THE HOMOLOGY OF HEISENBERG LIE ALGEBRAS OVER
FIELDS OF CHARACTERISTIC TWO

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Abstract. The generating function of the Betti numbers of the Heisenberg
Lie algebra over a field of characteristic 2 is calculated using discrete Morse
theory.

The Heisenberg Lie algebra of dimension $2n + 1$, denoted by $\mathfrak{h}_n$, is the vector
space with basis $B = \{ z, x_1, \ldots, x_n, y_1, \ldots y_n \}$ where the only non-zero Lie products
of basis elements are

$$[x_i, y_i] = -[y_i, x_i] = z.$$

In this paper the Betti numbers of the homology of $\mathfrak{h}_n$ over a field of characteristic 2
is computed with the aid of algebraic discrete Morse theory from [Skö]. The
notation from [Skö] will be freely used.

Theorem 1. The generating function of the Betti numbers of the Heisenberg Lie
algebra over a field of characteristic 2 is

$$\sum_{i \geq 0} \dim_k H_i(\mathfrak{h}_n) t^i = \frac{(1 + t^3)(1 + t)^{2n} + (t + t^2)(2t)^n}{1 + t^2}$$

When the ground field of $\mathfrak{h}_n$ has characteristic 0, Santharoubane [San83] has
shown that

$$\dim_k H_i(\mathfrak{h}_n) = \binom{2n}{i} - \binom{2n}{i - 2},$$

(the need for the ground field to have characteristic 0 is not explicitly mentioned).

Let us first recall the construction of the Chevalley–Eilenberg complex $V$ of $\mathfrak{h}_n$,
whose homology is the homology of $\mathfrak{h}_n$: the complex $V$ is given by

$$0 \longrightarrow \bigwedge^{2n+1} \mathfrak{h}_n \longrightarrow \cdots \longrightarrow \bigwedge^i \mathfrak{h}_n \longrightarrow \cdots \longrightarrow \bigwedge^2 \mathfrak{h}_n \longrightarrow \bigwedge \mathfrak{h}_n \longrightarrow 0$$

with the differential

$$\tilde{d}(w_1 \wedge \cdots \wedge w_n) = \sum_{i < j} (-1)^{i+j}[w_i, w_j] \wedge w_1 \wedge \cdots \wedge \hat{w_i} \wedge \cdots \wedge \hat{w_j} \wedge \cdots \wedge w_n$$

for $w_i \in B$.

The $p$-th homology (with trivial coefficients) of $\mathfrak{h}_n$, can now be obtained as the $p$-th homology group of the complex $V$.

If $I = \{i_1, \ldots, i_s\}$ is a subset of $[n]$, we will use the notation $x_I$ for the element
$x_{i_1} \wedge \cdots \wedge x_{i_s}$, (and similarly for $y_I$).

Proof. The result is proved by constructing a Morse matching $M$ on the digraph
$G_V$, and showing that when $\pi$ is the projection coming from the splitting homotopy
of $M$, we have that $\pi(V)$ has trivial differential.

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The decomposition of the Chevalley–Eilenberg complex we will use is the obvious; we consider the basis for $V$ given by $\{z \wedge x_I \wedge y_J, x_I \wedge y_J \mid I, J \subseteq [n]\}$.

Let the matching $M$ consist of the following edges in $G_V$:

$$x_I \wedge y_J \to z \wedge x_{I\setminus\{k\}} \wedge y_{J\setminus\{k\}}$$

whenever $\max(I^c \cap J^c) < \max(I \cap J)$ and $k = \max(I \cap J)$.

There are now two kinds of unmatched elements: first the elements $z \wedge x_I \wedge y_J$, with $\max(I^c \cap J^c) < \max(I \cap J)$, and then the elements $x_I \wedge y_J$, with $\max(I^c \cap J^c) > \max(I \cap J)$.

When $x_I \wedge y_J \in M^+$, there is exactly one element $z \wedge x_I \wedge y_J$ with $x_I \wedge y_J \to z \wedge x_I \wedge y_J$, that is not in $M^0$, which implies that there can be no directed cycle in the graph $G_V$. Together with the fact that for all edges in $G_V$ the corresponding component of the differential is an isomorphism, this implies that $M$ is a Morse matching.

We will now see that the differential in $\pi(V)$ is zero. For an element $z \wedge x_I \wedge y_J \in M^0$ it is obvious that $d\pi(z \wedge x_I \wedge y_J) = \pi d(z \wedge x_I \wedge y_J) = 0$. For $x_I \wedge y_J \in M^0$ with $m = \max(I^c \cap J^c)$ we get that

$$\pi(x_I \wedge y_J) = x_I \wedge y_J + \sum_{i \in I \setminus J} x_{I \setminus \{i\}} \wedge y_{J \cup \{i\}},$$

from which it is easily seen that $d\pi(x_I \wedge y_J) = 0$. From [Skö, Theorem 1] now follows that the $i$-th Betti number is equal to the number of unmatched vertices in homological degree $i$.

For the computation of the generating function we introduce the elements $u_i = x_i \wedge y_i$, and we begin by counting the critical vertices $z \wedge x_I \wedge y_J \wedge u_K$ and $x_I \wedge y_J \wedge u_K$ when $I \cup J = L$ for a fixed set $L \subseteq [n]$.

If $L = [n]$, the critical vertices are all $z \wedge x_I \wedge y_J$ and $x_I \wedge y_J$ and they contribute with $(1 + t)(2t)^n$ toward the homology.

If $L \neq [n]$, then the critical vertices of the form $z \wedge x_I \wedge y_J \wedge u_K$ are those with $\max([n] \setminus (I \cup J)) \in K$ so they contribute with $t^3(2t)^{|L|}(1 + t^2)^{n-|L|-1}$ toward the homology. The critical vertices of the form $x_I \wedge y_J \wedge u_K$ are those with $\max([n] \setminus (I \cup J)) \notin K$ and thus contribute with $(2t)^{|L|}(1 + t^2)^{n-|L|-1}$ toward the homology.

Summing up we get

$$f(t) = (1 + t)(2t)^n + (1 + t^3) \sum_{L \subseteq [n]} (2t)^{|L|}(1 + t^2)^{n-|L|-1}$$

$$= (1 + t)(2t)^n + (1 + t^3) \sum_{i=0}^{n-1} \binom{n}{i} (2t)^i(1 + t^2)^{n-i-1}$$

$$= (1 + t)(2t)^n + (1 + t^3)(1 + t^2)^{-1}((1 + 2t + t^2)^n - (2t)^n)$$

$$= \frac{(1 + t)(1 + t^2)(2t)^n}{1 + t^2} + \frac{(1 + t^3)(1 + t)^{2n} - (1 + t^3)(2t)^n}{1 + t^2}$$

$$= \frac{(1 + t^3)(1 + t)^{2n} + (t + t^2)(2t)^n}{1 + t^2}$$

$\square$

References

[San83] L. J. Santharoubane, Cohomology of Heisenberg Lie algebras, Proc. Amer. Math. Soc. 87 (1983), no. 1, 23–28. MR 84b:17010
[Skö] Emil Sköldberg, Combinatorial discrete Morse theory from an algebraic viewpoint, Preprint, Stockholm University.
