A time-inconsistent Dynkin game: from intra-personal to inter-personal equilibria

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Abstract
This paper studies a nonzero-sum Dynkin game in discrete time under non-exponential discounting. For both players, there are two intertwined levels of game-theoretic reasoning. First, each player looks for an intra-personal equilibrium among her current and future selves, so as to resolve time inconsistency triggered by non-exponential discounting. Next, given the other player’s chosen stopping policy, each player selects a best response among her intra-personal equilibria. A resulting inter-personal equilibrium is then a Nash equilibrium between the two players, each of whom employs her best intra-personal equilibrium with respect to the other player’s stopping policy. Under appropriate conditions, we show that an inter-personal equilibrium exists, based on concrete iterative procedures along with Zorn’s lemma. To illustrate our theoretical results, we investigate a two-player real options valuation problem where two firms negotiate a deal of cooperation to initiate a project jointly. By deriving inter-personal equilibria explicitly, we find that coercive power in negotiation depends crucially on the impatience levels of the two firms.

Keywords Dynkin games · Time inconsistency · Non-exponential discounting · Intra-personal equilibrium · Inter-personal equilibrium · Alternating fixed-point iterations

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1 Introduction

In dynamic optimisation, *time inconsistency* is the self-conflicting situation where the same agent at different times (i.e., the current and future selves) cannot agree on a “dynamically optimal strategy” that is good for the entire planning horizon. A long-standing approach to resolving time inconsistency is Strotz’*consistent planning* [41]: An agent should take her future selves’ disobedience into account, so as to find a strategy that none of her future selves will have an incentive to deviate from. Essentially, such a strategy is an *intra-personal* equilibrium—an equilibrium established internally within the agent, among her current and future selves.

The investigation of intra-personal equilibria, particularly their mathematical definitions and characterisations, has been a main focus of the literature on time inconsistency. This includes the classical framework in discrete time that relies on a straightforward backward sequential optimisation detailed in Pollak [36], as well as the more recent development in continuous time that employs the spike variation technique introduced in Ekeland and Lazrak [15]. The latter has led to vibrant research on time-inconsistent stochastic control, including Ekeland and Pirvu [17], Ekeland et al. [16], Björk et al. [7], Björk et al. [6], Yong [45], among many others. Lately, marked progress has been made for time-inconsistent optimal stopping, along two different paths. One is to extend the spike variation technique from stochastic control to optimal stopping, as carried out in Ebert et al. [14] and Christensen and Lindensjö [8, 9]. The other path is the iterative approach developed in Huang and Nguyen-Huu [20], Huang et al. [21], Huang and Yu [23], which circumvents spike variations via a fixed-point perspective. Let us also mention the recent work by Bayraktar et al. [4] which builds a connection between different concepts of equilibria in these two paths.

A natural question follows all the developments: How does the intra-personal reconciliation within one single agent integrate into the interaction among multiple (non-cooperative) agents? Intuitively, there should be two levels of game-theoretic reasoning—the *inner* level where each agent looks for time-consistent strategies her future selves will actually follow, and the *outer* level where each agent chooses her best strategy (among time-consistent ones) in response to other agents’ strategies. A resulting *inter-personal* equilibrium should then be a Nash equilibrium among all the agents, each of whom is restricted to choose time-consistent strategies (i.e., her intra-personal equilibria). To the best of our knowledge, such inter-personal equilibria built from intra-personal ones have not been properly formulated and studied in the literature. The crucial question is *whether* and *how* different agents’ respective intra-personal equilibria can ultimately forge an inter-personal equilibrium among all agents. This paper sheds new light on this through a time-inconsistent Dynkin game.

A Dynkin game involves two players interacting through their stopping strategies. The zero-sum version of the game, introduced in Dynkin [13], has been substantially studied along various directions, including Dynkin [13] and Neveu [34, Sect. VI-6] (discrete time), Bismut [5], Lepeltier and Maingueneau [28] and Morimoto [31] (continuous time), Yasuda [44], Rosenberg et al. [37] and Touzi and Vieille [43] (randomised strategies), Cvitanić and Karatzas [10] (non-Markovian settings) and Bayraktar and Yao [2] (model uncertainty), among others. Many of the studies not only show that a Nash equilibrium between the two players exists, but provide concrete constructions. By contrast, the nonzero-sum version of the game has received
relatively less attention; see the early investigations by Morimoto [32], Ohtsubo [35] and Nagai [33], as well as more recent ones by Hamadène and Zhang [19], Laraki and Solan [27] and De Angelis et al. [11], among others. Remarkably, all the developments above assume that the two players optimise their expected payoff/cost under exponential discounting (including the case of no discounting), which readily ensures time consistency.

In this paper, we consider a nonzero-sum Dynkin game where the state process $X$ is a discrete-time strong Markov process taking values in a Polish space $X$. Each player chooses to stop at the first entrance time of some Borel subset $S$ of $X$; this will be called a stopping policy. The Dynkin game is in general time-inconsistent as we allow the two players to take general discount functions that satisfy only a log-superadditivity condition, i.e., (2.4) below. This condition captures decreasing impatience, a widely observed feature of empirical discounting, and readily covers numerous non-exponential discount functions in behavioural economics; see the discussion below (2.4).

As time inconsistency arises under non-exponential discounting, each player, when given the other’s chosen stopping policy $T$, needs to find an intra-personal equilibrium $S$ among her current and future selves. Following the fixed-point approach in Huang and Nguyen-Huu [20], we define each player’s intra-personal equilibrium as a fixed point of an operator which encodes the aforementioned inner level of game-theoretic reasoning (Definition 2.1). To achieve an inter-personal equilibrium, a minimal requirement is that each player should attain her inner-level equilibrium simultaneously—that is, the following situation should materialise: $S$ is player 1’s intra-personal equilibrium given player 2’s stopping policy $T$, and $T$ is player 2’s intra-personal equilibrium given player 1’s stopping policy $S$. In this case, we say that $(S, T)$ is a soft inter-personal equilibrium (Definition 2.3). To further refine this “soft” definition, we note that each player, when following the Nash equilibrium idea, should not be satisfied with an arbitrary intra-personal equilibrium, but aim at the best one under an appropriate optimality criterion. Reminiscently of the “optimal equilibrium” concept proposed in Huang and Zhou [24, 25], we say that an intra-personal equilibrium is optimal if it generates larger values than any other intra-personal equilibrium, for not only the current but all future selves (Definition 2.4). This immediately brings about a stronger notion for an inter-personal equilibrium: $(S, T)$ is said to be a sharp inter-personal equilibrium if both players attain their best inner-level equilibrium simultaneously, i.e., $S$ is player 1’s optimal intra-personal equilibrium given player 2’s stopping policy $T$, and $T$ is player 2’s optimal intra-personal equilibrium given player 1’s stopping policy $S$ (Definition 2.6).

The focus of this paper is to establish the existence of inter-personal equilibria, soft and sharp, through concrete iterative procedures. First, we develop for each player an individual iterative procedure, i.e., (3.4) below, that directly leads to her optimal intra-personal equilibrium (Theorem 3.5). This procedure can be viewed as an improvement to those in Huang and Nguyen-Huu [20] and Huang et al. [21] which lead to intra-personal equilibria, but not necessarily the optimal ones. Next, we devise an alternating iterative procedure, i.e., (4.2) below, in which the two players take turns to perform the individual iterative procedure repetitively. In each iteration, one player, given the other’s stopping policy determined in the previous iteration,
performs the individual iterative procedure and then updates her policy to the optimal intra-personal equilibrium obtained; see Sect. 2.1 for details. Under appropriate conditions, this alternating iterative procedure converges and the limit, denoted by \((S_\infty, T_\infty)\), is a soft inter-personal equilibrium (Theorem 4.2). While it is tempting to believe that \((S_\infty, T_\infty)\) is in fact sharp, in view of its structure revealed in Theorem 4.2, this is in general not the case: We demonstrate explicitly that \((S_\infty, T_\infty)\) is sharp in Example 4.3, but only soft in the slightly modified Example 4.4. In other words, the general existence of sharp inter-personal equilibria is still in question. Assuming additionally that the state process \(X\) has transition densities, we are able to upgrade the construction of \((S_\infty, T_\infty)\) and apply Zorn’s lemma appropriately, which yields the desired result that a sharp inter-personal equilibrium must exist (Theorem 4.8).

Note that Theorems 4.2 and 4.8 hinge on a supermartingale condition, i.e., (3.13) below. As shown in Sect. 4.3, when the supermartingale condition fails, there may exist no inter-personal equilibrium, either soft or sharp; see Proposition 4.11 in particular. Let us point out that similar supermartingale conditions were also imposed in some studies on classical (time-consistent) nonzero-sum Dynkin games (e.g. Morimoto [32] and Ohtsubo [35]) to facilitate the existence of a Nash equilibrium.

As an application, we study the negotiation between two firms (or countries) in Sect. 5. Suppose that each firm wants the other to take unfavourable terms, as a way to obtain a larger payoff. A firm either waits until the other gives in and takes the larger payoff, or gives in to the other and accepts the unfavourable terms. Intuitively, one can purposely demonstrate a strong determination not to give in at the first place, so as to coerce the other firm into giving in in the negotiation. By computing explicitly the sharp inter-personal equilibrium between the two firms (Propositions 5.2 and 5.4, Corollary 5.5), we find that whether this strategy of coercion works depends on the impatience levels of the two firms: If a firm is less impatient than the other, using the coercion strategy always makes the other give in; on the other hand, if the firm is significantly more impatient than the other, it will simply perform the coercion to no avail. See in particular the discussion below Corollary 5.5 for details.

The rest of the paper is organised as follows. Section 2 introduces the model setup, formulates intra- and inter-personal equilibria and collects preliminary results. Section 3 develops an individual iterative procedure that directly yields a player’s optimal intra-personal equilibrium. Under a supermartingale condition, the monotonicity of this procedure is also established. Section 4 devises an alternating iterative procedure, from which we prove the existence of soft and sharp inter-personal equilibria. Examples are presented to demonstrate the alternating iterative procedure and the necessity of the supermartingale condition. Finally, Sect. 5 applies our analysis to the negotiation between two firms, relating coercive power to impatience level.

2 The model and preliminaries

Set \(\mathbb{N}_0 := \{0, 1, 2, \ldots \}\). Let us consider a time-homogeneous strong Markov process \(X = (X_t)_{t\in\mathbb{N}_0}\) taking values in a Polish space \(\mathbb{X}\). We denote by \(\mathcal{B}\) the Borel \(\sigma\)-algebra of \(\mathbb{X}\). On the path space \(\Omega\), the set of all functions mapping \(\mathbb{N}_0\) to \(\mathbb{X}\), let \((\mathcal{F}_t)_{t\in\mathbb{N}_0}\) be the filtration generated by \(X\), and \(\mathcal{T}\) the set of all \((\mathcal{F}_t)_{t\in\mathbb{N}_0}\)-stopping times. In
addition, we consider $\mathcal{F}_\infty := \bigcup_{t \in \mathbb{N}_0} \mathcal{F}_t$. For any $x \in \mathbb{X}$, we denote by $X^x$ the process $X$ with initial value $X_0 = x$, by $\mathbb{P}_x$ the probability measure on $(\Omega, \mathcal{F}_\infty)$ generated by $X^x$ (i.e., the law of $(X^x_t)_{t \in \mathbb{N}_0}$), and by $\mathbb{E}_x$ the expectation under $\mathbb{P}_x$.

Consider a nonzero-sum Dynkin game where the two players maximise their respective expected payoffs, determined jointly by their stopping strategies. Specifically, for $i \in \{1, 2\}$, given the stopping time $\sigma \in \mathcal{T}$ chosen by the other player, player $i$ at the current state $x \in \mathbb{X}$ selects a stopping time $\tau \in \mathcal{T}$ to maximise her expected discounted payoff

$$J_i(x, \tau, \sigma) := \mathbb{E}_x[F_i(\tau, \sigma)],$$  \hspace{1cm} (2.1)

where

$$F_i(\tau, \sigma) := \delta_i(\tau) f_i(X_\tau) 1_{\{\tau < \sigma\}} + \delta_i(\sigma) g_i(X_\sigma) 1_{\{\tau > \sigma\}} + \delta_i(\tau) h_i(X_\tau) 1_{\{\tau = \sigma\}}, \quad \forall \tau, \sigma \in \mathcal{T}. \hspace{1cm} (2.2)$$

Here, $\delta_i : \mathbb{N}_0 \to [0, 1]$ is player $i$’s discount function, assumed to be strictly decreasing with $\delta(0) = 1$, and $f_i, g_i, h_i : \mathbb{X} \to \mathbb{R}_+$ are player $i$’s payoff functions, assumed to be Borel-measurable. Note that we allow $\tau, \sigma \in \mathcal{T}$ to take the value $+\infty$. For any $\omega \in \{\tau = \sigma = +\infty\}$, we simply define $F_i(\tau, \sigma)(\omega) := \limsup_{t \to \infty} \delta_i(t) h_i(X_i(\omega))$.

To ensure that $J_i(x, \tau, \sigma)$ in (2.1) is well defined, we impose throughout the paper that

$$\mathbb{E}_x\left[\sup_{t \in \mathbb{N}} \delta_i(t) \left( f_i(X_t) + g_i(X_t) + h_i(X_t) \right) \right] < \infty, \quad \forall x \in \mathbb{X}. \hspace{1cm} (2.3)$$

As mentioned in the introduction, the vast literature on Dynkin games mostly assumes exponential discounting, i.e., $\delta_i(t) = e^{-\beta_i t}$ for some $\beta_i > 0$. Empirical studies (e.g. Thaler [42] and Loewenstein and Thaler [29]), on the other hand, have found that individuals do not normally discount exponentially. In this paper, the only standing assumption on $\delta_i$, $i \in \{1, 2\}$, is that

$$\delta_i(s) \delta_i(t) \leq \delta_i(s + t), \quad \forall s, t \in \mathbb{N}_0. \hspace{1cm} (2.4)$$

This in particular captures decreasing impatience, a widely observed feature of empirical discounting. Numerous non-exponential discount functions in behavioural economics, such as hyperbolic, generalised hyperbolic and pseudo-exponential discount functions, readily satisfy (2.4); see the discussion below Huang and Nguyen-Huu [20, Assumption 3.12] for details.

In a one-player stopping problem, it is well understood that non-exponential discounting induces time inconsistency: An optimal stopping strategy derived at the current state $x \in \mathbb{X}$ may no longer be optimal at a subsequent state $y \neq x$. In other words, the current and future selves cannot agree on a “dynamically optimal stopping strategy” that is good for the entire planning horizon; see e.g. Huang and Nguyen-Huu [20, Sect. 2.2] for an explicit demonstration. Strotz’ consistent planning [41] is a long-standing approach to resolving time inconsistency: Knowing that her future selves may overturn her current plan, an agent selects the best present action taking
the future disobedience as a constraint; the resulting strategy is a (subgame perfect) Nash equilibrium from which no future self has an incentive to deviate.

In our Dynkin game, thanks to the time-homogeneous Markovian setup, we assume that each player decides to stop or to continue depending on her current state \( x \in X \). That is, each player stops at the first entrance time of some \( S \in \mathcal{B} \), defined by

\[
\rho_S := \inf \{ t \geq 0 : X_t \in S \}.
\]

For convenience, we often call \( S \in \mathcal{B} \) a \textit{stopping policy}. This corresponds to a “pure strategy” in economic terms; see Remark 4.10 below for discussions on the use of pure and randomised strategies. For \( i \in \{1, 2\} \), given the other player’s stopping policy \( T \in \mathcal{B} \), player \( i \) is faced with time inconsistency among her current and future selves (as explained above), and needs to find an equilibrium stopping policy at the \textit{intra-personal} level. Following Huang and Zhou [25, Sect. 2.1] (or Huang and Nguyen-Huu [20, Sect. 3.1]), Strotz’ consistent planning boils down to the current self’s game-theoretic reasoning: “Given that my future selves will follow the policy \( S \in \mathcal{B} \), what is the best policy today in response to that?” The best policy is determined by comparing the payoff \( J_i(x, 0, \rho_T) \) of immediate stopping and the payoff \( J_i(x, \rho_S^+, \rho_T) \) of continuation, where

\[
\rho_S^+ := \inf \{ t > 0 : X_t \in S \}
\]

is the first hitting time to \( S \). This leads to the stopping policy

\[
\Theta^T_i(S) := \{ x \in S : J_i(x, 0, \rho_T) \geq J_i(x, \rho_S^+, \rho_T) \} \cup \{ x \notin S : J_i(x, 0, \rho_T) > J_i(x, \rho_S^+, \rho_T) \} \in \mathcal{B}.
\]

We can consider \( \Theta^T_i : \mathcal{B} \to \mathcal{B} \) as an \textit{improving} operator for player \( i \): Given the other player’s stopping policy \( T \in \mathcal{B} \), \( \Theta^T_i \) improves the present policy \( S \in \mathcal{B} \) of player \( i \) to \( \Theta^T_i(S) \in \mathcal{B} \).

For each player, we define an equilibrium at the \textit{intra-personal} level (i.e., among the player’s current and future selves) in the same spirit as Huang and Zhou [25, Definition 2.2] and Huang and Nguyen-Huu [20, Definition 3.7].

**Definition 2.1** For \( i \in \{1, 2\} \), \( S \in \mathcal{B} \) is called player \( i \)’s \textit{intra-personal equilibrium} with respect to \( T \in \mathcal{B} \) if \( \Theta^T_i(S) = S \). We denote by \( \mathcal{E}^T_i \) the set of all player \( i \)’s intra-personal equilibria with respect to \( T \in \mathcal{B} \).

**Remark 2.2** The above fixed-point definition of an intra-personal equilibrium was introduced in Huang and Nguyen-Huu [20] and followed by Huang et al. [21], Huang and Zhou [25] and Huang and Yu [23], among others. Note that there is a slightly different formulation in Huang and Zhou [24]: If we follow [24], particularly (2.5) therein, \( \Theta^T_i(S) \) in (2.5) above needs to be modified as

\[
\bar{\Theta}^T_i(S) := \{ x \in X : J_i(x, 0, \rho_T) \geq J_i(x, \rho_S^+, \rho_T) \}.
\]
Observe from (2.5) and (2.6) that the equilibrium condition $\Theta_i^T(S) = S$ in Definition 2.1 is slightly weaker than $\tilde{\Theta}_i^T(S) = S$ as in [24, Definition 2.3]. The present paper uses the slightly weaker definition because it conforms more closely to the Nash equilibrium idea—one deviates to a new policy only when it is strictly better than the current one; see the explanations before (3.6) in Huang and Nguyen-Huu [20]. Moreover, the weaker definition facilitates the search for intra-personal equilibria as it allows the explicit construction in Proposition 3.2 below.

Based on Definition 2.1, we introduce the first kind of equilibria at the inter-personal level (i.e., between the two players)—the soft inter-personal equilibria.

**Definition 2.3** We say $(S, T) \in \mathcal{B} \times \mathcal{B}$ is a soft inter-personal equilibrium (for the Dynkin game) if $S \in \mathcal{E}_1^T$ and $T \in \mathcal{E}_2^S$ (i.e., $\Theta_1^T(S) = S$ and $\Theta_2^S(T) = T$). We denote by $\mathcal{E}$ the set of all soft inter-personal equilibria.

Essentially, $(S, T) \in \mathcal{E}$ means that both players simultaneously attain an equilibrium at the intra-personal level, given the other player’s stopping policy: $S$ is player 1’s intra-personal equilibrium with respect to player 2’s policy $T$, and $T$ is player 2’s intra-personal equilibrium with respect to player 1’s policy $S$.

As emphasised in Huang and Zhou [24, 25], Strotz’ consistent planning is a two-phase procedure: An agent first determines the strategies that she will actually follow over time (phase I), and then chooses the best one among them (phase II). In our Dynkin game, phase I amounts to each player finding her intra-personal equilibria (with respect to the other player’s stopping policy); phase II is then the search for an optimal intra-personal equilibrium defined as below.

**Definition 2.4** For $i \in \{1, 2\}$ and $T \in \mathcal{B}$, the value function associated with $S \in \mathcal{E}_i^T$ is defined by

$$U_i^T(x, S) := J_i(x, 0, \rho_T) \vee J_i(x, \rho_S^+, \rho_T), \quad x \in \mathcal{X}.$$  

We say $S \in \mathcal{E}_i^T$ is player i’s optimal intra-personal equilibrium with respect to $T \in \mathcal{B}$ if for any $R \in \mathcal{E}_i^T$,

$$U_i^T(x, S) \geq U_i^T(x, R) \quad \text{for all } x \in \mathcal{X}.$$  

We denote by $\hat{\mathcal{E}}_i^T$ the set of all player i’s optimal intra-personal equilibria with respect to $T$.

**Remark 2.5** Thanks to $S \in \mathcal{E}_i^T$, $U_i^T(x, S)$ defined above coincides with $J_i(x, \rho_S, \rho_T)$. Indeed, by $\Theta_i^T(S) = S$ (due to $S \in \mathcal{E}_i^T$) and (2.5), we have

$$J_i(x, \rho_S, \rho_T) = J_i(x, 0, \rho_T) \geq J_i(x, \rho_S^+, \rho_T) \quad \text{for } x \in S,$$

$$J_i(x, \rho_S, \rho_T) = J_i(x, \rho_S^+, \rho_T) \geq J_i(x, 0, \rho_T) \quad \text{for } x \notin S.$$  

That is, $J_i(x, \rho_S, \rho_T) = J_i(x, 0, \rho_T) \vee J_i(x, \rho_S^+, \rho_T) = U_i^T(x, S)$ for all $x \in \mathcal{X}$. 

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Definition 2.4 follows the “optimal equilibrium” notion introduced in Huang and Zhou [24]. It is a rather strong optimality criterion, as it requires a (subgame perfect Nash) equilibrium to dominate any other equilibrium on the entire state space—a rare occurrence in game theory. Nonetheless, for the one-player optimal stopping problem under non-exponential discounting, as long as the discount function satisfies (2.4), the existence of an optimal equilibrium has been established first in discrete time (Huang and Zhou [24]) and then in continuous time (see Huang and Zhou [25] and Huang and Wang [22] for diffusion models, and Bayraktar et al. [4] for a continuous-time Markov chain model).

Based on Definition 2.4, we introduce the second kind of equilibria at the inter-personal level (i.e., between the two players)—the sharp inter-personal equilibria.

**Definition 2.6** We say \((S, T) \in B \times B\) is a sharp inter-personal equilibrium (for the Dynkin game) if \(S \in \hat{E}^T_1\) and \(T \in \hat{E}^S_2\). We denote by \(\hat{E}\) the set of all sharp inter-personal equilibria.

A sharp inter-personal equilibrium, compared with a soft one in Definition 2.3, conforms more closely to the Nash equilibrium concept. Given the other player’s policy, what a player aims at should not be an arbitrary agreement among her current and future selves (as stipulated in Definition 2.3), but the agreement that is best-rewarding—the one that generates the largest possible value for every incarnation of herself in time. In other words, our time-inconsistent Dynkin game involves two levels of game-theoretic reasoning. Player 1 wants to find the best response to player 2’s policy at the inter-personal level, while maintaining an agreement among her current and future selves at the intra-personal level; player 2 does the same in response to player 1’s policy. In the end, each player chooses an optimal intra-personal equilibrium with respect to the other player’s policy, leading to a sharp inter-personal equilibrium for the Dynkin game.

It is worth noting that our definition of a sharp inter-personal equilibrium covers as a special case the standard Nash equilibrium in a time-consistent Dynkin game.

**Remark 2.7** In the time-consistent case of exponential discounting, a Nash equilibrium for the Dynkin game is defined as a pair of stopping times \((\hat{\tau}, \hat{\sigma})\) such that \(\hat{\tau}\) is player 1’s optimal stopping time given that player 2 employs \(\hat{\sigma}\), while \(\hat{\sigma}\) at the same time is player 2’s optimal stopping time given that player 1 employs \(\hat{\tau}\). In a time-homogeneous setting, let \(\hat{S}\) (resp. \(\hat{T}\)) denote the stopping region associated with \(\hat{\tau}\) (resp. \(\hat{\sigma}\)). In view of Huang and Nguyen-Huu [20, Proposition 3.11], \(\hat{S}\) is readily player 1’s intra-personal equilibrium with respect to \(\hat{T}\). Moreover, by the argument in Huang and Wang [22, Remark 2.12], \(\hat{S}\) is in fact player 1’s optimal intra-personal equilibrium with respect to \(\hat{T}\)—namely, \(\hat{S} \in \hat{E}^\hat{T}_1\). By the same token, we have \(\hat{T} \in \hat{E}^\hat{S}_2\). It then follows that \((\hat{S}, \hat{T})\) is a sharp inter-personal equilibrium.

The above means that in the classical time-consistent case, a Nash equilibrium \((\hat{\tau}, \hat{\sigma})\) in a time-homogeneous model is automatically a sharp inter-personal equilibrium, once we re-state \((\hat{\tau}, \hat{\sigma})\) using their respective stopping regions.
2.1 Problem formulation

This paper aims to establish the existence of soft and sharp inter-personal equilibria, using concrete iterative procedures. Although the existence and construction of each player’s intra-personal equilibria is well understood (based on the one-player results in Huang and Nguyen-Huu [20] and Huang and Zhou [24]), it is unclear whether the two players’ respective intra-personal equilibria can be coordinated properly to form an inter-personal equilibrium, either soft or sharp.

We tackle this in two steps. First, we look into the one-player problem more closely, developing for each player an individual iterative procedure that directly brings about her optimal intra-personal equilibrium (Theorem 3.5). Next, we devise an alternating iterative procedure in which the two players take turns to perform the individual iterative procedure:

1. With respect to player 1’s initial policy $S_0 \in B$, player 2 performs the individual iterative procedure to get an optimal intra-personal equilibrium $T_0 \in B$.
2. With respect to player 2’s policy $T_0 \in B$, player 1 performs the individual iterative procedure to get an optimal intra-personal equilibrium $S_1 \in B$.
3. With respect to player 1’s policy $S_1 \in B$, player 2 performs the individual iterative procedure to get an optimal intra-personal equilibrium $T_1 \in B$.

The hope is that this alternating iterative procedure will ultimately converge, with the limit $(S_\infty, T_\infty)$ being a soft, or even sharp, inter-personal equilibrium. This is investigated in detail in Sect. 4, with affirmative results established in Theorems 4.2 and 4.8.

2.2 Preliminaries

We collect two technical results that will be useful throughout the paper. The first one concerns the convergence of first entrance and hitting times.

**Lemma 2.8** Let $(S_n)_{n \in \mathbb{N}}$ be a monotone sequence in $B$. For any $\omega \in \Omega$, there exists $N \in \mathbb{N}$ such that $\rho_{S_n}(\omega) = \rho_{S_\infty}(\omega)$ for all $n \geq N$, where

$$S_\infty := \begin{cases} \bigcup_{n \in \mathbb{N}} S_n, & \text{if } (S_n) \text{ is nondecreasing}, \\ \bigcap_{n \in \mathbb{N}} S_n, & \text{if } (S_n) \text{ is nonincreasing}. \end{cases}$$

The same result holds with $\rho$ replaced by $\rho^+$.

**Proof** Fix $\omega \in \Omega$. If $(S_n)$ is nondecreasing, set $t := \rho_{S_\infty}(\omega)$. Without loss of generality, assume $t < \infty$. Since $X_t(\omega) \in S_\infty = \bigcup_{n \in \mathbb{N}} S_n$, there exists $N \in \mathbb{N}$ such that $X_t(\omega) \in S_n$ for all $n \geq N$. Hence $\rho_{S_n}(\omega) = t$ for all $n \geq N$. If there exists $n^* \geq N$ such that $\rho_{S_{n^*}}(\omega) < t$, then $\rho_{S_n}(\omega) \leq \rho_{S_{n^*}}(\omega) < t$, a contradiction. We thus conclude $\rho_{S_n}(\omega) = t = \rho_{S_\infty}(\omega)$ for all $n \geq N$. On the other hand, if $(S_n)$ is nonincreasing, set $t := \lim_{n \to \infty} \rho_{S_n}(\omega)$. Without loss of generality, assume $t < \infty$. Then there exists $N \in \mathbb{N}$ such that $\rho_{S_n}(\omega) = t$ for all $n \geq N$. Hence $X_t(\omega) \in S_n$ for all $n \geq N$ and thus
$X_t(\omega) \in S_\infty = \bigcap_{n \in \mathbb{N}} S_n$. This implies $\rho_{S_\infty}(\omega) \leq t$. Since $\rho_{S_n}(\omega) \geq t$ by definition, we conclude $\rho_{S_\infty}(\omega) = t = \rho_{S_n}(\omega)$ for all $n \geq N$. The same arguments as above hold with $\rho$ replaced by $\rho^+$.

The next result states that any stopping policy containing an intra-personal equilibrium $R$ must be dominated by $R$. This kind of result was first established for one-player stopping problems in Huang and Zhou [25, Lemma 3.1], and is now extended to a Dynkin game setting.

**Lemma 2.9** Fix $i \in \{1, 2\}$ and assume $h_i \leq g_i$. Then for any $R, S \in \mathcal{B}$ with $R \subseteq S$ and $R \in \mathcal{E}_i^T$ for some $T \in \mathcal{B}$,

$$J_i(x, \rho_R^+, \rho_T) \geq J_i(x, \rho_S^+, \rho_T) \quad \text{for all } x \in \mathbb{X}.$$  

**Proof** Consider $A := \{\omega \in \Omega : \rho_S^+ = \rho_T < \rho_R^+\}$ and $B := \{\omega \in \Omega : \rho_S^+ < \rho_T \land \rho_R^+\}$. For any $x \in \mathbb{X}$, by (2.1) and (2.2),

$$J_i(x, \rho_R^+, \rho_T) - J_i(x, \rho_S^+, \rho_T) = \mathbb{E}_x[(1_A + 1_B)(F_i(\rho_R^+, \rho_T) - F_i(\rho_S^+, \rho_T))].$$  

By the assumption that $g_i \geq h_i$,

$$\mathbb{E}_x[1_A(F_i(\rho_R^+, \rho_T) - F_i(\rho_S^+, \rho_T))] = \mathbb{E}_x[1_A(\delta_i(\rho_T)g_i(X_{\rho_T}) - \delta_i(\rho_T)h_i(X_{\rho_T}))] \geq 0. \quad (2.8)$$

On the other hand,

$$\mathbb{E}_x[1_B(F_i(\rho_R^+, \rho_T) - F_i(\rho_S^+, \rho_T))] = \mathbb{E}_x[1_B(\mathbb{E}_x[F_i(\rho_R^+, \rho_T)|\mathcal{F}_{\rho_S^+}] - F_i(\rho_S^+, \rho_T))]. \quad (2.9)$$

In view of (2.2), (2.4) and the nonnegativity of $f_i, g_i$ and $h_i$,

$$1_B\mathbb{E}_x[F_i(\rho_R^+, \rho_T)|\mathcal{F}_{\rho_S^+}] \geq 1_B\delta_i(\rho_S^+)\mathbb{E}_x[\delta_i(\rho_R^+ - \rho_S^+)f_i(X_{\rho_R^+})1_{\rho_R^+<\rho_T} + \delta_i(\rho_T^+ - \rho_S^+)g_i(X_{\rho_T})1_{\rho_T^+>\rho_T}]$$

$$+ \delta_i(\rho_R^+ - \rho_S^+)h_i(X_{\rho_R^+})1_{\rho_R^+=\rho_T}|\mathcal{F}_{\rho_S^+}] = 1_B\delta_i(\rho_S^+)J_i(X_{\rho_S^+}, \rho_R^+, \rho_T), \quad (2.10)$$

where the equality follows from the strong Markov property of $X$. By (2.10) and the fact that $1_B F_i(\rho_S^+, \rho_T) = 1_B\delta_i(\rho_S^+)f_i(X_{\rho_S^+}) = 1_B\delta_i(\rho_S^+)J_i(X_{\rho_S^+}, 0, \rho_T)$, (2.9) implies

$$\mathbb{E}_x[1_B(F_i(\rho_R^+, \rho_T) - F_i(\rho_S^+, \rho_T))] \geq \mathbb{E}_x[1_B\delta_i(\rho_S^+)J_i(X_{\rho_S^+}, \rho_R^+, \rho_T) - J_i(X_{\rho_S^+}, 0, \rho_T)]. \quad (2.11)$$
On the set $B$, we deduce from $\rho_S^+ < \rho_R^+$ that $X_{\rho_S^+} \notin R$. In view of $R \in E_i^T$ and (2.5), this implies $J_i(X_{\rho_S^+}, 0, \rho_T) \leq J_i(X_{\rho_R^+}, \rho_T)$. Hence we obtain from (2.11) that $\mathbb{E}_x[1_B(F_i(\rho_R^+, \rho_T) - F_i(\rho_S^+, \rho_T))] \geq 0$. On the strength of this and (2.8), we conclude from (2.7) that $J_i(x, \rho_R^+, \rho_T) - J_i(x, \rho_S^+, \rho_T) \geq 0$. □

There is an intriguing message here—a smaller intra-personal equilibrium is more rewarding. Indeed, for any $R,S \in E_T$ with $R \subseteq S$, Lemma 2.9 asserts that for any $x \in X$, $J_i(x, \rho_R^+, \rho_T) \geq J_i(x, \rho_S^+, \rho_T)$. This “ranking by size” will play a crucial role in Theorem 3.5 below, where an optimal intra-personal equilibrium is derived. Economically, this ranking simply reflects a player’s decreasing impatience, which is captured by (2.4); see the detailed discussion below Huang and Zhou [25, Corollary 3.2].

3 The one-player analysis

In this section, we first focus on developing an iterative procedure for each player that directly leads to her intra-personal equilibrium. Next, under a supermartingale condition, we establish the monotonicity of this iterative procedure.

For $i \in \{1, 2\}$, we introduce for any fixed $T \in B$ the operator $\Phi_i^T : B \to B$ by

$$\Phi_i^T(S) := S \cup \{x \notin S : J_i(x, 0, \rho_T) > V_i^T(x, S)\},$$

where

$$V_i^T(x, S) := \sup_{1 \leq \tau \leq \rho_S^+} \mathbb{E}_x[F_i(\tau, \rho_T)]$$

for any $x \in X$ and $S \in B$. (3.2)

We first note that $V_i^T(x, \cdot)$ converges along nondecreasing sequences of stopping policies.

Lemma 3.1 Fix $i \in \{1, 2\}$. Let $(S_n)_{n \in \mathbb{N}}$ be a nondecreasing sequence in $B$. Then for any $T \in B$, we have $V_i^T(x, S_n) \downarrow V_i^T(x, S_\infty)$ for all $x \in X$, with $S_\infty := \bigcup_{n \in \mathbb{N}} S_n$.

Proof Fix $x \in X$. By the definition of $V_i^T$ in (3.2), $\lim_{n \to \infty} V_i^T(x, S_n) \geq V_i^T(x, S_\infty)$. To show the converse inequality, let $\tau_n \in T$ with $1 \leq \tau_n \leq \rho_S^+$ be a $\frac{1}{n}$-optimiser of $V_i^T(x, S_n)$, for each $n \in \mathbb{N}$. Then we get

$$V_i^T(x, S_n) - V_i^T(x, S_\infty) \leq \mathbb{E}_x[F_i(\tau_n, \rho_T)] + 1/n - \mathbb{E}_x[F_i(\tau_n \wedge \rho_S^+, \rho_T)]$$

$$= \mathbb{E}_x[F_i(\tau_n, \rho_T) - F_i(\tau_n \wedge \rho_S^+, \rho_T)] + 1/n. \tag{3.3}$$

For each $\omega \in \Omega$, by Lemma 2.8, there is $N \in \mathbb{N}$ such that $\tau_n(\omega) \leq \rho_S^+(\omega) = \rho_S^*(\omega)$ for all $n \geq N$. In particular, $\tau_n(\omega) = (\tau_n \wedge \rho_S^*)(\omega)$ for all $n \geq N$. This together with (2.3) implies that we may apply the dominated convergence theorem to show that the expectation in (3.3) converges to 0. Hence we conclude from (3.3) that $\lim_{n \to \infty} V_i^T(x, S_n) - V_i^T(x, S_\infty) \leq 0$. □
For any fixed $T \in \mathcal{B}$, we perform an iterative procedure by starting with the empty set and applying the operator $\Phi_i^T$ repetitively. This will give an intra-personal equilibrium for player $i$.

**Proposition 3.2** Fix $i \in \{1, 2\}$ and assume $h_i \leq g_i$. For any $T \in \mathcal{B}$, let $(S^n_i(T))_{n \in \mathbb{N}}$ be the nondecreasing sequence in $\mathcal{B}$ defined by

$$S^n_i(T) := \Phi_i^T(\emptyset), \quad S_2^n(T) := \Phi_i^T(S^{n-1}_i(T)) \quad \text{for } n \geq 2,$$

where $\Phi_i^T : \mathcal{B} \to \mathcal{B}$ is defined in (3.1). Then we have

$$\Gamma_i(T) := \bigcup_{n \in \mathbb{N}} S^n_i(T) \in \mathcal{E}_i^T. \quad (3.5)$$

Moreover, $T \cap S^n_i(T) = \emptyset$ for all $n \in \mathbb{N}$; in particular, $T \cap \Gamma_i(T) = \emptyset$.

**Proof** Fix $x \in \Gamma_i(T)$. If $x \in S^1_i(T) = \Phi_i^T(\emptyset)$, it follows directly from (3.1) that $J_i(x, 0, \rho_T) > V_i^T(x, \emptyset) = V_i^T(x, \Gamma_i(T)) \geq J_i(x, \rho_{\Gamma_i(T)}^+, \rho_T)$, which shows that $x \in \Theta_i^T(\Gamma_i(T))$. If instead $x \notin S^1_i(T)$, since $(S^n_i(T))_{n \in \mathbb{N}}$ is by definition nondecreasing, there must exist $n \in \mathbb{N}$ such that $x \in S^n_i(T) \setminus S_{n+1}^i(T)$. Thanks again to (3.1), this yields $J_i(x, 0, \rho_T) = V_i^T(x, S^n_i(T)) \geq V_i^T(x, \Gamma_i(T)) \geq J_i(x, \rho_{\Gamma_i(T)}^+, \rho_T)$, so that $x \in \Theta_i^T(\Gamma_i(T))$ follows. Hence we conclude that $\Gamma_i(T) \subseteq \Theta_i^T(\Gamma_i(T))$.

It remains to show the converse inclusion. Let us fix $x \notin \Gamma_i(T)$. We claim that $x \notin \Theta_i^T(\Gamma_i(T))$, i.e.,

$$J_i(x, 0, \rho_T) \leq J_i(x, \rho_{\Gamma_i(T)}^+, \rho_T). \quad (3.6)$$

Assume to the contrary that (3.6) fails so that

$$\Delta := \{y \notin \Gamma_i(T) : J_i(y, 0, \rho_T) > J_i(y, \rho_{\Gamma_i(T)}^+, \rho_T)\} \neq \emptyset. \quad (3.7)$$

Consider

$$\alpha := \sup_{y \in \Delta} (J_i(y, 0, \rho_T) - J_i(y, \rho_{\Gamma_i(T)}^+, \rho_T))^+ > 0. \quad (3.8)$$

As $\delta_i(1) < \delta_i(0) = 1$, we can take $x^* \in \Delta$ such that

$$J_i(x^*, 0, \rho_T) - J_i(x^*, \rho_{\Gamma_i(T)}^+, \rho_T) > \frac{1 + \delta_i(1)}{2} \alpha. \quad (3.9)$$

Observe that we must have $x^* \notin T$ and thus $\rho_T > 0$ $\mathbb{P}_{x^*}$-a.s. Indeed, if $x^* \in T$, then

$$J_i(x^*, 0, \rho_T) - J_i(x^*, \rho_{\Gamma_i(T)}^+, \rho_T) = h_i(x) - g_i(x) \leq 0,$$

which contradicts (3.9). Moreover, since $x^* \notin \Gamma_i(T)$ implies $x^* \notin S^n_i(T)$ for all $n \in \mathbb{N}$, we deduce from (3.1) that $J_i(x^*, 0, \rho_T) \leq V_i^T(x^*, S^n_i(T))$ for all $n \in \mathbb{N}$. By Lemma 3.1, this implies

$$J_i(x^*, 0, \rho_T) \leq V_i^T(x^*, \Gamma_i(T)). \quad (3.10)$$
Let $\rho^* \in T$ with $1 \leq \rho^* \leq \rho^+_{\Gamma_i(T)}$ be a $\frac{1-\delta_i(1)}{2}\alpha$-optimiser of $V^T_i(x^*, \Gamma_i(T))$. Consider the sets

$$A := \{ \omega \in \Omega : \rho^*(\omega) < \rho^+_{\Gamma_i(T)}(\omega), \rho^*(\omega) \leq \rho_T(\omega), X_{\rho^*}(\omega) \notin (\Gamma_i(T) \cup \Delta) \},$$

$$B := \{ \omega \in \Omega : \rho^*(\omega) < \rho^+_{\Gamma_i(T)}(\omega), \rho^*(\omega) \leq \rho_T(\omega), X_{\rho^*}(\omega) \in \Delta \setminus \Gamma_i(T) \}.$$

By (3.9) and (3.10),

$$1 + \frac{\delta_i(1)}{2}\alpha < J_i(x^*, 0, \rho_T) - J_i(x^*, \rho^+_{\Gamma_i(T)}, \rho_T) \leq \mathbb{E}_{x^*}[F_i(\rho^*, \rho_T) - F_i(\rho^+_{\Gamma_i(T)}, \rho_T)] + \frac{1 - \delta_i(1)}{2}\alpha \leq \mathbb{E}_{x^*}[(1_A + 1_B)(F_i(\rho^*, \rho_T) - F_i(\rho^+_{\Gamma_i(T)}, \rho_T))] + \frac{1 - \delta_i(1)}{2}\alpha. \quad (3.11)$$

By (2.4) and the nonnegativity of $f_i$, $g_i$ and $h_i$, we can argue as in (2.10) to get

$$\mathbb{E}_{x^*}[(1_A + 1_B)F_i(\rho^+, \rho_T)] = \mathbb{E}_{x^*}[(1_A + 1_B)\mathbb{E}_{x^*}[F_i(\rho^+_{\Gamma_i(T)}, \rho_T) | \mathcal{F}_{\rho^*}]] \geq \mathbb{E}_{x^*}[(1_A + 1_B)\delta_i(\rho^*)J_i(X_{\rho^*}, \rho^+_{\Gamma_i(T)}, \rho_T)].$$

Thanks to this and the fact that $(1_A + 1_B)F_i(\rho^*, \rho_T) = (1_A + 1_B)\delta_i(\rho^*)J_i(X_{\rho^*}, 0, \rho_T)$, (3.11) yields

$$\frac{1 + \delta_i(1)}{2}\alpha < \mathbb{E}_{x^*}[(1_A + 1_B)\delta_i(\rho^*)(J_i(X_{\rho^*}, 0, \rho_T) - J_i(X_{\rho^*}, \rho^+_{\Gamma_i(T)}, \rho_T))]$$

$$\leq \mathbb{E}_{x^*}[1_B\delta_i(\rho^*)(J_i(X_{\rho^*}, 0, \rho_T) - J_i(X_{\rho^*}, \rho^+_{\Gamma_i(T)}, \rho_T))] + \frac{1 - \delta_i(1)}{2}\alpha \leq \delta_i(1)\alpha + \frac{1 - \delta_i(1)}{2}\alpha = \frac{1 + \delta_i(1)}{2}\alpha,$$

where the second inequality follows from $X_{\rho^*} \notin \Delta$ on $A$ and the definition of $\Delta$ in (3.7), and the third inequality is due to $X_{\rho^*} \in \Delta$ on $B$, the definition of $\alpha$ in (3.8) and $\rho^* \geq 1$ by definition. The above inequality is clearly a contradiction, and we thus conclude that (3.6) holds. That is, we have shown that $(\Gamma_i(T))^c \subseteq (\Theta_i(T))^c$, or simply $\Gamma_i(T) \supseteq \Theta_i(T)$. This brings the final conclusion that $\Gamma_i(T) = \Theta_i(T)$, i.e., $\Gamma_i(T) \in \mathcal{E}_T$.

Finally, for any $x \in T$, since $\rho_T = 0 \mathbb{P}_x$-a.s., we get for all $R \in \mathcal{B}$ that

$$J_i(x, 0, \rho_T) = h_i(x) = \sup_{1 \leq \tau \leq \rho^+_K} \mathbb{E}_x[F_i(\tau, \rho_T)] = V^T_i(x, R). \quad (3.12)$$

In view of (3.4), taking $R = \emptyset$ in (3.12) shows that $x \notin S^1_i(T)$. Using $x \notin S^1_i(T)$ and taking $R = S^1_i(T)$ in (3.12), we in turn obtain $x \notin S^2_i(T)$. Applying (3.12) recursively...
in the same way then gives \( x \notin S^n_i(T) \) for all \( n \in \mathbb{N} \). Therefore we conclude that \( T \cap S^n_i(T) = \emptyset \) for all \( n \in \mathbb{N} \). \( \square \)

**Remark 3.3** The idea of the iterative procedure (3.4) was initially inspired by Christensen and Lindensjö [8, Sect. 3], while its specific construction is partially borrowed from Bayraktar et al. [4, Theorem 2.2].

We next go one step further to claim that \( \Gamma(T) \) in (3.5) is in fact an *optimal* intra-personal equilibrium for player \( i \). To this end, we need the following auxiliary result: Being included in an intra-personal equilibrium is an invariant relation under the operator \( \Phi^T_i \).

**Lemma 3.4** Fix \( i \in \{1, 2\}, T \in \mathcal{B} \) and \( R \in \mathcal{E}^T_i \). For any \( S \in \mathcal{B} \) with \( S \subseteq R \), we have \( \Phi^T_i(S) \subseteq R \).

**Proof** Suppose that there exists \( x \in \Phi^T_i(S) \setminus R \) for some \( S \in \mathcal{B} \) with \( S \subseteq R \). With \( x \in \Phi^T_i(S) \) but \( x \notin S \), (3.1) gives

\[
J_i(x, 0, \rho_T) > V^T_i(x, S) \geq V^T_i(x, R) \geq J_i(x, \rho_R^+, \rho_T),
\]

where the second and third inequalities follow directly from the definition of \( V^T_i \) in (3.2). As \( x \notin R \), this shows that \( x \in \Theta^T_i(R) \), from which we conclude that \( \Theta^T_i(R) \neq R \). This contradicts the assumption that \( R \in \mathcal{E}^T_i \). \( \square \)

Now we are ready to present the main result of this section.

**Theorem 3.5** Fix \( i \in \{1, 2\} \) and assume \( h_i \leq g_i \). For any \( T \in \mathcal{B} \), \( \Gamma_i(T) \) defined in (3.5) belongs to \( \hat{\mathcal{E}}^T_i \).

**Proof** By Proposition 3.2, \( \Gamma_i(T) \in \mathcal{E}^T_i \). Recall \( (S^n_i(T))_{n \in \mathbb{N}} \) defined in (3.4). For any \( R \in \mathcal{E}^T_i \), Lemma 3.4 directly implies that \( S^1_i(T) = \Phi^T_i(\emptyset) \subseteq R \). By applying Lemma 3.4 recursively, we obtain \( S^n_i(T) = \Phi^T_i (S^{n-1}_i(T)) \subseteq R \) for all \( n \geq 2 \). This readily shows that \( \Gamma_i(T) = \bigcup_{n \in \mathbb{N}} S^n_i(T) \subseteq R \). By Lemma 2.9, this implies that \( J_i(x, \rho_{\Gamma_i(T)}^+, \rho_T) \geq J_i(x, \rho_R^+, \rho_T) \) and thus \( U^T_i(x, \Gamma_i(T)) \geq U^T_i(x, R) \), for all \( x \in \mathbb{I} \). As \( R \in \mathcal{E}^T_i \) is arbitrarily chosen, we conclude that \( \Gamma_i(T) \in \hat{\mathcal{E}}^T_i \). \( \square \)

### 3.1 Monotonicity with respect to \( T \in \mathcal{B} \)

So far, we have fixed \( T \in \mathcal{B} \) (the other player’s stopping policy) and constructed a corresponding optimal intra-personal equilibrium \( \Gamma_i(T) \) in (3.5). By viewing \( T \in \mathcal{B} \) as a variable, we now show that the map \( T \mapsto \Gamma_i(T) \) is monotone under appropriate conditions.

**Lemma 3.6** Fix \( i \in \{1, 2\} \). Assume \( f_i \leq h_i \leq g_i \) and that

\[
(\delta(t)g_i(X^x_t))_{t \geq 0}
\]

is a supermartingale for all \( x \in \mathbb{I} \). 

(3.13)
Then for any \( T, R \in B \) with \( T \subseteq R \),
\[
J_i(x, \tau, \rho_T) \leq J_i(x, \tau, \rho_R), \quad \forall x \in \mathbb{X} \text{ and } \tau \in T.
\]
(3.14)

Hence \( V_i^T(x, S) \leq V_i^R(x, S) \) for all \( x \in \mathbb{X} \) and \( S \in B \). Moreover, we have
\[
\Phi_i^T(S) \supseteq \Phi_i^R(S'), \quad \forall S, S' \in B \text{ with } S \supseteq S'.
\]
(3.15)

Proof Given \( x \in \mathbb{X} \) and \( \tau \in T \), consider the sets
\[
A := \{ \omega \in \Omega : \tau = \rho_R < \rho_T \},
\]
\[
B := \{ \omega \in \Omega : \rho_R < \tau \land \rho_T \}.
\]
Observe that
\[
\mathbb{E}_x \left[ 1_B \left( F_i(\tau, \rho_T) - F_i(\tau, \rho_R) \right) \right] = \mathbb{E}_x \left[ 1_B \left( \delta_i(\tau \land \rho_T) g_i(X_{\tau \land \rho_T}) - \delta_i(\rho_R) g_i(X_{\rho_R}) \right) \right] \leq 0.
\]
where the first inequality is due to \( f_i \leq h_i \leq g_i \) and the last inequality follows from \((\delta_i(t) g_i(X_i))_{t \geq 0}\) being a supermartingale. By the above inequality and
\[
\mathbb{E}_x \left[ 1_A \left( F_i(\tau, \rho_T) - F_i(\tau, \rho_R) \right) \right] = \mathbb{E}_x \left[ 1_A \left( \delta_i(\tau) f_i(X_{\tau}) - \delta_i(\tau) h_i(X_{\tau}) \right) \right] \leq 0
\]
thanks to \( f_i \leq h_i \), we conclude that
\[
J_i(x, \tau, \rho_T) - J_i(x, \tau, \rho_R) = \mathbb{E}_x \left[ (1_A + 1_B) \left( F_i(\tau, \rho_T) - F_i(\tau, \rho_R) \right) \right] \leq 0.
\]

Next, fix \( S, S' \in B \) with \( S \supseteq S' \). For any \( x \in \Phi_i^R(S') \), if \( x \in S \), then \( x \in \Phi_i^T(S) \) by definition. Hence we assume \( x \notin S \) in the following. With \( x \in \Phi_i^R(S') \setminus S', (3.1) \) gives
\[
J_i(x, 0, \rho_R) > V_i^R(x, S') \geq V_i^T(x, S),
\]
(3.16)
where the last inequality follows from (3.14) and (3.2). Note that we must have \( x \notin R \). Indeed, if \( x \in R \), then \( \rho_R = 0 \) \( \mathbb{P}_x \)-a.s. and thus
\[
J_i(x, 0, \rho_R) = h_i(x) \leq g_i(x) = \sup_{1 \leq \tau \leq \rho^+_S} \mathbb{E}_x[F_i(\tau, \rho_R)] = V_i^R(x, S'),
\]
which contradicts the first inequality in (3.16). As \( x \notin R \) implies \( \rho_T \geq \rho_R > 0 \) \( \mathbb{P}_x \)-a.s., we observe that
\[
J_i(x, 0, \rho_T) = f_i(x) = J_i(x, 0, \rho_R) > V_i^T(x, S),
\]
where the inequality follows from (3.16). This along with \( x \notin S \) yields \( x \in \Phi_i^T(S) \). We therefore conclude that \( \Phi_i^R(S') \subseteq \Phi_i^T(S) \). \( \square \)

Corollary 3.7 Fix \( i \in \{1, 2\} \). Assume \( f_i \leq h_i \leq g_i \) and (3.13). Then for any \( T, R \in B \) with \( T \subseteq R \), we have \( \Gamma_i(T) \supseteq \Gamma_i(R) \), with \( \Gamma_i(\cdot) \) defined in (3.5).
**Proof** In view of (3.5), we have \( \Gamma_i(T) = \bigcup_{n \in \mathbb{N}} S^n_i(T) \) and \( \Gamma_i(R) = \bigcup_{n \in \mathbb{N}} S^n_i(R) \) with \( (S^n_i(T))_{n \in \mathbb{N}} \) and \( (S^n_i(R))_{n \in \mathbb{N}} \) defined in (3.4). Hence it suffices to show that \( S^n_i(T) \supseteq S^n_i(R) \) for all \( n \in \mathbb{N} \). By (3.15), \( S^1_i(T) = \Phi^T_i(\emptyset) \supseteq \Phi^R_i(\emptyset) = S^1_i(R) \). Using this and (3.15) again, we get \( S^2_i(T) = \Phi^T_i(S^1_i(T)) \supseteq \Phi^R_i(S^1_i(R)) = S^2_i(R) \). Applying (3.15) recursively in the same way then yields \( S^n_i(T) \supseteq S^n_i(R) \) for all \( n \in \mathbb{N} \). \( \square \)

The monotonicity of \( T \mapsto \Gamma_i(T), \ i \in \{1,2\} \), will play a crucial role in Theorem 4.2 below, contributing to the convergence of an alternating iterative procedure performed jointly by players 1 and 2.

### 4 The existence of inter-personal equilibria

In this section, we design an alternating iterative procedure to be performed jointly by the two players. As shown in Theorem 4.2 below, this procedure converges to a soft inter-personal equilibrium that is almost sharp. By a probabilistic modification of this iterative procedure and an appropriate use of Zorn’s lemma, we establish the existence of sharp inter-personal equilibria in Theorem 4.8 below. Explicit examples are presented to illustrate the procedure and the necessity of a supermartingale condition.

First, we observe that Theorem 3.5 already provides a sufficient condition for the existence of a sharp inter-personal equilibrium.

**Lemma 4.1** For each \( i \in \{1,2\} \), assume \( h_i \leq g_i \). If \((S, T) \in \mathcal{B} \times \mathcal{B}\) satisfies

\[
\Gamma_1(T) = S \quad \text{and} \quad \Gamma_2(S) = T,
\]

then \((S, T) \in \hat{\mathcal{E}}.\)

**Proof** By Theorem 3.5, \( S = \Gamma_1(T) \in \hat{\mathcal{E}}^T_1 \) and \( T = \Gamma_2(S) \in \hat{\mathcal{E}}^S_2 \). So \((S, T) \in \hat{\mathcal{E}}.\) \( \square \)

### 4.1 Construction of soft inter-personal equilibria

In order to achieve (4.1), we let the two players take turns to perform the *individual* iterative procedure (3.4). As the next result shows, such alternating iterations do converge, and the limit is guaranteed to be a soft inter-personal equilibrium.

**Theorem 4.2** For each \( i \in \{1,2\} \), assume \( f_i \leq h_i \leq g_i \) and (3.13). Let \((S_n, T_n)_{n \in \mathbb{N}_0}\) be a sequence in \( \mathcal{B} \times \mathcal{B} \) defined by \( S_0 := \emptyset \) and

\[
T_n := \Gamma_2(S_n) \quad \text{and} \quad S_{n+1} := \Gamma_1(T_n), \quad \forall n \in \mathbb{N}_0.
\]

Then \((S_n)\) is nondecreasing and \((T_n)\) is nonincreasing. By taking \( S_\infty := \bigcup_{n \in \mathbb{N}_0} S_n \) and \( T_\infty := \bigcap_{n \in \mathbb{N}_0} T_n \), we have \((S_\infty, T_\infty) \in \mathcal{E}\) with

\[
\Gamma_1(T_\infty) = S_\infty \quad \text{and} \quad \Gamma_2(S_\infty) \subseteq T_\infty.
\]
Proof Because $S_0 = \emptyset \subseteq S_1$, applying Corollary 3.7 for player 2 yields the inclusion $T_0 = \Gamma_2(S_0) \supseteq \Gamma_2(S_1) = T_1$. With $T_0 \supseteq T_1$, applying Corollary 3.7 for player 1 implies $S_1 = \Gamma_1(T_0) \subseteq \Gamma_1(T_1) = S_2$. Again, by $S_1 \subseteq S_2$, applying Corollary 3.7 for player 2 gives $T_1 = \Gamma_2(S_1) \supseteq \Gamma_2(S_2) = T_2$. Repeating this procedure for players 1 and 2 recursively, we see that $(S_n)$ is nondecreasing and $(T_n)$ is nonincreasing.

Next, let us show that $\Theta_1^{T_n}(S_n) = S_n$. Fix $x \in S_n = \bigcup_{n \in \mathbb{N}_0} S_n$. There exists $N \in \mathbb{N}$ such that $x \in S_{n+1} = \Gamma_1(T_n)$ for all $n > N$. By the fact that $\Gamma_1(T_n) \in \mathcal{E}_1^{T_n}$ (thanks to Proposition 3.2), $J_1(x, 0, \rho_{T_n}) \geq J_1(x, \rho_{T_n}^+, \rho_{T_n})$ for all $n \geq N$. As $n \to \infty$, (2.3) allows us to use the dominated convergence theorem so that we may conclude from Lemma 2.8 that

$$J_1(x, 0, \rho_{T_\infty}) \geq J_1(x, \rho_{S_\infty}^+, \rho_{T_\infty}), \quad (4.4)$$

which in turn implies $x \in \Theta_1^{T_\infty}(S_\infty)$. Hence $S_\infty \subseteq \Theta_1^{T_\infty}(S_\infty)$. On the other hand, for any $x \notin S_\infty = \bigcup_{n \in \mathbb{N}_0} S_n$, we have $x \notin S_{n+1} = \Gamma_1(T_n)$ for all $n \in \mathbb{N}$. Thanks again to the fact that $\Gamma_1(T_n) \in \mathcal{E}_1^{T_n}$, $x \notin \Gamma_1(T_n)$ implies $J_1(x, 0, \rho_{T_n}) \leq J_1(x, \rho_{S_{n+1}}^+, \rho_{T_n})$ for all $n \in \mathbb{N}$. As $n \to \infty$, we can argue as in (4.4) to get $J_1(x, 0, \rho_{T_\infty}) \leq J_1(x, \rho_{S_\infty}^+, \rho_{T_\infty})$, which implies $x \notin \Theta_1^{T_\infty}(S_\infty)$. Hence $(S_\infty)^c \subseteq (\Theta_1^{T_\infty}(S_\infty))^c$, or $\Theta_1^{T_\infty}(S_\infty) \subseteq S_\infty$. We thus conclude that $\Theta_1^{T_\infty}(S_\infty) = S_\infty$, i.e., $S_\infty \in \mathcal{E}_1^{T_\infty}$. Arguments similar to the above yield $\Theta_2^{S_\infty}(T_\infty) = T_\infty$, i.e., $T_\infty \in \mathcal{E}_2^{S_\infty}$. This readily shows that $(S_\infty, T_\infty) \in \mathcal{E}$.

As $S_n \subseteq S_\infty$ by construction for all $n \in \mathbb{N}$, $\Gamma_2(S_n) \subseteq \Gamma_2(S_\infty) = T_n$ for all $n \in \mathbb{N}$ by Corollary 3.7, which in turn gives $\Gamma_2(S_\infty) \subseteq \bigcap_{n \in \mathbb{N}_0} T_n = T_\infty$. Similarly, as $T_n \supseteq T_\infty$ by construction for all $n \in \mathbb{N}$, Corollary 3.7 implies for all $n \in \mathbb{N}$ that $\Gamma_1(T_\infty) \supseteq \Gamma_1(T_n) = S_{n+1}$, which in turn gives $\Gamma_1(T_\infty) \supseteq \bigcup_{n \in \mathbb{N}} S_{n+1} = S_\infty$. Finally, recall that $S_\infty \in \mathcal{E}_1^{T_\infty}$. This together with Lemma 3.4 implies that $\Gamma_1(T_\infty) \subseteq S_\infty$. We then conclude that $\Gamma_1(T_\infty) = S_\infty$. \hfill \Box

The next example illustrates the alternating iterative procedure (4.2) explicitly.

Example 4.3 Let $X = \{x_0, x_1, x_2, \ldots\}$ contain countably many states and assume

$$\mathbb{P}_{x_{n+1}}[X_1 = x_n] = 1 \quad \text{for } n = 0, 1, 2, \ldots,$$

$$\mathbb{P}_{x_0}[X_1 = x_0] = 1 - \varepsilon \quad \text{and} \quad \mathbb{P}_{x_0}[X_1 = x_1] = \varepsilon \quad \text{for some } \varepsilon \in [0, 1). \quad (4.5)$$

Take $M > 1$ such that

$$\delta_2(2) < 1/M < \delta_2(1). \quad (4.6)$$

Additionally, take $L > 1$ and consider for the two players the payoff functions

$$f_1(x_n) = 1 \quad \text{and} \quad g_1(x_n) = L \quad \text{for } n = 0, 1, 2, \ldots, \quad (4.7)$$

$$f_2(x_0) = 0, \quad f_2(x_n) = 1 \quad \text{for } n = 1, 2, \ldots, \quad (4.8)$$

$$g_2(x_n) = M \quad \text{for } n = 0, 1, 2, \ldots, \quad (4.9)$$
while $h_1$ (resp. $h_2$) is allowed to be any function that satisfies $f_1 \leq h_1 \leq g_1$ (resp. $f_2 \leq h_2 \leq g_2$) on $\mathbb{X}$.

For $\varepsilon \in [0, 1)$ small enough, we claim that the alternating iterative procedure (4.2) gives rise to

\[
S_0 = \emptyset, \quad T_0 = \{x_1, x_2, \ldots\},
S_1 = \{x_0\}, \quad T_1 = \{x_2, x_3, \ldots\},
S_2 = \{x_0, x_1\}, \quad T_2 = \{x_3, x_4, \ldots\},
\ldots
S_n = \{x_0, x_1, \ldots, x_{n-1}\}, \quad T_n = \{x_{n+1}, x_{n+2}, \ldots\}.
\]

(4.10)

First, starting with $S_0 = \emptyset$, we deduce from (4.8) that for any $S \in \mathcal{B} = 2^\mathbb{X}$,

\[
V_2^{S_0}(x_n, S) = \sup_{1 \leq \tau \leq \rho_S^+} \mathbb{E}_{x_n}[F_2(\tau, \rho_{S_0})]
\]

\[
\begin{cases}
0 = f_2(x_n) = J_2(x_n, 0, \rho_{S_0}) & \text{for } n = 0, \\
1 = f_2(x_n) = J_2(x_n, 0, \rho_{S_0}) & \text{for } n = 1, 2, \ldots
\end{cases}
\]

(4.11)

This implies $\Phi_2^{S_0}(\emptyset) = \{x_1, x_2, \ldots\}$ and $\Phi_2^{S_0}(\{x_1, x_2, \ldots\}) = \{x_1, x_2, \ldots\}$, which in turn yields $T_0 := \Gamma_2(S_0) = \{x_1, x_2, \ldots\}$. Next, thanks to (4.5) and (4.7), for any $S \in 2^\mathbb{X}$,

\[
V_1^{T_0}(x_n, S) = \sup_{1 \leq \tau \leq \rho_S^+} \mathbb{E}_{x_n}[F_1(\tau, \rho_{T_0})]
\]

\[
\begin{cases}
(1 - \varepsilon)\delta_1(1) + \varepsilon L(1 + (1 - \varepsilon) + (1 - \varepsilon)^2 + \cdots) \\
= (1 - \varepsilon)\delta_1(1) + \frac{\varepsilon L}{1 + \varepsilon} < 1 = f_1(x_n) = J_1(x_n, 0, \rho_{T_0}) & \text{for } n = 0,
\end{cases}
\]

(4.12)

where the inequality $(1 - \varepsilon)\delta_1(1) + \frac{\varepsilon L}{1 + \varepsilon} < 1$ holds as $\varepsilon \in [0, 1)$ is small enough.

This implies $\Phi_1^{T_0}(\emptyset) = \{x_0\}$ and $\Phi_1^{T_0}(\{x_0\}) = \{x_0\}$, so that $S_1 := \Gamma_1(T_0) = \{x_0\}$. Now thanks to (4.5), (4.8) and (4.9), for any $S \in 2^\mathbb{X}$ such that $x_0 \notin S$,

\[
V_2^{S_1}(x_n, S) = \sup_{1 \leq \tau \leq \rho_S^+} \mathbb{E}_{x_n}[F_2(\tau, \rho_{S_1})]
\]

\[
\begin{cases}
g_1(x_n) \geq h_1(x_n) = J_1(x_n, 0, \rho_{T_0}) & \text{for } n \geq 1,
\end{cases}
\]

(4.13)

where we use (4.6) in the last two lines. This implies $\Phi_2^{S_1}(\emptyset) = \{x_2, x_3, \ldots\}$ and $\Phi_2^{S_1}(\{x_2, x_3, \ldots\}) = \{x_2, x_3, \ldots\}$ so that $T_1 := \Gamma_2(S_1) = \{x_2, x_3, \ldots\}$. By similar arguments as above, we can derive $S_n$ and $T_n$ in (4.10) for all $n \geq 2$. By Theorem 4.2,
\((S_\infty, T_\infty) := (\bigcup_{n \in \mathbb{N}} S_n, \bigcap_{n \in \mathbb{N}} T_n) = (\emptyset, \emptyset)\) is a soft inter-personal equilibrium and satisfies (4.3). Observe that for any \(n = 0, 1, 2, \ldots\),

\[
V_2^X(x_n, \emptyset) = \sup_{1 \leq \tau \leq \rho^+} \mathbb{E}_{x_n}[F_2(\tau, \rho^+) = g_2(x_n) \geq h_2(x_n) = J_2(x_n, 0, \rho^x).
\]

It follows that \(\Phi_2^X(\emptyset) = \emptyset\) so that \(\Gamma_2(S_\infty) = \Gamma_2(X) = \emptyset = T_\infty\). That is, we have a stronger version of (4.3) where the inclusion there is an equality. Hence by Lemma 4.1, \((S_\infty, T_\infty) = (\emptyset, \emptyset)\) is in fact a sharp inter-personal equilibrium.

In view of (4.3) and Lemma 4.1, the soft inter-personal equilibrium \((S_\infty, T_\infty)\) constructed in Theorem 4.2 is nearly a sharp one. It is natural to ask whether the inclusion in (4.3) is actually an equality (as in Example 4.3), so that \((S_\infty, T_\infty)\) is sharp in general. The next example shows that this is in general not the case: the inclusion in (4.3) can be strict and \((S_\infty, T_\infty)\) may fail to be sharp.

**Example 4.4** Let us extend the state space in Example 4.3 by including two additional states, i.e., \(X = \{x_0, x_1, x_2, \ldots\} \cup \{y, z\}\). The transition probabilities are specified as in (4.5), as well as

\[
\mathbb{P}_y[X_1 = x_n] = p_n > 0 \quad \text{with} \quad \sum_{n=0}^{\infty} p_n = 1 \quad \text{and} \quad \mathbb{P}_z[X_1 = y] = 1. \quad (4.14)
\]

Take \(M > 1\) such that (4.6) holds. Assume additionally that \(\delta_2 : [0, \infty) \to [0, 1]\) satisfies \(\delta_2(1)^2 < \delta_2(2)\). Take \(L > 1\) and define \(f_i\) and \(g_i\), \(i \in \{1, 2\}\), as in (4.7)–(4.9) on \(\{x_0, x_1, x_2, \ldots\}\), along with

\[
f_2(y) = M\delta_2(1), \quad f_2(z) \in (M\delta_2(1)^2 \lor \delta_2(2), M\delta_2(2)),
\]

\[
g_2(y) = g_2(z) = M, \quad (4.15)
\]

while \(f_1\) and \(g_1\) are allowed to take arbitrary nonnegative values on \(\{y, z\}\) as long as \(f_1 \leq g_1\). Also, \(h_1\) (resp. \(h_2\)) is allowed to be any function such that \(f_1 \leq h_1 \leq g_1\) (resp. \(f_2 \leq h_2 \leq g_2\)) on \(X\).

For \(\varepsilon \in [0, 1)\) small enough, we claim that the alternating iterative procedure (4.2) gives rise to

\[
S_0 = \emptyset, \quad T_0 = \{x_1, x_2, \ldots\} \cup \{y, z\},
\]

\[
S_1 = \{x_0\}, \quad T_1 = \{x_2, x_3, \ldots\} \cup \{y, z\},
\]

\[
S_2 = \{x_0, x_1\}, \quad T_2 = \{x_3, x_4, \ldots\} \cup \{y, z\},
\]

\[
\vdots
\]

\[
S_n = \{x_0, x_1, \ldots, x_{n-1}\}, \quad T_n = \{x_{n+1}, x_{n+2}, \ldots\} \cup \{y, z\}. \quad (4.16)
\]
First, note that the relations (4.11)–(4.13) remain true in our current setting. Now, starting with \( S_0 = \emptyset \), we deduce from (4.14), (4.8) and (4.15) that for any \( S \in \mathcal{B} = 2^X \),

\[
V^S_2(y, S) = \sup_{1 \leq \tau \leq \rho^+_S} \mathbb{E}_y[F_2(\tau, \rho_S)] < \delta_2(1) \leq f_2(y) = J_2(y, 0, \rho_S),
\]

\[
V^S_2(z, S) = \sup_{1 \leq \tau \leq \rho^+_S} \mathbb{E}_z[F_2(\tau, \rho_S)] \leq \max[\delta_2(1) f_2(y), \delta_2(2)]
\]

\[
= \max\{M \delta_2(1)^2, \delta_2(2)\} < f_2(z) = J_2(z, 0, \rho_S).
\]

These two inequalities together with (4.11) show that \( \Phi^S_2(\emptyset) = \{x_1, x_2, \ldots \} \cup \{y, z\} \) and \( \Phi^S_2(\{x_1, x_2, \ldots \} \cup \{y, z\}) = \{x_1, x_2, \ldots \} \cup \{y, z\} \), which immediately implies that \( T_0 := \Gamma_2(S_0) = \{x_1, x_2, \ldots \} \cup \{y, z\} \). Next, since \( \{y, z\} \subseteq T_0 \), we have for any \( S \in 2^X \) that \( V^S_1(x, S) = \sup_{1 \leq \tau \leq \rho^+_S} \mathbb{E}_x[F_1(\tau, \rho_0)] \geq g_1(x) \geq h_1(x) = J_1(x, 0, \rho_0) \) for \( x \in \{y, z\} \). This together with (4.12) implies that as \( \epsilon \in [0, 1) \) is small enough, \( \Phi^S_1(\emptyset) = \{x_0\} \) and \( \Phi^S_1(\{x_0\}) = \{x_0\} \) so that \( S_1 := \Gamma_1(T_0) = \{x_0\} \). Thanks to (4.14), (4.5), (4.8) and (4.15), for any \( S \in 2^X \) such that \( x_0 \notin S \),

\[
V^S_1(y, S) = \sup_{1 \leq \tau \leq \rho^+_S} \mathbb{E}_y[F_2(\tau, \rho_S)] \leq \delta_2(1) M = f_2(y) = J_2(y, 0, \rho_S),
\]

\[
V^S_1(z, S) = \sup_{1 \leq \tau \leq \rho^+_S} \mathbb{E}_z[F_2(\tau, \rho_S)] \leq \max \left\{ \delta_2(1) f_2(y), \delta_2(2) \left( (1 - O(\epsilon)) + O(\epsilon) M \right) \right\}
\]

\[
< f_2(z) = J_2(z, 0, \rho_S),
\]

where the last inequality holds as \( \epsilon \in [0, 1) \) is small enough, thanks to (4.15). The above two inequalities along with (4.13) imply \( \Phi^S_1(\emptyset) = \{x_2, x_3, \ldots \} \cup \{y, z\} \) and \( \Phi^S_1(\{x_2, x_3, \ldots \}) = \{x_2, x_3, \ldots \} \cup \{y, z\} \); so \( T_1 := \Gamma_2(S_1) = \{x_2, x_3, \ldots \} \cup \{y, z\} \). By similar arguments as above, we can derive \( S_n \) and \( T_n \) in (4.16) for all \( n \geq 2 \). Hence \( (S_\infty, T_\infty) := (\bigcup_{n \in \mathbb{N}} S_n, \bigcap_{n \in \mathbb{N}} T_n) = (\{x_0, x_1, x_2, \ldots \}, \{y, z\}) \).

Now one can easily check that \( V^S_\infty(x_n, \emptyset) = g_2(x_n) \geq h_2(x_n) = J_2(x_n, 0, \rho_{S_\infty}) \) for all \( n = 0, 1, 2, \ldots \). Moreover, due to (4.14),

\[
V^S_\infty(y, \emptyset) = \sup_{1 \leq \tau \leq \rho^+_\emptyset} \mathbb{E}_y[F_2(\tau, \rho_\emptyset)] \leq \delta_2(1) M = f_2(y) = J_2(y, 0, \rho_{S_\infty}),
\]

\[
V^S_\infty(z, \emptyset) = \sup_{1 \leq \tau \leq \rho^+_\emptyset} \mathbb{E}_z[F_2(\tau, \rho_\emptyset)] \leq \delta_2(2) M > f_2(z) = J_2(z, 0, \rho_{S_\infty}).
\]

Using (3.1), we conclude that \( \Phi^S_\infty(\emptyset) = \emptyset \). By (3.5), this gives \( \Gamma_2(S_\infty) = \emptyset \subseteq T_\infty \). Note that the inclusion in (4.3) is strict, so that we can no longer conclude from Lemma 4.1 that \( (S_\infty, T_\infty) \) is a sharp inter-personal equilibrium. In fact, \( (S_\infty, T_\infty) \) is not sharp. Recall from Theorem 3.5 that \( \emptyset = \Gamma_2(S_\infty) \in \hat{E}_2^{S_\infty} \). Then it can be checked
directly that \( T_\infty = \{y, z\} \in \mathcal{E}_2^S \), but \( T_\infty \notin \hat{\mathcal{E}}_2^S \). Specifically, \( T_\infty \) is strictly dominated by \( \emptyset \) at the state \( z \), as (4.15) implies that

\[
U_2^S(z, T_\infty) = J_2(z, 0, \rho_{S_\infty}) \lor J_2(z, \rho_{T_\infty}^+, \rho_{S_\infty}) = f_2(z) \lor M_2(1) \leq U_2^S(z, \emptyset).
\]

As \( T_\infty \notin \hat{\mathcal{E}}_2^S \), \( (S_\infty, T_\infty) = (\{x_0, x_1, x_2, \ldots\}, \{y, z\}) \) is not a sharp inter-personal equilibrium.

4.2 General existence of sharp inter-personal equilibria

In view of Example 4.4, the soft inter-personal equilibrium constructed in Theorem 4.2 need not be a sharp one. Thus the general existence of a sharp inter-personal equilibrium is still in question. To resolve this, we assume appropriate regularity for \( X \).

**Assumption 4.5** \( X \) has transition densities \( (p_t)_{t \geq 1} \) with respect to a measure \( \mu \) on \( (X, \mathcal{B}) \). That is, for each \( t = 1, 2, \ldots, p_t : X \times X \to \mathbb{R}_+ \) is a Borel-measurable function such that

\[
P_x[X_t \in A] = \int_A p_t(x, y) \mu(dy), \quad \forall x \in X \text{ and } A \in \mathcal{B}.
\]

**Remark 4.6** When \( X \) is at most countable, Assumption 4.1 is trivially satisfied with \( \mu \) being the counting measure. When \( X \) is uncountable, the literature is focused on the case \( X = \mathbb{R}^d \) for some \( d \geq 1 \). In this case, many discrete-time Markov processes \( X \) fulfil Assumption 4.1 (with \( \mu \) being Lebesgue measure). This includes in particular \( X \) defined by \( X_{t+1} := G(X_t, Z_t), t \in \mathbb{N}_0 \), where \( G \) is a Borel-measurable function and \( Z_t \) is a random variable independent of \( X \) such that \( G(x, Z_t) \) admits a probability density function for all \( x \in X \). This formula is commonly used in practical simulation of Markov processes; see e.g. Sart [38, Sect. 3] and Akakpo and Lacour [1, Sect. 5].

The next result, as a direct consequence of Kosorok [26, Lemma 6.5], will also play a crucial role.

**Lemma 4.7** Let \( \mu \) be a measure on \( (X, \mathcal{B}) \). For any \( A \subseteq X \), there exists a maximal Borel minorant of \( A \) under \( \mu \), defined as a set \( A^\mu \in \mathcal{B} \) with \( A^\mu \subseteq A \) such that for any \( A' \in \mathcal{B} \) with \( A' \subseteq A \), \( \mu(A' \setminus A^\mu) = 0 \).

Now we are ready to present the general existence of a sharp inter-personal equilibrium.

**Theorem 4.8** Suppose Assumption 4.5 holds. For \( i \in \{1, 2\} \), assume \( f_i \leq h_i \leq g_i \) and (3.13). Then there exists a sharp inter-personal equilibrium.
Proof Consider the set

\[ A := \{ (S, T) \in \mathcal{E} : \Gamma_1(T) \supseteq S \text{ and } \Gamma_2(S) \subseteq T \}. \]

By Theorem 4.2, \( A \neq \emptyset \). Let us define a partial order on \( A \) as follows: for any \((S, T)\) and \((S', T')\) in \( A \),

\[ (S, T) \succeq (S', T') \quad \text{if } S \supseteq S' \text{ and } T \subseteq T'. \quad (4.17) \]

Step 1: We show that every totally ordered subset of \( A \) has an upper bound in \( A \). Let \( (S_\alpha, T_\alpha)_{\alpha \in I} \) be a subset of \( A \) that is totally ordered, where \( I \) is a generic index set that may be uncountable. Consider \( S_0 := \bigcup_{\alpha \in I} S_\alpha \) and \( T_0 := \bigcap_{\alpha \in I} T_\alpha \). Recall the measure \( \mu \) in Assumption 4.5. By Lemma 4.7, there exists a maximal Borel minorant of \( T_0 \) under \( \mu \), which we denote by \( T_0^\mu \). For any \( T \in \mathcal{B} \) with \( T \subseteq T_0 \), since \( \mu(T \setminus T_0^\mu) = 0 \), we deduce from Assumption 4.5 that

\[ \mathbb{P}_x[X_t \in T \setminus T_0^\mu] = \int_{T \setminus T_0^\mu} p_T(x, y) \mu(dy) = 0 \]

for all \( x \in \mathcal{X} \) and \( t \in \mathbb{N} \). It follows that

\[ \mathbb{P}_x[X_t \in T \setminus T_0^\mu \text{ for some } t \in \mathbb{N}] = 0, \quad \forall x \in \mathcal{X}, \]

whenever \( T \in \mathcal{B} \) and \( T \subseteq T_0 \). \( (4.18) \)

On the other hand, observe that

\[ \Gamma_1(T) \supseteq S_0 \quad \text{for any } T \in \mathcal{B} \text{ with } T \subseteq T_0, \]

\[ \Gamma_2(S) \subseteq T_0 \quad \text{for any } S \in \mathcal{B} \text{ with } S \supseteq S_0. \quad (4.19) \]

Indeed, for any \( T \in \mathcal{B} \) with \( T \subseteq T_0 = \bigcap_{\alpha \in I} T_\alpha \), Corollary 3.7 and the definition of \( A \) yield \( \Gamma_1(T) \supseteq \Gamma_1(T_\alpha) \supseteq S_\alpha \) for all \( \alpha \in I \), which implies \( \Gamma_1(T) \supseteq S_0 \). Similarly, for any \( S \in \mathcal{B} \) with \( S \supseteq S_0 = \bigcup_{\alpha \in I} S_\alpha \), Corollary 3.7 and the definition of \( A \) give \( \Gamma_2(S) \subseteq \Gamma_2(S_\alpha) \subseteq T_\alpha \) for all \( \alpha \in I \), which implies \( \Gamma_2(S) \subseteq T_0 \).

Now define

\[ S_1 := \Gamma_1(T_0^\mu) \supseteq S_0 \quad \text{and} \quad T_1 := \Gamma_2(S_1) \subseteq T_0. \]

Note that the first inclusion follows from \( T_0^\mu \in \mathcal{B} \), \( T_0^\mu \subseteq T_0 \) and (4.19). As \( T_0^\mu \in \mathcal{B} \) implies \( S_1 := \Gamma_1(T_0^\mu) \in \mathcal{B} \), we deduce from \( S_1 \in \mathcal{B} \), \( S_1 \supseteq S_0 \) and (4.19) that the second inclusion above holds. With \( T_1 \in \mathcal{B} \) (thanks to \( S_1 \in \mathcal{B} \)) and \( T_1 \subseteq T_0 \), (4.18) gives \( \mathbb{P}_x[X_t \in T_1 \setminus T_0^\mu \text{ for some } t \in \mathbb{N}] = 0 \) for all \( x \in \mathbb{R}^d \). This readily implies

\[ \rho_{T_1 \cup T_0^\mu} = \rho_{T_0^\mu} \quad \mathbb{P}_x\text{-a.s., for } x \notin T_1 \cup T_0^\mu. \quad (4.20) \]

We claim that \( \Gamma_1(T_1 \cup T_0^\mu) = \Gamma_1(T_0^\mu) \). By the definition of \( \Gamma_1 \) in (3.5), it suffices to show \( S_1^n(T_1 \cup T_0^\mu) = S_1^n(T_0^\mu) \) for all \( n \in \mathbb{N} \). First, as \( T_1 = \Gamma_2(S_1) \), the last assertion of Proposition 3.2 implies \( T_1 \cap S_1 = \emptyset \). With \( S_1 = \Gamma_1(T_0^\mu) = \bigcup_{n \in \mathbb{N}} S_1^n(T_0^\mu) \), we obtain

\[ T_1 \cap S_1^n(T_0^\mu) = \emptyset, \quad \forall n \in \mathbb{N}. \quad (4.21) \]
Now for $n = 1$, (3.15) implies $S^1_1(T_1 \cup T_0^\mu) = \Phi_1(T_1 \cup T_0^\mu) = \Phi_1(T_1 \cup T_0^\mu) = S^1_1(T_0^\mu)$. For any $x \in S^1_1(T_0^\mu)$, due to $T_0^\mu \cap S^1_1(T_0^\mu) = \emptyset$ (by Proposition 3.2) and (4.21), we must have $x \notin T_1 \cup T_0^\mu$. Observe that

$$J_1(x, 0, \rho_{T_1 \cup T_0^\mu}) = J_1(x, 0, \rho_{T_0^\mu}) > V_1^{T_0^\mu}(x, \emptyset) = V_1^{T_1 \cup T_0^\mu}(x, \emptyset),$$  

where the first and third equalities follow from (4.20) and the inequality from the definition of $S^1_1(T_0^\mu) = \Phi_1(T_0^\mu)$ in (3.1). This shows $x \in \Phi_1(T_1 \cup T_0^\mu)$. Hence we obtain $S^1_1(T_0^\mu) \subseteq S^1_1(T_1 \cup T_0^\mu)$ and conclude that $S^1_1(T_1 \cup T_0^\mu) = S^1_1(T_0^\mu)$.

Now suppose that $S^k_1(T_1 \cup T_0^\mu) = S^k_1(T_0^\mu)$ for some $k \geq 1$. Thanks to (3.15) again, $S^{k+1}_1(T_1 \cup T_0^\mu) \subseteq S^{k+1}_1(T_0^\mu)$. Fix $x \in S^{k+1}_1(T_0^\mu)$. By using Proposition 3.2 and (4.20) as above, we get $x \notin T_1 \cup T_0^\mu$. If $x \in S^1_1(T_1 \cup T_0^\mu)$, then $x \in S^{k+1}_1(T_1 \cup T_0^\mu)$ trivially, by the definition of $S^{k+1}_1(T_1 \cup T_0^\mu)$ in (3.4). If $x \notin S^1_1(T_1 \cup T_0^\mu) = S^1_1(T_0^\mu)$, then by the definition of $S^{k+1}_1(T_0^\mu)$,

$$J_1(x, 0, \rho_{T_1 \cup T_0^\mu}) > V_1^{T_0^\mu}(x, S^k_1(T_0^\mu)).$$

By (4.20) and the above inequality, we may argue similarly as in (4.22) to get

$$J_1(x, 0, \rho_{T_1 \cup T_0^\mu}) = J_1(x, 0, \rho_{T_0^\mu}) > V_1^{T_0^\mu}(x, S^k_1(T_0^\mu)) = V_1^{T_1 \cup T_0^\mu}(x, S^k_1(T_1 \cup T_0^\mu)) = V_1^{T_1 \cup T_0^\mu}(x, S^k_1(T_1 \cup T_0^\mu)), $$

where the second equality stems from $S^k_1(T_0^\mu) = S^k_1(T_1 \cup T_0^\mu)$. It then follows that $x \in S^{k+1}_1(T_1 \cup T_0^\mu)$. Hence we obtain $S^{k+1}_1(T_0^\mu) \subseteq S^{k+1}_1(T_1 \cup T_0^\mu)$ and conclude that $S^{k+1}_1(T_1 \cup T_0^\mu) = S^{k+1}_1(T_0^\mu)$. By induction, this shows that $S^n_1(T_0^\mu) = S^n_1(T_1 \cup T_0^\mu)$ for all $n \in \mathbb{N}$, as desired.

By Corollary 3.7 and $\Gamma_1(T_1 \cup T_0^\mu) = \Gamma_1(T_0^\mu)$,

$$S_2 := \Gamma_1(T_1) \supseteq \Gamma_1(T_1 \cup T_0^\mu) = \Gamma_1(T_0^\mu) = S_1,$$

$$T_2 := \Gamma_2(S_2) \subseteq \Gamma_2(S_1) = T_1.$$

Now by defining $S_{n+1} := \Gamma_1(T_n)$ and $T_{n+1} := \Gamma_2(S_{n+1})$ for all $n \geq 3$, we can follow the same argument as in the proof of Theorem 4.2 to show that $(S_n)$ is non-decreasing, $(T_n)$ is non-increasing, and $(S_\infty, T_\infty) \in A$ with $S_\infty := \bigcup_{n \in \mathbb{N}} S_n$ and $T_\infty := \bigcap_{n \in \mathbb{N}} T_n$. By construction, $S_\infty \supseteq S_0 \supseteq S_\alpha$ and $T_\infty \subseteq T_0 \subseteq T_\alpha$ for all $\alpha \in I$. Hence $(S_\infty, T_\infty) \in A$ is an upper bound for $(S_\alpha, T_\alpha)_{\alpha \in I}$.

**Step 2: Applying Zorn’s lemma.**

As every totally ordered subset of $A$ has by Step 1 an upper bound in $A$, Zorn’s lemma implies that there exists a maximal element in $A$, denoted by $(\bar{S}, \bar{T}) \in A$. We claim that $(\bar{S}, \bar{T}) \in \hat{E}$. Set $S_0 := \bar{S}$, $T_0 := \bar{T}$ and define

$$S_{n+1} := \Gamma_1(T_n), \quad T_{n+1} := \Gamma_2(S_{n+1}), \quad \forall n \geq 0.$$
Thanks to $\Gamma_1(T_0) \supseteq S_0$ and $\Gamma_2(S_0) \subseteq T_0$ (as $(S_0, T_0) = (\overline{S}, \overline{T}) \in A$), we may apply Corollary 3.7 recursively to show that $(S_n)$ is nondecreasing and $(T_n)$ is non-increasing. Then by the same argument as in the proof of Theorem 4.2, we obtain $(S_\infty, T_\infty) \in A$ with $S_\infty := \bigcup_{n \in \mathbb{N}_0} S_n$ and $T_\infty := \bigcap_{n \in \mathbb{N}_0} T_n$. By construction, $S_\infty \supseteq S_0 = \overline{S}$ and $T_\infty \subseteq T_0 = \overline{T}$. But since $(\overline{S}, \overline{T})$ is a maximal element of $A$ (under the partial order (4.17)), we must have $S_\infty = S_0$ and $T_\infty = T_0$. This in particular implies $S_1 = S_0$ and $T_1 = T_0$, so that

$$\Gamma_1(\overline{T}) = \Gamma_1(T_0) = S_1 = S_0 = \overline{S},$$

$$\Gamma_2(\overline{S}) = \Gamma_2(S_0) = \Gamma_2(S_1) = T_1 = T_0 = \overline{T}.$$  

By Lemma 4.1, this readily implies that $(\overline{S}, \overline{T}) \in \hat{\mathcal{E}}$. \hfill \qed

**Remark 4.9** In view of (2.1) and (2.2), the condition $f_i \leq h_i \leq g_i$ (in Theorems 4.2 and 4.8) encourages each player to wait/continue until the other player stops, so as to obtain a larger reward. Consequently, each player faces the tradeoff between the potential (generous) gain from outlasting the other player and the cost of waiting that increases with time (due to discounting and possible loss of opportunity). Thus our Dynkin game exemplifies the “war of attrition” in game theory. The negotiation example in Sect. 5 below well demonstrates this “war”: Each firm intends to wait until the other firm gives in so as to seal the best deal, while subject to the impact of discounting and the varying cost of project initiation.

**Remark 4.10** In a classical (time-consistent) nonzero-sum Dynkin game, the condition $f_i \leq h_i \leq g_i$ ensures that a Nash equilibrium, as a tuple of pure stopping times $(\tau^*, \sigma^*)$, exists; see e.g. Hamadène and Zhang [19]. Without the condition $f_i \leq h_i \leq g_i$, a Nash equilibrium $(\tau^*, \sigma^*)$ need not exist, as shown in Laraki and Solan [27]. One needs to consider randomised strategies to possibly establish the existence of a Nash equilibrium, as a tuple of randomised strategies. Still, in some cases, only an $\varepsilon$-Nash equilibrium is known to exist; see e.g. Shmaya and Solan [39], Ferenstein [18], Laraki and Solan [27].

In this paper, as we assume $f_i \leq h_i \leq g_i$ (cf. Theorems 4.2 and 4.8), our focus on pure strategies is consistent with the literature. If we drop the condition $f_i \leq h_i \leq g_i$, many arguments will no longer hold and we expect the use of randomised strategies to become indispensable. Randomised strategies for time-inconsistent stopping problems have recently been proposed and analysed by Bayraktar et al. [3] in discrete time and by Christensen and Lindensjö [9] in continuous time. It is of interest as future research to modify their definitions and allow randomised strategies in our Dynkin game.

### 4.3 Discussion on the supermartingale condition

It is worth noting that while the supermartingale condition (3.13) is required in Theorems 4.2 and 4.8, it does not play a role in Theorem 3.5. Because the one-player iterative procedure (3.4) is by construction monotone, it converges without the need of any other condition. It is much more complicated for the two-player alternating iterative procedure (4.2) to converge. The monotonicity of (3.4) only ensures that each
iteration (performed by one of the two players) converges to a stopping policy, but says nothing about whether the two resulting sequences of policies (one sequence for each player) will actually converge. It is the supermartingale condition (3.13) that brings about the monotonicity for these two sequences of policies (on the strength of Corollary 3.7), leading to an inter-personal equilibrium between the two players.

When (3.13) fails, the monotonicity in Corollary 3.7 no longer holds in general, and there may exist no inter-personal equilibrium, soft or sharp. To demonstrate this, consider a three-state model

$$\mathbb{X} = \{a, b, c\} \quad \text{with} \quad \mathbb{P}_x[X_1 = y] \begin{cases} = 0 & \text{for} \ (x, y) = (a, c), \\ > 0 & \text{otherwise}. \end{cases}$$

(4.23)

Given $M > 0$, define the payoff functions by

$$
\begin{align*}
&f_1(a) = 1, \quad g_1(a) = M^2, \quad f_2(a) = M, \quad g_2(a) = M + 1, \\
&f_1(b) = M, \quad g_1(b) = M + 1, \quad f_2(b) = 1, \quad g_2(b) = 2, \\
&f_1(c) = 1, \quad g_1(c) = 2, \quad f_2(c) = M^2, \quad g_2(c) = M^2 + 1,
\end{align*}
$$

(4.24)

and

$$h_i(x) = \frac{1}{2} (f_i(x) + g_i(x)), \quad \forall x \in \mathbb{X} \text{ and } i \in \{1, 2\}.$$  

(4.25)

**Proposition 4.11** Under (4.23)–(4.25), if $M > 0$ is large enough, (3.13) is violated and there exists no soft inter-personal equilibrium.

**Proof** Let $p_{xy} := \mathbb{P}_x[X_1 = y]$ for all $x, y \in \mathbb{X}$. Then

$$
\mathbb{E}_x[\delta_1(1)g_1(X_1)] = \delta_1(1)(p_{ca}M^2 + p_{cb}(M + 1) + p_{cc} \times 2) > 2 = g_1(c),
$$

where the inequality holds if $M > 0$ is large enough. This readily shows that (3.13) is violated.

For any $S, T \in \mathcal{B} = 2^\mathbb{X}$ and $x \in \mathbb{X}$, we deduce from the definitions of $f_i$, $h_i$, $g_i$, $i \in \{1, 2\}$, that for each $i \in \{1, 2\}$,

$$J_i(x, \rho^+_S, \rho_T) = k_0 + k_1M + k_2M^2 \quad \text{for some } k_0 \in [0, 2] \text{ and } k_1, k_2 \in [0, 1].$$

(4.26)

Moreover,

$$\rho_T > 0 \iff k_0 < 2, k_1 < 1 \text{ and } k_2 < 1.$$  

(4.27)

By (4.26) and (4.27), it can be checked that

$$
\begin{align*}
&\mathcal{E}_2^{\mathbb{X}} = \mathcal{E}_2^{[a,c]} = \mathcal{E}_2^{[c]} = \emptyset, \quad \text{but } \mathcal{E}_1^{\emptyset} = \{\{b\}\}, \\
&\mathcal{E}_2^{[a,b]} = \mathcal{E}_2^{[a]} = \mathcal{E}_2^{[\emptyset]} = \{\{c\}\}, \quad \text{but } \mathcal{E}_1^{[c]} = \{\{b\}\}, \\
&\mathcal{E}_2^{[b,c]} = \{\{a\}\}, \quad \text{but } \mathcal{E}_1^{[a]} = \emptyset, \\
&\mathcal{E}_2^{[b]} = \{\{a, c\}\}, \quad \text{but } \mathcal{E}_1^{[a,c]} = \emptyset.
\end{align*}
$$

(4.28)
This readily shows that there exists no \((S, T) \in 2^X \times 2^X\) such that \(S \in \mathcal{E}^T_1\) and \(T \in \mathcal{E}^S_2\), i.e., there exists no soft inter-personal equilibrium.

In the following, we show the derivation of \(\mathcal{E}^T_1^{[c]} = \{\{b\}\}\) in detail; all other identities in (4.28) can be proved in a similar manner. From the definitions of \(f_1\) and \(g_1\), we have \(J_1(b, 0, \rho_{[c]}^+) = f_1(b) = M\) and \(J_1(b, \rho_{[b]}^+, \rho_{[c]}) = k_0 < 2\). Hence for \(M > 0\) large enough,

\[
b \in \{x \in X : J_1(x, 0, \rho_{[c]}) > J_1(x, \rho_{[b]}^+, \rho_{[c]})\} = \Theta_1^{[c]}(\emptyset),
\]

which implies \(\Theta_1^{[c]}(\emptyset) \neq \emptyset\). Also, as \(J_1(b, \rho_{[b]}^+, \rho_{[c]}) = k_0 < 2\), we can similarly conclude that for \(M > 0\) large enough,

\[
b \in \{x \in \{b, c\} : J_1(x, 0, \rho_{[c]}) > J_1(x, \rho_{[b]}^+, \rho_{[c]})\} \subseteq \Theta_1^{[c]}(\{a\}),
\]

which implies that \(\Theta_1^{[c]}(\{a\}) \neq \{a\}\). Note that

\[
J_1(c, 0, \rho_{[c]}) = h_1(c) < g_1(c) = J_1(c, \rho_{[b]}^+, \rho_{[c]}), \quad \forall S \in 2^X. \tag{4.29}
\]

Taking \(S = \{c\}\), \(S = \{b, c\}\) and \(S = X\) in (4.29) immediately shows that we have \(c \notin \Theta_1^{[c]}(\{c\})\), \(c \notin \Theta_1^{[c]}(\{b, c\})\) and \(c \notin \Theta_1^{[c]}(X)\), respectively. We then conclude that \(\Theta_1^{[c]}(\{c\}) \neq \{c\}\), \(\Theta_1^{[c]}(\{b, c\}) \neq \{b, c\}\) and \(\Theta_1^{[c]}(X) \neq X\). Now observe that \(J_1(b, \rho_{[b]}^+, \rho_{[c]})\) and \(J_1(a, \rho_{[b]}^+, \rho_{[c]})\) are both of the form \(k_0 + k_1 M\) with \(k_1 < 1\) (recall (4.27)). Thus for \(M > 0\) large enough,

\[
J_1(b, 0, \rho_{[c]}) = f_1(b) = M > k_0 + k_1 M = J_1(b, \rho_{[b]}^+, \rho_{[c]}),
\]

\[
J_1(a, 0, \rho_{[c]}) = f_1(a) = 1 < k_0 + k_1 M = J_1(a, \rho_{[b]}^+, \rho_{[c]}). \tag{4.30}
\]

This together with (4.29) implies that \(\Theta_1^{[c]}(\{b\}) = \{b\}\). Since \(J_1(a, \rho_{[a,b]}^+, \rho_{[c]})\) (resp. \(J_1(a, \rho_{[a,c]}^+, \rho_{[c]})\)) is also of the form \(k_0 + k_1 M\), the inequality in (4.30) shows that for \(M > 0\) large enough, \(a \notin \Theta_1^{[c]}(\{a, b\})\) (resp. \(a \notin \Theta_1^{[c]}(\{a, c\})\)). Hence we conclude that \(\Theta_1^{[c]}(\{a, b\}) \neq \{a, b\}\) and \(\Theta_1^{[c]}(\{a, c\}) \neq \{a, c\}\). In view of the above derivations, \(\{b\}\) is the only intra-personal equilibrium for player 1 with respect to player 2’s policy \(c\), i.e., \(\mathcal{E}^T_1^{[c]} = \{\{b\}\}\). \(\square\)

**Remark 4.12** In (4.28), \(\mathcal{E}^T_i\) is a singleton for \(i \in \{1, 2\}\) and all \(T \in 2^X\). The single element in \(\mathcal{E}^T_i\) is trivially the optimal intra-personal equilibrium, which can be recovered by \(\Gamma_i(T)\) in (3.5) thanks to Theorem 3.5. In other words, (4.28) implies that

\[
\Gamma_2(X) = \Gamma_2([a, c]) = \Gamma_2([c]) = \emptyset, \quad \Gamma_2([a, b]) = \Gamma_2([a]) = \Gamma_2(\emptyset) = \{c\},
\]

\[
\Gamma_2([b, c]) = \{a\}, \quad \Gamma_2([b]) = \{a, c\},
\]

\[
\Gamma_1(\emptyset) = \Gamma_1([c]) = \{b\}, \quad \Gamma_1([a]) = \Gamma_1([a, c]) = \emptyset. \tag{4.31}
\]

This clearly shows that the monotonicity of \(T \mapsto \Gamma_i(T)\) fails: Despite the inclusion \(\{c\} \subseteq \{b, c\} \subseteq X\), we have \(\Gamma_2([c]) = \Gamma_2(X) = \emptyset \subseteq \{a\} = \Gamma_2([b, c])\).
Remark 4.13 Another way to interpret (4.31) is that the alternating iterative procedure (4.2) will never converge, failing to provide any soft inter-personal equilibrium. Specifically, (4.31) indicates that the alternating iterations will always lead to loops, as listed below where player 1’s stopping policies are underlined and player 2’s stopping policies are doubly underlined.

1. $\emptyset \rightarrow \{c\} \rightarrow \{b\} \rightarrow \{a,c\} \rightarrow \emptyset \rightarrow \cdots$
2. $\{a\} \rightarrow \{c\} \rightarrow \{b\} \rightarrow \{a,c\} \rightarrow \emptyset \rightarrow \{c\} \rightarrow \cdots$
3. $\{b\} \rightarrow \{a,c\} \rightarrow \emptyset \rightarrow \{c\} \rightarrow \{b\} \rightarrow \cdots$
4. $\{c\} \rightarrow \emptyset \rightarrow \{b\} \rightarrow \{a,c\} \rightarrow \emptyset \rightarrow \{b\} \rightarrow \cdots$
5. $\{a,b\} \rightarrow \{c\} \rightarrow \{b\} \rightarrow \{a,c\} \rightarrow \emptyset \rightarrow \{c\} \rightarrow \cdots$
6. $\{a,c\} \rightarrow \emptyset \rightarrow \{b\} \rightarrow \{a,c\} \rightarrow \emptyset \rightarrow \{c\} \rightarrow \{b\} \rightarrow \cdots$
7. $\{b,c\} \rightarrow \{a\} \rightarrow \emptyset \rightarrow \{c\} \rightarrow \{b\} \rightarrow \{a,c\} \rightarrow \emptyset \rightarrow \cdots$
8. $\{a,b,c\} \rightarrow \emptyset \rightarrow \{b\} \rightarrow \{a,c\} \rightarrow \emptyset \rightarrow \{c\} \rightarrow \{b\} \rightarrow \cdots$

Remark 4.14 As $\delta$ is not specified in Proposition 4.11, the result admits an interesting implication for classical (time-consistent) nonzero-sum Dynkin games. To see this, let $\delta$ be an exponential discount function so that there is no time inconsistency. Proposition 4.11 shows that even when the state process $X$ is time-homogeneous and payoff functions are as simple as (4.24) and (4.25), the Dynkin game has no “time-homogeneous” Nash equilibrium—a Nash equilibrium as a tuple of two stopping regions $(S,T)$, one for each player. Indeed, if such a Nash equilibrium existed, it would be a sharp inter-personal equilibrium under Definition 2.6. If a Nash equilibrium in fact exists, it must be of a more complicated form (which is also suggested by the constructions in Hamadène and Zhang [19] and Laraki and Solan [27]).

5 Application: Negotiation with diverse impatience

In this section, we apply our theoretical results in Sect. 4 to a two-player real options valuation problem. The vast literature on real options, see e.g. McDonald and Siegel [30], Dixit and Pindyck [12, Chaps. 5–7] and Smith and Nau [40], among many others, focuses on a single firm’s corporate decision making and in particular on the optimal timing of a project’s initiation. In contrast to this, we study two firms’ joint decision making on their cooperation to initiate a project together, embedding real options valuation in a nonzero-sum Dynkin game.

Consider two firms who would like to cooperate to initiate a new project, such as entering a new market or developing a new product. Each firm has a proprietary skill/technology so that only when they cooperate can the project be successfully carried out. Once the project is initiated, it will generate a fixed total revenue $R > 0$. The cost of initiation $X$, on the other hand, evolves stochastically and is modelled by a discrete-time Markov process on a binomial tree as follows. There exist $u > 1$ and $p \in (0, 1)$ such that $X$ takes values in

$$\mathcal{X} = \{u^i : i = 0, \pm 1, \pm 2, \ldots\} \quad (5.1)$$
and satisfies
\[ \mathbb{P}_x[X_1/x = u] = p \quad \text{and} \quad \mathbb{P}_x[X_1/x = 1/u] = 1 - p, \quad \forall x \in \mathbb{X}, \]
which readily specifies the dynamics of \( X \) as it is Markov. Assume additionally that \( X \) is a submartingale, which corresponds to the condition \( p \geq \frac{1}{u+1} \). That is, the cost \( X \) has a tendency to increase over time, which incentivises the two firms to strike a deal of cooperation sooner rather than later.

In negotiating such a deal, each firm, leveraging on its proprietary skill/technology, insists on taking a fixed (risk-free) larger share
\[ N \in (R/2, R) \]
of the total revenue \( R > 0 \), while demanding the other firm to take the smaller share
\[ K := R - N \in (0, R/2) \]
of revenue and additionally incur the stochastic (risky) cost \( X \). Each firm either waits until the other gives in and takes the larger payoff \( N \), or gives in to the other and takes the smaller payoff \( (K - X_\tau)^+ \), where \( \tau \) denotes the firm’s (random) time to give in. This can be formulated in our Dynkin game framework as
\[
\begin{align*}
 f_1(x) &= f_2(x) = (K - x)^+ \quad \text{and} \quad g_1(x) = g_2(x) = N, \quad \forall x \in \mathbb{X}.
\end{align*}
\]
If the two firms happen to give in at the same time, they realise that both of them cannot endure any delay of a deal and will quickly agree on a deal that is more mutually beneficial. This corresponds to the requirement \( f_i \leq h_i \leq g_i, i \in \{1, 2\} \). In addition, we model the time preferences of the firms using a hyperbolic discount function, i.e., for \( i \in \{1, 2\} \),
\[
\delta_i(t) = \frac{1}{1 + \beta_i t},
\]
where \( \beta_i > 0 \) is a constant that represents the level of impatience of firm \( i \).

To facilitate the investigation of inter-personal equilibria between the two firms, we introduce a random walk \( Y \) defined on some probability space \( (\Omega, \bar{\mathcal{F}}, \mathbb{P}) \) such that
\[
P[Y_{t+1} - Y_t = 1] = p \quad \text{and} \quad P[Y_{t+1} - Y_t = -1] = 1 - p, \quad \forall t \in \mathbb{N}_0.
\]
Consider
\[
\xi := \inf\{t \geq 0 : Y_t = 0\}
\]
and define, for \( i \in \{1, 2\},
\[
\alpha^n_i := E_n \left[ \frac{1}{1 + \beta_i \xi} \right], \quad \forall n \in \mathbb{N}, \quad (5.2)
\]
where $E_n$ denotes the expectation under $P$ conditioned on $Y_0 = n$. Note that $\alpha_n, n \in \mathbb{N}$, can be computed explicitly. For instance,

$$\alpha^1_i = \sum_{k=1}^{\infty} \frac{(2k-1)p^{k-1}(1-p)^k}{2k-1} \frac{1}{1+\beta_i(2k-1)}.$$  \hfill (5.3)

**Lemma 5.1** For $i \in \{1, 2\}$, we have $\Gamma_i(\emptyset) = (0, y_i^*] \cap \mathbb{X}$, where

$$y_i^* := \min \left( \left[ \frac{1-\alpha_i^1}{1-u-\alpha_i^1} K, \infty \right] \cap \mathbb{X} \right).$$  \hfill (5.4)

**Proof** Observe from (2.1) and (2.2) that

$$J_i(x, \tau, \rho_i) = \mathbb{E}_x[F_i(\tau, \rho_i)] = \mathbb{E}_x[\delta_i(\tau) f_i(X_\tau)] = \mathbb{E}_x \left[ \frac{(K-X_\tau)^+}{1+\beta_i \tau} \right], \quad \forall x \in \mathbb{X}.$$  

Hence the one-player stopping analysis in Huang and Zhou [24, Sect. 5] applies to our current setting. The same arguments as therein (in particular [24, Proposition 5.5]) show that $(0, y_i^*] \cap \mathbb{X}$, with $y_i^*$ given as in (5.4), is player $i$’s unique optimal intra-personal equilibrium with respect to $\emptyset$, i.e., $\hat{\mathcal{E}}^0_i = \{(0, y_i^*] \cap \mathbb{X} \}$. As $\Gamma_i(\emptyset)$ belongs to $\hat{\mathcal{E}}^0_i$ by Theorem 3.5, it must coincide with $(0, y_i^*] \cap \mathbb{X}$. \hfill \Box

With the aid of Lemma 5.1, we now show that the alternating iterative procedure (4.2) always leads to a sharp inter-personal equilibrium. Let us divide our investigation into two cases, depending on the impatience levels of the two firms: $\beta_1 \leq \beta_2$ (Proposition 5.2) and $\beta_1 > \beta_2$ (Proposition 5.4).

**Proposition 5.2** Suppose $\beta_1 \leq \beta_2$. Then the alternating iterative procedure (4.2) terminates after one iteration and gives a sharp inter-personal equilibrium. That is,

$$(S_\infty, T_\infty) = (S_0, T_0) = (\emptyset, (0, y_2^*] \cap \mathbb{X}) \in \hat{\mathcal{E}}.$$  

**Proof** Note from (5.3) that $\beta_1 \leq \beta_2$ implies $\alpha^1_1 \geq \alpha^1_2$. By (5.4), this yields $y_1^* \leq y_2^*$. Following (4.2), we have $S_0 := \emptyset$ and $T_0 := \Gamma_2(S_0) = \Gamma_2(\emptyset) = (0, y_2^*] \cap \mathbb{X}$, where the last equality is due to Lemma 5.1. Now in view of Corollary 3.7, Lemma 5.1 and Proposition 3.2,

$$\Gamma_1(T_0) \subseteq \Gamma_1(\emptyset) = (0, y_1^*] \cap \mathbb{X},$$

$$\Gamma_1(T_0) \cap ((0, y_2^*] \cap \mathbb{X}) = \Gamma_1(T_0) \cap T_0 = \emptyset.$$  \hfill (5.5)

As $y_1^* \leq y_2^*$, the above two relations entail $\Gamma_1(T_0) = \emptyset = S_0$. This together with $\Gamma_2(S_0) = T_0$ shows that $(S_0, T_0) \in \hat{\mathcal{E}}$, thanks to Lemma 4.1. \hfill \Box

**Remark 5.3** For the case $\beta_1 \geq \beta_2$, by starting with $T_0 = \emptyset$ and switching the roles of $S_n$ and $T_n$ in (4.2), we may repeat the same arguments in the above proof to show that $T_0$ and $S_0 := \Gamma_1(T_0) = (0, y_1^*] \cap \mathbb{X}$ satisfy $\Gamma_2(S_0) = \emptyset = T_0$, which then yields $(S_0, T_0) = ((0, y_1^*] \cap \mathbb{X}, \emptyset) \in \hat{\mathcal{E}}$.
Proposition 5.4 Suppose $\beta_1 > \beta_2$. Then the alternating iterative procedure (4.2) falls into the following three cases:

- **Case 1:** $S_1 = \emptyset$. Then $(S_{\infty}, T_{\infty}) = (\emptyset, (0, y_1^*) \cap X)$ is in $\mathcal{E}$.
- **Case 2:** $S_1 \neq \emptyset$ and $T_n = \emptyset$ for some $n > 1$. Then $(S_{\infty}, T_{\infty}) = ((0, y_1^*) \cap X, \emptyset)$ is in $\mathcal{E}$.
- **Case 3:** $S_1 \neq \emptyset$ and $T_n \neq \emptyset$ for all $n > 1$.
  - If (4.2) terminates in finitely many steps, there exist $y_2^* < c \leq y_1^*$ and $0 < d \leq y_2^*$ such that $(S_{\infty}, T_{\infty}) = ((c, y_1^*) \cap X, (0, d) \cap X)$ is in $\mathcal{E}$.
  - If (4.2) continues indefinitely, then $(S_{\infty}, T_{\infty}) = ((0, y_1^*) \cap X, \emptyset)$ is in $\mathcal{E}$.

**Proof** Note from (5.3) and (5.4) that $\beta_1 > \beta_2$ implies $y_1^* \geq y_2^*$. If $y_1^* = y_2^*$ (which happens when $\beta_1 - \beta_2 > 0$ is sufficiently small), the same arguments as in the proof of Proposition 5.2 apply, and we end up with Case 1. In the rest of the proof, we assume $y_1^* > y_2^*$. Following (4.2), $S_0 := \emptyset$ and $T_0 := \Gamma_2(S_0) = \Gamma_2(\emptyset) = (0, y_2^* \cap X)$, where the last equality is due to Lemma 5.1. If $S_1 := \Gamma_1(T_0) = \emptyset = S_0$, (4.2) terminates with $\Gamma_1(T_0) = S_0$ and $\Gamma_2(S_0) = T_0$. Hence $(S_{\infty}, T_{\infty}) = (S_0, T_0) = (\emptyset, (0, y_2^* \cap X) \in \mathcal{E}$ by Lemma 4.1. Alternatively, if $S_1 \neq \emptyset$ but $T_0 = \emptyset$ for some $n > 1$, following the argument in Remark 5.3 gives $(S_{n+1}, T_{n+1}) = ((0, y_1^*) \cap X, \emptyset) \in \mathcal{E}$.

It remains to deal with the situation where $S_1 \neq \emptyset$ and $T_n \neq \emptyset$ for all $n \geq 1$. By using Corollary 3.7, Lemma 5.1 and Proposition 3.2 as in (5.5), we get

$$S_1 := \Gamma_1(T_0) \subseteq \Gamma_1(\emptyset) = (0, y_1^*) \cap X,$$

$$S_1 \cap ((0, y_2^* \cap X) = \Gamma_1(T_0) \cap T_0 = \emptyset. \quad (5.6)$$

As $S_1 \neq \emptyset$ and $y_2^* < y_1^*$, the above two inequalities imply $S_1 = A \cap X$ for some nonempty $A \subseteq (y_2^*, y_1^*)$. With $X$ being a submartingale, the same argument as in the second half of the proof of Huang and Zhou [24, Lemma 5.1] can be repeated here, showing that the set $A$ must be connected, i.e., $A = [c_1, c_1']$ for some $c_1, c_1' \in X$ with $y_2^* < c_1 \leq c_1' \leq y_1^*$. Now by the same argument as in [24, Lemma 5.3], $c_1' \in X$ needs to be larger than or equal to $1 - \alpha_1 / u - \alpha_1$ as otherwise $S_1 = [c_1, c_1'] \cap X$ would not belong to $\mathcal{E}_T^{T_0}$ (which would contradict Proposition 3.2). We then conclude that $c_1' = y_1^*$ and thus $S_1 = [c_1, y_1^*] \cap X$. Next, since $T_1 := \Gamma_2(S_1) \neq \emptyset$ and $T_1 \subseteq T_0 = (0, y_2^* \cap X$, we must have $T_1 = B \cap X$ for some nonempty $B \subseteq (0, y_2^*)$. Again by the same argument as in the second half of the proof of [24, Lemma 5.1], $B$ is connected. Also, we must have $\inf B = 0$ because otherwise $T_1 = B \cap X \notin \mathcal{E}_2^{S_1}$ (which would contradict Proposition 3.2). Thus $T_1 = (0, d_1') \cap X$ for some $d_1' \in X$ with $0 < d_1' \leq y_2^*$. As $(S_n)$ is nondecreasing and $(T_n)$ is nonincreasing and $T_n \cap S_n = \Gamma_2(S_n) \cap S_n = \emptyset$ (by Proposition 3.2), there exist nonincreasing sequences $(c_n)$ and $(d_n)$ of positive reals with $d_n < c_n$ such that

$$S_n = [c_n, y_1^*] \cap X \quad \text{and} \quad T_n = (0, d_n) \cap X, \quad \forall n \in \mathbb{N}.$$ 

Now if there exists $n^* \in \mathbb{N}$ with $c_{n^*} = c_{n^*+1}$ or $d_{n^*-1} = d_{n^*}$, we get $\Gamma_2(S_{n^*}) = T_{n^*}$ and $\Gamma_1(T_{n^*}) = S_{n^*}$ so that $(S_{\infty}, T_{\infty}) = (S_{n^*}, T_{n^*}) \in \mathcal{E}$ by Lemma 4.1. If there exists no such $n^* \in \mathbb{N}$, the iterative procedure (4.2) continues indefinitely with $c_n \downarrow 0$ and $d_n \downarrow 0$, leading to $(S_{\infty}, T_{\infty}) = ((0, y_1^*) \cap X, \emptyset)$, which is in $\mathcal{E}$ by Remark 5.3. \qed
Finally, let us explore the more extreme situation where one firm is highly impatient while the other is highly patient.

**Corollary 5.5** If $\beta_1 > 0$ is sufficiently large and $\beta_2 > 0$ is sufficiently small, the alternating iterative procedure (4.2) yields $(S_\infty, T_\infty) = ((0, y_1^*) \cap X, \emptyset) \in \mathcal{E}$.

**Proof** Take $\beta_1 > \overline{\beta} := \frac{u}{u-1}(\frac{N}{K} - 1) + \frac{1}{u-1} > 0$. With $\beta_1$ fixed, in view of (5.3) and (5.1), there exists $\beta^* > 0$ small enough such that for $\beta_2 < \beta^*$, we have $\frac{1 - \alpha_2^1}{u - \alpha_2^1} < \frac{1 - \alpha_1^1}{u - \alpha_1^1}$ and the interval

$$
\left( \frac{1 - \alpha_2^1}{u - \alpha_2^1} K \right) \text{ contains at least two points in } X.
$$

(5.7)

Define $\beta := p\left(\frac{N}{K} - 1\right) > 0$ and take $\beta_2 < \min\{\beta^*, \beta\}$. Note that $\beta_1 > \overline{\beta} > \beta > \beta_2$, where the second inequality is due to $p < 1$. With $\overline{\beta} > \beta_2$, we deduce from (5.3), (5.4) and (5.7) that $y_1^* > y_2^*$ and the interval

$$(y_2^*, y_1^*) \text{ contains at least one point in } X.
$$

(5.8)

Following (4.2), $S_0 := \emptyset$ and $T_0 := T_2(\emptyset) = \Gamma_2(\emptyset) = (0, y_2^*) \cap X$, where the last equality is due to Lemma 5.1. By (5.8), $u y_2^* \in X$ must belong to $(y_2^*, y_1^*)$. Recall from (5.6) that

$$
S_1 \subseteq [u y_2^*, y_1^*) \cap X.
$$

(5.9)

For any $x \in [u y_2^*, y_1^*) \cap X$, observe that

$$
V_1^{T_0}(x, \emptyset) = \sup_{1 \leq \tau \leq \rho_T} \mathbb{E}_x[F_1(\tau, T_0)] \leq \frac{N}{1 + \beta_1} < K - \frac{1}{u} - K < \frac{1 - \alpha_1^1}{u - \alpha_1^1}K
$$

$$
\leq K - x = J_1(x, 0, \rho_{T_0}),
$$

(5.10)

where the second inequality follows from $\beta_1 > \overline{\beta}$, the third inequality is due to $u > 1$ and $0 < \alpha_1^1 < 1$ (recall (5.2)), and the last inequality stems from $x \in X$, $x < y_1^*$ and the definition of $y_1^*$ in (5.4). This shows that $[u y_2^*, y_1^*) \cap X \subseteq \Phi_1^{T_0}(\emptyset)$. Then in view of (3.4) and (3.5), $S_1 := \Gamma_1(T_0) \supseteq \Phi_1^{T_0}(\emptyset) \supseteq [u y_2^*, y_1^*) \cap X$. In particular, $S_1 \neq \emptyset$ as it must contain $u y_2^*$. If $T_n = \emptyset$ for some $n > 1$, Case 2 of Proposition 5.4 immediately gives $(S_\infty, T_\infty) = ((0, y_1^*) \cap X, \emptyset) \in \mathcal{E}$ as desired. Hence we assume $T_n \neq \emptyset$ for all $n \in \mathbb{N}$ in the rest of the proof.

As argued below (5.6), $T_1 := \Gamma_2(S_1) = (0, d_1] \cap X$ for some $d_1 \in X$ that satisfies $0 < d_1 \leq y_2^*$. We claim that $d_1 < y_2^*$. First, observe from (5.4) that $y_2^* < u \frac{1 - \alpha_1^1}{u - \alpha_2^1} K < K$. Consider the function

$$
\eta(x) := \frac{p(N - K) + x(1 - \frac{1}{u})}{K - x} \quad \text{for } x \in [0, K).
$$

(5.11)
Because $\eta'(x) > 0$ for all $x \in [0, K)$, we have $\beta = \frac{N}{K} - 1 = \eta(0) < \eta(x)$ for every $x \in (0, K)$. Recalling $\beta_2 < \beta$, we obtain $\beta_2 < \eta(x)$ for all $x \in (0, K)$. By direct calculation, this is equivalent to

$$(1 - p) \frac{K - x/u}{1 + \beta_2} + p \frac{N}{1 + \beta_2} > K - x, \quad \forall x \in (0, K). \quad (5.11)$$

Assume to the contrary that $d_1 = y_2^*$. Then

$$J_2(d_1, \rho_{T_1}, \rho_{S_1}) = (1 - p) \frac{K - d_1/u}{1 + \beta_2} + p \frac{N}{1 + \beta_2} > K - d_1 = J_2(d_1, 0, \rho_{S_1}),$$

where the inequality stems from $d_1 = y_2^* < K$ and (5.11). Thus $T_1 = (0, d_1] \cap \mathbb{X}$ is not in $E_2^0$, a contradiction to $T_1 = \Gamma_2(S_1) \in E_2^0$ (by Proposition 3.2). With $d_1 < y_2^*$ established, we conclude that $T_1 \subset T_0$. For any $x \in [ud_1, y_2^*) \cap \mathbb{X}$, the same calculation as in (5.10) (with $T_0$ replaced by $T_1$) yields $V_{T_1}^T(x, \emptyset) < J_1(x, 0, \rho_{T_1})$. By the same arguments as below (5.10), this gives $S_2 := \Gamma_1(T_1) \supseteq \Phi_1^T(\emptyset) \supseteq [ud_1, y_2^*) \cap \mathbb{X}$; in particular, $ud_1 \in S_2$. As $S_2 \supseteq S_1$ by construction, we deduce from $ud_1 \in S_2$, (5.9) and $d_1 < y_2^*$ that $S_2 \supseteq S_1$. By the same arguments as above, we can show recursively that $T_n \subset T_{n-1}$ and $S_{n+1} \supseteq S_n$ for all $n > 1$. Case 3 of Proposition 5.4 then implies $(S_\infty, T_\infty = ((0, y_1^*) \cap \mathbb{X}, \emptyset) \in \mathcal{E}$. \hfill \Box

Propositions 5.2 and 5.4, along with Corollary 5.5, admit interesting economic interpretations. Intuitively, a firm can demonstrate a strong determination not to give in so as to coerce the other firm into giving in in the negotiation. Whether this strategy will work depends on the impatience levels of the two firms. For the case where firm 1 is less impatient than firm 2 (i.e., $\beta_1 \leq \beta_2$), Proposition 5.2 shows that when firm 1 insists that it will never give in (i.e., $S_0 = \emptyset$), firm 2 indeed gives in by taking the stopping policy $T_0 = (0, y_2^*) \cap \mathbb{X}$, and $(S_0, T_0) = (\emptyset, (0, y_2^*) \cap \mathbb{X})$ is already a sharp inter-personal equilibrium. For the case where firm 1 is more impatient than firm 2 (i.e., $\beta_1 > \beta_2$), Proposition 5.4 shows that the negotiation becomes more complicated and firm 1’s coercion does not necessarily work. In particular, if firm 1 is sufficiently impatient and firm 2 is sufficiently patient, Corollary 5.5 shows that firm 1’s coercion must fail, and fail in a dramatic way—the coercer becomes coerced. While firm 1 started with $S_0 = \emptyset$, trying to coerce firm 2 into giving in, the alternating game-theoretic reasoning (4.2) eventually led to the sharp inter-personal equilibrium $((0, y_1^*) \cap \mathbb{X}, \emptyset)$. That is, it is firm 1 which ultimately gives in by taking the stopping policy $((0, y_1^*) \cap \mathbb{X}$, while firm 2 in the end decides not to stop at all.

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