A new three parameter Fréchet model with mathematical properties and applications

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1. Introduction and motivation

The extreme value theory (E.V.T) was originally proposed by Fréchet [1] and Fisher and Tippett [2] then completed by Von Mises [3], Gnedenko [4], Mead and Abd-Eltawab [5] and Von Mises [6]. The E.V.T is usually used for modelling the extreme phenomena such as the extreme floods, maximum sizes of ecological populations, the size of freak waves, the amounts of large insurance losses, equity risks, day to day market risk, side effects of some drugs, survival time. large wildfires, repair times. Let $Z_1, Z_2, \ldots, Z_n$ be a random sample (rs), which presents – at the same time – a finite sequence of independent and identically distributed (iid) random variables (rvs) with common cumulative distribution function (CDF). One of the most interesting statistics is the sample maximum

$$S^{(z)}_{(n)} = Z = \max \{Z_i \mid i = 1, 2, \ldots, n\}$$

then, the E.V.T is interested in the behaviour of $S^{(z)}_{(n)}$ as the sample size $n$ increases to infinity, then

$$\Pr \left( S^{(z)}_{(n)} \leq Z \right) = \Pr \left( Z_1 \leq z, Z_2 \leq z, \ldots, Z_n \leq z \right)$$

$$= \Pr (Z_1 \leq z) \Pr (Z_2 \leq z) \cdots \Pr (Z_n \leq z)$$

$$= G^n (z).$$

Suppose there are sequences of constants $A^{(z)}_{(n)} > 0$ and $\alpha^{(z)}_{(n)}$ such that

$$\Pr \left( S^{(z)}_{(n)} - B^{(z)}_{(n)} \leq z \right) \rightarrow G (z) \text{ as } n \rightarrow \infty,$$

then if $G(z)$ is a non-degenerate CDF then it will belong to one of the three following fundamental types of classic E.V.T family:

- Type I extreme value distribution (Gumbel distribution);
- Type II extreme value distribution (Fréchet distribution);
- Type III extreme value distribution (Weibull distribution).

In this paper we introduce a new extreme value distribution based on the the Fréchet (Fr) model and the generalized odd generalized exponential $G$ (GOGE)$G$ family. A rv $T$ is said to have the Fr distribution if its probability density function (PDF) and cumulative distribution function (CDF) are given by

$$g_\lambda (Z) = \lambda z^{-\lambda - 1} \exp \left( -z^{-\lambda} \right) \mathbb{1}_{(z > 0)} (\lambda > 0),$$

and

$$G_\lambda (Z) = \exp \left( -z^{-\lambda} \right) \mathbb{1}_{(z > 0)}.$$

Based on Alizadeh et al. [7], the CDF and PDF of the GOGE$G$ family are given, respectively, by

$$F_{\alpha, \beta} (Z) = \begin{cases} 1 - \exp \left[ \frac{-G_\Phi (Z^\alpha)}{1 - G_\Phi (Z^\alpha)} \right] \mathbb{1}_{(\alpha, \beta > 0)} \\
\end{cases} \mathbb{1}_{(\alpha, \beta > 0)}$$

$$f_{\alpha, \beta} (Z) = \alpha \beta g_\Phi (Z) G_\Phi (Z)^{\alpha - 1} \exp \left[ -G_\Phi (Z^\alpha) \right] \left[ 1 - G_\Phi (Z^\alpha) \right]^{\beta - 1} \mathbb{1}_{(\alpha, \beta > 0)}.$$
where $G_{\phi}(z)$ is the baseline CDF depending on a parameter vector $\phi$ and $g_{\phi}(z) = (d/dz)G_{\phi}(z)$ is its corresponding PDF and $\alpha, \beta$ are two additional shape parameters. Using (3) and (2) the CDF of the GOGEFr can be derived as

$$F_{\alpha,\beta,\lambda}(z) = \left(1 - \exp \left\{ -\frac{\alpha z^{-\lambda}}{1 - \exp[-\alpha z^{-\lambda}]} \right\} \right)^\beta \left|_{(\alpha,\beta,\lambda) > 0} \right.$$

(5)

the corresponding PDF of (5) can be expressed as

$$f_{\alpha,\beta,\lambda}(z) = \frac{\alpha \beta \lambda z^{-(\lambda+1)} [1 - \exp[-\alpha z^{-\lambda}]]^2}{[1 - \exp[-\alpha z^{-\lambda}]]^2 \exp[-\alpha z^{-\lambda}] \left[1 - \exp[-\alpha z^{-\lambda}]ight]^{\beta - 1}}$$

(6)

Henceforth, $Z \sim \text{GOGEFr}(\alpha, \beta, \lambda)$ denotes a rv having the PDF in (6). The HRF of Z can be calculated using the well-known relationship $f_{\alpha,\beta,\lambda}(z)/(1 - F_{\alpha,\beta,\lambda}(z))$. Plots of the GOGEFr HRF at some parameters values are presented in Figure 2 to show the flexibility of the new model. For simulating data from the GOGEFr model, if $U \sim U(0,1)$, then

$$z_u = \left\{ -\ln \left( \frac{1 - u^{\frac{1}{\lambda}}}{1 - (1 - u^{\frac{1}{\lambda}})} \right) \right\}^{\frac{1}{\lambda}}$$

(7)

has CDF (5). Now, we provide a useful representation for (6). Using the series expansion

$$(1 - s)^{\lambda} = \sum_{\omega=0}^{\infty} \frac{(-1)^{\omega} \Gamma(1 + c)}{\omega! \Gamma(1 + c - \omega)} s^\omega | |_{|s| < 1 \text{ and } c > 0 \text{ realnon—integer}}$$

the PDF of the GOGEFr density in (6) can be re-expressed as

$$f(z) = \frac{\alpha \beta \lambda z^{-(\lambda+1)}}{[1 - \exp[-\alpha z^{-\lambda}]]^2} \sum_{i=0}^{\infty} \frac{(-1)^{i+1} (i + 1)^{\ell_1} \Gamma (1 + \ell_1)}{\ell_1!} \times \left[1 - \exp\left[-\alpha z^{-\lambda}\right]\right]^{-(\ell_1+2)} \times \left(\frac{\beta - 1}{i}\right) \exp\left[-\alpha z^{-\lambda}\right]^{a(\ell_1+1)}.$$  

By applying the series expansion $(1 - s)^{\lambda}$ we have

$$f(z) = \sum_{\ell_1,\ell_2=0}^{\infty} \zeta(\ell_1,\ell_2) h_{\alpha(\ell_1+\ell_2)}(z),$$

(8)

where the PDF $h_{\alpha(\ell_1+\ell_2)}(z)$ related to Fr model with shape parameter $\lambda$ and scale parameter $\alpha$ and the CDF $\zeta(\ell_1,\ell_2) = \frac{\alpha \beta (\ell_1+1)^{\ell_1}}{\ell_1! [\alpha (1 + \ell_1 + \ell_2)]} [\ell_2 + \sum_{i=0}^{\infty} (-1)^i (\ell_1+1)^{i+1} \left(\frac{\beta - 1}{i}\right)]$ and

$$\zeta(\ell_1,\ell_2) = \alpha \beta \frac{(\ell_1+1)^{\ell_1}}{\ell_1! [\alpha (1 + \ell_1 + \ell_2)]} [\ell_2 + \sum_{i=0}^{\infty} (-1)^i (\ell_1+1)^{i+1} \left(\frac{\beta - 1}{i}\right)].$$

By integrating (8), we have

$$F(z) = \sum_{\ell_1,\ell_2=0}^{\infty} \zeta(\ell_1,\ell_2) h_{\alpha(\ell_1+\ell_2)}(z),$$

where the CDF $H_{\alpha(\ell_1+\ell_2)}(z)$ related to Fr model with scale parameter $\sqrt[\alpha]{(1 + \ell_1 + \ell_2)}$ and shape parameter $\lambda$. Figure 1 shows that the new density function can take a unimodal, symmetric and right skewed shapes. Figure 2 shows that the HRF of the new model can be increasing-constant ($\alpha = \beta = \lambda = 1$) or decreasing ($\alpha = 0.75, \beta = 5.5, \lambda = 0.1$) or increasing ($\alpha = 10, \beta = 3, \lambda = 2$) or upside-down ($\alpha = 0.5, \beta = 500, \lambda = 0.75$) or constant ($\alpha = 1.15, \beta = 0.95, \lambda = 1$). Based on Figure 2 we note that the new model can be used in modelling data which has the mentioned HRFs.

Recently, some new useful contributions were added for expanding the E.V.T such as the exponentiated-Fr (Exp-Fr) by Nadarajah and Kotz [8] who presented useful theoretical result, Nadarajah and Gupta [9] introduced and studied the beta-Fr (B-Fr), Nadarajah and Kotz [10] discussed the sociological models based on Fr rv's, Zaharim et al. [11] applied the Fr model for analysing the wind speed data, Barreto-Souza et al. [12] and Mubarak [13] studied the Fr progressive-type-II censored data, Krishna et al. [14] proposed and studied the Marshall–Olkin-Fr (MO-Fr) and derived some new important properties of the new model along with some applications to extreme value data, Mahmoud and Mandaouh [15] studied the transmuted-Fr (T-Fr), da Silva et al. [16] proposed the gamma extended-Fr (GamE-Fr) with some new real extreme data, Yousof et al. [17] studied the transmuted exponentiated generalized-Fr (TEG-Fr), Mead et al. [18] studied the beta exponential-Fr (BE-Fr), Korkmaz et al. [19] studied the odd Lindley-Fr (OL-Fr) with application to survival times, exceedances of flood peaks and breaking stress of carbon fibres data. Yousof et al. [20] derived the transmuted Topp-Leone-Fr (TTL-Fr) with application and characterizations. Yousof et al. [21] introduced the Topp Leone Generated-Fr (TLG-Fr) and used it for modelling the breaking stress of carbon fibres and strength of glass fibres data set, also they investigated a new regression model for prediction propose. Yousof et al. [22] proposed the Odd log-logistic-Fr (OLL-Fr) and derived its characterizations based on a simple relationship between two truncated moments and in terms of the reverse hazard function also introduced a new regression model based on the new model.

We are motivated to introduce the GOGEFr model for the following reasons:
(1) Expanding the E.V. T by introduce new useful versions of the Fr model with only three parameters as well as deriving its statistical properties (see Section 2).

(2) Introducing some real applications in many fields such as breaking stress and strengths data sets (see Section 6).

(3) Introducing a Fr model with decreasing, increasing, upside-down, increasing-constant and constant HRF (see Figure 2).

2. Mathematical properties

2.1. Some stochastic properties

**Theorem 2.1:** Suppose \( Z_1 \sim \text{GOGEFr}(\alpha_1, \beta_1, \lambda) \) and \( Z_2 \sim \text{GOGEFr}(\alpha_2, \beta_2, \lambda) \). Then \( Z_1 \) is stochastically smaller than \( Z_2 \) if \( \alpha_1 > \alpha_2 \) and \( \beta_1 > \beta_2 \).

Note that for any \( \alpha_1 > \alpha_2 \),

\[
\exp \left[-\alpha_1 z^{-\lambda} \right] > \exp \left[-\alpha_2 z^{-\lambda} \right].
\]

This is true for both integer and fractional values of \( \alpha_1 \) and \( \alpha_2 \). After some algebra, we get the following. Since for \( \alpha_1 > \alpha_2 \) we have

\[
\exp \left[-\alpha_1 z^{-\lambda} \right] > \exp \left[-\alpha_2 z^{-\lambda} \right],
\]

then

\[
\{1 - \exp \left[-\alpha_1 z_1^{-\lambda} \right]\} < \{1 - \exp \left[-\alpha_2 z_2^{-\lambda} \right]\},
\]

\[
\frac{\exp \left[-\alpha_1 z_1^{-\lambda} \right]}{\exp \left[-\alpha_2 z_2^{-\lambda} \right]} > \frac{1 - \exp \left[-\alpha_1 z_1^{-\lambda} \right]}{1 - \exp \left[-\alpha_2 z_2^{-\lambda} \right]}.
\]

and

\[
\frac{-\exp \left[-\alpha_1 z_1^{-\lambda} \right]}{-\exp \left[-\alpha_2 z_2^{-\lambda} \right]} < \frac{1 - \exp \left[-\alpha_1 z_1^{-\lambda} \right]}{1 - \exp \left[-\alpha_2 z_2^{-\lambda} \right]}.
\]

Rest of the proof follows immediately from here
Figure 2. Plots of the GOGEFr HRF at some parameters value.
is the incomplete gamma function, and \( \psi(z) \) is given by

\[
\psi(z) = \sum_{n=0}^{\infty} \frac{\Gamma(n+1)}{n!} \left( z^{-n} - 1 \right).
\]

2.2. Moments

The \( r \)th ordinary moment of \( Z \) is given by

\[
\mu_r = \mathbb{E}(Z^r) = \int_{-\infty}^{\infty} t^r f(z) \, dz.
\]

Then we obtain

\[
\mu_r^{(r)} |_{(\lambda > r)} = \Gamma \left( 1 - \frac{r}{\lambda} \right) \sum_{\ell_1, \ell_2 = 0}^{\infty} \xi(\ell_1, \ell_2) \gamma(1 + \ell_1 + \ell_2),
\]

where

\[
\Gamma(1 + \varphi) = \varphi! = \prod_{h=0}^{\varphi-1} (\varphi - h) = \int_0^\infty t^\varphi \exp(-t) \, dt,
\]

is the complete gamma function.

2.3. Incomplete moment

The \( r \)th incomplete moment, say \( \varphi_r(z) \), of \( Z \) can be derived, from (8), as

\[
\varphi_r(z) = \int_{-\infty}^{z} t^r f(z) \, dz
\]

\[
= \sum_{\ell_1, \ell_2 = 0}^{\infty} \xi(\ell_1, \ell_2) \gamma(1 + \ell_1 + \ell_2) \gamma \left( 1 - \frac{r}{\lambda} \right),
\]

where \( \gamma(\varphi, u) \)

\[
\gamma(\varphi, u) |_{(\varphi \neq 0, -1, -2, \ldots)} = \int_0^u t^{\varphi-1} \exp(-t) \, dt
\]

\[
= \frac{u^\varphi}{\varphi} \left( 1 - \Psi_1(\varphi, \varphi + 1; -u) \right)
\]

\[
= \sum_{\zeta=0}^{\infty} \frac{(-1)^\zeta}{\zeta! (\varphi + \zeta)} u^{\varphi+\zeta},
\]

is the incomplete gamma function, and \( \Psi_1(\cdot, \cdot, \cdot) \) is a confluent hypergeometric function. The first incomplete moment can be calculated by setting \( r = 1 \) in

\[
\varphi_1(z) \text{ as }
\]

\[
\varphi_1(t) = \sum_{\ell_1, \ell_2 = 0}^{\infty} \xi(\ell_1, \ell_2) \gamma(1 + \ell_1 + \ell_2) \gamma \left( 1 - \frac{r}{\lambda} \right) \bigg|_{(\lambda \geq r)}
\]

2.4. The moment generating function (MGF)

The MGF \( M_Z(t) = \mathbb{E}(e^{tZ}) \) of \( Z \) can be derived from Equation (4) as

\[
M_Z(t) |_{(\lambda > r)} = \sum_{\ell_1, \ell_2 = 0}^{\infty} \frac{t^\ell}{\ell!} \xi(\ell_1, \ell_2) \gamma(1 + \ell_1 + \ell_2) \gamma \left( 1 - \frac{r}{\lambda} \right) \cdot \Gamma \left( 1 - \frac{r}{\lambda} \right),
\]

Another alternative method for deriving the MGF can be introduced by the Wright generalized hypergeometric function (WHGF) which is defined by

\[
(c) \Psi \left[ \begin{array}{c} a_1, A_1, \ldots, a_c, A_c \end{array} \bigg| \begin{array}{c} b_1, B_1, \ldots, b_v, B_v \end{array} \bigg| z \right]
\]

\[
= \sum_{n=0}^{\infty} \prod_{j=1}^{c} \Gamma \left( a_j + A_j n \right) z^n \prod_{j=1}^{v} \Gamma \left( b_j + B_j n \right) n!
\]

Then, the MGF of (1) can be defined as

\[
M_Z(t) = (1) \Psi \left[ \begin{array}{c} (1, \frac{1}{2}) \end{array} \bigg| \begin{array}{c} (1) \end{array} \bigg| t \right].
\]

Combining expressions (8) and (10), we obtain the MGF of the GOGEFr as

\[
M_Z(t) = \sum_{\ell_1, \ell_2 = 0}^{\infty} \xi(\ell_1, \ell_2) \cdot \gamma \left( 1 - \frac{r}{\lambda} \right) \bigg|_{(\lambda \geq r)}
\]

\[
\times \left( (1) \Psi \left[ \begin{array}{c} (1, \frac{1}{2}) \end{array} \bigg| \begin{array}{c} (1) \end{array} \bigg| t \right] \right). \tag{11}
\]

Equations (9) and (11) can be easily evaluated by scripts of the Matlab, Maple and Mathematica platforms.

2.5. Residual life and reversed residual life functions

The \( n \)th moment of the residual life, say

\[
\mathbb{E}((Z - t)^n) |_{(Z > t \text{ and } n=1,2,\ldots)},
\]

uniquely determines the CDF \( F(z) \). The \( n \)th moment of the residual life of \( Z \) is given by

\[
\mathbb{E}(z^n) = \frac{\int_{-\infty}^{\infty} (z - t)^n \, dF(z)}{1 - F(t)}.
\]
Therefore

\[ z_n(t|_{\ell > n}) = \frac{1}{1 - F(t)} \sum_{\ell_1, \ell_2=0}^{\infty} a_{\ell_1, \ell_2} \left[ \alpha(1 + \ell_1 + \ell_2) \right]^{\ell_1} \times \Gamma \left( 1 - \frac{n}{\lambda}, \left[ \alpha(1 + \ell_1 + \ell_2) \right] \left( \frac{t}{\ell_1} \right) \right), \]

where

\[ a_{\ell_1, \ell_2} = \frac{\xi(\ell_1, \ell_2)}{\Gamma(\ell_1)} \sum_{r=0}^{n} (-t)^{n-r} \binom{n}{r}, \]

and

\[ \Gamma(\varphi, u) |_{\ell > 0} = \int_0^\infty t^{\varphi-1} \exp(-t) dt, \]

and

\[ \Gamma(\varphi, u) = \Gamma(\omega) = \gamma(\varphi, u). \]

The Rényi entropy of a rv \( Z \) defined by

\[ z_n(t) = E[(t - Z)^n] |_{\ell > 0} \text{ when } n=1,2,\ldots \]

uniquely determines the CDF \( F(z) \). We obtain

\[ z_n(t) = \frac{\int_0^t (t - Z)^n dF(z)}{F(t)}. \]

Then, the nth moment of the reversed residual life of \( Z \) becomes

\[ z_n(t|_{\ell > n}) = \frac{1}{F(t)} \sum_{\ell_1, \ell_2=0}^{\infty} A_{\ell_1, \ell_2} \left[ \alpha(1 + \ell_1 + \ell_2) \right]^{\ell_1} \times \gamma \left( 1 - \frac{n}{\lambda}, \left[ \alpha(1 + \ell_1 + \ell_2) \right] \left( \frac{t}{\ell_1} \right) \right), \]

where

\[ A_{\ell_1, \ell_2} = \frac{\xi(\ell_1, \ell_2)}{\Gamma(\ell_1)} \sum_{r=0}^{n} (-t)^{n-r} \binom{n}{r}. \]

### 2.6. Entropies

The Rényi entropy of a rv \( Z \) represents a measure of variation of the uncertainty. The Rényi entropy is defined by

\[ R_\theta(Z) |_{\ell > 0 \text{ and } \theta \neq 1} = \frac{\log \int_{-\infty}^{\infty} f(z)^\theta dz}{1 - \theta}. \]

Using the PDF (6), we can write

\[ f(z)^\theta = \sum_{\ell_1, \ell_2=0}^{\infty} \xi(\ell_1, \ell_2)^{\theta(\ell_1+\ell_2)} \times \exp\left[ -\left[ \alpha(\ell_1 + \ell_2 + \theta) \right] z^{-\lambda} \right], \]

where

\[ \xi(\ell_1, \ell_2) = (\alpha\beta\lambda)^{\theta(\ell_1+\ell_2)} \left( \frac{-\ell_1+\ell_2}{\ell_1!} \right)^{\ell_2} \times \sum_{j=0}^{\infty} (-1)^j (j + \theta)^{\ell_1} \left( \frac{\theta(\beta - 1)}{j} \right) \]

Then, the Rényi entropy of the GOGEFr model is given by

\[ R_\theta(Z) |_{\ell > 0 \text{ and } \theta \neq 1} = \frac{\log \int_{-\infty}^{\infty} f(z)^\theta dz}{1 - \theta}, \]

where

\[ \xi(\ell_1, \ell_2) = \int_0^{\infty} z^{-\theta(\ell_1+\ell_2)} \exp\left[ -\left[ \alpha(\ell_1 + \ell_2 + \theta) \right] z^{-\lambda} \right] dz. \]

The Shannon entropy of a random variable \( Z \), say \( SE \), is defined by

\[ SE = E\left[ -\log f(Z) \right]. \]

It is the special case of the Rényi entropy, \( R_0(Z) |_{\ell > 0 \text{ and } \theta \neq 1} \), when \( \theta \uparrow 1 \).

### 2.7. Order statistics

Let \( Z_1, Z_2, \ldots, Z_n \) be a rs from the GOGEFr distribution and let \( Z_{1:n}, Z_{2:n}, \ldots, Z_{n:n} \) be the corresponding order statistics. The PDF of the \( r \)th order statistic, say \( Z_{i:n} \), can be written as

\[ f_{i:n}(z) = B^{-1} (i, n - i) \sum_{j=0}^{n-i} (-1)^j \times \binom{n-i}{j} f(z)^i d^{j+i-1}(z), \]

where \( B(\cdot, \cdot) \) is the beta function. Substituting (1) and (2) in Equation (12) and using a power series expansion, we have

\[ f(z) F(z)^{j+i-1} = \sum_{\omega, \ell_2=0}^{\infty} \theta_{\omega, \ell_2} h_{\omega}(\omega + \ell_2 + 1) f(z) \]

\[ \times \sum_{\ell_1=0}^{\infty} (-1)^j (j + \theta)^{\ell_1} \left( \frac{\theta(\beta - 1)}{j} \right) \]
where
\[
\theta_{\omega,t_2} = \frac{\alpha \beta (-1)^{\omega + t_2}}{\omega! [\alpha (\omega + \ell_1 + 1)]} \left( -\frac{\omega + 2}{\ell_2} \right) \times \sum_{i=0}^{\infty} (-1)^j (l + 1)^\omega \frac{\beta (i + j - 1)}{i}.
\]

Then, the PDF of $Z_{\omega_n}$ can be expressed as
\[
f_{\omega_n}(z) = \sum_{j=0}^{n-i} \sum_{\omega,t_2=0}^{\infty} (-1)^j B^{-1}(i, 1 + n - i) \times \left( \frac{n-i}{j} \right) \theta_{\omega,t_2} h_{\omega}(\omega+t_2+1)(z).
\]

The PDF of the GOGEFr order statistics is a mixture of Fr densities. Based on the last equation, the moments of $Z_{\omega_n}$ can be obtained as
\[
E(\frac{Z_{\omega_n}^k}{1+\delta}) = \Gamma \left( 1 - \frac{\delta}{\lambda} \right) \times \sum_{\omega,t_2=0}^{\infty} \sum_{j=0}^{n-i} (-1)^j \frac{\alpha (\omega + \ell_1 + 1)}{B^{-1}(i, 1 + n - i)} \times \left( \frac{n-i}{j} \right) \theta_{\omega,t_2}.
\]

### 2.8. Quantile spread (QS) order

The QS ($\psi_Z(\pi)$) of the rv $Z \sim$ GOGEFr($\alpha, \beta, \lambda$) having the CDF (5) is given by
\[
\psi_Z(\pi) = \left[ F^{-1}(\pi) \right] - \left[ F^{-1}(1 - \pi) \right],
\]
which implies
\[
\psi_Z(\pi) = \left[ S^{-1}(1 - \pi) \right] - \left[ S^{-1}(\pi) \right],
\]
where $F^{-1}(\pi) = S^{-1}(1 - \pi)$ and $S(\cdot) = 1 - F(\cdot)$ is the survival function. Let $Z_1$ and $Z_2$ be two random variables following the GOGEFr model with quantile spreads $\psi_{Z_1}$ and $\psi_{Z_2}$, respectively. Then $Z_1$ is called smaller than $Z_2$ in quantile spread order, denoted as
\[
Z_1 \leq_{[\psi]} Z_2,
\]
if
\[
\psi_{Z_1}(\pi) \leq \psi_{Z_2}(\pi) \mid_{\pi \in \left( \frac{1}{2}, 1 \right)}. 
\]

The following properties of the QS order can be obtained:

- The order $\leq_{[\psi]}$ is location-free
  
  $Z_1 \leq_{[\psi]} Z_2$ if $(Z_1 + \zeta) \leq_{[\psi]} Z_2 \mid \zeta \in \mathbb{R}.$

- The order $\leq_{[\psi]}$ is adilative
  
  $Z_1 \leq_{[\psi]} \zeta Z_1$ whenever $\zeta \geq 1$ and $Z_2 \leq_{[\psi]} \zeta Z_2 \mid \zeta \geq 1.$

- Let $F_{Z_1}$ and $F_{Z_2}$ be symmetric, then
  
  $Z_1 \leq_{[\psi]} Z_2$ if, and only if $F_{Z_1}^{-1}(\pi) \leq F_{Z_2}^{-1}(\pi), \forall \pi \in \left( \frac{1}{2}, 1 \right).$

- The order $\leq_{[\psi]}$ implies ordering of the mean absolute deviation around the median, say $\sigma (Z_1) \mid (i=1,2),$ 
  
  $\sigma (Z_1) = E\left[ |Z_1 - \text{Median}(Z_1)| \right]$ and
  
  $\sigma (Z_2) = E\left[ |Z_2 - \text{Median}(Z_2)| \right],$
  
  where
  
  $Z_1 \leq_{[\psi]} Z_2$ implies $\sigma (Z_1) \leq_{[\psi]} \sigma (Z_2).$

Finally
\[
Z_1 \leq_{[\psi]} Z_2 \text{ if and only if } -Z_1 \leq_{[\psi]} -Z_2.
\]

### 2.9. Numerical analysis for some measures

The mean ($E(Z)$), variance ($\text{Var}(Z)$), skewness ($\text{Ske}(Z)$) and kurtosis ($\text{Ku}(Z)$) measures of the GOGEFr distribution can be calculated from the ordinary moments using well-known relationships. The numerical values in Table 1 indicate that the $\text{Ske}(Z)$ of the GOGEFr distribution is always positive and can range in the interval $(0.2178, 3.0163)$. The spread for its $\text{Ku}(Z)$ is much larger ranging from $3.0053$ to $21.1039$. For all values of $\alpha$ (where $\alpha = 1, 5, 20, 50, 100, 200$) and fixed $\beta = 50$ and $\lambda = 2$, the $\text{Ske}(Z) = 0.706448$ and the $\text{Ku}(Z) = 3.876657$. For all values of $\alpha$ (where $\alpha = 3, 10, 15, 20$) and fixed $\beta = 5$ and $\lambda = 5$, the $\text{Ske}(Z) = 0.21780$ and the $\text{Ku}(Z) = 3.0053$.

### 2.10. 3D plots for Ske(Z) and Ku(Z)

In this subsection, we will plot the skewness and kurtosis measures using the quantile new model. The Bowley’s skewness measure is given by
\[
\text{Ske}(Z) = \frac{Q_{\left( \frac{1}{4} \right)} - 2Q_{\left( \frac{1}{2} \right)} + Q_{\left( \frac{3}{4} \right)}}{Q_{\left( \frac{1}{4} \right)} - Q_{\left( \frac{3}{4} \right)}},
\]
and the Moors’s kurtosis measure is
\[
\text{Ku}(Z) = \frac{Q_{\left( \frac{2}{3} \right)} - Q_{\left( \frac{1}{3} \right)} - Q_{\left( \frac{1}{3} \right)} + Q_{\left( \frac{2}{3} \right)}}{Q_{\left( \frac{1}{3} \right)} - Q_{\left( \frac{2}{3} \right)}}.
\]
Plots of skewness and kurtosis of the new model is presented in Figures 3 and 4. This plot indicates that both measures depend very much on the shape parameters $\alpha$ and $\beta$. 
Table 1. \( E(Z), \text{Var}(Z), \text{Ske}(Z) \) and \( \text{Ku}(Z) \) of the GOGEFr distribution.

| \( \alpha \) | \( \beta \) | \( \lambda \) | Mean     | Variance    | Skewness | Kurtosis |
|----------|----------|----------|----------|------------|----------|----------|
| 1        | 50       | 2        | 2.214869 | 0.075831   | 0.706448 | 3.876657 |
| 5        | 4.952597 | 0.379154 | 0.706448 | 3.876657   | 3.876657 |
| 20       | 9.905194 | 1.516617 | 0.706448 | 3.876657   | 3.876657 |
| 50       | 15.66149 | 3.791543 | 0.706448 | 3.876657   | 3.876657 |
| 100      | 22.14869 | 7.583086 | 0.706448 | 3.876657   | 3.876657 |
| 200      | 31.32297 | 15.16617 | 0.706448 | 3.876657   | 3.876657 |
| 3        | 5        | 5        | 1.502522 | 0.016697   | 0.21780  | 3.0053   |
| 10       | 1.911598 | 0.027026 | 0.21780  | 3.0053     | 3.0053   |
| 15       | 2.073074 | 0.03178  | 0.21780  | 3.0053     | 3.0053   |
| 20       | 1.911598 | 0.027026 | 0.21780  | 3.0053     | 3.0053   |

3. Simple type copula

3.1. Via Morgenstern (Mor) family

First, we start with CDF for Mor family of two random variables \((Z_1, Z_2)\) which has the following form

\[
F_{\theta}(z_1, z_2) = \{1 + \theta [1 - F_{1}(z_1)][1 - F_{2}(z_2)]\} \times F_{1}(z_1) F_{2}(z_2),
\]

setting

\[
F_{1}(z_1) = F_{\alpha_1, \beta_1, \lambda}(z_1) = \left(1 - \exp \left\{ -\exp \left[ -\alpha_1 z_1^{\lambda} \right] \right\} \right)^{\beta_1},
\]

and

\[
F_{2}(z_2) = F_{\alpha_2, \beta_2, \lambda}(z_2) = \left(1 - \exp \left\{ -\exp \left[ -\alpha_2 z_2^{\lambda} \right] \right\} \right)^{\beta_2},
\]

then we have a seven-dimension parameter model.

3.2. Via Clayton copula

3.2.1. The bivariate extension

The bivariate extension via Clayton copula can be considered as a weighted version of the Clayton copula, which is of the form

\[
C(u, v) = u^{-(\alpha_1 + \beta_2)} + v^{-(\alpha_1 + \beta_2)} - 1^{-(\alpha_1 + \beta_2)}.
\]

This is indeed a valid copula. Next, let us assume that \(Z \sim \text{GOGEFr}(\alpha_1, \beta_1, \lambda)\) and \(Y \sim \text{GOGEFr}(\alpha_2, \beta_2, \lambda)\). Then, setting

\[
u = F(z) = \left(1 - \exp \left\{ -\exp \left[ -\alpha_1 z^{\lambda} \right] \right\} \right)^{\beta_1},
\]

Figure 3. 3D plot for skewness of the new model when \(\lambda = 2.25\).

Figure 4. 3D plot for kurtosis of the new model when \(\lambda = 2.25\).
and
\[ v = F(y) = \left(1 - \exp \left(-\exp \left[-\alpha_2 y^{-\lambda} \right] \right) \right)^{\beta_2}, \]
the associated bivariate CDF of the GOGEFr type distribution will be
\[ H(z_1, z_2) = \left[ \begin{array}{c} \left(1 - \exp \left(-\alpha_1 z_1^{-\lambda} \right) \right)^{-\beta_1(1+\delta_2)} \left(1 - \exp \left(-\alpha_2 z_2^{-\lambda} \right) \right)^{-\beta_2(1+\delta_2)} \\ + \left(1 - \exp \left(-\alpha_1 z_1^{-\lambda} \right) \right)^{\frac{1}{\lambda}} \left(1 - \exp \left(-\alpha_2 z_2^{-\lambda} \right) \right)^{\frac{1}{\lambda}} \end{array} \right]. \]

Note: Depending on the specific baseline CDF, one may construct various bivariate GOGEFr type models in which \((\delta_1 + \delta_2) \geq 0\).

### 3.2.2. The multivariate extension

A straightforward \(d\)-dimensional extension from the above will be
\[ H(z_1, z_2, \ldots, z_d) = \left[ \sum_{i=1}^{d} \left(1 - \exp \left(-\alpha_i z_i^{-\lambda} \right) \right)^{-\beta_i(1+\delta_2)} \right]^{-\frac{1}{\lambda}} - d + 1. \]

Further future works could be allocated for studying the bivariate and the multivariate extensions of the GOGEFr model.

### 4. Estimation

Let \(z_1, \ldots, z_n\) be a \(\text{rs}\) from the GOGEFr distribution with parameters \(\alpha, \beta, \lambda\). Let \(\Theta = (\alpha, \beta, \lambda)\) be the \(3 \times 1\) parameter vector. For determining the MLE of \(\Theta\), we have the log-likelihood function
\[ \ell(\alpha, \beta, \lambda) = n \log \alpha + n \log \beta + n \log \lambda + n \lambda \log - (\lambda + 1) \sum_{i=1}^{n} \log z_i - 2 \sum_{i=1}^{n} \log \left[1 - s_i^{(\alpha, \lambda)}\right]. \]

### 5. Simulations

We perform a Monte Carlo (MC) simulation study to verify the finite sample behaviour of the MLEs of \(\alpha, \beta, \lambda\). In each replication, a \(\text{rs}\) of size \(n\) where \(n = 50, 100, 250, 500\) and 1000) is drawn from \(z \sim \text{GOGEFr}(\alpha, \beta, \lambda)\) and the conjugate gradient method has been used for maximizing the total log-likelihood function. The GOGEFr random number generation is performed using the inversion method via (7). Table 2 lists the averages of the estimates (AEs) and mean square errors (MSEs) of the MLEs of the model parameters. The results in both tables indicate that the MSEs decrease when the sample size increases as expected under first-order asymptotic theory.

### 6. Real data modelling

This section presents two applications of the new distribution using real data sets. We shall compare the fit of the new distribution with the Poisson Burr X Fr (PBXFr) (Yousuf et al. (2017)), Weibull Fr (WFr) (Affify et al. [23]), exponentiated Fr (EFR) (Nadarajah and Kotz [8]), Kumaraswamy Fr (KwFr) (Mead and Abd-Eltawab [5]),

### Table 2. The AVs and MSEs based on 1000 simulations.

| \(n\)  | \(\alpha = 2.5, \beta = 5.5\) and \(\lambda = 2\) | \(\alpha = 3, \beta = 20\) and \(\lambda = 3.5\) |
|--------|---------------------------------|---------------------------------|
|        | AVs    | MSE    | AVs    | MSE    |
| 50     | \(\alpha\) | 2.891540519 | 1.80370491 | \(\alpha\) | 3.406603384 | 0.29932676 |
|        | \(\beta\) | 5.813742365 | 0.963315616 | \(\beta\) | 20.663365021 | 0.59633811 |
|        | \(\lambda\) | 2.503305047 | 1.173086191 | \(\lambda\) | 3.781045612 | 0.29325471 |
| 100    | \(\alpha\) | 2.818616655 | 1.153440705 | \(\alpha\) | 3.435668889 | 0.20995832 |
|        | \(\beta\) | 5.713702301 | 0.661073035 | \(\beta\) | 20.397700301 | 0.30548321 |
|        | \(\lambda\) | 2.395119874 | 1.005273523 | \(\lambda\) | 3.637014903 | 0.22340361 |
| 250    | \(\alpha\) | 2.610646671 | 0.790881432 | \(\alpha\) | 3.198473321 | 0.10481737 |
|        | \(\beta\) | 5.59082872 | 0.119432729 | \(\beta\) | 20.174441904 | 0.10052342 |
|        | \(\lambda\) | 2.255671987 | 0.332263043 | \(\lambda\) | 3.542253882 | 0.10212930 |
| 500    | \(\alpha\) | 2.52402446 | 0.003265512 | \(\alpha\) | 3.104156544 | 0.003401343 |
|        | \(\beta\) | 5.010431501 | 0.013409144 | \(\beta\) | 20.023244115 | 0.01450342 |
|        | \(\lambda\) | 2.11320380 | 0.010022742 | \(\lambda\) | 3.500873701 | 0.004975512 |
| 1000   | \(\alpha\) | 2.500121144 | 0.00015034 | \(\alpha\) | 3.003397020 | 0.000490531 |
|        | \(\beta\) | 5.500124326 | 0.000740431 | \(\beta\) | 20.002531012 | 0.000352345 |
|        | \(\lambda\) | 2.000891873 | 0.001333511 | \(\lambda\) | 3.500723549 | 0.000232534 |
beta Fr (BFr) (Barreto-Souza et al. [12]), transmuted Fr (TFr) (Mahmoud and Mandouh [15]), gamma extended Fr (GEFr) (Silva et al. [24]), Marshall–Olkin Fr (MOFr) (Krishna et al. [14]), and Fr (Fréchet [1]) distributions. The PDFs of the competitive model are available in statistical literature. The unknown parameters of the above PDFs are all positive real numbers except for the T-Fr and TMOFr distribution for which $|\lambda| \leq 1$. The 1st data set consists of 100 observations of breaking stress of carbon fibres given by Nichols and Padgett [25] (0.920, 0.9280, 0.9971, 1.0610, 1.117, 1.1620, 1.183, 1.187, 1.1920, 1.196, 1.2130, 1.215, 1.2199, 1.220, 1.2240, 1.225, 1.2280, 1.237, 1.244, 1.259, 1.2610, 1.263, 1.276, 1.310, 1.3210, 1.3290, 1.3310, 1.337, 1.351, 1.359, 1.388, 1.4080, 1.449, 1.4497, 1.450, 1.459, 1.471, 1.475, 1.477, 1.480, 1.489, 1.501, 1.507, 1.515, 1.530, 1.5304, 1.533, 1.544, 1.552, 1.556, 1.5620, 1.566, 1.585, 1.586, 1.599, 1.602, 1.6140, 1.6160, 1.617, 1.625, 1.6280, 1.6840, 1.7110, 1.7180, 1.733, 1.7380, 1.7430, 1.7590, 1.777, 1.7940, 1.799, 1.806, 1.816, 1.8160, 1.8280, 1.830, 1.884, 1.892, 1.944, 1.972, 1.9840, 1.987, 2.02, 2.0304, 2.0290, 2.0350, 2.0370, 2.0430, 2.0460, 2.0590, 2.111, 2.115, 2.165, 2.686, 2.778, 2.972, 3.054, 3.863, 5.3060).

The 2nd data set consists of 63 observations of the strengths of 1.5 cm glass fibres (see Smith and Nayler [26]) (1.312, 1.314, 1.479, 1.552, 1.700, 1.803, 1.861, 1.865, 1.944, 1.958, 1.966, 1.997, 2.006, 2.021, 2.027, 2.055, 2.063, 2.098, 2.140, 2.179, 2.224, 2.240, 2.253, 2.270, 2.272, 2.274, 2.301, 2.301, 2.359, 2.382, 2.382, 2.426, 2.434, 2.435, 2.478, 2.490, 2.511, 2.514, 2.535, 2.554, 2.566, 2.570, 2.586, 2.629, 2.633, 2.642, 2.648, 2.684, 2.697, 2.726, 2.770, 2.773, 2.800, 2.809, 2.818, 2.821, 2.821, 2.848, 2.880, 2.809, 2.818, 2.821, 2.848, 2.880, 2.954, 3.012, 3.067, 3.084, 3.090, 3.096, 3.128, 3.233, 3.433, 3.585, 3.585). Many other data sets can be found in Afify et al. [23], Korkmaz et al. [19], Abd El-Bar and Ragab [27], Hamedani et al. [28], Ahsanul Haq et al. [29], Basheer [30], Korkmaz et al. [31], Mukhtar [32], Alizadeh et al. [33,34], Yousof et al. [35–37] and Korkmaz et al. [38].

In order to compare the distributions, we consider the following criteria: the $-2\hat{\ell}(\alpha, \beta, \lambda)$ (Maximized

![Figure 5. TTT plots. (a) The 1st data set. (b) The 2nd data set.](image)
Table 5. The statistics AIC, BIC, HQIC and CAIC values for glass fibre data.

| Model  | AIC    | BIC    | HQIC   | CAIC   |
|--------|--------|--------|--------|--------|
| GOGEFr | 48.7   | 55.1   | 51.2   | 49.1   |
| TMOFr  | 56.5   | 65.0   | 59.8   | 57.1   |
| BFr    | 68.6   | 77.2   | 72.0   | 69.3   |
| GEFr   | 69.6   | 78.1   | 72.9   | 70.3   |
| Fr     | 97.7   | 102    | 99.4   | 97.9   |
| MOFr   | 101.7  | 108.2  | 102.6  | 100.5  |

Log-Likelihood), AIC (Akaike Information Criterion), CAIC (Consistent AIC), BIC (Bayesian IC) and HQIC (Hannan-Quinn IC). These statistics are given by

\[ AIC = 2 \left[ -\hat{\ell}(\alpha, \beta, \lambda) + \kappa(p) \right], \]

\[ BIC = 2 \left[ -\hat{\ell}(\alpha, \beta, \lambda) + \frac{1}{2} \kappa(p) \log(n) \right], \]

\[ HQIC = 2 \left[ -\hat{\ell} + \kappa(p) \log(\log(n)) \right], \]

and

\[ CAIC = 2 \left[ -\hat{\ell}(\alpha, \beta, \lambda) + \frac{n \kappa(p)}{n - \kappa(p) - 1} \right], \]

where \( \hat{\ell}(\alpha, \beta, \lambda) \) denotes the log-likelihood function evaluated at the MLEs, \( \kappa(p) \) is the number of model parameters and \( n \) is the sample size. The model with minimum

Table 6. MLEs and their SEs for glass fibre data.

| Model      | Estimates                      |
|------------|--------------------------------|
| GOGEFr(\alpha, \beta, \lambda) | 0.18699 (0.0292) 640.377 (549.54) 0.7340 (0.1046) |
| BFr(\alpha, \beta, \lambda, a, b) | 2.0518 (0.986) 0.6466 (0.163) 15.0756 (12.057) 36.9397 (22.649) |
| MOFr(\lambda, \theta, a, b) | 0.65 (0.049) 6.8744 (0.596) 376.268 (246.832) 0.1499 (0.302) |
| GEFr(\alpha, \beta, \lambda, a, b) | 1.6625 (0.952) 0.7421 (0.197) 32.112 (17.397) 13.2688 (9.967) |
| Tr(\alpha, \beta, a) | 1.3068 (0.034) 2.7898 (0.165) 0.1298 (0.208) |
| MOFr(\alpha, \beta, a) | 1.5441 (0.226) 2.3876 (0.253) 0.4816 (0.252) |
| Fr(\alpha, \beta) | 1.264 (0.059) 2.888 (0.234) |

Figure 6. Estimated CDFs. (a) The 1st data set. (b) The 2nd data set.

Figure 7. Estimated PDFs. (a) The 1st data set. (b) The 2nd data set.
Figure 8. Estimated HRFs. (a) The 1st data set. (b) The 2nd data set.

Figure 9. P–P plots. (a) The 1st data set. (b) The 2nd data set.

Figure 10. Kaplan–Meier Survival Plots. (a) The 1st data set. (b) The 2nd data set.
values for these statistics could be chosen as the best model to fit the data.

Total time test (TTT) plot (see Figure 3) for the two real data sets is presented in Figure 5, these plot indicates that the empirical HRFs of the the two data sets are increasing. Tables 3 and 5 compare the GOGEGFr model with other important competitive Fr distributions. Tables 4 and 6 give the MLEs and their standard errors (SEs) for the two data sets. The GOGEGFr model gives the lowest values for the AIC, BIC, HQIC and CAIC statistics (values in bold line) among all fitted Fr models. Figures 6–10, respectively, display the plots of estimated CDFs, PDFs, HRFs, P–P and Kaplan–Meier survival plots for the 1st and 2nd data. These plots reveal that the proposed distribution yields a sufficient fit for both data sets.

7. Conclusions

A new three-parameter extension of the Fréchet model is proposed and studied. Some of its fundamental statistical properties such ordinary and incomplete moments, moments generating functions, order statistics, quantile spread ordering, Rényi, Shannon and q-entropies, some stochastic properties, residual and reversed residual life functions are derived. A simple type Copula-based construction via Morgenstern family and via Clayton copula is used to derive many bivariate and multivariate extensions of the new model. We assessed the performance of the maximum likelihood estimators using a simulation study. The importance of the new model is shown via two applications to real data sets. We hope that the new model can be used in modelling data such as the extreme floods, maximum sizes of ecological populations, the size of freak waves, the amounts of large insurance losses, equity risks; day to day market risk, side effects of drugs, survival times, large wildfires and repair times.

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