SEYMOUR’S CONJECTURE ON
2-CONNECTED GRAPHS OF LARGE PATHWIDTH

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Abstract. We prove the conjecture of Seymour (1993) that for every apex-forest $H_1$ and outerplanar graph $H_2$ there is an integer $p$ such that every 2-connected graph of pathwidth at least $p$ contains $H_1$ or $H_2$ as a minor. An independent proof was recently obtained by Dang and Thomas (arXiv:1712.04549).

1. Introduction

Pathwidth is a graph parameter of fundamental importance, especially in graph structure theory. The pathwidth of a graph $G$ is the minimum integer $k$ for which there is a sequence of sets $B_1,\ldots,B_n \subseteq V(G)$ such that $|B_i| \leq k + 1$ for each $i \in [n]$, for every vertex $v$ of $G$, the set $\{i \in [n] : v \in B_i\}$ is a non-empty interval, and for each edge $vw$ of $G$, some $B_i$ contains both $v$ and $w$.

In the first paper of their graph minors series, Robertson and Seymour [8] proved the following theorem.

1.1. For every forest $F$, there exists a constant $p$ such that every graph with pathwidth at least $p$ contains $F$ as a minor.

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The constant $p$ was later improved to $|V(F)| - 1$ (which is best possible) by Bienstock, Robertson, Seymour, and Thomas [1]. A simpler proof of this result was later found by Diestel [6].

Since forests have unbounded pathwidth, 1.1 implies that a minor-closed class of graphs has unbounded pathwidth if and only if it includes all forests. However, these certificates of large pathwidth are not 2-connected, so it is natural to ask for which minor-closed classes $C$, does every 2-connected graph in $C$ have bounded pathwidth?

In 1993, Paul Seymour proposed the following answer (see [5]). A graph $H$ is an apex-forest if $H - v$ is a forest for some $v \in V(H)$. A graph $H$ is outerplanar if it has an embedding in the plane with all the vertices on the outerface. These classes are relevant since they both contain 2-connected graphs with arbitrarily large pathwidth. Seymour conjectured the following converse holds.

**1.2.** For every apex-forest $H_1$ and outerplanar graph $H_2$ there is an integer $p$ such that every 2-connected graph of pathwidth at least $p$ contains $H_1$ or $H_2$ as a minor.

Equivalently, 1.2 says that for a minor-closed class $C$, every 2-connected graph in $C$ has bounded pathwidth if and only if some apex-forest and some outerplanar graph are not in $C$.

The original motivation for conjecturing 1.2 was to seek a version of 1.1 for matroids (see [3]). Observe that apex-forests and outerplanar graphs are planar duals (see 2.1). Since a matroid and its dual have the same pathwidth (see [7] for the definition of matroid pathwidth), 1.2 provides some evidence for a matroid version of 1.1.

In this paper we prove 1.2. An independent proof was recently obtained by Dang and Thomas [3]. We actually prove a slightly different, but equivalent version of 1.2. Namely, we prove that there are two unavoidable families of minors for 2-connected graphs of large pathwidth. We now describe our two unavoidable families.

A binary tree is a rooted tree such that every vertex is either a leaf or has exactly two children. For $\ell \geq 0$, the complete binary tree of height $\ell$, denoted $\Gamma_\ell$, is the binary tree with $2^\ell$ leaves such that each root to leaf path has $\ell$ edges. It is well known that $\Gamma_\ell$ has pathwidth $\lceil \ell/2 \rceil$. Let $\Gamma_\ell^+$ be the graph obtained from $\Gamma_\ell$ by adding a new vertex adjacent to all the leaves of $\Gamma_\ell$. See Figure 1. Note that $\Gamma_\ell^+$ is a 2-connected apex-forest, and its pathwidth grows as $\ell$ grows (since it contains $\Gamma_\ell$).

![Figure 1.](image-url) Complete binary trees with an extra vertex adjacent to all the leaves.
Our second set of unavoidable minors is defined recursively as follows. Let $\nabla_1$ be a triangle with a root edge $e$. Let $H_1$ and $H_2$ be copies of $\nabla_\ell$ with root edges $e_1$ and $e_2$. Let $\nabla$ be a triangle with edges $e_1$, $e_2$ and $e_3$. Define $\nabla_{\ell+1}$ by gluing each $H_i$ to $\nabla$ along $e_i$ and then declaring $e_3$ as the new root edge. See Figure 2. Note that $\nabla_\ell$ is a 2-connected outerplanar graph, and its pathwidth grows as $\ell$ grows (since it contains $\Gamma_{\ell-1}$).

![Figure 2. Universal outerplanar graphs. The root edges are dashed.](image)

The following is our main theorem.

1.3. For every integer $\ell \geq 1$ there is an integer $p$ such that every 2-connected graph of pathwidth at least $p$ contains $\Gamma_\ell^+$ or $\nabla_\ell$ as a minor.

In Section 2, we prove that every apex-forest is a minor of a sufficiently large $\Gamma_\ell^+$ and every outerplanar graph is a minor of a sufficiently large $\nabla_\ell$. Thus, Theorem 1.3 implies Seymour’s conjecture.

We actually prove the following theorem, which by 1.1, implies 1.3.

1.4. For all integers $\ell \geq 1$, there exists an integer $k$ such that every 2-connected graph $G$ with a $\Gamma_k$ minor contains $\Gamma_\ell^+$ or $\nabla_\ell$ as a minor.

Our approach is different from that of Dang and Thomas [3], who instead observe that by the Grid Minor Theorem [9], one may assume that $G$ has bounded treewidth but large pathwidth. Dang and Thomas then apply the machinery of ‘non-branching tree decompositions’ that they developed in [4] to prove 1.2.

The rest of the paper is organized as follows. Section 2 proves the universality of our two families. In Sections 3 and 4, we define ‘special’ ear decompositions and prove that special ear decompositions always yield $\Gamma_\ell^+$ or $\nabla_\ell$ minors. In Section 5, we prove that a minimal counterexample to 1.4 always contains a special ear decomposition. Section 6 concludes with short derivations of our main results.

2. Universality

This section proves some elementary (and possibly well-known) results. We include the proofs for completeness.
2.1. Outerplanar graphs and apex-forests are planar duals.

Proof. Let $G$ be an apex-forest, where $G - v$ is a forest. Consider an arbitrary planar embedding of $G$. Note that every face of $G$ includes $v$ (otherwise $G - v$ would contain a cycle). Let $G^*$ be the planar dual of $G$. Let $f$ be the face of $G^*$ corresponding to $v$. Since every face of $G$ includes $v$, every vertex of $G^*$ is on $f$. So $G^*$ is outerplanar.

Conversely, let $G$ be an outerplanar graph. Consider a planar embedding of $G$, in which every vertex is on the outerface $f$. Let $G^*$ be the planar dual of $G$. Let $v$ be the vertex of $G^*$ corresponding to $f$. If $G^* - v$ contained a cycle $C$, then some vertex of $G$ on the ‘inside’ of $C$ would not be on $f$. Thus $G^* - v$ is a forest, and $G^*$ is an apex-forest. □

We now show that Theorem 1.3 implies Seymour’s conjecture, by proving two universality results.

2.2. Every apex-forest on $n \geq 2$ vertices is a minor of $\Gamma_{n-1}^+$. 2.2 is a corollary of the following.

2.3. Every tree with $n \geq 1$ vertices is a minor of $\Gamma_{n-1}$, such that each branch set includes a leaf of $\Gamma_{n-1}$.

Proof. We proceed by induction on $n$. The base case $n = 1$ is trivial. Let $T$ be a tree with $n \geq 2$ vertices. Let $v$ be a leaf of $T$. Let $w$ be the neighbour of $v$ in $T$. By induction, $T - v$ is a minor of $\Gamma_{n-2}$, such that each branch set includes a leaf of $\Gamma_{n-2}$. In particular, the branch set for $w$ includes some leaf $x$ of $\Gamma_{n-2}$. Let $y$ and $z$ be the leaf vertices of $\Gamma_{n-1}$ adjacent to each leaf of $\Gamma_{n-2}$. Let $y$ and $z$ be the branch set of $v$. For each leaf $u \neq x$ of $\Gamma_{n-2}$, if $u$ is in the branch set of some vertex of $T - v$, then extend this branch set to include one of the new leaves in $\Gamma_{n-1}$ adjacent to $u$. Now $T$ is a minor of $\Gamma_{n-1}$, such that each branch set includes a leaf of $\Gamma_{n-1}$. □

Our second universality result is for outerplanar graphs.

2.4. Every outerplanar graph on $n \geq 2$ vertices is a minor of $\nabla_{n-1}$. 2.4 is a corollary of the following.

2.5. Every outerplanar triangulation $G$ on $n \geq 3$ vertices is a minor of $\nabla_{n-1}$, such that for every edge $vw$ on the outerface of $G$, there is a non-root edge on the outerface of $\nabla_{n-1}$ joining the branch sets of $v$ and $w$.

Proof. We proceed by induction on $n$. The base case, $G = K_3$, is easily handled as illustrated in Figure 3. Let $G$ be an outerplanar triangulation with $n \geq 4$ vertices.
Every such graph has a vertex $u$ of degree 2, such that if $\alpha$ and $\beta$ are the neighbours of $u$, then $G - u$ is an outerplanar triangulation and $\alpha \beta$ is an edge on the outerface of $G - u$. By induction, $G - u$ is a minor of $\nabla_{n-2}$, such that for every edge $vw$ on the outerface of $G - u$, there is a non-root edge $v'w'$ on the outerface of $\nabla_{n-2}$ joining the branch sets of $v$ and $w$. In particular, there is a non-root edge $\alpha'\beta'$ of $\nabla_{n-2}$ joining the branch sets of $\alpha$ and $\beta$. Note that $\nabla_{n-1}$ is obtained from $\nabla_{n-2}$ by adding, for each non-root edge $pq$ on the outerface of $\nabla_{n-2}$, a new vertex adjacent to $p$ and $q$. Let the branch set of $u$ be the vertex $u'$ of $\nabla_{n-1} - V(\nabla_{n-2})$ adjacent to $\alpha'$ and $\beta'$. Thus $\nabla_{n-1}$ contains $G$ as a minor. Every edge on the outerface of $G$ is one of $ua$ or $u\beta$, or is on the outerface of $G - u$. By construction, $u'\alpha'$ is a non-root edge on the outerface of $\nabla_{n-1}$ joining the branch sets of $u$ and $\alpha$. Similarly, $u'\beta'$ is a non-root edge on the outerface of $\nabla_{n-1}$ joining the branch sets of $u$ and $\beta$. For every edge $vw$ on the outerface of $G$, where $vw \notin \{ua, u\beta\}$, if $z$ is the vertex in $\nabla_{n-1} - V(\nabla_{n-2})$ adjacent to $\alpha'$ and $\beta'$, extend the branch set of $v$ to include $z$. Now $zu'$ is an edge on the outerface of $\nabla_{n-1}$ joining the branch sets for $v$ and $w$. Thus for every edge $vw$ on the outerface of $G$, there is a non-root edge of $\nabla_{n-1}$ joining the branch sets of $v$ and $w$. \qed

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3.png}
\caption{Proof of 2.5 in the base case.}
\end{figure}

3. Binary Ear Trees

Henceforth, all graphs in this paper are finite and simple. In particular, after contracting an edge, we suppress parallel edges and loops. If $H$ and $G$ are graphs, then $H \cup G$ is the graph with $V(H \cup G) = V(H) \cup V(G)$ and $E(H \cup G) = E(H) \cup E(G)$. If $H$ is a subgraph of $G$, then an $H$-ear is a path in $G$ with its two ends in $H$ but with no internal vertex in $H$.

A binary ear tree in a graph $G$ is a pair $(T, \mathcal{P})$, where $T$ is a binary tree, and $\mathcal{P} = \{P_x : x \in V(T)\}$ is a collection of paths of $G$ satisfying

1. $|V(P_x)| \geq 3$ for all $x \in V(T)$,
2. for all distinct $x, y$ in $V(T)$ such that $y$ is not a child of $x$, $P_x$ and $P_y$ are internally-disjoint,
3. for all $x, y, z \in V(T)$ such that $y$ and $z$ are the children of $x$, $P_y$ and $P_z$ are $P_x$-ears, and $P_x$ has an end that is not an end of $P_y$ nor of $P_z$.

The main result of this section is the following.

3.1. For every integer $\ell \geq 1$, if $G$ has a binary ear tree $(T, \mathcal{P})$ such that $T \simeq \Gamma_{3\ell+2}$, then $G$ contains $\Gamma^\ell$ or $\nabla_\ell$ as a minor.
Before starting the proof, we first set up notation for a Ramsey-type result that we will need.

If \( p \) and \( q \) are vertices of a tree \( T \), then let \( pTq \) denote the unique \( pq \)-path in \( T \). If \( T' \) is a subdivision of a tree \( T \), the vertices of \( T' \) coming from \( T \) are called original vertices and the other vertices of \( T' \) are called subdivision vertices. Given a colouring of the vertices of \( T = \Gamma_n \) with colours \{red, blue\}, we say that \( T \) contains a red subdivision of \( \Gamma_k \), if it contains a subdivision \( T' \) of \( \Gamma_k \) such that all the original vertices of \( T' \) are red, and for all \( a, b \in V(T') \) with \( b \) a descendent of \( a \), the path \( aTb \) is descending. (Here a path is descending if it is contained in a path that starts at the root.) Define \( R(k, \ell) \) to be the minimum integer \( n \) such that every colouring of \( \Gamma_n \) with colours \{red, blue\} contains a red subdivision of \( \Gamma_k \) or a blue subdivision of \( \Gamma_\ell \). We will use the following easy result.

3.2. \( R(k, \ell) \leq k + \ell \) for all integers \( k, \ell \geq 0 \).

Proof. We proceed by induction on \( k + \ell \). As base cases, it is clear that \( R(k, 0) = k \) and \( R(0, \ell) = \ell \) for all \( k, \ell \). For the inductive step, assume \( k, \ell \geq 1 \) and let \( T \) be a \{red, blue\}-coloured copy of \( \Gamma_{k+\ell} \). By symmetry, we may assume that the root \( r \) of \( T \) is coloured red. Let \( T_1 \) and \( T_2 \) be the components of \( T - r \), both of which are copies of \( \Gamma_{k+\ell-1} \). If \( T_1 \) or \( T_2 \) contains a blue subdivision of \( \Gamma_{\ell} \), then so does \( T \) and we are done. By induction, \( R(k - 1, \ell) \leq k - 1 + \ell \), so both \( T_1 \) and \( T_2 \) contain a red subdivision of \( \Gamma_{k-1} \). Add the paths from \( r \) to the roots of these red subdivisions. We obtain a red subdivision of \( \Gamma_k \), as desired.

We now prove 3.1.

Proof of 3.1. Let \( t \) be a non-leaf vertex of \( T \). Let \( u \) and \( v \) be the children of \( t \). Let \( u_1 \) and \( u_2 \) be the ends of \( P_u \). Let \( v_1 \) and \( v_2 \) be the ends of \( P_v \). We say that \( t \) is nested if \( u_1Ptu_2 \subseteq v_1Pt'v_2 \) or \( v_1Pt'v_2 \subseteq u_1Ptu_2 \). If \( t \) is not nested, then \( t \) is split. See Figures 4 and 5. Regarding split and nested as colours, by 3.2, \( T \) contains a split subdivision of \( \Gamma_{\ell+1} \) or a nested subdivision of \( \Gamma_{2\ell+1} \).

![Figure 4. Examples of a nested vertex \( t \) with a path \( P_t \) in a binary ear tree.](image)

3.3. If \( T \) contains a split subdivision \( T' \) of \( \Gamma_{\ell+1} \) with no subdivision vertices, then \( \bigcup_{t \in V(T')} P_t \) contains \( \nabla_\ell \) as a minor.
Subproof. We first set-up a stronger induction hypothesis. Let \( \nabla^- \) be the graph obtained from \( \nabla \) by deleting its root edge \( xy \). Define the root path of \( \nabla \) as \( xzy \), where \( z \) is the unique common neighbour of \( x \) and \( y \). Let \( P \) be a path in a graph \( G \). We say that a \( \nabla^- \) minor in \( G \) is rooted on \( P \) if there are two edges \( e \) and \( f \) of \( P \) such that the root path of \( \nabla^- \) is obtained from \( P \) by contracting all edges of \( P \) except for \( e \) and \( f \). We prove the stronger result that \( \bigcup_{t \in V(T^1)} P_t \) contains a \( \nabla^-_{\ell+1} \) minor rooted on \( P_r \), where \( r \) is the root of \( T^1 \).

We proceed by induction on \( \ell \). The case \( \ell = 0 \) is clear since \( \nabla^-_1 \) is just a path with three vertices. For the inductive step, let \( a \) and \( b \) be the children of \( r \) and let \( T^1_a \) and \( T^1_b \) be the subtrees of \( T^1 \) rooted at \( a \) and \( b \). By induction, \( G_a := \bigcup_{t \in V(T^1_a)} P_t \) contains a \( \nabla^-_\ell \) minor \( H_a \) rooted on \( P_a \), and \( G_b := \bigcup_{t \in V(T^1_b)} P_t \) contains a \( \nabla^-_\ell \) minor \( H_b \) rooted on \( P_b \).

We prove that \( G_a \) and \( G_b \) are vertex-disjoint, except possibly at a vertex of \( P_a \cap P_b \) (there is at most one such vertex since \( r \) is split). For each \( v \in V(G_a) \), let \( a(v) \) be the highest vertex of \( T^1_a \) such that \( v \in V(P_{a(v)}) \). If \( a(v) \neq a \) and \( v \) is an end of \( P_{a(v)} \), then by property (iii) of binary ear trees, \( v \in V(P_w) \) where \( w \) is the parent of \( a(v) \) in \( T^1_a \). However, this contradicts the maximality of \( a(v) \). Therefore, for each \( v \in V(G_a) \), \( a(v) = a \) or \( v \) is an internal vertex of \( P_{a(v)} \). Define \( b(v) \) analogously for each \( v \in V(G_b) \) and suppose \( v \in V(G_a) \cap V(G_b) \). If \( a(v) \neq a \), then \( v \) is an internal vertex of \( P_{a(v)} \). But then \( a(v), b(v) \) contradicts (ii). By symmetry, \( a(v) = a \) and \( b(v) = b \). Thus, \( v \in V(P_a) \cap V(P_b) \), as desired.

Let \( a_1 \) and \( a_2 \) be the ends of \( P_a \), and \( b_1 \) and \( b_2 \) be the ends of \( P_b \). By symmetry, we may assume that the ordering of these points along \( P \) is either \( a_1, b_1, a_2, b_2 \) or \( a_1, a_2, b_1, b_2 \). Let \( a'_1, a'_2 \), and \( b'_1 \) be the vertices of \( P \) immediately following \( a_1, a_2, \) and \( b_1 \) respectively. In the first case let \( e = a_1 a'_1 \) and \( f = a_2 a'_2 \). In the second case let \( e = a_1 a'_1 \) and \( f = b_1 b'_1 \). In both cases, \( H_a \cup H_b \) extends to a \( \nabla^-_{\ell+1} \) minor rooted on \( P_r \) by contracting all edges of \( E(P_r) \setminus \{ e, f \} \). See Figure 6. Since \( \nabla^-_{\ell+1} \) contains a \( \nabla_\ell \) minor, we are done.

We will handle the case when \( T^1 \) has subdivision vertices at the very end of the proof. For now we switch to the case that \( T \) contains a nested subdivision \( T^2 \) of \( T_{2\ell+1} \). Orient each path in \( \mathcal{P} \) inductively as follows. Let \( r \) be the root of \( T \) and orient \( P_r \) arbitrarily. If \( P_s \) has already been oriented and \( t \) is a child of \( s \), then orient \( P_t \) so that \( P_s \cup P_t \) does not contain a directed cycle. Consider each path in \( \mathcal{P} \) to be oriented from left to right, and thus with left and right ends.
We may consider an original vertex of \(T^2\) to be a vertex of \(T_{2\ell+1}\) (and vice versa). Let \(t\) be a non-leaf, original vertex of \(T^2\) and let \(u\) and \(v\) be the children of \(t\) in \(T_{2\ell+1}\). Define \(t\) to be left-bad if at least one of the left ends of \(P_u\) or \(P_v\) is equal to the left end of \(P_t\). Define \(t\) to be left-good if \(t\) is not left-bad. Define right-good and right-bad analogously. Let \(T_{2\ell}\) be the tree obtained from \(T_{2\ell+1}\) by deleting all leaves. By 3.2, \(T_{2\ell}\) contains a left-good subdivision of \(\Gamma_\ell\) or a left-bad subdivision of \(\Gamma_\ell\). By (iii), no vertex of \(T_{2\ell}\) can be both left-bad and right-bad. Therefore, \(T_{2\ell}\) contains a left-good subdivision \(T^3\) of \(\Gamma_\ell \simeq \Gamma_\ell\).

As in the previous case, we assume that all vertices of \(T^3\) are original vertices, and then reduce to this case at the end of the proof. Let \(t\) be a non-leaf vertex of \(T^3\) and \(u\) and \(v\) be the children of \(t\) in \(T^3\). Let \(f(t)\) be the first vertex of \(P_t\) that is a left end of either \(P_u\) or of \(P_v\). Let \(e(t)\) be the last edge of \(P_t\) incident to a left end of either \(P_u\) or \(P_v\). If \(t\) is a leaf vertex of \(T^3\), then let \(f(t)\) be any internal vertex of \(P_t\) and \(e(t)\) be the last edge of \(P_t\) incident to \(f(t)\). Let \(H := \bigcup_{t \in V(T^3)} P_t\) and \(M := \{e(t) : t \in V(T^3)\}\).

Since every vertex of \(T^3\) is nested, \(H\) contains two components \(H_{\text{left}}\) and \(H_{\text{right}}\) such that \(H_{\text{left}}\) contains all left ends of \(\{P_t : t \in V(T^3)\}\) and \(H_{\text{right}}\) contains all right ends of \(\{P_t : t \in V(T^3)\}\). It is easy to see that \(H_{\text{left}}\) contains a subdivision \(T^4_\ell\) of \(\Gamma_\ell\) whose set of original vertices is \(\{f(t) : t \in V(T^3)\}\); see Figure 7. By construction, each leaf of \(T^4\) is incident to an edge in \(M\). Also, \(H_{\text{right}}\) is clearly connected. Therefore, after contracting all edges of \(H_{\text{right}}\), \(T^4 \cup M \cup H_{\text{right}}\) contains a \(\Gamma_\ell^e\) minor.

To complete the proof, it suffices to consider the case that \(T^1\) or \(T^3\) contain subdivision vertices. Let \(T'\) be either \(T^1\) or \(T^3\). Suppose that \(x\) and \(y\) are original vertices of \(T'\) with \(y\) a descendant of \(x\), and \(x, s(1), \ldots, s(k), y\) is a path in \(T'\) such that \(s(i)\) is a subdivision vertex for all \(i \in [k]\). It is easy to see that there exists a path \(P'_y\) in \(P_y \cup \bigcup_{i \in [k]} P_{s(i)}\) such
that $P'_y$ has the same ends as $P_{s(1)}$ and $P'_y \subseteq P'_y$; see Figure 8. Thus, we may replace $P'_y$ by $P'_y$ and suppress all vertices $s(1), \ldots, s(k)$ in $T'$. \hfill \Box

4. Binary Pear Trees

In order to prove our main theorem, we need something slightly more general than binary ear trees, which we now define. A binary pear tree in a graph $G$ is a pair $(T, B)$, where $T$ is a binary tree, and $B = \{(P_x, Q_x) : x \in V(T)\}$ is a collection of pairs of paths of $G$ satisfying

(i) $P_x \subseteq Q_x$ and $|V(P_x)| \geq 3$ for all $x \in V(T), \ldots$
(ii) for all \(x, y \in V(T)\), if \(Q_y\) contains an internal vertex of \(Q_x\), then \(y\) is a descendent of \(x\) or of the sibling of \(x\),

(iii) for all \(x, y, z \in V(T)\) such that \(y\) and \(z\) are the children of \(x\), \(Q_y\) and \(Q_z\) are \(P_x\)-ears, \(P_x\) has an end that is not an end of \(Q_y\) nor of \(Q_z\), no internal vertex of \(P_y\) is in \(Q_z\), and no internal vertex of \(P_z\) is in \(Q_y\).

Note that if \((T, \{P_x : x \in V(T)\})\) is a binary ear tree, then \((T, \{(P_x, P_x) : x \in V(T)\})\) is a binary pear tree. We now prove the following converse.

4.1. If \(G\) has a binary pear tree \((T, \mathcal{B})\), then \(G\) has a minor \(H\) such that \(H\) has a binary ear tree \((T, \mathcal{P})\).

Proof. Let \((T, \mathcal{B})\) be a binary pear tree in \(G\), where \(\mathcal{B} = \{(P_v, Q_v) : v \in V(T)\}\). We prove the stronger result that there exist \(H\) and \((T, \{P'_v : v \in V(T)\})\) such that \(H\) is a minor of \(G\), \((T, \{P'_v : v \in V(T)\})\) is a binary ear tree in \(H\), and \(P'_v = P_v\) for all leaves \(v\) of \(T\). Arguing by contradiction, suppose that this is not true. Among all counterexamples, choose \((G, (T, \mathcal{B}))\) such that \(|E(G)|\) is minimum. This clearly implies that \(|V(T)| > 1\). For \(e \in E(G)\), let \(\mathcal{B}/e := \{(P_v/e, Q_v/e) : v \in V(T)\}\).

Let \(y\) and \(z\) be leaves of \(T\) with a common parent \(x\). Let \(I\) be the set of vertices that are an internal vertex of \(Q_y\) or of \(Q_z\). By (ii), \(V(Q_v) \cap I = \emptyset\) for all \(v \in V(T)\) \(\setminus \{y, z\}\). Therefore, if \(e \in E(Q_y) \cap E(Q_z)\), then \((G/e, \mathcal{B}/e)\) is a smaller counterexample. Thus, \(Q_y\) and \(Q_z\) are edge-disjoint. If \(e \in E(Q_y) \setminus E(P_y)\), then \((G/e, \mathcal{B}/e)\) is a smaller counterexample, unless \(e\) is a \(Q_z\)-ear or \(e\) has one end on \(P_x\) and the other end on \(Q_z\).

Suppose \(e \in E(Q_y) \setminus E(P_y)\) is a \(Q_z\)-ear. The unique cycle \(C_e\) in \(Q_z \cup \{e\}\) must contain an edge of \(P_z\). Otherwise, we obtain a smaller counterexample by deleting \(e\) from \(G\) and rerouting \(Q_y\) through \(C_e\). Since \(e\) is not incident to an internal vertex of \(P_z\), \(E(P_z) \subseteq E(C_e)\). Let \(P'_y\) be the unique path in \(P_y \cup Q_y \cup \{e\}\) containing \(e\) and having the same ends as \(P_y\). Since \(P_x\) and \(P'_x\) have the same ends, we can use \(Q_x\) to extend \(P'_y\) to a path \(Q'_y\) with the same ends as \(Q_y\). Observe that \(Q_y\) contains a \(P_y\)-ear \(Q'_y\) such that \(Q'_y\) has the same ends as \(e\) and \(P_z \subseteq Q'_y\). There is also a unique \(P_x\)-ear \(Q'_x\) in \(Q_y \cup P_x\) such that \(P'_y \subseteq Q'_x\). Let \(\mathcal{B}' := \{(P''_v, Q''_v) : v \in V(T)\}\), where \((P''_v, Q''_v) = (P_v, Q_v)\) if \(v \in V(T) \setminus \{x, y, z\}\), and \((P''_v, Q''_v) = (P'_v, Q'_v)\) if \(v \in \{x, y, z\}\). By (iii), at least one end of \(e\) is not an end of \(P_x\). Therefore, at least one end of \(Q'_y\) is not an end of \(P'_x\), so \((T, \mathcal{B}')\) is a pear tree in \(G\). However, clearly \(Q'_x \cup Q'_y \cup Q'_z\) must avoid some edge \(f\) of \(P_z\). Therefore, \((G' \setminus f, \mathcal{B}')\) is a smaller counterexample.

It follows that every edge \(e\) of \(E(Q_y) \setminus E(P_y)\) has one end on \(P_x\) and the other end on \(Q_z\). By symmetry, every edge \(f\) of \(E(Q_z) \setminus E(P_z)\) has one end on \(P_x\) and the other end on \(Q_y\). In particular, \(|E(Q_y) \setminus E(P_y)| \leq 2\) and \(|E(Q_z) \setminus E(P_z)| \leq 2\).

Suppose \(Q_y = P_y\) and \(Q_z = P_z\). Recall that \(I\) is the set of vertices that are an internal vertex of \(Q_y\) or of \(Q_z\). Let \(G^- = G - I\) and \(T^- = T - \{y, z\}\). By minimality, \(G^-\) has a minor \(H^-\) such that \(H^-\) has a binary ear tree \((T; \{P_v^- : v \in V(T^-)\})\) such that \(P_v^- = P_v\) for all leaves \(v\) of \(T^-\). Since \(x\) is a leaf of \(T^-\), we have \(P_x^- = P_x\). Let \(H^- = H^- \cup P_y \cup P_z\) and \((T; \{P_v^+ : v \in V(T)\})\) be such that \(P_v^+ = P_v^+\) for all
By symmetry, the last case is end on. By (iii), there must exist deleting $f$. May assume that disjoint. Thus, on $f$ have a common end not on $f$. Thus, on $f$ have a common end not on $f$. Thus.

If $e \in E(Q_y) \setminus E(P_y)$ and $E(Q_z) \setminus E(P_z) = \emptyset$, then $e$ has one end on $P_x$ and the other end on $Q_z$. Since $Q_z = P_z$, this contradicts (iii).

By symmetry, the last case is $|E(Q_y) \setminus E(P_y)| \in \{1, 2\}$ and $|E(Q_z) \setminus E(P_z)| \in \{1, 2\}$. By (iii), there must exist $e \in E(Q_y) \setminus E(P_y)$ and $f \in E(Q_z) \setminus E(P_z)$ such that $e$ and $f$ have a common end not on $P_x$. Let $x_e$ be the end of $e$ on $P_x$ and $x_f$ be the end of $f$ on $P_x$. If $x_e = x_f$, then $e = f$, which contradicts the fact that $Q_y$ and $Q_z$ are edge-disjoint. Thus, $x_e \neq x_f$. Since $e$ is not a $Q_z$-ear and $f$ is not a $Q_y$-ear, by symmetry we may assume that $x_e$ is not an end of $P_z$. We then obtain a smaller counterexample by deleting $f$ from $G$ and replacing $Q_z$ by $(Q_z \setminus \{f\}) \cup \{e\}$. 

5. Finding Binary Pear Trees

The main result of this section is the following.

5.1. For all integers $\ell \geq 1$ and $k \geq 9\ell^2 + 9\ell + 1$, if $G$ is a minor-minimal $2$-connected graph containing a subdivision of $\Gamma_k$ and $T^1$ is a binary tree of height at most $3\ell + 2$, then either $G$ contains $\Gamma_k^+ \P^\pm$ as a minor, or $G$ contains a binary pear tree $(T^1, B)$.

We proceed via a sequence of lemmas.

5.2. If $G$ is a minor-minimal $2$-connected graph containing a subdivision of $\Gamma_k$, then every subdivision of $\Gamma_k$ in $G$ is a spanning tree.

Proof. Let $T$ be a subdivision of $\Gamma_k$ in $G$. We use the well-known fact that for all $e \in E(G)$, at least one of $G\setminus e$ or $G/e$ is $2$-connected. Therefore, if some edge $e$ of $G$ has an end not in $V(T)$, then $G\setminus e$ or $G/e$ is a $2$-connected graph containing a subdivision of $\Gamma_k$, which contradicts the minor-minimality of $G$. 

For a vertex $v$ in a rooted tree $T$, let $T_v$ be the subtree of $T$ rooted at $v$.

5.3. Let $1 \leq \ell \leq k$ and let $T$ be a $\Gamma_k$ with root $r$. Suppose that a non-empty subset of vertices of $T$ are marked. Then

(i) $T$ contains a subdivision of $\Gamma_\ell$, all of whose leaves are marked, or
(ii) there exist a vertex $v \in V(T)$ and a child $w$ of $v$ such that $T_w$ has at least one marked vertex but $T_w$ has none, and $w$ is at distance at most $\ell$ from $r$.

Proof. A vertex $v$ in $T$ is good if there is a marked vertex in $T_v$, and is bad otherwise. Let $T'$ be the subtree of $T$ induced by vertices at distance at most $\ell$ from $r$ in $T$. If each leaf of $T'$ is good, then for each such leaf $u$ we can find a marked vertex $m_u$ in $T_u$,
and \( T' \cup \bigcup \{ uT_m : u \text{ leaf of } T' \} \) is a \( \Gamma_k \) subdivision with all leaves marked, as required by (i). Now assume that some leaf \( u \) of \( T' \) is bad. Let \( w \) be the bad vertex closest to \( r \) on the \( rTu \) path. Since some vertex in \( T \) is marked, \( r \) is good. Thus \( w \neq r \). Moreover, the parent \( v \) of \( w \) is good, by our choice of \( w \). Also, \( w \) is at distance at most \( \ell \) from \( r \). Therefore, \( v \) and \( w \) satisfy (ii). \( \square \)

Our main technical tools are 5.4 and 5.5 below, which are lemmas about 2-connected graphs \( G \) containing a subdivision \( T \) of \( \Gamma_k \) as a spanning tree. In order to state them, we need to introduce some definitions and notation that we will use in this setting. For each vertex \( v \in V(G) \), let \( h(v) \) be the number of original non-leaf vertices on the path \( vT_w \), where \( w \) is any leaf of \( T_v \). We stress the fact that subdivision vertices are not counted when computing \( h(v) \). We also use the shorthand notation \( \text{Out}(v) := V(G) \smallsetminus V(T_v) \) when \( G \) and \( T \) are clear from the context. For \( X, Y \subseteq V(G) \), we say that \( X \) sees \( Y \) if \( xy \in E(G) \) for some \( x \in X \) and \( y \in Y \). If \( P \) is a path with ends \( x \) and \( y \), and \( Q \) is a path with ends \( y \) and \( z \), then let \( PQ \) be the walk that follows \( P \) from \( x \) to \( y \) and then follows \( Q \) from \( y \) to \( z \).

A path \( P \) of \( G \) is \( (x, a, y) \)-special if \( |V(P)| \geq 3 \), and \( x, y \) are the ends of \( P \), and \( a \) is a child of \( x \) such that \( V(P) \smallsetminus \{x, y\} \subseteq V(T_a) \) and \( y \notin V(T_a) \). A vertex \( w \) is safe for an \( (x, a, y) \)-special path \( P \) if \( w \) satisfies the following properties:

- the parent \( v \) of \( w \) is in \( V(P) \smallsetminus \{x, y\} \);
- \( h(v) \geq h(x) - 2\ell \);
- \( V(P) \cap V(T_w) = \emptyset \);
- \( V(T_w) \) does not see \( \text{Out}(a) \smallsetminus \{x\} \), and
- if \( v \) is an original vertex and \( u \) is its child distinct from \( w \), then either \( V(P) \cap V(T_u) \neq \emptyset \) or \( V(T_u) \) does not see \( \text{Out}(a) \smallsetminus \{x\} \).

5.4. Let \( 1 \leq \ell \leq k \). Let \( G \) be a minor-minimal 2-connected graph containing a subdivision of \( \Gamma_k \). Let \( T \) be a subdivision of \( \Gamma_k \) in \( G \), \( v \in V(T) \) with \( h(v) \geq 3\ell + 1 \), and \( w \) be a child of \( v \). Then, either \( G \) contains a \( \Gamma_k \) minor, or there is a \((v_0, w_0, v'_0)\)-special path \( P \) and two distinct safe vertices for \( P \) such that:

- \( V(P) \subseteq V(T_w) \),
- \( h(v_0) \geq h(v) - \ell \),
- \( V(T_{v_0}) \) sees \( \text{Out}(w) \smallsetminus \{v\} \),
- \( V(T_{u_0}) \) does not see \( \text{Out}(w) \smallsetminus \{v\} \), and
- \( V(T_{u_0}) \) sees \( \text{Out}(v_0) \) if \( v_0 \) is an original vertex and \( u_0 \) is its child distinct from \( w_0 \).

Proof. By 5.2, \( T \) is a spanning tree of \( G \). Color red each vertex of \( T_w \) that sees a vertex in \( \text{Out}(w) \smallsetminus \{v\} \). Observe that there is at least one red vertex. Indeed, \( V(T_w) \) must see \( \text{Out}(w) \smallsetminus \{v\} \), for otherwise \( v \) would be a cut vertex separating \( V(T_w) \) from \( \text{Out}(w) \smallsetminus \{v\} \) in \( G \).

Let \( T_w \) be the complete binary tree obtained from \( T_w \) by iteratively contracting each edge of the form \( pq \) with \( p \) a subdivision vertex and \( q \) the child of \( p \) into vertex \( q \).
Declare $q$ to be colored red after the edge contraction if at least one of $p, q$ was colored red beforehand.

If $\tilde{T}_w$ contains a subdivision of $\Gamma_\ell$ with all leaves colored red, then so does $T_w$. Therefore, $G$ contains $\Gamma_\ell^+$ as a minor, because $\text{Out}(w)$ induces a connected subgraph of $G$ which is vertex-disjoint from $V(T_w)$ and which sees all the leaves of $T_w$. Thus, by 5.3, we may assume there is a vertex $\tilde{v}_0$ of $\tilde{T}_w$ and a child $\tilde{w}_0$ of $\tilde{v}_0$ with $h(\tilde{w}_0) \geq h(w) - \ell$ such that $T_{\tilde{v}_0}$ has at least one red vertex but $T_{\tilde{w}_0}$ has none. Going back to $T_w$, we deduce that there is a vertex $v_0$ of $T_w$ and a child $w_0$ of $v_0$ with $h(w_0) \geq h(w) - \ell$ such that $T_{v_0}$ has at least one red vertex but $T_{w_0}$ has none. To see this, choose $v_0$ as the deepest red vertex in the preimage of $\tilde{v}_0$. Note that $v_0$ or $w_0$ could be subdivision vertices.

If $v_0$ is an original vertex, let $u_0$ denote the child of $v_0$ distinct from $w_0$. Since $v_0$ is not a cut vertex of $G$, one of the two subtrees $T_{u_0}$ and $T_{w_0}$ sees $\text{Out}(v_0)$. If $T_{u_0}$ does not see $\text{Out}(v_0)$, then $T_{u_0}$ has no red vertex and $T_{w_0}$ sees $\text{Out}(v_0)$. Therefore, by exchanging $u_0$ and $w_0$ if necessary, we guarantee that the following two properties hold when $u_0$ exists.

\[
T_{u_0} \text{ sees } \text{Out}(v_0) \quad \text{ and } \quad T_{w_0} \text{ has no red vertex. (1)}
\]

We iterate this process in $T_{w_0}$. Color blue each vertex of $T_{u_0}$ that sees a vertex in $\text{Out}(w_0) \setminus \{v_0\}$. There is at least one blue vertex, since otherwise $v_0$ would be a cut vertex of $G$ separating $V(T_{w_0})$ from $\text{Out}(w_0) \setminus \{v_0\}$. Defining $\tilde{T}_{w_0}$ similarly as above, if $\tilde{T}_{w_0}$ contains a subdivision of $\Gamma_\ell$ with all leaves colored blue, then $G$ has a $\Gamma_\ell^+$ minor. Applying 5.3 and going back to $T_{w_0}$, we may assume there is a vertex $v_1$ of $T_{w_0}$ and a child $w_1$ of $v_1$ with $h(w_1) \geq h(w_0) - \ell$ such that $T_{v_1}$ has at least one blue vertex but $T_{w_1}$ has none.

We now define the $(v_0, w_0, v_0')$-special path $P$, and identify two distinct safe vertices for $P$. To do so, we will need to consider different cases. In all cases, the end $v_0'$ will be a vertex of $\text{Out}(w_0) \setminus \{v_0\}$ seen by a (carefully chosen) blue vertex in $T_{v_1}$, thus $v_0' \notin V(T_{w_0})$, and the path $P$ will be such that $V(P) \setminus \{v_0, v_0'\} \subseteq V(T_{w_0})$. Note that the end $v_0'$ of $P$ satisfies $h(v_0') \geq h(v) - \ell$, as desired.

Before proceeding with the case analysis, we point out the following property of $G$. If $st$ is an edge of $G$ such that $G/st$ contains a subdivision of $\Gamma_k$, then $G/st$ is not 2-connected by the minor-minimality of $G$, and it follows that $\{s, t\}$ is a cutset of $G$. Note that this applies if $st$ is an edge of $T$ such that at least one of $s, t$ is a subdivision vertex, or if $st$ is an edge of $E(G) \setminus E(T)$ linking two subdivision vertices of $T$ that are on the same subdivided path of $T$. This will be used below.

**Case 1.** $v_1$ is a subdivision vertex:
In this case, $v_1$ is the unique blue vertex in $T_{v_1}$. Let $v_0'$ be a vertex of $\text{Out}(w_0) \setminus \{v_0\}$ seen by the blue vertex $v_1$. Since $v_1$ is not a cut vertex of $G$, there is an edge $st$ with $s \in V(T_{w_1})$ and $t \in \text{Out}(v_1)$. Note that $t \in V(T_{w_0}) \cup \{v_0\}$, since $T_{w_1}$ has no blue vertex.

**Case 1.1.** There is at least one original vertex on the path $v_1Ts$:
Let $q$ be the first original vertex on the path $v_1Ts$. Let $s_1$ denote a child of $q$ not on
the $qT_s$ path. Let $q'$ be the first original vertex distinct from $q$ on the $qT_s$ path if any, and otherwise let $q' := s$ (note that possibly $q' = q = s$). Let $s_2$ be a child of $q'$ not on the $qT_s$ path, and distinct from $s_1$ if $q' = q$. As illustrated in Figure 9, define

$$P := v_0TtsTv_1v'_0.$$ 

Observe that $V(P) \setminus \{v_0, v'_0\} \subseteq V(T_{w_0})$, by construction. Observe also that the parent $q'$ of $s_2$ satisfies $h(q') \geq h(q) - 1 = h(v_1) - 1 \geq h(v_0) - \ell - 1 \geq h(v_0) - 2\ell$. It can be checked that $s_1, s_2$ are two distinct safe vertices for $P$, as desired.

**Case 1.2.** All vertices of the path $v_1T_s$ are subdivision vertices:

In particular, $w_1$ is a subdivision vertex. We show that the unique child $q$ of $w_1$ is an original vertex, and therefore $s = w_1$. Indeed, assume not, and let $q'$ denote its child. Since $v_1$ is not a cut vertex of $G$ but $\{v_1, w_1\}$ is a cutset of $G$, we deduce that $w_1$ sees a vertex $w'_1$ in $\text{Out}(v_1)$ and that $V(T_w)$ does not see $\text{Out}(v_1)$. Similarly, because $w_1$ is not a cut vertex of $G$ but $\{w_1, q\}$ is a cutset of $G$, we deduce that $wq_1 \in E(G)$ and that $V(T_{q'})$ does not see $\text{Out}(w_1)$. Since $q$ is not a cut vertex, some vertex $q'' \in V(T_{q'})$ sees $\text{Out}(q)$, and hence sees $w_1$ (since $V(T_{q'})$ does not see $\text{Out}(v_1)$). But then, because of the edges $q''w_1$ and $w_1w'_1$, we see that $\{v_1, q\}$ cannot be a cutset of $G$. It follows that $G/v_1q$ is 2-connected and contains a $\Gamma_k$ minor, contradicting our assumption on $G$.

Hence, $q$ is an original vertex, and $s = w_1$. Since $w_1$ is not a cut vertex of $G$, there is an edge linking $V(T_q)$ to $\text{Out}(w_1)$. Since $\{v_1, w_1\}$ is a cutset of $G$, this edge links some vertex $s' \in V(T_q)$ to $v_1$.

Let $s_1$ denote a child of $q$ not on the $qT_s'$ path. Let $q'$ be the first original vertex distinct from $q$ on the $qT_s'$ path if any, and otherwise let $q' := s'$ (note that possibly $q' = s' = q$). Let $s_2$ be a child of $q'$ not on the $qT_s'$ path, and distinct from $s_1$ if $q' = q$. As illustrated in Figure 9, define

$$P := v_0Ttw_1T_s'v_1v'_0.$$ 

Again, note that $V(P) \setminus \{v_0, v'_0\} \subseteq V(T_{w_0})$ by construction. Observe also that the parent $q'$ of $s_2$ satisfies $h(q') \geq h(q) - 1 = h(v_1) - 1 \geq h(v_0) - \ell - 1 \geq h(v_0) - 2\ell$. It is easy to see that $s_1, s_2$ are two distinct safe vertices for $P$, as desired.

**Case 2.** $v_1$ is an original vertex:

Let $u_1$ denote the child of $v_1$ distinct from $w_1$. If $T_{u_1}$ has no blue vertex, then $v_1$ is the unique blue vertex in $T_{v_1}$. Let $v'_0$ be a vertex of $\text{Out}(w_0) \setminus \{v_0\}$ seen by the blue vertex $v_1$. Define

$$P := v_0Tv_1v'_0.$$ 

Clearly, $V(P) \setminus \{v_0, v'_0\} \subseteq V(T_{w_0})$, and $u_1, w_1$ are two distinct safe vertices for $P$.

Next, assume that $T_{u_1}$ has a blue vertex. In this case, we need to define an extra pair $v_2, w_2$ of vertices. Observe that $h(u_1) \geq h(w_0) - \ell \geq h(w) - 2\ell = h(v) - 2\ell - 1 \geq \ell$. Let $\tilde{T}_{u_1}$ be the tree obtained from $T_{u_1}$, as before. Again, if $\tilde{T}_{u_1}$ contains a subdivision of $\Gamma_\ell$ all of whose leaves are blue, then $G$ contains an $\Gamma^+_{\ell}$ minor. Thus, by 5.3, we may assume there is a vertex $v_2$ of $T_{u_1}$ and a child $w_2$ of $v_2$ with $h(w_2) \geq h(u_1) - \ell = h(w_1) - \ell$ such that $T_{v_2}$ has at least one blue vertex, but $T_{w_2}$ has none.
Case 2.1. $v_2$ is a subdivision vertex:
Here, $v_2$ is the unique blue vertex in $T_{v_2}$. Let $v'_0$ be a vertex of $\text{Out}(w_0) \setminus \{v_0\}$ seen by $v_2$. As illustrated in Figure 10, define
\[ P := v_0Tv_2v'_0. \]
Observe that $V(P) \setminus \{v_0, v'_0\} \subseteq V(T_{w_0})$ by construction, and that $w_1, w_2$ are two distinct safe vertices for $P$.

Case 2.2. $v_2$ is an original vertex:
Let $u_2$ be the child of $v_2$ distinct from $w_2$. Let $b_2$ denote a blue vertex in $V(T_{u_2}) \cup \{v_2\}$, distinct from $v_2$ if possible. Let $v'_0$ be a vertex of $\text{Out}(w_0) \setminus \{v_0\}$ seen by the blue vertex $b_2$. Define
\[ P := v_0Tb_2v'_0. \]
Again, $V(P) \setminus \{v_0, v'_0\} \subseteq V(T_{w_0})$ by construction.

If $b_2 \neq v_2$, then $P$ intersects $V(T_{u_2})$. If $b_2 = v_2$, then $P$ avoids $V(T_{u_2})$, and $V(T_{u_2})$ has no blue vertex. That is, $V(T_{u_2})$ does not see $\text{Out}(w_0) \setminus \{v_0\}$. Using these observations, one can check that $w_1, w_2$ are two distinct safe vertices for $P$ in both cases; see Figure 10. \qed

5.5. Let $1 \leq \ell \leq k$. Let $G$ be a minor-minimal 2-connected graph containing a subdivision of $\Gamma_k$ and let $T$ be a subdivision of $\Gamma_k$ in $G$. Let $S$ be an $(x, a, y)$-special path with $h(x) \geq 5\ell + 1$. Let $w$ be a safe vertex for $S$ and let $v \in V(S)$ denote the parent of $w$ in $T$. Then, either $G$ contains a $\Gamma_\ell^+$ minor, or there is a $(v_0, w_0, v'_0)$-special path $P$, two distinct safe vertices $w_1, w_2$ for $P$, and an $S$-ear $Q$ such that:

(a) $V(P) \subseteq V(T_w)$,
(b) $h(v_0) \geq h(x) - 3\ell$,
(c) $V(T_{w_0})$ does not see $\text{Out}(w) \setminus \{v\}$,
(d) $P \subseteq Q$. 

Figure 9. Path $P$ and the safe vertices $s_1, s_2$. Cases 1.1 and 1.2
Proof. By 5.2, we are done. Applying 5.4 on vertex \( P \) path \( V \)
vertex. It remains to extend \( S \) st \( v \), since \( w \) is a safe vertex for \( u \) is the child of \( v \) distinct from \( w \) if \( v \) is an original vertex with children \( u \), \( w \), the path \( S \) is disjoint from \( V(T_a) \), and \( e \) links \( V(T_a) \) to \( \text{Out}(v) \).

\( V(Q) \setminus V(P) \subseteq \text{Out}(w_0) \setminus \{v_0\} \),
\( V(Q) \subseteq V(T_a) \cup \{x\} \),
\( V(Q) \cap V(T_{w_i}) = \emptyset \) for \( i = 1, 2 \), and
\( \text{Out}(Q) \setminus E(T) \) and no end of \( e \) is in \( V(T_w) \), then \( v \) is an original vertex with children \( u \), \( w \), the path \( S \) is disjoint from \( V(T_a) \), and \( e \) links \( V(T_a) \) to \( \text{Out}(v) \).

First assume that \( v'_0 \notin V(T_{w_0}) \). Then \( v'_0 \in \text{Out}(w_0) \cap V(T_{w_0}) \). Recall that \( V(T_{v_0}) \setminus V(T_{w_0}) = V(T_{w_0}) \cup \{v_0\} \) sees \( \text{Out}(w) \setminus \{v\} = V(T_a) \cup \text{Out}(v) \). Suppose that there is an edge \( st \in E(G) \) with \( s \in V(T_{w_0}) \cup \{v_0\} \) and \( t \in \text{Out}(v) \). Note that \( t \in V(T_a) \cup \{x\} \), since \( w \) is a safe vertex for \( S \). Let \( v' \) be the closest ancestor of \( t \) in \( T \) that lies on \( S \). Note that \( v' \in V(T_a) \cup \{x\} \). Define

\[ Q_1 := vt v'_0 P v_0 T s t v' \]

Next, suppose that there is no such edge \( st \). Then, there must be an edge \( st \) with \( s \in V(T_{w_0}) \cup \{v_0\} \) and \( t \in V(T_w) \). In particular, \( u \) exists. If the path \( S \) intersects \( V(T_a) \), then let \( v' \) be a vertex in \( V(S) \cap V(T_a) \) that is closest to \( t \) in \( T \). Define

\[ Q_2 := vt v'_0 P v_0 T s t v' \].
Otherwise, we have $V(S) \cap V(T_u) = \emptyset$. Since $w$ is a safe vertex for $S$, $V(T_u)$ does not see \text{Out}(a) \setminus \{x\}$ in this case. If $V(T_u)$ sees $\text{Out}(v)$, then let $s't'$ be an edge with $s' \in V(T_u)$
and $t' \in \text{Out}(v)$, and let $v'$ be the closest ancestor of $t'$ in $T$ that lies on $S$. Note that both $t'$ and $v'$ lie in $V(T_a) \cup \{x\}$. Define

$$Q_3 := vTv'_0Pv_0Ts't'Tv'.$$

Otherwise, $V(T_u)$ does not see $\text{Out}(v)$. Since $v$ is not a cut vertex in $G$, we deduce that $V(T_u)$ sees $\text{Out}(v)$. As we already know that neither $V(T_{w_0})$ nor $V(T_{w_0}) \cup \{v_0\}$ sees $\text{Out}(v)$, there is an edge $s''t'' \in E(G)$ with $s'' \in V(T_w) \setminus V(T_{w_0})$ and $t'' \in \text{Out}(v)$. Again, since $w$ is safe for $S$, we know that $t'' \in V(T_a) \cup \{x\}$. Let $v'$ be the closest ancestor of $t''$ in $T$ that lies on $S$. Note that $v' \in V(T_a) \cup \{x\}$. Define

$$Q_4 := vTsTv_0Pv'_0Tst's't'Tv'.$$

Next, suppose that $V(T_{u_0})$ does not see $\text{Out}(v)$. If $V(T_{u_0})$ sees $V(T_u)$, then let $st$ be an edge with $s \in V(T_{u_0})$ and $t \in \text{Out}(v)$. Observe that $t \in V(T_a) \cup \{x\}$ since $w$ is safe for $S$. Let $v'$ be the closest ancestor of $t$ in $T$ that lies on $S$. Note that $v' \in V(T_a) \cup \{x\}$ as well. Define

$$Q_5 := vTv_0Pv'_0TstTv'.$$

Next, suppose that $V(T_{u_0})$ does not see $\text{Out}(v)$. Since $v$ is not a cut vertex in $G$, we deduce that $V(T_{u_0})$ sees $\text{Out}(v)$. As we already know that neither $V(T_{w_0})$ nor $V(T_{w_0})$ sees $\text{Out}(v)$, there is an edge $s''t'' \in E(G)$ with $s'' \in V(T_u) \setminus V(T_{w_0}) \cup \{v_0\}$ and $t'' \in \text{Out}(v)$. Again, since $w$ is safe for $S$, $t'' \in V(T_a) \cup \{x\}$. Let $v'$ be the closest ancestor of $t''$ in $T$ that lies on $S$. Note that both $t'$ and $v'$ lie in $V(T_a) \cup \{x\}$. Define

$$Q_6 := vTv_0Pv'_0TstTs't'Tv'.$$

Next, suppose that $V(T_{u_0})$ does not see $\text{Out}(v)$. Since $v$ is not a cut vertex in $G$, we deduce that $V(T_{u_0})$ sees $\text{Out}(v)$. As we already know that neither $V(T_{w_0})$ nor $V(T_{w_0})$ sees $\text{Out}(v)$, there is an edge $s''t'' \in E(G)$ with $s'' \in (V(T_u) \setminus V(T_{w_0})) \cup \{v_0\}$ and $t'' \in \text{Out}(v)$. Again, since $w$ is safe for $S$, $t'' \in V(T_a) \cup \{x\}$. Let $v'$ be the closest ancestor of $t''$ in $T$ that lies on $S$. Note that $v' \in V(T_a) \cup \{x\}$. Define

$$Q_7 := vTv_0Pv'_0TstTs't'Tv'.$$

We are done with the cases where $V(T_{u_0})$ sees $\text{Out}(v)$ or $V(T_u)$. Next, assume that $V(T_{u_0})$ sees neither of these two sets. Since $V(T_{u_0})$ sees $\text{Out}(v_0)$, there is an edge $st$ with $s \in V(T_{u_0})$ and $t \in V(T_u) \setminus V(T_{w_0})$. Recall that $V(T_{u_0})$ sees $\text{Out}(w) \setminus \{v\}$. Since neither $V(T_{u_0})$ nor $V(T_{u_0})$ sees $\text{Out}(w) \setminus \{v\}$, we conclude that $v_0$ sees $\text{Out}(w) \setminus \{v\}$. If $v_0$ sees $\text{Out}(v)$, then let $v_0t'$ be an edge with $t' \in \text{Out}(v)$. Let $v'$ be the closest ancestor of $t'$ in $T$. As before, $\{t', v'\} \subseteq V(T_a) \cup \{x\}$. Define

$$Q_8 := vTstTv_0Pv_0Ts't'Tv'.$$

Otherwise, $v_0$ sees $V(T_u)$. Let $v_0t'$ be an edge with $t' \in V(T_u)$. If $S$ intersects $V(T_u)$, then let $v'$ be a vertex in $V(S) \cap V(T_a)$ that is closest to $t'$ in $T$. Define

$$Q_9 := vTstTv_0Pv_0t'Tv'.$$
Otherwise, $V(S) \cap V(T_u) = \emptyset$. Since $w$ is a safe vertex for $S$, we know that $V(T_u)$ does not see $\text{Out}(a) \setminus \{x\}$ in this case. If $V(T_u)$ sees $\text{Out}(v)$, then let $s''t''$ be an edge with $s'' \in V(T_u)$ and $t'' \in \text{Out}(v)$ and let $v'$ be the closest ancestor of $t''$ in $T$ that lies on $S$. Note that both $t''$ and $v'$ lie in $V(T_u) \cup \{x\}$. Define 

$$Q_{11} := vtstT_vT'_0Pt'v't''Tv'.$$

Otherwise, $V(T_u)$ does not see $\text{Out}(v)$. Since $v$ is not a cut vertex in $G$, we deduce that $V(T_w)$ sees $\text{Out}(v)$. As we already know that neither $V(T_{u_0})$ nor $V(T_{v_0}) \cup \{v_0\}$ sees $\text{Out}(v)$, there is an edge $s''t'' \in E(G)$ with $s'' \in V(T_w) \setminus V(T_{v_0})$ and $t'' \in \text{Out}(v)$. Again, since $w$ is safe for $S$, $t'' \in V(T_u) \cup \{x\}$. Let $v'$ be the closest ancestor of $t''$ in $T$ that lies on $S$. Note that $v' \in V(T_a) \cup \{x\}$. Define

$$Q_{12} := vtT_v'Pv'_0TstT_sT''Tv'.$$

One can check that for all $i \in [12]$, if we set $Q = Q_i$, then $Q$ is an $S$-ear satisfying properties (d)–(h).

We now prove 5.1 using 5.4 and 5.5.

**Proof of 5.1.** Let $T$ be a subdivision of $\Gamma_k$ in $G$, which is a spanning tree of $G$ by 5.2. Also, $G$ has no $\Gamma_k^+$ minor (otherwise, we are done). As before, for $v \in V(G)$, we use $h(v)$ to denote the height of $v$ with respect to $T$. The depth of $x \in V(T^1)$, denoted $d(x)$, is the number of edges in $xT^1r$, where $r$ is the root of $T^1$.

We prove the stronger statement that $G$ contains a binary pear tree $(T^1, \{(P_x, Q_x) : x \in V(T^1)\})$ such that:

1. for all $x \in V(T^1)$, the path $P_x$ is a $(v_x, w_x, v'_x)$-special path having two distinguished safe vertices and such that $h(v_x) \geq k - 3d(x) - \ell$; for all $x, y, z \in V(T^1)$ such that $y$ and $z$ are the children of $x$, the two distinguished safe vertices of $P_x$ are denoted $s_{xyz}$ and $s_{xzx}$;
2. for all $x, y \in V(T^1)$, $v_x$ is an ancestor of $v_y$ in $T$ if and only if $x$ is an ancestor of $y$ in $T^1$;
3. for all $x, y \in V(T^1)$ such that $y$ is a child of $x$, the paths $P_y$ and $Q_y$ are obtained by applying 5.5 on $P_x$ with safe vertex $s_{xyz}$;
4. for all $y, z \in V(T^1)$ such that $y$ and $z$ are siblings, no vertex of $Q_z$ meets $T_{w_y}$, and no vertex of $Q_y$ meets $T_{w_z}$;
5. for all leaves $x$ of $T^1$, $V(T_{w_x})$ and $\bigcup_{p \in V(T^1) \setminus \{x\}} V(Q_p)$ are disjoint.

The proof is by induction on $|V(T^1)|$. For the base case $|V(T^1)| = 1$, the tree $T^1$ is a single vertex $x$. Applying 5.4 with $v$ the root of $T$ and $w$ a child of $v$ in $T$, we obtain a $(v_x, w_x, v'_x)$-special path $P_x$ and two distinct safe vertices for $P_x$. Let $Q_x := P_x$. Then $(T^1, \{(P_x, Q_x)\})$ is a binary pear tree in $G$. Observe that $d(x) = 0$ and $h(v_x) \geq h(v) - \ell = k - \ell$, thus (1) holds. Properties (2)–(5) hold vacuously since $x$ is the only vertex of $T^1$. 


Next, for the inductive case, assume $|V(T^1)| > 1$. Let $x$ be a vertex of $T^1$ with two children $y, z$ that are leaves of $T^1$. Applying induction on the binary tree $T^1 - \{y, z\}$, we obtain a binary pear tree $(T^1 - \{y, z\}, \{(P_p, Q_p) : p \in V(T^1 - \{y, z\})\})$ in $G$ satisfying the claim.

Note that $d(x) \leq 3\ell + 1$, and thus $h(v_x) \geq k - 3\ell d(x) - \ell \geq (9\ell^2 + 9\ell + 1) - 3\ell(3\ell + 1) - \ell \geq 5\ell + 1$. Let $s_{xy}$ and $s_{xz}$ denote the two distinguished safe vertices of $P_x$. Let $v_{xy}$ and $v_{xz}$ denote their respective parents in $T$. First, apply 5.5 with the path $P_x$ and safe vertex $s_{xy}$, giving a $(v_y, w_y, v'_y)$-special path $P_y$ with two distinct safe vertices, and a $P_x$-ear $Q_y$ satisfying the properties of 5.5. Next, apply 5.5 with the path $P_x$ and safe vertex $s_{xz}$, giving a $(v_z, w_z, v'_z)$-special path $P_z$ with two distinct safe vertices, and a $P_x$-ear $Q_z$ satisfying the properties of 5.5.

Observe that, by property (b) of 5.5, $h(v_y) \geq h(v_z) - 3\ell \geq k - 3\ell d(x) - 4\ell = k - 3\ell d(y) - \ell$, and similarly $h(v_z) \geq k - 3\ell d(z) - \ell$. Thus, property (1) is satisfied. Clearly, property (2) and property (3) are satisfied as well. To establish property (4), it only remains to show that no vertex of $Q_z$ meets $T_{w_y}$, and that no vertex of $Q_y$ meets $T_{w_z}$. By symmetry it is enough to show the former, which we do now.

Arguing by contradiction, assume that $Q_z$ meets $T_{w_y}$. Since $V(T_{w_y}) \subseteq V(T_{s_{xy}})$ and $V(Q_z) \cap V(T_{s_{xy}}) = \emptyset$ (by property (g) of 5.5), and since the two ends of $Q_z$ are on $Q_z$, we see that the two ends of $Q_z$ are outside $V(T_{w_y})$. Thus, at least two edges of $Q_z$ have exactly one end in $V(T_{w_y})$, and there is an edge $st$ which is not an edge of $T$ (i.e. $st \neq v_yw_y$). By symmetry, $s \in V(T_{w_y})$ and $t \notin V(T_{w_y})$.

Clearly, $s \notin V(T_{s_{xz}})$ since $V(T_{w_y}) \subseteq V(T_{s_{xy}})$, and $V(T_{s_{xy}}) \cap V(T_{s_{xz}}) = \emptyset$. Moreover, $t \notin V(T_{s_{xz}})$, since $V(T_{s_{xz}}) \subseteq \text{Out}(s_{xy}) \setminus \{v_{xy}\}$ and since $V(T_{w_y})$ does not see $\text{Out}(s_{xy}) \setminus \{v_{xy}\}$ by property (c) of 5.5. Since $st$ is an edge of $Q_z$ not in $T$ with neither of its ends in $V(T_{s_{xz}})$, it follows from property (h) of 5.5 that $v_{xz}$ is an original vertex with children $u_{xz}$ and $s_{xz}$; the path $P_x$ avoids $V(T_{u_{xz}})$; and the edge $st$ has one end in $V(T_{s_{xz}})$ and the other in $\text{Out}(v_{xz})$. (We remark that we do not know which end is in which set at this point.)

First, suppose $s_{xy} = u_{xz}$. Then $v_{xy} = v_{xz}$. Since $s \in V(T_{w_y}) \subseteq V(T_{s_{xy}})$ and $s_{xy} = u_{xz}$, we deduce that $s \in V(T_{u_{xz}})$ and $t \in \text{Out}(v_{xz})$ in this case. However, $V(T_{w_y})$ does not see $\text{Out}(s_{xy}) \setminus \{v_{xy}\}$ (by property (c) of 5.5), and $t \in \text{Out}(v_{xz}) \subseteq \text{Out}(u_{xz}) \setminus \{v_{xz}\} = \text{Out}(s_{xy}) \setminus \{v_{xy}\}$, a contradiction.

Next, assume that $s_{xy} \neq u_{xz}$. Then $s_{xy} \notin V(T_{u_{xz}})$, because the parent $v_{xz}$ of $s_{xy}$ is on the path $P_x$, and $P_x$ avoids $V(T_{u_{xz}})$. Since $s_{xy} \notin V(T_{s_{xz}})$ and $s_{xy} \neq v_{xz}$, it follows that $s_{xy} \notin \text{Out}(v_{xz})$. Since $s \in V(T_{w_y}) \subseteq V(T_{s_{xy}})$ and since $s_{xy}$ is not an ancestor of $v_{xz}$ (otherwise $V(T_{s_{xy}})$ would contain $v_{xz}$, which is on the path $P_x$), we deduce that $V(T_{s_{xy}}) \subseteq \text{Out}(v_{xz})$, and thus $s \in \text{Out}(v_{xz})$. It then follows that $t \in V(T_{u_{xz}})$. Observe that $u_{xz}$ is neither an ancestor of $v_{xz}$ (otherwise $V(T_{u_{xz}})$ would contain $v_{xz}$, which is on the path $P_x$), nor a descendent of $s_{xy}$ (otherwise $V(T_{s_{xy}})$ would contain $v_{xz}$ since $u_{xz} \neq s_{xy}$, which is a vertex of $P_x$). Hence, we deduce that $V(T_{u_{xz}}) \subseteq \text{Out}(s_{xy}) \setminus \{v_{xy}\}$. However, the edge $st$ then contradicts the fact that $V(T_{w_y})$ does not see $\text{Out}(s_{xy}) \setminus \{v_{xy}\}$.
(c.f. property (c) of 5.5). Therefore, \( V(Q_z) \cap V(T_{w_y}) = \emptyset \), as claimed. Property (4) follows.

We now verify property (5). First, we show (5) holds for the leaf \( y \) of \( T^1 \). Note that \( V(T_{w_y}) \subseteq V(T_{s_{xy}}) \subseteq V(T_{w_y}) \). Thus, \( V(T_{w_y}) \) and \( \bigcup_{p \in V(T^1) \setminus \{x,y,z\}} V(Q_p) \) are disjoint by induction and property (5) for the leaf \( x \) of \( T^1 - \{y, z\} \). Since \( V(T_{w_y}) \subseteq V(T_{s_{xy}}) \) and \( V(T_{s_{xy}}) \cap V(Q_x) = \emptyset \) (by property (g) of 5.5), we deduce that \( V(T_{w_y}) \cap V(Q_x) = \emptyset \). Moreover, \( V(T_{w_y}) \cap V(Q_z) = \emptyset \), by property (4) shown above. This proves property (5) for the leaf \( y \) of \( T^1 \), and also for the leaf \( z \) by symmetry.

Every other leaf \( q \) of \( T^1 \) is also a leaf in \( T^1 - \{y, z\} \). By induction, \( V(T_{w_q}) \) and \( \bigcup_{p \in V(T^1) \setminus \{q,y,z\}} V(Q_p) \) are disjoint. Moreover, \( V(T_{w_q}) \) and \( V(T_{v_x}) \) are disjoint, by property (2). Since \( V(Q_y) \) and \( V(Q_z) \) are contained in \( V(T_{v_x}) \) (by property (f) of 5.5) and \( V(T_{w_y}) \subseteq V(T_{v_x}) \), it follows that \( V(T_{w_q}) \) and \( V(Q_y) \cup V(Q_z) \) are also disjoint. Property (5) follows.

To conclude the proof, it only remains to verify that \( (T^1, \{(P_p, Q_p) : p \in V(T^1)\}) \) is a binary pear tree in \( G \). Property (i) of the definition of binary pear trees is clearly satisfied.

We now show that property (ii) holds. Using induction and symmetry, it is enough to prove the following two properties.

(A) For every \( p \in V(T^1) \), if \( Q_y \) contains an internal vertex of \( Q_p \), then \( y \) is a descendent of \( p \) or of the sibling of \( p \).

(B) For every \( p \in V(T^1) \), if \( Q_p \) contains an internal vertex of \( Q_y \), then \( p \in \{y, z\} \).

First, we show (A). Suppose that \( Q_y \) contains an internal vertex of \( Q_p \) for some \( p \in V(T^1) \). If \( p \in \{x, y, z\} \) we are done, thus assume \( p \notin \{x, y, z\} \). Note that \( V(Q_y) \subseteq V(T_{w_y}) \cup \{v_x\} \), by property (f) of 5.5. By induction, \( V(T_{w_y}) \) is disjoint from \( \bigcup_{q \in V(T^1) \setminus \{x,y,z\}} V(Q_q) \), by property (5) for the leaf \( x \) of \( T^1 - \{y, z\} \). Thus, the only vertex that the paths \( Q_y \) and \( Q_p \) can have in common is \( v_x \). In particular, \( v_x \) is an internal vertex of \( Q_p \). Now, since \( v_x \in V(Q_x) \), we see that \( Q_x \) also contains an internal vertex of \( Q_y \), namely \( v_x \). Since property (ii) holds for \( x \) and \( p \) (by induction), it follows that \( x \) is a descendent of \( p \) or of the sibling of \( p \). Since \( y \) is a child of \( x \), the same holds for \( y \), as desired. This proves (A).

Next, we show (B). Suppose that \( Q_p \) contains an internal vertex of \( Q_y \), for some \( p \in V(T^1) \). Recall that \( V(Q_y) \subseteq V(T_{w_y}) \cup \{v_x\} \). Note also that \( v_x \) cannot be an internal vertex of \( Q_y \), since \( v_x \in V(P_z) \) and \( Q_y \) is a \( P_z \)-ear. Hence, all internal vertices of \( Q_y \) are in \( V(T_{w_y}) \). Since \( V(T_{w_y}) \) and \( V(Q_q) \) are disjoint for all \( q \in V(T^1) \setminus \{x,y,z\} \) (by induction, using property (3) on the leaf \( x \) of \( T^1 - \{y, z\} \)), it follows that \( p \in \{x, y, z\} \). Furthermore, \( p \neq x \) because \( Q_y \) is a \( P_z \)-ear, and \( V(Q_x) \setminus V(P_x) \subseteq \text{Out}(w_x) \setminus \{v_x\} \) (by property (e) of 5.5). Hence, \( p \in \{y, z\} \), as desired. This shows (B).
Finally, to verify property (iii), it only remains to verify (iii) for \( x \) and its two children \( y, z \). The paths \( Q_y \) and \( Q_z \) are \( P_x \)-ears, by construction. Moreover, no internal vertex of \( P_y \) meets \( Q_z \) since \( V(Q_z) \cap V(T_{w_y}) = \emptyset \) by (4), and similarly no internal vertex of \( P_z \) meets \( Q_y \). Finally, the end \( v'_x \) of \( P_x \) is not an end of \( Q_y \), since \( V(Q_y) \subseteq V(T_{w_y}) \cup \{v_x\} \) (by property (f) of 5.5), and since \( v'_x \notin V(T_{w_y}) \cup \{v_x\} \). By symmetry, \( v'_x \) is not an end of \( Q_z \), either. Therefore, property (iii) is satisfied. This concludes the proof that \((T^1, \{(P_p, Q_p) : p \in V(T^1)\})\) is a binary pear tree in \( G \).

\[ \square \]

6. Proof of Main Theorems

We have the following quantitative version of 1.4.

**6.1.** For all integers \( \ell \geq 1 \) and \( k \geq 9\ell^2 + 9\ell + 1 \), every 2-connected graph \( G \) with a \( \Gamma_k \) minor contains \( \Gamma_\ell^+ \) or \( \nabla_\ell \) as a minor.

**Proof.** Among all 2-connected graphs containing \( \Gamma_k \) as a minor, but containing neither \( \Gamma_\ell^+ \) nor \( \nabla_\ell \) as a minor, choose \( G \) with \( |E(G)| \) minimum. Since \( \Gamma_k \) has maximum degree 3, \( G \) contains a subdivision of \( \Gamma_k \). Therefore, \( G \) is a minor-minimal 2-connected graph containing a subdivision of \( \Gamma_k \). By 5.1, \( G \) has a binary pear tree \((T^1, \mathcal{B})\), with \( T^1 \simeq \Gamma_{3\ell+2} \). By 4.1, \( G \) has a minor \( H \) such that \( H \) has a binary ear tree \((T^1, \mathcal{P})\), with \( T^1 \simeq \Gamma_{3\ell+2} \). By 3.1, \( H \) contains \( \Gamma_\ell^+ \) or \( \nabla_\ell \) as a minor, and hence so does \( G \).

\[ \square \]

We have the following quantitative version of 1.3.

**6.2.** For every integer \( \ell \geq 1 \), every 2-connected graph \( G \) of pathwidth at least \( 2^{9\ell^2+9\ell+2} - 2 \) contains \( \Gamma_\ell^+ \) or \( \nabla_\ell \) as a minor.

**Proof.** As mentioned in Section 1, Bienstock et al. [1] proved that for every forest \( F \), every graph with pathwidth at least \( |V(F)| \) contains \( F \) as a minor. Let \( k := 9\ell^2 + 9\ell + 1 \). Note that \( |V(\Gamma_k)| = 2^{k+1} - 1 \). By assumption, \( G \) has pathwidth at least \( 2^{k+1} - 2 \). Thus \( G \) contains \( \Gamma_k \) as a minor. The result follows from 6.1.

\[ \square \]

Finally, we have the following quantitative version of 1.2.

**6.3.** For every apex-forest \( H_1 \) and outerplanar graph \( H_2 \), if \( \ell := \max\{|V(H_1)|, |V(H_2)|, 2\} - 1 \) then every 2-connected graph \( G \) of pathwidth at least \( 2^{9\ell^2+9\ell+2} - 2 \) contains \( H_1 \) or \( H_2 \) as a minor.

**Proof.** By 6.2, \( G \) contains \( \Gamma_\ell^+ \) or \( \nabla_\ell \) as a minor. In the first case, by 2.2, \( H_1 \) is a minor of \( \Gamma_{|V(H_1)|-1}^+ \) and thus of \( G \) (since \( \ell \geq |V(H_1)| - 1 \)). In the second case, by 2.4, \( H_2 \) is a minor of \( \nabla_{|V(H_2)|-1} \) and thus of \( G \) (since \( \ell \geq |V(H_2)| - 1 \)).

\[ \square \]

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