Almost sure central limit theorem for self-normalized products of the some partial sums of $\rho^-$-mixing sequences

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Abstract

Let $\{X_n\}_{n \in \mathbb{N}}$ be a strictly stationary $\rho^-$-mixing sequence of positive random variables, under the suitable conditions, we get the almost sure central limit theorem for the products of the some partial sums $\left(\prod_{k=1}^n S_{nk}\right)^{\frac{\beta}{n\mu}}$, where $\beta > 0$ is a constant, and $E(X) = \mu$, $S_{nk} = \sum_{j=1}^k X_j - X_i$, $1 \leq i \leq k$, $V_k^2 = \sum_{j=1}^k (X_j - \mu)^2$.

MSC: 60F15

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1 Introduction and main result

In 1988, Brosamler [1] and Schatte [2] proposed the almost sure central limit theorem (ASCLT) for the sequence of i.i.d. random variables. On the basis of i.i.d., Khurelbaatar and Grzegorz [3] got the ASCLT for the products of the some partial sums of random variables. In 2008, Miao [4] gave a new form of ASCLT for products of some partial sums.

Theorem A ([4]) Let $\{X_n\}_{n \in \mathbb{N}}$ be a sequence of i.i.d. positive square integrable random variables with $E(X_1) = \mu$, $\text{Var}(X_1) = \sigma^2 > 0$ and the coefficient of variation $\gamma = \frac{\sigma}{\mu}$. Denote the $S_{nk} = \sum_{j=1}^k X_j - X_i$, $1 \leq i \leq k$. Then, for $\forall x \in \mathbb{R}$,

$$
\lim_{N \to \infty} \frac{1}{\log N} \sum_{n=1}^N \frac{1}{N} \left[ \left( \prod_{k=1}^n S_{nk} \right)^{\frac{1}{\sigma^2}} \leq x \right] = F(x) \quad \text{a.s.,}
$$

where $F(\cdot)$ is the distribution function of the random variables $e^N$, $N$ is a standard normal random variable.

For random variables $X, Y$, define

$$
\rho^-(X, Y) = 0 \lor \sup_{f, g \in \mathcal{C}} \frac{\text{Cov}(f(X), g(Y))}{(\text{Var}(f(X)))^{\frac{1}{2}} (\text{Var}(g(Y)))^{\frac{1}{2}}},
$$

where the sup is taken over all $f, g \in \mathcal{C}$ such that $E(f(X))^2 < \infty$ and $E(g(Y))^2 < \infty$, and $\mathcal{C}$ is a class of functions which are coordinatewise increasing.
Definition ([5]) A sequence \( \{X_n\}_{n \in \mathbb{N}} \) is called \( \rho^- \)-mixing, if

\[
\rho^-(s) = \sup \{ \rho^-(S, T); S, T \subset \mathbb{N}, \text{dist}(S, T) \geq s \} \to 0, \quad s \to \infty,
\]

where

\[
\rho^-(S, T) = 0 \vee \sup \left\{ \frac{\text{Cov}(f(X_i, i \in S), g(X_j, j \in T))}{\sqrt{\text{Var}[f(X_i, i \in S)]} \sqrt{\text{Var}[g(X_j, j \in T)]}} \mid f, g \in \mathcal{C} \right\},
\]

\( \mathcal{C} \) is a class of functions which are coordinatewise increasing.

The precise definition of \( \rho^- \)-mixing random variables was introduced initially by Zhang and Wang [5] in 1999. Obviously, \( \rho^- \)-mixing random variables include NA and \( \rho^* \)-mixing random variables, which have a lot of applications, their limit properties have aroused wide interest recently, and a lot of results have been obtained by many authors. In 2005, Zhou [6] proved the almost central limit theorem of the \( \rho^- \)-mixing sequence. The almost sure central limit theorem for products of the partial sums of \( \rho^- \)-mixing sequences was given by Tan [7] in 2012. Because the denominator of the self-normalized partial sums contains random variables, this brings about difficulties to the study of the self-normalized form limit theorem of the \( \rho^- \)-mixing sequence. At present, there are very few results of this kind. In this paper, we extend Theorem A, and get the almost sure central limit theorem for self-normalized products of the some partial sums of \( \rho^- \)-mixing sequences.

Throughout this paper, \( a_n \sim b_n \) means \( \lim_{n \to \infty} \frac{a_n}{b_n} = 1 \), and \( C \) denotes a positive constant, which may take different values whenever it appears in different expressions, and \( \log x = \ln(x \vee e) \). We assume \( \{X_n\}_{n \in \mathbb{N}} \) is a strictly stationary sequence of \( \rho^- \)-mixing random variables, and we denote \( Y_i = X_i - \mu \).

For every \( 1 \leq i \leq k \leq n \), define

\[
\tilde{Y}_{mi} = -\sqrt{m} \mathbb{I}(Y_i < -\sqrt{m}) + Y_i \mathbb{I}(|Y_i| \leq \sqrt{m}) + \sqrt{m} \mathbb{I}(Y_i > \sqrt{m}),
\]

\[
T_{k,n} = \sum_{i=1}^{k} \tilde{Y}_{mi}, \quad \bar{V}_n^2 = \sum_{i=1}^{n} Y_i^2, \quad \bar{V}_n^2 = \sum_{i=1}^{n} \tilde{Y}_{mi}^2,
\]

\[
\bar{V}_{n,1}^2 = \sum_{i=1}^{n} \tilde{Y}_{mi}^2 \mathbb{I}(Y_i \geq 0), \quad \bar{V}_{n,2}^2 = \sum_{i=1}^{n} \tilde{Y}_{mi}^2 \mathbb{I}(Y_i < 0),
\]

\[
\sigma_n^2 = \text{Var}(T_{n,k}), \quad \delta_n^2 = \mathbb{E}(\bar{V}_n^2), \quad \delta_{n,1}^2 = \mathbb{E}(\bar{V}_{n,1}^2), \quad \delta_{n,2}^2 = \mathbb{E}(\bar{V}_{n,2}^2).
\]

apparently, \( \delta_n^2 = \delta_{n,1}^2 + \delta_{n,2}^2 \), \( \mathbb{E}(\bar{V}_n^2) = n \delta_n^2 = n \delta_{n,1}^2 + n \delta_{n,2}^2 \).

Our main theorem is as follows.

Theorem 1 Let \( \{X_n\}_{n \in \mathbb{N}} \) be a strictly stationary \( \rho^- \)-mixing sequence of positive random variables with \( E X = \mu > 0 \), and for some \( r > 2 \), we have \( 0 < E|X|^r < \infty \). Denote \( S_{k,i} = \sum_{j=1}^{k} X_j - X_i \), \( 1 \leq i \leq k \) and \( Y = X - \mu \). Suppose that

\( (a_1) \ \) \( \mathbb{E}(Y^2 \mathbb{I}(Y \geq 0)) > 0, \ \mathbb{E}(Y^2 \mathbb{I}(Y < 0)) > 0, \)

\( (a_2) \ \) \( \sigma_t^2 = \mathbb{E}X^2 + 2 \sum_{k=0}^{\infty} \mathbb{C}(X_1, X_k) > 0, \sum_{k=0}^{\infty} |\mathbb{C}(X_1, X_k)| < \infty, \)

\( (a_3) \ \) \( \sigma^2_k \sim \beta^2 k \delta_n^2 \) for some \( \beta > 0, \)

\( (a_4) \ \) \( \rho^-(n) = O(\log^{-d} n), \exists d > 1. \)
Suppose $0 \leq \alpha < \frac{1}{2}$, and let

$$d_k = \frac{\exp(\log^\alpha k)}{k}, \quad D_n = \sum_{k=1}^{n} d_k,$$

(1)

then, for $\forall x \in \mathbb{R}$, we have

$$\lim_{n \to \infty} \frac{1}{D_n} \sum_{k=1}^{n} d_k \mathbb{I}\left( \prod_{i=1}^{k} S_{k,i} \leq \left( \frac{k}{k-1} \right)^{\frac{\alpha}{2}} \right) = F(x) \quad \text{a.s.,}$$

(2)

where $F(\cdot)$ is the distribution function of the random variables $e^\mathcal{N}$, $\mathcal{N}$ is a standard normal random variable.

**Corollary 1** By [8], (2) remains valid if we replace the weight sequence $\{d_k, k \geq 1\}$ by any $\{d^*_k, k \geq 1\}$ such that $0 \leq d^*_k \leq d_k$, $\sum_{k=1}^{\infty} d^*_k = \infty$.

**Corollary 2** If $\{X_n, n \geq 1\}$ is a sequence of strictly stationary independent positive random variables then one has (a3) and $\beta = 1$.

## 2 Some lemmas

We will need the following lemmas.

**Lemma 2.1** ([7]) Let $\{X, X_n\}_{n \in \mathbb{N}}$ be a strictly stationary sequence of $\rho^{-}$-mixing random variables with $EX_1 = 0$, $0 < \sigma^2 < \infty$, $\sigma^2 = EX_1^2 + 2 \sum_{k=2}^{\infty} \text{Cov}(X_1, X_k) > 0$ and $\sum_{k=2}^{\infty} \text{Cov}(X_1, X_k) < \infty$, then, for $0 < p < 2$, we have

$$\frac{S_n}{n^{\frac{p}{2}}} \to 0, \quad \text{a.s.,} \quad n \to \infty.$$

**Lemma 2.2** ([9]) Let $\{X, X_n\}_{n \in \mathbb{N}}$ be a sequence of $\rho^{-}$-mixing random variables, with

$$EX_n = 0, \quad E|X_n|^q < \infty, \quad \forall n \geq 1, q \geq 2,$$

then there is a positive constant $C = C(q, \rho^{-}(\cdot))$ only depending on $q$ and $\rho^{-}(\cdot)$ such that

$$E\left( \max_{1 \leq j \leq n} |S_j|^q \right) \leq C \left( \sum_{i=1}^{n} E|X_i|^q + \left( \sum_{i=1}^{n} EX_i^2 \right)^{\frac{q}{2}} \right).$$

**Lemma 2.3** ([10]) Suppose that $f_1(x)$ and $f_2(y)$ are real, bounded, absolutely continuous functions on $\mathbb{R}$ with $|f_1'(x)| \leq C_1$ and $|f_2'(y)| \leq C_2$, then, for any random variables $X$ and $Y$, $\quad |\text{Cov}(f_1(X), f_2(Y))| \leq C_1 C_2 \left( -\text{Cov}(X, Y) + 8 \rho^{-}(X, Y) \|X\|_{2,1} \|Y\|_{2,1} \right)$,

where $\|X\|_{2,1} = \int_{0}^{\infty} (P(|X| > x))^{\frac{1}{2}} dx$. 
Lemma 2.4 Let $\{\xi, \xi_i\}_{i \in N}$ be a sequence of uniformly bounded random variables. If $\exists \delta > 1$, $\rho^-(n) = O(\log^{-\delta} n)$, there exist constants $C > 0$ and $\varepsilon > 0$, such that

$$|E_{\xi} \xi_l| \leq C \left( \rho^-(k) + \left( \frac{k}{l} \right)^{\varepsilon} \right), \quad 1 \leq 2k < l, \tag{3}$$

then

$$\lim_{n \to \infty} \frac{1}{D_n} \sum_{k=1}^{n} d_k \xi_k = 0, \quad \text{a.s.}$$

Proof See the proof of Theorem 1 in [7]. □

Lemma 2.5 If the assumptions of Theorem 1 hold, then

$$\lim_{n \to \infty} \frac{1}{D_n} \sum_{k=1}^{n} d_k \left[ \frac{T_{k,k} - E(T_{k,k})}{\beta \delta_k \sqrt{k}} \right] \leq x = \Phi(x) \quad \text{a.s., } \forall x \in \mathbb{R}, \tag{4}$$

$$\lim_{n \to \infty} \frac{1}{D_n} \sum_{k=1}^{n} d_k \left[ f \left( \frac{V_{k,l}^2}{k \delta_k^2} \right) - E f \left( \frac{V_{k,l}^2}{k \delta_k^2} \right) \right] = 0 \quad \text{a.s., } l = 1, 2, \tag{5}$$

where $d_k$ and $D_k$ is defined as (1) and $f$ is real, bounded, absolutely continuous function on $\mathbb{R}$.

Proof Firstly, we prove (4), by the property of $\rho^-$-mixing sequence, we know that $\{\bar{Y}_{ni}\}_{i \geq 1; n \leq N}$ is a $\rho^-$-mixing sequence; using Lemma 2.1 in [7], the condition (a2), (a3), and $\beta > 0, \delta_k^2 \rightarrow E Y^2 > 0$, it follows that

$$\frac{T_{k,k} - E(T_{k,k})}{\beta \delta_k \sqrt{k}} \xrightarrow{d} \mathcal{N}, \quad k \to \infty,$$

hence, for any $g(x)$ which is a bounded function with bounded continuous derivative, we have

$$E g \left( \frac{T_{k,k} - E(T_{k,k})}{\beta \delta_k \sqrt{k}} \right) \rightarrow E g(\mathcal{N}), \quad k \to \infty,$$

by the Toeplitz lemma, we get

$$\lim_{n \to \infty} \frac{1}{D_n} \sum_{k=1}^{n} d_k E \left[ g \left( \frac{T_{k,k} - E(T_{k,k})}{\beta \delta_k \sqrt{k}} \right) \right] = E(g(\mathcal{N})).$$

On the other hand, from Theorem 7.1 of [11] and Sect. 2 of [12], we know that (4) is equivalent to

$$\lim_{n \to \infty} \frac{1}{D_n} \sum_{k=1}^{n} d_k g \left( \frac{T_{k,k} - E(T_{k,k})}{\beta \delta_k \sqrt{k}} \right) = E(g(\mathcal{N})) \quad \text{a.s.,}$$
hence, to prove (4), it suffices to prove
\[
\lim_{n \to \infty} \frac{1}{D_n} \sum_{k=1}^{n} d_k \left[ g \left( \frac{T_{k,k} - E(T_{k,k})}{\beta \delta \sqrt{k}} \right) - E \left( g \left( \frac{T_{k,k} - E(T_{k,k})}{\beta \delta \sqrt{k}} \right) \right) \right] = 0 \quad \text{a.s.,} \tag{6}
\]

noting that
\[
\xi_k = g \left( \frac{T_{k,k} - E(T_{k,k})}{\beta \delta \sqrt{k}} \right) - E \left( g \left( \frac{T_{k,k} - E(T_{k,k})}{\beta \delta \sqrt{k}} \right) \right),
\]

for every \(1 \leq 2k < l\), we have
\[
|E\xi_k\xi_l| = \left| \text{Cov} \left( \frac{T_{k,k} - E(T_{k,k})}{\beta \delta \sqrt{k}}, \frac{T_{l,l} - E(T_{l,l})}{\beta \delta \sqrt{l}} \right) \right|
\leq \text{Cov} \left( \frac{T_{k,k} - E(T_{k,k})}{\beta \delta \sqrt{k}}, \frac{T_{l,l} - E(T_{l,l})}{\beta \delta \sqrt{l}} \right) - \text{Cov} \left( \frac{T_{l,l} - E(T_{l,l}) - (T_{2k,l} - E(T_{2k,l}))}{\beta \delta \sqrt{l}} \right)
+ \text{Cov} \left( \frac{T_{l,l} - E(T_{l,l}) - (T_{2k,l} - E(T_{2k,l}))}{\beta \delta \sqrt{l}} \right)
= I_1 + I_2. \tag{7}
\]

First we estimate \(I_1\); we know that \(g\) is a bounded Lipschitz function, i.e., there exists a constant \(C\) such that
\[
|g(x) - g(y)| \leq C|x - y|
\]
for any \(x, y \in R\), since \(\{\bar{Y}_m\}_{n \geq 1, i \leq n}\) also is a \(\rho^*\)-mixing sequence; we use the condition \(\delta_l^2 \to E(Y_l^2) < \infty, l \to \infty\), and Lemma 2.2, to get
\[
I_1 \leq C \frac{E|T_{2k,l} - ET_{2k,l}|}{\sqrt{l}} \leq C \sqrt{E(T_{2k,l} - ET_{2k,l})^2} \leq C \sqrt{\frac{\delta^2}{k \sqrt{l}}} \leq C \sqrt{\frac{\delta^2}{k}} \leq C \frac{k}{l}. \tag{8}
\]

Next we estimate \(I_2\); by Lemma 2.2, we have
\[
\text{Var} \left( \frac{T_{k,k} - ET_{k,k}}{\beta \delta \sqrt{k}} \right) \leq \frac{C}{k} \text{Var}(T_{k,k} - ET_{k,k})
\leq \frac{C}{k} \sum_{i=1}^{k} E(Y_{ki} - E(Y_{ki}))^2 \leq \frac{C}{k} \sum_{i=1}^{k} E(Y_{ki})^2 \leq C \cdot k \leq C
\]
and
\[
\text{Var}\left( \frac{T_{ij} - ET_{ij} - (T_{2k,l} - ET_{2k,l})}{\beta \delta_i \sqrt{l}} \right) \leq \frac{C}{l} \text{Var}\left( \frac{T_{ij} - ET_{ij} - (T_{2k,l} - ET_{2k,l})}{\beta \delta_i \sqrt{l}} \right)
\]
\[
\leq \frac{C}{l} \sum_{i=2k+1}^l \mathbb{E}(\bar{Y}_i - E\bar{Y}_i)^2 \leq \frac{C}{l} \left( \sum_{i=1}^l EY_i^2 \right)
\]
\[
\leq C l \leq C.
\]

By the definition of a \( \rho^- \)-mixing sequence, \( EY^2 < \infty \), and Lemma 2.3, we have
\[
I_2 \leq \left( -\text{Cov}\left( \frac{T_{k,k} - ET_{k,k} - (T_{2k,l} - ET_{2k,l})}{\beta \delta_k \sqrt{k}} \right), \frac{T_{ij} - ET_{ij} - (T_{2k,l} - ET_{2k,l})}{\beta \delta_i \sqrt{l}} \right)
\]
\[
+ \frac{8 \rho^-}{\beta \delta_k \sqrt{k}} \left| \frac{T_{k,k} - ET_{k,k}}{\beta \delta_k \sqrt{k}} \right|_{2,1} \left| \frac{T_{ij} - ET_{ij} - (T_{2k,l} - ET_{2k,l})}{\beta \delta_i \sqrt{l}} \right|_{2,1}
\]
\[
\leq C \rho^- (k) \left( \text{Var}\left( \frac{T_{k,k} - ET_{k,k}}{\beta \delta_k \sqrt{k}} \right) \right) \frac{1}{2} \left( \text{Var}\left( \frac{T_{ij} - ET_{ij} - (T_{2k,l} - ET_{2k,l})}{\beta \delta_i \sqrt{l}} \right) \right) \frac{1}{2}
\]
\[
+ \frac{8 \rho^- (k)}{\beta \delta_k \sqrt{k}} \left| \frac{T_{k,k} - ET_{k,k}}{\beta \delta_k \sqrt{k}} \right|_{2,1} \left| \frac{T_{ij} - ET_{ij} - (T_{2k,l} - ET_{2k,l})}{\beta \delta_i \sqrt{l}} \right|_{2,1}
\]

By \( \|X\|_{2,1} \leq r/(r - 2)\|X\|_r \), \( r > 2 \) (see p. 254 of [10] or p. 251 of [13]), Minkowski inequality, Lemma 2.2, and the Hölder inequality, we get
\[
\left| \frac{T_{k,k} - ET_{k,k}}{\beta \delta_k \sqrt{k}} \right|_{2,1} \leq \frac{r}{r - 2} \left( \frac{T_{k,k} - ET_{k,k}}{\beta \delta_k \sqrt{k}} \right)_r
\]
\[
= \frac{r}{r - 2} \frac{1}{\beta \delta_k \sqrt{k}} \left( E|T_{k,k} - ET_{k,k}|^r \right)^{1/r}
\]
\[
\leq C \frac{1}{\sqrt{k}} \left( \sum_{i=1}^k E|\bar{Y}_i|^r + \left( \sum_{i=1}^k EY_i^2 \right)^{r/2} \right)^{1/r}
\]
\[
\leq C \frac{1}{\sqrt{k}} \left( k + k^{r/2} \right)^{1/r} \leq C,
\]

similarly
\[
\left| \frac{T_{ij} - ET_{ij} - (T_{2k,l} - ET_{2k,l})}{\beta \delta_i \sqrt{l}} \right|_{2,1} \leq C.
\]

Hence
\[
I_2 \leq C \rho^- (k).
\]

Combining with (7)–(9), (3) holds, and by (a4), Lemma 2.4, (6) holds, then (4) is true.
Secondly, we prove (5); for $\forall k \geq 1$, $\eta_k = f(\tilde{V}_{ik}^2/(k\delta_{k,1}^2)) - E(f(\tilde{V}_{ik}^2/(k\delta_{k,1}^2)))$, we have

$$|E\eta_k| = \left| \text{Cov}(f\left(\frac{\tilde{V}_{ik}^2}{k\delta_{k,1}^2}\right), f\left(\frac{\tilde{V}_{il}^2}{l\delta_{l,1}^2}\right)) \right|$$

$$\leq \left| \text{Cov}(f\left(\frac{\tilde{V}_{ik}^2}{k\delta_{k,1}^2}\right), f\left(\frac{\tilde{V}_{il}^2}{l\delta_{l,1}^2}\right) - f\left(\frac{\sum_{i=2k+1}^t \tilde{V}_{i}^2 I(Y_i \geq 0)}{l\delta_{l,1}^2}\right)) \right|$$

$$+ \left| \text{Cov}(f\left(\frac{\tilde{V}_{ik}^2}{k\delta_{k,1}^2}\right), f\left(\frac{\sum_{i=2k+1}^t \tilde{V}_{i}^2 I(Y_i \geq 0)}{l\delta_{l,1}^2}\right)) \right|$$

$$= J_1 + J_2,$$  \hspace{1cm} (10)

by the property of $f$, we know

$$J_1 \leq C\left( E\left( \sum_{i=1}^{2k} \tilde{V}_{ik}^2 I(Y_i \geq 0) \right) / I \right) \leq C\left( \frac{k}{l} \right).$$ \hspace{1cm} (11)

Now we estimate $J_2$,

$$\text{Var}\left(\frac{\tilde{V}_{ik}^2}{k\delta_{k,1}^2}\right) = \text{Var}\left(\frac{\sum_{i=1}^{k} \tilde{V}_{ik}^2 I(Y_i \geq 0)}{k\delta_{k,1}^2}\right)$$

$$\leq \frac{C}{k^2} E\left( \sum_{i=1}^{k} \tilde{V}_{ik}^2 I(Y_i \geq 0) \right)^2$$

$$= \frac{C}{k^2} E\left( \sum_{i=1}^{k} \tilde{V}_{ik}^2 I(Y_i \geq 0) - E\left( \sum_{i=1}^{k} \tilde{V}_{ik}^2 I(Y_i \geq 0) \right) + E\left( \sum_{i=1}^{k} \tilde{V}_{ik}^2 I(Y_i \geq 0) \right) \right)^2$$

$$\leq \frac{C}{k^2} E\left( \sum_{i=1}^{k} \tilde{V}_{ik}^2 I(Y_i \geq 0) - E\left( \tilde{V}_{ik}^2 I(Y_i \geq 0) \right) \right)^2$$

$$+ \frac{C}{k^2} \left( \sum_{i=1}^{k} E(\tilde{V}_{ik}^2 I(Y_i \geq 0)) \right)^2$$

$$\leq \frac{C}{k^2} \sum_{i=1}^{k} E\tilde{V}_{ik}^4 I(Y_i \geq 0) + \frac{C}{k^2} (kE(\tilde{V}_{ik}^2 I(Y_i \geq 0)))^2$$

$$\leq \frac{C}{k^2} \sum_{i=1}^{k} Ek(Y_i)^2 \leq C,$$

and similarly $\text{Var}(\sum_{i=2k+1}^t \tilde{V}_{i}^2 I(Y_i \geq 0)/(l\delta_{l,1}^2)) \leq C$. On the other hand, we have

$$\left\| \frac{\tilde{V}_{ik}^2}{k\delta_{k,1}^2} \right\|_{2,1} \leq \frac{r}{r-2} \cdot \frac{C}{k} \left( E|\tilde{V}_{ik}^2| \right)^{1/r}$$

$$\leq \frac{C}{k} \left( E\left( \sum_{i=1}^{k} (\tilde{V}_{ik}^2 I(Y_i \geq 0) - E(\tilde{V}_{ik}^2 I(Y_i \geq 0))) \right)^r + \sum_{i=1}^{k} E(\tilde{V}_{ik}^2 I(Y_i \geq 0)) \right)^{1/r}$$

$$\leq \frac{C}{k} \left( \sum_{i=1}^{k} E(\tilde{V}_{ik}^2 I(Y_i \geq 0) - E(\tilde{V}_{ik}^2 I(Y_i \geq 0))) \right)^r$$
hence, combining with (11) and (12), (3) holds, and by Lemma 2.4, (5) holds.

Let $C_k$

\[ \text{Proof of Theorem } 1 \]

Thus, by Lemma 2.3, we have

\[ I_2 \leq C \left\{ -\text{Cov} \left( \frac{\tilde{Y}_{k,1}^2 I(Y_i \geq 0)}{\delta_{k,1}^2}, \sum_{i=2k+1}^i \tilde{Y}_{k,1}^2 I(Y_i \geq 0) \right) \right\} + 8 \rho^* \left( \frac{\tilde{Y}_{k,1}^2}{\delta_{k,1}}, \sum_{i=2k+1}^i \tilde{Y}_{k,1}^2 I(Y_i \geq 0) \right) \cdot \left\| \sum_{i=2k+1}^i \tilde{Y}_{k,1}^2 I(Y_i \geq 0) \right\|_{2,1} \]

\[ \leq C \left\{ \rho^* \left( k \cdot \text{Var} \left( \frac{\tilde{Y}_{k,1}^2}{\delta_{k,1}} \right) \right) \cdot \text{Var} \left( \sum_{i=2k+1}^i \tilde{Y}_{k,1}^2 I(Y_i \geq 0) \right) \right\} \]

\[ \leq C \rho^* (k), \]  

(12)

hence, combining with (11) and (12), (3) holds, and by Lemma 2.4, (5) holds.

\[ \square \]

3 Proof of Theorem 1

Let $C_{k,i} = \frac{S_{k,i}}{(k-1)n}$, hence, (2) is equivalent to

\[ \lim_{n \to \infty} \frac{1}{D_n} \sum_{k=1}^n d_k \left( \frac{\mu}{BV_k} \sum_{i=1}^k \log C_{k,i} \leq x \right) = \Phi(x) \quad \text{a.s.} \]  

(13)

So we only need to prove (13), for a fixed $k$, $1 \leq k \leq n$ and $\forall \varepsilon > 0$; we have

\[ \lim_{k \to \infty} P \left( \bigcup_{m=k}^{\infty} \left\{ \frac{X_i}{m} \geq \varepsilon \right\} \right) = \lim_{k \to \infty} P \left\{ \frac{X_i}{k} \geq \varepsilon \right\} = \lim_{k \to \infty} P \left\{ |X_i| \geq \varepsilon k \right\} = 0, \]
therefore, by Theorem 1.5.2 in [14], we have
\[
\frac{X_i}{k} \to 0 \quad \text{a.s.} \quad k \to \infty,
\]
on the unanimous establishment of \(i\).

By Lemma 2.1, for some \(\frac{1}{3} < p < 2\), and enough large \(k\), we have
\[
\sup_{1 \leq i \leq k} |C_{k,i} - 1| \leq \frac{\sum_{i=1}^{k} (X_i - \mu)}{(k-1)\mu} + \sup_{1 \leq i \leq k} \frac{|X_i|}{(k-1)\mu} + \frac{1}{k-1}
\]
\[
\leq \left| \frac{S_k - k\mu}{k^{1/p}} \right| \leq C k^{\frac{1}{p} - 1},
\]
by \(\log(1 + x) = x + O(x^2), \quad x \to 0\), we get
\[
\left| \frac{\mu}{\beta \delta_k \sqrt{1 + \epsilon} k} \sum_{i=1}^{k} \ln C_{k,i} - \frac{\mu}{\beta \delta_k \sqrt{1 + \epsilon} k} \sum_{i=1}^{k} (C_{k,i} - 1) \right|
\]
\[
\leq \frac{C \mu}{\beta \delta_k \sqrt{1 + \epsilon} k} \sum_{i=1}^{k} (C_{k,i} - 1)^2
\]
\[
\leq \frac{C}{k^{\frac{1}{p} - 1}} \to 0 \quad \text{a.s.,} \quad k \to \infty,
\]
and then, for \(\delta > 0\) and every \(\omega\), there exists \(k_0 = k_0(\omega, \delta, x)\); when \(k > k_0\), we have
\[
I \left\{ \frac{\mu}{\beta \delta_k \sqrt{1 + \epsilon} k} \sum_{i=1}^{k} (C_{k,i} - 1) \leq x - \delta \right\}
\]
\[
\leq I \left\{ \frac{\mu}{\beta \delta_k \sqrt{1 + \epsilon} k} \sum_{i=1}^{k} \log C_{k,i} \leq x \right\}
\]
\[
\leq I \left\{ \frac{\mu}{\beta \delta_k \sqrt{1 + \epsilon} k} \sum_{i=1}^{k} (C_{k,i} - 1) \leq x + \delta \right\}, \quad (14)
\]
under the condition \(|X_i - \mu| \leq \sqrt{k}, 1 \leq i \leq k\), we have
\[
\mu \sum_{i=1}^{k} (C_{k,i} - 1) = \sum_{i=1}^{k} S_{k,i} - (k-1)\mu \frac{k}{k-1} = \sum_{i=1}^{k} Y_i = \sum_{i=1}^{k} \bar{Y}_{k,i} = T_{k,i}, \quad (15)
\]
furthermore, by (14) and (15), for any given \(0 < \epsilon < 1, \delta > 0\), when \(k > k_0\), we obtain
\[
I \left( \frac{\mu}{\beta \delta_k \sqrt{1 + \epsilon} k} \sum_{i=1}^{k} \log C_{k,i} \leq x \right)
\]
\[
\leq I \left( \frac{T_{k,i}}{\beta \delta_k \sqrt{k(1 + \epsilon)}} \leq x + \delta \right) + I(\bar{V}_k^2 > (1 + \epsilon)k\delta_k^2)
\]
\[
+ I \left( \bigcup_{i=1}^{k} (|X_i - \mu| > \sqrt{k}) \right), \quad x \geq 0,
\]
\[ I \left( \frac{\mu}{\beta} \sqrt{V_k} \sum_{i=1}^{k} \log C_{ij} \leq x \right) \]
\[ \leq I \left( \frac{T_{ki}}{\delta_k \beta \sqrt{k(1-\varepsilon)}} \leq x + \delta \right) + I \left( \bar{V}_k^2 < (1-\varepsilon)k\delta_k^2 \right) \]
\[ + I \left( \bigcup_{i=1}^{k} \{|X_i - \mu| > \sqrt{k}\}, \quad x < 0, \right) \]
\[ I \left( \frac{\mu}{\beta} \sqrt{V_k} \sum_{i=1}^{k} \log C_{ij} \leq x \right) \]
\[ \geq I \left( \frac{T_{ki}}{\delta_k \beta \sqrt{k(1-\varepsilon)}} \leq x - \delta \right) - I \left( \bar{V}_k^2 < (1-\varepsilon)k\delta_k^2 \right) \]
\[ - I \left( \bigcup_{i=1}^{k} \{|X_i - \mu| > \sqrt{k}\}, \quad x \geq 0, \right) \]
\[ I \left( \frac{\mu}{\beta} \sum_{i=1}^{k} \log C_{ij} \leq x \right) \]
\[ \geq I \left( \frac{T_{ki}}{\delta_k \beta \sqrt{k(1+\varepsilon)}} \leq x - \delta \right) - I \left( \bar{V}_k^2 > (1+\varepsilon)k\delta_k^2 \right) \]
\[ - I \left( \bigcup_{i=1}^{k} \{|X_i - \mu| > \sqrt{k}\}, \quad x < 0. \right) \]

Therefore, to prove (13), for any 0 < \varepsilon < 1, \delta_1 > 0, it suffices to prove

\[ \lim_{n \to \infty} \frac{1}{D_n} \sum_{k=1}^{n} d_k I \left( \frac{T_{ki}}{\beta \delta_k \sqrt{k}} \leq \sqrt{1 \pm \varepsilon x \pm \delta_1} \right) = \Phi(\sqrt{1 \pm \varepsilon x \pm \delta_1}) \quad \text{a.s.,} \quad (16) \]
\[ \lim_{n \to \infty} \frac{1}{D_n} \sum_{k=1}^{n} d_k I \left( \bigcup_{i=1}^{k} \{|X_i - \mu| > \sqrt{k}\} \right) = 0 \quad \text{a.s.,} \quad (17) \]
\[ \lim_{n \to \infty} \frac{1}{D_n} \sum_{k=1}^{n} d_k I \left( \bar{V}_k^2 > (1+\varepsilon)k\delta_k^2 \right) = 0 \quad \text{a.s.,} \quad (18) \]
\[ \lim_{n \to \infty} \frac{1}{D_n} \sum_{k=1}^{n} d_k I \left( \bar{V}_k^2 < (1-\varepsilon)k\delta_k^2 \right) = 0 \quad \text{a.s.} \quad (19) \]

Firstly, we prove (16), by \( E(Y^2) < \infty \), we know \( \lim_{x \to \infty} x^2 P(|Y| > x) = 0 \), and by \( E(Y) = 0 \),

it follows that

\[ |E(T_{ki})| = \left| E \left( \sum_{i=1}^{k} \tilde{Y}_{ki} \right) \right| = |kE\tilde{Y}_{k1}| \]
\[ \leq k |E(Y1(|Y| > \sqrt{k}))| + k^2 E(|Y| > \sqrt{k})) \]
\[ \leq \sqrt{k} E(Y1(|Y| > \sqrt{k})) + k^2 P(|Y| > \sqrt{k}) = o(\sqrt{k}), \]
so, combining with $\delta_k^2 \to E(Y^2) < \infty$, for any $\alpha > 0$, when $k \to \infty$, we have

$$ I \left( \frac{T_{k,i} - ET_{k,i}}{\beta \delta_k \sqrt{k}} \leq \sqrt{1 \pm \varepsilon x \pm \delta_1 - \alpha} \right) $$

$$ \leq I \left( \frac{T_{k,i}}{\beta \delta_k \sqrt{k}} \leq \sqrt{1 \pm \varepsilon x \pm \delta_1} \right) $$

$$ \leq I \left( \frac{T_{k,i} - ET_{k,i}}{\beta \delta_k \sqrt{k}} \leq \sqrt{1 \pm \varepsilon x \pm \delta_1 + \alpha} \right), $$

thus, by (4), we get

$$ \lim_{n \to \infty} \frac{1}{D_n} \sum_{k=1}^{n} d_k I \left( \frac{T_{k,i}}{\beta \delta_k \sqrt{k}} \leq \sqrt{1 \pm \varepsilon x \pm \delta_1} \right) $$

$$ \geq \Phi(\sqrt{1 \pm \varepsilon x \pm \delta_1} - \alpha), $$

$$ \lim_{n \to \infty} \frac{1}{D_n} \sum_{k=1}^{n} d_k I \left( \frac{T_{k,i}}{\beta \delta_k \sqrt{k}} \leq \sqrt{1 \pm \varepsilon x \pm \delta_1} \right) $$

$$ \leq \Phi(\sqrt{1 \pm \varepsilon x \pm \delta_1 + \alpha}) \quad \text{a.s.,} \quad (20) $$

letting $\alpha \to 0$ in (20) and (21), (16) holds.

Now, we prove (17); by $E(Y^2) < \infty$, we know $\lim_{x \to \infty} x^2 P(|Y| > x) = 0$, such that

$$ \text{EI} \left( \bigcup_{i=1}^{k} (|Y_i| > \sqrt{k}) \right) \leq \sum_{i=1}^{k} P(|Y_i| > \sqrt{k}) \leq kP(|Y| > \sqrt{k}) \to 0, \quad k \to \infty, $$

by the Toeplitz lemma, we get

$$ \lim_{n \to \infty} \frac{1}{D_n} \sum_{k=1}^{n} d_k \text{EI} \left( \bigcup_{i=1}^{k} (|Y_i| > \sqrt{k}) \right) \to 0 \quad \text{a.s.,} \quad (22) $$

hence, to prove (17), it suffices to prove

$$ \lim_{n \to \infty} \frac{1}{D_n} \sum_{k=1}^{n} d_k \left( \text{I} \left( \bigcup_{i=1}^{k} (|Y_i| > \sqrt{k}) \right) - \text{E} \left[ \text{I} \left( \bigcup_{i=1}^{k} (|Y_i| > \sqrt{k}) \right) \right] \right) \to 0 \quad \text{a.s.,} \quad (23) $$

writing

$$ Z_k = \text{I} \left( \bigcup_{i=1}^{k} (|Y_i| > \sqrt{k}) \right) - \text{E} \left[ \text{I} \left( \bigcup_{i=1}^{k} (|Y_i| > \sqrt{k}) \right) \right], $$
for every $0 \leq 2k < l$, so by the definition of $\rho^-$-mixing sequence, we have

$$E_{|\mathcal{Z}_n| \mathcal{Z}_n}| = \left| \text{Cov}\left( \mathbf{1}\left( \bigcup_{i=1}^{k} (|Y_i| > \sqrt{k}) \right), \mathbf{1}\left( \bigcup_{i=1}^{l} (|Y_i| > \sqrt{l}) \right) \right) \right|$$

$$\leq \left| \text{Cov}\left( \mathbf{1}\left( \bigcup_{i=1}^{k} (|Y_i| > \sqrt{k}) \right), \mathbf{1}\left( \bigcup_{i=1}^{l} (|Y_i| > \sqrt{l}) \right) - \mathbf{1}\left( \bigcup_{i=2k+1}^{l} (|Y_i| > \sqrt{l}) \right) \right) \right|$$

$$+ \left| \text{Cov}\left( \mathbf{1}\left( \bigcup_{i=1}^{k} (|Y_i| > \sqrt{k}) \right), \mathbf{1}\left( \bigcup_{i=2k+1}^{l} (|Y_i| > \sqrt{l}) \right) \right) \right|$$

$$\leq E\left| \mathbf{1}\left( \bigcup_{i=1}^{l} (|Y_i| > \sqrt{l}) \right) - \mathbf{1}\left( \bigcup_{i=2k+1}^{l} (|Y_i| > \sqrt{l}) \right) \right|$$

$$+ \rho^-(k) \sqrt{\text{Var}\left( \mathbf{1}\left( \bigcup_{i=1}^{k} (|Y_i| > \sqrt{k}) \right) \right) \text{Var}\left( \mathbf{1}\left( \bigcup_{i=2k+1}^{l} (|Y_i| > \sqrt{l}) \right) \right)}$$

$$\leq E\left[ \mathbf{1}\left( \bigcup_{i=1}^{2k} (|Y_i| > \sqrt{l}) \right) \right] + C \rho^-(k)$$

$$\leq \sum_{i=1}^{k} P(|Y_i| > \sqrt{l}) + C \rho^-(k)$$

$$\leq k P(|Y| > \sqrt{l}) + C \rho^-(k)$$

$$\leq C \left( \frac{k}{l} + \rho^-(k) \right),$$

so by Lemma 2.4, (23) holds. And combining with (22), we know that (17) holds.

Next, we prove (18); by $E(\bar{V}_{k}^2) = k \delta_{k}^2$, $\bar{V}_{k}^2 = \bar{V}_{k,1}^2 + \bar{V}_{k,2}^2$, $E(\bar{V}_{k,1}^2) = k \delta_{k,1}^2$, and $\delta_{k,1} \leq \delta_{k}^2$, $l = 1, 2$, we have

$$1(\bar{V}_{k}^2 > (1 + \varepsilon) k \delta_{k}^2) = 1(\bar{V}_{k}^2 - E(\bar{V}_{k}^2) > \varepsilon k \delta_{k}^2)$$

$$\leq 1(\bar{V}_{k,1}^2 - E(\bar{V}_{k,1}^2) > \varepsilon k \delta_{k,1}^2) + 1(\bar{V}_{k,2}^2 - E(\bar{V}_{k,2}^2) > \varepsilon k \delta_{k,1}^2)$$

$$\leq 1(\bar{V}_{k,1}^2 > \left( 1 + \frac{\varepsilon}{2} \right) k \delta_{k,1}^2) + 1(\bar{V}_{k,2}^2 > \left( 1 + \frac{\varepsilon}{2} \right) k \delta_{k,2}^2),$$

therefore, by the arbitrariness of $\varepsilon > 0$, to prove (18), it suffices to prove

$$\lim_{n \to \infty} \frac{1}{D_n} \sum_{k=1}^{n} d_{k,1} 1(\bar{V}_{k,1}^2 > \left( 1 + \frac{\varepsilon}{2} \right) k \delta_{k,1}^2) = 0 \quad \text{a.s. } l = 1, 2,$$  \hfill (24)

when $l = 1$, for given $\varepsilon > 0$, let $f$ be a bounded function with bounded continuous derivative such that

$$1(x > 1 + \varepsilon) \leq f(x) \leq 1\left( x > 1 + \frac{\varepsilon}{2} \right),$$  \hfill (25)
under the condition
\[ E(\bar{V}_{k,1}^2) = k\delta_{k,1}^2, \quad E(Y^2) < \infty, \quad E(Y^2 I(Y \geq 0)) > 0, \]

by the Markov inequality, and Lemma 2.2, we get
\[
P\left(\bar{V}_{k,1}^2 > \left(1 + \frac{\epsilon}{2}\right) k\delta_{k,1}^2\right)
= P\left(\bar{V}_{k,1}^2 - E(\bar{V}_{k,1}^2) > \frac{\epsilon}{2} k\delta_{k,1}^2\right)
\leq C \frac{E(\bar{V}_{k,1}^2 - E(\bar{V}_{k,1}^2))^2}{k^2}
\leq C \frac{\sum_{i=1}^{k} E(\bar{Y}_{k,1}^2 I(\bar{Y}_{k,1} \geq 0))^2}{k^2}
\leq C \frac{E\left(\sum_{i=1}^{k} E(\bar{Y}_{k,1}^2 I(\bar{Y}_{k,1} \geq 0))^2\right)}{k}
\leq C \frac{E(\bar{Y}_{k,1}^4 I(\bar{Y}_{k,1} \geq 0))}{k}
\leq C \frac{E(\bar{Y}_{k,1}^4 I(0 \leq \bar{Y} \leq \sqrt{k}))}{k}
\leq C \frac{E(\bar{Y}_{k,1}^4 I(Y > \sqrt{k}))}{k},
\]

because \( E(Y^2) < \infty \) implies \( \lim_{x \to \infty} x^2 P(|Y| > x) = 0 \), we have
\[
EY^4 I(0 \leq Y \leq \sqrt{k}) = \int_{0}^{\infty} P(|Y| I(0 \leq Y \leq \sqrt{k}) \geq t) t^3 dt
\leq C \int_{0}^{\sqrt{k}} P(|Y| \geq t) t^3 dt
= \int_{0}^{\sqrt{k}} o(1) t dt = o(1),
\]

thus, combining with (26),
\[
P\left(\bar{V}_{k,1}^2 > \left(1 + \frac{\epsilon}{2}\right) k\delta_{k,1}^2\right) \to 0, \quad k \to \infty.
\]

Therefore, from (5), (25) and the Toeplitz lemma
\[
0 \leq \frac{1}{D_n} \sum_{k=1}^{n} d_k I\left(\bar{V}_{k,1}^2 > \left(1 + \frac{\epsilon}{2}\right) k\delta_{k,1}^2\right)
\leq \frac{1}{D_n} \sum_{k=1}^{n} d_k f\left(\bar{V}_{k,1}^2 / k\delta_{k,1}^2\right)
= \frac{1}{D_n} \sum_{k=1}^{n} d_k E\left(f\left(\bar{V}_{k,1}^2 / k\delta_{k,1}^2\right)\right)
\leq \frac{1}{D_n} \sum_{k=1}^{n} d_k E\left(1\left(\bar{V}_{k,1}^2 > \left(1 + \frac{\epsilon}{2}\right) k\delta_{k,1}^2\right)\right)
+ \frac{1}{D_n} \sum_{k=1}^{n} d_k E\left(\bar{V}_{k,1}^2 / k\delta_{k,1}^2\right)
\leq \frac{1}{D_n} \sum_{k=1}^{n} d_k E\left(\bar{V}_{k,1}^2 > \left(1 + \frac{\epsilon}{2}\right) k\delta_{k,1}^2\right)
+ \frac{1}{D_n} \sum_{k=1}^{n} d_k E\left(\bar{V}_{k,1}^2 / k\delta_{k,1}^2\right)
\to 0 \quad a.s., \quad k \to \infty,
\]

hence, (24) holds for \( l = 1 \). Similarly, we can prove (24) for \( l = 2 \), so (18) is true. By similar methods used to prove (18), we can prove (19), this completes the proof of Theorem 1.
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All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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