Littlewood-Paley decompositions and Besov spaces related to symmetric cones

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Abstract

Starting from a Whitney decomposition of a symmetric cone Ω, analog to the dyadic partition \([2^j, 2^{j+1})\) of the positive real line, in this paper we develop an adapted Littlewood-Paley theory for functions with spectrum in Ω. In particular, we define a natural class of Besov spaces of such functions, \(B^{p,q}_ν\), where the role of usual derivation is now played by the generalized wave operator of the cone \(\Delta(\frac{\partial}{\partial x})\). Our main result shows that \(B^{p,q}_ν\) consists precisely of the distributional boundary values of holomorphic functions in the Bergman space \(A^{p,q}_ν(T_Ω)\), at least in a “good range” of indices \(1 \leq q < q_{ν,p}\). We obtain the sharp \(q_{ν,p}\) when \(p \leq 2\), and conjecture a critical index for \(p > 2\). Moreover, we show the equivalence of this problem with the boundedness of Bergman projectors \(P_ν: L^{p,q}_ν \to A^{p,q}_ν\), for which our result implies a positive answer when \(q_{ν,p} < q < q_{ν,p}\). This extends to general cones previous work of the authors in the light-cone. Finally, we conclude the paper with a finer analysis in light-cones, for which we establish a link between our conjecture and the cone multiplier problem. Moreover, using recent work by Tao, Vargas and Wolff, we improve in dimension 3 the range of \(q\)'s for which the Bergman projection is bounded.

1 Introduction

Let \(Ω\) be an irreducible symmetric cone in a Euclidean vector space \(V\) of dimension \(n\), endowed with an inner product \((\cdot | \cdot)\) for which the cone \(Ω\) is self-dual. We can identify \(V\) with \(\mathbb{R}^n\), by endowing the latter with such inner product. We denote by \(T_Ω = V + iΩ\) the corresponding tube domain in the complexification of \(V\), which we

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may also identify with \( \mathbb{C}^n \). As in the text [13], we shall write the rank and determinant associated with a cone by

\[ r = \text{rank } \Omega, \quad \text{and} \quad \Delta(x) = \det x, \quad x \in V. \]

Two examples of the above situation are the light-cones and the cones of positive definite symmetric matrices. The first ones are defined in \( \mathbb{R}^n \), for \( n \geq 3 \), by

\[ \Lambda_n = \{ y = (y_1, y') \in \mathbb{R}^n : y_1^2 - |y'|^2 > 0, \ y_1 > 0 \}. \]

These are symmetric cones of rank 2 with determinant given by the Lorentz form \( \Delta(y) = y_1^2 - |y'|^2 \). The cones \( \text{Sym}_+(r, \mathbb{R}) \) of positive definite symmetric matrices, have rank \( r \) and the usual determinant for matrices. In this last case, the underlying vector space \( V \) is the space of symmetric matrices \( \text{Sym}(r, \mathbb{R}) \), with dimension \( n = \frac{r(r+1)}{2} \), and with a Euclidean norm defined by the Hilbert-Schmidt inner product. We observe that this does not coincide with the canonical inner product in the usual identification between \( V \) and \( \mathbb{R}^n \).

The goal of this paper is to present, in the general setting of symmetric cones, a special Littlewood-Paley decomposition adapted to the geometry of \( \Omega \). From this we shall obtain new results in analytic problems, such as the boundedness of Bergman projectors and the characterization of boundary values for Bergman spaces in the tube domain \( T_\Omega \). To describe our setting, we shall denote by \( \mathcal{S}_\Omega \) the space of Schwartz functions \( f \in \mathcal{S}(\mathbb{R}^n) \) with \( \text{Supp} \hat{f} \subset \overline{\Omega} \), and normalize the Fourier transform by

\[ \hat{f}(\xi) = \mathcal{F}f(\xi) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-i(x|\xi)} f(x) \, dx, \quad \xi \in \mathbb{R}^n. \]

Our key tool will be the following special decomposition for functions in \( \mathcal{S}_\Omega \)

\[ f = \sum_j f * \tilde{\psi}_j, \quad \forall \ f \in \mathcal{S}_\Omega, \quad (1.1) \]

where \( \tilde{\psi}_j \) are test functions supported on “frequency blocks” \( B_j \), and these form a suitable Whitney covering on the cone \( \Omega \). In analogy with the dyadic decomposition of the half-line \( (0, \infty) \) (i.e., the 1-dimensional cone), the sets \( B_j \) are constructed from the homogeneous structure of \( \Omega \), as the “balls” \( B_j = \{ \xi \in \Omega : d(\xi, \xi_j) < 1 \} \), obtained from a \( G \)-invariant distance \( d \) and a \( d \)-lattice \( \{ \xi_j \} \) in \( \Omega \). These will turn out to be the right sets for the discretization of many operators in the cone, since functions which appear in their multiplier expressions, such as \( \Delta(\xi) \) or \( (\xi|y) \) (for fixed \( y \in \Omega \)), remain essentially constant when \( \xi \in B_j \).

A characteristic example of this situation is the generalized wave operator on the cone: \( \Box = \Delta(\frac{1}{i} \frac{\partial}{\partial x}) \), which is the differential operator of degree \( r \) defined by the equality:

\[ \Delta(\frac{1}{i} \frac{\partial}{\partial x})[e^{i(x|\xi)}] = \Delta(\xi)e^{i(x|\xi)}, \quad \xi \in \mathbb{R}^n. \quad (1.2) \]
This corresponds, in cones of rank 1 and 2, to

\[ \Box = \frac{1}{i} \frac{d}{dx} \text{ in } (0, \infty), \quad \text{and} \quad \Box = -\frac{1}{4} \left( \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \cdots - \frac{\partial^2}{\partial x_n^2} \right) \text{ in } \Lambda_n. \]

The Littlewood-Paley decomposition (1.1) provides a formal “discretization” of the action of \( \Box \) on functions with spectrum in \( \Omega \):

\[ \Box f = \mathcal{F}^{-1}(\Delta(\xi) \hat{f}(\xi)) = \sum_j \Delta(\xi_j) f \ast \psi_j \ast \mathcal{F}^{-1}(m_j), \quad f \in \mathcal{S}_\Omega, \]

where \( \{m_j\} \) is a uniformly bounded family of multipliers.

From these facts it is natural to introduce a new family of Besov-type spaces, \( B_{\nu}^{p,q} \), adapted to the Littlewood-Paley decomposition (1.1). These are defined as the equivalence classes of tempered distributions, which have finite seminorms

\[ \|f\|_{B_{\nu}^{p,q}} = \left[ \sum_j \Delta^{-\nu}(\xi_j) \|f \ast \psi_j\|_p^q \right]^{\frac{1}{q}}. \quad (1.3) \]

Our first result shows that these spaces satisfy analogous properties to the one-dimensional homogeneous Besov spaces, with the role of usual derivation played by the wave operator \( \Box \). We warn the reader that, for convenience in the applications that follow, we are using a non-standard normalization of indices in our definition of \( \| \cdot \|_{B_{\nu}^{p,q}} \) (compared, e.g., with [14]).

**Theorem 1.4** Let \( \nu \in \mathbb{R} \) and \( 1 \leq p, q < \infty \). Then

1. \( B_{\nu}^{p,q} \) is a Banach space and does not depend on the choice of \( \{\psi_j\} \) and \( \{\xi_j\} \).
2. \( \Box : B_{\nu}^{p,q} \to B_{\nu+q}^{p,q} \) is an isomorphism of Banach spaces.
3. If \( p, q > 1 \), then \( (B_{\nu}^{p,q})^* \) is isomorphic to \( B_{-\nu/q}^{p'/q'} \) with the usual duality pairing.

The rest of the paper is devoted to applications of this theory to two open problems involving the class of Bergman spaces. In this paper, a weighted mixed-norm version of these spaces is defined by the integrability condition:

\[ \|F\|_{L_{\nu}^{p,q}} := \left[ \int_{\Omega} \left( \int_{\mathbb{R}^n} |F(x + iy)|^p dx \right)^{\frac{2}{p}} \Delta^{\nu - \frac{n}{q}}(y) dy \right]^{\frac{1}{q}} < \infty. \quad (1.5) \]

Thus, when \( 1 \leq p, q < \infty \) and \( \nu \in \mathbb{R} \), we denote by \( A_{\nu}^{p,q}(T_\Omega) \) the closed subspace of \( L_{\nu}^{p,q} \) consisting of holomorphic functions in the tube \( T_\Omega \). We observe that these spaces are non null only when \( \nu > \frac{n}{r} - 1 \) (see, e.g., [3]). The usual \( A^p \) space corresponds to \( p = q \) and \( \nu = \frac{n}{r} \). To simplify notation we shall write \( A_{\nu}^{p} = A_{\nu}^{p,p} \), and similarly \( L_{\nu}^{p} = L_{\nu}^{p,p} \).

Two main questions concerning these spaces will be studied here:
1. The characterization of boundary values of functions in \( A^{p,q}_\nu \), as distributions in the Besov spaces \( B^{p,q}_\nu \).

2. The existence of bounded extensions into \( L^{p,q}_\nu \) spaces for the (weighted) Bergman projector, that is, the orthogonal projector \( P_\nu: L^2_\nu \to A^2_\nu \).

Regarding the first question, it has been known for some decades the relation, in the 1-dimensional setting, between boundary values of Bergman functions and homogeneous Besov spaces (see e.g. [18, 10], or the lecture notes [3]). To see this in higher dimensions, and restricted to tube domains over cones \( T_\Omega \), one writes a holomorphic function \( F \in A^{p,q}_\nu \) in terms of its Fourier-Laplace transform:

\[
F(z) = \mathcal{L}g(z) = \int_{\Omega} e^{i(z|\xi)} g(\xi) \, d\xi, \quad z \in T_\Omega, \tag{1.6}
\]

for some distribution \( g \) supported in \( \overline{\Omega} \). Observe that the new distribution \( f = F^{-1} g \) plays the role of a “Shilov boundary value” for \( F \), and hence it is a natural candidate to belong to \( B^{p,q}_\nu \). Now, by Theorem 1.4 this is equivalent to \( F^{-1}(e^{-(y^1)} \chi_\Omega) \) having a finite \( B^{p,q}_{\nu,r} \) norm, which by explicit computation can only happen when \( q \) is below a certain critical index

\[
\tilde{q}_{\nu,p} = \frac{\nu + \frac{n}{r} - 1}{(\frac{n}{r'} - 1)_+}
\]

(with \( \tilde{q}_{\nu,p} = \infty \), if \( \frac{n}{r} \leq p' \)). This constitutes our first result, whose detailed justification will be presented in sections 3.4 and 4.1.

**Theorem 1.7** Let \( \nu > \frac{n}{r} - 1 \), \( 1 \leq p < \infty \) and \( 1 \leq q < \tilde{q}_{\nu,p} \). Then, for every \( F \in A^{p,q}_\nu \) there exists a (unique) tempered distribution \( f \in \mathcal{S}'(\mathbb{R}^n) \) such that \( f = \sum_j f \ast \psi_j \) in \( \mathcal{S}'(\mathbb{R}^n) \), \( \|f\|_{B^{p,q}_\nu} < \infty \) and \( F = \mathcal{L} f \). Moreover we have

1. \( \lim_{y \to 0} F(\cdot + iy) = f, \) both in \( \mathcal{S}'(\mathbb{R}^n) \) and \( B^{p,q}_\nu \);

2. \( \|f\|_{B^{p,q}_\nu} \leq C \|F\|_{A^{p,q}_\nu}, \) for all \( F \in A^{p,q}_\nu \).

The converse result is more interesting, and turns out to be equivalent to the second of the questions posed above. We only have a partial answer, for which we need to introduce two new critical indices

\[
q_\nu = \frac{\nu + \frac{n}{r} - 1}{\frac{n}{r} - 1}, \quad q_{\nu,p} = \min\{p, p'\} q_\nu.
\]

Observe that in 1-dimension the three indices are equal to \( \infty \), while in general we have the ordering

\[ 2 < q_\nu \leq q_{\nu,p} \leq \tilde{q}_{\nu,p} \leq \tilde{q}_{\nu,p}. \]

The role of these new indices will be clarified later in relation with the Bergman projectors.
\textbf{Theorem 1.8} Let $\nu > \frac{n}{r} - 1$, $1 \leq p < \infty$ and $1 \leq q < q_{\nu,p}$. Given a distribution $f \in S'(\mathbb{R}^n)$ such that $f = \sum_j f \ast \psi_j$ and $\|f\|_{B^p,q} < \infty$, then the holomorphic function $F = \mathcal{L}\hat{f}$ belongs to $A^p_{\nu,q}$, and moreover, there exists a constant $C > 0$ so that

$$\frac{1}{C} \|f\|_{B^p,q} \leq \|\mathcal{L}\hat{f}\|_{A^p_{\nu,q}} \leq C \|f\|_{B^p,q}, \quad f \in B^p_{\nu,q}.$$ 

This theorem is sharp for $1 \leq p \leq 2$, in the sense that for each $q \geq q_{\nu,p} = pq_{\nu}$ there is a distribution with $\|f\|_{B^p,q} < \infty$ and $\|\mathcal{L}\hat{f}\|_{L^p,q} = \infty$. We shall present these examples in §4.4. When $p > 2$ we will construct similar examples, but only for values of $q \geq q_{\nu,2} = 2q_{\nu}$, leaving open the question when $p'q_{\nu} \leq q < \min\{2q_{\nu}, \tilde{q}_{\nu,p}\}$. New positive results in dimension 3 can be obtained using restriction theorems, and will be presented in the Appendix.

Finally, we turn to the second application of our theory, the boundedness of Bergman projectors in $L^p_{\nu,q}$. This is a challenging question which has been open for many years, and which still is not completely solved. The three indices defined above correspond to three steps of difficulty for this question. For instance, a trivial counterexample shows that $P_{\nu}$ can only be bounded in $L^p_{\nu,q}$ for $q_{\nu,p} < q < q_{\nu,p}$. This follows from the fact that the Bergman kernel belongs to $L^{p',q'}(T_{\Omega})$ only when $q < \tilde{q}_{\nu,p}$ (see §4.3 below). From the other two indices, the smallest one gives the natural range $q_{\nu} < q < q_{\nu}$ for boundedness of the positive operator $P_{\nu}^+$, obtained by replacing the Bergman kernel with its absolute value $|B_{\nu}(z,w)|$ [1, 6]. Finally, $q_{\nu,p}$ appears as the interpolation index between $q_{\nu}$ and $q_{\nu,2} = 2q_{\nu}$, giving the latter the sharp range of boundedness in the spaces $L^2_{\nu,q}$ [5]. The results in this last paper, which combine the Plancherel identity with a suitable “discretization of multipliers” in light-cones, have been the germ of the Littlewood-Paley decomposition we are introducing here. Our main contribution to this problem is, besides an extension to general symmetric cones, a direct formulation in terms of Littlewood-Paley inequalities, which allows further improvements as those considered in the Appendix. We gather these results in our next theorem, which may be stated only for $q \geq 2$ by self-adjointness of $P_{\nu}$.

\textbf{Theorem 1.9} Let $\nu > \frac{n}{r} - 1$, $1 \leq p < \infty$ and $2 \leq q < q_{\nu,p}$. Then, the inequality

$$\|\mathcal{L}\hat{f}\|_{L^p,q} \leq C \|f\|_{B^p,q}, \quad f \in S_{\Omega},$$

holds true if and only if $P_{\nu}$ can be boundedly extended from $L^p_{\nu,q}$ onto $A^p_{\nu,q}$. In particular, $P_{\nu}$ is bounded in $L^p_{\nu,q}$ for all $1 \leq p < \infty$ and $q_{\nu,p} < q < q_{\nu,p}$. Moreover, $P_{\nu}$ does not admit bounded extensions to $L^p_{\nu,q}$ when:

1. $1 \leq p \leq 2$ and $q \geq q_{\nu,p};$

2. $2 < p < \infty$ and $q \geq \min\{2q_{\nu}, \tilde{q}_{\nu,p}\}$. 

versions of it: in light-cones, as well as their analogs in spheres. In particular, square-function counterexamples compared with [5], first by considering the limit case s when 1 ≤ with the corresponding dual interval. We point out that our results produce new boundedness of Bergman projectors to very challenging questions related to inequalities like theorems in the 3-dimensional light-cone. Morally, this reduces the problem of boundedness to very challenging questions related to inequalities like

\[ \left( \sum_j \|f * \psi_j\|_p^{s/p} \right)^{1/s} \leq C \|f\|_p, \quad f \in \mathcal{S}(\mathbb{R}^n) \]  

for indices 2 ≤ s ≤ p. The inequality holds trivially for s = p (by interpolation between s = ∞ and s = p = 2), while any improvement in s smaller than p will directly imply boundedness of the Bergman projection outside the hexagonal region. In particular, going down to s = 2 will fill the gap in Figure 1.1 up the vertical line q = 2q_v. Variants of such inequalities (typically with s = 2) have been widely studied in light-cones, as well as their analogs in spheres. In particular, square-function versions of it:

\[ \| \sum_j f \ast \psi_j \|_{p'} \leq C \left( \sum_j |f \ast \psi_j|^2 \right)^{1/2_{p'}}, \quad f \in \mathcal{S}_\Omega, \]  

are intimately related with the cone multiplier problem and the Bochner-Riesz means. We shall show in the Appendix how such results lead to a slight improvement of our theorems in the 3-dimensional light-cone. Morally, this reduces the problem of boundedness of Bergman projectors to very challenging questions related to inequalities like

Figure 1.1: Region of boundedness of \( P_\nu \).

Figure 1.1 illustrates the regions of boundedness, unboundedness, and the open gap where for the moment no answer is known (compare with [5]). In the particular case of \( L^p_\nu \)-spaces (\( p = q \)) the gap becomes 1 + q_v ≤ p < \( \min\{2q_v, q_v + \frac{n}{n-1}\} \), together with the corresponding dual interval. We point out that our results produce new counterexamples compared with [5], first by considering the limit cases when 1 ≤ p ≤ 2 and q ≥ 2, and second by removing the region with q ≥ q_v,2 when \( p > 2 \). The counterexamples presented here are besides valid for general symmetric cones.

We do not wish to conclude this introductory section without mentioning our approach to the open question when \( p, q > 2 \). Our proofs provide sufficient conditions which are variants of the elementary inequality

\[ \left( \sum_j \|f \ast \psi_j\|_{p'}^{s/p} \right)^{1/s} \leq C \|f\|_p, \quad f \in \mathcal{S}(\mathbb{R}^n) \]  

for indices 2 ≤ s ≤ p. The inequality holds trivially for s = p (by interpolation between s = ∞ and s = p = 2), while any improvement in s smaller than p will directly imply boundedness of the Bergman projection outside the hexagonal region. In particular, going down to s = 2 will fill the gap in Figure 1.1 up the vertical line q = 2q_v. Variants of such inequalities (typically with s = 2) have been widely studied in light-cones, as well as their analogs in spheres. In particular, square-function versions of it:

\[ \| \sum_j f \ast \psi_j \|_{p'} \leq C \left( \sum_j |f \ast \psi_j|^2 \right)^{1/2_{p'}}, \quad f \in \mathcal{S}_\Omega, \]  

are intimately related with the cone multiplier problem and the Bochner-Riesz means. We shall show in the Appendix how such results lead to a slight improvement of our theorems in the 3-dimensional light-cone. Morally, this reduces the problem of boundedness of Bergman projectors to very challenging questions related to inequalities like
(1.10) or (1.11), where the complex analysis formalism has been completely removed.

Finally, we conclude by mentioning that a simplified version of our results, specialized to the case when $p = 2$ (for which the use of Plancherel is available), has been published separately in [4]. We also refer to the survey paper [7] for complementary information concerning the critical indices and the relation with Hardy-type inequalities.

2 Whitney decompositions on the cone

In this section we introduce the notation and a list of technical results on symmetric cones, mostly taken from the text [13]. We also give a detailed construction of the “Whitney decomposition” adapted to the analysis of the problems stated above. The main lines and applications of such constructions appear in previous papers: see [5] for the light-cone, and [4] for general symmetric cones.

2.1 Background on symmetric cones

Let $\Omega$ be a fixed symmetric cone in a real Euclidean vector space $V$, endowed with the inner product $(\cdot | \cdot)$. That is, $\Omega$ is a homogeneous open convex cone which is self-adjoint with respect to $(\cdot | \cdot)$. Let $G(\Omega)$ be the group of linear transformations of the cone, and $G$ its identity component. By definition, $G(\Omega)$ acts transitively on $\Omega$. Further, it is well known that there is a solvable subgroup $T$ of $G$ acting simply transitively on $\Omega$. That is, every $y \in \Omega$ can be written uniquely as $y = te$, with $t \in T$ and a fixed $e \in \Omega$. This gives an identification $\Omega \equiv T = G/K$, where $K$ is a maximal compact subgroup of $G$. Moreover, $K = \{ g \in G : ge = e \} = G \cap O(V)$. All these properties can be found in the first chapter of the text [13].

It is well-known that for every symmetric cone $\Omega$, its underlying vector space $V$ can be endowed with a multiplication rule which makes it a Euclidean Jordan algebra with identity element $e$. With such multiplication, $\Omega$ coincides with the set $\{ x^2 : x \in V \}$ of all squares in $V$. We may assume that the inner product in $V$ is given by $(x|y) = (xy|e) = \text{tr}(xy)$ (see [13, Ch. III]). The reader less familiar with these concepts can think on the example of positive-definite symmetric matrices, which we present in 2.2 below.

Suppose now that the cone is irreducible, has rank $r$ and its underlying space has dimension $n$. A precise form for the group $T$ can be obtained from the Jordan algebra structure of $V$. Following [13, Ch. VI], we let $\{ c_1, \ldots, c_r \}$ denote a fixed Jordan frame in $V$, and $V = \bigoplus_{1 \leq i \leq j \leq r} V_{i,j}$ its associated Peirce decomposition. Then $T$ may be taken as the corresponding solvable Lie group, which factors as the semidirect product $T = NA = AN$ of a nilpotent subgroup $N$ (of lower triangular matrices),
and an abelian subgroup $A$ (of diagonal matrices). The latter takes the explicit form

$$A = \{ P(a) : a = \sum_{i=1}^{r} a_i c_i, \ a_i > 0 \},$$

where $P$ is the quadratic representation of $V$. This leads as well to the classical decompositions of the semisimple Lie group $G = NAK$ and $G = KAK$.

Still following [13, Ch. VI], we shall denote by $\Delta_1(x), \ldots, \Delta_r(x)$ the principal minors of $x \in V$, with respect to the fixed Jordan frame $\{c_1, \ldots, c_r\}$. These are invariant functions under the group $N$:

$$\Delta_k(nx) = \Delta_k(x), \quad n \in N, \quad x \in V, \quad k = 1, \ldots, r,$$

and satisfy a homogeneity relation under $A$

$$\Delta_k(P(a)x) = a_1^2 \cdots a_k^2 \Delta_k(x), \quad \text{if} \quad a = a_1 c_1 + \ldots + a_r c_r.$$

The determinant function $\Delta(y) = \Delta_r(y)$ is also invariant under $K$, and moreover, satisfies the formula

$$\Delta(g y) = \Delta(g e) \Delta(y) = \text{Det}(g)^{-r} \Delta(y). \quad (2.1)$$

It follows from this formula that an invariant measure in $\Omega$ is given by $\Delta(y)^{-3/2} \, dy$.

Finally, we recall a version of Sylvester’s Theorem for symmetric cones, which allows to write these as:

$$\Omega = \{ x \in V : \Delta_k(x) > 0, \ k = 1, \ldots, r \}.$$

**Example 2.2 The cone of positive-definite symmetric matrices**

We describe the above concepts for the cone $\Omega = \text{Sym}_+(r, \mathbb{R})$, contained in the vector space $V = \text{Sym}(r, \mathbb{R})$. The Jordan algebra structure in $V$ corresponds to the symmetric product $X \circ Y = \frac{1}{2}(XY + YX)$, with the usual identity matrix $e = I$.

A standard Jordan frame is the set $D_j$ of diagonal matrices all of whose entries are 0 except for the $j$-th one equal to 1. The Peirce decomposition in $V$ is just the decomposition of a symmetric matrix in terms its $(i, j)$ entries.

In this example, the automorphism group $G(\Omega)$ can be identified with $\text{Gl}(r, \mathbb{R})$ via the adjoint action:

$$g \in \text{Gl}(r, \mathbb{R}), \ Y \in \text{Sym}(r, \mathbb{R}) \mapsto g \cdot Y = gYg^* \in \text{Sym}(r, \mathbb{R}). \quad (2.3)$$

Then, the group $T$ consists in the lower triangular matrices in $\text{Gl}(r, \mathbb{R})$, and the factorization $Y = t \cdot I$ is precisely the Gauss decomposition of a symmetric matrix. The subgroup $N$ consists of all triangular matrices in $\text{Gl}(r, \mathbb{R})$ with 1’s on the diagonal, while $A$ is given by the diagonal matrices $P(a) = \text{diag} \{ a_1, \ldots, a_r \}$. Finally, the associated principal minors are the usual principal minors from linear algebra, that is, the determinants of the $k \times k$ symmetric matrices obtained by restriction to the first $k$ coordinates. One verifies easily with this example the homogeneity properties with respect to $N$ and $A$ stated above.
2.2 The invariant metric and the covering lemma

With the identification $\Omega \equiv G/K$, the cone can be regarded as a Riemannian manifold with the $G$-invariant metric defined by

$$\langle \xi, \eta \rangle_y := (t^{-1}\xi|t^{-1}\eta)$$

if $y = te$ and $\xi, \eta$ are tangent vectors at $y \in \Omega$. We shall denote by $d(\cdot, \cdot)$ the corresponding distance, and by $B_\delta(\xi)$ the ball centered at $\xi$ of radius $\delta$. Note that, for each $g \in G$, the invariance implies $B_\delta(g\xi) = gB_\delta(\xi)$.

We shall need some weak local invariance properties of the quantities that we have defined on the cone. One consequence is the possibility of obtaining a Whitney-type decomposition for general symmetric cones in terms of invariant balls. Part of this material was already presented in [4].

**Lemma 2.4** Let $\delta > 0$. Then there is a constant $\gamma = \gamma(\delta, \Omega) > 0$ such that

$$d(\xi, \xi') \leq \delta \Rightarrow \frac{1}{\gamma} \leq \frac{\Delta_k(\xi)}{\Delta_k(\xi')} \leq \gamma, \quad k = 1, \ldots, r.$$

**Proof:** By invariance of the metric and the forms $\Delta_k$ under $N$, we may assume $\xi' = P(a)e$. Further, since

$$\frac{\Delta_k(\xi)}{\Delta_k(P(a)e)} = \frac{\Delta_k(P(a)^{-1}\xi)}{\Delta_k(e)},$$

we may even assume $\xi' = e$. Now, the estimations above and below for $\Delta_k$ in a ball $B_\delta(e)$ follow easily from the continuity of $\xi \mapsto \Delta_k(\xi)$, and a compactness argument. 

The next lemma states the local equivalence between two Riemannian metrics. The proof follows from standard arguments (see, e.g., [19, 9–22]).

**Lemma 2.5** Let $\delta_0 > 0$ be fixed. Then, there exist two constants $\eta_1 > \eta_0 > 0$, depending only on $\delta_0$ and $\Omega$, so that for every $0 < \delta \leq \delta_0$ we have

$$\{||\xi - e| < \eta_0\delta\} \subset B_\delta(e) \subset \{||\xi - e| < \eta_1\delta\}.$$

We can now estimate the volume of an invariant ball. Recall that the invariant measure in $\Omega$ is given by

$$\text{meas} \ (B) = \int_B \Delta(\xi)^{-\frac{n}{2}} \ d\xi, \quad B \subset \Omega \ \text{measurable}.$$

Therefore, from the previous results it follows that, for all $y \in \Omega$ and $0 < \delta \leq \delta_0$,

$$\text{meas} \ (B_\delta(y)) = \text{meas} \ (B_\delta(e)) \sim \text{Vol} \ (B_\delta(e)) \sim \delta^n,$$

where the equivalences denoted by “$\sim$” are modulo constants depending only on $\Omega$ and the fixed number $\delta_0$. Observe, however, that this estimate cannot hold uniformly in $\delta_0 \gg 1$, since the invariant measure is in general not doubling. We can now prove a covering lemma which will be of crucial importance for the rest of the paper.
**Lemma 2.6**: Whitney Decomposition of the cone. Let $\delta > 0$ and $R \geq 2$. Then, there exist sequences of points $\{\xi_j\}_j$ in $\Omega$ such that

(i) $\{B_\delta(\xi_j)\}_j$ is a disjoint family in $\Omega$;

(ii) $\{B_{R\delta}(\xi_j)\}_j$ is a covering of the cone $\Omega$.

Moreover, for each such sequence the balls $\{B_{R\delta}(\xi_j)\}_j$ have the finite intersection property. That is, if $\delta, R \leq R_0$, then there exists an integer $N = N(R_0, \Omega)$ so that at most $N$ of these balls can intersect an arbitrary set $E \subset \Omega$ with diameter

$$\text{diam} (E) = \sup\{d(\xi, \eta) : \xi, \eta \in E\} \leq R_0\delta.$$  \hspace{1cm} (2.7)

**Proof**: Consider $\{\xi_j\}_j$ a maximal subset of $\Omega$ (under inclusion) among those with the property that their elements are distant at least $2\delta$ from one another. Let us denote $B'_j$ the balls $B_\delta(\xi_j)$. They are pairwise disjoint, while, by maximality, the balls $\{B_j = B_{2\delta}(\xi_j)\}_j$ cover $\Omega$. Note also that, necessarily, the set $\{\xi_j\}_j$ is countable.

For the finite overlapping, let $E$ be a set as in (2.7). Denote by $J$ the set of indices $\{j : B_{R\delta}(\xi_j) \cap E \neq \emptyset\}$, and fix a point $\xi \in B_{R\delta}(\xi_{j_0}) \cap E$ for some $j_0 \in J$. Then, the condition on the diameter gives

$$\bigcup_{j \in J} B_\delta(\xi_j) \subset B_{(2R_0+1)\delta}(\xi).$$

Now, by disjointness and invariance of the measure we have

$$|J| \text{ meas } (B_\delta(e)) = \text{ meas } (\bigcup_{j \in J} B'_j) \leq \text{ meas } (B_{(2R_0+1)\delta}(\xi)) = \text{ meas } (B_{(2R_0+1)\delta}(e)).$$

Thus, the remarks preceding the lemma give us a bound for $N$ depending only on $\Omega$ and $R_0$.

\[\square\]

**Remark 2.8** 1. A sequence of points $\{\xi_j\}_j$ with the above properties will be called a $(\delta, R)$-lattice of the cone. Observe that one can always define an associated partition by letting

$$E_1 = B_1, \ldots, E_j = B_j \setminus E_{j-1}, \ldots$$

We shall call $\{E_j\}_j$ a Whitney decomposition of $\Omega$.

2. If $\{\xi_j\}_j$ is a $(\delta, R)$-lattice, then so is $\{\xi_j^{-1}\}_j$. Indeed, this follows from the fact that $y \mapsto y^{-1}$ is an isometry of the cone (see Chapter III of [13]). Therefore, $B_\delta(\xi_j^{-1}) = B_\delta(\xi_j)^{-1}$, and the conditions of Lemma 2.6 hold.

3. One can look at the sequences $\{\xi_j\}_j$ and $\{\xi_j^{-1}\}_j$ as a couple of dual lattices. In fact, $(\xi_j|\xi_j^{-1}) = r$, while using $\Delta(y^{-1}) = \Delta(y)^{-1}$ we also have Vol $(B_j) \sim \Delta(\xi_j)^{-\frac{\gamma}{r}}$ and Vol $(B_j^{-1}) \sim \Delta(\xi_j)^{-\frac{\gamma}{r}}$. Moreover, from the next lemma it will follow that actually $(\xi_j|y) \sim 1$ when $\xi \in B_j$ and $y \in B_j^{-1}$.
Lemma 2.9 Let $\delta > 0$. There exists $\gamma = \gamma(\Omega, \delta) > 0$ such that, for $y \in \overline{\Omega}$ and $\xi, \xi' \in \Omega$ with $d(\xi, \xi') \leq \delta$, then

$$\frac{1}{\gamma} \leq \frac{(\xi|y)}{(\xi'|y)} \leq \gamma.$$  

(2.10)

In particular, $\frac{1}{\gamma} \leq |\xi|/|\xi'| \leq \gamma$, when $d(\xi, \xi') \leq \delta$.

Proof: By continuity it suffices to show (2.10) for $y \in \Omega$. Using invariance under $G$ (and the fact that $G = G^*$), we may assume that $y = e$. To show that $(\xi'|e) \leq \gamma(\xi|e)$, let us write $\xi = kP(a)e$, for $k \in K$ and $a = a_1c_1 + \ldots + a_rc_r$. Then the new vector $\xi'' = P(a)^{-1}k^{-1}\xi$ belongs to the fixed ball $B_\delta(e)$. Therefore, we have

$$(\xi'|e) = (P(a)\xi''|e) \leq \sqrt{r}\|P(a)\|\|\xi''\| \leq \gamma\|P(a)\|,$$

where the last bound appears because $B_\delta(e)$ is a compact set. Now $P(a)$ has eigenvalues $a_i^2$ and $a_ia_j$, and hence

$$\|P(a)\| = \max\{a_i^2, a_ia_j\} \leq \sum_{i=1}^{r} a_i^2 = (P(a)e|e) = (\xi|e).$$  

(2.11)

Finally, let us remark that $(\xi|e)$ is equivalent to $|\xi|$. Indeed, $(\xi|e) \leq \sqrt{r}|\xi|$ by Schwarz inequality. Conversely, for $\xi = P(a)e$, we have

$$|\xi| = |\sum_{j=1}^{r} a_j^2c_j| \leq \sum_{j=1}^{r} a_j^2 = (\xi|e).$$

Lemma 2.12 For every $g \in G$ we have

$$\|g\| \leq |ge| \leq \sqrt{r}\|g\|.$$

Proof: Write $g = kP(a)h$, for some $h, k \in K$ and $a = a_1c_1 + \ldots + a_rc_r$. Then, as in (2.11)

$$\|P(a)\| \leq (a_1^4 + \ldots + a_r^4)^{\frac{1}{2}} = |P(a)e| = |ge|.$$

Thus,

$$\frac{|ge|}{|e|} \leq \|g\| = \|P(a)\| \leq |ge|.$$  

\[\square\]
2.3 Integrals on $\Omega$

To conclude with this preliminary section, we list basic some facts concerning integrals in the cone. Following [13], we define the generalized power function in $\Omega$ by

$$\Delta_s(x) = \Delta_{s_1}^{s_1} \Delta_{s_2}^{s_2} \cdots \Delta_{s_r}^{s_r}(x), \quad s = (s_1, s_2, \ldots, s_r) \in \mathbb{C}^r, \quad x \in \Omega,$$

where $\Delta_k$ are the principal minors with respect to a fixed Jordan frame $\{c_1, \ldots, c_r\}$. In particular, $\Delta_s(x) = a_1^{s_1} \cdots a_r^{s_r}$ when $x = a_1 c_1 + \cdots + a_r c_r$. The lemmas from the previous section justify the following discretization of integrals which we shall use often below.

**Proposition 2.13** Let $0 < \delta, R \leq R_0$ be fixed, and $\{\xi_j\}_j$ be a $(\delta, R)$-lattice with associated Whitney decomposition $\{E_j\}_j$. Then, for every $s \in \mathbb{R}^r$ there exists a positive constant $C$ such that, for any $y \in \Omega$ and for any non-negative function $f$ on the cone, we have

$$\frac{1}{C} \sum_j e^{-\gamma(y|\xi_j)} \Delta_s(\xi_j) \int_{E_j} f(\xi) \frac{d\xi}{\Delta(\xi)^{\frac{n}{r}}},$$

$$\leq \int_{\Omega} f(\xi) e^{-\gamma(y|\xi)} \Delta_s(\xi) \frac{d\xi}{\Delta(\xi)^{\frac{n}{r}}} \leq C \sum_j e^{-\frac{1}{\gamma}(y|\xi_j)} \Delta_s(\xi_j) \int_{E_j} f(\xi) \frac{d\xi}{\Delta(\xi)^{\frac{n}{r}}}.$$

where $\gamma = \gamma(R_0, \Omega)$ is a constant as in (2.10).

We shall also need the gamma function in $\Omega$ defined from the generalized powers. That is, given $s = (s_1, s_2, \ldots, s_r) \in \mathbb{C}^r$, one lets

$$\Gamma_{\Omega}(s) = \int_{\Omega} e^{-\langle \xi, e \rangle} \Delta_s(\xi) \frac{d\xi}{\Delta(\xi)^{\frac{n}{r}}}.$$  \hfill (2.14)

This integral is known to converge absolutely if and only if $\Re s_j > (j-1) \frac{n/r-1}{r-1}$, for all $j = 1, \ldots, r$. Moreover, in such case

$$\Gamma_{\Omega}(s) = (2\pi)^{\frac{n}{2r}} \prod_{j=1}^r \Gamma(s_j - (j-1) \frac{n/r-1}{r-1}),$$  \hfill (2.15)

where $\Gamma$ is the classical gamma function in $\mathbb{R}^+$ [13, Ch. VII]. We shall denote $\Gamma_{\Omega}(s) = \Gamma_{\Omega}(s)$ when $s = (s, \ldots, s)$. The next formula defines the Laplace transform of a generalized power, and can be found in [13, p. 124].

**Lemma 2.16** For $y \in \Omega$ and $s = (s_1, s_2, \ldots, s_r) \in \mathbb{C}^r$ with $\Re s_j > (j-1) \frac{n/r-1}{r-1}$, $j = 1, \ldots, r$, then

$$\int_{\Omega} e^{-\langle \xi, y \rangle} \Delta_s(\xi) \frac{d\xi}{\Delta(\xi)^{\frac{n}{r}}} = \Gamma_{\Omega}(s) \Delta_s(y^{-1}).$$
**Remark 2.17** We will sometimes write the above quantity \(\Delta_s(y^{-1})\) in terms of the rotated Jordan frame \(\{c_r, \ldots, c_1\}\). That is, if we denote by \(\Delta_j^*\), \(j = 1, \ldots, r\), the principal minors with respect to this new frame, then

\[
\Delta_s(y^{-1}) = [\Delta_s^*(y)]^{-1}, \quad \forall \ s = (s_1, \ldots, s_r) \in \mathbb{C}^r,
\]

where we have set \(s^* := (s_r, \ldots, s_1)\) (see [13, p. 127]).

Our last lemmas have to do with global and local integrability of generalized powers. The first one is a simple consequence of our last result and the Plancherel formula (see also [5]).

**Lemma 2.18** Let \(\alpha \in \mathbb{R}\), and define

\[
I_\alpha(y) = \int_{\mathbb{R}^n} |\Delta(x + iy)|^{-\alpha} \, dx, \quad y \in \Omega.
\]

Then, \(I_\alpha\) is finite if and only if \(\alpha > \frac{2n}{r} - 1\). In this case, \(I_\alpha(y) = c(\alpha) \Delta(y)^{-\alpha + \frac{2n}{r}}\).

We next establish the critical index for local integrability at the origin.

**Lemma 2.19** Let \(\alpha \in \mathbb{R}\) and \(g_\alpha(\xi) = \frac{e^{-\langle\xi,\xi\rangle}}{\Delta(\xi(1 + |\log \Delta(\xi)|)^\beta)}\). Then, \(g_\alpha\) is integrable if and only if \(\alpha > 1\).

**Proof:** This is a simple exercise using Gaussian coordinates (see Chapter VI of [13]). Indeed, with the notation in [13], the integral of \(g_\alpha\) is equal to

\[
c_r \int_{(0,\infty)^r} \frac{e^{-\sum u_j^2}}{(1 + 2|\log u_r|)^\alpha} \left[ \prod_{j=1}^{r-1} u_j^{(r-j)d-1} \right] \, du_1 \cdots du_r = c'_r \int_0^\infty \frac{e^{-u_r^2}}{(1 + 2|\log u_r|)^\alpha} \, du_r. \quad \Box
\]

Finally, we conclude with the critical index for integrability at infinity.

**Lemma 2.20** Let \(\alpha, \delta \in \mathbb{R}\), \(\beta > -1\) and \(g_{\alpha,\beta,\delta}(y) = \frac{\Delta^\beta(y)}{\Delta^\alpha(y + e^{(1+\beta)\Delta(y+e)})}\). Then, \(g_{\alpha,\beta,\delta}\) is integrable if and only if \(\alpha - \beta > \frac{2n}{r} - 1\) or \(\alpha - \beta = \frac{2n}{r} - 1\) and \(\delta > 1\).

**Proof:** This time we use the “polar coordinates” of the cone

\[
y = k(e^{t_1}c_1 + \ldots + e^{t_r}c_r), \quad t_1 < t_2 < \ldots < t_r, \quad k \in K
\]

(see [13, pag. 105]). Then, \(\Delta(y + e) = \prod_{j=1}^{r}(e^{t_j} + 1)\), and

\[
\int_{\Omega} g_{\alpha,\beta,\delta}(y) \, dy = c \int_{-\infty}^\infty \int_{-\infty}^{t_r} \cdots \int_{-\infty}^{t_1} \frac{e^{(t_1 + \ldots + t_r)(\frac{n}{r} + \beta)}}{\prod_{j=1}^{r}(e^{t_j} + 1)^\alpha (1 + \sum_{j=1}^{r} \log (1 + e^{t_j}))^\delta} \, dt_1 \cdots dt_r,
\]
where \( d = \dim V_{j,k} = 2(\frac{n}{r} - 1)/(r - 1) \). For the necessary condition, we can consider only the case when \( \alpha - \beta = 2\frac{n}{r} - 1 \) and \( \delta = 1 \). Moreover, we restrict the region of integration so that \( \mathrm{sh} \ t \geq ce^t \), and obtain

\[
I \geq c \int_2^\infty \int_2^{2r-1} \cdots \int_2 \frac{e^{(t_1+\cdots+t_r)(1-\frac{\beta}{2})}}{1+t_r} \prod_{j<k} e^{\frac{d}{2}(t_k-t_j)} \ dt_1 \cdots dt_r
\]

\[
\geq c' \int_2^\infty e^{\frac{r}{2}(1-\frac{\beta}{2})} \frac{e^{d(r-1)t_r}}{1+t_r} \ dt_r = \infty.
\]

To estimate from above, we use the bound

\[
\prod_{j<k} \mathrm{sh} \ \left( \frac{t_k-t_j}{2} \right) \leq \prod_{j<k} e^{\frac{(t_k-t_j)}{2}} = \prod_{j=1}^r e^{(j-1-\frac{r-1}{2})t_j}.
\]

Then the integral \( I \) is bounded by the product

\[
\prod_{j=1}^{r-1} \int_{-\infty}^{+\infty} e^{d(j-1)+\beta+1)t_j} \frac{dt_j \times \int_{-\infty}^{+\infty} e^{d(r-1)+\beta+1)t_r} \frac{dt_r}{(1+e^{t_r})^\alpha (1+\log(1+e^{t_r}))^\beta dt_r}.
\]

Each integral is convergent at \( -\infty \) since \( \beta + 1 > 0 \). We use the conditions on \( \alpha \), \( \beta \) and \( \delta \) to conclude easily for the integrability at \( +\infty \).

\[
3 \text{ Besov spaces with spectrum in } \Omega
\]

3.1 The Littlewood-Paley decomposition

Through the rest of the paper, \( \{\xi_j\} \) will be a fixed \( (\delta, R) \)-lattice in \( \Omega \) with \( \delta = \frac{1}{2} \) and \( R = 2 \). We can easily construct a smooth partition of the unity associated with the covering \( B_j = B_1(\xi_j) \). For this, we choose a real function \( \varphi_0 \in C_c^\infty(B_2(e)) \) such that

\[
0 \leq \varphi_0 \leq 1, \quad \text{and} \quad \varphi_0|_{B_1(e)} \equiv 1.
\]

We write each point \( \xi_j = g_j e \), for some fixed \( g_j \in G \) (which, for simplicity, we take self-adjoint). Then, we can define \( \varphi_j(\xi) := \varphi_0(g_j^{-1}\xi) \), so that

\[
\varphi_j \in C_c^\infty(B_2(\xi_j)), \quad 0 \leq \varphi_j \leq 1 \quad \text{and} \quad \varphi_j|_{B_1} \equiv 1. \quad (3.1)
\]

We assume that \( \xi_0 = e \), so that there is no ambiguity of notations. By the finite intersection property, there exists a constant \( c > 0 \) such that

\[
\frac{1}{c} \leq \Phi(\xi) := \sum_j \varphi_j(\xi) \leq c.
\]
\textbf{PROPOSITION 3.2} In the conditions above, let $\hat{\psi}_j = \varphi_j / \Phi$. Then

1. $\hat{\psi}_j \in C^\infty_c (B_2(\xi_j))$;

2. $0 \leq \hat{\psi}_j \leq 1$, and $\sum_j \hat{\psi}_j(\xi) = 1$, $\forall \xi \in \Omega$;

3. $\psi_j$ are uniformly bounded in $L^1(\mathbb{R}^n)$; in particular, there exists a constant $C > 0$ such that

$$\|f \ast \psi_j\|_p \leq C \|f\|_p, \quad \forall f \in L^p(\mathbb{R}^n), \quad \forall j, \ 1 \leq p \leq \infty. \quad (3.3)$$

\textbf{PROOF:} The first two statements are clear. For the last one, note first that

$$\|\psi_j\|_{L^1} = \|\mathcal{F}^{-1}(\varphi_0(g_j^{-1} \cdot) / \Phi)\|_{L^1} = \|\mathcal{F}^{-1}(\varphi_0(\Phi(g_j \cdot)))\|_{L^1}.$$ 

Now, when $\xi \in B_2(e)$ we can write

$$\Phi(g_j \xi) = \sum_{k \in J_j} \varphi_0(g_k^{-1} g_j \xi),$$

where $J_j = \{k : B_2(\xi_k) \cap B_2(\xi_j) \neq 0\}$ is a finite set with at most $N$ elements by the finite intersection property. Further, we claim that the following uniform estimate holds true:

$$\|g_k^{-1} g_j\| \leq C, \quad \text{when} \ k \in J_j, \ \forall j. \quad (3.4)$$

Indeed, since $d(g_k^{-1} g_j e, e) = d(\xi_j, \xi_k) \leq 4$, by Lemmas 2.12 and 2.9,

$$\|g_k^{-1} g_j\| \sim |g_k^{-1} g_j e| \sim |e| \sim 1.$$

From (3.4) the proposition follows easily. Indeed, integrating by parts we have

$$\mathcal{F}^{-1}(\varphi_0(\Phi(g_j \cdot)))(x) = \int_{B_2(e)} e^{i(x, \xi)} \frac{\varphi_0(\xi)}{\Phi(g_j \xi)} d\xi = \int_{B_2(e)} e^{i(x, \xi)} \frac{D^L(\varphi_0(\Phi(g_j \cdot)))(\xi)}{(-|\xi|^2)^L} d\xi, \quad (3.5)$$

where $D^L$ denotes a power of the Laplacian. All functions $D^L(\varphi_0/\Phi(g_j \cdot))$ are bounded. Thus, choosing $L = 0$ for $|x| \leq 1$, and $L > \frac{n}{2}$ for $|x| > 1$, we can majorize $\mathcal{F}^{-1}(\varphi_0/\Phi(g_j \cdot))$ uniformly in $j$ by an integrable function, and this establishes the result. \qed

In this paper we shall mainly be concerned with Besov-type seminorms derived from the couple $\{\xi_j, \psi_j\}$ as in (1.3). That is, for $\nu \in \mathbb{R}$, $1 \leq p, q \leq \infty$, and $f \in S'(\mathbb{R}^n)$ we let

$$\|f\|_{B^\nu_{p,q}} := \begin{cases} \left[ \sum_j \Delta^{-\nu}(\xi_j) \|f \ast \psi_j\|_p^q \right]^{1/q} & \text{if } q < \infty \smallskip \sup_j \Delta^{-\nu}(\xi_j) \|f \ast \psi_j\|_p & \text{if } q = \infty. \end{cases} \quad (3.6)$$
We shall make use of the fact that these seminorms do not actually depend on the choice of the lattice \( \{ \xi_j \} \) or the test functions \( \psi_j \). Moreover, they can as well be defined with test functions which are not normalized as in the previous proposition. That is, we may replace \( \{ \psi_j \} \) by any family
\[
\hat{\psi}_j(\xi) := \hat{\chi}(g_j^{-1} \xi),
\]
(3.7)
defined from an arbitrary \( \hat{\chi} \in C_c^\infty(B_4(e)) \) so that \( 0 \leq \hat{\chi} \leq 1 \) and \( \hat{\chi} \) is identically 1 in \( B_2(e) \). These and other elementary equivalences are stated and proved in the following lemma.

**Lemma 3.8** Let \( \{ \xi_j, \psi_j \} \) be as at the beginning of this section, and fix \( \nu \in \mathbb{R} \) and \( 1 \leq p, q \leq \infty \). Then, for any other \((\delta, R)\)-lattice \( \{ \tilde{\xi}_j \} \) with associated Littlewood-Paley functions \( \{ \tilde{\psi}_j \} \), and for any family \( \{ \chi_j \} \) as in (3.7), we have the equivalences
\[
\| f \ast \psi_j \|_q^p \sim \| f \ast \tilde{\psi}_j \|_q^p, \\
\| f \ast \psi_j \|_q^p \sim \| f \ast \chi_j \|_q^p,
\]
for all \( f \in S'(\mathbb{R}^n) \). Moreover, when \( g \in G \) and \( q < \infty \), it holds the equivalence
\[
\sum_j \Delta^{-\nu}(\xi_j) \| (f \circ g) \ast \tilde{\psi}_j \|_q^p \sim \Delta(g \mathbf{e})^{-\frac{n-\nu}{2}} \sum_j \Delta^{-\nu}(\xi_j) \| f \ast \tilde{\psi}_j \|_q^p.
\]

**Proof:** We just consider the case \( q < \infty \), the modifications for \( q = \infty \) being trivial. For the first part, we can write, for each \( j \),
\[
\hat{\psi}_j = \sum_k \hat{\psi}_j \hat{\psi}_k,
\]
where the index \( k \) runs through a set \( J_j = \{ k : B_{R\delta}(\tilde{\xi}_k) \cap B_2(\xi_j) \neq \emptyset \} \) of at most \( N = N(\delta, R, \Omega) \) elements. Then, using (3.3) and Lemma 2.4 we have
\[
\sum_j \Delta^{-\nu}(\xi_j) \| f \ast \psi_j \|_p^q \leq C \sum_j \sum_{k \in J_j} \Delta^{-\nu}(\xi_j) \| f \ast \tilde{\psi}_k \|_p^q \\
\leq C' \sum_k \Delta^{-\nu}(\tilde{\xi}_k) \| f \ast \tilde{\psi}_k \|_p^q.
\]
The converse inequality follows similarly. For the second equivalence in the lemma, the fact that \( \hat{\chi}_j \hat{\psi}_j = \tilde{\psi}_j \) implies immediately the left inequality. A similar use of the finite intersection property as we did above gives the right hand side.

Finally, for our last statement, it is sufficient to prove an inequality of the form \( \lesssim \), the converse inequality \( \gtrsim \) following after replacing \( g \) by its inverse. Now,
using a first change of variables, and the fact that the determinant of the transformation \(g\) in \(\mathbb{R}^n\) is equal to \(\Delta(ge)^{\frac{n}{2}}\), we are linked to consider the \(L^p\)-norms of the functions

\[
\Delta(ge)^{-(1+\frac{1}{p})\frac{n}{2}} f \ast (\psi_j \circ g^{-1}) = \Delta(ge)^{-(1+\frac{1}{p})\frac{n}{2}} \sum_k f \ast \psi_k \ast (\psi_j \circ g^{-1}).
\]

For each fixed \(j\) this last sum has at most \(N\) terms, since the Fourier transform of \(\psi_k \ast (\psi_j \circ g^{-1})\) is non zero only if \(d(\xi_k, g^*\xi_j) < 4\). So

\[
\|f \ast (\psi_j \circ g^{-1})\|_p \leq C\Delta(ge)^{\frac{n}{2}} \sum_{k:d(\xi_k, \xi_j) < 4} \|f \ast \psi_k\|_p,
\]

the factor \(\Delta(ge)^{\frac{n}{2}}\) appearing as the determinant of the transformation \(g\) in the computation of the \(L^1\) norm of \(\psi_j \circ g^{-1}\). Now, when \(d(g^*\xi_k, \xi_j) < 4\), then \(\Delta(\xi_j)\) is equivalent to \(\Delta(g^*\xi_k) = \Delta(ge)\Delta(\xi_k)\). Thus, we conclude

\[
\sum_j \Delta^{-\nu}(\xi_j) \|f \ast (\psi_j \circ g^{-1})\|_p^q \leq C\Delta(ge)^{\frac{n}{2}-\nu} \sum_j \sum_{k:d(\xi_k, \xi_j) < 4} \Delta^{-\nu}(\xi_k) \|f \ast \psi_k\|_p^q.
\]

We get the required inequality multiplying by \(\Delta(ge)^{-q(1+\frac{1}{p})\frac{n}{2}}\) and summing first in the \(j\) indices.

\[\square\]

Recall now that \(S_\Omega\) denotes the space of Schwartz functions \(f\) on \(\mathbb{R}^n\) with \(\text{Supp } \hat{f} \subset \overline{\Omega}\). The next proposition gives the Littlewood-Paley decomposition \(f = \sum_j f \ast \psi_j\) for functions in \(f \in S_\Omega\), and relates it with the Besov space norm.

**Proposition 3.9** Every \(f \in S_\Omega\) admits a Littlewood-Paley decomposition \(f = \sum_j f \ast \psi_j\) with convergence in \(S(\mathbb{R}^n)\). Further, for every \(\nu \in \mathbb{R}\), \(1 \leq p, q \leq \infty\) there is a constant \(C = C(p, q, \nu) > 0\) and an integer \(\ell = \ell(p, q, \nu) \geq 0\) so that

\[
\|f\|_{B^\nu_{p,q}} = \left[ \sum_j \Delta^{-\nu}(\xi_j) \|f \ast \psi_j\|_p^q \right]^{\frac{1}{q}} \leq Cp_\ell(\hat{f}) < \infty, \quad \forall f \in S_\Omega,
\]

where \(p_\ell(\varphi) = \sup_{|\alpha| \leq \ell} \sup_{\xi \in \mathbb{R}^n} (1 + |\xi|)^\ell |\partial^\alpha \varphi(\xi)|\) denotes a Schwartz seminorm.

The proof depends on a lemma which gives appropriate estimates for test functions in \(S_\Omega\).

**Lemma 3.11** Let \(N, M \geq 0\). Then, there is a constant \(C = C(N, M) > 0\) and an integer \(\ell = \ell(N, M) \geq 0\) such that for every \(f \in S_\Omega\)

1. \(|\hat{f}(\xi)| \leq Cp_\ell(\hat{f}) \Delta^M(\xi) (1 + |\xi|)^{-N}, \quad \forall \xi \in \Omega;\)
2. If $1 \leq p \leq \infty$, \( \|f * \psi_j\|_p \leq C p_\ell(\hat{f}) \Delta(\xi_j)^{M + \frac{n}{p'\ell}} (1 + |\xi_j|)^{-N}, \quad \forall j. \)

**PROOF of Lemma 3.11:**

For the first statement, it suffices to show that for every \( f \in S_\Omega \) and \( M \geq 1 \) there is \( M' \geq 1 \) so that

\[
|\hat{f}(\xi)| \leq C_{pM'}(\hat{f}) \Delta^M(\xi), \quad \text{whenever} \quad \Delta(\xi) \leq 1, \quad \xi \in \Omega. \tag{3.12}
\]

Indeed, we then write it for \( D^N f \) to get the full statement. So, let us prove (3.12). Let \( \xi \in \Omega \) be fixed, and choose \( \xi_0 \in \partial \Omega \) so that \( \text{dist}(\xi, \partial \Omega) = |\xi - \xi_0| \). Since \( \text{Supp} \ \hat{f} \subset \overline{\Omega} \), we have \( \partial^\alpha \hat{f}(\xi_0) = 0 \), for every multi-index \( \alpha \). Thus, given \( M \geq 1 \) there is a constant \( C = C(M) \) such that \( |\hat{f}(\xi)| \leq C_{pM}(\hat{f}) |\xi - \xi_0|^M \). We claim that \( |\xi - \xi_0| \leq \Delta(\xi)^{\frac{1}{2}} \), which will clearly establish (3.12).

To show our claim, we may assume that \( \xi = P(a)e \), where \( a = a_1c_1 + \ldots + a_rc_r \). Suppose also that \( a_1 = \min\{a_1, \ldots, a_r\} \). Then

\[
|\xi - \xi_0| = \text{dist} (\xi, \partial \Omega) \leq |\xi - (a_2^2c_2 + \ldots + a_r^2c_r)| = a_1^2 \leq (a_1^2 \cdots a_r^2)^{\frac{1}{2}} = \Delta(\xi)^{\frac{1}{2}}.
\]

Let us now prove the second statement in Lemma 3.11. It is sufficient to prove the same inequality, with the system \( \chi_j \) instead of \( \psi_j \). Given \( f \in S_\Omega \), we proceed as in (3.5):

\[
f * \chi_j(x) = \int_{\Omega} e^{i(x|\xi)} \hat{f}(\xi) \hat{\chi}(g_j^{-1}\xi) \, d\xi
= \Delta^\frac{n}{\ell}(\xi_j) \int_{\Omega} e^{i(g_j x|\xi)} \hat{f}(g_j \xi) \hat{\chi}(\xi) \, d\xi
= \Delta^\frac{n}{\ell}(\xi_j) \int_{B_2(e)} e^{i(g_j x|\xi)} \frac{D^L(\hat{f}(g_j \xi) \hat{\chi}(\xi))}{(-|g_j x|^2)^L} \, d\xi. \tag{3.13}
\]

The estimates in the first part, together with Lemmas 2.4, 2.9 and 2.12, imply that, on the invariant ball \( B_2(e) \),

\[
|D^L(\hat{f}(g_j \xi))| \leq C (1 + \|g_j\|)^{2L} \sum_{|\alpha| \leq 2L} |(\partial^\alpha \hat{f})(g_j \xi)|
\leq C' \ p_\ell(\hat{f}) \ \Delta^M(\xi_j) \frac{\Delta^M(\xi_j)}{(1 + |\xi_j|)^N} \frac{1}{(1 + |g_j x|^2)^L}, \quad x \in \mathbb{R}^n.
\]

for some integer \( \ell = \ell(M, N, L) \). Therefore

\[
|f * \chi_j(x)| \leq C \ p_\ell(\hat{f}) \ \Delta^\frac{n}{\ell}(\xi_j) \ \Delta^M(\xi_j) \frac{\Delta^M(\xi_j)}{(1 + |\xi_j|)^N} \frac{1}{(1 + |g_j x|^2)^L}, \quad x \in \mathbb{R}^n.
\]

Taking \( L^p \)-norms and changing variables, we conclude with

\[
\|f * \chi_j\|_p \leq C \ p_\ell(\hat{f}) \ \Delta^\frac{n}{p'}(\xi_j) \frac{\Delta^M(\xi_j)}{(1 + |\xi_j|)^N}. \tag{3.14}
\]
**Proof of Proposition 3.9:** Once we show the convergence of the series $\sum_j f \ast \psi_j$, the fact that the sum equals $f$ is immediate from $\text{Supp} \hat{f} \subset \overline{\Omega}$ and $\sum_j \hat{\psi}_j = \chi_\Omega$. Now, from the previous lemma and Proposition 2.13, we have

$$\sum_j \| \hat{f} \hat{\psi}_j \|_\infty = \sum_j \| \hat{f}(g_j \cdot) \frac{\varphi_0}{\Phi(g_j \cdot)} \|_\infty$$

$$\leq C_f \sum_j \Delta^M(\xi_j) (1 + |\xi_j|)^{-N} \leq C'_f \int_\Omega \frac{\Delta^M(\xi)}{(1 + |\xi|)^N} \frac{d\xi}{\Delta(\xi)^{\frac{q}{p'}}},$$

where the last integral is finite for $N, M$ large enough. A similar argument applies to $\| (1 + |\xi|) L^\alpha(\hat{f} \hat{\psi}_j) \|_\infty$, establishing our claim.

For the second assertion in the proposition, we use the second estimate in Lemma 3.11. Assuming $q < \infty$ (otherwise the estimate is trivial) we have

$$\| f \|_{B^{\nu, \gamma}_{p, q}} \leq C_{p, q}(\hat{f})^q \sum_j \Delta^{-\nu}(\xi_j) \frac{\Delta(\xi_j)^{Mq + \frac{2}{p'}}}{(1 + |\xi_j|)^{Nq}}$$

$$\leq C'_{p, q}(\hat{f})^q \int_\Omega \frac{\Delta(\xi)^{Mq + \frac{2}{p'} - \nu}}{(1 + |\xi|)^{Nq}} \frac{d\xi}{\Delta(\xi)^{\frac{q}{p'}}},$$

which is finite for a sufficiently large choice of $N, M$.

$\Box$

We observe that we have strongly used the assumption on the support of $\hat{f}$. The next proposition gives the sharp region of indices $\nu, p, q$ for which general Schwartz functions have finite $B^{\nu, \gamma}_{p, q}$-seminorms. We state this fact separately since such conditions will appear in the sequel in relation with the index $\hat{q}_{\nu, p}$.

**Proposition 3.16** Let $\nu \in \mathbb{R}$, $1 \leq p, q \leq \infty$ be such that

$$\frac{q}{p'} + \frac{n}{r} > \nu + \frac{n}{r} - 1$$

(or $\frac{1}{p'} + \frac{n}{r} \geq \nu$ when $q = \infty$). Then, there exist $C = C(p, q, \nu) > 0$ and an integer $\ell = \ell(p, q, \nu) \geq 0$ so that:

$$\| f \|_{B^{\nu, \gamma}_{p, q}} \leq C p(\hat{f}) < \infty, \quad \forall f \in \mathcal{S}(\mathbb{R}^n).$$

Moreover, this property can hold for all $f \in \mathcal{S}(\mathbb{R}^n)$ only if $\nu, p, q$ satisfy (3.17).

**Proof:** As before we assume $q < \infty$, with obvious modifications when $q = \infty$. First one observes that, when $f \in \mathcal{S}(\mathbb{R}^n)$, the conclusion of Lemma 3.11 is still valid with $M = 0$. Thus, a similar reasoning as in (3.15) gives

$$\| f \|_{B^{\nu, \gamma}_{p, q}} \leq C_{p, q}(\hat{f}) \left[ \int_\Omega \frac{\Delta^{\frac{2}{p'} - \nu}(\xi) d\xi}{(1 + |\xi|)^N} \frac{d\xi}{\Delta(\xi)^{\frac{q}{p'}}} \right]^{\frac{1}{q}}, \quad f \in \mathcal{S}(\mathbb{R}^n),$$

where the last integral is finite for $N, M$ large enough. A similar argument applies to $\| (1 + |\xi|) L^\alpha(\hat{f} \hat{\psi}_j) \|_\infty$, establishing our claim.

For the second assertion in the proposition, we use the second estimate in Lemma 3.11. Assuming $q < \infty$ (otherwise the estimate is trivial) we have

$$\| f \|_{B^{\nu, \gamma}_{p, q}} \leq C_{p, q}(\hat{f})^q \sum_j \Delta^{-\nu}(\xi_j) \frac{\Delta(\xi_j)^{Mq + \frac{2}{p'}}}{(1 + |\xi_j|)^{Nq}}$$

$$\leq C'_{p, q}(\hat{f})^q \int_\Omega \frac{\Delta(\xi)^{Mq + \frac{2}{p'} - \nu}}{(1 + |\xi|)^{Nq}} \frac{d\xi}{\Delta(\xi)^{\frac{q}{p'}}},$$

which is finite for a sufficiently large choice of $N, M$.

$\Box$
and this integral is finite under the condition \( \frac{q}{p} r > \nu + \frac{a}{r} - 1 \). To show that this condition is critical, take any \( f \in S(\mathbb{R}^n) \) such that \( \hat{f} \) is identically 1 in the Euclidean ball centered at 0 of radius 1. Then, for such an \( f \), one has the bound from below

\[
\|f\|_{B^{p,q}_\nu} \geq c \sum_{j:|\xi_j|<\delta} (\xi_j)^{-\nu} \|\chi_j\|_p^q \geq c'' \int_{\xi \in \Omega; |\xi|<\delta} \frac{d\xi}{(\xi)^{\frac{a}{r} - \frac{\nu}{q}}},
\]

and this last integral is infinity unless \( \frac{q}{p} r > \nu + \frac{a}{r} - 1 \).

\[\square\]

### 3.2 Properties of Besov spaces

Given a closed set \( F \subset \mathbb{R}^n \), we shall denote by \( S'_F = S'_F(\mathbb{R}^n) \) the space of tempered distributions with Fourier transform supported in \( F \). Recall the expression of the “seminorm” \( \|f\|_{B^{p,q}_\nu} \) in (1.3), and observe that a distribution \( f \in S'_\Omega \) satisfies \( \|f\|_{B^{p,q}_\nu} = 0 \) if and only if \( f \in S'_\partial \Omega \). This leads to the following natural definition of Besov spaces with spectrum in \( \Omega \):

**Definition 3.19** Given \( \nu \in \mathbb{R}, 1 \leq p,q < \infty \), we define \( B^{p,q}_\nu \) as the space of equivalence classes of tempered distributions

\[ B^{p,q}_\nu = \{ f \in S'_\Omega : \|f\|_{B^{p,q}_\nu} < \infty \} / S'_\partial \Omega. \]

It follows right away from Lemma 3.8 that \( B^{p,q}_\nu \) does not depend on the choice of \( \{\psi_j\} \) or the lattice \( \{\xi_j\} \). Moreover, \( B^{p,q}_\nu \) is invariant under the action of \( G \), and

\[
\|f \circ g\|_{B^{p,q}_\nu} \sim \Delta(ge)^{-\frac{a}{r} - \frac{\nu}{q}} \|f\|_{B^{p,q}_\nu}.
\]

Before collecting in the next proposition other basic properties of the spaces \( B^{p,q}_\nu \), we make a minor comment on the notation used below.

**Notation 3.21** Throughout this paper, the standard action of tempered distributions over Schwartz functions will be denoted by

\[
(f, \varphi) = \int f \varphi, \quad f \in S' (\mathbb{R}^n), \varphi \in S(\mathbb{R}^n).
\]

For convenience, we shall often use the anti-linear pairing:

\[
\langle f, \varphi \rangle := (f, \overline{\varphi}) = \int_{\mathbb{R}^n} f \overline{\varphi}, \quad f \in S' (\mathbb{R}^n), \varphi \in S(\mathbb{R}^n).
\]

This has the notational advantage of a simple Plancherel identity: \( \langle f, \varphi \rangle = \langle \hat{f}, \varphi \rangle \), leading to a natural pairing between \( f \in S'_\Omega (\mathbb{R}^n) \) and \( \varphi \in S_\Omega \) (rather than using \( (f, \varphi) = (\hat{f}, \varphi(- \cdot)) \), which requires to deal with \( \varphi \in S_\Omega \)).
With the above considerations we have the following lemma.

**Lemma 3.22** Let \( \nu \in \mathbb{R} \), \( 1 \leq p \leq \infty \) and \( 1 < q < \infty \). Then, there exists \( \ell = \ell(\nu, p, q) \geq 0 \) so that, for every distribution \( f \in S'(\mathbb{R}^n) \) with \( \| f \|_{B^{p,q}_\nu} < \infty \), we have

\[
|\langle f, \varphi \rangle| \leq C \| f \|_{B^{p,q}_\nu} \| \varphi \|_{B^{p',q'}_{-\nu q'/q}} \leq C' \| f \|_{B^{p,q}_\nu} p\ell(\varphi), \quad \forall \varphi \in S_{\Omega}. (3.23)
\]

When \( q = 1 \) or \( q = \infty \), the same holds replacing \( \| \varphi \|_{B^{p',q'}_{-\nu q'/q}} \) by \( \| \varphi \|_{B^{p',q'}_{-\nu q'/q}} \).

**Proof:**

Remember that \( \hat{\psi}_j = \hat{\psi} \hat{\chi}_j \) for all \( j \). Therefore, using Proposition 3.9 and Lemma 3.8

\[
|\langle f, \varphi \rangle| \leq \sum_j |\langle f \ast \psi_j, \varphi \ast \chi_j \rangle| \leq \sum_j \| f \ast \psi_j \|_p \| \varphi \ast \chi_j \|_{p'} \leq C \| f \|_{B^{p,q}_\nu} \| \varphi \|_{B^{p',q'}_{-\nu q'/q}}. (3.24)
\]

Finally, observe that \( \| \varphi \|_{B^{p',q'}_{-\nu q'/q}} = \| \varphi \|_{B^{p',q'}_{-\nu (1-q')}} \), which, for \( \varphi \in S_{\Omega} \), is bounded by a Schwartz seminorm by Proposition 3.9. The modifications for \( q = 1, \infty \) are obvious. \( \square \)

**Proposition 3.25** Let \( 1 \leq p, q < \infty \) and \( \nu \in \mathbb{R} \). Then

1. \( B^{p,q}_\nu \) is a Banach space.

2. The space \( D_{\Omega} := \{ f \in S(\mathbb{R}^n) : \text{Supp} \hat{f} \text{ is compact in } \Omega \} \) is dense in \( B^{p,q}_\nu \). Moreover, for every class \( f + S_{\Omega}' \) in \( B^{p,q}_\nu \), the series \( \sum_j f \ast \psi_j \) converges to (the class of) \( f \) in the space \( B^{p,q}_\nu \).

**Proof:**

Suppose \( \{ f_m \}_m \) is a Cauchy sequence of distributions in \( S_{\Omega}' \) for the \( B^{p,q}_\nu \)-seminorm. Then, from the previous lemma it follows that \( \langle f_m, \varphi \rangle \) converges for every \( \varphi \in S_{\Omega} \), and moreover, it defines a continuous (anti-linear) functional in \( S_{\Omega} \). We can extend it to \( S_{\Omega} \oplus S_{\Omega}' \) by letting it be identically zero in the second summand, and finally extend it to the whole Schwartz space \( S(\mathbb{R}^n) \) by the Hahn-Banach theorem. This gives a tempered distribution \( f \in S_{\Omega}' \), which in particular satisfies

\[
f \ast \psi_j(x) = \lim_{m \to \infty} f_m \ast \psi_j(x), \quad \forall x \in \mathbb{R}^n, \quad \forall j.
\]

Therefore, by Fatou’s lemma

\[
\| f \|_{B^{p,q}_\nu} \leq \lim_{m \to \infty} \| f_m \|_{B^{p,q}_\nu} < \infty,
\]

and in a similar fashion \( \lim_{m \to \infty} \| f - f_m \|_{B^{p,q}_\nu} = 0 \). This shows that \( B^{p,q}_\nu \) is a Banach space.
For the density, let \( f \) be a fixed distribution in \( \mathcal{S}'_\Omega \) with \( \| f \|_{B_{p,q}^\nu} < \infty \). We shall show that \( f \) is the \( B_{p,q}^\nu \)-limit of the partial sums of the series \( \sum_j f \ast \psi_j \). Remark that each finite sum belongs to \( L^p(\mathbb{R}^n) \), and therefore can be approached by a Schwartz function with Fourier transform supported in a compact set of \( \Omega \), justifying the density of \( \mathcal{D}_\Omega \). Now, it is easily seen that partial sums (for any order) constitute a Cauchy sequence in \( B_{p,q}^\nu \). Since \( B_{p,q}^\nu \) is a Banach space, they converge to a distribution \( u \in \mathcal{S}'_\Omega \). It remains to show that \( u \) and \( f \) belong to the same equivalence class in \( \mathcal{S}'_\Omega / \mathcal{S}'_{\partial\Omega} \), which is an immediate consequence of the fact that \( f \ast \psi_j = u \ast \psi_j \).

\[ \square \]

For the duality of the spaces \( B_{p,q}^\nu \), recall the Hölder type inequality in (3.23):

\[ |(\tilde{f}, g)| = |(f, g)| \leq C \| f \|_{B_{p',q'}^{-\nu/q'}} \| g \|_{B_{p,q}^\nu}, \quad g \in \mathcal{D}_\Omega, \]

valid for every \( f \in \mathcal{S}'_\Omega \) with \( \| f \|_{B_{p',q'}^{-\nu/q'}} < \infty \). Observe that \( g \in \mathcal{D}_\Omega \mapsto (\tilde{f}, g) \) is a linear functional in \( \mathcal{D}_\Omega \) which depends only on the equivalence class \( f + \mathcal{S}'_{\partial\Omega} \). Thus, by the above inequality and the density of \( \mathcal{D}_\Omega \), it defines a continuous linear functional \( \Phi_f \) in \( B_{p,q}^\nu \). Further, if \( \Phi_f = 0 \), then \( (\tilde{f}, g) = 0, \forall g \in \mathcal{D}_\Omega \), and necessarily \( f \in \mathcal{S}'_{\partial\Omega} \). Thus, the correspondence

\[ f + \mathcal{S}'_\Omega \in B_{p',q'}^{-\nu/q'} \longrightarrow \Phi_f \in (B_{p,q}^\nu)^* \quad (3.26) \]

is well-defined and injective.

**Proposition 3.27** Let \( \nu \in \mathbb{R} \) and \( 1 \leq p, q < \infty \). Then, the mapping in (3.26) is an anti-linear isomorphism of Banach spaces.

**Proof:** By the previous comments it suffices to show that for every \( \Phi \in (B_{p,q}^\nu)^* \), there exists a distribution \( f \in \mathcal{S}'_\Omega \) such that

\[ \Phi(g) = (\tilde{f}, g), \quad \forall g \in \mathcal{D}_\Omega \quad \text{and} \quad \| f \|_{B_{p',q'}^{-\nu/q'}} \leq C \| \Phi \|. \quad (3.28) \]

Now, since \( \mathcal{D}_\Omega \subset \mathcal{S}_\Omega \hookrightarrow B_{p,q}^\nu \), by Hahn-Banach we can extend continuously \( \Phi \) to \( \mathcal{S}(\mathbb{R}^n) \), and find a tempered distribution \( f \in \mathcal{S}'_\Omega \) such that \( \Phi(g) = (\tilde{f}, g), \forall g \in \mathcal{D}_\Omega \).

We now claim that each \( f \ast \psi_j \), which a priori is only a smooth function with polynomial growth, does belong to \( L^p(\mathbb{R}^n) \), and moreover, the sequence of their \( L^p \) norms belongs to the suitable space of sequences. Indeed, for every finite sequence \( g_j \in \mathcal{S}(\mathbb{R}^n) \) with \( \sum_j \Delta(\xi_j)^{-\nu} \| g_j \|_p^q \leq 1 \) we have

\[ |\sum_j (f \ast \psi_j, g_j)| = |\Phi(\sum_j g_j \ast \psi_j)| \leq \| \Phi \| \sum_j g_j \ast \psi_j \|_{B_{p,q}^\nu} \leq C \| \Phi \|. \]

The constant \( C \) depends only on the number \( N \) in the finite intersection property, the constant \( \gamma \) related to the variation of the function \( \Delta \) inside an invariant ball of radius 2, and the \( L^1 \) norm of the \( \psi_j \)'s. Since the constant is independent of the finite set of indices, (3.28) follows. We do not give the details of the proof, since it is completely analogous to the one of Lemma 3.8.
Let us remark that, for two classes of tempered distributions \( f + S'_{\partial \Omega} \) in \( B^{p,q}_{\nu} \) and \( g + S'_{\partial \Omega} \) in \( B^{p',q'}_{-\nu q'/q} \), the duality pairing can also be expressed as

\[
\Phi_f(g + S'_{\partial \Omega}) = \sum_j \langle f \ast \psi_j, g \ast \chi_j \rangle,
\]

where the series converges absolutely by (3.23). This representation is sometimes convenient, and of course, independent on the choice of \( \{\psi_j, \chi_j\} \).

### 3.3 The \( \Box \) operator and Besov multipliers

Next we describe some analytic properties of the spaces \( B^{p,q}_{\nu} \). The first one concerns the role of the generalized wave operator \( \Box \) (introduced in (1.2)) as a natural isomorphism between these spaces. Below we shall be interested in fractional and negative powers of \( \Box \), which can be defined by the rule

\[
\Box^\beta f = \mathcal{F}^{-1}(\Delta^\beta \hat{f}),
\]

at least for distributions \( f \in S'(\Omega) \) so that \( \text{Supp} \hat{f} \) is compact in \( \Omega \). Our next result is a more general version than (2) in Theorem 1.4.

**Proposition 3.31** Let \( \nu, \beta \in \mathbb{R} \) and \( 1 \leq p, q < \infty \). Then there is a constant \( C > 0 \) such that for every distribution \( f \in S'(\mathbb{R}^n) \) with \( \text{Supp} \hat{f} \) compact in \( \Omega \)

\[
\frac{1}{C} \|f\|_{B^{p,q}_{\nu}} \leq \|\Box^\beta f\|_{B^{p,q}_{\nu+\beta}} \leq C\|f\|_{B^{p,q}_{\nu}}.
\]

In particular, \( \Box^\beta \) extends to an isomorphism \( \Box^\beta : B^{p,q}_{\nu} \to B^{p,q}_{\nu+\beta} \).

**Proof:** Indeed, given \( f \in S'(\mathbb{R}^n) \) with \( \text{Supp} \hat{f} \) compact in \( \Omega \) we have

\[
\|\Box^\beta f\|_{B^{p,q}_{\nu+\beta}} = \sum_j \Delta(\xi_j)^{-(\nu+\beta)} \|\mathcal{F}^{-1}(\hat{f} \hat{\psi}_j \Delta^\beta)\|_{\ell_p}^q \leq \sum_j \Delta(\xi_j)^{-(\nu+\beta)} \|f \ast \psi_j\|_p \|\mathcal{F}^{-1}(\hat{\chi}_j \Delta^\beta)\|_{\ell_p}^q.
\]

Using \( \Delta(g\xi) = \Delta(g e) \Delta(\xi) \), for \( g \in G \), we have

\[
\|\mathcal{F}^{-1}(\hat{\chi}_j \Delta^\beta)\|_1 = \Delta^\beta(\xi_j) \|\mathcal{F}^{-1}(\hat{\chi} \Delta^\beta)\|_1 = c_\beta \Delta^\beta(\xi_j),
\]

from which (3.32) follows easily. To extend \( \Box^\beta \) to the space \( B^{p,q}_{\nu} \) one proceeds by density. More precisely, given any \( f \in S'_\Omega \) with \( \|f\|_{B^{p,q}_{\nu}} < \infty \), we denote by \( \Box^\beta f \) a representative from the equivalence class of \( \sum_j \Box^\beta(f \ast \psi_j) \), which by (3.32) (and Proposition 3.25) is a Cauchy series in the \( B^{p,q}_{\nu+\beta} \)-seminorm. Observe that \( \Box^\beta S'_{\partial \Omega} = 0 \) (or its equivalence class), while by uniqueness of the extension, \( \Box^\beta \) does not depend on the Littlewood-Paley functions \( \{\psi_j\} \).
A further step in the previous idea leads to a functional calculus in $B_{p,q}^\nu$ based on the operator $\Box$. Let $m \in C^\infty(0, \infty)$ be a Mihlin-type multiplier in 1 dimension. That is, there is a constant $C = C(m)$ so that

$$\sup_{\xi > 0} |\xi|^k |m^{(k)}(\xi)| \leq C, \quad \forall k = 0, 1, \ldots$$  \hspace{1cm} (3.33)

Then, it makes sense to define the operator $m(\Box)$ by

$$m(\Box)(f) = \mathcal{F}^{-1}(m(\Delta)\hat{f}),$$

at least for $f \in S'(\mathbb{R}^n)$ with Supp $\hat{f}$ compact in $\Omega$. Observe that a typical example is given by the imaginary powers $m(\xi) = \xi^{i\gamma}$ for $\gamma \in \mathbb{R}$. Then we have the following:

**Proposition 3.34** Let $\nu \in \mathbb{R}$, $1 \leq p, q < \infty$ and $m \in C^\infty(0, \infty)$ satisfying (3.33). Then, there is a constant $C = C(m)$ such that

$$\|m(\Box)(f)\|_{B_{p,q}^\nu} \leq C_m \|f\|_{B_{p,q}^\nu}, \quad f \in D_\Omega.$$  

In particular, $m(\Box)$ extends to a bounded operator in $B_{p,q}^\nu$.

**Proof:** For the proof it suffices to estimate

$$\|m(\Box)(f) * \psi_j\|_p \leq \|f * \psi_j\|_p \|\mathcal{F}^{-1}(m(\Delta)\hat{\chi}_j)\|_1, \quad \text{for } f \in D_\Omega. \hspace{1cm} (3.35)$$

Now,

$$\|\mathcal{F}^{-1}(m(\Delta)\hat{\chi}_j)\|_1 = \|\mathcal{F}^{-1}(m(\Delta(\xi_j)\Delta)\hat{\chi})\|_1 \leq C p_m(m(\Delta(\xi_j)\Delta)\hat{\chi}) \hspace{1cm} (3.36)$$

$$\leq C' \sup_{\xi \in B(0,4)} \sum_{s=0}^\ell \Delta(\xi_j)^s |m^{(s)}(\Delta(\xi_j)\Delta)| \leq C''.$$  

Thus, raising (3.35) to the $q^{th}$ power and summing we conclude easily.

**Remark 3.37**  
1. The previous proposition holds as well under the milder hypothesis $m \in C^{n+1}(0, \infty)$ and (3.33) for $k = 0, 1, \ldots, n + 1$. This follows from the fact that the key inequality (3.36) is actually valid with $\ell = n + 1$.

2. A similar result can be proved with higher dimensional multipliers. More precisely, given $m \in C^\infty((0, \infty)^r)$ satisfying

$$\sup_{\xi \in (0,\infty)^r} \left| \xi_1^{\alpha_1} \cdots \xi_r^{\alpha_r} \frac{\partial m}{\partial \xi_1^{\alpha_1} \cdots \partial \xi_r^{\alpha_r}}(\xi) \right| \leq C_\alpha, \quad \forall \alpha = (\alpha_1, \ldots, \alpha_r) \geq 0,$$

we can define the operator

$$T_m f = \mathcal{F}^{-1}(m(\Delta_1, \ldots, \Delta_r)\hat{f}), \quad \text{if } f \in D_\Omega.$$  

Then, an analogous proof gives $\|T_m f\|_{B_{p,q}^\nu} \leq C \|f\|_{B_{p,q}^\nu}$. 
3. In the previous examples $m$ is a Fourier multiplier belonging to $M_p$ for all $1 < p < \infty$. More general examples of multipliers for $B^{p,q}_\nu$ can be constructed as follows. Let $\{m_j\}_j$ be a uniformly bounded family of multipliers in $M_p(\mathbb{R}^n)$ with $\text{Supp} \, m_j$ contained in a fixed compact set of $\Omega$. Let $m(\xi) = \sum_j m_j(g_j^{-1}\xi)$ and $T_m f = \mathcal{F}^{-1}(m\hat{f})$. Then $\|T_m f\|_{B^{p,q}_\nu} \leq C \|f\|_{B^{p,q}_\nu}$. In particular, we may take $m_j = \varepsilon_j \hat{\psi}$, where $\varepsilon_j = \pm 1$, and conclude that $m_\varepsilon = \sum_j \varepsilon_j \psi_j$ is a multiplier for $B^{p,q}_\nu$. Observe however, that letting $\varepsilon_j = 1$, the function $m_\varepsilon = \chi_\Omega \notin M_p(\mathbb{R}^n)$ for any $p \neq 2$.

### 3.4 Fourier-Laplace extensions

It is well-known that to every distribution supported in a closed cone $\overline{\Omega}$ we can associate an analytic function in the tube domain $T_\Omega$ via the Fourier-Laplace integral. More precisely, this is given by:

$$Lg(z) = (g, e^{i(z|\xi)}) = \int_\Omega e^{i(z|\xi)} g(\xi) \, d\xi, \quad z \in T_\Omega,$$

which makes sense for compactly supported distributions $g$ in $\Omega$, and can also be given a meaning for all $g \in \mathcal{S}'(\mathbb{R}^n)$ with $\text{Supp} \, g \subset \Omega$ (see [16, Ch. VII]).

In this section we wish to describe the analytic functions associated with (classes of) distributions in our Besov spaces $B^{p,q}_\nu$. To avoid dealing with equivalence classes, it is convenient to restrict the indices $\nu, p, q$ so that $B^{p,q}_\nu$ can be embedded in the usual space of tempered distributions.

**Lemma 3.38** Let $\nu > 0, 1 \leq p < \infty, 1 \leq q < \tilde{q}_{\nu,p}$. Then, for every $f \in \mathcal{S}'_{\overline{\Omega}}$ with $\|f\|_{B^{p,q}_\nu} < \infty$, the series $\sum_j f \ast \psi_j$ converges in the space $\mathcal{S}'(\mathbb{R}^n)$. Moreover, the correspondence

$$B^{p,q}_\nu \longrightarrow \mathcal{S}'(\mathbb{R}^n)$$

$$f + \mathcal{S}_{\partial \Omega} \longmapsto f^\sharp = \sum_j f \ast \psi_j$$

is continuous, injective, and does not depend on the Littlewood-Paley functions $\{\psi_j\}$.

**Proof:** The proof of the convergence of the series is completely analogous to that of Lemma 3.22. In fact, using the Hölder-type inequality in (3.23) we can write

$$\sum_j |\langle f \ast \psi_j, \varphi \rangle| \leq C \|f\|_{B^{p,q}_\nu} \|\varphi\|_{B^{p',q'}_{\nu',q}} \quad \varphi \in \mathcal{S}(\mathbb{R}^n). \quad (3.39)$$

Now, from $q < \tilde{q}_{\nu,p}$ and Proposition 3.16 we obtain

$$\|\varphi\|_{B^{p',q'}_{\nu',q}} \leq C p_\varepsilon(\tilde{\varphi}) < \infty, \quad \forall \varphi \in \mathcal{S}(\mathbb{R}). \quad (3.40)$$

Using the previous two formulas and the density of $\mathcal{D}_\Omega$ it is customary to verify the last statement of the lemma.
REMARK 3.41 From now on, whenever we restrict to the indices \( \nu > 0, 1 \leq p < \infty \) and \( 1 \leq q < \tilde{q}_{\nu,p} \) as in the previous lemma, we shall identify \( B_{\nu}^{p,q} \) with the corresponding space \( (B_{\nu}^{p,q})^\# \) of tempered distributions. Observe that all these distributions have finite order, are supported in \( \overline{\Omega} \) and satisfy the Littlewood-Paley decomposition \( f = \sum_j f \ast \psi_j \).

For the purposes of this paper, we shall rather speak of Fourier-Laplace extensions for distributions \( f \) with spectrum in \( \overline{\Omega} \). That is, we define

\[
\mathcal{E} f(z) = \mathcal{L} \hat{f}(z) = (\hat{f}, e^{i(z|\xi)}) = \int_{\Omega} e^{i(z|\xi)} \hat{f}(\xi) \, d\xi, \quad z \in \mathcal{T}_\Omega,
\]

which we shall call the Fourier-Laplace extension of \( f \), and which defines a holomorphic function in the tube domain \( \mathcal{T}_\Omega \). For distributions \( f \in B_{\nu}^{p,q} \) this takes the form

\[
\tilde{\mathcal{E}} f(z) = \mathcal{E} \left( \sum_j f \ast \psi_j \right)(z) = \sum_j \mathcal{L}(\hat{f} \psi_j)(z), \quad z \in \mathcal{T}_\Omega,
\]  

(3.42)

where as we shall see below, the last series converges uniformly in compact sets of \( \mathcal{T}_\Omega \).

We stick to the notation \( \tilde{\mathcal{E}} f \) to recall that we are choosing the special representative \( \tilde{f} = \sum_j f \ast \psi_j \) from the equivalence class \( f + S'_{\partial \Omega} \). Observe that in general \( \tilde{\mathcal{E}}(S'_{\partial \Omega}) = 0 \), while \( \mathcal{E}(S'_{\partial \Omega}) \) is not. The main result about Fourier-Laplace extensions is the following proposition.

**PROPOSITION 3.43** Let \( \nu > 0, 1 \leq p < \infty, 1 \leq q < \tilde{q}_{\nu,p} \). Then for every distribution \( f = \sum_j f \ast \psi_j \in B_{\nu}^{p,q} \) the series in (3.42) converges uniformly on compact sets to a holomorphic function in \( \mathcal{T}_\Omega \), and moreover

\[
|\tilde{\mathcal{E}} f(x + iy)| \leq C \Delta(y)^{-\frac{1}{q} - \frac{\nu}{q}} \| f \|_{B_{\nu}^{p,q}}, \quad x + iy \in \mathcal{T}_\Omega. \tag{3.44}
\]

In addition, for each \( y \in \Omega \), the distributions \( \tilde{\mathcal{E}} f(\cdot + iy) \) satisfy

\[
\tilde{\mathcal{E}}(f)(\cdot + iy) = \sum_j \tilde{\mathcal{E}}(f)(\cdot + iy) \ast \psi_j, \quad \text{in} \ S'(\mathbb{R}^n)
\]

and

\[
\| \tilde{\mathcal{E}} f(\cdot + iy) \|_{B_{\nu}^{p,q}} \leq C \| f \|_{B_{\nu}^{p,q}}, \quad \lim_{y \to 0} \| \tilde{\mathcal{E}} f(\cdot + iy) - f \|_{B_{\nu}^{p,q}} = 0. \tag{3.45}
\]

**PROOF:** In the first part we shall only prove the pointwise convergence and (3.44). The proof can be easily adapted to obtain uniform convergence on compact sets. Since \( B_{\nu}^{p,q} \) is invariant by the action of \( G \) as well as by translations in the \( x \) variable,
we can reduce to the case \( y = ie \) and \( x = 0 \), using (3.20) to prove (3.44) in the general case. Now, by definition of \( \mathcal{E} \) and following the same steps as in (3.23), we can write

\[
\sum_j |\mathcal{E}(f * \psi_j)(ie)| = \sum_j |\langle f * \psi_j, \mathcal{F}^{-1}(\hat{\chi}_j e^{-|\cdot|}) \rangle| \\
\leq C \|f\|_{B_{p,q}} \|\mathcal{F}^{-1}(\chi e^{-|\cdot|})\|_{B_{p',q'/q}}.
\]

So, it suffices to compute the norm of \( h = \mathcal{F}^{-1}(\chi e^{-|\cdot|}) = \Gamma_{\Omega}(\nu) e^{-\frac{1}{\nu}|\cdot|} \).

Now, proceeding as in (3.13) with the function \( h \) we obtain the estimate

\[
|h * \psi_j(x)| \leq C \Delta^{\frac{n}{4}}(\xi_j) \left(1 + |\xi_j|\right)^{2n} e^{-\frac{1}{4}\gamma|\xi_j|} \leq C' \Delta^{\frac{n}{4}}(\xi_j) e^{-\frac{1}{4}\gamma|\xi_j|} \left(1 + |g_j x|^2\right)^n.
\]

(3.46)

Taking \( L' \)-norms and summing, we are led to

\[
\|h\|_{B_{p',q'}} \leq C \left[ \int_{\Omega} \Delta(\xi)^{\nu\frac{q'}{q} + \frac{q'}{p} - \frac{n}{2}} e^{-\frac{1}{4}\gamma|\xi|} \frac{d\xi}{\Delta(\xi)^{\nu/4}} \right]^\frac{1}{\nu'}.
\]

By Lemma 2.16 this integral is finite provided \( q < \tilde{q}_{\nu,p} \).

For the second part, fix \( y \in \Omega \) and \( j \), and use the Dominated Convergence Theorem with \( \sum_k |\mathcal{E}(f * \psi_k)(x + iy)| \leq C_y \) (from the first part) to write

\[
\left(\mathcal{E}(f)(\cdot + iy) * \psi_j\right)(x) = \sum_k \left(\mathcal{E}(f * \psi_k)(\cdot + iy) * \psi_j\right)(x) \\
= \sum_k \int_{\Omega} \hat{f}(\xi) \hat{\psi}_k(\xi) e^{-iy|\xi|} \hat{\psi}_j(\xi) e^{i(x|\xi|)} d\xi \\
= \int_{\Omega} e^{i(x+iy|\xi|)} \hat{f}(\xi) \hat{\psi}_j(\xi) d\xi = \mathcal{E}(f * \psi_j)(x + iy).
\]

Summing in \( j \) and using again the previous step it follows that

\[
\mathcal{E}(f)(x + iy) = \sum_j \left(\mathcal{E}(f)(\cdot + iy) * \psi_j\right)(x),
\]

converging uniformly and absolutely in \( x \), and hence also in \( \mathcal{S}'(\mathbb{R}^n) \).

Let us finally prove the statements in (3.45). First of all,

\[
\|\mathcal{E} f(\cdot + iy)\|_{B_{p,q}}^q = \sum_j \Delta^{\nu}(\xi_j) \|\mathcal{F}^{-1}(\hat{\psi}_j e^{-|y|})\|_p^q \\
\leq \sum_j \Delta^{\nu}(\xi_j) \|f * \psi_j\|_p^q \|\mathcal{F}^{-1}(\hat{\chi}_j e^{-|y|})\|_1^q \leq C \|f\|_{B_{p,q}}^q,
\]

where in the last step we have estimated the \( L^1 \)-norm by a Schwartz seminorm

\[
\|\mathcal{F}^{-1}(\hat{\chi}_j e^{-|y|})\|_1 = \|\mathcal{F}^{-1}(\hat{\chi} e^{-|y|})\|_1 \leq C' \rho(\hat{\chi} e^{-|y|}) \\
\leq C' (1 + |g_j y|) e^{-\frac{1}{2\gamma}(g_j y)} \leq C'', \quad \forall j, \quad \forall y \in \Omega.
\]
Finally, for the convergence, use the density to write \( f = g + h \), with \( g \in D_\Omega \) and \( h \) with a small \( B^p,q_\nu \)-norm. It is well known that \( \mathcal{E}g(z) \) is smooth up to the boundary with convergence of \( \mathcal{E}g(\cdot + iy) \) to \( g \) in the \( S(\mathbb{R}^n) \)-topology (and by Proposition 3.16 also in the \( B^p,q_\nu \)-topology). The convergence for \( f \) then follows from a standard \( \varepsilon/3 \) argument. 

\[ \blacksquare \]

**Remark 3.48** Let us finally remark that the index \( \tilde{q}_{\nu,p} \) in the previous proposition is optimal. Indeed, from the duality theorem, the continuity of the linear form \( f \mapsto \mathcal{E}f(ie) \) will imply that \( \mathcal{F}^{-1}(\chi_\Omega e^{-|\xi|^2}) \) belongs to \( B^p, q' \), \( \nu < \tilde{q}_{\nu,p} \). This continues to be the case after convolution with \( \mathcal{F}^{-1}(e^{\xi} \varphi) \), for any \( \varphi \in C^\infty_c(\mathbb{R}^n) \) identically 1 in a neighborhood of 0. Thus, it follows from Proposition 3.16 that we must have \( q < \tilde{q}_{\nu,p} \).

## 4 Bergman spaces and projectors

In this section we shall show Theorems 1.7 and 1.8, as well as the boundedness of the Bergman projector announced in the introduction. Heuristically, the correspondence between holomorphic functions \( F \) in the Bergman space \( A^p,q_\nu \) and distributions \( f \) in \( B^p,q_\nu \), is given by the Fourier-Laplace formula:

\[
F(z) = \mathcal{E}f(z) = \mathcal{L}\hat{f}(z) = \int_\Omega e^{i(z|\xi)} \hat{f}(\xi) d\xi, \quad z \in T_\Omega.
\]

The distribution \( f \) plays the role of a Shilov boundary value for the holomorphic function \( F \). The main result in this section is the equivalence of norms \( \|F\|_{A^p,q_\nu} \sim \|f\|_{B^p,q_\nu} \), which follows from a suitable discretization of the integral above using the Whitney decomposition in §2. Several technical estimates will appear in this process, involving gamma integrals in \( \Omega \) and Littlewood-Paley inequalities as in (1.10), forcing us at some point to assume further restrictions in the indices \( \nu, p, q \). The sharp range of parameters for the equivalence of these two norms is still an open question, related to finer problems in Harmonic Analysis such as restriction and cone multipliers (see the Appendix for a further discussion on these matters).

Before going into the proof of the theorems we need some preliminaries. Recall that a holomorphic function \( F \in \mathcal{H}(T_\Omega) \) belongs to the Hardy space \( H^2(T_\Omega) \) when

\[
\|F\|_{H^2} = \sup_{y \in \Omega} \|F(\cdot + iy)\|_{L^2(\mathbb{R}^n)} < \infty.
\]

The following result is known as the *Paley-Wiener Theorem* for Hardy spaces (see Chapter III of [20]):

**Proposition 4.1** A function \( F \in H^2(T_\Omega) \) if and only if \( F = \mathcal{L}\hat{f} \) for some \( f \in L^2(\mathbb{R}^n) \) with \( \text{Supp} \ \hat{f} \subset \overline{\Omega} \). In this case, \( \|F\|_{H^2} = \|f\|_{L^2(\mathbb{R}^n)} \).
We will use the previous result in combination with the next one, whose proof is a simple modification of the one presented in [5] (see also [15]).

**Proposition 4.2** Let $\nu > \frac{n}{r} - 1$ and $1 \leq p, q < \infty$. Then, the norms of the spaces $A_{p,q}^{\nu}$ are complete. Moreover, the intersection $H^2(T_\Omega) \cap A_{p,q}^{\nu}$ is dense in $A_{p,q}^{\nu}$.

Our last preliminary result will be the starting point in the discretization steps to follow.

**Proposition 4.3** Let $\nu > \frac{n}{r} - 1$, $1 \leq p, q < \infty$. Then, for every lattice $\{y_j\}$ in $\Omega$ there exists $c > 0$ such that

$$\frac{1}{c} \| F \|_{A_{p,q}^{\nu}} \leq \left( \sum_j \Delta^\nu(y_j) \| F(\cdot + iy_j) \|_p^q \right)^{\frac{1}{q}} \leq c \| F \|_{A_{p,q}^{\nu}}, \quad \forall F \in A_{p,q}^{\nu}(T_\Omega). \quad (4.4)$$

**Proof:**

The proof relies on the following elementary lemma:

**Lemma 4.5** Let $1 \leq p, q < \infty$. Then, for every $F \in H(T_\Omega)$ and $y_0 \in \Omega$ we have

$$\| F(\cdot + iy_0) \|_p \leq C \left[ \int_{B(y_0,1)} \| F(\cdot + iy) \|_p^q \frac{dy}{\Delta^\nu(y)} \right]^{\frac{1}{q}}, \quad (4.6)$$

where the constant depends only on $p, q$.

**Proof of Lemma 4.5:**

By homogeneity, we may assume $y_0 = e$. Let us consider first the case $p \leq q$. Then, by the mean value property for subharmonic functions,

$$|F(x + ie)|_p \leq c \int_{B(e,1)} \int_{|x'| \leq 1} |F(x + x' + iy)|_p \, dx' \, dy.$$

Thus, integrating in $x$ and using Hölder’s inequality we obtain

$$\| F(\cdot + ie) \|_p^p \leq c \int_{B(e,1)} \| F(\cdot + iy) \|_p^p \, dy \leq c \left( \int_{B(e,1)} \| F(\cdot + iy) \|_p^q \, dy \right)^{\frac{p}{q}}.$$

Suppose instead that $q \leq p$. Then, the mean value property gives

$$|F(x + ie)|_p \leq c \left( \int_{B(e,1)} \int_{|x'| \leq 1} |F(x + x' + iy)|^q \, dx' \, dy \right)^{\frac{1}{q}}.$$

A new integration in $x$ and Minkowski’s inequality gives the same result. \qed
Continuing with the proposition, we assume for simplicity that \( \{y_j\} \) is a \((\frac{1}{2}, 2)\)-lattice, that is, \( \{B_1(y_j)\}_j \) covers \( \Omega \) with the finite intersection property. The right hand side of (4.4) follows from a discretization of the integral on \( \Omega \) defining \( \|F\|_{A_{p,q}^e}^q \) as in Proposition 2.13. We then use the previous lemma to conclude. For the left hand side we just need a kind of converse for (4.6), where as before we can assume \( \|F\|_{A_{p,q}^e}^q \) as in Proposition 2.13. We then use the previous lemma to conclude. For the left hand side of (4.4) follows from a discretization of the integral on \( \Omega \) defining \( \|F\|_{A_{p,q}^e}^q \) as in Proposition 2.13. We then use the previous lemma to conclude. For the left hand side we just need a kind of converse for (4.6), where as before we can assume \( y_0 = e \). But this follows from the fact that, when \( F \in A_{p,q}^e \), the function \( y \to \|F(\cdot + iy)\|_p \) is monotonic in \( \Omega \) (see, e.g., [5, 15]), and therefore

\[
\left[ \int_{B(e \ell, 1)} \|F(\cdot + iy)\|_p^q \frac{dy}{\Delta^2(y)} \right]^{\frac{1}{q}} \leq C \|F(\cdot + ic)\|_p,
\]

for some constant \( c = c(\Omega) > 0 \). This establishes the proposition.

\[\square\]

### 4.1 The proof of Theorem 1.7

We wish to show that every \( F \in A_{p,q}^e \) can be written as \( F = \tilde{E}f \) for some distribution \( f \in B_{p,q}^e \). Suppose first that \( F \) belongs to the dense set \( H^2(T\Omega) \cap A_{p,q}^e \), so that, by Proposition 4.1, \( F = \mathcal{L}\hat{f} \) for some function \( f \in L^2(\mathbb{R}^n) \) with \( \text{Supp} \ f \subset \overline{\Omega} \). We shall show the inequality \( \|f\|_{B_{p,q}^e} \leq C \|F\|_{A_{p,q}^e} \). Observe that, since \( f = \sum_j f * \psi_j \) in \( L^2 \) (hence in \( \mathcal{S}'(\mathbb{R}^n) \)), it will follow from this and the definition of \( \tilde{E} \) that \( \tilde{E}f = \mathcal{L}\hat{f} \).

To prove the inequality of norms, let us first denote \( y_j = \xi_j^{-1} \) the dual lattice of \( \{\xi_j\} \). Then, Young's inequality gives

\[
\|f * \chi\|_p = \|\mathcal{F}^{-1}(\hat{f}(\xi)e^{-(y_j|\xi|)}\hat{\chi}_j(\xi))e^{(y_j|\xi|)}\|_p \leq \|\mathcal{F}^{-1}(\hat{f}e^{-(y_j|\xi|)})\|_p \|\mathcal{F}^{-1}(\hat{\chi}_j)e^{(y_j|\xi|)}\|_1.
\]

Since \( \xi_j^{-1} = g_j^{-1}e \) and \( g_j \) is self-adjoint, we observe that the last factor is actually constant,

\[
\|\mathcal{F}^{-1}(\hat{\chi}_j)e^{(y_j|\xi|)}\|_1 = \|\mathcal{F}^{-1}(\hat{\chi}e^{(y_j|\xi|)}\|_1 = c_1 < \infty.
\]

This leads to the estimate

\[
\|f\|_{B_{p,q}^e}^q \leq c \sum_j \Delta^{-\nu}(\xi_j) \|f * \chi\|_p^q \\
\leq c' \sum_j \Delta^{\nu}(y_j) \|\mathcal{F}^{-1}(\hat{f}e^{-(y_j|\xi|)})\|_p^q \\
 = c' \sum_j \Delta^{\nu}(y_j) \|F(\cdot + iy_j)\|_p^q \leq c'' \|F\|_{A_{p,q}^e}^q.
\]

(4.7)

where in the last inequality we have used Proposition 4.3.

For general \( F \in A_{p,q}^e \) one proceeds by density. Approximating with \( \{F_m\} \) in \( H^2(T\Omega) \cap A_{p,q}^e \), we obtain a corresponding sequence of functions \( f_m \in L^2(\mathbb{R}^n) \), which
by (4.7) is a Cauchy sequence in $B_p^\nu$. Then, by completeness of this space and Lemma 3.38, $f_m$ converges (in $B_p^\nu$ and in $S'$) to a distribution $f \in B_p^\nu$. Moreover, $\|f\|_{B_p^\nu} \leq C \|F\|_{A_p^\nu}$. It remains to prove that $F = \tilde{E}(f)$, for which we can use the continuity of the pointwise evaluation functional $f \mapsto \tilde{E}(f)$ in (3.44). Indeed, for each $z \in T_\Omega$ we have

$$\tilde{E}(f)(z) = \lim_{m \to \infty} \tilde{E}(f_m)(z) = \lim_{m \to \infty} \hat{L}(f_m)(z) = \lim_{m \to \infty} F_m(z) = F(z).$$

Finally, the convergence

$$\lim_{y \to 0} y \in \Omega F(\cdot + iy) = f,$$

in $B_p^\nu$ and $S'(\mathbb{R}^n)$, is just a consequence of Proposition 3.43 and Lemma 3.38. This completes the proof of the theorem.

\subsection*{4.2 The proof of Theorem 1.8}

We start with a preliminary result which is the dual version of (1.10), quoted in the introduction.

\begin{lemma}
Let $1 \leq p < \infty$ and let $1 \leq s \leq p = \min\{p, p'\}$. Then there exists a constant $C$ such that, for every sequence of functions $f_j \in L^p(\mathbb{R}^n)$ satisfying $\text{Supp} \hat{f}_j \subset B_2(\xi_j)$, we have the inequality

$$\|\sum_j f_j\|_p \leq C \left(\sum_j \|f_j\|_p^s\right)^{\frac{1}{s}}. \quad (4.9)$$

\end{lemma}

\begin{proof}
It is sufficient to prove the stronger inequality

$$\|\sum_j f_j \ast \chi_j\|_p \leq \left(\sum_j \|f_j\|_p^s\right)^{\frac{1}{s}},$$

valid for all sequences of functions in $L^p$. The $\chi_j$ are chosen as in §3.1 with their $L^1$ norm uniformly bounded, and their Fourier transform supported in $B_4(\xi_j)$ and identically 1 on the ball $B_2(\xi_j)$. For this last inequality, the proof is immediate when $s = 1$ by Minkowski’s inequality, as well as for $s = p = 2$ by the finite intersection property of the balls. We interpolate between these two cases to conclude.

\end{proof}

\begin{remark}
Our proof for Theorem 1.8 depends directly on the previous simple inequality, in which unfortunately the best exponent $s$ for each fixed $p$ seems not to be known (ideally, $s = 2$ would be the best possible). The reader can track down in our proof below that any improvement over $s = p$, for a fixed $p$, will end up in the validity of the theorem for all $q < sq_\nu$ and the same $p$. For this reason we state below a result, where a more general inequality than (4.9) is assumed to hold true. We shall discuss in the Appendix the validity of such inequalities in the case of light-cones.

\end{remark}
\textbf{Proposition 4.11} Let $\nu > \frac{q}{p'} - 1$, $1 \leq p, s < \infty$. Assume there exist numbers $\mu, \delta \geq 0$ and a constant $C = C(\mu, \delta) > 0$ such that

$$\|\sum_j f_j\|_p \leq C \left[ \sum_j \Delta^{-\mu}(\xi_j) e^{\delta(\xi_j|e)} \|f_j\|_p^s \right]^{\frac{1}{s}}$$

holds for every finite sequence $\{f_j\} \subset L^p(\mathbb{R}^n)$ satisfying $\text{Supp } \hat{f}_j \subset B_2(\xi_j)$. Then, for every index

$$q < \min \left\{ s \frac{q_\nu}{q_0}, s \frac{\nu - (p - 1)}{\mu}, \tilde{q}_{\nu, p} \right\},$$

and for every distribution $f$ with $\|f\|_{B^{p,q}_0} < \infty$, the function $F = \mathcal{E}(\sum_j f \ast \psi_j)$ belongs to $A^{p,q}_\nu$, and moreover,

$$\|F\|_{A^{p,q}_\nu} \lesssim \|f\|_{B^{p,q}_0}.$$

Using Lemma 4.8, it is clear that Proposition 4.11 suffices to establish Theorem 1.8. Indeed, the lemma implies the assumption of the proposition with $\mu = \delta = 0$ and $s = p_t = \min\{p, p'\}$. In this case $q_\mu = 1$, so if we assume the validity of the proposition the condition on $q$ simplifies into $q < \min\{sq_\nu, \tilde{q}_{\nu, p}\} = p_0q_\nu$, as stated in Theorem 1.8.

We turn now to the proof of Proposition 4.11. We shall show that, for every $f \in D_\Omega$, the function $F(z) := \mathcal{E}f(z) = \mathcal{L}\hat{f}(z)$ belongs to $A^{p,q}_\nu(T_1)$, and $\|F\|_{A^{p,q}_\nu} \leq C\|f\|_{B^{p,q}_0}$. This will be enough to conclude, since in the general case one can proceed by density. We will need an intermediate result, which we will comment later.

\textbf{Lemma 4.14} Let $1 \leq p, s < \infty$, and assume that (4.12) holds for some $\mu, \delta \geq 0$. Then, for every $f \in D_\Omega$ and $y \in \Omega$, the function $F(\cdot + iy) = \mathcal{E}f(\cdot + iy)$ belongs to $L^p(\mathbb{R}^n)$. Moreover,

$$\|F(\cdot + iy)\|_p \lesssim \Delta^{-\frac{p}{2}}(y)\|f\|_{B^{p,s}_\nu},$$

with constants independent of $f$ or $y \in \Omega$.

\textbf{Proof:} By homogeneity (see Lemma 3.8), it is sufficient to prove (4.15) when $y = \eta e$, for some fixed $\eta > 0$ to be chosen below. Now, given $f \in D_\Omega$, let us define $g \in D_\Omega$ by $\hat{g} = \hat{f} e^{-\eta(\xi_j|e)}$. Then, applying the assumption to $g = \sum_j g \ast \psi_j$, we obtain

$$\|g\|_p = \|F(\cdot + i\eta e)\|_p \lesssim \left( \sum_j \Delta^{-\mu}(\xi_j) e^{\delta(\xi_j|e)} \|F^{-1}(\hat{f} \psi_j e^{-\eta(\xi_j|e)}\|_p^s \right)^{\frac{1}{s}}$$

$$\lesssim \left( \sum_j \Delta^{-\mu}(\xi_j) e^{\delta(\xi_j|e)} \|f \ast \psi_j\|_p^s \|F^{-1}(e^{-\eta(\xi_j|e)}\chi_j)\|_1 \right)^{\frac{1}{s}}$$

$$\lesssim \left( \sum_j \Delta^{-\mu}(\xi_j) \|f \ast \psi_j\|_p^s \right)^{\frac{1}{s}}.$$

In the last step we have used the fact, from (3.47), that $\|F^{-1}(e^{-\eta(\xi_j|e)}\chi_j)\|_1$ can be bounded, up to some constant, by $e^{-\gamma\eta(\xi_j|e)}$. We conclude by choosing any $\eta$ larger than $\delta/\gamma$. 
Let us go back to the proof of the proposition. Given \( f \in D_\Omega \), and \( F(z) := \mathcal{E} f(z) = \mathcal{L} f(z) \), the previous lemma applied to \( F^{-1}(\hat{f} e^{-\gamma(yz)}) \) gives us

\[
\|F(\cdot + i2y)\|_p \lesssim \Delta^{-\frac{n}{s}}(y) \left[ \sum_j \Delta^{-\mu}(\xi_j) \|F^{-1}(\hat{f} \psi_j e^{-\gamma(yz)})\|_p^s \right]^{\frac{1}{2}} \\
\lesssim \Delta^{-\frac{n}{s}}(y) \left[ \sum_j \Delta^{-\mu}(\xi_j) e^{-\gamma(y\xi_j)} \|f \ast \psi_j\|_p^s \right]^{\frac{1}{2}},
\]

where once again we have used (3.47). Thus,

\[
I := \int_\Omega \|F(\cdot + iy)\|_p^q \Delta^{-\frac{n}{s}}(y) dy \\
\lesssim \int_\Omega \Delta^{-\frac{n}{s}}(y) \left( \sum_j \Delta^{-\mu}(\xi_j) e^{-\gamma(y\xi_j)} \|f \ast \psi_j\|_p^s \right) \Delta^{-\frac{n}{s}}(y) dy.
\]

When \( q \leq s \) we directly conclude

\[
I \lesssim \sum_j \Delta^{-\frac{n}{s}}(\xi_j) \|f \ast \psi_j\|_p^q \int_\Omega \Delta^{-\frac{n}{s}}(y) e^{-\gamma(y\xi_j)} \Delta^{-\frac{n}{s}}(y) dy,
\]

where the gamma integral equals a multiple of \( \Delta(\xi_j)^{\frac{n}{s} - \nu} \), whenever \( \nu - \frac{n}{s} > 1 \). This leads to one of the conditions stated in (4.13).

Suppose now that \( q > s \). Then we multiply and divide the summands by \( \Delta_\mathbf{t}(\xi_j) \), for some multi-index \( \mathbf{t} = (t_1, \ldots, t_r) \in \mathbb{R}^r \) to be chosen below. After applying Hölder’s inequality, we obtain

\[
I \lesssim \int_\Omega \Delta^{-\frac{n}{s}}(y) \left[ \sum_j \Delta^{-\frac{n}{s}}(\xi_j) e^{-\gamma(y\xi_j)} \|f \ast \psi_j\|_p^q \Delta_\mathbf{t}(\xi_j) \right] \\
\times \left[ \sum_j \Delta_\mathbf{t}(\xi_j) e^{-\gamma(y\xi_j)} \right] \Delta(y)^{\nu - \frac{n}{s}} dy.
\]

According to Proposition 2.13, the last bracket can be transformed into a gamma integral, which in order to be finite requires the following condition in the indices

\[
t_j(q/s)' > (j - 1) \frac{n/r - 1}{r - 1}, \quad \text{for all} \ j = 1, \ldots, r.
\]

Thus, replacing the expression inside the brackets by a multiple of \( \Delta_\mathbf{t}^{*}(q/s)'(y) \), we have

\[
I \lesssim \sum_j \Delta^{-\frac{n}{s}}(\xi_j) \|f \ast \psi_j\|_p^q \Delta_\mathbf{t}(\xi_j) \int_\Omega \Delta_\mathbf{t}^{*}(\cdot + \mu)(y) e^{-\gamma(y\xi_j)} \Delta(y)^{\nu - \frac{n}{s}} dy \\
\lesssim \sum_j \Delta^{-\nu}(\xi_j) \|f \ast \psi_j\|_p^q.
\]
In the last step we have computed again the gamma integral, which gives a finite multiple of \( \Delta_{(t+\mu)^{\frac{q}{s} - \nu}}(\xi_j) \) if we impose the condition in the indices:

\[
-\frac{q}{s} (t_r - (j-1) + \mu) + \nu > (j-1) \frac{n/r - 1}{r-1}, \quad \text{for all } j = 1, \ldots, r.
\]

Therefore, the \( t_j \)'s must be chosen so that:

\[
\frac{1}{(q/s)'} \frac{j-1}{r-1} \left( \frac{p}{r} - 1 \right) < t_j < \frac{1}{q/s} \left( \nu - \frac{r-j}{r-1} \left( \frac{p}{r} - 1 \right) \right) - \mu, \quad j = 1, \ldots, r.
\]

Using \( \frac{1}{(q/s)'} = 1 - \frac{1}{q/s} \), we see that this is only possible when

\[
\frac{j-1}{r-1} \left( \frac{p}{r} - 1 \right) + \mu < \frac{1}{q/s} \left( \nu - \frac{p}{r} - 1 + 2 \frac{j-1}{r-1} \left( \frac{p}{r} - 1 \right) \right), \quad j = 1, \ldots, r.
\]

Solving for \( q/s \), this forces us to have

\[
\frac{q}{s} < \min_{1 \leq j \leq r} \frac{\nu - \left( \frac{p}{r} - 1 \right) + 2 \frac{j-1}{r-1} \left( \frac{p}{r} - 1 \right)}{\mu + \frac{j-1}{r-1} \left( \frac{p}{r} - 1 \right)} = \begin{cases} \frac{q\nu}{q\mu}, & \nu > 2\mu + \frac{p}{r} - 1; \\ \frac{\nu - \left( \frac{p}{r} - 1 \right)}{\mu}, & \nu \leq 2\mu + \frac{p}{r} - 1. \end{cases}
\]

This is precisely the range of \( \nu \) and \( q \) assumed in (4.13) so the theorem is completely proved.

We conclude this subsection with some equivalent versions of the sufficient condition (4.12) when \( \mu > 0 \). These are convenient expressions which we shall relate in the Appendix with known inequalities in light-cones.

**Proposition 4.16** Let \( 1 \leq p, s < \infty \) and \( \mu > 0 \). Then the following properties are equivalent:

1. There exists \( \delta > 0 \), and a constant \( C_\delta > 0 \) such that

\[
\| \sum_j f_j \|_p \leq C_\delta \left[ \sum_j \Delta^{-\mu}(\xi_j) e^{\delta(\xi_j \cdot e)} \| f_j \|_p^s \right]^\frac{1}{s},
\]

for every finite sequence \( \{ f_j \} \) in \( L^p(\mathbb{R}^n) \) satisfying \( \text{Supp } \hat{f}_j \subset B_2(\xi_j) \).

2. There exists a constant \( C > 0 \) such that

\[
\| \sum_j f_j \|_p \leq C \left[ \sum_j \Delta^{-\mu}(\xi_j) \| f_j \|_p^s \right]^\frac{1}{s},
\]

for every finite sequence \( \{ f_j \} \) in \( L^p(\mathbb{R}^n) \) satisfying \( \text{Supp } \hat{f}_j \subset B_2(\xi_j) \cap H_0 \), where \( H_0 \) is the band in between two hyperplanes \( H_0 = \{ \frac{1}{2} < (e \cdot \xi) < 2 \} \).
3. There exists a constant $C > 0$ such that

$$\| \mathcal{E}f(\cdot + iy) \|_p \leq C \Delta^{-\frac{\mu}{2}}(y) \| f \|_{B_{\mu}^p}, \quad \forall f \in D_\Omega, \quad \forall y \in \Omega. \quad (4.19)$$

**Proof:** We have already proved in Lemma 4.14 that $(4.17) \Rightarrow (4.19)$. To prove that $(4.19) \Rightarrow (4.18)$, take a corresponding sequence $\{f_j\}$ as in (4.18). Define the functions $\hat{g}_j = e^{(e_1)} \hat{f}_j$, and apply the inequality (4.19) for $y = e$ to the function $g = \sum_j g_j \in D_\Omega$. Then,

$$\| \sum_j f_j \|_p = \| \mathcal{F}^{-1}(\hat{g}e^{-e_1}) \|_p \leq C \left[ \sum_k \Delta^{-\mu}(\xi_k) \| \mathcal{F}^{-1}(\hat{g}\psi_k) \|_p^s \right]^\frac{1}{s}. $$

Using the finite intersection property, and the fact that the $L^p$-norms of $f_j$ and $g_j$ are comparable, we obtain easily the right hand side of (4.18).

It remains to prove that $(4.18) \Rightarrow (4.17)$. To do this, we are going to slice the cone with hyperplanes, and then apply an scaled version of (4.18) to the restrictions of $\sum_j f_j$ to the bands

$$H_k = \{ 2^{k-1} < (\xi \mid e) < 2^{k+1} \}, \quad k \in \mathbb{Z}. $$

To do this argument precise, we select a sequence of smooth 1-variable functions $\{\rho_k\}$ so that $\text{Supp} \rho_k \subset (2^{k-1}, 2^{k+1})$ and $\sum_{k \in \mathbb{Z}} \rho_k \equiv 1 \in (0, \infty)$. We let $\hat{f}_{j,k}(\xi) = \rho_k((\xi \mid e)) \hat{f}_j(\xi)$, so that

$$\text{Supp} \hat{f}_{j,k} \subset B_2(\xi_j) \cap H_k \quad \text{and} \quad \| f_{j,k} \|_p \leq C \| f_j \|_p. \quad (4.20) $$

By Minkowski’s inequality we can write

$$I = \| \sum_j f_j \| \leq \sum_{k \in \mathbb{Z}} \| \sum_{j \in J_k} f_{j,k} \| =: \sum_{k \in \mathbb{Z}} \| F_k \|_p, $$

where the sets of indices $J_k$ are defined so that $B_2(\xi_j) \cap H_k \neq \emptyset$. In order to estimate the norm of $F_k$, we must first perform a dilation by $\delta = 2^{-k}$ so that, replacing $F_k$ with $F_k^{(\delta)} = \delta^\mu F_k(\delta \cdot)$, we do not change the $L^p$-norms and the Fourier transform is now supported in $H_0$. Thus, we are in conditions of applying (4.18):

$$\| F_k \|_p = \| \sum_\ell F_k^{(\delta)} * \psi_\ell \|_p \leq C \left[ \sum_\ell \Delta^{-\mu}(\xi_\ell) \| F_k^{(\delta)} * \psi_\ell \|_p^s \right]^\frac{1}{s}. $$

Now, observe that each $f_{j,k}^{(\delta)}$ has Fourier transform supported in $B_2(\delta \xi_j)$, and $\{\delta \xi_j\}$ is still a $(\frac{1}{2}, 2)$-lattice in the cone. Thus, by the finite intersection property, the set of indices $j$ for which $B_2(\delta \xi_j)$ intersects a fixed set $B_2(\xi_\ell)$ has at most $N = N(\Omega)$ elements, independently of $\ell$ and $\delta$. Thus,

$$\| F_k \|_p \leq \sum_\ell \Delta^{-\mu}(\xi_\ell) \sum_j \| f_{j,k}^{(\delta)} * \psi_\ell \|_p^s. $$
Now, changing the order of sums, restricting \( \ell \) to the bounded set of indices

\[
\tilde{J}_j = \{ \ell : B_2(\xi_{\ell}) \cap B_2(\delta \xi_j) \neq \emptyset \},
\]

and using that \( \Delta(\xi_{\ell}) \sim \Delta(\delta \xi_j) \) for such indices, we obtain

\[
\|F_k\|_p^s \lesssim \sum_{j} \Delta^{-\mu}(\delta \xi_j) \sum_{\ell \in \tilde{J}_j} \|f^{(\ell)}_{j,k}\|_p^s \|\psi_{\ell}\|_1^s
\leq \delta^{-\mu} \sum_{j} \Delta^{-\mu}(\xi_{\ell}) \|f_{j,k}\|_p^s \lesssim 2^{k\mu r} \sum_{j \in \tilde{J}_k} \Delta^{-\mu}(\xi_{\ell}) \|f_{j,k}\|_p^s;
\]

where in the last step we have also used (4.20). Thus, raising to the power \( 1/s \) and summing in \( k \), we have shown that

\[
I \lesssim \sum_{k \in \mathbb{Z}} \left[ \sum_{j \in \tilde{J}_k} \Delta^{-\mu}(\xi_{\ell}) \|f_{j,k}\|_p^s \right]^{1/s} 2^{k\mu r}.
\]

Multiplying and dividing by \( e^{2k} \), and applying H"older we obtain

\[
I \lesssim \left[ \sum_{k \in \mathbb{Z}} \sum_{j \in \tilde{J}_k} \Delta^{-\mu}(\xi_{\ell}) e^{s2k} \|f_{j,k}\|_p^s \right]^{1/s} \left[ \sum_{k \in \mathbb{Z}} 2^{k\mu rs'/s} e^{-s'2k} \right]^{1/s}.
\]

The last term is a finite constant when \( \mu > 0 \), while in the first factor we can replace \( e^{s2k} \) by \( e^{\eta (\xi_{\ell}|e)} \), for a sufficiently large \( \eta \). To conclude it remains only to show that, for each fixed \( j \), the set of all \( k \in \mathbb{Z} \) such that \( H_k \) intersects \( B_2(\xi_j) \) contains at most \( N = N(\Omega) \) elements. To see this observe, from Lemma 2.9, that each such \( k \) must satisfy

\[
\frac{2^{k-1}}{\gamma} < (\xi_{\ell}|e) < 2^{k+1},
\]

or equivalently \( \frac{1}{2\gamma} (\xi_{\ell}|e) < 2^k < 2\gamma (\xi_{\ell}|e) \). Taking logarithms we see that this is only possible for a constant number of such \( k \)’s. The proof of Proposition 4.16 is then complete.

\[\square\]

### 4.3 Boundedness of Bergman projectors in \( L^{p,q}_\nu \)

In this section we shall prove that the norm equivalence between \( A^{p,q}_\nu \) and \( B^{p,q}_\nu \) is equivalent to the boundedness of the Bergman projector \( P_\nu \) in \( L^{p,q}_\nu \) spaces.

Recall that the Bergman projector \( P_\nu \) is defined for functions \( F \in L^2_\nu(T_\Omega) \) as

\[
P_\nu F(x + i y) = \int_{\mathbb{R}^n} \int_{\Omega} B_\nu(x - u + i(y + v)) F(u + iv) \Delta(v)^{\nu - \frac{n}{\tau}} \, dv du,
\]

where the Bergman kernel has the well-known expression

\[
B_\nu(z - w) = d(\nu) \Delta^{-(\nu + \frac{n}{\tau})}(z - w)/i = c_\nu \int_\Omega e^{i(z - w)\xi} \Delta(\xi)^{\nu} \, d\xi, \quad z, w \in T_\Omega,
\]

\[\text{(4.21)}\]
for some positive constants $c_\nu, d(\nu)$ (see, e.g., Chapter XIII of [13]). It is clear that $P_\nu F(z)$ defines a holomorphic function in $T_\Omega$ whenever the integral in (4.21) converges absolutely. The following lemma shows that this is the case exactly when $F \in L_{\nu,q}^{p,q}$ and $q < \tilde{q}_{\nu,p}$. This elementary fact also gives us a trivial range of unboundedness for $P_\nu$ (see [6]).

**Lemma 4.23** Let $\nu > \frac{n}{r} - 1$ and $1 \leq p < \infty$. Then

$$B_\nu(z + i\epsilon) \in L_{\nu,q}^{p,q}(T_\Omega) \iff q < \tilde{q}_{\nu,p} := \frac{\nu + \frac{n}{r} - 1}{(\frac{n}{r} - 1)^+}, \quad (4.24)$$

Moreover, if $q \leq \tilde{q}_{\nu,p}$ or $q \geq \tilde{q}_{\nu,p}$ then $P_\nu$ does not admit bounded extensions into $L_{\nu,q}^{p,q}$.

**Proof:** The first statement is an elementary application of Lemma 2.18 to the formula in (4.22). For the second statement test with $F(z) = \Delta^{-\nu + \frac{n}{r}}(\Im z)Q(\Re \epsilon)(z)$, where $Q(\Re \epsilon)$ is a closed polydisk in $T_\Omega$ centered at $i\epsilon$. Then

$$P_\nu F(z) = c_n B_\nu(z + i\epsilon), \quad z \in T_\Omega,$$

by the mean value property for (anti)-holomorphic functions. Therefore, if $q \geq \tilde{q}_{\nu,p}$, $P_\nu$ cannot be bounded into $L_{\nu,q}^{p,q}$, and by self-adjointness neither into $L_{\nu,q}^{p,q}$.

We pass now to the study of boundedness of $P_\nu$ in $L_{\nu,q}^{p,q}$ when $\tilde{q}_{\nu,p} < q < \tilde{q}_{\nu,p}$. This is a difficult open question for which only partial results are known (see [5] for the light-cone, and [4] for the simpler case $L_{\nu,q}^{2,q}$). We prove here the following equivalence between this problem and the kind of estimates for Fourier-Laplace integrals that we have considered. We will see that it is an easy consequence of our previous study.

**Theorem 4.25** Let $\nu > \frac{n}{r} - 1$ and $2 \leq q < \tilde{q}_{\nu,p}$. Then, the Bergman projector $P_\nu$ is bounded in $L_{\nu,q}^{p,q}$ if and only if there exists a constant $C$ such that

$$\|\mathcal{L}\hat{f}\|_{L_{\nu,q}^{p,q}} \leq C\|f\|_{B_{\nu,q}^{p,q}}, \quad f \in \mathcal{D}_\Omega. \quad (4.26)$$

**Proof of the necessity condition:** Let $2 \leq q < \tilde{q}_{\nu,p}$ and assume that the projector $P_\nu$ is bounded in $L_{\nu,q}^{p,q}$. We want to compute $\|\mathcal{L}\hat{f}\|_{L_{\nu,q}^{p,q}}$ for $f \in \mathcal{D}_\Omega$. It is sufficient to test it on functions in $L_{\nu,q}^{p,q} \cap L_{\nu,2}^{2}$. Moreover, since $\mathcal{L}\hat{f} \in A_{\nu,q}^{p,q}$ and the projection is self-adjoint (hence, bounded in $L_{\nu,q}^{p,q}$), we can as well test it on functions which are in $A_{\nu,q}^{p,q} \cap A_{\nu,2}^{2}$. Such functions may be written as $\tilde{E}g = \mathcal{L}\hat{g}$, with $g = \sum_j g * \psi_j \in B_{\nu,q}^{p,q}$, and we know from Theorem 1.8 that, for this range of exponents, the norm of $\mathcal{L}\hat{g}$ in $A_{\nu,q}^{p,q}$ is equivalent to the norm of $g$ in $B_{\nu,q}^{p,q}$. So it is sufficient to prove that

$$\left| \int_{\mathbb{R}^n} \int_{\Omega} \mathcal{L}\hat{f}(x + iy) \mathcal{L}\hat{g}(x + iy) \Delta(y)^{\nu - \frac{n}{r}} dy \cdot dx \right| \leq C\|f\|_{B_{\nu,q}^{p,q}} \|g\|_{B_{\nu,q}^{p,q}},$$
for some constant $C$ which does not depend on $f$ and $g$. Using the Paley-Wiener Theorem for $A^2_\nu$ (see, e.g., [13, p. 260]), we know that the left hand side is equal to

$$\left| \int_{\Omega} \hat{f}(\xi) \overline{\hat{g}(\xi)} \frac{d\xi}{\Delta(\xi)^\nu} \right|^\nu.$$

Then, Plancherel’s Theorem and the definition of the fractional powers of $\Box$ tell us that this is the usual duality pairing between $f$ and $\Box^{-\nu}g$. As a consequence, it is bounded by

$$C \|f\|_{B^p,q_\nu} \|\Box^{-\nu}g\|_{B^{p',q'}_{-\nuq'/q}}.$$

To conclude, we use Proposition 3.31, which gives the equivalence of the norm $\|\Box^{-\nu}g\|_{B^{p',q'}_{-\nuq'/q}}$ with the norm $\|g\|_{B^{p',q'}_{\nu}}$. This finishes the proof of this direction.

For the other direction, we will prove a little more. We will show that $P_\nu$ is always bounded from $L^{p,q}_\nu$ into a new holomorphic function space $B^{p,q}_\nu := \tilde{E}(B^{p,q}_\nu)$, consisting of Fourier-Laplace transforms of distributions in $B^{p,q}_\nu$. Then one concludes easily from there and the identification $B^{p,q}_\nu = A^{p,q}_\nu$ in the previous section. To make these statements precise we begin with the definition of $B^{p,q}_\nu$.

**Definition 4.27** Given $\nu > 0$, $1 \leq p < \infty$ and $1 \leq q < \tilde{q}_{\nu,p}$, we define the holomorphic function space

$$B^{p,q}_\nu(T_\Omega) := \left\{ F = \tilde{E}f = \sum_j L(\hat{f}\hat{\psi}_j) : f \in S'_\Omega \text{ with } \|f\|_{B^{p,q}_\nu} < \infty \right\},$$

endowed with the norm $\|F\|_{B^{p,q}_\nu} = \|f\|_{B^{p,q}_\nu}$.

By Proposition 3.43, it follows that $\tilde{E} : B^{p,q}_\nu \to B^{p,q}_\nu$ is an isomorphism of Banach spaces, $B^{p,q}_\nu$ is continuously embedded in $H(T_\Omega)$ and its functions satisfy the inequality

$$|F(x + iy)| \leq C \Delta(y)^{-\frac{1}{2} - \frac{\nu}{p}} \|f\|_{B^{p,q}_\nu}, \quad x + iy \in T_\Omega.$$

In this context, Theorems 1.7 and 1.8 can be written as well as

$$A^{p,q}_\nu \subset B^{p,q}_\nu \text{ when } 1 \leq q < \tilde{q}_{\nu,p}, \quad \text{ and } \quad A^{p,q}_\nu = B^{p,q}_\nu \text{ when } 1 \leq q < q_{\nu,p}.$$

Observe also that $A^{p,q}_\nu$ is a dense subspace of $B^{p,q}_\nu$ (since $E(D_\Omega) \subset A^{p,q}_\nu$), but in general is not closed. In fact, examples in the next subsection show that the inclusion is strict whenever $q \geq \min\{2, p\} q_{\nu}$. We now prove the announced statement, which allows to conclude for the proof of Theorem 4.25.
Proposition 4.28 Let $\nu > \frac{n}{r} - 1$, $1 \leq p < \infty$ and $2 \leq q < q_{\nu,p}$. Then, with the previous notation, $P_{\nu}$ extends as a bounded operator from $L^{p,q}_{\nu}$ into $B^{p,q}_{\nu}$. That is, for every $F \in L^{p,q}_{\nu}$ there exists $g \in B^{p,q}_{\nu}$ such that $P_{\nu}F = \mathcal{E}g$ with
\[
\|P_{\nu}F\|_{B^{p,q}_{\nu}} = \|g\|_{B^{p,q}_{\nu}} \leq C\|F\|_{L^{p,q}_{\nu}}, \quad F \in L^{p,q}_{\nu}(T_{\Omega}).
\] (4.29)

Remark 4.30 Observe that for $F \in A^{p}_{\nu} \cap A^{p,q}_{\nu}$, $P_{\nu}F = F = \mathcal{E}f$, and therefore, (4.29) actually generalizes the inequality $\|f\|_{B^{p,q}_{\nu}} \leq C\|F\|_{A^{p,q}_{\nu}}$ in Theorem 1.7.

Proof: It is enough to show (4.29) for functions $F$ in the dense set $L^{p}_{\nu} \cap L^{p,q}_{\nu}$, proceeding otherwise as in the last part of §4.1. Since $P_{\nu}$ is a projector, for such functions we will have $P_{\nu}F \in A^{p}_{\nu}$, and therefore, by the Paley-Wiener Theorem for $A^{p}_{\nu}$ there exists a unique function $\hat{g} \in L^{2}(\Omega; \Delta^{-\nu}(\xi)d\xi)$ such that
\[
P_{\nu}F(z) = \mathcal{L}\hat{g}(z) = \int_{\Omega} e^{i(x+iy)\xi} \hat{\varphi}(\xi) d\xi, \quad z = x + iy \in T_{\Omega}.
\] (4.31)

Observe that $g = \sum_{j} g \ast \psi_{j}$ in $S'(\mathbb{R}^{n})$ (since $\hat{g} = \sum_{j} \hat{\psi}_{j}$ in $L^{2}(\Omega; \Delta^{-\nu}(\xi)d\xi)$), and therefore $P_{\nu}F = \mathcal{L}\hat{g} = \mathcal{E}g$. Thus, to prove $P_{\nu}F \in B^{p,q}_{\nu}$ we just need to bound $\|g\|_{B^{p,q}_{\nu}}$. From the duality of Besov spaces (Proposition 3.27), it follows that it is sufficient to prove that
\[
|\langle g, \varphi \rangle| \leq C\|F\|_{L^{p,q}_{\nu}}\|\varphi\|_{B^{p',q'}_{-\nu/q}}, \quad \varphi \in D_{\Omega}.
\]

As in the previous proof, we use the fact that
\[
\langle g, \varphi \rangle = \int_{\Omega} \int_{\Omega} \hat{\varphi}(x+iy) \mathcal{L}\hat{h}(x+iy) \Delta(y)^{\nu-\frac{p'}{q'}}dydx
\]
\[
= \int_{\Omega} \int_{\Omega} F(x+iy) \mathcal{L}\hat{h}(x+iy) \Delta(y)^{\nu-\frac{p'}{q'}}dydx,
\]
with $h = \Box^{\nu}\varphi$. So,
\[
|\langle g, \varphi \rangle| \leq C\|F\|_{L^{p,q}_{\nu}}\|\mathcal{E}h\|_{A^{p',q'}_{\nu}}.
\]

To conclude, we use Theorem 1.8 applied to $h$, and Proposition 3.31 as before to have the equivalence of the norm $\|\varphi\|_{B^{p',q'}_{-\nu/q}}$ with the norm $\|h\|_{B^{p',q'}_{\nu}}$.

As a corollary, we can use the previous section to extend the range of exponents for which the Bergman projector is bounded.

Corollary 4.32 If $\nu > \frac{n}{r} - 1$, $1 \leq p < \infty$ and $q'_{\nu,p} < q < q_{\nu,p}$, then $P_{\nu}$ admits a bounded extension to $L^{p,q}_{\nu}$. That is, there exists a constant $C > 0$ such that
\[
\|P_{\nu}F\|_{L^{p,q}_{\nu}} \leq C\|F\|_{L^{p,q}_{\nu}}, \quad \forall F \in L^{p,q}_{\nu}.
\]
4.4 Necessary conditions

In this section we shall construct counter-examples to Theorem 1.8, for some values of $\nu, p, q$ above the critical indices. That is, we shall show that the space $B_{p,q}^{\nu,0}$, of Fourier-Laplace extensions of $B_{p,q}^{\nu}$, cannot be embedded into $A_{p,\infty}^{\nu}$. Our examples are actually stronger and show that $B_{p,q}^{\nu}$ cannot be even embedded into $A_{p,\infty}^{\nu}$, that is, there does not exist a constant $C$ such that, for all $f \in B_{p,q}^{\nu}$, one has the inequality

$$\int_{\mathbb{R}^n} |F(x + ie)^p \, dx \leq C \|f\|_{B_{p,q}^{\nu}}, \quad \text{for} \quad F = \tilde{E}(f). \quad (4.33)$$

Observe that this indeed contradicts Theorem 1.8 since, by Lemma 4.5, the integral on the left hand side is always smaller than $\|F\|_{A_{p,q}^{\nu}}$. Our results are stated in the following proposition.

**Proposition 4.34** Let $\nu > \frac{n}{r} - 1$, $1 \leq p < \infty$ and $1 \leq q < \tilde{q}_{\nu,p}$. Then, there cannot exist a constant $C$ such that the inequality (4.33) is valid for all $f \in B_{p,q}^{\nu}$ in the two following cases:

(a) $1 \leq p \leq 2$ and $q \geq q_{\nu,p}$;

(b) $2 < p < \infty$ and $q \geq \min\{2q_{\nu}, \tilde{q}_{\nu,p}\}$.

**Proof:** We shall use a different method for (a) and (b). The first one is based on an explicit holomorphic function, and the second on a Rademacher argument with Littlewood-Paley inequalities. For the first part, we shall find $F \in H(T_\Omega)$ such that

$$\int_{\mathbb{R}^n} |F(x + ie)^p \, dx = \infty \quad \text{and} \quad \|\Box F\|_{L_{p,q}^{\nu}} < \infty,$$

for all $q \geq pq_{\nu}$ (with $q < \tilde{q}_{\nu,p}$). This gives a contradiction with (4.33), since in case that held, we would conclude

$$\|F(\cdot + ie)\|_p \lesssim \|f\|_{B_{p,q}^{\nu}} \sim \|\Box f\|_{B_{p,q}^{\nu}} \leq C \|\tilde{E}(\Box f)\|_{L_{p,q}^{\nu}} = C\|\Box F\|_{L_{p,q}^{\nu}} < \infty.$$

At this point, an example involving the $\Box$ operator may seem a bit cumbersome, but it is actually quite natural since the boundedness of the Bergman projection turns out to be equivalent to the existence of generalized Hardy inequalities. For more on this direction we refer to [5] (in the case of the light-cone), and to the survey paper [7]. Our specific example will be the holomorphic function

$$F(z) = \Delta((z + ie)/i)^{-\alpha} (1 + \log \Delta((z + ie)/i))^{-\frac{1}{r}}, \quad z \in T_\Omega,$$

where we choose $\alpha = (\frac{2n}{r} - 1)/p$. We are using the standard convention $\log \Delta(z/i) = \sum_{j=1}^r \log[\Delta_{j-1}(z/i)]$, since in this case $\Re[\Delta_j(z/i)] > 0$, for each $z \in T_\Omega$ and $j = 1, \ldots, r$ (see, e.g., the discussion in [15, §7]). We also remark that, since $|\Delta((z +
\[ |\Delta(y + e)| \geq \Delta(y + e) > 1, \] the expression under the power \( \frac{1}{p} \) has positive real part, and defines a holomorphic function.

To compute the first integral we estimate the denominator of \( F(z) \) using the elementary facts

\[ |1 + \log \Delta((x + ie)/i)| \leq r \frac{\pi}{2} + 1 + \log |\Delta(x + ie)| \quad \text{and} \quad |\Delta(x + ie)| \leq \Delta(x + e), \]

when \( x \in \Omega \). This leads to the expression

\[ \int_{\mathbb{R}^n} |F(x + i e)|^p dx \geq C \int_{\Omega} \frac{dx}{|\Delta(x + e)|^{\frac{2n}{p} - 1} (1 + \log \Delta(x + e))} = \infty, \]

by Lemma 2.20.

For the second integral we first calculate \( \square F(z) \). Observe that around each \( z_0 \in T_\Omega \) there is a neighborhood \( U \) so that \( F(z) = g_{z_0}(\Delta((z + ie)/i)) \), \( z \in U \), where \( g_{z_0}(w) = w^{-\gamma} (1 + \log w)^{-1/p} \) is a function of 1 complex variable with a determination of the log depending on \( z_0 \). We remark that functions corresponding to two points \( z_0, z_1 \) will only differ by constants which are irrelevant for our estimates below, and for this reason we shall drop the subindex in \( g \).

We can now compute \( \square F(z) \) using the formula

\[ \square [g(\Delta(z/i))] = \frac{(Bg)(\Delta(z/i))}{\Delta(z/i)}, \quad z \in T_\Omega, \tag{4.35} \]

where \( B = b(w \frac{d}{dw}) \) is the 1 variable differential operator of degree \( r \) given by the Bernstein polynomial \( b(\lambda) = (-1)^r \lambda (\lambda + \frac{d}{2}) \cdots (\lambda + (r-1) \frac{d}{2}) \). One can verify the equality (4.35) directly, using the Taylor series of \( g \) and \( \square [\Delta^n(z/i)] = b(n)\Delta^{n-1}(z/i) \) (see, e.g., [13, p. 142]). Thus, an easy computation of the derivatives of \( g(w) \) leads to the expression

\[ |\square F(z)| \leq C |\Delta((z + ie)/i)|^{-(\alpha + 1)} (1 + \log \Delta(y + e))^{-1/p}, \quad z = x + iy \in T_\Omega, \]

where we have also used \( |\Delta(u + iv)| \geq \Delta(v), v \in \Omega \). Now, we can apply Lemmas 2.18 and 2.20 to estimate the integral

\[
\int_{\Omega} \left( \int_{\mathbb{R}^n} |\square F(x + iy)|^p \, dx \right)^{\frac{1}{p'}} \Delta^{\nu-q-\frac{\gamma}{p'}}(y) \, dy \\
\leq C' \int_{\Omega} \left( \int_{\mathbb{R}^n} \frac{dx}{|\Delta(x + i(y + e))|^{(\alpha+1)p}} \right)^{\frac{1}{p'}} \Delta^{\nu-q-\frac{\gamma}{p'}}(y) \, dy \\
= C' \int_{\Omega} \frac{\Delta^{\nu-q-\frac{\gamma}{p'}}(y)}{\Delta(y + e)^{(\alpha+1)p-\frac{\gamma}{p'}}} \, dy \\
= C' \int_{\Omega} \frac{\Delta^{\nu-q-\frac{\gamma}{p'}}(y)}{\Delta(y + e)^{(\alpha+1)p-\frac{\gamma}{p'}}} \, dy.
\]
and observe that the last quantity is finite for every $q \geq p(1 + \frac{\nu}{p-1})$.

Let us now pass to the second type of counter-examples, and obtain case (b) in the proposition. We may assume $q > 2$. We know from Proposition 4.16 that the inequality (4.33) implies the existence of a constant $C$ such that

$$\| \sum f_j \|_p^q \leq C \sum \Delta(\xi_j)^{-\nu} \| f_j \|_p^q,$$

(4.36)

for any finite sequence $\{f_j\}$ of Schwartz functions satisfying $\text{Supp} \hat{f}_j \subset B_{\frac{1}{2}}(\xi_j)$ and restricted to indices $j$ so that $|\xi_j| < 1$. Let us prove that this implies very easily the necessity of $q < 2q_\nu$ (which was already announced in [5] for light-cones). Indeed, let us take $f_j = \varepsilon_j a_j e^{i(\xi_j \cdot \cdot)} f$, with $\{\varepsilon_j\}$ a sequence of Rademacher functions, and the support of $\hat{f}$ a small neighborhood of 0. Taking the mean over all $\varepsilon_j$'s and using Khintchine inequalities, we find that

$$\left[ \sum |a_j|^2 \right]^{\frac{1}{2}} \leq C \left[ \sum \Delta(\xi_j)^{-\nu} |a_j|^q \right]^{\frac{1}{q}},$$

with perhaps a different constant $C$, independent of the sequence $\{a_j\}$. Now, choosing $a_j = \Delta(\xi_j)^{\frac{\nu}{q-2}}$ and using $q > 2$, this implies that

$$\sum_{j:|\xi_j| < 1} \Delta(\xi_j)^{\frac{\nu}{q-2}} \leq C < \infty.$$

Using Proposition 2.13, this is equivalent to the fact that

$$\int_{|\xi| < 1} \Delta(\xi)^{\frac{\nu}{q-2}} \frac{d\xi}{\Delta(\xi)^{\frac{\nu}{p}}} < \infty,$$

which, in turn, is equivalent to the condition $q < 2q_\nu$.

\[\square\]

5 Appendix: extension of the range on light-cones

In this additional section we just focus on the previous problems for the particular case of the light cone $\Lambda_n$. As we shall see, the sufficient conditions given in Proposition 4.16 can be written in terms of Littlewood-Paley inequalities in $\mathbb{R}^n$, removing any dependence on complex coordinates. One of such inequalities will lead us to a variant of the so-called “cone multiplier problem”, for which recent results of Tao-Vargas and T. Wolff will provide us with some positive answers to our question. We observe here that our sufficient conditions are “essentially necessary”. In fact, Proposition 4.16 states that they are necessary for the boundedness of the projector with other
values of the parameters. So negative results for them should also give new regions of unboundedness for the Bergman projector. In this way, our approach to this problem ends up in a series of challenging questions in Harmonic Analysis, very closely related to the present restriction and multiplier problems for the cone. One can as well pose the corresponding questions for spheres, which even in the 2 dimensional case seem to be unknown at present.

Before particularizing for light-cones, we recall the situation after the results in this paper. In Figure 1.1 we show the regions of boundedness for the Bergman projector $P_\nu$ in $L^{p,q}_\nu$ spaces for a general symmetric cone. In the “blank region”, our main contribution up to now is Theorem 4.25, which gives the equivalence between boundedness of $P_\nu$ in $L^{p,q}_\nu$ and the inequality

$$\|\mathcal{L}_\nu \hat{f}\|_{L^{p,q}_\nu} \lesssim \|f\|_{B^{p,q}_\nu}, \quad f \in \mathcal{D}_\Omega.$$  

(5.1)

By self-adjointness of $P_\nu$ and the constraints explained in the introduction, we only study this question in the ranges of indices

$$\nu > \frac{n}{2} - 1, \quad 1 \leq p < \infty, \quad 2 \leq q < \tilde{q}_{\nu,p}.$$  

We know the validity of (5.1) for all $2 < q < \min\{p, p'\}_\nu$, and we know the sharpness of the right index when $1 \leq p \leq 2$, so we shall also restrict our study to $p > 2$. In this case, and after the results in §4.4, the region of uncertainty for the validity of (5.1) reduces to

$$p'q_\nu \leq q < \min\{2q_\nu, \tilde{q}_{\nu,p}\}.$$  

Our main contribution here is to show that one can conclude positively for a small neighborhood around the left endpoint of this interval.

### 5.1 A Whitney covering for the light-cone

From now on, we let $\Omega = \Lambda_n$ denote the light-cone in $\mathbb{R}^n$ with $n \geq 3$. We first describe an explicit Whitney decomposition for $\Lambda_n$.

A natural candidate for a lattice in the light-cone is constructed as follows. For every $j \geq 1$, take a maximal $2^{-j}$-separated sequence $\{\omega_k^{(j)}\}_{k=1}^{k_j}$ in the sphere $S^{n-2} \subset \mathbb{R}^{n-1}$, with respect to the Euclidean distance (so that $k_j \sim 2^{j(n-2)}$). Then, define the following grid of points in $\Lambda_n$:

$$\xi_{\ell,j,k} = \left(2^\ell, 2^\ell \sqrt{1 - 2^{-2j}} \omega_k^{(j)}\right), \quad \ell \in \mathbb{Z}, \quad j \geq 1, \quad k = 1, \ldots, k_j,$$

(5.2)

and the corresponding sets

$$E_{\ell,j,k} = \left\{(\tau, \xi') \in \Lambda_n : 2^{\ell-1} < \tau < 2^{\ell+1}, \quad 2^{-2j-2} < 1 - \frac{|\xi'|^2}{\tau^2} < 2^{-2j+2}, \quad \text{and} \quad \left|\frac{\xi'}{|\xi'|} - \omega_k^{(j)}\right| \leq \delta 2^{-j}\right\},$$
where the constant $\delta$ is chosen in such a way that the regions cover the cone. The geometric picture in $\mathbb{R}^3$ is as follows: the sets $E_{j,k}^\ell$ are truncated conical shells of height $\sim 2^\ell$, of thickness $\sim 2^{\ell-2j}$, and further decomposed into $k_j \sim 2^j$ sectors, all of equal arc-length $\sim 2^{\ell-j}$. This is the usual decomposition of $\Lambda_n$ for the study of cone multipliers (see [17]). The next proposition proves that the regions $E_{j,k}^\ell$ are very close from a Whitney covering of the cone.

**Proposition 5.3** With the notation above, the grid $\{\xi_{j,k}^\ell\}$ is a lattice in $\Lambda_n$. Moreover, there exist $0 < \eta_1 < \eta_2$ such that the corresponding family of invariant balls satisfy

- (a) $\{B_{\eta_1}(\xi_{j,k}^\ell)\}$ is disjoint in $\Omega$;
- (b) $\{B_{\eta_2}(\xi_{j,k}^\ell)\}$ is a covering of $\Omega$;
- (c) $B_{\eta_1}(\xi_{j,k}^\ell) \subset E_{j,k}^\ell \subset B_{\eta_2}(\xi_{j,k}^\ell)$.

**Proof:**

In view of the definition in §2.2, it suffices to find two fixed (invariant) balls $B \subset \bar{B}$, centered at $e$ and such that

$$g_{j,k}^\ell(B) \subset E_{j,k}^\ell \subset g_{j,k}^\ell(\bar{B}),$$

(5.4)

for some fixed automorphisms of the cone $g_{j,k}^\ell$ mapping $e$ into $\xi_{j,k}^\ell$. Using dilations as well as rotations with axis $e$, it is sufficient to prove this for the points $(1, \sqrt{1-2^{-2j}}, 0) = g_j e$. Then, an elementary exercise shows that the corresponding set $E_j$ is such that

$$E_j(c) \subset E_j \subset E_j(\tilde{c}),$$

where $E_j(c)$ is the set of all $\xi$ for which

$$c^{-1} < \frac{(\xi|e)}{(g_j|e)} < c, \quad c^{-1} < \frac{\Delta(\xi)}{\Delta(g_j|e)} < c, \quad c^{-1} < \frac{\Delta_1(\xi)}{\Delta_1(g_j|e)} < c.$$

Here $\Delta_1(\xi) = \xi_1 - \xi_2$, and the constants $c, \tilde{c}$ are independent of $j$. Let us finally prove that $g_j^{-1}(E_j(c))$ is contained in a ball and contains a ball, centered at $e$ and with radii independent of $j$. From the invariance properties of the quantities involved, this last set consists of $\xi$’s for which

$$c^{-1} < (g_j\xi|e) < c, \quad c^{-1} < \Delta(\xi) < c, \quad c^{-1} < \Delta_1(\xi) < c.$$

Using the explicit value $(g_j\xi|e) = \xi_1 + \sqrt{1-2^{-2j}} \xi_2$, it is an elementary exercise to find two such balls with radii independent of $j$. $\square$
From this choice of points, and from the second condition in Proposition 4.16, we get the following result. To simplify notation, we write $E_{j,k} = E_{j,k}^0$.

**PROPOSITION 5.5 : Weak sufficient condition.**

Let $1 \leq p, s < \infty$. Suppose that for some $\mu > 0$ there exists $C_\mu$ such that

$$\| \sum_{k=0}^{k_j} f_k \|_p \leq C_\mu \left( \sum_{k=0}^{k_j} \| f_k \|_p^s \right)^{1/s}, \quad \forall j \geq 1,$$

(5.6)

for every sequence $\{ f_k \}$ satisfying $\text{Supp} \hat{f}_k \subset E_{j,k}$. Then $P_\nu$ is bounded in $L^{p,q}_\nu$ for all $q$ such that

$$\frac{q}{s} < \min \left\{ \frac{q_\nu}{q_\mu}, \frac{\nu - (\frac{n}{2} - 1)}{\mu} \right\}.$$

(5.7)

**PROOF:** Using Proposition 4.16, we are reduced to prove that the assumption implies the inequality

$$\| \sum_{j=0}^{k_j} \sum_{k=0}^{k_j} f_{j,k} \|_p \leq C_{\mu'} \left[ \sum_{j=0}^{k_j} 2^{j\nu \frac{q_\nu}{q_\mu}} \sum_{k=0}^{k_j} \| f_{j,k} \|_p^s \right]^{1/s},$$

with $\mu'$ perhaps larger, but arbitrarily close from $\mu$. Here the Fourier transforms of the functions $f_{j,k}$ are contained in the regions $E_{j,k}$. To prove the previous inequality, start using Minkowski’s inequality in $j$, apply (5.7) in each block with fixed $j$, and conclude with Hölder’s inequality. 

**REMARK 5.8** The natural conjecture in order to fill the whole “blank region” in Figure 1.1 is that (5.6) holds, with $s = 2$ and any $\mu > 0$, within the range $2 < p < \frac{2n}{n-2}$. This range of $p$ coincides with the conjecture for the $\Lambda(p)$-set problem of $\mathbb{Z}^n$-points in spheres (see [9, 5.5]). In particular, it contains the conjectured range for the cone multiplier in $\mathbb{R}^n$ and for Bochner-Riesz in $\mathbb{R}^{n-1}$: $2 < p < \frac{2(n-1)}{n-2}$. When $n = 3$ the latter right end-point is $p = 4$, for which almost orthogonality techniques can be applied to obtain some partial results (see next subsection). We observe finally that, by Proposition 4.16, our condition (5.6) is necessary for the boundedness of $P_\mu$ in $L^{p,s}_\mu$ when $\mu > \frac{n}{r} - 1$.

### 5.2 Restriction techniques and new results for light-cones

From now on we restrict to $p > 2$. We shall study almost orthogonality and restriction theorem techniques that can imply our sufficient condition

$$\| \sum_{k=0}^{k_j} f_k \|_p \leq C_\epsilon \left( \sum_{k=0}^{k_j} \| f_k \|_p^2 \right)^{1/2},$$

(5.9)
for $\varepsilon$ as close to 0 as possible, and the Fourier transforms of $f_k$ supported in $E_{jk}$. A simple application of Minkowski’s inequality shows that (5.9) is implied by the square function estimate
\[ \| \sum_{k=0}^{k_j} f_k \|_p \leq C_{E_{jk}} 2^{j\varepsilon} \left( \sum_{k=0}^{k_j} |f_k|^2 \right)^{\frac{1}{2}} \|_p. \]  
(5.10)

When $n = 3$ and $p = 4$ this has been widely studied in relation with the cone multiplier problem. The analogous question for the 2-dimensional disk (that is, for a horizontal section of the cone) has a well-known positive answer with $\varepsilon = 0$, following from a simple geometric argument due to Córdoba and Fefferman [12]. This same argument is known to be less sharp in the 3-dimensional cone, where additional overlapping leads only to $\varepsilon = \frac{1}{4}$ (see [17]). Such estimate does not produce new results on the boundedness of Bergman projectors, as one can check easily through the numerology in our previous subsection. One must beat the exponent $\varepsilon = \frac{1}{4}$ to obtain some improvement in our problem.

In this direction there are more recent works by Bourgain [8] and Tao-Vargas [21], where this exponent has been lowered to $\varepsilon = \frac{1}{4} - \tau$, for a small $\tau > 0$. This improvement makes use of the so-called bilinear restriction estimates $R^* (2 \times 2 \to q)$. That is, finding the smallest value of $q \leq 2$ for which:
\[ \| \widehat{g_1 d\sigma_1} \widehat{g_2 d\sigma_2} \|_q \lesssim \| g_1 \|_2 \| g_2 \|_2, \]  
(5.11)

when $g_1, g_2$ are smooth functions with 1-separated supports and $d\sigma$ is the surface measure of the truncated cone $\Lambda'_3 = \{(|\xi'|, \xi') \in \mathbb{R}^3 : 1 < |\xi'| < 2\}$. More precisely, we cite the following theorem from [21], adapted to our notation.

**Theorem 5.12 :** see Theorem 5.1 in [21]. If $\kappa > 0$ is such that $R^* (2 \times 2 \to 2 - \kappa)$ holds for all $g_1, g_2$ with unit separated support in $\Lambda'_3$, then for all $\tau < \kappa/(16 - 4\kappa)$ the inequality (5.10) holds with $\varepsilon = \frac{1}{4} - \tau$.

Tao and Vargas are able to go down to $q > 2 - \frac{8}{121}$ in (5.11), obtaining in (5.10) all $\varepsilon > \frac{1}{4} - \frac{1}{238}$. The recent sharp results for (5.11) given by T. Wolff in [22], valid for all $q > 2 - \frac{11}{3}$, improve the estimate in (5.10) to all $\varepsilon > \frac{1}{4} - \frac{1}{44}$. Whether one can go down this index in (5.10) seems to be an open question, in spite of the sharpness of Wolff’s theorem in cone restriction. From this discussion and a straightforward computation of the numerology we conclude the following

**Corollary 5.13** In $\mathbb{R}^3$ and for $p = 4$, the Bergman projector $P_\nu$ is bounded in $L^{4,q}_\nu$ for all $2 \leq q < (\frac{1}{2} + \frac{1}{27}) q_\nu$, whenever $\nu > \frac{1}{2} + \frac{3}{11}$.

By interpolation, this allows to give a positive answer in a part of the blank region of Figure 1.1. We do not give a precise description of this new region since it will certainly be improved in the future.
We conclude by observing that in higher dimensions $n > 3$, the sharp index for bilinear restriction due to Wolff is $q > 2 - \frac{n-2}{n}$. We do not know if this has any implication for the inequality (5.10), since the above cited result of Tao-Vargas is only 3-dimensional.

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