On the matrices of given rank in a large subspace

Clément de Seguins Pazzis∗†

December 2, 2010

Abstract
Let \( V \) be a linear subspace of \( M_{n,p}(K) \) with codimension lesser than \( n \), where \( K \) is an arbitrary field and \( n \geq p \). In a recent work of the author, it was proven that \( V \) is always spanned by its rank \( p \) matrices unless \( n = p = 2 \) and \( K \cong \mathbb{F}_2 \). Here, we give a sufficient condition on \( \text{codim} \ V \) for \( V \) to be spanned by its rank \( r \) matrices for a given \( r \in [1, p - 1] \). This involves a generalization of the Gerstenhaber theorem on linear subspaces of nilpotent matrices.

AMS Classification: 15A30
Keywords: matrices, rank, linear combinations, dimension, codimension.

1 Introduction
In this paper, \( K \) denotes an arbitrary field, \( n \) a positive integer and \( M_n(K) \) the algebra of square matrices of order \( n \) with entries in \( K \). For \( (p, q) \in \mathbb{N}^2 \), we also let \( M_{p,q}(K) \) denote the vector space of matrices with \( p \) rows, \( q \) columns and entries in \( K \). Two linear subspaces \( V \) and \( W \) of \( M_{p,q}(K) \) will be called equivalent when there are non-singular matrices \( P \) and \( Q \) respectively in \( \text{GL}_p(K) \) and \( \text{GL}_q(K) \) such that \( W = PVQ \).

In a recent work of the author [10], the following proposition was a major tool for generalizing a theorem of Atkinson and Lloyd [1] to an arbitrary field:

∗Professor of Mathematics at Lycée Privé Sainte-Geneviève, 2, rue de l’École des Postes, 78029 Versailles Cedex, FRANCE.
†e-mail address: dsp.prof@gmail.com
Proposition 1. Let $n$ and $p$ denote positive integers such that $n \geq p$. Let $V$ be a linear subspace of $M_{n,p}(\mathbb{K})$ such that $\text{codim} V < n$, and assume $(n, p, \# \mathbb{K}) \neq (2, 2, 2)$ or $\text{codim} V < n - 1$. Then $V$ is spanned by its rank $p$ matrices.

The exceptional case of $M_2(\mathbb{F}_2)$ is easily described:

Proposition 2. Let $V$ be a linear hyperplane of $M_2(\mathbb{F}_2)$. Then:

- either $V$ is equivalent to $\mathfrak{sl}_2(\mathbb{F}_2) = \{ M \in M_2(\mathbb{F}_2) : \text{tr} M = 0 \}$ and then $V$ is spanned by its rank 2 matrices;
- or $V$ is equivalent to the subspace $T^+_2(\mathbb{F}_2)$ of upper triangular matrices, and then $V$ is not spanned by its rank 2 matrices.

Proof. Consider the orthogonal $V^\perp$ of $V$ for the non-degenerate symmetric bilinear form $b : (A, B) \mapsto \text{tr}(AB)$. Then $V^\perp$ contains only one non-zero matrix $C$. Either $C$ has rank 2, and it is equivalent to $I_2$, hence $V$ is equivalent to $\mathfrak{sl}_2(\mathbb{F}_2)$; or $C$ has rank 1, it is equivalent to $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ hence $V$ is equivalent to $T^+_2(\mathbb{F}_2)$.

In the first case, the three non-singular matrices $I_2$, $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ span $\mathfrak{sl}_2(\mathbb{F}_2)$. In the second one, $T^+_2(\mathbb{F}_2)$ has only two non-singular matrices, which obviously cannot span it.

Here, we wish to give a similar result for the rank $r$ matrices, still assuming that $\text{codim} V < n$. Our main results follow:

Theorem 3. Let $n \geq p$ be integers and $V$ be a linear subspace of $M_{n,p}(\mathbb{K})$ with $\text{codim} V < n$. Let $r \in [1, p]$ and $s \in [0, r]$. Then every rank $s$ matrix of $V$ is a linear combination of rank $r$ matrices of $V$, unless $n = p = r = \# \mathbb{K} = 2$ and $\text{codim} V = 1$.

This has the following easy corollary (which will be properly proven later on):

Corollary 4. Let $n \geq p$ be integers and $V$ be a linear subspace of $M_{n,p}(\mathbb{K})$ with $\text{codim} V < n$. Then, for every $r \in [1, p]$, the subspace $V$ contains a rank $r$ matrix.

Theorem 5. Let $n \geq p$ be integers and $V$ be a linear subspace of $M_{n,p}(\mathbb{K})$ with $\text{codim} V < n$. Let $r \in [1, p - 1]$. If $\text{codim} V \leq \left( \frac{r^2}{2} \right) - 2$, then $V$ is spanned by its rank $r$ matrices.
Notice that this has the following nice corollary (for which a much more elementary proof exists):

**Corollary 6.** Let $H$ be a linear hyperplane of $M_n(\mathbb{K})$, with $n \geq 2$. Then $H$ is spanned by its rank $r$ matrices, for every $r \in [1,n]$, unless $(n,r,\#\mathbb{K}) = (2,2,2)$.

We will also show that there exists a linear subspace of $M_{n,p}(\mathbb{K})$ with codimension $(r+2) - 2$ which is not spanned by its rank $r$ matrices, hence the upper bound $(r+2) - 2$ in the above theorem is tight. The proof of these results will involve an extension of the Flanders theorem to affine subspaces (see Section 3 of [9]) and a slight generalization of the famous Gerstenhaber theorem [4] on linear subspaces of nilpotent matrices.

### 2 Proving the main theorems

#### 2.1 Proof of Proposition 1

For the sake of completeness, we will recall here the proof of Proposition 1 already featured in [10]. This is based on the following theorem of the author, slightly generalizing earlier works of Dieudonné [3], Flanders [4] and Meshulam [7]:

**Theorem 7.** Given positive integers $n \geq p$, let $V$ be an affine subspace of $M_{n,p}(\mathbb{K})$ containing no rank $p$ matrix. Then codim $V \geq n$.

If in addition codim $V = n$ and $(n,p,\#\mathbb{K}) \neq (2,2,2)$, then $V$ is a linear subspace of $M_{n,p}(\mathbb{K})$.

**Proof of Proposition 1.** Assume that $V$ is not spanned by its rank $p$ matrices. Then there would be a linear hyperplane $H$ of $V$ containing every rank $p$ matrix of $V$. Choosing $M_0 \in V \setminus H$, it would follow that the affine subspace $M_0 + H$, which has codimension in $M_{n,p}(\mathbb{K})$ lesser than or equal to $n$, contains no rank $p$ matrix. However $M_0 + H$ is not a linear subspace of $M_{n,p}(\mathbb{K})$, which contradicts the above theorem.

#### 2.2 Proof of Theorem 3 and Corollary 4

We discard the case $n = p = r = \#\mathbb{K} = 2$ and codim $V = 1$, which has already been studied in the proof of Proposition 2.
Let us now prove Theorem 3. Let \( A \) be a rank \( s \) matrix of \( V \). Replacing \( V \) with an equivalent subspace, we lose no generality assuming that \( A \) has the form
\[
A = \begin{bmatrix} A_1 & 0 \end{bmatrix} \quad \text{for some} \quad A_1 \in M_{n,r}(\mathbb{K}).
\]
Denote by \( W \) the linear space consisting of those matrices \( M \in M_{n,r}(\mathbb{K}) \) such that \( [M \ 0] \in V \). Then the rank theorem shows that \( \text{codim}_{M_{n,r}(\mathbb{K})} W \leq \text{codim}_{M_{n,p}(\mathbb{K})} V < n \). Notice that the situation \( n = r = 2 \) may not arise, hence Proposition 1 shows that \( W \) is spanned by its rank \( r \) matrices. In particular, the matrix \( A_1 \) is a linear combination of rank \( r \) matrices of \( W \), hence \( A \) is a linear combination of rank \( r \) matrices of \( V \). This proves Theorem 3.

Let us now turn to Corollary 4. Denote by \( V' \) the linear subspace of \( V \) consisting of its matrices with all columns zero starting from the second one. Then \( \dim V' \geq n - \text{codim} V > 0 \), hence \( V' \neq \{0\} \), which proves that \( V \) contains a rank 1 matrix \( M \). Then, for every \( r \in [1, p] \), Theorem 3 shows that \( M \) is a linear combination of rank \( r \) matrices of \( V \), hence \( V \) must contain at least one rank \( r \) matrix!

2.3 Proof of Theorem 5

We will start from an observation that is similar to the one that lead to Proposition 1. Let \( V \) be a linear subspace of \( M_{n,p}(\mathbb{K}) \), let \( r \in [1, p-1] \) and assume that \( V \) is not spanned by its rank \( r \) matrices. Then there would be a linear hyperplane \( H \) of \( V \) containing every rank \( r \) matrix of \( V \). By Theorem 3, the subspace \( H \) must also contain every matrix of \( V \) with rank lesser than or equal to \( r \). Choosing arbitrarily \( M_0 \in V \setminus H \), it would follow that the (non-linear) affine subspace \( M_0 + H \) contains only matrices of rank greater than \( r \) and has dimension \( \dim V - 1 \).

Conversely, assume there exists an affine subspace \( \mathcal{H} \) of \( M_{n,p}(\mathbb{K}) \) which contains only matrices of rank greater than \( r \) (notice then that \( 0 \notin \mathcal{H} \)), and let \( H \) denote its translation vector space. Then \( H \) must contain every rank \( r \) matrix of the linear space \( V' := \text{span} \mathcal{H} \), therefore \( V' \), which has dimension \( \dim H + 1 \), is not spanned by its rank \( r \) matrices.

Theorem 5 will thus come from the following result (applied to \( k = r + 1 \)), which generalizes a theorem of Meshulam to an arbitrary field and rectangular matrices (Meshulam tackled the case of an algebraically closed field and the one of \( \mathbb{R} \), and he restricted his study to square matrices).

**Theorem 8.** Let \( n \geq p \geq k \) be positive integers. Denote by \( h(n, p, k) \) the largest
dimension for an affine subspace \( V \) of \( M_{n,p}(\mathbb{K}) \) satisfying

\[
\forall M \in V, \quad \text{rk} \, M \geq k.
\]

Then

\[
h(n,p,k) = np - \left( \frac{k+1}{2} \right).
\]

Inequality \( h(n,p,k) \geq np - \left( \frac{k+1}{2} \right) \) is obtained as in [8] by considering the affine subspace \( H \) consisting of all \( n \times p \) matrices of the form

\[
\begin{bmatrix}
I_r + T & \? \\
? & ?
\end{bmatrix}
\]

with \( T \in T_k^{++}(\mathbb{K}) \),

where \( T_k^{++}(\mathbb{K}) \) denotes the set of strictly upper triangular matrices of \( M_k(\mathbb{K}) \).

Obviously \( \text{codim}_{M_{n,p}(\mathbb{K})} H = \text{codim}_{M_k(\mathbb{K})} T_k^{++}(\mathbb{K}) = \left( \frac{k+1}{2} \right) \), whilst, judging from its left upper block, every matrix of \( H \) has a rank greater than or equal to \( k \).

In order to prove that \( h(n,p,k) \leq np - \left( \frac{k+1}{2} \right) \), we let \( V \) be an arbitrary affine subspace of \( M_{n,p}(\mathbb{K}) \) such that \( \forall M \in V \), \( \text{rk} \, M \geq k \), and we prove that \( \dim V \leq np - \left( \frac{k+1}{2} \right) \). Proceeding by downward induction on \( k \), we may assume furthermore that \( V \) contains a rank \( k \) matrix. We then lose no generality assuming that \( V \) contains the matrix \( J_k := \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix} \). Denote by \( V \) the translation vector space of \( V \) and consider the linear subspace \( W \) of \( M_k(\mathbb{K}) \) consisting of those matrices \( A \) for which \( \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \) belongs to \( V \). Then the rank theorem shows \( \text{codim}_{M_{n,p}(\mathbb{K})} V \geq \text{codim}_{M_k(\mathbb{K})} W \). The assumptions on \( V \) show that \( I_k + W \) contains only nonsingular matrices. Since \( W \) is a linear subspace, this shows that for any \( M \in W \), the only possible eigenvalue of \( M \) in the field \( \mathbb{K} \) is 0. The proof will thus be finished should we establish the next theorem:

For \( M \in M_n(\mathbb{K}) \), we let \( \text{Sp}(M) \) denote the set of its eigenvalues in the field \( \mathbb{K} \).

**Theorem 9** (Generalized Gerstenhaber theorem). Let \( V \) be a linear subspace of \( M_n(\mathbb{K}) \) such that \( \text{Sp}(M) \subseteq \{0\} \) for every \( M \in V \). Then \( \dim V \leq \left( \frac{n}{2} \right) \).

Note that this implies the Gerstenhaber theorem on linear subspaces of nilpotent matrices [2] [5] [6], and that this is equivalent to it when \( \mathbb{K} \) is algebraically closed. Moreover, for \( \mathbb{K} = \mathbb{R} \), the proof is easy by intersecting \( V \) with the space of symmetric matrices of \( M_n(\mathbb{K}) \) (see [8]). Our proof for an arbitrary field will
use a brand new method. For \( i \in [1, n] \) and \( M \in M_n(\mathbb{K}) \), we let \( L_i(M) \) denote the \( i \)-th row of \( M \). We set
\[
R_i(V) := \{ M \in V : \forall j \in [1, n] \setminus \{ i \}, L_j(M) = 0 \}.
\]

**Proposition 10.** Let \( V \) be a linear subspace of \( M_n(\mathbb{K}) \) such that \( \text{Sp}(M) \subset \{0\} \) for every \( M \in V \). Then \( R_i(V) = \{0\} \) for some \( i \in [1, n] \).

**Proof.** We prove this by induction on \( n \). Assume the claim holds for any subspace of \( (n - 1) \times (n - 1) \) matrices satisfying the assumptions, and that it fails for \( V \). Denote by \( W \) the linear subspace of \( V \) consisting of its matrices with a zero last row. We decompose every \( M \in W \) as \( M = \begin{bmatrix} K(M) & ? \\ 0 & 0 \end{bmatrix} \). Notice that \( K(W) \) is a linear subspace of \( M_{n-1}(\mathbb{K}) \) satisfying the assumptions of Proposition 10. By the induction hypothesis, there is an integer \( i \in [1, n - 1] \) such that \( R_i(K(W)) = \{0\} \). However, \( R_i(V) \neq \{0\} \), hence \( V \) contains the elementary matrix \( E_{i,n} \) (i.e. the one with entry 1 at the spot \((i,n)\), and for which all the other entries are zero). Conjugating \( V \) with a permutation matrix, this generalizes as follows: for every \( k \in [1, n] \), there is an integer \( f(k) \in [1, n] \) such that \( E_{f(k),k} \in V \). We may then find an \( f \)-cycle, i.e. a list \((i_1, \ldots, i_p)\) of pairwise distinct integers such that \( f(i_1) = i_2, f(i_2) = i_3, \ldots, f(i_{p-1}) = i_p \) and \( f(i_p) = i_1 \). Hence \( V \) contains the matrix \( M := E_{i_1,i_p} + \sum_{k=1}^{p-1} E_{i_{k+1},i_k} \). However \( 1 \in \text{Sp}(M) \) (consider the vector with entry 1 in every \( i_k \) row, and zero elsewhere), contradicting our assumptions.

**Proof of Theorem 9.** Again, we use an induction process. The result is trivial when \( n = 0 \) or \( n = 1 \). Assume \( n \geq 2 \) and the results holds for subspaces of \( M_{n-1}(\mathbb{K}) \). Let \( V \subset M_n(\mathbb{K}) \) be as in Theorem 9. Using Proposition 10, we lose no generality assuming that \( R_n(V) = \{0\} \) (we may reduce the situation to this one by conjugating \( V \) with a permutation matrix). Consider the linear subspace \( W \) of \( V \) consisting of its matrices which have the form
\[
M = \begin{bmatrix} A(M) & 0 \\ L(M) & \alpha(M) \end{bmatrix}
\]
where \( A(M) \in M_{n-1}(\mathbb{K}), \ L(M) \in M_{1,n-1}(\mathbb{K}) \) and \( \alpha(M) \in \mathbb{K} \). Then the rank theorem shows that \( \dim V \leq (n - 1) + \dim W \). For every \( M \in W \), one has \( \text{Sp}(M) \subset \{0\} \) hence \( \alpha(M) = 0 \) and \( \text{Sp} A(M) \subset \),
\{0\}. Since \( R_n(V) = \{0\} \), this yields \( \dim A(W) = \dim W \), whilst the induction hypothesis shows that \( \dim A(W) \leq \binom{n-1}{2} \). We conclude that

\[
\dim V \leq (n - 1) + \binom{n - 1}{2} = \binom{n}{2}.
\]

\[ \square \]

**Remark 1.** Proceeding by induction and using Proposition 10, it can even be proven that under the assumptions of Theorem 9, there is a permutation matrix \( P \in \text{GL}_n(\mathbb{K}) \) such that \( (P V P^{-1}) \cap T_n^{-}(\mathbb{K}) = \{0\} \), where \( T_n^{-}(\mathbb{K}) \) denotes the space of lower triangular matrices in \( M_n(\mathbb{K}) \). This would immediately yield Theorem 9.

This completes the proof of Theorem 5.

**References**

[1] M.D. Atkinson, S. Lloyd, Large spaces of matrices of bounded rank, *Quart. J. Math. Oxford (2)*, 31 (1980), 253-262.

[2] R. Brualdi, K. Chavey, Linear spaces of Toeplitz and nilpotent matrices, *J. Combin. Theory Ser A*, 63 (1993), 65-78.

[3] J. Dieudonné, Sur une généralisation du groupe orthogonal à quatre variables, *Arch. Math.*, 1 (1949), 282-287.

[4] H. Flanders, On spaces of linear transformations with bounded rank, *J. Lond. Math. Soc.*, 37 (1962), 10-16.

[5] M. Gerstenhaber, On Nilalgebras and Linear Varieties of Nilpotent Matrices (I), *Amer. J. Math.*, 80 (1958), 614-622.

[6] B. Mathes, M. Omladič, H. Radjavi, Linear spaces of nilpotent matrices, *Linear Algebra Appl.*, 149 (1991), 215-225.

[7] R. Meshulam, On the maximal rank in a subspace of matrices, *Q. J. Math.*, *Oxf. II*, 36 (1985), 225-229.

[8] R. Meshulam, On two extremal matrix problems, *Linear Algebra Appl.*, 114/115 (1989), 261-271.
[9] C. de Seguins Pazzis, The affine preservers of non-singular matrices, *Arch. Math.*, 95 (2010) 333-342.

[10] C. de Seguins Pazzis, The classification of large spaces of matrices of bounded rank, *ArXiv preprint* [http://arxiv.org/abs/1004.0298](http://arxiv.org/abs/1004.0298)