Absolute Lower Bound on the Bounce Action

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The decay rate of a false vacuum is determined by the minimal action solution of the tunnelling field: bounce. In this Letter, we focus on models with scalar fields which have a canonical kinetic term in $N(>2)$ dimensional Euclidean space, and derive an absolute lower bound on the bounce action. In the case of four-dimensional space, we show the bounce action is generically larger than $24/\lambda_{\text{cr}}$, where $\lambda_{\text{cr}} \equiv \max[-4V(\phi)/|\phi|^4]$ with the false vacuum being at $\phi = 0$ and $V(0) = 0$. We derive this bound on the bounce action without solving the equation of motion explicitly. Our bound is derived by a quite simple discussion, and it provides useful information even if it is difficult to obtain the explicit form of the bounce solution. Our bound offers a sufficient condition for the stability of a false vacuum, and it is useful as a quick check on the vacuum stability for given models. Our bound can be applied to a broad class of scalar potential with any number of scalar fields. We also discuss a necessary condition for the bounce action taking a value close to this lower bound.

I. INTRODUCTION

The stability condition of a vacuum is one of the important constraints on viable models of particle physics. Even in the standard model, it gives a nontrivial constraint on the Higgs boson mass and the top quark mass [1]. Furthermore, physics beyond the standard model often introduces additional scalar fields, and they could destabilize the standard model vacuum by giving a deeper vacuum. In these situations, the standard model vacuum is a false vacuum and its lifetime should be longer than the age of the Universe.

The lifetime of a false vacuum in quantum field theory can be calculated by using Coleman’s semiclassical method [2]. In this method, the decay rate of a false vacuum per volume is evaluated as $\Gamma \sim V \sim A e^{-S}$ where $A$ is a prefactor and $S$ is the action for a nontrivial solution of the equation of motion which gives the minimal action. Such a solution is called a bounce solution. To obtain the bounce solution, we have to solve the equation of motion of scalar fields with an appropriate boundary condition. However, it is not always easy to obtain the explicit solution of the equation of motion. In particular, we have to solve a large number of coupled equations of motion if we consider some model with a large number of scalar fields such as the landscape scenario [3, 4].

It is convenient if we can discuss a possible range of the minimal bounce action value without solving the equation of motion explicitly. In this context, for example, a generic upper bound on the minimal bounce action is discussed in Refs. [5, 6]. A lower bound is discussed in Ref. [7] which focuses on quartic scalar potential, and Ref. [8] which reduces the problem to the effective single scalar problem.

In this Letter, we derive a generic lower bound on the minimal bounce action, which can be applied to a broad class of scalar potential with any number of scalar fields. Our bound can be derived by using a quite simple discussion which is based on the Lagrange multiplier method. The bound has a simple form, and it provides a sufficient condition for the stability of a false vacuum. Therefore, even if it is difficult to obtain the explicit form of the bounce solution, our bound is useful as a quick check on the stability of the false vacuum. In section II, we discuss the lower bound on the minimal bounce action. In section III, we compare our lower bound with the actual value or the upper bound for some representative examples.

II. AN LOWER BOUND ON THE BOUNCE ACTION

Here, we derive an absolute lower bound on the bounce action. We consider $m$ scalar fields with the canonical kinetic term in $N$-dimensional Euclidean space. The action is given by

$$S[\bar{\phi}] = T[\bar{\phi}] - U[\bar{\phi}],$$

$$T = \int d^N x \sum_{a=1}^{m} \sum_{i=0}^{N-1} \frac{1}{2} \left( \frac{\partial \phi_a}{\partial x_i} \right)^2,$$

$$U = \int d^N x \ U(\phi),$$

where $\bar{\phi} \equiv (\phi_1, \phi_2, ..., \phi_m)$ and $U$ is the inverted potential: $U(\Phi) \equiv -V(\Phi)$ with $V(\Phi)$ being the actual one. Throughout this Letter, we set the false vacuum at $\bar{\phi} = 0$ (and $V(\bar{0}) = 0$) without loss of generality.

Let us consider $N = 4$ dimensional Euclidean space for a while. If $\phi$ is a solution of equation of motion, it stabilizes the action. Considering the rescaling of the
Euclidean space coordinates $\phi(x) \rightarrow \phi(\xi x)$, we have the following relation

$$\frac{\partial S[\phi(\xi x)]}{\partial \xi} \bigg|_{\xi=1} = -2T + 4U = 0,$$

which leads to

$$S = \frac{T}{2}. \quad (5)$$

Thus, the problem of finding the minimal action solution can be reduced to that of finding the minimal kinetic energy solution. Since the minimal action bounce solution is known to be $O(N)$ symmetric for $N > 2$ and even for multiscalar cases $[9–12]$, we consider an $O(4)$ symmetric bounce whose radial coordinate is $r$. With $O(4)$ symmetry, the kinetic energy $T$ is given by

$$T \equiv \sum_{a=1}^{m} \int_{0}^{\infty} dr \, \pi^2 r^3 \dot{\phi}_a^2. \quad (6)$$

The equation of motion is

$$\ddot{\phi}_a + \frac{3}{r} \dot{\phi}_a + \frac{\partial U}{\partial \phi_a} = 0 \quad (a = 1, \cdots, m), \quad (7)$$

where we denote the “dot” as a derivative with respect to $r$.

To discuss the minimum kinetic energy $T$, we define a class of bounce solutions. We characterize them by two parameters: field difference $\Delta \phi_a \equiv \phi_a(0) - \phi_a(\infty)$ and potential difference $\Delta U \equiv U[\phi(0)] - U[\phi(\infty)]$. Our first goal is to derive a lower bound for such a class of solution. We can easily see $\Delta \phi_a$ and $\Delta U$ are functional of $\dot{\phi}_a$’s. By multiplying $\dot{\phi}_a$ to Eq. (7) and integrating from zero to infinity, we obtain

$$\left[ \sum_{a=1}^{m} \frac{\dot{\phi}_a^2}{2} + U \right]_{r=0}^{r=\infty} = \Delta U = \sum_{a=1}^{m} \int_{0}^{\infty} dr \, \frac{3 \dot{\phi}_a^2}{r}. \quad (8)$$

On the other hand,

$$\Delta \phi_a = - \int_{0}^{\infty} dr \dot{\phi}_a, \quad (9)$$

holds. To consider the minimization problem on $T$ with fixed $\Delta \phi_a$ and $\Delta U$, we introduce the Lagrange multiplier $\alpha_a$ and $\beta$, and define $\hat{T}$ as

$$\hat{T}[\phi, \{\alpha\}, \beta] = T[\phi] + \sum_{a=1}^{m} 2\alpha_a \left( \Delta \phi_a + \int_{0}^{\infty} dr \dot{\phi}_a \right) - \beta \left( \Delta U - \sum_{a=1}^{m} \int_{0}^{\infty} dr \frac{3 \dot{\phi}_a^2}{r} \right). \quad (10)$$

An extremum condition $\delta \hat{T}/\delta \dot{\phi}_a = 0$ gives

$$\dot{\phi}_a = - \frac{\alpha_a r}{\pi^2 r^4 + 3\beta} \quad (a = 1, \cdots, m). \quad (11)$$

In the above solution, the Lagrange multiplier $\alpha_a$ and $\beta$ are determined from the constraints $\int_{0}^{\infty} dr \dot{\phi}_a = - \Delta \phi_a$ and $\sum_{a=1}^{m} \int_{0}^{\infty} dr (3/r) \dot{\phi}_a^2 = \Delta U$ as

$$\alpha_a = \frac{24 \Delta \phi_a |\Delta \phi_a|^2}{\Delta U}, \quad \beta = \frac{12 |\Delta \phi_a|^4}{\Delta U^2}, \quad (12)$$

where $|\Delta \phi| = \sqrt{\sum_{a=1}^{m} \Delta \phi_a^2}$ is known to be $O(N)$ symmetric for $N > 2$ and even for multiscalar cases $[9–12]$. At this point, the solution Eqs. (11, 12) is just an extremum, and it is not clear whether this point is the global minimum or not. To check this point, let us see $\hat{T}$ again with Eq. (12).

$$\hat{T}[\phi, \{\alpha\}, \beta] = \frac{12 |\Delta \phi_a|^4}{\Delta U^2} + \sum_{a=1}^{m} \int_{0}^{\infty} \left( \frac{\pi^2 r^4 + 3\beta}{r} \right) \left( \dot{\phi}_a + \frac{\alpha_a r}{\pi^2 r^4 + 3\beta} \right)^2. \quad (13)$$

The above equations tell us that the solution Eqs. (11, 12) does give the global minimum on $T$ for fixed $\Delta \phi_a$ and $\Delta U$.

Then, we can write the following inequality on the bounce action $S$ (if it exists) by using $|\Delta \phi|$ and $\Delta U$ as

$$S \geq \frac{24}{\lambda_\phi(\Delta \phi_a)}, \quad \lambda_\phi(\Delta \phi_a) \equiv \frac{4 \Delta U}{|\Delta \phi_a|^4}. \quad (14)$$

The above inequality is saturated if and only if $\dot{\phi}_a = - \alpha_a r / (\pi^2 r^4 + 3\beta)$ holds. To calculate this bound, we need $\Delta \phi_a$, i.e., $\phi_a(r = 0)$. Although we do not know about $\Delta \phi_a$ unless we explicitly solve the equation of motion, we can set a bound on the minimal action even without solving the equation of motion. Suppose there exists $\lambda(> 0)$ such that

$$-U(\phi_a) = V(\phi_a) \geq -\frac{\lambda}{4} |\phi|^4. \quad (15)$$

We can find $\lambda$ for the potential such that $V(\phi)/|\phi|^4$ is bounded below. Then, we can define $\lambda_{cr}$ as

$$\lambda_{cr} \equiv \text{max} \left( -4V(\phi_0) / |\phi|^4 \right). \quad (16)$$

We can see this $\lambda_{cr}$ is the minimum of a set of $\lambda$ which satisfy Eq. (15). Then, the bounce action has an absolute lower bound

$$S \geq \frac{24}{\lambda_{cr}}. \quad (17)$$

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1 One may be interested in the potential which realizes the bounce solution $\phi = - \alpha_a r / (\pi^2 r^4 + 3\beta)$. The explicit form of the potential for a single scalar field case is $U = \Delta U(\phi) = \Delta \phi - (4/3\pi) \sin(\pi \phi/\Delta \phi) + (1/6\pi) \sin(2\pi \phi/\Delta \phi)$. This potential give the minimum of the bound Eq. (14). However, we do not know an example which saturates the bound Eq. (17). Thus, the bound Eq. (17) may be weaker than Eq. (14).
because $\lambda_\phi \leq \lambda_c$ is satisfied for any value of $\phi_a$. As a reference, the Fubini instanton \[13\], which is a bounce solution with negative quartic potential $V = (\lambda_4/4)\phi^4$, has $S = 8\pi^2/3\lambda_4 \simeq 26.3/\lambda_4 > 24/\lambda_c$ because $\lambda_c = \lambda_4$ holds in this case. To derive the above bound Eq. (17), we do not need the explicit form of the bounce solution. Although the bound Eq. (17) may be weaker than Eq. (14), we can derive Eq. (17) only from the information of the potential. In $N(>2)$ dimensional case, the same procedure gives the lower bound:

$$S \geq \frac{4(N(N-1)(N-2))^{\frac{N-2}{2}}}{N ! (N/2)^{N-2}} \sin(2\pi/N) N \left(\frac{1}{\lambda_N}\right)^{\frac{N-2}{2}}$$

with $\lambda_N \equiv N \Delta U/(\Delta \phi)^{\frac{2N}{N-2}}$.

One may be interested in the condition in which the lower bound Eq. (17) becomes close to the actual value. As long as the true vacuum and the false vacuum are not degenerated, our method can give a good estimation on the lower bound of the decay rate. For detailed discussion, see the Appendix.

So far, we have derived a lower bound on the bounce action. Here let us comment on an upper bound on the bounce action, see the Appendix.

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**III. COMPARISONS WITH THE ACTUAL VALUE**

In this section, we discuss several explicit examples in four-dimensional space. We will see the consistency in four-dimensional space.

\[\text{footnote}{2} \text{The gauge dependence and renormalization scale dependence are canceled by considering loop corrections} \[14, 15].\]

\[\text{footnote}{3} \text{Strictly speaking, this condition should be imposed in} \phi > H_0 \text{ region.}\]
of Eq. (17), and furthermore, see that the lower bound Eq. (17) becomes close to the actual value of the bounce action in many cases. In this sense, the lower bound Eq. (17) is a quite useful tool to estimate the value of the bounce action when an explicit calculation is difficult.

A. Single scalar field

![Graph: The minimal bounce action and the lower bound.](image1)

The blue solid line shows the lower bound which is given in Eq. (17). The green dotted line is taken from Fig. 1 in Ref. [6].

![Graph: The ratio between the green dotted line and the blue solid line in Fig. 1.](image2)

The first example is a single scalar field theory with a polynomial potential:

\[ V(\phi) = \frac{1}{2} M^2 \phi^2 - \frac{1}{3} A \phi^3 + \frac{1}{4} \lambda_4 \phi^4. \]  

(24)

This potential gives us good insight into a relationship between our bound Eq. (17) and the minimal bounce action \( S \). As discussed in Ref. [6], we can parametrize the minimal bounce action as

\[ S = \frac{9 M^2}{2 A^2} \hat{S}(\kappa), \quad \kappa \equiv \frac{9 \lambda_4 M^2}{8 A^2}. \]  

(25)

Here \( \hat{S} \) is a function which only depends on \( \kappa \). According to the definition given in Eq. (16), \( \lambda_{cr} \) is calculated as

\[ \lambda_{cr} = \frac{2 A^2}{9 M^2} - \lambda_4. \]  

(26)

By using this \( \lambda_{cr} \), we obtain the bound on \( \hat{S}(\kappa) \) as

\[ \hat{S}(\kappa) \geq \frac{24}{1 - 4\kappa}. \]  

(27)

Ref. [6] gives the numerical result of \( \hat{S}(\kappa) \) by calculating the bounce configuration, and we show a comparison between the result of Ref. [6] and the bound Eq. (27) in Fig. 1 and Fig. 2. We can see that our bound becomes close for large negative \( \kappa \). In this regime, the bounce solution is well described by the Fubini instanton [13]. On the other hand, our bound departs from the numerical value of the minimal bounce action if \( \kappa \) is close to \( 1/4 \), in this regime, the false and true vacua are almost degenerate and the bounce solution is well described by thin-wall approximation. There exists a potential barrier between the true vacuum and false vacuum.

B. Multi scalar fields

The second example is a polynomial potential with multiscalar field \( \phi_1, \ldots, \phi_m \). We consider a term up to the quartic interaction, and parametrize it as follows

\[ V = \sum_i M^2 \mu_i \phi_i^2 + \sum_{i,j,k} M \gamma_{ijk} \phi_i \phi_j \phi_k + \sum_{ijkl} \lambda_{ijkl} \phi_i \phi_j \phi_k \phi_l, \]  

(28)

where \( M \) is a mass scale which does not affect the value of classical action and \( \mu_i, \gamma_{ijk}, \lambda_{ijkl} \) denote some dimensionless coupling. Here we do not calculate the bounce configuration explicitly. Instead of the explicit calculation, we estimate a lower and upper bound on the bounce action. The upper bound is estimated by the straight line method described in the later part of Sec. II. We define the ratio between the upper and lower bound as

\[ R = \frac{S_{upper}}{S_{lower}}. \]  

We compare the result with the bound Eq. (27). In this regime, the false and true vacua are almost degenerate and the bounce solution is well described by thin-wall approximation. There exists a potential barrier between the true vacuum and false vacuum.
We calculate the ratio $R$ by taking $\mu$’s, $\gamma$’s, and $\lambda$’s as random variables as in Ref. [4]. The ranges of the parameters are taken as

$$0 < \mu_i < 1, \quad -\frac{1}{m} < \gamma_{ij} < \frac{1}{m}, \quad -\frac{1}{m} < \lambda_{ijkl} < \frac{1}{m}. \quad (29)$$

We take the range of $\gamma$ and $\lambda$ so that the theory remains stable against loop corrections. We generate 1000 parameter points, and show the distribution of $R$ in Tab. 1. This result shows the lower bound Eq. (17) becomes close to the actual value of the bounce action in the case of a large number of scalar fields. This feature can be understood as follows. As we have seen in the previous single scalar example, $\lambda_{tt}$ depends on quartic coupling and the cubic coupling square (see Eq. (26)). In the present case, the typical value of quartic coupling is $1/m$ and that of cubic coupling square is $1/m^2$. With larger $m$, quartic coupling becomes more and more relevant and the bounce action becomes close to our lower bound.

**C. MSSM**

The last example is the MSSM. Supersymmetric models introduce a lot of scalar partners of the standard model fermions, and sometimes they destabilize the standard model-like vacuum. For example, Ref. [16] discussed a vacuum stability in a direction of the third generation slepton with large $\tan \beta$. The scalar potential for the up-type Higgs $H_u$, the left-handed stau $\tilde{\tau}_L$, and the right-handed stau $\tilde{\tau}_R$ is given as

$$V = (m_{H_u}^2 + \mu^2)|H_u|^2 + m_{\tilde{\tau}_L}^2|\tilde{\tau}_L|^2 + m_{\tilde{\tau}_R}^2|\tilde{\tau}_R|^2$$

$$+ \frac{g_2^2}{8}(|\tilde{\tau}_L|^2 + |H_u|^2)^2 + \frac{g_2^2}{8}(|\tilde{\tau}_L|^2 - 2|\tilde{\tau}_R|^2 - |H_u|^2)^2$$

$$+ \frac{g_2^2 + g_1^2}{8} \delta_H |H_u|^4 + y_t^2|\tilde{\tau}_L|^2$$

$$- (y_t \mu H_u^2 \tilde{\tau}_R + h.c.). \quad (30)$$

Here we do not consider the down-type Higgs $H_d$ because its VEV is suppressed by $1/\tan \beta$. $\delta_H$ expresses a radiative correction from the top quark and the stop, and its typical value is $\delta_H \simeq 1$. A cubic term $H_u^2 \tilde{\tau}_R$ in the last line destabilizes the standard model-like vacuum. Its coupling constant is proportional to $\mu \tan \beta$. In Fig. 3 we show a comparison between the lower bound on the bounce action which is given in Eq. (17) and Ref. [16]. The lower bound on the bounce action $S$ is 400 at the blue line, and the standard model-like vacuum is sufficiently stable in the lower right region of the blue line. By using the result in Ref. [16], in Fig. 3 we show the red line on which $S = 400$ is satisfied. We can see our bound Eq. (17) is consistent with the result of Ref. [16].

To discuss the stability in the upper left region, Eq. (17) is not enough in general. However, Fig. 3 shows that the sufficient stability condition by the blue line only differs by 5% from the upper bound on $\tan \beta$ by the red line. This means that Eq. (17) gives a good estimation on the upper bound of $\tan \beta$. Actually, Figs. 1, 2 show the lower bound on the bounce action gives a good estimation on the actual value unless the true and false vacua are degenerated. Such a degenerated situation is a special situation in the sense that it requires a tuning of the parameters or an approximate symmetry between two vacua. Thus, we can expect that our discussion is useful to discuss more complicated models.

**IV. CONCLUSION**

In this Letter, we derived a generic lower bound Eq. (17) on the bounce action by using a quite simple discussion with the Lagrange multiplier. Our bound can be applied to a broad class of scalar potential with any number of scalar field. Necessary information to derive this bound is only $\lambda_{tt}$ which is defined by Eq. (16). In particular, our bound provides useful information for a model with a large number of scalar fields such as the landscape scenario because we do not need the explicit form of the bounce solution. By using this result, in Eq. (22) we derived a sufficient condition of the stable vacuum of the Universe for a general scalar potential. The bound Eq. (22) can be used as a quick check on the stability of a false vacuum in a broad class of models. As we discussed in section III the lower bound Eq. (17) gives a good estimation on the actual value in many cases. We also investigated a condition for when the bounce action becomes close to the lower bound. As long as two vacua
are not almost degenerated the minimal bounce action can be close to the lower bound. We have seen this feature in some representative examples.

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Appendix A: Criteria for kinetic bounce solution

Here, we discuss the situation in which our bounce becomes close to the actual value of the minimal bounce action. First, let us look at large \( r \) behaviour of the field configuration which gives the lower bound (see Eq. (11) in the main text). It is given by

\[
\phi \sim \frac{1}{r^2}, \quad \text{(for } r^2 \gg \Delta \phi^2 / \Delta U). \tag{A1}
\]

Note that the bounce solution behaves as \( \phi \sim e^{-mr} \) for large \( r \) if there exists a mass term around the false vacuum, and the above solution is a solution of equation of motion without potential term:

\[
\ddot{\phi} + \frac{3}{r} \dot{\phi} = 0. \tag{A2}
\]

We denote this class of bounce solutions as “kinetic bounce solution”. Thus, it seems that if the potential term becomes ineffective in large \( r \), the bounce action can take a value close to the lower bound. Below, we will show this is actually the case. In addition, we derive a necessary condition for that the bounce dynamics at large \( r \) is dominated by the kinetic term. This condition is necessary in order for the bounce action to be close to the lower bound.

To see a condition for that the kinetic term dominates, it is instructive to consider the following simple (linear+flat) potential:

\[
V(\phi) = \begin{cases} 
0 & (\phi < \phi_\ast), \\
(\phi_\ast - \phi)F & (\phi > \phi_\ast),
\end{cases} \tag{A3}
\]

where \( \phi_\ast(>0) \) and \( F(>0) \) are some constants. The minimal \( \lambda \), which satisfies \( V + \lambda \phi^4 / 4 \geq 0 \), is given by

\[
\lambda_{\text{cr}} = \left( \frac{3}{4\phi_\ast} \right)^3 F. \tag{A4}
\]

We define “false vacuum” at \( \phi = 0 \). The bounce solution is uniquely determined and given by

\[
\phi(r) = \begin{cases} 
r^2 r^2/8 + \phi_\ast & (r < r_\ast), \\
\phi_\ast \left( \frac{r}{r_\ast} \right)^2 & (r > r_\ast),
\end{cases} \tag{A5}
\]

with

\[
r_\ast^2 = 8 \frac{\phi_\ast}{F}. \tag{A6}
\]

Note that for \( \phi < \phi_\ast \), the kinetic term fully determines the dynamics. To distinguish the bounce action which will be discussed later, we denote the bounce action under the potential Eq. (A3) as \( S_0 \). The bounce action \( S_0 \) is slightly larger than the lower bound \( 24/\lambda_{\text{cr}} \) and given by

\[
S_0 = \frac{45\pi^2}{16\lambda_{\text{cr}}} \approx 27.7. \tag{A7}
\]

Next, let us add a potential term \( V_+ \) in \( 0 \leq \phi < \phi_\ast \) region, and see how \( V_+ \) changes the value of the minimal action \( S \). If the potential energy satisfies \( |V_+| \ll \dot{\phi}^2 \), the bounce solution obeys \( \ddot{\phi} + (3/3) \dot{\phi} = 0 \). In this case, the kinetic energy at \( r > r_\ast \) is written as

\[
\dot{\phi}^2(r) \sim \frac{1}{4} \lambda_{\text{cr}} \phi^3(r) \phi_\ast. \tag{A8}
\]

The condition \( |V_+| \ll \dot{\phi}^2 \) is broken at \( \phi = \phi_c \) such that

\[
\frac{1}{4} \lambda_{\text{cr}} \phi_c^3 \phi_\ast \sim V_+(\phi_c). \tag{A9}
\]

We define radius \( r_c \) such that \( \phi(r_c) = \phi_c \). By definition, \( r_c \) is larger than \( r_\ast \). We split the bounce action \( S \) as

\[
S = S_{r>r_c} + S_{r<r_c}, \tag{A10}
\]

where \( S_{r>r_c} \equiv \int_{r_c}^{\infty} dr \pi^2 r^3 [\dot{\phi}^2 + 2U(\phi)] \) and \( S_{r<r_c} \equiv S - S_{r>r_c} \). For \( r > r_c \), we cannot neglect the potential term in the equation of motion.

First, let us see an effect on \( S_{r<r_c} \). By using Eq. (A5) and Eq. (A9), we can estimate the ratio of \( r_\ast \) over \( r_c \) and \( \phi_\ast \) over \( \phi_c \) as

\[
\frac{r_\ast}{r_c} \sim \left( \frac{V_+(\phi_c)}{\Delta U} \right)^{1/6}, \quad \frac{\phi_\ast}{\phi_c} \sim \left( \frac{V_+(\phi_c)}{\Delta U} \right)^{-1/3}. \tag{A11}
\]

As long as \( V_+(\phi_c) \ll \Delta U \), by a tiny shift of initial position \( \phi(0) \rightarrow \phi(0) + \delta \) with \( \delta \sim \phi_c \ll \phi(0) \), the kinetic energy at \( \phi_c \) changes by a factor and we will obtain a bounce solution. In this case, \( S_{r<r_c} \) remains almost the same:

\[
\frac{S_{r<r_c}}{S_0} - 1 \sim \mathcal{O} \left( \frac{V_+(\phi_c)}{\Delta U} \right)^{1/3}. \tag{A12}
\]

Next, let us see an effect on \( S_{r>r_c} \). We denote the maximal value of the potential inside \( \phi_c \) to be \( V_{\text{max}}(\phi_c) = \max \{ V(\phi) | 0 \leq \phi \leq \phi_c \} \). A typical mass scale at \( 0 \leq \phi \leq \phi_c \) may be given by \( m_{\text{typ}}^2 \sim V_{\text{max}}/\phi_c^2 \). Then, the bounce action inside \( \phi_c \) would be estimated as

\[
S_{r>r_c} \sim T_{r>r_c} \sim \int_{r_c}^{r_c + m_{\text{typ}}^{-1}} dr \, r^3 \dot{\phi}^2 \sim m_{\text{typ}}^2 \phi_c^3 r_c^3 \sim \frac{m_{\text{typ}}^2 r_c^3}{\phi_c} S_0. \tag{A13}
\]
We can see as long as
\[ \frac{m_{\text{typ}} r^2}{r_c} \ll 1, \tag{A14} \]
the contribution from \( S_{\lambda \rightarrow r_c} \) is suppressed. The left hand side of Eq. (A14) is small if we consider small and mild shape of \( V_+ \). We conclude a necessary condition for the minimal action which is close to the lower bound is
\[ V_{\text{max}} \ll \Delta U. \tag{A15} \]
If this condition is violated, the minimal action \( S \) is significantly deviated from \( S_0 \). Now, let us generalize the previous discussion. For given potential \( V(\phi) \), we can define \( \lambda_{\text{cr}} \) as a minimal \( \lambda \) with
\[ V(\phi) + \frac{1}{4} \lambda_{\text{cr}}^2 \phi^4 \geq 0. \tag{A16} \]
We also define \( \Delta U \) and \( \Delta \phi \) at the point where the equality holds:
\[ \Delta U \equiv \frac{1}{4} \lambda_{\text{cr}} \Delta \phi^4 = -V(\Delta \phi). \tag{A17} \]
We can also define \( \phi_{\text{c}} \) as the maximal value of \( \phi \) with
\[ V(\phi_{\text{c}}) = \frac{1}{4} \lambda_{\text{cr}}^2 \phi_{\text{c}}^4 \Delta \phi. \tag{A18} \]
Then, \( V_{\text{max}} \) is given by a maximal value of \( V(\phi) \) in \( \phi < \phi_{\text{c}} \). As before, we define \( r^2 \equiv \Delta \phi^2 / \Delta U \), \( r_c \equiv r (V(\phi_c)/\Delta U)^{-1/6} \) and \( m_{\text{typ}}^2 \equiv V_{\text{max}}/\phi_{\text{c}}^2 \). Then, if the condition Eq. (A14) does not hold, bounce action will deviate from the lower bound. Thus, this condition can be regarded as a necessary condition for the bounce action to have a value close to the lower bound.

The condition Eq. (A14) characterizes a smallness of the potential barrier. This is because if \( V_{\text{max}} \) is small, \( m_{\text{typ}} \) also becomes small. In addition, if \( V_+ \) is small, \( r_c \) becomes large. And if the barrier is relatively large, the bounce action will deviate from the lower bound 24/\( \lambda \).

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[1] G. Degrassi, S. Di Vita, J. Elias-Miro, J. R. Espinosa, G. F. Giudice, G. Isidori, and A. Strumia, “Higgs mass and vacuum stability in the Standard Model at NNLO,” *JHEP* **08** (2012) 098. [arXiv:1205.6497 [hep-ph]]

[2] S. R. Coleman, “The Fate of the False Vacuum. I. Semiclassical Theory,” *Phys. Rev.* **D15** (1977) 2929–2936. [Erratum: *Phys. Rev.D16*,1248(1977)].

[3] B. Greene, D. Kagan, A. Masoumi, D. Mehta, E. J. Weinberg, and X. Xiao, “Tumbling through a landscape: Evidence of instabilities in high-dimensional moduli spaces,” *Phys. Rev.**D88* no. 2, (2013) 026005. [arXiv:1303.4428 [hep-th]]

[4] M. Dine and S. Paban, “Tunneling in Theories with Many Fields,” *JHEP* **10** (2015) 088 [arXiv:1506.06428 [hep-th]]

[5] I. Dasgupta, “Estimating vacuum tunneling rates,” *Phys. Lett. B394* (1997) 116–122 [arXiv:hep-ph/9610403 [hep-ph]]

[6] U. Sarid, “Tools for tunneling,” *Phys. Rev. D58* (1998) 085017 [arXiv:hep-ph/9804308 [hep-ph]]

[7] A. Aravind, D. Lorschbough, and S. Paban, “Lower bound for the multifield bounce action,” *Phys. Rev. D89* no. 10, (2014) 103535 [arXiv:1401.1230 [hep-th]]

[8] A. Masoumi, K. D. Olum, and J. M. Wachter, “Approximating tunneling rates in multi-dimensional field spaces,” [arXiv:1702.00356 [gr-qc]]

[9] S. R. Coleman, V. Glaser, and A. Martin, “Action Minima Among Solutions to a Class of Euclidean Scalar Field Equations,” *Commun. Math. Phys.* **58** (1978) 211.

[10] O. Lopes, “Radial symmetry of minimizers for some translation and rotation invariant functionals,” *Journal of differential equations* **124** no. 2, (1996) 378–389

[11] J. Byeon, L. Jeanjean, and M. Mari˘s, “Symmetry and monotonicity of least energy solutions,” *Calculus of Variations and Partial Differential Equations* **36** no. 4, (2009) 481–492 [arXiv:0806.0299 [math.AP]]

[12] K. Blum, M. Honda, R. Sato, M. Takimoto, and K. Tobioka, “O(N) Invariance of the Multi-Field Bounce,” *JHEP* **05** (2017) 109 [arXiv:1611.04570 [hep-th]]

[13] S. Fubini, “A New Approach to Conformal Invariant Field Theories,” *Nuovo Cim.* **A34** (1976) 521

[14] M. Endo, T. Moroi, M. M. Nojiri, and Y. Shoji, “On the Gauge Invariance of the Decay Rate of False Vacuum,” *Phys. Lett. B771* (2017) 281–287 [arXiv:1703.09304 [hep-ph]]

[15] M. Endo, T. Moroi, M. M. Nojiri, and Y. Shoji, “False Vacuum Decay in Gauge Theory,” *JHEP* **11** (2017) 074 [arXiv:1704.03492 [hep-ph]]

[16] J. Hisano and S. Sugiyama, “Charge-breaking constraints on left-right mixing of stau’s,” *Phys. Lett.* **B696** (2011) 92–96 [arXiv:1011.0260 [hep-ph]] [Erratum: *Phys. Lett.B719*,472(2013)].