VANISHING THEOREM FOR TRANSVERSE DIRAC OPERATORS ON RIEMANNIAN FOLIATIONS

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Abstract. We obtain a vanishing theorem for the half-kernel of a transverse Spin\(^c\) Dirac operator on a compact manifold endowed with a transversely almost complex Riemannian foliation twisted by a sufficiently large power of a line bundle, whose curvature vanishes along the leaves and is transversely non-degenerate at any point of the ambient manifold.

Introduction

Let \(X\) be a compact manifold of dimension \(2n\) equipped with an almost complex structure \(J : TX \to TX\), \(E\) a Hermitian vector bundle on \(X\), and \(g_X\) a Riemannian metric on \(X\). Assume that the almost complex structure \(J\) is compatible with \(g_X\). Consider a Hermitian line bundle \(L\) over \(X\) endowed with a Hermitian connection \(\nabla_L\) such that its curvature \(R^L = (\nabla^2)^2\) is non-degenerate. Thus, \(\omega = \frac{i}{2\pi} R^L\) is a symplectic form on \(X\). One can construct canonically a Spin\(^c\) Dirac operator \(D_k\) acting on

\[\Omega^{0,*}(X, E \otimes L^k) = \bigoplus_{q=0}^n \Omega^{0,q}(X, E \otimes L^k),\]

the direct sum of spaces of \((0, q)\)-forms with values in \(E \otimes L^k\).

Under the assumption that \(J\) is compatible with \(\omega\), Borthwick and Uribe [2] proved that, for sufficiently large \(k\),

\[\text{Ker} \ D_k^- = 0,\]

where \(D_k^-\) denotes the restriction of \(D_k\) to \(\Omega^{0,\text{odd}}(X, E \otimes L^k)\). This result generalizes the famous Kodaira vanishing theorem for the cohomology of the sheaf of sections of a holomorphic vector bundle twisted by a large power of a positive line bundle. It has interesting applications in geometric quantization (see [2] and references therein).

In [18], Ma and Marinescu gave a proof of the Borthwick-Uribe result, which uses only the Lichnerowicz formula for the Spin\(^c\) Dirac operator. They also show that, if we put

\[m = \inf_{u \in T^{(1,0)}_{x,X}, x \in X} \frac{R^L_x(u, \bar{u})}{|u|^2} > 0,\]

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then there exists $C > 0$ such that, for $k \in \mathbb{N}$, the spectrum of $D_k^2$ is contained in the set $\{0\} \cup (2km - C, +\infty)$.

Let us mention that Braverman [3, 4] generalized the Borthwick-Urbe vanishing theorem to the case when the almost complex structure $J$ is compatible with $g_X$, the curvature $R^C$ is non-degenerate and $J$-invariant, but not necessarily compatible with $J$. In [19], Ma and Marinescu gave a proof of this result by the methods of [18].

Our main purpose is to obtain an analogue of the Borthwick-Urbe vanishing theorem for a transverse Spin$^c$ Dirac operator on a compact manifold endowed with a transversely almost complex Riemannian foliation. Our considerations are based on the approach of Ma-Marinescu [18]. So we also state a Lichnerowicz type formula for a transverse Dirac operator on a compact foliated manifold $(M, \mathcal{F})$, which, as we strongly believe, will be of independent interest.

The transverse Dirac operators for Riemannian foliations were introduced in [7]. This paper mainly deals with the transverse Dirac operators acting on basic sections (see also [8, 9, 11, 12, 13] and references therein). The index theory of transverse Dirac operators was studied in [6]. Finally, spectral triples defined by transverse Dirac-type operators on Riemannian foliations and related noncommutative geometry were considered in [15, 16, 17].

The paper is organized as follows. In Section 1, we introduce transverse Dirac operators and formulate our main results, the vanishing theorem for a transverse Spin$^c$ Dirac operator on a compact manifold endowed with a transversely almost complex Riemannian foliation, Theorem 3, and the Lichnerowicz formula for transverse Dirac operators, Theorem 4. The proof of the vanishing theorem is given in Section 2. Sections 3 and 4 contain the proof of the Lichnerowicz formula and related results.

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1. Preliminaries and main results

1.1. Transverse Dirac operators. Let $M$ be a compact manifold equipped with a Riemannian foliation $\mathcal{F}$ of even codimension $q$ and $g_M$ a bundle-like metric on $M$. Let $T^H_x M = T_x \mathcal{F}^\perp$. $T^H M$ is a smooth vector subbundle of $TM$ such that

\begin{equation}
TM = T^H M \oplus T \mathcal{F}.
\end{equation}

There is a natural isomorphism $T^H M \cong Q = TM/T \mathcal{F}$. Denote by $P_H$ (resp. $P_{\mathcal{F}}$) the orthogonal projection operator of $TM$ on $T^H M$ (resp. $T \mathcal{F}$) associated with the decomposition (1).

The Riemannian metric $g_M$ gives rise to a metric connection $\nabla$ in $T^H M$ (called the transverse Levi-Civita connection), which is defined as follows.
Denote by $\nabla^L$ the Levi-Civita connection defined by $g_M$. Then we have
\begin{equation}
\begin{aligned}
\nabla_XN &= P_H[X,N], \\ X &\in C^\infty(M, TM), \\ N &\in C^\infty(M, T^HM)
\end{aligned}
\end{equation}
\begin{equation}
\nabla_XN = P_H\nabla^L_XN, \\ X &\in C^\infty(M, T^HM), \\ N &\in C^\infty(M, T^HM).
\end{equation}

It turns out that $\nabla$ depends only on the transverse part of the metric $g_M$ and preserves the inner product of $T^HM$.

For any $x \in M$, denote by $Cl(Q_x)$ the Clifford algebra of $Q_x$. Recall that, relative to an orthonormal basis $\{f_1, f_2, \ldots, f_q\}$ of $Q_x$, $Cl(Q_x)$ is the complex algebra generated by 1 and $f_1, f_2, \ldots, f_q$ with relations

\[ f_\alpha f_\beta + f_\beta f_\alpha = -2\delta_{\alpha\beta}, \quad \alpha, \beta = 1, 2, \ldots, q. \]

The transverse Clifford bundle $Cl(Q)$ is the $\mathbb{Z}_2$-graded vector bundle over $M$ whose fiber at $x \in M$ is $Cl(Q_x)$. This bundle is associated to the principal $SO(q)$-bundle $O(Q)$ of oriented orthonormal frames in $Q$, $Cl(Q) = O(Q) \times_{O(q)} Cl(\mathbb{R}^q)$. Therefore, the transverse Levi-Civita connection $\nabla$ induces a natural leafwise flat connection $\nabla^{Cl(Q)}$ on $Cl(Q)$ which is compatible with the multiplication and preserves the $\mathbb{Z}_2$-grading on $Cl(Q)$. If $\{f_1, f_2, \ldots, f_q\}$ is a local orthonormal frame in $T^HM$, and $\omega^\gamma_{\alpha\beta}$ is the coefficients of the connection $\nabla$: $\nabla f_\alpha f_\beta = \sum_\gamma \omega^\gamma_{\alpha\beta} f_\gamma$, then
\begin{equation}
\nabla^{Cl(Q)} = f_\alpha + \frac{1}{4} \sum_{\gamma=1}^q \omega^\gamma_{\alpha\beta} (f_\beta) c(f_\gamma),
\end{equation}
where $c(a)$ denotes the action of an element $a \in Cl(Q)$ on $C^\infty(M, Cl(Q))$ by pointwise left multiplication.

A transverse Clifford module is a complex vector bundle $\mathcal{E}$ on $M$ endowed with an action of the bundle $Cl(Q)$. We will denote the action of $a \in C^\infty(M, Cl(Q))$ on $s \in C^\infty(M, \mathcal{E})$ as $c(a)s \in C^\infty(M, \mathcal{E})$.

A transverse Clifford module $\mathcal{E}$ is called self-adjoint if it endowed with a leafwise flat Hermitian metric such that the operator $c(f): \mathcal{E}_x \rightarrow \mathcal{E}_x$ is skew-adjoint for any $x \in M$ and $f \in Q_x$.

Any transverse Clifford module $\mathcal{E}$ carries a natural $\mathbb{Z}_2$-grading $\mathcal{E} = \mathcal{E}_+ \oplus \mathcal{E}_-$ (see, for instance, [1]).

A connection $\nabla^\mathcal{E}$ on a transverse Clifford module $\mathcal{E}$ is called a Clifford connection if it is compatible with the Clifford action, that is, for any $f \in C^\infty(M, T^HM)$ and $a \in C^\infty(M, Cl(Q))$,

\[ [\nabla^\mathcal{E}_f, c(a)] = c(\nabla^{Cl(Q)} f a). \]

Example 1. Assume that $\mathcal{F}$ is transversely oriented and the normal bundle $Q$ is spin. Thus the $SO(q)$ bundle $O(Q)$ of oriented orthonormal frames in $Q$ can be lifted to a $Spin(q)$ bundle $O'(Q)$ so that the projection $O'(Q) \rightarrow O(Q)$ induces the covering projection $Spin(q) \rightarrow SO(q)$ on each fiber.

Let $F(Q), F_+(Q), F_-(Q)$ be the bundles of spinors

\[ F(Q) = O'(Q) \times_{Spin(q)} S, \quad F_{\pm}(Q) = O'(Q) \times_{Spin(q)} S_{\pm}. \]
Since dim $Q = q$ is even \( \text{End} F(Q) \) is as a bundle of algebras over \( M \) isomorphic to the Clifford bundle \( Cl(Q) \). So \( F(Q) \) is a self-adjoint transverse Clifford module. The transverse Levi-Civita connection \( \nabla \) lifts to a leafwise flat Clifford connection \( \nabla^{F(Q)} \) on \( F(Q) \).

More generally, one can take a Hermitian vector bundle \( W \) equipped with a leafwise flat Hermitian connection \( \nabla^{W} \). Then \( F(Q) \otimes W \) is a transverse Clifford module: the action of \( a \in C^\infty(M, Cl(Q)) \) on \( C^\infty(M, F(Q) \otimes W) \) is given by \( c(a) \otimes 1 \) (\( c(a) \) denotes the action of \( a \) on \( C^\infty(M, F(Q)) \)). The product connection \( \nabla^{F(Q) \otimes W} = \nabla^{F(Q)} \otimes 1 + 1 \otimes \nabla^{W} \) on \( F(Q) \otimes W \) is a Clifford connection.

**Example 2.** Another example of a self-adjoint transverse Clifford module associated with a transverse almost complex structure on \( (M, F) \), a transverse Clifford module \( \Lambda^{0,*} \), is described in Section 1.2.

Let \( E \) be a self-adjoint transverse Clifford module equipped with a leafwise flat Clifford connection \( \nabla^{E} \). We will identify the bundle \( Q \) and \( Q^{*} \) by means of the metric \( g_{M} \) and define the operator \( D^{E} \) acting on the sections of \( E \) as the composition

\[
C^\infty(M, E) \xrightarrow{\nabla^{E}} C^\infty(M, Q^{*} \otimes E) = C^\infty(M, Q \otimes E) \xrightarrow{c} C^\infty(M, E).
\]

This operator is odd with respect to the natural \( \mathbb{Z}_{2} \)-grading on \( E \). If \( f_{1}, \ldots, f_{q} \) is a local orthonormal frame for \( T^{H}M \), then

\[
D^{E} = \sum_{\alpha=1}^{q} c(f_{\alpha}) \nabla^{E} f_{\alpha}.
\]

Let \( \tau \in C^\infty(M, T^{H}M) \) be the mean curvature vector field of \( F \). If \( e_{1}, e_{2}, \ldots, e_{p} \) is a local orthonormal frame in \( TF \), then

\[
\tau = \sum_{i=1}^{p} P_{H}(\nabla^{L}_{e_{i}} e_{i}).
\]

The transverse Dirac operator \( D^{E} \) is defined as

\[
D^{E} = D^{E} - \frac{1}{2} c(\tau) = \sum_{\alpha=1}^{q} c(f_{\alpha}) \left( \nabla^{E}_{f_{\alpha}} - \frac{1}{2} g_{M}(\tau, f_{\alpha}) \right).
\]

Denote by \( (\cdot, \cdot)_{x} \) the inner product in the fiber \( E_{x} \) over \( x \in M \). Then the inner product in \( L^{2}(M, E) \) is given by the formula

\[
(s_{1}, s_{2}) = \int_{M} (s_{1}(x), s_{2}(x))_{x} \omega_{M}, \quad s_{1}, s_{2} \in L^{2}(M, E),
\]

where \( \omega_{M} = \sqrt{\text{det} g_{M} \, dx} \) denotes the Riemannian volume form on \( M \). As shown in [17], the transverse Dirac operator \( D^{E} \) is formally self-adjoint in \( L^{2}(M, E) \).
1.2. The vanishing theorem. As above, let \( M \) be a compact manifold equipped with a Riemannian foliation \( \mathcal{F} \) of even codimension \( q \), \( g_M \) a bundle-like metric on \( M \). Consider a Hermitian line bundle \( L \) equipped with a leafwise flat Hermitian connection \( \nabla^L \).

The curvature of \( \nabla^L \) is an imaginary valued 2-form \( R^L = (\nabla^L)^2 \) on \( M \). Since \( \nabla^L \) is leafwise flat, \( R^L \) vanishes on \( TF \), and, therefore, defines a 2-form \( R^L \) on \( Q \). If this form is non-degenerate, then it is a symplectic form on \( Q \).

Let \( J : Q \to Q \) be an almost complex structure, which is compatible with \( g_Q \) and \( R^L \). The almost complex structure \( J \) defines canonically an orientation of \( Q \) and induces a splitting \( Q \otimes \mathbb{C} = Q^{(1,0)} \oplus Q^{(0,1)} \), where \( Q^{(1,0)} \) and \( Q^{(0,1)} \) are the eigenbundles of \( J \) corresponding to the eigenvalues \( i \) and \( -i \) respectively. We also have the corresponding decomposition of the complexified conormal bundle \( Q^* \otimes \mathbb{C} = Q^{(1,0)*} \oplus Q^{(0,1)*} \) and the decomposition of the exterior algebra bundles \( \Lambda(Q^* \otimes \mathbb{C}) = \oplus_{p,q} \Lambda^p Q^{(1,0)*} \otimes \Lambda^q Q^{(0,1)*} \). The transverse Levi-Civita connection \( \nabla \) can be written as

\[
\nabla = \nabla^{(1,0)} + \nabla^{(0,1)} + A,
\]

where \( \nabla^{(1,0)} \) and \( \nabla^{(0,1)} \) are the canonical Hermitian connections on \( Q^{(1,0)} \) and \( Q^{(0,1)} \) respectively and \( A \in C^\infty(T^*M \otimes \text{End}(Q)) \), which satisfies \( JA = -AJ \).

Consider a self-adjoint transverse Clifford module

\[
\Lambda^{0,*} = \Lambda^{even} Q^{(0,1)*} \oplus \Lambda^{odd} Q^{(0,1)*}.
\]

The action of any \( f \in Q \) with decomposition \( f = f_{1,0} + f_{0,1} \in Q^{(1,0)} \oplus Q^{(0,1)} \) on \( \Lambda^{0,*} \) is defined as

\[
c(f) = \sqrt{2}(\varepsilon f_{1,0}^* - i f_{0,1}),
\]

where \( \varepsilon f_{1,0}^* \) denotes the exterior product by the covector \( f_{1,0}^* \) dual to \( f_{1,0} \), \( i f_{0,1} \) the interior product by \( f_{0,1} \). This module has a natural leafwise flat Clifford connection \( \nabla^{\Lambda^{0,*}} \). The associated transverse Dirac operator \( D^{\Lambda^{0,*}} \) can be called the transverse Spin\(^c \) Dirac operator.

One can also consider a Hermitian vector bundle \( W \) equipped with a leafwise flat Hermitian connection \( \nabla^W \). Then one get the twisted transverse Clifford module \( \mathcal{E} = \Lambda^{0,*} \otimes W \) equipped with a product leafwise flat Hermitian connection \( \nabla^\mathcal{E} \) and the associated transverse Spin\(^c \) Dirac operator \( D_{\Lambda^{0,*} \otimes W} \).

Consider the transverse Spin\(^c \) Dirac operator

\[
D_k = D_{\Lambda^{0,*} \otimes W \otimes L^k} : C^\infty(M, \Lambda^{0,*} \otimes W \otimes L^k) \to C^\infty(M, \Lambda^{0,*} \otimes W \otimes L^k).
\]

Let \( D_k^- \) denote the restriction of \( D_k \) to the space \( C^\infty(M, \Lambda^{odd} Q^{(0,1)*} \otimes W \otimes L^k) \). Put

\[
m = \inf_{u \in Q^{(1,0)}, x \in M} \frac{R^L_x(u, \bar{u})}{|u|^2} > 0.
\]
Theorem 3. There exists $C > 0$ such that for $k \in \mathbb{N}$, the spectrum of $D_k^2$ is contained in the set $\{0\} \cup (2km - C, +\infty)$. For sufficiently large $k$

$$\ker D_k^{-} = 0.$$ 

The proof of this theorem will be given in Section 2.

1.3. The Lichnerowicz formula. In this Section, we will formulate the Lichnerowicz formula for a transverse Dirac operator, which will play a crucial role in the proof of Theorem 3.

Denote by $\mathcal{R}$ the integrability tensor (or curvature) of $T^HM$. It is an element of $C^\infty(M, \Lambda^2 T^*HM \otimes T\mathcal{F})$ given by

$$\mathcal{R}_x(f_1, f_2) = -P_F[f_1, f_2](x), \quad x \in M, \quad f_1, f_2 \in T_x^HM,$$

where, for any $f \in T_x^HM$, $\tilde{f} \in C^\infty(M, T^HM)$ denotes any vector field, which coincides with $f$ at $x$.

Since the Levi-Civita connection $\nabla^L$ is torsion-free, for any $f_1, f_2 \in C^\infty(M, T^HM)$, we have

$$\nabla f_1 f_2 - \nabla f_2 f_1 - [f_1, f_2] = \mathcal{R}(f_1, f_2). \quad (4)$$

Let $R$ be the curvature of the transverse Levi-Civita connection $\nabla$. By definition, $R$ is a section of $\Lambda^2 T^*M \otimes \text{End}(T^HM)$ given by the formula

$$R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}, \quad X, Y \in C^\infty(M, TM).$$

If $(B, g_B)$ is a local model for the foliation and $R^B$ is the curvature of $g_B$, then, for any $f_1, f_2, f_3 \in TB$ with the corresponding horizontal lifts $f_1^H, f_2^H, f_3^H \in T^HM$, we have

$$R(f_1^H, f_2^H, f_3^H) = [R^B(f_1, f_2), f_3]^H + P_H([\mathcal{R}(f_1^H, f_2^H), f_3^H]).$$

Denote by $R^E$ the curvature of the Clifford connection $\nabla^E$. By definition, $R^E$ is a section of $\Lambda^2 T^*_HM \otimes \text{End}(\mathcal{E})$ given by the formula

$$R^E(f_1, f_2) = \nabla^E f_1 \nabla^E f_2 - \nabla^E f_2 \nabla^E f_1 - \nabla^E_{[f_1, f_2]}.$$ 

It can be written as

$$R^E = c(R) + R^{E/S},$$

where $c(R) \in C^\infty(M, \Lambda^2 T^*_HM \otimes \text{Cl}(Q))$ is determined by the curvature $R$ of $\nabla$: If $\{f_1, f_2, \ldots, f_q\}$ is a local orthonormal frame in $T^HM$, then

$$c(R)(f_1, f_2) = \frac{1}{4} \sum_{\alpha, \beta} (R(f_1, f_2)f_{\alpha}, f_{\beta})c(f_{\alpha})c(f_{\beta}),$$

and $R^{E/S} \in C^\infty(M, \Lambda^2 T^*_HM \otimes \text{End}_{\text{Cl}(Q)}(\mathcal{E}))$ is the twisting curvature of $\mathcal{E}$.

Denote by $(\nabla^E_X)^*$ the formal adjoint of the operator $\nabla^E_X$ with $X \in C^\infty(M, T^HM)$ in $L^2(M, \mathcal{E})$. Observe the following formula:

$$(\nabla^E_X)^* = -\nabla^E_X - \text{div} X. \quad (5)$$
where \( \operatorname{div} X \in C^\infty(M) \) denotes the divergence of \( X \). If \( e_1, e_2, \ldots, e_p \) is a local orthonormal frame in \( TF \) and \( f_1, \ldots, f_q \) is a local orthonormal basis of \( T^H M \), then

\[
\operatorname{div} X = \sum_{k=1}^p g_M(e_k, \nabla e_k X) + \sum_{\beta=1}^q g_M(f_{\beta}, \nabla f_{\beta} X).
\]

In particular, it is easy to see that

\[
\operatorname{div} f_{\alpha} = -g_M(\tau + \sum_{\beta=1}^q \nabla f_{\beta} f_{\beta}, f_{\alpha}).
\]

Let \( f_1, \ldots, f_q \) be a local orthonormal basis of \( T^H M \). Define the transverse scalar curvature \( K \) as

\[
K = \sum_{\alpha, \beta} g(R(f_{\alpha}, f_{\beta}) f_{\alpha}, f_{\beta}).
\]

**Theorem 4.** The following formula holds:

\[
(D_E)^2 = \sum_{\alpha=1}^q (\nabla f_{\alpha})^* \nabla f_{\alpha} - \frac{1}{2} \sum_{\alpha=1}^q c(f_{\alpha}) c(\nabla f_{\alpha} \tau) - \frac{1}{4} \|\tau\|^2 + \frac{K}{4} + \frac{1}{2} \sum_{\alpha, \beta} c(f_{\alpha}) c(f_{\beta}) [\mathcal{R} (f_{\alpha}, f_{\beta}) - \nabla \mathcal{R} (f_{\alpha}, f_{\beta})],
\]

where \( f_1, \ldots, f_q \) is a local orthonormal basis of \( T^H M \).

The proof of this theorem and related results will be given in Section 3.

2. PROOF OF THE VANISHING THEOREM

The purpose of this Section is to give the proof of Theorem 3. This proof will make an essential use of Theorem 4 whose proof will be given later, in Section 3. First, we introduce some notation.

For any \( x \in M \), define the skew-symmetric linear map \( K_x : Q_x \to Q_x \) by the formula

\[
i R^E(v, w) = g_Q(v, K_x w), \quad v, w \in Q_x.
\]

The eigenvalues of \( K_x \) are purely imaginary: \( \pm i \mu_j(x), j = 1, 2, \ldots, l \) with \( \mu_j(x) > 0 \). Define

\[
\lambda(x) = \operatorname{Tr}^+ K_x = \mu_1(x) + \ldots + \mu_l(x), \quad m(x) = \min_j \mu_j(x).
\]

Observe that

\[
m = \min_{x \in M} m(x).
\]

Denote

\[
c(\mathcal{R}) = \frac{1}{2} \sum_{\alpha, \beta} c(f_{\alpha}) c(f_{\beta}) \nabla \mathcal{R} (f_{\alpha}, f_{\beta})
\]
\[
\begin{align*}
\Delta^V \otimes L^k &= \sum_{\alpha=1}^q (\nabla^V_{f_\alpha} \otimes L^k)^* \nabla^V_{f_\alpha} \\
&\geq k(\lambda u, u) - C\|u\|^2
\end{align*}
\]
for any \(u \in C^\infty(M, V \otimes L^k)\).

Proof. Consider the twisted transverse Clifford module \(\Lambda^0,\ast \otimes V \otimes L^k\) and the associated twisted transverse Spin^c Dirac operator \(D_{\Lambda^0,\ast \otimes V \otimes L^k}\). By Theorem \(\text{4}\), we have
\[
D_{\Lambda^0,\ast \otimes V \otimes L^k}^2 = \Delta^\Lambda^0,\ast \otimes V \otimes L^k - \frac{1}{2} \sum_{\alpha=1}^q (c(f_\alpha)c(\nabla f_\alpha \tau) \otimes 1) + \frac{K}{4} + c(R^V) - c(\mathcal{R}) + kc(R^L),
\]
where \(f_1, \ldots, f_q\) is a local orthonormal basis of \(T^HM\). From \(\text{3}\), we have
\[
\|\nabla^\Lambda^0,\ast \otimes V \otimes L^k u\|^2 = \|\nabla^V \otimes L^k u\|^2 + \frac{1}{16} \sum_\gamma \omega_{\alpha\beta}^\gamma c(f_\beta)c(f_\gamma)u|^2.
\]
It can be shown (see, for instance, \(\text{3}\) Lemma 7.10) that, for any \(u \in (V \otimes L^k)_x \subset (\Lambda^0,\ast \otimes V \otimes L^k)_x\)
\begin{equation}
(6)
\begin{align*}
c(R^L)u &= -\lambda u.
\end{align*}
\end{equation}
Using \(\text{5}\), we get, for any \(u \in C^\infty(M, V \otimes L^k) \subset C^\infty(M, \Lambda^0,\ast \otimes V \otimes L^k)\),
\[
0 \leq \|D_{\Lambda^0,\ast \otimes V \otimes L^k} u\|^2 \leq \|D^V \otimes L^k u\|^2 + C\|u\|^2 - (c(\mathcal{R})u, u) - k(\lambda u, u),
\]
that completes the proof. \(\square\)

By Theorem \(\text{4}\) we have
\[
D_k^2 = \Delta^{\Lambda^0,\ast \otimes W \otimes L^k} - \frac{1}{2} \sum_{\alpha=1}^q c(f_\alpha)c(\nabla f_\alpha \tau) - \frac{1}{4} \|\tau\|^2 + \frac{K}{4} + c(R^W) - c(\mathcal{R}) + kc(R^L),
\]
where \( f_1, \ldots, f_q \) is a local orthonormal basis of \( T^H M \). Therefore, for any \( u \in C^\infty(M, \Lambda^{0,*} \otimes W \otimes L^k) \), we have
\[
\|D_k u\|^2 \geq (\Delta^{\Lambda^{0,*} \otimes W \otimes L^k}(\mathcal{R}) u, u) + k(c(R\mathcal{C}) u, u) - C\|u\|^2.
\]
By Lemma 5, it follows that
\[
((\Delta^{\Lambda^{0,*} \otimes W \otimes L^k} - c(R\mathcal{C})) u, u) \geq k(\lambda u, u) - C\|u\|^2.
\]
So we see that
\[
\|D_k u\|^2 \geq k(\lambda u, u) + k(c(R\mathcal{C}) u, u) - C\|u\|^2.
\]
Finally, by [3, Proposition 7.5], we have
\[
(c(R\mathcal{C}) u, u)_x \geq -((\lambda(x) - 2m(x))\|u\|^2, \ u \in (\Lambda^{odd} Q^{(0,1)} \otimes W \otimes L^k)_x).
\]
Therefore, for \( u \in C^\infty(M, \Lambda^{odd} Q^{(0,1)} \otimes W \otimes L^k) \), we get
\[
\|D_k u\|^2 \geq 2k(mu, u) - C\|u\|^2,
\]
that immediately completes the proof of Theorem 3.

3. Proof of the Lichnerowicz formula

In this Section, we derive the Lichnerowicz formula for a transverse Dirac operator given in Theorem 4. We start with a computation of \((D_c')^2\):
\[
(D_c')^2 = \frac{1}{2} \left( \sum_{\alpha=1}^{q} c(f_\alpha) \nabla_{f_\alpha}^\xi \right) \left( \sum_{\beta=1}^{q} c(f_\beta) \nabla_{f_\beta}^\xi \right) + \left( \sum_{\beta=1}^{q} c(f_\beta) \nabla_{f_\beta}^\xi \right) \left( \sum_{\alpha=1}^{q} c(f_\alpha) \nabla_{f_\alpha}^\xi \right)
\]
\[
= \frac{1}{2} \sum_{\alpha, \beta} (c(f_\alpha)c(f_\beta) + c(f_\beta)c(f_\alpha)) \nabla_{f_\alpha}^\xi \nabla_{f_\beta}^\xi + \frac{1}{2} \sum_{\alpha, \beta} c(f_\beta)c(f_\alpha)(\nabla_{f_\beta}^\xi \nabla_{f_\alpha}^\xi - \nabla_{f_\alpha}^\xi \nabla_{f_\beta}^\xi)
\]
\[
+ \frac{1}{2} \sum_{\alpha, \beta} [c(f_\alpha)c(\nabla_{f_\alpha}^\xi f_\beta) \nabla_{f_\beta}^\xi + c(f_\beta)c(\nabla_{f_\beta}^\xi f_\alpha) \nabla_{f_\alpha}^\xi].
\]
For the first term, we get
\[
\frac{1}{2} \sum_{\alpha, \beta} (c(f_\alpha)c(f_\beta) + c(f_\beta)c(f_\alpha)) \nabla_{f_\alpha}^\xi \nabla_{f_\beta}^\xi = -\sum_{\alpha} (\nabla_{f_\alpha}^\xi)^2.
\]
For the second term, we get
\[
\frac{1}{2} \sum_{\alpha, \beta} c(f_\beta)c(f_\alpha)(\nabla_{f_\beta}^\xi \nabla_{f_\alpha}^\xi - \nabla_{f_\alpha}^\xi \nabla_{f_\beta}^\xi)
\]
\[
\frac{1}{2} \sum_{\alpha,\beta} c(\beta)c(\alpha) R^\varepsilon(\beta, \alpha) + \frac{1}{2} \sum_{\alpha,\beta} c(\beta)c(\alpha) \nabla^\varepsilon_{[\beta, \alpha]}.
\]

Let \( \nabla_{\beta, \alpha} = \sum_\gamma a^\gamma_{\alpha \beta} f_\gamma \). Since \( \nabla \) is compatible with the metric, we have \( a^\gamma_{\alpha \beta} = -a^\beta_{\alpha \gamma} \). Thus we get

\[
\frac{1}{2} \sum_{\alpha,\beta} [c(\alpha) c(\nabla_{\beta, \alpha}) \nabla^\varepsilon_{\beta} + c(\beta)c(\nabla_{\alpha, \beta}) \nabla^\varepsilon_{\alpha}]
\]

\[
= \frac{1}{2} \sum_{\alpha,\beta,\gamma} [a^\gamma_{\alpha \beta} c(\alpha) c(\gamma) \nabla^\varepsilon_{\beta} + a^\gamma_{\alpha \beta} c(\beta)c(\gamma) \nabla^\varepsilon_{\alpha}]
\]

\[
= -\frac{1}{2} \sum_{\alpha,\beta,\gamma} [a^\beta_{\alpha \gamma} c(\alpha) c(\gamma) \nabla^\varepsilon_{\beta} + a^\alpha_{\beta \gamma} c(\beta)c(\gamma) \nabla^\varepsilon_{\alpha}]
\]

\[
= -\frac{1}{2} \sum_{\alpha,\gamma} [c(\alpha) c(\nabla_{\beta, \alpha}) \nabla^\varepsilon_{\beta} + \sum_{\beta,\gamma} c(\beta)c(\gamma) \nabla^\varepsilon_{\beta, \alpha}]
\]

\[
= -\sum_{\alpha,\beta} c(\alpha) c(\beta) \nabla^\varepsilon_{\beta, \alpha}.
\]

From the last three identities, we get

\[
(D^\varepsilon)^2 = -\sum_\alpha (\nabla^\varepsilon_{\alpha})^2 + \frac{1}{2} \sum_{\alpha,\beta} c(\beta)c(\alpha) R^\varepsilon(\beta, \alpha)
\]

\[
+ \frac{1}{2} \sum_{\alpha,\beta} c(\beta)c(\alpha) \nabla^\varepsilon_{\beta, \alpha} - \sum_{\alpha,\beta} c(\alpha)c(\beta) \nabla^\varepsilon_{\beta, \alpha}.
\]

Consider the last two terms in this identity. Using (1), we get

\[
\frac{1}{2} \sum_{\alpha,\beta} c(\beta)c(\alpha) \nabla^\varepsilon_{\beta, \alpha} - \sum_{\alpha,\beta} c(\alpha)c(\beta) \nabla^\varepsilon_{\beta, \alpha}
\]

\[
= \frac{1}{2} \sum_{\alpha,\beta} c(\beta)(c(\alpha)(\nabla^\varepsilon_{\beta, \alpha} - \nabla^\varepsilon_{\beta, \alpha} - \nabla^\varepsilon_{R(\beta, \alpha)})
\]

\[
- \sum_{\alpha,\beta} c(\alpha)c(\beta) \nabla^\varepsilon_{\beta, \alpha}
\]

\[
= -\frac{1}{2} \sum_{\alpha,\beta} c(\alpha)c(\beta) + c(\beta)c(\alpha) \nabla^\varepsilon_{\beta, \alpha}
\]

\[
- \frac{1}{2} \sum_{\alpha,\beta} c(\beta)c(\alpha) \nabla^\varepsilon_{R(\beta, \alpha)}
\]

\[
= \sum_\alpha \nabla^\varepsilon_{\beta, \alpha} - \frac{1}{2} \sum_{\alpha,\beta} c(\beta)c(\alpha) \nabla^\varepsilon_{R(\beta, \alpha)}.
\]
By (3), we also get

\[
\sum_{\alpha=1}^{q} (\nabla_{f_{\alpha}}^{\varepsilon})^* \nabla_{f_{\alpha}}^{\varepsilon} = - \sum_{\alpha=1}^{q} (\nabla_{f_{\alpha}}^{\varepsilon})^2 + \nabla_{\tau}^{\varepsilon} + \nabla_{\sum_{\alpha}^{\varepsilon}}^{\varepsilon} f_{\alpha}. \tag{7}
\]

Thus, we arrive at the formula

\[
(D_{\varepsilon}')^2 = \sum_{\alpha=1}^{q} (\nabla_{f_{\alpha}}^{\varepsilon})^* \nabla_{f_{\alpha}}^{\varepsilon} - \frac{1}{2} \sum_{\alpha,\beta} c(f_{\alpha})c(f_{\beta}) [R^{\varepsilon}(f_{\alpha}, f_{\beta}) - \nabla_{R(f_{\alpha}, f_{\beta})}^{\varepsilon}] \tag{8}
\]

Taking into account (8), we get

\[
(D_{\varepsilon})^2 = (D_{\varepsilon}')^2 - \frac{1}{2} \sum_{\alpha=1}^{q} [c(f_{\alpha})c(\tau) + c(\tau)c(f_{\alpha})] \nabla_{f_{\alpha}}^{\varepsilon}
- \frac{1}{2} \sum_{\alpha=1}^{q} c(f_{\alpha})c(\nabla_{f_{\alpha}}^{\varepsilon} \tau) - \frac{1}{4} ||\tau||^2
= (D_{\varepsilon}')^2 - \frac{1}{2} \sum_{\alpha=1}^{q} c(f_{\alpha})c(\nabla_{f_{\alpha}}^{\varepsilon} \tau) - \frac{1}{4} ||\tau||^2. \tag{9}
\]

Finally, we use the formula

\[
\frac{1}{2} \sum_{\alpha,\beta} c(f_{\alpha})c(f_{\beta}) R^{\varepsilon}(f_{\alpha}, f_{\beta}) = K \frac{1}{4} + \frac{1}{2} \sum_{\alpha,\beta} c(f_{\alpha})c(f_{\beta}) R^{\varepsilon/S}(f_{\alpha}, f_{\beta}),
\]

that completes the proof of Theorem 4.

There is a natural action of \( Cl(Q_x) \) on \( \Lambda Q_x \) given by the formula

\[
c(f) = \varepsilon f^* - i_f, \quad f \in Q_x, \tag{10}
\]

where \( \varepsilon f^* \) denotes the exterior product by the covector \( f^* \in Q_x^* \) dual to \( f \), \( i_f \) the interior product by \( f \).

Recall that the symbol map \( \sigma : Cl(Q_x) \to \Lambda Q_x \) is defined by

\[
\sigma(a) = c(a)1, \quad a \in Cl(Q_x)
\]

and the quantization map \( \mathbf{c} = \sigma^{-1} : \Lambda Q_x \to Cl(Q_x) \) is given by

\[
\mathbf{c}(f_1 \wedge f_2 \wedge \ldots f_k) = c(f_{i_1})c(f_{i_2})\ldots c(f_{i_k}),
\]

where \( \{f_1, f_2, \ldots, f_q\} \) is an orthonormal base in \( Q_x \). These maps satisfy

\[
\sigma(\mathbf{c}(v)c(\omega)) = c(v)\omega, \quad v \in Q_x, \quad \omega \in \Lambda(Q_x). \tag{11}
\]
So, for any $\omega_1 \in \Lambda^i Q_x$ and $\omega_2 \in \Lambda^j Q_x$ we have

$$\sigma(c(\omega_1)c(\omega_2)) = \omega_1 \wedge \omega_2 \mod \Lambda^{i+j-2} Q_x.$$ 

By (10) and (11), we have

$$\sigma(\sum_{\alpha=1}^{q} c(f_\alpha)c(\nabla_{f_\alpha}\tau)) = \sum_{\alpha=1}^{q} c(f_\alpha)\nabla_{f_\alpha}\tau = \sum_{\alpha=1}^{q} (\varepsilon_{f_\alpha^*} - i_{f_\alpha})\nabla_{f_\alpha}\tau.$$ 

Recall the following lemma.

**Lemma 6.** Let $f_1, f_2, \ldots, f_q$ be a local orthonormal basis of $THM$ and $f_1^*, f_2^*, \ldots, f_q^*$ be the dual basis of $TH^*M$. Then on $C^\infty(M, \Lambda^0 T^*M)$ we have

$$d_H = \sum_{\alpha=1}^{q} \varepsilon_{f_\alpha^*}\nabla_{f_\alpha}, \quad d_H^* = -\sum_{\alpha=1}^{q} i_{f_\alpha} \nabla_{f_\alpha} + i_\tau.$$ 

By Lemma 6 we have

$$\sigma(\sum_{\alpha=1}^{q} c(f_\alpha)c(\nabla_{f_\alpha}\tau)) = d_H\tau + d_H^*\tau - \|\tau\|^2.$$ 

Assume that the bundle-like metric $g_M$ on $M$ satisfies the assumption: the mean curvature form $\tau$ is a basic one-form. As shown by Dominguez [5], such a bundle-like metric exists for any Riemannian foliation. Under this assumption, we have [14] (see also [20]): $d\tau = 0$. This fact implies that

$$\sigma(\sum_{\alpha=1}^{q} c(f_\alpha)c(\nabla_{f_\alpha}\tau)) = d_H\tau + d_H^*\tau - \|\tau\|^2 \in C^\infty(M, \Lambda^0 T^*M) = C^\infty(M),$$

and, therefore,

$$\sum_{\alpha=1}^{q} c(f_\alpha)c(\nabla_{f_\alpha}\tau) = d_H^*\tau - \|\tau\|^2.$$ 

So we come to the following consequence (cf. [7]):

**Theorem 7.** Assume that the bundle-like metric $g_M$ on $M$ satisfies the assumption: the mean curvature form $\tau$ is a basic one-form. Then we have:

$$(D_E)^2 = \sum_{\alpha=1}^{q} (\nabla^E_{f_\alpha}\tau)^* \nabla^E_{f_\alpha} \tau - \frac{1}{2} d_H^*\tau + \frac{1}{4} \|\tau\|^2 + \frac{K}{4} + \frac{1}{2} \sum_{\alpha, \beta} c(f_\alpha)c(f_\beta)[R^E/S(f_\alpha, f_\beta) - \nabla_{R(f_\alpha, f_\beta)}],$$

where $f_1, \ldots, f_q$ is a local orthonormal basis of $THM$.

4. **Transversal Bochner formula**

In this Section, we derive the Lichnerowicz formula for the transverse Laplacian on a compact manifold $M$ equipped with a Riemannian foliation $\mathcal{F}$, which can be naturally called a Bochner formula.
4.1. The transverse signature and Laplace operators. Suppose that
\((M, \mathcal{F})\) is a compact Riemannian foliated manifold equipped with a bundle-like metric \(g_M\). The decomposition \((1)\) induces a bigrading on \(\Lambda^* T^* M\):

\[
\Lambda^k T^* M = \bigoplus_{i=0}^{k} \Lambda^{i,k} T^* M,
\]

where

\[
\Lambda^{i,j} T^* M = \Lambda^i T^* F^* \otimes \Lambda^j T^* H^*.
\]

In this bigrading, the de Rham differential \(d\) can be written as

\[
d = d_F + d_H + \theta,
\]

where \(d_F\) and \(d_H\) are first order differential operators (the tangential de Rham differential and the transversal de Rham differential accordingly), and \(\theta\) is a zero order differential operator.

The transverse signature operator is a first order differential operator in \(C^\infty(M, \Lambda^* T^* H^*)\) given by

\[
D_H = d_H + d^*_H,
\]

and the transversal Laplacian is a second order transversally elliptic differential operator in \(C^\infty(M, \Lambda^* T^* H^*)\) given by

\[
\Delta_H = d_H d^*_H + d^*_H d_H.
\]

**Theorem 8.** Let \(f_1, \ldots, f_q\) be a local orthonormal basis of \(T^* H M\). Then we have the following formula

\[
\Delta_H = \sum_{\alpha=1}^{q} \nabla_{f_\alpha}^* \nabla f_\alpha + \sum_{\alpha=1}^{q} \varepsilon_f f_\alpha \nabla f_\alpha^* - \sum_{\alpha, \beta} \varepsilon_{f_\alpha} \nabla_{f_\alpha}^* (R(f_\alpha, f_\beta) - \nabla R(f_\alpha, f_\beta)).
\]

We give two proofs of Theorem 8. The first proof derives the theorem from Theorem 4 whereas the second proof is direct and makes no use of Theorem 4.

4.2. The first proof. Consider a transverse Clifford module \(\mathcal{E} = \Lambda^* T^* H M^*\) which equipped with a natural leafwise flat Clifford connection and the corresponding transverse Dirac operator \(D_{\Lambda^* T^* H M^*}\) acting in \(C^\infty(M, \Lambda^* T^* H M^*)\). The Clifford action of \(Cl(Q)\) on \(\mathcal{E}\) is defined by the formula \((11)\). By Lemma 6 and (10), we have

\[
\frac{\partial}{\partial r} + \frac{\partial}{\partial r} = \frac{1}{2} (\varepsilon_{r^*} + i_r).
\]

By \((12)\), it follows that

\[
\Delta_H = \left( D_{\Lambda^* T^* H M^*} + \frac{1}{2} (\varepsilon_{r^*} + i_r) \right)^2 - d^2_H - (d^*_H)^2.
\]

By \((12)\), it follows that

\[
\Delta_H = D^2_{\Lambda^* T^* H M^*} - d^2_H - (d^*_H)^2 + \frac{1}{2} \left( D_{\Lambda^* T^* H M^*} (\varepsilon_{r^*} + i_r) + (\varepsilon_{r^*} + i_r) D_{\Lambda^* T^* H M^*} \right)
\]
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\[ + \frac{1}{4} (\varepsilon_\tau i_\tau + i_\tau \varepsilon_\tau). \]

By Theorem 4, it follows that

\[ D^2_{A_{T^H}M^*} = \sum_{\alpha=1}^{q} \nabla^*_f a \nabla f_\alpha - \frac{1}{2} \sum_{\alpha=1}^{q} (\varepsilon f_\alpha - i f_\alpha) (\varepsilon \nabla f_\alpha - i \nabla f_\alpha) - \frac{1}{4} \| \tau \|^2 \]

\[ + \frac{1}{2} \sum_{\alpha,\beta} (\varepsilon f_\alpha - i f_\alpha) (\varepsilon f_\beta - i f_\beta) [R_{A_{T^H}M^*} (f_\alpha, f_\beta) - \nabla R_{(f_\alpha, f_\beta)}]. \]

As in the classical case, we have

(13) \[ \frac{1}{2} \sum_{\alpha,\beta} \varepsilon f_\alpha \varepsilon f_\beta R_{A_{T^H}M^*} (f_\alpha, f_\beta) = 0, \]

(14) \[ \frac{1}{2} \sum_{\alpha,\beta} i f_\alpha i f_\beta R_{A_{T^H}M^*} (f_\alpha, f_\beta) = 0. \]

The following lemma seems to be well known, but we didn’t find an appropriate reference.

**Lemma 9.** We have

\[ d^2_H = - \frac{1}{2} \sum_{\alpha,\beta} \varepsilon f_\alpha \varepsilon f_\beta \nabla R_{(f_\alpha, f_\beta)} \]

and

\[ (d^*_H)^2 = - \frac{1}{2} \sum_{\alpha,\beta} i f_\alpha i f_\beta \nabla R_{(f_\alpha, f_\beta)} - \sum_{\alpha} i f_\alpha i \nabla f_\alpha. \]

**Proof.** (1) By Lemma 6, we have

\[ d^2_H \omega = \sum_{\alpha,\beta} f_\alpha \wedge \nabla f_\alpha f_\beta \wedge \nabla f_\beta \omega + \sum_{\alpha,\beta} f_\alpha \wedge f_\beta \wedge \nabla f_\alpha \nabla f_\beta \omega. \]

As above, write \( \nabla f_\alpha f_\beta = \sum_{\gamma} a_{\alpha\beta}^{\gamma} f_\gamma, \) where \( a_{\alpha\beta}^{\gamma} = -a_{\alpha\beta}^{\gamma}. \) Then, for the first term, we have

\[ \sum_{\alpha,\beta} f_\alpha \wedge \nabla f_\alpha f_\beta \wedge \nabla f_\beta \omega = \sum_{\alpha,\beta,\gamma} a_{\alpha\beta}^{\gamma} f_\alpha \wedge f_\gamma \wedge \nabla f_\beta \omega \]

\[ = - \sum_{\alpha,\beta,\gamma} a_{\alpha\beta}^{\gamma} f_\alpha \wedge f_\gamma \wedge \nabla f_\beta \omega \]

\[ = - \sum_{\alpha,\gamma} f_\alpha \wedge f_\gamma \wedge \nabla f_\alpha f_\beta \omega \]

\[ = - \frac{1}{2} \sum_{\alpha,\gamma} f_\alpha \wedge f_\gamma \wedge (\nabla f_\alpha f_\beta - \nabla f_\alpha f_\beta) \omega \]

\[ = - \frac{1}{2} \sum_{\alpha,\gamma} f_\alpha \wedge f_\gamma \wedge (\nabla (f_\alpha f_\gamma) + R_{(f_\alpha, f_\gamma)}) \omega \]
For the second term, we use the definition of the curvature $R$ and \cite{13}:

$$
\sum_{\alpha,\beta} f_\alpha \wedge f_\beta \wedge \nabla f_\alpha \nabla f_\beta \omega = \frac{1}{2} \sum_{\alpha,\beta} f_\alpha \wedge f_\beta \wedge (\nabla f_\alpha \nabla f_\beta - \nabla f_\beta \nabla f_\alpha) \omega \\
= \frac{1}{2} \sum_{\alpha,\beta} f_\alpha \wedge f_\beta \wedge (\nabla [f_\alpha, f_\beta] + R(f_\alpha, f_\beta)) \omega \\
= \frac{1}{2} \sum_{\alpha,\beta} f_\alpha \wedge f_\beta \wedge [f_\alpha, f_\beta] \omega.
$$

(2) Similarly, using Lemma \cite{6} we get

$$
(d_H^*)^2 = \sum_{\alpha,\beta} i_{f_\alpha} i_{\nabla f_\alpha} f_\beta \nabla f_\beta + \sum_{\alpha,\beta} i_{f_\alpha} i_{f_\beta} \nabla f_\alpha \nabla f_\beta - \sum_{\alpha} (i_{\tau} i_{f_\alpha} \nabla f_\alpha + i_{f_\alpha} \nabla f_\alpha i_{\tau}).
$$

Repeating the same arguments as above, we obtain

$$
\sum_{\alpha,\beta} i_{f_\alpha} i_{\nabla f_\alpha} f_\beta \nabla f_\beta + \sum_{\alpha,\beta} i_{f_\alpha} i_{f_\beta} \nabla f_\alpha \nabla f_\beta = -\frac{1}{2} \sum_{\alpha,\beta} i_{f_\alpha} i_{f_\beta} \nabla R(f_\alpha, f_\beta).
$$

For the third term, we have

$$
\sum_{\alpha} (i_{\tau} i_{f_\alpha} \nabla f_\alpha + i_{f_\alpha} \nabla f_\alpha i_{\tau}) = \sum_{\alpha} (i_{\tau} i_{f_\alpha} + i_{f_\alpha} i_{\tau}) \nabla f_\alpha + i_{f_\alpha} \nabla f_\alpha i_{\tau} \\
= \sum_{\alpha} i_{f_\alpha} i_{\nabla f_\alpha} \tau.
$$

\[\square\]

By \cite{9} and Lemma \cite{8} it follows that

$$
D_{\Lambda^H M^*}^2 - d_H^2 - (d_H^*)^2 = \sum_{\alpha=1}^q \nabla f_\alpha \nabla f_\alpha - \frac{1}{2} \sum_{\alpha=1}^q (\varepsilon f_\alpha - i_{f_\alpha}) (\varepsilon \nabla f_\alpha - i_{\nabla f_\alpha} \tau - \frac{1}{4} \|\tau\|^2 - \sum_{\alpha,\beta} \varepsilon f_\alpha i_{f_\beta} [R_{\Lambda^H M^*}(f_\alpha, f_\beta) - \nabla R(f_\alpha, f_\beta)] + \sum_{\alpha} i_{f_\alpha} i_{\nabla f_\alpha} \tau.
$$

Recall that $D_{\Lambda^H M^*}$ is given by the formula

$$
D_{\Lambda^H M^*} = \sum_{\alpha=1}^q (\varepsilon f_\alpha - i_{f_\alpha}) \left( \nabla_{\Lambda^H M^*} f_\alpha - \frac{1}{2} g_M (\tau, f_\alpha) \right)
$$

Using the identity $(\varepsilon_u - i_u)(\varepsilon_v + i_v) + (\varepsilon_v + i_v)(\varepsilon_u - i_u) = 0$ for any $u$ and $v$, we get

$$
D_{\Lambda^H M^*}(\varepsilon \tau + i_{\tau}) + (\varepsilon \tau + i_{\tau}) D_{\Lambda^H M^*}
$$
\[
\sum_{\alpha=1}^{q} (\varepsilon f_\alpha - i f_\alpha) \left[ \nabla_{f_\alpha}^{\Lambda T H} M^* - \frac{1}{2} g_M(\tau, f_\alpha), \varepsilon \tau^* + i \tau \right] = \sum_{\alpha=1}^{q} (\varepsilon f_\alpha - i f_\alpha) (\varepsilon \nabla_{f_\alpha} \tau^* + i \nabla_{f_\alpha} \tau).
\]

The above identities and the formula \( \varepsilon \tau^* i \tau + i \tau \varepsilon \tau^* = \| \tau \|^2 \) immediately complete the proof.

4.3. **A direct proof.** Here we indicate a direct proof of Theorem 8. By Lemma 6, we have

\[
d_H^* d_H = - \sum_{\alpha, \beta} i f_\alpha \varepsilon f_\beta \nabla_{f_\alpha} \nabla_{f_\beta} - \sum_{\alpha, \beta} i f_\alpha \varepsilon f_\beta \nabla_{f_\alpha} \nabla_{f_\beta} + \sum_{\alpha=1}^{q} i \tau \varepsilon f_\alpha \nabla_{f_\alpha}
\]

and

\[
d_H d_H^* = - \sum_{\alpha, \beta} \varepsilon f_\alpha i f_\beta \nabla_{f_\alpha} \nabla_{f_\beta} - \sum_{\alpha, \beta} \varepsilon f_\alpha i f_\beta \nabla_{f_\alpha} \nabla_{f_\beta} + \sum_{\alpha=1}^{q} \varepsilon f_\alpha \nabla_{f_\alpha} i \tau.
\]

From these identities, it follows that

\[
\Delta_H = - \sum_{\alpha, \beta} (i f_\alpha \varepsilon f_\beta + \varepsilon f_\alpha i f_\beta) \nabla_{f_\beta} - \sum_{\alpha, \beta} (i f_\alpha \varepsilon f_\beta + \varepsilon f_\alpha i f_\beta) \nabla_{f_\beta} \nabla_{f_\beta}
\]

\[
+ \sum_{\alpha=1}^{q} (i \tau \varepsilon f_\alpha \nabla_{f_\alpha} + \varepsilon f_\alpha \nabla_{f_\alpha} i \tau).
\]

Writing \( \nabla_{f_\alpha} f_\beta = \sum_{\gamma} a^{\gamma}_{\alpha \beta} f_\gamma \), where \( a^{\gamma}_{\alpha \beta} = -a^{\beta}_{\alpha \gamma} \), we get

\[
\sum_{\alpha, \beta} (i f_\alpha \varepsilon f_\beta + \varepsilon f_\alpha i f_\beta) \nabla_{f_\beta} = \sum_{\alpha, \beta} a^{\gamma}_{\alpha \beta} (i f_\alpha \varepsilon f_\gamma + \varepsilon f_\alpha i f_\gamma) \nabla_{f_\beta}
\]

\[
= - \sum_{\alpha, \beta, \gamma} a^{\beta}_{\alpha \gamma} (i f_\alpha \varepsilon f_\gamma + \varepsilon f_\alpha i f_\gamma) \nabla_{f_\beta}
\]

\[
= - \sum_{\alpha, \gamma} (i f_\alpha \varepsilon f_\gamma + \varepsilon f_\alpha i f_\gamma) \nabla_{f_\alpha} f_\gamma
\]

\[
= - \sum_{\alpha, \gamma} (i f_\alpha \varepsilon f_\gamma + \varepsilon f_\gamma i f_\alpha) \nabla_{f_\alpha} f_\gamma
\]

\[
= - \sum_{\alpha, \gamma} (\varepsilon f_\gamma i f_\alpha - \varepsilon f_\alpha i f_\gamma) \nabla_{f_\alpha} f_\gamma
\]

\[
= - \nabla \sum_{\alpha} \nabla_{f_\alpha} f_\alpha - \sum_{\alpha, \gamma} \varepsilon f_\alpha i f_\gamma \nabla_{f_\alpha} f_\gamma - \nabla f_\gamma f_\alpha
\]

\[
= - \nabla \sum_{\alpha} \nabla_{f_\alpha} f_\alpha - \sum_{\alpha, \gamma} \varepsilon f_\alpha i f_\gamma \nabla_{f_\alpha} f_\gamma + \mathcal{R}(f_\alpha, f_\gamma).
\]
We also have
\[
\sum_{\alpha,\beta} (i_{f_\alpha} \varepsilon_{f_\beta} + \varepsilon_{f_\alpha} i_{f_\beta}) \nabla_{f_\alpha} \nabla_{f_\beta} = \sum_{\alpha,\beta} (i_{f_\alpha} \varepsilon_{f_\beta} + \varepsilon_{f_\beta} i_{f_\alpha}) \nabla_{f_\alpha} \nabla_{f_\beta}
\]

\[
+ \sum_{\alpha,\beta} (\varepsilon_{f_\alpha} i_{f_\beta} - \varepsilon_{f_\beta} i_{f_\alpha}) \nabla_{f_\alpha} \nabla_{f_\beta}
\]

\[
= \sum_{\alpha} (\nabla_{f_\alpha})^2 + \sum_{\alpha,\beta} \varepsilon_{f_\alpha} i_{f_\beta} (\nabla_{f_\alpha} \nabla_{f_\beta} - \nabla_{f_\beta} \nabla_{f_\alpha})
\]

\[
= \sum_{\alpha} (\nabla_{f_\alpha})^2 + \sum_{\alpha,\beta} \varepsilon_{f_\alpha} i_{f_\beta} (\nabla_{[f_\alpha,f_\beta]} + R(f_\alpha,f_\beta)).
\]

Taking into account (7), we immediately complete the proof of Theorem 8.

REFERENCES

[1] Berline, N., Getzler, E., Vergne, M.: Heat kernels and Dirac operators. Springer-Verlag, Berlin, 1992.
[2] Borthwick, D.; Uribe, A.: Almost complex structures and geometric quantization. Math. Res. Lett. 3, 845–861 (1996)
[3] Braverman, M.: Vanishing theorems for the kernel of a Dirac operator. Preprint arXiv:math.DG/9805127
[4] Braverman, M.: Vanishing theorems on covering manifolds, Tel Aviv Topology Conference: Rothenberg Festschrift (1998), 1–23, Contemp. Math., 231, Amer. Math. Soc., Providence, RI, 1999.
[5] Dominguez, D.: Finiteness and tenseness theorems for Riemannian foliations. Amer. J. Math. 120, 1237–1276 (1998)
[6] Douglas, R. G.; Glazebrook, J. F.; Kamber, F. W.; Yu, G.: Index formulas for geometric Dirac operators in Riemannian foliations. K-Theory 9, 407–441 (1995)
[7] Glazebrook, J. F.; Kamber, F. W.: Transversal Dirac families in Riemannian foliations. Comm. Math. Phys. 140, 217–240 (1991)
[8] Glazebrook, J. F.; Kamber, F. W.: On spectral flow of transversal Dirac operators and a theorem of Vafa-Witten. Ann. Global Anal. Geom. 9, 27–35 (1991)
[9] Glazebrook, J. F.; Kamber, F. W.: Secondary invariants and chiral anomalies of basic Dirac families. Differential Geom. Appl. 3, 285–299 (1993)
[10] Guillemin, V.; Uribe, A.: The Laplace operator on the nth tensor power of a line bundle: eigenvalues which are uniformly bounded in n. Asymptotic Anal. 1, 105–113 (1988)
[11] Jung, S. D.: The first eigenvalue of the transversal Dirac operator. J. Geom. Phys. 39, 253–264 (2001)
[12] Jung, S. D.; Ko, Y. S.: Eigenvalue estimates of the basic Dirac operator on a Riemannian foliation. Taiwanese J. Math. 10, 1139–1156 (2006)
[13] Jung, S. D.: Eigenvalue estimates for the basic Dirac operator on a Riemannian foliation admitting a basic harmonic 1-form. J. Geom. Phys. 57, 1239–1246 (2007)
[14] Kamber, F. W.; Tondeur, Ph. Foliations and metrics. Differential geometry (College Park, Md., 1981/1982), 103–152, Progr. Math., 32, Birkhauser Boston, Boston, MA, 1983.
[15] Kordyukov, Yu. A.: Noncommutative spectral geometry of Riemannian foliations. Manuscripta Math. 94, 45–73 (1997)
[16] Kordyukov, Yu. A.: Egorov’s theorem for transversally elliptic operators on foliated manifolds and noncommutative geodesic flow. Math. Phys. Anal. Geom. 8, 97–119 (2005)
Kordyukov, Yu. A.: The Egorov theorem for transverse Dirac type operators on foliated manifolds, submitted.

Ma, X.; Marinescu, G.: The Spin$^c$ Dirac operator on high tensor powers of a line bundle. Math. Z. 240, 651–664 (2002)

Ma, X.; Marinescu, G.: The first coefficients of the asymptotic expansion of the Bergman kernel of the Spin$^c$ Dirac operator. Internat. J. Math. 17, 737–759 (2006)

Tondeur, Ph. Geometry of foliations. Birkhäuser Verlag, Basel, 1997.

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