Towards a Definitive Measure of Repetitiveness

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Abstract

Unlike in statistical compression, where Shannon’s entropy is a definitive lower bound, no such a clear measure exists for the compressibility of repetitive sequences other than the uncomputable Kolmogorov’s complexity. Since statistical entropy does not capture repetitiveness, ad-hoc measures like the size \(z\) of the Lempel-Ziv parse are frequently used to estimate it. Recently, a more principled measure, the size \(\gamma\) of the smallest attractor of a string \(S[1..n]\), was introduced. Measure \(\gamma\) lower bounds all the previous relevant ones (e.g., \(z\)), yet \(S\) can be represented and indexed within space \(O(\gamma \log(n/\gamma))\), which also upper bounds most measures. While \(\gamma\) is certainly a better measure of repetitiveness, it is NP-complete to compute, and it is not known if \(S\) can always be represented in \(O(\gamma)\) space.

In this paper we study a smaller measure, \(\delta \leq \gamma\), which can be computed in linear time. We show that \(\delta\) captures better the concept of compressibility in repetitive strings: We prove that, for some string families, it holds \(\gamma = \Omega(\delta \log n)\). Still, we can build a representation of \(S\) of size \(O(\delta \log(n/\delta))\), which supports direct access to any \(S[i]\) in time \(O(\log(n/\delta))\) and finds the occurrences of any pattern \(P[1..m]\) in time \(O(m \log n + \text{occ} \log^2 n)\) for any constant \(\epsilon > 0\). Further, such representation is worst-case optimal because, in some families, \(S\) can only be represented in \(\Omega(\delta \log n)\) space. We complete our characterization of \(\delta\) by showing that \(\gamma, z\) and other measures of repetitiveness are always \(O(\delta \log(n/\delta))\), but in some string families, the smallest context-free grammar is of size \(g = \Omega(\delta \log^2 n / \log \log n)\). No such a lower bound is known to hold for \(\gamma\).

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1 Introduction

The recent sharp rise in the amount of data we aim to handle \cite{45} is driving research into compressed data representations that can be used directly in compressed form \cite{35}. Interestingly, much of today’s fastest-growing data is highly repetitive, which enables space reductions of orders of magnitude \cite{21}: genome collections, versioned text and software
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Repositories, periodic sky surveys, and other sources produce data where each element in the collection is very similar to others.

Since an important fraction of the data of interest consists of sequences, text indexes are important actors in this research. These are data structures that offer fast pattern matching (and possibly other more sophisticated capabilities) over a collection of strings. Though compressed text indexes are already mature [36], offering fast pattern searching within space close to the statistical entropy of the string collection, such kind of entropy is unable to capture repetitiveness [32, 35]. Achieving orders-of-magnitude space reductions requires instead to resort to other kinds of compressors, such as Lempel-Ziv [33], grammar compression [30], run-length compressed Burrows-Wheeler Transform [21], and others. Various compressed indexes have been built on those methods; Gagie et al. [21] give a thorough review.

Unlike statistical compression, where Shannon’s notion of entropy [44] gives a clear lower bound to what compressors can achieve, a similar notion capturing repetitiveness has been elusive. Beyond Kolmogorov’s complexity [31], which is uncomputable, repetitiveness is measured in ad-hoc terms, as the results of what specific compressors can achieve. A list of various such measures on a given string \( S[1..n] \) follows:

**Lempel-Ziv compression** [33] parses \( S \) into a sequence of “phrases”, each of which has appeared previously in \( S \). The associated measure is the number \( z \) of phrases produced. The measure can be computed in \( O(n) \) time [42].

**Bidirectional macro schemes** [46] extend Lempel-Ziv in the sense that the source of each phrase may precede or follow it, as long as no circular dependences are introduced. The associated measure is the size \( b \) of the smallest parsing. It holds \( b \leq z = O(b \log(n/b)) \) [20], but computing \( b \) is NP-complete [22].

**Grammar-based compression** [30] builds a context-free grammar that generates \( S \) and only \( S \). The associated measure is the size \( g \) of the smallest grammar (i.e., sum of the lengths of the right-hands of the rules). It holds \( z \leq g = O(z \log(n/z)) \) and, while it is NP-complete to compute \( g \), \( O(\log(n/z)) \)-approximations to \( g \) can be computed in linear time [43, 12, 23].

**Run-length grammar compression** [38] is similar but it allows rules of the form \( A \rightarrow B^t \) (\( t \) repetitions of \( B \)) of constant size. The associated measure is the size \( g_{rl} \) of the smallest run-length grammar, and it holds \( z/2 \leq g_{rl} \leq g \) and \( g_{rl} = O(b \log(n/b)) \) [20].

**Collage systems** [28] extend run-length grammars by allowing truncation: in constant space we can refer to a prefix or a suffix of another nonterminal. The associated measure, \( c \), is also NP-complete to compute. It holds \( c \leq g_{rl} \) and \( c = O(z \log z) \) [25].

**The Burrows-Wheeler-Transform (BWT)** [10] is a permutation of \( S \) that makes it more compressible. The associated measure is \( r \), the number of maximal equal-letter runs in the BWT, and it is computed in linear time via the suffix array [24]. It is incomparable with \( z \) [39] and it holds \( r \geq b/2 \) [20] and \( g_{rl} = O(r \log(n/r)) \) [21].

**CDAWGs** [9] are automata that recognize every substring of \( S \). The associated measure of repetitiveness is \( c \), the size of the smallest such automata, which is built in linear time [9] and is always larger than \( r \), \( g \), and \( z \) [3, 2].

An important improvement to this situation is the recent introduction of the concept of **string attractor** [27], which yields a more principled measure based on combinatorial properties of the string. An attractor \( \Gamma \) is a set of positions in \( S \) such that any substring of \( S \) must have a copy covering a position in \( \Gamma \). The size \( \gamma \) of the smallest attractor is shown to asymptotically lower bound all the listed repetitiveness measures \( (z, b, g, g_{rl}, c, r, e) \). Various results since then [27, 37, 40, 14] showed that efficient access and searches...
can be supported within \( O(\gamma \log(n/\gamma)) \) space, and that \( g_{r1} = O(\gamma \log(n/\gamma)) \). Previous results support random access to \( S \), or indexed searches on \( S \), within space \( O(z \log(n/z)) \) \[4, 6, 13, 10, O(g) \] \[15, 16, 13, 5, O(g_{dr}) \] \[21, O(r) \] or \( O(r \log(n/r)) \) \[34, 3, 21 \], and \( O(c) \) \[12, 2 \], no one improving the space \( O(\gamma \log(n/\gamma)) \) within which one can offer efficient access \[27 \] and indexing \[37, 14 \].

Using indexes based on \( \gamma \) is not exempt of problems, however. Computing it is NP-hard \[27 \] and therefore one has to resort to approximations like \( z \), in which case the representation is only guaranteed to be of size \( O(z \log(n/z)) \). While this problem has been recently sidestepped \[14 \], it is still unclear whether \( \gamma \) is the definitive measure of repetitiveness. In particular, it is unknown whether one can represent \( S \) within \( O(\gamma) \) space (while this is possible in \( O(b) \) space), nor whether the space \( O(\gamma \log(n/\gamma)) \) is the best one we can aim for.

**Our contributions.** In this paper we study a new measure of repetitiveness, \( \delta \), which arguably captures better the concept of compressibility in repetitive strings and is more convenient to deal with. Although this measure was already introduced in a stringology context \[11 \] and used to build indexes of size \( O(\gamma \log(n/\gamma)) \) without knowing \( \gamma \) \[14 \], its properties and full potential had not been explored. It always holds that \( \delta \leq \gamma \), and \( \delta \) can be computed in \( O(n) \) time \[14 \]. We show that, for some string families, it holds \( \gamma = \Omega(\delta \log n) \), that is, \( \delta \) can be asymptotically strictly smaller than \( \gamma \). Still, we show how to build a representation of \( S \) of size \( O(\delta \log(n/\delta)) \), which supports direct access to any \( S[i] \) in time \( O(\log(n/\delta)) \) and finds the \( occ \) occurrences of any pattern \( P[1..n] \) in time \( O(m \log n + occ \log^6 n) \) for any constant \( \epsilon > 0 \). We also show how to reduce block trees \[4 \] to size \( O(\delta \log(n/\delta)) \) while supporting various relevant operations on \( S \). Therefore, we obtain less space and the same time performance of previous results based on \( \gamma \) \[27, 37, 40 \], though the most recent index \[14 \] is faster. This also shows that string representations of size \( O(\gamma \log(n/\gamma)) \) are not space-optimal. Further, we show that our representations using \( O(\delta \log(n/\delta)) \) space are worst-case optimal because, in some string families, \( S \) can only be represented in \( \Omega(\delta \log n) \) space; such a result is unknown on attractors. We complete our characterization of \( \delta \) by proving that \( \gamma, b, z, \) and \( c \) are always \( O(\delta \log(n/\delta)) \), but in some string families, the smallest context-free grammar is of size \( g = \Omega(\delta \log^2 n/\log \log n) \). No such lower bound is known to hold on \( \gamma \).

## 2 Basic Concepts

### Strings and texts

A *string* is a sequence \( S[1..\ell] = S[1]S[2] \cdots S[\ell] \) of *symbols*. The symbols belong to an *alphabet* \( \Sigma \), which is a finite subset of the integers. A *substring* \( S[i] \cdots S[j] \) of \( S \) is denoted \( S[i..j] \). A *suffix* of \( S \) is a substring of the form \( S[i..\ell] \), and a *prefix* is a substring of the form \( S[1..i] \). The juxtaposition of strings and/or symbols represents their concatenation, and the exponentiation denotes the iterated concatenation. The *length* of \( S \) is written as \( |S| = \ell \). The *reverse* of \( S[1..\ell] \) is \( S^{rev} = S[\ell]S[\ell-1] \cdots S[1] \).

We will index a string \( T[1..n] \), called the *text*. We assume our text to be terminated by the special symbol \( T[n] = $ \), the smallest in the alphabet, which appears nowhere else in \( T \).

### Karp-Rabin signatures

*Karp-Rabin fingerprinting* \[25 \] assigns a string \( S[1..\ell] \) the signature \( \kappa(S) = (\sum_{i=1}^{\ell} S[i] \cdot c^{i-1}) \mod \mu \) for suitable integers \( c > 1 \) and prime \( \mu \). It is possible to build a signature formed
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by a pair of functions \( \langle \kappa_1, \kappa_2 \rangle \) guaranteeing no collisions between substrings of \( S[1..n] \), in \( O(n \log n) \) expected time \(^7\).

Model of computation

We use the RAM model with word size \( w = \Omega(\log n) \), allowing classic arithmetic and bit operations on words in constant time. Our logarithms are to the base 2 by default.

Attractors

An attractor \(^{27}\) \( \Gamma = \{ p_1, \ldots, p_\gamma \} \) for \( S[1..n] \) is a set of positions \( p_i \in [1..n] \) such that any substring \( S[i..j] \) has at least one copy \( S[i'..j'] \) that contains some attractor position \( p_i \) \(^{27}\). We make this copy explicit with the function \( f[i..j] = [i'..j'] \), arbitrarily choosing some copy.

3 Measure \( \delta \)

Measure \( \delta \) is defined by Christiansen et al. \(^{14}\) Sec. 5.1, but it is also related to the expression \( d_k(w)/k \), used by Raskhodnikova et al. \(^{41}\) to approximate \( z \). We summarize what is known about it.

\(^\uparrow\) Definition 1. Let \( S(k) \) be the total number of distinct substrings of length \( k \) in \( S \). Then

\[ \delta = \max \{ S(k)/k, \ k \geq 1 \}. \]

\(^\uparrow\) Lemma 2. It always holds \( \delta \leq \gamma \).

Proof. Since every length-\( k \) substring of \( S \) must have a copy containing an attractor position, there can be at most \( k\gamma \) distinct such substrings, that is, \( S(k)/k \leq \gamma \) for all \( k \) \(^{14}\) Lem. 5.6. \( \blacklozenge \)

\(^\uparrow\) Lemma 3. It always holds \( z = O(\delta \log(n/\delta)) \)

Proof. This is proved by Raskhodnikova et al. \(^{41}\) Lem. 5]: if we set \( \ell_0 = n/\delta \), we get that \( z \leq 4(\delta \log(n/\delta) + \delta) \). \( \blacklozenge \)

\(^\uparrow\) Lemma 4. Measure \( \delta \) can be computed in \( O(n) \) time and space from \( S[1..n] \).

Proof. This is done by Christiansen et al. \(^{14}\) Lem. 5.7], by using the suffix tree of \( S \). \( \blacklozenge \)

4 Lower Bounds in Terms of \( \delta \)

In this section we prove lower bounds in terms of the measure \( \delta \). First, we show that there exist string families where \( \delta = o(\gamma) \); further, \( \delta \) can be smaller by up to a logarithmic factor. Second, we prove that there are text families that cannot be encoded in \( O(\delta) \) space; indeed our \( O(\delta \log(n/\delta)) \) representation in the next section is worst-case optimal. Third, we prove that there are text families that cannot be represented with a context-free grammar of size \( O(\delta \log n) \); almost a logarithmic-factor separation exists.
4.1 Lower bounds on attractors

Consider the family of strings $S_n[i..n]$, where $S[i] = b$ if $i$ is of the form $2^j$ for some integer $j \geq 0$, and $S[i] = a$ otherwise. The family are then the nonempty prefixes of the infinite string $S_\infty = bbabaabaaaaab...$. We first show that this family has measure $\delta = O(1)$.

Lemma 5. It holds $\delta \leq 4$ for the family of strings $\{S_n, n \geq 1\}$.

Proof. For every $j$, it holds that every pair of consecutive $bs$ in $S_n[2^j + 1..n]$ is at distance more than $2^j$. Therefore, the only distinct substrings of length $2^j$ in $S_n[2^j + 1..n]$ are of the form $a^i b a^{2^j-i-1}$, for $i = 0, \ldots, 2^j - 1$. It then follows that all the distinct substrings of length $2^j$ in $S_n$ can be those starting up to position $2^j$, $S_n[i..i+2^j-1]$ for $i$ in $1..2^j$, or the other strings $a^i b a^{2^j-i-1}$ already mentioned, for a total of $S_n(2^j) \leq 2^{j+1}$. In general, since $S_n(k)$ is monotonic, we have $S_n(k) \leq S_n(2^{\lceil \log k \rceil}) \leq 2^{\lceil \log k \rceil+1} < 4k$. By definition of $\delta$, we then have $\delta(S_n) \leq 4$ for every $n$.

This is sufficient to show that there are string families where $\delta = o(\gamma)$, as shown next.

Theorem 6. There exists a string family for which $\gamma = \Omega(\delta \log n)$.

Proof. Consider the same family $S_n$ we have defined. For every $j$, there is a unique occurrence of the substring $ba2^{j-1}b$, and therefore any attractor must have an element inside that occurrence. In $S_n$, those substrings overlap by one symbol (a $b$), and therefore, at best, a single attractor element could belong to two such unique occurrences, by choosing every other occurrence of $b$. Since $S_n$ contains $1 + \lceil \log n \rceil$ occurrences of $b$, any attractor must have at least $\lceil (1 + \lceil \log n \rceil)/2 \rceil > (\log n)/2$ elements. (We also need one attractor element within the maximum run of $a$s, but it can be obtained for free in some cases, so we do not count it.) In total, $\gamma > (\log n)/2$ for $S_n$, whereas $\delta \leq 4$, thus $\gamma > (\delta/8) \log n$.

Note that this does not only happen if $\delta = O(1)$. We could create a family of strings $S_n^a,b$ for distinct pairs of characters $a$ and $b$, and then define strings $S_n = S_n^a b_1 \ldots S_n^m b_m$ over alphabets of size $2m$. Those have $\delta = O(m)$ and $\gamma = \Omega(m \log n)$.

4.2 Lower bounds on text entropy

We now show that there are text families that cannot be encoded in $O(\delta \log n)$ space, that is $O(\delta \log n)$ bits. It is not known if the same occurs with $\gamma$.

Consider a variant of our family where the position of every $b$ is perturbed without leaving its area. The family $S_n^a$ are the prefixes of length $n$ of infinite strings of the family $S^*$, which are all as except for $bs$ placed as follows: the first and second $bs$ are placed at positions $S^*[1]$ and $S^*[2]$ and then, for $j \geq 3$, the $j$th $b$ is placed anywhere in $S^*[3 \cdot 2^{j-3} + 1 \ldots 2^{j-1}]$.

Lemma 7. The family of strings $S_n^a$, for any $n \geq 1$, needs $\Omega(\log^2 n)$ bits to be encoded.

Proof. In our definition of $S^*$, every $j$th $b$ can choose among $2^{j-3}$ positions, and each combination of choices generates a different string in any $S_n^a$. It follows that, for $n$ of the form $2^i$, $|S_n^a| = 2^{2i/3}$. Any encoding that distinguishes the strings in $S_n^a$ then needs $\log |S_n^a| = i^2/2 - 5i/2 + 3$ bits, which is $\Omega(i^2) = \Omega(\log^2 n)$.

Theorem 8. There exist text families that need $\Omega(\delta \log n)$ space to be encoded.
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Proof. We already have that $\Omega(\log^2 n)$ bits are needed to encode $S^*_n$. On the other hand, the measure $\delta$ for any string in $S^*_n$ is still constant. Starting from position $2^{j+1}$, the distance between two consecutive $b$s is at least $2^j + 2$. Therefore, the distinct substrings of length $2^j$ are either those that start before position $2^{j+1}$ or those of the form $a^i b a 2^{j-1} b$. In total, there are at most $2^{j+1} - 1 + 2^j < 2^j 2$ distinct substrings of length $2^j$. By the monotonicity of this measure, the number of distinct substrings of length $k$ is at most that for length $2[\log k]$, which is less than $3 \cdot 2^{[\log k]} < 6k$. Therefore, $\delta < 6$ for every member of $S^*_n$, and thus we need $\Omega(\delta \log n)$ bits to encode the family $S^*_n$. ▶

4.3 Lower bounds on smallest grammars

A consequence of our observations on the family $S^*_n$ is that it cannot be represented with a grammar of size $g = O(\delta \log n)$, where such a result is not known for $\gamma$.

- Theorem 9. There are string families where the smallest grammar representing them is of size $\Omega(\delta \log^2 n/\log \log n)$.

Proof. Consider the same family $S^*_n$ as before, which needs $\Omega(\log^2 n)$ bits to be represented. If we could encode it with a grammar of size $g$, each grammar element would be a nonterminal that could be encoded with $O(\log g)$ bits. Therefore, our grammar representation would require $O(g \log g)$ bits. Since this must be $\Omega(\log^2 n)$, it follows that $g = \Omega(\log^2 n/\log \log n)$ for any grammar of size $g$ encoding $S^*_n$. Since $\delta = O(1)$ for this family, it follows that $g = \Omega(\delta \log^2 n/\log \log n)$.

5 Block Trees in $\delta$-Bounded Space

The block tree $[4]$ is a data structure designed to represent repetitive strings $S[1..n]$ while offering efficient access and other operations on $S$. It is shown to require $O(z \log (n/z))$ space. In this section we show that the block tree is easily tuned to use $O(\delta \log (n/\delta))$ space while retaining its functionality.

This means, in particular, that we can represent $S[1..n]$ in $O(\delta \log (n/\delta))$ space. Together with the lower bounds of Section $[4]$ this also implies that $\Theta(\delta \log (n/\delta))$ is a tight worst-case asymptotic measure of the text entropy.

5.1 Block trees

Given integer parameters $r$ and $s$, the first level of the block tree divides $S$ into $s$ equal-sized blocks (assume for simplicity that $n = s \cdot r^t$ for some integer $t$) $[4]$. Blocks are then classified into marked and unmarked. A block $B$ is unmarked if its leftmost occurrence $L$ in $S$ does not overlap $B$. Unmarked blocks are replaced by a pointer to the pair of blocks $B_1 : B_2$, that contain $L$, and the offset $\epsilon \geq 0$ where $L$ starts inside $B_1$. Marked blocks are divided into $r$ equal-sized sub-blocks and processed similarly in the next level. The level where the blocks become of length below $\log_s n$ is the last one, and its blocks store their plain string content using $O(\log n)$ bits. The height of the block tree is then $h = O \left( \log_{r \log^2 n} \left( \frac{n}{s \log \log n} \right) \right) = O \left( \log_{r \log \log^2 n} \left( \frac{n \log \sigma}{\log n} \right) \right) \subseteq O(\log (n/s))$.

1 If not, we simply pad $S$ with spurious symbols at the end; whole spurious blocks are not represented. The extra space incurred is only $O(rh)$ for a block tree of height $h$. The actual construction $[4]$ uses instead blocks of sizes $[n/s]$ and $[n/s]$. 
We now prove that there are only few marked blocks of length \( r \) in the block tree. For a fixed \( r \), the space is in \( O(\delta \log \log(n/\delta)) \) as in the previous section, and the height is in \( O(\log(n/\delta)) \).

Let us call level \( k \) of the block tree the one where blocks are of length \( r^k \). In level \( k \), then, \( S \) is covered regularly with blocks \( B = S[r^k(i - 1) + 1..r^k] \) of length \( r^k \) (though not all of them are present in the block tree). For a fixed \( k \), consider the leftmost occurrences \( L = S[\ell..\ell + r^k - 1] \) of the \( S(k) \) distinct substrings of length \( r^k \) of \( S \). From the blocks of level \( k \), those that exist and are marked are the ones intersecting the leftmost occurrence \( L \) of some unmarked block. The key result is that, even if we consider the leftmost occurrences \( L \) of every substring in \( S(k) \), the number of marked blocks would still be \( O(\delta) \) per level.

**Lemma 10.** The total number of marked blocks of length \( r^k \) in the block tree is \( O(\delta) \).

**Proof.** Consider all the \( r^k \) text positions \( p \) belonging to a marked block \( B \). Then the long substring \( E = S[p - 2 \cdot r^k .. p + 2 \cdot r^k - 1] \) centered at \( p \) contains the leftmost occurrence \( L \) intersected by the block \( B \). All those long substrings \( E \) must be distinct, because they contain a leftmost occurrence of a distinct substring \( L \) (at different offsets within \( E \)): if two long substrings are equal, then one of them does not contain the leftmost occurrence of any distinct substring \( L \). Note also that no two blocks produce the same positions for long substrings \( E \).

Therefore, there are at most \( S(4r^k) \) long substrings \( E \). Since each position \( p \) inside a block \( B \) induces a distinct long substring \( E \), and each marked block \( B \) contributes \( r^k \) distinct positions \( p \) because it is disjoint from the other blocks, it follows that there are at most \( S(4r^k)/r^k \) marked blocks of length \( r^k \). Since \( S(4r^k)/r^k = 4 \cdot S(4r^k)/(4r^k) \leq 4\delta \), the total number of marked blocks of length \( r^k \) is at most \( 4\delta \).

Since the block tree has at most \( 4\delta \) marked blocks per level, it has at most \( 4\delta r \) blocks in every level except the first. This yields the following result.
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**Theorem 11.** Let \( S[1..n] \), over alphabet \([1..\sigma]\), have compressibility measure \( \delta \). Then the block tree of \( S \), with parameters \( r \) and \( s \), is of size \( O \left( s + \delta r \log \frac{n \log \sigma}{s \log n} \right) \) and height \( h = O \left( \log \frac{n \log \sigma}{s \log n} \right) \).

Actually, the original block tree construction \([4]\), without optimizations, can also be bounded in the same way.

**Lemma 12.** The result of Lemma \([3]\) also holds for the original block tree construction.

**Proof.** In the original construction, every pair of blocks \( B_1 : B_2 \) of any level \( k \) such that the leftmost occurrence \( L \) of \( B_1 : B_2 \) overlaps \( B_1 : B_2 \), is marked regardless of whether the blocks will be pointed or not by unmarked blocks. Still, just as in the proof of Lemma \([2]\) we can see that if a marked block \( B \) of length \( r^k \) is inside a sequence of blocks \( B^{-} \cdot B \cdot B^{+} \), then either \( B^{-} \cdot B \) or \( B \cdot B^{+} \) overlap the leftmost occurrence \( L \) of some string of length \( 2r^k \), and thus the extended strings \( E = S[p - 3 \cdot r^k .. p + 3 \cdot r^k - 1] \), for all \( p \) inside \( B \), are all unique because they contain a leftmost occurrence of a distinct string at different positions. The rest proceeds analogously as in Lemma \([2]\) obtaining a bound of \( 6\delta \) per level \( k \).

As a final note, we remark that, if we know \( \gamma \), we can build the block tree with parameters \( s = \gamma \) and \( r = O(1) \), thereby obtaining height \( h = O(\log(n/\gamma)) \) and size \( O(\gamma + \delta \log(n/\delta)) \subseteq O(\delta \log(n/\delta)) \) (in the next section we prove \( \gamma = O(\delta \log(n/\delta)) \)). It is then possible to access any string position, as well as support rank and select, within the same time that previous work obtain using space \( O(\gamma \log(n/\gamma)) \) \([37, 40]\).

## 6 Upper Bounds in Terms of \( \delta \)

In this section we show that \( \delta \log(n/\delta) \) upper bounds several other measures of compressibility.

### 6.1 \( z = O(\delta \log(n/\delta)) \)

Consider the \( O(\delta \log(n/\delta)) \) leaves of the block tree. Each such leaf \( B \) is either a single letter at the last level, for which we can create a Lempel-Ziv phrase formed by one explicit letter, or has a pointer \( \langle B_1 : B_2, \epsilon \rangle \) to blocks containing a leftward occurrence of \( B \). We can then replace \( B \) by a Lempel-Ziv phrase pointing to the text area of \( \langle B_1 : B_2 \rangle[1 + \epsilon .. |B| + \epsilon] \). Therefore, we cover \( T \) with a parsing of size \( O(\delta \log(n/\delta)) \). Since Lempel-Ziv is the optimal parse among those where the phrases point leftward [ref], it holds \( z = O(\delta \log(n/\delta)) \). This also implies then that \( b = O(\delta \log(n/\delta)) \).

### 6.2 \( c = O(\delta \log(n/\delta)) \)

In a collage system \([29]\) we have the normal rules of run-length context-free grammars plus prefix and suffix productions, \( A \rightarrow \{B\}B \) or \( A \rightarrow B[^\epsilon] \), meaning that the expansion of \( A \) is the first or last \( t \) symbols of the expansion of \( B \), respectively. Given the block tree, we can create a collage system with one nonterminal per node. If the block \( B \) is marked and has two children \( \langle B_1, B_2 \rangle \), then the corresponding rule is \( B \rightarrow B_1B_2 \). If, instead, \( B \) is unmarked, with leftward pointer \( \langle B_1, B_2, \epsilon \rangle \), we create the rule \( B \rightarrow B_1[^\epsilon][B_2] \). For the blocks \( B \) of the last level corresponding to a symbol \( a \), we create the rule \( B \rightarrow a \). Since \( c \) is the size of the smallest collage system, we have \( c = O(\delta \log(n/\delta)) \).
\section{\textbf{6.3 $\gamma = O(\delta \log(n/\delta))$}}

This is already implied because $\gamma = O(\min(z, c))$. Here we work on a more general form, relating (a generalized version of) $\delta$ to the notion of $k$-attractor \cite{27}:

\> \textbf{Definition 13.} A $k$-attractor of a string $T \in \Sigma^n$ is a set of positions $\Gamma_k \subseteq \{1 \ldots n\}$ such that every substring $T[i \ldots j]$ with $i \leq j < i + k$ has at least one occurrence $T[i' \ldots j'] = T[i \ldots j]$ with $j'' \in [i' \ldots j']$ for some $j'' \in \Gamma_k$. We denote by $\gamma_k$ the size of the smallest $k$-attractor.

By the above definition, a string attractor is an $n$-attractor, and $\gamma = \gamma_n$. Such $k$-attractors were studied more in detail by Kempa et al. \cite{26}, who provided optimization and approximation algorithms to compute them. While for the case $k = n$ a logarithmic approximation is easy to achieve (e.g., $z$ is one), for general $k$ the only $O(\log k)$-approximation algorithm is based on a reduction to set-cover and runs in cubic time \cite{26}. We now provide a new approximation based on the following notion:

\> \textbf{Definition 14.} Let $T(j)$ be the number of distinct substrings of length $j$ in $T$. We define the measure $\delta_k$ as

\[ \delta_k = \max\{T(j)/j, \ 1 \leq j \leq k\}. \]

Note that $\delta = \delta_n$, and that $k' \leq k''$ implies $\delta_{k'} \leq \delta_{k''}$ (even if $k', k'' > n$). We have the following relation, which extends Lemma 2.

\> \textbf{Lemma 15.} For any $k$, it holds that $\delta_k \leq \gamma_k$.

\textbf{Proof.} Clearly $k' \leq k''$ implies $\gamma_{k'} \leq \gamma_{k''}$, since a $k''$-attractor is also a $k'$ attractor, by definition. Let now $j \leq k$. The number $T(j)$ of distinct substrings of length $j$ in $T$ satisfies $T(j) \leq \gamma_j \cdot j \leq \gamma_k \cdot j$. This yields $\gamma_k \geq T(j)/j$ for all $j = 1, \ldots, k$, that is, $\gamma_k \geq \max\{T(j)/j, \ 1 \leq j \leq k\} = \delta_k$.

\> \textbf{Theorem 16.} Let $\gamma_k$ be the size of a smallest $k$-attractor. Then,

\[ \gamma_k = O(p_k) \]

where $p_k = O(\delta_{4k} \cdot \min(\log k, \log(n/\delta_{4k})))$. Moreover, we can compute a $k$-attractor of size $O(p_k)$ in $O(n \cdot \min(\log k, \log(n/\delta_{4k})))$ time and linear space.

\textbf{Proof.} We first define a $k$-attractor $\Gamma_k$ of size $O(\delta_{4k} \cdot \min(\log k, \log(n/\delta_{4k})))$. Note that $\delta_{4k}$ can be computed in linear time and space using a generalized version of Lemma 4 (in the lemma we already compute all $T(j)$, for $j = 1, \ldots, n$; to compute $\delta_{4k}$, we just use $T(1), \ldots, T(4k)$).

We add to $\Gamma_k$ the $\delta_{4k} + 2$ positions $E = \{1, 1 \cdot [n/\delta_{4k}], 2 \cdot [n/\delta_{4k}], \ldots, \delta_{4k} \cdot [n/\delta_{4k}], n\}$. These positions capture all substrings of length more than $[n/\delta_{4k}]$. Next, we show how to capture substrings shorter than $k' = \min(k, n/\delta_{4k})$.

Consider the modified $\Gamma$-tree described above, where we stop at level $M = \lfloor \log_2 k' \rfloor$, that is, we consider chosen blocks of length at most $2^M$. For each chosen block $B$ of length $2^t$ (at levels $t = 1, \ldots, M$), we add to $\Gamma_k$ the positions corresponding to $B[1], B[2^t],$ and $B[2^t - 1]$.

The same argument used above shows that $|\Gamma_k| = O(\delta_{4k} \log k')$. Consider a chosen block $B$ of length $2^t$. Each position $i$ of $B$ can be associated with a string $B_i$ of length $2^{t+2}$ centered in $i$. As seen before, at level $t$ all strings $B_i$ (for all blocks $B$) must be distinct, therefore the total number of chosen blocks is at most $T(2^{t+2})/2^t \leq \delta_{4t} \leq \delta_{4k}$. Since we insert three attractor positions per chosen block, we obtain $|\Gamma_k| = O(\delta_{4k} \log k') = O(\delta_{4k} \min(\log k, \log(n/\delta_{4k})))$.\[\]
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We now show that $\Gamma_k$ is, indeed, a string attractor. Consider any substring $T[i..j]$ shorter than $k'$. Right-extend the substring to the next length being a power of two: $T[i..i + 2^e - 1]$, where $e$ is the smallest integer such that $2^e \geq j - i + 1$. If $i + 2^e - 1 > n$, then the string can be left-extended similarly, and the following argument still holds. Consider the leftmost occurrence $S$ of $T[i..i + 2^e - 1]$. By the way we defined chosen blocks, $S$ is completely covered by adjacent and non-overlapping chosen blocks of length $2^e$. Moreover, since one out of $2^e - 1$ positions of those chosen blocks is an attractor position, it follows that the occurrence of $T[i..j]$, of length at least $2^e - 1$, lying inside $S$ touches at least one of those attractor positions.

Let us now build $\Gamma_k$ efficiently. We first initialize $\Gamma_k$ with the positions in $E$, in linear time. To compute the other positions, we build the suffix tree of $T$ in $O(n)$ time and space [17]. We then traverse the tree, computing for each node the leftmost occurrence in its subtree, in postorder, also in $O(n)$ time. Finally, we traverse the top part of the tree, stopping at (and including) the first nodes whose string depths exceed $k'$. For each such node $v$ with parent $v'$, with string depths $\ell$ and $\ell'$, respectively, we consider all the values $j$ such that $\ell' < 2^j \leq \ell$. For each such value $j$ and leftmost occurrence $T[i]$ associated with $v$, we consider the window $T[i..i + 2^j - 1]$. We compute the starting positions of the (at most) two chosen blocks of length $2^j$ intersecting the window and add their first, middle, and last positions to $\Gamma_k$ (as seen above). To avoid inserting duplicates in $\Gamma_k$ we first mark all the positions to insert in a bitvector $B[1..n]$ and later collect them. The total time is $O(n \log k')$, dominated by the $O(\log k')$ values of $j$ to consider for each suffix tree node in the top part of the tree. ▶

In particular, since $\delta_k \leq \gamma$ for any $k$, the above theorem implies that we can compute a $k$-attractor of size $O(\gamma \log k)$ in $O(n \log k)$ time. Since $\delta_n = \delta_n = \delta$ and $\gamma_n = \gamma$, we also obtain that in $O(n \log(n/\delta))$ worst-case time we can build a string attractor of size $O(\delta \log(n/\delta))$. In particular, we obtain the relation $\gamma = O(\delta \log(n/\delta))$.

7 Text Indexing in $\delta$-Bounded Space

We now show that not only efficient access of $S$ can be supported within $O(\delta \log(n/\delta))$ space, but also text indexing, that is, efficiently listing all the positions in $S$ where a pattern $P[1..m]$ appears. In this section we speak of a text $T[1..n]$ instead of a string $S[1..n]$.

Our index builds on top of a slight variant of the block tree of the previous sections, with $r = 2$, $s = \delta$, and stopping only when the leaves are of length 1. This block tree is of size $O(\delta \log(n/\delta))$ and of height $O(\log(n/\delta))$.

To build the index, we follow the same ideas of the “universal index” [37], whose space will be improved without affecting its search time complexities. That index builds on a variant of block trees designed for attractors: the $\Gamma$-tree has a first level with $\gamma$ equal-sized blocks, and at any other level $k$, it marks the blocks that are at distance $< 2^k$ from an attractor position. Unmarked blocks $B$ then point to some copy of $B$ that crosses an attractor position (the blocks overlapping that copy are marked by definition). In the $\Gamma$-tree pointers can go leftward or rightward, not necessarily to a leftmost occurrence. The space of the $\Gamma$-tree is always $\Theta(\gamma \log(n/\gamma))$, which we now know, by Theorem [11] that is never asymptotically smaller than that of block trees with parameters $r = 2$ and $s = \delta$.

This index will need to compute Karp-Rabin fingerprints $\kappa(T[i..j])$ in time $O(\log(n/\delta))$. This is done on block trees by using the same algorithm described for the $\Gamma$-tree.

Lemma 17. Let $T[1..n]$ have compressibility measure $\delta$, and let $\kappa$ be a Karp–Rabin function. Then we can store a data structure of size $O(\delta \log(n/\delta))$ supporting the computation of $\kappa$ on any substring of $T$ in $O(\log(n/\delta))$ time.
Proof. The structure is the described block tree variant, with some further fields. We store \( \kappa(T[1..2^k]) \) at the \( i \)th top-level block, for all \( i \) and \( k = \lceil \log(n/\delta) \rceil \). We also store \( \kappa(B) \) for each block \( B \) stored in the tree and, for the unmarked blocks \( B \) pointing to \( B_1 : B_2 \) with offset \( \epsilon \), we also store \( \kappa(B_1[1 + \epsilon..]) \). Navarro and Prezza [37, Lem. 1] show that this information is sufficient to compute \( \kappa(T[i..j]) \) by spending \( O(1) \) time in each level of the \( \Gamma \)-tree; their proof holds verbatim for the block tree.

We also borrow the following concept [37, Lem. 2].

\[ \text{Lemma 18. Any substring } T[i..j] \text{ of length at least 2 either overlaps two consecutive represented blocks or is completely inside an unmarked block.} \]

Proof. The leaves of the block tree, read left to right, partition \( T \) into a sequence of represented blocks. The leaves are either unmarked blocks or level-0 blocks of length 1. Since \( T[i..j] \) is of length at least 2, if it is not completely inside an unmarked block, it cannot be contained in a leaf, and then it must cross a boundary between two represented blocks.

We now divide the possible occurrences of \( P[1..m] \) in \( T \) into primary (those overlapping two consecutive represented blocks) and secondary (those inside an unmarked block). Their technique [37, Sec. 3] applies verbatim to our structure: Primary occurrences are found using a grid of \( (s-1) \times (s-1) \), where \( s = O(\delta \log(n/\delta)) \) is the number of leaves in the block tree, which finds the \( \text{occ}_p \) primary occurrences in time \( O((m + \text{occ}_p) \log^\epsilon s) \), for any constant \( \epsilon > 0 \). The ranges to search for in the grid are obtained using their following result [37, Lem. 3].

\[ \text{Lemma 19. Let } X \text{ be a sorted set of suffixes of } T, \text{ and } \kappa \text{ a Karp–Rabin function. If one can extract a substring of length } \ell \text{ from } T \text{ in time } f_e(\ell) \text{ and compute } \kappa \text{ on it in time } f_h(\ell), \text{ then one can build a data structure of size } O(|X|) \text{ that obtains the lexicographic ranges in } X \text{ of the } m-1 \text{ suffixes of a given pattern } P \text{ in worst-case time } O(m(f_h(m) + \log m) + f_e(m)), \text{ provided that } \kappa \text{ is collision-free among substrings of } T \text{ whose lengths are powers of two.} \]

Since in our case \( f_e(m) = O(m \log(n/\delta)) \) and \( f_h(m) = O(\log(n/\delta)) \), we can find all the ranges to search for in time \( O(m \log mn/\delta)) \).

The \( \text{occ}_s \) secondary occurrences are obtained exactly as they do [37, Sec. 3.2], in time \( O((\text{occ}_p + \text{occ}_s) \log(n/\delta)) \). We then obtain the following result.

\[ \text{Theorem 20. Let } T[1..n] \text{ have compressibility measure } \delta \text{. Then there exists a data structure of size } O(\delta \log(n/\delta)) \text{ such that the occurrences of any pattern } P[1..m] \text{ in } T \text{ can be located in time } O(m \log n + \text{occ} \log^\epsilon n), \text{ for any constant } \epsilon > 0. \]

8 Conclusions

We have made an important step towards establishing the right measure of repetitiveness for a string \( S[1..n] \). Compared with the most principled prior measure, the size \( \gamma \) of the smallest attractor of \( S \), the measure \( \delta \) we propose has several important advantages:

1. It lower bounds the previous measure, \( \delta \leq \gamma \), and can be computed in linear time, while finding \( \gamma \) is NP-hard.
2. We can always encode \( S \) in \( O(\delta \log(n/\delta)) \) space, and this is worst-case optimal: there are text families needing that much space. Instead, no text family is known to need \( \omega(\gamma) \) space.
3. Measures \( \gamma, b, c, \) and \( z \) are upper bounded by \( O(\delta \log(n/\delta)) \), but there are text families where the smallest context-free grammar is of size \( g = \Omega(\delta \log^2 n/\log \log n) \). This lower bound is not known to hold on \( \gamma \).
The encodings using $O(\delta \log(n/\delta))$ space support direct access and indexed searches, with the same complexities obtained within attractor-bounded space, $O(\gamma \log(n/\gamma))$. An exception is a very recent work [14], which obtains better time complexity.

An ideal compressibility measure for repetitive sequences should be always reachable and worst-case optimal, apart from being practical to compute and lower-bound all the relevant compressors. Measure $\delta \log(n/\delta)$ satisfies the first two conditions, whereas $\gamma$ satisfies the last two. We then believe that $\delta$ is better than $\gamma$ in this sense, because $\gamma \log(n/\gamma)$ is not worst-case optimal, and computing $\gamma$ is NP-hard. Note that we do not know if one can always encode a string within $O(\gamma)$ space; if this was the case, then $\gamma$ would be the ideal measure except for being hard to compute. This fascinating quest is then still open.

On the more practical side, it would be interesting to obtain faster indexes within $O(\delta \log(n/\delta))$ space. The one we presented here obtains $O(m \log n + \text{occ} \log' n)$ search time. Within $O(\gamma \log(n/\gamma))$ space, instead, it is possible to search in time $O(m + (\text{occ} + 1) \log' n)$ [13]. Another challenge is to compute or approximate $\delta$ efficiently in external memory, that is, assuming we have $O(\delta \log(n/\delta))$ main memory space, but not $O(n)$. This would allow us indexing very large strings that, in plain form, do not fit in main memory.

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