On Locating Chromatic Number of Cubic Graph with Tree Cycle, $C_{n,2n,n}$, for $n=3,4,5$

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Abstract. Let $G = (V(G),E(G))$ be a connected graph and $c$ is coloring of graph $G$. Let $\Pi = \{C_1, C_2, ..., C_k\}$, where $C_i$ is the partition of the vertex in $G$ which is colored $i$ with $1 \leq i \leq k$. The representation $\nu$ for $\Pi$ is called the color code, denoted $C_\Pi(\nu)$ is an ordered pair with $k$-element namely, $C_\Pi(\nu) = (d(\nu, C_1), d(\nu, C_2), ..., d(\nu, C_k))$, where $d(\nu, C_i) = \min\{d(\nu, x) | x \in C_i\}$ for $1 \leq i \leq k$. If every vertex in $G$ have different color code, the $c$ is locating coloring. The minimum number of colors used in $G$ is called chromatic locating, notated by $\chi_l(G)$. In this paper, we will determine the locating coloring of graph cubic $C_{n,2n,n}$ for $n=3,4,5$.

1. Introduction

Chartrand, Salehi, and Zhang [4] introduced the concept of graph partition dimension in 1998. Many studies have been devoted to determine the partition dimension of graphs, alike in [3]. Introduced the locating-chromatic number of graphs, this notion is a combined concept between graph partition dimension and graph coloring.

Chartrand et al. [5] determined the locating-chromatic numbers for some well-known classes: path, cycle, complete multipartite graphs and double stars. The locating-chromatic number of a cycle $C_n$ is 3 if $n$ is odd and 4 if $n$ is even.

Furthermore, Welyyanti et al. expand definition of the locating chromatic number of disconnected graph[7]. One of the results of that study is the locating chromatic number of graph with two components[8] and two homogenous components[9].

The locating chromatic number of firecrackers graphs was studied by Asmiati et al in 2012 [1]. The locating chromatic number for a graph with a dominant vertex in 2015 was determined by Welyyanti, et al [6]. Furthermore, Asmiati, et al examined the locating chromatic number from amalgamation of star graphs in 2011 [2]. Based on the description above, to get new results, we are interested in studying the problem of determining the chromation of a cubic graph of $C_{n,2n,n}$.

Let us begin to state the following lemma and corollary which are useful to obtain our main results.

**Lemma 2.1.** [5] let $c$ be a locating coloring in a connected graph $G$. If $u$ and $v$ are distinct vertex of $G$ such that $d(u, w) = d(v, w)$ for all $w \in V(G) \setminus \{u, v\}$, then $c(u) \neq c(v)$. In particular, if $u$ and $v$ are adjacent to the same vertex, then $c(u) \neq c(v)$.
Corollary 2.2 \cite{5} if $G$ is a connected graph containing a vertex adjacent to $k$ leaves, then $\chi_L(G) \geq k + 1$.

A cubic graph denoted by $C_{n_1, n_2, \ldots, n_k}$ is a cubic graph formed from $k$ cycle $C_{n_i}^i$ by $i = 1, 2, \ldots, k$, where $C_{n_i}^i = v_{i1} v_{i2}, \ldots, v_{i(n_i)} v_{i1}$ is a circular graph to $i$ with $n_i$ vertices. Given a cycle graph with $n$ vertex written as $C_n$, with $n \geq 3$. Then formed a cubic graph consisting of tree cycle graph namely $C_n^1, C_n^2, C_n^3$ with $n \geq 3$.

The set of vertex on cubic graph $C_{n_2n,n}$ is $V(C_{n_2n,n}) = \{v_{1i}, v_{3i} \mid 1 \leq i \leq n\} \cup \{v_{2j} \mid 1 \leq j \leq 2n\}$ and the many vertices on the cubic $C_{n_2n,n}$ is $|V(C_{n_2n,n})| = 2n + 2n = 4n$. To get the edges of cubic graph, the following operations are performed \cite{7}.

1. The relationship of the vertex at $C_n^1$ with $C_{2n}^2$ namely with adding edge $\{v_{1i}, v_{2j}\}$ with $v_{1i} \in V(C_n^1)$ and $v_{2j} \in V(C_{2n}^2)$ for $1 \leq i \leq n$ and $j = 2i - 1$.

2. The relationship of the vertex at $C_{2n}^2$ with $C_n^3$ namely with adding edge $\{v_{2j}, v_{3i}\}$ with $v_{2j} \in V(C_{2n}^2)$ and $v_{3i} \in V(C_n^3)$ for $j = 2i$ and $1 \leq i \leq n$.

It is obtained that the edge set of cubic graph $C_{n_2n,n}$ is $E(C_{n_2n,n}) = \left(\bigcup_{i=1,3} E(C_n^i)\right) \cup \left(\bigcup_{i=2} E(C_{2n}^i)\right)$ for $1 \leq i \leq n, j = 2i$. The set of edge of graph $C_n^1, C_{2n}^2, C_n^3$ is as follows:

$E(C_n^1) = \{v_{1i}v_{1(i+1)}\}$ for $1 \leq i \leq n - 1$ and $v_{1i}v_{1n}$

$E(C_{2n}^2) = \{v_{2i}v_{2(i+1)}\}$ for $1 \leq i \leq 2n - 1$ and $v_{2i}v_{2n}$

$E(C_n^3) = \{v_{3i}v_{3(i+1)}\}$ for $1 \leq i \leq n - 1$ and $v_{3i}v_{3n}$

The number of edge of graph $C_{n_2n,n}$ namely $2(n - 1) + (2n - 1) + n + n = 4n - 3 + n + n = 6n - 3$. Ased on the description above, a cubic graph $C_{n_2n,n}$ with $n \geq 3$ can be constructed as shown in Figure 1.

![Figure 1. A Cubic Graph $C_{n_2n,n}$.](image-url)
Main Results

Theorem 2.1 If $G_{n,2n,n}$ is cubic graph with $3 \leq n \leq 5$, then $\chi_L(G_{n,2n,n}) = 5$.

Proof. If we have 4-locating coloring, then $G_{n,2n,n}$ will have two dominant with same color. It means $G_{n,2n,n}$ will have two vertices with same color code. Consequently, $\chi_L(G_{n,2n,n}) \geq 5$.

Let $c$ be a locating coloring in a graph cubic $G_{n,2n,n}$ and $\Pi = \{C_1, C_2, ..., C_5\}$ is a partition of $V(G_{n,2n,n})$ into the color class $C_i$. Consider the following cases:

- for $n = 3$. We will show if $C_{3,6,3}$ have a 4-coloring vertex then $C_{3,6,3}$ will have at least two vertex with the same color code in cubic $C_{3,6,3}$. Without loss generality, let $v_1, v_2, v_3$ be colored 1 and the neighboring vertices are colored 2,3 and 4. Furthermore, $v_4, v_5, v_6$ are colored with color 2 and the adjacent vertex $v_1, v_2, v_3$ are colored with color 1,3 and 4. $v_7, v_8$ are colored with color 3 and the adjacent vertex $v_1, v_2$ are colored with color 1,2,3 and the other vertex are colored with any color combination with 4 colors, then there will be two dominant vertex of the same color. So it must $\chi_L(C_{3,6,3}) \geq 5$.

Consider Figure 2. a 4-coloring vertex for $C_{3,6,3}$ there are two vertex that have the same color code are $\Pi(x_2) = \Pi(x_6) = (1,0,2,1)$, it must be $\chi_L(C_{3,6,3}) \geq 5$

Next, we will prove the upper bounds of the locating chromatic number of $C_{3,6,3}$ It will be shown that $\chi_L(C_{3,6,3}) \leq 5$. Suppose that the construction of the vertex coloring of a cubic graph $C_{3,6,3}$ is defined by $c : V(C_{3,6,3}) \to \{1,2,3,4,5\}$ such that:

\[
\begin{align*}
    c(v_1) = c(v_2) &= c(v_3) = c(v_5) = c(v_6) = 1 \\
    c(v_4) = c(v_7) &= c(v_8) = 2 \\
    c(v_{10}) = 3 \\
    c(v_9) &= c(v_{11}) = 4 \\
    c(v_{12}) &= 5
\end{align*}
\]

Based on this construction, the following color classes are obtained:

- $S_1 = \{v_{11}, v_{23}, v_{25}, v_{32}\}$
- $S_2 = \{v_{12}, v_{22}, v_{24}, v_{26}\}$
From the definition of vertex coloring above, it can be seen that each adjacent vertex is colored with a different color. It can be seen in Figure 3 vertex coloring with 5 colors as follows:

\[ S_3 = \{x_{13}\} \]
\[ S_4 = \{x_{21}, x_{31}\} \]
\[ S_5 = \{x_{33}\} \]

The color code for each vertex on graph \( C_{3,6,3} \) to \( \Pi \) is as follows:

\[ C_\Pi(x_{11}) = (d(x_{11}, S_1), d(x_{11}, S_2), d(x_{11}, S_3), d(x_{11}, S_4), d(x_{11}, S_5)) = (0,1,1,1,3) \]
\[ C_\Pi(x_{23}) = (d(x_{23}, S_1), d(x_{23}, S_2), d(x_{23}, S_3), d(x_{23}, S_4), d(x_{23}, S_5)) = (0,1,2,2,3) \]
\[ C_\Pi(x_{25}) = (0,1,1,2,2) \]
\[ C_\Pi(x_{24}) = (1,0,2,3,2) \]
\[ C_\Pi(x_{32}) = (0,1,3,1,1) \]
\[ C_\Pi(x_{13}) = (1,1,0,2,3) \]
\[ C_\Pi(x_{12}) = (1,0,1,2,4) \]
\[ C_\Pi(x_{26}) = (1,0,2,1,1) \]
\[ C_\Pi(x_{22}) = (1,0,3,1,2) \]
\[ C_\Pi(x_{21}) = (1,1,2,0,2) \]
\[ C_\Pi(x_{31}) = (1,1,4,0,1) \]
\[ C_\Pi(x_{33}) = (1,1,3,1,0) \]

Therefore, Figure 3 has 5 coloring for graph \( C_{3,6,3} \) with a different color code, by \( c \) is the coloring of the locating of \( C_{3,6,3} \), so we get \( \chi_L(C_{3,6,3}) \leq 5 \). It can be seen from the proofs \( \chi_L(C_{3,6,3}) \leq 5 \) and \( \chi_L(C_{3,6,3}) \geq 5 \). It can be concluded \( \chi_L(C_{3,6,3}) = 5 \).

- for \( n = 4 \). We will show if \( C_{4,8,4} \) have a 4-coloring vertex then \( C_{4,8,4} \) will have at least two vertex with the same color code in \( C_{4,8,4} \). Without loss of generality, let \( x_{11}, x_{23}, x_{27}, x_{31}, x_{33} \) be colored 1 and the neighboring that vertices are colored 2 and 3. Furthermore, \( x_{12}, x_{22}, x_{24}, x_{26}, x_{28} \) are colored with color 2 and the adjacent of vertex \( x_{12}, x_{22}, x_{24}, x_{26}, x_{28} \) are colored with color 1,3 and 4. \( x_{13} \) is colored with color 3 and the adjacent of vertex \( x_{13} \) is...
colored with 1, 2, and 4. $x_{14}, x_{21}, x_{25}, x_{32}, x_{34}$ are colored with color 4 and the adjacent of that vertices $x_{14}, x_{21}, x_{25}, x_{32}, x_{34}$ are colored with 1, 2, 3 and the other vertex are colored with any color combination with 4 colors, then there will be two dominant vertex of the same color. So it must $\chi_L(C_{4,8,4}) \geq 5$.

Figure 4. A Coloring with 4 colors for $C_{4,8,4}$

Consider Figure 4 a 4-coloring vertex for $C_{n,2n,n}$ there are two vertices that have the same color code are $G_1(x_{11}) = G_1(x_{27}) = (0,1,2,1)$, it must be $\chi_L(C_{4,8,4}) \geq 5$ and each vertex has a degree of 3. Next, we will prove the upper bounds of the locating chromatic number of $C_{4,8,4}$.

Suppose that the construction of the vertex coloring of a cubic graph $C_{4,8,4}$ is defined by $c : V(C_{4,8,4}) \rightarrow \{1,2,3,4,5\}$ such that:

$c(x_{11}) = c(x_{23}) = c(x_{25}) = c(x_{32}) = c(x_{34}) = 1$
$c(x_{12}) = c(x_{22}) = c(x_{24}) = c(x_{26}) = c(x_{28}) = 2$
$c(x_{13}) = 3$
$c(x_{14}) = c(x_{21}) = c(x_{25}) = c(x_{31}) = c(x_{33}) = 4$
$c(x_{34}) = 5$

Based on this construction, the following color classes are obtained:

$S_1 = \{x_{11}, x_{23}, x_{25}, x_{27}, x_{32}\}$
$S_2 = \{x_{12}, x_{22}, x_{24}, x_{26}, x_{28}\}$
$S_3 = \{x_{13}\}$
$S_4 = \{x_{14}, x_{21}, x_{31}, x_{33}\}$
$S_5 = \{x_{34}\}$
Figure 5. A Coloring with 5 colors for $C_{4,8,4}$

It will be shown that $\chi_L(C_{4,8,4}) \leq 5$. It can be seen in Figure 5 vertex coloring with 5 colors as follows.

The color code for each vertex on graph $C_{4,8,4}$ to $\Pi$ is as follows:

$C_\Pi(x_{11}) = (d(x_{11}, S_1), d(x_{11}, S_2), d(x_{11}, S_3), d(x_{11}, S_4), d(x_{11}, S_5)) = (0,1,2,1,3)$

$C_\Pi(x_{23}) = (d(x_{23}, S_1), d(x_{23}, S_2), d(x_{23}, S_3), d(x_{23}, S_4), d(x_{23}, S_5)) = (0,1,2,2,3)$

$C_\Pi(x_{27}) = (0,1,2,1,2)$  $C_\Pi(x_{32}) = (0,1,3,1,2)$

$C_\Pi(x_{12}) = (1,0,1,2,4)$  $C_\Pi(x_{22}) = (1,0,3,1,2)$

$C_\Pi(x_{29}) = (1,0,3,1,4)$  $C_\Pi(x_{13}) = (2,1,0,1,4)$

$C_\Pi(x_{14}) = (1,2,1,0,3)$  $C_\Pi(x_{21}) = (1,1,3,0,2)$

$C_\Pi(x_{31}) = (1,1,4,0,1)$  $C_\Pi(x_{33}) = (1,1,3,0,1)$

$C_\Pi(x_{34}) = (2,1,4,1,0)$

Therefore, Figure 5 has 5-coloring vertex for graph $C_{3,6,3}$ with a different color code, by $c$ is the coloring of the locating of $C_{4,8,4}$, so we get $\chi_L(C_{4,8,4}) \leq 5$. It can be seen from the proofs $\chi_L(C_{4,8,4}) \leq 5$ and $\chi_L(C_{4,8,4}) \geq 5$ it can be concluded $\chi_L(C_{4,8,4}) = 5$.

- for $n = 5$. We will show if $C_{5,10,5}$ have a 4-coloring vertex then $C_{5,10,5}$ will have at least two vertex with the same color code in $C_{5,10,5}$. Without loss of generality, let $x_{11}, x_{13}, x_{27}, x_{32}, x_{35}$ be colored 1 and the neighboring that vertices are colored 2,3 and 4. Furthermore, $x_{12}, x_{14}, x_{22}, x_{24}, x_{26}, x_{28}, x_{210}$ are colored with color 2 and the adjacent of vertex $x_{12}, x_{14}, x_{22}, x_{24}, x_{26}, x_{28}, x_{210}$ are colored with color 1,3 and 4. $x_{15}, x_{33}$ are colored with color 3 and the adjacent of vertex $x_{15}, x_{33}$ are colored with 1,2, and 4. $x_{21}, x_{25}, x_{29}, x_{31}, x_{34}$ are colored with color 4 and the adjacent of vertex $x_{21}, x_{25}, x_{29}, x_{31}, x_{34}$
are colored with 1,2,3 and the other vertex are colored with any color combination with 4 colors, then there will be two dominant vertex of the same color. So it must $\chi_L(C_{5,10,5}) \geq 5$.

**Figure 6.** Coloring with 4 colors for $C_{5,10,5}$

Consider Figure 6 a 4-coloring vertex for $C_{5,10,5}$ there are two vertices that have the same color code are $C_{[1]}(x_{11}) = C_{[1]}(x_{32}) = (0,1,1,1)$, it must be $\chi_L(C_{5,10,5}) \geq 5$ and each vertex has a degree of 3. Next, we will prove the upper limit of the chromatic number of location $C_{n,2n,n}$. It will be shown that $\chi_L(C_{5,10,5}) \leq 5$.

Suppose that the construction of the vertex coloring of a cubic graph $C_{5,10,5}$ is defined by $c : V(C_{5,10,5}) \to \{1,2,3,4,5\}$ such that:

- $c(x_{11}) = c(x_{13}) = c(x_{23}) = c(x_{25}) = c(x_{27}) = c(x_{32}) = c(x_{34}) = 1$
- $c(x_{12}) = c(x_{14}) = c(x_{22}) = c(x_{24}) = c(x_{26}) = c(x_{28}) = c(x_{210}) = 2$
- $c(x_{15}) = 3$
- $c(x_{21}) = c(x_{25}) = c(x_{29}) = c(x_{31}) = c(x_{33}) = 4$
- $c(x_{35}) = 5$

Based on this construction, the following color classes are obtained:

- $S_1 = \{x_{11}, x_{23}, x_{25}, x_{27}, x_{32}, x_{34}\}$
- $S_2 = \{x_{12}, x_{22}, x_{24}, x_{26}, x_{28}, x_{210}\}$
- $S_3 = \{x_{15}\}$
- $S_4 = \{x_{21}, x_{25}, x_{29}, x_{31}, x_{33}\}$
- $S_5 = \{x_{35}\}$

From the definition of vertex coloring above, it can be seen that each adjacent vertex is colored with a different color. It can be seen in Figure 7 vertex coloring with 5 colors as follows:
Figure 7. A Coloring with 5 colors for $C_{5,10,5}$

The color code for each vertex on graph $C_{5,10,5}$ to $\Pi$ is as follows:

$C_\Pi(x_{11}) = (d(x_{11}, S_1), d(x_{11}, S_2), d(x_{11}, S_3), d(x_{11}, S_4), d(x_{11}, S_5)) = (0,1,1,1,3)$

$C_\Pi(x_{13}) = (d(x_{13}, S_1), d(x_{13}, S_2), d(x_{13}, S_3), d(x_{13}, S_4), d(x_{13}, S_5)) = (0,1,2,1,4)$

$C_\Pi(x_{12}) = (1,0,2,2,4)$  $C_\Pi(x_{33}) = (0,1,4,1,2)$

$C_\Pi(x_{23}) = (0,1,3,2,3)$  $C_\Pi(x_{32}) = (0,1,4,1,5)$

$C_\Pi(x_{12}) = (1,0,2,2,4)$  $C_\Pi(x_{14}) = (1,0,1,2,4)$

$C_\Pi(x_{22}) = (1,0,3,1,2)$  $C_\Pi(x_{24}) = (1,0,4,1,3)$

$C_\Pi(x_{210}) = (2,0,2,1,1)$  $C_\Pi(x_{15}) = (1,1,0,1,3)$

$C_\Pi(x_{21}) = (1,1,2,0,2)$  $C_\Pi(x_{25}) = (1,1,3,0,4)$

$C_\Pi(x_{34}) = (1,1,3,0,1)$  $C_\Pi(x_{35}) = (1,1,3,1,0)$

Therefore, Figure 7 has 5 coloring for $C_{5,10,5}$ with a different color code, by $c$ is the coloring of the locating of $C_{5,10,5}$ so we get $\chi_L(C_{5,10,5}) \leq 5$. It can be seen from the proofs $\chi_L(C_{5,10,5}) \leq 5$ and $\chi_L(C_{5,10,5}) \geq 5$ it can be concluded $\chi_L(C_{5,10,5}) = 5$.

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References
[1] Asmiati, H. Assiyatun, E. T. Baskoro, D. Suprijanto, R. Simanjuntak and S. Uttunggadewa. 2012.
    The locating-chromatic number of firecracker graphs. Far East Journal of Mathematical Sciences. 63(1):11-23.
[2] Asmiati, Assiyatun H, Baskoro E.T. 2011.Locating-Chromatic of Amalgamation of Stars. ITB J
Sci.43A: 1-8.

[3] Chartrand, G., Salehi, E., Zhang, P. 2000. The Partition Dimesion of a Graph. Aequationes Mathematicae. 59:45-54.

[4] Chartrand, G., Zhang, P., Salehi E., 1998, On The Partition Dimension of Graph, Congr. Numer., 130, 157-168.

[5] Chartrand, G., Erwin, D., Henning, M.A., Slater, P.J., Zhang, P., 2002, The Locating-Chromatic Number of a Graph, Bull. Inst. Combin. Appl., 36, 89-101.

[6] Welyyanti, D., Baskoro, E.T., Simanjuntak, R., Utunggadewa. 2015. On Locating-Chromatic Number Of Graph with Dominat Vertices, Procedia Computer Science, 74 89-92.

[7] Welyyanti, D., Baskoro, E.T., Simanjuntak, R., Utunggadewa, S. 2014. The locating-Chromatic Number of Disconnected Graph. Far East Journal of Mathematical Science. 94(2):169-182.

[8] Welyyanti D, Simanjuntak R, Utunggadewa S, Baskoro ET 2016 Locating-chromatic number for a graph of two componentAIP Conf. Proc.1707 020024 10.1063/1.4940825

[9] Welyyanti, D, Simanjuntak R, Utunggadewa S, Baskoro ET. 2017.. Locating-Chromatic Number for a Graph of Two Homogenous Component. IOP Conf Proc. 893: 01204.