Enhanced $\mathcal{N} = 8$ Supersymmetry of

ABJM Theory on $\mathbb{R}^8$ and $\mathbb{R}^8/\mathbb{Z}_2$

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ABSTRACT

The ABJM theory refers to superconformal Chern-Simons-matter theory with product gauge group $U_L \times U_R$ and level $+k, -k$, respectively. The theory is a candidate for worldvolume dynamics of M2-branes sitting at $\mathbb{C}^4/\mathbb{Z}_k$. By utilizing monopole operators, we prove that ABJM theory gets enhanced $\mathcal{N} = 8$ supersymmetry and SO(8) R-symmetry at Chern-Simons levels $k = 1, 2$. We first show that the ABJM Lagrangian can be written in a manifestly SO(8) invariant form up to certain extra terms. We then show that upon integrating out Chern-Simons gauge fields these extra terms vanish precisely at levels $k = 1, 2$. Utilizing monopole operators at these levels, we identify new $\mathcal{N} = 2$ supersymmetry. We demonstrate that they combine with the manifest $\mathcal{N} = 6$ supersymmetry to close on-shell on $\mathcal{N} = 8$ supersymmetry. We finally show that the ABJM scalar potential is SO(8) invariant.
1 Introduction

Aharony, Bergman, Jafferis and Maldacena [1] proposed a three-dimensionl superconformal field theory as a microscopic description for worldvolume dynamics of multiple M2-branes on $SU(4) \times U(1)$ R-symmetric and $\mathcal{N} = 6$ superconformal M2-branes. Hereafter referred as ABJM theory, it is defined by a gauged linear sigma model: eight scalar and fermion fields in the bifundamental representation of quiver gauge group $\mathcal{G} = G_1 \times G_2$ coupled to Chern-Simons gauge theory. Therefore, the ABJM theory is characterized by two integer-valued parameters: the Chern-Simons level $k$ and rank of the gauge group rank($\mathcal{G}$). It was proposed [1] that ABJM theory is holographically dual to Type IIA string theory on AdS$_4 \times \mathbb{CP}^3$ in the planar limit of both rank($\mathcal{G}$) and $k$ infinite while holding 't Hooft coupling $\lambda \equiv (\text{rank}(\mathcal{G})/k)$ fixed and large. At finite $k$, the holographic dual is described most appropriately by M theory on AdS$_4 \times S^7/Z_k$. The proposal of [1] provides a Type IIA string or M-theory counterpart of the much studied AdS/CFT correspondence [2] between Type IIB string on AdS$_5 \times S^5$ and four-dimensional $\mathcal{N} = 4$ super Yang-Mills theory. Interestingly, there are strong indications that the ABJM theory is integrable, both at weak coupling [3], [4] and strong coupling [5] regimes.

Built upon this holography, it was further anticipated in [1] that the ABJM CFT at Chern-Simons levels $k = 1, 2$ actually has $\mathcal{N} = 8$ supersymmetry and $SO(8)$ R-symmetry which are the symmetries of coincident M2 branes on $\mathbb{R}^{1,2} \times \mathbb{R}^8$ or $\mathbb{R}^{1,2} \times (\mathbb{R}^8/Z_2)$, respectively. The purpose of this paper is to prove that the ABJM theory, for all possible rank of gauge groups, has enhanced $\mathcal{N} = 8$ superconformal symmetry and $SO(8)$ R-symmetry at Chern-Simons level $k = 1, 2$. Our proof relies crucially on utilizing so-called 3-algebra structure and monopole operators inherent in this theory. Therefrom, if the Chern-Simons level $k$ takes the value 1 or 2, a set of highly nontrivial algebraic identities follows among the matter fields. Utilizing these identities, we show that the ABJM theory possesses extra $\mathcal{N} = 2$ supersymmetry that combines with the existing $\mathcal{N} = 6$ supersymmetry to the fully enhanced $\mathcal{N} = 8$ supersymmetry and $SO(8)$ R-symmetry.

A feature of the ABJM theory is that the gauge dynamics, governed solely by the Chern-Simons term, is trivial. The Chern-Simons term merely induces braiding statistics to the matter fields. Consequently, operators built solely from the gauge potential such as holonomy and magnetic monopole operators $W_K$ would not carry any dynamics or scaling dimension, though they transform in nontrivial representations $R$ under $\mathcal{G}$ [6]. Upon coupling matter fields to the Chern-Simons gauge field, gauge invariant operators are constructible not just from matter fields alone but also by attaching the holonomy or magnetic monopole operators $W_K$ to them. Made entirely out of gauge potential, the monopole operators are singlets under internal rigid symmetries such as R-symmetry. As such, monopole operators can produce gauge invariant...
operators with a rich variety of the R-symmetry representations. Recently, through the study of superconformal index, it was shown that gauge invariant operators containing the monopole operators \( W_R \) are indispensable for confirming the AdS/CFT correspondence between the ABJM theory and the M-theory at finite \( k \) \cite{7}.

Another feature of ABJM theory is that high degree of supersymmetry restricts permissible gauge groups, as well as representations of matter contents. In applications to specific problems, it is useful to formulate the ABJM theory in terms of the Lie algebra \( \mathfrak{g} \) of the gauge group \( \mathfrak{g} \) and representation \( R \) of matter fields. On the other hand, in a formulation that aims at incorporating all possible gauge groups and matter contents compatible with \( \mathcal{N} = 6 \) supersymmetry, it would be more convenient and unifying to use an algebraic structure that underlies all ABJM theories. It was found in \cite{8} that the pertinent algebraic structure of the ABJM theory is so-called hermitian 3-algebra \( A_3(\mathbb{C}) \). In this formulation, classification of permissible gauge groups and representations for \( \mathcal{N} = 6 \) supersymmetry was carried out in \cite{9}. An infinite class of them were found, among which the smallest rank \( \mathfrak{g} = \text{SO}(4) = \text{SU}(2) \times \text{SU}(2) \) is found identical to the Bagger-Lambert-Gustavsson (BLG) theory \cite{10}. The BLG theory, however, is known to have real 3-algebra \( A_3(\mathbb{R}) \) and \( \mathcal{N} = 8 \) supersymmetry. This calls for better understanding under what other choices of the ABJM theory parameters would exhibit the maximally enhanced \( \mathcal{N} = 8 \) supersymmetry and SO(8) symmetry.

Our proof of enhanced symmetries constitutes in showing that, by utilizing the three-algebra \( A_3(\mathbb{C}) \) and the monopole operators \( W \), the ABJM theory at Chern-Simons levels \( k = 1, 2 \) is expressible as a ‘trial’ BLG theory, where the original real 3-algebra \( A_3(\mathbb{R}) \) is replaced by the hermitian 3-algebra \( A_3(\mathbb{C}) \). In this way, the \( \mathcal{N} = 8 \) supersymmetry and the SO(8) R-symmetry become manifest. Here, ‘trial’ refers to the triality of the SO(8) group.

We should point out that, though details differ somewhat, the symmetry enhancement at \( k = 1, 2 \) works for the non-relativistic ABJM theory \cite{11} — the non-relativistic reduction of the ABJM theory, where only holonomy and monopole operators are known to generate physically nontrivial correlators \cite{12}. In fact, this theory illustrates in a clean manner intimate relations among symmetry enhancement between the ABJM and the non-ABJM fields, trivial braiding statistics for \( k = 1, 2 \) and bound-states of M-theory momentum modes. Details will be related to a separate paper.

This paper is organized as follows. In section 2, we summarize key ideas and provide a roadmap of our proof. In section 3, we illustrate these key ideas and roadmap for abelian gauge group. In section 4, we present details of hermitian 3-algebra \( A_3(\mathbb{C}) \) inherent to the ABJM theory. Also, in section 5, we present properties of monopole operator. In particular, we pay attention to the general covariance property, which will play a prominent role for foregoing
In section 6, we lay down details of closure among so-called the ABJM fields and the non-ABJM fields – composites made of the ABJM fields and the rank-2 monopole operators. In section 7, we first identify novel $\mathcal{N} = 2$ supersymmetry that act between the ABJM and the non-ABJM fields. Combining them with the manifest $\mathcal{N} = 6$ supersymmetry yields the maximal $\mathcal{N} = 8$ supersymmetry we are after. In this section, we check explicitly on-shell closure of the $\mathcal{N} = 8$ supersymmetry. In section 8, utilizing the similar reasonings, we show that the ABJM scalar potential is in fact identical to the BLG scalar potential. This demonstrate SO(8) symmetry of the ABJM scalar potential. By $\mathcal{N} = 8$ supersymmetry, the Yukawa interactions also have SO(8) symmetry. In appendix A, we recall SO(8) gamma matrices and several relevant Fierz identities. In appendix B, we also recall SO(1,2) gamma matrices. In appendix C, we summarize branching rule of SO(8) to SU(4)$\times$U(1). In appendix D, we provide Fierz identities of $\mathcal{N} = 6$ supersymmetry, of the new $\mathcal{N} = 2$ supersymmetry and hence of the full $\mathcal{N} = 8$ supersymmetry. In appendix, we explain triality rotated, so-called trial BLG theory.

## 2 Roadmap and Key Ideas

In this section, we shall outline key ideas used and a roadmap to our proof.

### 3-algebra

Since we shall heavily use the 3-algebra formulation throughout, we here summarize its emergence in the BLG and the ABJM theories. As recalled above, underlying algebraic structure of the BLG theory was identified with the real 3-algebra $\mathcal{A}_3(\mathbb{R})$. Its structure constants $f^{bcd}_a$ are real-valued and totally antisymmetric in $b, c, d$. The structure was so restrictive that the only finite-dimensional choice of the gauge group $G$ is $SU_L(2) \times SU_R(2) = SO(4)$. To have more general gauge groups, it became clear one would have to relax the 3-algebra structure. But it seemed impossible to do so while keeping all the global symmetries of the BLG theory intact. A solution to this difficulty was proposed by ABJM [1], where the SO(8) R-symmetry is given up and only the SU(4)$\times$U(1) part of it is kept manifest. The resulting ABJM theories traded an infinite class of admissible $G$ with reduced $\mathcal{N} = 6$ supersymmetry and SU(4) R-symmetry.

As recalled above, algebraic structure underlying all admissible ABJM theories is the hermitian 3-algebra $\mathcal{A}_3(\mathbb{C})$ [8]. Its structure constants $f^{bc}_d$ are antisymmetric in their two upper

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1Note that metric structure of the 3-algebra is not needed for equations of motion and for closing the $\mathcal{N} = 8$ supersymmetry variations, but is imperative for Lagrangian formulation.
and two lower indices, respectively, and hermitian in the sense that

\[ f^{*bc}_{\text{da}} = f^{\text{da}}_{bc}. \]  

(2.1)

In this formulation, we do not need to assume a metric on the 3-algebra since we can use complex conjugation to raise and lower indices. Even though we have no metric, we do have a trace-form and we can express the ABJM action using this trace-form. We will refer to the 3-algebra without a metric structure as hermitian 3-algebra \( A_3(\mathbb{C}) \). In this way, all admissible ABJM theories (that includes the BLG theory as one of them) are unified in a single framework of the 3-algebra \( A_3(\cdot) \).

The classification of [9] may be viewed as a consequence of the hermitian 3-algebra structure and the fundamental identity therein. For \( \mathcal{N} = 6 \), there is an ABJM theory for every hermitian 3-algebra. A hermitian 3-algebra in turn corresponds to a choice of the gauge group \( \mathcal{G} \) based on a semi-simple Lie group. In this paper, shall we consider ABJM theories that correspond to hermitian 3-algebra, viz. semi-simple Lie group. There can also exists global \( U(1) \times U(1) \) symmetry, corresponding to conserved baryon numbers, modulo global identifications of center elements. In that case, these \( U(1) \)s can be gauged. The resulting theory is the ABJM theory originally proposed [1].

**rank-2 monopole operators**

In 3-algebra, we have gauge indices \( a, b, \ldots = 1, \cdots, \text{dim} A_3 \) associated with 3-algebra generators \( T^a \) and their complex conjugates that we denote as \( T_a \). The monopole operator that will be useful for us are those with two gauge indices up or two indices down, \( W^{ab} \) and \( W_{ab} \), respectively. These rank-2 monopole operators can be used to turn the ABJM scalar field \( Z^A_a \) into a field \( Z^A_a = W^{ab}Z^A_b \) and similarly for the ABJM fermion fields. Here \( A \) is an index transforming in the fundamental representation of the global \( SU(4) \) R-symmetry of the ABJM theory. With the rank-2 monopole operators at hand, there are two ways to move the 3-algebra indices of the ABJM fields up or down. The first is attaching the rank-2 monopole operator as described above. The second is to take complex conjugate of the ABJM fields. Note that the complex conjugation acts by raising and lowering both gauge and R-symmetry indices, so the scalar field \( Z^A_A \) is the complex conjugated field of \( Z^A_a \), etc. Summarizing, starting from the matter field \( Z^A_a \), we can construct \( Z^Aa \) or \( Z_a^A \) by attaching the monopole operator or by complex conjugation, respectively.

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2The hermitian 3-algebra \( A_3(\mathbb{C}) \) without metric structure can also be found in [13].

3The hermitian 3-algebra \( A_3(\mathbb{C}) \) is a generalization of the real 3-algebra \( A_3(\mathbb{R}) \). In particular, this also implies that the Nambu 3-bracket is also a realization of the hermitian 3-algebra.
Attaching a monopole operator to a local field renders the composite a non-local operator since the monopole operator depends in general on the Dirac string. If the Dirac-Schwinger-Zwanziger quantization condition is obeyed, the Dirac string is unobservable and the monopole operator becomes a local field configuration. Moreover, the monopole operator is covariantly constant. Below we shall demonstrate this explicitly for the abelian ABJM theory and find that, only for Chern-Simons levels \( k = 1 \) and 2, the composite operators are local field configurations. This fits nicely with the fact that only at levels \( k = 1, 2 \) can we expect to have enhanced supersymmetry and R-symmetry. This is our first evidence that monopole operators should play some role in symmetry enhancement of ABJM theory.

**roadmap**

Denote vector, spinor and cospinor representations of \( \text{SO}(8) \) as \( 8_v, 8_s, 8_c \), and their basis indices by \( I, \alpha, \dot{\alpha} = 1, \ldots, 8 \), respectively. In the hermitian BLG theory, matter fields are \( 8_v \) for \( X^I_a \) and \( 8_s \) for \( \psi^\alpha_a \). The hermitian BLG theory is then defined by Chern-Simons term and the gauged matter Lagrangian

\[
\mathcal{L}_{\text{matter}} = -\frac{1}{2} D_{\mu} X^I_a D^\mu X^I_a - \frac{1}{12} X^I_a X^J_b X^K_c X^f_d f^{bc} f_{adf}^e f_{ef} \\
+ \frac{i}{2} \bar{\psi}^{\alpha a} \gamma^\mu D_\mu \psi^\alpha_a + \frac{i}{4} \bar{\psi}^{\alpha a} \Gamma_{I\alpha\dot{\alpha}} \Gamma_{f\dot{\beta}I} X^I_b X^c X^d X^f_{bcda}.
\]

(2.2)

We next use the triality of \( \text{SO}(8) \) group and map the original fields to triality-rotated fields. This way, we can construct two new trial hermitian BLG theories. In all these theories, the Chern-Simons term is universal since it is unaffected by the \( \text{SO}(8) \) triality. We are interested in the theory obtained by the following triality transformation:

\[
(8_v, 8_s, 8_c) \rightarrow (8_s, 8_c, 8_v); \quad (I, \alpha, \dot{\alpha}) \rightarrow (\alpha, \dot{\alpha}, I).
\]

(2.3)

After the transformation, the matter Lagrangian reads

\[
\mathcal{L}_{\text{matter}} = -\frac{1}{2} D_{\mu} X^\alpha_a D^\mu X^\alpha_a - \frac{1}{12} X^\alpha_a X^\beta_b X^\gamma_c X^d_f f^{bc} f_{da} f_{ef}^g \\
+ \frac{i}{2} \bar{\psi}^{\alpha a} \gamma^\mu D_\mu \psi^\alpha_a + \frac{i}{4} \bar{\psi}^{\alpha a} \Gamma_{I\alpha\dot{\alpha}} \Gamma_{f\dot{\beta}I} X^I_b X^c X^d X^f_{bcda}.
\]

(2.4)

viz. the matter fields are \( \text{SO}(8) \) spinors and cospinors \( 8_s, 8_c \) and the supersymmetry is \( \text{SO}(8) \) vector \( 8_v \) (see appendix E). The Lagrangian (2.4) is the one related to the ABJM Lagrangian. To show this, we break \( \text{SO}(8) \) to \( \text{SO}(6) \times \text{SO}(2) \equiv \text{SU}(4) \times U(1) \) and decompose the \( \text{SO}(8) \) spinor and cospinor fields as

\[
X^\alpha_a = \begin{pmatrix} Z^A_A \\ Z_a^A \end{pmatrix}, \quad X^\alpha_a = \begin{pmatrix} Z^A_A \\ Z_a^A \end{pmatrix}
\]
\[ \psi_{a a} = \begin{pmatrix} \psi_A^a \\ -\psi_A^{a a} \end{pmatrix}, \quad \psi^{a a} = \begin{pmatrix} \psi_A^{a a} \\ -\psi_A^a \end{pmatrix}. \] (2.5)

We also split the SO(8) gamma matrices into SO(6) and SO(2) gamma matrices as \( \Gamma_I = (\Gamma_M, \Gamma_X) \) and denote by \( \Sigma_M^{AB} \) and \( \Sigma_M^{AB} \) the off-diagonal blocks in \( \Gamma_M \). The details are collected in Appendix C. The fields \( Z^A_a \) and \( \psi_{A a} \) as well as their hermitian conjugates are the ABJM scalar and fermion fields, where upper \( A \) is fundamental and lower \( A \) is anti-fundamental of SU(4). \(^4\) The fields \( Z^A a \) and \( \psi_A^a \) are not the ABJM fields — we refer them as ‘non-ABJM fields’. Our strategy is to relate the non-ABJM fields to the ABJM fields by means of the monopole operators \( W^{ab}, W_{ab} \), since these operators are the unique tensors that can raise or lower indices gauge covariantly.

After the decomposition, we find the matter Lagrangian as

\[
\mathcal{L}_{\text{matter}} = -D_\mu Z^A_a D^\mu Z^a_A - \bar{\psi} A a \gamma^\mu D_\mu \psi A a \\
+ i \left( - \bar{\psi} A a \psi B a Z^B B d + 2 \bar{\psi} A a \psi C a Z^C A d \right) f^{bc da} \\
- \left( \frac{1}{2} \varepsilon^{ABCD} \bar{\psi} B b Z^A c Z^C d \psi B c + \frac{1}{2} \varepsilon^{ABCD} \bar{\psi} B b Z^A c Z^C d \psi C b \right) f^{bc da} \\
- \frac{2}{3} \left( f^{ab} f^{gh} f^{ef} - \frac{1}{2} f^{ab} f^{eh} f^{fg} \right) Z^A a Z^B b Z^C c Z^D d + \ldots
\]  

(2.6)

The terms shown depend only on the ABJM fields and hence yields the ABJM Lagrangian. The ellipses denote all other terms that involve the non-ABJM fields. Under what conditions will the ellipses vanish identically and the trial BLG theory become identical to the ABJM theory? We find that this is so if the following set of algebraic identities hold:

\[
\left( Z^A_c Z^d_A + Z^{Ad}_A Z^{Ac}_C \right) f^{bc da} = 0
\]

\[
\left( Z^{Ab}_A Z^B B_d + Z^{Ad}_A Z^{Bc}_B \right) f^{bc da} = 0
\]

\[
\left( Z^{Ab}_A Z^B B_c - Z^{Ac}_C Z^B B_b \right) f^{bc da} = 0
\]

\[
\psi^{Ab}_A \left( Z^{Bc}_B Z^{Ad}_A + Z^{Bd}_B Z^{Ac}_C \right) f^{bc da} = 0
\]

\[
\psi^{Ab}_A \left( Z^{Bc}_B Z^{Ad}_A + Z^{Bd}_B Z^{Ac}_C \right) f^{da bc} = 0.
\]  

(2.7)

We also find the correspondence between the sextet scalar potential in the ABJM theory and the potential in the generalized trial BLG theory, as demonstrated in section 8. The correspondence between the ABJM and generalized trial BLG Yukawa coupling terms can be shown. \(^5\)

\(^4\)Equivalently, they are spinor and cospinor of SO(6).
If the ABJM Lagrangian is SO(8) invariant, the identities (2.7) should hold in some sense\textsuperscript{5} and we can express the ABJM Lagrangian in the manifestly SO(8) invariant form as a generalized trial BLG Lagrangian. We shall show that (2.7) originate from the flatness condition of the gauge field strengths

\[
\tilde{F}_{\mu\nu}^a + \tilde{F}_{\mu\nu}^a = 0. \tag{2.8}
\]

and that the identities (2.7) are all related to (2.8) by \( \mathcal{N} = 6 \) supersymmetry.

To show that there is \( \mathcal{N} = 8 \) supersymmetry, it is not enough to just show that the Lagrangian can be written in an SO(8) invariant form. Indeed, we will find that we need a few more identities of a similar type in order to have closure of \( \mathcal{N} = 8 \) supersymmetry variations on the ABJM equations of motion.

Incidentally, the above algebraic identities may be interpreted as constraining the matter fields\textsuperscript{6} \( Z_a^A \)'s. This may be an indication of the feature of the ABJM theory that the true degrees of freedom scales as \( N^{3/2} \), not as \( N^2 \).

3 Prelude: abelian ABJM theory

3.1 linear sigma model

To appreciate the symmetry enhancement clearer, we first study the abelian ABJM theory. Here, of course, the 3-algebra structure is not essential. We start with (2+1)-dimensional linear sigma model over the target space \( \mathbb{C}^4 \). There are four complex scalar fields \( Z_A \) and their complex conjugates \( (Z_A)^* = Z_A \). They transform as \( 4, \overline{4} \) under SU(4) of the target space. This linear sigma model corresponds to bosonic part of the ABJM theory with gauge group U(1) \( \times \) U(1) at Chern-Simons level \( k = 1 \), as we will see in the next section. The action reads

\[
L_{\text{matter}} = -\int d^3x \partial_\mu Z^A \partial^\mu Z_A. \tag{3.1}
\]

The sigma model is invariant under U(4) = SU(4) \( \times \) U(1) transformations:

\[
\delta Z^A = \omega^A_B Z^B, \tag{3.2}
\]

\textsuperscript{5}The symmetry enhancement can not be seen in the classical Lagrangian where \( k \) is just an overall factor multiplying the whole Lagrangian. But if we integrate out the gauge field then these identities will hold for levels \( k = 1, 2 \).

\textsuperscript{6}If we take the viewpoint that the (non-dynamical) gauge field is put on-shell and expressed as a composite field in terms of the matter fields.
Here,

\[(\omega^*)_B^A + \omega_B^A = 0 \quad (3.3)\]

are anti-hermitian matrices, generating SU(4) transformations by the traceless parts and U(1) transformation by the trace part. In total, there are 16 real parameters.

The sigma model (3.1) has more symmetries. It is also invariant under the transformations

\[\delta Z^A = \omega^{AB} Z_B \quad (3.4)\]

described by 6 complex parameters related by

\[
\begin{align*}
\omega^{AB} + \omega^{BA} &= 0, \\
\omega^{*AB} + \omega^{BA} &= 0.
\end{align*} \quad (3.5)
\]

These transformations do not close among themselves. However, when combined with the above SU(4) \(\times\) U(1) transformations, they are closed and generate SO(8) symmetry group with \(28 = 16 + 6 \cdot 2\) real parameters.

To see the SO(8) symmetry better, we elaborate here somewhat technical but fairly straightforward discussion regarding how part of the SO(8) transformations not contained in SU(4) \(\times\) U(1) acts on \(\mathbf{8_v}\) and \(\mathbf{8_s}\) representations of SO(8). The results obtained here will be useful later. Acting on a \(\mathbf{8_v}\) representation \(V_I (I = 1, \ldots, 8)\), an infinitesimal SO(8) transformation is given by

\[\delta V_I = \omega_{IJ} V_J \quad (3.6)\]

where \(\omega_{IJ}\) is anti-hermitian and has real components (in other words, it is antisymmetric). We decompose \(\mathbf{8_v}\) into a six-dimensional part \(V^M (M = 1, \ldots, 6)\) and a two-dimensional part \(V = v^7 + iv^8\). The metric being Kronecker deltas, we do not distinguish upper or lower SO(8) or SO(6) indices. The SO(2) parameter is \(\omega^{78}\) and the SO(6) parameters are \(\omega^{MN}\). We are mainly interested in the SO(8) rotations that mix SO(6) with SO(2). These rotations are parametrized by \(\omega^M := \omega^{M7} + i\omega^{M8}\) and act on the SO(8) vector as

\[
\begin{align*}
\delta V^M &= \frac{1}{2} (\omega^M V^* + \omega^{*M} V) \\
\delta V &= \omega^M V^M \\
\delta V^* &= \omega^{*M} V^M.
\end{align*} \quad (3.7)
\]

An SO(8) Dirac spinor decomposes into Weyl \(X_\alpha\) and anti-Weyl spinor \(\psi_\dot{\alpha}\). These in turn decompose into Weyl spinors of SO(6). We define these Weyl components as

\[X_\alpha = \begin{pmatrix} Z^A \\ Z_A \end{pmatrix} \quad (3.8)\]
\[ \psi_\alpha = \begin{pmatrix} \psi^A \\ -\psi_A \end{pmatrix}. \] 

(3.9)

On the SO(8) R-symmetry Dirac spinor \[^7\]

\[ \Xi = \begin{pmatrix} X_\alpha \\ \psi_{\dot{\alpha}} \end{pmatrix} \] 

(3.10)

an infinitesimal SO(8) transformation acts as

\[ \delta \Xi = -\frac{1}{2} \omega^{MX} \Gamma^{MX} \Xi. \] 

(3.11)

Here, the normalization is fixed by how the vector index of gamma matrices transforms (as a direct consequence of the Clifford algebra),

\[ [\Gamma_{IJ}, \Gamma_K] = -4 \delta_{K[I} \Gamma_{J)]. \] 

(3.12)

One can view this as the invariance condition of the gamma matrices where all its indices are transformed. Explicitly, we find the variations as

\[ \delta Z_A = \frac{i}{2} \omega^{M} \Sigma^{M,AB} Z_B \]

\[ \delta Z_A = \frac{i}{2} \omega^{+M} \Sigma_{AB}^M Z_B, \] 

(3.13)

\[ \delta \psi^A = \frac{i}{2} \omega^{+M} \Sigma^{M,AB} \psi_B \]

\[ \delta \psi_A = \frac{i}{2} \omega^{M} \Sigma_{AB}^M \psi_B. \] 

(3.14)

### 3.2 gauging U(1) symmetry

**Chern-Simons gauging:**

We now gauge the U(1) symmetry by introducing a flat one-form gauge field \( b \). We then define the covariant derivative

\[ DZ^A := dZ^A + i b Z^A \] 

(3.15)

and consider the gauged linear sigma model

\[ -\int d^3 x \left( D_\mu Z^A D^\mu Z_A + \frac{k}{2\pi} b \wedge d a \right). \] 

(3.16)

\[^7\]It is important that this is R-symmetry spinor as opposed to spacetime spinor. In particular, \( Z \) is commuting bosonic field.
Here, $a$ is a Lagrange multiplier one-form gauge field that constrains $b$ to be flat, $db = 0$. This model equals to the bosonic part of the abelian ABJM action at integer-valued Chern-Simons level $k$.

We can integrate out $a$, setting $[1, 15]$

$$k \ b = d \ \sigma .$$

This gives back the linear sigma model modulo the orbifold identification

$$Z^A \simeq e^{\frac{2\pi i}{k}} Z^A .$$

In general, this identification breaks SO(8) down to SU(4) × U(1). At $k = 1, 2$, however, the SO(8) symmetry is retained. If the $Z_k$ orbifolding is SO(8) invariant, it should commute with the transformation

$$Z^A \to Z^A + \omega^{AB} Z_B .$$

This implies that

$$Z^A \to Z^A + \omega^{AB} e^{-\frac{4\pi i}{k} Z_B}$$

should also be a symmetry. This singles out the Chern-Simons coefficient $k$ to 1, 2.

**monopole operators:**

Notice that SO(8) symmetry cannot act in this simple way were the gauge field not integrated out. The transformation

$$Z^A \to Z^A + \omega^{AB} Z_B$$

would not be gauge covariant since $Z^A$ and $Z_A$ are oppositely charged with respect to the gauge field $b$. The remedy for this is to redefine the scalar fields by attaching monopole operators to these fields in such a way that all equations transform covariantly under the U(1) gauge transformations. The monopole operator that we have at our disposal is of the form

$$T_k = e^{i\sigma} .$$

From the Chern-Simons term, we also see that this operator carries also $k$ unit of electric charge. Thus, the gauge transformations act as

$$T_k \to e^{ik\alpha} T_k$$
\[ Z_A^A \to e^{i\alpha Z_A^A} \]
\[ Z_A^A \to e^{-i\alpha Z_A^A} \]  
(3.23)

At level \( k = 1 \), we can make the field redefinitions
\[ Z_A^A \to Z_A^A \]
\[ Z_A^A \to T_1 T_1 Z_A^A. \]  
(3.24)

At level \( k = 2 \), we can also make the field redefinitions
\[ Z_A^A \to Z_A^A \]
\[ Z_A^A \to T_2 Z_A^A. \]  
(3.25)

On these redefined fields, the SO(8) transformation acts in a gauge covariant way. Important observation is that, for \( k > 2 \), no such local field redefinition is possible. Therefore, this is another way to see that we can have enhanced SO(8) symmetry only for \( k = 1, 2 \).

The Chern-Simons coefficient \( k = 1, 2 \) is also special for a seemingly different reason. Consider two external probes charged electrically under the gauge fields \( a \) and the \( b \), respectively. Upon encircling one of the probes around the other once, we pick up the Aharonov-Bohm phase \( \exp(2\pi i/k) \) as braiding statistics. For \( k = 1 \), the phase is trivial and braiding statistics is bosonic. For \( k = 2 \), the phase is \( \pi \) and braiding statistics is fermionic. For \( k > 2 \), the braiding statistics is anyonic. By the same argument, we see that the composite we formed above would retain the field statistics unchanged for \( k = 1, 2 \) but not so for \( k > 2 \).

**local versus nonlocal:**

The reason we have these monopole operators at our disposal comes from the Chern-Simons action. Consider the monopole operator
\[ \exp i\sigma(x) := \exp \left( i \int_x^\infty d\sigma(x) \right). \]  
(3.26)

Naively, one could think that operators of the form \( \exp(i\sigma(x)/\ell) \) is also feasible, where \( \ell \) is an arbitrary integer. However, this is not so because \( \sigma \) is a compact pseudo-scalar defined over the period \( 2\pi \). This means that \( \int \sigma(x)/\ell \equiv (2\pi/\ell)\mathbb{Z} \) when we integrate over a closed contour. Therefore, \( \exp \int \sigma(x)/\ell \) will be path-dependent, and hence non-local unless \( \ell = 1 \).

Not only being local, the monopole operator or products of it is also covariantly constant. Recalling that the monopole operator \( T_k \) carries an electric charge of \( k \) unit, the covariant derivative acting on it is defined by
\[ D_\mu T_k = (\partial_\mu - ikb_\mu) T_k = (i\partial_\mu \sigma - ikb_\mu) e^{i\sigma}. \]  
(3.27)
We see that this indeed vanishes by the defining relation of the dual scalar field, \( kb = d \sigma \). This shows that \( T_k \) is covariantly constant. Notice that this property holds for any \( k \).

Using these properties, we can put \( Z^A \) and \( Z_A \) fields on equal footing by attaching appropriate monopole operators to them. So, \( Z^A \) carries an electric charge of one unit, while \( (T_k)^n Z_A \) carries an electric charge of \( nk - 1 \). From the above analysis, we see that these two (composite) fields are local operators and, as discussed above, can carry equal electric charge when \( k = 1 \) and \( n = 2 \) or \( k = 2 \) and \( n = 1 \), but none for \( k > 2 \).

4 The ABJM theory

4.1 hermitian 3-algebra

The ABJM theory is isomorphic to Hermitian 3-algebras up to possible \( U(1) \) factors in the gauge group. As said, instead of studying the ABJM theory for each possible gauge group separately, it is convenient to utilize the 3-algebra formulation that puts all the possible gauge groups on equal footing. The only property of the gauge groups we need is then the corresponding fundamental identity of the 3-algebra.

so(4):

The simplest example of a 3-algebra is that of gauge group \( \mathfrak{g} = SU_L(2) \times SU_R(2) = SO(4) \). This corresponds to a real (which of course also is hermitian) 3-algebra. To see this, we note the following gamma matrix identity among the SO(4) gamma matrices \( \gamma_a \) and the chirality matrix \( \gamma \):

\[
\gamma_a \gamma_b \gamma_c - \gamma_b \gamma_c \gamma_a = 2 \epsilon_{abcd} \gamma_d. \tag{4.1}
\]

In the Weyl representation, the 3-algebra generators \( T^a \) sit in the gamma matrices as

\[
\gamma_a = \begin{pmatrix} 0 & (T^a)^i_{i'} \\ (T_a)^j_{j'} & 0 \end{pmatrix} \tag{4.2}
\]

Here upper (lower) indices \( i \) and \( i' \) are (anti)fundamental of \( SU_L(2) \) and \( SU_R(2) \), respectively. The gamma matrix identity above amounts to the 3-algebra

\[
T^a T_b T^c - T^b T_c T^a = f^{ab}_{\quad cd} T^d \tag{4.3}
\]

with real structure constants \( f^{ab}_{\quad cd} = 2 \epsilon_{abcd} \). Note that SO(4) also happen to have the metric \( \delta_{ab} \) that we can use to raise and lower indices. It is related to the epsilon tensors of \( SU_L(2) \times SU_R(2) \).
as
\[ \delta_{ab}(T^a)_i^j(T^b)_i^j = 2\epsilon_i^j\epsilon_i^j. \] (4.4)

We also have
\[ (T^a)_i^{j'}(T_a)^{j'} = 2\delta_i^{j'}\delta_i^{j'}. \] (4.5)

For generic ABJM gauge groups there is no such invariant tensor that we can use to raise and lower indices. What we can use instead are monopole operators.

**generalizations:**

We now generalize the SO(4) 3-algebra by keeping some of the structure of it but dropping the constraints of having real structure constants and a metric. We denote the complex 3-algebra generators by \( T^a \). We define complex conjugation as
\[ T^a = T_a. \] (4.6)

The 3-bracket maps three elements into a new element
\[ [T^a, T^b; T^c] = f^{abc}_{cd} T^d. \] (4.7)

Here the structure constants \( f^{abc}_{cd} \) are complex-valued. The 3-bracket has the properties
\[
\begin{align*}
[T^a, T^b; T^c] &= -[T^b, T^a; T^c] \\
[\lambda T^a, T^b; T^c] &= \lambda[T^a, T^b; T^c] \\
[T^a, T^b; \lambda T^c] &= \lambda^*[T^a, T^b; T^c].
\end{align*}
\] (4.8)

The 3-bracket obeys the so-called fundamental identity. The fundamental identity is best understood as a property of the derivation
\[ \delta = [\cdot, T^b; T^a] \omega^a_b, \] (4.9)

Here \( \omega^a_b \) is an anti-hermitian matrix:
\[ \omega^{*a}_b = -\omega^b_a. \] (4.10)

The derivation property is
\[ \delta[T^e, T^d; T^c] = [\delta T^e, T^d; T^c] + [T^e, \delta T^d; T^c] + [T^e, T^d; \delta T^c]. \] (4.11)

Using (4.9), this amounts to the fundamental identity:
\[ [[[T^e, T^d; T^c], T^b; T^a] = [[[T^e, T^b; T^a], T^d; T^c] + [T^e, [T^d, T^b; T^a]; T^c] - [T^e, T^d; [T^c, T^a, T^b]].] (4.12)

In terms of the structure constants, the identity reads
\[ f^{ed}_{cf} f^{fb}_{ag} = f^{eb}_{af} f^{fd}_{cg} + f^{db}_{af} f^{ef}_{cg} - f^{*ca}_{bf} f^{ed}_{fg}. \] (4.13)
inner product:

We also introduce inner product $\langle \cdot , \cdot \rangle$ such that

\[
\langle T^a, T^b \rangle = \delta^a_b \\
\langle T^a, T^b \rangle = \langle T^b, T^a \rangle^* \\
\langle T^a, T^b \rangle = \delta_{ab} \langle T_b, T_a \rangle.
\] (4.14)

By expanding a field $X$ in the 3-algebra basis $X = X_a T^a$, the last property can also be phrased as

\[
\langle X, Y \rangle = \langle Y^*, X^* \rangle \quad \text{for} \quad X = X_a T^a, \quad Y = Y_a T^a.
\] (4.15)

This may be taken as defining equation of the hermitian conjugate. Moreover, the inner product has the invariance property

\[
\langle \delta T^a, T^b \rangle + \langle T^a, \delta T^b \rangle = 0.
\] (4.16)

Using (4.9), we get

\[
f^{*ab}_{\;cd} = f^{cd}_{\;ab}.
\] (4.17)

One can also check that this condition can be written as

\[
\langle X, [Y; Z; U] \rangle = \langle [X; U; Z], Y \rangle.
\] (4.18)

We note that (4.12), (4.18) generalize the corresponding equations for totally antisymmetric 3-brackets introduced originally for the BLG theory. To get the corresponding fundamental identity and inner product invariance condition for totally antisymmetric 3-bracket, we just need to replace $[\cdot , \cdot , \cdot ]$ by totally antisymmetric 3-bracket $[\cdot , \cdot , \cdot ]$.

### 4.2 matrix realization of hermitian 3-algebra

matrix realization:

A matrix realization of the 3-algebra $\mathcal{A}_3(\cdot)$ is provided by

\[
[X, Y; Z] := XZ^\dagger Y - YZ^\dagger X \\
\langle X, Y \rangle := \text{tr}(XY^\dagger).
\] (4.19)

The matrix-valued fields $X, Y, Z$ are expanded as $X = X_a T^a$ etc., where $T^a$ is a basis of $(M \times N)$ matrices and $T_a$ are their hermitian conjugates. The 3-bracket is then a map from $M \times N$ matrices
to itself – the first requirement of an algebra. Moreover, the bracket satisfies the fundamental identity \((4.12)\). Hence, it is a realization of the 3-algebra \(\mathcal{A}_3(\cdot)\), called the Lie 3-algebra \(\mathcal{A}_3(\mathfrak{g})\).

An explicit solution to the fundamental identity can also be realized in terms of the generators \(t^\alpha\) of the associated semi-simple Lie algebra \(\mathfrak{g}\) as \([8]\)

\[
f^{ab}_{\cd} = (t^\alpha)^a_d (t_\alpha)^b_c
\]

where \((t^\alpha)^a_b\) are the generators in the bi-fundamental representation. The index \(\alpha\) is lowered by the inverse of Killing form \(\kappa^{\alpha\beta}\) of the Lie algebra \(\mathfrak{g}\). This realization does not in general satisfy antisymmetry with respect to \(a, b\) or \(c, d\) indices. Imposing this property restricts possible choices of the Lie algebras \(\mathfrak{g}\) and hence the Lie group \(\mathfrak{G}\). With the Lie group \(\mathfrak{G} = \mathfrak{G}_L \otimes \mathfrak{G}_R\), \(a, b, c, d\) ranges over \(1, \ldots, \text{rank}(\mathfrak{G}_L)\text{rank}(\mathfrak{G}_R)\) and \(\alpha\) ranges over \(1, \ldots, \dim(\mathfrak{G}_L) + \dim(\mathfrak{G}_R)\).

**Similarity Transformations:**

We can consider two types of similarity transformations of the Lie algebra generators associated with the 3-algebra. The first type is

\[
(t^\alpha)^a_b \rightarrow U^a_c (t^\alpha)^c_d U^{\dagger d}_b = U^\alpha_\beta (t^\beta)^a_b
\]

where \(U^a_b U^{\dagger b}_c = \delta^a_c\). The second type is

\[
(t^\alpha)^a_b \rightarrow U_{bc} (t^\alpha)^c_d U^{da}
\]

where \(U^{ab}_e U_{bc} = \delta^a_c\). Both types of transformations leave the Killing form \(\kappa^{\alpha\beta}\) invariant, and hence the 3-algebra structure constants are invariant. Explicitly,

\[
f^{ab}_{\cd} = f^{efg}_h U^a_e U^b_f U^{\dagger g}_c U^{\dagger h}_d
\]

and

\[
f^{ab}_{\cd} = f^{efg}_h U^{ga}_b U^{hb}_e U^{cf}_d
\]

respectively. Notice that the first type of transformations form a closed group, while the second is not. However, the total sum of the two types again forms a closed transformation group, which we denote as \(\hat{G}\).

The first type of similarity transformation means that the 3-algebra is invariant under the unitary transformation

\[
T^a \rightarrow T^b U^a_b
\]
The infinitesimal version of this invariance condition leads to the fundamental identity. Namely if we write
\[ U^a_{\ b} = \delta^a_{\ b} + \Omega^a_{\ b} \] (4.26)
we find that
\[ \delta f^{bc}_{\ da} = 0 \] (4.27)
where we define
\[ \delta f^{bc}_{\ da} = \Omega^b_{\ e f^{ec}_{\ da}} + \Omega^c_{\ e f^{be}_{\ da}} - \Omega^e_{\ d f^{bc}_{\ ea}} - \Omega^e_{\ a f^{bc}_{\ de}}. \] (4.28)

To make the connection with the fundamental identity, we just write out
\[ \Omega^b_{\ a} = \omega^d_{\ c f^{bc}_{\ da}}. \] (4.29)

The second type of similarity transformation is the transformation we shall use repeatedly in later sections.

### 4.3 ABJM theory in hermitian 3-algebra

We now describe the ABJM theory in 3-algebra formulation and arrive at (2.6).

**Lagrangian:**

In 3-algebra formulation, the covariant derivative is given by
\[ iD_\mu Z_a := i\partial_\mu Z_a + Z_b \tilde{A}_\mu^{\ b}, \quad D_\mu \Psi_a := \partial_\mu \Psi_a + \Psi_b \tilde{A}_\mu^{\ b}, \] (4.29)
where
\[ \tilde{A}_\mu^{\ b} \equiv A_\mu^{\ d c f^{bc}_{\ da}}. \] (4.30)

Our gauge fields are anti-Hermitian:
\[ A_\mu^{\ b a} = -A_\mu^{\ a b} \quad \text{equivalently} \quad A_\mu^{\ b a} = -A_\mu^{\ a b}. \] (4.31)

To translate the action to the more familiar Lie algebra formulation, we use some properties of the 3-algebra of the previous subsection. We just use the matrix realization (4.19). We also define gauge fields of the two Lie groups \( G_L, G_R \) associated with the 3-algebra by
\[ A_\mu^L = A_\mu^{\ d c T^c T_d}, \quad A_\mu^R = A_\mu^{\ d c T_T^c}. \] (4.32)
With these steps, we find the followings. First, the Chern-Simons term in the 3-algebra formulation turns into two Chern-Simons terms in Lie algebra formulation:

$$\frac{k}{2\pi}e^{i\nu\lambda}\text{Tr}(A^L_\mu\partial_\nu A^L_\lambda + \frac{2i}{3}A^L_\mu A^L_\nu A^L_\lambda) - \frac{k}{2\pi}e^{i\nu\lambda}\text{Tr}(A^R_\mu\partial_\nu A^R_\lambda + \frac{2i}{3}A^R_\mu A^R_\nu A^R_\lambda).$$ (4.33)

Second, the gauge covariant derivatives acting on matter fields are given by

$$iD_\mu Z^A = i\partial_\mu Z^A - A^A_\mu Z^A + Z^A A^R_\mu$$ (4.34)

and similarly for fermions. Third, the Yukawa-like terms are given by

$$\psi^{Aa}\psi^{Ab}Z^{B^c}Z^{d^B} = \text{Tr}(\psi^{Aa}\psi^{Aa}Z^{A^c}Z^{d^A}) - \text{Tr}(\psi^{Aa}Z^{A^c}Z^{d^A})$$ (4.35)

e etc. The same works for the scalar potential terms. This shows that the ABJM action (2.6) in 3-algebra formulation is identical to the ABJM action in Lie algebra formulation, as demonstrated first in [8].

**on-shell \( \mathcal{N} = 6 \) supersymmetry:**

For later use, we here enlist \( \mathcal{N} = 6 \) supersymmetry transformations of the ABJM theory in the 3-algebra formulation. They are

$$\delta Z^A_a = -ic^{AB}\psi^B_{Ba}$$
$$\delta \psi^{Aa} = \gamma^\nu\epsilon_{AB}D_\mu Z^B_a - \left(\epsilon_{AB}Z^B_a Z^C_c Z^d_d + \epsilon_{BC}Z^B_b Z^C_c Z^d_d\right)f^{bc da}$$
$$\delta \tilde{\Lambda}^b_a = \left(\epsilon^{AB}\gamma_\mu Z^A_a\psi^{Cd} - \epsilon^{AB} Y\eta Z^B_a\psi_B^{Cd}\right)f^{bc da}. (4.36)$$

The closure relations read

$$[\delta \eta, \delta \epsilon]Z^A_a = -2ic^{AB}\gamma^\mu\eta^M D_\mu Z^A_a + \tilde{\Lambda}^b_a Z^A_a,$$
$$[\delta \eta, \delta \epsilon]\psi^{Aa} = -2ic^{AB}\gamma^\mu\eta^M D_\mu \psi^{Aa} + \tilde{\Lambda}^b_a \psi^{Ab} + i\epsilon^{M} Y\epsilon^{MN}\gamma^\mu E^{Aa}$$
$$[\delta \eta, \delta \epsilon]\tilde{\Lambda}^b_a = -2ic^{AB}\gamma^\mu\eta^M \tilde{F}_{\nu}^{b} a - D_\mu \tilde{\Lambda}^b_a$$ (4.37)

with the gauge parameter

$$\tilde{\Lambda}^b_a = 2ic^{M} (\gamma^{MN})_{A}^{B} \eta^{N} Z^{A} c^{d} f^{bc da}. (4.38)$$

The equations of motion needed to close the supersymmetry on-shell are \( E_{Aa} = 0 \) with

$$E_{Aa} = \gamma^\nu D_\mu \psi^{Aa} + \left(\psi_{Ab}Z^C_c Z^d_d - 2\psi_{Bb}Z^B_b Z^C_c Z^d_d + \epsilon_{ABCD}Z^B_b Z^C_c Z^d_d\right)f^{bc da}$$ (4.39)

for the fermions and

$$\tilde{F}_{\nu}^{b} a = -\epsilon^{MN}\gamma^\lambda Z^{A} d^{A} Z^C_c Z^{C^d} Z^d_d - \tilde{\eta} \gamma^\lambda Y\psi^{Ac} f^{bc da}$$ (4.40)

for the gauge field.
5 Monopole Operator and Gauge Covariance

In this section, we shall introduce monopole operator which will play a central role in the foregoing discussions. Consider for definiteness the gauge group $G_L = SU(M)$, $G_R = SU(N)$. We start with infinitesimal gauge transformations

$$\delta \tilde{A}_\mu^b a = -D_\mu \Lambda_a^b$$

and

$$\delta Z_a^b = Z_b \Lambda_a^b$$ (5.1)

on gauge field and matter fields, respectively, where

$$\tilde{\Lambda}_a^b = \Lambda_{cd} f^{bc} a$$ (5.2)

and $\Lambda_{c d}$ is any antihermitian matrix.

The scalar fields in the Lie algebra and the 3-algebra basis are related by

$$Z_i^a = Z_a (T^a)_i$$ (5.3)

and similarly for the fermion fields. Here $i, \alpha$ are indices of $M, \overline{N}$, respectively. Complex conjugate field is

$$(Z^*)_i^\alpha = Z_i^a (T^a)_i^\alpha.$$ (5.4)

Gauge transformation with gauge group element $(g^L, g^R)$ acts on the bi-fundamental matter field as

$$Z_i^a \rightarrow (g^L)^i_j Z_j^a (g^R)^{\beta \alpha}.$$ (5.5)

5.1 nonabelian monopole operators

We now introduce monopole operators [6]. The monopole operator that transforms in the fundamental representations of $G_L = U(M)$ and $G_R = U(N)$ are denoted as $W^L$ and $W^R$, respectively.

$$(W^L)_i \rightarrow (W^L)_j (g^L)^i_j$$

and

$$(W^R)_\alpha \rightarrow (W^R)_\beta (g^R)^{\beta \alpha}.$$ (5.6)

Utilizing them, it is possible to obtain composite fields transforming differently. For example, one can form a gauge singlet composite of the bi-fundamental field $Z$ and monopole operators:

$$(W^L)_i Z^a_\alpha (W^R)^{\alpha} = Z_a (W^L)_i (T^a)_i^\alpha (W^R)^{\alpha}.$$ (5.7)
Obviously, such an operation does not bring the matter field outside the 3-algebra $A_3$, so the composite must again be some linear combination of 3-algebra generators. As such, we define the monopole operator of defining representation in 3-algebra formulation as

$$W^a \equiv (W^L)_i (T^a)^i_\alpha (W^{R\dagger})^\alpha.$$  \hspace{3cm} (5.8)

Therefore,

$$Z = W^a Z_a$$  \hspace{3cm} (5.9)

will be the above gauge singlet composite. Associated with $W^a$, there is also the monopole operator $W_a = W^{*a}$ transforming in the complex conjugate representation.

We can also form composites of other representations than the bi-fundamental, but again the resulting composite operator must be some linear combination of 3-algebra generators. In fact, in order to extend $N = 6$ supersymmetry to $N = 8$ supersymmetry, we may need the monopole operators of higher representations \[18\]. The most general monopole operator in the Lie algebra and in the 3-algebra basis are related each other as

$$W^{a_1\ldots a_k} = W^{\alpha_1\ldots\alpha_k}_{i_1\ldots i_k} (T^{a_1})^{i_1}_{\alpha_1} \ldots (T^{a_k})^{i_k}_{\alpha_k}.$$  \hspace{3cm} (5.10)

It turns out sufficient to consider symmetric rank-2 representations, $W^{ab}$ and $W_{ab}$. We note that these monopole operators can act to lower and raise gauge indices in a covariant way. For example, by attaching these monopole operators, we have

$$Z^{Aa} = W^{ab} Z^A_b, \quad Z_{Aa} = W_{ab} Z^b_A.$$  \hspace{3cm} (5.11)

Beware these operations are different from complex conjugation $Z^*_a = Z^a$ etc. In particular, the SU(4) representation is not affected by attaching the monopole operators.

Under gauge transformations, the rank-2 monopole operators transform as

$$\delta W^{ab} = - W^{cb} \tilde{\Lambda}^a_c - W^{ac} \tilde{\Lambda}^b_c$$

$$\delta W_{ab} = \tilde{\Lambda}_a^c W_{cb} + \tilde{\Lambda}_b^c W_{ac}.$$  \hspace{3cm} (5.12)

Moreover, they have the properties

$$W_{ac} W^{cb} = \delta^b_a$$

$$W_{ab} = W_{ba}$$

$$W^{ab} = W^{ba}.$$  \hspace{3cm} (5.13)

In the Lie algebra formulation, the relevant monopole operator is the one in bi-fundamental representations

$$W^\alpha_i = (W^{R\dagger})^\alpha (W^L)_i$$
They are related to the rank-2 monopole operators $W^{ab}, W_{ab}$ by

$$W^{ab} T_b = W T^a W$$

$$W_{ab} T^b = W^a T_b W^b.$$  (5.15)

### 5.2 general covariance

So far, we focused primarily on the representation contents of the monopole operators. In general, the monopole operators of a given representation are nonlocal. For the symmetric rank-2 representations, by the Dirac quantization condition, the monopole operator turns out a local operator only if the Chern-Simons level takes values $k = 1$ or 2. This locality condition leads to an important condition to the gauge field strength, which plays an essential role in foregoing considerations concerning supersymmetry enhancement. Much like the abelian case, invisibility of Dirac string implies that the monopole operator is covariantly constant:

$$D_\mu W_{cb} \equiv \partial_\mu W_{cb} + \tilde{A}_\mu^d W_{db} + \tilde{A}_\mu^d W_{cd} = 0.$$  (5.16)

From this it follows that

$$W^{ac}[D_\mu, D_\nu] W_{cb} = 0$$  (5.17)

and this amounts to the following flatness condition for the field strength

$$\tilde{F}_{\mu\nu}^a+b + \tilde{F}_{\mu\nu}^a+b = 0.$$  (5.18)

Here, we defined

$$\tilde{F}_{\mu\nu}^a+b = W^{ac} W_{bd} \tilde{F}_{\mu\nu}^d.$$  (5.19)

A few remarks are in order. First, for level $k = 1$, we should in principle also be able to bring all matter fields into gauge singlets using $W_a$ and $W^{a}$ monopole operators. However, this does not give us any nice identity for the field strength. Instead, what we get is $F_{\mu\nu,a}b W_b = 0$. However, we can not conclude from this any identity for $F_{\mu\nu}$ itself. It would be interesting to analyze how to use $W^{a}$ and $W_a$ to see supersymmetry and R-symmetry enhancement for level $k = 1$. In our approach, we shall be using $W_{ab}$ and $W^{ab}$ for both $k = 1$ and $k = 2$.

Second, expanding $F_{\mu\nu} = F_{\mu\nu,a}a$ in the Lie algebra generators, one might be tempted to conclude from (5.18) that the Lie algebra generators are invariant under the similarity transformation induced by the monopole operator

$$(t^a)^b_a = -W_{ac} (t^a)^c_d W^{db}.$$  (5.20)
This is not right because the gauge field strength cannot be varied independently of the monopole operator. Therefore (5.18) does not imply (5.20). In fact, (5.20) is not even gauge covariant since the generators do not transform under the gauge transformations whereas the monopole operators do transform in general. On the other hand, if we assume (5.20), we find the BLG theory as the only solution for which

$$W_{ab} = \delta_{ab},$$

the Kronecker delta of the $\text{SO}(4) = \text{SU}_L(2) \times \text{SU}_R(2)$ gauge group (which is invariant, $\delta \delta_{ab} = \Lambda_{ca} \delta_{cb} + \Lambda_{cb} \delta_{ac} = \Lambda_{ba} + \Lambda_{ab} = 0$) and $(t^\alpha)_{ab} = (-t^\alpha)_{ba}$ are the antisymmetric generators of $\text{SO}(4)$ gauge group. This is one of many indications that supersymmetry enhancement for the ABJM theory is highly nontrivial than one might naively extrapolate from the BLG theory.

6 Closure among ABJM and non-ABJM fields

6.1 closure relation and gauge condition

As far as $\mathcal{N} = 6$ and $\text{SU}(4)$ symmetry variations (let us denote variations as $\delta$) are concerned, since ABJM fields and non-ABJM fields do not mix, we do not need to consider the quantities

$$\Omega^b_{a} \equiv W^{bc} \delta_{0a} W_{ca}, \quad (6.1)$$

which encodes variation of the monopole operator. On the other hand, when we explore possible $\mathcal{N} = 8$ and $\text{SO}(8)$ symmetry enhancement, we must consider these quantities since the ABJM and non-ABJM fields mix each other. A priori, this indicates that we need to find explicit expression of $\Omega^b_{a}$. This, however, turned out extremely difficult. Fortuitously, we never need the explicit expression, as we now explain below.

It is easy to see why $\Omega^b_{a}$ is needed when we mix the ABJM and non-ABJM fields. Let us assume that

$$\delta Z^A_a = \hat{\delta} Z^A_a, \quad (6.2)$$

where $\hat{\delta}$ denotes any variation that does not involve $\Omega^b_{a}$ explicitly. We then get

$$\delta Z^{AA} = \hat{\delta} Z^{AA} - \Omega^b_{a} Z^A_b. \quad (6.3)$$

On the other hand, there is no good reason why ABJM fields should be treated any differently from non-ABJM fields. What we call ABJM and non-ABJM fields is really a matter of convention. Therefore, there is no reason we should not have $\Omega^b_{a}$ dependent terms in the variations of the ABJM fields. Let us therefore treat ABJM and non-ABJM fields on equal footing and take the general ansatz for the variations of the fields as

$$\delta Z^A_a = \hat{\delta} Z^A_a + \gamma \Omega^b_{a} Z^A_b.$$
\[ \delta Z^{Aa} = \hat{\delta} Z^{Aa} + (\gamma - 1)\Omega^a_{\; b}Z^{Ab}. \]  
(6.4)

Here \( \gamma \) could a priori be any real number. We then have

\[ \delta(D_\mu Z^A_a) = \hat{\delta}(D_\mu Z^A_a) + \gamma \Omega^b_a D_\mu Z^A_b. \]  
(6.5)

From the left-hand side, we get

\[ \delta A^b_{\mu a} = \hat{\delta} A^b_{\mu a} + \gamma \Omega^b_a D_\mu Z^A_b. \]  
(6.6)

Any symmetry variations should close among themselves. This requirement has an interesting consequence when it is applied to variations that mixes ABJM and non-ABJM fields. We get no restriction on \( \gamma \) as long as we consider variations that do not mix ABJM and non-ABJM fields. Let us therefore consider SO(8) variations that mix these fields. We can also consider \( \mathcal{N} = 8 \) variations but the steps are essentially the same. The variations take the form

\[ \delta Z^A_a = \omega^{AB} Z^B_a + \gamma Z^A_a \Omega^b_a, \]
\[ \delta Z^A_a = \omega^{AB} Z^a_B + (\gamma - 1)\Omega^d_b Z^{Ab}, \]
\[ \delta Z^A_a = -\omega^{AB} Z^B_a + (1 - \gamma)Z^A_a \Omega^b_a. \]  
(6.7)

More general variation may be considered such as \( \hat{\delta} Z^A_a = \omega^{AB} Z^b_a z^A_b + \omega^{AB} \; z^{AB} \; z^{Bb} + \ldots \) but the conclusion will anyway be the same. Since \( Z^A_a \) and \( Z^A_a \) transform the same under the gauge group and the second terms on the right hand side of the variations rotates gauge indices only, it motivates to have \( \gamma = (1 - \gamma) \), viz. \( \gamma = 1/2 \). We now show explicitly that this is indeed the necessary condition for the closure.

The closure among these variations reads

\[ [\delta \eta, \delta \omega] = \delta [\eta, \omega]. \]  
(6.8)

We get

\[ [\delta \eta, \delta \omega] Z^A_a = [\eta, \omega]^A_{\; B} Z^B_a \]
\[ + (1 - \gamma)\Omega^b_a \omega^{AB} Z^B_b + \gamma \Omega^b_a \eta^{AB} Z^B_b \]
\[ - (1 - \gamma)\Omega^b_a \omega^{AB} Z^B_b - \gamma \Omega^b_a \omega^{AB} Z^B_b \]
\[ + (\gamma^2 - \gamma) Z^A_b [\Omega^b_a, \Omega^b_c] + \gamma Z^A_b \Omega^b_a. \]  
(6.9)

Here, we have used the variation

\[ \delta \eta \Omega^b_a = -\Omega^b_a \Omega^d_a \eta^d_c + W^b_c \delta \eta \Omega^c_a. \]  
(6.10)

We also made the assumption that the variations close on the monopole operator

\[ [\delta \eta, \delta \epsilon] W_{ab} = \delta [\eta, \omega] W_{ab}. \]  
(6.11)
We now see that we can have the closure relation provided we set
\[ \gamma = \frac{1}{2}, \] (6.12)
since in this case the mixed transformation terms cancel each other. The remaining terms read
\begin{align*}
[\delta_\eta, \delta_\omega] Z_a^A &= [\eta, \omega]^A_b Z_a^B \\
&+ Z_b^A \left( \frac{1}{2} \Omega^b_{[\eta, \omega]} Z^a_a - \frac{1}{4} [\Omega_\eta, \Omega_\omega]^b_a \right). \quad (6.13)
\end{align*}

Here, \( \Omega \)s form a closed algebra
\[ [\Omega_\eta, \Omega_\omega] = \Omega_{[\eta, \omega]} \] (6.14)
due to the fact that \( \Omega \)s are homomorphism of SO(8) to \( \hat{\mathcal{G}} \). Comparing with (6.7), we see that the closure relation is up to a gauge transformation:
\[ [\delta_\eta, \delta_\omega] Z_a^A = \delta_{[\eta, \omega]} Z_a^A + \delta_{\text{gauge}} Z_a^A \] (6.15)
where the gauge parameter is given by \(-\frac{1}{4} \Omega^b_{[\eta, \omega]}\).

The result we found on \( \gamma \) is very interesting. It means that we find a gauge variation with gauge parameter
\[ \Lambda^b_a = \frac{1}{2} \Omega^b_a \] (6.16)
induced from the SO(8) variations. This gauge variation can be offset by making another gauge variation. This is the lucky circumstance that makes it possible to study variations that mix ABJM and non-ABJM fields without having to solve the tremendously difficult problem of finding an explicit expression for \( \Omega^b_a \) or of the variation of the monopole operator itself.

Having seen that \( \frac{1}{2} \Omega \) is just a gauge parameter, we can just drop all \( \Omega \)-dependent terms from our variations from the outset.

### 6.2 combining gauge covariance with \( \mathcal{N} = 6 \) supersymmetry

We can use \( \mathcal{N} = 6 \) supersymmetry to vary the identity (5.18) and get new identities. We can vary \( \tilde{F}_{\mu \nu} \) either by varying its on-shell expression (4.40), or we can compute the variation induced by variation of the gauge field as
\[ \delta_e \tilde{F}_{\mu \nu} = D_\mu \delta_e \tilde{A}_\nu - D_\nu \delta_e \tilde{A}_\mu. \] (6.17)
Both computations give the same result when the fields are put on-shell. The latter approach is the quicker, and it gives the result
\[ \delta \tilde{F}_{\mu \nu}^b_a = -i \tilde{\epsilon}^{AB} \gamma_\nu D_\mu (\psi_{[Ac} Z_{B]}^d f^{bc}_{da} + (\text{a.h.c.}). \] (6.18)

where (a.h.c) means that we should make the result antihermitian by adding the anti-hermitian conjugate term. Instead of computing the supersymmetry variation of \( \tilde{A}_{\mu a}^b = \tilde{A}_{\mu}^d W^{bc} W_{da} \), we use the former approach and compute the variation of the on-shell field strength \( \tilde{F}_{\mu \nu}^b_a \)
\[ \tilde{F}_{\mu \nu}^b_a = -\epsilon_{\mu \nu \lambda} \left( Z^A \gamma_\lambda D_\lambda \psi_{[Ac} Z_{B]}^d - D_\lambda Z^A \psi_{[Ac} Z_{B]}^d - i \tilde{\psi}_{[c}^A \gamma_\lambda \psi_{A]}^d f^{bc}_{da} \right). \] (6.19)

Then we can make a supersymmetry variation of the on-shell field strength. The result we get then is
\[ \delta \epsilon \tilde{F}_{\mu \nu}^b_a = -i \tilde{\epsilon}^{AB} \gamma_\nu D_\mu (\psi_{[Ac} Z_{B]}^d f^{bc}_{da} + (\text{a.h.c.}). \] (6.20)

Now the \( \mathcal{N} = 6 \) supersymmetry variation of the identity \( \tilde{F}^{bc}_{da} \) reads
\[ \tilde{\epsilon}^{AB} \gamma_\nu D_\mu (\psi_{[Ac} Z_{B]}^d + \psi_{[A}^d Z_{B]}^c ) f^{bc}_{da} + (\text{a.h.c.}) = 0. \] (6.21)

\( \tilde{\epsilon}^{AB} \) and its conjugate are arbitrary, so we find the equations
\[ \gamma_\nu D_\mu (\psi_{[Ac} Z_{B]}^d + \psi_{[A}^d Z_{B]}^c ) f^{bc}_{da} = 0. \] (6.22)

From this equation it follows that
\[ D_\mu (\psi_{[Ac} Z_{B]}^d + \psi_{[A}^d Z_{B]}^c ) f^{bc}_{da} = 0. \]

To understand this we note that an equation \( \gamma_\nu D_\mu \psi - \gamma_\mu D_\nu \psi = 0 \) implies \( \gamma^\lambda D_\mu \psi = 0 \) upon contracting by \( \gamma^\nu \). Second if we contract by \( \gamma^\mu \) we find \( -D_\nu \psi - \gamma^\nu (\gamma^\mu D_\mu \psi) = 0 \). Hence \( D_\nu \psi = 0 \). The covariant derivative only acts on gauge indices, not on spinor indices. Since there is no independent covariantly constant spinor, we find six identities
\[ (\psi_{[Ac} Z_{B]}^d + \psi_{[A}^d Z_{B]}^c ) f^{bc}_{da} = 0 \] (6.23)

one for each choice of the antisymmetric indices \( [AB] \). The right-hand side is zero since there is no non-trivial spinor of the same quantum number as the left-hand side.

It turns out (6.23) is the supersymmetry variation of the identity:
\[ (Z^A Z^d_c + Z^{Ad} Z_{Ac}) f^{bc}_{da} = 0. \] (6.24)
Again we could have added a supersymmetric invariant to the right hand side, but there is no such an invariant which is also gauge covariant and has the same dimension. To show this identity, take $N = 6$ supersymmetry transformation of (6.24):

$$
0 = -\epsilon^{AB} \left( \psi_{Bc} Z^d_A + \psi^d_B Z_{Ac} \right) + \epsilon_{AB} \left( Z^A_c \psi^{Bd} + Z^{Ad} \psi^B_c \right) + \frac{1}{2} \left( Z^A_c Z^d_A + Z^{Ad} Z_{Ac} \right) \left( \Omega^e_c f^{bc}_{da} - \Omega^c_d f^{be}_{ca} \right). 
$$  
(6.25)

To get (6.23) from this, we need to show that the third line vanishes. We note that $\Omega$ is a Lie algebra element, and hence we can pull out one 3-algebra structure constant from it as

$$
\Omega^b_a = \omega^d_c f^{bc}_{da} 
$$  
(6.26)

or we may directly use the fundamental identity (4.27) $\delta f^{bc}_{da} = 0$. Either way, we can rewrite the third line as

$$
\frac{1}{2} \left( Z^A_c Z^d_A + Z^{Ad} Z_{Ac} \right) \left( \Omega^b_c f^{ce}_{da} - \Omega^c d f^{be}_{ac} \right) 
$$  
(6.27)

and this vanishes by the identity (6.24).

This result is in concordance with the fact that $\Omega$-terms should play no important role in our equations.

7 $\mathcal{N} = 8$ Supersymmetry

We require any $\mathcal{N} = 8$ supersymmetry variations be such they reproduce BLG variations for BLG gauge groups (that means SO(4) and such, for which $W_{ab} = \delta_{ab}$ and $f^{bc}_{da} = f_{bcda}$ real and totally antisymmetric). We also require gauge covariance. We then find $\Omega$ terms that contribute a gauge variation with gauge parameter $\frac{1}{2} \Omega$. We off-set these by a supplementary gauge variation. Then we end up with the following ansatz for $\mathcal{N} = 8$ supersymmetry variations (for levels $k = 1, 2$),

$$
\delta Z_{Aa} = \epsilon^{AB} \Psi^B_a - \epsilon \Psi_{Aa}
$$

$$
\delta \psi_{Aa} = \gamma^\mu \epsilon_{AB} D_\mu Z^B_a + \gamma^\mu \epsilon D_\mu Z_{Aa} + \epsilon_{AB} Z^B_c Z^C_d Z^C_A \epsilon^d_{cd} f^{bc}_{da} - i \epsilon Z_{Ab} Z^B_c Z^d_f f^{bc}_{da} + \frac{i}{3} \epsilon^* \epsilon_{ABCD} Z^B_c Z^C_d Z^D_f f^{bc}_{da},
$$

$$
\delta \tilde{A}_\mu^A = \left( -\epsilon^{AB} \gamma_\mu \Psi^B_A + \epsilon_{AB} \gamma_\mu Z^A_c \Psi^B_d \right)
$$
Much is surely getting fixed in these supersymmetry transformations by the requirement that it reproduces the BLG transformation rules in certain limits. We go through that argument in detail in Appendix using triality. Gauge covariance then dictates how to put the gauge 3-algebra indices, at least to a large extent. Still some ambiguities remain. We will see how that ambiguity is cured by having associated identities in section 7.1.

It is also worth of noting that the supersymmetry transformations (7.1) involve terms of baryon number $\Delta Q_B = 0, \pm 1$. In M-theory, the baryon number is related to the Kaluza-Klein momentum around the M-theory circle. Upon dimensional reduction, there may be a priori an infinite tower of fields carrying multiple Kaluza-Klein momentum. The fact that only fields with $\Delta Q_B = 0, \pm 1$ and none with $\Delta Q_B \geq 2$ appear implies that higher momentum modes are bound-states of these elementary modes.

### 7.1 closing $\mathcal{N} = 2$ supersymmetry

The most general ansatz for the $\mathcal{N} = 2$ supersymmetry variations such that they reduce to BLG variations for BLG gauge groups are given by a 3-parameter family (we denote the three parameters as $a$, $b$ and $d$ respectively):

$$
\delta Z_{Aa} = -\gamma^\mu \psi_{Aa},
\delta \psi_{Aa} = i \gamma^\mu D_\mu Z_{Aa} - i e \left( a Z_{Ab} Z_{Bc}^d B_d + b Z_{Bc}^d Z_{cA}^d + (1 - a - b) Z_{Bc} B_c Z_{cA}^d \right) f^{bc}_{da}^a + i \epsilon^* \phi_{Aa} Z_{Bc}^d Z_{cA}^d + i \epsilon^* \phi_{Bc} Z_{Aa}^d Z_{cA}^d f^{bc}_{da}^a,
\delta \tilde{A}_{\mu b a} = \left( -\gamma^\mu Z_{cA}^d \psi_{Bc}^d - \gamma^\mu \psi_{cA}^d Z_{Bc}^d \right) f^{bc}_{da}^a + \epsilon^* \phi_{Aa} Z_{Bc}^d Z_{cA}^d + \epsilon^* \phi_{Bc} Z_{Aa}^d Z_{cA}^d f^{bc}_{da}^a.
$$

Eventually, we will see that all three parameters are traded for the three identities. At present, the only identity we can make use of, is identity in (6.24). We then find that the following variations

$$
\delta Z_{Aa} = -\gamma^\mu \psi_{Aa},
\delta \psi_{Aa} = i \gamma^\mu D_\mu Z_{Aa} - i e Z_{Ab} Z_{Bc}^d B_d - (1 - c) Z_{Bc}^d Z_{Bc} B_c (1 - c) Z_{Bc}^d Z_{Bc} B_c f^{bc}_{da}^a + i \epsilon^* \phi_{Bc} Z_{Aa}^d Z_{cA}^d + \epsilon^* \phi_{Bc} Z_{Aa}^d Z_{cA}^d f^{bc}_{da}^a,
\delta \tilde{A}_{\mu b a} = \left( -\gamma^\mu Z_{cA}^d \psi_{Bc}^d - (1 - c) Z_{Bc}^d Z_{Bc} B_c + \epsilon^* \phi_{Aa} Z_{Bc}^d Z_{cA}^d - (1 - c) \psi_{cA}^d Z_{Bc} B_c \right) f^{bc}_{da}^a
$$

close on some equations of motion. More precisely, they close on the one parameter set of equations of motion

$$
0 = \gamma^\mu D_\mu \psi_{Aa}
$$
\[ +c\left(2Z_{Ab}Z_c^B\psi_B^d + \psi_{Ab}Z_c^BZ_B^d\right) f^{bc}_{da} - (1-c)\left(2Z_{Ab}\psi_{Bc}Z_B^d + \psi_{Ab}Z_{Bc}Z_B^d\right) f^{bc}_{da} + \frac{1}{3}\epsilon_{ABCD}\left(2\psi_b^BZ_c^CZ^Dd + Z_b^BZ_c^C\psi^Dd\right) f^{bc}_{da}. \]  

(7.4)

Of course, we can not really get different results since we use just one and the same supersymmetry variation, and the dependence on the parameter \(c\) is fake, because we have the identity (6.24). So the equations of motion must not depend on the parameter \(c\). This implies that

\[ \left(Z_c^B\psi_B^d + \psi_{Bc}Z_B^d\right) f^{bc}_{da} = 0. \]  

(7.5)

We have generated a new identity! Now that we have this identity, we can go back to our ansatz and make it slightly more general

\[ \delta Z_{Aa} = -\bar{\epsilon}\psi_{Aa} \]

\[ \delta\psi_{Aa} = i\gamma^\mu\epsilon D\mu Z_{Aa} - i\epsilon Z_{Ab}(cZ_c^BZ_B^d - (1-c)Z^BdZ_Bc) f^{bc}_{da} + \frac{i}{3}\epsilon^*_{ABCD}Z_b^BZ_c^CZ^Dd f^{bc}_{da}, \]

\[ \delta\tilde{\lambda}_{\mu a} = \left(-\bar{\epsilon}\gamma_{\mu}Z^A_A^d\psi_{Aa} - \bar{\epsilon}^*\gamma_{\mu}\psi_{Aa}^A \right. \]

\[ + d\bar{\epsilon}\gamma_{\mu}\left(Z_c^B\psi_{Aa}^d + Z^Ad\psi_{Ac}\right) + d\bar{\epsilon}^*\gamma_{\mu}\left(\psi_{Ac}^A + \psi_A^Ad\psi_{Ac}\right) \]  

(7.6)

by allowing for two parameters \(d\) and \(c\) that need no longer be correlated due to our two identities (6.24) and (7.5). Again, we can carry out the closure computation but this time when we demand the closure equation does not depend on any choice of parameters (since the dependence on parameters in the variations is fake due to our identities), we find yet another identity

\[ \left(\psi_b^CZ_{Bc}Z_B^d - Z_{Bc}Z_b^B\psi^Cd\right) f^{bc}_{da} = 0 \]  

(7.7)

that will be very important for us below.

It would be desirable to have no ambiguity in the \(\mathcal{N} = 2\) supersymmetry variations. So far we have been able to explain only two of three parameters, namely the parameters \(a\) and \(d\). At the same time we have derived an identity (7.7) that seems to fit nowhere. Now let us be bold and just make a supersymmetry variation of an identity

\[ \left(Z_{Ab}Z_c^BZ_B^d + Z_{Bb}Z_c^BZ_A^d\right) f^{bc}_{da} = 0 \]  

(7.8)

that would be a most desirable identity, which we have not yet derived. What we then find is nothing but the identity (7.7). To see this requires a few further steps, but due to its importance, let us show it in detail. Supersymmetry variation gives us

\[ 0 = \Sigma_M^{\alpha} \left(\psi_b^DZ_c^BZ_B^d - Z_b^BZ_{Bc}\psi^Dd\right) \]
Then identity $Z_{Bb} Z^{Dd} \psi^{Bd} + \psi^{Bd} Z_{Bb}^{Dd} = 0$ follows from the identity Eq (6.23). Hence, we are left with the identity in (7.7). Consequently, we have now derived the identity (7.8), just make an inverse supersymmetry variation of (7.7)!

Now we have totally eliminated all ambiguity there was in our ansatz for the $\mathcal{N} = 2$ supersymmetry variations, all three parameters have been traded for corresponding identities. We can then go through our 'identity generating' mechanism a last time, computing $[\delta_\eta, \delta_\epsilon] \psi_{Aa}$ with three arbitrary parameters, and demand the outcome of that computation be independent of any parameters. This way we generate one new identity

$$\psi_{Bb} \left( Z_{Ac} Z^{Bd} + Z_{Bc}^{Ad} \right) f^{bcda} = 0. \quad (7.10)$$

Given these identities, we now find the following closure relations for the $\mathcal{N} = 2$ supersymmetry variations,

$$[\delta_\eta^{(2)}, \delta_\epsilon^{(2)}] Z_{Aa} = -2i \varepsilon^X \eta^X D_\mu Z_{Aa} + \tilde{\Lambda}^{(22)b} a Z_{Ab}$$

$$[\delta_\eta^{(2)}, \delta_\epsilon^{(2)}] \psi_{Aa} = -2i \varepsilon^X \eta^X D_\mu \psi_{Aa} + \tilde{\Lambda}^{(22)b} a \psi_{Aa} + 2 \varepsilon^X \gamma^X \gamma^Y E^{(22)}_{Aa} - 2 \varepsilon^8 \eta^7 E^{(22)}_{Aa},$$

$$[\delta_\eta^{(2)}, \delta_\epsilon^{(2)}] \tilde{A}_\mu^{(2)} a = -2i \varepsilon^X \eta^X \tilde{F}_{\nu \mu} a - D_\mu \tilde{\Lambda}^{(22)b} a$$

with gauge parameter

$$\tilde{\Lambda}^{(22)b} a = 4 \varepsilon^8 \eta^7 Z_c Z^{d} f^{bcda} \quad (7.11)$$

and we have closure on the ABJM equations of motion after we make use of all identities we have obtained so far.

### 7.2 commuting \(\mathcal{N} = 6\) and \(\mathcal{N} = 2\) supersymmetries

Making an $\mathcal{N} = 6$ supersymmetry variation of identity (7.5) we obtain three new identities\(^8\)

$$Z^{[A} D_\mu Z^{B]} + (D_\mu Z^{[B]} Z^{A]} = 0 \quad (7.14)$$

$$Z^{[A} Z^{B]} Z^{C]} + Z^{[B} Z^{C]} Z^{C]} + Z^{[A} Z^{B]} Z^{C]} = 0 \quad (7.15)$$

$$\psi_{A} [\gamma^\mu \psi_{B}] = 0. \quad (7.16)$$

\(^8\)To understand how we can get three new identities instead of just one, we note that an equation of the form

$$\gamma^\mu \varepsilon_M V_\mu + \varepsilon_M U = 0 \quad (7.13)$$

with $\varepsilon_M$ arbitrary, implies that $U = 0$ and $V_\mu = 0$ separately.
To be able to close supersymmetry and show SO(8) invariance, we must have two more identities. These are
\[
\begin{align*}
(Z_{Ab}Z_{Bc}Z^d_C - Z_{Cb}Z_{Ac}Z^d_B) f^{bc}_{\phantom{bc}da} &= 0, \\
(Z_{Ab}Z_{Bc}Z^d_C - Z_{Cb}Z_{Ac}Z^d_B) f^{bc}_{\phantom{bc}da} &= 0.
\end{align*}
\]
(7.17)
By contracting the first equation by the totally independent spinor \(\psi^C_a\), we easily can see that the result vanishes by using identities (6.23), (7.7). As an unnecessary extra check we can also contract the left-hand side by \(Z^C_a\) and again get zero by identity (7.8). Now we have more than shown that this identity holds. The second identity is proved the same way, by contracting by \(\psi_a^C\).

Let us make an \(\mathcal{N} = 6\) supersymmetry variation of the first identity. Expanding \(\Sigma^M_{AB} \Sigma^N_{CD}\) using Fierz relations in appendix, we find the supersymmetry variation gives just one single set of identities,
\[
\left(\psi^C_A Z_{[Ac} Z^d_{B]} - Z_{Ab} Z_{Bc} \psi^C_d\right) f^{bc}_{\phantom{bc}da} = 0.
\]
(7.18)
Using the same method as above, but applied to mixed supersymmetry variations, we generate the following new identities
\[
\begin{align*}
(Z_{Ab} \psi^{[Dc} Z^b_{B]} - \psi^{[Db} Z^b_{B]} Z^d_A) f^{bc}_{\phantom{bc}da} &= 0, \\
(Z_{Ab} \psi^{[Dc} Z^b_{B]} + \psi^{[Db} Z^b_{B]} Z^d_A) f^{bc}_{\phantom{bc}da} &= 0.
\end{align*}
\]
(7.19)
Let us now compute closure among these supersymmetries, commuting an \(\mathcal{N} = 2\) and an \(\mathcal{N} = 6\) variation. Given the above identities we get
\[
\begin{align*}
([\delta^{(2)}_\eta, \delta^{(6)}_\epsilon] + [\delta^{(6)}_\eta, \delta^{(2)}_\epsilon]) Z_{Aa} &= \tilde{\Lambda}^b_{\phantom{b}a} Z_{Ab}, \\
([\delta^{(2)}_\eta, \delta^{(6)}_\epsilon] + [\delta^{(6)}_\eta, \delta^{(2)}_\epsilon]) \psi_{Aa} &= \tilde{\Lambda}^b_{\phantom{b}a} \psi_{Ab} - (\epsilon_{AB} \eta - \eta_{AB} \epsilon) E^B_a, \\
([\delta^{(2)}_\eta, \delta^{(6)}_\epsilon] + [\delta^{(6)}_\eta, \delta^{(2)}_\epsilon]) \tilde{A}^{b}_{\mu a} &= -D_{\mu} \tilde{\Lambda}^{b}_{\phantom{b}a}
\end{align*}
\]
(7.20)
with gauge parameter
\[
\tilde{\Lambda}^{(62) b}_{\phantom{b}a} = \left( \epsilon \eta_{AB} Z^A_{c} Z_{B} + \epsilon^* \eta^{AB} Z_{Ac} Z^d_B \right) f^{bc}_{\phantom{bc}da} - (\epsilon \leftrightarrow \eta)
\]
(7.21)
and we have closure on the ABJM fermionic equation of motion \(E_{Aa} = 0\).

---

9 In practice this means we compute \(\delta^{(6)}_\epsilon (a Z_{A} Z^B + b Z^B Z_A Z_B + (1 - a - b) Z_{B} Z^B Z_A)\) and require the result be independent of \(a\) and \(b\).
8 Manifestly SO(8) invariant ABJM scalar potential

The ABJM sextic potential is most nicely expressed using 3-brackets. It can then be expressed as

\[ V_{\text{ABJM}} = \frac{2}{3} \left( \left\| [Z^A, Z^B; Z^C] \right\|^2 - \frac{1}{2} \left\| [Z^A, Z^B; Z^A] \right\|^2 \right) \] (8.1)

where we define

\[ \left\| X \right\|^2 = \langle X, X \rangle \] (8.2)

and SU(4) indices are contracted. We note that in this notation all SU(4) indices are up-stairs despite some of them are being contracted. Anytime we find an SU(4) index down-stairs in this notation, that will correspond to a non-ABJM field – a field with a monopole operator attached.

For the sake of completeness, let us list a few equivalent ways of expressing the sextic potential. We have the following alternative expressions

\[ V = \frac{2}{3} \left\| [Z^A, Z^B; Z^C] + \alpha [Z^D, Z^A; Z^D] S^A_C \right\|^2 \]

in the 3-algebra language, where we can choose \( \alpha = 1 \) or \( \alpha = \frac{1}{3} \). In the matrix realization of the 3-algebra, we find the potential expressed as

\[ V = -\frac{1}{3} \text{tr} \left( Z^A Z_A Z^B Z_B Z_C Z_C + Z_A Z^A Z^B Z_C Z_C + 4 Z^A Z_C Z^B Z_A Z_C Z_C - 6 Z^A Z_C Z^B Z_B Z_C Z_A \right) \] (8.4)

and as it should, this vanishes when the matrices are commuting.

To establish this let us first consider the first term in the ABJM potential and just apply the identity (7.17), which in terms of 3-brackets reads

\[ [Z^A, Z^B; Z^C] = [Z_C, Z^A; Z_B]. \] (8.5)

Again, notice that the right hand side involves two non-ABJM fields, viz. two monopole operators. We then get

\[ \langle [Z^A, Z^B; Z^C], [Z^A, Z^B; Z^C] \rangle = \langle [Z^A, Z^B; Z^C], [Z_C, Z^A; Z_B] \rangle = -\langle [Z^A, Z^B; Z^C], [Z^A, Z_C; Z_B] \rangle \] (8.6)
and we can continue from here as

\[-\left\langle [Z^A; Z^B; Z^C], [Z^A; Z^C; Z_B] \right\rangle = -\left\langle [Z^B; Z^A; Z^C], [Z^A; Z^C; Z_B] \right\rangle = \left\langle [Z^A; Z^C; Z_B], [Z^A; Z^C; Z_B] \right\rangle.\]  

(8.7)

Of course it is not true that

\[[Z^A, Z^B; Z^C] = -[Z^A, Z^C; Z_B]\]  

(8.8)

For this to be true we must contract by something antisymmetric in BC. However, there is no way to really tell whether this is the case or not by just looking at the first term – this term behaves in all respects just as if the 3-bracket had been totally antisymmetric.

For the second term we have by identities

\[[Z^A, Z^B; Z^A] = -[Z^A, Z_A; Z_B].\]  

(8.9)

Hence the terms are totally antisymmetric.

We now ask whether the ABJM potential can be written in the manifestly SO(8) invariant form of hermitian BLG theory

\[V_{BLG} = \frac{1}{12} \|[Z^A, Z^B; Z^C]\|^2\]  

(8.10)

where \(Z^\alpha\) are chosen to be real SO(8) spinors, and where we do not distinguish \(Z^\alpha\) from \(Z_\alpha\). Expanding them out as \(Z^\alpha = (Z^A, Z_A)\), (where \(Z_A\) has a monopole operator attached. In terms of indices, \(Z^\alpha_a = (Z^A_a, Z_Aa)\), we get

\[V_{BLG} = \frac{1}{6} \left( \|[Z^A, Z^B; Z_C]\|^2 + \|[Z^A, Z^B; Z^C]\|^2 + 2\|[Z^A, Z_B; Z^C]\|^2 \right)\]  

(8.11)

but due to the above result obtained from identity (7.17), we can write this as

\[V_{BLG} = \frac{1}{6} \left( \left\langle [Z^A, Z^B; Z_C],[Z^A, Z^B; Z_C] \right\rangle + 3 \left\langle [Z^A, Z^B; Z^C][Z^A, Z^B; Z^C] \right\rangle \right)\]  

(8.12)

Next we use the fundamental identity (4.12) (this is really the same algebraic structure as in BLG theory, only that the ABJM 3-algebra is a refined version of the BLG 3-algebra, where some care must be taken with respect to how the generators are ordered inside the 3-product) together with the trace invariance condition (4.18) (again this is the same trace invariance condition as in BLG theory, the only difference is that here care must be taken with respect to the ordering of elements), and can derive the following trace identity,

\[\langle [X, Y; Z], [U, V; W] \rangle = \langle [X, W; V], [U, Z; Y] \rangle\]
By applying this identity we derive the identity

\[
\langle [Z^A, Z^B; Z^C], [Z^A, Z^B; Z^C] \rangle = \langle [Z^A, Z^C; Z^B], [Z^A, Z^C; Z^B] \rangle \\
- \langle [Z^B, Z^C; Z^B], [Z^A, Z^C; Z^A] \rangle \\
+ \langle [Z^A, Z^B; Z^A], [Z_C, Z^B; Z_C] \rangle
\] (8.13)

Now we rewrite the last term as

\[
[Z_C, Z^B; Z_C] = [-Z^C, Z^B; Z^C]
\] (8.15)

using identity (7.17), and the second term as

\[
[Z^B, Z^C; Z^B] = [Z^B, Z_C; Z_C]
\] (8.16)

again using (7.17). Using this, we can write the trace identity (8.14) in the form

\[
\langle [Z^A, Z^B; Z^C], [Z^A, Z^B; Z^C] \rangle = \langle [Z^A, Z^C; Z^B], [Z^A, Z^C; Z^B] \rangle \\
- 2 \langle [Z^B, Z^C; Z^B], [Z^A, Z^C; Z^A] \rangle .
\] (8.17)

Substituting this expression into the hermitian BLG potential, we find that this becomes equal to the ABJM potential. This establishes the sought-for SO(8) invariance of the ABJM scalar potential.

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A SO(8) gamma matrices

Here $\Gamma^I$ are SO(8) gamma matrices in the Weyl basis

$$\Gamma^I = \begin{pmatrix} 0 & \Gamma^{I\alpha\beta} \\ \Gamma^{I\dot{\alpha}\dot{\beta}} & 0 \end{pmatrix}$$ (A.1)

They can be chosen to have real components and are then antisymmetric

$$(\Gamma^I)^T = -\Gamma^I.$$ (A.2)

The charge conjugation matrix is then

$$\Omega = \begin{pmatrix} \delta_{\alpha\beta} & 0 \\ 0 & \delta_{\dot{\alpha}\dot{\beta}} \end{pmatrix}$$ (A.3)

and its inverse is

$$\Omega^{-1} = \begin{pmatrix} \delta_{\alpha\beta} & 0 \\ 0 & \delta_{\dot{\alpha}\dot{\beta}} \end{pmatrix}$$ (A.4)

Since invariant tensors with two equal indices (that is $\delta_{IJ}$, $\delta_{\alpha\beta}$ and $\delta_{\dot{\alpha}\dot{\beta}}$ in SO(8) are thus identity matrices, we can put all SO(8) indices downstairs. We define the chirality matrix

$$\Gamma = \Gamma^{12...8}$$ (A.5)

These gamma matrices have properties

$$\Gamma^2 = 1$$

$$\{\Gamma, \Gamma^I\} = 0$$

$$\Gamma^T = \Gamma$$

$$(\Gamma^I)^T = -\Gamma^I$$

$$(\Gamma^{IJ})^T = -\Gamma^{IJ}$$

$$(\Gamma^{IJK})^T = \Gamma^{IJK}$$

$$(\Gamma^{IJKL})^T = \Gamma^{IJKL}$$ (A.6)

and duality

$$\Gamma^{I_1...I_m} = \frac{1}{(8-m)!} \epsilon^{I_1...I_mI_{m+1}...I_8} \Gamma^{I_{m+1}...I_{m+1}}$$

$$\Gamma^{I_{8}...I_{m}} = \frac{1}{(m-1)!} \epsilon^{I_1...I_mI_{m+1}...I_8} \Gamma^{I_1...I_{m-1}}$$ (A.7)
Defining $\eta = \epsilon^\dagger$, we find the Fierz identity


given by

\begin{align}
16\epsilon\eta &= - (\eta\epsilon) - (\eta\Gamma\epsilon)\Gamma \\
&\quad - (\eta\Gamma\epsilon)\Gamma^I + (\eta\Gamma^I\epsilon)\Gamma^I \\
&\quad - \frac{1}{2}(\eta\Gamma^{IJ}\epsilon)\Gamma^{IJ} + \frac{1}{2}(\eta\Gamma^I\epsilon)\Gamma^{IJ} \\
&\quad + \frac{1}{6}(\eta\Gamma^{IJK}\epsilon)\Gamma^{IJK} - (\eta\Gamma^{IJ}\epsilon)\Gamma^{IJK} \\
&\quad - \frac{1}{24}(\eta\Gamma^{IJKL}\epsilon)\Gamma^{IJKL}
\end{align}

(A.8)

For chiral spinors

\begin{align}
\Gamma\epsilon &= \epsilon \\
\Gamma\eta &= \eta
\end{align}

(A.9)

we have

\begin{equation}
\eta^{I_1\cdots I_{odd}}\epsilon = 0
\end{equation}

(A.10)

and get the Fierz identity

\begin{align}
\epsilon\eta &= \frac{1}{16} \left[ - \eta\epsilon + \frac{1}{2} \eta\Gamma_{IJ}\epsilon\Gamma_{IJ} - \frac{1}{24} \eta\Gamma_{IJKL}\epsilon\Gamma_{IJKL} \right] (1 + \Gamma).
\end{align}

(A.11)

and consequently

\begin{align}
16(\epsilon\eta - \eta\epsilon) &= \eta\Gamma_{IJ}\epsilon\Gamma_{IJ} \frac{1 + \Gamma}{2}.
\end{align}

(A.12)

**B SO(1,2) gamma matrices**

We let $\gamma^\mu$ denote gamma matrices and $c$ charge conjugation. These have properties

\begin{align}
c^T &= -c \\
(\gamma^\mu)^T &= -c\gamma^\mu c^{-1}
\end{align}

(B.1)

We have the Fierz identity

\begin{equation}
\epsilon\eta = - \frac{1}{2} (\eta\epsilon + (\eta\gamma^\mu)\gamma_\mu).
\end{equation}

(B.2)

An explicit realization is

\begin{equation}
\gamma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \gamma^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\end{equation}

(B.3)
and
\[ c = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \]  
(B.4)

Since also
\[
\begin{align*}
(\gamma_\mu)^\dagger &= \gamma_0 \gamma_\mu \gamma_0 \\
(\gamma_\mu)^T &= -c \gamma_\mu c^{-1}
\end{align*}
\]  
(B.5)

and we understand that the choice
\[ c = \gamma_0 \]  
(B.6)

amounts to gamma matrices with real components, for instance we could take them as specified explicitly above.

In such a basis, Majorana spinors also have real components since the majorana condition
\[ \Psi = \psi^T c \]  
(B.7)

amounts to the condition
\[ \Psi^\dagger = \psi^T \]  
(B.8)

if we define \( \Psi = \psi^\dagger \gamma_0 \).

C Reducing SO(8) to SU(4) x U(1)

To reduce BLG theory to ABJM theory we want to reduce the symmetry as
\[ \text{SO}(8) \to \text{SO}(6) \times \text{SO}(2) = \text{SU}(2) \times \text{U}(1) . \]  
(C.1)

We represent the SO(8) gamma matrices
\[ \Gamma^I = (\Gamma^M, \Gamma^7, \Gamma^8) \]  
(C.2)

where
\[
\begin{align*}
\Gamma^M &= \Sigma^M \otimes \sigma^1 \\
\Gamma^7 &= 1 \otimes \sigma^2 \\
\Gamma^8 &= \Sigma \otimes \sigma^1
\end{align*}
\]  
(C.3)
and

\[
\begin{align*}
\sigma^1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
\sigma^2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}
\end{align*}
\]

(C.4)

Here \( \Sigma^M \) are hermitian SO(6) gamma matrices that we represent as

\[
\Sigma^M = \begin{pmatrix} 0 & \Sigma^{MAB} \\ \Sigma_{AB} & 0 \end{pmatrix}
\]

(C.5)

where \( A \) is Weyl index of SO(6), its chirality being distinguished by the placing up and down respectively. Hermiticity amounts to the condition

\[
\Sigma^{*MAB} = -\Sigma_{AB}.
\]

(C.6)

We also define

\[
\Sigma = \begin{pmatrix} \delta^A_B & 0 \\ 0 & -\delta^A_B \end{pmatrix}.
\]

(C.7)

We use index notation as follows. The spinor and co-spinor are decomposed as

\[
\begin{align*}
\bar{\xi}_\alpha &= \begin{pmatrix} \xi^A \\ \bar{\xi}_A \end{pmatrix}, \\
\xi_{\dot{\alpha}} &= \begin{pmatrix} \bar{\xi}^A \\ -\bar{\xi}_A \end{pmatrix}.
\end{align*}
\]

(C.8)

Accordingly, matrices (linear maps on the space of these vectors) are represented as

\[
\begin{align*}
M_{\alpha\dot{\beta}} &= \begin{pmatrix} M^A_B & M^{AB} \\ M_{AB} & M_A^B \end{pmatrix}, \\
M_{\dot{\alpha}\beta} &= \begin{pmatrix} M^A_B & M^{AB} \\ M_{AB} & M_A^B \end{pmatrix}, \\
M_{\alpha\beta} &= \begin{pmatrix} M^A_B & M^{AB} \\ M_{AB} & M_A^B \end{pmatrix}, \\
M_{\dot{\alpha}\dot{\beta}} &= \begin{pmatrix} M^{AB} & M_{AB} \\ M_A^B & M_A^B \end{pmatrix}
\end{align*}
\]

(C.9)

and these in turn sit in an SO(8) matrix

\[
\begin{pmatrix} M_{\alpha\beta} & M_{\alpha\dot{\beta}} \\ M_{\dot{\alpha}\beta} & M_{\dot{\alpha}\dot{\beta}} \end{pmatrix}
\]

(C.10)
that maps a spinor \((\xi_\alpha, \xi_\dot{\alpha})^T\) into a new spinor with the same spinor index structure.

For the reduction we also need

\[
\Gamma_{IJ} = (\Gamma_{MN}, \Gamma_{M7}, \Gamma_{M8}, \Gamma_{78}) = (\Sigma_{MN} \otimes 1, \Sigma_M \otimes i\sigma^3, \Sigma_M \Sigma \otimes 1, -\Sigma \otimes i\sigma^3) \tag{C.11}
\]

We define the hermitian SO(8) chirality matrix as

\[
\Gamma = i^{1..8} = 1 \otimes \sigma^3. \tag{C.12}
\]

It is convenient to define supersymmetry parameter

\[
\epsilon^{AB} = \epsilon^M \Sigma^{M,AB} \tag{C.13}
\]

where \(\epsilon^M\) is a real component spinor. This will have the property

\[
(\epsilon^{AB})^* = -\epsilon_{AB} \tag{C.14}
\]

We have that

\[
\Sigma_{AB}^M = \frac{1}{2} \epsilon_{ABCD} \Sigma^{M,CD} = \Sigma^{*M,BA} \tag{C.15}
\]

and

\[
\Sigma_{AB}^M \Sigma^{M,CD} = -4 \delta_{AB}^D \Sigma_{AB}^M \Sigma^{M,CD} = -2 \epsilon_{ABCD}. \tag{C.16}
\]

## D Some more useful relations

The \(\mathcal{N}=8\) Fierz identity is

\[
\epsilon_i \bar{\eta}_J - \eta_i \bar{\epsilon}_J = -\bar{\epsilon}_i [\eta_J + \bar{\epsilon}(\gamma^\mu \eta_J) \gamma_\mu], \tag{D.1}
\]

### D.1 \(\mathcal{N}=6\)

Fierz identities read

\[
\Sigma_{AB}^M \Sigma^{N,CD} = -(\Sigma^{MN})_{[A} \delta_{B]}^F \epsilon_{[E]B]CD - \frac{1}{3} \delta^{MN} \epsilon_{ABCD}
\]

\[
\Sigma_{AB}^M \Sigma^{N,CD} = -2(\Sigma^{MN})_{[A} \delta_{B]}^F \epsilon_{[E]B]CD - \frac{2}{3} \delta^{MN} \delta_{AB}. \tag{D.2}
\]
D.2 \( \mathcal{N} = 2 \)

Fierz identities read

\[
\begin{align*}
\varepsilon \overline{\eta} - \eta \varepsilon &= (\varepsilon \gamma^{\nu} \eta_{\nu}) \gamma_{\nu} - 2\overline{\varepsilon} [8 \eta_{7}] \\
\varepsilon^{*} \overline{\eta}^{*} - \eta^{*} \varepsilon^{*} &= (\varepsilon \gamma^{\nu} \eta_{\nu}) \gamma_{\nu} + 2\overline{\varepsilon} [8 \eta_{7}] \\
\varepsilon \overline{\eta}^{*} - \eta \varepsilon^{*} &= (\varepsilon \gamma^{\nu} \eta_{\nu} - \varepsilon_{8} \gamma^{\nu} \eta_{8} + 2\overline{\varepsilon} (8 \gamma^{\nu} \eta_{7})) \gamma_{\nu}
\end{align*}
\]

(D.3)

and then we have

\[
\begin{align*}
\overline{\varepsilon} \gamma^{\mu} \eta - \overline{\eta} \gamma^{\mu} \varepsilon &= 2\overline{\varepsilon} X \gamma^{\mu} \eta^{X} \\
\overline{\varepsilon} \eta - \overline{\eta} \varepsilon &= -4\overline{\varepsilon} [8 \eta_{7}] \\
\overline{\varepsilon}^{*} \eta - \overline{\eta}^{*} \varepsilon &= 0.
\end{align*}
\]

(D.4)

D.3 \( \mathcal{N} = 8 \)

Fierz identities are those for \( \mathcal{N} = 6 \) and \( \mathcal{N} = 2 \) plus the mixed ones,

\[
\begin{align*}
\varepsilon_{M} \overline{\eta} - \eta_{M} \varepsilon &= \frac{1}{2} \left( -\overline{\varepsilon} M \eta^{*} + \overline{\varepsilon} M \gamma^{*} \eta_{\mu} - (\varepsilon \leftrightarrow \eta) \right) \\
\varepsilon \overline{\eta}_{M} - \eta \varepsilon_{M} &= \frac{1}{2} \left( \overline{\varepsilon} M \eta + \overline{\varepsilon} M \gamma \eta_{\mu} - (\varepsilon \leftrightarrow \eta) \right)
\end{align*}
\]

(D.5)

E \quad BLG theory

The matter content in BLG theory is eight scalar fields \( X_{I} \) and eight fermions \( \psi_{\alpha} \) where \( I \) transforms as a vector and \( \alpha \) as a chiral spinor of the global internal symmetry group SO(8). We denote SO(8) gamma matrices as \( \Gamma^{I} \) and SO(1,2) gamma matrices as \( \gamma^{\mu} \). We define the chirality matrix of SO(8) as

\[
\Gamma = \Gamma^{1...8}.
\]

(E.1)

We denote by \( c \) the charge conjugation matrix in SO(1,2). The charge conjugation matrix of SO(8) can be chosen to be unity. The fermions are constrained by

\[
\begin{align*}
\Gamma \psi &= -\psi \\
\overline{\psi} &= \psi^{T} c
\end{align*}
\]

(E.2)

Here \( \overline{\psi} = \psi^{\dagger} \gamma^{0} \). If we let \( \gamma^{0} = c \) this is the SO(8) Majorana-Weyl spinor condition \( \psi^{\dagger} = \psi^{T} \), that is all components are real. We let \( \varepsilon_{\alpha} \) denote a supersymmetry parameter,

\[
\Gamma \varepsilon = \varepsilon
\]

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which will then also have real components. We let \( A_\mu \) denote a non-dynamical gauge field and define covariant derivative as \( D_\mu = \partial_\mu + A_\mu \). In these conventions the \( \mathcal{N} = 8 \) supersymmetry transformations read

\[
\begin{align*}
\delta X_I &= i \varepsilon^I \Gamma_{I\alpha\dot{\alpha}} \psi_\alpha \\
\delta \psi_\alpha &= -\gamma^\mu \Gamma_{I\alpha\dot{\alpha}} \varepsilon_\alpha D_\mu X_I + \frac{1}{6} \Gamma_{I\alpha\dot{\alpha}} \Gamma_{J\alpha\dot{\beta}} \Gamma_{K\beta\dot{\gamma}} \varepsilon_\beta [X_I, X_J, X_K] \\
\delta A_\mu &= -i \varepsilon_\alpha \gamma_\mu \Gamma_{I\alpha\dot{\alpha}} [\cdot, \psi_{\dot{\beta}}, X_{\dot{\alpha}}]
\end{align*}
\] (E.4)

They close on-shell. In particular the fermionic equation of motion reads

\[
\gamma^\mu D_\mu \psi - \frac{1}{2} \Gamma_{IJ} [\psi, X_I, X_J] = 0. \tag{E.5}
\]

**E.1 Trial BLG theory**

We can use triality of SO(8) to rotate \( \mathbf{8}_s, \mathbf{8}_c, \mathbf{8}_v \). We want to do this in such a way that the ABJM SO(6) R-symmetry is embedded in SO(8) in such a way that we have the decomposition rules

\[
\begin{align*}
\mathbf{8}_s &\rightarrow \mathbf{4} + \mathbf{4}' \\
\mathbf{8}_c &\rightarrow \mathbf{4} + \mathbf{4}' \\
\mathbf{8}_v &\rightarrow \mathbf{6} + \mathbf{1} + \mathbf{1}
\end{align*} \tag{E.6}
\]

To this end we make the following triality rotation of matter fields and supersymmetry parameters,

\[
\begin{align*}
X_I &\rightarrow X_{\alpha} \\
\psi_\alpha &\rightarrow \psi_{\dot{\alpha}} \\
\varepsilon_\alpha &\rightarrow \varepsilon_{I}.
\end{align*} \tag{E.7}
\]

The BLG theory is then mapped to a trial theory where supersymmetry transformations read

\[
\begin{align*}
\delta X_{\alpha\dot{\alpha}} &= i \varepsilon_I \Gamma_{I\alpha\dot{\alpha}} \psi_{\dot{\alpha}} \\
\delta \psi_{\alpha\dot{\alpha}} &= -\gamma^\mu \Gamma_{I\alpha\dot{\alpha}} \varepsilon_I D_\mu X_{\alpha\dot{\alpha}} + \frac{1}{6} \Gamma_{I\alpha\dot{\alpha}} \Gamma_{J\alpha\dot{\beta}} \Gamma_{K\beta\dot{\gamma}} \varepsilon_\beta [X_\alpha, X_\beta, X_\gamma] \\
\delta A_\mu &= -i \varepsilon_I \gamma_\mu \Gamma_{I\alpha\dot{\alpha}} [\cdot, \psi_{\dot{\beta}}, X_{\dot{\alpha}}]
\end{align*} \tag{E.8}
\]

To understand this, one just re-labels indices and defines

\[
\begin{align*}
\Gamma_{I\alpha\dot{\alpha}} &= \Gamma_{\alpha/\dot{\alpha}} = \Gamma_{\alpha\dot{\alpha}} \\
\Gamma_{I\alpha\dot{\alpha}} &= \Gamma_{\alpha/\dot{\alpha}} = \Gamma_{\dot{\alpha}\alpha}.
\end{align*} \tag{E.9}
\]
To relate to the ABJM supersymmetry transformations, we decompose

\[
X_{\alpha a} = \left( \begin{array}{c} Z^A_a \\ Z^A_{\dot{A}} \end{array} \right),
\psi_{\dot{\alpha} a} = \left( \begin{array}{c} \psi^{\dot{A}}_a \\ -\psi_{\dot{A} a} \end{array} \right)
\]

(E.10)

into Weyl spinors of SO(6) and we let

\[
\epsilon_I = (\epsilon^M, \epsilon^7, \epsilon^8).
\]

(E.11)

A Majorana-Weyl spinor \(X\) of SO(8) is subject to

\[
X^\dagger = X^T.
\]

(E.12)

We introduce a complex supersymmetry parameter

\[
\varepsilon \equiv \epsilon^7 + i\epsilon^8
\]

(E.13)

We can parametrize the six supersymmetries by the supersymmetry parameters

\[
\varepsilon^{AB} \equiv \epsilon^M \Sigma^{MAB},
\varepsilon_{AB} \equiv \frac{1}{2} \epsilon^{ABCD} \epsilon^{CD}
\]

(E.14)

These supersymmetry variations become

\[
\delta Z^A_a = -i\epsilon^{AB} \psi_{Ba},
\delta Z^A_{\dot{A}} = i\bar{\epsilon}_{AB} \psi^{Ba},
\delta \psi^{\dot{A}}_a = -\gamma^\mu \epsilon_{AB} D^B \epsilon^B_{\dot{C}} Z^C_B Z^C_{\dot{D}} f^{\dot{D} d a c} + \epsilon^{BC} Z^B_B Z^C_{\dot{C}} Z^C_{\dot{D}} f^{d c a b} - i\epsilon^{ABCD} \epsilon^{CD} f^{d c a b},
\delta \psi_{\dot{A} a} = \gamma^\mu \epsilon_{AB} D^B \psi^B_{\dot{A} a} - \epsilon^{BC} Z^B_B Z^C_{\dot{C}} Z^C_{\dot{D}} f^{d c a b} + \epsilon^{BC} Z^B_B Z^C_{\dot{C}} Z^C_{\dot{D}} f^{d c a b}
\]

(E.15)

We also have two more supersymmetries in trial BLG theory, parametrized by \(\varepsilon\) and \(\varepsilon^*\). These are

\[
\delta Z^A_a = \bar{\epsilon} \psi^A_a,
\delta Z^A_{\dot{A}} = -\bar{\epsilon} \psi_{\dot{A} a},
\delta \psi^{\dot{A}}_a = -i\gamma^\mu \epsilon^{+ AB} D^B_{\mu} Z^A_a + i\epsilon^{+ BC} Z^B_{\dot{B}} Z^C_{\dot{C}} f^{d a c b} Z^B_{\dot{B}} - i\epsilon^{+ ABCD} Z^B_B Z^C_{\dot{C}} Z^D_{\dot{D}} f^{d a c b},
\delta \psi_{\dot{A} a} = i\gamma^\mu \epsilon^{+ AB} D^B_{\mu} \psi^B_{\dot{A} a} + i\epsilon^{+ BC} Z^B_{\dot{B}} Z^C_{\dot{C}} f^{d a c b} + i\epsilon^{+ ABCD} Z^B_B Z^C_{\dot{C}} Z^D_{\dot{D}} f^{d a c b}
\]

(E.16)

Now we wrote these BLG supersymmetry variations in an ABJM notation but they are gauge covariant, and close on-shell, only when the structure constants \(f^{\dot{B} c d a}\) are real and totally antisymmetric, and indices are raised by \(\delta^{ab}\).
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