BILINEAR RESTRICTION ESTIMATES FOR SURFACES OF CODIMENSION BIGGER THAN ONE

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Abstract. In connection with the restriction problem in $\mathbb{R}^n$ for hypersurfaces including the sphere and paraboloid, the bilinear (adjoint) restriction estimates have been extensively studied. However, not much is known about such estimates for surfaces with codimension (and dimension) larger than one. In this paper we show sharp bilinear $L^2 \times L^2 \rightarrow L^q$ restriction estimates for general surfaces of higher codimension. In some special cases, we can apply these results to obtain the corresponding linear estimates.

1. Introduction and statement of results

For a smooth hypersurface $S$ such as the sphere or paraboloid in $\mathbb{R}^n$, $n \geq 3$, the $L^p$-$L^q$ boundedness of the (adjoint) restriction operator (or the extension operator) $\hat{f}d\sigma$ has been extensively studied since the late 1960s. Here $d\sigma$ denotes the induced Lebesgue measure on $S$. Especially, when $S$ is the sphere, it is conjectured by E. M. Stein (cf. [25]) that $\hat{f}d\sigma$ should map $L^p(S)$ boundedly to $L^q(\mathbb{R}^n)$, precisely when $q \geq \frac{n+1}{n-1}p'$ and $q > \frac{2n}{n-1}$. Since then, a large amount of literature has been devoted to this problem. Over the last couple of decades, the bilinear and multilinear approaches have proven to be quite effective, and substantial progress has been made through these approaches. We refer the reader to [9, 11, 15] for the most recent developments.

On the other hand, when the dimension of the manifold is one, namely, when the associated surface is a curve, the restriction estimate is by now fairly well understood [4–6, 26]. However, not much is known about the intermediate cases, namely, when the codimension $k$ of the manifold is between 1 and $n - 1$. The restriction problem for quadratic surfaces of codimension $k \geq 2$ was first studied by Christ [12] and Mockenhaupt [20]. They also considered the problem in a more general setting and found some necessary conditions on the curvature and codimension of the surface. For some surfaces they also established the optimal $L^2 \rightarrow L^q$ linear estimates, which may be regarded as generalizations of the Stein-Tomas restriction theorem (see also [8]). Although there are some known cases in which the $L^p$-$L^q$ boundedness is completely characterized (see for example [21, 32]), for most surfaces with codimension bigger than one, the current state of the restriction problem is hardly beyond that of the Stein-Tomas theorem.

In this paper, we are concerned with restriction estimates for surfaces of codimension $k \geq 2$. To be more specific, let us set $k \geq 1$ and $I = [-1, 1]$. Let $\Phi : I^d \rightarrow \mathbb{R}^k$ be a smooth
function given by
\[ \Phi(\xi) = (\varphi_1(\xi), \varphi_2(\xi), \ldots, \varphi_k(\xi)). \]
The adjoint restriction operator (the extension operator) \( E = E_\Phi \) for the surface \((\xi, \Phi(\xi)) \in \mathbb{R}^d \times \mathbb{R}^k\) is defined by
\[ E f(x, t) = \int_{\mathbb{R}^d} e^{2\pi i (x \cdot \xi + t \Phi(\xi))} f(\xi) d\xi, \quad (x, t) \in \mathbb{R}^d \times \mathbb{R}^k. \]

Specific examples of such operators with \(2 \leq k \leq d - 2\) can be found in [2, 3, 12, 20, 23]. (Also, see Section 5.)

There are some classes of surfaces for which the optimal \(L^2-L^q\) boundedness of \(E\) is well understood. In fact, using a Knapp type example it is easy to see that \(E\) may be bounded from \(L^p\) to \(L^q\) only if \(d + 2k \leq d(1 - 1/p)\). Hence, the best possible \(L^2-L^q\) bound is that for \(q = \frac{2(d+2k)}{d} \). Christ [12] and Mockenhaupt [20] showed that this is true for a class of surfaces satisfying a suitable curvature condition. In particular, let \(M\) be a linear map from \(\mathbb{R}^k\) to the space of \(d \times d\) symmetric matrices and suppose that
\[ \int_{S^{d-1}} |\det M(t)|^{-\gamma} d\sigma(t) < \infty \] for \(\gamma = \frac{k}{d}\). Then it was proven in [20] that the extension operator \(E\) defined by \(\Phi = \xi^t M(t) \xi\) is bounded from \(L^2\) to \(L^2\left(\frac{2k+2d}{d}\right)\).

In order to obtain estimates for some \(q < \frac{2(d+2k)}{d}\) and \(p > 2\), it seems necessary to consider methods other than the \(TT^*\) argument which solely relies on the decay estimate for the Fourier transform of the surface measure. For this reason we wish to consider the bilinear restriction estimates for surfaces of codimension greater than 1 and try to obtain the best possible estimates.

Let \(S_1, S_2\) be closed cubes contained in \(I^d\) and define
\[ E_i f(x, t) = \int_{S_i} e^{2\pi i (x \cdot \xi + t \Phi(\xi))} f(\xi) d\xi, \quad i = 1, 2. \]
Let us consider the estimate
\[ \|E_1 f E_2 g\|_{L^q(\mathbb{R}^{d+k})} \leq C \|f\|_{L^p(\mathbb{R}^d)} \|g\|_{L^p(\mathbb{R}^d)}. \] (1.1)

For the elliptic surfaces, bilinear estimates can be thought of as a generalization of linear estimates, since a linear restriction estimate follows from the corresponding bilinear one by an argument involving a Whitney decomposition. (See e.g. [30].) The advantage of the bilinear estimates is that a wider range of boundedness is possible than for the linear estimate, provided that a separation condition holds between the supports of the functions \(f, g\). For surfaces with codimension 1, the sharp bilinear (adjoint) restriction estimate for the cone was obtained by Wolff [34], and for the paraboloid the corresponding estimate was proved by Tao [27]. The bilinear approach has also been applied to the restriction problem for hyperbolic surfaces: for the saddle surface in \(\mathbb{R}^3\), Vargas [31] and, independently, Lee [18] proved the bilinear estimate by extending Tao’s method. From these bilinear restriction estimates the corresponding linear ones have been obtained as well.

\[ \text{For more general negatively curved surfaces in } \mathbb{R}^3 \text{ and higher dimensions, Lee [18] showed the bilinear restriction estimates. However, in higher dimensions the linear estimate could not be deduced from the bilinear one, because the separation condition needed to prove the bilinear estimate for hyperbolic surfaces was more complex than that for the elliptic surfaces.} \]
In order to state our results, we first introduce some notations. For \( \nu_1, \nu_2 \in I^d \), we define the \( k \times d \) matrix \( D(\nu_1, \nu_2) \) by

\[
D(\nu_1, \nu_2) = \begin{pmatrix}
\nabla \varphi_1(\nu_2) - \nabla \varphi_1(\nu_1) \\
\vdots \\
\nabla \varphi_k(\nu_2) - \nabla \varphi_k(\nu_1)
\end{pmatrix}.
\]

Here \( \nabla \varphi_j \) is a row vector. Let \( H\varphi \) denote the Hessian of \( \varphi \) and \( D^t(\nu_1, \nu_2) \) be the transpose of \( D(\nu_1, \nu_2) \). The following is our main theorem.

**Theorem 1.1.** Let \( t = (t_1, \ldots, t_k) \), \( k \geq 1 \). Suppose that, for \( \nu \in S_1 \cup S_2 \) and \( |t| = 1 \),

\[
(1.2) \quad \det \left( \sum_{i=1}^k t_i H\varphi_i(\nu) \right) \neq 0
\]

and, for \( \nu_1 \in S_1, \nu_2 \in S_2, |t| = 1 \) and for \( \nu = \nu_1, \nu_2 \),

\[
(1.3) \quad \det \left[ D(\nu_1, \nu_2) \left( \sum_{j=1}^k t_j H\varphi_j(\nu) \right)^{-1} D^t(\nu_1, \nu_2) \right] \neq 0.
\]

Then, for \( q > \frac{d+k}{d+k+1} \) and \( \frac{1}{p} + \frac{d+k}{d+k+1} + \frac{1}{2q} < 1 \), the estimate (1.1) holds.

As special cases of Theorem 1.1 one can deduce the known bilinear restriction theorems for the elliptic surfaces in [27] and the negatively curved ones in [18, 31].

Let us set

\[
M(t, \nu_1, \nu_2, \nu) := \begin{pmatrix}
0 \\
D^t(\nu_1, \nu_2) \\
\sum_{i=1}^k t_i H\varphi_i(\nu)
\end{pmatrix}.
\]

Assuming the condition (1.2), it is easy to see that (1.3) is equivalent to

\[
(1.4) \quad \det M(t, \nu_1, \nu_2, \nu) \neq 0
\]

for \( \nu_1 \in S_1, \nu_2 \in S_2, |t| = 1 \) and for \( \nu = \nu_1, \nu_2 \). The condition (1.4) may seem rather complicated, but such a condition appears naturally when one considers the bilinear \( L^2 \times L^2 \rightarrow L^2 \) estimate. When \( k = 1 \), it is closely related to the “rotational curvature”. (See [18] for more details.) The necessity of the condition (1.4) will become clear in the course of the proof of Proposition 1.3 below.

From the condition (1.3) it follows that the matrix \( D(\nu_1, \nu_2) \) has rank \( k \). So, the vectors \( \{ \nabla \varphi_i(\nu_2) - \nabla \varphi_i(\nu_1) : i = 1, \ldots, k \} \) are linearly independent. This means \( d \geq k \). If \( d = k \), then (1.4) implies (1.3), but otherwise (1.4) may hold without (1.3) being satisfied.

In fact, it is possible to obtain a local version (Theorem 1.2 below) of Theorem 1.1 which holds under a weaker assumption. Let \( n_1, \ldots, n_{d-k} \) be orthonormal vectors which

\^\footnote{One can use the block matrix formula \( \det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(D) \det(A - BD^{-1}C) \).}

\^\footnote{We consider these vectors as row vectors.}
are perpendicular to the span of \( \{ \nabla \varphi_i(\nu_2) - \nabla \varphi_i(\nu_1) : i = 1, \ldots, k \} \) and set
\[
N(\nu_2, \nu_1) = \begin{pmatrix}
    n_1 \\
    \vdots \\
    n_{d-k}
\end{pmatrix}.
\]

Then we can replace the condition (1.4) with
\[
\det \left[ N(\nu_2, \nu_1) \left( \sum_{i=1}^{k} t_i H \varphi_i(\nu) \right) N^t(\nu_2, \nu_1) \right] \neq 0
\]
whenever \( \nu_1 \in S_1, \nu_2 \in S_2, |t| = 1 \) and \( \nu = \nu_1, \nu_2 \). It is easy to see that the value of this determinant is independent of the particular choice of orthonormal vectors \( n_1, \ldots, n_{d-k} \), and that the condition (1.5) is equivalent to (1.4) under the assumption (1.2). If we have (1.5) instead of (1.3), then we don’t need (1.2) to get (1.6) for any \( \alpha > 0 \). More precisely, we have

**Theorem 1.2.** Suppose that, for any \( \nu_1 \in S_1, \nu_2 \in S_2 \), the vectors \( \nabla \varphi_i(\nu_2) - \nabla \varphi_i(\nu_1), i = 1, \ldots, k \), are linearly independent and that (1.5) holds for \( \nu_1 \in S_1, \nu_2 \in S_2, |t| = 1 \) and for \( \nu = \nu_1, \nu_2 \). Then, for any \( \alpha > 0 \), there is a constant \( C_\alpha \) such that
\[
\| E_1 f E_2 g \|_{L_{d+k+\alpha}(Q_R)} \leq C_\alpha R^\alpha \| f \|_2 \| g \|_2,
\]
where \( Q_R \) is a cube of sidelength \( R \gg 1 \).

However, to obtain the global estimates \( L^2 \times L^2 \rightarrow L^q \), for \( q > \frac{d+3k}{d+k} \), we need to impose a decay condition on the Fourier transform of the surface measure, since it is needed to apply the epsilon removal lemma \[11\]. Under the condition (1.2) such a decay estimate follows from the stationary phase method.

For \( q \geq 2 \), the estimate (1.1) is relatively easier to prove under the conditions (1.2), (1.3). The following may be thought of as a generalization of Theorem 2.3 in [30] (see also Theorem 4.2 in [21]) which is concerned with elliptic hypersurfaces. A generalization to general hypersurfaces had already been observed in [18]. As a byproduct this gives estimates for the endpoint cases of \( (p, q) \) satisfying \( \frac{1}{p} + \frac{d+3k}{d+k} \frac{1}{2q} = 1 \), \( q \geq 2 \).

**Proposition 1.3.** Suppose the condition (1.4) holds for \( \nu_1 \in S_1, \nu_2 \in S_2 \) and \( |t| = 1 \). Then, for \( q \geq 2 \) and \( \frac{1}{p} + \frac{d+3k}{d+k} \frac{1}{2q} < 1 \), the estimate (1.1) holds.

**Remark 1.4.** In the proof of the above results we may assume that the aforementioned conditions hold uniformly, by breaking up the extension operator by decomposing \( S_1, S_2 \) into sufficiently small pieces. That is to say, there is a constant \( c > 0 \) such that for \( \nu \in S_1 \cup S_2 \) and \( |t| = 1 \),
\[
\det \left( \sum_{i=1}^{k} t_i H \varphi_i(\nu) \right) \geq c
\]

\[\text{Indeed, if } H, N, D \text{ are matrices of size } d \times d, (d-k) \times d, k \times d, \text{ respectively, such that } ND^t = 0, \text{ det } H \neq 0, \text{ and rank}(N^t D^t) = d, \text{ then det}(NHN^t) \neq 0 \text{ if and only if det}(DH^{-1}D^t) \neq 0 \text{ because } \begin{pmatrix} NH \\ D \end{pmatrix} (N^t D^t) = \begin{pmatrix} N \nu N^t \\ 0 \\ DD^t \end{pmatrix} \text{ and } \begin{pmatrix} N \\ DH^{-1} \end{pmatrix} (N^t D^t) = \begin{pmatrix} NN^t \\ 0 \\ DD^{-1} D^t \end{pmatrix}.\]
and, for $\nu_1, \nu'_1 \in S_1, \nu_2, \nu'_2 \in S_2, |t| \sim 1$ and for $\nu \in S_1 \cup S_2$,

\begin{equation}
(1.8) \quad \left| \det \left[ D(\nu_1, \nu_2) \left( \sum_{j=1}^{k} t_j H_{\varphi_j}(\nu) \right)^{-1} D(\nu'_1, \nu'_2) \right] \right| \geq c.
\end{equation}

The same holds also for the conditions (1.4) and, for $\nu$.

\section{Necessary conditions for (1.1).}

By modifying the examples in [28] with some specific surfaces we see that (1.1) cannot hold in general, unless

\begin{equation}
(1.9) \quad q \geq \frac{d + k}{d},
\end{equation}

\begin{equation}
(1.10) \quad \frac{1}{p} + \frac{d + 3k}{d + k} \frac{1}{2q} \leq 1,
\end{equation}

\begin{equation}
(1.11) \quad \frac{2(d - k)}{p} + \frac{d + 3k}{q} \leq 2d.
\end{equation}

In fact, (i) (1.9) is necessary for (1.1) to hold under (1.2), and (ii) so is (1.10) under the assumption that the matrix $D(\nu_1, \nu_2)$ has rank $k$ for $\nu_j \in S_j, j = 1, 2$. However, in general, (1.11) is not necessarily required for (1.1), but as is well known there are various $\Phi$ satisfying (1.2) and (1.3) for which (1.1) fails if $\frac{2(d - k)}{p} + \frac{d + 3k}{q} > 2d$. We show (i) and (ii) in the following paragraphs.

\section{(i).}

By making use of the stationary phase method together with the condition (1.2) it is not difficult to see that, with suitable choice of $x_0$, there is a cube $Q$ of sidelength $R \gg 1$ such that $|E_1(e^{-2\pi i x_0 \xi} \psi)| \sim |E_2 \psi(x)| \sim R^{-\frac{d}{2}}$ on $Q$ provided that supports of $\psi_1$, $\psi_2$ are small enough. We insert these into (1.1) to see $R^{-\frac{d}{2}} R^{-\frac{d}{2}} R^{\frac{d + k}{q}} \lesssim 1$, from which we get (1.9) by letting $R \to \infty$.

\section{(ii).}

For $j = 1, 2$, let $\Sigma_j$ be the surface $\{ (\xi, \Phi(\xi)) : \xi \in S_j \}$, and denote by $d\sigma_j$ the induced Lebesgue measure on $\Sigma_j$. To see (1.9) it is more convenient to consider $f \to \int d\sigma_j$, instead of dealing with the operator $E_j$. Also, let $\nu_j$ be the center of cube $S_j$ and let $\zeta_j = (\nu_j, \Phi(\nu_j)) \in \Sigma_j, j = 1, 2$. The normal space $N_j$ to $\Sigma_j$ at $\zeta_j$ is spanned by $n_{j,i} = (-\nabla \varphi_i(\nu_j), e_i), i = 1, 2, \ldots, k,$ where $e_i \in \mathbb{R}^k$ is the usual unit vector with its $i$-th entry being equal to 1. Clearly, these vectors are linearly independent because $D(\nu_1, \nu_2)$ has rank $k$. Let $p_n, n = 1, \ldots, d - k$, be an orthonormal basis of the orthogonal complement of $\text{span}\{n_{j,i}, i = 1, 2, \ldots, k, j = 1, 2\}$. Let us set, for $j = 1, 2$,

\begin{equation}
\Lambda_j = \{ \xi \in \Sigma_j : |(\xi - \zeta_j) \cdot n_{3-j,i}| \leq \delta, |(\xi - \zeta_j) \cdot p_n| \leq \delta^\frac{1}{2}, i = 1, \ldots, k, n = 1, \ldots, d - k \}.
\end{equation}

Now, we set $f_j = \chi_{\Lambda_j}, j = 1, 2$. Then it is easy to see $|\int f_j d\sigma_j(x,t)| \gtrsim \delta^{\frac{d+k}{2}}, j = 1, 2$, provided that

$|\langle x,t \rangle \cdot n_{\ell,i}| \lesssim c\delta^{-1}, |\langle x,t \rangle \cdot p_n| \lesssim c\delta^{-\frac{1}{2}}, i = 1, \ldots, k, \ell = 1, 2, n = 1, \ldots, d - k$.

\begin{footnote}
This can also be shown by making use of a wave packet decomposition (see Lemma 1.2) and randomization.
\end{footnote}
with sufficiently small $c > 0$. (For example, see the proof Lemma 4.2.) Since (1.1) implies
\[ \| f 1d\sigma_1 f 2d\sigma_2\|_q \lesssim \| f 1\|_p \| f 2\|_p, \]
we get $\delta^{d+k-\frac{\delta+1}{2q}} \leq C\delta^{d+k}$ and (1.10) by letting $\delta \to 0$.

**Restriction to complex surfaces.** Using the above theorem we can obtain a bilinear restriction estimate for complex quadratic surfaces. To define the (Fourier) extension operator for a complex surface we first distinguish the dot product and the inner product for complex variables, and define an auxiliary product $\circ$. For $z, w \in \mathbb{C}^m$, we define $z \cdot w$, $(z, w)$, $z \circ w$ by
\[ z \cdot w = \sum_{j=1}^{m} z_j w_j, \quad (z, w) = \sum_{j=1}^{m} z_j \bar{w}_j, \quad z \circ w = \text{Re} \langle z, w \rangle, \]
respectively. Hence, if $z = x + iy$ and $w = u + iv$ for $x, u, v \in \mathbb{R}^m$, then $z \circ w = x \cdot u + y \cdot v$.

If we identify $\mathbb{C}^m$ with $\mathbb{R}^{2m}$ in the usual way, then $z \circ w$ is just the inner product on $\mathbb{R}^{2m}$.

Let $n \geq 1$ be an integer and let $D$ be a real symmetric invertible matrix. Then we define the complex quadratic surface $\gamma \subset \mathbb{C}^{n+1}$ by
\[ \gamma(z) = \left( z, \frac{1}{2} z^t Dz \right), \quad z \in \mathbb{C}^n. \]

Now we define the extension operator $E_\gamma f$ by
\[ E_\gamma f(w) = \int_{\mathbb{C}^n} e^{2\pi i [w \gamma(z)]} f(z) \, dz, \quad w \in \mathbb{C}^{n+1} \]
where we have written $dz$ for $dx \, dy$, $z = x + iy$. The operator $E_\gamma f$ is an extension operator for surfaces of codimension 2 in $\mathbb{R}^{2n}$, which is given by $(x, y, \frac{1}{2} \text{Re} (x + iy)^t D(x + iy)), \frac{1}{2} \text{Im} (x + iy)^t D(x + iy)), x, y \in \mathbb{R}^n$. From Theorem 1.1 we can establish the following.

**Corollary 1.5.** Let $S_1, S_2$ be closed cubes in $\mathbb{C}^n$. Suppose that, for any $z_1 \in S_1$ and $z_2 \in S_2$,
\[ |(z_2 - z_1)^t D(z_2 - z_1)| \neq 0. \]

Then, whenever $f$, $g$ are supported on $S_1$, $S_2$, respectively, for $q > \frac{n+3}{2n+1}$ and $\frac{1}{p} + \frac{1}{q} + \frac{3}{2n+1} > 1$, there is a constant $C$ such that
\[ \| E_\gamma f E_\gamma g \|_{L^q(\mathbb{C}^{n+1})} \leq C \| f \|_{L^p(\mathbb{C}^n)} \| g \|_{L^p(\mathbb{C}^n)}. \]

This theorem can also be stated without using the complex number notation, but the use of the complex number notation makes it easier to derive the linear estimates from the bilinear one. The condition (1.13) in $\mathbb{C}^2$ can be contrasted with that in $\mathbb{R}^2$. If $S_1, S_2 \subset \mathbb{R}^2$ and the eigenvalues of $D$ have the same sign, then the condition (1.13) is always valid if dist$(S_1, S_2) \neq 0$. But, when $S_1, S_2 \subset \mathbb{C}^2$, the condition (1.13) may fail even if the separation condition is satisfied. For instance, if $D$ is the $2 \times 2$ identity matrix, the condition (1.13) becomes $|(v_1 - w_1)^2 + (v_2 - w_2)^2| \geq 1$ with $z_1 = (v_1, v_2)$ and $z_2 = (w_1, w_2)$. Since we may factorize $(v_1 - w_1)^2 + (v_2 - w_2)^2$ as $[(v_1 - w_1) + i(v_2 - w_2)][(v_1 - w_1) - i(v_2 - w_2)]$, the expression $|(v_1 - w_1)^2 + (v_2 - w_2)^2|$ may vanish even if dist$(S_1, S_2) \geq 1$. When $D$ has eigenvalues with different signs, this phenomenon may occur even when $S_1, S_2 \subset \mathbb{R}^2$; for instance, if $D$ is the $2 \times 2$ diagonal matrix with diagonal entries 1 and $-1$, then we have $x \cdot Dx = x_1^2 - x_2^2 = (x_1 + x_2)(x_1 - x_2)$. This real-variable case was studied by Lee [18] and
Vargas [31]. In the special case that the surface is two-dimensional they could deduce a linear estimate from the bilinear one.

By adapting their argument, we can obtain the following linear estimate.

**Theorem 1.6.** Let \( n = 2 \) and \( \gamma \) be given by (1.12) with a nonsingular real symmetric matrix \( D \). Then, for \( q > \frac{10}{3} \) and \( \frac{1}{p} + \frac{2}{q} < 1 \),

\[
\| E_\gamma f \|_{L^q(C^n)} \leq C \| f \|_{L^p(C^2)}
\]
whenever \( f \) is supported in a bounded set.

By analogy with the corresponding problem for the paraboloid (elliptic or hyperbolic) in \( \mathbb{R}^3 \), it may be conjectured that (1.14) holds if and only if \( q > 3 \) and \( \frac{1}{p} + \frac{2}{q} \leq 1 \). Theorem 1.6 extends the known \((p,q)\) range for the operator \( E_\gamma f \) when \( D \) is a nonsingular real symmetric matrix. This result is an analog of the adjoint Fourier restriction estimates for the hyperbolic paraboloid in \( \mathbb{R}^3 \), which is known to hold for the same range of \( p, q \).

As a special case of the results by Christ (see Lemma 4.3 in [12]) and Mockenhaupt (Theorem 2.11, [20]), it was previously known that \( E f \) maps \( L^2(\mathbb{R}^4) \) boundedly to \( L^4(\mathbb{R}^6) \). Also, the slightly stronger Lorentz space estimate \( \| E f \|_{L^{4,2}(\mathbb{R}^6)} \leq C \| f \|_{L^2(\mathbb{R}^4)} \) can be deduced by applying Theorem 1.1 in [7]. It is quite likely that the multilinear approach will yield further progress on these problems. We hope to return to this problem in the near future.

**Notation.** We adopt the usual convention to let \( C \) or \( c \) represent strictly positive constants, whose value may vary from line to line. But these constants will always be independent of \( f \), for instance. We write \( A \lesssim B \) or \( B \gtrsim A \) to mean \( A \leq CB \), and \( A \sim B \) means both \( A \lesssim B \) and \( B \lesssim A \).

2. \( L^{\frac{4(d+k)}{d+k}} \times L^{\frac{4(d+k)}{d+k}} \to L^2 \) estimates and proof of Proposition 1.3

In this section we show Proposition 1.3. Our proof here is different from that in [30]. Instead of making use of the boundedness of the averaging operator, we directly exploit the oscillatory decay estimate which is concealed in the averaging operator. For this we need the following lemma.

**Lemma 2.1** ([14, Section 1.1]). Let \( a \in C^\infty_c(\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^N) \) and set

\[
T_\lambda f(x) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^N} e^{i\lambda \phi(x,y,\theta)} a(x,y,\theta) \, d\theta \, f(y) \, dy,
\]

where \( \phi \) is a smooth function on the support of \( a \). Suppose \( \det \begin{pmatrix} \phi_{\theta\theta} & \phi_{\thetay} & \phi_{\theta\eta} & \phi_{\eta\theta} & \phi_{\eta\eta} \\ \phi_{\thetay} & \phi_{yy} & \phi_{\theta\eta} & \phi_{\eta\theta} & \phi_{\eta\eta} \\ \phi_{\theta\eta} & \phi_{\eta\theta} & \phi_{\eta\eta} \end{pmatrix} \neq 0 \) on the support of \( a \) whenever \( \phi_\theta = 0 \). Then, \( \| T_\lambda f \|_2 \lesssim \lambda^{-\frac{d+N}{2}} \| f \|_2 \).

**Proof of Proposition 1.3.** By interpolation with the trivial \( L^1 \times L^1 \to L^\infty \) estimate, it suffices to show

\[
\| E_1 f_1 E_2 f_2 \|_2 \lesssim \| f_1 \|_{L^{\frac{4(d+k)}{d+k}}(\mathbb{R}^4)} \| f_2 \|_{L^{\frac{4(d+k)}{d+k}}(\mathbb{R}^6)}.
\]

For fixed \( \xi_2 \), set

\[
\Phi(\xi_1, \eta_1) = \Phi(\xi_1) + \Phi(\xi_2) - \Phi(\eta_1) - \Phi(\xi_1 + \xi_2 - \eta_1)
\]
where $\delta$ is the delta function. Its composition is well defined, since the vectors $\nabla \varphi_i(\nu_2) - \nabla \varphi_i(\nu_1)$, $i = 1, \ldots, k$, are linearly independent.

By Plancherel’s theorem
\[
\|E_1 f_1 E_2 f_2\|_2^2 = \iint \int \delta(\xi_1 + \xi_2 - \eta_1 - \eta_2, \Phi(\xi_1) + \Phi(\xi_2) - \Phi(\eta_1) - \Phi(\eta_2))
\times f_1(\xi_1) f_2(\xi_2) \bar{f}_1(\eta_1) \bar{f}_2(\eta_2) \, d\xi_1 d\xi_2 d\eta_1 d\eta_2
\]
\[
= \iint \int \delta(\Phi^{\xi_2}(\xi_1, \eta_1)) f_1(\xi_1) \bar{f}_1(\eta_1) f_2(\xi_2) \bar{f}_2(\eta_2) \, d\xi_1 d\xi_2 d\eta_1 d\eta_2,
\]
where $f_1, f_2$ are assumed to be supported in $S_1, S_2$, respectively. We claim that
\[
\|E_1 f_1 E_2 f_2\|_2 \lesssim \|f_1\|_{p,1} \|f_2\|_1 \|\bar{f}_1\|_{p,1} \|\bar{f}_2\|_{\infty},
\]
where $p = \frac{d+1}{2}$. Here $\|f\|_{r,s}$ denotes the norm of Lorentz space $L^{r,s}$. For this we may obviously assume that the functions $f_1, \bar{f}_1, f_2, \bar{f}_2$ are nonnegative. In order to show (2.1) it suffices to prove
\[
|I^{\xi_2}(f, g)| \lesssim \|f\|_{p,1} \|g\|_{p,1}.
\]

Let $\psi$ be a smooth function with compact Fourier support contained in $B(0, 1)$ such that $\bar{\psi} = 1$ on $B(0, 1/2)$. Since $h(0) = \lim_{j \to \infty} 2^{jk} \int_{\mathbb{R}^k} \psi(2^j x) h(x) \, dx$ for any Schwartz function $h$, we have $\delta = \lim_{j \to \infty} 2^{jk} \hat{\psi}(2^j)$. So, we may write
\[
\delta = \sum_{j=-\infty}^{\infty} [2^{(j+1)k} \psi(2^{j+1} x) - 2^{jk} \psi(2^j x)] = \sum_{j=-\infty}^{\infty} 2^{jk} \eta(2^j x)
\]
where $\eta(x) := 2^k \psi(2^j) - \psi(x)$. By the choice of $\psi$ we see that the Fourier support of $\eta$ is contained in $\{\xi : 1/2 < |\xi| \leq 2\}$. We decompose $I^{\xi_2}(f, g)$ by making use of the above decomposition of $\delta$ to get
\[
I^{\xi_2}(f, g) = \sum_{j=-\infty}^{\infty} I_j(f, g),
\]
where
\[
I_j(f, g) := 2^{jk} \int \int \eta(2^j \Phi^{\xi_2}(\xi_1, \eta_1)) f(\xi_1) g(\eta_1) \, d\xi_1 d\eta_1.
\]
It should be noted that we are assuming that $f, g$ are supported on $S_1$ and $\xi_1 + \xi_2 - \eta_1 \in S_2$. Using Fourier transform we write $I_j(f_1, \bar{f}_2)$ as
\[
I_j(f, g) = 2^{jk} \int \left( \int \hat{\eta}(\tau) e^{2^j \Phi^{\xi_2}(\xi_1, \eta_1)} d\tau \right) f(\xi_1) g(\eta_1) \, d\xi_1 d\eta_1.
\]
Now, we will apply Lemma 2.1 to the double integral inside the parentheses. If we set $\phi(\xi_1, \eta_1, \tau) = \tau : \Phi^{\xi_2}(\xi_1, \eta_1)$, then
\[
\left| \det \begin{pmatrix} \phi_{\tau}^{\nu} & \phi_{\xi}^{\nu} \\ \phi_{\eta,\tau}^{\nu} & \phi_{\xi,\eta}^{\nu} \end{pmatrix} \right| = \left| \det \begin{pmatrix} 0 & D(\eta_1 + \xi_2 - \eta_1) \\ D(\eta_1, \xi_1 + \xi_2 - \eta_1) & \sum_{j=1}^{k} \tau_j \bar{H} \varphi_j(\xi_1 + \xi_2 - \eta_1) \end{pmatrix} \right|.
\]
So, by the condition (1.2), the last expression does not vanish since $|\tau| \sim 1$. Hence, by Lemma 2.1 it follows that

$$|I_j(f, g)| \lesssim 2^{j+\epsilon}|f||g|.$$ 

On the other hand, we have the trivial bound $|I_j(f, g)| \sim 2^{2j}||f||_1||g||_1$. Now we may use a summation method (usually called Bourgain’s summation trick) to obtain (2.2).

Considering $(f_1, f_2, f_3, f_4) \mapsto \|Ef_1 Ef_2\|_2^2$ as a quadrilinear mapping (replacing $f_1$, $f_2$ on the left-hand side by $f_3$ and $f_4$, respectively), we apply M. Christ’s multilinear trick [13]. By symmetry and interpolation we get the estimates

$$\left|\int \int Ef_1 Ef_2 Ef_3 Ef_4 \, dx \, dt\right| \lesssim \prod_{j=1}^4 \|f_j\|_{p_j, 1}$$

for $(1/p_1, 1/p_2, 1/p_3, 1/p_4)$ contained in the convex hull of the four points

$$v_1 = (1/p, 1/p, 1, 0), \quad v_2 = (1/p, 1/p, 0, 1), \quad v_3 = (1, 0, 1/p, 1/p), \quad v_4 = (0, 1, 1/p, 1/p)$$

which is contained in the 3-plane $\Pi = \{u_1 + u_2 + u_3 + u_4 = 1 + \frac{2}{p}\}$. The convex hull has a nonempty interior in $\Pi$, because $\det(v_1, v_2, v_3, v_4) \neq 0$ as long as $1/p \neq 1/2$. Hence we may apply the multilinear trick to get

$$\|Ef_1 Ef_2\|_2^2 \lesssim \|f_1\|_{\frac{4(4+p)}{2(p+1)+2}} \|f_2\|_{\frac{4(4+p)}{2(p+1)+2}} \|\bar{f}_1\|_{\frac{4(4+p)}{2(p+1)+2}} \|\bar{f}_2\|_{\frac{4(4+p)}{2(p+1)+2}}.$$ 

This completes the proof of the proposition. □

3. Transversality and the Curvature Conditions

In this section we prove several lemmas that will play crucial roles in proving Theorem 1.1. These lemmas are related to the curvature conditions.

For $R \gg 1$ and $\nu \in S_1 \cup S_2$, we set

$$\pi_\nu = \left\{(x, t) : |x + \left(\sum_{j=1}^k t_j \nabla \varphi_j(\nu)\right)| \leq R^{1/2}\right\}, \quad R^\delta \pi_\nu = \pi_\nu + O(R^{1/2+\delta}).$$

Here, for any set $A \subset \mathbb{R}^{d+k}$ and $\rho > 0$, $A + O(\rho) = \{u \in R^{d+k} : \text{dist}(u, A) \leq CR^{1/2}\}$.

**Lemma 3.1.** Suppose that the vectors $\nabla \varphi_j(\nu_2) - \nabla \varphi_j(\nu_1)$, $1 \leq j \leq k$, are linearly independent for all $\nu_1 \in S_1$ and $\nu_2 \in S_2$. Then, there is a constant $C$ such that

$$\pi_{\nu_1} \cap \pi_{\nu_2} \subset B(0, CR^{1/2}).$$

**Proof.** Since the set $\{\nabla \varphi_j(\nu_2) - \nabla \varphi_j(\nu_1)\}_{j=1}^k$ is linearly independent for all $\nu_1 \in S_1$ and $\nu_2 \in S_2$, the map $(t_1, \ldots, t_k) \mapsto (t_1, \ldots, t_k)^T D(\nu_1, \nu_2)$ is injective. So, by continuity and compactness it follows that there is a constant $C$ such that, for all $\nu_1 \in S_1$ and $\nu_2 \in S_2$,

$$|(t_1, \ldots, t_k)^T D(\nu_1, \nu_2)| \geq C|(t_1, \ldots, t_k)|.$$

If $(x, t) \in \pi_{\nu_1} \cap \pi_{\nu_2}$, then $|x + \left(\sum_{j=1}^k t_j \nabla \varphi_j(\nu_i)\right)| \leq R^{1/2}$ for $i = 1, 2$. This gives $|(t_1, \ldots, t_k)^T D(\nu_1, \nu_2)| \leq 2R^{1/2}$. Hence, the above inequality yields $|(t_1, \ldots, t_k)| \leq CR^{1/2}$. So, we also get $|x| \leq CR^{1/2}$. This completes the proof. □
As it was already shown in [18, 31], a simple transversality condition between the two wave packets is not enough to obtain a bilinear estimate beyond the range of the linear $L^2 \to L^q$ estimate. So, we need to consider the Fourier supports of the wave packets to put a restriction on the permissible wave packets. This makes the geometry of the associated wave packets more favorable.

For given $\nu_1 \in S_1$ and $\nu'_2 \in S_2$ we define $\Pi_{\nu_1, \nu'_2}$ by

$$
(3.1) \Pi_{\nu_1, \nu'_2} = \{ \nu'_1 \in S_1 : \nu'_1 + \nu'_2 - \nu_1 \in S_2, \Phi(\nu_1) + \Phi(\nu'_1 + \nu'_2 - \nu_1) = \Phi(\nu'_1) + \Phi(\nu'_2) \}.
$$

Since $\{\nabla \varphi_j(\nu_2) - \nabla \varphi_j(\nu_1)\}_{j=1}^k$ are linearly independent, by the implicit function theorem we may assume that $\Pi_{\nu_1, \nu'_2}$ is a smooth $(d-k)$-dimensional surface. We now set

$$
\Gamma_{\nu_1, \nu'_2}(R) = \bigcup_{\nu'_1 \in \Pi_{\nu_1, \nu'_2}} R^\delta \pi_{\nu'_1},
$$

which is a $O(R^{\frac{1}{2}+\delta})$ neighborhood of the conical set with $k$ null directions. The transversality between $\Gamma_{\nu_1, \nu'_2}$ and the opposite plates $\pi_{\nu_2}$ is important. Such a transversality is made precise in the following (see Figure 1):

**Lemma 3.2.** Let $0 < \delta \ll 1, u \in \mathbb{R}^{d+k}$ and set

$$
\overline{\Gamma}_{\nu_1, \nu'_2}(R, R^\delta) = \{(x, t) \in \Gamma_{\nu_1, \nu'_2}(R) : R^{1-\delta} \leq |(x, t)| \leq CR\}. \]

Suppose that the conditions (1.2) and (1.3) hold. Then, if $S_1$ and $S_2$ are sufficiently small, there exist a constant $C$, independent of $\nu_1, \nu'_2$, $R$, and a vector $u \in \mathbb{R}^{d+k}$ such that for some $u' \in \mathbb{R}^{d+k}$,

$$
\overline{\Gamma}_{\nu_1, \nu'_2}(R, R^\delta) \cap (R^\delta \pi_{\nu_2} + u) \subset B(u', CR^{\frac{1}{2}+C\delta}). \]

---

\footnote{We may need to assume that $S_1$ and $S_2$ are small enough.}
Then it is easy to see that
\[ \Gamma \subset B_{\varepsilon}(0) \] and
\[ \Gamma \Gamma \subset B_{\varepsilon}(0) \] for some \( \varepsilon > 0 \). After scaling it is sufficient to show that the intersection of the two sets
\[ \Gamma_1 = \left\{ \Phi^{\nu_1,\nu_2}_1(\nu, t) : \nu \in \Pi_1^{\nu_1,\nu_2}, \ R^{-\delta} \leq |t| \leq C \right\} + O(R^{-\frac{1}{2}+\delta}) \]
and
\[ \Gamma_2 = \left\{ \Phi^{\nu_1,\nu_2}_2(\nu, t) : \nu \in \Pi_2^{\nu_1,\nu_2}, \ R^{-\delta} \leq |t| \leq C \right\} + O(R^{-\frac{1}{2}+\delta}) \]
is contained in a ball of radius \( CR^{-\frac{1}{2}+C\delta} \). For \( j \geq -C \), let us set
\[ \Gamma_j = \left\{ \Phi^{\nu_1,\nu_2}_j(\nu, t) : \nu \in \Pi_j^{\nu_1,\nu_2}, \ 2^{-j-1} \leq |t| \leq 2^{-j} \right\} + O(R^{-\frac{1}{2}+\delta}). \]
Using homogeneity and a dyadic decomposition in \( t \) for \( \Gamma_1 \), the matter can be reduced to the case \( 2^{-1} \leq |t| \leq 1 \). That is to say,
\[ \Gamma_0^j(R^{-\frac{1}{2}+\delta}) \cap \Gamma_2(R^{-\frac{1}{2}+\delta}) \subset B(u, C_0R^{-\frac{1}{2}+\delta}) \]
for some \( u \) and \( C_0 > 0 \). In fact, using scaling we see \[ \Gamma_0^j(R^{-\frac{1}{2}+\delta}) \cap \Gamma_2(R^{-\frac{1}{2}+\delta}) \] is contained in a ball of radius \( C_0R^{-\frac{1}{2}+\delta} \). Since \( \Gamma_1 \subset \bigcup_{-2^{j-1} \leq t \leq 2^{j}} \Gamma_j^j \), \( \Gamma_1 \cap \Gamma_2(R^{-\frac{1}{2}+\delta}) \) is contained in the union of as many as \( \sim \log R \) such balls of radius \( C_0R^{-\frac{1}{2}+\delta} \). This union of balls is obviously contained in a ball of radius \( CR^{-\frac{1}{2}+C\delta} \) since the set \( \Gamma_1 \cap \Gamma_2(R^{-\frac{1}{2}+\delta}) \) is connected.

Since we may assume that \( S_1 \) and \( S_2 \) are sufficiently small, in order to show \[ \Gamma_0^j(R^{-\frac{1}{2}+\delta}) \cap \Gamma_2(R^{-\frac{1}{2}+\delta}) \] it is enough to show that the tangent spaces of the surfaces \( \Phi^{\nu_1,\nu_2}_j : \Pi_1^{\nu_1,\nu_2} \times \{ 2^{-1} \leq |t| \leq 1 \} \to \mathbb{R}^d \) and \( \{ (\sum_{j=1}^k t_j \nabla \varphi_j(\nu_2), t) : |t| \leq C \} \) are uniformly transversal to each other. In fact, since all the underlying sets are compact, by continuity it is enough to check this at each point.

Let \( u_0 = \Phi^{\nu_1,\nu_2}_j(\nu_0, t_0) \) for \( \nu_0 \in \Pi_1^{\nu_1,\nu_2} \) and \( 2^{-1} \leq |t_0| \leq 1 \). Let \( \nu_1, \cdots, \nu_{d-k} \) be orthonormal vectors spanning the tangent space \( T_{u_0} \Pi_1^{\nu_1,\nu_2} \). Then the tangent space of the parametrized surface \( \Phi^{\nu_1,\nu_2}_j : \Pi_1^{\nu_1,\nu_2} \times \{ 2^{-1} \leq |t| \leq 1 \} \to \mathbb{R}^d \) at \( u_0 \) is spanned by the vectors
\[ (\nabla \varphi_1(\nu_0), -1, 0, \ldots, 0), (\nabla \varphi_2(\nu_0), 0, -1, 0, \ldots, 0), \cdots, (\nabla \varphi_k(\nu_0), 0, \ldots, 0, -1) \]

\[ \text{Here we change variables } (x, t) \to 2^{-j}(x, t), \text{ apply } (3.2), \text{ and reverse the change of variables.} \]
and

\begin{equation}
\left(v_i\left(\sum_{j=1}^{k} t_{0,j} H \varphi_j(\nu_0)\right), 0, \ldots, 0\right), \quad i = 1, \ldots, d - k.
\end{equation}

On the other hand, the \(k\)-dimensional plane \(\left\{ (-\sum_{j=1}^{k} t_j \nabla \varphi_j(\nu_2), t) : |t| \leq C \right\}\) is spanned by

\begin{equation}
(\nabla \varphi_1(\nu_2), -1, 0, \ldots, 0), \ (\nabla \varphi_2(\nu_2), 0, -1, 0, \ldots, 0), \ldots, (\nabla \varphi_k(\nu_2), 0, \ldots, 0, -1).
\end{equation}

Hence it suffices to show that these \(d + k\) vectors are linearly independent, or equivalently that the determinant of the matrix with these vectors as row vectors is nonzero. After Gaussian elimination it is enough to show

\begin{equation}
\det \left( V \left( \sum_{j=1}^{k} t_{0,j} H \varphi_j(\nu_0) \right) \right) \neq 0
\end{equation}

where \(V\) is the \((d - k) \times d\) matrix having \(v_1, \ldots, v_{d-k}\) as its row vectors. Now by (3.1) we note that the vectors \(v_1, \ldots, v_{d-k}\) are orthogonal to the span of the vectors

\(\nabla \varphi_j(\nu_0 + \nu_2 - \nu_1) - \nabla \varphi_j(\nu_0), \quad j = 1, \ldots, k.\)

Under the assumption that \(S_2\) is small enough, we may replace \(D(\nu_0, \nu_2)\) with \(D(\nu_0, \nu_0 + \nu_2 - \nu_1)\). For simplicity we set \(\tilde{\nu}_2 = \nu_0 + \nu_2 - \nu_1\). Since \(\left( \sum_{j=1}^{k} t_{0,j} H \varphi_j(\nu_0) \right)\) is invertible, we need only show that

\(\det A \neq 0,\)

where

\[A = \left( \begin{array}{cc} V & D(\nu_0, \nu_2) \left( \sum_{j=1}^{k} t_{0,j} H \varphi_j(\nu_0) \right)^{-1} \end{array} \right).\]

Since \(VD'(\nu_0, \nu_2) = 0\), we note that the matrix \(A \left(V^t \ D'(\nu_0, \tilde{\nu}_2) \right)\) equals

\[
\begin{pmatrix}
I_{d-k} & 0 \\
D(\nu_0, \tilde{\nu}_2) \left( \sum_{j=1}^{k} t_{0,j} H \varphi_j(\nu_0) \right)^{-1} V^t & D(\nu_0, \tilde{\nu}_2) \left( \sum_{j=1}^{k} t_{0,j} H \varphi_j(\nu_0) \right)^{-1} D'(\nu_0, \tilde{\nu}_2)
\end{pmatrix}.
\]

This matrix is clearly invertible thanks to (1.3). Hence, so is the matrix \(A\). This completes the proof. \(\Box\)

In what follows we show that the following version of Lemma 3.2 holds, where we assume (1.5) instead of (1.3), dropping the condition (1.2).

**Lemma 3.3.** Suppose that, for any \(\nu_1 \in S_1, \ \nu_2 \in S_2, \ \nabla \varphi_i(\nu_2) - \nabla \varphi_i(\nu_1), \ i = 1, \ldots, k\) are linearly independent and (1.5) holds for \(\nu_1 \in S_1, \ \nu_2 \in S_2, \ |t| = 1\) and for \(\nu = \nu_1, \nu_2\). If \(S_1\) and \(S_2\) are sufficiently small, there is a constant \(C\), independent of \(\nu_1, \nu_2, \ R, \ and \ u\) such that, for some \(u' \in \mathbb{R}^{d+1},\)

\[
\Gamma_{1,\nu_1,\nu_2}^{\nu_1,\nu_2}(R, R^d) \cap \left( R^d \pi_{\nu_2} + u \right) \subset B(u', CR^{1/2} + C\delta).
\]

\(\text{hWe may assume that there is a } c > 0 \text{ such that } |\det \left[ N(\nu_2, \nu_1) \left( \sum_{i=1}^{k} t_i H \varphi_i(\nu) \right) N'(\nu, \nu_1) \right] | > c \text{ for } \nu_1 \in S_2 \text{ and } \nu_2 \in S_2 \text{ (see Remark 1.4).}\)
Proof. It is sufficient to show that \((4.1)\) holds. As before, under the assumption that \(S_2\) is small enough, replacing \(D(\nu_0, \nu_2)\) with \(D(\nu_0, \tilde{\nu}_2)\), \(\tilde{\nu}_2 = \nu_0 + \nu'_2 - \nu_1\). We need only show that
\[
\det \left( \frac{\mathbf{V} \left( \sum_{j=1}^{k} t_{0,j} H \varphi_j(\nu_0) \right)}{\mathbf{D}(\nu_0, \tilde{\nu}_2)} \right) \neq 0.
\]
Since vectors \(\mathbf{v}_1, \ldots, \mathbf{v}_{d-k}\) are orthogonal to the row vectors of \(D(\nu_0, \tilde{\nu}_2)\), by multiplying the nonsingular matrix \((\mathbf{V}^t, \mathbf{D}^t(\nu_0, \tilde{\nu}_2))\) to the matrix inside the determinant from the right, we see that the above is equivalent to
\[
\det \left( \mathbf{V} \left( \sum_{j=1}^{k} t_{0,j} H \varphi_j(\nu_0) \right) \mathbf{V}^t \right) \mathbf{V} \left( \sum_{j=1}^{k} t_{0,j} H \varphi_j(\nu_0) \right) \mathbf{D}(\nu_0, \tilde{\nu}_2) \mathbf{D}^t(\nu_0, \tilde{\nu}_2) = 0.
\]
Since the matrix \(\mathbf{D}(\nu_0, \tilde{\nu}_2)\mathbf{D}^t(\nu_0, \tilde{\nu}_2)\) is nonsingular, it is clear that the above is equivalent to \(\det [\mathbf{V} \left( \sum_{j=1}^{k} t_{0,j} H \varphi_j(\nu_0) \right) \mathbf{V}^t] = 0\), which is the condition \((1.5)\).

\[\square\]

4. Proof of Theorem [1, 11]

In this section we will prove Theorem [1, 11]. Our proof is similar to that in [18] (also see [27]). To prove Theorem [1, 11] we need only show that, for \(p > \frac{d + 3k}{d + k}\),
\[
\|E_1 f E_2 g\|_p \leq C \|f\|_2 \|g\|_2
\]

since we can obtain the desired conclusion by interpolating this estimate with the trivial estimate \(\|E_1 f E_2 g\|_\infty \leq \|f\|_1 \|g\|_1\). By an \(\epsilon\)-removal argument [11, 28], it is sufficient to show that \((4.1)\) holds for any \(\alpha > 0\). In fact, by the assumption that \(\sum_{j=1}^{k} t_{0,j} H \varphi_j(\nu)\) is nonsingular for \(\nu \in \text{supp} f \cup \text{supp} g\) as long as \(|t| = 1\), it follows that
\[
|E_\kappa(a_\kappa)(x, t)| \lesssim (|x| + |t|)^{-\frac{d}{2}}, \quad \kappa = 1, 2,
\]
where \(a_1, a_2\) are smooth bump functions which vanish on the supports of \(f\) and \(g\), respectively. This can be shown by the stationary phase method. Hence, the arguments in [11, 28] work here without modification.

Proposition 4.1. Let \(0 < \delta \ll 1\). If \((1.6)\) holds, then for any \(\epsilon > 0\)
\[
\|E_1 f E_2 g\|_{L^\frac{d+3k}{d+\delta}(Q_R)} \leq C \epsilon R^{(\alpha(1-\delta), C\delta)+\epsilon} \|f\|_2 \|g\|_2
\]

with \(C\) independent of \(\delta\).

By iterating finitely many times the implication in Proposition 4.1 we can easily obtain the estimate \((1.6)\) for any \(\alpha > 0\).

4.1. Wave packet decomposition. In this section we decompose the function \(E f\) into wave packets. Let \(R \gg 1\). We denote by
\[
\mathcal{L} = \mathcal{L}(R) := R^{1/2} \mathbb{Z}^d, \quad \mathcal{V} = \mathcal{V}(R) := R^{-1/2} \mathbb{Z}^d.
\]
Let \(\psi\) be a nonnegative Schwartz function such that \(\hat{\psi}\) is supported on \(B(0, 1)\) and \(\sum_{k \in \mathbb{Z}^d} \psi(-k) = 1\). Also, let \(\zeta\) be a smooth function supported on \(B(0, 1)\) and \(\sum_{k \in \mathbb{Z}^d} \zeta(-k) = 1\).
For $\ell \in \mathcal{L}$, $\nu \in \mathcal{V}$ we set $\psi_\ell(x) := \psi\left(\frac{x-\ell}{R}\right)$, $\zeta_\nu(\xi) =: \zeta(\xi)$, and for a given function $f$, we define $f_{\ell,\nu}$ by

$$f_{\ell,\nu} = \mathcal{F}\left(\psi_\ell \mathcal{F}^{-1}(\zeta_\nu f)\right),$$

where $\mathcal{F}$, $\mathcal{F}^{-1}$ denote Fourier transform and the inverse Fourier transform, respectively. Then, it follows that $f = \sum_{\nu \in \mathcal{V}} \sum_{\ell \in \mathcal{L}} f_{\ell,\nu}$. Hence we may write

$$E f = \sum_{\nu \in \mathcal{V}} \sum_{\ell \in \mathcal{L}} E f_{\ell,\nu}.$$  

**Lemma 4.2.** If $|t| \lesssim R$, then

$$|E f_{\ell,\nu}(x, t)| \leq C_N \left(1 + R^{-\frac{d}{2}} |x - \ell + \sum_{j=1}^k t_j \nabla \varphi_j(\nu)|\right)^{-N} M(f)(\ell)$$

for all $N \geq 0$. Here, $Mf$ is the Hardy-Littlewood maximal function of $f$.

**Proof.** Since $f_{\ell,\nu}$ is supported in $B(\nu, 3R^{-1/2})$, multiplying by a harmless smooth bump function $\tilde{\chi}$ supported in $B(0, 5)$ and satisfying $\tilde{\chi} = 1$ on $B(0, 3)$, we may write

$$E f_{\ell,\nu}(x, t) = \int K(x - z, t) \psi_\ell(z) \mathcal{F}^{-1}(f_{\ell,\nu})(z) dz,$$

where $K(x, t) = \int e^{2\pi i (x \cdot \xi + t \cdot \Phi(\xi))} \chi\left(R^{1/2}(\xi - \nu)\right) d\xi$. Changing variables $\xi \to R^{-1/2}(\xi + \nu)$,

$$K(x, t) = R^{-d/2} e^{2\pi i x \cdot \nu} \int e^{2\pi i (R^{-1/2} x \cdot \xi + t \cdot \Phi(R^{-1/2} \xi + \nu))} \chi(\xi) d\xi.$$

Since $|t| \lesssim R$, $\nabla \xi(R^{-1/2} x \cdot \xi + t \cdot \Phi(R^{-1/2} \xi + \nu)) = R^{-1/2} \left(x + \sum_{j=1}^k t_j \nabla \varphi_j(\nu)\right) + O(1)$. This follows by Taylor’s expansion. Hence, by repeated integration by parts we get

$$|K(x, t)| \leq C_N R^{-d/2} \left(1 + R^{-1/2} |x + \sum_{j=1}^k t_j \nabla \varphi_j(\nu)|\right)^{-N}.$$

Once this is established, (4.3) follows by a standard argument. See [18] for the details. \(\square\)

From the above lemma we see that $E f_{\ell,\nu}$ is essentially supported on

$$\pi_{\ell,\nu} = \pi + (\ell, 0).$$

If $\pi = \pi_{\ell,\nu}$, we define $\nu(\pi) = \nu$, which may be considered as the (generalized) direction of $\pi$.

The following is the main lemma of this section.

**Lemma 4.3.** Let $R \gg 1$. Then, $E f$ can be rewritten as

$$E f(x, t) = \sum_{(\ell,\nu) \in \mathcal{L} \times \mathcal{V}} c_{\ell,\nu} P_{\ell,\nu}(x, t)$$

and $c_{\ell,\nu}$, $P_{\ell,\nu}$ satisfy the following:

(i) $\mathcal{F}(P_{\ell,\nu}(\cdot, t))$ is supported in the disc $D(\nu, CR^{-1/2})$. 

(ii) If $|t| \lesssim R$, then for any $N \geq 0$
\[ |P_{\ell,\nu}(x,t)| \leq C_N R^{-d/4} \left(1 + R^{-1/2} |x - \ell| + \left( \sum_{j=1}^{k} t_j |\nabla \varphi_j(\nu)| \right) \right)^{-N}. \]

(iii) \( (\sum_{(\ell,\nu) \in \mathcal{L} \times \mathcal{V}} |c_{\ell,\nu}|^2)^{1/2} \lesssim \|f\|_2. \)

(iv) If $|t| \lesssim R$, then $\|\sum_{(\ell,\nu) \in \mathcal{W}} P_{\ell,\nu}(\cdot,t)\|_2^2 \lesssim \# \mathcal{W}$ for any $\mathcal{W} \subset \mathcal{L} \times \mathcal{V}$.

Proof. We define $c_{\ell,\nu}$ and $P_{\ell,\nu}$ by
\[ c_{\ell,\nu} = R^{d/4} M(\mathcal{F}^{-1} f_{\nu})(\ell), \quad P_{\ell,\nu}(x,t) = c_{\ell,\nu}^{-1} E f_{\ell,\nu}(x,t) \]
where $M$ denotes the Hardy-Littlewood maximal function. Then we have (4.5) from (4.2).

Since $Ef_{\ell,\nu}(\cdot,y) = \mathcal{F}^{-1}(e^{2\pi i y \cdot} f_{\ell,\nu})$, $Ef_{\ell,\nu}(\cdot,y)$ has a Fourier support contained in $\text{supp} \ f_{\ell,\nu}$, which is in turn contained in $D(\nu, CR^{-1/2})$. Thus (i) follows and so does (ii) from Lemma 4.2.

In order to show (iii), note that
\[ (4.6) \quad \sum_{(\ell,\nu) \in \mathcal{L} \times \mathcal{V}} |c_T|^2 = R^{d/2} \sum_{(\ell,\nu) \in \mathcal{L} \times \mathcal{V}} M(\mathcal{F}^{-1}(\zeta_{\nu} f))(\ell)^2. \]

Since $\zeta_{\nu} f$ is supported on $B(\nu, CR^{1/2})$, $M(\mathcal{F}^{-1}(\zeta_{\nu} f))(x) \sim M(\mathcal{F}^{-1}(\zeta_{\nu} f))(x')$ if $|x - x'| \lesssim R^{1/2}$. Hence, from the Hardy-Littlewood maximal theorem and the Plancherel theorem we have that, for each $\nu$,
\[ R^{d/2} \sum_{(\ell,\nu) \in \mathcal{L} \times \mathcal{V}} |M(\mathcal{F}^{-1}(\zeta_{\nu} f))(\ell)|^2 \lesssim \int |M(\mathcal{F}^{-1}(\zeta_{\nu} f))(x)|^2 dx \lesssim \|\zeta_{\nu} f\|_2^2. \]

Combining this and (4.6) we obtain $\sum_{(\ell,\nu) \in \mathcal{L} \times \mathcal{V}} |c_{\ell,\nu}|^2 \lesssim \sum_{\nu \in \mathcal{V}} \|\zeta_{\nu} f\|_2^2 \lesssim \|f\|_2^2$, and (iii).

Finally, we consider (iv). Since $\sum_{(\ell,\nu) \in \mathcal{W}} P_{\ell,\nu}(\cdot,t)$ is Fourier-supported in $D(\nu, CR^{-1/2})$, which have bounded overlap as $\nu$ varies over $\mathcal{V}$. By Plancherel’s theorem,
\[ \left\| \sum_{(\ell,\nu) \in \mathcal{W}} P_{\ell,\nu}(\cdot,t) \right\|_2^2 \lesssim \sum_{\nu \in \mathcal{V}} \left\| \sum_{(\ell,\nu) \in \mathcal{W}} P_{\ell,\nu}(\cdot,t) \right\|_2^2. \]

From (ii) it is easy to see that $\left\| \sum_{(\ell,\nu) \in \mathcal{W}} P_{\ell,\nu}(\cdot,t) \right\|_2^2 \lesssim \# \{\ell : (\ell,\nu) \in \mathcal{W}\}$. Hence, combining this with the above gives (iv).

4.2. Dyadic pigeonholing and reduction. From now on we will prove Proposition 4.1.

For simplicity we set
\[ p_0 = \frac{d + 3k}{d + k}. \]

By translation invariance we may assume that $Q_R$ is centered at the origin. Let
\[ \mathcal{W}_i \subset \{ (\ell,\nu) \in \mathcal{L} \times \mathcal{V} : \nu \in S_i + O(R^{-\frac{1}{2}}) \}, \quad i = 1, 2. \]
By Lemma 4.3 and the standard reduction with pigeonholing, which may only cause a loss of \((\log R)^C\) (see [18, 27]), the matter is reduced to showing

\[
\left\| \sum_{\omega_1 \in W_1} P_{\omega_1} \sum_{\omega_2 \in W_2} P_{\omega_2} \right\|_{L^{p_0}(Q_R)} \lesssim (R^{1-\delta} + R^C) (#W_1 #W_2)^{\frac{1}{2}},
\]

whenever \(P_{\omega_1}, P_{\omega_2}\) satisfy (i), (ii), (iv) in Lemma 4.3. Here \(A \lesssim B\) means \(A \leq C \cdot R^r B\) for any \(\epsilon > 0\).

By a further pigeonholing argument we specify the associated quantities in dyadic scales. Let \(Q\) be a collection of almost disjoint cubes of the same sidelength \(\sim R^{1/2}\), which cover \(Q_R\). For each \(q \in Q\) we define

\[
W_j(q) = \{\omega_j \in W_j : \pi_{\omega_j} \cap R^\delta q \neq \emptyset\}.
\]

For dyadic numbers \(\rho_1, \rho_2\) with \(1 \leq \rho_1, \rho_2 \leq R^{100d}\), we define

\[
Q(\rho_1, \rho_2) = \{q \in Q : \rho_j \leq \#W_j(q) < 2\rho_j, \ j = 1, 2\}.
\]

For \(\omega \in W_1 \cup W_2\), we set

\[
\lambda(\omega; \rho_1, \rho_2) = \# \{q \in Q(\rho_1, \rho_2) : \pi_{\omega} \cap R^\delta q \neq \emptyset\}.
\]

For dyadic numbers \(1 \leq \lambda \leq R^{100d}\) we define

\[
W_j[\lambda; \rho_1, \rho_2] = \{\omega_j \in W_j : \lambda(\omega_j; \rho_1, \rho_2) < 2\lambda\}, \ j = 1, 2.
\]

By a standard pigeonhole argument, it is sufficient to show

\[
\left( \sum_{q \in Q(\rho_1, \rho_2)} \left\| \sum_{\omega_1 \in W_1[\lambda_1; \rho_1, \rho_2]} P_{\omega_1} \sum_{\omega_2 \in W_2[\lambda_2; \rho_1, \rho_2]} P_{\omega_2} \right\|_{L^{p_0}(Q(q))}^{p_0} \right)^{1/p_0} \lesssim (R^{1-\delta} + R^C) (#W_1 #W_2)^{1/2}.
\]

For the rest of the proof we assume that \(q \in Q(\rho_1, \rho_2)\), \(\omega_1 \in W_1[\lambda_1; \rho_1, \rho_2]\) and \(\omega_2 \in W_2[\lambda_2; \rho_1, \rho_2]\) if it is not mentioned otherwise. So, the above sums are denoted simply by \(\sum_q\), \(\sum_{\omega_1}\), and \(\sum_{\omega_2}\), respectively.

4.3. Induction argument. For brevity let us put

\[
\Delta = \bigcup_{q \in Q(\rho_1, \rho_2)} q.
\]

Let \(\{B\}\) be a collection of almost disjoint cubes of the same sidelength \(R^{1-\delta}\), which cover \(Q_R\). Then

\[
\text{LHS of (4.9)} \leq \sum_B \left\| \sum_{\omega_1} P_{\omega_1} \sum_{\omega_2} P_{\omega_2} \right\|_{L^{p_0}(\Delta \cap B)}.
\]

We define a relation \(\sim\) between \(\omega_1\) (or \(\omega_2\)) and the cubes in \(\{B\}\). For each \(\omega \in W_1[\lambda_1; \rho_1, \rho_2] \cup W_2[\lambda_2; \rho_1, \rho_2]\), we define \(B^*(\omega) \in \{B\}\) to be the cube which maximizes the quantity

\[
\# \{q \in Q(\rho_1, \rho_2) : \pi_{\omega} \cap R^\delta q \neq \emptyset; \ q \cap B \neq \emptyset\}.
\]

Then the relation \(\sim\) is defined as follows:

\[
\omega \sim B \text{ if } B \cap 10B^*(\omega) \neq \emptyset.
\]
Here $10B^*(\omega)$ is the cube which has the same center as $B^*(\omega)$ and sidelength 10 times as large as that of $B^*(\omega)$. Using this relation we divide the sum into three parts to get

$$\sum_B \left\| \sum_{\omega_1} P_{\omega_1} \sum_{\omega_2} P_{\omega_2} \right\|_{L^p(\Delta \cap B)} \leq \sum_B \left\| \sum_{\omega_1 \omega_1 \sim B} P_{\omega_1} \sum_{\omega_2 \omega_2 \sim B} P_{\omega_2} \right\|_{L^p(\Delta \cap B)}$$

$$+ \sum_B \left\| \sum_{\omega_1 \omega_1 \sim B} P_{\omega_1} \sum_{\omega_2 \omega_2 \sim B} P_{\omega_2} \right\|_{L^p(\Delta \cap B)} + \sum_B \left\| \sum_{\omega_1 \omega_1 \sim B} P_{\omega_1} \sum_{\omega_2} P_{\omega_2} \right\|_{L^p(\Delta \cap B)}.$$

We will first show that

$$(4.12) \sum_B \left\| \sum_{\omega_1 \omega_1 \sim B} P_{\omega_1} \sum_{\omega_2 \omega_2 \sim B} P_{\omega_2} \right\|_{L^p(\Delta \cap B)} \lesssim R^{(1-\alpha)(\#W_1 \#W_2)^{1/2}}.$$

By applying the hypothesis $(1.6)$, $(iv)$ in Lemma 4.3 and the Cauchy-Schwarz inequality,

$$\sum_B \left\| \sum_{\omega_1 \omega_1 \sim B} P_{\omega_1} \sum_{\omega_2 \omega_2 \sim B} P_{\omega_2} \right\|_{L^p(\Delta \cap B)} \leq CR^{(1-\alpha)} \left( \sum_B \# \{\omega_j : \omega_j \sim B\} \right)^{1/2}.$$

From the definition of the relation $\sim$ it is clear that $\# \{B : \omega_j \sim B\} \leq C$. Hence, for $j = 1, 2$

$$\sum_B \# \{\omega_j : \omega_j \sim B\} = \sum_{\omega_j} \# \{B : \omega_j \sim B\} \lesssim W_j.$$

By inserting this into the previous inequality, we get $(4.12)$.

Now, to prove $(4.9)$ it is enough to show

$$\left\| \sum_{\omega_1 \omega_1 \sim B} P_{\omega_1} \sum_{\omega_2} P_{\omega_2} \right\|_{L^p(\Delta \cap B)} \lesssim R^{C \delta \#(W_1 \#W_2)^{1/2}}$$

and

$$(4.13) \left\| \sum_{\omega_1 \omega_1 \sim B} P_{\omega_1} \sum_{\omega_2} P_{\omega_2} \right\|_{L^p(\Delta \cap B)} \lesssim R^{C \delta \#(W_1 \#W_2)^{1/2}}.$$

The proofs of these two estimates are similar. So, we will only prove $(4.13)$. By Plancherel’s theorem, $\|Ef(\cdot, t)\|_2 \leq \|f\|_2$ for all $t \in \mathbb{R}^k$. Integration in $t$ gives $\|Ef\|_{L^2(Q_R)} \lesssim R^{k} \|f\|_2$. By the Schwarz inequality it follows that

$$\|E_1fE_2g\|_{L^2(Q_R)} \lesssim R^{k} \|f\|_2 \|g\|_2.$$

Combining this with $(iv)$ in Lemma 4.3 yields

$$(4.14) \left\| \sum_{\omega_1 \omega_1 \sim B} P_{\omega_1} \sum_{\omega_2} P_{\omega_2} \right\|_{L^1(\Delta \cap B)} \lesssim R^{k} \#(W_1 \#W_2)^{1/2}.$$

Hence, the $(4.13)$ follows from interpolation between $(4.14)$ and

$$(4.15) \left\| \sum_{\omega_1 \omega_1 \sim B} P_{\omega_1} \sum_{\omega_2} P_{\omega_2} \right\|_{L^2(\Delta \cap B)} \lesssim R^{C \delta R^{-\frac{1}{2\alpha^2}} \#(W_1 \#W_2)^{1/2}}.$$

Now it remains to show the $L^2$-estimate $(4.15)$. 
4.4. $L^2$ estimate. To prove (4.15) it suffices to show

\begin{equation}
\sum_{q \in Q(\rho_1, \rho_2); q \subset 2B} \left\| \sum_{\omega_1 \sim B} P_{\omega_1} \sum_{\omega_2} P_{\omega_2} \right\|_{L^2(q)}^2 \leq R^{C^\delta} R^{-(d-k)/2} \# W_1 \# W_2.
\end{equation}

For $j = 1, 2$, let us set

$$W_j(q) = \{ \omega_j \in W_{i; \rho_1, \rho_2} : \omega_j \cap R^q \neq \emptyset \}, \quad W_j^{\rho_1}(q) = \{ \omega_j \in W_j(q) : \omega_j \sim B \}.$$  

Then by (ii) in Lemma 4.3 we may discard some harmless terms, whose contributions are $O(R^{-C^\delta})$. Hence, it suffices to show

\begin{equation}
\sum_{q \in Q(\rho_1, \rho_2); q \subset 2B} \left\| \sum_{\omega_1 \in W_1^{\rho_1} B(q)} P_{\omega_1} \sum_{\omega_2(q)} P_{\omega_2} \right\|_{L^2(q)}^2 \leq R^{C^\delta} R^{-(d-k)/2} \# W_1 \# W_2.
\end{equation}

By using Plancherel’s theorem we write

$$\left\| \sum_{\omega_1 \in W_1^{\rho_1} B(q)} P_{\omega_1} \sum_{\omega_2(q)} P_{\omega_2} \right\|_{L^2(q)}^2 = \sum_{\omega_1 \in W_1^{\rho_1} B(q)} \sum_{\omega_2(q)} \sum_{\omega_1 \in W_1^{\rho_1} B(q)} \sum_{\omega_2(q)} \left\langle \hat{P}_{\omega_1}, \hat{P}_{\omega_2}, \hat{P}_{\omega_1}, \hat{P}_{\omega_2} \right\rangle.$$  

Let us write $\omega_j = (\ell_j, \nu_j), \omega_j' = (\ell_j', \nu_j'), j = 1, 2$. For any $\nu_1 \in S_1, \nu_2 \in S_2$, we define $W_1^{\rho_1} B(q; \nu_1, \nu_2)$ by

$$W_1^{\rho_1} B(q; \nu_1, \nu_2) = \{ \omega_1' = (\ell_1', \nu_1') : \nu_1' \in \Pi_{\nu_1, \nu_2} + O(R^{-1/2}) \}.$$  

Then $\hat{P}_{\omega_1} * \hat{P}_{\omega_2}$ is supported on the $O(R^{-1/2})$-neighborhood of the point $(\nu_1 + \nu_2, \Phi(\nu_1) + \Phi(\nu_2))$. So the inner product $\left\langle \hat{P}_{\omega_1}, \hat{P}_{\omega_2}, \hat{P}_{\omega_1}, \hat{P}_{\omega_2} \right\rangle$ vanishes unless

$$\nu_1 + \nu_2 = \nu_1' + \nu_2' + O(R^{-1/2}), \quad \Phi(\nu_1) + \Phi(\nu_2) = \Phi(\nu_1') + \Phi(\nu_2') + O(R^{-1/2}).$$

Thus, for given $\nu_1$ and $\nu_2$, we see that $\nu_1'$ is contained in a $O(R^{-1/2})$-neighborhood of $\Pi_{\nu_1, \nu_2}$, which is defined by (3.1). Once $\nu_1, \nu_1'$ and $\nu_2, \nu_2'$ are given, then there are only $O(1)$ many $\nu_2$, since $\nu_2$ should be in a $O(R^{-1/2})$-neighborhood of the point $\nu_1 + \nu_1' - \nu_2'$. Therefore,

$$\left\| \sum_{\omega_1 \in W_1^{\rho_1} B(q)} P_{\omega_1} \sum_{\omega_2(q)} P_{\omega_2} \right\|_{L^2(q)}^2 \leq R^{-(d-k)/2} \sum_{\omega_1 \in W_1^{\rho_1} B(q)} \sum_{\omega_2(q)} \# W_1^{\rho_1} B(q; \nu_1, \nu_2),$$

where we also used

$$\left| \left\langle P_{\omega_1}, P_{\omega_1'}, P_{\omega_2}, P_{\omega_2'} \right\rangle \right| \lesssim R^{-(d-k)/2}.$$  

This follows from (ii) in Lemma 4.3 and the transversality between $\pi_{\omega_1}$ ($\pi_{\omega_1'}$) and $\pi_{\omega_2}$ ($\pi_{\omega_2'}$), respectively. Hence, (4.17) follows if we show

\begin{equation}
\max_{q \subset 2B, \nu_1, \nu_2} \# W_1^{\rho_1} B(q; \nu_1, \nu_2) \sum_{q \in Q(\rho_1, \rho_2); q \subset 2B} \# W_2(q) \# W_2(q) \lesssim R^{C^\delta} \# W_1 \# W_2.
\end{equation}
We will prove (4.18), assuming for the moment that
\[
\max_{q \in 2B, \nu_1, \nu_2} \# W_1^{\nu B}(q; \nu_1, \nu_2) \lesssim R^{C_{\delta}} \frac{\# W_2}{\lambda_1 \rho_2}.
\]

To this end it is enough to show
\[
\sum_{q \in \mathcal{Q}(\rho_1, \rho_2), q \in 2B} \# W_1^{\nu B}(q) \# W_2(q) \lesssim \lambda_1 \rho_2 \# W_1.
\]

Recalling \(\# W_2(q) \lesssim \rho_2\), we see that the left hand side is bounded by
\[
C \rho_2 \sum_{q \in \mathcal{Q}(\rho_1, \rho_2)} \# W_1(q).
\]

Changing the order of summation, we see this in turn is bounded by \(C \rho_2 \sum_{\omega \in \mathcal{Q}(\rho_1, \rho_2)} \# \{ q \in \mathcal{Q}(\rho_1, \rho_2) : \pi_{\omega_1} \cap R^\delta q \} \lesssim \lambda_1\), the desired inequality (4.18) follows.

### 4.5. Proof of (4.19)

Fix \(q \subset 2B, \nu_1 \in S_1\) and \(\nu_2 \in S_2\). Let us consider the set
\[
S := \{ (\tilde{q}, \omega_1, \omega_2) \in \mathcal{Q}(\rho_1, \rho_2) \times W_1^{\nu B}(q, \nu_1, \nu_2) \times W_2 : \pi_{\omega_1} \cap R^\delta \tilde{q} \neq \emptyset, \pi_{\omega_2} \cap R^\delta \tilde{q} \neq \emptyset, \operatorname{dist}(\tilde{q}, q) \geq R^{1-\delta} \}.
\]

To prove (4.19) it suffices to show
\[
R^{-C_{\delta}} \lambda_1 \rho_2 \# W_1^{\nu B}(q; \nu_1, \nu_2) \lesssim \# S \lesssim R^{C_{\delta}} \# W_2.
\]

For the lower bound it is enough to show that, for each \(\omega_1 \in W_1^{\nu B}(q; \nu_1, \nu_2)\),
\[
\# \{ (\tilde{q}, \omega_2) \in \mathcal{Q}(\rho_1, \rho_2) \times W_2 : \pi_{\omega_1} \cap R^\delta \tilde{q} \neq \emptyset, \pi_{\omega_2} \cap R^\delta \tilde{q} \neq \emptyset, \operatorname{dist}(\tilde{q}, q) \geq R^{1-\delta} \} \gtrsim R^{1-\delta} \lambda_1 \rho_2.
\]

By (4.8) \(\omega_1\) contains as many as \(O(\lambda_1)\) cubes \(\tilde{q}\) in \(\mathcal{Q}(\rho_1, \rho_2)\).

Let \(B^* (\omega_1) \in \mathcal{Q}\) be the cube which maximizes the quantity given by (4.11) with \(\omega = \omega_1\). Since \(\omega_1 \sim B\), it follows from the definition of the relation \(\sim\) that \(\operatorname{dist}(B^* (\omega_1), B) \gtrsim R^{1-\delta}\). Since \(\pi_{\omega_1} + O(R^{1/2+\delta})\) can be covered by \(R^{C_{\delta}}\) cubes \(B\), by a simple pigeonholing argument we get
\[
\# \{ \tilde{q} \in \mathcal{Q}(\rho_1, \rho_2) : \pi_{\omega_1} \cap R^\delta \tilde{q} \neq \emptyset, \operatorname{dist}(\tilde{q}, q) \geq R^{1-\delta} \} \gtrsim R^{-C_{\delta}} \lambda_1.
\]

Next, for the upper bound it suffices to show that, for any \(\omega_2 \in W_2\),
\[
\# \{ (\tilde{q}, \omega_1) \in \mathcal{Q}(\rho_1, \rho_2) \times W_1^{\nu B}(q; \nu_1, \nu_2) : \pi_{\omega_1} \cap R^\delta \tilde{q} \neq \emptyset, \pi_{\omega_2} \cap R^\delta \tilde{q} \neq \emptyset, \operatorname{dist}(\tilde{q}, q) \gtrsim R^{1-\delta} \} \lesssim R^{C_{\delta}}.
\]

Let \(z_0\) be the center of \(q\). Then, by the definition of \(W_1^{\nu B}(q_0; \nu_1, \nu_2)\), it follows that
\[
\bigcup_{\omega_1 \in W_1^{\nu B}(q; \nu_1, \nu_2)} \pi_{\omega_1} \subset \Gamma^F_{\nu_1, \nu_2} (C R^{1/2+\delta}) + z_0.
\]

If \(\omega_2 \in W_2\), then it follows from Lemma 3.2 that the intersection
\[
\pi_{\omega_2} \cap \left( \bigcup_{\omega_1 \in W_1^{\nu B}(q; \nu_1, \nu_2)} \pi_{\omega_1} \right)
\]

\footnote{Recall that we are assume assuming \(q \in \mathcal{Q}(\rho_1, \rho_2)\), \(\omega_1 \in W_1[\lambda_1; \rho_1, \rho_2]\) and \(\omega_2 \in W_2[\lambda_1; \rho_1, \rho_2]\).}
is contained in a cube of sidelength $O(R^{1/2+\delta})$. Thus, there are at most $O(R^{C\delta})$ choices of balls $\tilde{q} \in Q(\rho_1, \rho_2)$ such that $(\tilde{q}, \omega_1)$ is contained in the set in (4.21). On the other hand, since $\text{dist}(\tilde{q}, q) \gtrsim R^{1-C\delta}$, we have

$$\text{#}\{w_1 \in W_{\nu}^{s,B}(q; \nu_1, \nu_2) : \pi_{\nu} \cap R^{\delta} \tilde{q} \neq \emptyset, \ \pi_{\omega_1} \cap R^{\delta} q \neq \emptyset\} \lesssim R^{C\delta}.$$ 

To see this, by scaling it is enough to check that the map $S \ni \nu \mapsto \sum_{i=1}^k t_j \nabla \varphi_i(\nu)$ is one-to-one whenever $|t| = 1$. But this follows from the condition (1.2) if we take $S_1$ to be small enough. Thus we obtain the claim (4.21). Hence, we also have (4.9), which finishes the proof of Proposition 4.1. This completes the proof of Theorem 1.1.

Proof of Corollary 1.5. Thanks to Lemma 3.3, the line of argument in the proof of Theorem 1.1 works without modification except that we need to show (4.22). However, to prove (4.22) we don’t need to show $S_1 \ni \nu \mapsto \sum_{i=1}^k t_j \nabla \varphi_i(\nu)$ is one-to-one. Instead, as is clear after rescaling it is enough to show that $\Pi^{\nu_1, \nu_2} \ni \nu \mapsto \sum_{i=1}^k t_j \nabla \varphi_i(\nu)$ is one-to-one. Let $t_1, \ldots, t_{d-k}$ be a set of vectors spanning the tangent space of $\Pi^{\nu_1, \nu_2}$ at $\nu_0$. Then the above follows if we show that the matrix

$$(t_1^i, \ldots, t_{d-k}^i) \left( \sum_{i=1}^k t_j H \varphi_i(\nu_0) \right)$$

has rank $d - k$ for $|t| = 1$. In fact, $t_1, \ldots, t_{d-k}$ are almost normal to the span of $\{\nabla \varphi_i(\nu_2) - \nabla \varphi_i(\nu_1) : i = 1, \ldots, k\}$. These vectors are close to $n_1, \ldots, n_{d-k}$. Hence, assuming that $S_1$ and $S_2$ are small enough, the above follows if we show $N(\nu_2, \nu_1) \sum_{i=1}^k t_j H \varphi_i(\nu_0)$ has rank $d - k$. This clearly follows from (1.3). 

5. Restriction Estimates for Complex Surfaces

In this section we provide the proofs of Corollary 1.5 and Theorem 1.6. In what follows we set $k = 2, d = 2n$.

Proof of Corollary 1.5. Let $\varphi_1, \varphi_2$ be given by $\frac{1}{2} z^t Dz = \varphi_1 + i \varphi_2$ so that

$$\varphi_1(x, y) = \frac{1}{2} (x^t D x - y^t D y), \ \varphi_2(x, y) = x^t D y, \ (x, y) \in \mathbb{R}^n \times \mathbb{R}^n.$$ 

In order to prove Corollary 1.5 we need only to show that the condition (1.13) implies the assumptions in Theorem 1.1.

Let us set $z_j = x_j + iy_j \in \mathbb{C}^n$ for $j = 1, 2$, $\delta_x = x_2 - x_1$, and $\delta_y = y_2 - y_1$. Then a computation shows that the associated matrix $M(t, z_1, z_2, z)$ is given by

$$M(t, z_1, z_2, z) = \begin{pmatrix} 0 & 0 & \delta_x^t D & -\delta_y^t D \\ 0 & 0 & \delta_y^t D & \delta_x^t D \\ D \delta_x & D \delta_y & t_1 D & t_2 D \\ -D \delta_y & D \delta_x & t_2 D & -t_1 D \end{pmatrix}.$$ 

Note that

$$\sum_{j=1}^2 t_j H \varphi_j = \begin{pmatrix} t_1 D & t_2 D \\ t_2 D & -t_1 D \end{pmatrix}.$$
Then, it is easy to see that the inverse of \(\sum_{j=1}^{2} t_j H \varphi_j\) is \((t_1^2 + t_2^2)^{-1} \begin{pmatrix} t_1 D^{-1} & t_2 D^{-1} \\ -t_2 D^{-1} & t_1 D^{-1} \end{pmatrix}\). So, the assumption (1.2) holds. Hence, it suffices to show that (1.13) implies (1.3). By the block matrix formula we only need to check

\[
\det \begin{pmatrix} \delta_x^i D & -\delta_y^i D \\ \delta_y^i D & \delta_x^i D \end{pmatrix} \begin{pmatrix} t_1 D^{-1} & t_2 D^{-1} \\ -t_2 D^{-1} & t_1 D^{-1} \end{pmatrix} \begin{pmatrix} D \delta_x & D \delta_y \\ -D \delta_y & D \delta_x \end{pmatrix} \neq 0.
\]

By a direct computation it is not difficult to see that the left-hand side equals

\[-(t_1^2 + t_2^2)((\delta_x^i D \delta_x - \delta_y^i D \delta_y)^2 + 4(\delta_x^i D \delta_y)^2)\]

Since \((z_2 - z_1)^t D(z_2 - z_1) = \delta_x^i D \delta_x - \delta_y^i D \delta_y + 2i\delta_x^i D \delta_y\), it is now clear that (1.13) implies (1.3). \hfill \Box

**Proof of Theorem 1.6.** From the bilinear estimate we can get the linear estimate by adapting the arguments in [18, 30, 31]. Since \(D\) is nonsingular and symmetric, by making use of linear transforms we may assume that

\[D = \begin{pmatrix} 1 & 0 \\ 0 & \pm 1 \end{pmatrix},\]

and so we have either \(\Phi(z_1, z_2) = z_1^2 + z_2^2 = (z_1 + iz_2)(z_1 - iz_2)\) or \(\Phi(z_1, z_2) = z_1^2 - z_2^2 = (z_1 + z_2)(z_1 - z_2)\). By a linear change of variables the problem can be further reduced to showing Proposition 1.6 when \(\Phi(z_1, z_2) = z_1 z_2\).

The following is an immediate consequence of Theorem 1.1 and the translation invariance of the bilinear estimate.

**Lemma 5.1.** Let \(\Phi(z_1, z_2) = z_1 z_2\) and \(Q_1, Q_2 \subset \mathbb{C}^2\) be closed cubes. Assume that

\[2^4 \geq |z_1 - w_1| \geq 2^{-1}, \text{ and } 2^4 \geq |z_2 - w_2| \geq 2^{-1}\]

whenever \((z_1, z_2) \in Q_1\) and \((w_1, w_2) \in Q_2\). If \(\text{supp}(f) \subset Q_1\) and \(\text{supp}(g) \subset Q_2\), then for \(q > \frac{10}{3}\) and \(\frac{1}{p} + \frac{5}{3q} < 1\)

\[\|Ef Eg\|_{q/2} \leq C_{p,q} \|f\|_p \|g\|_p.\]

In the next lemma the hypothesis of ‘nonvanishing rotational curvature’ is weakened to the usual separation condition. But then, for the conclusion to hold, the pair \((1/p, 1/q)\) needs to satisfy a more restrictive condition. This lemma is an analog of Proposition 4.1 in [18].

**Lemma 5.2.** Let \(Q_1, Q_2\) be closed cubes in \(\mathbb{C}^2\) such that \(\text{dist}(Q_1, Q_2) \geq 1\). If \(\text{supp}(f) \subset Q_1\) and \(\text{supp}(g) \subset Q_2\), then there is a constant \(C_{p,q}\) such that

\[\|Ef Eg\|_{q/2} \leq C_{p,q} \|f\|_p \|g\|_p\]

if \(\frac{1}{p} + \frac{2}{q} < 1\), \(q > \frac{10}{3}\), or \(\|E\chi_F E\chi_G\|_{q/2} \lesssim \|f\|_p, \|g\|_p\) if \(\frac{1}{p} + \frac{2}{q} = 1\), \(q > \frac{10}{3}\).

\[1\text{In fact, the product of the three matrices is equal to } \begin{pmatrix} t_1 & -t_2 \\ t_2 & t_1 \end{pmatrix} \begin{pmatrix} \delta_x^i D \delta_x - \delta_y^i D \delta_y & 2\delta_x^i D \delta_y \\ 2\delta_x^i D \delta_y & \delta_y^i D \delta_y - \delta_x^i D \delta_x \end{pmatrix}.\]
By translation it is clear that in Lemma 5.1 and Lemma 5.2 the same estimate holds with $Q_1, Q_2$ replaced by $Q_1 + a, Q_2 + a$, respectively, for any $a \in \mathbb{C}^2$. It is possible to prove the strong-type estimate $\|E_{X_F} E_{X_G}\|_{q/2} \lesssim \|f\|_p \|g\|_p$ for $\frac{1}{p} + \frac{2}{q} = 1$, $q > \frac{10}{3}$ by making use of the asymmetric estimates which are obtained in the course of proof of Proposition 1.3, and the bilinear interpolation (see e.g. [10, Sec. 3.13, 5(b)]). However, we have decided not to include the details here, because it does not seem to have any consequences for linear estimates.

Proof of Lemma 5.2. By interpolation it suffices to consider the case $\frac{10}{3} < q \leq 4$ and $p \leq q$. By decomposition of the domains, followed by translation and scaling, we may assume that $Q_1 = H_1 \times K$ and $Q_2 = H_2 \times K$, where $\text{dist}(H_1, H_2) \geq 2^{-1}$ and $K$ is the unit cube in $\mathbb{C}$, centered at the origin.

By a Whitney decomposition, we get
\[
(K \times K) \setminus D = \bigcup_{j > 1} \bigcup_{(k, k') : I_k^j \sim I_{k'}^j} I_k^j \times I_{k'}^j,
\]
where $D = \{(z_2, w_2) : z_2 = w_2\}$, and $\{I_k^j\}_k$ are the dyadic cubes in $\mathbb{C}$ of sidelength $2^{-j}$, and as usual the notation $I_k^j \sim I_{k'}^j$ means that the parent cubes of $I_k^j$ and $I_{k'}^j$ are adjacent, while $I_k^j$ and $I_{k'}^j$ are not.

Let us set
\[
f_k^j(z_1, z_2) = x_{I_k^j}(z_2)f(z_1, z_2), \quad g_k^j(w_1, w_2) = x_{I_k^j}(w_2)g(w_1, w_2).
\]
Then, since the cubes $I_k^j \times I_{k'}^j$ are almost disjoint, we may write
\[
Ef = Eg = \sum_j \sum_{(k, k') : I_k^j \sim I_{k'}^j} E(f_k^j) E(g_{k'}^j).
\]
Since $q > \frac{10}{3}$, we get
\[
\|Ef \cdot Eg\|_{q/2} \leq \sum_j \sum_{I_k^j \sim I_{k'}^j} E(f_k^j) E(g_{k'}^j) \|f_k^j\|_{q/2} \lesssim \sum_j \left( \sum_{I_k^j \sim I_{k'}^j} \|E(f_k^j) E(g_{k'}^j)\|_{q/2} \right)^{2/q},
\]
where the last inequality follows from Lemma 6.1 in [30]. Here, we used the fact that for each fixed $j$ the supports of the Fourier transforms of $E(f_k^j) E(g_{k'}^j)$ have uniformly bounded overlap as $(k, k')$ varies, provided that $I_k^j \sim I_{k'}^j$. This is a consequence of the Whitney decomposition. We now claim that if $I_k^j \sim I_{k'}^j$, then
\[
\|E(f_k^j) E(g_{k'}^j)\|_{q/2} \lesssim 2^{4j \left( \frac{1}{p} + \frac{2}{q} - 1 \right)} \|f_k^j\|_p \|g_{k'}^j\|_p
\]
when $\frac{1}{p} + \frac{2}{q} < 1$, $q > \frac{10}{3}$. This is an easy consequence of a translated version of Lemma 5.1. Assuming this for the moment, we will finish the proof. Since $q \geq p$, for $\frac{1}{p} + \frac{5}{3q} < 1$, we have
\[
\|E(f_k^j) E(g_{k'}^j)\|_{q/2} \lesssim 2^{4j \left( \frac{1}{p} + \frac{2}{q} - 1 \right)} \|f_k^j\|_p \|g_{k'}^j\|_p
\]
4 > q > \frac{10}{3}$, we have
\[
\|Ef \chi_F\|_{q/2} \leq \sum_j 2^{4j}\left(\frac{3}{4} - 1\right) \left( \sum_{I_k^j} \|f_k^j\|^2 \|g_k^j\|^2 \right)^{2/q}
\]
\[
\lesssim \sum_j 2^{4j}\left(\frac{3}{4} - 1\right) \left( \sum_{I_k^j} \|f_k^j\|^{1/p} \left( \sum_j \|g_k^j\|^{1/p} \right)^{1/p} \right) \lesssim \sum_j 2^{4j}\left(\frac{3}{4} - 1\right) \|f\|_p\|g\|_p.
\]

Now take $f = \chi_F$ and $g = \chi_G$ for measurable sets $F, G$ contained in $V_1, V_2$, respectively. Fix $p, q$ with $4 > q > \frac{10}{3}$, $\frac{1}{p} + \frac{2}{q} = 1$, and choose $p_1$ and $p_2$ such that $\frac{1}{p_1} + \frac{5}{3q} < 1, j = 1, 2$, and
\[
\frac{1}{p_1} + \frac{2}{q} - 1 = \eta, \quad \frac{1}{p_2} + \frac{2}{q} - 1 = -\eta
\]
for some small $\eta > 0$. Then by applying the last estimate for $p = p_1$ and $p = p_2$, we obtain
\[
\|E\chi_F E\chi_G\|_{q/2} \lesssim \sum_j \min\{2^{4j}\eta |F|^\eta \|G\|^{1-\eta} |F|^{-\eta} \|G\|^{1-\eta}\}
\]
\[
\lesssim |F|^{1-\eta} G|^{1-\eta} = |F|^{1/p} G|^{1/p}.
\]
This shows the estimate $\|E\chi_F E\chi_G\|_{q/2} \lesssim \|f\|_p\|g\|_p$ for $\frac{1}{p} + \frac{2}{q} = 1, 4 > q > \frac{10}{3}$.

Now it remains to show (5.1). Clearly, $I_k^j$ and $I_k^j$ are contained in a ball of radius $2^{2-j}$ and dist$(I_k^j, I_k^j) \geq 2^{1-j}$. Hence, by a change of variables,
\[
E(f_k^j)(w) = 2^{-2j} E(f_k^j(\cdot, 2^{-j} \cdot))(w, 2^{-j}w_2, 2^{-j}w_3),
\]
\[
E(g_k^j)(w) = 2^{-2j} E(g_k^j(\cdot, 2^{-j} \cdot))(w, 2^{-j}w_2, 2^{-j}w_3).
\]

Then we see that $supp f_k^j(\cdot, 2^{-j} \cdot) \subset H_1 \times \tilde{I}_1$ and $g_k^j(\cdot, 2^{-j} \cdot) \subset H_2 \times \tilde{I}_2$ if dist$(\tilde{I}_1, \tilde{I}_2) \geq 2^{-1}$ and $\tilde{I}_1, \tilde{I}_2$ are contained in a ball of radius $\lesssim 2^3$. The assumption of Lemma 5.1 is satisfied with $f = f_k^j(\cdot, 2^{-j} \cdot)$ and $g = g_k^j(\cdot, 2^{-j} \cdot)$. Hence we may apply it to $E(f_k^j(\cdot, 2^{-j} \cdot)) E(f_k^j(\cdot, 2^{-j} \cdot))$ and get (5.1). This completes the proof. \(\Box\)

Once Lemma 5.2 is established, the usual argument in [30], used to deduce linear estimates from bilinear ones, works without modification. We omit the details. \(\Box\)

Acknowledgements. We would like to thank Andreas Seeger and the anonymous referee for bringing the reference [1] and [8] to our attention.

References

[1] D. Alvarez, Bounds for some Kakeya-type maximal functions. Ph.D. thesis, University of California, Berkeley, 1997.
[2] J.-G. Bak, S.H. Ham, Restriction of the Fourier transform to some complex curves, J. Math. Anal. Appl. 409 (2014), 1107–1127.
[3] J.-G. Bak, S.H. Lee, Restriction of the Fourier transform to a quadratic surface in $\mathbb{R}^n$, Math. Z. 247 (2004), no. 2, 409–422.
[4] J.-G. Bak, D.M. Oberlin, A. Seeger, Two endpoint bounds for generalized Radon transforms in the plane, Rev. Math. Iberoamericana 18 (2002), 231–247.
[5] J. Bak, J. Lee, S. Lee, Restriction of Fourier transforms to curves and related oscillatory integrals, *Amer. J. Math.* **131** (2009), 277–311.

[6] J. Bak, J. Lee, S. Lee, Restriction of Fourier transforms to curves: An endpoint estimate with affine arclength measure, *J. Reine Angew. Math.* **682** (2013), 167–205.

[7] J.-G. Bak, A. Seeger, Extensions of the Stein-Tomas theorem, *Math. Res. Lett.* **18** (2011), no. 4, 767–781.

[8] A. Banner, Restriction of the Fourier transform to quadratic submanifolds. Diss. Princeton University, 2002.

[9] J. Bennett, A. Carbery, T. Tao, On the multilinear restriction and Kakeya conjectures, *Acta Math.*, **196** (2006), no. 2, 261–302.

[10] J. Bergh and J. Löfström, *Interpolation spaces. An introduction*, Grundlehren der Mathematischen Wissenschaften, No. 223. Springer-Verlag, Berlin-New York, 1976.

[11] J. Bourgain, L. Guth, Bounds on oscillatory integral operators based on multilinear estimates, *Geom. Funct. Anal.* **21** (2011), no. 6, 1239–1295.

[12] M. Christ, Restriction of the Fourier transform to submanifolds of low codimension. Ph.D. thesis, University of Chicago, 1982.

[13] , On the restriction of the Fourier transform to curves: endpoint results and the degenerate case, *Trans. Amer. Math. Soc.* **287** (1985), 223–238.

[14] A. Greenleaf, A. Seeger, Oscillatory and Fourier integral operators with degenerate canonical relations, Special issue: Proceedings of the El Escorial Conference 2000, *Publications Mathématiques* (2002), 93–141.

[15] L. Guth, A restriction estimate using polynomial partitioning, *J. Amer. Math. Soc.* **29** (2016), no. 2, 371–413.

[16] S. Krantz, Function theory of several complex variables, 2nd ed., Wadsworth and Brooks, 1992.

[17] S.H. Lee, Improved bounds for Bochner-Riesz and maximal Bochner-Riesz operators, *Duke Math. J.* **122** (2004), no. 1, 205–232.

[18] , Bilinear restriction estimates for surfaces with curvatures of different signs, *Trans. Amer. Math. Soc.* **358** (2006), no. 8, 3511–3533.

[19] , On pointwise convergence of the solutions to Schrödinger equations in $\mathbb{R}^2$, *Int. Math. Res. Not.* **2006**, Art. ID 32597, 21pp.

[20] G. Mockenhaupt, Bounds in Lebesgue spaces of oscillatory integral operators, Habilitation thesis, Universität Siegen, 1996.

[21] A. Moyua, A. Vargas, L. Vega, Restriction theorems and maximal operators related to oscillatory integrals in $\mathbb{R}^3$, *Duke Math. J.* **96** (1999), 547–574.

[22] D.M. Oberlin, Some convolution inequalities and their applications, *Trans. Amer. Math. Soc.* **354** (2002), no. 6, 2541–2556.

[23] D.M. Oberlin, A restriction theorem for a $k$-surface in $\mathbb{R}^n$, *Canad. Math. Bull.* **48** (2005), no. 2, 260–266.

[24] R. Oberlin, Bounds for Kakeya-type maximal operators associated with $k$-planes, *Math. Res. Lett.* **14** (2007), no. 1, 87–97.

[25] E.M. Stein, *Harmonic Analysis*, Princeton University Press, 1993.

[26] B. Stovall, Uniform estimates for Fourier restriction to polynomial curves in $\mathbb{R}^d$, *Amer. J. Math.* **138** (2016), no. 2, 449–471.

[27] T. Tao, A sharp bilinear restrictions estimate for paraboloids, *Geom. Funct. Anal.* **13** (2003), no. 6, 1359–1384.

[28] T. Tao, A. Vargas, A bilinear approach to cone multipliers. I. Restriction estimates, *Geom. Funct. Anal.* **10** (2000), no. 1, 185–215.

[29] , A bilinear approach to cone multipliers. II. Restriction estimates, *Geom. Funct. Anal.* **10** (2000), no. 1, 216–258.

[30] T. Tao, A. Vargas, L. Vega, A bilinear approach to the restriction and Kakeya conjectures, *J. Amer. Math. Soc.* **11** (1998), 967–1000.
[31] A. Vargas, Restriction theorems for a surface with negative curvature, *Math. Z.* **249** (2005), no. 1, 97–111.

[32] T. Wolff, An improved bound for Kakeya type maximal functions, *Rev. Mat. Iberoamericana*, **11** (1995), no. 3, 651–674.

[33] _______, A mixed norm estimate for the X-ray transform, *Rev. Mat. Iberoamericana*, **14** (1998), no. 3, 561–600.

[34] _______, A sharp bilinear cone restriction estimate, *Ann. of Math.* **153** (2001), no. 3, 661–698.

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