Lectures on Unification

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Abstract

In these lectures we review the motivation, principles of and (circumstantial) evidence for the program of unification of the fundamental forces. In an appendix, we review the group theory pertinent to the program.

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1 Introduction

The standard model of the electromagnetic, weak and strong interactions is the corner stone of elementary particle physics\cite{1, 2}. It is a lagrangian field theory of quark, lepton and gauge bosons degrees of freedom with the spontaneous breakdown of electroweak symmetry achieved by an elementary higgs scalar potential. While the standard model is enormously successful at present day energies, it is likely to be the “low”-energy limit of a more complete and perhaps simpler description of these interactions — a description which derives from the experience of the standard model, in the sense of being a lagrangian field theory, being a gauge theory and which uses the key concepts of spontaneous symmetry breaking. Grand unification\cite{3, 4, 5, 6}, where in the standard model gauge symmetry is embedded in larger symmetries is such a program and is a subject of the present set of lectures. Another key unification is that of particle types, viz., particles of bosonic and fermionic types. Supersymmetry\cite{7, 8, 9} is the symmetry that treats these degrees of freedom on the same footing and may be combined with lagrangian field theory. In particular, modern approaches to unification simultaneously require grand unification as well as unification of bosonic and fermionic statistics and is called supersymmetric unification and is the framework within which the present discussion will take place. These symmetries, however, must be broken or hidden since there is no (direct) evidence for such unification.

In these lectures we will review the motivation, principles and circumstantial evidence for the program of unification of the fundamental forces, with the exception of the gravitational forces. The aim of the lectures at this school is to bring the participants up to date with the current status of research in the areas covered at the school assuming as little as possible. We will mention virtually all the central notions that enter the construction of the unification program, in italics. We note, however that many of the preliminaries are already presented in standard textbooks\cite{10, 11} and we will frequently refer the reader to them for tracing the primary sources. The relevant group theory is presented in an appendix and is a summary of results discussed elsewhere\cite{12, 13}. 
2 The Standard Model

At all length scales probed thus far at high energy accelerators, there has been no evidence to suggest that the fundamental constituents of matter, namely the quarks and the leptons are anything but point-like. The quarks come in the varieties of up, down, charm, strange, top and bottom and the leptons come in the varieties of electron and its neutrino, the muon and its neutrino and the tau lepton and its neutrino. Of these the leptons do not participate in the strong interactions and the neutrinos alone are electrically neutral. Furthermore, the weak interactions are known to violate parity, in that the left- and right- chiral projections of these particles do not participate in the weak interactions on par. The quarks themselves are never seen isolated in nature and are confined to reside in hadronic matter although at very high energies and on very short time scales there is indubitable evidence for their existence.

All the forces listed so far result from the exchange of vector bosons, viz., quanta of fields that transform as vectors under the Lorentz transformations. The vector particles themselves are introduced via the gauge principle: the gauge principle dictates that the underlying Lagrangian field theory for the interactions is invariant under gauge transformations of the local kind which in turn implies the existence of a covariant derivative, schematically written as $\partial_\mu - igA_\mu$, which brings in the vector fields of interest. The number of gauge fields fields is equal to the number of infinitesimal generators of the gauge symmetry. The photon, $(\gamma)$ responsible for the long range electromagnetic interactions based on the symmetry $U(1)$, the one-dimensional unitary group is massless and exists in asymptotic states. On the other hand the weak interactions which are short range are mediated by the exchange of massive vector particles, the $W^\pm$ and the $Z^0$. Finally the strong interactions mediated by the massless gluons, $g$ rendered short ranged by a yet to be discovered mechanism for the confinement of colour quantum number that is carried by the gluon (indeed, as it is by the quarks). The gluons are the gauge bosons of the underlying $SU(3)$ colour gauge group and eight fields have to be introduced corresponding to the number of infinitesimal generators. The quarks come in three colors and transform as triplets under the color gauge group, whereas the leptons are singlets under this gauge group and do not participate.
in the strong interactions at the tree-level. [In the following we will be suppress the color indices assuming that they are correctly summed over.]

The manner in which particles are coupled to the gauge fields is dictated by which representation of the relevant gauge groups they lie in. The principle of gauge invariance also dictates the manner in which particles interact between themselves. Only those couplings between the particles are allowed which are left invariant by the action of a gauge transformation. This picture, thus, requires us to specify the transformation properties of the matter fields under the gauge group SU(2) × U(1). In particular, the left-handed projections of the u and d quarks, \( q_L^T \equiv [u_L \ d_L]^T \) transforms as a doublet under SU(2) and carries the hypercharge 1/3, whereas the the right handed projections \( u_R \) and \( d_R \) transform as singlets under SU(2) and carry hypercharges 4/3 and \(-2/3\) respectively. Mathematically this would correspond to a term in the Lagrangian density that would look like:

\[
\bar{q}_L i \gamma^\mu (\partial_\mu - \frac{ig}{2} T^i A^i_\mu - \frac{ig'}{6} B^\mu) q_L + \bar{u}_R i (\partial_\mu - \frac{2}{3} ig' B^\mu) u_R + \bar{d}_R i (\partial_\mu + \frac{1}{3} ig' B^\mu) d_R
\]

(1)

Analogously we have the lepton doublets \([\nu_L \ e_L]^T\) which transform as a doublet and with hypercharge \(-1\) whereas the right-handed projections \( \nu_R \) and \( e^R \) transform as singlets and carry hypercharge of 0 and \(-2\) respectively. We note here that the right handed neutrino is completely inert with respect to the standard model gauge group and may even be left out of the spectrum.

A consistent picture arises when the electromagnetic and weak interactions are considered simultaneously in an electroweak framework based on a group SU(2) × U(1) [where SU(2) (or more generally SU(N) is the group of \(2 \times 2\) (or more generally \(N \times N\) unitary matrices) which is then broken spontaneously by the Higgs mechanism when a standard model Higgs doublet of scalar fields \( \phi^T = [\phi^+ \ \phi^0]^T \) is introduced to produce U(1) of electromagnetism, and in the process turns three of the gauge bosons, now named \( W^\pm \) and \( Z^0 \), massive. The higgs mechanism occurs when a quartic potential is introduced for the doublet and when the classical potential turns into one by the arrangement of specific relations between the mass parameter and the quartic coupling wherein the ground state is the asymmetric minimum. More precisely the
The higgs potential is written down as:

\[ L_\phi = \left| \left( \partial_\mu - ig T^i A^i_\mu \right) \phi \right|^2 - \frac{\mu^2}{2} \phi^\dagger \phi - \lambda (\phi^\dagger \phi)^2 \]  \hspace{1cm} (2)

These transformation properties then specify the nature of the kinetic energy terms of the standard model particles, as we saw for the quarks, leptons and the higgs fields. Finally the kinetic energy terms of the gauge bosons themselves:

\[ F^{\mu\nu}_i = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + g \epsilon_{ijk} A^j_\mu A^k_\nu, B^{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \]  \hspace{1cm} (3)

For the non-abelian gauge fields the kinetic energy involves self-couplings of the gauge fields, a feature not present in electrodynamics. This has a crucial implication for the strong interactions — asymptotic freedom is a property arising from this feature. We must also note that the parameters in the Lagrangian above themselves are “running” coupling constants whose evolution is governed by the renormalization group equations. In particular, for the gauge couplings, in the standard model, we have the one-loop evolution equations for the couplings:

\[ \frac{d\alpha_s^2}{d \ln Q^2} = \frac{1}{4\pi} \left[ 11 - 4F/3 \right] \alpha_s^2 \]

\[ \frac{d\alpha_g^2}{d \ln Q^2} = \frac{1}{4\pi} \left[ 22/3 - 4F/3 \right] \alpha_g^2 \]

\[ \frac{d\alpha_{g'}^2}{d \ln Q^2} = \frac{1}{4\pi} \left[ -20F/9 \right] \alpha_{g'}^2 \]  \hspace{1cm} (4)

where \( Q \) is the momentum and \( F \) is the number of families. We note here that the the quadratic Casimirs of the representations in which the gauge bosons and fermions enter the final expressions since they represent the summing over the intermediate particle states in the one-loop computation of the beta functions.

Electroweak symmetry is broken when \( \mu^2 \) is chosen negative with \( \lambda > 0 \). In particular, it is possible to arrange the parameters to yield the vacuum expectation value to the electrically neutral component: \( \sqrt{2}v = \left[ 0(-\mu^2/\lambda)^{1/2} \right]^T \). It is then possible to work through the details of the higgs mechanism to produce expressions for the masses of the \( W^\pm \) and \( Z \) bosons:

\[ m_W = gv/2, \quad m_Z = \sqrt{g^2 + g'^2}v/2 \]  \hspace{1cm} (5)
One electrically neutral scalar higgs boson is left behind after spontaneous symmetry breaking. Furthermore, from experiment, we have the relation for $v$ in terms of the Fermi constant $G_f = 1.17 \cdot 10^{-5}$ GeV$^{-2}$, $v = 2^{-1/4}G_F^{-1/2} = 246.2$ GeV.

The transformation properties also constrain the interactions between the particles themselves; the term that we add to the Lagrangian must be invariant under the combined gauge group. Such terms imply Yukawa couplings between the higgs doublet and left- and right-handed matter fields. Gauge invariance allows terms of the type:

$$\bar{q}_L \phi q_R + \text{H.C.}$$

When spontaneous symmetry breaking occurs then the vacuum expectation value of the Higgs field multiplied by the Yukawa coupling gives rise to an effective mass term for the standard model fermions. Note that the absence of the right handed neutrino implies that the neutrino is massless in the standard model. Furthermore, at the perturbative level, the absence of lepton (and baryon) number violating couplings rules out the possibility of Majorana type masses which can be added to the lagrangian to provide a mass to the left-handed neutrino. We also note that we would finally have to sum over all the families. Since all the quarks have non-zero masses, once the electroweak symmetry is broken, the quarks may mix amongst themselves, viz., that the “flavor” eigenbasis would now not correspond to the “mass” eigenbasis. This would then be accounted for in the standard model by the Cabibbo-Kobayashi-Maskawa mechanism which is also rich enough to contain a single CP violating phase. We do not discuss this any further except to note that the standard model falls into the class of “milliweak” CP violating models which is yet to be confirmed experimentally.

3 Grand Unification

A compelling goal of theoretical physics is to replace what are the engineering aspects of the standard model by a fundamental theory; for example, arbitrary parameters of the standard model, hitherto fixed by experiment, would then be explained as consequences of a unified and
symmetric structure. Such a theory would then make a whole host of predictions and simplifications of our understanding of fundamental phenomena. Indeed, it would be very pleasing if the seemingly arbitrary pattern of $SU(3) \times SU(2) \times U(1)$ were to be aesthetically situated into an elegant framework. It is possible to envisage a scenario wherein this is embedded in a larger group $G$, which would be the basis of the gauge invariance of a theory manifest above a unification scale $M_G$. The evolution of the standard model gauge couplings does provide some credence to this belief as we describe in one of the following subsections. Below $M_G$, $G$ would be spontaneously broken via the Higgs and possibility some other mechanism to a sub-group large enough to contain the standard model (in a multi-step scenario), which would then be further broken down to the standard model gauge group at various stages.

3.1 $SU(5)$

Earliest examples of grand unification were provided by those based on the groups $SU(4) \times SU(2) \times SU(2)$, $SU(5)$ and $SO(10)$. In fact, the unitary group $SU(5)$ does admit the standard model gauge group as a maximal subgroup and is an ideal candidate for unification. Indeed, it is the smallest group large enough to contain the standard model gauge group. This may be simply seen from erasing one of the inner dots of the Dynkin diagram of the Lie algebra of $SU(5)$. These properties and other group theoretic results maybe read off from Slansky’s tables, the essential mathematical steps recounted in the book by Cahn and summarized in the appendix.

In this instance, we find that the standard model gauge group’s Lie algebra is a maximal subalgebra of $SU(5)$ [$SU(4) \times U(1)$ being the other maximal subalgebra], obtained by erasing one of the external dots of the Dynkin diagram of $SU(5)$]. Furthermore, when we consider the smallest representations of $SU(5)$ namely the $\overline{5}$- and 10- dimensional representations. Their branching rules under the standard model gauge group $SU(3) \times SU(2) \times U(1)$ are given by

$$(1,2)(-3) + (\overline{3},1)(2), \text{ and } (1,1)(6) + (\overline{3},1)(-4) + (3,2)(1)$$

(7)

respectively. These may easily seen to be precisely the quantum numbers of one standard model family. In particular, they correspond to the left handed lepton doublet, right handed down-type
quark singlet (conjugate), the right handed electron (conjugate), the right handed up-type quark singlet (conjugate) and the left handed quark doublet, respectively. Among other things, this would imply that transitions are possible between quark and lepton states [proton decay problem] and mass relations between various fermions, now unified into irreducible representations of the groups.

It may also be seen that a 5-dimensional scalar multiplet can accommodate the electroweak doublet but the electroweak singlet, colored triplet must be very massive in order to prevent rapid proton decay [14]. The 24 dimensional representation may also be considered, with the branching rules: \((1,1)(0) + (1,3)(0) + (3,2)(-5) + (3,2)(5) + (8,1)(0)\). The singlet component is interesting, since a higgs scalar in the 24-dimensional representation can be used to break \(SU(5)\) down to the standard model. The 24-dimensional representation is also the adjoint of \(SU(5)\) which contains the gauge bosons of the unified group. Sure enough under the standard model gauge group, we find candidates for the electro-weak bosons, namely the \((1,1)(0)\) and the \((1,3)(0)\) and for the gluons, the \((8,1)(0)\). The rest must become supermassive associated with the scale \(M_G\).

The theory is specified by writing down the terms in the Lagrangian that couple these fields. In particular, we see that Yukawa couplings may be written down for the fermions in the 5 and 10 and the 5 dimensional scalar. Indeed, one may then compute the tensor products of these irreducible representations and find in the sum of irreducible representations a piece that is invariant (i.e., a singlet) under \(SU(5)\).

### 3.2 Charge Quantization

The fact that standard model fermions of differing hypercharges are accommodated into irreducible representations of \(SU(5)\) implies there is a basis for relating the hypercharge assignments of those fermions that are in the same multiplet. For instance, when we consider the electro-weak doublet and the down-type anti-quark that lie in the same 5 it implies that the action of the same diagonal hypercharge generator produces eigenvalues of their respective hypercharges.
This in turn implies that charge is now quantized. Furthermore, we have the result that the normalization of the hypercharge generator is now related to the normalization of the diagonal generators of $SU(2)$ and $SU(3)$.

The seemingly arbitrary choice of gauge couplings in the standard model would also have to be replaced by a unique gauge coupling in the event of unification into $SU(5)$. However, we must first fix the normalization of the hypercharge generator of the standard model, 	extit{vis a vis} the generator that is embedded in $SU(5)$. We recall the relations:

$$\sin^2 \theta_w = \frac{e^2}{g^2} = \frac{g'^2}{g'^2 + g^2}$$

(8)

In the standard model, we have the Gell-Mann-Nishijima type relation:

$$Q = T_3 + Y/2$$

(9)

However, in $SU(5)$, if we consider the $SU(3)_C$ subgroup to lie in the upper $3 \times 3$ diagonal sub-group and $SU(2)$ (weak-isospin) to lie in the lower $2 \times 2$ diagonal sub-group, then $T_3 = \text{diagonal}(0 0 1 - 1)/2$ and the hypercharge would be proportional to $Y' = \text{diagonal}(-2 -2 -2 3 3)/(2\sqrt{15})$. If we have to correctly produce the electric charge assignments to the $\mathbf{5}$, then we would have to define $Q = T_3 + \sqrt{5/3}Y'$. This then gives us the required boundary condition that $g' = \sqrt{3/5}g$ at the unification scale.

### 3.3 Coupling Constant Unification

A unification scale $M_G \sim 10^{16}\text{GeV}$ is suggested by gauge coupling unification, above which physics would be described by a grand unified theory $\mathfrak{G}$ based on a gauge group $G$. Indeed, the arrival at the structure of fundamental interactions from renormalization group flow has a predecessor in the example of asymptotic freedom in deep inelastic scattering experiments and thus gauge coupling unification is an extremely encouraging sign that grand unified theories are the right step for a theory of fundamental interactions. The evolution equations we consider are precisely those we encountered earlier eq.(4). These equations provide the following system of
relations between the two inputs at low-energies $\alpha$ and $\alpha_s$ and the unification scale, the value of the unified coupling constant $\alpha_G$ and the value of $\sin^2 \theta_w$ at low-energies.

\[
M_G = Q_0 \exp\left(\frac{2\pi}{11\alpha}(1 - \frac{8}{3} (\alpha/\alpha_s))\right)
\]
\[
\sin^2 \theta_w = 1/6 + 5/9(\alpha/\alpha_s)
\] (10)
\[
1/\alpha_G = 3/8(1/\alpha - 1/(6\pi)(32/3F - 22) \ln M_G/Q_0).
\] (11)

With the fairly accurately known inputs for $\alpha = 1/128$ and $\alpha_s = 0.12$ at $Q_0 = M_Z \sim 92$GeV, we find the results $M_G \sim 1.2 \cdot 10^{15}$GeV, $\sin^2 \theta_w \sim 0.20$ and $\alpha_G \sim 1/41$.

3.4 Complexity of Representations

In the choice of gauge groups there are many theoretical restrictions and furthermore in the choice of the representations that could be of possible utility in model building. One important property of the standard model that singles out certain groups is the fact that the weak interactions violate parity. This implies the existence of chiral fermions and the fact that left- and right-handed chiral projections are assigned to inequivalent representations of $SU(2)$. When viewed in the context of unification, this implies that the representations we can use for accommodating standard model fermions must be complex, where the present notion of complexity implies that the image of a group element in the representation and that of its complex conjugate element cannot be made equal by a similarity transformation using an element of the representation. It has been shown that the only groups that admit complex representations are $SU(n), n \geq 3, SO(4n + 2)$ and $E_6$.

3.5 $SO(10)$ and Anomaly Cancellation

The seemingly arbitrary assignments of a standard model fermion to representations of $SU(5)$ finds a natural resolution when we consider an even larger gauge symmetry, viz., $SO(10)$. It may be easily seen from the Dynkin diagram structure of the algebra of $SO(10)$ that $SU(5)$ is a subalgebra, with $SU(5) \times U(1)$ being a maximal subalgebra. We may either choose the $SU(5)$
as it stands as the Georgi-Glashow SU(5) or alternatively we can choose a linear combination of one of the diagonal generators of SU(5) and the additional U(1) of the maximal subalgebra as hypercharge. The latter corresponds to the so-called flipped unification, wherein the word “flipped” refers to the flipping of assignments of certain particles to representations of the SU(5), which we will not discuss here. SO(10) is in a class of groups that admit so-called spinor representations of dimension of dimension 16 in this case. The branching rules for certain interesting and important representations of SO(10) under SU(5) × U(1) read:

\[
10 = 5(2) + 5(-2) \\
16 = 1(-5) + 5(3) + 10(-1) \\
45 = 1(0) + 10(4) + 10(-4) + 4(0) \\
126 = 1(-10) + 5(-2) + 10(-6) + 10(6) + 45(2) + 45(-2)
\] (12)

One may easily gather from here that the 16-dimensional representation in indeed the correct candidate for a standard model generation and in addition contains a candidate for a right-handed neutrino, which is an SU(5) singlet. The 10-dimensional representation on the other hand contains candidates for SU(2) doublets that lie in the SU(5) 5-dimensional representations. For pedagogical purposes we have also included the branching rules of the 45- of SO(10) which would contain the gauge bosons of SU(5) and U(1) which might result for a direction in a scalar 45- obtaining a vacuum expectation value. The branching rules of the 126- are given so as to provide a discussion of Majorana masses for neutrinos in the following subsection.

Since the rank of SO(10) [viz., the number of diagonal generators] is one larger than the number of mutually commuting generators of the standard model gauge group, it is possible to find a U(1) gauge boson, associated with the secondary breakdown of the gauge symmetry SU(5) × U(1). However, it is entirely likely that a single step breaking of the gauge symmetry takes place in which event it might be worthwhile to consider the branching rules of the representation under the other maximal subalgebra SU(4) × SU(2) × SU(2):

\[
10 = (1, 2, 2) + (6, 1, 1)
\]
\[16 = (4, 2, 1) + (\bar{1}, 1, 2)\]
\[45 = (1, 3, 1) + (1, 1, 3) + (15, 1, 1) + (6, 2, 2)\]
\[126 = (6, 1, 1) + (\bar{10}, 3, 1) + (10, 1, 3) + (15, 2, 2)\] (13)

It would be instructive to think of the assignments of the standard model fermions to the various multiplets of $SU(4) \times SU(2) \times SU(2)$: such a model is manifestly left-right symmetric. However, in order to be compatible with phenomenology, it would be necessary to break one of the $SU(2)$ and part of the $SU(4)$ down to $U(1)$ hypercharge and $SU(3)$ respectively. Here we also have the interesting identification of the broken diagonal generator of $SU(4)$ with lepton number.

Another outstanding feature of the standard model is the possible appearance of gauge anomalies, associated with triangle diagrams with axial vector currents at one of the vertices of the triangle. The assignments of hypercharges in the standard model from phenomenology just serves the purpose of cancelling the possible anomalies which also calls in the presence of the color quantum number. This mystery is not resolved even in the case of $SU(5)$ unification in which the particle assignments merely rearrange the miraculous cancellation of the standard model. However the embedding of the gauge symmetry into $SO(10)$ provides a raison d’être for the cancellation. This has to do with the fact that in order to evaluate the anomaly, one encounters the following trace

\[\text{Tr} \lambda_{ij} \{\lambda^k, \lambda^{mn}\}\]

where the $\lambda_{ij} (= - \lambda_{ji})$ represent the generators of $SO(10)$ in a Cartesian basis. This result must necessarily be proportional to a 6-index tensor which does not exist for any orthogonal group with the exception of $SO(6)$. Thus the representations of $SO(n), n \neq 6$ are anomaly free.

### 3.6 Neutrino Masses

Note that whereas in the standard model, the field content forbids a Dirac mass for the neutrinos since the right handed neutrino is absent and Majorana mass is forbidden by the conservation of lepton number. In grand unified models, neither of these principles is respected and a wide
variety of possibilities exists for the generation of neutrino masses. However, far from being arbitrary, it should be possible to uncover information regarding the structure of unified theories from accurate determination of small and eventually large neutrino masses and mixing angles, \textit{viz.}, neutrino masses may be viewed as bearing an imprint on the structure of grand unification and the nature of the breakdown of unification\cite{15}.

One pedagogical example we consider is one wherein the right-handed neutrino receives a Majorana mass of the type $\nu_R^R \nu_R^R < 126 >$ when the 126- dimensional representation of $SO(10)$ receives a vacuum expectation value at a supermassive scale, to break $SO(10)$ to $SU(5)$. This can be seen, when we consider the branching rules under $SU(5) \times U(1)$, we find that 126 has an $SU(5)$ singlet component from eq.(12). This gives the Majorana mass. This Majorana mass is necessary to make the see-saw mechanism function to give a supermassive mass to the right handed neutrino while making the left handed component sufficiently light and preserving mass relations for the Dirac masses.

It is commonly stated that the Majorana mass must necessarily result from a $\Delta L = 2$ vertex, which means that the component acquiring the vev must break lepton number. This is seen by considering the branching rules of the 16 as well as the 126 under $SU(4) \times SU(2) \times SU(2)$, wherein we consider the first of the $SU(2)$ to be $SU(2)_L$. The right handed neutrino lies in the $(\bar{4}, 1, 2)$ while the direction of interest from the 126 lies in the $(10, 1, 3)$ component. The branching rules of the 10 of $SU(4)$ under $SU(3) \times U(1)$ read $10 = 1(2) + 3(2/3) + 6(-2/3)$, while that of the $\bar{T}$ reads $\bar{T} = 1(-1) + 3(1/3)$; clearly one may then have a Yukawa coupling $\bar{T} \cdot \bar{T} \cdot 10$. The $SU(2)$ algebra will admit a coupling between the 2- dimensional representation in which the fermions are acocomodated and the 3- dimensional representation in which the scalar lies. Furthermore, when the $SU(3)$ singlet direction in the 10- dimensional representation of $SU(4)$ acquires a vev, lepton number is broken proving the result that the Majorana mass requires lepton number to be broken.
3.7 Hierarchy Problem

The presence of disparate scales in the theory, $M_G$ and the weak scale $M_W \sim 174$ GeV, expected to be separated by more than ten orders of magnitude, would render the mass of the Higgs scalar of the electro-weak model $\sim M_W$, unnatural-natural. Should the Higgs scalar be elementary, then one manner in which it would remain naturally at the weak scale is due to cancellation of divergences as in supersymmetric unified models\[7, 8\]. This is further discussed in the next section.

4 Supersymmetric unification

*This section is extracted from a recent review article[16] and is sufficiently detailed to serve as a self-contained discussion of the subject.*

Supersymmetry is the unique symmetry that has non-trivial commutation relations with the generators of the Lorentz group. Supersymmetries enjoy non-trivial anti-commutation relations amongst each other. Their action on representations of the supersymmetry algebra interchange the statistics between the members. Linear representations of the supersymmetry algebra in relativistic field theory are realized in the Wess-Zumino model\[9\]. Important representations include chiral multiplets and vector multiplets, which form the basis of the extension of the standard model to various supersymmetric versions of the standard model. Since supersymmetry is not manifest in nature, it must be broken, either spontaneously or explicitly. It appears that the second option is more favored, certainly more popular, wherein supersymmetry is broken explicitly but softly. The requirement of soft supersymmetry breaking is in accordance with the requirement of the well-known properties of supersymmetric models including the cancellation of quadratic mass divergences for scalars.

In the context of grand unified model building, the existence of scales $M_W$ and $M_G$ separated by several orders of magnitude renders the mass of the elementary Higgs of the standard model unstable and would drive it to the unification scale, without an un-natural fine tuning
of parameters of the Lagrangian. The cancellation of quadratic divergences in manifestly and softly-broken supersymmetric theories renders supersymmetric versions of grand unified models attractive candidates for unification. The program of writing down a supersymmetric version of the standard model, which is then embedded in a grand unified scheme, [alternatively a supersymmetric version of a grand unified scheme] may be realized by replacing every matter and Higgs field, by a chiral superfield whose members carry the same gauge quantum numbers, and by replacing every gauge field, by a vector super-multiplet. Supersymmetry also requires that the standard model Higgs doublet is replaced by two Higgs multiplets. This in turn leads to the introduction of another parameter \( \tan \beta \) which is defined as the ratio of the vacuum expectation values of these two Higgs fields, \( v_2/v_1 \) where \( v_2 \) and \( v_1 \) are the vacuum expectation value of the Higgs fields that provide the mass for the up-type quark and the down-type and charged leptons respectively. All the interactions of the resulting model may then be written down once the superpotential is specified. Note that gauge invariance and supersymmetry allow the existence of a large number of couplings in the effective theory that would lead to proton decay at unacceptably large rates. An ad hoc symmetry called R-parity is imposed on the resulting model which eliminates these undesirable couplings and such a version has received the greatest attention for supersymmetry search. More recently models have been and are being considered where R-parity is partially broken in order to study the implications to collider searches. However such models are constrained by bounds on flavor changing neutral currents as well as by the standard CKM picture, also as it applies to CP violating phases.

In what follows we recall some of the essential successes of the recent investigations\[^1\] in the theory of supersymmetric unification. This was spurred by the confrontation of the ideas of unification by the precision measurements of the gauge couplings of the standard model at the LEP\[^2\]. A highly simplified understanding of this feature may be obtained from a glance at the one-loop evolution equation for the standard model gauge couplings, more correctly the gauge couplings of the minimal supersymmetric standard model assuming that the effective supersymmetry scale is that of the weak scale, with \( t = \ln \mu \): \( \frac{d\alpha_i}{dt} = \frac{\alpha_i^2}{2\pi} b_i \), \( b_1 = 33/5 \), \( b_2 = \)}
1, $b_3 = -3$, where we have assumed three generations. One may then integrate these equations to obtain:

$$\frac{1}{\alpha_i(M_Z)} = \frac{1}{\alpha_i(M_G)} + \frac{b_i}{2\pi} \ln \frac{M_G}{M_Z}.$$ 

One may then use the accurately known value of $\alpha_{em}(M_Z) = 1/128$, with the identity $1/\alpha_{em} = 5/3\alpha_1 + 1/\alpha_2$ which accounts for the normalization imposed by unification, and the values of $\alpha_3(M_Z) \approx 0.12$ to solve for the unification scale $M_G$ and the unified coupling constant $\alpha_G \equiv \alpha_{1,2,3}(M_G)$. One then has a prediction for $\sin^2 \theta_w$ at the weak scale which comes out in the experimentally measured range. Sophisticated analysis around this highly simplified picture up to two and even three loops taking into account the Yukawa couplings of the heaviest generation which contribute non-trivially at the higher orders, threshold effects, etc., vindicate this picture of gauge coupling unification which today provides one of the strongest pieces of circumstantial evidence for grand unification\[19\].

Predictions arising from (supersymmetric) unification such as for the mass of the top-quark have been vindicated experimentally. It turns out that unification based on $SO(10)$ is a scheme with great predictive power not merely in the context of top-quark mass but also with implications for the rest of the superparticle spectrum. The primary requirement that is imposed is that the heaviest generation receives its mass from a unique coupling in the superpotential $h_{16.16.10}$ where the $16$ contains a complete generation and the complex $10$ the two electroweak doublets\[20\]. When the Yukawa couplings of the top and b-quarks and the $\tau$-lepton are evolved down to the low energy and $\tan \beta$ pinned down from the accurately known $\tau$-mass, one has a unique prediction for the b and top-quark masses for a given value of $h$. If $h$ is chosen so as to yield $m_b(m_b)$ in its experimental range, the top-quark mass is uniquely determined up to these uncertainties. Now $\tan \beta \simeq m_t/m_b$ and the top-b hierarchy is elegantly explained in terms of this ratio coming out large naturally.

It is truly intriguing that this picture yields a top-quark mass in its experimental range, with $\alpha_S$ in the range of the LEP measurements despite the complex interplay between the evolution equations involved, the determination of the unification scale, running of QCD couplings below the weak scale. Note that this requires that the top-Yukawa coupling must also come out of order unity at $M_Z$. It is also worth noting that due to the nature of the evolution equations
and competition between the contributions to these from the gauge and Yukawa couplings, this number $m_t(m_t)$ lies near a quasi-fixed point of its evolution, *viz*, there is some insensitivity to the initial choice of $h[21]$. Moreover, if the $SO(10)$ unification condition is relaxed to an $SU(5)$ one where only the b-quark and τ-lepton Yukawa couplings are required to unify at $M_G$, $m_t(m_t)$ comes out in the experimental range while preserving $m_b(m_b)$ in its experimental range for $\tan\beta$ near unity. In this event also the top-quark Yukawa coupling lies near a quasi-fixed point which is numerically larger compensating for the smaller value of $\sin\beta$ that enters the expression for its mass: $m_t = h_t \sin\beta 174$ GeV. Another interesting connection arises in this context between the values of the Yukawa couplings at unification and that of the gauge coupling when one-loop finiteness and reduction of couplings is required: such a program also yields top-quark masses in the experimental range[22].

Besides the vindication of top-quark discovery predicted by susy guts, another strong test takes shape in the form of its prediction for the scalar spectrum. In the MSSM the mass of the lightest scalar is bounded at tree level by $M_Z$ since all quartic couplings arise from the D-term in the scalar potential. The presence of the heavy top-quark enhances the tree-level mass, but the upper bound in these models is no larger than 140 GeV.

Other predictions for softly-broken susy models arise when a detailed analysis of the evolution equations of all the parameters of the model are performed and the ground state carefully analyzed. In the predictive scheme with $SO(10)$ unification, the model is further specified by $M_{1/2}$, $m_0$ and $A$, the common gaugino, scalar and tri-linear soft parameters[8]. It turns out that in this scheme $M_{1/2}$ is required to come out to be fairly large, at least $\sim 500$ GeV implying a lower bound on the gluino mass of a little more than a TeV and providing a natural explanation for the continuing absence of observation of susy particles from scenarios based on radiative electro-weak symmetry breaking[24]. [An extensive study of the NMSSM with $SO(10)$ conditions has also been performed[24. Considerably greater freedom exists when the $SO(10)$ boundary condition is relaxed[25]. In summary many predictions and consistency of the MSSM and its embedding in a unified framework have been vindicated; however it is important to continue theoretical
investigations and checks to the consistency of these approaches and extensions to include the lighter generations \[26\].

5 Gravitation

The final frontier that still remains to be explored is a framework within which a consistent incorporation of the gravitational interactions is successful. Whereas it has not been possible to replace the Einstein theory by a quantum version due to bad ultra-violet behaviour, supergravity possesses improved ultra-violet properties \[7\]. String theories \[27\] often contain supergravity in their low energy spectrum and as a result supersymmetric unification is a favored candidate for these reasons as well.

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Appendix: Review of Some Group Theory

Much of the discussion presented below are those elements of group theory required for
unification. Furthermore, our discussion will only be confined to the theory of Lie algebras, or
in other words, the generators of the Lie groups of interest. In what follows are the discussions
of the Cartan subalgebra, the roots of a Lie algebra, the Killing form, the metric on the algebra,
the notions of positive and simple roots, the Cartan matrix, the translation to Dynkin diagrams,
the restrictions on the entries of the Cartan matrix, the notion of weights in the root-space of
a representation, the highest weight and the Weyl dimension formula. Finally we present the
formulae for computing the maximal sub-algebras and the branching rules for representations,
thus completing the list of topics required for a discussion on unification. The discussion closely
follows that of Cahn and should be viewed as a handbook to Slansky’s Tables.

The language is developed for the algebra $SU(3)$ as an extension of the familiar $SU(2)$
angular momentum algebra. $SU(2)$ is the group of $2 \times 2$ unitary matrices and is homomorphic
to $SO(3)$, the rotation group in three dimensions. It is characterized by the three operators
$T_{1,2,3}$ and may be related to the Pauli matrices and satisfy the commutation relations:

\[
[T_i, T_j] = i \epsilon_{ijk} T_k
\]  \hspace{1cm} (14)

Note that the group of $2 \times 2$ unitary matrices is obtained by

\[
\exp i \theta_i T_i.
\]  \hspace{1cm} (15)

Furthermore, the existence of continuous symmetries in field theory implies the existences of
conserved currents.

It is customary to define the combinations $T_{\pm} = T_1 \pm T_2$, which are the familiar raising
and lowering operators. It is possible to diagonalize only one of the operators $T_i$ and customarily it
is chosen to be $T_3$. In terms of these redefined operators, the commutation relations now read:

\[
[T_+, T_-] = 2T_Z, \quad [T_3, T_{\pm}] = \pm T_{\pm}
\]  \hspace{1cm} (16)

Furthermore, one defines the quadratic operator $T^2 = (T^+ \cdot T^- + T^- \cdot T^+)/2 + T_3^2$. It is also
well know that one may define a basis for higher angular momentum states as an eigenstate of
and $T_3$ in terms of the quantum numbers $(j, m)$ with $-j \leq m \leq j$ and the state is then $(2j + 1)$-degenerate. This is an example of a higher dimensional representation of the angular momentum algebra. One may string out the entire $(2j + 1)$-dimensional basis vectors on a line and the lowering and raising operators cause transitions between these states.

$SU(3)$ is the smallest algebra which shows a structure rich enough to be extended to the remainder of the semi-simple algebras namely the classical series and the exceptional series. This algebra may be defined in terms of 8 linearly independent operators. Conventionally these 8 may be represented by the Gell-Mann matrices. Of these, two may be simultaneously diagonalized. These simultaneously diagonalizable which are called $T_z$ and $Y$ operators span the Cartan sub-algebra of the original algebra. These two are proportional to the two diagonal Gell-Mann matrices $\lambda_3$ and $\lambda_8$ respectively. The remaining 6 operators are named $T_\pm$, $V_\pm$ and $U_\pm$ and are equal to $(\lambda_1 \pm i\lambda_2)/2$, $(\lambda_4 \pm i\lambda_5)/2$ and $(\lambda_6 \pm i\lambda_7)/2$ respectively. We stick to these choices of linearly independent operators since they naturally generalize the raising and lowering operators of the $SU(2)$ algebra.

One may then work out the commutation relations between the 8 linearly independent operators knowing their representations in terms of the Gell-Mann matrices. For instance, we may list the commutation relations enjoyed by $T^+$ with system of operators we have chosen:

\[
\begin{align*}
[t_+, t_+] &= 0, & [t_+, t_-] &= -2t_2, & [t_+, t_z] &= t_+, & [t_+, u_+] &= -v_+ \\
[t_+, u_-] &= 0, & [t_+, v_+] &= 0, & [t_+, v_-] &= u_-, & [t_+, y] &= 0
\end{align*}
\]

(17)

Since we are working with a Lie algebra, if we take an arbitrary linear combination of our 8 operators and consider its commutation relation with a fixed operator out of the 8, we produce a different linear combination of the original 8 operators. Corresponding to each of the 8 original operators $X_i$, we would find 8 different linear combinations. Our knowledge of linear algebra teaches us that we may therefore represent each of these by $8 \times 8$ matrices, we call $\text{ad}X_i$ and is called the adjoint representation. By fixing an order for the operators $X_i$, one may produce explicit representations for $\text{ad}X_i$. In particular, if we fix the order of the $X_i$ to be $T_+, T_-, T_z, U_+, U_-, V_+, V_-, Y$, the representation for $\text{ad}(aT_z + bY)$ is an $8 \times 8$ diagonal matrix.
with the diagonal entries \(a, -a, 0, (-a/2 + b), (a/2 - b), (a/2 + b), (-a/2 - b), 0\). The original choice of the linearly independent basis is now justified; each of them that is not in the Cartan subalgebra is now called a root vector and the corresponding diagonal entries is called the root. Generalization to other algebras may be performed by considering the operators that lie in the Cartan-subalgebra and the remainder broken up into \((\dim G - \dim H)\) root vectors.

It is now possible to associate to the adjoint representation the Killing form:

\[(X_i, X_j) = \text{Tr} \, \text{ad} \, X_i X_j.\]  \hspace{1cm} (18)

It turns out that for our choice of linearly independent vectors, the only non-zero answers occur for

\[(t_z, t_z) = 3, (y, y) = 4 \hspace{1cm} (19)\]
\[(t_+, t_-) = 6, (v_+, v_-) = 6, (u_+, u_-) = 6\]

In short, we have the root vectors \(\alpha_i(k), i = 1, 2, 3\) of the algebra with roots \(\pm a, \pm (-a/2 + b), \pm (a/2 + b)\), respectively when we chose the vector in the Cartan subalgebra \(k = at_z + bY\). [Note that \(\alpha_3 = \alpha_1 + \alpha_2\).] Corresponding to these three roots are the vectors in the Cartan subalgebra \(h_{\alpha_i}, i = 1, 2, 3, h_{\alpha_1} = t_z/3, h_{\alpha_2} = -t_z/6 + y/4, h_{\alpha_3} = t_z/6 + y/4\) such that \(\alpha_i(k) = (h_{\alpha_i}, k)\).

Now, we may define the scalar product on the space of roots with the definition:

\[<\alpha, \beta> = (h_{\alpha}, h_{\beta}).\]  \hspace{1cm} (20)

In particular, for the system of roots \(\alpha_i\), we have \(<\alpha_i, \alpha_1> = 1/3, <\alpha_1, \alpha_2> = -1/6, <\alpha_1, \alpha_3> = 1/6, <\alpha_2, \alpha_3> = 1/6\). This can be expressed geometrically as vectors of equal length \(1/\sqrt{3}\), with \(\alpha_1\) and \(\alpha_2\) at an angle of \(120^0\) and \(\alpha_1\) and \(\alpha_3\) at an angle of \(60^0\). These may be represented as non-orthogonal vectors in a two-dimensional plane. Generalization to higher algebras would entail the representation of roots in a space whose dimension is equal to \(\dim H\).

What we can observe from the eq. (17) and the definition of the Killing form and the structure of the roots and the associated root vectors, is that the commutation relation between root-vector \(e_\alpha\) of the root \(\alpha\) and \(e_{-\alpha}\) of the root \(-\alpha\) yields the element of the Cartan algebra \(h_\alpha\) multiplied by
the Killing form \((e_\alpha, e_{-\alpha})\), the commutation relation between a root-vector and an element of the Cartan algebra produces an eigenvalue equation for the same root-vector, where the eigenvalue is the root in question and finally, a commutation relation between two root-vectors yields an expression that is non-zero only if the sum of the two roots associated with the root-vectors is itself a root: 
\[ [e_\alpha, e_\beta] = N_{\alpha\beta} e_{\alpha+\beta}, \]
if \(\alpha + \beta\) is itself a root, or zero otherwise.

These properties may be simply generalized for a larger and more abstract (semi-simple) Lie algebra. However the generalization itself imposes severe restrictions on the nature of the root-space. In order to discuss the generalization, we will first of all discuss the higher dimensional representations of the \(SU(3)\) algebra, having encountered thus far, the fundamental representation and the adjoint representation. A representation is obtained when we have for each element of the algebra a linear transformation (i.e., a matrix) on a vector space (i.e., column vectors) that preserves the commutation relations. Note that for members of the Cartan subalgebra we can simultaneously diagonalize the associated matrices and the column vectors \(\phi^a\) can be so chosen such that

\[ H_i \phi^a = \lambda_i^a \phi^a \]  

In the case of the fundamental representation of \(SU(3)\) with 
\(T_z = \text{diag}(1/2, -1/2, 0)\) and 
\(Y = \text{diag}(1/3, 1/3, -2/3)\), the weight vectors are 
\(\phi^a = [1, 0, 0]^T\), \(\phi^b = [0, 1, 0]^T\) and \(\phi^c = [0, 0, 1]^T\), and with 
\(H = aT_z + bY\) we find 
\(H \phi^a = (a/2 + b/3)\phi^a = (2\alpha_1/3 + \alpha_2/3)\phi^a\), 
\(H \phi^b = (-a/2 + b/3)\phi^b = (-\alpha_1/3 + \alpha_2/3)\phi^b\) and 
\(H \phi^c = (-2b/3)\phi^c = (2\alpha_1/3 + \alpha_2/3)\phi^c\). The eigenvalues of the eigenvalue equations \(M^i, i = a, b, c\) above are known as the weights of the representation and the corresponding eigenvectors \(\phi^i, i = a, b, c\) and are known as the weight vectors corresponding to that weight. We have also shown that it is possible to express the weights as linear combinations of the roots. It may then be shown that that image of the root vectors \(e_\alpha\) in the matrix representation \(E_\alpha\) when acting on \(\phi^a\), produces a weight vector corresponding to the weight \(M^a + \alpha\), unless \(E_\alpha \phi^a = 0\). Thus the \(E_\alpha\) play the role of raising operators and the \(E_{-\alpha}\) as lowering operators. Just as we may represent the root-vectors in a two-dimensional plane, the weight-vectors may also be represented by points on the same two-dimensional plane. Another
way of expressing this is to say that the weight-vectors, in general, are a linear combination of
the root-vectors. The action of the raising and lowering operators associated with a fixed root
would be take one to another unless the action terminates, just as in the case of the SU(2)
algebra the action of the lowering operator would be to cause transitions between the states
in a multiplet unless \( m = -j \). In particular, if we have a weight \( M \) that lies in the string
\( M + p\alpha, ..., M, M - m\alpha \), then the following relations hold:

\[
m + p = \frac{2 < M + p\alpha, \alpha >}{< \alpha, \alpha >} \\
m - p = \frac{2 < M, \alpha >}{< \alpha, \alpha >}
\]

(22)

Just as we have reflection symmetry about the origin in SU(2) algebra, for larger algebras we
have a richer symmetry structure which is known as the Weyl Group.

One may then use the defining properties of Lie algebras to deduce many of the properties of
weight vectors in general. The multiplicity of states with a fixed weight may in principle exceed
one. However for the adjoint representation it is unity for each root with the exception of the
Cartan sub-algebra. Furthermore it turns out that for the root-system, the following identity
has to be respected:

\[
\frac{< \alpha, \beta >^2}{< \alpha, \alpha > < \beta, \beta >} = \frac{mn}{4}
\]

(23)

where \( m \) and \( n \) are integers [this follows from an important and interesting property of the roots
that if \( \alpha \) is a root, \( 2\alpha \) cannot be a root]. However, the left hand side may be seen to be nothing
but \( \cos^2 \theta \) where \( \theta \) is the angle between the root-vectors \( \alpha \) and \( \beta \). This then implies that \( \cos^2 \theta \)
can be 0, 1/4, 1/2 and 3/4.

Now we describe further characteristics of the SU(3) root system which by now may have
already become evident: while there are 6 roots, 3 or them are negatives of the other 3. Of
these only two are linearly independent. One then considers a certain ordering of these roots to
define the notion of a positive root. In the present case, these may seen to be \( \alpha_1, -\alpha_2 \) and
\( \alpha_3 \) [simply put, these roots are the ones where the coefficient of \( a \) is positive when we consider
the commutation relations of the root vectors with \( aT_3 + bY \)]. Out of these, \( \alpha_1 \) can be written

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as $\alpha_3 + (-\alpha_2)$. Then we are led to the definition of a simple root as one that cannot be written as a sum of two positive roots. In the present example, $-\alpha_2$ and $\alpha_3$ are the simple roots.

All the information regarding the algebra can then be expressed in terms of the Cartan matrix. The Cartan matrix is defined as the matrix whose elements are given by

$$A_{ij} = \frac{2 \langle \alpha_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle}$$

In the case of $SU(3)$, we see quite simply that the diagonal elements are 2 and the off-diagonal elements are equal and each is $-1$. $SU(3)$ belongs to what is known as the classical series of algebras and in particular to the one wherein all the simple roots are of equal length [this property is called simply laced and the off-diagonal elements are equal]. Detailed study of the properties of the root-systems of algebras in general also shows that simple roots can come in almost two lengths. Thus the angles between the roots and their lengths completely characterize the algebra. Given these the Cartan Matrix may be written down for any algebra and must be subject to the constraints of the root-system. Besides the classical series on which there is no restriction on the number of simple roots, viz., no restriction on the dimension of the Cartan subalgebra, there is the exceptional series all of which are known. It turns out that the root-systems of the classical series are in one-to-one correspondence with the algebra of the infinitesimal generators of the unitary, orthogonal [of even and odd order] and symplectic groups. These are the $A_n$, $B_n$, $D_n$ and $C_n$ series. The exceptional series consists of $G_2$, $F_4$, $E_{6,7,8}$ which have been documented in several references. Of particular interest to us will be the unitary, orthogonal and the $E$-exceptional series.

The Cartan matrix language for treating the Lie algebras may be translated into what are known as the Dynkin diagrams. The Dynkin diagrams code the information by representing the simple roots by dots of two types [if the roots are of unequal length] and the $i$ and $j$ roots are joined by the larger number of the entries $A_{ij}$ or $A_{ji}$.

The Dynkin diagram technique makes it very simple to study the subalgebras by erasing dots out of the Dynkin diagrams (or their extended versions). The extension of the Dynkin diagrams in order to evaluate the maximal subalgebras is performed using standard techniques.
Corresponding to each representation, one may define the *Dynkin* label of the representation \( \Lambda \):

\[
\Lambda_i = \frac{2 < \Lambda, \alpha_i >}{< \alpha_i, \alpha_i >}
\] (25)

Multiplication of the vector \( \Lambda_i \) by the inverse of the Cartan matrix, which is known as the *metric tensor* on the root space, express the element of the representation as a linear combination of the simple roots.

Given the Dynkin label of the highest weight, one may then evaluate the dimensionality of the representation to which it belongs by use of the Weyl dimension formula which reads:

\[
dim R = \prod_{\alpha > 0} \frac{< \alpha, \Lambda + \delta >}{< \alpha, \delta >},
\] (26)

where \( \delta = (\sum_{\alpha > 0} \alpha)/2 \).

The tensor product of irreducible representations breaks up into a sum of irreducible representations. In particular, for the *SU*(\( n \)) algebras, the Young’s tableaux method allows one to compute the sum in a straightforward manner. For other algebras, there are methods to perform the computations and in particular, the Dynkin labels also allow one to figure out the product representations.

A final application of the Dynkin labels allow us to study the branching rules of a representation under its subalgebras. The branching rules for many interesting groups are catalogued in the primary sources.
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