Integrable Structure of Conformal Field Theory II. 
Q-operator and DDV equation

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Abstract

This paper is a direct continuation of [1] where we begun the study of the integrable structures in Conformal Field Theory. We show here how to construct the operators $Q_{\pm}(\lambda)$ which act in highest weight Virasoro module and commute for different values of the parameter $\lambda$. These operators appear to be the CFT analogs of the $Q$ - matrix of Baxter [2], in particular they satisfy famous Baxter’s $T - Q$ equation. We also show that under natural assumptions about analytic properties of the operators $Q(\lambda)$ as the functions of $\lambda$ the Baxter’s relation allows one to derive the nonlinear integral equations of Destri-de Vega (DDV) [3] for the eigenvalues of the $Q$-operators. We then use the DDV equation to obtain the asymptotic expansions of the $Q$ - operators at large $\lambda$; it is remarkable that unlike the expansions of the $T$ operators of [1], the asymptotic series for $Q(\lambda)$ contains the “dual” nonlocal Integrals of Motion along with the local ones. We also discuss an intriguing relation between the vacuum eigenvalues of the $Q$ - operators and the stationary transport properties in boundary sine-Gordon model. On this basis we propose a number of new exact results about finite voltage charge transport through the point contact in quantum Hall system.
1. Introduction

Existence of an infinite set of mutually commuting local Integrals of Motion (IM) is the characteristic feature of an integrable quantum field theory (IQFT). Therefore simultaneous diagonalization of these local IM is the fundamental problem of IQFT. In the case of infinite-size system this problem reduces to finding mass spectrum and factorizable S-matrix associated with IQFT; much progress in this direction has been made during the last two decades (see e.g. [4] for a review). On the other hand, for a finite-size system (say, with the spatial coordinate compactified on a circle of circumference $R$) this problem becomes highly nontrivial and so far its solution is known to a very limited extent. Most important progress here has been made with the help of so called Thermodynamic Bethe Ansatz (TBA) approach [5], [6]. TBA allows one to find the eigenvalues associated with the ground state of the system (in particular the ground-state energy) in terms of solutions of nonlinear integral equation (TBA equation). However it is not clear how the combination of thermodynamic and relativistic ideas which is used in traditional derivation of the TBA equation can be extended to include the excited states.

The above diagonalization problem is very similar to that treated in solvable lattice models. In the lattice theory very powerful algebraic and analytic methods of diagonalization of the Baxter’s families of commuting transfer-matrices are known [2], [7]; these methods are further developed in Quantum Inverse Scattering Method (QISM) [8], [9]. Of course many IQFT can be obtained by taking continuous limits of solvable lattice models and the method based on commuting transfer-matrices can be used to solve these QFT. This is essentially the way how IQFT are treated in the QISM. However, for many IQFT (notably, for most of IQFT defined as perturbed CFT [10]) the associated solvable lattice models are not known. Besides, it seems to be conceptually important to develop the above methods directly in continuous QFT, in particular, to find continuous QFT versions of the Baxter’s commuting transfer-matrices.

This problem was addressed in our recent paper [1] where we concentrated attention on the case of Conformal Field Theory (CFT), more specifically on $c < 1$ CFT. We should stress here that although the structure of the space of states and the energy spectrum in CFT are relatively well understood the diagonalization of the full set of the local IM remains very nontrivial open problem. In [1] we have constructed an infinite set of operator valued functions $T_j(\lambda)$, where $j = 0, \frac{1}{2}, 1, \frac{3}{2}, ...$ and $\lambda$ is a complex variable. These operators
(we will exhibit their explicit form in Sect. 2) act invariantly in irreducible highest weight Virasoro module $V_\Delta$ and they commute between themselves for any values of $\lambda$, i.e.

$$T_j(\lambda) : \quad V_\Delta \to V_\Delta, \quad [T_j(\lambda), T_j'(\lambda')] = 0.$$  

(1.1)

The operators $T_j(\lambda)$ are defined in terms of certain monodromy matrices associated with $2j + 1$ dimensional representations of quantum algebra $U_q(sl_2)$ where

$$q = e^{i\pi \beta^2} .$$  

(1.2)

and $\beta$ is related to the Virasoro central charge as

$$c = 13 - 6(\beta^2 + \beta^{-2}) .$$  

(1.3)

Evidently, the operators $T_j(\lambda)$ are CFT versions of the commuting transfer-matrices of the Baxter’s lattice theory. We will still call these operators “transfer-matrices” although the original meaning of this term [7] apparently is lost. As we have shown in [1], in CFT the operators $T_j(\lambda)$ enjoy particularly simple analytic properties, namely they are entire functions of $\lambda^2$ with an essential singularity at $\lambda^2 = \infty$ and their asymptotic behavior near this point is described in terms of the local IM. Therefore the operators $T_j(\lambda)$ can be thought of as the generating functions for the local IM since all the information about their eigenvalues is contained in the eigenvalues of $T_j(\lambda)$. The operators $T_j(\lambda)$ are shown to obey the “fusion relations” which for any rational value of $\beta^2$ in (1.2) provide a finite system of functional equations for the eigenvalues of these operators. For the ground-state eigenvalue (in CFT it corresponds to a primary state) these functional equations turn out to be equivalent to the TBA equations; in general case they provide modified TBA equations suitable for the excited states. Interesting but somewhat inconvenient feature of this approach is that the resulting TBA equations depend on $c$ in a very irregular manner (they depend on the arithmetic properties of the rational number $\beta^2$) whereas the resulting eigenvalues of $T_j(\lambda)$ are expected to be smooth functions of $c$.

Another powerful method known in the lattice theory is based on the so-called $Q$-operator. This method was introduced by Baxter in his original study of 8-vertex model [2]. One of its advantages is that it is not limited to the cases when $q$ is a root of unity. In this paper (which is a sequel to [1]) we introduce the analog of $Q$-operator directly in CFT and study its properties. The $Q$-operators (in fact we will define two $Q$-operators,
\( Q_{\pm}(\lambda) \) are defined again as the traces of certain monodromy matrices, this time associated with infinite-dimensional representations of so called “\( q \)-oscillator algebra”. The operators \( Q_{\pm}(\lambda) \) obey the Baxter’s functional relation

\[
T(\lambda)Q(\lambda) = Q(q\lambda) + Q(q^{-1}\lambda),
\]

where \( T(\lambda) \equiv T_{\frac{1}{2}}(\lambda) \). This construction is presented in Sect.2 where also the most important properties of the \( Q \)-operators are discussed.

In the lattice theory the Baxter’s relation (1.4) is known to be a powerful tool for finding the eigenvalues of the transfer-matrices \( \bar{T} \), the knowledge about analytic properties of the \( Q \)-operator being a key ingredient in this approach. Our construction of the \( Q \)-operators as the traces of the monodromy matrices makes it natural to assume that they enjoy very simple analytic properties similar to those of the operators \( T_{j}(\lambda) \): up to overall power-like factors they are entire functions of \( \lambda^2 \) with the following asymptotic at \( \lambda^2 \to -\infty \)

\[
\log Q_{\pm}(\lambda) \sim M (-\lambda^2)^{\frac{1}{2-2\beta^2}},
\]

where \( M \) is a constant (which will be actually calculated in Sect.3). Using these properties we show that the eigenvalues of the \( Q \)-operators satisfy the Destry-de Vega (DDV) equation \[3\]. This is done in Sect.3. The DDV equation can be solved exactly in the limit \( \Delta \to +\infty \), where \( \Delta \) is the Virasoro highest weight in (1.1), and we analyze in the Sect.3 the vacuum eigenvalues of the operators \( Q(\lambda) \) in this limit. In Sect.4 we further study the properties of the \( Q \)-operators and formulate our basic conjectures about their analytic characteristics. The exact asymptotic expansions of the \( Q \) and \( T \) operators at \( \lambda^2 \to \infty \) are proposed here. We observe that unlike the asymptotic expansion of \( T(\lambda) \) the large \( \lambda^2 \) expansion of \( Q(\lambda) \) contains both local and nonlocal IM and that the operators \( Q(\lambda) \) obey remarkable duality relation with respect to the substitution \( \beta^2 \to \beta^{-2} \). Although the results of this section have somewhat conjectural status we support them by explicit study of the eigenvalues of the \( Q \) operators at the “free fermion point” \( \beta^2 = 1/2 \). In Sect.5 we discuss the relation of the \( Q \) operators to the characteristics of stationary non-equilibrium states in so called boundary sine-Gordon model \[11\] (see also \[12\]); these states attracted lately much attention in relation to the finite-voltage current through the point contact in a quantum Hall system \[13\], \[14\], \[15\], \[16\]. Possible directions of further studies are discussed in Sect.6.
2. The Q-operators

In this section we will introduce the operators $Q_{\pm}(\lambda)$ which satisfy (1.4). We start with a brief review of the definitions and results of [1].

Let $\varphi(u)$ be a free chiral Bose field, i.e. the operator-valued function

$$\varphi(u) = iQ + iP + \sum_{n \neq 0} \frac{a_{-n}}{n} e^{inu}. \quad (2.1)$$

Here $P, Q$ and $a_n, n = \pm 1, \pm 2, \ldots$ are operators which satisfy the commutation relations of the Heisenberg algebra

$$[Q, P] = \frac{i}{2} \beta^2; \quad [a_n, a_m] = \frac{n}{2} \beta^2 \delta_{n+m,0}. \quad (2.2)$$

with real $\beta$. The variable $u$ is interpreted as a complex coordinate on $2D$ cylinder of a circumference $2\pi$. As follows from (2.1) the field $\varphi(u)$ is a quasi-periodic function of $u$, i.e.

$$\varphi(u + 2\pi) = \varphi(u) + 2\pi iP. \quad (2.3)$$

Let $\mathcal{F}_p$ be the Fock space, i.e. the space generated by a free action of the operators $a_n$ with $n < 0$ on the vacuum vector $|p\rangle$ which satisfies

$$a_n |p\rangle = 0, \quad \text{for} \quad n > 0;$$

$$P |p\rangle = p |p\rangle. \quad (2.4)$$

The composite field

$$-\beta^2 T(u) =: \varphi'(u)^2 : + (1 - \beta^2) \varphi''(u) + \frac{\beta^2}{24} \quad (2.5)$$

is called the energy-momentum tensor; it is periodic function of $u$ and its Fourier modes

$$L_n = \int_{-\pi}^{\pi} \frac{du}{2\pi} \left[ T(u) + \frac{c}{24} \right] e^{inu} \quad (2.6)$$

generate the Virasoro algebra with the central charge (1.3) [17], [18]. It is well known that for generic values of the parameters $\beta$ and $p$ the Fock space $\mathcal{F}_p$ realizes an irreducible highest weight Virasoro module $\mathcal{V}_\Delta$ with the highest weight $\Delta$ related to $p$ as

$$\Delta = \left( \frac{p}{\beta} \right)^2 + \frac{c - 1}{24}. \quad (2.7)$$
For particular values of these parameters, when null-vectors appear in $F_p$, $\mathcal{V}_\Delta$ is obtained from $F_p$ by factoring out all the invariant subspaces. The space

$$\hat{F}_p = \bigoplus_{n=-\infty}^{\infty} F_{p+n\beta^2}$$  \hspace{1cm} (2.8)$$

admits the action of the exponential fields

$$V_\pm(u) = :e^{\pm 2\varphi(u)} :.$$  \hspace{1cm} (2.9)$$

Also, let $E$, $F$ and $H$ be a canonical generating elements of the algebra $U_q(sl(2))$ [14], i.e.

$$[H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = \frac{q^H - q^{-H}}{q - q^{-1}},$$  \hspace{1cm} (2.10)$$

where $q$ is given by (1.2). Let $j$ be a non-negative integer or half-integer number. We denote $\pi_j$ an irreducible $2j + 1$ dimensional matrix representation of $U_q(sl(2))$ so that

$$E_j \equiv \pi_j(E), \quad F_j \equiv \pi_j(F) \quad \text{and} \quad H_j \equiv \pi_j(H)$$

are $(2j + 1) \times (2j + 1)$ matrices which satisfy the relations (2.10).

The “transfer-matrices” $T_j(\lambda)$, $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ are defined as [1]

$$T_j(\lambda) = \text{tr}_\pi \left[ e^{2\pi i PH_j} \mathcal{P} \exp \left( \lambda \int_0^{2\pi} K_j(u) \, du \right) \right]$$

$$\equiv \text{tr}_\pi \left[ e^{2\pi i PH_j} \sum_{n=0}^{\infty} \lambda^n \int_{2\pi \geq u_1 \geq u_2 \geq \ldots \geq u_n \geq 0} K_j(u_1) K_j(u_2) \ldots K_j(u_n) \, du_1 du_2 \ldots du_n \right].$$  \hspace{1cm} (2.11)$$

Here

$$K_j(u) = V_-(u) \, q^\frac{H_j}{2} \, E_j + V_+(u) \, q^{-\frac{H_j}{2}} \, F_j$$

and $\mathcal{P}$ denotes the operator ordering along the integration path. Obviously,

$$T_0(\lambda) = I,$$  \hspace{1cm} (2.12)$$

where $I$ is the identity operator. Although the exponentials in (2.11) act from one component of the sum (2.8) to another it is easy to see that the operators (2.11) invariantly act in the Fock space $F_p$. Another obvious property of $T_j$ is that these operators are in fact the functions of $\lambda^2$ as the traces of the odd-order terms in (2.11) vanish.

As is explained in [1] the operators (2.11) form the commuting family, i.e. they satisfy the relations (1.1). They also obey the “fusion relations”

$$T(\lambda) T_j(q^{j+\frac{1}{2}} \lambda) = T_{j-\frac{1}{2}}(q^{j+1} \lambda) + T_{j+\frac{1}{2}}(q^j \lambda).$$  \hspace{1cm} (2.13)$$
Together with (2.12) this relations allow one to express recurrently any of the “higher-spin” operators $T_j(\lambda)$ with $j = 1, \frac{3}{2}, 2, ...$ in terms of the basic one

$$T(\lambda) \equiv T_{\frac{1}{2}}(\lambda).$$

(2.14)

In the case $j = \frac{1}{2}$ it is easy to evaluate the traces in (2.11) and obtain the power-series expansion

$$T(\lambda) = 2\cos(2\pi P) + \sum_{n=1}^{\infty} \lambda^{2n} G_n,$$

(2.15)

where the coefficients define an infinite set of basic nonlocal IM

$$G_n = q^n \int_{2\pi \geq u_1 \geq u_2 \geq ... \geq u_{2n}} \left( e^{2i\pi P} V_-(u_1)V_+(u_2)V_-(u_3)...V_+(u_{2n}) + e^{-2i\pi P} V_+(u_1)V_-(u_2)V_+(u_3)V_-(u_{2n}) \right) du_1du_2...du_{2n}.$$

(2.16)

It follows from (1.1) that these nonlocal IM commute among themselves

$$[G_n, G_m] = 0.$$

The operator-product expansion

$$V_+(u)V_-(u') = (u - u')^{-2\beta^2} \left( 1 + O(u - u') \right), \quad u - u' \to 0$$

(2.17)

shows that the expressions (2.11) and (2.16) can be taken literally only if

$$0 < \beta^2 < \frac{1}{2},$$

(2.18)

for otherwise the integrals in (2.16) diverge. In what follows we will call the region (2.18) the Semi-classical Domain (SD). In fact one can define the operators $G_n$ outside the SD by analytic continuation in $\beta^2$. A convenient way to do that is to transform the ordered integrals in (2.16) to contour integrals. For example $G_1$ can be written as

$$G_1 = (q^2 - q^{-2})^{-1} \int_0^{2\pi} du_1 \int_0^{2\pi} du_2 \left\{ (qe^{-2\pi i P} - q^{-1}e^{2\pi i P}) \times V_-(u_1 + i0)V_+(u_2 - i0) + (qe^{2\pi i P} - q^{-1}e^{-2\pi i P}) V_+(u_1 + i0)V_-(u_2 - i0) \right\}.$$

(2.19)

Obviously, (2.19) does not contain divergent integrals and for $\beta^2$ in SD coincides with the ordered integral in (2.16). Similar representation exists for higher $G_n$, and thus the
operator $T(\lambda)$ can be defined outside SD through (2.15). Of course the operators $G_n$ and $T(\lambda)$ obtained this way exhibit singularities at $\beta^2 = \beta_n^2$

$$\beta_n^2 = \frac{2n - 1}{2n}, \quad n = 1, 2, 3, \ldots,$$

where the integrals (2.16) develop logarithmic divergences. In order to define the operators $T_j(\lambda)$ at the singular points (2.20) some renormalization is needed but we do not discuss it here (see however our analysis of the case $\beta^2 = 1/2$ in Sect.4).

As is mentioned in the Introduction, the power series (2.15) defines $T(\lambda)$ as an entire function of $\lambda^2$ with an essential singularity at $\lambda^2 \to \infty$. Its asymptotic expansion near this essential singularity can be expressed in terms of local IM as

$$\log T(\lambda) \simeq m \lambda^{\frac{1}{1-\beta^2}} I - \sum_{n=1}^{\infty} C_n \lambda^{\frac{1-2n}{1-\beta^2}} I_{2n-1},$$

where $I_{2n-1}$ is the basic set of commutative local IM as defined in [1]; the operators $I_{2n-1}$ can be written as the integrals

$$I_{2n-1} = \int_0^{2\pi} \frac{du}{2\pi} T_{2n}(u),$$

where the local densities $T_{2n}(u)$ are particular normal ordered polynomials of $\partial_u \varphi(u), \ldots, \partial_u^{2n} \varphi(u)$ (see [1] for details and for our convention about the normalization of $I_{2n-1}$). The numerical coefficients $m$ and $C_n$ in the expansion (2.21) will be calculated exactly in Sect.3.

As follows from (1.1) all these local IM commute with the nonlocal IM $G_n$.

As is known [20], [21], [22], the local IM $I_{2n-1}$ defined in [1] do not change under the substitution $\beta^2 \to \beta^{-2}$, if we simultaneously make the replacement $\varphi(u) \to \beta^{-2} \varphi(u)$

$$I_{2n-1}\{\beta^2, \varphi(u)\} = I_{2n-1}\{\beta^{-2}, \beta^{-2} \varphi(u)\}.$$

Evidently the nonlocal IM do change and so there exists an infinite set of “dual” nonlocal IM $\tilde{G}_n$ which are obtained from (2.16) by just this substitution, i.e.

$$\tilde{G}_n = \tilde{q}^n \int_{2\pi \geq u_1 \geq u_2 \geq \ldots \geq u_{2n} \geq 0} \left( \tilde{q}^{2P} U_-(u_1)U_+(u_2)U_-(u_3)\ldots U_+(u_{2n})+ \tilde{q}^{-2P} U_+(u_1)U_-(u_2)U_+(u_3)\ldots U_-(u_{2n}) \right) du_1 du_2 \ldots du_{2n},$$

where

$$U_\pm(u) =: e^{\pm \frac{\lambda^2}{\beta^2} \varphi(u)};$$

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and
\[ \tilde{q} = e^{\frac{i\pi}{\beta^2}}. \]  

(2.25)

Of course if \( \beta^2 < 2 \) the analytic continuation described above is needed to define \( \tilde{G}_n \). However it is possible to check that the operators \( \tilde{G}_n \) thus defined commute among themselves and commute with all the nonlocal IM \( G_n \) and local IM \( I_{2m-1} \)

\[ [G_n, G_m] = [\tilde{G}_n, G_m] = [\tilde{G}_n, I_{2m-1}] = 0. \]

(2.26)

Now we are in position to define the \( Q \)-operators and to describe their basic properties. Let \( \mathcal{E}_+, \mathcal{E}_- \) and \( \mathcal{H} \) be operators which satisfy the commutation relations of so called “q-oscillator algebra”,

\[ q \mathcal{E}_+ \mathcal{E}_- - q^{-1} \mathcal{E}_- \mathcal{E}_+ = \frac{1}{q - q^{-1}}, \quad [\mathcal{H}, \mathcal{E}_\pm] = \pm 2 \mathcal{E}_\pm \]

(2.27)

and let \( \rho \) be any representation of this algebra such that the trace

\[ Z(p) = tr_\rho[e^{2\pi i p \mathcal{H}}] \]

(2.28)

exists for complex \( p \) belonging to the upper half plane, \( \Im p > 0 \). Then one can define two operators

\[ A_\pm(\lambda) = Z^{-1}(\pm P)tr_\rho[e^{±2i\pi P \mathcal{H}}P \exp(\lambda \int_0^{2\pi} du(V_-(u)q^{±\frac{2i}{\beta^2}} \mathcal{E}_\pm + V_+(u)q^{±\frac{2i}{\beta^2}} \mathcal{E}_\pm))] , \]

(2.29)

where again the symbol \( P \) denotes the operator ordering along the integration domain. As we are interested in the action of these operators in \( F_p \) the operator \( P \) in (2.29) can be substituted for its eigenvalue \( p \). Strictly speaking the definition (2.29) makes sense only if \( \Im m p > 0 \) for \( A_+ \) and if \( \Im m p < 0 \) for \( A_- \). However these operators can be defined for all complex \( p \) (except for some set of singular points on the real axis) by analytic continuation in \( p \). Then it is easy to see that \( A_- \) can be obtained from \( A_+ \) by substitution

\[ P \rightarrow -P, \quad \varphi(u) \rightarrow -\varphi(u) . \]

(2.30)

The operators (2.29) can be written as the power series

\[ A_\pm(\lambda) = 1 + \sum_{n=1}^\infty \sum_{\{\sigma_i = \pm 1\}} \lambda^{2n} a_{2n}(\sigma_1, \ldots, \sigma_{2n} | \pm P) J_{2n}(\mp \sigma_1, \mp \sigma_2, \ldots, \mp \sigma_{2n}) , \]

(2.31)
where

\[ J_{2n}(\sigma_1, \ldots, \sigma_{2n}) = q^n \int_{2\pi u_1, \ldots, u_{2n} \geq 0} V_{\sigma_1}(u_1)V_{\sigma_2}(u_2) \cdots V_{\sigma_{2n}}(u_{2n}) \, du_1 \cdots du_{2n} \]

and

\[ a_n(\sigma_1, \ldots, \sigma_{2n})|P) = Z^{-1}(P) \, tr\left(e^{2\pi i P \mathcal{H}} \mathcal{E}_{\sigma_1} \mathcal{E}_{\sigma_2} \cdots \mathcal{E}_{\sigma_{2n}}\right). \tag{2.32} \]

It is easy to see that the coefficients \[ (2.32) \] are completely determined by the commutation relations \[ (2.27) \] and the cyclic property of the trace and so the operators \[ (2.29) \] do not depend on the particular choice of the representation \( \rho \). The operator coefficients in the power series \[ (2.31) \] can be expressed in terms of the basic nonlocal IM \[ (2.16) \]. It is not difficult to calculate first few terms of this series

\[ A_{\pm}(\lambda) = 1 - \lambda^2 \frac{G_1}{4 \sin a \sin(a \pm x)} - \lambda^4 \left\{ \frac{G_2}{4 \sin 2a \sin(2a \pm x)} \right\} - \lambda^6 \left\{ \frac{G_3}{4 \sin 3a \sin(3a \pm x)} \right\} + \lambda^8 \left\{ \frac{G_4}{64 \sin a \sin 2a \sin 3a \sin(a \pm x) \sin(2a \pm x) \sin(3a \pm x)} \right\} + O(\lambda^8), \tag{2.33} \]

where

\[ x = 2\pi P, \quad a = \pi \beta^2. \]

For further references it is convenient to introduce an alternative set of nonlocal IM defined as coefficients in the expansion

\[ \log A_{\pm}(\lambda) = - \sum_{n=1}^{\infty} y^{2n} H_n, \tag{2.34} \]

where

\[ y = \beta^{-2} \Gamma(1 - \beta^2) \lambda. \tag{2.35} \]

These coefficients are, of course, algebraically dependent on those in \[ (2.16) \]. For example,

\[ H_1 = \frac{\beta^4}{4\pi \Gamma(1 - \beta^2) \sin(2\pi P + \pi \beta^2)} \frac{\Gamma(\beta^2)}{\Gamma(1)} G_1. \tag{2.36} \]

Define also a new set of “dual” nonlocal IM \( \tilde{H}_n \)

\[ \tilde{H}_n\{\beta^2, \varphi(u)\} = H_n\{\beta^{-2}, \beta^{-2}\varphi(u)\}, \tag{2.37} \]
The operators \( Q_{\pm}(\lambda) \) are defined as
\[
Q_{\pm}(\lambda) = \lambda^{\pm 2P/\beta^2} A_{\pm}(\lambda) .
\] (2.38)

Like the operators \( T_j(\lambda) \) above the operators (2.38) act in a Fock space \( F_p \)
\[
Q_{\pm}(\lambda) : F_p \to F_p .
\] (2.39)

The operators \( Q_{\pm}(\lambda) \) exhibit remarkable properties. Here we simply list some of them leaving the proofs to the other paper.

i. The operators \( Q_{\pm}(\lambda) \) commute among themselves and with all the operators \( T_j(\lambda) \),
\[
\begin{align*}
&\left[ Q_{\pm}(\lambda), Q_{\pm}(\lambda') \right] = 0 , \\
&\left[ Q_{\pm}(\lambda), T_j(\lambda') \right] = 0 .
\end{align*}
\] (2.40)

ii. The operators \( Q_{\pm}(\lambda) \) satisfy the equation (1.4), i.e.
\[
T(\lambda)Q_{\pm}(\lambda) = Q_{\pm}(q\lambda) + Q_{\pm}(q^{-1}\lambda) .
\] (2.41)

The equation (1.4) can be thought of as the finite-difference analog of the second order differential equation so we expect it to have two linearly independent solutions. As \( T(\lambda) \) is a single-valued function of \( \lambda^2 \), i.e. it is periodic function of \( \log \lambda^2 \), the operators \( Q_{\pm}(\lambda) \) are just two “Bloch-wave” solutions to the equation (1.4). The operators \( Q_{\pm}(\lambda) \) satisfy the “quantum Wronskian” condition
\[
Q_+(q^{1/2}\lambda)Q_-(q^{-1/2}\lambda) - Q_+(q^{-1/2}\lambda)Q_-(q^{1/2}\lambda) = 2i \sin(2\pi P) .
\] (2.42)

iii. The “transfer-matrices” \( T_j(\lambda) \) can be expressed in terms of \( Q_{\pm}(\lambda) \) as
\[
2i \sin(2\pi P) T_j(\lambda) = Q_+(q^{j+1/2}\lambda)Q_-(q^{-j-1/2}\lambda) - Q_+(q^{-j-1/2}\lambda)Q_-(q^{j+1/2}\lambda) .
\] (2.43)

In view of this equation the operators \( Q_{\pm}(\lambda) \) appear more fundamental then the transfer-matrices \( T_j(\lambda) \). We will see more support to this idea below.

Let us just briefly sketch the derivation [23] of the functional relation of this section. The main idea is to consider more general \( T \)-operators \( T_j^+(\lambda) \) defined as in (2.11), but associated with the infinite dimensional representation \( \pi^+ \) of \( U_q(sl(2)) \) with arbitrary (complex) highest weight \( 2j \). Note that if \( j \) takes a non-negative integer or half-integer value then the matrices \( \pi_j^+(E) \), \( \pi_j^+(F) \) and \( \pi_j^+(H) \) have a block-triangular form with two
diagonal blocks, one equivalent to the $(2j + 1)$-dimensional representation $\pi_j$ and the other being the highest weight representation $\pi_{j-1}^+$. In this way we obtain the following simple relation

$$T_j^+(\lambda) = T_j(\lambda) + T_{j-1}^+(\lambda), \quad j = 0, 1/2, 1, 3/2, \ldots.$$ (2.44)

Next, the operator $T_j^+(\lambda)$ enjoy the following remarkable factorization property

$$2i \sin(2\pi P) T_j^+(\lambda) = Q_+^{\lambda}(q^{j+\frac{1}{2}}) Q_-^{\lambda}(q^{-j-\frac{1}{2}}),$$ (2.45)

which is proved explicitly by using decomposition properties of the tensor product of two representations of the $q$-oscillator algebra (the latter are also representations of the Borel sub-algebra of $U_q(\hat{sl}(2))$ with respect to the co-multiplication from $U_q(\hat{sl}(2))$. Then the functional relations (2.42), (2.43) trivially follow from (2.44) and (2.45) above, while the remaining relations (2.40), (2.41) are simple corollaries of these two.

Although our considerations here were specific to the continuous theory similar results hold for $Q$-matrix of the lattice theory as well. This is quite obvious since the structure of the functional equations is determined merely by the decomposition properties of products of representations of $U_q(\hat{sl}(2))$ associated with the monodromy matrices. The details of these calculations will be given in [23].

The operators $Q_\pm(\lambda)$ take particularly simple form when applied to the space $F_p$ with $2p$ equals some integer which we denote $N$. In these cases the “quantum Wronskian” (2.42) is equal to zero and the solutions $A_+^{\lambda}$ and $A_-^{\lambda}$ (2.38) coincide. For $2p = N$ all the coefficients (2.32) are readily calculated

$$a_{2n}(\sigma_1, \ldots, \sigma_{2n} | N/2) = (q - q^{-1})^{-2n},$$ (2.46)

and the operators (2.38) can be written as

$$A_\pm^{\lambda}|_{p=N/2} = \sum_{n=0}^{\infty} \frac{\mu^{2n}}{(n!)^2} q^n \mathcal{P} \left\{ \left[ \int_0^{2\pi} \frac{du}{2\pi} V_+(u) \right]^n \left[ \int_0^{2\pi} \frac{du}{2\pi} V_-(u) \right]^n \right\}|_{p=N/2},$$ (2.47)

where

$$\mu = -i \frac{\pi \lambda}{\sin(\pi \beta^2)}.$$ (2.48)

and, as before, the symbol $\mathcal{P}$ denotes “$u$-ordering”, i.e. the operators $V_\pm(u)$ with greater $u$ are placed to the left. For $p = N/2$ the exponentials $V_\pm(u)$ are $2\pi$-periodic functions of $u$ and so the integration contours in (2.47) close. The series (2.47) is closely related to
the boundary state in so called Boundary Sine-Gordon model (with the bulk mass equal zero) \[1\]. In particular, its vacuum-vacuum matrix element \( A^{(\text{vac})}_\pm(\lambda) \), defined as

\[
A_\pm(\lambda) \mid p \rangle = A^{(\text{vac})}_\pm(\lambda) \mid p \rangle ,
\]

coincides for \( p = N/2 \) with the one-dimensional Coulomb gas partition function

\[
A^{(\text{vac})}(\lambda)\mid p = N/2 \equiv Z_N(\mu) = \sum_{n=0}^{\infty} \frac{\mu^{2n}}{(n!)^2} \int_0^{2\pi} \frac{du_1}{2\pi} \cdots \int_0^{2\pi} \frac{du_n}{2\pi} \int_0^{2\pi} \frac{dv_1}{2\pi} \cdots \int_0^{2\pi} \frac{dv_n}{2\pi} \times 
\]

\[
e^{iN\sum_{i=1}^{n} (v_i - u_i)} \prod_{i \neq j} \left| 4 \sin \left( \frac{u_i - u_j}{2} \right) \sin \left( \frac{v_i - v_j}{2} \right) \right|^{2\beta^2} \prod_{i,j} \left| 2 \sin \left( \frac{u_i - v_j}{2} \right) \right|^{-2\beta^2} .
\]

As was shown in \[24\] the series \[2.50\] defines an entire function of \( \mu^2 \) with the asymptotic behavior \( \log Z_N(\mu) \sim \text{const} (\mu^2)^{2-2\beta^2} \). In fact, it is easy to show that this result implies not only to the vacuum eigenvalue \[2.49\] but also to any matrix element of the operator \( A \). We conclude that for \( p = N/2 \) the operators \( A_\pm(\lambda) \) are entire functions of \( \lambda^2 \) and they enjoy the asymptotic form

\[
\log A_\pm(\lambda) \sim M (-\lambda^2)^{1-2\beta^2}, \quad \lambda^2 \to -\infty .
\]

We should stress again that the relations \[2.47\] and \[2.50\] hold only for \( 2p = N \). For non-integer \( 2p \) they do not hold. Nonetheless we find it natural to assume that the above analytic properties of the operators \( A_\pm(\lambda) \) as the functions of \( \lambda^2 \) hold for any \( p \). In the following Sections we use the functional equation \[1.4\] together with this analyticity assumption to derive various asymptotic expansions for the operators \( A_\pm(\lambda) \).

### 3. Destri-de Vega equation

Now let us turn to the eigenvalue problem for the operators \( Q_\pm(\lambda) \)

\[
Q_\pm(\lambda) \mid \alpha \rangle = Q_\pm^0(\lambda) \mid \alpha \rangle ,
\]

where \( \mid \alpha \rangle \in \mathcal{F}_p \). In this section we consider only the case of \( \beta^2 \) in SD \[2.18\]. Let us concentrate attention on one of the operators \[2.38\], say \( Q(\lambda) \equiv Q_+(\lambda) \); the problem for \( Q_-(\lambda) \) can be solved then by the substitution \[2.30\].
As follows from (2.41) any eigenvalue $Q(\lambda)$ of $Q(\lambda)$ satisfies the Baxter’s functional equation
\[
T(\lambda)Q(\lambda) = Q(q\lambda) + Q(q^{-1}\lambda) ,
\]
where $T(\lambda)$ is the corresponding eigenvalue of the operator $T(\lambda)$. Denoting $A(\lambda)$ the eigenvalue of the operator $A_+(\lambda)$ in (2.38) one can rewrite (3.2) as
\[
T(\lambda)A(\lambda) = e^{2\pi ip}A(q\lambda) + e^{-2\pi ip}A(q^{-1}\lambda) .
\]

Although this equation looks as just a relation between two unknown functions in fact it imposes severe restrictions on these functions provided their analytic properties are known. Motivated by the relation to the Coulomb gas partition function (2.50) discussed in Sect. 2 we accept here the following assumptions about analytic properties of the eigenvalues $A(\lambda)$.

Let $\beta^2$ be restricted to SD and let $\Im m p = 0$. Then

(i) **Analyticity.** The functions $A(\lambda)$ and $T(\lambda)$ are entire functions of the complex variable $\lambda^2$.

(ii) **Location of zeroes.** Zeroes of the function $A(\lambda)$ in the $\lambda^2$-plane are either real or occur in complex conjugated pairs. For any given eigenvalue $A(\lambda)$ there are only finite number of complex or real negative zeroes. Real zeroes accumulate toward $+\infty$ in the $\lambda^2$. For the vacuum eigenvalues all zeroes are real and if $2p > -\beta^2$ they are all positive.

(iii) **Asymptotic behavior.** The leading asymptotic behavior of $A(\lambda)$ for large $\lambda^2$ is given by (2.51) with some constant $M$.

If $0 < \beta^2 < \frac{1}{2}$ an entire function $A(\lambda)$ with the asymptotic behavior (2.51) is completely determined by its zeroes $\lambda^2_k$, $k = 0, 1, \ldots$; it can be represented by a convergent product
\[
A(\lambda) = \prod_{k=0}^{\infty} \left(1 - \frac{\lambda^2}{\lambda^2_k}\right) ,
\]
where the normalization condition
\[
A(0) = 1
\]
is taken into account.

1 In lattice theory the equation (3.3) (which appears there “decorated” with some non-universal factors in the right hand side) leads to the Bethe Ansatz equations which completely determine the eigenvalues.

2 In fact one can prove this property for $T(\lambda)$, see [7].
Actually, one important remark about the normalization (2.38), (3.3) of the operators $A$ must be made here. For given eigenvalue $A(\lambda)$ the positions of its zeroes $\lambda_k$ depend on $p$ and for special values of this parameter one of these zeroes, say $k_0$, can happen to be at zero, $\lambda_{k_0} = 0$. For example, for the vacuum eigenvalue $A_{vac}(\lambda)$ this happens at $2p = -\beta^2 - n$; $n = 0, 1, 2, ...$, as brief inspection of (2.33) and (3.21) below shows. Obviously, at these values of $2p$ the normalization of the operators $A$ as in (2.38), (3.5) is not suitable as under this normalization all the terms in the power series $A(\lambda) = 1 + \sum_{n=1}^{\infty} \lambda^{2n} A_n$ except the first one would diverge. Of course this formal singularity can be eliminated by renormalization of $A$ with appropriate $p$-dependent factor. As our results below are not sensitive to this subtlety, in what follows we use the normalization (2.38), (3.5) (which we find technically very convenient) and just assume that $p$ does not take these “dangerous” values (i.e. that all $\lambda_k \neq 0$).

Introduce the function
\[
a(\lambda) = e^{4\pi ip} \frac{A(\lambda q)}{A(\lambda q^{-1})} .
\]
Setting $\lambda^2 = \lambda_k^2$ in (3.3) and remembering that $T(\lambda)$ has no singularities at finite $\lambda^2$ one obtains the Bethe-Ansatz type equations for the positions of the zeroes $\lambda_k^2$
\[
a(\lambda_k) = -1 .
\]
This is an infinite set of transcendental equations for the infinite set of unknowns $\lambda_k^2$. It can be transformed into a coupled system of a single non-linear integral equation and a finite set of equation (3.7) only for those zeroes which do not belong to the positive real axis in the $\lambda^2$-plane. This system is known as Destry-de Vega (DDV) equation [3]. In the Appendix A we show that under the assumptions (i)-(iii) given above the infinite set of equations (3.7) is equivalent to
\[
\begin{align*}
  i \log a(\theta) &= -\frac{2\pi p}{\beta^2} + 2M \cos \frac{\pi \xi}{2} e^\theta + i \sum_a' \log S(\theta - \theta_a) - 2G \ast \Im m \log (1 + a(\theta - i0)) \\
  a(\theta_a) &= -1 ,
\end{align*}
\]
where we have used new variables
\[
\lambda = e^{\frac{\theta}{1+i\xi}} , \quad \lambda_a = e^{\frac{\theta_a}{1+i\xi}} , \quad \beta^2 = \frac{\xi}{1+\xi}
\]
and $\lambda^2_a$ denote zeroes of $A(\lambda)$ lying outside the positive real axis of $\lambda^2$. The star in (3.8) denotes the convolution

$$A \ast B(\theta) = \int_{-\infty}^{\infty} d\theta' A(\theta - \theta') B(\theta')$$

and

$$S(\theta) = \exp\left\{-i \int_{0}^{\infty} \frac{d\nu}{\nu} \sin(\nu\theta) \frac{\sinh\left(\frac{\pi\nu(1 + \xi)}{2}\right)}{\cosh\left(\frac{\pi\nu}{2}\right) \sinh\left(\frac{\pi\nu\xi}{2}\right)}\right\},$$

$$G(\theta) = \delta(\theta) + \frac{1}{2\pi i} \partial_{\theta} \log S(\theta).$$

Note that the function $S(\theta)$ coincide with the soliton-soliton scattering amplitude for the Sine-Gordon model [25]. The equation (3.8) for the vacuum eigenvalues (in fact, more general equation obtained by replacing $e^{\theta}$ in the r.h.s. by $2\sinh\theta$) was originally derived from the Bethe ansatz equations associated with XXZ lattice model by taking appropriate continuous limit which leads to the sine-Gordon field theory [3]. Here we obtain (the “mass-less” version of) this equation directly in continuous field theory, bypassing any direct reference to the lattice theory.

Given a solution of (3.8) the function $\log A(\lambda)$ can be calculated as [3]

$$\log A(\lambda) = -i \int_{C_{\nu}} \frac{d\nu}{\nu} \frac{g(\nu)}{\cosh \pi\nu/2 \sinh \pi\nu\xi/2} \left(-\lambda^2\right)^{i\nu(1+\xi)/2},$$

where the function $g(\nu)$ is defined as

$$g(\nu) = \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} \Im m \log \left(1 + a(\theta - i0)\right) e^{-i\nu\theta}. \quad (3.13)$$

The integration contour $C_{\nu}$ in (3.12) goes along the line $\Im m \nu = -1 - \epsilon$ with arbitrary small positive $\epsilon$.

It is instructive to study the vacuum eigenvalue $A^{(\text{vac})}(\lambda)$ of $A(\lambda)$, defined by (2.49), in the limit $p \to +\infty$. Note that the parameter $p$ enters the DDV equation (3.8) exactly the same way the external gauge field (which couples to the soliton charge) appears in the sine-Gordon DDV equation [26]. The DDV equation (3.8) is greatly simplified in the limit $p \to +\infty$. According to the assumption (ii) above all zeroes $\lambda^2_k$ in this case are real and positive and so one can drop the term $i \sum_{a} \log S(\theta - \theta_a)$ in the r.h.s. of the integral equation (3.8). Moreover, in the limit $p \to +\infty$ this equation reduces to a linear equation of Winer-Hopf type (see Appendix A for some details)

$$-\frac{\pi p}{\beta^2} + M \cos \frac{\pi \xi}{2} e^{\theta} - \int_{-\infty}^{B(\nu)} \frac{d\theta'}{2\pi i} \partial_{\theta'} \log S(\theta - \theta') \Im m \log \left(1 + a^{(\text{vac})}(\theta')\right) = 0,$$
provided one assumes that

$$B(p) = \frac{1 + \xi}{2} \log \lambda_0^2 \sim \text{const} \log p, \quad p \to +\infty,$$  \hspace{1cm} (3.15)

where $\lambda_0^2$ is a minimal of the zeroes $\lambda_k^2$.

The equation (3.14) can be solved by the standard technique [27]. As a result one obtains

$$g^{(\text{vac})}(\nu)|_{p \to +\infty} \sim \frac{i p \sqrt{\pi}}{4 \nu} \frac{\Gamma(1 - i \nu(1 + \xi)/2)}{\Gamma(3/2 - i \nu/2) \Gamma(1 - i \nu \xi/2)} e^{i \delta \nu} \left( \lambda_0^2 \right)^{-i \nu(1 + \xi)/2},$$  \hspace{1cm} (3.16)

where $g^{(\text{vac})}(\nu)$ is related to $a^{(\text{vac})}(\theta)$ through (3.13)

$$
\left( \lambda_0^2 \right)^{(1+\xi)/2} \bigg|_{p \to +\infty} \sim \frac{\Gamma(\frac{\xi}{2}) \Gamma(\frac{1}{2} - \frac{\xi}{2})}{\sqrt{\pi} M} e^{\delta p}, \hspace{1cm} (3.17)
$$

and

$$2\delta = (1 + \xi) \log(1 + \xi) - \xi \log \xi.$$

Note that (3.17) supports the assumption (3.13). The equation (3.12) gives

$$\log A^{(\text{vac})}(\lambda)|_{p \to +\infty} \sim
-\frac{p}{2\pi^2 \xi} \int_{C_\nu} \frac{d\nu}{\nu^2} \frac{\Gamma(1 - i \nu(1 + \xi)/2) \Gamma(1 + i \nu \xi/2)}{\Gamma(-\frac{1}{2} + i \nu/2) \Gamma(1 - i \nu \xi/2)} e^{i \delta \nu} \left( -\frac{\lambda^2}{\lambda_0^2} \right)^{i \nu(1+\xi)/2},$$  \hspace{1cm} (3.18)

where integration contour $C_\nu$ go along the line $\text{Im} \nu = -1 - \epsilon$. The integrand $\sigma(\nu)$ in (3.18) has the simple asymptotic behavior in whole complex $\nu$-plane except positive imaginary axis

$$\sigma(\nu) \sim \frac{p \sqrt{\beta^2}}{\sinh \left( \pi \nu(1 + \xi)/2 \right)} \left( -\frac{\lambda^2}{\lambda_0^2} \right)^{i \nu(1+\xi)/2}, \quad \nu \to \infty$$

and we can work out the integral (3.18) for $|\lambda^2| < |\lambda_0^2|$ by closing the integration contour $C_\nu$ at the infinity in the lower half plane $\text{Im} \nu < -1$ and calculating the residues of the poles at $\nu = -2in/(1 + \xi)$, $n = 1, 2, \ldots$. The result can be written in the form

$$\log A^{(\text{vac})}(\lambda)|_{p \to +\infty} = -\sum_{n=1}^{\infty} y^{2n} H_n^{(\text{vac})}|_{p \to \infty},$$  \hspace{1cm} (3.19)
where \( y \) is given by \((2.35)\) and the above residue calculations give the coefficients \( H_n^{(v)}|_{p \to \infty} \) explicitly. According to \((2.34)\) this coefficients coincide with the large \( p \) asymptotic of the vacuum eigenvalues of the nonlocal IM \( H_n \). In particular, we find

\[
H_1^{(v)}|_{p \to +\infty} \sim \frac{\beta^4 \Gamma(\beta^2) \Gamma(1/2 - \beta^2)}{2\sqrt{\pi}} \frac{\beta^2 - 1}{p^2}. \tag{3.20}
\]

On the other hand an exact expression for \( H_1^{(v)} \) for all values of \( p \) is known explicitly from \((2.36)\)

\[
H_1^{(v)} = \frac{\beta^4 \Gamma(\beta^2) \Gamma(1 - 2\beta^2)}{\Gamma(1 - \beta^2 + 2\beta^2)} \frac{\Gamma(\beta^2 + 2p)}{\Gamma(1 - \beta^2 + 2p)}, \tag{3.21}
\]

where we have used the known vacuum eigenvalue

\[
G_1^{(v)} = \frac{4\pi^2 \Gamma(1 - 2\beta^2)}{\Gamma(1 - \beta^2 - 2p) \Gamma(1 - \beta^2 + 2p)}. \tag{3.22}
\]

of the nonlocal IM \( G_1 \) and \((2.36)\). Comparing the asymptotic of \((3.21)\) with \((3.20)\) one obtains exactly the coefficients \( M \) which enters the leading asymptotic behaviors \((2.51)\) of the operators \( A(\lambda) \):

\[
M = \frac{\Gamma(\xi/2) \Gamma(1/2 - \xi/2)}{\sqrt{\pi}} \left( \Gamma(1 - \beta^2) \right)^{1+\xi}. \tag{3.23}
\]

Calculating further coefficients in \((3.19)\) we find the large \( p \) asymptotic of the eigenvalues \( H_n^{(v)} \)

\[
H_n^{(v)}|_{p \to +\infty} \sim \frac{\Gamma(n\beta^2) \Gamma(-1/2 + n(1 - \beta^2))}{2\sqrt{\pi n!}} \left( \beta^2 \right)^{2n} p^{1 - 2n + 2n\beta^2}. \tag{3.24}
\]

It is not obvious at all how this asymptotic can be obtained directly from the definition of nonlocal IM \( H_n \) in terms of the ordered integrals \((2.10)\) through \((2.15)\) and \((2.34)\).

The large \( p \) asymptotic of vacuum eigenvalues of the dual nonlocal IM \((2.37)\) is obtained from \((3.24)\) by the substitution \( \beta^2 \to \beta^{-2} \), \( p \to \beta^{-2} \)

\[
\widetilde{H}_n^{(v)}|_{p \to +\infty} \sim \frac{\Gamma(n\beta^{-2}) \Gamma(-1/2 + n(1 - \beta^{-2}))}{2\sqrt{\pi n!}} \left( \beta^2 \right)^{-2n} p^{1 - 2n + 2n\beta^{-2}}. \tag{3.25}
\]

For completeness we also present here the large \( p \) asymptotic of the eigenvalues of the local IM

\[
I_{2n-1}^{(v)}|_{p \to +\infty} \sim \left( \beta^2 \right)^{-n} p^{2n}, \tag{3.26}
\]

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which follow directly from the definition of the operators $I_{2n-1}$, see [1]. Note that this asymptotic hold for any eigenvalues (not just the vacuum ones) of the local IM.

Let us now consider the large $\lambda$ behavior of (3.18). To obtain large $\lambda$ asymptotic expansion of $A^{(\text{vac})}(\lambda)$ one can close the integral in (3.18) in the upper half-plane $\Im m \geq -1$. Then (3.18) can be represented as a sum of infinitely many terms associated with the residues of the poles of the integrand located at $\Im m \geq -1$, plus the integral over the large circle. Note that the contribution of the large circle in fact diverges. Correspondingly, the sum of the residues gives only asymptotic series expansion for $A^{(\text{vac})}(\lambda)$, with zero radius of convergence. This asymptotic expansion has the form

$$A^{(\text{vac})}(\lambda) \simeq C^{(\text{vac})}(\beta^2, p) \left( -y^2 \right)^{-\frac{2p}{\beta^2}} \exp \left\{ \sum_{n=0}^{\infty} B_n \left( -y^2 \right)^{\frac{1-2n}{2+2\beta^2}} I^{(\text{vac})}_{2n-1} \right\} \times \exp \left\{ -\sum_{n=1}^{+\infty} (-1)^n \left( -y^2 \right)^{-\frac{n}{\beta^2}} \tilde{H}^{(\text{vac})}_n \right\}.$$ \hspace{1cm} (3.27)

where the variable $y$ is related to $\lambda$ as in (2.35),

$$C^{(\text{vac})}(\beta^2, p) \big|_{p \to +\infty} \sim (\beta^2)^{-\frac{2p}{\pi^2}} \left( \frac{2p}{e} \right)^{\frac{2p}{\beta^2}-2p}.$$ \hspace{1cm} (3.28)

and

$$B_n = \frac{(-1)^{n+1}}{2\sqrt{\pi} \left( 1 - \beta^2 \right) n!} \Gamma \left( \frac{2n-1}{2} \right) \Gamma \left( \frac{2n-1}{2-2\beta^2} \right) \left( \frac{2n-1}{2-2\beta^2} \right) \left( \beta^2 \right)^{n\beta^2+n-\beta^2}.$$ \hspace{1cm} (3.29)

Strictly speaking, here we have derived the expansion (3.27) in the limit $p \to +\infty$, so the quantities $I^{(\text{vac})}_{2n-1}$ and $\tilde{H}^{(\text{vac})}_n$ in (3.27) denote the $p \to +\infty$ asymptotic of the corresponding vacuum eigenvalues given by (3.26) and (3.25). However in the next section we adopt additional analyticity assumptions about the operators $A(\lambda)$ which allow for derivation of the same asymptotic expansion (3.27) for arbitrary $p$. Moreover, it is natural to expect that the expansion (3.27) holds not only for the vacuum eigenvalues but for the whole operator $A(\lambda)$; in that case of course the vacuum eigenvalues $I^{(\text{vac})}_{2n-1}$ and $\tilde{H}^{(\text{vac})}_n$ in (3.27) must be replaced by the operators $I_{2n-1}$ and $\tilde{H}_n$ themselves. 19
4. Conjectures: exact asymptotic expansions and duality

Consider the following operator-valued function of the complex variable $\nu$:

$$\Psi(\nu) = \frac{2\sqrt{\pi} \left( \Gamma(1 - \beta^2) \right)^{-i\nu(1+\xi)}}{\Gamma(i\nu/2) \Gamma(-1/2 + i\nu/2) \Gamma(-i\nu(1 + \xi)/2)} \times \int_{-\infty}^{0} \frac{d\lambda^2}{\lambda^2} \left( -\lambda^2 \right)^{-i\nu(\xi+1)/2} \log A(\lambda).$$

(4.1)

The function $\Psi(\nu)$ provides a continuous set of IM parameterized by the variable $\nu$. Indeed, the operators $A(\lambda)$ with different values of $\lambda$ as well as their arbitrary linear combinations commute among themselves and with all local and nonlocal IM. As we shall see below the function $\Psi(\nu)$ is remarkable in many respects, in particular, it can be thought as an analytic continuation of the local IM $I_{2n-1}$ to arbitrary complex values of their index $n$.

The integral (4.1) converges only for $2\beta^2 - 2 < \Im \nu < -1$, however the definition of $\Psi(\nu)$ can be extended to the whole $\nu$-plane by means of the analytic continuation. The latter can be done in various ways. For instance, using the product representation (3.4) which holds for any eigenvalue $A(\lambda)$ of the operator $A(\lambda)$ one can write the corresponding eigenvalue of $\Psi(\nu)$ as a generalized Dirichlet series of the form:

$$\Psi(\nu) = 2\sqrt{\pi} \frac{\Gamma(i\nu(1 + \xi)/2) \left( \Gamma(1 - \beta^2) \right)^{-i\nu(1+\xi)}}{\Gamma(i\nu/2) \Gamma(-1/2 + i\nu/2)} \sum_{k=0}^{\infty} \left( \lambda^2_k \right)^{-i\nu(1+\xi)/2}.$$

(4.2)

For $0 < \beta^2 < 1/2$ this series converges absolutely for $\Im \nu < -1$ so that (4.2) defines an analytic function of $\nu$ in the half plane $\Im \nu < -1$ which can be then analytically continued to the whole complex $\nu$-plane by the standard technique of the analytic continuation of Dirichlet series [28]. We expect that the function $\Psi(\nu)$ defined in this way enjoy the following remarkable analytic properties

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3. In writing (4.1) we make a technical assumption that all zeroes $\lambda_k$ of all eigenvalues of $A$ are away from zero, see the remark in Sect.3.

4. The definition (4.1) contains an ambiguity in the choice of the branches of the logarithm. In (4.2) this translates into the choice of the phases of the roots $\lambda^2_k$. This ambiguity, however, does not affect the following arguments. For definiteness one may assume that $|\arg \lambda^2_k| < \pi$ when $\lambda^2_k$ lies outside the negative real axis of $\lambda^2$, while for real negative $\lambda^2_k$ the corresponding entry in the sum (4.2) is replaced by $\cosh \left( \pi \nu(1 + \xi)/2 \right) |\lambda_k|^{-i\nu(1+\xi)}$. 

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Conjecture 1. For real $p$ and $0 < \beta^2 \leq \frac{1}{2}$ the function $\Psi(\nu)$ is an entire function of the complex variable $\nu$.

These simple analytic properties makes $\Psi(\nu)$ extremely convenient for studying eigenvalues of the $A$ and $T$ operators. Converting the integral transform in (4.1) one expresses $A(\lambda)$ as

$$\log A(\lambda) = -\frac{i}{4\pi^2} \int_{C_\nu} \frac{d\nu}{\nu} \left( \Gamma(1 - i\nu(1 + \xi)/2) \Gamma(i\nu\xi/2) \Gamma(-1/2 + i\nu/2) \times \left( \Gamma(1 - \beta^2) \right)^{i\nu(1+\xi)} \Psi(\nu) \left( -\lambda^2 \right)^{i\nu(1+\xi)/2} \right),$$

(4.3)

where the integration contour $C_\nu$ is the same as in (3.12). At the same time, the corresponding eigenvalue of the transfer matrix $T(\lambda)$ reads

$$T(\lambda) = A(q^{\frac{1}{2}}\lambda) + A^{-1}(q^{-\frac{1}{2}}\lambda),$$

(4.4)

where

$$\log A(\lambda) = 2\pi iP + \int_{C_\nu} \frac{d\nu}{\nu} \left( \Gamma(1 - i\nu(1 + \xi)/2) \Gamma(-1/2 + i\nu/2) \times \left( \Gamma(1 - \beta^2) \right)^{i\nu(1+\xi)} \Psi(\nu) \left( -\lambda^2 \right)^{i\nu(1+\xi)/2} \right).$$

(4.5)

The values of the function $\Psi(\nu)$ at special points on the imaginary $\nu$-axis (where the Gamma-functions in (4.3) and (4.5) display poles) are of particular interest. For example, it is not difficult to see that the values $\Psi(-2in(1-\beta^2))$, $n = 1, 2, \ldots, \infty$, are related to the eigenvalues of the nonlocal IM $H_n$. Using (3.4) and (2.34) one obtains for the latter

$$H_n = n^{-1} \left( \beta^{-2}\Gamma(1 - \beta^2) \right)^{-2n} \sum_{k=0}^{\infty} \lambda_k^{-2n}.$$

(4.6)

Then it follows from (4.2) that

$$\Psi(-2in(1-\beta^2)) = \frac{2 \sqrt{n!} \left( \beta^2 \right)^{-2n}}{\Gamma(n\beta^2) \Gamma(-\frac{1}{2} + n(1 - \beta^2))} H_n.$$

(4.7)

For other special values on the imaginary $\nu$-axis we will adopt the following

Conjecture 2. For real $p$ and $0 < \beta^2 \leq \frac{1}{2}$ the operator $\Psi(\nu)$ has the following special values on the imaginary $\nu$-axis

$$\Psi((2n - 1)i) = (\beta^2)^n I_{2n-1}, \quad n = 0, 1, \ldots,$$

(4.8)
\[ \Psi(2in(\beta^{-2} - 1)) = \frac{2^{\frac{2n}{\beta^2} + 1}}{\Gamma(n\beta^{-2}) \Gamma(-\frac{1}{2} + n(1 - \beta^{-2}))} \tilde{H}_n , \quad n = 1, 2, \ldots , \quad (4.9) \]

\[ \Psi(0) = P , \quad (4.10) \]

where \( \mathbf{I}_{2n-1} \) and \( \tilde{H}_n \) denote the local IM \((2.23)\) and the dual nonlocal IM \((2.37)\) respectively.

Note that \((4.8)\) with \( n = 0 \) reads

\[ \Psi(-i) = \mathbf{I}_{-1} \equiv \mathbf{I} , \quad (4.11) \]

where \( \mathbf{I} \) is the identity operator. The relations \((4.8)-(4.10)\) together with the conditions \((2.23), (2.37)\) makes it natural to assume that the operator \( \Psi \) satisfy the following duality condition:

**Conjecture 3.**

\[ \Psi\{\nu, \beta^2, \varphi(u)\} = (\beta^2)^{1-i\nu} \Psi\{\nu, \beta^{-2}, \beta^{-2}\varphi(u)\} . \quad (4.12) \]

An initial motivation for the above conjectures came from the study of the large \( p \) asymptotic of the vacuum eigenvalues of \( \mathbf{A}(\lambda) \) in the previous section. Indeed, comparing \((3.18)\) and \((4.3)\) one obtains

\[ \Psi^{(\text{vac})}(\nu)|_{p \to +\infty} \sim p^{1-i\nu} , \quad (4.13) \]

which with an account of \((3.26)\) and \((3.25)\) obviously satisfy all the statements of the Conjectures 1-3. Further motivations and justifications of these conjectures are discussed below.

We are now ready to derive exact asymptotic expansions of the eigenvalues for large \( \lambda^2 \). This is achieved by calculating the integrals \((4.3)\) and \((4.5)\) as formal sums over residues in the upper half plane \( \Im m \nu \geq -1 \). Using \((1.8)\) one thus has from \((4.3)\) for \( \lambda^2 \to \infty \)

\[ \log \mathbf{A}(\lambda) \simeq i \, m \, \mathbf{I} \left( -\lambda^2 \right)^{\frac{1}{2-2\beta^2}} - i \sum_{n=1}^{\infty} (-1)^n \, C_n \left( -\lambda^2 \right)^{\frac{1-2n}{2-2\beta^2}} \lambda^{(1-2n)(1+\xi)} \, \mathbf{I}_{2n-1} , \quad (4.14) \]

where

\[ m = \frac{2\sqrt{\pi} \, \Gamma\left( \frac{1}{2} - \frac{\xi}{2} \right)}{\Gamma\left( 1 - \frac{\xi}{2} \right)} \left( \Gamma\left( 1 - \beta^2 \right) \right)^{1+\xi} \quad (4.15) \]

\footnote{Like in \((3.26)\), the asymptotic \((4.13)\) holds for all eigenvalues of \( \Psi \), not just the vacuum ones.}
and

\[ C_n = \frac{\sqrt{\pi} (1 + \xi)}{n!} \left( \beta^2 \right)^n \frac{\Gamma((n - \frac{1}{2})(1 + \xi))}{\Gamma(1 + (n - \frac{1}{2})\xi)} \left( \Gamma(1 - \beta^2) \right)^{-(2n-1)(1+\xi)}. \]  \hspace{1cm} (4.16)

It follows then that the large \( \lambda \) asymptotic behavior of the operator \( T \) is given by \( (2.21) \), where the numerical coefficients \( m \) and \( C_n \) are given by the formulas \( (4.15) \) and \( (4.16) \). The asymptotic expansion \( (2.21) \) holds for

\[ -\pi < \arg \lambda^2 < \pi. \]  \hspace{1cm} (4.17)

Similarly one can get the asymptotic expansion for the operator \( A(\lambda) \) \( (4.3) \). Define the operator

\[ C = \left( \beta^2 \right)^{-\frac{2P}{\beta^2}} \left( 2 e^{i\partial_\nu \log \Psi(0)-1} \right)^{\frac{2P}{\beta^2}-2P}, \]  \hspace{1cm} (4.18)

which does not depend on the spectral parameter \( \lambda \). It is also convenient to use the variable \( y \) given by \( (2.35) \) instead of the variable \( \lambda \) and exhibit all arguments of \( A(iy) \):

\[ A\{iy, \beta^2, \varphi(u)\} \equiv \exp \left( -\sum_{n=1}^{\infty} (-1)^n \left( y^2 \right)^n H_n \right). \]  \hspace{1cm} (4.19)

Then, from \( (1.3) \) one has for \( |y| \rightarrow \infty, \ -\pi < \arg y^2 < \pi \)

\[ A\{iy, \beta^2, \varphi(u)\} \simeq C\{\beta^2, \varphi(u)\} \left( y^2 \right)^{-\frac{P}{\beta^2}} \exp \left\{ \sum_{n=0}^{\infty} B_n \left( y^2 \right)^{\frac{1-2n}{2-2\beta^2}} I_{2n-1} \right\} \times A\{iy^{-\frac{1}{\beta^2}}, \beta^{-2}, \beta^{-2}\varphi(u)\}, \]  \hspace{1cm} (4.19)

where the coefficients \( B_n \) are given by \( (3.29) \) and

\[ A\{iy^{-\frac{1}{\beta^2}}, \beta^{-2}, \beta^{-2}\varphi(u)\} \equiv \exp \left( -\sum_{n=1}^{\infty} (-1)^n \left( y^2 \right)^{-\frac{1}{\beta^2}} \tilde{H}_n \right). \]  \hspace{1cm} (4.20)

The explicit form of the operator \( C \) in \( (1.19) \) is not determined by the above calculations. For the vacuum eigenvalue its \( p \rightarrow +\infty \) asymptotic is given by \( (3.28) \). Moreover, one can show that \( (2.4) \)

\[ C^{(\text{vac})}(\beta^2, 0) = \sqrt{\beta^2}. \]  \hspace{1cm} (4.21)

We have the following conjecture about the exact form of this coefficient for all values of \( p \) and \( 0 < \beta^2 \leq \frac{1}{2} \)

\[ C^{(\text{vac})}(\beta^2, p) = \sqrt{\beta^2} \frac{\Gamma(1 + 2p\beta^{-2})}{\Gamma(1 + 2p)}. \]  \hspace{1cm} (4.22)
The formula for the asymptotic expansion (4.19) is essentially equivalent to the Conjectures 1,2 given above. In fact, the coefficients in front of powers of $\lambda$ in (4.19) are in one-to-one correspondence with the values (4.8) while any singularity of $\Psi(\nu)$ in the upper half plane would bring in some additional terms in (4.19).

The asymptotic expansions (2.21) and (4.19) are in remarkable agreement with the numerical calculation through the (modified) TBA equation. We postpone a detailed description of these calculations to a separate publication but just mention some the results here. In [1] we calculated numerically a few coefficients in (2.21) for two vacuum states ($\Delta = -1/5$ and $\Delta = 0$) in the $M_{2,5}$ CFT ($c = -22/5$) and found an excellent agreement with the corresponding exact eigenvalues of the local IM given explicitly in [1] (up to I, inclusive). Another numerical result concerning the part of (4.19) containing the dual nonlocal IM is mentioned in Sect.5.

As an additional support to our conjectures consider the case

$$\beta^2 = \frac{1}{2},$$

where the eigenvalues of $T$ and $A$-operators can be calculated explicitly. Moreover, the eigenvalues of the local IM can be independently found using the fermionic representation. The value $\beta^2 = \frac{1}{2}$ does not lie in SD (2.18) therefore the results of Sect.2 do not apply directly. In particular, the definitions (2.11) and (2.29) for $T(\lambda)$ and $A(\lambda)$ requires a renormalization since the nonlocal IM $G_{2n}$ in (2.15) and (2.33) diverge logarithmically at $\beta^2 = \frac{1}{2}$. It turns out that this renormalization affects the functional equation (3.3). To see this consider the expressions for the vacuum eigenvalues to within the first order in $\lambda^2$

$$T^{(vac)}(\lambda) = 2 \cos(2\pi p) + \lambda^2 G_1^{(vac)} + O(\lambda^4),$$

$$A^{(vac)}(\lambda) = 1 - \frac{\lambda^2}{4 \cos(2\pi p)} \left( G_1^{(vac)} - 2\pi^2 \sin(2\pi p) \right) + O(\lambda^4),$$

where

$$G_1^{(vac)} = 2 \cos(2\pi p) \left( 2\mathcal{C} - \pi\psi\left(1 - 2p\right) - \pi\psi\left(1 - 2p\right) \right),$$

$\psi(x) = \partial_x \log \Gamma(x)$ is the logarithmic derivative of the gamma-function and $\mathcal{C}$ is a (non-universal) renormalization constant depending on the ultraviolet cutoff. A simplest way to obtain these expressions is to set $\beta^2 = \frac{1}{2} - \epsilon$, $\epsilon \to 0$ in (3.21) and (3.22); that gives an analytic regularization of the divergent integrals with the value of $\mathcal{C}$

$$\mathcal{C} = \frac{\pi}{2\epsilon} + \pi\psi(1).$$
With the above accuracy in $\lambda^2$ the eigenvalues (4.23) (4.24) satisfy a “renormalized” functional equation

$$T(\lambda)A(\lambda) = e^{2\pi ip - i\pi^2 \lambda^2} A(q\lambda) + e^{-2\pi ip + i\pi^2 \lambda^2} A(q^{-1}\lambda),$$

(4.27)

where $q = \exp(i\pi/2)$. Using the lattice regularization (i.e., considering discrete approximations to the $\mathcal{P}$-exponents in (2.11) and (2.29) and then tending the number of partitions to infinity) one can show that the functional equation (4.27) is, in fact, exact in the sense that it is valid for arbitrary eigenvalues $T(\lambda)$ and $A(\lambda)$ to all order in $\lambda^2$.

The functional equation (4.27) completely determine the eigenvalues $T(\lambda)$ and $A(\lambda)$ provided one assumes them to be entire functions of $\lambda^2$. One can obtain

$$T(\lambda) = T^{(\text{vac})}(\lambda) \prod_{k=1}^{L} F(i\lambda, p, n_k^+, n_k^-) F(i\lambda, -p, n_k^-, n_k^+),$$

(4.28)

$$A(\lambda) = A^{(\text{vac})}(\lambda) \prod_{k=1}^{L} F(\lambda, p, n_k^+, n_k^-),$$

(4.29)

where

$$F(\lambda, p, n^+, n^-) = \frac{(2p - n^- + \frac{1}{2} - \pi \lambda^2)(2p + n^+ - \frac{1}{2})}{(2p + n^+ - \frac{1}{2} - \pi \lambda^2)(2p - n^- + \frac{1}{2})}$$

(4.30)

and $n_1^+, n_2^+, \cdots, n_L^+$ are two finite sequences of non-negative integers, $1 \leq n_1^+ < \cdots < n_L^+$, which uniquely specifies certain vector in the Fock space $\mathcal{F}_{p L}$; this vector has the Virasoro weight

$$\Delta\{p; n_1^+, \ldots, n_L^+, n_1^-, \ldots, n_L^-\} = \Delta^{(\text{vac})}(p) + \sum_{k=1}^{L} (n_k^+ + n_k^- - 1).$$

(4.31)

For the vacuum eigenvalues $L = 0$ and

$$T^{(\text{vac})}(\lambda) = \frac{2\pi e^{2c \lambda^2}}{\Gamma(1/2 + 2p + \pi \lambda^2) \Gamma(1/2 - 2p + \pi \lambda^2)},$$

(4.32)

$$A^{(\text{vac})}(\lambda) = e^{-c \lambda^2} \frac{\Gamma(2p + 1/2)}{\Gamma(2p + 1/2 - \pi \lambda^2)}.$$
It follows then from (4.2) that

\[ \Psi(\nu) = 2^{i\nu - 1} (i\nu - 1) \left\{ \zeta(i\nu, 2p + \frac{1}{2}) + E_L(i\nu, p) \right\}, \]

\[ E_L(s, p) = \sum_{k=1}^{L} \left\{ (2p + \frac{1}{2} - n_k^-)^{-s} - (2p - \frac{1}{2} + n_k^+)^{-s} \right\}, \]

where \( \zeta(s, \alpha) \) is the generalized zeta function defined as the analytic continuation of the series

\[ \zeta(s, \alpha) = \sum_{n=0}^{\infty} \left( \alpha + n \right)^{-s}, \quad \Re s > 1 \]

(4.35)
to the whole complex plane of the variable \( s \). This function is analytic everywhere in the \( s \)-plane except the point \( s = 1 \), where it has a simple pole. Therefore the eigenvalues (4.34) are entire functions of \( \nu \) in agreement with our conjecture.

The values of (4.34) at the integer points on the imaginary \( \nu \)-axis follow from the formula

\[ \zeta(-m, \alpha) = -\frac{B_{m+1}(\alpha)}{m+1}, \quad m = 0, 1, \ldots, \]

(4.36)
where \( B_m(\alpha) \) are the Bernoulli polynomials \([31]\). Note, in particular, that

\[ \Psi(-i) = 1, \quad \Psi(0) = p, \]

(4.37)
in agreement with (1.11) and (4.10). Further, the values of (4.34) at \( \nu = i(2n - 1) \), \( n = 0, 1, \ldots, \) conjectured in (4.8), perfectly match the eigenvalues of the local IM, which for \( \beta^2 = \frac{1}{2} \) can be independently obtained from the explicit expression for the local IM through free fermion fields

\[ I_{2n-1}^{(vac)} = 2^{-n} B_{2n}(2p + \frac{1}{2}) \].

(4.38)
The equation (4.9) can not be tested since the eigenvalues of the dual nonlocal IM \( \tilde{H}_n \) are not generally known independently except for the vacuum eigenvalue of \( \tilde{H}_1^{(vac)} \). For the latter one we have

\[ \Psi^{(vac)}(2i) = 2^{-3} B_3(2p + \frac{1}{2}) = p \left( p - \frac{1}{4} \right) \left( p + \frac{1}{4} \right). \]

(4.39)
in a precise agreement with (1.9) and the relation which is dual to (3.21).

Finally note that the asymptotic expansions of (4.28) and (4.29) agree with (2.21) and (4.19) provided one calculate the latter in the limit \( \beta^2 = \frac{1}{2} - \epsilon, \epsilon \to 0 \), and identifies the divergent part of the coefficient in the leading asymptotic of (2.21) and (1.19) with the renormalization constant \( C \) in (1.26), (4.32) and (4.33).
5. Vacuum eigenvalues of the Q - operators and non-equilibrium states in boundary sine-Gordon model

As is explained in Sect.2 the vacuum eigenvalue \( A^{(vac)}(\lambda) \) for integer \( 2p = N \) coincides with the Coulomb gas partition function (2.50). In fact it was already observed in [32] that the Coulomb gas partition function (2.50) with \( p = 0 \) satisfies the functional equation (1.4). The Coulomb gas partition function is obviously related to the finite temperature theory of mass-less bose field \( \Phi(t,x) \) on the half plane \( x < 0 \) with interaction at the boundary; its action is

\[
A = \frac{1}{4\pi\beta^2} \int_{-\infty}^{\infty} dt \int_{-\infty}^{0} dx \left( \Phi_t^2 - \Phi_x^2 \right) + \frac{\kappa}{\beta^2} \int_{-\infty}^{\infty} dt \cos \left( \Phi(t,0) + Vt \right). \tag{5.1}
\]

Here \( \beta^2, \kappa \) and \( V \) are parameters. This model finds interesting applications in dissipative quantum mechanics [33], [34], [35], [36]. As was discussed in [13], [14] it also describes the universal current through the point contact in quantum Hall system. At nonzero driving potential \( V \) and arbitrary temperature \( T \) the system (5.1) develops a stationary non-equilibrium state with

\[
J_B \equiv \frac{1}{2\pi} \langle \Phi_x(t,x) \rangle = -\kappa \langle \sin \left( \Phi(t,0) + Vt \right) \rangle \neq 0. \tag{5.2}
\]

This quantity is interpreted as the backscattering current through the point contact. In general case (5.2) is non-equilibrium expectation value and as such it requires non-equilibrium methods for its computation [13]. Note that if the driving potential is continued to pure imaginary values

\[
V = V_N \equiv 2\pi i N \beta^{-2} \beta \tag{5.3}
\]

We obtained this result independently, along with more general statement (3.3), before the paper [32] appeared.

The composite field \( \cos(\Phi + Vt) \) in (5.1) is assumed to be canonically normalized with respect to its short-distance behavior, i.e. \( \cos \left( \Phi(t) + Vt \right) \cos \left( \Phi(t') + Vt' \right) \sim 2^{-1} \left( i(t - t') + 0 \right)^{-2\beta^2} \), as is conventional in conformal perturbation theory. This is why no ultraviolet cutoff will appear in the matrix elements below.

The voltage \( V \) and current \( J_B \) (5.2) differs in normalization from the real voltage \( V^{(phys)} \) and total current \( J^{(phys)} \) in the Hall system

\[
V^{(phys)} = e^{-1} V, \quad J^{(phys)} = \frac{e}{h} \beta^2 \left( V + J_B \right),
\]

where \( e \) and \( h \) are the electron charge and Plank's constant. Also, \( \beta^2 \) coincides with the fractional filling of the Luttinger state in a Hall bar and the temperature is measured in energy units.
with integer $N$, one can make the Wick rotation $t \rightarrow -i\tau$ in (5.1) and formally define the Matsubara partition function

$$Z_N = \int [D\Phi] \exp(-A_M),$$

(5.4)

where

$$A_M = \frac{1}{4\pi\beta^2} \int_0^{\beta^2/T} d\tau \int_{-\infty}^{0} dx \left( \phi^2_{\tau} + \phi^2_x \right) - \frac{\kappa}{\beta^2} \int_0^{\beta^2/T} d\tau \cos \left( \Phi(0, \tau) + 2\pi N \beta^{-2} T \tau \right),$$

(5.5)

and the functional integral is taken over Euclidean fields $\Phi$ which satisfy the Matsubara condition $\Phi(\tau + \beta^2/T, x) = \Phi(\tau, x)$. It is easy to see that up to overall constant (the partition function (5.4) with $\kappa = 0$) (5.4) coincides with the series (2.50) with

$$\mu = \frac{\kappa}{2T} \left( \frac{\beta^2}{2\pi T} \right)^{-\beta^2}.$$ 

(5.6)

Moreover, it is possible to show, using non-equilibrium methods, that for $V = V_N$ (5.3) the expectation values of $e^{\pm i\Phi(t,0)}$ in (5.2) can be calculated in the Matsubara theory (5.5) as

$$\langle e^{\pm i\Phi} \rangle = \int [D\Phi] e^{\pm i\Phi(\tau,0)} e^{-A_M} / Z_N(\mu).$$

(5.7)

Of course equilibrium state can not support nonzero current and indeed it is not difficult to show that

$$\left. \{ e^{V\tau} \langle e^{i\Phi} \rangle - e^{-V\tau} \langle e^{-i\Phi} \rangle \} \right|_{V=V_N} = 0.$$ 

(5.8)

Nonetheless it is natural to expect that the non-equilibrium expectation value (5.2) can be obtained by some kind of analytic continuation of (5.7) back to real $V$. Needless to say this analytic continuation is ambiguous.

In [15] the current (5.2) is calculated exactly (for integer values of $\beta^{-2}$) using the Boltzmann equation with the distribution function of charge carriers determined through Thermodynamic Bethe Ansatz technique. A conjecture is proposed about exact current for arbitrary $\beta^2$ and $V$ in the recent paper [16]. It was suggested there to define the “partition function” $Z_{2p}(\mu)$ for $V = 4i\pi p \beta^{-2} T$ as certain analytic continuation of $Z_N$ which uses infinite sum expressions for the coefficients in (2.50) obtained with the help of
Jack polynomials \cite{24}. It is conjectured in \cite{16} that the current (5.2) can be expressed in terms of $Z_{2p}(\mu)$ as

$$
J_B(V, \mu, \beta^2) = i\pi T \mu \partial_{\mu} \log \frac{Z_{2p}(\mu)}{Z_{-2p}(\mu)}, \quad p = -\frac{i\beta^2 V}{4\pi T}.
$$

(5.9)

This conjecture agrees with the earlier conjecture for the linear conductance in \cite{24}. In fact, various checks performed in \cite{16} suggest that $Z_{2p}$ thus defined satisfies the functional relation (3.3). It is therefore very plausible that $Z_{2p}$ coincides with the vacuum eigenvalue $A^{(\text{vac})}_+(\lambda)$.

Indeed, we found that the result of \cite{29} is in complete agreement with the following formula\cite{11}

$$
J_B(V, \mu, \beta^2) = i\pi T \lambda \partial_{\lambda} \log \frac{A^{(\text{vac})}_+(\lambda)}{A^{(\text{vac})}_-(\lambda)},
$$

(5.10)

where

$$
\lambda = i \frac{\sin(\pi \beta^2)}{\pi} \mu, \quad p = -\frac{i\beta^2 V}{4\pi T}.
$$

Namely, the expression for $J_B$ obtained in \cite{15} can be written in the form (5.10) with $A_{\pm}(\lambda)$ solving the functional equation (3.3). Note that for $2p = N$ both $A^{(\text{vac})}_+$ and $A^{(\text{vac})}_-$ coincide with $Z_N$, so each of them defines certain analytic continuation of $Z_N$ to complex $N$. It is suggestive to note though that analytic properties of $A^{(\text{vac})}_+$ and $A^{(\text{vac})}_-$ as the functions of $p$ are different, namely $A^{(\text{vac})}_+$ is analytic at $\Re e 2p > -\beta^2$ whereas $A^{(\text{vac})}_-$ is analytic at $\Re e 2p < \beta^2$. There is no clear notion of partition function for a non-equilibrium state, and therefore it is remarkable that $A^{(\text{vac})}_+$ (and $A^{(\text{vac})}_-$ as well) admits interpretation as an “equilibrium-state” partition function of the system similar to (5.1) but with additional boundary degree of freedom described by the “q-oscillator” $E_{\pm}, \mathcal{H}$ in (2.29).

According to (4.19) the conjecture (5.10) implies in particular that the backscattering current (5.2) satisfies the following “strong-week barrier” duality relation

$$
J_B\left(V, \mu, \beta^2\right) \simeq -V - \beta^{-2} J_B\left(\beta^2 V, C\mu^{-\frac{1}{\beta^2}}, \beta^{-2}\right),
$$

(5.11)\footnote{We acknowledge a private communication with P. Fendley, F. Lesage and H. Saleur who explained to us how the definition of $Z_{2p}$ in \cite{16} must be understood.}

\footnote{Again, we have arrived at (5.10) independently, before the paper \cite{16} appeared. However we were significantly influenced by the conjecture about the universal conductance proposed in \cite{24} and by the results of \cite{13}.}
(remarkably, the factor containing the local IM in (4.19) cancels in the ratio $A_+/A_-$) which generalizes similar relation obtained in [15] for the case $T = 0$. The constant $C = C(\beta^2)$ in (5.11) reads explicitly:

$$C(\beta^2) = \left( \Gamma(1 + \beta^2) \right)^{\frac{\beta^2}{2}} \Gamma(1 + \beta^{-2}).$$  \hspace{1cm} (5.12)

Conventionally, one introduces the renormalized coupling parameter $X$ [13], [24]

$$X^2 = 2^{1-2\beta^2} \frac{\Gamma(1 + \beta^2)}{\Gamma(1/2 + \beta^2)} \mu^2,$$  \hspace{1cm} (5.13)

defined in such a way that

$$\partial_V J_B(V, X, \beta^2) \bigg|_{V=0} = -X^2 + O(X^4), \quad X^2 \to 0.$$  \hspace{1cm} (5.14)

According to the formula (5.11) the current admit the following decompositions:

$$J_B(V, X, \beta^2) = -\sum_{n=1}^{\infty} F_n(V, \beta^2) \ X^{2n}$$
$$\simeq -V + \beta^{-2} \sum_{n=1}^{\infty} F_n(\beta^2 V, \beta^{-2}) \ (\beta^{-2} K)^n \ X^{-\frac{2n}{\beta^2}},$$  \hspace{1cm} (5.15)

where $F_n$ are certain functions and the constant $K = K(\beta^2)$ is

$$K(\beta^2) = \beta^2 \left( \frac{\pi}{4} \right)^{\frac{1}{2} + \frac{1}{2\beta^2}} \left\{ \frac{\Gamma^3(1 + \beta^2)}{\Gamma(1/2 + \beta^2)} \right\}^{\frac{1}{\beta^2}} \frac{\Gamma^3(1 + \beta^{-2})}{\Gamma(1/2 + \beta^{-2})}. \hspace{1cm} (5.16)$$

Note that this constant determines the leading $X \to \infty$ asymptotic of the backscattering current,

$$J_B \simeq -V + 2\beta^{-2}T \ K(\beta^2) \ \sinh\frac{V}{2T} \ \frac{\Gamma(\beta^{-2} + i \frac{V}{2\pi T}) \Gamma(\beta^{-2} - i \frac{V}{2\pi T})}{\Gamma^2(\beta^{-2})} \ X^{-\frac{2}{\beta^2}} + O(X^{-\frac{4}{\beta^2}}).$$  \hspace{1cm} (5.17)

\footnote{In fact, exactly this duality relation (for the nonlinear mobility in associated dissipative quantum mechanics problem) was proposed a while ago in [35]. The arguments in [35] are based on “instanton” description of the hopping amplitudes in the strong barrier limit; from general point of view this description could be regarded as an approximation. Therefore it looks quite remarkable to us that this relation is indeed exact. See also the discussion in [15].}
We should stress that the formula (5.10) is a conjecture and therefore any checks would be valuable. As was mentioned above this formula agrees for integer $\beta^{-2}$ with the TBA solution obtained in [15]. Although the calculations in [15] are also based on a conjecture (exact applicability of the Boltzmann approximation to integrable systems), one could check (5.10) against numerical results obtained from the TBA solution. We have done that for $\beta^{-2} = 3$ and found an excellent agreement with (5.17). In particular our numerical value for the constant $K$ is

$$K_{num}(1/3) = 3.35485280612..., \quad (5.18)$$

(note that this number is slightly off from the numerical result for the same constant given in [37]) which is in agreement with the exact value

$$K(1/3) = \frac{3\sqrt{2}\pi^4}{5} \left( \frac{2\Gamma(7/6)}{\sqrt{3}} \right)^{1/2} = 3.35485280611990... \quad (5.19)$$

Completely rigorous check can be made in the case $\beta^{-2} = 2$, where the formula (5.10) agrees with the explicit calculations in the free-fermion theory [13], [15]

$$J_B(V, X, \frac{1}{2}) = -\frac{4TX^2}{i\pi} \left( \psi\left( \frac{1}{2} + \frac{2X^2}{\pi^2} + \frac{iV}{4\pi T} \right) - \psi\left( \frac{1}{2} + \frac{2X^2}{\pi^2} - \frac{iV}{4\pi T} \right) \right), \quad (5.20)$$

where $\psi(x) = \partial_x \log \Gamma(x)$. Another interesting limiting case is the classical limit $\beta^2 \to 0$. In this limit the eigenvalues $A_{\pm}^{(vac)}(\lambda)$ reduce to (see Appendix B):

$$A_{\pm}^{(vac)}(\lambda)\big|_{\beta^2 \to 0} \to 2^{\pm\rho} \Gamma(1 \pm \rho) X^{\mp \rho} J_{\pm \rho}(\sqrt{2}X), \quad (5.21)$$

where the variables

$$X = \sqrt{2} \beta^{-2} \lambda, \quad \rho = 2\beta^{-2} \rho$$

are kept fixed when $\beta^2 \to 0$. Substituting this into (5.10) one obtains after a little algebra

$$J_B(V, X, 0) = -V + \frac{2T \sinh \frac{V}{2T}}{I_{\rho}(\sqrt{2}X) I_{-\rho}(\sqrt{2}X)}, \quad \rho = -\frac{iV}{2\pi T}, \quad (5.22)$$

where $I_{\rho}(x)$ is modified Bessel function. As is known the field theory (5.1) can be interpreted in terms of dissipative quantum mechanics of a single particle in periodic potential [33], [34], [35], [36]. As $\beta^2$ plays the role of the Planck constant, in the limit $\beta^2 \to 0$ this reduces to the theory of classical Brownian particle at finite temperature $T$. In Appendix
C we study associated Fokker-Planck equation and calculate the classical current. This gives additional support to the conjecture \((5.10)\).

As was explained above, the vacuum eigenvalue \(A_{(\text{vac})}^{(\lambda)}(\lambda)\) at \(p = 0\) coincides with the partition function of \((5.1)\) at the temperature \(T\) and zero voltage \(V = 0\). Therefore the asymptotic expansion \((4.19), (4.20)\) specialized for the vacuum eigenvalue with \(p = 0\) is essentially the low temperature expansion for the associated free energy. In particular, the leading asymptotic of the heat capacitance \(C(X, \beta^2)\) of the point contact in the quantum Hall system at \(T \sim \text{const } X^{-\frac{1}{1-\beta^2}} \rightarrow 0\) and zero voltage \(V\) reads

\[
C(X, \beta^2) \sim \frac{\Gamma\left(\frac{1}{2-2\beta^2}\right)\Gamma\left(\frac{2-3\beta^2}{2-2\beta^2}\right)}{6 \sqrt{\pi} \beta^2} \left\{ \frac{\sqrt{\pi} \Gamma^3(1+\beta^2)}{2 \beta^4 \Gamma\left(\frac{1}{2} + \beta^2\right)} \right\} X^{-\frac{1}{1-\beta^2}}, \quad 0 < \beta^2 < \frac{2}{3},
\]

\[
C(X, \beta^2) \sim \frac{(1-\beta^2)^2 \Gamma^2(1+\frac{1}{\beta^2})\Gamma\left(\frac{3\beta^2-2}{2\beta^2}\right)}{\sqrt{\pi} \beta^4 \Gamma(2 - \frac{1}{\beta^2})} \left\{ \frac{\sqrt{\pi} \Gamma^3(1+\beta^2)}{2 \Gamma\left(\frac{1}{2} + \beta^2\right)} \right\} \beta^{\frac{2}{\beta^2}} X^{-\frac{2}{\beta^2}}, \quad \frac{2}{3} < \beta^2 < 1.
\]

Likewise, \((2.21)\) determines the low temperature expansion for the impurity free energy in the \(s = 1/2\) anisotropic Kondo problem.

6. Discussion

In this paper we have studied further how the powerful apparatus of the Yang-Baxter theory of integrability can be brought about directly in continuous Quantum Fields Theory. We have constructed the operators \(Q_{\pm}(\lambda)\), which are field theoretic versions of the \(Q\) - matrix of Baxter, directly in Conformal Field Theory. The \(Q\) - operators are constructed as the traces of certain monodromy matrices associated with the infinite - dimensional spaces - the representation spaces of “q-oscillator algebra” \((2.27)\). It is worth mentioning that our construction is not specific for the continuous theory - the \(Q\)- matrix of lattice theory admits similar representation (we will give the details elsewhere). We also found that the \(Q\) - operators thus constructed satisfy the remarkable relations \((2.40), (2.41), (2.42)\) and \((2.43)\). These relations allow one to employ the powerful machinery of nonlinear integral DDV equations to study the eigenvalues of the \(Q\) - operators in the highest weight Virasoro module. We have used the DDV approach to derive (under some analyticity conjectures) various asymptotic expansions for both \(Q\) and \(T\) operators. We also observed a remarkable (although somewhat puzzling) relation between the vacuum eigenvalues of the \(Q\) - operators and the stationary transport characteristics in boundary sine-Gordon model.
(the later also relate to the kinetic properties of one-dimensional quantum particle coupled to a dissipative environment). In this paper we did not present the derivation of our basic relations $(2.40), (2.41), (2.42)$ and $(2.43)$; this gap will be filled in the forthcoming paper [23].

Clearly, further study of the $Q$ and $T$ operators is desirable. First, almost all the discussion in this paper concerns the “Semi-classical Domain” $\beta^2 < 1/2$ (the case $\beta^2 = 1/2$ is studied explicitly through the free fermion theory in the Sect.4). However, the most interesting CFT (notably, the unitary CFT) lay outside this domain. We have argued in the Sect.2 that the $Q$ operators can be defined for $1 > \beta^2 > 1/2$ as well, but there are reasons to believe that outside the SD the analytic properties of both $Q$ and $T$ operators undergo significant (and very interesting) changes. We are planning to extend our analysis to the domain $1 > \beta^2 > 1/2$ in the future.

The analysis in this paper is concerned explicitly with the conformal field theories (more precisely, the chiral sectors of a CFT). One can argue however that the operators $Q_{\pm}(\lambda)$ (as well as the $T$ operators of [1]) can be defined in non-conformal integrable QFT obtained by perturbing the CFT with the local operator $\Psi_{1,3}$, the most important relations $(2.40), (2.41), (2.42)$ and $(2.43)$ remaining intact, and the most significant modifications being in analytic properties of these operators (they develop essential singularities at $\lambda^2 \to 0$ in the perturbed case). The $Q$ operators and associated (modified) DDV equations can be used then to study the finite-size spectra of these non-conformal theories. We will report some preliminary results in this direction in the forthcoming paper [38].

And finally it seems extremely desirable to get more understanding about the relation of the $Q$ - operators to the non-equilibrium properties of the boundary sine-Gordon model discussed in Sect.5 above. In particular it seems important to find a physical interpretation to the “$q$ - oscillator” degrees of freedom which evidently play central role in our construction of the $Q$ - operators.

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7. Appendix A.

In this Appendix we give details of the derivation of the DDV equation (3.8). For any eigenvalue \( A(\lambda) \) satisfying (3.3) consider the function \( a(\lambda) \) defined by (3.6). For real values of \( p \) the property (ii) of the function \( A(\lambda) \) (given in Sect.3 above the formula (3.4)) implies

\[
a(\lambda)^* = a(\lambda^*)^{-1},
\]

where the star denotes the complex conjugation. Next, it follows from leading asymptotic (4.16) of \( A(\lambda) \) at large \( \lambda \) that

\[
\log a(\lambda) \sim -2i M \cos(\pi \xi/2) \lambda^{\xi+1}, \quad \lambda^2 \to \infty, \quad |\arg \lambda^2| < 2\pi \beta^2,
\]

where \( \beta^2 = \frac{\xi}{1+\xi} \). Moreover, the function \( a(\lambda) \) obviously remains finite for small \( \lambda^2 \)

\[
a(\lambda) = 4\pi i p + O(\lambda^2), \quad \lambda^2 \to 0.
\]

The function \( a(\lambda) \) satisfies the Bethe-Ansatz type equations (3.7). The product representation (3.4) implies

\[
\log a(\lambda) - 4\pi i p = \sum_{k=0}^{\infty} F(\lambda \lambda_k^{-1}),
\]

where

\[
F(\lambda) = \log \frac{1 - \lambda^2 q^2}{1 - \lambda^2 q^{-2}}.
\]

The sum in the RHS of (7.4) can be written as a contour integral

\[
\log a(\lambda) - 4\pi i p = f(\lambda) + \int_C \frac{d\mu}{2\pi i} F(\lambda \mu^{-1}) \partial_\mu \log(1 + a(\mu)),
\]

where

\[
f(\lambda) = \sum_a F(\lambda \lambda_a^{-1})
\]

denotes a finite sum including only those zeroes \( \lambda_a^2 \) which do not lie on the positive real axis in the \( \lambda^2 \)-plane. The contour \( C \) goes from \(+\infty\) to zero above the positive real axis.
then winds around zero and returns to infinity below the positive real axis in the \( \lambda^2 \)-plane. Integrating by parts (boundary terms vanish due to (7.2), since \( 0 < \xi < 1 \) for \( \beta^2 \) in SD (2.18)) one obtains

\[
\begin{align*}
[f(\lambda)] &= f(\lambda) - \int_0^\infty \frac{d\mu}{2\pi i \mu} \lambda \partial_\lambda F(\lambda \mu^{-1}) \left\{ \log (1 + a(\mu + i0)) - \log (1 + a(\mu - i0)) \right\} \\
&= \int_0^\infty \frac{d\mu}{2\pi i \mu} \lambda \partial_\lambda F(\lambda \mu^{-1}) \Im \log (1 + a(\mu - i0)) - \int_0^\infty \frac{d\mu}{2\pi i \mu} \lambda \partial_\lambda F(\lambda \mu^{-1}) \log a(\mu),
\end{align*}
\]

where we have used (7.1).

Introducing new variables \( \theta, \theta' \) and \( \theta_k \) as

\[
\lambda = e^{\frac{\theta}{1+\xi}}, \quad \mu = e^{\frac{\theta'}{1+\xi}}, \quad \lambda_k = e^{\frac{\theta_k}{1+\xi}}
\]

and recalling that \( q = e^{\frac{\theta}{1+\xi}} \), one obtains

\[
[f(\theta)] = f(\theta) - 2i \int_{-\infty}^{\infty} d\theta' \ R(\theta - \theta') \Im \log (1 + a(\theta' - i0)) + \int_{-\infty}^{\infty} d\theta' \ R(\theta - \theta') \log a(\theta').
\]

(7.10)

where

\[
R(\theta) = \frac{i}{2\pi(1+\xi)} \lambda \partial_\lambda F(\lambda).
\]

With the standard notation for the convolution (3.10), the equation (7.10) can be written as

\[
K \ast \log a(\theta) = 4\pi ip + f(\theta) - 2i \ R \ast \Im \log (1 + a(\theta - i0)),
\]

(7.12)

where

\[
K(\theta) = \delta(\theta) - R(\theta).
\]

(7.13)

Let us now apply the inverse of the integral operator \( K \) to both sides of (7.12); in this one has to add an appropriate zero mode of \( K \) to the r.h.s. of the resulting equation to make it consistent with the asymptotic conditions (7.2) and (7.3). In this way one obtains the integral equation (3.8) in the main text. As is noted there, this integral equation has to be complemented by a finite number of the transcendental equations (3.7) for the roots \( \lambda^2_a \) lying outside the positive real axis of \( \lambda^2 \).

For the vacuum eigenvalues with \( 2p > -\beta^2 \) all the roots \( \lambda^2_k \) are real and positive and the term \( i \sum \phi_a \log S(\theta - \theta_a) \) in the r.h.s. of (3.8) is absent. Let \( \lambda^2_0 \) be the minimal of the zeroes \( \lambda^2_k \). Introduce the function \( a^{(vac)}(\lambda) \) related to \( A^{(vac)}(\lambda) \) as in (3.9) and denote

\[
B(p) = \frac{1 + \xi}{2} \log \lambda^2_0.
\]

(7.14)
The function \( \log \left( 1 + a^{(\text{vac})}(\theta) \right) \) is analytic in the strip \(-\pi \xi < \Im \theta < \pi \xi\) with the branch cut along the positive real axis from \(B(p)\) to infinity. So it is analytic for real \(\theta < B(P)\), where it obeys the relation

\[
\log a(\theta) = 2 \Im \log \left( 1 + a(\theta - i0) \right),
\]

which follows from \((7.1)\). Note that the infinitesimal shift \(-i0\) here is not essential. Taking \((7.15)\) into account one can rewrite the \((3.8)\) for \(\theta < B(p)\) as

\[
\frac{-\pi p}{\beta^2} + M \cos \frac{\pi \xi}{2} e^\theta - \int_{-\infty}^{B(P)} \frac{d\theta'}{2\pi i} \partial_\theta \log S(\theta - \theta') \Im \log \left( 1 + a^{(\text{vac})}(\theta') \right)
= \int_{B(p)}^{+\infty} \frac{d\theta'}{2\pi i} \partial_\theta \log S(\theta - \theta') \Im \log \left( 1 + a^{(\text{vac})}(\theta' - i0) \right).
\]

Let us make technical assumption

\[
B(p) \sim \text{const} \log p \quad \text{as} \quad p \to +\infty.
\]

Then one can show that the r.h.s. of \((7.16)\) decreases at the large positive \(p\) and therefore can be dropped in the leading approximation at \(p \to +\infty\). This brings \((7.16)\) to the linear integral equation \((3.14)\) of the Winer-Hopf type.

8. Appendix B

In this appendix we consider the functional equations for the eigenvalues of \(T\) and \(Q\) operators in the classical limit \(\beta^2 \to 0\), where they reveal a remarkable connection with the theory of classical Liouville equation \([13]\). We start with the functional equations for the eigenvalues \(T_j(\lambda)\) of the operators \((2.11)\), which follow from \((2.13)\); they can be written in the form \([1]\)

\[
T_j(q^{\frac{1}{2}} \lambda) T_j(q^{-\frac{1}{2}} \lambda) = 1 + T_{j-\frac{1}{2}}(\lambda) T_{j+\frac{1}{2}}(\lambda)
\]

Consider limiting values of the eigenvalues \(T_j(\lambda)\) and \(Q_{\pm}(\lambda)\), when

\[
2\lambda = \beta^2 e^\sigma, \quad j = \frac{\tau}{\pi \beta^2}, \quad 2p = \beta^2 \rho, \quad \beta^2 \to 0,
\]

\[\text{The fact that the functional relations associated with the TBA equations have many features in common with the classical Liouville equation is well known to experts } [26].\]
and the variables $\sigma$, $\tau$ and $\rho$ are kept fixed. We assume that in this limit

$$T_j(\lambda) \to \frac{2}{\pi \beta^2} e^{-\phi(\sigma, \tau)}, \quad Q_{\pm}(\lambda) \to (\beta^2)^{\pm \rho} Q_{\pm}(\sigma), \quad (8.3)$$

where $\phi(\sigma, \tau)$ and $Q_{\pm}(\sigma)$ are smooth functions of their arguments. Then it is easy to see that the functional equation (8.1) becomes the classical Liouville equation

$$(\partial^2_\sigma + \partial^2_\tau) \phi(\sigma, \tau) = e^{2\phi(\sigma, \tau)} \quad (8.4)$$

for the field $\phi(\sigma, \tau)$ in the Euclidean space. It should be complemented by the periodical boundary conditions

$$\phi(\sigma + i\pi, \tau) = \phi(\sigma, \tau), \quad (8.5)$$

since $T_j(\lambda)$ in (8.3) is a single-valued function of $\lambda^2$. The limiting form of the functional equation (2.43) reads

$$e^{-\phi(\sigma, \tau)} = (4i\rho)^{-1} \left( Q_+(\sigma + i\tau)Q_-(\sigma - i\tau) - Q_+(\sigma - i\tau)Q_-(\sigma + i\tau) \right), \quad (8.6)$$

whereas (2.42) becomes the ordinary Wronskian condition

$$\partial_\sigma Q_+(\sigma) \ Q_-(\sigma) - Q_+(\sigma) \ \partial_\sigma Q_-(\sigma) = 2\rho. \quad (8.7)$$

for the function $Q_{\pm}(\sigma)$.

One can recognize in (8.6), (8.7) the general local solution of the Liouville equation (8.4) satisfying the condition

$$e^{-\phi(\sigma, \tau)} = \tau + O(\tau^2), \quad \tau \to 0. \quad (8.8)$$

The Baxter’s relation (1.4) reduces to the second order differential equation

$$(\partial^2_\sigma + w(\sigma)) \ Q(\sigma) = 0, \quad (8.9)$$

with a periodic potential

$$w(\sigma + i\pi) = w(\sigma), \quad (8.10)$$

which is determined by the leading asymptotic of the eigenvalue $T(\lambda) \equiv T_{\frac{1}{2}}(\lambda)$ in the limit (8.2)

$$T(\lambda) = 2 + (\pi \beta^2) w(\sigma) + O(\beta^4) \quad (8.11)$$
The functions $Q_{\pm}(u)$ are just two linearly independent Bloch-wave solutions
\[ Q_{\pm}(\sigma + i\pi) = e^{\pm\pi i\rho} Q_{\pm}(\sigma) . \tag{8.12} \]
to the second order differential equation (8.9). Note that (8.12) imply the periodicity (8.3) of the solutions (8.6) for the Liuoville equation (8.4).

It is illustrative to apply the above limiting procedure to the vacuum eigenvalues of $T$ and $Q$ operators. Notice that all the vacuum eigenvalues of the nonlocal IM (2.16) entering the series expansion (2.15) remain finite in the limit (8.2). Therefore only two first terms of the expansion
\[ T^{(\text{vac})}(\lambda) = 2 \cos(2\pi p) + \lambda^2 G_1^{(\text{vac})} + O(\lambda^4) . \tag{8.13} \]
contribute to (8.11). Using (3.22) one thus obtains
\[ w(\sigma) = e^{2\sigma} - \rho^2 \tag{8.14} \]
The solutions to the differential equation
\[ (\partial_\sigma^2 + e^{2\sigma} - \rho^2) Q_{\pm}(\sigma) = 0 , \]
satisfying the conditions (8.7), (8.12) have the forms
\[ Q_{\pm}(\rho) = \Gamma(1 \pm \rho) J_{\pm\rho}(e^{\sigma}) , \]
where $J_{\sigma}(x)$ is the conventional Bessel function.

The eigenvalues $T_j^{(\text{vac})}(\lambda)$ in the limit (8.2) are then expressed from (8.3) through the corresponding solution (8.6) of the Liouville equation
\[ T_j^{(\text{vac})}(\lambda) \rightarrow \frac{2}{\pi \beta^2} e^{-\phi(\sigma,\tau)} \]
\[ = \frac{1}{2i \beta^2 \sin \pi \rho} \left( J_{\rho}(e^{\sigma+i\tau}) J_{-\rho}(e^{\sigma-i\tau}) - J_{\rho}(e^{\sigma-i\tau}) J_{-\rho}(e^{\sigma+i\tau}) \right) . \tag{8.15} \]
It would be interesting to check this formula against the solutions to the TBA equations in the limit $\beta^2 \rightarrow 0$. 

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9. Appendix C

As is discussed in [33] the field theory (5.1) describes a dissipative quantum mechanics of one dimensional particle if one interprets the boundary value \( \Phi(t) \equiv \Phi(t,0) \) as the position of the particle, the bulk degrees of freedom \( \Phi(t,x), \ x \neq 0 \) playing the role of thermostat. As \( \beta^2 \) enters (5.1) as the Planck’s constant, in the limit \( \beta^2 \to 0 \) this problem reduces to the one of classical Brownian particle which is described by the Langevin equation

\[
\frac{1}{2\pi} \dot{\Phi}(t) = -\kappa \sin(\Phi(t) + Vt) + \xi(t), \quad (9.1)
\]

where \( \dot{\Phi} = \partial_t \Phi(t) \), and \( \xi(t) \) is the white noise

\[
\langle \xi(t)\xi(t') \rangle = \frac{T}{\pi} \delta(t-t') . \quad (9.2)
\]

We are interested in the limiting value of the average velocity

\[
J_B(V) = \langle \dot{\Phi} \rangle_{t \to \infty} . \quad (9.3)
\]

The time dependence of the potential term in (9.1) can be eliminated by the substitution

\[
z(t) = \Phi(t) + Vt , \quad (9.4)
\]

which transforms (9.1) to the form

\[
\dot{z} = V - 2\pi \kappa \sin(z) + 2\pi \xi . \quad (9.5)
\]

The driving force \( V \) appears here because the transformation (9.4) brings us to the frame which moves with the velocity \( -V \) with respect to the thermostat. The current (9.3) then is

\[
J_B(V) = -V + J(V) , \quad J(V) = \langle \dot{z} \rangle_{t \to \infty} . \quad (9.6)
\]

The probability distribution \( P(z) \) associated with the stochastic process (9.5) satisfies the Fokker-Planck equation

\[
\partial_t P = 2\pi T \partial_z \{ (-\nu + \sqrt{2X} \sin z) P + \partial_z P \} , \quad (9.7)
\]

where

\[
\nu = \frac{V}{2\pi T} , \quad \sqrt{2X} = \frac{\kappa}{T} . \quad (9.8)
\]
The suitable solution of (9.7), which describes the stationary drift has the form (see e.g. [39])

\[ P(z) = N^{-1} P_0(z) , \quad P_0(z) = e^{\sqrt{2}X \cos(z) + \nu z} \int_z^{z+2\pi} \frac{dy}{2\pi} e^{-\sqrt{2}X \cos(y) - \nu y} , \quad (9.9) \]

where

\[ N = \int_0^{2\pi} dz \ P_0(z) \quad (9.10) \]

is the normalization constant. The current \( J \) can be then expressed as

\[ J = 2\pi T N^{-1} \left ( 1 - e^{-2\pi\nu} \right ) . \quad (9.11) \]

The integral (9.10) is calculated explicitly with the result

\[ J = \frac{2T \sinh(\pi\nu)}{I_{i\nu}(\sqrt{2}X) \ I_{-i\nu}(\sqrt{2}X)} , \quad (9.12) \]

where \( I_{\rho}(x) \) is the modified Bessel function. With the account (9.6), (9.8), (5.6), (5.13), this agrees perfectly with (5.22).
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