Disorder on the landscape

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Abstract. Disorder on the string theory landscape may significantly affect dynamics of eternal inflation leading to the possibility for some vacua on the landscape to become dynamically preferable over others. We systematically study effects of a generic disorder on the landscape, starting by identifying a sector with built-in disorder—a set of de Sitter vacua corresponding to compactifications of the type IIB string theory on Calabi–Yau manifolds with a number of warped Klebanov–Strassler throats attached randomly to the bulk part of the Calabi–Yau. Further, we derive a continuum limit of the vacuum dynamics equations on the landscape. Using methods of the dynamical renormalization group we determine the late-time behavior of the probability distribution for an observer to measure a given value of the cosmological constant. We find the diffusion of the probability distribution to significantly slow down in sectors of the landscape where the number of nearest-neighboring vacua for any given vacuum is small. We discuss the relation of this slowdown with the phenomenon of Anderson localization in disordered media.

Keywords: string theory and cosmology, inflation, physics of the early universe

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1. Introduction and summary

As we know, the Universe is expanding with acceleration in the present epoch of its evolution [1]. It looks like that (1) acceleration becomes noticeable at redshifts $z \sim 1$ and that (2) the magnitude of the acceleration itself does not strongly depend on the redshift $z$ and is extremely small in natural units. Indeed, the matter density associated with this acceleration is of the order of $10^{-29}$ g cm$^{-3}$, 120 orders of magnitude smaller than the Planckian density. It remains unclear what is the meaning of this huge gap in the spectrum of the underlying fundamental theory.

The physical nature of the source of this acceleration is also unclear. In general relativity we have learned that the expansion of a Universe filled with any known type of matter should decelerate. Is Nature trying to tell us that gravity itself should be modified at $z \sim 1$ [2] or is there simply an additional degree of freedom [3] with a very long relaxation time and with a special equation of state $\epsilon_A \approx -p_A$? This problem, arguably one of the most important ones in modern theoretical cosmology, is known as the cosmological constant problem.

An attempt to resolve the cosmological constant problem stems from using anthropic arguments [4]. The logic behind anthropic reasoning is simple: universes (or Hubble patches) with negative cosmological constant very rapidly collapse, reaching an AdS (anti-de Sitter) big crunch singularity. Universes with large positive cosmological constant, on the other hand, expand so rapidly that the large-scale structure does not have enough time to form. In other words, observers able to measure the value of cosmological constants...
do not appear in universes with large positive and negative cosmological constants, and the bare fact of our existence automatically implies that the value of the cosmological constant should be small in our Universe. The statement that ‘the cosmological constant is very small’ is true since, if it were wrong, there would be no observer to measure the cosmological constant.

We have to introduce a multiverse (see, e.g., [5])—an immensely large collection of Hubble patches with different values of cosmological constants. Following anthropic reasoning, the issue of the smallness of the cosmological constant of our Hubble patch is thus resolved.

An important argument in favor of the anthropic principle is that such a system of Hubble patches can be realized on the string theory landscape of metastable vacua [6] populated by eternal inflation. Even then, one might be left unhappy with an explanation of the cosmological constant’s smallness based on the anthropic principle, since many questions remain unresolved. For example, it is still unclear how to properly define a gauge-invariant probability for a given observer to measure a given value of the cosmological constant (recent discussion of the measurement problem can be found in [7, 8]) or how to determine the distribution function of vacua within the string theory landscape [4].

Instead of following the temptation of anthropic reasoning, one could try to establish a dynamical vacuum selection principle on the landscape such that dynamics of the theory itself could automatically guarantee that vacua with low positive values of the cosmological constant are preferable.

One such dynamical selection principle on the landscape is based on the analogue of Anderson localization [10]–[12]. In condensed matter theory, Anderson localization was first discovered as a phenomenon of electron wavefunction localization inside a semiconductor, provided that the disorder (for example, impurities or defects) of the effective potential that electrons are experiencing is sufficiently strong. Physics of Anderson localization in a semiconductor is related to the existence of impurities, called localization centers, with potentials so strong that they bind the Bloch wavefunction of the electron propagating in a disordered medium.

Since Anderson localization is a general phenomenon taking place in disordered media, it is plausible that, with a significant amount of ‘disorder’ on the string theory landscape, it may localize the wavefunction of the Universe in vacua with low cosmological constant [10, 11], thus providing an important dynamical selection principle. In this paper, we qualitatively study the effects of disorder on the string theory landscape.

Instead of solving the Wheeler–De Witt equation with disordered potential [10, 11] and in order to capture effects of eternal inflation important for populating vacua on the landscape we consider vacuum dynamics equations [14]:

\[ \frac{dP_i}{d\tau} = \sum_j (\Gamma_{ji}P_j - \Gamma_{ij}P_i) = \sum_j H_{ij}P_j. \]

4 The latter problem, as well as the problem of calculating correlation functions weighted over such a distribution function, turns out to be NP-hard [9].

5 The original publications, where the effect of Anderson localization was introduced, include Anderson [13] and Mott and Twose [13]. The suppression of conductivity in one-dimensional disordered systems was originally proven by diagrammatic methods in Berezinsky [13] and Alvikosov and Ryzhkin [13].
Equation (1.1) describes eternal inflation and dynamics of tunneling between de Sitter vacua on the landscape. In this expression \( \Gamma_{ij} \) are tunneling rates (inverse characteristic times of transition) between vacua \( i \) and \( j \). Due to tunneling between different vacua on the landscape, probabilities \( P_i \) for an observer to measure a given value of the cosmological constant \( \Lambda_i \) in her Hubble patch (i.e. to find herself in a given vacuum \( i \) on the landscape) evolve as time passes. If the tunneling rates \( \Gamma_{ij} \) have significant amounts of disorder, an analogue of Anderson localization may appear: there may exist vacua, ‘localization centers’, such that the probability for an observer to live in such a vacuum is higher.

The long-time (large but finite \( \tau \) ) behavior of the probability distribution function \( P_i(\tau) \) is physically the most relevant to understand. We find it in three steps:

- First we construct a continuum limit of the vacuum dynamics equations (1.1). This is possible when the number of vacua on the landscape is sufficiently large and tunneling rates \( \Gamma_{ij} \) do not strongly fluctuate as functions of the position on the landscape (labeled by the index \( i \)).

- Then we average over disorder. The reason for us to do this averaging is the following. When disorder is weak, one can treat its effects by means of perturbation theory (see appendix A). Perturbed eigenstates and eigenvalues of the operator \( H_{ij} \) in (1.1) are given by the integrals over the moduli space of the landscape. Averaging over the moduli space of the landscape is equivalent to averaging over disorder, and the latter is technically much simpler to perform than the former.

- Finally, we apply dynamical renormalization group (RG) methods to find the long (but finite) \( \tau \) behavior of the distribution function \( P_i(\tau) \) on a landscape with disorder.

The main result of our paper is that the diffusion of the probability to measure a given value of the cosmological constant in a given Hubble patch may indeed be suppressed in the parts of the landscape where the number of nearest neighbors for any given vacuum is small.

This paper is organized as follows. We begin by identifying a possible source of disorder on the string theory landscape in section 2. We argue that compactifying the type IIB string theory on a special class of Calabi–Yau (CY) manifolds will induce disorder on the string theory landscape. We consider Calabi–Yau manifolds which have large numbers of both long and short Klebanov–Strassler (KS) throats. Disordering comes from the fact that the lengths of Klebanov–Strassler throats and the points of their attachment to the bulk Calabi–Yau space may be essentially random. As a by-product, we show that it is possible to have eternal inflation of a stochastic type in the KKLMMT set-up [15] with multiple Klebanov–Strassler throats.

In section 3 we derive the continuum limit of the vacuum dynamics equations (1.1) by only using the fact that the number of vacua on the landscape (or a given sector of the landscape) is very large. In section 4 we study the continuum limit of (1.1), which describes the dynamics of eternal inflation on a disordered landscape. In particular, we determine the late-time (large but finite \( \tau \) ) behavior of the probability distribution \( P_i(\tau) \) by using dynamical renormalization group methods for various values of \( N \)—effective dimensionality of the given island on the landscape. Section 5 is devoted to discussion and conclusions.
2. Eternal inflation in a scenario with multiple Klebanov–Strassler throats

In this section we apply the idea of vacuum dynamics to the inflaton field in a string theory set-up. We start with the set-up as laid down in [15] where the open string modulus on a mobile $D3$-brane plays the role of the inflaton field. This class of models belongs to the $D3$–anti-$D3$-brane inflation models. The $D3/D7$-brane model of inflation [16] is another model of inflation where one of the open string moduli plays the role of inflaton. There are also other models of inflation within string theory, where some other moduli, in particular closed string moduli, play the role of inflaton. Some specific models include Kähler moduli inflation [17,18], complex structure moduli inflation [19], racetrack models [20] and axion modulus inflation [21] and [22] which used the $\alpha'$-correction to the Kähler potential [23] to produce a metastable $dS_4$ vacuum instead of anti-$D3$-branes.

As mentioned before, we confine ourselves to the $D3$–anti-$D3$-brane inflation models. We consider a generic type IIB compactification on a Calabi–Yau orientifold in the presence of flux. Locally at a few places the geometry of the compactification is known: these local patches are denoted by the Klebanov–Strassler throats [24]. The $D3$-brane travel between various Klebanov–Strassler throats is equivalent to hopping between different de Sitter vacua within the string landscape. We are interested in showing how disorder gets introduced once the $D3$-brane starts traversing between various throats. This begs for an actual calculation of the tunneling rate of the inflaton field between various throats. The calculation of the tunneling rate of the inflaton field from the first principle of string theory needs a proper understanding of the dynamics of the inflaton field in the bulk Klebanov–Strassler geometry after incorporation of the back-reaction. We will discuss this aspect in a future publication [25]. Here we confine ourselves to the more qualitative feature of the problem.

We first show that it is possible for the inflaton field to reach the value within a single Klebanov–Strassler throat, where stochasticity takes over the classical motion. This is shown by proving the existence of a robust upper bound on the value of the inflaton field in such a throat. This helps us define a probability distribution of the inflaton field over the stringy landscape of vacua. In the next step, we estimate the maximum possible number of larger Klebanov–Strassler throats for Calabi–Yau compactification in the presence of flux.

2.1. An upper bound on the inflaton field in a single-throat scenario

In [26], it has been shown that the open string modulus, which plays the role of inflaton, has a maximum range in warped $D$-brane inflation models. We will summarize their result here. In particular, we note that the upper bound on the open string modulus is independent of the details of the compact manifold and hence it is pretty much robust.

Let us assume a warped compactification to four spacetime dimensions in type IIB string theory set-up [27], with the ten-dimensional line element

$$
\begin{align*}
&dS^2_{10} = e^{2A(r)} \eta_{\mu\nu} \, dx^\mu \, dx^\nu + e^{-2A(r)} (dr^2 + r^2 \, d\Omega_5) \\
&\equiv G_{MN} \, dx^M \, dx^N \\
&\equiv G_{\mu\nu} \, dx^\mu \, dx^\nu + G_{mn} \, dy^m \, dy^n \\
&\equiv G_{\mu\nu} \, dx^\mu \, dx^\nu + e^{-2A} g_{mn} \, dy^m \, dy^n,
\end{align*}
$$

(2.1) (2.2) (2.3)
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Figure 1. The locations $r_{\text{IR}}$ and $r$ of the $D3$ and anti-$D3$-brane, respectively, inside a single Klebanov–Strassler throat. It also shows the distance $r_{\text{UV}}$ from the tip of the throat where the throat merges with the bulk Calabi–Yau. The origin $r = 0$ is located at the tip of the throat.

where $\eta_{\mu\nu} = \text{diag}(-1,1,1,1)$ and $\tilde{g}_{mn}$ is the metric on a six-dimensional Calabi–Yau manifold $\mathcal{M}_6$. Our convention for the indices are $M,N = 0,\ldots,9$, $\mu,\nu = 0,\ldots,3$ and $m,n = 4,\ldots,9$. The radial coordinate $r$ of $\mathcal{M}_6$ is defined by $r^2 = (y^4)^2 + \cdots + (y^9)^2$.

The internal space is a Klebanov–Strassler throat or deformed conifold which is a cone over a five-dimensional Sasaki–Einstein base manifold $\Omega_5$ [24]. This conical throat is assumed to be smoothly joined into the rest of the compact six-dimensional Calabi–Yau manifold $\delta V^w_6$. The full metric over this compact Calabi–Yau is not known: however, locally at the throat it is known completely, i.e. the warp factor $e^{2A}$ is known and it is a complicated function of $r$, closed string coupling $g_s$ and fluxes. The origin of the coordinates $r$ inside the throat is at the deformed tip of the conifold. We have turned on the following fluxes:\footnote{We have also turned on the RR 0-form $C_0$.}

$$
\frac{1}{(2\pi)^2\alpha'} \int_A F_3 = M; \quad \frac{1}{(2\pi)^2\alpha'} \int_B H_3 = -K,
$$

where $M$, $K$ are integers, $A$ is the $S^3$ at the tip of the KS throat, $B$ is a 3-cycle, Poincaré dual of $A$. In (2.4) the field strengths $F_3 = dC_2$ and $H_3 = dB_2$, where $C_2$ and $B_2$ are the RR and NSNS 2-forms, respectively; see, for example, the reviews [28,29] for more discussions.

Suppose that the throat merges smoothly into the rest of the Calabi–Yau at some distance $r_{\text{UV}}$ measured from the tip. The anti-$D3$-brane is placed very near the tip, $r_{\text{IR}} \sim 0$, see figure 1. When $r_{\text{IR}} \ll r \ll r_{\text{UV}}$, the warped throat can be locally approximated as $\text{AdS}_5 \times \Omega_5$, with the warp factor

$$
e^{-4A} = \left( \frac{R}{r} \right)^4,
$$

(2.5)
where $R$ is the radius of curvature of AdS$_5$. Since the background in (2.1) is generated by fluxes of amount $MK$, we have the relation \[ \frac{R^4}{\alpha'} = 4\pi g_s MK \frac{\pi^3}{\text{Vol}(\Omega_5)}. \] (2.6)

Here $\text{Vol}(\Omega_5)$ denotes the dimensionless volume of the space $\Omega_5$ with unit radius. Under the AdS approximation of the throat, we can think of the background as the near-horizon geometry of $N$ D3-branes, where $N = MK$.

If $V_6^w$ denotes the total warped volume of the compact six-dimensional space and $\kappa_{10}$ the ten-dimensional Newton constant, the four-dimensional Planck mass, $M_P$, is \[ M_P^2 = \frac{V_6^w}{g_s^2 \kappa_{10}^2}, \] (2.7)

where $\kappa_{10}^2 = \frac{1}{2}(2\pi)^7(\alpha')^4$. The warped volume of the internal space is \[ V_6^w = \int d^6y \sqrt{\tilde{g}_6} e^{-4A}, \] (2.8)

where $\tilde{g}_6 = \text{det} \tilde{g}_{mn}$.

We can formally split the volume of the internal space into the volume of the throat, $V^w_{\text{throat}}$, plus the rest of the Calabi–Yau, $\delta V_6^w$: \[ V_6^w = V^w_{\text{throat}} + \delta V_6^w. \] (2.9)

Under AdS approximation, the throat contribution is \[ V^w_{\text{throat}} = \text{Vol}(\Omega_5) \int_0^{r_{UV}} dr r^5 e^{-4A(r)} \]

\[ = \frac{1}{2}\text{Vol}(\Omega_5) R^4 r_{UV}^2 \]

\[ = 2\pi^4 g_s MK \alpha'^2 r_{UV}^2, \] (2.10)

where, in the last step, we have used the relation (2.6). Note that (2.10) is independent of $\text{Vol}(\Omega_5)$. Since the bulk volume is model-dependent and we are looking for a fairly robust upper bound of the inflaton field, we use the fact $V_6^w > V^w_{\text{throat}}$, so that we can write (2.7) as \[ M_P^2 > \frac{V^w_{\text{throat}}}{g_s^2 \kappa_{10}^2}. \] (2.11)

If $r$ denotes the mobile D3-brane radial modulus with respect to the anti-D3-brane inside the throat, the canonically normalized inflaton field is $\phi = \sqrt{T_3}r$, where $T_3 = 1/(2\pi)^3(\alpha'^2 g_s)$ is the physical D3-brane tension. The maximum allowed value of the inflaton field is thus $\phi_{\text{max}} = \sqrt{T_3}r_{UV}$. Hence we can write \[ \left( \frac{\phi_{\text{max}}}{M_P} \right)^2 \simeq \frac{T_3 r_{UV}^2}{M_P^2} < \frac{T_3 g_s^2 \kappa_{10}^2 r_{UV}^2}{V^w_{\text{throat}}}. \] (2.12)

Using (2.6) and the values of $T_3$ and $\kappa_{10}$ in (2.12), we get the maximum allowed value of the inflaton field in four-dimensional Planck units, in a Klebanov–Strassler throat [26] \[ \left( \frac{\phi_{\text{max}}}{M_P} \right)^2 < \frac{4}{MK}. \] (2.13)

Thus, for example, for $MK = 100$, $\phi_{\text{max}} \simeq 0.2M_P$. 

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Now we should find whether it is possible to have eternal inflation with $\phi_{\text{max}} \simeq 0.2M_P$ within this set-up. The distance $r$ between the D3-brane and the anti-D3-brane after an appropriate canonical normalization $\phi = \sqrt{T}r$ plays the role of the inflaton in the four-dimensional effective field theory. The effective potential for the inflaton has the form

$$V(\phi) = \frac{4\pi^2 \phi_0^4}{M_K} \left( 1 - \frac{\phi_0^4}{M_K \phi^4} \right),$$

(2.14)

where $\phi_0 = \sqrt{T}r_0$ denotes the location of the anti-D3-brane. Around $\phi_0$, i.e. near the tip of the throat, the geometry deviates from AdS$_5 \times S^5$. From (2.14), we get the slow-roll parameters

$$\epsilon(\phi) = \frac{M_P^2}{2} \left( \frac{V'}{V} \right)^2 \approx \frac{8}{(MK)^2} \frac{M_P^2 \phi_0^8}{\phi^{10}},$$

(2.15)

$$\eta(\phi) = M_P^2 \frac{V''}{V} \approx -\frac{20}{MK} \frac{M_P^2 \phi_0^4}{\phi^6}.$$  

(2.16)

If the conditions $\epsilon, \eta \ll 1$ are satisfied, then we have an inflationary dynamics in our set-up. Among the two conditions, (2.16) is the more restrictive one. From there, we have

$$\phi \gg \left( \frac{20}{MK} M_P^2 \phi_0^4 \right)^{1/6}.$$  

(2.17)

Therefore, if the vacuum expectation value of $\phi$ is large enough, then we have inflationary dynamics.

During one Hubble time $\Delta t \sim H^{-1}$ the value of the inflaton field decreases by

$$\Delta \phi \sim \dot{\phi} \Delta t \sim \frac{1}{3H^2} \frac{\partial V}{\partial \phi} \sim \frac{M_P^2}{8\pi} \phi_0^4 \sim \frac{M_P}{8\pi} \sqrt{2\epsilon(\phi)} = \frac{M_P^2}{2\pi MK} \phi_0^4.$$  

(2.18)

Now let us take the quantum fluctuations of the inflaton field into account. During the same Hubble time the quantum fluctuations $\delta \phi$ of the field $\phi$ are generated; for the fluctuations with the wavelength $l \sim k^{-1} \sim H^{-1}$ we have the following estimation for the amplitude:

$$|\delta \phi| \sim \frac{H}{2\pi} \sim \sqrt{\frac{2V}{3\pi M_P^2}} \sim \sqrt{\frac{8\pi \phi_0^4}{3M_P^2 MK}} = \sqrt{\frac{8\pi}{3MK} \frac{\phi_0^2}{M_P}}.$$  

(2.19)

Therefore, if $\phi$ reaches the value

$$\phi_{\text{stoch}} \sim \left( \frac{M_P^3 \phi_0^2}{10\sqrt{MK}} \right)^{1/5} \sim 10^{-2}M_P$$  

(2.20)

or higher, the stochastic force becomes more important than the classical and the inflation enters the eternal regime [32]. Comparing (2.13) with (2.20), we find that the maximum value of the inflaton field allowed by string theory in a single Klebanov–Strassler throat is at least ten times bigger in magnitude than what is needed to trigger the stochastic
force. Since the bound obtained in (2.13) is much more robust and independent of the details of the throat, we conclude that it is possible to have eternal inflation even with a single Klebanov–Strassler throat.

Finally, let us note that the regime of eternal brane inflation is, in a sense, opposite to the one usually considered in the literature (see, for example, [34] and references therein): the regime of deterministic slow-roll or DBI inflation corresponding to relatively small values of $\phi$ and ending with a $D3$–anti-$D3$ pair annihilation at the tip of the throat. Sufficiently short KS throats where the $D3$-brane quickly approaches the tip of the throat effectively play a role of sinks on the multithroat sector of the landscape that we consider. Once the $D3$-brane enters such a throat, eternal inflation comes to the end for an observer living on the brane.

2.2. An upper bound on the number of large Klebanov–Strassler throats

In this section we will be interested in estimating the number of longer Klebanov–Strassler throats in a generic type IIB Calabi–Yau flux compactification. In particular, we would like to get an idea about the maximum number of such longer throats. A statistical analysis for the number and length of such throats following the principles laid down in [9] was done in [35]. Our discussion will be more general than this analysis.

Let us assume that the compact Calabi–Yau manifold has $N_L = \sum_i N_i$ number of large and $\sum_a n_a$ number of small Klebanov–Strassler throats. The radii of curvature $R_i$ of the former class of throats are large compared to the latter ones, $r_a$. We also assume that $N_i \gg n_a$ as the larger throats have small curvatures and hence the classical supergravity analysis is valid. From (2.7), (2.9) and (2.10), it follows that

$$M_P^2 = \frac{\text{Vol}(\Omega_5)}{2g_s^2\kappa_{10}^2} \left[ \sum_i N_i R_i^4 (r_{UV})_i^2 + \sum_a n_a r_a^4 (r_{UV})_a^2 + \delta V_w^6 \right].$$

(2.21)

Here $(r_{UV})_i$ and $(r_{UV})_a$ are the distances of the larger and smaller throats, respectively, measured from the tip of the respective throats, where they merge smoothly with the rest of the Calabi–Yau manifold. From (2.21), it follows that

$$\frac{\text{Vol}(\Omega_5)}{2g_s^2\kappa_{10}^2} \sum_i N_i R_i^4 (r_{UV})_i^2 < M_P^2.$$  

(2.22)

For estimation, let us assume that all the larger throats have almost the same radii of curvature, $R_i \simeq \bar{R}$ (see figure 2). Moreover, we also assume that their merging points in the bulk of the Calabi–Yau are almost the same and equal to some average value $(r_{UV})_i \simeq \bar{r}_{UV}$. Since $N_L = \sum_i N_i$ is the total number of larger throats, we obtain

$$\frac{N_L}{2g_s^2\kappa_{10}^2} \text{Vol}(\Omega_5) \bar{R}^4 \bar{r}_{UV}^2 < M_P^2.$$  

(2.23)

According to [33], the maximum possible value $\phi_{\text{max}}$ of the inflaton field in brane inflation scenarios with warping is such that the regime of eternal inflation is not allowed. The estimation of $\phi_{\text{max}}$ by Chen et al is based on the rough estimation of the volume of the Calabi–Yau as $V_6^w \sim L^6$ and subsequent requirement that $R < L$ (as follows from the value of the four-dimensional Planckian mass). The estimation of the CY6 volume (2.10) and (2.11) taking warping of the throat into account is more accurate.
Using (2.6) and the relation $\bar{\phi}_{\text{max}} = \sqrt{T_3 \bar{r}_{\text{UV}}}$, where $\bar{\phi}_{\text{max}}$ is the average value of the maximum bound of the inflaton field inside the larger throats, we have

$$N_L < \left(\frac{M_P}{\bar{\phi}_{\text{max}}}\right)^2 \frac{4}{M\bar{K}}$$

(2.24)

where $\bar{M}$, $\bar{K}$ are some average values of the fluxes inside the larger throats.

The larger throats have smaller curvature, so in their presence we can trust the semiclassical supergravity analysis. Their presence makes the dynamics of the inflaton over the string theory landscape more interesting. In particular, one major dynamical problem is to find out the tunneling rates of the inflaton field between various larger Klebanov–Strassler throats. One can address this problem in the language of four-dimensional effective field theory. In general, the bulk Calabi–Yau geometry can be quite complicated and we expect that the classical traversing of the $D3$-brane from one throat to another may be impossible. In that case, we can look for the tunneling rate due to barrier penetration following references [36]–[38].

3. Continuum limit of vacuum dynamics equations. Tunneling graph

In the previous section we have identified one source of disorder on the string theory landscape: a sector on the landscape corresponding to a certain class of manifolds string theory can be compactified to, namely Calabi–Yau manifolds with a number of warped Klebanov–Strassler throats attached to the bulk part of the Calabi–Yau manifold (see figure 2). Since KS throats can be attached randomly to the bulk CY and have random lengths, dynamics of eternal inflation on the sector of the landscape will be affected by this randomness.

Let us now consider the vacuum dynamics equations (1.1) on a generic landscape (or a sector of the landscape).
To simplify the discussion, consider first a part of the landscape which consists only of de Sitter vacua, i.e. vacua with positive cosmological constants. We want to understand the dynamics of tunneling between de Sitter vacua.

The vacuum dynamics equations (1.1) define the dynamics of probabilities $P_i$ to measure a given value of the cosmological constant in a given Hubble patch (i.e. for an observer to find herself in a de Sitter vacuum $i$). In general, the index $i$ enumerates all possible de Sitter vacua on the landscape or just vacua within a single (sub)sector of the landscape we are interested in. For example, $i$ can label vacuum states with different numbers of fluxes, such as in the Bousso–Polchinski landscape [39], or different KS throats in the sector of the landscape corresponding to tunneling between separate KS throats, such as in the case discussed in section 2.

The tunneling rates between de Sitter vacua $i$ and $j$ are given by $\Gamma_{ij}$. These tunneling rates are not necessarily given by the Coleman–de Luccia instantons: for an example, on the sector of the landscape considered in section 2, $\Gamma_{ij}$ are given by inverse characteristic times of $D3$-brane travel between different KS throats. Many other sectors on the string theory landscape may exist due to a non-trivial dynamics of volume and complex structure moduli of the CY manifold. In the following, we do not need to know the tunneling rates $\Gamma_{ij}$ explicitly. By studying the properties of the solutions of vacuum dynamics equations (1.1) will let us draw lessons of physics interest which hold in general.

The general solution of the system (1.1) can be represented as a sum [40]:

$$P_i(\tau) = \sum_{j=1}^{n_{\text{max}}} A_{ij} e^{-\omega_{ij} \tau}, \quad (3.1)$$

where $A_{ij}$ are constants of integration, $n_{\text{max}}$ is the total number of vacua on the landscape and $\omega_{ij}$ are complex (in the general case) functions of tunneling rates $\Gamma_{ij}$. More precisely, the system of $n_{\text{max}}$ first-order differential equations (1.1) can be transformed to a single $n_{\text{max}}$th-order differential equation

$$\sum_{j=1}^{n_{\text{max}}} a_j \frac{d^j P_i}{d\tau^j} = 0, \quad (3.2)$$

In (3.2) $a_j$ are known functions of the tunneling rates $\Gamma_{ij}$. Then, the rates $\omega_{ij}$ are given by $n_{\text{max}}$ roots of the $n_{\text{max}}$th-order polynomial

$$\sum_{j=1}^{n_{\text{max}}} a_j (-)^j \omega_{ik}^j = 0, \quad \forall i, k = 1, \ldots, n_{\text{max}}. \quad (3.3)$$

Clearly, when the number of vacua $n_{\text{max}}$ is very large, finding the spectrum of $\omega_{ij}$’s by directly solving the polynomial equation (3.3) is hopeless. However, when the tunneling rates $\Gamma_{ij}$ are weakly fluctuating functions of the position $i$ on the landscape (and $n_{\text{max}} \gg 1$) the vacuum dynamics equations (1.1) have a continuum limit. In this limit we can adopt powerful methods of the theory of partial differential equations and functional analysis.

To determine how the continuum limit of the vacuum dynamics equations (1.1) emerges, we notice that (1.1) define dynamics of tunneling (hopping) on a graph (see

\[8\] On the ‘multithroat’ sector of the landscape discussed in section 2.2 it is not very large, but still we can take $n_{\text{max}}$ of the order of 100.
The tunneling graph of vacua. The island of vacua denoted by ‘green’ is well separated from ‘red’ vacua because the tunneling rates $\Gamma_{25}$ and $\Gamma_{52}$ are small.

The dimensionality $N$ of the island of ‘red’ vacua is 1 (two nearest neighbors exist for any given ‘red’ vacuum), while $N = 2$ for the island of ‘green’ vacua.

For simplicity, we will first consider the quasi-one-dimensional case $N = 1$ (two nearest neighbors for any given vacuum on the island). In this case, exactly two adjacent vacua exist for any given vacuum $i$ on the island and tunneling rates to other vacua from $i$ are suppressed. Following the terminology introduced in [12] such islands are denoted as quasi-one-dimensional islands on the landscape. The system of equations (1.1) reduces to

$$\frac{d P_i}{d \tau} = \Gamma_{i+1,i} P_{i+1} - \Gamma_{i,i+1} P_i + \Gamma_{i-1,i} P_{i-1} - \Gamma_{i,i-1} P_i.$$  

First of all, let us suppose that tunneling rates $\Gamma_{i,j+1}$ are slightly fluctuating near some average value $\bar{\Gamma}$, so that they can be represented as

$$\Gamma_{i,j+1} = \bar{\Gamma} + \delta \Gamma^s_{i,j+1} + \delta \Gamma^a_{i,j+1},$$  

where

$$\delta \Gamma^s_{i,j+1} = \frac{1}{2} (\delta \Gamma_{i,j+1} + \delta \Gamma_{i+1,j})$$  

is the symmetric part of the tunneling rate and

$$\delta \Gamma^a_{i,j+1} = \frac{1}{2} (\delta \Gamma_{i,j+1} - \delta \Gamma_{i+1,j})$$  

is its antisymmetric part. Apparently, this situation can be realized in the multithroat scenario we have discussed in section 2—for this, it is necessary that the lengths of
different KS throats and the distances between them in the interior CY were not very much different.

Let us rewrite the system (3.4) as

\[ \frac{dP_i}{d\tau} = \bar{\Gamma} \left( P_{i+1} - 2P_i + P_{i-1} \right) + \Delta, \]

where

\[ \Delta = \delta \Gamma^s_{i,i+1} (P_{i+1} - P_i) - \delta \Gamma^s_{i,i-1} (P_i - P_{i-1}) + \delta \Gamma^a_{i+1,i} (P_{i+1} + P_i) - \delta \Gamma^a_{i,i-1} (P_i + P_{i-1}). \]

It is apparent from (3.8) that the continuum limit of (3.4) exists and it is realized as a diffusion equation

\[ \frac{\partial P}{\partial \tau} = D \frac{\partial^2 P}{\partial n^2}; \quad D = \bar{\Gamma} \delta n^2, \]

(3.10)

when \( \Delta \) is set to zero. For non-vanishing \( \Delta \neq 0 \), (3.10) has additional terms and it is

\[ \frac{\partial P}{\partial \tau} = \frac{\partial}{\partial n} \left( (D + \delta D^s(n)) \frac{\partial P}{\partial n} \right) + 2 \frac{\partial}{\partial n} (\delta D^a(n) P), \]

(3.11)

where \( \delta D^s = \delta \Gamma^s \delta n^2, \delta D^a = \delta \Gamma^a \delta n^2 \) and all higher-order terms with respect to \( \delta n \) are neglected.

The physical meanings of symmetric and antisymmetric fluctuating parts \( \delta \Gamma^s_{ij}, \delta \Gamma^a_{ij} \) of the tunneling rate are clear. Equation (3.4) describes a Brownian motion of a single ‘particle’ in a random medium\(^9\). The symmetric part of the tunneling rate \( \delta \Gamma^s_{ij} \) corresponds to a fluctuation of the diffusion coefficient \( D \), while the antisymmetric part \( \delta \Gamma^a_{ij} \) corresponds to a random force acting on the particle while propagating in the medium. Since symmetric and antisymmetric fluctuation parts are independent we can separately analyze the ‘random force’ and effects associated with the fluctuations of the diffusion coefficient \( D \).

Let us now include AdS sinks into the discussion and also take into account the effects coming from finite volumes of expanding de Sitter bubbles. To understand the dynamics of the volume-weighted probability \([7, 40, 41]\) we need to add extra terms \( 3H_i P_i \) (if \( \tau \) is the world time of the four-dimensional observer) to the right-hand side of (1.1). The continuum limit of the vacuum dynamics equations for a quasi-one-dimensional island on the landscape with AdS sinks then takes the form

\[ \frac{\partial P}{\partial \tau} = \frac{\partial}{\partial n} \left( (D + \delta D^s(n)) \frac{\partial P}{\partial n} \right) + 2 \frac{\partial}{\partial n} (\delta D^a(n) P) + (3H(n) - \Gamma^s(n)) P, \]

(3.12)

where \( \Gamma^s(n) \) is the rate of tunneling from a given vacuum \( n \) to an AdS sink.

\(^9\) Technically, this means that the motion of the particle is determined by the Langevin equation \( \frac{\partial n}{\partial \tau} = \eta(n, \tau) + F(n) \), where \( \eta \) is a random force with correlation properties of a white noise \( \langle \eta(n, \tau) \eta(n, \tau') \rangle = 2(D + \delta D^s(n)) \delta(\tau - \tau') \). Equation (3.11) is the corresponding Fokker–Planck equation.
By performing a similar analysis for quasi-2d \((N = 2)\), quasi-3d \((N = 3)\), etc, islands on the landscape one finds the continuum limit of the vacuum dynamics equations:

\[
\frac{\partial P}{\partial \tau} = D \Delta P + \sum_d^N \left( \frac{\partial}{\partial n_d} \left( \delta D^s_d \frac{\partial P}{\partial n_d} \right) + 2 \frac{\partial}{\partial n_d} \left( \delta D^a_d P \right) \right)
\]

if AdS sinks and the volume effects of dS bubbles are neglected and

\[
\frac{\partial P}{\partial \tau} = D \Delta P + \sum_d^N \left( \frac{\partial}{\partial n_d} \left( \delta D^s_d \frac{\partial P}{\partial n_d} \right) + 2 \frac{\partial}{\partial n_d} \left( \delta D^a_d P \right) \right) + (3H - \Gamma_s)P
\]

in the general case. The main difference from the quasi-one-dimensional case \((N = 1)\) is that the diffusion coefficients \(\delta D^s_d\) and \(\delta D^a_d\) are now vectors; the diffusion of the ‘particle’ is anisotropic (with \(\delta D^s_d\) being the fluctuating anisotropic diffusion coefficient), while the random force \(\delta D^a_d\) acting on the ‘particle’ is a vector. Effects due to the fluctuating diffusion coefficient and random force will be treated separately in the next two sections.

As a warm up before studying effects of randomness on the landscape, let us briefly discuss the continuum limit of the Bousso–Polchinski (BP) landscape \([39]\). The BP landscape consists of a discrete set of dS vacua with different values of the cosmological constant \(\Lambda\) defined by the formula

\[
\Lambda = -\Lambda_0 + \frac{1}{2} \sum_k q_k^2 n_k^2,
\]

where \(\Lambda_0\) is a bare vacuum energy, \(n_k\) are the numbers of units of a given flux \(k\) and \(q_k\) are the associated topological charges.

The tunneling rates between different dS vacua are written in terms of Coleman–De Luccia (CDL) instantons \([38], [42]–[45]\):

\[
\Gamma_{ij} = A_{ij} e^{-B_{ij}},
\]

where \(A_{ij} \sim 1\) and \(B_{ij}\) are corresponding CDL actions. For adjacent vacua the action \(B_{i,i-1}\) can be estimated \([45]\) as

\[
B_{i,i-1} \approx \frac{27\pi^2}{8} \frac{1}{n_i^3 q_i^2}.
\]

The formula (3.17) is only valid when the tunneling exponent is large, i.e. at small energies, when the values of the cosmological constant in adjacent vacua are relatively small compared to the Planckian density \([45]\). As we can see from (3.17), the transitions between adjacent vacua happen faster at higher \(n_i\), while at low energies they proceed slower. Another important feature of the BP landscape is that the transition downwards (towards lower energies) is always more probable than the transition upwards, from lower to higher energies, because tunneling rates for adjacent vacua are related as

\[
\Gamma_{i-1,i} = \Gamma_{i,i-1} \exp (S_i - S_{i-1}) = \Gamma_{i,i-1} \exp (\delta S_i),
\]

where

\[
S_i \sim 1/\Lambda_i
\]

is the Gibbons–Hawking entropy.

\[\text{For simplicity we consider a flat spacetime limit.}\]
Let us consider transitions between vacua different by the number of quanta of a single flux \( n_i \rightarrow n_i - 1 \). For the symmetric and antisymmetric parts of the tunneling rate \( \Gamma_{ij} \) between adjacent dS vacua we have for \( q_n \ll \Lambda_i \)

\[
\Gamma_{i,i-1}^s = \frac{1}{2}(\Gamma_i + \Gamma_{i-1}) \approx A \exp\left(-B_i(1 + \frac{1}{2}\delta S_i)\right)
\]

and

\[
\Gamma_{i,i-1}^a = \frac{1}{2}(\Gamma_i - \Gamma_{i-1}) \approx A \exp\left(-B_i(1 + \frac{1}{2}\delta S_i)\right)
\]

where we took \( \delta S_i \ll 1 \).

Therefore, the continuum version of the vacuum dynamics equations for this sub-sector of the BP landscape (tunneling between vacuum states different only by the number of quanta of a single flux) is

\[
\frac{\partial P}{\partial \tau} = A \frac{\partial}{\partial n} \left( A e^{-B_i(1 + \frac{1}{2}\delta S_i)} \frac{\partial P}{\partial n} \right) + A \frac{\partial}{\partial n} \left( e^{-B_i(1 + \frac{1}{2}\delta S_i)} \cdot \delta S_i \right).
\]

This equation describes diffusion under the action of external ‘force’ \( F = A e^{-B_i(1 + \frac{1}{2}\delta S_i)} \) directed from higher \( n_i \) (vacua with larger values of the cosmological constant) towards lower \( n_i \) (vacua with lower values of the cosmological constant).

Equation (3.22) describes a single quasi-one-dimensional \((N = 1)\) sector of the BP landscape. It generalizes straightforwardly to other possible transition channels between vacua with different fluxes turned on. The entire BP landscape is described by a similar equation with \( N \gg 1 \) and all possible transition channels taken into account.

### 4. Effects of disorder on the landscape

In this section we analyze effects of disorder on the landscape. In particular, we study solutions of the vacuum dynamics equations for the non-volume-weighted measure with disorder built into the tunneling rates \( \Gamma_{ij} \). We are especially interested to find how this disorder affects long-time (large but finite \( \tau \)) behavior of the probability distribution \( P(\vec{n}, \tau) \) for a four-dimensional observer to measure a given value of the cosmological constant \( \Lambda_n \) on the sector of the landscape discussed in section 2. To determine this behavior, we pursue the following strategy. First, we suppose that tunneling rates \( \Gamma_{ij} \) are not too strongly fluctuating functions of the position on the landscape (index \( i \) labeling vacua on the island); since the number of vacua on the landscape (or an island on the landscape) is considered to be very large, we use a continuous version of the vacuum dynamics equations (3.13) derived in section 3. Second, we study asymptotic properties of the general solution of the vacuum dynamics equations by averaging over the disorder present in tunneling rates \( \Gamma_{ij} \) and by using dynamical renormalization group methods.

#### 4.1. Random environment

We begin analyzing (3.11) and (3.13) by studying the influence of the ‘random environment’ (fluctuating antisymmetric parts of tunneling rates \( \Gamma_{ij}^a \)) on the dynamics of the probability distribution \( P(\vec{n}, \tau) \) in the presence of disorder on the landscape.
Namely, we consider the $N$-dimensional diffusion equation

$$\frac{\partial P}{\partial \tau} = D \nabla^2 P + 2 \sum_{d=1}^{N} \frac{\partial}{\partial n_d} (\delta D^a_d P),$$

(4.1)

where the ‘random force’ $\delta D^a_d$ has the correlation properties

$$\langle \delta D^a_{d_1}(\vec{n})\delta D^a_{d_2}(\vec{n}') \rangle = \sigma \delta_{d_1,d_2} \delta(\vec{n} - \vec{n}'),$$

(4.2)

$$\langle \delta \vec{D}^a(\vec{n}) \rangle = 0$$

(4.3)

of the white noise. Here $\sigma$ is a constant denoting disorder strength.

Such correlation properties correspond to a multithroat sector of the landscape discussed in section 2. In particular, the absence of any disorder corresponds to the situation when KS throats have equal lengths and are attached equidistantly to the bulk CY. Weak disorder corresponds to slight fluctuations of their lengths and distances between them.

Strictly speaking, condition (4.3) is not applicable, for example, for the BP landscape, where transitions towards smaller $n$ are always more probable. This gives rise to the correlation properties

$$\langle \delta \vec{D}^a(\vec{n}) \rangle \neq 0$$

(4.4)

to be discussed in a future publication.

We study properties of the solution of (4.1) using methods of dynamical renormalization group analysis\(^{11}\).

Equation (4.1) (with correlation properties (4.2) and (4.3) of $\delta \vec{D}^a$) can be considered as a Fokker–Planck equation describing a random walk of a particle in a random environment (characterized by the coefficients $\delta D^a_d(\vec{n})$). Indeed, if the particle is moving according to the Langevin equation:

$$\frac{\partial \vec{n}(\tau)}{\partial \tau} = \vec{\eta}(\tau) + \delta \vec{D}^a(\vec{n}),$$

(4.5)

where the noise $\vec{\eta}(\tau)$ is correlated as

$$\langle \eta^{b_1}(\tau)\eta^{b_2}(\tau') \rangle = D \delta^{b_1,b_2} \delta(\tau - \tau'),$$

(4.6)

$$\langle \vec{\eta} \rangle = 0,$$

(4.7)

then (4.1) defines the probability to find the particle in the position $\vec{n}$ at a given time $\tau$. Apart from the random noise $\vec{\eta}$ the particle is affected by the force $\delta \vec{D}^a$ in any point of the medium.

\(^{11}\) Non-Gaussian correlation properties different from (4.2) give rise to irrelevant terms in the effective action (see (4.12) below) and do not affect late-time behavior of the probability distribution $P(\vec{n}, \tau)$. 
To analyze random walks in a random environment, we will closely follow \[46\]. Let us hence introduce the Martin–Siggia–Rose (MSR) generating functional \[47\]:

\[
Z[J] = \int \mathcal{D}P \delta \mathcal{D}^a \delta \left( \frac{\partial P}{\partial \tau} - D \Delta P - 2 \sum_d \frac{\partial}{\partial n_d} (\delta D^a d P) \right) \\
\times \exp \left( \int d\tau d^N n J(\tau, \vec{n}) P(\tau, \vec{n}) \right) \exp \left( - \int d^N n \frac{(\delta \vec{D}^a)^2}{\sigma} \right)
\]

(4.8)

and average over the disorder present in \(\delta D^a\).

By introducing Lagrangian multipliers and transforming the functional \(\delta\)-function into the exponent, the latter can be rewritten in terms of MSR field variables \(\bar{P}\) and \(P\) \[47, 48\]:

\[
Z[J] = \int \mathcal{D}\bar{P} \mathcal{D}P \delta \mathcal{D}^a \exp(S^a_{\text{MSR}}[\bar{P}, P, \delta D^a]) \\
\times \exp \left( \int d\tau d^N n J(\tau, \vec{n}) P(\tau, \vec{n}) \right) \exp \left( - \int d^N n \frac{(\delta \vec{D}^a)^2}{\sigma} \right),
\]

(4.9)

where

\[
S^a_{\text{MSR}}[\bar{P}, P, \delta D^a] = i \int d\tau d^N n \bar{P} \left( \frac{\partial P}{\partial \tau} - D \Delta P - 2 \sum_{d=1}^N \frac{\partial}{\partial n_d} (D^a d P) \right).
\]

(4.10)

Integrating out the Gaussian disorder \(\delta D^a\) we find

\[
Z[J] = \int \mathcal{D}\bar{P} \mathcal{D}P \exp(S^a_{\text{eff}}[\bar{P}, P]) \exp \left( \int d\tau d^N n J(\tau, \vec{n}) P(\tau, \vec{n}) \right),
\]

(4.11)

where the effective action is

\[
S^a_{\text{eff}} = i \int d\tau d^N n \bar{P} \left( \frac{\partial P}{\partial \tau} - D \Delta P \right) \\
- \frac{\sigma}{2} \sum_{d=1}^N \int d\tau' d^N n P(\vec{n}, \tau) \frac{\partial}{\partial n_d} \bar{P}(\vec{n}, \tau) P(\vec{n}, \tau') \frac{\partial}{\partial n_d} \bar{P}(\vec{n}, \tau').
\]

(4.12)

The bare propagator of the field \(P\) in the momentum representation is

\[
G_0(\omega, q) = \frac{1}{-i\omega + Dq^2},
\]

where \(\omega\) and \(q\) are the canonically conjugate momentum variables of \(\tau\) and \(\vec{n}\), respectively. Hence, in the absence of disorder, the probability distribution \(P(\vec{n}, \tau)\) is spreading out according to the linear diffusion law

\[
\langle \vec{n}^2(\tau) \rangle = 2ND\tau.
\]

(4.14)

In order to find the late-time behavior of \(P(\vec{n}, \tau)\), one should, in principle, work it out from (4.12) perturbatively in \(\sigma\) using standard diagrammatic techniques. However, using simple scaling arguments we can determine the late-time behavior of the probability distribution \(P(\vec{n}, \tau)\). From the quadratic part of the action, by multiplying momentum \(q\) by a factor of \(l\), we immediately find that the effective frequency \(\omega\) rescales as
The MSR propagator and the vertex proportional to the strength of the disorder.

\[ \omega' = \omega l^z = \omega l^2, \]  
while the Fourier components of the MSR fields \( \tilde{P}(\omega, \vec{q}) \) and \( \tilde{P}(\omega, \vec{q}) \) rescale as \( \tilde{P}\zeta \) and \( \tilde{P}\zeta \), where \( \zeta \zeta = l^{N+1} \). The same argument for the interaction term proportional to \( \sigma \) in the effective action (4.12) shows that the disorder strength \( \sigma \) renormalizes as \( \sigma' = l^{2-N} \sigma \). Therefore, disorder should be irrelevant for islands on the landscape with \( N > 2 \) and the linear diffusion law (4.14) is not affected by disorder at long timescales.

To understand what happens at \( N \leq 2 \) we have to generalize these naive scaling arguments into a complete renormalization group analysis (expansion in the disorder strength \( \sigma \) near \( N = 2 \)). Diagrammatic rules following from the form of the MSR action (4.12) are given by the bare propagator \( iG_0(\omega, \vec{q}) = (\omega - iDq^2)^{-1} \) and the vertex \( \sim -\sigma/2 \) represented in figure 4.

At the one-loop level the propagator remains unchanged [46], but the vertex renormalizes as

\[
\frac{d\sigma}{dl} = (2 - N)\sigma - \frac{1}{4\pi}\sigma^2 + O(\sigma^3). \tag{4.15}
\]

The RG flow for the dynamical exponent \( z \) is given by

\[
\frac{dz}{dl} = 2 + \frac{1}{8\pi^2}\sigma^2 + O(\sigma^3). \tag{4.16}
\]

\( \sigma = 0, z = 2 \) is a fixed point of the renormalization group flow. A second-order analysis shows that it is the only fixed point for \( N > 2 \) [46]. Therefore, disorder does not affect the late-time asymptotics of the probability distribution \( P(\vec{n}, \tau) \) for large \( N \) and the mean-square displacement \( \langle \vec{n}^2(\tau) \rangle \) behaves according to the linear diffusion law (4.14) at large \( \tau \).

At \( N = 2 - \epsilon \), long-time evolution is determined by a new fixed point

\[
\sigma = 4\pi\epsilon + O(\epsilon^2), \tag{4.17}
\]
\[
z = 2 + 2\epsilon^2 + O(\epsilon^3). \tag{4.18}
\]

The mean-square displacement

\[
\langle \vec{n}^2(\tau) \rangle = -\frac{d}{dq^2} \frac{d}{dq^2} \int d\omega e^{-i\omega\tau} G(\omega, \vec{q}) \bigg|_{\vec{q}=0} \tag{4.19}
\]

behaves as

\[
\langle \vec{n}^2(\tau) \rangle \sim \tau^{2/z} \sim \tau^{1-\epsilon^2}. \tag{4.20}
\]
For $N < 2$ the diffusion process is slow and $\epsilon = 1$ (quasi-one-dimensional islands on the landscape) is a special case treated separately below.

Similarly, one can show that for $N = 2$ the mean-square displacement behaves as

$$\langle \vec{n}^2(\tau) \rangle = 4D_R\tau \left( 1 + \frac{4}{\log \tau} + \cdots \right), \quad (4.21)$$

where $D_R$ is the renormalized diffusion coefficient, i.e. diffusion is linear with a small logarithmic correction.

For $N > 2$ and at weak disorder $\sigma \simeq 0$, the long-time behavior is determined by the trivial fixed point $\sigma = 0$ corresponding to the usual linear diffusion law.

It is interesting to note that the system of islands on the landscape in fact introduces a physical meaning to the RG $\epsilon$-expansion. This is because the number of nearest neighbors for a vacuum in a given island can fluctuate within the island. The Hausdorff dimensionality of the island (see section 3) can easily have non-integer values if the tunneling graph of the island is fractal-like. In this case, the long-time behavior of the mean-square displacement is given by

$$\langle \vec{n}^2(\tau) \rangle \sim \tau^{1-(2-N)^2} = \tau^{-3+4N-N^2} \quad (4.22)$$

for $1 < N < 2$ and the diffusion of the probability distribution is anomalous.

The case $N = 1$ turns out to be the most interesting. If we naively substituted $\epsilon = 1$ into the formula (4.20), we would expect the diffusion process to be frozen even for weak disorder. This turns out to be too premature. Indeed, Sinai showed by using ergodic methods [49] that the diffusion process does not completely stop but proceeds very slowly instead according to the logarithmic law

$$\langle n^2(\tau) \rangle \sim \log^4 \tau. \quad (4.23)$$

Notice that this behavior is independent of the disorder strength (the result holds for an arbitrarily weak disorder) and the disorder potential$^{12}$.

The overall conclusion we come to in this section is the following. For islands on the landscape with $N < 2$ of the tunneling graph, the diffusion of the probability to measure a given value of the cosmological constant in a given Hubble patch is suppressed according to the formula (4.20) for any amount of the ‘environment’ disorder $\sigma$. More precisely, it is suppressed logarithmically for quasi-one-dimensional islands ($N = 1$) on the landscape and as a slow power law for $1 < N < 2$. For quasi-two-dimensional islands ($N = 2$), the linear diffusion law acquires small logarithmic corrections, and for $N > 2$ the diffusion of the probability distribution function $P$ evolves according to the linear diffusion law.

### 4.2. Random anisotropic diffusion

We now turn to the discussion of effects due to fluctuations of the anisotropic diffusion coefficient $\delta \vec{D}$. Consider the following diffusion equation:

$$\frac{\partial P}{\partial \tau} = \sum_d^N \frac{\partial}{\partial n^d} \left( (D + \delta D^d)(n) \frac{\partial P}{\partial n^d} \right), \quad (4.24)$$

$^{12}$ Non-exponential behavior of $P_i(\tau)$ was also reported in [50] in a different context (there the situation was considered when the transition probability between vacua depends on the age of each vacuum).
where fluctuating anisotropic diffusion coefficients \( \delta D^s_d \) have the correlation properties\(^{13}\)

\[
\langle \delta D^s_{d_1}(\vec{n})\delta D^s_{d_2}(\vec{n}') \rangle = \Delta \delta_{d_1,d_2}\delta(\vec{n} - \vec{n}'),
\]

\[
\langle \delta D^s_s(\vec{n}) \rangle = 0.
\]

In principle, it is necessary to consider only the case \( \delta D^s < D \), so these correlation properties should be modified. However, in the renormalization group analysis this modification will introduce higher-order (i.e. irrelevant) terms in the effective action and we will neglect them in the present paper.

The Martin–Siggia–Rose generating functional now has the form

\[
Z[J] = \int \mathcal{D}\bar{P}\mathcal{D}\tilde{D}^s\delta \left( \frac{\partial P}{\partial \tau} - \sum_d \frac{\partial}{\partial n_d^2} \left( (D + \delta D^s_d(n)) \frac{\partial P}{\partial n_d^2} \right) \right) \times \exp \left( \int d\tau d^N n J(\tau, \vec{n}) P(\tau, \vec{n}) \right) \exp \left( -\int d^N n \frac{(\delta \tilde{D}^s)^2}{\Delta} \right)
\]

\[
= \int \mathcal{D}\bar{P}\mathcal{D}P\mathcal{D}\delta \tilde{D}^s \exp(S_{\text{MSR}}^s[\bar{P}, P, \delta \tilde{D}^s]) \times \exp \left( \int d\tau d^N n J(\tau, \vec{n}) P(\tau, \vec{n}) \right) \exp \left( -\int d^N n \frac{(\delta \tilde{D}^s)^2}{\Delta} \right),
\]

where

\[
S_{\text{MSR}}^s[\bar{P}, P, \delta \tilde{D}^s] = i \int d\tau d^N n \bar{P} \left( \frac{\partial P}{\partial \tau} - \sum_d \frac{\partial}{\partial n_d^2} \left( (D + \delta D^s_d(n)) \frac{\partial P}{\partial n_d^2} \right) \right).
\]

Since the disorder is Gaussian, we can readily integrate it out to find

\[
Z[J] = \int \mathcal{D}P \mathcal{D}P \exp(S_{\text{eff}}^s[P, P]) \exp \left( \int d\tau d^N n J(\tau, \vec{n}) P(\tau, \vec{n}) \right),
\]

where the effective action is

\[
S_{\text{eff}}^s = i \int d\tau d^N n \bar{P} \left( \frac{\partial P}{\partial \tau} - DP \right)
\]

\[
- \frac{\Delta}{2} \sum_{d,e} \int d\tau d\tau' d^N n \left( \frac{\partial \bar{P}(\tau, \vec{n})}{\partial n_d} \frac{\partial P(\tau, \vec{n})}{\partial n_d} \frac{\partial P(\tau', \vec{n})}{\partial n_e} \frac{\partial P(\tau', \vec{n})}{\partial n_e} \right) = 0.
\]

Simple scaling analysis (analogous to that of section 4.1) shows that \( \Delta = 0 \) is a fixed point of the renormalization group flow. Rescaling the length \( \vec{n} \) by a factor of \( l \) and requiring that the bare propagator stays invariant will allow us to verify that the dynamical exponent (the one which determines how the frequency \( \omega' = \omega l^z \) is rescaled) is again \( z = 2 \), and the rescaling law for Fourier components of the MSR fields is the same as we found in section 4.1. Disorder strength is renormalized as \( \Delta' = \Delta l^{-N} \). We therefore conclude that a weak disorder in anisotropic diffusion coefficients is irrelevant for any \( N \). One can confirm this conclusion by applying exact renormalization group analysis.

\(^{13}\) We again consider only the multithroat sector on the landscape described in section 2. Correlation properties of \( D^s_d \) are different for the BP landscape (\( \langle \delta D^s_s(\vec{n}) \rangle \neq 0 \)) and will be considered elsewhere.
4.3. Random isotropic diffusion: Hermitian case

According to the renormalization group analysis, random fluctuations of the anisotropic diffusion coefficients $\delta D^a_i$ do not affect the character of diffusion of the probability distribution $P(\vec{n}, \tau)$ in the limit $\tau \to \infty$ for arbitrary $N$. This, however, does not mean that the dynamics of the probability distribution on the landscape is not influenced by disorder $\delta D^a_i$ at intermediate timescales. To get some idea for what we might expect at intermediate $\tau$, let us consider the Fokker–Planck equation

$$\frac{\partial P}{\partial \tau} = \hat{H} P,$$

(4.32)

with the following Hermitian\(^\text{14}\) Hamiltonian [12]

$$\hat{H} = D \Delta + \partial_d (V(\vec{n}) \cdot \partial_d),$$

(4.33)

where $V(\vec{n})$ is a random Gaussian-distributed potential. This is a special case of (4.24) with only the isotropic piece (i.e. independent of $d$ directions) of the anisotropic diffusion coefficients $\delta D^a_i$ non-vanishing.

For Hermitian $\hat{H}$ the solution of the Fokker–Planck equation (4.32) can be expanded in terms of a complete set of orthogonal eigenfunctions of the Hamiltonian (4.33), and all corresponding eigenvalues will be real\(^\text{15}\). The solution of (4.32) has the form (below we take $V \ll D$, but the generalization for arbitrary $V$ is straightforward)

$$P(\vec{n}, \tau) = \sum_{k=0}^{\infty} c_k \psi_k(\vec{n}) e^{-E_k(\tau - \tau_0)},$$

(4.34)

where $\psi_k$ and $E_k$ are eigenfunctions and eigenvalues of the following Schrödinger equation:

$$\partial_n^2 \psi_k + \left( \frac{E_k}{\sqrt{D}} - W(\vec{n}) \right) = 0$$

(4.35)

and the effective potential $W(\vec{n})$ is given by

$$W(\vec{n}) = \frac{(\partial_n V)^2}{4 D^2} - \frac{\partial_n^2 V}{2 D}.$$  

(4.36)

Since the effective potential $W(\vec{n})$ in (4.35) has a supersymmetric form, all eigenvalues $E_n$ are positive definite. If the ‘volume’ $\int d^N n$ of the moduli space of the landscape is finite, the ground state $\psi_0$ (normalized to $\psi_0(\vec{n}) = 1$) is trivial and has zero energy $E_0 = 0$. As $\tau \to \infty$ (or, more accurately, at $\tau \gg E_1^{-1}$) only the first term corresponding to the trivial ground state survives in the expansion (4.34).

Let us now average over disorder, apply the RG machinery used in previous sections and reproduce this asymptotic behavior. After introducing the Martin–Siggia–Rose functional and integrating out the Gaussian disorder

$$\langle V(\vec{n}) V(\vec{n}') \rangle = \mu \delta(\vec{n} - \vec{n}'),$$

(4.37)

\(^\text{14}\) With respect to the scalar product $(\Psi', \Psi) = \int d^N n \Psi(\vec{n}) \Psi(\vec{n})$.

\(^\text{15}\) Let us return from the continuum limit (3.11) back to the discrete set of the vacuum dynamics equations (3.8), where the matrix $H_{ij}$ has a general form, i.e. it is neither normal nor (what is even more restrictive) Hermitian. In this case, the set of eigenfunctions of the operator $H_{ij}$ may be incomplete.
where $\mu$ is a constant, of the random potential $V(\vec{n})$ we find the effective action of the form similar to (4.31). Scaling analysis again shows that disorder of $V(\vec{n})$ is irrelevant for arbitrary effective numbers of dimensions $N$. The physical reason for this is now clear: the Hamiltonian (4.33) has a trivial extended eigenstate with $E_0 = 0$ defining asymptotic behavior of the probability distribution as $\tau \to \infty$.

Late-time (large but finite $\tau$) behavior is determined by eigenstates $\psi_k(\vec{n})$ with higher $k > 0$. Also, if the ground state $\psi_0(\vec{n})$ is not normalizable (such as in the case of an infinite landscape of vacua), higher $k$ eigenstates are relevant in determining the late-time asymptotics.

The following picture [51] emerges from the analysis of the nonlinear sigma model corresponding to the Martin–Siggia–Rose functional (4.31). For $N = 1$ (quasi-one-dimensional islands on the landscape) all eigenstates $\psi_k(n)$ with $k > 0$ are localized near the so-called localization centers $n = n_i$ in the sense that wavefunctions $\psi_k$ are exponentially peaked at $n = n_i$. Localization centers correspond to the lowest local minima of the superpotential $W(\vec{n})$.

The width of $\psi_k$ is called the localization length $\xi_k$. At $N = 1$ it behaves as

$$\xi_k \sim E_k^{-1} \sim k^{-2},$$

(4.38)
i.e. it grows at small energies and reaches $\infty$ for $E = E_0 = 0$. For small $\tau$, the localized eigenstates with higher energy have a significant contribution to the probability distribution (4.34), thus affecting the character of diffusion.

Behavior of the correlation function at intermediate times can be estimated as follows. For simplicity, let us suppose that all eigenstates are localized near the same localization center $n_i$ (e.g. in the case when a single very deep minimum of the superpotential $W(n)$ is relevant), so that

$$\psi_k(n) \simeq \exp \left( -\frac{|\vec{n} - \vec{n}_i|}{\xi_k} \right).$$

(4.39)

We therefore have

$$\langle (\vec{n} - \vec{n}_i)^2 \rangle \simeq \frac{\int d^N l d^N n \, (\vec{n} - \vec{n}_i)^2 \exp \left( -\frac{|\vec{n} - \vec{n}_i|}{\xi_k} - \sum_k E_k l_k^2 \tau \right)}{\int d^N l d^N n \, \exp \left( -\frac{|\vec{n} - \vec{n}_i|}{\xi_k} - \sum_k E_k l_k^2 \tau \right)}$$

(4.40)

$$= \frac{\int dE \, g(E) \, \frac{\xi^{N+2}}{g(E)} \, e^{-E \tau}}{\int dE \, g(E) \, \xi^{N} \, e^{-E \tau}},$$

(4.41)

where $E_1$ is the energy of the first excited state$^{16}$ and $g(E)$ is the density of states and is proportional to

$$g(E) \propto E^{(N-2)/2}.$$  

(4.42)

For $N = 1$ and $\tau \ll E_1^{-1}$ we find

$$\langle (n - n_i)^2 \rangle \sim \tau^2.$$  

(4.43)

This means that the diffusion is slowed down in one dimension due to the presence of localized states. The slowdown is not very strong, since the density of states grows with decreasing energy. The density of states $g(E)$ is depicted in figure 5 for arbitrary $N$.

$^{16}$ It is inversely proportional to the ‘volume’ of the moduli space of the island and determines the longest timescale in the problem.
Figure 5. Densities of states \( g(E) \) for \( N = 1 \), \( N = 2 \) and \( N \geq 3 \). In the \( N = 1 \) case (a) all states \( \psi(E, n) \) are localized (gray region); discreteness of the spectrum should be taken into account for small \( E \). Also, in the \( N = 2 \) case (b) all states are localized and the density of states remains constant. Finally, for \( N \geq 3 \) (case (c)) there exists a mobility edge \( E^* \) such that all states with \( E > E^* \) are localized, while states with \( E < E^* \) are not.

Similarly for \( N = 2 \) (quasi-two-dimensional case), all eigenstates \( \psi_k(\vec{n}) \) with \( k > 0 \) are localized near localization centers but the localization length now grows much faster as \( E \to 0 \):

\[
\xi(E) \sim e^{1/E}.
\]

However, the density of states now remains constant as \( E \to 0 \), so diffusion slows down more effectively.

Finally, for \( N > 2 \) there exists a mobility edge \( E^* \) such that all states with \( E_k < E^* \) are extended, while states with \( E_k > E^* \) are localized. The localization length in this case behaves as

\[
\xi(E) \sim \frac{1}{(E - E^*)^{1/(N-2)}}
\]

near the threshold \( E = E^* \). Linear diffusion asymptotics

\[
\langle \vec{n}^2(\tau) \rangle \sim \tau
\]
is also reproduced when $\tau \gg (E^*)^{-1} \gg E_1^{-1}$. The diffusion process of the probability distribution is less effective at $(E^*)^{-1} \gg \tau \gg E_1^{-1}$: mean-square displacement behaves as

$$\langle (\vec{n}(\tau) - \vec{n}_i)^2 \rangle \sim \tau^{2/(N-2)}.$$  \hspace{1cm} (4.47)

5. Conclusions and discussion

In this paper, we systematically study the dynamics of eternal inflation on the string theory landscape in the presence of disorder. Due to a non-trivial dynamics of moduli fields, many sources of disorder may exist in different sectors of the landscape. We identify one such sector with built-in disorder in section 2. This sector is established by compactifying type IIB string theory on a class of Calabi–Yau manifolds with a number of warped Klebanov–Strassler throats attached to the bulk part of the Calabi–Yau (see figure 2). KS throats can be attached randomly to the bulk CY and can have random lengths.

Eternal inflation on such a sector of the landscape has a mechanism different from the one realized on the Bousso–Polchinski sector [39]. Our four-dimensional Universe is localized on a $D3$-brane located in one of the KS throats, and the effective value of the cosmological constant for the four-dimensional observer living on the brane is essentially determined by the distance $r$ between the $D3$-brane and the tip of the KS throat (more accurately, this distance determines the value of the effective inflaton field $\phi = \sqrt{T_3 r}$ detected by the four-dimensional observer). The $D3$-brane travels in the throat: as a result, the four-dimensional Hubble patch, the one our observer lives in, inflates—this is the KKLMMT scenario [15] in a nutshell. Eternal inflation (as we proved in section 2, it is possible in a KKLMMT set-up) corresponds to strong fluctuations of the inflaton field, while the distance $r$ of the $D3$-brane from the tip of the throat is large. The $D3$-brane can leave the given KS throat and enter other KS throats, with different lengths, i.e. different scales of the cosmological constant measured by a four-dimensional observer. By construction, KS throats (with random lengths) are randomly attached to the bulk CY and this disorder influences the dynamics of eternal inflation.

After identifying a possible source of disorder on the string theory landscape, we consider the vacuum dynamics equations (1.1) describing the dynamics of eternal inflation on a disordered sector of the landscape (or entire landscape). One particular goal of our study is to understand how disorder influences the late-time ($\tau$ large but finite) behavior of the probability $P_i(\tau)$ to measure a given value of the cosmological constant $\Lambda_i$ in a given four-dimensional Hubble patch.

Discussing eternal inflation on the landscape, usually one is interested in the behavior of $P_i(\tau)$ at $\tau \rightarrow \infty$. If the set of eigenfunctions of the operator $\hat{H}$ is complete, $P_i(\tau \rightarrow \infty)$ behavior is determined by the eigenvalue $E_0$ of $\hat{H}$ with the lowest possible real part. One notes that it always exists since the spectrum of $\hat{H}$ is bounded from below for the compact landscape. Generally, $E_0$ is complex and its real part can be negative. If this is so, $P(\vec{n}, \tau)$ itself as well as correlation functions weighted over this probability distribution are never gauge-invariant, since they explicitly depend on $\tau$. If the operator $\hat{H}$ is Hermitian and supersymmetric (such as for the case discussed in section 4.3), the ground state corresponds to $E_0 = 0$ and the asymptotics $P(\tau \rightarrow \infty)$ is stationary. For the case of isotropic diffusion discussed in section 4.3 the form of the lowest eigenstate is trivial (constant). At weak disorder the behavior of $P(\tau = \infty)$ does not depend on
Disorder on the landscape

initial conditions for eternal inflation on such a landscape, on reparameterizations of \( \tau \) and disorder (see section 4.1).

Although it is physically important to know \( P_i(\tau = \infty) \) asymptotics on the landscape, finite time behavior of \( P_i(\tau) \) is also of great physical relevance. Indeed, because the moduli space of the string theory landscape is huge, albeit compact, the late-time asymptotics \( P_i(\tau = \infty) \) is reached at timescales roughly estimated as \( \tau \gg \tau_1 \sim n_{\text{max}} \), where \( n_{\text{max}} \) is the number of vacua on the landscape, i.e. the volume of its compact moduli space. This timescale is extremely long, so it is plausible that our own Hubble patch was generated in the epoch \( \tau \ll \tau_1 \sim n_{\text{max}} \).

To determine the behavior of \( P_i(\tau) \) at large but finite \( \tau \) in the presence of disorder on the landscape, our strategy was to first construct the continuum limit of the vacuum dynamics equations (1.1) and then determine the asymptotic behavior of the probability distribution \( P_i(\tau) \) by applying dynamical renormalization group methods.

It is easy to understand why studying the continuum limit of equations (1.1) is appropriate if we consider a landscape with Hermitian \( \hat{H} \). In this case, dynamics of \( P_i(\tau) \) is completely determined by real eigenvalues and the form of eigenstates of the operator \( \hat{H} \) (see section 4.3). At intermediate timescales \( \tau \ll E_1^{-1} \) (here \( E_1 \) is the first excited eigenstate) the spectrum of \( \hat{H} \) can be approximately considered continuous since the energy gap between eigenstates is of the order of the inverse volume \( 1/n_{\text{max}} \) of the moduli space of the landscape (or the part of the landscape under consideration).

The use of dynamical RG methods is appropriate once recalling why RG is applicable in a generic quantum field theory (QFT): in the \( k \to 0 \) limit only renormalizable interactions or, in other words, only relevant terms in the effective action of the QFT survive. Similarly, for a time-dependent problem only relevant terms survive in the effective Martin–Siggia–Rose action [47] describing the dynamics of \( P_i(\tau) \) (see section 4).

The results of our weak disorder analysis are the following. In the presence of disorder on the landscape

(1) large \( \tau \) behavior of the probability distribution \( P_i(\tau) \) is strongly modified on islands of the landscape with \( N < 3 \), where \( N \) is the Hausdorff dimension of the tunneling graph corresponding to the given island. There exists a non-trivial fixed point in the RG flow corresponding to finite disorder (see section 4.1) which defines this anomalous behavior.

(2) Large \( \tau \) behavior of the probability distribution \( P_i(\tau) \) for islands with \( N > 3 \) is not affected by weak disorder since the only fixed point of the RG flow corresponds to the absence of any disorder (again see section 4.1).

(3) On sectors of the landscape, where \( \hat{H} \) is Hermitian, finite \( \tau \) behavior is always affected by weak disorder for arbitrary \( N \), and diffusion of the probability \( P_i(\tau) \) is generally suppressed for any \( N \). The reason for this suppression is the existence of localized eigenstates of the operator \( \hat{H} \), i.e. Anderson localization on sectors of the landscape with disorder.

\[ 17 \] In this respect, let us again emphasize that we study the behavior of the non-volume-weighted measure at \( \tau \to \infty \). The behavior of the volume-weighted measure is very different: for example, it is much more likely that our Hubble patch is generated at late times if probabilities are given by integrating over volume-weighted measure.
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The situation in the presence of strong disorder on the string theory landscape is much harder to understand qualitatively. Infinitely strong disorder corresponds to a fixed point of the renormalization group flow of the MSR action. The physics of the infinite disorder case is such that the diffusion of the probability distribution function $P_i(\tau)$ is suppressed completely: as $\tau \to \infty$, the mean-square displacement $\langle n^2(\tau) \rangle$ approaches a constant for arbitrary $N$. This fixed point is believed to be unstable with respect to inverse disorder corrections. At $N = 1$ this follows from Sinai’s result [49] showing that the mean-square displacement diverges as $\tau \to \infty$ for large but finite disorder. Any proof of this statement is absent for islands with $N > 1$.

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Appendix A. Why average over disorder?

In this appendix we will explain why it is necessary to average over disorder to determine the late-time dynamics of the probability distribution, giving us a way to measure a given value of the cosmological constant in a given Hubble patch. To understand the physical meaning of averaging over disorder, let us consider a particular realization of the disorder on the quasi-one-dimensional ($N = 1$) landscape. This system is described by the continuous version of the vacuum dynamics equations

$$\frac{\partial P}{\partial \tau} = \hat{H}P, \quad (A.1)$$

where the operator $\hat{H}$ is Hermitian. In this case, the solution of (A.1) can be represented as a sum over the complete set of orthogonal eigenstates of the operator $\hat{H}$.

Since the moduli space of the island is compact, the distribution function $P(n, \tau)$ reaches its equilibrium asymptotics $P_0(n)$ in finite time $\tau \sim E_1^{-1}$, where $E_1$ is the eigenvalue of the first excited eigenstate of the operator $\hat{H}$. The asymptotics $P_0(n)$ can be solved from the equation

$$\frac{\partial}{\partial n} \left( (D + \delta D^a(n)) \frac{\partial P_0(n)}{\partial n} \right) = 0. \quad (A.2)$$

18 For $N > 1$ this is only possible if the coefficients $\delta D^a_1 = 0$, while the diffusion is isotropic, i.e. $\delta D^a_1 = \delta D \cdot \hat{I}$, where $\hat{I} = \text{diag}(1, 1, \ldots)$; the latter condition is always realized on the quasi-one-dimensional island.
The solution does not depend on $n$ nor disorder and is

$$P_0(n) = \frac{1}{\int dn} = \frac{1}{n_{\text{max}}}.$$  \hspace{1cm} (A.3)

The general solution of (A.1) has the form

$$P(n, \tau) = \sum_k c_k \phi_k(n) e^{-E_k \tau},$$  \hspace{1cm} (A.4)

where $\phi_k$ and $E_k$ are given by the Schrödinger equation

$$D\phi_k'' + (\delta D^s(n) \phi_k)' = E_k \phi_k.$$  \hspace{1cm} (A.5)

The coefficients $c_k$ are defined by the initial conditions for the probability distribution $P(n, \tau = 0)$.

If the disorder on the landscape is weak, i.e. $\delta D^s \ll D$, the behavior of the probability distribution $P(n, \tau)$ at finite time can be determined by means of perturbation theory.

Since the moduli space of the given island is compact, the spectrum of eigenvalues $E_k$ is discrete. In the leading approximation (i.e. in the absence of any disorder), the spectrum of eigenstates is given by

$$E_k^{(0)} = \frac{D \pi^2 k^2}{n_{\text{max}}^2},$$  \hspace{1cm} (A.6)

where $n_{\text{max}}$ is the maximum allowed index on the island (i.e. the volume of its moduli space). The eigenfunctions are

$$|k\rangle = \phi_k^{(0)}(n) = c_k \cos \left( \frac{kn}{n_{\text{max}}} \right).$$  \hspace{1cm} (A.7)

In the first subleading approximation, disorder gives rise to a spectrum shift

$$\delta E_k^{(1)} = \langle k | \delta \hat{H} | k \rangle \equiv \int dn c_k^2 \cos \left( \frac{kn}{n_{\text{max}}} \right) \delta \hat{H} \cos \left( \frac{kn}{n_{\text{max}}} \right),$$  \hspace{1cm} (A.8)

and the following modification in the form of eigenstates (we suppose that the spectrum of states is not degenerate; the generalization for the case with degeneracy is trivial):

$$\delta \phi_k^{(1)} = \sum_{k' \neq k} \langle k' | \delta \hat{H} | k \rangle \left( E_k^{(0)} - E_{k'}^{(0)} \right) \phi_{k'}^{(0)},$$  \hspace{1cm} (A.9)

where the interaction Hamiltonian is

$$\delta \hat{H}_\phi = \frac{\partial}{\partial n} \left( \delta D^s(n) \frac{\partial}{\partial n} \phi_k \right).$$  \hspace{1cm} (A.10)

We find

$$\delta E_k^{(1)} = -c_k^2 \frac{k^2}{n_{\text{max}}^2} \int dn \delta D^s(n) \sin^2 \left( \frac{kn}{n_{\text{max}}} \right),$$  \hspace{1cm} (A.11)

$$\delta \phi_k^{(1)} = \sum_{k' \neq k} c_k c_{k'}^2 \int dn' k' \delta D^s(n) \sin \left( \frac{kn'}{n_{\text{max}}} \right) \sin \left( \frac{k'n'}{n_{\text{max}}} \right) \cos \left( \frac{k'n}{n_{\text{max}}} \right),$$  \hspace{1cm} (A.12)
The main conclusion we draw is the following. As we can see from the expressions (A.11) and (A.12), after taking the small disorder on the landscape into account, the probability distribution \( P(n, \tau) \) acquires small corrections having the form of integrals of disorder over the moduli space of the island, i.e. averaging of one particular realization of disorder over the overall volume of the moduli space. By ergodicity, this is equivalent to averaging over all possible realizations of the disorder.

Appendix B. Schwinger–Keldysh diagrammatic technique

In appendices B and C we mostly follow the excellent lecture notes by Kamenev [52] and the classical book by Kadanoff and Baym [53], where the reader can find a much deeper overview of the Schwinger–Keldysh diagrammatic methods.

The Schwinger–Keldysh diagrammatics is used to deal with quantum fields out of equilibrium. The crucial difference between the equilibrium diagrammatic (Feynman) methods and the Schwinger–Keldysh diagrammatics is that amplitudes and partition functions in the latter case are calculated along a closed time contour in the complex plane. The contour starts from \( t = -\infty \), goes to some fixed moment of time \( t = t_0 \) and then goes back to the infinite past (see figure B.1).

For simplicity, let us suppose that we have a theory with a single real scalar field \( \chi(x) \).

The major consequence of calculating amplitudes along the closed time contour is an effective doubling of degrees of freedom. In Schwinger–Keldysh diagrammatic technique, instead of a single field \( \chi(x) \) we introduce two Keldysh fields \( \chi^+(x) \) and \( \chi^-(x) \) defined on the lower and upper sides of the closed time contour. Because the contour is connected, the fields \( \chi^+, \chi^- \) are not independent. In particular, the correlator \( \langle \chi^+(x)\chi^-(x) \rangle \) is non-vanishing.

After introducing the Keldysh indices \( a, b = \{+, -\} \), the generating functional for the Green functions has the form

\[
Z[J^a_\chi] = \int \mathcal{D}\chi^a \exp \left[ i \left( S[\chi^a] + \int d^4x \sqrt{-g(x)J^a_\chi(x)\chi^a(x)} \right) \right]. \quad (B.1)
\]

The correlation functions of the fields \( \chi^a \) are given by functional derivatives with respect to sources at \( J^a_\chi = 0 \). For example, the two-point correlation function of \(+\) and \(-\) Keldysh fields is

\[
iG^{+-}(x, x') \equiv \langle \chi^+(x)\chi^-(x') \rangle = \frac{1}{2} \frac{\delta^2 Z[J^+(x), J^-(x')]}{\delta J^+(x)\delta J^-(x')} \bigg|_{J^+, J^- = 0}. \quad (B.2)
\]

Not all of the four possible two-point Green functions\(^{20}\) are independent. Due to causality constraints (see [53]) we have an identity

\[
G^{++}(x, y) + G^{--}(x, y) = G^{+-}(x, y) + G^{-+}(x, y). \quad (B.3)
\]

\(^{19}\) It is not necessary for the contour to start from infinity. It is actually not suitable to use such a contour for many physical situations [54]. However, using an infinitely long contour is more convenient, since one keeps the preparation of the initial state well under control by adiabatically slowly turning on the interaction term.

\(^{20}\) I.e. the Green functions \( iG^{++} = \langle \chi^+(x)\chi^+(x') \rangle \), \( iG^{--} = \langle \chi^-(x)\chi^-(x') \rangle \), \( iG^{+-} = \langle \chi^-(x)\chi^+(x') \rangle \) and \( iG^{-+} = \langle \chi^+(x)\chi^-(x') \rangle \).
It is therefore possible to simplify the analysis of perturbation theory by doing the Keldysh rotation
\[
\chi_{cl}(x) = \frac{1}{\sqrt{2}}(\chi^+(x) + \chi^-(x)), \tag{B.4}
\]
\[
\chi_q(x) = \frac{1}{\sqrt{2}}(\chi^+(x) - \chi^-(x)), \tag{B.5}
\]
since only the Green functions \(\langle \chi_{cl}(x)\chi_q(x') \rangle \equiv iG^R(x, x')\), \(\langle \chi_q(x)\chi_{cl}(x') \rangle \equiv iG^A(x, x')\) and \(\langle \chi_{cl}(x)\chi_{cl}(x') \rangle = iG^K(x, x')\) are non-zero. \(G^R(x, x')\) and \(G^A(x, x')\) are retarded and advanced Green functions, respectively. \(G^K\) is called the Keldysh Green function. The Green function \(i\langle \chi_q(x)\chi_q(x') \rangle\) remains zero non-perturbatively because the identity \(B.3\) holds in all orders of the perturbative expansion \[52\).

The field \(\chi_{cl}(x)\) is usually denoted as ‘classical’ while the field \(\chi_q(x)\) as ‘quantum’, since among saddle points of the effective action there is always a solution such that \(\chi_q(x) = 0\) and \(\chi_{cl}(x)\) satisfy the classical equations of motion \[52\).

Let us now discuss the information carried by the Keldysh Green functions. For a free massive scalar field \(\chi(x)\) we have
\[
G^{++}(k) = (k^2 - m^2 + i\epsilon)^{-1} - 2\pi i n(k)\delta(k^2 - m^2), \tag{B.6}
\]
\[
G^{--}(k) = -(k^2 - m^2 + i\epsilon)^{-1} - 2\pi i n(k)\delta(k^2 - m^2), \tag{B.7}
\]
\[
G^{+-}(k) = -2\pi i(\theta(k^0) + n(k))\delta(k^2 - m^2), \tag{B.8}
\]
where \(n(k)\) is the occupation number for a given mode with momentum \(k\) and \(\theta(k)\) is a step function.

The simplest way to verify \(B.6\)–\(B.8\) is to use the WKB (Wentzel–Kramers–Brillouin) approximation. For example, the limit \(x \to x'\) of the \(G^{++}\) Green function (neglecting the vacuum contribution) is given by
\[
\langle \chi^+(x)\chi^+(x) \rangle = \int \frac{d^3k}{(2\pi)^3} 2\omega_k n(k) = 2\pi \int \frac{d^4k}{(2\pi)^4} n(k)\delta(\omega_k^2 - k^2 - m^2). \tag{B.9}
\]
Recalling that \(\langle \chi^+(x)\chi^+(x) \rangle = iG^{++}(x, x)\) we come to the expression \(B.6\).

Finally, by performing the Keldysh rotation \(B.5\) yields
\[
G^K(k) = -2\pi i(1 + 2n(k))\delta(k^2 - m^2), \tag{B.10}
\]
\[
G^R(k) = (k^2 - m^2 + i0)^{-1}, \tag{B.11}
\]
\[
G^A(k) = (k^2 - m^2 - i0)^{-1}. \tag{B.12}
\]
We conclude that the Keldysh Green function \(G^K\) carries information about the distribution function \(n(k)\), while the retarded and advanced Green functions \(G^R, G^A\) give...
the spectrum of particles (and are independent of the distribution function \(n(k)\)). This separation is only valid for systems close enough to a thermal equilibrium\(^{21}\).

**Appendix C. Quasi-classical Keldysh action, Martin–Siggia–Rose diagrammatics**

The trivial saddle point of the generating functional (B.1) rewritten in terms of quantum and classical Keldysh fields (B.5) is determined by the equations

\[
\frac{\delta S}{\delta \chi_{\text{cl}}} = 0 \rightarrow \chi_{q} = 0, \tag{C.1}
\]

\[
\frac{\delta S}{\delta \chi_{q}} = 2\mathcal{O}_{R}[\chi_{\text{cl}}] \chi_{\text{cl}} = 0, \tag{C.2}
\]

where \(\mathcal{O}_{R}\) is a retarded operator describing the dynamics of the classical Keldysh field, while the fields \(\chi_{\text{cl}}, \chi_{q}\) were introduced in the previous appendix. The trivial saddle point is \(S = 0\) with \(Z = 1\)\(^{22}\). In fact, the condition \(Z = 1\) holds even if one considers fluctuations near this saddle point \(^{52}\).

To consider the quasi-classical limit, fluctuations of \(\chi_{q}\) should be allowed near the classical trajectory. Let us keep terms only up to second order in \(\chi_{q}\) in the Keldysh action. The semiclassical action will then take the following form:

\[
S_{\text{scI}} = 2 \int \int dt\, dt' \left[ \chi_{q}[G^{-1}]^{\text{K}} \chi_{q} + (\chi_{q}[G^{-1}]^{\text{R}}[\chi_{\text{cl}}] + \text{c.c.}) \right], \tag{C.3}
\]

where c.c. denotes the complex conjugation.

One simple way to treat this semiclassical theory is to use the Hubbard–Stratonovich transformation introducing the auxiliary stochastic field \(\xi(t)\) and decoupling the quadratic term in the quasi-classical equation. One will find that the resulting action is linear with respect to \(\chi_{q}\), i.e. the integration over \(\chi_{q}\) leads to the functional \(\delta\)-function and to the stochastic Langevin equation:

\[
\mathcal{O}_{\text{R}}[\phi_{\text{cl}}] \phi_{\text{cl}}(t) = \xi(t), \tag{C.4}
\]

where

\[
\langle \xi(t)\xi(t') \rangle = \frac{i}{2}[G^{-1}]^{\text{K}}(t, t'). \tag{C.5}
\]

Another way to deal with the semiclassical theory is to integrate out the \(\chi_{q}\) field, since its contribution to the action is quadratic. The result is the theory of the classical Keldysh field:

\[
S[\chi_{\text{cl}}] = 2 \int_{-\infty}^{\infty} dt\, dt' \chi_{\text{cl}}(\mathcal{O}^{A}[\chi_{\text{cl}}]G^{A})[G^{\text{K}}]^{-1}(G^{\text{R}}\mathcal{O}_{\text{R}}[\chi_{\text{cl}}])\chi_{\text{cl}}. \tag{C.6}
\]

If nonlinearities with respect to \(\chi_{\text{cl}}\) are neglected, the theory is a free field theory but with a complicated propagator. Using the first quantized version of this theory, one can show

\(^{21}\) Far from equilibrium the situation is not so trivial anymore, but it is possible to show that imaginary parts of the Green functions still carry information about the occupation numbers in the corresponding modes \(^{52}\).

\(^{22}\) It is possible that the Keldysh generating functional can have other saddle points (similar to instanton trajectories in the equilibrium situation) but they have zero contribution to the partition function \(Z\).
that dynamics of the probability $P(x, t)$ to find a particle excitation in the point $x$ at time $t$ is governed by the Fokker–Planck equation. This is not surprising since the variation of the classical limit of the Schwinger–Keldysh equation gives the Langevin equation (C.4).

In fact, the opposite procedure is possible: one can start from a Langevin/Fokker–Planck equation (or, generally speaking, with any equation describing diffusive dynamics) and construct a classical limit of the corresponding Schwinger–Keldysh diagrammatic technique. This procedure was first introduced by Martin, Siggia and Rose [47].

Consider the Fokker–Planck equation

$$\frac{\partial P}{\partial \tau} = \hat{H}P = D\Delta P + \delta\hat{H}P.$$  \hspace{1cm} (C.7)

The corresponding ‘generating functional’ can be written as a functional $\delta$-function

$$Z = \int \mathcal{D}P \delta \left( \frac{\partial P}{\partial t} - \hat{H}P \right),$$  \hspace{1cm} (C.8)

which, after introducing the auxiliary field $\bar{P}$, acquires the form

$$Z = \int \mathcal{D}P \mathcal{D}\bar{P} \exp \left( i \int dt \, d\vec{n} \, \bar{P} \left( \frac{\partial P}{\partial t} - \hat{H}P \right) \right).$$  \hspace{1cm} (C.9)

There exists a remarkable similarity between (C.9) and the Schwinger–Keldysh generating functional describing the quasi-classical approximation of the quantum non-equilibrium dynamics. For example, in the Martin–Siggia–Rose diagrammatic technique, the field $\bar{P}$ plays the same role as the quantum field $\chi_q$ in the Schwinger–Keldysh framework.

The diagrammatic expansion of the generating functional (C.9) is not of much use, since we are not interested in correlation functions of $P$ or $\bar{P}$: $P$ is itself the probability measure for a random walk/diffusion process governed by the corresponding Fokker–Planck equation. Observables in the problem under consideration (such as the mean-square displacement $\langle \vec{n}^2(\tau) \rangle$) are related to the physics of this random walk. However, the correlation functions of $\vec{n}$ can be determined from the correlation functions in the momentum representation generated by (C.9) by differentiating over momenta. For example, we have

$$\langle \vec{n}^2(\tau) \rangle = -\frac{d}{dq^4} \frac{d}{dq^a} \int d\omega \, e^{-i\omega \tau} G(\omega, q) \bigg|_{q=0},$$  \hspace{1cm} (C.10)

where $G(\omega, q)$ is the Fourier component of the Green function $\langle PP \rangle$. This is due to the correspondence between the Langevin equation describing the dynamics of the observable $\vec{n}$ and the Fokker–Planck equation describing the dynamics of the probability distribution $P$ of the diffusion process.

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