Predictions from extra dimensions.

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**Abstract:** By representing the electroweak gauge symmetry group $SU(2) \times U(1)$ by a hypertorus $S_2 \times S_1$, the electroweak mixing angle and the fine structure constant are predicted. By representing neutrinos as oscillating spheroid perturbations on the same hypertorus, the number of neutrino families and the neutrino mixing matrix are predicted.

**Keywords:** Kaluza-Klein Theory (Higher-dimensional Gravity), Weinberg-Salam Electroweak Model, Beyond the Standard Model
1 Introduction

General relativity describes the gravitational force as being due to the curvature of space and time[1]. Quantum physics describes quantum forces as being due to the exchange of intermediate fields. How these two different descriptions of forces are combined at the quantum level has eluded physics. Can general relativity be applied to quantum physics to predict experimentally measured quantities?

2 Electroweak Mixing Angle

Measurements of the electroweak interaction have revealed the massless electromagnetic photon $A_\mu$ and the massive neutral weak boson $Z_\mu$ to be mixtures of the neutral hypercharge field $B_\mu$, and the neutral third component of the weak isospin field $W_{3\mu}$[2][3][4]

$$\begin{pmatrix} A_\mu \\ Z_\mu \end{pmatrix} = \begin{pmatrix} \cos \theta_W & \sin \theta_W \\ -\sin \theta_W & \cos \theta_W \end{pmatrix} \begin{pmatrix} B_\mu \\ W_{3\mu} \end{pmatrix}. \quad (2.1)$$

The standard model of particle physics offers no established explanation for the value of this Weinberg electroweak mixing angle $\theta_W$.

Gravity has been explained as due to the curvature of spacetime. The Kaluza-Klein model proposed that electromagnetism is due to the skewing of an extra spacelike dimension curved to a small radius with the coupling strength of electromagnetism inversely related to the curvature radius[5][6]. The model has been extended to extra gauge interactions by including extra dimensions with spatial symmetries matching the gauge symmetries[7]. Begin with the usual spacetime dimensions $x^\mu$ described by the spacetime metric $g_{\mu\nu}(x)$. Add extra dimensions $y^m$ described by extra dimensions...
metric \( h_{mn}(y) \), Killing vectors \( K_{am}(y) \), perturbed vector fields \( A_{a\mu}(x) \), and perturbed scalar fields \( S_{mn}(x) \)

\[
g_{\mu\nu}(x) \rightarrow \left[ \begin{array}{cc}
g_{\mu\nu}(x) + (h_{mn}(y) + S_{mn}(x)) & K_{em}(y) A_{e\mu}(x) \\
K_{bm}(y) A_{b\nu}(x) & h_{mn}(y) + S_{mn}(x)
\end{array} \right]
\]

(2.2)

Perhaps the mixing angle is specific to our universe, and other universes in the multiverse have different mixing angles and different interactions[8], or perhaps the mixing angle can offer insight into physics beyond the standard model. Can the mixing angle be predicted from physics beyond the standard model such as theories with extra dimensions?

### 3 Interactions and Geometries

The electroweak gauge group can be described by the gauge transformations of the product group \( SU(2) \times U(1) \). The one transformation of the group \( U(1) \) can be trivially described by the rotation of the one dimensional circular space \( S_1 \). The three transformations of the group \( SU(2) \) can be described by the three rotations of the one dimensional complex projective line \( CP_1 \), or the two dimensional real projective Riemann sphere \( P_2 \). The nonorientable Riemann sphere with constant curvature can be described by the orientable two dimensional real sphere \( S_2 \) but with its antipodes identified. The total integration of the local curvature is relevant. Double covering the Riemann sphere does not alter the predicted mixing angle.

Consider the closed three dimensional real space described by coordinates \( y_m = (y_1, y_2, y_3) \) constructed from a circle \( S_1 \) with radius \( r_1 \) and described by an circular angle \( 0 < y_1 < 2\pi \), around which is revolved a two dimensional surface of a sphere \( S_2 \) with radius \( r_2 \) with circle radius larger than sphere radius \( r_1 > r_2 \) described by latitude angle \( 0 < y_2 < \pi \) and longitude angle \( 0 < y_3 < 2\pi \) with the hypertorus metric

\[
h_{mn}(r_1, r_2) = \begin{pmatrix}
(r_1 - r_2 \sin y_2 \cos y_3)^2 & 0 & 0 \\
0 & (r_2)^2 & 0 \\
0 & 0 & (r_2 \sin y_2)^2
\end{pmatrix} 
\]

(3.1)

with the hypertorus Jacobian integration factor

\[
\sqrt{|h|} = (r_1 - r_2 \sin y_2 \cos y_3) r_2^2 \sin y_2.
\]

(3.2)

From the metric is computed the three dimensional spatial volume

\[
H(r_1, r_2) = \int_0^{2\pi} dy_1 \int_0^\pi dy_2 \int_0^{2\pi} dy_3 \sqrt{|h|}.
\]

(3.3)

The traditional Einstein-Hilbert action consists of a Ricci scalar term. The addition of a Ricci tensor squared term has been studied by others[9][10]. From the metric are computed the usual quantities of general relativity, Affine connection, Riemann tensor, and the Ricci curvature tensor \( R_m^n \) which describes the curvature of space[11]. Calculate the Ricci tensor squared

\[
R_m^n R^n_m = \frac{r_2^2 \sin y_2 (1 + \sin^4 y_2) (r_1 - 2 r_2 \sin y_2 \cos y_3)^2}{r_1 - r_2 \sin y_2 \cos y_3} + 4 \sin^3 y_2 \cos^2 y_3 (r_1 - r_2 \sin y_2 \cos y_3)^3.
\]

(3.4)
Integrate the Ricci tensor squared for the Lagrangian

$$L(r_1, r_2) = \frac{1}{16\pi G} \int_0^{2\pi} dy_1 \int_0^{\pi} dy_2 \int_0^{2\pi} dy_3 \sqrt{|h|} R^n_m R^m_n.$$ (3.5)

For this computation without fixed constant parameters, only the ratio of the two radii $r_1/r_2$ matters, not the magnitude of each radius. To find the extremal radii ratio of the ground state of the curved space, the extremal radii ratio is computed by holding constant the spatial volume while minimizing the Lagrangian using the Lagrange multiplier method

$$\frac{\partial H}{\partial r_1} \frac{\partial L}{\partial r_2} - \frac{\partial H}{\partial r_2} \frac{\partial L}{\partial r_1} = 0.$$ (3.6)

The extremal radii ratio for minimum curvature is computed

$$r_1/r_2 = 1.1808.$$(3.7)

The extremal radii ratio represents the ground state of the unperturbed space. The electroweak vector fields represent the perturbations of the ground state. Consider the mixing angle as a measure of the hypercharge circle radius and the weak sphere radius dependence on each other to perturbation. A vanishing mixing angle would indicate full independent nonmixing between hypercharge and weak third component. A unit tangent mixing angle would indicate an equal sharing between the two vector fields. The electroweak mixing angle is predicted to be

$$\sin^2 \theta_W^{\text{theory}} = \sin^2 (\arctan P) = 0.2324.$$ (3.8)

The electroweak mixing angle runs with the energy scale. The measured mixing angle is further complicated by renormalized quantum corrections and uncertainties of standard model parameters which slightly alter the angle. At the low energy scale, the electroweak mixing angle is experimentally measured to be[12]

$$\sin^2 \theta_W(m = 0)^{\text{experiment}} = 0.23867 \pm 0.00016.$$ (3.9)

At the energy scale of the $Z^0$ boson mass, the electroweak mixing angle with modified minimal subtraction is experimentally measured to be[13]

$$\sin^2 \theta_W(m = m_{Z^0})^{\text{experiment}} = 0.23122 \pm 0.00003.$$ (3.10)

This predicted electroweak mixing angle agrees with the measured electroweak mixing angle better than one percent.

4 Neutrino Flavors

Unlike spin and charge, neutrino flavor carries no conserved quantum numbers. Neutrinos $\psi_i$ interact in their flavor eigenstates $\nu_e, \nu_\mu, \nu_\tau$ and propagate in their mass eigenstates $\nu_1, \nu_2, \nu_3$ allowing neutrinos to oscillate among eigenstates described by the unitary matrix $U_{ij}$. No generally accepted explanation exists why the number of neutrino eigenstates equals three. Consider the Lagrangian neutrino kinetic term, weak interaction term, and mass term

$$\mathcal{L} = \bar{\psi}_i \gamma^\mu \left(1 - \gamma^5\right) \left[i \partial_\mu + g Z^0_{\mu} U_{ij}\right] \psi_j + m_i \bar{\psi}_i \psi_i$$ (4.1)
with spinor indices hidden. Extend the neutrino spinor field $\psi$ beyond 4-spacetime to include extra dimensions

$$ \psi(x^\mu) \rightarrow \psi(x^\mu, y^m). \quad (4.2) $$

Perturbations $p_i(y^m)$ in the extra dimensions can act independently of the 4-spacetime dimensions

$$ \psi_i(x^\mu, y^m) \rightarrow \psi_i(x^\mu, p_i(y^m)). \quad (4.3) $$

Let three extra dimensions form a hypertorus product space of a circle $S_1$ with coordinate $y_1$ and a sphere $S_2$ with coordinates $y_2, y_3$. Consider neutrinos as small perturbations of spheres oscillating between elongated prolate spheroids and flattened oblate spheroids. A sphere with three orthogonal rotational axes allows three independent orthogonal oscillating spheroids. These perturbations carry no conserved quantities such as linear or rotational momenta which would inhibit neutrino oscillations. The number of theoretically predicted independent perpendicular perturbations

$$ N_{\text{theory}} = 3 \quad (4.4) $$

agrees well with the number of the experimentally measured neutrino flavors

$$ N_{\text{experiment}} = 3 \quad (4.5) $$

although small numbers carry little statistical significance.

5 Oscillating Spheroid Perturbations

Let 4-spacetime be expanded with three extra spatial dimensional coordinates $y^m = (y^1, y^2, y^3)$, with the same hypertorus product space metric

$$ h_{mn} = \begin{pmatrix} (r_1 - r_2 \sin y_2 \cos y_3)^2 & 0 & 0 \\ 0 & (r_2)^2 & 0 \\ 0 & 0 & (r_2 \sin y_2)^2 \end{pmatrix} \quad (5.1) $$

with the same hypertorus product space Jacobian integration factor

$$ \sqrt{|h|} = (r_1 - r_2 \sin y_2 \cos y_3) r_2^2 \sin y_2. \quad (5.2) $$

Perturb the sphere radius

$$ r_2 \rightarrow r_2 + p_i \quad (5.3) $$

with three small amplitude, perpendicular spheroids $p_i(y^m)$ oscillating between elongated prolate and flattened oblate. To preserve the surface area of the perturbed oscillating spheroids, the expansion or contraction of the polar radius of the prolate spheroid compared with that of a unit amount in the oblate equatorial radius is about $q = 2$.\[14\]. To avoid creating electric charge from momentum around the circle, let the three perpendicular oscillating spheroids all travel in both directions around the
circle coordinate $y^1$ with the approximations
\[
p_1 = \left[ q \cos(r_2y_2) \cos^2 t - \sin(r_2y_2)(1 - \cos^2 t) \right] \times \left[ \sin(2r_1y_1 - t) + \sin(2r_1y_1 + t) \right]/2 \tag{5.4}
\]
\[
p_2 = \left[ q \cos(r_2(y_1 + y_3) + \phi_2) \sin(r_2y_2) \cos^2 t - \sin(r_2(y_1 + y_3) + \phi_2)(1 - \cos^2 t) \right] \times \left[ \sin(2r_1y_1 - t + \phi_2) + \sin(2r_1y_1 + t + \phi_2) \right]/2 \tag{5.5}
\]
\[
p_3 = \left[ q \sin(r_2(y_1 + y_3) + \phi_3) \sin(r_2y_2) \cos^2 t - \cos(r_2(y_1 + y_3) + \phi_3)(1 - \cos^2 t) \right] \times \left[ \sin(2r_1y_1 - t + \phi_3) + \sin(2r_1y_1 + t + \phi_3) \right]/2 . \tag{5.6}
\]

The perturbations have relative ring phase shifts $\phi_2, \phi_3$ to allow their maximums to be independently set which will be minimized. The overlapping perturbations will be integrated with their time independent perturbation counterparts
\[
p_1' = \left[ q \cos(r_2y_2) - \sin(r_2y_2) \right] \sin(2r_1y_1) \tag{5.7}
\]
\[
p_2' = \left[ q \cos(r_2(y_1 + y_3) + \phi_2) \sin(r_2y_2) - \sin(r_2(y_1 + y_3) + \phi_2) \right] \sin(2r_1y_1 + \phi_2) \tag{5.8}
\]
\[
p_3' = \left[ q \sin(r_2(y_1 + y_3) + \phi_3) \sin(r_2y_2) - \cos(r_2(y_1 + y_3) + \phi_3) \right] \sin(2r_1y_1 + \phi_3) . \tag{5.9}
\]

Provide uniform perturbations by integrating the perturbations for perturbation normalization factors
\[
a_t = \left( \int_0^{2\pi} dt \int_0^{2\pi} dy_1 \int_0^{2\pi} dy_2 \int_0^{2\pi} dy_3 \sqrt{|h|} |p_i p_i| \right)^{1/2} . \tag{5.10}
\]

The variations in the perturbation normalization factors around the region of interest are typically a few percent. The normalization factors have an insignificant effect on the result.

### 6 Neutrino Mixing Matrix

The Pontecorvo-Maki-Nakagawa-Sakata mixing matrix, which henceforth will be called the neutrino mixing matrix, describes the transition between neutrino eigenstates. No generally accepted explanation exists why the elements of the neutrino mixing matrix have the values they hold.\textsuperscript{[15]} Complicating the marriage between general relativity and quantum physics is that general relativity lives in real numbers, while quantum physics enjoys complex numbers. Although the neutrino mixing matrix is a complex-valued unitary matrix, a real-valued orthogonal matrix will be calculated. Perhaps the complexity is a small effect due to another physical mechanism. Can perturbations of oscillating spheroids explain the neutrino mixing matrix?

Imagine the set of three oscillating spheroids being the three mass eigenstates. Differentiate the extra dimension perturbations
\[
\partial p_i = -\partial y_i p_i + \partial y_2 p_i + \partial y_3 p_i \tag{6.1}
\]
\[
\partial p_i' = -\partial y_i p_i' + \partial y_2 p_i' + \partial y_3 p_i' . \tag{6.2}
\]

Substitute the perturbations and discard higher order terms $p_i' \partial p_j \to 0$
\[
(r_2 + p_i')(r_2 + \partial p_j) - r_2^2 + \text{other combinations} 
\approx (p_i + p_i' + p_j' + \partial p_i + \partial p_j + \partial p_i' + \partial p_j') r_2 . \tag{6.3}
\]
The sphere radius \( r_2 \) is a constant. Integrate the overlapping perturbations kinetic matrix

\[
K_{ij} = \frac{1}{(a_i a_j + a_j a_i)/2} \int_0^{2\pi} dt \int_0^{2\pi} dy_1 \int_0^\pi dy_2 \int_0^{2\pi} dy_3 \sqrt{|h|} \times \\
\times (p_i + p_j + p'_i + p'_j + \partial p_i + \partial p_j + \partial p'_i + \partial p'_j) r_2. 
\]

Find the relative ring phase shifts \( \phi_2, \phi_3 \) that minimize the overlapping kinetic matrix squared \( K_{ij}K^{ji} \). A minimum of relative ring phase shifts is found near

\[
\phi_2 = -0.43\pi, \quad \phi_3 = +0.53\pi. 
\]

Compute the phase-shifted overlap of the oscillating spheroids. Normalize the overlapping kinetic matrix with its first element \( |K_{11}| = 1 \)

\[
\frac{K_{ij}}{|K_{11}|} = \begin{pmatrix}
-1.000 & -0.403 & -0.436 \\
-0.403 & -0.573 & 0.176 \\
-0.436 & 0.176 & 0.143
\end{pmatrix}. 
\]

Although this matrix is symmetric, it is not unitary which a good neutrino mixing matrix needs to be. Use QR decomposition with the Gram Schmidt method to transform the kinetic matrix \( K_{ij} \) through a series of approximations into an orthogonal neutrino mixing matrix \( U_{ij} \)

\[
QR \left( \frac{K_{ij}}{|K_{11}|} \right) = \begin{pmatrix}
-0.86 & -0.35 & -0.37 \\
0.02 & -0.75 & 0.66 \\
-0.51 & 0.56 & 0.65
\end{pmatrix}. 
\]

Take absolute values which only are measured by experiment. Since the original matrix was symmetric, its QR decomposition can be transposed. Exchange the second and third columns and rows, which were arbitrarily assigned to the three perturbations

\[
|U_{ij}|_{\text{theory}} = \begin{pmatrix}
0.86 & 0.51 & 0.02 \\
0.37 & 0.65 & 0.66 \\
0.35 & 0.56 & 0.75
\end{pmatrix}. 
\]

Notice the diagonal preference of the matrix. Many experiments have contributed measurements to the neutrino mixing matrix. Compare with the Nu-Fit 2.3 values for the 3\( \sigma \) confidence limits on the experimentally measured absolute values of the neutrino mixing matrix \( U_{ij} \)

\[
|U_{ij}|_{\text{experiment}} = \begin{pmatrix}
0.799 & 0.844 & 0.516 & 0.582 & 0.141 & 0.156 \\
0.242 & 0.494 & 0.467 & 0.678 & 0.639 & 0.774 \\
0.284 & 0.521 & 0.490 & 0.695 & 0.615 & 0.754
\end{pmatrix}. 
\]

Most of the elements of the theoretically predicted neutrino mixing matrix appear within these confidence limits of the experimentally measured neutrino mixing matrix. The small upper right element \( U_{13} \) is usually associated with CP violating effects.

7 Fine Structure Constant

Consider the fine structure constant

\[
\alpha = \frac{e^2}{4\pi \varepsilon_0 \hbar c}. 
\]
In Kaluza-Klein theory, electric charge is the momentum in the extra dimension quantized in the circumference, hence electric charge is related to the inverse of the curvature of the extra dimension. Compare the squared inverse curvature of the hypertorus ring with that of a hypersphere of same dimension. Use the same radii ratio \( r_1/r_2 = 1.1808 \) of ring radius to sphere radius, and the unit sphere radius \( r_2 = 1 \). The three dimensional hypertorus Ricci scalar curvature \( R_T \) is

\[
R_T = \frac{(1 + \sin^2 y_2)(r_1 + 2r_2 \sin y_2 \cos y_3)}{r_1 + r_2 \sin y_2 \cos y_3} + \frac{2 \sin y_2 \cos y_3 (r_1 + r_2 \sin y_2 \cos y_3)}{r_2} \tag{7.2}
\]

with the same hypertorus Jacobian integration factor \( \sqrt{|h|} \). Integrate the hypertorus Ricci scalar curvature \( R_T \)

\[
T = \int_0^{2\pi} dy_1 \int_0^\pi dy_2 \int_0^{2\pi} dy_3 \sqrt{|h|} R_T = 279.7 r_2^3. \tag{7.3}
\]

Compare with the three dimensions of the hypersphere which has the Ricci scalar curvature \( R_S \)

\[
R_S = 2 \left( \sin^2 y_1 \sin^2 y_2 + \sin^2 y_1 + 1 \right) \tag{7.4}
\]

and the hypersphere Jacobian integration factor

\[
\sqrt{|h_S|} = r_2^3 \sin^2 y_1 \sin y_2. \tag{7.5}
\]

Integrate the hypersphere Ricci scalar curvature \( R_S \)

\[
S = \int_0^\pi dy_1 \int_0^\pi dy_2 \int_0^{2\pi} dy_3 \sqrt{|h_S|} R_S = 88.83 r_2^3. \tag{7.6}
\]

Note the difference of angles and integration limit. With vacuum electric permittivity \( \varepsilon_0 = 1 \) and this calculation being relativistically quantum \( \hbar c = 1 \), calculate the fine structure constant

\[
\alpha_{\text{theory}} = \frac{e^2}{4\pi \varepsilon_0 \hbar c} = \frac{1}{4\pi} \left( \frac{S}{T} \right)^2 = \frac{1}{124.6}. \tag{7.7}
\]

The fine structure constant runs with the energy scale. At the low energy scale, the fine structure constant is experimentally measured to be\[19\]

\[
\alpha(m = 0)_{\text{experiment}} = \frac{1}{137.035999139 \pm 0.000000031}. \tag{7.8}
\]

At the momentum transfer energy scale of the \( Z^0 \) boson mass, the effective fine structure constant is experimentally measured to be\[20\]

\[
\alpha(m = m_{Z^0})_{\text{experiment}} = \frac{1}{128.936 \pm 0.046}. \tag{7.9}
\]

Theory agrees with experiment at the \( Z^0 \) boson mass scale within a few percent although disagrees enormously with the precise experimental error confidence limits.
8 Conclusions and Discussions

The electroweak mixing angle was predicted within an accuracy of about 1 percent. The number of neutrino flavors was predicted precisely although with small number statistics. Most of the elements of the neutrino mixing matrix were predicted within 3$\sigma$ of their experimental measurements. The fine structure constant was predicted within a few percent. Decreasing the statistical significance of these predictions is the number of trials performed with different assumptions. Perhaps general relativity with extra dimensions can offer predictive value for quantum physics.

References

[1] A. Einstein, *The Foundation of the General Theory of Relativity*, Sitzungsber. Preuss. Akad. Wiss. 1915 (1915) pg. 778, 799, 831, 844
[2] S. Glashow, *Partial-symmetries of weak interactions*, Nucl. Phys. 22 (1961) pg. 579
[3] S. Weinberg, *A Model of Leptons*, Phys. Rev. Lett. 19 (1967) pg. 1264
[4] A. Salam, J. C. Ward, *Gauge Theory of Weak and Electromagnet Interactions*, Phys. Lett. 13 (1964) pg. 168
[5] T. Kaluza, *On the Unification Problem in Physics*, Sitzungsber. Preuss. Akad. Wiss. 1921 (1921) pg. 966
[6] O. Klein, *The Atomicity of Electricity as a Quantum Theory Law*, Nature 118 (1926) pg. 516
[7] B. S. DeWitt, *Les Houches Summer School 1963* (Gordon and Breach, Science Publishers, Inc., New York U.S.A. (1965), pg. 139
[8] J. F. Donoghue, *The Multiverse and Particle Physics*, Ann. Rev. Nucl. and Part. Sci. 66 (2016) pg. 1
[9] B. Li et al., *Cosmology of Ricci-tensor-squared gravity in the Palatine variational approach*, Phys. Rev. D 76 (2007) 104047
[10] N. Ohta et al., *Gauges and functional measures in quantum gravity II: higher-derivative gravity*, Eur. Phys. J. C 77 (2017) pg. 611
[11] S. Weinberg, *Gravitation and Cosmology*, Wiley, New York, U.S.A. (1972)
[12] J. Erler and M. J. Ramsey-Musolf, *Weak mixing angle at low energies*, Phys. Rev. D 72 (2005) 073003
[13] M. Tanabashi et al. (Particle Data Group), *Review of Particle Physics*, Phys. Rev. D 98 (2018) 030001 pg. 163
[14] W. Beyer, *CRC Standard Mathematical Tables*, CRC Press, Boca Raton, FL, U.S.A. (1987) pg. 131
[15] R. N. Mohapatra and A. Y. Smirnov, *Neutrino mass and new physics*, Annu. Rev. Nucl. Part. Sci. 56 (2006) pg. 569
[16] W. Cheney and D. Kincaid, *Linear Algebra: Theory and Applications*, Jones & Bartlett Publishers, Burlington MA, U.S.A. (2009)
[17] sagemath; A=Matrix(CDF,[[1,1,1],[1,1,1],[1,1,1]]); G,M=A.gram_schmidt()
[20] H. Burkhardt and B. Pietrzyk, *Update of the hadronic contribution to the QED vacuum polarization*, Phys. Lett. B 513 (2001) pg. 46