Anisotropic instability in a higher order gravity theory

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Abstract. We study a metric cubic gravity theory considering odd-parity modes of linear inhomogeneous perturbations on a spatially homogeneous Bianchi type I manifold close to the isotropic de Sitter spacetime. We show that in the regime of small anisotropy, the theory possesses new degrees of freedom compared to General Relativity, whose kinetic energy vanishes in the limit of exact isotropy. From the mass dispersion relation we show that such theory always possesses at least one ghost mode as well as a very short-time-scale (compared to the Hubble time) classical tachyonic (or ghost-tachyonic) instability. In order to confirm our analytic analysis, we also solve the equations of motion numerically and we find that this instability is developed well before a single e-fold of the scale factor. This shows that this gravity theory, as it is, cannot be used to construct viable cosmological models.

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1 Introduction

Modified gravity models have been introduced in theoretical physics for different aims. General covariance allows for an infinite number of geometrical Lagrangian densities built only out of the Riemann tensor, its covariant derivatives and its contractions. The first and simplest example is for sure the Einstein-Hilbert action, which was giving the original 1916 Einstein equations \[\text{1, 2}\], and up to now, with the addition of a cosmological constant \(\Lambda\) introduced by Einstein, too \[\text{3}\], it is considered to be the theory of gravitational interactions valid both at small and large (cosmological) scales.

However, modifications of General Relativity (GR) beyond the Ricci scalar \(R\) and \(\Lambda\), were studied in the context of efforts to obtain a renormalizable theory of gravity \[\text{4}\]. But probably the biggest success in application to observable effects was achieved by the introduction of such a modification in cosmology in the context of inflation \[\text{5–7}\]. Since then, modified gravity theories were considered not only as purely theoretical constructions, but they have become the basis for building falsifiable models able to describe high curvature regimes in gravity and cosmology \[\text{8–13}\].

More recently, after the discovery of the late time accelerated expansion of our Universe, purely geometrical modified gravity theories like the \(f(R)\) gravity \[\text{14}\] have been also considered as alternative to scalar field (quintessence) models in GR in order to describe present dark energy in the Universe not by \(\Lambda\), but through infra-red modifications of GR, see the review \[\text{15}\]. This has also shed new light on other, more general modifications of gravity like scalar-tensor theories \[\text{16, 17}\] including \(f(R)\) gravity as a particular case, vector-tensor theories \[\text{18, 19}\], massive gravity \[\text{20–22}\], bi-gravity \[\text{23}\] and so on.
All these modifications of gravity tend to share, as a common feature, the property of appearance of new degrees of freedom besides the standard massless gravitational waves, even in the absence of matter. These new degrees of freedom in general lead to cosmological low-curvature phenomenology completely different from the standard Λ-CDM model, unless some mechanism exists to effectively screen them like the chameleon mechanism [24]. In addition, for all purely geometrical modifications of GR without torsion and non-metricity known by now, apart from $f(R)$ gravity satisfying the conditions $f'(R) > 0$, $f''(R) > 0$, these degrees of freedom appear to be tachyons or ghosts, or even ghost-tachyons.

Among these modifications of gravity, there recently appeared a model which imposes, as a defining condition, that it leads to equations of motion which are only of the second order on maximally symmetric spacetimes [25, 26]. Thus, the theory is required to possess the same number of degrees of freedom as GR on these spacetimes. This condition seem not to agree with the Lovelock condition which requires second order differential equations on any background [27]. As a result, several authors found Lagrangians not reducing to the Lovelock result in four space-time dimensions. In fact, at least a Lagrangian cubic in the Riemann tensor (and its contractions), known as Einsteinian cubic gravity (ECG) has been introduced in [25, 28]. This model was later generalized to all orders in the Riemann tensor in [26].

However, since ECG does not satisfy the Lovelock theorem, one should expect that it generically possesses other propagating modes. On a FLRW spacetime without matter and with second order equations of motion for the scale factor and for perturbations, these modes can be tensor perturbations (massive gravitational waves) only. This fact could be problematic, but if the mass of these extra degrees of freedom is large enough, then the model would still be viable as an effective low-energy theory. If so, the above mentioned defining prescription on maximally symmetric spacetimes could be enough to ensure good behavior of higher order gravity theories. That is why in this paper, we will try to understand what happens to the extra degrees of freedom, and in particular, we will address the issue whether these extra degrees of freedom make the maximally symmetric de Sitter solution unstable or not.

For this purpose, we will find it convenient to study ECG solutions for small inhomogeneous perturbations on a plane-symmetric spatially homogeneous Bianchi type I spacetime. We consider this particular manifold because, as we will show later on, it possesses a smooth limit to an isotropic and homogeneous FLRW background. The Bianchi-I spacetime itself can be thought of as a strong tensor perturbation (gravitational wave) with the infinite wavelength superimposed on a FLRW background. Therefore, if we use solutions for perturbations on it to study what happens to the extra degrees of freedom as the Bianchi-I metric becomes more and more isotropic, this consideration will be effectively beyond the linear order with respect to the limiting FLRW background. To arrive such a goal, we will find it sufficient to study only the odd-parity-modes subset of all perturbation variables. We find that, on a general Bianchi type I manifold, three odd-parity modes propagate (instead of the single one present in GR). This fact is in agreement with the Lovelock theorem. In other words, solutions of ECG for perturbations on a generic background do not behave as in GR. In fact, we find that for any Bianchi I type background solution in ECG, at least one of its three perturbation modes is always a ghost.

However, these anisotropic degrees of freedom are not present on a FLRW background. Therefore, we want to know how the ECG theory behaves in the FLRW limit. We show the existence of such a FLRW limit, i.e. the existence of a Bianchi type I background solution of the ECG equations of motion which evolves in time more and more towards a FLRW isotropic solution. We will call it the isotropic limit of Bianchi-I solutions. This solution is
important as it will show what happens to the three anisotropic modes as the background becomes more and more isotropic.

In this isotropic limit, we then study the no-ghost conditions for the three modes, and find that at all times either one or two modes are ghosts. We then proceed to study the dispersion relations for all the modes in this isotropic limit. We find that at the leading order, the dynamics of the ghost(s) decouple from the other modes, and one (of two) modes become either tachyonic or ghost-tachyonic (we will see more clearly what we mean by this later on). Anyhow, in both these cases, a strong classical instability arises, which exponentially grows in a short time (much less than in a single e-fold).

For this reason, we believe that the maximally symmetric FLRW spacetime — the de Sitter one — cannot be considered as a stable, ghost-free solution of the equations of motion. Rather, as long as the background is not exactly de Sitter, even if the would-be FLRW background equations of motion are stable, tensor perturbations (at least their odd-parity modes) will have extra and/or ghost degrees of freedom which will tend to grow exponentially. Therefore, as long as one does not find a cure to such a behavior, the ECG theory could not lead to a viable cosmological model.

This paper is organized as follows. In section 2 we introduce the ECG Lagrangian and discuss small inhomogeneous odd-parity perturbations of a homogeneous Bianchi-I metric. In section 3, we establish the existence of an isotropic limit solution to the Bianchi-I metric. Then, we expand the Lagrangian up to the second order in odd-parity modes in section 4. Introducing a new Lagrangian multiplier, we find that there exist three degrees of freedom for perturbations. After diagonalizing the kinetic matrix, we make a canonical field redefinition in the section 5 to study the mass dispersion relation. In this section we show that there always exists at least one ghost and tachyonic or ghost-tachyonic instability. In section, 6 we solve the equation of motion numerically with respect to the number of scale factor e-folds \( N \), and show that the instability is developed much before even single e-fold number. Finally in section 7, we present our concluding remarks.

2 The Lagrangian density

The Lagrangian density of the ECG contains the following linear combination of cubic in the Riemann and Ricci tensors terms [28]

\[
\mathcal{L} = 12\mathcal{L}_1 + \mathcal{L}_2 - 8\mathcal{L}_3 + 2\mathcal{L}_4 + 4\mathcal{L}_5 + 8\mathcal{L}_6 - 4\mathcal{L}_7, \tag{2.1}
\]

where

\[
\begin{align*}
\mathcal{L}_1 &= R^{\alpha\beta\gamma\delta} R_{\mu\nu\beta\gamma} R^{\mu\nu\alpha\gamma}, \\
\mathcal{L}_2 &= R^{\alpha\beta\gamma\delta} R_{\mu\nu\gamma\delta} R^{\mu\nu\alpha\beta}, \\
\mathcal{L}_3 &= R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} R^{\mu\nu\alpha\beta}, \\
\mathcal{L}_4 &= R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} R, \\
\mathcal{L}_5 &= R_{\alpha\beta\gamma\delta} R^{\alpha\gamma} R^{\beta\delta}, \\
\mathcal{L}_6 &= R^{\alpha\beta} R^{\beta\delta} R^{\delta\alpha}, \\
\mathcal{L}_7 &= R^{\alpha\beta} R^{\beta\alpha} R,
\end{align*}
\]

and total gravitational action reads as

\[
S = \int d^4x \sqrt{-g} \left[ \frac{M_p^2}{2} (R - 2\Lambda) + \beta \frac{\mathcal{L}}{M_p^2} \right]. \tag{2.9}
\]
However, the scalar $L$, not belonging to any of the Lovelock scalars, leads to non-trivial contributions in four dimensions, and in turn, this new theory of gravity is then, if seen from the Lovelock theorem point of view, necessarily of higher order. Nonetheless, in vacuum and on a spatially flat FLRW background, it is rather easy to show that the following background equations of motion hold

$$\frac{\ddot{a}}{a^2} + \frac{16\beta}{M_P^4} \frac{\dot{a}}{a^6} - \frac{\Lambda}{3} = 0,$$

(2.10)

$$\left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}\right) \left(M_P^4 + 48\beta \frac{\dot{a}}{a^4}\right) = 0.$$

(2.11)

These equations of motion imply a de Sitter solution $a \propto e^{H_0 t}$, where $H_0 = \frac{\dot{a}}{a}$ and the constant value of $H_0$ is related to the value of the bare cosmological constant $\Lambda$. On top of that, considering linear perturbation theory, we find that only the two different polarization of the tensor modes, i.e. $h_{ij} = \sum_{\lambda=+,\times} H_\lambda^n e^{i_{ij}}$, do propagate, and their reduced action can be written as

$$S = \frac{M_P^4 + 48\beta H_0^2}{8M_P^2} \sum_{\lambda} \int dt \, d^3x \, a^3 \left[ \dot{H}_{\lambda}^2 - \frac{1}{a^2} (\partial_i H_\lambda)^2 \right].$$

(2.12)

Indeed, this theory satisfies the defining conditions that on maximally symmetric spacetimes, the equations of motion are of second order and only tensor modes propagate, as in GR.

This result also shows that, as long as $M_P^4 + 48\beta H_0^2 > 0$, the tensor modes are well behaved on the de Sitter solution. This necessary condition automatically excludes the particular value of $H_0$ which would make the second factor in eq. (2.11) vanish. Furthermore, this condition together with eq. (2.10) imply that the de Sitter solution exists provided $\Lambda > 2H_0^2 > 0$. Therefore, there are no stable de Sitter solutions without a bare positive cosmological constant in this gravity theory. The condition $\Lambda > 2H_0^2$ is very strong, as this fact indicates that in general there is no mechanism to end inflation within the theory itself, at least at the classical level. This immediately puts a very serious obstacle to the construction of viable cosmological models in this theory. Still it is possible to avoid it, i.e., like it was done in [5], by waiving the requirement that the equation of motion for the scale factor should be the second order (but still keeping second order equations for perturbations!) and by adding the $R^2$ term to the total Lagrangian density in the r.h.s. of (2.9). However, as we shall see later on, the real problem of this theory is that this de Sitter solution is unstable due to the presence of perturbation instabilities.

2.1 The metric

Let us consider a homogeneous plane-symmetric Bianchi type I background

$$ds^2 = -dt^2 + a^2 \, dx^2 + b^2 \, \delta_{ij} dy^i dy^j,$$

(2.13)

and let us focus on odd-parity modes of perturbations upon it (dubbed odd modes below for brevity):

$$ds^2 = -dt^2 + a^2 \, dx^2 + b^2 \, \delta_{ij} dy^i dy^j + 2bV_i dt \, dy^i + 2ab \, (\partial_x W_i) \, dx \, dy^i + \frac{b}{2} \, (\partial_i Z_j + \partial_j Z_i) \, dy^i dy^j.$$
For an odd modes gauge transformation, we get
\[ \xi^\alpha = (0, 0, \xi^i), \]  
where
\[ \delta^{ij} \partial_j V_i = 0 = \delta^{ij} \partial_j W_i = 0 = \delta^{ij} \partial_j Z_i = 0 = \partial_i \xi^i, \]
(2.15)
because the metric in the \( y \)-subspace is Euclidean. Then we can consider a decomposition for odd modes as follows
\[ V_i = V(t, x) \frac{\epsilon_{ij} \delta^{jk} \partial_k Y(y)}{b}, \]  
(2.17)
\[ W_i = W(t, x) \frac{\epsilon_{ij} \delta^{jk} \partial_k Y(y)}{b}, \]  
(2.18)
\[ Z_i = Z(t, x) \frac{\epsilon_{ij} \delta^{jk} \partial_k Y(y)}{b}, \]  
(2.19)
\[ \xi^i = \xi^V(t, x) \frac{\delta^{il} \epsilon_{lj} \delta^{jk} \partial_k Y(y)}{b}, \]  
(2.20)
where \( Y \) satisfies the equation \( \delta^{ij} \partial_i \partial_j Y = -q^2 Y \), and \( \epsilon_{12} = 1 = -\epsilon_{21} \). In fact, we find
\[ \delta^{ij} \partial_j V_i = V(t, x) \frac{\epsilon_{ij} \delta^{jk} \partial_k Y(y)}{b} = V(t, x) \frac{\epsilon_{ij} \delta^{jk} \partial_k Y(y)}{b} = 0, \]
(2.21)
amatically.

For an odd modes gauge transformation, we have
\[ \Delta \delta g_{ij} = -\xi_{i,j} - \xi_{j,i} = -(\xi_{i,j} - \Gamma^\alpha_{ij} \xi^\alpha) - (\xi_{j,i} - \Gamma^\alpha_{ji} \xi^\alpha) \]
\[ = -\xi_{i,j} - \xi_{j,i} + 2\Gamma^l_{ij} \xi^l = -\xi_{i,j} - \xi_{j,i}. \]
(2.22)
This leads to
\[ \Delta Z = -2\xi^V/b. \]
(2.23)

So we set the odd-mode flat gauge for which
\[ Z = 0, \]
(2.24)
or
\[ ds^2 = -dt^2 + a^2 dx^2 + b^2 \delta_{ij} dy^i dy^j + 2V(t, x) \frac{\epsilon_{ij} \delta^{jk} \partial_k Y(y)}{b} dt dy^i + 2a \partial_x W(t, x) \frac{\epsilon_{ij} \delta^{jk} \partial_k Y(y)}{b} dx dy^i. \]
(2.25)
In the limit \( b \to a \), we get back to a FLRW manifold. So we have a smooth transition to a homogeneous and isotropic universe. In the following we will only consider the vacuum case. In this study we perform the same analysis as in [29]. In particular, in GR one would expect that only one odd mode is propagating, namely one of the two polarizations of gravitational waves.

### 3 The background and the FLRW limit

In this section, we show the existence of a smooth isotropic limit for Bianchi-I background solutions in ECG gravity. We need such a solution in order to understand the behavior of
the extra modes, present in the theory, in the smooth isotropic FLRW limit of a Bianchi type I metric. If there were no such stable isotropic limit, this would not be possible.

In order to achieve this goal, we try then to solve the equations of background motion iteratively in the isotropic limit. In particular, we have two differential equations for the two background variables $a$ and $b$, and when $b \to a$ we have a background which reduces to FLRW, whose solution describes the de Sitter expansion. Therefore we can try to find an iterative solution of the following kind

$$a = a_0 + a_1 + \ldots, \quad \text{(3.1)}$$
$$b = a_0 + b_1 + \ldots, \quad \text{(3.2)}$$

where $a_1, b_1 \ll a_0$, and

$$a_0 \propto e^{H_0 t}, \quad \text{(3.3)}$$

where $H_0$ is given as the solution of the first Friedmann equation, namely eq. (2.10). Since we know that $a_0$ satisfies the equations of motion for the FLRW background, we can linearize the background equations for $a_1$ and $b_1$ and find their dynamics. In principle, one can continue further to the next order with the condition that $a_2, b_2 \ll a_1, b_1 \ll a_0$, etc. If it is possible to build iteratively such a solution, then indeed we can construct an anisotropic Bianchi-I background which approximates an isotropic FLRW one.

Actually, we have three equations of motion in Bianchi-I corresponding to the lapse equation and the two equations of motion for the fields $a$ and $b$. However, because of the Bianchi identities, only two of them are independent. At the lowest order, as we already know by construction of such a solution, one finds that the $a$-equation of motion implies

$$a_0 \dot{a}_0 - \dot{a}_0^2 = 0, \quad \text{(3.4)}$$

which is automatically satisfied on de Sitter. This is the analogue of the GR equation $\dot{H} = -4\pi G (\rho + p) = 0$, in the presence of only a cosmological constant. Among all family of de Sitter solutions, we impose here the condition

$$M^2_P + 48\beta H_0^4 > 0, \quad \text{(3.5)}$$

otherwise we would have that the tensor modes on FLRW either become (massless) ghosts (for $\beta < -\frac{1}{48} M^2_P/H_0^4$) or they get strongly coupled (for $\beta = -\frac{1}{48} M^2_P/H_0^4$). In the following we will not consider this nonphysical situation. Furthermore, for obvious reasons, we will consider the case $\beta \neq 0$. At the lowest order the $b$-equation of motion (see the appendix A for more details) does not add any new information.

Now that we have set the zeroth order solution, we can proceed to find the deviations from exact de Sitter by studying the next variables $a_1$ and $b_1$.

At the first order in the variables $a_1$ and $b_1$, we get from the $\Lambda$-equation

$$\dot{a}_1 + 2\dot{b}_1 = H_0 (a_1 + 2b_1). \quad \text{(3.6)}$$

The solution for the previous equation can be written as

$$b_1 = -\frac{1}{2} a_1 + C_1 e^{H_0 t}, \quad \text{(3.7)}$$

where the last term containing $C_1$ can be included into a renormalization of the background lowest order solution for which $a_0 \propto e^{H_0 t}$. Therefore we can set

$$b_1 = -\frac{1}{2} a_1. \quad \text{(3.8)}$$
The $b$-equation instead gives
\[ \ddot{a}_1 + H_0 \dot{a}_1 - 2H_0^2 a_1 = 0, \tag{3.9} \]
which is solved by
\[ a_1 = -2C_1 e^{-2H_0 t} + C_2 e^{H_0 t}. \tag{3.10} \]
Once more the second solution can be reabsorbed into the lowest order term, so that we are left with
\[ a_1 = -2C_1 e^{-2H_0 t} = -\frac{2C_1}{a_0^2}, \tag{3.11} \]
\[ b_1 = C_1 e^{-2H_0 t} = \frac{C_1}{a_0^2}. \tag{3.12} \]
Indeed we find that, for a large range of values of $C_1$, $a_1, b_1 \ll a_0$ when $a_0 > 1$ as expected during inflation. The other remaining equation of motion is automatically satisfied at the same order, because of Bianchi identities.

At the next order one finds from the $\Lambda$-equation
\[ [(2\ddot{b}_2 + \dot{a}_2) - H_0(2b_2 + a_2)] a_0^5 \left( \frac{M_P^4}{48} + \beta H_0^4 \right) + \frac{387C_1^2 H_0}{2} \left( \frac{M_P^4}{2064} + \beta H_0^4 \right) = 0, \]
which is solved by
\[ a_2 = -2b_2 + \frac{3C_1^2}{4a_0^5} \left( \frac{M_P^4 + 2064\beta H_0^4}{M_P^4 + 48\beta H_0^4} \right), \tag{3.13} \]
where we have once more discarded a term proportional to $e^{H_0 t}$. The $b$-equation of motion gives
\[ b_2 = -\frac{C_1^2}{4a_0^5} \left( \frac{M_P^4 + 2064\beta H_0^4}{M_P^4 + 48\beta H_0^4} \right), \tag{3.14} \]
where we have discarded two terms proportional to $e^{H_0 t}$ and $e^{-2H_0 t}$ respectively as renormalizations of the previous $a_0$ and $a_1$ solutions. Therefore we find
\[ a_2 = -5b_2. \]

In this way one we have found a solution for which $(a_2, b_2) \propto a_0^{-5} \ll (a_1, b_1) \propto a_0^{-2} \ll a_0 \propto e^{H_0 t}$, and one can continue building up such a solution order by order in inverse powers of $a_0$. Finally, we have shown the existence of a solution (at least up to second order) for which $a_0$ grows exponentially in time, and $\lim_{a_0 \to \infty} b(t) = a(t) = a_0(t)$. Up to the second order (but we may continue further on), we find that this approximate solution can be written as
\[ a = a_0 \left[ 1 - \frac{2C_1}{a_0^2} + \frac{5C_1^2}{4a_0^5} \left( \frac{M_P^4 + 2064\beta H_0^4}{M_P^4 + 48\beta H_0^4} \right) + O(C_1^3/a_0^9) \right], \tag{3.15} \]
\[ b = a_0 \left[ 1 + \frac{C_1}{a_0^2} - \frac{C_1^2}{4a_0^5} \left( \frac{M_P^4 + 2064\beta H_0^4}{M_P^4 + 48\beta H_0^4} \right) + O(C_1^3/a_0^9) \right]. \tag{3.16} \]
4 Degrees of freedom for odd modes

We expand the Lagrangian density at the second order in perturbations for a general Bianchi-I background for the theory under consideration. Then we will consider the limit for $b \to a$. Since the background is homogeneous, we can expand the perturbation variables in Fourier modes also for the $x$ coordinate, with a basis which satisfies $\partial_2^2 \tilde{Y}(x) = -k^2 \tilde{Y}(x)$. Here we will consider only the constraints coming from the local behavior of linear perturbations. For this aim, for simplicity, but without lack of generality, we can focus on one single Fourier mode with the wave vector $\vec{K} = (k, \vec{q})$.

We have three background equations of motion (but only two of them are independent). On imposing the background equations of motion on the second order Lagrangian density $L$, we notice that the following terms (which are new compared to GR) come out, namely

$$L = \frac{6\beta k^2 q^2}{M_P^2 b^2} (ab - ba) \dot{W} \left( \ddot{W} - \frac{2}{a} \dot{V} \right) + \frac{6\beta q^2 b (k^2 b^2 - 2q^2 a^2)}{M_P^2 b^4 a^2} (ab - ba) \dot{V}^2 + \ldots, \quad (4.1)$$

These terms are absent in GR, because $\beta = 0$ identically. Then in GR, $V$ would represent a Lagrange multiplier which can be integrated out leaving a reduced action for a single propagating mode, $W$, which would have a standard kinetic term. It is interesting to notice that in the exact isotropic limit i.e. $b \propto a$, the above terms in the action vanish, hence the mode $V$ becomes a Lagrangian multiplier on FLRW.

However, for this new theory, i.e. $\beta \neq 0$, the presence of these two terms in the non-isotropic case suggests that new degrees of freedom will arise in general. The highest-derivative terms responsible for the presence of the new modes, on the other hand, tend to vanish in the isotropic limit, so that we need to understand what happens to such degrees of freedom in this limit.

We can rewrite the previous terms in the following equivalent way

$$L = \frac{6\beta k^2 q^2}{M_P^2 b^2} (ab - ba) \dot{W} \left( \ddot{W} - \frac{2}{a} \dot{V} \right) - \frac{12\beta q^4 b (ab - ba)}{M_P^2 b^4} \dot{V}^2 + \ldots, \quad (4.2)$$

and the terms above can be rewritten as

$$L' = \frac{6\beta k^2 q^2}{M_P^2 b^2} (ab - ba) \left[ 2\zeta \left( \dot{W} - \frac{1}{a} \dot{V} \right) - \zeta^2 \right] - \frac{12\beta q^4 b (ab - ba)}{M_P^2 b^4} \dot{V}^2 + \ldots, \quad (4.3)$$

where we have included a new Lagrange multiplier $\zeta$, whose equation of motion (algebraic in $\zeta$ itself) is

$$\zeta = \ddot{W} - \frac{1}{a} \dot{V}. \quad (4.4)$$

In fact, on replacing it inside the new Lagrangian density $L'$, we get the original Lagrangian density $L$. In other words, we have two equivalent Lagrangian densities which lead to the same classical equations of motion.

After integrating by parts the term involving $\dot{W}$, we get a term of the form $\dot{\zeta} \ddot{W}$, so that in the new obtained Lagrangian density $L'$ we can represent now the kinetic term for the fields $\psi_i = (\zeta, W, V)$ as

$$L' = K_{ij} \dot{\psi}_i \dot{\psi}_j + \ldots, \quad (4.5)$$
where $K_{ij} = K_{ji}$, $K_{11} = 0 = K_{13}$, but $K_{12} \neq 0$. In the exact isotropic limit, the terms $K_{12}$, $K_{23}$, $K_{33}$ all vanish, making the field $V = \psi_3$ a Lagrange multiplier. We now try to diagonalize the matrix $K_{ij}$ by the following field redefinition

$$\zeta = F_1,$$

$$W = \Gamma_1 F_1 + F_2,$$

$$V = \Gamma_2 F_1 + \Gamma_3 F_2 + F_3.$$  

This transformation is in general well defined (as shown in the appendix B), even in the isotropic limit, for example

$$\Gamma_3 = \frac{k^2 b}{q^2 a^2} (\dot{a} b - \dot{a} b),$$

and its determinant is equal to unity. After this field redefinition, it is possible to write down the new diagonal kinetic matrix $A_{ij}$ as follows

$$L' = A_{ij} \dot{F}_i \dot{F}_j + \ldots,$$

with three different diagonal elements. Investigation of the positivity of these diagonal elements is sufficient for understanding whether the theory has ghosts or not. In fact, we find

$$A_{11} = g_1 = - \frac{K_{33} K_{12}^2}{K_{22} K_{33} - K_{23}^2},$$

$$A_{22} = g_2 = \frac{K_{22} K_{33} - K_{23}^2}{K_{33}},$$

$$A_{33} = g_3 = K_{33} - \frac{12 (\dot{a} b - \dot{a} b) \beta b q^4}{b^4 M_P^2},$$

from which it is clear that $g_1 g_2 = -K_{12}^2 < 0$. Therefore, no matter what the evolution is, there will be always at least one ghost mode in the odd sector. If also $K_{33} < 0$, then two ghosts will be present. Furthermore, both $g_1$ and $g_3$ tend to vanish in the isotropic limit, whereas $g_2$ does not. This was expected, as on FLRW we should apparently get only one odd propagating mode.

For the solution approaching FLRW which was found in the previous section, we find

$$g_1 = -\frac{11664 k^2 q^2 H_0^3 \beta^2 C_1^2}{a_0^5 M_P^2 (M_P^4 + 48 \beta H_0^4)} + O(C_1^3/a_0^5),$$

$$g_2 = \frac{k^2 q^2 a_0 (M_P^4 + 48 \beta H_0^4)}{4 M_P^2} + O(C_1/a_0^7),$$

$$g_3 = \frac{108 q^4 H_0^3 \beta C_1}{a_0^4 M_P^2} + O(C_1^2/a_0^7).$$

By investigating these expressions we can see that, for all allowed values of $\beta$, $g_1 < 0$, whereas $g_2 > 0$, i.e. the field $F_2$ is always well behaved. In fact, since $F_2$ is the only odd mode which seems to exist on exact FLRW, it should represent a tensor mode. If $C_1 \beta < 0$, then $F_3$ represents a (second) ghost. We will distinguish the two cases which depend on the sign of $\beta C_1$. If $\beta C_1 > 0$, then $F_2$ and $F_3$ are never ghost degrees of freedom in the isotropic limit. If one takes the exact FLRW limit, one re-obtains the standard equation of motion for one polarization (the cross one) for the de Sitter cosmological tensor modes, as also shown in the appendix B.
5 Mass dispersion relations

In the following we want to study behavior of the perturbations variables, in particular we want to see what happens to them in the isotropic limit.

5.1 One ghost case

In this case we choose $\beta C_1 > 0$, that is $g_3 > 0$, whereas $A_{11} < 0$, so that only one ghost exists. In order to study canonically normalized fields on FLRW, we can make another field redefinition by imposing

$$F_1 = \frac{a_0^{3/2}}{\sqrt{-2A_{11}}} Z_1,$$

$$F_2 = \frac{a_0^{3/2}}{\sqrt{2A_{22}}} Z_2,$$

$$F_3 = \frac{a_0^{3/2}}{\sqrt{2A_{33}}} Z_3,$$

so that the Lagrangian density for the perturbations can be rewritten as

$$\mathcal{L}_{\text{odd}} = \frac{a_0^3}{2} \left[ -\dot{Z}_1^2 + \dot{Z}_2^2 + \dot{Z}_3^2 + B_{ij} (\dot{Z}_i Z_j - Z_i \dot{Z}_j) - \mu_{ij} Z_i Z_j \right].$$

Now it is clear that in the isotropic limit, when the both $A_{11}$ and $A_{33}$ tend to vanish, then in general other terms in the Lagrangian density might become larger and larger, in particular the mass term of such modes. If this mass will be positive, then the mode would become very massive, but if such a mass is negative this mode (if not a ghost) would become extremely unstable. In such a situation, as we shall see later on, the theory develops a short-time instability, i.e. an instability which cannot be neglected in a Hubble time.

In order to see what happens to the stability of perturbations by evaluating their mass-dispersion relation, we should take equations for them which look as follows

$$\ddot{Z}_1 + \mu_{1j} Z_j + 2B_{1j} \dot{Z}_j + \ldots = 0,$$

$$\ddot{Z}_2 - \mu_{2j} Z_j - 2B_{2j} \dot{Z}_j + \ldots = 0,$$

$$\ddot{Z}_3 - \mu_{3j} Z_j - 2B_{3j} \dot{Z}_j + \ldots = 0,$$

and consider the isotropic limit for the anti-symmetric matrix $B_{ij}$ and for the symmetric matrix $\mu_{ij}$. One can see that at the lowest order in isotropy, one finds

$$B_{12} = -\frac{H_0}{2} + O(C_1/a_0^3),$$

$$B_{13} = \frac{3kH_0 \sqrt{3\beta C_1}}{a_0^{3/2} q \sqrt{M_p^4 + 48 \beta H_0^4}} + O[(\beta C_1/a_0^3)^{3/2}],$$

$$B_{23} = -\frac{a_0^{3/2} k \sqrt{3(M_p^4 + 48 \beta H_0^4)}}{72 \sqrt{\beta C_1} H_0 q} + O[(\beta C_1/a_0^3)^{1/2}].$$
Then the eigenvalues of the matrix $\mu_{ij}$ determine the mass eigenvalues of the modes. We find that in the isotropic limit the elements of the matrix $\mu_{ij}$ reduce to:

\[
\mu_{11} = -\frac{(M^2_\phi + 48 \beta H^4_0) a^3_0}{216 H^0_0 \beta C_1} - \frac{(M^2_\phi + 1248 \beta H^4_0)}{72 H^0_0 \beta} (k^2 + g^2)/a^2_0,
\]

(5.11)

\[
\mu_{22} = \frac{q^2}{a_0^2} + O(C_1/a_0^3),
\]

(5.12)

\[
\mu_{33} = -\frac{(k^2 + g^2)(M^2_\phi + 48 \beta H^4_0) a^3_0}{432 C_1 q^2 H^0_0 \beta} - \frac{[(1128 k^2 + 348 q^2) H^4_0 \beta + M^4_\phi (3 k^2 + q^2)]}{144 H^0_0 a^2_0 \beta q^2} a^2_0 + 72 (k^2 + q^2)^2 H^0_0 \beta,
\]

(5.13)

\[
\mu_{12} = -\frac{q^2}{a_0^2} + \frac{5}{2} H^2_0 + O(C_1/a_0^3),
\]

(5.14)

\[
\mu_{13} = \frac{a_0^{3/2} k \sqrt{3(M^2_\phi + 48 \beta H^4_0)}}{36 \sqrt{\beta C_1 q}} + O[(\beta C_1/a_0^3)^{1/2}],
\]

(5.15)

\[
\mu_{23} = -\frac{a_0^{3/2} k \sqrt{3(M^2_\phi + 48 \beta H^4_0)}}{16 \sqrt{\beta C_1 q}} + O[(\beta C_1/a_0^3)^{1/2}].
\]

(5.16)

In other words, we can see that the mass matrix can be approximated as

\[
\mu_{ij} = \frac{M^2_\phi + 48 \beta H^4_0}{432 H^0_0 \beta C_1} \begin{pmatrix}
\frac{-2}{O(C_1/a_0^3)} & O(C_1/a_0^3) & O(\sqrt{\beta C_1/a_0^3}) \\
O(C_1/a_0^3) & O(C_1/a_0^3) & O(\sqrt{\beta C_1/a_0^3}) \\
O(\sqrt{\beta C_1/a_0^3}) & O(\sqrt{\beta C_1/a_0^3}) & \frac{k^2 + q^2}{q^2}
\end{pmatrix},
\]

(5.17)

so that to the lowest order, the modes $Z_1$ and $Z_3$ have negative self-coupling terms. For the ghost mode, this is actually a good point, because it would make it stable. However the $Z_3$ mode, which is not a ghost, tends to be strongly unstable in the isotropic limit. Notice that $(k^2 + q^2)/q^2 > 1$, so that this problem takes place at any scale (and gets worse when $q/k \rightarrow 0$).

As already noticed above, the reason why e.g. the term $\mu_{33}$ becomes larger and larger in the isotropic limit is due to the fact that the coefficient of $F^2_{33}$ tends to vanish in the same limit.

This instability is purely classical, so that we do not need to invoke any quantum particle production. That is due to the fact that $Z_3$ becomes a tachyon, its mass growing exponentially but towards more and more negative values. Thus, we expect to have an exponentially growing instability when we solve the equations of motion.

### 5.2 Two ghosts case

Along the same lines as in the previous section, in this case we consider $C_2 = -C_1$, together with $\beta C_2 > 0$. In this case, we find that in the isotropic limit both $g_1$ and $g_3$ become negative, so that there are actually two ghost degrees of freedom. We can make a further

---

1In fact, for a stable harmonic oscillator we have $L = \dot{x}^2 - \omega^2 x^2$, whereas for a stable ghost we should have $L_g = -\dot{x}^2 + \omega^2 x^2 = -L$, because both $L$ and $L_g$ lead to the same equations of motion.
field redefinition

\[ F_1 = \frac{a_{0}^{3/2}}{\sqrt{2A_{11}}} Z_1, \]  
\[ F_2 = \frac{a_{0}^{3/2}}{\sqrt{2A_{22}}} Z_2, \]  
\[ F_3 = \frac{a_{0}^{3/2}}{\sqrt{2A_{33}}} Z_3, \]

which is convenient in the isotropic limit, so that the Lagrangian density for the perturbations can be rewritten as

\[ \mathcal{L}_{\text{odd}} = \frac{a_{0}^{3}}{2} \left[ -Z_1^2 + Z_2^2 - Z_3^2 + B_{ij}(\dot{Z}_iZ_j - Z_i\dot{Z}_j) - \mu_{ij} Z_iZ_j \right]. \]

Then we consider the isotropic limit for the anti-symmetric matrix \( B_{ij} \) and for the symmetric matrix \( \mu_{ij} \), which then become functions of \( a_{0} \), the wave numbers \( q \) and \( k \), the parameters of the action, and finally of \( C_2 \). One can see that at the lowest order in isotropy, one finds

\[ B_{12} = \frac{H_0}{2} + O(C_2/a_{0}^3), \] 
\[ B_{13} = \frac{3kH_0\sqrt{3\beta C_2}}{a_{0}^{3/2}q\sqrt{M^4_p + 48\beta H_0^4}} + O[(\beta C_2/a_{0}^3)^{1/2}], \] 
\[ B_{23} = -\frac{a_{0}^{3/2}k\sqrt{3(M^4_p + 48\beta H_0^4)}}{72\sqrt{3}\beta C_2H_0q} + O[(\beta C_2/a_{0}^3)^{1/2}]. \]

Then the eigenvalues of the matrix \( \mu_{ij} \) determine the mass eigenvalues of the modes. We find that in the isotropic limit the element of matrix \( \mu_{ij} \) reduce to:

\[ \mu_{11} = \frac{(M^4_p + 48\beta H_0^4)a_{0}^3}{216H_0^2\beta C_2} - \frac{(M^4_p + 1248\beta H_0^4)}{72H_0^2\beta} \] 
\[ \mu_{22} = \frac{q^2}{a_{0}^3} + O(C_2/a_{0}^3), \] 
\[ \mu_{33} = -\frac{(k^2 + q^2)(M^4_p + 48\beta H_0^4)a_{0}^3}{432C_2q^2H_0^2\beta} \] 
\[ + \frac{[(1128k^2 + 348q^2)H_0^4\beta + M^4_p(3k^2 + q^2)]a_{0}^3 + 72(k^2 + q^2)^2H_0^2\beta}{144H_0^2a_{0}^{2\beta}q^2}, \]
\[ \mu_{12} = \frac{q^2}{a_{0}^3} - \frac{5}{2}H_0^2 + O(C_2/a_{0}^3), \] 
\[ \mu_{13} = -\frac{a_{0}^{3/2}k\sqrt{3(M^4_p + 48\beta H_0^4)}}{36\sqrt{3}\beta C_2q} + O[(\beta C_2/a_{0}^3)^{1/2}], \] 
\[ \mu_{23} = -\frac{a_{0}^{3/2}k\sqrt{3(M^4_p + 48\beta H_0^4)}}{16\sqrt{3}\beta C_2q} + O[(\beta C_2/a_{0}^3)^{1/2}]. \]
In this case, we can see that the mass matrix can be approximated as

\[ \mu_{ij} = \frac{M_P^4 + 48 \beta H_0^4}{432 H_0^4} a_0^3 \frac{\beta C_2}{a_0^3} \begin{pmatrix} 2 & \mathcal{O}(C_2/a_0^3) & \mathcal{O}(\sqrt{\beta C_2/a_0^3}) \\ \mathcal{O}(C_2/a_0^3) & \mathcal{O}(C_2/a_0^3) & \mathcal{O}(\sqrt{\beta C_2/a_0^3}) \\ \mathcal{O}(\sqrt{\beta C_2/a_0^3}) & \mathcal{O}(\sqrt{\beta C_2/a_0^3}) & -\frac{k^2+q^2}{q^2} \end{pmatrix}, \]  

so that to the lowest order the modes \( Z_1 \) and \( Z_3 \) have self-coupling terms of opposite signs. For the ghost mode \( Z_1 \), a positive squared-mass diagonal element \( \mu_{11} \) corresponds to a tachyonic instability. In this case a negative mass for \( Z_3 \), which is now also a ghost, would instead make it stable. In any case, the whole system, because of the tachyonic mass for the ghost, tends to be unstable. As in the one ghost case, this instability is purely classical, so that we do not need to invoke any quantum particle production, which could of course contribute to produce an additional instability. However, even from a pure classical level, the background will be unstable. Thus, we expect to have an exponentially growing instability in this case, too, when we solve the equations of motion.

6 Numerical integration

We want to show numerically that the instability studied analytically in the previous section develops well before even one single e-folding in general. This implies we should expect the de Sitter solution to be unstable before the effective field theory approach breaks down. In order to check the characteristic time of such an instability, in the following, we will solve numerically the full equations of motion for the two Lagrangian densities described in the eqs. (5.4) and (5.21), assuming

\[ a \approx a_0 - \frac{2C_1}{a_0^2}, \]  

\[ b \approx a_0 + \frac{C_1}{a_0^2}. \]  

On replacing \( q = \bar{q}H_0 \), \( k = \bar{k}H_0 \), and \( H_0 = \alpha M_P \), we integrate the equations of motion with respect to the number of e-folds \( N \equiv \ln a_0/a_{0,\text{ini}} \) variable by adding the extra equation of motion \( a_0 = a_0 \), so that the system of ODEs becomes autonomous, i.e. explicitly independent of \( N \). We have considered typical values for both parameters and initial conditions \( Z_{i,\text{ini}} = 10^{-6}, \dot{Z}_{i,\text{ini}} = 0, a_{0,\text{ini}} = 1 \). The results are shown in figure 1 and they confirm the analytical prediction for the existence of a classical instability. Here, we have chosen these initial conditions which looks sensible. Indeed we want to start from a universe which is close to a FLRW one, and see what happens next to the perturbations variables. Even when we change the initial conditions, we get a similar unstable behavior. This provides additional support for a generic exponential growth of such an instability, making the FLRW behavior nonviable. We have checked that extending the expansion of solutions for \( a \) and \( b \) to the order \( \mathcal{O}(C_2^2/a_0^6) \) does not change numerical results qualitatively.

7 Conclusions

After introducing the ECG theory [25, 26] a generalization of this theory has been also proposed and its cosmology has been studied at the background level [30]. In [30], it is proposed that this theory can explain both early universe inflationary era and the late-time
acceleration of the universe. It was shown that the theory does not possess any ghost modes on a FLRW background. However, as we show here, this is not enough to say a theory is viable or not. One has to investigate its linear perturbations — and not only on a FLRW background — to show that the theory does not possess any instabilities or strong coupling behavior.

Since ECG is a higher order theory in the curvature tensor, as predicted by the Lovelock theorem, it should contain extra degrees of freedom. On the other hand, in this theory only two degrees of freedom propagate on a FLRW manifold in the absence of matter, as in GR. In order to look at the nature of other degrees of freedom which present in ECG and whose existence is predicted by the Lovelock theorem, we have studied linear perturbations on a homogeneous anisotropic Bianchi type I spacetime in this gravity theory. Such consideration introduces an anisotropy (with the infinite length scale) already at the background level compared to the isotropic FLRW metric. We showed the existence of a vacuum solution for Bianchi-I background which smoothly approaches the de Sitter metric in the isotropic limit, and for which anisotropy decreases with time as $a_0^{-3}$, where $a_0 \propto e^{H_0 t}$. Therefore, we can study the FLRW limit of a Bianchi-I manifold. As it was known in literature, we re-obtained that on a FLRW background, the ECG theory should satisfy the no-ghost condition $M_1^3 + 48\beta H_0^2 > 0$ for tensor modes of perturbations. Together with the background Friedman equation, it is interesting to see that this theory always requires $\Lambda > 2H_0^2 > 0$ that indicates the absence of stable de Sitter solutions without a bare positive cosmological constant of the order of $H_0^2$.

However, studying inhomogeneous perturbations on a Bianchi-I background in the regime of small anisotropy, we have expanded the Lagrangian density up to the second order in odd-parity perturbations and have found the existence of new degrees of freedom (compared to GR), confirming the predictions of the Lovelock theorem. The kinetic term of these new degrees vanishes in the exact isotropic limit. Thus, consideration of linear perturbations on a Bianchi-I manifold, even in its isotropic limit, makes possible to go beyond the linear order with respect to a FLRW background and to obtain non-perturbative results. We find

(a) Classical instability in the single ghost case, that is $\beta C_1 > 0$. (b) Classical instability in the double ghost case, that is $\beta C_1 < 0$.

**Figure 1.** Classical instability present in the theory in the isotropic limit, when $b \to a$. In the left panel, we have set $H_0/M_0 = 10^{-2}$, $q = k = 10H_0$, $\beta = 0.1 = C_1$. We can see that the non-ghost mode $Z_3$ is exponentially unstable and its growth makes the other modes grow exponentially. In the right panel, in the case with two ghosts, we have set $H_0/M_0 = 10^{-2}$, $q_3 = 10H_0$, $k = 10H_0$, $\beta = 0.1 = -C_1$. Here we can see that the ghost mode $Z_1$ is exponentially unstable and its growth makes the other modes grow exponentially, too.
that in total, as for odd modes, three degrees of freedoms are present. Then we diagonalize the kinetic matrix and redefine fields into canonically normalized ones in order to study the speed of their propagation and mass dispersion relations.

We find that for any parameter of ECG theory, there always exists at least one ghost which propagates in the background. Furthermore, we also find that one of the modes always acquire a tachyonic instability with time-scale much shorter than the Hubble time. We checked numerically that the instability grows well before one single e-fold. The instability present here is pure classical, thus the quantum particle production needs not be studied.

This study shows that ECG theory has a classical instability in the isotropic de Sitter limit, which is present at all scales and is developed even before one e-fold. Thus, without curing this problem, this theory cannot be considered as a viable theory of gravity and cannot be used to construct internally consistent isotropic cosmological models. Note finally that this phenomenon is very similar to that arising in $f(R, G)$ theory of gravity, where $G$ is the Gauss-Bonnet invariant, and making it not viable, too, apart from some specific exceptional cases [31].

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**A Background equations of motion**

We write in the following the equations of motion for the background. We have three equations of motion, which can be written as follows

\[ E_1 \equiv -\frac{\Lambda}{6} + \frac{b^2}{6 b^2} + \frac{16 b^2 \beta}{M_p^2 b^2} + \frac{8 b^2 \beta}{M_p^2 b^2} + \frac{32 b^4 \beta}{M_p^4 b^2} + \frac{\beta a}{M_p^2 b^2 a} - \frac{8 b^2 \beta}{M_p^4 b^2 a} + \frac{8 b^2 \beta}{M_p^4 b^2 a} = 0, \]  

\[ E_2 \equiv \frac{b}{3 b} + \frac{8 b^2 \beta a}{M_p^2 b^2 a} + \frac{8 b^2 \beta a}{M_p^2 b^2 a} + \frac{24 b^4 \beta}{M_p^4 b^2 a} - \frac{8 b^2 \beta}{M_p^4 b^2 a} + \frac{8 b^2 \beta}{M_p^4 b^2 a} - \frac{8 b^2 \beta}{M_p^4 b^2 a} + \frac{16 b^3 \beta}{M_p^4 b^2 a} = 0, \]  

\[ E_3 \equiv \frac{8 a \beta}{M_p^2 b^2 a} + \frac{16 a \beta}{M_p^2 b^2 a} + \frac{24 \beta a b}{M_p^2 b^2 a} + \frac{24 \beta a b}{M_p^2 b^2 a} + \frac{8 \beta a b}{M_p^2 b^2 a} + \frac{8 \beta a b}{M_p^2 b^2 a} + \frac{48 \beta a b}{M_p^2 b^2 a} + \frac{8 \beta a b}{M_p^2 b^2 a} = 0, \]
Therefore, once both it is in general regular, in particular in the isotropic limit. We have in general that

\[
\frac{24b^2a^2}{M_0^2b^3M_0^2b^3} - \frac{16b^4a^4}{M_0^2b^3M_0^2b^3} + \frac{b}{3b} - \frac{32b^3a}{M_0^2b^3M_0^2b^3} - \frac{16b^2a^2}{M_0^2b^3M_0^2b^3} - \frac{32b^3a}{M_0^2b^3M_0^2b^3} + \frac{16b^2b}{M_0^2b^3M_0^2b^3}
\]

We will write here the field redefinition which diagonalizes the kinetic matrix for a general Bianchi-I manifold

\[
\Gamma_1 = \frac{24(ba - \dot{a}b) - 24b^2a^2 - 24b^2a^2}{\Delta_1},
\]

\[
\Gamma_2 = \frac{24(ba - \dot{a}b) - 24b^2a^2}{\Delta_1},
\]

\[
\Gamma_3 = \frac{24(ba - \dot{a}b)}{\Delta_1},
\]

\[
\Delta_1 = q^2a^2b^2k^4 + \beta \left( 48q^2 \left( \frac{b^2 + 3}{2} \right) b^2 + \frac{b}{2} \left( q^2 - 13b^2 \right) b + \frac{1}{2} q^2b^2 + b^4 \right) a^5
\]

These equations in the text will be denoted as \( \Lambda, \dot{a}, \text{ and } \dot{b} \). These equations of motion are not independent equations of motion, in fact we have the following identity

\[
\dot{E_1} + \left( \frac{\dot{a}}{a} + \frac{2b}{b} \right) E_1 - \frac{\dot{a}}{a} E_2 - \frac{b}{b} E_3 = 0.
\]

As long as we are not in the exact FLRW limit, then we can solve these equations for \( \Lambda, \dot{a}, \text{ and } \dot{b} \). These equations of motion are not independent equations of motion, in fact we have the following identity

\[
\dot{E_1} + \left( \frac{\dot{a}}{a} + \frac{2b}{b} \right) E_1 - \frac{\dot{a}}{a} E_2 - \frac{b}{b} E_3 = 0.
\]

This relation states that \( E_3 \) can be written in terms of \( E_1 \), its time derivative and \( E_2 \). Therefore, once both \( E_1 \) and \( E_2 \) are satisfied, then automatically also \( E_3 \) will be.

### B Regular field redefinition

We will write here the field redefinition which diagonalizes the kinetic matrix for a general Bianchi-I solution for the theory under consideration. We write it explicitly here to show it is in general regular, in particular in the isotropic limit. We have in general that \( K_{23} \propto (ab - \dot{a}b)^2 \), whereas both \( K_{12} \) and \( K_{33} \) are proportional to \( (ab - \dot{a}b) \) linearly. Then we obtain, without any approximation, i.e. for a general Bianchi-I manifold

\[
K_{12} = \frac{24(ba - \dot{a}b) - 24b^2a^2}{\Delta_1},
\]

\[
K_{23} = \frac{24(ba - \dot{a}b) - 24b^2a^2}{\Delta_1},
\]

\[
K_{33} = \frac{24(ba - \dot{a}b)}{\Delta_1},
\]

\[
\Delta_1 = q^2a^2b^2k^4 + \beta \left( 48q^2 \left( \frac{b^2 + 3}{2} \right) b^2 + \frac{b}{2} \left( q^2 - 13b^2 \right) b + \frac{1}{2} q^2b^2 + b^4 \right) a^5
\]

and we can see that in the exact isotropic limit, we find that \( \Gamma_1, \Gamma_2, \text{ and } \Gamma_3 \) all vanish as

\[
\lim_{a,b \to a_0} \Delta_1 = -q^2a^2(M_0^2 + 48 \beta H_0^2).
\]
Then, after performing the field redefinition, we find

\[ g_1 = \frac{144 a^4 b^2 \beta^2 k^2 (ab - b\dot{a})^2 a^4}{M_P^4 \Delta_1}, \]  
\[ g_2 = -\frac{k^2 \Delta_1}{4a^4 b^4 M_P^2}, \]  
\[ g_3 = -\frac{12 (ab - b\dot{a}) \dot{b} q^4 \beta}{M_P^2 b^4}. \]  

(B.6) \hspace{1cm} (B.7) \hspace{1cm} (B.8)

B.1 Exact FLRW limit

We discuss here the exact limit for which \( b \to a \) in the Lagrangian density for the fields \( F_i \). Although this limit should not be made exactly, as we would loose information regarding the propagating fields \( F_1 \) and \( F_2 \), we only want to show here that we can get back the FLRW result for the cross polarization of gravitational waves. In fact, in this case the Lagrangian density reduces to

\[
\mathcal{L} = \frac{M_P^4}{M_P^2} + 48 \beta H_0^4 \left[ \frac{a_0 q^2 k^2}{4} \dot{F}_2^2 - \frac{q^2 k^2}{4} (\dot{F}_2 F_3 - \dot{F}_3 F_2) - \frac{(q^2 - 2 H_0^2 a_0^2) q^2 k^2}{4a_0} F_2^2 \right. \\
+ \frac{q^2 k^2 H_0}{2} F_2 F_3 + \frac{(k^2 + q^2) q^2}{4a_0} F_3^2 \left. \right],
\]  
\[
(B.9)
\]

and any term including the field \( F_1 \) disappears in this exact FLRW limit. After integrating by parts the term \( F_2 \dot{F}_3 \) term, we find that the field \( F_3 \) becomes a Lagrange multiplier, which can be integrated out (at least in this wrong limit) to give

\[
F_3 = -\frac{a_0 k^2 (H_0 F_2 - \dot{F}_2)}{k^2 + q^2}.
\]  
\[
(B.10)
\]

On performing the field redefinition

\[
F_2 = \frac{\sqrt{2} a_0 \sqrt{k^2 + q^2}}{2kq^2} f_2,
\]  
\[
(B.11)
\]

then we get, as expected, the standard propagation for a gravitational wave in the de Sitter background for this theory, namely

\[
\mathcal{L} = \frac{(M_P^4 + 48 \beta H_0^4) a_0^3}{8M_P^2} \left[ f_2^2 - \frac{k^2 + q^2}{a_0^2} f_2^2 \right].
\]  
\[
(B.12)
\]

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