On nonlinear boundary value problem corresponding to
N-dimensional inverse spectral problem

Y.Sh. Ilyasov\textsuperscript{a,*}, N. F. Valeev\textsuperscript{a}

\textsuperscript{a}Institute of Mathematics of UFRC RAS, 112, Chernyshevsky str., 450008 Ufa, Russia

Abstract

We establish a relationship between an inverse optimization spectral problem for N-dimensional Schrödinger equation \(-\Delta \psi + q\psi = \lambda\psi\) and a solution of the nonlinear boundary value problem \(-\Delta u + q_0 u = \lambda u - u^{-1}, \ u > 0, \ u|_{\partial\Omega} = 0\). Using this relationship, we find an exact solution for the inverse optimization spectral problem, investigate its stability and obtain new results on the existence and uniqueness of the solution for the nonlinear boundary value problem.

Keywords: Schrödinger operator, inverse spectral problem, nonlinear elliptic equations; 35P30, 35R30, 35J65, 35J10, 35J60

1. Introduction

This paper is concerned with the inverse spectral problem for the operator of the form

\[ \mathcal{L}_q \phi := -\Delta \phi + q\phi, \quad x \in \Omega, \]  

subject to the Dirichlet boundary condition

\[ \phi|_{\partial\Omega} = 0. \]  

Here \(\Omega\) is a bounded domain in \(\mathbb{R}^N, \ N \geq 1\), the boundary \(\partial\Omega\) is of class \(C^{1,1}\). We assume that \(q \in L^p(\Omega)\), where

\[ p \in \begin{cases} [2, +\infty) & \text{if} \ N < 4, \\ (2, +\infty) & \text{if} \ N = 4, \\ [N/2, +\infty) & \text{if} \ N > 4. \end{cases} \]  

Under these conditions, \(\mathcal{L}_q\) with domain \(D(\mathcal{L}_q) := W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega)\) defines a self-adjoint operator (see, e.g., [6, 10]) so that its spectrum consists of an infinite sequence of eigenvalues

\textsuperscript{*}Corresponding author

Email addresses: ilyasov02@gmail.com (Y.Sh. Ilyasov), valeevnf@mail.ru (N. F. Valeev)
\(\{\lambda_i(q)\}_{i=1}^{\infty}\), repeated according to their finite multiplicity and ordered as \(\lambda_1(q) < \lambda_2(q) \leq \ldots\). Furthermore, the principal eigenvalue \(\lambda_1(q)\) is a simple and isolated.

The recover of the potential \(q(x)\) from a knowledge of the spectral data \(\{\lambda_i(q)\}_{i=1}^{\infty}\) is a classical problem and, beginning with the celebrated papers by Ambartsumyan [1] in 1929, Borg in 1946 [3], Gel’fand & Levitan [7] in 1951, it received a lot of attention; see, e.g., surveys [4, 12]. It is well known that a knowledge of the single spectrum \(\{\lambda_i(q)\}_{i=1}^{\infty}\) is insufficient to determine the potential \(q(x)\); see, e.g., [3, 7].

In this work we deal with an inverse problem where given finite set of eigenvalues: \(\{\lambda_i\}_{i=1}^{m}, m < +\infty\). Having only finite spectral data, the inverse problem possesses infinitely many solutions. Thus additional conditions have to be imposed in order to make the problem well-posed. To overcome this difficulty, we assume that an approximation \(q_0\) of the potential \(q\) is known. Under this assumption, it is natural to consider the following inverse optimization spectral problem: for a given \(q_0\) and \(\{\lambda_i\}_{i=1}^{m}, m < +\infty\), find a potential \(\hat{q}\) closest to \(q_0\) in a prescribed norm, such that \(\lambda_i = \lambda_i(\hat{q})\) for all \(i = 1, \ldots, m\).

In the present paper, we study the following simplest variant of this problem:

\[(P) : \text{For a given } \lambda \in \mathbb{R} \text{ and } q_0 \in L^p(\Omega), \text{ find a potential } \hat{q} \in L^p(\Omega) \text{ such that } \lambda = \lambda_1(\hat{q}) \text{ and}\]
\[
\|q_0 - \hat{q}\|_{L^p} = \inf \{\|q_0 - q\|_{L^p} : \lambda = \lambda_1(q), q \in L^p(\Omega)\}. \tag{1.4}
\]

It turns out that this problem is related to the following logistic nonlinear boundary value problem:

\[
\left\{
\begin{array}{ll}
-\Delta u + q_0 u = \lambda u - u^{\frac{p+1}{p}}, & x \in \Omega, \\
u > 0, & x \in \Omega, \\
|u|_{\partial \Omega} = 0.
\end{array}
\right. \tag{1.5}
\]

Our first main result is as follows.

**Theorem 1.** Assume \(\Omega\) is a bounded connected domain in \(\mathbb{R}^N\) with a \(C^{1,1}\)-boundary \(\partial \Omega\). Let \(q_0 \in L^p(\Omega)\) be a given potential, where \(p\) satisfies (1.3). Then, for any \(\lambda > \lambda_1(q_0)\),

1° there exists a unique potential \(\hat{q} \in L^p(\Omega)\) such that \(\lambda = \lambda_1(\hat{q})\) and (1.4) is satisfied;

2° there exists a weak positive solution \(\hat{u} \in W^{1,2}_0(\Omega)\) of (1.5) such that

\[\hat{q} = q_0 + \hat{u}^{2/(p-1)} \text{ a.e. in } \Omega.\]

Furthermore, \(\hat{u} \in C^{1,\beta}(\overline{\Omega})\) for some \(\beta \in (0, 1)\) and \(\phi_1(\hat{q}) = \hat{u}/\|\hat{u}\|_{L^2}\).

Using the relationship between (P) and (1.5) stated in Theorem 1, we are able to prove the following theorem on the uniqueness of the solution for (1.3).

**Theorem 2.** Assume that

\[
\begin{align*}
2 < \gamma &\leq 4 \quad \text{if } N < 4, \\
2 < \gamma &< 4 \quad \text{if } N = 4, \\
2 < \gamma &\leq \frac{2N}{N-2} \quad \text{if } N > 4.
\end{align*} \tag{1.6}
\]
Then, for any \( q_0 \in L^p(\Omega) \) with \( p \geq \frac{2}{\gamma - 2} \) and any \( \lambda > \lambda_1(q_0) \), the boundary value problem

\[
\begin{aligned}
-\Delta u + q_0 u &= \lambda u - u^{\gamma - 1}, \quad x \in \Omega, \\
u \geq 0, \quad u \not\equiv 0, \quad x \in \Omega, \\
u|_{\partial \Omega} &= 0,
\end{aligned}
\] (1.5)

has at most one weak solution.

The existence of a solution for (1.5) follows in a standard way cf. [2]. In the case \( q_0 \in L^\infty(\Omega) \), there are various proofs of the uniqueness of the solution for (1.5); see, e.g., [2, 5] and the references given there. However, as far as we know, the uniqueness in the case of an unbounded potential \( q_0 \in L^p(\Omega) \) has not been proven before.

It should be emphasized that Theorem 1 also can be seen as a new method of proving the existence and uniqueness of a solution for nonlinear boundary value problems. Indeed, the finding of the minimizer \( \hat{q} \) of constrained minimization problem (1.4) also implies the existence of the solution \( \hat{u} = (\hat{q} - q_0)^{(p-1)/2} \) for (1.5), whereas the uniqueness of \( \hat{u} \) follows from the uniqueness of the minimizer of (1.4), as will be shown below.

This paper is organised as follows. Section 2 contains some preliminaries. In Section 3, we give the proofs of Theorems 1 and 2. In Section 4, using nonlinear problem (1.5), we investigate stability properties of inverse optimization spectral problem \((P)\). Section 5 contains some remarks and open problems.

2. Preliminaries

In what follows, we denote by \( \langle \cdot, \cdot \rangle \) and \( \| \cdot \|_{L^2} \) the scalar product and the norm in \( L^2(\Omega) \), respectively; \( W^{1,2}(\Omega) \), \( W^{2,2}(\Omega) \) are usual Sobolev spaces; \( W^{1,2}_0 := W^{1,2}_0(\Omega) \) is the closure of \( C_0^\infty(\Omega) \) in the norm

\[
\|u\|_1 = \left( \int_\Omega |\nabla u|^2 \, dx \right)^{1/2}.
\]

By a standard criterion (see, e.g., [6], Theorem 1.4. p. 306), assumption (1.3) implies that \( L_q \) with domain \( D(L_q) := W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega) \) is self-adjoint on \( L^2(\Omega) \). Moreover, \( L_q \) is a semibounded operator so that the principal eigenvalue satisfies

\[
-\infty < \lambda_1(q) = \inf_{\phi \in W^{1,2}_0 \setminus \{0\}} \frac{\int_\Omega |\nabla \phi|^2 \, dx + \int_\Omega q \phi^2 \, dx}{\int_\Omega \phi^2 \, dx},
\] (2.1)

where the minimum attained at eigenfunction \( \phi_1 \in W^{1,2}_0 \setminus \{0\} \). The regularity of solutions for elliptic equations (see, e.g., Lemma B 3 in [13]) implies that \( \phi_1 \in W^{2,q}(\Omega) \) for any \( q \geq 2 \) and therefore by the Sobolev theorem, \( \phi_1 \in C^{1,\alpha}(\Omega) \) for any \( \alpha \in (0, 1) \). Furthermore, in view of (1.3), we may apply the weak Harnack inequality (see Theorem 5.2. in [14]) and obtain, in a standard fashion (see, e.g., Theorem 8.38 in [13]), that the principal eigenvalue \( \lambda_1(q) \) is simple and \( \phi_1 > 0 \) in \( \Omega \).
Lemma 1. \( \lambda \) analytically depends on \( \varepsilon \) for some constants \( q \). Indeed, assume (2.5) is true. Since (1.3), by the Sobolev theorem, the embedding \( \| D^\lambda \|_{L^2} \) is analytic of type \( (A) \) (see [10], p. 16) and therefore, by Theorem X.12 in [11], \( \mathcal{L}_{q_0+\varepsilon q} \) is an analytic family in the sense of Kato. Hence by Theorem X.8 in [11], \( \lambda_1 (q_0 + \varepsilon q) \) is an analytic function of \( \varepsilon \) near 0 and \( \phi_1 (q_0 + \varepsilon q) \) analytically depends on \( \varepsilon \) near 0 as a function of \( \varepsilon \) with values in \( L^2 \).

**Lemma 1.** \( \lambda_1 (q) \) is a continuously differentiable map in \( L^p \) with the Fréchet-derivative

\[
D\lambda_1 (q) (h) = \frac{1}{\| \phi_1 (q) \|_{L^2}} \int_\Omega \phi_1^2 (q) h \, dx, \quad \forall q, h \in L^p. \tag{2.3}
\]

**Proof.** Set \( \| \phi_1 (q) \|_{L^2} = 1 \). Observe,

\[
\frac{d}{d\varepsilon} \lambda_1 (q + \varepsilon h)_{|\varepsilon=0} = \int_\Omega \phi_1^2 (q) h \, dx, \quad \forall q, h \in L^p. \tag{2.4}
\]

Indeed, testing equation \( \mathcal{L}_{q+\varepsilon h} \phi_1 (q + \varepsilon h) = \lambda_1 (q + \varepsilon h) \phi_1 (q + \varepsilon h) \) by \( \phi_1 (q) \) and integrating by parts one obtains

\[
\int_\Omega \phi_1 (q + \varepsilon h) (-\Delta \phi_1 (q)) \, dx + \int_\Omega (q + \varepsilon h) \phi_1 (q + \varepsilon h) \phi_1 (q) \, dx = \lambda_1 (q + \varepsilon h) \int_\Omega \phi_1 (q + \varepsilon h) \phi_1 (q) \, dx.
\]

By the above, all terms in this equality are differentiable with respect to \( \varepsilon \). Thus we have

\[
- \int_\Omega \Delta \phi_1 (q) \frac{d}{d\varepsilon} \phi_1 (q + \varepsilon h)_{|\varepsilon=0} \, dx + \int_\Omega h \phi_1^2 (q) \, dx + \int_\Omega h \frac{d}{d\varepsilon} \phi_1 (q + \varepsilon h)_{|\varepsilon=0} \phi_1 (q) \, dx = \frac{d}{d\varepsilon} \lambda_1 (q + \varepsilon h)_{|\varepsilon=0} \int_\Omega \phi_1^2 (q) \, dx + \lambda_1 (q) \int_\Omega \frac{d}{d\varepsilon} \phi_1 (q + \varepsilon h)_{|\varepsilon=0} \phi_1 (q) \, dx.
\]

Since \( \phi_1 (q) \) is an eigenfunction of \( \mathcal{L}_q \), this implies (2.4).

To conclude the proof of the lemma, it is sufficient to show that

\[
\phi_1 (\cdot) \in C (L^p; W^{1,2}_0 (\Omega)). \tag{2.5}
\]

Indeed, assume (2.5) is true. Since (1.3), by the Sobolev theorem, the embedding \( W^{1,2}_0 (\Omega) \subset L^p \) is continuous. Hence the map \( \phi_1 (\cdot) : L^p \to W^{1,2}_0 (\Omega) \cap L^2 \) is continuous and therefore the norm of Gateaux derivative \( D\lambda_1 (q) \) continuously depends on \( q \in L^p \). This implies that \( \lambda_1 (q) \) is continuously differentiable in \( L^p \).
To prove (2.5), let us first show that $\lambda_1(q)$ defines a continuous map in $L^p$. Suppose, contrary to our claim, that there is a sequence $(q_n)$ such that $q_n \to q$ in $L^p$ as $n \to \infty$ and $|\lambda_1(q_n) - \lambda_1(q)| > \epsilon$ for some $\epsilon > 0$, $n = 1, 2, \ldots$. Consider Rayleigh’s quotient

$$
\lambda_1(q_n) \equiv R_{q_n}(\phi_1(q_n)) := \frac{\|\phi_1(q_n)\|^2_2 + \int_0^1 q_n \phi_1^2(q_n) \, dx}{\|\phi_1(q_n)\|^2_{L^2}} \quad n = 1, 2, \ldots \tag{2.6}
$$

It is easily seen that

$$
R_{q_n}(\phi_1(q)) \to R_q(\phi_1(q)) = \lambda_1(q) \quad \text{as } n \to \infty. \tag{2.7}
$$

Hence and since $\lambda_1(q_n) = R_{q_n}(\phi_1(q_n)) \leq R_{q_n}(\phi_1(q))$, we conclude that $\lambda_1(q_n) = R_{q_n}(\phi_1(q_n)) < C_0 < \infty$, where $C_0 < \infty$ does not depend on $n = 1, 2, \ldots$. Due to homogeneity of $R_{q_n}(\phi_1(q_n))$ we may assume that $\|\phi_1(q_n)\|^2_2 = 1$ for all $n$. Hence the Banach-Alaoglu and Sobolev theorems imply that there is a subsequence, which we again denote by $(\phi_1(q_n))$, such that $\phi_1(q_n) \rightharpoonup \tilde{\phi}$ as $n \to \infty$ weakly in $W^{1,2}$ and strongly in $L^q$ for $q \in [2, 2^*)$, where $2^* = 2N/(N - 2)$ if $N > 2$ and $2^* = +\infty$ if $N \leq 2$. Observe that if $\tilde{\phi} = 0$, then $\|\phi_1(q_n)\|_{L^2} \to 0$ and $\int_0^1 q_n \phi_1^2(q_n) \to 0$ as $n \to \infty$, which implies by (2.6) that $\lambda_1(q_n) \to +\infty$. We get a contradiction. Thus $\tilde{\phi} \neq 0$ and therefore $|\lambda_1(q_n)| < C$, where $C < \infty$ does not depend on $n = 1, 2, \ldots$. Hence, in view of (2.7), we conclude that

$$
\lambda_1(q) = R_q(\phi_1(q)) \leq R_q(\tilde{\phi}) \leq \liminf_{n \to \infty} R_{q_n}(\phi_1(q_n)) \leq \lim_{n \to \infty} R_{q_n}(\phi_1(q)) = \lambda_1(q),
$$

which contradicts to our assumption. Thus, indeed, the map $\lambda_1(\cdot) : L^p \to \mathbb{R}$ is continuous.

Take $q, q_0 \in L^p$. Then

$$-
\Delta(\phi_1(q_0) - \phi_1(q)) + q_0(\phi_1(q_0) - \phi_1(q)) - \lambda_1(q_0)(\phi_1(q_0) - \phi_1(q)) +$$

$$(q_0 - q)(\phi_1(q) - (\lambda_1(q_0) - \lambda_1(q))\phi_1(q) = 0.)$$

Testing this equation by $(\phi_1(q) - \phi_1(q_0))$ and integrating by parts, we obtain

$$
\|\phi_1(q_0) - \phi_1(q)\|^2_1 + \int_\Omega q_0(\phi_1(q_0) - \phi_1(q))^2 \, dx - \lambda_1(q_0)\int_\Omega (\phi_1(q_0) - \phi_1(q))^2 \, dx +$$

$$
\int_\Omega (q_0 - q)(\phi_1(q) - \phi_1(q_0)) \, dx - (\lambda_1(q_0) - \lambda_1(q)) \int_\Omega \phi_1(q)(\phi_1(q_0) - \phi_1(q)) \, dx = 0.
$$

Let $q_k \to q_0$ in $L^p$ as $k \to \infty$. We may assume that $\|\phi_1(q_k)\|_1 = 1, k = 1, 2, \ldots$. Set $t_k := \|\phi_1(q_k) - \phi_1(q_0)\|_1$, $\psi_k := (\phi_1(q_k) - \phi_1(q_0))/t_k$, $k = 1, 2, \ldots$. Then $0 < t_k < 1 + \|\phi_1(q_0)\|_1 := C_1 < +\infty$, $\|\psi_k\|_1 = 1$ and

$$
t_k \left(1 + \int_\Omega q_0 \psi_k^2 \, dx - \lambda_1(q_0)\int_\Omega \psi_k^2 \, dx \right) =$$

$$
- \int_\Omega (q_0 - q_k)(\phi_1(q_k)\psi_k \, dx + (\lambda_1(q_0) - \lambda_1(q_k)) \int_\Omega \phi_1(q_k)\psi_k \, dx, \quad k = 1, 2, \ldots \tag{2.8}
$$
By Hölder’s inequality and the Sobolev theorem, we have
\[
\left| \int_{\Omega} (q_k - q_0) \phi_1(q_k) \psi_k \, dx \right| \leq \|q_k - q_0\|_{L^p} \|\phi_1(q_k)\|_{L^{\frac{2p}{p+1}}} \|\psi_k\|_{L^{\frac{2p}{p-1}}} \leq C_2 \|q_k - q_0\|_{L^p},
\]
and
\[
\left| (\lambda_1(q_k) - \lambda_1(q_0)) \int_{\Omega} \phi_1(q_k) \psi_k \, dx \right| \leq C_2 |\lambda_1(q_k) - \lambda_1(q_0)|,
\]
where \(C_2 < +\infty\) does not depend on \(k = 1, 2, \ldots\). Hence and from (2.8) it follows that \(t_k := \|\phi_1(q_k) - \phi_1(q_0)\|_1 \to 0\) as \(q_k \to q_0\) in \(L^p\). Thus we get (2.9).

**Lemma 2.** \(\lambda_1(q)\) is strictly concave functional in \(L^p\).

**Proof.** Let \(q_1, q_2 \in L^p \setminus 0\). Denote \(\phi'_1 := \phi_1(tq_1 + (1-t)q_2)\). Assume that \(\|\phi'_1\|_{L^2} = 1\), \(t \in [0, 1]\). Then due to (2.1) we have
\[
\lambda_1(tq_1 + (1-t)q_2) = \int_{\Omega} |\nabla \phi'_1|^2 \, dx + \int_{\Omega} |(tq_1 + (1-t)q_2)|\phi'_1|^2 \, dx =
\]
\[
t(t \int_{\Omega} |\nabla \phi'_1|^2 \, dx + \int_{\Omega} q_1|\phi'_1|^2 \, dx) + (1-t)\left( \int_{\Omega} |\nabla \phi'_1|^2 \, dx + \int_{\Omega} q_2|\phi'_1|^2 \, dx \right) >
\]
\[
t\lambda_1(q_1) + (1-t)\lambda(q_2), \quad \forall t \in (0, 1),
\]
which yields the proof.

3. **Proof of the main results**

**Proof of Theorem 1.**

Let \(q_0 \in L^p\) and \(\lambda > \lambda_1(q_0)\). Consider the constrained minimization problem
\[
\hat{Q} = \min \{ Q(q) : q \in M_\lambda \},
\]
where \(Q(q) := \|q_0 - q\|_{L^p}^p\) for \(q \in L^p(\Omega)\) and
\[
M_\lambda := \{ q \in L^p(\Omega) : \lambda \leq \lambda_1(q) \}.
\]
Notice that \(M_\lambda \neq \emptyset\). Indeed, \(\lambda = \lambda_1(q_0 + \lambda - \lambda_1(q_0)), \forall \lambda \in \mathbb{R}\) and thus \(q_0 + \lambda - \lambda_1(q_0) \in M_\lambda\) for any \(\lambda > \lambda_1(q_0)\). Moreover, by Lemma 2 \(M_\lambda\) is convex. Hence, by coerciveness of \(Q : L^p \to \mathbb{R}\) there exists a minimizer \(\hat{q} \in M_\lambda\) of (3.1). Since the strong inequality \(\lambda > \lambda_1(q_0)\), it follows that \(\hat{q} \neq 0\). The convexity of \(M_\lambda\) and \(Q\) entails that \(\hat{q}\) is unique and that
\[
\hat{q} \in \partial M_\lambda = \{ q \in L^p : \lambda = \lambda_1(q) \}.
\]
This concludes the proof of assertion (1o) of Theorem 1.

Evidently \(Q\) is \(C^1\)-functional in \(L^p\). Hence, in view of Lemma 1 the Lagrange multiplier rule implies
\[
\mu_1 DQ(\hat{q})(h) + \mu_2 D\lambda_1(\hat{q})(h) = 0, \quad \forall h \in L^p,
\]
(3.2)
where $\mu_1, \mu_2$ such that $|\mu_1| + |\mu_2| \neq 0, \mu_1 \geq 0, \mu_2 \leq 0$. Thus by (2.3) we deduce

$$
\int_{\Omega} (-\mu_1 p(q_0 - \hat{q})|q_0 - \hat{q}|^{p-2} + \mu_2 \phi_1^2(q)) h \, dx = 0, \; \forall h \in L^p,
$$

(3.3)

where $\|\phi_1^2(q)\|_{L^2} = 1$. Arguing by contradiction, it is easily to conclude that $\mu_1 > 0, \mu_2 < 0$. Thus we have

$$
(q_0 - \hat{q})|q_0 - \hat{q}|^{p-2} = \mu \phi_1^2(q) \; \text{a.e. in } \Omega,
$$

where $\mu = \mu_2/(p\mu_1) < 0$. Notice that $\phi_1 > 0$ in $\Omega$. Hence $q_0 < \hat{q}$ a.e. in $\Omega$ and

$$
\hat{q} = q_0 + \nu \phi_1^{2/(p-1)}(\hat{q}) \; \text{a.e. in } \Omega,
$$

(3.4)

where $\nu := (-\mu)^{1/(p-1)} > 0$. Substituting this into (1.1) yields

$$
-\Delta \phi_1(\hat{q}) + q_0 \phi_1(\hat{q}) = \lambda \phi_1(\hat{q}) - \nu \phi_1^{p/(p-1)}(\hat{q}).
$$

(3.5)

Thus, indeed, $\hat{u} = \nu^{p/(p-1)} \phi_1(\hat{q})$ satisfies (1.5). Moreover, $\hat{q} = q_0 + \hat{u}^{2/(p-1)}$ a.e. in $\Omega$. This concludes the proof of (2').

**Proof of Theorem 2.** Since (1.5) is obtained from (1.5) by replacing $\gamma = \frac{2p}{p-1}$, it is sufficient to prove the uniqueness of the solution for (1.5).

First we prove

**Lemma 3.** Let $u \in W_0^{1,2}$ be a nonnegative weak solution of (1.5). Then the function $\tilde{q} := q_0 + u^{2/(p-1)}$ is a local minimum point of $Q$ in $M_\lambda$.

**Proof.** Let $u \in W_0^{1,2}$ be a nonnegative weak solution of (1.5). The regularity solutions for elliptic equations (see, e.g., Lemma B.3 in [15]) implies that $u \in W^{2,q}(\Omega)$ for any $q \geq 2$ and therefore $u \in C^{1,\alpha}(\Omega)$ for any $\alpha \in (0, 1)$. By the weak Harnack inequality (see Theorem 5.2. in [14]) it follows that $u > 0$ in $\Omega$. This implies that $u$ is an eigenfunction of $\mathcal{L}_{\tilde{q}}$ corresponding to the principal eigenvalue $\lambda = \lambda_1(\tilde{q})$, i.e., $u = \phi_1(\tilde{q})$. In view of Lemma 1, by the Lusternik theorem [3] the tangent space of $\partial M_\lambda$ at $q \in \partial M_\lambda$ can be expressed as follows

$$
T_q(\partial M_\lambda) := \{h \in L^p : D\lambda_1(q)(h) \equiv \int_{\Omega} u^2 \cdot h \, dx = 0\}.
$$

(3.6)

From this

$$
D_q Q(q)|_{q=q_0+u^{2/(p-1)}}(h) = p \int_{\Omega} u^2 \cdot h \, dx = 0, \; \forall h \in T_q(\partial M_\lambda).
$$

Hence and since

$$
D_{qq} Q(q)|_{q=q_0+u^{2/(p-1)}}(h, h) = p(p-1) \int_{\Omega} u^{2(p-2)/p-1} \cdot h^2 \, dx > 0, \; \forall h \in T_q(\partial M_\lambda),
$$

we obtain that

$$
Q(\tilde{q} + h) > Q(\tilde{q}),
$$

for any $h \in T_q(\partial M_\lambda)$ with sufficient small $\|h\|_{L^p}$. \qed
Let us conclude the proof of Theorem 2. By Theorem 1 we know that (1.5) possess a solution \( \hat{u} \) such that the functional \( Q \) admits a global minimum at \( \hat{q} = q_0 + \hat{u}^2/(p-1) \) on \( M_\lambda \). Assume that there exists a second weak solution \( \bar{u} \) of (1.5). Then by Lemma 3 \( \bar{q} = q_0 + \bar{u}^2/(p-1) \) is a local minimum point of \( Q \) in \( M_\lambda \). However, due to strict convexity of \( \lambda_1(q) \) and \( Q(v) \) this is possible only if \( \bar{q} = \hat{u} \).

4. Stability results

In this section, we prove that the solution \( \hat{q} \) of the inverse optimization spectral problem (P) is stable with respect to variation of \( q_0 \) and \( \lambda \).

Let \( q_0 \in L^p, \lambda > \lambda_1(q_0) \). Denote by \( \hat{q}(\lambda, q_0) \) the unique solution of (P) obtained by Theorem 1 and denote by \( \hat{u}(\lambda, q_0) \) the corresponding solution of (1.5).

**Proposition 1.** (i) For any \( \lambda \in (\lambda_1(q_0), +\infty) \), the map \( \hat{u}(\lambda, \cdot) : L^p \to W_0^{1,2} \) is continuous and thus \( \hat{q}(\lambda, q_0) \) continuously depends on \( q_0 \) as a map from \( L^p \) to \( L^p \).

(ii) For any \( q_0 \in L^p \), the map \( \hat{\cdot}(\cdot, q_0) : (\lambda_1(q_0), +\infty) \to W_0^{1,2} \) is continuous and thus \( \hat{q}(\lambda, q_0) \) continuously depends on \( q_0 \) as a map from \( L^p \) to \( L^p \). Moreover, \( \hat{q}(\lambda, q_0) \to q_0 \) in \( L^p \) as \( \lambda \downarrow \lambda_1(q_0) \).

**Proof.** We give the proof of (i) only for the case \( N \geq 3 \); the other cases are left to the reader. Let \( \lambda \in (\lambda_1(q_0), +\infty) \). Assume \( q_n \to q_0 \) in \( L^p \) as \( n \to \infty \). By the above, \( \lambda_1(q) \) continuously depends on \( q \in L^p \). Thus for sufficiently large \( n \) we have \( \lambda > \lambda_1(q_n) \).

We claim that the sequence \( \|\hat{u}(\lambda, q_n)\|_1 \), \( n = 1, 2, \ldots \) is bounded and separated from zero. Let \( t_n := \|\hat{u}(\lambda, q_n)\|_1, v_n := \hat{u}(\lambda, q_n)/t_n, n = 1, 2, \ldots \). Since \( \|v_n\|_1 = 1, n = 1, 2, \ldots \), by the Banach-Alaoglu and Sobolev theorems we may assume that \( v_n \to v \) weakly in \( W^{1,2} \) a.e. in \( \Omega \) and strongly in \( L^q \), \( 2 \leq q < 2N/(N-2) \) for some \( v \in W^{1,2} \). It follows from (1.5)

\[
\|v_n\|_1 + \int_\Omega q_nv_n^2 \, dx - \lambda \int_\Omega v_n^2 \, dx + \int_\Omega \frac{2}{p-1} \int_\Omega \frac{2p}{v_n^2} \, dx = 0, \quad n = 1, 2, \ldots \tag{4.1}
\]

By Hölder’s inequality

\[
|\int_\Omega q_nv_n^2 \, dx| \leq \|q_n\|_{L^p}\|v_n\|_{L^{2p/(p-1)}}, \quad n = 1, 2, \ldots,
\]

where, in view of (1.3), we have \( 2 < 2p/(p-1) < 2N/(N-2) \). Suppose that \( v = 0 \). Then

\[
\|v_n\|_1 + \int_\Omega q_nv_n^2 \, dx - \lambda \int_\Omega v_n^2 \, dx + \int_\Omega \frac{2}{p-1} \int_\Omega \frac{2p}{v_n^2} \, dx \geq \|v_n\|_1 + \int_\Omega q_nv_n^2 \, dx - \lambda \int_\Omega v_n^2 \, dx \to 1,
\]

as \( n \to +\infty \), which contradicts to (4.1). Thus \( v \neq 0 \) and therefore by (4.1) the sequence \( t_n \) is bounded. Assume, by contradiction, that \( t_n \to 0 \). Then passing to the limit in (1.5) yields

\[-\Delta v + q_0 v = \lambda v.\]
From the above, it follows that \( v \geq 0, v \neq 0 \). Thus \( v \) is an eigenfunction corresponding to the principal eigenvalue of \( L_{q_0} \). However, by the assumption \( \lambda > \lambda_1(q_0) \) and we get a contradiction.

Thus the claim is proving and we may assume that \( \hat{u}(\lambda, q_n) \rightarrow \bar{u} \) weakly in \( W^{1,2} \), \( \hat{u}(\lambda, q_n) \rightarrow \bar{u} \) a.e. in \( \Omega \) and strongly in \( L^q, 2 \leq q < 2N/(N - 2) \) for some \( \bar{u} \in W^{1,2} \setminus 0 \). Since \( \hat{u}(\lambda, q_n) > 0 \) in \( \Omega \), we conclude that \( \bar{u} \geq 0 \) a.e. in \( \Omega \). Passing to the limit in (1.5) we obtain

\[
-\Delta \bar{u} + q_0 \bar{u} = \lambda \bar{u} - \bar{u}^{\frac{p+1}{p}}.
\]

Due to the uniqueness of solution of (1.5), it follows that \( \bar{u} = \hat{u}(\lambda, q_0) \). Furthermore, this implies that \( \hat{u}(\lambda, q_n) \rightarrow \hat{u}(\lambda, q_0) \) strongly in \( W^{1,2} \). Thus we have proved that the map \( \hat{u}(\lambda, \cdot) : L^p \rightarrow W^{1,2}_0 \) is continuous.

Under assumption (1.3), we have a continuous embedding \( W^{1,2} \subset L^{2p/(p-1)} \). Hence \( \hat{u}^{2/(p-1)}(\lambda, q_n) \rightarrow \hat{u}^{2/(p-1)}(\lambda, q_0) \) strongly in \( L^p \) and thus \( \hat{q}(\lambda, q_n) = q_0 + \hat{u}^{2/(p-1)}(\lambda, q_n) \) strongly converges to \( \hat{q}(\lambda, q_0) = q_0 + \hat{u}^{2/(p-1)}(\lambda, q_0) \) in \( L^p \) as \( n \rightarrow +\infty \). This concludes the proof of (i).

The proof for the first part of (ii) is similar to (i). To prove that \( \hat{q}(q_0, \lambda) \rightarrow q_0 \) in \( L^p \) as \( \lambda \downarrow \lambda_1(q_0) \), it is remained to show that for \( \lambda = \lambda_1(q_0) \) problem (1.5) may has only zero solution. Suppose, contrary to our claim, that there exists a positive solution \( u \) of (1.5) for \( \lambda = \lambda_1(q_0) \). Then testing the equation in (1.5) by \( \phi_1(q_0) \) and integrating by parts we obtain

\[
\int_{\Omega} u(-\Delta \phi_1(q_0)) \, dx + \int_{\Omega} q_0 u \phi_1(q_0) \, dx = \lambda_1(q_0) \int_{\Omega} u \phi_1(q_0) \, dx - \int_{\Omega} u^{\frac{p+1}{p}} \phi_1(q_0) \, dx,
\]

which implies that \( \int_{\Omega} u^{\frac{p+1}{p}} \phi_1(q) \, dx = 0 \). However this is possible only if \( u \equiv 0 \) in \( \Omega \).

**5. Conclusion remarks and open problems**

Notice that if \( \lambda \leq \lambda_1(q_0) \), then nonlinear boundary value problem has no solution (see, e.g., [2]). However, the existence of solution of (P) in the case \( \lambda < \lambda_1(q_0) \) is unknown.

Since for various \( p \) equation (1.5) has different solutions, the answer on the inverse optimization spectral problem (P) essentially depends on the prescribed norm \( \| \cdot \|_{L^p} \). We are unable to offer criteria necessary to identify preferred norms. However, it should be noted that a similar problem about choosing a suitable norm has already been encountered in the literature on the theory of inverse problems (see, e.g., [9, 12]).

It is an open problem to solve the \( m \)-parametric inverse optimization spectral problem (\( P^m \)): Given \( \lambda_1, \ldots, \lambda_m \in \mathbb{R} \) and \( q_0 \in L^p(\Omega) \). Find a potential \( \tilde{q} \in L^p(\Omega) \) such that \( \lambda_i = \lambda_i(\tilde{q}), i = 1, \ldots, m \) and

\[
\|q_0 - \tilde{q}\|_{L^p} = \inf \left\{ \|q_0 - q\|_{L^p} : \lambda_i = \lambda_i(q), i = 1, \ldots, m, q \in L^p(\Omega) \right\}.
\]

The above inference of the nonlinear boundary value problem (1.5) can be formally generalized to be applicable to problem (\( P^m \)). In that case, one can obtain the following system of nonlinear equations.
\[
-\Delta u_i + q_0 u_i = \lambda_i u_i - (\sum_{j=1}^{m} \mu_j u_j^2)^{\frac{p}{p-1}} u_i, \quad i = 1, 2, \ldots, m,
\]
\[
u_i|_{\partial \Omega} = 0, \quad i = 1, 2, \ldots, m.
\]

where \( \mu_i \geq 0, \ i = 1, 2, \ldots, m \) are some constants so that

\[
\hat{q} = q_0 + (\sum_{j=1}^{m} \mu_j u_j^2)^{\frac{1}{p-1}}.
\]

However, we do not know how to justify this approach. Moreover, as far as we know, the existence and uniqueness of solution for (5.1) with \( q_0 \in L^p \) is also an open problem. Nevertheless, it would be useful to verify (5.2) numerically.

References

[1] V. Ambarzumian, "Uber eine frage der eigenwerttheorie. Zeitschrift für Physik A Hadrons and Nuclei 53 (9) (1929) 690-695.

[2] H. Brezis, & L. Oswald, "Remarks on sublinear elliptic equations. Nonlinear Analysis: Theory, Methods & Applications 10 (1) (1986) 55-64.

[3] G. Borg, "Eine umkehrung der Sturm-Liouvilleleschen eigenwertaufgabe. Acta Mathematica, 78 (1) (1946) 1-96.

[4] K. Chadan, D. Colton, L. Päivärinta, and W. Rundell, An introduction to inverse scattering and inverse spectral problems. Society for Industrial and Applied Mathematics. 1997.

[5] J. I. Díaz, J. E. Saá, "Existence et unicité de solutions positives pour certaines équations elliptiques quasilinéaires." CR Acad. Sci. Paris Sér. I Math, 305(12) (1987) 521-524.

[6] D. E. Edmunds, W. D. Evans, Spectral theory and differential operators. Vol. 15. Oxford: Clarendon Press, 1987.

[7] I. M. Gel’fand, B. M. Levitan, "On the determination of a differential equation from its spectral function. Izvestiya Rossiiskoi Akademii Nauk. Seriya Matematicheskaya, 15 (4) (1951) 309-360.

[8] Lusternik, L. "Sur les extrémés relatifs des fonctionnelles." Matematicheskii Sbornik, 41 (3) (1934) 390-401.

[9] M. Marletta, R. Weikard, "Weak stability for an inverse Sturm-Liouville problem with finite spectral data and complex potential, Inverse Problems, 21 (4) (2005) 1275-1290

[10] M. Reed, B. Simon, Methods of modern mathematical physics. vol. 2. Functional analysis. New York: Academic, 1980.
[11] M. Reed, B. Simon., Methods of modern mathematical physics. vol. 4. Functional analysis. New York: Academic, 1980.

[12] A.M. Savchuk, A.A. Shkalikov, Recovering of a potential of the Sturm-Liouville problem from finite sets of spectral data. Spectral Theory and Differential Equations, Amer. Math. Soc. Transl. 233 (2) (2014) 211–224

[13] D. Gilbarg, N. S. Trudinger, Elliptic partial differential equations of second order. Springer, 2015.

[14] N. S. Trudinger, Linear elliptic operators with measurable coefficients. Annali della Scuola Normale Superiore di Pisa Classe di Scienze 27 (2) (1973) 265-308.

[15] M. Struwe, Variational methods. Vol. 31999. Berlin etc.: Springer, 1990.