Cauchy magnetic field component and magnitude distribution studied by the zero-field muon spin relaxation technique

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Zero-field muon spin relaxation (ZF-μSR) data for dilute spin magnetic systems have been widely interpreted with what is called a Kubo-Toyabe form based on a Lorentzian distribution of local field components. We derive here the proper magnetic field magnitude distribution using independent and uncorrelated component distributions. Our result is then compared to the previously accepted formula for ZF-μSR. We discuss the origins of the magnetic field component and magnitude distributions. Further we found that after rescaling the magnetic field, the differences that are amenable to experimental examination are quite small, although the interpretations behind them are quite different.

Zero-field Muon Spin Relaxation (ZF-μSR)[1] has been used long and widely to study the local magnetic field distribution in the rare-earth metallic alloys, spin glasses, heavy fermion systems, superconducting compounds and other magnetic systems.[2] It has been believed[3] that in the case of the dilute limit of sparse magnetic moments, the local field component distribution is Cauchy or Lorentzian-like:

\[ g(B_i) = \frac{1}{\pi} \frac{\alpha}{\alpha^2 + B_i^2} \]

where \( B_i = B_x, B_y \) or \( B_z \), and \( \alpha \) is the half-width at half-maximum (HWHM).

On the other hand, from studies of systems with randomly distributed dilute magnetic impurities it was concluded that the internal field magnitude distribution is expected to be:[4–7]

\[ P(|B|) = \frac{4}{\pi} \frac{\Gamma}{(\Gamma^2 + |B|^2)^2} \]

Consequently the ZF-μSR relaxation function takes the Lorentzian Kubo-Toyabe form

\[ G_z(t) = \frac{1}{3} + \frac{2}{3}(1 - \lambda t) e^{-\lambda t} \]

where \( \lambda = \gamma \mu \Gamma \), and \( \gamma \mu \) is the gyromagnetic ratio of the muon \( 2\pi(13.554 \text{ kHz/G}) \).

In the following calculation, we will prove that the magnetic field magnitude distribution corresponding to the component distribution (1) is:

\[ P_c(|B|) = \frac{12 |B| \arctan \sqrt{2 \left( \frac{|B|}{\alpha} \right)^2 + \left( \frac{|B|}{\alpha} \right)^4}}{\alpha^2 \pi^2 \left( \left( \frac{|B|}{\alpha} \right)^2 + 3 \right) \sqrt{2 + \left( \frac{|B|}{\alpha} \right)^2}} \]

If the components of the magnetic field are independent and uncorrelated with each other then the three dimensional probability distribution \( p(B) = g(B_x)g(B_y)g(B_z) \). Note that this probability distribution is not isotropic. In fact the three dimensional distribution as a product of individual component distributions is only isotropic for the case of gaussian individual component distributions.

The magnitude distribution \( P_c(|B|) \) may be written as an integral over the \( p(B) \):

\[ P_c(|B|) = \int_0^\pi \int_0^{2\pi} g(|B| \sin \theta \cos \varphi) g(|B| \sin \theta \sin \varphi) g(|B| \cos \theta) |B|^2 \sin \theta d\varphi d\theta \]

Inserting Eq. (1) into the expression of \( P_c(|B|) \) above, assuming \( u = \frac{|B|}{\alpha} \)

\[ P_c(|B|) |B|^2 = f(u)du \]
Putting this result back into Eq. (7), we get

\[
f(u) = \frac{2u^2}{\pi^2} \int_0^\infty \frac{\sin \theta d\theta}{(1 + u^2 \cos^2 \theta)(1 + \frac{1}{4}u^2 \sin^2 \theta)\sqrt{1 + u^2 \sin^2 \theta}}
\]

\[
= \frac{4u^2}{\pi^2} \int_0^1 \frac{d t}{(1 + u^2 t^2)(1 + \frac{1}{4}u^2 t^2)\sqrt{1 + u^2 - u^2 t^2}}
\]

\[
= \frac{12u \arctan \sqrt{2u^2 + u^4}}{\pi^2(u^2 + 3)\sqrt{2 + u^2}}
\]

(where \( t = \cos \theta \))

By using the identity Eq. (6), we finally get the magnetic field magnitude distribution Eq. (4). A study of this field distribution discloses two important features:

a) At low field, the field magnitude distribution, Eq. (9), is asymptotic to \( \frac{\alpha B^2}{\pi} \), i.e. proportional to \( |B|^2 \).

This is the same asymptotic behavior shown by Eq. (2), a result expected for a variety of models. However, it should be noted that it is difficult to obtain a specific heat which is linear in \( T \) at low temperatures as observed experimentally with a field magnitude distribution which is proportional to \( |B|^2 \) while using the molecular-field model with the spins of the impurities quantized along the local field and no overall preferred spin direction. On the other hand, there exist other physical models which would give at low field a constant field-magnitude asymptote. These models would yield a linear low temperature \( T \) dependence for the specific heat.

b) It can be determined that the Wronskian of Eq. (4) and Eq. (2) (even the rescaled Eq. (2)) is nonzero which implies that they are indeed independent i.e. \( P_c(|B|) \neq C \cdot P(D \cdot |B|) \) where \( C \) and \( D \) are constant arbitrary constants. Nevertheless, it is interesting to compare these two distributions numerically. If one starts from Eq. (2) and uses \( v = \frac{|B|}{\alpha} \), one obtains

\[
P(|B|)d|B| = h(v)dv
\]

where

\[
h(v) = \frac{4}{\pi} \frac{v^2}{(1 + v^2)^2}
\]

\[
j(u) = \frac{4}{\pi} \frac{u^2}{1 + (\frac{u}{1.476})^2} \frac{1}{1.476}
\]

where \( j(u) \) is just a rescaled distribution of \( h(v) \) for \( u = 1.476v \).

A plot showing the similarity and difference between \( f(u) \) and \( j(u) \) is shown in Fig. 1.

It is interesting to study Eq. (2) further. Suppose we divide it by \( 4\pi |B|^2 \) to change it to a probability density distribution (we assume here the field is isotropic) and do a double integral in the Cartesian coordinate system as follows:

\[
P_x(B_x) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{\Gamma}{\pi^2 (\Gamma^2 + B_x^2 + B_y^2 + B_z^2)^2} dB_y dB_z
\]

\[
= \frac{1}{\pi} \frac{\Gamma}{\Gamma^2 + B_x^2}
\]

(14)

If we take \( \Gamma = \alpha \), the above equation is exactly Eq. (1). At first glance, this is quite puzzling since we started from the same distribution and obtained a totally different magnitude distribution: Eq. (4). A careful study of all the derivation procedures above, leads us to conclude that: there is not a one to one mapping or correspondence between the component distribution and magnitude distribution; to some extent, the two seemingly paradoxical results are due to different causalities; the three component distributions behind the magnitude distribution Eq. (4) are assumed to be independent and uncorrelated, on the other hand, the component distribution Eq. (14) arises from the magnitude distribution assuming isotropy.

It can be seen from figure 1 that although the form of our new result is analytically quite different from the previous one which is widely used in ZF-\( \mu \)SR spectroscopy,
numerically they are amazingly close to each other. As far as the relaxation function is concerned, an experimental determination of which field magnitude distribution is correct should be essentially impossible.

A commonly-used procedure[4] which yields the Lorentzian field magnitude distribution of Eq. (2) is to assume a RKKY(Ruderman-Kittel-Kasuya-Yosida) interaction among the dilutely distributed spins and to use the MRF(Mean Random Field) approximation. Further this produces the distribution of the vector field $\mathbf{B}$ (not just the individual component distributions). Even for an isotropic vector field distribution, it rarely occurs that this distribution factors into 3 uncorrelated field component distributions. Our approach to this problem is logically straightforward and independent of any physical models, the only assumption here is the use of an appropriate field component distribution.

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