APPLICATIONS OF SPHERICAL TWIST FUNCTORS TO LIE ALGEBRAS ASSOCIATED TO ROOT CATEGORIES OF PREPROJECTIVE ALGEBRAS

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Abstract. Let $\Lambda_Q$ be the preprojective algebra of a finite acyclic quiver $Q$ of non-Dynkin type and $D^b(\text{rep}^n\Lambda_Q)$ be the bounded derived category of finite dimensional nilpotent $\Lambda_Q$-modules. We define spherical twist functors over the root category $\mathcal{R}_{\Lambda_Q}$ of $D^b(\text{rep}^n\Lambda_Q)$ and then realize the Weyl group associated to $Q$ as certain subquotient of the automorphism group of the Ringel-Hall Lie algebra $\mathfrak{g}(\mathcal{R}_{\Lambda_Q})$ of $\mathcal{R}_{\Lambda_Q}$ induced by spherical twist functors. We also present a conjectural relation between certain Lie subalgebras of $\mathfrak{g}(\mathcal{R}_{\Lambda_Q})$ and $\mathfrak{g}(\mathcal{R}_Q)$, where $\mathfrak{g}(\mathcal{R}_Q)$ is the Ringel-Hall Lie algebra associated to the root category $\mathcal{R}_Q$ of $Q$.

1. Introduction

Let $Q$ be a finite (connected) acyclic quiver of non-Dynkin type and $\Lambda_Q$ the preprojective algebra. The preprojective algebra $\Lambda_Q$ plays important roles in studying topics related to the Kac-Moody Lie algebra $\mathfrak{g}_Q$ of $Q$ and its enveloping algebra. Firstly, Lusztig ([13], [14]) gave a geometric realization of the positive part $U_Q^+$ of the enveloping algebra of the Kac-Moody Lie algebra $\mathfrak{g}_Q$ and then constructed a basis of $U_Q^+$ called semicanonical basis indexed by irreducible components of the variety $\Lambda_v$ of nilpotent $\Lambda_Q$-modules with dimension vector $v$. Secondly, the preprojective algebra $\Lambda_Q$ is independent of orientations of the quiver $Q$. Thirdly, the preprojective algebra $\Lambda_Q$ is derived 2-Calabi-Yau, i.e. the derived category $D^b(\text{rep}^n\Lambda_Q)$ is a 2-Calabi-Yau category ([8],[12]). Geiss, Leclerc and Schroeer ([8]) applied this property to prove the cluster multiplication between evaluation forms. In [20], Shiraishi, Takahashi, and Wada defined the spherical twist functor over $D^b(\text{rep}^n\Lambda_Q)$ and showed the spherical twist functor can be identified with simple reflections in the corresponding Weyl group on the level of the Grothendieck group $K_0(D^b(\Lambda_Q))$ based on the property of derived 2-Calabi-Yau.

The notion of reflection functors for the categories of representations of quivers was introduced by Bernstein, Gelfand, and Ponomarev [2] whose aim was to obtain a simple and elegant proof of Gabriel’s theorem. Applying the BGP-reflection functors

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in the root categories to the Lie algebra model, Xiao, Zhang and Zhu [23] obtained the well-known Weyl group action on the Kac-Moody Lie algebras by explicit formula. However, this construction does not work in general since BGP-reflection functors can only be defined for sinks or sources of quivers. Deng, Ruan and Xiao in [5] generalized this construction to arbitrary vertices for the star-shaped quivers associated with weighted projective lines. However, they can not give a realization of the composition of two simple reflections, because BGP-reflection functors and mutation functors given in [5] are functors between different categories.

The aim of this paper is to realize simple reflections in the Weyl group $W_Q$ of the Kac-Moody Lie algebra $g_Q$ as automorphisms of a certain Lie algebra, and then realize the Weyl group $W_Q$ as the subquotient of the automorphism group of the corresponding Lie algebra. The idea is illuminated by [20], where each simple reflection of $W_Q$ can be lifted to a spherical twist functor of the bounded derived category $D^b(rep^n A_Q)$.

The paper is organized as follows. In Section 2, we introduce some preliminary results, including the Ringel-Hall Lie algebra, the triangulated hull of an orbit category, etc. As we know, $D^b(rep^n A_Q)$ is a 2-Calabi-Yau category, so we can define a spherical twist functor $T_{S_i}$ associated to each simple $A_Q$-module $S_i$, which induces an autoequivalence on the orbit category $D^b(rep^n A_Q)/[2]$. Note that $D^b(rep^n A_Q)/[2]$ may not be a triangulated category in general, using Keller’s construction of triangulated hull of orbit categories [11], we obtain the triangulated hull $R_{A_Q}$ of the orbit category $D^b(rep^n A_Q)/[2]$. In Section 3, we extend the autoequivalence induced by a spherical twist functor $T_{S_i}$ to a triangle autoequivalence $\tilde{T}_{2, S_i}$ of the triangulated hull $R_{A_Q}$, which we still call the spherical twist functor. Similar to [20], the spherical twist functor $\tilde{T}_{2, S_i}$ is just the simple reflections associated to the simple root $\alpha_i$ on the Grothendieck group level. On the other hand, Peng and Xiao ([16]) constructed the Ringel-Hall Lie algebras associated to 2-periodic triangulated categories. Applying this construction to the triangulated hull $R_{A_Q}$, we obtain a Ringel–Hall Lie algebra $\mathfrak{g}(R_{A_Q})$, even the root category is not proper anymore [22]. We show that the spherical twist functor $\tilde{T}_{2, S_i}$ gives rise to an automorphism $\Phi_i$ of $\mathfrak{g}(R_{A_Q})$ in Section 3. Thus the Weyl group $W_Q$ is a quotient of the subgroup $\text{Aut}_0(\mathfrak{g}(R_{A_Q}))$, where $\text{Aut}_0(\mathfrak{g}(R_{A_Q}))$ is the subgroup of the automorphism group of $\mathfrak{g}(R_{A_Q})$ generated by $\Phi_i$. In Section 4, we explore connections between the Lie subalgebra $\mathfrak{g}_0(R_Q)$ of $\mathfrak{g}(R_Q)$ and the Lie subalgebra $\mathfrak{g}_0(R_{A_Q})$ of $\mathfrak{g}(R_{A_Q})$ generated by simple objects $\hat{u}_{S_i}$, $\hat{u}_{S_i[1]}$ and Cartan elements $\frac{h_i}{\delta(S_i)}$ for $i \in Q_0$. We show that the positive (resp. negative) part $\mathfrak{n}_0^+(R_Q)$ (resp. $\mathfrak{n}_0^-(R_Q)$) is isomorphic to the positive (resp. negative) part $\mathfrak{n}_0^+(R_{A_Q})$ (resp. $\mathfrak{n}_0^-(R_{A_Q})$). Since the triangulated hull $R_{A_Q}$ is not proper, there are more elements in $\mathfrak{g}_0(R_{A_Q})$ than in $\mathfrak{g}_0(R_Q)$. Let $\mathcal{I}$ be the ideal generated by elements $\hat{u}_{E_i} - \hat{u}_{E_i[1]}$, $i \in Q_0$, and $\mathfrak{h}' \subset \mathfrak{h}$ be the center of $\mathfrak{g}_0(R_Q)$. Then $\mathfrak{g}_0(R_{A_Q})/(\mathcal{I} + \mathfrak{h}')$ is either 0 or isomorphic to $\mathfrak{g}_0(R_Q)/\mathfrak{h}'$. 

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2. Preliminaries

2.1. Root category. Let $\Lambda_Q$ be the preprojective algebra of an acyclic quiver $Q = (Q_0, Q_1)$ over a field $k$, where $Q$ is not of Dynkin type, then $\Lambda_Q$ is of infinite dimensional with finite global dimension equal to 2. Let $\text{mod}\Lambda_Q$ be the category of finite dimensional $\Lambda_Q$-modules, and $D^b(\Lambda_Q)$ the bounded derived category of $\text{mod}\Lambda_Q$. Denote by $[1]$ the shift functor. Define the orbit category $D^b(\text{rep}^n\Lambda_Q)/[2]$ of $D^b(\Lambda_Q)$ to be the category having the same objects with $D^b(\text{rep}^n\Lambda_Q)$, and morphism spaces are given by

$$\text{Hom}_{D^b(\Lambda_Q)/[2]}(X, Y) := \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{D^b(\Lambda_Q)}(X, Y[2i]).$$

Let $\text{Proj}\Lambda_Q$ be the subcategory of all finitely generated projective $\Lambda_Q$-modules. It is well known that there is a triangle equivalence $p : D^b(\Lambda_Q) \to K^b(\text{Proj}\Lambda_Q)$ taking every object to its projective resolution. Denote by $\text{rep}^n\Lambda_Q$ the finite dimensional nilpotent $\Lambda_Q$-modules. Note that $\text{rep}^n\Lambda_Q$ is an abelian subcategory of $\text{mod}\Lambda_Q$, so we can consider the bounded derived category $D^b(\text{rep}^n\Lambda_Q)$ of $\text{rep}^n\Lambda_Q$. The image $p(D^b(\text{rep}^n\Lambda_Q))$ is a Hom-finite triangulated subcategory of $K^b(\text{Proj}\Lambda_Q)$ since $D^b(\text{rep}^n\Lambda_Q)$ is Hom-finite.

Let us recall Keller’s construction of the triangulated hull of a root category [11].

Denote $C_{dg}(\text{Proj}\Lambda_Q)$ by the DG category of bounded complexes of finitely generated projective $\Lambda_Q$-modules. Let $\mathcal{A}$ be the smallest DG subcategory of $C_{dg}(\text{Proj}\Lambda_Q)$ consisting of objects of $p(D^b(\text{rep}^n\Lambda_Q))$. Then

$$D^b(\text{rep}^n\Lambda_Q) \simeq p(D^b(\text{rep}^n\Lambda_Q)) = H^0(\mathcal{A}).$$

Let $\mathcal{B}$ be the DG category having the same objects with $\mathcal{A}$, and the morphism spaces in $\mathcal{B}$ are defined by

$$\mathcal{B}(X, Y) := \bigoplus_{i \in \mathbb{Z}} \mathcal{A}(X, Y[2i])$$

Then we have that

$$D^b(\text{rep}^n\Lambda_Q)/[2] \simeq H^0\mathcal{A}/[2] = H^0\mathcal{B}.$$ Let $D(\mathcal{B})$ be the derived category of DG-modules of $\mathcal{B}$, Keller showed that the perfect category (for definition see Remark 2.1) $\text{Per}\mathcal{B} \subset D\mathcal{B}$ is a triangulated hull of $D^b(\text{rep}^n\Lambda_Q)/[2]$ such that the composition functor

$$\pi : D^b(\text{rep}^n\Lambda_Q) \to D^b(\text{rep}^n\Lambda_Q)/[2] \simeq H^0(\mathcal{B}) \hookrightarrow \text{Per}\mathcal{B}$$

is a triangle functor.

**Remark 2.1.** The perfect category $\text{Per}\mathcal{B}$ is defined to be the smallest full triangulated subcategory of $D(\mathcal{B})$ containing $H^0(\mathcal{B})$ and closed under direct factors. Note that $H^0(\mathcal{B})$ is Hom-finite, it follows that $\text{Per}\mathcal{B}$ is also Hom-finite.
2.2. The relative homotopy category of 2-periodic complexes. Consider the relative homotopy category $K_2(\text{Proj}\Lambda Q)$ of 2-periodic complexes of finitely generated projective $\Lambda Q$-modules. The following result comes from [15].

**Theorem 2.2.** $K_2(\text{Proj}\Lambda Q)$ is a triangulated category with the shift functor defined for complex category. And the functor $\Delta : K^b(\text{Proj}\Lambda Q) \to K_2(\text{Proj}\Lambda Q)$ takes every complex $P^\bullet$ to

$$\bigoplus_i P_{2i} \mapsto \bigoplus_i P_{2i+1}$$

is exact.

Since $\Delta$ commutes with the shift functor [2], we have an induced functor $\Delta_2 : K^b(\text{Proj}\Lambda Q)/[2] \to K_2(\text{Proj}\Lambda Q)$. Moreover $\Delta_2$ is fully faithful (cf. [3, Lemma 3.1]).

**Remark 2.3.** Following from the construction of the triangulated hull and the fact $H^0(B) \hookrightarrow K^b(\text{Proj}\Lambda Q)/[2] \hookrightarrow K_2(\text{Proj}\Lambda Q)$, there is a fully faithful triangle functor $\iota : \text{Per}B \to K_2(\text{Proj}\Lambda Q)$.

According to Remark 2.1, we will regard the triangulated hull $\text{Per}B$ as the Hom-finite triangulated subcategory of $K_2(\text{Proj}\Lambda Q)$ in the following sections and denote it by $\mathcal{R}_{\Lambda Q}$.

In $K_2(\text{Proj}\Lambda Q)$, we need the notion of tensor products.

**Definition 2.4.** For $N^\bullet : N_0 \xrightarrow{d_N^0} N_1 \in K_2(\text{Proj}\Lambda Q)$ and a 2-periodic complex of finite dimensional vector spaces $V^\bullet : V_0 \xrightarrow{d_V^1} V_1$, define the tensor product $V^\bullet \otimes N^\bullet$ by

$$V^\bullet \otimes N^\bullet : (V_0 \otimes N_0) \oplus (V_1 \otimes N_1) \xrightarrow{d^0} (V_1 \otimes N_0) \oplus (V_0 \otimes N_1)$$

where $d^0 = \begin{bmatrix} d_V^0 \otimes 1 & -1 \otimes d_N^1 \\ 1 \otimes d_N^0 & d_V^1 \otimes 1 \end{bmatrix}$ and $d^1 = \begin{bmatrix} d_V^1 \otimes 1 & 1 \otimes d_N^0 \\ -1 \otimes d_N^1 & d_V^0 \otimes 1 \end{bmatrix}$.

**Remark 2.5.** It is easy to check that $V^\bullet \otimes N^\bullet$ belongs to $K_2(\text{Proj}\Lambda Q)$. In some sense, the definition of the tensor product of 2-periodic complexes is just the usual tensor product of complexes. Indeed, as $\mathbb{Z}_2$-graded vector spaces, $V^\bullet \otimes N^\bullet$ is the tensor product of $V_0 \oplus V_1[-1]$ with $N_0 \oplus N_1[-1]$, but the differential $d^0$ is given by the restriction of the differential $d_{V'} \otimes d_{N'}$, where

$$V' := V_0 \xrightarrow{d_V^1} V_1 \xrightarrow{d_V^1} V_0,$$

and $d^1$ is the restriction of the differential $d_{V''} \otimes d_{N''}$, where

$$V'' := V_1 \xrightarrow{d_V^1} V_0 \xrightarrow{d_V^1} V_1,$$

and

$$N' := N_0 \xrightarrow{d_N^1} N_1 \xrightarrow{d_N^1} N_0,$$

and

$$N'' := N_1 \xrightarrow{d_N^1} N_0 \xrightarrow{d_N^1} N_1.$$
Proposition 2.6. For $P$ and $Q$ in $K^b(\text{Proj} A_Q)$, we have that
\[ \Delta(P^* \otimes Q^*) = \Delta(P^*) \otimes \Delta(Q^*). \]

Proof. Without loss of generality, we may assume that $P^* = 0 \to P_0 \to \cdots \to P_n \to 0$ and $Q^* = 0 \to Q_0 \to \cdots \to Q_m \to 0$. Then:
\[ \Delta(P^*): \bigoplus \bigoplus P_{2i} \xrightarrow{(d_i^2)} P_{2i+1} \]

and
\[ \Delta(Q^*): \bigoplus \bigoplus Q_{2i+1}. \]

Then for $r \in \mathbb{N}$, as graded vector spaces, we have that
\[ \Delta(P^* \otimes Q^*)_0 = \bigoplus_{s+t=2r} P_s \otimes Q_t = \bigoplus_{i} P_{2i} \otimes \bigoplus_{j} Q_{2j} + \bigoplus_{i} \bigoplus_{j} Q_{2j+1} \]

So $\Delta(P^* \otimes Q^*)_0 = (\Delta(P^*) \otimes \Delta(Q^*))_0$. Similarly, we can show it for $\Delta(P^* \otimes Q^*)_1$.

For $x_i \otimes y_j \in \Delta(P^* \otimes Q^*)$, if $i$ and $j$ are both even, then $d_{\Delta}(x_i \otimes y_j) = d_{P^*}(x_i \otimes y_j) = d(x_i) \otimes y_j + (-1)^{|x_i|}x_i \otimes d(y_j)$. On the other hand,
\[ d_{\Delta \otimes \Delta}(x_i \otimes y_j) = \begin{bmatrix} d_{\Delta(P)} \otimes 1 & 1 \otimes d_{\Delta(Q)} \\ 1 \otimes d_{\Delta(Q)} & d_{\Delta(P)} \otimes 1 \end{bmatrix} \begin{bmatrix} x_i \otimes y_j \\ 0 \end{bmatrix} = d_{\Delta(P)}(x_i) \otimes y_j + (-1)^{|x_i|}x_i \otimes d_{\Delta(Q)}(y_j). \]
Hence, the differentials of both sides are the same. \hfill \square

2.3. Euler form. Let $A$ be the path algebra $kQ$ of the acyclic quiver $Q$ and $A_Q$ the preprojective algebra of $Q$. Denote $S_i$ by the simple $A$-modules associated to $i \in Q_0$. Also, they are simple $A_Q$-modules. Let $\langle \cdot, \cdot \rangle_A$ be the Euler form of $A$ on the Grothendieck group $K_0(A)$ of the category $\text{mod} A$ of finite dimensional $A$-modules (also the derived category $D^b(\text{mod} A)$). Denote by $\hat{M}$ the class of $M \in \text{mod} A$ in $K_0(A)$, then
\[ \langle \hat{M}, \hat{N} \rangle_A := \sum_{i \in \mathbb{Z}} (-1)^i \dim_k \text{Hom}_{D^b(A)}(M, N[i]). \]

And the symmetric Euler form $(-,-)_A$ is given by
\[ (\hat{M}, \hat{N})_A = \langle \hat{M}, \hat{N} \rangle_A + \langle \hat{N}, \hat{M} \rangle_A. \]
The Euler form $\langle \cdot, \cdot \rangle_{A_Q}$ of $A_Q$ is defined similarly. Note that $D^b(\text{rep}^n A_Q)$ is a 2-Calabi-Yau category, we have the following result. To simplify notations, we will denote $|V|$ by the dimension of the vector space $V$.

Lemma 2.7. For all simple modules $S_i, S_j$, $i, j \in Q_0$, we have that
\[ \langle \hat{S}_i, \hat{S}_j \rangle_{A_Q} = (\hat{S}_i, \hat{S}_j)_A. \]

In particular, $\langle \cdot, \cdot \rangle_{A_Q}$ is a symmetric bilinear form.
Proof. If \( i = j \), then
\[
< \hat{S}_i, \hat{S}_i >_{A_Q} = |\text{Hom}_{A_Q}(S_i, S_i)| - |\text{Ext}^1_{A_Q}(S_i, S_i)| + |\text{Ext}^2_{A_Q}(S_i, S_i)| = 2.
\]
by direct calculations.

If \( i \neq j \), we know that \( |\text{Ext}^1_{A_Q}(S_i, S_j)| = \#\{\alpha : i \to j | \alpha \in Q_1\} = a_{ij} \). Note that the quiver \( Q \) is acyclic, either \( a_{ij} = 0 \) or \( a_{ji} = 0 \). Then we have that
\[
< \hat{S}_i, \hat{S}_i >_{A_Q} = |\text{Hom}_{A_Q}(S_i, S_j)| - |\text{Ext}^1_{A_Q}(S_i, S_j)| + |\text{Ext}^2_{A_Q}(S_i, S_j)| = -(a_{ij} + a_{ji}).
\]
On the other hand,
\[
|\text{Hom}_{A}(S_i, S_j)| = \delta_{ij} \text{ and } |\text{Ext}^2_{A}(S_i, S_j)| = 0
\]
for any \( i, j \), then \( < \hat{S}_i, \hat{S}_i >_{A_Q} = (\hat{S}_i, \hat{S}_i)_A = 2 \). If \( i \neq j \), \( < \hat{S}_i, \hat{S}_j >_{A_Q} = (\hat{S}_i, \hat{S}_j)_A = -(a_{ij} + a_{ji}) \). The proof is completed. \( \square \)

Note that the composition \( D^b(\text{rep}^n_{A_Q}) \xrightarrow{\pi} \mathcal{R}_{A_Q} \subset K_2(\text{Proj}_{A_Q}) \) is a triangle functor and \( K_0(D^b(\text{rep}^n_{A_Q})) \cong \mathbb{Z}Q_0 \cong K_0(K_2(\text{Proj}_{A_Q})) \), it follows that there is an isomorphism \( \eta : K_0(\mathcal{R}_{A_Q}) \cong \mathbb{Z}Q_0 \) by sending \( \hat{S}_i \) to \( \alpha_i \).

To construct a Kac-Moody Lie algebra of \( \mathcal{R}_{A_Q} \), we need a bilinear form \((-|-)_{\mathcal{R}_{A_Q}}\) on the Grothendieck group \( K_0(\mathcal{R}_{A_Q}) \). Define a bilinear form on \( K_0(\mathcal{R}_{A_Q}) \) by
\[
(\hat{M}|\hat{N})_{\mathcal{R}_{A_Q}} := \dim_k \text{Hom}_{\mathcal{R}_{A_Q}}(M, N) - \dim_k \text{Hom}_{\mathcal{R}_{A_Q}}(M, N[1]).
\]
for \( \hat{M}, \hat{N} \) in \( K_0(\mathcal{R}_{A_Q}) \).

Remark 2.8. Note that \( \mathcal{R}_{A_Q} \) is 2-periodic, the bilinear form above is well-defined, i.e. for any triangle \( N' \to N \to N'' \to N'[1] \) in \( \mathcal{R}_{A_Q} \) and \( M \in \mathcal{R}_{A_Q} \), we have
\[
(\hat{M}|\hat{N})_{\mathcal{R}_{A_Q}} := (\hat{M}|\hat{N'})_{\mathcal{R}_{A_Q}} + (\hat{M}|\hat{N''})_{\mathcal{R}_{A_Q}}.
\]
Moreover, the bilinear form is symmetric. Indeed, by using the fact
\[
\text{Hom}_{\mathcal{R}_{A_Q}}(S_i, S_j) = \text{Hom}_{D^b(\mathcal{A}_Q)}(S_i, S_j) \oplus \text{Hom}_{D^b(\mathcal{A}_Q)}(S_i, S_j[2]),
\]
and
\[
\text{Hom}_{\mathcal{R}_{A_Q}}(S_i, S_j[1]) = \text{Hom}_{D^b(\mathcal{A}_Q)}(S_j, S_i[1]),
\]
it follows that
\[
(\hat{S}_i|\hat{S}_j)_{\mathcal{R}_{A_Q}} = < \hat{S}_i, \hat{S}_j >_{A_Q}
\]
and \( < -, -, >_{A_Q} \) is symmetric by Lemma 2.7.

Corollary 2.9. The triangle functor \( \pi : D^b(\text{rep}^n_{A_Q}) \to \mathcal{R}_{A_Q} \) induces an isotropy \( \pi^* : K_0(\mathcal{A}_Q) \to K_0(\mathcal{R}_{A_Q}) \).

Proof. We know that the map \( \pi^*(\hat{S}_i) = \hat{S}_i \) is an isomorphism of groups by Remark 2.8. Also Remark 2.8 tells us that \( < \hat{S}_i, \hat{S}_j >_{A_Q} = (\hat{S}_i|\hat{S}_j)_{\mathcal{R}_{A_Q}} \). \( \square \)
2.4. **Integral Hall algebra.** Fix a finite field $k = \mathbb{F}_q$. Let $R$ be a finitary algebra over $k$ and mod$R$ the category of finite dimensional right $R$-modules. The integral Ringel–Hall algebra $\mathcal{H}(R)$ of $R$ is by definition the free abelian group with the basis consisting of all isoclasses $[M]$ for $M \in \text{mod}R$. The multiplication is given by

$$[M][N] := \sum_{[L]} g_{MN}^L[L],$$

where $g_{MN}^L$ is the number of submodules $X$ of $L$ such that $X \cong N$ and $L/X \cong M$.

Denote by $C\mathcal{H}(R)$ the subalgebra of $\mathcal{H}(R)$ generated by $[S_i]$, $i \in \mathbb{Q}_0$, which is the so-called composition algebra of $\mathcal{H}(R)$. The following proposition is given in Proposition 5.1 of [4].

**Proposition 2.10.** Let $R$ be a finitary $k$-algebra and let $R'$ be a factor algebra of $R$. Then there are epimorphisms $\mathcal{H}(R) \rightarrow \mathcal{H}(R')$ and $C\mathcal{H}(R) \rightarrow C\mathcal{H}(R')$ of $\mathbb{Z}$-algebras.

2.5. **The Ringel-Hall Lie algebra.** Recall the definition of the Ringel-Hall Lie algebra of a root category $\mathcal{R}$ following [16]. By ind$\mathcal{R}$ we denote the set of all isoclasses of indecomposable objects in $\mathcal{R}$.

Given $X, Y, L \in \text{Obj}(\mathcal{R})$, we define

$$W(X, Y; L) := \{(f, g, h) \in \text{Hom}_\mathcal{R}(Y, L) \times \text{Hom}_\mathcal{R}(L, X) \times \text{Hom}_\mathcal{R}(X, Y[1])\}$$

$$Y \xrightarrow{f} L \xrightarrow{g} X \xrightarrow{h} Y[1] \text{ is a triangle },$$

$$V(X, Y; L) := W(X, Y; L)/(\text{Aut}(Y) \times \text{Aut}(X)).$$

where the action of $\text{Aut}(Y) \times \text{Aut}(X)$ on $W(X, Y; L)$ is given by

$$(a, c).(f, g, h) = (fa, c^{-1}g, a[1]^{-1}hc)$$

for $(f, g, h) \in W(X, Y; L)$ and $(a, c) \in \text{Aut}(X) \times \text{Aut}(Y)$. In the following, we will denote by $(f, g, h)$ the orbit of $(f, g, h)$ in $V(X, Y; L)$.

Let $\text{Hom}_\mathcal{R}(Y, L)_X$ be the subset of $\text{Hom}_\mathcal{R}(Y, L)$ consisting of morphisms $f : Y \rightarrow L$ such that $\text{cone}f \cong X$. Consider the action of $\text{Aut}(Y)$ on $\text{Hom}_\mathcal{R}(Y, L)_X$ by $a.f = fa$, the orbit space is denoted by

$$\text{Hom}_\mathcal{R}(Y, L)_X^* := \text{Hom}_\mathcal{R}(Y, L)_X/\text{Aut}(Y).$$

Dually, consider the subset $\text{Hom}_\mathcal{R}(L, X)_Y[1]$ of $\text{Hom}_\mathcal{R}(L, X)$ and $\text{Aut}(X)$ acts on $\text{Hom}_\mathcal{R}(L, X)_Y[1]$ by $c.g = gc^{-1}$, we have another orbit space $\text{Hom}_\mathcal{R}(L, X)_Y[1]^*$. 

**Proposition 2.11** ([21]). $|V(X, Y; L)| = |\text{Hom}_\mathcal{R}(Y, L)_X^*| = |\text{Hom}_\mathcal{R}(L, X)_Y[1]^*|$

Let $\mathfrak{h}$ be the subgroup of $K_0(\mathcal{R}) \otimes \mathbb{Q}$ generated by $\frac{h}{d(M)}$ with $M \in \text{ind}\mathcal{R}$ and $d(M) = \dim_k \text{End} M/\text{rad(End} M)$. One can naturally extend the symmetric Euler form $(-|-)_\mathcal{R}$ to $\mathfrak{h} \times \mathfrak{h}$. Let $\mathfrak{n}$ be the free abelian group with the basis $\{u_X | X \in \text{ind}\mathcal{R}\}$. Let

$$\mathfrak{g}(\mathcal{R}) = \mathfrak{h} \oplus \mathfrak{n}$$

be a direct sum of $\mathbb{Z}$-modules. The Lie operation is given as follows:
for any indecomposable objects $X, Y \in \mathcal{R}$,
\[
[u_X, u_Y] = \sum_{L \in \text{ind}\mathcal{R}} (F^L_X - F^L_Y) u_L - \delta_{X,Y[1]} \frac{h_X}{d(X)},
\]
where $F^L_X := |V(X, Y; L)|$.

- for any objects $X, Y \in \mathcal{R}$ with $Y$ indecomposable
  \[
  [h_X, u_Y] = (h_X|h_Y) \mathcal{R} u_Y = -[u_Y, h_X]
  \]
- $[h, h] = 0$.

Consider the quotient algebra
\[
\mathfrak{g}(\mathcal{R})_{(q-1)} = \mathfrak{g}(\mathcal{R})/(q-1) \mathfrak{g}(\mathcal{R}).
\]

Then by Peng-Xiao [16] we know that $\mathfrak{g}(\mathcal{R})_{(q-1)}$ is a Lie algebra over $\mathbb{Z}/(q-1)$.

Recall that $A$ is the path algebra $kQ$ of the acyclic quiver $Q$ of non-Dynkin type and $A_Q$ is the preprojective algebra. Since $A$ is hereditary, the root category $\mathcal{R}_A$ is triangle equivalent to the homotopy category $K_2(\text{Proj} A)$ of 2-periodic complexes of finitely generated projective $A$-modules. Recall that $\mathcal{R}_{AQ}$ is the triangulated hull of $D^b(\text{rep}^n A_Q)/[2]$. Since $\mathcal{R}_A$ and $\mathcal{R}_{AQ}$ are both Hom-finite $k$-linear triangulated category with the shift functor $[1]$, we have the corresponding Ringel-Hall Lie algebras $\mathfrak{g}(\mathcal{R}_A)_{q-1} = \mathfrak{h}(A) \oplus \mathfrak{n}(A)$ and $\mathfrak{g}(\mathcal{R}_{AQ})_{q-1}$ over $\mathbb{Z}/(q-1)$.

Set $h_i := \frac{h_i}{d(A_i)}$, consider the Lie subalgebra $\mathfrak{g}_0(\mathcal{R}_A)$ (resp. $\mathfrak{g}_0(\mathcal{R}_{AQ})$) of $\mathfrak{g}(\mathcal{R}_A)_{(q-1)}$ (resp. $\mathfrak{g}(\mathcal{R}_{AQ})_{(q-1)}$) generated by $\{u_{S_i}, u_{S_i[1]}, h_j | j \in Q_0\}$ (resp. $\{\check{u}_{S_i}, \check{u}_{S_i[1]}, h_j | j \in Q_0\}$), and $\mathfrak{n}_0^+(A)$ (resp. $\mathfrak{n}_0^+(\mathcal{R}_{AQ})$) generated by $\{u_{S_i[j]} \in Q_0\}$ (resp. $\{\check{u}_{S_i[j]} \in Q_0\}$).

By $\check{u}_{S_j}$ we mean $\check{u}_{\Delta_{\text{red}}(S_j)}$, i.e., the image of $S_j$ in $H^0(\mathcal{B}) \subset K^0_2(\text{Proj} A_Q)$.

3. Spherical Twist Functors and Weyl Groups

3.1. Construction of spherical twist functors. Since $D^b(\text{rep}^n A_Q)$ is a 2-Calabi-Yau category, every simple object is a 2-spherical object (i.e., $\mathbb{R}\text{Hom}(S_i, S_i) \cong k \oplus k[-2]$.) By [19], for a simple $A_Q$-module $S$, let $P_S$ be the minimal projective resolution of $S$, then the spherical twist functor $T_S : D^b(\text{rep}^n A_Q) \to D^b(\text{rep}^n A_Q)$ is given by
\[
T_S(X) = \text{cone}(\text{Hom}^*(P_S, X) \otimes P_S \xrightarrow{ev} X)
\]
where $\text{Hom}^*(P_S, X)$ is the usual complex of vector spaces and $ev$ is the natural evaluation map, which is a cochain map. For any chain map $f : X \to Y$,
\[
T_S(f) = \begin{bmatrix} f & 0 \\ 0 & f_\ast \otimes 1 \end{bmatrix}.
\]
Moreover, $T_S$ is a triangle autoequivalence on $D^b(\text{rep}^n A_Q)$ (cf. [19, Proposition 2.10]).

Remark 3.1.

(i) The spherical twist functor $T_S$ is well defined. Note that $\text{Hom}^*(P_S, X) \otimes P_S$ is a bounded complex since both $P_S$ and $X$ are bounded. Furthermore, we have that
H^*(\text{Hom}^*(P_S, X) \otimes P_S) \cong H^*(\text{Hom}^*(P_S, X)) \otimes H^*(P_S) has finite total dimension, so \text{Hom}^*(P_S, X) \otimes P_S belongs to \text{D}^b(\text{rep}^n A_Q), it follows that the cone \text{T}_S(X) lies in \text{D}^b(\text{rep}^n A_Q).

(ii) The spherical twist functor \text{T}_S : \text{D}^b(\text{rep}^n A_Q) \rightarrow \text{D}^b(\text{rep}^n A_Q) is induced by a DG functor of some DG categories. Indeed, we have known that \text{A} is the DG enhancement of \text{D}^b(\text{rep}^n A_Q), that is \text{H}^0(\text{A}) \simeq \text{D}^b(\text{rep}^n A_Q). Moreover, there is a DG functor

\text{T}_S' : \text{A} \rightarrow \text{A}, \ P_X \mapsto \text{cone}(ev : \text{Hom}^*(P_S, P_X) \otimes P_S \rightarrow P_X),

sending \( f \in \text{Hom}^i(P_X, P_Y) \) to

\[ \text{T}_S'(f) := \begin{bmatrix} f & 0 \\ 0 & (-1)^i f_s \otimes 1 \end{bmatrix}. \]

Therefore, if we regard \text{D}^b(\text{rep}^n A_Q) as \text{H}^0(\text{A}), then \text{T}_S = \text{H}^0(\text{T}_S').

**Example 3.2.** For \( i \in \mathbb{Q}_0 \), let \( S_i \) be the simple \( A_Q \)-module. The minimal projective resolution \( P_{S_i} \) of \( S_i \) is given by

\[ P_{S_i} := 0 \rightarrow P_i \rightarrow \bigoplus_{l(h_i) = i} P_{s(h_i)} \rightarrow P_i \rightarrow 0 \]

Then \( \text{Hom}^*(P_{S_i}, S_i) = k \oplus k[-2], \) and \( ev = (f_0, f_2) : P_{S_i} \oplus P_{S_i}[-2] \rightarrow S_i, \) where \( f_0 \in \text{Hom}_{A_Q}(S_i, S_i) \) and \( f_1 \in \text{Ext}^2_{A_Q}(S_i, S_i) \) are nontrivial. Hence \( \text{T}_S(S_i) = \text{cone}(ev) \cong S_i[-1]. \)

Set \( a_{ji} = \text{dim} \text{Ext}^1_{A_Q}(S_i, S_j). \) Since \( \text{Hom}^*(P_{S_i}, S_j) \otimes P_{S_i} = P_{S_i}^{\oplus a_{ji}}[-1], \) it follows that \( \text{T}_S(S_j) = I_{ji}, \) where \( I_{ji} \) is an extension of \( S_i^{\oplus a_{ji}} \) by \( S_j, \) i.e., \( I_{ji} = (V_i \oplus V_j, h \in \mathbb{Q}_1), \)

where \( V_i = k^{\oplus a_{ji}}, V_j = k, h_l = (0, \cdots, 0, 1, 0, \cdots, 0) : V_i \rightarrow V_j \) with \( 1 \) in the \( l \)-th term, if \( t(h_i) = j, \ 1 \leq l \leq a_{ji}, \) and \( h_l = 0 \) for all \( l. \) Note that \( I_{ji} \) is indecomposable.

We recall some useful properties for spherical twist functors from [20, Proposition 6.8].

**Proposition 3.3.** Let \( S \) be a spherical object in \( \text{D}^b(\text{rep}^n A_Q). \)

(i) For an integer \( l \in \mathbb{Z}, \) we have that \( \text{T}_S[l] \cong \text{T}_S. \)

(ii) We have that \( \text{T}_S(S) = S[-1]. \)

Note that \( \text{T}_S' \) commutes with shift functor [2], we have an induced functor \( \text{T}_{2,S} : \text{B} \rightarrow \text{B} \). We have the following diagram commute.

\[
\begin{array}{ccc}
\text{D}^b(\text{rep}^n A_Q) & \xrightarrow{T_S} & \text{D}^b(\text{rep}^n A_Q) \\
\downarrow p & & \downarrow p \\
\text{H}^0(\text{A}) & \xrightarrow{\text{H}^0(T'_S)} & \text{H}^0(\text{A}) \\
\downarrow F & & \downarrow F \\
\text{H}^0(\text{B}) & \xrightarrow{\text{H}^0(T'_{2,S})} & \text{H}^0(\text{B}).
\end{array}
\]
Denote by $T_{2,S}$ the induced functor $H^0(T'_{2,S})$, then $T_{2,S} : H^0(\mathcal{B}) \to H^0(\mathcal{B})$ is an equivalence.

To construct a functor $\bar{T}_{2,S} : K_2(\text{Proj}A_Q) \to K_2(\text{Proj}A_Q)$ extending the induced twist functor $T_{2,S} : H^0(\mathcal{B}) \to H^0(\mathcal{B})$, for $P^\bullet, Q^\bullet \in K_2(\text{Proj}A_Q)$, we define $B_2(P^\bullet, Q^\bullet)$ to be a 2-periodic complexes of vector spaces as follows:

$$B_2(P^\bullet, Q^\bullet) : \text{Hom}_{A_Q}(P^\bullet, Q^\bullet) \xrightarrow{d^0} \text{Hom}_{A_Q}(P^\bullet, Q^\bullet[1])$$

where $\text{Hom}_{A_Q}(P^\bullet, Q^\bullet) = \text{Hom}_{A_Q}(P_0, Q_0) \oplus \text{Hom}_{A_Q}(P_1, Q_1)$, $d^0$ is defined by

$$d^0(f_0) = d^0_2(f_0) - f_0d^1_p, \quad d^0(f_1) = d^1_2(f_1) - f_1d^0_p,$$

for $f = (f_0, f_1) \in \text{Hom}_{A_Q}(P^\bullet, Q^\bullet)$, and $d^1$ is given by

$$d^1(g_0) = d^1_2(g_0) + g_0d^1_p, \quad d^1(g_1) = d^1_2(g_1) + g_1d^0_p,$$

for $g = (g_0, g_1) \in \text{Hom}_{A_Q}(P^\bullet, Q^\bullet[1])$. It can be checked that $d^0d^1 = 0$ and $d^1d^0 = 0$.

Note if $P^\bullet$ and $Q^\bullet$ are in the image of the compression functor $\Delta : K^b(\text{Proj}A_Q) \to K_2(\text{Proj}A_Q)$, then $B_2(P^\bullet, Q^\bullet)$ is precisely the compression of $\text{Hom}^\bullet(P^\bullet, Q^\bullet)$ by definition.

Define the functor $\bar{T}_{2,S} : K_2(\text{Proj}A_Q) \to K_2(\text{Proj}A_Q)$ by

$$\bar{T}_{2,S}(Q^\bullet) = \text{Cone}(ev : B_2(\Delta(P_S), Q^\bullet) \otimes \Delta(P_S) \to Q^\bullet).$$

for any object $Q^\bullet \in K_2(\text{Proj}A_Q)$,

$$\bar{T}_{2,S}(f) := \begin{bmatrix} f & 0 \\ 0 & f \otimes 1 \end{bmatrix} : \text{cone}(ev) \to \text{cone}(ev'),$$

for any $f \in \text{Hom}_{K_2(\text{Proj}A_Q)}(Q^\bullet, Q^\bullet)$. Because $f \otimes 1 : B_2(\Delta(P_S), Q^\bullet) \otimes \Delta(P_S) \to B_2(\Delta(P_S), Q^\bullet) \otimes \Delta(P_S)$ is also a morphism of 2-periodic complexes and $f \circ ev = ev' \circ (f \otimes 1)$, $\bar{T}_{2,S}(f) : \text{cone}(ev) \to \text{cone}(ev')$ is well defined.

**Lemma 3.4.** The functor $\bar{T}_{2,S} : K_2(\text{Proj}A_Q) \to K_2(\text{Proj}A_Q)$ restricts to a functor $T_{2,S} : H^0(\mathcal{B}) \to H^0(\mathcal{B})$ along the fully faithful functor $\Delta_2 : K^b(\text{Proj}A_Q)/[2] \to K_2(\text{Proj}A_Q)$.

**Proof.** By the definitions of $\bar{T}_{2,S}$ and $T_{2,S}$, they are the same in morphism spaces. It suffices to show $\bar{T}_{2,S}(\Delta_2(P^\bullet)) = \Delta_2(T_{2,S}(P^\bullet)) = \Delta(T_{2,S}(P^\bullet))$ for each object $P^\bullet$ in $H^0(\mathcal{B})$. Note that $\Delta(\text{Hom}^\bullet(P_S, P^\bullet) \otimes P_S) = \Delta(\text{Hom}^\bullet(P_S, P^\bullet)) \otimes \Delta(P_S) = B_2(\Delta(P_S), \Delta(P^\bullet)) \otimes \Delta(P_S)$ by Proposition 2.6, then the evaluation map

$$ev' : B_2(\Delta(P_S), \Delta(P^\bullet)) \otimes \Delta(P_S) \to \Delta(P^\bullet)$$

in $K_2(\text{Proj}A_Q)$ is precisely $\Delta(ev : \text{Hom}^\bullet(P_S, P^\bullet) \otimes P_S \to P^\bullet)$. Thus we have $\bar{T}_{2,S}(\Delta(P^\bullet)) = \text{cone}(ev') = \text{cone}(\Delta(ev)) = \Delta(\text{cone}(ev)) = T_{2,S}(P^\bullet)$, which completes the proof. \qed
Next, we will prove the following proposition. Our strategy is firstly to show \( \tilde{T}_{2,S} : K_2(\text{Proj}\,\Lambda_Q) \rightarrow K_2(\text{Proj}\,\Lambda_Q) \) is a triangle functor. Due to the above Lemma 3.4 and the fact that \( \mathcal{R}_{\Lambda_Q} \) is the smallest triangulated subcategory of \( K_2(\text{Proj}\,\Lambda_Q) \) containing \( H^0(\mathcal{B}) \), it follows that \( \tilde{T}_{2,S} \) sends \( \mathcal{R}_{\Lambda_Q} \) to \( \mathcal{R}_{\Lambda_Q} \).

**Proposition 3.5.** The restriction functor \( \tilde{T}_{2,S}|_{\mathcal{R}_{\Lambda_Q}} : \mathcal{R}_{\Lambda_Q} \rightarrow \mathcal{R}_{\Lambda_Q} \) of the functor \( \tilde{T}_{2,S} \) along the triangulated hull \( \mathcal{R}_{\Lambda_Q} \) is a triangle equivalence.

In order to prove Proposition 3.5, we introduce a DG category \( \mathcal{C} \):

\( \text{Obj}(\mathcal{C}) \) = all 2-periodic complexes of finitely generated projective \( \Lambda_Q \)-modules

For \( M^\bullet, N^\bullet \in \mathcal{C} \), the morphism space is defined by

\[
\text{Hom}_\mathcal{C}(M^\bullet, N^\bullet) := \bigoplus_{i \in \mathbb{Z}_2} \text{Hom}_{\Lambda_Q}(M^\bullet, N^\bullet[i]),
\]

where \( \text{Hom}_{\Lambda_Q}(M^\bullet, N^\bullet) = \text{Hom}_{\Lambda_Q}(M_0, N_0) \oplus \text{Hom}_{\Lambda_Q}(M_1, N_1) \), and the differential \( d \) of \( \text{Hom}_\mathcal{C}(M^\bullet, N^\bullet) \) is given by

\[
d(f) := d_N f - (-1)^i f d_M,
\]

for \( f \in \text{Hom}_\mathcal{C}^i(M^\bullet, N^\bullet) \). The composition map \( \text{Hom}_\mathcal{C}^i(N^\bullet, L^\bullet) \otimes_k \text{Hom}_\mathcal{C}^i(M^\bullet, N^\bullet) \rightarrow \text{Hom}_\mathcal{C}^i(M^\bullet, L^\bullet) \) is given by \( f^i \otimes g^j \mapsto f^i g^j \) for \( i, j = 0, 1 \). It can be checked that \( \mathcal{C} \) is a DG category. And it is clear that \( H^0(\mathcal{C}) = K_2(\text{Proj}\,\Lambda_Q) \).

Define a functor \( \tilde{T}_2 : \mathcal{C} \rightarrow \mathcal{C} \) by

\[
\tilde{T}_2(M^\bullet) := \text{cone}(ev : B_2(\Delta(P_S), M^\bullet) \otimes \Delta(P_S) \rightarrow M^\bullet),
\]

for \( f \in \text{Hom}_\mathcal{C}^i(M^\bullet, N^\bullet) \),

\[
\tilde{T}_2(f) := \begin{bmatrix} f & 0 \\ 0 & (-1)^i f_* \otimes 1 \end{bmatrix} \in \text{Hom}_\mathcal{C}(\tilde{T}_2(X), \tilde{T}_2(Y))^i.
\]

**Lemma 3.6.** The functor \( \tilde{T}_2 : \mathcal{C} \rightarrow \mathcal{C} \) defined as above is a DG functor. In particular, the homotopy functor \( H^0(\tilde{T}_2) \) is the functor \( \tilde{T}_{2,S} : K_2(\text{Proj}\,\Lambda_Q) \rightarrow K_2(\text{Proj}\,\Lambda_Q) \).

**Proof.** We need to show \( (\tilde{T}_2)_{X,Y} : \text{Hom}_\mathcal{C}^i(X, Y) \rightarrow \text{Hom}_\mathcal{C}^i(\tilde{T}_2(X), \tilde{T}_2(Y)) \) is a strict morphism for any \( X, Y \in \text{Obj}(\mathcal{C}) \), i.e., \( d_{\text{Hom}_\mathcal{C}^i(\tilde{T}_2(X), \tilde{T}_2(Y))}(\tilde{T}_2)_{X,Y} = (\tilde{T}_2)_{X,Y}d_{\text{Hom}_\mathcal{C}^i(X,Y)}. \)

Namely, for any \( f \in \text{Hom}_\mathcal{C}^i(X, Y) \),

\[
d_{\tilde{T}_2(X)} \tilde{T}_2(f) - (-1)^i \tilde{T}_2(f)d_{\tilde{T}_2(X)} = \begin{bmatrix} d(f) & 0 \\ 0 & (-1)^{i+1}(df)_* \otimes 1 \end{bmatrix}
\]

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where \( d(f) = d_Y f - (-1)^i f d_X \).

\[
\begin{align*}
\text{LHS} &= \begin{bmatrix} d_Y & \text{ev} \\ 0 & -d_{B_2(\Delta(P_S), \Delta(P_S))} \end{bmatrix} \begin{bmatrix} f \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ (-1)^i f_* \otimes 1 \end{bmatrix} \\
&= (-1)^i \begin{bmatrix} f \\ 0 \end{bmatrix} \begin{bmatrix} f \\ 0 \end{bmatrix} \begin{bmatrix} -d_{B_2(\Delta(P_S), \Delta(P_S))} \end{bmatrix} \\
&= \begin{bmatrix} d_Y f - (-1)^i f d_X \\ 0 \end{bmatrix} \begin{bmatrix} (\text{ev} \cdot (f_* \otimes 1) - f \cdot \text{ev}) \\ (f_* \otimes 1) \end{bmatrix} \\
&= \begin{bmatrix} d_Y f - (-1)^i f d_X \\ 0 \end{bmatrix} \begin{bmatrix} (d_{B_2(\Delta(P_S), \Delta(P_S))}(f_* \otimes 1) - (-1)^i f d_X) \end{bmatrix} \\
&= \begin{bmatrix} d_Y f - (-1)^i f d_X \\ 0 \end{bmatrix} \begin{bmatrix} (d_{B_2(\Delta(P_S), \Delta(P_S))}(f_* \otimes 1) - (-1)^i (f d_X)_* \otimes 1) \end{bmatrix} \\
&= \text{RHS}.
\end{align*}
\]

The second statement is clear from the definition of \( \bar{T}_2 \).

So \( \bar{T}_{2,S} : K_2(\text{Proj}A_Q) \to K_2(\text{Proj}A_Q) \) is a triangle functor by [24, Theorem 4.4.41], which implies the restriction \( \bar{T}_{2,S}|_{\mathcal{R}_A} : \mathcal{R}_A \to \mathcal{R}_A \) is also a triangle functor.

**Proof of Proposition 3.5**

**Proof.** We only need to show \( \bar{T}_{2,S}|_{\mathcal{R}_A} = \mathcal{R}_A \to \mathcal{R}_A \) is an equivalence. Note that \( \bar{T}_{2,S}|_{H^0B} = T_{2,S} \) and \( T_{2,S} \) is an equivalence, hence \( \bar{T}_{2,S}|_{\mathcal{R}_A} \) is fully faithful and dense on \( H^0B \). By the definition of the triangulated hull \( \mathcal{R}_A = \text{Per}B \) (which is the smallest triangulated subcategory containing \( H^0B \) and closed under direct summands), it follows that the functor \( \bar{T}_{2,S}|_{\mathcal{R}_A} \) is fully faithful by using Five lemma continuously. It can be showed in the same way that \( \bar{T}_{2,S}|_{\mathcal{R}_A} \) is dense.

In the following sections, we will simply denote by \( \bar{T}_{2,S} \) the restriction functor \( \bar{T}_{2,S}|_{\mathcal{R}_A} \), and call it the spherical twist functor.

### 3.2. Isomorphisms of Lie algebras induced by spherical twist functors

Fix a vertex \( i \in Q_0 \), the spherical twist functor \( \bar{T}_{2,S_i} : \mathcal{R}_A \to \mathcal{R}_A \) (for simplicity, denote \( \bar{T}_i := \bar{T}_{2,S_i} \)) naturally induces a morphism

\[
\Phi_i : g(\mathcal{R}_A)_{q-1} \to g(\mathcal{R}_A)_{q-1} : u_X \mapsto u_{\bar{T}_i(X)}, h_Y \mapsto h_{\bar{T}_i(Y)}.
\]

**Proposition 3.7.** For any simple \( \Lambda_Q \)-module \( S_i \), the spherical twist functor \( \bar{T}_{2,S_i} : \mathcal{R}_A \to \mathcal{R}_A \) induces an isomorphism \( \Phi_i : g(\mathcal{R}_A)_{q-1} \to g(\mathcal{R}_A)_{q-1} \) of Lie Algebras.

**Proof.** First, we need to show that \( \Phi_i \) is a homomorphism of Lie algebras.

For \( X, Y \in \text{ind}\mathcal{R}_A \), \( \Phi_i([\bar{u}_X, \bar{u}_Y]) = \sum_L (F_{L}^{T_i} X - F_{L}^{T_i} Y) \bar{u}_{T_i L} + \delta_{X,Y} [h_{\bar{T}_i(X)}, h_{\bar{T}_i(Y)}] \). On the other hand, \( [\bar{u}_{T_i X}, \bar{u}_{T_i Y}] = \sum_L (F_{L}^{T_i} X - F_{L}^{T_i} Y) \bar{u}_L + \delta_{X,Y} [h_{\bar{T}_i(X)}, h_{\bar{T}_i(Y)}] \). Since
\[ \tilde{T}_i : \mathcal{R}_{A_Q} \to \mathcal{R}_{A_Q} \] is a triangle equivalence, we have \( Y \xrightarrow{f} L \xrightarrow{g} X \xrightarrow{h} Y[1] \) is a triangle iff \( \tilde{T}_i Y \xrightarrow{\tilde{T}_i(f)} \tilde{T}_i L \xrightarrow{\tilde{T}_i(g)} \tilde{T}_i X \xrightarrow{\tilde{T}_i(h)} \tilde{T}_i Y[1] \) is a triangle. Moreover,

\[
(f, g, h) \sim^{\text{Aut}(Y)} (f', g', h') \iff (\tilde{T}_i(f), \tilde{T}_i(g), \tilde{T}_i(h)) \sim^{\text{Aut}(\tilde{T}_i Y)} (\tilde{T}_i(f'), \tilde{T}_i(g'), \tilde{T}_i(h')),
\]

for \( \tilde{T}_i \) preserves isomorphisms and \( \tilde{T}_i(id) = id \). Therefore, we have

\[
\frac{\text{Hom}_{\mathcal{R}_{A_Q}}(Y, L)_X}{\text{Aut}(Y)} = \frac{\text{Hom}_{\mathcal{R}_{A_Q}}(\tilde{T}_i Y, \tilde{T}_i L)_\tilde{T}_i X}{\text{Aut}(\tilde{T}_i Y)},
\]

which implies \( F^L_{Y,X} = F^L_{\tilde{T}_i Y, \tilde{T}_i X} \), it follows that

\[
[\hat{u}_{\tilde{T}_i X}, \hat{u}_{\tilde{T}_i Y}] = \sum_{L' \cong \tilde{T}_i L} (F^L_{Y,X} - F^L_{Y,X}) \hat{u}_{\tilde{T}_i L} + \delta_{\tilde{T}_i X, \tilde{T}_i Y[1]} \frac{h_{\tilde{T}_i X}}{d(\tilde{T}_i X)}.
\]

Note \( X \cong Y \) iff \( \tilde{T}_i X \cong \tilde{T}_i Y \), then \( \Phi_i(\hat{u}_X, \hat{u}_Y) = [\hat{u}_{\tilde{T}_i X}, \hat{u}_{\tilde{T}_i Y}] = [\Phi_i(\hat{u}_X), \Phi_i(\hat{u}_Y)] \).

For the other hand, \( [h_{\tilde{T}_i X}, \hat{u}_{\tilde{T}_i Y}] = (h_{\tilde{T}_i X} [h_{\tilde{T}_i Y}])_{\mathcal{R}_{A_Q}} \hat{u}_{\tilde{T}_i Y} \). Since the triangle equivalence \( \tilde{T}_i \) induces an isotropy on the Grothendieck group \( K_0(\mathcal{R}_{A_Q}) \) with respect to \((-1)_{\mathcal{R}_{A_Q}} \), we have \( (h_X | h_{\tilde{T}_i Y})_{\mathcal{R}_{A_Q}} = \tilde{T}_i^{-1}(h_{\tilde{T}_i X} [h_{\tilde{T}_i Y}])_{\mathcal{R}_{A_Q}} \), which gives \( \Phi_i([h_X, \hat{u}_Y]) = [h_{\tilde{T}_i X}, \hat{u}_{\tilde{T}_i Y}] \).

So \( \Phi_i \) is a homomorphism of Lie algebras. Then it is an isomorphism and the inverse is induced by the quasi-inverse of \( \tilde{T}_i \).

Recall that \( g_0(\mathcal{R}_{A_Q}) \) is the Lie subalgebra of \( g(\mathcal{R}_{A_Q})_{(q-1)} \) generated by \( \hat{u}_{S_j}, \hat{u}_{S_j}[1] \) and \( h_j \) for all \( j \in Q_0 \). From Example \( 3.2 \), we know that \( T_{2,S_i}(S_i) = S_i[-1] \) and \( T_{2,S_i}(S_j) = I_{ji} \) if \( j \neq i \) for simple \( A_Q \)-module \( S_j \). Then \( \Phi_i : g_0(\mathcal{R}_{A_Q}) \to g(\mathcal{R}_{A_Q}) \) is given by

\[
\Phi_i(\hat{u}_{S_i}) = \hat{u}_{S_i}[1], \quad \Phi_i(\hat{u}_{S_i}[1]) = \hat{u}_{S_i},
\]

\[
\Phi_i(\hat{u}_{S_j}) = u_{I_{ji}}, \quad \Phi_i(\hat{u}_{S_j}[1]) = u_{I_{ji}[1]}, \quad \text{if } j \neq i.
\]

When \( i \in Q_0 \) is a source, \cite[Theorem 2.1]{23} stated that there exists a functor \( R(S^-) \) on root categories inducing an isomorphism \( \tilde{\phi}_i \) of Kac-Moody Lie algebras, which is a lifting of Weyl group actions on root system. Namely, \( \tilde{\phi}_i(e_i) = -f_i \), and for \( j \neq i \), \( \tilde{\phi}_i(e_j) = ad(e_i)^{a_{ij}}(e_j) \), etc. However, as for \( \Phi_i \), this does not hold.

**Remark 3.8.** Even if \( i \in Q_0 \) is a source, \( \Phi_i \) still can not be restricted to an automorphism of the Lie subalgebra \( g_0(\mathcal{R}_{A_Q}) \). Indeed,

\[
[I_{ji}] \neq \sum_{r=0}^{a_{ij}} (-1)^r [S_i]^{(a_{ji}-r)}[S_j][S_i]^{(r)} = [I_{ji}] - [P_{ji}],
\]

where \( P_{ji} \in \text{rep}^n A_Q \) is an extension of \( S_i^{\oplus a_{ji}} \) by \( S_j \), i.e. \( P_{ji} = (V_j \oplus V_i, h \in \mathbb{Q}_1) \) with \( V_j = k, V_i = k^{a_{ji}}, h_l = (0, \cdots, 0, 1, 0, \cdots, 0)^l : V_j \to V_i \) if \( s(h_l) = j, 1 \leq l \leq a_{ji} \). Thus, \( \Phi_i(S_j) \not\in g_0(\mathcal{R}_{A_Q}) \).
**Remark 3.9.** The spherical twist functor $T_{S_i}$ on $D^b(\text{rep}^n A_Q)$ is the right derived functor of the reflection functor $\Sigma_i$ defined in [1, Section 2.2]. Indeed, it suffices to show that $H^0(T_{S_i}(M)) = \Sigma_i(M)$ for any $M \in \text{rep}^n A_Q$. Denoted by $\text{hd}_i M$ the $S_i$-isotropic component of the head of $M \in \text{rep}^n A_Q$. For $M = (\bigoplus_{j \in Q_0} M_j, M_h)$ such that $\text{hd}_i M = 0$, then the following sequence

$$M_i \xrightarrow{(\epsilon(h) M_h)} \bigoplus_{h : i \rightarrow j} M_j \xrightarrow{(M_h)} M_i$$

is exact at the last term, i.e. the map $M_{\text{out}} := (M_h) : \bigoplus_{h : i \rightarrow j} M_j \rightarrow M_i$ is surjective. On the other hand, the complex $\text{Hom}(P_{S_i}, M)$ of vector spaces is precisely the sequence (3.1). Hence, for $M$ with trivial $i$-head, $\text{Hom}(P_{S_i}, M) \otimes P_{S_i}$ is quasi-isomorphic to $(M_i \rightarrow \text{Ker} M_{\text{out}}) \otimes P_{S_i}$, which is equal to $(V_0 \oplus V_1[-1]) \otimes P_{S_i}$, here $V_0$ (resp. $V_1$) is the 0-th (resp. 1-th) cohomology of the complex $M_i \rightarrow \text{Ker} M_{\text{out}}$. Then $T_{S_i}(M) = \text{cone}((f_0, f_1) : (S_i^0 \oplus S_i^1[-1]) \rightarrow M)$, with $f_0 = (d_1, \cdots, d_{v_1}) : S_i^0 \rightarrow M$ and $f_1 = (c_1, \cdots, c_{v_1}) : S_i^1 \rightarrow M_i$, where $d_1, \cdots, d_{v_1}$ forms a basis for $\text{Hom}_{\Lambda Q}(S_i, M)$ and $c_1, \cdots, c_{v_1}$ forms a basis for $\text{Ext}^1_{\Lambda Q}(S_i, M)$. Denote by $M'$ the extension of $M$ by $S_i^1$ representing the class $f_1$ in $\text{Ext}^1_{\Lambda Q}(S_i^0, M)$, then $T_{S_i}(M) = \text{cone}(f_0 : S_i^0 \rightarrow M')$. Note that $f_0$ is injective and $\dim M_i - v_0 = \dim \text{Ker} M_{\text{out}}$, it follows that $T_{S_i}(M)$ is quasi-isomorphic to $\Sigma_i(M)$.

For $M \in \text{rep}^n A_Q$ with trivial $i$-head such that $M$ is an extension of $N$ by $S_i^0$. So there is a short exact sequence as follows:

$$0 \rightarrow N \rightarrow M \rightarrow S_i^0 \rightarrow 0.$$  

Applying $T_{S_i}$, we have a triangle

$$T_{S_i}(N) \rightarrow T_{S_i}(M) \rightarrow T_{S_i}(S_i^0) \rightarrow T_{S_i}(N)[1].$$

Take $H$ to the above triangle and note $T_{S_i} = S_i[-1]$ and $T_{S_i}(N) = \Sigma_i(N)$, it follows that $H^0(T_{S_i}(M)) \cong H^0(T_{S_i}(N))$. Since $\Sigma_i(M) = \Sigma_i(N)$, we conclude that $H^0(T_{S_i}(M)) \cong \Sigma_i(M)$ for any $M \in \text{rep}^n A_Q$.

### 3.3. Realization of Weyl groups

Let $W_Q$ be the Weyl group of the associated Kac-Moody Lie algebra $\mathfrak{g}_Q$. Namely, $W_Q$ is the subgroup of $\text{Aut}(\mathbb{Z}Q_0)$ generated by simple reflections $s_i, i \in Q_0$, where $s_i(\lambda) = \lambda - (\lambda, \alpha_i)\alpha_i$ for $\lambda \in \mathbb{Z}Q_0$ and $\{\alpha_i\}$ is the set of simple roots.

Recall that we have an isomorphism $\eta : K_0(\mathcal{R}_{\Lambda Q}) \rightarrow \mathbb{Z}Q_0, \tilde{S}_i \mapsto \alpha_i$. Let $T_i : K_0(\mathcal{R}_{\Lambda Q}) \rightarrow K_0(\mathcal{R}_{\Lambda Q})$ be the induced map of the spherical twist functor $\tilde{T}_{2, S_i}$ of simple $\Lambda Q$-module $\tilde{S}_i$. Then, we have the following

**Proposition 3.10.** For any $i \in I$, the following identity holds in $\text{Aut}(\mathbb{Z}Q_0)$:

$$s_i = \eta T_i \eta^{-1}.$$
That is, the following diagram commutes

\[
\begin{array}{ccc}
K_0(\mathcal{R}_{AQ}) & \xrightarrow{T_i} & K_0(\mathcal{R}_{AQ}) \\
\downarrow{\eta} & & \downarrow{\eta} \\
\mathbb{Z}Q_0 & \xrightarrow{s_i} & \mathbb{Z}Q_0.
\end{array}
\]

**Proof.** It suffices to check \( s_i = \eta T_i \eta^{-1} \) for \( \alpha_j, j \in Q_0 \). \( \eta T_i \eta^{-1}(\alpha_i) = \eta T_i(\hat{S}_i) = \eta(-\hat{S}_i) = -\alpha_i \). For \( j \neq i \), \( \eta T_i \eta^{-1}(\alpha_j) = \eta T_i(\hat{S}_j) = \eta(L_{ji}) = \alpha_j + a_{ji}\alpha_i \). Combining with \( s_i(\alpha_i) = -\alpha_i \) and \( s_i(\alpha_j) = \alpha_j - (\alpha_j, \alpha_i)\alpha_i = \alpha_j + a_{ji}\alpha_i \), where \(-a_{ji} = < \hat{S}_j, \hat{S}_i >_{AQ} = (\hat{S}_j, \hat{S}_i)_A \), we complete the proof. \( \square \)

In the sequel, we will regard \( T_i \) as \( s_i \) in \( W_Q \).

Let \( \text{Br}(\mathcal{R}_{AQ}) \) be the subgroup of the autoequivalence group \( \text{Auteq}(\mathcal{R}_{AQ}) \) generated by \( T_{2,S_i}, i \in Q_0 \). It can be seen that the map \( \epsilon : \text{Br}(\mathcal{R}_{AQ}) \rightarrow W_Q \) taking any autoequivalence to its restriction on \( K_0(\mathcal{R}_{AQ}) \) is a surjective group homomorphism following from Proposition 3.10. On the other hand, there is a surjective group homomorphism \( \xi : \text{Br}(\mathcal{R}_{AQ}) \rightarrow \text{Aut}_0(\mathfrak{g}(\mathcal{R}_{AQ})) \), \( T_{2,S_i} \mapsto \Phi_i \), where \( \text{Aut}_0(\mathfrak{g}(\mathcal{R}_{AQ})) \) is the subgroup of \( \text{Aut}(\mathfrak{g}(\mathcal{R}_{AQ})) \) generated by \( \Phi_i \).

**Theorem 3.11.** There exists a group homomorphism

\[
\epsilon' : \text{Aut}_0(\mathfrak{g}(\mathcal{R}_{AQ})) \rightarrow W_Q, \quad \Phi_i \mapsto s_i.
\]

Moreover, \( \epsilon' \) is surjective.

**Proof.** Let \( \text{Ker}\xi \) be the kernel of \( \xi : \text{Br}(\mathcal{R}_{AQ}) \rightarrow \text{Aut}_0(\mathfrak{g}(\mathcal{R}_{AQ})) \), we claim that for any \( w \in \text{Ker}\xi, \epsilon(w) = 0 \). Note that for each element \( w \in \text{Br}(\mathcal{R}_{AQ}) \), \( w \) is a product of \( T_{2,S_i} \), write \( w = T_{2,S_{i_1}}T_{2,S_{i_2}}\cdots T_{2,S_{i_r}} \). Then for \( w \in \text{Ker}\xi \), by the definition of \( \Phi_i \), we have \( w(X) \cong X \) for \( X \in \text{ind}\mathcal{R}_{AQ} \), it follows that \( T_{i_1}T_{i_2}\cdots T_{i_r}([S_i]) = [S_i] \) for any \( i \in Q_0 \). Hence \( \epsilon(w)(\alpha_i) = \eta T_{i_1}T_{i_2}\cdots T_{i_r}\eta^{-1}(\alpha_i) = \alpha_i \) for \( i \in Q_0 \), which means that \( \epsilon(w) = 1 \). Then, there exists a group homomorphism \( \epsilon' : \text{Aut}_0(\mathfrak{g}(\mathcal{R}_{AQ})) \rightarrow W_Q \) such that the following diagram commutes.

\[
\begin{array}{ccc}
\text{Br}(\mathcal{R}_{AQ}) & \xrightarrow{\xi} & \text{Aut}_0(\mathfrak{g}(\mathcal{R}_{AQ})) \\
\downarrow{\epsilon} & & \downarrow{\epsilon'} \\
W_Q & \xleftarrow{\epsilon'} & \\
\end{array}
\]

\( \square \)

**Remark 3.12.** As shown in [1, Section 5], the reflection functor \( \Sigma_i \) is the simple reflection \( s_i \in W_Q \) at the crystal level in some sense. However the reflection functor \( \Sigma_i \) may not be well defined at the weight \( \alpha_i \). Because \( s_i(\alpha_i) = -\alpha_i \) which is not a positive root anymore. In order to overcome it, we extend the reflection functor \( \Sigma_i \) on \( \text{rep}^n AQ \) to the spherical twist functor \( T_i \) on the triangulated hull \( \mathcal{R}_{AQ} \) of \( D^b(\text{rep}^n AQ) \), and reach the result Theorem 3.11.
4. Relations between \( g_0(\mathcal{R}_A)_{(q-1)} \) and \( g_0(\mathcal{R}_{AQ})_{(q-1)} \)

4.1. Relations between \( n_+^0(\mathcal{R}_A)_{(q-1)} \) and \( n_+^0(\mathcal{R}_{AQ})_{(q-1)} \). Let \( k \) be the finite field with \(|k| = q \). Recall that \( g_0(\mathcal{R}_A) \) (resp. \( g_0(\mathcal{R}_{AQ}) \)) is a subalgebra of \( g(\mathcal{R}_A)_{(q-1)} \) (resp. \( g(\mathcal{R}_{AQ})_{(q-1)} \)) generated by \( u_{S_j}, u_{S_j}[1], h_j, j \in Q_0 \) (resp. \( \hat{u}_{S_j}, \hat{u}_{S_j}[1], h_j, j \in Q_0 \)), and \( n_0^+ (\mathcal{R}_A) \) (resp. \( n_0^+ (\mathcal{R}_{AQ}) \)) generated by \( u_{S_j}, j \in Q_0 \) (resp. \( \hat{u}_{S_j}, j \in Q_0 \)).

For \( M, N, L \in \text{rep}^n A_Q \), denote by \( \text{Hom}_{A_Q}(N, L) \) the subset of \( \text{Hom}_{A_Q}(N, L) \) consisting of all injective homomorphisms whose \( \text{cokernel} \) is isomorphic to \( M \). Then for any \( f_0 \in \text{Hom}_{A_Q}(N, L)_M \), we have a triangle in \( D^b(\text{rep}^n A_Q) \) as follows:

\[
N \xrightarrow{f_0} L \longrightarrow M \longrightarrow N[1].
\]

Hence its image under the functor \( \pi : D^b(\text{rep}^n A_Q) \rightarrow \mathcal{R}_{AQ} \) is also a triangle in \( \mathcal{R}_{AQ} \). So \( \text{Hom}_{A_Q}(N, L)_M \subset \text{Hom}_{\mathcal{R}_{AQ}}(N, L)_M \) by regarding \( f_0 \) as \( (f_0, 0) \), because \( \text{Hom}_{\mathcal{R}_{AQ}}(N, L) \cong \text{Hom}_{A_Q}(N, L) \oplus \text{Ext}_{A_Q}^2(N, L) \).

Set

\[
\text{Hom}_{\mathcal{R}_{AQ}}(N, L)_M := \{ f \in \text{Hom}_{\mathcal{R}_{AQ}}(N, L) \mid \text{cone} f \cong M \}.
\]

Example 4.1. Take the acyclic quiver \( Q \) to be the Kronecker quiver

\[
1 \xrightarrow{\beta_1} \beta_2 \xrightarrow{2}
\]

Let us compute \([S_1][S_2]\) in \( \text{rep}^n A_Q \) and in \( \mathcal{R}_{AQ} \) respectively.

\[
[S_1][S_2] = \sum_{\lambda \in k} [E_\lambda] + [E'] + [S_1 \oplus S_2]
\]

in the Hall algebra of \( \text{rep}^n A_Q \), where \( E_\lambda \) and \( E' \) are as follows:

\[
E_\lambda : \begin{array}{c}
1 \\
\lambda \\
k
\end{array} \xrightarrow{1} \begin{array}{c}
k \\
\lambda \\
k
\end{array} \quad \text{and} \quad \begin{array}{c}
E' : \\
0 \\
k
\end{array} \xrightarrow{0} \begin{array}{c}
k \\
1
\end{array}.
\]

On the other hand, \( S_2 \longrightarrow L \longrightarrow S_1 \xrightarrow{w} S_2[1] \) is a triangle in \( \mathcal{R}_{AQ} \) if and only if \( L[1] \cong \text{cone} w \). Note \( w \in \text{Hom}_{\mathcal{R}_{AQ}}(S_1, S_2[1]) = \text{Ext}^1_{\mathcal{R}_{AQ}}(S_1, S_2) \), then \( L \in D^b(\text{rep}^n A_Q) \), actually \( L \) is an extension of \( S_1 \) by \( S_2 \). \( L \cong S_1 \oplus S_2 \) iff \( w = 0 \), then

\[
F_{S_1, S_2} = \frac{\text{Hom}_{\mathcal{R}_{AQ}}(S_2, S_1 \oplus S_2)_{S_1}}{\text{Aut}_{\mathcal{R}_{AQ}}(S_2)} = \frac{\text{Hom}_{\mathcal{R}_{AQ}}(S_2, S_2)_0}{\text{Aut}_{\mathcal{R}_{AQ}}(S_2)} = 1.
\]

If \( L \cong E_\lambda \), Applying \( \text{Hom}_{\mathcal{R}_{AQ}}(S_2, -) \) to the short exact sequence \( S_2 \hookrightarrow E_\lambda \rightarrow S_1 \), we have a long exact sequence

\[
0 \rightarrow \text{Ext}^1_{\mathcal{R}_{AQ}}(S_2, E_\lambda) \rightarrow \text{Ext}^1_{\mathcal{R}_{AQ}}(S_2, S_1) \xrightarrow{w_\lambda} \text{Ext}^2_{\mathcal{R}_{AQ}}(S_2, S_2) \rightarrow \text{Ext}^2_{\mathcal{R}_{AQ}}(S_2, E_\lambda) \rightarrow 0
\]

Since \( w_\lambda \neq 0 \), we have that \( w_\lambda \) is an isomorphism. Hence

\[
\text{Hom}_{\mathcal{R}_{AQ}}(S_2, E_\lambda) = \text{Hom}_{\mathcal{R}_{AQ}}(S_2, E_\lambda).
\]
Then

\[
F_{S_1, S_2}^{E_1} = \frac{\text{Hom}_{\mathcal{R}_{AQ}}(S_2, E_1)_{S_1}}{\text{Aut}_{\mathcal{R}_{AQ}}(S_2)} = \frac{\text{Hom}_{\mathcal{A}_Q}(S_2, E_1)_0}{\text{Aut}_{\mathcal{A}_Q}(S_2)} = 1.
\]

Similarly, we have that \(F_{S_1, S_2}^{E_1'} = 1\). So

\[
[S_1][S_2] = \sum_{\lambda \in k} [E_{\lambda}] + [E'] + [S_1 \oplus S_2]
\]

in the Hall algebra of \(\mathcal{R}_{AQ}\).

Let \(M, N\), be indecomposable \(\mathcal{A}_Q\)-modules, and \(L\) an extension of \(M\) by \(N\). We may not be able to compute \(\text{Hom}_{\mathcal{R}_{AQ}}(N, L)\). However we still can show that \(F_{M,N}^L\) (\(\text{rep}^n \mathcal{A}_Q\)) \(\equiv \text{Ext}_{\mathcal{R}_{AQ}}^1(M, N)\) (mod \((q - 1))\), where \(L\) is an indecomposable \(\mathcal{A}_Q\)-module. Here \(F_{M,N}^L\) means the cardinality of \(\frac{\text{Hom}_{\mathcal{A}_Q}(N, L)_M}{\text{Aut}_{\mathcal{A}_Q}(N)}\) in the given category \(\mathcal{A}\).

**Lemma 4.2.** Let \(M, N \in \text{rep}^n \mathcal{A}_Q\) be indecomposable, and \(L \in \mathcal{R}_{AQ}\) indecomposable such that

\[
N \xrightarrow{f} L \rightarrow M \rightarrow N[1]
\]

is a triangle in \(\mathcal{R}_{AQ}\), then \(L \in \text{rep}^n \mathcal{A}_Q\), and

\[
\text{Hom}_{\mathcal{R}_{AQ}}(N, L)_M = \{(f_0, f_1)|\text{cone} f_0 \cong M\} = \text{Hom}_{\mathcal{A}_Q}(N, L)_M \oplus \text{Ext}_{\mathcal{A}}^2(N, L).
\]

**Proof.** Note that if \(N \xrightarrow{f} L \xrightarrow{g} M \xrightarrow{h} N[1]\) is a triangle in \(\mathcal{R}_{AQ}\) with \(M, N \in \text{rep}^n \mathcal{A}_Q\), then \(h \in \text{Hom}_{\mathcal{R}_{AQ}}(M, N[1]) = \text{Ext}_{\mathcal{A}_Q}^1(M, N)\), since \(\text{gl.dim} \mathcal{A}_Q = 2\), it follows that \(L[1] \cong \text{cone} h \in \text{D}^b(\text{rep}^n \mathcal{A}_Q)\) must be an extension of \(M\) by \(N\). Moreover, for any element \((f_0, f_1)\) of \(\text{Hom}_{\mathcal{R}_{AQ}}(N, L)_M, f_0 \neq 0\). Indeed, if \(f_0 = 0\) then there is a triangle in \(\mathcal{R}_{AQ}\),

\[
N \xrightarrow{f_1} L \xrightarrow{g} M \xrightarrow{h} N[1].
\]

But the functor \(\pi : \text{D}^b(\text{rep}^n \mathcal{A}_Q) \rightarrow \mathcal{R}_{AQ}\) is a triangle functor and \(f_1 \in \text{Hom}_{\text{D}^b(\mathcal{A}_Q)}(N, L[2])\), the above triangle must be the image of the triangle \(N \xrightarrow{f_1} M \rightarrow \text{cone} f_1 \rightarrow N[1]\) in \(\text{D}^b(\text{rep}^n \mathcal{A}_Q)\). Hence \(M[2l] \cong \text{cone} f_1 \in \text{D}^b(\text{rep}^n \mathcal{A}_Q)\) for some integer \(l\), which implies that \(g = 0\) or \(h = 0\). So \(N \cong M[-1] \oplus L[2]\) or \(L[2] \cong N \oplus M[2l]\), this is a contradiction.

Let \(f = (f_0, f_1) \in \text{Hom}_{\mathcal{R}_{AQ}}(N, L)_M\), then we have a triangle in \(\mathcal{R}_{AQ}\)

\[
N \xrightarrow{(f_0, f_1)} L \xrightarrow{(g_0, g_1)} M \xrightarrow{h} N[1].
\]

then \((f_0, f_1)h[-1] = (f_0h[1], f_1h[-1]) = 0\), this implies that \(f_0h[-1] = 0\). Hence there exists \(v = (v_0, v_1) \in \text{Hom}_{\mathcal{R}_{AQ}}(L, L)\) such that \(vf = f_0\). So we have a morphism of
triangles
\[\begin{array}{ccc}
N & \xrightarrow{(f_0,f_1)} & L \\
\downarrow{id} & & \downarrow{(v_0,v_1)} \\
N & \xrightarrow{f_0} & L \\
\rightarrow & & \xrightarrow{\text{cone}(f_0)} \\
& & N[1]
\end{array}\]  

If \(v_0 \in \text{End}_{\mathcal{A}_Q}(L)\) is a unit, then the two triangles above are isomorphic and we get that \(\text{cone}f_0 \cong M\).

If \(v_0 \in \text{End}_{\mathcal{A}_Q}(L)\) is not a unit, then \(v_0^r = 0\) for some positive integer \(r\) because \(\text{End}_{\mathcal{A}_Q}(L)\) is a local ring. By direct computation, \((v_0,v_1)^{2r} = 0\). Therefore, we have a morphism as follows between the two triangles above
\[\begin{array}{ccc}
N & \xrightarrow{(f_0,f_1)} & L \\
\downarrow{id} & & \downarrow{(v_0,v_1)^{2r}} \\
N & \xrightarrow{f_0} & L \\
\rightarrow & & \xrightarrow{\text{cone}(f_0)} \\
& & N[1]
\end{array}\]  

Then \(f_0 = v^{2r}f = 0\), this contradicts to \(f_0 \neq 0\).

Therefore, we have shown that \(\text{Hom}_{\mathcal{R}_\mathcal{A}_Q}(N,L)_X \subset \{(f_0,f_1) | \text{cone}f_0 \cong M\}\).

On the other hand, let \(f = (f_0,f_1) \in \text{Hom}_{\mathcal{R}_\mathcal{A}_Q}(N,L)\) such that \(\text{cone}f_0 \cong M\). Consider the triangle
\[\begin{array}{ccc}
N & \xrightarrow{(f_0,f_1)} & L \\
\downarrow{id} & & \downarrow{(v_0,v_1)^{2r}} \\
N & \xrightarrow{f_0} & L \\
\rightarrow & & \xrightarrow{\text{cone}(f_0)} \\
& & N[1]
\end{array}\]  

Applying the functor \(\text{Hom}_{\mathcal{A}_Q}(-,L)\) to the short exact sequence \(0 \rightarrow N \xrightarrow{f_0} L \xrightarrow{g_0} M \rightarrow 0\), we get a long exact sequence
\[\cdots \rightarrow \text{Ext}^2_{\mathcal{A}_Q}(L,L) \rightarrow \text{Ext}^2_{\mathcal{A}_Q}(Y,L) \rightarrow 0,\]

since \(\text{Ext}^3_{\mathcal{A}_Q}(X,L) = 0\). Therefore, for any \(f_1 \in \text{Ext}^2_{\mathcal{A}_Q}(Y,L)\), there exists \(v_1 \in \text{Ext}^2(L,L)\) such that \(f_1 = v_1f_0\). Note that \(L\) is indecomposable and \(\text{Ext}^2_{\mathcal{A}_Q}(L,L) \subset \text{rad}(\text{Hom}_{\mathcal{R}_\mathcal{A}_Q}(L,L))\), so \((id,v_1)\) is an isomorphism in \(\text{Hom}_{\mathcal{R}_\mathcal{A}_Q}(L,L)\). Then the following two triangles are isomorphic
\[\begin{array}{ccc}
N & \xrightarrow{f_0} & L \\
\downarrow{id} & & \downarrow{(id,v_1)} \\
N & \xrightarrow{(f_0,f_1)} & L \\
\rightarrow & & \xrightarrow{\text{cone}(f_0)} \\
& & N[1]
\end{array}\]  

which implies \(\text{cone}f \cong M\) and \(f \in \text{Hom}_{\mathcal{R}_\mathcal{A}_Q}(N,L)_M\). The proof is completed. \(\square\)

The automorphism group \(\text{Aut}_{\mathcal{A}_Q}(N)\) can be naturally embedded into the automorphism group \(\text{Aut}_{\mathcal{R}_\mathcal{A}_Q}(N)\) by \(f_0 \mapsto (f_0,0)\), and the extension group \(\text{Ext}^2_{\mathcal{A}_Q}(N,N)\) also can be regarded as a subgroup of \(\text{Aut}_{\mathcal{R}_\mathcal{A}_Q}(N)\) by \(f_1 \mapsto (1,f_1)\). Furthermore,
Aut_{AQ}(N) acts on Ext^2_{AQ}(N, N) by composition, and for any \((f_0, f_1)\) in Aut_{\mathcal{R}_{AQ}}(N), \((f_0, f_1) = (f_0, 0)(1, f_0^{-1}f_1) \in \text{Aut}_{AQ}(N) \times \text{Ext}^2_{AQ}(N, N)\). Hence we have that

\[
\text{Aut}_{\mathcal{R}_{AQ}}(N) = \text{Aut}_{AQ}(N) \times \text{Ext}^2_{AQ}(N, N).
\]

**Lemma 4.3.** For \(M, N, L\) indecomposable in \(\text{rep}^n AQ\), we have that

\[
|\text{Ext}^2_{AQ}(N, N)| \cdot \frac{\text{Hom}_{\mathcal{R}_{AQ}}(N, L)_M}{\text{Aut}_{AQ}(N)} = |\text{Ext}^2_{AQ}(N, L)| \cdot |\text{Hom}_{AQ}(N, L)_M^*|.
\]

**Proof.** By Lemma 4.2 we know that \(\text{Hom}_{\mathcal{R}_{AQ}}(N, L)_M \cong \text{Hom}_{AQ}(N, L)_M \oplus \text{Ext}^2_{AQ}(N, L)_M\), it follows that \(f_0\) is injective for any \(f = (f_0, f_1) \in \text{Hom}_{\mathcal{R}_{AQ}}(N, L)_M\). Thus, the action of \(\text{Aut}_{\mathcal{R}_{AQ}}(N)\) on \(\text{Hom}_{\mathcal{R}_{AQ}}(N, L)_M\) is free. Indeed, if \((f_0, f_1)(u, v) = (f_0, f_1)\), then \(f_0u = f_0\) and \(f_0v + f_1u = f_1\), which implies \(u = id_N\) and \(v = 0\). Naturally, as a subgroup of \(\text{Aut}_{\mathcal{R}_{AQ}}(N)\), \(\text{Aut}_{AQ}N\) acts on \(\text{Hom}_{\mathcal{R}_{AQ}}(N, L)_M\) freely. Then we have a surjection

\[
E := \frac{\text{Hom}_{\mathcal{R}_{AQ}}(N, L)_M \oplus \text{Ext}^2_{AQ}(N, L)}{\text{Aut}_{AQ}(N)} \rightarrow \frac{\text{Hom}_{\mathcal{R}_{AQ}}(N, L)_M}{\text{Aut}_{AQ}(N)}
\]

with fiber \(\text{Ext}^2_{AQ}(N, L)\).

On the other hand, since \(\text{Aut}_{\mathcal{R}_{AQ}}(N) = \text{Aut}_{AQ}(N) \times \text{Ext}^2_{AQ}(N, N)\), we have another surjection

\[
E = \frac{\text{Hom}_{\mathcal{R}_{AQ}}(N, L)_M}{\text{Aut}_{AQ}(N)} \rightarrow \frac{\text{Hom}_{\mathcal{R}_{AQ}}(N, L)_M}{\text{Aut}_{AQ}(N) \times \text{Ext}^2_{AQ}(N, N)}
\]

with fiber \(\text{Ext}^2_{AQ}(N, N)\). We complete the proof. \(\square\)

Since the integral Hall algebra \(\mathcal{H}(AQ)\) of the preprojective algebra is an associative algebra, we can define a Lie algebra \(n^+(AQ)\) to be the free abelian group with the basis \([M]\) for \(M \in \text{rep}^n AQ\), and the Lie bracket is given by \([[M], [N]] = [M][N] - [N][M]\). Also, we can consider the Lie subalgebra \(n_0^+(AQ)\) of \(n^+(AQ)\) generated by \([S_i]\) for \(i \in I\).

**Proposition 4.4.** There is an isomorphism of Lie algebras:

\[
\phi_0 : n^+_0(\mathcal{R}_{AQ}) \rightarrow n^+_0(AQ), \ u_i \mapsto [S_i].
\]

**Proof.** By Lemma 4.3 and Lemma 4.2, we have an algebra homomorphism from \(\phi : n^+(\mathcal{R}_{AQ}) \rightarrow n^+(AQ)\), \(u_i \mapsto [M]\), which is obviously a bijection. Thus, \(\phi\) maps \(n^+_0(\mathcal{R}_{AQ})\) to \(n^+_0(AQ)\) bijectively, which is exactly \(\phi_0\). \(\square\)

Let us recall Lusztig’s categorification of the positive half part of the enveloping algebra \(U^+\) of the associated Kac-Moody Lie algebra \(g_Q\) using constructible functions \([14]\). Fix a dimension vector \(v \in \mathbb{N}Q_0\), \(G_v := \prod_{i \in I} GL(v_i; \mathbb{C})\). Consider the representation space \(E_v = \prod_{h \in Q_0} \text{Hom}(\mathbb{C}^{v(h)}_v, \mathbb{C}^{v(h)})\) consisting of all \(\mathbb{C}Q\)-modules with dimension vector \(v\). Note \(E_v\) is an affine variety over complex field \(\mathbb{C}\). Denote by
$M(E_v)$ the $\mathbb{Q}$-vector space of all constructible $G_v$-equivariant functions $f : A_v \to \mathbb{Q}$ (A constructible function is a finite sum of constant functions on a constructible subset of $E_v$ and $f$ is said to be $G_v$-equivariant if $f$ is constant on $G_v$-orbits). Set $M(E) := \bigoplus_{v \in \mathcal{N}} M(E_v)$, it has a standard convolution product $\ast$. Namely, for $f \in M(E_v)$, $g \in M(E_{v''})$, set $v = v' + v''$, define
\[ f \ast g(x) := \sum_{x' \in E_{v'}, x'' \in E_{v''}} \chi(V(x', x''; x)) f(x') g(x'') \]
where $V(x', x''; x) = \{ L \subset x | L \cong x'', x/L \cong x' \}$, which is a locally closed subset of $E_v$, and $\chi(V)$ is the Euler character of a variety $V$ (see [10, Lemma 3.3]).

Consider the $\mathbb{Q}$-linear subalgebra $M_0(E)$ of $M(E)$ generated by $f_i = 1_{O(S_i)}(i \in \mathcal{N})$, where $O(S_i)$ is the $G_v$-orbit of $S_i$. By [14, Proposition 10.20], there is an isomorphism
\[ \kappa : U^+ \longrightarrow M_0(E), \ e_i \mapsto f_i, \]
where $U^+$ is the $\mathbb{Q}$-algebra $U^+$ generated by $e_i$, $i \in \mathcal{N}$ and subject to the Serre relations
\[ (4.1) \sum_{l=0}^{N} (-1)^l \binom{N}{l} e_i^{N-l} e_j f_i = 0, \]
for $i \neq j$, where $N = 1 + a_{ij} + a_{ji}$, $a_{ij} = \dim \text{Ext}^1_A(S_i, S_j)$.

Let $U_+^\mathcal{Z}$ be the $\mathbb{Z}$-linear subalgebra of $M_0(E)$ generated by $f_i$, $i \in \mathcal{N}$. Note Serre relations (4.1) are $\mathbb{Z}$-coefficients, hence $U_+^\mathcal{Z}$ is a $\mathbb{Z}$-algebra generated by $f_i$, $i \in \mathcal{N}$ and subject to
\[ \sum_{l=0}^{N} (-1)^l \binom{N}{l} f_i^{N-l} f_j f_i = 0, \]
for $i \neq j$, where $N = 1 + a_{ij} + a_{ji}$. Finally, we define a $\mathbb{Z}/(q - 1)$-algebra by setting
\[ \overline{U_+^\mathcal{Z}} := U_+^\mathcal{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/(q - 1). \]
It is clear that $\overline{U_+^\mathcal{Z}}$ is a $\mathbb{Z}/(q - 1)$-algebra with generators $\bar{f}_i$, $i \in \mathcal{N}$, subject to
\[ \sum_{l=0}^{N} (-1)^l \binom{N}{l} \bar{f}_i^{N-l} \bar{f}_j \bar{f}_i = 0, \]
for $i \neq j$, where $N = 1 + a_{ij} + a_{ji}$.

We introduce some notations.
\[ \binom{m}{l} := \frac{m!}{l!(m-l)!}, \quad [l]_q := \frac{q^l - 1}{q - 1}, \quad \binom{m}{l}_q := \frac{[m]_q!}{[l]_q![m-l]_q!}. \]

Fix a dimension vector $v \in \mathbb{N}Q_0$. Set
\[ \mathfrak{N}_v := \{ \lambda = l_1i_1 + l_2i_2 + \cdots + l_ti_t | i_1, \cdots, i_t \in I, \sum_{j=1}^t l_ji_j = v, l_j \in \mathbb{N} \} \]
and \( \mathcal{N} := \bigcup_{v \in \mathbb{N}^\lambda} \mathcal{N}_v \). For \( \lambda = l_1 i_1 + l_2 i_2 + \cdots + l_t i_t \in \mathcal{N}_v \) and \( M \in \text{mod}A \), to compute \([S_{i_1}]^{l_1} [S_{i_2}]^{l_2} \cdots [S_{i_t}]^{l_t} \in C^H(A) \), we first compute \( S_\lambda := [S_{i_1}]^{l_1}|S_{i_2}]^{l_2} \cdots [S_{i_t}]^{l_t} \). Denote by \( F^M_\lambda \) the number of times of \([M] \) in \([S_{i_1}]^{l_1}|S_{i_2}]^{l_2} \cdots [S_{i_t}]^{l_t} \), then

\[
[S_{i_1}]^{l_1} [S_{i_2}]^{l_2} \cdots [S_{i_t}]^{l_t} = \prod_{j=1}^{t} (l_j)! \sum_{[M]} F^M_\lambda [M].
\]

Let

\[
V(\lambda; M)_q := \{ L_t \subset \cdots \subset L_1 = M/L_j/L_{j+1} \cong S_{i_j}^{q \lambda} \}.
\]

Then the cardinality \(|V(\lambda; M)_q|\) is exactly the number \( F^M_\lambda \) from the definition of Hall product.

On the other hand, for \( f_1^i \in M_0(E_0) \), note \( f_i^l = \prod_{j=1}^{l-1} \chi(\text{Gr}(j, j + 1)) \), hence \( \prod_{j=1}^{l-1} \chi(\text{Gr}(j, j + 1)) \). Here \( \text{Gr}(m, n) \) means the Grassmannian variety of \( m \)-dimensional vector subspaces of the vector space \( \mathbb{C}^n \). Set

\[
V(\lambda; M)_C := \{ x_t \subset \cdots \subset x_1 = x | x_j \in \text{modCQ}, x_j/x_{j+1} \cong S_{i_j}^{q \lambda} \}.
\]

Then \( f_1^i f_2^i \cdots f_t^i = \prod_{j=1}^{t} (l_j)! \sum_{[M]} \chi(V(\lambda; M)_C)1_{\mathcal{O}(M)} \).

**Lemma 4.5.** Regard \( q = |k| \) as an indeterminate, then for \( \lambda \in \mathcal{N}_v \) and \( M \in \text{mod}A \), we have that \( F^M_\lambda \in \mathbb{Z}[q] \).

**Proof.** We compute \( F^M_\lambda = |V(\lambda; M)| \) by induction on \(|v| \) for \( v \in \mathbb{N}^\lambda, \lambda \in \mathcal{N}_v \), and \( M \in \text{mod}A \). If \(|v| = 1 \), then \( S_\lambda = S_i \) for some \( i \in I \). It is obvious that \( F^M_\lambda \) is a constant function. Assume the statements holds for \(|v| < m \), for \(|v| = m \) and \( \lambda = l_1 i_1 + \cdots + l_t i_t \in \mathcal{N}_v \), consider all submodules \( N \) of \( M \) such that \( N \cong S_{i_t}^{q \lambda} \), define an equivalent relation by \( N \sim N' \) iff \( M/N \cong M/N' \). Denote the class of \( N \) by \( \bar{N} \) and the set of all equivalent classes by \( \Omega \). Then

\[
F^M_\lambda = \sum_{N \in \Omega} [d_{l_t - a_N}] q F^{M/N}_{\lambda - l_t}\cdot
\]

where \( d_M = \dim \bigcap_{l(h) = l_t} \ker h \), \( a_N = \dim N \bigcap \sum_{l(h) = l_t} n \ker h \). Note \( \lambda - l_t i_t \in \mathcal{N}_v \) with \(|v'| < |v| \), by induction, \( F^{M/N}_{\lambda - l_t} \in \mathbb{Z}[q] \). It is well-known that \([d_{l_t - a_N}] q \in \mathbb{Z}[q] \). Hence, \( F^M_\lambda \in \mathbb{Z}[q] \). \( \square \)

Due to Lemma 4.5, the variety \( V(\lambda; M)_q \) can be counted by a polynomial, we have \( \chi_c(V(\lambda; M)_C) = F^M_\lambda(1) \) for any \( \lambda \in \mathcal{N}_v \) and \( M \in \text{mod}A \), which can be deduced from [17, Proposition 6.1] or [18, Lemma 8.1].

**Lemma 4.6 ( [18]).** Suppose there exists a rational function \( P_X(t) \in \mathbb{Q}(t) \) such that \( P_X(|k|) = |X(k)| \) for almost all reductions \( k \) of the ring \( R \) over which \( X \) is defined. Then \( P_X(t) \in \mathbb{Z}[t] \) is actually a polynomial with integer coefficients, and
\[ P_X(1) = \chi_c(X), \text{ the Euler characteristic of } X \text{ in singular cohomology with compact support.} \]

Note \( V(\lambda; M)_C \) is a locally closed subset defined over an algebraically closed field \( \mathbb{C} \), the compactly supported Euler characteristic \( \chi_c(V(\lambda; M)_C) \) is equal to the Euler characteristic \( \chi(V(\lambda; M)_C) \). For details, see page 141 in [7].

Set \( \overline{\text{CH}}(A) := \text{CH}(A)/(\text{CH}(A) \cap (q - 1)\mathcal{H}(A)), \overline{\text{CH}}(A_Q) := \text{CH}(A_Q)/(\text{CH}(A_Q) \cap (q - 1)\mathcal{H}(A_Q)). \)

**Lemma 4.7.** The morphism \( \kappa_1 : \overline{U_{Z}} \longrightarrow \overline{\text{CH}}(A) \) sending \( f_i \) to \([S_i]\) is an isomorphism of \( \mathbb{Z}/(q - 1) \)-algebras.

**Proof.** First, we should show that \( \kappa_1 \) is well-defined. Namely, \( \overline{\text{CH}}(A) \) admits the Serre relations

\[
\sum_{l=0}^{N} (-1)^l \binom{N}{l} [S_i]^{N-l}[S_j][S_i]^l = 0 \pmod{(q - 1)\mathcal{H}(A)},
\]

for \( i \neq j \) and \( a_{ij} + a_{ji} \neq 0 \), where \( N = 1 + a_{ij} + a_{ji} \).

Fix \( i \neq j \in I \), denote by \( a_{M,l} \) the multiple of \([M]\) in \([S_i]^{N-l}[S_j][S_i]^l\) for \( M \in \text{mod}A \). Note that \( Q \) is an acyclic quiver, then either \( a_{ij} = 0 \) and \( a_{ji} \neq 0 \), or \( a_{ij} \neq 0 \) and \( a_{ji} = 0 \). In the first case, say there are \( a_{ji} \) arrows \( \tilde{h} \) from \( j \to i \). \( a_{M,l} \neq 0 \) iff \( V_M := \sum_{\tilde{h} \in S^\oplus I} \text{im}\tilde{h} \subset S_i^{\oplus I} \). If so, \( a_{M,l} = [l]_q [N-l]_q [l-d_M]_q \), here \( d_M = \dim V_M \leq a_{ji} < N = 1 + a_{ji} + a_{ji} \). Then, we have that

\[
\sum_{l=0}^{N} (-1)^l \binom{N}{l} [S_i]^{N-l}r[M] = \sum_{[M]} \sum_{l=0}^{N} (-1)^l \binom{N}{l} [l]_q [N-l]_q [l-d_M]_q [M] \\
\equiv \sum_{[M]} \sum_{l=0}^{N} (-1)^l N! \binom{N-d_M}{l} [M] \pmod{(q - 1)\mathcal{H}(A))} \\
\equiv 0 \pmod{(q - 1)\mathcal{H}(A)).
\]

In the second case, say there are \( a_{ij} \) arrows \( h_r \) from \( i \) to \( j \). Then \( a_{M,l} \neq 0 \) iff \( S_i^{\oplus I} \subset \bigcap_{h_r} \text{Ker}h_r =: W_M \). If so, \( a_{M,l} = [l]_q [N-l]_q [d_M]_q \), here \( d_M = \dim W_M \geq 1 \)
for \( \dim \Ker h_r \geq a_{ij}, \, 1 \leq r \leq a_{ij} \). Thus, we have that

\[
\sum_{l=0}^{N} (-1)^l \binom{N}{l} S_i^{N-l} S_j S_i^l = \sum_{[M]} \sum_{l=0}^{N} (-1)^l \binom{N}{l} [l]q! [N-l]q! \cdot \frac{d_M}{l} q[M] = \sum_{[M]}\sum_{l=0}^{N} (-1)^l N! \frac{d_M}{l} [M] \pmod{(q-1)\mathcal{H}(A)}
\]

\[
\equiv 0 \pmod{(q-1)\mathcal{H}(A)}.
\]

Now, we will show that \( \kappa_1 \) is injective. Suppose \( \sum_{\lambda \in \mathbb{N}} b_{\lambda}[S_{i_1}]^{l_1}[S_{i_2}]^{l_2} \cdots [S_{i_t}]^{l_t} = 0 \) (finite sum) in \( \overline{CH}(A) \), then \( \sum_{\lambda \in \mathbb{N}} b_{\lambda} \prod_{j=1}^{t} [l_j] q! F_{\lambda}^M[M] \equiv 0 \pmod{(q-1)} \) for finitely many \([M] \). Note that \( F_{\lambda}^M \equiv F_{\lambda}^M(1) \pmod{(q-1)} \) and \( F_{\lambda}^M(1) = \chi_c(V(\lambda; M)) \), it follows that \( \sum_{\lambda \in \mathbb{N}} b_{\lambda} \prod_{j=1}^{t} [l_j] q! \chi_c(V(\lambda; M)) \equiv 0 \pmod{(q-1)} \). Then \( \sum_{\lambda \in \mathbb{N}} b_{\lambda} f_{i_1}^{l_1} f_{i_2}^{l_2} \cdots f_{i_t}^{l_t} = 0 \) in \( U_{\mathbb{Z}} \), the proof is completed. \( \square \)

According to [14, Lemma 12.11], we have the following

**Lemma 4.8.** The morphism \( \kappa_2 : \overline{U_{\mathbb{Z}}} \longrightarrow \overline{CH}(A_Q) \) sending \( f_i \) to \( [S_i] \) is a surjective homomorphism of \( \mathbb{Z}/(q-1) \)-algebras.

**Proof.** It suffices to show that for \( i \neq j \in I \), there are the following relations

\[
\sum_{l=0}^{N} (-1)^l \binom{N}{l} [S_i]^{N-l}[S_j][S_i]^l \equiv 0 \pmod{(q-1)\mathcal{H}(A_Q)}.
\]

in \( \overline{CH}(A_Q) \), where \( N = 1 + a_{ij} + a_{ji} \).

If there are \( m := a_{ij} + a_{ji} \) arrows \( h_r \) from \( i \rightarrow j \) and \( a_{ij} + a_{ji} \) arrows \( h_r \) from \( j \rightarrow i \). Denote by \( a_{M,l} \) the multiple of \([M]\) in \([S_i]^{N-l}[S_j][S_i]^l \) for \( M = (V, (x_h)_{h \in Q_1}) \in \text{rep}^nA_Q \).

Set \( V_M := \sum h_r \text{im} h_r \) and \( W_M := \bigcap h_r \text{Ker} h_r \), then \( a_{M,l} \neq 0 \) iff \( V_M \subset S_i^{\otimes l} \subset W_M \). If so, then \( a_{M,l} = [l]q! [N-l]q! \frac{d_M - d_{M'}}{l - d_M} q \), here \( d_M = \dim W_M \geq 1 \), \( d_M = \dim V_M \). Note that there is a sequence exact except at the fourth item \( V_j^m \)

\[
0 \longrightarrow W_M \leftarrow V_1 \xrightarrow{\left( x_h \right)} V_j \xrightarrow{\left( x_h \right)} V_M \longrightarrow 0,
\]
it follows that \( \dim W_M - \dim V_M \geq \dim V_i - \dim V_j^m = N - m = 1 \). Then we have that

\[
\sum_{l=0}^{N} (-1)^l \binom{N}{l} [S_i]^{N-l}[S_j][S_i]^l
\]

\[
= \sum_{[M]} \sum_{l=0}^{N} (-1)^l \binom{N}{l} [l]_q! [N - l]_q! \binom{d_M - d_M}{l - d_M} [M]
\]

\[
\equiv \sum_{[M]} \sum_{l=0}^{N} (-1)^l N! \binom{d_M - d_M}{l - d_M} [M] \pmod{(q - 1)\mathcal{H}(\Lambda_Q))}
\]

\[
\equiv 0 \pmod{(q - 1)\mathcal{H}(\Lambda_Q)).
\]

Notice that \( A \) can be regarded as a quotient algebra of \( \Lambda_Q \), then there is an epimorphism \( \text{Res} : \mathcal{H}(\Lambda_Q) \to \mathcal{H}(A) \) according to Proposition 2.10, which induces an epimorphism \( \theta : \mathcal{H}(\Lambda_Q)/(q - 1)\mathcal{H}(\Lambda_Q) \to \mathcal{H}(A)/(q - 1)\mathcal{H}(A) \). Furthermore, simple \( \Lambda_Q \)-modules are precisely simple \( A \)-modules, then the epimorphism \( \theta \) maps \([S_i] \) to \([S_i] \) for \( i \in Q_0 \). Hence, the following diagram commutes.

\[
\begin{array}{ccc}
U_Z^+ & \xrightarrow{\kappa_2} & \overline{\mathcal{H}(\Lambda_Q)} \\
\downarrow{\kappa_1} & & \downarrow{\theta} \\
\overline{\mathcal{H}(A)} & \xleftarrow{\theta} & \mathcal{H}(A)
\end{array}
\]

**Theorem 4.9.** There is an isomorphism of algebras \( \varphi^+ : \mathfrak{n}_0^+(\Lambda_Q) \to \mathfrak{n}_0^+(A) \).

**Proof.** By Lemma 4.7, \( \kappa_1 = \theta \kappa_2 \) is an isomorphism, we obtain that \( \kappa_2 \) is injective. Since \( \kappa_2 \) is surjective, \( \kappa_2 \) is also an isomorphism. Then

\[
\theta = \kappa_1 \kappa_2^{-1} : \overline{\mathcal{H}(\Lambda_Q)} \to \overline{\mathcal{H}(A)}
\]

is an isomorphism. Moreover, \( \overline{\mathcal{H}(\Lambda_Q)} \) (resp. \( \overline{\mathcal{H}(A)} \)) is the universal enveloping algebra of \( \mathfrak{n}_0^+(\Lambda_Q) \) (resp. \( \mathfrak{n}_0^+(A) \)), it follows that the isomorphism \( \text{Res} : \overline{\mathcal{H}(\Lambda_Q)} \to \overline{\mathcal{H}(A)} \) gives an isomorphism \( \varphi^+ : \mathfrak{n}_0^+(\Lambda_Q) \to \mathfrak{n}_0^+(A) \). \( \square \)

Similarly, we have an isomorphism \( \varphi^- : \mathfrak{n}_0^-\Lambda_Q) \to \mathfrak{n}_0^-(A) \). Combining Theorem 4.9 with Proposition 4.4, we reach the following

**Theorem 4.10.** As Lie algebras, \( \mathfrak{n}_0^+(\mathcal{R}_{\Lambda_Q}) \) (resp. \( \mathfrak{n}_0^-\mathcal{R}_{\Lambda_Q}) \) is isomorphic to \( \mathfrak{n}_0^+(\Lambda_Q) \) (resp. \( \mathfrak{n}_0^-(\Lambda_Q) \)).

4.2. The Ringel-Hall Lie algebra of a spherical object. Now we consider a certain Lie subalgebra of \( g(\mathcal{R}_{\Lambda_Q}) \) supported on one single vertex. Let \( \text{thick}(S_i) \) be the smallest triangulated subcategory of \( D^b(\text{rep}^{\Lambda_Q}) \) containing the simple module \( S_i \) and stable under direct summands. Note \( S_i \) is a 2-spherical object and \( \text{thick}(S_i) \) is
an algebraic triangulated category (for $D^b(\text{rep}^n A_Q)$) is algebraic triangulated category and its triangulated subcategories are also algebraic from [24, Lemma 3.5.8]), hence thick($S_i$) is triangle equivalent to the derived DG category $D_{fd}(\Gamma_i)$ of DG $\Gamma_i$-modules generated by the unique simple object $k[t]/(t)$, where $\Gamma_i = k[t]$ with deg($t$) = −1, see [6, Theorem 2.2]. According to [6, Theorem 3.2], the orbit category $D_{fd}(\Gamma_i)/[2]$ admits a canonical triangle structure, hence $D_{fd}(\Gamma_i)/[2] \simeq \text{thick}(S_i)/[2]$ is a triangulated subcategory of $\mathcal{R}_{AQ}$. Let $\mathcal{L}_i$ be the Ringel–Hall Lie algebra of thick($S_i$)/[2], then $\mathcal{L}_i$ is a Lie subalgebra of $\mathfrak{g}(\mathcal{R}_{AQ})$.

Let $\triangle$ be the cyclic quiver with 2 vertices and $T_2$ be the category of finitely generated nilpotent $k\triangle$-modules. Then $D_{fd}(\Gamma_i)/[2]$ is triangulated equivalent to the cluster tube $D^b(T_2)/\tau \circ [-1]$ of rank 2 by [6, Lemma 4.6]. Hence each indecomposable objects in $D_{fd}(\Gamma_i)/[2]$ is of the form $\langle n \rangle$ or $\langle -n \rangle$, where $\langle n \rangle$ (resp. $\langle -n \rangle$) is the unique indecomposable $k\triangle$-module of length $n$ with socle the simple module corresponding to vertex 1 (resp. 2). By [6, Theorem 4.14], $\mathcal{L}_i$ is the Lie algebra over $\mathbb{Z}/(q - 1)$ with the basis $\{h_i, \hat{u}_{i,(n)}, n \in \mathbb{Z}^\times\}$ and structure constants given by (in the following $x, y \in \mathbb{N}$, $m, n \in \mathbb{Z}^\times$ and $h_i := \frac{h_i}{\delta(S_i)}$)

(i) $[\hat{u}_{i,(m)}, \hat{u}_{i,(n)}] = 0$ for $m$ and $n$ even;

(ii) $[\hat{u}_{i,(m), \hat{u}_{i,(n)}]} = 0$ for $m$ and $n$ both odd of the same sign;

(iii) $[\hat{u}_{i,(2x), \hat{u}_{i,(2y-1)}]} = \begin{cases} \hat{u}_{i,(2(x+y)-1)} + \hat{u}_{i,(2(y-x)-1)}, & \text{for } x < y, \\ \hat{u}_{i,(2(x+y)-1)} - \hat{u}_{i,(2(y-x)+1)}, & \text{for } x \geq y; \end{cases}$

(iv) $[\hat{u}_{i,(2x), \hat{u}_{i,(2y+1)}]} = \begin{cases} -\hat{u}_{i,(2(x+y)+1)} - \hat{u}_{i,(2(y-x)-1)}, & \text{for } x < y, \\ -\hat{u}_{i,(2(x+y)+1)} + \hat{u}_{i,(2(y-x)+1)}, & \text{for } x \geq y; \end{cases}$

(v) $[\hat{u}_{i,(-2x), \hat{u}_{i,(2y-1)}]} = \begin{cases} -\hat{u}_{i,(2(x+y)-1)} - \hat{u}_{i,(2(y-x)-1)}, & \text{for } x < y, \\ -\hat{u}_{i,(2(x+y)-1)} + \hat{u}_{i,(2(y-x)+1)}, & \text{for } x \geq y; \end{cases}$

(vi) $[\hat{u}_{i,(-2x), \hat{u}_{i,(2y+1)}]} = \begin{cases} \hat{u}_{i,(2(x+y)+1)} + \hat{u}_{i,(2(y-x)-1)}, & \text{for } x < y, \\ \hat{u}_{i,(2(x+y)+1)} - \hat{u}_{i,(2(y-x)+1)}, & \text{for } x \geq y; \end{cases}$

(vii) $[\hat{u}_{i,(2x-1), \hat{u}_{i,(2y+1)}]} = \begin{cases} \hat{u}_{i,(2(x+y)+2)} - \hat{u}_{i,(-2(x-y)+2)} - \hat{u}_{i,(2(y-x))}, & \text{for } x < y, \\ \hat{u}_{i,(2(y-x))} - \hat{u}_{i,(-2(x-y)+2)} + \hat{u}_{i,(2(x-y))} - \hat{u}_{i,(2(y-x))}, & \text{for } x > y; \end{cases}$

(viii) $[h_i, \hat{u}_{i,(n)}] = \begin{cases} 0, & \text{for } n \text{ even}, \\ 2\hat{u}_{i,(n)}, & \text{for } n \text{ positive odd}, \\ -2\hat{u}_{i,(n)}, & \text{for } n \text{ negative odd}. \end{cases}$

We replace the symmetric Euler form in the last term above by its half, which is precisely the bilinear form $\langle -\rangle_{\mathcal{R}_{AQ}}$ we used.

**Example 4.11.** Let $\hat{u}_{i,(1)}$ represent $\hat{u}_{S_i}$ and $\hat{u}_{i,(-1)}$ represent $\hat{u}_{S_i[-1]}$. Then

$$[\hat{u}_{S_i}, \hat{u}_{S_i[-1]}] = -h_i + \hat{u}_{E_i} - \hat{u}_{E_i[-1]}$$
where \( E_i[1] = \text{cone}(t_i : S_i \to S_i[2]) \in D^b(\text{rep}^n A_Q) \). Comparing with the identity
\[
[u_{i,(1)}, u_{i,(−1)}] = −h_i + u_{i,(2)} − u_{i,(−2)},
\]
we know \( u_{i,(2)} = \hat{u}_{E_i} \) and \( u_{i,(−2)} = \hat{u}_{E_i}[1] \). Indeed, to compute \([S_i][S_i[1]], \) we consider the triangle in \( R_{A_Q} \)
\[
S_i[1] \to L \to S_i \xrightarrow{f} S_i[2],
\]
where \( f \in \text{Hom}_{D^b A_Q}(S_i, S_i) \oplus \text{Hom}_{D^b A_Q}(S_i, S_i[2]) \). Note \( \text{Hom}_{A_Q}(S_i, S_i) \cong k \cong \text{Ext}^2_{A_Q}(S_i, S_i) \), for any \( 0 \neq f = (f_0, f_2), (f_0, f_2) \sim (1, 0) \under Aut_{R_{A_Q}}(S_i)-\text{action}, \) if \( f_0 \neq 0 \). So \( L \cong 0 \) and \( F^0_{S_i, S_i[1]} = 1 \); if \( f_0 = 0 \), then \( (0, f_2) \sim (0, 1) \), so \( L[1] \cong \text{cone}(t_i : S_i \to S_i[2]) \in D^b(\text{rep}^n A_Q) \) and \( F^2_{S_i, S_i[1]} = 1 \). Therefore
\[
[S_i][S_i[1]] = 1 + [E_i].
\]
Similarly, we can show
\[
[S_i[1]][S_i] = 1 + [E_i[1]].
\]

**Remark 4.12.** \( g_0(R_A) \) is not isomorphic to \( g_0(R_{A_Q}) \) as Lie algebras because the root category \( R_{A_Q} \) is not proper, there are more elements belonging to \( g_0(R_{A_Q}) \). Indeed, the last example 4.11 shows that \( \hat{u}_{E_i} - \hat{u}_{E_i[1]} = [\hat{u}_{S_i}, \hat{u}_{S_i[1]}] + h_i \in g_0(R_{A_Q}) \) does not belong to \( g_0(R_A) \), since \( \hat{E}_i \) and \( E_i[1] \) are zero objects in \( K_0(R_{A_Q}) \).

Let \( \mathcal{I} \) be an ideal of \( g_0(R_{A_Q}) \) generated by \( \hat{u}_{E_i} - \hat{u}_{E_i[1]} \), \( i \in Q_0 \). Let \( \mathfrak{S}_{i,0} \) be the Lie subalgebra of \( \mathfrak{S}_i \) generated by \( \hat{u}_{i,(±1)} \) and \( h_i \); \( \mathcal{I}_i \) be the ideal of \( \mathfrak{S}_{i,0} \) generated by \( \hat{u}_{i,(2)} - \hat{u}_{i,(−2)} \). Denote by \( \bar{u}_{i,1} \) the image of \( \hat{u}_{i,1} \) in \( \mathfrak{S}_{i,0}/\mathcal{I}_i \). Then we have the following

**Lemma 4.13.** The Lie algebra \( \mathfrak{S}_{i,0}/\mathcal{I}_i \) is the Lie algebra over \( \mathbb{Z}/(q − 1) \) with a basis
\[
\{\bar{h}_i, \bar{u}_{i,(1)}, \bar{u}_{i,(−1)}\}
\]
subject to relations
\[
[\bar{h}_i, \bar{u}_{i,(±1)}] = ±2\bar{u}_{i,(±1)} \quad \text{and} \quad [\bar{u}_{i,(1)}, \bar{u}_{i,(−1)}] = \bar{h}_i
\]
Equivalently, \( \mathfrak{S}_i/\mathcal{I}_i \) is the Lie algebra \( \mathfrak{sl}_2 \).

**Proof.** Compute with given relations of \( \mathfrak{S}_i \), we have
\[
[u_{i,(1)}, u_{i,(−1)}] = \bar{h}_i + u_{i,(2)} - u_{i,(−2)} = \bar{h}_i.
\]
and
\[
[h_i, u_{i,(±1)}] = ±2u_{i,(±1)}.
\]
Therefore there exists an algebra homomorphism \( \Psi_i \) from \( \mathfrak{sl}_2 \) to \( \mathfrak{S}_{i,0}/\mathcal{I}_i \), which is clearly surjective. On the other hand, using the defining relations (i)-(viii) of \( \mathfrak{S}_i \), we can deduce that \( \hat{u}_{i,(1)} \notin \mathcal{I}_i \). So \( \Psi_i \) must be injective. \( \square \)
It is well known that $g_0(\mathcal{R}_A)$ is a Kac-Moody Lie algebra over $\mathbb{Z}/(q - 1)$. Namely, $g_0(\mathcal{R}_A)$ is the Lie algebra over $\mathbb{Z}/(q - 1)$ with generators $\{u_{S_i}, u_{S_i[1]}, h_i | i \in Q_0\}$ subject to the following relations:

$$
[h_i, h_j] = 0, \quad [h_i, u_{S_j}] = (\hat{S}_i, \hat{S}_j)_A u_{S_j}, \quad [h_i, u_{S_j[1]}] = -(\hat{S}_i, \hat{S}_j)_A u_{S_j[1]},
$$

$$ad(u_{S_j})^{1+c_{ij}}(u_{S_j}) = 0, \quad ad(u_{S_j[1]})^{1+c_{ij}}(u_{S_j[1]}) = 0, \text{ if } i \neq j,
$$

where $c_{ij} = \dim \text{Ext}_A^1(S_i, S_j) + \dim \text{Ext}_A^1(S_j, S_i)$. Note that we have shown that $n^+_0(\mathcal{R}_{AQ})$ is isomorphic to $n^+_0(A)$ as Lie algebras in Theorem 4.10. As a result, we have an algebra homomorphism

$$
\Psi : g_0(\mathcal{R}_A) \rightarrow g_0(\mathcal{R}_{AQ})/\mathcal{I}.
$$

Since the underlying diagram of the quiver $Q$ is connected, the ideal Ker$\Psi$ is either $g_0(\mathcal{R}_A)$ or contained in the center of $g_0(\mathcal{R}_A)$ by [9, Proposition 1.7].

Let $h' \subset h$ be the center of $g_0(\mathcal{R}_A)$, then $h'$ in $g_0(\mathcal{R}_{AQ})$ is also a part of the center of $g_0(\mathcal{R}_{AQ})$. Because $g_0(\mathcal{R}_A)$ and $g_0(\mathcal{R}_A)$ share same killing form $(-|-)_A = (-|-)_{\mathcal{R}_{AQ}}$.

**Theorem 4.14.** Assume $g_0(\mathcal{R}_{AQ})/\mathcal{I} \neq 0$, then

$$
\Psi : g_0(\mathcal{R}_A)/h' \rightarrow g_0(\mathcal{R}_{AQ})/(\mathcal{I} + h'), \quad u_{S_i} \mapsto \hat{u}_{S_i}, \quad u_{S_i[1]} \mapsto \hat{u}_{S_i[1]} \text{ and } h_i \mapsto h_i.
$$

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