Second law for an autonomous information machine connected with multiple baths

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March 29, 2018

Abstract

In an Information machine system’s dynamics gets affected by the attached information reservoir. Second law of thermodynamics can be apparently violated for this case. In this article we have derived second law for an information machine, when a system is connected to multiple heat baths along with a work source and a single information reservoir. Here a sequence of bits written on a tape is considered as an information reservoir. We find that the bath entropy production during a time interval is restricted by the change of Shannon entropy of the joint system (system + information reservoir) during that interval. We have also given several examples where this law can be applicable.

Keywords: information processing, exact results, stochastic processes

1 Introduction

Second law is a fundamental law in thermodynamics and always valid on an average [1-3]. According to the law average entropy production is always positive. Recent development of fluctuation theorems dictates that it is possible to find negative entropy production for an individual event for any duration although their probabilities are exponentially small. However validity of the second law was questioned by Maxwell even more than a century ago, when he proposed a thought experiment involving an intelligent being known as Maxwell demon [4]. In this gedanken experiment only by knowing the velocity of gas particles confined in a box, a demon can separate them into hotter (consists of faster molecules) and colder (consists of slower molecules) part without doing any work. Half a century later, Szilard proposed another thought experiment [5] where he showed that it is always possible to extract heat from single heat bath and perform useful work cyclically when a gas molecule, confined in a box, is treated by a certain protocol which involves measurement of the state of the system.

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To understand these puzzles, the last century has witnessed several wonderful research works establishing the connection between information theory and thermodynamics. In fact one needs to take into account the cost of information during the process and above all, the process would be completed only when the information contained in the memory register of the demon will be erased. Now, according to Landauer, one need to do at least \( k_B T \ln 2 \) work to erase one bit of information (\( k_B \) represents Boltzmann constant). Hence the second law is saved when one takes into account the effect of information.

There are mainly two cases in the framework of information processing when the second law is apparently violated. In the first one, measurement is performed and depending on the measurement outcome, the protocol is altered. In another approach the information contained in an information reservoir is changed when it is allowed to interact sequentially with the system. A sequence of bits written on a tape can be considered as an information reservoir. Recently the second law is derived for the second type of approach. It is showed that when a system is connected to a heat bath, work source and information reservoir, the second law is given by

\[
\beta W \leq \mathcal{H}(\Pi_\tau) - \mathcal{H}(\Pi_0). 
\] (1)

Here \( \mathcal{H} \) represents the Shannon entropy of the joint system consisting of interacting bit and system. If that joint distribution at any time \( t \) is given by \( \Pi_t \) then corresponding Shannon entropy is denoted by \( \mathcal{H}(\Pi_t) = - \sum \Pi_t \ln \Pi_t \) where the sum is done over all possible states. The above equation dictates that the average extracted work \( W \) during the time interval \( 0 \leq t < \tau \) is restricted by the change of Shannon entropy of the joint system during that interval. Motivated by this we would like to study an information machine which is connected to multiple baths and to derive corresponding second law. First we have described the model and derived our result. Then we compare the result with the earlier studies. After that we have given several examples where this law can be applicable. Finally we prove our result numerically taking a simple model.

2 The Model

The model consists of a demon(system) which is attached with an information reservoir and a work source. A tape, where the information is written as discrete symbols, acts as an information reservoir. The input tape is represented by a sequence of symbols \( x_1, x_2, x_3, \ldots, x_N \) which is taken from a finite set \( \chi \) (for binary symbols \( \chi = (0, 1) \)). Each symbol then interacts one by one sequentially with the demon. As a result the demon state is going through internal states \( s_1, s_2, s_3, \ldots, s_N \) which is taken from a finite set \( S \). On the other hand, after interaction, the outgoing tape consists of another sequence of symbols \( y_1, y_2, y_3, \ldots, y_N \) which are elements of same set \( \chi \). Total entropy of the incoming tape is

\[
H(X^N) = - \sum_{x^N \in \chi} P(x^N) \ln P(x^N). 
\] (2)
Here, $P(x^N) = P(x_1, x_2, x_3, ..., x_N)$ is the probability distribution of that sequence of the input. Now, If the input sequence is correlated then

$$H(X^N) = - \sum_{x^N \in X} P(x^N) \ln P(x^N)$$

$$= - \sum_{x^N \in X} P(x^N) \ln [P(x_N|x^{N-1})...P(x_3|x_2,x_1)P(x_2|x_1)P(x_1)]$$

$$= - \sum_{x^N \in X} P(x^N) \ln P(x_N|x^{N-1}) - ... - \sum_{x^N \in X} P(x^N) \ln P(x_3|x_2,x_1)$$

$$- \sum_{x^N \in X} P(x^N) \ln P(x_2|x_1) - \sum_{x^N \in X} P(x^N) \ln P(x_1)$$

$$= - \sum_{x^N \in X} P(x^N) \ln P(x_N|x^{N-1}) - ... - \sum_{x_3,x_2,x_1} P(x_3,x_2,x_1) \ln P(x_3|x_2,x_1)$$

$$- \sum_{x_2,x_1} P(x_2,x_1) \ln P(x_2|x_1) - \sum_{x_1} P(x_1) \ln P(x_1)$$

$$= \sum_{n=1}^{N} H(X_n|X^{n-1}). \quad (3)$$

Where $H(X_n|X^{n-1}) = - \sum_{x^n} P(x^n) \ln P(x_n|x^{n-1})$. Similarly if $P(y^N) = P(y_1, y_2, y_3, ..., y_N)$ represents the probability distribution of the output sequence of the tape, then its entropy is given by

$$H(Y^N) = - \sum_{y^N \in Y} P(y^N) \ln P(y^N) = \sum_{n=1}^{N} H(Y_n|Y^{n-1}). \quad (4)$$

The demon can interact with one symbol of the tape at a time and its interaction time is taken as $\tau$. During that time, the joint state of demon and tape evolves with time. As an example, in $n^{th}$ interaction interval in between time $(n-1)\tau \leq t < n\tau$, the input joint state $(x_n, s_n)$ evolves and finally reaches to $(y_n, s_{n+1})$. In the beginning of the next cycle, the demon state does not change but the tape is advanced by one unit. As a result the next cycle starts from the joint state $(x_{n+1}, s_{n+1})$. After time $\tau$, the state $(x_{n+1}, s_{n+1})$ evolves to $(y_{n+1}, s_{n+2})$ and this process continues until the tape has been passed completely.

Note that there are two types of dynamics; One is discrete and only deals with the input and output states ( $(x_1, s_1) \rightarrow (y_1, s_2); (x_2, s_2) \rightarrow (y_2, s_3); ...$ ). Another one is continuous and deals how a output state is evolved from the input state during the time interval $\tau$. Consider the states of the joint system of demon and interacting tape are taken form a product set $(\chi \times S)$ that contains $M + 1$ elements which are denoted by $(s_0, s_1, ..., s_M)$. Now at the starting of $n^{th}$ cycle the input state is related to the joint state by $(x_n, s_n) \equiv \sigma^{(n-1)\tau}$. Note that the superscript only denotes the time at which the state appears. The state $\sigma^{(n-1)\tau}$ then evolves according to the dynamics and finally at $n\tau^-$ it reaches to another element, say $\sigma^{n\tau^-} \equiv (y_n, s_{n+1})$. At time $n\tau$ the tape is forwarded by one unit and the bit state is changed. As a result, the next cycle starts from $(x_{n+1}, s_{n+1}) \equiv \sigma^{n\tau}$ which is again an element of that set and the process continues. It can be mentioned that if the output and next input symbol (bit for binary sequence)
Figure 1: Schematic diagram to describe the joint states of demon and information reservoir. The energy difference between the states $\sigma_i$ and $\sigma_{i-1}$ is denoted by $E_i$ and when transition occurs between these states, energy is exchanged only with the bath having inverse temperature $\beta_i$.

is same then the actual state does not change at the time of this switching, if not, then it starts from another element of $\sigma$. Next we will describe the dynamics and the evolution of the joint system in a particular interval in detail.

The energy difference between the states $\sigma_0$ and $\sigma_1$ is $E_1$ (fig.1). Similarly for state $\sigma_1$ and $\sigma_2$ it is $E_2$ and so on. Note that, we have taken $\sigma_0$ as ground state and corresponding energy is taken as 0. Each of these consecutive states can exchange energy with only one bath and transition can only happen between the consecutive states. As an example, transition between $\sigma_0$ and $\sigma_1$ can only happen by exchanging heat from a bath with inverse temperature $\beta_1$. Similarly for $\sigma_1$ and $\sigma_2$ the transition can happen when heat is exchanged from bath with inverse temperature $\beta_2$ and so on. There is no other transition possible from or to $\sigma_1$. Hence all the states form a linear chain and only back and forth transition is possible.

3 Derivation of Second Law in presence of multiple baths

Before proceed further, lets take the problem in a simpler way. We consider a three level system A, B, C. The allowed transition is such that, heat will flow from a bath having temperature $T_1$ when transition occurs from A to B or vice-versa. Similarly for any transition between B and C heat is exchanged with the bath, which is at temperature $T_2$. i.e, $A \rightleftharpoons B \rightleftharpoons C$. Note that the allowed transitions form a linear chain and there are no loop. Even, only one bath is allowed to exchange heat for any transition between two given states. In long time limit, when the system will reach to a steady state, the
probability density of each state does not change with time. For any transition between A and B, which can only happen with the exchange of heat with bath $T_1$ (Note that there is no alternative route connecting A and B through other states or baths), the rate of transition between them will only depend on energy difference $E_{AB}$ and bath temperature $T_1$ and is given by $R_{AB}/R_{BA} = \exp(-E_{AB}/k_BT_1)$ [17], where $R_{AB}$ denotes the transition rate from B to A while $E_{AB}$ denotes their energy difference. In this respect we know that when system is connected to a single bath, the transition rate is valid in the non-equilibrium regime. During any relaxation process the system eventually reaches to equilibrium and then detail balance is restored. Similarly, for the above case, one can take these rates in the non-equilibrium regime to take the system ultimately into a steady state where probability distribution will not change and current will be zero.

Note that this problem is completely different from the problem where heat can be exchanged with any bath for any pair of states, when the system is connected with multiple baths simultaneously where in steady state heat flows from one bath to another continuously.

Now we come back to our problem. If the system is kept as it is, at large time limit the joint system will eventually reach to the steady state. Then the probability distribution will not alter with time. Now according to the construction of the model, the probability distribution of two successive levels at steady state will be related by

$$
\Pi_s(\sigma_1) = \exp^{-\beta_1 E_1} \Pi_s(\sigma_0),
\Pi_s(\sigma_2) = \exp^{-\beta_2 E_2} \Pi_s(\sigma_1),
\Pi_s(\sigma_3) = \exp^{-\beta_3 E_3} \Pi_s(\sigma_2),
$$

and so on. Then the steady state distribution for any state ($\sigma_i$) can be written as

$$
\Pi_s(\sigma_i) = \exp^{-\beta_i E_i} \Pi_s(\sigma_{i-1}) = \exp^{-\left(\beta_i E_i + \ldots + \beta_2 E_2 + \beta_1 E_1\right)} \Pi_s(\sigma_0) = \frac{\exp^{-\sum_{j=1}^{i} \beta_j E_j}}{Z}.
$$

(5)

Where in the last line, we have used the normalization condition and $Z = 1 + \sum_{i=1}^{M} e^{-\sum_{j=1}^{i} \beta_j E_j}$. As we already know that as the time increases, $\Pi_t$ (the probability density at any time $t$) will approach monotonically towards $\Pi_s$ and their distance will reduce (Here for notational simplicity we have taken $0 \leq t < \tau$. However it will be true for any interval $(n-1)\tau \leq t < n\tau$). Hence one can write

$$
D(\Pi_\tau||\Pi_s) \leq D(\Pi_0||\Pi_s).
$$

(6)

Here the Kullback-Leibler divergence is defined as $D(\Pi_{t_1}||\Pi_{t_2}) = \sum_{i=0}^{M} \Pi_{t_1}(\sigma_i) \ln \frac{\Pi_{t_1}(\sigma_i)}{\Pi_{t_2}(\sigma_i)}$. We can easily expand the above inequality and rewrite it to the form:

$$
\sum_{i=0}^{M} [\Pi_\tau(\sigma_i) - \Pi_0(\sigma_i)] \ln \frac{1}{\Pi_s(\sigma_i)} \leq \mathcal{H}(\Pi_\tau) - \mathcal{H}(\Pi_0),
$$

(7)

where $\mathcal{H}$ represents Shannon entropy which have been already defined earlier. Therefore the right hand side represents change of Shannon entropy during the evolution. Again
we have
\[ \ln \frac{1}{\Pi_s(\sigma_i)} = \ln Z + \sum_{j=1}^{i} \beta_j E_j. \] (8)

Then left hand side of eq.7 becomes
\[
\sum_{i=0}^{M} [\Pi_{\tau}(\sigma_i) - \Pi_0(\sigma_i)] \ln \frac{1}{\Pi_i(\sigma_i)} \\
= \sum_{i=1}^{M} [\Pi_{\tau}(\sigma_i) - \Pi_0(\sigma_i)] \sum_{j=1}^{i} \beta_j E_j \\
= \sum_{i=1}^{M} [\Pi_{\tau}(\sigma_i) - \Pi_0(\sigma_i)] \beta_1 E_1 + \sum_{i=2}^{M} [\Pi_{\tau}(\sigma_i) - \Pi_0(\sigma_i)] \beta_2 E_2 \\
+ \ldots + \sum_{i=M}^{M} [\Pi_{\tau}(\sigma_i) - \Pi_0(\sigma_i)] \beta_M E_M \\
= \sum_{i=1}^{M} [\Pi_{\tau}(\sigma_i) - \Pi_0(\sigma_i)] \beta_j E_j. \] (9)

Note that \( \sum_{i=0}^{M} [\Pi_{\tau}(\sigma_i) - \Pi_0(\sigma_i)] \) represents the net change of probability of all the states above \((\sigma_j)\) including \((\sigma_j)\) during the time of operation \(0 \leq t < \tau\). As all the allowed transitions form a linear chain, these net probability change can only happen if same amount of transition happen from \((\sigma_{j-1})\) state to \((\sigma_j)\). Now for each of these transition \(E_j\) amount of heat will be absorbed from the bath \(\beta_j\). One can write average amount of heat that is absorbed from this bath along the evolution during time \(\tau\) as
\[ q_j = \sum_{i=j}^{M} [\Pi_{\tau}(\sigma_i) - \Pi_0(\sigma_i)] E_j. \] (10)

Hence one can rewrite eq.7 as
\[ \beta_1 q_1 + \beta_2 q_2 + \ldots + \beta_M q_M \leq \mathcal{H}(\Pi_{\tau}) - \mathcal{H}(\Pi_0). \] (11)

This is the second law for each individual cycle. The left side of the equation is related to the bath entropy production while the right side represents entropy change of the joint system. Now if the system is connected with a single bath and all the transitions are happening with the energy exchange with this bath then we can write \(\beta_1 = \beta_2 = \ldots = \beta_M = \beta\) and the above equation will simply reduce to
\[ \beta q \leq \mathcal{H}(\Pi_{\tau}) - \mathcal{H}(\Pi_0). \] (12)

where \(q\) represents total heat absorbed from the bath. In [20] it is assumed that each energy level is associated to a work source, as a result, the amount of heat absorbed in each transition from the single bath is equal to same amount of work extraction. Then the above result will be reduced to Eq.11 as obtained in [20]. Note that in this model transition between any two states is allowed. However in our model, we have restricted it to accommodate multiple baths which act simultaneously on the system. To understand the applicability of our model, we consider different examples in next section. But before
going there, we will try to relate the left hand side of eq. 11 with the entropy change of the tape. Taking the notation of discrete process for \( n \)th cycle eq.11 will take the form

\[
\beta_1 q_1(n) + \beta_2 q_2(n) + \ldots + \beta_M q_M(n) \leq H(Y_n, S_{n+1}) - H(X_n, S_n).
\]

(13)

Here \( q_i(n) \) represents heat absorbed from the \( i \)th bath in \( n \)th cycle. Now for \( N \) cycles, the above inequality will take the form

\[
\sum_{n=1}^{N} \left[ \beta_1 q_1(n) + \beta_2 q_2(n) + \ldots + \beta_M q_M(n) \right] \leq \sum_{n=1}^{N} \left[ H(Y_n, S_{n+1}) - H(X_n, S_n) \right]
\]

\[
= \sum_{n=1}^{N} \left[ H(Y_n|S_{n+1}) - H(X_n|S_n) \right] + \sum_{n=1}^{N} \left[ H(S_{n+1}) - H(S_n) \right]
\]

\[
= \sum_{n=1}^{N} \left[ H(Y_n|S_{n+1}) - H(X_n|S_n) \right] + H(S_{N+1}) - H(S_1)
\]

\[
\approx \sum_{n=1}^{N} \left[ H(Y_n|S_{n+1}) - H(X_n|S_n) \right].
\]

(14)

In the last line, it is assumed that \( N \) is very large compared to the total number of joint states \( N \gg M \). As a result, the contribution of the system (demon) entropy (which will be order of \( \ln M \)) will be negligible compared to the other terms. Note that, the right hand side of the above equation is not equal to the entropy change of the tape which is given by

\[
H(Y^N) - H(X^N) = \sum_{n=1}^{N} \left[ H(Y_n|Y^{n-1}) - H(X_n|X^{n-1}) \right].
\]

(15)

Hence our result differ from the earlier result \[21,22\], where the entropy production rate of the bath is restricted by the change of these Shannon entropy rate (which includes all the correlation present in a stream of bits) between input tape and the processed output tape. Note that, in \[21,22\] only the input and output bit stream are considered to calculate the entropy production rate without concerning the detailed methodology of how the output bit is processed. However the correlation in the output may be implicitly contain the information of how it is processed. This flow is pointed out in \[20\] and the second law is derived consequently by making these approach more compact.

It is generally assumed that the input sequence of the tape is not correlated with the demon state i.e, \( P(x_n|s_n) = P(x_n) \) then the right hand side of eq. (14) becomes

\[
\sum_{n=1}^{N} \left[ H(Y_n|S_{n+1}) - H(X_n|S_n) \right] = \sum_{n=1}^{N} \left[ H(Y_n|S_{n+1}) - H(X_n) \right].
\]

(16)

Even for this case it is not also equal to the entropy change of the tape. For simplicity we take uncorrelated input sequence and try to find out the differences.
3.1 uncorrelated input sequence

If the input sequence does not have any correlation, then \( P(x^N) = P(x_N) \ldots P(x_3)P(x_2)P(x_1) \) and the total entropy of the input tape now becomes

\[
H(x^N) = \sum_{n=1}^{N} H(X_n) = NH(X).
\]

where \( H(X_n) = -\sum_{x_n} P(x_n) \ln P(x_n) \). In the last step, it is assumed that the individual probability of each element in a particular position of the sequence (say \( n^{th} \)) is independent of its position. Again, eq. (14) can be rewritten in the form:

\[
\sum_{n=1}^{N} [\beta_1 q_1(n) + \beta_2 q_2(n) + \ldots + \beta_M q_M(n)] \\
\leq \sum_{n=1}^{N} [H(Y_n|S_{n+1}) - H(X_n|S_n)] + H(S_{n+1}) - H(S_1) \\
= \sum_{n=1}^{N} [H(Y_n) - H(X_n)] - \sum_{n=1}^{N} I(S_{n+1}, Y_n) + H(S_{n+1}) - H(S_1). \tag{18}
\]

In last line it is again assumed that the input sequence is uncorrelated with the demon states. \( I(S_{n+1}, Y_n) \) represents the correlation between \( S_{n+1} \) and \( Y_n \) and is given by

\[
I(S_{n+1}, Y_n) = \sum_{s_{n+1}} \sum_{y_n} P(s_{n+1}, y_n) \ln \left( \frac{P(s_{n+1}, y_n)}{P(s_{n+1})P(y_n)} \right). \tag{19}
\]

Note that, \( I(S_{n+1}, Y_n) \) is always positive. Neglecting the contribution of demon state (which becomes zero for large \( N \)) the above inequality shows that our bound is more compact compared to the earlier one [21][22] (although the derivation has been done there for single heat bath, we are comparing the other part except the bath entropy).

Now if the demon performs in steady state, then there is no need to concern about each individual cycle (the average heat absorbed from \( i^{th} \) bath in \( n^{th} \) cycle \( q_i(n) \) will be independent of the cycle i.e, \( q_i(n) = q_i \)). On the other hand entropy change of demon will also be zero. Then the second law for uncorrelated independent sequence becomes

\[
\beta_1 q_1 + \beta_2 q_2 + \ldots + \beta_M q_M \leq H(Y) - H(X) - \frac{1}{N} \sum_{n=1}^{N} I(S_{n+1}, Y_n)). \tag{20}
\]

For large \( \tau \), the joint system may reach steady state with the baths, so that the probability distributions of joint states do not change with time. For this case that probability distributions will take the form as shown in eq.5. Now if the energy of each joint state can be written as sum of two terms, i.e, \( E(\xi, \sigma) = E(\xi) + E(\sigma) \), then the total probability density can be expressed in terms of product of demon state probability and tape state probability. For this case, the correlation after the evolution at \( \tau \) between demon state and tape state \( I(S_{n+1}, Y_n) \) will be zero. Hence in steady state, the second law for uncorrelated independent sequence and for large \( \tau \) takes the form:

\[
\beta_1 q_1 + \beta_2 q_2 + \ldots + \beta_M q_M \leq H(Y) - H(X). \tag{21}
\]
Note that, for this case only the right hand side is equal to the entropy change of the tape.

4 Examples

4.1 Example 1

First we consider the Maxwell refrigerator model [17]. In this model, a two level system is coupled with an information bath and two thermal baths. A simple binary tape is taken as an information bath. Hence depending on the system(demon) state and the bit state there will be four joint states 0d, 1d, 0u and 1u. Each bit can interact with the demon for a time \( \tau \) before the next bit arrives. The incoming bit can change its state only while it is interacting with the demon. After the interaction, for a time \( \tau \), it retains its last state as an output and the tape is forwarded. The rule for the transitions during the interaction time is as follows: When transition takes place with the exchange of heat with hot bath \( T_h \), the bit state does not change, i.e., transition between 0u and 0d, similarly between 1u and 1d energy is exchanged with bath \( T_h \). But for 0d and 1u energy is exchanged with the bath \( T_c \). No other transition is permitted.

Hence the joint states form a linear chain and they are connected by the allowed transition 0u \( \Leftrightarrow \) 0d \( \Leftrightarrow \) 1u \( \Leftrightarrow \) 1d. Therefore we can apply our model for this case. Note that, in this model 0d state can exchange energy with bath \( T_h \) in one end and it can also exchange energy with bath \( T_c \) in another end. If the system is kept as it is, at long time limit, the joint system will eventually reaches to a steady state with the baths. Then the probability distribution will not change with time and can be determined by transition rate. Corresponding density will be

\[
\begin{align*}
P_s(1u) &= e^{E/T_h} P_s(1d), \\
P_s(0d) &= e^{E/T_c} P_s(1u), \\
P_s(0u) &= e^{-E/T_h} P_s(0d).
\end{align*}
\]

Then the second law for each cycle will be \( \beta_1 q_1 + \beta_2 q_2 + \beta_3 q_3 \leq H(\Pi_r) - H(\Pi_0). \) Note that for this case, first and third bath is same i.e., \( \beta_1 = \beta_3 = \frac{1}{k_B T_h} \) and second bath is denoted by \( \beta_2 = \frac{1}{k_B T_c}. \) Then heat absorbed from hot and cold bath is given by \( q_h = q_1 + q_3 \) and \( q_c = q_2. \) When the incoming bits are uncorrelated, then, in the steady state the second law will be reduced to

\[
\frac{q_h}{T_h} + \frac{q_c}{T_c} \leq k_B \left[ H(Y) - H(X) - \frac{1}{N} \sum_{n=1}^{N} I(S_{n+1}, Y_n) \right]. \tag{22}
\]

4.2 Example 2

In next autonomous information machine model [18] there is an additional work source along with the information reservoir and two heat baths. Here a three level system is taken. Hence depending on bit state and demon(system) state there will be six joint states. For any transition between the energy levels A and B, \( E_1 \) amount of energy is exchanged with bath \( T_c \). This is true for any transition between B and C. Note that
During these transitions bit state is not changed. However, transition between A and C is restricted and depends on the interacting bit. When transition occurs from (to) C0 to (from) A1, E amount of energy is absorbed (released) from (to) bath $T_h$ and $w$ amount of work is done on (extracted) from the system. But transition between C1 and A0 is restricted. Hence the allowed transitions, from one state to another, form a linear chain and is given by $A_0 \rightleftharpoons B_0 \rightleftharpoons C_0 \rightleftharpoons A_1 \rightleftharpoons B_1 \rightleftharpoons C_1$. Therefore we can apply eq.11 also for this case. Similar to the earlier example, all the heat exchanged with the bath $T_c$ can be summed up to $q_c$ and heat absorbed during the transition C0 and A1 is taken as $q_h$. Then the second law takes the form:

$$\frac{q_h}{T_h} + \frac{q_c}{T_c} \leq k_B \left[H(Y) - H(X) - \frac{1}{N} \sum_{n=1}^{N} I(S_{n+1}, Y_n) \right].$$

(23)

As average energy of the demon does not change here, hence the first law becomes $q_h + q_c = W$, where $W$ represents work extraction. On the other hand, there is no work source in the earlier example and first law takes the form $q_h + q_c = 0$, although second law is same.

### 4.3 Example 3

Now if we take $E_1 = 0$, then the above problem will be reduced to the Maxwell demon model [11]. For any transition from A to B or B to C and vice versa, no energy is exchanged. Hence, there is no significance of bath $T_c$. Then the above second law will be reduced to

$$q_h \leq k_B T_h \left[H(Y) - H(X) - \frac{1}{N} \sum_{n=1}^{N} I(S_{n+1}, Y_n) \right].$$

(24)

As total heat absorption $q_h$ will be equal to work extraction $W$ then

$$W \leq k_B T_h \left[H(Y) - H(X) - \frac{1}{N} \sum_{n=1}^{N} I(S_{n+1}, Y_n) \right].$$

(25)

Note that $k_B[H(Y) - H(X)]$ denotes the entropy change of the tape. As $I$ is always positive, maximum extractable work becomes less than that was previously thought [11] while writing same amount of information on tape. On the other hand, erasing same amount of information we need to do more work to compensate the term $I$.

### 5 Numerical Results

In this section we will prove our results numerically by considering the second example. The demon consists of three states A, B and C. We have already mentioned that for any transition between the level A and B or B and C, $E_1$ amount of energy is exchanged with the bath $T_c$ and bit state does not change. However, transition between A and C is restricted. During the transition from C0 to A1, $E$ amount of energy is absorbed from the bath $T_h$ and $w$ amount of work is extracted. The reverse will happen for transition
from A1 to C0. However transition is not allowed between A0 and C1. Let's define the weight parameter

\[ E = \tanh \left( \frac{E}{2T_h} \right). \]  (26)

Note that \(-1 \leq E \leq 1\). Consider \(\delta\) represents excess of 0 in the input tape compared to the 1, i.e.,

\[ \delta = p(0) - p(1). \]  (27)

Here \( p(0) \) and \( p(1) \) denotes the probability of 0 and 1 respectively in the incoming bit stream. Note that \(-1 \leq \delta \leq 1\). As the consecutive bits are uncorrelated with each other, the Shannon entropy of the incoming tape becomes

\[ H(X) = -p(0) \ln p(0) - p(1) \ln p(1). \]  (28)

Similarly, if the probability to get 0 and 1 in the outgoing bit stream are represented by \( p'(0) \) and \( p'(1) \); then the corresponding Shannon entropy will be

\[ H(Y) = -p'(0) \ln p'(0) - p'(1) \ln p'(1). \]  (29)

Then entropy change of the tape is given by

\[ \Delta S = k_B (H(Y) - H(X)). \]  (30)

As \( q_h \) and \( q_c \) represents average heat absorption to the hot bath \( T_h \) and cold bath \( T_c \) per unit cycle in steady state, then bath entropy production will be

\[ S_B = -\frac{q_h}{T_h} - \frac{q_c}{T_c}. \]  (31)

The hidden entropy generated due to the correlation of output bits and the demon per unit cycle in steady state is given by

\[ \Delta S_{cor} = -\frac{k_B}{N} \sum_{n=1}^{N} I(S_{n+1}, Y_n) = -k_B I(S_{n+1}, Y_n). \]  (32)

Hence the second law for this case can be rewritten as

\[ S_{tot} = S_B + \Delta S + \Delta S_{cor} \geq 0. \]  (33)

In numerical simulation we have set \( T_h = 1.0, T_c = 0.5 \) and \( E_1 = 0.5 \). We have also set Boltzmann constant \( k_B = 1 \) and the interaction time of the each bit with the demon \( \tau = 0.4 \). In fig.2 we have plotted total entropy production \( S_{tot} \) for different parameter set \( \delta \) and \( E \). We have found that it remain always positive. Hence, it proves the second law. Moreover the figure clearly indicates the reversible region, where \( S_{tot} \to 0 \), by simply connecting the white dots.

The relative error \( R_{err} \) between the the apparent entropy production \( S_{app} = S_B + \Delta S \) and \( S_{tot} \) is defined as

\[ R_{err} = \frac{S_{app} - S_{tot}}{S_{tot}}. \]  (34)

When the joint system behaves reversibly entropy production is zero and \( R_{err} \) becomes undefined. The imaginary line connecting white dots in the fig.3 denotes where these phenomena is occurring. Form the figure we have found that for quite large region \( R_{err} \) can take value greater than 10% even it can exceed 50% in certain region (\( \delta \sim 1 \) and high \( E \)). Hence we can not neglect \( S_{cor} \) term and \( S_{tot} \) represents the proper bound.
Figure 2: Histogram of $S_{tot}$ for different values of $\delta$ and $E$ at $T_h = 1.0$, $T_c = 0.5$, $E_1 = 0.5$, $\tau = 0.4$.

Figure 3: Histogram of $R_{err}$ for different values of $\delta$ and $E$ at $T_h = 1.0$, $T_c = 0.5$, $E_1 = 0.5$, $\tau = 0.4$.

6 Conclusions

In this paper we have derived the second law for an information machine connected with multiple baths. First we have derived the second law for each individual cycle. Then we sum them up to get the net inequality. We have found that this inequality is tighter compared to the earlier results which only includes entropy change of the tape between its output and input sequences. Moreover we have shown several examples where this
law can be applicable. Finally we have numerically shown that the correction term is quite significant and hence should not be neglected.

Acknowledgements

SR thanks A M Jayannavar for broad discussions throughout the work. SR also thanks Deepak Dhar for useful discussions.

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