TWO EQUIVALENT CONDITIONS FOR MUCKENHOUPT WEIGHTS

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Abstract. We present a new characterization of Muckenhoupt $A_\infty$-weights whose logarithm is in VMO($\mathbb{R}$) in terms of vanishing Carleson measures on $\mathbb{R}^2_+$ and vanishing doubling weights on $\mathbb{R}$. This also gives a novel description of strongly symmetric homeomorphisms on the real line (a subclass of quasisymmetric homeomorphisms introduced recently in [19]) without using quasiconformal extensions.

1. Introduction

A locally integrable non-negative measurable function $\omega$ on $\mathbb{R}$ is called a weight. We say that $\omega$ is a doubling weight if there exists a constant $\rho$ such that

$$\rho^{-1}\omega(J) \leq \omega(I) \leq \rho\omega(J)$$

for any adjacent bounded intervals $I$ and $J$ of length $|I| = |J|$. Here, $\omega(I) = \int_I \omega(x)dx$. We call the optimal value of such $\rho$ the doubling constant for $\omega$. Moreover, a doubling weight $\omega$ is called vanishing if $\omega(I)/\omega(J) \to 1$ as $|I| = |J| \to 0$. We say that $\omega$ is a Muckenhoupt $A_\infty$-weight (abbreviated to $A_\infty$-weight) (see [6, Chapter 6] for more information) if for any $\varepsilon > 0$ there exists some $\delta > 0$ such that

$$|E| \leq \delta|I| \Rightarrow \omega(E) \leq \varepsilon\omega(I)$$

whenever $I \subset \mathbb{R}$ is a bounded interval and $E \subset I$ a measurable subset. Naturally an $A_\infty$-weight is doubling. Fefferman and Muckenhoupt [5] gave this a direct computation, and they also provided an example of a function that satisfies the doubling condition but not $A_\infty$.

For a weight function $\omega$ on the real line $\mathbb{R}$, we define a sense-preserving homeomorphism $h = h_\omega : \mathbb{R} \to \mathbb{R}$ given by $h(x) = h(0) + \int_0^x \omega(t)dt$. The homeomorphism $h$ is called strongly quasisymmetric if $\omega$ is an $A_\infty$-weight. In particular, $\log \omega \in \text{BMO}(\mathbb{R})$, the space of functions of bounded mean oscillation on the real line (see section 2 for precise definition). This subclass of quasisymmetric homeomorphisms and its Teichmüller space were much investigated (see [1, 2, 4, 17, 21]) because of their great importance in the application to harmonic analysis and elliptic operator theory (see [3, 4, 11, 16]). In particular, it was proved that a sense-preserving homeomorphism $h$ is strongly quasisymmetric if and only if

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it can be extended to a quasiconformal homeomorphism of $\mathbb{R}_+^2$ onto itself whose Beltrami coefficient $\mu$ induces a Carleson measure $|\mu(z)|^2y^{-1}dxdy$ on $\mathbb{R}_+^2$. Moreover, a strongly quasisymmetric homeomorphism $h$ is said to be strongly symmetric if the $A_{\infty}$-weight $\omega$ satisfies that $\log \omega \in \text{VMO}(\mathbb{R})$, the space of functions of vanishing mean oscillation on the real line (see section 2 for precise definition). This class was first studied in [19] when Shen discussed the characterizations of VMO-Teichmüller space on the real line and its complex Banach manifold structure. Later, it was investigated further in [23, 24]. In particular, it was proved that $h$ is strongly symmetric if and only if it can be extended to a quasiconformal homeomorphism of $\mathbb{R}_+^2$ onto itself whose Beltrami coefficient $\mu$ induces a vanishing Carleson measure $|\mu(z)|^2y^{-1}dxdy$ on $\mathbb{R}_+^2$.

Here, a measure $\lambda(x,y)dxdy$ on $\mathbb{R}_+^2$ is called a Carleson measure if

$$\|\lambda\|_{c^2} = \sup_{I \subset \mathbb{R}} \frac{1}{|I|} \int_0^{|I|} \lambda(x,y)dxdy < \infty.$$  

Furthermore, if a Carleson measure $\lambda$ satisfies

$$\lim_{\delta \to 0} \sup_{|I| \leq \delta} \frac{1}{|I|} \int_0^{|I|} \lambda(x,y)dxdy = 0,$$

we call $\lambda$ a vanishing Carleson measure.

**Remark 1.** The strongly quasisymmetric homeomorphisms and its Teichmüller space are usually defined and discussed on the unit circle $\mathbb{S}$. Due to the conformal invariance of $A_{\infty}$-weights and Carleson measures, we have the same theory on the real line $\mathbb{R}$. On the other hand, a sense-preserving homeomorphism $g$ on $\mathbb{S}$ is called strongly symmetric if it is absolutely continuous such that $\log g'$ belongs to VMO($\mathbb{S}$), the space of functions of vanishing mean oscillation on $\mathbb{S}$. This class was introduced in [14] when Partyka studied eigenvalues of quasisymmetric automorphisms determined by VMO functions. It was investigated further later in [21, 22] during their study of BMO theory of the universal Teichmüller space. Since the logarithmic derivative does not have conformal invariance, the notion of strongly symmetric homeomorphism on the real line is different from the one on the unit circle. In fact, the space of strongly symmetric homeomorphisms on $\mathbb{S}$ is smaller than that on $\mathbb{R}$ in the sense of conformal automorphism (see [24]).

The purpose of the present paper is to give a new description of strongly symmetric homeomorphisms without using quasiconformal extensions (see Theorem 1 below). Before stating the theorem we need to introduce some notations. For $z = (x, y) \in \mathbb{R}_+^2$, let $I_z = \{s \mid |s - x| < y/2\}$, and let $I_z^-$ and $I_z^+$ be the left and right half of the interval $I_z$, respectively. For a doubling weight $\omega$ on $\mathbb{R}$, we introduce a geometric quantity

$$2^{-1}\frac{\omega(I_z)}{\omega(I_z^+)\omega(I_z^-)^{1/2}},$$

which is actually the ratio between the arithmetic mean and the geometric mean of the densities of $\omega$ over the intervals $I_z^+$ and $I_z^-$, that is $\omega(I_z^+) / |I_z^+|$ and $\omega(I_z^-) / |I_z^-|$, and
then define
\[
\eta(z) = \log \frac{2^{-1} \omega(I_z)}{\omega(I_{z^+})^{1/2} \omega(I_{z^-})^{1/2}}.
\]

In [8] it was shown that the doubling weight \( \omega \) is an \( A_{\infty} \)-weight if and only if \( \eta(z) y^{-1} dx \) is a Carleson measure on \( \mathbb{R}^2_+ \). This expresses the close connection between \( A_{\infty} \)-weights (or strongly quasisymmetric homeomorphisms) and Carleson measures. We consider to what extent one can extend this result to \( A_{\infty} \)-weights whose logarithm is in \( \text{VMO}(\mathbb{R}) \) (or strongly symmetric homeomorphisms) and prove the following.

**Theorem 1.** For an \( A_{\infty} \)-weight \( \omega \) on \( \mathbb{R} \), the function \( \log \omega \in \text{VMO}(\mathbb{R}) \) if and only if the Carleson measure \( \eta(z) y^{-1} dx dy \) is vanishing on \( \mathbb{R}^2_+ \) and the doubling weight \( \omega \) is vanishing on \( \mathbb{R} \).

**Remark 2.** Let \( \tilde{\eta}(z) = |1 - \omega(I_z^+)/\omega(I_z^-)|^2 \). Since the weight \( \omega \) is doubling, it is easy to compute that \( \tilde{\eta}(z) \) and \( \eta(z) \) are comparable with comparison constant depending only on doubling constant for \( \omega \). Then, the Carleson measure \( \eta(z) y^{-1} dx dy \) is vanishing on \( \mathbb{R}^2_+ \) if and only if the Carleson measure \( \tilde{\eta}(z) y^{-1} dx dy \) is. We consider \( \lambda_\delta(\omega) = \sup_{0 < y < \delta} \tilde{\eta}(z) \). Then the doubling weight \( \omega \) is vanishing on \( \mathbb{R} \) (namely, \( \eta(z) \to 0 \) uniformly for \( x \in \mathbb{R} \) as \( y \to 0 \)) if and only if \( \lim_{\delta \to 0} \lambda_\delta(\omega) = 0 \). Further, if the rate of convergence of \( \lambda_\delta(\omega) \) satisfies the following condition
\[
\int_0^1 \frac{\lambda_\delta(\omega)}{\delta} d\delta < \infty,
\]
then the Carleson measure \( \tilde{\eta}(z) y^{-1} dx dy \) is vanishing on \( \mathbb{R}^2_+ \). In fact,
\[
\int_{I(x_0, t)} \int_0^t \tilde{\eta}(z) \frac{dx dy}{y} \leq \int_{I(x_0, t)} \int_0^t \lambda_\delta(\omega) \frac{dx dy}{y} = t \int_0^t \lambda_\delta(\omega) dy,
\]
which implies that the Carleson measure \( \tilde{\eta}(z) y^{-1} dx dy \) is vanishing on \( \mathbb{R}^2_+ \) by the assumption (1).

**Remark 3.** The condition that \( \omega \) is an \( A_{\infty} \)-weight with \( \log \omega \in \text{VMO}(\mathbb{R}) \) implies that the doubling weight \( \omega \) is vanishing on \( \mathbb{R} \), which we may obtain by examining the proof of [18, Lemma 3.3] in the unit circle case.

The paper is structured as follows: in Section 2, we give some basic definitions and results on BMO functions, VMO functions and Muckenhoupt weights. Section 3 is devoted to the proof of Theorem 1. In the final Appendix section, we provide an alternative explanation on the vanishing Carleson measure in Theorem 1.

2. Preliminaries

Let \( I_0 \) be any interval on the real line \( \mathbb{R} \). A locally integrable function \( u \in L^1_{\text{loc}}(I_0) \) is said to have bounded mean oscillation (abbreviated to BMO) if
\[
\|u\|_{\text{BMO}(I_0)} = \sup_{|I|} \frac{1}{|I|} \int_I |u(x) - u_I| \, dx < \infty,
\]
where the supremum is taken over all bounded intervals \( I \) of \( I_0 \), \( u_I \) is the average of \( u \) on the interval \( I \), namely,
\[
u_I = \frac{1}{|I|} \int_I u(x) \, dx.
\]
The set of all \( \text{BMO} \) functions on \( I_0 \) is denoted by \( \text{BMO}(I_0) \). However, \( \| \cdot \|_{\text{BMO}(I_0)} \) is not a true norm since it is trivial that the \( \text{BMO} \)-norm of any constant function is 0. In fact, this is regarded as a Banach space with norm \( \| \cdot \|_{\text{BMO}(I_0)} \) modulo constants. Moreover, if \( u \) also satisfies the condition
\[
\lim_{|I| \to 0} \frac{1}{|I|} \int_I |u(x) - u_I| \, dx = 0,
\]
we say \( u \) has \textit{vanishing mean oscillation} (abbreviated to \( \text{VMO} \)). The set of all \( \text{VMO} \) functions on \( I_0 \) is denoted by \( \text{VMO}(I_0) \). This is a closed subspace of \( \text{BMO}(I_0) \). Functions of bounded mean oscillation in \( \mathbb{R}^n \) were first introduced by John and Nirenberg in [10] and they applied them to smoothness problems in partial differential equation. The \textit{John–Nirenberg inequality} for \( \text{BMO} \) functions (see [6, VI.2], [20, IV.1.3]) asserts that there exists two universal positive constants \( C_1 \) and \( C_2 \) such that for any \( \text{BMO} \) function \( u \), any bounded interval \( I \) of \( I_0 \), and any \( \lambda > 0 \), it holds that
\[
\frac{1}{|I|} \left| \left\{ t \in I : |u(t) - u_I| \geq \lambda \right\} \right| \leq C_1 \exp \left( \frac{-C_2 \lambda}{\| u \|_{\text{BMO}(I_0)}} \right). \quad (2)
\]

We say that the weight \( \omega \) is a \textit{Muckenhoupt} \( A_p \)-weight (abbreviated to \( A_p \)-weight) [13] for \( p > 1 \) if there exists a constant \( C_p(\omega) \geq 1 \) such that
\[
( \frac{1}{|I|} \int_I \omega(x) \, dx ) \left( \frac{1}{|I|} \int_I \left( \frac{1}{\omega(x)} \right)^{p-1} \, dx \right)^{\frac{1}{p-1}} \leq C_p(\omega) \quad (3)
\]
for any bounded interval \( I \subset \mathbb{R} \). We call the optimal value of such \( C_p(\omega) \) the \( A_p \)-constant for \( \omega \). It is known that \( \bigcup_{p<\infty} A_p = A_\infty \) and \( A_p \subset A_q \) for \( p < q \).

The Jensen inequality implies that
\[
\exp \left( \frac{1}{|I|} \int_I \log \omega(x) \, dx \right) \leq \frac{1}{|I|} \int_I \omega(x) \, dx. \quad (4)
\]

Another characterization of \( A_\infty \)-weights can be given by the reverse Jensen inequality. Namely, \( \omega \geq 0 \) belongs to the class of \( A_\infty \)-weights if and only if there exists a constant \( C_\infty(\omega) \geq 1 \) such that
\[
\frac{1}{|I|} \int_I \omega(x) \, dx \leq C_\infty(\omega) \exp \left( \frac{1}{|I|} \int_I \log \omega(x) \, dx \right) \quad (5)
\]
for every bounded interval \( I \subset \mathbb{R} \) (see [9]). We call the optimal value of such \( C_\infty(\omega) \) the \( A_\infty \)-constant for \( \omega \).

D. Sarason [15, Theorem 2] gave a characterization of \( \text{VMO} \) functions by means of \( A_2 \)-weights:
Proposition 2. Let $\omega$ be a weight function with $\log \omega \in BMO(\mathbb{R})$. Then, $\log \omega$ belongs to $VMO(\mathbb{R})$ if and only if

$$\lim_{\delta \to 0} \sup_{|I| < \delta} \left( \frac{1}{|I|} \int_I \omega(x) dx \right) \left( \frac{1}{|I|} \int_I \frac{1}{\omega(x)} dx \right) = 1.$$  

Here, (6) may be thought of as a limit $A_2$-condition. Inspired by Proposition 2, Mitsis [12] pushed the analogy between $A_p$-weights and $A_\infty$-weights further by replacing the limit $A_2$-condition with the following so-called asymptotic reverse Jensen inequality (see (7) below):

Proposition 3. Let $\omega$ be a weight function with $\log \omega \in BMO(\mathbb{R})$. Then, $\log \omega$ belongs to $VMO(\mathbb{R})$ if and only if

$$\lim_{\delta \to 0} \sup_{|I| < \delta} \left( \frac{1}{|I|} \int_I \omega(x) dx \right) \exp \left( - \frac{1}{|I|} \int_I \log \omega(x) dx \right) = 1.$$  

More precisely, Mitsis [12] proved sufficiency with respect to nonatomic measures (covering Lebesgue measure) by performing a dyadic decomposition of the involved interval and omitted the detailed proof of necessity by pointing out it is a standard argument involving the John-Nirenberg inequality for nonatomic measures. Proposition 3 is relevant to the proof of Theorem 1 in the following section. For the completeness of our paper, we complement the proof of necessity and give sufficiency a simple proof involving only elementary measure theoretic considerations.

Remark 4. Put it differently, Propositions 2 and 3 imply that $A_2$-condition and $A_\infty$-condition coincide if one restricts to weights which tend to be constant on arbitrarily small intervals.

Proof of Proposition 3. Let $u = \log \omega$. Suppose $u \in VMO(\mathbb{R})$. Then, for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any bounded interval $I_0 \subset \mathbb{R}$ with $|I_0| \leq \delta$ we have $\|u\|_{BMO(I_0)} < \min\{C_2\varepsilon/(2C_1), C_2/2\}$. Here, $C_1$ and $C_2$ are constants in [2]. Then, for any bounded interval $I \subset I_0$, the John-Nirenberg inequality yields that

$$\frac{1}{|I|} \int_I e^{u(x)} - u_I dx = \frac{1}{|I|} \int_0^\infty |\{x \in I : |u(x) - u_I| \geq \lambda\}| e^{\lambda} d\lambda + 1 \leq C_1 \int_0^\infty \exp \left( - \frac{C_2 \lambda}{\|u\|_{BMO(I_0)}} \right) e^{\lambda} d\lambda + 1 \leq C_1 \frac{C_1}{C_2 - \|u\|_{BMO(I_0)}} + 1 < \varepsilon + 1.$$  

By the Jensen inequality, we see

$$1 \leq \left( \frac{1}{|I|} \int_I \omega(x) dx \right) \exp \left( - \frac{1}{|I|} \int_I \log \omega(x) dx \right) \leq \left( \frac{1}{|I|} \int_I \omega(x) dx \right) \left( \frac{1}{|I|} \int_I \frac{1}{\omega(x)} dx \right).$$
and moreover, by (8) and \( ab \leq \frac{1}{4}(a + b)^2 \) for \( a, b \geq 0 \), we have

\[
\left( \frac{1}{|I|} \int_I \omega(x) dx \right) \left( \frac{1}{|I|} \int_I \frac{1}{\omega(x)} dx \right) = \left( \frac{1}{|I|} \int_I \int_I e^{u(x) - u_I} \omega(x) dx \right) \left( \frac{1}{|I|} \int_I \int_I e^{u_I - u(x)} dx \right) \\
\leq \frac{1}{4} \left( \frac{1}{|I|} \int_I \int_I e^{u(x) - u_I} dx + \frac{1}{|I|} \int_I \int_I e^{u_I - u(x)} dx \right)^2 \\
= \frac{1}{4} \left( \frac{1}{|I|} \int_I e^{\int_I (u(x) - u_I) dx} + \frac{1}{|I|} \int_I e^{-\int_I (u(x) - u_I) dx} \right)^2 \\
\leq \frac{1}{4} \left( \frac{1}{|I|} \int_I e^{\int_I (u(x) - u_I) dx + 1} \right)^2 \leq \frac{1}{4}(\varepsilon + 2)^2.
\]

Thus, (7) is satisfied. This gives the proof of necessity.

Suppose (7) holds. To show \( u = \log \omega \in \text{VMO}(\mathbb{R}) \), we use a strategy of measure theory in [15, Lemma 3]. Let \( I \) be a bounded interval in \( \mathbb{R} \) such that

\[
\left( \frac{1}{|I|} \int_I \omega(x) dx \right) \exp \left( -\frac{1}{|I|} \int_I \log \omega(x) dx \right) = 1 + \varepsilon^3 
\]

for \( 0 < \varepsilon < 1/2 \). Assume

\[
u_I = \frac{1}{|I|} \int_I \log \omega(x) dx = 0.
\]

Then, we have

\[
\frac{1}{|I|} \int_I \omega(x) dx = 1 + \varepsilon^3.
\]

Let \( F \) be the set where \( e^{-\varepsilon} < \omega < e^{\varepsilon} \) and \( E = I - F \). We have

\[
(1 + \varepsilon^3)|I| = \int_E (\omega(x) - \log \omega(x)) dx + \int_F (\omega(x) - \log \omega(x)) dx \\
\geq (e^{-\varepsilon} + \varepsilon)|E| + |F| \\
\geq (1 + \frac{1}{4}\varepsilon^2)|E| + |F| \\
= |I| + \frac{1}{4}\varepsilon^2|E|,
\]

which implies \( |E| \leq 4\varepsilon|I| \) and \( |F| \geq (1 - 4\varepsilon)|I| \). Thus,

\[
\int_E \omega(x) dx = (1 + \varepsilon^3)|I| - \int_F \omega(x) dx \\
\leq (1 + \varepsilon)|I| - e^{-\varepsilon}|F| \\
\leq (1 + \varepsilon)|I| - (1 - \varepsilon)(1 - 4\varepsilon)|I| \\
< 6\varepsilon|I|.
\]
On the other hand, by (10) we have

$$-\int_E \log \omega(x) dx = \int_F \log \omega(x) dx \leq \varepsilon |F| \leq \varepsilon |I|.$$ 

Noting that $|\log \omega| \leq \varepsilon$ on $F$, and $|\log \omega| \leq \omega - \log \omega$ generally, we conclude that

$$\int_I |\log \omega(x)| dx = \int_E |\log \omega(x)| dx + \int_F |\log \omega(x)| dx$$

$$\leq \int_E (\omega(x) - \log \omega(x)) dx + \varepsilon |F|$$

$$\leq 6\varepsilon |I| + \varepsilon |I| + \varepsilon |I| = 8\varepsilon |I|.$$ 

Combined with (10), this implies that

$$\int_I \left| \frac{1}{|I|} \int_I |u(x) - u_I| dx < 8\varepsilon \right. \quad (11)$$

for $u = \log \omega$. If $\log \omega$ does not satisfy (10), then we write $\log \omega = (\log \omega - a) + a$ with $a = \frac{1}{|I|} \int_I \log \omega(x) dx$. Since $\log \omega - a$ satisfies (9) and (10) holds with $\log \omega$ replaced by $\log \omega - a$, we conclude from (11) that

$$\int_I \left| \frac{1}{|I|} \int_I |u(x) - a| - (u - a)_I| dx < 8\varepsilon. \right.$$ 

Thus, (11) holds for every $u = \log \omega$ that satisfies (9). Consequently, (7) implies $u = \log \omega \in \text{VMO}(\mathbb{R})$. This completes the proof of Proposition 3. \hfill \Box

3. PROOF OF THEOREM 1

In this section, we focus on the proof of Theorem 1.

For any $x_0 \in \mathbb{R}$ and $t > 0$, we set $I(x_0,t) = \{x \mid |x - x_0| < t/2\}$ and set

$$A(x_0,t) = \frac{1}{t} \int_0^t \int_{I(x_0,t)} \eta(z) \frac{dx dy}{y}.$$ 

By dividing the integral by $dy$ over $[0,t]$ into those on dyadic intervals and then by changing the variables, we obtain

$$A(x_0,t) = \sum_{k=0}^{\infty} \frac{1}{t} \int_{t/2^{k+1}}^{t/2^k} \int_{I(x_0,t)} \log \frac{2^{-1} \omega(I_z)}{\omega(I_z^+)^{1/2} \omega(I_z^-)^{1/2}} \frac{dx dy}{y}$$

$$= \sum_{k=1}^{\infty} \frac{1}{t} \int_{t/2}^{t} \int_{I(x_0,t)} \log \frac{2^{-1} \omega(x - y/2^k, x + y/2^k)}{\omega(x - y/2^k, x + y/2^k)^{1/2} \omega(x, x + y/2^k)^{1/2}} \frac{dx dy}{y}. $$
We now decompose $A(x_0, t)$ into several parts for convenience of computation. By rearranging the order of the following sum:

$$\sum_{k=1}^{N} \log \frac{2^{-N} \omega(x - y/2^k, x + y/2^k)}{\omega(x - y/2^k, x) \omega(x + y/2^k)}$$

and by using further observation on the ratio of the first term:

$$\frac{2^{-N} \omega(x - y/2, x + y/2)}{\omega(x - y/2^k, x) \omega(x + y/2^k)} = \omega(x - y/2, x + y/2) / y$$

we see that $N$-th partial sum of the series $A(x_0, t)$ can be written as

$$\sum_{k=1}^{N} \frac{1}{t} \int_{t/2}^{t} \frac{1}{I(x_0, t)} \log \frac{2^{-1} \omega(x - y/2^k, x + y/2^k)}{\omega(x - y/2^k, x) \omega(x + y/2^k)} \frac{dx dy}{y}$$

$$= \frac{1}{t} \int_{t/2}^{t} \int_{I(x_0, t)} \log \frac{\omega(x - y/2, x + y/2)}{y} \frac{dx dy}{y}$$

$$- \frac{1}{2t} \left( \int_{t/2}^{t} \int_{I(x_0, t)} \log \frac{\omega(x - y/2^k, x)}{y} \frac{dx dy}{y} + \int_{t/2}^{t} \int_{I(x_0, t)} \log \frac{\omega(x + y/2^k)}{y} \frac{dx dy}{y} \right)$$

$$+ \sum_{k=1}^{N-1} \frac{1}{t} \int_{t/2}^{t} \int_{I(x_0, t)} \log \frac{\omega(x - y/2^{k+1}, x + y/2^{k+1})}{\omega(x - y/2^k, x) \omega(x + y/2^k)} \frac{dx dy}{y}$$

$$= A_1 - A_2 + A_3.$$

We remark that the above term $A_3$ involves three intervals $(x - y/2^{k+1}, x + y/2^{k+1})$, $(x - y/2^k, x)$, $(x, x + y/2^k)$ of each one having the same length. Moreover, we notice that

$$A_1 - A_2 = (A_1 - \log 2 \log \omega_{I(x_0, t)}) - (A_2 - \log 2 \log \omega_{I(x_0, t)}) + \log 2 (\log \omega_{I(x_0, t)} - (\log \omega)_{I(x_0, t)})$$

$$= \widehat{A}_1 - \widehat{A}_2 + A_4,$$

Here, $\omega_I$ is the average of $\omega$ on the interval $I$, namely, $\omega_I = \frac{1}{|I|} \int_I \omega(x)dx$.

Now we see

$$A(x_0, t) = \widehat{A}_1 - \lim_{N \to \infty} \widehat{A}_2 + \lim_{N \to \infty} A_3 + A_4.$$
Proof. Since $\omega$ is an $A_{\infty}$-weight, we conclude by the reverse Hölder inequality that there exists some $\delta > 0$ such that $\omega \in L^{1+\delta}$ locally on $\mathbb{R}$ (see [6, Corollary VI.6.10]). This in particular implies the Hardy-Littlewood maximal function for $\omega$, denoted by $M\omega$, is integrable locally on $\mathbb{R}$ (see [6, Theorem I.4.3]), so is $\log M\omega$. On the other hand, we see by the definition of $M\omega$ that
\[
\log \frac{\omega(x - y/2^N, x)}{y/2^N} \leq \log M\omega(x), \quad x \in \mathbb{R}
\]
for any integer $N$ and by the Lebesgue differentiation Theorem that
\[
\lim_{N \to \infty} \log \frac{\omega(x - y/2^N, x)}{y/2^N} = \log \omega(x)
\]
for almost every $x \in \mathbb{R}$. Then, applying the Lebesgue dominated convergence Theorem we have
\[
\lim_{N \to \infty} \int_{I(x_0,t)} \log \frac{\omega(x - y/2^N, x)}{y/2^N} \, dx = (\log \omega)_{I(x_0,t)},
\]
and similarly, we have
\[
\lim_{N \to \infty} \int_{I(x_0,t)} \log \frac{\omega(x + y/2^N)}{y/2^N} \, dx = (\log \omega)_{I(x_0,t)}.
\]
These imply that $\lim_{N \to \infty} \mathcal{A}_2 = 0$. □

Lemma 4. Let the doubling weight $\omega$ be vanishing, namely, for any $\varepsilon > 0$ there exists some $\delta > 0$ such that
\[
(1 + \varepsilon)^{-1} \leq \frac{\omega(x_0, x_0 + t)}{\omega(x_0 - t, x_0)} \leq 1 + \varepsilon
\]
for every $x_0 \in \mathbb{R}$ and for every $t \in (0, \delta)$. Then,
\[
(a) \quad \frac{(1 + \varepsilon)^n - 1}{\varepsilon(1 + \varepsilon)^{n-1}} \leq \frac{\omega(x_0 - nt, x_0 + nt)}{\omega(x_0 - t, x_0 + t)} \leq \frac{(1 + \varepsilon)^n - 1}{\varepsilon},
\]
\[
(b) \quad (1 + \varepsilon)^{-1} \leq \frac{\omega(x_1, x_1 + t)}{\omega(x_0 - t, x_0)} \leq 1 + \varepsilon
\]
are satisfied for every integer $n$ and for any $x_1 \in (x_0 - t, x_0)$, respectively.

Proof. We estimate $\omega(x_0 - nt, x_0 + nt)$ from both above and below in terms of $\omega(x_0 - t, x_0 + t)$.
\[
\omega(x_0 - nt, x_0 + nt) = \sum_{k=1}^{n} \omega(x_0 - kt, x_0 - (k-1)t) + \omega(x_0 + (k-1)t, x_0 + kt)) \leq \sum_{k=1}^{n} (1 + \varepsilon)^{k-1} (\omega(x_0 - t, x_0) + \omega(x_0, x_0 + t)) = \frac{(1 + \varepsilon)^n - 1}{\varepsilon} \omega(x_0 - t, x_0 + t),
\]
and similarly,
\[
\omega(x_0 - nt, x_0 + nt) \\
\geq \sum_{k=1}^{n} ((1 + \varepsilon)^{-1})^{k-1} \omega(x_0 - t, x_0 + t) \\
= \frac{(1 + \varepsilon)^n - 1}{\varepsilon (1 + \varepsilon)^{n-1}} \omega(x_0 - t, x_0 + t),
\]
from which we obtain the statement (a).

To prove the statement (b) we use the property of the mediant. It says assuming
\[
(1 + \varepsilon)^{-1} \leq \frac{b}{a}, \quad \frac{d}{c} \leq 1 + \varepsilon
\]
we have
\[
(1 + \varepsilon)^{-1} \leq \frac{b + d}{a + c} \leq 1 + \varepsilon.
\]
Since
\[
(1 + \varepsilon)^{-1} \leq \frac{\omega((x_0 + x_1)/2, x_1 + t)}{\omega(x_0 - t, (x_0 + x_1)/2)}, \quad \frac{\omega(x_1, (x_0 + x_1)/2)}{\omega((x_0 + x_1)/2, x_0)} \leq 1 + \varepsilon,
\]
by the property of the mediant, the statement (b) is proved.

\[\square\]

**Claim 2.** If the doubling weight \( \omega \) is vanishing on \( \mathbb{R} \), then it holds that
\[
\lim_{t \to 0} \hat{A}_1 = 0
\]
uniformly for \( x_0 \in \mathbb{R} \).

**Proof.** We separate \( \hat{A}_1 \) into two parts:
\[
\hat{A}_1 = \frac{1}{t} \int_{t/2}^{t} \int_{I(x_0,t)} \log \frac{\omega_I(x,y)}{\omega_I(x_0,t)} \frac{dxdy}{y} \\
= \frac{1}{t} \int_{t/2}^{t} \int_{I(x_0,t)} \log \frac{\omega_I(x,y)}{\omega_I(x,t)} \frac{dxdy}{y} + \frac{1}{t} \int_{t/2}^{t} \int_{I(x_0,t)} \log \frac{\omega_I(x,y)}{\omega_I(x_0,t)} \frac{dxdy}{y}.
\]

Since \( \omega \) is vanishing, for any arbitrarily small \( \varepsilon > 0 \) there exists some \( \delta > 0 \) such that \([12]\) holds for every \( x_0 \in \mathbb{R} \) and for every \( t \in (0, \delta) \). We suppose \( t \in (0, \delta) \) in the following.

First, we estimate the first term in the last line of \([13]\). We set \( N = 1/\sqrt{\varepsilon} \) (we may adjust \( \varepsilon \) so that \( N \) becomes an integer). Then,
\[
\left| \int_{t/2}^{t} \log \frac{\omega_I(x,y)}{\omega_I(x,t)} \frac{dy}{y} \right| \leq \sum_{m=1}^{N} \int_{\frac{t}{2}(1 + \frac{m}{N})}^{\frac{t}{2}(1 + \frac{m+1}{N})} \left| \log \omega_I(x,y) - \log \omega_I(x,t) \right| \frac{dy}{y}.
\]
We note that as \( y \in \left[ \frac{t}{2} \left( 1 + \frac{m-1}{N} \right), \frac{t}{2} \left( 1 + \frac{m}{N} \right) \right] \), the difference value \( |\log \omega_{I(x,y)} - \log \omega_{I(x,t)}| \) is less than the maximum of

\[
\left| \log \left( \frac{1}{2} \left( 1 + \frac{m-1}{N} \right) \int_{x - \frac{t}{4} \left( 1 + \frac{m}{N} \right)}^{x + \frac{t}{4} \left( 1 + \frac{m}{N} \right)} \omega(u) du \right) - \log \omega_{I(x,t)} \right| \leq \log \left( \frac{1}{2} \left( 1 + \frac{m}{N} \right) \int_{x - \frac{t}{4} \left( 1 + \frac{m}{N} \right)}^{x + \frac{t}{4} \left( 1 + \frac{m}{N} \right)} \omega(u) du \right) - \log \omega_{I(x,t)}
\]  \( (14) \)

and

\[
\left| \log \left( \frac{1}{2} \left( 1 + \frac{m}{N} \right) \int_{x - \frac{t}{4} \left( 1 + \frac{m}{N} \right)}^{x + \frac{t}{4} \left( 1 + \frac{m}{N} \right)} \omega(u) du \right) - \log \omega_{I(x,t)} \right|
\]  \( (15) \)

We assume that \((14)\) is bigger than \((15)\) and continue our computation (the other case can be treated similarly). Then, by the statement \((a)\) in Lemma 4, we have

\[
\sum_{m=1}^{N} \left[ \log(1 + \frac{1}{N}) + \left| \log \omega_{I(x, \frac{t}{2N} \times (N+m))} - \log \omega_{I(x, \frac{t}{2N} \times 2N)} \right| \right]
\]

We assume \( \log \omega_{I(x, \frac{t}{2N} \times (N+m))} \) is bigger than \( \log \omega_{I(x, \frac{t}{2N} \times 2N)} \) and continue the following estimate (the other case can be treated similarly). Then, by the statement \((a)\) in Lemma 4 we have

\[
\sum_{m=1}^{N} \frac{1}{N} \left( \log \omega_{I(x, \frac{t}{2N} \times (N+m))} - \log \omega_{I(x, \frac{t}{2N} \times 2N)} \right)
\]

\[
\leq \sum_{m=1}^{N} \frac{1}{N} \left[ \log \left( \frac{(1 + \varepsilon)^{N+m} - 1}{(N+m)\varepsilon} \omega_{I(x, \frac{t}{2N})} \right) - \log \left( \frac{(1 + \varepsilon)^{2N} - 1}{2N\varepsilon(1 + \varepsilon)^{2N-1} \omega_{I(x, \frac{t}{2N})}} \right) \right]
\]

\[
\leq \log(1 + \varepsilon)^{2N-1} + \max_{1 \leq m \leq N} \left| \log \left( \frac{(1 + \varepsilon)^{N+m} - 1}{(N+m)\varepsilon} \right) - \log \left( \frac{(1 + \varepsilon)^{2N} - 1}{2N\varepsilon} \right) \right|
\]

This can be arbitrarily small as \( \varepsilon \) is sufficiently small. Indeed, it tends to 0 as \( \varepsilon \to 0 \), and in particular, for any fixed \( m \in [1, N] \), the absolute value in the last line tends to 0 as \( \varepsilon \to 0 \). Thus, the first term in the last line of \((13)\) tends to 0 uniformly for \( x_0 \in \mathbb{R} \) as \( t \to 0 \).

Next, we consider the second term in the last line of \((13)\). By using the statement \((b)\) in Lemma 4, we see that
\[
\left| \frac{1}{t} \int_{t/2}^{t} \int_{I(x_0,t)} \log \frac{\omega_{I(x,t)}}{\omega_{I(x_0,t)}} \frac{dxdy}{y} \right|
\]
\[
\leq \frac{1}{t} \int_{I(x_0,t)} \log \frac{\omega_{I(x,t)}}{\omega_{I(x_0,t)}} dx \times \log 2
\]
\[
= \frac{1}{t} \int_{I(x_0,t)} \log \frac{\omega(x-t/2, x+t/2)}{\omega(x_0-t/2, x_0+t/2)} dx \times \log 2
\]
\[
\leq \varepsilon \times \log 2.
\]

Thus, the second term in the last line of (13) is bounded by a constant multiple of \(\varepsilon\). This completes the proof of Claim 2. \(\square\)

Claim 3. If the doubling weight \(\omega\) is vanishing on \(\mathbb{R}\), then it holds that
\[
\lim_{t \to 0} \lim_{N \to \infty} A_3 = 0
\]
uniformly for \(x_0 \in \mathbb{R}\).

Proof. Set
\[
2F_k(y) = 2 \int_{I(x_0,t)} \log \frac{\omega(x-y/2^{k+1}, x+y/2^{k+1})}{\omega(x-y/2^{k}, x+y/2^{k})} dx.
\]
(16)

Then,
\[
\lim_{N \to \infty} A_3 = \lim_{N \to \infty} \sum_{k=1}^{N-1} \frac{1}{t} \int_{t/2}^{t} F_k(y) \frac{dy}{y}.
\]
(17)

Assume that \(\omega\) is vanishing, namely, for any arbitrarily small \(\varepsilon > 0\) there exists some \(\delta > 0\) such that (12) holds for every \(x_0 \in \mathbb{R}\) and for every \(t \in (0, \delta)\). We suppose again \(t \in (0, \delta)\) in the following. We now estimate the ratio \(|F_k(y)/y|\). The expression (16) for \(2F_k(y)\) is changed in form step by step in the following for convenience of the estimate. We first notice that
\[
2F_k(y) = 2 \int_{I(x_0,t)} \log \omega(I(x, y/2^k)) dx - \int_{I(x_0,t)} \log \omega(x-y/2^k, x) dx
\]
\[
- \int_{I(x_0,t)} \log \omega(x, x+y/2^k) dx.
\]

By the change of the variables, we make the integrands in the second and third terms are same as the first one, namely,
\[
\int_{I(x_0,t)} \log \omega(x-y/2^k, x) dx + \int_{I(x_0,t)} \log \omega(x, x+y/2^k) dx
\]
\[
= \int_{x_0 + \frac{t}{2} - \frac{y}{2^{k+1}}}^{x_0 + \frac{t}{2} + \frac{y}{2^{k+1}}} \log \omega(I(x, y/2^k)) dx + \int_{x_0 - \frac{t}{2} + \frac{y}{2^{k+1}}}^{x_0 - \frac{t}{2} - \frac{y}{2^{k+1}}} \log \omega(I(x, y/2^k)) dx.
\]
Then, by rearranging the interval of integration, $2F_k(y)$ is divided into four terms:

$$2F_k(y) = \left( \int_{x_0-\frac{t}{2}}^{x_0-\frac{t}{2}+\frac{\pi t}{2}} \log \omega(I(x, y/2^k)) dx - \int_{x_0-\frac{t}{2}}^{x_0-\frac{t}{2}+\frac{\pi t}{2}} \log \omega(I(x, y/2^k)) dx \right)$$

$$+ \left( \int_{x_0+\frac{t}{2}-\frac{\pi t}{2}}^{x_0+\frac{t}{2}} \log \omega(I(x, y/2^k)) dx - \int_{x_0+\frac{t}{2}-\frac{\pi t}{2}}^{x_0+\frac{t}{2}} \log \omega(I(x, y/2^k)) dx \right).$$

Finally, by the change of the variables again, we have

$$2F_k(y) = \int_{x_0-\frac{t}{2}+\frac{\pi t}{2}}^{x_0-\frac{t}{2}} \log \omega(I(x, y/2^k)) \omega(x-y/2^k, x) dx + \int_{x_0+\frac{t}{2}-\frac{\pi t}{2}}^{x_0+\frac{t}{2}} \log \omega(I(x, y/2^k)) \omega(x-y/2^k, x) dx.$$

By using the statement (b) in Lemma 4 as above, we have that

$$\left| \log \frac{\omega(I(x, y/2^k))}{\omega(x-y/2^k, x)} \right| \leq \log(1 + \varepsilon) \leq 4\varepsilon \quad (18)$$

which implies that

$$\frac{|F_k(y)|}{y} \leq 2^{-(k-1)}\varepsilon. \quad (19)$$

Lastly, we deal with $\lim_{N \to \infty} |A_3|$. By (17),

$$\lim_{N \to \infty} |A_3| \leq \lim_{N \to \infty} \sum_{k=1}^{N-1} \frac{1}{t} \int_{t/2}^{t} |F_k(y)| \frac{dy}{y} \leq \sum_{k=1}^{\infty} \varepsilon \cdot \frac{\varepsilon}{2^k} = \varepsilon.$$

This completes the proof of Claim 3. \qed

**Claim 4.** It holds that

$$\lim_{t \to 0} A_4 = 0$$

uniformly for $x_0 \in \mathbb{R}$ if and only if the function $\log \omega \in \text{VMO}(\mathbb{R})$.

**Proof.** This is a direct result of Proposition 3 \qed

**4. Appendix: An alternative explanation on Theorem 1**

Let $\Gamma_t(x)$ be the truncated cone on $\mathbb{R}_+^2$ with vertex $x \in \mathbb{R}$, height $t > 0$ and angle $\pi/2$, namely,

$$\Gamma_t(x) = \{ z = (u, y) \in \mathbb{R}_+^2 \mid |u - x| < y, \ 0 < y < t \}.$$

We define the area function

$$A_t(x) = \left( \int_{\Gamma_t(x)} \eta(u, y) \frac{dudy}{y^2} \right)^{1/2}.$$
For any interval $I$ on $\mathbb{R}$ of length $|I| = t$, we consider the average of $A_t^2(x)$ on $I$

$$(A_t^2)_I = \frac{1}{|I|} \int_I A_t^2(x)dx.$$  

Then, the vanishing property of Carleson measure in Theorem I can be understood as that of the area function.

**Proposition 5.** Let the doubling weight $\omega$ be an $A_\infty$-weight, and vanishing on $\mathbb{R}$. Then, the Carleson measure $\eta(z)y^{-1}dxdy$ is vanishing on $\mathbb{R}^2_+$ if and only if

$$\lim_{\delta \to 0} \sup_{|t|=\delta} (A_t^2)_I = 0.$$  

**Proof.** For any $x_0 \in \mathbb{R}$, let $I = \{ x \mid |x - x_0| < t/2 \}$. Then, we separate $(A_t^2)_I$ as follows.

$$(A_t^2)_I = \frac{1}{t} \int_I \left[ \int_0^t \left( \int_{-y}^{x+y} \eta(u, y)du \right) \right] dx$$

$$= \frac{1}{t} \int_I \left[ \int_0^t \left( \eta(u, y)du \right) \int_0^{\frac{t}{2}} \left( \eta(x + y, y) - \eta(x, y) \right) \frac{dxdy}{y} \right] dx$$

$$+ \frac{1}{t} \int_0^t \int_I \left( \int_{-y}^{x+y} \eta(u, y)du \right) \frac{dxdy}{y} + \frac{1}{t} \int_0^t \int_I \left( \int_{-y}^{x+y} \frac{\partial}{\partial y} \eta(u, y)du \right) \frac{dxdy}{y}$$

$$+ \frac{2}{t} \int_0^t \int_I \eta(x, y) \frac{dxdy}{y}.$$

Thus, we only need to show each of $B_1$, $B_{21}$, $B_{22}$ and $B_3$ tends to 0 uniformly for any $x_0 \in \mathbb{R}$ as $t \to 0$ under the assumption about the vanishing property of the doubling weight $\omega$.

Noting that $\omega$ is vanishing, we conclude by the statement (b) in Lemma II as above that for arbitrarily small ($\ll 1/10$) $\varepsilon > 0$ there exists some $\delta > 0$ such that

$$\left| \log \frac{\omega(u - t/2, u + t/2)}{\omega(u - t/2, u)^{1/2}\omega(u, u + t/2)^{1/2}} \right|$$

$$= \log \frac{1}{2} \left( \frac{\sqrt{\omega(u - t/2, u)} + \sqrt{\omega(u, u + t/2)}}{\omega(u, u + t/2)} \right)$$

$$\leq \log(1 + \varepsilon)^{1/2} \leq \varepsilon/2$$

as $t \in (0, \delta)$, which yields that

$$|B_1| \leq \frac{1}{t} \int_I \lim_{y \to y^+} \frac{1}{y} \int_{x-y}^{x+y} \left| \log \frac{\omega(u - y/2, u + y/2)^{1/2}}{\omega(u - y/2, u)^{1/2}\omega(u, u + y/2)^{1/2}} \right| du dx$$

(20)
Here, for the last inequality we used the statement (see [7, Theorem 2.20])

\[
\leq 2 \times \varepsilon + 2 \times \varepsilon = 2\varepsilon.
\]

Therefore, we have \( \lim_{t \to 0} B_1 = 0 \) uniformly for any \( x_0 \in \mathbb{R} \).

Now we deal with the term \( B_{21} \) (the term \( B_{22} \) can be treated similarly).

\[
|B_{21}| \leq \frac{1}{t} \int_{0}^{t} \int_{I} \log \left( \frac{\omega(x + \frac{y}{2}, x + \frac{3}{2}y)}{\omega(x - \frac{y}{2}, x + \frac{y}{2})} \cdot \frac{\omega(x - \frac{y}{2}, x + \frac{3}{2}y)}{\omega(x + y, x + \frac{3}{2}y)} \right) dxdy.
\]

Then, by using the same trick as that when we estimate the ratio \( |F_k(y)/y| \) in Claim 3, we obtain

\[
|B_{21}| \leq \frac{1}{t} \int_{0}^{t} (\varepsilon y + \varepsilon t/2 + \varepsilon y) \frac{dxdy}{y} = 2\varepsilon.
\]  \hspace{1cm} (21)

It remains to take care of the term \( B_3 \). We notice that \( \int_{x-y}^{x+y} \frac{\partial}{\partial y} \eta(u, y) du \) can be regarded as the difference \( G(x+y) - G(x-y) \), where \( G(x) \) is an absolutely continuous function on the interval \([x-y, x+y] \). Noting that \( G(x) \) is a bounded variation, we have the estimate (see [7, Theorem 2.20])

\[
\frac{1}{y} \int_{I} |G(x+y) - G(x-y)| dx \leq \text{Var}(G([x-y, x+y])) \]  \hspace{1cm} (22)

for small \(|I|\). From the classical real analysis theory, we obtain

\[
2\text{Var}(G([x-y, x+y])) = 2 \int_{x-y}^{x+y} \left| \frac{\partial}{\partial y} \eta(u, y) \right| du \]  \hspace{1cm} (23)

Here, for the last inequality we used the statement (b) again.

To finish the proof of Proposition [5] it suffices to show that \( \int_{x-y}^{x+y} \frac{\omega(u, u+\varepsilon/2)}{\omega(u, u+y/2)} du \) and \( \int_{x-y}^{x+y} \frac{\omega(u-y/2, u)}{\omega(u, u+y/2)} du \) are uniformly bounded for small intervals. By the statements (a) and
(b) in Lemma 4, we compute
\[
\int_{x-y}^{x+y} \omega(u + y/2) \omega(u + y/2) du \leq \frac{1 + \varepsilon}{\omega(x_0 - y/4, x_0 + y/4)} \int_{x-y/2}^{x+y/2} \omega(u) du \tag{24}
\]
\[
\leq (1 + \varepsilon)^2 \frac{\omega(x_0 - y, x_0 + y)}{\omega(x_0 - y/4, x_0 + y/4)}
\]
\[
\leq (1 + \varepsilon)^2 \frac{(1 + \varepsilon)^4 - 1}{\varepsilon} < 8.
\]

The other term \( \int_{x-y}^{x+y} \omega(u+y/2, u) du \) can be estimated similarly.

(24) together with (22) and (23) implies

\[
|B_3| \leq \frac{1}{t} \int_0^t \left( \frac{1}{y} \int_I |G(x+y) - G(x-y)| dx \right) dy \leq 2\varepsilon.
\]

This completes the proof of Proposition 5. \( \square \)

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