Global attractors and their upper semicontinuity for a structural damped wave equation with supercritical nonlinearity on $\mathbb{R}^N$

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Abstract

The paper investigates the existence of global attractors and their upper semicontinuity for a structural damped wave equation on $\mathbb{R}^N$:

$$u_{tt} - \Delta u + (-\Delta)^\alpha u_t + u_t + u + g(u) = f(x), \quad (x,t) \in \mathbb{R}^N \times \mathbb{R}^+,$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \mathbb{R}^N,$$

where $\alpha \in (1/2, 1)$ is called a dissipative index. We propose a new method based on the harmonic analysis technique and the commutator estimate to exploit the dissipative effect of the structural damping $(-\Delta)^\alpha u_t$ and to overcome the essential difficulty: "both the unbounded domain $\mathbb{R}^N$ and the supercritical nonlinearity cause that the Sobolev embedding loses its compactness"; Meanwhile we show that there exists a supercritical index $p_\alpha \equiv \frac{N+4}{N-4}$ depending on $\alpha$ such that when the growth exponent $p$ of the nonlinearity $g(u)$ is up to the supercritical range: $1 \leq p < p_\alpha$; (i) the IVP of the equation is well-posed and its solution is of additionally global smoothness when $t > 0$; (ii) the related solution semigroup possesses a global attractor $A_\alpha$ in natural energy space for each $\alpha \in (1/2, 1)$; (iii) the family of global attractors $\{A_\alpha\}_{\alpha \in (1/2, 1)}$ is upper semicontinuous at each point $\alpha_0 \in (1/2, 1)$.

Keywords: Structural damped wave equation; unbounded domain; well-posedness; global attractor; upper semicontinuity.

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1 Introduction

In this paper, we investigate the well-posedness, the existence of global attractors and their upper semicontinuity to the following structural damped wave equation on $\mathbb{R}^N$:

$$u_{tt} - \Delta u + (-\Delta)^\alpha u_t + u_t + u + g(u) = f(x), \quad (x,t) \in \mathbb{R}^N \times \mathbb{R}^+,$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \mathbb{R}^N,$$

where $\alpha \in (1/2, 1)$ is called a dissipative index.

It is well known that when $\Omega \subset \mathbb{R}^N$ is a bounded domain, the fractional damping $(-\Delta)^\alpha u_t$ (which is especially called a structural damping as $\alpha \in (1/2, 1]$ [cf. [6, 7]]) has a regularizing effect to the solutions of the IBVP of Eq. (1.1), i.e., it makes them be of additionally global (when $\alpha \in (0, 1)$) or partial (when...
\( \alpha = 1 \) smoothness when \( t > 0 \). For example, Chueshov \[8, 9\] studied more general Kirchhoff wave model with structural/strong nonlinear damping

\[
u_{tt} - \phi(\|\nabla u\|^2)\Delta u + \sigma(\|\nabla u\|^2)(-\Delta)^\alpha u_t + g(u) = f(x), \tag{1.3}
\]

with either \( \alpha \in (1/2, 1) \) (cf. \[8\]) or \( \alpha = 1 \) (cf. \[9\]) on a bounded domain \( \Omega \) with Dirichlet boundary condition. When \( \alpha = 1 \), he found a supercritical exponent \( p^{**} \equiv \frac{N+4}{(N-4)\alpha} \) depending on the dissipative index \( \alpha \) and showed that when the growth exponent \( p \) of the nonlinearity \( g(u) \) is up to supercritical range: \( 1 \leq p < p^{**} \), the IBVP of Eq. (1.3) is still well-posed, as well as its solutions possess additionally partial regularity when \( t > 0 \); Moreover the related solution semigroup \( S(t) \) has a finite-dimensional ‘partially strong’ global attractor in natural phase space \( X = (H_0^1 \cap L^{p+1})(\Omega) \times L^2(\Omega) \). Recently, Ding, Yang and Li \[11\] removed this ‘partially strong’ restriction for the global attractor.

When \( \alpha \in (1/2, 1) \) and \( \sigma(s) \equiv 1 \), Eq. (1.3) becomes

\[
u_{tt} - \phi(\|\nabla u\|^2)\Delta u + (-\Delta)^\alpha u_t + g(u) = f(x), \tag{1.4}
\]

Yang, Ding and Li \[27\] found a supercritical growth exponent \( p_\alpha \equiv \frac{N+4}{(N-4)\alpha} \) depending on the dissipative index \( \alpha \) and showed that when the growth exponent \( p \) of the nonlinearity \( g(u) \) is up to supercritical range: \( 1 \leq p < p_\alpha \), not only is the IBVP of Eq. (1.4) well-posed, but also its solutions possess additionally global regularity when \( t > 0 \) (rather than partial one as \( \alpha = 1 \) case); Furthermore, the related solution semigroup \( S(t) \) possesses a global and an exponential attractor in phase space \( X = (H_0^1 \cap L^{p+1})(\Omega) \times L^2(\Omega) \). These results improve those in \[8\] to some extent. For the related researches on this topic on a bounded domain, one can see \[4, 11, 28, 29\] and references therein.

However for unbounded domain \( \Omega \), such as \( \Omega = \mathbb{R}^N \), the case becomes much more complex. One major difficulty is the loss of the compactness of the Sobolev embedding, which is indispensable for obtaining the existence of global attractor. Recently, Yang and Ding \[26\] studied the well-posedness and longtime dynamics for the strongly damped Kirchhoff wave model on \( \mathbb{R}^N \):

\[
u_{tt} - M(\|\nabla u\|^2)\Delta u - \Delta u_t + u_t + g(x, u) = f(x), \tag{1.5}
\]

which includes the strongly damped wave Eq. (1.1) (taking \( \alpha = 1 \) there) as its special case by taking \( M(s) \equiv 1 \). By using the tail cut-off method (cf. \[23\]), the authors obtained the existence of global and exponential attractors in phase space \( \mathcal{H} = H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N) \) provided that the nonlinearity \( g(x, u) \) is of at most critical growth \( p^* \equiv \frac{N+2}{N-2} \). These results extend the previous ones on this topic in \[25\].

Roughly speaking, the tail cut-off method introduced in \[26\] is that the authors firstly split \( \mathbb{R}^N \) into the union of a ball with radius \( R \) and its complement: \( \mathbb{R}^N = B_R \cup B_R^c \), then establish the tail estimate on \( B_R^c \) such that it is sufficiently small when \( R \) suitable large; Secondly, by fixing \( R \) they both establish the regularizing estimate and use the compactness of the Sobolev embedding on \( B_R \) to get the asymptotical compactness of the related solution semigroup. Unfortunately, the above technique is not valid for the structural damping case: \( (-\Delta)^\alpha u_t \), with \( \alpha \in (1/2, 1) \), because there exists an essential difference between the spectrum of the operator \(-\Delta\) on a bounded domain \( \Omega \) and that on \( \mathbb{R}^N \), which leads to that one neither uses the limiting process of the approximating solution sequence on the bounded balls \( B_R \) to get the existence and the regularity of the solutions of problem (1.1)-(1.2), nor proceeds the tail estimate as done in \[26\] because integral by parts fails in this case for the appearance of the structural damping. Therefore, it needs to provide a new method to overcome these difficulties arising from both the structural damping and unbounded domain \( \mathbb{R}^N \). The desired aim is to obtain the corresponding results as in a bounded domain as in \[27\].

In the present paper, based on the Littlewood-Paley theory we put forward a double truncation method (rather than letting \( B_R \to \mathbb{R}^N \) as \( R \to \infty \) as done before) and use the localization in frequency of harmonic
analysis technique to make full use of the dissipative effect of the structural damping $(-\Delta)\alpha u_t$ and establish the well-posedness of problem (1.1)-(1.2) together with the additionally global regularity of its solutions when $t > 0$. Furthermore, we introduce the commutator estimate to conquer the difficulty in tail cut-off estimate arising from the structural damping and the essential difficulty: ‘both the unbounded domain $\mathbb{R}^N$ and the supercritical nonlinearity cause that the compactness of Sobolev embedding is no longer true’. What’s more, we use the harmonic analysis technique to solve the robustness of the global attractors on the dissipative index.

The main contribution of this paper is that it realizes the desired aim. In the concrete, there exists a supercritical index $p_\alpha \equiv \frac{\alpha}{\alpha + 2\alpha} \leq p$ depending on $\alpha$ such that when the growth exponent $p$ of the nonlinearity $g(u)$ is up to the supercritical range: $1 \leq p < p_\alpha$,

(i) the IVP of Eq. (1.1) is well-posed and its solution is of additionally global smoothness when $t > 0$; (see Theorem 2.6)

(ii) the related solution semigroup possesses a global attractor $A_\alpha$ in natural energy space for each $\alpha \in (1/2, 1)$; (see Theorem 3.4)

(iii) the family of global attractors $\{A_\alpha\}_{\alpha \in (1/2, 1)}$ is upper semicontinuous at each point $\alpha_0 \in (1/2, 1)$, i.e., for any neighborhood $U$ of $A_{\alpha_0}$, $A_\alpha \subset U$ when $|\alpha - \alpha_0| \ll 1$. (see Theorem 4.2)

We mention that there have been several remedies to save the loss of compactness of the Sobolev embedding on the unbounded domain. One of them is working in weighted Sobolev spaces (cf. \cite{12, 16, 19, 30} and references therein). For example, Savostianov \cite{20} studied the infinite-energy solutions of the semilinear wave equation with fractional damping in an unbounded domain of $\mathbb{R}^3$

$$u_{tt} - \Delta u + \lambda_0 u + \gamma (I - \Delta)^{\frac{\alpha}{2}} u_t + f(u) = g.$$ \hspace{1cm} (1.6)

Under the critical nonlinearity assumption:

$$|f'(s)| \leq C(1 + |s|^4) \text{ and } f(s)s \geq -M,$$

he proved the existence, uniqueness and extra regularity of the infinite-energy solutions, and established the existence of locally compact attractor of the corresponding solution semigroup. For the related researches on the longtime dynamics of a nonlinear evolution equation in weighted Sobolev spaces, one can see \cite{12, 18, 31} and references therein.

Another approach developed for damped wave equation is working in the usual Sobolev spaces. For example, by using the property of finite speed of propagation, the Strichartz estimate and a suitable semigroup decomposition, Feireisl \cite{13, 14} established the existence of global attractor for the damped semilinear wave equation on $\mathbb{R}^3$:

$$u_{tt} - \Delta u + d(x)u_t + au + f(x, u) = g(x)$$ \hspace{1cm} (1.7)

provided that $|f_s(x, s)| \leq C(1 + |s|^q), q < 4$.

By combining the decomposition of the solution semigroup with the suitable cut-off functional, Belleri and Pata \cite{3} proved the existence of global attractor for the strongly damped semilinear wave equation on $\mathbb{R}^3$:

$$u_{tt} - \Delta u_t - \Delta u + g(x, u) + \phi(x)u_t = f(x)$$ \hspace{1cm} (1.8)

provided that the nonlinearity $g(x, u)$ is of subcritical growth 3. Then in critical nonlinearity case, namely, the growth rate of $g(x, u)$ can reach to 5, Conti, Pata and Squassina \cite{10} obtained the existence of global attractor for Eq. (1.8) (replacing $\phi(x)u_t$ there by more general $\phi(x, u_t)$).

Obviously, in all the above mentioned researches, the growth exponent of the nonlinearity reaches at most to critical. To the best of the authors’ knowledge, this paper is the first one to establish the existence of global attractors and their robustness on the dissipative index in supercritical nonlinearity case on $\mathbb{R}^N$. 3
The paper is arranged as follows. In Section 2, we discuss the well-posedness of problem \((1.1)-(1.2)\) and the additionally global smoothness of its solutions when \(t > 0\). In Section 3, we study the existence of global attractor for each \(\alpha \in (1/2, 1)\). In Section 4, we investigate upper semicontinuity of the family of global attractors \(\{A_\alpha\}_{\alpha \in (1/2, 1)}\).

## 2 Well-posedness

We begin with the following abbreviations:

\[ L^p = L^p(\mathbb{R}^N), \quad H^{s,p} = H^{s,p}(\mathbb{R}^N), \quad \dot{H}^{s,p} = \dot{H}^{s,p}(\mathbb{R}^N), \quad \int = \int_{\mathbb{R}^N}, \quad \| \cdot \|_p = \| \cdot \|_{L^p}, \quad \| \cdot \| = \| \cdot \|_2, \]

with \(s \in \mathbb{R}, p > 1\), where \(H^{s,p}\) is the Bessel potential space equipped with the norm

\[
\|f\|_{H^{s,p}} = \|(I - \Delta)^{s/2} f\|_p = \|\mathcal{F}^{-1}[(1 + |\xi|^2)^{s/2}\mathcal{F} f]\|_p,
\]

and where \(I\) denotes the identity operator, \(\dot{f} = \mathcal{F}(f)\) and \(\ddot{f} = \mathcal{F}^{-1}(f)\) are the Fourier transformation and the Fourier inverse transformation of \(f\), respectively. And \(\dot{H}^{s,p}\) is the Riesz potential space equipped with the semi-norm

\[
\|f\|_{\dot{H}^{s,p}} = \|(\Delta)^{s/2} f\|_p = \|\mathcal{F}^{-1}[|\xi|^s \mathcal{F} f]\|_p,
\]

where the operators \((I - \Delta)^{s/2}\) and \((\Delta)^{s/2}\) are called Bessel potential and Riesz potential, respectively.

The notation \((\cdot, \cdot)\) for \(L^2\)-inner product will also be used for the notation of duality pairing between the dual spaces. We use the same letter \(C\) to denote different positive constants, and use \(C(\cdot, \cdot)\) to denote positive constants depending on the quantities appearing in parenthesis. The sign \(H_1 \hookrightarrow H_2\) denotes that the functional space \(H_1\) continuously embeds into \(H_2\) and \(H_1 \hookrightarrow \hookrightarrow H_2\) denotes that \(H_1\) compactly embeds into \(H_2\).

Define the phase spaces

\[ \mathcal{H}_\alpha = H^{1+\alpha} \times H^{\alpha}, \quad \mathcal{H} = (H^1 \cap L^{p+1}) \times L^2, \quad \mathcal{H}_{-\alpha} = H^{\alpha} \times H^{-\alpha}, \]

which are equipped with usual graph norms, for instance,

\[
\| (u, v) \|_{\mathcal{H}}^2 = \|u\|_{H^1}^2 + d_0 \|u\|_{p+1}^2 + \|v\|^2, \quad \forall (u, v) \in \mathcal{H},
\]

where

\[
d_0 = \begin{cases} 0, & \text{if } 1 \leq p \leq p^* \equiv \frac{N+2}{(N-2)+}, \\ 1, & \text{if } p^* < p < p_\alpha \equiv \frac{N+4\alpha}{(N-4\alpha)+}. \end{cases}
\]

Obviously,

\[ \mathcal{H}_\alpha \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{H}_{-\alpha} \]

for each \(\alpha \in (1/2, 1)\).

Define the smooth radial function

\[
\hat{\varphi}(\xi) = \begin{cases} 1, & \text{if } |\xi| \leq 1, \\ 0, & \text{if } |\xi| \geq 2, \end{cases}, \quad \int_{\mathbb{R}^N} \hat{\varphi}(\xi) d\xi = 1, \quad \hat{\varphi}(\xi) = \hat{\varphi}(\frac{\xi}{2t}),
\]

where \(\hat{\varphi} = \mathcal{F}(\varphi)\). Define the operator \(S_t\):

\[
(S_t u)(x) = \mathcal{F}^{-1}(\hat{\varphi}_t \hat{u})(x) = (\varphi_t * u)(x),
\]

where \(\varphi_t(x) = 2^{4N} \varphi(2^t x) \in L^1\).
Lemma 2.1. Let $B$ be a ball. Then for any non-negative integer $k$, any couple of real $(a, b)$, with $b \geq a \geq 1$, and any function $u \in L^a$, there exists a positive number $C$ such that

$$\sup_{\lambda B} \hat{u} \in \lambda B \Rightarrow \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^b} \leq C^{k+1} \lambda^{k+N(\frac{1}{b} - \frac{1}{a})} \|u\|_{L^a}. \quad (2.2)$$

Lemma 2.2. (Properties of the operator $S_l$) (i) For any $m \in \mathbb{R}$, if $u \in H^m$, then $S_l u \in H^m$, $\forall n \geq m$, and

$$\|S_l u\|_{H^n} \leq C(l, m, n) \|S_l u\|_{H^m},$$

especially,

$$\|S_l u\|_{H^n} \leq \|u\|_{H^m} \text{ and } \lim_{l \to \infty} \|S_l u - u\|_{H^m} = 0. \quad (2.3)$$

(ii) If $u \in L^q$ with $1 \leq q < \infty$, then $S_l u \in L^p, \forall p \geq q$, and

$$\|S_l u\|_p \leq C(l, p, q) \|S_l u\|_q,$$

especially,

$$\|S_l u\|_q \leq \|\varphi_0\|_q \text{ and } \lim_{l \to \infty} \|S_l u - u\|_q = 0,$$

where $\varphi_0 \equiv \|\varphi\|_1$.

(iii) $$(S_l u, v) = (u, S_l v), \ \forall u, v \in L^2.$$  

Proof. (i) For any $n \geq m$,

$$\|S_l u\|_{H^n} = \|(1 + |\xi|^2)^{n-m} \hat{\varphi}_l (\hat{\varphi}_l \hat{u})\|_{H^n} \leq C(l, m, n) \|(1 + |\xi|^2)^{\frac{n}{2}} \hat{\varphi}_l (\hat{\varphi}_l \hat{u})\|_{H^n} = C(l, m, n) \|S_l u\|_{H^m}.$$  

Obviously, $C(l, m, n) = 1$ when $n = m$. By the Plancherel theorem,

$$\|S_l u - u\|_{H^m}^2 = \int (1 + |\xi|^2)^m (\hat{\varphi}_l - 1)^2 |\hat{u}|^2 d\xi \leq C \int_{|\xi| > 2^l} (1 + |\xi|^2)^m |\hat{u}|^2 d\xi \to 0 \text{ as } l \to \infty.$$  

(ii) By formula (2.2) (taking that $B$ is a unit ball, $k = 0, \lambda = 2^{l+1}, a = q, b = p$ there), we have

$$\|S_l u\|_p \leq C(l, p, q) \|u\|_q.$$  

Taking account of $\varphi, \varphi_l \in L^1$, we obtain

$$\|S_l u\|_q = \||\varphi_l * u\|_q \leq \|\varphi_l\|_1 \|u\|_q = \|\varphi_1\|_1 \|u\|_q \leq \varphi_0 \|u\|_q.$$  

By the density of $C_0^\infty(\mathbb{R}^N)$ in $L^q$ we have that for any $\epsilon > 0$, there exists a function $g \in C_0^\infty(\mathbb{R}^N)$ satisfying

$$\|u - g\|_q < \frac{\epsilon}{2(\varphi_0 + 1)}.$$  

Since $H^m \hookrightarrow L^q$ for $m$ suitably large and (2.3), there must exist a $L > 0$ such that

$$\|S_l g - g\|_q \leq C \|S_l g - g\|_{H^m} < \frac{\epsilon}{2} \text{ as } l > L.$$  

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Therefore,
\[ \|S_t u - u\|_q \leq \|S_t u - S_t g\|_q + \|S_t g - g\|_q + \|g - u\|_q \]
\[ \leq (\varphi_0 + 1)\|u - g\|_q + \|S_t g - g\|_q < \epsilon, \text{ as } l > L. \]

(iii) For any \(u, v \in L^2\),
\[ (S_t u, v) = (F^{-1}(\hat{\varphi}_t \hat{u}), v) = (\hat{u}, \hat{\varphi}_t \hat{v}) = (u, S_t v). \]

\[ \square \]

Let the phase space \(X_\alpha = H^{2\alpha+1} \times H^{2\alpha}\), with \(\alpha \in (1/2, 1)\). By Lemma 2.2,
\[ \|(S_t u_0, S_t u_1)\|_{X_{\alpha}} \leq C_0(l)\|(u_0, u_1)\|_{\mathcal{H}}, \quad \|(S_t u_0, S_t u_1)\|_{\mathcal{H}} \leq (\varphi_0 + 1)\|(u_0, u_1)\|_{\mathcal{H}}, \]
\[ \|S_t f\| \leq \|f\|, \quad \lim_{t \to \infty} \|(S_t u_0, S_t u_1, S_t f) - (u_0, u_1, f)\|_{\mathcal{H} \times L^2} = 0, \quad (2.4) \]
where and in the following \(C_0(l)\) denotes a positive constant depending on \(l\).

**Lemma 2.3.** [21] Let \(X, B\) and \(Y\) be three Banach spaces, \(X \hookrightarrow B \hookrightarrow Y\),
\[ W = \{u \in L^p(0, T; X)|u_t \in L^1(0, T; Y)\}, \text{ with } 1 \leq p < \infty, \]
\[ W_1 = \{u \in L^\infty(0, T; X)|u_t \in L^r(0, T; Y)\}, \text{ with } r > 1. \]

Then \(W \hookrightarrow \hookrightarrow L^p(0, T; B), W_1 \hookrightarrow \hookrightarrow C([0, T]; B)\).

**Assumption 2.4.** (i) \(g \in C^1(\mathbb{R})\), with \(g(0) = 0\). Either
\[ |g'(s)| \leq C(1 + |s|^{p-1}) \text{ and } g'(s) > -C_0 \quad \text{for some } 1 \leq p \leq p^*, \]
or else
\[ C_1|s|^{p-1} - C_0 \leq g'(s) \leq C(1 + |s|^{p-1}) \quad \text{for some } p^* < p < p_\alpha \equiv \frac{N + 4\alpha}{(N - 4\alpha)^+}, \]
where \(C_1 > 0, 0 < C_0 < 1\) and \(a^+ = \max\{0, a\}\).

(ii) \(f \in L^2, \xi(u) = (u_0, u_1) \in \mathcal{H}, \|(u_0, u_1)\|_{\mathcal{H}} \leq R_0\) for some positive constant \(R_0\).

**Remark 2.5.** It follows from Assumption 2.4(i) that
\[ |g(s_1) - g(s_2)| \leq C(1 + |s_1|^{p-1} + |s_2|^{p-1})|s_1 - s_2|, \quad |g(s)| \leq C(|s| + |s|^p), \]
\[ g(s)s \geq \frac{d_0 C_1}{p}|s|^{p+1} - C_0 s^2, \quad G(s) \geq \frac{d_0 C_1}{p(p+1)}|s|^{p+1} - \frac{C_0}{2} s^2, \]
\[ g(s)s - G(s) + \frac{C_0}{2} s^2 \geq 0 \]
for \(1 \leq p < p_\alpha\) (which means \(H^{2\alpha} \hookrightarrow L^{p+1}\)), where and in the following \(G(s) = \int_0^s g(\tau)d\tau\).

**Theorem 2.6.** Let Assumption 2.4 be valid. Then problem (1.1)-(1.2) admits a unique solution \(u(= u^\alpha)\) for each \(\alpha \in (1/2, 1)\), with \((u, u_t) \in L^\infty(\mathbb{R}^+; \mathcal{H}) \cap C_u(\mathbb{R}^+; \mathcal{H})\), and the solution possesses the following
properties:

(i) (Dissipativity)

\[ \|u(t)\|_{H^1}^2 + \|u(t)\|_{p+1}^2 + \|u_t(t)\|^2 \leq C(R_0, \|f\|)e^{-\kappa t} + C\|f\|^2, \quad t > 0, \]

\[ \int_0^\infty \|u_t(\tau)\|_{H^1}^2 \, d\tau \leq C(R_0, \|f\|). \]  \hspace{1cm} (2.5)

(ii) (Additional regularity as \( t > 0 \)) For any \( a > 0, T > a > 0, \)

\[ (u_t, u_{tt}) \in L^\infty(a, T; H^\alpha \times H^{-\alpha}) \cap L^2(a, T; H^1 \times L^2), \]

\[ \|u_t(t)\|_{H^\alpha}^2 + \|u_{tt}(t)\|_{H^{-\alpha}}^2 + \int_t^{t+1} (d_0 \int |u(\tau)|^{p-1} |u_t(\tau)|^2 \, dx + \|u_t(\tau)\|_{H^1}^2 + \|u_{tt}(\tau)\|^2) d\tau \]

\[ \leq (1 + \frac{1}{t^2})C(R_0, \|f\|), \quad t > 0. \]  \hspace{1cm} (2.6)

Moreover, \( u \in L^\infty(a, T; H^{1+\alpha}) \cap L^2(t, t + 1; H^2) \) and

\[ \|u(t)\|_{H^{1+\alpha}}^2 + \int_t^{t+1} \|u(\tau)\|_{H^2}^2 d\tau \leq \left(1 + \frac{1}{(t - \frac{\alpha}{\alpha+1})^{1-\alpha}}\right) C(\alpha)C(R_0, \|f\|), \quad t > \frac{a}{2}, \]  \hspace{1cm} (2.7)

where \( C(\alpha) = \frac{C_{\alpha}}{1-\alpha} \frac{2}{1-\alpha} \).

(iii) (Lipshitz stability and quasi-stability in weaker space \( H_{-\alpha} \))

\[ \|z, z_t(t)\|_{H_{-\alpha}}^2 \leq C(T)\|(z, z_t)(0)\|_{H_{-\alpha}}, \quad t \in [0, T], \]  \hspace{1cm} (2.8)

and

\[ \|z, z_t(t)\|_{H_{-\alpha}}^2 \leq Ce^{-\kappa t}\|(z, z_t)(0)\|_{H_{-\alpha}}^2 + \int_0^t e^{-\kappa(t-\tau)}\|(z, z_t)(\tau)\|_{L^2 \times H^{-2\alpha}}^2 \, d\tau, \quad t > 0, \]  \hspace{1cm} (2.9)

where and in the following \( \kappa \) denotes a small positive constant, and where \( z = u - v, u, v \) are two solutions of problem (1.1), (1.2) corresponding to initial data \( (u_0, u_1) \) and \( (v_0, v_1) \) with \( \|(u_0, u_1)\|_{H^1 + \|v_0, v_1\|_{H}} \leq R_0, \) respectively.

Proof. We first consider the following auxiliary problem on \( \mathbb{R}^N \):

\[ \begin{cases} u_{tt} - \Delta u + (-\Delta)^{\alpha} u_t + u_t + u + S(t, S_t u) = f^l, \\ u_t(0) = u_0^l, \quad u_t(0) = u_1^l, \end{cases} \]  \hspace{1cm} (2.10)

where \( (u_0^l, u_1^l) = (S_t u_0, S_t u_1), f^l = S_t f \). Rewrite this problem as the equivalent form on \( \mathbb{R}^N \):

\[ \begin{cases} \frac{d}{dt}U^l(t) = BU^l(t) + F(u^l), \\ U^l(0) = U_0^l, \end{cases} \]  \hspace{1cm} (2.11)

where

\[ U^l = \begin{pmatrix} u^l \\ u_t^l \end{pmatrix}, \quad B = \begin{pmatrix} 0 & I \\ \Delta - I & -(-\Delta)^{\alpha} - I \end{pmatrix}, \quad F(u^l) = \begin{pmatrix} 0 \\ -S_t(g(S_t u^l)) + f^l \end{pmatrix}, \quad U_0^l = \begin{pmatrix} u_0^l \\ u_1^l \end{pmatrix}. \]
We show that problem (2.11) admits a unique global solution $U^t$ for each $l > 0$. By Theorem 2.1 in [15], we know that the following linear problem on $\mathbb{R}^N$:

$$\begin{align*}
\frac{d}{dt} U^t(t) &= B U^t(t), \\
U^t(0) &= U_0^t
\end{align*}$$

possesses a unique solution $U^t \in C_0(\mathbb{R}^+; X_\alpha)$ for each $U_0^t \in X_\alpha$. Define the solution operator $\Sigma(t): X_\alpha \to X_\alpha$, $\Sigma(t)U_0^t = U^t(t)$, $\forall t \in \mathbb{R}^+$. Then $\{\Sigma(t)\}_{t \geq 0}$ constitutes a semigroup on $X_\alpha$ and a simple calculation shows that

$$\|\Sigma(t)U_0^t\|_{X_\alpha} \leq \|U_0^t\|_{X_\alpha}, \forall U_0^t \in X_\alpha,$$

$i.e.$, $\|\Sigma(t)\|_{L(X_\alpha)} \leq 1$, $\forall t > 0$. (2.12)

Let the space

$$Z = C([0, T]; X_\alpha)$$

equipped with the norm $\|U\|_Z = \max_{0 \leq t \leq T} \|U(t)\|_{X_\alpha}$, and

$$X_{R,T} = \{U \in Z | \|U\|_Z \leq 2R, U(0) = U_0^t\}, \text{ with } R > C_0(l)R_0 + \|f\|,$$

equipped with the metric $d(U, V) = \|U - V\|_Z$. Obviously, $X_{R,T}$ is a complete metric space. For problem (2.11) we define the operator $\mathbb{T}: X_{R,T} \to Z$, for any $U^t \in X_{R,T}$,

$$\mathbb{T}U^t(t) = \Sigma(t)U_0^t + \int_0^t \Sigma(t - \tau)F(u^t(\tau))d\tau, \ t \in (0, T].$$

(2.13)

Now, we show that $\mathbb{T}$ is a contraction mapping from $X_{R,T}$ to itself for $T$ suitably small.

(i) By Lemma 2.2 formulas (2.12), (2.13) and the Sobolev embedding $H^{2\alpha+1} \hookrightarrow H^{2\alpha} \hookrightarrow L^{p+1}$ for $1 \leq p < p_\alpha$, we have

$$\begin{align*}
\|\mathbb{T}U^t\|_{X_\alpha} &\leq \|U_0^t\|_{X_\alpha} + T\|S_l(g(S_l u^t)) - f^l\|_{C([0,T];H^{2\alpha})} \\
&\leq \|U_0^t\|_{X_\alpha} + C(l)T\left(\|g(S_l u^t)\|_{C([0,T];l^2)} + \|f^l\|\right) \\
&\leq \|U_0^t\|_{X_\alpha} + C(l)T\left(\|S_l u^t\|_{C([0,T];l^2)} + \|S_l u^t\|_{C([0,T];l^{2p})} + \|f^l\|\right) \\
&\leq \|U_0^t\|_{X_\alpha} + C(l)T\left(\|S_l u^t\|_{C([0,T];l^2)} + \|S_l u^t\|_{C([0,T];l^{p+1})} + \|f^l\|\right) \\
&\leq \|U_0^t\|_{X_\alpha} + C(l)T\left(\|u^t\|_{C([0,T];H^{2\alpha+1})} + \|f^l\|\right) \\
&\leq R + C_1(l)T(2R)^p \leq 2R, \ t \in [0, T]
\end{align*}$$

(2.14)

for $0 < T < (2C_1(l)(2R)^{p-1})^{-1}$, which means that $\mathbb{T}: X_{R,T} \to X_{R,T}$.

(ii) Similar to the proof of (2.14) and by the Hölder inequality, we have

$$\begin{align*}
\|\mathbb{T}U^t - \mathbb{T}V^t\|_Z &\leq CT\|S_l(g(S_l u^t)) - S_l(g(S_l v^t))\|_{C([0,T];H^{2\alpha})} \\
&\leq C(l)T\|g(S_l u^t) - g(S_l v^t)\|_{C([0,T];l^2)} \\
&\leq C(l)T\left(\|S_l u^t - S_l v^t\|_{C([0,T];l^2)} \\
&\quad + (\|S_l u^t\|_{C([0,T];l^{2(p+1)})}^{p-1} + \|S_l v^t\|_{C([0,T];l^{2(p+1)})}^{p-1})\|S_l u^t - S_l v^t\|_{C([0,T];l^{p+1})}\right) \\
&\leq C(l)T\left(\|u^t - v^t\|_{C([0,T];l^2)} \\
&\quad + (\|S_l u^t\|_{C([0,T];l^{p+1})}^{p-1} + \|S_l v^t\|_{C([0,T];l^{p+1})}^{p-1})\|u^t - v^t\|_{C([0,T];l^{p+1})}\right) \\
&\leq C_2(l)(2R)^{p-1}\|U^t - V^t\|_Z \leq \frac{1}{2}\|U^t - V^t\|_Z
\end{align*}$$
for \(0 < T < (2C_2(l)(2R)^{p-1})^{-1}\). Taking
\[
T : 0 < T < \min \left\{ \frac{1}{2C_1(l)(2R)^{p-1}}, \frac{1}{2C_2(l)(2R)^{p-1}} \right\},
\]
then \(T\) is a contraction mapping from \(X_{R,T}\) to itself. By the Banach fixed point theorem, the mapping \(T\) has a unique fixed point, i.e., problem (2.11) has a unique solution \(U^l \in C([0, T]; X_\alpha)\). Let \([0, T_i]\) be the maximal interval of existence of the solution \(U^l\), i.e., \(U^l \in C([0, T_i]; X_\alpha)\). By the standard arguments we know that if
\[
\sup_{0 \leq t < T_i} \|U^l(t)\|_{X_{\alpha}} < \infty,
\]
then \(T_i = \infty\).

In order to prove \(T_i = \infty\), we first give some a priori estimates for \(U^l\). Using the multiplier \(u_t^l + \epsilon u^l\) in Eq. (2.10), we get
\[
\frac{d}{dt} H(\xi_u^l(t)) + \Phi(\xi_u^l(t)) = 0,
\]
where \(\xi_u^l = (u^l, u_t^l)\),
\[
H(\xi_u^l) = \frac{1}{2} \left( \|u_t^l\|^2 + \|u^l\|_{H^1}^2 + 2 \int G(s_t u^l) dx - 2(f^l, u^l) \right) + \epsilon \left( (u^l, u_t^l) + \frac{1}{2} \|u_t^l\|_{H^1}^2 + \|u^l\|^2 \right),
\]
\[
\Phi(\xi_u^l) = \|u_t^l\|_{L^2}^2 + (1 - \epsilon) \|u^l\|^2 + \epsilon \left( \|u^l\|_{H^1}^2 + \|u^l\|^2 + (g(s_t u^l), s_t u^l) - (f^l, u^l) \right).
\]

It follows from Remark 2.5 that
\[
H(\xi_u^l) \geq \kappa (\|u_t^l\|^2 + \|u^l\|_{H^1}^2 + d_0 \|s_t u^l\|_{p+1}^{p+1} - C \|f^l\|^2),
\]
\[
\Phi(\xi_u^l) - \epsilon H(\xi_u^l) \geq 0
\]
for \(\epsilon > 0\) suitably small, and by (2.4),
\[
H(\xi_u^l(0)) \leq C \left( \|u_t^l\|^2 + \|u^l\|_{H^1}^2 + \|s_t u^l\|_{p+1}^{p+1} + \|f^l\|^2 \right) \leq C(R_0) + C \|f\|^2.
\]

Inserting (2.17) into (2.16) gives
\[
\frac{d}{dt} H(\xi_u^l(t)) + \epsilon H(\xi_u^l(t)) \leq C \|f\|^2,
\]
\[
\|u_t^l(t)\|_{H^1}^2 + d_0 \|s_t u^l(t)\|_{p+1}^{p+1} + \|u^l(t)\|^2 \leq C(R_0, \|f\|) e^{-\kappa t} + C \|f\|^2, \quad t > 0.
\]
Letting \(\epsilon = 0\) in (2.16), integrating the resulting expression over \((0, t)\) and using (2.18), we have
\[
\int_0^t \|u_t^l(\tau)\|_{H^1}^2 d\tau \leq C(R_0, \|f\|), \quad t > 0.
\]

The combination of (2.18) and (2.19) means that formula (2.5) uniformly holds for \(U^l(t)\), and the control constants in the right hand side are independent of \(l\).

Using the multiplier \((-\Delta)^{2\alpha} u_t^l + \epsilon (-\Delta)^{2\alpha} u^l\) in Eq. (2.10) yields
\[
\frac{d}{dt} H_1(\xi_u^l) + \Phi_1(\xi_u^l) = (f^l - s_t(g(s_t u^l)), (-\Delta)^{2\alpha} u_t^l + \epsilon (-\Delta)^{2\alpha} u^l),
\]
(2.20)
The combination of (2.16) and (2.24) gives
\[ H_1(\xi_{u'}) = \frac{1}{2} \left( \|u_t'\|^2_{H^{2\alpha}} + \|u_t'^2\|_{H^{2\alpha+1}} + \|u_t'^2\|_{H^{2\alpha}} \right) 
+ \epsilon \left( \|(-\Delta)^{2\alpha} u_t' + u_t'\|^2_{H^{2\alpha}} \right) 
\sim \|u_t'^2\|_{H^{2\alpha+1}} + \|u_t'^2\|_{H^{2\alpha}}, \] (2.21)
\[ \Phi_1(\xi_{u'}) = \|u_t'^2\|_{H^{2\alpha}} + (1 - \epsilon)\|u_t'^2\|_{H^{2\alpha}} + \epsilon \left( \|u_t'^2\|_{H^{2\alpha+1}} + \|u_t'^2\|_{H^{2\alpha}} \right) \]
for \( \epsilon \) suitably small. By Lemmas 2.1 and 2.2
\[ \left| \left( S_t(g(S_t u')), (-\Delta)^{2\alpha} u_t' \right) \right| \leq \|u_t'^2\|_{H^{2\alpha}} \|S_t(g(S_t u'))\|_{H^{2\alpha}} \]
\[ \leq C(l) \|u_t'^2\|_{H^{2\alpha}} \|g(S_t u')\| \]
\[ \leq C(l) \|u_t'^2\|_{H^{2\alpha}} (\|S_t u'\|^2 + \|S_t u'\|_{2p}^p), \] (2.22)
\[ \leq \frac{1}{4} \|u_t'^2\|_{H^{2\alpha}}^2 + C(l) (\|S_t u'\|^2 + \|S_t u'\|_{2p}^p). \] (2.23)
By the similar argument as to the estimate (2.22), we have
\[ \left| \left( S_t(g(S_t u')), (-\Delta)^{2\alpha} u_t' \right) \right| \leq \frac{1}{4} \|u_t'^2\|_{H^{2\alpha+1}} + C(l) (\|S_t u'\|^2 + \|S_t u'\|_{2p}^p), \]
\[ \|f', (-\Delta)^{2\alpha} u_t' + \epsilon (-\Delta)^{2\alpha} u_t'\| \leq \frac{1}{4} (\|u_t'^2\|_{H^{2\alpha+1}} + \|u_t'^2\|_{H^{2\alpha+1}}) + C(l) \|f'\|^2. \] (2.23)
Inserting (2.22), (2.23) into (2.20) and using (2.18), (2.21) turn out
\[ \frac{d}{dt} H_1(\xi_{u'}(t)) + \kappa H_1(\xi_{u'}(t)) \leq C(l, R_0, \|f\|). \] (2.24)
Let
\[ \Psi(\xi_{u'}) = H(\xi_{u'}) + H_1(\xi_{u'}). \]
The combination of (2.16) and (2.24) gives
\[ \frac{d}{dt} \Psi(\xi_{u'}) + \kappa \Psi(\xi_{u'}) \leq C(l, R_0, \|f\|), \]
\[ \|\xi_{u'}(t)\|^2_{H^{2\alpha}} = \|u_t'(t)\|^2_{H^{2\alpha+1}} + \|u_t'(t)\|^2_{H^{2\alpha}} \leq C(l, R_0, \|f\|), \quad t > 0, \]
which means that formula (2.15) holds and hence \( T_1 = +\infty. \)
For any \( \psi \in L^2 \cap L^{p+1} \), by Lemma 2.2 and (2.18),
\[ \|g(S_t u') \|_{L^2} \leq C(\|g(S_t u')\|_{L^2} \|\psi\|_2 + \|S_t u'\|_{2p}^p \|\psi\|_p) \leq C(\|u_t'\| + \|S_t u'\|_{2p}^p), \]
\[ \|g(S_t u')\|_{L^2 + L^{1+\frac{1}{p}}} \leq C(\|u_t'\| + \|S_t u'\|_{2p}^p), \]
\[ \|S_t g(S_t u')\|_{H^{2\alpha}} \leq C(\|g(S_t u')\|_{L^2} \|\psi\|_2 + \|S_t u'\|_{2p}^p), \]
where we have used the facts: \( L^2 + L^{1+\frac{1}{p}} = (L^2 \cap L^{p+1})' \) and \( H^{2\alpha} \hookrightarrow (L^2 \cap L^{p+1}) \) for \( 1 \leq p < p_\alpha \). So by Eq. (2.10) and (2.18),
\[ \int_t^{t+1} \|u_{tt}^l(\tau)\|^2_{H^{2\alpha}} d\tau \leq C(R_0, \|f\|), \quad t > 0. \] (2.26)
Owing to \((u^l, u^l_t, u^l_{tt}) \in C(\mathbb{R}^+; H^{2\alpha+1} \times H^{2\alpha} \times L^2)\), differentiating Eq. (2.10) with respect to \(t\) and letting \(v^l = u^l_t\), we see that \(v^l\) solves

\[
v^l_{tt} + (I - \Delta)v^l + (I + (\Delta)^{\alpha})v^l_t + S_l(g'(S_l v^l)S_l v^l) = 0. \tag{2.27}
\]

Using the multiplier \((I - \Delta)^{-\alpha}v^l_t + \epsilon v^l\) in (2.27) gives

\[
\frac{1}{2}\frac{d}{dt}H_2(\xi_{v^l}(t)) + \|(I + (\Delta)^{\alpha})^{1/2}(I - \Delta)^{-\alpha/2}v^l\|_2^2 + \epsilon \left(\|v^l\|_{2}^2 + \int S_l(g'(S_l u^l_t)S_l v^l) v^l dx\right)
= \epsilon\|v^l_t\|^2 - \int S_l(g'(S_l u^l_t)S_l v^l)(I - \Delta)^{-\alpha}v^l_t dx,
\tag{2.28}
\]

where \(\xi_{v^l}(t) = (v^l(t), v^l_t(t))\),

\[
H_2(\xi_{v^l}) = \|v^l\|_{2}^2 + \|v^l_t\|_{2}^2 + \epsilon \left(\|v^l\|_{2}^2 + \|v^l_t\|_{2}^2 + 2(v^l_t, v^l)\right) \sim \|v^l\|_{2}^2 + \|v^l_t\|_{2}^2,
\tag{2.29}
\]

for \(\epsilon > 0\) suitably small, and where the equivalent constants are independent of \(l\). By Lemma 2.2

\[
\int S_l(g'(S_l u^l_t)S_l v^l) v^l dx = \int g'(S_l u^l)(S_l v^l)^2 dx 
\geq C_1 d_0 \int |S_l u^l|^{p-1}(S_l v^l)^2 dx - C_0 \|S_l v^l\|^2,
\tag{2.30}
\]

and

\[
\left| \int S_l(g'(S_l u^l)S_l v^l)(I - \Delta)^{-\alpha}v^l_t dx \right| = \left| \int g'(S_l u^l)S_l v^l S_l((I - \Delta)^{-\alpha}v^l_t) dx \right| 
\leq C \|S_l v^l\| L \left| S_l((I - \Delta)^{-\alpha}v^l_t) \right| 
+ \left\{ \frac{C}{2} \left| S_l u^l \right|_{p+1}^2 \left| S_l v^l \right|_{p+1}^2 \left| S_l((I - \Delta)^{-\alpha}v^l_t) \right|_{p+1}, \quad 1 \leq p \leq p^*, \right.
\]

\[
+ \left\{ \frac{C}{2} \int |S_l u^l|^{p-1}|S_l v^l|^2 dx + C(\epsilon) \int |S_l u^l|^{p-1}|S_l((I - \Delta)^{-\alpha}v^l_t)|^2 dx, \quad p^* < p < p_0 \right. 
\leq C \|S_l v^l\|_{2}^2 + \frac{d_0 C_1}{2} \int |S_l u^l|^{p-1}|S_l v^l|^2 dx + \|v^l_t\|^2 + C(\epsilon, R_0, \|f\|)(\|S_l v^l\|^2 + \|v^l_t\|_{2}^2),
\tag{2.31}
\]

where we have used the Sobolev embedding \(H^{2\alpha-\delta} \hookrightarrow L^{p+1}\) for \(\delta > 0 < \delta \ll 1\) and the formula

\[
\|S_l((I - \Delta)^{-\alpha}v^l_t)\|_{p+1} \leq C \|S_l((I - \Delta)^{-\alpha}v^l_t)\|_{H^{2\alpha-\delta}} \leq C \|S_l v^l\|_{H^{2\alpha-\delta}} \leq \epsilon \|v^l_t\|^2 + C(\epsilon) \|v^l_t\|_{H^{2\alpha}}^2.
\]

Inserting (2.30)-(2.31) into (2.28) and using (2.29) we obtain

\[
\frac{d}{dt}H_2(\xi_{v^l}) + \kappa \left( H_2(\xi_{v^l}) + 2\|v^l\|^2 + \|v^l_t\|_{2}^2 \right) + d_0 \int |S_l u^l|^{p-1}|S_l v^l|^2 dx
\leq C(R_0, \|f\|)(\|u^l_t\|^2 + \|u^l_{tt}\|_{H^{2\alpha}}^2).
\tag{2.32}
\]
When $0 \leq t \leq 1$, multiplying (2.32) by $t^2$ yields
\[
\frac{d}{dt} \left( t^2 H_2(\xi_{i\ell}) \right) + \kappa t^2 \left( H_2(\xi_{i\ell}) + 2\|v_{t\ell}\|^2 \right) \leq C(R_0, \|f\|)(\|u_{t\ell}\|^2 + \|u_{t\ell\ell}\|^2_{H^{-2\alpha}}) + Ct(\|v_{t\ell}\|^2_{H^\alpha} + \|v_{t\ell}\|^2_{H^{-\alpha}}) \\
\leq \kappa t^2\|v_{t\ell}\|^2 + C(\kappa, R_0, \|f\|)(\|u_{t\ell}\|^2_{H^\alpha} + \|u_{t\ell\ell}\|^2_{H^{-2\alpha}}),
\] (2.33)
where we have used the fact
\[
Ct\|v_{t\ell}\|^2_{H^{-\alpha}} \leq Ct\|v_{t\ell}\|^2_{H^{-2\alpha}} \leq \kappa t^2\|v_{t\ell}\|^2 + C(\kappa)(\|v_{t\ell}\|^2_{H^{-2\alpha}}).
\]
Applying the Gronwall inequality to (2.33) over $(0, t)$, with $0 < t \leq 1$, and utilizing (2.19), (2.26), we have
\[
t^2H_2(\xi_{i\ell}(t)) \leq C(R_0, \|f\|)\int_0^1 (\|u_{t\ell}(\tau)\|^2_{H^\alpha} + \|u_{t\ell\ell}(\tau)\|^2_{H^{-2\alpha}}) d\tau \leq C(R_0, \|f\|),
\]
\[
\|u_{t\ell}(t)\|^2_{H^\alpha} + \|u_{tt\ell}(t)\|^2_{H^{-\alpha}} \leq \frac{1}{t^2}C(R_0, \|f\|), \quad 0 < t \leq 1.
\] (2.34)
When $t \geq 1$, applying the Gronwall inequality to (2.32) over $(1, t)$ and using (2.34) at $t = 1$, we obtain
\[
\|u_{t\ell}(t)\|^2_{H^\alpha} + \|u_{tt\ell}(t)\|^2_{H^{-\alpha}} \leq H_2(\xi_{i\ell}(1))e^{-\kappa(t-1)} + C \int_1^t e^{-\kappa(t-\tau)}(\|u_{t\ell}(\tau)\|^2_{H^\alpha} + \|u_{tt\ell}(\tau)\|^2_{H^{-2\alpha}}) d\tau \\
\leq C(R_0, \|f\|).
\] (2.35)
Integrating (2.32) over $(t, t+1)$, with $t > 0$, gives
\[
\int_t^{t+1} (d_0 \int |S_lu_{t\ell}(\tau)|^{p-1}|S_lu_{t\ell}(\tau)|^2 d\tau + \|u_{t\ell}(\tau)\|^2_{H^1} + \|u_{tt\ell}(\tau)\|^2) d\tau \leq (1 + \frac{1}{t^2})C(R_0, \|f\|).
\] (2.36)
The combination of (2.34), (2.36) means that estimate (2.6) uniformly holds for $U^t$.

Using the multiplier $-\Delta u^t$ in Eq. (2.10) we get
\[
\frac{1}{2} \frac{d}{dt} H_3(u^t) + \|u^t\|^2_{H^2} + \|u^t\|^2_{H^1} + \{S_l(g(S_lu^t)), -\Delta u^t\} = (f^t - u_{tt^\ell}, -\Delta u^t) + (u^t, u_{t\ell}), \]
where
\[
H_3(u^t) = \|u^t\|^2_{H^{1+\alpha}} + \|u^t\|^2_{H^1} + \|u^t\|^2 \sim \|u^t\|^2_{H^{1+\alpha}}.
\]
Since $g'(s) > -C_0$,
\[
\{S_l(g(S_lu^t)), -\Delta u^t\} = (g'(S_lu^t))(\nabla S_lu^t) \nabla S_lu^t \geq -C_0\|u^t\|^2_{H^1},
\]
we get
\[
\frac{d}{dt} H_3(u^t) + \|u^t\|^2_{H^2} + \|u^t\|^2_{H^1} + \|u^t\|^2 \leq C\|u_{tt\ell}\|^2 + C(R_0, \|f\|).
\] (2.37)
Multiplying (2.37) by $(t - \frac{\alpha}{2})^{1-\alpha}$ with $\frac{\alpha}{2} \leq t \leq 1 + \frac{\alpha}{2}, 0 < \alpha < 1$, we get
\[
\frac{d}{dt} \left( (t - \frac{\alpha}{2})^{1-\alpha} H_3(u^t) \right) + (t - \frac{\alpha}{2})^{1-\alpha}\|u^t\|^2_{H^2} \leq C\|u_{tt\ell}\|^2 + \frac{C(t - \frac{\alpha}{2})^{1-\alpha}}{1-\alpha}\|u^t\|^2_{H^{1+\alpha}} + C(R_0, \|f\|).
\]
By the interpolation theorem,
\[
\frac{C}{1-\alpha}(t - \frac{a}{2})^{\frac{\alpha}{1-\alpha}} \|u_t\|^2_{H^{1+\alpha}} \leq \frac{C}{1-\alpha}(t - \frac{a}{2})^{\frac{\alpha}{1-\alpha}} \|u_t\|^2_{H^2} \|u_t\|^2_{H^{1+\alpha}} \leq \frac{(t - \frac{a}{2})^{\frac{1}{1-\alpha}}}{2} \|u_t\|^2_{H^2} + C(\alpha) \|u_t\|^2_{H^1},
\]
where \(C(\alpha) = (\frac{C\alpha}{1-\alpha})^{\frac{\alpha}{1-\alpha}}\), so
\[
\frac{d}{dt} \left( (t - \frac{a}{2})^{\frac{\alpha}{1-\alpha}} H_3(u_t) \right) + \kappa(t - \frac{a}{2})^{\frac{\alpha}{1-\alpha}} H_3(u_t) \leq C \|u_t\|^2 + C(\alpha) \|u_t\|^2_{H^2} + C(R_0, \|f\|),
\]
\[
\|u_t(t)\|^2_{H^{1+\alpha}} \leq C(\alpha) C(R_0, \|f\|) (t - \frac{a}{2})^{\frac{1}{1-\alpha}} \|u_t\|^2_{H^2} + C(R_0, \|f\|), \quad a/2 < t - a/2. \tag{2.38}
\]
Applying the Gronwall lemma to (2.37) over \((1+a/2, t)\) and making use of (2.36) and (2.38) at \(t = 1+a/2\), we get
\[
\|u_t(t)\|^2_{H^{1+\alpha}} \leq C\|u_t(1+a/2)\|^2_{H^{1+\alpha}} + C \int_{1+a/2}^{t} e^{-\kappa(t-\tau)} \left( \|u_{tt}(\tau)\|^2 + C(R_0, \|f\|) \right) d\tau \leq C(\alpha) C(R_0, \|f\|) \left( 1 + (t - \frac{a}{2})^{\frac{1}{1-\alpha}} \right). \tag{2.39}
\]
Integrating (2.37) over \((t, t+1)\), with \(t > a/2\), and exploiting (2.38)-(2.39), we obtain
\[
\int_t^{t+1} \|u_t(\tau)\|^2_{H^2} d\tau \leq C(\alpha) C(R_0, \|f\|) \left( 1 + (t - \frac{a}{2})^{\frac{1}{1-\alpha}} \right). \tag{2.40}
\]

The combination (2.38)-(2.40) implies that estimate (2.7) uniformly holds for \(U^1(t)\).

By estimates (2.18)-(2.19) and (2.25), we have (subsequence if necessary)
\[
u_t \rightarrow u \text{ weakly}^* \text{ in } L^{\infty}(0, T; H^1),
\]
\[
g(S_t u_t) \rightarrow \vartheta \text{ weakly}^* \text{ in } L^{\infty}(0, T; L^2 + L^{p+1}),
\]
\[
u_t \rightarrow u_t \text{ weakly}^* \text{ in } L^{\infty}(0, T; L^2) \cap L^2(0, T; H^\alpha). \tag{2.41}
\]

If \(\vartheta = g(u)\), then \(u_t\) satisfies that for any \(\phi \in H^1 \cap L^{p+1}\),
\[
(u_t, \phi) + \int_0^t \left[ (\nabla u_t, \nabla \phi) + ((-\Delta)^{\frac{\alpha}{2}} u_t, (-\Delta)^{\frac{\alpha}{2}} \phi) + (u_t + u, \phi) + (S_t(g(S_t u_t)) - f_t, \phi) \right] d\tau = (u_1, \phi).
\]
Letting \(l \rightarrow \infty\), we get
\[
(u_t, \phi) + \int_0^t \left[ (\nabla u, \nabla \phi) + ((-\Delta)^{\frac{\alpha}{2}} u_t, (-\Delta)^{\frac{\alpha}{2}} \phi) + (u_t + u, \phi) + (g(u) - f, \phi) \right] d\tau = (u_1, \phi),
\]
where we have used the fact:
\[
\lim_{l \rightarrow \infty} \int_0^t (S_t(g(S_l u_t)) - g(u), \phi) d\tau = \lim_{l \rightarrow \infty} \int_0^t \left[ (g(S_l u_t), S_t \phi - \phi) + (g(S_l u_t) - g(u), \phi) \right] d\tau = 0.
\]
Therefore, \(u\) is the weak solution of problem (1.1)-(1.2). By the lower semi-continuity of weak* limit, estimates (2.5)-(2.7) hold for \(u\).
In order to show \( \vartheta = g(u) \), it is enough to prove

\[ S_iu^l \to u \text{ a.e. in } [0, T] \times \mathbb{R}^N. \]

Let \( B_R \) be a ball in \( \mathbb{R}^N \) with center zero and radius \( R \in \mathbb{N}^+ \). Estimates (2.18) and (2.19) show that the sequence \( \{ S_iu^l, S_ju^l \} \) is uniformly bounded in the space

\[ L^\infty(0, T; H^1(B_R) \cap L^{p+1}(B_R)) \times L^\infty(0, T; L^2(B_R)) \cap L^2(0, T; H^\alpha(B_R)) \].

By Lemma 2.3 (subsequence if necessary)

\[ S_iu^l \to u \text{ in } C([0, T]; L^2(B_R)) \text{ and a.e. in } [0, T] \times B_R. \]

By the standard diagonal argument, we can extract a subsequence (still denoted by itself) such that

\[ S_iu^l \to u \text{ a.e. in } [0, T] \times \mathbb{R}^N, \]

which combining with (2.41) implies that \( \vartheta = g(u) \).

Let \( u, v \) be two solutions of problem (1.1)-(1.2) corresponding to initial data \((u_0, u_1)\) and \((v_0, v_1)\) with \( \| (u_0, u_1) \|_H + \| (v_0, v_1) \|_H \leq R_0 \), respectively. Then \( z = u - v \) solves

\begin{align*}
&z_{tt} + (-\Delta)^\alpha z_t - \Delta z + z_t + z + g(u) - g(v) = 0, \quad (2.42) \\
u(0) = u_0 - v_0, \quad u_t(0) = u_1 - v_1.
\end{align*}

Using the multiplier \((I - \Delta)^{-\alpha}z_t + \varepsilon z\) in Eq. (2.42) yields

\begin{align*}
\frac{1}{2} \frac{d}{dt} H_2(\xi_z) + \| (I + (-\Delta)^\alpha)^{1/2} (I - \Delta)^{-\alpha/2} z_t \|^2 + \varepsilon \left( \| z \|_{H^1}^2 + (g(u) - g(v), z) \right)
= & \varepsilon \| z_t \|^2 - (g(u) - g(v), (I - \Delta)^{-\alpha} z_t), \quad (2.43)
\end{align*}

where \( \xi_z(t) = (z(t), z_t(t)) \) and

\[ H_2(\xi_z(t)) = \| z_t \|^2_{H^{-\alpha}} + \| z \|^2_{H^{1-\alpha}} + \varepsilon \left( \| z \|^2_{H^\alpha} + \| z_t \|^2 + 2(z, z_t) \right) \sim \| z \|^2_{H^\alpha} + \| z_t \|^2_{H^{-\alpha}} \]

for \( \varepsilon > 0 \) suitably small. When \( 1 \leq p < p_\alpha \), similar to the proofs of (2.30)-(2.31) we have

\begin{align*}
(g(u) - g(v), z) & \geq d_0 C_2 \int (|u|^{p-1} + |v|^{p-1}) |z|^2 dx - C \| z \|^2, \quad (2.44) \\
\left| (g(u) - g(v), (I - \Delta)^{-\alpha} z_t) \right| & \leq \varepsilon \left( d_0 C_2 \int (|u|^{p-1} + |v|^{p-1}) |z|^2 dx + \frac{1}{2} \| z \|^2_{H^1} + \| z_t \|^2 \right) + C(\varepsilon) \| z \|^2 + \| z_t \|^2_{H^{-2\alpha}} \quad (2.45)
\end{align*}

for some \( C_2 > 0 \). Inserting (2.44)-(2.45) into (2.43) arrives at

\[ \frac{d}{dt} H_2(\xi_z) + \kappa \left( \| z \|^2_{H^1} + \| z_t \|^2 \right) \leq C(\| z \|^2 + \| z_t \|^2_{H^{-2\alpha}}). \quad (2.46) \]

Using the Sobolev embedding \( H^1 \times L^2 \hookrightarrow H^\alpha \times H^{-\alpha} \) and applying the Gronwall lemma to (2.46), one obtains (2.9). Then one directly obtains (2.8) for \( H^\alpha \times H^{-\alpha} \hookrightarrow L^2 \times H^{-2\alpha}. \)
Remark 2.7. (i) The control constants in Theorem 2.6 are independent of $\alpha$ except $C(\alpha)$. A simple calculation shows that $\lim_{\alpha \to 1} C(\alpha) = \infty$, which means that estimate (2.7) does not hold for $\alpha = 1$.

(ii) Although by carefully choosing $\alpha$ and $t$ in (2.7), for example $t \geq \frac{a}{2} + \left(\frac{C\alpha}{1-\alpha}\right)^{\alpha}$, we can balance the blowup of the constant $C(\alpha)$ and obtain

$$\|u\|_{H^{1+\alpha}} \leq C(R_0, \|g\|), \quad t \geq \frac{a}{2} + \left(\frac{C\alpha}{1-\alpha}\right)^{\alpha}.$$ \[ But the additional regularity for $u$ still fails when $\alpha \to 1^-$ for $\frac{a}{2} + \left(\frac{C\alpha}{1-\alpha}\right)^{\alpha} \to +\infty$. \]

When $1 \leq p < p_\alpha$, based on Theorem 2.6 we define the solution operator $S^\alpha(t) : \mathcal{H} \to \mathcal{H}$,

$$S^\alpha(t)(u_0, u_1) = \xi_{u^\alpha}(t) = (u^\alpha(t), u^\alpha_1(t)), \quad (u_0, u_1) \in \mathcal{H},$$

where $u^\alpha$ is the solution of problem (1.1)-(1.2), and the family of solution operators $\{S^\alpha(t)\}_{t \geq 0}$ constitutes a semigroup on $\mathcal{H}$ for each $\alpha \in (1/2, 1)$, which is Lipschitz continuous in weaker space $\mathcal{H}_{-\alpha}$. By the interpolation and standard argument as in [27], one easily knows that $\{S^\alpha(t)\}_{t \geq 0}$ is Hölder continuous in phase space $\mathcal{H}$ for each $\alpha \in (1/2, 1)$.

3 Global attractor

In this section, we study the existence of global attractor of the dynamical system $(S^\alpha(t), \mathcal{H})$ for each $\alpha \in (1/2, 1)$. For brevity, we denote $S^\alpha(t)$ by $S(t)$ and $(u^\alpha, u^\alpha_1)$ by $(u, u_1)$, respectively.

It follows from (2.5) that the dynamical system $(S(t), \mathcal{H})$ possesses a bounded absorbing set

$$B_0 = \{(u, v) \in \mathcal{H} \mid \|u\|_{H^1}^2 + \|u\|_{p+1}^2 + \|v\|^2 \leq R_1^2\}$$

for $R_1$ suitably large, so there exists a positive constant $t_0$ such that $S(t)B_0 \subseteq B_0$ for $t \geq t_0$. Let

$$\mathcal{B} = \left[ \bigcup_{t \geq t_0+1} S(t)B_0 \right]_{\mathcal{H}_{-\alpha}},$$

where $[\bigcup]_{\mathcal{H}_{-\alpha}}$ denotes the closure in $\mathcal{H}_{-\alpha}$. Obviously, $\mathcal{B}$ is a forward invariant absorbing set and bounded in $\mathcal{H}_\alpha$ (see (2.1), (2.6) and (2.7)). Then the solution $u$ corresponding to the initial data $(u_0, u_1) \in \mathcal{B}$ satisfies

$$\|u(t)\|_{H^\alpha}^2 + \int_0^t \|u_\tau\|_{H^\alpha}^2 d\tau + \int_t^{t+1} (\int_0^\tau |u_\tau|^p + \|u_\tau\|_{H^1}^2 + \|u_\tau\|_{L^2}^2) d\tau \leq C(R_0, \|f\|), \quad \forall t > 0,$$

$$\int_0^t (g(u), u_\tau) d\tau \leq C \int_0^t \left( \|u\|_{\mathcal{H}_{-\alpha}} + \|u_\tau\|_{H^1} \right)^{p} d\tau \leq C(t), \quad \forall t > 0,$$

i.e., $g(u)u_\tau \in L^1([0, t] \times \mathbb{R}^N)$. So by the technique used in [22], we can use the multiplier $u_\tau$ in Eq. (1.1) and the energy equality

$$E(\xi_\alpha(t)) + \int_t^s (\|u_\tau\|_{H^\alpha}^2 + \|u_\tau\|_{L^2}^2) d\tau = E(\xi_\alpha(s)), \quad \forall t > s \geq 0$$

holds, where $\xi_u = (u, u_\tau)$,

$$E(u, v) = \frac{1}{2}(\|v\|^2 + \|u\|_{H^1}^2 + \|u\|^2) + \int G(u)dx - (f, u).$$
We construct the function

\[
K_0(s) = \begin{cases} 
0, & 0 \leq s \leq 1, \\
 s - 1, & 1 < s \leq 2, \\
 1, & s > 2,
\end{cases}
\]

and let

\[
K_\delta(s) = (\rho_\delta * K_0)(s) = \int_\mathbb{R} \rho_\delta(s-y) K_0(y) dy,
\]

where \( \rho_\delta(s) \) is the standard mollifier on \( \mathbb{R} \) with \( \text{supp} \rho_\delta \subset [-\delta, \delta] \). Obviously,

\[
K_\delta \in C^\infty(\mathbb{R}), \quad 0 \leq K_\delta(s) \leq 1, \quad K_\delta(s) = 0 \text{ as } 0 \leq s < 1; \quad K_\delta(s) = 1 \text{ as } s > 2,
\]

with \( 0 < \delta \ll 1 \). Let \( \psi(x) = K_\delta(\frac{|x|}{R}) \). A simple calculation shows that

\[
\psi(x) = 0 \text{ as } |x| < R, \quad 0 \leq \psi(x) \leq 1 \text{ and } |\nabla \psi(x)| \leq \begin{cases} 
CR, & R \leq |x| \leq 2R, \\
0, & \text{others},
\end{cases}
\]

where the constant \( C \) is independent of \( R \). Hence, we have

\[
\|
\nabla \psi\n\|_p \leq CR^{-1 + \frac{1}{p}}, \quad p \geq 1.
\]

In light of \( \psi \in L^\infty \), by the Sobolev embedding theorem we have

\[
\|
\psi^{\eta} \phi\n\|_{2^N} \leq C\|
\psi\n\|_{H^{\frac{1}{2}+\eta}} \leq C\|
\psi\n\|_{H^1} \leq CR^{-\frac{1}{p}},
\]

where \( \Lambda = (-\Delta)^{\frac{1}{2}} \). Using the similar procedure as to the estimate (3.2), we have

\[
\|
\psi\n\|_{H^{s, q}} \leq C\|
\psi\n\|_{H^{1, \frac{1-s-\frac{N}{q}}{q}}} \leq CR^{-\frac{s-\frac{N}{q}}{q}}, \quad \forall s \leq 1, q \geq 1.
\]

Lemma 3.1. [17] Assume that \( \Lambda = (-\Delta)^{\frac{1}{2}}, 0 < s < 1, s_1, s_2 \in [0, s], s = s_1 + s_2 \) and \( p, p_1, p_2 \in (1, +\infty) \) satisfying

\[
\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2},
\]

then

\[
\|
\Lambda^s(fg) - f\Lambda^s g - g\Lambda^s f\n\|_p \leq C\|
\Lambda^{s_1} f\n\|_{p_1}\|
\Lambda^{s_2} g\n\|_{p_2}.
\]

Moreover, the inequality holds for \( s_1 = 0, p_1 = \infty, \) i.e.,

\[
\|
\Lambda^s(fg) - f\Lambda^s g - g\Lambda^s f\n\|_p \leq C\|
\phi\n\|_\infty\|
\Lambda^s g\n\|_p.
\]

Lemma 3.2. Let \( u \) be the solution of problem (1.1) - (1.2) as shown in Theorem 2.6 Then

\[
\|
\psi \Lambda^s u\n\| \leq C(\|
\psi\n\phi\n\| \|
\nabla u\n\|) + CR^{-\frac{1}{q}}\|
\nabla u\n\|_{H^1}.
\]
Proof. Since
\[ \|\psi\Lambda^\alpha u\| \leq \|\Lambda^\alpha(\psi u) - \psi\Lambda^\alpha u\| + \|\Lambda^\alpha(\psi u)\|, \]  
(3.5)
by Lemma 3.1 (taking \( s_1 = \alpha, s_2 = 0 \) there), the Hölder inequality and (3.1)-(3.2), we have
\[ \|\Lambda^\alpha(\psi u) - \psi\Lambda^\alpha u\| \leq \|u\Lambda^\alpha\psi\| + C\|\Lambda^\alpha\psi\|_{2N/\alpha} \|u\|_{2N/\alpha} \leq C\|\psi\|_{H^{\alpha,2N}} \|u\|_{2N/\alpha} \leq CR^{-\frac{2}{\alpha}} \|u\|_{H^1}, \]
\[ \|\Lambda^\alpha(\psi u)\| = \|\psi u\|_{H^{\alpha}} \leq C\|\psi u\|_{H^{\alpha}} + \|\psi\Lambda^\alpha u\|_{H^{\alpha}} \leq C\|\psi u\|_{H^{1}} + \|\psi\nabla u\|_{H^{1}} + CR^{-1} \|u\|. \]
Inserting above inequalities into (3.5) yields estimate (3.4).

Lemma 3.3. Let \( S(t)(u_0, u_1) = (u(t), u_t(t)) \), with \( (u_0, u_1) \in \mathcal{B} \). Then for any \( \epsilon > 0 \), there exist positive constants \( K = K(\epsilon) \) and \( T_0 = T_0(R_1) \) such that
\[ \|(u(t), u_t(t))\|_{H(\mathcal{B}^C_{R_1})} < \epsilon \quad \text{as} \quad R \geq K, \ t \geq T_0, \]
where \( \mathcal{B}_R \) is the ball centered at zero with radius \( R \) in \( \mathbb{R}^N \), \( \mathcal{B}^C_{R} = \mathbb{R}^N \setminus \mathcal{B}_R \).

Proof. Using the multiplier \( \psi^2(u_t + \epsilon u) \) in Eq. (1.1) and making use of the boundedness of \( (u, u_t) \) in \( H_{\alpha} \), we have
\[
\frac{d}{dt}H_4(\xi_u) + \Phi_2(\xi_u) = -2(\psi\nabla\psi(u_t + \epsilon u), \nabla u) + \left( \Lambda^\alpha u_t, \psi^2\Lambda^\alpha(u_t + \epsilon u) - \Lambda^\alpha(\psi^2(u_t + \epsilon u)) \right) \\
\leq \epsilon^2\|\psi\nabla u\|^2 + C(\|u_t\nabla\psi\|^2 + \|u\nabla\psi\|^2) \\
+ \left( \|\psi^2\Lambda^\alpha u_t - \Lambda^\alpha(\psi^2 u_t)\| + \epsilon\|\psi^2\Lambda^\alpha u - \Lambda^\alpha(\psi^2 u)\| \right) \|\Lambda^\alpha u_t\| \\
\leq \epsilon^2\|\psi\nabla u\|^2 + C(\|u_t\nabla\psi\|^2 + \|u\nabla\psi\|^2) + CR^{-\frac{2}{\alpha}}(\|u_t\|_{H^{\alpha}}^2 + \|u\|_{H^1}^2), \quad (3.6)
\]
where
\[
H_4(\xi_u) = \frac{1}{2} \left( \|\psi u_t\|^2 + \|\psi\nabla u\|^2 + \|\psi u\|^2 + 2 \int \psi^2(G(u) - f u)dx \right) \\
+ \epsilon \left( \|\psi^2u_t\|^2 + \|\psi\Lambda^\alpha u_t\|^2 \right) ,
\]
\[
\Phi_2(\xi_u) = \{1 - \epsilon\}\|\psi u_t\|^2 + \epsilon\|\psi\Lambda^\alpha u_t\|^2 + \epsilon(\|\psi\nabla u\|^2 + \|\psi u\|^2 + \int \psi^2 g(\psi)dxu - (\psi^2 u, f)),
\]
and where we have used Lemma 3.1 (with \( s_1 = \alpha, s_2 = 0 \) there), and formula (3.3) to get the estimates:
\[
\|\psi^2\Lambda^\alpha u_t - \Lambda^\alpha(\psi^2 u_t)\| \leq \|u_t\Lambda^\alpha\psi^2\| + C\|\Lambda^\alpha\psi\|_{2N/\alpha} \|u_t\|_{2N/\alpha} \\
\leq C\|\psi\|_{H^{\alpha}} \|\psi\|_{H^{\alpha}} \|u_t\|_{H^{\alpha/2}} \leq CR^{-\alpha/2} \|u_t\|_{H^1}, \quad (3.7)
\]
and by the similar argument as to the estimate (3.7), we obtain
\[
\|\psi^2\Lambda^\alpha u - \Lambda^\alpha(\psi^2 u)\| \leq CR^{-\alpha/2} \|u\|_{H^1}.
\]
By Lemma 3.2 and Remark 2.5 we have
\[
H_4(\xi_u) \geq \kappa(\|\psi u_t\|^2 + \|\psi u\|^2 + \|\psi\nabla u\|^2 + d_0 \int \psi^2|u|^{p+1}dx) - C\|\psi f\|^2, \quad (3.8)
\]
\[
\Phi_2(\xi_u) - \psi(\|\psi\nabla u\|^2 + H_4(\xi_u)) \\
\geq \kappa(\|\psi u_t\|^2 + \|\psi\Lambda^\alpha u_t\|^2 + \|\psi\nabla u\|^2 + \|\psi u\|^2) - CR^{-\frac{2}{\alpha}} \|u\|_{H^1}^2, \quad (3.9)
\]
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for $\epsilon > 0$ suitably small. Inserting (3.9) into (3.6) and using (3.1), (3.8), we have

\[
\frac{d}{dt}H_4(\xi_u) + \kappa H_4(\xi_u) \leq CR^{-\frac{\alpha}{2}}(\|u\|_{H^1}^2 + \|u_t\|_{H^{1/2}}^2) + C\|\psi f\|^2,
\]

\[
H_4(\xi_u) \leq CH_4(\xi_u(0))e^{-\kappa t} + C(R^{-\frac{\alpha}{2}} + \|f\|_{L^2(B_{2R}^\alpha)})^2. \tag{3.10}
\]

The combination of (3.8) and (3.10) implies the conclusion of Lemma 3.3.

\[\square\]

**Theorem 3.4.** Let Assumption 2.4 be valid. Then the solution semigroup $S(t)(= S^\alpha(t))$ possesses a global attractor $A(= A_\alpha)$ in $H$ for each $\alpha \in (1/2, 1)$.

**Proof.** It is enough to show that $S(t)$ is asymptotically compact on $\mathcal{B}$ with respect to $H$-topology. Let $\{(u^n_0, u^n_1)\}$ be a bounded sequence in $\mathcal{B}$, $S(t)(u^n_0, u^n_1) = (u^n(t), u^n_t(t))$. Applying (2.9) to

\[w^{m,n}(t) = u^m(t + t_m - T) - u^n(t + t_n - T), \text{ with } t_m > t_n > T > 0, t \geq 0,
\]

and making use of the fact $L^2 \hookrightarrow H^{-2}$, we obtain

\[
\|(w^{m,n}, w^{m,n}_t)(t)\|^2_{H^{-\alpha}} \leq Ce^{-\kappa t} + C \sup_{0 \leq s \leq T} \|(w^{m,n}, w^{m,n}_t)(s)\|_{L^2 \times L^2}^2.
\]

Taking $t = T$ yields

\[
\|(u^m(t_m) - u^n(t_n), u^m_t(t_m) - u^n_t(t_n))\|^2_{H^{-\alpha}} \leq Ce^{-\kappa T} + C \left( \sup_{0 \leq s \leq T} \|(w^{m,n}, w^{m,n}_t)(s)\|_{L^2(B_{2R}^\alpha) \times L^2(B_{2R}^\alpha)}^2 + \sup_{0 \leq s \leq T} \|(w^{m,n}, w^{m,n}_t)(s)\|_{L^2(B_{2R}) \times L^2(B_{2R})}^2 \right).
\]

By Lemma 2.3,

\[
\Pi_1 = \{u \in L^\infty(0, T; H^{1+\alpha}(B_{2R})) \mid u_t \in L^2(0, T; H^\alpha(B_{2R}))\} \hookrightarrow C([0, T]; L^2(B_{2R})),
\]

\[
\Pi_2 = \{u_t \in L^\infty(0, T; H^\alpha(B_{2R})) \mid u_{tt} \in L^2(0, T; L^2(B_{2R}))\} \hookrightarrow C([0, T]; L^2(B_{2R})),
\]

thus the subsequence

\[
\{(u^m(t_m), u^m_t(t_m))\}
\]

is precompact in $C([0, T]; L^2(B_{2R}) \times L^2(B_{2R}))$.

Therefore, for any $\epsilon > 0$, fixing $T : Ce^{-\kappa T} < \epsilon/4$, there must exist a $N_0 > 0$ such that when $m, n \geq N_0$, $t_m - T > T_0, t_n - T > T_0$, and

\[
\|(u^m(t_m) - u^n(t_n), u^m_t(t_m) - u^n_t(t_n))\|_{H^{-\alpha}} < \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} < \epsilon;
\]

i.e., the semigroup $S(t)$ is asymptotically compact on $\mathcal{B}$ with respect to the topology $H^{-\alpha}$. Taking account of the boundedness of $\mathcal{B}$ in $H^\alpha(\hookrightarrow H \hookrightarrow H^{-\alpha})$, by the interpolation one easily sees that $S(t)$ is asymptotically compact on $\mathcal{B}$ with respect to $H$-topology. Therefore, $(S(t), H)$ possesses a global attractor $A$, and $A \subset \mathcal{B}$ is bounded in $H^\alpha$. \[\square\]
4 Upper semicontinuity of the global attractors

Lemma 4.1. Let $X,Y$ be two Banach spaces, $X \hookrightarrow Y$, and the semigroup $\{S^\alpha(t)\}_{t \geq 0}$ has a bounded absorbing set $B_\alpha$ and a global attractor $A_\alpha$ in $X$ for each $\alpha \in I$ (a subset of $\mathbb{R}$). Assume that the following assumptions hold:

(i) the union $\bigcup_{\alpha \in I} B_\alpha$ is bounded in $X$;
(ii) for any sequences $\{\alpha_n\} \subset I, \{x_n\} \subset \bigcup_{\alpha \in I} B_\alpha$ and $t_n \to \infty$, the sequence $\{S^{\alpha_n}(t_n)x_n\}$ is precompact in $X$;
(iii) for any sequences $\{\alpha_n\} \subset I$ with $\alpha_n \to \alpha_0$ and $\{x_n\} \subset X$ with $x_n \to x_0$ in $X$,
$$\lim_{n \to \infty} \|S^{\alpha_n}(t)x_n - S^{\alpha_0}(t)x_0\|_Y = 0, \ \forall t \geq 0.$$ Then
$$\lim_{\alpha \to \alpha_0} \text{dist}_X(A_\alpha, A_{\alpha_0}) = 0.$$

Theorem 4.2. Let Assumption 2.2 be valid, with $1 \leq p < p_\alpha \equiv \frac{N + 4\alpha}{(N - 4\alpha)^+}$, $\alpha \in I \equiv (1/2, 1)$. Then the family of global attractors $\{A_\alpha\}_{\alpha \in I}$ as shown in Theorem 3.4 is upper semicontinuous at the point $\alpha_0$, i.e.,
$$\lim_{\alpha \to \alpha_0} \text{dist}_H(\{A_\alpha\}, A_{\alpha_0}) = 0. \tag{4.1}$$

Proof. Without loss of generality we assume that $\alpha \in [\gamma, \gamma_1] \equiv \Gamma(\subset (1/2, 1))$ for $\alpha \to \alpha_0$, where $\gamma \equiv \alpha_0 - \eta(> 0), \gamma_1 \equiv \alpha_0 + \eta$, with $\eta : 0 < \eta < \min\{\alpha_0 - 1/2, \alpha_0/3, (1 - \alpha_0)/3\}$.

Similar to the proof of (2.26), for the approximate solution $u^\alpha(= u^{\alpha,I})$ of the auxiliary problem (2.10), the following estimate holds:
$$\int_0^{t+1} \|u^{\alpha}_{tt}(\tau)\|_{H^{-2}}^2 d\tau \leq C(R_0, \|f\|), \ \forall t > 0, \alpha \in \Gamma.$$ When $1 \leq p < p_\gamma(\equiv \frac{N + 4\gamma}{(N - 4\gamma)^+})$, similar to the proofs of (2.6) and (2.7) (replacing the multiplier $(I - \Delta)^{-\alpha}u^\alpha_t + \epsilon u^\alpha$ after Eq. (2.27) by $(I - \Delta)^{-\gamma}v^\alpha_t + \epsilon v^\alpha$) one easily obtains that
$$\|u^{\alpha}_t(t)\|^2_{H^{\alpha_n}} + \|u^{\alpha}_{tt}(t)\|^2_{H^{-\gamma}} + \int_t^{t+1} (\|u^{\alpha}_t(\tau)\|^2_{H^{1+\gamma}} + \|u^{\alpha}_{tt}(\tau)\|^2) d\tau \leq \left(1 + \frac{1}{t^2}\right) C(R_0, \|f\|) \tag{4.2}$$ and
$$\|u^\alpha(t)\|^2_{H^{1+\gamma}} \leq \|u^\alpha(t)\|^2_{H^{1+\alpha}} \leq \left(1 + \frac{1}{(t - \frac{\epsilon}{t})^{1-\alpha}}\right) C(\alpha) C(R_0, \|f\|), \ \ t > \alpha. \tag{4.3}$$

A simple calculation shows that $C(\alpha) = (\frac{C_\alpha}{t - \frac{\epsilon}{t}})^{1-\alpha} \leq C_3(\alpha_0)$ for $\alpha \in \Gamma$. So by the lower semicontinuity of weak limit, estimates (4.2) and (4.3) (replacing the control constant $C(\alpha)$ there by $C_3(\alpha_0)$) uniformly (w.r.t. $\alpha \in \Gamma$) hold for the weak solutions $u^\alpha$ of problem (1.1)-(1.2).

(i) It follows from estimate (2.5) that the family of dynamical systems $(S^\alpha(t), \mathcal{H}), \alpha \in \Gamma$ possesses in $\mathcal{H}$ a common absorbing set
$$B_1 \equiv \{(u,v) \in \mathcal{H}, \|(u,v)\|_\mathcal{H} \leq R_1\} \text{ for } R_1 > C(\|f\|),$$
and there exists a $t_1 = t(R_1, \|f\|)$ such that $\bigcup_{\alpha \in \Gamma} S^\alpha(t_1)B_1 \subset B_1$ for $t \geq t_1$. Let
$$B_1 = \bigcup_{\alpha \in \Gamma} \bigcup_{t \geq t_1 + 2} S^\alpha(t)B_1 \subset B_1.$$
Then the set $\mathcal{B}_1$ is bounded in $\mathcal{H}_\alpha$ (see estimates (4.2)-(4.3)). Moreover, $\mathcal{B}_1$ is a common forward invariant absorbing set of the family of dynamical systems $(S^\alpha(t), \mathcal{H}), \alpha \in \Gamma$. And $\mathcal{B}_1 = \bigcup_{\alpha \in \Gamma} \mathcal{B}_1$.

(ii) For any sequences $\{\alpha_n\} \subset \Gamma$, $\{\xi_{t_n}\} \subset B_1(\equiv \bigcup_{\alpha \in \Gamma} \mathcal{B}_1)$ and $t_n \to \infty$, the sequence $\{S^{\alpha_n}(t_n)\xi_{u_n}\}$ is precompact in $\mathcal{H}$.

Indeed, let $S^\alpha(t)(u_0, u_1) = (u^\alpha(t), u^\alpha_t(t))$, with $(u_0, u_1) \in \mathcal{B}_1$ and $\alpha \in \Gamma$. It follows from Lemma 3.3 that for any $\epsilon > 0$, there exist positive constants $K_2 = K_2(\epsilon) > 0$ and $T_2 = T_2(R_1) > 0$ such that for all $\alpha \in \Gamma$,

$$
\|(u^\alpha(t), u^\alpha_t(t))\|_{\mathcal{H}(B^\alpha_{K_2})} < \epsilon \text{ as } K \geq K_2, \ t \geq T_2.
$$

So there exists a $N > 0$ such that when $n, m \geq N$, $t_n, t_m \geq T_2$ and $\alpha \in \Gamma$,

$$
\|S^{\alpha_n}(t_n)\xi_{u_n}\|_{\mathcal{H}(B^\alpha_{K_2})} < \epsilon.
$$

Therefore,

$$
\|S^{\alpha_n}(t_n)\xi_{u_n} - S^{\alpha_m}(t_m)\xi_{u_m}\|_{\mathcal{H}} \leq \|S^{\alpha_n}(t_n)\xi_{u_n} - S^{\alpha_m}(t_m)\xi_{u_m}\|_{\mathcal{H}(B^\alpha_{K_2})} + 2\epsilon, \ \forall n, m \geq N,
$$

which, combining with the boundedness of $\mathcal{B}_1$ in $\mathcal{H}_\gamma$ and $\mathcal{H}_\gamma(B_{K_2}) \hookrightarrow \mathcal{H}(B_{K_2})$ for $1 \leq p < p_\gamma$, implies that the sequence $\{S^{\alpha_n}(t_n)\xi_{u_n}\}$ is precompact in $\mathcal{H}$.

(iii) When $1 \leq p < p_\gamma$, for any sequences $\{\alpha_n\} \subset \Gamma$ with $\alpha_n \to \alpha_0$ and $\{\xi_{u_n}\} \subset \mathcal{H}(\hookrightarrow \mathcal{H}_{-1/2})$ with $\xi_{u_n} \to \xi$ in $\mathcal{H}$, we have

$$
\lim_{n \to \infty} \|S^{\alpha_n}(t)\xi_{u_n} - S^{\alpha_0}(t)\xi\|_{\mathcal{H}_{-1/2}} = 0, \ \forall t \geq 0. \quad (4.4)
$$

Indeed, obviously formula (4.4) holds for $t = 0$. We show that formula (4.4) holds for any $t > 0$. Let

$$
S^{\alpha_n}(t)\xi_{u_n} = \xi_{w^{\alpha_n}(t)} = (u^{\alpha_n}(t), u^{\alpha_n}_t(t)), \quad S^{\alpha_0}(t)\xi = \xi_{w^{\alpha_0}(t)} = (u^{\alpha_0}(t), u^{\alpha_0}_t(t)),
$$

then $z(t) = u^{\alpha_n}(t) - u^{\alpha_0}(t)$ solves

$$
\begin{cases}
\dot{z}_t + (\Delta)^{\alpha_n} z_t + z_t - \Delta z + z + [(-\Delta)^{\alpha_n} - (-\Delta)^{\alpha_0}] u^{\alpha_n}_t + g(u^{\alpha_n}) - g(u^{\alpha_0}) = 0, \\
\xi(0) = (z, z_t) = 0 = \xi_{u_n} - \xi.
\end{cases} \quad (4.5)
$$

Using the multiplier $(I - \Delta)^{-\gamma} z_t + \epsilon z$ in Eq. (4.5) turns out

$$
\frac{d}{dt}H_5(\xi_z) + \| (I + (\Delta)^{\alpha_n})^{\gamma} (I - \Delta)^{-\gamma} z_t \|_2^2 - \epsilon \| z_t \|_2^2 + \epsilon \| z \|_2^2_{H^{\alpha_n}} + \epsilon (g(u^{\alpha_n}) - g(u^{\alpha_0}), z) = - (g(u^{\alpha_n}) - g(u^{\alpha_0}), (I - \Delta)^{-\gamma} z_t) - \| (-\Delta)^{\alpha_n} - (-\Delta)^{\alpha_0} u^{\alpha_n}_t, (I - \Delta)^{-\gamma} z_t + \epsilon z),
$$

where $\xi_z = (z, z_t)$, and

$$
H_5(\xi_z(t)) = \frac{1}{2} \left( \| z_t \|_{H^{\alpha_n}}^2 + \| z \|_{H^{1-\gamma}}^2 + \epsilon \| z \|_{H^{\alpha_n}}^2 + \| z \|_2^2 \right) + \epsilon (z, z_t) \sim \| z \|_{H^{\alpha_n}}^2 + \| z_t \|_{H^{1-\gamma}}^2
$$
for $\epsilon > 0$ suitably small. Obviously,

$$
\beta_1 \| \xi_z \|_{H_{-\gamma}} \leq \| z \|_{H^{\alpha_n}} + \| z_t \|_{H^{-\gamma}} \leq \beta_2 \| \xi_z \|_{H}, \quad (4.7)
$$

$$
\| (I + (-\Delta)^{\alpha_n})^{\gamma} (I - \Delta)^{-\gamma} z_t \|_{H^{\alpha_n}-\gamma} \sim \| z_t \|_{H^{\alpha_n}-\gamma},
$$

20
with $\beta_1, \beta_2 > 0$. Taking account of the Sobolev embedding $H^{\alpha_n - \gamma} \hookrightarrow H^{1-2\gamma}$ for $\eta < \alpha_0 - 1/2$, we have

$$\frac{d}{dt} H_5(\xi_z(t)) + \kappa H_5(\xi_z(t)) \leq C \|((-\Delta)^{\alpha_n} - (-\Delta)^{\alpha_0})u_t^{\alpha_0}\|_{H^{-1}}^2 + C \|z_t\|_{H^{1-\gamma}}^2 + C \|z\|_{H^{1-\gamma}}^2$$

where $H_5(\xi_z(t))$ is defined as

$$H_5(\xi_z(t)) = \int_{\mathbb{R}^N} \xi_z(\xi, t) \, d\xi.$$

So when $1 \leq p < p_\gamma$ ($< p_{\alpha_0}$), inserting (4.8) and (2.44)-(2.45) (replacing $u, v$ there by $u^{\alpha_n}, u^{\alpha_0}$, respectively) into (4.6), we obtain

$$\frac{d}{dt} H_5(\xi_z(t)) + \kappa H_5(\xi_z(t)) \leq C \frac{\|z\|_{H^{1-\gamma}}^2}{\|z\|_{H^{1-\gamma}}} + C \|z_t\|_{H^{1-\gamma}}^2 + C \|z\|_{H^{1-\gamma}}^2.$$

By (2.7) and (4.2), $u_t^{\alpha_0} \in L^2(0, t; H^{\alpha_0}) \cap C_w([0, t]; H)$, and

$$\frac{d}{dt} H_5(\xi_z(t)) + \kappa H_5(\xi_z(t)) \leq C \frac{\|z\|_{H^{1-\gamma}}^2}{\|z\|_{H^{1-\gamma}}} + C \|z_t\|_{H^{1-\gamma}}^2 + C \|z\|_{H^{1-\gamma}}^2.$$

Therefore, by Lemma 4.1, the family of global attractors $\{A_\alpha\}$ is upper semicontinuous at the point $\alpha_0$, i.e., formula (4.11) holds.
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