Duality and symmetry in chiral Potts model

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Abstract. We discover an Ising-type duality in the general \( N \)-state chiral Potts model, which is the Kramers–Wannier duality of a planar Ising model when \( N = 2 \). This duality relates the spectrum and eigenvectors of one chiral Potts model at a low temperature (of small \( k' \)) to those of another chiral Potts model at a high temperature (of \( k'^{-1} \)). The \( \tau^{(2)} \) model and chiral Potts model on the dual lattice are established alongside of the dual chiral Potts models. With the aid of this duality relation, we exact a precise relationship between the Onsager-algebra symmetry of a homogeneous superintegrable chiral Potts model and the \( sl_2 \)-loop-algebra symmetry of its associated spin-(\( N - 1 \))/2 XXZ chain through the identification of their eigenstates.

Keywords: algebraic structures of integrable models, rigorous results in statistical mechanics, solvable lattice models, symmetries of integrable models

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1. Introduction

The Kramers–Wannier duality is a reflective symmetry in statistical physics, which relates a two-dimensional square lattice Ising model at a low temperature to another Ising model at a high temperature [33]. In this paper, we have found an Ising-type duality in the \(N\)-state chiral Potts model (CPM), which is the Kramers–Wannier duality of the usual Ising model when \(N = 2\). In the field of solvable statistical models, the \(N\)-state CPM has proved important and particularly challenging, due to the fact that for \(N = 2\) it reduces to the Ising model, the free energy of which was calculated by Onsager in 1944. The model was originally formulated as an \(N\)-state one-dimensional Hamiltonian [31, 32], then as a two-dimensional classical lattice model in statistical mechanics [2, 19, 39]. The free energy was first obtained for an infinite lattice using the properties of the free energy and its derivatives [7]. Then in 1990 the functional relations of [18, 20] were used to calculate the free energy more explicitly as a double integral [9, 10]. The interactions of CPM are defined by (local) Boltzmann weights depending on a temperature-like parameter \(k'\), which is small at low temperatures and large at high temperatures. The system displays
ferromagnetic order with a critical temperature below which the boundary conditions are relevant even for an infinitely large lattice. The order parameter has recently been proved by Baxter [16,17]. Furthermore, results in [1,8] about the free energy in the superintegrable CPM have strongly suggested the existence of duality in the theory of a chiral Potts model parallel to the Kramers–Wannier duality in the Ising model. In the present paper we will show that it is indeed the case. Here, we study the general inhomogeneous CPM of a finite size $L$ with a (skewed) boundary condition $r$ ($\in \mathbb{Z}_N$). It is known that the chiral Potts transfer matrix, $T$ or $\tilde{T}$, with rapidities in the $k'$-curve $W$, carries a quantum number of $\mathbb{Z}_N$-charge $Q$, (see (2.7), (2.12) and (2.16) in this paper). The duality of CPM relates two chiral Potts transfer matrices, $T$ and $T^*$, with the same eigenvalue spectrum, where $T$ is one over a $k'$-curve $W$ in the $Q$ sector with the boundary condition $r$, and $T^*$ is another one over the $k'−1$-curve $W^*$ in the $Q^*$ sector with the boundary condition $r^*$, when the charge and boundary condition are interchanged, $(Q^*, r^*) = (r, Q)$. The duality is established upon the correspondence of rapidity curves, $W$ and $W^*$, about the dual Boltzmann weights, and a similar isomorphism of $(r, Q)$- and $(r^*, Q^*)$ quantum spaces about ‘ordered and disordered fields’ (Theorem 3.1 in the content). Indeed, the dual Boltzmann weights are connected by the relation of the Fourier transform. Furthermore, under the dual correspondence of rapidities and quantum spaces, the $\tau^{(2)}$ models associated with two dual CPM are equally identified. In a special superintegrable case, the equal partition functions of the dual chiral Potts models are in agreement with the duality discussion of Baxter in [8] where the vertical-interfacial tension was computed. Similarly, we can form the general chiral Potts model over the dual lattice, as well as that for the face $\tau^{(2)}$ model, by using the underlying duality symmetry. In the homogeneous superintegrable case, there are two types of degeneracy symmetries about $\tau^{(2)}$ states in the study of CPM. One is the Onsager-algebra symmetry derived from the chiral Potts-$\mathbb{Z}_N$-spin Hamiltonian, the other is the $sl_2$-loop-algebra symmetry induced from a twisted spin-$(N − 1)/2$ XXZ chain which is equivalent to the $\tau^{(2)}$ model [41,44,46]. We observe that the Onsager-algebra symmetries of two superintegrable $\tau^{(2)}$ models intertwine under the dual correspondence of rapidities and quantum spaces in the duality of CPM. With the aid of this duality, all the $\tau^{(2)}$-degeneracy symmetries are unified in a common underlying ($\otimes sl_2$) structure at $k' = 0, \infty$ for the eigenspace.

This paper is organized as follows. In section 2, we recall some basic facts of the $\tau^{(2)}$ model and CPM. For the purpose of this paper, the results are formulated in the most general case, the inhomogeneous CPM with a skewed boundary condition. Much of the work in this section could be a paraphrase of those done before [1,9,11,13,18,20,24,31,40,43,46,48,49], but in a more general form suited for the discussion of this paper; some results will be simply stated with adequate citations at important places. In section 2.1, we first recall the main definitions in CPM and briefly discuss the functional relations and Bethe relation of the theory. Then we illustrate the results on the inhomogeneous superintegrable CPM, among which is a special periodic case which appeared recently in [42]. In section 2.2, we represent a detailed study of the homogeneous CPM with an arbitrary superintegrable vertical rapidity and the boundary condition, an extended superintegrable version for those in [1,11,13,31]. By explicit formulae about the chiral Potts transfer matrix and energy form of the $\mathbb{Z}_N$-spin Hamiltonian, the Onsager-algebra symmetry and its induced $sl_2$-loop-algebra structure of the $\tau^{(2)}$ eigenspace are thoroughly discussed here. Section 3 is devoted to the duality in
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CPM. First we discuss the $\tau^{(2)}$ duality in section 3.1. By studying the $\tau^{(2)}$-face-model, we discover the duality of the $\tau^{(2)}$ model through a correspondence of dual rapidity curves and quantum spaces. Based on the duality of rapidity curves for the dual Boltzmann weights, we verify the duality relation of CPM in section 3.2. In section 3.3, we incorporate the underlying duality symmetry of CPM into the formulation of chiral Potts model of the dual lattice, as well as that of the $\tau^{(2)}$-face-model. In section 3.4, we justify the CPM duality for $N=2$ in agreement with the usual Kramers–Wannier duality of the Ising model [6,33]. Then in section 3.5 we illustrate the consistency of the duality and quantum spin chain Hamiltonian in the homogeneous CPM. In particular, the Onsager-algebra symmetry of the dual homogeneous superintegrable models are identified under the duality transformation. In section 4, we first recall the definition of a general inhomogeneous XXZ chain with the quantum group $U_q(sl_2)$ and an arbitrary skew boundary condition; then we derive the associated affine algebra $U_q(sl_2)$, and the root-of-unity-symmetry generators when $q^N = 1$ for the XXZ chains with cyclic $U_q(sl_2)$ representation. In section 4.1, using the argument in [50] about the homogeneous case, we demonstrate the equivalent relation between the inhomogeneous $\tau^{(2)}$ models and XXZ chains with the $U_q(sl_2)$-cyclic representation. In section 4.2, we study the relationship between all degeneracy symmetries of a homogeneous superintegrable $\tau^{(2)}$ model for odd $N$. By the equivalence between the $\tau^{(2)}$, $\tau_F^{(2)}$ model and homogeneous spin-$(N-1)/2$ XXZ chains which carry the $sl_2$-loop-algebra symmetry, we find that the $\tau^{(2)}$-Bethe states can be identified with the highest or lowest weight vectors of the Onsager-algebra-Hamiltonian generators in a $\tau^{(2)}$ eigenspace. The canonical basis at $k' = 0, \infty$ in the $\tau^{(2)}$ eigenspace provides a unified structure for both Onsager-algebra and $sl_2$-loop-algebra symmetry about the $\tau^{(2)}$ degeneracy. Finally we close in section 5 with a concluding remark.

Notation: in this paper, we use the following standard notations. For a positive integer $N$ greater than one, $C^N$ denotes the vector space of $N$-cyclic vectors with the canonical base $|\sigma\rangle, \sigma \in \mathbb{Z}_N (:= \mathbb{Z}/N\mathbb{Z})$. We fix the $N$th root of unity $\omega = e^{2\pi i/N}$ and the Weyl $C^N$-operators $X,Z$:

$$X|\sigma\rangle = |\sigma + 1\rangle, \quad Z|\sigma\rangle = \omega^\sigma|\sigma\rangle \quad (\sigma \in \mathbb{Z}_N),$$

satisfying $X^N = Z^N = 1$ and the Weyl relation: $XZ = \omega^{-1}ZX$. The Fourier basis $\{|\kappa\rangle\}$ of $\{|\sigma\rangle\}$ is defined by

$$|\kappa\rangle = \frac{1}{\sqrt{N}} \sum_{\sigma=0}^{N-1} \omega^{-k\sigma}|\sigma\rangle, \quad |\sigma\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \omega^{\sigma k}|\kappa\rangle, \quad \sigma \in \mathbb{Z}_N; \quad (1.1)$$

with the corresponding Weyl operators, $\hat{X}|\kappa\rangle = |\kappa + 1\rangle, \hat{Z}|\kappa\rangle = \omega^k|\kappa\rangle$. Then the following equality holds:

$$(X, Z) = (\hat{Z}, \hat{X}^{-1}). \quad (1.2)$$
2. $\tau^{(2)}$ model and chiral Potts model

The $L$-operator of the $\tau^{(2)}$ model [18,20,48,49] is the two-by-two matrix expressed by Weyl $C^N$-operators $X,Z$ or $\hat{X},\hat{Z}$ in (1.2):

$$L(t) = \begin{pmatrix} 1-t\frac{c}{b}X, & \left(\frac{1}{b} - \omega\frac{a}{b}\right)Z \\ -t\left(\frac{1}{b} - \frac{a}{b}\right)X^{-1}, & -\frac{1}{b} + \omega\frac{a}{b}X \end{pmatrix}$$

with non-zero complex parameters $a,b,a',b',c$. It is known that the above $L$-operator satisfies the YB equation:

$$R(t/t') \left( L(t) \bigotimes_{\text{aux}} 1 \right) \left( 1 \bigotimes_{\text{aux}} L(t') \right) = \left( 1 \bigotimes_{\text{aux}} L(t') \right) \left( L(t) \bigotimes_{\text{aux}} 1 \right) R(t/t')$$

for the asymmetry six-vertex $R$ matrix:

$$R(t) = \begin{pmatrix} t\omega - 1 & 0 & 0 & 0 \\ 0 & t - 1 & \omega - 1 & 0 \\ 0 & \omega - 1 & (t-1)\omega & 0 \\ 0 & 0 & 0 & t\omega - 1 \end{pmatrix}.$$
where \( p_\ell, p'_\ell \) \((1 \leq \ell \leq L)\) all lie in the same rapidity curve of CPM for a temperature-like parameter \( k' \)\((\neq 0)\), i.e., a curve \( \mathfrak{W} = \mathfrak{W}_{k'}(= \mathfrak{W}_{k',k}) \), \( \mathfrak{W}_{\pm 1}, \mathfrak{W}_{\pm 1} \) consisting of elements \( (x, y, \mu) \in \mathbb{C}^3 \) with the equation

\[
\begin{align*}
\mathfrak{W}_{k'}: kx^N &= 1 - k'\mu^{-N}, \quad k'y^N = 1 - k'\mu^N, \quad (k'^2 \neq 1, k^2 + k'^2 = 1); \\
\mathfrak{W}_1: x^N + y^N &= 1, \quad \mu^N = 1; \\
\mathfrak{W}_{-1}: x^N &= 1 + \mu^{-N}, \quad y^N = 1 + \mu^N; \\
\mathfrak{W}_{\pm 1}: x^N + y^N &= 0, \quad \mu^N = \pm 1 \quad \text{respectively}
\end{align*}
\]

(see, e.g., \([1, 18, 48, 49]\)). Note that, for a given \( k' \neq \pm 1 \), \( \mathfrak{W}_{k',k} \) is isomorphic to \( \mathfrak{W}_{k',-k} \) via the transformation \( (x, y, \mu) \mapsto ((-1)^{1/N}x, (-1)^{1/N}y, \mu) \). Hereafter we shall write \( \mathfrak{W}_{k'} \) to represent one of these two curves if no confusion could arise; and we write the \( \tau^{(2)} \) model \((2.5)\) with parameters \((2.6)\) by \( \tau(t):= \mathfrak{W}_1(t; \{p_\ell\}, \{p'_\ell\}) \).

The spectral parameter \( t \) of \( \tau^{(2)} \) will be identified with \( x_qy_q \) for a generic rapidity \( q \) of a curve \( \mathfrak{W} \) in \((2.7)\):

\[
t(t=t_q) = x_qy_q.
\]

Then \( x^N \) is related to \( t^N \) by a quadratic relation, which defines a hyperelliptic curve \( W = W_{k'}, W_{\pm 1}, \mathfrak{W}_{\pm 1} \) of lower genus with the coordinates \((t, \lambda)\) \(([49] \quad (2.13) \quad (2.16))\):

\[
\begin{align*}
W_{k'}: t^N &= \frac{(1 - k'\lambda)(1 - k'\lambda^\dagger)}{k^2}, \quad (\lambda := \mu^N, \lambda^\dagger = \lambda^{-1}), \\
W_{-1}: t^N &= (1 + \lambda)(1 + \lambda^\dagger), \quad (\lambda := \mu^N, \lambda^\dagger = \lambda^{-1}), \\
W_1: t^N &= \lambda\lambda^\dagger, \quad (\lambda := x^N, \lambda^\dagger = 1 - \lambda), \\
\mathfrak{W}_{\pm 1}: t^N &= \lambda\lambda^\dagger, \quad (\lambda := x^N, \lambda^\dagger = -\lambda).
\end{align*}
\]

In the case \( \mathfrak{W}_{\pm 1} \), only the odd \( N \) case will be considered, as the even \( N \) case consists of two rational irreducible components. As in \([11] \quad (3.11) \quad (3.13), \quad [46] \quad \text{(proposition 2.1 and (2.31)) and } [48] \quad \text{(2.25)), one can construct } \tau^{(j)} \text{ matrices from the } L\text{-operator (2.1), with } \tau^{(0)} = 0, \tau^{(1)} = I \text{ and } \tau^{(2)} \text{ in (2.8), so that the fusion relation holds:}

\[
\begin{align*}
\tau^{(2)}(\omega^{i-1}t)\tau^{(j)}(t) &= \omega^rXz(\omega^{i-1}t)\tau^{(j-1)}(t) + \tau^{(j+1)}(t), \quad j \geq 1; \\
\tau^{(N+1)}(t) &= \omega^rXz(t)\tau^{(N-1)}(Ut) + u(t)I,
\end{align*}
\]

where \( z(t), u(t) = \alpha_q + \overline{\alpha}_q \) are defined by

\[
\begin{align*}
z(t) &= \prod_{\ell=1}^{L} \omega^{\mu_{p_\ell}^{(t)}\mu_{p'_\ell}^{(t)}}(t_{p_\ell}^{(t)} - t)(t_{p'_\ell}^{(t)} - t), \\
\alpha_q &= \prod_{\ell=1}^{L} \mu_{p_\ell}^{(N)}(y_{p_\ell}^{(N)} - x^N)(y_{p'_\ell}^{N} - x^N) \Big/ k^Ny_{p_\ell}^{N}y_{p'_\ell}^{N}, \quad \overline{\alpha}_q = \prod_{\ell=1}^{L} \mu_{p'_{\ell}}^{-N}(y_{p_\ell}^{N} - y^N)(y_{p'_\ell}^{N} - y^N) \Big/ k^Ny_{p_\ell}^{N}y_{p'_\ell}^{N}.
\end{align*}
\]

\(^1\) The curves \( \mathfrak{W}_1, \mathfrak{W}_{-1}, \mathfrak{W}_{\pm 1} \) here correspond to \( \mathfrak{W}_{\ell}, \mathfrak{W}_{\ell}', \mathfrak{W}_{\ell}'' \), respectively, in \([49] \quad (2.9)\).
2.1. Inhomogeneous chiral Potts model with a skewed boundary condition

With the rapidities \( p, q \in \mathbb{W} \) in (2.7), the Boltzmann weights of CPM are defined by

\[
\frac{W_{pq}(\sigma)}{W_{pq}(0)} = \left( \frac{\mu_p}{\mu_q} \right)^\sigma \prod_{j=1}^\sigma y_q - \omega^j x_p \quad \text{and} \quad \frac{\overline{W}_{pq}(\sigma)}{\overline{W}_{pq}(0)} = \left( \frac{\mu_p\mu_q}{\mu_p} \right)^\sigma \prod_{j=1}^\sigma \omega x_p - \omega^j y_q,
\]

(2.12)

which satisfy the star-triangle relation [3, 4, 19, 30, 38, 39]

\[
\sum_{\sigma=0}^{N-1} \overline{W}_{pq}(\sigma') \overline{W}_{pr}(\sigma-j) \overline{W}_{pq}(\sigma-j'') = R_{pqr} \overline{W}_{pq}(j-j') \overline{W}_{pr}(j-j'') \overline{W}_{pq}(j-j''') \tag{2.13}
\]

where \( R_{pqr} = f_{pq} f_{qr} / f_{pr} \) with \( f_{pq} = (\mathfrak{g}_p(q)/g_p(q))^{1/N} \), and

\[
g_p(q) := \prod_{n=0}^{N-1} \overline{W}_{pq}(n) = \left( \frac{\mu_p}{\mu_q} \right)^{(N-1)/2} \prod_{j=1}^{N-1} \left( \frac{x_p - \omega^j y_q}{x_q - \omega^j y_p} \right)^j,
\]

(2.14)

where \( \mathfrak{g}_p(q) := \det_N(\overline{W}_{pq}(i-j)) = N^{N/2} e^{i\pi(N-1)(N-2)/12} \prod_{j=1}^{N-1} \frac{(t_p - \omega^j t_q)^j}{(x_p - \omega^j x_q)(y_p - \omega^j y_q)} \) \cite{18} (2.44) and \cite{49} (2.24). Without loss of generality, we set \( W_{p,0}(0) = \overline{W}_{p,0}(0) = 1 \).

The \( N \)-cyclic Fourier bases defined by the Boltzmann weights (2.12) can also be expressed in terms of the Fourier bases: \( \sum_{\sigma=0}^{N-1} W_{pq}(\sigma)|\sigma\rangle = \sum_{k=0}^{N-1} W_{pq}(f(k)\hat{k}) \), \( \sum_{\sigma=0}^{N-1} \overline{W}_{pq}(\sigma)|\sigma\rangle = \sum_{k=0}^{N-1} \overline{W}_{pq}(f(k)\hat{k}) \). By \cite{18} (2.24), one finds

\[
\begin{align*}
W_{pq}^{(f)}(k) &= \frac{1}{\sqrt{N}} \sum_{\sigma=0}^{N-1} \omega^{k\sigma} \overline{W}_{pq}(\sigma), \\
\overline{W}_{pq}^{(f)}(0) &= \prod_{j=1}^{N-1} \frac{y_q - \omega^j x_p \mu_p \mu_q}{y_p - \omega^j x_q \mu_p \mu_q};
\end{align*}
\]

\[
W_{pq}^{(f)}(k) = \frac{1}{\sqrt{N}} \sum_{\sigma=0}^{N-1} \omega^{k\sigma} \overline{W}_{pq}(\sigma), \\
\overline{W}_{pq}^{(f)}(0) = \prod_{j=1}^{N-1} \frac{y_q - \omega^j x_p \mu_p \mu_q}{y_p - \omega^j x_q \mu_p \mu_q};
\]

(2.15)

The chiral Potts transfer matrix of a size \( L \) with the (skewed) boundary condition (2.4) and vertical rapidities \( \{p_\ell, p'_\ell\}_{\ell=1}^{L} \) are the \( \otimes \mathbb{C}^N \) operators defined by \cite{11, 18}

\[
T(q)_{\{\sigma\},\{\sigma'\}} = T(q; \{p_\ell\}, \{p'_\ell\})_{\{\sigma\},\{\sigma'\}} = \prod_{\ell=1}^{L} W_{p_\ell q}(\sigma_\ell - \sigma'_\ell) \overline{W}_{p'_\ell q}(\sigma_{\ell+1} - \sigma'_{\ell+1}),
\]

(2.16)

\[
\hat{T}(q)_{\{\sigma\},\{\sigma''\}} = \hat{T}(q; \{p_\ell\}, \{p'_\ell\})_{\{\sigma\},\{\sigma''\}} = \prod_{\ell=1}^{L} \overline{W}_{p_\ell q}(\sigma_\ell - \sigma''_\ell) W_{p'_\ell q}(\sigma'_\ell - \sigma''_{\ell+1}),
\]

which commute with the spin-shift operator \( X \). Here \( q \) is an arbitrary rapidity, \( \sigma_\ell, \sigma'_\ell \in \mathbb{Z}_N \), and the periodic vertical rapidities, \( p_{L+1} = p_1, p'_{L+1} = p'_1 \), are imposed. The star-triangle relation (2.13) yields the relation

\[
T(q) \hat{T}(r) = \left( \prod_{\ell=1}^{L} W_{p'_\ell q} f_{p'_\ell r} / f_{p_\ell q} f_{p'_\ell r} \right) T(r) \hat{T}(q), \quad \hat{T}(q) T(r) = \left( \prod_{\ell=1}^{L} W_{p'_\ell q} f_{p'_\ell r} / f_{p_\ell q} f_{p'_\ell r} \right) \hat{T}(r) T(q),
\]

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by which the following commutative relations hold for rapidities \( q, r, q', r' \):

\[
\tilde{T}(q)T(r)\tilde{T}(q')T(r') = \tilde{T}(q')T(r')\tilde{T}(q)T(r),
\]

\[
T(q)\tilde{T}(r)T(q')\tilde{T}(r') = T(q')\tilde{T}(r')T(q)\tilde{T}(r).
\]

Hence the matrices \( T(q), \tilde{T}(q) \) can be diagonalized by two invertible \( q \)-independent matrices \( P_B, P_W \), i.e. \( P_W^{-1}T(q)P_B, P_B^{-1}\tilde{T}(q)P_W \) are diagonal with the ‘eigenvalues’ of \( T, \tilde{T} \) as the diagonal entries ([18] (2.32)–(2.34), (4.46), [11] (2.10)–(2.13)):

\[
\tilde{T}_{\text{diag}}(q) = T_{\text{diag}}(q) \left( \prod_{\ell=1}^{L} \frac{f_{p\ell q}}{f_{p\ell q'}} \right) D \tag{2.17}
\]

where \( D \) is a \( q \)-independent diagonal matrix. In particular for the homogeneous case, there is one extra symmetry:

\[
\tilde{T}(q) = T(q)S_R = S_RT(q), \quad \text{when } p_\ell = p'_\ell = p \text{ for all } \ell, \tag{2.18}
\]

where \( S_R \) is the spatial translation operator \( (S_R)_{\sigma,\sigma'} = \prod_\ell \delta_{\sigma_\ell-1,\sigma'_\ell} \), equivalently \( S_R|j_1,\ldots,j_L\rangle = |j_2,\ldots,j_{L+1}\rangle \) or \( |j_1,\ldots,j_L|S_R = |j_0,j_1,\ldots,j_{L-1}\rangle \), with eigenvalues \( \omega^{-\ell_1L/2\pi\alpha}L/2 \left( \ell_R \in \mathbb{Z}_L \right) \). Then \( \{T(q),\tilde{T}(q)\}_{q \in \mathbb{R}} \) form a commuting family, with \( P_B = P_W \) and \( D = S_R \) in (2.17).

In the study of the inhomogeneous chiral Potts model ([11], [18] p 842), there are functional relations between \( T, \tilde{T} \) in (2.16) and \( \tau^{(2)}, \tau^{(2)} \) in (2.10). The \( \tau^{(2)}T \) relation is the relation between the \( \tau^{(2)} \) and \( T \) matrices ([11] (3.15), [18] (4.31), [49] (2.31)–(2.32)):

\[
\begin{align*}
\tau^{(2)}(t_q)T(Uq) &= \varphi_q T(q) + \omega^r \varphi_q XT(U^2q), \\
\tau^{(2)}(t_q)T(U^q) &= \varphi_q' XT(q) + \omega^r \varphi_q' T(U^2q),
\end{align*}
\tag{2.19}
\]

where the automorphism \( U(x,y,\mu) := (wx,y,\mu) \), \( U'(x,y,\mu) := (x,wy,\mu) \), and the functions \( \varphi_q, \varphi'_q, \varphi_q', \varphi'_q \) are defined by

\[
\begin{align*}
\varphi_q &= \prod_\ell \frac{(t_{p\ell q} - t_q)(y_{p\ell} - \omega x_q)}{y_{p\ell} y_{p\ell'}(x_{p\ell'} - x_q)}, \\
\varphi'_q &= \prod_\ell \frac{\omega \mu_{p\ell} \mu_{p\ell'} (t_{p\ell q} - t_q)(x_{p\ell'} - x_q)}{y_{p\ell} y_{p\ell'}(y_{p\ell'} - y_q)}, \\
\varphi'_q &= \prod_\ell \frac{\omega \mu_{p\ell} \mu_{p\ell'} (t_{p\ell q} - t_q)(x_{p\ell'} - y_q)}{y_{p\ell} y_{p\ell'}(y_{p\ell'} - y_q)}, \\
\varphi'_q &= \prod_\ell \frac{(t_{p\ell q} - t_q)(y_{p\ell} - y_q)}{y_{p\ell} y_{p\ell'}(x_{p\ell'} - y_q)}.
\end{align*}
\]

Similarly, one has the \( \tau^{(2)}T \) relation between \( \tau^{(2)} \) and \( \tilde{T} \) ([49] (2.33)):

\[
\begin{align*}
\tilde{T}(Uq)\tau^{(2)}(t_q) &= \varphi_{(p\ell'),(p\ell):q}\tilde{T}(q) + \omega^r \varphi_{(p\ell'),(p\ell):Uq} X\tilde{T}(U^2q), \\
\tilde{T}(U^q)\tau^{(2)}(t_q) &= \varphi_{(p\ell'),(p\ell):q} X\tilde{T}(q) + \varphi'_{(p\ell'),(p\ell):U^q} \tilde{T}(U^2y_q).
\end{align*}
\]

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Using (2.19) and (2.10), one finds the $\tau^{(j)}T$ relation ([18] (4.34)_{k=0}, [49] (2.34)):

$$\tau^{(j)}(t_q) = \sum_{m=0}^{j-1} \omega^m \varphi_q \cdots \varphi_{U^{j-1}q} \varphi_{U^{m+1}q} \cdots \varphi_{U^{j-1}q} \times T(q)T(U^{m}q)^{-1}T(U^{j}q)T(U^{m+1}q)^{-1}X^{j-m-1}.$$  

Other than the above $\tau^{(j)}T$- and $\tau^{(2)}T$-relations, there is the $T\bar{T}$-relation ([11] (3.1), [18] (2.36), [14] (13), [49] (4.7)):

$$\frac{T(q)\bar{T}(q_y,\omega^xq,\mu_q^{-1})}{r(q)h_j(q)} = \omega^j \zeta(t_q)z(\omega t_q) \cdots z(\omega^{j-1}t_q) \tau^{(N-j)}(\omega^j t_q) X_j$$  \hspace{1cm} (2.20)

where $z(t), \alpha_q$ are in (2.11), and $r(q), h_j(q)$ are defined by

$$r(q) = \prod_{\ell} \frac{N(x_{\ell} - x_q)(y_{\ell} - y_q)(t_{\ell} - t_N)}{(x_{\ell}^N - x_q^N)(y_{\ell}^N - y_q^N)(t_{\ell} - t_q)},$$

$$h_j(q) = \prod_{\ell=1}^{j-1} \frac{y_{\ell}y_{\ell}^*(x_{\ell} - \omega^m x_q)}{y_{\ell} - \omega^m x_q}(t_{\ell} - \omega^m t_q).$$

Then the functional relation of $T\bar{T}$ ([18] (4.40)) follows from (2.20) and the $\tau^{(j)}T$ relation. One can solve the eigenvalue problem of CPM using the whole set of functional relations [9, 40, 49]. First, we need to solve $\tau^{(2)}$ eigenvalues satisfying the following Bethe relation ([49] (3.7) (3.10)), a parallel version of the $\tau^{(2)}T$ relation (2.19):

$$\tau^{(2)}(t) = \omega^{-P_h}h^+(t) \frac{F(t)}{F(\omega t)} - \frac{\omega^{Q+P_h}h^-}(\omega t) \frac{F(\omega^2 t)}{F(\omega t)}$$

$$= \omega^{-P_h}h^+(t) \frac{F(t)}{F(\omega t)} + \omega^{P_h}h^+(\omega t) \frac{F(\omega^2 t)}{F(\omega t)}. $$ \hspace{1cm} (2.21)

through the Bethe polynomial $t^{P_h}F(t) = t^{P_h}F'(t)$ with $F(t) = \prod_{j=1}^{J}(1 + \omega v_j t)$ and $F'(t) = \prod_{j=1}^{J}(1 + \omega v_j' t)$. Here the $t$-functions $h^+(t)$ or $h^-(t)$ are obtained through the Wiener–Hopf splitting of $\alpha_q, \sigma_q$ in (2.11) (for the details, see [49] section (3.2)). The regular-function condition of $\tau^{(2)}(t)$ gives rise to the Bethe equation of $v_j$s or $v_j'$s (see the formulae in [49] (3.8), (3.11)). By (2.10), one then expresses the functions $\tau^{(j)}(t)$ ($j \geq 2$) in terms of the Bethe solution $F(t), F'(t)$ ([49] (3.9), (3.12)):

$$\tau^{(j)}(t) = \omega^{-1}(Q + P_h + r) F(t) F(\omega^j t)$$

$$\times \sum_{k=0}^{j-1} \omega^k \frac{h^+(t) \cdots h^+(\omega^{k+1}t) h^-(\omega^{j+1}t) \cdots h^-(\omega^j t) \omega^{-n(Q + P_h + r)}}{F(\omega^k t) F(\omega^{k+1} t)}$$

$$= \omega^{-1}(Q + P_h) F'(t) F'(\omega^j t)$$

$$\times \sum_{k=0}^{j-1} \omega^k \frac{h^-(t) \cdots h^-(\omega^{k+1}t) h^+(\omega^{j+1}t) \cdots h^+(\omega^j t) \omega^{k(Q - 2P_h + r)}}{F'(\omega^k t) F'(\omega^{k+1} t)}.$$
Using the above \( \tau^{(N)}(t) \) expressions and the \( T\hat{T} \) relation (2.20) for \( j = N \), one can derive the expression of eigenvalues of \( T(q)\), \( \hat{T}(q) \) through the functional relation method (for the details, see [49] section 4)\(^2\).

We now describe the formulae in the superintegrable case ([1, 11, 13], [49] section 4.3). In this paper, by the superintegrable\(^3\) inhomogeneous CPM, we mean the vertical rapidities \( \{p_\ell, p'_\ell\}_{\ell=1}^L \) satisfy the relation
\[
x_{p_\ell} = \omega^m y_{p_\ell}, \quad x_{p'_\ell} = \omega^n y_{p'_\ell}, \quad \mu_{p_\ell} \mu_{p'_\ell} = \omega^n
\] (2.22)
for some integers \( m, n \), which is equivalent to \( t_{p_\ell} = t_{p'_\ell} = \omega^m y_{p_\ell} y_{p'_\ell}, \mu_{p_\ell} \mu_{p'_\ell} = \omega^n \). Denote
\[
h(t) := \prod_{\ell=1}^L \left( 1 - \frac{t}{t_{p_\ell}} \right).
\]
The functions \( h^\pm(t), h^{t\pm}(t) \) in (2.21) are given by \( h^+(t) = h^{t+}(t) = h(t), h^-(t) = h^{t-}(t) = \omega^{(1 + 2m + n)L} h(t) \) with the \( \tau^{(2)} \) and \( \tau^{(N)} \) eigenvalues expressed by ([49] ((3.7)–(3.11)))
\[
\tau^{(2)}(t) = \omega^{-P_b} h(t) F(t) + \omega^{-P_b} h(\omega t) \frac{F(\omega^2 t)}{F(\omega t)},
\]
\[
\frac{\tau^{(N)}(t)}{F(t)²} = \omega^{-P_b} \sum_{k=0}^{N-1} \frac{h(t) \cdots h(\omega^{k-1} t) h(\omega^{k+1} t) \cdots h(\omega^{N-1} t) \omega^{-k(P_a + P_b)}}{F(\omega^k t) F(\omega^{k+1} t)} \tag{2.23}
\]
where the polynomial \( F(t) = \prod_{j=1}^J (1 + \omega v_j t) \) satisfies the Bethe equation ([49] (4.32)):
\[
\prod_{\ell=1}^L \frac{(t_{p_\ell}v_i + \omega^{-1})}{(t_{p_\ell}v_i + \omega^{-2})} = -\omega^{-P_a - P_b} \prod_{j=1}^J \frac{v_i - \omega^{-1}v_j}{v_i - \omega v_j}, \quad i = 1, \ldots, J. \tag{2.24}
\]
Note that the right-hand side of the above equation is equal to \( h(-\omega^{-1} v_i^{-1}) h(-\omega^{-2} v_i^{-1})^{-1} \). Here \( P_a, P_b \) are integers satisfying the relations ([49] ((4.36), (4.37)))\(^4\)
\[
0 \leq P_a + P_b \leq N - 1, \quad P_b - P_a \equiv Q + r + (1 + 2m + n)L \mod N; \quad P_a \equiv 0 \text{ or } P_b \equiv 0, \quad J + P_b \equiv (m + n)L + Q, mL + r \mod N. \tag{2.25}
\]
We define the \( t^N \) polynomial ([49] (4.24), (4.35)):
\[
P(t) = C t^{-P_a - P_b} \frac{\tau^{(N)}(t)}{F(t)^2} \tag{2.26}
\]
\[
= C \omega^{-P_b} \sum_{k=0}^{N-1} \frac{h(t) \cdots h(\omega^{k-1} t) h(\omega^{k+1} t) \cdots h(\omega^{N-1} t) (\omega^k t)^{(P_a + P_b)}}{F(\omega^k t) F(\omega^{k+1} t)}
\]

\(^2\) The argument and formulae about the Bethe equation of the \( \tau^{(2)} \) model and eigenvalues of chiral Potts transfer matrix with alternating rapidities in [49] are all valid in the inhomogeneous and skewed boundary condition case after a suitable modification of scalar coefficients as described in [11] and [18] p 842.

\(^3\) The superintegrable condition here is slightly different from the alternating superintegrable case in [49] (4.30), where the integers \( m, m' \) are assumed to be equal in the discussion of this paper.

\(^4\) A misprint occurred in the last formula in [49] (4.37) where \( J + P_b \equiv (m + 2m') L \) should be \( J + P_b \equiv mL \).
with $P(0) \neq 0$, where $C = \omega^{-(1+2m+n)(N-1)L/2} \prod_{\ell=1}^L t_{p_{\ell}}^{N-1}$. Using the coordinates $(t, \lambda)$ in (2.9), one can factorize $P(t)$ using a $\lambda$ function $G$:

$$P(t) = DG(\lambda)G(\lambda^d),$$

(2.27)

where $D$ is the $q$-independent function in (2.17). Then the eigenvalues of the normalized transfer matrices $V, \hat{V}$ of $T, \hat{T}$ with $\hat{V} = VD$ ([9] section 2, [49] (4.4)):

$$V(q) = T(q) \left( \mu_q^{(N(N-1)L)/2} \prod_{\ell=1}^L g_{p_{\ell}}(q)\tilde{g}_{p'_{\ell}}(q) \right)^{-1/N},$$

$$\hat{V}(q) = \hat{T}(q) \left( \mu_q^{(N(N-1)L)/2} \prod_{\ell=1}^L g_{p_{\ell}}(q)\tilde{g}_{p'_{\ell}}(q) \right)^{-1/N},$$

are expressed by ([49] (4.34))

$$V(q) = \zeta_0^{L/N} x q^a_p y q^b_p \mu_q^{-p_{\mu}} \prod_{\ell=1}^L \frac{F(t_{q_{\ell}})}{\prod_{k=1}^{N-1} (t_{q'_{\ell}} - \omega^k t_{q_{\ell}})^{k/N}} G(\lambda_q),$$

(2.28)

where $\zeta_0 = e^{i\pi (N-1)(N+4)/12}$ and $P_{\mu} \equiv \mathcal{r} \mod N$. In particular when $\mathcal{r} = 0$ and $m = n = 0$ in (2.22), an equivalent expression of the $T$ eigenvalue appeared in [42] section 3.2 through the algebraic-Bethe-ansatz method.

2.2. Onsager-algebra symmetry and the induced $sl_2$-loop-algebra structure in homogeneous superintegrable chiral Potts model

We now consider the homogeneous superintegrable case, i.e. $p = p_\ell = p'_{\ell}$ for all $\ell$ in (2.18). The $L$-operators (2.5) are all equivalent to

$$L_{\ell}(t) = \left( \begin{array}{cc} 1 - \omega^ntX & (1 - \omega^{1+m+n}X)Z \\ -t(1 - \omega^{m+n}X)Z^{-1} & -t + \omega^{1+2m+n}X \end{array} \right) \quad (t := \omega^nt_p^{-1}t) \quad \text{for all } \ell. \quad (2.29)$$

Using the above normalized spectral parameter $t$, the $\tau^{(2)}$ models for all $k'$ are the same when the vertical rapidities $p, p'$ lie in a curve $2G$ in (2.7). Express all polynomials in (2.23) in terms of $t$:

$$h(t) = h(t) = (1 - \omega^{-m}t)^L, \quad F(t) = F(t) = \prod_{j=1}^J (1 + \omega v_j t) \quad (v_j := \omega^{-m}t_pv_j).$$

(2.30)

The Bethe equation (2.24) becomes ([49] ((4.31), (4.32))):

$$\left( \frac{v_i + \omega^{-1-m}}{v_i - \omega^{-2-m}} \right)^L = -\omega^{-p_{-} - p_{\mu}} \prod_{j=1}^J \frac{v_i - \omega^{-1}v_j}{v_i - \omega v_j}, \quad i = 1, \ldots, J,$$

(2.31)

with $\tau^{(2)}$ eigenvalues (2.23) expressed by

$$\tau^{(2)}(t) = \omega^{-p_{-}}(1 - \omega^{-m}t)^L \frac{F(t)}{F(\omega t)} + \omega^{p_{\mu}}(1 - \omega^{1-m}t)^L \frac{F(\omega^2 t)}{F(\omega t)}.$$

(2.32)

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Normalize $P(t)$ in (2.26) by

$$P(t) = C_p P(t), \quad P(t) = \omega^{-P_b} \sum_{k=0}^{N-1} \frac{(1-t^N)^L(\omega^k t)^{-P_a + P_b}}{(1 - \omega^{-m+k} t)^L F(\omega^k t) F(\omega^{k+1} t)}$$

(2.33)

where $C_p = \omega^{-(1+2m+n)(N-1)L/2+m(P_a+P_b)}$. $\mathcal{P}(\alpha)$ has the polynomial criterion of $P(t)$ in (2.33), which can be regarded as an $t^N$-polynomial with the degree

$$m_E := \left[ \frac{(N-1)L - P_a - P_b - 2J}{N} \right]$$

with $P(0) \neq 0$. We shall denote the $\tau^{(2)}$ eigenspace with the eigenvalue (2.32) for a Bethe polynomial $F(t)$ in (2.33) by $\mathcal{E}_{\mathcal{F},P_a,P_b}$. Then the dimension of $\mathcal{E}_{\mathcal{F},P_a,P_b}$ is equal to $2^{m_E}$. One has the $\tau^{(2)}$-eigenspace decomposition of the quantum space:

$$\bigotimes L \mathbb{C}^N = \bigoplus \{ \mathcal{E}_{\mathcal{F},P_a,P_b} | F: \text{Bethe polynomial with quantum numbers } P_a, P_b \}.$$  

(2.34)

For simple notations, hereafter we assume the vertical rapidity $p$ of a homogeneous superintegrable CPM always in $\mathbb{M}_k$ ($k' \neq 0, \pm 1$) with $\mu_p (= \omega^{n/2})$ being an $N$th root of unity, equivalently $n \equiv 2n_0$ for some $n_0 \in \mathbb{Z}$, (a constraint required only in the even $N$ case). First we consider the case

$$p: (x_p, y_p, \mu_p) = (\eta^{1/2} \omega^m, \eta^{1/2} \omega^{m_0}) \in \mathbb{M}_{k'}, \quad \eta := \left( \frac{1-k'}{1+k'} \right)^{1/N},$$

(2.35)

where $k' \neq 0, \pm 1$, and $0 \leq m \leq N-1$ (see [1, 11, 13, 44, 47]). All $\tau^{(2)}, T, \hat{T}$ matrices commute with each other for $q \in \mathbb{M}_{k'}$. Using (2.14), (2.27), (2.28) and (2.33), one finds the following formulae of $T, \hat{T}$ eigenvalues:

$$T(q) = \alpha_1 N L \frac{R_m(x) L(1-x)^L}{R_m(y) L(1-x^N)^L} x^P y^\mu \omega^{-P_b} F(t) \frac{F(\omega^{-m})}{\omega^{m(N+1)} F(\omega^m)} \mathcal{G}(\lambda),$$

$$\hat{T}(q) = \alpha_1^{-1} N L \frac{R_m(x) L(1-x)^L}{R_m(y) L(1-x^N)^L} x^P y^\mu \omega^{-P_b} F(t) \frac{F(\omega^{m+1})}{F(\omega^m)} \mathcal{G}(\lambda),$$

(2.36)

where $\alpha_1 := (-1)^{mL} \omega^{(m+1)L-2mP_a-2n_0P_b}/2, \quad R_m(z) := (1-z^N)/\prod_{j=0}^{N-1-m}(1-\omega^j z), \quad \mu := \mu_q, \quad \lambda := \mu^N, \quad$ the variables $x, y, t$ are the normalized coordinates of $x_q, y_q, t_q$:

$$x := \omega^m x_p^{-1} x_q, \quad y := y_p^{-1} y_q, \quad t := \omega^m t_p^{-1} t_q,$$

(2.37)

and $\mathcal{G}(\lambda)$ is the $\lambda$ function on $W_{k'}$ in (2.9) to factorize the polynomial $P(t)$ in (2.33): $\mathcal{G}(\lambda) \mathcal{G}(\lambda^{-1}) = \frac{P(t)}{P(\omega^m)}$. Note that $\hat{T}(p) = 1$ and the total momentum $S_R (= D$ in (2.27)) is defined by

$$S_R = \omega^{-m(m+1)L+m(P_b-P_a)+2n_0P_b} \omega^{P_b} F(\omega t_p) \frac{F(\omega^{-m})}{F(t_p)}.$$  

(2.38)

$^5$ In the even-$N$ and odd-$n$ case, $\mu_p^\infty = -1$ with the $\eta$ in (2.35) changed to $\eta^{-1}$ in the definition of $p$. One may also discuss the Onsager-algebra symmetry of superintegrable CPM by changing $\eta$ here to $\eta^{-1}$ in the argument.

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The function $G(\lambda)$ is related to $G(\lambda_q)$ in (2.28) by

$$G(\lambda_q) = \alpha_q \epsilon^{-(1+2n_0)(N-1)L/2N} \left( 1 - \frac{x}{1 - y} \right)^{mL/N} \frac{\eta^{1/2} (L(N-1)-P_a-P_b)}{\omega^{P_b+m(P_b+P_a)}} F(\omega^{1+m\eta}) G(\lambda), \quad (2.39)$$

where $\lambda = \lambda_q$. Since the Boltzmann weights are finite when any of $x_q, y_q, \mu_q, \mu_q^{-1}$ tends to zero, it follows $P_a, P_b$ being non-negative integers, and $P_b + J - mL \leq P_a \leq (N - 1 - m)L - P_a - NmE - J$. In the case when $m = 0$, the formula (2.36) is the same as that in [1,8,11,13] and the relation (2.39) has been given in [49] section 4.3.6. Therefore $G(\lambda)$ is determined by the zeros of the $t^N$-polynomial $P(t)$, denoted by $t_i^N$, $i = 1, \ldots, m_E$, through $W_{k'}$ in (2.9), equivalently the curve

$$W_{k'}: \frac{(1 - k')^2}{4} w^2 = \frac{(1 - k')^2}{4} \lambda - t^N \quad \left( w := \frac{\lambda + 1}{\lambda - 1}, t := \omega^{mL_1} \right). \quad (2.40)$$

Indeed, $G(\lambda)$ is expressed by

$$G(\lambda) = \prod_{i=1}^{m_E} \left( \frac{(\lambda + 1) - (\lambda - 1)w_i}{2\lambda} \right) \quad (2.41)$$

where $w_i$s are solutions in equation (2.40) for $t^N = t_i^N$ ([1] (2.22), [13] (20)). There are two solutions of $w_i = \pm \sqrt{w}$, where Re$((1 - k')\sqrt{w}/2) > 0$ for real $k' > 0$. Any choice of $w_i = s_iw_i$ ($1 \leq i \leq m_E$) with $s_i = \pm$ gives rise to a $T$ (or $\overline{T}$) eigenvalue (2.36) with the norm-one eigenvector, denoted by $\overline{v}(s_1, \ldots, s_{m_E}; k')$. All such vectors form a basis of the $\tau^{(2)}$ eigenspace $E_{F,P_a,P_b}$ in (2.34):

$$E_{F,P_a,P_b} = \bigoplus_{s_i = \pm} C \overline{v}(s_1, \ldots, s_{m_E}; k'). \quad (2.42)$$

Note that, when $s_E = 0$, $E_{F,P_a,P_b}$ is of dimension 1 with $\overline{v}(s_1, \ldots, s_{m_E}; k')$ being the norm-one base element. Furthermore, the integers $P_a, P_b$ and $P_b$ in (2.36) are indeed quantum numbers of the $\tau^{(2)}$ model, depending only on the $\tau^{(2)}$ eigenvalue.

We now discuss the Onsager-algebra (OA) and $sl_2$-loop-algebra ($sl_2[z, z^{-1}]$) symmetry of a homogeneous superintegrable CPM with the boundary condition $r$. As $q$ tends to $p$ in $W_{k'}$, up to the first order with small $\epsilon$, we may set

$$x_q = \omega^m \eta^{1/2} (1 - 2k' \epsilon), \quad y_q = \eta^{1/2} (1 + 2k' \epsilon), \quad \mu_q = \omega^{m/2} (1 + 2(k' - 1) \epsilon). \quad (2.43)$$

Then $\overline{T}(q)$ near $p$ ([1] ((1.11)–(1.17)))\footnote{For the $m = k = n = 0$ case in [49] section 4.3, some misprints occurred in the formulæ of $T(q), V(x_q, y_q), S(\lambda_q)$ there, where $F(\eta^{-1} t_0), F(\omega^{n} \eta^{-1} t_0)$ should be $F(\eta^{-1} t_0)$, $F(\omega^{n} \eta^{-1} t_0)$, respectively.} is expressed by

$$\overline{T}(q) = \{ 1 + (N - 1 - 2m) E \} + \epsilon H(k') + O(\epsilon^2)$$

where the Hamiltonian $H(k') = H_0 + k'H_1$ is expressed by

$$H_0 = -2 \sum_{l=1}^{L} \sum_{j=1}^{N-1} \frac{\omega^{m} Z_{l+1}^j Z_{l+1}^j}{1 - \omega^{-j}} \quad H_1 = -2 \sum_{l=1}^{L} \sum_{j=1}^{N-1} \frac{\omega^{m+2n_0} X_{l}^j}{1 - \omega^{-j}} \quad (2.44)$$

\footnote{The variable $\epsilon$ is related to the $u$ in [1] (1.11) (or (3.23) in this paper) by $(-1)^{\tilde{m}} u = \epsilon$. Note that, in the even $N$ case when $\mu_q^N = -1$, the expression of $\overline{T}(q)$ near $p$ here still holds by changing $H(k')$ to $H(-k')$.}
with the boundary condition: \( Z_{L+1} = \omega^{-r}Z_1, \ X_{L+1} = X_1 \). By (2.42), one may regard \((2.46)\) as the \(H(k')\)-eigenvector decomposition of \(E_{F,P_a,P_b} \) with the \(H(k')\) eigenvalues:

\[
E(s_1, \ldots, s_mE; k') = 2P_m + NmE - (N - 1 - 2m)L \\
+ k' \left((N - 1 - 2m)L - NmE + 2(P_b - P_a - P_m)\right) \\
+ 2N \sum_{i=1}^{mE} \left(1 - k'\right) \frac{1}{2} (s_i \bar{w}_i).
\]

(2.45)

(When \(sE = 0\), \(\sum_{i=1}^{mE} \) in (2.45) and \(\prod_{i=1}^{mE} \) in (2.41) are defined to be 0, 1, respectively.) The matrices \(\tau^{(2)}(t)\) are the same for all \(k'\) when using the rescaled variable \(t\) in (2.29), hence commute with \(H(k')\) for all \(k'\), equivalently, \([\tau^{(2)}(t), H_0] = [\tau^{(2)}(t), H_1] = 0\).

Since the operators \(H_0, H_1\) in (2.44) satisfy the Dolan-Grady relation for the Onsager-algebra generators \([24, 31, 47]\), one obtains an OA representation on the \(\tau^{(2)}\) eigenspace \(E_{F,P_a,P_b} \) in (2.34) \([44, 47]\). By the general theory of OA representation, this Onsager-algebra symmetry is inherited from a \(sl_2\)-loop-algebra structure of \(E_{F,P_a,P_b} \) \([43]\). We now describe their explicit relationship. As was known in \([43]\), OA can be regarded as the Lie subalgebra of \(sl_2[z, z^{-1}]\) fixed by the involution, \(i: (e^+, h, z) \leftrightarrow (e^-, h, z^{-1})\), where \(e^\pm, h\) are the standard generators of \(sl_2\) with \(\left[e^\pm, e^-\right] = \pm 2e^\pm\); and a finite-dimensional representation of OA can be factored through an \(sl_2[z, z^{-1}]\) representation.

As before we denote by \(t_iN^\pm (i = 1, \ldots, mE)\) the zeros of the \(t^N\) polynomial \(P(t)\), and write \(
\cos \theta_i = \frac{1 + t_iN^+/1 - t_iN^-}{1 - t_iN^+/1 + t_iN^-},
\)

equivalently

\[
e^{i\theta_i} = \frac{1 + t_iN^2}{1 - t_iN^2}, \quad t_iN^\pm = \left(e^{i\theta_i} - 1, e^{i\theta_i} + 1\right)^2, \quad \text{Im}(t_iN^2) \geq 0.
\]

(2.46)

By \([44]\) (theorem 3 and (32), (33)), there exists a basis of \(E_{F,P_a,P_b}, \vec{B}(s_1, \ldots, s_mE)\) for \(s_i = \pm\) such that the generator \(H_0, H_1\) of the OA representation \(E_{F,P_a,P_b} \) can be expressed in the form

\[
H_0 = \alpha + k'\beta + N \sum_{i=1}^{mE} (e_i^+ + e_i^-) = \alpha + k'\beta + 2N \sum_{i=1}^{mE} J_i^x,
\]

\[
H_1 = \alpha + k'\beta + N \sum_{i=1}^{mE} (e^{i\theta}e_i^+ + e^{-i\theta}e_i^-) = \alpha + k'\beta + 2N \sum_{i=1}^{mE} (\cos \theta_i J_i^x - \sin \theta_i J_i^y),
\]

(2.47)

where \(e_i^\pm, h_i\) are the \(sl_2\)-generators for the basis elements \(\vec{B}(s_1, \ldots, s_i, \ldots, s_mE)\) acted only on the \(i\)th \(s_i = \pm\) as the spin-1/2 representation, and \(J_i^x := \frac{1}{2}(e_i^+ + e_i^-), J_i^y := \frac{1}{2}(e_i^+ - e_i^-), J_i^z := \frac{1}{2}h_i\). The OA representation (2.47) is induced from the \(sl_2[z, z^{-1}]\) structure of \(E_{F,P_a,P_b} \) by evaluating \(z\) at \(e^{i\theta_i}\)s:

\[
\vec{g}: sl_2[z, z^{-1}] \rightarrow \text{GL}(E_{F,P_a,P_b}), \quad e^\pm z^k \mapsto \sum_{i=1}^{mE} e_i^\pm e^{i\theta_i k}.
\]

(2.48)

By (2.45) and (2.47), \(H(k') = \alpha + k'\beta + N \sum_{i=1}^{mE} ((1 + k' \cos \theta) J_i^x - k' \sin \theta J_i^y)\), with eigenvalues \(\alpha + k'\beta + 2N \sum_{i=1}^{mE} \pm \sqrt{1 + k'^2 + 2k' \cos \theta_i}\), where \(\alpha = 2P_m + NmE - (N - 1 - 2m)L, \ \beta = (N - 1 - 2m)L - NmE + 2(P_b - P_a - P_m)\) and \(\sqrt{1 + k'^2 + 2k' \cos \theta_i} = (1 - k')\bar{\pi}_i/2\).
Furthermore, the $H(k')$ eigenvectors in (2.42) are related to $\tilde{b}(s_1, \ldots, s_{m_E})$ by
\begin{equation}
\tilde{v}(s_1, \ldots, s_{m_E}; k') = \frac{(\prod_{i=1}^{m_E} \sqrt{s_i} e^{i\varphi_i/2})}{(2i)^{m_E/2} s_1 \cdots s_{m_E}} \prod_{s_i} (\sqrt{s_i} e^{i\varphi_i/2}) \tilde{b}(s_1', \ldots, s_{m_E}')
\end{equation}
where $e^{i\varphi_i} = (1 + k' \cos \theta_i + i k' \sin \theta_i) / \sqrt{1 + k'^2 + 2k' \cos \theta_i}$, and we make the identification $\pm = \pm 1$, $\sqrt{\pm} = 1$, $\sqrt{\mp} = i$. Note that, for a given $k'$, there is a $(\oplus s l_2)$-algebra structure of $\mathcal{E}_{F,P_a,P_b}$ with the generators $e_{i,k'}^\pm, h_{i,k'}$ (or $J_{i,k'}^x, J_{i,k'}^y, J_{i,k'}^z$) acting on the basis elements $\tilde{v}(s_1, \ldots, s_{m_E}; k')$ only on the $i$th $s_i$. One may also introduce the $k'$th $s l_2[z, z^{-1}]$ structures of $\mathcal{E}_{F,P_a,P_b}$ by
\begin{equation}
\rho_{k'}: s l_2[z, z^{-1}] \rightarrow \text{GL}(\mathcal{E}_{F,P_a,P_b}), \quad e^z \mapsto \sum_{i=1}^{m_E} e_{i,k'}^\pm e^{i \theta_i k_i}.
\end{equation}
By (2.49), the following operators of $\mathcal{E}_{F,P_a,P_b}$ are identical:
\begin{equation}
J_i^x = \cos \varphi_i J_i^{x,k'} + \sin \varphi_i J_i^{y,k'}, \quad J_i^y = -\sin \varphi_i J_i^{x,k'} + \cos \varphi_i J_i^{y,k'}, \quad J_i^z = J_i^{z,k'}
\end{equation}
equivalently,
\begin{equation}
\cos \varphi_i J_i^x - \sin \varphi_i J_i^y = J_i^{x,k'}, \quad \sin \varphi_i J_i^x + \cos \varphi_i J_i^y = J_i^{y,k'}, \quad J_i^z = J_i^{z,k'},
\end{equation}
which in turn yield $H(k') = \alpha + k' \beta + N \sum_{i=1}^{m_E} \sqrt{1 + k'^2 + 2k' \cos \theta_i} J_i^{z,k'}$. In particular as $k'$ tends to $0$ or $\infty$, one obtains
\begin{equation}
J_i^x = J_{i,0}^x, \quad J_i^y = J_{i,0}^y, \quad J_i^z = J_{i,0}^z, \quad \cos \theta_i J_i^x - \sin \theta_i J_i^y = J_{i,\infty}^x, \quad \sin \theta_i J_i^x + \cos \theta_i J_i^y = J_{i,\infty}^y, \quad J_i^z = J_{i,\infty}^z.
\end{equation}
Using the above relations, one finds the equivalent $s l_2[z, z^{-1}]$ structure on $\mathcal{E}_{F,P_a,P_b}$ induced from $\rho_{k'}$ ($k' = 0, \infty$) and the standard $\varrho$ in (2.48):
\begin{equation}
\varrho = \rho_0 \cdot R, \quad \varrho = \rho_\infty \cdot R \cdot \iota \cdot \nu,
\end{equation}
where $R$ and (involutions) $\iota, \nu$ are $s l_2[z, z^{-1}]$ automorphisms defined by
\begin{equation}
R: (J_i^x, J_i^y, J_i^z, z) \mapsto (J_i^x, J_i^y, J_i^z, z), \quad \iota: (e^z, h, z) \mapsto (e^{-z}, -h, z^{-1}), \quad \nu: (e^z, e^{-z}, z^{-1}e^z, ze^{-z}, h) \mapsto (z^{-1}e^z, e^z, e^{-z}, -h).
\end{equation}
Later in section 4.2, we shall compare the above $s l_2[z, z^{-1}]$ structure of $\mathcal{E}_{F,P_a,P_b}$ with the $s l_2$-loop-algebra symmetry induced from the XXZ chain equivalent to the $\tau^{(2)}$ model in [50].

Remark. (1) The discussion of this subsection is still valid for the homogeneous superintegrable CPM with a vertical rapidity $p = (\eta^{1/2} \omega^{m+m'}, \eta^{1/2} \omega^{m'}, \omega^{n_0})$. Indeed, under the automorphism $(x, y, \mu) \mapsto (\omega^m x, \omega^m y, \mu)$ on both vertical and horizontal rapidities, the Boltzmann weights (2.12) and the chiral Potts transfer matrix (2.16) are unchanged, so are the values of $x, y, \mu$ coordinates in (2.37); hence the results and formulae of this subsection remain the same.

(2) The $O_{A}$ representation of $\mathcal{E}_{F,P_a,P_b}$ in (2.47) is indeed irreducible. By the theory of the $O_{A}$ representation ([44], [23] theorem 6), the evaluated values $e^{i \theta_i}$ in (2.46) for the $s l_2[z, z^{-1}]$ representation (2.48) satisfy the relation, $e^{i \theta_i} \neq 0, \pm 1$ and $e^{i \theta_i} \neq e^{\pm i \theta_j}$ ($i \neq j$), equivalently $t_i^N \neq 1, 0$, $t_i^N \neq t_j^N$ ($i \neq j$) for the roots of $P(t)$.

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3. Duality in chiral Potts model and $\tau^{(2)}$ model

In this section, we derive the duality in the $\tau^{(2)}$ model and the chiral Potts model. For simpler notations, we discuss only the alternating-rapidity case, i.e. the rapidities in (2.6) at all sites are the same:

$$p = p_t, \quad p' = p'_t \in \mathcal{W} \text{ for all } \ell$$

(3.1)

where $\mathcal{W}$ is a curve in (2.7). Note that with the same argument, all formulae in this section still hold for the inhomogeneous case after a suitable modification of scalar coefficients, as illustrated in section 2 in comparison with the parallel results in [49].

3.1. Duality in $\tau^{(2)}$ model and $\tau^{(2)}$-face-model

In the alternating-rapidity case (3.1), the parameter of $L_\ell$ in (2.1) is given by

$$(a, a', b, b', c) = (x_p, x_{p'}, y_p, y_{p'}, \mu_p \mu_{p'})$$

(3.2)

and we write the $\tau^{(2)}$ matrix in (2.8) simply by $\tau^{(2)}(t) = \tau^{(2)}(t; p, p')$. It is known that $\tau^{(2)}(t)$ can be written as the following product form ([18] ((3.44a) $\text{in } \mu, j = 0, j = 2$, (3.48)), [11] ((2.14)–(2.16))):

$$\tau^{(2)}(t; p, p')_{\{\sigma\}, \{\sigma''\}} = \prod_{\ell=1}^{L} U(\sigma_\ell, \sigma'_\ell | \sigma_{\ell+1}, \sigma''_{\ell+1}).$$

(3.3)

Here the factor $U(a, d|b, c)$ is defined by

$$U(a, d|b, c) = \sum_{m=0,1} \omega^{m(d-b)} (-\omega t)^{a-d-m} F_p(a-d, m) F_{p'}(b-c, m)$$

(3.4)

where $F_p(0,0) = 1$, $F_p(0,1) = -\omega t/y_p$; $F_p(1,0) = \mu_p/y_p$, $F_p(1,1) = -\omega x_p \mu_p/y_p$ and $F_p(a, m) = 0$ if $\alpha \neq 0, 1$. One may express (3.4) in terms of the ‘face variables’, $n := a-b$, $n' := d-c \in \mathbb{Z}_N$, and form the L operator of the $\tau^{(2)}$-face model:

$$L_\ell(t) = \left( \begin{array}{cc} \frac{1}{y_{p'}} & -t \frac{1}{y_{p'}} \frac{Z}{y_{p'}} \\ t \frac{\mu_p}{y_p} - x_p \mu_{p'} \frac{y_{p'}}{y_{p'}} & \frac{\mu_p}{y_p} + \frac{y_{p'}}{y_{p'}} - t \frac{\mu_p}{y_p} \frac{y_{p'}}{y_{p'}} + \omega x_p \mu_{p'} \frac{Z}{y_{p'}} \end{array} \right) X^{-1}, \quad 1 \leq \ell \leq L,$$

(3.5)

where $X, Z$ are the Weyl operators of the face-quantum space $C^N := \sum_{n \in \mathbb{Z}_N} C|n\rangle$, $X|n\rangle = |n+1\rangle$, $Z|n\rangle = \omega^{n'}|n\rangle$. Under the identification of $(X, Z)$ with $(\hat{X}, \hat{Z})$, the L-operator (3.5) is the same as the L-operator (2.1) with the parameter

$$(a^*, a'^*, b^*, b'^*, c^*) = \left( x' \mu_p, x_p \mu_{p'}, y_p, y_{p'}, \frac{1}{\mu_p} \right).$$

(3.6)

Therefore (3.5) satisfies the YB relation (2.2) and so is the monodromy matrix

$$L_1(t)L_2(t) \cdots L_L(t) = \left( \begin{array}{cc} A(t) & B(t) \\ C(t) & D(t) \end{array} \right).$$

(3.7)

The face $\tau^{(2)}$ operator with the boundary condition

$$n_{L+1} = n_1 - r \text{ (mod } N \text{)} \quad (r \in \mathbb{Z}_N)$$

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is the commuting family defined by

$$\tau^F_\ell(t) = \tau^{(2)}_\ell(t; p, p') = A(\omega t) + \omega^t D(\omega t).$$

(3.8)

It is easy to see that \( \tau^{(2)}_\ell(t) \) commute with the charge operator \( \mathbf{Z} := \prod_l Z_\ell \), whose eigenvalues are \( \omega^q \) for \( q \in \mathbb{Z}_N \).

Define the birational morphism of \( \mathbb{C}^3 \):

$$p = (x_p, y_p, \mu_p) \longrightarrow p^* = (x_{p^*}, y_{p^*}, \mu_{p^*}) := (i^{1/N} x_p \mu_p, i^{1/N} y_p \mu_p^{-1}, \mu_p^{-1})$$

(3.9)

which identifies the chiral Potts curves of \( k' \) and \( k'^{-1} \), \( \mathcal{W} \cong \mathcal{W}^* \), with \( \mathcal{W} \) in (2.7) and its dual curve defined by

$$\mathcal{W}^* = \begin{cases} \mathcal{W}_{1/k', k/k'} = \mathcal{W}_{1/k'}, & \text{if } \mathcal{W} = \mathcal{W}_k, \quad (k^2 \neq 1, 0), \\ \mathcal{W}^*_{\pm 1}, \mathcal{W}_{\pm 1} & \text{if } \mathcal{W} = \mathcal{W}_{\pm 1}, \mathcal{W}_{\pm 1} \text{ respectively} \end{cases}$$

(3.10)

where \( \mathcal{W}^*_{\pm 1}, \mathcal{W}_{\pm 1} \) are obtained by the substitution of variables, \( (x, y, \mu) = (i^{1/N} x, i^{1/N} y^*, \mu^*) \), in the equation of \( \mathcal{W}_{\pm 1}, \mathcal{W}_{\pm 1} \), respectively. Note that, for \( k' \neq \pm 1, p \leftrightarrow p^* \) is the canonical identification between \( \mathcal{W}_{k', k} \) and \( \mathcal{W}_{k', -k} \). For \( p \in \mathcal{W} \), the corresponding \( p^* \) in the dual curve \( \mathcal{W}^* \) will be called the dual rapidity of \( p \). By (3.5) and (3.6), the \( \mathbf{L} \)-operator of \( \tau^{(2)}_\ell(t; p, p') \) is gauge-equivalent to the \( \mathbf{L} \)-operator of \( \tau^{(2)}_\ell(t^*; p^*, p^*) \) via the diagonal matrix \( \text{dia}[1, i^{1/N}] \). Hence the face \( \tau^{(2)} \) model \( \tau^{(2)}_\ell(t; p, p') \) is equivalent to \( \tau^{(2)}_\ell(t^*; p^*, p^*) \) with the boundary condition \( r^* \) via the identification of quantum vector spaces:

$$\Phi: \mathbb{C}^N \rightarrow \mathbb{C}^N, \quad |n\rangle \mapsto \hat{n},$$

(3.11)

and the identical boundary and charge condition, \( (r, \mathbf{Q}) = (r^*, \mathbf{Q}^*) \):

$$\tau^{(2)}_\ell(t; p, p') = \Phi^{-1} \tau^{(2)}_\ell(t^*; p^*, p^*) \Phi, \quad t^* = (-1)^{1/N} t.$$

(3.12)

On the other hand, the \( \tau^{(2)} \) models, (3.3) and (3.8), are closely related through the correspondence of their quantum spaces:

$$\Theta_r(=\Theta) : \bigotimes_{\ell=1}^L \mathbb{C}^N \longrightarrow \bigotimes_{\ell=1}^L \mathbb{C}^N, \quad |\sigma_1, \ldots, \sigma_L\rangle \mapsto |n_1, \ldots, n_L\rangle, \quad n_\ell := \sigma_\ell - \sigma_{\ell+1},$$

where \( 1 \leq \ell \leq L \) with \( \sigma_{L+1} = \sigma_1 - r \). It is easy to see that the kernel of \( \Theta_r = \mathbb{C}^N \) with the image being the \( \mathbb{Z} \)-charge-\( r \) subspace. Indeed, the inverse of the vector \( |n_1, \ldots, n_L\rangle \) is expressed by

$$\Theta_r^{-1}(|n_1, \ldots, n_L\rangle) = \bigoplus_{Q=0}^{N-1} |Q; n_1, \ldots, n_L\rangle,$$

where \( |Q; n_1, \ldots, n_L\rangle = N^{-1/2} \sum_{|\sigma_1, \ldots, \sigma_L| \neq 0} \omega^{-Q\sigma_1} |\sigma_1, \ldots, \sigma_L\rangle \) with \( \sigma_\ell - \sigma_{\ell+1} = n_\ell \) [5]. For a fixed \( Q \), all \( |Q; n_1, \ldots, n_L\rangle \) with \( \sum_{\ell=1}^L n_\ell = r \) form a basis of the subspace

$$V_{r,Q} = \left\{ v = (v_{\sigma_1, \ldots, \sigma_L}) \in \bigotimes_{\ell=1}^L \mathbb{C}^N |\sigma_{L+1} = \sigma_1 - r, \quad X(v) = \omega^Q v \right\},$$

(3.13)
with the dual basis \(|Q; n_1, \ldots, n_L| = N^{-1/2} \sum_{\sigma_1=0}^{N-1} \omega^{Q \sigma_1} |\sigma_1, \ldots, \sigma_L|, \) \((\sigma_\ell - \sigma_{\ell+1} = n_\ell)\).

Under \(\Theta_r\), the vector space \(V_{r, Q}\) is isomorphic to the \(\Theta_r\) image, regarded as a subspace of the quantum space \(\bigotimes^L C^N\) with \(r \equiv Q\), and \(Q \equiv r\):

\[
W_{r, Q} = \left\{ w = (w_{n_1}, \ldots, w_{n_L}) \in \bigotimes^L C^N | n_{L+1} \equiv n_1 - r, \quad Z(w) = \omega^Q w \right\},
\]

and hence follows the isomorphism:

\[\Theta: V_{r, Q} \cong W_{r, Q}, \quad |Q; n_1, \ldots, n_L\rangle \mapsto |n_1, \ldots, n_L\rangle, \quad (r, Q) = (Q, r). \quad (3.14)\]

Using (3.3), (3.4), (3.5) and (3.8), one obtains

\[\langle Q; n_1, \ldots, n_L | \tau^{(2)}(t; p, p') | Q; n_1', \ldots, n_L' \rangle = \langle \langle n_1, \ldots, n_L | \tau_F^{(2)}(t; p, p') | n_1', \ldots, n_L' \rangle \]

where \(\sum_\ell n_\ell \equiv \sum_\ell n'_\ell \equiv r\), and \(\tau_F^{(2)}(t; p, p')\) with the boundary condition \(r \equiv Q\).

Equivalently, \(\tau^{(2)}(t; p, p')\) in the charge-\(Q\) sector with boundary condition \(r\) is equivalent to \(\tau_F^{(2)}(t; p, p')\) in the charge-\(Q\) sector with boundary condition \(r = Q\):

\[\tau^{(2)}(t; p, p') = \Theta^{-1} \tau_F^{(2)}(t; p, p') \Theta, \quad (3.15)\]

Combining (3.12) and (3.15), we have shown the following identical \(\tau^{(2)}\) models:

**Proposition 3.1.** For rapidities \(p, p'\) in a chiral Potts curve \(W\) in (2.7), let \(p^*, p'^*\) be the dual rapidities in the \(W^*\) defined by (3.10). Then \(\tau^{(2)}(t; p, p')\) in the charge-\(Q\) sector with boundary condition \(r\) is equivalent to \(\tau^{(2)}(t^*; p'^*, p'^*)\) in the charge-\(Q^*\) sector with boundary condition \(r^*\) when \((Q^*, r^*) = (r, Q)\). Indeed, \(\tau^{(2)}(t; p, p')\) on \(V_{r, Q}\) is similar to \(\tau^{(2)}(t^*; p'^*, p'^*)\) on \(V_{r^*, Q^*}\):

\[\tau^{(2)}(t; p, p') = \Psi^{-1} \tau_F^{(2)}(t^*; p'^*, p'^*) \Psi, \quad t^* = (-1)^{1/N} t,
\]

where \(\Psi\) is the isomorphism:

\[\Psi: V_{r, Q} \rightarrow V_{r^*, Q^*}, \quad |Q; n_1, \ldots, n_L\rangle \mapsto |\widehat{n}_1, \ldots, \widehat{n}_L\rangle, \quad \left(\sum_{\ell=1}^L n_\ell \equiv r\right) \quad (3.16)\]

with \((Q^*, r^*) = (r, Q)\).

**Remark.** There are two bases for the vector space \(V_{r, Q}\) in (3.13):

\[V_{r, Q} = \bigoplus_{n_1} C|Q; n_1, \ldots, n_L\rangle, \quad \left(\sum_{\ell=1}^L n_\ell \equiv r, n_{L+1} = n_1\right)\]

\[= \bigoplus_{n_1'} C|\widehat{n}_1', \ldots, \widehat{n}_L'\rangle, \quad \left(\sum_{\ell=1}^L n'_\ell \equiv Q, |\widehat{n}'_{L+1}\rangle = \omega^{-r n_1'} |\widehat{n}_1\rangle\right),
\]

which are related by

\[|\widehat{n}_1', \ldots, \widehat{n}_L'\rangle = N^{-(L-1)/2} \sum_{n_1} \omega^{\sum_{\ell=1}^\ell n'_\ell(n_1 + \ldots + n_{\ell-1})} |Q; n_1, \ldots, n_L\rangle;
\]

\[|Q; n_1, \ldots, n_L\rangle = N^{-(L-1)/2} \sum_{n_1'} \omega^{-r(n_1 + \ldots + n_{L-1}) n_1'} |\widehat{n}_1', \ldots, \widehat{n}_L\rangle.
\]

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Since the spatial translation \( S_R \) acts on \( V_{r,Q} \) by
\[ S_R|Q; n_1, \ldots, n_L \rangle = \omega^{-Qn_1}|Q; n_2, \ldots, n_L, n_{L+1} \rangle, \]
the isomorphism \( \Psi \) in (3.16) is \( S_R \)-equivariant, also described by
\[ \Psi: V_{r,Q} \longrightarrow V_{r',Q'}, \quad |n_1', \ldots, n_L' \rangle \mapsto \omega^{(Q-n'_1)Q'}|Q'; n_2', \ldots, n_L', n'_1 \rangle. \]

### 3.2. Duality relation in chiral Potts models

We now extend the duality of the \( \tau^{(2)} \) model to CPM. First we note that there exists a \( k' \leftrightarrow k^{-1} \) duality about the Boltzmann weights in (2.12). Indeed, for rapidities \( p, q \) in a chiral Potts curve \( \mathcal{W} \) in (2.7), the Fourier forms of Boltzmann weights (2.15) equal to Boltzmann weights for the dual rapidities \( p^*, q^* \in \mathcal{W}^* \):

\[ \frac{\prod_{pq}^{*}(k)}{\prod_{pq}^{*}(0)} = W_{p^*,q^*}(k), \quad \frac{W_{pq}(k)}{W_{pq}(0)} = \prod_{pq}^{*}(-k) \quad (k \in \mathbb{Z}_N). \]

(3.17)

Note that, when \( k' = \pm 1 \), \( \mathcal{W}^* \) can be identified with \( \mathcal{W} \) via the isomorphism, \( (i^{1/N}x^*, i^{1/N}y^*, \mu^*) \mapsto (x^*, y^*, \mu^*) \), preserving Boltzmann weights. The above relation describes the self-dual property of the Boltzmann weights at \( k' = \pm 1 \), in particular for the case \( \mathcal{W} = \mathcal{W}_{\pm 1} \) in [30]. Using (3.17), one finds the following expression of the chiral Potts transfer matrix (2.16) for \( \mathcal{W}^* \):

\[ \langle \hat{n}_1, \ldots, \hat{n}_L | T(q^*; p^*, p^*) | \hat{n}_1', \ldots, \hat{n}_L' \rangle = (N\prod_{pq}^{*}(0)\prod_{pq}^{*}(0))^{-L} \]

\[ \times \sum_{k_\ell,k_\ell'} \omega^{\sum_{\ell=1}^{L}(n_\ell k_\ell - n_\ell' k_\ell')} \prod_{\ell=1}^{L} \prod_{pq}^{*}(k_\ell - k_\ell') \prod_{pq}^{*}(-k_\ell+1+k_\ell'), \]

(3.18)

\[ \langle \hat{n}_1, \ldots, \hat{n}_L | T(q^*; p^*, p^*) | \hat{n}_1', \ldots, \hat{n}_L' \rangle = (N\prod_{pq}^{*}(0)\prod_{pq}^{*}(0))^{-L} \]

\[ \times \sum_{k_\ell,k_\ell'} \omega^{\sum_{\ell=1}^{L}(n_\ell k_\ell - n_\ell' k_\ell')} \prod_{\ell=1}^{L} \prod_{pq}^{*}(-k_\ell+k_\ell') \prod_{pq}^{*}(-k_\ell+1-k_\ell'), \]

where the indices \( k_\ell, k_\ell' \) are in \( \mathbb{Z}_N \) with the boundary condition, \( k_{L+1} = k_1 - r^*, k_{L+1}' = k_1' - r^* \). On the other hand, when \( \sum_{\ell=1}^{L} n_\ell = \sum_{\ell=1}^{L} n_\ell' = r \), one has

\[ \langle Q; n_1, \ldots, n_L | T(q; p, p') | Q; n_1', \ldots, n_L' \rangle \]

\[ = \frac{1}{N} \sum_{\sigma_1, \sigma_1'} \omega^{Q(\sigma_1 - \sigma_1')} \prod_{\ell=1}^{L} W_{pq}(\sigma_\ell - \sigma_\ell') W_{p'q}(\sigma_{\ell+1} - \sigma_\ell') \]

\[ = \frac{1}{N} \sum_{\sigma_1, \sigma_1', m_\ell, m_\ell'} \omega^{Q(\sigma_1 - \sigma_1') - \sum_{\ell=1}^{L} ((\sigma_\ell - \sigma_\ell')m_\ell + (\sigma_{\ell+1} - \sigma_\ell)m_\ell')} \prod_{\ell=1}^{L} W_{pq}(m_\ell) W_{p'q}(m_\ell') \]

\[ = \frac{1}{N} \sum_{m_\ell} \omega^{\sum_{\ell=1}^{L} m_\ell} \left( \prod_{\ell} W_{p'q}(m_\ell) \right) \]

\[ \times \left( \sum_{m_\ell} \omega^{\sum_{\ell=1}^{L} m_\ell} \prod_{\ell=1}^{L} W_{pq}(-m_\ell) \right), \]

\[ \text{doi:10.1088/1742-5468/2009/08/P08012} \]
Similarly, the same relation between the formula, up to a scale factor, can be identified with the first expression in (3.18) under the constraint \( r^* = Q \) by the change of indices \( k\ell - k\ell' = m\ell', k\ell - k\ell-1 = -m\ell + m\ell-1 \). Similarly, the same relation between \( \hat{T}(q;p,p') \) and \( \hat{T}(q^*;p^*,p^*) \) holds. Indeed, one finds

\[
\langle Q; n_1, \ldots, n_L | T(q;p,p') | Q; n_1', \ldots, n_L' \rangle = (\sum_{p,q} W_{p',q}(0) W_{p,q}(0))^{L} \langle \hat{\eta}_1, \ldots, \hat{\eta}_L | T(q^*;p^*,p^*) | \hat{\eta}_1', \ldots, \hat{\eta}_L' \rangle,
\]
\[
\langle Q; n_1, \ldots, n_L | \hat{T}(q;p,p') | Q; n_1', \ldots, n_L' \rangle = (\sum_{p',q} W_{p',q}(0) W_{p,q}(0))^{L} \langle \hat{\eta}_1, \ldots, \hat{\eta}_L | \hat{T}(q^*;p^*,p^*) | \hat{\eta}_1', \ldots, \hat{\eta}_L' \rangle.
\]

Using \( W_{p',q}(0) = \sum_{p,q} W_{p',q}(0) \), we have shown the following duality relation in CPM as an extension of the \( \tau^2 \) duality in Proposition 3.1.

**Theorem 3.1.** For a \( k' \) curve \( \mathcal{M} \) in (2.7) for \( k' \neq 0 \), let \( \mathcal{M}^* \) be the dual \( k'^{-1} \) curve defined in (3.10). Then the chiral Potts models with rapidities in \( \mathcal{M} \) and \( \mathcal{M}^* \) are equivalent when the \( Z_N \)-charge and the boundary condition are interchanged. More precisely, the chiral Potts transfer matrices over \( \mathcal{M} \) in the \( Q \) sector with the boundary condition \( r \) are similar to those over \( \mathcal{M}^* \) in the \( Q^* \) sector with the boundary condition \( r^* \) via the isomorphism (3.16) when \( (Q^*, r^*) = (r, Q) \):

\[
W_{p',q}(0) = -W_{p,q}(0)^{-L} \hat{T}(q;p,p') = W_{p,q}(0)^{-L} \Psi^{-1} T(q^*;p^*,p^*) \Psi,
\]

\[
W_{p',q}(0) = -W_{p,q}(0)^{-L} \hat{T}(q;p,p') = W_{p,q}(0)^{-L} \Psi^{-1} \hat{T}(q^*;p^*,p^*) \Psi
\]

where \( \Psi \) is the isomorphism in (3.16) and \( p, p', q \in \mathcal{M} \) with the corresponding dual \( p^*, p^*, q^* \in \mathcal{M}^* \).

**Remark.** By the same argument, Theorem 3.1 is also valid for inhomogeneous CPM with vertical rapidities \( \{p_1, \ldots, p_L\}, \{p'_1, \ldots, p'_L\} \). The relation (3.19) still holds, but with one shifting the indices of \( p^* \) by one, i.e. the operators \( T(q^*;\{p'_1, \ldots, p'_L\}, \{p_2, \ldots, p_{L+1}\}) \), \( \hat{T}(q^*;\{p'_1, \ldots, p'_L\}, \{p_2, \ldots, p_{L+1}\}) \); a similar statement also for the \( \tau^2 \) model in Proposition 3.1.

### 3.3. Chiral Potts model of \( \tau^{(2)} \)-face-model and chiral Potts model of the dual lattice

In this subsection, we form the chiral Potts model over the dual lattice, as well as that of the \( \tau^{(2)} \)-face-model, as in [8] section 5 for the superintegrable case. Introduce the \( \mathbb{C}^N \) basis:

\[
|\sigma^*\rangle^* = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \omega^{\sigma^* n} |n\rangle \in \mathbb{C}^N, \quad \sigma \in \mathbb{Z}_N,
\]

by which \( \Phi \) in (3.11) is defined by \( |\sigma^*\rangle^* \mapsto |\sigma^*\rangle \). As in (1.2), the \( \mathbb{C}^N \)-Weyl operators (\( X, Z \)) can be expressed by the basis \( |\sigma^*\rangle^* \) with \( (X^*, Z^*) = (Z, X^{-1}) \), where \( X^* |\sigma^*\rangle^* = |\sigma^* + 1\rangle^* \) and \( Z^* |\sigma^*\rangle^* = \omega^\sigma |\sigma^*\rangle^* \). Using the \( \mathbb{C}^N \) basis \( |\sigma^*\rangle^* \), one defines the chiral Potts model over the dual lattice as follows. The chiral Potts model \( T(q;p,p') \) or \( \hat{T}(q;p,p') \) in (2.16) with
rapidities in a curve \( \mathcal{W} \) in (2.7) is defined on a square lattice \( \Gamma \) by attaching a \( \mathbb{C}^N \)-quantum space over each vertex, (see figure 1 with \( \circ \) as vertices and \( \star \) as faces).

The dual lattice \( \Gamma^* \) of \( \Gamma \) is a square lattice whose vertices (or faces) are in one-to-one correspondence with the faces (vertices respectively) of \( \Gamma \). Using the same Boltzmann weights (2.12) with rapidities in \( \mathcal{W} \), the chiral Potts model, \( T^*(q;p,p') \), \( \hat{T}^*(q;p,p') \), is defined on the dual lattice \( \Gamma^* \) by attaching a \( \mathbb{C}^N \)-quantum space on each vertex of \( \Gamma^* \).

The transfer matrices are defined by

\[
T^*(q;\{p'_\ell\},\{p'_{\ell'}\}) = \prod_{\ell=1}^{L} W_{\ell q}(\sigma^*_{\ell} - \sigma^*_{\ell-1}) W_{\ell' q}(\sigma^*_{\ell} - \sigma^*_{\ell'})
\]

(3.20)

(see figure 2 about \( T^*(q^*;p^*,p'^*) \), \( \hat{T}^*(q^*;p^*,p'^*) \)). By (2.16) and (3.20), \( T^*(q;p,p') \), \( \hat{T}^*(q;p,p') \) are the same as \( T(q;p',p) \), \( \hat{T}(q;p',p) \), respectively, by identifying \( \mathbb{C}^N \) with \( \mathbb{C}^N \) via \( \Phi \) in (3.11):

\[
T^*(q;p,p') = \Phi^{-1} T(q;p',p) \Phi, \quad \hat{T}^*(q;p,p') = \Phi^{-1} \hat{T}(q;p',p) \Phi.
\]

(3.21)

Theorem 3.1 can be stated as the relation between a chiral Potts model in \( \mathcal{W} \) rapidities on a \( \Gamma \) lattice (in the \( Q \) sector and \( r \)-boundary condition) and the dual chiral Potts model in \( \mathcal{W}^* \) rapidities on a \( \Gamma^* \) lattice (in the \( Q^* \) sector and \( r^* \)-boundary condition) for \((Q,r) = (r^*,Q^*)\):

\[
W_{p,q}^{(f)}(0)^{-L} T(q;p,p') = W_{p',q^*}^{(f)}(0)^{-L} \Theta^{-1} T^*(q^*;p^*,p'^*) \Theta,
\]

\[
W_{p',q^*}^{(f)}(0)^{-L} \hat{T}(q;p,p') = W_{p^*,q^*}^{(f)}(0)^{-L} \Theta^{-1} \hat{T}^*(q^*;p^*,p'^*) \Theta
\]

(3.22)

where \( \Theta \) is the isomorphism defined in (3.14). Note that the above formula is still valid in the inhomogeneous case by changing \((p,p'),(p^*,p'^*)\) to \((\{p_\ell\},\{p'_{\ell'}\}),\{p^*_\ell\},\{p'^*_{\ell'}\})\), respectively.

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We now define the chiral Potts model, $T_F(q,p,p')$ or $\hat{T}_F(q,p,p')$, over the $\Gamma^*$ lattice for the $\tau^{(2)}$-face-model $\tau^{(2)}_F(t;p,p')$ in (3.8). As before, the L-operator (3.5) of $\tau^{(2)}_F(t;p,p')$ for $p,p' \in \mathcal{W}$ is equivalent to the L-operator (2.1) of $\tau^{(2)}(t^*;p^*,p^*)$ for the dual rapidities $p^*,p'^* \in \mathcal{W}^*$ via the isomorphism $\Phi$ in (3.11). Therefore, the chiral Potts model $T_F(q,p,p'),\hat{T}_F(q,p,p')$ with $p,p',q \in \mathcal{W}$ is identified with the CPM over the dual curve $\mathcal{W}^*$ via the dual map (3.9) so that the following relations hold:

$$T_F(q,p,p') = \Phi^{-1} T(q^*;p^*,p^*) \Phi, \quad \hat{T}_F(q,p,p') = \Phi^{-1} \hat{T}(q^*;p^*,p^*) \Phi,$$

extending that in (3.12). By (3.21), the above relations are the same as

$$T_F(q,p,p') = T^*(q^*,p^*,p^*), \quad \hat{T}_F(q,p,p') = \hat{T}^*(q^*,p^*,p'^*),$$

with the identical charge and boundary condition, $(Q,r) = (Q^*,r^*)$. By the duality relation (3.22), one obtains the extended relation of (3.15):

$$W_{p,q}^{(f)}(0)^{-L} T(q;p,p') = W_{p',q'}^{(f)}(0)^{-L} \Theta^{-1} T_F(q,p,p') \Theta,$$

$$W_{p',q'}^{(f)}(0)^{-L} \hat{T}(q;p,p') = W_{p,q}^{(f)}(0)^{-L} \Theta^{-1} T_F(q,p,p') \Theta,$$

with the constraint $(Q,r) = (r,Q)$.

### 3.4. Kramers–Wannier duality in Ising model

For CPM in the $N = 2$ case, we regain the Ising model with the following homogeneous parametrization of rapidities $p \in \mathcal{W}_{k'}$ for $k' \neq 0, \pm 1$ ([12] section 3):

$$a_p,b_p,c_p,d_p = -H(u_p); -H_1(u_p); \Theta_1(u_p); \Theta(u_p),$$

equivalently the coordinates of $\mathcal{W}_{k'}$ in (2.7) given by

$$x_p = -k'^{1/2} \sin u_p, \quad y_p = -k'^{1/2} \cos u_p, \quad \mu_p = k'^{1/2} \frac{1}{\sin u_p}.$$
Here the Jacobi theta functions are of modulus $k$ with the elliptical integrals $(K, K') = (K(k), K'(k'))$. The Boltzmann weights (2.12) are now expressed by

$$W_{pq}(0) = 1, \quad W_{pq}(1) = \frac{cnu + snupq}{cnu + pq} = k'scd(u - u + K);$$

$$\mathcal{W}_{pq}(0) = 1, \quad \mathcal{W}_{pq}(1) = \frac{k'(scd(u - u) - pq)}{dnu, cnu + pq} = k'scd(u - u).$$

where $scd(u) := sn(u/2)/(cn(u/2)dn(u/2)) = snu/(cnu + dnu)$. If $J$ and $\overline{J}$ are the usual dimensionless Ising model interaction coefficients, $W_{pq}(1) = \exp(-2J)$, $\mathcal{W}_{pq}(1) = \exp(-2\overline{J})$, and we find [6] (7.8.5),

$$\sinh 2J = \frac{sn}{cnu}, \quad \sinh 2\overline{J} = \frac{cnu}{k'snu} \quad (u = u - u).$$

Similarly, the coordinates of the rapidity $\rho^*$ in $\mathcal{W}_{1/k'}$ are expressed by

$$x_{\rho^*} = -\left( \frac{ik}{k'} \right)^{1/2} sn^*u^*, \quad y_{\rho^*} = -\left( \frac{ik}{k'} \right)^{1/2} cn^*u^*, \quad \mu_{\rho^*} = \left( \frac{1}{k'} \right)^{1/2} \frac{1}{dn^*u^*},$$

with the Boltzmann weights

$$W_{p^*q^*}(0) = 1, \quad W_{p^*q^*}(1) = k^{-1}scd^*(u^* - u^* + K^*) = \exp(-2J^*);$$

$$\mathcal{W}_{p^*q^*}(0) = 1, \quad \mathcal{W}_{p^*q^*}(1) = k^{-1}scd^*(u^* - u^*) = \exp(-2\overline{J}^*),$$

where the Jacobi theta functions are of modulus $ik/k'$ with the elliptical integrals $(K^*, K'^*) = (K(ik/k'), K'(ik/k'))$. When $p^*$ is the dual rapidity of $p$, one finds $u^* = k'u_p$ (equivalently $K^* = k'K$), and the equivalence of (3.17) with the duality of weights in the Ising model ([33], [6] (6.2.14a)):

$$\tanh J = e^{-2J}, \quad \tanh \overline{J} = e^{-2\overline{J}}.$$

The relation (3.22) is identified with the usual duality in the Ising model (see, e.g., [6] chapter 6). Note that the value $\tau = i(K/K')$ for modular $k'$ and the value $\tau^* = i(K^*/K'^*)$ for $k'^{-1}$ are related by $\tau^* = -\tau/(\tau + 1)$ when $k'$ are in a certain region, e.g. for real and positive $k'$.

### 3.5. Duality and Onsager-algebra symmetry in homogeneous chiral Potts model

In this section, we show the duality symmetry of homogeneous CPM compatible with the associated quantum spin chain Hamiltonian. In particular, we obtain the explicit relation between the duality and the Onsager-algebra symmetry in the superintegrable case. For a general homogeneous CPM (2.18), the quantum spin chain Hamiltonian is obtained from $\hat{T}(q)$ by letting $q \to p$ in $\mathcal{W}_{k'}$ with small $u$ up to first order:

$$\frac{x_q}{x_p} = 1 - 2k'\left( \frac{y_p}{x_p}\frac{N/2}{y_p} \right)^{N/2} u, \quad \frac{y_q}{y_p} = 1 + 2k'\left( \frac{x_p\mu_p^2}{y_p} \right)^{N/2} u, \quad \frac{\mu_q}{\mu_p} = 1 - 2k(x_py_p)^{N/2} u,$$

which in turn yields

$$\hat{T}(q) = 1 + 2Lu \sum_{j=1}^{N-1} \frac{(x_py_{p-1})^{j-(N/2)}}{1 - \omega^{-j}} + u\mathcal{H}(k'; p) + O(u^2).$$

$\text{doi:10.1088/1742-5468/2009/08/P08012}$
where $e^{2i\phi/N} = (-1)^{\pi j/N}x_p y_p$ and $e^{2i\tilde{\phi}/N} = (-1)^{\pi j/N}x_p y_p^2$ (cf. (1.10)–(1.17)). Note that in the superintegrable case (2.35), $u$ in (2.33) and the above $\mathcal{H}(k'; p)$ are related to $\epsilon$ in (2.43) and $H(k')$ in (2.44) by $(-1)^m u = \epsilon, (-1)^m H(k'; p) = H(k')$. Then the dual correspondence (3.9) between $\mathfrak{M}_{k'}$ and $\mathfrak{M}_{k-1}$ gives rise to the identification of $u$ and $\phi$, $\tilde{\phi}$ at $p$ with those at the dual rapidity $p^*$:

$$k' u = u^*, \quad e^{2i\phi/N} = e^{2i\tilde{\phi}/N}, \quad e^{2i\tilde{\phi}/N} = e^{2i\phi^*/N}.$$ 

Using (3.23) and the equality $\sum_{k=1}^{N-1} (N-k)\omega^{jk} = -N/(1-\omega^{-j})$, one finds

$$W_{pq}^{(f)}(0) = \sqrt{N} \left( 1 + 2 \sum_{j=1}^{N-1} \frac{(x_p y_p^{-1})^j - (N/2)}{1 - \omega^{-j}} \right).$$

By the second equality in (3.19), we obtain the explicit relation between Hamiltonian $\mathcal{H}(k'; p)$ and $\mathcal{H}(k'^{-1}; p^*)$: $\mathcal{H}(k'; p) = k' \Psi^{-1} \mathcal{H}(k'^{-1}; p^*) \Psi$. Indeed, by the relation between $\phi, \phi^*$ and $\tilde{\phi}, \phi^*$, one can verify this Hamiltonian relation directly by using (3.16) and the correspondence of local operators:

$$Z_{\ell} Z_{\ell+1}^{-1} = \Psi^{-1} X_{\ell} \Psi, \quad X_{\ell+1} = \Psi^{-1}(Z_{\ell} Z_{\ell+1}^{-1}) \Psi \quad (1 \leq \ell \leq L).$$

We now consider the homogeneous superintegrable case with the vertical rapidity $p$ in (2.35). As in section 2.2, the chiral Potts transfer matrix $T_p(q)$ and $\hat{T}_p(q)$ with $q \in \mathfrak{M}_{k'}$ are expressed by (2.36) using the variables $x, y, \mu$ and quantum numbers $P_a, P_b, P_\mu$. Under the duality transformation (3.9), $T_p, \hat{T}_p$ are related to the transfer matrices $T_{p^*}(q^*), \hat{T}_{p^*}(q^*)$ for $q^* \in \mathfrak{M}_{k'^{-1}}$ with the vertical rapidity $p^*$ defined by

$$p^*: (x_p, y_p, \mu_p) = (\eta^{1/2} \omega^m, \eta^{1/2} \omega^{-m}, \omega^m) \in \mathfrak{M}_{k'^{-1}},$$

where $\eta^* := \frac{1}{1+\bar{k}}^{1/2}/N$ and $n_0^* := -n_0$. By Remark (1) at the end of section 2.2, $T_{p^*}(q^*), \hat{T}_{p^*}(q^*)$ are again expressed by formulae in (2.36) with the variables $x^*, y^*, \mu^*$ and quantum numbers $P_a^*, P_b^*, P_\mu^*$, where the integer $m^*$ is defined by $m^* \equiv m + 2n_0 (\text{mod } N)$ and $0 \leq m^* \leq N - 1$. By Theorem 3.1 and the second relation in (2.25), $T_p(q), \hat{T}_p(q)$ are equivalent to $T_{p^*}(q^*), \hat{T}_{p^*}(q^*)$ via the linear isomorphism $\Psi$ (3.16) with $(Q, r) = (r^*, Q^*)$. Furthermore, the comparison of $x, y, \mu$ zero orders for both sides of (3.19) (including those from $W_{pq}^{(f)}(0) L, W_{pq}^{(g)}(0) L$ factor) in turn yields the identification of variables and quantum numbers:

$$x^* = \omega^m x, \quad y^* = \omega^m y, \quad \mu^* = \mu, \quad t^* = \omega^{2n_0} t, \quad P_a^* = P_a, \quad P_b^* = P_b, \quad P_\mu^* \equiv r, \quad P_\mu^* \equiv Q,$$

$$J^* = J, \quad m_\epsilon^* = m_\epsilon, \quad \alpha^*_1 = \alpha_1 \omega^{n_0 (P_b + P_a)},$$

as well as the identification of the Bethe polynomial, $F(t) = F^*(t^*)$, and their roots, $v_j = v_j \omega^{2n_0}$, together with $G^*(\lambda^*) = \lambda^m G(\lambda)$ and $\bar{w}_i = \bar{w}_i$. Note that by $F(\omega^m) = F^*(\omega^m^*)$, the formula (2.38) implies $S_R = S_R^*$, which is consistent with the $S_R$-equivariant-property of $\Psi$ in the remark of Proposition 3.1. Indeed, the isomorphism

\[ \text{doi:10.1088/1742-5468/2009/08/P08012} \]

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Ψ makes the identification of basis elements in (2.34):

\[
\Psi: \tilde{v}^\ast(s_1, \ldots, s_{m_E}; k') \mapsto \tilde{v}(s_1, \ldots, s_{m_E}; k')^{-1}.
\]  

(3.25)

The equivalence of \( \mathcal{H}(k'; p) \) and \( \mathcal{H}(k'^{-1}; p^*) \) now becomes the identification of Onsager-algebra generators:

\[
H_0 = \Psi^{-1} H_1^\ast \Psi, \quad H_1 = \Psi^{-1} H_0^\ast \Psi,
\]

(3.26)

where \((H_0, H_1)\) are defined in (2.44) for \((m, n_0), (m^*, n_0^*)\) and the boundary condition \(r, r^*\), respectively.

**Remark.** In the special superintegrable case when \(m = n_0 = 0\), the duality has been discussed by Baxter in [8], where the variables \(x, y, \mu\) in (2.1) and \(x_d, y_d, \mu_d\) in (5.4) there correspond to \(x, y^{-1}, y\mu^{-1}\) and \(x^*, y^*, y^*\mu^*\), respectively, in this paper.

### 4. Inhomogeneous XXZ chain of \(U_q(sl_2)\)-cyclic representation

For an arbitrary \(q\), the quantum group \(U_q(sl_2)\) is the \(C\) algebra generated by \(K^{\pm \frac{1}{2}}, e^\pm\) with \(K^{\frac{1}{2}} K^{-\frac{1}{2}} = K^{-\frac{1}{2}} K^{\frac{1}{2}} = 1\) and the relation

\[
K^{\frac{1}{2}} e^\pm K^{-\frac{1}{2}} = q^{\pm \frac{1}{2}} e^\pm, \quad [e^+, e^-] = \frac{K - K^{-1}}{q - q^{-1}}.
\]

(4.1)

Using \(U_q(sl_2)\), one constructs a two-parameter family of \(L\) operators:

\[
L(s) = \begin{pmatrix}
\rho^{-1} \nu^{1/2} s K^{1/2} - \nu^{-1/2} s^{-1} K^{1/2} & (q - q^{-1}) e^-

(q - q^{-1}) e^+ & \nu^{1/2} s K^{1/2} - \rho \nu^{-1/2} s^{-1} K^{1/2}
\end{pmatrix},
\]

(4.2)

with \(\rho, \nu \neq 0 \in C\), satisfying the YB equation:

\[
R_{6v}(s/s') \begin{pmatrix} L(s) \otimes 1 \\ 1 \otimes L(s') \end{pmatrix} = \begin{pmatrix} 1 \otimes L(s') \\ L(s) \otimes 1 \end{pmatrix} R_{6v}(s/s').
\]

(4.3)

for the symmetric six-vertex \(R\) matrix [29, 37]:

\[
R_{6v}(s) = \begin{pmatrix}
s^{-1} q - sq^{-1} & 0 & 0 & 0

0 & s^{-1} - s & q - q^{-1} & 0

0 & q - q^{-1} & s^{-1} - s & 0

0 & 0 & 0 & s^{-1} q - sq^{-1}
\end{pmatrix}.
\]

For a chain of size \(L\), we assign the \(L\)-operator \(L_\ell\) at the \(\ell\)th site with the parameter \(\rho_\ell, \nu_\ell\) in (4.2). The monodromy matrix

\[
\bigotimes_{\ell=1}^{L} L_\ell(s) = \begin{pmatrix} A(s) & B(s) \\ C(s) & D(s) \end{pmatrix}
\]

(4.4)

again satisfies YB (4.3), whose \(q^{-2r}\)-twisted trace

\[
t(s) = A(s) + q^{-2r} D(s),
\]

(4.5)

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form the commuting family in the tensor algebra \((\bigotimes U_q(sl_2))(s)\). The leading and lowest terms of the monodromy matrix:
\[
A_{\pm} = \lim_{s \to \pm \infty} \tilde{\nu}^{-1/2} \tilde{\rho}^{-1/2} A(s), \quad A_{\pm} = \lim_{s \to \pm \infty} \tilde{\nu}^{-1} (-s)^{-1} A(s),
\]
\[
B_{\pm} = \lim_{s \to \pm \infty} \tilde{\nu}^{(1/2)} (\pm s)^{(L-1)/2} B(s), \quad C_{\pm} = \lim_{s \to \pm \infty} \tilde{\nu}^{(1/2)} (\pm s)^{(L-1)/2} C(s), \quad D_{\pm} = \lim_{s \to \pm \infty} \tilde{\nu}^{-1/2} (-s)^{L/2} D(s),
\]
give rise to the quantum affine algebra \(U_q(\hat{sl}_2)\), where \(\tilde{\nu} := \prod_{\ell} \nu_{\ell}\) and \(\tilde{\rho} := \prod_{\ell} \rho_{\ell}\). Indeed the generators of \(U_q(\hat{sl}_2)\):
\[
k_0^{-1} = k_1 = A_{\pm}^2 = B_{\pm}^2, \quad e_1 = S^+ := C_+, \quad f_1 = S^- := B_-, \quad e_0 = T^- := B_+, \quad f_0 = T^+ := C_-,
\]
are expressed by
\[
A_{\pm} = D_{\pm} = K^{1/2} \otimes \cdots \otimes K^{1/2}, \quad A_{\pm} = D_{\pm} = K^{-1/2} \otimes \cdots \otimes K^{-1/2},
\]
\[
S^{\pm} = \sum_{i=1}^{L} \nu_i^{(1/2)} \prod_{j<i} \rho_j^{1/2} K^{1/2} \otimes \cdots \otimes K^{1/2} \otimes e^\pm \otimes K^{-1/2} \otimes \cdots \otimes K^{-1/2}, \quad L^{-1},
\]
\[
T^{\pm} = \sum_{i=1}^{L} \nu_i^{(1/2)} \prod_{j<i} \rho_j^{1/2} K^{1/2} \otimes \cdots \otimes K^{1/2} \otimes e^\pm \otimes K^{1/2} \otimes \cdots \otimes K^{1/2}.
\]
In the homogeneous case with the periodic boundary condition, i.e. \(r = 0, \nu = \nu_{\ell}, \rho = \rho_{\ell}\) for all \(\ell\), the quantum group \(U_q(\hat{sl}_2)\) also possesses a Hopf-algebra structure defined by
\[
\Delta(k_i) = k_i \otimes k_i, \quad i = 0, 1,
\]
\[
\Delta(\nu_i) = \nu_i \otimes e_i + e_i \otimes \nu_i, \quad \Delta(f_i) = k_i \otimes f_i + \nu_i f_i \otimes k_i,
\]
\[
\Delta(e_0) = \rho_0 e_0 + e_0 \otimes k_1, \quad \Delta(f_0) = \rho_0 f_0 + f_0 \otimes k_1.
\]
In particular, with \(\rho = 1, \nu = q^{d-2}\) and the spin-(\(d-1)/2\) (highest weight) representation of \(U_q(\hat{sl}_2)\) on \(C^d = \oplus_{k=0}^{d-1} C e^k\):
\[
K^{1/2}(e^k) = q^{(d-1-2k)/2} e^k, \quad e^\pm(e^k) = [k] e^{k-1}, \quad e^{-\pm}(e^k) = [d - 1] e^{k+1},
\]
where \([n] = (q^n - q^{-n})/(q - q^{-1})\) and \(e^+(e^0) = e^-(e^{d-1}) = 0\). \((4.5)\) gives rise to the transfer matrix of the well-known homogeneous XXZ chain of spin-(\(d-1)/2\) (see, e.g., \([34, 45, 46]\) and references therein).

Hereafter in this paper, we assume the anisotropic parameter \(q\) of the inhomogeneous model \((4.4)\) to be an \(N\)th primitive root of unity. There is a three-parameter family of \(U_q(sl_2)\)-cyclic representation on \(C^N\) (the space of cyclic \(N\) vectors), \(s_{\phi,\phi',\varepsilon}\), labelled by non-zero complex numbers \(\phi, \phi', \varepsilon\), and defined by
\[
K^{1/2}(e^\sigma) = q^{\phi + (\phi' - \phi)/2} e^\sigma, \quad e^\sigma = q^{\phi - \sigma} - q^{-\phi + \sigma} / (q - q^{-1}) (\sigma + 1), \quad e^{-\sigma} = q^{-\varepsilon} q^{\phi + \sigma} - q^{-\phi - \sigma} / (q - q^{-1}) (1 - \sigma). \quad (4.8)
\]

\(^8\) The definition \((4.8)\) in this paper is the same as \([50]\) (3.1) where \(|n\rangle\) is replaced by \(|\sigma\rangle\) here.
(see, e.g., [21, 25]), where \( \hat{\sigma} \) (\( \sigma \in \mathbb{Z}_N \)) are the Fourier bases of \( \mathcal{C}^N \) in (1.1). Applying the cyclic representation \( s_{\phi', \phi, \nu, \rho} \) on \( \mathcal{L}_N \) in (4.4), we form the monodromy matrix
\[
\bigotimes_{\ell=1}^L \mathcal{L}_\ell(s) = \left( \begin{array}{c|c}
A(s) & B(s) \\
\hline
C(s) & D(s) \end{array} \right), \quad \mathcal{L}_\ell(s) = s_{\phi', \phi, \nu, \rho} \mathcal{L}_\ell(s). \tag{4.9}
\]
The transfer matrices of the inhomogeneous XXZ chain of \( U_q(sl_2) \)-cyclic representation with parameters \( \{ \phi, \phi', \nu, \rho \} \) and with the boundary condition (2.4) are the \( \mathcal{L} \) \( \mathcal{C}^N \) operators:
\[
T(s) = A(s) + q^{-2r} D(s) = (\otimes \mathcal{L}s_{\phi', \phi, \nu, \rho}) t(s), \tag{4.10}
\]
which commute with \( K^{1/2} := \otimes q K^{1/2} \), the product of local \( K^{1/2} \) operators.

In the case when \( \phi, \phi' \) are integers for all \( \ell \), we write the basis elements in (4.8) by
\[
v^k(\equiv v^k) := |\phi + N - k\rangle, \quad v^k(\equiv v^k) := | - \phi' - 1 - k\rangle \quad (0 \leq k \leq N - 1). \tag{4.11}
\]
Then \( K(v^k) = q^\phi \delta - 2k v^k \) and \( K(v^k) = q^{\phi'} - 2k v^k \). With \( K = q^h \), the relation (4.8) is consistent with
\[
e^+(v^k) = q^h |k\rangle v^k, \quad h^+(v^k) = (h(v^k) - h(v^k)) e^+(v^k),
\]
\[
e^-(v^k) = q^{-h} [N - k - 1] v^k, \quad h^-(v^k) = (h(v^k) - h(v^k)) e^+(v^k),
\]
which in turn yield \( h(v^k) = h(v^k) + 2 \) and \( h(v^k) = h(v^k) - 2 = 0 \) for \( 1 \leq k \leq N - 1 \). Therefore \( v^k = v^k \), equivalently
\[
\phi, \phi' \in \mathbb{Z}, \quad \phi + \phi' + 1 \equiv 0 \pmod{N}, \tag{4.12}
\]
hence \( h(v^k) = -1 - 2k + c \epsilon N \) \( (1 \leq k \leq N - 1) \) for an integer \( c \). When \( \phi, \phi' \) satisfy the condition (4.12), one may define the normalized \( N \)th power of \( S^\pm, T^\pm, S^{\pm(N)} = S^{\pm N}/[N]! \),
\[
S^{\pm(N)} = \sum_{0 \leq k_1 < N, k_1 + \ldots + k_L = N} \left[ k_1 \right]! \ldots \left[ k_L \right]!
\]
\[
\times \bigotimes_{i=1}^L K^{-1/2} \left( \sum_{j<i} k_j \right) e_i^{\pm k_1} v_i^{k_1} / \left[ \sum_{j<i} k_j \right], \tag{4.13}
\]
\[
T^{\pm(N)} = \sum_{0 \leq k_1 < N, k_1 + \ldots + k_L = N} \left[ k_1 \right]! \ldots \left[ k_L \right]!
\]
\[
\times \bigotimes_{i=1}^L K^{1/2} \left( \sum_{j<i} k_j \right) e_i^{\pm k_1} v_i^{k_1} / \left[ \sum_{j<i} k_j \right].
\]
As in [46] section 4.2, the entries of (4.9) satisfy the ABCD algebra, which in turn yields
\[
\mathcal{A}(s) = \frac{N}{x^2 - s_2 q^{N+1}} \prod_{i=1}^N \mathcal{B}(s_i) \mathcal{A}(s) = \frac{N}{x^2 - s_2 q^{N+1}} \prod_{i=1}^N \mathcal{B}(s_i) \mathcal{A}(s),
\]
\[
\mathcal{D}(s) = \frac{N}{x^2 - s_2 q^{-N}} \prod_{i=1}^N \mathcal{B}(s_i) \mathcal{D}(s) = \frac{N}{x^2 - s_2 q^{-N+1}} \prod_{i=1}^N \mathcal{B}(s_i) \mathcal{D}(s),
\]
\[
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\]
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as well as another two formulae by interchanging the above \(A, B\) and \(D, C\), respectively, where \(s_i = x q^{(N+1-2i)/2}\) for \(i = 1, \ldots, N\). By multiplying \((\pm s_i) \hat{\tau}(L-1)\hat{\rho}^{(1/2)}(q - q^{-1})^{-1}\) to \(B(s)\) or \(C(s)\) in the above at a generic \(q\), then taking the limit of \(s_i^2\) at \(q^N = 1\), one arrives at

\[
\begin{align*}
[A(s), S^{\pm(N)}] &= \mp s^2 S^{\pm(N-1)} \hat{C}(s) \hat{\rho}^{-1} K^\mp 1/2, \\
[D(s), S^{\pm(N)}] &= \pm s S^{\pm(N-1)} \hat{C}(s) \hat{\rho} K^\mp 1/2, \\
[A(s), T^{\pm(N)}] &= \mp s^2 T^{\pm(N-1)} \hat{C}(s) \hat{\rho}^{-1} K^\mp 1/2, \\
[D(s), T^{\pm(N)}] &= \pm s T^{\pm(N-1)} \hat{C}(s) \hat{\rho} K^\mp 1/2,
\end{align*}
\]

which imply

\[
\begin{align*}
[T(s), S^{\pm(N)}] &= s^2 S^{\pm(N-1)} \hat{C}(s) \hat{\rho}^{-1} K^1/2 - \hat{\rho}^{-1} q^{-2r} K^{-1/2}, \\
[T(s), T^{\pm(N)}] &= -s^2 T^{\pm(N-1)} \hat{C}(s) \hat{\rho}^{-1} q^{-2r} K^{-1/2} - \hat{\rho}^{-1} K^{-1/2}.
\end{align*}
\]

In particular, under the constraint \(K = q^{-2r} \hat{\rho} = \pm 1\), \(T^{\pm(N)}\) and \(S^{\pm(N)}\) commute with \(T(s)\) in (4.10):

\[
[T(s), S^{\pm(N)}] = [T(s), T^{\pm(N)}] = 0, \quad \text{when } K = q^{-2r} \hat{\rho} = \pm 1. \tag{4.14}
\]

In this situation, with \(H^{(N)} = N^{-1} \sum_{i=1}^{L} 1 \otimes \cdots \otimes h_i \otimes 1 \otimes \cdots \otimes 1\), the operators, \(H^{(N)}, S^{\pm(N)}\) and \(T^{\pm(N)}\), generate an \(sl_2\)-loop-algebra representation:

\[
\begin{align*}
[H^{(N)}, S^{\pm(N)}] &= \pm 2S^{\pm(N)}, \\
[H^{(N)}, T^{\pm(N)}] &= \pm 2T^{\pm(N)}
\end{align*}
\]

with the Chevalley generators:

\[
-H_0 = H_1 = H^{(N)}, E_1 = S^{+(N)}, F_1 = S^{-(N)}, E_0 = T^{-(N)}, F_0 = T^{+(N)}, \tag{4.15}
\]

which are related to the mode basis by \(E_1 = e(0), F_1 = f(0), E_0 = f(1), F_0 = e(-1)\). Note that the representation \(s_{\phi,\epsilon,\phi',\epsilon'}\) in (4.12) is equivalent to the spin-\((N-1)/2\) representation in (4.7) with the basis \(u^k\)s corresponding to \(e^k\). In this situation, the algebraic-Bethe-ansatz method [29, 35, 36] can be employed to the diagonalization of the transfer matrix (4.10).

4.1. The equivalence of inhomogeneous \(\tau^{(2)}\) models and XXZ chains with \(U_q(sl_2)\)-cyclic representation

As in the homogeneous case [50], we now show the general inhomogeneous \(\tau^{(2)}\) models (2.5) are equivalent to XXZ chains in (4.10). For simplicity, we consider only the odd \(N(= 2M + 1)\) case\(^9\) where \(q\) is chosen to be the \(N\)th root of unity satisfying \(q^{-2} = \omega\), i.e. \(q := \omega^M\). One can express the cyclic representation (4.8) in terms of \(\hat{X}, \hat{Z}\):

\[
\begin{align*}
K^{1/2} &= q^{(\phi'-\phi)/2} \hat{Z}^{-1/2}, \\
e^+ &= q^\epsilon \left(q^{\phi+1} \hat{Z}^{1/2} - q^{-\phi-1} \hat{Z}^{-1/2}\right) \hat{X} / (q - q^{-1}) \\
e^- &= q^{-\epsilon} \left(q^{\phi'+1} \hat{Z}^{-1/2} - q^{-\phi'-1} \hat{Z}^{1/2}\right) \hat{X}^{-1} / (q - q^{-1}).
\end{align*}
\]

\(^9\) For an arbitrary \(N\), the argument in the homogeneous case [50] section 3.2 can also be applied to the general inhomogeneous case so that an XXZ chain (4.10) is equivalent to the sum of two copies of the same \(\tau^{(2)}\) model (2.5) via the identification of parameters (4.17).
By (1.2), the operators \( K^{-1}, K^{-1/2} e^\pm \) are represented in the form
\[
K^{-1} = q^{\phi - \phi'} X, \quad K^{-1/2} e^+ = -q^{((\phi - \phi')/2) + \varepsilon - 1} (1 - q^{2\phi - 2X}) Z^{-1} \frac{q - q^{-1}}{q - q^{-1}}, \\
K^{-1/2} e^- = q^{((\phi + \phi')/2) - \varepsilon + 1} (1 - q^{-2\phi' - 2X}) Z^{-1} \frac{q - q^{-1}}{q - q^{-1}}.
\] (4.16)

By setting \( t = s^2 \) and with the gauge transform \( \text{dia}[1, -sq] \), the modified \( L \)-operator, \(-s\nu_{\ell}^{-1/2} K_{\ell}^{-1/2} \mathcal{L}(s)\), of (4.9) is equivalent to
\[
\begin{pmatrix}
1 - \nu_{\ell}^{-1} q^{\phi - \phi'} X & \nu_{\ell}^{-1/2} q^{((\phi + \phi')/2) - \varepsilon} (1 - q^{-2\phi' - 2X}) Z^{-1} \\
-t\nu_{\ell}^{-1/2} q^{((\phi - \phi')/2) + \varepsilon} (1 - q^{2\phi + 2X}) Z^{-1} & -\nu_{\ell} + \rho_{\ell} q^{\phi - \phi'} X
\end{pmatrix},
\]
which is the same as \( L_{\ell}(t) \) in (2.5) with the identification of parameters
\[
a_{\ell} = \rho_{\ell} \nu_{\ell}^{-1} q^{((\phi - \phi')/2) - \varepsilon}, \quad \omega_{\ell} a'_{\ell} = \rho_{\ell}^2 \nu_{\ell}^{-1}, \quad b_{\ell} = \nu_{\ell}^{-1/2} q^{((\phi - \phi')/2) + \varepsilon},
\]
equivalently, \( q^{\varepsilon} = (\omega_{\ell} a_{\ell}/a_{\ell} b'_{\ell})^{1/4}, \quad q^{\varepsilon} = \omega_{\ell} a_{\ell}/b_{\ell}, \quad q^{\varepsilon} = b_{\ell} = a_{\ell}/c_{\ell}, \quad \rho_{\ell}^2 = \omega_{\ell} a_{\ell}/b_{\ell}, \quad \nu_{\ell} = 1/b_{\ell} b'_{\ell}. \) Therefore \( \tau^{(2)} \) models (2.5) are equivalent to XXZ chains (4.10) in the general inhomogeneous case, where the product of local operator \( K^{-1} \) corresponds to (a scalar multiple of) the spin-shift operator \( X \) in the \( \tau^{(2)} \) model. Note that with \( a', b', a, b, c \) in (2.6), the formula (4.17) establishes a relation between the chiral Potts rapidities and cyclic-representation parameters, which gives a scheme of reproducing the Boltzmann weights of CPM from the representation theory of \( U_q(sl_2) \) in [22].

### 4.2. The connection between Onsager-algebra symmetry of superintegrable \( \tau^{(2)} \) model and the \( sl_2 \)-loop-algebra of XXZ chain

In this section, we consider a special homogeneous XXZ chain case in (4.12), \( \nu = \nu_{\ell}, \rho = \rho_{\ell}, \varepsilon = \varepsilon_{\ell}, \phi = \phi_{\ell}, \phi' = \phi'_{\ell} \) for all \( \ell \) with integers \( \phi, \phi' \) satisfying \( \phi + \phi' + 1 \equiv 0(\text{mod } N) \) for odd \( N = 2M + 1 \). As before, \( q^{-2} = \omega \) with \( q = \omega^{-1/2}(=\omega^M) \). By (4.17) and identifying the spectral parameter \( t = s^2 \), the superintegrable \( \tau^{(2)} \) model (2.29) is equivalent to the homogeneous XXZ chain with \( \rho = \omega^{m-M}, \nu = 1, \phi \equiv -1 - m - 2n_0, \phi' \equiv m + 2n_0, \varepsilon = M \) in (4.9):
\[
\mathcal{L}_{\ell}(s) = \left( q^{1+2m} s K^{-1/2} - s^{-1} K^{1/2} \right) \left( q - q^{-1} \right) e^+ - s K^{1/2} - q^{-1-2m} s^{-1} K^{-1/2} \right) \left( q - q^{-1} \right) e^-
\]
for all \( \ell \). Here \( K, e^{\pm} \) are in (4.16), expressed by
\[
K^{1/2} = q^{(1+2m+2n_0)\tilde{Z}^{-1/2}}, \quad e^+ = q^{-1/2} \frac{(q^{-m-2n_0}\tilde{Z}^{1/2} - q^{m+2n_0}\tilde{Z}^{-1/2})}{q - q^{-1}} \tilde{X},
\]
\[
e^- = q^{1/2} \frac{(q^{1+m+2n_0}\tilde{Z}^{-1/2} - q^{-1-m-2n_0}\tilde{Z}^{1/2})}{q - q^{-1}} \tilde{X}^{-1},
\]
where \( \tilde{X}, \tilde{Z} \) are the local Weyl operators (1.2). We write the second Onsager-algebra operator \( \tilde{H}_1 \) of (2.44) in the form
\[
\tilde{H}_1 = -2 \sum_{\ell} S^z_{\ell}, \quad S^z(=S^z_{\ell}) := \sum_{j=1}^{N-1} \frac{\omega^{(m+2n_0)j} \tilde{Z}^j}{1 - \omega^{-j}},
\]
\[\text{doi:10.1088/1742-5468/2009/08/P08012} \]
By the equality \( \sum_{j=1}^{N-1} \omega^{kj}(1 - \omega^{-j})^{-1} = (N - 1 - 2k)/2 \) for \( 0 \leq k \leq N - 1 \), one finds

\[
S^z(e^k) = \frac{N - 1}{2} - k, \quad e^k := \{-m-2n_0+k\} \quad (k = 0, \ldots, N - 1).
\]

(4.19)

Using the basis \( e^k \)s, the operators \( K, e^\pm \) in (4.18) are now expressed by

\[
K^{1/2}(e^k) = q^{-S^z}e^k, \quad e^+(e^k) = q^{-1}[N - 1 - k]e^{k+1}, \quad e^-(e^k) = q^{1/2}[k]e^{k-1},
\]

(4.20)

which is similar to the spin-(\( N - 1 \))/2 \( U_q(sl_2) \) representation (4.20) via the linear isomorphism of \( C^N, e^k \mapsto q^{k/2}e^{N-k} \). Hence we have shown the following equivalent relation:

**Lemma 4.1.** The superintegrable \( \tau^{(2)} \) model (2.29) is equivalent to the homogeneous XXZ chain (4.18) with the spin-(\( N - 1 \))/2 \( U_q(sl_2) \) representation (4.20). Furthermore the Onsager-algebra operator \( -H_1/2 \) in (2.44) is the total \( S^z \) operator with \(-2S^z \equiv (1 + 2m + 4n_0)L + 2Q \).

In particular, when \( m = M \), the periodic XXZ chain (4.18) is equivalent to the usual spin-(\( N - 1 \))/2 chain for the representation (4.7) \( \alpha = N \), i.e. the homogeneous XXZ chain (4.2) and (4.8), with \( \phi = \phi' = M, \varepsilon = 0, \nu = \rho = 1 \) ([47] (4.5)–(4.9)), where the \( sl_2 \)-loop-algebra symmetry is known to exist in the sector \( 2S^z \equiv 0 \) (mod \( N \)) ([41, 46]). In the general case, by (4.14), the \( \tau^{(2)} \) model (2.29) possesses the \( sl_2 \)-loop-algebra symmetry in the sector \( K = q^{-2r}\rho^L = \pm 1 \), which by (2.25), is equivalent to

\[
2S^z \equiv 2r + (2m + 1)L \equiv 0 \Leftrightarrow P_b = P_a \equiv Q + 2n_0L - r \equiv 0 \pmod{N}.
\]

(4.21)

We now employ the algebraic-Bethe-ansatz techniques to determine the Bethe states as in the \( m = M, n_0 = r = 0 \) case ([46] section 3). The pseudo-vacuum of the \( \tau^{(2)} \) model (2.29) is defined by

\[
\Omega^+ (= \Omega^+_+) := \otimes_{j=1}^L e^{N-1} \quad \text{or} \quad \Omega^- (= \Omega^-_-) := \otimes_{j=1}^L e^0.
\]

(4.22)

where \( e^k \)s are the basis in (4.19). Then

\[
C(t)\Omega^+ = 0, \quad A(t)\Omega^+ = h(\omega^{-1}t)\Omega^+, \quad D(t)\Omega^+ = \omega^mLh(t)\Omega^+;
\]

\[
B(t)\Omega^- = 0, \quad A(t)\Omega^- = h(t)\Omega^-, \quad D(t)\Omega^- = \omega^{(1+m)L}h(\omega^{-1}t)\Omega^-,
\]

where \( A, B, C, D \) are the entries of the monodromy matrix (2.3), and \( h(t) \) is the polynomial in (2.30). One may study the eigenvalues and eigenstates of the \( \tau^{(2)} \) model using its ABCD-algebra structure ([46] section 2). The Bethe states, \( \psi^\pm(= \psi^\pm(\psi^\pm_1, \ldots, \psi^\pm_{2L})) \), are defined by\(^{10}\)

\[
\psi^+ = \prod_{j=1}^{J} B(-\omega\psi^+_j)^{-1}\Omega^+, \quad (P^+_a = 0, P^+_b \equiv mL + r - J);
\]

\[
\psi^- = \prod_{j=1}^{J} C(-\omega\psi^-_j)^{-1}\Omega^-, \quad (P^-_a \equiv -(1 + m)L - r - J, P^-_b = 0),
\]

(4.23)

\(^{10}\) The results in the \( m = n_0 = r = 0 \) case were given in [46] section 3.1, where the vector \( f_k \) is \( \sqrt{N}(-k) \) in this paper, and \( m, h_1(t)L, h_2(t)L, F(t), -t^{-1}_j \) in formulas (3.4)–(3.9) there corresponds respectively to \( J, h(t), h(t), F(t), \omega v_j \) in this paper.
where \( v_j^\pm \)'s form a solution of the Bethe equation \( (2.31) \) with the \( \tau^{(2)} \) eigenvalue \( (2.32) \). Note that the \( Z_N \) charges of the above \( \psi^+, \psi^- \) are \( Q = -(1 + m + 2n_0)L - J, -(m + 2n_0)L + J \), respectively. We now identify the Bethe states in \( (4.23) \) with the basis elements at \( k' = \infty \) in \( (2.42) \). By using \( (2.40) \), one finds the expression of \( H_1 \) eigenvalues from the formula \( (4.45) \) at \( k' = \infty \): \[
abla = (N - 1 - 2m)L + 2(P_b - P_a - P_\mu) + N \sum_{i=1}^{m_E} (s_i - 1). \tag{4.24}
abla
\]

The pseudo-vacuum \( \Omega^+ \) (or \( \Omega^- \)) is a state in \( \otimes^L \mathbb{C}^N \) with the maximum (minimum, respectively) \( -2S^\pm \). As in the case of the six-vertex model at roots of unity \([26]–[28]\), the Bethe state \( \psi^\pm \) in \( (4.23) \) is characterized as the vector with maximum or minimum \( -2S^\pm \) in \( \mathcal{E}_{F,P_a,P_b}^{\pm} \), by which the relation \( (4.24) \) in turn yields \( \psi^+ = \vec{v}(+, \ldots, +; \infty) \) or \( \psi^- = \vec{v}(-, \ldots, -; \infty) \). The condition \( (4.21) \) is characterized as the sector with \( P_a = P_a^\pm = 0, P_b = P_b^\pm = 0 \) in \( (4.23) \), where we define the Bethe states \( \psi^\pm \) with \( v_j = v_j^\pm \) for all \( j \). By a similar argument in the case of the spin-\((N - 1)/2 \) XXZ chain (see \([41] \) or \([46] \) section 4.2)\(^{11} \), the operators \( S^{\pm(N)}, T^{\pm(N)} \) in \( (4.13) \) and \( H^{(N)} = -2S^z/N \) on \( \mathcal{E}_{F,P_a,P_b} \) form an \( sl_2[z, z^{-1}] \) representation, having the evaluation polynomial \( P(t)/N \) for the \( t^N \) polynomial \( P(t) \) in \( (2.33) \). The \( sl_2 \)-loop-algebra generators are described by \( (4.15) \) when using \( \psi^+ \), and with the indices 0, 1 interchanged in the case \( \psi^- \). This \( sl_2[z, z^{-1}] \) representation is induced from a \((\oplus E \oplus sl_2) \) structure on \( \mathcal{E}_{F,P_a,P_b} \), then evaluating \( z \) on roots \( t_i^N \) of \( P(t) \), together with the spin-\(1/2 \) representation of \( sl_2 \). Note that, as the \( sl_2[z, z^{-1}] \) structures of \( \mathcal{E}_{F,P_a,P_b} \), the \( sl_2 \)-loop-algebra symmetry induced from the XXZ chain is different from \( \rho_{\infty} \) in \( (2.50) \) where the evaluation values \( e^{i\theta_i} \) (\( \neq t_i^N \)) are defined in \( (2.46) \). However, both \( sl_2[z, z^{-1}] \) representations share the same highest or lowest weight vector, \( \psi^\pm = \vec{v}(\ldots, \pm; \infty) \), hence they both give rise to the same underlying \((\oplus E \oplus sl_2) \) structure of \( \mathcal{E}_{F,P_a,P_b} \), determined by \( e^\pm z^k (0 \leq k < m_E) \) on the highest or lowest weight vector, and presented by the basis in \( (2.42) \) at \( k' = \infty \).

Using \( (3.12) \) and the duality \( (3.19) \), we may also connect the Onsager-algebra symmetry at \( k' = 0 \) with the \( sl_2 \)-loop-algebra symmetry of the XXZ chain through the \( \tau^{(2)} \)-face-model. The \( L \)-operator \( (3.5) \) of a homogeneous superintegrable \( \tau^{(2)} \)-face-model with a vertical rapidity \( p \) \( (2.35) \) is expressed by \( c^N = \sum_{n \in Z_N} C[n])\)-Weyl operators:

\[
L_\ell(t) = \left( \begin{array}{c}
1 - tZ \\
-\omega_{2n_0} t (1 - \omega^m Z) X^{-1} \end{array} \right) \left( t = \omega^m \frac{t}{L_p} \right), \tag{4.25}
\]

for all \( \ell \). Through the dual map \( (3.9) \) and the linear isomorphism \( (3.11) \), the above model is equivalent to the homogeneous superintegrable \( \tau^{(2)} \) model with the vertical rapidity \( p^* \) and boundary condition \( r^* = r \), having the \( L \) operator \( L_\ell(t^*) \) in \( (2.29) \) where \( \ell, m, n_0 \) \( t \) are replaced by \( m^* (= m + 2n_0), n_0^* (= -n_0) \), and \( t^* (= \omega^m t^* p^{-1} = \omega^{2n_0} t_{p^*}) \), respectively. By Lemma 4.1, the above \( \tau^{(2)} \) model is equivalent to the homogeneous spin-\((N - 1)/2 \) XXZ chain \( (4.20) \) via the identification \( (4.19) \) for \( m^*, n_0^* \). Equivalently, the \( \tau^{(2)} \)-face-model \( (4.25) \)

\(^{11} \) The variable \( t = q^2 \) in \([46] \) Theorem 4 differs from \( t \) in this paper by the factor \( q \), but with the same \( t^N \)-polynomial: \( P(t) \) in this paper \( = P_{\beta t}(t^N) \) in \([46] \) \( (4.32) \).
is the homogeneous XXZ chain defined by the L operator

$$\mathcal{L}_F(s) = \left( q^{1 + 2m + 4n_0}sK^{-1/2} - s^{-1}K^{1/2} \right) \left( q - q^{-1} \right) e^+ - \left( q^{1 + 2m - 4n_0}s^{-1}K^{-1/2} \right) \left( q - q^{-1} \right) e^-,$$

and the spin-($N - 1)/2$ representation (4.20) of $K, e^\pm$ on $\mathbb{C}^N$, where $e^{k} := \{-m + k\}$ (4.27) where $S^z := \sum_{j=1}^{N-1} \frac{\omega_j^m \omega_j^m}{1 - \omega_j^m}$. The second Onsager-algebra generator $H_1$ for $\hat{T}_F(q; p, p)$ in section 3.3 becomes $H_1 = -2\sum_{\ell} S^z_\ell$. Through the isomorphism $\Theta$ in (3.21), the first Onsager-algebra generator $H_0$ in (4.24) is identified with $H_1$, $H_0 = \Theta^{-1}H_1\Theta$, by (3.26).

Indeed $\Theta^{-1}S^z_\ell\Theta$ is equal to the $\ell$th local operator in $H_0$. By (3.15), the charge and boundary condition of $\tau^{(2)}_F$ and $\tau^{(2)}$ are interchanged, and we will make the identification, $(r, \mathcal{Q}) = (Q, r)$, in a later discussion. By the algebraic-Bethe-ansatz method, the $\tau^{(2)}_F$ model (4.25) has the pseudo-vacuum (4.22) and Bethe states $\phi^\pm := \phi^+(v^+_1, \ldots, v^+_J)$:

$$\begin{align*}
\phi^+ &= \prod_{j=1}^{J} \mathcal{B}(-\omega v^+_j)^{-1} \Omega^+,
\phi^- &= \prod_{j=1}^{J} \mathcal{C}(-\omega v^-_j)^{-1} \Omega^-,
\end{align*}$$

where $m^* = m + 2n_0$, $A, B, C, D$ are the entries of monodromy matrix (3.7), and $v^+_j$s satisfy the Bethe equation (2.31) for $\tau^{(2)}_F(t) = \tau^{(2)}_F(t)$ expressed by (2.32) via (3.15). Here we use the equivalence (3.12) and the relation of quantum numbers in (3.24). The Bethe state $\phi^+, \phi^-$ in (4.28) are with the boundary condition $rting (1 + m^*)L - J, -2\theta \Theta$ in $\Theta(\mathcal{E}_{2 \tau^+, \tau^+, \tau^+})$. Then follows $\Theta^{-1}(\phi^+ \equiv \vec{v}(+, \ldots, +; 0), \Theta^{-1}(\phi^- \equiv \vec{v}(-, \ldots, -; 0)$ by (3.25). In the sector (4.21), equivalently $P_a = \vec{P}_a^+ = 0, P_b = \vec{P}_b^- = 0$ in (4.28), the Bethe states $\phi^\pm$ for $v_j = v^+_j$ correspond to $\vec{v}(\ldots, \pm, \ldots; 0) \in \mathcal{E}_{F, P_a, P_b}$. Indeed as before, they are the highest and lowest weight vectors of two $\mathfrak{s}_2^2[z, z^{-1}]$ representations on $\mathcal{E}_{F, P_a, P_b}$, one from the $\mathfrak{s}_2^2$-loop-algebra symmetry of the XXZ chain using the evaluation polynomial $\mathcal{P}(t)/\mathcal{N}$, the other by $\rho_0$ in (2.50) with $e^{\theta_0}(\not=t^\mathcal{N})$ in (2.46) as the evaluation values. The decomposition of $\mathcal{E}_{F, P_a, P_b}$ in (4.22) at $k' = 0$ provides the underlying $(\oplus \mathfrak{s}_2^2)$ structure for both $\mathfrak{s}_2^2[z, z^{-1}]$ structures. In this situation, the Bethe states $\psi^\pm, \phi^\pm$ in (2.23) and (4.28) belong to the same $\tau^{(2)}$ eigenspace $\mathcal{E}_{F, P_a, P_b}$ with $\psi^\pm = \vec{v}(\ldots, \pm, \ldots; \infty)$, $\phi^\pm = \vec{v}(\ldots, \pm, \ldots; 0)$. Indeed, the $(\oplus \mathfrak{s}_2^2)$ structure of $\mathcal{E}_{F, P_a, P_b}$ at $k' = 0, \infty$ of the $\mathfrak{s}_2^2$-loop-algebra symmetry are identified under the correspondence, $\vec{v}(s_1, \ldots, s_{m^*}; \infty) \mapsto \vec{v}(s_1, \ldots, s_{m^*}; 0)$. We now summarize the results of this section as follows:

**Proposition 4.1.** (1) For odd $N$ and a rapidity $p = (x_p, y_p, \mu_p)$ with $x_p = \omega^m y_p, \mu_p = \omega^m$, the homogeneous superintegrable $\tau^{(2)}_F$ model (2.29) (or $\tau^{(2)}_F$ model (4.25)) with the vertical rapidity $p$ is equivalent to the homogeneous spin-($N - 1)/2$ XXZ chain (4.18) (4.26) resp.) with the $U_q(\mathfrak{s}_2^2)$ representation (4.20) via the basis (4.19) ((4.27) resp.).

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(II). The Bethe states of the $\tau^{(2)}$ model, $\psi^\pm$ in (4.23), are equal to $\tilde{v}(\ldots, \pm, \ldots; \infty)$ in $\mathcal{E}_{\tau_+^\pm, \tau_-^\pm}$ with maximum or minimum $-2S^z(=H_1)$. Through the isomorphism $\Theta$ in (3.14), the Bethe states of the $\tau_F^{(2)}$ model, $\phi^\pm$ in (4.28), corresponds to the vector in $\mathcal{E}_{\tau_F^{\pm}, \tilde{P}_a^\pm, \tilde{P}_b^\pm}$ with maximum or minimum $-2S^z(=H_0)$, $\Theta^{-1}(\phi^\mp) = \tilde{v}(\ldots, \pm, \ldots; 0)$.

(III). In the sector (4.21), equivalently $P_a = P_a^\pm = \tilde{P}_a^\pm = 0$, $P_b = P_b^\pm = \tilde{P}_b^\pm = 0$, the Bethe states, $\psi^\pm$ and $\Theta^{-1}(\phi^\mp)$, with $v_j = v_j^\pm$ in (4.23) and (4.28) belong to the same $\tau^{(2)}$ eigenspace $\mathcal{E}_{P_a, P_b}$.

Remark. (1) The $P_a^\pm, P_b^\pm, \tilde{P}_a^\pm, \tilde{P}_b^\pm$ in (4.23) and (4.28) provide the four cases in the last two constraints in (2.25). Indeed, the cases, $P_b^- = 0$, and $\tilde{P}_b^- = 0$, correspond to those in (2.25) with $P_b \equiv 0$, $J + P_b \equiv (m + 2n_0)L + Q, mL + r$, respectively. Hence the Bethe vectors (4.23) of the $\tau^{(2)}$ model derived from the algebraic-Bethe-ansatz cover only one ‘half’ of states, not the whole theory as previously indicated in [46] section 3.1. The other half of the states are those corresponding to Bethe states of the $\tau_F^{(2)}$ model in (4.28).

(2) We consider only the odd $N$ case in Proposition 4.1. However, the pseudo-vacuum discussion in (4.19), (4.27) and (4.22) about the ground state is valid for an arbitrary $N$ with $F = 1$, $P_a$ and $P_b$ in (4.23) or (4.28) for $J = 0$. Therefore the Bethe states’ characterization in (II) is likely true for all $N$, hence a more complete theory about the $sl_2$-loop-algebra symmetry of $\mathcal{E}_{P_a, P_b}$ would be expected by a general argument.

5. Concluding remarks

We establish a Ising-type duality relation in $N$-state CPM as a generalization of the usual Kramers–Wannier duality of the Ising model when $N = 2$. The approach is based on the functional relation method in CPM. We first find the duality of the $\tau^{(2)}$ model under a dual correspondence of $k', k^{−1}$ rapidities and quantum spaces for the temperature-like parameter $k'$, under the constraint of interchanging the $\mathbb{Z}_N$-charge and skewed boundary condition. Then the duality of CPM follows from the dual connection between Boltzmann weights at $k'$ and $k^{−1}$. The method is carried out by calculations in the Fourier transform about the transfer matrix, as the duality discussion in [8] on a special superintegable case about the vertical-interfacial tension of CPM. We can incorporate this duality into the CPM and $\tau^{(2)}$ model over the dual lattice, as well as that of the face $\tau^{(2)}$ model. The duality in this work not only establishes a complete theory about the duality symmetry of a general CPM, but also provides a useful means for gaining insights on the structure, generally not available within the superintegrable case alone. In the homogeneous superintegrable case, the duality symmetry fits nicely with the Onsager-algebra and $sl_2$-loop-algebra symmetry about the degeneracy of the $\tau^{(2)}$ model. In particular, the analysis of Bethe states leads to the understanding of eigenvectors in CPM in the sector (4.21) through the algebraic-Bethe-ansatz method for the odd $N$ case. The approach apparently works also in the general situation as suggested in remark (2) of Proposition 4.1. A more complete theory

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about eigenvectors of CPM for all sectors and an arbitrary $N$ is now under consideration along this line. In view of the fundamental importance of Kramers–Wannier duality in the study of the Ising model, regardless of the complicated nature of techniques in CPM, further development on the duality found in the present work would be expected, especially concerning the structure at the critical $k' = \pm 1$ case. This problem is outside the scope of this paper and will be considered elsewhere.

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References

[1] Albertini G, McCoy B M and Perk J H H, Eigenvalue spectrum of the superintegrable chiral Potts model, 1989 Integrable System in Quantum Field Theory and Statistical Mechanics (Advanced Studies in Pure Mathematics vol 19) (Boston, MA: Kinokuniya Academic, Academic) pp 1–55

[2] Au-Yang H, McCoy B M, Perk J H H, Tang S and Yan M L, Commuting transfer matrices in chiral Potts models: solutions of the star–triangle equations with genus >1, 1987 Phys. Lett. A 123 219

[3] Au-Yang H, McCoy B M, Perk J H H and Tang S, Solvable models in statistical mechanics and Riemann surfaces of genus greater than one, 1988 Algebraic Analysis vol 1, ed M Kashiwara and T Kawai (San Diego, CA: Academic) pp 29–40

[4] Au-Yang H and Perk J H H, Onsager’s star–triangle equation: master key to integrability, 1989 Integrable System in Quantum Field Theory and Statistical Mechanics (Advanced Studies in Pure Mathematics vol 19) (Boston, MA: Kinokuniya Academic, Academic) pp 57–94

[5] Au-Yang H and Perk J H H, Eigenvectors the superintegrable model I: $\mathfrak{sl}_2$ generators, 2008 J. Phys. A: Math. Theor. 41 275201 [arXiv:0710.5257]

[6] Baxter R J, 1982 Exactly Solved Models in Statistical Mechanics (New York: Academic)

[7] Baxter R J, Free energy of the solvable chiral Potts model, 1988 J. Stat. Phys. 52 639

[8] Baxter R J, Superintegrable chiral Potts model: thermodynamic properties, an ‘Inverse’ model, and a simple associated Hamiltonian, 1989 J. Stat. Phys. 57 1

[9] Baxter R J, Chiral Potts model: eigenvalues of the transfer matrix, 1990 Phys. Lett. A 146 110

[10] Baxter R J, Calculation of the eigenvalues of the transfer matrix of the chiral Potts model, 1991 Proc. Fourth Asia-Pacific Physics Conf. (Seoul 1990) vol 1 (Singapore: World-Scientific) pp 42–58

[11] Baxter R J, Chiral Potts model with skewed boundary conditions, 1993 J. Stat. Phys. 73 461

[12] Baxter R J, Corner transfer matrices of the chiral Potts model II: The triangular lattice, 1993 J. Stat. Phys. 70 535

[13] Baxter R J, Interfacial tension of the chiral Potts model, 1994 J. Phys. A: Math. Gen. 27 1837

[14] Baxter R J, The ‘inversion relation’ method for obtaining the free energy of the chiral Potts model, 2003 Physica A 322 407 [arXiv:cond-mat/02121075]

[15] Baxter R J, Transfer matrix functional relation for the generalized $\tau_2(t_q)$ model, 2004 J. Stat. Phys. 117 1 [arXiv:cond-mat/0409493]

[16] Baxter R J, The order parameter of the chiral Potts model, 2005 J. Stat. Phys. 120 1 [arXiv:cond-mat/0501226]

[17] Baxter R J, Derivation of the order parameter of the chiral Potts model, 2005 Phys. Rev. Lett. 94 130602 [arXiv:cond-mat/0501227]

[18] Baxter R J, Bazhanov V V and Perk J H H, Functional relations for transfer matrices of the chiral Potts model, 1999 Int. J. Mod. Phys. B 14 803

[19] Baxter R J, Perk J H H and Au-Yang H, New solutions of the star–triangle relations for the chiral Potts model, 1988 Phys. Lett. A 128 138

[20] Bazhanov V V and Stroganov Yu G, Chiral Potts model as a descendant of the six-vertex model, 1990 J. Stat. Phys. 59 799

doi:10.1088/1742-5468/2009/08/P08012 34
The structure of quotients of the Onsager algebra by closed ideals, 2000 J. Phys. A: Math. Gen. 33 3275 [arXiv:math.QA/9911018]

Davies B, Onsager’s algebra and superintegrability, 1990 J. Phys. A: Math. Gen. 23 2245

Davies B, Onsager’s algebra and the Dolan–Grady condition in the non-self case, 1991 J. Math. Phys. 32 2945

DeConcini C and Kac V G, Representations of quantum groups at roots of unity, 1990 Operator Algebra, Unitary Representations, Enveloping Algebras, and Invariant Theory (Paris (1989) (Progress in Mathematics vol 92) (Boston, MA: Birkhäuser) pp 471–506

Deguchi T, Regular XXZ Bethe states at roots of unity—as highest weight vectors of the sl2 loop algebra at roots of unity, 2005 arXiv:cond-mat/0503564v3

Deguchi T, Fabricius K and McCoy B M, The sl2 loop algebra symmetry for the six-vertex model at roots of unity, 2001 J. Stat. Phys. 102 701 [arXiv:cond-mat/9912141]

Fabricius K and McCoy B M, Functional equations and fusion matrices for the eight vertex model, 2004 Publ. RIMS 40 905 [arXiv:cond-mat/0311122]

Faddeev L D, How algebraic Bethe Ansatz works for integrable models, 1998 Quantum Symmetries/Symmetries Quantiques (Proceedings of the Les Houches Summer School, Session LXIV) ed A Connes, K Gawedzki and J Zinn-Justin (Amsterdam: North-Holland) pp 149–208

Fateev V A and Zamolodchikov A B, Self-dual solutions of the star–triangle relations in the superintegrable chiral Potts model, 1985 Phys. Lett. A 92 37

von Gehlen G and Rittenberg R, Zn-symmetric quantum chains with infinite set of conserved charges and Zn zero modes, 1985 Nucl. Phys. B 257 351

Howes S, Kadanoff L P and den Nijs M, Quantum model for commensurate–incommensurate transitions, 1983 Nucl. Phys. B 215 169

Kramers H A and Wannier G H, Statistics of the two-dimensional ferromagnet, 1941 Phys. Rev. 60 252

Kirillov A N and Reshetikhin N Yu, Exact solution of the integrable XXZ Heisenberg model with arbitrary spin: I. The ground state and the excitation spectrum, 1987 J. Phys. A: Math. Gen. 20 1565

Korepin V E, Bogoliubov N M and Izegin A G, Quantum Inverse Scattering Method. Recent Developments (Cambridge Lecture Notes in Physics vol 151) ed J Hietarinta and C Montonen (Berlin: Springer) pp 61–119

Kulish P P and Sklyanin E K, Yang Baxter equation and representation theory, 1981 Lett. Math. Phys. 5 393

Matveev V B and Simonov A O, Some comments on the solvable chiral Potts model, 1990 Lett. Math. Phys. 19 179

McCoy B M, Perk J H H, Tang S and Sah C H, Commuting transfer matrices for the four-state self-dual chiral Potts model with a genus-three uniformizing Fermat curve, 1987 Phys. Lett. A 125 9

McCoy B M and Roan S S, Excitation spectrum and phase structure of the chiral Potts model, 1990 Phys. Lett. A 150 347

Nishino A and Deguchi T, The L(sl2) symmetry of the Bazhanov–Stroganov model associated with the superintegrable chiral Potts model, 2006 Phys. Lett. A 356 366 [arXiv:cond-mat/0605551]

Nishino A and Deguchi T, An algebraic derivation of the eigenspaces associated with an Ising-like spectrum of superintegrable chiral Potts model, 2008 arXiv:0806.1268

Roan S S, Onsager’s algebra, loop algebra and chiral Potts model, 1991 Preprint MPI 91–70 Max-Plank-Inst. für Math., Bonn

Roan S S, The Onsager algebra symmetry of τ(1)-matrices in the superintegrable chiral Potts model, 2005 J. Stat. Mech. P0007 [arXiv:cond-mat/0505698]

Roan S S, The Q-operator for root-of-unity symmetry in six vertex model, 2006 J. Phys. A: Math. Gen. 39 12303 [arXiv:cond-mat/0602375]

Roan S S, Fusion operators in the generalized τ(2)-model and root-of-unity symmetry of the XXZ spin chain of higher spin, 2007 J. Phys. A: Math. Theor. 40 1481 [arXiv:cond-mat/0607258]

Roan S S, The transfer matrix of superintegrable chiral Potts model as the Q-operator of root-of-unity XXZ chain with cyclic representation of Un(sl2), 2007 J. Stat. Mech. P09021 [arXiv:0705.2856]

Roan S S, On the equivalent theory of the generalized τ(2)-model and the chiral Potts model with two alternating vertical rapidities, 2007 arXiv:0710.2764

doi:10.1088/1742-5468/2009/08/P08012 35
[49] Roan S S, Bethe equation of $\tau^{(2)}$-model and eigenvalues of finite-size transfer matrix of chiral Potts model with alternating rapidities, 2008 J. Stat. Mech. P10001 [arXiv:0805.1585]

[50] Roan S S, On $\tau^{(2)}$-model in chiral Potts model and cyclic representation of quantum group $U_q(sl_2)$, 2009 J. Phys. A: Math. Theor. 42 072003 [arXiv:0806.0216]