A CARTESIAN PRESENTATION OF WEAK $n$-CATEGORIES

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Abstract. We propose a notion of weak $(n + k, n)$-category, which we call $(n + k, n)$-$\Theta$-spaces. The $(n + k, n)$-$\Theta$-spaces are precisely the fibrant objects of a certain model category structure on the category of presheaves of simplicial sets on Joyal’s category $\Theta_n$. This notion is a generalization of that of complete Segal spaces (which are precisely the $(\infty, 1)$-$\Theta$-spaces). Our main result is that the above model category is cartesian.

1. Introduction

In this note, we propose a definition of weak $n$-category, and more generally, weak $(n + k, n)$-category for all $0 \leq n < \infty$ and $-2 \leq k \leq \infty$, called $(n + k, n)$-$\Theta$-spaces. The collection of $(n + k, n)$-$\Theta$-spaces forms a category $\Theta_nSp_k^{fib}$, and there is a notion for a morphism in this category to be an equivalence. The category $\Theta_nSp_k^{fib}$ together with the given class of equivalences has the following desirable property: it is cartesian closed, in a way compatible with the equivalences. More precisely, we have the following.

1. The category $\Theta_nSp_k^{fib}$ is cartesian closed; i.e., it has products $Y \times Z$ and function objects $Z^Y$ for any pair of objects $Y, Z$ in $\Theta_nSp_k^{fib}$, so that $\Theta_nSp_k^{fib}(X \times Y, Z) \approx \Theta_nSp_k^{fib}(X, Z^Y)$.

2. If $f : X \to Y$ is an equivalence in $\Theta_nSp_k^{fib}$, then so are $f \times Z : X \times Z \to Y \times Z$ and $Z^f : Z^Y \to Z^X$.

The category $\Theta_nSp_k^{fib}$ will be defined as the full subcategory of fibrant objects in a Quillen model category $\Theta_nSp_k$. The underlying category of $\Theta_nSp_k$ is the category $sPSh(\Theta_n)$ of simplicial presheaves on a certain category $\Theta_n$. We equip this category with a model category structure, obtained as the Bousfield localization of the injective model structure on presheaves with respect to a certain set of morphisms $T_n,k$. We will prove that $\Theta_nSp_k$ is a cartesian model category, i.e., the model category structure is nicely compatible with the internal function objects of $sPSh(\Theta_n)$. Then $\Theta_nSp_k^{fib}$ is the full subcategory of fibrant objects in $\Theta_nSp_k$; equivalences in $\Theta_nSp_k^{fib}$ are just levelwise weak equivalences of presheaves.

For $n = 0$, the category $\Theta_0$ is the terminal category, so that $sPSh(\Theta_0)$ is the category of simplicial sets $Sp$. An $(\infty, 0)$-$\Theta$-space is precisely a Kan complex, and a $(k, 0)$-$\Theta$-space is precisely a $k$-truncated Kan complex, i.e., a Kan complex with homotopy groups vanishing above dimension $k$.

For $n = 1$, the category $\Theta_1$ is the category $\Delta$ of finite ordinals, so that $sPSh(\Theta_1)$ is the category of simplicial spaces. An $(\infty, 1)$-$\Theta$-space is precisely a complete Segal space, in the sense of [Rez01].

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The category \( \Theta_n \) which we use was introduced by Joyal [Joy97], as part of an attempt to define a notion of weak \( n \)-category generalizing the notion of quasicategory. Sketches of these ideas can be found in [Lei01] and [CL04]. The category \( \Theta_n \) has also been studied by Berger [BerC02, BerC07], with particular application to the theory of iterated loop spaces.

1.1. The categories \( \Theta_n \).

We will give an informal description of Joyal’s categories \( \Theta_n \) here, suitable for our purposes; our description is essentially the same as that given in [BerC07, §3]. It is most useful for us to regard \( \Theta_n \) as a full subcategory of \( \text{St}^n\text{-Cat} \), the category of strict \( n \)-categories. Thus, \( \Theta_0 \) is the full subcategory of \( \text{St}^0\text{-Cat} = \text{Set} \) consisting of the terminal object. The category \( \Theta_1 \) is the full subcategory of \( \text{St}^1\text{-Cat} \) consisting of object \([n]\) for \( n \geq 0 \), where \([n]\) represents the free strict 1-category on the diagram

\[
\begin{array}{cccc}
0 & \rightarrow & 1 & \rightarrow & \cdots & \rightarrow & (n-1) & \rightarrow & n \\
\end{array}
\]

Thus, \( \Theta_1 = \Delta \), the usual simplicial indexing category. The category \( \Theta_2 \) is the full subcategory of \( \text{St}^2\text{-Cat} \) consisting of objects which are denoted \([m](\theta_1, \ldots, \theta_m)\) for \( m, n_1, \ldots, n_m \geq 0 \). This represents the strict 2-category \( C \) which is “freely generated” by: objects \( \{0, 1, \ldots, m\} \), and morphism categories \( C(i-1, i) = [n_i] \). For instance, the object \([4](\theta_4, \theta_3, 1, 0, 0, 1)\) in \( \Theta_2 \) corresponds to the “free 2-category” on the following picture:

\[
\begin{array}{c}
0 \\
\downarrow \\
1 \\
\downarrow \\
2 \\
\rightarrow \\
\downarrow \\
3 \\
\rightarrow \\
\downarrow \\
4
\end{array}
\]

In general, the objects of \( \Theta_n \) are of the form \([m](\theta_1, \ldots, \theta_m)\), where \( m \geq 0 \) and the \( \theta_i \) are objects of \( \Theta_{n-1} \); this object corresponds to the strict \( n \)-category \( C \) “freely generated” by: objects \( \{0, \ldots, m\} \), and a strict \((n-1)\)-category of morphisms \( C(i-1, i) = \theta_i \). The morphisms of \( \Theta_n \) are functors between strict \( n \)-categories.

We make special note of objects \( O_0, \ldots, O_n \) in \( \Theta_n \). These are defined recursively by:

\( O_0 = [0], \) and \( O_k = [1](O_{k-1}) \) for \( k = 1, \ldots, n \). Thus, the object \( O_k \) in \( \Theta_n \) corresponds to the “freestanding \( k \)-cell” in \( \text{St}^n\text{-Cat} \).

1.2. Informal description of \( \Theta \)-spaces.

We will start by describing \( \Theta_n\text{Sp}^\text{fib}_\infty \), the category of \((\infty, n)\)-\( \Theta \)-spaces. Let \( \text{Sp} \) denote the category of simplicial sets. We will regard objects of \( \text{Sp} \) as “spaces”; the following definitions are perfectly sensible if objects of \( \text{Sp} \) are taken to be actual topological spaces (compactly generated).

An object of \( \Theta_n\text{Sp}^\text{fib}_\infty \) is a simplicial presheaf on \( \Theta_n \) (i.e., a functor \( \Theta_n^{\text{op}} \rightarrow \text{Sp} \)), satisfying three conditions:

(i) an injective fibrancy condition,
(ii) a Segal condition, and
(iii) a completeness condition.

A morphism \( f : X \rightarrow Y \) of \( \Theta_n\text{Sp}^\text{fib}_\infty \) is a morphism of simplicial presheaves; the morphism \( f \) is said to be an equivalence (or weak equivalence) if it is a “levelwise” weak equivalence.
of simplicial presheaves, i.e., if \( f(\theta) : X\theta \to Y\theta \) is a weak equivalence of simplicial sets for all \( \theta \in \text{ob}\Theta \).

The injective fibrancy condition (i) says that \( X \) has a right lifting property with respect to maps in \( sPSh(\Theta_n) \) which are both monomorphisms and levelwise weak equivalences; that is, \( X \) is fibrant in the injective model structure on \( sPSh(\Theta_n) \).

The Segal condition (ii) says that for all objects \( \theta \) of \( \Theta_n \), the space \( X(\theta) \) is weakly equivalent to an inverse limit of a certain diagram of spaces \( X(O_k) \); taken together with the injective fibrancy condition, this inverse limit is in fact the homotopy inverse limit of the given diagram. For \( n = 1 \), the Segal condition amounts to requiring that the “Segal map”

\[
X([m]) \to X([1]) \times X([0]) \times \cdots \times X([1]) \approx X(O_1) \times X(O_0) \times \cdots \times X(O_1)
\]

be a weak equivalence for all \( m \geq 2 \). As an example of how the Segal condition works for \( n = 2 \), the space \( X([4],[2],[3],[0],[1]) \) is required to be weakly equivalent to

\[
\left( X(O_2) \times X(O_2) \right) \times \left( X(O_2) \times X(O_2) \times X(O_2) \right) \times X(O_1) \times X(O_0)
\]

The completeness condition (iii) says that the space \( X(O_k) \) should behave like the “moduli space” of \( k \)-cells in a \((\infty,n)\)-category. That is, if the points of \( X(O_k) \) correspond to individual \( k \)-cells, such points should be connected by a path in \( X(O_k) \) if they represent “equivalent” \( k \)-cells, there should be a homotopy between paths for every “equivalence between equivalences”, and so on. It turns out that the way to enforce this is to require that, for \( k = 1, \ldots, n \), the map \( X(i_k) : X(O_{k-1}) \to X(O_k) \) which encodes “send a \((k-1)\)-cell to its identity \( k \)-morphism” should induce a weak equivalence of spaces

\[
X(O_{k-1}) \to X(O_k)^{\text{equiv}}.
\]

Here \( X(O_k)^{\text{equiv}} \) is the union of those path components of \( X(O_k) \) which consist of \( k \)-morphisms which are “\( k \)-equivalences”. Thus, the completeness condition asserts that the moduli space of \((k-1)\)-cells is weakly equivalent to the moduli space of \( k \)-equivalences.

The category \( \Theta_n \text{-Sp}_k \) of \((n+k,n)\)-\(\Theta\)-spaces is obtained by imposing an additional \((iv) \) \( k \)-truncation condition.

To state this, we need the moduli space \( X(\partial O_m) \) of “parallel pairs of \((m-1)\)-morphisms” in \( X \). This space is defined inductively as an inverse limit of the spaces \( X(O_m) \), so that

\[
X(\partial O_m) \overset{\text{def}}{=} \lim X(O_{m-1}) \to X(\partial O_{m-1}) \leftarrow X(O_{m-1}),
\]

with \( X(\partial O_0) = 1 \). Then the \( k \) truncation condition asserts that the fibers of \( X(O_n) \to X(\partial O_n) \) are \( k \)-types, i.e., have vanishing homotopy groups in all dimensions greater than \( k \).

The above definition is examined in detail in [11]

1.3. Presentations and enriched model categories. Our construction of a cartesian model category structure is a special case of a general procedure, which associates to certain kinds of model categories \( M \) a new model category \( M-\Theta\text{Sp} \); we may regard this as being analogous to the procedure which associates to a category \( V \) with finite products the category \( V-\text{Cat} \) of categories enriched over \( V \).

Specifically, suppose we are given a pair \((C, S)\) consisting of a small category \( C \) and a set \( S \) of morphisms in \( sPSh(C) \); this data is called a presentation, following the treatment of
\[ \text{[Dug01].} \] (Here, \( s\text{PSh}(C) \) denotes the category of presheaves of simplicial sets on \( C \).) Let \( M = s\text{PSh}(C)_{S}^{\text{inj}} \) denote the model category structure on \( s\text{PSh}(C) \) obtained by Bousfield localization of the injective model structure with respect to \( S \). We define a new presentation \((\Theta C, S_\Theta)\), and thus obtain a model category \( M-\Theta \text{Sp} \overset{\text{def}}{=} s\text{PSh}(\Theta C)_{S_\Theta}^{\text{inj}} \). The category \( \Theta C \) is a “wreath product” of \( \Delta \) with \( C \), as defined by \[ \text{[BerC07]} \] (see \( \S 3 \)), while the set \( S_\Theta \) consists of some maps built from elements of \( S \), together with certain “Segal” and “completeness” maps (this set is described in \( \S 8 \)).

Our main result is the following theorem.

1.4. \textbf{Theorem.} Let \( M = s\text{PSh}(C)_{S}^{\text{inj}} \) for some presentation \((C, S)\). If \( M \) is a cartesian model category, then \( M-\Theta \text{Sp} \) is also a cartesian model category.

This theorem is a straightforward generalization of the main theorem of \[ \text{[Rez01]} \], which proves the theorem for the special case \((C, S) = (1, \emptyset)\) (in which case \( M = \text{Sp} \), and thus \( M-\Theta \text{Sp} \) is the category of simplicial spaces with the complete Segal space model structure.)

The model categories for \((n+k, n)\)-\( \Theta \)-spaces are obtained iteratively, so that
\[
\Theta_{n+1}\text{Sp}_k \overset{\text{def}}{=} (\Theta_n\text{Sp}_k)-\Theta \text{Sp},
\]
starting with \( \Theta_0\text{Sp}_k = \text{Sp}_k \), where \( \text{Sp}_k \) is the Bousfield localization of \( \text{Sp} \) whose fibrant objects are Kan complexes which are \( k \)-types. Applying the theorem inductively shows that \( \Theta_n\text{Sp}_k \) are cartesian model categories. The category \( \Theta_n\text{Sp}_k^{\text{fib}} \) is defined to be the full subcategory of fibrant objects in the model category \( \Theta_n\text{Sp}_k \).

1.5. \textbf{The relationship between} \( M-\Theta \text{Sp} \) \textbf{and} \( M-\text{Cat} \). If \( M \) is a cartesian model category, then we may certainly consider \( M-\text{Cat} \), the category of small categories enriched over \( M \). Given an object \( X \) of \( M-\text{Cat} \), let \( hX \) denote the ordinary category whose objects are the same as \( X \), and whose morphism sets are given by \( hX(a, b) \overset{\text{def}}{=} hM(1, X(a, b)) \), where \( hM \) denotes the homotopy category of \( M \). Let us say that a morphism \( f : X \to Y \) of objects of \( M-\text{Cat} \) is a weak equivalence if

1. for each pair of objects \( a, b \) of \( X \), the induced map \( X(a, b) \to Y(fa, fb) \) is a weak equivalence in \( M \), and
2. the induced functor \( hX \to hY \) is an equivalence of 1-categories.

We can make the following conjecture.

1.6. \textbf{Conjecture.} Let \( M = s\text{PSh}(C)_{S}^{\text{inj}} \) for some presentation \((C, S)\), and suppose that \( M \) is a cartesian model category. Then there is a model category structure on \( M-\text{Cat} \) with the above weak equivalences, and a Quillen equivalence
\[
M-\text{Cat} \approx M-\Theta \text{Sp}.
\]

For the case of \( M = \text{Sp} \), the conjecture follows from theorems of Bergner \[ \text{[BerJ07a, BerJ07b]} \].

1.7. \textbf{Why is this a good notion of weak} \( n \)-\textbf{category?} We propose that \((n+k, n)\)-\( \Theta \)-spaces are a model for weak \((n+k, n)\)-categories. Some points in its favor are the following.

1. Our notion of \((\infty, 1)\)-\( \Theta \)-spaces is precisely what we called a complete Segal space in \[ \text{[Rez01]} \]. This is recognized as a suitable model for \((\infty, 1)\)-categories, due to work of Bergner \[ \text{[BerJ07b]} \] and Joyal-Tierney \[ \text{[JT07]} \].
(2) As noted above (1.5), the definition of \((n + k, n)\)-\(\Theta\)-spaces is a special case of a more general construction, which conjecturally models “homotopy theories enriched over a cartesian model category”. In particular, a consequence of our conjecture would be a Quillen equivalence

\[(\Theta_n^{\mathrm{Sp}_k})\text{-Cat} \approx (\Theta_n^{\mathrm{Sp}_k})\text{-}\Theta\text{Sp} = \Theta_{n+1}^{\text{Sp}_k}\).

That is, \((n + 1 + k, n + 1)\)-\(\Theta\)-spaces are (conjecturally) “the same” as categories enriched over \((n + k, n)\)-\(\Theta\)-spaces.

(3) Our notion satisfies the “homotopy hypothesis”. There is an evident notion of groupoid object in \(\Theta_n^{\text{Sp}_k}\), and one can show that the full subcategory of such groupoid objects models \(\text{Sp}_{n+k}\), the homotopy theory of \((n + k)\)-types (11.25).

(4) More generally, it is understood that \(n\)-tuply monoidal \(k\)-groupoids should correspond to “\(k\)-types of \(E_n\)-spaces”, where \(E_n\) is a version of the little \(n\)-cubes operad; furthermore, \(n\)-tuply groupal \(k\)-groupoids should correspond to “\(k\)-types of \(n\)-fold loop spaces” (see, for instance, [BD98, §3]). In terms of our models, \(n\)-tuply groupal \(k\)-groupoids are objects \(X\) of \(\Theta_n^{\text{Sp}_k}\) for which (i) \(X(O_j) \approx 1\) for \(j < n\), and (ii) \(X(O_n)^{\text{equiv}} \approx X(O_n)\), and one would conjecture that the full subcategory of such objects in \(\Theta_n^{\text{Sp}_k}\) should model \(k\)-types of \(n\)-fold loop spaces. That this is in fact the case is apparent from the results of Berger [BerC07].

As noted above, the theory of \(\Theta\)-spaces is consciously a generalization of the theory of complete Segal spaces, which is one of a family of models for \((\infty, 1)\)-categories based on simplicial objects. A reasonable approach to producing a generalization of these ideas is to use multi-simplicial objects; proposals for this include Tamsamani’s theory of weak \(n\)-categories [Tam99], the Segal \(n\)-categories of Hirschowitz and Simpson [HS01], and more recent work by Barwick and Lurie on multi-simplicial generalizations of complete Segal spaces (see for instance [Bar07] and [Lur09]). Although all these constructions appear to give good models for \((\infty, n)\)-categories, it is not clear to me that any of them result in a Quillen model category which satisfies all of the following: (i) it models the homotopy theory of \((\infty, n)\)-categories with the correct notion of equivalence, (ii) it is a cartesian model category, and (iii) it is a simplicial model category. It does appear that the Hirschowitz-Simpson model satisfies (i) and (ii), but it does not satisfy (iii). The multi-simplicial complete Segal space model of Barwick and Lurie does satisfy (i) and (iii), but does not appear to satisfy (ii) (when \(n > 1\)).

2. Cartesian model categories and cartesian presentations

2.1. Cartesian closed categories. A category \(V\) is said to be cartesian closed if it has finite products, and if for all \(X, Y \in \text{ob}V\) there is an internal function object \(Y^X\), which comes equipped with a natural isomorphism

\[V(T, Y^X) \approx V(T \times X, Y)\).

Examples of cartesian categories include the category of sets, and the category of presheaves of sets on a small category \(C\).

We will write \(\emptyset\) for some chosen initial object in a cartesian closed category \(V\).
2.2. Cartesian model categories. We will say that a Quillen model category $M$ is cartesian if it is cartesian closed as a category, if the terminal object is cofibrant, and if the following equivalent properties hold.

1. If $f: A \to A'$ and $g: B \to B'$ are cofibrations in $M$, then the induced map $h: A \times_{B \times B'} A' \times B \to A' \times B'$ is a cofibration; if in addition either $f$ or $g$ is a weak equivalence then so is $h$.

2. If $f: A \to A'$ is a cofibration and $p: X \to X'$ is a fibration in $M$, then the induced map $q: (X')^A \to (X')^A \times_{X \times X} X'$ is a fibration; if in addition either $f$ or $p$ is a weak equivalence then so is $q$.

(This notion is a bit stronger than Hovey’s notion of symmetric monoidal model category as applied to cartesian closed categories [Hov99, 4.2]; he does not require the unit object to be cofibrant, but rather imposes a weaker condition.)

2.3. Spaces. Let $Sp$ denote the category of simplicial sets, equipped with the standard Quillen model structure. We call this the model category of spaces. It is standard that $Sp$ is a cartesian model category.

We will often use topological flavored language when discussing objects of $Sp$, even though such are objects are not topological spaces but simplicial sets. Thus, a “point” in a “space” is really a 0-simplex of a simplicial set, a “path” is a 1-simplex, and so on.

2.4. Simplicial presheaves. Let $C$ be a small category, and let $sPSh(C)$ denote the category of simplicial presheaves on $C$, i.e., the category of contravariant functors $C^{op} \to Sp$.

A simplicial presheaf $X$ is said to be discrete if each $X(c)$ is a discrete simplicial set; the full subcategory of discrete objects in $sPSh(C)$ is equivalent to the category of presheaves of sets on $C$, and we will implicitly identify the two.

Let $F_C: C \to sPSh(C)$ denote the Yoneda functor; thus $F_C$ sends an object $c \in obC$ to the presheaf $F_C c = C(-, c)$. Observe that the presheaf $F_C c$ is discrete. When the context is understood we may write $F$ for $F_C$.

Let $\Gamma_C: sPSh(C) \to Sp$ denote the global sections functor, which sends a functor $X: C^{op} \to Sp$ to its limit. The functor $\Gamma$ is right adjoint to the functor $Sp \to sPSh(C)$ which sends a simplicial set $K$ to the constant presheaf with value $K$ at each object of $C$. Note that if $C$ has a terminal object $[0]$, then $\Gamma X \approx X([0])$. When the context is understood we may write $\Gamma X$ for $\Gamma_C X$.

For $X, Y$ in $sPSh(C)$, we write $\text{Map}_C(X, Y) \overset{\text{def}}{=} \Gamma(Y^X)$; this is called the mapping space. Thus, $sPSh(C)$ is enriched over $Sp$. Note that if $c \in obC$, then we have

$$X(c) \approx \Gamma(X^{F(c)}) \approx \text{Map}_C(F(c), X).$$

When the context is understood we may write $\text{Map}(X, Y)$ for $\text{Map}_C(X, Y)$.

2.5. Model categories for simplicial presheaves. Say that a map $f: X \to Y \in sPSh(C)$ is a levelwise weak equivalence if each map $f(c): X(c) \to Y(c)$ is a weak equivalence in $Sp$ for all $c \in obC$. There are two standard model category structures we can put on $sPSh(C)$ with these weak equivalences, called the projective and injective structures; they are Quillen equivalent to each other.

The projective structure is characterized by requiring that $f: X \to Y \in sPSh(C)$ be a fibration if and only if $f(c): X(c) \to Y(c)$ is one in $Sp$ for all $c \in obC$. We write $sPSh(C)^{proj}$...
for the category of presheaves of simplicial sets on \( C \) equipped with the projective model structure.

The **injective** structure is characterized by requiring that \( f: X \to Y \in sPSh(C) \) is a cofibration if and only if \( f(c): X(c) \to Y(c) \) is one in \( \text{Sp} \) for all \( c \in \text{ob} \, C \). We write \( sPSh(C)^{\text{proj}} \) for the category of presheaves of simplicial sets on \( C \) equipped with the projective model structure.

The identity functor provides a Quillen equivalence \( sPSh(C)^{\text{proj}} \rightleftarrows sPSh(C)^{\text{inj}} \).

Both the projective and injective model category structures are cofibrantly generated and proper, and have functorial factorizations. They are also both simplicial model categories.

Given object \( X, Y \) in \( sPSh(C)^{\text{inj}} \), we write \( \text{hMap}_C(X, Y) \) for the derived mapping space of maps from \( X \) to \( Y \). This is a homotopy type in \( \text{Sp} \), defined so that for any cofibrant approximation \( X^c \to X \) and fibrant approximation \( Y \to Y^f \), the derived mapping space \( \text{hMap}_C(X, Y) \) is weakly equivalent to the space of maps \( \text{Map}_C(X^c, Y^f) \). Note that in the above, we may pick either the injective or projective model category structures in order to make our replacements.

2.6. **The injective model structure.** The injective model structure has a few additional properties which are of importance to us. In particular,

1. every object of \( sPSh(C)^{\text{inj}} \) is cofibrant, and
2. every discrete object of \( sPSh(C)^{\text{inj}} \) is fibrant.

Furthermore, we have the following.

2.7. **Proposition.** The model category \( sPSh(C)^{\text{inj}} \) is a cartesian model category.

*Proof.* This is immediate from characterization (1) of cartesian model category. \( \square \)

2.8. **Presentations.** A **presentation** is a pair \( (C, S) \) consisting of a small category \( C \) and a set \( S = \{ s: S \to S' \} \) of morphisms in \( sPSh(C) \). We say that an object \( X \) of \( sPSh(C) \) is \( S \)-**local** if for all morphisms \( s: S \to S' \) in \( S \), the induced map

\[
\text{hMap}(s, X): \text{hMap}(S', X) \to \text{hMap}(S, X)
\]

is a weak equivalence of spaces. We say that a morphism \( f: A \to B \) in \( sPSh(C) \) is an \( S \)-**equivalence** if the induced map

\[
\text{hMap}(f, X): \text{hMap}(B, X) \to \text{hMap}(A, X)
\]

is a weak equivalence of spaces for all \( S \)-local objects \( X \). The collection of \( S \)-equivalences is denoted \( \overline{S} \); we have that \( S \subset \overline{S} \).

Let \( (C, S) \) be a presentation, let \( X \) be an object of \( sPSh(C) \), and let \( X \to X^f \) denote a fibrant replacement of \( X \) in the injective model structure. Since every object is cofibrant in the injective model structure, we have that \( X \) is \( S \)-local if and only if \( \text{Map}(S', X^f) \to \text{Map}(S, X^f) \) is a weak equivalence for all \( s \in S \).

2.9. **Cartesian presentations.** Let \( (C, S) \) be a presentation. Given an object in \( X \) of \( sPSh(C) \), we say it is \( S \)-**cartesian local** if for all \( s: S \to S' \) in \( S \), the induced map

\[
Y^s: Y^{S'} \to Y^S
\]

is a levelwise weak equivalence, where \( X \to Y \) is some choice of fibrant replacement in \( sPSh(C)^{\text{inj}} \).
2.10. Proposition. Let $X$ be an object of $sPSh(C)$, and choose some fibrant replacement $X \to Y$ in $sPSh(C)^{\text{inj}}$. Then $X$ is $S$-cartesian local if and only if for all $c \in \text{ob} C$, the function object $Y^{F(c)}$ is $S$-local.

Proof. Immediate from the isomorphism  
\[ Y^S(c) \approx \text{Map}(F(c), Y^S) \approx \text{Map}(S, Y^{F(c)}). \]
\[ \square \]

Observe that every $S$-cartesian local object is necessarily $S$-local, since $\text{Map}(S, Y) \approx \text{Map}(1, Y^S)$; however, the converse need not hold. We say that a presentation $(C, S)$ is a cartesian presentation if every $S$-local object is $S$-cartesian local.

2.11. Proposition. Let $(C, S)$ be a presentation. The following are equivalent.

1. $(C, S)$ is a cartesian presentation.
2. For all $S$-fibrant $X$ in $sPSh(C)$ and all $c \in \text{ob} C$, the object $X^{F(c)}$ is $S$-local.
3. For all $S: S \to S' \in S$ and all $c \in \text{ob} C$, the map $s \times \text{id}: S \times Fc \to S' \times Fc$ is in $\mathcal{S}$.

Proof. Immediate from [2.10]. \[ \square \]

2.12. Proposition. If $(C, S)$ is a cartesian presentation, then $f, g \in \mathcal{S}$ imply $f \times g \in \mathcal{S}$.

2.13. Localization. Given a presentation $(C, S)$, we write $sPSh(C)^{\text{proj}}_S$ and $sPSh(C)^{\text{inj}}_S$ for the model category structures on $sPSh(C)$ obtained by Bousfield localization of the original projective and injective model structures on $sPSh(C)$. These model categories are again Quillen equivalent to each other. We will set out the details in the case of the injective model structure.

2.14. Proposition. Given a presentation $(C, S)$ there exists a cofibrantly generated, left proper, simplicial model category structure on $sPSh(C)$ which is characterized by the following properties.

1. The cofibrations are exactly the monomorphisms.
2. The fibrant objects are precisely the injective fibrant objects which are $S$-local. (We call these the $S$-fibrant objects.)
3. The weak equivalences are precisely the $S$-equivalences.

Furthermore, we have that

4. A levelwise weak equivalence $g: X \to Y$ is an $S$-equivalence, and the converse holds if both $X$ and $Y$ are $S$-local.
5. An $S$-fibration $g: X \to Y$ is an injective fibration, and the converse holds if both $X$ and $Y$ are $S$-fibrant.

Proof. This is an example of [Hir03, Thm. 4.1.1], since $sPSh(C)^{\text{inj}}$ is a left proper cellular model category. \[ \square \]

We will write $sPSh(C)^{\text{inj}}_S$ for the above model structure, which is called the $S$-local injective model structure.

Observe that if $(C, S)$ and $(C, S')$ are two presentations on $C$ such that the $S$-local objects are precisely the same as the $S'$-local objects, then $sPSh(C)^{\text{inj}}_S = sPSh(C)^{\text{inj}}_{S'}$. 
2.15. Quillen pairs between localizations.

2.16. Proposition. Suppose that \((C, S)\) and \((D, T)\) are presentations, and that we have a Quillen pair \(G_\#: sPSh(C)^{\text{inj}} \rightleftarrows sPSh(D)^{\text{inj}}\) (with \(G_\#\) the left adjoint). Then

\[
G_\#: sPSh(C)^S \rightleftarrows sPSh(D)^T
\]

is a Quillen pair if and only if either of the two following equivalent statements hold.

1. For all \(s \in S\), \(G_\#s \in T\).
2. For all \(T\)-fibrant objects \(Y\) in \(sPSh(D)\), \(G^*Y\) is \(S\)-fibrant.

Proof. This is straightforward from the definitions. □

2.17. \(S\)-equivalences and homotopy colimits. The following proposition says that the class of \(S\)-equivalences is closed under homotopy colimits. We refer to [Hir03] for background on homotopy colimits.

2.18. Proposition. Let \(D\) be a small category, and let \((C, S)\) be a presentation. Suppose that \(\alpha: G \to H\) is a natural transformation of functors \(D \to sPSh(C)^{\text{inj}}\), and consider the induced map

\[
\text{hocolim}_D \alpha: \text{hocolim}_D G \to \text{hocolim}_D H
\]

on homotopy colimits, where these homotopy colimits are computed in the injective model structure on \(sPSh(C)\). If \(\alpha(d) \in \mathcal{S}\) for all \(d \in \text{ob}D\), then \(\text{hocolim}_D \alpha \in \mathcal{S}\).

Proof. In general, the map \(h\text{Map}_C(\text{hocolim}_D H, X) \to h\text{Map}(\text{hocolim}_D G, X)\) is weakly equivalent to the map \(\text{holim}_D h\text{Map}(H, X) \to \text{holim}_D h\text{Map}(G, X)\); the result follows by considering the case when \(X\) is \(S\)-local. □

We later use (in §6), we record the following fact.

2.19. Proposition. Let \(C\) be a small category, and let \(X\) be an object of \(sPSh(C)^{\text{inj}}\). Suppose we are given a finite set \(\mathcal{P}\) of subobjects \(K \subseteq X\) in \(sPSh(C)\). If \(\text{colim}_{K \in \mathcal{P}} K \to X\) is an isomorphism (regarding \(\mathcal{P}\) as a finite poset), then

\[
\text{hocolim}_{K \in \mathcal{P}} K \to X
\]

is a levelwise weak equivalence, where homotopy colimit is computed using the injective model structure.

Proof. Since \((\text{hocolim}_{K \in \mathcal{P}} K)(c) \approx \text{hocolim}_{K \in \mathcal{P}}(K(c))\), we can reduce to the case when \(C = 1\); that is, we may assume \(X\) is an object of \(\text{Sp}\).

Suppose that if \(X\) is a set, and \(\mathcal{P}\) is a collection of subsets of \(X\) such that \(\text{colim}_{K \in \mathcal{P}} K \approx X\). It is straightforward to show that for all \(K \in \mathcal{P}\), the map \(\text{colim}_{P \subseteq L} K \to L\) is a monomorphism, where \(P \subseteq L\) denotes the poset of proper subobjects of \(L\). The same therefore holds true for a collection of subobjects of a simplicial sets satisfying the same properties. Thus the functor \(\mathcal{P} \to \text{Sp}\) determined by the collection of subobjects of \(X\), is cofibrant in the projective model structure on \(sPSh(\mathcal{P}^{op})\), and so the colimit of this functor is the homotopy colimit. □

Finally, we record the following fact, which we use in §5. For a category \(C\) and an object \(A\) in \(C\), we write \(A \setminus C\) for the slice category of objects under \(A\) in \(C\).
2.20. **Proposition.** Let $C$ be a small category, and let $(D,S)$ be a presentation. Suppose that $\alpha: G \to H$ is a natural transformation of functors $sPSh(C) \to sPSh(D)$. Suppose the following hold.

1. The functors $X \mapsto (G(\varnothing) \to G(X)): sPSh(C)^{\text{inj}} \to G(\varnothing) \setminus sPSh(D)^{\text{inj}}$ and $X \mapsto (H(\varnothing) \to H(X)): sPSh(C)^{\text{inj}} \to H(\varnothing) \setminus sPSh(D)^{\text{inj}}$ are left Quillen functors.

2. The map $\alpha(\varnothing): G(\varnothing) \to H(\varnothing)$ is a monomorphism, and is in $\mathcal{S}$.

3. The maps $\alpha(Fc): G(Fc) \to H(Fc)$ are in $\mathcal{S}$ for all $c \in \text{ob} C$.

Then $\alpha(X) \in \mathcal{S}$ for all objects $X$ of $sPSh(C)$.

**Proof.** In the special case in which $\alpha(\varnothing): G(\varnothing) \to H(\varnothing)$ is an isomorphism, note that since (i) every object of $sPSh(C)$ is levelwise weakly equivalent to a homotopy colimit of some diagram of free objects, and (ii) left Quillen functors preserve homotopy colimits, the result follows using (2.15).

For the general case, factor $\alpha$ into $G \xrightarrow{\beta} K \xrightarrow{\gamma} H$ where $K(X) = G(X) \cup_{G(\varnothing)} H(\varnothing)$. The map $\beta(X)$ is a pushout of the $S$-local equivalence $\alpha(\varnothing)$ along the map $G(\varnothing) \to G(X)$ which is an injective cofibration by (i); thus $\beta(X) \in \mathcal{S}$. The special case described above applies to show that $\gamma(X) \in \mathcal{S}$. Thus, the composite $\alpha(X) \in \mathcal{S}$, as desired.

2.21. **Cartesian presentations give cartesian model categories.**

2.22. **Proposition.** The model category $sPSh(C)^{\text{inj}}_S$ is cartesian if and only if $(C,S)$ is a cartesian presentation. In particular, $sPSh(C)^{\text{inj}}_S$ is a cartesian model category if for all $S$-fibrant $Y$ and all $c \in \text{ob} C$, the object $Y^{F(c)}$ is $S$-fibrant.

**Proof.** It is clear that the terminal object is cofibrant in $sPSh(C)^{\text{inj}}_S$, so it suffices to show that $(C,S)$ is a cartesian presentation if and only if condition (1) of (2.2) holds. Let $f: A \to A'$ and $g: B \to B'$ be cofibrations in $sPSh(C)^{\text{inj}}_S$. It is clear that the map

$$h: A \times B \coprod_{A \times B} A' \times B \to A' \times B'$$

is a cofibration in any case, so it suffices to show that “$W$ is $S$-fibrant implies $W$ is $S$-cartesian fibrant” is equivalent to “$g \in \mathcal{S}$ implies $h \in \mathcal{S}$”. Since $sPSh(C)^{\text{inj}}$ is a cartesian model category, for a cofibration $g$ as above and all injective fibrant $W$ we have that $W^g$ is a levelwise weak equivalence if and only if $\text{Map}(h,W)$ is a weak equivalence. The result follows by considering the case of $S$-fibrant $W$.

2.23. **$k$-types.** Observe that $\text{Sp} \approx sPSh(1)^{\text{inj}}$. For any integer $k \geq -2$, let

$$\text{Sp}_k \overset{\text{def}}{=} sPSh(1)^{\text{inj}}_{\{\partial \Delta^{k+2} \to \Delta^{k+2}\}}.$$ 

This is called the model category of $k$-**types**. The fibrant objects are precisely the fibrant simplicial sets whose homotopy groups vanish in dimensions greater than $k$. This is a cartesian model category.

3. **The $\Theta$ construction**

The $\Theta$ construction was introduced by Berger in [1], where, with good cause, he calls it the “categorical wreath product over $\Delta$”; what we are calling $\Theta C$, he calls $\Delta \wr C$. 

3.1. The category $\Delta$. We write $\Delta$ for the standard category of finite ordinals; the objects are $[m] = \{0, 1, \ldots, m\}$ for $m \geq 0$, and the morphisms are weakly monotone maps. We will use the following transparent notation to describe particular maps in $\Delta$; we write

$$\delta^{k_0 k_1 \cdots k_m} : [m] \to [n]$$

for the function defined by $i \mapsto k_i$.

We call a morphism $\delta : [m] \to [n] \in \Delta$ an injection or surjection if it is so as a map of sets. We say that $\delta$ is sequential if

$$\delta(i - 1) + 1 \geq \delta(i) \quad \text{for all } i = 1, \ldots, m.$$ 

Observe that every surjection is sequential.

3.2. The category $\Theta C$. Let $C$ be a category. We define a new category $\Theta C$ as follows. The objects of $\Theta C$ are tuples of the form $([m], c_1, \ldots, c_m)$, where $[m]$ is an object of $\Delta$ and $c_1, \ldots, c_m$ are objects of $C$. It will be convenient to write $[m](c_1, \ldots, c_m)$ for this object, and to write $[0]$ for the unique object with $m = 0$.

Morphisms $[m](c_1, \ldots, c_m) \to [n](d_1, \ldots, d_n)$ are tuples $(\delta, \{f_{ij}\})$ consisting of

(i) a morphism $\delta : [m] \to [n]$ of $\Delta$, and

(ii) for each pair $i, j$ of integers such that $1 \leq i \leq m$, $1 \leq j \leq n$, and $\delta(i - 1) < j \leq \delta(i)$, a morphism $f_{ij} : c_i \to d_j$ of $C$.

In other words,

$$(\Theta C)([m](c_1, \ldots, c_m), [n](d_1, \ldots, d_n)) \approx \prod_{\delta : [m] \to [n]} \prod_{i=1}^{m} \prod_{j=\delta(i)-1+1}^{\delta(i)} C(c_i, d_j).$$

The composite

$$[m](c_1, \ldots, c_m) \xrightarrow{(\delta, \{f_{ij}\})} [n](d_1, \ldots, d_n) \xrightarrow{(e, \{g_{jk}\})} [p](e_1, \ldots, e_p)$$

is the pair $(e\delta, \{h_{ik}\})$, where $h_{ik} = g_{jk} f_{ij}$ for the unique value of $j$ for which $f_{ij}$ and $g_{jk}$ are both defined.

Pictorially, it is convenient to represent an object of $\Theta C$ as a sequence of arrows labelled by objects of $C$. For instance, $[3](c_1, c_2, c_3)$ would be drawn

$$0 \xrightarrow{c_1} 1 \xrightarrow{c_2} 2 \xrightarrow{c_3} 3$$

An example of a morphism $[3](c_1, c_2, c_3) \to [4](d_1, d_2, d_3, d_4)$ is the picture

$$0 \xrightarrow{c_1} 1 \xrightarrow{c_2} 2 \xrightarrow{c_3} 3$$

$$d_1 \xleftarrow{f_{11}} \xleftarrow{f_{12}} 2 \xleftarrow{f_{33}} d_3 \xrightarrow{f_{34}} d_4$$

where the dotted arrows describe the map $\delta^{0223} : [3] \to [4]$, and the squiggly arrows represent morphisms $f_{11} : c_1 \to d_1, f_{12} : c_1 \to d_2, f_{33} : c_3 \to d_3$ in $C$.

Observe that (as suggested by our notation) there are functors

$$[m] : C^{\times m} \to \Theta C.$$
for \( m \geq 0 \), defined in the evident way on objects, and which to a morphism \((g_i : c_i \to d_i)_{i=1,\ldots,m}\) assign the morphism \((\text{id},\{f_{ij}\})\) where \(f_{ii} = g_i\).

If \( C \) is a small category, then so is \( \Theta C \), and it is apparent that \( \Theta \) describes a 2-functor \( \text{Cat} \to \text{Cat} \).

### 3.3. A notation for morphisms in \( \Theta C \).

We use the following notation for certain maps in \( \Theta C \). Suppose \( (\delta, \{f_{ij}\}) : [m](c_1,\ldots,c_m) \to [n](d_1,\ldots,d_n) \) is a morphism in \( \Theta C \) such that for each \( i = 1,\ldots,m \), the sequence of maps \((f_{ij} : c_i \to d_j)_{j=\delta(i-1)+1,\ldots,\delta(i)}\) identifies \( c_i \) as the product of the \( d_j \)'s in \( C \). Then we simply write \( \delta \) for this morphism. Note that even if \( C \) is a category which does not have all products, this notation is always sensible if \( \delta \in \Delta \) is injective and sequential.

### 3.4. Remark.

If \( C \) is a category with finite products, morphisms in \( \Theta C \) amount to pairs \( (\delta, \{f_i\}_{i=1,\ldots,m}) \), where

\[
f_i : c_i \to d_{\delta(i-1)+1} \times \cdots \times d_{\delta(i)}.
\]

In this case, our special notation is to write \( \delta \) for \( (\delta, \{f_i\}_{i=1,\ldots,m}) \).

There is a variant of the \( \Theta \) construction which works when \( C \) is a monoidal category. If \( C \) is a monoidal category, we can define a category \( \Theta^{\text{mon}} \) with the same objects as \( \Theta C \), but with morphisms \([m](c_1,\ldots,c_m) \to [n](d_1,\ldots,d_n)\) corresponding to tuples \( (\delta, \{f_i\}_{i=1,\ldots,m}) \) where

\[
f_i : c_i \to d_{\delta(i-1)+1} \otimes \cdots \otimes d_{\delta(i)}.
\]

It seems likely that this variant notion should be useful for producing presentations of categories enriched over general monoidal model categories.

### 3.5. The categories \( \Theta_n \).

For \( n \geq 0 \) we define categories \( \Theta_n \) by setting \( \Theta_0 = 1 \) (the terminal category), and defining \( \Theta_n \) \( \overset{\text{def}}{=} \Theta \Theta_{n-1} \). One sees immediately that \( \Theta_1 \) is isomorphic to \( \Delta \).

### 3.6. Remark.

The category \( \Theta_n \) can be identified as a category of finite planar trees of level \( \leq n \) \([\text{Joy}97]\). The opposite category \( \Theta_n^{\text{op}} \) is isomorphic to the category of “combinatorial \( n \)-disks” in the sense of Joyal \([\text{Joy}97]\); see \([\text{CL}04\text{ Ch. 7]}\), \([\text{Ber}07]\).

### 3.7. \( \Theta \) and enriched categories.

If \( V \) is a cartesian closed category and \( \emptyset \) is an initial object of \( V \), it is straightforward to show that

1. for every object \( v \in \text{ob}V \), the product \( \emptyset \times v \) is an initial object of \( V \), and
2. for an object \( v \in \text{ob}V \), the set \( \text{hom}_V(v, \emptyset) \) is non-empty if and only if \( v \) is initial.

Suppose that \( V \) is a cartesian closed category with a chosen initial object \( \emptyset \). The tautological functor

\[
\tau : \Theta V \to V\text{-Cat}
\]

is defined as follows. For an object \([m](v_1,\ldots,v_m)\), we let \( C = \tau([m](v_1,\ldots,v_m)) \) be the \( V \)-category with object set \( C_0 = \{0,1,\ldots,m\} \), and with morphism objects

\[
C(p,q) = \begin{cases} 
\emptyset & \text{if } p > q, \\
1 & \text{if } p = q, \\
v_{p+1} \times \cdots \times v_q & \text{if } p < q.
\end{cases}
\]

The unique maps \( 1 \to C(p,p) \) define “identity maps”, and composition \( C(p,q) \times C(q,r) \to C(p,r) \) is defined in the evident way. It is clear how to define \( \tau \) on morphisms.
3.8. **Remark.** The functor $\tau$ is not fully faithful. For instance, there is a $V$-functor $f: \tau([1](\emptyset)) \to \tau([1](\emptyset))$ which on objects sends $0 \in [1]$ to $1 \in [1]$ and vice versa; this map $f$ is not in the image of $\tau$.

For a full subcategory $W$ of $V$, we will write $\tau: \Theta W \to V$ for the evident composite $\Theta W \to \Theta V \overset{\tau}{\to} V$. The category $\Theta W$ is fully faithful. For instance, there is a $V$-functor $\tau: \Theta W \to \Theta V$ which relates $\Theta(C)$ to $\Theta(V)$. The functor $\tau$ determines and is determined by morphisms $\delta: \{0, \ldots, m\} \to \{0, \ldots, n\}$. Given $\delta$, the functor $\tau$ determines and is determined by morphisms $f_{ij}: c_i \to d_j$ for $i = 1, \ldots, m$, $j = \delta(i) - 1 + 1, \ldots, \delta(i)$.

3.9. **Proposition.** ([BerC07, Prop. 3.5]). If $W$ is a full subcategory of $V$ which does not contain any initial objects of $V$, then $\tau: \Theta W \to V.$ is fully faithful.

**Proof.** The fact that only an initial object can map to an initial object in $V$ implies that for $c_i, d_j \in \text{ob} W$, a functor $F: \tau([m](c_1, \ldots, c_m)) \to \tau([n](d_1, \ldots, d_n))$ is necessarily given on objects by a weakly monotone function $\delta: \{0, \ldots, m\} \to \{0, \ldots, n\}$. Given $\delta$, the functor $F$ determines and is determined by morphisms $f_{ij}: c_i \to d_j$ for $i = 1, \ldots, m$, $j = \delta(i) - 1 + 1, \ldots, \delta(i)$.

3.10. **Corollary.** For each $n \geq 0$, the functor $\tau_n: \Theta_n \to \text{St-n-Cat}$ defined inductively as the composite

$$\Theta_n \overset{\Theta_{n-1}}{\to} \Theta(\text{St}-(n-1)-\text{Cat}) \overset{\tau}{\to} \text{St-n-Cat}$$

is fully faithful.

Thus, we can identify $\Theta_n$ with a full subcategory of $\text{St-n-Cat}$.

4. Presheaves of spaces over $\Theta C$

In the next few sections we will be especially concerned with the category $s\text{PSh}(\Theta C)$ of simplicial presheaves on $\Theta C$. In this section we describe two essential constructions. First, we describe an adjoint pair of functors $(T_#, T^*)$ between simplicial presheaves on $\Theta C$ and simplicial presheaves on $\Delta = \Theta 1$. Next, we describe a functor $V$, called the “intertwining functor”, which relates $\Theta(\text{sPSh}(C))$ and $\text{sPSh}(\Theta C)$.

4.1. **The functors $T^*$ and $T_#$.** Let $T: \Delta \to s\text{PSh}(\Theta C)$ be the functor defined by

$$(T[n])([m](c_1, \ldots, c_m)) \overset{\text{def}}{=} \Delta([m], [n]),$$

Observe that if $C$ has a terminal object $t$, then $T[n] \approx F_{\Theta C}[n](t, \ldots, t)$.

Let $T^*: s\text{PSh}(\Theta C) \to s\text{PSh}(\Delta)$ denote the functor defined by $(T^*X)[m] \overset{\text{def}}{=} \text{Map}_{s\text{PSh}(\Theta C)}(T[m], X)$. The functor $T^*$ preserves limits, and has a left adjoint $T_#: s\text{PSh}(\Delta) \to s\text{PSh}(\Theta C)$.

4.2. **Proposition.** On objects $X$ in $s\text{PSh}(\Delta)$, the object $T_#X$ is given by

$$(T_#X)[m](c_1, \ldots, c_m) \approx X[m].$$

**Proof.** A straightforward calculation. □

4.3. **Corollary.** The functor $T_#: s\text{PSh}(\Delta)^{\text{inj}} \to s\text{PSh}(\Theta C)^{\text{inj}}$ preserves small limits, cofibrations and weak equivalences; in particular, it is the left adjoint of a Quillen pair.

We will regard $T^*X$ as the “underlying simplicial space” of the object $X$ in $s\text{PSh}(\Theta C)$. 
4.4. The intertwining functor $V$. The intertwining functor 

$$V : \Theta(sPSh(C)) \to sPSh(\Theta C)$$

is a functor which extends the Yoneda functor $F_{\Theta C} : \Theta C \to sPSh(\Theta C)$; it will play a crucial role in what follows.

Recall [3.4] that since $sPSh(C)$ has finite products, a morphism $[m](A_1, \ldots, A_m) \to [n](B_1, \ldots, B_n)$ in $\Theta(sPSh(C))$ amounts to a a pair $(\delta, \{f_j\}_{j=1,\ldots,m})$ where $\delta : [m] \to [n]$ in $\Delta$, and

$$f_j : A_j \to \prod_{j=\delta(k-1)+1}^{\delta(k)} B_k \quad \text{in } sPSh(C).$$

On objects $[m](A_1, \ldots, A_m)$ in $\Theta(sPSh(C))$ the functor $V$ is defined by

$$(V[m](A_1, \ldots, A_m))(\{q\}(c_1, \ldots, c_q)) = \prod_{\delta \in \Delta([q],[m])} \prod_{i=1}^{q} \prod_{j=\delta(i-1)+1}^{\delta(i)} A_j(c_i).$$

To a morphism $(\sigma, \{f_j\}) : [m](A_1, \ldots, A_m) \to [n](B_1, \ldots, B_n)$ we associate the map of presheaves defined by

$$\prod_{\delta \in \Delta([q],[m])} \prod_{i=1}^{q} \prod_{j=\delta(i-1)+1}^{\delta(i)} A_j(c_i) \to \prod_{\delta' \in \Delta([q],[n])} \prod_{i=1}^{q} \prod_{k=\delta'(i-1)+1}^{\delta'(i)} B_k(c_i)$$

which sends the summand associated to $\delta$ to the summand associated to $\delta' = \sigma \delta$ by a map which is a product of maps of the form $f_j(c_i)$.

Observe that

$$(V[m](Fd_1, \ldots, Fd_m))(\{q\}(c_1, \ldots, c_q)) \approx \prod_{\delta : [q] \to [m]} \prod_{i=1}^{q} \prod_{j=\delta(i-1)+1}^{\delta(i)} C(c_i, d_j)$$

$$\approx (\Theta C)(\{q\}(c_1, \ldots, c_q), [m](d_1, \ldots, d_m))$$

$$\approx F_{\Theta C}[m](d_1, \ldots, d_m)(\{q\}(c_1, \ldots, c_q)).$$

Thus we obtain a natural isomorphism $\nu : F_{\Theta C} \to V(\Theta F_C)$ of functors $\Theta C \to sPSh(\Theta C)$.

In this paper, we are proposing the category $sPSh(\Theta C)$ as a model for $sPSh(C)$-enriched categories. In this light, the object $V[m](A_1, \ldots, A_m)$ of $sPSh(\Theta C)$ may be thought of as a model of the $sPSh(C)$-enriched category freely generated by the $sPSh(C)$-enriched graph

$$\xymatrix{(0) \ar[r]^-{A_1} & (1) \ar[r]^-{A_2} & \cdots \ar[r]^-{A_{m-1}} & (m-1) \ar[r]^-{A_m} & (m).}$$

The following proposition describes how the intertwining functor interacts with colimits.

Recall that for an object $X$ of a category $C$, $A \backslash X$ denotes the slice category of objects under $X$ in $C$.

4.5. Proposition. The intertwining functor $V : \Theta(sPSh(C)) \to sPSh(\Theta C)$ has the following properties. Fix $m, n \geq 0$ and objects $A_1, \ldots, A_m, B_1, \ldots, B_n$ of $sPSh(C)$.
(1) The map $(V\delta^0,\ldots,m, V\delta^{m+1},\ldots,m+1,n)$ which sends $V[m](A_1,\ldots,A_m) \amalg V[n](B_1,\ldots,B_n) \to V[m+1+n](A_1,\ldots,A_m,\emptyset,B_1,\ldots,B_n)$ is an isomorphism.

(2) The functor $X \mapsto V[m+1+n](A_1,\ldots,A_m, X,B_1,\ldots,B_n)$:

\[ \text{sPSh}(C) \to V[m+1+n](A_1,\ldots,A_m, \emptyset,B_1,\ldots,B_n) \backslash \text{sPSh}(\Theta C) \]

is a left adjoint.

Proof. For $p = 0,\ldots,q+1$, let $G(p) \subseteq \Delta([q],[m+1+n])$ be defined by

\[ G(p) = \begin{cases} \{\delta \mid \delta(0) \geq m+1\} & \text{if } p = 0, \\ \{\delta \mid \delta(p-1) \leq m, \delta(p) \geq m+1\} & \text{if } 1 \leq p \leq q, \\ \{\delta \mid \delta(q) \leq m\} & \text{if } p = q+1. \end{cases} \]

Thus the $G(p)$ determine a partition of the set $\Delta([q],[m+1+n])$. The coproduct which defines $V[m](A_1,\ldots,A_m, X,B_1,\ldots,B_n)(\theta)$ for $\theta = [q](c_1,\ldots,c_q)$ decomposes into factors according to this partition of $\Delta([q],[m+1+n])$. Under this decomposition, the factor corresponding to $p = 0$ is

\[ \prod_{\delta \in G(0)} \prod_{i=1}^{q} \prod_{j=\delta(i)-1+1}^{\delta(i)} B_{j-(m+1)}(c_i) \cong V[n](B_1,\ldots,B_n)(\theta), \]

the factor corresponding to $p = q+1$ is

\[ \prod_{\delta \in G(q+1)} \prod_{i=1}^{q} \prod_{j=\delta(i)-1+1}^{\delta(i)} A_{j}(c_i) \cong V[m](A_1,\ldots,A_m)(\theta), \]

while the factor corresponding to $p$ where $1 \leq p \leq q$ is

\[ \prod_{\delta \in G(p)} \left( \prod_{i=1}^{p} \prod_{j=\delta(i)-1+1}^{\min(\delta(i),m)} A_{j}(c_i) \right) \times X(c_p) \times \left( \prod_{i=p}^{q} \prod_{j=\max(\delta(i)-1,1),m} B_{j-(m+1)}(c_i) \right). \]

From this claim (1) is immediate, as is the observation that the functor described in (2) preserves colimits, and so has a right adjoint. □

4.6. Proposition. For all $m,n \geq 0$ and objects $A_1,\ldots,A_m, B_1,\ldots,B_n$ in $\text{sPSh}(C)$, the functor $X \mapsto V[m+1+n](A_1,\ldots,A_m, X,B_1,\ldots,B_n)$:

\[ \text{sPSh}(C) \to V[m+1+n](A_1,\ldots,A_m, \emptyset,B_1,\ldots,B_n) \backslash \text{sPSh}(\Theta C) \]

preserves cofibrations and weak equivalences, and thus is the left adjoint of a Quillen pair.

Proof. A straightforward calculation, using the decomposition given in the proof of (4.5). □

4.7. Remark. It can be shown that $V$ is the left Kan extension of $F_{\Theta C}$ along $\Theta F_C$. 
4.8. A product decomposition. We will need to make use of the following description of the product $V[1](A) \times V[1](B)$ in $sPSh(\Theta C)$.

4.9. Proposition. The map

$$\text{colim}(V[2](A, B) \xrightarrow{V^0} V[1](A \times B) \xrightarrow{V^0} V[2](B, A)) \rightarrow V[1](A) \times V[1](B),$$

induced by $(V^0, V^0) : V[2](A, B) \rightarrow V[1](A) \times V[1](B)$ and $(V^0, V^0) : V[2](B, A) \rightarrow V[1](A) \times V[1](B)$, is an isomorphism.

Proof. This is a straightforward calculation. To grok the argument, it may be helpful to contemplate the following diagram.

![Diagram](image)

4.10. Subobjects of $V[m](c_1, \ldots, c_m)$. Observe that $V[m](1, \ldots, 1) \approx T\# F[m]$; we write $\pi : V[m](A_1, \ldots, A_m) \rightarrow T\# F[m]$ for the map induced by projection to the terminal object in $sPSh(C)$. Given a subobject $f : K \subseteq F[m]$ in $sPSh(\Delta)$, we define $V_K(A_1, \ldots, A_m)$ to be the inverse limit of the diagram

$$V[m](A_1, \ldots, A_m) \xrightarrow{\pi} T\# F[m] \xleftarrow{T\# f} T\# K.$$

Explicitly, $V_K(A_1, \ldots, A_m)([q](c_1, \ldots, c_q))$ is the subobject of $V[m](A_1, \ldots, A_m)([q](c_1, \ldots, c_q))$ coming from summands associated to $\delta : [q] \rightarrow [m]$ such that $F\delta : F[q] \rightarrow F[m]$ factors through $K \subseteq F[m]$ in $sPSh(\Delta)$. Observe that $V_F[m](A_1, \ldots, A_m) \approx V[m](A_1, \ldots, A_m)$.

These subobjects will be used in §6.

4.11. Mapping objects. Given an object $X$ in $sPSh(\Theta C)$, an ordered sequence $x_0, \ldots, x_m$ of points in $X[0]$, and a sequence $c_1, \ldots, c_m \in \text{ob}C$, we define $M_X(x_0, \ldots, x_m)(c_1, \ldots, c_m)$ to be the pullback of the diagram

$$\{(x_0, \ldots, x_m)\} \xrightarrow{X[0]^{x \times m+1}} X[1] \xleftarrow{X[1]^{x \times m}} X[m](c_1, \ldots, c_m).$$

Allowing the objects $c_1, \ldots, c_m$ to vary gives us a presheaf $M_X(x_0, \ldots, x_m)$ in $sPSh(C^{x \times m})$. We will be especially interested in $M_X(x_0, x_1)$, an object of $sPSh(C)$, which we will refer to as a mapping object for $X$.

We can use the intertwining functor $V$ to get a fancier version of the mapping objects, as follows. Again, given $X$ in $sPSh(\Theta C)$ and $x_0, \ldots, x_m \in X[0]$, and also given objects $A_1, \ldots, A_m$ in $sPSh(C)$, we define $\tilde{M}_X(x_0, \ldots, x_q)(A_1, \ldots, A_q)$ to be the pullback of the diagram

$$\{(x_0, \ldots, x_m)\} \xrightarrow{X[0]^{x \times m+1}} \text{Map}(V[m](\emptyset, \ldots, \emptyset), X) \xleftarrow{\text{Map}(V[m](A_1, \ldots, A_m), X)} X.$$

where the right-hand map is induced by the maps $X\delta^a$. Allowing the objects $A_1, \ldots, A_m$ to vary gives us a functor $\tilde{M}_X(x_0, \ldots, x_m) : sPSh(C)^{x \times m} \rightarrow sPSh(\Theta C)$. Observe that

$$M_X(x_0, \ldots, x_m)(c_1, \ldots, c_m) \approx \tilde{M}_X(x_0, \ldots, x_m)(Fc_1, \ldots, Fc_m).$$
and that
\[
\widetilde{M}_X(x_0, \ldots, x_{m+1+n})(A_1, \ldots, A_m, \varnothing, B_1, \ldots, B_n)
\approx \widetilde{M}_X(x_0, \ldots, x_m)(A_1, \ldots, A_m) \times \widetilde{M}_X(x_{m+1}, \ldots, x_{m+n+1})(B_1, \ldots, B_n).
\]

4.12. Lemma. There is a natural isomorphism
\[
\widetilde{M}_X(x_0, x_1)(A) \approx \text{Map}_C(A, M_X(x_0, x_1)).
\]

Proof. This follows using the natural isomorphisms
\[
\widetilde{M}_X(x_0, x_1)(F c) \approx M_X(x_0, x_1)(c) \approx \text{Map}_C(F c, M_X(x_0, x_1))
\]
and the fact that \(V[1]: \text{sPSh}(C) \to V[1](\varnothing) \setminus \text{sPSh}(\Theta C)\) preserves colimits \(\text{(4.5)}\), which implies that \(A \mapsto \widetilde{M}_X(x_0, x_1)(A)\) takes colimits to limits. \(\Box\)

5. Segal objects

In this section, we examine the properties of a certain class of objects in \(\text{sPSh}(\Theta C)\), called Segal objects. In the case that \(C = 1\), these are the Segal spaces of \(\text{[Rez01]}\). We work with a fixed small category \(C\).

5.1. Segal maps and Segal objects. Let \(\text{SeC}\) denote the set of morphisms in \(\text{sPSh}(\Theta C)\) of the form
\[
\text{se}^{(c_1, \ldots, c_m)} \overset{\text{def}}{=} (F \delta^0, \ldots, F \delta^{m-1}) : G[m](c_1, \ldots, c_m) \to F[m](c_1, \ldots, c_m)
\]
where
\[
G[m](c_1, \ldots, c_m) \overset{\text{def}}{=} \text{colim}(F[1](c_1) \xleftarrow{F \delta^1} F[0] \xrightarrow{F \delta^0} \cdots \xrightarrow{F \delta^1} F[0] \xleftarrow{F \delta^0} F[1](c_m))
\]
for \(m \geq 2\) and \(c_1, \ldots, c_m \in \text{obC}\). It is straightforward to check that an injective fibrant \(X\) in \(\text{sPSh}(\Theta C)\) is \(\text{SeC}\)-fibrant if and only if each of the induced maps
\[
X[m](c_1, \ldots, c_m) \to \text{lim}(X[1](c_1) \xleftarrow{X \delta^1} X[0] \xrightarrow{X \delta^0} \cdots \xrightarrow{X \delta^1} X[0] \xleftarrow{X \delta^0} X[1](c_m))
\]
is a weak equivalence. Equivalently, an injective fibrant \(X\) is \(\text{SeC}\)-fibrant if and only if the evident maps
\[
M_X(x_0, \ldots, x_m)(c_1, \ldots, c_m) \to M_X(x_0, x_1)(c_1) \times \cdots \times M_X(x_m, x_m)(c_m)
\]
are weak equivalences.

A Segal object is a \(\text{SeC}\)-fibrant object in \(\text{sPSh}(\Theta C)^\text{inj}\), i.e., a fibrant object in \(\text{sPSh}(\Theta C)^\text{inj}_{\text{SeC}}\).

5.2. More maps of Segal type. For an object \([m](A_1, \ldots, A_m)\) of \(\Theta(\text{sPSh}(C))\), we obtain a map in \(\text{sPSh}(\Theta C)\) of the form
\[
\text{se}^{[m](A_1, \ldots, A_m)} : V_G[m](A_1, \ldots, A_m) \to V[m](A_1, \ldots, A_m).
\]
induced by \(\text{se}^{(1, \ldots, 1)} : G[m] \to F[m]\) in \(\text{sPSh}(\Delta)\), where \(V_G[m]\) is as defined in \(\text{[4.10]}\). Observe that
\[
V_G[m](A_1, \ldots, A_m) \approx \text{colim}(V[1](A_1) \xleftarrow{\delta^1} V[0] \xrightarrow{\delta^0} \cdots \xleftarrow{\delta^1} V[0] \xrightarrow{\delta^0} V[1](A_m)).
\]
Note also that if $\delta: [m] \to [n]$ in $\Delta$ is injective and sequential, then $F\delta: F[m] \to F[n]$ carries $G[m]$ into $G[n]$, and thus we obtain an induced map

$$\delta_*: V_{G[m]}(A_{\delta(1)}, \ldots, A_{\delta(m)}) \to V_{G[n]}(A_1, \ldots, A_m).$$

It is straightforward to check that $V_{G[m]}: s\text{PSh}(C)^{\times m} \to s\text{PSh}(\Theta C)$ satisfies formal properties similar to $V[m]: s\text{PSh}(C)^{\times m} \to s\text{PSh}(\Theta C)$. Namely,

1. for all $A_1, \ldots, A_m, B_1, \ldots, B_n$ objects of $s\text{PSh}(C)$, the map $(\delta^0, \ldots, \delta^m, \delta^{m+1}, \ldots, m+n)$ which sends

$$V_{G[m]}(A_1, \ldots, A_m) \amalg V_{G[n]}(B_1, \ldots, B_n) \to V_{G[m+n]}(A_1, \ldots, A_m, \emptyset, B_1, \ldots, B_n)$$

is an isomorphism, and

2. for all $m, n \geq 0$ and objects $A_1, \ldots, A_m, B_1, \ldots, B_n$ in $s\text{PSh}(C)$, the functor

$$X \mapsto V_{G[m+n]}(A_1, \ldots, A_m, X, B_1, \ldots, B_n): s\text{PSh}(C) \to V_{G[m+n]}(A_1, \ldots, A_m, \emptyset, B_1, \ldots, B_n) \setminus s\text{PSh}(\Theta C)$$

is a left Quillen functor.

We record the following fact.

5.3. Proposition. For all objects $[m](A_1, \ldots, A_m)$ of $\Theta(s\text{PSh}(C))$, we have

$$\text{se}[m](A_1, \ldots, A_m) \in \overline{\text{Se}C},$$

where $\overline{\text{Se}C}$ is the class of $\text{Se}C$-local equivalences.

Proof. We prove this by induction on $m \geq 0$. Let $D$ denote the class of objects $[m](A_1, \ldots, A_m)$ in $\Theta(s\text{PSh}(C))$ such that $\text{se}[m](A_1, \ldots, A_m) \in \overline{\text{Se}C}$.

We observe the following.

1. All objects of the form $[0]$ and $[1](A)$ are in $D$, since $\text{se}[0]$ and $\text{se}[1](A)$ are isomorphisms.

2. All objects of the form $[m](F c_1, \ldots, F c_m)$ for all $c_1, \ldots, c_m \in \text{ob}C$ are in $D$, since $\text{se}[m](F c_1, \ldots, F c_m) = \text{se}(c_1, \ldots, c_m)$. For $m \geq 1$ and $0 \leq j \leq m$, let $E_{m,j}$ denote the class of objects of the form $[m](A_1, \ldots, A_j, F c_{j+1}, \ldots, F c_m)$ or $[m](A_1, \ldots, A_j, \emptyset, F c_{j+2}, \ldots, F c_m)$, where $A_1, \ldots, A_j$ in $s\text{PSh}(C)$ and $c_{j+1}, \ldots, c_m \in \text{ob}C$. We need to prove that $E_{m,j} \subseteq D$ for all $m$. Observation (1) says that this is so for $m = 0$ and $m = 1$, while observation (2) says that $E_{0,0} \subseteq D$ for all $m$. The proof will be completed once we show that for all $m \geq 2$ and $1 \leq j \leq m$, $E_{m,j-1} \subseteq D$ implies $E_{m,j} \subseteq D$.

Consider the transformation $\alpha: G \to H$ and the $s\text{PSh}(C) \to s\text{PSh}(\Theta C)$ defined by the evident inclusion

$$\alpha: V_{G[m]}(A_1, \ldots, A_{j-1}, -, F c_{j+1}, \ldots, F c_m) \to V[m](A_1, \ldots, A_{j-1}, -, F c_{j+1}, \ldots, F c_m).$$

The functors $G$ and $H$ produce left Quillen functors $s\text{PSh}(C)^{\text{inj}} \to G(\emptyset)\setminus s\text{PSh}(\Theta C)^{\text{inj}}$ and $s\text{PSh}(C)^{\text{inj}} \to H(\emptyset)\setminus s\text{PSh}(\Theta C)^{\text{inj}}$, and it is clear from the explicit description of $V$ that $\alpha(\emptyset)$ is a monomorphism. Since $E_{m,j-1} \subseteq D$, we have that $\alpha(\emptyset) \in \overline{\text{Se}C}$ and $\alpha(F c_j) \in \overline{\text{Se}C}$ for all objects $c_j$ of $C$. Thus $(\text{[2.20]}$) applies to show that $\alpha(A_j) \in \overline{\text{Se}C}$ for all objects $A_j$ of $s\text{PSh}(C)$, and thus $[m](A_1, \ldots, A_j, F c_{j+1}, \ldots, F c_m) \in D$, as desired. \qed
5.4. **Corollary.** Let $X$ be a $\text{Se}_{C}$-fibrant object of $\text{sPSh}(\Theta C)$, let $x_{0}, \ldots, x_{m} \in X[0]$, and let $A_{1}, \ldots, A_{m}$ be objects of $\text{sPSh}(C)$. Then the map

\[
\text{Map}_{\Theta C}(V[m](A_{1}, \ldots, A_{m}), X) \to \text{Map}_{\Theta C}(V[1](A_{1}), X) \times \cdots \times \text{Map}_{\Theta C}(V[1](A_{m}), X)
\]

induced by $V\delta_{i}$ for $1 \leq i \leq m$ is a weak equivalence, and the map

\[
\bar{M}_{X}(x_{0}, \ldots, x_{m})(A_{1}, \ldots, A_{m}) \to \bar{M}_{X}(x_{0}, x_{1})(A_{1}) \times \cdots \times \bar{M}_{X}(x_{m-1}, x_{m})(A_{m})
\]

is a weak equivalence.

6. **$(\Theta C, \text{Se}_{C})$ is a cartesian presentation**

We now prove the following result.

6.1. **Proposition.** For any small category $C$, the pair $(\Theta C, \text{Se}_{C})$ is a cartesian presentation, and thus $\text{sPSh}(C)_{\text{inj}}^{\text{inj}}$ is a cartesian model category.

Our proof is an adaptation of the proof we gave in [Rez01, §10] for the case $C = 1$; it follows after (6.6) below.

6.2. **Covers.** Let $[m]$ be an object of $\Delta$, and let $K \subseteq F[m]$ be a subobject in $\text{sPSh}(\Delta)$. We say that $K$ is a **cover** of $F[m]$ if

(i) for all sequential $\delta: [1] \to [m]$, the map $F\delta: F[1] \to F[m]$ factors through $K$, and

(ii) there exists a (necessarily unique) dotted arrow making the diagram commute in every diagram of the form

\[
\begin{array}{ccc}
F[1] & \to & K \\
\downarrow_{F\delta m} & & \downarrow \\
F[n] & \to & F[m]
\end{array}
\]

It is immediate that

(0) the identity map $\text{id}: F[m] \to F[m]$ is a cover;

(1) the subobject $G[m] \subseteq F[m]$ generated by the images of the maps $F\delta_{i}^{-1,i}: F[1] \to F[m]$ is a cover (called the **minimal cover**);

(2) if $\delta: [p] \to [m]$ is sequential, and $K \subseteq F[m]$ is a cover, then the pullback $\delta^{-1}K \subseteq F[p]$ of $K$ along $F\delta$ is a cover of $F[p]$;

(3) if $\delta: [p] \to [m]$ and $\delta': [p] \to [n]$ are sequential, and $M \subseteq F[m]$ and $N \subseteq F[n]$ are covers, then the pullback $(\delta, \delta')^{-1}(M \times N)$ of $M \times N$ along $(F\delta, F\delta') : F[p] \to F[m] \times F[n]$ is a cover of $F[p]$.

6.3. **Covers produce $\text{Se}_{C}$-equivalences.** Recall that given a subobject $K \subseteq F[m]$ in $\text{sPSh}(\Delta)$, and a sequence $A_{1}, \ldots, A_{m}$ of $\text{sPSh}(C)$, we have defined (in §4.10) a subobject $V_{K}(A_{1}, \ldots, A_{m})$ of $V[m](A_{1}, \ldots, A_{m})$ in $\text{sPSh}(\Theta C)$.

6.4. **Proposition.** If $K \subseteq F[m]$ is a cover in $\text{sPSh}(\Delta)$, then $V_{K}(A_{1}, \ldots, A_{m}) \to V[m](A_{1}, \ldots, A_{m})$ is in $\text{Se}_{C}$ for all $A_{1}, \ldots, A_{m}$ objects of $\text{sPSh}(C)$. 
Proof. Since \( V_{G[m]}(A_1, \ldots, A_m) \to V[m](A_1, \ldots, A_m) \) is in \( \text{Sec}_C \) by \((5.3)\), it suffices to show that \( V_{G[m]}(A_1, \ldots, A_m) \to V_K(A_1, \ldots, A_m) \) is in \( \text{Sec}_C \) for covers \( K \subset F[m] \) which are proper inclusions. We will prove this using induction on \( \delta \).

Given a subobject \( K \subset F[m] \) in \( sPSh(\Delta) \), let \( \mathcal{P}_K \) denote the category whose objects are injective sequential maps \( \delta : [p] \to [m] \) such that \( F\delta \) factors through \( K \), and whose morphisms \( ([p] \to [m]) \to ([p'] \to [m]) \) are arrows \([p] \to [p'] \) in \( \Delta \) making the evident triangle commute. The category \( \mathcal{P}_K \) is a poset. For each \( \delta : [p] \to [m] \in \mathcal{P}_K \) we have a natural square

\[
\begin{array}{ccc}
V_{\delta^{-1}K}(A_1, \ldots, A_m) & \longrightarrow & V_{G[m]}(A_1, \ldots, A_m) \\
\downarrow & & \downarrow \\
V_F[p](A_1, \ldots, A_m) & \longrightarrow & V_K(A_1, \ldots, A_m)
\end{array}
\]

Observe that since \( \delta \) is a monomorphism, the map \( F[p] \to F[m] \) is a monomorphism; we have abused notation and written \( F[p] \) for this subobject.

We have that the maps

\[
\text{hocolim}_{\mathcal{P}_K} V_{\delta^{-1}K}(A_1, \ldots, A_m) \to V_{G[m]}(A_1, \ldots, A_m)
\]

and

\[
\text{hocolim}_{\mathcal{P}_K} V_F[p](A_1, \ldots, A_m) \to V_K(A_1, \ldots, A_m)
\]

are levelwise weak equivalences in \( sPSh(\Theta C) \) by \((2.19)\), since the corresponding maps from colimits over \( \mathcal{P}_K \) are isomorphisms. Since the inclusion \( K \subset F[m] \) is proper, \( p < m \) for all objects of \( \mathcal{P}_K \), and so each \( V_{\delta^{-1}K}(A_1, \ldots, A_m) \to V_F[p](A_1, \ldots, A_m) \in \text{Sec}_C \) by the induction hypothesis, the result follows using \((2.18)\). \( \square \)

6.5. **Proof that \((\Theta C, \text{Sec}_C)\) is cartesian.**

6.6. **Proposition.** If \( M \subset F[m] \) and \( N \subset F[n] \) are covers in \( sPSh(\Delta) \), then

\[
V_M(A_1, \ldots, A_m) \times V_N(B_1, \ldots, B_n) \to V[m](A_1, \ldots, A_m) \times V[n](B_1, \ldots, B_m)
\]

is an \( \text{Sec}_C \)-equivalence.

Proof. Let \( \mathcal{Q}_{m,n} \) denote the category whose objects are pairs of maps \( (\delta : [p] \to [m], \delta' : [p] \to [n]) \) in \( \Delta \) such that \( \delta \) and \( \delta' \) are surjective (and thus sequential), and \( (F\delta, F\delta') : F[p] \to F[m] \times F[n] \) is a monomorphism. The category \( \mathcal{Q}_{m,n} \) is a poset, and we have \( \text{colim}_{\mathcal{Q}_{m,n}} F[p] \to F[m] \times F[n] \) is an isomorphism in \( sPSh(\Delta) \). For each object \((\delta, \delta')\) of \( \mathcal{Q}_{m,n} \) there is a natural square

\[
\begin{array}{ccc}
V_{(\delta,\delta')^{-1}M \times N}(C_1, \ldots, C_p) & \longrightarrow & V_M(A_1, \ldots, A_m) \times V_N(B_1, \ldots, B_n) \\
\downarrow & & \downarrow \\
V[p](C_1, \ldots, C_p) & \longrightarrow & V[m](A_1, \ldots, A_m) \times V[n](B_1, \ldots, B_n)
\end{array}
\]

where \( C_i = A_{\delta(i)} \) or \( B_{\delta'(i)} \) according to whether \( \delta(i) > \delta(i - 1) \) or \( \delta'(i) > \delta'(i - 1) \). We have that \( \text{hocolim}_{\mathcal{Q}_{m,n}} V_{(\delta,\delta')^{-1}M \times N} \to V_M \times V_N \) and \( \text{hocolim}_{\mathcal{Q}_{m,n}} V_F[p] \to V_F[m] \times V_F[n] \) are levelwise weak equivalences in \( sPSh(\Theta C) \) by \((2.19)\). By \((6.4)\), each of the maps \( V_{(\delta,\delta')^{-1}M \times N} \to V_F[p] \) is in \( \text{Sec}_C \), and thus the result follows using \((2.18)\). \( \square \)
Now we can give the proof of the proposition stated at the beginning of the section.

**Proof of (6.1).** To prove that $(\Theta C, Se_C)$ is cartesian, it is enough to show that $se^{(c_1, \ldots, c_m)} \times F\theta : G[m](c_1, \ldots, c_m) \times F\theta \to F[m](c_1, \ldots, c_m) \times F\theta$ is in $Se_C$ for all $m \geq 2$, $c_1, \ldots, c_m \in \text{ob}C$, and $\theta \in \text{ob}\Theta C$. This is a special case of (6.6). \hfill \Box

6.7. **Presentations of the form $(\Theta C, Se_C \cup \mathcal{U})$.** Let $\mathcal{U}$ be a set of morphisms in $sPSh(\Theta C)$.

6.8. **Proposition.** The presentation $(\Theta C, Se_C \cup \mathcal{U})$ is cartesian if and only if for all $X$ in $sPSh(\Theta C)$ which are $(Se_C \cup \mathcal{U})$-fibrant, and for all $c \in \text{ob}C$, the function object $X^{F[1](c)}$ is $\mathcal{U}$-local.

**Proof.** By (2.11) and (6.1), it is enough to show that if $X$ is $(Se_C \cup \mathcal{U})$-fibrant, then $X^{F\theta}$ is $\mathcal{U}$-local for all $\theta \in \text{ob}\Theta C$. Since $X$ is $Se_C$-local, every $X^{F\theta}$ is weakly equivalent to a homotopy fiber product of the form $X^{F[1](c_1)} \times X \cdots \times X X^{F[1](c_m)}$, and thus the result follows. \hfill \Box

7. **Complete Segal objects**

In this section, we examine the properties of a certain class of Segal objects in $sPSh(\Theta C)$, called complete Segal objects. In the case that $C = 1$, these are precisely the complete Segal objects of $[\text{Rez01}]$. We show below that complete Segal objects are the fibrant objects of a cartesian model category, generalizing a result of $[\text{Rez01}, \S 12]$.

7.1. **The underlying Segal space of a Segal object.** Recall the Quillen pair $T_# : sPSh(\Delta) \rightleftarrows sPSh(\Theta C) : T^*$ of $[\text{L1}]$. Given an object $X$ of $sPSh(\Theta C)$, we will call $T^*X$ in $sPSh(\Delta)$ its **underlying simplicial space**; according to the following proposition, it is reasonable to call $T^*X$ the underlying Segal space of $X$ if $X$ is itself a Segal object.

7.2. **Proposition.** If $X$ is an $Se_C$-fibrant object in $sPSh(\Theta C)$, then $T^*X$ is an $Se_1$-fibrant object in $sPSh(\Theta C) = sPSh(\Delta)$. That is, $T^*X$ is a Segal space in the sense of $[\text{Rez01}]$.

**Proof.** The map

$$se^{[m](1, \ldots, 1)} : \text{colim}(V[1](1) \leftarrow V[0] \to \cdots \leftarrow V[1](1)) \to V[m](1, \ldots, 1)$$

is isomorphic to $T_#se^{(1, \ldots, 1)} : T_#G[m] \to T_#F[m]$. \hfill \Box

7.3. **The homotopy category of a Segal object.** Recall that if $X$ in $sPSh(\Delta)$ is a Segal space, then we define its homotopy category $hX$ as follows. The objects of $hX$ are points of $X[0]$, and morphisms are given by

$$hX(x_0, x_1) \overset{\text{def}}{=} \pi_0M_X(x_0, x_1).$$

It is shown in $[\text{Rez01}]$ that $hX$ is indeed a category; composition is defined and its properties verified using the isomorphisms $\pi_0M_X(x_0, \ldots, x_m) \approx hX(x_0, x_1) \times \cdots \times hX(x_{m-1}, x_m)$ which hold for a Segal space.

For an $Se_C$-fibrant object $X$ in $sPSh(\Theta C)$, we define its **homotopy category** $hX$ to be the homotopy category of $T^*X$. Explicitly, objects of $hX$ are points in $X[0]$, and morphisms are

$$hX(x_0, x_1) \overset{\text{def}}{=} \pi_0G_CM_X(x_0, x_1).$$
7.4. The enriched homotopy category of a Segal object. The homotopy category $hX$ described above can be refined to a homotopy category enriched over the homotopy category $h\text{PSh}(C)$ of presheaves of spaces on $C$. This $h\text{PSh}(C)$-enriched homotopy category is denoted $h\Delta_{\ast}$, and is defined as follows.

We take $\text{ob}h\Delta_{\ast} = \text{ob}hX$. Given objects $x_0, x_1$ of $hX$, recall that the function object of maps $x_0 \to x_1$ is the object $M_X(x_0, x_1)$ of $\text{PSh}(C)$. For objects $x_0, x_1$ in $hX$, we set $h\Delta_{\ast}(x_0, x_1) \overset{\text{def}}{=} M_X(x_0, x_1)$ viewed as an object in the homotopy category $h\text{PSh}(C)$. To make this a category, let $\Delta_{\ast}: C \to C^{\times m}$ denote the “diagonal” functor, and let $\Delta_{\ast}^m: \text{PSh}(C^{\times m}) \to \text{PSh}(C)$ denote the functor which sends $F \mapsto F\Delta_{\ast}$. The functor $\Delta_{\ast}^m$ preserves weak equivalences and products. Observe that since $X$ is a Segal object, there are evident weak equivalences

$$\Delta_{\ast}^m M_X(x_0, \ldots, x_m) \to M_X(x_0, x_1) \times \cdots \times M_X(x_{m-1}, x_m)$$

in $\text{PSh}(C)$. Thus we obtain “identity” and “composition” maps

$$1 \approx \Delta_{\ast}^m M_X(x_0) \to M_X(x_0, x_0),$$

$$M_X(x_0, x_1) \times M_X(x_1, x_2) \overset{\Delta_{\ast}^m M_X(x_0, x_1, x_2)}{\rightarrow} M_X(x_0, x_2)$$

in $h\text{PSh}(C)$, and it is straightforward to check that these make $h\Delta_{\ast}$ into an $h\text{PSh}(C)$-enriched category. Furthermore, we see that $h\text{PSh}(C)(1, h\Delta_{\ast}(x_0, x_1)) \approx hX(x_0, x_1)$.

7.5. Equivalences in a Segal object. Recall that if $X$ in $\text{PSh}(\Delta)$ is a Segal space, then we say that a point in $X[1]$ is an equivalence if it projects to an isomorphism in the homotopy category $hX$. We write $M_X^{\text{equiv}}(x_0, x_1)$ for the subspace of $M_X(x_0, x_1)$ consisting of path components which project to isomorphisms in $hX$, and we let $X^{\text{equiv}}$ denote the subspace of $X[1]$ consisting of path components which contain points from $M_X^{\text{equiv}}(x_0, x_1)$ for some $x_0, x_1 \in \text{ob}hX$. Thus $M_X^{\text{equiv}}(x_0, x_1)$ is the fiber of $(X^{\delta^0}, X^{\delta^1}): X^{\text{equiv}} \to X[0] \times X[0]$ over $(x_0, x_1)$; observe that the map $X^{\delta^0}: X[0] \to X[1]$ factors through $X^{\text{equiv}} \subseteq X[1]$.

These definitions transfer to Segal objects. Thus, if $X$ is a Segal object in $\text{PSh}(\Theta C)$, we say that a point in $\Gamma_C M_X(x_1, x_2)$ is an equivalence if it projects to an isomorphism in the homotopy category $hX$. We define $M_X^{\text{equiv}}(x_0, x_1)$ to be the subspace of $\Gamma_C M_X(x_0, x_1)$ consisting of path components which project to isomorphisms in $hX$. The space of equivalences $X^{\text{equiv}}$ is defined to be the subspace of $\Gamma_C X[1]$ consisting of path components which contain points from $M_X^{\text{equiv}}(x_0, x_1)$ for some $x_0, x_1 \in X[0]$; thus $M_X^{\text{equiv}}(x_0, x_1)$ is the fiber of $X^{\text{equiv}} \to X[0] \times X[0]$ over $(x_0, x_1)$. Observe that the map $X^{\delta^0}: X[0] \to \Gamma_C X[1]$ factors through $X^{\text{equiv}} \subseteq \Gamma_C X[1]$.

7.6. The set $\text{Cpt}_C$. Let $E$ be the object in $\text{PSh}(\Delta)$ which is the “discrete nerve of the free-standing isomorphism”, as in [Rez01, §6]. Let $p: E \to F[0]$ be the evident projection map, and let $i: F[1] \to E$ be the inclusion of one of the non-identity arrows. We recall the following result.

7.7. Proposition ([Rez01, Thm. 6.2]). If $X$ in $\text{PSh}(\Delta)$ is a Segal space, then the map $\text{Map}(i, X): \text{Map}(E, X) \to \text{Map}(F[1], X) \approx X[1]$ factors through $X^{\text{equiv}} \subseteq X[1]$ and induces a weak equivalence $\text{Map}(E, X) \to X^{\text{equiv}}$. 
We define $\text{Cpt}_C$ to be the set consisting of the single map 
\[ T\# p: T\# E \to T\# F[0]. \]
We say that $X$ in $s\text{PSh}(C)$ is a **complete Segal object** if it is $(\text{Se}_C \cup \text{Cpt}_C)$-fibrant. As a consequence of (7.7), we have the following.

7.8. **Proposition.** Let $X$ be a Segal object of $s\text{PSh}(\Theta C)$. The map $\text{Map}(T\# F[1], X) \to \text{Map}(T\# E, X) \approx T^* X[1]$ factors through $X^{\text{equiv}} \subseteq T^* X[1]$, and induces a weak equivalence $\text{Map}(T\# E, X) \to X^{\text{equiv}}$ of spaces. Thus, a Segal object $X$ is a complete Segal object if and only if $X[0] \to X^{\text{equiv}}$ is a weak equivalence of spaces.

7.9. **Remark.** In §10 we give another formulation of the completeness condition, in which the simplicial space $E$ is replaced by a smaller one $Z$, so that variants of (7.7) and (7.8) hold with $E$ replaced by $Z$. Either formulation works just as well for our purposes.

7.10. **Fully faithful maps.** If $X$ and $Y$ are Segal objects in $s\text{PSh}(\Theta C)$, we say that a map $f: X \to Y$ is **fully faithful** if for all $c \in \text{ob} C$ the square 
\[ \begin{array}{c}
X[1](c) \to Y[1](c) \\
\downarrow \quad \downarrow \\
X[0] \times X[0] \to Y[0] \times Y[0]
\end{array} \]
is a homotopy pullback square.

7.11. **Proposition.** Let $f: X \to Y$ be a map between Segal objects in $s\text{PSh}(\Theta C)$. The following are equivalent.

1. $f$ is fully faithful.
2. For all $c \in \text{ob} C$ and all $x_0, x_1$ points of $X[0]$, the map $M_X(x_0, x_1)(c) \to M_X(f x_0, f x_1)(c)$ induced by $f$ is a weak equivalence of spaces.
3. The induced map $hX \to hY$ of enriched homotopy categories is fully faithful, i.e., $hX(x_0, x_1) \to hY(x_0, x_1)$ is an isomorphism in $h\text{PSh}(C)$ for all points $x_0, x_1$ of $X[0]$.

7.12. **Proposition.** Suppose $X$ is a Segal object in $s\text{PSh}(C)$. Then the map $X \approx X^{T\# F[0]} \to X^{T\# F[1]}$ induced by $T\# F^{\delta_{00}}$ is fully faithful.

**Proof.** Observe that $T\# F[1] \approx V[1](1)$, and that the statement will be proved if we can show that for all $c \in \text{ob} C$, the square obtained by applying $\text{Map}_{\Theta C}(\cdot, X)$ to the square 
\[ \begin{array}{c}
V[1](Fc) \to V[1](Fc) \times V[1](1) \\
V[1](\emptyset) \to V[1](\emptyset) \times V[1](1)
\end{array} \]

\[ \begin{array}{c}
\text{proj} \quad \text{proj} \\
V[1](incl) \quad V[1](incl) \times \text{id}
\end{array} \]
is a homotopy pullback of spaces. Using the product decomposition \([4,9]\), we obtain a diagram

\[
\begin{align*}
V[2](F_c, 1) & \xleftarrow{V_δ^{012}} V[1](F_c \times 1) & \xrightarrow{V_δ^{012}} V[2](1, F_c) \\
V[2](∅, 1) & \xleftarrow{V_δ^{01}} V[1](∅ \times 1) & \xrightarrow{V_δ^{01}} V[2](1, ∅) \\
V[1](∅) & \xleftarrow{V_δ^{112}} V[1](∅) & \xrightarrow{V_δ^{112}} V[1](∅)
\end{align*}
\]

in which taking colimits of rows provides the diagram

\[
V[1](F_c) \times V[1](1) \xrightarrow{V[1](incl) \times id} V[1](∅) \times V[1](1) \xrightarrow{proj} V[1](∅).
\]

The horizontal morphisms of \([7,13]\) are monomorphisms, so the colimits of the rows are in fact homotopy colimits in \(sPSh(ΘC)\). Thus, it suffices to show that \(\text{Map}_{ΘC}(V[1](F_c), X)\) maps by a weak equivalence to the homotopy limit of \(\text{Map}_{ΘC}(−, X)\) applied to the above diagram. We claim that in fact that the evident projection maps induce weak equivalences from \(\text{Map}_{ΘC}(V[1](F_c), X)\) to the inverse limits of each of the columns of \(\text{Map}_{ΘC}(−, X)\) applied to the diagram.

This is clear for the middle column: the map \(V[1](∅ \times 1) \to V[1](∅)\) is an isomorphism, so the colimit of the middle column is isomorphic to \(V[1](F_c)\). We will show the proof for the left-hand column, leaving the right-hand column for the reader. Consider the diagram

\[
\begin{align*}
V[1](F_c) & \xrightarrow{V_δ^{01}} V[2](F_c, 1) & \xrightarrow{V_δ^{011}} V[1](F_c) \\
V[2](∅, 1) & \xrightarrow{V_δ^{12}} V[1](∅) & \xrightarrow{V_δ^{12}} V[1](∅) \\
V[0] & \xrightarrow{V_δ^{12}} V[1](1) & \xrightarrow{V_δ^{12}} V[0]
\end{align*}
\]

We want to show that \(\text{Map}_{ΘC}(−, X)\) carries the upper-right square to a homotopy pullback. The lower-right square is a pushout square (use the isomorphism \(V[2](∅, 1) \approx V[0] \amalg V[1](1))\), as is the outer square; thus they are homotopy pushouts (of spaces) since the vertical maps are monomorphisms. Applying \(\text{Map}_{ΘC}(−, X)\) to the diagram takes these two squares to homotopy pullbacks of spaces; this operation also takes the left-hand rectangle to a homotopy pullback of spaces, since \(X\) is a Segal object. Thus we can conclude that \(\text{Map}_{ΘC}(−, X)\) carries the upper-right square to a homotopy pullback of spaces, as desired. □

Say that a map \(f: X \to Y\) of spaces is a **homotopy monomorphism** if it is injective on \(π_0\), and induces a weak equivalence between each path component of \(X\) and the corresponding path component of \(Y\). Say a map \(f: X \to Y\) of objects of \(sPSh(C)\) is a homotopy monomorphism if each \(f(c): X(c) \to Y(c)\) is a homotopy monomorphism of spaces.
7.14. **Lemma.** If $X$ is a Segal object in $sPSh(\Theta C)$, then $X^{T_# i}: X^{T_# E} \to X^{T_# F[1]}$ is a homotopy monomorphism in $sPSh(\Theta C)$.

**Proof.** For $\theta \in \text{ob}\Theta C$, the map $X^{T_# E}(\theta) \to X^{T_# F[1]}(\theta)$ is isomorphic to
\[
\text{Map}(T_# E, X^{F\theta}) \to \text{Map}(T_# F[1], X^{F\theta}),
\]
which since $X^{F\theta}$ is a Segal object, is weakly equivalent to the map $(X^{F\theta})^{\text{equiv}} \to T^*(X^{F\theta})[1]$, which is a homotopy monomorphism. $\square$

7.15. **Proposition.** Suppose $X$ is a Segal object in $sPSh(\Theta C)$. Then the map $X^{T_# F[0]} \to X^{T_# E}$ is fully faithful.

**Proof.** It is straightforward to check that if $X \xrightarrow{f} Y \xrightarrow{g} Z$ are maps of Segal objects in $sPSh(C)$ such that $gf$ is fully faithful and $g$ is a homotopy monomorphism, then $f$ is fully faithful. Apply this observation to $X \to X^{T_# E} \to X^{T_# F[1]}$, using (7.12) and (7.14). $\square$

7.16. **Essentially surjective maps.** If $X$ and $Y$ are Segal objects in $sPSh(\Theta C)$, we say that a map $f: X \to Y$ is **essentially surjective** if the induced functor $hf: hX \to hY$ on homotopy categories (7.3) is essentially surjective, i.e., if every object of $hY$ is isomorphic to an object in the image of $hf$.

7.17. **Proposition.** Suppose $X$ is a Segal object in $sPSh(\Theta C)$. Then the map $X^{T_# F[0]} \to X^{T_# E}$ is essentially surjective.

**Proof.** Observe that since $T_#$ preserves products (4.3), the map $T^*(X^{T_# F[0]}): T^*(X^{T_# F[1]}) \to T^*(X^{T_# F[1]})$ is isomorphic to the map $(T^*X)^{\eta}: (T^*X)^{F[0]} \to (T^*X)^{F}$. Thus we are reduced to the case when $C = 1$, and $X$ is a Segal space, in which case the result follows from [Rez01, Lemma 13.9]. $\square$

7.18. **Lemma.** If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are maps of Segal objects in $sPSh(\Theta C)$ such that (i) $gf$ is fully faithful and (ii) $f$ is fully faithful and essentially surjective, then $g$ is fully faithful.

**Proof.** We need to show for all $y_0, y_1$ points of $Y[0]$ that $hY(y_0, y_1) \to hZ(gy_0, gy_1)$ is an isomorphism in $hsPSh(C)$. Since $f$ is essentially surjective, we may choose points $x_0, x_1$ in $X[0]$ so that $f.x_i \approx y_i$, $i = 0, 1$, as objects of $hY$. $\square$

7.19. **Proposition.** If $X$ is a Segal object in $sPSh(C)$, then the map $X^{T_# i}: X^{T_# E} \to X^{T_# F[1]}$ is fully faithful.

**Proof.** Apply (7.18) to $X \to X^{T_# E} \to X^{T_# F[1]}$, using (7.12), (7.15), and (7.17). $\square$

7.20. $(\Theta C, \text{Sec} \cup \text{Cpt}_C)$ is a cartesian presentation.

7.21. **Proposition.** $(\Theta C, \text{Sec} \cup \text{Cpt}_C)$ is a cartesian presentation.

**Proof.** By (6.3), it suffices to show that if $X$ is a complete Segal object, then $X^{F[1](c)}$ is Cpt$_C$-local for all $c \in \text{ob}C$. That is, we must show that $X^{F[1](c)} \approx \text{Map}_{\Theta C}(T_# F[0], X^{F[1](c)}) \to \text{Map}_{\Theta C}(T_# E, X^{F[1](c)})$ is a weak equivalence of spaces, or equivalently that
\[
(X^{T_# F[0]})[1](c) \to (X^{T_# E})[1](c)
\]
is a weak equivalence of spaces. This is immediate from the fact that $X^{T \# F[0]} \to X^{T \# E}$ is fully faithful (7.15) and the fact that $X[0] \approx X^{T \# F[0][0]} \to X^{T \# E[0]} \approx X^{\text{equiv}}$ is a weak equivalence, since $X$ is a complete Segal object. □

8. The presentation $(\Theta C, S_\Theta)$

In this section, we consider what happens when we start with a presentation $(C, S)$. In this case, we define a new presentation $(\Theta C, S_\Theta)$ which depends on $(C, S)$, by

$$S_\Theta \overset{\text{def}}{=} S_C \cup \text{Cpt}_C \cup V[1](S),$$

where $V[1](S) \overset{\text{def}}{=} \{ V[1](f) \mid f \in S \}$, and where $S_C$, and Cpt$_C$ are as defined in §5.1 and §7.6.

Say that two model categories $M_1$ and $M_2$ are equivalent if there is an equivalence $E: M_1 \to M_2$ of categories which preserves and reflects cofibrations, fibrations, and weak equivalences (this is much stronger than Quillen equivalence). If $M$ is a model category equivalent to one of the form $sPSh(C)_S$ for some presentation $(C, S)$, then we write

$$M-\Theta Sp \overset{\text{def}}{=} sPSh(\Theta C)_S.$$

We call $M-\Theta Sp$ the model category of $\Theta$-spaces over $M$.

In the rest of this section, we prove the following result, which is the the precise form of (1.4).

8.1. **Theorem.** If $(C, S)$ is a cartesian presentation, then $(\Theta C, S_\Theta)$ is a cartesian presentation, so that $sPSh(\Theta C)_S$ is a cartesian model category.

8.2. **$V[1](S)$-fibrant objects.** The $V[1](S)$-fibrant objects are precisely the injective fibrant objects whose mapping spaces are $S$-fibrant. Explicitly, we have the following.

8.3. **Proposition.** An injectively fibrant object $X$ in $sPSh(\Theta C)$ is $V[1]S$-fibrant if and only if for each $x_1, x_2 \in X[0]$, the object $M_X(x_1, x_2)$ is an $S$-fibrant object of $sPSh(C)$.

8.4. **Proof of (8.1).** It is clear that (8.1) follows from (6.8), (7.21), and the following.

8.5. **Proposition.** If $(C, S)$ is a cartesian presentation, then $(\Theta C, S_C \cup V[1](S))$ is a cartesian presentation.

In the remainder of the section we prove this proposition (8.5).

In light of (8.3) and (6.8), it is enough to show that if $(C, S)$ is a cartesian presentation and $X$ is $S_C \cup V[1](S)$-fibrant, and if $Y = X^{F[1](d)}$ for some $d \in \text{ob}C$, then $M_Y(g_0, g_1)$ is an $S$-fibrant object of $sPSh(C)$, for all points $g_0, g_1$ in $X^{F[1](d)[0]}$.

Let $c$ and $d$ be objects in $C$, and consider the following diagram in $sPSh(\Theta C)$.

\[
\begin{array}{ccc}
V[2](Fc, Fd) & \xleftarrow{\delta^2} & V[1](Fc \times Fd) & \xrightarrow{\delta^2} & V[2](Fd, Fc) \\
\downarrow & & \downarrow & & \downarrow \\
V[2](\emptyset, Fd) & \xleftarrow{\delta^2} & V[1](\emptyset) & \xrightarrow{\delta^2} & V[2](Fd, \emptyset)
\end{array}
\]
By (4.9), taking colimits along the rows gives the map
\[ f : V[1](Fd) \amalg V[1](Fd) \approx V[1](\varnothing) \times V[1](Fd) \to V[1](Fc) \times V[1](Fd) \]
induced by \( \varnothing \to Fc \). (Recall that \( V[1](\varnothing) \approx \{1\} \amalg \{1\} \).)

Now \( \text{Map}_{Gpd}(f, X) \) is isomorphic to the map
\[ (Y\delta^0, Y\delta^1) : Y[1](c) \to Y[0] \times Y[0] \]
whose fiber over \((g_0, g_1)\) is \( M_Y(g_0, g_1) \).

Applying \( \text{Map}_{sPSh(\Theta C)}(-, X) \) to the diagram (8.6) gives
\[ (8.7) \]
\[ X[2](c, d) \quad \xrightarrow{\text{Map}(V[1](Fc \times Fd), X)} \quad X[2](d, c) \]
\[ X[0] \times X[1](d) \quad \xrightarrow{X[0] \times X[0]} \quad X[1](d) \times X[0] \]
The space \( M_Y(g_0, g_1) \) is the pullback of the diagram obtained by taking fibers of each of the vertical maps of (8.7), over points \((x_{00}, g_0), (x_{01}, x_{11})\), and \((g_0, x_{11})\) respectively, where \( x_{ij} = (X\delta^j)(g_i) \). The vertical maps of (8.7) are fibrations of spaces, and thus the pullback of fibers is a homotopy pullback. Thus, it suffices to show that the fiber of each of the vertical maps, viewed as a functor of \( c \), is an S-fibrant object of \( sPSh(C) \).

We claim that these fibers, as presheaves on \( C \), are weakly equivalent to the presheaves \( M_X(x_{00}, x_{10}), (M_X(x_{00}, x_{11}))^{Fd}, \) and \( M_X(x_{01}, x_{11}) \) respectively. The objects \( M_X(x_{00}, x_{10}), M_X(x_{00}, x_{11}), \) and \( M_X(x_{01}, x_{11}) \) are S-fibrant by the hypothesis that \( X \) is \( V[1](S) \)-fibrant, since \((C, S)\) is a cartesian presentation, it follows that \((M_X(x_{00}, x_{11}))^{Fd} \) is S-fibrant. Thus, we complete the proof of the proposition by proving this claim.

The left-hand vertical arrow of (8.7) factors
\[ X[2](c, d) \quad \xrightarrow{(X\delta^{01}, X\delta^{12})} \quad X[1](c) \times X[0] \quad X[1](d) \quad \xrightarrow{(X\delta^0, X\delta^1)} \quad X[0] \times X[1](d). \]
The first map is a weak equivalence since \( X \) is \( \text{Se}_C \)-local, so it suffices to examine the fibers of the second map over \((x_{00}, g_1)\). It is straightforward to check that this fiber is isomorphic to \( M_X(x_{00}, x_{10})(c) \).

The right-hand vertical arrow of (8.7) is analysed similarly, so that its fibers are weakly equivalent to \( M_X(x_{01}, x_{11})(c) \).

For the middle vertical arrow of (8.7), (4.12) allows us to identify the fiber over \((x_{00}, x_{11})\) with
\[ \text{Map}_{sPSh(C)}(Fc \times Fd, M_X(x_{00}, x_{11})) \approx (M_X(x_{00}, x_{11}))^{Fd}(c). \]

9. Groupoid objects

Let \( \text{Gpd}_C \) be the set consisting of the morphism
\[ T_\# q : T_\# F[1] \to T_\# E. \]
We say that a Segal object is a Segal groupoid if it is \( \text{Gpd}_C \)-local; likewise, a complete Segal object is called a complete Segal groupoid if it is \( \text{Gpd}_C \)-local.

9.1. Lemma. If \( X \) is a Segal object in \( sPSh(\Theta C) \), then \( X \) is \( \text{Gpd}_C \)-local if and only if \( X^{T_\#} : X^{T_\#} E \to X^{T_\#} F[1] \) is a levelwise weak equivalence in \( sPSh(\Theta C) \).
Proof. The if part is immediate. To prove the only if part, note that for any Segal object $Y$, the map $Y^{T\#E} \to Y^{T\#F[1]}$ is fully faithful by (7.19). If $X$ is $s\text{Seg} \cup \text{Gpd}_C$-fibrant, then $X^{T\#i}[0]: X^{T\#E}[0] \to X^{T\#F[1][0]}$ is a weak equivalence of spaces, and it follows that $X^{T\#i}$ must be a levelwise weak equivalence in $s\text{PSh} (\Theta C)$. 

9.2. Proposition. The presentations $(\Theta C, s\text{Seg} \cup \text{Gpd}_C)$ and $(\Theta C, s\text{Seg} \cup \text{Cpt}_C \cup \text{Gpd}_C)$ are cartesian presentations. If $(C, S)$ is a cartesian presentation, then $(\Theta C, s\text{Seg} \cup \text{Gpd}_C \cup V[1]|S)$ and $(\Theta C, s\text{Seg} \cup \text{Cpt}_C \cup \text{Gpd}_C \cup V[1]|S)$ are cartesian presentations.

Proof. We only need to show that $(\Theta C, s\text{Seg} \cup \text{Gpd}_C)$ is a cartesian presentation; the other results follow using (7.21) and (8.1).

To show that $(\Theta C, s\text{Seg} \cup \text{Gpd}_C)$ is a cartesian presentation, we need to show (6.8) that if $X$ is $s\text{Seg} \cup \text{Gpd}_C$-fibrant, then $Y = X^{F[1]|c}$ is $\text{Gpd}_C$-local for all $c \in \text{ob} C$. The map $\text{Map}_{\Theta C}(T\#i, Y): \text{Map}_{\Theta C}(T\#E, Y) \to \text{Map}_{\Theta C}(T\#F[1], Y)$ is isomorphic to $(X^{T\#i}[1]|c): (X^{T\#E}[1]|c) \to (X^{T\#F[1]}[1]|c)$, which is a weak equivalence by (9.1).

Given a presentation $(C, S)$ with $M = s\text{PSh}(C)^{inj}_S$, let

$$\Theta_{\text{Gpd}}(C, S) = (\Theta C, S_\Theta \cup \text{Gpd}_C)$$

and

$$M\Theta_{\text{Gpd}} \overset{\text{def}}{=} s\text{PSh}(\Theta C)^{inj}_{S_\Theta \cup \text{Gpd}_C}.$$  

10. Alternate characterization of complete Segal objects

In the section we consider a characterization of the “completeness” property in the definition of a complete Segal space, which is a bit more elementary than the one given in [Rez01]. The results of this section are not needed elsewhere in this paper.

Let $E$ in $s\text{PSh}(\Delta)$ be the “discrete nerve” of the groupoid with two uniquely isomorphic objects $x$ and $y$, let $p: E \to F[0]$ denote the projection, and let $i: F[1] \to E$ denote the map which picks out the morphism from $x$ to $y$. In [Rez01, Prop. 6.2] it is shown that if $X$ is a Segal space, then $\text{Map}(i, X): \text{Map}(E, X) \to \text{Map}(F[1], X) \approx X[1]$ factors through a weak equivalence $\text{Map}(E, X) \to X^{\text{equiv}}$. From this, we see that a Segal space $X$ is complete if and only if $\text{Map}(p, X)$ is a weak equivalence [Rez01, Prop. 6.4].

The proof of [Rez01, Prop. 6.2] is long and technical. Also, the result is not entirely satisfying, because $E$ is an “infinite dimensional” object, in the sense that as a simplicial space it is constructed from infinitely many cells, which appear in all dimensions (see [Rez01, §11]). It is possible to replace $E$ with the finite subobject $E^{(k)}$ for $k \geq 3$ (see [Rez01 Prop. 11.1]), but this is also not very satisfying.

Here we prove a variant of [Rez01 Prop. 6.2] where $E$ is replaced by an object $Z$, which is a finite cell object. The idea is based on the following observation: in a category enriched over spaces, the homotopy equivalences $g: X \to Y$ are precisely those morphisms for which there exist morphisms $f, h: Y \to X$ and homotopies $\alpha: gf \sim 1_Y$ and $\beta: hg \sim 1_X$, and that for a given homotopy equivalence $g$ the “moduli space” of such data $(f, h, \alpha, \beta)$ is weakly contractible.

Define an object $Z$ in $s\text{PSh}(\Delta)$ to be the colimit of the diagram

$$F[3] \xrightarrow{(\delta^{02} \delta^{13})} F[1] \coprod F[1] \xrightarrow{\delta^{01} \delta^{00}} F[0] \coprod F[0].$$
Let $p: Z \to F[0]$ be the evident projection map, and let $i: F[1] \to Z$ be the composite of $\delta^{12}: F[1] \to F[3]$ with the quotient map $F[3] \to Z$.

10.1. **Proposition.** Let $X$ be a Segal space (i.e., an $\mathsf{S}e_1$-fibrant object of $\mathsf{sPSh}(\Delta)$). The map $\text{Map}(Z, X) \to \text{Map}(F[1], X)$ factors through $X^{\text{equiv}} \subseteq X[1]$, and induces a weak equivalence $\text{Map}(Z, X) \to X^{\text{equiv}}$ of spaces.

Thus, a Segal space $X$ is a complete Segal space if and only if the square

$$
\begin{array}{ccc}
X[0] & \xrightarrow{X^{\delta^{000}}} & X[3] \\
\downarrow^{(X^{\delta^0}, X^{\delta^0})} & & \downarrow^{(X^{\delta^{02}}, X^{\delta^{13}})} \\
X[0] \times X[0] & \xrightarrow{X^{\delta^{00}} \times X^{\delta^{00}}} & X[1] \times X[1] \\
\end{array}
$$

is a homotopy pullback.

**Proof.** Consider the following commutative diagram

$$
\begin{array}{ccc}
P & \xrightarrow{d} & T & \xrightarrow{e} & X^{\text{equiv}} \\
b \downarrow & & c \downarrow & & j \downarrow \\
Q & \xrightarrow{a} & X[3] & \xrightarrow{X^{\delta^{12}}} & X[1] \\
\downarrow^{(X^{\delta^{02}}, X^{\delta^{13}})} & & \downarrow^{X^{\delta^{1}} \times X^{\delta^{0}}} & & \downarrow^{(X^{\delta^{1}}, X^{\delta^{0}})} \\
X[0] \times X[0] & \xrightarrow{X^{\delta^{00}} \times X^{\delta^{00}}} & X[1] \times X[1] & \xrightarrow{X^{\delta^{0}} \times X^{\delta^{0}}} & X[0] \times X[0] \\
\end{array}
$$

Here, the objects $Q$, $T$, and $P$ are defined to be the pullbacks of the lower left, upper right, and upper left squares respectively; each of these squares is a homotopy pullback of spaces, since $(X^{\delta^{02}}, X^{\delta^{13}})$ and $j$ are fibrations. (The lower right square is in general not a pullback or a homotopy pullback.) The maps $b$, $c$, and $j$ are homotopy monomorphisms. Observe that $Q \approx \text{Map}(Z, X)$, and so we want to prove that $(X^{\delta^{12}})a$ factors through a weak equivalence $k: Q \to X^{\text{equiv}}$.

The result will follow by showing (i) that the horizontal map $(X^{\delta^{12}})a: Q \to X_1$ factors through the inclusion $j: X^{\text{equiv}} \to X_1$ by a map $k: Q \to X^{\text{equiv}}$ (and thus $b: P \to Q$ is a weak equivalence), and (ii) that the right hand rectangle is a homotopy pullback, i.e., that $T \approx \text{holim}(X_h \to X_0 \times X_0 \leftarrow X_1 \times X_1)$. Condition (ii) implies that $ed: P \to X^{\text{equiv}}$ is a weak equivalence, since it is a homotopy pullback of the identity map of $X_0 \times X_0$. Since $fb = ed$, it follows that $k$ is a weak equivalence, as desired.

The proof of (i) is straightforward. If $H$ is a point in $Q$, let $g \overset{\text{def}}{=} ((X^{\delta^{12}})a)(H)$ in $X[1]$. Then by construction the class $[g]$ in the homotopy category $hX$ admits both a left and a right inverse, and thus $g$ is a point of $X^{\text{equiv}}$. (See the discussion in [Rez01] §5.5.)

To prove (ii), let

$$
T' = \text{lim}(X^{\text{equiv}} (X^{\delta^{1}} \times X^{\delta^{0}})j) \xrightarrow{X^{\delta^{1}} \times X^{\delta^{0}}} X[1] \times X[1].
$$

Since $X^{\delta^{1}}$ and $X^{\delta^{0}}$ are fibrations, this is a homotopy pullback. We need to show that $t: T \to T'$ is a weak equivalence. Let $\pi': X[1] \times X[1] \to X[0] \times X[0] \times X[0] \times X[0]$, be the
map defined by
\[ \pi'(u, v) \overset{\text{def}}{=} ((X\delta^0)u, (X\delta^0)v, (X\delta^1)u, (X\delta^1)v). \]
Let \( \pi: T' \to (X[0])^4 \) be the composite of \( \pi' \) with the tautological map \( T' \to X[1] \times X[1]. \)
Note that both \( \pi \) and \( \pi t \) are fibrations of spaces.

Let \( x = (x_0, x_1, x_2, x_3) \) be a tuple of points in \( X[0] \). The fiber of \( \pi \) over \( x \) is the space
\[ T_x \overset{\text{def}}{=} M_X(x_0, x_2) \times M_x^{\text{equiv}}(x_1, x_2) \times M_X(x_1, x_3). \]
The fiber of \( \pi t \) over \( x \) is the limit
\[ T_x \overset{\text{def}}{=} \lim\left(M_X(x_0, x_1, x_2, x_3) \to M_X(x_1, x_2) \leftarrow M_X^{\text{equiv}}(x_1, x_2) \right). \]
To prove the proposition, we need to show that for all \( x \) the map \( t_x^*: T_x \to T'_x \) induced by \( t \) is a weak equivalence.

Given a point \( f \) in \( M_X(x_0, x_1) \), we write \( M_X(x_0, x_1)_f \) for the path component of \( M_X(x_0, x_1) \) containing \( f \). Given sequence of points \( f_i \) in \( M_X(x_{i-1}, x_i) \), we write \( M_X(x_0, \ldots, x_n)_{f_1, \ldots, f_n} \) for the path component of \( M_X(x_0, \ldots, x_n) \) which projects to \( M_X(x_0, x_1)_{f_1} \times \cdots \times M_X(x_{n-1}, x_n)_{f_n} \) under the Segal map. We claim that if \( f \in M_X(x_0, x_1), g \in M_X^{\text{equiv}}(x_1, x_2), \) and \( h \in M_X(x_2, x_3) \), then the maps
\[ \zeta : M_X(x_0, x_1, x_2, x_3)_{f, g, h} \to M_X(x_0, x_1, x_2)_{f, g} \times M_X(x_1, x_3)_{h, g} \]
and
\[ \eta : M_X(x_0, x_1, x_2)_{f, g} \times M_X(x_1, x_3)_{h, g} \to M_X(x_0, x_2)_{g, f} \times M_X(x_1, x_2)_{g} \times M_X(x_1, x_3)_{h, g} \]
are weak equivalences. This is a straightforward calculation, using the ideas of [Rez01, Prop. 11.6]. The map \( t_x^* \) is the disjoint union of maps \( \eta \zeta \) over the appropriate path components, and thus the proposition is proved.

11. \((n+k, n)\)-\(\Theta\)-spaces

In this section, we do three things. First, we make precise the “informal description” of \((n+k, n)\)-\(\Theta\)-spaces given in \ref{11.2}. Next, we identify the “discrete” \((\infty, n)\)-\(\Theta\)-spaces \ref{11.24}. Finally, we show that “groupoids” in \((n+k, n)\)-\(\Theta\)-spaces are essentially the same as \((n+k)\)-truncated spaces \ref{11.27}, thus proving the “homotopy hypothesis” for these models.

11.1. Functor associated to a presheaf on \(\Theta\). Given an object \( X \) of \( s\text{PSh}(\Theta_n) \), let \( \tilde{X} : s\text{PSh}(\Theta_n)^{\text{op}} \to \text{Sp} \) denote the functor defined by
\[ \tilde{X}(K) \overset{\text{def}}{=} \text{Map}_{\Theta_n}(K, X). \]
The construction \( X \mapsto \tilde{X} \) is the Yoneda embedding of \( s\text{PSh}(\Theta_n) \) into the category of \(\text{Sp}\)-enriched functors \( s\text{PSh}(\Theta_n)^{\text{op}} \to \text{Sp} \). The object \( X \) is recovered from the functor \( \tilde{X} \) by the formula \( X(\theta) \approx \tilde{X}(F\theta) \).
11.2. **The discrete nerve.** Given a strict $n$-category $C$, we define the discrete nerve of $C$ to be the presheaf of sets $\text{dnerve} C$ on $\Theta_n$ defined by

$$(\text{dnerve} C)(\theta) = \text{St-n-Cat}(\tau_n \theta, C).$$

Since we can regard presheaves of sets as a full subcategory of discrete presheaves of simplicial sets, we will regard $\text{dnerve}$ as a functor $\text{dnerve} : \text{St-n-Cat} \to s\text{PSh}(\Theta_n)$. This functor is fully faithful. Finally, note that there is a natural isomorphism $F \cong \text{dnerve} \tau$, where $\tau_n : \Theta_n \to \text{St-n-Cat}$ is the inclusion functor of (3.10), and $F : \Theta_n \to s\text{PSh}(\Theta_n)$ is the Yoneda embedding of $\Theta_n$.

11.3. **The suspension and inclusion functors.** For all $n \geq 1$ there is a suspension functor

$$\sigma : \Theta_{n-1} \to \Theta_n$$

defined on objects by $\sigma(\theta) \overset{\text{def}}{=} [1](\theta)$. Composing suspension functors gives functors $\sigma^k : \Theta_{n-k} \to \Theta_n$ for $0 \leq k \leq n$.

For all $n \geq 1$ there is an inclusion functor

$$\tau : \Theta_{n-1} \to \Theta_n$$

which is the restriction of the standard inclusion $\text{St-n-1-Cat} \to \text{St-n-Cat}$ to $\Theta_{n-1}$. Composing inclusion functors gives functors $\tau^k : \Theta_{n-k} \to \Theta_n$ for $0 \leq k \leq n$.

11.4. **The category $\Theta_n\text{Sp}_k$.** For $0 \leq n < \infty$, let $\mathcal{T}_{n,\infty}$ be the set of morphisms in $s\text{PSh}(\Theta_n)$ defined by

$$\mathcal{T}_{0,\infty} = \emptyset,$$

$$\mathcal{T}_{n,\infty} = \text{See}_{n-1} \cup \text{Cpt}_{n-1} \cup V[1](\mathcal{T}_{n-1,\infty}) \text{ for } n > 0.$$

If also given $-2 \leq k < \infty$, let $\mathcal{T}_{n,k}$ be the set of morphisms in $s\text{PSh}(\Theta_n)$ defined by

$$\mathcal{T}_{0,k} = \{\partial \Delta^{k+2} \to \Delta^{k+2}\},$$

$$\mathcal{T}_{n,k} = \text{See}_{n-1} \cup \text{Cpt}_{n-1} \cup V[1](\mathcal{T}_{n-1,k}) \text{ for } n > 0.$$

In the notation of §12, $\mathcal{T}_{n,k} = (\mathcal{T}_{n-1,k})_\Theta$ for $n > 0$.

11.5. **Proposition.** For all $0 \leq n < \infty$ and $-2 \leq k \leq \infty$, the presentation $(\Theta_n, \mathcal{T}_{n,k})$ is cartesian.

Proof. Immediate from (5.1). \qed

Let $\Theta_n\text{Sp}_k \overset{\text{def}}{=} s\text{PSh}(\Theta_n)_{\mathcal{T}_{n,k}}$; we call this the $(n + k, n)$-$\Theta$-space model category. We show that the fibrant objects of $\Theta_n\text{Sp}_k$ are precisely the $(n + k, n)$-$\Theta$-spaces described in §12.
11.6. **The structure of the sets** $T_{n,k}$. For $n \geq 0$ and $-2 \leq k < \infty$, we have

$$T_{n,\infty} = T^\text{Se}_n \cup T^\text{Cpt}_n$$

and

$$T_{n,k} = T_{n,\infty} \cup \{(V[1])^n(\partial \Delta^{k+2} \rightarrow \Delta^{k+2})\},$$

where

$$T^\text{Se}_n = \{(V[1])^k(se^{\theta_1,\ldots,\theta_r}) \mid 0 \leq k < n, \ r \geq 2, \ \theta_1, \ldots, \theta_r \in \text{ob}\Theta_{n-k}\}$$

and

$$T^\text{Cpt}_n = \{(V[1])^k(T[p]) \mid 0 \leq k < n\}.$$ 

11.7. **Proposition.** For $\theta_1, \ldots, \theta_r \in \text{ob}\Theta_{n-k}$, the map $(V[1])^k(se^{\theta_1,\ldots,\theta_r})$ is isomorphic to the map

$$\text{colim}(F\sigma^k[1](\theta_1) \leftarrow F\sigma^k[0] \rightarrow \cdots \leftarrow F\sigma^k[0] \rightarrow F\sigma^k[1](\theta_1) \rightarrow F\sigma^k[r]\theta_1, \ldots, \theta_r),$$

induced by applying $F\sigma^k$ to the maps $\delta^{i,1,i}:[1](\theta_i) \rightarrow [r](\theta_1, \ldots, \theta_r)$.

**Proof.** Immediate using (11.10) and (4.5). \qed

11.8. **The objects** $O_k$ and $\partial O_k$. Fix $n \geq 0$. We write $O_k$ for the discrete nerve of the free-standing $k$-cell in $\textbf{St}_{n\text{-Cat}}$. It follows that $O_k \approx F\sigma^k[0] \approx F[1][([1]([\cdots ([1])(0)])],$ where $\sigma^k: \Theta_{n-k} \rightarrow \Theta_n$. Note that our usage of $O_k$ here is slightly different than that described in the introduction, where $O_k$ was used to mean the object of $\Theta_n$, rather than the object of $\text{sPSh}(\Theta_n)$.

If $k > 0$, then the free-standing $k$-cell in $\textbf{St}_{n\text{-Cat}}$ is a $k$-morphism between two parallel $(k-1)$-cells. Let $s_k, t_k: O_{k-1} \rightarrow O_k$ denote the map between discrete nerves induced by the inclusion of the two parallel $(k-1)$-cells. Equivalently, $s_k$ and $t_k$ are the maps obtained by applying $\sigma^k$ to the maps $\delta^0, \delta^1: [0] \rightarrow [1]$ of $\Theta_{n-k}$.

Let $\partial O_k$ denote the maximal proper subobject of $O_k$: that is, $\partial O_k \subset O_k$ is the largest sub-$\Theta_n$-presheaf of $O_k$ which does not contain the “tautological section” $\iota \in O_k(\sigma^k[0])$. Let $e_k: \partial O_k \rightarrow O_k$ denote the inclusion.

11.9. **Proposition.** For $1 \leq k \leq n$, the map

$$\text{colim}(O_{k-1} \leftarrow e_{k-1} \partial O_{k-2} \Rightarrow e_{k-1} O_{k-1}) \rightarrow \partial O_k$$

defined by $s_k, t_k: O_{k-1} \rightarrow O_k$ is an isomorphism in $\text{sPSh}(\Theta_n)$.

By abuse of notation, we write $s_k, t_k: O_{k-1} \rightarrow \partial O_k$ for the inclusion of the two copies of $O_{k-1}$.

It is clear that $\partial O_k$ is isomorphic to the discrete nerve of the “free-standing pair of parallel $(k-1)$-cells”. Observe that $\partial O_0 = \varnothing$.

11.10. **Lemma.** For $\theta \in \text{ob}\Theta_{n-1}$, the object $V[1](F\theta) \approx F(\{1\}(\theta)) \approx F\sigma(\theta)$ as objects of $\text{sPSh}(\Theta_n)$.

**Proof.** A straightforward calculation. \qed
11.11. Proposition. For $1 \leq k < n$, the functor $V[1] : sPSh(\Theta_{n-1}) \to sPSh(\Theta_n)$ carries the diagram

$$s_k, t_k : O_{k-1} \Rightarrow O_k \leftarrow \partial O_k : e_k$$

up to isomorphism to the diagram

$$s_{k+1}, t_{k+1} : O_k \Rightarrow O_{k+1} \leftarrow \partial O_{k+1} : e_{k+1}.$$ 

Furthermore, $V[1](\varnothing) = V[1](\partial O_0) \approx \partial O_1$.

Proof. Again, a straightforward calculation using (11.10) and (4.5). □

11.12. Mapping objects between pairs of parallel $(k - 1)$-cells. Let $X$ be a $\mathcal{T}^\text{Se}_{n-1}$-fibrant object in $sPSh(\Theta_n)$. We call the space $X(O_k) = \text{Map}_{\Theta_n}(O_k, X)$ the moduli space of $k$-cells of $X$. We call the space $\tilde{X}(\partial O_k) = \text{Map}_{\Theta_n}(\partial O_k, X)$ the moduli space of pairs of parallel $(k - 1)$-cells. (These are the spaces denoted $X(O_k)$ and $X(\partial O_k)$ in the introduction.)

Observe that the maps $s_k, t_k : O_{k-1} \to \partial O_k$ determine an isomorphism

$$\tilde{X}(\partial O_k) \sim \tilde{X}(O_{k-1}) \times_{\tilde{X}(\partial O_{k-1})} \tilde{X}(O_{k-1}).$$

In particular, $\tilde{X}(\partial O_k) \to \tilde{X}(O_{k-1}) \times \tilde{X}(O_{k-1})$ is a monomorphism, so that a point of $\tilde{X}(\partial O_k)$ can be named by a suitable pair of points in $\tilde{X}(O_{k-1}).$

Suppose $1 \leq k \leq n$, and suppose given $(f_0, f_1) \in \tilde{X}(\partial O_k)$. We write $\map_X(f_0, f_1)$ for the object of $sPSh(\Theta_{n-k})$ defined by

$$\map_X(f_0, f_1)(\theta) \overset{\text{def}}{=} \lim (\tilde{X}(V[1]^k(F\theta))) \to \tilde{X}(V[1]^k(\varnothing)) \approx \tilde{X}(\partial O_k) \leftarrow \{(f_0, f_1)\}.$$ 

Observe that these objects can be obtained by iterating the mapping object construction of (11.11). In particular, if $(x_0, x_1) \in X[0] \times X[0] \approx \tilde{X}(\partial O_1)$, then $\map_X(x_0, x_1) \approx M_X(x_0, x_1)$ as objects of $sPSh(\Theta_{n-1}).$

11.13. Lemma. If $X$ is a $\mathcal{T}^\text{Se}_{n-1}$-fibrant object of $sPSh(\Theta_n)$, then $\map_X(f_0, f_1)$ is a $\mathcal{T}^\text{Se}_{n-k}$-fibrant object of $sPSh(\Theta_{n-k}).$

Proof. Immediate from the fact that $\mathcal{T}^\text{Se}_{n-1} \supseteq V[1]^k(\mathcal{T}^\text{Se}_{n-k}).$ □

11.14. The moduli space $X(O_k)^\text{equiv}$ of k-equivalences. Let $X$ be a $\mathcal{T}^\text{Se}_{n-1}$-fibrant object of $sPSh(\Theta_n)$, and suppose $1 \leq k \leq n$. Given a $k$-cell in $X$, i.e., a point $g$ in $\tilde{X}(O_k)$, let

$$b_0 = (\tilde{X}s_k)(g) \quad \text{and} \quad b_1 = (\tilde{X}t_k)(g)$$

be the “source” and “target” $(k - 1)$-cells of $g$, and let

$$a_0 = (\tilde{X}s_{k-1})b_0 = (\tilde{X}s_{k-1})b_1 \quad \text{and} \quad a_1 = (\tilde{X}t_{k-1})b_0 = (\tilde{X}t_{k-1})b_1$$

be the “source” and “target” $(k - 2)$-cells of $b_0$ and $b_1$. Let $Y = \map_X(a_0, a_1)$ as an object of $sPSh(\Theta_{n-k-1})$; by (11.13) the presheaf $Y$ is $\mathcal{T}^\text{Se}_{n-k+1}$-fibrant. Then $b_0$ and $b_1$ are “objects” of $Y$, (that is, points in $Y[0] = Y(O_0)$), and $g$ is a “1-cell” of $Y$, (that is, a point of $Y(O_1)$). Recall (7.3) that $g$ thus represents an element $[g]$ of the homotopy category $hY$ of $Y$.

Say that a $k$-cell $g$ of $X$ is a $k$-equivalence if it represents an isomorphism in the homotopy category $hY$ of $Y = \map_X(a_0, a_1)$. Let $X(O_k)^\text{equiv} \subseteq X(O_k)$ denote the union of path components of $\tilde{X}(O_k)$ which contain $k$-equivalences.
11.15. **Characterization of $\mathcal{T}^\text{Se}_n \cup \mathcal{T}^\text{Cpt}_n$-fibrant objects.** Recall the map $i: F[1] \to E$ of \[.\]

11.16. **Proposition.** For all $1 \leq k \leq n$, the map

$$\tilde{X}(V[1]^{k-1}(T\#i)) : \tilde{X}(V[1]^{k-1}(T\#E)) \to \tilde{X}(V[1]^{k-1}(T\#F[1])) \approx \tilde{X}(O_k)$$

factors through the subspace $\tilde{X}(O_k)^{\text{equiv}} \subseteq \tilde{X}(O_k)$ and induces a weak equivalence $\tilde{X}(V[1]^{k-1}(T\#E)) \to \tilde{X}(O_k)^{\text{equiv}}$.

**Proof.** Let $(a_0, a_1)$ be a point in $\tilde{X}(\partial O_{k-1})$ and let $Y = \text{map}_{\mathcal{T}^\text{Se}_n}(a_0, a_1)$. Since $Y$ is $\mathcal{T}^\text{Se}_{n-k}$-fibrant, it is in particular $\text{Se}_{\Theta_{n-k-1}}$-fibrant, and thus the map

$$\text{Map}_{\Theta_{n-k}}(T\#i, Y) : \text{Map}_{\Theta_{n-k}}(T\#E, Y) \to \text{Map}_{\Theta_{n-k}}(T\#F[1], Y) \approx T^* Y[1]$$

factors through $Y^{\text{equiv}} \subseteq T^* Y[1]$, and induces a weak equivalence $\text{Map}(T\#E, Y) \to Y^{\text{equiv}}$ of spaces $\boxed{11.16}$. Now consider the diagram

$$\begin{array}{ccc}
\tilde{X}(V[1]^{k-1}(T\#E)) & \xrightarrow{\tilde{X}(V[1]^{k-1}(T\#i))} & \tilde{X}(V[1]^{k-1}(T\#F[1])) \\
\downarrow & & \downarrow \\
\tilde{X}(V[1]^{k-1}(\emptyset)) & & \\
\end{array}$$

Over $(a_0, a_1) \in \tilde{X}(\partial O_{k-1}) \approx \tilde{X}(V[1]^{k-1}(\emptyset))$, the map induced by $\tilde{X}(V[1]^{k-1}(T\#i))$ on fibers is isomorphic to the map $\text{Map}_{\Theta_{n-k}}(T\#i, Y)$, and the result follows. \[.\]

11.17. **Corollary.** Let $X$ be a $\mathcal{T}^\text{Se}_n$-fibrant object of $s\text{PSh}(\Theta_n)$. Then $X$ is $\mathcal{T}^\text{Cpt}_n$-fibrant if and only if the maps $\tilde{X}(i_k) : \tilde{X}(O_{k-1}) \to \tilde{X}(O_k)^{\text{equiv}}$ are weak equivalences for $1 \leq k \leq n$.

**Proof.** Immediate from the structure of $\mathcal{T}^\text{Cpt}_n \boxed{11.16}$. \[.\]

Thus, the $\mathcal{T}^\text{Se}_n \cup \mathcal{T}^\text{Cpt}_n$-fibrant objects of $s\text{PSh}(\Theta_n)$ are precisely the $(\infty, n)$-$\Theta$-spaces. We record the following.

11.18. **Proposition.** If $X$ is a $(\infty, n)$-$\Theta$-space, and $(f_0, f_1)$ in $\tilde{X}(\partial O_k)$ is a pair of parallel $(k-1)$-cells of $X$, then $\text{map}_\chi(f_0, f_1)$ is an $(\infty, n-k)$-$\Theta$-space.

11.19. **Characterization of $k$-truncated objects.** Let $X$ be an $(\infty, n)$-$\Theta$-space (i.e., a $\mathcal{T}_{n,\infty} = \mathcal{T}^\text{Se}_n \cup \mathcal{T}^\text{Cpt}_n$-fibrant object in $s\text{PSh}(\Theta_n)$). Let $(f_0, f_1)$ be a point in $\tilde{X}(\partial O_n)$. Then $\text{map}_\chi(f_0, f_1)$ is an object of $s\text{PSh}(\Theta_0) \approx \text{Sp}$. Furthermore, if $K$ is a space, the fiber of $\tilde{X}(V[1]^n(K)) \to \tilde{X}(V[1]^n(\emptyset)) \approx \tilde{X}(\partial O_n)$ over $(f_0, f_1)$ is naturally isomorphic to $\text{Map}(K, \text{map}_\chi(f_0, f_1))$.

11.20. **Proposition.** A $\mathcal{T}_{n,\infty}$-fibrant object $X$ of $s\text{PSh}(\Theta_n)$ is $\mathcal{T}_{n,k}$-fibrant if and only if for all $(f_0, f_1)$ in $\tilde{X}(\partial O_n)$, the space $\text{map}_\chi(f_0, f_1)$ is $k$-truncated.

**Proof.** On fibers over $(f_0, f_1)$, the map $\tilde{X}(V[1]^n(\Delta^{k+2})) \to \tilde{X}(V[1]^n(\partial \Delta^{k+2}))$ induces the map $\text{Map}(\Delta^{k+2}, \text{map}_\chi(f_0, f_1)) \to \text{Map}(\partial \Delta^{k+2}, \text{map}_\chi(f_0, f_1))$ of spaces. \[.\]
11.21. **Rigid $n$-categories.** The following proposition characterizes the discrete $\mathcal{T}^\text{Se}_n$-fibrant objects of $s\text{PSh}(\Theta_n)$.

11.22. **Proposition.** The functor $\text{dnerve}$ induces an equivalence between $\text{St}$-$n$-$\text{Cat}$ and the full subcategory of discrete $\mathcal{T}^\text{Se}_n$-fibrant objects of $s\text{PSh}(\Theta_n)$.

**Proof.** A discrete presheaf $X$ is $\Theta$-fibrant if and only if $\text{Map}(s, X) : \text{Map}(S', X) \to \text{Map}(S, X)$ is an isomorphism for all $s : S \to S'$ in $\Theta$. It is clear that if $\Theta = \mathcal{T}^\text{Se}_n$, then this condition amounts to requiring that $X$ be in the essential image of $\text{dnerve}$.

Let $C$ be a strict $n$-category. We define the following notions for cells in $C$, by downwards induction.

1. Let $g : x \to y$ be a $k$-morphism in $C$. If $1 \leq k \leq n$, we say $g$ is a $k$-equivalence if there exist $k$-cells $f, h : y \to x$ in $C$ such that $gf \sim 1_y$ and $hg \sim 1_x$. If $k = n$, we say $g$ is a $k$-equivalence if it is a $k$-isomorphism.
2. Let $f, g : x \to y$ be two parallel $k$-cells in $C$. If $0 \leq k < n$, we say that $f$ and $g$ are equivalent, and write $f \sim g$, if there exists a $(k + 1)$-equivalence $h : f \to g$. If $k = n$, we say that $f$ and $g$ are equivalent if there are equal.

11.23. **Proposition.** Let $C$ be a strict $n$-category. The following are equivalent.

1. For all $1 \leq k \leq n$, every $k$-equivalence is an identity $k$-morphism.
2. For all $1 \leq k \leq n$, every $k$-isomorphism is an identity $k$-morphism.

**Proof.** It is clear that (1) implies (2). To show that (2) implies (1), use downward induction on $k$. \qed

By a rigid $n$-category, we mean a strict $n$-category $C$ satisfying either of the equivalent conditions of (11.23).

11.24. **Proposition.** Let $C$ be a strict $n$-category. The discrete nerve $\text{dnerve} C$ is an $(\infty, n)$-$\Theta$-space (i.e., is $\mathcal{T}^\text{Se}_n \cup \mathcal{T}^\text{Cpt}_n$-fibrant) if and only if $C$ is a rigid $n$-category.

**Proof.** Let $C$ be a strict $n$-category. By (11.23) $\text{dnerve} C$ is $\mathcal{T}^\text{Se}_n$-fibrant. It will also be $\mathcal{T}^\text{Cpt}_n$-fibrant if and only if $\check{X}(O_k)_{\text{equiv}} \to \check{X}(O_k)$ is an isomorphism for all $1 \leq k \leq n$, and the result follows from the observation that $\check{X}(O_k)_{\text{equiv}}$ corresponds precisely to the set of $k$-isomorphisms in $C$. \qed

11.25. **Groupoids and the homotopy hypothesis.** For $n \geq 0$, let

\[
\mathcal{T}^\text{Gpd}_0 \overset{\text{def}}{=} \emptyset, \\
\mathcal{T}^\text{Gpd}_n \overset{\text{def}}{=} \text{Gpd}_{\Theta_{n-1}} \cup V[1](\mathcal{T}^\text{Gpd}_{n-1}),
\]

using the definition of $\text{Gpd}_C$ of [9]. Explicitly, we have

\[
\mathcal{T}^\text{Gpd}_n \approx \{ V[1]^k(T\# q) \mid 0 \leq k < n \},
\]

where $T\# q : T\# F[1] \to T\# E$ is as in [9].

Let $\Theta_n \text{Gpd}_k \overset{\text{def}}{=} s\text{PSh}(\Theta_n)_{\mathcal{T}^\text{Gpd}}^{\text{inj}}$; we call this the $(n + k, n)$-$\Theta$-groupoid model category. The fibrant objects of $\Theta_n \text{Gpd}_k$ are called $(n + k, n)$-$\Theta$-groupoids; they form a full subcategory of the category of $(n + k, n)$-$\Theta$-spaces.
11.26. Proposition. Let \( n \geq 0 \). Let \( X \) be a \((\infty, n)\)-\( \Theta \)-space. The following are equivalent.

1. The object \( X \) is \( T^\text{Gpd}_n \)-fibrant.

2. For all \( 0 \leq k < n \), the maps \( \bar{X}(i_k) : \bar{X}(O_k) \to \bar{X}(O_{k+1}) \) are weak equivalences of spaces.

3. For all \( \theta \in \text{ob}\Theta_n \), the map \( X\theta : X[0] \to X\theta \) induced by \( \rho : \theta \to [0] \) is a weak equivalence of spaces.

Proof. It is clear from the definition of \( T^\text{Gpd}_n \) that (1) and (2) are equivalent. It is immediate that (3) implies (2); it remains to show that (1) implies (3), which we will show by induction on \( n \). Note that there is nothing to prove if \( n = 0 \). Since \( T^\text{Gpd}_n \supseteq \text{Gpd}_\Theta_{n-1} \), we see that the object \( T^*X \) is a \((\infty, 1)\)-\( \Theta \)-groupoid, which is to say, a groupoid-like complete Segal space, and thus all maps \( (T^*X)^\partial^\delta : (T^*X)[0] \to (T^*X)[m] \) are weak equivalences of spaces [Rez01, Cor. 6.6]. Therefore, \( X[0] \to X\theta \) is a weak equivalence for all \( \theta = [m][0], \ldots, [0] \), \( m \geq 0 \). Now consider the diagram

\[
\begin{array}{c}
X[0] \xrightarrow{\sim} X[m](0, \ldots, [0]) \xrightarrow{\sim} X[1](0)[0] \times_{X[0]} \cdots \times_{X[0]} X[m](0) \\
\downarrow a \quad \downarrow b \\
X[m](\theta_1, \ldots, \theta_N) \xrightarrow{\sim} X[1](\theta_1)[0] \times_{X[0]} \cdots \times_{X[0]} X[1](\theta_m)
\end{array}
\]

where \( \theta_1, \ldots, \theta_N \in \text{ob}\Theta_{n-1} \). To show that \( a \) is a weak equivalence, it suffices to show that \( b \) is, so it suffices to show that \( X[1](p) : X[1](0) \to X[1](\theta) \) is a weak equivalence of spaces for all \( \theta \in \Theta_{n-1} \). Consider the diagram

\[
\begin{array}{ccc}
X(V[1](F[0])) & \xrightarrow{X(V[1](F \rho))} & \bar{X}(V[1](F \rho)) \\
\downarrow \quad \downarrow & & \downarrow \\
\bar{X}(V[1](\mathcal{O})) & \xrightarrow{\bar{X}(V[1](\mathcal{O}))} & \bar{X}(V[1](\mathcal{O}))
\end{array}
\]

The map \( \bar{X}(V[1](F \rho)) \) is isomorphic to \( X[1](p) : X[1](0) \to X[1](\theta) \). Let \( (x_0, x_1) \) be a point in \( \bar{X}(\partial O_1) \approx \bar{X}(V[1](\mathcal{O})) \); the map induced on fibers over \( (x_0, x_1) \) by \( \bar{X}(V[1](F \rho)) \) is isomorphic to

\[
\text{map}_X(x_0, x_1)(p) : \text{map}_X(x_0, x_1)(0) \to \text{map}_X(x_0, x_1)(\theta).
\]

It is clear that \( \text{map}_X(x_0, x_1) \) is a \((\infty, n-1)\)-\( \Theta \)-groupoid, and thus \( \text{map}_X(x_0, x_1)(p) \) is a weak equivalence of spaces by the inductive hypothesis. □

Let \( c_* : \text{Sp} \xrightarrow{\sim} s\text{PSh}(\Theta_n) \) denote the adjoint pair where the left adjoint \( c_* \) sends a space \( X \) to the constant presheaf with value \( X \).

11.27. Proposition.

1. The adjoint pair \( c_* : \text{Sp} \xrightarrow{\sim} s\text{PSh}(\Theta_n) : \text{map}_X(x_0, x_1) \) is a Quillen equivalence.

2. For all \( -2 \leq k < \infty \) the adjoint pair \( c_* : \text{Sp}_{n+k} \xrightarrow{\sim} s\text{PSh}(\Theta_n)_{n+k} : \text{map}_X(x_0, x_1) \) is a Quillen equivalence.
Proof. We first consider (1). Observe that \(c_\#\) preserves cofibrations, and that caries all spaces to \(\mathcal{T}_{n,\infty} \cup \mathcal{T}_{n}^{\text{Gpd}}\)-fibrant objects by (11.27), and thus \(c_\#\) preserves weak equivalences. Therefore the pair is a Quillen pair, and it is a straightforward consequence of (11.27) that the natural map \(X \to c^*c_\#X\) is always weak equivalence, the natural map \(c_\#c^*Y \to Y\) is a weak equivalence for all \(\mathcal{T}_{n,\infty} \cup \mathcal{T}_{n}^{\text{Gpd}}\)-local objects \(Y\), and thus the pair is a Quillen equivalence.

The proof that we get a Quillen equivalence in (2) proceeds in the same way, once we observe that \(c_\#\) carries \(n+k\)-truncated spaces to \(\mathcal{T}_{n,k} \cup \mathcal{T}_{n}^{\text{Gpd}}\)-local objects. \(\square\)

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