On the differentiability conditions at spacelike infinity

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Abstract

We consider space-times which are asymptotically flat at spacelike infinity, \( i^0 \). It is well known that, in general, one cannot have a smooth differentiable structure at \( i^0 \), but have to use direction dependent structures. Instead of the oftenly used \( C^{>1} \)-differentiable structure, we suggest a weaker differential structure, a \( C^{1+} \) structure. The reason for this is that we have not seen any completions of the Schwarzschild space-time which is \( C^{>1} \) in both spacelike and null directions at \( i^0 \). In a \( C^{1+} \) structure all directions can be treated equal, at the expense of logarithmic singularities at \( i^0 \). We show that, in general, the relevant part of the curvature tensor, the Weyl part, is free from these singularities, and that the (rescaled) Weyl tensor has a certain symmetry property.
1 Introduction

When studying asymptotic properties of space-times, one often has to use what is called direction dependent structures. In [3], Ashtekar and Hansen examines space-times that are asymptotically flat at null and spacelike (or spatial) infinity. Using conformal completions, spacelike infinity is represented by a point, \( i^0 \), which is the vertex of \( I^+ \) and \( I^- \).

In general, the differential structure at \( i^0 \) cannot be smooth, and in [3] a so called \( C^{>1} \) differential structure is used. Roughly speaking, the (conformally rescaled) metric \( g_{ab} \) is well defined at \( i^0 \), the first derivatives of the metric are direction dependent at \( i^0 \), and the curvature components diverge like \( 1/r \) as one approaches \( i^0 \) (in spacelike directions). Here \( r \) is some suitable distance function with \( r(i^0) = 0 \). It is worth noting that in the definition of a \( C^{>1} \) differential structure, as given in [3], one examines direction dependent quantities by their limits at \( i^0 \) along spacelike directions only. As we will see, in the completions of the “obviously” asymptotically flat Schwarzschild space-time which are given by e.g. [3],[6], one has rescaled metrics which are \( C^{>1} \) at \( i^0 \) in this sense, but where the metric is only \( C^0 \) on \( I^+ \) and \( I^- \). In the following we assume that the reader is familiar with the concept of \( C^{>n} \) differentiable structures and functions as defined in [3] or [13], although we for completeness repeat some basic definitions in section 3.

There are examples of other conformal rescalings of the Schwarzschild space-time, [6],[10], where the rescaled metric is smooth at \( I^+ \) and \( I^- \), but where the derivatives of the metric do not have direction dependent limits at \( i^0 \) but rather diverges logarithmically. At first this may seem as a serious defect, since these logarithmic singularities can be expected to show up in the conformally rescaled Ricci and Riemann tensors. This is certainly the case, but as we will show below, for the physically relevant part, namely the Weyl tensor, these singularities disappear.

Below we start by comparing the two different situations mentioned above; this will serve as a motivation for the new differential structure that we then introduce. Finally we show that an expected singularity for the Weyl tensor is absent, and we also discuss some of the consequences of this.

2 Two completions of Schwarzschild space-time

As a motivation for the differentiability conditions introduced in the following section, we compare two different completions of the Schwarzschild space-time. These completions can be found in for instance [3],[6],[10]. In the Schwarzschild space-time, with standard coordinates \( t, r, \theta, \phi \) the metric
is
\[ ds^2 = (1 - \frac{2m}{r}) dt^2 - \frac{1}{1 - \frac{2m}{r}} dr^2 - r^2 d\Sigma^2, \quad d\Sigma^2 = d\theta^2 + \sin^2 \theta d\phi^2. \quad (1) \]

A Schwarzschild space-time is of course considered as asymptotically flat at spacelike infinity, \( i^0 \), as well as at null infinity, \( I \). Therefore, any reasonable definition of asymptotic flatness in these regions should clearly admit this space-time. For readers not familiar with the concept of asymptotic flatness we refer to [3] and [13] for definitions and discussions, although some definitions are included in the next section.

The first conformal rescaling we look at uses the so called Schmidt-Walker coordinates, [3], [6]. This gives us a “standard” completion of the Schwarzschild space-time, in which the new, unphysical metric \( \hat{g}_{ab} \) is \( C^0 \) only for spacelike directions. Here \( \hat{g}_{ab} = \Omega^2 g_{ab} \), where \( g_{ab} \) is the physical metric and \( \Omega \) is the conformal factor.

Namely, let us first define, for \( r > 2m \), the function \( f \) by
\[ f(r) = r + 2m \log(\frac{r}{2m} - 1). \quad (2) \]

Put \( r^* = f(r) \) and define \( \hat{v} \) and \( \hat{w} \) implicitly by
\[ r^* = \frac{1}{2} [f(\frac{1}{v}) + f(\frac{1}{w})], \quad t = \frac{1}{2} [f(\frac{1}{v}) - f(\frac{1}{w})]. \quad (3) \]

With \( \hat{\Omega} = \hat{v}\hat{w} \), the rescaled metric \( \hat{g}_{ab} = \hat{\Omega}^2 g_{ab} \) becomes
\[ d\hat{s}^2 = -\frac{1 - 2m/r}{(1 - 2m\hat{v})(1 - 2m\hat{w})} d\hat{v}d\hat{w} - \omega^2 \left( \frac{\hat{v} + \hat{w}}{2} \right)^2 d\Sigma^2, \quad (4) \]

where \( \omega = \omega(\hat{v}, \hat{w}) \). Here \( i^0 \) is of course the point \( \hat{v} = \hat{w} = 0 \). \( \hat{v} = 0 \) and \( \hat{w} = 0 \) corresponds to \( I^+ \) and \( I^- \) respectively. If we put \( \hat{t} = \frac{\hat{v} + \hat{w}}{2} \) and \( \hat{r} = \frac{\hat{v} - \hat{w}}{2} \), \( \omega \) can be brought to the form (see [3])
\[ \omega = 1 + m\hat{r}(1 - \frac{\hat{r}^2}{r^2}) \log(1 - \frac{\hat{r}^2}{r^2}) + O(\hat{r}^2). \]

This means that \( \omega \) takes a finite value (by continuity) on \( \mathcal{I} \) i.e., when \( \frac{\hat{r}}{r} = \pm 1 \), but that non-tangential derivatives do not exist there. Therefore, on \( \mathcal{I} \), \( \hat{g}_{ab} \) is defined but not differentiable. For spacelike directions approaching \( i^0 \), the situation is better. The metric \( \hat{g}_{ab} \) is \( C^0 \) at \( i^0 \); in particular \( \sqrt{\hat{\Omega}}\hat{R}_{abcd} \) has a regular direction dependent limit for these directions. Here \( \hat{R}_{abcd} \) is of course the Riemann tensor connected to \( \hat{g}_{ab} \).

The second completion is similar, but has different properties. In this completion, taken from [10] (see also remark in [3]), the metric will not be \( C^0 \) at \( i^0 \) in any direction. On the other hand, it will be smooth on \( \mathcal{I} \). Also,
the metric will have components where the derivatives diverges in a “mild” way, namely logarithmically with respect to a suitable radial parameter. This time, cf. [10], we put

\[ r^* = \left[ f\left( \frac{1}{\tilde{v}} \right) + f\left( \frac{1}{\tilde{w}} \right) \right], \quad t = \left[ f\left( \frac{1}{\tilde{v}} \right) - f\left( \frac{1}{\tilde{w}} \right) \right], \]

\[ \tilde{r} = \frac{\tilde{v} - \tilde{w}}{2}, \quad \tilde{t} = \frac{\tilde{v} + \tilde{w}}{2}, \quad \tilde{\Omega} = \tilde{v}\tilde{w}. \] (5)

The rescaled metric becomes

\[ ds^2 = -\frac{1 - 2m/r}{(1 - 2m\tilde{v})(1 - 2m\tilde{w})}d\tilde{v}d\tilde{w} - \left( \frac{2\tilde{v}\tilde{w}r}{\tilde{v} + \tilde{w}} \right)^2 \tilde{r}^2 d\Sigma^2. \] (6)

As shown in [10], \( \frac{\tilde{v}\tilde{w}}{\tilde{v} + \tilde{w}} \) has the following expansion in a neighbourhood of \( i^0 \);

\[ \frac{\tilde{v}\tilde{w}}{\tilde{v} + \tilde{w}} = 1 - 2m\frac{\tilde{v}\tilde{w}}{\tilde{v} + \tilde{w}} \log(2m(\tilde{v} + \tilde{w})) + O((\tilde{v} + \tilde{w})^2). \] (7)

This expression can be differentiated arbitrarily many times. In particular, \( \frac{\tilde{v}\tilde{w}}{\tilde{v} + \tilde{w}} \) is shown to be analytic on \( I^+ \) and \( I^- \). Thus in this completion, the metric is smooth on \( I^+ \) and \( I^- \), but it is not \( \mathcal{C}^0 \) at \( i^0 \). In fact, the metric is continuos at \( i^0 \) and has derivatives that blow up logarithmically (i.e. like \( \log \tilde{r} \) as \( \tilde{r} \to 0^+ \)) as one approaches \( i^0 \). Comparing with definition 3.4 below, we see that the metric is \( \mathcal{C}^{0+} \) at \( i^0 \).

We will now discuss some properties of these completions. Let us first recall that the Weyl tensor \( C_{abc}^d \) is conformally invariant, i.e., under a rescaling of the metric \( g_{ab} \to \tilde{g}_{ab} = \tilde{\Omega}^2 g_{ab} \), the Weyl tensor transforms like \( C_{abc}^d \to \tilde{C}_{abc}^d = C_{abc}^d \). Let us also recall that, for a space-time which is asymptotically flat at null infinity, \( \Omega^{-1} C_{abc}^d \) has a smooth limit at \( I \). (For a proof of this, see [12].) In our first completion, using the coordinates given by (3), the rescaled metric \( \tilde{g}_{ab} \) is not smooth on \( I \), so we can not say, a priori, anything about the limit of \( \tilde{\Omega}^{-1} \tilde{C}_{abc}^d \) there. Of course, in the special case of a Schwarzschild space-time, we can check directly that \( \tilde{\Omega}^{-1} \tilde{C}_{abc}^d \) happens to have a limit on \( I \). On the other hand is \( \tilde{R}_{abcd} \) singular on \( I \).

Similarly, for the second completion, using (5) where the metric is not \( \mathcal{C}^{>0} \) in spacelike directions at \( i^0 \), we expect that \( \sqrt{\tilde{\Omega}} \tilde{R}_{abcd} \) diverges logarithmically at \( i^0 \). Nevertheless, for the space-time considered, \( \sqrt{\tilde{\Omega}} \tilde{C}_{abcd} \) will have direction dependent limits at \( i^0 \).

Basically,

1. \( \tilde{g}_{ab} \) is \( \mathcal{C}^{>0} \) in spacelike directions from \( i^0 \), so that \( \lim_{i^0} \sqrt{\tilde{\Omega}} \tilde{C}_{abc}^d \) exists along spacelike directions.

2. \( \tilde{g}_{ab} \) is smooth on \( I \) so that \( \tilde{\Omega}^{-1} \tilde{C}_{abc}^d \) exists on \( I \).
By the conformal invariance we have that
\[ \sqrt{\tilde{\Omega}} \tilde{C}^d_{abc} = \sqrt{\frac{\bar{\Omega}}{\tilde{\Omega}}} \sqrt{\tilde{\Omega}} \tilde{C}^d_{abc} = \sqrt{\frac{\bar{v}}{\tilde{v}}} \sqrt{\tilde{w}} \sqrt{\tilde{\Omega}} \tilde{C}^d_{abc}. \] (8)
and
\[ \hat{\Omega}^{-1} \hat{C}^d_{abc} = \sqrt{\frac{\bar{v}}{\tilde{v}}} \sqrt{\tilde{w}} \sqrt{\tilde{\Omega}} \hat{\Omega}^{-1} \hat{C}^d_{abc} = \sqrt{\frac{\bar{v}}{\tilde{v}}} \sqrt{\tilde{w}} \sqrt{\tilde{\Omega}}^{-1} \hat{C}^d_{abc}. \] (9)

By examining \( \tilde{\bar{v}} \tilde{w} \) it is not hard to see that this expression has a (non-differentiable) non-zero limit as one approaches \( I \) or \( i^0 \). Thus, in addition to the limits above, we can also conclude that \( \lim_{\bar{v}} \sqrt{\tilde{\Omega}} \tilde{C}^d_{abc} \) exists and that \( \hat{\Omega}^{-1} \hat{C}^d_{abc} \) exists (by continuity) on \( I \), although this was not expected from the behaviour of the metrices \( \bar{g}_{ab} \) and \( \tilde{g}_{ab} \). Thus, for the Weyl tensor (at least in this case), one can allow the metric to be less regular than expected, and still have 'nice limits' at \( i^0 \). This idea will now be developed.

3 The differentiability conditions at \( i^0 \)

In this section we will discuss the differentiability conditions at spacelike infinity \( i^0 \), aiming at emphasizing the (physical) gravitational field rather than the metric. These changes will be motivated by the previous example and because, to our knowledge, there is no given example of a conformal rescaling of the Schwarzschild metric which makes the rescaled metric \( C^{>0} \) in both spacelike and null directions.

Since from now on we will deal mainly with conformally rescaled space-times, we adopt the convention that the rescaled metric (normally) will be written without hat/tilde. Thus we may have \( g_{ab} = \Omega^2 \bar{g}_{ab} \), where \( \Omega \) is the conformal factor and \( g_{ab} \) the physical metric. Accordingly, \( R_{abcd}, R_{ab} \) etc. will refer to quantities related to the unphysical metric \( g_{ab} \) and the associated derivative operator.

We believe that the interest should be focused, not on the metric, but rather on the Weyl tensor \( C_{abcd} \). The \( C^{>0} \) condition at \( i^0 \) is precisely what is needed in order to deduce that \( \sqrt{\Omega} R_{abcd} \) is regular direction dependent at \( i^0 \). However, starting with a (physical) vacuum space-time, the physical Ricci tensor \( R_{pab} \) is zero and all information about the space-time lies within the Weyl tensor. After the rescaling, the Weyl tensor remains the same, i.e., carries the same information (about the physical space-time) as before. In addition, due to the conformal transformation, we get, see [13], a non-physical Ricci tensor \( R_{ab} \). We argue that the condition that \( \sqrt{\Omega} R_{ab} \) is regular direction dependent at \( i^0 \) may be too strong and perhaps of secondary interest. Instead we should try to ensure that \( \sqrt{\Omega} C_{abcd} \) is regular direction dependent at \( i^0 \). (Of course, if \( \sqrt{\Omega} C_{abcd} \) is direction dependent but not \( \sqrt{\Omega} R_{ab}, \sqrt{\Omega} R_{abcd} \) will also fail to be direction dependent.)
Also, by the peeling property, $\Omega^{-1}C_{abcd}$ will exist on $\mathcal{I}$. This means that $\sqrt{\Omega}C_{abcd}$ is trivially zero on $\mathcal{I}$ and that the condition that $\sqrt{\Omega}C_{abcd}$ be direction dependent should be changed there, cf. the situation for the electromagnetic field in section \[3\].

In a way, we can regard the metric as a potential for the Weyl tensor. Thus we prefer to look at the “field” rather than the “potential”.

We will now recall the concepts of regularly direction dependent functions, asymptotic flatness and direction dependent differentiable structures of class $C^{>1}$, and also introduce the weaker structure of class $C^{1+}$. Thus definitions \[3.1\], \[3.2\] and \[3.5\] below are taken more or less from for instance \[3\] and \[13\].

The starting point is a manifold which is smooth everywhere except at one point $p$ where it is $C^1$ so that one at least has a tangent space there. Thus it is possible to talk about directions at $p$. This enables us to talk about direction dependent functions at $p$, i.e. functions $f$ for which the limit along any $C^1$-curve $\gamma$ ending at $p$ exists and depends only on the tangent direction to $\gamma$ at $p$. We write this as $\lim_p f = f(\eta)$ where $\eta$ is the tangent vector to $\gamma$ at $p$. We can then define what it means for a function to be regularly direction dependent:

**Definition 3.1** Let $M$ be a manifold which is $C^{\infty}$ everywhere except at a point $p$, where it is $C^1$. Let $f$ be a function which is direction dependent at $p$, and let $(U, \Psi)$ be a chart containing $p$, with coordinates $x^i$ so that $x^i(p) = 0$. In terms of these coordinates, let $F$ be constant on rays from the origin, so that $\lim_{\Psi(p)}(f \circ \Psi^{-1}) = F(\eta)$. $f$ is then said to be regular direction dependent (with respect to this chart) if, for all $m \geq 0$, $1 \leq k \leq m$, $1 \leq i_k, j_k \leq 4$,

$$
\lim_{\Psi(p)}(x^{i_1} \frac{\partial}{\partial x^{i_1}})...(x^{i_m} \frac{\partial}{\partial x^{i_m}})(f \circ \Psi^{-1}) = [(x^{i_1} \frac{\partial}{\partial x^{i_1}})...(x^{i_m} \frac{\partial}{\partial x^{i_m}})F](\eta)
$$

(10)

If $x^i(p) = \alpha^i$, $\alpha^i \neq 0$ for some $1 \leq i \leq 4$, $f$ is said to be regular direction dependent at $p$ if (10) holds with respect to the translated coordinates $\bar{x}^i = x^i - \alpha^i$.

If a function $f$ is regular direction dependent with respect to the chart $(U, \Psi)$ above, the same need not be true for another chart (containing $p$) in the $C^1$ atlas. For, let $(V, \Phi)$ be another chart with coordinates $y^i$, $y^i(p) = 0$ and suppose $f$ is the component, $f_k$, of a tensor field $t_a = f_j(dx^i)_a = \bar{f}_j(dy^i)_a$ say, so that $\bar{f}_k = f_j \frac{\partial y^j}{\partial x^i}$. Writing, for simplicity, $x^i \frac{\partial}{\partial x^i} f_k$ instead of $(x^i \frac{\partial}{\partial x^i}(f_k \circ \Psi^{-1})) \circ \Phi$, we see that even if $x^i \frac{\partial}{\partial x^i} f_k$ is direction dependent for all $i$ and $j$, $y^i \frac{\partial}{\partial y^i}(f_j \frac{\partial y^j}{\partial x^i}) = (y^i \frac{\partial}{\partial y^i}(f_j \frac{\partial y^j}{\partial x^i})) \frac{\partial x^i}{\partial y^i}$ need not be since we have no control over $\lim_p \frac{\partial x^i}{\partial y^i}$. Thus the $C^1$ structure is too “large” and we have to choose a smaller atlas, a $C^{>1}$ differential structure. The following definition is similar to the one given in \[3\].
Definition 3.2 Let $M$ be a manifold which is $C^\infty$ everywhere except at a point $p$, where it is $C^1$. Let $(U, \Psi), (V, \Phi)$ be two charts containing $p$, with coordinates $x^i$ and $y^i$ respectively. $(U, \Psi), (V, \Phi)$ are said to be $C^{>1}$-related at $p$ if for all $i, j$ and $k$, both
\[
\frac{\partial^2 y^i}{\partial x^j \partial x^k} \quad \text{and} \quad \frac{\partial^2 x^i}{\partial y^j \partial y^k}
\]
have regular direction dependent limits at $p$, in terms of the charts $(U, \Psi)$ and $(V, \Phi)$ respectively. A manifold $M$ is said to be $C^{>1}$ at a point $p$ if (it is otherwise smooth and if ) all charts containing $p$ are $C^{>1}$-related.
A function $f$ on $M \setminus \{p\}$ is said to be regularly direction dependent at $p$ (where $M$ is assumed $C^{>1}$) if $f$ is regular direction dependent in any chart containing $p$, and we then write $f \in C^{>-1}(p)$.
A tensor field on $M \setminus \{p\}$ is said to be regularly direction dependent at $p$ if its components in any chart are.
We have not introduced any metric in the definition of a $C^{>1}$ manifold $M$. Since a (non-zero) vector field on a $C^k$ manifold is at most only $C^{k-1}$, the metric $g_{ab}$ on a manifold which is $C^{>1}$ at a point $p$ is at most $C^{>0}$ there. By this we mean that the first derivatives of the components of the metric are regular direction dependent. In particular, the metric is continuous at $p$, and since a $C^{>1}$ manifold is $C^1$, we have a tangent space with metric at $p$. In the same way, any tensor field is said to be $C^{>0}$ at $p$ if it is continuous at $p$ and if the derivatives (with respect to some chart in the $C^{>1}$ atlas) are regular direction dependent. In order to give another motivation for the introduction of a weaker differentiable structure, the $C^{1+}$-structure, let us look again at the properties of the $C^{>1}$-structure.
In order to ensure that the property of being regularly direction dependent is independent of the choice of charts, $x^i$ and $y^i$ say, we imposed the condition that $\frac{\partial^2 y^i}{\partial x^j \partial x^k}$ and $\frac{\partial^2 x^i}{\partial y^j \partial y^k}$ both be regularly direction dependent. However, looking at the transformation rules for the components of a tensor field, the primary requirement is that $\lim_p y^m \frac{\partial^2 y^i}{\partial x^j \partial x^k} = 0$ and $\lim_p x^m \frac{\partial^2 x^i}{\partial y^j \partial y^k} = 0$.
The $C^{1+}$-condition will allow $\lim_p \frac{\partial^2 x^i}{\partial y^k \partial y^j}$ to diverge logarithmically at $p$, but keep the property
\[
\lim_p y^i \frac{\partial^2 x^l}{\partial y^k \partial y^j} = 0.
\]
Definition 3.3 Let $M$ be a manifold which is $C^\infty$ everywhere except at a point $p$, where it is $C^1$. Let $(U, \Psi), (V, \Phi)$ be two charts containing $p$, with coordinates $x^i$ and $y^i$ respectively. Let (via a translation if necessary) $x^i(p) = 0$, $y^i(p) = 0$ and put $\rho_x = \sqrt{(x^1)^2 + \ldots + (x^4)^2}$, $\rho_y = \sqrt{(y^1)^2 + \ldots + (y^4)^2}$.
$(U, \Psi), (V, \Phi)$ are said to be $C^{1+}$-related at $p$ if for all $i$

$$y^i = k^i(x)\rho_x^2 \log \rho_x + h^i(x)\rho_x^2 + c^i_j x^j \quad (13)$$

$$x^i = \tilde{k}^i(y)\rho_y^2 \log \rho_y + \tilde{h}^i(y)\rho_y^2 + \tilde{c}^i_j y^j \quad (14)$$

where each $k^i(\tilde{k}^i)$ is constant along rays, each $h^i(\tilde{h}^i)$ is regular direction dependent (in terms of the appropriate charts) and the $c^i_j(\tilde{c}^i_j)$ are constants.

A manifold $M$ is said to be $C^{1+}$ at a point $p$ if (it is otherwise smooth and if ) all charts containing $p$ are $C^{1+}$-related.

Regular direction dependent tensor field are defined as in definition $3.2$.

Note that if we do not allow the logarithmic terms, we will get the $C^{>1}$-structure.

Using a $C^{1+}$-structure, (non-zero) tensor fields can, in general, be at most $C^0$ at $i^0$. By this we mean the following.

**Definition 3.4** Let $M$ be a manifold which is $C^{1+}$ at a point $p$. Let $T^{a...b...c...d}$ be a tensor field on $M$. $T^{a...b...c...d}$ is said to be $C^{0+}$ at $p$ if its component with respect to any coordinates $x^i, x^j(p) = 0$ (in the $C^{1+}$-atlas ) can be written

$$T^{i...j...k...m} = k^{i...j...k...m}\rho_x \log \rho_x + h^{i...j...k...m}$$

where each $k^{i...j...k...m}$ is constant along rays and each $h^{i...j...k...m}$ is a $C^{>0}$-function.

Even if the differentiable structure at $p$ is only $C^{1+}$, there is an exceptional case where functions (and tensorfields) can have higher regularity, namely if the function has a zero of sufficient order at $p$. For instance, we say that a function $f$ is $C^{2+}$ at $p$ if it, in some chart, can be written as

$$f(y) = c_{ij} y^i y^j + h(y)\rho_y^3 + k(y)\rho_y^3 \log \rho_y \quad (15)$$

where the $c_{ij}$s are constants and again $k$ is constant along rays in the choosen chart and $h$ is a regularly direction dependent function.

The definition of an asymptotically flat space-time will be the same as in [3] or [13] except that we use the $C^{1+}$ structure. Thus, citing [13], we have

**Definition 3.5** A vacuum space-time $(M, g_{ab})$ is called asymptotically flat at null and spacelike infinity if there exists a space-time $(\tilde{M}, \tilde{g}_{ab})$, with $\tilde{g}_{ab} C^\infty$ everywhere except at a point $i^0$ where $\tilde{M}$ is $C^{1+}$ and $\tilde{g}_{ab}$ is $C^{0+}$, and a conformal factor $\Omega$ satisfying the following conditions:

1. $J^+(i^0) \cup J^-(i^0) = \tilde{M} \setminus M$

2. There exists an open neighbourhood $V$ of $\tilde{M} = \tilde{M} \setminus M$ such that the space-time $(V, \tilde{g}_{ab})$ is strongly causal.
3. $\Omega$ can be extended to a function on all of $\tilde{M}$ which is $C^2$ at $i^0$ and $C^\infty$ elsewhere.

4. (a) On $I^+ = \dot{J}^+(i^0) \setminus i^0$ and $I^- = \dot{J}^-(i^0) \setminus i^0$ we have $\Omega = 0$ and $\nabla_a \Omega \neq 0$. (b) We have $\Omega(i^0) = 0$, $\nabla_a \Omega(i^0) = 0$, and $\nabla_a \nabla_b \Omega(i^0) = -2g_{ab}(i^0)$.

5. (a) The map of null directions at $i^0$ into the space of integral curves of $n^a = \tilde{g}^{ab} \nabla_b \Omega$ on $I^+$ and $I^-$ is a diffeomorphism.

(b) For a smooth function, $\omega$, on $\tilde{M} \setminus i^0$ with $\omega > 0$ on $M \cup I^+ \cup I^-$ which satisfies $\nabla_a (\omega^4 n^a) = 0$ on $I^+ \cup I^-$, the vector field $\omega^{-1} n^a$ is complete on $I^+ \cup I^-$. For a detailed discussion of the meaning of these conditions, we refer to [13] and [3].

In the next section we will look at the Bianchi identity near $i^0$. Therefore we need to look at the curvature components connected to a $C^0$ metric. So, let $\{x^i\}$ be the coordinates of some chart with $x^i(i^0) = 0$ and put $\rho^2 = \sum_i (x^i)^2$. Then the components of the metric are

$$g_{ij} = \eta_{ij} + \rho \log \rho \ k_{ij}$$

(16)

where $\eta_{ij}$ are $C^\infty$ functions which have the Minkowski metric as limit at $i^0$ and where $k_{ij}$ are constant along rays (in terms of the coordinates) from $i^0$. From the definition of the Christoffel symbols it follows that we can write

$$\Gamma^\rho_{\mu\nu} = \log \rho \ k^\rho_{\mu\nu} + \gamma^\rho_{\mu\nu},$$

(17)

where the $k^\rho_{\mu\nu}$’s are constant along rays and where each $\gamma^\rho_{\mu\nu}$ is a regular direction dependent function, and that the curvature components can be written as

$$\rho R^\sigma_{\mu\nu\rho} = \log \rho k^\sigma_{\mu\nu\rho} + \gamma^\sigma_{\mu\nu\rho},$$

(18)

where again each $k^\sigma_{\mu\nu\rho}$ is constant along rays and the $\gamma^\sigma_{\mu\nu\rho}$ are regular direction dependent functions. If this is true in one chart, it is true in all (given the “right” differentiable structure). Our aim is to prove that the logarithmic term is absent for the Weyl tensor, i.e., that

$$\rho C^\sigma_{\mu\nu\rho} = \gamma^\sigma_{\mu\nu\rho}$$

(19)

for some regular direction dependent functions $\gamma^\sigma_{\mu\nu\rho}$.

It will be convenient to use the properties of the function $\rho$ above without referring to a particular chart. Chosing another chart $y^i$, $y^i(i^0) = 0$, and writing $\rho_y^2 = (y^1)^2 + \ldots (y^4)^2$, we find, using the properties in Definition 3.3,

$$\rho_y = k_1 \rho (1 + h \rho + k_2 \rho \log \rho),$$

(20)
where both \( k_1 \) and \( k_2 \) are constant along rays, and where \( h \) is a regular direction dependent function. We will therefore often consider functions \( r \) (on \( M \)) which can be written in the form (20), i.e., \( r = k_1 \rho (1 + h \rho + k_2 \rho \log \rho) \) in some (and therefore every) chart.

4 The Bianchi identity near \( i^0 \)

In this section we will study the asymptotic Bianchi equation, i.e. the Bianchi equation near \( i^0 \). We will assume that we have a space-time which is asymptotically flat at null and spacelike infinity in the sense of definition 3.5. We saw in the previous section, that if we have a \( C^{0+} \) metric at \( i^0 \), then the curvature tensor \( R_{abcd} \) behaves, near \( i^0 \), as

\[
\rho R_{abcd} = \log \rho k_{abcd} + h_{abcd},
\]

where \( k_{abcd} \) and \( h_{abcd} \) are two regular direction dependent tensor fields with \( k_{abcd} \) constant along rays. The same is of course true for the Weyl tensor \( C_{abcd} \). We will show that, for the Weyl tensor, there is no logarithmic term, i.e., we have \( rC_{abcd} = h_{abcd} \) near \( i^0 \). In fact we will show that

\[
r_3 \frac{C_{abcd}}{\Omega} = h_{abcd},
\]

where \( r \) is any function satisfying (20) and where \( \Omega \) is the conformal factor. Instead of using the Weyl tensor \( C_{abcd} \), we will use the Weyl spinor, and in order to show (24) we will need a direction dependent null tetrad and spin frame near \( i^0 \). As shown in appendix A, one can choose a direction dependent null tetrad \( N^a, L^a, M^a \) and an associated spin frame which has essentially the following properties: If we denote standard polar coordinates in Minkowski space-time with \( t, r, \theta, \varphi \) we can think of our tetrad, in the limit, as

\[
N^a = \frac{1}{\sqrt{2}} (\frac{\partial}{\partial t} + \frac{\partial}{\partial r})^a, \quad L^a = \frac{1}{\sqrt{2}} (\frac{\partial}{\partial t} - \frac{\partial}{\partial r})^a, \\
M^a = \frac{1}{\sqrt{2}r} (\frac{\partial}{\partial \theta} - \frac{i}{\sin \theta} \frac{\partial}{\partial \varphi})^a.
\]

One can also choose a radial parameter \( r \), essentially (i.e. in the limit) the “polar” \( r \) above such that

\[
\lim_{i^0} \nabla_a r = -N_a + L_a.
\]

We want to adopt the GHP formalism, [7], [11], so we now let \( (n^a, l^a, m^a, \bar{m}^a) \) be weighted vector fields (with weights \( \{-1,-1\}, \{1,1\}, \{1,-1\}, \{-1,1\} \) respectively), with \( n^a, l^a \) pointing along the directions of \( N^a, L^a \), and let \( (a^A, t^A) \)
be the spinor dyad corresponding to $(n^a, l^a, m^a, \bar{m}^a)$. We also define the weighted scalar $e$, which we require to be $C^{>0}$, of weight $\{1,1\}$ by

$$e = l^a N_a.$$  \hfill (25)

Thus we have, at $i^0$, 

$$\nabla_a r = -N_a + L_a = -en_a + e^{-1}l_a.$$ \hfill (26)

and therefore also

$$\lim_{i^0} m^a \nabla_a r = \lim_{i^0} \bar{m}^a \nabla_a r = 0,$$

$$\lim_{i^0} l^a \nabla_a r = -e, \quad \lim_{i^0} n^a \nabla_a r = e^{-1}. $$ \hfill (27)

Let us denote the space of spacelike and null directions at $i^0$ by $K$, so that $k \in K$ is any such direction. We then define derivative operators on $K$ as follows. Let $\eta^o \in C^{>-1}(i^0)$, i.e., $\eta^o$ is any regularly direction dependent quantity, and put

$$\eta(k) = \lim_{x \to i^0} \eta^o(x).$$ \hfill (28)

We then define the derivative operator $\partial_a$ on $K$ by

$$\partial_a \eta(k) = \lim_{x \to i^0} r \nabla_a \eta^o(x)$$ \hfill (29)

Note that although $\partial_a$ depends on the choice of function $r$, $\partial_a$ is independent of the components of the Christoffel symbol, $\Gamma^i_{jk}$ since $\lim_{i^0} r \Gamma^i_{jk} = 0$. Thus we can use a coordinate derivative on the RHS of (29) and get the same results as in flat space-time.

In the GHP formalism, we now denote the usual derivative operators by $\nabla^o, \nabla^{i^0}, \partial^o$ and $\partial^{i^0}$, so that e.g., on a zero-weighted scalar $\eta$, $\nabla^o \eta = l^a \nabla_a \eta$ etc. We then define, for any weighted $C^{>-1}(i^0)$ scalar $\eta^o$ the following derivative operators on $K$.

$$\nabla^o \eta(k) = \lim_{i^0} r \nabla^o \eta^o, \quad \nabla^{i^0} \eta(k) = \lim_{i^0} r \nabla^{i^0} \eta^o,$$

$$\partial^o \eta(k) = \lim_{i^0} r \partial^o \eta^o, \quad \partial^{i^0} \eta(k) = \lim_{i^0} r \partial^{i^0} \eta^o.$$ \hfill (30)

Using these derivative operators we find, in the limit at $i^0$, (again we refer to appendix A) i.e. on $K$ that

$$\nabla^o \partial_a = \nabla^{i^0} \partial_a = \partial^o A = \partial^{i^0} A = 0$$ \hfill (31)

$$\partial^o A = e^{-1} \partial A.$$ \hfill (32)
\[ \partial' o_A = -e v_A \]  
and that

\[ \lim r \rho = e, \quad \lim r \rho' = -e^{-1} \]  
\[ \lim r \kappa = \lim r \kappa' = \lim r \sigma = \lim r \sigma' = \lim r \tau = \lim r \tau' = \lim r e = \lim r e' = 0. \]  
The commutator relations become simply

\[ \slashed{\partial} \slashed{\partial}' - \slashed{\partial}' \slashed{\partial} = -e \slashed{\partial}' - e^{-1} \slashed{\partial}, \]  
\[ \slashed{\partial} \partial - \partial \slashed{\partial} = 0, \]  
\[ \partial \partial' - \partial' \partial = (p - q), \]  
plus the primed and conjugated versions.

Let us finally derive another expression for the operators \( \slashed{\partial} \) and \( \slashed{\partial}' \) on \( K \). Again referring to the polar coordinates \( t, r, \theta, \varphi \) in the tangent space at \( i^0 \), we may label the directions in \( K \) with \( \alpha, \theta, \varphi \) where

\[ \alpha = \frac{t}{r}. \]  
Thus \( \alpha = 1 \) (\( \alpha = -1 \)) for future (passed) directed null directions, and on \( K \) one finds that

\[ \slashed{\partial} = e(1 + \alpha) \frac{\partial}{\partial \alpha}, \]  
\[ \slashed{\partial}' = e^{-1}(1 - \alpha) \frac{\partial}{\partial \alpha}. \]  
We now return to the equation of interest, i.e. the Bianchi equation. We put

\[ \varphi_{ABCD} = \Omega^{-1} \Psi_{ABCD}. \]  
By virtue of the peeling property, \( \varphi_{ABCD} \) exists on \( \mathcal{I} \). From the discussion in the beginning of this section, we know that, near \( i^0 \),

\[ r \Psi_{ABCD} = \log r k_{ABCD} + h_{ABCD}, \]  
where \( r \) is of type \( \psi_0 \), and \( k_{ABCD}, h_{ABCD} \) are regular direction dependent (symmetric) spinor fields, where the components of \( k_{ABCD} \) can be taken to be constant along rays (in the chosen spin frame). From the definition of a \( C^2^+ \)-function it follows that \( \Omega \) has the property that for any chart (within
the $C^1$-structure) $\Omega$ takes the form $\Omega = r^2 k$, where $r$ satisfies \([24]\) and $k$ is a function which is constant along rays and which vanishes on $I$. Thus,

$$r^3 \frac{1}{\Omega} \Psi_{ABCD} = r^3 \varphi_{ABCD} = \log r \mu_{ABCD} + \gamma_{ABCD}. \quad (45)$$

Again, $\mu_{ABCD}$ and $\gamma_{ABCD}$ are regular direction dependent symmetric spinor fields where the components can be taken as for $k_{ABCD}$ and $h_{ABCD}$.

We will now show that $\mu_{ABCD} = 0$.

Let us first recall that, with $\hat{\Psi}_{ABCD}$ being the Weyl spinor in physical space-time $\hat{M}$, the Bianchi identity reads

$$\hat{\nabla}^A \hat{\Psi}_{ABCD} = 0. \quad (46)$$

Here $\hat{\nabla}_a$ is the derivative operator associated with the physical metric $\hat{g}_{ab}$. Under a conformal rescaling $\hat{g}_{ab} \rightarrow g_{ab} = \Omega^2 \hat{g}_{ab}$, the Weyl spinor is conformally invariant,

$$\Psi_{ABCD} = \hat{\Psi}_{ABCD}. \quad (47)$$

For a proof of this, see \([12]\).

In the rescaled space-time $M$, with derivative operator $\nabla_a$, the Bianchi identity is

$$\nabla^A \left( \frac{1}{\Omega} \Psi_{ABCD} \right) = 0. \quad (48)$$

Therefore with the notation adopted, we have

$$\nabla^A \varphi_{ABCD} = 0. \quad (49)$$

\(\check{\text{From (49) we thus have}}\)

$$\mu_{ABCD} = \frac{r^3}{\log r} \varphi_{ABCD} - \frac{1}{\log r} \gamma_{ABCD}. \quad (50)$$

We differentiate this equation, multiply with $r$ and get

$$r \nabla^A \mu_{ABCD} = \frac{3r^3}{\log r} (1 - \frac{1}{3 \log r}) \varphi_{ABCD} \nabla^A r$$

$$+ \frac{r^4}{\log r} \nabla^A \varphi_{ABCD} - r \nabla^A \left( \frac{\gamma_{ABCD}}{\log r} \right). \quad (51)$$

Taking the limit, the last term on the RHS will vanish, and by using (49) and (50) we get the following equation on $K$

$$\partial^A \mu_{ABCD} = 3 \nabla^A \mu_{ABCD}, \quad (52)$$
Let us now look at the LHS of equation (52). We have

$$\nabla^{AA'} = -e_{A'}^{A} + e^{-1}o^{A}o^{A'}$$  \tag{53}

at \(t^{0}\). In order to solve (52) (and (33)) we make the usual decomposition of \(\mu_{ABCD}\) and \(\gamma_{ABCD}\), so that

$$\mu_{0} = \mu_{ABCD} o^{A}o^{B}o^{C}o^{D}, \quad \gamma_{0} = \gamma_{ABCD} o^{A}o^{B}o^{C}o^{D}$$  \tag{54}

$$\mu_{1} = \mu_{ABCD} \iota^{A}o^{B}o^{C}o^{D}, \quad \gamma_{1} = \gamma_{ABCD} \iota^{A}o^{B}o^{C}o^{D}$$  \tag{55}

$$\mu_{2} = \mu_{ABCD} \iota^{A}\iota^{B}o^{C}o^{D}, \quad \gamma_{2} = \gamma_{ABCD} \iota^{A}\iota^{B}o^{C}o^{D}$$  \tag{56}

$$\mu_{3} = \mu_{ABCD} \iota^{A}\iota^{B}\iota^{C}o^{D}, \quad \gamma_{3} = \gamma_{ABCD} \iota^{A}\iota^{B}\iota^{C}o^{D}$$  \tag{57}

$$\mu_{4} = \mu_{ABCD} \iota^{A}\iota^{B}\iota^{C}\iota^{D}, \quad \gamma_{4} = \gamma_{ABCD} \iota^{A}\iota^{B}\iota^{C}\iota^{D}$$  \tag{58}

Let us now look at the LHS of equation (52). We have

$$\partial^{AA'} \mu_{ABCD} = \lim_{t^{0}} \varepsilon^{AE}r \nabla_{E}^{A'} \mu_{ABCD}$$

$$= \lim_{t^{0}}(o^{A} \iota^{E} - \iota^{A}o^{E})r \nabla_{E}^{A'} \mu_{ABCD}.$$

So, transvecting with \(o_{A'}\) and \(\iota_{A'}\) respectively, and using (53), we obtain

$$(o^{A}\partial^{'} - \iota^{A}\underline{\partial}) \mu_{ABCD} = 3e_{i}^{A} \mu_{ABCD}$$  \tag{61}

$$(o^{A}\underline{\partial}^{'} - \iota^{A}\partial) \mu_{ABCD} = 3e^{-1}o^{A} \mu_{ABCD}.$$  \tag{62}

We now contract these equations with \(o^{B}o^{C}o^{D}, o^{B}o^{C}\iota^{D}, o^{B}\iota^{C}\iota^{D}\) and \(\iota^{B}\iota^{C}\iota^{D}\). Together with Leibniz’ rule and \(\underline{\partial}^{(1)} - \partial^{(1)}\), we get

$$\underline{\partial} \mu_{1} - \partial^{'} \mu_{0} - e \mu_{1} = 0,$$  \tag{63}

$$\underline{\partial}^{'} \mu_{0} - \partial \mu_{1} - 2e^{-1} \mu_{0} = 0,$$  \tag{64}

$$\underline{\partial} \mu_{2} - \partial^{'} \mu_{1} = 0,$$  \tag{65}

$$\underline{\partial}^{'} \mu_{1} - \partial \mu_{2} - e^{-1} \mu_{1} = 0,$$  \tag{66}

$$\underline{\partial} \mu_{3} - \partial^{'} \mu_{2} + e \mu_{3} = 0,$$  \tag{67}

$$\underline{\partial}^{'} \mu_{2} - \partial \mu_{3} = 0,$$  \tag{68}

$$\underline{\partial} \mu_{4} - \partial^{'} \mu_{3} + 2e \mu_{4} = 0,$$  \tag{69}

$$\underline{\partial}^{'} \mu_{3} - \partial \mu_{4} + e^{-1} \mu_{3} = 0.$$  \tag{70}

Note that \(\underline{\partial}^{'}\) gives zero when acting on directions on \(I^{+}\), so that equation (63) gives \(\partial \mu_{3}(k) = 0, k \in I^{+}\). Since \(\mu_{3}\) has spin weight -1, this implies \(\mu_{3}(k) = 0, k \in I^{+}\). Applying this result to (64), we find, since \(\mu_{4}\) has negative spin weight, that also \(\mu_{4}(k) = 0, k \in I^{+}\). The same conclusion holds for \(\mu_{0}\) and \(\mu_{1}\) on \(I^{-}\). This means that \(\mu_{2}(k), k \in I^{+}\) is unaffected by the change of spin basis \((o^{A}, \iota^{A}) \rightarrow (o^{A} + \lambda \iota^{A}, \iota^{A})\) (and that \(\mu_{2}(k), k \in I^{-}\)
is invariant under \((o^A, o^A) \rightarrow (o^A, o^A + \lambda o^A))\). It is also clear that \(\mu_0, \mu_1, \mu_3\) and \(\mu_4\) are determined by \(\mu_2\).

Note that the above equations come naturally in pairs. Using this and the commutator relations we derive the following equations.

\[
(\mathbf{P} \mathbf{P}' - e^{-1} \mathbf{P} - e \mathbf{P}' + 2 - \partial \partial^r) \mu_0 = 0, \quad (71)
\]
\[
(\mathbf{P} \mathbf{P}' - e^{-1} \mathbf{P} - \partial \partial^r) \mu_1 = 0, \quad (72)
\]
\[
(\mathbf{P} \mathbf{P}' - e^{-1} \mathbf{P} - \partial \partial^r) \mu_2 = 0, \quad (73)
\]
\[
(\mathbf{P} \mathbf{P}' + e \mathbf{P}' - \partial \partial^r) \mu_3 = 0, \quad (74)
\]
\[
(\mathbf{P} \mathbf{P}' + 2e \mathbf{P}' + e^{-1} \mathbf{P} + 2 - \partial \partial^r) \mu_4 = 0. \quad (75)
\]

We now use (41) and (42) on (73). We get

\[
((1 - \alpha^2) \frac{\partial^2}{\partial \alpha^2} - 2\alpha \frac{\partial}{\partial \alpha} - \partial \partial^r) \mu_2 = 0. \quad (76)
\]

Writing the directions \(k\) as \(k = (\alpha, \theta, \varphi)\), where \((\theta, \varphi)\) are standard spherical coordinates (and \(-1 \leq \alpha \leq 1\)), we expand, for each \(\alpha, \mu_2\) in spherical harmonics. Thus

\[
\mu_2(\alpha, \theta, \varphi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} c_n^m(\alpha) Y_n^m(\theta, \varphi), \quad (77)
\]

where each \(c_n^m(\alpha)\) becomes smooth since \(\mu_2\) is assumed smooth. Using that \(\partial \partial^r Y_n^m = -n(n+1) Y_n^m\) and that the series may be differentiated termwise, we get the following equations for the \(c_n^m\)’s.

\[
\left((1 - \alpha^2) \frac{\partial^2}{\partial \alpha^2} - 2\alpha \frac{\partial}{\partial \alpha} + n(n+1)\right) c_n^m = 0, \quad (78)
\]

i.e., the Legendre differential equation.

For each \(n \in \mathbb{N}\), we have the Legendre polynomial \(p_n\) as solution. The Legendre functions of the second kind are not allowed since they diverge at \(\alpha = 1\) and \(\alpha = -1\). Note that this is an essential restriction compared to [3].

Let us write \(\mu_2(1, \theta, \varphi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} d_n^m Y_n^m(\theta, \varphi)\). Since \(p_n(1) = 1\) for all \(n\), we can write \(\mu_2\) as

\[
\mu_2(\alpha, \theta, \varphi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} d_n^m p_n(\alpha) Y_n^m(\theta, \varphi). \quad (79)
\]

We now return to equation (15). We differentiate this relation, multiply with \(r\), use (14) and (15) again and find

\[
\begin{align*}
\log r \{3 \nabla AA' \mu_{ABCD} - r \nabla AA' \mu_{ABCD} \} &= r \nabla AA' \gamma_{ABCD} - 3 \nabla AA' \gamma_{ABCD} + \nabla AA' \mu_{ABCD} \quad (80)
\end{align*}
\]
By (52), the bracket on the LHS of (80) is, in the limit, zero. Furthermore, by choosing \( \mu_{ABCD} \) such that the components in some dyad are constant along rays, we conclude that the LHS behaves like \( \log r \rho \log \rho \), i.e., goes to zero as \( r \to 0 \). Thus, taking the limit \( r \to 0 \), we get the equation

\[
3\nabla^{AA'} \gamma_{ABCD} - \partial^{AA'} \gamma_{ABCD} = \nabla^{AA'} \mu_{ABCD}.
\]  

(81)

A calculation, analogous to the above one yields

\[
\begin{align*}
\Phi_1 - \partial_1 \gamma_0 - e \gamma_1 &= -e \mu_1, \\
\Phi_2 = \partial \gamma_2 - 2e^{-1} \gamma_0 &= e^{-1} \mu_0, \\
\Phi_3 &= \partial_3 \gamma_3 = -e \mu_3, \\
\Phi_4 &= \partial_4 \gamma_4 = -e \mu_4.
\end{align*}
\]

(82) \( \Phi_1 \) \( \Phi_2 \) \( \Phi_3 \) \( \Phi_4 \) (83) \( \Phi_2 \) \( \Phi_3 \) \( \Phi_4 \) (84) \( \Phi_3 \) \( \Phi_4 \) (85) \( \Phi_4 \) (86) \( \Phi_4 \) (87) \( \Phi_4 \) (88) \( \Phi_4 \) (89)

Similarly we apply \( \Phi_1 \) to (84), \( \Phi_2 \) to (85), use (85) and (86) again. We obtain

\[
(\Phi_1 - e^{-1} \Phi - \partial \gamma_2) = \mu_2 - e \mu_2 + e^{-1} \mu_2.
\]

(90)

We expand \( \gamma_2 \) in spherical harmonics, so that \( \gamma_2(\alpha, \theta, \phi) \) takes the form

\[
\sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_n^m(\alpha) Y_n^m(\theta, \phi).
\]

(91)

We also use the expansion (79) for \( \mu_2 \). Since

\[
1 - e \Phi + e^{-1} \Phi = 1 + 2a \frac{d}{d\alpha},
\]

\( a_n^m \) must satisfy

\[
\{1 - \alpha^2 \} \frac{d^2}{d\alpha^2} - \frac{2}{\alpha} \frac{d}{d\alpha} + n(n + 1) \} a_n(\alpha) =
\]

\[
d_n^m(a_n^{p_n} + 2 \alpha \frac{d}{d\alpha} p_n(\alpha)).
\]

(92)

Let us look for solutions of the type \( a_n^m = v_n^m p_n \). The resulting equation can be written

\[
\frac{d}{d\alpha} \{1 - \alpha^2 \} p_n^2 \frac{d}{d\alpha} v_n^m = d_n^m \frac{d}{d\alpha} (\alpha p_n^2)
\]

(93)

which implies that

\[
a_n^m = v_n^m p_n = - \frac{d_n^m p_n}{2} \log(1 - \alpha^2) + C q_n + D p_n
\]

(94)

for some constants \( C \) and \( D \). Since \( \gamma_{ABCD} \) is assumed smooth, this implies that \( a_n^m \) must also be smooth. We note that \( \log(1 - \alpha^2) \) diverges logarithmically when \( \alpha \to 1 \) and \( \alpha \to -1 \). \( q_n \) also behaves in this way, but since \( p_n \) and \( q_n \) has the opposite parity, \( a_n^m \) can be defined for \( \alpha = 1 \) and \( \alpha = -1 \) only if \( d_n^m \) and \( C \) are zero. This means that \( \lim_{\alpha \to 0} \mu_{ABCD} = 0 \), and, by the properties of \( \mu_{ABCD} \), that \( \lim_{\alpha \to 0} \log r \mu_{ABCD} = 0 \).

We can now state the following theorem.
Theorem 1 Suppose \((\hat{M}, \hat{g}_{ab})\) is asymptotically flat at null and spacelike infinity in the sense of Definition 2.3, with the rescaled metric \(C_{\theta}^{0+}\) at \(i^0\). If \(r\) is any \(C>0\) function with \(r(i^0) = 0\), \(C_{abc}^d(\eta) = \lim_{\eta \to 0} r^3 C_{abc}^d\) exists and has the symmetry property \(C_{abc}^d(\eta) = C_{abc}^d(-\eta)\) where \(\eta\) is any spacelike or null vector in \(T_0 M\).

Using the direction dependent tetrad \(L^a, N^a, M^a, \overline{M}^a\) from appendix A so that \(L^a(\eta) = N^a(\eta), M^a(\eta) = \overline{M}^a(\eta)\), and also using our decomposition of the Weyl tensor, we have to show that

\[
\gamma_2(\eta) = \gamma_2(-\eta), \gamma_3(\eta) = \gamma_1(-\eta), \gamma_4(\eta) = \gamma_0(-\eta).
\]

(94)

To show that \(\gamma_2(\eta) = \gamma_2(-\eta)\), we note that from (82) with \(d_m^m = C = 0\), we have

\[
\gamma_2(\alpha, \theta, \varphi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} b_n^m p_n(\alpha) Y_n^m(\theta, \varphi).
\]

(95)

Therefore, the symmetry property follows from

\[
p_n(\alpha) = (-1)^n p_n(-\alpha)
\]

(96)

\[
Y_n^m(\pi - \theta, \varphi \pm \pi) = (-1)^n Y_n^m(\theta, \varphi),
\]

(97)

That \(\gamma_3(\eta) = \gamma_1(-\eta)\) then follows from the symmetry of the equations (84) and (87) (with \(\mu_{ABCD} = 0\)). Similarly (82) and (89) give \(\gamma_4(\eta) = \gamma_0(-\eta)\).

Note that using the alternative definitions \(C_{abc}^d(\eta) = \lim_{\eta \to 0} \Omega^{1/2} C_{abc}^d\) or \(C_{abc}^d(\eta) = \lim_{\eta \to 0} r C_{abc}^d\), we still have that the limits exist and that \(C_{abc}^d(\eta) = C_{abc}^d(-\eta), \eta \in T_0 M\).

To see this, we note that the factor \(\sqrt{\Omega}\) has the symmetry property \(\lim_{\eta \to 0} \sqrt{\Omega}(\eta) = \lim_{\eta \to 0} \sqrt{\Omega}(-\eta)\) which is obvious for null directions and which follows in spacelike directions if we let \(\eta\) be indicated by the corresponding timelike unit vector \(\eta^a\) at \(T_0\) and use \(\lim_{\eta \to 0} \sqrt{\Omega}(\eta) = \lim_{\eta \to 0} \frac{\partial \sqrt{\Omega}}{\partial \eta^a}(\eta) = \lim_{\eta \to 0} \frac{\eta^a}{\eta^{(L_a - N_a)}}(\eta)\) which is invariant under \(\eta^a \to -\eta^a\). Here again we have used the same tetrad as above, so that \((L_a - N_a)(\eta) = -(L_a - N_a)(-\eta)\).

5 The electromagnetic field near \(i^0\)

In this section we will discuss the difference in the differentiability conditions at \(i^0\) by comparing with the corresponding conditions on the electromagnetic field. In 3, with \(\Omega\) being the conformal factor and \(F_{ab}\) the electromagnetic field, the imposed condition is that \(\Omega F_{ab}\) is regular direction dependent at \(i^0\). As we will argue, this condition is too weak, i.e. it is necessary to include suitable requirements on \(T^+\) and \(T^-\). In this way, we again motivate the general importance of including the (limits at \(i^0\) of ) fields on \(T^+\) and \(T^-\).
Consider the three pairs
\[ E = -\frac{dr}{r^2}, \quad B = 0, \]  
\[ E = -\frac{t \cos \theta}{r^3} dr - \frac{1}{2} \frac{t \sin \theta}{r^3} r d\theta, \quad B = -\frac{1}{2} \frac{\sin \theta}{r^2} r \sin \theta d\phi, \]  
\[ E = -\frac{t}{2r} \log(\frac{r + t}{r - t}) - 1) \frac{2 \cos \theta}{r^2} dr - \left( \frac{t}{2r} \log(\frac{r + t}{r - t}) - \frac{t^2}{r^2} \right) \frac{\sin \theta}{r^2} r d\theta, \]  
\[ B = -\left( \frac{1}{2} \log(\frac{r + t}{r - t}) - \frac{rt}{l^2 - r^2} \right) \frac{\sin \theta}{r^2} r \sin \theta d\phi. \]

These are solutions to Maxwell’s equation in flat space-time, obtained in the following way. By making a standard conformal completion of the Minkowski space-time and taking the limit at \( i^0 \), in spacelike directions, to Maxwells equation using \( \Omega F_{ab} \), one get an equation on \( \tilde{K} \), the unit time-like hyperboloid in the tangent space at \( i^0 \). One get a family of solutions, essentially parametrized by the Legendre polynomials of the first and second kind. By taking the first few solutions on \( \tilde{K} \), we write down the corresponding solutions in our original spacetime.

The solution (98) is just the field from a point charge (placed at \( r = 0 \)). The field (99) can be thought of as a dipol where the dipol moment increases (linearly) with time. This may not seem very physical, and one can exclude, see [9], such solutions by imposing suitable conditions at timelike infinity, \( i^+ \). The solution (100) is singular at the light cone of the origin, \( t = 0, r = 0 \). It may be thought of as a charged sphere, expanding with the speed of light. We believe that this is also not a very physical solution, and by changing the conditions on the electromagnetic field near \( i^0 \), solutions of this type will be excluded.

We can equally well consider the fields (98)- (100) as fields in the tangent space at \( i^0 \), or in this case, as fields in the rescaled space-time by interpreting \( t, r, \theta, \varphi \) as polar coordinates with respect to \( i^0 \). In the standard completion of the Minkowski space-time, the rescaled space-time is again flat and the conformal factor \( \Omega \) is related to the polar coordinates above via \( \Omega = r^2 - t^2 \).

Thus in any spacelike direction, the condition that \( \lim_{\alpha} \Omega F_{ab} \) exists as a regular direction dependent tensor field at \( i^0 \) is equivalent to the same condition on \( \lim_{\alpha} r^2 F_{ab} \), since \( \Omega = r^2(1 - \frac{t^2}{r^2}) \) and the limit of \( (1 - \frac{t^2}{r^2}) \) along any spacelike direction is non-zero. On \( I^+ \) and \( I^- \), the fields (98) and (99) diverge like \( 1/r^2 \) so that \( r^2 F_{ab} \) has a non-zero limit along null directions. \( \Omega F_{ab} \) is trivially zero on \( I \) for these fields and imposes no restriction. Moreover, by continuity, \( \Omega F_{ab} \) exists and is zero on \( I \) also for the field given in (100), but, for this field, \( r^2 F_{ab} \) is not even defined there.

We conclude that, even if the condition imposed on \( \Omega F_{ab} \) is very natural in that it is completely coordinate independent, it might not be restrictive enough. A limiting condition on \( r^2 F_{ab} \) seems more appropriate, but is, as
it stands, formulated in coordinates. Below we will suggest a coordinate independent version.

Let us also note that solutions corresponding to pairs \( p_k, Y_l^k \) for higher \( k \), will all correspond to multipole moments with strength that increases in time, while solutions corresponding to \( q_k, Y_l^k \) will all be singular on \( \mathcal{I} \).

Also, in exactly the same way, we can start with a magnetic field on \( K \) and derive corresponding fields in \( M \).

In the case of a general asymptotically flat space-time, we consider electromagnetic fields \( F_{ab} \) with the following properties. For any \( C^\infty \) function \( r \) with \( r(\ell^0) = 0 \), the field \( r^2F_{ab} \) should be regularly direction dependent. We may analyse the situation by deriving the equations corresponding to (63)-(70) and (71)-(75). We use the tetrad from appendix A, and put \( \phi_{AB} = r^2\phi_{AB} \), where \( \phi_{AB} \) is defined by \( F_{ab} = \phi_{AB}\epsilon_{A'B'} + \epsilon_{AB}\phi_{A'B'} \). Using the source free Maxwell equation, \( \nabla_A\phi_{AB} = 0 \) and (29), we obtain, after a derivation analogous to the derivation on page 14, the following equations at \( \ell^0 \):

\[
\begin{align*}
\mathbf{P}\phi_1 - \partial'\phi_0 &= 0, \\
\mathbf{P}'\phi_0 - \partial\phi_1 - e^{-1}\phi_0 &= 0, \\
\mathbf{P}\phi_2 - \partial'\phi_1 + e\phi_2 &= 0, \\
\mathbf{P}'\phi_1 - \partial\phi_2 &= 0.
\end{align*}
\]

and the decoupled equations

\[
\begin{align*}
(\mathbf{P}\mathbf{P}' - e^{-1}\partial\partial')\phi_0 &= 0, \\
(\mathbf{P}' - e^{-1}\mathbf{P} - \partial\partial')\phi_1 &= 0, \\
(\mathbf{P}'\mathbf{P} + e\mathbf{P}' - \partial\partial')\phi_2 &= 0.
\end{align*}
\]

We see that equation (106) is the same as (73). Therefore \( \phi_1 \) takes the form

\[
\phi_1(\alpha, \theta, \varphi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} b_n^m p_n(\alpha) Y_n^m(\theta, \varphi)
\]

for some constants \( b_n^m \). Contributions from the Legendre functions of the second kind are excluded by the conditions on \( r^2F_{ab} \) on \( \mathcal{I}^+ \) and \( \mathcal{I}^- \). We also see that \( \phi_1 \) has the same antipodal property as \( \gamma_2 \) in the gravitational case.

Let us remark that we can also consider fields which behave like the field \( r^3\varphi_{ABCD} \) in (15), i.e., the situation when

\[
r^2\varphi_{AB} = \log r \mu_{AB} + \gamma_{AB}.
\]

where \( \mu_{AB} \) and \( \gamma_{AB} \) have the same properties as the corresponding fields in (15).

By the results of the previous section, we expect that \( \mu_{AB} \) can be taken smooth and non-zero at the expense of \( \gamma_{AB} \) blowing up at \( \mathcal{I}^+ \) and \( \mathcal{I}^- \). As an
example of such a field, with standard coordinates \((t, r, \theta, \varphi)\) in Minkowski
space-time, we take

\[
E = \frac{\log r}{r^2} \left( \frac{t}{r} \cos \theta \hat{r} + \frac{1}{2} \frac{t}{r} \sin \theta \hat{\theta} \right) + \\
\frac{1}{2} \frac{t}{r^3} \left( \log(1 - \frac{t^2}{r^2}) \cos \theta \hat{r} + (-1 + \frac{1}{2} \log(1 - \frac{t^2}{r^2}) - \frac{t^2/r^2}{1 - t^2/r^2}) \sin \theta \hat{\theta}, \right)
\]

\[
B = \frac{1}{2} \frac{r}{r^2} \sin \theta \hat{\varphi} + \frac{1}{2} \frac{1}{r^2} \left( \frac{1}{2} \log(1 - \frac{t^2}{r^2}) - \frac{t^2/r^2}{1 - t^2/r^2} \right) \sin \theta \hat{\varphi}. \tag{110}
\]

We see that the logarithmic part of both \(E\) and \(B\) have, suitable rescaled,
regular direction dependent limits at the origin, but that the next order
behaviour, i.e., the parts without the log terms diverge at the null cone
trough the origin.

6 Discussion

In this article we have studied space-times which are asymptotically flat
at spacelike infinity, \(i^0\). We have chosen to follow the ideas of [3]. The
differences are that we have used a weaker differentiable structure at \(i^0\) and
put more conditions on the limits along \(I^+\) and \(I^-\).

The conditions on \(I^+\) and \(I^-\) are motivated by peeling property, [12],
and by the behaviour of electromagnetic fields in flat space-time. These con-
ditions eliminate (in the case of electromagnetic fields) some of the fields that
are “obviously non-physical” as a whole, but which still have the expected
fall off in spacelike directions at \(i^0\).

The differentiable structure, the \(C^{1+}\) structure, allows, at least poten-
tially, the Weyl tensor to diverge in an unwanted way. If \(r\) is a parameter
satisfying (20), we know that \(r R_{abcd}^d\) is a regular direction dependent tensor
if we use a \(C^{>1}\) differentiable structure at \(i^0\). If we use only a \(C^{1+}\) structure,
\(r R_{abcd}^d\) may diverge like \(\log r\). We have shown that in the latter case, the
relevant part of \(r R_{abcd}^d\), the \((r \text{ times the})\ Weyl tensor, \(r C_{abc}^d\), is nevertheless
regular direction dependent. To conclude this, the limits along both \(I^+\) and
\(I^-\) must be considered. This may be related to the discussion in [4].

In the electromagnetic case, the situation is similar. If we consider fields
of the form (109), we can eliminate the logarithmic part only if we include
limits of the field along both future directed and passed directed null direc-
tions. For instance, we can combine (100) and (110) to get a field which is
smooth on the future null cone through the origin.

An interesting question is of course if there exists a completion of the
Schwarzschild space-time where the metric is \(C^{>0}\) in both spacelike and null
directions. If so, the antipodal property of theorem 3 will follow (by the
same argument as used here).
Another interesting question is the relation to the decomposition of the Weyl tensor into the electric and magnetic parts, \( E \). As discussed in \([3]\), the vanishing of \( B \), the magnetic part of the Weyl tensor, is a necessary condition in order to be able to define angular momentum at \( i^0 \). Using the tetrad of section \( i \), one finds that \( B_{ab}M^aM^b \) corresponds to the imaginary (and \( E_{ab}M^aM^b \) to the real) part of \( \gamma_2 \) in the expansion \([3, 2]\). If the imaginary part of \( \gamma_2 \) is zero for null directions, the same will hold for all directions. Since \( \gamma_2 \) on e.g. \( \mathcal{T}^+ \) is related to \( \Psi_2 \), there may be a connection to the positive mass theorems.

We also hope that the use of \( C^{1+} \) differential structures which in a sense allow logarithmic divergences at \( i^0 \), together with the symmetry property of the rescaled Weyl tensor can shed some light on the so called logarithmic ambiguities, \([5], [2]\) in the completions near \( i^0 \).

### 7 Appendix A, A tetrad near \( i^0 \).

In this appendix we will justify our choice of the direction dependent tetrad in section 4.

Choose any future directed, timelike vector \( v^a \in T_{i^0}M \), such that
\[ v^a v_a = 2. \]
We then take a chart with coordinates \( x^1 = t, x^2 = x, x^3 = y, x^4 = z \) containing \( i^0 \). By a translation and a linear transformation we may assume that \( x^i(i^0) = 0 \) and that the metric at \( i^0 \) has components \( \text{diag}(1, -1, -1, -1) \). We can also arrange so that \( v^a = \sqrt{2}\frac{\partial}{\partial x} \) at \( i^0 \).

Let us also define \( \rho \) via \( \rho^2 = t^2 + x^2 + y^2 + z^2 \) and \( r \) via \( r^2 = x^2 + y^2 + z^2 \). Recall that the metric \( g_{ab} \) has components \( g_{ij} \) that can be written
\[ g_{ij} = \eta_{ij} + \rho \log \rho \ k_{ij} \tag{111} \]
where each \( k_{ij} \) is constant along rays and the \( C^{1+} \)-functions \( \eta_{ij} \) have the Minkowski metric as limit at \( i^0 \).

We will now define the direction dependent null vectors \( N^a \) and \( L^a \). It will be convenient to adopt the convention that \( h \) stands for any \( C^{1+} \)-function which is zero at \( i^0 \), and that \( k \) stands for any function which is constant on rays. We then define \( N^a \) as
\[ N^a = \frac{1}{\sqrt{2}} ((\frac{\partial}{\partial t})^a + (1 + \alpha)(\frac{\partial}{\partial r})^a), \quad (\frac{\partial}{\partial r})^a = (\frac{x}{r} \frac{\partial}{\partial x} + \frac{y}{r} \frac{\partial}{\partial y} + \frac{z}{r} \frac{\partial}{\partial z})^a \tag{112} \]
where \( \alpha \) is chosen such that we get a null vector. Using our convention, we have
\[ (\frac{\partial}{\partial r})^a, (\frac{\partial}{\partial t})^a = 1 + h + \rho \log \rho k, \quad (\frac{\partial}{\partial t})^a, (\frac{\partial}{\partial r})^a = -1 + h + \rho \log \rho k. \]
and
\[ (\frac{\partial}{\partial t})^a, (\frac{\partial}{\partial r})^a = -1 + h + \rho \log \rho k. \]
Thus, the condition that \( N^a \) is a null vector implies that
\[ 1 + h + \rho \log \rho k + 2(1 + \alpha)(h + \rho \log \rho k) - (1 + \alpha)^2(1 + h + \rho \log \rho k) = 0. \tag{113} \tag{114} \]
This can be written
\[
\alpha^2(1+h+\rho \log \rho k) + 2\alpha(1+h+\rho \log \rho k) + h + \rho \log \rho k = 0 \tag{115}
\]
from which we conclude that \( \alpha \) is also of the form
\[
\alpha = h + \rho \log \rho k, \tag{116}
\]
i.e., \( \alpha \) is a \( C^0 \)-function with \( \alpha(i^0) = 0 \).

Similarly, we define \( L^a \) as
\[
L^a = (1 + \gamma) \sqrt{2} [\frac{\partial}{\partial t} - (1 + \beta)(\frac{\partial}{\partial r})^a], \tag{117}
\]
where both \( \beta \) and \( \gamma \) are \( C^0 \)-function that are zero at \( i^0 \), and where we first choose \( \beta \) so that \( L^a \) becomes a null vector, and then choose \( \gamma \) so that \( L^a N_a = 1 \).

To complete the tetrad, we must define a complex null vector \( M^a \) such that \( M^a L_a = M^a N_a = 0, M^a M_a = 0, M^a \bar{M}_A = -1 \). We define the imaginary part of \( M^a \) via
\[
\text{Im}(M_a) = \frac{1}{\sqrt{2}} \frac{u_a}{\sqrt{-g^{ab}u_a u_b}}, \quad u_a = \frac{y}{\sqrt{x^2 + y^2}}(dx)_a - \frac{x}{\sqrt{x^2 + y^2}}(dy)_a, \tag{118}
\]
and the real part of \( M^a \) via
\[
\text{Re}(M_a) = \frac{1}{\sqrt{2}} \frac{w_a}{\sqrt{-g^{ab}w_a w_b}}, \quad w_a = dx [a dy_b dz_c](\frac{\partial}{\partial r})^b u^c. \tag{119}
\]
We now have
\[
L^a v_a = 1, \quad N^a v_a = 1, \quad L^a + N^a = v^a \tag{120}
\]
at \( i^0 \).

In order to derive e.g. (32), we have to calculate
\[
\partial^o o^A = \lim_{r^0 \to r} r \partial^o o^A = \lim_{r^0 \to r} -r \rho \partial^o A, \tag{121}
\]
etc. where we have used that \( \partial^o o^A = -\rho \partial^o A, \tag{11} \). Thus we must calculate \( \lim_{r^0 \to r} -r \rho \), where the spin coefficient \( \rho \) is defined as \( \rho = m^a \bar{m}^b \nabla b l_a \). We have
\[
\rho = m^a \bar{m}^b \nabla b l_a = m^a \bar{m}^b \nabla b (e L_a) = em^a \bar{m}^b \nabla b L_a + L_a m^a \bar{m}^b \nabla b e = em^a \bar{m}^b \nabla b L_a. \tag{122}
\]
Writing \( L_a = \tilde{L}_a + h \rho \log \rho k, M_a = \tilde{M}_a + h \rho \log \rho k \) and \( M_a = \tilde{M}_a + h \rho \log \rho k \), where
\[
\tilde{N}^a = \frac{1}{\sqrt{2}}(\frac{\partial}{\partial t} + \frac{\partial}{\partial \rho})^a, \quad \tilde{L}^a = \frac{1}{\sqrt{2}}(\frac{\partial}{\partial t} - \frac{\partial}{\partial \rho})^a, \tag{123}
\]
\[
\tilde{M}^a = \frac{1}{\sqrt{2} \rho} (\frac{\partial}{\partial \theta} - i \frac{\sin \theta}{\partial \varphi})^a,
\]
and using that \( r\Gamma^i_{jk} = h + \rho \log k \), we find that
\[
 r\rho = e(r\tilde{M}_a M^b \partial_b \tilde{L}_a + h + \rho \log k),
\]
(124)
where \( \tilde{\partial}_a \) is the coordinate derivative. This means that we can use the results from flat space-time, i.e., use that \( r\tilde{M}_a M^b \partial_b \tilde{L}_a = 1 \) so that
\[
 \lim_{\rho^0} r\rho = e, \text{ i.e. } \partial' \alpha^A = -e\alpha^A
\]
(125)
at \( \rho^0 \).

The commutator relations are derived in a similar manner, for instance
\[
\begin{align*}
\lim_{\rho^0} r\phi - \phi &= \lim_{\rho^0} r\phi^o(r\phi^o) - r\phi^o(r\phi^o) = \lim_{\rho^0} -er\phi^o + \lim_{\rho^0} r^2\{\phi^o \phi^o - \phi^o \phi^o\} \\
&= -e\phi + \lim_{\rho^0} r(\phi^o + \phi^o - \phi^o - \phi^o - \phi^o - \phi^o - \phi^o - \phi^o - \phi^o) \\
&= -e\phi + e\phi = 0.
\end{align*}
\]
(126)
Here we have used that \( \lim_{\rho^0} r^2 \psi_1 = \lim_{\rho^0} r^2 \phi_{01} = 0 \), which follows directly from \( \lim_{\rho^0} r^2 R_{abcd} = 0 \). In the same way, we find the commutator relations
\[
\begin{align*}
\lim_{\rho^0} r\phi' - \phi' &= -e\phi' - e^{-1}\phi, \\
\lim_{\rho^0} r\phi = 0, \\
\partial' \alpha^A - \partial' \alpha^A &= (p - q),
\end{align*}
\]
(127-129)
plus the primed and conjugated versions.
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