Range-relaxed criteria for choosing the Lagrange multipliers in the
Levenberg-Marquardt method

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Abstract

In this article we propose a novel strategy for choosing the Lagrange multipliers in the
Levenberg-Marquardt method for solving ill-posed problems modeled by nonlinear operators
acting between Hilbert spaces. Convergence analysis results are established for the proposed
method, including: monotonicity of iteration error, geometrical decay of the residual, conver-
gence for exact data, stability and semi-convergence for noisy data. Numerical experiments are
presented for an elliptic parameter identification two-dimensional EIT problem. The perfor-
man of our strategy is compared with standard implementations of the Levenberg-Marquardt
method (using a priori choice of the multipliers).

Keywords. Nonlinear Ill-posed problems; Levenberg-Marquardt method, Lagrange multipliers.

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1 Introduction

In this article we address the Levenberg-Marquardt (LM) method [14, 18], which is a well established
iterative method for obtaining stable approximate solutions of nonlinear ill-posed operator equations
[6, 9] (see also the textbooks [7, 11] and the references therein).

The novelty of our approach consists in adopting a range-relaxed criteria for the choice of the
Lagrange multipliers in the LM method. Our approach is inspired in the recent paper [3], where a
range-relaxed criteria was proposed for choosing the Lagrange multipliers in the iterated Tikhonov
method for linear ill-posed problems.

With our strategy, the new iterate is obtained as the projection of the current one onto a level-set
of the linearized residual function. This level belongs to an interval (or range), which is defined by
the current (nonlinear) residual and by the noise level. As a consequence, the admissible Lagrange
multipliers (in each iteration) shall belong to a non-degenerate interval instead of being a single
value (see (4)). This fact reduces the computational burden of evaluating the multipliers. Moreover,
under appropriate assumptions, the choice of the above mentioned range enforces geometrical decay
of the residual (see (31)).

The resulting method (see Section 2) proves, in the preliminary numerical experiments (see
Section 4), to be more efficient than the classical geometrical choice of the Lagrange multipliers,
typically used in implementations of LM type methods.

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1.1 The model problem

The _exact case_ of the inverse problem we are interested in consists of determining an unknown quantity \( x \in X \) from the set of data \( y \in Y \), where \( X, Y \) are Hilbert spaces, and \( y \) is obtained by indirect measurements of the parameter \( x \), this process being described by the model

\[
F(x) = y, \tag{1}
\]

with \( F : D(F) \subseteq X \to Y \) being a non-linear ill-posed operator. In practical situations, one does not know the data exactly. Instead, an approximate measured data \( y^\delta \in Y \) satisfying

\[
\|y^\delta - y\| \leq \delta, \tag{2}
\]
is available, where \( \delta > 0 \) is the (known) noise level.

Standard methods for finding a solution of (1) are based in the use of _Iterative type_ regularization methods [1, 7, 10, 11, 12], which include the LM method, or _Tikhonov type_ regularization methods [7, 19, 22, 23, 24, 21].

1.2 The Levenberg-Marquardt method

In what follows we briefly revise the LM method, which was proposed separately by K. Levenberg [14] and D.W. Marquardt [18] for solving nonlinear optimization problems. The LM method for solving the nonlinear ill-posed operator equation (1) was originally considered in [6, 9], and is defined by

\[
x_k^{\delta+1} := \arg \min \left\{ \| y^\delta - F(x_k^\delta) - F'(x_k^\delta)(x - x_k^\delta) \|^2 + \alpha_k \| x - x_k^\delta \|^2 \right\}, \quad k = 0, 1, \ldots
\]

Here \( F'(z) : X \to Y \) is the Fréchet-derivative of \( F \) in \( z \in D(F) \), \( F'(z)^* : Y \to X \) is the corresponding adjoint operator and \( x_0^\delta \in X \) is some initial guess (possibly incorporating _a priori_ knowledge about the exact solution(s) of \( F(x) = y \)). Moreover, \( \{\alpha_k\} \) is a sequence of positive relaxation parameters (or Lagrange multipliers), aiming to guarantee convergence and stability of the iteration. This method can be summarized as follows

\[
x_{k+1}^\delta = x_k^\delta + h_k, \quad \text{whith} \quad h_k := (F'(x_k^\delta)^* F'(x_k^\delta) + \alpha_k I)^{-1} F'(x_k^\delta)^* (y^\delta - F(x_k^\delta)). \tag{3}
\]

In the sequel we address some previous convergence analysis results:

(i) For exact data (i.e., \( \delta = 0 \)) convergence is proved in [9, Theorem 2.2], provided the operator \( F \) satisfies adequate regularity assumptions, and \( \{\alpha_k\} \) satisfies the "exact" condition

\[
\| y^\delta - F(x_k^\delta) - F'(x_k^\delta) h_{k,\alpha_k} \|^2 = \theta \| y^\delta - F(x_k^\delta) \|^2, \tag{4}
\]

where \( h_{k,\alpha_k} = h_k(\alpha_k) \) is given by (3), and \( \theta < 1 \) is an appropriately chosen constant.\(^1\) In the case of inexact data (i.e., \( \delta > 0 \)), semi-convergence is proven if the iteration in (3) is stopped according to the discrepancy principle. The analysis presented in [9] depends on a nonlinearity assumption on the operator \( F \), namely the _strong Tangential Cone Condition_ (sTCC) [11].

(ii) In [2] a convergence analysis for a Kaczmarz version of the LM method, using constant sequence \( \{\alpha_k = \alpha\} \), is presented. The convergence proofs depend once again on a nonlinearity assumption on the operator \( F \), namely the _weak Tangential Cone Condition_ (wTCC) [7, 11, 10].

(iii) The algorithm REGINN is a Newton-like method for solving nonlinear inverse problems [20]. This iterative algorithm linearizes the forward operator around the current iterate and subsequently

\(^1\)It is well known (cf [8]) that \( \alpha_k \) is uniquely defined by (4).
applies a regularization technique in order to find an approximate solution to the linearized system, which in turn is added to the current iterate to provide an update. If \( wTCC \) holds true and the iteration is terminated by the discrepancy principle, then REGINN renders a regularization method in the sense of [7]. If Tikhonov regularization is used for approximating the solution of the linearized system, then REGINN becomes a variant of the LM method with a choice of the Lagrange multipliers performed \textit{a posteriori}. In this case, the resulting method is very similar to the one presented in [9], but with the difference that the equality in (4) is replaced by an inequality.

### 1.3 Criticism on the available choices of the Lagrange multipliers

Although the proposed choice of \( \{\alpha_k\} \) in [9] is performed \textit{a posteriori}, there is a severe drawback: the calculation of \( \alpha_k \) in (4) cannot be performed explicitly. Moreover, computation of accurate numerical approximations for \( \alpha_k \) is highly expensive.

For larger choices of the discrepancy constant, alternative parameter choice rules are discussed in [9], namely \( \alpha_k = \alpha \) a positive constant, or \( \alpha_k := \|F'(x_k^\delta)\|^2 \). However, the use of large values for discrepancy principle implies in the computation of small stopping indexes, meaning that LM iteration is interrupted before it can deliver the best possible approximate solution. On the other hand, the constant choice \( \{\alpha_k = \alpha\} \) also has an intrinsic disadvantage: although the calculation of \( \alpha \) demands no numerical effort, it does not lead to fast convergence of the sequence \( \{x_k^\delta\} \) (this is observed in the numerical experiments presented in [2]).

The Newton type method proposed in [20] also chooses the Lagrange multiplier within a range (see also [25]). However, differently from our criteria (9), this range is defined by a single inequality [20, Inequality (2.6)]. As a consequence, a regularization method (an inner iteration) is needed for the accurate computation of each multiplier.

In our method, the computation of \( \alpha_k \) requires knowledge about the noise level \( \delta > 0 \) and the \( wTCC \) constant \( \eta \in [0,1) \) (see Algorithm I). Other Newton type methods (with \textit{a posteriori} choice of \( \alpha_k \)) also have this characteristic, e.g., see [20, Lemma 3.2] and [9, proof of Theorems 2.2 and 2.3].

### 1.4 Outline of the manuscript

In Section 2 we state the basic assumptions and introduce the range-relaxed criteria for choosing the Lagrange multipliers. The algorithm for the corresponding LM type method is presented, and we prove some preliminary results, which guarantee that our method is well defined. In Section 3 we present the main convergence analysis results, namely: convergence for exact data, stability and semiconvergence results. In Section 4 numerical experiments are presented for the EIT problem in a 2D-domain. We compare the performance of our method with other implementations of the LM method using classical (\textit{a priori}) geometrical choices of the Lagrange multipliers. Section 5 is devoted to final remarks and conclusions.

### 2 Range-relaxed Levenberg-Marquardt method

In this section we introduce a range-relaxed criteria for choosing the Lagrange multipliers in the Levenberg-Marquardt (LM) method. Moreover, we present and discuss an algorithm for the resulting LM type method, here called the \textit{range-relaxed Levenberg-Marquardt} (rrLM) method.

We begin this section by introducing the main assumptions used in this manuscript. It is worth mentioning that these assumptions are commonly used in the analysis of iterative regularization methods for nonlinear ill-posed problems [7, 11, 21].
2.1 Main assumptions

Throughout this article we assume that the domain of definition $D(F)$ has nonempty interior, and that the initial guess $x_0 \in X$ satisfies $B_\rho(x_0) \subset D(F)$ for some $\rho > 0$. Additionally,

(A1) The operator $F$ and its Fréchet derivative $F'$ are continuous. Moreover, there exists $C > 0$ such that
\[
\|F'(x)\| \leq C, \quad x \in B_\rho(x_0).
\]

(A2) The wTCC holds at some ball $B_\rho(x_0)$, with $0 \leq \eta < 1$ and $\rho > 0$, i.e.,
\[
\|F(\bar{x}) - F(x) - F'(x)(\bar{x} - x)\|_Y \leq \eta \|F(\bar{x}) - F(x)\|_Y, \quad \forall x, \bar{x} \in B_\rho(x_0).
\]

(A3) There exists $x^* \in B_{\rho/2}(x_0)$ such that $F(x^*) = y$, where $y \in Rg(F)$ are the exact data satisfying (2), i.e., $x^*$ is an arbitrary solution (non necessarily unique).

2.2 A Levenberg-Marquadt type algorithm

In what follows we introduce an iterative method, which derives from the choice of Lagrange multipliers proposed in this manuscript (see Step [3.1] of the Algorithm I).

**Algorithm I: Range-relaxed Levenberg-Marquadt.**

\[0\] Choose an initial guess $x_0 \in X$; Set $k = 0$.

\[1\] Choose the positive constants $\tau$, $\varepsilon$ and $p$ such that
\[
\tau > \frac{1 + \eta}{1 - \eta}, \quad 0 < \varepsilon < \frac{\tau(1 - \eta) - (1 + \eta)}{\eta \tau}, \quad 0 < p < 1.
\]

\[2\] If $\|F(x_0) - y^\delta\| \leq \tau \delta$, then $k^* = 0$; Stop!

\[3\] For $k \geq 0$ do

\[3.1\] Compute $\alpha_k > 0$ and $h_k \in X$, such that
\[
h_k = (F'(x_k^\delta)^*F'(x_k^\delta) + \alpha_k I)^{-1}F'(x_k^\delta)^*(y^\delta - F(x_k^\delta))
\]
\[
\|y^\delta - F(x_k^\delta) - F'(x_k^\delta)h_k\| \in [c_k, d_k]
\]
where
\[
c_k = (1 + \varepsilon)\eta\|F(x_k^\delta) - y^\delta\| + (1 + \eta)\delta
\]
\[
d_k = p c_k + (1 - p)\|F(x_k^\delta) - y^\delta\|.
\]

\[3.2\] Set
\[
x_{k+1}^\delta = x_k^\delta + h_k.
\]

\[3.3\] If $\|F(x_k^\delta) - y^\delta\| \leq \tau \delta$, then $k^* = k$; Stop!

Else $k = k + 1$; Go to Step [3].

**Remark 2.1.** Due to (A2) and (7), it follows $\tau > 1$. Moreover, $[\tau(1 - \eta) - (1 + \eta)](\eta \tau)^{-1} > 0$. Consequently, the interval used to define $\varepsilon$ in (7) is non-degenerate.

**Remark 2.2.** For linear operators $F : X \to Y$, Assumption (A2) is trivially satisfied with $\eta = 0$. Thus, $c_k = \delta$, $d_k = p \delta + (1 - p)\|F(x_k^\delta) - y^\delta\|$ and (9) reduces to
\[
\|F x_k^\delta - y^\delta + F h_k^\delta\| = \|F x_{k+1}^\delta - y^\delta\| \in [c_k, d_k].
\]

Consequently, the rrLM method in Algorithm I generalizes the range-relaxed nonstationary iterated Tikhonov (rrNIT) method for linear ill-posed operator equations proposed in [3].
From now on we assume that $F'(x) \neq 0$ for $x \in B_\rho(x_0)$. Notice that this fact follows from Assumption (A2) provided $F$ is non-constant in $B_\rho(x_0)$.

The remaining of this section is devoted to verify that, under assumptions (A1), (A2) and (A3), Algorithm I is well defined (see Theorem 2.6). We open the discussion with Lemma 2.3, where a collection of preliminary results in Functional and Convex analysis is presented.

**Lemma 2.3.** Suppose $A : X \to Y$ ($A \neq 0$) is a continuous linear mapping, $\bar{z} \in X$, $b \in Y$ has a non-zero projection onto the closure of the range of $A$ and define, for $\alpha > 0$,

$$z_\alpha = \arg \min_{z \in X} \| A(z - \bar{z}) - b \|^2 + \alpha \| z - \bar{z} \|^2. \quad (13)$$

The following assertions hold

1. $z_\alpha = \bar{z} + (A^*A + \alpha I)^{-1}A^*b$;
2. $\alpha \mapsto \| A(z_\alpha - \bar{z}) - b \|$ is a continuous, strictly increasing function on $\alpha > 0$;
3. $\lim_{\alpha \to 0} \| A(z_\alpha - \bar{z}) - b \| = \inf_{z \in X} \| A(z - \bar{z}) - b \|$;
4. $\lim_{\alpha \to \infty} \| A(z_\alpha - \bar{z}) - b \| = \| b \|$;
5. $\| A(z_\alpha - \bar{z}) \| \geq \| b \| - \| A(z_\alpha - \bar{z}) - b \| \geq 0$;
6. $\alpha \leq ||A^*b||^2 \left[ ||b||(||b|| - ||A(z_\alpha - \bar{z}) - b||) \right]^{-1}$;
7. For $z \in X$ and $\alpha > 0$

$$\| z - \bar{z} \|^2 - \| z - z_\alpha \|^2 = \| z_\alpha - \bar{z} \|^2 + \frac{1}{\alpha} \left[ ||A(z_\alpha - \bar{z}) - b||^2 - ||A(z - \bar{z}) - b||^2 \right] + \frac{1}{\alpha} ||A(z - z_\alpha)||^2; \quad (14)$$

8. For $z \in X$, $z \neq \bar{z}$, and $\alpha > 0$

$$\alpha \geq \frac{||A(z_\alpha - \bar{z}) - b||^2 - ||A(z - \bar{z}) - b||^2}{\| z - \bar{z} \|^2}. \quad (15)$$

**Proof.** The proofs of items 1. and 5. are straightforward. For a proof of items 2. to 4. we refer the reader to [8]. The proofs of items 6. and 7. are adaptations of proofs presented in [3], and item 8. follows from item 7. \hfill $\Box$

The next Lemma provides an auxiliary estimate, which is used in the proof of Proposition 2.5. This proposition is fundamental for establishing that, as long as the discrepancy is not reached (see Step [3.3] of Algorithm I), two key facts hold true: (i) it is possible to find a pair $(\alpha_k, h_k \in X)$ solving (8), (9) in Step [3.1] of Algorithm I; (ii) for any sequence $\{x_k^\delta\}$ generated by Algorithm I, the iteration error $\| x_k^\delta - x_k \|$ is monotonically decreasing in $k$.

**Lemma 2.4.** Let Assumptions (A2) and (A3) hold. Then, for $x^\delta$ as in (A3) it holds

$$\| F(x) - y^\delta + F'(x)(x^\delta - x) \| \leq \eta \| F(x) - y^\delta \| + (1 + \eta)\delta, \forall x \in B_\rho(x_0).$$
Proposition 2.5. Let Assumptions (A2) and (A3) hold. Given \( x \in B_{\rho}(x_0) \), define
\[
(0, +\infty) \ni \alpha \mapsto \xi_\alpha := \arg \min_{\xi \in X} \|F(x) - y^{\delta} + F'(x)(\xi - x)\|^2 + \alpha \|\xi - x\|^2 \in X.
\]
1. For every \( \alpha > 0 \) it holds
\[
\|F'(x)\| \|\xi_\alpha - x\| \geq \|F(x) - y^{\delta}\| - \|F(x) - y^{\delta} + F'(x)(\xi_\alpha - x)\|. \tag{17}
\]
Additionally, if \( \|F(x) - y^{\delta}\| > \tau \delta \), define the scalars
\[
c := (1 + \varepsilon)\eta \|F(x) - y^{\delta}\| + (1 + \eta)\delta, \\
d := p \left[ (1 + \varepsilon)\eta \|F(x) - y^{\delta}\| + (1 + \eta)\delta \right] + (1 - p)\|F(x) - y^{\delta}\|,
\]
and the set \( J := \{ \alpha > 0 : \|F(x) - y^{\delta} + F'(x)(\xi_\alpha - x)\| \in [c, d] \} \). Then
2. \( J \) is a non-empty, non-degenerate interval;
3. For \( \alpha \in J \) and \( x^* \) as in (A3) it holds
\[
\|x^* - x\|^2 - \|x^* - \xi_\alpha\|^2 \geq \|\xi_\alpha - x\|^2. \tag{18}
\]

Proof. We adopt the notation: \( z = x^*, \ z_\alpha = \xi_\alpha, \ z = x, \ b = y^{\delta} - F(x) \) and \( A = F'(x) \).

Add 1.: Equation (17) follows from Lemma 2.3 (item 5.).

Add 2.: From the definition of \( \varepsilon \) and \( \tau \) in (7) it follows that
\[
c < \left[ \eta \tau + \tau (1 - \eta) - (1 + \eta) \right] \tau^{-1} \|F(x) - y^{\delta}\| + (1 + \eta)\delta \leq \|F(x) - y^{\delta}\|,
\]
(the last inequality follows from \( \delta \leq \tau^{-1} \|F(x) - y^{\delta}\| \)). Since \( d \) is a proper convex combination of \( c \) and \( \|F(x) - y^{\delta}\| \), we have
\[
c < d < \|F(x) - y^{\delta}\|. \tag{19}
\]
On the other hand, it follows from Lemma 2.4 that
\[
\|F(x) - y^{\delta} + F'(x)(x^* - x)\| \leq \eta \|F(x) - y^{\delta}\| + (1 + \eta)\delta < c. \tag{20}
\]
From (19), (20) it follows that
\[
\inf_z \|F(x) - y^{\delta} + F'(x)(z - x)\| < c < d < \|F(x) - y^{\delta}\|.
\]

Assertion 2. follows from this inequality and Lemma 2.3 (items 2., 3. and 4.).

Add 3.: From (20) and the assumption \( \alpha \in J \), we conclude that
\[
\|F(x) - y^{\delta} + F'(x)(x^* - x)\| < c \leq \|F(x) - y^{\delta} + F'(x)(\xi_\alpha - x)\|.
\]
Assertion 3. follows from this inequality and Lemma 2.3 (item 7.).
We are now ready to state and prove the main result of this section.

**Theorem 2.6.** Let Assumptions (A1), (A2) and (A3) hold. Then, Algorithm I is well defined, i.e., for \( k < k^* \) (the stopping index defined in Step [3.3]) there exists a pair \(( \alpha_k \in \mathbb{R}^+, h_k \in X)\) solving (8), (9). Moreover, \( k^* \) is finite and any sequence \( \{ x_k^\delta \} \) generated by this algorithm satisfies

\[
\| x^\delta - x_{k}^\delta \|^2 - \| x^\delta - x_{k+1}^\delta \|^2 \geq \| x_k^\delta - x_{k+1}^\delta \|^2, \ 0 \leq k < k^*.
\]

**(21)**

**Proof.** Let step \( k = 0 \) of Algorithm I be its initialization. We may assume \( \| F(x_0) - y^\delta \| > \tau \delta \) (otherwise the algorithm stops with \( k^* = 0 \), and the theorem is trivial).

We use induction for proving this result. For \( k = 0 \), it follows from Proposition 2.5 (item 2.) with \( x = x_0 \), the existence of \( (\alpha_0 \in \mathbb{R}^+, h_0 \in X)\) solving (8), (9). Moreover, it follows from Proposition 2.5 (item 3.) with \( x = x_0 \), that (21) holds for \( k = 0 \).

Assume by induction that Algorithm I is well defined up to step \( k_0 > 0 \), and that (21) holds for \( k = 0, \ldots, k_0 - 1 \). There are two possible scenarios to consider:

- **Case I:** \( \| F(x_{k_0}^\delta) - y^\delta \| \leq \tau \delta \).

In this case, the algorithm terminates at iteration \( k^* = k_0 \geq 1 \), concluding the proof.

- **Case II:** \( \| F(x_{k_0}^\delta) - y^\delta \| > \tau \delta \).

Due to the inductive assumption, \( \| x^\delta - x_{k_0} \| \leq \| x^\delta - x_{k_0-1} \| \leq \cdots \leq \| x^\delta - x_0 \| \). From (A3) follows

\[
\| x_{k_0}^\delta - x_k^\delta \| \leq \| x_{k_0}^\delta - x^\delta \| + \| x^\delta - x_k^\delta \| \leq 2\| x^\delta - x_0 \| < \rho.
\]

Hence, \( x_{k_0}^\delta \in B_{\rho}(x_0) \). Proposition 2.5 (item 2.) with \( x = x_{k_0}^\delta \), guarantees the existence of a pair \((\alpha_{k_0} \in \mathbb{R}^+, h_{k_0} \in X)\) solving (8), (9) as well as the existence of \( x_{k_0+1}^\delta \in X \). The validity of (21) for \( k = k_0 \) follows from Proposition 2.5 (item 3.) with \( x = x_{k_0}^\delta \).

In order to verify the finiteness of the stopping index \( k^* \), notice that, from Proposition 2.5 (item 1.) with \( x = x_{k}^\delta, \ \alpha = \alpha_k \) and \( \xi_k = x_{k+1}^\delta \), it follows

\[
\| F'(x_k^\delta) \| \| x_{k+1}^\delta - x_k^\delta \| \geq \| F(x_k^\delta) - y^\delta \| - \| F(x_{k_0}^\delta) - y^\delta + F'(x_{k_0}^\delta)(x_{k_0+1}^\delta - x_{k_0}^\delta) \|, \ 0 \leq \cdots \leq k^* - 1.
\]

From this inequality and the definition of \( c_k \) and \( d_k \) in Step [3.1], it follows that

\[
\| F'(x_k^\delta) \| \| x_{k+1}^\delta - x_k^\delta \| \geq \| F(x_k^\delta) - y^\delta \| - d_k = p[\| F(x_k^\delta) - y^\delta \| - c_k] = p[(1 - (1 + \varepsilon)\eta) \| F(x_k^\delta) - y^\delta \| - (1 + \eta)\delta], \ 0 \leq \cdots \leq k^* - 1.
\]

Since \( \| F(x_k^\delta) - y^\delta \| > \tau \delta, \ 0 \leq k < k^* \) and \( \varepsilon < \frac{1}{\eta} - 1 \) (see (7)), we obtain from the last inequality

\[
\| F'(x_k^\delta) \| \| x_{k+1}^\delta - x_k^\delta \| \geq p[(1 - (1 + \varepsilon)\eta) \tau - (1 + \eta)\delta] \delta = p\delta \eta \tau \left[ \frac{(1 - \eta) - (1 + \eta) - \varepsilon}{\eta \tau} \right],
\]

for \( k = 0, \ldots, k^* - 1 \). Now, Assumption (A1) implies

\[
\| x_{k+1}^\delta - x_k^\delta \| \geq \frac{p\delta \eta \tau}{C} \left[ \frac{(1 - \eta) - (1 + \eta) - \varepsilon}{\eta \tau} \right] =: \Psi > 0, \ k = 0, \ldots, k^* - 1.
\]

**(22)**

Adding up inequality (21) for \( k = 0, \ldots, k^* - 1 \) and using (22) we finally obtain

\[
\| x^\delta - x_0 \|^2 > \| x^\delta - x_0 \|^2 - \| x^\delta - x_{k_0} \|^2 > \sum_{k=0}^{k^*-1} \| x_k^\delta - x_{k+1}^\delta \|^2 > k^* \Psi^2,
\]

from where the finiteness of the stopping index \( k^* \) follows. \( \square \)
Remark 2.7. Assumption (A1) is used only once in the proof of Theorem 2.6, namely in the derivation of (22), which is used to prove finiteness of the stopping index $k^*$. 

Corollary 2.8. Let Assumptions (A1), (A2) and (A3) hold, and assume the data is exact, i.e., $\delta = 0$. Then, any sequence $\{x_k\}$ generated by Algorithm I satisfies

$$\sum_{k=0}^{\infty} \|x_k - x_{k+1}\|^2 < \infty. \tag{23}$$

Proof. Adding up inequality (21), we obtain

$$\|x^* - x_0\|^2 - \|x^* - x_{n+1}\|^2 > \sum_{k=0}^{n} \|x_k - x_{k+1}\|^2, \quad \forall n > 0$$

and the assertion follows.

We conclude this section obtaining an estimate for the Lagrange multipliers $\{\alpha_k\}$ defined in Step [3.1] of Algorithm I.

Proposition 2.9. Let Assumptions (A2) and (A3) hold. Then the Lagrange multipliers $\{\alpha_k\}$ in Algorithm I satisfy

$$\alpha_k \geq \rho^{-2} \varepsilon \eta \|F(x_k^\delta) - y^\delta\| \left[ (1 + \varepsilon) \eta \|F(x_k^\delta) - y^\delta\| + (1 + \eta) \delta \right]. \tag{24}$$

Proof. Take $\alpha = \alpha_k$, $z_\alpha = x_{k+1}^\delta$, $z = x^*$, $b = y^\delta - F(x_k^\delta)$ and $A = F'(x_k^\delta)$. Arguing as in the proof of Lemma 2.4 we obtain

$$\|A(z - \bar{z}) - b\| \leq \eta \|b\| + (1 + \eta) \delta. \tag{25}$$

On the other hand, it follows from Step [3.1] that

$$\|A(z_\alpha - \bar{z}) - b\| \geq (1 + \varepsilon) \eta \|b\| + (1 + \eta) \delta. \tag{26}$$

From (25) and (26) we obtain $\|A(z_\alpha - \bar{z}) - b\| - \|A(z - \bar{z}) - b\| \geq \varepsilon \eta \|b\|$. This last inequality together with (15) allow us to estimate

$$\alpha_k \geq \rho^{-2} [\|A(z_\alpha - \bar{z}) - b\|^2 - \|A(z - \bar{z}) - b\|^2] \geq \rho^{-2} [\|A(z_\alpha - \bar{z}) - b\| + \|A(z - \bar{z}) - b\|] \varepsilon \eta \|b\| \geq \rho^{-2} \varepsilon \eta \|b\| \|A(z_\alpha - \bar{z}) - b\|.$$

Estimate (24) follows from this inequality together with (26).

3 Convergence analysis

We open this section obtaining an estimate, which is similar in spirit to Lemma 2.3 (item 7.).

Lemma 3.1. Let Assumptions (A2) and (A3) hold. Then, for $x^*$ as in (A3) it holds

$$\|x^* - x_k^\delta\|^2 - \|x^* - x_{k+1}^\delta\|^2 \geq \|x_k^\delta - x_{k+1}^\delta\|^2 + 2\varepsilon \eta \alpha_k^{-1} \|F'(x_k^\delta)(x_{k+1}^\delta - x_k^\delta) + F(x_k^\delta) - y^\delta\| \|F(x_k^\delta) - y^\delta\|, \tag{27}$$

for $k = 0, \ldots, k^* - 1$. 

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Proof. The polarization identity yields
\[\|x^* - x_k^\ell\|^2 - \|x^* - x_{k+1}^\ell\|^2 = \|x_k^\ell - x_{k+1}^\ell\|^2 - 2\langle x_{k+1}^\ell - x_k^\ell, x_{k+1}^\ell - x^*\rangle.\] (28)

Adopting the notation \( A := F'(x_k^\ell), \ b := y^\delta - F(x_k^\ell) \), it follows from (8) and (12)
\[-\langle x_{k+1}^\ell - x_k^\ell, x_{k+1}^\ell - x^*\rangle = \alpha_k^{-1}\langle A^*(Ah_k - b), x_{k+1}^\ell - x^*\rangle = \alpha_k^{-1}\langle Ah_k - b, A[h_k - (x^* - x_k^\ell)]\rangle = \alpha_k^{-1}\left[\langle Ah_k - b, Ah_k - b\rangle - \langle Ah_k - b, A(x^* - x_k^\ell) - b\rangle\right]\geq \alpha_k^{-1}\left\| Ah_k - b\right\|^2 - \|Ah_k - b\|\|A(x^* - x_k^\ell) - b\| = \alpha_k^{-1}\|Ah_k - b\|\left(\|Ah_k - b\| - \|A(x^* - x_k^\ell) - b\|\right).\] (29)

However, from Lemma 2.4 (with \( x = x_k^\ell \) and Algorithm I (see (10) and (9)), it follows
\[\|A(x^* - x_k^\ell) - b\| \leq \eta\|b\| + (1 + \eta)\delta = c_k - \varepsilon\eta\|b\| \leq \|Ah_k - b\| - \varepsilon\eta\|b\|.\] (30)

Thus, inequality (27) follows substituting (29) and (30) in (28).

The following results are devoted to the analysis of the residuals \( y^\delta - F(x_k^\ell) \) for a sequence \( \{x_k^\ell\} \) generated by Algorithm I. In Proposition 3.2 we estimate the decay rate of the residuals. Moreover, in Proposition 3.4 we prove the summability of the series of squared residuals.

**Proposition 3.2.** Let Assumptions (A2) and (A3) hold. Then, for any sequence \( \{x_k^\ell\} \) generated by Algorithm I we have
\[\|y^\delta - F(x_{k+1}^\ell)\| \leq \Lambda\|y^\delta - F(x_k^\ell)\|,\] (31)
for \( k = 0, \ldots, k^* - 1 \). Here \( \Lambda := (C_1 + \eta)(1 - \eta)^{-1}, \ C_1 := p(C_0 - 1) + 1 \) and \( C_0 := (1 + \varepsilon)\eta + (1 + \eta)\tau^{-1} < 1 \). Additionally, if
\[\eta < \frac{p + \frac{p}{\tau}}{2 + p(1 + \varepsilon) - \frac{p}{\tau}},\] (32)
then \( \Lambda < 1 \), from where it follows \( k^* = O(|\ln \delta| + 1).^2\)

**Proof.** From Algorithm I (see (11)) and (A2), it follows
\[\|y^\delta - F(x_{k+1}^\ell)\| \leq \|y^\delta - F(x_k^\ell) - F'(x_k^\ell)h_k\| + \|F(x_k^\ell) + F'(x_k^\ell)h_k - F(x_{k+1}^\ell)\|\leq d_k + \eta\|F(x_k^\ell) - F(x_{k+1}^\ell)\|\leq d_k + \eta(\|y^\delta - F(x_k^\ell)\| + \|y^\delta - F(x_{k+1}^\ell)\|), \ 0 \leq k < k^*.\] (33)

On the other hand, Algorithm I (see (10)) implies \( c_k \leq C_0\|y^\delta - F(x_k^\ell)\|, 0 \leq k < k^* \). Consequently, \( d_k \leq C_1\|y^\delta - F(x_k^\ell)\|, 0 \leq k < k^* \). Substituting this inequality in (33), we obtain the estimate (31).

To prove the last assertion, observe that \( \Lambda < 1 \) iff (32) holds true. Moreover, from Algorithm I (see Step [3.3]) and (31) follows \( \tau\delta \leq \|y^\delta - F(x_k^\ell)\| \leq \Lambda^{k^*-1}\|y^\delta - F(x_0)\| \). Consequently, (32) implies \( k^* \leq (\ln \Lambda)^{-1}\ln \left(\tau\delta/\|y^\delta - F(x_0)\|\right) + 1 \), completing the proof.

**Remark 3.3.** Inequality (32) holds true if \( \eta < 1/3, \ p \) is sufficiently close to \( 1, \ \varepsilon \) is sufficiently close to zero and \( \tau \) is large enough. Notice that the condition \( \eta < 1/3 \) is not necessary for the convergence analysis devised in this manuscript.

^2Here \( k^* \) is the stopping index defined in Step [3.3] of Algorithm I.
Proposition 3.4. Let Assumptions (A1), (A2) and (A3) hold. Suppose that no noise is present in the data (i.e., δ = 0). Then, for any sequence \( \{x_k\} \) generated by Algorithm I we have
\[
\sum_{k=0}^{\infty} \|y - F(x_k)\|^2 < \infty.
\] (34)

Proof. From Lemma 2.3 (item 6.) with Proof. Remark 3.6. Due to (A2), given \( c = x_k, z_0 = x_{k+1}, b = y - F(x_k) \) and \( A = F'(x_k) \), follows
\[
\frac{1}{\alpha_k} \geq \frac{\|y - F(x_k)\|}{\frac{\|F'(x_k)^*(y - F(x_k))\|}{\|F'(x_k)\|^2}} \geq \frac{\|y - F(x_k)\| - \|F'(x_k)h_k\|}{\frac{\|y - F(x_k)\|}{C^2}}
\] (35)
(the last inequality follows from (A1)). Moreover, it follows from Algorithm I (see (9))
\[
\|y - F(x_k)\| - \|F'(x_k)h_k\| \geq \|y - F(x_k)\| - d_k \geq \rho(1 - (1 + \varepsilon)\eta)\|y - F(x_k)\|
\] (notice that \((1 - (1 + \varepsilon)\eta) > 0 \) due to (7)). From this inequality, (35) and (27) follows
\[
\|x^* - x_0\|^2 \geq \sum_{k=0}^{m} \frac{2m}{\alpha_k} \|F'(x_k)(x_{k+1} - x_k) + F(x_k) - y\| \|F(x_k) - y\|
\] \[
\geq \frac{2m(1 - (1 + \varepsilon)\eta)}{C^2} \sum_{k=0}^{m} \|F'(x_k)(x_{k+1} - x_k) + F(x_k) - y\| \|F(x_k) - y\|,
\] for all \( m \in \mathbb{N} \). Finally, (34) follows from (36) and the inequality \( \|F'(x_k)(x_{k+1} - x_k) + F(x_k) - y\| \geq c_k = (1 + \varepsilon)\eta\|F(x_k) - y\| \) (see Algorithm I, (9) and (10)).

Remark 3.5. An immediate consequence of Proposition 3.4 is the fact that \( \|F(x_k) - y\| \to 0 \) as \( k \to \infty \). It is worth noticing that (36) and Algorithm I also imply the summability of the series
\[
\sum_{k=0}^{\infty} \|F'(x_k)(x_{k+1} - x_k) + F(x_k) - y\| \|F(x_k) - y\|
\] (compare with [2, inequalities (18a), (18b), (18c)]).

In the sequel we address the first main result of this section (see Theorem 3.7), namely convergence of Algorithm I in the exact data case (i.e., δ = 0). To state this theorem we need the concept of \( x_0 - \text{minimal-norm solution of } (1) \), i.e., the unique \( x^* \in X \) satisfying \( \|x^* - x_0\| := \inf \{\|x^* - x_0\| : F(x^*) = y \text{ and } x^* \in B_p(x_0)\} \).

Remark 3.6. Due to (A2), given \( x^* \in B_{p/2}(x_0) \) a solution of (1) and \( z \in N(F'(x^*)) \), the element \( x^* + tz \in B_p(x_0) \) is also a solution of (1) for all \( t \in (-\frac{\delta}{2}, \frac{\delta}{2}) \).

Due to (A3), \( x^* \in B_{p/2}(x_0) \). Thus, the inequality \( \|x^* - x_0\|^2 \leq \|(x^* + tz) - x_0\|^2 \) holds for all \( t \in (-\frac{\delta}{2}, \frac{\delta}{2}) \) and all \( z \in N(F'(x^*)) \), from where we conclude
\[
x^* - x_0 \in N(F'(x^*))^\perp.
\] (37)

\footnote{Indeed, due to (A2) we have
\[
\|F(x^* + tz) - y\| = \|F(x^* + tz) - F(x^*)\| \leq \frac{1}{1 - \eta} \|F'(x^*)(x^* + tz - x^*)\| = \frac{|t|}{1 - \eta} \|F'(x^*)z\| = 0.
\]}

\footnote{The conclusion follows from the fact that \( \|x^* - x_0\|^2 \leq \|(x^* + t\zeta) - x_0\|^2, \forall \zeta \in (-\varepsilon, \varepsilon) \), implies \( (x^* - x_0, \zeta) = 0. \)
Theorem 3.7. Let Assumptions (A1), (A2) and (A3) hold. Suppose that no noise is present in the data (i.e., \( \delta = 0 \)). Then, any sequence \( \{x_k\} \) generated by Algorithm I either terminates after finitely many iterations with a solution of (1), or it converges to a solution of this equation as \( k \to \infty \). Moreover, if
\[
N(F'(x^1)) \subset N(F'(x)), \, \forall x \in B_\rho(x_0)
\]
holds, then \( x_k \to x^1 \) as \( k \to \infty \).

Proof. In what follows we adopt the notation \( A_k := F'(x_k) \), \( b_k := y - F(x_k) \). If for some \( k \in \mathbb{N} \), \( \|y - F(x_k)\| = 0 \), then \( x_k \) is a solution and Algorithm I stops with \( k^* = k \). Otherwise, \( \{x_k\}_{k \in \mathbb{N}} \) is a Cauchy sequence. Indeed, fix \( m < n \) and choose \( \bar{k} \in \{m, \ldots, n\} \) s.t.
\[
\|b_{\bar{k}}\| \leq \|b_k\| \text{ for all } k \in \{m, \ldots, n\}.
\]
From the triangle inequality and the polarization identity, it follows that for any \( x^* \) as in (A3)
\[
\frac{1}{2} \|x_n - x_m\|^2 \leq \|x_n - x_{\bar{k}}\|^2 + \|x_m - x_{\bar{k}}\|^2
\]
\[
= (\|x^* - x_n\|^2 - \|x^* - x_{\bar{k}}\|^2 + 2 \langle x_n - x_{\bar{k}}, x_{\bar{k}} - x^* \rangle)
\]
\[
+ (\|x^* - x_m\|^2 - \|x^* - x_{\bar{k}}\|^2 + 2 \langle x_m - x_{\bar{k}}, x_{\bar{k}} - x^* \rangle).
\]
Since the sequence \( \{\|x^* - x_n\|\}_{n \in \mathbb{N}} \) is non-negative and non-increasing (see (21)), it converges.

Therefore, the difference \( \|x^* - x_n\|^2 - \|x^* - x_{\bar{k}}\|^2 \) as well as \( \|x^* - x_m\|^2 - \|x^* - x_{\bar{k}}\|^2 \) both converge to zero as \( m \to \infty \). It remains to estimate the inner products in (40). Notice that
\[
|\langle x_n - x_{\bar{k}}, x_{\bar{k}} - x^* \rangle + \langle x_m - x_{\bar{k}}, x_{\bar{k}} - x^* \rangle| \leq |\langle x_n - x_m, x_{\bar{k}} - x^* \rangle|
\]
\[
\leq \sum_{k=m}^{n-1} |\langle x_{k+1} - x_k, x_{\bar{k}} - x^* \rangle|
\]
\[
= \sum_{k=m}^{n-1} \frac{1}{\alpha_k} |\langle A_k^* (A_k h_k - b_k), x_{\bar{k}} - x^* \rangle|
\]
\[
\leq \sum_{k=m}^{n-1} \frac{1}{\alpha_k} \|A_k h_k - b_k\| \|A_k (x_{\bar{k}} - x^*)\|,
\]
with \( h_k \) as in (8). However, from (39) and (A2) follows
\[
\|A_k (x_{\bar{k}} - x^*)\| \leq \|A_k (x_{\bar{k}} - x_k)\| + \|A_k (x^* - x_k)\|
\]
\[
\leq \|F(x_{\bar{k}}) - F(x_k) - A_k (x_{\bar{k}} - x_k)\| + \|F(x_{\bar{k}}) - F(x_k)\|
\]
\[
+ \|F(x^*) - F(x_k) - A_k (x^* - x_k)\| + \|F(x^*) - F(x_k)\|
\]
\[
\leq (\eta + 1) \|F(x_{\bar{k}}) - F(x_k)\| + (\eta + 1) \|y - F(x_k)\|
\]
\[
\leq 2(\eta + 1) \|y - F(x_k)\| + (\eta + 1) \|y - F(x_{\bar{k}})\|
\]
\[
\leq 3(\eta + 1) \|y - F(x_k)\|.
\]
Substituting this last inequality in (41), and using (27) (with \( x_k^\delta = x_k, \, y^\delta = y \)) we obtain
\[
|\langle x_n - x_{\bar{k}}, x_{\bar{k}} - x^* \rangle + \langle x_m - x_{\bar{k}}, x_{\bar{k}} - x^* \rangle| \leq 3(\eta + 1) \sum_{k=m}^{n-1} \frac{1}{\alpha_k} \|A_k h_k - b_k\| \|b_k\|
\]
\[
\leq \frac{3(\eta + 1)}{2\varepsilon \eta} \sum_{k=m}^{n-1} (\|x^* - x_k\|^2 - \|x^* - x_{k+1}\|^2)
\]
\[
= \frac{3(\eta + 1)}{2\varepsilon \eta} \left(\|x^* - x_m\|^2 - \|x^* - x_n\|^2\right) \to 0
\]
as $m \to \infty$. Thus, it follows from (40) that $\|x_n - x_m\| \to 0$ as $m \to \infty$, proving that $\{x_k\}_{k \in \mathbb{N}}$ is indeed a Cauchy sequence.

Since $X$ is complete, $\{x_k\}$ converges to some $x_\infty \in X$ as $k \to \infty$. On the other hand, $\|y - F(x_k)\| \to 0$ as $k \to \infty$ (see Remark 3.5). Consequently, $x_\infty$ is a solution of (1) proving the first assertion.

In order to prove the last assertion notice that, if (38) hold, then

$$ x_{k+1} - x_k = \alpha_k^{-1} A_k^*(A_k h_k - b_k) \in R(F'(x_k)^*) \subset N(F'(x_k)) \subset N(F'(x)^{+}) \cap N(F'(x)^{+}), \quad k = 0, 1, \ldots, $$

from where we conclude that $x_k - x_0 \in N(F'(x^1))^{+}$, $k \in \mathbb{N}$. Since $x^1 - x_0 \in N(F'(x^1))^{+}$ (see (37)), it follows that $x_k - x^1 \in N(F'(x^1))^{+}$, $k \in \mathbb{N}$. Consequently, $x_\infty - x^1 = \lim_k x_k - x^1 \in N(F'(x^1))^{+}$.

However, (A2) implies $\|F'(x^1)(x_\infty - x^1)\| \leq (1 + \eta)\|F(x_\infty) - F(x^1)\| = 0$, from what follows $x_\infty - x^1 \in N(F'(x^1))$. Thus, $x_\infty - x^1 = 0$. \qed

We conclude this section addressing the last two main results, namely: Stability (Theorem 3.9) and Semi-Convergence (Theorem 3.10). The following definition is quintessential for the discussion of these results.

**Definition 3.8.** A vector $z \in X$ is a successor of $x_k^\delta$ if

- $k < k^*$;
- There exists $(\alpha_k > 0, h_k \in X)$ satisfying (8), (9), such that $z = x_k^\delta + h_k$;

Notice that Theorem 3.7 guarantees that the sequence $\{x_k\}_{k \in \mathbb{N}}$ converges to a solution of $F(x) = y$ whenever $x_{k+1}$ is a successor of $x_k$ for every $k \in \mathbb{N}$. In this situation, we call $\{x_k\}_{k \in \mathbb{N}}$ a noiseless sequence.

**Theorem 3.9** (Stability). Let Assumptions (A1), (A2) and (A3) hold, and $\{\delta_j\}_{j \in \mathbb{N}}$ be a positive zero-sequence. Assume that the (finite) sequences $\{x_{k}^{\delta_{j}}\}_{0 \leq k \leq k^*(\delta_j)}$, $j \in \mathbb{N}$, are fixed,\(^5\) where $x_{k+1}^\delta$ is a successor of $x_k^\delta$. Then, there exists a noiseless sequence $\{x_k\}_{k \in \mathbb{N}}$ such that, for every fixed $k \in \mathbb{N}$, there exists a subsequence $\{\delta_{j_{m}}\}_{m \in \mathbb{N}}$ (depending on $k$) satisfying

$$ x_{\ell}^{\delta_{j_{m}}} \to x_\ell \quad \text{as} \quad m \to \infty, \quad \text{for} \quad \ell = 0, \ldots, k. $$

**Proof.** We use an inductive argument. Since $x_0^\delta = x_0$ for every $\delta \geq 0$, the assertion is clear for $k = 0$. Our main argument consists of repeatedly choosing a subsequence of the current subsequence. In order to avoid a notational overload, we denote a subsequence of $\{\delta_j\}_j$ again by $\{\delta_j\}_j$.

Suppose by induction that the assertion holds true for some $k \in \mathbb{N}$, i.e., that there exists a subsequence $\{\delta_j\}_j$ and $\{x_{\ell}^{\delta_j}\}_{\ell = 0}^k$ satisfying

$$ x_{\ell}^{\delta_j} \to x_\ell \quad \text{as} \quad j \to \infty, \quad \text{for} \quad \ell = 0, \ldots, k, $$

where $k < k^*(\delta_j)$ and $x_{\ell+1}^\delta$ is a successor of $x_\ell$, for $\ell = 0, \ldots, k - 1$. Since $x_{k+1}^\delta$ is a successor of $x_k^\delta$ (for each $\delta_j$), there exists (for each $\delta_j$) a positive number $\alpha_k^{\delta_j}$ such that $x_{k+1}^{\delta_j} = x_k^{\delta_j} + h_k^{\delta_j}$, with $h_k^{\delta_j}$ as in (8) and

$$ \|F(x_k^{\delta_j}) - y^{\delta_j} + F'(x_k^{\delta_j})h_k^{\delta_j}\| \in [c_k^{\delta_j}, d_k^{\delta_j}]. \quad (42) $$

Our next goal is to prove the existence of a successor $x_{k+1}$ of $x_k$ and of a subsequence $\{\delta_j\}_{j}$ of the current subsequence such that $x_{k+1}^{\delta_j} \to x_{k+1}$ as $j \to \infty$, ensuring that

$$ x_{\ell}^{\delta_j} \to x_\ell \quad \text{as} \quad j \to \infty, \quad \text{for} \quad \ell = 0, \ldots, k + 1, \quad (43) $$

\(^5\)Notice that the stopping index $k^*$ in Step \([3.3]\) depends on $\delta$, i.e., $k^* = k^*(\delta)$. 

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and completing the inductive argument. We divide this proof in 4 steps as follows:

Step 1. We find a vector $z \in X$ such that, for some subsequence $\{\delta_j\}_j$

$$h_k^{\delta_j} \to z \quad \text{as} \quad j \to \infty,$$  \hbox{(44)}

Step 2. We define

$$\alpha_k := \liminf_{j \to \infty} \alpha_k^{\delta_j}$$  \hbox{(45)}

and prove that $\alpha_k > 0$, which permit us to define $h_k$ as in (8) as well as $x_{k+1} := x_k + h_k$.

Step 3. We show that $h_k = z$, which ensures that $h_k^{\delta_j} \to h_k$.

Step 4. We validate that

$$\|h_k^{\delta_j}\| \to \|h_k\|, \quad \text{as} \quad j \to \infty,$$  \hbox{(46)}

which together with $h_k^{\delta_j} \to h_k$ proves that $h_k^{\delta_j} \to h_k$ and, consequently, $x_{k+1}^{\delta_j} \to x_{k+1}$.

Finally, we prove that $x_{k+1}$ is a successor of $x_k$, which validates (43).

Proof of Step 1. Since the sequence $\{h_k^{\delta_j}\}_{j \in \mathbb{N}}$ is bounded (see (21)), there exists a subsequence $\{\delta_j\}$ of the current subsequence, and a vector $z \in X$ such that (44) holds. Consequently,

$$A_k \delta_j h_k^{\delta_j} - h_k^{\delta_j} \to A_k z - b_k, \quad \text{as} \quad j \to \infty$$  \hbox{(47)}

(here $A_k^\delta = F'(x_k^\delta)$, $A_k = F'(x_k)$, $b_k^{\delta_j} = y^{\delta_j} - F(x_k^{\delta_j})$, $b_k = y - F(x_k)$).

Proof of Step 2. If $\alpha_k$ in (45) is not positive, we conclude from (A2)

$$\liminf_j \|A_k^\delta h_k^{\delta_j} - b_k^{\delta_j}\|^2 \leq \liminf_j T_{k,\delta_j,\alpha_k}(h_k^{\delta_j}) \leq \liminf_j T_{k,\delta_j,\alpha_k}(x^1 - x_k)$$

$$= \liminf_j \left(\|A_k^\delta (x^1 - x_k) - b_k^{\delta_j}\|^2 + \alpha_k^{\delta_j} \|x^1 - x_k\|^2\right)$$

$$= \|A_k (x^1 - x_k) - b_k\|^2 \leq \eta_2 \|b_k\|^2$$

(here $T_{k,\delta}(h) := \|F'(x_k^\delta)h - y^\delta + F(x_k^\delta)\|^2 + \alpha\|h\|^2$). This leads to the contradiction

$$c_k = \lim_j c_k^{\delta_j} \leq \liminf_j \|A_k^\delta h_k^{\delta_j} - b_k^{\delta_j}\| \leq \eta_2 \|b_k\| < c_k.$$

Thus $\alpha_k > 0$ holds. We define $T_{k,\alpha}(h) := T_{k,\delta,\alpha}(h)$ with $\delta = 0$, $h_k := \arg \min_{h \in X} T_{k,\alpha}(h)$, and $x_{k+1} := x_k + h_k$. In order to prove that $x_{k+1}$ is a successor of $x_k$, it is necessary to prove that

$$c_k \leq \|A_k h_k - b_k\| \leq d_k.$$  \hbox{(48)}

We first prove that $h_k = z$ (see Step 3).

Proof of Step 3. From (44), (47) and (45), it follows

$$T_{k,\alpha_k}(z) = \|A_k z - b_k\|^2 + \alpha_k \|z\|^2 \leq \liminf_j (\|A_k^\delta h_k^{\delta_j} - b_k^{\delta_j}\|^2 + \alpha_k^{\delta_j} \|h_k^{\delta_j}\|^2)$$

$$= \liminf_j T_{k,\delta_j,\alpha_k}(h_k^{\delta_j}) \leq \liminf_j T_{k,\delta_j,\alpha_k}(h_k) = T_{k,\alpha_k}(h_k).$$

Since $h_k$ is the unique minimizer of $T_{k,\alpha_k}$, we conclude that $h_k = z$. Thus, $h_k^{\delta_j} \to h_k$ as $j \to \infty$. The last inequalities also ensure that $\liminf_j T_{k,\delta_j,\alpha_k}(h_k^{\delta_j}) = T_{k,\alpha_k}(h_k)$. This guarantees the existence of a subsequence satisfying

$$\lim_{j \to \infty} T_{k,\delta_j,\alpha_k}(h_k^{\delta_j}) = T_{k,\alpha_k}(h_k).$$  \hbox{(49)}
Proof of Step 4. The goal is to validate (46), which, together with $h_k^{\delta_j} \to h_k$, imply $h_k^{\delta_j} \to h_k$. Consequently, (48) follows from (42). This ensures that $x_{k+1}$ is a successor of $x_k$ and validates (43), completing the proof of the theorem.

We first prove the existence of a constant $\alpha_{\text{max}, k}$ such that

$$\alpha_k^{\delta_j} \leq \alpha_{\text{max}, k} \quad \text{for all} \quad j \in \mathbb{N}. $$

Indeed, if such a constant did not exist, we could find a subsequence satisfying $\alpha_k^{\delta_j} \to \infty$ as $j \to \infty$. Thus, since

$$\alpha_k^{\delta_j} ||h_k^{\delta_j}||^2 \leq T_{k, \delta, \alpha_k}^{\delta_j}(h_k^{\delta_j}) \leq T_{k, \delta, \alpha_k}^{\delta_j}(0) = ||b_k^{\delta_j}||^2,$$

we would have,

$$\lim_{j \to \infty} \alpha_k^{\delta_j} ||h_k^{\delta_j}||^2 \leq ||b_k||^2 < \infty,$$

which would imply $h_k^{\delta_j} \to 0$. Consequently,

$$\lim_{j \to \infty} ||A_k^{\delta_j} h_k^{\delta_j} - b_k^{\delta_j}|| = ||b_k|| > d_k = \lim_{j \to \infty} \alpha_k^{\delta_j},$$

which would imply the contradiction $||A_k^{\delta_j} h_k^{\delta_j} - b_k^{\delta_j}|| > d_k^{\delta_j}$, for $j$ large enough.

Now we validate (46). This proof follows the lines of [17, Lemma 5.2]. Define

$$a_j := ||h_k^{\delta_j}||^2, \quad a := \limsup a_j, \quad c := ||h_k||^2, \quad re_j := ||A_k^{\delta_j} h_k^{\delta_j} - b_k^{\delta_j}||^2, \quad re := \liminf re_j.$$ 

As $||h_k|| \leq \lim inf ||h_k^{\delta_j}||$, it suffices to prove that $a \leq c$. Assume the contrary. From (49), there exists a number $N_1 \in \mathbb{N}$ such that

$$j \geq N_1 \implies T_{k, \delta_j, \alpha_k}^{\delta_j}(h_k^{\delta_j}) < T_{k, \alpha_k}(h_k) + \alpha_k \frac{a - c}{2}. \quad (50)$$

From definition of lim inf, there exist constants $N_2, N_3 \in \mathbb{N}$ such that

$$j \geq N_2 \implies re_j \geq re - \alpha_k(a - c)/6 \quad (51)$$

and

$$j \geq N_3 \implies \alpha_k^{\delta_j} \geq \alpha_k - \alpha_k(a - c)/6a. \quad (52)$$

Moreover, from definition of lim sup, we conclude that for each $M \in \mathbb{N}$ fixed, there exists an index $j \geq M$ such that

$$a_j \geq a - \alpha_k(a - c)/(6\alpha_{\text{max}, k}). \quad (53)$$

Therefore, for $M := \max\{N_1, N_2, N_3\}$, there exists an index $j \geq M$ such that

$$T_{k, \alpha_k}(h_k) \leq re + \alpha_k c = re + (\alpha_k - \alpha_k^{\delta_j}) a + \alpha_k^{\delta_j} (a - a_j) + \alpha_k^{\delta_j} a_j - \alpha_k(a - c)$$

$$\leq (re_j + \alpha_k \frac{1}{6}(a - c)) + \alpha_k \frac{1}{6}(a - c) + \alpha_k \frac{1}{6}(a - c) + \alpha_k \frac{1}{6} a_j - \alpha_k(a - c)$$

$$= re_j + \alpha_k \frac{1}{6} a_j - \alpha_k \frac{1}{6}(a - c) = T_{k, \delta_j, \alpha_k}^{\delta_j}(h_k^{\delta_j}) - \alpha_k \frac{1}{6}(a - c) < T_{k, \alpha_k}(h_k),$$

where the second inequality follows from (51), (52), (53), while the last inequality follows from (50). This leads to the obvious contradiction $T_{k, \alpha_k}(h_k) < T_{k, \alpha_k}(h_k)$, proving that $a \leq c$ as desired. Thus, (46) holds and the proof is complete. \qed
Theorem 3.10 (Regularization). Let Assumptions (A1), (A2) and (A3) hold, and \( \{\delta_j\}_{j \in \mathbb{N}} \) be a positive zero-sequence. Assume that the (finite) sequences \( \{x^\delta_{j_k}\}_{0 \leq k \leq k^*} \), \( j \in \mathbb{N} \), are fixed, where \( x^\delta_{j_k+1} \) is a successor of \( x^\delta_{j_k} \). Then, every subsequence of \( \{x^\delta_{k^*} \}_{j \in \mathbb{N}} \) has itself a subsequence converging strongly to a solution of (1).

Proof. Since any subsequence of \( \{\delta_j\}_{j \in \mathbb{N}} \) is itself a positive zero-sequence, it suffices to prove that \( \{x^\delta_{k^*} \}_{j \in \mathbb{N}} \) has a subsequence converging to a solution. We consider two cases:

Case 1. The sequence \( \{k^*(\delta_j)\}_{j \in \mathbb{N}} \) is bounded.
Thus, there exists a constant \( M \in \mathbb{N} \) such that \( k^*(\delta_j) \leq M \) for all \( j \in \mathbb{N} \). Thus, the sequence \( \{x^\delta_{k^*} \}_{j \in \mathbb{N}} \) splits into at most \( M+1 \) subsequences having the form \( \{x^\delta_{m} \}_{n \in \mathbb{N}} \), with fixed \( m \leq M \). Pick one of these subsequences. From Theorem 3.9, this subsequence has itself a subsequence (again denoted by \( \{x^\delta_{m} \}_{n \in \mathbb{N}} \)) converging to some \( x_m \in X \), i.e.,

\[
\lim_{n \to \infty} x^\delta_{n,k^*(\delta_m)} = \lim_{n \to \infty} x^\delta_{m} = x_m.
\]

Notice that \( x_m \) is a solution of (1). Indeed,

\[
\|y - F(x_m)\| = \lim_{n \to \infty} \|y - F(x^\delta_{n,k^*(\delta_m)})\|
\leq \lim_{n \to \infty} (\|y - y^\delta_{n}\| + \|y^\delta_{n} - F(x^\delta_{n,k^*(\delta_m)})\|)
\leq \lim_{n \to \infty} (\tau + 1) \delta_{n} = 0.
\]

Case 2. The sequence \( \{k^*(\delta_j)\}_{j \in \mathbb{N}} \) is not bounded.
Thus, there is a subsequence such \( k^*(\delta_j) \to \infty \) as \( j \to \infty \). Let \( \varepsilon > 0 \) be given and consider the noiseless sequence \( \{x_k\}_{k \in \mathbb{N}} \) constructed in last theorem. Since \( x_{k+1} \) is a successor of \( x_k \) for all \( k \in \mathbb{N} \), \( \{x_k\}_{k \in \mathbb{N}} \) converges to some solution \( x^* \) of (1) (see Theorem 3.7). Then, there exists \( M = M(\varepsilon) \in \mathbb{N} \) such that

\[
\|x_{M} - x^*\| < \frac{1}{2} \varepsilon.
\]

On the other hand, there exists \( J \in \mathbb{N} \) such that \( k^*(\delta_j) \geq M \), for \( j \geq J \). Consequently, it follows from the monotonicity of the iteration error (see Theorem 2.6) that

\[
j \geq J \implies \|x^\delta_{k^*(\delta_j)} - x^*\| \leq \|x^\delta_{M} - x^*\|.
\]

Moreover, it follows from Theorem 3.9, the existence of a subsequence \( \{\delta_{j_m}\} \) (depending on \( M(\varepsilon) \)) and the existence of \( N \in \mathbb{N} \) such that

\[
m \geq N \implies \|x^\delta_{j_m} - x_{M}\| < \frac{1}{2} \varepsilon.
\]

Consequently, for \( m \geq \max\{J, N\} \) (which simultaneously guarantees \( j_m \geq m \geq J \) and \( m \geq N \)), it holds

\[
\|x^\delta_{j_m} - x^*\| \leq \|x^\delta_{j_m} - x_{M}\| \leq \|x^\delta_{j_m} - x_{M}\| + \|x_{M} - x^*\| < \varepsilon.
\]

(54)

Notice that the subsequence \( \{\delta_{j_m}\} \) depends on \( \varepsilon \). We now construct an \( \varepsilon \)-independent subsequence using a diagonal argument: for \( \varepsilon = 1 \) in (54), there is a subsequence of \( \{\delta_j\} \) (called again \( \{\delta_j\} \)) and \( j_1 \in \mathbb{N} \) such that

\[
\|x^\delta_{j_1} - x^*\| < 1.
\]
Now, for $\varepsilon = 1/2$, there exists a subsequence $\{\delta_j\}$ of the previous one, and $j_2 > j_1$ such that
\[ \|x_{k^*(\delta_{j_2})}^{\delta_{j_2}} - x^*\| < 2^{-1}. \]
Arguing in this way, we construct a subsequence $\{\delta_{j_n}\}_{n \in \mathbb{N}}$ satisfying
\[ \|x_{k^*(\delta_{j_n})}^{\delta_{j_n}} - x^*\| < n^{-1}, \]
from what follows $\lim x_{k^*(\delta_{j_n})}^{\delta_{j_n}} = x^*$. \hfill \Box

**Remark 3.11.** If the solution of (1) referred in Theorem 3.9 were independent of the chosen subsequence, then any subsequence of $\{x_{k^*(\delta_j)}^{\delta_j}\}_{j \in \mathbb{N}}$ would have itself a subsequence converging to the same solution. This would be enough to ensure that the whole sequence $\{x_{k^*(\delta_j)}^{\delta_j}\}_{j \in \mathbb{N}}$ converges to $x^*$.

However, $x^*$ in the above proof depends on the noiseless sequence $\{x_n\}_{n \in \mathbb{N}}$ (whose existence is guaranteed by Theorem 3.9), which in turn depends on the fixed sequences $\{x_{k^*(\delta_j)}^{\delta_j}\}_{0 \leq k \leq k^*(\delta_j)}$, $j \in \mathbb{N}$. Consequently, if different subsequences of $\{\delta_j\}_{j \in \mathbb{N}}$ are chosen, the solution of (1) referred in Theorem 3.9 can be different.

**Corollary 3.12.** Under the assumptions of Theorem 3.9, the following assertions hold true:

1. The sequence $\{x_{k^*(\delta_j)}^{\delta_j}\}_{j \in \mathbb{N}}$ splits into convergent subsequences, each one converges to a solution of (1);

2. If $x^*$ in (A3) is the unique solution of (1) in $B_\rho(x_0)$, then $x_{k^*(\delta_j)}^{\delta_j} \to x^*$ as $j \to \infty$;

3. If the null-space condition (38) holds, then $\{x_{k^*(\delta_j)}^{\delta_j}\}_{j \in \mathbb{N}}$ converges to the $x_0$-minimal-norm solution $x^\dagger$ as $j \to \infty$.

**Proof.** The proof of Assertion 1. is straightforward. Assertion 2. follows from the fact that, if $x^*$ is the unique solution of (1) in $B_\rho(x_0)$, then $x_{k^*(\delta_j)}^{\delta_j} \to x^*$ as $j \to \infty$. To prove Assertion 3. notice that, if (38) holds then any noiseless sequence $\{x_n\}_{n \in \mathbb{N}}$ converges to $x^\dagger$ (Theorem 3.7). Thus, any subsequence of $\{x_{k^*(\delta_j)}^{\delta_j}\}_{j \in \mathbb{N}}$ has itself a subsequence converging to $x^\dagger$, and the proof follows. \hfill \Box

## 4 Numerical experiments

### 4.1 The model problem and its discretization

We test the performance of our method applying it to the non-linear and ill-posed inverse problem of EIT (Electrical Impedance Tomography) introduced by Calderón [5]. A survey article concerning this problem is [4].

Let $\Omega \subset \mathbb{R}^2$ be a bounded and simply connected Lipschitz domain. The EIT problem consists in applying different configurations of electric currents on the boundary of $\Omega$ and then reading the resulting voltages on the boundary of $\Omega$ as well. The objective is recovering the electric conductivity in the whole of set $\Omega$. This problem is governing by the variational equation

\[ \int_\Omega \gamma \nabla u \nabla \varphi = \int_{\partial \Omega} g \varphi \quad \text{for all } \varphi \in H^1_0(\Omega), \quad (55) \]
where \( g : \partial \Omega \to \mathbb{R} \) represents the electric current, \( \gamma : \Omega \to \mathbb{R} \) is the electric conductivity and \( u : \Omega \to \mathbb{R} \) represents the electric potential. Employing the Lax-Milgram Lemma, one can prove that, for each \( g \in L^2_0(\partial \Omega) := \{ v \in L^2(\partial \Omega) : \int_{\partial \Omega} v = 0 \} \) and \( \gamma \in L^\infty_2(\Omega) := \{ v \in L^\infty(\Omega) : v \geq c > 0 \text{ a.e. in } \Omega \} \) fixed, there exists a unique \( u \in H^1_0(\Omega) := \{ v \in H^1(\Omega) : \int_{\partial \Omega} v = 0 \} \) satisfying (55). The voltage \( f : \partial \Omega \to \mathbb{R} \) is the trace of the potential \( u \) (\( f = u|_{\partial \Omega} \)), which belongs to \( L^2_0(\partial \Omega) \).

For a fixed conductivity \( \gamma \in L^\infty_2(\Omega) \), the bounded linear operator \( \Lambda_\gamma : L^2_0(\partial \Omega) \to L^2_0(\partial \Omega) \), \( g \mapsto f \), which associates the electric current with the resulting voltage is the so-called Neumann-to-Dirichlet map (in short NtD). The forward operator associated with EIT is defined by

\[
\mathcal{F}(\gamma) = \Lambda_\gamma,
\]

with \( \mathcal{F} : L^\infty_2(\Omega) \subset L^\infty(\Omega) \to L(L^2_0(\partial \Omega), L^2_0(\partial \Omega)) \). The EIT inverse problem consists in finding \( \gamma \) in above equation for a given \( \Lambda_\gamma \). However, in practical situations, only a part of the data can be observed and therefore the NtD map is not completely available. One has to apply \( g \) to \( \mathcal{F}(\gamma) \).

Since an analytical solution of (55) is not available in general, the inverse problem needs to be solved with help of a computer. For this reason, we construct a triangulation for \( \Omega \), \( \tilde{\mathcal{V}} = \tilde{\mathcal{V}}(\Omega) \cap V \).

It is still unclear whether the forward operator associated with the continuous model of EIT, defined in (56), satisfies the tangential cone condition (6), but the version presented in the restricted set (57) guarantees this result, at least in a small ball around a solution, see [13]. The Fréchet derivative of \( F, \; F' : \text{int}(\tilde{\mathcal{V}}) \to L(V, (L^2(\partial \Omega))^d) \), satisfies \( F'(\gamma)h = (w_1|_{\partial \Omega}, \ldots, w_d|_{\partial \Omega}) \), where \( w_j \in H^1_0(\Omega) \) is the unique solution of

\[
\int_{\Omega} \gamma \nabla w_j \nabla \varphi = -\int_{\Omega} h \nabla u_j \nabla \varphi \quad \text{for all } \varphi \in H^1_0(\Omega),
\]

with \( u_j \) solving (55) for \( g = g_j \). The adjoint operator \( F'(\gamma)^* : (L^2(\partial \Omega))^d \to V \) is given by

\[
F'(\gamma)^* z = -\sum_{j=1}^d \nabla u_j \nabla \psi_{z_j},
\]

where \( z := (z_1, \ldots, z_d) \in (L^2(\partial \Omega))^d \) and for each \( j = 1, \ldots, d \), the vectors \( u_j \) and \( \psi_{z_j} \) are the unique solutions of (55) for \( g = g_j \) and \( g = z_j \) respectively.

In our numerical simulations we define \( \Omega := (0,1) \times (0,1) \) and supply the current-vector \((g_1, \ldots, g_d)\) with \( d = 8 \) independent currents: identifying the faces of \( \Omega \) with the numbers

\[
\sum_{i=1}^d \theta_i \chi_{T_i} \quad \text{with} \quad \{ \theta_1, \ldots, \theta_M \} \in \mathbb{R}^M.
\]

Equipped with an inner product defined in a very natural way, induced by the inner product in \( L^2(\partial \Omega) \), the space \((L^2_0(\partial \Omega))^d \) is a Hilbert space.

Notice that \( \text{span}\{\chi_{T_1}, \ldots, \chi_{T_M}\} \subset L^\infty(\Omega) \).
0, 1, 2, 3, we apply the currents
\[ g_{2m+k}(x) = \begin{cases} \cos(2k\pi x) & : \text{on the face } m \\ 0 & : \text{elsewhere on } \partial\Omega \end{cases} \]
for \( k = 1, 2 \). The exact solution \( \gamma^+ \) consists of a constant background conductivity 1 and an inclusion \( B \subset \Omega \) with conductivity 2:
\[ \gamma^+(x) := \begin{cases} 2 & : x \in B \\ 1 & : \text{otherwise} \end{cases} \]
The set \( B \) models two balls with radii equal 0.15 and center at the points (0.35, 0.35) and (0.65, 0.65). The data,
\[ y := (\Lambda_{\gamma^+} g_1, \ldots, \Lambda_{\gamma^+} g_d), \tag{60} \]
corresponding to the exact solution \( \gamma^+ \) are computed using the Finite Element Method (FEM). The problems (55) and (58) have been solved by FEM as well, but using a much coarser discretization mesh than the one used to generate the data for avoiding inverse crimes, see Figure 1.

It is well known that in this specific problem, undesirable instability effects may arise from an unfavorable selection of the geometry of the mesh. For avoiding this problem we employ a strategy using a weight-function \( \omega: \Omega \to \mathbb{R} \) to define the weighted-space \( L_2^\omega(\Omega) := \{ f : \Omega \to \mathbb{R} : \int_\Omega |f|^2\omega < \infty \} \). This alteration changes the evaluation of the adjoint operator (59) in the discretized setting, see [25] and [16, Subsection 5.1.2] for details. In the mentioned references, the authors use the weight-function
\[ \omega := \sum_{i=1}^M \beta_i \chi_{T_i} \quad \text{with} \quad \beta_i := \frac{\|F'((\gamma_0)\chi_{T_i})\|_{L_2^\omega(\partial\Omega)^d}}{|T_i|}, \]
where \( |T_i| \) is the area of triangle \( T_i \), and the initial iterate \( \gamma_0 \) is the constant 1 function.

In the notation of Section 1 we have \( F: D(F) \subset X := (\text{span}\{\chi_{T_1}, \ldots, \chi_{T_M}\}, \|\cdot\|_{L_2^\omega(\Omega)}) \to (L^2(\partial\Omega))^d =: Y \), where \( D(F) = X \cap L_2^\omega(\Omega) \). We define the relative error in the \( k \)-th iterate \( \gamma_k \) as
\[ E_k := 100 \frac{\|\gamma_k - \gamma^+\|_X}{\|\gamma^+\|_X}, \tag{61} \]
and use it to compare the quality of the reconstructions. Finally, we corrupt the simulated data \( y \) in (60) by adding artificially generated random noise, with a relative noise level \( \delta > 0 \),
\[ y^\delta = y + \delta \text{noi}\|y\|_Y, \tag{62} \]
where noi \( \in Y \) is a uniformly distributed random variable such that \( \|\text{noi}\|_Y = 1 \).
4.2 Implementation of the range-relaxed Levenberg-Marquardt method

Now, we turn to the problem of finding a pair \((\alpha_k > 0, h_k \in X)\) in accordance to Step [3.1] of Algorithm I.

An usual choice for the parameters \(\alpha_k\) is of geometric type, i.e., the parameters are defined \textit{a priori} by the rule \(\alpha_k = r\alpha_{k-1}\), where \(\alpha_0 > 0\) and \(0 < r < 1\) (the \textit{decreasing ratio}) are given. This method is usually very efficient if a good guess for the constant \(r\) is available. However, big troubles may arise if the decreasing ratio \(r\) is chosen either too large or too small. Indeed, on the one hand, if the constant \(r\) is too large \((r \approx 1)\), then the method becomes slow and the computational costs increase considerably; on the other hand, the Levenberg-Marquardt method becomes unstable in case \(r\) is chosen too small \((r \approx 0)\), see Figure 2 below.

![Figure 2: Geometric choice of the parameters \(\alpha_k\) for \(r = 0.1\) and \(r = 0.9\), with \(\alpha_0 = 2\). Noise level \(\delta = 0.1\\%\), \(\eta = 0.4\) and \(\tau = 1.3(1 + \eta)/(1 - \eta)\). LEFT: residual, RIGHT: iteration error.](image)

Notice that the \(\alpha_k\) defined by the geometric choice does not necessarily satisfy the problem in Step [3.1]. We propose a strategy for choosing the decreasing ratio \(r\) in each step, so that the resulting parameter \(\alpha_k\) (and the corresponding \(h_k\)) are in agreement with Step [3.1]. For the actual computation of the ratio \(r\) in the current step, we use information on the current iteration and past iterations as well. This is described in the sequel.

We adopt the notation \(H_k(\alpha) = \|y^\delta - F(\gamma_k) - F'(\gamma_k)h_\alpha\|, \alpha > 0\), where \(h_\alpha\) is given by

\[
h_\alpha = (F'(\gamma_k)^*F'(\gamma_k) + \alpha I)^{-1}F'(\gamma_k)^*(y^\delta - F(\gamma_k)).
\]

According to Step [3.1] in Algorithm I, we need to determine \(\alpha_k > 0\) such that \(H_k(\alpha_k) \in [c_k, d_k]\), where \(c_k\) and \(d_k\) are defined in (10) and (11) respectively. For doing that, we have employed the \textit{adaptive strategy} introduced in \cite{15}. This algorithm is based on the geometric method but allows adaptation of the decreasing ratio using \textit{a posteriori} information. First, we define the constants

\[
\hat{c}_k = p_1c_k + (1 - p_1)d_k \quad \text{and} \quad \hat{d}_k = p_2c_k + (1 - p_2)d_k,
\]

where \(0 < p_1 < p_2 < 1\). Notice that \([\hat{c}_k, \hat{d}_k] \subset [c_k, d_k]\).

Choose the initial parameter \(\alpha_0 > 0\); compute \(h_0 := h_{\alpha_0}\) and \(\gamma_1\) according to Algorithm I.
Choose the initial decreasing ratio $0 < r_0 < 1$, define $\alpha_1 = r_0 \alpha_0$; compute $h_1 := h_{\alpha_1}$ and $\gamma_2$ according to Algorithm I.

For $k \geq 1$, we define $\alpha_{k+1} = r_k \alpha_k$, where

$$r_k = \begin{cases} 
  a_1 r_{k-1}, & \text{if } c_{k-1} \leq H_{k-1}(\alpha_{k-1}) < \hat{c}_{k-1} \\
  a_2 r_{k-1}, & \text{if } \hat{d}_{k-1} < H_{k-1}(\alpha_{k-1}) \leq d_{k-1} \\
  r_{k-1}, & \text{if } H_{k-1}(\alpha_{k-1}) \in [\hat{c}_{k-1}, \hat{d}_{k-1}] 
\end{cases} \quad (65)$$

Here the constants $0 < a_2 < 1 < a_1$ play the role of correction factors, and are chosen \textit{a priori}.

The idea of the \textit{adaptive strategy} is to observe the behavior of the function $H_k$ and try to determine how much the parameter $\alpha_k$ should be decreased in the next iteration. For example, the number $H_k(\alpha_k)$ lying to the left of the smaller interval $[\hat{c}_k, \hat{d}_k]$ means that $\alpha_k$ was too small. We thus multiply the decreasing ratio $r_{k-1}$ by the number $a_1 > 1$, in order to increase it, and consequently, to decrease the parameter $\alpha_k$ slower than in the previous step, trying to hit $[\hat{c}_k, \hat{d}_k]$ in the next iteration. This algorithm is efficient in terms of computational cost: Like the geometric choice for $\alpha_k$, it requires only one minimization of a Tikhonov functional in each iteration. Further, the \textit{adaptive strategy} has the additional advantage of correcting the decreasing ratio if this ratio is either too large or too small.

An attentive reader could object that, in some iterations, the evaluated parameter $\alpha_k$ may lead to a number $H_k(\alpha_k)$ which does not belong to the interval $[c_k, d_k]$ defined in Step [3.1]. This is indeed possible! In this situation, we apply the secant method in order to recalculate $\alpha_k$ such that $H_k(\alpha_k) \in [c_k, d_k]$, before starting the next iteration. This is however an expensive task, since each step of the secant method demands the additional minimizations of Tikhonov functionals.

It is worth noticing that this situation has been barely observed in our numerical experiments, occurring only in the cases when either the initial decreasing ratio $r_0$ or the initial guess $\alpha_0$ are poorly chosen.

### 4.3 Numerical realizations

For the constant $\tau$ in (7) we use $\tau = 1.3(1 + \eta)/(1 - \eta)$, where $\eta = 0.4$ is the constant in (A2). Moreover, we choose $p = 0.1$ and $\varepsilon = 0.1[\tau(1 - \eta) - (1 + \eta)]/\eta \tau$ in (7). The constants in (64) are $p_1 = 1/3$ and $p_2 = 2/3$, while the constants in (65) are $a_1 = 2$ and $a_2 = 1/2$.

**First test (one level of noise):**

The goal of this test is to investigate the performance of our rrLM method with adaptive strategy (a posteriori) for computing the parameters, with respect of different choices of initial decreasing ratio $r_0$.

As observed in Figure 2, the performance of the LM method with geometric choice (a priori) of parameters is very sensitive to the choice of the (constant) decreasing ratio $r < 1$.

We implement the rrLM method (using \textit{adaptive strategy}) with $r_0 = 0.1$ and $r_0 = 0.9$. In Figure 3 the results of the the rrLM method are compared with the LM method using geometric choice of parameters (see top-left, top-right and bottom-left pictures).

- [GREEN] rrLM with $r_0 = 0.1$, reaches discrepancy with $k^* = 11$ steps;
- [MAGENTA] rrLM with $r_0 = 0.9$, reaches discrepancy with $k^* = 11$ steps;
- [RED] LM with $r = 0.9$, reaches discrepancy with $k^* = 36$ steps;
- [BLUE] LM with $r = 0.1$, does not reach discrepancy.

The noise level is $\delta = 0.1\%$. All methods are started with $\alpha_0 = 2$. The last picture in Figure 3 (bottom-right) shows the values of the linearized residual $H_k(\alpha_k)$ as well as the intervals $[c_k, d_k]$ (see (10) and (11)) for the rrLM with $r_0 = 0.9$. 

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Figure 3: First test: Noisy data, $\delta = 0.1\%$. TOP-LEFT: Residual. TOP-RIGHT: Relative iteration error. BOTTOM-LEFT: Parameter $\alpha_k$. BOTTOM-RIGHT: Linearized residual $H_k(\alpha_k)$ and the numbers $c_k$ and $d_k$ for the rrLM with $r_0 = 0.9$.

From this first test we draw the following conclusions:

- The rrLM method (using *adaptive strategy*) is robust with respect of the choice of the (initial) decreasing ratio. We tested two poor choices of initial decreasing ratios (namely $r_0 = 0.1$ and $r_0 = 0.9$); nevertheless the performance of the rrLM method in both cases is stable and numerically efficient. For rrLM method, the relative error obtained for $r_0 = 0.1$ is comparable to that obtained for $r_0 = 0.9$ (see top-right picture in Figure 3).

- We also tested the rrLM method (using *adaptive strategy*) and the LM method (using geometric choice of parameters) for $r = r_0 = 0.5$, which seems to be the "optimal" choice of constant decreasing ratio. In this case, both methods performed similarly. Moreover, the performance of the rrLM method (number of iterations and numerical effort) was similar to the ones depicted in Figure 3 using $r_0 = 0.1$ and $r_0 = 0.9$.

- The rrLM method (using *adaptive strategy*) "corrects" eventual poor choices of the decreasing ratio. If $r_0$ is too small, the *adaptive strategy* increases this ratio during the first iterations (GREEN curve in Figure 3) preventing instabilities (compare with the LM method using geometric choice of parameters — BLUE curve in Figure 3).

On the other hand, if $r_0$ is large (close to one), the *adaptive strategy* decreases this ratio during the first iterations (MAGENTA curve in Figure 3), preventing slow convergence (compare with the LM method using geometric choice of parameters — RED curve in Figure 3).
\[ k^*(N_k^*) \]

| \( \delta(\%) \) | \( k^*(N_k^*) \) | \( (r_0 = 0.9) \) | \( (r_0 = 0.5) \) | \( (r_0 = 0.1) \) |
|--------------|-----------------|----------------|----------------|----------------|
|              | rrLM | LM | rrLM | LM | rrLM | LM |
| 0.8          | 5(6) | 3(3) | 4(5) | 3(3) | 5(8) | 3(3) |
| 0.4          | 8(8) | 8(8) | 6(6) | 4(4) | 8(12) | 4(4) |
| 0.2          | 9(9) | 18(18) | 7(7) | 7(7) | 8(11) | Fails |
| 0.1          | 11(11) | 35(35) | 10(10) | 10(10) | 11(14) | Fails |

Table 1: Comparison between rrLM and LM methods: Computational effort.

| \( \delta(\%) \) | \( E_{k^*} \) | \( (r_0 = 0.9) \) | \( (r_0 = 0.5) \) | \( (r_0 = 0.1) \) |
|--------------|----------------|----------------|----------------|----------------|
|              | rrLM | LM | rrLM | LM | rrLM | LM |
| 0.8          | 82.6 | 82.7 | 82.8 | 81.5 | 82.8 | 80.9 |
| 0.4          | 79.7 | 79.7 | 79.5 | 79.7 | 79.6 | 79.5 |
| 0.2          | 76.5 | 76.5 | 76.3 | 76.6 | 76.4 | Fails |
| 0.1          | 71.5 | 72.9 | 71.6 | 71.7 | 72.1 | Fails |

Table 2: Comparison between rrLM and LM methods: Relative iterative error at the final iteration.

- The last picture in Figure 3 (bottom-right) shows that the linearized residual \( H_k(\alpha_k) \), computed using the adaptive strategy, satisfies (9) in Step [3.1] of Algorithm I. Consequently, this strategy provides a numerical realization of Algorithm I, which is in agreement with the theory devised in this article.

**Second test (several levels of noise):**

The goal of this test is twofold: (1st) We validate the regularization property (see Theorem 3.10 and Corollary 3.12) by choosing different levels of noise \( \delta > 0 \), and observing what happens when the noise level decreases; (2nd) We compare the numerical effort of the rrLM method (with adaptive strategy) with the LM method (with geometric choice of parameters).

In what follows we present a set of experiments with four different levels of noise \( \delta > 0 \) namely, \( \delta = 0.8\%, \delta = 0.4\%, \delta = 0.2\%, \delta = 0.1\% \). In each scenario above, we implemented the rrLM method (with adaptive strategy) as well as the LM method (with geometric choice of parameters).

For the implementation of the LM method with geometric choice of parameters we use the constant decreasing ratios: \( r_0 = 0.9 \), \( r_0 = 0.5 \) and \( r_0 = 0.1 \). For the implementation of the rrLM method we used the same choices of \( r_0 \) as starting value for \( r \) together with the adaptive strategy.

In all implementations \( \alpha_0 = 2 \) is used. Comparisons of these methods are presented in Tables 1 and 2. Three distinct indicators are used, namely

- Number of iterations to reach discrepancy \( k^* = k^*(\delta) \) (see Step [3.3]);
- Total number of Tikhonov functionals minimized for \( k = 0, \ldots, k^* - 1 \), denoted by \( N_k^* \);\(^{8}\)
- Relative iteration error at step \( k = k^* \), denoted by \( E_{k^*} \) (see (61)).\(^{9}\)

From this second test we draw the following conclusions:

- For both methods \( k^* \) increases and \( E_{k^*} \) decreases as \( \delta \) becomes smaller (validating the regularization property).
- For each fixed noise level \( \delta \), the values of \( E_{k^*} \) are similar for both methods.

\(^{8}\)The numbers \( k^* \) and \( N_k^* \) are always the same in the geometric choice (LM method), but \( N_k^* \) may be larger than \( k^* \) in the adaptive strategy (rrLM method).

\(^{9}\)It is worth noticing that the initial iteration error is \( E_0 = 87.39\% \) in all four scenarios above.
If the noise level is small ($\delta = 0.1\%$ and $\delta = 0.2\%$), the rrLM method is more efficient than the LM method for $r_0 = 0.9$. Both methods perform similarly for $r_0 = 0.5$. For $r_0 = 0.1$ the LM method fails to converge, while the rrLM method succeed in reaching the stopping criterium.

For higher levels of noise ($\delta = 0.4\%$ and $\delta = 0.8\%$), both methods perform similarly for $r_0 = 0.9$ and $r_0 = 0.5$. For $r_0 = 0.1$ the LM method converges faster than the rrLM method. This is due to the fact that rrLM needs to correct the initial guess for $\alpha_0 = 2$.

For levels of noise higher than 0.8%, the rrLM stops after 2 or less iterations (for different choices of $r_0$). Consequently, these experiments do not give relevant information about the performance of our method.

For the rrLM method, the values of $k^*$ and $N_{k^*}$ are identical in most of the scenarios of Table 1, i.e., only one Tikhonov functional is minimized in each step (this is the same numerical cost for one step of the LM method with geometric choice of parameters).

The last conclusion validates the adaptive strategy for computing the parameters $\alpha_k$ as an efficient alternative for the numerical implementation of Step [3.1] in Algorithm I.

5 Final remarks and conclusions

In this article we address the Levenberg-Marquardt method for solving nonlinear ill-posed problems and propose a novel range-relaxed criteria for choosing the Lagrange multipliers, namely: the new iterate is obtained as the projection of the current one onto a level-set of the linearized residual function; this level belongs to an interval (or range), which is defined by the current nonlinear residual and by the noise level (see Step [3.1] of Algorithm I).

The main contributions in this article are:

- We derive a complete convergence analysis: convergence (Theorem 3.7), stability (Theorem 3.9), semi-convergence (Theorem 3.10). We also prove monotonicity of iteration error (Theorem 2.6) and geometric decay of residual (Proposition 3.2). Moreover, we prove convergence to minimal-norm solution under additional null-space condition (38), in both exact and noisy data cases.

- We give a novel proof for the stability result, which uses non standard arguments. In the classical stability proof, since each Lagrange multiplier is uniquely defined by an (implicit) equation, the set of successors (Definition 3.8) of each $x^\delta_k$ is singleton. However, due to our range-relaxed criteria (9), each set of successors may contain infinitely many elements; consequently, the subsequences $\{\delta_jm\}_{m \in \mathbb{N}}$ obtained in Theorem 3.9 do depend on the iteration index $k$.

- We devise a numerical algorithm, based on the adaptive strategy (see Subsection 4.2), for implementing the range-relaxed criteria proposed in this article. Its main features are:

  - Efficiency in terms of computational cost: Like the LM with geometric (a priori) choice of parameters, it (almost always) requires only one minimization of a Tikhonov functional in each iteration.

  - Correction of the decreasing ratio if this ratio is either too large or too small.

  - The computed pairs $(\alpha_k, h_k)$ satisfy (9) for all $k > 0$, i.e., this algorithm provides a numerical realization of Algorithm I.

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