Post-Newtonian approximation of gravitational waves from the inspiral phase

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Abstract. We review the post-Newtonian (PN) expansion for the relativistic correction to the motion of a binary system of black holes. Using the Newtonian formalism, we derive the orbital equations for the 2nd order PN corrections, considering small velocities and weak fields. From a wave equation obtained by the post-Newtonian approximation, we find the $h_+$ and $h_\times$ polarization of the Gravitational Waves (GW) which could be detected by Laser Interferometer Gravitational Observatory (LIGO).

1. Introduction
A new window to explore our universe will be possible because of the GW discovery [1]. GW were predicted by Einstein in 1916 [2, 3], he described them as ripples in the fabric of space-time caused by violent and energetic processes in the Universe such as binary black hole collisions. On February 2016 the first detection of GW was announced. The detection of the binary black hole systems GW150914, GW151226 and GW170104 [4]-[6] was possible due to three fundamental scientific and technological achievements. First, the solution of Einstein’s equations through the PN approximation to the relativistic two-body problem and numerical relativity have allowed to obtain accurate GW waveforms during the coalescence of binary BH. Second, the Laser Interferometer Gravitational Observatory (LIGO) [7] detector, which is an extremely sensitive instrument that is able to measure the very tiny GW signals generated by distant astrophysical systems. Third, the data analysis algorithms, which aim to pull out GW signals from the measurements provided by LIGO.
A particular type of GW are produced by orbiting pairs of massive and dense objects, in circular orbits. In this process, the system evolves in three different phases, inspiral, merger and ringdown. In this work we stay in the inspiral phase describing the analytic PN expansion to find the first relativistic corrections to the equations of motion of a binary black holes system. The expansion assumes velocities to be small $v \ll 1$ and a weak field $\Phi$. Using Newtonian approximation, we derive the analytic expression for the second order correction 2PN \cite{8}. The 2PN approximation was useful to construct GW templates for binary systems which were injected in the S5 LIGO run \cite{9, 10}.

The present lecture notes are partially based on our paper \cite{11}, for a better description of a GW detection, we recommend review it. In section (II) we described the Newtonian approximation formalism. In section (III) we review the second order PN approximation to obtain the gravitational wave $h(t)$, in section (IV) we give some conclusions.

2. Gravitational Theory

2.1. General Relativity

The theory of General Relativity describes space and time as part of the same entity, their representation in any coordinate system is expressed by the line element as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu,$$

where $g_{\mu\nu}$ is the metric tensor. The representation of the geometry in the Newtonian spacetime is expressed by

$$ds^2 = (1+2\Phi) dt^2 - (1-2\Phi) \delta^{ij} dx^i dx^j,$$

where $\Phi$ is the Newtonian gravitational potential of classical Newtonian gravity.

In the General Relativity theory, the geometry of the spacetime, produced by the matter and energy is described by the Einstein field equations

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} R = \frac{8 \pi G}{c^4} T_{\mu\nu},$$

where $G_{\mu\nu}$ is the Einstein tensor, $R_{\mu\nu}$ and $R$ are the Ricci tensor and scalar respectively, $T_{\mu\nu}$ is the stress-energy density tensor and $G$ is the Newton's constant of gravitation.

2.2. Gravitational Radiation

If the spacetime is flat with a weak perturbation on it, we can describe it as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad |h_{\mu\nu}| \ll 1,$$

where $\eta_{\mu\nu}$ is the flat Minkowski metric and $h_{\mu\nu}$ is a perturbation to the metric. This $g_{\mu\nu}$ is replaced in equation (3) and the linearized Einstein equations are obtained, which must be invariant under coordinate transformation (gauge transformation) \cite{12}

$$\tilde{T}_{\mu\nu} = h_{\mu\nu} + 2 \partial_{(\mu} \xi_{\nu)},$$

where $\xi_{\nu}$ is so small that terms of order $O(h_{\mu\nu}(\partial_{\mu}\xi^\nu))$ and higher can be neglected of the perturbation. So, the linearized Einstein field equations become

$$\square \tilde{T}_{\mu\nu} + g_{\mu\nu} \partial^\rho \partial^\sigma \tilde{T}_{\rho\sigma} - \partial^\rho \partial_{\rho} \tilde{T}_{\mu\nu} - \partial^\rho \partial_{\mu} \tilde{T}_{\nu\rho} = -\frac{16 \pi G}{c^4} T_{\mu\nu},$$

where $\mu, \nu$ are index which run from 0 to 3 and $i, j$ run from 1 to 3.
where \( \Box \equiv \eta_{\mu\nu} \partial_{\mu} \partial_{\nu} \) is the D’Alambertian operator\(^2\). To further simplify this expression, it is convenient to use harmonic gauge such that \( \partial^\nu \bar{T}_{\mu\nu} = 0 \), which gives

\[
\Box \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}. \tag{7}
\]

If we make a measurement very far from the object, the energy-momentum tensor becomes \( T_{\mu\nu} = 0 \). Substituting the D’Alambertian operator, we obtain the wave equation in vacuum

\[
\left(-\frac{1}{c^2} \frac{\partial}{\partial t^2} + \nabla^2\right) \bar{h}_{\mu\nu} = 0, \tag{8}
\]

where \( \bar{h}_{\mu\nu} \) is the perturbation in the spacetime geometry whose solution is

\[
\bar{h}_{\mu\nu} = C_{\mu\nu} \exp(ik_{\lambda} x^{\lambda}), \tag{9}
\]

where \( C_{\mu\nu} \) is a symmetric tensor, transverse-traceless and constant tensor with \( C_{0\mu} = C_{\rho0} = 0 \) which contains the polarization of the wave and is called polarization tensor. The spatial components of the GW seen by an observer at a point far from a source, where the source is located at the origin in the \( xy \) plane and wave propagates along the \( z \)-direction are

\[
h_{ab}^{TT} = \begin{pmatrix}
  h_+ & h_\times \\
  h_\times & -h_+
\end{pmatrix}, \tag{10}
\]

where \( h_+ \) and \( h_\times \) are the plus and cross independent polarizations. Here \( TT \) represents Transverse-Traceless tensor because \( h_{i}^i = 0 \) and \( \kappa^\mu C_{\mu\nu} = 0 \), respectively. This shows that the Einstein field equations lead to an oscillatory activity of the spacetime geometry that travel at the speed of light [12].

If we consider \( T_{\mu\nu} \neq 0 \) into Eq. (7), the solution to the equation becomes

\[
\bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} \int d^4 x' G(x-x') T_{\mu\nu}(x'), \quad G(x-x') = -\frac{\delta(t_{ret} - t')}{{4\pi}|x-x'|}, \tag{11}
\]

where \( t_{ret} = t - \frac{1}{c}|x-x'| \) is the retarded time where the GW from source point \( \vec{x}' \) needs a certain time to arrive at the observer at position \( \vec{x} \), and \( G(x-x') \) is the Green function for the 3D wave equation.

When \( |\vec{x}| = r \) is greater than the distance of the source \( d \), we can make the expansion \(|x-x'| = r - \vec{x}' \cdot \vec{n} + ...\). So the solution to the wave equation is [13]

\[
\bar{h}_{\mu\nu} = \frac{4G}{r} \int d^3 x' T_{\mu\nu} \left(t - r + \vec{x}' \cdot \vec{n}, \vec{x}'\right). \tag{12}
\]

Applying Fourier transformation to the source we find

\[
T_{\mu\nu} \left(t - r + \vec{x}' \cdot \vec{n}, \vec{x}'\right) = \int \frac{d^4 k}{(2\pi)^4} \tilde{T}_{kl} (\omega, \vec{k}) e^{i\omega(t-r+\vec{x}' \cdot \vec{n}) + i\vec{k} \cdot \vec{x}'}, \tag{13}
\]

The frequency of the motion inside the source is \( \omega_s \sim v/d \). To integrate the Eq. (12) we can expand the exponent as

\[
e^{i\omega(t-r+\vec{x}' \cdot \vec{n})} = e^{-i\omega(t-r)} \left[1 - i\omega \vec{n} \cdot \vec{k} + \frac{1}{2} (-i\omega)^2 \vec{x}' \cdot \vec{n} \cdot \vec{k} + ...ight], \tag{14}
\]

\(^2\) This operator is described by temporal and spatial partial derivatives \(-\frac{1}{c^2} \frac{\partial}{\partial t^2} + \nabla^2\).
with this expansion into (13) we can integrate and obtain multipolar expansion, where the quadrupole moment is representative of gravitational waves in Eq. (12),

$\frac{d^2 M(t)^{ij}}{dt^2}$

(15)

where $M(t)^{ij}$ is the second mass moments. As we can see in this expression we have just spatial components in two dimension, this is because the velocities are small ($v \ll c$) the metric changes slowly, so we can neglect the time derivative of the metric such as the temporal components of $h_{00}$. In terms of the polarization tensors we have

$h_x = \frac{G}{r c^4} \left( \ddot{M}_{11} - \ddot{M}_{22} \right)$

$h_+ = \frac{2 G}{r c^4} \ddot{M}_{12}$

(16)

2.2.1. Newtonian approximation for a binary system

Coalescence of two black holes is the more intensive source of GW generations. The Newtonian order approximation for a binary inspiral signal is represented by two masses, $m_1$ and $m_2$, separated by a distance $R$ (assuming a circular orbit) and orbiting the reduced mass of the system $\mu = \frac{m_1 m_2}{m_1 + m_2}$. An orbital plane intersects the $y$-axis, making an angle $\iota$ with the $z$-axis [11]. Parametrizing the position for a circular orbit

$x_1 = R \frac{m_1}{m_1 + m_2} \hat{e}(t)$

$x_2 = R \frac{m_2}{m_1 + m_2} \hat{e}(t)$

where $\hat{e}(t) = [\cos(\omega t), \sin(\omega t) \cos(\iota), \sin(\omega t) \sin(\iota)]$ [13]. The components of the spatial tensor in Eq. (16) are

$M_{11} = R^2 \mu \cos^2(\omega t)$

$M_{22} = R^2 \mu \sin^2(\omega t) \cos^2(\iota)$

$M_{12} = R^2 \mu \cos(\omega t) \sin(\omega t) \cos^2(\iota)$

(17)

Introducing the spatial tensor in the polarization tensor, we obtain

$h_+(t) = \frac{4}{r} \left( \frac{G M}{c^2} \right)^{5/3} \left( \frac{\pi f(t)}{c} \right)^{2/3} \frac{1 + \cos^2 \iota}{2} \sin \phi$

(18)

$h_\times(t) = \frac{4}{r} \left( \frac{G M}{c^2} \right)^{5/3} \left( \frac{\pi f(t)}{c} \right)^{2/3} \cos \iota \sin \phi$

(19)

where $M = \mu^{3/5} M^{2/5}$. If we also define the characteristic radius $R_c = \frac{2GM}{c^2}$ and wavelength $\lambda_c = c/f_{gw}$, where $f_{gw}$ is the frequency of the GW, evaluating at the retarded time, with an arbitrary phase factor, we get

$h_+(t) = A \frac{1 + \cos^2 \iota}{2} \cos(2\pi f_{gw} t_{ret} + 2\phi)$

(20)

$h_\times(t) = A \cos \iota \sin(2\pi f_{gw} t_{ret} + 2\phi)$

(21)

where the phase and frequency are

$\phi(t) = -\frac{2c^3}{5GM} [\Theta(t)]^{-3/8} (t_c - t) + \phi_c$

(22)

$f_{gw}(t) = \frac{c^3}{8\pi GM} [\Theta(t)]^{-3/8}$

(23)
2.2.2. Post-Newtonian Approximation The Newtonian approximation has not sufficient accuracy to model the quasi-stationary circular orbits in a binary system, to increase accuracy we need the post-Newtonian theory, which adds corrections in the form of a power series in $\phi$ and $f$ (expanded around the dimensionless parameter $x_{\text{par}} = (\frac{Gm}{c^3})^{2/3}$). The phase and frequency time-evolution of the GW in the second order post-Newtonian (2PN) approximation of general relativity are \[14\]

$$
\phi_{gw}(t) = \phi_0 - \frac{1}{\eta} \left[ \Theta(t)^{\frac{2}{3}} + \left( \frac{3715}{8064} + \frac{55}{96} \right) \Theta(t)^{\frac{4}{3}} - \frac{3\pi}{4} \Theta(t)^{\frac{5}{3}} \right],
$$

$$
f_{gw}(t) = \frac{\mathcal{C}^3}{8\pi GM} \left[ \Theta(t)^{-\frac{3}{4}} + \left( \frac{743}{2688} + \frac{11}{32} \right) \Theta(t)^{-\frac{3}{2}} - \frac{3\pi}{10} \Theta(t)^{-\frac{7}{4}} \right]
+ \left( \frac{1855099}{14450688} + \frac{56975}{258048} + \frac{37}{2048} \right) \Theta(t)^{-\frac{5}{2}},
$$

where \( \Theta = \left( \frac{1}{4x_{\text{dim}}} \right)^4 \) is a dimensionless time variable defined as \( \Theta(t) := \frac{\mathcal{C}^3}{GM} (t_c - t) \) and \( \eta = \frac{(m_1 + m_2)^2}{m_1 m_2} \) is the symmetric mass ratio, and \( t_c \) is the time at which the coalescence takes place.

Note that these expressions are extensions for the phase and frequency of the GW in the classical Newtonian order solution presented in equations (22) and (23). It is important to mention that to discriminate the signal of a coalesce binary from noise detectors as LIGO, we need the corrections from the post-Newtonian approximation at higher PN orders. In this work just obtain the 2PN approximation which had been used in [11].

3. Gravitational wave strain signal: \( h(t) \)
Considering the reference frame of an interferometer-based detector, the GW strain produced by a source located at \((r, \theta, \varphi)\) is a linear superposition of the \( h_+ \) and \( h_x \) polarizations of the GW according to the detector’s response \[16\]

$$
h(t) = F_+ h_+(t) + F_x h_x(t), \tag{24}
$$

where \( F_+ \) and \( F_x \) are the so-called beam-pattern functions of the detector depending on two angles giving the direction of the source as seen from the detector and a polarization angle, where

\[
F_+ = -\frac{1}{2} (1 + \cos^2 \theta) \cos 2\varphi \cos 2\psi - \cos \theta \cos 2\varphi \sin 2\psi, \\
F_x = +\frac{1}{2} (1 + \cos^2 \theta) \cos 2\varphi \sin 2\psi - \cos \theta \sin 2\varphi \cos 2\psi. \tag{25}
\]

The angle \( \psi \) is the third Euler angle that translates from the detectors frame to the radiation frame. Therefore, the GW strain induced in an interferometer-based detector is

$$
h(t) = \frac{A(t)}{D} \cos (2\phi_{gw}(t) - \theta), \tag{26}
$$

where \( A(t) = -(2G\mu/c^4)(\pi GM f_{gw}(t))^{\frac{3}{2}} \) is the time-dependent quadrupolar amplitude, \( \theta \) is a phase angle comprising the inclination angle and detector’s response $\tan \theta = \frac{F_x}{F_+ (1 + \cos^2 \psi)/2}$ and \( D \) is the effective distance of the source

$$
D = \frac{r}{\sqrt{F_+^2 \left( 1 + \cos^2 \psi / 2 \right)^2 + F_x^2 \cos^2 \psi}}. \tag{27}
$$
The effective distance $D$ is related to real distance $r$ by the geometrical parameters that relates the source orientation to the detector’s orientation. Thus, $D$ is equal to $r$ when the source is optimally-oriented (the inclination angle is $i = 0$, i.e., the source is located on the $z$-axis on the observer’s line of sight; and the source location is at sky position $\theta = 0$ or $\pi/2$, i.e. above or below the zenith of the detector), and $D$ is greater than $r$ when the source is sub-optimally-oriented.

4. Conclusion

We obtain the second order post-Newtonian approximation for a binary system of two black holes in the inspiral phase at 2PN order. To obtain this approximation we expanded the solution of the Newtonian orbital equations considering negligible velocities $v \ll c$ and weak field $\Phi$. In this approximation we find a wave equation, the solution to this equation are $h_+$ and $h_\times$ which represents the polarizations of the GW. To discriminating the signal of a coalescing binary from the detector noise, we need relativistic corrections to those solutions in the form of a power series in $\phi$ and $f$. Future research in the detection of GW in the inspiral phase with LIGO will need the post-Newtonan approximation to higher orders than 2PN.

Acknowledgements

The authors would like to thank the support of the CONACyT grant No. 271904 and the CONACyT-AEM grants No. 248411 and 262847. CM want to thank PROSNI-UDG 2016 and UDG-CA-813.

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