Non-Gaussian Distributions in Extended Dynamical Systems

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December 31, 2021

Abstract

We propose a novel mechanism for the origin of non-Gaussian tails in the probability distribution functions (PDFs) of local variables in nonlinear, diffusive, dynamical systems including passive scalars advected by chaotic velocity fields. Intermittent fluctuations on appropriate time scales in the amplitude of the (chaotic) noise can lead to exponential tails. We provide numerical evidence for such behavior in deterministic, discrete-time passive scalar models. Different possibilities for PDFs are also outlined.

PACS numbers: 02.50Ey,05.40+j,47.27Qb
The explanation of the form of the probability distribution of the fluctuations in local variables in extended, dissipative, dynamical systems poses an interesting challenge. In various turbulence experiments [1, 2, 3, 4], Probability Distribution Functions (PDFs) of velocity gradients and passive scalars (temperature) are observed to be non-Gaussian; in particular, exponential tails have been seen whereas Gaussian distributions might be expected if one naively invokes the central limit theorem. Theoretically, a variety of explanations including an ingenious phenomenological model based on a nonlinear Fokker-Planck equation [5] and methods that rely on obtaining closure and moment balance relations [6, 7] have been proposed to explain exponential tails in the passive scalar problem. The moment balance approach has been used to provide good fits to experimental data [8]. In addition, several eddy diffusive models that yield exponential tails have also been studied [9].

In this letter we explore a new mechanism in which the nature of the temporal correlations of the fluctuations that couple to a diffusive field plays a crucial role. We focus on passive scalar models and show that non-Gaussian PDFs including exponential and stretched exponential behaviors can arise when local variables are coupled to fluctuations whose amplitude varies randomly on a time scale comparable to the intrinsic diffusive time scale. In experimental fluid systems, exponential tails could, therefore, arise if the velocity fluctuations exhibit intermittent behavior in their magnitude on relevant time scales.

We consider the diffusion of the temperature field $\theta(\vec{r}, t)$ advected by a flow characterized by the velocity $\vec{V}$ in the presence of an externally-imposed mean temperature gradient $\beta$. This is described by the passive scalar equa-
\[ \frac{\partial \theta}{\partial t} = \nu \nabla^2 \theta - \vec{V} \cdot \vec{\nabla} \theta. \]  

(1)

The coefficient \( \nu \) is the effective thermal diffusivity and the velocity is assumed to be incompressible: \( \nabla \cdot \vec{V} = 0 \).

The behavior of (1) in the presence of prescribed random velocity fields can be studied numerically directly in the continuum. However, the case of a velocity field arising from a deterministic chaotic dynamics is computationally prohibitive if one uses Navier-Stokes equations. Instead we consider 2-dimensional coupled map lattice models in discrete time and space which can be viewed as coarse-grained versions of the continuum equation with the lattice spacing of the order of the correlation length of the velocity fluctuations. The scalar \( \theta(i) \) is defined on each site \( i \) of a \( L \times L \) square lattice with a mean gradient \( \beta \) along the \( y \) direction and periodic boundary conditions in the \( x \) direction. The system evolves synchronously in discrete time \( n \) according to

\[ \theta_{n+1}(i) = \theta_n(i) + \nu \nabla^2_L \theta_n(i) - \vec{V}_n(i) \cdot \vec{\nabla}_L \theta_n(i) \]  

(2)

where \( \vec{\nabla}_L \) is a symmetric, lattice gradient. The incompressibility of the two-dimensional velocity field \( \vec{V}_n(i) = (u_n(i), v_n(i)) \) is enforced by obtaining it from a stream function \( \psi_n(i) \): \( u_n(i) = \gamma \nabla_y \psi_n(i) \); \( v_n(i) = -\gamma \nabla_x \psi_n(i) \). The velocity vanishes at both \( y = 1 \) and \( y = L \) boundaries. The parameter \( \gamma \) is introduced to adjust the variance \( \sigma_v \) of the velocity field since the (discrete time) model is unstable for large values of the variances \( \sigma_v \).

In our numerical simulations, for all the models we explored, the mean scalar profile is linear. If we expand \( \theta \) around the mean profile we obtain both
an additive noise term, $\beta v_n(i)$, and a convective (multiplicative) noise term. This separation also occurs in the continuum equation. We find that different regimes of behavior of the scalar PDFs can be characterized empirically by a single parameter $B$ that measures the relative strengths of convective and additive noise terms: $B = \sigma_\theta/\beta \xi_\theta$, where $\sigma_\theta$ is the variance and $\xi_\theta$ the characteristic length scale of $\theta$.

We first consider the regime $B < 1$ which occurs for $\sigma_v^2 \tau_c/\nu < 1$ where $\tau_c$ is the correlation time of the velocity field. Different models are defined by the dynamics of $V_n(i)$. When the velocity field has Gaussian amplitude fluctuations on a time scale $\tau_a$ comparable to the diffusive time scale $\tau_d = 1/\nu$ we obtain PDFs with exponential tails for $\theta$.

Model A: The easiest way to obtain amplitude fluctuations is to use a stochastic model and choose the stream function to be a product of two noise terms $\eta_n(i)\eta_n'(i)$ where $\eta_n'(i)$ is a white noise and $\eta_n(i)$ a Gaussian noise with a correlation time $\tau_a$, both having delta-function spatial correlations. The stream function and the velocity display two correlation times: (i) the mixing time $\tau_c$ governed by $\eta_n'(i)$ chosen to be one timestep and (ii) the time scale $\tau_a$ on which the amplitude varies. This model exhibits exponential tails in the PDF of $\theta(i)$ when $\tau_a \approx \tau_d$ (See Fig.1). On the other hand when $\tau_a \ll \tau_d$ the distributions are Gaussian.

Model B: We next provide numerical evidence to show that exponential tails also occur when the stream function is derived from a deterministic chaotic model provided it leads to amplitude fluctuations on the appropriate time scale. We investigate the following model: define the stream function in terms of an auxiliary variable $\phi_n(i)$ by $\psi_n(i) = \nabla_I^2 \phi_{2n}(i)$; let $\phi_n(i)$ be
updated according to $\phi_{m+1}(i) = \frac{1}{\beta} \sum_{k} F(\phi_{m}(k))$, where the sum includes the site $i$ and its nearest neighbours. The function $F$ is chosen to be $F(x) = (1-2a)x + 2ax^3$; we consider $a = 2$ for which the maximum Liapunov exponent is $\approx 0.95$ and the correlation length is measured to be one lattice spacing. The PDFs for $u,v$ display Gaussian tails (See Fig.2a). The PDF for $\theta(i)$ is shown in Fig.2b which clearly shows exponential tails over five decades (similar results persist for a range of values of $a$). The slope of the tails is proportional to $\sqrt{\tau_{d}/(\beta \sigma_{v})}$ as expected from dimensional considerations.

To demonstrate that the velocity field derived from this deterministic model displays intermittent amplitude fluctuations we compute the autocorrelation functions $C_{2}(n) = \langle v_{n}v_{0} \rangle - \langle v_{0} \rangle^2$ and $C_{4}(n) = \langle v_{n}^{2}v_{0}^{2} \rangle - \langle v_{0}^{2} \rangle^2$, where $\langle \ldots \rangle$ indicates time average. In Fig.3 we show $C_{4}(n)$ which shows two exponential decays associated with time scales $\tau_{c}/2 \approx 1$, $\tau_{a}/2 \approx 4$. The inset shows $C_{2}(n)$ which is expected to display a decay time $(1/\tau_{c} + 1/\tau_{a})^{-1} \approx 1.6$, consistent with the above interpretation. Note that in Fig.2b $\tau_{d}$ is comparable to $\tau_{a}$ and the kurtosis decreases with increasing $\tau_{d}$.

To elucidate this mechanism we present a mean-field-like toy model with only additive noise since the multiplicative noise term is small for $B < 1$. The model is described by the Langevin equations,

$$\dot{x} = -\nu x + y \eta_{1}, \quad (3)$$

$$\dot{y} = -\alpha y + \eta_{2}, \quad (4)$$

where the variable $x$ couples to a noise with an amplitude $y$ that fluctuates in time. The terms $\eta_{i}$ are assumed to be white noise with $\langle \eta_{i} \rangle = 0$ and $\langle \eta_{i}(t)\eta_{j}(t') \rangle = \sigma_{i}^{2}\delta_{ij}\delta(t - t')$. The crucial time scales are given by
\( \tau_a = 1/\alpha \) and \( \tau_d = 1/\nu \). It is convenient to introduce scaled variables \( x_0 = x/\sqrt{D_1D_2} \), and \( y_0 = y/\sqrt{D_2} \) where \( D_1 = \sigma_1^2/2\nu \), \( D_2 = \sigma_2^2/2\alpha \). Since \( y_0 \) is independent of \( x_0 \) one can obtain a stationary solution for the Fokker-Planck (FP) equation \([11]\) of the form \( P(x_0,y_0) = Q(x_0|y_0)R(y_0) \), where \( R(y_0) = [2\pi]^{-1/2} \exp(-y_0^2/2) \). The conditional probability \( Q(x_0|y_0) \) satisfies another FP equation, that can be solved iteratively for small \( \alpha/\nu \) \([12]\). For large \( x_0 \) we have obtained the first three terms of \( p(x_0) = \int P(x_0,y_0)dy_0 \) expanded as a formal power series: \( p(x_0) = \sum_{n=0}^{\infty} (\alpha/\nu)^np_n(x_0) \). The functions \( p_0, p_1, p_2 \) are combinations of modified Bessel functions that have exponential tails for large values of \( x_0 \) with \( p_0(x_0) = [\pi]^{-1}K_0(|x_0|) \), where \( K_0 \) is the modified Bessel function of order zero. Note that the exponential tail results from a linear FP equation due to a branch cut in the characteristic functions in contrast to Ref.5 in which a nonlinear FP equation is used with simple poles in the characteristic function. In the limit of very large \( \alpha/\nu \), \( p(x_0) \) becomes Gaussian. We compute the low-order moments and find that the kurtosis is given by \( K = 3 + \frac{6\nu}{\alpha+\nu} \); this interpolates between 9 for small \( \alpha/\nu \) and 3 for large \( \alpha/\nu \) corresponding to \( p_0 \) and the Gaussian distribution respectively. In Fig.4 we show the PDF of \( x_0 \) obtained from numerical simulations for \( \alpha = \nu \) for which \( K = 6 \) corresponding to that of an exponential distribution.

The form of the PDF depends not only on the parameter \( B \) but also on the nature of the velocity correlations. In addition to exponential tails discussed above other non-Gaussian behavior can occur in different models. For example, the introduction of another variable \( z \) using \( \dot{z} = -\mu z + x\eta_3 \) in the toy model (Eq.3,4) leads to a stretched exponential PDF for \( z \) with an exponent 2/3 for appropriate \( \alpha, \nu \) and \( \mu \). A similar generalization of the
(stochastic) model A can be constructed. We emphasize that the PDF of the variable depends on the distribution of the slowly varying amplitude of the noise but not on the distribution of the noise itself.

We now consider the model in which the velocity is obtained from $\psi_n(i) = \eta_n(i)$, where $\eta_n(i)$ is a Gaussian noise correlated over a time $\tau_c$ with a correlation length $\xi_v$. The PDFs of $\theta$ are Gaussian when $B$ is smaller than unity even for $\tau_c \approx \tau_d$. This is true for both the discrete time and the continuum model. However, for $B >\approx 1$ the PDFs are non-Gaussian and we study this case using the continuum model i.e., Eq.(1). The resultant PDFs are shown in Fig.5 for different parameters. (i) For $B >> 1$ the distributions (Fig.5a,5b,5c) clearly do not have exponential tails; however, strikingly, we could fit the PDFs very well with a modified Lorentzian with two parameters $(\kappa, \delta)$: $p(x) = C/(1 + \kappa x^2)^{1+\delta}$ where $C$ is fixed by normalization. This function is a variant of the form proposed by Sinai and Yakhot. (ii) For $B \approx 1$ the scalar PDFs appear to have exponential tails over a narrow range of parameters (this case corresponds to Fig.5d). However, the fit with a modified Lorentzian agrees over a wide range of $B$ values including this narrow range.

Since the convective noise term is not negligible we model the $B >\approx 1$ regime by a single variable with both an additive ($\zeta_1$) and a multiplicative ($\zeta_2$) noise term:

$$\dot{x} = -\nu x + \zeta_1 + \zeta_2 x. \quad (5)$$

When $\zeta_i$ are white noise with $<\zeta_i(t)> = 0$ and $<\zeta_i(t)\zeta_j(t')> = 2D_i\delta_{ij}\delta(t-t')$, a straightforward solution of the Fokker-Planck equation yields exactly the modified Lorentzian form for the PDF with $\kappa = D_2/D_1$, $\delta = \nu/2D_2$.
This provides another example of a simple toy problem that mimics the PDFs of passive scalars.

We note that the exponential tail regime for passive scalars in several experiments appears to correspond to the $B < 1$ regime in our model where intermittent velocity fluctuations lead to exponential tails. Experimentally such amplitude variations may be connected with coherent structures such as plumes and turbulent bursts. A quite different extended dynamical system in which our mechanism is operative is a Capacitive Josephson Junction Array driven by external dc and ac currents; numerical simulations in chaotic states indicate exponential tails in the PDFs of local junction voltages transverse to the external current. This can be ascribed to intermittent current fluctuations caused by random vortex motion. However, establishing the occurrence of such amplitude fluctuations starting from the underlying equations, e.g., Navier-Stokes equations with appropriate boundary conditions, remains a challenge.

This work was supported by the U. S. Department of Energy (Contract No. DE-F-G02-88ER13916A000). We acknowledge computer time on the Cray-YMP provided by The Ohio Supercomputer Center. CJ is grateful to Dr. Yu He for valuable discussions; RB thanks Jayesh for helpful conversations.

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1 Figure captions

Fig.1: PDF for the normalized fluctuation $X = \delta \theta / \sigma_\theta$ for Model A (stochastic) with $L = 48, \nu = 0.1, \gamma = 0.4, \beta = 0.1, \tau_a \approx 10$. The white noise terms are uniformly distributed between -1 and 1. The variances are $\sigma_\theta \approx 0.034, \sigma_v \approx 0.21$ and the kurtosis is 4.22.

Fig.2: Data for Model B (deterministic).
a) PDF for $v$ normalized by $\sigma_v$ has Gaussian tails. b) PDFs for the normalized fluctuation $X$ are shown for two parameters. The dashed line is drawn with the same slope as that of the tail. i) Upper curve (shifted up by two decades) is for $\nu = 0.1, \gamma = 0.5, \beta = 0.1$. The variances are: $\sigma_\theta \approx 0.045, \sigma_v \approx 0.2$ and the kurtosis is 4.62. ii) Lower curve is for $\nu = 0.05, \gamma = 0.25, \beta = 0.1$. The variances are $\sigma_\theta \approx 0.031, \sigma_v \approx 0.1$ and the kurtosis is 4.12. Data are obtained using 5 million points in intervals of 10 time steps at a site in the middle of a $48 \times 48$ lattice.

Fig.3: Correlation functions of the velocity used in Fig.2. A semilog plot of $C_4(n)$ vs $n$ is shown. The straight lines have slopes $\approx 1.3$ and $\approx 0.24$. The inset shows a semilog plot of $C_2(n)$ vs $n$; the straight line has a slope 0.69. See text for discussion. (The variance $\sigma_v$ is set to unity.)

Fig.4: PDF for $x_0 = x / \sigma_x$ in the toy model of Equations (3) and (4) with $\alpha = \nu$. The noise variables $\eta_1, \eta_2$ are uniformly distributed with $\sigma_1 = \sigma_2 = 1$ and the measured $\sigma_x = 0.5$.

Fig.5: PDFs for the passive scalar fluctuations with $B \gg 1$ using the continuum model Equation (1). The data are obtained with $L = 48, \nu = 0.1, \beta = 0.1$ using the discretizations $\delta t = 0.05, \delta l = 0.5$ and velocity correla-
tions $\xi \approx \delta l$ and $\tau_c = 2$. The upper three curves (a,b,c) are shifted up from the lowest curve (d) by 6,4 and 2 decades respectively. The curves a,b,c,d correspond to $\sigma_v = 1.4\gamma$ where $\gamma = (0.4, 0.35, 0.3, 0.2)$ with kurtosis values (15.6, 6.3, 4.1, 3.5). The dashed lines are modified Lorentzian (ML) fits for the data.