Angular width of Cherenkov radiation with inclusion of multiple scattering: an path-integral approach

Jian Zheng

CAS Key Laboratory of Geospace Environment and Department of Modern Physics,
University of Science and Technology of China,
Hefei, Anhui 230026, Peoples Republic of China and
IFSA Collaborative Innovation Center, Shanghai Jiao Tong University,
Shanghai 200240, Peoples Republic of China

Abstract

Visible Cherenkov radiation can offers a method of the measurement of the velocity of a charged particles. The angular width of the radiation is important since it determines the resolution of the velocity measurement. In this article, the angular width of Cherenkov radiation with inclusion of multiple scattering is calculated through the path-integral method, and the analytical expressions are presented. The condition that multiple scattering process dominates the angular distribution is obtained.

PACS numbers: PACS number: 41.60.Bq, 52.25.Os

Keywords: Cherenkov radiation, multiple scattering, path integral method

*Electronic address: jzheng@ustc.edu.cn
I. INTRODUCTION

The Cherenkov radiation can be emitted when a fast charged particle moves in a transparent medium with a speed of \( v \) greater than the light speed in the medium. The radiation appears mainly at an angle from the path of the particle given by \( \vartheta_0 = \arccos[1/(n\beta)] \), where \( \beta = v/c \) is the normalized speed, \( c \) is the light speed in vacuum, and \( n \) is the refractive index of the medium. If the speed \( v \) is not too close to unity, the measurement of \( \vartheta_0 \) offers a convenient method for the determination of \( v/c \), and hence the momentum and energy of the particle. Recent experiments show that this method can be applied to the study of energetic electrons generated in ultra-intense laser plasma interactions \[1,3\]. Conical forward THz emission from femosecond-laser-beam filamentation in air is also partially attributed to Cherenkov radiation by some authors \[4\]. For the case of simplicity, the motion of electrons is usually assumed uniform \[5\]. However, as an electron moves in a medium, it suffers multiple scattering with the nuclei in medium, and no longer moves uniformly. Multiple scattering changes the average direction of the particle and partially destroys the coherence of the radiation along the path, leading to a wider angular width of the Cherenkov emission cone and a little lower radiation intensity \[6,8\]. Although modern numerical simulations can be performed to address this topic, an analytical approach could provide a deeper insight. In this article, based upon the assumption that the scattering angles between energetic electrons and medium particles are small, we calculate the effect of multiple scattering on Cherenkov radiation with the method of path integral \[9\]. Only the effects of multiple scattering and diffraction will be considered here. Analytical formulas of the spectral power of the radiation are derived in certain circumstances.

II. RADIATION FROM ENERGETIC ELECTRONS WITH INCLUSION OF MULTIPLE SCATTERING

Radiation from a rapid moving charged particle can be well described with the classical theory of electrodynamics. The spectral intensity of the radiation emitted from a moving electron is given by \[10\]

\[
\frac{d^2 \mathcal{E}}{d\omega d\Omega} = \frac{e^2 \omega^2}{(2\pi)^2 c^3} \int_0^T dt \int_0^T dt' e^{i\omega(t-t')} \left[ \mathbf{v}(t) \cdot \mathbf{v}(t') - \frac{\omega^2}{k^2} \right] e^{-i\mathbf{k}[\mathbf{r}(t)-\mathbf{r}(t')]},
\]  

(1)
where \( v(t) \) is the velocity, \( r(t) \) is the trajectory, \(-e\) is the electron charge, and \( T \) is the time that the electron passes through a medium which is assumed a transparent slab in our article. The wave frequency \( \omega \) and wave number \( k \) in Eq. (1) satisfy the dispersion relation of electromagnetic waves in the medium,
\[
\frac{\omega}{k} = \frac{c}{n},
\]
where \( n \) is the refractive index of the transparent slab.

When the scattering angle is small, the velocity of the electron can be approximately written as,
\[
v(t) \simeq v(t) + v \left[ 1 - \frac{1}{2} \theta^2(t) \right] e_z. \tag{2}
\]
Here \( \theta(t) = \theta(x) e_x + \theta(y) e_y \) is an two-dimensional random vector describing the scattering angle which is assumed small, and \( \theta^2 = \theta_x^2 + \theta_y^2 \). In writing Eq. (2), we assume that the incident electron is along the \( z \)–direction, and already neglect the energy loss of the electron in the medium. With aid of Eq. (2), we have
\[
v(t) \cdot v(t') \simeq v^2 \left\{ 1 + \theta(t) \cdot \theta(t') - \frac{\theta^2(t) + \theta^2(t')}{2} \right\}, \tag{3}
\]
and
\[
\Phi(t) = \frac{1}{2} k v n_z \int_0^t \theta^2(\tau) d\tau - k v n_\perp \cdot \int_0^t \theta(\tau) d\tau, \tag{4a}
\]
where \( n = n_\perp + n_z e_z \) is the unit vector along the wave vector \( k \), \( n_z = \cos \vartheta \), and \( \vartheta \) is the angle between the radiation emission and the \( z \)–axis. Substituting Eqs. (3) and (4) into Eq. (1), the radiation intensity can be written as a functional of the random scattering angle \( \theta \),
\[
\frac{d^2 \mathcal{E}}{d\omega d\Omega} = \frac{e^2 \omega^2 e^2}{(2\pi)^2 c^3} \left( 1 - \frac{1}{\beta^2 n^2} \right) \int_0^T dt \int_0^T dt' e^{i \omega (1-n \beta \cos \vartheta) (t-t')} e^{i \Phi(t) - i \Phi(t')}
\]
\[
+ \frac{e^2 \omega^2 e^2}{(2\pi)^2 c^3} \int_0^T dt \int_0^T dt' [\theta(t) \cdot \theta(t')] e^{i \omega (1-n \beta \cos \vartheta) (t-t')} e^{i \Phi(t) - i \Phi(t')}
\]
\[
- \frac{e^2 \omega^2 e^2}{2(2\pi)^2 c^3} \int_0^T dt \int_0^T dt' \left[ \theta^2(t) + \theta^2(t') \right] e^{i \omega (1-n \beta \cos \vartheta) (t-t')} e^{i \Phi(t) - i \Phi(t')} \tag{5}
\]
Equation (5) has to be averaged over the random variables,
\[
\frac{d^2\mathcal{E}}{d\omega d\Omega} = \frac{e^2\omega^2v^2}{(2\pi)^2c^3} \left(1 - \frac{1}{\beta^2n^2}\right) \left| \left\langle \int_0^T e^{i\omega(1-n^2\cos\theta)t+i\Phi(t)} dt \right\rangle \right|^2 + e^2\omega^2v^2\left\langle \left| \int_0^T e^{i\omega(1-n^2\cos\theta)t+i\Phi(t)} dt \right| \right|^2 + e^2\omega^2v^2\left| \left\langle \int_0^T e^{i\omega(1-n^2\cos\theta)t+i\Phi(t)} dt \right\rangle \right|^2 - \frac{e^2\omega^2v^2}{(2\pi)^2c^3} \text{Re} \left\langle \left\langle \int_0^T e^{i\omega(1-n^2\cos\theta)t+i\Phi(t)} dt \right\rangle \left\langle \int_0^T e^{-i\omega(1-n^2\cos\theta)t-i\Phi(t)} dt \right\rangle \right\rangle,
\]
where \(\langle \cdots \rangle\) denotes the average. In regard to the random process related to multiple small-angle scattering in an isotropic medium, the transition probability of the scattering angle \(\theta\) obeys the Gaussian distribution,
\[
P(\theta, t) = \frac{1}{(q\pi t)^{\frac{3}{2}}} e^{-\theta^2/(qt)}.
\]
The average can then be performed through the path-integral approach [9],
\[
\langle f[\theta(t)] \rangle = \lim_{N \to \infty, N\Delta \to 0} \frac{1}{(q\pi \Delta)^{N/2}} \int d\theta_1 \cdots d\theta_N \exp \left\{-\sum_{n=0}^{N-1} \frac{(\theta_{n+1} - \theta_n)^2}{q\Delta} \right\} f(\theta).
\]
Given the slab thickness of \(\ell\), the time \(T\) can be considered as a constant within the range of \(qT \ll 1\), i.e.,
\[
T \approx \frac{\ell}{v}.
\]
We then have
\[
\frac{d^2\mathcal{E}}{d\omega d\Omega} = \frac{e^2\omega^2v^2}{(2\pi)^2c^3} \left(1 - \frac{1}{\beta^2n^2}\right) \left| \left\langle \int_0^T dt e^{i\omega(1-n^2\cos\theta)t+i\Phi(t)} \right\rangle \right|^2 + e^2\omega^2v^2\left\langle \left| \int_0^T dt e^{i\omega(1-n^2\cos\theta)t+i\Phi(t)} \right| \right|^2 + e^2\omega^2v^2\left| \left\langle \int_0^T dt e^{i\omega(1-n^2\cos\theta)t+i\Phi(t)} \right\rangle \right|^2 - \frac{e^2\omega^2v^2}{(2\pi)^2c^3} \text{Re} \left\langle \left\langle \int_0^T dt e^{i\omega(1-n^2\cos\theta)t+i\Phi(t)} \right\rangle \left\langle \int_0^T dt e^{-i\omega(1-n^2\cos\theta)t-i\Phi(t)} \right\rangle \right\rangle,
\]
It can be shown that [seeing in the Appendix]
\[
\langle e^{i\Phi(t)} \rangle = \frac{1}{\sinh \Omega t} \exp \left\{-i\frac{k\nu\eta}{2n_z} \left[ t - \frac{1}{\Omega} \tanh \Omega t \right] \right\},
\]
\[
\langle \theta_x(t)e^{i\Phi(t)} \rangle = \frac{n_x}{n_z} \cosh \Omega t - 1 \exp \left\{-i\frac{k\nu\eta}{2n_z} \left[ t - \frac{1}{\Omega} \tanh \Omega t \right] \right\},
\]
\[
\langle \theta_y(t)e^{i\Phi(t)} \rangle = \frac{n_y}{n_z} \cosh \Omega t - 1 \exp \left\{-i\frac{k\nu\eta}{2n_z} \left[ t - \frac{1}{\Omega} \tanh \Omega t \right] \right\},
\]
\[
\langle \theta^2(t)e^{i\Phi(t)} \rangle = \frac{q}{\Omega} \sinh \Omega t + \frac{n_z^2}{n_z^2} \left[ \cosh \Omega t - 1 \right]^2 \exp \left\{-i\frac{k\nu\eta}{2n_z} \left[ t - \frac{1}{\Omega} \tanh \Omega t \right] \right\}.
\]
where

\[ \Omega = \Omega_0 e^{-i\pi/4}, \]
\[ \Omega_0 = (q\omega)^{1/2}(n\beta/2)^{1/2} \cos^{1/2} \vartheta. \]

It is easily to show that the second and third terms on the right hand side of Eq. (9) is much smaller than the first term in the case of \( q/\omega \ll 1 \). Therefore, the radiation intensity can be approximately written as

\[ \frac{d^2 \mathcal{E}}{d\omega d\Omega} = \frac{e^2 v^2}{(2\pi)^2 c^3} \left( 1 - \frac{1}{n^2 \beta^2} \right) F(\vartheta), \]

where the function \( F(\vartheta) \) describes the angular distribution of the radiation

\[ F(\vartheta) = \left| \int_0^{T_\omega} d\xi \exp \left\{ i(1 - n\beta \cos \vartheta) \xi - i \frac{(n\beta/2) \sin^2 \vartheta}{\cos \vartheta} \left[ \xi - \frac{\omega}{\Omega_0} e^{i\pi/4} \tanh \left[ e^{-i\pi/4}(\Omega_0/\omega)\xi \right] \right] \cosh \left[ e^{-i\pi/4}(\Omega_0/\omega)\xi \right] \right\}^2 \right| \]

(11)

In real experiment, we usually have

\[ T_\omega = \frac{\ell}{v} \frac{2\pi c}{\lambda} = \frac{2\pi \ell}{\beta \lambda} \gg 1, \]

where \( \lambda \) is the wavelength of the radiation in vacuum, and \( \beta = v/c \) is the normalized charge velocity. Usually, we also have the condition

\[ \frac{q}{\omega} \ll 1. \]

The ratio of \( \Omega_0 \) to \( \omega \) is then a small number,

\[ (\Omega_0/\omega) = (q/\omega)^{1/2}(n\beta/2)^{1/2} \cos^{1/2} \vartheta \ll 1. \]

In this case, the amplitude of the integrand in Eq. (11) decays to the half of its maximum around

\[ \xi_0 = \frac{3^{1/3}}{(q/\omega)^{1/3}(n\beta)^{1/6} \cos^{1/6} \vartheta}. \]

(12)

If the parameter \( \xi_0 \) satisfies the condition

\[ \xi_0 \gg T_\omega, \]

(13)

we can make the approximation

\[ F(\vartheta) \simeq \left| \int_0^{T_\omega} d\xi \exp \left\{ i(1 - n\beta \cos \vartheta) \xi \right\} \right|^2 \]

\[ = 4 \frac{\sin^2[(1 - n\beta \cos \vartheta)T_\omega/2]}{(1 - n\beta \cos \vartheta)^2}. \]
We return to the collisionless limit, in which the angular width is

\[(\Delta \vartheta)_{\text{diff}} \sim \frac{1}{(T\omega)n\beta \sin \vartheta_0}.\]  \hspace{1cm} (14)

When the parameter \(\xi_0\) is much smaller than the upper integral limit of Eq. (11), i.e.,

\[\xi_0 \ll T\omega,\]  \hspace{1cm} (15)

or

\[\frac{3^{1/3}}{(q/\omega)^{1/3}(n\beta)^{1/6} \cos^{1/6} \vartheta} \ll T\omega\]

the function \(F(\vartheta)\) can be approximated as

\[F(\vartheta) \simeq \left| \int_0^\infty d\xi \exp \left\{ i(1 - n\beta \cos \vartheta)\xi - \frac{(q/\omega)(n\beta/2)^2 \sin^2 \vartheta}{3} \xi^3 \right\} \right|^2 \]
\[= \frac{1}{[(q/\omega)(n\beta/2)^2 \sin^2 \vartheta]^{2/3}} \left| \int_0^\infty d\xi \exp \left\{ i \frac{(1 - n\beta \cos \vartheta)}{[(q/\omega)(n\beta/2)^2 \sin^2 \vartheta]^{1/3}} \xi - \frac{1}{3} \xi^3 \right\} \right|^2. \hspace{1cm} (16)

In this case, the function \(F(\vartheta)\) is independent of the slab thickness, indicating that multiple scattering processes dominate the angular distribution of the radiation, and that only part of the electron trajectory can contribute the observed radiation. Introducing the function

\[H(x) = \left| \int_0^\infty e^{ixt - t^3/3} dt \right|^2, \hspace{1cm} (17)\]

the angular distribution function can be written as

\[F(\vartheta) = \frac{1}{[(q/\omega)(n\beta/2)^2 \sin^2 \vartheta]^{2/3}} H \left\{ \frac{(1 - n\beta \cos \vartheta)}{[(q/\omega)(n\beta/2)^2 \sin^2 \vartheta]^{1/3}} \right\}. \hspace{1cm} (18)\]

We plot the function \(H(x)\) in Fig. 1. As seen in this figure, the maximum of the function \(H(x)\) locates at \(x = 0\), indicating that the radiation emission reaches its peak at the angle \(\vartheta_0 = \arccos(1/n\beta)\).
This result means that multiple scattering nearly does not shift the peak position of the Cherenkov emission. The full width at the half maximum of the peak is about,

\[
(\Delta \vartheta)_{\text{ms}} \simeq \frac{(q/\omega)^{1/3}}{\sin^{1/3} \vartheta_0(n\beta)^{1/3}}.
\]  

(19)

Under the condition of Eq. (15), this angular width is significantly greater than that in the collisionless limit.

The spectral power of the Cherenkov radiation with inclusion of multiple scattering is already calculated through the path integral method [8]. In the case of

\[
\frac{n\beta - 1}{\sqrt{q/\omega}} \gg 1,
\]  

(20)

The spectral power is approximately given by,

\[
\frac{dP}{d\omega} = \frac{e^2 \omega \beta}{cn} \left[1 - (n\beta)^{-2}\right] - \frac{2e^2 q}{3\pi cn^{3/2} \left[1 - (n\beta)^{-2}\right]}.
\]  

(21)

We can see that the reduction of the Cherenkov radiation due to multiple scattering is proportional to the ratio of \( q/\omega \), which is usually very small. In the limit of

\[
\left|\frac{\beta n - 1}{\sqrt{q/\omega}}\right| \ll 1,
\]

the so-called Landau-Pomeranchuk-Migdal effect plays important role [11, 12],

\[
\frac{dP}{d\omega} = \frac{e^2 (q\omega)^{1/2}}{\pi cn^{5/2}}.
\]

In this case, the radiation is essentially due to bremsstrahlung [13].

**III. DISCUSSION AND SUMMARY**

We calculate the effect of multiple scattering on the Cherenkov radiation through the path-integral method. For the sake of simplicity, we make some important assumptions in our calculation. The first is the neglect of the energy loss of the charged particles, the second is the Gaussian transition probability of the scattering angle, seeing Eq. (7), and the third is that the root of mean square of the scattering angle is much less than one. These assumptions lead to a slight different results of ours from previous studies like Dedrick’s [6] and Bowler’s [7]. However, the main conclusions are the same.
Acknowledgments

This work is supported by the Natural Science Foundation of China (Grant Nos. 11175179 and 11475171).

Appendix: The calculation of the path integral

We briefly present the calculation of the path integral. Denoting

\[ \phi = i \alpha \int_0^t \theta^2(d\tau) + i \beta \int_0^t \theta(\tau)d\tau, \]  

(A.1)

where \( \theta(t) \) is an one-dimensional random variable whose transition probability is a gaussian, we have

\[
I \equiv \langle e^{i\phi} \rangle = \lim_{\Delta \to 0, N \to \infty} \frac{1}{(\pi q \Delta)^{N/2}} \int d\theta_1 \cdots \int d\theta_N \exp \left[ -\sum_{n=0}^{N-1} \frac{(\theta_{n+1} - \theta_n)^2}{q\Delta} + \mu \theta_N^2 + \eta \theta_N + i\alpha \sum_{n=1}^N \theta_n^2 + i\beta \sum_{n=1}^N \theta_n \right],
\]  

(A.2)

where \( \Delta = t/N \). It is easy to see that

\[
\langle e^{i\phi} \rangle = I_{\mu=0,\eta=0},
\]

\[
\langle \theta(t)e^{i\phi} \rangle = \left( \frac{\partial I}{\partial \eta} \right)_{\mu=0,\eta=0},
\]

\[
\langle \theta^2(t)e^{i\phi} \rangle = \left( \frac{\partial I}{\partial \mu} \right)_{\mu=0,\eta=0}.
\]

After some manipulations, we have

\[
\langle e^{i\phi} \rangle = \lim_{\Delta \to 0, N \to \infty} \frac{1}{(\pi \Delta)^{N/2}} \int d\theta_1 \cdots \int d\theta_N \exp \left\{ -\sum_{l,m=1}^N \theta_l A_{lm} \theta_m + 2 \sum_{m=1}^N B_m \theta_m \right\}
\]

where the matrix elements \( A_{nm} \) is given by

\[
A_{N,N} = 1 - \mu q \Delta - i\alpha q \Delta^2,
\]

\[
A_{n,n} = 2 - i\alpha q \Delta^2, \quad (1 \leq n \leq N - 1),
\]

\[
A_{n,n+1} = A_{n,n-1} = -1,
\]

\[
A_{n,m} = 0, \text{ otherwise},
\]

8
and the vector component $B_m$ is
\[
B_N = \frac{1}{2} \eta q^{1/2} \Delta^{1/2} + \frac{i}{2} \beta q^{1/2} \Delta^{3/2},
\]
\[
B_n = \frac{i}{2} \beta q^{1/2} \Delta^{3/2}, \quad (1 \leq n \leq N - 1).
\]
The integral can be accomplished with variable transformation,
\[
I = \lim_{N \to \infty} \frac{1}{\sqrt{\det A}} \exp \left( \sum_{l,m=0}^{N} B_l A^{-1}_{lm} B_m \right),
\]
where $A^{-1}_{lm}$ is the element of the inverse matrix of $A$. The calculations of $\sqrt{\det A}$ and $\sum_{l,m=0}^{N} B_l A^{-1}_{lm} B_m$ can be found in the article by E. W. Montroll [14]. The results are
\[
\lim_{N \to \infty} \det A = D(\tau = 0) = \cosh(\Omega t) - \frac{\mu q}{\Omega} \sinh(\Omega t),
\]
and
\[
\lim_{N \to \infty} \sum_{l,m=0}^{N} B_l A^{-1}_{lm} B_m = \frac{\beta^2 q}{4} \int_{0}^{t} \frac{d\tau}{D^2(\tau)} \left( \int_{\tau}^{t} D(\tau') d\tau' \right)^2
\]
\[
+ \frac{i}{2} \eta q \int_{0}^{t} d\tau D(\tau) \int_{0}^{\tau} \frac{d\tau'}{D^2(\tau')}
\]
\[
+ \frac{1}{4} \eta^2 q \int_{0}^{t} \frac{d\tau}{D^2(\tau)}.
\]
where the function $D(\tau)$ is defined as
\[
D(\tau) = \cosh[\Omega(t - \tau)] - \frac{\mu q}{\Omega} \sinh[\Omega(t - \tau)]
\]
and
\[
\Omega^2 = -i \alpha q.
\]

With these results, we have
\[
\langle e^{i\phi} \rangle = \frac{1}{\sqrt{\cosh \Omega t}} \exp \left\{ -\frac{\beta^2 q}{4\Omega^3} (\Omega t - \tanh \Omega t) \right\},
\]
\[
\langle \theta(t) e^{i\phi} \rangle = \frac{i}{2} \frac{\beta q}{\Omega^2} \left( 1 - \frac{1}{\cosh \Omega t} \right) \frac{1}{\cosh^{1/2} \Omega t} \exp \left\{ -\frac{\beta^2 q}{4\Omega^3} (\Omega t - \tanh \Omega t) \right\}
\]
\[
\langle \theta^2(t) e^{i\phi} \rangle = \left[ \frac{q}{2\Omega} \frac{\sinh \Omega t}{\cosh^{3/2} \Omega t} - \frac{\beta^2 q^2 (\cosh \Omega t - 1)^2}{4\Omega^4 \cosh^{5/2} \Omega t} \right] \exp \left\{ -\frac{\beta^2 q}{4\Omega^3} (\Omega t - \tanh \Omega t) \right\}.
\]

[1] F. Brandl, G. Pretzler, D. Habs, and E. Fill, Europhys. Lett. 61, 632 (2003).
[2] M. Mancossi, J. J. Santos, D. Batani, J. Faure, A. Debyale, V. T. Tikhonchuk, and V. Malka, Phys. Rev. Lett. 96, 125002 (2006).

[3] H. Habara, K. Ohta, K. A. Tanaka, G. R. Kumar, M. Krishnamurthy, S. Kahaly, S. Mondal, M. K. Bhuyan, R. Rajeev, and J. Zheng, Phys. Rev. Lett. 104, 055001 (2010).

[4] C. D’Amico, A. Houard, M. Franco, B. Prade, and A. Mysyrowicz, A. Couairon, V. T. Tikhonchuk, Phys. Rev. Lett. 98, 235002 (2007).

[5] J. Zheng, C. X. Yu, Z. J. Zheng, and K. A. Tanaka, Phys. Plasmas 12, 093105 (2005).

[6] K. G. Dedrick, Phys. Rev. 87, 891 (1952).

[7] M. G. Bower and M. D. Lay, Nuclear Instruments and Methods in Physics Research A 378, 468 (1996).

[8] J. Zheng, Phys. Plasmas 20, 013302 (2013).

[9] N. V. Laskin, A. S. Mazmanishvili, N. N. Nasonov, and N. F. Shul’ga, Sov. Phys. JETP 62, 438 (1985).

[10] L. D. Landau and E. M. Lifshitz, The Classical Theory of Fields (Pergamon Press, Oxford, 1975), 4th Edition, Sect. 66.

[11] L. D. Landau and I. J. Pomeranchuk, Dokl. Akad. Nauk SSSR 92, 535 (1953).

[12] A. B. Migdal, Phys. Rev. 103, 1811 (1956).

[13] S. Klein, Rev. Mod. Phys. 71, 1501 (1999).

[14] E. W. Montroll, Comm. Pure Appl. Math. 5, 415 (1952).