Cartan Connection for $h$-Matsumoto change

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Abstract

In the present paper, we have studied the Matsumoto change $\overline{L}(x, y) = \frac{L^2(x, y)}{L(x, y) - \beta(x, y)}$ with an $h$-vector $b_i(x, y)$. We have derived some fundamental tensors for this transformation. We have also obtained the necessary and sufficient condition for which the Cartan connection coefficients for both the spaces $F^n = (M^n, L)$ and $\overline{F}^n = (M^n, \overline{L})$ are same.

Keywords: Finsler space, Matsumoto change and $h$-vector.

1 Introduction

Let $M$ be an $n$-dimensional $C^\infty$ Manifold and $T_x M$ denotes the tangent space of $M$ at $x$. The tangent bundle of $M$ is the union of tangent space $TM := \bigcup_{x \in M} T_x M$. A function $L: TM \to [0, \infty)$ is called Finsler metric function if it has the following properties [8]

1. $L$ is $C^\infty$ on $TM \setminus \{0\}$,
2. For each $x \in M$, $L_x := L|_{T_x M}$ is a Minkowaski norm on $T_x M$.

The pair $(M^n, L)$ is then called a Finsler space. The normalized supporting element, metric tensor, angular metric tensor and Cartan tensor are defined by $l_i = \partial_i L$, $g_{ij} = \frac{1}{2} \partial_i \partial_j L^2$, $h_{ij} = L \partial_i \partial_j L$ and $C_{ijk} = \frac{1}{2} \partial_k g_{ij}$ respectively. The Cartan connection for the Finsler space $F^n$ is given by $(F_j^i, N_j^i, C_j^i)$. The $h$-covariant and $v$-covariant derivative of the tensor $T^i_j$ with respect to Cartan connection, are respectively given as follows:

$$T^i_{jk} = \delta_k T^i_j + T^r_j F^i_{rk} - T^i_r F^r_{jk},$$

$$T^i_{j|k} = \partial_k T^i_j + T^r_j C^i_{rk} - T^i_r C^r_{jk}. $$

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where $\delta_k$ is differential operator $\delta_k = \partial_k - N^r_k \partial_r$.

In 1984, C. Shibata [9] introduced the change $\overline{L} = f(L, \beta)$ as a generalization of Randers change, where $f$ is positively homogeneous function of degree one in $L$ and $\beta(x, y) = b_i(x)y^i$. This change is called $\beta$-change. An important class of $\beta$-change is Matsumoto change, given by $\overline{L}(x, y) = \frac{L^2}{L - \beta}$. If $L(x, y)$ reduces to a Riemannian metric then $\overline{L}(x, y)$ becomes Matsumoto metric. A famous example of Finsler space “A slope measure of a mountain with respect to time measure” was given by M. Matsumoto [7]. Due to his great contribution in Finsler geometry, this metric was named after him.

A. Tayebi et al. [12] and Bankteswar Tiwari et al. [13] discussed the Kropina change and generalized Kropina change respectively, for the Finsler space with $m^{th}$ root metric. In 2017, A. Tayebi et al. [11] obtained the condition for the Finsler space given by Matsumoto change to be projectively related with the original Finsler space.

The concept of $h$-vector $b_i$, was first introduced by H. Izumi [5], which is $v$-covariant constant with respect to Cartan connection and satisfies $LC^h_{ij} b_h = \rho h_{ij}$, where $\rho$ is a non-zero scalar function. He showed that the scalar $\rho$ depends only on positional coordinates i.e. $\partial_i \rho = 0$.

From the definition of $h$-vector, it is clear that it depends not only on positional coordinates, but also on directional arguments.

Gupta and Pandey [2, 4], discussed certain properties of Randers change and Kropina change with an $h$-vector. They [4] showed that If the $h$-vector is gradient then the scalar $\rho$ is constant, i.e. $\partial_i \rho = 0$. In 2016, Gupta and Gupta [1, 3] have analysed Finsler space subjected to $h$-exponential change.

In the present paper, we have studied a Finsler metric defined by

$$\overline{L}(x, y) = \frac{L^2(x, y)}{L(x, y) - b_i(x, y)y^i}, \quad (1.1)$$

where $b_i(x, y)$ is an $h$-vector in $(M^n, L)$.

The structure of this paper is as follows: In section 2, we have obtained the expressions for different fundamental tensors of the transformed Finsler space. In section 3, we have observed how the Cartan connection coefficients change due to Matsumoto change with an $h$-vector and also find the necessary and sufficient condition for which both connection coefficients would be same.

**Remark 1** H. S. Shukla et al. [10] also discussed Matsumoto change of Finsler metric by $h$-vector. Unfortunately, the results are wrong because of wrong computation in Lemma 1.1 of [10].
2 The Finsler space \( \mathcal{F}^n = (M^n, \mathcal{L}) \)

Let the Finsler space transformed by the Matsumoto change (1.1) with an \( h \)-vector, be denoted by \( \mathcal{F}^n = (M^n, \mathcal{L}) \). If we denote \( \beta = b_i(x, y)y^i \), then indicatory property of angular metric tensor yields \( \dot{\beta}/\beta = b_j \). Throughout this paper, we have barred the geometrical objects associated with \( \mathcal{F}^n \).

From (1.1), we get the normalized supporting element as

\[
\bar{L}_l = \frac{\tau}{(\tau - 1)} l_i + \frac{\tau^2}{(\tau - 1)^2} m_i, \tag{2.1}
\]

where \( \tau = \frac{L}{\beta} \) and \( m_i = b_i - \frac{1}{\tau} l_i \).

Remark 2 The covariant vector \( m_i \) satisfies the following relations

(i) \( m_i \neq 0 \) (ii) \( m^i = g^{ij} m_j \) (iii) \( m^i m_i = b^2 - \frac{1}{\tau^2 \tau} = m^2 \) (iv) \( m_i y^i = 0 \).

Differentiating equation (2.1) with respect to \( y^j \), and using the notation \( L_{ij} = \dot{\beta} l_i \) we get

\[
\bar{L}_{ij} = \frac{\tau(\tau + \rho \tau - 2)}{(\tau - 1)^2} L_{ij} + \frac{2\tau^2}{\beta (\tau - 1)^3} m_i m_j. \tag{2.2}
\]

Therefore, the angular metric tensor \( \mathcal{h}_{ij} \) is obtained as

\[
\mathcal{h}_{ij} = \frac{\tau^2 (\tau + \rho \tau - 2)}{(\tau - 1)^3} h_{ij} + \frac{2\tau^4}{(\tau - 1)^2} m_i m_j. \tag{2.3}
\]

The metric tensor \( g_{ij} = \mathcal{h}_{ij} + l_i l_j \) is given by

\[
g_{ij} = \frac{\tau^2 (\tau + \rho \tau - 2)}{(\tau - 1)^3} g_{ij} + \frac{\tau^2 (1 - \rho \tau)}{(\tau - 1)^3} l_i l_j + \frac{\tau^3}{(\tau - 1)^3} (m_i l_j + m_j l_i) + \frac{3\tau^4}{(\tau - 1)^4} m_i m_j, \tag{2.4}
\]

which can be rewritten as

\[
g_{ij} = p g_{ij} + p_1 l_i l_j + p_2 (m_i l_j + m_j l_i) + p_3 m_i m_j,
\]

where

\[
p = \frac{\tau^2 (\tau + \rho \tau - 2)}{(\tau - 1)^3}, \quad p_1 = \frac{\tau^2 (1 - \rho \tau)}{(\tau - 1)^3}, \quad p_2 = \frac{\tau^3}{(\tau - 1)^3}, \quad p_3 = \frac{3\tau^4}{(\tau - 1)^4}.
\]

The following lemma helps us to compute the inverse of metric tensor \( g_{ij} \).

Lemma 2.1 [6]: Let \( (m_{ij}) \) be a non-singular matrix and \( l_{ij} = m_{ij} + n_i n_j \). The elements \( t^{ij} \) of the inverse matrix, and the determinant of the matrix \( (l_{ij}) \) are given by

\[
t^{ij} = m^{ij} - (1 + n_k n^k)^{-1} n^i n^j, \quad det(l_{ij}) = (1 + n_k n^k) det(m_{ij})
\]

respectively, where \( m^{ij} \) are elements of the inverse matrix of \( (m_{ij}) \) and \( n^k = m^{ki} n_i \).
The inverse metric tensor of $\bar{F}^n$ can be derived as follows:

$$\bar{g}^{ij} = q g^{ij} + q_1 l^il^j + q_2 (l^im^i + m^im^j) + q_3 m^im^j,$$  \hspace{1cm} (2.5)

where

$$q = \frac{1}{p}, \quad q_1 = -\frac{1}{2} \frac{p_1p_3 - p_2^2}{(p_1 + p)p_3 - p_2^2} + \frac{2p_2^2p_3}{(3p + 2p_3m^2)(p_1 + p)p_3 - p_2^2},$$

$$q_2 = \frac{-2p_2p_3}{(3p + 2p_3m^2)(p_1 + p)p_3 - p_2^2}, \quad q_3 = -\frac{2p_3}{p(3p + 2p_3m^2)}.$$

The Cartan tensor $C_{ijk}$ is obtained by differentiating the equation (2.4) with respect to $y^k$, as follows:

$$C_{ijk} = p C_{ijk} + V_{ijk},$$  \hspace{1cm} (2.6)

where

$$V_{ijk} = K_1 (h_{ij} m_k + h_{jk} m_i + h_{ki} m_j) + K_2 m_i m_j m_k,$$

and

$$K_1 = \frac{\tau^3(\tau + 3\rho \tau - 4)}{2L(\tau - 1)^4}, \quad K_2 = \frac{6\tau^4}{\beta(\tau - 1)^5}.$$

**Remark 3** From above we can retrieve relations between the scalars as

$$\frac{\partial p}{\partial \tau} = -\frac{2L}{\tau^2} K_1, \quad \frac{\partial p_3}{\partial \tau} = -\frac{2L}{\tau^2} K_2,$$

$$K_1 = \frac{1}{2L} \left\{ p_2 + p_3 \left( \rho - \frac{1}{\tau} \right) \right\} \quad \text{and} \quad p_1 + p_2 \left( \rho - \frac{1}{\tau} \right) = 0.$$

From equation (2.5) and (2.6), we get the $(h)hv$-torsion tensor $\bar{C}_{ijk}^i$

$$\bar{C}_{ijk}^i = C_{ijk}^i + M_{jk}^i,$$  \hspace{1cm} (2.7)

where

$$M_{jk}^i = q K_1 (m_k h_{ij}^i + m_j h_{ik}^i) + (q_2 l^i + q_3 m^i) \left\{ 2K_1 m_j m_k + \frac{p}{L} \rho h_{jk} \right\}$$

$$\quad + \left\{ q m^i + (q_2 l^i + q_3 m^i)m^2 \right\} \left( K_2 m_j m_k + K_1 h_{jk} \right).$$

**3 Cartan Connection of the space $\bar{F}^n$**

The Cartan connection for a Finsler space $\bar{F}^n$ is given by the traid $(\bar{F}_{ijk}^i, \bar{N}_{j}^i, \bar{C}_{jk}^i)$. The $v$-connection coefficient $\bar{C}_{jk}^i$ is given by equation (2.7). Now, we are obtaining the $h$-connection coefficient $\bar{F}_{jk}^i$ and non-linear connection coefficient $\bar{N}_{j}^i$.

First, we will try to find canonical spray of the transformed space $\bar{F}^n$.

Differentiating equation (2.4) with respect to $x^k$, and using the definition of $h$-covariant derivative, we obtain
\[ \partial_k \bar{g}_{ij} = p \partial_k g_{ij} + p_t (l_l r F_{jk} + l_j r F_{ik}) + p_2 (\rho_k h_{ij} + l_i b_{jk} + l_j b_{ik} + m_r F_{jk} l_i + m_r F_{ik} l_j \\
+ m_i F_{jk} l_r + m_j F_{ik} l_r) + p_3 (m_i b_{jk} + m_j b_{ik} + m_i m_r F_{jk} + m_j m_r F_{ik}) \\
+ 2 (K_1 h_{ij} + 2 K_2 m_i m_j) (\beta_k + N_k^r m_r) + K_1 (h_{jr} N_k^r m_i + h_{ir} N_k^r m_j), \]
where \( \partial_k \rho = \rho_k \) and \( \beta_{ik} = \beta_k. \)

Applying Christoffel process with respect to indices \( i, j, k \) in above equation, we obtain the coefficient of Christoffel symbol as follows:

\[ \bar{\gamma}_{ijk} = p \gamma_{ijk} + \mathcal{G}_{ijk} \left\{ \frac{p_2}{2} \rho_k h_{ij} + (\beta_k + N_k^r m_r) B_{ij} \right\} + Q_i F_{jk} + Q_k F_{ij} + Q_j E_{ik} \]
\[ + (\bar{g}_{rj} - p g_{rj}) \left\{ \gamma_{rjk} + g^{rt} (C_{ikm} N_t^m - C_{tkm} N_i^m - C_{itm} N_k^m) \right\}, \]
where the symbol \( \mathcal{G}_{ijk} \) is defined as \( \mathcal{G}_{ijk} U_{i j k} = U_{ijk} - U_{jki} + U_{kij} \) and we have used the notation

\[ Q_i = p_j l_i + p_m m_i, \quad B_{ij} = K_i h_{ij} + K_2 m_i m_j, \quad 2 E_{ij} = b_{i j} + b_{j i}, \quad 2 F_{ij} = b_{i j} - b_{j i}. \]

**Remark 4** The tensors \( Q_i \) and \( B_{ij} \) satisfy the following

(i) \( \partial_j Q_i = B_{ij} \) \quad (ii) \( B_{ij} = B_{ji} \) \quad (iii) \( B_{ij} y^j = 0. \)

The Christoffel Symbol of second kind of the Finsler space \( F^n \) is given by

\[ \bar{\gamma}^i_{jk} = \gamma^i_{jk} + (g^{rt} - p g^{rt}) (C_{jkm} N_t^m - C_{tkm} N_j^m - C_{itm} N_k^m) \]
\[ + \bar{g}^{is} \mathcal{G}_{jsk} \left\{ \frac{p_2}{2} \rho_k h_{js} + (\beta_k + N_k^r m_r) B_{sj} \right\} + Q_j F_{sk} + Q_k F_{sj} + Q_s E_{jk} \].

Transvecting equation (3.3) by \( y^j y^k \) and using \( G^i = \frac{1}{2} \bar{\gamma}^i_{jk} y^j y^k \), we get

\[ \bar{G}^i = G^i + D^i, \]
where

\[ D^i = \frac{1}{2} \bar{\gamma}^{is} \left[ Q_s E_{so} + 2 p_2 LF_{so} \right]. \]

Thus, we have:

**Proposition 3.1** The spray coefficient of the transformed space is given by equation (3.4).

**Remark 5** In the subscript zero ‘o’ is used to denote the transvection by \( y^i \), i.e. \( F_{so} = F_{st} y^t \).

Differentiating equation (3.4) with respect to \( y^i \) and using \( \partial_j G^i = N^i_j \) and \( \partial_j \bar{G}^i = -2 \bar{g}^{is} \bar{C}^s_{rj} \), we get

\[ \bar{N}^i_j = N^i_j + D^i_j, \]
where
\[ D^j_i = \mathcal{G}^{ir} \left\{ -2D^m(pC_{mjr} + V_{mjr}) + Q_rE_{oij} + E_{oo}B_{rj} + q_2LF_{rj} + Q_j E_{ro} + \frac{p_2}{2} \rho_k \gamma^k h_{rj} \right\}. \] (3.7)

Thus, we have:

**Proposition 3.2** The non-linear connection coefficient of the transformed space is given by the equation (3.6).

Now, we are in a position to obtain the Cartan connection coefficient for the space \( F^n \). We know that the relation between the Christoffel symbol and Cartan connection coefficient is given by

\[ F^i_{jk} = \gamma^i_{jk} + g^{is}(C_{jkr}N^r_s - C_{skr}N^r_j - C_{jsr}N^r_k). \]

In view of equation (2.6), (3.3) and (3.6), we have

\[ \mathcal{F}^i_{jk} = \gamma^i_{jk} + \left( g^{it} - g^{jt} \right)(C_{ikm}N^m_t - C_{tkm}N^m_i - C_{itm}N^m_k) \]
\[ + \mathcal{G}^{is}\left\{ \mathcal{S}_{jsk}\left( \frac{p_2}{2} \rho_k h_{js} + (\beta_k + N^r_k m_r)B_{sj} \right) + Q_j F_{sk} + Q_k F_{sj} + Q_s E_{jk} \right\} \]
\[ + \mathcal{G}^{is}\left\{ (pC_{jkr} + V_{jkr})(N^r_s + D^r_s) - (pC_{skr} + V_{skr})(N^r_j + D^r_j) - (pC_{jsr} + V_{jsr})(N^r_k + D^r_k) \right\} \]

which can be simplified as

\[ \mathcal{F}^i_{jk} = \gamma^i_{jk} + \mathcal{G}^{is}\left\{ \mathcal{S}_{jsk}\left( \frac{p_2}{2} \rho_k h_{js} + \beta_k B_{js} - p C_{jsr}D^r_k - V_{jsr}D^r_k \right) + Q_j F_{sk} + Q_k F_{sj} + Q_s E_{jk} \right\}. \]

Above equation can be rewritten as

\[ \mathcal{F}^i_{jk} = \gamma^i_{jk} + D^i_{jk}, \] (3.8)

where

\[ D^i_{jk} = \mathcal{G}^{is}\left\{ \mathcal{S}_{jsk}\left( \frac{p_2}{2} \rho_k h_{js} + \beta_k B_{js} - p C_{jsr}D^r_k - V_{jsr}D^r_k \right) + Q_j F_{sk} + Q_k F_{sj} + Q_s E_{jk} \right\}. \] (3.9)

Hence, we have:

**Theorem 3.1** The relation between the Cartan connection coefficient of \( F^n \) and \( \mathcal{F}^n \) is given by equation (3.8).

**Remark 6** The tensors \( D^i_{jk} \), \( D^i_j \) and \( D^i \) are related as

(i) \( D^i_{jk} y^k = D^i_j \),

(ii) \( D^i_j y^j = 2D^i \),

(iii) \( \partial_j D^i = D^i_j \).

Now, we want to find the condition for which the Cartan connection coefficients for both spaces \( F^n \) and \( \mathcal{F}^n \) are same, i.e. \( F^i_{jk} = \mathcal{F}^i_{jk} \) then \( D^i_{jk} = 0 \), which implies \( D^i_j = 0 \), then \( D^i = 0 \). Therefore the equation (3.5) gives

\[ 2p_2LF_{io} + E_{oo}Q_i = 0, \]
which on transvection by $y^i$ gives $E_{oo} = 0$ and then $F_{io} = 0$. Differentiating $E_{oo} = 0$ partially with respect to $y^i$ gives $E_{io} = 0$. Therefore we have $E_{io} = 0 = F_{io}$, which implies $b_{i|o} = b_{o|i} = \beta_{|i} = 0$. Differentiating $\beta_{|i}$ partially with respect to $y^j$ and using the commutation formula $\partial_j(\beta_{|i}) - (\partial_j\beta)_{|i} = (\partial_r\beta)C^r_{i|io}$, we get

$$b_{ji} = -b_rC^r_{ij|o}.$$ (3.10)

This will give us $F_{ij} = 0$. Taking $h$-covariant derivative of $LC^r_{ij}b_r = \rho h_{ij}$ and using $\rho_{|k} = 0$, $L_{ik} = 0$ and $h_{ij|k} = 0$, we get

$$(C^r_{ij}b_r)_{|k} = \left(\frac{\rho}{L} h_{ij}\right)_{|k} = 0.$$ 

This gives

$$C^r_{ij}b_r|k + C^r_{ij|k}b_r = 0.$$ 

Transvecting by $y^k$ and using $b_{r|io} = 0$, we get $C^r_{ij|o}b_r = 0$ and then equation (3.10) gives $b_{ij} = 0$, i.e. the $h$-vector $b_i$ is parallel with respect to Cartan connection of $F^n$.

Conversely, If $b_{ij} = 0$ then we get $E_{ij} = F_{ij} = 0$ and $\beta_i = \beta_{|i} = b_{j|i} y^j = 0$. Then equation (3.5) reduces to $D^i = 0$. From $F_{ij} = 0$ we have $\rho_i = 0$, which implies $D^i_j = 0$. Therefore, from equation (3.9), we get $D^i_{jk} = 0$, which gives $F^i_{jk} = F^i_{jk}$. Thus, we have:

**Theorem 3.2** For the Matsumoto change with an $h$-vector, the Cartan connection coefficients for both spaces $F^n$ and $\overline{F^n}$ are the same if and only if the $h$-vector $b_i$ is parallel with respect to the Cartan connection of $F^n$.

Now, differentiating equation (3.6) with respect to $y^k$, and using $\dot{\partial}_k N^i_j = G^i_{jk}$, we obtain

$$\overline{G}^i_{jk} = G^i_{jk} + \dot{\partial}_k D^i_j,$$ (3.11)

where $G^i_{jk}$ are the Berwald connection coefficients.

Now, if the $h$-vector $b_i$ is parallel with respect to the Cartan connection of $F^n$, then by the Theorem 3.2, the cartan connection coefficients for both Finsler space $F^n$ and $\overline{F^n}$ are the same, i.e. $D^i_{jk} = 0$ which implies $D^i_j = 0$. Then from equation (3.11), we get $\overline{G}^i_{jk} = G^i_{jk}$.

Conversely, if $\overline{G}^i_{jk} = G^i_{jk}$ then, from equation (3.11), we have $\dot{\partial}_k D^i_j = 0$, which on transvecting by $y^j$ and using Remark 6, gives $D^i_k = 0$. Using the same procedure as in the Theorem 3.2, we get $b_{ij} = 0$, i.e. the $h$-vector $b_i$ is parallel with respect to Cartan connection of $F^n$.

Thus, we have:

**Theorem 3.3** For the Matsumoto change with an $h$-vector, the Berwald connection coefficients for both spaces $F^n$ and $\overline{F^n}$ are the same if and only if the $h$-vector $b_i$ is parallel with respect to the Cartan connection of $F^n$.
Conclusion

In the present paper, the Cartan connection of the changed Finsler space is discovered and with the condition (h-vector $b_i$ is parallel, i.e. $b_{ij} = 0$), the Cartan connection of both the spaces are same.

*For this transformation we can also find some geometric properties for the transformed Finsler space like the curvature tensor, torsion tensor, $T$-tensor etc.*

Gupta and Pandey [4] have proved that, “For the Kropina change with an h-vector, the Cartan connection coefficients for both spaces $F^n$ and $F^n'$ are the same if and only if the h-vector $b_i$ is parallel with respect to the Cartan connection of $F^n$.” We here observe that the Kropina change has finite number of terms whereas, Matsumoto change has infinite number of terms, although in both cases (finite and infinite) same result holds.

*The goal for future study in this area is to identify a class of change with an h-vector $b_i$ is parallel, for which the Cartan connection of both the Finsler space are same.*

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