Discrete vortices on spatially nonuniform two-dimensional electric networks

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Two-dimensional arrays of nonlinear electric oscillators are considered theoretically, where nearest neighbors are coupled by relatively small, constant, but non-equal capacitors. The dynamics is approximately reduced to a weakly dissipative defocusing discrete nonlinear Schrödinger equation with translationally non-invariant linear dispersive coefficients. Behavior of quantized discrete vortices in such systems is shown to depend strongly on the spatial profile of the inter-node coupling as well as on the ratio between time-increasing healing length and lattice spacing. In particular, vortex clusters can be stably trapped for a some initial period of time by a circular barrier in the coupling profile, but then, due to gradual dissipative broadening of vortex cores, they lose stability and suddenly start to move.

I. INTRODUCTION

Nonlinear complex wave fields are known to support quantized vortices in two and three spatial dimensions [1,8]. Vortices have also been studied in discrete systems (on lattices; see, e.g., [9–14] and citations therein). As far as weakly dissipative lattice dynamics is considered, among the most popular mathematical models are modifications of a discrete nonlinear Schrödinger equation (DNLSE) [9–23]. They arise in various scientific contexts (but mostly in nonlinear optics [12,13] and in physics of nonlinear metamaterials [14]), where we have nearly identical oscillators with their normal complex variables

\[ a_n(t) = A_n(t) \exp(-i \omega_0 t), \]

and with nonlinear frequency shifts \( g|A_n|^2 \ll \omega_0 \). The simplest form of DNLSE is

\[ i(\dot{A}_n + \gamma \omega_0 A_n) = g|A_n|^2 A_n - \frac{1}{2} \sum_{n'} c_{n,n'} A_{n'}, \tag{1} \]

where overdot means time derivative. A linear damping rate \( \gamma \omega_0 \) takes into account dissipative effects, with small \( \gamma = 1/Q \ll 1 \) being an inverse quality factor. Oscillators are weakly coupled by (real) coefficients \( c_{n,n'} \), (if coupling strength and/or nonlinearity level are not weak, then more complicated forms of DNLSE appear, including non-linearities in coupling terms [12,13]). In many interesting cases, multi-index \( n \) is a node \( n = (n_1, \ldots, n_d) \) of a simple regular lattice in one, two, or three spatial dimensions \( (d = 1, 2, 3) \). Besides electromagnetic artificially created structures [14], DNLSE has been successfully applied in nonlinear optics where it describes stationary regime of light propagation in waveguide arrays [12] (one-dimensional (1D) and two-dimensional (2D) cases, with time variable \( t \) replaced by propagation coordinate \( z \)).

The coupling coefficients \( c_{n,n'} \) are often considered as translationally invariant on the lattice and taking place between a few near neighbors. If they have a definite sign, then in the long-scale quasi-continuous limit we have either defocusing regime (when \( gc > 0 \)), or focusing one (when \( gc < 0 \)). Accordingly, different nonlinear coherent wave structures can take place in each case. In particular, in the most well-studied focusing regime, there are highly localized discrete solitons and discrete vortex solitons (see the above cited research works). In the defocusing regime, there are dark solitons, and besides that, discrete analogues of superfluid quantized vortices can be excited and interact with each other over long distances.

In this work, we consider discrete vortices, but in somewhat more complicated arrangements where coupling coefficients are not translationally invariant, \( c_{n+1,n'+1} \neq c_{n,n'} \), and the corresponding terms contain differences \( (A_n - A_{n'}) \) instead of \( (A_n + A_{n'}) \):

\[ i(\dot{A}_n + \gamma \omega_0 A_n) = g|A_n|^2 A_n + \frac{1}{2} \sum_{n'} c_{n,n'} (A_n - A_{n'}). \tag{2} \]

In general, equations (1) and (2) are not equivalent. Exception is for infinite and uniform lattices, where they are related to each other by a simple gauge transformation.

It is important that Eq. (2), with any coefficients \( c_{n,n'} \), admits a class of spatially uniform solutions,

\[ A_n = A_0 \exp [-\gamma \omega_0 t - ig|A_0|^2(1 - e^{-2\gamma \omega_0 t})/(2\gamma \omega_0)]. \tag{3} \]

However, spatial nonuniformity of couplings strongly affects vortex dynamics on the above background, since vortices are known to have highly de-localized phase gradients even if the amplitude variation (vortex core) is localized. Continuous quantized vortices on spatially nonuniform backgrounds have been extensively studied in application to trapped Bose-Einstein condensates (see, e.g., Refs. [2,8,20], and citations therein). The goal of this work is to investigate the effect of nonuniformity for vortices on discrete lattices within model (2). For simplicity, we consider below a square lattice and interactions between the nearest neighbors in the form

\[ c_{n,n'} = f(h|n + n'|/2), \tag{4} \]

where \( h \ll 1 \) is a lattice spacing, and \( f(x,y) \) is a sign-definite function varying on scales \( (\Delta x, \Delta y) \sim 1 \).

Eq. (2) with nonuniform couplings has been introduced recently in a formal manner as a three-dimensional (3D)
discrete system supporting long-lived vortex knots \[25\]. In the present work, as a possible physical implementation approximately corresponding to this equation, we theoretically suggest and analyze a specially designed electric circuit network. Implementation of discrete nonlinear dynamic systems in the form of 1D and 2D electric networks has a long and rich history \[50–53\], including even experimental simulations of the integrable Toda chain \[50–53\]. Major attention has been devoted to modulationally unstable systems. Here we consider a network possessing stable solutions \(3\). We adopt a scheme consisting of nonlinear oscillator circuits coupled by relatively small, non-equal capacitors. It will be shown below that nonlinear constant \(g\) and coupling coefficients \(c_{n,n'}\) appear both negative in this case, so the corresponding DNLSE is defocusing and appropriate for vortices. If instead of small capacitances, oscillators are coupled by large inductances, then a focusing DNLSE arises. That case has been already studied previously (on uniform lattices).

It should be noted that electric networks can be of macroscopic sizes and assembled of standard radiotechnical elements. Typical dispersive and nonlinear times can be about milliseconds, with the carrier frequency \(\omega_0/2\pi\) of order 1 MHz. Additional convenience of electric implementation is a potentially flexible controllability including arbitrary variation of coupling capacitances with time. Another important thing is that our electric scheme can be equally suitable for construction of 3D nonlinear lattices. A practical problem is only in a very big number of elements. So, to observe interesting nonlinear behavior of vortices, in a 2D lattice we need about \(N_{2D} \sim 10^3 - 10^4\) individual oscillators, while for a 3D lattice the required number is \(N_{3D} \sim 10^5 - 10^6\). Therefore, planar constructions seem more realistic at the moment.

This article is organized as follows. In section II we introduce the theoretical model and derive DNLSE together with the parameters. In section III, we generally analyze vortex motion in 2D case, with orientation on quasi-continuous limit. In section IV, we present some numerical results demonstrating nontrivial behavior of interacting vortices in discrete, spatially nonuniform, weakly dissipative 2D systems. In particular, it will be shown that depending on parameters, vortex clusters can be stably trapped for some initial period of time by a circular barrier in function \(f\) profile, but then, due to gradual dissipative broadening of vortex cores, they lose stability and suddenly start to move in a complicated manner, some of vortices penetrating the barrier. Finally, section V contains brief summary of the work.

II. MODEL DESCRIPTION AND BASIC EQUATIONS

In the beginning, we describe our simple scheme (see Fig. 1). Let each electric oscillator in the network consist of a coil with inductance \(L\) and small active resistance \(R_L\), connected in series to a voltage-dependent differential capacitance (varicap) \(C(V) = dq/dV\), where \(q\) is electric charge. A reverse-biased varactor diode is implied (perhaps in parallel with an ordinary capacitor), or something else. The varicap is characterized by a large shunt resistance \(R_C\) (for leakage current). The remaining end of the coil is connected to a d.c. bias voltage \(V_b\), while the remaining contact of the varicap is grounded. A voltage at the contact between the coil and the varicap is \(V_b + V_n(t)\). For a reverse-biased diode, experimental dependencies \(C(V_n)\) can be well expressed by formula

\[
C(V_n) = C_0 \left[ \mu + \frac{(1 - \mu)}{(1 + V_n/V_s)^\nu} \right],
\]

with fitting parameters \(C_0, V_s, \mu,\) and \(\nu\). In some research works, a different kind of variable capacitor is also considered, with \(C(V_n) = C_0(1 + V_n^2/V_s^2)\) \[62\]. Such a symmetric dependence is possible in devices using special nonlinear dielectric films \[63\]. In any case, accumulated electrostatic energy at the varicap is given by formula

\[
W(V_n) = \int_{V_n}^{V_b} C(u)udu.
\]

When expressed in terms of the electric charge \(q_n\), the voltage \(V_n\) is

\[
V_n = U(q_n) = C_0^{-1}[q_n + \alpha q_n^2 + \beta q_n^3 + \cdots].
\]

Equation of motion for a single oscillator circuit, with dissipative terms neglected, is \(L\ddot{q}_n + U(q_n) = 0\). The nonlinear frequency shift for such equation is known to be

\[
\Delta \omega = \omega_0(3\beta/8 - 5\alpha^2/12)q_0^2,
\]

with \(\omega_0 = 1/\sqrt{LC_0}\) and \(q_0\) being an amplitude of the main harmonics. For our purposes it will be important that \(\Delta \omega < 0\). The inverse quality factor of the oscillator is apparently

\[
\gamma = \left( R_L\sqrt{C_0/L} + R_C\sqrt{L/C_0} \right)/2.
\]
It is presumed very small (as we will see below, the most interesting things happening with vortices begin at $Q \gtrsim 10^4$). For example, with $L = 5.0 \times 10^{-4}$ H, $C_0 = 5.0 \times 10^{-10}$ F, $R_L < 0.1$ Ohm, and $R_C > 10^7$ Ohm, we have $\omega_0 = 2.0 \times 10^9$ rad/s, corresponding to a frequency about 0.3 MHz, and a sufficiently high quality factor $Q > 10^4$. Perhaps, even smaller values of $R_L$ and larger values of $R_C$ can be achieved at reasonably low temperatures, making $Q \gtrsim 10^6$.

There are also weak ordinary capacitors $C_{n,n'} \ll C_0$ inserted between points $V_n$ and $V_{n'}$. They unite individual oscillators into a whole network.

Equations of motion for the united system can be derived in a very simple manner. Indeed, electric current through the coil is $I_n$, while currents through the capacitors are $C(V_n)V_n$ and $C_{n,n'}(V_n - V_{n'})$. Leakage current parallel to varicap is $V_n/R_C$. Thus we obtain equations

$$C(V_n)V_n + \sum_{n'} C_{n,n'}(V_n - V_{n'}) + \frac{V_n}{R_C} = I_n. \quad (10)$$

A voltage difference at the coil is $\sum L I_n + R_L I_n$. In sum with $V_n + V_{n'}$ it should give $V_b$. Therefore we have the second sub-set of equations, closing the system,

$$L \dot{I}_n + V_n + R_L I_n = 0. \quad (11)$$

It is clear that our system admits a class of $n$-independent solutions related to Eq. (13), when each node oscillates as if there were no couplings.

It is not so obvious at first glance but can be easily checked that without dissipative terms containing active resistances $R_L$ and $R_C$, equations (10) - (11) correspond to a Lagrangian system with the Lagrangian function

$$L = \sum_n \left[ \frac{1}{2} C(V_n)V_n + \sum_{n'} C_{n,n'}(V_n - V_{n'}) \right]^2 - \sum_n W(V_n) - \sum_{n,n'} \frac{C_{n,n'}}{4} (V_n - V_{n'})^2. \quad (12)$$

Our next steps will be to rewrite it in terms of charges $q_n$, and then introduce a Hamiltonian description. It is convenient to adopt non-dimensionalization (voltage in units $V_*$, charge in units $C_0 V_*$, time in units $1/\omega_0$), formally corresponding to $L = 1$, $C_0 = 1$. Then, in the first order on small quantities $\bar{c}_{n,n'} = C_{n,n'}/C_0$, and retaining only main terms on oscillation amplitudes in the couplings, we have

$$L \approx \sum_n \left[ \frac{\bar{q}_n^2}{2} - \frac{q_n^2}{2} - \alpha \frac{q_n^3}{3} + \beta \frac{q_n^4}{4} \right] + \frac{q_n}{2} \sum_{n',n''} \bar{c}_{n,n''} [2(q_n - q_{n''})^2 - (q_n - q_{n''})^2]. \quad (13)$$

The canonical momenta for this Lagrangian are

$$p_n = \dot{q}_n + 2 \sum_{n,n''} \bar{c}_{n,n''} (\dot{q}_n - \dot{q}_{n''}). \quad (14)$$

Inverse relations, again with the first-order accuracy on $\bar{c}_{n,n''}$, are easily obtained as

$$\dot{q}_n \approx p_n - 2 \sum_{n,n''} \bar{c}_{n,n''} (p_n - p_{n''}). \quad (15)$$

As the result, the Hamiltonian function of weakly interacting oscillators takes the following form:

$$H \approx \sum_n \left[ \frac{p_n^2}{2} + \frac{\bar{q}_n^2}{2} + \alpha \frac{q_n^3}{3} + \beta \frac{q_n^4}{4} \right] - \frac{1}{4} \sum_{n,n'} \bar{c}_{n,n'} [2(p_n - p_{n'})^2 - (q_n - q_{n'})^2]. \quad (16)$$

According to general theory, there exists a weakly nonlinear canonical transform (depending on $\alpha$ and $\beta$),

$$q_n = \tilde{q}_n + \cdots, \quad p_n = \tilde{p}_n + \cdots,$$

such that combination $a_n = (\tilde{q}_n + i\tilde{p}_n)/\sqrt{2}$ (the normal complex variable) is related to the action-angle variables $S_n$ and $\phi_n$ by formula $a_n = \sqrt{S_n} \exp(i\phi_n)$. That transform excludes third-order terms from the partial Hamiltonians. Neglecting again nonlinearities in the couplings, we reduce the total Hamiltonian to the following expression:

$$H \approx \sum_n \left[ |a_n|^2 \right] + \frac{\beta}{2} |a_n|^4$$

$$- \frac{1}{4} \sum_{n,n'} \bar{c}_{n,n'} [2(a_n - a_{n'})^2 - (a_n^* - a_{n'}^*)^2] + \frac{3}{8} \sum_{n,n'} \bar{c}_{n,n'} [2(a_n - a_{n'})^2 + (a_n^* - a_{n'}^*)^2]. \quad (17)$$

In terms of $a_n$, Hamiltonian equations of motion are $i\dot{a}_n = \partial H/\partial a_n^*$. In the main approximation, $a_n$ behaves proportionally to $\exp(-it)$, since the nonlinearity and the couplings are weak. Therefore, the last double sum in Eq. (17) contains quickly oscillating quantities which are not important after averaging. Introducing slow envelopes $A_n = a_n \exp(it)$ and taking into account linear damping (not covered by Hamiltonian theory), we arrive at Eq. (2), with negative $c_{n,n'} = -\bar{c}_{n,n'}$. Nonlinear coefficient $g$, for physically relevant parameters in Eq. (14), appears also negative [for example, if $\mu = 0$, then $g = \nu(3 - \nu)/24$].

**III. ANALYSIS OF VORTEX MOTION IN 2D**

As far as our goal is to study vortices on 2D networks, it is convenient to introduce new complex variables $\psi_n(t)$ through the following substitution (compare to Ref. [25]):

$$A_n(t) = A_0 \psi_n(t) \exp[-\gamma t - i\varphi(t)], \quad (18)$$

where real $A_0$ is a typical amplitude at $t = 0$, and $\varphi(t) = gA_0^2(1 - e^{-2\gamma t})/(2\gamma)$. As the result, we reduce
our dissipative autonomous system to a non-autonomous Hamiltonian system,
\[ i\dot{\psi}_n = \sum_{n'} \bar{c}_{n,n'} \left( \psi_n - \psi_{n'} \right) + |gA_0^2|e^{-2\gamma\tau}(|\psi_n|^2 - 1)\psi_n. \] (19)

Let a typical value of \( \bar{c}_{n,n'} \) be \( \bar{c} \approx 1 \). For purposes of further analysis, we introduce a slow time \( \tau = \hbar^2 \bar{c} t \) and small parameters
\[ \delta = \gamma/(\hbar^2 \bar{c}) \ll 1, \quad \xi = (\hbar^2 \bar{c} / |gA_0^2|)^{1/2} \ll 1. \] (20)

Then Eq. (19) takes the following form,
\[ i\frac{d\psi_n}{d\tau} = \sum_{n'} \frac{F_{n,n'}}{2\hbar^2} (\psi_n - \psi_{n'}) + \frac{e^{-2\delta\tau}}{\xi^2}(|\psi_n|^2 - 1)\psi_n, \] (21)

where \( n' \) are the nearest neighbors for \( n \) on square lattice, \( F_{n,n'} = F(h |n + n'|/2), \) and \( F(r) \sim 1 \) is a non-negative function. In the continuous limit, the above equation reduces to a defocusing NLSE with spatially variable dispersion coefficient and time-dependent nonlinear coefficient,
\[ i\psi_\tau = -\frac{1}{2} \nabla \cdot [F(r)\nabla \psi] + \frac{e^{-2\delta\tau}}{\xi^2}(|\psi|^2 - 1)\psi. \] (22)

We are interested in vortices on constant background \( \psi_0 = 1 \). It is clear from the equation above that intervals \( \Delta \tau \sim 1 \) are typical vortex turnover times in the system, \( \xi \) is a typical relative healing length at \( \tau = 0 \), while
\[ \xi(r, \tau) = \xi e^{\delta\tau} \sqrt{F(r)} \] (23)

is a local relative vortex core width. Vortices described by Eq. (22) have been analyzed in Ref. [25] for 3D case. Applying similar analysis to 2D situation, we easily obtain that coordinates \( x_j \) and \( y_j \) of \( N \) “point” vortices are canonically conjugate quantities (up to vortex signs \( \sigma_j = \pm 1 \)). On not very long times and for small \( \xi \), when \( \xi \exp(\delta\tau) \ll 1 \), vortex motion is approximately described by a time-dependent Hamiltonian function (compare to Refs. [41, 42]),
\[ H = \sum_j \sigma_j^2 \mathcal{E}(r_j, \tau) + \frac{1}{2} \sum_{j,k} \sigma_j \sigma_k G(r_j, r_k), \] (24)

\[ \mathcal{E}(r, \tau) \approx \frac{1}{2} G(r - \mathbf{e}_z\mathbf{r}(\tau)/2, r + \mathbf{e}_z\mathbf{r}(\tau)/2), \] (25)

where the prime means omitting diagonal terms in the double sum determining pair interactions between vortices, \( \mathbf{e}_z \) is a unit vector, and a two-dimensional Green function \( G(r, r_1) \) satisfies equation
\[ -\nabla_r \cdot \frac{1}{F(r)} \nabla_r G(r, r_1) = 2\pi \delta_{\text{Dirac}}(r - r_1). \] (26)

The physical meaning of \( G(r, r_1) \) can be explained as follows. Let \( \psi = \sqrt{\rho} \exp(i\Phi) \) be the Madelung transform, and \( \mathbf{J} = \rho F(r) \nabla \Phi \) be a “current density” (in the hydrodynamic sense) for Eq. (22). In the “long-scale” hydrodynamic regime, away from vortex cores we have \( \rho \approx 1 \) and thus \( \nabla \cdot \mathbf{J} \approx 0 \), so a stream function \( \Theta \) exists for 2D vector field \( F(r) \nabla \Phi \). Since \( \Phi \) - field created by vortices is not single-valued and has singularities, it satisfies equation curl_{2D} \nabla \Phi = 2\pi \sum_j \sigma_j \delta_{\text{Dirac}}(r - r_j). \) Therefore we have a partial differential equation determining the stream function,
\[ -\nabla_r \cdot \frac{1}{F(r)} \nabla_r \Theta(r) = 2\pi \sum_j \sigma_j \delta_{\text{Dirac}}(r - r_j). \] (27)

So \( G(r, r_1) \) is a stream function created at point \( r \) by a vortex placed in point \( r_1 \). Expression for vortex Hamiltonian \( H \) then follows from appropriately regularized “kinetic energy” integral
\[ 2\pi H = \frac{1}{2} \int \frac{(\nabla \Theta)^2}{F} d^2 r. \] (28)

It follows from Eq. (28) that
\[ G(r_1, r_2) = \tilde{\theta}(r_1, r_2) - \sqrt{F(r_1)F(r_2)} \ln |r_1 - r_2|, \] (29)

with some smooth function \( \tilde{\theta}(r_1, r_2) \sim 1 \). Therefore the self-energy is
\[ \mathcal{E}(r, \tau) = \theta(r) - \frac{1}{2} F(r) \left[ \ln \left( \xi \sqrt{F(r)} \right) + \delta\tau \right], \] (30)

where \( \theta(r) = \tilde{\theta}(r, r)/2 \).

In particular, we may take circularly symmetric profile \( F(r) \), with \( r = \sqrt{x^2 + y^2} \), and roughly (with a logarithmic accuracy) estimate energy of a vortex cluster in the form of a regular \( N \)-polygon,
\[ E_N(r, \tau) \approx \frac{N}{2} F(r) [\Lambda(\tau) - (N - 1) \ln |r|], \] (31)

where \( \Lambda(\tau) = |\ln(1/\xi) - \delta\tau| \) is a logarithmically large quantity. It is not difficult to understand that if \( F(r) \) has a barrier at some finite \( r_b \), and \( N \) is not too large, then expression (31) may have a minimum at some \( 0 < r_\ast < r_b \). Thus, such a profile is able to trap vortex clusters.

Discreteness (finite \( \hbar \)) acts also to stabilize vortex configurations because, while \( \xi \exp(\delta\tau) \lesssim \hbar \), the lattice tends to create local minima (in inter-node vortex center positions) for the Hamiltonian corresponding to Eq. (24),
\[ \dot{\mathbf{r}} = \sum_{n,n'} \frac{F_{n,n'}}{4\hbar^2} |\psi_n - \psi_{n'}|^2 + \sum_n \frac{e^{-2\delta\tau}}{2\xi^2}(|\psi_n|^2 - 1)^2. \] (32)

So we can expect stable trapping of a few vortices of the same sign within a domain surrounded by the barrier. However, as time increases, function \( \Lambda(\tau) \) decreases, and therefore vortex configuration should suddenly become unstable at some moment. In the next section, we numerically verify such a scenario within Eq. (24).
IV. NUMERICAL RESULTS

Eq. (21) has been numerically simulated using a 4-th order Runge-Kutta scheme for time stepping. Function $F(\tau)$ was taken in a simple form: $F = 1$ if $r^2 < 1$, and $F = 4$ if $1 \leq r^2 < 3$; otherwise $F = 0$. That corresponds to using just two kinds of coupling capacitors $C_{n,n'}$. Thus we have a compact planar structure with a finite number of interacting degrees of freedom.

We present numerical results for $N = 2$, $N = 3$, and $N = 4$ vortices (Figs. 2, 3, and 4, respectively). The relevant parameters in these numerical experiments were: $h = 0.12$, $\xi = 0.05$, and $\delta = 0.04$. As initial states, we took non-symmetric vortex configurations corresponding to numerically found local minima of Hamiltonian (32).

The most regular dynamics was observed for $N = 2$, perhaps because the simplified continuous counterpart (24) is an integrable system in the case of two vortices (besides the Hamiltonian, the angular momentum is conserved). After initial quasi-static period of evolution [Fig. 2(a)], there was stage of oscillatory motion without orbiting [Fig. 2(b)]. Then, it was orbiting in anticlockwise azimuthal direction, with gradually widening cores [Figs. 2(c) and 2(d)]. Finally, wide vortices comparable to the whole system size were transformed to a wave struc-

Figure 2: An example of evolution of two vortices.

Figure 3: An example of evolution of three vortices.
ture propagating mainly clockwise [Figs. 2(e) and 2(f)]. The last stage was practically in linear regime because the effective nonlinear coefficient \( \exp(-2\delta\tau)/\xi^2 \) was very small at \( \tau \gtrsim 100 \).

Vortex clusters with \( 3 \leq N \leq 5 \) passed similar initial two stages in their evolution, but the subsequent dynamics was different. The first stage was again a stable, nearly static configuration, when vortex centers were motionless while their cores gradually broadened according to Eq. (23) [Figs. 3(a) and 4(a)]. The second stage was oscillation of vortices around their previous positions [Figs. 3(b) and 4(b)]. At the third stage, vortices lose stability and begin to move in a complicated manner, typically one or two of them at fast “external” orbits [Figs. 3(c) and 4(c)]. At the fourth stage, the external vortices quit the lattice producing strong short-scale non-vortical oscillations in it [Figs. 3(d) and 4(d)]. During a further evolution, some of the remaining vortices go to external orbits and leave the lattice in a similar manner, until one or two are present on a highly disturbed background (not shown).

Static initial configurations with \( N \geq 6 \) were not found with the given parameters. However, cases \( N = 6 \) and \( N = 6+1 \) (hexagon plus central vortex) were successfully simulated with \( h = 0.04, \xi = 0.025, \) and \( \delta = 0.02 \) (not shown). It should be noted that for this case the quality factor should be extremely high, since \( \gamma/gA_0^2 = \delta\xi^2 \sim 10^{-5} \), while \( gA_0^2 \sim 0.1 \). The dynamics was qualitatively similar to that described above. It is interesting that in the last case, the central vortex lost stability first and quickly passed to external orbit, crossing soon after that the system boundary.

V. SUMMARY

To summarize, in this work a general scheme of an electric network has been suggested which can be approximately described by a weakly dissipative defocusing discrete nonlinear Schrödinger equation of special kind, where coupling terms are not translationally invariant but spatially uniform background solutions exist. Discrete vortices in such systems have been analyzed and numerically simulated.

Of crucial importance is the quality factor of oscillator circuits. Numerical experiments have shown that nontrivial behavior of vortices is observable with \( Q \gtrsim 10^4 - 10^5 \). In practice such values could be achieved at sufficiently low temperatures when conductivity of metals as well as resistivity of dielectrics are both substantially higher than they are at the room temperature.

The system seems deserving more accurate future analysis which apparently should be based upon the original system of equations (10) and (11). In particular, direct numerical simulations of that equations could justify or somehow refine the present results obtained within DNLSE under many simplifying assumptions.

Figure 4: An example of evolution of four vortices.
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