MYERSON VALUE OF DIRECTED HYPERGRAPHS

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Abstract. This paper presents directed hypergraphs as cooperative networks and defines their Myerson value. The axiomatization of the Myerson value, i.e., strong component efficiency and fairness, has been demonstrated. Furthermore, the concept of safety, as defined by Li and Shan, has been modified, and the hyperedge safety condition is proven in terms of the Myerson value of directed hypergraphs.

1. Introduction

Game theory can be classified into types—cooperative and noncooperative. In cooperative game theory, all players cooperate with each other to increase their utility, thereby forming a coalition and generating worth. The cooperative game theory is employed in several analyses, including those of markets in economics, bill votes by political parties, and cost sharing. Recently, it has also found application in market design by way of auction and matching theories. The primary focus in the application of these theories concerns the sharing or distribution of the worth gained from a cooperation. To this end, we usually consider a transferable utility or TU game. In 1953, Shapley introduced the Shapley value based on the results of one of the most famous investigations concerning this topic. Although it corresponds to a method of fairly distributing the worth gained from a cooperation, it is assumed there exist no restrictions on the cooperation possibilities of the players. To solve this problem, in 1977, Myerson introduced a graph-restricted game by modifying the original game and defined the corresponding Shapley value, namely the Myerson value. Subsequently, several studies have been conducted concerning the Myerson value. An interesting topic, which has hitherto been seldom investigated in available literature, concerns the type of cooperative network. Initially, Myerson considered an undirected graph as a cooperative network. Therefore, Li–Shan considered the Myerson value for a directed graph, and Nouweland–Borm–Tijs considered the Myerson value for a hypergraph. Another topic of research concerns the characterization of the Myerson value. In 2001, Algaba–Bilbao–Borm–Lopez characterized the Myerson value for union stable structures. In 2004, Gomez—Gonzalez–Manuel—Owen—Pozo–Tejada calculated the Myerson value of splitting graphs for pure overhead games. In 2012, Kim–Hee introduced various concepts of betweenness and centrality. Moreover, they deduced the relationship between them and the Myerson value. In 2016, Beal—Casajus–Huettner studied the efficient egalitarian Myerson value citeBA. In 2021, Wang–Shan proved the decomposition property for the weighted Myerson value.

Notably, the stability of an edge is an important concept in the study of the Myerson value. Myerson suggested that an edge is stable if none of the connected players benefit from breaking it under the Myerson value. Moreover, he proved that all edges in a super additive game are stable. However, there nonetheless exist games wherein one’s payoff might affect those of others. To solve this problem, in 2020, Li—Shan defined the safety of a link and deduced the appropriate safety conditions in terms of the Myerson value. Their findings...
imply that all bridges pertaining to a convex game are safe. In addition, they demonstrated that if a game is strictly convex, the corresponding graph link is safe if and only if it represents a bridge. However, the strictly convex of the game is a bit strong. Therefore, the motivation of this study is to improve their results.

The proposed study considers a directed hypergraph, a natural generalization of an undirected graph, a directed graph, and undirected hypergraph, as a cooperative network. Therefore, directed hypergraphs represent the most common discrete-object types. However, the number of vertices containing an edge on a directed hypergraph usually exceeds 2. Moreover, each edge is oriented. Therefore, studies involving directed hypergraphs are challenging compared to those involving graphs. In fact, no research concerning the Myerson values for directed hypergraphs has been undertaken to date. Moreover, in this study, we identified hypergraphs considering a simplicial complex. Therefore, the findings of this study are applicable to topological homology.

The remainder of this paper is organized as follows. Section 2 presents the basic notions and properties of directed hypergraphs and the Myerson value. Section 3 describes the modifications made in this study to the concept of safety, as defined by Li—Shan, as well as the condition of hyperedge safety in terms of the Myerson value for directed hypergraphs. Notably, the statement of the proposed result is more comprehensive compared to that proposed by Li–Shan, while the corresponding proof is comparatively straightforward. Section 4 describes the calculation of the Myerson value for each player within a specific graph. Finally, Section 5 lists the major conclusions drawn from this study and identifies the scope for future research.

2. Preliminaries

Let $H = (N, E)$ represent a hypergraph, wherein $N$ denotes a set of players $N = \{1, \cdots, n\}$ and $E$ is a subset of $2^N$. Let $v_i$ and $e_i$ denote individual elements of $N$ and $E$, respectively. Moreover, $v_i$ and $e_i$, can be referred to as player and hyperedge, respectively. Because the object of this study corresponds to a directed hypergraph, every hyperedge $e \in E$ represents a directional relation between two non-empty subsets $A_e$ and $B_e$ of $N$. Accordingly, $A_e$ and $B_e$ represent the tail and head sets of $e$, respectively.

Some notations pertinent to graph theory can be described as follows.

**Definition 2.1.** For any two players $s$ and $t$, a path from $s$ to $t$ represents a sequence of players and hyperedges $(s = v_0, e_0, v_1, e_1, \cdots, e_{l-1}, v_l = t)$, where

\[
\begin{align*}
  s & \in A_{e_0}, t \in B_{e_{l-1}}, \\
  v_i & \in A_{e_i} \cap B_{e_{i+1}}
\end{align*}
\]

for any $i \in \{1, \cdots, l-1\}$.

**Definition 2.2.** For any two players $s$ and $t$, $P(s, t)$ denotes the set of paths from $s$ to $t$. Player $i$ is critical for $s$ and $t$ if $i \in p$ for any $p \in P(s, t)$.

If there exist different paths from $s$ to $t$ and $t$ to $s$, then $s$ and $t$ are assumed to be connected. Moreover, if any player pair in $N$ is connected, then $H = (N, E)$ can be assumed to be connected. However, it is noteworthy that if there exists a path from $s$ to $t$, a path from $t$ to $s$ need not exist necessarily.

**Definition 2.3.** An induced subgraph $H'$ of $H = (N, E)$ represents a pair of $(N_{H'}, E_{H'})$ that satisfy $N_{H'} \subseteq N$ and

\[
E_{H'} = \{e \mid e \in E, \ e \subseteq N_{H'}\}.
\]
A maximal connected induced subgraph is called the strong component of $H$, and $N/E$ denotes the set of all strong components in $H$. Therefore, the union of all strong components in $H$ equals $N$. A hyperedge $e$ is referred to as a bridge of $H$ if the strong component containing $e$ splits into disconnected sets upon the deletion of $e$.

The TU game represents a pair $(N, v)$, where $N$ denotes a set of players and $v : 2^N \to \mathbb{R}$ is a characteristic function with $v(\emptyset) = 0$. Each $S \subseteq 2^N$ is referred to as a coalition, and $v(S)$ represents the worth of $S$. $TU(N)$ represents the set of all TU games. Given $(N, v) \in TU(N), v$ can be referred to as super additive if $v(S \cup T) \geq v(S) + v(T)$ and convex if $v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$. Moreover, $v$ can be referred to as strictly super additive (strictly convex) if $\geq$ is replaced with $>$ in the above-described inequalities.

Now, let us allocate each player to the worth, thereby defining an allocation rule for any $i \in N$. A popular allocation rule is the Shapley value $Sh(N, v)$ that can be described as follows for any $i \in N$.

$$Sh_i(N, v) = \sum_{S \subseteq N \setminus \{i\}} \frac{s!(n-s-1)!}{n!}(v(S \cup \{i\}) - v(S)).$$

In the above expression, $s = |S|$. An important theorem concerning the Shapley value can be stated as follows.

**Theorem 2.4 ([9]).** The Shapley value can be uniquely determined as efficient, additive, symmetric, and conforming to the null-player property subject to the following conditions.

1. An allocation rule $f$ on $TU(N)$ is efficient if $\sum_{i \in N} f_i(N, v) = v(N)$ for each $(N, v) \in TU(N)$.

2. An allocation rule $f$ on $TU(N)$ is additive if $f(N, v + w) = f(N, v) + f(N, w)$ for all $(N, v), (N, w) \in TU(N)$.

3. An allocation rule $f$ on $TU(N)$ is symmetric if $f_i(N, v) = f_j(N, v)$ for each $(N, v) \in TU(N)$ and any two players $i, j \in N$ with $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \setminus \{i, j\}$.

4. An allocation rule $f$ on $TU(N)$ satisfies the null-player property if $f_i(N, v) = 0$ for each $(N, v) \in TU(N)$ and any $i \in N$ with $v(S) = v(S \cap \{i\})$ for all $S \subseteq N$.

To define the Myerson value, let us consider a graph game.

**Definition 2.5.** (1) A directed hypergraph game on $N$ represents a triplet $(N, v, E)$ such that $(N, v)$ is a TU game and $(N, E)$ is a directed hypergraph. The set of all directed hypergraph games are denoted by $DHG(N)$.

2. Given a directed hypergraph game $(N, v, E)$, the directed hypergraph-restricted game $(N, v^E)$ is defined by for all $S \subseteq N$; therefore,

$$v^E(S) = \sum_{T \in S \setminus E_S} v(T).$$

3. The Myerson value $\mu(N, v, E)$ can be expressed as

$$\mu(N, v, E) = Sh(M, v^E) = \sum_{S \subseteq N \setminus \{i\}} s!(n-s-1)! \frac{|v^E(S \cup \{i\}) - v^E(S)|}{n!}.$$
3. Characterization of Myerson value for directed hypergraph games

In accordance with extant research, the properties of the Myerson value can be modified as follows.

1. An allocation rule \( f \) on \( DHG(N) \) is called **strong component efficient** if
   \[
   \sum_{i \in T} f_i(N, v, E) = v(T)
   \]
   for any \((N, v, E) \in DHG(N)\) and any \( T \in N/E \).

2. An allocation rule \( f \) on \( DHG(N) \) is called **fair** if
   \[
   f_i(N, v, E) - f_i(N, v, E \setminus e) = f_j(N, v, E) - f_j(N, v, E \setminus e)
   \]
   for any \((N, v, E) \in DHG(N)\) and any \( e \in E \) with \( i \in A_e \) and \( j \in B_e \).

The Myerson value can be characterized using these properties.

**Theorem 3.1.** For any \((N, v, E) \in DHG(N)\), the Myerson value represents a unique allocation rule that satisfies the conditions of strong component efficiency and fairness.

**Proof.** Consider two distinct allocation rules \( h(N, v, E) \) and \( g(N, v, E) \) that satisfy the strong component efficiency and fairness conditions. Next, let us consider a directed hypergraph \( H = (N, E) \) comprising a minimal number of hyperedges such that \( h(N, v, E) \neq g(N, v, E) \).

Therefore, for any \( e \in E \), it follows that
\[
h(N, v, E \setminus e) = g(N, v, E \setminus e).
\]
In accordance with the fairness condition, for any \( i \in A_e \) and \( j \in B_e \), it follows that
\[
h_i(N, v, E) - h_i(N, v, E \setminus e) = h_j(N, v, E) - h_j(N, v, E \setminus e),
\]
which implies
\[
h_i(N, v, E) - h_j(N, v, E) = h_j(N, v, E \setminus e) - h_j(N, v, E \setminus e),
\]
\[
= g_i(N, v, E \setminus e) - g_j(N, v, E \setminus e),
\]
\[
= g_i(N, v, E) - g_j(N, v, E).
\]
Considering any other hyperedge \( e' \in E \) with \( j \in A_{e'} \) and \( k \in B_{e'} \), we get
\[
h_j(N, v, E) - h_k(N, v, E) = g_j(N, v, E) - g_k(N, v, E).
\]
Therefore, there exists a constant \( C_T \) such that for any \( T \in N/E \) and any \( i \in T \) we have
\[
h_i(N, v, E) - g_i(N, v, E) = C_T.
\]
In accordance with the strong component efficiency condition,
\[
0 = \sum_{i \in T} (h_i(N, v, E) - g_i(N, v, E)) = |T|C_T.
\]
Therefore, \( C_T = 0 \). This is obviously a contradiction.

Next, we prove that the Myerson value satisfies the strong component efficiency and fairness conditions. Let \( T_1, \cdots, T_m \) represent a strong component in \( H = (N, E) \). Therefore,
\[
\bigcup_{i=1}^m T_i = N.
\]
For any induced subgraph \( H' = (S, E_S) \subseteq H \), because it holds that
\[
S/E_S = \{S \cap T_1, S \cap T_2, \cdots, S \cap T_m\},
\]
we have

\[ v^E(S) = \sum_{i=1}^{m} v^E(S \cap T_i) \]

In accordance with the additivity condition of the Shapley value,

\[ Sh(N, v^E) = \sum_{i=1}^{m} Sh(N, v^E(S \cap T_i)). \]

Clearly, for each \( S \subseteq N \setminus \{i\} \), if \( i \in T_j \),

\[ v^E((S \cup \{i\}) \cap T_j) - v^E(S \cap T_j) = 0. \]

Therefore, if \( i \in T_j \)

\[ Sh_i(N, v^E(S \cap T_j)) = 0 \]

For each fixed \( T_k \in N/E \), the efficiency condition of the Shapley value requires that

\[ \sum_{i \in T_k} Sh_i(N, v^E(S \cap T_k)) = v^E(T_k \cap T_k) = v^E(T_k) = v(T_k). \]

Therefore,

\[ \sum_{i \in T_k} \mu_i(N, v, E) = \sum_{i \in T_k} Sh_i(N, v^E) \]

\[ = \sum_{i \in T_k} \sum_{j=1}^{m} Sh_i(N, v^E(S \cap T_j)) \]

\[ = \sum_{i \in T_k} Sh_i(N, v^E(S \cap T_k)) = v(T_k), \]

This confirms that the Myerson value satisfies the strong component efficiency condition.

Finally, it can be demonstrated that the Myerson value satisfies the fairness condition.

For any \( e \in E \) with \( i \in A_e \) and \( j \in B_e \), the following allocation rule can be defined.

\[ w(S) := v^E(S) - v^{E\setminus e}(S). \]

Furthermore, given a coalition \( S \), if \( i \notin S \) or \( j \notin S \), it follows that \( S/E_S = S/(E_S \setminus e) \) and

\[ w(S) = v^E(S) - v^{E\setminus e}(S) = \sum_{T \in S/E_S} v(T) - \sum_{T \in S/(E_S \setminus e)} v(T) = 0. \]

Therefore, the coalitions that contain both players \( i \) and \( j \) can be exclusively considered. In accordance with the symmetry and additivity conditions of the Shapley value, we have

\[ \mu_i(N, v, E) - \mu_i(N, v, E \setminus e) = Sh_i(N, v^E) - Sh_i(N, v^{E\setminus e}) \]

\[ = Sh_i(N, v^E - v^{E\setminus e}) \]

\[ = Sh_i(N, w) = Sh_j(N, w) \]

\[ = \mu_j(N, v, E) - \mu_j(N, v, E \setminus e). \]

This completes the proof. \[ \square \]
4. Safety of directed hyperedges with respect to the Myerson value

Based on the stability of the allocation rules defined by Myerson, the stability of the Myerson value for a directed hypergraph game can be defined as explained hereunder.

**Definition 4.1.** For any \((N, v, E) \in DHG(N)\), a hyperedge \(e\) can be considered *stable* with respect to the Myerson value if and only if

\[
\mu_k(N, v, E) \geq \mu_k(N, v, E \setminus e)
\]

for any \(k \in e\). Therefore, if a given edge is stable, the stability of the Myerson value follows.

In this context, stability implies that if a hyperedge \(e\) is deleted, any vertex containing \(e\) decreases the profit. Myerson proved that if a TU game is a super additive, the corresponding Myerson value is stable. Li–Shan defined safety considering a stronger definition of stability.

**Definition 4.2.** For any \((N, v, E) \in DHG(N)\), a hyperedge \(e\) can be considered *safe* with respect to the Myerson value if and only if

\[
\mu_k(N, v, E) > \mu_k(N, v, E \setminus e)
\]

for any \(k \in N\).

Therefore, safety implies that if a hyperedge \(e\) is deleted, all vertices in \(N\) decrease the profit. That is, if a hyperedge is not safe, there exists a player that increases the benefit from breaking the edge. In this study, we modify Li–Shan’s result and demonstrate the condition of safety with respect to the Myerson value. The following theorem can be stated.

**Theorem 4.3.** For any \((N, v, E) \in DHG(M)\), if the TU game \((N, v)\) is convex, a hyperedge \(e\) of the hypergraph \((N, E)\) represents a bridge if and only if it is safe.

**Proof.** Let \(e\) denote a hypergraph bridge. To prove that \(e\) is safe, it is sufficient to demonstrate that for any \(i \in N\),

\[
\sum_{S \subseteq N \setminus \{i\}} \frac{s!(n-s-1)!}{n!} (v^E(S \cup \{i\}) - v^E(S) - v^{E\setminus e}(S \cup \{i\}) + v^{E\setminus e}(S)).
\]

Therefore, it must be demonstrated that for any \(i \in N\) and any \(S \subseteq N \setminus \{i\}\),

\[
v^E(S \cup \{i\}) - v^{E\setminus e}(S \cup \{i\}) \geq v^E(S) - v^{E\setminus e}(S) \geq 0. \tag{4.1}
\]

In accordance with the super additivity and convexity definitions, if \((N, v)\) is convex, it follows that it is super additive. Therefore, \(v^E(S) \geq v^{E\setminus e}(S)\), which implies \(v^E(S) - v^{E\setminus e}(S) \geq 0\) for any subset \(S \subseteq N\).

If there exists a player \(k \in e\) such that \(k \notin S\), it follows that \(e \notin S\). Moreover, it is clear that (4.1) holds because \(v^E(S) - v^{E\setminus e}(S) = 0\); that is, \(v^E(S \cup \{i\}) - v^{E\setminus e}(S \cup \{i\}) \geq 0\).

Therefore, in this study, we exclusively considered the case wherein all players of \(e\) are contained by a strong component of \(S\). Let \(N/E = \{T_0, \ldots, T_m\}\). Therefore, we have \(S/E_S = \{T_0 \cap S, \ldots, T_m \cap S\}\). Without loss of generality, it can be considered that

\[
e \subseteq T_0 \cap S,
\]

and \(T_0 \cap S\) splits the two strong components \(T_0'\) and \(T_0''\) in \(E_{T_0 \cap S \setminus e}\). Depending on the strong component player in which \(i\) is included, one of the following cases holds (Figure 1).

Case 1. \(i \notin T_0\). Both sides of (4.1) correspond to \(v(T_0 \cap S) - v(T_0')\) and \(v(T_0'')\), and this completes the proof.

Case 2. \(i \in T_0\). This can be further divided into two cases.

Case 2.1. There exists a player in \(T_0 \cap S\) connecting \(i\).

Since \(e\) represents a bridge of \(T_0 \cap S\), all players that connect \(i\) in \(T_0 \cap S\) are contained within
either $T'_0$ or $T''_0$. Therefore, without loss of generality, it can be considered that all players that connect $i$ in $T_0 \cap S$ are contained within $T'_0$. Therefore,

$$v^E(S \cup \{i\}) - v^{E\setminus e}(S \cup \{i\}) = v((T_0 \cap S) \cup \{i\}) - v(T'_0 \cup \{i\}) - v(T''_0),$$
$$v^E(S) - v^{E\setminus e}(S) = v(T_0 \cap S) - v(T'_0) - v(T''_0).$$

Based on the convexity condition of $v$ for $T'_0 \cup T''_0$ and $T'_0 \cup \{i\}$, we have

$$v((T_0 \cap S) \cup \{i\}) + v(T'_0) \geq v(T_0 \cap S) + v(T'_0 \cup \{i\}).$$

This completes the proof.

Case 2.2. There exists no player in $T_0 \cap S$ that connects $i$. In this case, both sides of (4.1) correspond to $v(T_0 \cap S) - v(T'_0) - v(T''_0)$. This completes the proof.

Conversely, assuming $e$ is safe, it follows that for any $k \in \mathbb{N}$,

$$\mu_k(N, v, E) \geq \mu_k(N, v, E \setminus e).$$

Let us assume that $e$ does not represent a bridge. Therefore, it follows that there exists a strong component $T \in N/E$ such that

$$T \in N/(E \setminus e), \ e \subset T.$$ 

Based on the component efficiency and fairness conditions of the Myerson value, for any $i \in A_e$ and any $j \in B_e$,

$$\sum_{k \in T \setminus \{i,j\}} (\mu_k(N, v, E) - \mu_k(N, v, E \setminus e)) = 2(\mu_i(N, v, E \setminus e) - \mu_i(N, v, E)).$$

Considering the safety condition of $e$, $0 \geq \mu_i(N, v, E \setminus e) - \mu_i(N, v, E)$, it implies that

$$\mu_i(N, v, E) \leq \mu_i(N, v, E \setminus e).$$

This represents a contradiction. \hfill \Box

Remark 4.4. Li–Shan demonstrated that if a game is strictly convex, the corresponding graph link can be considered safe if and only if it represents a bridge. Therefore, the statement of this theorem is more accurate, and its proof is more straightforward compared to reported in [5].

5. Example

In this study, we calculated the Myerson value for each player depicted in Figure 2. Considering $H = (N, E)$ denotes the hypergraph in Figure 2, it can be stated that

$$N = \{1, 2, 3, 4, 5\},$$
$$E = \{e_1 = \{1, 2\}, e_2 = \{1, 2\}, e_3 = \{2, 3, 4\}, e_4 = \{1, 3, 4, 5\}\}.$$
where

\[
A_{e_1} = \{1\}, \quad B_{e_1} = \{2\},
\]
\[
A_{e_2} = \{2\}, \quad B_{e_2} = \{1\},
\]
\[
A_{e_3} = \{2, 3\}, \quad B_{e_3} = \{4\},
\]
\[
A_{e_4} = \{3, 4, 5\}, \quad B_{e_4} = \{1\}.
\]

Considering the characteristic function \(v(S) = |S|\) for any \(S \subseteq N\), it follows that

\[
\mu(N, v, E) = \left(\frac{28}{5}, \frac{23}{5}, \frac{23}{5}, \frac{23}{5}\right).
\]

Because every hyperedge of \(H\), except \(e_2\), represents a bridge, we verified Theorem 4.3 by deleting \(e_4\) and calculated the Myerson value for \((N, v, E \setminus e_4)\).

\[
\mu(N, v, E \setminus e_4) = (2, 2, 1, 1).
\]

Therefore, for any \(k \in N\), it follows that

\[
\mu_k(N, v, E) \geq \mu_k(N, v, E \setminus e_4).
\]

Therefore, \(e_4\) is safe. Similarly, the other edge can be proved to be safe. Conversely, if we delete \(e_2\) and calculate the Myerson value for \((N, v, E \setminus e_2)\), we get

\[
\mu(N, v, E \setminus e_2) = (5, 5, 5, 5).
\]

Therefore, for any \(k \in \{3, 4, 5\}\), we have \(\mu_k(N, v, E) < \mu_k(N, v, E \setminus e_2)\). This proves that \(e_2\) is not safe.

\[\text{Figure 2. Directed hypergraph}\]

6. Conclusions

This paper defines the Myerson value for a directed hypergraph game. Moreover, it demonstrates the axiomatizations of the Myerson value in terms of the properties of strong component efficiency and fairness. Accordingly, the Myerson value can be defined for all discrete objects. Li–Shan had previously demonstrated that an edge is safe if and only if it represents a bridge with a strictly convex graph game. Moreover, this safety condition requires that the above-mentioned strictly convex game condition is accompanied by the strongest superadditivity, convexity, and strict superadditivity. Therefore, a hyperedge can be considered safe if and only if it represents a bridge with a directed convex hypergraph game. Other interesting topics for future research in the same vein include an alternate allocation rule for a directed hypergraph game as well as another relation between the Myerson value and some topological notations, such as homology and genus.
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