A Lagrangian Stochastic Model of a Volcanic Eruption Column

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Abstract We present a Lagrangian stochastic model (LSM) of a volcanic plume in which the mean flow is provided by an integral plume model of the eruption column and fluctuations in the vertical velocity are modeled by a suitably constructed stochastic differential equation. This model is based on that of Bisignano and Devenish (2015, http://doi.org/10.1007/s10546-015-0055-3) with the mean flow provided by the integral plume model of Devenish (2013, http://doi.org/10.1016/j.jvolgeores.2013.02.015). The LSM is applied to the two eruptions considered by Costa et al. (2016, http://doi.org/10.1016/j.jvolgeores.2016.01.017) for the volcanic-plume intercomparison study with no ambient wind and to the hypothetical cases with ambient wind considered by Aubry et al. (2019, http://doi.org/10.1029/2019GL083975). Vertical profiles of the mass concentration computed from the LSM are compared with equivalent results from large eddy simulations (LES). The LSM captures the order of magnitude of the LES mass concentrations and aspects of their profiles: for example, in contrast with a standard integral plume model, that is, without fluctuations, the mass concentration computed from the LSM decays (to zero) toward the top of the plume consistent with the LES plumes. In the lower part of the plume, we show that the presence of ash leads to a peak in the mass concentration at the level at which there is a transition from a negatively buoyant jet to a positively buoyant plume. A quantitative analysis of the cases with ambient wind shows generally good agreement between the LSM and the LES for a number of quantities including the maximum concentration, the level at which it occurs, the maximum rise height, and the total erupted mass.

Plain Language Summary We present a simple one-dimensional model for a volcanic plume that accounts for fluctuations due to turbulence within the plume. It builds on existing integral models of volcanic plumes which are here used to provide the mean flow. We compare the results of this simple model with more sophisticated three-dimensional (3D) models and focus on vertical profiles of the mass concentration. In contrast with standard integral plume models, the concentration decays to zero at the top of the plume in qualitative agreement with the 3D data. Possible applications of the model include using the concentration profiles to initialize a long-range atmospheric dispersion model or a Bayesian inversion model. The model could also be used as a dynamic source model within an atmospheric dispersion model.

1. Introduction

Volcanic plumes represent the most powerful naturally occurring buoyant sources of airborne contaminants. The spread of volcanic ash downwind of the eruption column presents a significant hazard to aviation which motivates the development of mathematical models that enable its prediction. Models of the eruption column play a role in quantifying a volcanic source for a long-range dispersion model. While simple one-dimensional (1D) integral models of volcanic plumes (see, e.g., Costa et al., 2016, and references therein; Devenish, 2013; Glaze et al., 1997; Mastin, 2007; Woods, 1988, 1993) can provide the variation with height of bulk properties of the plume including the mass concentration, this latter quantity has the undesirable property that it tends to infinity at the top of the plume. The vertical profile of mass concentration produced by an integral model is then not suitable for initializing an atmospheric dispersion model such as the UK Met Office’s operational dispersion model NAME (Numerical Atmospheric-dispersion Modeling Environment) that is regularly used for modeling the dispersion of volcanic ash downwind of an eruption (e.g., Beckett et al., 2020). One motivation for this article is to present a model that can provide a more realistic vertical profile of the mass concentration.
A second motivation for this study is the development of a model for the explicit treatment of volcanic plumes within an atmospheric dispersion model such as NAME. The latter often takes a Lagrangian form as this provides a natural framework for modeling subgrid-scale features such as unresolved turbulence and point sources. In realistic applications of Lagrangian dispersion models, the resolved flow field is usually taken from a numerical weather prediction model and the unresolved turbulence is modeled by suitably constructed random increments. This class of model is known as a Lagrangian stochastic model (LSM): they can be rigorously formulated (Thomson, 1987) and have been successful at reproducing observations (e.g., Thomson & Wilson, 2012). By way of example, several studies have assessed the ability of NAME to model the long-range dispersion of volcanic ash in the atmosphere by making detailed comparisons with observations of the 2010 eruption of Eyjafjallajökul (e.g., Dacre et al., 2011; Devenish, Thomson, et al., 2012; Devenish, Francis, et al., 2012).

In the operational use of LSMS, it is usual for the model particles to move independently of each other through the flow field. Typically, many thousands of model particles are tracked and statistics such as the mean concentration are calculated from the ensemble of particles. The independence of the particles gives rise to an inherent difficulty in modeling a coherent process such as a volcanic plume in a single-particle LSM: in reality the motion of each fluid element depends on the buoyancy of all the fluid elements. This aspect of a buoyant flow will not arise naturally in a model in which all the model particles move independently.

Several authors have attempted to model simple Boussinesq plumes using a Lagrangian approach (e.g., Alessandrini et al., 2013; Anfossi et al., 1993; Heinz & van Dop, 1999; Luhr & Britter, 1992; Marro et al., 2014; Weil, 1994). In particular we consider the approach developed by Webster and Thomson (2002) and Bisignano and Devenish (2015) in which the mean flow is calculated from an integral plume model and the fluctuations are calculated using a suitably formulated stochastic differential equation (SDE). These models were developed for single-phase Boussinesq flows: here we extend this approach to the modeling of volcanic plumes. This forms the third and final motivation for this study, namely the development of a 1D mathematical model of a volcanic plume that incorporates fluctuations and any insight into the physics of the eruption column that this can provide.

In the next section we present the LSM which is formulated for a realistic atmosphere with non-uniform stability, ambient wind, and moisture. In Section 4, we consider numerical solutions of the LSM for the two cases considered in the volcanic-plume-model intercomparison project of Costa et al. (2016) with no ambient wind, the results of which are compared with equivalent large eddy simulations (LES). In Section 5, we compare numerical solutions of the LSM with equivalent LES for the hypothetical cases with ambient wind considered by Aubry et al. (2019). In both cases the LES results were computed with ASHEE, a compressible multiphase code (Cerminara, Esposti Ongaro, & Berselli, 2016).

2. Formulation of the Model

The model we present here is a combination of that of Devenish (2013), which is used to provide the mean flow of a volcanic plume, and that of Bisignano and Devenish (2015), on which the SDE for the vertical velocity fluctuations is based. As will be seen in Section 2.2 the mean velocity component affects the fluctuating velocity both directly and through the parameterization of the turbulence but not vice versa.

2.1. Integral Plume Model

The model of Devenish (2013) is here reformulated in terms of time, $t$, to facilitate coupling with an SDE for the vertical velocity fluctuations (see below). The governing equations take the form:

$$\frac{dQ_u}{dt} = Ev_p$$

(1)

$$\frac{dM_v}{dt} = (\rho_u - \rho_p)gb^2\bar{v}_p$$

(2)
\[ \frac{dM_i}{dt} = -Q_m \frac{dU_i}{dt} \quad i = x, y \]  

(3)

\[ \frac{dH}{dt} = \left( (1 - n_d) c_{pd} + n_d c_{pm} \right) T_a \bar{v}_p - g \rho_p \pi b^2 \bar{v}_p \bar{w}_p 
+ \left[ L_v + 273(c_{pv} - c_{pi}) \right] \frac{dQ}{dt} \]  

(4)

\[ \frac{dQ}{dt} = Eq_{ma} \bar{v}_p \]  

(5)

where \( Q_m = \rho_p \pi b^2 \bar{v}_p \) is the mass flux; \( Q_l = n_l Q_m \) is the total moisture flux (water vapor and liquid water; the model contains no ice); \( Q_i = n_i Q_m \) is the flux of liquid water; \( M_i = \sqrt{u^2 + v^2} \bar{w}_p \) are the horizontal components of the momentum flux; \( M_v = \bar{w}_p Q_m \) is the vertical component of the momentum flux; and \( H = c_{pm} T_p Q_m \) is the enthalpy flux. In Equations 1–5, and the expressions for the fluxes, \( \bar{v}_p = \sqrt{u^2 + v^2 + \bar{w}_p^2} \) is the total velocity in which \( u_p \) \((i = x, y)\) are the horizontal components of the plume velocity and \( \bar{w}_p \) is the vertical component of the plume velocity; \( n \) is a mass fraction and we note that \( n_i = n_v + n_l \); other symbols are listed in Table 1.

As is appropriate for a non-Boussinesq plume, the entrainment rate depends on the densities of both the plume and the ambient fluid (Morton, 1965; Ricou & Spalding, 1961; Rooney & Linden, 1996):

\[ E = 2 \pi b \sqrt{\rho_p \rho_a \bar{v}_p} \]  

(6)

### Table 1

| Symbol | Description | Units |
|--------|-------------|-------|
| \( Q_m \) | Mass flux | kg s\(^{-1}\) |
| \( Q_l \) | Total moisture flux | kg s\(^{-1}\) |
| \( Q_i \) | Liquid water flux | kg s\(^{-1}\) |
| \( M_i \) | Momentum flux \((i = x, y, z)\) | kg m s\(^{-2}\) |
| \( H \) | Enthalpy flux | J s\(^{-1}\) |
| \( \bar{v}_p \) | Total velocity of plume | m s\(^{-1}\) |
| \( \bar{w}_p \) | Vertical component of plume velocity | m s\(^{-1}\) |
| \( U_i \) | Horizontal components of ambient wind \((i = x, y)\) | m s\(^{-1}\) |
| \( g \) | Acceleration due to gravity | m s\(^{-2}\) |
| \( E \) | Entrainment rate | kg m s\(^{-1}\) |
| \( b \) | Plume radius | m |
| \( \rho_i \) | Density \((i = a, p)\) | kg m\(^{-3}\) |
| \( n_i \) | Mass fraction \((i = d, g, s, l, v, t)\) | kg kg\(^{-1}\) |
| \( T_i \) | Temperature \((i = a, p)\) | K |
| \( g \) | Acceleration due to gravity | m s\(^{-2}\) |
| \( q_i \) | Humidity \((i = a, p)\) | kg kg\(^{-1}\) |
| \( c_{pi} \) | Specific heat capacity at constant pressure \((i = p, d, v, l)\) | J kg\(^{-1}\) K\(^{-1}\) |
| \( L_v \) | Latent heat of vaporization at 0°C | J kg\(^{-1}\) |

Note. The subscripts \( x, y \) refer to the horizontal coordinates and \( z \) the vertical coordinate; \( a, p \) refer to ambient and plume, respectively; \( d, g, s, l, v, t \) refer to respectively dry gas, total gaseous phase, solid (ash) phase, liquid water, water vapor, and total moisture.
where \( u_e \) is the entrainment velocity. As the plume rises, sufficient heat may be transferred from the particulate material to the plume gas (assuming that the gas–solid mixture is approximately in thermal equilibrium) to allow the plume to rise due to buoyancy. Once \( \rho_p \leq \rho_v \), Equation 6 reduces to the familiar form \( E = 2\pi bp_u u_e \) (e.g., Mastin, 2007; Woods, 1988).

It is commonplace to assume that there are two entrainment mechanisms in a crosswind (e.g., Devenish, Rooney, Webster, et al., 2010; Hoult et al., 1969; Hoult & Weil, 1972; Webster & Thomson, 2002), one due to velocity differences normal to the plume axis and the other due to velocity differences parallel to the plume axis and that the two mechanisms are additive. Devenish, Rooney, Webster, et al. (2010) suggested that this additive entrainment assumption be an \( l^m \)-norm:

\[
u_e = \left( (\alpha \| \Delta u_h \|_1^m + (\beta \| \Delta u_v \|_\perp^m) \right)^{1/m}
\]

(7)

where \( \Delta u_h \) and \( \Delta u_v \) are the components of the relative velocity parallel to and perpendicular to the plume axis, respectively, \( \alpha \) and \( \beta \) are the entrainment coefficients associated with each entrainment mechanism and \( m \geq 1 \) is a tunable parameter. Throughout this study we take \( \alpha = 0.1, \beta = 0.5, \) and \( m = 3/2 \); these values are consistent with previous laboratory and numerical studies of purely buoyant plumes (e.g., Aubry, Carazzo, & Jellinek, 2017; Briggs, 1984; Devenish, Rooney, & Thomson, 2010; Devenish, Rooney, Webster, et al., 2010; Hoult & Weil, 1972; McNeal et al., 2019), with laboratory studies of jets with reversing buoyancy (Michaud-Dubuy et al., 2020) and with observations of volcanic eruptions (Aubry & Jellinek, 2018).

The plume density, bulk gas constant, and bulk specific heat capacity are calculated following Devenish (2013). The plume density is given by:

\[
\frac{1}{\rho_p} = \frac{n_g}{\rho_g} + \frac{n_s}{\rho_s} + \frac{n_l}{\rho_l}
\]

where \( n_g + n_s + n_l = 1 \) and \( n_g = n_a + n_v \). Above the lower part of the plume, the volume fraction of ash (and any liquid water) is sufficiently small that \( \rho_p \approx \rho_g / n_v \). The mass fraction of gas can be derived from:

\[
(1 - n_g - n_l) \rho_m = (1 - n_{g0}) \rho_{m0}
\]

where the subscript “0” indicates the value at the vent and no fallout of either solid material or liquid water is assumed. Assuming that the pressure inside the plume is equal to the ambient pressure, the gas density (which includes both dry air and water vapor) is given by:

\[
\rho_g = \frac{p_a}{R_p T_p}
\]

where \( p_a \) is the ambient pressure and \( R_p = q_{wp} R_v + (1 - q_{wp}) R_d \) is the bulk gas constant (Woods, 1993) for the plume in which \( R_v \) is the gas constant of water vapor and \( R_d \) is the gas constant of dry air. The bulk specific heat capacity (at constant pressure) is given by:

\[
c_{pp} = n_g c_{pd} + n_v c_{pv} + n_l c_{pl} + n_s c_{ps}
\]

where we assume that \( c_p \) of any phase is independent of temperature.

Phase changes between water vapor and liquid water are calculated following Devenish (2013): liquid water condensate is produced whenever the water vapor mixing ratio, \( r_v \), is larger than the saturation mixing ratio, \( r_{sat} \), that is, \( r_v = \max(r_v - r_{sat}, 0) \) where \( r_v \) is the liquid water mixing ratio and \( r_{sat} \) is the mixing ratio of the total water content. This can be expressed in terms of the mass fractions of water as:

\[
n_l = \max(n_l - n_{sat} r_{sat}, 0)
\]

which clearly allows for the possibility that liquid water can evaporate due to entrainment of dry air. Analytical expressions for \( r_{sat} \), which is a function of the dry pressure, \( p_{de} \), and the local temperature, \( T \), can be derived from the Clausius–Clapeyron equation on making use of \( r_{sat} = e_{sat} / p_{de} \) where \( e_{sat} \) is the saturation vapor
pressure and \( e = 0.62 \) is the ratio of the molecular mass of water vapor to dry air. For \(-35^\circ C \leq T \leq 35^\circ C\), a simpler expression is given by a modification of Tetens' empirical formula,

\[
e_{\text{sat}} = 6.112 \exp \left( \frac{17.65T}{T + 243.5} \right) \tag{8}
\]

which is accurate to within 0.3% (Emanuel, 1994, p. 117). (Note that Equation 8 requires that pressure is measured in hPa and \( T \) in degrees Celsius; to a good approximation \( p_d \approx p_w \).) Of course, one would expect much higher temperatures in a volcanic plume but condensation is not expected to occur until temperatures within the range \(-35^\circ C \leq T \leq 35^\circ C\) are encountered well above the plume source. Thus, for our purposes Equation 8 remains appropriate. We also assume that any liquid condensate that forms remains in the plume that is, the total water content is conserved.

2.2. Stochastic Differential Equation for Velocity Fluctuations

The form of the SDE for the vertical velocity fluctuations is based on that of Bisignano and Devenish (2015). However, unlike their model, we do not include an SDE for temperature fluctuations. In the construction of the model, it is useful to rewrite Equation 2 as an equation for the vertical velocity:

\[
\frac{d\bar{w}_p}{dt} = \frac{g(\rho_a - \rho_p)}{\rho_p} - \frac{\bar{v}_p \bar{w}_p E}{Q_m}. \tag{9}
\]

Equation 9 makes clear that the evolution of \( \bar{w}_p \) depends on the local buoyancy of the plume and that entrainment produces a deceleration of the plume. Because we expect the turbulence to scale with the mean velocity we would expect the evolution of the fluctuating vertical velocity, \( w_p' \), to be given by:

\[
\frac{dw_p'}{dt} = -\frac{\bar{v}_p w_p' E}{Q_m} \tag{10}
\]

where, for simplicity, we ignore any possible fluctuations in \( \rho_p \). We also assume that \( E \) depends only on the mean properties of the plume: we would not expect entrainment rates to vary much from one Lagrangian particle (or fluid parcel) to another as might be the case if \( E \) is allowed to be a function of both the mean and fluctuating components. Moreover, values of the entrainment coefficients have been calculated from the bulk properties of buoyant plumes. While Equation 10 represents the effect of entrainment on the fluctuating velocity, it does not describe the internal turbulence within the plume. For this we use the Langevin-type equation (formally an Ornstein–Uhlenbeck process) appropriate for inhomogeneous turbulence (Stohl & Thomson, 1999; Thomson, 1987).

\[
dw_p' = -\frac{w_p'}{T_L} dt + \frac{1}{2} \left( \frac{1}{w_p'} + \frac{w_p'}{\sigma_w^2} \right) d\sigma_w^2 + \frac{\sigma_w^2}{w_p \rho_p} d\rho_p + \sqrt{2\sigma_w^2} dW \tag{11}
\]

where \( w_p = \bar{w}_p + w_p' \), \( T_L \) is the time scale on which \( w_p' \) changes, \( \sigma_w^2 \) is the vertical-velocity variance, and \( dW \) is the increment of a Wiener process. This equation is derived assuming that the one-point joint velocity-density distribution is Gaussian and requiring consistency with the Eulerian flow statistics and Kolmogorov's similarity theory for homogeneous isotropic turbulence (Thomson, 1987). A derivation of this equation can be found in, for example, Stohl and Thomson (1999), Rodean (1996), and Wilson and Sawford (1996). Note that the penultimate term on the RHS is often neglected (Stohl & Thomson, 1999) but, as will be shown below, can be significant. Combining Equations 10 and 11, which are consistent with Bisignano and Devenish (2015), we get:

\[
dw_p' = -\frac{\bar{v}_p w_p' E}{Q_m} dt - \frac{w_p'}{T_L} dt + \frac{1}{2} \left( \frac{1}{w_p'} + \frac{w_p'}{\sigma_w^2} \right) d\sigma_w^2 + \frac{\sigma_w^2}{w_p \rho_p} d\rho_p + \sqrt{2\sigma_w^2} dW \tag{12}
\]
The model is completed by specifying the forms of $\sigma^2_w$ and $T_L$ which are both functions of $z$. We expect $\sigma_w$ to scale with $\bar{w}_p$ and $T_L$ to be related to the appropriate mean quantities in the problem. Hence, following Bisignano and Devenish (2015), we choose $T_L = b / (\bar{w}_p)$ and

$$\sigma_w = \gamma |\bar{w}_p|$$

(13)

where the value of $\gamma$ will be chosen with reference to LES data below. At the vent, $T_L$ is defined by the source radius and the exit velocity which is consistent with the eddy decorrelation time scale identified by Cerminara, Esposti Ongaro, and Neri (2016).

### 2.3. Numerical Solution of the LSM

The LSM presented in the previous section along with the integral model of Section 2.1, that is, Equations 1, 3–5, 9, and 12 are solved simultaneously using an Euler–Maruyama method (e.g., Kloeden & Platen, 1992, p. 305). The fluctuating velocity component is initially drawn from a Gaussian distribution with mean zero and variance $\sigma^2_w$. The results are computed following 100,000 independently moving particles (chosen to minimize the statistical noise without excessive computational cost). Integration is terminated for each particle when $\bar{w}_p$ becomes zero. The mass concentration per unit length, $C$, is calculated according to:

$$C(z) = \frac{Q_{m0}}{\Delta z N_p} \sum (\text{# particles in each box}) \Delta t$$

where $\Delta z$ is the depth of each box, $N_p$ is the number of particles, $N_t$ is the number of time steps, and $\Delta t$ is the time step (or residence time). In the results to be presented below, we scale the concentration by its initial value at the vent, $C_0 = Q_{m0}\bar{w}_{p0}$ where $\bar{w}_{p0}$ is the exit velocity (for both the LSM and the LES).

### 3. Large Eddy Simulation

The compressible multiphase model ASHEE, which is used to generate the results against which the LSM is compared in Sections 4.2 and 5 below, is based on the C++ libraries developed in OpenFOAM. The governing equations are solved using a finite volume method. The discretized equations are solved using a segregated approach that is second order in both space and time: the latter uses the Crank–Nicolson scheme while space is treated with a central differencing scheme. Subgrid turbulence is accounted for by a dynamic subgrid model without the need for any empirical parameters. The size of the grid cells varies with height (see, e.g., Suzuki, Costa, Cerminara, et al., 2016, Figure A1) with a minimum of eight grid cells per vent diameter. A Dirichlet boundary condition is imposed on the scalar fields that represent the ash and gas phases at the upwind edge of the domain while a Neumann boundary condition is imposed on the same fields at the downwind edge of the domain. The scalar fields cannot re-enter the domain. The data are averaged over 1,000 s starting 1,000 s after the start of the eruption. The LES plume region is defined as the subdomain where the mass fraction of a tracer is larger than 0.1% of its initial value; the sensitivity to this threshold is discussed in Suzuki, Costa, Cerminara, et al. (2016, Figures 6, 7, and 12 and attendant discussion). The maximum rise height of the LES plumes is determined from the mass flux: it is the level at which the mass flux falls below 1% of its value at the vent (for more details, see Cerminara, Esposti Ongaro, & Neri, 2016). The sensitivity to this threshold is shown in Suzuki, Costa, Cerminara, et al. (2016, Figure 3). Further details about the LES can be found in Cerminara, Esposti Ongaro, and Neri (2016) and Cerminara, Esposti Ongaro, and Berselli (2016).

The jet length scale, $L_M$, is the characteristic height at which the initial momentum-dominated jet becomes a buoyancy dominated plume. For a volcanic plume, $L_M$ is defined as (Cerminara, 2016, Section 3.6)

$$L_M = \frac{L_0}{(2\alpha \text{Re})^{1/2}}$$
where

\[ L_0 = \frac{Q_m}{\sqrt{\gamma p M_z}} = b \frac{\rho_p}{\rho_a} \frac{v_p}{w_p} \]

and \( \phi = h_p/h_a - 1 \) where \( h_p \) and \( h_a \) are the specific enthalpies of the plume and the environment, respectively. The plume properties and those of the environment are evaluated at the vent level. In the Boussinesq limit \( L_0 \) reduces to the vent radius for a vertically rising plume; in practice \( \approx \) ppw for a volcanic plume at the vent. In the same limit, \( L_M \) reduces to the jet length scale defined by Morton (1959). For forced plumes such as volcanic plumes, \( L_o \ll L_M \).

4. Vertically Rising Plumes: Application of the LSM to Two Case Studies

Results are presented for two eruptions: the weak and strong eruptions considered by Costa et al. (2016) without the ambient wind. The weak eruption is the January 26, 2011 Shinmoedake eruption in Japan that produced a plume that reached a maximum rise height of about 8 km above sea level (Hashimoto et al., 2012; Kozono et al., 2013; Suzuki & Koyaguchi, 2013). The strong eruption is based on the climactic phase of the Pinatubo eruption, Philippines, on June 15, 1991, during which the eruption column reached a maximum rise height of about 39 km above sea level (Costa et al., 2013; Holasek et al., 1996; Koyaguchi & Tokuno, 1993). The mass eruption rates (MERs) for each case are \( 1.5 \times 10^6 \) kg s\(^{-1} \) and \( 1.5 \times 10^9 \) kg s\(^{-1} \), respectively. The same initial conditions and profiles of the ambient quantities as used by Costa, Suzuki, Cerminara, et al. (2016) are also used here.

4.1. LES Results

The results of the LSM are compared with corresponding results from the LES of Cerminara, Esposti Ongaro, and Neri (2016) for the case of no ambient wind. Here we briefly provide some salient points about the LES results before presenting profiles of the turbulence intensity which will be used to determine the value of \( \gamma \) as defined by Equation 13. The LES results to be presented below are for the upwardly rising core of the plume.

At the vent level, the plume is a mixture of three components: water vapor, coarse particles, and fine particles. For the weak eruption, the mass fractions are 3 wt.%, 48.5 wt.%, and 48.5 wt.%, respectively; coarse particles have a diameter of 1 mm and fine particles a diameter of 62.5 \( \mu \)m. For the strong eruption the mass fractions are 5 wt.%, 47.5 wt.%, and 47.5 wt.%, respectively; coarse particles have a diameter of 500 \( \mu \)m and fine particles a diameter of 16 \( \mu \)m. The level of neutral buoyancy (that the plume first encounters during its initial rise), \( z_{eq} \), is 9.3 km above mean sea level (msl) for the weak eruption and 20.3 km (above msl) for the strong eruption (Cerminara, Esposti Ongaro, & Neri, 2016). For the weak eruption \( L_M \approx 0.35 \) km and for the strong eruption \( L_M \approx 4 \) km.

Figure 1 shows the variation with height of the turbulence intensity, \( \sigma_w / | \overline{v_p} | \), along the centerline of the LES plumes. (Note that \( \overline{v_p} \approx \overline{w_p} \) over most of the depth of the plume except at heights of order \( L_M \) and at the plume top where \( \overline{w_p} \) becomes zero.) It shows that in the region between \( L_M \) and \( z_{eq} \) – the positively buoyant part of the plume – a value of \( \gamma \) in the range \( 0.1 \leq \gamma \leq 0.3 \) is consistent with the weak and strong eruptions. Appropriately then we consider simulations of the LSM with values of \( \gamma \) in this range. Note that there is no further tuning of the LSM in the comparison with the LES data below or in Section 5 for a variety of plume statistics.
4.2. Comparison of the LSM With LES

The vertical profile of the mass concentration calculated from the LSM for three values of $\gamma$ is shown in Figure 2 for the case of no ambient wind both with and without turbulence. The latter case amounts to solving the integral model alone, that is, Equations 1–5, such that the concentration is given by $\pi b^2 \rho$. Figure 2 shows that turbulent fluctuations only have a small effect on the concentration profile in the lower part of the plume but that without turbulent fluctuations the mass concentration increases very rapidly toward the top of the plume. This behavior of the integral model is explained by the fact that as $\bar{w}_p \to 0$ at the top of the plume, $b \to \infty$ (since $Q_m$ grows monotonically with height). In the cases with turbulent fluctuations, particles have different values of $z$ when $\bar{w}_p = 0$ and this gives rise to the smooth peak which decays to zero at the top of the plume.

Figure 2 also shows the vertical profiles of the mass concentration for the LES plume (i.e., $\int_{S(z)} \rho_p(x, y, z) \, dx \, dy$ where $S(z)$ is a horizontal slice at height $z$ of the LES plume region with positive vertical velocity). Comparing the maximum rise height of the LES plume with those of the LSM plumes, we can see that, for both eruptions, the maximum rise height of the LSM plume increases with increasing $\gamma$. The relative error
between the maximum rise height of the LSM and LES plumes for the weak eruption is 21.1% for $\gamma = 0.1$, 14.6% for $\gamma = 0.2$, and 6.56% for $\gamma = 0.3$; for the strong eruption it is 0.99% for $\gamma = 0.1$, 9.75% for $\gamma = 0.2$, and 16.2% for $\gamma = 0.3$. Note that because a very small number of particles can rise significantly further than most, we define the maximum rise height of the LSM plumes to be the height at which $C/C_0$ falls below $10^{-2}$.

The total mass within the plume (i.e., $\int C(z) \, dz$) for the LSM plumes was calculated to be 315 Gg for the weak eruption and 314 Tg for the strong eruption. The variation of the total mass of the LSM plumes with $\gamma$ is less than 0.5% since increasing turbulence simply spreads the material out more (as shown by the declining prominence of the upper peak in the LSM profile and the increasing value of the maximum rise height). The total mass of the LES plume (combined coarse and fine ash mass fractions) is 356 Gg for the weak eruption and 405 Tg for the strong eruption. These results indicate that, at least for the total mass, the LSM compares better with the weak eruption than the strong eruption.

Turning now to the vertical profiles of the concentration statistics, Figure 2 shows that for the weak eruption the LES plume for the weak eruption exhibits a distinct lower peak and a broader higher peak; the profile for the strong eruption is dominated by the lower peak. The equivalent LSM plumes show dominant upper peaks for both the weak and strong eruptions but only incipient lower peaks especially for the weak eruption. It can be seen that the source of much of the difference between the LSM plumes and the LES plumes arises from the integral plume model itself. Clearly, as $\gamma$ increases, the profile of the LSM plume shows more divergence from the case with no turbulence (compare the red and blue lines in Figure 2). For the weak eruption, the relative error in the concentration profiles between the LSM and LES is generally 20%–40% reaching 60% around the lower peak but much larger values toward the top of the plume (though showing a marked decrease with increasing $\gamma$). For the strong eruption, the relative error is typically no more than 55% except around the pronounced lower peak of the LES plume and again toward the top of the plume. For both the weak and strong eruptions, the height of the upper peak of the LSM plume with $\gamma = 0.1$ is consistent with the level of maximum radial spread as found by Suzuki, Costa, Cerminara, et al. (2016); for the weak eruption $C/C_0$ peaks at just over 9 km for the LSM plume compared with just below 10 km as estimated from Figure 1 of Suzuki, Costa, Cerminara, et al. (2016); for the strong eruption $C/C_0$ peaks at approximately 31 km for the LSM plume compared with approximately 30 km as estimated from Figure 2 of Suzuki, Costa, Cerminara, et al. (2016).

The presence of the solid phase in the LES plumes can lead to partial column collapse, particle settling, and recirculation especially at $z \sim L_M$. This is particularly prevalent in the strong eruption and leads to the formation of the significant lower peak in the concentration profile as shown in Figure 2b (see also the discussion in Devenish and Cerminara 2018). However, since these processes are absent in the LSM, it does not explain why the LSM plume also exhibits a lower peak albeit less pronounced than for the LES plume. Figure 3 shows the LSM plumes with and without ash for both the weak and strong eruptions. In the absence of ash, the concentration does not exhibit a lower peak for either the weak or strong eruptions; this is particularly noteworthy for the strong eruption. The height of the lower peak occurs, to a good approximation, at a distance $L_M$ above the vent, the height associated with buoyancy reversal that is, a transition from negative to positive buoyancy. In the LES plume, this is the height at which partial column collapse tends to occur since material above this height is more likely to be carried aloft by the positive buoyancy. In the absence of ash, the plume is dominated by buoyancy from the vent upward and so no transition occurs.

The effect of the acceleration due to the change in density with height (the “density correction” as represented by the penultimate term on the RHS of Equation 12) is shown in Figure 3. It is immediately clear that for the weak eruption the density correction has no appreciable effect on the concentration profile whereas it does have an effect on the strong eruption indicating that density differences within the plume lead to a small but noticeable change in the concentration profile.

5. The Effect of the Ambient Wind

In this section, we consider the effect of the ambient wind on the evolution and morphology of the plume. We use the same LES cases as Aubry et al. (2019). Two locations are considered for the LES: Iceland and the Philippines which were chosen to provide typical atmospheric profiles for midlatitude and tropical eruptions; more details about both the profiles and the initial conditions are given in Aubry et al. (2019).
Ten simulations for each location were performed by varying the MERs at regular intervals on a logarithmic scale between $1.6 \times 10^5$ and $7.9 \times 10^7$ kg s$^{-1}$. The other independent eruption source parameters were fixed for all cases: the vent altitude, exit Richardson number (defined here as $(1 - \rho_p / \rho_o)g \delta_h / \bar{w}_p^2$), temperature, initial mixture density, and initial water vapor mass fraction were set to 1,500 m, $-3.16 \times 10^{-2}$, 1,200 K, $3.77$ kg m$^{-3}$, and 4 wt.$\%$, respectively.

The vertical profiles of the mass concentration computed from both the LSM and the LES for the 20 cases show considerable variation in terms of their agreement with each other: two illustrative cases are shown in Figure 4; the other cases can be found in the supporting information. As with the vertically rising cases above, the LSM inherits its main characteristics from the integral plume model. Fluctuations around the mean velocity provided by the integral model result in a concentration profile with two peaks, an indistinct lower peak, and a more distinct upper peak whose height and strength depend on the value of $\gamma$. The presence of the lower peak occurs for the same reasons as described in Section 4.2 and at a height of order $L_M$. The behavior of the upper peak is similar to that observed in Figure 2 for the cases with no ambient wind: for the LSM with $\gamma = 0.1$, the peak is very much in line with the height at which the concentration calculated from the integral model (the LSM with no turbulence) tends to infinity. For the LSM with $\gamma = 0.2$ and...
\( \gamma = 0.3 \), the peak becomes less distinct; the height at which the peak concentration occurs decreases with increasing \( \gamma \). Above the level of peak concentration, the concentration profile decays smoothly in qualitative agreement with the LES. The value of the maximum rise height increases with increasing \( \gamma \).

In Figures 5 and 6, we show the variation of the maximum rise height, the maximum concentration, the height at which the maximum concentration occurs, and the relative error in the total mass over the eruption column and its initial downwind spread. It should be noted that the level of agreement between the LSM and the LES is similar for both the Icelandic and Philippine cases. This indicates that both models are capable of modeling eruptions in the tropics as well as at midlatitudes. For both the Icelandic and Philippine cases, the relative error in the total mass erupted is (with the exception of one case) less than 20\% with more than half the simulations having a relative error of less than 10\% (with little systematic variation with \( Q_{m0} \)). For both sets of simulations, both the LES and the LSM show the maximum concentration increasing with MER: this is to be expected since the maximum concentration scales linearly with the initial mass flux. The height at which this peak concentration occurs increases with increasing MER for the simple reason that more powerful eruptions rise higher, are less affected by the ambient wind and so spread laterally at a higher level. Both the height of the peak concentration, \( z_{\text{spread}} \), and the maximum rise height, \( z_{\text{max}} \), of the LSM plumes tend to diverge most from the equivalent LES values for the largest values of \( Q_{m0} \); differences that were also noted by Aubry et al. (2019) when comparing integral plume models with LES. It should also be noted that larger eruptions tend to exhibit more pulsating behavior toward the top compared with

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figures.png}
\caption{The variation with MER for the Icelandic cases of (a) the maximum concentration, \( C_{\text{max}} \), (b) the height at which this occurs, \( z_{\text{spread}} \), (c) the maximum rise height, \( z_{\text{max}} \), and (d) the relative error in the total mass calculated as the absolute difference in the total mass between the LES and LSM relative to the LES value. In each figure the black line is the LES and the red line is the LSM with \( \gamma = 0.2 \). The error bars on the LSM results are the difference between the values calculated with \( \gamma = 0.1 \) and \( \gamma = 0.3 \); the error in the calculation of the total mass is negligible.}
\end{figure}
weaker eruptions, an effect that has also been reported by Suzuki, Costa, and Koyaguchi (2016); Suzuki, Costa, Cerminara, et al. (2016); and Costa et al. (2018). It has been averaged out in the LES results presented here. Moreover, the suspended flow in the lower part of the plume affects the entrainment rate and changes the effective initial conditions of the integral model (Cerminara, Esposti Ongaro, & Neri, 2016; Suzuki, Costa, & Koyaguchi, 2016). Numerical simulations (Suzuki, Costa, & Koyaguchi, 2016) have shown that the threshold at which this behavior starts to become prevalent occurs for $Q_m$ of order $10^7$ kg s$^{-1}$ (with an appropriate value of the exit velocity) and our results are consistent with this observation.

As the LSM inherits most of its characteristics from the integral plume model, it shares the same sensitivities to the parameters that define the integral plume model such as source temperature, exit velocity, and entrainment coefficients. The sensitivity of integral plume models to these parameters has been extensively studied (e.g., Costa et al., 2016, and references therein; Devenish, 2013, 2016). In Figure 7, we show the sensitivity of the LSM to variation of $\beta$, arguably the entrainment coefficient whose value is most unconstrained by experiment or observation. The maximum rise height of a Boussinesq single-phase plume in a uniform

![Figure 6](image_url)

**Figure 6.** As Figure 5 but for the Philippine cases.

![Figure 7](image_url)

**Figure 7.** The variation of $z_{\text{max}}$ (red) and $z_{\text{spread}}$ (blue) with $\beta$ calculated from the LSM with $\gamma = 0.2$ for the first Icelandic case for which $Q_m = 1.58 \times 10^5$ kg s$^{-1}$. The error bars are the difference between the values calculated with $\gamma = 0.1$ and $\gamma = 0.3$. 

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crossflow, whose magnitude is much larger than the vertical velocity of the plume, is proportional to $\beta^{-2/3}$ (see e.g., Devenish, Rooney, Webster, et al., 2010). Of course this relationship will not necessarily hold exactly in a realistic atmosphere where the wind speed varies with height but, as Figure 7 shows, it holds approximately.

6. Discussion

We have presented an LSM of a volcanic plume in which an integral plume model provides the mean flow and a suitably constructed SDE gives the fluctuating vertical velocity. The integral model is that of Devenish (2013) and the SDE is based on that used by Bisignano and Devenish (2015) for a Boussinesq plume with temperature fluctuations. We compared the mass concentration computed from the LSM with data from corresponding LES for the two eruptions considered in the volcanic-plume intercomparison study of Costa et al. (2016) and the LES of Aubry et al. (2019) for hypothetical, but realistic, eruptions in Iceland and the Philippines. The latter includes the effects of the ambient wind. It is emphasized that the only tuning of the LSM to the LES data is through the parameter $\gamma$ (as defined by Equation 13), which is an input to the LSM, and then only with reference to one set of LES data, namely that used for the intercomparison study.

We used a range of $\gamma$ values to provide an error estimate for the LSM results. The LSM captures the order of magnitude of the mass concentration and the general shape of its vertical profile. In qualitative agreement with the LES results, the LSM mass concentration decays to zero at the top of the plume. In contrast, the mass concentration computed from a standard integral plume model, that is, without fluctuations, tends to infinity at the top of the plume. We made a quantitative assessment of several aspects of the LSM and LES plumes including the maximum rise height, the maximum concentration, the level at which it occurs and the total mass. While there is considerable variation in the level of agreement, much of the error is inherited by the LSM from the integral plume model which provides the mean flow. For example, as with the integral plume models considered in Costa et al. (2016), the LSM compares relatively well with the weak eruption but not so well with the strong eruption. The reasons for this are complex but are likely to include a relatively small aspect ratio and a relatively small ratio of $z_{eq}$ to $L_M$ for the strong eruption compared with the weak eruption (Devenish & Cerminara, 2018). We showed that the presence of ash is responsible for a peak in the mass concentration at $z \sim L_M$, the height at which there is a transition from a negatively buoyant jet to a positively buoyant plume. Although for the LES plume the magnitude of this peak is likely to be determined by the complex flow features in the LES, their absence in the LSM and the standard integral plume model suggests that $L_M$ plays an important role in the LES plume as well.

Atmospheric dispersion models such as NAME, which is used operationally for predicting the airborne spread of volcanic ash, often initialize the source of Lagrangian particles (in effect the eruption column) as a uniform line source (with no explicit treatment of the dynamics of the eruption column). However, it is widely expected that most of the ash is released from the eruption column to the surrounding atmosphere toward the top of the eruption column. Simulations with NAME using non-uniform line sources, for example, a Gaussian distribution toward the top of the eruption column do lead to significant differences in the downwind dispersion of the ash (e.g., Devenish, Francis, et al., 2012; Devenish, Thomson, et al., 2012). However, the tails of a Gaussian distribution decay rapidly and mean that very little ash is released from the lower part of the eruption column; it is important in an operational context to ensure that ash is released over the depth of the eruption column to minimize the penalty of not predicting where ash is subsequently detected. The vertical concentration profiles produced by the LSM, as shown in Figures 2 and 4 and figures in supporting information, display a peak in concentration toward the top of the eruption column but with a substantial tail in the lower part of the plume. Suitably normalized, these profiles could be used as probability density functions to provide a more realistic distribution of ash for an atmospheric dispersion model such as NAME. A further application of these profiles is as prior distributions for Bayesian approaches to inversion modeling in which the source is reconstructed from a synthesis of observations and simulations from an atmospheric dispersion model such as NAME.

Another potential application of the LSM presented here is direct incorporation into an atmospheric dispersion model such as NAME thereby explicitly treating the dynamics of the eruption column within the dispersion simulation. The advantage of this approach is that model particles released at the vent would continue downwind of the eruption. The effects of the prevailing meteorology at the time of the eruption...
would then automatically affect the shape and height of the eruption column leading to a more realistic release of model particles.

Further development of the LSM will most likely be influenced by the more detailed simulations LES provides. As discussed in Bisignano and Devenish (2015) for single-phase Boussinesq plumes, “what may help facilitate such developments is a more rigorous framework for the construction of an LSM of buoyant plume rise.” That such a framework does not exist despite the existence of many such models could point to the inherent difficulty of applying a single-particle LSM (in which particles move independently of each other) to a coherent process such as a buoyant plume. Moreover, integral plume models are already averaged in some sense and adding fluctuations to these models is re-introducing information that has been removed by the averaging process. It may well then be inevitable that some heuristic element will remain albeit one that is physically motivated. These issues hold as much for LSMs of volcanic plumes as they do for LSMs of simpler Boussinesq plumes.

Data Availability Statement

The concentration data can be accessed at https://doi.org/10.5281/zenodo.3975944.

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