On some Coulomb gas integrals in higher dimensions

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We point out that there is a generalization to higher dimensions $d = 2h > 2$ of the two-dimensional Dotsenko-Fateev formula [1] for particular Coulomb gas conformal invariant integrals. These expressions represent structure constants of 3-point functions of vertex operators related to a higher dimensional generalization of the Liouville theory. The Coulomb gas formulae admit two different analytic continuations generalizing the DOZZ formula, i.e., the theory is not selfdual for $d > 2$.

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1. Introduction

Coulomb gas correlators with a background charge in higher dimensions $d = 2h$ have been considered by the authors [2] as a generalization of the Virasoro minimal theories for $c < 1$. Later we showed [3] that there is a hidden Virasoro algebra, intrinsic for this $2h$-dimensional conformal model. The generators and the vertex operators - interpreted as the primary fields of this algebra, were explicitly constructed in terms of the modes of a subcanonical logarithmic field

$$\frac{2}{(4\pi)^h \Gamma(h)} (-\Box)^h < \phi(x) \phi(0) > = \delta^{2h}(x).$$

It was demonstrated on an example that the factorization of the singular vectors of this Virasoro algebra leads, as in 2d, to differential equations for the 4-point correlators containing fields described by degenerate representations. In the simplest case this reproduced the second order Appel differential operators $D(x, y)$ which in 2d amount to a pair of hypergeometric equations.

As far as the Coulomb gas correlators are concerned the consideration in [3] extends to the region of parameters generalizing the $c > 25$ Virasoro theory, i.e., the Liouville theory. Recently the Liouville theory in higher dimensions has been studied in [4] in broader context. In particular the authors computed for the model on the $2h$ sphere $S^{2h}$ directly the 3-point function of semiclassical light charge vertex operators, generalizing the 2d derivation of [3].

In this note we first show that some of the Coulomb gas integrals representing the 3-point functions of vertex operators, namely the ones involving a screening charge of one and the same type, can be computed in higher dimensions as well. This provides an explicit expression for the particular conformally invariant multiple integral given by a generalized to $2h > 2$ Dotsenko-Fateev \( \Pi \) formula. Namely, let the parameters $p_a, a = 1, 2, 3, w$ satisfy

$$\sum_{a=1}^{3} p_a - 2(n - 1)w^2 = -2h \quad (1.1)$$

where $n$ is a positive integer. Then

$$I_n = \frac{1}{n!} \prod_{i=1}^{n} \int \frac{d^{2h}t_i}{\pi^h} \prod_{1 \leq i < j \leq n} |t_i - t_j|^{-4w^2} \prod_{a=1}^{3} \prod_{i=1}^{n} |x_a - t_i|^{2p_a} =$$

$$\prod_{c \neq a < b \neq c} |x_{ab}|^{2n((n-1)w^2-p_c-h)} \frac{1}{\gamma^h(-w^2)} \prod_{k=0}^{n-1} \frac{\gamma_h((k-n)w^2)}{\prod_{a=1}^{3} \gamma_h(-p_a + kw^2)} \quad (1.2)$$
where
\[ \gamma_h(x) := \frac{\Gamma(x)}{\Gamma(h-x)} = \frac{1}{\gamma_h(h-x)}. \] (1.3)

The condition (1.3) expressing the invariance of each of the integrals with respect to the (euclidean) conformal group in 2h-dimensional space-time is equivalent to the conservation condition for the charges of the vertex operators \( V_{\beta a}(x_a) = e^{2\beta_a \phi(x_a)} \) in the presence of one type of screening charges \( \int d^{2h}x V_w(x) \) with \( p_a = -2\beta_a w \); in general for N-point correlator
\[
\sum_{a=1}^{N} \beta_a + nw = Q_h(w) = \frac{h}{w} + w. \quad (1.4)
\]

The nonnegative integer \( n \) is the number of screening charges and \(-Q_h\) is the background charge. The parameter \( w \) will take two (real) values \( w = b, \frac{h}{b} \) and our notation for the 2d scale dimension of the vertex operators \( V_\beta \) and \( V_{Q_h-\beta} \) is
\[
2\Delta_h(\beta) = 2\beta(Q_h - \beta) = 2h\beta(Q(b) - \beta) = 2h\Delta(\beta) \quad (1.5)
\]
with
\[
\beta = \sqrt{h\bar{\beta}}, \quad b = \sqrt{h}; \quad \frac{1}{b} = \sqrt{h}, \quad Q_h(b) = \frac{h}{b} + b = \sqrt{h}Q(b) = \sqrt{h}(\frac{1}{b} + b). \quad (1.6)
\]

The vertex operators \( V_b \) and \( V_{\frac{h}{b}} \) have dimension \( 2h \). The integral (1.2) times the coordinate factor \( \prod_{c\neq a<b\neq c} |x_{ab}|^{-2\beta_a\beta_b} \) represents the 3-point function of scalar fields of dimensions given by (1.3).

The derivation of (1.2) is a straightforward generalization of the method exploited in \([3]\). For that one has to generalize to higher dimensions the basic duality formula of Baseilhac - Fateev (BF) \([4]\), which allows to derive a recursion relation in the number of integrations in (1.2).

Next we look at the possible analytic continuation of the 3-point constant in the r.h.s. of (1.2) beyond the restriction (1.4) on the three charges \( \beta_a, a = 1, ..., N = 3 \), with \( n \) becoming an arbitrary parameter, in an attempt to generalize the 2d DOZZ formula \([5], [8]\) for the Liouville theory structure constant. It turns out the two Coulomb gas constants computed with one of the two types of screening charges, either \( w = b \) or \( w = h/b \), give rise to two different unrelated for \( 2h > 2 \) expressions. The integrals \( \int d^{2h}x V_b(x) \) and \( \int d^{2h}x V_{\frac{h}{b}}(x) \) describe the interaction terms of two dual Liouville theories on the 2h sphere \( S^{2h} \) and the Coulomb gas correlators are recovered as residua of the Liouville correlators.
Accordingly we obtain two different expressions for the $h > 1$ analog of the light charges semiclassical limit $b \to 0$ of \[5\] and compare them with the result in \[4\].

In the last section 5 the extension of the 3-point constant in the region of parameters described by the value $c < 1$ of the intrinsic Virasoro algebra is also considered. We recall that this theory plays the role of ”matter” in the 2d Liouville gravity. Some data on the higher dimensional Liouville theory and details of computations are collected in the appendices.

2. Coulomb gas correlators in $2h$ dimensions

Consider the Coulomb gas integral representing the 3-point function with the third point taken at infinity

$$C_s(\beta_1, \beta_2, \beta_3) = \frac{(x_{12}^2)^{\Delta(\beta_1)+\Delta(\beta_2)-\Delta(\beta_3)}}{(x_{12}^2)^{\Delta(\beta_1)+\Delta(\beta_2)-\Delta(\beta_3)}} = \frac{(x_{12}^2)^{-2\beta_1\beta_2 I_s(\beta_1, \beta_2, \beta_3)(x_1, x_2)}}{D_s(\beta_1, \beta_2, \beta_3)(x_1, x_2)}$$

$$I_s(\beta_1, \beta_2, \beta_3)(x_1, x_2) = \int d\mu_s(t) D_s^{-2b^2}(t) \prod_{i=1}^{s} |t_i - x_1|^{2p_1} |t_i - x_2|^{2p_2}$$

where

$$d\mu_s(t) = \frac{1}{\pi^{hs_s!}} \prod_{i=1}^{s} d^{2h}(t_i), \quad D_s(t) = \prod_{1 \leq i < j \leq s} |t_i - t_j|^2.$$  \hspace{1cm} (2.1)

In (2.1)

$$p_a = -2\beta_a b = -2h_\beta b, \quad a = 1, 2$$

and the three charges satisfy the charge conservation condition (1.4) for $N = 3, n = s$.

This $2h$ Liouville Coulomb gas integral is computed recursively by means of a $2h$ analog of the BF formula \[7\]

$$\int d\mu(y) D_h(y) \prod_{i=1}^{n+m+1} \prod_{j=1}^{n+m+1} |y_i - t_j|^{2p_j} = \prod_{j=1}^{n+m+1} \frac{1}{\gamma_h(-p_j)} \frac{1}{\gamma_h(h(n + 1) + \sum_j p_j)} \times \prod_{i < j} |t_i - t_j|^{2h + 2p_i + 2p_j}$$

$$\int d\mu_m(u) D_h^m(u) \prod_{i=1}^{m} \prod_{j=1}^{n+m+1} |u_i - t_j|^{-2h - 2p_j}.$$ \hspace{1cm} (2.3)

This formula is derived comparing the two equivalent Coulomb gas realizations of a correlator of $N$ operators $V_{\beta_i}, i = 1,..N$, with different number $n$ and $m$, respectively, of screening charges of one and the same type $\int d^{2h} x V(x)$. In the second realization all
vertex operators are replaced by their dual counterparts \( V_{Q_{k-\beta_i}}, i = 1, \ldots N \) of the same scaling dimension; the two are related by a reflection factor

\[
V_{\beta_i} = r(\beta_i) V_{Q_{k-\beta_i}}.
\]

The requirement of existence of such two realizations i.e., the validity of the two charge conservation relations, both with nonnegative integer \( n \) and \( m \), imposes strong restrictions if \( m+n > 0 \). Namely, the number of points should be \( N = m+n+2 \) and the parameter \( b^2 \) is fixed to either \( b^2 = -2h \), or \( b^2 = -h/2 \) depending on which of the two types of screening charges is used; in both cases \( 2w^2 = -h \).

The two integral representations differ by the product \( \prod_{i=1}^{N} r(\beta_i) \) of reflection factors in (2.4). For the particular value of \( b^2 \) the factor \( r(\beta) \) is computed from the 3-point case \( N = n + m + 2 = 3 \) with \( n = 1, m = 0 \). This leads to (2.3) in which furthermore one of the points is taken to infinity and one of the \( N \) charges is replaced by its value from (1.4) in which \( 2w^2 = -h \).

Given (2.3) one computes the integral (2.1) following the steps in the \( h = 1 \) case. Namely, take all \( p_i = -h - b^2 \) in (2.3) with \( m = 0 \) and \( n = s - 1 \) getting \( D_s^{-h-2b^2}(t) \) for the overall factor in the r.h.s. Then represent the power \( D_s^{-2b^2}(t) \) in (2.4) by the integral in the l.h.s. of (2.3)

\[
D_s^{-2b^2}(t) = D_s^h(t) D_s^{-h-2b^2}(t) = D_s^h(t) \frac{\gamma_h(-sb^2)}{\gamma_s(-b^2)} \int d\mu_{s-1}(y) D_{s-1}^h(y) \prod_{i=1}^{s-1} \prod_{j=1}^{s} |t_j - y_i|^{-2h-2b^2}.
\]

Next apply (2.3) for \( n = s, m+n+1 = s-1+2, \) hence \( m = 0 \)

\[
\int d\mu_{s}(t) D_{s}^h(t) \prod_{j=1}^{s} \prod_{i=1}^{s-1} |t_j - y_i|^{-2h-2b^2} |t_j - x_1|^{2p_1} |t_j - x_2|^{2p_2}
\]

\[
= (x_{12}^2)^{h+p_1+p_2} \frac{\gamma_{s-1}(-b^2)}{\gamma_h(2h+p_1+p_2-(s-1)b^2)} 2^{\frac{s}{2}} \frac{1}{\gamma_h(-p_i)} D_{s-1}^{-h-2b^2}(y) \prod_{i=1}^{s-1} |y_i - x_1|^{2p_1-2b^2} |y_i - x_2|^{2p_2-2b^2}.
\]

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1 To be precise, it is more consistent to perform this derivation in the "matter" region, discussed in sect 5 below, obtained by \( b^2 \to -b^2 \); then one reproduces the same formula (2.3) for the particular values of the real parameter \( b: b^2 = 2h \), or \( h/2 \) respectively.
Combining (2.5) and (2.6) we get a recursion relation which is solved using that $I_0 = 1$.

\[ I_s(\beta_1, \beta_2, \beta_3)(x_1, x_2) = \gamma_h(-sb^2) \frac{(x_{12}^2)^{h+p_1+p_2}}{\gamma_h(-b^2) \gamma_h(-p_1) \gamma_h(-p_2) \gamma_h(-p_3 + (s-1)b^2)} I_{s-1}(\beta_1 + \frac{b}{2}, \beta_2 + \frac{b}{2}, \beta_3)(x_1, x_2) \]

(2.7)

\[ = (x_{12}^2)^{s-1} \frac{1}{\gamma_h(-b^2)} \prod_{k=0}^{s-1} \frac{\gamma_h((k-s)b^2)}{\gamma_h(2\beta_i b + kb^2)} \]

\[ = (x_{12}^2)^{s-b(\beta_{123} - 2\beta_3)} C_s(\beta_1, \beta_2, \beta_3). \]

This reproduces (1.2) with $w = b$; here $\beta_{123} = \sum_{i=1}^{3} \beta_i$. The Coulomb gas integral $I_s^{(h/b)}(\beta_1, \beta_2, \beta_3)$ with the dual screening charge, i.e., with charges satisfying (1.4) with $w = h/b$ is computed in the same way and gives an expression $\tilde{C}_s(\beta_1, \beta_2, \beta_3)$ analogous to (2.7) with $b \rightarrow h/b$.

3. Analytic continuation

Recall the double Gamma function $\Gamma_{w}(x) = \Gamma_{\frac{1}{w}}(x)$ satisfying the functional relations

\[ \frac{\Gamma_{w}(x + \epsilon)}{\Gamma_{w}(x)} = \sqrt{2\pi} \frac{(w^\epsilon (w^\epsilon x - \frac{1}{2}))}{\Gamma(w^\epsilon x)}, \epsilon = \pm 1. \]

(3.1)

It has poles at $x = -nw - m/w$, $n, m \in \mathbb{Z}_{\geq 0}$. Define with $Q_h = Q_h(w) = \frac{h}{w} + w$

\[ \Upsilon_{w}^{(h)}(x) := \frac{1}{\Gamma_{w}(x) \Gamma_{w}(Q_h - x)} = \Upsilon_{w}^{(h)}(Q_h - x). \]

(3.2)

It satisfies the functional relation

\[ \frac{\Upsilon_{w}^{(h)}(x + w)}{\Upsilon_{w}^{(h)}(x)} = w^{h-2wx} \gamma_h(xw). \]

(3.3)

The shift with $\frac{h}{w}$ is computed applying $h$ times the functional relation (3.1) for $\epsilon = -1$.

Exploiting (3.3) for $w = b$ we can express the products on (2.7) as

\[ \prod_{k=0}^{s-1} \frac{1}{\gamma_h(xb + kb^2)} = b^{sb(Q_h - 2x - sb)} \frac{\Upsilon_{b}^{(h)}(x)}{\Upsilon_{b}^{(h)}(x + sb)} \]

(3.4)
for \( x = 2\beta_i, i = 1, 2, 3 \). Furthermore one gets rid of the restriction on the three charges, replacing \( sb = Q_h - \beta_{123} \) in the r.h.s. of (3.4). On the other hand

\[
Res_{x=-sb} \frac{\Upsilon_b^{(h)}(b)}{\Upsilon_b^{(h)}(x)} = \frac{b^{h-1}}{\Gamma(h)} b^{sb(Q_h+sb)} \prod_{k=0}^{s-1} \gamma_h(-sb^2 + kb^2). \tag{3.5}
\]

Taking into account (3.4) and (3.5) one can analytically continue the Coulomb gas 3-point constant (2.7) for arbitrary \( \beta_i \) not restricted by the charge conservation condition (1.4) as

\[
C(\beta_1, \beta_2, \beta_3) = \frac{\Gamma(h)}{b^{h-1}} \left( -\frac{\pi^h \mu b^{2h-2b^2}}{b^{4h} \gamma_h(-b^2)} \right) \frac{\gamma_h(\beta_{123}, -\frac{\beta_{123}}{h})}{\beta_{123} - Q_h} \prod_{k=1}^{3} \frac{\Upsilon_b^{(h)}(2\beta_k)}{\Upsilon_b^{(h)}(\beta_{123} - 2\beta_k)} \frac{\Upsilon_b^{(h)}(b)}{\Upsilon_b^{(h)}(\beta_{123} - Q_h)} \tag{3.6}
\]

so that

\[
Res_{\beta_{123}-Q_h=-sb} C(\beta_1, \beta_2, \beta_3) = (-\pi^h \mu)^s C_s(\beta_1, \beta_2, \beta_3) \tag{3.7}
\]

where \( \mu \) is the Liouville coupling constant.

On the other hand starting from the 3-point Coulomb gas correlator \( I_s^{(h/b)}(\beta_1, \beta_2, \beta_3) \) with \( \beta_k \) satisfying the charge conservation condition with the second screening charge \( w = h/b \) in (1.4) we arrive at a different expression involving \( \Upsilon_w^{(h)} \) with \( w = \frac{h}{b} \)

\[
\tilde{C}(\beta_1, \beta_2, \beta_3) = \frac{\Gamma(h)}{(\frac{h}{b})^{h-1}} \left( -\frac{\pi^h \tilde{\mu}(\frac{h}{b})^{2h-2(b^2)}(b^2)}{(\frac{h}{b})^{4h} \gamma_h(-b^2)} \right) \frac{\gamma_h(\beta_{123}, -\frac{\beta_{123}}{h})}{\beta_{123} - Q_h} \prod_{k=1}^{3} \frac{\Upsilon_{\frac{h}{b}}^{(h)}(2\beta_k)}{\Upsilon_{\frac{h}{b}}^{(h)}(\beta_{123} - 2\beta_k)} \frac{\Upsilon_{\frac{h}{b}}^{(h)}(\frac{h}{b})}{\Upsilon_{\frac{h}{b}}^{(h)}(\beta_{123} - Q_h)} \tag{3.8}
\]

s.t.

\[
Res_{\beta_{123}-Q_h=-s\frac{h}{b}} \tilde{C}(\beta_1, \beta_2, \beta_3) = (-\pi^h \tilde{\mu})^s \tilde{C}_s(\beta_1, \beta_2, \beta_3) \tag{3.9}
\]

with \( \tilde{\mu} \) - the dual Liouville coupling constant in front of the interaction term \( \int d^2 h x V_\frac{h}{b}(x) \).

Thus, starting from the two Coulomb gas 3-point expressions we get different unrelated analytic continuations expressed by the two different functions \( \Upsilon_w^{(h)} = \Upsilon_{\sqrt{hw}}^{(h)} \) with \( \tilde{w} = \frac{b}{h} \), and \( \tilde{w} = 1/b \) respectively. This implies that the two Liouville theories with interaction terms defined by the two dual screening charges produce different correlators for \( h > 1 \) that are related by \( b, \mu \to \frac{b}{h}, \tilde{\mu} \); the theory is not selfdual for \( h > 1 \) - even if we impose some relation between the two coupling coinstants \( \mu \) and \( \tilde{\mu} \). The residues of the poles in (3.6) or (3.8) are consistent with the singularities of the corresponding factors obtained integrating out the zero modes of the field \( \phi \). The physical interpretation of the other poles (like e.g., the ones of (3.6) at \( \beta_{123} - Q_h = -s\frac{h}{b} \)) remains an open question.
4. The light charge semiclassical limit

Next we study the limit $b \to 0$ of the 3-point correlators for light charges $\beta_k = b\sigma_k$, s.t $\sigma_k$ are finite: their scaling dimensions become $2\triangle_h(b\sigma_k) \to 2h\sigma_k$. One has

$$\frac{\Upsilon^{(h)}_b(b\sigma)}{\Upsilon^{(h)}_b(b)} \to \frac{(h^{2-h}\Gamma(h))^{1-\sigma}}{\Gamma(\sigma)}, \quad (4.1)$$

$$\frac{\Upsilon^{(h)}_b(b\sigma)}{\Upsilon^{(h)}_b(b^2)} \to \frac{(b^{1-\sigma}\Gamma(h))}{\Gamma(h)} \cdot \quad (4.2)$$

We apply (4.1) to the properly normalized constant (3.6) getting (here $\sigma = \sigma_{123} = \sigma_1 + \sigma_2 + \sigma_3$)

$$C(b\sigma_1, b\sigma_2, b\sigma_3) \frac{\Upsilon^{(h)}_b((\sigma b - Q_h)) b^{-h\frac{2}{b^2}(Q_h - b\sigma)}}{\Upsilon^{(h)}_b((\sigma - 1)b)} \to \lambda^{\frac{b^2}{2} + 1-\sigma} b^{-2h^2 - 3} \Gamma(h)^{-1} \prod_{k=1}^{3} \frac{\Gamma((\sigma - 2\sigma_k))}{\Gamma(2\sigma_k)} \Gamma((\sigma - 1)), \quad (4.3)$$

where $\lambda^{\frac{b^2}{2} + 1-\sigma} = \left( \frac{-\mu^h \pi^h}{b^{4h}\gamma_h(-b^2)} \right)^{\frac{b^2}{2} + 1-\sigma} \to \left( \frac{\mu^h \pi^h}{b^{2(2h-1)}} \Gamma(h) \right)^{\frac{b^2}{2} + 1-\sigma} e^{h(\Psi(h) + \Psi(1))}$. Analogously using (4.2):

$$\tilde{C}(b\sigma_1, b\sigma_2, b\sigma_3) \frac{\Upsilon^{(h)}_b((\sigma b - Q_h)) \left( \frac{b}{h} \right)^{-h\frac{2}{b^2}(Q_h - b\sigma)}}{\Upsilon^{(h)}_b((\sigma - 1)b)} \to \tilde{\lambda}^{\frac{b^2}{2} + 1-\sigma} \left( \frac{b}{h} \right)^{1-4h} \prod_{k=1}^{3} \frac{\Gamma(h(\sigma - 2\sigma_k))}{\Gamma(2h\sigma_k)} \Gamma(h(\sigma - 1)), \quad (4.4)$$

where

$$\tilde{\lambda} = \frac{-\tilde{\mu}^h}{\left( \frac{b}{h} \right)^{4h}\gamma_h(-\left( \frac{b}{h} \right)^2)}. \quad (4.5)$$

The normalizing factor in the l.h.s. of (4.4) is equal to $1/\gamma_h((\sigma b - Q)\frac{b}{h})$; the one in the l.h.s. of (4.3) involves the shift $\Upsilon^{(h)}_b(x + \frac{b}{h})$ mentioned above.

The product of Gamma functions in the second formula (4.4) (with the factor $h$ appearing in their arguments) reproduces the direct classical computation in [4], generalizing
the one in [5]. However, the fixed area $A$ correlator considered in [4] is rather related to the limit of the properly normalized constant (3.6); it differs by a finite normalization factor from the l.h.s. of (4.3). For that we get

$$C^A(b\sigma_1,b\sigma_2,b\sigma_3) = (\mu A)^{\frac{h\sigma_i - Q h}{b}} \frac{C(b\sigma_1,b\sigma_2,b\sigma_3)}{\Gamma(\frac{2h\sigma_i - Q h}{b})} \times$$

$$\to \frac{e^{\frac{h(2h-1)}{b^2}} e^{h(\Psi(h) + \Psi(1))}}{b^4 \Gamma(h) \Gamma(2h)} \sqrt{\frac{\Gamma(2h)}{2\pi}} C_h(b) \times$$

$$\left(\frac{A}{\pi^h} \frac{\Gamma(2h)}{\Gamma(h)}\right)^{\frac{h\sigma_i - Q h}{b}} \prod_{k=1}^{3} \frac{\Gamma((\sigma - 2\sigma_k))}{\Gamma(2\sigma_k)} \Gamma((\sigma - 1)).$$

(4.6)

The constant $C_h(b)$ is given in appendix (A2), $C_1(b) = 1$. For $h = 1$ (4.4) reproduces the semiclassical fixed area correlator in [5]; $\Psi(1) = -C$ is the Euler constant.

The factor containing $A$ in the third line of (4.6) reproduces the corresponding factor in the expression written in [4] - where it appears in front of the ratio of $h$-dependent $\Gamma$ functions as in the r.h.s. of (1.4). We may choose the dual coupling constant $\tilde{\mu}$ as a function of $\mu$ so that the power of $\mu$ is identical in the two formulae (4.3) and (1.4), by some generalization of the $h = 1$ relation, $\tilde{\lambda} = \lambda^\frac{h}{b}$, which ensures the selfduality of the Liouville correlator in 2d. Then the (properly normalized as in (4.6)) second formula (4.4), would reproduce qualitatively the expression in [4].

To make connection with the notation in [4]: the parameters $b, Q$ and charges $\beta_i = b\sigma_i$ in [4] correspond to our $\bar{b}, Q = Q_h/\sqrt{h}$ and $\bar{\beta}_i = \bar{b}\sigma_i$ in (1.6), respectively.

5. Compact ("matter") region

The Coulomb gas representation for the correlator of the vertex operators $V_{e(M)}(x) = e^{2e \cdot i\chi(x)}$ in this region is given by changing in (2.1) $b^2 \to -b^2$ while $p_a = -2\beta_a b \to -2e_a b$; accordingly $-2\beta_a \frac{h}{b} \to 2e_a \frac{h}{b}$. The scaling dimension is

$$2\Delta^M(e) = 2e(e - e_0^{(h)}) = 2\bar{e}(\bar{e} - e_0), e_0^{(h)} = \frac{h}{b} - b = \sqrt{h}e_0 = \sqrt{h}(\frac{1}{b} - \bar{b})$$

(5.1)

so that the vertex operators of dimension $2h$ are $V_{-b}^{(M)}$ and $V_{b}^{(M)}$; the (new) parameter $b$ is again assumed real.
The analog of the 3-point constant (3.6) for arbitrary three charges $e_i$ reads

$$C^{(M)}(e_1, e_2, e_3) = (-\pi^h \mu_M)^{e_{123} - e_0^{(h)}} C_s^{(M)}(e_1, e_2, e_3)$$

$$= (-\pi^h \mu_M \frac{b^{-2b_{e_0}^{(h)}}}{\gamma_h(b^2)})^3 \prod_{k=1}^{3} \frac{\gamma^{(h)}_b(e_{123} - 2e_k + b)}{\gamma^{(h)}_b(2e_k + b)} \frac{\gamma^{(h)}_b(e_{123} - e_0^{(h)} + b)}{\gamma^{(h)}_b(b)}$$

s.t. $C^{(M)}(e_1, e_2, e_3) = 1$ for $e_{123} = e_0^{(h)}$, while for $e_{123} - bs = e_0^{(h)}$ (5.2) gives

$$C^{(M)}(e_1, e_2, e_3)_{e_{123} - e_0^{(h)} = sb} = (-\pi^h \mu_M)^{s} \frac{1}{\gamma_h(b^2)} \prod_{k=0}^{s-1} \frac{\gamma_h((b + s)k)^2}{\gamma_h(2e_kb - kb^2)}$$

i.e. it reproduces the Coulomb gas expression computed with $s$ screening charges $\int d^{2h}x V_{-b}^{(M)}(x)$. Analogously one can represent $\tilde{C}_M(e_1, e_2, e_3)$ by $\gamma_h^{(h)}$.

As in the $h = 1$ case [9], [10], the 3-point constant (5.3) is inverse to the Liouville one (3.4) with charges $\beta_i = e_i + b$ up to a product of $\gamma_h$ functions. For these values the sum of the ”matter” and Liouville scale dimensions

$$2\Delta(\beta) + 2\Delta^M(e) = 2h ,$$

so that $V_{\beta} V_e^{(M)}$ with $\beta = e + b$ are analogs of the ”tachion” fields of the 2d Liouville gravity. Analogous cancellation takes place for the product of the dual constants whenever the charges are related as $\beta_i = -e_i + \frac{b}{b}$. The sum of central charges of the intrinsic Virasoro algebra of [3] is as in the 2d case $1 + 6Q^2(b) + 1 - 6e_0^2(b) = 26$ (see appendix (A3)).

6. Final remark

The generalized BF formula (2.3) can be used to compute also some particular $2h$-dimensional integrals representing 3-point functions in the higher rank generalizations of the Liouville theory - the conformal $W$-theories, as it has been done in [11], [12] in the 2-dimensional case. One would also expect that analogously to the construction of the Virasoro algebra [3], intrinsic $W$-algebras can be constructed in these $2h$-dimensional models as well.

Acknowledgements

This research is partially supported by the Bulgarian NSF grant DN 18/1 and by the COST action MP-1405 QSPACE.
Appendix A1. The Liouville action in $2\hbar$ dimensions

The Liouville action and its dual

$$S^w = \frac{1}{(4\pi)^h \Gamma(h)} \int d^{2h}x (\phi(-\Box)^h \phi + 2Q_h(w) \phi \sqrt{gG}) + \mu_w \int d^{2h}x \sqrt{g} e^{2w\phi} \quad (A1.1)$$

with

$w = b$, $\mu_b = \mu$, or $w = \frac{h}{b}$, $\mu_{h/b} = \tilde{\mu}$

$Q_h(b) = Q_h(h/b)$. With more symmetric notation, rescaling parameters as in (1.6) and $\phi = \sqrt{h}\bar{\phi}$, (A1.1) is brought to the notation in [4]. On the $2\hbar$ sphere $S^{2h}$

$$\frac{1}{(4\pi)^h \Gamma(h)} \int d^{2h} \sqrt{gG}(x) = 1$$

and we assume that the reference metric is locally flat with the only singularity of the curvature related factor being localized at a point at infinity, i.e., $\sqrt{gG}(x) = (4\pi)^h \Gamma(h)\delta^{2h}(x - R_\infty)$. This term in the action is then equivalent to the insertion of vertex operator $V_{-Q_h} = e^{-2Q_h\phi}$. For more elaborated presentation see [4].

Appendix A2. Formula used in the derivation of (4.6)

Recall the integral representation of $\Gamma_b(x)$ entering the definition of $\Upsilon^{(h)}_b(x)$ in (3.2) (with $w = b$, $Q = \frac{1}{b} + b$)

$$\ln \Gamma_b(x) = \int_0^\infty \frac{dt}{t} \left( \frac{e^{-2tx} - e^{-Qt}}{(1 - e^{-2bt})(1 - e^{-2t/b})} - \frac{(Q - x)^2}{2} e^{-2t} - \frac{Q - x}{2t} \right) . \quad (A2.1)$$

We write down the formula for the shift $\Upsilon^{(h)}_b(x + \frac{h}{b})$ by the dual charge $\frac{h}{b}$

$$\frac{\Upsilon^{(h)}_b(x + \frac{h}{b})}{\Upsilon^{(h)}_b(x)} = \left( \frac{1}{b} \right)^{h-\frac{2h}{b}} \gamma_1(\frac{x}{b}) \frac{\prod_{k=1}^h \Gamma((Q_h - x - \frac{k}{b})\frac{1}{b})}{\Gamma((x + \frac{k}{b})\frac{1}{b})} . \quad (A2.2)$$

Applied to the factor in the l.h.s. of (4.3) i.e., for $x = \sigma b - Q_h$, it reads

$$\frac{\Upsilon^{(h)}_b(\sigma b - Q_h)b^{-h-\frac{2h}{b}(Q_h - \sigma b)}}{\Upsilon^{(h)}_b((\sigma - 1)b)} = \frac{1}{\gamma_1(\sigma - 1 - \frac{h}{b^2})} \prod_{k=1}^{h-1} \frac{\Gamma(2 - \sigma + \frac{h+k}{b^2})}{\Gamma((\sigma - 1 - \frac{h-k}{b^2})} . \quad (A2.3)$$

To obtain (4.6) one has to evaluate the asymptotic expansion for big arguments of $\Gamma(z)$ of the inverse of (A2.3) times $1/\Gamma(\sigma - 1 - \frac{h}{b^2})$ and combine it with the r.h.s. of (4.3). The remaining constant in (4.6) is $C_h(b) = \prod_{k=1}^{h-1} (\frac{h-k}{k+h})^{k/b^2}$. 

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Appendix A3. Some data on the intrinsic Virasoro algebra of the 2h model

We recall some data on the Virasoro algebra revealed in [3], now given in the Liouville region of parameters. There are two realizations of the generators \( L_n(x) \) constructed in terms of the modes \( b_n(x) \) of the field \( \phi(x) \) with central charges and eigenvalues of \( L_0 \) given respectively by:

\[
\begin{align*}
  i) \quad & c = 1 + 12hQ^2(\bar{b}) = 1 + 12 Q^2_n(\bar{b}) , \quad \bar{L}_0|\alpha > = 2h\triangle(\bar{\alpha})|\alpha > = 2\triangle_h(\alpha)|\alpha > \\
  ii) \quad & c = 1 + 6Q^2(\bar{b}) , \quad L_0|\alpha > = \triangle(\bar{\alpha})|\alpha > . \tag{A3.1}
\end{align*}
\]

In the 1-dim case \( h = 1/2 \) the two realizations coincide. In the ”matter” region, originally discussed in [3], \( b^2 \to -b^2 \) in the expressions for the central charge and formulae (5.1) replace the scaling dimensions above.

The action of the generators \( \bar{L}_n(z), n = -1, +1, 0 \) of the finite dimensional subalgebra in the first realization can be identified (see formula (3.9) of [3]) with \( (z \cdot P), (z \cdot K), D \), where \( P_\mu, K_\mu, D \) are the differential operators realizing (scalar) representations of the generators of translations, special conformal transformations and dilatations in the (euclidean) conformal group in \( 2h \) dimensions. The second realization with central charge \( c \) and eigenvalue of \( L_0 \) identical to the ones in the 1-dimensional conformal theory, is more convenient in the analysis of the degenerate representations and singular vectors.
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