Multiparty Protocol that Usually Shuffles

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Multiparty computation is raising importance because it’s primary objective is to replace any trusted third party in the distributed computation. This work presents two multiparty shuffling protocols where each party possesses a private input, agrees on a random permutation while keeping the permutation secret. The proposed shuffling protocols are based on permutation network, thereby data-oblivious. The first proposal is $n$-permute that permutes $n$ inputs in all $n!$ possible ways. $n$-permute network consists of $2 \log_2 n - 1$ layers, and in each layer there are $n/2$ gates. Our second protocol is $n_\pi$-permute shuffling that defines a permutation set $\Pi = \{\pi_1, \ldots, \pi_N\}$ where $|\Pi| < n!$, and the resultant shuffling is a random permutation $\pi_i \in \Pi$. The $n_\pi$-permute network contains leases number of layers compare to $n$-permute network. Let $n = n_1 n_2$, the $n_\pi$-permute network would define $2 \log_2 n_1 - 1 + \log_2 n_2$ layers.

The proposed shuffling protocols are unconditionally secure against malicious adversary who can corrupt at most $t < n/3$ parties. The probability that adversary can learn the outcome of $n$-permute is upper bound by $((n - t)!)^{-1}$. Whereas, the probability that adversary can learn the outcome of $n_\pi$-permute is upper bounded by $(f_\Pi(n_1 - \theta_1)^{n_2} 2^{\theta_2})^{-1}$, for some

Abbreviations: ABC, a black cat; DEF, doesn’t ever fret; GHI, goes home immediately.

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1 | INTRODUCTION

Shuffling is a process that produces a random permutation of an indistinguishable input sequence. Shuffling is secure if the underlying permutation remains secret. A multiparty shuffling (MPS) is a protocol where parties collaboratively execute the shuffling protocol, but no subset of colluded parties (up to a certain threshold) is unable to learn the underlying permutation. There are generally two variations of MPC. Firstly, each party possesses a secret input. Secondly, each party possesses the shares of other secrets. The first setting can easily be plugged into the second setting - where parties make a secret sharing of their secrets among the other parties. MPS is one of the primitive operations for many privacy-preserving applications, like anonymous communication \[1, 2\], personalized browsing \[3\], e-voting and e-auction \[4, 5\], online gaming \[6, 7, 8, 9\], private data outsourcing \[10\], and oblivious RAM designing \[11\].

MPS was primarily studied in anonymous communication, known as Mixnet. Chaum \[1\] introduced the Mixnet as a network consisting of a chain of servers, called mixing nodes. Each mixing node receives a batch of encrypted messages then decrypts (or re-encrypts) individual message, performs a random permutation, and forwards the batch to the next mixing node. The final output is unlinkable to the input when at least one of the mixing node’s permutation remains secret. Shuffling is the composition of permutations along the chain of the mixing nodes. Shuffling is verifiable \[12, 13\] if there is a mechanism to prove the correctness of the output sequence. Mixnet relies on the cryptographic-primitives, e.g., factorization or computational discrete logarithm problem. Therefore verifiable Mixnets \[14, 15, 16\] incur the additional cost of communication and computation. Moreover, there are some attacks like - traffic analysis \[17, 18\], insertion/deletion attack \[19\], which have shown some weakness in theunlinkability property.

Chaum \[20\] also proposed the DCnet \[18\] for anonymous communication. DCnet is free from cryptographic-primitives and eliminates the problem of traffic analysis. DCnet is primarily designed with honest parties. DCnet with adversary was addressed in \[21\]. Unfortunately, DCnet suffers from collision and jamming attacks \[21\].

Mixnet and DCnet are asynchronous in nature. This is an advantage as well as disadvantage of the system. The advantageous part is that - nodes operate independently. Every mixing node performs three operations: receives, processes, and forwards. Mixing nodes do not require any communicate with others during their execution. The mixing node only interacts with its predecessor and successor. The disadvantage is - failure of any node may result in the total failure of the system. To withstand the mixing node failure, Mixnet implements threshold cryptography \[22, 23\] that generates a public key for encryption where the corresponding private key is shared among the parties in a t-n (read t-out of n) threshold sharing scheme. Any subset of t parties can perform the Mixnet decryption. Bayer and Diaconis \[24\] reported that \(\frac{1}{2} \log n\) shuffles were sufficient to mix up a deck of \(n\) arranged cards. Bayer-Diaconis’s result intuitively amortizes the upper bound of the number of mixing nodes in the Mixnet.

MPS is designed as a data-oblivious algorithm where the operations are independent of the input data. Data-oblivious algorithms are classified into two categories:

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1 A multiparty protocol based on the Dining Cryptographers Problem.
• **Network Model**: A network model is a fixed wired network of oblivious switches (alternatively called gates) that transfers the input to the output. Examples of network models are sorting network \[25, 26\], permutation networks \[29\], etc.

• **Randomization Model**: Randomization model, on the other hand, hides the data access pattern of an algorithm by distributing every access instruction (e.g., read and write) over the whole memory block. Examples of randomization based models are ORAM \[11\], oblivious looked-up table \[27\], etc.

Most of the MPS protocols are based on the network model \[28, 12\].

### 1.1 Multiparty Shuffling and Data-Oblivious Algorithm

Let \( I = \{i_1, \ldots, i_n\} \) be the set of inputs taken randomly from a uniform distribution. For example, cipher-text of any IND-CCA or information-theoretic encryption. Sorting of \( I \) produces a random permutation of the input. Thus shuffling can be realized by oblivious sorting. Movahedi et al. \[28\] proposed an MPS protocol based sorting network \[30\].

Alternative approach is to route through a permutation network. A network with \( n \)-inputs \( n \)-output is rearrangeable if for all permutation \( \pi \) of \( \{1, \ldots, n\} \), there exist edge disjoint paths to connect any input \( i \) to the output \( \pi(i) \). A rearrangeable network consists of configurable switches of two inputs, called swap-gate. The swap-gate randomly swaps the inputs. A Beneš network with \( n \) inputs is an well-known \((2 \log n - 1)\) layers rearrangeable network, and able to permute the inputs in all possible \( n! \) ways \[31, 32\]. Some similar types of \( 2 \log n \) layers rearrangeable networks are baseline-baseline\(^{-1}\), omega-omega\(^{-1}\) \[33\].

### 1.2 Related work

Despite of several applications of MPS, to the best of our knowledge, MPS has not been explored to a great extent. Laur et al. \[34\] studied MPS in the context of database privacy. In \[34\], three shuffling mechanisms were proposed:

1. For \( n \) ordered element \( \langle x_1, \ldots, x_n \rangle \), parties define a permutation matrix \( M \) such that the sum of the elements along every rows and column is 1. Shuffling is computed as \( \langle x_1, \ldots, x_n \rangle \times M \)
2. Sorting \( n \) elements obliviously.
3. Share the elements to a trusted third party who shuffles and delivers the elements privately.

The above mechanisms were primarily designed with honest participants. Non-interactive zero-knowledge proofs \[35, 36\] were applied to ensure verifiability and correctness of shuffling.

Movahedi et al. \[28\] proposed a scalable protocol for MPS. In their protocol, parties generate random shared-seeds and pair the elements with the shared-seeds. Let \( (r, i) \) be a pair, where \( r \) and \( i \) are the random-seen and parties’ input, respectively. The protocol obliviously sorts the pairs according to the random shared-seeds and thereby obtains a random shuffling.

### 1.3 Our Contribution

In this paper, we propose two MPS protocols based on the rearrangeable network. We assume that the malicious adversary can corrupt up to \( t < \frac{n}{3} \) parties. Our first protocol has \( \frac{2}{5} (\log n - 1) \) layers, and in every layer, there are \( \frac{n}{2} \)
\[\mathbb{F}_p\] Finite field of order \(p\), where \(p\) is prime.

\(\mathcal{P} = \{P_1, \ldots, P_n\}\) A set of \(n\) mutually distrustful parties.

\(x_i \in \mathbb{F}_p\) \(P_i\)’s secret.

\([x]\) Secret sharing of \(x \in \mathbb{F}_p\).

\(x_{ij} \in [x_i]\) The \(j^{th}\) share of \(P_i\)’s secret. \(P_i\) privately communicates the share to party \(P_j\).

**TABLE 1** Notations and their definitions.

swap-gates. In the presence of \(t\) corrupted parties, the adversary can guess the permutation with probability \(\frac{1}{n-t!}\). When the number of inputs is large (e.g., \(n \geq 2^8\)), MPS becomes inefficient. The cost of shuffling is proportional to the number of swap-gates in the network.

In our second protocol, we reduce the number of layers at the cost of the number of permutations. Let the number of inputs be \(n = n_1n_2\). We design \(n_2\) rearrangeable networks, each with \(n_1\) inputs, followed by a Riffle\(^2\) network. In this design, a malicious adversary who corrupts at most \(t\) parties can guess the permutation with probability \(((n_1 - \theta_1))!n_22^{\theta_2})^{-1}\) for some positive \(\theta_2\).

We analyze the existing MPS by sorting network\([28]\). We find that sorting gates are costlier than the swap-gates used in the rearrangeable network.

Finally, we show that our proposed MPS protocols are unconditionally secure against \(t < n/3\) corrupted parties. The protocols are universally-composable and scalable with the number of inputs.

## 2 | PRELIMINARIES

In the rest of this paper, we use the symbols which are summarized in Table 1. Furthermore, we define round that comprises all the communications where every party sends one message to all other parties and performs some local computation. We assume that parties are synchronous, that is messages do not have an arbitrary delay. Every party has an identifier (index) which is known to all. We assume an authenticated-private channel between every pair of parties.

### 2.1 | Secret sharing

Let \(\mathcal{P}\) be a finite set of parties. Let \(D \notin \mathcal{P}\) be a distinguished party, called dealer, who possesses a secret \(x \in X\). A distribution scheme \(S = (S_S, S_R)\) is called \(t\)-\(n\) threshold secret sharing if:

1. \(S_S : X \times R \to \mathbb{F}^n\) be the sharing function, where \(R\) is a uniform distribution of randomness over some finite field extension \(\mathbb{F}^{t-1}\). The dealer samples \(r \in R\) uniformly at random, and shares the secret \(x\) according to \(S_S(x, r) \to (x_1, \ldots, x_n)\), where every \(x_i \in \mathbb{F}\). Dealer privately communicates the share \(x_i\) to player \(P_i\).

2. A set of the authorized party can reconstruct the secret \(x\). The size of the authorized set is parameterized by \(t\). Any subset of lesser than \(t\) parties is unable to reconstruct the secret. \(S_R : \mathbb{F}^t \to X\) is the reconstruction function. The probability of correct reconstruction with the authorized set is \(Pr[S_R(\cdot) = x_i] = 1\).

\(^2\)Riffle is a well-known technique for card shuffling.
2.2 | Shamir’s Secret Sharing (SSS)

Shamir \[37\] introduced a \( t \)-\( n \) threshold secret sharing scheme over the finite field \( \mathbb{F}_p \). Let \( \mathcal{P} \) be the set of parties, and the dealer possesses a secret \( x \in \mathbb{F}_p \). Dealer chooses \( a_0, \ldots, a_{t-1} \in \mathbb{F}_p \) randomly, sets a polynomial \( f(z) = x + \sum_{i=0}^{t-1} a_i z^i \), and computes the shares as \( \{f(1), \ldots, f(n)\} \). Dealer communicates \( f(i) \) to party \( P_i \) privately. The sharing is represented as \( S_S(x, a_0, \ldots, a_{t-1}) \rightarrow [x] \), where \( x_i \in [x] \) is the \( i \)th share and computed as \( x_i = f(i) \).

As \( n \) points on a polynomial of degree \( t-1 \) have been shared among the parties, any \( t \) points can redefine the polynomial \( f(x) \), thereby \( x = f(0) \). The reconstruction function is the interpolation of any \( t \) points on the polynomial, and defines as \( x = \sum_{i=1}^{t} \lambda_i f(i) \). Here \( \lambda \) is the Lagrange Interpolation of \( t \) distinct points.

2.3 | Verifiable secret sharing

In presence of the faulty party, where either the dealer or some party may not behave honestly, SSS fails to meet the correct output. The notion of Verifiable Secret Sharing (VSS) was introduced in \[38\], where the parties can verify the correctness of their shares that they have received from the dealer. A secret sharing protocol is verifiable if parties can verify the following without learning any additional information about the secret.

- The dealer distributes the valid shares.
- During reconstruction, reconstructor receives the correct shares from the respective parties.

Ben-Or et al. \[39\] and Chaum et al. \[42\] proposed the interactive VSS mechanism for unconditionally secure protocols with at most \( t < n/2 \) passive corruption and at most \( t < n/3 \) active corruption. Further, Rabin and Ben-Or \[41\] showed that in the presence of a broadcast channel, the upper bound of corrupted parties could be \( t < n/2 \), irrespective of the mode of corruption.

Our construction is based on Ben-Or et al. VSS mechanism \[39\]. We index the parties as \( P_1, \ldots, P_n \). The \( i \)th party receives the share as \( f(\omega^i) \), where \( \omega \in \mathbb{F}_p \) is the \( p \)th root of unity (i.e., \( \omega^p = 1 \)), and \( n < p \).

2.4 | Secure multiparty computation

Now we consider \( n \) parties \( \{P_1, \ldots, P_n\} \), each of them possesses a secret \( x_i \), wants to compute a publicly known function \( F(x_1, \ldots, x_n) \) on their private inputs in such a way that no party learns anything about others’ input, and the output is either known to all or none.

2.4.1 | Adversary models

The adversary is an entity that corrupts some of the parties to learn private information. The adversary may be either semi honest or malicious. In a semi-honest setting, the corrupted parties follow the protocol correctly. However, the adversary obtains all the internal states and the messages that are received by the corrupt parties. Semi-honest adversary is often called honest-but-curious or passive adversary.

On the other hand, malicious adversary controls the corrupted parties. The adversary determines the inputs of the corrupted parties. Moreover, the corrupted parties may deviate from the protocol arbitrarily. The only restriction in both the cases is - adversary cannot learns the randomness of the corrupted parties, which implies that parties can toss coins independently, and the outcome of the tosses are private.
An adversary structure is a subset of possible corrupted parties. The adversary is static if the subset of corrupted parties is chosen prior to the start of the protocol. On the other hand, the adversary structure is adaptive when the subset of corrupted parties is dynamic, i.e., the adversary corrupts the parties during the execution of the protocol.

Depending on the computational capacity, multiparty protocols are classified into two categories.

- **Computationally Bounded or Conditional:** Adversary runs in polynomial time, and is unable to solve certain hardness of the problem (e.g., factorization, computational DLP problems, etc.)
- **Computationally Unbounded or Unconditional:** Adversary may not be limited to polynomial running time. The adversary has unlimited computational power. For example, given two random elements, \( x \) and \( y \) form a set, the adversary's ability to distinguish the elements is negligible.

\( \text{SSS} \) is unconditionally secure. Given any \( t-1 \) shares \( (x_1, \ldots, x_{t-1}) \in \mathbb{F}_p \) and any random polynomial function \( G(\cdot) \) to compute the secret with, probability that adversary can compute the secret is \( \Pr[ G((x_1, \ldots, x_{t-1})) = x ] = \frac{1}{|\mathbb{F}_p|} \).

### 2.4.2 Security Definition

A multiparty computation \( F(x_1, \ldots, x_n) = y \) with \( n \) parties is a random mapping of \( n \)-private inputs \( (x_1, \ldots, x_n) \) to \( n \)-private outputs \( (y_1, \ldots, y_n) \), one for each party, such that a reconstruction function \( S_R(y_1, \ldots, y_n) \) remaps the output to \( y \). We refer such a process as functionality in the *Ideal* sense. The mapping is defined as \( F: ((0,1)^n)^p \rightarrow ((0,1)^n)^p \) where \( F = (f_1, \ldots, f_n) \) and every \( f_i \) realizes the functionality of party \( P_i \). For every input \( X = (x_1, \ldots, x_n) \), the output is a random mapping \( (f_1(x_1), \ldots, f_n(x_n)) \). Here \( x_i \) is the private input of party \( P_i \).

Let \( \Phi \) be a multiparty protocol that realizes functionality \( F \). The *view* of a party \( P_i \) during the execution of \( \Phi \) on input \( X = (x_1, \ldots, x_n) \) and security parameter \( t \) is denoted as \( \text{view}_i^\Phi(X, t) = (x_i, r_i, m^1_i, \ldots, m^n_i) \), where \( r_i \) is the local randomness for party \( P_i \) and \( m^j_i \) is the message received by party \( P_i \) from party \( P_j \). The output of party \( P_i \) is denoted as \( \text{output}_i^\Phi(X, t) \). The joint output of all parties is \( \text{output}^\Phi(X, t) \). All views and their subsequent outputs are random variables.

In the MPC, all outputs are indistinguishable. That is, one cannot distinguish the output; from the others. We represent the indistinguishably as

\[
\text{output}_1^\Phi(X, t) \overset{\text{d}}{=} \cdots \overset{\text{d}}{=} \text{output}_n^\Phi(X, t) \overset{\text{d}}{=} \text{output}^\Phi(X, t)
\]

**Definition:** *Security in semi-honest adversary:* Let \( F = (f_1, \ldots, f_n) \) be a functionality and \( t \) be the security parameter. The protocol \( \Phi \) securely computes \( F \) in the presence of static semi-honest adversary if there exist Probabilistic Polynomial Time (PPT) simulators \( S_1, \ldots, S_n \) for the \( P_1, \ldots, P_n \), respectively such that simulators do not learn any additional information than the views of the corresponding parties, and the output of the functionality \( F \). We formally denote:

\[
\{S_i(1^t, x_i, f_i(x_i)), F(X)\} \overset{\text{d}}{=} \{\text{view}_i^\Phi(X, t), \text{output}_i^\Phi(X, t)\}
\]

(1)

Let \( F \) be the functionality, and \( A \) be a PPT adversary. An *Ideal* execution of the functionality refers to the process where every party handovers his input to a trusted party, the trusted party computes the functionality \( F \), and returns the *output* to the corresponding party privately. An *Ideal* execution of party \( P_i \) is denoted as \( I deal_{t_i, A(z)}(X, t) \) where \( z \) is the input of the adversary.

A *Real* execution of the protocol \( \Phi \) refers to the process where parties execute the protocol \( \Phi \) by exchanging
messages over private channels. A Real execution of party $P_i$ is denoted as $\text{Real}_{\Phi, A_i}(X, t)$.

**Definition Security in malicious adversary:** Let $F$ be the functionality, $t$ be the security parameter, and $\Phi$ be the multiparty protocol. The protocol $\Phi$ is securely evaluating the functionality $F$ in the presence of malicious adversary if, for every Real execution, there exists a PPT simulator $S_A$ corresponds to the adversary such that the Ideal process with the simulator $S_A$ is equivalent to any Real execution with the adversary $A$ with the local randomness $z$.

$$\{\text{Ideal}_{F, S_A}(X, t)\} \equiv \{\text{Real}_{\Phi, A_i}(X, t)\}$$

(2)

The simulation models in Equation 1 and 2 provide the security definitions in the stand-alone paradigm.

### 2.4.3 Universal Composability of Cryptographic protocol

Universal composability (UC) is a general framework to describe and analyze the security properties of any cryptographic protocol. Protocols are modeled as a computation to be executed by some computational entities, called parties who communicate among themselves. Parties run the protocol on their local inputs and randomness. There is an additional computational entity, called adversary, who may control a subset of parties their respective communication channels. However, the adversary can not control the local randomness of individual parties.

Under the composition paradigm, UC defines the adversary as the Environment, denoted as $Z$. The Environment generates all the inputs, reads all outputs, and interacts with the real adversary in an arbitrary way.

**Definition UC-securely computation:** Let $F$ be the functionality, $t$ be the security parameter, and $\Phi$ be the multiparty protocol. The protocol $\Phi$ is UC-securely computable if, there does not exist any Environment $Z$ who can distinguish whether the execution is the Ideal functionality $F$ with the simulator $S$ or the Real in the presence of adversary $A$.

$$\{\text{Ideal}_{F, S_A}(X, t)\} \nmid \{\text{Real}_{\Phi, A_i}(X, t)\}$$

(3)

In the definition, view of the Environment includes all inputs and outputs of every party, except their randomnesses. The notation $S_A(z)$ denotes whatever the input of the simulator $S$ is known to Environment $Z$ or not.

**Definition Straight-line Black-box simulator:** The simulator is black-box if it only allows oracle access to the adversary. Such a simulator is straight-line if it interacts with the adversary in a state-full manner. That is, the simulator sends all the simulated messages of a round to the adversary and then proceeds to the next round.

**Theorem 1** (Kushilevitz et. al [43]) If a protocol is securely computable in the stand-alone model and has a straight-line black-box simulator, then the protocol is also UC-securely computable.

### 2.4.4 UC Hybrid model

Now, consider a protocol $\Phi$ that has $\eta$ sub-protocol invocations where each of the sub-protocol is already proven to be UC-securely computable. The modular composition theorem [44] allows to analyze the UC-security of the protocol $\Phi$ from the composability of the sub-protocols. Let $(\Phi_1, \ldots, \Phi_\eta)$ be the sub-protocols and $(F_1, \ldots, F_\eta)$ be the functionalities of the sub-protocols, respectively. The $(\Phi_1, \ldots, \Phi_\eta)$ hybrid model is defined as below:
**Definition** Hybrid model: Let \( F = (F_1, \ldots, F_\eta) \) be the functionality corresponding to the protocol \( \Phi \) having invocation \( \{\Phi_1, \ldots, \Phi_\eta\} \) sub-protocols, \( t \) be the security parameter. Protocol \( \Phi \) securely evaluates \( F \) if for every \( \text{Real} \) execution, there exists a PPT simulator \( S_A \) corresponding to the adversary such that no Environment \( Z \) can distinguish whether the execution is the \( \text{Ideal} \) process with the simulator \( S_A \) or the \( \text{Real} \) execution of the adversary \( A \). Here \( Z \) learns the inputs of all parties and the outputs of the corrupted parties and interacts with the adversary arbitrarily. We formally define:

\[
\text{Ideal}_{\Phi, S_A(Z)}^{F_1, \ldots, F_\eta}(X, t) \equiv \text{Real}_{\Phi, A(Z)}^{\Phi}(X, t)
\]  

(4)

### 2.5 Multiparty computation on SSS

Let \( \mathcal{P} \) be the set of \( n \) parties, and \([x], [y]\) be two shared secrets over the field \( \mathbb{F}_p \):

- **Addition**: Parties can compute \([x + y] \leftarrow \text{Add}([x], [y])\) locally.
- **Multiplication**: Parties can compute \([xy] \leftarrow \text{Mul}([x], [y])\) with one round of communication.
- **Constant Multiplication**: For a publicly known constant \( c \in \mathbb{F}_p \), parties can compute \([cy] \leftarrow \text{Mul}(c, [y])\) locally.

Damgård et al.\cite{45} further enhanced the multiparty functionalities as below:

- **Random Number Generation**: Parties generate a random share as \([r] \leftarrow \text{Rand}()\) with one round of communication.
- **Random Bit Generation**: Parties generate a random shared bit as \([b] \leftarrow \text{Rand}_2()\) where \( b \in \{0, 1\} \subseteq \mathbb{F}_p \) with two rounds of communication.
- **Inverse**: Let \([x]\) be a shared secret. Parties compute the inverse \([x^{-1}] \leftarrow \text{Inv}([x])\) with two rounds of communication.

| **BIT S Protocol** | **Rounds** | **Invocations of Mul** |
|---------------------|----------|------------------------|
| Damgård et al.\cite{45} | 38       | \(O(l)\)               |
| Nishide et al.\cite{46} | 25       | \(O(l)\)               |
| Veugen\cite{47}      | \(l + 8\) | \(O(l)\)               |
|                     | \(T\)†   | \(O(l)\)               |

**TABLE 2** The round complexity of \( \text{BIT S} \) operation. †Veugen\cite{47} proposed an efficient \( \text{BIT S} \) operation with pre-computed randomness to reduce the round complexity.

- **Bit Decomposition**: Bit decomposition function is a random mapping \( \text{BIT S} : \mathbb{F}_p \rightarrow (\mathbb{F}_p)^l \) where \( l = \log p \). Let \([x]\) be a shared secret, then \( ([b_{l-1}], \ldots, [b_0]) \leftarrow \text{BIT S}([x]) \) such that \( x = \sum_{i=0}^{l-1} 2^i b_i \) and \( b_i \in \{0, 1\} \). \( \text{BIT S} \) is the primitive functionality that is used to map any arithmetic circuit to Boolean circuitry. Bit decomposition is a constant round operation. However, it invokes \( O(l) \) \( \text{Mul} \) operations. Table 2 presents the round complexity of different bit decomposition protocols.

- **Random-swap**: Let \(([x], [y])\) be an ordered pair of two shared secrets. The Random-swap function swaps the

\[3\text{Multiplication operation requires at least } 2t - 1 \text{ parties.}\]
pair with probability 1/2. The parties toss a secret coin, and depending on the output the elements are swapped. Algorithm 1 describes the Random-swap operation. The MulS in Algorithm 1 are performed in parallel which incurs one round operation. Therefore, the complexity of Random-swap is three rounds of communication - two for Rand2(), and one for Mul.

- **Comparison**: Parties compare two shared secrets as \( b \leftarrow Com([x], [y]) \) (if \( x > y \) then \( b = 1 \) otherwise \( b = 0 \)). Comparison invokes bits as a sub-protocol followed by a Boolean circuit of \( O(l) \) depth. Therefore, \( Com \) is constant round, but \( O(l) \) rounds of Mul operation.

- **Reshare**: Let \( \mathcal{P}_1 = \{ P_1, \ldots, P_{n_1} \} \) and \( \mathcal{P}_2 = \{ \bar{P}_1, \ldots, \bar{P}_{n_2} \} \) be two different sets of parties. Let \( [x] \) has been shared among the parties of \( \mathcal{P}_1 \). Resharing refers to the functionality where parties in \( \mathcal{P}_1 \) construct another distribution \( [\bar{x}] \) and privately communicates to the parties of \( \mathcal{P}_2 \) such that the reconstruction of \( [x] \) and \( [\bar{x}] \) by \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \) respectively, are equal.

\[
\text{Algorithm 1: Random-swap}([x], [y])
\]
\[
\begin{align*}
\text{Rand}_2() & \rightarrow [b]; \\
\text{Add}([x], [y]) & \rightarrow [z]; \\
\text{Add}(\text{Mul}([b], [y]), \text{Mul}(\text{Add}(1 - [b]), [x])) & \rightarrow [\alpha]; \\
\text{Add}([z], [-\alpha]) & \rightarrow [\beta]; \\
\text{return} ([\alpha], [\beta]);
\end{align*}
\]

Algorithm 2: Reshare([x], \mathcal{P}_1, \mathcal{P}_2)

\[
\begin{align*}
*/ \text{ Let } x_i \in [x] \text{ be the share of } x \text{ possessed by } P_i \in \mathcal{P}_1 */ \\
\forall i, P_i \in \mathcal{P}_1 \text{ chooses a random polynomial } f_i(z) \text{ of degree } t - 1, \text{ such that } f_i(0) = x_i, \text{ and invokes VSS-shares with respect to } \mathcal{P}_2; \\
*/ \text{ let } P_i \in \mathcal{P}_2 \text{ receives the shares from } P_{i_1}, P_{i_2}, \ldots \in \mathcal{P}_2 */ \\
\text{On receiving the shares from } \mathcal{P}_1, \text{ each party } \bar{P}_i \in \mathcal{P}_2 \text{ computes } \bar{x}_i = \sum_{i=1}^{n_2} \beta_i f_i(x_j)
\end{align*}
\]

2.6 | Shuffling by Sorting

In this section we briefly discuss MPS based on sorting network proposed in [6]. Let \( I = (x_1, \ldots, x_n) \) be the secret inputs of \( n \) parties. For every input \( x_i \in I \), parties generates a random element \( r_i \leftarrow Rand() \) and form the tuple \((r_i, x_i)\). Parties then jointly execute a data-oblivious sorting network on the randomness \( r_1, \ldots, r_n \), which intuitively shuffles the sequence.

2.6.1 | Assumptions and Limitations

Sorting network is comprised of compare and swap gates. Multiparty comparison is costly, it invokes \( \text{B} I \text{TS} \) followed by a Boolean circuit of logarithmic depth. Therefore, every compare gate incurs \( O(\log p) \) round of complexity where \( \mathbb{F}_p \) be the underlying field. As data-oblivious sorting network typically contains \( O(n(\log n)^2) \) compare gates, the overall complexity of a shuffle network is \( O(n(\log n)^3) \).
| Number of Elements (n) | Size of n (bits) | $q = \frac{3}{2} n^2 \log n$ | Size of $q$ (bits) | $P(n, q)$ |
|------------------------|------------------|-----------------------------|------------------|-----------|
| 32                     | 5                | 7680                        | 13               | 0.0625    |
| 64                     | 6                | 36864                       | 16               | 0.0548    |
| 128                    | 7                | 172032                      | 18               | 0.0461    |
| 256                    | 8                | 786432                      | 20               | 0.0406    |

TABLE 3 Birthday Attack in MPS by sorting protocol [28]

In contrast, a permutation network is comprised of Random-swap and does not require any comparison. The Random-swap gate is computationally efficient that comparison. In our construction, we define the Random-swap gate with three rounds of complexity. A permutation network typically contains of $O(n \log n)$ Random-swap gates. Therefore, the complexity of permutation is $O(n \log n)$.

For any $16 (= 0.0625)$ independent runs of the shuffling algorithm [28] with 32 inputs, there is an overwhelming chance that one (or more) run would have at least one repetition in the randomness.

The protocol [28] operates on Compare-swap gates. In the design of the protocol, comparisons are performed over the field $\mathbb{F}_q$, whereas swapping are performed over another field $\mathbb{F}_p$, where $q < p$. Therefore, the output of every comparison i.e. $[b] \in \mathbb{F}_q$ has to be mapped to an equivalent $[b] \in \mathbb{F}_p$, where $b \in (0, 1)$. This share conversion from one domain to another incurs additional rounds of communication. As the sorting network is comprised of $O(n(\log n^2)$ Compare-swap gates, and for every gate invokes a share conversion, the overall round complexity of [28] is higher.

2.7 | Byzantine agreement and quorum in a large network

In a large network with many parties, the computation often becomes inefficient due to a large number of inter-party message passing. We often form quorums and distribute the computation among the quorums. Forming quorums with the faulty party is not trivial. We refer to the problem of Byzantine Agreement is presence of malicious adversary. Malicious party can view all the messages in a round before sending its own message of that round, and is state-full in the sense that party can remember all previous rounds. In a nutshell, Byzantine Agreement is a protocol that allows the honest parties to agree on a common binary string [40]. Byzantine Agreement protocol is used to form quorums. King et al. [48] and Dani et al. [49] protocols are used to generate $n$ number of good quorums among $n$ parties.

**Definition** Good Quorum [48, 49]: A $n$ party protocol, called Quorum-Gen, with at most $t < n/8$ malicious parties forms $n$ quorums each of them having $O(\log n)$ parties. The quorums are called good if no more than $(t/n + \delta)$ parties in each quorum are faulty, where $\delta$ is small. Moreover, the quorums are load-balanced in the sense that no party is mapped to more than $O(\log n)$ quorums.

2.8 | Permutation Network

A permutation network (or rearrangeable network) is a non-blocking network of switches that permutes $n$ inputs to all possible $n!$ ways. The building block of a permutation network is Random-swap gates. Every Random-swap gate has two inputs and two outputs. The gate randomly maps the input lines to the output lines, one to each output. Let $S$ be a permutation network with $n$ inputs, and $\pi$ be a random permutation, then there exists some configuration $C$ such
TABLE 4  the occurrences of different permutations in a 8-input Beneš network. One can read first row as - 8192 different permutations each occurs 8 times.

| No. of Permutations | No. of Occurrences |
|----------------------|--------------------|
| 8192                 | 8                  |
| 14336                | 16                 |
| 12288                | 32                 |
| 2048                 | 40                 |
| 2816                 | 64                 |
| 512                  | 128                |
| 128                  | 256                |

that the network outputs the permutation \(\pi\). If \((i, j)^{th}\) entry of \(C\) is 1, then the corresponding gate swaps the input, otherwise passes the inputs.

**Definition** \(n\)-permute: A \(n\)-permute network with \(n\)-inputs is capable of permuting the input sequence in all \(n!\) possible ways.

### 2.8.1 Beneš Network

Beneš network is a \(n\)-permute network\(^{50}\). The dimension \(d\) characterizes the Beneš network. A \(d\)-dimension Beneš network has \(n = 2^d\) input and output terminals. The network consists of \(K = \frac{n}{2}(2d - 1)\) gates arranged in \(2d - 1\) layers. Every gate in the network behaves as a Random-swap gate. The gates are assigned with a random bit \(b \in \{0, 1\}\). The gate swaps the inputs when the random bit is 1; otherwise, it passed the inputs. It is easy to observe that, every gate routes its outputs to the alternate halves of the network. For example, the first layer virtually divides the network into two horizontal halves and routes accordingly. The partitioning and routing are applied recursively on each half.

Let \(X = (x_1, \ldots, x_n)\) be the input sequence. After \(d\) layers of routing, an input element \(x_i \in X\) is permuted to any position, provided that the adjacent element \(x_{i+1}\) (in the input sequence) is always in the other half of the permuted sequence. To overrule this constraint, the network performs another \(d - 1\) layers of routing. The configuration of the network is captured in a matrix called a configuration matrix. Figure 1 shows a Beneš network with 8 input. The corresponding configuration of the network is shown in Equation 5. It is easy to observe that for any permutation \(\pi\) there exist multiple configurations. For example, Equation 5 presents two configurations, \(C\) and \(\bar{C}\), correspond to the 8 input Beneš network shown in Figure 1.

\[
C_S = \begin{bmatrix}
1 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 \\
1 & 0 & 1 & 1 & 0 \\
1 & 1 & 0 & 0 & 0
\end{bmatrix}, \quad \bar{C}_S = \begin{bmatrix}
1 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1
\end{bmatrix}
\]

Every gate in Beneš network takes two inputs and swaps the inputs with probability \(1/2\). This implies that the network is able to produce \(2^K\) possible outputs. Whereas, the total number of permutations with \(n\) inputs is \(n! < 2^K\),
FIGURE 1  An 8-input Beneš network and its routing corresponds to the configuration $C_8$ in Equation 5.

FIGURE 2  Left: The construction of a 3-input swap gate with three 2-input swap gates in a cycle. Right: The construction of a $(n + 1)$-permute network with one 3-input swap gate.

for $n > 2$. Therefore, the mapping from the set of all possible configurations $C = \{C_1, \ldots, C_{2^K}\}$ to the set of all possible permutations $\Pi = \{\pi_1, \ldots, \pi_n!\}$ is many to one. We find that the distribution of occurring the permutations are not uniform. Table 4 shows the distribution of occurring of different permutations of an 8-input Beneš network. The first column of Table 4 represents the number of permutations that occur by the number depicted in the second column.

2.8.2  |  Arbitrary Size Beneš Network

Beneš network has a limitation. The number of inputs is always a power of 2. An arbitrary size Beneš network was introduced by Chang and Melhel [51] and further optimized in [52]. They proposed a 3-input swap gate as shown in Figure 2 (Left). The arbitrary size $n$-permute network is recursively constructed by forming a $\lfloor \frac{n}{2} \rfloor$-permute and another $\lceil \frac{n}{2} \rceil$-permute networks. If $n$ is even, then the two networks are of equal size, otherwise one of the network contains odd number of inputs. An odd input permute network includes one 3-input swap gate. Figure 2 (Right) shows the construction of a $(n + 1)$-permute network, where $n$ is even.
2.9 \( n_\pi \)-permute Network

There are two reasons that motivate us to define \( n_\pi \)-permute network. Firstly, the structure of an arbitrary size Beneš network is asymmetric. A 3-permute swap gate is constructed with three 2-permute swap gates in a cycle. The design of an asymmetric network using crossbar-switches is inefficient. Therefore, we propose a \( n_\pi \)-permute network which is almost symmetric in structure. Secondly, the cost of a permutation network mainly depends on the size of the network, which is determined by the number of gates in the network. If the number of inputs is \( n \) then there are at most \( 2^{2 \log n} \) gates in the network. When \( n \) is large (say \( 64 \geq n \)), then the cost of permutation is relatively high. In our second construction, we propose another \( n_\pi \)-permute network that reduces the cost of the permutation.

The proposed \( n_\pi \)-permute network is a blocking network. That means, there are some permutations for which no configuration is defined in the network. Our design goal is not to design all possible permutations, but to maximize the number of permutations. Furthermore, \( n_\pi \)-permute network is primarily used for multiparty shuffling and may not be prescribed for permuting the inputs.

Definition \( n_\pi \)-permute: Let \( \Pi = \{\pi_1, \ldots, \pi_k\} \) be a subset of all permutations of \( n \) inputs. A \( n_\pi \)-permute network produces the permutation \( \pi_i \in \Pi \). The number of distinct permutations generated by \( n_\pi \)-permute network is upper bounded by \( k \).

We present two constructions of \( n_\pi \)-permute network. The first construction is based on arbitrary size Beneš network \[51\]. We call this network as symmetric \( n_\pi \)-permute. The second construction reduces the size of the network. Let \( n = n_1 n_2 \), then we construct \( n_2 \) number of \( n_1 \)-permute (or \( n_1 \)-permute) networks followed by a Riffle structure that mixes the individual permutations.

2.9.1 Symmetric \( n_\pi \)-permute Network

Symmetric \( n_\pi \)-permute network is of even size (i.e. \( n \) is even) and appears to be almost symmetric in structure. We propose a cross-connector that binds two \( n \)-permute (or \( n_\pi \)-permute) networks and forms a \( (2n + 2)_\pi \)-permute network. Figure 3 shows the binding technique of two networks. The green lines randomly draw one element from each upper and lowed \( n \)-permute networks and push the elements to gate \( G_2 \). On the other hand, gate \( G_1 \) pushes two random elements \( \alpha \) and \( \beta \), where \( \alpha \in \{x_1, x_2\} \) and \( \beta \in \{y_{n-1}, y_n\} \), to the upper and lower permute network, respectively. This drawing and pushing operations occur at the middle of the permute network. The following layers mixes the elements in such a way that \( \alpha \) and \( \beta \) always appear at the upper and lower half of the respective \( n \)-permute networks. In Figure 3 the blue gates are used to show all the possible mixing scenarios of \( \alpha \) and \( \beta \).

The proposed symmetric \( n_\pi \)-permute network cannot produce all possible permutations. Consider the Figure 3 it is easy to observe that \( \alpha \) and \( \beta \) never reach at gate \( G_1 \). Therefore, the final permutation never contains \( \alpha \) and \( \beta \) in the middle of the permuted string. We estimate the upper bound of the number of permutations produced by the symmetric \( n_\pi \)-permute network. We consider two equal sized \( n \)-permute (or \( n_\pi \)-permute) networks which are connected by a cross-connector. The total number of inputs becomes \( 2n + 2 \). Let \( N = 2n + 2 \), we find that all but two elements, one from \( \{x_1, x_2\} \) and other from \( \{y_{n-1}, y_n\} \), may permute at the middle of the output string of size \( N \). We recursively define the upper bound as \( f_1(N) \) as

\[
\begin{align*}
f_1(N) &= \begin{cases} 
N! & \text{when } N \text{ is a power of } 2 \\
N(N-1) \ldots \left(\frac{N}{2} + 1\right) f_1(N/2) & \text{when } N = 2n \text{ and } n \text{ is even} \\
2(N-2) \ldots \left(\frac{N}{2} + 1\right) f_1(N/2) & \text{when } N = 2n \text{ and } n \text{ is odd}
\end{cases}
\end{align*}
\]
2.9.2 | Reduced $n_{\pi}$-permute Network

In our second construction of $n_{\pi}$-permute network, we represent $n = n_1n_2$. We construct $n_2$ number of $n$-permute (or $n_1_{\pi}$-permute) networks and Mix the outputs of all $n_1_{\pi}$-permute networks using a Riffle structure. We call this permutation as reduced $n_{\pi}$-permute network.

Algorithm 3: Binary Riffle$(i,j,([x_i],...,[x_j]),d_2)$

\begin{verbatim}
mid \leftarrow \frac{i+j}{2} + 1;
if \text{d}_2 > 1 then
\begin{algorithm}
\text{Binary Riffle}(i, \text{mid} - 1, ([x_i],...,[x_{\text{mid} - 1}]), \text{d}_2/2);
\text{Binary Riffle}(\text{mid}, j, ([x_{\text{mid}}],...,[x_j]), \text{d}_2/2);
for \text{k} = 0 \text{ to} \text{mid} - 1 \text{ do}
\begin{algorithm}
\text{Random swap}(([x_{i+k}],[x_{\text{mid}+k}]));
\end{algorithm}
end for;
\end{algorithm}
else
\begin{algorithm}
\text{return};
\end{algorithm}
end if;
\end{verbatim}

Riffle:
This is one of the common technique for card shuffling. The deck of the card is divided into two halves. The halves are held in each hand and released so that the cards fall almost interleavely.

We emulate Binary Riffle to realize the Riffle operation. Binary Riffle divides the input sequence into two equal halves. A pair is formed by taking one element from each half. Subsequently, the elements are swapped ran-
domly. This operation is recursively applied on each half. Algorithm 3 describes the process of BinaryRiffle. In general, there would be $\log n$ recursions. However, we tailor the BinaryRiffle and run for $\log n^2$ recursions.

Figure 4 presents a reduced $n_\pi$-permute network with $n = 265$ inputs. We represent $n_1 = 2^6$ and $n_2 = 2^2$. There are 4 Beneš structures, each of having 64 inputs followed by a BinaryRiffle of having two recursions.

3 | MULTIPARTY SHUFFLING ALGORITHM

Let $\pi : \{1, \ldots, n\} \rightarrow \{1, \ldots, n\}$ denotes a permutation, and $\Pi$ be the subset of all $n!$ permutations. Let $X = (x_1, \ldots, x_n)$ be an ordered sequence. Shuffling is a random bijection that maps the sequence $X$ to another sequence $X_\pi = (x_{\pi(1)}, \ldots, x_{\pi(n)})$ where $\pi \in \Pi$. We denote shuffling as a mapping $S : \{X\} \xrightarrow{\pi} \{X\}$.

In MPS protocol, there are $n$ parties. Each party $P_i$ contains a secret input $x_i$. Parties agree on a random permutation $\pi \in \Pi$ and shuffle the sequence $X = (x_1, \ldots, x_n)$ to $X_\pi = (x_{\pi(1)}, \ldots, x_{\pi(n)})$ in such a way that no party learns the permutation $\pi$ and the inputs of other parties.

We present two MPS protocols. The first protocol, Shuffle-I, is a $n$-permute (or symmetric $n_\pi$-permute) network. The second protocol, Shuffle-II, is a reduced $n_\pi$-permute network. We apply either Beneš or arbitrary size Beneš
Algorithm 4: Shuffle-I

Input : Sequence of $n$ elements, $X = (x_1, \ldots, x_n)$

Output: A random permutation, $X_\pi$

begin

Preprocessing:

Parties agree on an $n$-input or symmetric $n_{\pi}$-permute network. Let the network be $S$ and public to all;

Parties execute Quorum-Gen, and form $n$ quorums $Q = \{Q_1, \ldots, Q_n\}$. Each gate $G_i \in S$ is assigned some quorum $Q_i$, where $i \equiv j \mod n$. Thus $G_1$ is assigned to $Q_1$, $G_2$ is assigned to $Q_2$, etc. As there are more gates than the number of quorums, some quorums are associated with multiple gates. Figure 1 shows the assignment of the quorums of an 8-permute Beneš network;

Every $Q_i=1, \ldots, n/2$ receives two shared-secrets: $[x_i]$ from party $P_{2i-1}$ and $[y_i]$ from parties $P_{2i}$, respectively;

Gate Operation:

At layer-1, the quorums invoke

$(([\alpha_i], [\beta_i]) \leftarrow \text{Random-swap}([x_i], [y_i])$ for $i = 1, \ldots, n/2$

Thus, the output-lines of gate $G_i$ (for $i = 1, \ldots, n/2$), contains the ordered pair $([\alpha_i], [\beta_i])$;

Resharing:

The output-lines of layer-1 are now forwarded to layer-2. Quorums at layer-1 re-share the outputs to the quorums at layer-2;

If $Q_i$ forwards the outputs to $Q_k^1$ and $Q_k^2$, respectively, then the parties in $Q_i$ re-share $[\alpha_i]$ to $Q_k^1$, and $[\beta_i]$ to $Q_k^2$. For example $Q_1$ re-shares $[\alpha_1]$ to $Q_2^{n+1}$ and $[\beta_1]$ to $Q_2^{n+1}$;

Routing:

Routing is performed layer by layer. In each layer, the quorums operate synchronously and forward the outputs to the next layer. On receiving the inputs from the preceding layer, quorums perform the gate operations and forward to the next layer;

structure to realize the $n$-permute network.

Definition $\zeta$-bit Unlinkability: The permutation network ($n$-permute or $n_{\pi}$-permute), in the presence of an adversary, is $\zeta$-bit unlinkable if the probability of an adversary to guess the permutation $\pi$ is less than $\frac{1}{2^\zeta}$, where $\zeta$ is the largest positive integer.

A MPS protocol, with adversary who can corrupt at most $t$ parties, should satisfy the following properties:

- $\zeta$-bit unlinkability: For a security parameter $t$, the protocol must satisfy $\zeta$-bit unlinkability.
- Uniformly Knowing: The corrupted party cannot stop the honest party to learn the output. We call this all-or-nothing where every party learns the permutation, or no party learns anything.

Definition A MPS protocol with $t < n/3$ corrupted parties is $t$-resistant, if the protocol guaranteed $\zeta$-bit unlinkability, and uniformly knowing properties.
3.1 | Shuffle-I

Our first protocol is a $n$-permute (or symmetric $n_\pi$-permute) shuffling. General, a rearrangeable non-blocking permutation network can be used for shuffling. In our design, we use either a Beneš or an arbitrary size Beneš or a symmetric $n_\pi$-permute network. The gates are replaced by Random-swap gates. Algorithm 4 describes the MPS protocol.

3.1.1 | Shuffle-I with passive adversary

Following we present the properties of Shuffle-I protocol.

ζ-bit Unlinkability: Shuffle-I generates all $n!$ permutations. We consider a passive adversary who can corrupt $t$ parties and learns their inputs. Therefore, the adversary can link those $t$ elements at the output, where the remaining $n - t$ elements are permuted in $f_1(n - t)$ possible ways. Therefore, the probability that the adversary can guess the permutation correctly is $\frac{1}{f_1(n - t)}$. Consequently, Shuffle-I defines the security parameter $\zeta = \lfloor \log (f_1(n - t)) \rfloor$.

Uniformly Knowing: Shuffle-I basically runs on the $n$-permute or symmetric $n_\pi$-permute structure. The permutation $\pi$ progresses as the inputs progress layer by layer. The gates of the structure are assigned to some quorum. In a particular layer, the quorums operate synchronously. Thus, if any quorum fails to deliver to the next layer, then the permutation is aborted. The final output is obtained when the inputs pass through all the layers of the network, otherwise aborts. Therefore, Shuffle-I guarantees all-or-nothing.

3.2 | Shuffle-II

The second protocol is based on reduced $n_\pi$-permute shuffling. The protocol is shown in Algorithm 5. When the number of inputs is large (say $2^8 \geq n$), shuffling often becomes inefficient. In that case, we propose a reduction in the number of layers of the network. Let the number of inputs be $n$. We represent $n$ as a product of two numbers, say $n = n_1n_2$. We form $n_2$ independent Shuffle-I structures, each with $n_1$ inputs. Finally, the outputs of all the Shuffle-I are mixed using BinaryRiffle.

3.2.1 | Shuffle-II with the passive adversary

Following, we present the properties of Shuffle-II protocol.

ζ-bit Unlinkability: A Shuffle-II structure is consisting of two blocks. The first block is Shuffle-I structure and the second block is the BinaryRiffle. Let there are $n_2$ Shuffle-I structures, each of them having $n_1$ inputs. We also assume that adversary knows $t$ inputs, and those $t$ elements are uniformly distributed over the input sequence. Without loss of generality, we consider that every sub-sequence of $n_1$ elements at input there are $t/n_2$ elements whose values are known to the adversary. Therefore, in every Shuffle-I there are $t/n_2$ elements at the output which are linkable, and the remaining $(n_1 - t/n_2)$ elements are permuted in $f_1(n_1 - t/n_2)$ possible ways. As there are $n_2$ Shuffle-I structures and all the Shuffle-I structure operates concurrently on different inputs set, the unknown elements are permuted in $(f_1(n_1 - t/n_2))^{n_2}$ possible ways.

The second block is the BinaryRiffle. A $n$-input Riffle is constructed with $n/2$ Random-swap gates and is capable to produce $2^\frac{n}{2}$ permutations. In our construction the output of the first block is fed to the BinaryRiffle.
Algorithm 5: Shuffle-II

Input : A sequence of \( n \) elements \( X = (x_1, \ldots, x_n) \)

Output: A random permutation \( X_\pi \)

begin

Preprocessing:
- Parties invoke Quorum-Gen, and form \( n \) quorums \( Q = \{ Q_1, \ldots, Q_n \} \);
- Parties agree on \( n = n_1 n_2 \) and defines \( n_2 \) Shuffle-I networks, each of having \( n_1 \) inputs. Parties further agree on a BinaryRiffle of log \( n_2 \) layers. Let the network be \( S^\Pi \), and is public;
- Every gate \( G \in S^\Pi \) is assigned to some quorum \( Q_i \);

Shuffling in Two Phases:
/* Shuffling is performed in two phases */

Phase-I
- For every subsequence of \( n_1 \), say \( (x_1, \ldots, x_{n_1}) \), \( (x_{n_1+1}, \ldots, x_{2n_1}) \), \( \ldots \), \( (x_{(n_2-1)n_1+1}, \ldots, x_{n_1 n_2}) \), parties invoke Shuffle-I independently on each subsequence. Thus Phase-I produces \( n_2 \) independent shuffles each with \( n_1 \) distinct elements;

Phase-II:
- Phase-II is a BinaryRiffle with log \( n_2 \) layers. The outputs of Shuffle-I are mixed using the Riffle;

Routing:
- Routing is performed layer by layer. In each layer, the quorums operate synchronously, and forward the output to the next layer. On receiving the inputs from the preceding layer, quorums performs the gate operation;

structure. Figure 1 shows a reduced 256\( _\pi \)-permute network with 4 numbers of 64-permute networks in the first block and a BinaryRiffle of depth 2 at the second block.

Now the input to a BinaryRiffle is a sequence of \( n \) elements where \( t \) elements are linkable. Let the structure has \( d_2 = \log n_2 \) layers. All the layers, except the last, independently permutes the sequence in \( 2^{2^{d_2-1}} \) possible ways. As \( t \) elements are linkable, the last layer permutes the sequence in \( 2^{2^{d_2-1}} \) possible ways. Therefore, a BinaryRiffle with \( d_2 \) layers permutes the input in \( 2^{(2^{d_2-1})/2} \) possible ways.

Finally, a reduced \( n_\pi \)-permute with \( t \) linkable inputs, is capable of permuting the input sequence in \( (f_\Pi(n_1 - t/n_2))^{n_2} 2^{(2^{d_2-1})/2} \) possible ways. Thus the probability of an adversary to guess the permutation is \( \frac{1}{(f_\Pi(n_1 - t/n_2))^{n_2} 2^{(2^{d_2-1})/2}} \). Consequently, Shuffle-II defines the security parameter \( \zeta = \lfloor \log (f_\Pi(n_1 - t/n_2))^{n_2} 2^{(2^{d_2-1})/2} \rfloor \).

Table 5 shows the estimation of \( \zeta \) for both Shuffle-I and Shuffle-II. We consider that \( \lfloor n/3 \rfloor \) players are corrupted. We measure the \( \zeta \)-bit unlinkability for \( n = 128, 256 \), and 512. In each of the cases, we design the reduced \( n_\pi \)-permute with \( n_2 = 2, 4 \), and 8 input Beneš networks where each Beneš network consists of \( n_1 = 64, 128, 255 \) inputs, respectively. It is easy to observe that \( \zeta \)-bit unlinkability property of Shuffle-I does not vary significantly from Shuffle-II.

Uniformly Knowing: This property of Shuffle-II directly follows from the protocol Shuffle-I.
In this section we present the UC-security proofs of the two protocols: Shuffle-I and Shuffle-II. Before proceed, we consider the following:

1. The adversary structure is static. However, adversary may be semi-honest or malicious.
2. The adversary is computationally unbounded. However, the simulator that corresponds to the adversary is probabilistic polynomially bounded.
3. Environment $Z$ acts as an interactive Turing machine. The interactive Turing machine has additional tapes which receive inputs during the execution of the Turing machine. This model basically simulates the real-life operation of the protocol, where the inputs are not known during the instantiation of the simulation. The interactive Turing machine models the straight-line black-box behavior of the simulator (see Theorem 1).
4. Let $\mathcal{P}_C \subset \mathcal{P}$ be a static subset of corrupted parties. Adversary is $\mathcal{P}_C$-limited if he can only corrupt the parties from the set $\mathcal{P}_C$. As the adversary structure is static, $\mathcal{P}_C$ is fixed and defined before the execution of the protocol. We set the cardinality of $\mathcal{P}_C$ equal to the security parameter of the protocol, i.e. $|\mathcal{P}_C| = t$.
5. Finally, we assume that Quorum-Gen has already formed $n$ quorums. For every gate $G_j$, let $Q_j$ be the associated quorum, where $i \equiv j \mod n$. We also assume that $Q_{C_i} \subset Q_i$ be the set of corrupted parties in quorum $Q_i$.

### 4.1 UC modeling of Shuffle-I and Shuffle-II

We model the protocols using modular composability model and prove the UC-security of the protocols. Here we define the Ideal functionality correspond to the sub-protocols used in Shuffle-I and Shuffle-II. Table 6 presents the list of sub-protocols, their dependencies, and the Ideal functionalities.

| (number of inputs, number of corruptions) | Shuffle-I | Shuffle-II |
|------------------------------------------|-----------|------------|
| $n = 128, t = 42$                        | $\zeta$   | $\zeta$ (when $n_1 = 64, n_2 = 2$) |
|                                         | 433       | 413        |
| $n = 256, t = 85$                        | $\zeta$   | $\zeta$ (when $n_1 = 64, n_2 = 4$) |
|                                         | 1026      | 908        |
| $n = 512, t = 170$                       | $\zeta$   | $\zeta$ (when $n_1 = 128, n_2 = 2$) |
|                                         | 2319      | 2074       |
|                                         |           | 2146       |
|                                         |           | 2224       |
TABLE 6 The sub-protocols and their Ideal functionalities. The dependencies of the sub-protocols are depicted depicts the invocation of sub-protocols.

| Protocol      | Ideal Functionality |
|---------------|---------------------|
| VSS-share [39]| F_{VSS-share}       |
| VSS-recons [39]| F_{VSS-recons}     |
| Reshare [53]  | F_{Reshare}         |
| Mul [39, 45]  | F_{Mul}             |
| Rand₂ [45]    | F_{Rand₂}           |
| Random-swap   | F_{RS}              |

The security proofs of VSS-share, VSS-recons, Mul, Rand₂, and Reshare protocols (using straight-line black-box simulator) were presented in [54, 45, 53]. We apply UC-hybrid model to prove the security of Random-swap and BinaryRiffle. Subsequently, we show that Shuffle-I and Shuffle-II are also UC-secure.

Algorithm 6: Functionality F_{RS}

begin
/* Let Q_i be the quorum associated with the gate G_j. */
Trusted party receives input \((a_1, b_1), \ldots, (a_n, b_n) \in (F_p)^2\) from \(P_1, \ldots, P_n \in Q_i\) respectively;
if \(P_i\) does not send the input then
  set \((a_i, b_i) \leftarrow (0, 0)\);
Trusted party tosses an unbiased coin and generates the randomness. Let \(r \in \{0, 1\}\) be the tossed value;
Trusted party computes
\[ a \leftarrow F_{VSS-recons}(a_1, \ldots, a_n) \quad \text{and} \quad b \leftarrow F_{VSS-recons}(b_1, \ldots, b_n) \]
if \((r == 1)\) then
  Trusted party invokes \((F_{VSS-share}(b), F_{VSS-share}(a))\);
else
  Trusted party invokes \((F_{VSS-share}(a), F_{VSS-share}(b))\);
end

4.1.1 Protocol Random-swap:

The Ideal functionality of Random-swap is given in algorithm 6. The Ideal function receives inputs from the parties, tosses a coin, swaps the inputs based on the toss, and invokes VSS-shares. Since VSS-share generates uniform and random shares of the input, the views of the parties are independent and random.

The composability of Random-swap in the F_{Mul} and F_{Rand₂} hybrid model is similar to the protocol in algorithm 1 except that every call to the real protocol Mul or Rand₂ is replaced by the call to the Ideal functionalities F_{Mul} or F_{Rand₂}, respectively. The simulation is given in algorithm 7. Let \(A\) be the adversary for protocol Random-swap. \(A\) interacts with \(n\) parties and accesses to \(n/2\) copies of the Ideal functionalities F_{Rand₂} and F_{Mul}. Given \(A\), the simulator
Algorithm 7: Environment \( Z \), with Adversary \( A \) and Simulator \( S_{RS} \)

begin
  /* Let \( Q_i \) be the quorum associated with the gate \( G_j \). */
  /* \( P_{C_i} \subseteq Q_i \) be the set of corrupted parties. */
  /* \( I_i = Q_i - P_{C_i} \) be the set of honest parties. */
  Simulation
  for every party \( P_i \in I_i \) do
    \( S_{RS} \) selects random \((x_i, y_i)\);
    \( S_{RS} \) writes \((P_i, (x_i, y_i))\) on the tape of \( Z \);
  for every party \( P_i \in P_{C_i} \) do
    \( S_{RS} \) obtains the input \((x_i, y_i)\) of party \( P_i \), and passes the input to \( A \);
    if \( A \) corrupts \( P_i \) then
      \( S_{RS} \) receives \((\tilde{x}_i, \tilde{y}_i)\) from \( A \);
      \( S_{RS} \) writes \((P_i, (\tilde{x}_i, \tilde{y}_i))\) on the tape of \( Z \);
    else
      \( S_{RS} \) writes \((P_i, (x_i, y_i))\) on the tape of \( Z \);
  for all \( Q_{ij}, i=1, \ldots, n/2 \) do
    \( S_{RS} \) invokes \([r] \leftarrow F_{Rand}()\) with respect to \( Q_{ij} \);
  for all \( Q_{ij}, i=1, \ldots, n/2 \) do
    \( S_{RS} \) computes:
    \[ [z_j] \leftarrow \text{Add}([x], [y]); \text{Addition is a local computation} \]
    \[ [\tilde{a}] \leftarrow \text{Add}([F_{Mul}(x), [r]).F_{Mul}(Y_{Add}(1 - [r]), [y])]); \]
    \[ [\tilde{b}] \leftarrow \text{Add}([z], [-\tilde{a}]); \]
    \( S_{RS} \) adds \((\tilde{a}_i, \tilde{b}_i)\) to the output of party \( P_i \) and writes \((P_i, (\tilde{a}_i, \tilde{b}_i))\) on the tape of \( Z \);
Let $Q_i$ be the quorum. Let $P_{C_i} \subseteq Q_i$ be the set of corrupted parties.

2. For every gate $G_i, i = 1, \ldots, n/2$, there are $n/2$ copies of Ideal functionalities $F^i_{\text{Rand}_2}$ and $F^i_{\text{Mul}_\text{Ideal}}$ corresponds to the parties $P_i$.

If $A$ corrupts a party $P_j \in P_{C_i}$ in some quorum $Q_i$, then $S_{RS}$ obtains the input $(x_i, y_i)$ from $P_j$ and passes the inputs to $A$. If $A$ instructs to corrupt the inputs, then $S_{RS}$ receives $(\tilde{x}_i, \tilde{y}_i)$ from $A$ manipulates the input of $P_j$. On the other hand, if $A$ does not corrupt $P_j$, then $S_{RS}$ randomly sets the input of $P_j$. Simulator $S_{RS}$ writes all inputs on the tape of $Z$.

For every quorum, $S_{RS}$ simulates the copy of the Ideal functionalities $F^i_{\text{Rand}_2}$ and $F^i_{\text{Mul}_\text{Ideal}}$ with their inputs. The output of the Ideal functionalities are written on the tape of $Z$.

Simulation of $\mu_1$ and $\text{Rand}_2$ in the $F_{\text{VSS-share}}$-hybrid modeling are secure $[43, 44]$. Both the protocols have calls to the Ideal functionality of $F_{\text{VSS-share}}$, which generates uniform and random shares over $\mathbb{F}_p$. Simulator writes every inputs and output on the tape of $Z$. As the outputs are random variables over the field $\mathbb{F}_p$, $Z$ is unable to distinguish - who writes on the tape? Is it the Ideal functionality or the simulator?

Algorithm 8: Functionality $F_{\mu_1}$

```plaintext
begin
  /* Let the network has $n = 2^d$ inputs. Then there are $\frac{n}{2}(2d - 1)$ Random-swap gates in the network. The input lines of the network are assigned to the gates labeled as $G_1, G_2, \ldots, G_{n/2}$. */

Network setup:
for $i = 1, \ldots, n/2$ do
  Parties $P_{2i-1}$ and $P_{2i}$ VSS-share their secret to quorum $Q_i$;
  if Party $P_{2i-1}$ (or $P_{2i}$) does not shares the secret then
    Set the secret as 0

Routing:
for every layers do
  for every quorum $Q_i$ do
    Trusted party receives $(|x_i|, |y_i|)$ from $Q_i$;
    Trusted party invokes $F_{RS}$;
    For first output line, trusted party invokes $F_{\text{VSS-share}}$ to quorum $Q^1_{k}$;
    For second output line, trusted party invokes $F_{\text{VSS-share}}$ to quorum $Q^2_{k}$;
end
end
```

4.1.2 Protocol Shuffle-I:

There are $n$ parties. The parties already form $n$ quorums where no party is in more than $O(\log n)$ quorums $[48]$. There exists a $n$-permute (or symmetric $n$-permute) structure with $n$ inputs. The structure contains at most $\frac{n}{2}(2 \log n - 1)$ gates which are arranged in $(2 \log n - 1)$ layers. Each gate $G_j$ is assigned to some quorum $Q_i$, where $i \equiv j \mod n$. The first layer contains $n/2$ gates, and are indexed as $G_1, \ldots, G_{n/2}$. The corresponding quorums are $Q_1, \ldots, Q_{n/2}$.
respectively. The gate $G_i$ (for $i = 1, \ldots, n/2$) receives the inputs from $P_{2i-1}$ (first input line) and $P_{2i}$ (second input line). Thus, $P_{2i-1}$ and $P_{2i}$ use \texttt{VSS-share} to distribute their secrets to the parties $P_j \in Q_i$. In the subsequent layers, the gates receive inputs from the preceding layer. Let gate $G_i$ deliver its first output line to the gate $G_k$ and second output line to $G_{k+1}$. Also, let for $Q_k$ and $Q_{k+1}$ be the corresponding quorums associate to $G_k$ and $G_{k+1}$, respectively. Every party $P_j \in Q_i$ uses \texttt{VSS-share} to distribute the outputs to quorums $Q_k$ (first output line) and $G_{k+1}$ (second output line), respectively.

The \textit{Ideal} functionality of \texttt{Shuffle-I} is shown in \textbf{Algorithm 3}. The functionality is defined in two phases: the network setup phase and the routing phase. The network setup phase receives the input from the parties, and the routing phase shuffles the input by traveling thorough the network. In every layer, the output of the gates are \texttt{VSS-share}ed to the next layer. Since \texttt{VSS-share} produces random shares of the secret, all the outputs are random and uniformly distributed over $\mathbb{F}_p$.

Let $A$ be the adversary who can corrupts $t < n/2$ parties. We assume that $\text{Quorum-Gen}$ produces good quorums where the majority of the parties in each quorum are honest. $A$ interacts with $n$ parties which are executing the protocol \texttt{Shuffle-I} with access to multiple copies of the \textit{Ideal} functionality $F_{RS}$. Given $A$, simulator $S_{S-I}$ simulates the real execution of protocol \texttt{Shuffle-I} for the adversary $A$ as follows:

1. Simulator $S_{S-I}$ executes the structure layer-by-layer. In each layer, the quorums operate synchronously. If some quorum fails to deliver within a fixed time, the protocol is aborted.
2. Let $Q_i$ be a quorum. The quorum may contain some corrupt parties. Let $P_i \subset Q_i$ be the subset of corrupted parties in $Q_i$. We consider that $|P_i| < |Q_i|/2$.
3. For $i = 1, \ldots, n/2$, there are $n/2$ copies of \textit{Ideal} functionalities $F_{RS}$ that correspond to quorum $Q_i$ in each layer.

For every gate of the first layer, simulator $S_{S-I}$ obtains the input from party $P_{2i-1}$ and $P_{2i}$. Since $G_i$ is assigned to quorum $Q_i$, the parties do the following:

- If $P_j \in Q_i$ is a corrupted player, then the simulator obtains the input $(x_j, y_j)$ from $P_j$ and passes the input $(x_j, y_j)$ to $A$. If $A$ instructs to corrupt the input to $(\tilde{x}_j, \tilde{y}_j)$, then the simulator sets the input as $(x_j, y_j) = (\tilde{x}_j, \tilde{y}_j)$.
- Simulator writes $(P_j, (x_j, y_j))$ on the tape of $Z$.
- If $P_j \in Q_i$ is not corrupted, then the simulator randomly sets the input as $(x_j, y_j)$, and writes $(P_j, (x_j, y_j))$ on the tape of $Z$.

The composability of \texttt{Shuffle-I} in $F_{S-I}$-hybrid model is similar to the protocol in \textbf{Algorithm 4} except that every call to the real protocol \texttt{Random-swap} is replaced with the call to the \textit{Ideal} functionalities $F_{RS}$. The simulation is given in \textbf{Algorithm 9}. Simulator $S_{S-I}$ writes every views of the parties on the tape of the environment. Since $F_{RS}$ outputs random variables over the field $F_p$, the environment is unable to distinguish - who writes on the tape? Is it the \textit{Ideal} functionality or the simulator?

### 4.1.3 Protocol \texttt{BinaryRiffle} and \texttt{Shuffle-II}:

The two protocols \texttt{BinaryRiffle} and \texttt{Shuffle-II} are similar to \texttt{Shuffle-I} except the underlying structure.

\textbf{Lemma 1} \texttt{BinaryRiffle} and \texttt{Shuffle-II} are UC-securely computable.

\textbf{Proof} \texttt{Shuffle-I} is UC-secure under the straight-line black-box simulation. The simulator $S_{S-I}$ is independent of the underline structure (network). If the underline structure is replaced by \texttt{BinaryRiffle} then it provides the UC-secure
Algorithm 9: Environment $Z$, with Adversary $A$ and Simulator $S_{S,I}$

begin

Network setup: for Party $P_j \in P$ do

if $P_j \in P_C$ then

$S_{S,I}$ obtains the secret $x_j$ of $P_j$ and passes to $A$;

if $A$ corrupts $P_j$ then

$A$ corrupts the secret of $P_j$ as $\bar{x}_j$ and $S_{S,I}$ resets $x_j = \bar{x}_j$.

else

$S_{S,I}$ randomly sets the input $x_j$ for party $P_j$;

for all $i = 1, \ldots, n/2$ do

$S_{S,I}$ calls $F_{VSS,share}^i$ with respect to the quorum $Q_j$;

Simulation:

for every layer do

for every $P_j \in Q_i$ do

if $P_j \in P_C$ then

$S_{S,I}$ obtains the input $(x_j, y_j)$ from $P_j$ and passes the input to $A$.

if $A$ corrupts $P_j$ then

$A$ corrupts the input as $(\bar{x}_j, \bar{y}_j)$ and $S_{S,I}$ resets $P_j$’s secret as $(x_j, y_j) = (\bar{x}_j, \bar{y}_j)$.

$S_{S,I}$ writes $(P_j, (x_j, y_j))$ on the tape of $Z$.

$S_{S,I}$ randomly sets the input as $(x_j, y_j)$ for $P_j$ and writes $(P_j, (x_j, y_j))$ on the tape of $Z$.

for all $i = 1, \ldots, n/2$ do

$S_{S,I}$ calls $F_{RS}^j$ /* This is a concurrent operation. */

if not the last layer then

if $P_j \in P_C$ and $A$ corrupts $P_j$ then

$S_{S,I}$ obtains the output $j$ of $P_j$ from $A$.

$Q_1^i$ and $Q_2^i$, respectively /* This is a concurrent operation. */

end

of the simulator $S_{BR}$. Similarly, the UC-security of Shuffle-II is obtained.

4.2 Why Permutation Network?

The prior MPS protocols (e.g. \[28, 12, 34\]) are based on sorting network. The commonly used sorting networks are having $\mathcal{O}((\log n)^2)$ layers \[24\]. Ajtai et al. proposed an $\mathcal{O}(\log n)$ layer sorting network for large number of inputs \[55\]. However, their protocol requires large number of inputs. Donald E. Knuth commented in \[56\] that -

The networks they constructed are not for practical interest, since many components were introduced just to save a factor of $\log n$; Batcher’s method is much better, unless $n$ exceeds the total memory capacity of all computers on earth!

Later on Leighton et al. proposed an $\mathcal{O}(\log n)$ layers network that usually sorted the input sequence with high probability. Their protocol \[30\] is based on the butterfly tournament protocol. Nevertheless, the protocol \[30\] defines a
sorting network with 7.44 log \( n \) layers.

A \( n \)-permute network produces a random permutation of the input sequence. \( n \)-permute networks (e.g. \( [56, 31, 29] \)) are having \( O(\log n) \) layers. Our implementation is based on Beneš network that has \( 2 \log n - 1 \) layers. Therefore, \( n \)-permute networks are more suitable than the sorting networks. However, \( n \)-permute network does not produce uniform distribution of permutations. That is, some of the permutations are more likely that the others. For example, Table 4 shows the distribution of \( 8 \)-permute network. We find that, for all possible configurations the mean of occurring a permutation is 26.0063 and the standard deviation is 1.622.

5 | CONCLUSION

In this paper, we have presented two MPS protocols. The protocols are based on permutation networks. The first MPS protocol emulates the Beneš network. We generalize the Beneš structure for any even number of inputs and design a multiparty protocol to shuffle the inputs. We find that when the number of inputs is large, say greater than \( 2^8 \), shuffling becomes inefficient. Our second MPS protocol is designed for large number of inputs. We propose a \( n_\pi \)-permute network to reduce the number of layers of the network. However, \( n_\pi \)-permute network produces less than \( n! \) permutations.

The building block of the shuffling protocol is Random-swap gate. We design a multiparty Random-swap gate that swaps the inputs with probability \( \frac{1}{2} \), without learning the inputs. Moreover, an observer can not distinguish whether the inputs are swapped or passed through. Each gate is operated by a quorum of parties. We consider that at most 1/3 of the parties may be corrupted.

Instead of sorting network, we use permutation network for shuffling. Unlike the Compare-swap gate in sorting network, the permutation network uses Random-swap gates. We find that Random-swap gate is more efficient than Compare-swap. There are three rounds of communication in the Random-swap gate, whereas Compare-swap incurs \( O(\log p) \) rounds of communications where \( p \) is the size of the underlying field.

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