Simplified Analytical Solution for Tapered Circular Elements on Homogeneous or Non-homogeneous Soil

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Abstract. This paper presents a simplified method to examine the response of circular tapered Euler-Bernoulli beam-columns. The Differential Transformation Method (DTM) is implemented to solve the differential equation (DE) that governs the response of the element. When conventional analytical approaches (i.e., discrete or continuum approaches) are used to solve the DE, and because of the introduction of the non-uniform cross-section and the soil non-homogeneity, the analysis becomes rather difficult and the solution complex to obtain. Here, the rather complex DE and corresponding boundary conditions (B.Cs.) are expressed as a system of linear algebraic expressions which solution is readily available. The proposed formulation includes the effects of i) semi-rigid connections and lateral restraints at the ends of the element, ii) an external transverse load, iii) flexible and short elements, iv) soil/element stiffness, and v) an elastic homogeneous or non-homogeneous Pasternak soil. Both static and buckling analysis can be carried out using the same formulation.

1. Introduction and Technical Background

The structural response of tapered circular elements is a topic of interest in various engineering-related fields (civil, mechanical, aeronautics, ocean engineering, etc.) [1, 2]. Several analytical and numerical methods have been proposed to study the static (i.e., determination of deformation, shear, and moment profiles) and stability (buckling) behavior of tapered bars under different scenarios of loading conditions [3–10]. In civil engineering, and in particular for geotechnical applications, these analyses are mainly focused on the interaction between the tapered element and the surrounding soil [11–17]. The use of circular tapered elements (i.e., tapered piles) for foundation purposes is a common practice. It is known that the mechanical response of tapered piles is superior than that of prismatic elements [12, 13, 18]. Studies conducted on this subject are, in general, formulated for i) either static or buckling analyses (not both), ii) homogeneous soils, or iii) classical rotational and lateral translation end-restraints (i.e., free, pinned, fixed, etc.).

This paper presents a simplified method to analyze circular tapered elements with generalized B.Cs. reacting against a non-homogeneous soil. The solution to the DE was found using the DTM. Solving this problem using other classical approaches is a very challenging task. Therefore, seeking alternative solutions for tapered circular elements is of great concern for practitioners. The proposed methodology is limited to circular elements and is assumed that both the soil and the beam-column element behave elastically.
2. Structural Model

2.1. Structural element

Consider the 2D Euler-Bernoulli tapered circular element reacting against an inhomogeneous soil shown in Figure 1(a). The element i) is connected at both ends by semi-rigid connections and linear transverse springs $k$ (units in force $\times$ length/radian) and $S$ (units in force/length), respectively; ii) has a length $L$ and stiffness $EI(x)$; iii) is reacting against an elastic soil; iv) is subjected to external axial and lateral loads at its ends ($P$, $V$, and $M$), and v) is subjected to an external distributed load of the form: $a + bx + cx^2$.

2.2. Governing equations

The governing DE of the circular tapered element is found by applying equilibrium to the element shown in Figure 2(a). From equilibrium:

$$\frac{dV(x)}{dx} = p(x) - q(x)$$

$$\frac{dM(x)}{dx} = V(x) + P\frac{dy}{dx}$$

where $M(x) = -EI(x)\frac{d^2y}{dx^2}$; $p(x) = K_s(x)y - k_g\frac{d^2y}{dx^2}$, and $q(x) = \gamma_1 + S_l x + t_1 x^2$. $K_s(x) = (K_o + m_h x)B(x)$ and $k_g$ are the first-parameter and second-parameter of the Pasternak soil, respectively. $m_h = 0$ represents a uniform variation of $K_s$ (homogeneous soil) and $m_h \neq 0$ a linear variation (inhomogeneous soil). The dimensions of $K_s$, $k_g$, and $m_h$ are force/length$^3$, force, and force/length$^4$, respectively. The diameter and variation of the second moment of inertia are expressed, respectively, as: $\phi(x) = 2r_1 \left[1 + \frac{x}{L}(m-1)\right]$ and $I(x) = \frac{\pi r_1^4}{4} [1 + \frac{x}{L}(m-1)]^4$. Where $m = r_b/r_t$ is the taper ratio, and $r_b$ and $r_t$ are the radius of the element at the bottom and top, respectively.

Deriving Equation (2) once and then substituting it into Equation (1), and after some mathematical manipulation, the following expression for the governing DE is found:

$$EI(x)\frac{d^4y}{dx^4} + 2E\frac{dI}{dx}\frac{d^3y}{dx^3} + \left[E\frac{d^2I}{dx^2} + (P - k_g)\right]\frac{d^2y}{dx^2} + (K_o + m_h x)\phi(x)y = q(x)$$
Letting $\xi = x/L$ and $Y = y/L$, the following normalized form of the DE is obtained:

$$I(\xi) \frac{d^4Y}{d\xi^4} + 2I'(\xi) \frac{d^3Y}{d\xi^3} + N(\xi) \frac{d^2Y}{d\xi^2} + \Lambda(\xi) \frac{dY}{d\xi} + Q(\xi) = 0 \quad (4)$$

Equation (4) can be rewritten in the following form:

$$I(\xi) \frac{d^4Y}{d\xi^4} + 2I'(\xi) \frac{d^3Y}{d\xi^3} + N(\xi) \frac{d^2Y}{d\xi^2} + \Lambda(\xi) \frac{dY}{d\xi} + Q(\xi) = 0 \quad (5)$$

where $I'(\xi) = dI/d\xi$, $I(\xi) = \frac{\pi r^4}{4} [1 + \xi(m-1)^2]$, $N(\xi) = \{3\pi r^4(m-1)^2[1+\xi(m-1)^2]+\frac{L^2}{E}(P-k_g)\}$,

$\Lambda(\xi) = \frac{2\pi r^4 L}{E}\{K_0 + [K_0(m-1) + m_h L]\xi + m_h L(m-1)\xi^2\}$, and $Q(\xi) = \frac{L^3}{E}(y_1 + S_1 L\xi + t_1 L^2 \xi^2)$.

The normalized B.C.s. of the proposed element are:

At $\xi = 0$

$$EI_a \frac{d^2Y}{L \frac{d\xi^2}{} - k_a \frac{dY}{d\xi} + M_a = 0 \quad (6)$$

At $\xi = 1$

$$-EI_b \frac{d^2Y}{L \frac{d\xi^2}{} - k_b \frac{dY}{d\xi} + M_b = 0 \quad (8)$$

$$\frac{d^3Y}{d\xi^3} + I_b \frac{d^2Y}{L \frac{d\xi^2}{} + (P+k_g) \frac{dY}{d\xi} - S_h \frac{dY}{d\xi} + V_h = 0 \quad (9)$$

where $I_a = I(\xi) = 0$, $I'_a = I'(\xi) = 0$, $I_b = I(\xi) = 1$, and $I'_b = I'(\xi) = 1$. For easy implementation, and because of the element linear variation in cross-section, the analysis presented herein is conducted by taking the coordinate $x = L/2$ as the reference point. The radius ($r_{eq}$) of the element at $x = L/2$ and the variation of the taper ratio are shown in Figure 2(b). Equation (5) is the resulting governing DE of the proposed tapered circular element. When some variables are not considered in the analysis, Equation (5) and Equations (6) - (9) can capture particular cases developed by other researchers [19–22].
Table 1. Functions $N(\xi)$, $\Lambda(\xi)$, and $Q(\xi)$

| Original/Transformed functions at $\xi = 0$ |
|--------------------------------------------|
| $N(\xi) = I''(\xi) + \frac{L^2}{E} [P - k_g]$ |
| $\tilde{N}(0) = N_0 = [3\pi r_0^4(m - 1)^2 + \frac{L^2}{E} (P - k_g)]$ |
| $\tilde{N}(1) = N_1 = [6\pi r_0^4(m - 1)^3]$ |
| $\tilde{N}(2) = N_2 = [3\pi r_0^4(m - 1)^4]$ |
| $\Lambda(\xi) = \frac{L^2}{E} (K_o + m_b\xi L) B(\xi)$ |
| $\tilde{\Lambda}(0) = \Lambda_0 = 2r_1 L^4 K_o / E$ |
| $\tilde{\Lambda}(1) = \Lambda_1 = 2r_1 L^4 [(m - 1) + m_b L] / E$ |
| $\tilde{\Lambda}(2) = \Lambda_2 = 2r_1 L^4 [(m - 1)m_b L] / E$ |
| $Q(\xi) = \frac{L^2}{E} (\gamma_1 + S_1 L \xi + t_1 L^2 \xi^2)$ |
| $\tilde{Q}(k) = \frac{L^2}{E} \left[ \gamma_1 \delta (k) + S_1 L \delta (k - 1) + t_1 L^2 \delta (k - 2) \right]$ |
| $\delta (k - n) = 1 \text{ if } k = n(*)$ |
| $\delta (k - n) = 0 \text{ if } k \neq n(*)$ |

2.3. Transformation of the DE and B.Cs.

The normalized DE and corresponding B.Cs. are solved implementing the DTM developed by [23]. For details about the DTM approach employed herein, and for the sake of brevity, the reader is referred to [22]. The solution to the DE in terms of the DTM is:

$$Y(\xi) = \tilde{Y}(0) + \tilde{Y}(1) \xi + \tilde{Y}(2) \xi^2 + \tilde{Y}(3) \xi^3 + \cdots + \tilde{Y}(m) \xi^m = \sum_{k=0}^{\infty} \tilde{Y}(k) \xi^k$$  \hspace{1cm} (10)

$$\tilde{Y}(k) = \frac{1}{k!} \left[ \frac{d^k Y(\xi)}{d\xi^k} \right]_{\xi=0}$$  \hspace{1cm} (11)

where $Y(\xi)$ is the transverse deflection of the element, and $\tilde{Y}(k)$ are the transformed coefficients. The transformed DE can be expressed as (Theorems listed in Table 1 from [22]):

$$\sum_{r=0}^{k} \tilde{I}(k - r)(r + 1)(r + 2)(r + 3)(r + 4) \tilde{Y}(r + 4) + 2 \sum_{r=0}^{k} \tilde{N}(k - r)(r + 1) \tilde{Y}(r + 1) + 2 \sum_{r=0}^{k} \tilde{\Lambda}(k - r) \tilde{Y}(r) = \tilde{Q}(k)$$  \hspace{1cm} (12)

where $\tilde{I}(k)$, $\tilde{N}(k)$, $\tilde{\Lambda}(k)$, and $\tilde{Q}(k)$ are the transformed functions of $I(\xi)$, $N(\xi)$, $\Lambda(\xi)$, and $Q(\xi)$, respectively. Using the properties of the Kronecker delta function ($\delta(\cdot)$ from Table 1), the recurrence form of the DE is rewritten as:

$$\tilde{Y}(k + 4) = \left\{ (L^2/E) \left[ \gamma_1 \delta (k) + S_1 L \delta (k - 1) + t_1 L^2 \delta (k - 2) \right] - \left[ \Lambda_2 \tilde{Y}(k - 2) + \Lambda_1 \tilde{Y}(k - 1) + \Lambda_0 \tilde{Y}(k) \right] \right\}$$

$$+ \left\{ (k - 1) k N_2 \tilde{Y}(k) + k (k + 1) N_1 \tilde{Y}(k + 1) + (k + 1) (k + 2) N_0 \tilde{Y}(k + 2) \right\} - \left\{ I_1 (k + 3)(k + 2)^2(k + 1) \tilde{Y}(k + 3) \right\}$$

$$+ I_2 (k + 3)(k + 2)(k + 1) k \tilde{Y}(k + 2) + I_3 (k + 4)(k + 1)(k - 1) \tilde{Y}(k + 1) + I_4 (k + 5) k (k - 1) \tilde{Y}(k) \right\} / I_o k_4$$  \hspace{1cm} (13)

where $I_o = \pi r_0^4 / 4$, $I_1 = 4(m - 1) \pi r_0^4 / 4$, $I_2 = 6(m - 1)^2 \pi r_0^4 / 4$, $I_3 = 4(m - 1)^3 \pi r_0^4 / 4$, $I_4 = (m - 1)^4 \pi r_0^4 / 4$, and $k_4 = (k + 4)(k + 3)(k + 2)(k + 1)$. Terms $N_0$, $N_1$, $N_2$, $\Lambda_0$, $\Lambda_1$, and $\Lambda_2$ are listed in Table 1. Here, $\tilde{Y}(-2) = \tilde{Y}(-1) = 0$. 

The B.C.s. at $\xi = 0$ are transformed as follows (Equation (11)):

$$2\ddot{Y}(2) - \frac{k_a}{EI_a/L}\ddot{Y}(1) + \frac{M_a}{EI_a/L} = 0$$

$$-6\ddot{Y}(3) - \frac{2\dot{Y}}{I_a}(2) - \frac{(P - k_g)}{EI_a/L^2}\ddot{Y}(1) - \frac{S_a\dot{Y}(0)}{EI_a/L^3} + \frac{V_a}{EI_a/L^2} = 0$$

At $\xi = 1$ are transformed as follows (Equation (10)):

$$-\sum_{k=0}^{m} k(k-1)\ddot{Y}(k) - \frac{k_b}{EI_b/L}\sum_{k=0}^{m} k\ddot{Y}(k) + \frac{M_b}{EI_b/L} = 0$$

$$\sum_{k=0}^{m} k(k-1)(k-2)\ddot{Y}(k) + \frac{\dot{Y}}{I_b}\sum_{k=0}^{m} k(k-1)\ddot{Y}(k) + \frac{(P + k_g)}{EI_b/L^2}\sum_{k=0}^{m} k\ddot{Y}(k) - \frac{S_b}{EI_b/L^3}\sum_{k=0}^{m} \dddot{Y}(k)$$

$$+ \frac{V_b}{EI_b/L^2} = 0$$

3. Proposed Solution

There exist several methods to solve the resulting set of equations (Equations (16) and (17)). Here, a simple procedure using the proposed formulation is suggested to conduct lateral deformation or buckling analysis.

**For static analyses:** set Equations (16) and (17) equal to $T1$ and $T2$, respectively. Now, assume a value of $\ddot{Y}(0)$ and determine the value of $\ddot{Y}(1)$ that makes $T1 = 0$ and the value that makes $T2 = 0$. Repeat the procedure for different values of $\ddot{Y}(0)$. With these coordinates ($\ddot{Y}(0), \ddot{Y}(1)$), plot functions $T1$ and $T2$. Here, the intersection of these two functions is the solution that satisfies both B.C.s. Once $\dddot{Y}(0)$ and $\dddot{Y}(1)$ are known, determine $\dddot{Y}(2) - \dddot{Y}(m)$. The element transverse deflection is determined from Equation (10), and the moment and shear profiles from $M(\xi) = -(EI(\xi)/L)\frac{d^2Y}{d\xi^2}$ and $V(\xi) = -(EI(\xi)/L^2)\frac{dY}{d\xi} - (P - k_g)\frac{dY}{d\xi}$, respectively. These steps can be easily carried out with available commercial software or a spreadsheet such as excel.

**For stability analyses:** because coefficients $\dddot{Y}(2) - \dddot{Y}(m)$ can be expressed in terms of $\dddot{Y}(0)$ and $\dddot{Y}(1)$, expressions $T1$ and $T2$ can be arranged in terms of $\dddot{Y}(0)$ and $\dddot{Y}(1)$ as follows:

$$\begin{bmatrix} T1 \\ T2 \end{bmatrix} = \begin{bmatrix} T11 & T12 \\ T21 & T22 \end{bmatrix} \begin{bmatrix} \dddot{Y}(0) \\ \dddot{Y}(1) \end{bmatrix}$$

Terms $T11$ and $T21$ are found by letting $\dddot{Y}(0) = 1$ and $\dddot{Y}(1) = 0$, and then computing $T1$ and $T2$. Here, $T11 = T1$ and $T21 = T2$. In an analogous manner, terms $T12$ and $T22$ are found by letting $\dddot{Y}(0) = 0$ and $\dddot{Y}(1) = 1$. Now, $T12 = T1$ and $T22 = T2$. The buckling loads are found by forcing the determinant of the matrix to vanish (i.e., $Det = T11 * T22 - T21 * T12$). The critical load, $P_{cr1}$, is taken as the lowest buckling load.

4. Parametric study

A parametric study is conducted to investigate the effect of the taper ratio ($m$), soil non-homogeneity (i.e., linear distribution of $K_s$ with length), and intermediate values of rotational and lateral translation end-restraints on the static and stability behavior of circular tapered piles, and over a wide range of soil stiffness ratios $a = \sqrt{\frac{K_{eq}L}{EI_{eq}}}$. The taper ratios investigated are $m = 1$ (prismatic element), $m = 0.5$, and $m = 0.3$. Even though the proposed formulation includes the effect of the second-parameter of the soil, this parameter is not investigated herein. This analysis is part of an ongoing study.
results from the model are compared with results from finite element analysis (SAP2000). In SAP2000, the element was subdivided into 100 segments for stability analyses and into 500 segments for static analyses to achieve the desired accuracy. In SAP2000, the shear deformation was neglected by assuming a very high value of the shear rigidity of the element.

Figure 3. Example 1 - Head deflection of a pile subjected to a horizontal load (a and b) and a bending moment (c and d).

Example 1: Pile-head transverse deflection of a circular tapered pile.
Figure 3 shows the normalized pile-head transverse deflection of a flexible tapered pile ($L/B > 40$) subjected to lateral loads at its head. The pile is free at the bottom and partially inhibited against rotation at its head. As anticipated, and regardless of the value of $m$, the normalized pile-head deflection, $y_{x=0}B^2K_{eq}/M$, decreases as $k_a$ increases, and increases as the normalized soil stiffness parameter, $E/(K_{eq}B)$ or $E/(m_hB^2)$, increases. The results from the proposed formulation agree well with those obtain from finite element analysis.

Example 2: Critical load of tapered pile on homogeneous or non-homogeneous elastic soil.
Figure 4 shows the critical load ($P_{crit}$) of a free-free and a clamped-clamped tapered pile with a soil non-homogeneity index of $F = 0, 0.5$, and $F = 1$. In general, it is noted that $P_{crit}$ is highly influenced by the taper ratio, head-boundary condition, and soil non-homogeneity. From Figures 4(a)-(f), it is observed that, as soil non-homogeneity increases (i.e., moves from $F = 1$ towards $F = 0$), $P_{crit}$ significantly decreases for prismatic elements ($m = 1$) and increases for the tapered ones ($m ≠ 1$). This effect is more notorious at high values of soil stiffness parameter $\alpha$. These results highlight the contribution of the upper portion of the soil on $P_{crit}$. In this portion of the pile, the effective width reacting against the soil is greater in tapered piles than in those of prismatic ones. As for the previous example, the results show good agreement with those from SAP2000.

Example 3: Critical load of a tapered pile partially inhibited against lateral translation at its head.
Figure 5 shows the $P_{crit}$ of a tapered pile clamped at the base and clamped with sidesway partially inhibited at its head. Figures 5(a)-(c) show the variation of $P_{crit}$ for a pile in a homogeneous soil. It is
Figure 4. Example 2 - Variation of $P_{crit}$ with soil non-homogeneity and taper ratio for a free-free and a fixed-fixed pile.

Figure 5. Example 3 - Variation of $P_{crit}$ with soil non-homogeneity, taper ratio, and pile-head lateral transverse stiffness.
observed that $P_{crit}$ for a prismatic element increases as the magnitude of the lateral transverse stiffness, $S$, increases. However, the increase, if any, for taper ratios $m = 0.5$ and $m = 0.3$ is much less pronounced, indicating that the contribution of $S$ on $P_{crit}$ is minimal, and the elastic stability capacity of the element is provided mainly by the surrounding soil. On the other hand, the effect of $S$ on $P_{crit}$ varies, and depends on $m$ and $F$, when the pile is embedded in a non-homogeneous soil (Figures 5(d)-(f)). For instance, $P_{crit}$ for tapered piles is significantly greater than that of prismatic piles, and its magnitude depends on both, the variation of the non-uniform cross-section ($m$-value) and the soil non-homogeneity.

5. Summary and Conclusions
A simplified, analytical method to carry out static or stability analyses of circular tapered Euler-Bernoulli elements on a Pasternak foundation was presented. The rather complex governing DE and corresponding B.C.s. were solved using the DTM. These equations were converted, without major difficulties, into a set of linear algebraic expressions which solution was simple to find. The effects of i) semi-rigid connections and lateral translation restraints at the ends of the element, ii) an external transverse load, iii) long and short elements, iv) soil/element stiffness, and v) soil inhomogeneity were included in the proposed formulation. Three examples were provided to show the versatility of the method and its accuracy. The results, just by using a single segment, were in good agreement with those obtained using finite element analysis. From the parametric study, it was found that the response of circular tapered beam-columns is highly influenced by the taper ratio, soil non-homogeneity, and end-boundary conditions.

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