First betti numbers of Kähler manifolds with weakly pseudoconvex boundary

Brian Weber

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1 Introduction

We use certain connections between pseudoconvexity and harmonic function theory to obtain topological constraints on both a Kähler manifold and its boundary, if the boundary is weakly pseudoconvex. A boundary component $L \subset \partial K$ of a complex manifold $(K, J)$ is pseudoconvex (or weakly pseudoconvex) if it has a plurisuperharmonic defining function, meaning a differentiable defining function $f$ with $\sqrt{-1} \partial \overline{\partial} f \geq 0$, and is strongly pseudoconvex if $\sqrt{-1} \partial \overline{\partial} f > 0$. We consider aspects of both the real and complex geometry of $K$, so it will be convenient to use $-\frac{1}{2} dJd\phi = \sqrt{-1} \partial \overline{\partial} \phi$ for functions $\phi$.

A Green’s function on a complete manifold will mean any function $G$ defined on the compliment of a point $x$ so that $\triangle G = -\delta_x$ in the distributional sense. A complete manifold is called non-parabolic if it admits a Green’s function that is bounded on one side, and parabolic otherwise. This definition applies to manifolds-with-boundary, assuming the boundary is compact, by requiring, for instance, Neumann boundary conditions; therefore ends of manifolds (connected unbounded domains with compact boundary) may themselves be referred to as parabolic or non-parabolic. It is known that a complete manifold with one non-parabolic end is non-parabolic.

The theory of non-parabolic ends can be applied to Kähler manifolds-with-boundary assuming the pseudoconvexity of its boundary: the Kähler metric on a neighborhood $U$ of a pseudoconvex boundary component can be extended to make a complete end, by choosing an appropriate potential function. Any such metric is non-parabolic in a strong sense; in the terminology of Section 2 an end formed this way is distinguishable.

Positive, bounded, non-constant harmonic functions exist on Riemannian manifolds with at least two non-parabolic ends (eg. [12]), so after extending the pseudoconvex boundaries of $K$ we obtain bounded, non-constant harmonic functions. A simple argument shows that an harmonic function $h$ obtained in this way is actually pluriharmonic, meaning $\partial \overline{\partial} h = 0$. A particular consequence is that all boundary components have a defining func-
tion with zero Levi form. In addition, $\partial h$ clearly carries non-zero Dolbeault cohomology in $H^{1,0}(K)$. Less trivially, we assert that $Jdh$ carries non-trivial de Rham cohomology.

Throughout, we shall make the following assumption about our manifolds:

$$(K^m, J, \omega_0)$$ is a Kähler manifold-with-boundary of complex dimension $m$, with $n$ many non-parabolic ends (possibly $n = 0$ or $\infty$). If $\{L_i\}_{i=1}^l$ are its boundary components (possibly $l = \infty$), then each $L_i$ is compact, smooth, and has a defining function $f_i$ defined in a neighborhood $U_i$ of $L_i$ so that $\sqrt{-1} \partial \bar{\partial} f_i \geq 0$ on $U_i$. We require the $U_i$ be disjoint, and that a constant $\epsilon > 0$ exist so that each $f_i$ satisfies $|df_i| > \epsilon$, and so that $U_i$ contains an $\epsilon$-tubular neighborhood around $L_i$.

In short, our manifolds are weakly pseudoconvex, with compact boundary components, and, when there are infinitely many boundary components, a uniformity property on the gradients of the defining functions and on the sizes of their domains of definition. Our main results are that the topology of $K$, the topology of the $L_i$, and the CR structure of the $L_i$ have some constraints. Our main technical result is the following:

**Proposition 1.1** (cf. Proposition 2.4) Assume $(K, J, \omega_0)$ satisfies ($\ast$) and $l + n \geq 2$. Then for each $i \leq l$, a non-constant pluriharmonic function $h_i : K \to [0, 1]$ exists with $h_i = 1$ on $L_i$ and $h_i = 0$ on $L_j$ for $j \neq i$.

A real-valued function $f$ is called pluriharmonic if $\partial \bar{\partial} f = 0$. This depends on the complex structure only. It is noteworthy that a pluriharmonic function is harmonic with respect to any compatible Kähler metric. The existence of these $h_i$ can be used to prove that some cohomology classes are non-trivial. Let $b^{p,q}(K) = \dim \mathbb{C} H^{p,q}_{DB}(K)$ and $b^p(K) = \dim \mathbb{R} H^p_{DR}(K)$ be the dimensions of the respective Dolbeault and de Rham groups.

**Theorem 1.2** (cf. Theorem 3.5) Assume $(K, J, \omega_0)$ satisfies ($\ast$) and that $l \geq 1$. Then $b^{1,0}(K) \geq l - 1$ and $b^1(K) \geq l - 1$. If in addition $n \geq 1$, then $b^{1,0}(K) \geq l$ and $b^1(K) \geq l$.

**Theorem 1.3** (cf. Theorem 3.6) Assume $(K, J, \omega_0)$ satisfies ($\ast$) and $l \geq 1$. If $l + n \geq 2$, then each pseudoconvex boundary component $L_i$ has $b^1(L_i) \geq 1$, and has a pluriharmonic defining function.

In the case of complex dimension 1, Theorem 1.3 is obvious and Theorem 1.2 is not much more difficult. Of course if $K$ is any complex 1-manifold with non-trivial boundary then $H^{1,0}(K) \neq 0$, since any non-constant harmonic function $h$ provides a representative (namely $\partial h$) of a non-trivial $H^{1,0}$ class. Less trivially, Theorem 1.2 says $\dim H^1(K) \geq l - 1$, although in dimension 1 this can be proved with a relative homology sequence.

Nevertheless it is instructive to see how the proofs of Theorems 1.2 and 1.3 work in dimension 1, as the general case is no more difficult once Proposition 1.1 is accepted. Let $K$
be a compact, complex 1-manifold with smooth boundary components \( \{ L_i \}_{i=1}^l \) (these are automatically pseudoconvex). Let \( h_i \) be the harmonic function with \( h_i = 1 \) on \( L_i \) and \( h_i = 0 \) on \( L_j \) when \( j \neq i \). In the 1-dimensional case we have \( 2\sqrt{-1} \partial \bar{\partial} f = \Delta f \) for functions \( f \), so harmonic functions are pluriharmonic. Since also \( -dJdf = 2\sqrt{-1} \partial \bar{\partial} f \), each of the 1-forms \( Jdh_i \) represents a class in \( H^1(K) \).

To prove that this class is non-trivial, assume on the contrary that a function \( f_i \) exists with \( -Jdh_i = df_j \). A computation shows that the function \( z_i = h_i + \sqrt{-1} f_i \) is holomorphic, and sends \( K \) to the strip \( \{ 0 \leq Re(z_i) \leq 1 \} \subset \mathbb{C} \). The boundary of \( K \) is mapped to the union of lines \( \{ Re(z_i) = 0 \} \cup \{ Re(z_i) = 1 \} \), and by the open mapping theorem the image of \( z_i \) has no other boundary. However \( z_i \) has no poles (as \( dh_i \) and therefore \( dz_i \) are bounded), so the image of \( z_i \) in \( \mathbb{C} \) is compact, has non-empty interior, and has boundary in \( \{ Re(z_i) = 0 \} \cup \{ Re(z_i) = 1 \} \). This is an impossibility, so therefore \( 0 \neq [\bar{J}dh_i] \in H^1(K) \). For Theorem 1.3, simply note that by restricting \( Jdh_i \) to a collar neighborhood of \( L_i \) and applying the same argument, we obtain a non-trivial class in \( H^1([0,\epsilon] \times L_i) \approx H^1(L_i) \).

We present a few corollaries of our main theorems.

**Corollary 1.4** Assume \((K,J,\omega_0)\) satisfies \((\ast)\) and has one boundary component \(L\) with \( b^1(L) = 0 \). Then \( L \) is the only boundary component of \( K \), all ends of \((K,J,\omega_0)\) are parabolic, and \( b^1(K) = 0 \). If \((K,J,\omega_0)\) has no parabolic ends, then \( b^{2m-1}(K) = 0 \). If \( K \) has complex dimension 2 and no parabolic ends, then \( \chi(K) \geq 1 \).

**Pf** If a non-parabolic end exists or if another pseudoconvex boundary component exists, Theorem 1.3 implies \( b^1(L) > 0 \), a contradiction. If \( b^1(K) \neq 0 \), we can pass to the universal cover \( \tilde{K} \) of \( K \), with the lifted Kähler structure \((\tilde{J},\tilde{\omega}_0)\). It is easily seen that \( \tilde{K} \) continues to satisfy \((\ast)\). Since \( \pi_1(L) \) is finite, each component of the pre-image of \( L \) is compact. Therefore the pre-image of \( L \) has infinitely many components, each of which is compact. Letting \( \tilde{L} \subset \partial \tilde{K} \) be one of these components, Theorem 1.3 applied to \( \tilde{K} \) implies that \( b^1(\tilde{L}) > 0 \), again an impossibility, proving that \( b^1(K) = 0 \).

Finally assume \( K \) has no parabolic ends; then \( K \) has no ends. Poincaré duality gives \( H^1(K) \approx H^2m-1(K,L) \) and \( H^2m-2(L) \approx H^1(L) = \{ 0 \} \), so the relative homology sequence gives \( H^{2m-1}(K,L) \approx H^{2m-1}(K) \). Therefore \( b^{2m-1}(K) = b^{2m-1}(K,L) = b^1(K) = 0 \). In the 2-dimensional case this means \( b^1(K) = b^3(K) = 0 \), so

\[
\chi(K) = 1 - b^1(K) + b^2(K) - b^3(K) = 1 + b^2(K) \geq 1.
\] (1)

\[\square\]

An end of a Riemannian manifold is called asymptotically locally Euclidean (ALE) if it is diffeomorphic to a quotient of \( \mathbb{R}^k \setminus B(1) \) by a finite subgroup of \( O(k) \) (or of \( U(k)/2 \) in the Kähler case), and so that \(|\text{Rm}| = o(r^{-2})\) where \( r \) is the distance to some fixed point. Theorem 1.3 can be used to show that a Kähler manifold of complex dimension at least 2 (whether it is of finite type or not) that has an ALE end has only one ALE end. We
point out that this is also implied by the statement of Theorem 4.2 of [12], and, assuming
\( K \) has finite type, by the theorems of Kohn-Rossi [9] and Kohn [7] that are discussed in the
remarks below.

**Corollary 1.5** Assume \( (K, J, \omega_0) \) has complex dimension at least 2, satisfies (*)
and has an ALE end. Then every other end of \( K \) is parabolic, and \( H^1(K) = 0 \). If \( K \) has no
parabolic ends, then \( H^{n-1}(K) = 0 \). If \( K \) has no parabolic ends and complex dimension 2,
then \( \chi(K) \geq 1 \).

**Proof** We can assume \( K' \subset K \) is an ALE end so that the boundary of \( K \setminus K' \) is diffeomorphic to a quotient of an \( (n-1) \)-sphere and is geometrically locally convex, and therefore pseudoconvex. Corollary [1.4] applied to \( K \setminus K' \) then provides the conclusion. □

**Remark.** The effect of boundary pseudoconvexity on cohomology groups has been
studied extensively. Hilbert space methods were developed in Kohn [7] [8], Andreotti-Vesentini [1],
and Hörmander [4] for the purpose of solving \( \overline{\partial} \)-Neumann problems and non-
homogeneous \( \partial \)-problems. One result was a proof that the pseudoconvexity of subdomains
of \( \mathbb{C}^n \) or of Stein manifolds gives rise to strong cohomological vanishing theorems. In addition,
a Hodge decomposition on compact complex manifolds-with-boundary holds in a given
bidegree, provided the boundary satisfies a certain pseudoconvexity condition (which in any
bidegree is implied by strong pseudoconvexity; see [9]). That is, if \( \Delta \) is the \( \partial \)-Laplacian,
there is a compact operator \( G \) so that
\[
\bigwedge^{p,q} = \Delta \circ G \left( \bigwedge^{p,q} \right) \oplus \mathcal{H}^{p,q},
\]
where \( \bigwedge^{p,q} \) is the space of \( C^\infty \) forms of the indicated bidegree, and \( \mathcal{H}^{p,q} \)
is the space of harmonic \((p,q)\)-forms.

The hypotheses of the present theorems differ from those in [7], [8], and [9] in that we
require just non-negativity of eigenvalues of the Levi form instead of positivity, and we do
not require that the closure of the manifold be compact, although we require the manifold
be Kähler rather than Hermitian.

**Remark.** Among other uses, Kohn-Rossi [10] used the Hodge decomposition \( \bigwedge^{p,q} \) to solve
a number of boundary value problems, one of which was the following: if \( K \) is a compact
complex manifold whose boundary satisfies an appropriate convexity condition (the Levi
form has everywhere one positive or \( n \) negative eigenvalues), if \( f \) is a function on \( \partial K \) that
satisfies a certain compatibility condition (namely that \( \overline{\partial}_b f = 0 \), where \( \overline{\partial}_b \)
is the restriction of the \( \overline{\partial} \)-operator to the boundary), and if \( f \) is orthogonal to the restriction of \( \mathcal{H}^{n,n-1} \)
to the boundary, then \( f \) is the restriction to \( \partial K \) of a holomorphic function \( F \) on \( K \). As a corollary,
they proved that a compact Hermitian manifold, all of whose boundary components are
strictly pseudoconvex, actually has a connected boundary.

To see this with the Kohn-Rossi method, first note that strict pseudoconvexity implies
\( \mathcal{H}^{n,n-1} \) is finite dimensional, by the previous remark. The existence of a single non-constant
holomorphic function on \( A \) is implied by Theorem 9.1 of [7] (see also [3]). By taking powers
of this function, we see that the vector space of holomorphic functions in infinite dimensional. Consider the subspace \( \{ A^i \}_{i=0}^{\infty} \) spanned by powers of \( A \). To any bounded function \( F = c_i A^i \), construct the function \( g : \partial M \to \mathbb{C} \) by multiplying \( F|_{\partial M} \) by different constants on each component of \( \partial M \). By the finite-dimensionality of \( \mathcal{H}^{n,n-1} \), we can choose the \( c_i \) so that \( g \) is orthogonal to the restriction of \( \mathcal{H}^{n,n-1} \) to \( \partial M \). Since also \( \bar{\partial} g = 0 \), we can extend \( g \) to a holomorphic function \( G \) on \( K \). Then \( G/F \) will be a meromorphic function that is locally constant on the boundary, and therefore constant by unique continuation. This is impossible unless the boundary is connected.

**Organization.** In section 2 we review the material on harmonic function theory used in the proof of Proposition 1.1. Our main concern is with exactly how harmonic functions with 2-sided bounds are constructed—the method is termed compact exhaustion and appears for instance in [11] and [12]. We also introduce the useful notion of distinguishability, which is a strengthened form of non-parabolicity. We conclude with an example showing that distinguishability is strictly stronger than non-parabolicity. Section 3 contains the proofs of Proposition 1.1 and of Theorems 1.2 and 1.3.

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## 2 Parabolic and non-parabolic ends of Riemannian manifolds

The literature on harmonic function theory on complete manifolds is very large. Here we review some well-known results that will be useful later, and introduce the notion of the **distinguishability** of an end, which means, roughly speaking, that the end can be separated from the rest of the manifold by a bounded harmonic function. We show that this notion is strictly stronger than non-parabolicity. In this section we are concerned only with Riemannian, not Kähler, structures.

A function \( G_x : M \to \mathbb{R} \) is called a Green’s function at the point \( x \) if \( \triangle G_x = -\delta_x \) in the sense of distributions. A complete manifold is called parabolic if it admits no positive Green’s function, and non-parabolic otherwise. These definitions are equally good on manifolds with compact boundary, with Green’s functions made to satisfy Neumann conditions on boundary components. With \( c_{n-1} \) the area of the unit \((n-1)\)-sphere, the Green’s functions (at the

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1 Any opinions, findings and conclusions or recommendations in this material are those of the author(s) and do not necessarily reflect the view of the National Science Foundation.
origin) of the flat manifolds $\mathbb{R}^n$ are
\[ G_x(y) = \begin{cases} \frac{-1}{2\pi} \log |x - y| & \text{if } n = 2 \\ \frac{1}{(n-2)c_{n-1}} |x - y|^{2-n} & \text{if } n \neq 2. \end{cases} \]
(3)

Therefore $\mathbb{R}^n$ is parabolic when $n = 2$ and non-parabolic when $n > 2$.

Related to parabolicity is the notion of capacity. Given any set $\Omega \subset M$ with compact closure, we define its capacity $\text{Cap}(\Omega)$ to be an infimum of Dirichlet integrals:
\[ \text{Cap}(\Omega) = \inf_{\varphi} \int_M |\nabla \varphi|^2 \]
(4)
where the infimum is over all $\varphi \in C^{0,1}_c(M)$ with $\varphi \geq 1$ on $\Omega$. Assuming $\Omega$ is a smooth domain and $\text{Cap}(\Omega) > 0$, the infimum is obtained by a Lipschitz function $\varphi$ with $\varphi = 1$ on $\Omega$, $\Delta \varphi = 0$ outside $\Omega$, and $\varphi \to 0$ along some (but not necessarily every) sequence of points that diverges to infinity. If $\text{Cap}(\Omega) = 0$, a minimizing sequence will converge to a constant function. The connection between capacity and parabolicity is the following proposition, which can be found for instance in [5], and also follows from (2) of Proposition 1.2 in [13].

**Proposition 2.1** A Riemannian manifold $(M, g)$ is non-parabolic if and only if it has a sub-domain with compact closure and positive capacity.

A geometric phenomenon totally absent on $\mathbb{R}^n$, $n \neq 1$, is the possibility of separating unbounded sets with domains of compact closure. If $\Omega$ is a domain with pre-compact boundary, we call any unbounded component of $M \setminus \Omega$ an end of $M$ with respect to $\Omega$. We shall refer to a connected, unbounded subset $M'$ as an end if $\partial M'$ is compact, and usually leave the domain $\Omega$ implicit. With this terminology, any open, complete manifold is an end, and it is possible that and end may have two or more non-intersecting subsets that are themselves distinct ends. This terminology is therefore somewhat imprecise, but is sufficient for our purposes.

Capacity, and therefore parabolicity and non-parabolicity, can be understood to be a property of an end. If $M'$ is an end with non-empty boundary, we define its capacity to be
\[ \text{Cap}(M') = \inf_{\varphi} \int_{M'} |\nabla \varphi|^2 \]
(5)
where the infimum is taken over $C^{0,1}(M')$ functions $\varphi$ of compact support with $\varphi = 1$ on $\partial M'$. If $\partial M'$ is smooth and $\text{Cap}(M') > 0$, the capacity is realized by some harmonic $C^\infty_0(M')$ function $\varphi$ with $\varphi = 1$ on the boundary. We call end end $M'$ non-parabolic if $\text{Cap}(M') > 0$, and parabolic if $\text{Cap}(M') = 0$. Clearly a manifold is non-parabolic if it has one non-parabolic end. There is also the the following alternative characterization (cf. Proposition 1.2 of [13]).
Lemma 2.2  An end $M'$ of $M$ is non-parabolic if and only if a superharmonic function $f$ exists on $M'$ with \( \inf_{\partial M'} f > 0 \) and $f \to 0$ along some sequence of points that diverges to infinity.

Proof The proof here is similar to that in [13]; we go through it because some details will be used later. First assume the stated function $f$ exists. After multiplying by \( (\inf_{\partial M'} f)^{-1} \) we can assume $f \geq 1$ on $\partial M'$. Let $\varphi_i$ be a minimizing sequence for $f$. A simple argument (which we omit) states that we can replace $\varphi_i$ by $\min \{ 1, \varphi_i \}$ to obtain a function with smaller Dirichlet integral, and that we can replace $\varphi_i$ with a harmonic function with the same boundary values, and also obtain a function with strictly smaller Dirichlet integral. Therefore we assume the $\varphi_i$ are harmonic, have compact support, satisfy $\varphi = 1$ on $\partial M'$, and that $\Omega_i = \text{supp} \varphi_i$ is an exhaustion of $M'$ by compact sets. Since $f$ is superharmonic and $f \geq \varphi_i$ on each $\partial \Omega_i$, we have $\varphi_i \leq f$. Since $0 \leq \varphi_i \leq 1$, a subsequence will converge to a harmonic function $\varphi$, and we retain $f \geq \varphi$. The Dirichlet integrals $\int |\nabla \varphi_i|^2$ decrease monotonically and converge to $\int |\nabla \varphi|^2$, so that $\varphi$ is non-constant and has a finite (but non-zero) Dirichlet integral.

Conversely, assume $M'$ is non-parabolic. We may assume $M'$ has a smooth boundary, as shrinking $M'$ increases its capacity. Letting $\Omega_i$ be a compact exhaustion of $M'$ so that $\partial M' \subset \partial \Omega_i$, let $\varphi_i$ be harmonic functions with $\varphi_i = 1$ on $\partial M'$ and $\varphi_i = 0$ on $\partial \Omega_i \setminus \partial M'$. We have $\int_{\partial M'} \frac{\partial \varphi_i}{\partial n} = \int_{M'} |\nabla \varphi_i|^2 > \text{Cap}(M') > 0$, and (essentially by the Hopf lemma) that $\int |\nabla \varphi_i|^2$ decreases monotonically. Defining $\varphi = \lim_i \varphi_i$, we have $\int_{M'} |\nabla \varphi|^2 \geq \text{Cap}(M')$, so that $\varphi$ is non-constant, harmonic, and bounded above by 1. By these properties and because $\varphi = 1$ on $\partial M'$, $\varphi$ obtains a strict minimum at infinity. We wish to prove $\inf_{M'} \varphi = 0$. Setting $\epsilon = \inf \varphi$, then $\tilde{\varphi} = \frac{\varphi}{\epsilon}$ is a harmonic function equal to 1 on the boundary, and $\tilde{\varphi} \to 0$ along some subsequence that diverges to $\infty$. Using $\tilde{\varphi}$ as a barrier and following the argument of the previous paragraph, we have that actually $\varphi = \lim_i \varphi_i \leq \tilde{\varphi}$, which means $\inf \varphi = 0$. \(\square\)

We shall call an end $M'$ distinguishable if a positive harmonic function $\varphi$ exists on $M'$ with $\varphi = 1$ on $\partial M'$ and $\varphi \to 0$ along every sequence of points in $M'$ that diverges to $\infty$. The following lemma is essentially obvious.

Lemma 2.3  An end $M'$ of a manifold is distinguishable if and only if there is a positive superharmonic function $f : M' \to \mathbb{R}$ with $\inf_{\partial M'} f > 0$ and so that $f \to 0$ along every sequence of points in $M'$ that diverges to infinity.

Proof This follows after the constructing a harmonic function $\varphi$ as in Lemma 2.2, by noting that (after possibly multiplying $f$ by a constant), we have $f \geq \varphi > 0$. \(\square\)

The importance of distinguishability comes from the following lemma, which states that, on a manifold-with-boundary with compact boundary, distinguishable ends can be separated from the rest of the manifold with harmonic functions, provided at least one other non-parabolic end exists. The first assertion in the following proposition is well known (eg. [13]). The second assertion is new.
Proposition 2.4 (Separation of distinguishable ends) Assume $(M, g)$ is a smooth Riemannian manifold-with-boundary, with smooth boundary. If $M$ has at least two non-parabolic ends, then there exists a non-constant harmonic function $\varphi : M \to \mathbb{R}$ with $0 < \varphi < 1$. If, in addition, $M'$ is a distinguishable non-parabolic end, a number $\delta' > 0$ can be chosen so that if $\delta \in (0, \delta')$ and $\Omega_{\delta} \equiv \{ \varphi > 1 - \delta \}$, we have that $\Omega_{\delta} \subset M'$ and that $M' \setminus \Omega_{\delta}$ has compact closure.

**Proof** The method for proving the first assertion is compact exhaustion. Namely let $f'$, $f''$ be the superharmonic barrier functions on the non-parabolic ends $M'$, $M''$, respectively, that are guaranteed by Lemma 22 or 23. Let $M_i$ be an exhaustion of $M$ by smooth, pre-compact domains, each of which separates $M'$ and $M''$. If $\partial M$ is non-empty, assume $\partial M \subset M_i$. Let $\varphi_i : M_i \to \mathbb{R}$ be the harmonic function with $\varphi_i = 1$ on $(\partial M_i \cap \partial M') \setminus \partial M$, $\varphi_i = 0$ on $\partial M_i \setminus M'$, and $\varphi_i$ satisfies von Neumann conditions on $\partial M$. Set $\varphi = \lim_i \varphi_i$. Since $\varphi_i \geq 1 - f' \to f''$ on $M'$ and $\varphi_i \leq f''$ on $M''$, the same holds for $\varphi$; therefore $\varphi$ is not constant.

We can prove that on $M \setminus M'$, $\varphi$ is bounded strictly below 1, unless possibly $M'$ is the only non-parabolic end. By the maximum principle we have $\sup_{M \setminus M'} \varphi_i = \sup_{\partial M'} \varphi_i$. Taking $i \to \infty$, the same holds for $\varphi$. On the other hand, $\varphi \leq 1$ so the strong maximum principle implies that either $\varphi \equiv 1$ or else $\varphi < 1$ on $K$. By the compactness of $\partial M'$, we have either $\varphi \equiv 1$ on $K$ or else $\sup_{M \setminus M'} \varphi = \sup_{\partial M'} \varphi < 1 - \delta$ for all sufficiently small $\delta$.

Finally, assume $M'$ is distinguished. We can assume the upper barrier $f'$ satisfies $f'(x_i) \setminus 0$ along any sequence $x_i$ in $M'$ that diverges to infinity. Now choose the $M_i$ so that $f' < 2^{-i}$ on $M' \cap M_i$; clearly $M \setminus M_i$ is compact. Since $\varphi \geq 1 - f' > 1 - 2^{-i}$ on $M' \cap M_i$, we can choosing a large enough $i$ so that $1 - 2^{-i} > \sup_{\partial M'} \varphi$. Putting $\delta = 2^{-i}$, the set $\Omega_{\delta} = \{ \varphi > 1 - \delta \}$ is therefore a subset of $M'$. Clearly $M' \setminus \Omega_{\delta}$ is compact as $M_i \subset \Omega_{\delta} \subset M_j$ (strict inclusion) when $2^{-j} < \delta < 2^{-i}$.

We close this section with an example of an end that is non-parabolic but not distinguishable. Let $\mathbb{E}^2$ be $\mathbb{R}^2$ a flat metric $g_F$ and let $\mathbb{H}^2$ be $\mathbb{R}^2$ with a hyperbolic metric $g_H$. Then $\mathbb{E}^2$ is parabolic with Green’s function given by $\log \left( \sqrt{e-1} \right)$, where $r = \text{dist}(x, y)$. Let $x_i$ (resp. $y_i$) be a sequence of points in $\mathbb{E}^2$ (resp. $\mathbb{H}^2$) with $x_i \to \infty$ (resp. $y_i \to \infty$), and attach $\mathbb{E}^2$ to $\mathbb{H}^2$ by removing small balls $B_{x_i}(\delta_i/2)$ from $\mathbb{E}^2$ and $B_{y_i}(\delta_i/2)$ from $\mathbb{H}^2$ ($\delta_i$ is a sequence of positive numbers), and gluing the ends of a cylinder to each pair of corresponding boundary components. Label this manifold $(M, g)$, where $g$ is chosen so the metrics on $\mathbb{E}^2 \setminus \bigcup_i B_{x_i}(\delta_i)$ and $\mathbb{H}^2 \setminus \bigcup_i B_{y_i}(\delta_i)$ are unchanged.

The resulting manifold $(M, g)$ clearly has a single end. That $M$ is non-parabolic follows, for instance, from Theorem 2.1 of [4] with $p = 2$, after noting that the metric on the hyperbolic part of $M$ makes the volume growth of balls exponential.

Let $B_E$ be the unit ball about the origin on the Euclidean part of $M$ (we assume this does not intersect any of the $B_{x_i}(\delta_i)$), and let $M' = M \setminus B_E$ be the end with respect to $B_E$. We will construct a family of lower barrier functions $F_\eta$ with the property that any positive harmonic function $\varphi$ on $M$ with $\varphi \geq 1$ on $B_E$ has $\varphi \geq F_\eta$ when $\eta < \eta_0$, and then we shall
show that a limiting function \( F = \lim_{\eta \to \eta_0} F_\eta \) exists and that \( F_\eta \) is asymptotically nonzero along some diverging sequences.

Consider the following family of functions defined a.e. on \( \mathbb{E}^2 \):

\[
F_i(x) = 1 - \eta \log(|x|) + \sum_{a=1}^{i} c_a \log(|x-x_a|).
\]

If the \( c_a \) converge to zero fast enough (say \( c_a = a^{-2} \) and \( x_a \) has coordinates \((a,0)\)), then the sum converges. Note that \( \Delta F_\eta \) is zero aside from a delta function of weight \(-\eta\) at the origin and delta functions of decreasing but positive weights at the \( x_i \). Let \( D_\eta = \{ F_\eta > 0 \} \setminus B_E \).

This set is pre-compact when \( \eta > \eta_0 \equiv \sum_i c_i \). We can prove that whenever \( \eta > \eta_0 \), we have \( \varphi \geq 1 + \sum_i c_i \log(1 + |x_i|) \).

Let \( \omega = \omega_0 + dJ\varphi(f) \) (8) is positive if \( \varphi'' \leq 0 \) and \( \varphi' \geq 0 \). If \( \varphi' \) approaches infinity sufficiently quickly as \( t \to 0 \), the corresponding metric is complete. One obvious choice is \( \varphi(t) = \log(t) \); this gives a complete manifold with constant negative bisectional curvature at infinity. Another possibility is \( \varphi(t) = -t^{-\alpha}/\alpha \) for \( \alpha > 0 \); in this case the bisectional curvature decays to zero like \( O(r^{-2}) \), where \( r \) is the \( \omega \)-distance from a fixed point.

3 Pluriharmonic functions on Kähler manifolds with pseudoconvex boundary

It is known that a Kähler metric near a pseudoconvex boundary component can be made complete by the choice of an appropriate potential function—in fact this is a defining feature; see [15]. Let \( f \) be a positive defining function for the pseudoconvex boundary component \( L \subset K \). If \( \varphi : \mathbb{R}^+ \to \mathbb{R}^+ \) is a twice-differentiable function then

\[
dJ \varphi(f) = \varphi''(f) df \wedge Jdf + \varphi'(f) dJdf
\]

Note that \( df \wedge Jdf \) is always non-positive and that since \( f \) is pseudoconcave, \( dJdf \) is non-negative. Thus the form

\[
\omega = \omega_0 + dJ \varphi(f)
\] (8)

is positive if \( \varphi'' \leq 0 \) and \( \varphi' \geq 0 \). If \( \varphi' \) approaches infinity sufficiently quickly as \( t \to 0 \), the corresponding metric is complete. One obvious choice is \( \varphi(t) = \log(t) \); this gives a complete manifold with constant negative bisectional curvature at infinity. Another possibility is \( \varphi(t) = -t^{-\alpha}/\alpha \) for \( \alpha > 0 \); in this case the bisectional curvature decays to zero like \( O(r^{-2}) \), where \( r \) is the \( \omega \)-distance from a fixed point.
Lemma 3.1 Assume \((K, J, \omega_0)\) satisfies \((*)\) of the introduction. Then a positive smooth function \(f : K \to \mathbb{R}\) exists so that \(f\) is non-strictly plurisuperharmonic, and agrees with \(f_i\) on a neighborhood of \(L_i\). Further, for \(\alpha \geq 0\), the \((1,1)\)-form

\[
\omega = -\frac{1}{\alpha}dJ df^{-\alpha} + \omega_0
\]

is a Kähler form whose associated metric is complete near any boundary component \(L_i\) (when \(\alpha = 0\), we take \(\omega = -dJd \log f + \omega_0\)).

**Pf** Setting \(f_i = \infty\) where it was otherwise undefined, by \((*)\) there is a number \(\delta\) so that \(f = \inf\{\delta, f_1, \ldots, f_l\}\) is Lipschitz (even if \(l = \infty\)). When \(f < \delta\) then \(f\) is smooth and \(dJ df > 0\). Now we can smooth \(f\) in any way that leaves it unaffected on a neighborhood of each \(L_i\) by replacing \(f\) by a function \(\psi(f)\). We let \(\psi\) be a smooth increasing function with \(\psi(t) = t\) when \(t < \delta/4\), \(\psi(t) = 3\delta/8\) when \(t > \delta/2\), and \(\psi''(t) < 8/\delta\) when \(-\delta < t < -\delta/2\), then \(\psi(f)\) is smooth, (non-strictly) plurisuperharmonic, and agrees with each \(f_i\) on some neighborhood of \(L_i\), as desired.

Finally we show the resulting manifold is complete. Choose \(s, S\) so \(0 < s < S < \delta/2\), and let \(\gamma(t)\) be a path in \(\{s \leq f \leq S\}\) from a point in \(\{f = S\}\) to a point in \(\{f = s\}\). We have

\[
|\dot{\gamma}|^2 = \omega(\dot{\gamma}, J \dot{\gamma}) = -(1 + \alpha)f^{-2-\alpha}(df \wedge J df)(\gamma, J \gamma) + f^{-1-\alpha}(dJ df)(\gamma, J \gamma) + \omega_0(\gamma, J \gamma) \\
\geq (1 + \alpha)f^{-2-\alpha}(df(\dot{\gamma}))^2 = \frac{4(1 + \alpha)}{\alpha^2} \left(\frac{df}{dt}\right)^2.
\]

We can assume \(f \circ \gamma\) is \(C^1\) and decreasing, so the length of \(\gamma\) is estimated from below by

\[
\int |\dot{\gamma}| dt \geq \frac{2\sqrt{1 + \alpha}}{\alpha} \int_{f=S}^{f=s} \frac{df}{dt} dt = \frac{2\sqrt{1 + \alpha}}{\alpha} (s^{-\frac{4}{\alpha}} - S^{-\frac{4}{\alpha}}).
\]

If \(\alpha = 0\) the appropriate expression with logarithms is obvious. When \(\alpha \geq 0\), the length of \(\gamma\) therefore grows unboundedly as its terminal point approaches \(\partial K\) at \(s = 0\).

A real-valued function \(h\) is called pluriharmonic when \(\sqrt{-1} \partial \bar{\partial} h = 0\). This depends only on the complex structure, so, notably, a pluriharmonic function is harmonic with respect to any compatible Kähler metric. In this section we show that the existence of more than one pseudoconvex boundary component on \((K, J, \omega_0)\) allows the construction of pluriharmonic functions. We shall be careful to observe the distinction between the original metric \(g_0 = \omega_0(\cdot, J \cdot)\), and a choice of a complete metric \(g = \omega(\cdot, J \cdot)\) given by Lemma 3.1.

Any end that comes from a pseudoconvex boundary component is distinguishable. To see this, note that \(f_i > 0\) in \(U_i\) and \(f_i \nrightarrow 0\) along any sequence in \(U_i\) that diverges in the
For each \( h \) constructed: 
\[
\triangle_g f_i = -\frac{dJd_f^i \wedge \omega^{n-1}}{\omega^n} < 0
\]
so that \( f_i \) is superharmonic. Thus the end is distinguishable by Lemma 2.3.

**Proposition 3.2** Assume \((K, J, \omega_0)\) satisfies \((\ast)\) of the introduction. There exists a positive pluriharmonic function \( h_i \) on \((K, J, \omega_i)\) with \( h_i = 1 \) on \( L_i \) and \( h_i = 0 \) on \( L_j \) when \( j \neq i \). In particular, \( dJdh_i = d^*Jdh_i = 0 \).

**PF** For each \( i \) let \( V_i \) be a neighborhood of \( L_i \) so that \( V_i \) does not intersect any boundary component of \( K \) besides \( L_i \). By Propositions 3.1 and 2.4, a harmonic function \( h_i \) exists on \((K, J, \omega)\) that limits to 1 along any divergent sequence in \( V_i \) and limits to 0 along any unbounded sequence in \( V_i \) for all \( j \neq i \).

We first prove the Dirichlet integral of \( h_i \) is finite. To see this, recall how the \( h_i \) are constructed: \( h_i = \lim_{R \to \infty} h_{i, R} \) where \( h_{i, R} \) is the harmonic function on the large ball \( B_p(R) \) with \( h_{i, R} = 1 \) on \( \partial(B_p(R) \cap V_i) \) and \( h_{i, R} = 0 \) on \( \partial(B_p(R) \setminus V_i) \). Then

\[
\int_K |\nabla h_{i, R}|^2 = \int_{K \setminus V_i} |\nabla h_{i, R}|^2 + \int_{V_i} |\nabla (1 - h_{i, R})|^2
\]

\[
= \int_{\partial V_i} h_{i, R}\frac{\partial h_{i, R}}{\partial \hat{n}} - \int_{\partial V_i} (1 - h_{i, R})\frac{\partial (1 - h_{i, R})}{\partial \hat{n}}
\]

\[
= \int_{\partial V_i} \frac{\partial h_{i, R}}{\partial \hat{n}}
\]

where \( \hat{n} \) is the outward pointing normal of \( V_i \). Since \( \partial V_i \) is compact and since \( \partial h_{i, R}/\partial \hat{n} \) is uniformly bounded by the Cheng-Yau gradient estimate \([2]\), the Dirichlet integral \( \int |\nabla h_{i, R}|^2 \) is uniformly bounded. Since \( h_{i, R} \to h_i \) as \( R \to \infty \) in (at least) the \( C^1 \) sense, we have that \( \int |\nabla h_i|^2 \) is finite by Fatou’s lemma.

Let \( \langle \cdot, \cdot \rangle \) denote the \( L^2 \) inner product on a Riemannian manifold. If \( \eta \) is any \( p \)-form and \( \varphi \) is a \( C^\infty \) function, a computation gives

\[
|d\varphi \wedge \eta|^2 + |d^* \varphi \wedge \eta|^2 = \langle \varphi^2 \eta, \Delta \eta \rangle - 2 \langle d\varphi \wedge \eta, \varphi \wedge \eta \rangle + 2 \langle \iota_{d\varphi} \eta, \varphi \wedge \eta \rangle.
\]

If \( \eta \) is harmonic, then by replacing \( \varphi \) by \( \varphi^2 \) and using a Hölder inequality we easily conclude

\[
|d\varphi \wedge \eta|^2 + |d^* \varphi \wedge \eta|^2 \leq 4 |d\varphi \wedge \eta|^2 + 4 |\iota_{d\varphi} \eta|^2.
\]

It follows that if \( \eta \) is also bounded (or square-integrable, or increases like \( o(r^2) \)), then \( d\eta = d^* \eta = 0 \) (compare with \([10]\), Lemma 3.1). This is proved, in the usual way, by letting \( \varphi_k \) be a cutoff function with \( \varphi_k \equiv 1 \) in \( B_p(2^k) \), \( \varphi_k \equiv 0 \) outside \( B_p(2^{k+1}) \), and with \( |d\varphi_k| \leq 2^{-k+1} \), and then sending \( k \to \infty \).
Finally let $\eta = Jdh_i$. Above we proved that $|\eta|^2 = |dh_i|^2$ is integrable. In addition, we have $\Delta Jdh_i = J\Delta dh_i = 0$. This is due to the Kähler condition, and can be seen from the Bochner formula on 1-forms:

$$\Delta = -\Delta_g + \text{Ric}$$

(14)

(where $\Delta_g$ is the rough Laplacian), by noting that both $\Delta_g$ and Ric commute with $J$. Therefore we have proven that $dJdh_i = d^*Jdh_i = 0$. □

**Lemma 3.3** A function $h_i$ constructed above is non-constant provided that $(K,J,\omega_0)$ has more than one pseudoconvex boundary component, or has one pseudoconvex boundary component and at least one non-parabolic end.

**Pf** Obvious by construction. □

**Proposition 3.4** If the original manifold $(K,J,\omega_0)$ has only parabolic ends or has no ends, then $0 = \sum_i Jdh_i$. If $(K,J,\omega_0)$ has a non-parabolic end, there are no relations among the $Jdh_i$.

**Pf** Let $c = (c_1, \ldots, c_l)$ and consider the function $h_c = \sum_{i=1}^l c_i h_i$ on $(K, g_0)$. Since $h_c$ is harmonic and takes the value $c_i$ on $L_i$, it cannot be constant unless all the $c_i$ are equal. If there is a non-parabolic end $M'$ on the original manifold, a function $\varphi'$ on $M'$ exists with $\varphi \to 0$ along some subsequence of points in $M'$, and (by construction) we have $h_c < C\varphi'$ where $C = \sum c_i / \sup_{\partial M'} \varphi'$, so $h_c$ also converges to 0 along some sequence. If all the ends of $(K,J,\omega_0)$ are parabolic, then $h_c$ is constant when all the $c_i$ are equal. This can be seen by noting that $(K,g_0)$ is parabolic, so $\bigcup_i L_i$ has zero capacity, which implies that the function $h_c$, being 1 on $\bigcup_i L_i = \partial K$, must have zero Dirichlet integral. □

**Theorem 3.5** If $h_c = \sum_i c_i h_i$ is not constant and takes its maximum on $\partial K$, then $Jdh_c$ represents a non-trivial class in $H^1_{DR}(K)$.

**Pf** Recall that $h_i$ (if non-constant) distinguishes the boundary component $L_i$ in the sense that given any neighborhood $U_i$ of $L_i$, a number $\delta > 0$ exists so that $\{h_i > 1 - \delta\}$ is a neighborhood of $L_i$ contained in $U_i$. Since $h_c$ obtains its maximum on those $L_i$ for which $c_i = \sup_j \{c_j\}$, which we can take to be 1, by Proposition 2.4 there is a number $\delta$ so that some component of $\{h_c > 1 - \delta\}$ is a pre-compact neighborhood that is bounded away from all other boundary components of $K$.

For convenience, give the function $h_c$ the name $x$. For an argument by contradiction, suppose $x$ is exact, meaning a function $y : K \to \mathbb{R}$ exists with $dy = -Jdx$. Setting $z = x + \sqrt{-1}y$ and recalling that on functions we have $\bar{\partial} = \frac{1}{2} (d + \sqrt{-1}Jd)$, a computation gives $\bar{\partial} z = 0$, so $z : K \to \mathbb{C}$ is holomorphic. Let $V$ be a pre-compact component of $\{x \geq 1 - \delta\} \subset K$. The image of $V$ under $z$ lies in the strip $\{1 - \delta \leq x \leq 1\} \subset \mathbb{C}$.
By the open mapping theorem, the boundary of the image lies in the union of the lines 
\( \{x = 1 - \delta\} \cup \{x = 1\} \), meaning the image of \( \{x \geq 1 - \delta\} \subset \mathbb{C} \) under \( z \) is relatively open in 
\( \{1 - \delta \leq x \leq 1\} \subset \mathbb{C} \). By continuity the image is also closed, so the image of \( \{x \geq 1 - \delta\} \subset K \) is precisely \( \{1 - \delta \leq x \leq 1\} \subset \mathbb{C} \). However, this implies that \( z \) has a pole on the interior of 
\( K \), an impossibility since both \( x \) and \( y \) are of class \( C^1 \).

**Proof of Theorem 1.2** By Theorem 3.5 there is a linear map from the Hilbert space 
\( V \) generated by \( \{h_1, \ldots, h_l\} \) to \( \mathcal{H}_{DR}(K) \). If \( V \neq \{0\} \), then Proposition 3.4 states that the 
kernel is 1-dimensional if \( K \) has no non-parabolic ends, and zero-dimensional if there is at 
least one non-parabolic end.

**Theorem 3.6** If \( h_i \) is not constant, then \( b^1(L_i) \geq 1 \).

**Proof** Since \( \nabla_0 h_i \), the gradient of \( h_i \) in the \( \omega_0 \)-metric, is non-zero near \( L_i \) by the Hopf 
Lemma, the isotopy lemma guarantees a \( \delta_i \) so that \( V_i = \{h_i \geq 1 - \delta_i\} \) is diffeomorphic to 
\( L_i \times (1 - \delta_i, 1) \). Thus \( V_i \) and \( L_i \) have the same de Rham cohomology. The proof that \( Jdh_i \) 
represents a nontrivial class in \( \mathcal{H}_{DR}(V_i) \) is identical to the proof in Theorem 3.5.

**Proof of Theorem 1.3** A condition that \( h_i \) be non-constant, from Proposition 3.4, is 
that \( l \geq 1 \) and \( l + n \geq 2 \).

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