Hamiltonian reductions of free particles under polar actions of compact Lie groups

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Abstract

Classical and quantum Hamiltonian reductions of free geodesic systems of complete Riemannian manifolds are investigated. The reduced systems are described under the assumption that the underlying compact symmetry group acts in a polar manner in the sense that there exist regularly embedded, closed, connected submanifolds meeting all orbits orthogonally in the configuration space. Hyperpolar actions on Lie groups and on symmetric spaces lead to families of integrable systems of spin Calogero-Sutherland type.
1 Introduction

In the theory of integrable systems one of the basic facts is that many interesting models arise as Hamiltonian reductions of certain canonical ‘free’ systems that can be integrated ‘obviously’ due to their large symmetries. For example, the celebrated Sutherland model of interacting particles on the line, defined classically by the Hamiltonian

$$\mathcal{H}(q,p) = \frac{1}{2} \sum_{k=1}^{n} p_k^2 + \nu(\nu - 1) \sum_{1 \leq i < j \leq n} \frac{1}{\sin^2(q_i - q_j)}, \quad (1.1)$$

can be viewed as a reduction of the canonical geodesic system on the group $SU(n)$. The underlying symmetry is given by the conjugation action of the group on itself, and the model (1.1) results (in the center of mass frame) by fixing the Noether charges of this symmetry in a very special manner. The Hamiltonian reduction can be performed both at the classical [1] and at the quantum mechanical [2] level. For reasons of representation theory, in the latter case one obtains the model with integer values of the coupling constant $\nu$. The derivation of the Sutherland model (1.1) by Hamiltonian reduction has been generalized to obtain other integrable models of (spin) Calogero-Moser-Sutherland type [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13].

The most direct generalizations [6, 7] result by replacing $SU(n)$ with an arbitrary connected, compact simple Lie group, and fixing the Noether charges arbitrarily. Then the reductions relying on the conjugation symmetry lead to spin Sutherland type models with interaction potentials determined by the corresponding root system. Classically, the phase spaces of these models contain additional (‘spin’) degrees of freedom besides the cotangent bundle of the reduced configuration space representing particle coordinates. Quantum mechanically, the Hamiltonian acts on multicomponent wave functions as opposed to the scalar wave functions of the spinless Sutherland type models. These wave functions correspond to vector valued spherical functions on the symmetry group, much studied in harmonic analysis.

Other generalizations start from geodesics systems on symmetric spaces. The pioneering contributions in this line of research are due to Olshanetsky ad Perelomov [3, 4]. For recent developments, see e.g. [11, 12, 13] and references therein.

In the examples alluded to the starting point is always a connected, complete Riemannian manifold equipped with an isometric action of a compact symmetry group that admits flat sections. A section, in the sense of Palais and Terng [14], for an isometric action of a group $G$ is a regularly embedded, closed, connected submanifold that meets each $G$-orbit at least once and it does so orthogonally at every intersection point with an orbit. An action that admits sections is called polar, and the polar actions with flat sections are called hyperpolar. For example, the conjugation action of a connected compact Lie group on itself is hyperpolar, with the sections being the maximal tori. Polar actions, especially on symmetric spaces, have been much studied in the literature [15, 16, 17, 18].

The aim of this paper is to present some general results concerning Hamiltonian symmetry reduction under polar actions of compact Lie groups. Our motivation comes from our interest in integrable systems, for which examples of such reductions have already proved useful as mentioned above. On the one hand, we realized that both the classical and the quantum Hamiltonian reduction can be easily treated in a unified manner in the setting of polar actions.
On the other hand, it turns out that the reduced systems arising from hyperpolar actions on Lie groups and on symmetric spaces yield, generically, spin Calogero-Sutherland type integrable models. This article may serve as a step towards exploring these models systematically.

The organization of the paper and the main results are as follows. After recalling some background material in Chapter 2, we deal with the classical Hamiltonian reduction of the geodesic motion focusing on the dense, open submanifold of regular elements in the configuration space. The main result in Chapter 3 is Theorem 3.1, which characterizes the reduced system arising at an arbitrary value of the momentum map. This result may be obtained also by specializing more general results of Hochgeherner \[10, 19\] on reduced cotangent bundles. We give a simple, self-contained proof of it directly in the framework of polar actions. Chapter 4 is devoted to quantum Hamiltonian reduction. In essence, this amounts to restricting the Laplace-Beltrami operator of the Riemannian manifold to certain equivariant wave functions. Here our main result is Theorem 4.5, stating both the explicit form of the reduced Laplace-Beltrami operator and its self-adjointness on a suitable domain. In Chapter 5 we collect some examples of hyperpolar actions, which lead to spin Calogero-Sutherland type models as will be detailed elsewhere.

Although some of the results that we derive, or their special cases, are already known, to our knowledge they have not yet been treated from the unifying point of view of polar actions. We thought it worthwhile to systematize and extend the existing results from this perspective. An important feature of our work is that we present the classical and quantum Hamiltonian reduction as two sides of the same coin, showing both the similarities and the differences between the corresponding classical and quantum reduced systems.

2 Basic assumptions and definitions

Let us consider a smooth, connected, complete Riemannian manifold \( Y \) with metric \( \eta \) and a compact Lie group \( G \) that acts as a symmetry group of \((Y, \eta)\). It is a very nice situation when the group action admits sections as defined and described in detail by Palais and Terng \[14\]. By definition, a section \( \Sigma \) is a connected, closed, regularly embedded, smooth submanifold of \( Y \) that meets every \( G \)-orbit and it does so orthogonally at every intersection point of \( \Sigma \) with an orbit. As was already mentioned, the group actions permitting sections are called polar, and hyperpolar if the sections are flat in the induced metric. (Some authors relax the definition in various ways, e.g. the section is sometimes required to be an immersed submanifold only.)

Denote by \( \hat{Y} \) the open, dense, \( G \)-invariant submanifold of \( Y \) consisting of the \( G \)-regular points of principal orbit type\(^1\), i.e., those points \( y \in Y \) whose isotropy subgroups, \( G_y \), are the smallest possible for the given \( G \)-action, \( G \ni g \mapsto \phi_g \in \text{Diff}(Y) \). Since the isotropy subgroups of all elements of \( \hat{Y} \) are conjugate in \( G \), the space of orbits \( \hat{Y}/G \) is a smooth manifold, carrying a naturally induced Riemannian metric that we call \( n_{\text{red}} \). For a section \( \Sigma \), denote by \( \hat{\Sigma} \) a connected component of the manifold \( \hat{\Sigma} := \hat{Y} \cap \Sigma \). It is known that \( \hat{Y}/G \) is connected, and the restriction of the natural projection \( \pi : \hat{Y} \to \hat{Y}/G \) to \( \hat{\Sigma} \) is an isometric diffeomorphism, with the metric on \( \hat{\Sigma} \) induced from its being a submanifold of \( Y \). The isotropy subgroups of all

\(^1\)We call the points of \( \hat{Y} \) simply ‘regular’. For a general reference on group actions, see e.g. \[20\]. Note also that in the notion of manifold we include the second axiom of countability.
elements of $\hat{\Sigma}$ are the same, and for a fixed section we define $K := G_y$ for $y \in \hat{\Sigma}$.

It is also worth noting that if a section exists, then there is a unique section through every $y \in \hat{Y}$, which is the image of the orthogonal complement of $T_y(G.y)$ in $T_yY$ by means of the geodesic exponential map. Moreover, one has the generalized Weyl group $W(\Sigma) = N(\Sigma)/K$, where $N(\Sigma)$ is the subgroup of $G$ consisting of the elements that map $\Sigma$ to $\Sigma$, including $K$ as a normal subgroup of finite index. The group $W(\Sigma)$ permutes the connected components of $\hat{\Sigma}$.

A very simple example is the standard action of $SO(2)$ on the plane for which the sections are the straight lines through the origin. In this case $\Sigma \setminus \hat{\Sigma}$ is just the origin, and $W \simeq \mathbb{Z}_2$.

For our purpose, we choose a $G$-invariant, positive definite scalar product, $B$, on $G := \text{Lie}(G)$ and often identify $G$ with its dual $G^*$ by means of $B$. We have the Lie algebras $G_y := \text{Lie}(G_y)$, in particular $K := \text{Lie}(K)$, and the corresponding orthogonal decompositions,

$$G = G_y \oplus G^*_y,$$

induced by $B$. These give rise to the identifications $(G_y)^* \simeq G_y$, $(G^*_y)^* \simeq G_y$. We decompose the tangent space $T_yY$ into vertical and horizontal subspaces as

$$T_yY = V_y \oplus H_y \quad \text{with} \quad V_y := T_y(G.y), \quad H_y := V_y^\perp,$$

where orthogonality is defined by $\eta_y$. By using $\eta_y$, we identify $T_yY$ with its dual $T^*_yY$, whereby $V_y^* \simeq V_y$ and $H_y^* \simeq H_y$. At the regular points we also have

$$H_y = T_y\hat{\Sigma} = T_y\Sigma \quad \forall y \in \hat{\Sigma}.$$

For any $\zeta \in G$ denote by $\zeta_Y$ the associated vector field on $Y$ and introduce the map

$$\mathcal{L}_y : G \to T_yY, \quad \mathcal{L}_y : \zeta \mapsto \zeta_Y(y).$$

The restricted map

$$\tilde{\mathcal{L}}_y : G_y^* \to V_y$$

is a linear isomorphism, and we also introduce its inverse

$$\tilde{A}_y := (\tilde{\mathcal{L}}_y)^{-1} : V_y \to G_y^*$$

as well as the ‘mechanical connection’ $A_y : T_yY \to G$ that extends $\tilde{A}_y$ by zero on $H_y$. We then define the ‘inertia operator’ $I_y \in \text{End}(G_y^*)$ by requiring

$$\eta_y(\mathcal{L}_y\xi, \mathcal{L}_y\zeta) = B(I_y\xi, \zeta) \quad \forall \xi, \zeta \in G_y^*.$$

The inertia operator is symmetric and positive definite with respect to the restriction of the scalar product $B$ to $G_y^*$. It enjoys the equivariance property

$$I_{\phi(y)} \circ M_y = M_y \circ I_y,$$

where $M_y : G_y^* \to G_{\phi(y)}$ is given by $M_y(\xi) = \text{Ad}_y(\xi)$. 

3
The lifted action of $G$ to the cotangent bundle $T^*Y$ is Hamiltonian with respect to the canonical symplectic structure, and the associated momentum map

$$\psi : T^*Y \to G^*$$ \hfill (2.9)

is given by the formula

$$\psi(\alpha_y) = \mathcal{L}_{\alpha_y}(\alpha_y) \quad \forall \alpha_y \in T^*_yY,$$ \hfill (2.10)

which implies that $\psi(\alpha_y) \in G^*_y \simeq G^*_y$, where $G^*_y \subset G^*$ is the annihilator of $G_y$.

## 3 Classical Hamiltonian reduction

The geodesic motion on a Riemannian manifold $(Y, \eta)$ can be modeled as a Hamiltonian system on $T^*Y$ and it is an important problem to describe its symmetry reductions built on isometric, proper group actions. This problem is at the moment still open, due to difficulties originating from the singularities of $Y/G$. Restricting to the principal orbit type one always (independently of having or not having sections) obtains a smooth 'reduced configuration space', $\tilde{Y}_{\text{red}} := \tilde{Y}/G$, and therefore the symmetry reductions of $T^*\tilde{Y}$ are much easier to characterize. In fact, Hochgerner [10, 19] gave a general analysis of cotangent bundle reduction under assuming a single isotropy type for the action on the configuration space. The result in Theorem 3.1 below follows as the special case of his result when the $G$-action admits sections. For convenience, because the general case is rather involved, we present a direct proof in our restricted setting.

It proves convenient to utilize the ‘shifting trick’ of symplectic reduction [21] in order to characterize the (singular) Marsden-Weinstein reductions of the geodesic system on $T^*\tilde{Y}$. That is to say, we take a coadjoint orbit $O$ of $G$, such that $-O \subset \psi(T^*\tilde{Y})$, and start from the extended Hamiltonian system

$$(\tilde{P}^\text{ext}, \Omega^\text{ext}, \mathcal{H}^\text{ext})$$ \hfill (3.1)

defined as follows. The extended phase space is

$$\tilde{P}^\text{ext} := T^*\tilde{Y} \times O = \{(\alpha_y, \xi) \mid \alpha_y \in T^*_y\tilde{Y}, \ y \in \tilde{Y}, \ \xi \in O\}$$ \hfill (3.2)

endowed with the product symplectic structure,

$$\Omega^\text{ext}(\alpha_y, \xi) = (d\theta_{\tilde{Y}})(\alpha_y) + \omega(\xi),$$ \hfill (3.3)

where $\theta_{\tilde{Y}}$ is the canonical one-form of $T^*\tilde{Y}$ and $\omega$ is the symplectic form of the orbit $O$. The extended Hamiltonian is just the kinetic energy,

$$\mathcal{H}^\text{ext}(\alpha_y, \xi) := \frac{1}{2} \eta^*_y(\alpha_y, \alpha_y),$$ \hfill (3.4)

where $\eta^*_y$ is the metric on $T^*_y\tilde{Y}$ corresponding to the metric $\eta_y$ on $T_y\tilde{Y}$. Let us denote the natural diagonal action of $G$ on $P^\text{ext}$ by $\phi^\text{ext}_g$ for all $g \in G$. This action operates by combining the cotangent lift of the original action $\phi_y^g$ with the coadjoint action on $O$, and therefore (with $\psi$ in (2.10)) the associated momentum map is furnished by

$$\Psi : \tilde{P}^\text{ext} \to G^*, \quad \Psi(\alpha_y, \xi) = \psi(\alpha_y) + \xi.$$ \hfill (3.5)
Our problem is to describe the reduced Hamiltonian system at the value $\Psi = 0$, denoted as
\[
(\tilde{P}_{\text{red}}, \Omega_{\text{red}}, \mathcal{H}_{\text{red}}) \quad \text{with} \quad \tilde{P}_{\text{red}} = \tilde{P}^\text{ext} / /_{0} G := \tilde{P}^\text{ext}_{\Psi = 0} / G.
\] (3.6)

Referring to the notations of Chapter 2, notice that the principal isotropy group $K \subset G$ acts naturally on $O$, and the momentum map for this action is the projection $G^* \ni \xi \mapsto \xi|_K \in K^*$. Then introduce the (singular) reduced coadjoint orbit
\[
(O_{\text{red}}, \omega_{\text{red}}) \quad \text{with} \quad O_{\text{red}} = O / /_{0} K \simeq (O \cap K^\perp) / K,
\] (3.7)
where in the last equality we identified $G^*$ with $G$ by means of $B$. The reduced orbit is a stratified symplectic space in general, i.e., a union of smooth symplectic manifolds (see [21]). The reduced configuration space $\tilde{Y}_{\text{red}}$ is equipped with the inherited Riemannian metric $\eta_{\text{red}}$, and we denote by $\eta^*_{\text{red}}$ the corresponding scalar product on the fibers of $T^*\tilde{Y}_{\text{red}}$. Finally, let $\theta_{\tilde{Y}_{\text{red}}}$ be the natural one-form on $T^*\tilde{Y}_{\text{red}}$.

**Theorem 3.1.** [10] Suppose that the $G$-action on $(Y, \eta)$ admits sections and fix a connected component $\hat{\Sigma}$ of the regular elements of a section $\Sigma$. Then, using the notations introduced above, for any orbit $-O \subset \psi(T^*\tilde{Y})$, the reduced phase space (3.6) can be identified as
\[
\tilde{P}_{\text{red}} = T^*\tilde{Y}_{\text{red}} \times O_{\text{red}} = \{(p_q, [\xi]) | p_q \in T_q\tilde{Y}_{\text{red}}, q \in \tilde{Y}_{\text{red}}, [\xi] \in O_{\text{red}}\}
\] (3.8)
equipped with the product (stratified) symplectic structure,
\[
\Omega_{\text{red}}(p_q, [\xi]) = (d\theta_{\tilde{Y}_{\text{red}}})(p_q) + \omega_{\text{red}}([\xi]).
\] (3.9)
The reduced Hamiltonian arising from the kinetic energy of the geodesic motion on $\tilde{Y}$ reads
\[
\mathcal{H}_{\text{red}}(p_q, [\xi]) = \frac{1}{2} \eta^*_{\text{red}}(p_q, p_q) + \frac{1}{2} B(T^{-1}_{y(q)} [\xi], [\xi]),
\] (3.10)
where $\eta_{\text{red}}$ is the induced metric on $\tilde{Y}_{\text{red}}$, $y(q) \in \Sigma$ projects to $q \in \tilde{Y}_{\text{red}}$, $[\xi] = K, \xi \subset O \cap K^\perp$, and $T^{-1}_{y(q)} \in GL(K^\perp)$ is the $K$-equivariant inertia operator (2.7).

**Proof.** To describe the reduced system (3.6), the key point is to introduce the following submanifold $S \subset \tilde{P}^\text{ext}_{\Psi = 0}$,
\[
S = \{ (\alpha_y, \xi) \in \tilde{P}^\text{ext}_{\Psi = 0} | y \in \hat{\Sigma} \}.
\] (3.11)
All $G$ orbits in $\tilde{P}^\text{ext}_{\Psi = 0}$ intersect $S$ and the ‘residual gauge transformations’ corresponding to this submanifold are provided by the group $K$. Indeed, an element of $G$ can map some element of $S$ into $S$ only if it belongs to $K$, and the elements of $K$ map each element of $S$ into $S$. As a consequence, the reduced phase space enjoys the property
\[
\tilde{P}_{\text{red}} = \tilde{P}^\text{ext}_{\Psi = 0} / G = S / K.
\] (3.12)
The reduced symplectic structure and the reduced Hamiltonian will be obtained simply by solving the momentum map constraint $\psi(\alpha_y) + \xi = 0$ on $S$. For this, let us consider the orthogonal decompositions
\[
\alpha_y = \alpha_y^H + \alpha_y^V, \quad \xi = \xi_K + \xi_{K^\perp}
\] (3.13)
corresponding to (2.2) and to \( \mathcal{G} = \mathcal{K} \oplus \mathcal{K}^\perp \) (2.1). It is easy to see that the momentum map constraint is completely solved on \( S \) by
\[
\xi_\mathcal{K} = 0 \quad \text{and} \quad y \in \hat{\Sigma}, \quad \alpha_y^H \in H_y^\ast : \text{arbitrary}
\] (3.14)
together with the formula
\[
\alpha_y^V = -\hat{A}_y^\ast(\xi_\mathcal{K}^\perp),
\] (3.15)
where we identified \( \mathcal{G} \) with \( \mathcal{G}^* \) by \( \mathcal{B} \) and used the mechanical connection \( \hat{A}_y : V_y \rightarrow \mathcal{K}^\perp \) (2.6). Notice that \( \alpha_y^H \in H_y^\ast \cong T_y^\ast \hat{\Sigma} \) (2.3) can be naturally regarded as belonging to the cotangent bundle \( T^\ast \hat{\Sigma}, \) and the parametrization of \( S \) by the variables \( (\alpha_y^H, \xi_\mathcal{K}^\perp) \) yields the identification
\[
S \cong T^\ast \hat{\Sigma} \times (\mathcal{O} \cap \mathcal{K}^\perp) = \{ (\alpha_y^H, \xi_\mathcal{K}^\perp) \mid \alpha_y^H \in T_y^\ast \hat{\Sigma}, \ y \in \hat{\Sigma}, \ \xi_\mathcal{K}^\perp \in \mathcal{O} \cap \mathcal{K}^\perp \}. \quad (3.16)
\]
This is a \( K \)-equivariant identification since \( \hat{A}_y^\ast \) is a \( K \)-equivariant map and \( \alpha_y^H \) is \( K \)-invariant since \( K \) is the isotropy group of all \( y \in \hat{\Sigma} \) (and the ‘slice representation’ is always trivial at the regular points). Now the first important point is that in terms of the identification (3.16) the pull-back \( \Omega^\ast |_S \) of \( \Omega^\ast \) to \( S \) becomes
\[
\Omega^\ast |_S(\alpha_y^H, \xi_\mathcal{K}^\perp) = (d\theta_\Sigma)(\alpha_y^H) + \omega|_{\mathcal{O} \cap \mathcal{K}^\perp}(\xi_\mathcal{K}^\perp),
\] (3.17)
where \( \theta_\Sigma \) is the natural one-form on \( T^\ast \hat{\Sigma} \). Since \( T^\ast \hat{\Sigma} \) is a model of \( T^\ast \hat{Y}_{\text{red}} \), we obtain the statement of the theorem concerning the reduced symplectic structure. The second important point is that the restriction of the Hamiltonian (3.4) gives, in terms of the variables \( (\alpha_y^H, \xi_\mathcal{K}^\perp) \), the following function
\[
\mathcal{H}^\ast |_S(\alpha_y^H, \xi_\mathcal{K}^\perp) = \frac{1}{2} \eta_y^\ast(\alpha_y^H, \alpha_y^H) + \frac{1}{2} \eta_y^\ast(\hat{A}_y^\ast(\xi_\mathcal{K}^\perp), \hat{A}_y^\ast(\xi_\mathcal{K}^\perp)).
\] (3.18)
The first term represents the kinetic energy of a particle on \( (\hat{Y}_{\text{red}}, \eta_{\text{red}}) \) modeled by the submanifold \( \Sigma \) of \( \hat{Y} \). It follows directly from the definitions given in Chapter 2 that the second term can be rewritten as
\[
\eta_y^\ast(\hat{A}_y^\ast(\xi_\mathcal{K}^\perp), \hat{A}_y^\ast(\xi_\mathcal{K}^\perp)) = \mathcal{B}(\mathcal{T}_y^{-1}\xi_\mathcal{K}^\perp, \xi_\mathcal{K}^\perp),
\] (3.19)
which yields a well-defined function on \( \hat{P}_{\text{red}} \) since (by (2.8)) the inertia operator \( \mathcal{T}_y \) is \( K \)-equivariant for \( y \in \hat{\Sigma} \). Q.E.D.

Remark 3.2. Identifying \( \hat{Y}_{\text{red}} \) with \( \Sigma \), as in the proof, we can also model the reduced phase space as \( T^\ast \hat{\Sigma} \times \mathcal{O}_{\text{red}} \). Moreover, we can view this as the factor space of \( T^\ast \hat{\Sigma} \times \mathcal{O}_{\text{red}} \) under the natural, diagonal action of the finite group \( \mathcal{W}(\Sigma) \). Here, \( \mathcal{W}(\Sigma) = N(\Sigma)/K \) acts on \( \mathcal{O}///_0K \) since \( K \) is a normal subgroup of \( N(\Sigma) \). The Hamiltonian gives rise to a \( \mathcal{W}(\Sigma) \)-invariant function on \( T^\ast \hat{\Sigma} \times \mathcal{O}_{\text{red}} \). Therefore, all objects belonging to the reduced system can be interpreted as \( \mathcal{W}(\Sigma) \)-invariants of a system associated with the (regular) part of the slice \( \Sigma \). This is very familiar in the examples of (spin) Calogero type models studied e.g. in [7][10][11][12].

Remark 3.3. The first term of the reduced Hamiltonian (3.10) is just the kinetic energy on \( (\hat{Y}_{\text{red}}, \eta_{\text{red}}) \), while the second term can be viewed as potential energy depending also on the ‘spin’ variables belonging to \( \mathcal{O}_{\text{red}} \). According to (3.19), the second term can be expressed also in terms of the dual of the mechanical connection, and in certain cases the function \( q \mapsto \hat{A}_y^\ast(q) \) yields a so-called dynamical \( r \)-matrix [10][11]. In some exceptional cases [1][12][13] \( \mathcal{O}_{\text{red}} \) consists of a single point, which means that no spin degrees of freedom appear in the reduced dynamics.
4 Quantum Hamiltonian reduction

Below we first recall the standard quantization of the Hamiltonian system \((P^{\text{ext}},\Omega^{\text{ext}},H^{\text{ext}})\) defined similarly to (3.1) with \(P^{\text{ext}} = T^*Y \times \mathcal{O}\). Then we impose the analogue of the ‘first class constraints’ \(\Psi = 0\) on the Hilbert space of this system to obtain a reduced quantum system. Finally, we comment on the relation between the outcome of this ‘first quantize then reduce’ procedure and the structure of the reduced classical system given in Theorem 3.1. Our consideration are close to those in the paper [22], but in the context of polar actions we can say more about the reduced systems.

Consider a unitary representation

\[
\rho: G \to U(V) \quad (4.1)
\]
on a finite dimensional complex Hilbert space \(V\) with scalar product \((\ ,\ )_V\), and the associated Lie algebra representation

\[
\rho': G \to u(V) \quad (4.2)
\]
where \(u(V)\) is the Lie algebra of anti-hermitian operators on \(V\). The representation \(\rho\) can be viewed as the quantum mechanical analogue of the coadjoint orbit \(O\) that features at the classical level. The standard quantum mechanical analogue of \(P^{\text{ext}}\) is the Hilbert space \(L^2(Y,V,d\mu_Y)\) consisting of the \(V\)-valued square integrable functions on \(Y\) with the scalar product

\[
(f_1,f_2)_V = \int_Y (f_1,f_2)_V \ d\mu_Y, \quad (4.3)
\]
where \(d\mu_Y\) is the measure associated with the Riemannian metric \(\eta\) on \(Y\). Denote by \(\Delta^0_Y\) the Laplace-Beltrami operator \(\Delta_Y\) of \((Y,\eta)\) on the domain \(C^\infty_c(Y,V) \subset L^2(Y,V,d\mu_Y)\) containing the smooth \(V\)-valued functions of compact support. It is well-known\(^2\) that \(\Delta^0_Y\) is essentially self-adjoint, and \(-\frac{1}{2}\) times its closure yields the Hamilton operator corresponding to the classical Hamiltonian \(H^{\text{ext}}\).

We need the equivariant functions \(\mathcal{F} \in C^\infty(Y,V)^G\) that satisfy by definition

\[
\mathcal{F} \circ \phi_g = \rho(g) \circ \mathcal{F} \quad \forall g \in G. \quad (4.4)
\]
The space of functions \(C^\infty(\Sigma,V^K)^W\) can be defined similarly, where \(V^K\) is the subspace of \(K\)-invariant vectors in \(V\), on which \(W\) acts since \(W = N(\Sigma)/K\). From now on we assume that \(\dim(V^K) > 0\). It is easy to see that the restriction of functions on \(Y\) to \(\Sigma\) gives rise to an injective map\(^3\)

\[
C^\infty(Y,V)^G \to C^\infty(\Sigma,V^K)^W. \quad (4.5)
\]
Note also that any function \(\mathcal{F} \in C^\infty(Y,V)^G\) is uniquely determined by its restriction to a connected, open component \(\hat{\Sigma} \subset \Sigma\). Then introduce the linear space

\[
\text{Fun}(\hat{\Sigma},V^K) := \{ f \in C^\infty(\hat{\Sigma},V^K) \ | \ \exists \mathcal{F} \in C^\infty_c(Y,V)^G, \ f = \mathcal{F}|_{\Sigma} \}. \quad (4.6)
\]

\(^2\)See e.g. [23], paragraph II.3.7, and references therein. We do not include the Ricci scalar in the quantum Hamiltonian, but its inclusion would not cause any extra difficulty in our arguments.

\(^3\)This map is known to be surjective [14] if \(\dim(V) = 1\). It would be interesting to generalize this result, and another important question is to find all cases for which \(\dim(V) > \dim(V^K) = 1\) like in the examples in [2, 9].
By its isomorphism with $C_c^\infty(Y,V)^G \subset L^2(Y,V,d\mu_Y)$, $\text{Fun}(\hat{\Sigma},V^K)$ becomes a pre-Hilbert space and we denote its closure by $\overline{\text{Fun}(\hat{\Sigma},V^K)}$. It is not difficult to verify the natural isometric isomorphism

$$\overline{\text{Fun}(\hat{\Sigma},V^K)} \simeq L^2(Y,V,d\mu_Y)^G,$$

and it is also worth remarking that $\text{Fun}(\hat{\Sigma},V^K)$ contains $C_c^\infty(\hat{\Sigma},V^K)$.

Because of (4.7), a natural quantum mechanical analogue of the classical Hamiltonian reduction is obtained by taking the reduced Hilbert space to be $\text{Fun}(\hat{\Sigma},V^K)$. The reduced Hamilton operator results from $\Delta_Y$ on account of the following simple observation. There exists a unique linear operator $\Delta_{\text{eff}} : \text{Fun}(\hat{\Sigma},V^K) \to \text{Fun}(\hat{\Sigma},V^K)$ (4.8) defined by the property

$$\Delta_{\text{eff}} f = (\Delta_Y F)|_{\hat{\Sigma}}, \quad \text{for} \quad f = F|_{\hat{\Sigma}}, \quad F \in C_c^\infty(Y,V)^G.$$

In other words, the ‘effective Laplace-Beltrami operator’ $\Delta_{\text{eff}}$ is the restriction of $\Delta_Y$ to $C_c^\infty(Y,V)^G$, which is well-defined because the metric $\eta$ is $G$-invariant. Next we present the explicit formula of $\Delta_{\text{eff}}$.

### 4.1 The effective Laplace-Beltrami operator

A convenient local decomposition of the Laplace-Beltrami operator into ‘radial’ and ‘orbital’ (angular) parts is always applicable if one has a local, orthogonal ‘cross section’ of the $G$-orbits on a Riemannian $G$-manifold [24]. Upon restriction to $G$-equivariant functions, the orbital part can be calculated explicitly, and in our case we can apply this decomposition over $\hat{Y}$ since we are dealing with a polar action. Before describing the result, we need some further notations.

Thinking of $\hat{\Sigma}$ as the (smooth part of the) reduced configuration space, denote the elements of $\hat{\Sigma}$ by $q$ and consider the $G$ orbit $G.q$ through any $q \in \hat{\Sigma}$. Both $\hat{\Sigma}$ and $G.q$ are regularly embedded submanifolds of $Y$ and by their embeddings they inherit Riemannian metrics, $\eta_{\hat{\Sigma}}$ and $\eta_{G.q}$, from $(Y,\eta)$. Let $\Delta_{\hat{\Sigma}}$ and $\Delta_{G.q}$ denote the Laplace-Beltrami operators defined on the respective Riemannian manifolds $(\hat{\Sigma},\eta_{\hat{\Sigma}})$ and $(G.q,\eta_{G.q})$. Introduce the smooth density function $\delta : \hat{\Sigma} \to \mathbb{R}_{>0}$ by

$$\delta(q) := \text{volume of the Riemannian manifold } (G.q,\eta_{G.q}).$$

Of course, the volume is understood with respect to the measure belonging to $\eta_{G.q}$. (By the same formula, $\delta$ can also be defined as a $W$-invariant function on $\Sigma$.) Referring to (2.7), let us define the function $J : \hat{\Sigma} \to \text{End}(K^\perp)$ by

$$J := I|_{\hat{\Sigma}},$$

and notice that, because of the $G$-symmetry, the inertia operator $J(q)$ carries the same information as the metric $\eta_{G.q}$. Denote by $\{T_\alpha\}$ and $\{T^\beta\}$ some fixed dual bases of $K^\perp$ with respect to the scalar product $B$,

$$B(T_\alpha, T^\beta) = \delta^\beta_\alpha.$$

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In fact (see also Remark 4.3 below), one has
\[ \delta(q) = C |\det b_{\alpha,\beta}(q)|^{\frac{1}{2}} \quad \text{with} \quad b_{\alpha,\beta}(q) = \mathcal{B}(\mathcal{J}(q)T_\alpha, T_\beta) \tag{4.13} \]
and a \( q \)-independent constant \( C > 0 \), whose value could be given but is not important for us.

**Proposition 4.1.** On \( \text{Fun}(\bar{\Sigma}, V^K) \) \( (4.6) \) the effective Laplace-Beltrami operator \( (4.9) \) can be expressed in the following form:
\[ \Delta_{\text{eff}} = \delta^{-\frac{1}{2}} \circ \Delta_{\bar{\Sigma}} \circ \delta^{\frac{1}{2}} - \delta^{-\frac{1}{2}} \Delta_{\bar{\Sigma}}(\delta^{\frac{1}{2}}) + b^{\alpha,\beta}(q) \rho'(T_\alpha) \rho'(T_\beta). \tag{4.14} \]
More explicitly, this means that for any \( q \in \bar{\Sigma} \) and \( f \in \text{Fun}(\bar{\Sigma}, V^K) \), one has
\[ (\Delta_{\text{eff}} f)(q) = \delta^{-\frac{1}{2}}(q)(\Delta_{\bar{\Sigma}}(\delta^{\frac{1}{2}} f))(q) - \delta^{-\frac{1}{2}}(q)(\Delta_{\bar{\Sigma}}(\delta^{\frac{1}{2}}))(q) f(q) + b^{\alpha,\beta}(q) \rho'(T_\alpha) \rho'(T_\beta) f(q), \quad (4.15) \]
where the linear operator on \( V^K \) in the last term contains the inverse \( b^{\alpha,\beta}(q) = \mathcal{B}(\mathcal{J}^{-1}(q)T_\alpha, T_\beta) \) of the matrix \( b_{\alpha,\beta}(q) \) \( (4.13) \).

**Proof.** Take an arbitrary function \( F \in C^\infty(Y, V) \) and consider its restrictions
\[ f := F|_{\bar{\Sigma}} \quad \text{and} \quad F_q := F|_{G \cdot q} \quad \forall q \in \bar{\Sigma}. \quad (4.16) \]
Then \( (\Delta_Y F)(q) \) can be found with the aid of the following well-known formula.

**Lemma 4.2.** Using the above notations,
\[ (\Delta_Y F)(q) = (\Delta_{\text{rad}} f)(q) + (\Delta_{G \cdot q} F_q)(q) \quad (4.17) \]
with the radial part of \( \Delta_Y \) given by
\[ \Delta_{\text{rad}} = \delta^{-\frac{1}{2}} \circ \Delta_{\bar{\Sigma}} \circ \delta^{\frac{1}{2}} - \delta^{-\frac{1}{2}} \Delta_{\bar{\Sigma}}(\delta^{\frac{1}{2}}). \quad (4.18) \]

The statement of Lemma 4.2 is quite standard (see e.g. [24]), and one can also verify it in a direct manner by simply writing out the Laplace-Beltrami operator in such local coordinates, \( \{ y^\alpha \} = \{ q^i \} \cup \{ z^\alpha \} \), around \( q \in \bar{\Sigma} \) that are composed of some coordinates \( q^i \) on \( \bar{\Sigma} \) and coordinates \( z^\alpha \) around the origin of the coset space \( G/K \) according to the \( G \)-equivariant diffeomorphism
\[ \bar{\Sigma} \times G/K \cong \bar{Y} \quad (4.19) \]
defined by \( \bar{\Sigma} \times G/K \ni (q, gK) \mapsto \phi_g(q) \in \bar{Y} \). The coordinates can be chosen by taking advantage of the local diffeomorphism
\[ K^+ \ni z = z^\alpha T_\alpha \mapsto e^z K \in G/K, \quad (4.20) \]
and the metric on \( Y \) is then represented by \( g_{\alpha,\beta}(q, z) := \eta_{(q,z)}(\partial_{y^\alpha}, \partial_{y^\beta}) \) satisfying
\[ g_{i,j}(q, z) = g_{i,j}(q, 0), \quad g_{i,\alpha}(q, z) = 0, \quad g_{\alpha,\beta}(q, 0) = b_{\alpha,\beta}(q) \quad (4.21) \]
with \( b_{\alpha,\beta}(q) \) in (4.13). On account of this block-diagonal structure, the local expression
\[
\Delta_Y \longleftrightarrow \frac{1}{\sqrt{g}} \partial_y^a \circ \sqrt{g} g^{ab} \circ \partial_y^b, \quad g := |\det g_{a,b}|
\] (4.22)
separates into the sum of two terms, which yield the two terms in (4.17). Since \( f \in \text{Fun}(\hat{\Sigma}, V^K) \) corresponds to \( F \in C^\infty_c(Y, V^G) \), to prove the proposition it is now enough to calculate \( \Delta_{G.q} F_q \) for equivariant functions \( F_q \in C^\infty_c(G.q, V^G) \). The result of this latter calculation can presumably be also found in the literature, but one can also compute it directly by using the exponential coordinates on \( G.q \cong G/K \) introduced above. One finds that
\[
(\Delta_{G.q} F_q)(q) = b_{\alpha,\beta}(q) \rho'(T_\alpha) \rho'(T_\beta) f(q) \quad \text{if} \quad F_q \in C^\infty(G.q, V^G),
\] (4.23)
which completes the proof of the proposition. Q.E.D.

**Remark 4.3.** The coset space \( G/K \) carries a \( G \)-invariant Haar measure, which is unique up to a multiplicative constant. The Haar measure is associated with a \( G \)-invariant differential form of top degree on \( G/K \). This differential form is uniquely determined by its value at the origin \( K \in G/K \). Upon the identification \( G/K \cong G.q \), the origin becomes \( q \), and the value of the \( G \)-invariant volume form associated with the metric \( \eta_{G.q} \) gives at the origin
\[
|\det b_{\alpha,\beta}(q)|^{\frac{1}{2}} (dz^1 \wedge dz^2 \wedge \cdots \wedge dz^m)_q, \quad m := \dim(G/K).
\] (4.24)

**4.2 The reduced quantum system**

The effective Laplace-Beltrami operator \( \Delta_{\text{eff}} \) (4.14) can be shown to be essentially self-adjoint on the domain \( \text{Fun}(\hat{\Sigma}, V^K) \subset \overline{\text{Fun}(\hat{\Sigma}, V^K)} \simeq L^2(Y, V, d\mu_Y)^G \). In order to relate the reduced Hilbert space to the Riemannian metric on the smooth part of the reduced configuration manifold, \((\hat{\Sigma}_{\text{red}}, \eta_{\text{red}}) \simeq (\hat{\Sigma}, \eta_{\hat{\Sigma}})\), the following lemma is needed.

**Lemma 4.4.** The complement of the dense, open submanifold \( \hat{Y} \subset Y \) of principal orbit type has zero measure with respect to \( d\mu_Y \).

**Proof.** It is known [20] that the non-principal orbits of a given type fill lower-dimensional regular submanifolds in \( Y \) and at most countably many different types of orbits can occur. Since the measure \( d\mu_Y \) is smooth, and \( Y \) is second countable, this implies (see e.g. [26], page 529) that \( Y \setminus \hat{Y} \) has measure zero. Q.E.D.

To proceed further, we also need the following integration formula:
\[
\int_{\hat{Y}} (F_1, F_2)_V d\mu_Y = \int_Y (F_1, F_2)_V d\mu_Y = \int_{\Sigma} (f_1, f_2)_V \delta d\mu_\Sigma, \quad F_i \in C^\infty_c(Y, V)^G, \quad f_i = F_i|_{\Sigma} \quad (4.25)
\]
The first equality is guaranteed by Lemma 4.4. The second equality holds since, as is standard to show, the measure on \( \hat{Y} \cong \hat{\Sigma} \times G/K \) takes the product form
\[
d\mu_{\hat{Y}} = (\delta d\mu_\Sigma) \times d\mu_{G/K},
\] (4.26)
where $d\mu_{G/K}$ is the probability Haar measure on $G/K$, $d\mu_{\Sigma}$ is the measure on $\Sigma$ associated with the Riemannian metric $\eta_\Sigma$, the density $\delta$ is defined in (1.10); and $(\mathcal{F}_1, \mathcal{F}_2)_V$ is $G$-invariant. The integration formula and the fact that $\text{Fun}(\Sigma, V^K)$ contains $C^\infty_c(\Sigma, V^K)$, in association with $C^\infty_c(\hat{Y}, V)^G \subset C^\infty_c(Y, V)^G$, together imply that

$$\text{Fun}(\Sigma, V^K) \simeq L^2(\Sigma, V^K, \delta d\mu_{\Sigma}).$$  \hspace{1cm} (4.27)

By transforming away the factor $\delta$ from the ‘induced measure’ $\delta d\mu_{\Sigma}$, we obtain the final result.

**Theorem 4.5.** Using the above notations, the reduction of the quantum system defined by the closure of $-\frac{1}{2}\Delta_Y$ on $C^\infty_c(Y, V) \subset L^2(Y, V, d\mu_Y)$ leads to the reduced Hamilton operator $-\frac{1}{2}\Delta_{\text{red}}$ given by

$$\Delta_{\text{red}} = \delta^\frac{1}{2} \circ \Delta_{\text{eff}} \circ \delta^{-\frac{1}{2}} = \Delta_{\Sigma} - \delta^{-\frac{1}{2}}(\Delta_{\Sigma}\delta^\frac{1}{2}) + b^{\alpha, \beta}(T_{\alpha})\rho'(T_{\beta}).$$ \hspace{1cm} (4.28)

This operator is essentially self-adjoint on the dense domain $\delta^\frac{1}{2}\text{Fun}(\Sigma, V^K)$ in the reduced Hilbert space identified as $L^2(\Sigma, V^K, d\mu_{\Sigma})$.

**Proof.** The multiplication operator $U : f \mapsto \delta^\frac{1}{2}f$ is an isometry from $L^2(\Sigma, V^K, \delta d\mu_{\Sigma})$ to $L^2(\Sigma, V^K, d\mu_{\Sigma})$. Plainly, Proposition 4.1 and Lemma 4.4 imply that $\Delta_{\text{red}} = U \circ \Delta_{\text{eff}} \circ U^{-1}$ is a symmetric operator on the dense domain $U(\text{Fun}(\Sigma, V^K)) = \delta^\frac{1}{2}\text{Fun}(\Sigma, V^K) \subset L^2(\Sigma, V^K, d\mu_{\Sigma})$. The essential self-adjointness of $\Delta_{\text{red}}$ can be traced back to the essential self-adjointness of $\Delta_Y$ on $C^\infty_c(Y, V)$. More details on this last point are provided in [23]. Q.E.D.

Let us now compare the result of the quantum Hamiltonian reduction given by Theorem 4.5 with the classical reduced system in Theorem 3.1. First, the classical kinetic energy clearly corresponds to $-\frac{1}{2}\Delta_{\Sigma}$. Formally, the second term of the classical Hamiltonian $H_{\text{red}}$ (3.10) corresponds the third term of $-\frac{1}{2}\Delta_{\text{red}}$ (1.28). This term can be interpreted as potential energy if $\dim(V^K) = 1$, otherwise it is a ‘spin dependent potential energy’. As represented by the second term in (1.28), an extra ‘measure factor’ appears at the quantum level in general, which has no trace in $H_{\text{red}}$. This term gives a constant or a non-trivial contribution to the potential energy depending on the concrete examples.

In the quantum Hamiltonian reduction we started from the full configuration space $Y$, while classically we have restricted our attention to $\hat{Y}$ from the beginning. In some sense, the outcome of the quantum Hamiltonian reduction can nevertheless be viewed as a quantization of the reduced classical system of Theorem 3.1 because $Y \setminus \hat{Y}$ has zero measure. However, this delicate correspondence needs further investigation (see also [27]). The structure of the full (singular) reduced phase space $P_{\text{red}}$ coming from $T^*Y$ should be explored, too, since it is clear that the reduced geodesic flows may leave $\hat{P}_{\text{red}} \subset P_{\text{red}}$ in certain cases.

It should be stressed that at the abstract level, on account of the natural identifications (4.7) and (4.27), the reduced Hilbert space is simply provided by the $G$-singlets $L^2(Y, V, d\mu_Y)^G$. In some cases (for example if $Y$ is a compact Lie group) $\Delta_Y$ possesses pure point spectrum that can be determined from known results (such as the Peter-Weyl theorem) in harmonic analysis. In such cases finding the spectrum of the reduced Hamilton operator becomes a problem in branching rules, since it requires finding the above $G$-singlets among the eigensubspaces of $\Delta_Y$. 


5 Examples related to spin Sutherland type models

We here recall from [17, 18] a class of important hyperpolar actions on compact Lie groups. By subjecting them to the classical and quantum Hamiltonian reduction as described in this paper, one may obtain, and solve, a large family of spin Sutherland type integrable models. Details on some of these models will be reported elsewhere.

Let \( Y \) be a compact, connected, semisimple Lie group carrying the Riemannian metric induced by a multiple of the Killing form. Take the ‘reduction group’ \( G \) to be any symmetric subgroup of \( Y \times Y \). That is to say, \( G \) is any subgroup which is pointwise fixed by an involutive automorphism \( \sigma \) of \( Y \times Y \) and contains the connected component of the full subgroup fixed by \( \sigma \). Consider the following action of \( G \) on \( Y \):

\[
\phi_{(a,b)}(y) := ayb^{-1} \quad \forall y \in Y, \ (a, b) \in G \subset Y \times Y. \tag{5.1}
\]

This action is known to be hyperpolar (see [17, 18] and the references there). The sections are provided by certain tori, \( A \subset Y \). In fact, \( A \) is the exponential of an Abelian subalgebra \( A \) of the correct dimension lying in the subspace \( (T_e(G.e))^\perp \) of \( T_e Y \). The underlying reasons behind the appearance of Sutherland type models in association with these actions are the exponential parametrization of \( \Sigma = A \) together with the decomposition of the Lie algebra of \( Y \) into joint eigensubspaces of \( A \). One can illustrate this by the particular examples to which we now turn.

First, consider \( \sigma(y_1, y_2) = (y_2, y_1) \). Then \( G = \{(a, a) | a \in Y \} \simeq Y \) and the action (5.1) is just the adjoint action of \( Y \) on itself, for which the sections are the maximal tori of \( Y \). The associated spin (and in exceptional cases spinless) Sutherland models were studied in [2, 7].

Second, take any non-trivial automorphism \( \theta \) of \( Y \) and set \( \sigma(y_1, y_2) := (\theta(y_2), \theta^{-1}(y_1)) \). Now \( G = \{(\theta(a), a) | a \in Y \} \simeq Y, \tag{5.2} \)

and (5.1) yields the action of \( Y \) on itself by \( \theta \)-twisted conjugations, \( \phi_{(\theta(a), a)}(y) = \theta(a)ya^{-1} \). The interesting cases are when \( \theta \) corresponds to a Dynkin diagram symmetry of \( Y \). Some of the resulting generalized spin Sutherland models have been described in [11, 28].

Third, suppose that \( \theta_1 \) and \( \theta_2 \) are two involutive automorphisms of \( Y \) and let \( K_1 \) and \( K_2 \) be corresponding symmetric subgroups of \( Y \), i.e., \( (Y, K_j) \) are symmetric pairs for \( j = 1, 2 \). By taking \( \sigma(y_1, y_2) := (\theta_1(y_1), \theta_2(y_2)) \), one obtains

\[
G = K_1 \times K_2 \subset Y \times Y, \tag{5.3}
\]

and (5.1) becomes the so-called Hermann action on \( Y \). Besides this action, the induced action of \( K_1 \) on \( Y/K_2 \) is also hyperpolar. The resulting spin Sutherland type models have not yet been explored systematically, apart from the case [12] of the isotropy action of \( K_1 \) on the symmetric space \( Y/K_1 \) arising under \( K_1 = K_2 \). Since they include, in fact, interesting spinless cases, we shall return to this class of models elsewhere. It could be also worthwhile to investigate the reduced systems induced by other polar actions given in [17, 18].

Because of the compactness of \( Y \), in the above cases the corresponding Sutherland models involve trigonometric potential functions. Hyperbolic analogues of these models can be derived [13] by starting from non-compact semisimple Lie groups \( Y \). This requires a slight extension...
of the theory of polar actions, so as to cover suitable actions on pseudo-Riemannian manifolds. Rational degenerations of the trigonometric models can be obtained by Hamiltonian reduction in those cases for which the $G$-action has a fixed point, $p$, by using that in those cases the representation of $G$ on $T_p Y$ inherits the polar property of the original action. It is an important open problem whether the formalism of Hamiltonian reduction under polar actions may be extended in such a way to incorporate also the elliptic Calogero-Sutherland type models.

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