Mekler’s construction and generalized stability

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Mekler’s construction

- Let $p > 2$ be prime.
- Let $T$ be any theory in a finite relational language.
- [Mekler’81] A uniform construction of a group $G(\mathcal{M})$ for every $\mathcal{M} \models T$, a theory $T^*$ of all groups $\{G(\mathcal{M}) : \mathcal{M} \models T\}$ and an interpretation $\Gamma$ of $T$ in $T^*$ s.t.:
  - $T^*$ is a theory of nilpotent groups of class 2 and of exponent $p$,
  - if $G \models T^*$, then $\exists \mathcal{M} \models T$ s.t. $G(\mathcal{M}) \equiv G$,
  - For $\mathcal{M}, \mathcal{N} \models T$, $\mathcal{M} \equiv \mathcal{N} \iff G(\mathcal{M}) \equiv G(\mathcal{N})$,
  - $\Gamma(G(\mathcal{M})) \cong \mathcal{M}$.

- Idea:
  - Bi-interpret $\mathcal{M}$ with a nice graph $C$.
  - Define a group $G(C)$ generated freely by the vertices of $C$, imposing that two generators commute $\iff$ they are connected by an edge in $C$.

- This kind of coding of graphs is known in probabilistic group theory, recursion theory, etc.
What model-theoretic properties are preserved?

- This is not a bi-interpretation (e.g., the resulting group is never $\omega$-categorical), however some model-theoretic tameness properties are known to be preserved.

- [Mekler ’81] For any cardinal $\kappa$, $\text{Th}(\mathcal{M})$ is $\kappa$-stable $\iff$ $\text{Th}(G(\mathcal{M}))$ is $\kappa$-stable.

- [Baudisch, Pentzel ’02] $\text{Th}(\mathcal{M})$ is simple $\iff$ $\text{Th}(G(\mathcal{M}))$ is simple.

- [Baudisch ’02] Assuming stability, $\text{Th}(\mathcal{M})$ is CM-trivial $\iff$ $\text{Th}(G(\mathcal{M}))$ is CM-trivial.

- We investigate what further properties from Shelah’s classification are preserved.
\textbf{$k$-dependent theories}

- We fix a complete theory $T$ in a language $\mathcal{L}$. For $k \geq 1$ we define:

\textbf{Definition [Shelah]}

- A formula $\phi(x; y_1, \ldots, y_k)$ is \textit{k-dependent} if there are no infinite sets $A_i = \{a_{i,j} : j \in \omega\} \subseteq M_{y_i}$, $i \in \{1, \ldots, k\}$ in a model $M$ of $T$ such that $A = \prod_{i=1}^{n} A_i$ is \textit{shattered} by $\phi$, where “$A$ shattered” means: for any $s \subseteq \omega^k$, there is some $b_s \in M_x$ s.t. $M \models \phi(b_s; a_{1,j_1}, \ldots, a_{k,j_k}) \iff (j_1, \ldots, j_k) \in s$.
- $T$ is \textit{k-dependent} if all formulas are $k$-dependent.
- $T$ is \textit{strictly $k$-dependent} if it is $k$-dependent, but not $(k - 1)$-dependent.

- $T$ is 1-dependent $\iff$ $T$ is NIP.
- 1-dependent $\subsetneq$ 2-dependent $\subsetneq \ldots$ as witnessed by e.g. the theory of the random $k$-hypergraph.
Problem. Are there strictly $k$-dependent fields, for $k > 1$?

Conjecture. There are no simple strictly $k$-dependent fields, for $k > 1$.

[Hempel '15] Let $K$ be an infinite field.

1. If $\text{Th}(K)$ is $n$-dependent, then $K$ is Artin-Schreier closed.
2. If $K$ is a PAC field which is not separably closed, then $\text{Th}(K)$ is not $k$-dependent for any $k \in \omega$.

(2) is due to Parigot for $k = 1$, and if $K$ is pseudofinite, by Beyarslan $K$ interprets the random $k$-hypergraph for all $k \in \omega$. 
**k-dependent groups**

- Let $T$ be a theory and $G$ a type-definable group (over $\emptyset$), and $A \subseteq \mathbb{M}$ a small subset.
- Let $G_A^{00}$ be the minimal type-definable over $A$ subgroup of $G$ of bounded index.

**Fact**

$T$ is NIP $\implies G_A^{00} = G_\emptyset^{00}$ for all small $A$.

**Example**

Let $G := \bigoplus_\omega \mathbb{F}_p$. Let $\mathcal{M} := (G, \mathbb{F}_p, 0, +, \cdot)$ with $\cdot$ the bilinear form $(a_i) \cdot (b_i) = \sum_i a_i b_i$ from $G$ to $\mathbb{F}_p$.

Then $G$ is 2-dependent and $G_A^{00} = \{ g \in G : \bigcap_{a \in A} g \cdot a = 0 \}$ — gets smaller when enlarging $A$.

**Fact**

[Shelah] Let $T$ be 2-dependent. Then for a suitable cardinal $\kappa$, if $\mathcal{M} \prec \mathbb{M}$ is $\kappa$-saturated and $|B| < \kappa$, then $G_{M \cup B}^{00} = G_M^{00} \cap G_{A \cup B}^{00}$ for some $A \subseteq M$, $|A| < \kappa$.

- This can be viewed as a trace of modularity.
Mekler’s construction preserves \( k \)-dependence

- No examples of strictly \( k \)-dependent groups for \( k > 2 \) were known.

**Theorem**

[C., Hempel ’17] For any \( k \in \omega \), \( \text{Th}(\mathcal{M}) \) \( k \)-dependent \( \iff \) \( \text{Th}(G(\mathcal{M})) \) is \( k \)-dependent.

- Applying Mekler’s construction to the random \( k \)-hypergraph, we get:

**Corollary**

For every \( k \in \omega \), there is a strictly \( k \)-dependent pure group \( G_k \) (moreover, \( \text{Th}(G_k) \) simple by Baudisch).
A proof for NIP, 1

- For a complete theory $T$, its stability spectrum is the function
  \[ f_T(\kappa) := \sup \{|S_1(M)| : M \models T, |M| = \kappa \}. \]

- \[ \text{ded}(\kappa) := \sup \{|I| : I \text{ is a linear order with a dense subset of size } \kappa \}. \]

**Fact**

*Shelah* Let the language of $T$ be countable.

1. If $T$ is NIP, then $f_T(\kappa) \leq (\text{ded } \kappa)^{\aleph_0}$ for all infinite cardinals $\kappa$.
2. If $T$ has IP, then $f_T(\kappa) = 2^\kappa$ for all infinite cardinals $\kappa$.

- Assuming GCH, $\text{ded } \kappa = 2^\kappa$ for all $\kappa$. On the other hand:
  - *Mitchell* For every cardinal $\kappa$ with $\text{cf}(\kappa) > \aleph_0$, there is a forcing extension of the model of ZFC such that $(\text{ded } \kappa)^{\aleph_0} < 2^\kappa$. 
The actual result in the original paper of Mekler is:

Fact

\[ f_{\text{Th}(G(\mathcal{M}))}(\kappa) \leq f_{\text{Th}(\mathcal{M})}(\kappa) + \aleph_0 \text{ for all infinite cardinals } \kappa. \]

Hence if \( \text{Th}(\mathcal{M}) \) is NIP, then \( f_{\text{Th}(G(\mathcal{M}))}(\kappa) \leq (\text{ded } \kappa)^{\aleph_0} \) for all \( \kappa \), in all models of ZFC.

Combining with Mitchell and using Schoenfield’s absoluteness, \( \text{Th}(G(\mathcal{M})) \) is NIP.

Admittedly this is somewhat esoteric, and more importantly doesn’t generalize to \( k > 1 \).
Characterization of $k$-dependence

- We want a formula-free characterization of $k$-dependence (in \( \text{Th}(G(\mathcal{M})) \) we understand automorphisms, but not formulas).
- Let \( \kappa := |T|^+ \).

Fact

\( T \) is NIP \iff for every (\( \emptyset \)-)indiscernible sequence \( (a_i : i \in \kappa) \) and \( b \) of finite tuples in \( \mathbb{M} \), there is some \( \alpha \in \kappa \) such that \( (a_i : i > \alpha) \) is indiscernible over \( b \).

- What is the analogue for \( k \)-dependence?
Generalized indiscernibles

- $T$ is a theory in a language $\mathcal{L}$, $\mathbb{M} \models T$.

Definition

Let $I$ be an $\mathcal{L}_0$-structure. Say that $\bar{a} = (a_i : i \in I)$, with $a_i$ a tuple in $\mathbb{M}$, is $I$-indiscernible over $C \subseteq \mathbb{M}$ if for all $i_1, \ldots, i_n$ and $j_1, \ldots, j_n$ from $I$:

\[
\text{qftp}_{\mathcal{L}_0} (i_1, \ldots, i_n) = \text{qftp}_{\mathcal{L}_0} (j_1, \ldots, j_n) \implies \text{tp}_{\mathcal{L}} (a_{i_1}, \ldots, a_{i_n}/C) = \text{tp}_{\mathcal{L}} (a_{j_1}, \ldots, a_{j_n}/C).
\]

- For $\mathcal{L}_0$-structures $I, J$, say that $(b_j : j \in J)$ is based on $(a_i : i \in I)$ over $C$ if for any finite set $\Delta$ of $\mathcal{L} (C)$-formulas and any $(j_0, \ldots, j_n)$ from $J$ there is some $(i_1, \ldots, i_n)$ from $I$ s.t.

\[
\text{qftp}_{\mathcal{L}_0} (j_1, \ldots, j_n) = \text{qftp}_{\mathcal{L}_0} (i_1, \ldots, i_n) \text{ and } \text{tp}_{\Delta} (b_{j_1}, \ldots, b_{j_n}) = \text{tp}_{\Delta} (a_{i_1}, \ldots, a_{i_n}).
\]

- We say that $I$-indiscernibles exist if for any $\bar{a}$ indexed by $I$ there is an $I$-indiscernible based on it.
Connection to structural Ramsey theory

Implicitly used by Shelah already in the classification book, made explicit by Scow and others.

Definition
Let $K$ be a class of finite $\mathcal{L}_0$-structures. For $A, B \in K$, let $\binom{B}{A}$ be the set of all $A' \subseteq B$ s.t. $A' \cong A$.

$K$ is Ramsey if for any $A, B \in K$ and $k \in \omega$ there is some $C \in K$ s.t. for any coloring $f : \binom{C}{A} \to k$, there is some $B' \in \binom{C}{B}$ s.t. $f \upharpoonright \binom{B'}{A}$ is constant.

Classical Ramsey theorem $\iff$ the class of finite linear orders is Ramsey.

Fact
Let $K$ be a Fraïssé class, and let $I$ be its limit. If $K$ is Ramsey, then $I$-indiscernibles exist.
Ordered random hypergraph indiscernibles

Fact

[Nesetril, Rödl '77,'83] For any \( k \in \omega \), the class of all finite ordered \( k \)-hypergraphs is Ramsey.

- Fix \( k \in \omega \). Modifying their proof, we have existence of \( \mathcal{G} \)-indiscernibles for \( \mathcal{G} = (P_1, \ldots, P_k, R(x_1, \ldots, x_k), <) \) the ordered \( k \)-partite random hypergraph (where \( P_1 < \ldots < P_k \)).
- Let \( \mathcal{O} = (P_1, \ldots, P_k, <) \) denote the reduct of \( \mathcal{G} \).
- Of course, \( (a_g : g \in \mathcal{G}) \) is \( \mathcal{O} \)-indiscernible \( /C \) implies it is \( \mathcal{G} \)-indiscernible \( /C \).
- Clarifying Shelah,

Fact

[C., Palacin, Takeuchi '14] TFAE:

1. \( T \) is \( k \)-dependent.

2. For any \( (a_g : g \in \mathcal{G}) \) and \( b \), with \( a_g, b \) finite tuples in \( \overline{M} \), if \( (a_g : g \in \mathcal{G}) \) is \( \mathcal{G} \)-indiscernible over \( b \) and \( \mathcal{O} \)-indiscernible (over \( \emptyset \)), then it is \( \mathcal{O} \)-indiscernible over \( b \).
Mekler’s construction in more detail, 1

- A graph (binary, symmetric, irreflexive relation) $C$ is *nice* if:
  - $\exists a \neq b$,
  - $\forall a \neq b \exists c \left( R(a, c) \land \neg R(b, c) \right)$,
  - no triangles or squares.

**Fact**

*Any structure in a finite relational language is bi-interpretable with a nice graph.*

- Let $G \models \text{Th}(G(C))$, where $G(C)$ is generated freely by the vertices of $C$, and two generators commute $\iff$ they are connected by an edge in $C$s.
- We consider the following $\emptyset$-definable equivalence relations on $G$, each refining the previous one:
  - $g \sim h \iff C_G(g) = C_G(h)$,
  - $g \approx h \iff \exists r \in \omega, c \in Z(G) \text{ s.t. } g = h^r c$.
  - $g \equiv_Z h \iff gZ(G) = hZ(G)$.
Mekler’s construction in more detail, 2

- $g \in G$ is of type $q$ if $\exists q$-many $\sim$-classes in $[g]_{\sim}$.
- $g$ is isolated if $[g]_{\sim} = [g]_{\equiv_Z}$.
- $G$ can be partitioned into the following $\emptyset$-definable set:
  - non-isolated elements of type 1 — type $1^\nu$,
  - isolated elements of type 1 — type $1^\iota$,
  - elements of type $p$,
  - elements of type $p - 1$.
- For every $g \in G$ of type $p$, the elements of $G$ commuting with it are:
  - elements $\sim$-equivalent to $g$,
  - an element $b$ of type $1^\nu$ together with the elements $\sim$-equivalent to $b$.
- Such a $b$ is called a handle of $g$, and is definable from $g$ up to $\sim$-equivalence.
Definition
A set $X \subseteq G$ is a \textit{transversal} if $X = X_\nu \sqcup X_p \sqcup X_\iota$, where:

1. $X_\nu$: representatives for each $\sim$-class of elements of type $1^\nu$ in $G$;

2. $X_p$: representatives of $\sim$-classes of \textit{proper} (i.e. not a product of any elements of type $1^\nu$) elements of type $p$, maximal with the property that if $Y \subseteq X_p$ is a finite of elements with the same handle, then $Y$ is independent modulo the subgroup generated by all elements of type $1^\nu$ and $Z(G)$;

3. $X_\iota$: representatives of $\sim$-classes of proper elements of type $1^\iota$, maximal independent modulo the subgroup generated by all elements of types $1^\nu$ and $p$ in $G$, together with $Z(G)$. 
Mekler’s construction in more detail, 4

- $C = (V, R)$ is interpreted in $G$ as $\Gamma(G)$:
  - $V = \{g \in G : g \text{ is of type } 1^\nu, g \notin Z(G)\} / \approx$,
  - $([g]_\approx, [h]_\approx) \in R \iff g, h \text{ commute}$.

- For $X$ a transversal of $G$, $\Gamma(X_\nu)$ is isomorphic to $C$.

- Let $G \models \text{Th}(G(C))$ and $X$ a transversal of $G$. There is a subgroup (elementary abelian $p$-group) $H$ of $Z(G)$ s.t. $G \cong \langle X \rangle \times H$.

- There is some canonicity about this choice: $\langle X \rangle' = G'$ for any transversal $X$ of $G$. 
Mekler’s construction in more detail, summarizing

- For any partial transversal $X'$ and any linearly independent over $G'$ subset $H'$ of $Z(G)$, we can find a transversal $X \supseteq X'$ and a maximal set $H \supseteq H'$ s.t. $G = \langle X \rangle \times \langle H \rangle$.

- **Lemma.** Both conditions on $X'$ and $H'$ are type-definable.

- If $Y, Z \subseteq X$ and $h : Y \to Z$ is a bijection respecting the $1^\nu$-, $p$-, and $1^\iota$-parts and the handles, and $\text{tp}_\Gamma(Y_\nu) = \text{tp}_\Gamma(h(Y_\nu))$, then $\text{tp}_G(Y) = \text{tp}_G(h(Y))$.

- Moreover, assuming saturation, $h$ extends to an automorphism of $G$ by gluing it with any automorphism of $\langle H \rangle$. 
Sketch of the proof, 1

Let \( G \models \text{Th}(G(\mathcal{M})) \) be a monster model, and \( \phi(x; y_1, \ldots, y_k) \) not \( k \)-dependent.

Choose a transversal \( X \) and \( H \subseteq Z(G) \) s.t. \( G = \langle X \rangle \times \langle H \rangle \).

Compactness: a very large witness \( (a_g : g \in \mathcal{G}) \) to the failure of \( k \)-dependence, shattered by \( \phi \).

For cardinality reasons, may assume \( a_g = t(\bar{x}_g, \bar{h}_g) \) for some \( \mathcal{L}_G \)-term \( t \) and \( \bar{x}_g \) from \( X \) and \( \bar{h}_g \) from \( H \).

Can close under handles and, changing the formula, replace the original shattered set by \( (\bar{x}_g \bar{h}_g : g \in \mathcal{G}) \).

Using type-definability of partial transversals, etc. and existence of \( \mathcal{G} \)-indiscernibles, can assume \( (\bar{x}_g \bar{h}_g : g \in \mathcal{G}) \) is \( \mathcal{O} \)-indiscernible (possibly changing the transversal to some \( X', H' \)).

As \( (\bar{x}_g \bar{h}_g : g \in \mathcal{G}) \) is shattered, can choose \( b = s(\bar{y}, \bar{k}) \in G \) with \( \bar{y} \in X', \bar{k} \in H' \) s.t. \( \phi(b; y_1, \ldots, y_k) \) cuts out exactly the edge relation of the random \( k \)-hypergraph \( \mathcal{G} \).
Using existence of $\mathcal{G}$-indiscernibles again, can assume that $(\bar{x}_g \bar{h}_g : g \in \mathcal{G})$ is $\mathcal{G}$-indiscernible over $b$ (needs some argument, replacing $X', H'$ by some $X'', H''$).

Using that $\text{Th}(\langle X \rangle)$ and $\text{Th}(\langle H \rangle)$ are $k$-dependent by assumption (hence $\mathcal{G}$-indiscernibility collapses to $\mathcal{O}$-indiscernibility in them by the characterization above), can build an automorphism of $\mathcal{G}$ (glueing separate automorphisms of $\langle X'' \rangle$ and $\langle H'' \rangle$ together by the lemma above) $\sigma$ such that:

- for some finite tuples of indices $\bar{g}, \bar{h}$ of the same type in $\mathcal{O}$, but not in $\mathcal{G}$, $\sigma$ fixes $b$ and sends $(\bar{x}_g \bar{h}_g : g \in \bar{g})$ to $(\bar{x}_h \bar{h}_h : h \in \bar{h})$.

— contradiction to the choice of $b$. 
Other results and directions

Theorem

\[ C., \ Hempel \ '17 \] \( \text{Th}(\mathcal{M}) \) is NTP<sub>2</sub> \( \iff \) \( \text{Th}(\text{G}(\mathcal{M})) \) is NTP<sub>2</sub>.

- Problem.
  - Are there pseudofinite strictly \( k \)-dependent groups?
  - Are there \( \omega \)-categorical strictly \( k \)-dependent groups?