Vector bundles and connections on Riemann surfaces with projective structure

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Abstract
Let \( B_g(r) \) be the moduli space of triples of the form \((X, K_X^{1/2}, F)\), where \( X \) is a compact connected Riemann surface of genus \( g \), with \( g \geq 2 \), \( K_X^{1/2} \) is a theta characteristic on \( X \), and \( F \) is a stable vector bundle on \( X \) of rank \( r \) and degree zero. We construct a \( T^*B_g(r) \)-torsor \( \mathcal{H}_g(r) \) over \( B_g(r) \). This generalizes on the one hand the torsor over the moduli space of stable vector bundles of rank \( r \), on a fixed Riemann surface \( Y \), given by the moduli space of algebraic connections on the stable vector bundles of rank \( r \) on \( Y \), and on the other hand the torsor over the moduli space of Riemann surfaces given by the moduli space of Riemann surfaces with a projective structure. It is shown that \( \mathcal{H}_g(r) \) has a holomorphic symplectic structure compatible with the \( T^*B_g(r) \)-torsor structure. We also describe \( \mathcal{H}_g(r) \) in terms of the second order matrix valued differential operators. It is shown that \( \mathcal{H}_g(r) \) is identified with the \( T^*B_g(r) \)-torsor given by the sheaf of holomorphic connections on the theta line bundle over \( B_g(r) \).

Keywords  Projective structure · Differential operator · Algebraic connection · Oper · Torsor

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1 Introduction

Let $X$ be a compact connected Riemann surface of genus $g$, with $g \geq 2$, and $\mathcal{N}_X(r)$ the moduli space of stable vector bundles on $X$ of rank $r$ and degree zero. Let $\mathcal{C}_X(r)$ be the moduli space of rank $r$ holomorphic connections $(E, D)$ on $X$ such that the underlying holomorphic vector bundle $E$ is stable. It has a forgetful map to $\mathcal{N}_X(r)$ that sends any $(E, D)$ to $E$, and $\mathcal{C}_X(r)$ is an algebraic torsor over $\mathcal{N}_X(r)$ for the holomorphic cotangent bundle $T^*\mathcal{N}_X(r)$. Moreover, $\mathcal{C}_X(r)$ has an algebraic symplectic structure which is compatible with the $T^*\mathcal{N}_X(r)$-torsor structure.

There is another $T^*\mathcal{N}_X(r)$-torsor that one can construct. To each bundle $E \in \mathcal{N}_X(r)$ we associate the Quillen determinant line for its $\bar{\partial}$ operator, and this defines a holomorphic line bundle $L$ over $\mathcal{N}_X(r)$. Then consider the sheaf of pointwise connections $\text{Conn}(L)$ over $\mathcal{N}_X(r)$ (i.e., the sheaf on $\mathcal{N}_X(r)$ whose sections are holomorphic connections on $L$). This is again a $T^*\mathcal{N}_X(r)$-torsor over $\mathcal{N}_X(r)$.

The surprise, established in [9, 10] is that there is a canonical isomorphism between these two torsors. The isomorphism is constructed by taking $C^\infty$ sections on both sides that at the first sight do not seem to have much to do with each other, but have the same data encoding the torsor. For $\mathcal{C}_X(r)$, we associate to each bundle its unitary connection given by the Narasimhan–Seshadri theorem; for $\text{Conn}(L)$ the Chern connection for the Quillen metric on $L$. There is a holomorphic version of this equivalence by sections, expanded in [10], in which the line bundle $L$ is restricted to the complement $\mathcal{U}$ of the theta divisor in $\mathcal{N}_X(r)$ associated to a theta characteristic $K_X^{1/2}$ of $X$, so the canonical trivialization of $L$ over $\mathcal{U}$ produces an integrable algebraic connection on $L|_{\mathcal{U}}$. On the other hand, a holomorphic connection on any bundle $E \in \mathcal{U}$ is obtained as the asymptotic data of a canonical meromorphic section of $(E \otimes K_X^{1/2}) \boxtimes (E^* \otimes K_X^{1/2})$ over $X \times X$, with singularities on the diagonal. There is a holomorphic isomorphism between $\text{Conn}(L)$ and $\mathcal{C}_X(r)$ that maps the section of $\text{Conn}(L)|_{\mathcal{U}}$ given be the trivialization of $L|_{\mathcal{U}}$ to the section of $\mathcal{C}_X(r)|_{\mathcal{U}}$ given by the above canonical section. In fact, not only is there an isomorphism, but it is obtained by linking together rather special and hitherto unrelated objects on each side of the equivalence, in several different ways.
The holomorphic version allows us, of course to move not only the bundle, but the base curve as well, and the question we are examining in this paper is whether we can again have two a priori inequivalent holomorphic $T^*B_g(r)$ torsors over the space $B_g(r)$ of pairs (bundles, curves) actually turn out to be the same. The answer turns out to be yes. As a bonus, we have a hereto undefined symplectic structure for one of the torsors.

To explain with more details, let $P_g$ denote the moduli space of Riemann surfaces of genus $g$ equipped with a projective structure. It has a natural map to the moduli space $M_g$ of Riemann surfaces of genus $g$ that simply forgets the projective structure, and $P_g$ is a holomorphic torsor over $M_g$ for the holomorphic cotangent bundle $T^*M_g$. Also, $P_g$ has a holomorphic symplectic structure which is compatible with the $T^*M_g$-torsor structure.

To combine the torsors $P_g$ and $C_X(r)$ into a single one, let $B_g(r)$ be the moduli space of triples of the form $(X, K_X^{1/2}, F)$, where

- $X$ is a compact connected Riemann surface of genus $g$, with $g \geq 2$,
- $K_X^{1/2}$ is a theta characteristic on $X$ (the holomorphic cotangent bundle of $X$ is denoted by $K_X$), and
- $F$ is a stable vector bundle over $X$ of rank $r$ and degree zero.

Fix a theta characteristic $K_X^{1/2}$ on a compact Riemann surface $X$ of genus $g$. Given a stable vector bundle $F$ on $X$ of rank $r$ and degree zero, we construct a certain quotient space of the space of all holomorphic connections on the first order jet bundle $J^1(F \otimes (K_X^{1/2})^*)$; this quotient space is denoted by $D(F)$ (see (2.34) and Corollary 3.2). Let $H_g(r)$ denote the moduli space of objects of the form $(X, K_X^{1/2}, F, D)$, where $(X, K_X^{1/2}, F) \in B_g(r)$ and $D \in D(F)$. It has a natural map

$$\gamma : H_g(r) \longrightarrow B_g(r), \quad (X, K_X^{1/2}, F, D) \longmapsto (X, K_X^{1/2}, F).$$

We prove the following (see Theorem 5.4 and Corollary 8.2):

**Theorem 1.1**

1. $H_g(r)$ is a torsor over $B_g(r)$ for the holomorphic cotangent bundle $T^*B_g(r)$.
2. $H_g(r)$ has an algebraic symplectic structure $\Omega_{H_g(r)}$.
3. The symplectic form $\Omega_{H_g(r)}$ on $H_g(r)$ is compatible with the $T^*B_g(r)$-torsor structure of $H_g(r)$.
4. There is a holomorphic line bundle $L$ on $H_g(r)$, and a holomorphic connection $\nabla^L$ on $L$, such that the curvature of $\nabla^L$ is the symplectic form $\Omega_{H_g(r)}$.

Consider the standard subbundle

$$F \otimes K_X^{1/2} = F \otimes (K_X^{1/2})^* \otimes K_X \subset J^1(F \otimes (K_X^{1/2})^*).$$

For any holomorphic connection $\mathbb{D}$ on $J^1(F \otimes (K_X^{1/2})^*)$, the second fundamental form of $F \otimes K_X^{1/2}$ for $\mathbb{D}$ is the identity map of $F$ (see Corollary 3.2). This property is very similar to the defining property of opers. We recall that opers were introduced by Beilinson and Drinfeld [4, 5]. Their motivation came from the works of Drinfeld and Sokolov [15, 16]. We note that the study of opers within geometry and mathematical physics has received much attention in the recent times.

There is a natural divisor $\Theta \subset B_g(r)$ consisting of all $(X, K_X^{1/2}, F)$ such that $H^0(X, F \otimes K_X^{1/2}) \neq 0$. The line bundle on $B_g(r)$ defined by $\Theta$ will be denoted by $\mathcal{L}$. Let

$$\mathcal{C}(\mathcal{L}) \longrightarrow B_g(r).$$
be the holomorphic fiber bundle defined by the sheaf of holomorphic connection on \( L \). So the space of holomorphic sections of \( \mathcal{C}(L) \) over an open subset \( U \subset B_g(r) \) is the space of all holomorphic connections on \( L|_U \). This \( \mathcal{C}(L) \) is an algebraic torsor over \( B_g(r) \) for the holomorphic cotangent bundle \( T^*B_g(r) \).

We prove the following (see Theorem 8.1):

**Theorem 1.2** There is a canonical algebraic isomorphism of \( T^*B_g(r) \)-torsors

\[ \mathcal{H} : \mathcal{H}_g(r) \longrightarrow \mathcal{C}(L) . \]

Projective structures on a Riemann surface \( X \) are defined by giving a holomorphic coordinate atlas on \( X \) such that all the transition functions are Möbius transformations. Projective structures on a Riemann surface \( X \) are identified with holomorphic ordinary differential operators \( D \) on \( X \) of order two such that

- the symbol of \( D \) is the constant function 1 on \( X \), and
- the sub-leading term of \( D \) vanishes identically (equivalently, the GL(2, \( \mathbb{C} \))–local system of \( X \) defined by the sheaf of solutions of \( D \) is actually a SL(2, \( \mathbb{C} \))–local system).

The above mentioned space \( \mathcal{H}_g(r) \) admits a similar description in terms of the second order matrix valued differential operators. To explain this, for any \( (X, K_{1/2}^{1/2}, F) \in B_g(r) \), let

\[ \tilde{\mathcal{D}}'(X, K_{1/2}^{1/2}, F) \subset H^0(X, \text{Diff}^2(X, K_{1/2}^{1/2}, F \otimes K^{3/2})) \]

be the locus of holomorphic differential operators whose symbol is \( \text{Id}_F \in H^0(X, \text{End}(F)) \). The vector space \( H^0(X, \text{End}(F) \otimes K^{2/2}) \) acts freely on \( \tilde{\mathcal{D}}'(X, K_{1/2}^{1/2}, F) \) (but this action is not transitive).

We prove the following (see Theorem 6.4):

**Theorem 1.3** There is a canonical bijection between \( \mathcal{D}(F) \) and the quotient space

\[ \tilde{\mathcal{D}}'(X, K_{1/2}^{1/2}, F)/H^0(X, \text{ad}(F) \otimes K^{2/2}) . \]

There is a natural holomorphic projection \( \mathcal{H}_g(r) \longrightarrow \mathcal{P}_g \) (see Proposition 9.1), using which we may pullback, to \( \mathcal{H}_g(r) \), the symplectic 2-form on \( \mathcal{P}_g \). On the other hand, using isomonodromic deformations, the symplectic form on \( \mathcal{C}_X(r) \) produces a holomorphic 2-form on \( \mathcal{H}_g(r) \). It is natural to ask whether the symplectic form on \( \mathcal{H}_g(r) \) in Theorem 1.1(2) is a combination of these two 2-forms (see Conjecture 9.3 for a precise formulation).

The result of this paper, as well as those of [9, 10], have a flavor of geometric quantization: a symplectic manifold projects to a manifold of half the dimension, equipped with a line bundle with a connection. The fibers of the projection are Lagrangians, and the curvature of the connection is the symplectic form. We are unaware, however, of any concrete link to geometric quantization.

### 2 Stable r-opers and their properties

#### 2.1 Stable r-opers

Let \( X \) be a compact connected Riemann surface. Equivalently, \( E \) is an irreducible smooth projective curve defined over \( \mathbb{C} \). Any vector bundle on \( X \) will be algebraic (equivalently, holomorphic, by GAGA [23]) unless explicitly specified otherwise.
The holomorphic cotangent (respectively, tangent) bundle will be denoted by $K_X$ (respectively, $TX$). We shall assume throughout that

$$\text{genus}(X) = g \geq 2.$$  

An algebraic connection on a vector bundle $E$ over $X$ is a first order algebraic differential operator

$$D : E \to E \otimes K_X$$

such that the symbol of $D$—which is an algebraic section of $\text{Hom}(E, E \otimes K_X) \otimes TX = \text{End}(E)$—is $\text{Id}_E$, equivalently, $D$ satisfies the Leibniz identity (see [1]). An algebraic connection on $E$ is automatically flat because $\Omega^1 \otimes TX = 0$.

If $D$ is an algebraic connection on $E$, and $S \subset E$ is an algebraic subbundle of $E$, then consider the composition of homomorphisms

$$S \hookrightarrow E \xrightarrow{D} E \otimes K_X \xrightarrow{q \otimes \text{Id}_{K_X}} (E/S) \otimes K_X,$$

where

$$q : E \to E/S$$

is the quotient map. This composition of homomorphisms is $O_X$–linear, and hence it corresponds to an algebraic section

$$DS \in H^0(X, \text{Hom}(S, E/S) \otimes K_X).$$

This homomorphism $DS$ is called the second fundamental form of $S$ for the connection $D$.

A vector bundle $V$ on $X$ is called stable if

$$\frac{\text{degree}(W)}{\text{rank}(W)} < \frac{\text{degree}(V)}{\text{rank}(V)}$$

for every algebraic subbundle $0 \neq W \subset V$.

**Definition 2.1** For a positive integer $r$, a $r$–oper on $X$ is a triple $(E, S, D)$, where

- $E$ is a vector bundle on $X$ of rank $2r$,
- $D$ is an algebraic connection on $E$, and
- $S \subset E$ is an algebraic subbundle of rank $r$,

such that the second fundamental form $DS$ defined in (2.2) is an isomorphism.

A $r$–oper $(E, S, D)$ is called stable if the vector bundle $S$ of rank $r$ is stable.

Using the isomorphism $DS : S \to (E/S) \otimes K_X$ in Definition 2.1, we may identify $E/S$ with $S \otimes TX$. Invoking this isomorphism the second fundamental form $DS$ is made into the identity map of $S$. Henceforth, for any $r$–oper we shall always execute the option of using this isomorphism.

Fix a line bundle $K^{1/2}_X$ on $X$ of degree $g - 1$ such that $K^{1/2}_X \otimes K^{1/2}_X = K_X$; also fix an algebraic isomorphism of $K^{1/2}_X \otimes K^{1/2}_X$ with $K_X$. Such a line bundle is called a theta characteristic on $X$. The dual line bundle $(K^{1/2}_X)^*$ will be denoted by $K_X^{-1/2}$; also $(K^{1/2}_X)^{\otimes n}$ (respectively, $(K^{1/2}_X)^{-\otimes n}$) will be denoted by $K^n_X$ (respectively, $K^{-n/2}_X$) for every $n \geq 1$.

Take a $r$–oper $(E, S, D)$ on $X$. Define the vector bundle $F := S \otimes K_X^{-1/2}$, so we have

$$S = F \otimes K^{1/2}_X.$$  

(2.3)
Then from the isomorphism $D_S$ in Definition 2.1 we have
\[ E/S = S \otimes (K_X)^* = F \otimes K_X^{1/2} \otimes (K_X)^* = F \otimes K_X^{-1/2}. \] (2.4)
Since $E$ admits an algebraic connection, we have
\[ \text{degree}(E) = 0 \] (2.5)
[1, p. 202, Proposition 18(i)]. Using (2.5), (2.4) and (2.3) it follows that
\[ 0 = \text{degree}(E) = \text{degree}(S) + \text{degree}(E/S) \]
\[ = \text{degree}(F \otimes K_X^{1/2}) + \text{degree}(F \otimes K_X^{-1/2}) = 2 \cdot \text{degree}(F). \]
Hence we have degree($F$) = 0.

For any vector bundle $W$ on $X$, the first jet bundle $J^1(W)$ fits into the following short exact sequence of vector bundles on $X$:
\[ 0 \rightarrow W \otimes K_X \rightarrow J^1(W) \rightarrow J^0(W) = W \rightarrow 0. \] (2.6)

**Lemma 2.2** Let $(E, S, D)$ be a $r$–oper on $X$. Then the first jet bundle $J^1(E/S)$ is canonically identified with the vector bundle $E$. Also, $E$ is identified with $J^1(S \otimes TX)$.

**Proof** Using the flat connection $D$ we shall construct a homomorphism
\[ \varphi : E \rightarrow J^1(E/S). \] (2.7)
For this, take any point $x \in X$ and any element $v \in E_x$ in the fiber over $x$. Let $\tilde{v}$ be the unique flat section of $E$ (for the connection $D$), defined on a simply connected analytic open neighborhood $U \subset X$ of $x$, such that $\tilde{v}(x) = v$. Now restrict the section $q(\tilde{v}) \in H^0(U, E/S)$, where $q$ is the quotient map in (2.1), to the first order infinitesimal neighborhood of $x$; let $\tilde{v}' \in J^1(E/S)_x$ be the element obtained this way from $q(\tilde{v})$. The map $\varphi$ in (2.7) sends any $v \in E_x, x \in X$, to $\tilde{v}' \in J^1(E/S)_x$ constructed above from it. The homomorphism $\varphi$ is evidently holomorphic. Hence $\varphi$ is algebraic by GAGA.

We describe an alternative construction of the map $\varphi$ in (2.7). The sheaf $E$ has a locally constant subsheaf $\mathbb{E} \subset E$ defined by the connection $D$. Consider the composition of homomorphism of sheaves
\[ \mathbb{E} \hookrightarrow E \rightarrow E/S \rightarrow J^1(E/S); \]
note that the above homomorphism of sheaves $E/S \rightarrow J^1(E/S)$, which sends a section of $E/S$ to the section of $J^1(E/S)$ corresponding to it, is not $O_X$–linear. Tensoring the above homomorphism $\mathbb{E} \rightarrow J^1(E/S)$ with $O_X$, and using the natural $O_X$–module structure of $J^1(E/S)$ together with the fact that $\mathbb{E} \otimes_O O_X = E$, we get $\varphi$ in (2.7).

The map $\varphi$ fits in the following commutative diagram of homomorphisms
\[ \begin{array}{ccc}
0 & \rightarrow & S \\
\downarrow \varphi' & & \downarrow \varphi \\
0 & \rightarrow & (E/S) \otimes K_X \\
\end{array} \rightarrow J^1(E/S) \rightarrow E/S \rightarrow 0 \] (2.8)
where $\varphi'$ is the restriction of $\varphi$ to the subbundle $S$, and the exact sequence at the bottom of (2.8) is the one in (2.6) for $W = E/S$. It is straightforward to check that the homomorphism $\varphi'$ in (2.8) actually coincides with the second fundamental form $D_S$ defined in (2.2). Since $D_S$ is an isomorphism, from the commutativity of (2.8) it follows immediately that $\varphi$ constructed in (2.7) is an isomorphism.
Using the isomorphism $E/S \sim S \otimes TX$ in (2.4), the isomorphism $\varphi$ identifies $E$ with $J^1(S \otimes TX)$.

Lemma 2.3 Let $(E, S, D)$ be a $r$–oper on $X$. Let $D^1 := (\varphi^{-1})^* D$ be the connection on $J^1(E/S)$, where $\varphi$ is the isomorphism in (2.8). Then the second fundamental form of $(E/S) \otimes KX \hookrightarrow J^1(E/S)$ for the connection $D^1$ is the identity map of $(E/S) \otimes KX$.

**Proof** This follows immediately from the construction of $\varphi$, the commutativity of the diagram in (2.8) and the fact that $\varphi'$ coincides with the second fundamental form $D_S$ defined in (2.2).

Proposition 2.4 Let $F$ be a stable vector bundle on $X$ of rank $r$ and degree zero. Then there is a vector bundle $E$ on $X$ of rank $2r$, and an algebraic connection $D$ on $E$, such that

1. $S := F \otimes KX^{1/2}$ is a subbundle of $E$, and
2. the triple $(E, S, D)$ is a stable $r$–oper. In particular, $E = J^1(S \otimes TX)$ by Lemma 2.2.

**Proof** The stable vector bundle $F$ of degree zero admits an algebraic connection [1, p. 203, Proposition 19], [25] (in fact $F$ admits a unique algebraic connection whose monodromy representation is unitary [22]). Fix an algebraic connection $D^F$ on $F$.

Now consider the holomorphic vector bundle

$$E := F \otimes J^1(KX^{-1/2})$$

Since $J^1(KX^{-1/2})$ is indecomposable of degree zero, it admits an algebraic connection [1, p. 203, Proposition 19], [25]. In fact, a projective structure on $X$ produces an algebraic connection on $J^1(KX^{-1/2})$ [20]; the definition of projective structure is recalled in Sect. 9.1.

Let $D^J$ be an algebraic connection on $J^1(KX^{-1/2})$. Consider the natural inclusion map

$$KX^{1/2} = KX^{-1/2} \otimes KX \hookrightarrow J^1(KX^{-1/2})$$

(see (2.6)). Let

$$D^J(KX^{1/2}) : KX^{1/2} \rightarrow KX^{-1/2} \otimes KX = KX^{1/2}$$

be the second fundamental form of this subbundle $KX^{1/2} \hookrightarrow J^1(KX^{-1/2})$ for the above connection $D^J$ on $J^1(KX^{-1/2})$. This homomorphism $D^J(KX^{1/2})$ in (2.10) is a nonzero constant scalar multiple of the identity map of $KX^{1/2}$; the scalar is nonzero because $KX^{1/2}$ does not admit any holomorphic connection as its degree is nonzero.

The above algebraic connections $D^F$ and $D^J$, on $F$ and $J^1(KX^{-1/2})$ respectively, together produce the algebraic connection

$$D := D^F \otimes \text{Id}_{J^1(KX^{-1/2})} + \text{Id}_F \otimes D^J$$

(2.11) on $F \otimes J^1(KX^{-1/2}) = E$. The inclusion map $\iota$ in (2.9) produces an inclusion map

$$S := F \otimes KX^{1/2} \overset{\text{Id}_F \otimes \iota}{\rightarrow} F \otimes J^1(KX^{-1/2}) = E$$

The second fundamental form $D_S$ (see (2.2)) of $S$ for the connection $D$ in (2.11) coincides with

$$\text{Id}_F \otimes D^J(KX^{1/2}) : F \otimes KX^{1/2} \rightarrow F \otimes (KX^{-1/2} \otimes KX) = F \otimes KX^{1/2}$$
where $D^f (K_X^{1/2})$ is the homomorphism in (2.10). Since $D^f (K_X^{1/2})$ is a nonzero scalar multiple of the identity map of $K_X^{1/2}$, we now conclude that the second fundamental form $D_S$ is a nonzero scalar multiple of the identity map of $F \otimes K_X^{1/2}$, in particular, $D_S$ is an isomorphism. Therefore, the triple $(E, S, D)$ is a stable $r$-oper on $X$. \hfill \Box

**Proposition 2.5** Let $F$ be a stable vector bundle on $X$ of degree zero, and let $V$ be any vector bundle on $X$. Then there is an algebraic isomorphism

$$h : F \otimes J^1(V) \longrightarrow J^1(F \otimes V)$$

such that the exact sequence in (2.6) for $W = F \otimes V$ coincides with the exact sequence in (2.6) for $W = V$ tensored with $F$.

**Proof** As in the proof of Proposition 2.4, fix an algebraic connection $D^f$ on $F$. For any point $x \in X$, and any $v \in F_x$, let $\widehat{v}$ be the unique flat section of $F$, defined on a simply connected analytic open neighborhood of $x \in X$, such that $\widehat{v}(x) = x$ (see the proof of Lemma 2.2). Take any element $w \in J^1(V)_x$, so $w$ is a section of $V$ defined on the first order infinitesimal neighborhood of $x$. Therefore, $\widehat{v} \otimes w$ is a section of $F \otimes V$ defined on the first order infinitesimal neighborhood of $x$, where $\widehat{v}$ is the restriction of $\widehat{v}$ to the first order infinitesimal neighborhood of $x$. Now we have a map

$$h(x) : F_x \otimes J^1(V)_x \longrightarrow J^1(F \otimes V)_x$$

that sends any $v \otimes w \in F_x \otimes J^1(V)_x$, to the element $\widehat{v} \otimes w$ of $J^1(F \otimes V)_x$ constructed above. It is straightforward to check that $h(x)$ is an isomorphism. Moreover, we get a holomorphic isomorphism

$$h : F \otimes J^1(V) \longrightarrow J^1(F \otimes V) \quad (2.12)$$

which coincides with $h(x)$ for any $x \in X$. This holomorphic isomorphism is algebraic using GAGA [23].

The isomorphism $h$ in (2.12) has the following alternative description. Let $\mathbb{F} \subset F$ be the locally constant subsheaf defined by the connection $D^f$. Now we have a natural map

$$\mathbb{F} \otimes_{\mathcal{O}} J^1(V) \longrightarrow J^1(F \otimes V).$$

It extends uniquely to a homomorphism $F \otimes J^1(V) \longrightarrow J^1(F \otimes V)$ and this extended homomorphism coincides with $h$.

From the above construction of $h$ we conclude that the following diagram of homomorphisms is commutative

$$\begin{array}{ccc}
0 & \longrightarrow & F \otimes (V \otimes K_X) \\
\downarrow & & \downarrow h \\
0 & \longrightarrow & (F \otimes V) \otimes K_X \\
\end{array} \quad (2.13)$$

where the top one is the exact sequence in (2.6) for $W = V$ tensored with $F$ and the bottom one is the exact sequence in (2.6) for $W = F \otimes V$. This completes the proof. \hfill \Box

For any $F$ as in Proposition 2.5, consider the short exact sequence

$$0 \longrightarrow F \otimes K_X^{-1/2} \otimes K_X = F \otimes K_X^{1/2} \longrightarrow J^1(F \otimes K_X^{1/2}) \longrightarrow F \otimes K_X^{-1/2} \longrightarrow 0$$

in (2.6) for $W = F \otimes K_X^{-1/2}$. It corresponds to an extension class

$$c(F) \in H^1(X, \text{Hom}(F \otimes K_X^{-1/2}, F \otimes K_X^{1/2})) = H^1(X, \text{End}(F) \otimes K_X). \quad (2.15)$$
Since \( H^1(X, K_X) = H^0(X, \mathcal{O}_X)^* = C^* = C \), we have
\[
\text{Id}_F \otimes 1 \in H^0(X, \text{End}(F)) \otimes H^1(X, K_X) \hookrightarrow H^1(X, \text{End}(F) \otimes K_X); \quad (2.16)
\]
the above map \( H^0(X, \text{End}(F)) \otimes H^1(X, K_X) \to H^1(X, \text{End}(F) \otimes K_X) \) is injective because \( H^0(X, \text{End}(F)) = \mathbb{C} \) (recall that \( F \) is stable).

**Corollary 2.6** Let \( F \) be a stable vector bundle on \( X \) of degree zero. Then the cohomology class \( c(F) \) in (2.15) coincides with \( \text{Id}_F \otimes 1 \) in (2.16).

**Proof** Consider the short exact sequence
\[
0 \longrightarrow K_X^{-1/2} \otimes K_X = K_X^{1/2} \longrightarrow J^1(K_X^{-1/2}) \longrightarrow K_X^{-1/2} \longrightarrow 0
\]
in (2.6) for \( W = K_X^{-1/2} \). The corresponding extension class coincides with
\[
1 \in H^1(X, \text{Hom}(K_X^{-1/2}, K_X^{1/2})) = H^1(X, K_X) = C
\]
[20]. Hence the extension class of the tensor product of the above exact sequence with \( F \)
\[
0 \longrightarrow F \otimes K_X^{1/2} \longrightarrow F \otimes J^1(K_X^{-1/2}) \longrightarrow F \otimes K_X^{-1/2} \longrightarrow 0
\]
is \( \text{Id}_F \otimes 1 \in H^1(X, \text{End}(F) \otimes K_X) = H^1(X, \text{Hom}(F \otimes K_X^{-1/2}, F \otimes K_X^{1/2})) \) in (2.16). Therefore, the proof is completed using the isomorphism \( h \) in (2.13) together with the commutativity of the diagram in (2.13). \( \square \)

### 2.2 An equivalence relation

Take a vector bundle \( F \) on \( X \) of rank \( r \) and degree zero; it need not be stable. Assume that the jet bundle \( J^1(F \otimes K_X^{-1/2}) \) has an algebraic connection
\[
D_1 : J^1(F \otimes K_X^{-1/2}) \longrightarrow J^1(F \otimes K_X^{-1/2}) \otimes K_X. \quad (2.17)
\]
Consider the subbundle
\[
F \otimes K_X^{1/2} = F \otimes K_X^{-1/2} \otimes K_X \subset J^1(F \otimes K_X^{-1/2})
\]
(see (2.6)). The second fundamental form of this subbundle for the connection \( D_1 \) is a homomorphism
\[
S(F, D_1) : F \otimes K_X^{1/2} \longrightarrow (J^1(F \otimes K_X^{-1/2})/(F \otimes K_X^{1/2})) \otimes K_X
\]
\[
= F \otimes K_X^{-1/2} \otimes K_X = F \otimes K_X^{1/2}
\]
(see (2.2) and (2.6)).

Now assume that \( J^1(F \otimes K_X^{-1/2}) \) admits an algebraic connection \( D_1 \) for which the second fundamental form \( S(F, D_1) \) in (2.18) is the identity map of \( F \otimes K_X^{1/2} \). Note that for such an algebraic connection \( D_1 \), the triple \( (J^1(F \otimes K_X^{-1/2}), F \otimes K_X^{1/2}, D_1) \) is a \( r \)--oper (see Definition 2.1, (2.3) and (2.4)).

Let
\[
\text{Conn}(J^1(F \otimes K_X^{-1/2})) \quad (2.19)
\]
denote the space of all algebraic connections on the vector bundle \( J^1(F \otimes K_X^{-1/2}) \).
Definition 2.7 Let

$$\tilde{C}(F) \subset \text{Conn}(J^1(F \otimes K_X^{-1/2}))$$

be the space of all algebraic connections $D$ on $J^1(F \otimes K_X^{-1/2})$ with the property that the corresponding second fundamental form $S(F, D)$ in (2.18) is the identity map of $F \otimes K_X^{1/2}$.

We note that $\tilde{C}(F)$ is nonempty by the assumption on $F$. The space $\text{Conn}(J^1(F \otimes K_X^{-1/2}))$ in (2.19) is an affine variety; more precisely, it is an affine space (or torsor) for the vector space $H^0(X, \text{End}(J^1(F \otimes K_X^{-1/2}) \otimes K_X))$. The above defined $\tilde{C}(F)$ is also an affine space. To describe this, take any $\psi \in H^0(X, \text{End}(J^1(F \otimes K_X^{-1/2}) \otimes K_X))$, so it is a homomorphism $\psi : J^1(F \otimes K_X^{-1/2}) \rightarrow J^1(F \otimes K_X^{-1/2}) \otimes K_X$. Let

$$H : H^0(X, \text{End}(J^1(F \otimes K_X^{-1/2}) \otimes K_X)) \rightarrow H^0(X, \text{Hom}(F \otimes K_X^{1/2}, F \otimes K_X^{-1/2}) \otimes K_X)$$

(2.20)

be the homomorphism that sends any $\psi$ as above to the following composition of homomorphisms

$$F \otimes K_X^{1/2} \hookrightarrow J^1(F \otimes K_X^{-1/2}) \xrightarrow{\psi} J^1(F \otimes K_X^{-1/2}) \otimes K_X \rightarrow F \otimes K_X^{1/2} \otimes K_X$$

(see (2.14) for the above injection and projection). The above defined $\tilde{C}(F)$ is an affine space for the vector subspace

$$\text{kernel}(H) \subset H^0(X, \text{End}(J^1(F \otimes K_X^{-1/2}) \otimes K_X)),$$

(2.21)

where $H$ is constructed in (2.20).

From the above property of $\tilde{C}(F)$ it follows that $(J^1(F \otimes K_X^{-1/2}), F \otimes K_X^{1/2}, D)$ is a $r$–oper for every $D \in \tilde{C}(F)$.

We shall see in Corollary 3.2 that $\tilde{C}(F) = \text{Conn}(J^1(F \otimes K_X^{-1/2}))$ when the vector bundle $F$ is stable.

For notational convenience, define

$$S := F \otimes K_X^{1/2} \quad \text{and} \quad E := J^1(F \otimes K_X^{-1/2})$$

(2.22)

(see (2.3), (2.4) and Lemma 2.2), so $S$ is an algebraic subbundle of $E$ by (2.6) (see (2.23) below). Note that we have $\text{End}(S) = \text{End}(F) \otimes \text{End}(K_X^{1/2}) = \text{End}(F)$.

As in (2.14), setting $W = F \otimes K_X^{-1/2}$ in (2.6), we get a short exact sequence

$$0 \rightarrow F \otimes K_X^{-1/2} \otimes K_X \rightarrow S \xrightarrow{1} J^1(F \otimes K_X^{-1/2})$$

$$\quad = E \xrightarrow{q_0} F \otimes K_X^{-1/2} \rightarrow S \otimes TX \rightarrow 0$$

(2.23)

(see (2.22)). Consequently, there is a natural inclusion map

$$\psi : H^0(X, \text{End}(S \otimes K_X^{1/2})) = H^0(X, \text{Hom}(S \otimes TX, S \otimes K_X)) \rightarrow H^0(X, \text{End}(E \otimes K_X))$$

(2.24)

that sends any homomorphism $\alpha : S \otimes TX \rightarrow S \otimes K_X$ to the following composition of homomorphisms

$$E \xrightarrow{q_0} S \otimes TX \xrightarrow{\alpha} S \otimes K_X \xrightarrow{i \otimes \text{Id}_{K_X}} E \otimes K_X,$$

where $q_0$ and $i$ are the homomorphisms in (2.23).
The space of all algebraic connections on $E$ is an affine space modeled on the vector space $H^0(X, \text{End}(E) \otimes K_X)$; recall that $E$ is assumed to admit an algebraic connection (see (2.17), (2.22)). In view of the homomorphism $\psi$ in (2.24), for any algebraic connection $D$ on $E$ and any $\alpha \in H^0(X, \text{End}(S) \otimes K_X^{\otimes 2})$, we get an algebraic connection $D + \psi(\alpha)$ on $E$. If $D \in \tilde{C}(F)$ (see Definition 2.7), then it can be shown that

$$D + \psi(\alpha) \in \tilde{C}(F).$$

(2.25)

Indeed, the restrictions of $D$ and $D + \psi(\alpha)$ to the subbundle $S$ in (2.23) coincide. Therefore, the second fundamental form of $S$ for the connection $D$ coincides with the second fundamental form of $S$ for the connection $D + \psi(\alpha)$. Hence (2.25) holds. Consequently, the vector space $H^0(X, \text{End}(S) \otimes K_X^{\otimes 2})$ acts on $\tilde{C}(F)$; the action of any $\alpha \in H^0(X, \text{End}(S) \otimes K_X^{\otimes 2})$ sends any $D \in \tilde{C}(F)$ to $D + \psi(\alpha)$. Let

$$\tilde{C}(F) \times H^0(X, \text{End}(S) \otimes K_X^{\otimes 2}) \longrightarrow \tilde{C}(F)$$

(2.26)

be this action. The action in (2.26) is evidently free; however, the action is not transitive (the dimension of $H^0(X, \text{End}(S) \otimes K_X^{\otimes 2})$ is in fact smaller than that of $\tilde{C}(F)$).

Let $\text{ad}(S) \subset \text{End}(S)$ be the subbundle of rank $r^2 - 1$ given by the sheaf of endomorphisms of $S$ of trace zero. Note that we have

$$\text{End}(S) = \text{ad}(S) \oplus \mathcal{O}_X;$$

(2.27)

the inclusion map $\mathcal{O}_X \hookrightarrow \text{End}(S)$ sends a locally defined holomorphic function $f$ on $X$ to the locally defined endomorphism of $S$ that maps any locally defined section $s$ of $E$ to $f \cdot s$. It is evident that $\text{ad}(S) = \text{ad}(F)$ (see (2.22)). The decomposition of $\text{End}(S)$ in (2.27) produces a decomposition

$$H^0(X, \text{End}(S) \otimes K_X^{\otimes 2}) = H^0(X, \text{ad}(S) \otimes K_X^{\otimes 2}) \oplus H^0(X, K_X^{\otimes 2}).$$

(2.28)

Consider the action of $H^0(X, \text{End}(S) \otimes K_X^{\otimes 2})$ on $\tilde{C}(F)$ in (2.26). In view of (2.28), from this action we obtain an action of $H^0(X, \text{ad}(S) \otimes K_X^{\otimes 2})$ on $\tilde{C}(F)$.

**Definition 2.8** Define $\mathcal{C}(F)$ to be the quotient space $\tilde{C}(F)/H^0(X, \text{ad}(S) \otimes K_X^{\otimes 2})$ for the above action of $H^0(X, \text{ad}(S) \otimes K_X^{\otimes 2})$ on the space $\tilde{C}(F)$ in Definition 2.7.

From (2.28) it follows immediately that $H^0(X, K_X^{\otimes 2})$ acts freely on $\mathcal{C}(F)$. It is evident that

$$\mathcal{C}(F)/H^0(X, K_X^{\otimes 2}) = \tilde{C}(F)/H^0(X, \text{End}(S) \otimes K_X^{\otimes 2}).$$

We saw that $\tilde{C}(F)$ is an affine space for the vector subspace kernel($\mathbf{H}$), where $\mathbf{H}$ is the homomorphism in (2.20) (see (2.21)). Consequently, $\mathcal{C}(F)$ is an affine space for the quotient vector space kernel($\mathbf{H}$)/$H^0(X, \text{ad}(S) \otimes K_X^{\otimes 2})$.

We have a homomorphism

$$\sigma : \text{End}(F) \otimes K_X = \text{Hom}(F \otimes K_X^{-1/2}, F \otimes K_X^{1/2}) \longrightarrow \text{End}(J^1(F \otimes K_X^{-1/2}))$$

(2.29)

that sends any locally defined homomorphism $\eta : F \otimes K_X^{-1/2} \longrightarrow F \otimes K_X^{1/2}$ to the following composition of (locally defined) homomorphisms:

$$J^1(F \otimes K_X^{-1/2}) \xrightarrow{q_0} F \otimes K_X^{-1/2} \xrightarrow{\eta} F \otimes K_X^{1/2} \xrightarrow{\iota} J^1(F \otimes K_X^{-1/2}),$$

where $\iota$ and $q_0$ are the homomorphisms in (2.23).
Lemma 2.9 Let $F$ be a stable vector bundle on $X$ of rank $r$ and degree zero. The homomorphism

$$\hat{\sigma} : H^0(X, \text{End}(F) \otimes K_X) \oplus \mathbb{C} \longrightarrow H^0(X, \text{End}(J^1(F \otimes K_X^{-1/2}))) ,$$

that sends any $(v, c) \in H^0(X, \text{End}(F) \otimes K_X) \oplus \mathbb{C}$ to $\sigma(v) + c \cdot \text{Id}_{J^1(F \otimes K_X^{-1/2})}$, where $\sigma$ is the homomorphism in (2.29), is an isomorphism.

Proof The vector bundle $F \otimes K_X^{-1/2}$ does not admit any algebraic connection because $\text{deg}(F \otimes K_X^{-1/2}) = r(1 - g) \neq 0$ [1, p. 202, Proposition 18(i)] (recall that $g \geq 2$). The statement that $F \otimes K_X^{-1/2}$ does not admit any algebraic connection is equivalent to the statement that the short exact sequence in (2.23) does not split algebraically [1]. Indeed, an algebraic splitting homomorphism

$$D_1 : J^1(F \otimes K_X^{-1/2}) \longrightarrow (F \otimes K_X^{-1/2}) \otimes K_X$$

for (2.23) such that $D_1 \circ \iota = \text{Id}$, where $\iota$ is the homomorphism in (2.23), defines an algebraic differential operator of order one $\tilde{D}_1 : F \otimes K_X^{-1/2} \longrightarrow (F \otimes K_X^{-1/2}) \otimes K_X$ whose symbol is $\text{Id}_{F \otimes K_X^{-1/2}}$, in other words, $\tilde{D}_1$ is an algebraic connection on $F \otimes K_X^{-1/2}$; conversely, the homomorphism $D_2 : J^1(F \otimes K_X^{-1/2}) \longrightarrow (F \otimes K_X^{-1/2}) \otimes K_X$ corresponding to any algebraic connection on $F \otimes K_X^{-1/2}$ produces an algebraic splitting of (2.23).

Take any endomorphism $\rho : J^1(F \otimes K_X^{-1/2}) \longrightarrow J^1(F \otimes K_X^{-1/2})$ over $X$. Consider the composition of homomorphisms

$$F \otimes K_X^{1/2} \overset{\iota}{\longrightarrow} J^1(F \otimes K_X^{-1/2}) \overset{\rho}{\longrightarrow} J^1(F \otimes K_X^{-1/2}) \overset{q_0}{\longrightarrow} F \otimes K_X^{-1/2} ,$$

(2.30)

where $\iota$ and $q_0$ are the homomorphisms in (2.23). Both the vector bundles $F \otimes K_X^{1/2}$ and $F \otimes K_X^{-1/2}$ are stable because $F$ is so. We have

$$\frac{\text{deg}(F \otimes K_X^{1/2})}{\text{rank}(F \otimes K_X^{1/2})} > \frac{\text{deg}(F \otimes K_X^{-1/2})}{\text{rank}(F \otimes K_X^{-1/2})}$$

since $g \geq 2$. Therefore, there is no nonzero homomorphism from $F \otimes K_X^{1/2}$ to $F \otimes K_X^{-1/2}$. In particular, the composition of homomorphisms in (2.30) vanishes identically.

Now let $\rho' := \rho\big|_{F \otimes K_X^{1/2}} : F \otimes K_X^{1/2} \longrightarrow F \otimes K_X^{1/2}$ be the restriction of $\rho$ to the subbundle $F \otimes K_X^{1/2}$ in (2.23). Since $F \otimes K_X^{1/2}$ is stable, there is a $c \in \mathbb{C}$ such that $\rho' = c \cdot \text{Id}_{F \otimes K_X^{1/2}}$. Define the endomorphism over $X$

$$\rho_1 := \rho - c \cdot \text{Id}_{J^1(F \otimes K_X^{-1/2})} : J^1(F \otimes K_X^{-1/2}) \longrightarrow J^1(F \otimes K_X^{-1/2}) .$$

(2.31)

So we have $\rho_1(F \otimes K_X^{1/2}) = 0$. This implies that there is a homomorphism

$$\rho_2' : J^1(F \otimes K_X^{-1/2})/(F \otimes K_X^{1/2}) = F \otimes K_X^{-1/2} \longrightarrow J^1(F \otimes K_X^{-1/2})$$

such that

$$\rho_1 = \rho_2' \circ q_0 ,$$

(2.32)

where $q_0$ and $\rho_1$ are the homomorphisms in (2.23) and (2.31) respectively. Now $q_0 \circ \rho_2' \in H^0(X, \text{End}(F \otimes K_X^{-1/2}))$ coincides with $c' \cdot \text{Id}$ for some $c' \in \mathbb{C}$, because $F \otimes K_X^{-1/2}$ is
stable. Next we note that if \( c' \neq 0 \), then \( \frac{1}{c'} \rho_2' : F \otimes K_X^{-1/2} \longrightarrow J^1(F \otimes K_X^{-1/2}) \) is an algebraic splitting of the short exact sequence in (2.23). Since the short exact sequence in (2.23) does not admit an algebraic splitting (this was shown earlier), we conclude that \( c' = 0 \).

Given that \( c' = 0 \), it is deduced that there is a homomorphism

\[ \rho_2 : F \otimes K_X^{-1/2} \longrightarrow F \otimes K_X^{1/2} \]

such that \( \rho_2' = \iota \circ \rho_2 \), where \( \iota \) and \( \rho_2' \) are the homomorphisms in (2.23) and (2.32) respectively. Therefore, from (2.32) and the definition of \( \sigma \) in (2.29) it follows that

\[ \rho_1 = \iota \circ \rho_2 \circ q_0 = \sigma(\rho_2). \]

So from (2.31) we know that \( \rho = \sigma(\rho_2) + c \cdot \text{Id}_{J^1(F \otimes K_X^{-1/2})} \).

As in Lemma 2.9, \( F \) is a stable vector bundle on \( X \) of rank \( r \) and degree zero. Let

\[ \text{Aut}(J^1(F \otimes K_X^{-1/2})) \]

be the group of all algebraic automorphisms of \( J^1(F \otimes K_X^{-1/2}) \); it is a complex affine algebraic group, in fact, it is a nonempty Zariski open subset of the affine space \( H^0(X, \text{End}(J^1(F \otimes K_X^{-1/2}))) \). More precisely

\[ \text{Aut}(J^1(F \otimes K_X^{-1/2})) \subset H^0(X, \text{End}(J^1(F \otimes K_X^{-1/2}))) \]

is the nonzero locus of the polynomial function

\[ H^0(X, \text{End}(J^1(F \otimes K_X^{-1/2}))) \longrightarrow \mathbb{C}, \quad A \longmapsto \det A \]

(note that \( \det A \) is a constant function on \( X \)).

From Lemma 2.9 it follows immediately that

\[ \text{Aut}(J^1(F \otimes K_X^{-1/2})) = H^0(X, \text{End}(F) \otimes K_X) \times \mathbb{G}_m, \quad (2.33) \]

where \( \mathbb{G}_m = \mathbb{C}^* = \mathbb{C} \setminus \{0\} \) is the multiplicative group.

The group \( \text{Aut}(J^1(F \otimes K_X^{-1/2})) \) has a natural action on \( \text{Conn}(J^1(F \otimes K_X^{-1/2})) \) defined in (2.19). The action of any \( T \in \text{Aut}(J^1(F \otimes K_X^{-1/2})) \) sends any algebraic connection \( D \) to the algebraic connection given by the composition \( (T \otimes \text{Id}_{K_X}) \circ D \circ T^{-1} \) of operators.

The isomorphism in (2.33) yields the following:

**Corollary 2.10** Let \( F \) be a stable vector bundle on \( X \) of rank \( r \) and degree zero. The action of \( \text{Aut}(J^1(F \otimes K_X^{-1/2})) \) on \( \text{Conn}(J^1(F \otimes K_X^{-1/2})) \) preserves the subvariety \( \tilde{C}(F) \) in Definition 2.7.

The action of \( \text{Aut}(J^1(F \otimes K_X^{-1/2})) \) on \( \tilde{C}(F) \) descends to an action of \( \text{Aut}(J^1(F \otimes K_X^{-1/2})) \) on the quotient space \( C(F) \) in Definition 2.8.

**Proof** The first statement is straightforward. Take any \( D \in \text{Conn}(J^1(F \otimes K_X^{-1/2})) \) and \( T \in \text{Aut}(J^1(F \otimes K_X^{-1/2})) \). From (2.33) it follows immediately that the second fundamental forms of the subbundle \( F \otimes K_X^{1/2} \subset J^1(F \otimes K_X^{-1/2}) \) (see (2.23)) for the two connections \( D \) and \((T \otimes \text{Id}_{K_X}) \circ D \circ T^{-1}\) coincide; see also Corollary 3.2. This implies that the action of \( \text{Aut}(J^1(F \otimes K_X^{-1/2})) \) on \( \text{Conn}(J^1(F \otimes K_X^{-1/2})) \) preserves \( \tilde{C}(F) \).

Take any \( D \in \text{Conn}(J^1(F \otimes K_X^{-1/2})) \), \( T \in \text{Aut}(J^1(F \otimes K_X^{-1/2})) \) and

\[ B \in H^0(X, \text{ad}(S) \otimes K_X^\otimes). \]
From (2.33) it follows that
\[(T \otimes \text{Id}_{K_X}) \circ (D + B) \circ T^{-1} = (T \otimes \text{Id}_{K_X}) \circ D \circ T^{-1} + B.\]

Therefore, the translation action of $H^0(X, \text{ad}(S) \otimes K_X^{\otimes 2})$ on $\tilde{C}(F)$ and the action of $\text{Aut}(J^1(F \otimes K_X^{-1/2}))$ on $\tilde{C}(F)$ commute. The second statement of the corollary follows from this commuting property. $\square$

In the next section we will put structures on the quotient space obtained from Corollary 2.10
\[\mathcal{D}(F) := \mathcal{C}(F)/\text{Aut}(J^1(F \otimes K_X^{-1/2})).\]  

(2.34)

3 The space of algebraic connections

Take a stable vector bundle $F$ on $X$ of rank $r$ and degree zero.

As before, $K_X^{1/2}$ is a theta characteristic on $X$. From Proposition 2.4 and Lemma 2.3 we know that $J^1(F \otimes K_X^{-1/2})$ admits an algebraic connection $D_1$ for which the second fundamental form $S(F, D_1)$ in (2.18) is the identity map of $F \otimes K_X^{1/2}$.

Recall that the space $\text{Conn}(J^1(F \otimes K_X^{-1/2}))$ in (2.19) is an affine space modeled on the vector space $H^0(X, \text{End}(J^1(F \otimes K_X^{-1/2})) \otimes K_X)$. Let
\[0 \longrightarrow F \otimes K_X^{3/2} = F \otimes K_X^{1/2} \otimes K_X \overset{\lambda \otimes \text{Id}_{K_X}}{\longrightarrow} J^1(F \otimes K_X^{-1/2}) \otimes K_X \]  

(3.1)

\[\quad \overset{q_0 \otimes \text{Id}_{K_X}}{\longrightarrow} F \otimes K_X^{-1/2} \otimes K_X = F \otimes K_X^{1/2} \longrightarrow 0\]

be the short exact sequence of vector bundles obtained by tensoring the short exact sequence in (2.23) by $K_X$.

Lemma 3.1 Take any
\[\Psi \in H^0(X, \text{Hom}(J^1(F \otimes K_X^{-1/2}), J^1(F \otimes K_X^{-1/2}) \otimes K_X)).\]

Then
\[\Psi(F \otimes K_X^{1/2}) \subset F \otimes K_X^{3/2},\]

where $F \otimes K_X^{1/2}$ (respectively, $F \otimes K_X^{3/2}$) is the subbundle of $J^1(F \otimes K_X^{-1/2})$ (respectively, $J^1(F \otimes K_X^{-1/2}) \otimes K_X$) in (2.23) (respectively, (3.1)).

Proof Take any algebraic homomorphism
\[\Psi : J^1(F \otimes K_X^{-1/2}) \longrightarrow J^1(F \otimes K_X^{-1/2}) \otimes K_X.\]

Consider the composition of homomorphisms
\[F \otimes K_X^{1/2} \hookrightarrow J^1(F \otimes K_X^{-1/2}) \overset{\Psi}{\longrightarrow} J^1(F \otimes K_X^{-1/2}) \otimes K_X \overset{q_0 \otimes \text{Id}_{K_X}}{\longrightarrow} F \otimes K_X^{-1/2} \otimes K_X = F \otimes K_X^{1/2}.\]  

(3.2)
where $q_0 \otimes \text{Id}_{K_X}$ is the homomorphism in (3.1). Since the vector bundle $F$—and hence $F \otimes K_X^{-1/2}$—is stable, any nonzero endomorphism of $F \otimes K_X^{-1/2}$ is in fact an isomorphism.

We will now show that the composition of homomorphisms in (3.2) is the zero homomorphism.

To prove this by contradiction, assume that the composition of homomorphisms in (3.2) is nonzero. As observed above, this implies that the composition of homomorphisms in (3.2) is an isomorphism. Consequently, $\Psi(F \otimes K_X^{-1/2})$ is a subbundle of $J^1(F \otimes K_X^{-1/2}) \otimes K_X$. Now consider the subbundle

$$\Psi(F \otimes K_X^{-1/2}) \otimes TX \subset J^1(F \otimes K_X^{-1/2}) \otimes K_X \otimes TX = J^1(F \otimes K_X^{-1/2}).$$

(3.3)

The condition that the composition of homomorphisms in (3.2) is an isomorphism implies that the homomorphism

$$\Psi(F \otimes K_X^{-1/2}) \otimes TX \xrightarrow{q_0'} F \otimes K_X^{-1/2},$$

where $q_0'$ is the restriction of the projection $q_0$ in (2.23) (see (3.3)), is an isomorphism. Consequently, the subbundle $\Psi(F \otimes K_X^{-1/2}) \otimes TX \subset J^1(F \otimes K_X^{-1/2})$ in (3.3) produces an algebraic splitting of the short exact sequence in (2.23). But it was observed in the proof of Lemma 2.9 that the short exact sequence in (2.23) does not split algebraically.

In view of the above contradiction we conclude that the composition of homomorphisms in (3.2) is the zero homomorphism. This proves the lemma.

The following two results are deduced using Lemma 3.1.

**Corollary 3.2** As in Lemma 3.1, $F$ is a stable vector bundle on $X$ of degree zero. The space $\tilde{C}(F)$ in Definition 2.7 actually coincides with $\text{Conn}(J^1(F \otimes K_X^{-1/2}))$ in (2.19).

**Proof** Take any algebraic connection $D$ on $J^1(F \otimes K_X^{-1/2})$, and any

$$\Psi \in H^0(X, \text{Hom}(J^1(F \otimes K_X^{-1/2}), J^1(F \otimes K_X^{-1/2}) \otimes K_X)).$$

So $D + \Psi$ is an algebraic connection on $J^1(F \otimes K_X^{-1/2}).$ Consider the second fundamental forms of the subbundle $F \otimes K_X \subset J^1(F \otimes K_X^{-1/2})$ for the two connections $D$ and $D + \Psi.$ From Lemma 3.1 it follows immediately that these two second fundamental forms actually coincide. Since $J^1(F \otimes K_X^{-1/2})$ admits an algebraic connection $D_1$ for which the second fundamental form $S(F, D_1)$ in (2.18) is the identity map of $F \otimes K_X^{-1/2}$ (see Proposition 2.4), for every algebraic connection $D'$ on $J^1(F \otimes K_X^{-1/2})$ the second fundamental form $S(F, D')$ in (2.18) is the identity map of $F \otimes K_X^{-1/2}.$ This completes the proof.

**Lemma 3.3** The vector space $H^0(X, \text{Hom}(J^1(F \otimes K_X^{-1/2}), J^1(F \otimes K_X^{-1/2}) \otimes K_X))$ fits in the following short exact sequence:

$$0 \to H^0(X, \text{End}(F) \otimes K_X^{-2}) \to H^0(X, \text{Hom}(J^1(F \otimes K_X^{-1/2}), J^1(F \otimes K_X^{-1/2}) \otimes K_X)) \to H^0(X, \text{End}(F) \otimes K_X) \to 0.$$

**Proof** Let

$$\text{Hom}_F(J^1(F \otimes K_X^{-1/2}), J^1(F \otimes K_X^{-1/2}) \otimes K_X) \subset \text{Hom}(J^1(F \otimes K_X^{-1/2}), J^1(F \otimes K_X^{-1/2}) \otimes K_X)$$
be the subbundle defined by the sheaf of homomorphisms from $J^1(F \otimes K_X^{-1/2})$ to $J^1(F \otimes K_X^{-1/2}) \otimes K_X$ that take the subbundle $F \otimes K_X^{1/2}$ in (2.23) to the subbundle $F \otimes K_X^{3/2}$ in (3.1). Let

\[ \text{Hom}(F \otimes K_X^{-1/2}, F \otimes K_X^{3/2}) = \text{End}(F) \otimes K_X^{\otimes 2} \]

\[ \mapsto \text{Hom}_P(J^1(F \otimes K_X^{-1/2}), J^1(F \otimes K_X^{-1/2}) \otimes K_X) \]  

(3.4)

be the homomorphism that sends any locally defined homomorphism

\[ \eta : F \otimes K_X^{-1/2} \longrightarrow F \otimes K_X^{3/2} \]

to the following composition of (locally defined) homomorphisms:

\[ J^1(F \otimes K_X^{-1/2}) \stackrel{q_0}{\longrightarrow} F \otimes K_X^{-1/2} \overset{\eta}{\longrightarrow} F \otimes K_X^{3/2} \overset{\iota \otimes \text{Id}_{K_X}}{\longrightarrow} J^1(F \otimes K_X^{-1/2}) \otimes K_X, \]

where $q_0$ and $\iota \otimes \text{Id}_{K_X}$ are the homomorphisms in (2.23) and (3.1) respectively.

We have a natural surjective homomorphism

\[ \sigma_1 : \text{Hom}_P(J^1(F \otimes K_X^{-1/2}), J^1(F \otimes K_X^{-1/2}) \otimes K_X) \longrightarrow \text{Hom}(F \otimes K_X^{1/2}, F \otimes K_X^{3/2}) \oplus \text{Hom}(F \otimes K_X^{-1/2}, F \otimes K_X^{1/2}) = (\text{End}(F) \otimes K_X)^{\otimes 2} \]

that sends any homomorphism to the induced homomorphism from the subbundle (respectively, quotient bundle) in (2.23) to the subbundle (respectively, quotient bundle) in (3.1). Consequently, we have a short exact sequence of holomorphic vector bundles on $X$

\[ 0 \longrightarrow \text{End}(F) \otimes K_X^{\otimes 2} \overset{j}{\longrightarrow} \text{Hom}_P(J^1(F \otimes K_X^{-1/2}), J^1(F \otimes K_X^{-1/2}) \otimes K_X) \overset{\sigma_1}{\longrightarrow} (\text{End}(F) \otimes K_X)^{\otimes 2} \longrightarrow 0, \]

(3.5)

where $j$ is the homomorphism in (3.4).

Let

\[ 0 \longrightarrow H^0(X, \text{End}(F) \otimes K_X^{\otimes 2}) \longrightarrow H^0(X, \text{Hom}_P(J^1(F \otimes K_X^{-1/2}), J^1(F \otimes K_X^{-1/2}) \otimes K_X)) \longrightarrow H^0(X, (\text{End}(K) \otimes K_X)^{\otimes 2}) \longrightarrow H^1(X, \text{End}(F) \otimes K_X^{\otimes 2}) \]

(3.6)

be the long exact sequence of cohomologies corresponding to the exact sequence in (3.5). By Serre duality,

\[ H^1(X, \text{End}(F) \otimes K_X^{\otimes 2}) = H^0(X, \text{End}(F) \otimes TX)^* = 0, \]

(3.7)

because $\text{End}(F)$ is semistable of degree zero (recall that $F$ is stable) and degree($TX$) < 0 (recall that $g \geq 2$); the unitary flat connection on $F$, [22], induces a unitary flat connection on $\text{End}(F)$ and hence $\text{End}(F)$ is polystable, in particular, $\text{End}(F)$ is semistable. From Lemma 3.1 we have

\[ H^0(X, \text{Hom}_P(J^1(F \otimes K_X^{-1/2}), J^1(F \otimes K_X^{-1/2}) \otimes K_X)) \]

\[ = H^0(X, \text{Hom}(J^1(F \otimes K_X^{-1/2}), J^1(F \otimes K_X^{-1/2}) \otimes K_X)). \]

Consequently, the lemma follows from (3.6) and (3.7).
4 Infinitesimal deformations

The space of all infinitesimal deformations of a compact Riemann surface $X$ is identified with $H^1(X, TX)$. By Serre duality we have

$$H^1(X, TX)^* = H^0(X, K_X^\otimes 2).$$

For a holomorphic vector bundle $V$ on $X$, the infinitesimal deformations of $V$, keeping $X$ fixed, are parametrized by $H^1(X, \text{End}(V))$. By Serre duality,

$$H^1(X, \text{End}(V))^* = H^0(X, \text{End}(V) \otimes K_X).$$

Consider the holomorphic vector bundle

$$\text{Diff}^i_X(V, V) := \text{Hom}(J^i(V), V) \quad (4.1)$$
on $X$ given by the sheaf of all holomorphic differential operators of order $i$ from $V$ to itself. Take the dual of the exact sequence in (2.6) for $W = V$

$$0 \longrightarrow V^* \stackrel{\iota}{\longrightarrow} J^1(V)^* \longrightarrow V^* \otimes TX \longrightarrow 0. \quad (4.2)$$

Tensoring it with $V$ the following short exact sequence of holomorphic vector bundles on $X$ is obtained

$$0 \longrightarrow \text{Diff}^0_X(V, V) = \text{End}(V) \longrightarrow \text{Diff}^1_X(V, V) \stackrel{\sigma_0}{\longrightarrow} \text{End}(V) \otimes TX \longrightarrow 0; \quad (4.2)$$

the above projection $\sigma_0$ coincides with the symbol map. Using the natural inclusion $\mathcal{O}_X \subset \text{End}(V)$ (see (2.27)), we have $TX \subset \text{End}(V) \otimes TX$. Now define the Atiyah bundle for $V$

$$\text{At}(V) := \sigma_0^{-1}(TX) \subset \text{Diff}^1_X(V, V), \quad (4.3)$$

where $\sigma_0$ is the projection in (4.2) [1]. The exact sequence in (4.2) produces the short exact sequence of holomorphic vector bundles on $X$

$$0 \longrightarrow \text{Diff}^0_X(V, V) = \text{End}(V) \longrightarrow \text{At}(V) \stackrel{\sigma}{\longrightarrow} TX \longrightarrow 0, \quad (4.4)$$

where $\sigma$ is the restriction of $\sigma_0$ (in (4.2)) to the subbundle $\text{At}(V)$; this exact sequence is known as the Atiyah exact sequence for $V$ (see [1]).

The space of all infinitesimal deformations of the pair $(X, V)$ is known to be identified with $H^1(X, \text{At}(V))$ (see [13, p. 1413, Proposition 4.3]). The homomorphism

$$H^1(X, \text{At}(V)) \longrightarrow H^1(X, TX) \quad (4.5)$$

produced by the projection $\sigma$ in (4.4) coincides with the forgetful map that sends an infinitesimal deformation of $(X, V)$ to the infinitesimal deformation of $X$ obtained from it by simply forgetting the vector bundle. The homomorphism

$$H^1(X, \text{End}(V)) \longrightarrow H^1(X, \text{At}(V))$$

induced by the homomorphism $\text{End}(V) \longrightarrow \text{At}(V)$ in (4.4) coincides with the map that sends an infinitesimal deformation of $V$ to the infinitesimal deformation of $(X, V)$ associated to it that keeps the Riemann surface fixed.

From the construction of $\text{At}(V)$ in (4.3) it follows immediately that the dual vector bundle $\text{At}(V)^*$ is a quotient of $\text{Diff}^1_X(V, V)^* = J^1(V) \otimes V^*$ (see (4.1)). To describe this quotient, consider the subbundle $V \otimes K_X \subset J^1(V)$ in (2.6). Using it, we have

$$\text{ad}(V) \otimes K_X \subset \text{End}(V) \otimes K_X = (V \otimes K_X) \otimes V^* \subset J^1(V) \otimes V^*, \quad (4.6)$$
where \( \text{ad}(V) \) as before is the sheaf of trace zero endomorphisms of \( V \). From (4.3) it is deduced that

\[
\text{At}(V)^* = (J^1(V) \otimes V^*) / (\text{ad}(V) \otimes K_X) = \text{Diff}_{X}^1(V, V)^* / (\text{ad}(V) \otimes K_X),
\]

where the quotient is by the subbundle in (4.6); note that \( \text{Diff}_{X}^1(V, V)^* = J^1(V) \otimes V^* \). In view of (4.7), by Serre duality we have

\[
H^1(X, \text{At}(V))^* = H^0 \left( X, \frac{J^1(V) \otimes V^*}{\text{ad}(V) \otimes K_X} \otimes K_X \right).
\]

As in Sect. 2.1., let \( K_{X}^{1/2} \) be a theta characteristic on \( X \). Since the collection of theta characteristics on \( X \) is a discrete set, there is unique way to move the theta characteristic when \( X \) moves over a family of Riemann surfaces parametrized by a simply connected space. A consequence of this observation will be explained now.

Consider the Atiyah exact sequence

\[
0 \rightarrow \mathcal{O}_X \rightarrow \text{At}(K_X^{1/2}) \rightarrow TX \rightarrow 0
\]
in (4.4) for \( V = K_X^{1/2} \). Let

\[
0 = H^0(X, TX) \rightarrow H^1(X, \mathcal{O}_X) \xrightarrow{h} H^1(X, \text{At}(K_X^{1/2})) \xrightarrow{\xi} H^1(X, TX) \rightarrow H^2(X, \mathcal{O}_X) = 0
\]

be the long exact sequence of cohomologies associated to it. The above observation implies that there is a canonical homomorphism

\[
\phi : H^1(X, TX) \rightarrow H^1(X, \text{At}(K_X^{1/2}))
\]

that sends an infinitesimal deformation of \( X \) to the corresponding infinitesimal deformation of the pair \( (X, K_X^{1/2}) \). In particular, we have \( \xi \circ \phi = \text{Id}_{H^1(X, TX)} \), where \( \xi \) is the homomorphism in (4.9).

An alternative description of the homomorphism \( \phi \) in (4.10) is the following. Let \( \{U_i\}_{i \in I} \) be a covering of \( X \) by open subsets, and let

\[
\lambda_{i,j} : U_{ij} = U_i \cap U_j \rightarrow TU_{ij}, \quad i, j \in I,
\]

be a 1-cocycle giving an element of \( H^1(X, TX) \). Then \( \lambda_{i,j} \) acts on \( H^0(U_{ij}, K_X^{1/2}) \) by Lie derivation. To define the operation Lie derivation, take a locally defined holomorphic vector field \( v \) and a locally defined holomorphic section \( s \) of \( K_X^{1/2} \); so \( s \otimes s \) is a locally defined holomorphic 1-from on \( X \). Now define \( L_v s \) by the equation

\[
L_v s = s \otimes L_v(s) = \frac{1}{2}(i_v d(s \otimes s) + d((s \otimes s)(v))).
\]

Using this action by Lie derivation, \( \{\lambda_{i,j}\} \) is considered as a 1-cocycle with values in \( \text{At}(K_X^{1/2}) \). The corresponding element of \( H^1(X, \text{At}(K_X^{1/2})) \) is the image, under the homomorphism \( \phi \) in (4.10), of the cohomology class of \( \{\lambda_{i,j}\} \in H^1(X, TX) \).

**Lemma 4.1** For any holomorphic vector bundle \( V \) on \( X \), there is a natural isomorphism

\[
\zeta : H^1(X, \text{At}(V)) \sim H^1(X, \text{At}(V \otimes K_X^{-1/2})).
\]
Fix $X, V$ and a theta characteristic $K^{1/2}_X$ on $X$. Take any infinitesimal deformation $v \in H^1(X, \text{At}(V))$ of $(X, V)$. It corresponds to a family of curves $\mathcal{X} \to \text{Spec } \mathbb{C}[\epsilon]/\epsilon^2$ together with a holomorphic vector bundle $\mathcal{V} \to \mathcal{X}$. The fiber of $\mathcal{X}$ over $\text{Spec } \mathbb{C}$ is $X$. Let $v' \in H^1(X, TX)$ be the image of $v$ under the homomorphism $H^1(X, \text{At}(V)) \to H^1(X, TX)$ in (4.5). The image of $v'$ under the homomorphism $\phi$ in (4.10) gives a relative theta line bundle $K^{1/2}$, whose restriction to $X$ is the chosen theta characteristic $K^{1/2}_X$. Now the pair $(\mathcal{X}, \mathcal{V} \otimes (K^{1/2}))$ corresponds to an element $v'' \in H^1(X, \text{At}(V \otimes K_X^{-1/2}))$.

The map

$$\zeta : H^1(X, \text{At}(V)) \to H^1(X, \text{At}(V \otimes K_X^{-1/2})), \ v \mapsto v''$$

can be described in terms of the cocycles as follows. Let $\{U_i\}_{i \in I}$ be a covering of $X$ by open subsets, and let

$$\delta_{i,j} : U_{ij} = U_i \cap U_j \to \text{At}(V)|_{U_{ij}}, \ i, j \in I,$$

be a 1-cocycle giving an element $c \in H^1(X, \text{At}(V))$. So

$$\sigma \circ \delta_{i,j} : U_{ij} = U_i \cap U_j \to TU_{ij}, \ i, j \in I,$$

is a 1-cocycle with values in $TX$, where $\sigma$ is the projection in (4.4). Its cohomology class in $H^1(X, TX)$ coincides with the image of $c$ by the homomorphism in (4.5). For any holomorphic sections

$$v \in H^0(U_{ij}, V|_{U_{ij}}), \ \kappa \in H^0(U_{ij}, K_X^{-1/2}|_{U_{ij}}),$$

define $\Delta_{ij}(v \otimes \kappa) = \delta_{i,j}(v) \otimes \kappa + v \otimes L_{\sigma \circ \delta_{i,j}} \kappa$, where $L_{\sigma \circ \delta_{i,j}}$ is the Lie derivation by the vector field $\sigma \circ \delta_{i,j}$ (Lie derivation is defined in (4.11)); recall that $\delta_{i,j}$ is a first order holomorphic differential operator $V|_{U_{ij}} \to V|_{U_{ij}}$. It is straightforward to check that

$$\Delta_{ij}((f v) \otimes \kappa) = \Delta_{ij}(v \otimes (f \kappa))$$

for any holomorphic function $f$ on $U_{ij}$. Consequently, we have

$$\Delta_{ij} \in H^0(U_{ij}, \text{At}(V \otimes K_X^{-1/2})).$$

The homomorphism $\zeta$ takes the cohomology class $c$ of $\{\delta_{i,j}\}$ to the cohomology class of $\{\Delta_{ij}\}$ in $H^1(X, \text{At}(V \otimes K_X^{-1/2}))$. $\square$

Now take a triple $(X, V, D)$, where $X$ is a compact Riemann surface of genus $g$, $V$ is a holomorphic vector bundle on $X$, and $D$ is a holomorphic connection on $V$. We recall that holomorphic connections on $V$ are precisely the holomorphic splittings of the Atiyah exact sequence in (4.4) [1]. The connection $D$ on $V$ produces a holomorphic differential operator of order one

$$\tilde{D} : \text{At}(V) \to \text{End}(V) \otimes K_X$$

which is constructed as follows. Let $p_1 : \text{At}(V) \to \text{End}(V)$ be the holomorphic projection defined by the holomorphic splitting of (4.4) given by $D$. Let

$$D' : \text{End}(V) \to \text{End}(V) \otimes K_X$$

be the holomorphic connection on $\text{End}(V)$ induced by the connection $D$ on $V$. Then

$$\tilde{D} = D' \circ p_1.$$
Note that \( \bar{D} \) is not \( \mathcal{O}_X \)-linear; it is a differential operator of order one.

Let

\[
\mathcal{A}_* : \mathcal{A}_0 := \text{At}(V) \xrightarrow{\bar{D}} \mathcal{A}_1 := \text{End}(V) \otimes K_X
\]

be the two term complex of sheaves on \( X \), where \( \bar{D} \) is the differential operator in (4.12), and \( \mathcal{A}_i \) is at the \( i \)-th position. The infinitesimal deformations of the triple \( (X, V, D) \) are parametrized by the first hypercohomology

\[
\mathbb{H}^1(\mathcal{A}_*),
\]

where \( \mathcal{A}_* \) is the complex constructed in (4.13) [13, p. 1415, Proposition 4.4].

Consider the homomorphisms of complexes

\[
\begin{array}{c}
0 \\ \downarrow \\ \mathcal{A}_* : \text{At}(V) \xrightarrow{\bar{D}} \text{End}(V) \otimes K_X \\
\downarrow \text{id} \\
\text{At}(V) & \longrightarrow & 0
\end{array}
\]

It induces homomorphisms

\[
\mathbb{H}^1(\mathcal{A}_*) \longrightarrow H^1(X, \text{At}(V)) \text{ and } H^0(X, \text{End}(V) \otimes K_X) \longrightarrow \mathbb{H}^1(\mathcal{A}_*).
\]

The first homomorphism corresponds to the forgetful map that sends an infinitesimal deformation of the triple \( (X, V, D) \) to the infinitesimal deformation of the pair \( (X, V) \) obtained from it by simply forgetting the connection. The second homomorphism corresponds to the map that sends an infinitesimal deformation of the connection \( D \) to the infinitesimal deformation of the triple \( (X, V, D) \) produced by it by keeping the pair \( (X, V) \) fixed.

### 5 Affine space structure

As before, \( F \) is a stable vector bundle on \( X \) of rank \( r \) and degree zero. Set \( V = F \otimes K_X^{-1/2} \), and consider the vector space

\[
H^0 \left( X, \frac{J^1(F \otimes K_X^{-1/2}) \otimes F^* \otimes K_X^{1/2}}{\text{ad}(F \otimes K_X^{-1/2}) \otimes K_X} \right)
\]

constructed in (4.8). Recall the quotient space \( \mathcal{D}(F) \) in (2.34).

**Proposition 5.1** The space \( \mathcal{D}(F) \) in (2.34) is an affine space modeled on the above vector space

\[
H^0 \left( X, \frac{J^1(F \otimes K_X^{-1/2}) \otimes F^* \otimes K_X^{1/2}}{\text{ad}(F \otimes K_X^{-1/2}) \otimes K_X} \right).
\]

**Proof** Using the natural inclusion map \( \iota \) in (2.23), we have

\[
\text{ad}(F \otimes K_X^{-1/2}) \otimes K_X^{\otimes 2} \hookrightarrow \text{End}(F \otimes K_X^{-1/2}) \otimes K_X^{\otimes 2}
\]

\[
= \text{End}(F) \otimes K_X^{\otimes 2}
\]

\[
= (F \otimes K_X^{1/2}) \otimes (F^* \otimes K_X^{3/2}) \xrightarrow{\iota \otimes \text{Id}} J^1(F \otimes K_X^{-1/2}) \otimes F^* \otimes K_X^{3/2}
\]
Consequently, the vector space in (5.1) coincides with the quotient space
\[
0 \longrightarrow \ad(F \otimes K_X^{-1/2}) \otimes K_X^2 \longrightarrow J^1(F \otimes K_X^{-1/2}) \otimes F^* \otimes K_X^{3/2} \longrightarrow J^1(F \otimes K_X^{-1/2}) \otimes F^* \otimes K_X^{1/2} \otimes K_X \longrightarrow 0.
\]
(5.2)

Since \( F \) is stable, we have
\[
H^1(X, \ad(F \otimes K_X^{-1/2}) \otimes K_X^2) = H^0(X, \ad(F) \otimes TX)^* = 0
\]
because \( \ad(F) \otimes TX \) is polystable of negative degree (recall that \( g \geq 2 \)). Therefore, the long exact sequence of cohomologies associated to this short exact sequence in (5.2) gives a short exact sequence
\[
0 \longrightarrow H^0(X, \ad(F \otimes K_X^{-1/2}) \otimes K_X^2) \longrightarrow H^0(X, J^1(F \otimes K_X^{-1/2}) \otimes F^* \otimes K_X^{3/2}) \longrightarrow H^0(X, \ad(F) \otimes K_X^2) \longrightarrow 0.
\]
(5.3)

Consequently, the vector space in (5.1) coincides with the quotient space
\[
Q(F) =: \frac{H^0(X, J^1(F \otimes K_X^{-1/2}) \otimes F^* \otimes K_X^{3/2})}{H^0(X, \ad(F) \otimes K_X^2)} = \frac{H^0(X, J^1(F \otimes K_X^{-1/2}) \otimes F^* \otimes K_X^{3/2})}{H^0(X, \ad(F) \otimes K_X^2)}.
\]
(5.4)

Now, we have a natural inclusion map
\[
H^0(X, J^1(F \otimes K_X^{-1/2}) \otimes F^* \otimes K_X^{3/2}) \cong H^0(X, \Hom(F \otimes K_X^{-2/2}, J^1(F \otimes K_X^{-1/2}) \otimes K_X)) \cong H^0(X, \Hom(J^1(F \otimes K_X^{-1/2}), J^1(F \otimes K_X^{-1/2}) \otimes K_X))
\]
(5.5)

that sends any homomorphism \( \eta : F \otimes K_X^{-1/2} \longrightarrow J^1(F \otimes K_X^{-1/2}) \otimes K_X \) over \( X \) to
\[
\eta \circ q_0 : J^1(F \otimes K_X^{-1/2}) \longrightarrow J^1(F \otimes K_X^{-1/2}) \otimes K_X,
\]
where \( q_0 \) is the projection in (2.23). Using the inclusion map in (5.5), the natural action of the vector space \( H^0(X, \Hom(J^1(F \otimes K_X^{-1/2}), J^1(F \otimes K_X^{-1/2}) \otimes K_X)) \) on the affine space \( \Conn(J^1(F \otimes K_X^{-1/2})) \) (defined in (2.19)) restricts to an action of \( H^0(X, J^1(F \otimes K_X^{-1/2}) \otimes F^* \otimes K_X^{3/2}) \) on \( \Conn(J^1(F \otimes K_X^{-1/2})) \).

Recall that Corollary 3.2 says that \( \Conn(J^1(F) \otimes K_X^{-1/2}) = \tilde{\mathcal{C}}(F) \). Therefore, the vector space \( H^0(X, J^1(F \otimes K_X^{-1/2}) \otimes F^* \otimes K_X^{3/2}) \) acts on \( \tilde{\mathcal{C}}(F) \).

It can be shown that the above action of \( H^0(X, J^1(F \otimes K_X^{-1/2}) \otimes F^* \otimes K_X^{3/2}) \) on \( \tilde{\mathcal{C}}(F) \) produces an action of \( H^0(X, J^1(F \otimes K_X^{-1/2}) \otimes F^* \otimes K_X^{3/2}) \) on the quotient space \( \mathcal{C}(F) \) in Definition 2.8. Indeed, the action of \( H^0(X, J^1(F \otimes K_X^{-1/2}) \otimes F^* \otimes K_X^{3/2}) \) on \( \tilde{\mathcal{C}}(F) \), and also the quotient map \( \tilde{\mathcal{C}}(F) \longrightarrow \mathcal{C}(F) \), are both constructed using the action
of $H^0(X, \text{End}(J^1(F \otimes K_X^{-1/2})) \otimes K_X)$ on $\tilde{C}(F)$; as $S = F \otimes K_X^{1/2}$ (see (2.22)), we have $\text{End}(S) = \text{End}(F) = \text{End}(F \otimes K_X^{-1/2})$. Since the group $H^0(X, \text{End}(J^1(F \otimes K_X^{-1/2})) \otimes K_X)$ is abelian, the two actions on $\tilde{C}(F)$ commute, and hence we get an action of $H^0(X, J^1(F \otimes K_X^{-1/2}) \otimes F^* \otimes K_X^{3/2})$ on $C(F)$.

For the above action of $H^0(X, J^1(F \otimes K_X^{-1/2}) \otimes F^* \otimes K_X^{3/2})$ on $C(F)$, it is evident that the subspace

\[ H^0(X, \text{ad}(F) \otimes K_X^{3/2}) = H^0(X, \text{ad}(F \otimes K_X^{-1/2}) \otimes K_X^{3/2}) \subset H^0(X, J^1(F \otimes K_X^{-1/2}) \otimes F^* \otimes K_X^{3/2}) \]

in (5.3) acts trivially. Indeed, this follows from the fact that $C(F)$ is the quotient of $\tilde{C}(F)$ by the action of $H^0(X, \text{ad}(F) \otimes K_X^{3/2})$. Thus we have an action on $C(F)$ of the quotient space $Q(F)$ in (5.4). This action of $Q(F)$ on $C(F)$ is free, because

- the action of $H^0(X, \text{Hom}(J^1(F \otimes K_X^{-1/2}), J^1(F \otimes K_X^{-1/2}) \otimes K_X))$ on $\text{Conn}(J^1(F \otimes K_X^{-1/2}))$ is free, and
- $Q(F)$ and $C(F)$ are constructed by quotienting with the same subspace, namely

\[ H^0(X, \text{ad}(F) \otimes K_X^{3/2}) \subset H^0(X, J^1(F \otimes K_X^{-1/2}) \otimes F^* \otimes K_X^{3/2}) \]

(see (5.4) and Definition 2.8).

Finally from the structure of $\text{Aut}(J^1(F \otimes K_X^{-1/2}))$ shown in (2.33) it follows that the above action of $Q(F)$ on $C(F)$ produces an action of $Q(F)$ on the quotient space $D(F)$ in (2.34).

Using Lemma 3.3 it follows that the above action of $Q(F)$ on $D(F)$ is transitive. This action is also free. Indeed, this follows from (2.33) and the above observation that the action of $Q(F)$ on $C(F)$ is free.

We noted earlier that $Q(F)$ is identified with the vector space in (5.1). Therefore, the above action of $Q(F)$ on $D(F)$ produces an action, on $D(F)$, of the vector space in (5.1). Now the proposition follows from the above properties of the action of $Q(F)$ on $D(F)$. \[\square\]

Proposition 5.1 and Lemma 4.1 together give the following:

**Corollary 5.2** The space $D(F)$ is an affine space modeled on the complex vector space $H^1(X, \text{At}(F))^*$. 

**Proof** From Lemma 4.1 we know that $H^1(X, \text{At}(F))^* = H^1(X, \text{At}(F \otimes K_X^{-1/2}))^*$. From (4.8) it follows that

\[
H^1(X, \text{At}(F \otimes K_X^{-1/2}))^* = H^0\left(X, \left(\frac{J^1(F \otimes K_X^{-1/2}) \otimes F^* \otimes K_X^{1/2}}{\text{ad}(F \otimes K_X^{-1/2}) \otimes K_X}\right) \otimes K_X\right).
\]

Now Proposition 5.1 completes the proof. \[\square\]

**Remark 5.3** The affine space structure $D(F)$, in Corollary 5.2, for the vector space $H^1(X, \text{At}(F))^*$ is canonical in the sense that it does not involve making any choice. As a consequence this pointwise affine space structure extends to any smooth algebraic family of smooth projective curves.

Let $\mathcal{M}_g^0$ denote the moduli space of irreducible smooth complex projective curves of genus $g$, with $g \geq 2$, equipped with a theta characteristic. It is a smooth orbifold of complex
dimension $3g - 3$. We note that $\mathcal{M}_g^0$ is not connected; the loci of curves with an odd theta characteristic and curves with an even theta characteristic are disconnected. For any fixed $r \geq 1$, let
\[ \beta : B_g(r) \rightarrow \mathcal{M}_g^0 \]  
be the moduli space of triples of the form $(X, K_X^{1/2}, F)$, where

- $X$ is a compact connected Riemann surface of genus $g$,  
- $K_X^{1/2}$ is a theta characteristic on $X$, and  
- $F$ is a stable vector bundle on $X$ of rank $r$ and degree zero.

The map $\beta$ in (5.6) sends any $(X, K_X^{1/2}, F)$ to $(X, K_X^{1/2})$ by forgetting $F$. The moduli space $B_g(r)$ is a smooth orbifold of complex dimension $3g + r^2(g - 1) - 2$. Note that $B_g(r)$ is not connected as $\mathcal{M}_g^0$ is not connected. Let
\[ \gamma : \mathcal{H}_g(r) \rightarrow B_g(r) \]  
be the moduli space of quadruples of the form $(X, K_X^{1/2}, F, D)$, where

- $X$ is a compact connected Riemann surface of genus $g$,  
- $K_X^{1/2}$ is a theta characteristic on $X$,  
- $F$ is a stable vector bundle on $X$ of rank $r$ and degree zero, and  
- $D \in \mathcal{D}(F)$ (see (2.34)).

The projection $\gamma$ in (5.7) sends any $(X, K_X^{1/2}, F, D)$ to $(X, K_X^{1/2}, F)$ by simply forgetting $D$. The moduli space $\mathcal{H}_g(r)$ is a smooth orbifold of complex dimension $2(3g + r^2(g - 1) - 2)$.

**Theorem 5.4** The algebraic fiber bundle $\gamma : \mathcal{H}_g(r) \rightarrow B_g(r)$ in (5.7) is an algebraic affine bundle modeled on the holomorphic cotangent bundle $T^*B_g(r)$. In other words, $\mathcal{H}_g(r)$ is an algebraic torsor over $B_g(r)$ for $T^*B(r)$.

**Proof** From Proposition 2.4 and Corollary 3.2 we know that $\gamma$ is surjective.

Since the space of infinitesimal deformations of a pair $(X, K_X^{1/2}, F) \in B_g(r)$ are parametrized by $H^1(X, \text{At}(F))$ (see (4.5) and (4.10)), we conclude that
\[ T^*_{(X, K_X^{1/2}, F)} B_g(r) = H^1(X, \text{At}(F))^* . \]

Therefore, the theorem follows from Corollary 5.2. As mentioned in Remark 5.3, the canonical nature of the action of $H^1(X, \text{At}(F))^*$ on $\mathcal{D}(F)$ in Corollary 5.2 ensures that it extends to any given family of Riemann surfaces equipped with a theta characteristic and a stable vector bundle of rank $r$ and degree zero. \qed

On $B_g(r)$ there is a natural reduced divisor $\Theta$ which is defined as follows:
\[ \Theta := \{(X, K_X^{1/2}, F) \in B_g(r) \mid H^0(X, F \otimes K_X^{1/2}) \neq 0\} \subset B_g(r) . \]  

Since $\chi(X, F \otimes K_X^{1/2}) = 0$ (Riemann–Roch theorem), it follows that $H^0(X, F \otimes K_X^{1/2}) \neq 0$ if and only if $H^1(X, F \otimes K_X^{1/2}) \neq 0$. For any $(X, K_X^{1/2}, F) \in \Theta$, we have
\[ (X, K_X^{1/2}, F^*) \in \Theta , \]  

because $H^0(X, F^* \otimes K_X^{1/2}) = H^1(X, F \otimes K_X^{1/2})^* \neq 0$ by Serre duality.
6 Differential operators, integral kernels and $r$-opers

6.1 Differential operators and $r$-opers

As before, take any $\langle X, K_X^{1/2}, F \rangle \in B_{s}(r)$ (see (5.6)). Let $D$ be an algebraic connection on $E := J^{1}(F \otimes K_X^{-1/2})$; it exists by Proposition 2.4. We know that the triple $(E, F \otimes K_X^{1/2}, D)$ is a $r$–oper (see Corollary 3.2 and Proposition 2.4), where $F \otimes K_X^{1/2}$ is considered as a subbundle of $E$ using $i$ in (2.23). Consider the automorphism

$$\varphi : E \rightarrow J^{1}(F \otimes K_X^{-1/2}) \sim J^{1}(F \otimes K_X^{-1/2}) = E$$

constructed in (2.7). It should be clarified that the construction of $\varphi$ uses $D$ in an essential way; so $\varphi$ need not be the identity map of $E$. Imitating the construction of $\varphi$ in (2.7) we will construct a homomorphism

$$\varphi_2 : E \rightarrow J^{1}(F \otimes K_X^{-1/2}) \rightarrow J^{2}(F \otimes K_X^{-1/2}).$$

To construct $\varphi_2$, take any $x \in X$ and any $v \in E_x$. As in the proof of Lemma 2.2, let $\hat{v}$ be the unique flat section of $E$ for the connection $D$, defined on a simply connected open neighborhood $U \subset X$ of $x$, such that $\hat{v}(x) = v$. So $q_0(\hat{v}) \in H^{0}(U, F \otimes K_X^{-1/2})$, where $q_0$ is the projection in (2.23). Now restrict $q_0(\hat{v})$ to the second order infinitesimal neighborhood of $x$; let $\hat{v}'_2 \in J^{2}(F \otimes K_X^{-1/2})$, be the element obtained this way from $q_0(\hat{v})$. The homomorphism $\varphi_2$ sends any $v \in E_x$, $x \in X$, to $\hat{v}'_2 \in J^{2}(F \otimes K_X^{-1/2})$, constructed above from it.

We describe an alternative construction of the map $\varphi_2$ in (6.2) along the line of the alternative construction of the map $\varphi$ in (2.7) in the proof of Lemma 2.2. Let $\mathcal{E} \subset E = J^{1}(F \otimes K_X^{-1/2})$ be the locally constant subsheaf corresponding to the integrable connection $D$. We have the composition of homomorphism of sheaves

$$\mathcal{E} \hookrightarrow J^{1}(F \otimes K_X^{-1/2}) \xrightarrow{q_0} F \otimes K_X^{-1/2} \xrightarrow{k} J^{2}(F \otimes K_X^{-1/2}),$$

where $q_0$ is the projection in (2.23), and the above map $F \otimes K_X^{-1/2} \rightarrow J^{2}(F \otimes K_X^{-1/2})$ is the natural homomorphism that sends a section of $F \otimes K_X^{-1/2}$ to its second jet (note that this map is not $O_X$–linear). Tensoring the above homomorphism $\mathcal{E} \rightarrow J^{2}(F \otimes K_X^{-1/2})$ with $O_X$, and using the natural $O_X$–module structure of $J^{2}(F \otimes K_X^{-1/2})$ together with the fact that $\mathcal{E} \otimes O_X = E$, we get $\varphi_2$ in (6.2).

Let

$$0 \rightarrow F \otimes K_X^{1/2} \otimes K_X^{2} \xrightarrow{i} J^{2}(F \otimes K_X^{-1/2}) \xrightarrow{q_0} J^{1}(F \otimes K_X^{-1/2}) \rightarrow 0$$

be the canonical short exact sequence of jet bundles. The homomorphism

$$\varphi_2 \circ \varphi^{-1} : J^{1}(F \otimes K_X^{-1/2}) \rightarrow J^{2}(F \otimes K_X^{-1/2}),$$

where $\varphi_2$ and $\varphi$ are the homomorphisms in (6.2) and (6.1) respectively, satisfies the equation

$$q_2 \circ (\varphi_2 \circ \varphi^{-1}) = \text{Id}_{J^{1}(F \otimes K_X^{-1/2})},$$

which is satisfied in (6.4).
where \( q_2 \) is the projection in (6.3). In other words, \( \varphi_2 \circ \varphi^{-1} \) gives a holomorphic splitting of the short exact sequence in (6.3). Therefore, there is a unique holomorphic homomorphism

\[
\Delta_D : J^2(F \otimes K_X^{-1/2}) \longrightarrow F \otimes K_X^{3/2}
\]

(6.5)
such that

- kernel(\( \Delta_D \)) = image(\( \varphi_2 \circ \varphi^{-1} \)) = image(\( \varphi_2 \)), and
- \( \Delta_D \circ \iota_2 = \text{Id}_{F \otimes K_X^{3/2}} \), where \( \iota_2 \) is the homomorphism in (6.3).

We note that the homomorphism \( \Delta_D \) in (6.5) defines a holomorphic differential operator of order two

\[
\Delta_D \in H^0(X, \text{Diff}_X^2(F \otimes K_X^{-1/2}, F \otimes K_X^{3/2}))
\]

(6.6)
from \( F \otimes K_X^{-1/2} \) to \( F \otimes K_X^{3/2} \). The symbol of any holomorphic differential operator of order two from \( F \otimes K_X^{-1/2} \) to \( F \otimes K_X^{3/2} \) is a holomorphic section of

\[
\text{Hom}(F \otimes K_X^{-1/2}, F \otimes K_X^{3/2}) \otimes (TX)^{\otimes 2} = \text{End}(F).
\]

From the above equation \( \Delta_D \circ \iota_2 = \text{Id}_{F \otimes K_X^{3/2}} \) it follows immediately that the symbol of the differential operator \( \Delta_D \) in (6.6) is actually \( \text{Id}_{F} \in H^0(X, \text{End}(F)) \).

For any \( A \in H^0(X, \text{Diff}_X^2(F \otimes K_X^{-1/2}, F \otimes K_X^{3/2})) \) and

\[
B \in H^0(X, \text{Diff}_X^3(F \otimes K_X^{-1/2}, F \otimes K_X^{3/2}))
\]

\[
= H^0(X, \text{Hom}(F \otimes K_X^{-1/2}, F \otimes K_X^{3/2})) = H^0(X, \text{End}(F) \otimes K_X^2),
\]

we have

\[
A + B \in H^0(X, \text{Diff}_X^2(F \otimes K_X^{-1/2}, F \otimes K_X^{3/2})).
\]

Also, the symbol of \( A + B \) coincides with the symbol of \( A \), because \( B \) is a lower order differential operator.

Two holomorphic differential operators \( A_1, A_2 \in H^0(X, \text{Diff}_X^2(F \otimes K_X^{-1/2}, F \otimes K_X^{3/2})) \) will be called equivalent if

\[
A_1 - A_2 \in H^0(X, \text{ad}(F) \otimes K_X^2) \subset H^0(X, \text{End}(F) \otimes K_X^2).
\]

(6.7)

**Definition 6.1** The space of all equivalence classes of differential operators

\[
A \in H^0(X, \text{Diff}_X^2(F \otimes K_X^{-1/2}, F \otimes K_X^{3/2}))
\]
such that the symbol of \( A \) is \( \text{Id}_F \in H^0(X, \text{End}(F)) \) will be denoted by \( \tilde{\mathcal{D}}(X, K_X^{1/2}, F) \).

The above construction of \( \Delta_D \) in (6.6) from \( D \) gives the following:

**Lemma 6.2** For any \( (X, K_X^{1/2}, F) \in \mathcal{B}_X(r) \), there is a natural map

\[
\Psi : \mathcal{D}(F) \longrightarrow \tilde{\mathcal{D}}(X, K_X^{1/2}, F),
\]

where \( \mathcal{D}(F) \) and \( \tilde{\mathcal{D}}(X, K_X^{1/2}, F) \) are constructed in (2.34) and Definition 6.1 respectively.
Proof  Let $\widetilde{D}'(X, K_X^{1/2}, F)$ denote the space of all holomorphic differential operators $A \in H^0(X, \text{Diff}_X^2(F \otimes K_X^{-1/2}, F \otimes K_X^{3/2}))$ such that the symbol of $A$ is $\text{Id}_F \in H^0(X, \text{End}(F))$. The construction of $\Delta_D$ in (6.6) from $D$ clearly gives a map

$$\Psi' : \tilde{\mathcal{C}}(F) \longrightarrow \widetilde{D}'(X, K_X^{1/2}, F)$$

(see Definition 2.7). Now consider the quotient space $\tilde{\mathcal{C}}(F)$ of $\tilde{\mathcal{C}}(F)$ in Definition 2.8. Take two element $D_1, D_2 \in \tilde{\mathcal{C}}(F)$ that give the same element of $\tilde{\mathcal{C}}(F)$. It is straightforward to check that the differential operators $\Psi'(D_1)$ and $\Psi'(D_2)$ are equivalent. Therefore, $\Psi'$ gives a map

$$\Psi'' : \tilde{\mathcal{C}}(F) \longrightarrow \tilde{\mathcal{D}}(X, K_X^{1/2}, F).$$

The map $\Psi''$ clearly factors through the quotient $\mathcal{D}(F)$ of $\tilde{\mathcal{C}}(F)$ in (2.34). Hence $\Psi''$ produces a map $\Psi$ as in the statement of the lemma. $\square$

We will construct an inverse of the map $\Psi$ in Lemma 6.2.

For a holomorphic vector bundle $W$ on $X$, there is a canonical commutative diagram of homomorphisms

$$
\begin{array}{cccccc}
0 & \rightarrow & W \otimes K_X^2 & \xrightarrow{\iota_2} & J^2(W) & \xrightarrow{q_2} & J^1(W) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow b & & \downarrow & & \\
0 & \rightarrow & J^1(W) \otimes K_X & \xrightarrow{\iota'} & J^1(J^1(W)) & \xrightarrow{q'} & J^1(W) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
W \otimes K_X & = & W \otimes K_X & & & & & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & & & & & \\
\end{array}
$$

(6.8)

where the rows and columns are exact; the top (respectively, bottom) row is the one as in (6.3) (respectively, (2.6)), while the left column is (2.6) tensored with $K_X$. The map $b$ is tautological; it follows from the definition of jet bundles. Now set

$$W = F \otimes K_X^{-1/2}$$

in (6.8), where $F$ is a stable vector bundle of rank $r$ and degree zero on $X$, and $K_X^{1/2}$ is a theta characteristic on $X$. Let

$$\Delta_0 : J^2(F \otimes K_X^{-1/2}) \longrightarrow F \otimes K_X^{-1/2} \otimes K_X^2 = F \otimes K_X^{3/2}$$

be a holomorphic homomorphism such that

$$\Delta_0 \circ \iota_2 = \text{Id}_{F \otimes K_X^{3/2}},$$

(6.9)

where $\iota_2$ is the homomorphism in (6.8). In other words,

$$\Delta_0 \in H^0(X, \text{Diff}_X^2(F \otimes K_X^{-1/2}, F \otimes K_X^{3/2}))$$
and the symbol of $\Delta_0$ is $\text{Id}_F \in H^0(X, \text{End}(F))$ (it is equivalent to the equation in (6.9)). From (6.9) it follows that $\Delta_0$ gives a holomorphic splitting of the top row in (6.8). Consequently, there is a unique holomorphic homomorphism

$$D' : J^1(F \otimes K_X^{-1/2}) \longrightarrow J^2(F \otimes K_X^{-1/2})$$

such that $\Delta_0 \circ D' = 0$ and

$$q_2 \circ D' = \text{Id}_{J^1(F \otimes K_X^{-1/2})}, \quad (6.10)$$

where $q_2$ is the projection in (6.8).

Now consider the composition of homomorphisms

$$D := b \circ D' : J^1(F \otimes K_X^{-1/2}) \longrightarrow J^1(J^1(F \otimes K_X^{-1/2})),$$

where $b$ is the homomorphism in (6.8). Since the diagram in (6.8) is commutative, from (6.10) it follows that

$$q' \circ D = q' \circ b \circ D' = \text{Id}_{J^1(F \otimes K_X^{-1/2})} \circ q_2 \circ D' = \text{Id}_{J^1(F \otimes K_X^{-1/2})},$$

where $q'$ is the projection in (6.8). In other words, $D$ in (6.11) gives a holomorphic connection on the vector bundle $J^1(F \otimes K_X^{-1/2})$. The differential operator $J^1(F \otimes K_X^{-1/2}) \longrightarrow J^1(F \otimes K_X^{-1/2}) \otimes K_X$ for the connection $D$ is the unique $O_X$–linear homomorphism

$$\widehat{D} : J^1(J^1(F \otimes K_X^{-1/2})) \longrightarrow J^1(F \otimes K_X^{-1/2}) \otimes K_X$$

that satisfies the following conditions:

- kernel($\widehat{D}$) = image($D$), and
- $\widehat{D} \circ i' = \text{Id}_{J^1(F \otimes K_X^{-1/2})} \otimes K_X$ where $i'$ is the homomorphism in (6.8).

From Corollary 3.2 we know that any holomorphic connection on $J^1(F \otimes K_X^{-1/2})$ gives an element of $\mathcal{C}(F)$ (see Definition 2.8). Hence the holomorphic connection $D$ in (6.11) gives an element

$$D_{\Delta_0} \in \mathcal{D}(F), \quad (6.12)$$

where $\mathcal{D}(F)$ is the quotient of $\mathcal{C}(F)$ defined in (2.34).

**Lemma 6.3** For any $(X, K_X^{1/2}, F) \in \mathcal{B}_q(r)$ (see (5.6)), there is a natural map

$$\Phi : \widetilde{\mathcal{D}}(X, K_X^{1/2}, F) \longrightarrow \mathcal{D}(F),$$

where $\widetilde{\mathcal{D}}(X, K_X^{1/2}, F)$ and $\mathcal{D}(F)$ are constructed in Definition 6.1 and (2.34) respectively.

**Proof** The above construction of $D_{\Delta_0}$ (in (6.12)) from $\Delta_0$ produces a map

$$\Phi' : \widetilde{\mathcal{D}}'(X, K_X^{1/2}, F) \longrightarrow \mathcal{D}(F), \quad (6.13)$$

where $\widetilde{\mathcal{D}}'(X, K_X^{1/2}, F)$ is defined in the proof of Lemma 6.2.

Take $\Delta_1, \Delta_2 \in \widetilde{\mathcal{D}}'(X, K_X^{1/2}, F)$ such that $\Delta_1$ is equivalent to $\Delta_2$ (see (6.7)). Let $D_1$ (respectively, $D_2$) be the holomorphic connection on $J^1(F \otimes K_X^{-1/2})$ corresponding to $\Delta_1$ (respectively, $\Delta_2$); see (6.11). Since

$$\Delta_1 - \Delta_2 \in H^0(X, \text{ad}(F) \otimes K_X^2),$$
we have

\[ D_1 - D_2 = \Delta_1 - \Delta_2 \in H^0(X, \text{ad}(F) \otimes K_X^2). \]

Hence \( D_1 \) and \( D_2 \) give the same element of \( C(F) \) defined in Definition 2.8. This implies that \( \Phi' \) in (6.13) produces a map \( \Phi \) as in the statement of the lemma. 

\begin{proof}
In view of the explicit nature of the maps \( \Psi \) and \( \Phi \), this is a verification by straightforward computations. We omit the details.
\end{proof}

**Theorem 6.4** The two maps \( \Psi \) and \( \Phi \), constructed in Lemma 6.2 and Lemma 6.3 respectively, are inverses of each other.

**Proof** In view of the explicit nature of the maps \( \Psi \) and \( \Phi \), this is a verification by straightforward computations. We omit the details. 

\section*{6.2 Integral kernels and differential operators}

For \( i = 1, 2 \), let

\[ q_i : X \times X \rightarrow X \]

be the projection to the \( i \)-th factor. For holomorphic vector bundles \( A, B \) on \( X \), the holomorphic vector bundle \((q_1^* A) \otimes (q_2^* B)\) on \( X \times X \) will be denoted by \( A \boxtimes B \). Let

\[ \Delta := \{(x, x) \mid x \in X\} \subset X \times X \]

be the reduced diagonal divisor.

For holomorphic vector bundles \( A, B \) on \( X \), and a nonnegative integer \( d \), we will construct a torsion sheaf on \( X \times X \) supported on the divisor \((d + 1)\Delta \subset X \times X \). Let \( r_A \) and \( r_B \) be the ranks of \( A \) and \( B \) respectively.

Consider the holomorphic vector bundles

\[ A \boxtimes (B^* \otimes K_X) \quad \text{and} \quad A \boxtimes (B^* \otimes K_X) \otimes O_{X \times X}((d + 1)\Delta) \]

on \( X \times X \). We note that \( A \boxtimes (B^* \otimes K_X) \) is a subsheaf of \( A \boxtimes (B^* \otimes K_X) \otimes O_{X \times X}((d + 1)\Delta) \) because \((d + 1)\Delta \) is an effective divisor on \( X \times X \). So we have a short exact sequence of coherent sheaves on \( X \times X \)

\[ 0 \rightarrow A \boxtimes (B^* \otimes K_X) \rightarrow A \boxtimes (B^* \otimes K_X) \otimes O_{X \times X}((d + 1)\Delta) \rightarrow Q_d(A, B) := \frac{A \boxtimes (B^* \otimes K_X) \otimes O_{X \times X}((d + 1)\Delta)}{A \boxtimes (B^* \otimes K_X)} \rightarrow 0; \tag{6.14} \]

the support of the above quotient sheaf \( Q_d(A, B) \) in (6.14) is \((d + 1)\Delta \). The direct image

\[ K_d(A, B) := q_1_* Q_d(A, B) \tag{6.15} \]

is a holomorphic vector bundle on \( X \) of rank \( r_A r_B(d + 1) \). It is known that

\[ K_d(A, B) = \text{Hom}(J^d(B), A) = \text{Diff}^d_X(B, A), \tag{6.16} \]

where \( K_d(A, B) \) is constructed in (6.15) (see [19, Part 4, Ch. 6], [6, Sect. 2.1], [8, p. 25, (5.1)], [7, Sect. 3.1, p. 1314]).
For $d \geq 1$, the sheaf $Q_d(A, B)$ in (6.14) fits in the following short exact sequence of sheaves on $X \times X$

$$0 \longrightarrow Q_{d-1}(A, B) := \frac{A \boxtimes (B^* \otimes K_X) \otimes \mathcal{O}_{X \times X}(d \Delta)}{A \boxtimes (B^* \otimes K_X)} \longrightarrow Q_d(A, B)$$

$$:= \frac{A \boxtimes (B^* \otimes K_X) \otimes \mathcal{O}_{X \times X}((d + 1) \Delta)}{A \boxtimes (B^* \otimes K_X)} \longrightarrow \frac{A \boxtimes (B^* \otimes K_X) \otimes \mathcal{O}_{X \times X}(d \Delta)}{A \boxtimes (B^* \otimes K_X)} \longrightarrow 0. \quad (6.17)$$

The above sheaf $\frac{A \boxtimes (B^* \otimes K_X) \otimes \mathcal{O}_{X \times X}((d + 1) \Delta)}{A \boxtimes (B^* \otimes K_X)}$ is supported on the reduced divisor $\Delta$. Taking direct image of the short exact sequence in (6.17) by the projection $q_1$ we get the following short exact sequence holomorphic vector bundles on $X$

$$0 \longrightarrow q_1*Q_{d-1}(A, B) = K_{d-1}(A, B) \longrightarrow q_1*Q_d(A, B) = K_d(A, B)$$

$$\xrightarrow{p_0} q_1* \left( \frac{A \boxtimes (B^* \otimes K_X) \otimes \mathcal{O}_{X \times X}((d + 1) \Delta)}{A \boxtimes (B^* \otimes K_X)} \right) \longrightarrow 0. \quad (6.18)$$

Poincaré adjunction formula says that $\mathcal{O}_{X \times X}(\Delta)\big|_\Delta$ is the normal bundle of $\Delta$ [18, p. 146]. So $\mathcal{O}_{X \times X}(\Delta)\big|_\Delta = TX$, using the identification of $\Delta$ with $X$ defined by $x \mapsto (x, x)$. Therefore, we have

$$q_1* \left( \frac{A \boxtimes (B^* \otimes K_X) \otimes \mathcal{O}_{X \times X}((d + 1) \Delta)}{A \boxtimes (B^* \otimes K_X)} \right) = \text{Hom}(B, A) \otimes (TX)^\otimes d.$$ 

Hence the isomorphism in (6.16) and the projection $p_0$ in (6.18) together produce a homomorphism

$$\text{Diff}^d_X(B, A) \longrightarrow \text{Hom}(B, A) \otimes (TX)^\otimes d.$$ 

The homomorphism of global sections corresponding to it

$$H^0(X, \text{Diff}^d_X(B, A)) \longrightarrow H^0(X, \text{Hom}(B, A) \otimes (TX)^\otimes d) \quad (6.19)$$

is the symbol map on the global differential operators.

Let $F$ be a stable vector bundle on $X$ of rank $r$ and degree zero. Using the isomorphism in (6.16), the space of holomorphic differential operators $H^0(X, \text{Diff}^2_X(F \otimes K_X^{-1/2}, F \otimes K_X^{3/2}))$ has the following isomorphism:

$$H^0(X, \text{Diff}^2_X(F \otimes K_X^{-1/2}, F \otimes K_X^{3/2})) \overset{\sim}{\longrightarrow} H^0(3\Delta, (F \otimes K_X^{3/2}) \boxtimes (F^* \otimes K_X^{3/2}) \otimes \mathcal{O}_{X \times X}(3\Delta))\big|_{3\Delta}. \quad (6.20)$$

Now, for any

$$s \in H^0(3\Delta, (F \otimes K_X^{3/2}) \boxtimes (F^* \otimes K_X^{3/2}) \otimes \mathcal{O}_{X \times X}(3\Delta))\big|_{3\Delta},$$

let

$$s_0 := s\big|_{\Delta} \in H^0(\Delta, (F \boxtimes F)\big|_{\Delta}) = H^0(X, \text{End}(F)) \quad (6.21)$$

be the restriction of it to $\Delta \subset 3\Delta$; note that $((K_X^{3/2} \boxtimes K_X^{3/2}) \otimes \mathcal{O}_{X \times X}(3\Delta))\big|_{\Delta}$ is canonically trivialized, because $\mathcal{O}_{X \times X}(\Delta)\big|_{\Delta} = TX$ (after identifying $\Delta$ with $X$). Then $s_0$ coincides with the symbol of the differential operator

$$D_x \in H^0(X, \text{Diff}^2_X(F \otimes K_X^{-1/2}, F \otimes K_X^{3/2})).$$
corresponding to the above section \( s \) (see (6.19) and (6.20)).

Consider the natural short exact sequence

\[
0 \longrightarrow K_{\Delta}^{\otimes 2} = K_X^{\otimes 2} \longrightarrow O_{3\Delta} \longrightarrow O_{2\Delta} \longrightarrow 0.
\]

Tensoring it with \((F \otimes K_X^{3/2}) \boxtimes (F^* \otimes K_X^{3/2}) \otimes O_{X \times X} (3\Delta)\), and then taking global sections, we see that the vector space

\[
H^0(\Delta, ((F \otimes K_X^{3/2}) \boxtimes (F^* \otimes K_X^{3/2}) \otimes O_{X \times X} (3\Delta))|_{3\Delta} \otimes K_{\Delta}^{\otimes 2})
\]

is a subspace of \(H^0(3\Delta, ((F \otimes K_X^{3/2}) \boxtimes (F^* \otimes K_X^{3/2}) \otimes O_{X \times X} (3\Delta))|_{3\Delta})\). Take two sections

\[
s, t \in H^0(3\Delta, ((F \otimes K_X^{3/2}) \boxtimes (F^* \otimes K_X^{3/2}) \otimes O_{X \times X} (3\Delta))|_{3\Delta}).
\]

Then the corresponding differential operators (see (6.20))

\[
D_s, D_t \in H^0(X, \text{Diff}^2_X (F \otimes K_X^{-1/2}, F \otimes K_X^{3/2}))
\]

are equivalent (see (6.7)) if and only if

\[
s - t \in H^0(X, \text{ad}(F) \otimes K_X^{\otimes 2}) \subset H^0(X, \text{End}(F) \otimes K_X^{\otimes 2}).
\]

Consequently, we obtain the following description of \(\tilde{D}(X, K_X^{1/2}, F)\) (see Definition 6.1) in terms of integral kernels: \(\tilde{D}(X, K_X^{1/2}, F)\) is identified with the quotient of

\[
\{s \in H^0(3\Delta, ((F \otimes K_X^{3/2}) \boxtimes (F^* \otimes K_X^{3/2}) \otimes O_{X \times X} (3\Delta))|_{3\Delta} \mid s|_{\Delta} = H^0(X, \text{End}(F))\}
\]

by the subspace \(H^0(X, \text{ad}(F) \otimes K_X^{\otimes 2})\) of it. Using this description of \(\tilde{D}(X, K_X^{1/2}, F)\), Theorem 6.4 gives the following description of \(\mathcal{H}_g(r)\) in terms of the integral kernels.

**Corollary 6.5** The moduli space \(\mathcal{H}_g(r)\) in (5.7) is identified with the space of quadruples of the form \((X, K_X^{1/2}, F, s)\), where \((X, K_X^{1/2}, F) \in \mathcal{B}_g(r)\) (see (5.6)) and

\[
s \in \frac{H^0(3\Delta, ((F \otimes K_X^{3/2}) \boxtimes (F^* \otimes K_X^{3/2}) \otimes O_{X \times X} (3\Delta))|_{3\Delta}}{H^0(X, \text{ad}(F) \otimes K_X^{\otimes 2})}
\]

such that \(s_0 := s|_{\Delta} = \text{Id}_F\) (see (6.21)).

### 7 A canonical holomorphic section

Recall the projection \(\gamma\) in (5.7) and the effective divisor \(\Theta\) in (5.8). In this section we we will construct a holomorphic map on the complement of the theta divisor

\[
\mathbb{S} : \mathcal{B}_g(r) \setminus \Theta \longrightarrow \mathcal{H}_g(r)
\]

such that \(\gamma \circ \mathbb{S} = \text{Id}_{\mathcal{B}_g(r) \setminus \Theta}\).

Take any \((X, K_X^{1/2}, F) \in \mathcal{B}_g(r) \setminus \Theta\). Since

\[
H^0(X, F \otimes K_X^{1/2}) = 0 = H^1(X, F \otimes K_X^{1/2}),
\]

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it follows that
\[ H^0(X \times X, (F \otimes K_X^{1/2}) \otimes (F^* \otimes K_X^{1/2}) \otimes O_{X \times X} (\Delta)) = H^0(X, \text{End}(F \otimes K_X^{1/2})) = \text{End}(F) \]
(7.1)
(see [9, Remark 2.4], [10, p. 8]). As in [10, (3.6)], let
\[ \beta_F \in H^0(X \times X, (F \otimes K_X^{1/2}) \otimes (F^* \otimes K_X^{1/2}) \otimes O_{X \times X} (\Delta)) \]
be the section that corresponds to \( \text{Id}_F \in H^0(X, \text{End}(F)) \) by the isomorphism in (7.1). The line bundle over \( 2\Delta \)
\[ K_X^{1/2} \otimes K_X^{1/2} \otimes O_{X \times X} (\Delta)|_{2 \Delta} \rightarrow 2\Delta \]
has a natural trivialization given by a section
\[ \sigma_0 \in H^0(2\Delta, K_X^{1/2} \otimes K_X^{1/2} \otimes O_{X \times X} (\Delta)|_{2 \Delta}) \]
(7.3)
[10, (3.8)], [12, p. 688, Theorem 2.2]. Let
\[ \hat{\beta}_F \in H^0(2\Delta, (F \otimes F^*)|_{2 \Delta}) \]
be the section defined by the equation \( \beta_F|_{2 \Delta} = \hat{\beta}_F \otimes \sigma_0 \), where \( \beta_F \) is the section in (7.2) [10, (3.9)]. The restriction of \( \hat{\beta}_F \) to \( \Delta \subset 2\Delta \) is \( \text{Id}_F \) [10, (3.10)]; the holomorphic connection on \( F \) given by \( \hat{\beta}_F \) will be denoted by \( \hat{\beta}_F \).

Any holomorphic connection on \( X \) is integrable, because \( \Omega^2_{X,0} = 0 \). Therefore, using the above holomorphic connection \( \hat{\beta}_F \), for any simply connected open subset \( U \subset X \), the restriction \( F|_{U} \) is canonically identified with the trivialized holomorphic vector bundle \( F_{x_0} \times U \rightarrow U \), for any point \( x_0 \in U \). More precisely, this identification of vector bundles is constructed by taking parallel translations of \( F_{x_0} \), for the integrable connection \( \hat{\beta}_F \), along paths originating from \( x_0 \). Consequently, the two holomorphic vector bundles \( q_1^* F \) and \( q_2^* F \) are holomorphically identified over an analytic neighborhood of \( \Delta \subset X \times X \). The restriction of this isomorphism to \( 2\Delta \) coincides with the isomorphism
\[ (q_1^* F)|_{2 \Delta} \sim (q_2^* F)|_{2 \Delta} \]
(7.5)
given by \( \hat{\beta}_F \) in (7.4). Therefore, we have an extension of the isomorphism in (7.5) to an isomorphism
\[ (q_1^* F)|_{m \Delta} \sim (q_2^* F)|_{m \Delta} \]
for every \( m \geq 2 \), which is given by \( \hat{\beta}_F \). In particular, we get an isomorphism
\[ f_3 : (q_1^* F)|_{3 \Delta} \sim (q_2^* F)|_{3 \Delta} \]
(7.6)
extending the isomorphism in (7.5).

Using the isomorphism \( f_3 \) in (7.6), the section \( \beta_F|_{3 \Delta} \) in (7.2) becomes a section
\[ \beta_{3,F} := \beta_F|_{3 \Delta} \in H^0(3\Delta, \left( (F \otimes F^* \otimes K_X^{1/2} \otimes K_X^{1/2}) \otimes O_{X \times X} (\Delta) \right)|_{3 \Delta}) \]
\[ = H^0(3\Delta, \left( (K_X^{1/2} \otimes K_X^{1/2}) \otimes q_1^* \text{End}(F) \otimes O_{X \times X} (\Delta) \right)|_{3 \Delta}) \] .

Now using the trace homomorphism
\[ q_1^* \text{End}(F) \rightarrow q_1^* O_X = O_{X \times X}, \ w \mapsto \frac{1}{r} \text{trace}(w) \]
the above section $\beta_{3,F}$ produces a section
\[ \sigma_1 \in H^0(3\Delta, (K_{X}^{1/2} \boxtimes K_{X}^{1/2} \otimes \mathcal{O}_{X \times X}(\Delta))|_{3\Delta}); \] (7.7)
the restriction of $\sigma_1$ to $2\Delta \subset 3\Delta$ coincides with the section $\sigma_0$ in (7.3).

Now define
\[ \gamma_F := (\beta_F)|_{3\Delta} \otimes (\sigma_1) \in H^0\left(3\Delta, \left(\left((F \boxtimes K_{X}^{1/2}) \boxtimes (F^* \otimes K_{X}^{1/2})\right) \otimes \mathcal{O}_{X \times X}(3\Delta)\right)|_{3\Delta}\right) \]
\[ = H^0\left(\begin{array}{l} X, q_1* \left(\left((F \boxtimes K_{X}^{1/2}) \boxtimes (F^* \otimes K_{X}^{1/2})\right) \otimes \mathcal{O}_{X \times X}(3\Delta)\right)|_{3\Delta}\right) \right) \] (7.8)
where $\beta_F$ and $\sigma_1$ are the sections constructed in (7.2) and (7.7) respectively; here the restriction of the projection $q_1$ to $3\Delta \subset X \times X$ is also denoted by $q_1$.

We have
\[ \left(\left((F \boxtimes K_{X}^{1/2}) \boxtimes (F^* \otimes K_{X}^{1/2})\right) \otimes \mathcal{O}_{X \times X}(3\Delta)\right)|_{3\Delta} \]
\[ = Q_2(F \boxtimes K_{X}^{1/2}, F \boxtimes K_{X}^{-1/2}) \]
(see (6.14)), which implies that
\[ q_1* \left(\left((F \boxtimes K_{X}^{1/2}) \boxtimes (F^* \otimes K_{X}^{1/2})\right) \otimes \mathcal{O}_{X \times X}(3\Delta)\right)|_{3\Delta} \]
\[ = K_2(F \boxtimes K_{X}^{1/2}, F \boxtimes K_{X}^{-1/2}) \]
(see (6.15)). Therefore, from (6.16) it follows that
\[ H^0\left(\begin{array}{l} X, q_1* \left(\left((F \boxtimes K_{X}^{1/2}) \boxtimes (F^* \otimes K_{X}^{1/2})\right) \otimes \mathcal{O}_{X \times X}(3\Delta)\right)|_{3\Delta}\right) \right) \]
\[ = H^0\left(\begin{array}{l} X, K_2(F \boxtimes K_{X}^{1/2}, F \boxtimes K_{X}^{-1/2})\right) \right) \]
\[ = H^0\left(\begin{array}{l} X, \text{Diff}_X^2(F \boxtimes K_{X}^{-1/2}, F \boxtimes K_{X}^{3/2})\right) \right) . \]

Using this isomorphism, the section $\gamma_F$ constructed in (7.8) gives a holomorphic differential operator
\[ \tilde{\gamma}_F \in H^0(X, \text{Diff}_X^2(F \boxtimes K_{X}^{-1/2}, F \boxtimes K_{X}^{3/2})). \] (7.9)

From the symbol homomorphism constructed in (6.19) it can be shown that the symbol of the differential operator $\tilde{\gamma}_F$ in (7.9) is
\[ \text{Id}_F \in H^0(X, \text{Hom}(F \boxtimes K_{X}^{-1/2}, F \boxtimes K_{X}^{3/2}) \otimes (TX)^{\otimes 2}) = H^0(X, \text{End}(F)) . \]

Indeed, in view of the construction of $\gamma_F$ in (7.8), this is a consequence of the following two facts:

- The restriction of the section $\tilde{\beta}_F$ in (7.4) to $\Delta \subset 2\Delta$ is $\text{Id}_F$, and
- the restriction of the section $\sigma_1$ in (7.7) to $\Delta \subset 3\Delta$ is the constant function 1 on $\Delta$ (recall that the restriction of $\sigma_1$ to $2\Delta \subset 3\Delta$ coincides with $\sigma_0$ in (7.3)).

Recall the space of differential operators $\tilde{D}'(X, K_{X}^{1/2}, F)$ defined in the proof of Lemma 6.2. Since the symbol of the differential operator $\tilde{\gamma}_F$ in (7.9) is $\text{Id}_F$, it is an element of $\tilde{D}'(X, K_{X}^{1/2}, F)$. Therefore, $\tilde{\gamma}_F$ gives an element of the quotient space $\tilde{D}(X, K_{X}^{1/2}, F)$ of $\tilde{D}'(X, K_{X}^{1/2}, F)$ (see Definition 6.1). Let
\[ \tilde{\gamma}'_F \in \tilde{D}(X, K_{X}^{1/2}, F) \]
(7.10)
be the element given by $\tilde{\gamma}_F$.

The above construction is summarized in the following lemma.

**Lemma 7.1** There is a natural holomorphic map
\[ \mathbb{S} : B_g(r) \setminus \Theta \longrightarrow \mathcal{H}_g(r) \]
such that $\gamma \circ \mathbb{S} = \text{Id}_{B_g(r) \setminus \Theta}$, where $\gamma$ is the projection in (5.7).
Proof Take any \((X, K_X^{1/2}, F) \in B_g(r)\setminus \Theta\). From Theorem 6.4 we know that
\[(X, K_X^{1/2}, F, \hat{\gamma}_F^r) \in \mathcal{H}_g(r),\]
where \(\hat{\gamma}_F^r\) is constructed from \((X, K_X^{1/2}, F)\) in (7.10). Let
\[\mathcal{S} : B_g(r) \setminus \Theta \longrightarrow \mathcal{H}_g(r)\]
be the map that sends any \((X, K_X^{1/2}, F) \in B_g(r)\setminus \Theta\) to \((X, K_X^{1/2}, F, \hat{\gamma}_F^r)\). It is evident that \(\gamma \circ \mathcal{S} = \text{Id}_{B_g(r)\setminus \Theta}\).

The above map \(\mathcal{S}\) can also be described using Corollary 6.5.

8 A canonical isomorphism of torsors and a symplectic structure

8.1 A canonical isomorphism of torsors

Let
\[\mathcal{L} := \mathcal{O}_{B_g(r)}(\Theta) \longrightarrow B_g(r)\]
be the line bundle corresponding to the reduced divisor \(\Theta\) in (5.8). Consider the short exact sequence
\[0 \longrightarrow \mathcal{O}_{B_g(r)} = \text{Diff}^0_X(\mathcal{L}, \mathcal{L}) \longrightarrow \text{At}(\mathcal{L}) := \text{Diff}^1_X(\mathcal{L}, \mathcal{L}) \longrightarrow T^*B_g(r) \longrightarrow 0,\]
where \(\zeta\) is the symbol map. We note that (8.2) coincides with the Atiyah exact sequence for the line bundle \(\mathcal{L}\) in (8.1) (see (4.4)). Let
\[0 \longrightarrow T^*B_g(r) \longrightarrow \text{At}(\mathcal{L})^* \longrightarrow \mathcal{O}_{B_g(r)} \longrightarrow 0\]
be the dual of the sequence in (8.2). Define
\[\mathcal{C}(\mathcal{L}) := \eta^{-1}(\text{image}(\mathbf{1})) \subset \text{At}(\mathcal{L})^*,\]
where \(\eta\) is the projection in (8.3), and \(\mathbf{1}\) is the section of \(\mathcal{O}_{B_g(r)}\) given by the constant function 1 on \(B_g(r)\). Let
\[\rho : \mathcal{C}(\mathcal{L}) \longrightarrow B_g(r)\]
be the restriction to \(\mathcal{C}(\mathcal{L})\) of the natural projection \(\text{At}(\mathcal{L})^* \longrightarrow B_g(r)\). From (8.3) it follows that \(\mathcal{C}(\mathcal{L})\) is an algebraic torsor over \(B_g(r)\) for the holomorphic cotangent bundle \(T^*B_g(r)\). Giving a holomorphic section of \(\mathcal{C}(\mathcal{L})\) over an open subset \(U \subset B_g(r)\) is equivalent to giving a holomorphic connection on the line bundle \(\mathcal{L}|_U \longrightarrow U\).

Recall from Theorem 5.4 the algebraic torsor \(\gamma : \mathcal{H}_g(r) \longrightarrow B_g(r)\) for the holomorphic cotangent bundle \(T^*B_g(r)\). Using Lemma 7.1 it can be deduced that the two torsors \(\mathcal{C}(\mathcal{L})\) (constructed in (8.5)) and \(\mathcal{H}_g(r)\), over \(B_g(r)\) for the holomorphic cotangent bundle \(T^*B_g(r)\), are naturally identified when restricted to the open subset \(B_g(r)\setminus \Theta \subset B_g(r)\), where \(\Theta\) is the divisor in (5.8). To explain this, define
\[\mathcal{H}_g^0(r) := \gamma^{-1}(B_g(r)\setminus \Theta) \subset \mathcal{H}_g(r)\]
\[\mathcal{C}(\mathcal{L})^0 := \rho^{-1}(B_g(r)\setminus \Theta) \subset \mathcal{C}(\mathcal{L})\]
where \(\gamma\) and \(\rho\) are the projections in (5.7) and (8.5) respectively. Since the restriction of \(\mathcal{L}\) to \(B_g(r)\setminus \Theta \subset B_g(r)\) is the trivial bundle \(\mathcal{O}_{B_g(r)\setminus \Theta}\), there is a natural integrable algebraic
connection $\nabla^0$ on the restriction of $\mathcal{L}$ to $B_g(r) \setminus \Theta$ which is given by the de Rham differential $d$ on $\mathcal{O}_{B_g(r)\setminus\Theta}$. This connection $\nabla^0$ produces an algebraic splitting of the Atiyah exact sequence for $\mathcal{L}|_{B_g(r)\setminus\Theta}$ (see (8.2)), which in turn produces an algebraic splitting, over $B_g(r) \setminus \Theta$, of the short exact sequence in (8.3). Hence we get an algebraic section

$$G : B_g(r) \setminus \Theta \longrightarrow \mathcal{C}(\mathcal{L})^0$$

(8.8)

of the bundle $\rho : \mathcal{C}(\mathcal{L})^0 \longrightarrow B_g(r) \setminus \Theta$ in (8.7). To explain the construction of $G$, if

$$s_0 : \mathcal{O}_{B_g(r)\setminus\Theta} \longrightarrow \text{At}(\mathcal{L})^*|_{B_g(r)\setminus\Theta}$$

is the splitting homomorphism for (8.3) over $B_g(r) \setminus \Theta$, then $G(z) = s_0(1(z))$ for all $z \in B_g(r) \setminus \Theta$, where $1$ as before is the section of $\mathcal{O}_{B_g(r)\setminus\Theta}$ given by the constant function $1$ on $B_g(r) \setminus \Theta$.

Consider $\mathcal{H}_g^0(r)$ and $\mathcal{C}(\mathcal{L})^0$ constructed in (8.6) and (8.7) respectively. We have a map

$$H : \mathcal{H}_g^0(r) \longrightarrow \mathcal{C}(\mathcal{L})^0$$

(8.9)

that sends any $v \in \mathcal{H}_g^0(r)$ to

$$G(\gamma(v)) + (v - S(\gamma(v)))$$

(8.10)

where $S$, $G$ and $\gamma$ are constructed in Lemma 7.1, (8.8) and (5.7) respectively; note that $v - S(\gamma(v))$ in (8.10) is an element of $T^*_{\gamma(v)}B_g(r)$ (recall that $\mathcal{H}_g(r)$ is a torsor for $T^*B_g(r)$ by Theorem 5.4), so the sum in (8.10) is an element of $\mathcal{C}(\mathcal{L})^0$ because $\mathcal{C}(\mathcal{L})^0$ is a torsor over $B_g(r) \setminus \Theta$ for $T^*(B_g(r) \setminus \Theta)$ (see (8.4) and (8.7)).

It is evident that $H$ in (8.9) is an algebraic isomorphism of torsors over $B_g(r) \setminus \Theta$ for $T^*(B_g(r) \setminus \Theta)$.

**Theorem 8.1** *The algebraic isomorphism $H$ in (8.9), of torsors over $B_g(r) \setminus \Theta$ for $T^*(B_g(r) \setminus \Theta)$, extends to an algebraic isomorphism of $T^*B_g(r)$-torsors

$$\mathcal{H} : \mathcal{H}_g(r) \longrightarrow \mathcal{C}(\mathcal{L})$$

over entire $B_g(r)$.*

**Proof** Consider the submersion $\beta$ in (5.6). The kernel of the differential of $\beta$

$$d\beta : TB_g(r) \longrightarrow \beta^*TM_g^\theta$$

will be denoted by $T_\beta$; in other words, $T_\beta$ is the relative tangent bundle for the projection $\beta$. So we have the short exact sequence of vector bundles

$$0 \longrightarrow \beta^*TM_g^\theta \stackrel{(d\beta)^*}{\longrightarrow} T^*B_g(r) \longrightarrow T_\beta^* := (T_\beta)^* \longrightarrow 0$$

over $B_g(r)$. Let $\mathcal{W} \longrightarrow B_g(r)$ be an algebraic torsor over $B_g(r)$ for $T^*B_g(r)$. Then

$$\mathcal{W}/(\beta^*TM_g^\theta) \longrightarrow B_g(r)$$

(8.11)

is an algebraic torsor over $B_g(r)$ for the vector bundle $T_\beta^*$.

Substitute the two $T^*B_g(r)$-torsors $\mathcal{H}_g(r)$ and $\mathcal{C}(\mathcal{L})$ in place of the above $T^*B_g(r)$-torsor $\mathcal{W}$. The construction in (8.11) produces two $T_\beta^*$-torsors over $B_g(r)$ from $\mathcal{H}_g(r)$ and $\mathcal{C}(\mathcal{L})$; these two $T_\beta^*$-torsors will be denoted by $\hat{\mathcal{H}}_g(r)$ and $\hat{\mathcal{C}}(\mathcal{L})$ respectively. The restrictions of $\hat{\mathcal{H}}_g(r)$ and $\hat{\mathcal{C}}(\mathcal{L})$ to the Zariski open subset

$$B_g(r) \setminus \Theta \subset B_g(r)$$
will be denoted by \( \hat{\mathcal{H}}_g^0(r) \) and \( \hat{\mathcal{C}}(\mathcal{L})^0 \) respectively.

The isomorphism \( \hat{H} \) in (8.9) produces an algebraic isomorphism of \( T_{\beta}^* \)-torsors over \( B_g(r) \setminus \Theta \)
\[
\hat{H} : \hat{\mathcal{H}}_g^0(r) \longrightarrow \hat{\mathcal{C}}(\mathcal{L})^0.
\] (8.12)

For each point \( z \in \mathcal{M}_g^0 \), consider the restriction of the isomorphism \( \hat{H} \) in (8.12) to the complement \( \beta^{-1}(z) \setminus (\beta^{-1}(z) \cap \Theta) \), where \( \beta \) is the projection in (5.6). This restriction coincides with the isomorphism constructed in [10, Lemma 3.1]. Therefore, from [10, Corollary 4.5] we know that \( \hat{H} \) in (8.12) extends to an algebraic isomorphism of \( T_{\beta}^* \)-torsors
\[
\hat{H}' : \hat{\mathcal{H}}_g(r) \longrightarrow \hat{\mathcal{C}}(\mathcal{L}).
\] (8.13)

over entire \( B_g(r) \). Let
\[
\mathbb{G} := \{(y, \hat{H}'(y)) \mid y \in \hat{\mathcal{H}}_g(r)\} \subset \hat{\mathcal{H}}_g(r) \times B_g(r) \hat{\mathcal{C}}(\mathcal{L})
\] (8.14)
be the graph of the map \( \hat{H}' \) in (8.13).

Consider the natural quotient map
\[
\mathcal{H}_g(r) \times B_g(r) \mathcal{C}(\mathcal{L}) \xrightarrow{q_1} (\mathcal{H}_g(r)/(\beta^* T^* \mathcal{M}_g^0)) \times B_g(r) (\mathcal{C}(\mathcal{L})/(\beta^* T^* \mathcal{M}_g^0)) = \hat{\mathcal{H}}_g(r) \times B_g(r) \hat{\mathcal{C}}(\mathcal{L}).
\]
The inverse image
\[
\mathcal{G}_1 := q_1^{-1}(\mathbb{G}) \subset \mathcal{H}_g(r) \times B_g(r) \mathcal{C}(\mathcal{L}),
\]
where \( \mathcal{G} \) is constructed in (8.14), is an algebraic torsor over \( B_g(r) \) for the vector bundle \( \beta^* T^* \mathcal{M}_g^0 \otimes \mathbb{G}^2 \). Let
\[
\mathcal{G} := \mathcal{G}_1/(\beta^* T^* \mathcal{M}_g^0) \xrightarrow{q} B_g(r)
\] (8.15)
be the quotient for the diagonal action of \( \beta^* T^* \mathcal{M}_g^0 \); so for any \( z \in B_g(r) \), two elements \( (x, y) \) and \( (x', y') \) of the fiber \( \mathcal{G}_1 \) give the same element of the fiber \( \mathcal{G}_z \) if and only if there is an element \( v \in (\beta^* T^* \mathcal{M}_g^0)_z = T_{\beta(z)}^* \mathcal{M}_g^0 \) such that \( x' = x + v \) and \( y' = y + v \). Therefore, we have a map
\[
v : \mathcal{H}_g(r) \times B_g(r) \mathcal{G} \longrightarrow \mathcal{C}(\mathcal{L})
\]
that sends any \( (h, (x, y)) \), where \( h \in \mathcal{H}_g(r)_z \), \( (x, y) \in \mathcal{G}_z \) and \( z \in B_g(r) \), to
\[
y + (h - x) \in \mathcal{C}(\mathcal{L})_z
\]
(note that \( h - x \in T_x^* B_g(r) \)). Using this map \( v \), the space of holomorphic sections of \( \mathcal{G} \) over any open subset \( U \subset B_g(r) \) is identified with the space of holomorphic isomorphisms \( \mathcal{H}_g(r)|_U \longrightarrow \mathcal{C}(\mathcal{L})|_U \), of the torsor over \( U \) for \( T^* U \), that induce the isomorphism \( \hat{H}'|_U \) in (8.13) of \( T_{\beta}^*|_U \)-torsors.

We note that \( \mathcal{G} \) is an algebraic torsor over \( B_g(r) \) for the vector bundle \( \beta^* T^* \mathcal{M}_g^0 \), where \( \beta \) is the projection in (5.6), as follows: For any \( z \in B_g(r) \), take \( (x, y) \in \mathcal{G}_z \) and \( v \in (\beta^* T^* \mathcal{M}_g^0)_z = T_{\beta(z)}^* \mathcal{M}_g^0 \); then we have
\[
(x, y) + v = (x + v, y - v).
\]

The moduli space \( B_g(r) \) has an algebraic involution
\[
\mathcal{I}_B : B_g(r) \longrightarrow B_g(r), \quad (X, K_X^{1/2}, F) \longmapsto (X, K_X^{1/2}, F^*).
\] (8.16)
This involution $\mathcal{I}_B$ preserves the divisor $\Theta$ in (5.8) (see (5.9)). Hence the action $\mathbb{Z}/2\mathbb{Z}$ on $B_g(r)$ given by $\mathcal{I}_B$ lifts to the line bundle $\mathcal{L}$ in (8.1). Consequently, the involution $\mathcal{I}_B$ lifts to an involution

$$\mathcal{I}_C : \mathcal{C}(\mathcal{L}) \rightarrow \mathcal{C}(\mathcal{L})$$

(8.17)
of $\mathcal{C}(\mathcal{L})$.

We will now describe the relationship between $\mathcal{I}_C$ and the $T^*B_g(r)$-torsor structure of $\mathcal{C}(\mathcal{L})$. Take any $y \in B_g(r)$, $z \in \mathcal{C}(\mathcal{L})_y$ and $w \in T^*_y B_g(r)$. Then we have

$$\mathcal{I}_C(z + w) = \mathcal{I}_C(z) - (d\mathcal{I}_B)_{\mathcal{I}_B(y)}^*(w),$$

(8.18)

where $(d\mathcal{I}_B)_{\mathcal{I}_B(y)}^* : T^*_y B_g(r) \rightarrow T^*_{\mathcal{I}_B(y)} B_g(r)$ is the dual of the differential $(d\mathcal{I}_B)_{\mathcal{I}_B(y)} : T_{\mathcal{I}_B(y)} B_g(r) \rightarrow T_{\mathcal{I}_B(y)} B_g(r)$ of the map $\mathcal{I}_B$ at the point $\mathcal{I}_B(y)$.

Since $\bigwedge^2 J^1(K_X^{-1/2}) = K_X \otimes (K_X^{-1/2})^\otimes 2 = \mathcal{O}_X$ (see (2.6)), it follows that $J^1(K_X^{-1/2}) = J^1(K_X^{-1/2})^*$. Let $F$ be a stable vector bundle on $X$ of rank $r$ and degree zero. Since $F$ admits a holomorphic connection (see the proof of Proposition 2.4), from Proposition 2.5 we conclude that

$$J^1(F^* \otimes K_X^{-1/2}) = F^* \otimes J^1(K_X^{-1/2}) = F^* \otimes J^1(K_X^{-1/2})^* = J^1(F \otimes K_X^{-1/2})^*.$$  

Fixing an isomorphism of $J^1(F^* \otimes K_X^{-1/2})$ with $J^1(F \otimes K_X^{-1/2})^*$ we conclude that any holomorphic connection on $J^1(F \otimes K_X^{-1/2})$ produces a holomorphic connection on $J^1(F^* \otimes K_X^{-1/2})^* = J^1(F^* \otimes K_X^{-1/2})$. It is straightforward to check that this produces a bijection

$$M_F : \mathcal{D}(F) \rightarrow \mathcal{D}(F^*),$$

where $\mathcal{D}(F)$ is constructed in (2.34). This map $M_F$ does not depend on the choice of the isomorphism of $J^1(F^* \otimes K_X^{-1/2})$ with $J^1(F \otimes K_X^{-1/2})^*$; this is because in the construction of $\mathcal{D}(F)$ quotienting by Aut$(F \otimes K_X^{-1/2})$ was executed. Now we have an algebraic involution

$$\mathcal{I}_H : \mathcal{H}_g(r) \rightarrow \mathcal{H}_g(r), \ (X, K_X^{1/2}, F, D) \mapsto (X, K_X^{1/2}, F^*, M_F(D)),$$

(8.19)

where $M_F$ is the map constructed above. It is evident that

$$\gamma \circ \mathcal{I}_H = \mathcal{I}_B \circ \gamma,$$

where $\gamma$ and $\mathcal{I}_B$ are the maps in (5.7) and (8.16) respectively.

We will describe the relationship between $\mathcal{I}_H$ and the $T^*B_g(r)$-torsor structure of $\mathcal{H}_g(r)$. Take any $y \in B_g(r)$, $z \in \mathcal{H}_g(r)_y$ and $w \in T^*_y B_g(r)$. Then we have

$$\mathcal{I}_H(z + w) = \mathcal{I}_H(z) - (d\mathcal{I}_B)_{\mathcal{I}_B(y)}^*(w),$$

(8.20)

where $(d\mathcal{I}_B)_{\mathcal{I}_B(y)}^*$ is the homomorphism in (8.18).

We note that the projection $\beta$ in (5.6) satisfies the equation

$$\beta \circ \mathcal{I}_B = \beta.$$

In view of this, from (8.18) and (8.20) we conclude the following:

1. The involution $\mathcal{I}_B$ lifts to an involution

$$\mathcal{I}_G : \mathcal{G} \rightarrow \mathcal{G}$$

(8.21)
of $\mathcal{G}$ constructed in (8.15). For any $z \in B_g(r)$, take $(x, y) \in \mathcal{G}_z$; the map $\mathcal{I}_G$ sends $(x, y)$ to $(\mathcal{I}_H(x), \mathcal{I}_C(y))$, where $\mathcal{I}_H$ and $\mathcal{I}_C$ are the involutions in (8.19) and (8.17) respectively.
(2) For any $v \in (\beta^* T^* \mathcal{M}_g^0)_{\mathfrak{z}} = T^*_{\beta(\mathfrak{z})} \mathcal{M}_g^0$, 
\[ \mathcal{I}_G((x, y) + v) = \mathcal{I}_G((x, y)) + v; \quad (8.22) \]

recall that $\mathcal{G}$ is a torsor for $\beta^* T^* \mathcal{M}_g^0$.

Take any point 
\[ t := (X, K_X^{1/2}) \in \mathcal{M}_g^0. \]

Let 
\[ \mathcal{G}^t := \mathcal{G}|_{\beta^{-1}(t)} \xrightarrow{q'} \mathcal{B}^t := \beta^{-1}(t) \quad (8.23) \]

be the restriction, where $q'$ is the restriction of the projection $q$ in (8.15), and $\beta$ is the projection in (5.6). So $\mathcal{G}^t$ is an algebraic torsor over $\mathcal{B}^t$ for the trivial vector bundle 
\[ \mathcal{V}^t := \mathcal{B}^t \times T^*_t \mathcal{M}_g^0 \rightarrow \mathcal{B}^t \]

over $\mathcal{B}^t$ with fiber $T^*_t \mathcal{M}_g^0$; this is because $\mathcal{G}$ is a torsor for $\beta^* T^* \mathcal{M}_g^0$.

Let 
\[ \mathcal{I}^t_B := \mathcal{I}_B|_{\mathcal{B}^t} : \mathcal{B}^t \rightarrow \mathcal{B}^t \]

be the restriction of $\mathcal{I}_B$ in (8.16) to the subvariety $\mathcal{B}^t$ in (8.23).

We note that the isomorphism classes of algebraic $\mathcal{V}^t$–torsors over the variety $\mathcal{B}^t$ in (8.23) are parametrized by 
\[ H^1(\mathcal{B}^t, \mathcal{V}^t) = H^1(\mathcal{B}^t, \mathcal{O}_{\mathcal{B}^t}) \otimes T^*_t \mathcal{M}_g^0. \]

Let 
\[ c \in H^1(\mathcal{B}^t, \mathcal{O}_{\mathcal{B}^t}) \otimes T^*_t \mathcal{M}_g^0 \quad (8.24) \]

be the class of the $\mathcal{V}^t$–torsor $\mathcal{G}^t$ in (8.23). From (8.22) it follows immediately that 
\[ (\mathcal{I}^t_B)^* c = c. \quad (8.25) \]

Consider the determinant map 
\[ \widetilde{\delta} : \mathcal{B}^t \rightarrow \text{Pic}^0(X), \quad F \longmapsto \bigwedge^r F. \]

The corresponding homomorphism 
\[ \widetilde{\delta}^* : H^1(\text{Pic}^0(X), \mathcal{O}_{\text{Pic}^0(X)}) \rightarrow H^1(\mathcal{B}^t, \mathcal{O}_{\mathcal{B}^t}) \quad (8.26) \]

is an isomorphism. Indeed, Pic($\mathcal{B}^t$) = Pic(Pic$^0(X)$) $\oplus \mathbb{Z}$ [14, p. 57, Theorem D] when $r \geq 2$ and $(g, r) \neq (2, 2)$ (when $g = 2 = r$, the quotient Pic($\mathcal{B}^t$)/Pic(Pic$^0(X)$) is a finite group), so using the exact sequence of cohomologies 
\[ H^1(Y, 2\pi \sqrt{-1} \cdot \mathbb{Z}) \rightarrow H^1(Y, \mathcal{O}_Y) \rightarrow H^1(Y, \mathcal{O}_Y^+) \rightarrow H^2(Y, 2\pi \sqrt{-1} \cdot \mathbb{Z}) \]

associated to the exponential sequence 
\[ 0 \rightarrow 2\pi \sqrt{-1} \cdot \mathbb{Z} \rightarrow \mathcal{O}_Y \xrightarrow{\exp} \mathcal{O}_Y^+ \rightarrow 0 \]

on a complex variety $Y$ we conclude that $\widetilde{\delta}^*$ in (8.26) is an isomorphism. We also note that the involution 
\[ \text{Pic}^0(X) \rightarrow \text{Pic}^0(X), \quad \xi \mapsto \xi^* \]
acts on $H^1(\text{Pic}^0(X), \mathcal{O}_{\text{Pic}^0(X)})$ as multiplication by $-1$. In other words, no nonzero element of $H^1(\text{Pic}^0(X), \mathcal{O}_{\text{Pic}^0(X)})$ is fixed by this involution. In view of these, from (8.25) we conclude that

$$c = 0.$$  

So from (8.24) we conclude that the $\mathcal{V}'$–torsor $\mathcal{G}'$ in (8.23) is the trivial $\mathcal{V}'$–torsor

$$\mathcal{V}' = \mathcal{B}' \times T^*_g M^0_g \rightarrow \mathcal{B}'$$

Restrict $H$ (constructed in (8.9)) to $\mathcal{B}' \subset \mathcal{B}_g(r)$; denote this restriction by $H'$. From the above isomorphism of $\mathcal{G}'$ with the trivial $\mathcal{V}'$–torsor $\mathcal{V}'$ it follows that $H'$ is a meromorphic function on $\mathcal{B}'$ with values in the vector space $T^*_g M^0_g$ (recall that $\mathcal{V}' := \mathcal{B}' \times T^*_g M^0_g$). This meromorphic function is evidently regular on the complement $\mathcal{B}' \setminus (\mathcal{B}' \cap \Theta)$, where $\Theta$ is constructed in (5.8). From the construction of $H$ it is straightforward to deduce that $H'$ has a pole of order at most one on the divisor $\mathcal{B}' \cap \Theta \subset \mathcal{B}'$. On the other hand, we know that

$$H^0(\mathcal{B}', \mathcal{O}_{\mathcal{B}'}(\mathcal{B}' \cap \Theta)) = H^0(\mathcal{B}', \mathcal{O}_{\mathcal{B}'}) = \mathbb{C}$$

[3, p. 169, Theorem 2]. Consequently, the section $H'$ over $\mathcal{B}' \setminus (\mathcal{B}' \cap \Theta)$ extends to entire $\mathcal{B}'$ as a regular section. From this it follows immediately that the isomorphism $H$ of torsors over $\mathcal{B}_g(r) \setminus \Theta$ for $T^*(\mathcal{B}_g(r) \setminus \Theta)$, extends to an algebraic isomorphism of $T^*\mathcal{B}_g(r)$-torsors $\mathcal{H}_g(r) \rightarrow \mathcal{C}(\mathcal{L})$ over entire $\mathcal{B}_g(r)$. □

### 8.2 A holomorphic symplectic form

Recall the holomorphic line bundle $\mathcal{L} \rightarrow \mathcal{B}_g(r)$ in (8.1). The space $\mathcal{C}(\mathcal{L})$ in (8.5) has a canonical holomorphic symplectic structure. We will briefly recall the construction of this symplectic form on $\mathcal{C}(\mathcal{L})$.

From the construction of $\mathcal{C}(\mathcal{L})$ in (8.4) it follows immediately that there is a tautological holomorphic splitting

$$\rho^*\text{At}(\mathcal{L}) = \rho^*\mathcal{O}_{\mathcal{B}_g(r)} \oplus \rho^*\mathcal{T}\mathcal{B}_g(r) = \mathcal{O}_{\mathcal{C}(\mathcal{L})} \oplus \rho^*\mathcal{T}\mathcal{B}_g(r),$$

where $\rho$ is constructed in (8.5). This decomposition of $\rho^*\text{At}(\mathcal{L})$ gives a holomorphic projection

$$f_0 : \rho^*\text{At}(\mathcal{L}) \rightarrow \mathcal{O}_{\mathcal{C}(\mathcal{L})}.$$  \hspace{1cm} (8.27)

Let

$$0 \rightarrow \mathcal{O}_{\mathcal{C}(\mathcal{L})} \rightarrow \text{At}(\rho^*\mathcal{L}) \xrightarrow{\tilde{\xi}'} \mathcal{T}\text{Conn}(\mathcal{L}) \rightarrow 0$$  \hspace{1cm} (8.28)

be the Atiyah exact sequence for $\rho^*\mathcal{L}$ (see (4.4)). We also have a tautological projection

$$h_0 : \text{At}(\rho^*\mathcal{L}) \rightarrow \rho^*\text{At}(\mathcal{L})$$

such that the diagram

$$\begin{array}{ccc}
\text{At}(\rho^*\mathcal{L}) & \xrightarrow{h_0} & \rho^*\text{At}(\mathcal{L}) \\
\downarrow{\tilde{\xi}'} & & \downarrow{\rho^*\xi} \\
\mathcal{T}\text{Conn}(\mathcal{L}) & \xrightarrow{d\rho} & \rho^*\mathcal{T}\mathcal{B}_g(r)
\end{array}$$

is commutative, where $\xi$ and $\tilde{\xi}'$ are the projections in (8.2) and (8.28) respectively, and $d\rho$ is the differential of the projection $\rho$ in (8.5); see [11, (3.9)] [9, Sect. 3] for the construction.
of $h_0$. The composition of homomorphisms

$$f_0 \circ h_0 : \text{At}(\rho^* \mathcal{L}) \longrightarrow \mathcal{O}_\mathcal{L},$$

(8.29)

where $f_0$ is constructed in (8.27), gives a holomorphic splitting of the Atiyah exact sequence in (8.28). Hence $f_0 \circ h_0$ defines a holomorphic connection on $\rho^* \mathcal{L}$; see [11, Proposition 3.3].

The curvature $\Omega_L$ of the holomorphic connection $f_0 \circ h_0$ on $\rho^* \mathcal{L}$ in (8.29) is a closed algebraic 2–form on $\text{Conn}(\mathcal{L})$. This algebraic 2–form $\Omega_L$ is symplectic. (See [9, Sect. 3], [11].)

Recall the holomorphic $T^* B_g(r)$–torsor structure of $\text{Conn}(\mathcal{L})$. The above symplectic form $\Omega_{\mathcal{L}}$ is compatible with the $T^* B_g(r)$–torsor structure. This means that for any locally defined holomorphic section of the projection $\rho$ in (8.5)

$$B_g(r) \supset U \xrightarrow{s} \text{Conn}(\mathcal{L})$$

and any holomorphic 1–form $\omega \in H^0(U, T^*U)$, we have

$$s^* \Omega_{\mathcal{L}} + d\omega = (s + \omega)^* \Omega_{\mathcal{L}};$$

(8.30)

note that $y \mapsto s(y) + \omega(y)$ is a holomorphic section, over $U$, of the projection $\rho$ in (8.5).

**Corollary 8.2**

1. The moduli space $\mathcal{H}_g(r)$ in (5.7) has a canonical algebraic symplectic structure $\Omega_{\mathcal{H}_g(r)}$.
2. The symplectic form $\Omega_{\mathcal{H}_g(r)}$ on $\mathcal{H}_g(r)$ is compatible with the $T^* B_g(r)$–torsor structure of $\mathcal{H}_g(r)$ obtained in Theorem 5.4.
3. There is a holomorphic line bundle $\mathcal{L}$ on $\mathcal{H}_g(r)$ and a holomorphic connection $\nabla^\mathcal{L}$ on $\mathcal{L}$ such that the curvature of $\nabla^\mathcal{L}$ is the symplectic form $\Omega_{\mathcal{H}_g(r)}$.

**Proof** Using the isomorphism $\mathcal{H}$ in Theorem 8.1, the above algebraic symplectic form $\Omega_{\mathcal{L}}$ on $\text{Conn}(\mathcal{L})$ (see (8.30)) produces an algebraic symplectic form

$$\Omega_{\mathcal{H}_g(r)} := \mathcal{H}^* \Omega_{\mathcal{L}}$$

on $\mathcal{H}_g(r)$.

The compatibility condition in the statement (2) says that for any locally defined holomorphic section of the projection $\gamma$ in (5.7)

$$B_g(r) \supset U \xrightarrow{s} \mathcal{H}_g(r)$$

and any holomorphic 1–form $\omega \in H^0(U, T^*U)$, the equality

$$s^* \Omega_{\mathcal{H}_g(r)} + d\omega = (s + \omega)^* \Omega_{\mathcal{H}_g(r)}$$

(8.31)

holds; note that $y \mapsto s(y) + \omega(y)$ is a holomorphic section, over $U$, of the projection $\gamma$ in (5.7). Now, (8.31) follows immediately from (8.30).

We recall that $\Omega_{\mathcal{L}}$ is the curvature of the holomorphic connection $f_0 \circ h_0$ (see (8.29)) on the holomorphic line bundle $\rho^* \mathcal{L}$. Therefore, $\Omega_{\mathcal{H}_g(r)}$ is the curvature of the holomorphic connection

$$\nabla^\mathcal{L} := \mathcal{H}^*(f_0 \circ h_0)$$

on the holomorphic line bundle $\mathcal{L} := \mathcal{H}^* \rho^* \mathcal{L}$. □
9 Isomonodromy and symplectic form

9.1 Projective structure on a Riemann surface

Let \( Y \) be a compact connected Riemann surface. A holomorphic coordinate chart on \( Y \) is a pair of the form \((U, \varphi)\), where \( U \subset Y \) is an open subset and \( \varphi: U \rightarrow \mathbb{C}P^1 \) is a holomorphic embedding. A holomorphic coordinate atlas on \( Y \) is a collection of holomorphic coordinate charts \( \{(U_i, \varphi_i)\}_{i \in I} \) such that \( Y = \bigcup_{i \in I} U_i \). A projective structure on \( Y \) is given by a holomorphic coordinate atlas \( \{(U_i, \varphi_i)\}_{i \in I} \) satisfying the following condition: For every \( i, j \in J \times J \) with \( U_i \cap U_j \neq \emptyset \), and every connected component \( V_c \subset U_i \cap U_j \), there is a \( \tau^c_{j,i} \in \text{PSL}(2, \mathbb{C}) \) such that the map \( (\varphi_j \circ \varphi_i^{-1})|_{\varphi_i(V_c)} \) is the restriction, to \( \varphi_i(V_c) \), of the automorphism of \( \mathbb{C}P^1 \) given by \( \tau^c_{j,i} \). Recall that \( \text{PSL}(2, \mathbb{C}) = \text{Aut}(\mathbb{C}P^1) \).

Two holomorphic coordinate atlases \( \{(U_i, \varphi_i)\}_{i \in I} \) and \( \{(U_i, \varphi_i)\}_{i \in J'} \) satisfying the above condition are called equivalent if their union \( \{(U_i, \varphi_i)\}_{i \in I \cup J'} \) also satisfies the above condition. A projective structure on \( Y \) is an equivalence class of holomorphic coordinate atlases satisfying the above condition.

Giving a projective structure on \( Y \) is equivalent to giving a holomorphic Cartan geometry on \( Y \) for the pair of groups \( (\text{PSL}(2, \mathbb{C}), B) \), where \( B \subset \text{PSL}(2, \mathbb{C}) \) is the Borel subgroup

\[
B := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}(2, \mathbb{C}) \mid c = 0 \right\} ;
\]

see [24] for Cartan geometry. Let \( \tilde{B} \subset \text{SL}(2, \mathbb{C}) \) be the Borel subgroup that projects to \( B \) defined above. Giving a holomorphic Cartan geometry on \( Y \) for the pair of groups \( (\text{SL}(2, \mathbb{C}), \tilde{B}) \) is equivalent to giving a projective structure on \( Y \) together with a theta characteristic on \( Y \) [20].

For \( g \geq 2 \), let

\[ \delta: \mathcal{P}_g \rightarrow \mathcal{M}_g^0 \tag{9.1} \]

be the moduli space of triples of the form \((X, K_X^{1/2}, P)\), where

- \( X \) is a compact connected Riemann surface of genus \( g \),
- \( K_X^{1/2} \) is a theta characteristic on \( X \), and
- \( P \) is a projective structure on \( X \).

The map \( \delta \) in (9.1) sends any \((X, K_X^{1/2}, P)\) to \((X, K_X^{1/2})\).

**Proposition 9.1** The moduli space \( \mathcal{H}_g(r) \) in (5.7) admits a natural projection

\[ f: \mathcal{H}_g(r) \rightarrow \mathcal{P}_g , \]

where \( \mathcal{P}_g \) is defined in (9.1).

**Proof** Take any \((X, K_X^{1/2}, F, D) \in \mathcal{H}_g(r) \). Let

\[ s \in H^0\left(3\Delta, \left( (F \otimes K_X^{3/2}) \boxtimes (F^* \otimes K_X^{3/2}) \otimes \mathcal{O}_{X \times X}(3\Delta) \right)_{|3\Delta} \right) \]

be the element corresponding to it given by Corollary 6.5. Restricting \( s \) to \( 2\Delta \subset 3\Delta \), we get a section

\[
\begin{align*}
s_1 \in H^0\left(2\Delta, \left( (F \otimes K_X^{3/2}) \boxtimes (F^* \otimes K_X^{3/2}) \otimes \mathcal{O}_{X \times X}(3\Delta) \right)_{|2\Delta} \right) \\
= H^0\left(2\Delta, \left( (F \otimes F^*) \boxtimes ((K_X^{3/2} \boxtimes K_X^{3/2}) \otimes \mathcal{O}_{X \times X}(3\Delta)) \right)_{|2\Delta} \right) .
\end{align*}
\]
Since \(((K^{3/2}_X \boxtimes \tilde{K}^{3/2}_X) \otimes \mathcal{O}_{X \times X}(3\Delta))|_{2\Delta}\) is canonically trivialized \[12, p. 688, Theorem 2.2\] (this was noted in \(7.3\)), there is a unique section \[s'_1 \in H^0(2\Delta, (F \boxtimes F^*|_{2\Delta})\] such that \(s_1 = s'_1 \otimes t\), where \(t\) is the section of \(((K^{3/2}_X \boxtimes \tilde{K}^{3/2}_X) \otimes \mathcal{O}_{X \times X}(3\Delta))|_{2\Delta}\) that trivializes it. On the other hand, \(s_1|_{\Delta} = \text{Id}_F\) (see Corollary \(6.5\)), which implies that \(s'_1|_{\Delta} = \text{Id}_F\), because the restriction of \(t\) to \(\Delta\) is the constant function 1 on \(X\) using the identification of \(\Delta\) with \(X\) given by \(x \mapsto (x, x)\). Consequently, \(s'_1\) defines a holomorphic connection on \(F\). This holomorphic connection in turn provides an extension of the section \(s'_1\) to a section \[s_2 \in H^0(3\Delta, (F \boxtimes F^*)|_{3\Delta})\]; we note that using parallel translations, for the integrable connection on \(F\) defined by \(s'_1\), we get a holomorphic isomorphism between \(q_1^*F\) and \(q_2^*F\) over an analytic neighborhood of \(\Delta \subset X \times X\) (recall that \(q_i\) is the projection of \(X \times X\) to the \(i\)-th factor). Now, invoking the isomorphism \[s_2 : (q_1^*F)|_{3\Delta} \longrightarrow (q_2^*F)|_{3\Delta},\] the element \(s\) in \((9.2)\) becomes \[\frac{H^0(3\Delta, \left((F \otimes F^* \otimes K^{3/2}_X) \boxtimes (K^{3/2}_X) \otimes \mathcal{O}_{X \times X}(3\Delta)\right)|_{3\Delta})}{H^0(X, \text{ad}(F) \boxtimes \tilde{K}^{3/2}_X)}\]. Composing with the trace homomorphism \(F \otimes F^* = \text{End}(F) \longrightarrow \mathcal{O}_X\) defined by \(A \mapsto \frac{1}{r} \text{trace}(A)\), the above element \(s\) gives a section \[\phi_2 \in H^0(3\Delta, \left((K^{3/2}_X) \boxtimes K^{3/2}_X) \otimes \mathcal{O}_{X \times X}(3\Delta)\right)|_{3\Delta})\]. This section \(\phi_2\) defines a projective structure on \(X\) \[12, p. 688, Theorem 2.2\]. Therefore, we have a map \[f : \mathcal{H}_g(r) \longrightarrow \mathcal{P}_g\] that sends any \((X, K^{1/2}_X, F, D) \in \mathcal{H}_g(r)\) to \((X, K^{1/2}_X, \phi_2) \in \mathcal{P}_g\), where \(\phi_2\) is constructed above from \(D\). \(\square\)

### 9.2 Isomonodromy

For any compact Riemann surface \(X\) of genus \(g\), with \(g \geq 2\), let \(\mathcal{D}_X(r)\) denote the moduli space of pairs of the form \((F, D)\), where \(F\) is a stable vector bundle on \(X\) of rank \(r\) and degree zero, and \(D\) is a holomorphic connection on \(F\).

**Proposition 9.2** For the map \(f\) in Proposition 9.1, the fiber over any \((X, K^{1/2}_X, P) \in \mathcal{P}_g\) is canonically identified with the moduli space \(\mathcal{D}_X(r)\).

**Proof** Take any \((X, K^{1/2}_X, F, D) \in \mathcal{H}_g(r)\) in the fiber of \(f\) over \((X, K^{1/2}_X, P) \in \mathcal{P}_g\). As seen before, \(s'_1\) in \((9.3)\) defines a holomorphic connection on \(F\). So we get a map from the fiber of \(f\) over \((X, K^{1/2}_X, P)\) to \(\mathcal{D}_X(r)\) that sends any \((X, K^{1/2}_X, F, D)\) to \((F, s'_1)\).

Conversely, take any \((F, D) \in \mathcal{D}_X(r)\). Then \(D\) gives a section \[\widehat{s}_D \in H^0(3\Delta, (F \boxtimes F^*)|_{3\Delta})\].
On the other hand, the projective structure $P$ gives a section

$$P' \in H^0(3\Delta, ((K^{3/2}_X \otimes K^{3/2}_X) \otimes O_{X \times X}(3\Delta))|_{3\Delta})$$

[12, p. 688, Theorem 2.2]. So we have

$$\hat{s}_D \otimes P' \in H^0\left(3\Delta, \left((F \otimes K^{3/2}_X) \boxtimes (F^* \otimes K^{3/2}_X) \otimes O_{X \times X}(3\Delta)\right)|_{3\Delta}\right).$$

Consider the image of $\hat{s}_D \otimes P'$

$$\tilde{\hat{s}}_D \otimes P' \in H^0\left(3\Delta, \left((F \otimes K^{3/2}_X) \boxtimes (F^* \otimes K^{3/2}_X) \otimes O_{X \times X}(3\Delta)\right)|_{3\Delta}\right) / H^0(X, \text{ad}(F) \otimes K^{3/2}_X)$$

in the quotient space. Now $(X, K^{1/2}_X, F, \tilde{\hat{s}}_D \otimes P')$ gives an element of $\mathcal{H}_g(r)$ in the fiber of $f$ over $(X, K^{1/2}_X, P) \in \mathcal{P}_g$ (see Corollary 6.5).

In view of Proposition 9.2, the isomonodromy condition for integrable holomorphic connections defines a holomorphic foliation

$$\mathcal{F} \subset T\mathcal{H}_g(r)$$

which gives the following decomposition:

$$T\mathcal{H}_g(r) = \mathcal{F} \oplus \text{kernel}(df),$$

where $df : T\mathcal{H}_g(r) \rightarrow f^*TP_g$ is the differential of the projection $f$ in Proposition 9.1. Consequently, the differential $df$ identifies $\mathcal{F}$ with $f^*TP_g$.

We note that $\mathcal{D}_X(r)$ has a natural holomorphic symplectic form [2, 17]. Also, $\mathcal{P}_g$ has a holomorphic symplectic form which is constructed using the monodromy representation associated to any projective structure [2, 17, 21]. Therefore, using the decomposition in (9.5) we obtain two closed holomorphic 2-forms on $\mathcal{H}_g(r)$: one is given by the symplectic form on $\mathcal{D}_X(r)$ and the other is given by the symplectic form on $\mathcal{P}_g$.

We end with the following conjecture:

**Conjecture 9.3** The holomorphic symplectic form on $\mathcal{H}_g(r)$ in Corollary 8.2(1) is a constant linear combination of the above two holomorphic 2-forms on $\mathcal{H}_g(r)$.

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