Classes of exact wavefunctions for general time-dependent Dirac Hamiltonians in 1+1 dimensions

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Abstract

In this work we construct two classes of exact solutions for the most general time-dependent Dirac Hamiltonian in 1+1 dimensions. Some problems regarding to some formal solutions in the literature are discussed. Finally the existence of a generalized Lewis-Riesenfeld invariant connected with such solutions is discussed.

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The interest in solving problems involving time-dependent systems has attracted the attention of physicists since a long time. This happens due to its applicability for the understanding of many problems in quantum optics, quantum chemistry and others areas of physics [1]-[11]. In particular we can cite the case of the electromagnetic field intensities in a Fabry-Pérot cavity [1]. In fact this kind of problem still represents a line of investigation which attracts the interest of physicists [10][11]. However, as has been observed recently by Landim and Guedes [12], the most part of these works deal with nonrelativistic systems. So they tried to bridge this gap, by studying the problem of a fermion under the presence of a time-dependent Lorentz vector linear potential in two-dimensional space-time. For this they used the so called Lewis-Riesenfeld invariant operator, in order to guess the wavefunction which solves the problem. Unfortunately, they just presented a formal solution for the problem under question. In fact, if one consider that in the most part of the works treating the non-relativistic time-dependent systems, the solution is not formal only for a limited number of time-dependence for the potential parameters, it is a little bit strange that in [12] there is no limitation over those parameters. In view of these comments, as a first example, we are going to complete the program initiated by Landim and Guedes, by solving the problem up to the end, showing that there exists really some restrictions for a complete solution of this problem. After that, we will show that there exists at least two classes of solutions for the time-dependent Dirac equation in 1+1 dimensions. This is done considering the most general combination of Lorentz structures for the potential matrix. Then we show that the problem studied in [12] is only a particular case of one of those classes. Finally we construct the Lewis-Riesenfeld invariant operators which have our solutions as eigenfunctions.

Let us begin by presenting the Dirac equation in 1+1 dimensions, in the presence of a time-dependent potential for a fermion of rest mass $m$,

$$H \psi(q, t) = \frac{i}{\hbar} \frac{\partial \psi(q, t)}{\partial t}, \quad H = \alpha p + \beta m + V_V(q, t) + \beta V_S(q, t) + \alpha \beta V_P(q, t),$$

where we used $c = \hbar = 1$ and $p$ is the momentum operator. $\alpha$ and $\beta$ are Hermitian square matrices satisfying the relations $\alpha^2 = \beta^2 = 1, \{\alpha, \beta\} = 0$. Here we choose $\alpha = \sigma_1$ and $\beta = \sigma_3$, where $\sigma_1$ and $\sigma_3$ are the $2 \times 2$ standard Pauli matrices. In this way $\psi$ is a two-component spinor. Furthermore the
subscripts for the terms of potential denote their properties under a Lorentz transformation: $V$ for the time component of the two-vector potential, $S$ and $P$ for the scalar and pseudoscalar terms, respectively. The absence of the space component of the two-vector potential is due to the possibility of its elimination through a gauge-like transformation.

In the particular case considered by Landim and Guedes [12], one has

$$H = \alpha p + \beta m + f(t) \eta,$$

(2)

As suggested by them, we will use as an Ansatz for the solution the following spinor

$$\psi = \left( \begin{array}{c} M_1(t) \\ M_2(t) \end{array} \right) e^{i \eta(t) \eta},$$

(3)

so that the corresponding coupled equations are

\begin{align*}
i \dot{M}_1 &= [(\dot{\eta} + f) \eta + m] M_1 + \eta M_2, \\
i \dot{M}_2 &= [(\dot{\eta} + f) \eta - m] M_2 + \eta M_1,
\end{align*}

(4)

where the dot denotes differentiation with respect to $t$. Imposing that $\eta(t) = \int_t^t f(\lambda) d\lambda$ we eliminate the spatial dependence of the above equations so obtaining

\begin{align*}
i \dot{M}_1 &= m M_1 + \eta M_2, \\
i \dot{M}_2 &= -m M_2 + \eta M_1.
\end{align*}

(5)

Now we simplify even more the above equations if we perform the following identifications

$$M_1 \equiv G_1 e^{-i m t}, \quad M_2 \equiv G_2 e^{i m t},$$

(6)

getting

$$\dot{G}_1 = -\eta_1 G_2, \quad \dot{G}_2 = -\eta_2 G_1,$$

(7)
where \( \eta_1 \equiv \eta e^{2imt} \) and \( \eta_2 \equiv \eta e^{-2imt} \). By deriving the each one of the above equations it is easy to conclude that one gets the following second-order equations

\[
\ddot{G}_i - \left( \frac{\ddot{\eta}_i}{\eta_i} \right) \dot{G}_i + \eta^2 G_i = 0, \quad i = 1, 2. \tag{8}
\]

At this point it is convenient to rescale the \( G_i \) functions as \( G_i = \sqrt{\eta_i} g_i \), one gets finally that

\[
\ddot{g}_i + \left[ \frac{1}{2} \left( \frac{\ddot{\eta}_i}{\eta_i} \right) - \left( \frac{\dot{\eta}_i}{\eta_i} \right)^2 + \eta^2 \right] g_i = 0. \tag{9}
\]

Note that in general these last equations are not somewhat straightforward to solve as asserted in [12]. However we note that for a time exponentially decaying force, these equations takes the form a Schroedinger equation for the Morse potential in the time variable, without the usual boundary conditions of such kind of equation.

¿From now on we will present an extension of the class of time-dependent relativistic systems with exact solutions. For this purpose we begin with the complete Hamiltonian (1), and make the more general Ansatz

\[
\psi = \begin{pmatrix} M_1(t) e^{i F_1(q,t)} \\ M_2(t) e^{i F_2(q,t)} \end{pmatrix}. \tag{10}
\]

Thus we obtain

\[
i \dot{M}_1 = \left[ \dot{\hat{F}}_1 + V_V + V_S + m \right] M_1 + (F'_2 - V_P) e^{i(F_2-F_1)} M_2,
\]

\[
i \dot{M}_2 = \left[ \dot{\hat{F}}_2 + V_V - V_S - m \right] M_2 + (F'_1 + V_P) e^{-i(F_2-F_1)} M_1, \tag{11}
\]

where the prime denotes differentiation with respect to \( q \). As by construction \( M_i \) do not depend on \( q \), one sees that it is mandatory to get rid of such a dependence in the above equations. As a consequence two classes of solution emerge, corresponding to \( F_1 = F_2 \) or \( F_1 = -F_2 \).
Let us analyze the class where $F_1 = F_2 = F$. In this case the exponential factor appearing in the equations disappears and, consequently the only way of getting the off-diagonal terms independent of $q$, is by imposing that

$$V_P = V_P(t), \quad F = \theta_1(t) \cdot q.$$ \hspace{1cm} (12)

On the other hand, the effect of this condition over the diagonal terms results that

$$V_S = V_S(t), \quad V_V = -\dot{\theta}_1(t) \cdot q + \theta_2(t).$$ \hspace{1cm} (13)

Note that this class allows the treatment of systems that are at most linear in the spatial coordinate, and includes the system proposed by Landim and Guedes as a particular case. The general equations to be solved in this class is now given by

$$i \dot{M}_1 = (\chi_+ + m) M_1 + \eta_- M_2,$$

$$i \dot{M}_2 = (\chi_- - m) M_2 + \eta_+ M_1,$$ \hspace{1cm} (14)

with $\chi_\pm \equiv \theta_2 \pm V_S$ and $\eta_\pm \equiv \theta_1 \pm V_P$. The above equations can now be decoupled giving

$$i \ddot{G}_1 - \left[ \chi_+ + \chi_- + i \left( \frac{\dot{\eta}_1}{\eta_1} \right) \right] \dot{G}_1 - \left[ \dot{\chi}_+ - \left( \frac{\dot{\eta}_1}{\eta_1} \right) \chi_- - i \eta_+ \eta_- + i \chi_+ \chi_- \right] G_1 = 0,$$

$$i \ddot{G}_2 - \left[ \chi_+ + \chi_- + i \left( \frac{\dot{\eta}_2}{\eta_2} \right) \right] \dot{G}_2 - \left[ \dot{\chi}_- - \left( \frac{\dot{\eta}_2}{\eta_2} \right) \chi_+ - i \eta_+ \eta_- + i \chi_+ \chi_- \right] G_2 = 0,$$ \hspace{1cm} (15)

where we made the same definition as above for the $G_i$ functions and now $\eta_1 \equiv \eta_- e^{2imt}$ and $\eta_2 \equiv \eta_+ e^{-2imt}$. It is easy to verify that the above equations recall that appearing in (8) in their particular case. Once again it is in general not solvable, showing that we have got a formal solution, so that if one want to solve the problem until the end, one must to look for particular cases where these equations have explicit solutions.

¿From now on we analyze the class where $F_1 = -F_2 = F$. In this new class, the exponential factor appearing in the equations holds and, consequently the only way of getting the off-diagonal terms independent of $q$ in
the equations (11), is by imposing that
\[ F' = -V_P (q,t). \] (16)

Now, the condition of spatial independency of the diagonal terms implies that
\[ V_V = V_V (t), \quad V_S = V_S (q,t) = \gamma (t) - \dot{F} (q,t), \] (17)
where \( \gamma (t) \) is an arbitrary function of the time. In this case the equation decouple giving simply
\[ i \dot{M}_1 = (V_V + \gamma + m) M_1, \quad i \dot{M}_2 = (V_V - \gamma - m) M_2. \] (18)

Now it is easy to obtain the solution of these equations. These are given by
\[ M_1 (t) = M_1 (0) e^{-i \{ m t + \int [V_V (\lambda) + \gamma (\lambda)] d\lambda \}}, \] (19)
\[ M_2 (t) = M_2 (0) e^{i \{ m t - \int [V_V (\lambda) - \gamma (\lambda)] d\lambda \}}. \]

At this point some comments are in order. From above we easily conclude that for this second class of solutions, one really obtains arbitrary non-formal solutions, provided that the remaining integrals can be done. Here it is important to remark that this new class of systems allows arbitrary dependence on the spatial and time variables, in contrast to the previous class of systems.

In what follows, we try to construct an invariant operator of the Lewis and Riesenfeld type [13], which have the above classes of solutions as eigenfunctions. It could be used, as done in [12], in order to suggest the form of the spinor presented as an Ansatz in this work. The invariant obviously must satisfy the equation
\[ \frac{dI}{dt} = \frac{\partial I}{\partial t} - i [I, H] = 0, \] (20)
and can be written as
\[ I = \mathcal{A} (t) \ p + \mathcal{F} (q,t), \] (21)
where \( A \) and \( F \) are \( 2 \times 2 \) matrices, whose elements, in order to guarantee the validity of the above equation (20), must obey the following set of coupled differential equations

\[
\begin{align*}
A_{11} &= A_{22} = \text{const.}, \quad A_{12} = A_{21} = \text{const.}, \\
\Lambda'_- &= -A_{11} V'_p, \quad \Sigma_- = A_{12} \left( m + V_S \right), \\
\dot{\Sigma}_+ &= \Lambda_+ + \Lambda_- - A_{11} V'_p, \\
-i \dot{\Sigma}_- &= 2 \Lambda_+ - i A_{12} V'_p - i A_{11} V'_S, \\
i \dot{\Lambda}_+ &= 2 \Sigma_- V_P + i \Sigma'_- + i A_{12} V'_V + 2 \Lambda_- \left( m + V_S \right), \\
-i \dot{\Lambda}_- &= i \Sigma'_+ + i A_{11} V'_P + i A_{12} V'_S - 2 \Lambda_+ \left( m + V_S \right),
\end{align*}
\]

(22)

where we defined that \( \Lambda_{\pm} \equiv \frac{F_{12} \mp F_{21}}{2} \) and \( \Sigma_{\pm} \equiv \frac{F_{11} \pm F_{22}}{2} \). On the other hand, we are looking for invariant operators which have the Dirac spinors as eigenfunctions, which implies into further conditions over the elements of the matrices \( A \) and \( F \). This lead us to the following conditions if we consider the eigenvalue equation for the invariant operator acting over the spinor (10) introduced in this work

\[
\begin{align*}
A_{11} (F'_1 + F'_2) + F_{11} + F_{22} &= \chi_1 (t), \\
(A_{12} F'_1 + F_{21}) (A_{12} F'_2 + F_{12}) &= \chi_2 (t),
\end{align*}
\]

(23)

with the functions \( \chi_{1,2} \) being arbitrary functions of time. It is not difficult to show that for the first class one can obtain a solution, which includes that proposed in [12] as a particular case, given by

\[
\begin{align*}
A_{12} &= A_{21} = F_{12} = F_{21} = 0, \quad A_{11} = A_{22} \equiv A, \quad F_{11} = F_{22} \equiv F, \\
V_S &= V_S (t), \quad V_P = V_P (t), \quad V_V = \frac{\dot{\xi}}{A} q + \chi (t).
\end{align*}
\]

(24)
For the second class of solutions however, as can be easily realised from the above set of coupled nonlinear equations (22) and the constraints coming from (23), it is quite difficult to extract simple solutions. For this reason we were not able to verify if there exist a relativistic invariant which is linear in the momentum, responsible for the generalized solutions here presented. In fact, as it is typical of nonlinear equations, different solutions of it could lead to independent sets of wavefunctions. Finally it is interesting to make some comments, the first one being the observation that the extension of the solutions discussed in this work, by introducing a time-dependent mass is quite simple, as can be seen due to its appearance combined with time-dependent potentials $V_S$ and $V_V$. On the other hand, the case of the so called Dirac oscillator [14]is not directly solvable from the second class of solutions, this happens because when $V_P$ is linear in the coordinate, it implies into the need of a $V_S$ quadratic in $q$, in order to guarantee the exact solution of the form guessed in this work.

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