Correlation energy of fractional-quantum-Hall-effect systems of composite fermions

Piotr Sitko and Lucjan Jacak

Institute of Physics, Technical University of Wrocław,
Wyb. Wyspiańskiego 27, 50-370 Wrocław, Poland.

Abstract

We find the RPA correlation energy and collective modes of the systems of composite fermions corresponding to the fillings of $1/3$, $1/5$, $2/5$ in the fractional quantum Hall effect. It is verified that transmutations to composite fermions do not change the ground-state energy.
1. Introduction

The fractional quantum Hall effect (FQHE) is originally explained within the idea of the Laughlin wave function [1]. An alternative approach was proposed by Jain [2, 3] who noted that in two-dimensional (2D) systems the antisymmetricity of the many-particle wave function is not an unambiguous criterion of quantum statistics. The antisymmetricity is held by a class of quantum-statistics particles called composite fermions [2]. An interchange of two composite fermions produces a phase factor of \( e^{i(2p+1)\pi} \) (\( p \) - an integer number). Note that this is also the case of the Laughlin wave function.

Let us consider 2D electron gas in the external magnetic field. Statistics transmutations to composite fermions are given by Chern-Simons gauge field which is equivalent to attaching an even number (2p) of flux quanta to each electron [4, 5]:

\[
H = \frac{1}{2m} \sum_{i=1}^{N} \left( \mathbf{p}_i + \frac{e}{c} \mathbf{A}_i + \frac{e}{c} \mathbf{A}^{ex}_i \right)^2
\]

where

\[
\mathbf{A}_i = -\frac{2p\hbar c}{e} \hat{z} \times \sum_{j \neq i} \frac{(\mathbf{r}_i - \mathbf{r}_j)}{|\mathbf{r}_i - \mathbf{r}_j|^2},
\]

\( \hat{z} \) - a perpendicular to the plane unit vector. Replacing the sum of point fluxes by an average flux one finds the average statistical field \( B^s = -2p\hbar c/e\rho \) which reduces the effective field acting on electrons to \( B^{ef} = B^{ex} + B^s \). We predict an analog of the quantum Hall effect when, effectively, \( n \) Landau levels are completely filled, i.e. \( B^{ef} = \frac{1}{n} \frac{\hbar c}{e} \rho \). Hence, \( B^{ex} = 2p\hbar c/n e \rho \), which means that from the point of view of the external magnetic field the lowest Landau level is filled in the fraction \( \nu = \frac{n}{2p+1} \). The RPA calculations confirm that the system (1) exhibits the fractional quantum Hall effect [4, 6].

Considering the Hartree-Fock approximation of the system (1) one finds the ground-state energy [3, 4]:

\[
\frac{< H >}{N} = \frac{1}{2} E_F \left( 1 + 3p^2 + \frac{2p}{n} \right) - \frac{p^2}{n}.
\]

where \( E_F = \frac{2\pi\hbar^2}{m} \) - the Fermi energy of the 2D electron gas. The energy of electrons in the external magnetic field \( B^{ex} = \frac{2p\hbar c}{n} e \rho \) is \( \frac{1}{2} N E_F (2p + \frac{1}{n}) \). Comparing with the result
(3) one finds the cost of the transmutation to composite fermions:

\[ \frac{\Delta E}{N} = \frac{1}{2} E_F [2p^2 + (p - 1)^2 (1 - \frac{1}{n})] \]  

which is always greater than zero and increases rapidly with \( p \).

One of the crucial ideas supporting the existence of new realisations of quantum statistics in 2D systems is the braid group argument [8]. However, composite fermions have the same braid group representation as fermions. Hence, it is reasonable to expect transmutations to composite fermions not to change the ground-state energy. In this paper we verify the result (3) by calculating the RPA correlation energy for filling fractions 1/3, 1/5 and 2/5. We find also dispersion relations of collective modes at these fractions.

2. RPA correlation energy

The Hamiltonian \( H \) can be separated into two parts: \( H = H_0 + H_1 \) where \( H_0 = \frac{1}{2m} \sum_i (P_i + A_{ie}^f)^2 \) is treated as the unperturbed term and \( H_1 \) is the interaction Hamiltonian [8, 9]. In this paper we assume that \( B_{ef} = \nabla \times A_{ef} = B_{ex} + B_s = \frac{1}{n} \frac{hc}{e} \rho \) and in the ground state one has \( n \) completely filled Landau levels. Let us define the current density:

\[ j(r) = \frac{1}{2m} \sum_j \left\{ P_j + \frac{e}{c} A_{ej}^f, \delta(r - r_j) \right\} \]  

where braces denote an anticommutator, \( j \) is the vector part of \( j^\mu \) with \( \mu = 0, x, y \). We define \( j^0 \) as density fluctuations: \( j^0 = \sum_j \delta(r - r_j) - \rho \).

It can be shown that the correlation energy (if omitting three-body contributions) is given by [10, 11]

\[ E_c^{RPA} = -\frac{1}{2} \hbar L^2 \int \frac{dq}{(2\pi)^2} \int_0^\infty \frac{d\omega}{\pi} \int_0^1 \frac{d\lambda}{\lambda} \text{Im} \text{tr}(\lambda V(q))(D_{\lambda}^{RPA}(q, \omega) - D_0(q, \omega)) \]  

where \( D_{\lambda}^{RPA} \) is the correlation function of effective field currents:

\[ D_{\mu \nu}^{RPA}(r, r') = -\frac{i}{\hbar} < T[j^\mu(r), j^{\nu'}(r')] > \]
given within the random-phase approximation (with the coupling constant \( \lambda \)):

\[
D^{RPA}_\lambda(q, \omega) = [I - \lambda D_0(q, \omega)V(q)]^{-1}D_0(q, \omega). \tag{8}
\]

The interaction matrix \( V \) is obtained from the Hamiltonian \( H_1 \). We choose \( q = q \hat{x} \) and the Coulomb gauge which reduce the problem to \( 2 \times 2 \) \( D^{RPA}_{\mu \nu} \) matrix (\( \mu = 0, y \)) \[12\]. Taking \( \omega^{ef} = \frac{eB^{ef}}{cm} \) and \( a_0^{ef} = \sqrt{\frac{\hbar c}{eB^{ef}}} \) to be frequency and length units, respectively one finds \[4, 12\]:

\[
V(q) = \frac{4p\pi}{q^2} \begin{pmatrix} 2pn & -iq \\ iq & 0 \end{pmatrix}. \tag{9}
\]

As in the case of anyons \[13\] we have

\[
D_0(q, \omega) = \frac{n}{2\pi} \begin{pmatrix} q^2 \Sigma_0 & -iq \Sigma_1 \\ iq \Sigma_1 & \Sigma_2 \end{pmatrix} \tag{10}
\]

where

\[
\Sigma_j = \sum_{m=1}^{\infty} \sum_{l=0}^{n-1} (\omega)^2 - (m - l - i\eta)^2 \frac{l!}{m!} x^{m-l-1}[L_{m-l}^j(x)]^2 \times [(m - l - x)L_{m-l}^j(x) + 2x\frac{dL_{m-l}^j(x)}{dx}]^j
\]

and \( x = \frac{q^2}{2} \) \( (L_{m-l}^j - \text{Laguerre polynomials}) \). Then one obtains:

\[
D^{RPA}(q, \omega) = \frac{n}{2\pi \det} \begin{pmatrix} q^2 \Sigma_0 & -iq \Sigma_s \\ iq \Sigma_s & \Sigma_p \end{pmatrix} \tag{12}
\]

where \( \det = \det(I - D^0V) = (1-2pn\Sigma_1)^2 - (2pn)^2 \Sigma_0(1 + \Sigma_2), \Sigma_s = \Sigma_1 - 2pn\Sigma_1^2 + 2pn\Sigma_0\Sigma_2, \Sigma_p = (2pn)^2 \Sigma_1^2 + \Sigma_2 - (2pn)^2 \Sigma_0 \Sigma_2 \).

Collective modes are determined by the poles of the correlation function \( D^{RPA} \). In plotting Fig. 1-3 we have used simpler relation finding zeros of the determinant \( \det \). In units of \( \hbar \omega^{ef} \) the correlation energy can be expressed as follows \[10\]:

\[
E^{RPA}_c = \frac{N}{2n} \int_0^\infty dq \int_0^\infty d\omega \frac{d\omega}{\pi} \text{Im}([\ln \det + \text{tr}(V(q)D_0)]) \tag{13}
\]

which is equal to

\[
E^{RPA}_c = \frac{N}{2n} \int_0^\infty dq \int_0^\infty d\omega \frac{d\omega}{\pi} \text{Im}([\ln \det + 2pn(2pn\Sigma_0 + 2\Sigma_1))]. \tag{14}
\]
Following the results of Hanna and Fetter \[10\] we can write:

\[
\int_0^\infty dx \int_0^\infty \frac{d\omega}{\pi} \text{Im}\Sigma_0(x, \omega) = -\frac{1}{2} \sum_{m=1}^\infty \frac{1}{m} + \frac{1}{2} (S_n - 1)
\]  

(15)

\(S_n = \sum_{j=1}^n \frac{1}{j}\) and

\[
\int_0^\infty dx \int_0^\infty \frac{d\omega}{\pi} \text{Im}\Sigma_1(x, \omega) = 0.
\]  

(16)

In the following section we will calculate the remaining integral numerically using dispersion relations of collective modes.

3. The results for \(n = 1\) and \(n = 2\)

Let us consider first the case of the one filled Landau level \((n = 1)\), then double poles in the determinant \(det\), generally appearing in the expression \(\Sigma_1^2 - \Sigma_0 \Sigma_2\), cancel out and we have an infinite set of modes with shortwavelength behaviour like \(\omega_m(q \to \infty) = m\) – Fig.1-2. One can see that at the frequency of \(2pn + 1\) (in units of \(\omega_{ef}\) and thus at \(\omega_{ex} = \frac{eB_{ex}}{cm}\)) the pole at \(q \to 0\) is degenerated which was first predicted by Lopez and Fradkin \[5\]. It can be shown that \[10\]:

\[
\int_0^\infty \frac{d\omega}{\pi} \text{Im} \ln det = \sum_{m=1}^\infty (\omega_m - m) = \sum_{m=1}^\infty \Delta \omega_m
\]  

(17)

and then

\[
\frac{E^{RPA}_c}{N} = \frac{1}{2} \sum_{m=1}^\infty \left( \int \Delta \omega_m(x) dx - (2p)^2 \frac{1}{2m} \right).
\]  

(18)

The integrals have been calculated numerically using \(k\)-point Gauss-Laquerre integration \[14\]. It was verified that the summation over \(m\) converges well and the sums have been truncated at \(2k\) terms. The results for filling fractions \(1/3, 1/5\) are given in Table I and show that large Hartre-Fock contributions are canceled out.
Table I. RPA correlation energies for $\nu = 1/3$, 1/5, 2/5 (in respective units of $\hbar \omega_c^F$). To obtain approximated value we have used a functional dependence $A + B k^{-2}$. The cost of the transmutation to composite fermions $\frac{\Delta E}{N}$ combines Hartree-Fock energy and RPA correlation energy.

In the case of two occupied Landau levels ($n = 2$) the doubles poles in the determinant $det$ are present. In contrast to the result of Ref.\[5\] every root higher than first is splitted into two \[10\] (for $m > 1$, $\omega_m^- (q \to \infty) = m = \omega_m^+ (q \to \infty)$ – Fig. 3). Again at $\omega_c^{ex}$ one finds at $q = 0$ different splitting, however, into three roots. One has

$$\int_0^\infty \frac{d \omega}{\pi} \text{Im} \ln det = \Delta \omega_1 + \sum_{m=2}^\infty (\Delta \omega_m^- + \Delta \omega_m^+)$$

and the correlation energy is then given by

$$\frac{E^{RPA}}{N} = \frac{1}{4} \left( \int \Delta \omega_1 (x) dx - 8 p^2 \right) + \frac{1}{4} \sum_{m=2}^\infty \left[ \int (\Delta \omega_m^- (x) + \Delta \omega_m^+ (x)) dx - (4p)^2 \frac{1}{2m} \right] + p^2.$$  \quad (20)

The result for the filling of 2/5 is given in Table I, as in the case of 1/3 the cost of the transmutation to composite fermions is close to zero.

4. Conclusions

The FQHE systems of composite fermions are considered within the random-phase approximation. In agreement with the results of Lopez and Fradkin \[5\] the double
degeneracy of the collective mode at $\omega_{\text{ex}}$ (at $q = 0$) for one filled Landau level is found – Fig. 1-2. However, in contrast to Ref. [5], for two filled Landau levels we have the additional splitting into two roots of each mode ($m \neq 1$) [10]. At $\omega_{\text{ex}}$ one finds at $q \to 0$ the splitting into three roots – Fig. 3.

The RPA correlation energies at the fractions $1/3$, $1/5$, $2/5$ are obtained verifying the Hartree-Fock results [6]. For $p = 1$ (the minimal transmutation, $\nu = 1/3$, $2/5$) it is shown that the cost of the transmutation to composite fermions is close to zero. However, for $p = 2$ ($\nu = 1/5$) the cost becomes significantly negative. It is suspected that such discrepancy comes from omitting three-body contributions in the RPA calculations [10]. For $p > 1$ the logarithmic interaction in the potential matrix $V(q)$ dominates other contributions which affects the RPA result. Nevertheless, the obtained results strongly suggest the cost of transmutations to composite fermions to be zero.

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