SOME OSTROWSKI TYPE INEQUALITIES VIA RIEMANN-LIOUVILLE FRACTIONAL INTEGRALS FOR $h$--CONVEX FUNCTIONS

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ABSTRACT. In this paper, some Ostrowski type inequalities via Riemann-Liouville fractional integrals for $h$--convex functions, which are super-multiplicative or super-additive, are given. These results not only generalize those of [21,25], but also provide new estimates on these types of Ostrowski inequalities for fractional integrals.

1. Introduction

Let $f : I \to \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an interval, be a mapping differentiable in the interior $I^\circ$ of $I$, and let $a, b \in I^\circ$ with $a < b$. If $|f'(x)| \leq M$ for all $x \in [a, b]$, then

\[
(1.1) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t) \, dt \right| \leq M(b-a) \left[ \frac{1}{4} + \frac{(x-a+b)^2}{(b-a)^2} \right], \quad \forall x \in [a, b].
\]

This is the well-known Ostrowski inequality (see [19] or [18] page 468), which gives an upper bound for the approximation of the integral average $\frac{1}{b-a} \int_a^b f(t) \, dt$ by the value $f(x)$ at point $x \in [a, b]$. In recent years, a number of authors have written about generalizations, extensions and variants of such inequalities (see [1,7,14,15,16,21]).

Let us recall definitions of some kinds of convexity as follows.

Definition A. [11] We say that $f : I \to \mathbb{R}$ is a Godunova-Levin function or that $f$ belongs to the class $Q(I)$ if $f$ is non-negative and for all $x, y \in I$ and $t \in (0,1)$, one has

\[
f(tx + (1-t)y) \leq \frac{f(x)}{t} + \frac{f(y)}{1-t}.
\]

Definition B. [11] We say that $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a $P$--function or that $f$ belongs to the class $P(I)$ if $f$ is non-negative and for all $x, y \in I$ and $t \in [0,1]$, one has

\[
f(tx + (1-t)y) \leq f(x) + f(y).
\]

Definition C. [13] We say that $f : (0,\infty) \to [0,\infty]$ is $s$--convex in the second sense, or that $f$ belongs to the class $K^2_s$, if for all $x, y \in (0,b]$, $t \in [0,1]$ and for some fixed $s \in (0,1]$, one has

\[
f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y).
\]

Definition D. [26] Let $h : J \subseteq \mathbb{R} \to \mathbb{R}$ be a positive function. We say that $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is $h$--convex, or that $f$ belongs to the class $SX(h,I)$, if $f$ is non-negative and for all $x, y \in I$ and $t \in [0,1]$, one has

\[
(1.2) \quad f(tx + (1-t)y) \leq h(t)f(x) + h(1-t)f(y).
\]

If inequality (1.2) is reversed, then $f$ is said to be $h$--concave, i.e. $f \in SV(h,I)$.

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If \( h(t) = t \), then all non-negative convex functions belong to \( SX(h, I) \) and all non-negative concave functions belong to \( SV(h, I) \); if \( h(t) = \frac{1}{t} \), then \( SX(h, I) = Q(I) \); if \( h(t) = 1 \), then \( SX(h, I) \supseteq P(I) \); and if \( h(t) = t^s \) for \( s \in (0, 1] \), then \( SX(h, I) \supseteq K_s^2 \).

**Remark 1.** \( [26] \) Let \( h \) be a non-negative function such that \( h(t) \geq t \) for all \( t \in (0, 1) \). If \( f \) is a non-negative convex function on \( I \), then for \( x, y \in I \), \( t \in (0, 1) \), one has

\[
(1.3) \quad f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y) \leq h(t)f(x) + h(1 - t)f(y).
\]

So, \( f \in SX(h, I) \). Similarly, if the function \( h \) has the property: \( h(t) \leq t \) for all \( t \in (0, 1) \), then any non-negative concave function \( f \) belongs to the class \( SV(h, I) \).

**Definition E.** \( [26] \) We say that \( h : J \rightarrow \mathbb{R} \) is a super-multiplicative function, if for all \( x, y \in J \), one has

\[
h(xy) \geq h(x)h(y).
\]

**Definition F.** \( [2] \) We say that \( h : J \rightarrow \mathbb{R} \) is a super-additive function, if for all \( x, y \in J \), one has

\[
h(x + y) \geq h(x) + h(y).
\]

For recent results concerning \( h \)-convex functions see \([5, 23, 25, 26]\) and references therein. More recently, Tunc \( [25] \) established some new Ostrowski type inequalities for the class of \( h \)-convex functions which are super-multiplicative or super-additive.

We then recall some definitions and mathematical preliminaries of fractional calculus theory which will be used throughout this paper.

**Definition G.** Let \( f \in L_1[a, b] \). The Riemann-Liouville integrals \( J_{a+}^\alpha f \) and \( J_{b-}^\alpha f \) of order \( \alpha > 0 \) with \( a \geq 0 \) are defined by

\[
J_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a
\]

and

\[
J_{b-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b,
\]

respectively, where \( \Gamma(\alpha) = \int_0^\infty e^{-u}u^{\alpha-1}du \). Here, \( J_{a+}^\alpha f(x) = J_{b-}^\alpha f(x) = f(x) \).

In the case of \( \alpha = 1 \), the fractional integral reduces to the classical integral. For some recent results connected with fractional integral inequalities we refer the reader to the papers \([3, 4, 6, 10, 12, 17, 20, 22]\) and the reference cited therein. In \([21]\), Set established some new Ostrowski type inequalities for \( s \)-convex functions in the second sense via Riemann-Liouville fractional integral.

Motivated by these results, in the present paper, we establish some Ostrowski type inequalities via Riemann-Liouville fractional integrals for \( h \)-convex functions, which are super-multiplicative or super-additive. So, new estimates on these types of Ostrowski inequalities via fractional integrals are provided and the results of \([21, 25]\) are generalized.

2. Ostrowski type fractional integral inequalities for \( h \)-convex functions

To prove our main theorems, we need the following identity established by Set in \([21]\):

**Lemma 1.** Let \( f : [a, b] \rightarrow \mathbb{R} \) be a differentiable mapping on \( (a, b) \) with \( a < b \). If \( f' \in L_1[a, b] \), then for all \( x \in [a, b] \) and \( \alpha > 0 \), one has

\[
\left( \frac{(x-a)^\alpha + (b-x)^\alpha}{b-a} \right) f(x) - \frac{\Gamma(\alpha + 1)}{(b-a)} \left[ J_{x-}^\alpha f(a) + J_{x+}^\alpha f(b) \right] = \frac{(x-a)^{\alpha+1}}{b-a} \int_0^1 t^\alpha f'(tx + (1-t)a) dt - \frac{(b-x)^{\alpha+1}}{b-a} \int_0^1 t^\alpha f'(tx + (1-t)b) dt.
\]
Using this lemma, we can obtain the following fractional integral inequalities for \( h \)-convex functions.

**Theorem 1.** Let \( h : J \subseteq \mathbb{R} \to \mathbb{R} \) ([0, 1] \subseteq J) be a non-negative and super-multiplicative function, \( h(t) \geq t \) for \( 0 \leq t \leq 1 \), \( f : [a, b] \subseteq \mathbb{R} \to \mathbb{R} \) be a differentiable mapping on \((a, b)\) with \( a < b \) such that \( f' \in L_{1}[a, b] \). If \( |f'| \) is \( h \)-convex on \([a, b] \) and \( |f'(x)| \leq M \), \( x \in [a, b] \), then the following inequalities for fractional integrals with \( \alpha > 0 \) hold:

\[
\left| \frac{(x-a)^\alpha (b-x)^\alpha}{b-a} f(x) - \frac{\Gamma(\alpha+1)}{(b-a)} \left[ J_{x}^{\alpha} f(a) + J_{x}^{\alpha} f(b) \right] \right| 
\leq \frac{M (x-a)^{\alpha+1} + (b-x)^{\alpha+1}}{b-a} \int_{0}^{1} \left| t^{\alpha} h(t) + t^{\alpha} h(1-t) \right| dt
\]

(2.2)

\[
\leq \frac{M (x-a)^{\alpha+1} + (b-x)^{\alpha+1}}{b-a} \int_{0}^{1} \left[ t^{\alpha} h(t) + t^{\alpha} h(1-t) \right] dt
\]

(2.3)

**Proof.** From (2.1) and since \( |f'| \) is \( h \)-convex, we have

\[
\left| \frac{(x-a)^\alpha (b-x)^\alpha}{b-a} f(x) - \frac{\Gamma(\alpha+1)}{(b-a)} \left[ J_{x}^{\alpha} f(a) + J_{x}^{\alpha} f(b) \right] \right| 
\leq \frac{(x-a)^{\alpha+1} + (b-x)^{\alpha+1}}{b-a} \int_{0}^{1} t^{\alpha} \left| f'(tx + (1-t)a) \right| dt + \frac{(b-x)^{\alpha+1}}{b-a} \int_{0}^{1} t^{\alpha} \left| f'(tx + (1-t)b) \right| dt
\]

\[
\leq \frac{(x-a)^{\alpha+1} + (b-x)^{\alpha+1}}{b-a} \int_{0}^{1} t^{\alpha} h(t) \left| f'(x) \right| + t^{\alpha} h(1-t) \left| f'(a) \right| dt
\]

\[
+ \frac{(b-x)^{\alpha+1}}{b-a} \int_{0}^{1} t^{\alpha} h(t) \left| f'(x) \right| + t^{\alpha} h(1-t) \left| f'(b) \right| dt
\]

\[
\leq \frac{M (x-a)^{\alpha+1} + (b-x)^{\alpha+1}}{b-a} \int_{0}^{1} t^{\alpha} h(t) + t^{\alpha} h(1-t) dt + \frac{M (b-x)^{\alpha+1}}{b-a} \int_{0}^{1} t^{\alpha} h(t) + t^{\alpha} h(1-t) dt,
\]

which completes the proof of (2.2).

By using the additional properties of \( h \) in the assumptions, we further have

\[
\int_{0}^{1} t^{\alpha} h(t) + t^{\alpha} h(1-t) dt \leq \int_{0}^{1} \left[ h(t^\alpha) h(t) + h(t^\alpha) h(1-t) \right] dt
\]

(2.4)

\[
\leq \int_{0}^{1} \left[ h(t^{\alpha+1}) + h(t^{\alpha}(1-t)) \right] dt.
\]

Hence, the proof of (2.3) is complete.

**Remark 2.** We note that in the proof of (2.2) we do not use the additional super-multiplicative property of \( h \) and the condition \( "h(t) \geq t" \) for \( 0 \leq t \leq 1 \). In Theorem 7 if we choose \( \alpha = 1 \), then (2.3) reduces the inequality \( [25, (2.1)] \), i.e.,

\[
\left| f(x) - \frac{1}{b-a} \int_{a}^{b} f(u) du \right| \leq \frac{M (x-a)^{2} + (b-x)^{2}}{b-a} \int_{0}^{1} \left[ h(t^2) + h(t-t^2) \right] dt,
\]

which can be better than the inequality \( [1.1] \) provide that \( h \) is chosen such that

\[
\int_{0}^{1} \left[ h(t^2) + h(t-t^2) \right] dt < \frac{1}{2}.
\]

In Theorem 7 if we choose \( h(t) = t \), then (2.2) and (2.3) reduce the inequality in \( [24, \text{ Corollary 1}] \).
In the next corollary, we will also make use of the Beta function of Euler type, which is defined as

\[ \beta(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} \, dt = \frac{\Gamma(x) \Gamma(y)}{\Gamma(x+y)}, \quad \forall \, x, y > 0. \]

**Corollary 1.** If we choose \( h(t) = t^s \), \( s \in (0, 1] \), in Theorem [7], then we have

\[
\left| \left( \frac{(x-a)^\alpha + (b-x)^\alpha}{b-a} \right) f(x) - \frac{\Gamma(\alpha+1)}{(b-a)} [J_{x-} \! f(a) + J_{x+} \! f(b)] \right| \\
\leq \frac{M}{b-a} \left[ 1 + \frac{\Gamma(\alpha+1) \Gamma(s+1)}{\Gamma(\alpha+s+1)} \right] \frac{(x-a)^{\alpha+1} + (b-x)^{\alpha+1}}{\alpha + s + 1} \\
\leq \frac{M}{b-a} \left[ 1 + \frac{\Gamma(\alpha+1) \Gamma(s+1)}{\Gamma(\alpha+s+1)} \right] \frac{(x-a)^{\alpha+1} + (b-x)^{\alpha+1}}{\alpha + s + 1},
\]

due to the fact that

\[
\int_0^1 \left[ h \left( t^{\alpha+1} \right) + h \left( t^\alpha (1-t) \right) \right] dt = \int_0^1 t^{\alpha(\alpha+1)} dt + \int_0^1 t^\alpha (1-t)^\alpha dt \\
= \frac{1}{\alpha s + s + 1} + \frac{\Gamma(\alpha+1) \Gamma(s+1)}{\Gamma(\alpha+s+2)} = \frac{1}{\alpha s + s + 1} \left[ 1 + \frac{\Gamma(\alpha+1) \Gamma(s+1)}{\Gamma(\alpha+s+1)} \right].
\]

The first inequality is the same as the one established in [24, Theorem 7].

**Theorem 2.** Let \( h : J \subseteq \mathbb{R} \rightarrow \mathbb{R} \) \((0, 1] \subseteq J\) be a non-negative and super-additive function, and \( f : (a, b) \subseteq \mathbb{R} \) be a differentiable mapping on \((a, b)\) with \( a < b \) such that \( f' \in L_1(a, b) \). If \(|f'|^q\) is \( h\)-convex on \([a, b]\), \( p, q > 1, \frac{1}{p} + \frac{1}{q} = 1 \), and \(|f'(x)| \leq M, \, x \in [a, b] \), then the following inequality for fractional integrals with \( \alpha > 0 \) holds:

\[
\left| \left( \frac{(x-a)^\alpha + (b-x)^\alpha}{b-a} \right) f(x) - \frac{\Gamma(\alpha+1)}{(b-a)} [J_{x-} \! f(a) + J_{x+} \! f(b)] \right| \\
\leq \frac{M}{(1 + p\alpha)^{\frac{1}{p}} (b-a)} \left( \int_0^1 [h(t) + h(1-t)] \, dt \right)^{\frac{1}{q}} \\
\leq \frac{M}{(1 + p\alpha)^{\frac{1}{p}} (b-a)} h^{\frac{1}{q}}(1).
\]

**Proof.** From Lemma [11] and using the well-known Hölder’s inequality, we have

\[
\left| \left( \frac{(x-a)^\alpha + (b-x)^\alpha}{b-a} \right) f(x) - \frac{\Gamma(\alpha+1)}{(b-a)} [J_{x-} \! f(a) + J_{x+} \! f(b)] \right| \\
\leq \frac{(x-a)^{\alpha+1}}{b-a} \left( \int_0^1 t^\alpha |f'(tx + (1-t)a)| \, dt \right)^{\frac{1}{p}} + \frac{(b-x)^{\alpha+1}}{b-a} \left( \int_0^1 t^\alpha |f'(tx + (1-t)b)| \, dt \right)^{\frac{1}{q}} \\
\leq \frac{(x-a)^{\alpha+1}}{b-a} \left( \int_0^1 t^\alpha \, dt \right)^{\frac{1}{p}} \left( \int_0^1 |f'(tx + (1-t)a)|^q \, dt \right)^{\frac{1}{q}} \\
+ \frac{(b-x)^{\alpha+1}}{b-a} \left( \int_0^1 t^\alpha \, dt \right)^{\frac{1}{q}} \left( \int_0^1 |f'(tx + (1-t)b)|^q \, dt \right)^{\frac{1}{q}}.
\]

Since \(|f'|^q\) is \( h\)-convex and \(|f'(x)| \leq M\), we get

\[
\int_0^1 |f'(tx + (1-t)a)|^q \, dt \leq \int_0^1 [h(t) |f'(x)|^q + h(1-t) |f'(a)|^q] \, dt
\]
By simple computation, we have

\[ \text{Corollary 2.} \] in Theorem 2, if we choose \( \alpha \) and similarly

\[ \int_0^1 |f'(tx + (1-t)b)|^q dt \leq M^q \int_0^1 [h(t) + h(1-t)] dt. \]

Using these results, we complete the proof of \((2.5)\).

By using the super-additive property of \( h \) in the assumptions, we further have

\[ \int_0^1 [h(t) + h(1-t)] dt \leq \int_0^1 h(1) dt = h(1). \]

Hence, the proof of \((2.6)\) is complete. \( \square \)

**Remark 3.** We note that in the proof of \((2.5)\) we do not use the additional super-additive property of \( h \). In Theorem 2, if we choose \( h(t) = t \), then \((2.5)\) reduces the inequality in \([24, \text{Corollary 2}]\); in Theorem 3, if we choose \( \alpha = 1 \), then \((2.5)\) becomes

\[ (2.7) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{M \left[ (x-a)^2 + (b-x)^2 \right]}{(1+p)^\frac{1}{2}} \left( \int_0^1 [h(t) + h(1-t)] dt \right)^\frac{1}{2}, \]

which can be better than the inequality \((1.1)\) provide that \( p, q \) and \( h \) are chosen such that

\[ \left( \int_0^1 [h(t) + h(1-t)] dt \right)^\frac{1}{2} < \frac{1}{2} (1+p)^\frac{1}{2}. \]

**Corollary 2.** If we choose \( h(t) = t^s, s \in (0,1] \), in Theorem 2 then we have

\[ \left| \frac{(x-a)^\alpha + (b-x)^\alpha}{b-a} f(x) - \frac{\Gamma(\alpha+1)}{(b-a)} \left[ J_{x}^\alpha f(a) + J_{x}^\alpha f(b) \right] \right| \leq \frac{M}{(1+pa)^\frac{1}{s}} \left( \frac{2}{s+1} \right)^\frac{1}{s} \frac{(x-a)^{\alpha+1} + (b-x)^{\alpha+1}}{b-a}, \]

due to the fact that

\[ \int_0^1 [h(t) + h(1-t)] dt = \frac{2}{s+1}. \]

This is the inequality established in \([24, \text{Theorem 8}]\).

**Theorem 3.** Let \( h : J \subseteq \mathbb{R} \to \mathbb{R} \) be a non-negative and super-multiplicative function, \( h(t) \geq t \) for \( 0 \leq t \leq 1 \), \( f : [a,b] \subseteq [0,\infty) \to \mathbb{R} \) be a differentiable mapping on \((a,b)\) with \( a < b \) such that \( f' \in L_1 [a,b] \). If \(|f'|^q\) is \( h\)-convex on \([a,b]\), \( q \geq 1 \) and \(|f'(x)| \leq M, x \in [a,b]\), then the following inequalities for fractional integrals with \( \alpha > 0 \) hold:

\[ \left| \frac{(x-a)^\alpha + (b-x)^\alpha}{b-a} f(x) - \frac{\Gamma(\alpha+1)}{(b-a)} \left[ J_{x}^\alpha f(a) + J_{x}^\alpha f(b) \right] \right| \leq \frac{M}{(1+\alpha)^{1-\frac{1}{q}}} \frac{(x-a)^{\alpha+1} + (b-x)^{\alpha+1}}{b-a} \left( \int_0^1 \left[ t^\alpha h(t) + t^\alpha h(1-t) \right] dt \right)^\frac{1}{q}, \]

\[ (2.8) \]
Corollary 3. We choose the multiplicative property of □. The proof of (2.9) is complete.

We note that in the proof of Remark 4. Using these inequalities, we complete the proof of (2.8).

The first inequality is the same as the one established in [21, Theorem 9].

\[ (2.9) \quad \leq \frac{M}{(1 + \alpha)^{1 - \frac{1}{\alpha}} \frac{(x - a)^{\alpha + 1} + (b - x)^{\alpha + 1}}{b - a}} \left( \int_0^1 \left[ h(t^{\alpha + 1}) + h(t^{\alpha}(1 - t)) \right] dt \right)^{\frac{1}{\alpha}}. \]

**Proof.** From Lemma 1 and using the well-known power mean inequality, we have

\[ \left| \frac{(x - a)^{\alpha} + (b - x)^{\alpha}}{b - a} \right| f(x) - \frac{\Gamma(\alpha + 1)}{(b - a)} \left[ J_{x-}^\alpha f(a) + J_x^\alpha f(b) \right] \leq \frac{(x - a)^{\alpha + 1}}{b - a} \int_0^1 t^\alpha |f'(tx + (1 - t)a)| dt + \frac{(b - x)^{\alpha + 1}}{b - a} \int_0^1 t^\alpha |f'(tx + (1 - t)b)| dt \]

\[ \leq \frac{(x - a)^{\alpha + 1}}{b - a} \left( \int_0^1 t^\alpha dt \right)^{1 - \frac{1}{\alpha}} \left( \int_0^1 t^\alpha |f'(tx + (1 - t)a)|^q dt \right)^{\frac{1}{q}} + \frac{(b - x)^{\alpha + 1}}{b - a} \left( \int_0^1 t^\alpha dt \right)^{1 - \frac{1}{\alpha}} \left( \int_0^1 t^\alpha |f'(tx + (1 - t)b)|^q dt \right)^{\frac{1}{q}}. \]

Since \(|f'|^q| is \ h-convex on \ [a, b] and \ |f'(x)| \leq M, we get

\[ \int_0^1 t^\alpha |f'(tx + (1 - t)a)|^q dt \leq \int_0^1 [t^\alpha h(t) |f'(x)|^q + t^\alpha h(1 - t) |f'(a)|^q] dt \]

\[ \leq M^q \int_0^1 [t^\alpha h(t) + t^\alpha h(1 - t)] dt \]

and similarly

\[ \int_0^1 t^\alpha |f'(tx + (1 - t)b)|^q dt \leq M^q \int_0^1 [t^\alpha h(t) + t^\alpha h(1 - t)] dt. \]

Using these inequalities, we complete the proof of (2.8).

By using the additional properties of \( h \) in the assumptions, we further have (2.4). Hence, the proof of (2.9) is complete. \( \square \)

**Remark 4.** We note that in the proof of (2.8) we do not use the additional super-multiplicative property of \( h \) and the condition "\( h(t) \geq t \) for \( 0 \leq t \leq 1 \). In Theorem 3 if we choose \( \alpha = 1 \), then (2.9) reduces the inequality \[25\] (2.4); in Theorem 4 if we choose \( h(t) = t \), then (2.8) and (2.9) reduce the inequality in \[24\] Corollary 3.

**Corollary 3.** If we choose \( h(t) = t^s \), \( s \in (0, 1) \), in Theorem 3, then we have

\[ \left| \frac{(x - a)^{\alpha} + (b - x)^{\alpha}}{b - a} \right| f(x) - \frac{\Gamma(\alpha + 1)}{(b - a)} \left[ J_{x-}^\alpha f(a) + J_x^\alpha f(b) \right] \leq \frac{M}{(1 + \alpha)^{1 - \frac{1}{\alpha}} \frac{(x - a)^{\alpha + 1} + (b - x)^{\alpha + 1}}{b - a}} \left( \int_0^1 \left[ h(t^{\alpha + 1}) + h(t^{\alpha}(1 - t)) \right] dt \right)^{\frac{1}{\alpha}} \]

\[ \leq \frac{M}{(1 + \alpha)^{1 - \frac{1}{\alpha}} \frac{(x - a)^{\alpha + 1} + (b - x)^{\alpha + 1}}{b - a}} \left( \int_0^1 \left[ \frac{\Gamma(\alpha + 1) \Gamma(s + 1)}{\Gamma(\alpha s + s + 1)} \right]^{\frac{1}{\alpha}} (x - a)^{\alpha + 1} + (b - x)^{\alpha + 1} \right)^{\frac{1}{\alpha}} \frac{1}{\alpha s + 1}. \]

The first inequality is the same as the one established in [21, Theorem 9].

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