Abstract: The calculation of the magnetic moment of the deuteron in the framework of the Bethe–Salpeter approach is performed. The relativistic corrections are calculated analytically and estimated numerically. It is shown that the main contributions are due to partial waves with positive energies and P–waves. A comparison with the non-relativistic schemes of calculations including mesonic exchange currents is made.

1 Introduction

The relativistic description of the deuteron has a long history. This problem still remains present although non-relativistic schemes of calculations are widely used to analyze reactions with the deuteron. There are different approaches which take into account relativistic effects. They can be separated into three groups: (i) the Bethe–Salpeter approach [1], (ii) a reduction of the Bethe–Salpeter equation (quasi-potential, light front dynamics), and (iii) prescriptional treatments.

The Bethe–Salpeter approach is the most consistent one. However, it contains some difficulties that do not allow to establish a direct link to the non-relativistic calculations – e.g. the absence of a non-relativistic reduction of arbitrary kernels or the problem to interpret the abnormal parity states.

Non-relativistic calculations with mesonic exchange currents give a reasonable description of the electromagnetic processes of the deuteron (e.g. elastic scattering and electro disintegration). It is important to note that those *mesonic exchange currents treated in a non–relativistic framework* are partially included in the above mentioned relativistic calculations (i) and (ii) without additional assumptions. This has been demonstrated e.g. in ref. [2] for the electro disintegration of the deuteron in the formalism of light front dynamics.

Here, we present the investigation of a simple electromagnetic property of the deuteron, the magnetic moment, in the framework of the Bethe–Salpeter approach, and show the connection to the results of a non-relativistic calculation.

2 Basic definitions

The main object of the Bethe–Salpeter approach is the vertex function $\Gamma(P,p)$ which obeys the Bethe–Salpeter equation, here written within the matrix formalism:

$$
[\Gamma(P,p)C] = i \int \frac{d^4k}{(2\pi)^4} v(p,k) \Gamma^{(1)}(\frac{P}{2} + k) \left[ S^{(1)}(\frac{P}{2} - k) \Gamma^{(2)}(P - k) \right],
$$

(1)
where \( C \) is charge conjugation matrix, \( v(p, k)\Gamma^{(1)} \otimes \Gamma^{(2)} \) is the interaction kernel, and \( S^{(i)}, \tilde{S}^{(i)} \) are Fermi propagators. The vertex function for the deuteron can be decomposed into basis states of definite values of orbital momentum \( L \), spin \( S \) and \( \rho \)-spin \( \rho \) \[3\]. \[4\]. The general form in the rest frame is given by

\[
\Gamma_M(P, p) = \sum_{\alpha} g_\alpha(p_0, |p|) \Gamma^\alpha_M(p),
\]  

(2)

where \( g_\alpha \) are radial functions, and \( \Gamma^\alpha_M(p) \) are spin - angular momentum functions.

In order to exhibit the \( \rho \)-spin dependence, the spin – angular momentum functions \( \Gamma^\alpha_M(p) \) may be replaced via \( \Gamma^\alpha_M(p) \equiv \Gamma^\tilde{\alpha}_M(p) \), where

\[
\begin{align*}
\Gamma^\tilde{\alpha}_{M}^{++}(p) &= \frac{\hat{p}_2 + m}{\sqrt{2E(m + E)}} \frac{1 + \gamma_0}{2} \tilde{\Gamma}^\alpha_M(p) \frac{\hat{p}_1 - m}{\sqrt{2E(m + E)}}, \\
\Gamma^\tilde{\alpha}_{M}^{--}(p) &= \frac{\hat{p}_1 - m}{\sqrt{2E(m + E)}} \frac{-1 + \gamma_0}{2} \tilde{\Gamma}^\alpha_M(p) \frac{\hat{p}_2 + m}{\sqrt{2E(m + E)}}, \\
\Gamma^\tilde{\alpha}_{M}^{+-}(p) &= \frac{\hat{p}_2 + m}{\sqrt{2E(m + E)}} \frac{1 + \gamma_0}{2} \tilde{\Gamma}^\alpha_M(p) \frac{\hat{p}_1 - m}{\sqrt{2E(m + E)}}, \\
\Gamma^\tilde{\alpha}_{M}^{-+}(p) &= \frac{\hat{p}_1 - m}{\sqrt{2E(m + E)}} \frac{1 - \gamma_0}{2} \tilde{\Gamma}^\alpha_M(p) \frac{\hat{p}_2 + m}{\sqrt{2E(m + E)}},
\end{align*}
\]  

(3)

with \( \tilde{\alpha} \in \{L, S, J\} \), \( m \) the nucleon mass, \( \hat{p} = \gamma_\mu p^\mu \), and \( \tilde{\Gamma}^\alpha_M \) given by

\[
\begin{align*}
\tilde{\alpha} & \quad \sqrt{8\pi} \tilde{\Gamma}^\alpha_M \\
3S_1 & \quad -\frac{1}{\sqrt{2}} \left[ \tilde{\xi}_M + \frac{3}{2}(\hat{p}_1 - \hat{p}_2)(p\xi_M)|p|^{-2} \right] \\
3D_1 & \quad \frac{\sqrt{2}}{2} \left[ \frac{3}{2}\hat{\xi}_M(\hat{p}_1 - \hat{p}_2) - (p\xi_M) \right] |p|^{-1} \\
3P_1 & \quad \frac{\sqrt{3}}{2}(p\xi_M)|p|^{-1} \\
1P_1 & \quad \frac{\sqrt{8\pi}}{\Gamma^\alpha_M} \\

\end{align*}
\]

There are eight states in the deuteron channel (instead of two in the non-relativistic case), viz. \( 3S_1^{++}, 3D_1^{++}, 3S_1^{--}, 3D_1^{--}, 3P_1^e, 3P_1^o, 1P_1^e, 1P_1^o \) (notation: \( 2S+1L_J^{(\rho_2)} \)). The normalization condition for these functions can be written as:

\[
P_+ + P_- = 1,
\]

\[
P_+ = P_{3S_1^{++}} + P_{3D_1^{++}}, \\
P_- = P_{3S_1^{--}} + P_{3D_1^{--}} + P_{3P_1^e} + P_{1P_1^e} + P_{3P_1^o} + P_{1P_1^o},
\]  

(4)

introducing pseudo-probabilities \( P_\alpha \) that are negative for the states \( 3S_1^{--}, 3D_1^{--}, 3P_1^e, 3P_1^o, 1P_1^e, 1P_1^o \), and positive for \( 3S_1^{++}, 3D_1^{++} \) \[3\]. The calculation with realistic vertex functions give the following values:

\[
\begin{array}{cccc}
& 3D_1^{++} & 3D_1^{--} & 3P_1^e + 3P_1^o & 1P_1^e + 1P_1^o \\
[\%] & 4.8 & -6 \cdot 10^{-4} & -0.88 \cdot 10^{-2} & -2.5 \cdot 10^{-2} \\
[\%] & 5.1 & -3.4 \cdot 10^{-4} & -9 \cdot 10^{-2} & -2.4 \cdot 10^{-2}
\end{array}
\]

It is obvious that the main contribution to the normalization is due to the states with positive energies, and the contribution of the \( P \)-states is larger than that of the negative energies states by at least one order of magnitude.
3 The calculation of the magnetic moment

The matrix element of the electromagnetic current used to evaluate the magnetic moment of the deuteron is given by

\[
\langle P'M'|J_x|PM \rangle = \frac{ie}{2\pi^2 M} \int d^4p \ Tr \left\{ \Gamma_{M'}(P',p') S^{(1)} \left( \frac{P'}{2} + p' \right) \left[ \gamma_1 F_1^{(s)}(q^2) - \frac{\gamma_1 \hat{q} - \hat{q} \gamma_1}{4m} F_2^{(s)}(q^2) \right] \Psi_M(P,p) \right\},
\]

where \( \Psi_M(P,p) = S^{(1)}(\frac{P}{2} + p) \Gamma_M(P,p) S^{(2)}(\frac{P}{2} - p) \), the isoscalar form factors of the nucleons are given by \( F_{1,2}^{(s)}(q^2) \), and \( p' = p + q/2 \), \( P' = P + q \). The magnetic moment then is evaluated via

\[
\mu_D = \frac{1}{e M} \lim_{\eta \to 0} \sqrt{\frac{\langle M' = +1|J_x|M = 0 \rangle}{\eta \sqrt{1 + \eta}}},
\]

with \( \eta = -q^2/4M^2 \). Since the integral (3) vanishes at \( q^2 = 0 \), we expand the integrand of (3) in powers of \( \sqrt{\eta} \). This is done in the Breit system. To this end the vertex functions need to be transformed from the rest system to the Breit system using the following formulae.

\[
\Gamma_M(P^{(B)},p^{(B)}) = \Lambda(P^{(B)}) \Gamma_M(P^{(0)},p^{(0)}) \Lambda^{-1}(P^{(B)}),
\]

\[
\Gamma_M(P'^{(B)},p'^{(B)}) = \Lambda^{-1}(P^{(B)}) \Gamma_M(P^{(0)},p'^{(0)}) \Lambda(P^{(B)}),
\]

where \( \Lambda(P^{(B)}) = (M + \hat{P}^{(B)} \gamma_0)/\sqrt{2M(E^{(B)} + M)} \), and the vectors \( P^{(B)}, P'^{(B)}, p^{(B)}, p'^{(B)} \) are connected with the respective vectors in the rest system by

\[
P^{(B)} = \mathcal{L} P^{(0)}, \quad p^{(B)} = \mathcal{L} p^{(0)}, \quad P'^{(B)} = \mathcal{L}^{-1} P^{(0)}, \quad p'^{(B)} = \mathcal{L}^{-1} p^{(0)}.
\]

The Lorentz transformation matrix \( \mathcal{L} \) is of the form

\[
\mathcal{L} = \begin{pmatrix}
\sqrt{1 + \eta} & 0 & 0 & -\sqrt{\eta} \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-\sqrt{\eta} & 0 & 0 & \sqrt{1 + \eta}
\end{pmatrix}.
\]

From (4) it follows that \( p'^{(0)} = \mathcal{L} p^{(0)} = \mathcal{L} p^{(B)} + \frac{1}{2} q^{(B)} = \mathcal{L}^2 p^{(0)} + \frac{1}{2} \mathcal{L} q^{(B)} \). To simplify notation the vector \( p^{(0)} \) will be denoted as \( p^{(0)} \equiv p = (p_0, p_x, p_y, p_z) \). The components of the vector \( p'^{(0)} \equiv p' \) are then given by

\[
p'_0 = (1 + 2\eta)p_0 - 2\sqrt{\eta} \sqrt{1 + \eta} p_z - M \eta,
\]

\[
p'_x = p_x, \quad p'_y = p_y,
\]

\[
p'_z = (1 + 2\eta)p_z - 2\sqrt{\eta} \sqrt{1 + \eta} p_0 + M \sqrt{\eta} \sqrt{1 + \eta}.
\]

With the help of the transformations (4), the integral (3) may then be written as

\[
\langle P'M'|J_x|PM \rangle = \frac{ie}{2\pi^2 M} \int d^4p \ Tr \left\{ \Gamma_{M'}(P^{(0)},p') S^{(1)} \left( \frac{P^{(0)}}{2} + p' \right) \times \right.
\]

\[
\left. \times \Lambda(P^{(B)}) \left[ \gamma_1 F_1^{(s)}(q^2) - \frac{\gamma_1 \hat{q} - \hat{q} \gamma_1}{4m} F_2^{(s)}(q^2) \right] \Lambda(P^{(B)}) \Psi_M(P^{(0)},p) \left[ \Lambda^{-1}(P^{(B)}) \right]^2 \right\},
\]

(10)
where the wave function $\Psi_M(P^{(0)}, p)$ and the vertex function $\bar{\Gamma}_{M'}(P^{(0)}, p')$ are taken in the deuteron rest frame.

It is obvious from (11) that the sources of $\sqrt{\eta}$ terms may appear in (i) the matrix $\Lambda(P^{(B)})$, (ii) the propagator $S^{(1)}(\frac{P^{(0)}}{2} + p')$, and (iii) the vertex function $\bar{\Gamma}_{M'}(P^{(0)}, p')$. In detail this reads for the matrix (i)

$$\Lambda(P^{(B)})[\gamma_1 F_1^{(s)}(q^2) - \frac{\gamma_1 \hat{q} - \hat{q} \gamma_1}{4m} F_2^{(s)}(q^2)] \Lambda(P^{(B)}) = \frac{1}{2} (\gamma_1 + \sqrt{\eta} \gamma_1 \gamma_3 \gamma_0 - \frac{\kappa}{4m} (\gamma_1 \hat{q} - \hat{q} \gamma_1)), \quad [\Lambda^{-1}(P^{(B)})]^{2} = 1 + \sqrt{\eta} \gamma_0 \gamma_3, \quad (11)$$

and for the propagator (ii)

$$S^{(1)}(\frac{P^{(0)}}{2} + p') = S^{(1)}(\frac{P^{(0)}}{2} + p) \left[ 1 + \sqrt{\eta} \left( \frac{4M p_z}{(\frac{P^{(0)}}{2} + p)^2 - m^2} \right) \right] - \sqrt{\eta} \frac{2p_z \gamma_0 + (M - 2p_0) \gamma_3}{(\frac{P^{(0)}}{2} + p)^2 - m^2}. \quad (12)$$

The corrections due to the vertex function (iii) may be separated into ($\alpha$) corrections due to the radial wave functions, and ($\beta$) due to the spin – angular momentum function. This will be demonstrated in the following using only one component of the vertex function (4), i.e.

$$\bar{\Gamma}_{M'}(P^{(0)}, p') = g(p_0', |p'|) \bar{\Gamma}_{M'}(p'). \quad (13)$$

For the radial part we get the following expansion

$$g(p_0', |p'|) = g(p_0, |p|) + \sqrt{\eta} p_z \left\{ -2 \frac{\partial}{\partial p_0} + \frac{M - 2p_0 \partial}{|p| \partial |p|} \right\} g(p_0, |p|). \quad (14)$$

The spin – angular momentum part can be written (e.g. for the $^3D_{1}^{++}$ state) as

$$\bar{\Gamma}_{M'}(p) = N(E')(m - \hat{p}_1) \bar{\Gamma}(p', \xi)(m + \hat{p}_2), \quad (15)$$

where

$$\hat{p}_1 = \hat{p}_1 + \sqrt{\eta} (M - 2p_0) \left( \frac{p_z}{E} \gamma_0 - \gamma_3 \right),$$

$$\hat{p}_2 = \hat{p}_2 + \sqrt{\eta} (M - 2p_0) \left( \frac{p_z}{E} \gamma_0 + \gamma_3 \right), \quad (16)$$

$$N(E') = \frac{1}{2E'(m + E')} = \frac{1}{2E(m + E)} \left( 1 - \sqrt{\eta} \frac{M - 2p_0}{E^2(m + E)} (m + 2E)p_z \right), \quad (17)$$

and corrections to $\bar{\Gamma}(p', \xi)$ can be calculated using (16)–(17).

### 4 Results of calculations

The general formula for magnetic moment can be written as

$$\mu = \mu_+ + \mu_- + \mu_{2-}$$

$$\mu_+ = (\mu_p + \mu_n) (P_{3S_{1}^{++}} + P_{3D_{1}^{++}}) - \frac{3}{2} (\mu_p + \mu_n - \frac{1}{2}) P_{3D_{1}^{++}} + R_+,$$

$$\mu_{2-} = -(\mu_p + \mu_n) P_{3S_{1}^{--}} + P_{3S_{1}^{--}} + \frac{1}{2} (\mu_p + \mu_n) P_{3D_{1}^{--}} - \frac{5}{4} P_{3D_{1}^{--}} + R_{2-},$$

$$\mu_{1-} = -\frac{1}{2} (\mu_p + \mu_n) (P_{3P_{1}^{--}} + P_{3P_{1}^{--}}) + \frac{1}{4} (P_{3P_{1}^{--}} + P_{3P_{1}^{--}}) - \frac{1}{2} (P_{3P_{1}^{++}} + P_{3P_{1}^{--}}) + R_{1-},$$

4
where \( R_a \) are the relativistic correction terms, viz.

\[
\begin{align*}
R_+ &= -\frac{1}{3}(\mu_p + \mu_n)H_1^{3s_1^{++}} - \frac{1}{2}H_2^{3s_1^{++}} - \frac{m}{M}H_3^{3S_1^{++}} - \left(1 - \frac{2m}{M}\right)G_1^{3S_1^{++}} - \\
&\quad -\frac{1}{6}(\mu_p + \mu_n - 3)H_1^{3D_1^{++}} - \frac{1}{2}H_2^{3D_1^{++}} - \frac{m}{M}H_3^{3D_1^{++}} + \left(1 - \frac{2m}{M}\right)G_2^{3D_1^{++}} + \\
&\quad + \frac{\sqrt{2}}{3}\left[\mu_p + \mu_n - (1 + \frac{m}{M})\right]H_1^{3s_1^{++}, D_1^{++}}, \\
R_{2-} &= -\frac{1}{6}\left[\mu_p + \mu_n - (1 - \frac{2m}{M})\right]H_1^{3s_i^{--}} - \frac{m}{M}H_2^{3s_i^{--}} + \frac{m}{M}H_3^{3s_i^{--}} - \\
&\quad -\frac{1}{6}\left[\mu_p + \mu_n - 1 + \frac{4m}{M}\right]H_1^{3d_i^{--}} - \frac{m}{M}H_2^{3d_i^{--}} + \frac{m}{M}H_3^{3d_i^{--}} + \frac{3}{4}(1 - \frac{2m}{M})P_3D_i^{--} + \\
&\quad + \frac{\sqrt{2}}{3}\left[\mu_p + \mu_n - (1 - \frac{m}{M})\right]H_1^{3s_i^{--}, D_i^{--}}, \\
R_{1-} &= \frac{1}{2}\left(1 - \frac{2m}{M}\right)\left[\mu_p + \mu_n + \frac{1}{2}\right]\left(P_3p_i^+ + P_3p_i^+\right) - [\mu_p + \mu_n + 1]H_4^{3p^3_1, p_i^+} + \\
&\quad + \frac{1}{2}\left(1 - \frac{2m}{M}\right)\left(P_1p_i^+ + P_1p_i^+\right) - 2H_4^{3p^3_1, p_i^+} + \\
&\quad + G_4^{3p^3_1, p_i^+} + G_4^{3p^3_1, p_i^+} + G_5^{3p^3_1, p_i^+} + G_6^{3p^3_1, p_i^+},
\end{align*}
\]

and \( H_i^{\alpha, \alpha'} (H_i^{\alpha, \alpha} \equiv H_i^s) \) and \( G_i^s \) are integrals of the form \( \frac{i}{4\pi^2M} \int dp_0|p|^2dp \) with the integrands given by (compare with [4])

\[
\begin{align*}
H_i^{\alpha, \alpha'} &= \left(1 - \frac{m}{E}\right)[\phi_\alpha(p_0, |p|)\phi_{\alpha'}(p_0, |p|)] \times \left\{ \begin{array}{ll}
(E - M/2), & \text{for } \alpha(\alpha') = 3 S_i^{++}, D_i^{++}, \\
(E + M/2), & \text{for } \alpha(\alpha') = 3 S_i^{--}, D_i^{--},
\end{array} \right. \\
H_i^s &= \left(\frac{1}{E}(E - \frac{M}{2})\right)[\phi_\alpha(p_0, |p|)]^2 \times \left\{ \begin{array}{ll}
(E - M/2), & \text{for } \alpha = 3 S_i^{++}, D_i^{++}, \\
(E + M/2), & \text{for } \alpha = 3 S_i^{--}, D_i^{--},
\end{array} \right. \\
H_i^s &= \left(\frac{p_0^2}{E}\right)[\phi_\alpha(p_0, |p|)]^2, \quad H_i^{\alpha, \alpha'} = \left(\frac{m}{E}p_0\right)[\phi_\alpha(p_0, |p|)\phi_{\alpha'}(p_0, |p|)] \\
G_i^{3s_1^{++}} &= \frac{2E + 4m + 3M}{12E} (E - \frac{M}{2})[\phi_3s_1^{++}(p_0, |p|)]^2 \\
G_i^{3d_1^{++}} &= \frac{E - 4m + 3M}{12E} (E - \frac{M}{2})[\phi_3d_1^{++}(p_0, |p|)]^2 \\
G_i^{\alpha, \alpha'} &= \left(-\frac{2\sqrt{2}m^2}{ME}p_0\right)[\phi_\alpha(p_0, |p|)\phi_{\alpha'}(p_0, |p|)] \\
G_5^{3p^3_1, p_i^+} &= \left((\mu_p + \mu_n)\frac{\sqrt{2}M}{2} - \frac{\sqrt{2}M}{2}(1 - \frac{4m^2}{M^2})\right)[\phi_3p_i^+(p_0, |p|)\phi_1p_i^+(p_0, |p|)] \\
G_6^{3p^3_1, p_i^+} &= \left((\mu_p + \mu_n)\frac{\sqrt{2}M}{2} - \frac{\sqrt{2}M}{2}(1 - \frac{m^2}{E^2}(1 + \frac{4p_0^2}{M^2}))\right)[\phi_3p_i^+(p_0, |p|)\phi_1p_i^+(p_0, |p|)]
\end{align*}
\]

The relativistic corrections \( R_a \) can be estimated numerically for a separable model [4]. They are smaller than the dominant terms by at least one order of magnitude. Taking into account different orders of pseudo-probabilities one gets

\[
\mu_d = \mu_{NR} + \Delta \mu
\]
\[ \mu_{NR} = (\mu_p + \mu_n) + \frac{3}{2}(\mu_p + \mu_n - \frac{1}{2})P_{3D_1}^{++}, \]
\[ \Delta \mu = R_+ + \Delta \mu_{1-} \]
\[ \Delta \mu_{1-} = -\frac{3}{2}(\mu_p + \mu_n)(P_{3p_1} + P_{3p_1'}) - (\mu_p + \mu_n)(P_{1p_1} + P_{1p_1'}) - \frac{1}{4}(P_{3p_e} + P_{3p_o}) - \frac{1}{2}(P_{1p_e} + P_{1p_o}), \]

The contribution \( R_+ \) is negative and \( \Delta \mu_{1-} \) is positive. An estimation gives the following results
\[ R_+/\mu_{NR} = -(5.8 - 11) \times 10^{-2} \%, \quad \Delta \mu_{1-}/\mu_{NR} = 0.2 \% \] (19)

5 Conclusion

We have shown that the expression for the magnetic moment in the Bethe–Salpeter approach can be written in a form closer to non-relativistic calculations. The additional terms in equation (18) can be considered as relativistic corrections to the non-relativistic formula.

The experimental value of the magnetic moment is known with a high accuracy: \( \mu_{\text{exp}} = 0.857406(1) \). The non-relativistic value reflects only the D-state probability. Whereas in the relativistic corrections \( P \)-states play the dominant role.

The magnitude of the corrections can be compared with the contributions of mesonic exchange currents to the magnetic moment as extracted from ref. [8]. The main contribution is due to the pair term, which leads to \( \Delta \mu/\mu_{NR} = 0.21 - 0.22\% \) for different forms of the Bonn potential. The same size of this correction as compared to (19) may be considered as an indication that both corrections are of the same physical origin.

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References

[1] E.E.Salpeter and H.A.Bethe, Phys. Rev. C84, 1232 (1951).
[2] B. Desplanques et.al. Preprint IPNO/TH 94-78
[3] M.J.Zuilhof, J.A.Tjon Phys.Rev. C6,2369 (1980).
[4] J.J.Kubis, Phys. Rev. D6, 547 (1972).
[5] K.Kazakov, private communication. We are grateful to K. Kazakov for providing us with this numbers prior to publication
[6] N. Honzava, S. Ishida, Phys. Rev. C45, 47 (1992).
[7] G.Rupp, J.A.Tjon. Phys. Rev. C 37,1729 (1988).
[8] V.V. Burov, V.N. Dostovalov, S.E. Suskov. Elementary Particles and Atomic Nuclei 23(3),721 (1992).