External Potential Flow Around Multiple Aerofoils

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Abstract. This chapter presents a fast boundary integral equation method for computing uniform potential flow past multiple aerofoils. The presented fast multipole-based iterative solution procedure requires only $O(mn \ln n)$ operations where $m$ is the number of aerofoils and $n$ is the number of nodes in the discretization of each aerofoil's boundary. We demonstrate the performance of our methods on several numerical examples.

Keywords. Uniform potential flow; Multiply connected regions; Generalized Neumann kernel; Fast multipole method.

1 Introduction

In this chapter, we consider the two dimensional, steady-state, irrotational flow around multiple aerofoils of general shape. We assume that the fluid is incompressible and free from viscosity and the boundaries of the aerofoils are stationary and impervious. The problem will be solved using a fast boundary integral method based on the boundary integral equation with the generalized Neumann kernel presented in [9]. The integral equation will be solved using the fast method presented in [15] which is based on discretizing the integral equation using the Nyström method with the trapezoidal rule then solving the obtained linear system by the generalized minimal residual (GMRES) method [16]. The GMRES method will converge significantly faster since the eigenvalues of the coefficient matrix of the linear system are clustered around 1 (see [12, 13, 14]). Each iteration of the GMRES method requires a matrix-vector product which can be computed using the Fast Multipole Method (FMM) in $O(mn)$ operations where $m$ is the number of aerofoils and $n$ is the number of nodes in the discretization of each aerofoil’s boundary. Computing the right-hand side of the integral equation requires applying the FFT for each of the $m$ aerofoils which requires $O(mn \ln n)$ operations. Thus, the complexity of the presented method is $O(mn \ln n)$.

Three numerical examples will be presented. The numerical results illustrate that the present method has the ability to handle regions with very high connectivity and complex geometry.

2 Notations and auxiliary material

We consider an unbounded multiply connected regions $G$ in the extended complex plane $\overline{\mathbb{C}}$ exterior to $m \geq 1$ simply connected regions $G_j$, $j = 1, 2, \ldots, m$. We assume that the region $G$ is filled with an irrotational incompressible fluid flow and the bounded regions $G_j$, $j = 1, 2, \ldots, m$, represents $m$ aerofoils or obstacles in the flow path. We assume that the
boundaries $\Gamma_j := \partial G_j$ of the aerofoils are smooth closed Jordan curves. The orientation of the whole boundary $\Gamma := \partial G = \bigcup_{j=1}^{m} \Gamma_j$ is such that $G$ is always on the left of $\Gamma$, i.e., the curves $\Gamma_1, \ldots, \Gamma_m$ always have clockwise orientations (see Fig. 1).

![Figure 1: An unbounded multiply connected region $G$ of connectivity $m$.](image)

The curve $\Gamma_j$ is parametrized by a $2\pi$-periodic twice continuously differentiable complex function $\eta_j(t)$ with non-vanishing first derivative

$$\dot{\eta}_j(t) = d\eta_j(t)/dt \neq 0, \quad t \in J_j := [0, 2\pi], \quad j = 1, 2, \ldots, m. \quad (1)$$

We define the total parameter domain $J$ as the disjoint union of the intervals $J_j$. Hence, a parametrization of the whole boundary $\Gamma$ is defined as the complex function $\eta$ defined on $J$ by

$$\eta(t) := \begin{cases} 
\eta_1(t), & t \in J_1, \\
\vdots \\
\eta_m(t), & t \in J_m.
\end{cases} \quad (2)$$

The definition of the function $\eta(t)$ in (2) means that for a given real number $\hat{t} \in [0, 2\pi]$, to evaluate $\eta(\hat{t})$, we should know in advance the interval $J_j$ to which $\hat{t}$ belongs, i.e., we should know the boundary $\Gamma_j$ contains the point $\eta(\hat{t})$, then we compute $\eta(\hat{t}) = \eta_j(\hat{t})$.

The real kernel $N$ defined by

$$N(s, t) = \frac{1}{\pi} \text{Im} \left( \frac{\dot{\eta}(t)}{\eta(t) - \eta(s)} \right). \quad (3)$$

is known as the Neumann kernel. It is special case of the generalized Neumann kernel with $A = 1$. The kernel $N$ is continuous with

$$N(t, t) = \frac{1}{2\pi} \text{Im} \frac{\dot{\eta}(t)}{\dot{\eta}(t)}. \quad (4)$$

The real kernel $M$ defined by

$$M(s, t) = \frac{1}{\pi} \text{Re} \left( \frac{\dot{\eta}(t)}{\eta(t) - \eta(s)} \right), \quad (5)$$

has a cotangent singularity type. When $s, t \in J_j$ are in the same parameter interval $J_j$, then

$$M(s, t) = -\frac{1}{2\pi} \cot \frac{s - t}{2} + M_1(s, t) \quad (6)$$

with a continuous kernel $M_1$ which takes on the diagonal the values

$$M_1(t, t) = \frac{1}{2\pi} \text{Re} \frac{\dot{\eta}(t)}{\dot{\eta}(t)}. \quad (7)$$

See [3, 5, 8, 14, 17, 18] for more details.
We define the Fredholm integral operator with the kernel $N$ and the singular operator with the kernel $M$ by

$$N\mu := \int J_N(s,t)\mu(t)dt, \quad (8)$$

$$M\mu := \int J_M(s,t)\mu(t)dt. \quad (9)$$

The integral in (9) is a principal value integral.

3 The external potential flow problem

Suppose that $F(z)$ is the complex potential and $W(z) = F'(z)$ is the complex velocity of the flow where $z = x + iy \in G \cup \Gamma$. The associated velocity field $(u, v)$ is given, in complex form, by the relation

$$u(x, y) - iv(x, y) = W(z).$$

The velocity potential $\phi(x, y)$ and the stream function $\psi(x, y)$ associated with the flow are defined by

$$\phi(x, y) + i\psi(x, y) = F(z).$$

The families of curves

$$\phi(x, y) = \text{constant}, \quad \psi(x, y) = \text{constant}$$

are known as the equi-potential curves and the stream lines, respectively, [4, p. 98].

The complex potential $F(z)$ can be written in the form

$$F(z) = e^{-i\alpha}z - if(z) + \sum_{j=1}^{m} \frac{\chi_j}{2\pi i} \log(z - a_j) + c, \quad z \in G \cup \Gamma, \quad (10)$$

where $f(z)$ is an analytic function in $G$ with $f(\infty) = 0$, $c$ is a complex constant and $\chi_j$ is the circulation of the fluid along the boundary component $\Gamma_j$ (Note that the boundaries $\Gamma_j$ are assumed to be clockwise oriented). The complex velocity $W(z)$ is given by

$$W(z) = e^{-i\alpha} - if'(z) + \sum_{j=1}^{m} \frac{\chi_j}{2\pi i} \frac{1}{z - a_j}, \quad z \in G \cup \Gamma. \quad (11)$$

It is clear from (11) that knowing the function $f'(z)$ is sufficient to know the velocity function $W(z)$. For the potential function $F(z)$, the constant $c$ in (10) has no effect on the velocity field. Hence, to determine the potential function $F(z)$, it is only required to determine the auxiliary function $f(z)$. Then, the stream function is given by

$$\psi(x, y) = \text{Im}[e^{-i\alpha}z] - \text{Re}[f(z)] - \sum_{j=1}^{m} \frac{\chi_j}{2\pi} \ln |z - z_j| + \text{constant} \quad (12)$$

and the velocity potential is given by

$$\phi(x, y) = \text{Re}[e^{-i\alpha}z] + \text{Im}[f(z)] + \sum_{j=1}^{m} \frac{\chi_j}{2\pi} \text{arg}(z - z_j) + \text{constant}. \quad (13)$$
4 The integral equation with the Neumann kernel

The boundary values of the analytic function \( f(z) \) in (10) are given by

\[
f(\eta(t)) = \gamma(t) + h(t) + i\mu(t)
\]

where the function \( \gamma(t) \) is defined on \( J \) by

\[
\gamma = \text{Im}[e^{-i\alpha \eta}] - \sum_{j=1}^{m} \frac{\chi_j}{2\pi} \ln |\eta - z_j|,
\]

the function \( \mu \) is the unique solution of the integral equation

\[
\mu - N\mu = -M\gamma,
\]

and the function \( h \) is given by

\[
h = [M\mu - (I - N)\gamma]/2.
\]

The function \( h \) is a piecewise constant real-valued function, i.e.,

\[
h(t) = \begin{cases} h_1, & t \in J_1, \\ \vdots & \\ h_m, & t \in J_m, \end{cases}
\]

with real constants \( h_1, \ldots, h_m \).

In view of (14), the values of the function \( f(z) \) can be calculated for \( z \in G \) by the Cauchy integral formula

\[
f(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\gamma + h + i\mu}{\eta - z} d\eta.
\]

Then, the stream function can be computed from (12).

5 Numerical examples

Since the functions \( A_j \) and \( \eta_j \) are 2\pi-periodic, a reliable procedure for solving the integral equation (16) numerically is by using the Nyström method with the trapezoidal rule [1]. Thus solving the integral equation reduces to solving an \( mn \times mn \) linear system where \( m \) is the multiplicity of the multiply connected region and \( n \) is the number of nodes in the discretization of each boundary component. Since the integral equation (16) is uniquely solvable, then for sufficiently large \( n \), the obtained linear system is also uniquely solvable [1]. See [6, 7, 9, 10, 11] for more details.

In this chapter, the MATLAB function \texttt{FBIE} presented in [15] will be used to solve the integral equation (16) and compute the function \( h \) in (17). The MATLAB function \texttt{FBIE} is based on discretizing the integral equation (16) using the Nyström method with the trapezoidal rule then solving the obtained linear system by the MATLAB function \texttt{gmres}. The function \texttt{gmres} can be used with a matrix-vector product function, i.e., it is not necessary to have an explicit form of the coefficient matrix of the linear system. In [15], the matrix-vector product function for the coefficient matrix of our linear system was defined using the function \texttt{zfmm2dpart} in the MATLAB toolbox \texttt{FMMLIB2D} developed by Greengard and Gimbutas [2]. Thus, the obtained linear systems will be solved in \( O(mn) \) operation. However, computing the right-hand side of the integral equation requires applying the \texttt{FFT} for each of the \( m \) boundary components which requires \( O(mn \ln n) \) operations. Thus, the complexity of the presented method is \( O(mn \ln n) \).
By solving the integral equation (16) numerically, we obtain an approximation to the boundary values of the function $f$ by (14). Then an approximation $f_n(z)$ to the values of the function $f(z)$ for points $z \in G$ will be computed using the Cauchy integral formula (19). The integral in (19) is discretized by the trapezoidal rule. The FMM will be used for fast computing of the values of $f_n(z)$. See [15] for more details.

**Example 1.** The region $G$ is an unbounded multiply connected region exterior to 15 smooth Jordan curves (see Fig. 2).

![Figure 2](image)

Figure 2: Numerical results for Example 1 obtained with $n = 2048$ (total number of nodes is 30720). The streamlines are shown for $\alpha = 0$ and for zero circulations along 10 aerofoils and non-zero circulations along 5 aerofoils (the circulation along each aerofoil is shown inside the curve).

**Example 2.** The region $G$ is an unbounded multiply connected region exterior to 110 smooth Jordan curves (see Fig. 3).

![Figure 3](image)

Figure 3: Numerical results for Example 2 obtained with $n = 2048$ (total number of nodes is 225280). The streamlines are shown for $\alpha = 0$ and for circulations $-5$ along the cylindrical aerofoils and zero circulations along the other aerofoils.

**Example 3.** The region $G$ is an unbounded multiply connected region exterior to 2000 circles (see Fig. 4).
Figure 4: Numerical results for Example 3 obtained with $n = 1024$ (total number of nodes is 2048000). The streamlines are shown for $\alpha = 0$ and the circulations along each aerofoil is an arbitrary number between $-1$ and $1$.

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