SEMILINEAR ELLIPTIC EQUATIONS WITH HARDY POTENTIAL AND
SUBCRITICAL SOURCE TERM

PHUOC-TAI NGUYEN

DEPARTAMENTO DE MATEMÁTICAS, PONTIFICIA UNIVERSIDAD CATÓLICA DE CHILE
CASILLA 307, CORREO 2, SANTIAGO, CHILE

Contents

1. Introduction 2
2. Preliminaries 7
  2.1. Weak $L^p$ spaces 8
  2.2. Green and Martin kernels 8
  2.3. Some result on linear equations 9
3. Nonlinear equations with source term 10
  3.1. Equivalent formulation 10
  3.2. Nondecreasing source 11
4. Power source 13
  4.1. Subcritical case 13
5. Subcriticality and sublinearity 20
  5.1. Subcriticality 20
  5.2. Sublinearity 24
  5.3. Supercritical case 25
References 26

Abstract. Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^N$ and $\delta(x) = \text{dist}(x, \partial\Omega)$. Assume
$\mu \in \mathbb{R}$, $\nu$ is a nonnegative finite measure on $\partial\Omega$ and $g \in C(\Omega \times \mathbb{R}_+)$. We study positive
solutions of

$$(P) - \Delta u + \frac{\mu}{\delta^2} u = g(x, u) \text{ in } \Omega, \quad \text{tr}^*(u) = \nu.$$ 

Here $\text{tr}^*(u)$ denotes the normalized boundary trace of $u$ which was recently introduced by M.
Marcus and P. T. Nguyen in [16]. We focus on the case $0 < \mu < C_H(\Omega)$ (the Hardy constant
for $\Omega$) and provide some qualitative properties of solutions of $(P)$. When $g(x, u) = u^q$ with
$q > 1$, we prove that there is a critical value $q^*$ (depending only on $N$, $\mu$) for $(P)$ in the
sense that if $1 < q < q^*$ then $(P)$ admits a solution under a smallness assumption on $\nu$,
but if $q \geq q^*$ this problem admits no solution with isolated boundary singularity. Existence
result is then extended to a more general setting where $g$ is subcritical (see (1.27)). We also
investigate the case where the $g$ is linear or sublinear and give some existence results for $(P)$.

Key words: Hardy potential, Martin kernel, normalized boundary trace, source terms.

2000 Mathematics Subject Classification: 35J60, 35J75, 35J10

Email address: nguyenphuoctai.hcmup@gmail.com.

1
1. Introduction

This paper concerns a study of weak solutions of semilinear elliptic equation with Hardy potential and source term

\( -\Delta u - \frac{\mu}{\delta^2} u = g(x, u) \)

in a \( C^2 \) bounded domain \( \Omega \), where \( \mu \in \mathbb{R}, \delta(x) = \text{dist}(x, \partial\Omega) \) and \( g \in C(\Omega \times \mathbb{R}_+) \) and \( g(x, 0) = 0 \).

Henceforth, we will use the notations \( L_\mu := \Delta + \frac{\mu}{\delta^2} \) and \((g \circ u)(x) := g(x, u(x))\).

**Definition 1.1.** (i) A function \( u \) is called a (weak) solution (resp. subsolution, supersolution) of \( (1.1) \) if \( u \geq 0 \), \( u \in L^1_{\text{loc}}(\Omega) \), \( g \circ u \in L^1_{\text{loc}}(\Omega) \) and

\[-L_\mu u = g \circ u \ (\text{resp.} -L_\mu u \leq g \circ u, -L_\mu u \geq g \circ u)\]

in the sense of distributions.

(ii) A function \( u \) is an \( L_\mu \)-harmonic function (resp. \( L_\mu \)-subharmonic, \( L_\mu \)-superharmonic) if \( u \in L^1_{\text{loc}}(\Omega) \) and

\[-L_\mu u = 0 \ (\text{resp.} -L_\mu u \leq 0, -L_\mu u \geq 0)\]

in the sense of distributions.

Boundary value problem with measures for \( (1.1) \) with \( \mu = 0 \) and \( g \circ u = u^q \), i.e. the problem,

\( -\Delta u = u^q \text{ in } \Omega, \quad u = \nu \text{ on } \partial\Omega. \)

was first considered by Bidault-Véron and Vivier in [6]. They established estimates involving classical Green and Poisson kernels for \( -\Delta \) and applied these estimates to obtain an existence result in the subcritical case, i.e. \( 1 < q < q_c := \frac{N+1}{N-1} \). Then Bidaut-Véron and Yarur [8] reconsidered this type of problem in a more general setting and provided necessary and sufficient condition for the existence of a solution of \( (1.2) \). In [11] Chen, Felmer and Véron investigated \( (1.1) \) with \( \mu = 0 \) and \( g \) satisfying a subcritical condition. Their approach makes use of Schauder fixed point theorem, essentially based on estimates related to weighted Marcinkiewicz spaces. Recently, Véron et al. [7] provided new criteria for the existence of weak solutions of problem \( (1.2) \) and extended those results to the case where \( \Delta \) is replaced by \( L_\mu \).

When \( \mu \neq 0 \), the study of \( (1.1) \) relies strongly on the investigation of the linear equation

\( -L_\mu u = 0. \)

Equation \( (1.3) \) with \( \mu < 0 \), and more generally Schrödinger equations \( -\Delta u - V(x)u = 0 \) where \( V \) is a nonnegative potential, was studied by Ancona [1, 2], Marcus [15], Ancona and Marcus [3] and by Véron and Yarur [21]. The case \( \mu > 0 \) was considered by Bandle et al. [4, 5], Marcus and Nguyen [16] and by Gkikas and Véron [14] in connection with the optimal constant \( C_H(\Omega) \) in Hardy’s inequality, namely

\( C_H(\Omega) = \inf_{H^1(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla u|^2 \, dx}{\int_\Omega (u/\delta)^2 \, dx}. \)

It is well known (see [10, 17]) that \( C_H(\Omega) \in (0, \frac{1}{2}] \) and \( C_H(\Omega) = \frac{1}{4} \) when \( \Omega \) is convex. Moreover the infimum is achieved if and only if \( C_H(\Omega) < 1/4. \)

Let \( \phi \geq 0 \) in \( \Omega \) and \( p \geq 1 \), we denote by \( L^p(\Omega; \phi) \) the space of all function \( v \) on \( \Omega \) satisfying

\( \int_\Omega |v|^p \phi \, dx < \infty. \)

We denote by \( \mathcal{M}(\Omega; \phi) \) the space of Radon measures \( \tau \) on \( \Omega \) satisfying

\( \int_\Omega \phi \, d\tau < \infty \)

and by \( \mathcal{M}^+(\Omega; \phi) \) the nonnegative cone of \( \mathcal{M}(\Omega; \phi) \). When \( \phi \equiv 1 \), we use the usual notations \( \mathcal{M}(\Omega) \) and \( \mathcal{M}^+(\Omega) \). We also denote by \( \mathcal{M}(\partial\Omega) \) the space of finite measures on \( \partial\Omega \) and by \( \mathcal{M}^+(\partial\Omega) \) the nonnegative cone of \( \mathcal{M}(\partial\Omega) \).
Let $G_\mu$ and $K_\mu$ be the Green and the Martin kernels for $-L_\mu$ in $\Omega$ respectively (see [16] for more detail). Denote by $G_\mu$ and $K_\mu$ the associated operators defined by

$$G_\mu[\tau](x) = \int_\Omega G_\mu(x, y)d\tau(y) \quad \forall \tau \in \mathcal{M}(\Omega).$$

(1.5)

$$K_\mu[\nu](x) = \int_{\partial\Omega} K_\mu(x, z)d\nu(z) \quad \forall \nu \in \mathcal{M}(\partial\Omega).$$

(1.6)

Put

$$\alpha_\pm := \frac{1 \pm \sqrt{1 - 4\mu}}{2}. \tag{1.7}$$

Let $\lambda_{\mu,1}$ be the first eigenvalue of $-L_\mu$ in $\Omega$ and denote by $\varphi_{\mu,1}$ the corresponding eigenfunction normalized by $\int_{\Omega} (\varphi_{\mu,1}/\delta)^2 dx = 1$ (see [10]). If $\mu \in (0, C_H(\Omega))$ then $\lambda_{\mu,1} > 0$ and by [12] (see also [19]), there exists a constant $c_1 > 0$ such that

$$c_1^{-1} \delta^{\alpha_+} \leq \varphi_{\mu,1} \leq c_1 \delta^{\alpha_+} \quad \text{in } \Omega. \tag{1.8}$$

For $\beta > 0$, put

$$\Omega_\beta = \{x \in \Omega : \delta(x) < \beta\}, \quad D_\beta = \{x \in \Omega : \delta(x) > \beta\}, \quad \Sigma_\beta = \{x \in \Omega : \delta(x) = \beta\}.$$ 

When dealing with boundary value problem associated to (1.1) with $\mu > 0$ one encounters the following difficulties:

- The first one is due to the fact that every positive $L_\mu$-harmonic function has classical measure boundary trace zero (see [16, Corollary 2.11]). Therefore, classical boundary trace no longer plays a role in describing the boundary behavior of $L_\mu$-harmonic function or solutions of (1.1).

- The second one stems from the invalidity of the classical Keller-Osserman estimate, as well as the lack of a universal upper bound for solutions of (1.1). Moreover, contrast to the case of nonnegative absorption nonlinearity, $K_\mu[\nu]$ is a subsolution of

$$-L_\mu u = g \circ u \quad \text{in } \Omega, \quad \text{tr}^*(u) = \nu \tag{1.9}$$

and therefore it is no longer an upper bound for solutions of (1.9).

In order to overcome the first difficulty, we shall employ the notion of normalized boundary trace which is defined as follows:

**Definition 1.2.** A positive function $u$ possesses a normalized boundary trace if there exists a measure $\nu \in \mathcal{M}^+(\partial\Omega)$ such that

$$\lim_{\beta \to 0} \int_{\Sigma_\beta} \frac{|u - K_\mu[\nu]|}{\beta^{\alpha_+}} dS = 0. \tag{1.10}$$

The normalized boundary trace of $u$ is denoted by $\text{tr}^*(u)$.

In the above definition, we use the notation $dS = d\mathbb{H}_{N-1}$ where $\mathbb{H}_{N-1}$ denotes the Hausdorff measure. This notion is introduced by Marcus and Nguyen [16] in the case $\mu \in (0, C_H(\Omega))$. It is worth mentioning that if $\mu \in (0, C_H(\Omega))$ then $\lambda_{\mu,1} > 0$ and hence $\varphi_{\mu,1}$ is a positive $L_\mu$-superharmonic function. This fact, together with a classical result of Ancona [2], implies the existence of $L_\mu$ harmonic functions and guarantees the validity of Representation theorem (see [16]). Normalized boundary trace turns out to be a more appropriate notion to investigate the problem

$$-L_\mu u + u^q = 0 \quad \text{in } \Omega, \quad \text{tr}^*(u) = \nu. \tag{1.11}$$
More precisely, when $\mu \in (0, C_H(\Omega))$, they showed that there exists a critical exponent
\begin{equation}
q^* = q^*(N, \mu) := \frac{N + \alpha_+}{N - 1 - \alpha_-}.
\end{equation}
for (1.11). This means that if $1 < q < q^*$, for every positive finite boundary measure $\nu$, (1.11) admits a unique positive solution while if $q \geq q^*$ there exists no positive solution of (1.11) with $\nu$ being a Dirac measure. Stability result was also discussed in the case $1 < q < q^*$. Problem (1.11) with $u^q$ replaced by a more general nonlinearity $f(u)$ was then investigated by Gkikas and Véron [14] in a slightly different setting. When $f(u) = |u|^{q-1}u$, they provided a necessary and sufficient condition in terms of Besov capacity for solving (1.11) in the supercritical case, i.e. $q \geq q^*$.

Because of the second difficulty, we mainly deal with the minimal solution of (1.9) which possesses several exploitable properties. This solution is constructed due to sub-supersolutions theorem that is established in Section 3. Observe that $K_\mu[\nu]$ is a subsolution of (1.9). Hence in order to prove the existence of a minimal solution of (1.9), it is sufficient to find a supersolution of (1.9) which dominates $K_\mu[\nu]$.

Throughout the present paper, we assume that $\mu \in (0, C_H(\Omega))$. We now introduce the definition of solutions of (1.9).

\textbf{Definition 1.3.} (i) A nonnegative function $u$ is called a (weak) solution of (1.9) if $u$ is a solution of (1.1) and has normalized boundary trace $\nu$.

(ii) Let us define the space of admissible test function as follows:

$$X(\Omega) = \{ \zeta \in C^2(\Omega) : \delta^\alpha - L_\mu \zeta \in L^\infty(\Omega), \delta^{-\alpha} \zeta \in L^\infty(\Omega) \}.$$  

A function $\zeta \in X(\Omega)$ is called an admissible test function for (1.9).

Notice that $\varphi_{\mu,1} \in X(\Omega)$. More properties of $X(\Omega)$ can be found in [16, Section 2.4]. Using this space, we establish integral formulation for weak solutions of (1.9). This is stated in the following result.

\textbf{Theorem A.} Let $\nu \in \mathcal{M}^+(\partial \Omega)$. The following statements are equivalent:

(i) $u$ is a solution of (1.9),

(ii) $g \circ u \in L^1(\Omega; \delta^\alpha)$ and

\begin{equation}
u = G_\mu[g \circ u] + K_\mu[\nu].\end{equation}

(iii) $u \in L^1(\Omega; \delta^{-\alpha}-)$, $g \circ u \in L^1(\Omega; \delta^{-\alpha})$ and

\begin{equation}-\int u L_\mu \zeta dx = \int (g \circ u) \zeta dx - \int K_\mu[\nu] L_\mu \zeta dx \ \forall \zeta \in X(\Omega).
\end{equation}

Under some additional assumptions on $g$, we obtain existence result for (1.9).

\textbf{Theorem B.} Let $g(x, r)$ be a nondecreasing continuous function with respect to $r$ for every $x \in \Omega$ and $\nu \in \mathcal{M}^+(\partial \Omega)$ with $\|\nu\|_{\mathcal{M}^+(\partial \Omega)} = 1$. Assume that there exist numbers $c_2 > 0$, $c_3 > 0$, $0 \leq r_1 < r_2 \leq \infty$ and a positive function $\ell$ such that

\begin{equation}g(x, rs) \leq \ell(r) g(x, s) \ \forall s \geq 0, r > 0, x \in \Omega.
\end{equation}

\begin{equation}\ell(1 + c_2 c_3 r^{-1} \ell(r)) \leq c_2 \ \forall r \in (r_1, r_2),
\end{equation}

\begin{equation}G_\mu[g \circ (K_\mu[\nu])] \leq c_3 K_\mu[\nu] \ \text{a.e. in } \Omega.
\end{equation}
1. existence. For any \( \varrho \in (r_1, r_2) \) the problem
\[
-L_\mu u = g \circ u \quad \text{in } \Omega, \quad \text{tr}^*(u) = \varrho
\]
adopts a minimal solution \( u_{\varrho\nu} \) in the sense that if \( v \) is a solution of (1.18) then \( u_{\varrho\nu} \leq v \).

2. Estimates. This solution satisfies
\[
\varrho K_\mu[v] \leq u_{\varrho\nu} \leq c_4 \varrho K_\mu[v]
\]
where \( c_4 = c_4(c_2, c_3, \ell, \varrho) \).

3. Nontangential convergence. For \( \nu \)-a.e. \( z \in \partial \Omega \), there holds
\[
\lim_{x \to z} \frac{u_{\varrho\nu}(x)}{K_\mu[v](x)} = \varrho \quad \text{non tangentially.}
\]

Remark. As a model, we can take \( g(x, u) = \ln^\beta (u + 1)u^q \) with small \( \beta \geq 0 \) and \( q > 0 \).

In the next results, we focus on the pure power case, namely the problem
\[
(D_\nu) \quad -L_\mu u = u^q \quad \text{in } \Omega, \quad \text{tr}^*(u) = \nu
\]
where \( q > 0 \) and \( \nu \in \mathfrak{M}^+(\partial \Omega) \). We shall establish some estimates related Green and Martin operators and a necessary condition for the existence of solutions of \( (D_\nu) \) in the case \( q > 1 \).

**Proposition C.** Let \( q > 0 \) and \( \nu \in \mathfrak{M}^+(\partial \Omega) \). Then there exists a positive constant \( c_5 = c_5(N, \mu, q, \Omega) \) such that
\[
\mathcal{G}_\mu[K_\mu[v]^q] \leq c_5 \|\nu\|_{\mathfrak{M}^+(\partial \Omega)}^{-1} K_\mu[v].
\]
Furthermore, if \( q > 1 \) and problem \( (D_\nu) \) admits a solution then there holds
\[
\mathcal{G}_\mu[K_\mu[v]^q] \leq \frac{1}{q-1} K_\mu[v].
\]

Remark. It is worth mentioning that (1.21) with \( q > 1 \) and (1.22) with an inexplicit multiplier were proved in [7, Theorem 4.1]. In this paper we employ the method in [8] to prove (1.21) for \( q > 0 \) and apply the idea in [9] to point out that the multiplier can be explicitly chosen as \( \frac{1}{q-1} \).

The next results reveal that \( q^* \) is the critical exponent for \( (D_\nu) \). More precisely, in the subcritical case, namely \( 1 < q < q^* \), \( (D_\nu) \) admits a solution under smallness assumption on the boundary datum while in the supercritical case, i.e. \( q \geq q^* \), this problem possesses no solution with isolated boundary singularity.

For \( z \in \partial \Omega \), we denote by \( \delta_z \) the Dirac measure concentrated at \( z \). Existence and nonexistence results when \( 0 < q < q^* \), \( q \neq 1 \) is given as follows.

**Theorem D.** Let \( q \in (0, q^*) \), \( q \neq 1 \) and \( \nu \in \mathfrak{M}^+(\partial \Omega) \) with \( \|\nu\|_{\mathfrak{M}^+(\partial \Omega)} = 1 \). For \( q > 0 \), consider the problem
\[
(D_{\varrho\nu}) \quad -L_\mu u = u^q \quad \text{in } \Omega, \quad \text{tr}^*(u) = \varrho \nu.
\]

1. **Case \( q > 1 \).** There exists a threshold value \( \varrho^* \in \mathbb{R}_+ \) for \( (D_{\varrho\nu}) \) such that
   (i) If \( \varrho \in (0, \varrho^*) \) then problem \( (D_{\varrho\nu}) \) admits a minimal solution \( u_{\varrho\nu} \). Moreover, if \( \varrho \in (0, \varrho^*) \), \( u_{\varrho\nu} \) satisfies (1.19) and (1.20).
   (ii) If \( \varrho > \varrho^* \) then there exists no solution of \( (D_{\varrho\nu}) \).

In addition, if \( \{\varrho_n\} \) be a nondecreasing sequence converging to \( \varrho^* \) then the sequence \( \{u_{\varrho_n\nu}\} \) converges to \( u_{\varrho^*\nu} \) in \( L^1(\Omega; \delta^{-\alpha-}) \) and in \( L^q(\Omega; \delta^{\alpha+}) \).

(ii) If \( \varrho > \varrho^* \) then there exists no solution of \( (D_{\varrho\nu}) \).
2. Case \( q \in (0,1) \). For every \( q > 0 \) problem \((D_{\nu q})\) admits a minimal solution \( u_{\nu q} \) which satisfies \((1.19)\) and \((1.20)\). Moreover, \( \lim_{q \to 0} u_{\nu q} = 0 \) and \( \lim_{q \to \infty} u_{\nu q} = \infty \).

In any case, if \( \nu = \delta_z \) with \( z \in \partial \Omega \) then there holds

\[
(1.23) \quad \lim_{x \to z} \frac{u_{\nu q}}{K_\mu(x,z)} = q.
\]

**Remark.** It is worth noticing that in absorption case \((1.11)\), if \( 1 < q < q^* \), there are two types of solution with isolated boundary singularity: weakly singular solutions \( u_{\nu z} \) (the solution of \((1.11)\) with \( \nu = \delta_z \)) and strongly singular solution \( u_{\nu \infty,z} \). Actually, \( u_{\nu \infty,z} \) is the limit of the sequence \( u_{\nu z} \) as \( q \to \infty \). This limiting process can not be executed in the source case since \((D_{\nu \delta_z})\) admits no solution if \( q > q^* \) due to Theorem D.

We next give a stability result.

**Theorem E.** Let \( q \in (0,q^*) \), \( q \neq 1 \) and \( \{\nu_n\} \) is a sequence of measures in \( \mathcal{M}^+(\partial \Omega) \) which converges weakly to \( \nu \in \mathcal{M}^+(\partial \Omega) \). If \( q > 1 \), assume, in addition, that

\[
(1.24) \quad \sup_n \|\nu_n\|_{\mathcal{M}(\partial \Omega)} \leq q^*.
\]

For each \( n \), let \( u_{\nu_n} \) be a solution of \((D_{\nu_n})\). Then, up to a subsequence, \( \{u_{\nu_n}\} \) converges to a solution \( u_{\nu} \) of \((D_{\nu})\) in \( L^1(\Omega;\delta^{\alpha-}) \) and in \( L^q(\Omega;\delta^{\alpha+}) \).

Existence and stability result in the case \( q = 1 \) is stated in the following theorem in which \( \lambda_{\mu,1} \) is the first eigenvalue of \(-L_\mu\) in \( \Omega \).

**Theorem F.** Let \( \nu \in \mathcal{M}^+(\partial \Omega) \). For \( \kappa > 0 \), consider the problem

\[
(E^\kappa_{\nu}) \quad -L_\mu u = \kappa u \quad \text{in } \Omega; \quad \text{tr}^*(u) = \nu.
\]

There exists a number \( \kappa^* \in (0,\lambda_{\mu,1}) \) such that

(i) If \( \kappa \in (0,\kappa^*) \) then problem \((E^\kappa_{\nu})\) admits a minimal solution \( u_{\kappa,\nu} \). Moreover, \( u_{\kappa,\nu} \) satisfies \((1.20)\).

Assume \( \{\nu_n\} \) is a sequence of measures in \( \mathcal{M}^+(\partial \Omega) \) which converges weakly to \( \nu \in \mathcal{M}^+(\partial \Omega) \) and for each \( n \) denote by \( u_{\kappa,\nu_n} \) a solution of \((E^\kappa_{\nu_n})\). Then, up to a subsequence, \( \{u_{\kappa,\nu_n}\} \) converges to a solution \( u_{\kappa,\nu} \) of \((E^\kappa_{\nu})\) in \( L^1(\Omega;\delta^{\alpha-}) \).

(ii) If \( \kappa \geq \kappa^* \) then \((E^\kappa_{\nu})\) admits no solution.

Furthermore, problem \((E^\kappa_{\nu})\) admits no solution.

Note that the assumption that \( g(x,r) \) is nondecreasing with respect to \( r \) is crucial to obtain the existence in Theorems D and F. A natural question arises: "Does the existence results still hold if the monotonicity condition is dropped?". Positive answer to this question is given in the next two theorems where a more general weighted source term \( g(x,r) \) is involved. More precisely, we consider the case \( g(x,r)(x) = \delta(x)^\gamma \tilde{g}(u(x)) \) with \( \gamma > -1 - \alpha_+ \) and \( \tilde{g} : \mathbb{R}_+ \to \mathbb{R}_+ \) being continuous. In this framework, the critical value is defined as follows:

\[
(1.25) \quad q^*_\gamma = q^*_\gamma(N,\mu,\gamma) := \frac{N + \alpha_+ + \gamma}{N - 1 - \alpha_-}.
\]

Clearly \( q^*_0 = q^* \).

Theorem G gives existence result for the problem

\[
(1.26) \quad -L_\mu u = \delta^\gamma \tilde{g}(u) \quad \text{in } \Omega; \quad \text{tr}^*(u) = \nu.
\]
Theorem G. Let \( \nu \in \mathcal{M}^+(\partial \Omega) \) such that \( \|\nu\|_{\mathcal{M}(\partial \Omega)} = 1 \). Assume that
\[
\Lambda_0 := \int_1^{\infty} s^{-1-q_1^\ast} \tilde{g}(s) ds < +\infty,
\]
(1.27)
\[
\tilde{g}(s) \leq \Lambda_1 s^{q_1} + \theta \quad \forall s \in [0,1] \quad \text{for some } q_1 > 1, \Lambda_1 > 0, \theta > 0.
\]
(1.28)

There exist \( \theta_0 > 0 \) and \( \varrho_0 > 0 \) depending on \( N, \mu, \gamma, \Lambda_0, \Lambda_1 \) and \( q_1 \) such that for every \( \theta \in (0, \theta_0) \) and \( \varrho \in (0, \varrho_0) \) problem (1.26) admits a nonnegative solution.

Remark. We say that \( g \) is subcritical if \( \tilde{g} \) satisfies (1.27).

The case where \( g \) is linear or sublinear is treated in the following theorem.

Theorem H. Let \( \nu \in \mathcal{M}^+(\partial \Omega) \) such that \( \|\nu\|_{\mathcal{M}(\partial \Omega)} = 1 \). Assume that
\[
\tilde{g}(s) \leq \Lambda_2 s^{q_2} + \theta \quad \forall s \geq 0
\]
for some \( q_2 \in (0,1], \Lambda_2 > 0 \) and \( \theta > 0 \).
\[
\text{(1.29)}
\]
In (1.29), if \( q_2 = 1 \) we assume in addition that \( \Lambda_2 \) is small enough. Then for any \( \varrho > 0 \), (1.26) admits a nonnegative solution.

Note that in Theorem H, when \( q_2 < 1 \), smallness assumption on \( \theta \) is not required.

When \( \tilde{g} \) does not satisfy (1.27), there is no solution with an isolated boundary singularity.
This is stated in the following Theorem where we assume that \( \tilde{g} \) is nondecreasing.

Theorem I. Assume \( \tilde{g} \) is a nondecreasing function such that
\[
\int_a^{\infty} s^{-1-q_1^\ast} \tilde{g}(s) ds = \infty \quad \text{for some } a > 0.
\]
(1.30)

Then for every \( \varrho > 0 \) and \( z \in \partial \Omega \) there exists no positive solution of
\[
-L_\mu u = \delta^\gamma \tilde{g}(u) \quad \text{in } \Omega, \quad \text{tr}^\ast(u) = \varrho \delta_z
\]
(1.31)

Remark. If \( g(u) = u^q \) then (1.30) is satisfied if and only if \( q \) belongs to supercritical range, i.e. \( q \geq q_1^\ast \). We notice that Theorem I was obtained in [7] for the case \( \gamma = 0 \) and \( g(u) = u^q \).

Interesting existence results for \( (D_\rho \nu) \) in the supercritical case are also provided in [7].

The plan of the paper is as follows. In Section 2 we give some results concerning Green and Martin kernels and boundary value problem for linear equations with Hardy potential. Theorems A and B are proved in Section 3. It is noteworthy that main ingredients in proving Theorem A is a generalization of Herglotz-Doob to \( L_\mu \)-superharmonic functions and theory of Schrödinger linear equations. Theorem B is established using a sub-supersolutions theorem. The proof of Proposition C and Theorems D-F are provided in Section 4. Finally, in Section 5 we present the proof of the existence result for a more general class of source terms (Theorems G and H) and demonstrate the nonexistence result in the supercritical case (Theorem I).

Acknowledgements The author is grateful to Q. H. Nguyen for his useful comments.

2. Preliminaries

Throughout this paper we assume that \( 0 < \mu < C_H(\Omega) \).
2.1. Weak $L^p$ spaces. Denote $L^p_w(\Omega; \tau)$, $1 \leq p < \infty$, $\tau \in \mathcal{M}^+(\Omega)$, the weak $L^p$ space (or Marcinkiewicz space) defined as follows: a measurable function $f$ in $\Omega$ belongs to this space if there exists a constant $c$ such that
\begin{equation}
\lambda_f(a; \tau) := \tau\{x \in \Omega : |f(x)| > a\} \leq ca^{-p}, \quad \forall a > 0.
\end{equation}
The function $\lambda_f$ is called the distribution function of $f$ (relative to $\tau$). For $p \geq 1$, denote
\begin{equation*}
L^p_w(\Omega; \tau) = \{ f \text{ Borel measurable} : \sup_{a>0} a^p \lambda_f(a; \tau) < \infty \},
\end{equation*}
\begin{equation}
\|f\|_{L^p_w(\Omega; \tau)} = (\sup_{a>0} a^p \lambda_f(a; \tau))^{1/p}.
\end{equation}
The $\|\cdot\|_{L^p_w(\Omega; \tau)}$ is not a norm, but for $p > 1$, it is equivalent to the norm
\begin{equation}
\|f\|_{L^p_w(\Omega; \tau)} = \sup \left\{ \int_\Omega |f|^p d\tau : \omega \subset \Omega, \omega \text{ measurable}, 0 < \tau(\omega) < \infty \right\}.
\end{equation}

More precisely,
\begin{equation}
\|f\|_{L^p_w(\Omega; \tau)} \leq \|f\|_{L^p_w(\Omega; \tau)} \leq \frac{p}{p-1} \|f\|_{L^p_w(\Omega; \tau)}.
\end{equation}
When $\tau = \delta^a dx$, for simplicity, we use the notation $L^p_w(\Omega; \delta^a)$. Notice that, for every $\alpha > -1$, $L^p_w(\Omega; \delta^a dx) \subset L^r(\Omega; \delta^a)$, $\forall r \in [1, p)$.

From (2.2) and (2.4), one can derive the following estimate which is useful in the sequel. If $u \in L^p_w(\Omega; \delta^\alpha)$ ($\alpha > -1$) then
\begin{equation}
\int_{\{u \geq s\}} \delta^a dx \leq s^{-p} \|u\|_{L^p_w(\Omega; \delta^a)}^p.
\end{equation}

2.2. Green and Martin kernels. Let $G_\mu$ be the Green kernel for the operator $-L_\mu$ in $\Omega \times \Omega$ and denote by $\mathcal{G}_\mu$ the associated operator defined by (1.5). It was shown in [16] that for every $\tau \in \mathcal{M}(\Omega; \delta^\alpha)$, $|G_\mu[\tau]| < \infty$ a.e. in $\Omega$. Denote by $K_\mu$ the Martin kernel for $-L_\mu$ in $\Omega$ and by $\mathcal{K}_\mu$ the Martin operator defined by (1.6).

In what follows the notation $f \sim g$ means: there is a positive constant $c$ such that $c^{-1}f < g < cf$ in the domain of the two functions.

By [13, Theorem 4.11] and [16] (see also [14]),
\begin{equation}
G_\mu(x, y) \sim \min \left\{ |x - y|^{2-N}, \delta(x)^\alpha + \delta(y)^\alpha + |x - y|^{2\alpha - N} \right\}, \forall x, y \in \Omega, x \neq y.
\end{equation}

\begin{equation}
K_\mu(x, z) \sim \delta(x)^\alpha |x - z|^{2\alpha - N}, \forall x \in \Omega, z \in \partial \Omega.
\end{equation}

By combining (2.6), (2.7) and the estimates of [20, Lemma 2.3.2], we obtain the following.

**Proposition 2.1.** Let $\xi \in [0, 1]$. There exist constants $c_i = c_i(N, \mu, \beta, \Omega)$ ($i = 6, 7, 8$) such that,
\begin{equation}
\|\mathcal{G}_\mu[\tau]\|_{L^\infty(\Omega, \delta^{\alpha+})} \leq c_6 \|\tau\|_{\mathcal{M}(\Omega, \delta^{\alpha})}, \forall \tau \in \mathcal{M}(\Omega, \delta^{\alpha}),
\end{equation}
with $\beta > \max\{-1, -2 + 2\alpha_+\}$,
\begin{equation}
\|\mathcal{G}_\mu[\tau]\|_{L^\infty(\Omega, \delta^{\alpha})} \leq c_7 \|\tau\|_{\mathcal{M}(\Omega, \delta^{\alpha})}, \forall \tau \in \mathcal{M}(\Omega, \delta^{\alpha}), \beta > -1.
\end{equation}

\begin{equation}
\|\mathcal{K}_\mu[\nu]\|_{L^\infty(\partial \Omega, \delta^{\alpha})} \leq c_8 \|\nu\|_{\mathcal{M}(\partial \Omega)}, \forall \nu \in \mathcal{M}(\partial \Omega), \beta > -1.
\end{equation}
Proof. By (2.6), for every $\varsigma \in [0,1]$ there exists a constant $c_6'$ such that
\[
G_\mu(x,y) \leq c_6'\delta(x)^{\varsigma\alpha+}\delta(y)^{\varsigma\alpha+}|x-y|^{2-N-2\alpha+} \quad \forall x \neq y.
\]

By proceeding as in the proof of [20, Lemma 2.3.3], we obtain (2.8).

We next prove (2.9). If $\delta(y) \leq 2|x-y|$ then
\[
G_\mu(x,y) \leq c_7'\delta(x)^{\alpha+}\delta(y)^{\alpha+}|x-y|^{2-N} \leq c_7'2^{\alpha+}\delta(x)^{\alpha+}|x-y|^{1+\alpha-N} \quad \forall x \neq y.
\]

If $\delta(y) > 2|x-y|$ then from the inequality $|\delta(x) - \delta(y)| \leq |x-y|$ we deduce that $\delta(x) \geq \delta(y) - |x-y| > |x-y|$. Therefore
\[
G_\mu(x,y) \leq c_7''|x-y|^{2-N} \leq c_7''\delta(x)^{\alpha+}|x-y|^{1+\alpha-N}.
\]

Combining (2.11) and (2.12) yields
\[
G_\mu(x,y) \leq c_8'\delta(x)^{\alpha+}|x-y|^{1+\alpha-N} \quad \forall x \neq y
\]
where $c_8' = \max\{c_7'', c_7'2^{\alpha+}\}$. Using similar argument as in the proof of [20, Lemma 2.3.3], we deduce (2.9).

Estimate (2.10) was already obtained in [16, Proposition 2.8].

2.3. Some result on linear equations. In this subsection, we recall some results concerning boundary value problem for non-homogeneous linear equation.

(2.14) \[-L_\mu u = \tau.\]

Definition 2.2. (i) A function $u$ is a solution of (2.14) if $u \in L^1_{\text{loc}}(\Omega)$ and (2.14) is understood in the sense of distributions.

(ii) Let $\tau \in \mathcal{M}(\Omega;\delta^{\alpha+})$ and $\nu \in \mathcal{M}^+(\partial\Omega)$. A function $u$ is a solution of
\[
(2.15) \quad -L_\mu u = \tau \quad \text{in } \Omega, \quad \text{tr}^*(u) = \nu,
\]
if $u$ is a solution of (2.14) and $u$ admits normalized boundary trace $\nu$.

Definition 2.3. A nonnegative $L_\mu$-superharmonic function is called an $L_\mu$-potential if its largest $L_\mu$-harmonic minorant is zero.

The following results, which can be found in [16, Theorem I], is crucial in proving Theorem A.

Proposition 2.4. (i) If $\tau = 0$ then problem (2.15) has a unique solution, $u = K_\mu[v]$. If $u$ is a non-negative $L_\mu$-harmonic function and $\text{tr}^*(u) = 0$ then $u = 0$.

(ii) If $\tau \in \mathcal{M}^+(\Omega;\delta^{\alpha+})$ then $\mathcal{G}_\mu[\tau]$ has normalized trace zero. Thus $\mathcal{G}_\mu[\tau]$ is a solution of (2.15) with $\nu = 0$.

(iii) Let $u$ be a positive $L_\mu$-subharmonic function. If $u$ is dominated by an $L_\mu$-superharmonic function then $L_\mu u \in \mathcal{M}^+(\Omega;\delta^{\alpha+})$ and $u$ has a normalized boundary trace. In this case $\text{tr}^*(u) = 0$ if and only if $u \equiv 0$.

(iv) Let $u$ be a positive $L_\mu$-superharmonic function. Then there exist $\nu \in \mathcal{M}^+(\partial\Omega)$ and $\tau \in \mathcal{M}^+(\Omega;\delta^{\alpha+})$ such that
\[
(2.16) \quad u = \mathcal{G}_\mu[\tau] + K_\mu[v].
\]
In particular, $u$ is an $L_\mu$-potential if and only if $\text{tr}^*(u) = 0$.

(v) For every $\nu \in \mathcal{M}^+(\partial\Omega)$ and $\tau \in \mathcal{M}^+(\Omega;\delta^{\alpha+})$, problem (2.15) has a unique positive solution. The solution is given by (2.16). Moreover, there exists a positive constant $c_9 = c_9(N,\mu,\Omega)$ such that
\[
(2.17) \quad \|u\|_{L^1(\Omega;\delta^{-\alpha-})} \leq c_9(\|\tau\|_{\mathcal{M}(\Omega;\delta^{\alpha+})} + \|\nu\|_{\mathcal{M}(\partial\Omega)}).
\]
(vi) \( u \) is a solution of \((2.15)\) if and only if \( u \in L^1(\Omega; \delta^{-\alpha-}) \) and

\[
-\int_{\Omega} u L_\mu \zeta \, dx = \int_{\Omega} \zeta \, d\tau - \int_{\Omega} K_\mu[u] L_\mu \zeta \, dx \quad \forall \zeta \in X(\Omega).
\]

For easy reference, we present a potential theoretic result which serves to prove Theorem B.

**Theorem 2.5.** Let \( w_1 \) be a positive \( L_\mu \)-potential and \( w_2 \) be a positive \( L_\mu \)-harmonic function with \( \nu = \text{tr}^*(w_2) \). Assume that \( \frac{w_1}{w_2} \) satisfies the local Harnack inequality. Then for \( \nu \text{-a.e.} \ z \in \partial \Omega, \)

\[
\lim_{x \to z} \frac{w_1(x)}{w_2(x)} = 0 \quad \text{non-tangentially.}
\]

This Proposition Can be obtained by combining the Fatou convergence theorem [1, Theorem 1.8] and the fact that if a function satisfies the Harnack inequality, fine convergence at the boundary (in the sense of [1]) implies non-tangential convergence (for more details, see [3]).

3. **Nonlinear equations with source term**

In this section, we deal with nonlinear equations involving source term

\[
- L_\mu u = g \circ u
\]

in \( \Omega \) where \( 0 < \mu < C_H(\Omega) \) and \( g : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+ \) is continuous.

3.1. **Equivalent formulation.** For \( z \in \partial \Omega \), denote by \( n_z \) the outward unit normal vector to \( \partial \Omega \) at \( z \). We recall below a geometric property of \( C^2 \) domains (see [20]).

**Proposition 3.1.** There exists \( \beta_0 > 0 \) such that for every point \( x \in \overline{\Omega}_{\beta_0} \), there exists a unique point \( \sigma_x \in \partial \Omega \) such that \( x = \sigma_x - \delta(x)n_{\sigma_x} \). The mappings \( x \mapsto \delta(x) \) and \( x \mapsto \sigma_x \) belong to \( C^2(\overline{\Omega}_{\beta_0}) \) and \( C^1(\overline{\Omega}_{\beta_0}) \) respectively. Moreover, \( \lim_{x \to \sigma(x)} \nabla \delta(x) = -n_{\sigma_x} \).

For \( D \Subset \Omega \), let \( G^D_\mu \) and \( K^D_\mu \) be the Green and Poisson kernels of \( -L_\mu \) in \( D \) respectively. Denote by \( G^D_\mu \) and \( K^D_\mu \) the corresponding Green and Poisson operators in \( D \).

We prove below main properties of solutions of \((1.9)\).

**Proof of Theorem A.**

(i) \(\mapsto\) (ii). Assume \( u \) is a positive solution of \((1.9)\). Put \( \tau = g \circ u \) and denote \( \tau_\beta = \tau|_{D_\beta} \) and \( \lambda_\beta = u|_{\Sigma_\beta} \) for \( \beta \in (0, \beta_0) \). Consider the boundary value problem

\[
- L_\mu u = \tau_\beta \quad \text{in} \ D_\beta, \quad v = \lambda_\beta \quad \text{on} \ \Sigma_\beta.
\]

This problem admits a unique solution \( v_\beta \) (the uniqueness is derived from [5, Lemma 2.1] since \( \mu < C_H(\Omega) \)). Therefore \( v_\beta = u|_{D_\beta} \). We have

\[
u|_{D_\beta} = v_\beta = G^D_\mu(\cdot, y)(g \circ u)(y) \, dy = G^D_\mu[\tau_\beta] + K^D_\mu[\lambda_\beta].
\]

It follows that

\[
\int_{D_\beta} G^D_\mu(\cdot, y)(g \circ u)(y) \, dy = G^D_\mu[\tau_\beta] \leq u|_{D_\beta}.
\]

Letting \( \beta \to 0 \), we get

\[
\int_{\Omega} G_\mu(\cdot, y)(g \circ u)(y) \, dy \leq u.
\]
Fix a point \( x_0 \in \Omega \) such that \( u(x_0) < \infty \). Keeping in mind that \( G(x_0, y) > c_{\alpha_0} \delta(y)^{\alpha_+} \) for every \( y \in \Omega \), we deduce from (3.2) that \( g \circ u \in L^1(\Omega; \delta^{\alpha+}) \). Thanks to Proposition 2.4 (v), we obtain (1.13).

(ii) \( \implies \) (i). Assume \( u \) is a function such that \( g \circ u \in L^1(\Omega; \delta^{\alpha+}) \) and (1.13) holds. By Proposition 2.4 (i) \(-L_\mu \mathbb{K}_\mu [\nu] = 0\), which implies that \( u \) is a solution of (3.1). On the other hand, since \( g \circ u \in L^1(\Omega; \delta^{\alpha+}) \), we deduce from Proposition 2.4 (ii) that \( \text{tr}^* (\mathbb{G}_\mu [g \circ u]) = 0 \). Consequently, \( \text{tr}^* (u) = \text{tr}^* (\mathbb{K}_\mu [\nu]) = \nu \).

(i) \( \implies \) (iii). Assume \( u \) is a positive solution of (1.9). From the implication (i) \( \implies \) (ii), we deduce that \( u \in L^1(\Omega; \delta^{-\alpha-}) \) and \( g \circ u \in L^1(\Omega; \delta^{\alpha+}) \). Hence, by Proposition 2.4 (vi), \( u \) satisfies (1.14).

(iii) \( \implies \) (i). This implication follows straightforward from Proposition 2.4 (vi). \( \square \)

3.2. Nondecreasing source. We start with an existence result for (3.1) in presence of sub and super solutions.

**Theorem 3.2.** Let \( g \in C(\Omega \times \mathbb{R}_+) \), \( g(x, r) \) be nondecreasing with respect to \( r \) for any \( x \in \Omega \). Assume that there exist a subsolution \( V_1 \) and a supersolution \( V_2 \) of (3.1) such that \( 0 \leq V_1 \leq V_2 \) in \( \Omega \). Then there exists a solution \( u \) of (3.1) which satisfies \( V_1 \leq u \leq V_2 \) in \( \Omega \).

Moreover, if \( V_1 = \mathbb{K}_\mu [\nu] \) for some \( \nu \in \mathcal{M}^+(\partial \Omega) \) and \( g \circ V_2 \in L^1(\Omega; \delta^{\alpha+}) \) then \( u \) is the minimal solution of (1.9) in the sense that \( u \leq v \) for every solution \( v \) of (1.9).

**Lemma 3.3.** Let \( D \subseteq \Omega \), \( f \in L^1(D) \), \( f \geq 0 \) and \( \eta \in L^1(\partial D) \), \( \eta \geq 0 \). Then there exists a unique solution of

\[
-L_\mu u = f \quad \text{in} \quad D, \quad u = \eta \quad \text{on} \quad \partial D.
\]

**Proof.** We start with the case \( f \in L^\infty(D) \) and \( \eta = 0 \). Let us consider the functional

\[
\mathcal{J}(v) = \frac{1}{2} \int_D \left( |\nabla v|^2 - \frac{\mu}{\delta^2} v^2 \right) dx - \int_D fv dx
\]

over the space \( H^1_0(D) \). It can be verified that \( \mathcal{J} \) is a convex and semicontinuous on \( H^1_0(D) \). Furthermore, since \( \mu < C_H(\Omega) \), it follows that the functional \( \mathcal{J} \) is coercive. Therefore the problem \( \min_{H^1_0(D)} \mathcal{J}(v) \) admits a solution \( v \in H^1_0(D) \). The minimizer \( v \) is the unique weak solution of (3.3).

If \( f \in L^1(D) \) then we can approximate it by an increasing sequence \( \{f_m\} \subseteq L^\infty(D) \). Let \( v_m \) be the solution of (3.3) with \( \eta = 0 \) and \( f \) replaced by \( f_m \). By comparison principle [5, Lemma 2.1], \( \{v_m\} \) increases and therefore \( v := \lim_{m \to \infty} v_m \) is a solution of (3.3) with \( \eta = 0 \).

We next consider the case \( \eta \in L^1(\partial D) \). Let \( v \) be a solution of (3.3) with \( \eta = 0 \) then \( u = v + \mathbb{P}_{\mu}[\eta] \) is a solution of (3.3). The uniqueness follows from the comparison principle. \( \square \)

**Proof of Theorem 3.2.** Put \( u_0 = V_1 \) and \( \eta_\beta = V_1|_{\Sigma_\beta} \) for \( \beta \in (0, \beta_0) \). For \( n \geq 1 \), consider the problem

\[
-L_\mu u = g \circ u_{n-1} \quad \text{in} \quad D_\beta, \quad u = \eta_\beta \quad \text{on} \quad \partial D_\beta.
\]

For each \( n \geq 1 \), by Lemma 3.3 there exists a unique solution \( u_{\beta,n} \) of (3.4). Moreover, since \( g(x, r) \) is nondecreasing with respect to \( r \) for every \( x \in \Omega \), by applying comparison principle, we deduce that \( V_1 \leq u_{\beta,n} \leq u_{\beta,n+1} \leq V_2 \) in \( D_\beta \). Therefore \( u_\beta := \lim_{n \to \infty} u_{\beta,n} \) is a solution of (3.1) in \( D_\beta \) which satisfies \( V_1 \leq u_\beta \leq V_2 \) in \( D_\beta \). Moreover,

\[
u_\beta = \mathbb{G}_{\mu}^{D_\beta}[g \circ u_\beta] + \mathbb{P}_{\mu}^{D_\beta}[\eta_\beta].
\]
For $0 < \beta' < \beta < \beta_0$, by the comparison principle, $u_{\beta,1} \leq u_{\beta',1}$ in $D_\beta$. By the monotonicity assumption on $g$, it follows that $u_{\beta,n} \leq u_{\beta',n}$ in $D_\beta$ for every $n > 1$. Therefore $V_1 \leq u_\beta \leq u_{\beta'} \leq V_2$ in $D_\beta$ and hence $u := \lim_{\beta \downarrow \beta_0} u_\beta$ is a solution of (3.1) in $\Omega$ satisfying $V_1 \leq u \leq V_2$.

In the case $V_1 = \mathbb{K}_\mu[\nu]$, formulation (3.5) becomes

$$u_\beta = \mathbb{G}_\mu^{D_\beta}[g \circ u_\beta] + \mathbb{K}_\mu[\nu].$$

Since $0 \leq g \circ u_\beta \leq g \circ V_2 \in L^1(\Omega; \delta^{\alpha^+})$, it follows that

$$\lim_{\beta \downarrow 0} \mathbb{G}_\mu^{D_\beta}[g \circ u_\beta] = \mathbb{G}_\mu[g \circ u].$$

Letting $\beta \downarrow 0$ in (3.6), we infer that $u$ satisfies (1.13), namely $u$ is a solution of (1.9). Notice that in the above argument, $u$ is independent of $V_2$. Hence, if $v$ is a solution of (1.9) then $v \geq \mathbb{K}_\mu[\nu]$ and $g \circ v \in L^1(\Omega; \delta^{\alpha^+})$; consequently $u \leq v$.

**Proof of Theorem B.** We first notice that since $g \circ (\mathbb{K}_\mu[\nu]) \in L^1_{\text{loc}}(\Omega)$ and $\mathbb{G}_\mu[g \circ (\mathbb{K}_\mu[\nu])] \leq \infty$, it follows that $g \circ (\mathbb{K}_\mu[\nu]) \in L^1(\Omega; \delta^{\alpha^+})$ due to a similar argument as in the proof of Theorem A. It is easy to see that $\mathbb{K}_\mu[\nu]$ is a subsolution of (3.1). For $\rho \in (r_1, r_2)$, we look for a supersolution of $v$ of the form

$$v = \rho \mathbb{K}_\mu[\nu] + c_2 \mathbb{G}_\mu[g \circ (\rho \mathbb{K}_\mu[\nu])],$$

where $c_2$ will be made precise latter on. By (1.15) and (1.17), we obtain

$$v \leq \rho(1 + c_2 c_3 g^{-1}(\rho) \mathbb{K}_\mu[\nu]).$$

The monotonicity property of $g$ implies

$$g \circ v \leq g \circ (\rho(1 + c_2 c_3 g^{-1}(\rho) \mathbb{K}_\mu[\nu])).$$

By (1.15),

$$g \circ v \leq \ell(1 + c_2 c_3 g^{-1}(\rho)) g \circ (\rho \mathbb{K}_\mu[\nu]).$$

In light of (1.16), we deduce

$$g \circ v \leq c_2 g \circ (\rho \mathbb{K}_\mu[\nu]) = -L_\mu v.$$

This means $v$ is a supersolution of (3.1).

We apply Theorem 3.2 to derive that problem (1.18) admits a minimal solution $u_{\rho \nu}$ satisfying

$$\rho \mathbb{K}_\mu[\nu] \leq u_{\rho \nu} \leq g \mathbb{K}_\mu[\nu] + c_2 \mathbb{G}_\mu[g \circ (\mathbb{K}_\mu[\nu])].$$

Estimates (1.19) follows straightforward from (1.17) and (3.10) with $c_4 = 1 + c_2 c_3 g^{-1}(\rho)$.

We next prove (1.20). Due to (1.13), it is sufficient to prove that for $\nu$-a.e. $z \in \partial \Omega$,

$$\lim_{x \rightarrow z} \frac{\mathbb{G}_\mu[g \circ u_{\rho \nu}](x)}{\mathbb{K}_\mu[\nu](x)} = 0 \quad \text{non tangentially}.$$

To obtain (3.11), we shall employ Theorem 2.5. Since $\mathbb{K}_\mu[\nu]$ is a positive $L_\mu$-harmonic function satisfying local Harnack inequality, we only need to show that:

(i) $\mathbb{G}_\mu[g \circ u_{\rho \nu}]$ is a positive $L_\mu$-potential.

(ii) $\mathbb{G}_\mu[g \circ u_{\rho \nu}]$ satisfies Harnack inequality.

Since $g \circ u_{\rho \nu} \in L^1(\Omega; \delta^{\alpha^+})$, $\text{tr}^*(\mathbb{G}_\mu[g \circ u_{\rho \nu}])$ and hence (i) follows from Proposition 2.4 (iv).

By (1.19), we infer that $u_{\rho \nu}$ satisfies the local Harnack inequality. Since $u_{\rho \nu}$ can be written under the form (1.13), it follows that $\mathbb{G}_\mu[g \circ u_{\rho \nu}]$ satisfies this inequality too. Hence (ii) is verified. By invoking Theorem 2.5, we get (3.11). \qed
4. Power source

In this section, we focus on the equation
\begin{equation}
-\mathcal{L}_\mu u = u^q \quad \text{in } \Omega
\end{equation}

4.1. Subcritical case. We start with a lemma the proof of which is an adaptation of an idea in [6].

**Lemma 4.1.** Assume $0 < q < q^*$ and $z \in \partial \Omega$. Then there exists a constant $c_{10} = c_{10}(N, \mu, q, \Omega)$ such that
\begin{equation}
\mathcal{G}_\mu[K_\mu(z, z)^q](x) \leq c_{10}|x-z|^N + (N-1-\alpha-)q K_\mu(x, z) \quad \forall x \in \Omega.
\end{equation}

**Proof.** By (2.7) and (2.13), there exists a constant $c_{11}$ such that for every $x \in \Omega$,
\begin{equation}
\mathcal{G}_\mu[K_\mu(z, z)^q](x) \leq c_{11}\delta(x)^{\alpha_+} \int_{\Omega} |x-y|^{1+\alpha_-N}|y-z|^q(1+\alpha_-N) dy.
\end{equation}

Put
\[ D_1 = \Omega \cap B(x, |x-z|/2), \quad D_2 = \Omega \cap B(z, |x-z|/2), \quad D_3 = \Omega \setminus (D_1 \cup D_2), \]
and
\[ I_i := \int_{D_i} |x-y|^{1+\alpha_-N}|y-z|^q(1+\alpha_-N) dy, \quad i = 1, 2, 3. \]

For every $x \in D_1$, $|x-z| \leq 2|y-z|$, therefore
\begin{equation}
I_1 \leq c_{12}|x-z|^q(1+\alpha_-N) \int_{D_1} |x-y|^{1+\alpha_-N} dy \leq c'_{12}|x-y|^{1+\alpha_-N} dy.
\end{equation}

For every $x \in D_2$, $|x-z| \leq 2|x-y|$, hence
\begin{equation}
I_2 \leq c_{13}|x-z|^{1+\alpha_-N} \int_{D_2} |y-z|^{1+\alpha_-N} dy \leq c'_{13}|x-y|^{1+\alpha_-N} dy.
\end{equation}

For every $x \in D_2$, $|x-y| \geq 3|y-z|$, therefore
\begin{equation}
I_3 \leq c_{14} \int_{D_3} |y-z|^{1+\alpha_-N-(N-1-\alpha_-)} dy \leq c'_{14}|x-y|^{1+\alpha_-N-(N-1-\alpha_-)} dy.
\end{equation}

Combining (4.3)-(4.6), we obtain
\begin{equation}
\mathcal{G}_\mu[K_\mu(z, z)^q](x) \leq c_{11}(c'_{12} + c'_{13} + c'_{14}) \delta(x)^{\alpha_+} |x-y|^{1+\alpha_-N-(N-1-\alpha_-)} q.
\end{equation}

Estimate (4.2) follows straightforward from (2.7) and (4.7).

**Proposition 4.2.** Assume $0 < q < q^*$ and $\nu$ is a positive finite measure on $\partial \Omega$. Then $\mathbb{K}_\mu[\nu] \in L^q(\Omega; \delta^{\alpha_+})$ and there exists a constant $c_{15}$ depending on $N, \mu, q, \Omega$ such that
\begin{equation}
\mathcal{G}_\mu[\mathbb{K}_\mu[\nu]] \leq c_{15}||\nu||_{L^q(\partial \Omega)}^{q-1} \mathbb{K}_\mu[\nu].
\end{equation}

**Remark.** We notice that (4.8) was proved in [7] for the case $q > 1$.

**Proof.** We may assume that $||\nu||_{L^\infty(\partial \Omega)} = 1$ (if it is not the case, one can replace $\nu$ by $\nu/||\nu||_{L^\infty(\partial \Omega)}$). We first consider the case $q \geq 1$. From (2.10) and the fact that $L^q_\mu(\Omega, \delta^{\alpha_+}) \subset L^q(\Omega, \delta^{\alpha_+})$, we deduce that $\mathbb{K}_\mu[\nu] \in L^q(\Omega; \delta^{\alpha_+})$. It follows from (1.6) and Jensen’s inequality that
\[ \mathbb{K}_\mu[\nu](x)^q \leq \int_{\partial \Omega} K_\mu(x, z)^q d\nu(z). \]
Consequently
\[ G_\mu[K_\mu[\nu]](x) \leq \int_{\partial \Omega} G_\mu[K_\mu(\cdot, z)] d\nu(z). \]

By Lemma 4.1, since \( N + \alpha_+ - (N - 1 - \alpha_-)q > 0 \),
\[ G_\mu[K_\mu[\nu]](x) \leq c_{10} \int_{\partial \Omega} |x - z|^{N + \alpha_+ - (N - 1 - \alpha_-)q} K_\mu(x, z) d\nu(z) \]
\[ \leq c_{10}(\text{diam}(\Omega))^{N + \alpha_+ - (N - 1 - \alpha_-)q} \nu_\mu[\nu](x). \]

Thus we obtain (4.8).

If \( 0 < q < 1 \) then
\[ G_\mu[K_\mu[\nu]] \leq G_\mu[1 + K_\mu[\nu]] = G_\mu[1] + G_\mu[K_\mu[\nu]]. \]

From the case \( q = 1 \), we deduce that
\[ G_\mu[K_\mu[\nu]] \leq G_\mu[1] + c_{15} K_\mu[\nu] \]

By the estimates \( G_\mu[1] \leq c_{16} K_\mu[\nu] \), where \( c_{16} = c_{16}(N, \mu, \Omega) \), we conclude (4.8). \[ \square \]

**Lemma 4.3.** Let \( f \in L^1(\Omega; \delta^{\alpha_+}) \), \( f \geq 0 \), \( \nu \in \mathfrak{M}^+(\partial \Omega) \), \( \nu \neq 0 \) and \( \phi \in C^1([0, \infty)) \) be a concave, nondecreasing function such that \( \phi(1) \geq 0 \) and \( \phi' \) is bounded. Let \( \varphi \) be a positive function in \( L^1_{\text{loc}}(\Omega) \) such that \( -L_\mu \varphi \geq f \). Then
\[ \phi' \left( \frac{\varphi}{K_\mu[\nu]} \right) f \in L^1(\Omega; \delta^{\alpha_+}). \]

**Proof.** Since \( -L_\mu \varphi \geq f \geq 0 \), by Proposition 2.4, there exist \( \tau \in \mathfrak{M}^+(\Omega; \delta^{\alpha_+}) \) and \( \lambda \in \mathfrak{M}^+(\partial \Omega) \) such that
\[ \varphi = G_\mu[f] + G_\mu[\tau] + K_\mu[\lambda]. \]

Put \( \psi = K_\mu[\nu] \). Let \( \{f_n\} \) and \( \{\tau_n\} \) be two sequences in \( C^\infty_c(\Omega) \) such that \( \{f_n\} \) converges to \( f \) in \( L^1(\Omega; \delta^{\alpha_+}) \) and \( \{\tau_n\} \) converges to \( \tau \) in the weak sense of \( \mathfrak{M}^+(\Omega; \delta^{\alpha_+}) \). Let \( \{\nu_n\} \) and \( \{\lambda_n\} \) be two sequences in \( C^1(\partial \Omega) \) converging to \( \nu \) and \( \lambda \) respectively in the weak sense of \( \mathfrak{M}^+(\partial \Omega) \).

Put \( \varphi_n = G_\mu[f_n] + G_\mu[\tau_n] + K_\mu[\lambda_n] \) and \( \psi_n = K_\mu[\nu_n] \). By the bootstrap argument, one can prove that \( \varphi_n, \psi_n \in C^3(\Omega) \) for every \( n \in \mathbb{N} \). By [16] \( \{G_\mu[f_n]\}, \{G_\mu[\tau_n]\}, \{K_\mu[\lambda_n]\} \) and \( \{K_\mu[\nu_n]\} \) converge to \( G_\mu[f], G_\mu[\tau], K_\mu[\lambda] \) and \( K_\mu[\nu] \) respectively in \( L^1(\Omega) \). As a consequence, up to a subsequence, \( \{\varphi_n\} \) and \( \{\psi_n\} \) converge to \( \varphi \) and \( \psi \) respectively a.e. in \( \Omega \). Therefore, for \( n \) large enough, \( \psi_n > 0 \).

Due to [9, Lemma 5.3],
\[ -\Delta \left[ \psi_n \phi \left( \frac{\varphi_n}{\psi_n} \right) \right] \geq \phi' \left( \frac{\varphi_n}{\psi_n} \right) \left( -\Delta \varphi_n \right) + \left[ \phi \left( \frac{\varphi_n}{\psi_n} \right) - \frac{\varphi_n}{\psi_n} \phi' \left( \frac{\varphi_n}{\psi_n} \right) \right] \left( -\Delta \psi_n \right). \]

It follows that
\[ -L_\mu \left[ \psi_n \phi \left( \frac{\varphi_n}{\psi_n} \right) \right] \geq \phi' \left( \frac{\varphi_n}{\psi_n} \right) \left( -L_\mu \varphi_n \right) + \left[ \phi \left( \frac{\varphi_n}{\psi_n} \right) - \frac{\varphi_n}{\psi_n} \phi' \left( \frac{\varphi_n}{\psi_n} \right) \right] \left( -L_\mu \psi_n \right). \]

Consequently,
\[ -L_\mu \left[ \psi_n \phi \left( \frac{\varphi_n}{\psi_n} \right) \right] \geq \phi' \left( \frac{\varphi_n}{\psi_n} \right) f_n. \]
Then for every nonnegative function $\zeta \in X(\Omega)$, there holds

\begin{equation}
-\int_{\Omega} \psi_n \phi \left( \frac{\varphi_n}{\psi_n} \right) L_\mu \zeta dx \geq \int_{\Omega} \phi' \left( \frac{\varphi_n}{\psi_n} \right) f_n \zeta dx.
\end{equation}

We see that

\begin{equation}
0 \leq \psi_n \phi \left( \frac{\varphi_n}{\psi_n} \right) \leq \psi_n \left( \phi(0) + \phi'(0) \frac{\varphi_n}{\psi_n} \right) = c_{17} (\psi_n + \varphi_n).
\end{equation}

By (2.9) and (2.10), $\{\varphi_n\}$ and $\{\psi_n\}$ are uniformly bounded in $L^p(\Omega; \delta^{-\alpha})$ for $p \in (1, \frac{N-a}{N-1-\alpha})$. Due to Holder inequality, $\{\varphi_n\}$ and $\{\psi_n\}$ are uniformly integrable with respect to $\delta^{\alpha} dx$. In view of Vitali theorem $\{\varphi_n\}$ and $\{\psi_n\}$ converge to $\varphi$ and $\psi$ in $L^1(\Omega; \delta^{-\alpha})$ respectively. By (4.13) and dominated convergence theorem we deduce that

$$
\psi_n \phi \left( \frac{\varphi_n}{\psi_n} \right) \to \psi \phi \left( \frac{\varphi}{\psi} \right) \quad \text{in} \quad L^1(\Omega; \delta^{-\alpha}).
$$

Due to Fatou lemma, by sending $n \to \infty$ in (4.12), we obtain (4.9) and (4.10).

\section*{Theorem 4.4.}

Let $q > 1$ and $\nu \in \mathfrak{M}^+(\partial \Omega)$, $\nu \neq 0$. If problem $(D_\nu)$ admits a solution then (1.22) holds.

\textbf{Proof.} Since (1.9) admits a solution $u$ then by Theorem A, $u^q \in L^1(\Omega; \delta^{\alpha})$ and (1.13) holds. Consequently $\mathbb{K}_\mu[\nu]^q \in L^1(\Omega; \delta^{\alpha})$. Now applying Lemma 4.3 with $f = u^q$, $\varphi = u$ and

$$
\phi(s) = \begin{cases} 
1 - s^{1-q}, & \text{if } s \geq 1, \\
\frac{q-1}{s-1} & \text{if } 0 \leq s < 1
\end{cases}
$$

we obtain the following estimate in the weak sense

\begin{equation}
-L_\mu \left[ \mathbb{K}_\mu[\nu] \phi \left( \frac{u}{\mathbb{K}_\mu[\nu]} \right) \right] \geq \left( \frac{u}{\mathbb{K}_\mu[\nu]} \right)^{-q} u^q \geq \mathbb{K}_\mu[\nu]^q.
\end{equation}

Put

$$
\Psi := \mathbb{K}_\mu[\nu] \phi \left( \frac{u}{\mathbb{K}_\mu[\nu]} \right) \quad \text{and} \quad \tilde{\Psi} := \mathbb{G}_\mu[\mathbb{K}_\mu[\nu]^q].
$$

Then $\Psi$ is an $L_\mu$-superharmonic function and by Proposition 2.4, $\Psi$ admits a nonnegative normalized boundary trace. By Kato lemma (see [20]), $(\tilde{\Psi} - \Psi)_+$ is an $L_\mu$-subharmonic function and $\text{tr}^* ((\tilde{\Psi} - \Psi)_+) = 0$. It follows that $(\tilde{\Psi} - \Psi)_+ = 0$ and hence $\tilde{\Psi} \leq \Psi$ in $\Omega$. This means

$$
\mathbb{G}_\mu[\mathbb{K}_\mu[\nu]^q] \leq \mathbb{K}_\mu[\nu] \phi \left( \frac{u}{\mathbb{K}_\mu[\nu]} \right) \leq \frac{1}{q-1} \mathbb{K}_\mu[\nu].
$$

\section*{Proof of Proposition C.}

This theorem follows straightforward from Lemma 4.1 and Theorem 4.4.

\section*{Proposition 4.5.}

Assume $0 < q < q^*$, $q \neq 1$ and $\nu \in \mathfrak{M}^+(\partial \Omega)$ such that $\|\nu\|_{\mathfrak{M}(\partial \Omega)} = 1$.

(i) If $q > 1$ then there exists a positive number $\varrho_0 > 0$ depending on $N, \mu, q, \Omega$ such that for every $\varrho \in (0, \varrho_0)$ problem $(D_{\varrho \nu})$ admits a minimal solution $u_{\varrho \nu}$.

(ii) If $q \in (0, 1)$ then for every $\varrho > 0$ problem $(D_{\varrho \nu})$ admits a minimal solution $u_{\varrho \nu}$.

In any case $u_{\varrho \nu}$ satisfies (1.19) and (1.20).
Proof. We shall apply Theorem B to deduce the existence of a solution of \((D_{\varrho \nu})\). One can verify that the functions \(g(x, s) = s^q\) and \(\ell(s) = s^q\) with \(q > 0\) satisfy (1.15). From Proposition 4.2 we deduce that condition (1.17) is fulfilled with the constant \(c_{18}\). For such \(g\) and \(\ell\), condition (1.16) is valid if one can find a positive constant \(c_{19}\) such that

\[
1 + c_{18}c_{19}q^{q-1} \leq c_{18}^q.
\]

If \(q > 1\) then there exists \(\varrho_0 = \varrho_0(q, c_{15})\) and \(c_{18} = c_{18}(q)\) such that (4.15) holds true. If \(q < 1\) then for every \(q \in [1, \infty)\) one can choose \(c_{18} = c_{18}(q, c_{15})\) such that (4.15) holds. Hence, by Theorem B, there exists a minimal solution \(u_{\varrho \nu}\) of problem \((D_{\varrho \nu})\) if \(q > 1\) and \(q \in (0, \varrho_0)\) or if \(q < 1\) and \(q \in [1, \infty)\).

We next consider the case \(q < 1\) and \(0 < q < 1\). Let \(u\) be a solution of the problem \((D_{\nu})\) then \(v \geq \varrho\mathcal{K}_{\mu}[\nu]\). Due to Theorem 3.2, we deduce the existence of a minimal solution \(u_{\varrho \nu}\) of problem \((D_{\varrho \nu})\) satisfying \(\varrho\mathcal{K}_{\mu}[\nu] \leq u_{\varrho \nu} \leq v\).

Thus, if \(q < 1\), for any \(\varrho > 0\), \(u_{\varrho \nu}\) satisfies (1.19) and (1.20).

Lemma 4.6. Let \(0 < q \neq 1\) and \(\nu \in \mathfrak{M}^+(\partial \Omega)\). If \(u\) is a solution of \((D_{\nu})\) then there is a constant \(c_{19} = c_{19}(N, \mu, q, \Omega)\) such that

\[
\|u\|_{L^1(\Omega, \delta^{\alpha})} + \|u\|_{L^1(\Omega, \delta^{\alpha-})} \leq c_{19}(1 + \|\nu\|_{\mathfrak{M}(\partial \Omega)}).
\]

Proof. Indeed, by taking \(\zeta = \varphi_{\mu, 1}\) in the formulation satisfied by \(u\), we obtain

\[
\lambda_{\mu, 1} \int_{\Omega} u\varphi_{\mu, 1} dx = \int_{\Omega} \mu \varphi_{\mu, 1} dx + \lambda_{\mu, 1} \int_{\Omega} \mathcal{K}_{\mu}[\nu]\varphi_{\mu, 1} dx \quad \forall \zeta \in X(\Omega).
\]

Case 1: \(q > 1\). By Young inequality, we get

\[
\int_{\Omega} u\varphi_{\mu, 1} dx \leq (2\lambda_{\mu, 1})^{-1} \int_{\Omega} u\varphi_{\mu, 1} dx + (2\lambda_{\mu, 1})^{\frac{1}{q}} \int_{\Omega} \varphi_{\mu, 1} dx.
\]

By (4.17) and (4.18), we obtain

\[
\int_{\Omega} u\varphi_{\mu, 1} dx \leq 2\lambda_{\mu, 1} \int_{\Omega} \mathcal{K}_{\mu}[\nu]\varphi_{\mu, 1} dx \leq (2\lambda_{\mu, 1})^{\frac{1}{q}} \int_{\Omega} \varphi_{\mu, 1} dx.
\]

Since the second term on the left hand-side of (4.19) is nonnegative and by (1.8), we get

\[
\|u\|_{L^1(\Omega, \delta^{\alpha})} \leq c_{19}(2\lambda_{\mu, 1})^{\frac{1}{q}} \int_{\Omega} \delta^{\alpha} dx \leq c_{20}.
\]

On the other hand, we derive from (1.13), (2.9) and (2.10) that

\[
\|u\|_{L^1(\Omega, \delta^{\alpha-})} \leq c_{21}(\|u\|_{L^1(\Omega, \delta^{\alpha})} + \|\nu\|_{\mathfrak{M}(\partial \Omega)}).
\]

Combining (4.20) and (4.21), we obtain (4.16).

Case 2: \(q \in (0, 1)\). By Young inequality, we have

\[
\int_{\Omega} u^q \varphi_{\mu, 1} dx \leq \frac{\lambda}{2} \int_{\Omega} u\varphi_{\mu, 1} dx + (2\lambda_{\mu, 1})^{\frac{1}{q} - \frac{1}{q}} \int_{\Omega} \varphi_{\mu, 1} dx.
\]

Consequently,

\[
\int_{\Omega} u\varphi_{\mu, 1} dx \leq (2\lambda_{\mu, 1})^{-1} \int_{\Omega} \varphi_{\mu, 1} dx + 2 \int_{\Omega} \mathcal{K}_{\mu}[\nu]\varphi_{\mu, 1} dx.
\]

Therefore

\[
\|u\|_{L^1(\Omega, \delta^{\alpha-})} \leq c_{22}(1 + \|\nu\|_{\mathfrak{M}(\partial \Omega)}).
\]

Combining (4.25) and (4.23) leads to (4.16). \(\square\)
Theorem 4.7. Assume \( q \in (1, q^*) \). Let \( \nu \in M^+(\partial \Omega) \) with \( \|\nu\|_{M(\partial \Omega)} = 1 \). Then there exists a threshold value \( \varrho^* \in \mathbb{R}_+ \) for \((D_{\varrho \nu})\) such that

(i) If \( \varrho \in (0, \varrho^*) \) then problem \((D_{\varrho \nu})\) admits a minimal solution \( u_{\varrho \nu} \). If \( \varrho \in (0, \varrho^*) \) then \( u_{\varrho \nu} \) satisfies (1.19) and (1.20).

Moreover \( \{u_{\varrho \nu}\} \) is an increasing sequence which converges, as \( \varrho \to \varrho^* \), to the minimal solution \( u_{\varrho^* \nu} \) of \((D_{\varrho^* \nu})\) in \( L^1(\Omega; \delta^{-\alpha -}) \) and in \( L^1(\Omega; \delta^\alpha+) \).

(ii) If \( \varrho > \varrho^* \) then there exists no solution of \((D_{\varrho \nu})\).

Proof. Put

\[ A := \{ \varrho > 0 : (D_{\varrho \nu}) \text{ admits a solution} \} \quad \text{and} \quad \varrho^* := \sup A. \]

By Proposition 4.5, \((D_{\varrho \nu})\) admits a solution for \( \varrho > 0 \) small, therefore \( A \neq \emptyset \). Moreover, from Theorem 4.4, we deduce that \( \varrho^* \) is finite.

We shall show that \((0, \varrho^*) \subset A \). For this purpose, we have to show that if \( 0 < \varrho < \varrho^* \) and \( A \ni \varrho' < \varrho^* \) then \( \varrho \in A \). Since \( \varrho' \in A \), due to Theorem 4.4, there exists a minimal solution \( u_{\varrho' \nu} \) of \((D_{\varrho' \nu})\) which is greater than \( \varrho \mathbb{K}_\mu[n] \). By Theorem 3.2, problem \((D_{\varrho \nu})\) admits a minimal solution \( u_{\varrho \nu} \), i.e. \( \varrho \in A \).

Next we prove that \( \varrho^* \in A \), namely problem \((D_{\varrho^* \nu})\) admits a solution. Let \( \{\varrho_n\} \) be an increasing sequence converging to \( \varrho^* \). For each \( n \), let \( u_{\varrho_n \nu} \) be a solution of \((D_{\varrho_n \nu})\). Then \( u_{\varrho_n \nu} \in L^1(\Omega; \delta^{-\alpha -}) \cap L^q(\Omega; \delta^\alpha+) \) and it satisfies the formula

\[ (4.24) \quad -\int_\Omega u_{\varrho_n \nu} L \mu \zeta dx = \int_\Omega u_{\varrho_n \nu}^q \zeta dx - q_n \int_\Omega \mathbb{K}_\mu[n] L \mu \zeta dx \quad \forall \zeta \in X(\Omega). \]

It follows from Lemma 4.6 that the sequence \( \{u_{\varrho_n \nu}\} \) is uniformly bounded in \( L^1(\Omega; \delta^\alpha+) \) and hence by local regularity for elliptic equations [18] there exists a subsequence, still denoted by the same notation, such that \( \{u_{\varrho_n \nu}\} \) converges a.e. to a function \( u_{\varrho^* \nu} \). From Theorem A, there holds

\[ (4.25) \quad u_{\varrho_n \nu} = G_\mu[u_{\varrho_n \nu}^q] + \varrho_n \mathbb{K}_\mu[n]. \]

Thanks to Proposition 2.1, \( \{u_{\varrho_n \nu}\} \) is uniformly bounded in \( L^{q_1}(\Omega; \delta^{-\alpha -}) \) and in \( L^{q_2}(\Omega; \delta^\alpha+) \) where \( 1 < q_1 < \frac{N-\alpha -}{N-1-\alpha} \) and \( q < q_2 < q^* \). We invoke Holder inequality to infer that \( \{u_{\varrho_n \nu}\} \) and \( \{u_{\varrho_n \nu}^q\} \) are uniformly integrable with respect to \( \delta^{-\alpha -} dx \) and \( \delta^\alpha+ dx \) respectively. As a consequence, \( \{u_{\varrho_n \nu}\} \) converges to \( u_{\varrho^* \nu} \) in \( L^1(\Omega; \delta^{-\alpha -}) \) and \( \{u_{\varrho_n \nu}^q\} \) converges to \( u_{\varrho^* \nu}^q \) in \( L^1(\Omega; \delta^\alpha+) \). Letting \( n \to \infty \) in (4.24) implies

\[ (4.26) \quad -\int_\Omega u_{\varrho^* \nu} L \mu \zeta dx = \int_\Omega u_{\varrho^* \nu}^q \zeta dx - \varrho^* \int_\Omega \mathbb{K}_\mu[n] L \mu \zeta dx \quad \forall \zeta \in X(\Omega). \]

We infer from Theorem A that \( u_{\varrho^* \nu} \) is a solution of \((D_{\varrho^* \nu})\).

Notice that, in light of Theorem 3.2 and the above argument, one can prove that \( \{u_{\varrho_n \nu}\} \) is an increasing sequence converging to the minimal solution \( u_{\varrho^* \nu} \) of \((D_{\varrho^* \nu})\) in \( L^1(\Omega; \delta^{-\alpha -}) \) and in \( L^1(\Omega; \delta^\alpha+) \).

We next show that for each \( \varrho \in (0, \varrho^*) \), there exists a minimal solution \( u_{\varrho \nu} \) of \((D_{\varrho \nu})\) which satisfies (1.19). Take \( \varrho' = \frac{\varrho + \varrho^*}{2} \) and let \( u_{\varrho' \nu} \) be a solution of \((D_{\varrho' \nu})\). We next apply (4.10) with \( \nu \) replaced by \( \varrho' \nu, \varphi = u_{\varrho' \nu}, f = u_{\varrho' \nu}^q \) and

\[ \phi(s) = \begin{cases} s(1 + \varepsilon s^{q-1})^{\frac{1}{q-1}}, & \text{if } s \geq 1, \\ \left(\frac{s}{\varrho'}\right)^q + \left(\frac{\varrho}{\varrho'}\right)^q - \left(\frac{s}{\varrho'}\right)^q & \text{if } 0 \leq s \leq 1 \end{cases} \]
We get
\[-L_\mu \left( K_\mu [g'_\nu] \phi \left( \frac{u_{g'_\nu}}{K_\mu [g'_\nu]} \right) \right) \geq g' \left( \frac{u_{g'_\nu}}{K_\mu [g'_\nu]} \right) u_{g'_\nu} = \left( K_\mu [g'_\nu] \phi \left( \frac{u_{g'_\nu}}{K_\mu [g'_\nu]} \right) \right)^q .\]

Therefore
\[\Psi = K_\mu [g'_\nu] \phi \left( \frac{u_{g'_\nu}}{K_\mu [g'_\nu]} \right)\]
is a supersolution of (4.1). Moreover \(\Psi \geq \varrho K_\mu [\nu]\). By Theorem 3.2 there exists a minimal solution \(u_{g'_\nu}\) of \((D_{g'_\nu})\) such that
\[\varrho K_\mu [\nu] \leq u_{g'_\nu} \leq \Psi .\]
This implies
\[\varrho K_\mu [\nu] \leq u_{g'_\nu} \leq \varepsilon^{-\frac{1}{q-1}} g' K_\mu [\nu] .\]

Therefore we get (1.19) with \(c_4 = \varrho^{-1} g' \varepsilon^{-\frac{1}{q-1}}\). Finally, (1.20) can be obtained by a similar argument as in the proof of Theorem B.

**Proof of Theorem D.** Part (1) follows from Theorem 4.7. Part (2) follows from Proposition 4.5 (ii).

If \(\nu = \varrho \delta_2\), by (1.13) and (1.19), we obtain

\[\rho \leq \frac{g_{\rho \delta_2}(x)}{K_\mu (x, z)} \leq \rho + \frac{G_\mu [K_\mu (\cdot, z)](x)}{K_\mu (x, z)} .\]

Since \(q < q^*\), it follows from Lemma 4.1 that
\[\lim_{x \to z} \frac{G_\mu [K_\mu (\cdot, z)](x)}{K_\mu (x, z)} = 0 .\]

Thus, by (4.27), we conclude (1.23).

**Proof of Proposition E.** If \(q > 1\), assumption (1.24) guarantees the existence of a solution \(u_{\nu_n}\) of \((D_{\nu_n})\). Moreover, since \(\{\nu_n\}\) converges weakly to \(\nu\), it follows that \(\|\nu\|_{\mathcal{H}(\partial \Omega)} \leq q^*\). Due to Lemma 4.6, the sequence \(\{u_{\nu_n}\}\) is uniformly bounded in \(L^q(\Omega; \delta^{a+})\). Employing a similar argument as in the proof of Theorem 4.7, we obtain the convergence in \(L^1(\Omega; \delta^{-a-})\) and in \(L^q(\Omega; \delta^{a+})\).

If \(q \in (0, 1)\), due to Lemma 4.6, we obtain the convergence in \(L^1(\Omega; \delta^{-a-})\).

We next consider the case \(q = 1\).

**Lemma 4.8.** Let \(\kappa > 0\) and \(u\) be a positive solution of
\[-L_\mu u = \kappa u \quad \text{in } \Omega .\]
Then \(u\) satisfies the Harnack inequality; i.e., for every \(a \in (0, 1)\), there exists a constant \(c_{24} = c_{24}(N, \mu, q, \Omega)\) such that for every \(x \in \Omega\),
\[\sup_{B(x, a\delta(x))} u \leq c_{24} \inf_{B(x, a\delta(x))} u .\]

**Proof.** Equation (4.28) can be written as follows
\[-\Delta u = \left( \frac{\mu}{\delta^2} + \kappa \right) u \quad \text{in } \Omega .\]

Take arbitrarily \(a \in (0, 1)\) and \(x_0 \in \Omega\). Put \(d = \frac{a+1}{2} \delta(x_0)\) and \(M := \max_{B(x_0, d)} u\). Put \(y_0 = d^{-1} x_0\), \(\Omega_d = d^{-1} \Omega\), \(\delta_d(y) = \text{dist}(y, \partial \Omega_d)\) with \(y \in \Omega_d\).
We define
\[ v_d(y) := M^{-1}u(dy), \quad \forall y \in \Omega_d. \]
Clearly, \( \max_{B(y_0;1)} v_d = 1 \) and due to (4.30) we deduce that \( v_d \) is a solution of
\[(4.31)\]
\[-\Delta v_d = Vv_d \quad \text{in} \ \Omega_d.\]
where
\[ V(y) := \frac{\mu}{\delta_d(y)^2} + d^2 \kappa. \]

One can find a positive constant \( c_{25} \) such that \( V(y) \leq c_{25}\delta_d(y)^{-2} \) for every \( y \in \Omega_d \). Notice that \( B(y_0;1) \subset \Omega_d \) and for every \( y \in B(y_0;1) \), there holds
\[ \delta_d(y) \geq \frac{1-a}{1+a}. \]

Hence \( 0 \leq V \leq c_{26} \) in \( B(y_0;1) \) where \( c_{26} = c_{26}(a,\mu) \). By applying Harnack inequality, we deduce that there is a constant \( c_{27} = c_{27}(a,\mu,N,\Omega) \) such that
\[ \sup_{B(y_0;\frac{2a}{a+1})} v_d \leq c_{27} \inf_{B(y_0;\frac{2a}{a+1})} v_d. \]
Thus we obtain (4.29).

**Proof of Theorem F.** Put
\[ \kappa_0 = \min\{1, \|G_\mu[1]\|_{L^\infty(\Omega)}^{-1}\}. \]

**Claim 1.** For any \( \kappa \in (0,\kappa_0) \) there exists a minimal solution \( u_{\kappa,\nu} \) of problem \((E_\nu^\kappa)\).

Fix \( q \in (1,q^*) \) such that \( \kappa < (q \|G_\mu[1]\|_{L^\infty(\Omega)})^{-1} \) and let \( q^* \) be the threshold value for \((D_{\varphi_\nu})\).

Put \( \varrho = \|\nu\|_{W^{1,1}(\Omega)} > 0 \) and \( \tilde{\nu} = \nu/\varrho \).

We first assume that \( \varrho \in (0,\varrho^*) \) and let \( \tilde{u}_\nu \) be the minimal solution of the problem \((D_\nu)\).
By Young inequality, we get
\[ u_\nu^q + 1 \geq u_\nu + \frac{1}{q} \geq \kappa(u_\nu + G_\mu[1]). \]

It follows that
\[ -L_\mu(u_\nu + G_\mu[1]) \geq \kappa(u_\nu + G_\mu[1]). \]

Therefore \( u_\nu + G_\mu[1] \) is a super solution of the equation
\[(4.32)\]
\[-L_\mu u = \kappa u. \]
Clearly \( K_\mu[\nu] \) is a subsolution of (4.32). By Theorem 3.2 there is a minimal solution \( \underline{u}_{\kappa,\nu} \) of \((E_\nu^\kappa)\) which satisfies \( K_\mu[\nu] \leq \underline{u}_{\kappa,\nu} \leq u_\nu + G_\mu[1] \). Since \( G_\mu[1] \leq c_{16}K_\mu[\nu] \), we infer that \( \underline{u}_{\kappa,\nu} \) satisfies (1.19) and (1.20).

If \( \varrho \geq \varrho^* \) then there exists \( m > 0 \) such that \( \varrho/m \in (0,\varrho^*) \). Let \( \underline{u}_{\kappa,m} \) be a solution of \((E_\mu^\kappa)\).

Put \( \underline{u}_{\kappa,\nu} = \frac{m\underline{u}_{\kappa,m}}{2} \) then by the linearity, we deduce that \( \underline{u}_{\kappa,\nu} \) is the minimal solution of \((E_\nu^\kappa)\) with satisfies (1.19) and (1.20).

**Claim 2.** There exists a number \( \kappa^* \in (0,\lambda_{\mu,1}) \) such that
(i) If \( \kappa \in (0,\kappa^*) \) then \((E_\nu^\kappa)\) admits a solution.
(ii) If \( \kappa > \kappa^* \) then \((E_\nu^\kappa)\) admits no solution.

Put \( B := \{ \kappa > 0 : (E_\nu^\kappa) \text{ admits a solution} \} \) and denote \( \kappa^* := \sup B \). We shall show that \( (0,\kappa^*) \subset B \). Take \( \kappa' \in B \) and let \( u_{\kappa',\nu} \) be the minimal solution of \((E_\nu^{\kappa'})\). For any \( \kappa \in (0,\kappa') \), \( u_{\kappa,\nu} \) and \( K_\mu[\nu] \) are respectively super and subsolution of \((E_\nu^{\kappa})\) such that \( K_\mu[\nu] \leq u_{\kappa,\nu} \). Then
by Theorem 3.2 there exists a minimal solution $u_{\kappa, \nu}$ of $(E_{\kappa, \nu}^\alpha)$ satisfying $K_{\mu, \nu} \leq u_{\kappa, \nu} \leq u_{\kappa', \nu}$ in $\Omega$. Hence $\kappa \in B$.

By Lemma 4.8, $G_{\mu}[u_{\kappa, \nu}]$ satisfies local Harnack inequality. Hence, we deduce from Theorem 2.5 that, for $\nu$-a.e. $z \in \partial \omega$, there holds
\[
\lim_{x \to z} \frac{G_{\mu}[u_{\kappa, \nu}](x)}{K_{\mu}[\nu](x)} = 0.
\]
Consequently, (1.20) remains valid with $u_{\nu}$ replaced by $u_{\kappa, \nu}$.

Now let $\nu \in \mathcal{M}^+(\partial \Omega)$, $\kappa \in B$ and denote by $u_{\kappa, \nu}$ a solution of $(E_{\kappa}^\alpha)$. Then by Theorem A,
\[
- \int_\Omega u_{\kappa, \nu} L_{\mu} \zeta dx = \kappa \int_\Omega u_{\kappa, \nu} \zeta dx - \int_\Omega K_{\mu}[\nu] L_{\mu} \zeta dx \quad \forall \zeta \in X(\Omega).
\]
Taking $\zeta = \varphi_{\mu, 1}$, we obtain
\[
(4.33) \quad \lambda_{\mu, 1} \int_\Omega u_{\kappa, \nu} \varphi_{\mu, 1} dx = \kappa \int_\Omega u_{\kappa, \nu} \varphi_{\mu, 1} dx + \lambda_{\mu, 1} \int_\Omega K_{\mu}[\nu] \varphi_{\mu, 1} dx,
\]
which implies that $\kappa < \lambda_{\mu, 1}$. Consequently, $\kappa^* \leq \lambda_{\mu, 1}$.

We show that $\lambda_{\mu, 1} \notin B$ by contradiction. Indeed, suppose that there exists $\nu \in \mathcal{M}^+(\partial \Omega)$ such that the problem
\[
(4.34) \quad -L_{\mu} u = \lambda_{\mu, 1} u \quad \text{in } \Omega, \quad \text{tr}^*(u) = \nu
\]
admits a solution $\hat{u}$. Take $\varphi_{\mu, 1}$ as a test function in the weak formulation satisfied by $\hat{u}$, we deduce $\nu \equiv 0$, which is a contradiction.

Now let $\kappa \in (0, \kappa^*)$ and assume $\{\nu_n\}$ is a sequence of measures in $\mathcal{M}^+(\partial \Omega)$ which converges weakly to $\nu \in \mathcal{M}^+(\partial \Omega)$. Let $u_{\kappa, \nu_n}$ be a solution of $(E_{\kappa}^\alpha)$. By (4.33), we deduce
\[
\|u_{\kappa, \nu_n}\|_{L^1(\Omega, \delta^{\alpha-})} \leq c_{28}(\lambda_{\mu, 1} - \kappa)^{-1} \|\nu_n\|_{\mathcal{M}(\partial \Omega)} \leq c_{29}(\lambda_{\mu, 1} - \kappa)^{-1} \|\nu\|_{\mathcal{M}(\partial \Omega)}.
\]
By a similar argument as in the proof of Theorem 4.7, we deduce that, up to a subsequence, $\{u_{\kappa, \nu_n}\}$ converges to a solution $u_{\kappa, \nu}$ of $(E_{\kappa}^\alpha)$ in $L^1(\Omega, \delta^{-\alpha-})$. □

**Remark.** (i) If $\kappa > 0$ small then $u_{\kappa, \nu}$ satisfies (1.19). Moreover, if $\nu = \rho \delta_x$ with $\rho > 0$, $z \in \partial \Omega$ then $u_{\kappa, \rho \delta_x}$ satisfies (1.23).

(ii) A question remains open: "Is $\kappa^* = \lambda_{\mu, 1}$?" In case that this equality holds true then $(E_{\kappa}^\alpha)$ admits no solution. Otherwise, if $\kappa^* < \lambda_{\mu, 1}$, $(E_{\kappa}^\alpha)$ admits a solution.

## 5. Subcriticality and sublinearity

In this section, we assume that $(g \circ u)(x) = \delta(x') \tilde{g}(u(x))$ where $\gamma > -1 - \alpha_+$ and $\tilde{g} \in C(\mathbb{R}_+)$, $\tilde{g}(0) = 0$. The proof of Theorems H and I is an adaptation of the idea in [11]. A distinct feature of this approach is that convexity and monotonicity hypotheses of $g$ can be relaxed while these properties are crucial in other methods.

### 5.1. Subcriticality.** Let $\{g_n\}$ be a sequence of $C^1$ nonnegative functions defined on $\mathbb{R}_+$ such that
\[
(5.1) \quad g_n(0) = \tilde{g}(0), \quad g_n \leq g_{n+1} \leq \tilde{g}, \quad \sup_{\mathbb{R}_+} g_n = n \quad \text{and} \quad \lim_{n \to \infty} \|g_n - \tilde{g}\|_{L^\infty(\mathbb{R}_+)} = 0.
\]
Put
\[
(5.2) \quad \tilde{\gamma} = \min\{\alpha_+ + \gamma, -\alpha_-\} > -1.
\]
In preparation for proving Theorem G, we establish the following lemma:
Lemma 5.1. Let $\nu \in M^+(\partial \Omega)$ such that $\|\nu\|_{M(\partial \Omega)} = 1$ and $\{g_n\} \subset C^1(\mathbb{R}^+)$ be a sequence satisfying (5.1). Assume (1.27) and (1.28) are satisfied. Then there exist $\bar{\lambda}, \theta_0 > 0$ and $\varphi_0 > 0$ depending on $\Lambda_0$, $\Lambda_1$, $N$, $\mu$, $\gamma$ and $q_1$ such that for every $\theta \in (0, \theta_0)$ and $\varphi \in (0, \varphi_0)$ the following problem

\[(5.3)\]

admits a nonnegative $v_n \in L^{q^*_w}(\Omega; \delta^{\alpha+\gamma}) \cap L^\infty(\Omega; \delta^{-\alpha})$ satisfying

\[(5.4)\]

Proof. We shall use Schauder fixed point theorem to show the existence of positive solutions of (5.3). For $n \in \mathbb{N}$, define the operator $S_n$ by

\[(5.5)\]

Set

\[(5.6)\]

Step 1: Estimate on $g_n(v + \varphi K^\mu [\nu])$ in $L^1(\Omega; \delta^{\alpha+\gamma})$ for $v \in L^{q^*_w}(\Omega; \delta^{\alpha+\gamma}) \cap L^\infty(\Omega; \delta^{-\alpha})$.

For $\lambda > 0$, set $A_\lambda = \{x \in \Omega : v + \varphi K^\mu [\nu] > \lambda\}$ and $a(\lambda) = \int_{A_\lambda} \delta^{\alpha+\gamma}dx$. We write

\[(5.7)\]

We first estimate $I$ from above. We see that

\[I = a(1)g_n(1) + \int_{\lambda}^\infty a(s)dg_n(s)\]

Since (1.27) holds, it was proved in [11, Lemma 3.1] that there exists an increasing sequence of real positive number \{\ell_j\} such that

\[(5.8)\]

Consequently,

\[(5.9)\]

Observe that

\[\int_{\lambda}^\infty a(s)dg_n(s) = \lim_{j \to \infty} \int_{\ell_j}^{\ell_j} a(s)dg_n(s)\]

On the other hand, by (2.5) one gets, for every $s > 0$,

\[(5.10)\]
where \( c_i = c_i(N, \mu, \Omega) \) with \( i = 31, 32 \). Using (5.10), we obtain

\[
a(1)g_n(1) + \int_1^{\ell_j} a(s)dg_n(s)
\leq c_{31}(M_1(v) + c_{32})q^\gamma g_n(1) + c_{31}(M_1(v) + c_{32})q^\gamma \int_1^{\ell_j} s^{-q^\gamma}dg_n(s)
\leq c_{31}(M_1(v) + c_{32})q^\gamma \ell_j^{-q^\gamma}g_n(\ell_j) + c_{31}q^\gamma(M_1(v) + c_{32})q^\gamma \int_1^{\ell_j} s^{-1-q^\gamma}g_n(s)ds.
\]

By virtue of (5.8), letting \( j \to \infty \) yields

\[
I \leq c_{31}q^\gamma(M_1(v) + c_{32})q^\gamma \int_1^{\infty} s^{-1-q^\gamma}g_n(s)ds \leq c_{33}A_0M_1(v)q^\gamma + c_{33}A_0q^\gamma
\]

where \( c_{33} = c_{33}(N, \mu, \Omega) \).

To handle the remaining term \( II \), without lost of generality, we assume \( q_1 \in (1, \frac{N+\gamma}{N-1-\alpha-\gamma}) \).

Since \( \tilde{g} \) satisfies condition (1.28) and \( g_n \leq \tilde{g} \), it follows that \( g_n \) satisfies this condition too.

Hence

\[
II \leq A_1 \int_{A_1^c} (v + \varrho\mathcal{K}_{\mu}[\nu])q^\gamma \delta^{\alpha+\gamma}dx + \theta \int_{A_1^c} \delta^{\alpha+\gamma}dx
\]

\[
\leq A_1 c_{34} \int_{\Omega} v q^\gamma \delta^{\alpha+\gamma}dx + A_1 c_{34} q^\gamma + c_{34}\theta
\]

where \( c_i = c_i(N, \mu, \Omega), i = 33, 34 \).

Combining (5.7), (5.11) and (5.12) yields

\[
\|g_n(v + \varrho\mathcal{K}_{\mu}[\nu])\|_{L^1(\Omega; \delta^{\alpha+\gamma})} \leq c_{33}A_0M_1(v)q^\gamma + c_{35}A_1M_2(v)q^\gamma + c_{34}\theta + d_\varrho
\]

where \( d_\varrho = c_{33}A_0q^\gamma + c_{34}A_1q^\gamma \).

**Step 2:** Estimates related to \( M_1, M_2 \) and \( M \).

From (2.9), we have

\[
M_1(S_n(v)) = \|G_\mu[\delta^{\gamma}g_n(v + \varrho\mathcal{K}_{\mu}[\nu])]\|_{L^\alpha_{\mu}(\Omega; \delta^{\alpha+\gamma})}
\leq c_7 \|g_n(v + \varrho\mathcal{K}_{\mu}[\nu])\|_{L^1(\Omega; \delta^{\alpha+\gamma})}.
\]

It follows that

\[
M_1(S_n(v)) \leq c_7c_{33}A_0M_1(v)q^\gamma + c_7c_{35}A_1M_2(v)q^\gamma + c_7c_{34}\theta + c_7d_\varrho.
\]

Applying (2.9), we get

\[
M_2(S_n(v)) = \|G_\mu[\delta^{\gamma}g_n(v + \varrho\mathcal{K}_{\mu}[\nu])]\|_{L^1(\Omega; \delta^{\alpha+\gamma})}
\leq c_{36} \|g_n(v + \varrho\mathcal{K}_{\mu}[\nu])\|_{L^1(\Omega; \delta^{\alpha+\gamma})},
\]

which implies

\[
M_2(S_n(v)) \leq c_{36}c_{33}A_0M_1(v)q^\gamma + c_{36}c_{35}A_1M_2(v)q^\gamma + c_{36}c_{34}\theta + c_{36}d_\varrho.
\]

Consequently,

\[
M(S_n(v)) \leq c_{37}A_0M_1(v)q^\gamma + c_{38}A_1M_2(v)q^\gamma + c_{40}\theta + c_{38}d_\varrho.
\]
where $c_{37} = (c_7 + c_{36})c_{33}$, $c_{38} = (c_7 + c_{36})c_{35}$, $c_{39} = c_7 + c_{36}$, $c_{40} = (c_7 + c_{36})c_{34}$. Therefore if $M(S_n(v)) \leq \lambda$ then

$$M(S_n(v)) \leq c_{37}A_0\theta^q + c_{38}A_1\theta^q + c_{40}\theta + c_{39}d\rho.$$

Since $q_0^*>1$ and $q_1>1$, there exist $\theta_0>0$ and $\theta_0>0$ such that for any $\rho \in (0, \theta_0)$ and $\theta \in (0, \theta_0)$ the equation

$$c_{37}A_0\lambda^{q^*} + c_{38}A_1\lambda^{q^*} + c_{40}\lambda + c_{39}d\rho = \lambda$$

admits a largest root $\bar{\lambda}>0$. Therefore,

$$M(v) \leq \bar{\lambda} \implies M(S_n(v)) \leq \bar{\lambda}.$$

**Step 3:** We apply Schauder fixed point theorem to our setting.

Set

$$\mathcal{O} = \{ \phi \in L^1_+(\Omega) : M(\phi) \leq \bar{\lambda} \}.$$

Clearly, $\mathcal{O}$ is a convex subset of $L^1(\Omega)$. We shall show that $\mathcal{O}$ is a closed subset of $L^1(\Omega)$. Indeed, let $\{\phi_m\}$ be a sequence in $\mathcal{O}$ converging to $\phi$ in $L^1(\Omega)$. Obviously, $\phi \geq 0$. We can extract a subsequence, still denoted by $\{\phi_m\}$, such that $\phi_m \rightharpoonup \phi$ a.e. in $\Omega$. Consequently, by Fatou’s lemma, $M_i(\phi) \leq \liminf_{m \to \infty} M_i(\phi_m)$ for $i = 1, 2$. It follows that $M(\phi) \leq \bar{\lambda}$. So $\phi \in \mathcal{O}$ and therefore $\mathcal{O}$ is a closed subset of $L^1(\Omega)$.

In light of (5.13) and (5.18), $S_n$ is well-defined in $\mathcal{O}$ and $S_n(\phi) \subset \mathcal{O}$.

We observe that $S_n$ is continuous. Indeed, if $\phi_m \rightharpoonup \phi$ as $m \to \infty$ in $L^1(\Omega)$ then $g_n(\phi_m + \rho K_{\mu}[v]) \to g_n(\phi + \rho K_{\mu}[v])$ as $m \to \infty$ in $L^1(\Omega; \delta^{q^*+\gamma})$. By (2.9), $S_n(\phi_m) \to S_n(\phi)$ as $m \to \infty$ in $L^1(\Omega)$.

We next show that $S_n$ is a compact operator. Let $\{\phi_m\} \subset \mathcal{O}$ and for each $n$ put $\psi_m = S_n(\phi_m)$. Hence $\{\Delta \psi_n\}$ is uniformly bounded in $L^q(G)$ for every compact subset $G \subset \Omega$. Therefore $\{\psi_m\}$ is uniformly bounded in $W^{1,q}(G)$. Consequently, there exists a subsequence, still denoted by $\{\psi_m\}$, and a function $\psi$ such that $\psi_m \rightharpoonup \psi$ a.e. in $\Omega$. By dominated convergence theorem, $\psi_m \to \psi$ in $L^1(\Omega)$. Thus $S_n$ is compact.

By Schauder fixed point theorem there is a function $v_n \in L^1_+(\Omega)$ such that $S_n(v_n) = v_n$ and $M(v_n) \leq \bar{\lambda}$ where $\bar{\lambda}$ is independent of $n$. Due to Proposition 2.4, $\text{tr}^*(v_n) = 0$ and $v_n$ is a nonnegative solution of (5.3). Moreover, there holds

$$-\int_{\Omega} v_n L_{\mu} \zeta \, dx = \int_{\Omega} \delta^* g_n(v_n + \rho K_{\mu}[v]) \zeta \, dx \quad \forall \zeta \in X(\Omega).$$

**Proof of Theorem G.** Let $\theta \in (0, \theta_0)$ and $\rho \in (0, \theta_0)$. For each $n$, set $u_n = v_n + \rho K_{\mu}[v]$ where $v_n$ is the solution constructed in Lemma 5.1. Then $\text{tr}^*(u_n) = g \nu$ and

$$-\int_{\Omega} u_n L_{\mu} \zeta \, dx = \int_{\Omega} \delta^* g_n(u_n) \zeta \, dx - \rho \int_{\Omega} K_{\mu}[v] L_{\mu} \zeta \, dx \quad \forall \zeta \in X(\Omega).$$

Since $\{v_n\} \subset \mathcal{O}$, the sequence $\{g_n(v_n + \rho K_{\mu}[v])\}$ is uniformly bounded in $L^1(\Omega; \delta^{q^*+\gamma})$ and the sequence $\{\nu v_n\}$ is uniformly bounded in $L^1(\Omega; \delta^{q^*+\gamma})$ for every compact subset $G \subset \Omega$. As a consequence, $\{\Delta v_n\}$ is uniformly bounded in $L^1(\Omega)$. By regularity result [18] for elliptic equations, there exists a subsequence, still denoted by $\{v_n\}$, and a function $v$ such that $v_n \rightharpoonup v$ a.e. in $\Omega$. Therefore $u_n \rightharpoonup u$ a.e. in $\Omega$ with $u = v + \rho K_{\mu}[v]$ and $g_n(u_n) \to g(u)$ a.e. in $\Omega$.

We show that $u_n \to u$ in $L^1(\Omega; \delta^{-\alpha})$. Since $\{v_n\}$ is uniformly bounded in $L^q(\Omega; \delta^q)$, by (2.10), we derive that $\{u_n\}$ is uniformly bounded in $L^1(\Omega; \delta^{-\alpha})$. Due to Holder inequality, $\{u_n\}$ is uniformly integrable with respect to $\delta^{-\alpha} \, dx$. We invoke Vitali’s convergence theorem to derive that $u_n \to u$ in $L^1(\Omega; \delta^{-\alpha})$. 


We next prove that \( g_n(u_n) \to \tilde{g}(u) \) in \( L^1(\Omega; \delta^{\alpha+\gamma}) \). For \( \lambda > 0 \) and \( n \in \mathbb{N} \) set \( B_{n,\lambda} = \{ x \in \Omega : u_n > \lambda \} \) and \( b_n(\lambda) = \int_{B_{n,\lambda}} \delta^{\alpha+\gamma}dx \). For any Borel set \( E \subset \Omega \),

\[
\int_E g_n(u_n)\delta^{\alpha+\gamma}dx = \int_{E \cap B_{n,\lambda}} g_n(u_n)\delta^{\alpha+\gamma}dx + \int_{E \cap B_{n,\lambda}^c} g_n(u_n)\delta^{\alpha+\gamma}dx \\
\leq \int_{B_{n,\lambda}} g_n(u_n)\delta^{\alpha+\gamma}dx + \Theta_\lambda \int_E \delta^{\alpha+\gamma}dx \\
\leq b_n(\lambda)g_n(\lambda) + \int_\lambda^\infty b_n(s)dg_n(s) + \Theta_\lambda \int_E \delta^{\alpha+\gamma}dx.
\]

where \( \Theta_\lambda := \sup_{[0,\lambda]} g \). By proceeding as in the proof of Lemma 5.1, we deduce

\[
b_n(\lambda)g_n(\lambda) + \int_\lambda^\infty b_n(s)dg_n(s) \leq c_{41} \int_\lambda^\infty s^{-1-q^*_\gamma}g_n(s)ds \leq c_{41} \int_\lambda^\infty s^{-1-q^*_\gamma}\tilde{g}(s)ds
\]

where \( c_{41} \) depends on \( N, \mu, \gamma \) and \( \Omega \). Note that the term on the right hand-side of (5.22) tends to 0 as \( \lambda \to \infty \). Take arbitrarily \( \varepsilon > 0 \), there exists \( \lambda > 0 \) such that the right hand side of (5.22) is smaller than \( \frac{\varepsilon}{2} \). Fix such \( \lambda \) and put \( \eta = \frac{\varepsilon}{2\Theta_\lambda} \). Then, by (5.21),

\[
\int_E \delta(x)^{\alpha+\gamma}dx \leq \eta \implies \int_E g_n(u_n)\delta(x)^{\alpha+\gamma}dx < \varepsilon.
\]

Therefore the sequence \( \{g_n(u_n)\} \) is uniformly integrable with respect to \( \delta^{\alpha+\gamma}dx \). Due to Vitali convergence theorem, we deduce that \( g_n(u_n) \to \tilde{g}(u) \) in \( L^1(\Omega; \delta^{\alpha+\gamma}) \).

Finally, by sending \( n \to \infty \) in each term of (5.20) we obtain

\[
-\int_\Omega uL_\mu \zeta dx = \int_\Omega \delta^\gamma \tilde{g}(u)\zeta dx - \varrho \int_\Omega \mathbb{K}_\mu[v]L_\mu \zeta dx \quad \forall \zeta \in \mathcal{X}(\Omega).
\]

By Theorem A, \( u \) is a nonnegative weak solution of (1.26). \( \square \)

5.2. Sublinearity. In this subsection we deal with the case where \( g \) is sublinear.

**Lemma 5.2.** Let \( \nu \in \mathcal{M}^+(\partial\Omega) \) such that \( \|\nu\|_{\mathcal{M}(\partial\Omega)} = 1 \) and \( \{g_n\} \subset C^1(\mathbb{R}_+) \) be a sequence satisfying (5.1). Assume (1.29) is satisfied. Then for every \( \varrho > 0 \) problem (5.3) admits a nonnegative solution \( v_n \) satisfying

\[
\|v_n\|_{L^1(\Omega;\delta^\gamma)} \leq \tilde{\lambda}
\]

where \( \tilde{\gamma} \) is as in (5.2) and \( \tilde{\lambda} \) depends on \( \Lambda_2, q_2, N, \mu \).

**Proof.** The proof is similar to that of Lemma 5.1, also based on Schauder fixed point theorem. So we point out only the main modifications. Let \( S_n \) be the operator defined in (5.5). Fix \( q_3 \in (1, \frac{N+\gamma}{N-1-\alpha-}) \) and put

\[
N(v) = \|v\|_{L^{q_3}(\Omega;\delta^\gamma)}, \quad \forall v \in L^{q_3}(\Omega;\delta^\gamma).
\]
Combining (2.9), (2.10) and (1.29) leads to
\[ N(S_n(v)) \leq c_4^2 \|g_n(v + \phi K_\mu[v])\|_{L^1(\Omega; \delta^{\alpha+\gamma})} \]
\[ \leq c_4^2 \int_\Omega A_2(v + \phi K_\mu[v])v^2 \delta^{\alpha+\gamma} dx + c_4^2 \int_\Omega \delta^{\alpha+\gamma} dx \]
\[ \leq c_{44} \Lambda_2 \int_\Omega v^{q_2} \delta^{\alpha+\gamma} dx + c_{43}(q_2 + \theta) \]
\[ \leq c_{44} \Lambda_2 N(v)^{q_2} + c_{43}(q_2 + \theta) \]
where \( c_i = c_i(N, \mu, \Omega, q_2) \) (42 \( i \leq 44 \)). Therefore, if \( N(v) \leq \lambda \) for some \( \lambda > 0 \) then
\[ N(S_n(v)) \leq c_{44} \Lambda_2 \lambda^{q_2} + c_{43}(q_2 + \theta) \]

Consider the following algebraic equation
\[ c_{44} \Lambda_2 \lambda^{q_2} + c_{43}(q_2 + \theta) = \lambda \] (5.25)
If \( q_2 < 1 \) then for any \( \rho > 0 \) (5.25) admits a unique positive root \( \tilde{\lambda} \). If \( q_2 = 1 \) then for \( \Lambda_2 \) small such that \( c_{44} \Lambda_2 < 1 \) and \( \rho > 0 \) equation (5.25) admits a unique positive root \( \hat{\lambda} \). Therefore,
\[ N(v) \leq \tilde{\lambda} \Rightarrow N(S_n(v)) \leq \tilde{\lambda} \] (5.26)
By proceeding as in the proof of Lemma 5.1, we can prove that \( S_n \) is a continuous, compact operator from the closed, convex set
\[ \tilde{\mathcal{O}} = \{ v \in L^1_+(\Omega) : N(v) \leq \tilde{\lambda} \} \]
into itself. Thus by appealing to Schauder fixed point theorem, we see that there exists a function \( v_n \in L^1_+(\Omega) \) such that \( S_n(v_n) = v_n \) and \( N(v_n) \leq \tilde{\lambda} \) with \( \tilde{\lambda} \) being independent of \( n \).

By Proposition 2.4, \( \text{tr}^*(v_n) = 0 \) and \( v_n \) is a nonnegative solution of (5.3). Moreover (5.19) holds.

**Proof of Theorem H.** Let \( v_n \) be the solution of (5.3) constructed in Lemma 5.2. Put \( u_n = v_n + \phi K_\mu[v] \) then \( u_n \) satisfies (5.20). By a similar argument as in the proof of Theorem G, there exists a subsequence, still denoted by \( \{ u_n \} \) and a function \( u \) such that \( u_n \rightarrow u \) a.e. in \( \Omega \). Since \( \{ v_n \} \subset \tilde{\mathcal{O}} \), it follows that \( \{ v_n \} \) is uniformly bounded in \( L^{q_3}(\Omega; \delta^{-\alpha-}) \), so is \( \{ u_n \} \).

By Holder inequality, \( \{ u_n \} \) is uniformly integrable in \( L^1(\Omega; \delta^{-\alpha-}) \). Due to (1.29), \( \{ g_n(u_n) \} \) is uniformly integrable in \( L^1(\Omega; \delta^{\alpha+\gamma}) \). Vitali convergence theorem implies that \( u_n \rightarrow u \) in \( L^1(\Omega; \delta^{-\alpha-}) \) and \( g_n(u_n) \rightarrow \tilde{g}(u) \) in \( L^1(\Omega; \delta^{\alpha+\gamma}) \). Letting \( n \rightarrow \infty \) in (5.20), we conclude that \( u \) is a nonnegative solution of (1.9) satisfying (1.13). \( \square \)

**5.3. Supercritical case. Proof of Theorem I.** Suppose by contradiction that for some \( \rho > 0 \) and \( z \in \partial \Omega \) there exists a positive solution \( u \) of (1.31). Then by Theorem A, \( \tilde{g}(u) \in L^q(\Omega; \delta^{\alpha+\gamma}) \) and
\[ u = \mathbb{G}_\mu[\delta^\gamma \tilde{g}(u)] + \phi K_\mu(\cdot, z). \]
This, along with (2.7) and the monotonicity assumption, implies
\[
\int_{\Omega} \delta(x)^{\alpha+\gamma} \tilde{g}(u) dx \geq \int_{\Omega} \delta(x)^{\alpha+\gamma} \tilde{g}(K_\mu(x,z)) dx
\]
\[
\geq \int_{\Omega} \delta(x)^{\alpha+\gamma} \tilde{g}(\delta(x)^{\alpha+}|x-z|^{2\alpha-N}) dx
\]
\[
\geq c_{45} \int_{\{x\in\Omega: \delta(x)\geq \frac{1}{2}|x-z|\}} |x-z|^{\alpha+\gamma} \tilde{g}(|x-z|^{1+\alpha-N}) dx.
\]
Fix \(r_0 > 0\) such that
\[
C_{r_0} := \left\{ x : |x| \leq r_0, \delta(x) \geq \frac{1}{2}|x-z| \right\} \subset \left\{ x \in \Omega : \delta(x) \geq \frac{1}{2}|x-z| \right\}.
\]
Then
\[
\int_{\Omega} \delta(x)^{\alpha+\gamma} \tilde{g}(u) dx \geq c_{45} \int_{C_{r_0}} |x-z|^{\alpha+\gamma} \tilde{g}(|x-z|^{1+\alpha-N}) dx
\]
\[
\geq c_{46} \int_{a^*}^{\infty} s^{-1-q^*} \tilde{g}(s) ds
\]
for some \(a^* > 0\). By assumption (1.30), \(\tilde{g}(u) \notin L^1(\Omega; \delta^{\alpha+\gamma})\), which leads to a contradiction. \(\square\)

REFERENCES

[1] A. Ancona, Theoré du potentiel sur les graphes et les variétés, in Ecole d’été de Probabilités de Saint-Flour XVIII-1988, Springer Lecture Notes in Math, 1427 (1990), 1-112.
[2] A. Ancona, Negatively curved manifolds, elliptic operators and the Martin boundary, Ann. of Math. (2), 125 (1987), 495-536.
[3] A. Ancona and M. Marcus, Positive solutions of a class of semilinear equations with absorption and schrodinger equations, J. Math. Pures et Appl. 104, 587-618 (2015).
[4] C. Bandle, V. Moroz and W. Reichel, Boundary blowup type sub-solutions to semilinear elliptic equations with Hardy potential, J. London Math. Soc. 2 (2008), 503-523.
[5] C. Bandle, V. Moroz and W. Reichel, Large solutions to semilinear elliptic equations with Hardy potential and exponential nonlinearity, Around the research of Vladimir Maz’ya. II, 1-22, Int. Math. Ser. (N. Y.), 12, Springer, New York, 2010.
[6] M. F. Bidaut-Véron and L. Vivier, An elliptic semilinear equation with source term involving boundary measures: the subcritical case, Rev. Mat. Iberoamericana 16 (2000), 477-513.
[7] M. F. Bidaut-Véron, G. Hoang, Q. H. Nguyen, L. Véron, An elliptic semilinear equation with source term and boundary measure data: the supercritical case, Journal of Functional Analysis 269 (2015), 1995-2017.
[8] M.F. Bidaut-Véron and C. Yarur, Semilinear elliptic equations and systems with measure data: existence and a priori estimates, Adv. Differential Equations 7 (2002), 257-296
[9] H. Brezis and X. Cabré, Some simple nonlinear PDE’s without solutions, Boll. Unione Mat. Italiana, 8 (1998), 223-262.
[10] H. Brezis and M. Marcus, Hardy inequalities revisited, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 25 (1997) 217-237.
[11] H. Chen, P. Felmer and L. Véron, Elliptic equations involving general subcritical source nonlinearity and measures, arxiv.org/abs/1409.3067 (2014).
[12] J.Dávila and L. Dupaigne, Hardy-type inequalities, J. Eur. Math. Soc. 6 (2004), 335-365.
[13] S. Filippas, L. Moschini and A. Tertikas, Sharp two-sided heat kernel estimates for critical Schrodinger operators on bounded domains, Commun. Math. Phys. 273 (2007), 237-281.
[14] K. T. Gikas and L. Véron, Boundary singularities of solutions of semilinear elliptic equations with critical Hardy potentials, Nonlinear Anal. 121, 4695-4674 (2015).
[15] M. Marcus, Complete classification of the positive solutions of \(-\Delta u + u^q = 0\), Jl. d’Anal. Math. 117 (2012), 187-220.
[16] M. Marcus and P. T. Nguyen, *Moderate solutions of semilinear elliptic equations with Hardy potential.* (to appear in Ann. Inst. H. Poincaré Anal. Non Linéaire).

[17] M. Marcus, V. J. Mizel and Y. Pinchover, *On the best constant for Hardy’s inequality in \( \mathbb{R}^N \),* Trans. Amer. Math. Soc. **350** (1998), 3237-3255.

[18] G. Mingione, *The Calderón-Zygmund theory for elliptic problems with measure data,* Ann. SNS Pisa Cl. Sci. **6** (2007), 195-261.

[19] M. Marcus and I. Shafrir, *An eigenvalue problem related to Hardy’s \( L^p \) inequality,* Ann. Scuola. Norm. Sup. Pisa. Cl. Sc. **29** (2000), 581-604.

[20] M. Marcus and L. Véron, *Nonlinear second order elliptic equations involving measures,* De Gruyter Series in Nonlinear Analysis and Applications, 2013.

[21] L. Véron and C. Yarur, *Boundary value problems with measures for elliptic equations with singular potentials,* J. Funct. Anal. **262** (2012) 733-772.