_{j,m}^{n} + V_{j+1,m}^{n} + V_{j,m-1}^{n} - 2V_{j,m}^{n} + V_{j,m+1}^{n} \right] \\
+ \sigma \beta \left[ V_{j-1,m}^{n-1} - 2V_{j,m}^{n-1} + V_{j+1,m}^{n-1} + V_{j,m-1}^{n-1} - 2V_{j,m}^{n-1} + V_{j,m+1}^{n-1} \right] + \sigma \lambda F_{j,m}^{n}.
$$

Taking into account the boundary conditions, we obtain the full matrix expression

$$
U^{n+1} - \sigma \beta (AV^{n+1} + V^{n+1}A^T) = U^{n+1} + \sigma (1 - 2\beta) (AV^{n} + V^{n}A^T) \\
+ \sigma \beta (AV^{n-1} + V^{n-1}A^T) + \sigma \lambda F^{n} \quad (8)
$$

where

$$
A = \begin{pmatrix}
-2 & 2 & 0 & \cdots & 0 \\
1 & -2 & 1 & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & 1 & -2 & 1 \\
0 & \cdots & \cdots & 0 & 2 & -2
\end{pmatrix}
$$

$U^{n} = (U^{n}_{j,m})_{0 \leq j,m \leq J}$ and $V = (V^{n}_{j,m})_{0 \leq j,m \leq J}$.

Next, for $Q \in \mathcal{M}_{(J+1)^2}(\mathbb{R})$, we denote $\mathcal{L}_{Q}$ the linear operator which associates to $X \in \mathcal{M}_{(J+1)^2}(\mathbb{R})$ its image $\mathcal{L}_{Q}(X) = QX + XQ^T$. Thus, equation (??) will be written as

$$
U^{n+1} - \sigma \beta \mathcal{L}_{A}(V^{n+1}) = U^{n+1} + \sigma (1 - 2\beta) \mathcal{L}_{A}(V^{n}) + \sigma \beta \mathcal{L}_{A}(V^{n-1}) + \sigma \lambda F^{n}. \quad (9)
$$

Now, applying similar techniques as previously we obtain

$$
V^{n,\beta}_{j,m} = qU^{n,\alpha}_{j,m} - \frac{\kappa}{h^2} \left( U^{n,\alpha}_{j-1,m} - 2U^{n,\alpha}_{j,m} + U^{n,\alpha}_{j+1,m} + U^{n,\alpha}_{j,m-1} - 2U^{n,\alpha}_{j,m} + U^{n,\alpha}_{j,m+1} \right).
$$

Let $\delta = \frac{1}{h^2}$. This results in the following equation

$$
\beta V^{n+1}_{j,m} + (1 - 2\beta) V^{n}_{j,m} + V^{n-1}_{j,m} = q \left[ \alpha U^{n+1}_{j,m} + (1 - 2\alpha) U^{n}_{j,m} + \alpha U^{n-1}_{j,m} \right] \\
- \delta \kappa \left[ U^{n+1}_{j-1,m} - 2U^{n+1}_{j,m} + U^{n+1}_{j+1,m} + U^{n+1}_{j,m-1} - 2U^{n+1}_{j,m} + U^{n+1}_{j,m+1} \right] \\
- 2\alpha U^{n+1}_{j,m} - 2 \left( 1 - 2\alpha \right) U^{n-1}_{j,m} - 2\alpha U^{n-1}_{j,m} \\
\alpha U^{n+1}_{j,m+1} + (1 - 2\alpha) U^{n+1}_{j,m+1} + \alpha U^{n-1}_{j,m+1} \\
\alpha U^{n+1}_{j,m-1} + (1 - 2\alpha) U^{n+1}_{j,m-1} + \alpha U^{n-1}_{j,m-1} \\
- 2\alpha U^{n+1}_{j,m} - 2 \left( 1 - 2\alpha \right) U^{n-1}_{j,m} - 2\alpha U^{n-1}_{j,m} \\
\alpha U^{n+1}_{j,m+1} + (1 - 2\alpha) U^{n+1}_{j,m+1} + \alpha U^{n-1}_{j,m+1},
$$

or equivalently,

$$
\beta V^{n+1}_{j,m} - q\alpha U^{n+1}_{j,m} \\
+ \delta \kappa \left[ U^{n+1}_{j-1,m} - 2U^{n+1}_{j,m} + U^{n+1}_{j+1,m} + U^{n+1}_{j,m-1} - 2U^{n+1}_{j,m} + U^{n+1}_{j,m+1} \right] \\
= - (1 - 2\beta) V^{n}_{j,m} - \beta V^{n-1}_{j,m} + \alpha U^{n+1}_{j,m} + \alpha U^{n-1}_{j,m} - \delta \kappa \left[ U^{n+1}_{j-1,m} - 2U^{n+1}_{j,m} + U^{n+1}_{j+1,m} + U^{n+1}_{j,m-1} - 2U^{n+1}_{j,m} + U^{n+1}_{j,m+1} \right] \\
+ \left[ U^{n-1}_{j-1,m} - 2U^{n-1}_{j,m} + U^{n-1}_{j+1,m} + U^{n-1}_{j,m-1} - 2U^{n-1}_{j,m} + U^{n-1}_{j,m+1} \right].
$$
Now, applying the boundary conditions, we obtain
\[
\beta V^{n+1} - \alpha [qU^{n+1} - \delta \kappa L_A (U^{n+1})] = (1 - 2\alpha) (qU^n - \delta \kappa L_A (U^n)) + \alpha (qU^{n-1} - \delta \kappa L_A (U^{n-1})) - (1 - 2\beta) V^n - \beta V^{n-1}.
\]
(10)

As a result, we obtain finally the following discrete coupled system.
\[
\begin{cases}
U^{n+1} - \sigma \beta L_A (V^{n+1}) = U^{n-1} + \sigma (1 - 2\beta) L_A (V^n) + \sigma \beta L_A (V^{n-1}) + \sigma \lambda F^n \\
\beta V^{n+1} - \alpha [qU^{n+1} - \delta \kappa L_A (U^{n+1})] = (1 - 2\alpha)(qU^n - \delta \kappa L_A (U^n)) + \alpha (qU^{n-1} - \delta \kappa L_A (U^{n-1})) - (1 - 2\beta) V^n - \beta V^{n-1}
\end{cases}
\]
(11)

3 Solvability of the discret method

In this section we will examine the solvability of the discrete scheme. The main idea is by transforming the system (11) to an equality of the form \((U^{n+1}, V^{n+1}) = \phi (U^n, V^n, U^{n-1}, V^{n-1})\) with an appropriate function \(\phi\). We prove precisely that \(\phi\) can be expressed with general Lyapunov-Sylvester form. Next using general properties of such operators we prove that \(\phi\) is an isomorphism. In most studies, even recent ones such as [2] the authors proved that \(\phi\) or some modified versions may be contractive by inserting translation-dilation parameters leading to fixed point theory. In the present context it seems that such transformation is not possible as \(\|\phi\|\) may be greater than 1 and thus no contraction may occur. To overcome this problem we come back to differential calculus and topological properties.

**Theorem 1** The system (11) is uniquely solvable whenever \(U^0\) and \(U^1\) are known.

The proof of this result is based on the following preliminary lemma.

**Lemma 2** Let \(E\) be a finite dimensional vector space on \(\mathbb{R}\) (or \(\mathbb{C}\)), and \((\phi_n)\) be a sequence of endomorphisms converging uniformly to an invertible endomorphism \(\phi\). Then there exist \(n_0\) such that the endomorphism \(\phi_n\) is invertible for all \(n \geq n_0\).

**Proof.** Consider the endomorphism \(\phi\) on \(M_{(J+1)^2} (\mathbb{R}) \times M_{(J+1)^2} (\mathbb{R})\) defined by
\[
\phi (X,Y) = (X - \sigma \beta L_A (Y), \beta Y - \alpha \Gamma (X))
\]
where \(\Gamma (X) = qX - \delta \kappa L_A (X)\). To prove Theorem 2, we show that \(\phi\) is a one to one, and so \(\ker \phi\) is reduced to 0. Indeed,
\[
\phi (X,Y) = 0 \iff (X - \sigma \beta L_A (Y), \beta Y - \alpha [qX - \delta \kappa L_A (X)]) = (0,0).
\]
Which is equivalent to
\[
\begin{cases}
X = \sigma \beta L_A (Y) \\
\beta Y = \alpha \Gamma (X)
\end{cases}
\]
(13)

for \(\beta \neq 0\) we get
\[
Y = \sigma \alpha L_A (Y)
\]
(14)
Choosing \( l = o(h^{s+4}) \) (which is always possible), the operator \( K = I - \sigma \alpha \Gamma L_A \) tends uniformly to \( I \) whenever \( h \) tends to zero. Indeed, denote

\[
W = \sigma \alpha (qA - \delta \kappa A^2) = \frac{2l}{h^2} \alpha \left( qA - \frac{1}{h^2} \delta \kappa A^2 \right).
\]

We have

\[
K(X) = X - \sigma \alpha \Gamma L_A(X) = Y - WX - X W^T + 2 \sigma \alpha \delta \kappa AX A^T.
\]

thus,

\[
\| (K - I) X \| = \| WX + X W^T + 2 (\sigma \alpha \delta \kappa) AX A^T \| \\
\leq 2 \| W \| \| X \| + 32 \sigma \alpha \delta \kappa \| X \|
\]

Since we have \( \| W \| \leq 4 \sigma |\alpha| [\| q \| + 4 \delta |\kappa|] \), we obtain,

\[
\| (K - I) X \| \leq 8 \sigma \alpha [\| q \| + 8 \delta |\kappa|] \| X \|
\]

For \( l = o(h^{4+s}) \) this implies that,

\[
\| (K(X) - I(X)) \| \leq 16 \alpha [\| q \| h^{2+s} + 8 h^s |\kappa|] \| X \|
\]

Consequently the operator \( K \) converge uniformly to the identity whenever \( h \) tends towred 0 and \( l = o(h^{4+s}) \), withs \( s > 0 \). Thus, using Lemma 1 \( \phi \) is invertible for \( l,h \) small enough with \( l = o(h^{4+s}) \).

For \( \beta = 0 \) we obtain the system

\[
\begin{align*}
U^{n+1} &= U^{n-1} + \sigma L_A (V^n) + \sigma \lambda F^n \\
V^n &= \alpha \Gamma (U^{n+1}) + (1 - 2\alpha) \Gamma (U^n) + \alpha \Gamma (U^{n-1})
\end{align*}
\]

and thus,

\[
U^{n+1} - \sigma \alpha \Gamma L_A (U^{n+1}) = \sigma [(1 - 2\alpha) \Gamma L_A (U^n) + U^{n-1} + \alpha \Gamma L_A (U^{n-1}) + \lambda F^n]
\]

For the same assumption on \( l \) and \( h \) as above the same operator \( K(X) = X - \sigma \alpha \Gamma L_A (X) \) tends toward the identity as \( h \) tends to 0.

4 Consistency

The consistency of the proposed method will be proved by evaluating the local truncation error arising from the scheme introduced for the discretization of the system.

Applying Taylor Taylor’s expansion, and assuming that \( u \) and \( v \) to be sufficiently differentiable, we get

\[
U^{n+1}_{j,m} = u + l \frac{\partial u}{\partial t} + \frac{l^2}{2} \frac{\partial^2 u}{\partial t^2} + \frac{l^3}{6} \frac{\partial^3 u}{\partial t^3} + \frac{l^4}{24} \frac{\partial^4 u}{\partial t^4} + ...
\]

Similarly,

\[
U^{n-1}_{j,m} = u - l \frac{\partial u}{\partial t} + \frac{l^2}{2} \frac{\partial^2 u}{\partial t^2} - \frac{l^3}{6} \frac{\partial^3 u}{\partial t^3} + \frac{l^4}{24} \frac{\partial^4 u}{\partial t^4} + ...
\]
Hence,

\[ \frac{U_{j,m}^{n+1} - U_{j,m}^{n-1}}{2l} = \frac{\partial u}{\partial t} + \frac{l^2 \partial^3 u}{6 \partial t^3} + \ldots \quad (19) \]

Next, we get also

\[
\begin{align*}
V_{j-1,m}^{n+1} &= \left[ v - l \frac{\partial v}{\partial t} + \frac{l^2 \partial^2 v}{2 \partial t^2} - \frac{l^3 \partial^3 v}{6 \partial t^3} + \frac{l^4 \partial^4 v}{24 \partial t^4} \right] \\
&\quad - h \left[ \frac{\partial v}{\partial x} - l \frac{\partial^2 v}{\partial t \partial x} + \frac{l^2 \partial^3 v}{2 \partial t^2 \partial x} - \frac{l^3 \partial^4 v}{6 \partial t^3 \partial x} + \frac{l^4 \partial^5 v}{24 \partial t^4 \partial x} \right] \\
&\quad + \frac{h^2}{2} \left[ \frac{\partial^2 v}{\partial x^2} - l \frac{\partial^3 v}{\partial t \partial x^2} + \frac{l^2 \partial^4 v}{2 \partial t^2 \partial x^2} - \frac{l^3 \partial^5 v}{6 \partial t^3 \partial x^2} + \frac{l^4 \partial^6 v}{24 \partial t^4 \partial x^2} \right] \\
&\quad - \frac{h^3}{6} \left[ \frac{\partial^3 v}{\partial x^3} - l \frac{\partial^4 v}{\partial t \partial x^3} + \frac{l^2 \partial^5 v}{2 \partial t^2 \partial x^3} - \frac{l^3 \partial^6 v}{6 \partial t^3 \partial x^3} + \frac{l^4 \partial^7 v}{24 \partial t^4 \partial x^3} \right] \\
&\quad + \frac{h^4}{24} \left[ \frac{\partial^4 v}{\partial x^4} - l \frac{\partial^5 v}{\partial t \partial x^4} + \frac{l^2 \partial^6 v}{2 \partial t^2 \partial x^4} - \frac{l^3 \partial^7 v}{6 \partial t^3 \partial x^4} - \frac{l^4 \partial^8 v}{24 \partial t^4 \partial x^4} \right] + \ldots \quad (20)
\end{align*}
\]

and

\[
V_{j-1,m}^n = v - h \frac{\partial v}{\partial x} + \frac{h^2 \partial^2 v}{2 \partial x^2} - \frac{l^3 \partial^3 v}{6 \partial x^3} + \frac{l^4 \partial^4 v}{24 \partial x^4} + \ldots \quad (21)
\]

and finally,

\[
\begin{align*}
V_{j-1,m}^{n+1} &= \left[ v + l \frac{\partial v}{\partial t} + \frac{l^2 \partial^2 v}{2 \partial t^2} + \frac{l^3 \partial^3 v}{6 \partial t^3} + \frac{l^4 \partial^4 v}{24 \partial t^4} \right] \\
&\quad - h \left[ \frac{\partial v}{\partial x} + l \frac{\partial^2 v}{\partial t \partial x} + \frac{l^2 \partial^3 v}{2 \partial t^2 \partial x} + \frac{l^3 \partial^4 v}{6 \partial t^3 \partial x} + \frac{l^4 \partial^5 v}{24 \partial t^4 \partial x} \right] \\
&\quad + \frac{h^2}{2} \left[ \frac{\partial^2 v}{\partial x^2} + l \frac{\partial^3 v}{\partial t \partial x^2} + \frac{l^2 \partial^4 v}{2 \partial t^2 \partial x^2} + \frac{l^3 \partial^5 v}{6 \partial t^3 \partial x^2} + \frac{l^4 \partial^6 v}{24 \partial t^4 \partial x^2} \right] \\
&\quad - \frac{h^3}{6} \left[ \frac{\partial^3 v}{\partial x^3} + l \frac{\partial^4 v}{\partial t \partial x^3} + \frac{l^2 \partial^5 v}{2 \partial t^2 \partial x^3} + \frac{l^3 \partial^6 v}{6 \partial t^3 \partial x^3} + \frac{l^4 \partial^7 v}{24 \partial t^4 \partial x^3} \right] \\
&\quad + \frac{h^4}{24} \left[ \frac{\partial^4 v}{\partial x^4} + l \frac{\partial^5 v}{\partial t \partial x^4} + \frac{l^2 \partial^6 v}{2 \partial t^2 \partial x^4} + \frac{l^3 \partial^7 v}{6 \partial t^3 \partial x^4} + \frac{l^4 \partial^8 v}{24 \partial t^4 \partial x^4} \right] + \ldots \quad (22)
\end{align*}
\]

Thus,

\[
\begin{align*}
V_{j-1,m}^{n,\beta} &= v + \beta l^2 \frac{\partial^2 v}{\partial t^2} + \beta \frac{l^4 \partial^4 v}{12 \partial t^4} \\
&\quad + h \left[ \frac{\partial v}{\partial x} - \beta l^2 \frac{\partial^3 v}{\partial t^2 \partial x} - \beta \frac{l^4 \partial^5 v}{12 \partial t^4 \partial x} \right] \\
&\quad + \frac{h^2}{2} \left[ \frac{\partial^2 v}{\partial x^2} + \beta l^2 \frac{\partial^4 v}{\partial t^2 \partial x^2} + \beta \frac{l^4 \partial^6 v}{12 \partial t^4 \partial x^2} \right] \\
&\quad + \frac{h^3}{6} \left[ - \frac{\partial^3 v}{\partial x^3} - \beta l^2 \frac{\partial^5 v}{\partial t^2 \partial x^3} - \beta \frac{l^4 \partial^7 v}{12 \partial t^4 \partial x^3} \right] \\
&\quad + \frac{h^4}{24} \left[ \frac{\partial^4 v}{\partial x^4} + \beta l^2 \frac{\partial^6 v}{\partial t^2 \partial x^4} + \beta \frac{l^4 \partial^8 v}{12 \partial t^4 \partial x^4} \right] + \ldots \quad (23)
\end{align*}
\]
Similarly,

\[ V_{j,m}^{n,\beta} = v + \beta l^2 \frac{\partial^2 v}{\partial t^2} + \beta \frac{l^4}{12} \frac{\partial^4 v}{\partial t^4} + \ldots \]  

(24)

We have also

\[ V_{j+1,m}^{n,\beta} = v + \beta l^2 \frac{\partial^2 v}{\partial t^2} + \beta \frac{l^4}{12} \frac{\partial^4 v}{\partial t^4} + \ldots \]

(25)

Finally,

\[ \frac{V_{j,m-1}^{n,\beta} - 2V_{j,m}^{n,\beta} + V_{j,m+1}^{n,\beta}}{h^2} = \left[ \frac{\partial^2 v}{\partial y^2} + \beta l^2 \frac{\partial^4 v}{\partial t^2 \partial y^2} + \beta \frac{l^4}{12} \frac{\partial^6 v}{\partial t^4 \partial y^2} \right] + \ldots \]

(26)

Now,

\[
\frac{2l}{h^2} \left[ \frac{V_{j,m}^{n,\beta} - 2V_{j,m}^{n,\beta} + V_{j,m+1}^{n,\beta}}{h^2} \right] = \frac{\partial u}{\partial t} - \Delta v + l^2 \frac{\partial^2 u}{6 \partial t^2} - \beta l^2 \frac{\partial^2}{\partial t^2} (\Delta v) - \frac{h^2}{12} \left( \frac{\partial^4 v}{\partial x^4} + \frac{\partial^4 v}{\partial y^4} \right)
+ \beta l^2 \frac{\partial^2}{\partial t^2} \left( \frac{\partial^4 v}{\partial x^4} + \frac{\partial^4 v}{\partial y^4} \right) - \beta \frac{l^4}{12} \left[ \frac{\partial^6 v}{\partial t^4 \partial x^2} + \frac{\partial^6 v}{\partial t^4 \partial y^2} \right]
- \beta \frac{l^4}{12} \frac{h^2}{12} \left[ \frac{\partial^8 v}{\partial t^4 \partial x^4} + \frac{\partial^8 v}{\partial t^4 \partial y^4} \right] + \ldots
\]

(27)

We now examine the second equation in (4). Applying the same calculus as above, we get

\[ v = (qv - \kappa (\Delta u)) - \beta l^2 \frac{\partial^2 v}{\partial t^2} + q\alpha l^2 \frac{\partial^2 u}{\partial t^2} - \kappa \alpha l^2 \frac{\partial (\Delta u)}{\partial t^2} - \kappa \frac{h^2}{12} \left( \frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4} \right) + o (l^2 + h^2) \]

It results from above that the principal part of the first equation in system (4) is

\[ L_{u,v}^1 (t, x, y) = \beta l^2 \frac{\partial^2 v}{\partial t^2} - \alpha q l^2 \frac{\partial^2 u}{\partial t^2} - \alpha \kappa l^2 \frac{\partial^2 (\Delta v)}{\partial t^2}
- \kappa \frac{l^2}{12} \left( \frac{\partial^4 v}{\partial x^4} + \frac{\partial^4 v}{\partial y^4} \right) + o (l^2 + h^2). \]

(28)
The principal part of the local error truncation due to the second equation is
\[
L^2_{u,v} (t, x, y) = \beta \frac{l^2}{2} \frac{\partial^2}{\partial t^2} (v - q u - \kappa \alpha \Delta u) - \kappa h^2 \frac{1}{12} \left( \frac{\partial^4 v}{\partial x^4} + \frac{\partial^4 v}{\partial y^4} \right) + o \left( l^2 + h^2 \right).
\] (29)

As a result, we get the following lemma.

**Lemma 3** The numerical method is consistent with an order 2 in space and time.

**Proof.** It is clear that the two operators \( L^1_{u,v} \) and \( L^2_{u,v} \) tend towards 0 as \( l \) and \( h \) tend to 0, which ensures the consistency of the method. Furthermore, the method is consistent with an order 2 in time and space. ■

### 5 Stability and convergence

The stability of the discrete scheme will be evaluated using Lyapunov criterion which states that a linear system \( L (x_{n+1}, x_n, x_{n-1}, ...) = 0 \) is stable in the sense of Lyapunov if for any bounded initial solution \( x_0 \), the solution \( x_n \) remains bounded for all \( n \geq 0 \). In this section we prove precisely the following result.

**Lemma 4** The solution \( (U^n, V^n) \) is bounded independently of \( n \) whenever the initial solution \( (U^0, V^0) \) is.

**Proof.** We proceed by recurrence on \( n \). Assume \( \|(U_0, V_0)\| \leq \eta \) for some positive \( \eta \). The system (11) can be written on the form
\[
\begin{aligned}
U^{n+1} - \sigma \beta L_A (V^{n+1}) &= U^{n-1} + \sigma (1 - 2\beta) L_A (V^n) + \sigma \beta L_A (V^{n-1}) + \sigma \lambda F^n. \\
\beta V^{n+1} &= \alpha \Gamma (U^{n+1}) + (1 - 2\alpha) \Gamma (U^n) + \alpha \Gamma (U^{n-1}) - (1 - 2\beta) V^n - \beta V^{n-1}.
\end{aligned}
\] (30)

The last equation gives,
\[
\beta L_A (V^{n+1}) = \alpha L_A \Gamma (U^{n+1}) + (1 - 2\alpha) L_A \Gamma (U^n) + \alpha L_A \Gamma (U^{n-1}) - (1 - 2\beta) L_A (V^n) - \beta L_A (V^{n-1}).
\]

Substituting in the first one, we obtain
\[
K (U^{n+1}) = \sigma (1 - 2\alpha) L_A \Gamma (U^n) + \sigma \alpha L_A \Gamma (U^{n-1}) + U^{n-1} + \sigma \lambda F^n.
\] (31)

Next, recall that
\[
|F^n_{j,m}| = \frac{1}{4} \left( (U^n_{j+1,m} - U^n_{j-1,m})^2 + (U^n_{j,m+1} - U^n_{j,m-1})^2 \right).
\]

Thus,
\[
|F^n_{j,m}| \leq \frac{1}{2} \left[ (U^n_{j+1,m})^2 + (U^n_{j-1,m})^2 + (U^n_{j,m+1})^2 + (U^n_{j,m-1})^2 \right]
\]
and consequently,
\[
\|F^n\|_2 \leq 2 \|U^n\|_2^2.
\] (32)
Finally, (31) yields that

\[ \| K(U^{n+1}) \| \leq \sigma |1 - 2\alpha| \| \mathcal{L}_A\Gamma(U^n) \| + \sigma |\alpha| \| \mathcal{L}_A\Gamma(U^{n-1}) \| + \| U^{n-1} \| + \sigma |\lambda| \| F^n \|. \]

Setting \( \omega = |q| + 8\delta |\kappa| \), we obtain

\[ \| K(U^{n+1}) \| \leq 8\omega \sigma |1 - 2\alpha| \| U^n \| + [1 + 8\omega \sigma |\alpha|] \| U^{n-1} \| + 2\sigma |\lambda| \| U^n \|^2. \]  

We now evaluate \( \| V^{n+1} \| \). Applying \( \Gamma \) for the first equation in the system (30), we get

\[
\Gamma(U^{n+1}) = \sigma \beta \Gamma \mathcal{L}_A(V^{n+1}) + \Gamma(U^{n-1}) + \sigma (1 - 2\beta) \Gamma(\mathcal{L}_A(V^n)) + \sigma \beta \Gamma(\mathcal{L}_A(V^{n-1})) + \sigma \lambda \Gamma(F^n).
\]

By replacing in the second equation of (13) we obtain

\[
\beta K(V^{n+1}) = (1 - 2\beta)[(\sigma \alpha \Gamma(\mathcal{L}_A(V^n)) - V^n] \]

\[
+ \beta [(\sigma \alpha \Gamma(\mathcal{L}_A(V^{n-1})) - V^{n-1}] + 2\alpha \Gamma(U^{n-1}) + (1 - 2\alpha) \Gamma(U^n) + \sigma \alpha \lambda \Gamma(F^n),
\]

We get from (34) and (32),

\[ \| \beta K(V^{n+1}) \| \leq |1 - 2\beta| [8\sigma |\alpha| \omega + 1] \| V^n \| \]

\[ + |\beta| [8\sigma |\alpha| \omega + 1] \| V^{n-1} \| + 2 |\alpha| \omega \| U^{n-1} \| \]

\[ + (1 - 2\alpha) \omega \| U^n \| + 2\sigma |\alpha| \omega \| U^n \|^2. \]

Now coming back to (30) and applying boundary conditions, we get

\[ U^{-1} = U^0 + I\tilde{\varphi} \quad \text{and} \quad V^{-1} = qU^0 + \tilde{\psi} \]

where,

\[ \tilde{\varphi} = -q\Delta \varphi + \kappa \Delta^2 \varphi - \lambda |\nabla \varphi|^2 \]

and

\[ \tilde{\psi} = -(lq^2 + \kappa) \Delta \varphi + 2l\kappa q \Delta^2 \varphi - l\kappa^2 \Delta^3 \varphi - \lambda lq |\nabla \varphi|^2 + \lambda l\kappa \Delta (|\nabla \varphi|^2). \]

Hence,

\[ \| U^{-1} \| \leq \| U^0 \| + l \| \tilde{\varphi} \| \quad \text{and} \quad \| V^{-1} \| \leq |q| \| U^0 \| + \| \tilde{\psi} \|. \]

Now, the Lyapunov criterion for stability states exactly that

\[ \forall \varepsilon > 0, \exists \eta > 0 : \| (U^0, V^0) \| \leq \eta \Rightarrow \| (U^n, V^n) \| \leq \varepsilon, \quad \forall n \geq 0. \]

For \( n = 1 \), and any \( \varepsilon \) given such that \( \|(U^1, V^1)\| \leq \varepsilon \), we seek an \( \eta > 0 \) for which \( \|(U^0, V^0)\| < \eta \). By direct substitution in (33), for \( n = 0 \), we obtain

\[ \| K(U^1) \| \leq 8\omega \sigma |1 - 2\alpha| \| U^0 \| + [1 + 8\omega \sigma |\alpha|] \| U^{-1} \| + 2\sigma |\lambda| \| U^0 \|^2. \]
From \(37\), we obtain
\[
\| \mathcal{K}(U^1) \| \leq 2\sigma |\lambda| \|U^0\|^2 + (1 + 8\omega\sigma |1 - 2\alpha| + |\alpha|) \|U^0\| + l (1 + 8\sigma |\alpha| \omega) \|\tilde{\varphi}\|.
\]
Observing that,
\[
|1 - 2\alpha| + |\alpha| \leq (1 + 3 |\alpha|),
\]
we get
\[
\| \mathcal{K}(U^1) \| \leq 2\sigma |\lambda| \|U^0\|^2 + 8\omega\sigma (1 + 3 |\alpha|) \|U^0\| + l (1 + 8\sigma |\alpha| \omega) \|\tilde{\varphi}\|.
\]
Next choosing \(l = o(h^{4+s})\) small enough, we obtain
\[
\|U^1\| \leq 4\sigma |\lambda| \|U^0\|^2 + 16\omega\sigma (1 + 3 |\alpha|) \|U^0\| + 2l (1 + 8\sigma |\alpha| \omega) \|\tilde{\varphi}\|.
\]
Now, for \(\varepsilon > 0\), we seek \(\eta > 0\) such that
\[
4\sigma |\lambda| \eta^2 + 16\omega\sigma (1 + 3 |\alpha|) \eta + 2l (1 + 8\sigma |\alpha| \omega) \|\tilde{\varphi}\| < \varepsilon
\]
or otherwise,
\[
4\sigma |\lambda| \eta^2 + 16\omega\sigma (1 + 3 |\alpha|) \eta + 2l (1 + 8\sigma |\alpha| \omega) \|\tilde{\varphi}\| - \varepsilon < 0.
\]
The discriminant of the last inequality is
\[
\Delta' = 64 (\omega\sigma (1 + 3 |\alpha|))^2 - 4\sigma |\lambda| (2l (1 + 8\sigma |\alpha| \omega) \|\tilde{\varphi}\| - \varepsilon).
\]
For the same assumption on \(l\) and \(h\) as above,
\[
\Delta' \sim 64 (\omega\sigma (1 + 3 |\alpha|))^2 + 4\sigma |\lambda| \varepsilon > 0.
\]
Consequently, there are two zeros \(\eta_1 < \eta_2\) of the inequality above. Furthermore, replacing \(\eta\) with 0 we get a negative quantity, thus \(\eta_1 < 0 < \eta_2\). As a result, \(\eta_2\) is the good candidate. Now, choosing \(\|(U^0, V^0)\| \leq \eta_2\), we get immediately \(\|U^1\| < \varepsilon\).

Next, already with \(n = 0\), we get similarly to the previous case
\[
\|\beta \mathcal{K}(V^1)\| \leq |1 - 2\beta| [8\sigma |\alpha| \omega + 1] \|V^0\| + |\beta| [8\sigma |\alpha| \omega + 1] \|V^{-1}\| + 2|\alpha| \omega \|U^{-1}\| + |1 - 2\alpha| \omega \|U^0\| + 2\sigma\alpha\lambda\omega \|U^0\|^2.
\]
Choosing \(l = o(h^{4+s})\) small enough as above, and \(\mu = 8\sigma |\alpha| \omega + 1\), we obtain
\[
\frac{|\beta|}{2} \|V^1\| \leq \|\beta \mathcal{K}(V^1)\| \leq \mu |1 - 2\beta| \|V^0\| + \mu |\beta| \|V^{-1}\| + 2|\alpha| \omega \|U^{-1}\| + |1 - 2\alpha| \omega \|U^0\| + 2\sigma\alpha\lambda\omega \|U^0\|^2.
\]
Next, recall that
\[
\|U^{-1}\| = \|U^0\| + \|\tilde{\varphi}\| \quad \text{and} \quad \|V^{-1}\| \leq |q| \|U^0\| + l \|\tilde{\psi}\|,
\]
we get
\[
|\beta| \|V^1\| \leq 2\mu |1 - 2\beta| \|V^0\| + 2\mu |\beta| \left(|q| \|U^0\| + \|\tilde{\psi}\|\right) + 4|\alpha| \omega \left((|U^0| + l \|\tilde{\varphi}\|) + 2|1 - 2\alpha| \omega \|U^0\| + 4\sigma\alpha\lambda\omega \|U^0\|^2.
\]

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Consider also the endomorphism $\phi$ and its continuous, then

\begin{align*}
|\beta| \|V^1\| &\leq 4\sigma \alpha \lambda \|U_0\|^2 + 2\mu |1 - 2\beta| \|V_0\| \\
&+ 2(\mu |q| |\beta| + \omega (2 |\alpha| + |1 - 2\alpha|)) \|U_0\| \\
&+ 2\mu |\beta| \|\bar{\psi}\| + 4 |\alpha| \omega \|\bar{\phi}\|.
\end{align*}

Now, proceeding as for $U^1$, we prove that for all $\varepsilon > 0$, there is an $\eta_2 > 0$ satisfying $\|V^1\| < \varepsilon$ whenever $\|(C^0, V^0)\| \leq \eta_2$. Finally $\eta = \min(\eta_1, \eta_2)$ answers the question.

Assume now that $(U^k, V^k)$ is bounded for $k = 1, 2, ..., n$ by $\varepsilon_1$ whenever $(U^0, V^0)$ is bounded by $\eta$ and let $\varepsilon > 0$. We shall prove that it is possible to choose $\eta$ satisfying $\|(U^{n+1}, V^{n+1})\| \leq \varepsilon$.

**Lemma 5 (Banach)** Let $E, F$ be two Banach spaces and $\phi : E \to F$ be linear. If $\phi$ is continuous and bijective then $\phi$ is a homeomorphism.

Consider as above the endomorphism $\phi$ on $M_{(J+1)^2} (\mathbb{R}) \times M_{(J+1)^2} (\mathbb{R})$ defined by

$$
\phi(X, Y) = (X - \sigma \beta L_A(Y), \beta Y - \alpha [qX - \delta \kappa L_A(X)]).
$$

Consider also

$$
f_1(X, Y, Z, W) = X + \alpha \beta L_A(Y) + \sigma (1 - 2\beta) + \sigma \beta L_A(Z) + \sigma \lambda W
$$

and

$$
f_2(X, Y, Z, T) = \alpha \Gamma(X) - \beta Y + (1 - 2\alpha) \Gamma(Z) - (1 - 2\beta) T.
$$

We obtain thus

$$
\phi(U^{n+1}, V^{n+1}) = (U^{n+1} - \sigma \beta L_A(V^{n+1}), \beta V^{n+1} - \alpha \Gamma(U^{n+1})) = (f_1(U^{n-1}, V^{n-1}, V^n, F^n), f_2(U^{n-1}, V^{n-1}, U^n, V^n)).
$$

We have already proved that $\phi$ is one to one for $l$ and $h$ small enough. Since $\phi$ is linear and continuous, then $\phi$ has a continuous inverse function. So, $\phi$ is a homeomorphism on $M_{(J+1)^2} (\mathbb{R}) \times M_{(J+1)^2} (\mathbb{R})$. Furthermore

$$
(U^{n+1}, V^{n+1}) = \phi^{-1}(f_1(U^{n-1}, V^{n-1}, V^n, F^n), f_2(U^{n-1}, V^{n-1}, U^n, V^n)).
$$

As $\phi^{-1}$ is continuous, there exists $C > 0$ such that

$$
\|(U^{n+1}, V^{n+1})\| = \|\phi^{-1}(f_1(U^{n-1}, V^{n-1}, V^n, F^n), f_2(U^{n-1}, V^{n-1}, U^n, V^n))\| \\
\leq C \|f_1(U^{n-1}, V^{n-1}, V^n, F^n), f_2(U^{n-1}, V^{n-1}, U^n, V^n))\|.
$$

By choosing $\|(U^k, V^k)\| \leq \eta$, for all $k = 0, 1, ... n$, we get

$$
\|f_1(U^{n-1}, V^{n-1}, V^n, F^n)\| \leq 2\sigma |\lambda| \eta^2 + (1 + 8\sigma (1 + 3|\beta|)) \eta.
$$

Now, it suffices to prove that there exists $\eta > 0$ for which

$$(1 + 8\sigma (1 + 3|\beta|)) \eta + 2\sigma |\lambda| \eta^2 \leq \varepsilon \quad \Leftrightarrow \quad 2\sigma |\lambda| \eta^2 + (1 + 8\sigma (1 + 3|\beta|)) \eta - \varepsilon \leq 0.$$

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The discernments is
\[ \Delta = (1 + 8\sigma (1 + 3|\beta|))^2 + 8\sigma |\lambda| \varepsilon > 0. \]
Hence, there are two zeros of the last equality
\[ \eta_1 = \frac{-(1 + 8\sigma (1 + 3|\beta|)) - \sqrt{\Delta}}{4\sigma |\lambda|} \]
and
\[ \eta_2 = \frac{-(1 + 8\sigma (1 + 3|\beta|)) + \sqrt{\Delta}}{4\sigma |\lambda|}. \]
It is straightforward that \(0 \in \eta_1, \eta_2\). Hence \(\eta_2 > 0\). Now for \(f_2\) we get
\[ \|f_2(U^{n-1}, V^{n-1}, U^n, V^n)\| \leq \omega (1 + 3|\alpha|) + (1 + 3|\beta|) \eta. \]
For
\[ \eta \leq \eta_3 = \frac{\varepsilon}{\omega (1 + 3|\alpha|) + (1 + 3|\beta|)}, \]
we obtain \(\|V^{n+1}\| \leq \varepsilon\). Finally, choosing \(\eta\) the minimum between \(\eta_2\) and \(\eta_3\) the criterion is proved.

**Lemma 6** As the numerical scheme is consistent and stable, it is then convergent.

### 6 Numerical Implementations

We propose in this section to present some numerical examples to validate the theoretical results developed previously. The error between the exact solutions and the numerical ones via an \(L_2\) discrete norm will be estimated. The matrix norm used will be
\[ \|X\|_2 = \left( \frac{1}{N} \sum_{i,j=1}^{N} |X_{ij}|^2 \right)^{1/2} \]
for a matrix \(X = (X_{ij}) \in \mathcal{M}_{N+2}\mathbb{C}\). Denote \(u^n\) the net function \(u(x, y, t^n)\) and \(U^n\) the numerical solution. We propose to compute the discrete error
\[ Er = \max_n \|U^n - u^n\|_2 \quad (39) \]
on the grid \((x_i, y_j), 0 \leq i, j \leq J + 1\) and to validate the convergence rate of the proposed schemes we propose to compute the proportion
\[ C = \frac{Er}{l^2 + h^2}, \]

We consider the inhomogeneous problem
\[
\begin{cases}
\frac{\partial u}{\partial t} = \Delta v + \lambda |\nabla u|^2 + g(x, y, t), & (x, y, t) \in \Omega \times (t_0, +\infty) \\
v = qu - \kappa \Delta u, & (x, y, t) \in \Omega \times (t_0, +\infty) \\
(u, v)(x, y, t_0) = (\varphi, \psi)(x, y), & (x, y) \in \overline{\Omega} \\
\nabla (u, v)(x, y, t) = 0, & (x, y, t) \in \partial \Omega \times (t_0, +\infty)
\end{cases}
\quad (40)
\]
on the rectangular domain \(\Omega = [-1, 1] \times [-1, 1]\), where
\[ g(x, y, t) \]
\[
= Ke(t) [e(t) [C^4(x)S^2(x)C^6(y) + C^6(x)C^4(y)S^2(y)] - [C(x)S^2(x)C(y)S^2(y)]] \quad (41)
\]
\[
- [C(x)S^2(x)C(y)S^2(y)] \quad (42)
\]
Table 1: Error estimates for $l = o(h^{4.01})$

| $J$ | $N$   | $Er^2$       |
|-----|-------|--------------|
| 10  | 640   | $1.25.10^{-5}$|
| 12  | 1320  | $2.46.10^{-6}$|
| 14  | 2450  | $6.14.10^{-7}$|
| 16  | 4183  | $1.84.10^{-7}$|
| 18  | 6707  | $6.38.10^{-8}$|
| 20  | 10233 | $2.46.10^{-8}$|
| 22  | 14997 | $1.04.10^{-8}$|
| 24  | 21258 | $4.76.10^{-9}$|
| 25  | 25039 | $3.29.10^{-9}$|
| 30  | 52015 | $6.36.10^{-10}$|

and the exact solution

$$(u, v)(x, y, t) = (e(t)C^3(x)C^3(y), ...),$$

with

$C(x) = \cos\left(\frac{\pi x}{2}\right), \quad S(x) = \sin\left(\frac{\pi x}{2}\right), \quad e(t) = e^{-9\pi^4 t/2} \quad \text{and} \quad K = \frac{9\pi^4}{2}.$

In the following tables, numerical results are provided. We computed for different space and time steps the discrete $L_2$-error estimates defined by (39). The time interval is $[0, 1]$ for a choice $t_0 = 0$ and $T = 1$. The following results are obtained for different values of the parameters $J$ (and thus $h$), $l$ ((and thus $N$). The parameters $\alpha$ and $\beta$ are fixed to $\alpha = \frac{1}{3}$ and $\beta = \frac{1}{5}$. We just notice that some variations done on these latter parameters have induced an important variation in the error estimates which explains their effect as they calibrates the position of the approximated solution around the exact one. The parameters $\kappa$, $\lambda$ and $\kappa$ have the role of viscosity-type coefficients and fixed to the values $\kappa = 1$, $\lambda = -2\pi^2$ and $q = \frac{11\pi^2}{2}$. The following tables outputs the error estimates relatively to the discrete $L^2$–norm defined above for different values of the space and time steps. First, we provide estimates when the optimal condition $l = o(h^{4+s}), s > 0$ is fulfilled. $s$ is fixed to 0.01 for the first table. Next, to control more the effect of such assumption which is due to the presence of a second order Laplacian in the original problem, we tested the convergence of the scheme at some orders less than the optimal power fixed to 4. The second table provides the estimates with a slightly sub-critical power $4 - s, s > 0$ small enough. Here also $s$ is fixed to 0.01, and finally in the third table, we tested the discrete scheme for a strong sub-critical power.

### 7 Conclusion

This paper investigated the solution of the well-known Kuramoto-Sivashinsky equation in two-dimensional case by applying a two-dimensional finite difference discretization. The original equation is a 4-th order partial differential equation. Thus, in a first step it was recasted into a system of second order partial differential equations by applying a reduction order. Next, the continuous
| $J$ | $N$   | $Er2$       |
|-----|-------|-------------|
| 10  | 616   | $1,35.10^{-8}$ |
| 12  | 1273  | $2,55.10^{-6}$ |
| 14  | 2355  | $6,65.10^{-7}$ |
| 16  | 4012  | $2,00.10^{-7}$ |
| 18  | 6419  | $6,96.10^{-8}$ |
| 20  | 7973  | $4,06.10^{-8}$ |
| 22  | 14295 | $1,14.10^{-8}$ |
| 24  | 20228 | $5,26.10^{-9}$ |
| 25  | 23806 | $3,64.10^{-9}$ |
| 30  | 49273 | $7,09.10^{-10}$ |

Table 2: Error estimates for $l = o(h^{3.99})$

| $J$ | $N$   | $Er2$       |
|-----|-------|-------------|
| 10  | 128   | $3,10.10^{-4}$ |
| 12  | 220   | $8,82.10^{-6}$ |
| 14  | 350   | $2,99.10^{-8}$ |
| 16  | 523   | $1,17.10^{-8}$ |
| 18  | 746   | $9,59.10^{-6}$ |
| 20  | 1024  | $2,45.10^{-6}$ |
| 22  | 1364  | $1,26.10^{-6}$ |
| 24  | 1772  | $6,84.10^{-7}$ |
| 25  | 2004  | $5,14.10^{-7}$ |
| 30  | 3468  | $1,34.10^{-6}$ |
| 50  | 16137 | $3,96.10^{-9}$ |

Table 3: Error estimates for $l = o(h^{3.01})$
A system of simultaneous coupled PDEs has been transformed into an algebraic discrete system involving a generalized Lyapunov-Sylvester type operators. Solvability, consistency, stability and convergence are then established by applying well-known methods such as Lax-Richtmyer equivalence theorem and Lyapunov Stability and by examining the topological properties of the obtained Lyapunov-Sylvester type operators. The method was finally improved by developing a numerical example.

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