A REMARK ON CONVOLUTION PRODUCTS FOR QUIVER HECKE ALGEBRAS

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Abstract. In this paper, we investigate a connection between convolution products for quiver Hecke algebras and tensor products for quantum groups. We give a categorification of the natural projection $\pi_{\lambda, \mu} : V_\lambda(\lambda)^\vee \otimes_A V_\lambda(\mu)^\vee \twoheadrightarrow V_\lambda(\lambda + \mu)^\vee$ sending the tensor product of the highest weight vectors to the highest weight vector in terms of convolution products. When the quiver Hecke algebra is symmetric and the base field is of characteristic 0, we obtain a positivity condition on some coefficients associated with the projection $\pi_{\lambda, \mu}$ and the upper global basis, and prove several results related to the crystal bases. We then apply our results to finite type $A$ using the homogeneous simple modules $S^T$ indexed by one-column tableaux $T$.

INTRODUCTION

Quiver Hecke algebras (or Khovanov-Lauda-Rouquier algebras) were introduced to give a categorification of the half of a quantum group [14, 15, 22]. These algebras have special graded quotients, called cyclotomic quiver Hecke algebras, which categorify irreducible highest weight integrable modules [6]. Cyclotomic quiver Hecke algebras of affine type $A$ are isomorphic to blocks of cyclotomic Hecke algebras [4, 22]. In this sense, quiver Hecke algebras are a vast generalization of Hecke algebras in the direction of categorification. When quiver Hecke algebras are symmetric and the base field is of characteristic 0, they have a geometric realization using quiver varieties which yields that the upper global basis (dual canonical basis) corresponds to the isomorphism classes of self-dual simple modules [23, 24], and they also give a monoidal categorification for some quantum cluster algebras [8]. The study of quiver Hecke algebras has been one of the main research themes on quantum groups in a viewpoint of categorification.

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Let $U_q(g)$ be the quantum group associated with a generalized Cartan matrix $A$ and $R(\beta)$ a quiver Hecke algebra associated with $A$ and $\beta \in Q_+$, where $Q_+$ is the positive cone of the root lattice $Q$. The convolution product $\circ$ and the restriction functor $\text{Res}$ give twisted bialgebra structures on the Grothendieck groups $[R\text{-proj}]$ and $[R\text{-gmod}]$, where $R\text{-proj}$ (resp. $R\text{-gmod}$) is the category of finitely generated projective (resp. finite-dimensional) graded $R$-modules. It was proved that there are bialgebra isomorphisms $\ U_q^- (g) \simeq [R\text{-proj}]$ and $U_q^- (g)^\vee \simeq [R\text{-gmod}]$, where $U_q^- (g)^\vee$ is the dual of $U_q^- (g)$. Similarly, cyclotomic quiver Hecke algebras give a categorification for irreducible highest weight modules. For a dominant integral weight $\Lambda$, let $V_q(\Lambda)$ be the irreducible highest weight $U_q(g)$-module with highest weight $\Lambda$ and $R_\Lambda(\beta)$ the cyclotomic quiver Hecke algebras of $R(\beta)$ corresponding to $\Lambda$. It was proved that there exist $U_A^- (g)$-module isomorphisms $V_A(\Lambda)^\vee \simeq [R_\Lambda\text{-proj}]$ and $V_A(\Lambda)^\vee \simeq [R_\Lambda\text{-gmod}]$. Here $V_A(\Lambda)^\vee$ is the dual of $V_A(\Lambda)$, and $R_\Lambda\text{-proj}$ (resp. $R_\Lambda\text{-gmod}$) is the category of finitely generated projective (resp. finite-dimensional) graded $R_\Lambda$-modules.

In this paper, we investigate a connection between convolution products for quiver Hecke algebras and tensor products for quantum groups. For dominant integral weights $\lambda$ and $\mu$, we consider a chain of homomorphisms

$$U^-_A (g) \xrightarrow{\Phi_\lambda} U_A^- (g) \otimes U_A^- (g) \xrightarrow{p_\lambda \otimes p_\mu} V_A(\lambda) \otimes V_A(\mu),$$

where $\Phi_\lambda$ is defined in (3.1) and $p_\lambda$ is the natural projection defined in (1.3). We interpret the above homomorphisms in terms of the restriction functor $\text{Res}$ in a viewpoint of categorification, which yields our main result, that is a categorification of the surjective homomorphism

$$\pi_{\lambda, \mu} : V_A(\lambda)^\vee \otimes V_A(\mu)^\vee \rightarrow V_A(\lambda + \mu)^\vee,$$

where $\pi_{\lambda, \mu}$ is the natural projection sending the tensor product of the highest weight vectors to the highest weight vector. More precisely, we prove that

$$\pi_{\lambda, \mu} ([M] \otimes [N]) = q^{(\beta_2, \lambda)} [M \circ N]$$

for $M \in R^\lambda(\beta_1)$-gmod and $N \in R^\mu(\beta_2)$-gmod (Corollary 3.4). For types $A_\infty$ and $A^{(1)}_n$, the same connection was investigated in terms of affine Hecke algebras and upper global bases in [18]. Note also that a categorification of tensor products of irreducible highest weight modules was studied in [25].

When $R$ is symmetric, we study further with upper global bases and crystal bases. We assume that $R$ is symmetric and the base field is of characteristic 0. Let $B(\lambda)$ be a crystal of $V_q(\lambda)$ and $b_\lambda$ the highest weight vector of $B(\lambda)$. For $b \in B(\lambda)$, we denote by $G_A^{\text{up}}(b)$ the member of the upper global base $B_{\text{up}}(\lambda)$ corresponding to $b$ and by $L(b)$ the self-dual simple
for $A_{b''}(q) \in \mathbb{Q}(q)$, then we prove that
\[A_{b''}(q) = q^{(\beta_2, \lambda)}[L(b) \circ L(b') : L(b'')]_q,\]
which yields the positivity $A_{b''}(q) \in \mathbb{Z}_{\geq 0} [q^\pm]$ (Corollary 3.6). Let $C_{\lambda_1, \ldots, \lambda_r}$ be the connected component of $B(\lambda_1) \otimes \cdots \otimes B(\lambda_r)$ containing $b_{\lambda_1} \otimes \cdots \otimes b_{\lambda_r}$. We then prove that if

(i) $b_1 \otimes \cdots \otimes b_r \in C_{\lambda_1, \ldots, \lambda_r}$,
(ii) $\text{hd}(L(b_1) \circ \cdots \circ L(b_r))$ is simple,

then
\[\text{hd}(L(b_1) \circ \cdots \circ L(b_r)) \simeq L(b_1 \otimes \cdots \otimes b_r)\]
up to a grading shift (Theorem 3.7). When applying this result to a pair of real simple $R$-modules, we have more results. Suppose that $L(b_1)$ and $L(b_2)$ are real. We then compute the degree $\Lambda(L(b_1), L(b_2))$ of the R-matrix and related quantities in terms of weights and dominant integral weights (Corollary 3.8). Moreover, if there is $b_1' \otimes b_2' \in C_{\lambda_1', \lambda_2'}$ such that $\lambda_1 + \lambda_2 = \lambda_1' + \lambda_2'$ and $b_1 \otimes b_2$ is crystal equivalent to $b_1' \otimes b_2'$, then we show that
\[\text{Hom}_{R-\text{gmod}}(q^d L(b_1) \circ L(b_2), L(b_2') \circ L(b_1')) \simeq k,\]
where $d$ is given in Theorem 3.10. As a consequence, we have a couple of equivalent conditions to the condition that $L(b_1)$ and $L(b_2)$ strongly commute (Corollary 3.11).

As an application, we apply our results to finite type $A$. We compute several examples with the homogeneous simple $R^{\lambda_i}$-modules $S^T$ indexed by one-column tableaux $T$, which were introduced in [4, 17]. In Theorem 4.5, we also explain that the graded decomposition number between $S^{T_r} \circ \cdots \circ S^{T_1}$ and simple $R$-modules in terms of the Kazhdan-Lusztig polynomials by using the results on the entries of the transition matrix between the standard monomial and upper global bases of the irreducible $U_q(\mathfrak{gl}_n)$-modules given in [2]. Note that these entries also appear in composition multiplicities of the standard modules for the finite $W$-algebras/shifted Yangians [3].

The paper is organized as follows. In Section 1, we review global bases and crystal bases of quantum groups. In Section 2, we recall the categorification using quiver Hecke algebras. In Section 3, we investigate a connection between tensor products and convolution products, and provide the main theorems. In Section 4, we apply our main results to finite type $A$. 

$R$-module corresponding to $b$. For $b \in B(\lambda)$ and $b' \in B(\mu)$, if we write
\[\pi_{\lambda, \mu}(G_{\lambda}(b) \otimes G_{\mu}(b')) = \sum_{b'' \in B(\lambda+\mu)} A_{b''}(q)G_{\lambda+\mu}(b'')\]
1. Quantum groups

1.1. Quantum groups. Let \( I \) be an index set. A Cartan datum \((A, P, \Pi, P^\vee, \Pi^\vee)\) consists of

1. a matrix \( A = (a_{ij})_{i,j \in I} \), called the symmetrizable generalized Cartan matrix, satisfying
   
   (a) \( a_{ii} = 2 \) for \( i \in I \) and \( a_{ij} \in \mathbb{Z}_{\leq 0} \) for \( i \neq j \),
   
   (b) \( a_{ij} = 0 \) if and only if \( a_{ji} = 0 \),
   
   (c) there exists a diagonal matrix \( D = \text{diag}(d_i \mid i \in I) \) such that \( DA \) is symmetric, and \( d_i \)'s are relatively prime positive integers,

2. a free abelian group \( P \), called the weight lattice,

3. \( \Pi = \{ \alpha_i \mid i \in I \} \subset P \), called the set of simple roots,

4. \( P^\vee = \text{Hom}_\mathbb{Z}(P, \mathbb{Z}) \), called the coweight lattice,

5. \( \Pi^\vee = \{ \beta_i \in P^\vee \mid i \in I \} \), called the set of simple coroots, satisfying

   (a) \( \langle h_i, \alpha_j \rangle = a_{ij} \) for \( i, j \in I \),

   (b) \( \Pi \) is linearly independent over \( \mathbb{Q} \),

   (c) for each \( i \in I \), there exists \( \Lambda_i \in P \), called the fundamental weight, such that

   \[
   \langle h_i, \Lambda_j \rangle = \delta_{ji} \text{ for all } j \in I.
   \]

Set \( \mathfrak{h} = \mathbb{Q} \otimes \mathbb{Z} P \). We fix a nondegenerate symmetric bilinear form \((\cdot, \cdot)\) on \( \mathfrak{h}^* \) satisfying

\[
(\alpha_i, \alpha_j) = d_i a_{ij} \quad (i, j \in I), \quad \text{and} \quad \langle h_i, \lambda \rangle = \frac{2\langle \alpha_i, \lambda \rangle}{(\alpha_i, \alpha_i)} \quad (\lambda \in \mathfrak{h}^*, \ i \in I).
\]

Set

\[
\mathbb{Q} = \bigoplus_{i \in I} \mathbb{Z} \alpha_i, \quad \mathbb{Q}_+ = \bigoplus_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i, \quad \mathbb{Q}_- = -\mathbb{Q}_+.
\]

We call \( \mathbb{Q} \) (respectively, \( \mathbb{Q}_+ \)) the root lattice (respectively, the positive root lattice). We define

\( \text{ht}(\beta) = \sum_{i \in I} k_i \) for \( \beta = \sum_{i \in I} k_i \alpha_i \in \mathbb{Q}_+ \), and denote by \( P^+ = \{ \lambda \in P \mid \langle h_i, \lambda \rangle \geq 0 \text{ for all } i \in I \} \) the set of dominant integral weights.

Let \( U_q(\mathfrak{g}) \) be the quantum group associated with the Cartan datum \((A, P, P^\vee \Pi, \Pi^\vee)\), which is a \( \mathbb{Q}(q) \)-algebra generated by \( f_i, e_i \) \((i \in I)\) and \( q^h \) \((h \in P)\) with certain defining relations (see [5, Chater 3] for details). Let \( U_q(\mathfrak{g})_{\beta} = \{ x \in U_q(\mathfrak{g}) \mid q^h x q^{-h} = q^{(h, \beta)} x \text{ for all } h \in P^\vee \} \). Then we have \( U_q(\mathfrak{g}) = \bigoplus_{\beta \in \mathbb{Q}^\vee} U_q(\mathfrak{g})_{\beta} \) and \( U_q^-(\mathfrak{g}) = \bigoplus_{\beta \in \mathbb{Q}_-} U_q(\mathfrak{g})_{\beta} \). In the sequel, for a \( P \)-graded vector space \( V = \bigoplus_{\mu \in P} V_\mu \), we denote by

\[
V^\vee := \bigoplus_{\mu \in P} \text{Hom}_{\mathbb{Q}(q)}(V_\mu, \mathbb{Q}(q))
\]

the restricted dual of \( V \). We often write \( \langle v, f \rangle = f(v) \) for \( f \in V^\vee \) and \( v \in V \) in pairing notation. The \( \mathbb{Q}(q) \)-algebra homomorphisms (comultiplications) \( \Delta_{\pm} : U_q(\mathfrak{g}) \rightarrow U_q(\mathfrak{g}) \otimes_{\mathbb{Q}(q)} U_q(\mathfrak{g}) \).
$U_q(\mathfrak{g})$ are defined by

\[
\Delta_+(f_i) = f_i \otimes t_i^{-1} + 1 \otimes f_i, \quad \Delta_+(e_i) = e_i \otimes 1 + t_i \otimes e_i, \quad \Delta_+(q^h) = q^h \otimes q^h,
\]

\[
\Delta_-(f_i) = f_i \otimes 1 + t_i \otimes f_i, \quad \Delta_-(e_i) = e_i \otimes t_i^{-1} + 1 \otimes e_i, \quad \Delta_-(q^h) = q^h \otimes q^h,
\]

where $t_i = q^{d_i}$. For homogeneous elements $x, y, z, w \in U^{-}_q(\mathfrak{g})$, we define

\[(x \otimes y) \cdot (z \otimes w) = q^{-(\text{wt}(y), \text{wt}(z))}xz \otimes yw,
\]

which gives another $\mathbb{Q}(q)$-algebra structure on $U^{-}_q(\mathfrak{g}) \otimes_{\mathbb{Q}(q)} U^{-}_q(\mathfrak{g})$. Then we have a $\mathbb{Q}(q)$-algebra homomorphism $\Delta_n : U^{-}_q(\mathfrak{g}) \to U^{-}_q(\mathfrak{g}) \otimes_{\mathbb{Q}(q)} U^{-}_q(\mathfrak{g})$ with respect to the above multiplication, which is given by

\[
\Delta_n(f_i) = 1 \otimes f_i + f_i \otimes 1 \quad \text{for } i \in I.
\]

We often write $\Delta_n(x) = x_{(1)} \otimes x_{(2)}$ for $x \in U^{-}_q(\mathfrak{g})$ in Sweedler’s notation. We set

\[
U^{-}_q(\mathfrak{g})^\vee := \bigoplus_{\beta \in \mathbb{Q}^-} U^{-}_q(\mathfrak{g})^\vee_{\beta},
\]

and define, for $f, g \in U^{-}_q(\mathfrak{g})^\vee$ and $u \in U^{-}_q(\mathfrak{g})$,

\[
(f \cdot g)(u) := (f \otimes g)(\Delta_n(u)) = f(u_{(1)})f(u_{(2)}), \quad \text{where } \Delta_n(u) = u_{(1)} \otimes u_{(2)}.
\]

Let $A = \mathbb{Z}[q, q^{-1}]$, and $U^{-}_A(\mathfrak{g})$ denote the subalgebra of $U^{-}_q(\mathfrak{g})$ generated by $f_i^{(n)} := f_i^n /[n]_![n]_!$ for $i \in I$ and $n \in \mathbb{Z}_{\geq 0}$, where

\[
q_i = q^{d_i}, \quad [n]_i = \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}, \quad [n]_! = \prod_{k=1}^{n} [k]_i.
\]

Let

\[
U^{-}_A(\mathfrak{g})^\vee := \{ f \in U^{-}_q(\mathfrak{g})^\vee : f(U^{-}_A(\mathfrak{g})) \subset A \},
\]

which forms an $A$-subalgebra of $U^{-}_q(\mathfrak{g})^\vee$.

A $U_q(\mathfrak{g})$-module is called integrable if $M = \bigoplus_{\lambda \in \mathcal{P}} M_\lambda$, where $M_\lambda := \{ v \in M : q^h v = q^{(h, \lambda)_+} v \}$ for all $h \in P^\vee$, dim$_{\mathbb{Q}(q)} M_\mu < \infty$, and the actions of $e_i$ and $f_i$ on $M$ are locally nilpotent for all $i \in I$. We denote by $\mathcal{O}_{\text{int}}(\mathfrak{g})$ the category of integrable left $U_q(\mathfrak{g})$-modules $M$ such that there exist finitely many $\lambda_1, \ldots, \lambda_m$ with $\text{wt}(M) \subset \bigcup_j (\lambda_j - Q_+^+)$. Let $t$ be the $\mathbb{Q}(q)$-algebra anti-involution on $U_q(\mathfrak{g})$ given by $t(f_i) = e_i$, $t(e_i) = f_i$ and $t(q^h) = q^h$ for $i \in I$ and $h \in P^\vee$. Then the restricted dual $M^\vee$ of an integrable module $M$ has the left $U_q(\mathfrak{g})$-module structure defined by

\[(xf)(m) := f(t(x)m) \quad \text{for } f \in M^\vee, m \in M \text{ and } x \in U_q(\mathfrak{g}).\]
Note that, using pairing notation, we have $\langle m, xf \rangle = \langle t(x)m, f \rangle$. For $\lambda \in \mathbb{P}^+$, let $V_q(\lambda)$ be the irreducible highest weight module of highest weight $\lambda$, and let $u_\lambda$ denote a highest weight vector of $V_q(\lambda)$. Note that, as $U_q(\mathfrak{g})$-modules,

$$V_q(\lambda) \simeq V_q(\lambda)^\vee. \tag{1.2}$$

We set

$$V_\lambda(\lambda) := U_\lambda(\mathfrak{g})u_\lambda, \quad V_\lambda(\lambda)^\vee := \{ \phi \in V_q(\lambda)^\vee \mid \phi(V_\lambda(\lambda) \subset A) \}. \tag{1.3}$$

Consider the natural projection map $p_\lambda : U_q^- (\mathfrak{g}) \rightarrow V_q(\lambda)$ given by $p_\lambda(x) = xu_\lambda$. Restricting to $U_\lambda(\mathfrak{g})$, we have

$$p_\lambda|_{U_\lambda(\mathfrak{g})} : U_\lambda(\mathfrak{g}) \rightarrow V_\lambda(\lambda). \tag{1.4}$$

We simply write $p_\lambda$ for $p_\lambda|_{U_\lambda(\mathfrak{g})}$. Restricting the dual map $i_\lambda := p_\lambda^\vee : V_q(\lambda)^\vee \hookrightarrow U_q^- (\mathfrak{g})$ to $V_\lambda(\lambda)^\vee$, we have the following

$$i_\lambda : V_\lambda(\lambda)^\vee \hookrightarrow U_\lambda(\mathfrak{g})^\vee \subset U_q^- (\mathfrak{g})^\vee. \tag{1.5}$$

For $U_q(\mathfrak{g})$-modules $M$ and $N$, we define $U_q(\mathfrak{g})$-module actions on $M \otimes_{\mathbb{Q}(q)} N$ by

$$x \cdot (m \otimes n) := \Delta_{\pm}(x)(m \otimes n) \quad \text{for } x \in U_q(\mathfrak{g}) \text{ and } m \otimes n \in M \otimes_{\mathbb{Q}(q)} N,$$

which is denoted by $M \otimes_{\pm} N$ as a $U_q(\mathfrak{g})$-module. Note that, as a $U_q(\mathfrak{g})$-module,

$$M^\vee \otimes_{\pm} N^\vee \simeq (M \otimes_{\pm} N)^\vee, \tag{1.5}$$

where the isomorphisms are given by

$$\langle f \otimes g, v \otimes w \rangle := f(v)g(w) \quad \text{for } f \in M^\vee, g \in N^\vee, v \in M, \text{ and } w \in N.$$

### 1.2. Global bases and crystal bases.

Let us recall the notions of global bases and crystal bases briefly (see [10, 11], [5, Chap. 4 and 6] for details). Let $A_0$ be the subring of $\mathbb{Q}(q)$ consisting of rational functions which are regular at $q = 0$. For each $i \in I$, let us denote by $\hat{f}_i$ and $\hat{e}_i$ the Kashiwara operators on $U_q^- (\mathfrak{g})$ given in [10, (3.5.1)], and set

$$L(\infty) := \sum_{t \in \mathbb{Z}_{\geq 0}, i_1, \ldots, i_t \in I} A_0 \hat{f}_{i_1} \cdots \hat{f}_{i_t} 1 \subset U_q^- (\mathfrak{g}), \quad \overline{L(\infty)} := \{ \overline{x} \in U_q^- (\mathfrak{g}) \mid x \in L(\infty) \},$$

$$B(\infty) := \{ \hat{f}_{i_1} \cdots \hat{f}_{i_t} 1 \mod qL(\infty) \mid l \in \mathbb{Z}_{\geq 0}, i_1, \ldots, i_t \in I \} \subset L(\infty)/qL(\infty),$$

where $^{-} : U_q(\mathfrak{g}) \xrightarrow{\sim} U_q(\mathfrak{g})$ is the $\mathbb{Q}$-algebra automorphism given by $\overline{e}_i = e_i$, $\overline{f}_i = f_i$, $q^\overline{e} = q^{-h}$, and $\overline{f} = q^{-1}$. Then the triple $(U \otimes_{\mathbb{Z}} U_\lambda(\mathfrak{g})), L(\infty), \overline{L(\infty)}$ is balanced; i.e., the natural map $(U \otimes_{\mathbb{Z}} U_\lambda(\mathfrak{g})) \cap L(\infty) \cap \overline{L(\infty)} \rightarrow L(\infty)/qL(\infty)$ is a $\mathbb{Q}$-linear isomorphism. Denote the inverse by $G_{\text{low}}$. The set

$$B_{\text{low}}(\infty) := \{ G_{\text{low}}(b) \in U_\lambda(\mathfrak{g}) \mid b \in B(\infty) \}$$
forms an $\mathbb{A}$-basis of $U_{\mathbb{A}}^{\sim}(\mathfrak{g})$ and called the lower global basis of $U_{\mathfrak{g}}^{\sim}$. Then we have the dual basis
\[ B^{\text{up}}(\infty) := \{ G^{\text{up}}(b) \in U_{\mathbb{A}}^{\sim}(\mathfrak{g})^{\vee} \mid b \in B(\infty) \}, \]
which forms an $\mathbb{A}$-basis of $U_{\mathbb{A}}^{\sim}(\mathfrak{g})^{\vee}$ called the upper global basis of $U_{\mathfrak{g}}^{\sim}$; i.e.,
\[ \langle G^{\text{low}}(b), G^{\text{up}}(b') \rangle = \delta_{b,b'} \text{ for } b, b' \in B(\infty). \]

Let $\lambda \in P^+$ and set
\[ L^{\text{low}}(\lambda) := \sum_{l \in \mathbb{Z}_{\geq 0}, i_1, \ldots, i_l \in I} \mathbb{A}_0 \tilde{f}_{i_1} \cdots \tilde{f}_{i_l} u_{\lambda} \subset V_{\lambda}(\lambda), \quad \overline{L^{\text{low}}(\lambda)} := \{ \overline{v} \in V_{\lambda}(\lambda) \mid v \in L^{\text{low}}(\lambda) \} \]

\[ B(\lambda) := \{ \tilde{f}_{i_1} \cdots \tilde{f}_{i_l} u_{\lambda} \mod q L^{\text{low}}(\lambda) \mid l \in \mathbb{Z}_{\geq 0}, i_1, \ldots, i_l \in I \} \subset L^{\text{low}}(\lambda)/q L(\lambda), \]
where $\gamma : V_{\lambda}(\lambda) \to V_{\lambda}(\lambda)$ is the $\mathbb{Q}$-linear automorphism given by $\overline{x u_{\lambda}} := \overline{x} u_{\lambda}$. Denote the inverse of the $\mathbb{Q}$-linear isomorphism $(\mathbb{Q} \otimes_{\mathbb{Z}} V_{\lambda}(\lambda)) \cap L^{\text{low}}(\lambda) \cap \overline{L^{\text{low}}(\lambda)} \to L^{\text{low}}(\lambda)/q L^{\text{low}}(\lambda)$ by $G^{\text{low}}_{\lambda}$. Then the set
\[ B^{\text{low}}(\lambda) := \{ G^{\text{low}}_{\lambda}(b) \mid b \in B(\lambda) \} \]
forms an $\mathbb{A}$-basis of $V_{\lambda}(\lambda)$ called the lower global basis of $V_{\lambda}(\lambda)$. Similarly, we have the dual basis
\[ B^{\text{up}}(\lambda) := \{ G^{\text{up}}_{\lambda}(b) \in V_{\lambda}(\lambda)^{\vee} \mid b \in B(\lambda) \}, \]
which forms an $\mathbb{A}$-basis of $V_{\lambda}(\lambda)^{\vee}$ called the upper global basis of $V_{\lambda}(\lambda)$; i.e.,
\[ \langle G^{\text{low}}_{\lambda}(b), G^{\text{up}}_{\lambda}(b') \rangle = \delta_{b,b'} \text{ for } b, b' \in B(\lambda). \]

The global basis of $U_{\mathfrak{g}}^{\sim}$ and that of $V_{\lambda}(\lambda)$ are compatible in the following sense:

**Theorem 1.1.** ([10]) We have
\[ p_{\lambda} L(\infty) = L^{\text{low}}(\lambda). \]

The induced surjective map $\overline{p}_{\lambda} : L(\infty)/q L(\infty) \to L^{\text{low}}(\lambda)/q L^{\text{low}}(\lambda)$ is a bijection between $\{ b \in B(\infty) \mid \overline{p}_{\lambda}(b) \neq 0 \}$ and $B(\lambda)$. If $\overline{p}_{\lambda}(b) \neq 0$, then $p_{\lambda} G^{\text{low}}_{\lambda}(b) = G^{\text{low}}_{\lambda}(b) u_{\lambda} = G^{\text{low}}_{\lambda}(\overline{p}_{\lambda}(b))$. Taking duals, the map $i_{\lambda}$ induces an injective $\mathbb{A}_0$-linear map from $L^{\text{up}}(\lambda)$ to $L^{\text{up}}(\infty)$ and for $b \in B(\lambda)$ we have
\[ i_{\lambda} G^{\text{up}}_{\lambda}(b) = G^{\text{up}}_{\lambda}(\overline{i}_{\lambda}(b)), \]
where $\overline{i}_{\lambda}$ denotes the inverse of $\overline{p}_{\lambda}$ on $B(\lambda)$.

In the sequel, we will omit the maps $i_{\lambda}$, $\overline{i}_{\lambda}$ and identify $G^{\text{up}}_{\lambda}(\overline{i}_{\lambda}(b))$ with $G^{\text{up}}_{\lambda}(b)$ if there is no afraid of confusion.

For $M \in \mathcal{O}_{\text{int}}(\mathfrak{g})$, a pair $(L, B)$ of a free $\mathbb{A}_0$-submodule $L$ of $M$ and a $\mathbb{Q}$-basis $B$ of $L/q L$ is called a lower crystal base of $M$ if $L$ and $B$ are stable under the Kashiwara operator $\tilde{e}_i, \tilde{f}_i$ with some conditions (see [5, Chap. 4]). The set $B$ equips with an $I$-colored oriented graph structure by setting $b \overset{i}{\longrightarrow} \tilde{f}_i b$ for $b, \tilde{f}_i(b) \in B$. When $(L_1, B_1)$ and $(L_2, B_2)$ are crystal bases of
$M_1$ and $M_2$ in $\mathcal{O}_{\text{int}}(g)$, the pair $(L_1 \otimes_{k_0} L_2, B_1 \otimes B_2)$ is a crystal basis of $M_1 \otimes M_2$, where $B_1 \otimes B_2 := \{ b_1 \otimes b_2 \mid b_i \in B_i, \ (i = 1, 2) \} \subset L_1 \otimes_{k_0} L_2 / q(L_1 \otimes_{k_0} L_2)$ ([10]). The Kashiwara operator on $B_1 \otimes B_2$ can be described explicitly as follows.

**Definition 1.2.** For $i \in I$, $b_1 \in B_1$ and $b_2 \in B_2$,

$$\text{wt}(b_1 \otimes b_2) = \text{wt}(b_1) + \text{wt}(b_2),$$

$$\varepsilon_i(b_1 \otimes b_2) = \max(\varepsilon_i(b_1), \varepsilon_i(b_2) - \langle h_i, \text{wt}(b_1) \rangle),$$

$$\varphi_i(b_1 \otimes b_2) = \max(\varphi_i(b_1) + \langle h_i, \text{wt}(b_2) \rangle, \varphi_i(b_2)).$$

$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i(b_1) \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ b_1 \otimes \tilde{e}_i(b_2) & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases}$$

$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i(b_1) \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes \tilde{f}_i(b_2) & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2). \end{cases}$$

In particular, the pair $(L^{\text{low}}(\lambda_1) \otimes_{k_0} \cdots \otimes_{k_0} L^{\text{low}}(\lambda_r), B(\lambda_1) \otimes \cdots \otimes B(\lambda_r))$ is a crystal basis of $V_q(\lambda_1) \otimes \cdots \otimes V_q(\lambda_r)$ for $\lambda_k \in \mathbb{P}^+ \ (k = 1, \ldots, r)$. We set

$$C_{\lambda_1, \ldots, \lambda_r} := \text{the connected component of } B(\lambda_1) \otimes \cdots \otimes B(\lambda_r) \text{ containing } b_{\lambda_1} \otimes \cdots \otimes b_{\lambda_r}.$$

Let $B, B'$ be crystals. For $b \in B$ and $b' \in B'$, we say that $b$ is crystal equivalent to $b'$, which is denoted by $b \simeq b'$, if there exists an isomorphism between $C(b)$ and $C(b')$ sending $b$ to $b'$, where $C(b)$ and $C(b')$ are the connected components of $B$ and $B'$ containing $b$ and $b'$ respectively.

We recall the following proposition ([9, 10], [16, Proposition 3.29]), which plays an important role later.

**Proposition 1.3.** Let $\lambda_k \in \mathbb{P}^+$ and $b_k \in B(\lambda_k)$ for $k = 1, \ldots, r$, and set $\lambda = \sum_{k=1}^r \lambda_k$. If $b_1 \otimes \cdots \otimes b_r \in C_{\lambda_1, \ldots, \lambda_r}$, then

$$\pi_{\lambda_1, \ldots, \lambda_r}(G_{\lambda_1}^{\text{up}}(b_1) \otimes \cdots \otimes G_{\lambda_r}^{\text{up}}(b_r)) \equiv G_{\lambda}^{\text{up}}(b_1 \otimes \cdots \otimes b_r) \mod qL^{\text{up}}(\lambda),$$

where $\pi_{\lambda_1, \ldots, \lambda_r} : V_q(\lambda_1)^\vee \otimes \cdots \otimes V_q(\lambda_r)^\vee \to V_q(\lambda)^\vee$ is the dual map of the $U_q(g)$-module homomorphism $\iota_{\lambda_1, \ldots, \lambda_r} : V_q(\lambda) \to V_q(\lambda_1) \otimes \cdots \otimes V_q(\lambda_r)$ which is given by $\iota_{\lambda_1, \ldots, \lambda_r}(u_\lambda) = (u_{\lambda_1} \otimes \cdots \otimes u_{\lambda_r})$.

2. Quiver Hecke algebras

2.1. Quiver Hecke algebras. Let $k$ be a field and let $(A, P, \Pi, P^\vee, \Pi^\vee)$ be a Cartan datum. Let us take a family of polynomials $(Q_{i,j})_{i,j \in I}$ in $k[u, v]$ which satisfies

1. $Q_{i,j}(u, v) = Q_{j,i}(v, u)$ for any $i, j \in I$,
We denote by $S_{\beta}$ and $i$ the transposition of $Q_{\beta}$. For a $Z$-algebra generated by the elements $\{e_1, \ldots, e_n\}$ with $\deg e_i = 0$, $\deg x_k = (\alpha_{v_k}, \alpha_{v_k})$, and $\deg \tau_1 = -(\alpha_{v_k}, \alpha_{v_k+1})$.

For a $Z$-graded algebra $A$ over $k$, let us denote by $A$-$\text{Proj}$ the category of graded left modules over $A$ with homogeneous homomorphism. Let us denote by $A$-$\text{proj}$ (respectively,
A-gmod) the full subcategory of \(A\)-Mod consisting of finitely generated projective graded \(A\)-modules (respectively, finite-dimensional graded \(A\)-modules). Set \(R\)-proj := \(\bigoplus_{\beta \in \mathbb{Q}_+} R(\beta)\)-proj and \(R\)-gmod := \(\bigoplus_{\beta \in \mathbb{Q}_+} R(\beta)\)-gmod.

For \(M \in R(\beta)\)-gmod, the \(q\)-character of \(M\) is given by
\[
\text{ch}_q(M) = \sum_{\nu \in I^\beta} \dim_q(e(\nu)M)\nu,
\]
where \(\dim_q V := \sum_{n \in \mathbb{Z}} q^n \dim V_n\) for a \(\mathbb{Z}\)-graded vector space \(V = \bigoplus_{k \in \mathbb{Z}} V_k\). For an \(R(\beta)\)-module \(M = \bigoplus_{k \in \mathbb{Z}} M_k\), we define \(qM = \bigoplus_{k \in \mathbb{Z}} (qM)_k\), where \((qM)_k := M_{k-1}\) for all \(k \in \mathbb{Z}\).

We call \(q\) the grading shift functor on the category of graded \(R(\beta)\)-modules.

For \(\beta, \gamma \in \mathbb{Q}_+\), we set \(e(\beta, \gamma) := \sum_{\nu_1 \in I^\beta, \nu_2 \in I^\gamma} e(\nu_1 * \nu_2)\), where \(\nu_1 * \nu_2\) is the concatenation of \(\nu_1\) and \(\nu_2\). The induction functor is defined as
\[
\text{Ind}_{\beta, \gamma} : R(\beta) \otimes R(\gamma)\text{-Mod} \to R(\beta + \gamma)\text{-Mod}
\]
\[
V \mapsto R(\beta + \gamma)e(\beta, \gamma) \otimes_{R(\beta) \otimes R(\gamma)} V.
\]

We often write
\[
M \circ N = \text{Ind}_{\beta, \gamma}(M \otimes N),
\]
for \(M \in R(\beta)\)-Mod and \(N \in R(\gamma)\)-Mod, which is called the convolution product of \(M\) and \(N\). We denote by \(M \triangledown N\) and \(M \triangledown N\) the head and the socle of \(M \circ N\), respectively. For simple modules \(M, N \in R\)-gmod, we say that \(M\) and \(N\) strongly commute if \(M \circ N\) is simple. We say that a simple module \(L\) is real if \(L\) strongly commutes with itself. The restriction functor
\[
\text{Res}_{\beta, \gamma} : R(\beta + \gamma)\text{-Mod} \to R(\beta) \otimes R(\gamma)\text{-Mod}
\]
is defined as
\[
\text{Res}_{\beta, \gamma}(W) := e(\beta, \gamma)W \simeq e(\beta, \gamma)R(\beta + \gamma) \otimes_{R(\beta + \gamma)} W
\]
for \(W \in R(\beta + \gamma)\)-Mod. Note that \(\text{Res}_{\beta, \gamma}\) is a right adjoint of \(\text{Ind}_{\beta, \gamma}\).

Let \(\psi\) be the algebra anti-involution on \(R(\beta)\) which fixes the generators. Then \(\psi\) gives a left \(R\)-module structure on the linear dual \(M^* = \text{Hom}_k(M, k)\) of a module \(M \in R\)-gmod. We call \(M\) is self-dual if \(M \simeq M^*\). The Khovanov-Lauda paring \((\cdot, \cdot)(14, (2.43))\)
\[
(\cdot, \cdot) : [R\text{-proj}] \times [R\text{-gmod}] \to \mathbb{A}
\]
is defined by
\[
([P], [M]) := \dim_q P^\psi \otimes_R M
\]
for \( P \in \text{R-proj} \) and \( M \in \text{R-gmod} \). Here \( P^\psi \) is the right \( R \)-module induced from \( P \) with the actions twisted by \( \psi \). It also defines a paring \([\text{R-proj}] \otimes [\text{R-proj}] \times [\text{R-gmod}] \otimes [\text{R-gmod}] \to A\) by

\[
([X_1] \otimes [X_2], [Y_1] \otimes [Y_2]) := ([X_1], [Y_1])([X_2], [Y_2])
\]

for \( X_1, X_2 \in [\text{R-proj}], Y_1, Y_2 \in [\text{R-gmod}] \).

**Lemma 2.1.** ([14, Proposition 3.3]) For \( P \in \text{R-proj} \) and \( M, N \in \text{R-gmod} \), we have

\[
([P], [M \circ N]) = ([\text{Res}(P)], [M] \otimes [N]).
\]

The Grothendieck groups \([\text{R-proj}]\) and \([\text{R-gmod}]\) admit \( A \)-algebra structures with the multiplication given by the functor \( \text{Ind} := \bigoplus_{\beta, \gamma} \text{Ind}_{\beta, \gamma} \).

**Theorem 2.2.** ([14, 15, 22]) There exists an \( A \)-algebra isomorphism

\[
\gamma : U^-_A(\mathfrak{g}) \xrightarrow{\sim} [\text{R-proj}]
\]

We identify \([\text{R-gmod}]\) with \([\text{R-proj}]^\vee\) as an \( A \)-module using the above pairing \((, ,)\). Then The dual map of \( \gamma \)

\[
\gamma^\vee : [\text{R-gmod}] \xrightarrow{\sim} U^-_A(\mathfrak{g})^\vee
\]

is an \( A \)-algebra isomorphism.

Note that, for \( u \in U^-_A(\mathfrak{g}) \) and \( M \in \text{R-gmod} \),

\[
(\gamma(u), [M]) = \langle u, \gamma^\vee[M] \rangle.
\]

In the sequel, we identify \( U^-_A(\mathfrak{g}) \) (respectively, \( U^-_A(\mathfrak{g})^\vee \)) with \([\text{R-proj}]\) (respectively, \([\text{R-gmod}]\)) via the isomorphism \( \gamma \) (respectively, \( \gamma^\vee \)).

**Lemma 2.3.** ([14, Proposition 3.2]) The functor \( \text{Res} \) categorify \( \Delta_\lambda \). That is, under the isomorphism \( \gamma \), the map \( \text{Res} \) on \([\text{R-proj}]\) corresponds to the map \( \Delta_\lambda \) on \( U^-_A(\mathfrak{g}) \).

### 2.2. Cyclotomic quotients.

For \( \Lambda \in \mathbb{P}^+ \) and \( \beta \in \mathbb{Q}_+ \), let \( I^\Lambda_\beta \) be the two-sided ideal of \( R(\beta) \) generated by the elements \( \{x^{(h_{\nu, \mu}, \Lambda)}e(\nu) \mid \nu \in I^\beta\} \). The **cyclotomic quiver Hecke algebra** \( R^\Lambda(\beta) \) is defined as the quotient algebra

\[
R^\Lambda(\beta) := R(\beta)/I^\Lambda_\beta.
\]

We define the functors \( F^\Lambda_i \) and \( E^\Lambda_i \) by

\[
F^\Lambda_i M := R^\Lambda(\beta + \alpha_i)e(\alpha_i, \beta) \otimes_{R^\Lambda(\mu)} M, \quad \text{and} \quad E^\Lambda_i M := e(\alpha_i, \beta - \alpha_i)M,
\]
for $M \in R^\Lambda(\beta)$-Mod. Then the actions of the Chevalley generators are given by the following:

$$
\begin{align*}
R^\Lambda(\beta)\text{-proj} & \xrightarrow{F_i^\Lambda} R^\Lambda(\beta + \alpha_i)\text{-proj} \\
R^\Lambda(\beta)\text{-gmod} & \xrightarrow{q_i^{(1-(h_i,\Lambda-\beta))}E_i^\Lambda} R^\Lambda(\beta + \alpha_i)\text{-gmod}
\end{align*}
$$

Theorem 2.4. ([6]) There exists an $\mathbb{A}$-module isomorphism

$$(2.3) \quad [R^\Lambda\text{-proj}] \xrightarrow{\sim} V_\mathbb{A}(\Lambda).$$

Taking the dual, we have an $\mathbb{A}$-module isomorphism

$$(2.3) \quad [R^\Lambda\text{-gmod}] \xrightarrow{\sim} V_\mathbb{A}(\Lambda)^\vee.$$

Under these isomorphisms, we have

$$p_\Lambda = [R^\Lambda \otimes_R -] \quad \text{and} \quad i_\Lambda = [\text{infl}_\Lambda]$$

where $R^\Lambda \otimes_R - := \bigoplus_{\beta \in \mathbb{Q}^+} (R^\Lambda(\beta) \otimes_{R(\beta)} -)$ : $R$-proj $\rightarrow R^\Lambda$-proj, and $\text{infl}_\Lambda : R^\Lambda$-gmod $\rightarrow$ $R$-gmod is the fully faithful functor induced by the surjective ring homomorphism $R(\beta) \rightarrow R^\Lambda(\beta)$.

In the sequel, we omit the map $i_\Lambda$ on $V_\mathbb{A}(\Lambda)$ (respectively, on $B(\Lambda)$) and the functor $\text{infl}_\Lambda$ if no confusion arises. Note that there exists a pairing

$$[R^\Lambda\text{-proj}] \times [R^\Lambda\text{-gmod}] \rightarrow \mathbb{A}$$

$$(P, M) \mapsto \dim_q P^\psi \otimes_{R^\Lambda} M,$$

by which $[R^\Lambda\text{-proj}]$ and $[R^\Lambda\text{-gmod}]$ are dual to each other.

Proposition 2.5. [1, Proposition 2.21] Let $M \in R^\Lambda$-gmod and $N \in R^\mu$-gmod. Then the convolution $M \circ N$ is an $R^{\Lambda+\mu}$-module.

3. Tensor products and convolution products

3.1. Tensor products.

The following lemma is known, but we include a proof for the convenience of the reader.

Lemma 3.1 (cf. [16, Lemma 2.5]). For $x \in U_q^-(\mathfrak{g})$ with $\Delta_\mathfrak{a}(x) = x_{(1)} \otimes x_{(2)}$, we have

$$\Delta_-(x) = x_{(1)} q^{-\xi(\text{wt}(x_{(2)}))} \otimes x_{(2)},$$
where $\xi : h^* \to h$ is the $\mathbb{Q}$-linear isomorphism determined by

$$\langle \xi(\lambda), \mu \rangle = (\lambda, \mu) \text{ for all } \lambda, \mu \in h^*.$$  

**Proof.** We shall use induction on $ht(x)$. As it is trivial when $ht(x) = 0$, we assume that the assertion holds for $x$ with $ht(x) > 0$.

Let $i \in I$. By the definition of $\Delta_n$, we have

$$\Delta_n(f_i x) = \Delta_n(f_i) \Delta_n(x) = f_i x_{(1)} \otimes x_{(2)} + q^{(\alpha_i, \text{wt}(x_{(1)}))} x_{(1)} \otimes f_i x_{(2)}.$$  

On the other hand, by the induction hypothesis,

$$\Delta_n(f_i x) = (f_i \otimes 1 + t_i \otimes f_i)(x_{(1)} q^{-\xi(wt(x_{(2)}))} \otimes x_{(2)})$$  

$$= f_i x_{(1)} q^{-\xi(wt(x_{(2)}))} \otimes x_{(2)} + t_i x_{(1)} q^{-\xi(wt(x_{(2)}))} \otimes f_i x_{(2)}$$  

$$= f_i x_{(1)} q^{-\xi(wt(x_{(2)}))} \otimes x_{(2)} + q^{(\alpha_i, \text{wt}(x_{(1)}))} x_{(1)} t_i q^{-\xi(wt(x_{(2)}))} \otimes f_i x_{(2)}$$  

$$= f_i x_{(1)} q^{-\xi(wt(x_{(2)}))} \otimes x_{(2)} + q^{(\alpha_i, \text{wt}(x_{(1)}))} x_{(1)} q^{-\xi(-\alpha_i + wt(x_{(2)}))} \otimes f_i x_{(2)}.$$  

Therefore, the assertion holds for $f_i x$ which complete the proof. 

Let $\lambda \in P^+$. We now consider the $\mathbb{Q}(q)$-linear map

$$\Phi_{\lambda} : U_q^-(g) \to U_q^-(g) \otimes_{\mathbb{Q}(q)} U_q^-(g)$$  

defined by

$$\Phi_{\lambda}(x) = q^{-\xi(wt(x_{(2)}))} x_{(1)} \otimes x_{(2)},$$

where $\Delta_n(x) = x_{(1)} \otimes x_{(2)}$. Note that

$$\Phi_{\lambda} \vert_{U_A^-} : U_A^-(g) \to U_A^-(g) \otimes_A U_A^-(g).$$

We simply write $\Phi_{\lambda}$ instead of $\Phi_{\lambda} \vert_{U_A^-}$ if there is no afraid of confusion.

Define

$$(U_q^-(g) \otimes_{\mathbb{Q}(q)} U_q^-(g))^\vee := \{ f \in (U_q^-(g) \otimes_{\mathbb{Q}(q)} U_q^-(g))^\vee \mid f(U_A^-(g) \otimes_A U_A^-(g)) \subseteq A \}.$$  

By restricting the $\mathbb{Q}(q)$-linear isomorphism

$$U_q^-(g)^\vee \otimes_{\mathbb{Q}(q)} U_q^-(g)^\vee \simeq (U_q^-(g) \otimes_{\mathbb{Q}(q)} U_q^-(g))^\vee$$

which is given by $\langle f \otimes g, x \otimes y \rangle := f(x)g(x)$ for $f, g \in U_q^-(g)^\vee$ and $x, y \in U_q^-(g)$ on the $A$-lattice $U_A^-(g)^\vee \otimes_A U_A^-(g)^\vee \subseteq U_q^-(g)^\vee \otimes_{\mathbb{Q}(q)} U_q^-(g)^\vee$, we obtain an $A$-module isomorphism

$$U_A^-(g)^\vee \otimes_A U_A^-(g)^\vee \iso (U_q^-(g) \otimes_{\mathbb{Q}(q)} U_q^-(g))^\vee.$$  

(3.2)
Similarly, by restricting (1.5) on the $A$-lattice $V_A(\lambda)^{\vee} \otimes_A V_A(\mu)^{\vee} \subset V_q(\lambda)^{\vee} \otimes_+ V_q(\mu)^{\vee}$, we obtain an $A$-module isomorphism
\begin{equation}
V_A(\lambda)^{\vee} \otimes_A V_A(\mu)^{\vee} \longrightarrow (V_q(\lambda) \otimes_- V_q(\mu))^A,
\end{equation}
where $(V_q(\lambda) \otimes_- V_q(\mu))^A := \{ f \in (V_q(\lambda) \otimes_- V_q(\mu))^A \mid f(V_A(\lambda) \otimes_A V_A(\mu)) \subset A \}$. In the sequel, we will identify the spaces in (3.2) and (3.3) respectively, without mentioning the isomorphisms.

Now, restricting the dual map $(U_q^{-}(g) \otimes_{Q(q)} U_q^{-}(g))^{\vee} \xrightarrow{(\Phi_{\lambda})^{\vee}} U_q^{-}(g)^{\vee}$ to $(U_q^{-}(g) \otimes_{Q(q)} U_q^{-}(g))^{\vee} \simeq U_A^{-}(g)^{\vee} \otimes_A U_A^{-}(g)^{\vee}$, we have
\begin{equation}
\Psi_{\lambda} := (\Phi_{\lambda})^{\vee} : U_A^{-}(g)^{\vee} \otimes_A U_A^{-}(g)^{\vee} \longrightarrow U_A^{-}(g)^{\vee}
\end{equation}
Let $\lambda, \mu \in P^+$. We define a $U_q(g)$-module homomorphism
\begin{equation*}
\iota_{\lambda, \mu} : V_q(\lambda + \mu) \rightarrow V_q(\lambda) \otimes_- V_q(\mu)
\end{equation*}
by $\iota_{\lambda, \mu}(u_{\lambda+\mu}) = u_{\lambda} \otimes u_{\mu}$. By Lemma 3.1 and (3.1), for $x \in U_A^{-}(g)$, we have
\begin{equation}
x \cdot (u_{\lambda} \otimes u_{\mu}) = \Delta_{-}(x)(u_{\lambda} \otimes u_{\mu}) = \Phi_{\lambda}(x)(u_{\lambda} \otimes u_{\mu}) \in V_A(\lambda) \otimes_A V_A(\mu).
\end{equation}
Restricting $\iota_{\lambda, \mu}$ to $V_A(\lambda + \mu)$, we have
\begin{equation}
\iota_{\lambda, \mu} \vert_{V_A(\lambda+\mu)} : V_A(\lambda + \mu) \rightarrow V_A(\lambda) \otimes_A V_A(\mu) \subset V_q(\lambda) \otimes_- V_q(\mu).
\end{equation}
We simply write $\iota_{\lambda, \mu}$ instead of $\iota_{\lambda, \mu} \vert_{V_A(\lambda+\mu)}$ if there is no afraid of confusion. Similarly, restricting the dual map $(V_q(\lambda) \otimes_- V_q(\mu))^{\vee} \xrightarrow{(\iota_{\lambda, \mu})^{\vee}} V_q(\lambda + \mu)^{\vee}$ to $(V_q(\lambda) \otimes_- V_q(\mu))^{\vee} \simeq V_A(\lambda)^{\vee} \otimes_A V_A(\mu)^{\vee}$, we have
\begin{equation}
\pi_{\lambda, \mu} := (\iota_{\lambda, \mu})^{\vee} : V_A(\lambda)^{\vee} \otimes_A V_A(\mu)^{\vee} \longrightarrow V_A(\lambda + \mu)^{\vee}.
\end{equation}

We now consider the following commutative diagram:
\begin{equation}
\begin{array}{ccc}
U_q^{-}(g) & \xrightarrow{\Phi_{\lambda}} & U_q^{-}(g) \otimes U_q^{-}(g) \\
\downarrow p_{\lambda+\mu} & & \downarrow \iota_{\lambda, \mu} \\
V_q(\lambda + \mu)^{\vee} & \xrightarrow{\pi_{\lambda, \mu}} & V_A(\lambda) \otimes_- V_q(\mu)
\end{array}
\end{equation}
Note that, since $\iota_{\lambda, \mu}$ is injective, so is $\Phi_{\lambda}$. Restricting (3.8) to the $A$-lattices, by (1.3), (3.1) and (3.6), we have the following:
\begin{equation}
\begin{array}{ccc}
U_A^{-}(g) & \xrightarrow{\Phi_{\lambda}} & U_A^{-}(g) \otimes U_A^{-}(g) \\
\downarrow p_{\lambda+\mu} & & \downarrow \iota_{\lambda, \mu} \\
V_A(\lambda + \mu)^{\vee} & \xrightarrow{\pi_{\lambda, \mu}} & V_A(\lambda) \otimes_A V_A(\mu)
\end{array}
\end{equation}
Taking the dual of (3.8) and restricting it to the $A$-lattices, by (1.4), (3.4) and (3.7), we have the following commutative diagram:

$$\begin{array}{ccc}
U^-_A(g)^\vee & \xrightarrow{\Psi_\lambda} & U^-_A(g)^\vee \otimes_h U^-_A(g)^\vee \\
\downarrow^{i_{\lambda+\mu}} & & \downarrow^{\pi_{\lambda,\mu}} \\
V_h(\lambda + \mu)^\vee & \xrightarrow{V_h(\lambda)^\vee \otimes_h V_h(\mu)^\vee} & 
\end{array}$$

3.2. Convolution products.

For $\lambda \in \mathbb{P}^+$ and $M \in R(\beta)$-Mod, we define

$$\text{Res}_\lambda(M) := \bigoplus_{\beta_1, \beta_2 \in \mathbb{Q}^+, \beta_1 + \beta_2 = \beta} q^{(\beta_2, \lambda)} \text{Res}_{\beta_1, \beta_2}(M).$$

Thanks to [14, Proposition 2.19], $\text{Res}_\lambda$ takes projective modules to projective modules. Thus we have the functors

$$\text{Res}_\lambda : R\text{-proj} \rightarrow (R \otimes R)\text{-proj},$$

which give the $A$-linear maps at the level of Grothendieck groups

$$[\text{Res}_\lambda] : [(R \otimes R)\text{-proj}] \simeq [R\text{-proj}] \otimes [R\text{-proj}].$$

**Lemma 3.2.** The functors $\text{Res}_\lambda$ categorify $\Phi_\lambda$. That is, for any $P \in R\text{-proj}$ we have

$$[\text{Res}_\lambda P] = \Phi_\lambda([P]).$$

**Proof.** It follows from Lemma 2.3 and Lemma 3.1. \qed

**Theorem 3.3.** For $M \in R(\beta_1)$-gmod and $N \in R(\beta_2)$-gmod, we have

$$\Psi_\lambda([M] \otimes [N]) = q^{(\beta_2, \lambda)} [M \circ N].$$

**Proof.** By the definition, we have $\langle u, \Phi_\lambda(m \otimes n) \rangle = (\Phi_\lambda(u), m \otimes n)$ for any $u \in U^-_A(g)$ and $m, n \in U^-_A(g)^\vee$. As we identify $U^-_A(g) \simeq [R\text{-proj}]$ and $[R\text{-gmod}] \simeq U^-_A(g)^\vee$ via $\gamma$ and $\gamma^\vee$ given in Theorem 2.2 respectively, it follows from (2.2) that

$$\langle [P], \Phi_\lambda([M] \otimes [N]) \rangle = (\Phi_\lambda([P]), [M] \otimes [N])$$

for any $P \in R\text{-proj}$ and $M, N \in R\text{-gmod}$. By Lemma 2.1 and Lemma 3.2, for any $P \in R\text{-proj}$, $M \in R(\beta_1)$-gmod and $N \in R(\beta_2)$-gmod, we have

$$\langle [P], \Psi_\lambda([M] \otimes [N]) \rangle = (\Phi_\lambda([P]), [M] \otimes [N]) = ([\text{Res}_\lambda P], [M] \otimes [N])$$

$$= (q^{(\beta_2, \lambda)} [\text{Res}_{\beta_1, \beta_2}(P)], [M] \otimes [N])$$

$$= ([P], q^{(\beta_2, \lambda)} [M \circ N]),$$

which implies the assertion. \qed
Then we have the following corollary.

**Corollary 3.4.** For $M \in R^\lambda(\beta_1)-\text{gmod}$ and $N \in R^\mu(\beta_2)-\text{gmod}$, we have

$$\pi_{\lambda,\mu}([M] \otimes [N]) = q^{(\beta_2,\lambda)}[M \circ N]$$

*Proof.* The assertion follows from (3.10), Proposition 2.5 and Theorem 3.3. \(\square\)

3.3. **Symmetric case.** From now on, we assume that the base field $k$ is of characteristic 0 and $R$ is symmetric; that is, the parameter $Q_{i,j}(u,v)$ is a polynomial in $u - v$ for any $i, j \in I$.

Then it was proved in [23, 24] that the set of self-dual simple $R$-modules in $R$-gmod corresponds to the upper global basis of $U_{\lambda}^- (g)^\vee$ under the isomorphsim $\gamma^\vee$. Combining this with the connection between tensor products and convolution products, we have interesting theorems below. For $b \in B(\infty)$, we denote by $L(b)$ the self-dual simple R-module corresponding to $G_{\lambda}^\text{up} (b)$. For $b \in B(\lambda)$, we simply use the same notation $L(b)$ for the self-dual simple $R^\lambda$-module corresponding to $G_{\lambda}^\text{up} (b)$ if there is no afraid of confusion.

**Theorem 3.5.** Let $M \in R^\lambda(\beta_1)-\text{gmod}$ and $N \in R^\mu(\beta_2)-\text{gmod}$. Suppose that

$$\pi_{\lambda,\mu}([M] \otimes [N]) = \sum_{b \in B(\lambda+\mu)} A_b(q) [L(b)]$$

for $A_b(q) \in \mathbb{Q}(q)$. Then, for $b \in B(\lambda + \mu)$, we have

1. $A_b(q) \in \mathbb{Z}_{\geq 0}[q, q^{-1}]$,
2. $[M \circ N : L(b)]_q = q^{-(\beta_2,\lambda)} A_b(q)$.

*Proof.* As the set of simple $R$-modules corresponds to the upper global basis of $U_{\lambda}^- (g)^\vee$, the assertion follows from Corollary 3.4. \(\square\)

We have the following corollary immediately.

**Corollary 3.6.** For $b \in B(\lambda)$ and $b' \in B(\mu)$, we write

$$\pi_{\lambda,\mu}(G_{\lambda}^\text{up}(b) \otimes G_{\mu}^\text{up}(b')) = \sum_{b'' \in B(\lambda+\mu)} A_{b''}(q) G_{\lambda+\mu}^\text{up}(b'')$$

for $A_{b''}(q) \in \mathbb{Q}(q)$. Then, we prove that

1. $A_{b''}(q) \in \mathbb{Z}_{\geq 0}[q, q^{-1}]$,
2. $[L(b) \circ L(b') : L(b'')]_q = q^{-(\beta_2,\lambda)} A_{b''}(q)$ for $b'' \in B(\lambda + \mu)$.

Let us recall that $C_{\lambda_1,\ldots,\lambda_r}$ denotes the connected component of the crystal $B(\lambda_1) \otimes \cdots \otimes B(\lambda_r)$ containing the highest weight vector $u_{\lambda_1} \otimes \cdots \otimes u_{\lambda_r}$ for $\lambda_1, \ldots, \lambda_r \in \mathbb{P}^+$. 
Theorem 3.7. Let $\lambda_k \in \mathbb{P}^+$ and $b_k \in B(\lambda_k)$ for $k = 1, \ldots, r$. We set $\beta_k := \lambda_k - \text{wt}(b_k)$. If

(i) $b_1 \otimes \cdots \otimes b_r \in C_{\lambda_1, \ldots, \lambda_r}$,

(ii) $\text{hd}(L(b_1) \circ \cdots \circ L(b_r))$ is simple,

then we have

$$q^t \text{hd}(L(b_1) \circ \cdots \circ L(b_r)) \simeq L(b_1 \otimes \cdots \otimes b_r)$$

where $t = \sum_{1 \leq i < j \leq r} (\beta_j, \lambda_i)$.

Proof. It was shown in [8, Theorem 4.2.1] that for $M \in R\text{-gmod}$, if $\text{hd}(M)$ is simple and $q^{-s}(\text{hd}(M))$ is self-dual for some $s \in \mathbb{Z}$, then in the Grothendieck group $[R\text{-gmod}]$

$$[M] = [\text{hd}(M)] + \sum_k q^s [S_k]$$

for some self-dual simple modules $S_k$ and some $s_k > s$.

We choose an integer $c$ such that $q^c \text{hd}(L(b_1) \circ L(b_2) \circ \cdots \circ L(b_r))$ is self-dual. Then we have

$$q^c [L(b_1) \circ L(b_2) \circ \cdots \circ L(b_r)] = q^c [\text{hd}(L(b_1) \circ L(b_2) \circ \cdots \circ L(b_r))] + \sum_k q^{c_k} [S_k]$$

for some self-dual simple modules $S_k$ and some $c_k > 0$ in $[R^\lambda\text{-gmod}]$, where $\lambda = \lambda_1 + \cdots + \lambda_r$.

In other word, we know that $q^c [\text{hd}(L(b_1) \circ L(b_2) \circ \cdots \circ L(b_r))]$ belongs to the upper global basis and

$$q^c [L(b_1) \circ L(b_2) \circ \cdots \circ L(b_r)] \equiv q^c [\text{hd}(L(b_1) \circ L(b_2) \circ \cdots \circ L(b_r))] \mod qL^\text{up}(\lambda),$$

since the upper global basis is an $A_0$-basis of $L^\text{up}(\lambda)$.

On the other hand, by Corollary 3.4 and Proposition 1.3 we have

$$q^t [L(b_1) \circ \cdots \circ L(b_r)] = \pi_{\lambda_1, \ldots, \lambda_r} ([L(b_1)] \otimes [L(b_2)] \otimes \cdots \otimes [L(b_r)])$$

$$\equiv [L(b_1 \otimes b_2 \otimes \cdots \otimes b_r)] \mod qL^\text{up}(\lambda).$$

Hence we conclude that $c = t$ and

$$q^t \text{hd}(L(b_1) \circ \cdots \circ L(b_r)) \simeq L(b_1 \otimes \cdots \otimes b_r),$$

as desired. \qed

Recall that for each pair of nonzero modules $M$ and $N$ in $R\text{-gmod}$, there exists an integer $\Lambda(M, N)$ and a distinguished nonzero homomorphism $r_{M, N} : M \circ N \to q^{-\Lambda(M, N)} N \circ M$, called the $R$-matrix. For the definition and properties of $R$-matrices, we refer [8, Section 2.2, 3.1, 3.2]. Set

$$\overline{\Lambda}(M, N) := \frac{1}{2} (\Lambda(M, N) + (\text{wt}M, \text{wt}N)), \quad \mathfrak{b}(M, N) := \frac{1}{2} (\Lambda(M, N) + \Lambda(N, M)).$$
If $M$ and $N$ are simple $R$-modules and one of them is real, then $M$ and $N$ strongly commute if and only if $\vartheta(M, N) = 0$ ([8, Lemma 3.2.3]).

**Corollary 3.8.** Let $\lambda_1, \lambda_2 \in \mathbb{P}^+$ and $b_1 \otimes b_2 \in C_{\lambda_1, \lambda_2}$. We set $\beta_k := \lambda_k - \text{wt}(b_k) \in \mathbb{Q}_+$ for $k = 1, 2$. Suppose that one of $L(b_1)$ and $L(b_2)$ is real. Then we have

1. $q^{(\beta_2, \lambda_1)} L(b_1) \nabla L(b_2) \simeq L(b_1 \otimes b_2)$,
2. $\Lambda(L(b_1), L(b_2)) = (\beta_2, \lambda_1)$ and $\Lambda(L(b_1), L(b_2)) = (\beta_2, 2\lambda_1 - \beta_1)$,
3. if there are $\lambda'_1, \lambda'_2 \in \mathbb{P}^+$ and $b'_2 \otimes b'_1 \in C_{\lambda'_2, \lambda'_1}$ such that $L(b_1) \simeq L(b'_1)$ and $L(b_2) \simeq L(b'_2)$, then we have

$$\vartheta(L(b_1), L(b_2)) = (\beta_1, \lambda'_2) + (\beta_2, \lambda_1) - (\beta_1, \beta_2) = (\lambda_1, \lambda'_2) - (\text{wt}(b_1), \text{wt}(b'_2)).$$

In particular, $L(b_1)$ and $L(b_2)$ strongly commute if and only if $(\lambda_1, \lambda'_2) = (\text{wt}(b_1), \text{wt}(b'_2))$.

**Proof.** (1) As either of $L(b_1)$ and $L(b_2)$ is real, $L(b_1) \circ L(b_2)$ has a simple head by [7, Theorem 3.2]. Thus it follows from Theorem 3.7.

(2) By (1), we know that $q^{(\beta_2, \lambda_1)} L(b_1) \nabla L(b_2)$ is self-dual. It follows that $(\beta_2, \lambda_1) = \Lambda(L(b_1), L(b_2))$ by [8, Lemma 3.1.4]. The second follows from it by the definition.

(3) It is straightforward to prove from (1) and (2). \hfill \Box

**Lemma 3.9.** Let $V$ and $U$ be nonzero modules over a ring $A$, and $f : V \rightarrow U$ be a nonzero $A$-module homomorphism. Suppose that $V$ has a unique maximal submodule $M$. Then the induced homomorphism

$$\overline{f} : V/M \rightarrow U/f(M)$$

is nonzero.

**Proof.** Without loss of generality, we may assume that $f$ is surjective. Then, it suffices to show that $f(M) \neq U$.

We assume that $f(M) = U$. Let $x \in V \setminus M$ and take $y \in M$ such that $f(x) = f(y)$. We set $v := x - y \in V$. Then we have $v \notin M$, $f(v) = 0$ and $Av = V$ since $Av \subset M$. Thus $\text{Im}(f) = f(V) = f(Av) = 0$, which is a contradiction to nonzeroness of $f$. \hfill \Box

**Theorem 3.10.** Let $\Lambda, \lambda_1, \lambda_2, \lambda'_1, \lambda'_2 \in \mathbb{P}^+$ with $\Lambda = \lambda_1 + \lambda_2 = \lambda'_1 + \lambda'_2$, and let $b_k \in B(\lambda_k)$, $b'_k \in B(\lambda'_k)$ for $k = 1, 2$. We set $\beta_k := \lambda_k - \text{wt}(b_k)$ and $\beta'_k := \lambda'_k - \text{wt}(b'_k)$ for $k = 1, 2$. Suppose that

(i) $b_1 \otimes b_2 \in C_{\lambda_1, \lambda_2}$,
(ii) $b_1 \otimes b_2 \simeq b'_1 \otimes b'_2$,
(iii) one of $G^u(b_1)$ and $G^u(b_2)$ (resp. one of $G^u(b'_1)$ and $G^u(b'_2)$) is real.

Then we have

1. $\text{Hom}_{R-\text{mod}}(q^d L(b_1) \circ L(b_2), L(b'_2) \circ L(b'_1)) \simeq k$, where $d = (\beta_2, \lambda_1) + (\beta'_2, \lambda'_1) - (\beta'_1, \beta_2)$,
(2) For any nonzero homomorphism $\varphi \in \text{Hom}_{R\text{-gmod}}(q^d L(b_1) \circ L(b_2), L(b'_2) \circ L(b'_1))$, we have

$$q^d \text{Im}(\varphi) \simeq q^d L(b_1) \nabla L(b_2) \simeq L(b'_2) \Delta L(b'_1).$$

Proof. As $b_1 \otimes b_2 \in C_{\lambda_1, \lambda_2}$ and $b_1 \otimes b_2 \simeq b'_1 \otimes b'_2$, the connected component of $B(\lambda'_1) \otimes B(\lambda'_2)$ containing $b'_1 \otimes b'_2$ should be isomorphic to $B(\Lambda)$. Thus, $b'_1 \otimes b'_2$ is contained in $C_{\lambda'_1, \lambda'_2}$ since $C_{\lambda'_1, \lambda'_2}$ is a unique connected component of $B(\lambda'_1) \otimes B(\lambda'_2)$ which is isomorphic to $B(\Lambda)$. It follows from Corollary 3.8 that

$$q^{(\beta_2, \lambda_1)} L(b_1) \nabla L(b_2) \simeq L(b_1 \otimes b_2) \simeq L(b'_1 \otimes b'_2) \simeq q^{(\beta'_2, \lambda'_1)} L(b'_1) \nabla L(b'_2).$$

As $L(b'_1) \nabla L(b'_2) \simeq q^{-\Lambda(L(b'_1), L(b'_2))} L(b'_2) \Delta L(b'_1)$ [7, Theorem 3.2], we have a chain of homomorphisms

$$q^d L(b_1) \circ L(b_2) \twoheadrightarrow q^d L(b_1) \nabla L(b_2) \xrightarrow{\sim} L(b'_2) \Delta L(b'_1) \twoheadrightarrow L(b'_2) \circ L(b'_1),$$

where

$$d = (\beta_2, \lambda_1) - (\beta'_2, \lambda'_1) + \Lambda(L(b'_1), L(b'_2)) = (\beta_2, \lambda_1) + (\beta'_2, \lambda'_1) - (\beta'_1, \beta'_2)$$

by Corollary 3.8. Moreover the composition of the homomorphisms in (3.12) is nonzero. Thus we conclude that

$$\dim_k(\text{Hom}_{R\text{-gmod}}(q^d L(b_1) \circ L(b_2), L(b'_2) \circ L(b'_1))) \geq 1.$$
f can be obtained from the composition of the homomorphisms in (3.12). Therefore, we complete the proof.

\[ \square \]

Corollary 3.11. Let \( \lambda_1, \lambda_2 \in \mathbb{P}^+ \), \( b_1 \otimes b_2 \in C_{\lambda_2, \lambda_1} \) and \( b_2 \otimes b_1 \in C_{\lambda_1, \lambda_2} \). Suppose that one of \( L(b_1) \) and \( L(b_2) \) is real. Then the followings are equivalent.

1. \( b_1 \otimes b_2 \simeq b_2 \otimes b_1 \).
2. \( L(b_1) \) and \( L(b_2) \) strongly commute.
3. \( (\lambda, \mu) = (\text{wt}(b_1), \text{wt}(b_2)) \).

Proof. Since (2) and (3) are equivalent by Corollary 3.8 (2), it suffices to show that (1) and (2) are equivalent.

If (1) holds, then it follows from Theorem 3.10 that \( L(b_1) \nabla L(b_2) \simeq L(b_1) \Delta L(b_2) \) up to a grading shift, which implies (2) by [7, Corollary 3.9].

Suppose that (2) holds. By Corollary 3.8 (1), we have

\[
L(b_1 \otimes b_2) \simeq L(b_1) \nabla L(b_2) = L(b_1) \circ L(b_2) \simeq L(b_2) \circ L(b_1) = L(b_2) \nabla L(b_1) \simeq L(b_2 \otimes b_1)
\]

up to grading shifts. Hence we have

\[
G_{\lambda_1 + \lambda_2}^{\text{up}}(\pi_{\lambda, \mu}(b_1 \otimes b_2)) = G_{\lambda_1 + \lambda_2}^{\text{up}}((\pi_{\mu, \lambda}(b_2 \otimes b_1))
\]

and hence \( b_1 \otimes b_2 \simeq b_2 \otimes b_1 \), as desired. \[ \square \]

Remark 3.12. In the case of Corollary 3.11, we actually obtain that

\[
d = (\beta_1, \lambda_2) + (\beta_2, \lambda_1) - (\beta_1, \beta_2) = \langle L(b_1), L(b_2) \rangle = 0,
\]

where \( d \) is the integer given in Theorem 3.10.

4. Applications to finite type A

Let \( \lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_l > 0) \) be a Young diagram of size \( |\lambda| = \sum_{k=1}^l \lambda_k \). We denote by \( \lambda' \) the conjugate of \( \lambda \), and write \( \emptyset \) for the Young diagram of 0. We will depict a Young diagram using English convention. Let \( \mathcal{P}_n := \{ \lambda \mid \ell(\lambda) \leq n \} \), where \( \ell(\lambda) \) is the number of rows of \( \lambda \). A semistandard tableau \( T \) of shape \( \lambda \in \mathcal{P}_n \) is a filling of boxes of \( \lambda \) with entries in \( \{1, 2, \ldots, n\} \) such that (i) the entries in rows are weakly increasing from left to right, (ii) the entries in columns are strictly increasing from top to bottom. We write \( \text{sh}(T) = \lambda \). Let \( \text{SST}(\lambda) \) be the set of all semistandard tableaux of shape \( \lambda \). Similarly, a standard tableau of shape \( \lambda \) is a filling of boxes of \( \lambda \) with \( \{1, \ldots, |\lambda|\} \) such that (i) each number is used exactly once, (ii) the entries in rows and columns increase from left to right and from top to bottom, respectively. Let us denote by \( \text{ST}(\lambda) \) the set of all standard tableaux of shape \( \lambda \). We write \( b = (p, q) \in T \) if \( b \) is a box of \( T \) at the \( p \)-th row and the \( q \)-th column, and set \( T(b) \) to
be the entry in the box $b$. For example, if $n = 8$ and $\lambda = (10, 8, 4, 2)$, then $T(1, 3) = 2$, $\lambda' = (4, 4, 3, 2, 2, 2, 2, 1, 1)$ and the following is a semistandard tableau of shape $\lambda$.

\[
\begin{array}{cccccccc}
1 & 1 & 2 & 3 & 3 & 3 & 5 & 6 & 7 & 8 \\
2 & 3 & 3 & 4 & 4 & 6 & 6 & 7 \\
4 & 5 & 5 & 7 \\
6 & 8
\end{array}
\]

We assume that the Cartan matrix $A$ is of type $A_{n-1}$ with $I = \{1, 2, \ldots, n-1\}$, the field $k$ is of characteristic 0 and the quiver Hecke algebra $R$ is symmetric. Note that, in type $A_{n-1}$, any quiver Hecke algebra is isomorphic to a symmetric quiver Hecke algebra. Let $\lambda$ be a Young diagram with $\ell(\lambda) < n$ and $\mu := \lambda'$. We write $\mu = (\mu_1, \ldots, \mu_r)$ and set $\Lambda := \sum_{k=1}^r \Lambda_{\mu_k}$. Then it was known that $\text{SST}(\lambda)$ has a $U_q(\mathfrak{g})$-crystal structure and

$$\text{SST}(\lambda) \simeq B(\Lambda)$$

as crystals (see [13], [5, Chapter 7] for details). Moreover, considering the tensor product of crystals, we have the crystal embedding

$$\text{SST}(\lambda) \rightarrow \text{SST}(\mu'_1) \otimes \text{SST}(\mu'_{r-1}) \otimes \cdots \otimes \text{SST}(\mu'_1)$$

sending $T$ to $T_r \otimes T_{r-1} \otimes \cdots \otimes T_1$, where $T_k$ is the $k$-th column of $T$ from the left. For $T \in \text{SST}(\lambda)$, we denote by $L(T)$ the self-dual simple $R$-module corresponding to $T$ under the categorification.

For $k, l \in I$, there is an explicit combinatorial description of a crystal isomorphism

\begin{equation}
\sigma : \text{SST}(k') \otimes \text{SST}(l') \xrightarrow{\sim} \text{SST}(l') \otimes \text{SST}(k')
\end{equation}

which was explained in [21, Section 3.5]. Note that $\sigma$ comes from the combinatorial $R$-matrix if we regard $\text{SST}(k')$ and $\text{SST}(l')$ as realizations of the Kirillov-Reshetikhin crystals $B^{k,1}$ and $B^{l,1}$ of type $A_{n-1}^{(1)}$. Let us explain the crystal isomorphism $\sigma$ briefly. Let $B = \{00, 10, 01, 11\}$ and define a $U_q(A_1)$-crystal structure by

$$\tilde{e}(00) = \tilde{e}(11) = \tilde{e}(10) = \emptyset, \quad \tilde{e}(01) = 10, \quad \tilde{f}(00) = \tilde{f}(11) = \tilde{f}(01) = \emptyset, \quad \tilde{f}(10) = 01.$$ 

Note that $B \simeq B(1) \oplus B(0)^{\oplus 2}$. We identify $\bigcup_{0 \leq k_1, k_2 \leq n} \text{SST}(k_1') \otimes \text{SST}(k_2')$ with $B^{\otimes n}$ as follows. For $T \in \text{SST}(k')$ and $a = 1, \ldots, n$, we write $a \in T$ when $a$ appears in $T$ as an entry. Then we have the following bijection

\begin{equation}
b : \bigcup_{0 \leq k_1, k_2 \leq n} \text{SST}(k_1') \otimes \text{SST}(k_2') \xrightarrow{\sim} B^{\otimes n}
\end{equation}
defined by \( b(T_1 \otimes T_2) = b_n \otimes b_{n-1} \otimes \cdots \otimes b_1 \), where
\[
b_n = \begin{cases} 
0 & \text{if } a \notin T_1 \text{ and } a \notin T_2, \\
10 & \text{if } a \in T_1 \text{ and } a \notin T_2, \\
01 & \text{if } a \notin T_1 \text{ and } a \in T_2, \\
11 & \text{if } a \in T_1 \text{ and } a \in T_2.
\end{cases}
\]

Then, for \( k \leq l \), the crystal isomorphism \( \sigma \) in (4.1) can be described as
\[
\sigma(T_1 \otimes T_2) = b^{-1}(e^{l-k}b(T_1 \otimes T_2)).
\]

Let \( k \in I \). Every vector of the crystal \( B(\Lambda_k) \) is extremal, which tells that \( R^{\Lambda_k}(\beta) \) is a simple algebra for any \( \beta \in \Lambda_k - \text{wt}(B(\Lambda_k)) \). For \( T \in \text{SST}(k') \), we denote by \( S^T \) the self-dual simple \( R^{\Lambda_k} \)-module. Moreover, \( S^T \) is real since it is a determinantal module [12, Proposition 4.2]. The simple module \( S^T \) can be realized in terms of standard tableaux as follows [4, Section 5]. Let \( T \in \text{SST}(k') \) and set \( \xi_T := (t_k, t_{k-1} - k + 1, \ldots, t_1 - 1) \) and \( m := |\xi_T| \), where \( t_a \) is the entry in the \( a \)-th box of \( T \) from the top. For \( S \in \text{ST}(\xi_T) \) and a box \( b = (p, q) \in S \), we define \( \text{res}(b) = q - p + k \). Then we have the residue sequence of \( S \)
\[
\text{res}(S) = (\text{res}(b_n), \text{res}(b_{m-1}), \ldots, \text{res}(b_1)) \in I^{\Lambda_k - \text{wt}(T)}
\]
where \( b_k \) is the box of \( S \) with entry \( k \). We set
\[
S^T := \bigoplus_{S \in \text{ST}(\xi_T)} kS
\]
and define the action of the quiver Hecke algebra by
\[
x_i S = 0, \quad \tau_j S = \begin{cases} 
s_j S & \text{if } s_j S \text{ is standard}, \\
0 & \text{otherwise}
\end{cases}, \quad e(\nu) S = \begin{cases} 
S & \text{if } \nu = \text{res}(S), \\
0 & \text{otherwise},
\end{cases}
\]
where \( s_j S \) is the tableau obtained from \( S \) by exchanging the entries \( j \) and \( j + 1 \). It is easy to check that \( S^T \) is a self-dual simple \( R^{\Lambda_k} \)-module and \( \text{ch}_q S^T = \sum_{S \in \text{ST}(\xi_T)} \text{res}(S) \).

**Theorem 4.1.** Let \( \lambda \) be a two-columns Young diagram with \( \ell(\lambda) < n \), and \( T \in \text{SST}(\lambda) \). We set \( T_k \) to be the \( k \)-th column of \( T \) for \( k = 1, 2 \), and write \( \sigma(T_2 \otimes T_1) = (T'_2 \otimes T'_1) \), where \( \sigma \) is given in (4.1). We set \( \beta_k = \Lambda_{\text{sh}(T_k)} - \text{wt}(T_k) \) and \( \beta'_k = \Lambda_{\text{sh}(T'_k)} - \text{wt}(T'_k) \) for \( k = 1, 2 \).

1. We have
\[
\text{Hom}_{R^{\text{gmod}}}(q^d S^{T_2} \circ S^{T_1}, S^{T'_1} \circ S^{T'_2}) = k,
\]
where \( d = (\beta_1, \Lambda_{\text{sh}(T_2)}) + (\beta'_1, \Lambda_{\text{sh}(T'_2)}) - (\beta'_1, \beta'_2) \).
2. If \( T_1 = T'_2 \) and \( T_2 = T'_1 \), then \( S^{T_1} \) and \( S^{T_2} \) strongly commute.

**Proof.** It follows from Theorem 3.10 and Corollary 3.11. \( \square \)
Example 4.2. We suppose that \( n = 4 \). Let \( \lambda = (2, 1) \) and

\[
T = \begin{bmatrix} 2 & 4 \\ 3 & \end{bmatrix}.
\]

Then, \( T \in \text{SST}(\lambda) \), \( \mu := \lambda' = (2, 1) \), and

\[
T_1 = \begin{bmatrix} 2 & \\ 3 & \end{bmatrix}, \quad T_2 = \begin{bmatrix} 4 & \end{bmatrix}.
\]

From the bijection (4.2), we have \( \tilde{e}b(T_2 \otimes T_1) = \tilde{e}(10 \otimes 01 \otimes 01 \otimes 00) = 10 \otimes 01 \otimes 10 \otimes 00 \), which tells \( \sigma(T_2 \otimes T_1) = T'_2 \otimes T'_1 \), where

\[
T'_1 = \begin{bmatrix} 3 & \end{bmatrix}, \quad T'_2 = \begin{bmatrix} 2 & \\ 4 & \end{bmatrix}.
\]

Thus, \( \xi_{T_1} = (1, 1), \xi_{T_2} = (3), \xi_{T'_1} = (2), \) and \( \xi_{T'_2} = (2, 1) \), which tells

\[
\text{ch}_q S^{T_1} = (1, 2), \quad \text{ch}_q S^{T_2} = (3, 2, 1), \quad \text{ch}_q S^{T'_1} = (2, 1), \quad \text{ch}_q S^{T'_2} = (3, 1, 2) + (1, 3, 2).
\]

By Theorem 4.1, we obtain \( d = (\alpha_1 + \alpha_2, \Lambda_1) + (\alpha_1 + \alpha_2, \Lambda_2) - (\alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3) = 1 \) and

(4.3) \[ \text{Hom}_{R_{g\text{-mod}}}(qS^{T_2} \circ S^{T_1}, S^{T'_1} \circ S^{T'_2}) = k. \]

We may also construct directly a nonzero homomorphism in (4.3). Let us consider the following chain of nonzero homomorphisms:

\[
qL(3) \circ S^{T'_1} \circ S^{T_1} \xrightarrow{r_1 \circ \text{id}} S^{T'_1} \circ L(3) \circ S^{T_1} \to S^{T'_1} \circ L(3) \nabla S^{T_1},
\]

where \( r_1 \) is the \( R \)-matrix with \( \text{deg}(r_1) = 1 \) and \( L(3) \) is the self-dual simple module of \( R(\alpha_3) \). It is easy to check that the composition of the above homomorphisms is nonzero and

(4.4) \[ \text{Im} r_1 \simeq qL(3) \nabla S^{T'_1} \simeq qS^{T_2}, \quad L(3) \nabla S^{T_1} \simeq S^{T'_2}. \]

Under the identification (4.4), restricting the projection to \( \text{Im}(r_1 \circ \text{id}) \), we have a nonzero homomorphism

\[
qS^{T_2} \circ S^{T_1} \to S^{T'_1} \circ S^{T'_2}.
\]

Moreover, one can prove directly that \( S^{T_2} \circ S^{T_1} \) is simple, which implies (4.3).

Example 4.3. Suppose that \( n = 5 \). Let \( \lambda = (2, 2, 1, 1) \) and

\[
T = \begin{bmatrix} 1 & 3 \\ 3 & 5 \\ 4 & \\ 5 & \end{bmatrix}.
\]
Then $\mu := \lambda' = (4, 2)$, and

\[
T_1 = \begin{array}{c}
1 \\
3 \\
4 \\
5 \\
\end{array}, \quad T_2 = \begin{array}{c}
3 \\
5 \\
\end{array}.
\]

Using (4.2), we have

\[
e_2^2 b(T_2 \otimes T_1) = e_2^2(11 \otimes 01 \otimes 11 \otimes 00 \otimes 01) = 11 \otimes 10 \otimes 11 \otimes 00 \otimes 10,
\]

which implies $\sigma(T_2 \otimes T_1) = T_1 \otimes T_2$. Note that $\xi_{T_1} = (1, 1, 1)$, $\xi_{T_2} = (3, 2)$ and

\[
\text{ch}_q S^{T_1} = (2, 3, 4), \quad \text{ch}_q S^{T_2} = (2, 1, 4, 3, 2) + (2, 4, 1, 3, 2) + (4, 2, 1, 3, 2) + (4, 2, 3, 1, 2).
\]

By Theorem 4.1,

$S^{T_2} \circ S^{T_1}$ is simple.

**Remark 4.4.** Let

\[
T_1 = \begin{array}{c}
1 \\
3 \\
4 \\
5 \\
\end{array}, \quad T_2 = \begin{array}{c}
2 \\
4 \\
\end{array}.
\]

One can show that $T_2 \otimes T_1 \in C_{\Lambda_2, \Lambda_4}$ and $T_1 \otimes T_2 \in C_{\Lambda_4, \Lambda_2}$, but $T_2 \otimes T_1 \not\cong T_1 \otimes T_2$. Since

\[
e_2^2 b(T_2 \otimes T_1) = e_2^2(01 \otimes 11 \otimes 01 \otimes 10 \otimes 01) = 10 \otimes 11 \otimes 10 \otimes 10 \otimes 01,
\]

we have $\sigma(T_2 \otimes T_1) = T_2' \otimes T_1'$, where

\[
T_1' = \begin{array}{c}
1 \\
4 \\
\end{array}, \quad T_2' = \begin{array}{c}
2 \\
3 \\
4 \\
5 \\
\end{array}.
\]

An algorithm was given in [20] for computing the entries of the transition matrices between standard monomials and global basis via the embedding

\[
V_q(\Lambda) \longrightarrow V_q(\Lambda_1) \otimes V_q(\Lambda_2) \otimes \cdots \otimes V_q(\Lambda_{n-1}) \otimes V_q(\Lambda_n),
\]

where $\Lambda = \sum_{k=1}^{n-1} m_k \Lambda_k$. Moreover, it turned out in [2] that the entries of the transition matrix can be computed in terms of the Kazhdan-Lusztig polynomials. Note that the entries also appear in composition multiplicities of the standard modules for the finite $W$-algebras/shifted Yangians [3]. Let $\lambda$ be a Young diagram with $\ell(\lambda) < n$, and $\mu = \lambda'$. We write $\mu = (\mu_1, \ldots, \mu_r)$ and let $\Lambda := \sum_{k=1}^{r} \Lambda_{\mu_k}$. Take $T_k \in \text{SST}(\mu_k)$ for $k = 1, \ldots, r$ and set $T := T_1 \ast \cdots \ast T_r$. 

to be the column strict tableau obtained by concatenating $T_k$'s. Via the projection $\pi : V_\nu(\Lambda_{\mu_1}) \otimes \cdots \otimes V_\nu(\Lambda_{\mu_r}) \to V(\Lambda)$, we write

\begin{equation}
\pi(G_{\Lambda_{\mu_1}}^\text{up}(T_r) \otimes \cdots \otimes G_{\Lambda_{\mu_1}}^\text{up}(T_1)) = \sum_{T' \in \text{SST}(\Lambda)} A_{T,T'}(q) G_{\Lambda}^\text{up}(T'),
\end{equation}

for some $A_{T,T'}(q) \in \mathbb{Z}[q, q^{-1}]$. Then, $A_{T,T'}(q)$ is computed in [2, Theorem 26] as follows:

\begin{equation}
A_{T,T'}(q) = (-q)^{\ell(T_r)-\ell(T')} \sum_{z \in D_{\nu_T} \cap S_{\nu_T} d_T S_{\nu'}} (-1)^{\ell(z)+\ell(d_T)} P_{zw|\Lambda|, d_T w|\Lambda|} (q^{-2}).
\end{equation}

We explain the notations appeared in (4.6) briefly. We refer the reader to [2] for precise definitions and notations. $P_{x,y}(t)$ is the Kazhdan-Lusztig polynomials associated with the symmetric group. For $\nu \in \mathbb{Z}_r^\geq$, let $D_\nu$ be the set of minimal length $S_\nu \backslash S_{[\nu]}$-coset representatives. For a column-strict tableau $T$, $\nu_T$ is the content of $T$, $\gamma_T$ is the column reading of $T$ from top to bottom and from right to left, and $d_T$ is a unique element in $D_{\nu_T}$ such that $\gamma_T d_T^{-1}$ is weakly decreasing. We remark that upper crystal bases dealt in [2] are bases at $q = \infty$ while we use bases at $q = 0$. Combining this with Theorem 3.3 and Theorem 3.7, we have the following.

**Theorem 4.5.** Let $\lambda$ be a Young diagram with $\ell(\lambda) < n$ and $\mu := \lambda'$. We write $\mu = (\mu_1, \ldots, \mu_r)$, and set $\Lambda := \sum_{k=1}^r \Lambda_{\mu_k}$. For $T_k \in \text{SST}(\mu_k)$ ($k = 1, \ldots, r$) and $T' \in \text{SST}(\lambda)$, we have

\begin{equation}
[S^T \circ \cdots \circ S^{T_1} : L(T')]_q = q^{-\sum_{1 \leq a < b \leq r} (\beta_a, \Lambda_{\mu_0})} A_{T,T'}(q),
\end{equation}

where $\beta_k = \Lambda_{\mu_k} - \text{wt}(T_k)$, $T := T_1 \ast \cdots \ast T_r$ is the column strict tableau obtained by concatenating $T_k$'s, and $A_{T,T'}(q)$ is given in (4.6).

**Example 4.6.** Let $n = 5$, $\lambda = (4, 3, 1)$, $\mu = \lambda' = (3, 2, 2, 1)$ and

\begin{align*}
T & = \begin{array}{cccc}
1 & 1 & 3 & 5 \\
2 & 2 & 4 & \\
3 & & & \\
\end{array}, & S & = \begin{array}{cccc}
1 & 1 & 2 & 5 \\
2 & 3 & 4 & \\
3 & & & \\
\end{array}.
\end{align*}

$T_k$ and $S_k$ are the $k$-th column of $T$ and $S$ from the left respectively. Then we have $\xi_{T_1} = \xi_{T_2} = (0), \xi_{T_3} = (2, 2), \xi_{T_4} = (4), \xi_{S_1} = (0), \xi_{S_2} = (1), \xi_{S_3} = (2, 1), \xi_{S_4} = (4)$, and

\begin{align*}
& \text{ch}_q S^{T_3} = (2, 3, 1, 2) + (2, 1, 3, 2), & \text{ch}_q S^{T_2} = (4, 3, 2, 1), \\
& \text{ch}_q S^{S_2} = (2), & \text{ch}_q S^{S_3} = (3, 1, 2) + (1, 3, 2), & \text{ch}_q S^{S_4} = (4, 3, 2, 1).
\end{align*}
Note that $S^{T_1}$, $S^{T_2}$ and $S^{S_1}$ are trivial. Using the table of the entries of the transition matrix computed in [2, Table 1] or [20, Section 5], we have

$$A_{T,T'}(q) = \begin{cases} 1 & \text{if } T' = T, \\ 0 & \text{otherwise}, \end{cases} \quad A_{S,S'}(q) = \begin{cases} q & \text{if } S' = T, \\ 1 & \text{if } S' = S, \\ 0 & \text{otherwise}, \end{cases}$$

Since

$$\sum_{1 \leq a < b \leq 4} (\beta_a, \Lambda_{\mu_b}) = 1, \quad \sum_{1 \leq a < b \leq 4} (\gamma_a, \Lambda_{\mu_b}) = 2,$$

where $\beta_k = \Lambda_{\mu_k} - \text{wt}(T_k)$ and $\gamma_k = \Lambda_{\mu_k} - \text{wt}(S_k)$, we have

$$[S^{T_1} \circ S^{T_3} \circ S^{T_2} \circ S^{T_1}] = [S^{T_4} \circ S^{T_3}] = q^{-1}[L(T)],$$

$$[S^{S_4} \circ S^{S_3} \circ S^{S_2} \circ S^{S_1}] = [S^{S_4} \circ S^{S_3} \circ S^{S_2}] = q^{-2}(q[L(T)] + [L(S)]).$$

Thus, $q(S^{T_4} \circ S^{T_3})$ is a self-dual simple $R$-module. We may find directly how to embed $L(T)$ to $S^{S_4} \circ S^{S_3} \circ S^{S_2}$ using the homomorphism

$$S^{S_4} \circ S^{S_2} \circ S^{S_1} \xrightarrow{id \circ r_2} S^{S_4} \circ S^{S_3} \circ S^{S_2}$$

where $r_2$ is the $R$-matrix. It is easy to show that $\deg r_2 = 0$, $S^{T_4} \simeq S^{S_4}$ and $\text{Im } r_2 \simeq S^{T_3}$. Hence we have the embedding

$$q^{-1}L(T) \simeq S^{T_4} \circ S^{T_3} \simeq \text{Im}(id \circ r_2) \mapsto S^{S_4} \circ S^{S_3} \circ S^{S_2}.$$
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