COMPACTNESS OF THE $\bar{\partial}$-NEUMANN PROBLEM ON CONVEX DOMAINS

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Abstract. The $\bar{\partial}$-Neumann operator on $(0,q)$-forms $(1 \leq q \leq n)$ on a bounded convex domain $\Omega$ in $\mathbb{C}^n$ is compact if and only if the boundary of $\Omega$ contains no complex analytic (equivalently: affine) variety of dimension greater than or equal to $q$.

1. Introduction.

Let $\Omega$ denote a bounded pseudoconvex domain in $\mathbb{C}^n$. For $1 \leq q \leq n$, let $L^2(0,q) \Omega$ denote the space of $(0,q)$-forms with square integrable coefficients, with the norm $\|\sum' a_I d\bar{z}_I\|^2 = \sum' \int_{\Omega} |a_I|^2 dV(z)$, where the prime indicates the summation over strictly increasing $q$-tuples. The $\bar{\partial}$-Neumann operator $N_q$ is the (bounded) inverse of the (unbounded) self-adjoint, surjective operator $\bar{\partial}\bar{\partial}^* + \bar{\partial}^* \bar{\partial}$. We refer the reader to [FK], [Ko], [Kr2], and the recent survey [BS3] for background on the $\bar{\partial}$-Neumann problem.

Compactness of the $\bar{\partial}$-Neumann problem is a basic property with many useful consequences. In the case of domains with smooth boundary, it implies global regularity of the $\bar{\partial}$-Neumann problem (in the sense of preservation of the $L^2$-Sobolev spaces), see [KN]. Also, the Fredholm theory for Toeplitz operators is a direct consequence of the compactness of the $\bar{\partial}$-Neumann problem ([V], [HI]). In a related context, whether or not the $\bar{\partial}$-Neumann problem is compact has ramifications for certain $C^*$-algebras of operators naturally associated with a domain in $\mathbb{C}^n$; compare for example [Sa].

Catlin [Ca2] proved compactness of the $\bar{\partial}$-Neumann operator on smoothly bounded domains whose boundary satisfies Property (P). The boundary of a domain $\Omega$ satisfies Property (P) if for every positive number $M$ there is a plurisubharmonic function $\lambda \in C^\infty(\Omega)$ with $0 \leq \lambda \leq 1$, such that for all $z \in \partial\Omega$ and $w \in \mathbb{C}^n$, $\sum_{\alpha,\beta=1}^n (\partial^2 \lambda(z)/\partial z_\alpha \partial \bar{z}_\beta)w_\alpha \bar{w}_\beta \geq M|w|^2$ ([Ca2]). Property (P) was studied systematically under the name of $B$-regularity in [Si] (in the context of arbitrary compact sets in $\mathbb{C}^n$). It was recently observed that Catlin’s result remains true when no boundary smoothness at all is assumed: the $\bar{\partial}$-Neumann problem is compact on a bounded pseudoconvex domain whose boundary is a $B$-regular set ([St]). (Under the additional assumption that the domain is hyperconvex, compactness of the $\bar{\partial}$-Neumann problem had been shown earlier in [HI].)

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In this article, we discuss compactness of the $\bar{\partial}$-Neumann problem on bounded convex domains. We obtain a complete characterization of compactness by the absence from the boundary of complex analytic (equivalently: affine) varieties of appropriate dimensions. A closely related (in fact, in the context of convex domains, equivalent) question is that of compact solution operators for $\bar{\partial}$. For $1 \leq q \leq n$, consider $\bar{\partial}$ as an unbounded operator from $L^2_{(0,q-1)}(\Omega)$ to $L^2_{(0,q)} \cap \ker \bar{\partial}$. A bounded linear operator $S_q$ from $L^2_{(0,q)} \cap \ker \bar{\partial}$ to $L^2_{(0,q-1)}(\Omega)$ is called a solution operator for $\bar{\partial}$ on $(0,q)$-forms if $\bar{\partial}S_q u = u$ for all $u \in L^2_{(0,q)} \cap \ker \bar{\partial}$.

The following terminology will be convenient: an affine variety of dimension $q$ is a (relatively) open subset of a complex affine subspace of $\mathbb{C}^n$ of dimension $q$.

Theorem 1.1. Let $\Omega$ be a bounded convex domain in $\mathbb{C}^n$. Let $1 \leq q \leq n$. The following are equivalent:

1. There exists a compact solution operator for $\bar{\partial}$ on $(0,q)$-forms.
2. The boundary of $\Omega$ does not contain any affine variety of dimension greater than or equal to $q$.
3. The boundary of $\Omega$ does not contain any analytic variety of dimension greater than or equal to $q$.
4. The $\bar{\partial}$-Neumann operator $N_q$ is compact.

The implication (4) $\Rightarrow$ (1) holds in general (and is well known): compactness of the $\bar{\partial}$-Neumann operator implies compactness of the canonical solution operator. In fact, the formula $N_q = (\bar{\partial}^* N_q)^* (\bar{\partial}^* N_q) + (\bar{\partial}^* N_{q+1})^* (\bar{\partial}^* N_{q+1})$ (see [R], [FK]) shows that $N_q$ is compact if and only if the canonical solution operators $\bar{\partial}^* N_q$ and $\bar{\partial}^* N_{q+1}$ are compact. Note that in statement (1), it is the same to say that the canonical solution operator is compact, since compactness is preserved by projection onto the orthogonal complement of the kernel of $\bar{\partial}$.

Henkin and Iordan [HI] recently showed that (on a bounded convex domain) the $\bar{\partial}$-Neumann operators are compact for $1 \leq q \leq n$ if there are no one-dimensional analytic varieties contained in the boundary of the domain (see [Si] for the smooth case). It has been known for some time that analytic discs in the boundary of a smooth pseudoconvex domain in $\mathbb{C}^2$ obstruct compactness of the $\bar{\partial}$-Neumann operator. Specific examples for failure of compactness are given in [Li] and [Kr2]. These examples are pseudoconvex Reinhardt domains in $\mathbb{C}^n$. Theorem 5 in [Sa] implies that on a pseudoconvex Reinhardt domain, compactness of the canonical solution operator on $(0,1)$-forms is incompatible with analytic discs on the boundary. To what extent analytic varieties in the boundary obstruct compactness of the $\bar{\partial}$-Neumann problem on “general” domains seems to be open. On the other hand, it is known that obstructions to compactness of the $\bar{\partial}$-Neumann problem can be more subtle than analytic varieties in the boundary (so that the above characterization is false without some assumption on the domain): Matheos [Mt] has recently shown that there exist smooth bounded pseudoconvex Hartogs domains in $\mathbb{C}^2$ without discs in their boundaries, whose $\bar{\partial}$-Neumann operators are nonetheless not compact.

We remark that we make no explicit assumption on smoothness of the boundary. However, convexity implies that the boundary is Lipschitz (see e.g. [Mz], §1.1.8).

The remainder of the paper is organized as follows. In Section 2, we briefly discuss the
equivalence of conditions (2) and (3) in Theorem 1.1. This may be part of the folklore, but we include the (simple) argument for completeness. Section 3 contains the proof that (2) implies (4). The proof of Theorem 1.1 is completed in Section 4, where we show that (1) implies (2). We conclude the paper with some additional remarks in Section 5.

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2. Varieties in the Boundary of a Convex Domain.

In this section we show that (2) implies (3) in Theorem 1.1 (\((3) \Rightarrow (2)\) is trivial). This may be viewed as the simplest manifestation of the general principle that on the boundaries of convex domains, questions of orders of contact with analytic varieties are determined by the orders of contact with affine subspaces ([Mc], [BS2], [Y]). Related observations may be found in [N], [Ch]. Complex manifolds in general (but smooth) pseudoconvex boundaries are studied in [BF].

We first observe the following: if \(V\) is a \(q\)-dimensional variety in \(\mathbb{C}^n\), then its convex hull \(\hat{V}\) contains an affine variety of dimension \(q\). This is clear if \(n = 1\). For general \(n\), the observation follows by induction on the dimensions as follows. If \(\hat{V}\) has non-empty interior (in \(\mathbb{C}^n\)), we are done. If the interior of \(\hat{V}\) is empty, \(V\) is contained in a real hyperplane (since then there are no \(2n\) line segments with end points in \(V\) which are linearly independent over \(\mathbb{R}\)). After a suitable change of coordinates, this hyperplane is \(\{x_n = 0\}\). By the open mapping property of non-constant holomorphic functions, applied to the restriction of the function \(z_n\) to (the regular part of) \(V\), \(V\) is contained in the complex hyperplane \(\{z_n = 0\}\). This completes the induction.

To prove that (2) implies (3) in Theorem 1.1, assume now that \(b\Omega\) contains a \(q\)-dimensional analytic variety \(V\). Let \(p_0\) be a regular point of \(V\) so that near \(p_0\), \(V\) is a \(q\)-dimensional complex manifold, and assume without loss of generality that a supporting hyperplane for \(\Omega\) at \(p_0\) is given by \(\{x_n = 0\}\). The argument in the previous paragraph shows that if \(V_{p_0}\) is the intersection of \(V\) with a small neighborhood of \(p_0\), then \(V_{p_0} \subseteq \{z_n = 0\}\). Consequently, the convex hull of \(V_{p_0}\) is likewise contained both in \(\overline{\Omega}\) and in \(\{z_n = 0\} \subseteq \{x_n = 0\}\), hence in \(b\Omega\). But by the above observation, this convex hull contains a \(q\)-dimensional affine variety.

3. Sufficient Conditions for Compactness of \(N_q\).

**Proposition 3.1.** Let \(\Omega\) be a bounded pseudoconvex domain in \(\mathbb{C}^n\). Assume that for every positive number \(M\), there exists a neighborhood \(U\) of \(b\Omega\) and a \(C^2\)-smooth function \(\lambda\) on \(U\), \(0 \leq \lambda \leq 1\), such that for all \(z \in U\), the sum of any \(q\) (equivalently: the smallest \(q\)) eigenvalues of the Hermitian form \(\left(\partial^2 \lambda(z)/\partial z_\alpha \partial \overline{z}_\beta\right)_{\alpha, \beta=1}^{n}\) is at least \(M\). Then the \(\partial\)-Neumann operator \(N_q\) on \(\Omega\) is compact.

Note that for \(q = 1\), the above condition on the Hessian of \(\lambda\) reduces to the condition that appears in Property (P). For \(q > 1\), this condition does not imply that \(\lambda\) is plurisubharmonic (while it still implies that \(\lambda\) is subharmonic).
To prove Proposition 3.1, we first note (following [H], p. 137) that the condition on \( \lambda \) implies

\[
\sum_I' \sum_{\alpha,\beta=1}^n \frac{\partial^2 \lambda(z)}{\partial z_\alpha \partial \bar{z}_\beta} f_{\alpha I} f_{\beta I} \geq M \| f \|^2
\]

for all \( z \in U \) and \( f \in \Lambda^{(0,q)}_z \), where \( \Lambda^{(0,q)}_z \) denotes the space of \((0,q)\)-forms at \( z \), and the prime indicates summation over increasing \((q-1)\)-tuples \( I \). (3.1) can be seen by using a frame (at \( z \)) where the Hessian of \( \lambda \) is diagonalized. The proof of Proposition 3.1 is now based on [Ca3], Theorem 2.1 (see also [BS3], Section 2, for a somewhat different approach to this type of estimate) and [Ca2], proof of Theorem 1. The fact that no boundary smoothness is assumed necessitates working on smooth subdomains and using a regularization procedure for the forms involved that was introduced in [St]. The details of this argument are carried out in [St], proof of Corollary 3, to which we refer the reader.

In order to show that (2) implies (4) in Theorem 1.1, it now suffices to show that if the boundary of a bounded convex domain contains no affine varieties of dimension \( q \) or greater, then the assumption in Proposition 3.1 is satisfied. This can be done by suitably generalizing the arguments in [Si], Proposition 2.4 that cover the case \( q = 1 \): the Choquet theory has to be done for a cone of functions that reflects the condition on the Hessian used in Proposition 3.1 (rather than for the cone of plurisubharmonic functions). We also need a substitute for the fact that boundary points of a convex domain \( \Omega \) are peak points for the algebra of functions holomorphic on \( \Omega \) and continuous on \( \overline{\Omega} \) when there are no varieties of positive dimension in the boundary of \( \Omega \). We now develop the necessary ideas.

**Proposition 3.2.** Let \( X \) be a compact convex subset of \( \mathbb{C}^n \), let \( z_0 \in X \), and let \( 1 \leq q \leq n \). Then there exists a complex affine subspace \( L \) of dimension \( \leq q - 1 \) through \( z_0 \) such that \( X \cap L \) is a peak set if and only if \( X \) contains no affine variety of dimension \( \geq q \) through \( z_0 \).

**Proof.** The \( \Rightarrow \) direction follows easily from the maximum modulus principle. To prove the reverse direction, we need Glicksberg’s peak set theorem (cf. [G2], pp. 58) which says that \( E \) is a peak set if and only if \( \nu_E \in H(X)^+ \) for all finite regular Borel measures \( \nu \in H(X)^+ \), where \( \nu_E \) is the restriction of \( \nu \) to \( E \).

We argue by induction. The case \( n = 1 \) is clear (but see the case \( q = n \) below). Assume the conclusion for \( n - 1 \), \( n \geq 2 \). We need to establish it for \( n \). Since there is no affine variety of dimension \( \geq q \) through \( z_0 \) and contained in \( X \), \( z_0 \) is a boundary point. Without loss of generality, assume that \( z_0 \) is the origin and \( X \subseteq \{ \text{Re } z_n \geq 0 \} \). Set \( g(z) = \exp(-\sqrt{z_n}) \) (where the square root is the principal branch). Let \( J = X \cap \{ z_n = 0 \} \). If \( q = n \), let \( L = \{ z_n = 0 \} \). Then \( J \) is a peak set: the function \( g(z) \) is a peak function.
Now assume that $1 \leq q \leq n - 1$. Then $J$ is a compact convex subset of $\mathbb{C}^{n-1}$, and by the induction assumption, there is a complex affine subspace $L \subseteq \{ z_n = 0 \} \cong \mathbb{C}^{n-1}$ of dimension $\leq q - 1$ such that $J \cap L$ is a peak set for $H(J)$. We now show that $X \cap L = J \cap L$ is a peak set for $H(X)$. Let $\nu$ be a finite regular Borel measure on $X$, $\nu \in H(X)^\perp$. Thus we have for any holomorphic polynomial $f$ and positive integer $m$ that $\int_X f \cdot g^m \, d\nu = 0$ (note that $g$, although not itself analytic in a neighborhood of $X$, is in $H(X)$). Letting $m \to \infty$, we obtain that $\int_J f \, d\nu = 0$. The convex set $J$ is polynomially convex, so the holomorphic polynomials are dense in $H(J)$ by the Oka-Weil approximation theorem. Consequently, $\nu_J \in H(J)^\perp$ and hence $\nu_{J \cap L} \in H(J)^\perp$, by Glicksberg’s theorem. Using Glicksberg’s theorem in the other direction, we conclude that $J \cap L$ is a peak set for $H(X)$. This completes the induction and the proof of Proposition 3.2.

For an open set $U \subset \mathbb{C}^n$, denote by $P_q(U)$ the set of continuous functions $\lambda$ on $U$ such that for any $z \in U$ and orthonormal set of vectors $\{ t_1, \ldots, t_q \}$ in $\mathbb{C}^n$, the function
\[
\zeta = (\zeta_1, \ldots, \zeta_q) \in \mathbb{C}^q \mapsto \lambda(z + \zeta_1 t_1 + \ldots + \zeta_q t_q)
\]
is subharmonic on $\{ \zeta \in \mathbb{C}^q; \; z + \zeta_1 t_1 + \ldots + \zeta_q t_q \in U \}$. That is, $P_q(U)$ consists of the continuous functions on $U$ that are subharmonic on each $q$-dimensional complex affine subspace. In particular, $P_1(U)$ is the set of all continuous plurisubharmonic functions and $P_n(U)$ is the set of all continuous subharmonic functions. $P_q(U)$ is a convex cone in $C(U)$ that is closed under taking the pointwise maximum of finitely many of its elements. Note that each function in $P_q(U)$ is a locally uniform limit of $C^\infty$-smooth elements in $P_q$ of slightly smaller open sets: this follows from the usual mollifier argument. Finally, it is not hard to check that $-\sum_{j=1}^{q-1} |z_j|^2 + (q - 1) \sum_{j=q}^n |z_j|^2 \in P_q(\mathbb{C}^n)$.

We now return to a bounded pseudoconvex domain $\Omega$. We denote by $P_q(b\Omega)$ the closure in $C(b\Omega)$ of functions that are in $P_q$ in a neighborhood of $b\Omega$. A probability measure $\mu$ on $b\Omega$ is said to be a $P_q$-measure for $z \in b\Omega$ if
\[
(3.2) \quad \lambda(z) \leq \int_{b\Omega} \lambda \, d\mu, \quad \lambda \in P_q(b\Omega).
\]

We refer the reader to [G1], Chapter 1 for a treatment of these measures in an abstract context, and for the elements of Choquet theory. In particular, $P_q(b\Omega)$ satisfies the properties (1.1)-(1.3) in [G1].

Let $\Omega$ be a bounded convex domain, and $z_0$ a boundary point through which there is no affine variety of dimension $q$ or higher that is contained in $b\Omega$. We claim that the only $P_q$-measure for $z_0$ is the point mass at $z_0$. Note that there is also no affine variety of dimension $\geq q$ through $z_0$ that is contained in $\overline{\Omega}$ (this is a special case of the argument at the end of Section 2). By Proposition 3.2, there is a complex affine subspace $L$ of dimension $\leq q - 1$ such that $b\Omega \cap L$ is a peak set for $H(\overline{\Omega})$. Let $f$ be the corresponding (weak) peak function. Because $f \in H(\overline{\Omega})$, $|f| \in P_q(b\Omega)$. (3.2) now shows that any $P_q$-measure $\nu$ for $z_0$ is supported on $b\Omega \cap L$. In suitable coordinates, we may assume that $z_0$ is the origin and $L \subseteq \{ z_q = \ldots = z_n = 0 \}$. In (3.2), take now $\lambda = -\sum_{j=1}^{q-1} |z_j|^2 + (q - 1) \sum_{j=q}^n |z_j|^2$. We
already know that the support of $\nu$ is contained in $L$, where $\lambda$ reduces to $-\sum_{j=1}^{q-1} |z_j|^2$. We thus obtain from (3.2) that the support of $\nu$ consists of the point $z_0$.

We next invoke Edwards’ theorem ([G1], Theorem 1.2): for every continuous function $u$ on $b\Omega$ and $z \in b\Omega$, $\inf\{\int_{\Omega} u \, d\mu; \mu$ is a $P_q$-measure for $z\} = \sup\{\lambda(z); \lambda \in P_q(b\Omega), \lambda \leq u$ on $b\Omega\}$. Because all $P_q$-measures have point support, the theorem gives

$$u(z) = \sup\{\lambda(z); \lambda \in P_q(b\Omega), \lambda \leq u$ on $b\Omega\}$$

for every function $u \in C(b\Omega)$. For $M > 0$, let $u_M(z) = -M|z|^2$. It follows from (3.3) and a compactness argument similar to the proof of Dini’s theorem that $u_M$ can be approximated uniformly on $b\Omega$ by functions in $P_q(b\Omega)$, hence by functions that are smooth and in $P_q$ in a neighborhood of $b\Omega$. In particular, there exists a neighborhood $U$ of $b\Omega$ and a function $\lambda \in P_q(U) \cap C^2(U)$ such that $0 \leq \lambda + M|z|^2 \leq 1$ on $U$ (after shrinking $U$ if necessary). The sum of the smallest $q$ eigenvalues of the Hessian of $\lambda + M|z|^2$ is at least $qM \geq M$ (because $\lambda \in P_q(U)$, the sum of the $q$ smallest eigenvalues of the Hessian of $\lambda$ is non-negative). Therefore $\Omega$ satisfies the assumptions in Proposition 3.1, and the proof that (2) implies (4) in Theorem 1.1 is complete.

4. Necessary Conditions for the Compactness of Solution Operators to $\overline{\partial}$.

In this section, we prove the implication $(1) \Rightarrow (2)$ of Theorem 1.1. One of the main tools in the proof is the Ohsawa-Takegoshi extension theorem ([OT], [O2]). We also use an idea that comes from [Ca1] (see also [DP]).

We first prove an auxiliary lemma. Denote by $K_{\Omega}(z,w)$ the Bergman kernel function of a domain $\Omega$.

**Lemma 4.1.** Let $\Omega$ be a bounded convex domain in $\mathbb{C}^n$.

(1) For any $p_0 \in b\Omega$ and $p_1 \in \Omega$, there exist constants $C > 0$ and $\delta_0 > 0$ such that

$$K_{\Omega}(p_\delta, p_\delta) \geq CK_{\Omega}(p_{2\delta}, p_{2\delta})$$

for any $\delta \in (0, \delta_0)$, where $p_\delta = p_0 + \delta(p_1 - p_0)/\|p_1 - p_0\|$. 

(2) For any sequence $\{p_j\} \in \Omega$ converging to $p_0 \in b\Omega,$

$$\lim_{j \to \infty} \frac{K_{\Omega}(z, p_j)}{\sqrt{K_{\Omega}(p_j, p_j)}} = 0,$$

locally uniformly on $\Omega$.

**Proof.** (1) Let $U$ be a ball with center $p_0$ and radius $r$ the minimum of $\text{dist}(p_1, b\Omega)$ and $\|p_1 - p_0\|/2$. Let $\vec{n} = (p_1 - p_0)/\|p_1 - p_0\|$. It is easy to see from the convexity of $\Omega$ that $T_\delta(\Omega \cap U) \subseteq \Omega$ for $0 < \delta < \|p_1 - p_0\|/2$, where $T_\delta(z) = z + \delta \vec{n}$. Let $\delta_0 = r/2$.

Then for $0 \leq \delta \leq \delta_0,$

$$K_{\Omega}(p_\delta, p_\delta) \geq CK_{\Omega \cap U}(p_\delta, p_\delta) = CK_{T_\delta(\Omega \cap U)}(p_{2\delta}, p_{2\delta}) \geq CK_{\Omega}(p_{2\delta}, p_{2\delta}),$$
where the first inequality follows by localization of the kernel ([JP], Theorem 6.3.5), and the last inequality holds because $T_\delta(\Omega \cap U) \subseteq \Omega$.

(2) This part of the lemma is implicit in work of Pflug (see [JP, §7.6]) and Ohsawa [O1] on the completeness of the Bergman metric. We recall the proof for the reader’s convenience. Without loss of generality, assume that $\Omega$ contains the origin. It suffices to establish pointwise convergence: Vitali’s theorem (note that $K_{\Omega}(\cdot, p_j)/\sqrt{K_{\Omega}(p_j, p_j)}$ has norm 1) then implies that the convergence is locally uniform. For $z \in \Omega$, let $f(w) = K_{\Omega}(z, w)$. Then $\|f(w) - f(tw)\|_\Omega \to 0$ as $t \to 1^-$. Now fix $t$, $0 < t < 1$. Then

$$\frac{|f(p_j)|}{\sqrt{K_{\Omega}(p_j, p_j)}} \leq \frac{|f(p_j) - f(tp_j)|}{\sqrt{K_{\Omega}(p_j, p_j)}} + \frac{|f(tp_j)|}{\sqrt{K_{\Omega}(p_j, p_j)}} \leq \|f(w) - f(tw)\|_\Omega + \frac{|K_{\Omega}(z, tp_j)|}{\sqrt{K_{\Omega}(p_j, p_j)}}.$$ 

The domain $\Omega$ is convex and so satisfies an outer cone condition. Therefore, $K_{\Omega}(p_j, p_j) \to \infty$ as $j \to \infty$ (see e.g. [JP], Theorem 6.1.17). Thus, letting first $j \to \infty$, then $t \to 1^-$, we obtain part (2) of Lemma 4.1.

We are now in a position to prove the implication $(1) \Rightarrow (2)$ in Theorem 1.1.

Proof of $(1) \Rightarrow (2)$. Arguing indirectly, we assume that there exists a compact solution operator $S_q$ on $(0, q)$-forms and $\partial \Omega$ contains an affine variety of dimension $q$. (Thus $q \leq n - 1$.) After an affine transformation, we may assume that $\{(z', 0) \in \mathbb{C}^n; |z'| < 2\} \subseteq \partial \Omega$, where $z' = (z_1, \ldots, z_q)$. Let $z'' = (z_{q+1}, \ldots, z_n)$.

Let $\Omega_1 = \{z'' \in \mathbb{C}^{n-q}, (0, z'') \in \Omega\}$. It follows from the convexity of $\Omega$ that $\Omega_1$ is a (non-empty) convex domain in $\mathbb{C}^{n-q}(z'')$. Let $\Omega_2 = \{z'' \in \mathbb{C}^{n-q}; 2z'' \in \Omega_1\}$. Then $\{z' \in \mathbb{C}^q; |z'| < 1\} \times \Omega_2 \subseteq \Omega$: every point in this set is the midpoint of a line segment joining a point in $\{|z'| < 2\} \times \{0\}$ to a point in $\{0\} \times \Omega_1$.

Let $p_0$ be a point in $\Omega_2$ and let $p_j = p_0/j$, $j \in \mathbb{N}$. Let

$$f_j(z'') = \frac{K_{\Omega_1}(z'', p_j)}{\sqrt{K_{\Omega_1}(p_j, p_j)}}.$$

Then $\|f_j\|_{\Omega_1} = 1$. We have

$$\|f_j(z'')\|^2_{\Omega_2} = \frac{K_{\Omega_1}(\cdot, p_j)}{K_{\Omega_1}(p_j, p_j)} \geq \frac{K_{\Omega_2}(p_j, p_j)}{K_{\Omega_2}(p_j, p_j)} = 2^{-2(n-q)} \frac{K_{\Omega_1}(p_j, p_j)}{K_{\Omega_1}(2p_j, 2p_j)} \geq C,$$

for $j$ large enough. The first inequality follows because $K_{\Omega_1}(p_j, p_j) \leq (K_{\Omega_2}(p_j, p_j))^{1/2}$ $\|K_{\Omega_1}(\cdot, p_j)\|_{\Omega_2}$ (obtained by applying the reproducing property of $K_{\Omega_2}(p_j, \cdot)$ to the function $K_{\Omega_1}(\cdot, p_j)$). The last equality follows from the transformation formula of the Bergman
kernel. The last inequality follows from (4.1). On the other hand, by (4.2), $f_j \to 0$ locally uniformly on $\Omega_1$. Consequently, no subsequence of $\{f_j\}$ can converge in $L^2(\Omega_2)$.

By the Ohsawa-Takegoshi extension theorem [OT] (see also [O2]), there exist $L^2$-holomorphic functions $F_j(z', z'')$ on $\Omega$ such that $F_j(0, z'') = f_j(z'')$ and $\|F_j\|_{\Omega} \leq C$. We now use an idea from [Ca1] (compare also [DP]). Let $\alpha_j = F_j(z', z'') dz_1 \wedge \cdots \wedge d\bar{z}_q$. Then $\bar{\partial} \alpha_j = 0$, $\|\alpha_j\|_{L^2(\Omega)} \leq C$. Let $g_j = S_q \alpha_j$. Denote by $\widehat{g}_j$ the form obtained from $g_j$ by discarding terms containing a $d\bar{z}_j$ with $q + 1 \leq j \leq n$. For $z'' \in \Omega_2$ fixed, we can think of the forms $\alpha_j$ and $\widehat{g}_j$ as $(0, q)$ and $(0, q - 1)$-forms respectively, in the variables $z' = (z_1, \ldots, z_q)$, $|z'| < 1$. Note that we still have $\bar{\partial}_{\bar{\nu}} \widehat{g}_j = \alpha_j$, where $\bar{\partial}_{\bar{\nu}}$ denotes $\bar{\partial}$ in the variables $z'$. Let $\langle \cdot, \cdot \rangle$ be the standard pointwise inner product on forms in $\mathbb{C}^q$. Let $\chi \in C_0^\infty(-\infty, \infty)$ be a cut-off function such that $0 \leq \chi \leq 1$ when $t \leq 1/2$ and $\chi = 0$ when $t \geq 3/4$. Let $\beta = \chi(|z'|) d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_q$. It follows from the mean value property of holomorphic functions that for $z'' \in \Omega_2$,

$$|f_j(z'') - f_k(z'')| = C \int_{|z'| < 1} |\alpha_j - \alpha_k, \beta| dV(z')$$
$$= C \int_{|z'| < 1} |\widehat{g}_j - \widehat{g}_k, \beta| dV(z') \quad (\beta \text{ is the formal adjoint of } \bar{\partial}_{\bar{\nu}})$$
$$\leq C \left\{ \int_{|z'| < 1} |\widehat{g}_j - \widehat{g}_k|^2 dV(z') \right\}^{\frac{1}{2}}$$

Therefore, after integrating in $z''$,

$$\|f_j - f_k\|_{\Omega_2} \leq C \|\widehat{g}_j - \widehat{g}_k\|_{L^2(0, q - 1)}(\Omega) \leq C \|g_j - g_k\|_{L^2(0, q - 1)}(\Omega) \quad \text{as } j, k \to \infty.$$ 

Since $\{f_j\}$ has no subsequence that converges in $L^2(\Omega_2)$, $\{g_j\}$ has no subsequence that converges in $L^2(0, q - 1)(\Omega)$, contradicting the compactness of $S_q$. This completes the proof that (1) implies (2) in Theorem 1.1.

5. Further Remarks.

1) The arguments in Section 4 can be localized by using suitable cut-off functions in $z''$ as well; it is enough to control the geometry locally. One then needs a lemma to the following effect: Let $U_1$ and $U_2$ be neighborhoods of a boundary point $p_0$ of a bounded pseudoconvex domain $\Omega$, $U_1 \subset \subset U_2$. Then $K_\Omega(w, w)$ and $\|K_\Omega(\cdot, w)\|_{\Omega \cap U_2}$ are comparable, uniformly for $w \in \Omega \cap U_1$. This can be shown by applying the reproducing property of $K_{\Omega \cap U_2}$ to $K_\Omega(\cdot, w)$ and using that $K_{\Omega \cap U_2}(w, w)$ and $K_\Omega(w, w)$ are comparable. Also, compactness of the $\bar{\partial}$-Neumann problem is a local property: if every boundary point has the property that a compactness estimate holds for forms supported near the point, then the $\bar{\partial}$-Neumann problem is compact. This shows that Theorem 1.1 holds on domains that are locally convexifiable.
2) It is noteworthy that in the proof that (1) implies (2) in Theorem 1.1, we have only used that there is a compact solution operator to $\bar{\partial}$ on the $(0, q)$-forms with holomorphic coefficients.

3) On smooth bounded convex domains there is a hierarchy of regularity for the $\bar{\partial}$-Neumann problem which can be described in terms of the contact with the boundary of affine complex varieties. $N_q$ is subelliptic if and only if the order of contact with the boundary of $q$-dimensional affine complex varieties is bounded from above ([Ca3], [Mc], [Y]); $N_q$ is compact if and only if the boundary contains no $q$-dimensional affine varieties (Theorem 1.1); finally, $N_q$ is globally regular regardless of whether or not $b\Omega$ contains analytic varieties ([BS1]).

4) We have stated our results for $(0, q)$-forms, rather than $(p, q)$-forms, as the index $p$ plays no rôle in solving $\bar{\partial}$.

5) To prove compactness of $N_q$, we have used (the analogue, for $(0, q)$-forms, of) Property (P), see Proposition 3.1. Our work shows that for convex domains, this property is actually equivalent to compactness of $N_q$. On general pseudoconvex domains, Property (P) still seems to be the only systematic way to derive compactness of the $\bar{\partial}$-Neumann problem, but (to quote from [BS3]) “it is not yet understood how much room there is between Property (P) and compactness”.

6) In the proof of Lemma 4.1, we have used the fact that on a bounded convex domain $K_{\Omega}(z, z) \to \infty$ as $z \to b\Omega$. While this is sufficient for the proof of Lemma 4.1, it is interesting to note that for convex domains, there is the (optimal) lower estimate $K_{\Omega}(z, z) \geq C/(\text{dist}(z, b\Omega))^2$. This can be shown by the Ohsawa-Takegoshi extension theorem (see [OT], [O2]) and the fact that the estimate is true for bounded convex domains in the plane.

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