Transformation Digroups

Keqin Liu
Department of Mathematics
The University of British Columbia
Vancouver, BC
Canada, V6T 1Z2

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Abstract
We introduce the notion of a transformation digroup and prove that every digroup is isomorphic to a transformation digroup.

The purpose of this paper is to show how to choose a class of non-bijective transformations on a Cartesian product of two sets to define a transformation digroup on the Cartesian product. The main result of this paper is that every digroup is isomorphic to a transformation digroup.

The notion of a digroup we shall use in this paper was introduced in Chapter 6 of [6]. Its special case, which is the notion of a digroup with an identity, was introduced independently by people who work in different areas of mathematics ([1], [3] and [5]).

After reviewing some basic definitions about digroups in Section 1, we introduce in Section 2 the notion of a symmetric digroup on the Cartesian product $\Delta \times \Gamma$, where $\Delta$ and $\Gamma$ are two sets. If $|\Delta| = 1$, then a symmetric digroup on $\Delta \times \Gamma$ becomes the symmetric group on the set $\Gamma$, where $|\Delta|$ denotes the cardinality of $\Delta$. In Section 3 we prove that every digroup is isomorphic a subdigroup of a symmetric digroup, which is a better counterpart of Cayley’s Theorem in the context of digroups.

1 The Notion of a Digroup

The following definition of a digroup is a version of Definition 6.1 of [6].

Definition 1.1 A nonempty set $G$ is called a digroup if there are two binary operations $\cdot$ and $\cdot$ on $G$ such that the following three properties are satisfied.

(i) (The Diassociative Law) The two operations $\cdot$ and $\cdot$ are diassociative, that is,

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z = x \cdot (y \cdot z),$$

(1)
\[(x \cdot y) \cdot z = x \cdot (y \cdot z), \tag{2}\]
\[(x \cdot y) \cdot z = (x \cdot y) \cdot z = x \cdot (y \cdot z) \tag{3}\]

for all \(x, y, z \in G\).

(ii) **(Bar-unit)** There is an element \(e\) of \(G\) such that
\[x \cdot e = e = e \cdot x \tag{4}\]
for all \(x \in G\).

(iii) **(One-sided Inverses)** For each element \(x\) in \(G\), there exist two elements \(x_e^{\ell}\) and \(x_e^{r}\) of \(G\) such that
\[x_e^{\ell} \cdot x = e = x \cdot x_e^{r}. \tag{5}\]

The diassociative law was introduced by J. -L. Loday to study Leibniz algebras. An element \(e\) satisfying (4) is called a **bar-unit**, and the set of all bar-units is called the **halo** (\([7]\)). The two elements \(x_e^{\ell}\) and \(x_e^{r}\) are called a **left inverse** and a **right inverse** of \(x\) with respect to the bar-unit \(e\), respectively (\([6]\)). The binary operations \(\cdot\) and \(\cdot\) are called the **left product** and the **right product**, respectively. We also use \((G, \cdot, \cdot)\) to signify that \(G\) is a digroup with the left product \(\cdot\) and the right product \(\cdot\).

Both the left inverse \(x_e^{\ell}\) and the right inverse \(x_e^{r}\) of an element \(x\) with respect to a bar-unit \(e\) is unique (Proposition 6.1 in \([6]\)), but \(x_e^{\ell}\) is generally not equal to \(x_e^{r}\) (Proposition 6.2 in \([6]\)).

The next proposition shows that Definition 1.1 of a digroup is independent of the choice of the bar-unit \(e\).

**Proposition 1.1** Let \(G\) be a digroup. If \(\alpha\) is a bar-unit of \(G\), then every element \(x\) of \(G\) has both the left inverse \(x_{\alpha}^{\ell}\) and the right inverse \(x_{\alpha}^{r}\) with respect to \(\alpha\); that is,
\[x_{\alpha}^{\ell} \cdot x = \alpha = x \cdot x_{\alpha}^{r}. \tag{6}\]

**Proof** By Definition 1.1 there is a bar-unit \(e\) of \(G\) such that
\[x_{e}^{\ell} \cdot x = e = x \cdot x_{e}^{r}. \tag{7}\]

It follows that
\[(\alpha \cdot x_{e}^{\ell}) \cdot x = \alpha = x \cdot (x_{e}^{r} \cdot \alpha), \tag{8}\]
which proves that

\[ x^{-1}_{\alpha} = \alpha \cdot x^{-1}_e \quad \text{and} \quad x^{-1}_r = x^{-1}_e \cdot \alpha. \]

\[ \square \]

If \( G \) is a digroup, then the \textbf{halo} of \( G \) is denoted by \( \bar{h}(G) \); that is

\[ \bar{h}(G) := \{ \alpha \mid \alpha \in G \text{ and } x \cdot \alpha = x = \alpha \cdot x \text{ for all } x \in G \}. \]

**Definition 1.2** Let \( (G, \cdot, \cdot) \) be a digroup. An element \( e \) of \( G \) is called an \textbf{identity} of \( G \) if \( e \in \bar{h}(G) \) and

\[ e \cdot x = x \cdot e \quad \text{for all } x \in G. \]

By Proposition 6.2 in [6], if \( e \) is a bar-unit of \( G \), then \( e \) is an identity of a digroup \( G \) if and only if \( x^{-1}_e = x^{-1}_e \) for all \( x \in G \). Example 6 in [6] gives a digroup which does not have an identity.

We now introduce the notion of a subdigroup.

**Definition 1.3** Let \( H \) be a subset of a digroup \( (G, \cdot, \cdot) \). If \( H \cap \bar{h}(G) \neq \emptyset \) and \( H \) is itself a digroup under the two binary operations of \( G \), we say that \( H \) is a \textbf{subdigroup} of \( G \). The notation \( H \leq \text{d} G \) is used to indicate that \( H \) is a subdigroup of \( G \).

The following definition introduces two important subdigroups of a digroup.

**Definition 1.4** Let \( (G, \cdot, \cdot) \) be a digroup.

(i) The set

\[ Z^t(G) := \{ z \in G \mid z \cdot x = x \cdot z \text{ for all } x \in G \} \]

is called the \textbf{target center} of \( G \).

(ii) The set

\[ Z^s(G) := \{ z \in G \mid x \cdot z = z \cdot x \text{ for all } x \in G \} \]

is called the \textbf{source center} of \( G \).

One can check that the target center \( Z^t(G) \) is a subdigroup of a digroup \( G \) and

\[ \bar{h}(G) \subseteq Z^t(G). \]

If \( G \) is a digroup with an identity, then the source center \( Z^s(G) \) is a subdigroup of \( G \).

We finish this section with the definition of an isomorphism between digroups.
Definition 1.5 If $G$ and $\bar{G}$ are digroups, then a map $\varphi$ from $G$ to $\bar{G}$ is called an isomorphism if $\varphi$ is bijective and

$$\varphi(x * y) = \varphi(x) * \varphi(y)$$

for all $x, y \in G$ and $* = \leftarrow, \rightarrow$.

2 Symmetric Digroups

Let $T(\Delta \times \Gamma)$ be the set of all maps from $\Delta \times \Gamma$ to $\Delta \times \Gamma$, where $\Delta$ and $\Gamma$ are two sets. Then $T(\Delta \times \Gamma)$ is a semigroup with the identity $1$ under the product:

$$fg := f \cdot g,$$

where $1$ is the identity map, and the product $f \cdot g$ is the composite of $g$ and $f$ ($f$ following $g$):

$$(f \cdot g)(x) := f(g(x)) \quad \text{for } x \in \Delta \times \Gamma.$$

Definition 2.1 An element $\ell$ of $T(\Delta \times \Gamma)$ is called a $\ell$-map on $\Delta \times \Gamma$ if there exists an element $(s, f) \in \Delta \times Sym\Gamma$ such that

$$\ell(k, i) = (s, f(i)) \quad \text{for } (k, i) \in \Delta \times \Gamma,$$

where $Sym\Gamma$ is the symmetric group on $\Gamma$.

It is clear that the element $(s, f)$ in Definition 2.1 is determined uniquely by the $\ell$-map $\ell$. Hence, $\ell$ in Definition 2.1 is also called the $\ell$-map on $\Delta \times \Gamma$ induced by $(s, f) \in \Delta \times Sym\Gamma$. We shall use the notation $\ell_{s,f}$ to indicate that the $\ell$-map induced by $(s, f)$. Thus, we have

$$\ell_{s,f}(k, i) = (s, f(i)) \quad \text{for } (k, i) \in \Delta \times \Gamma. \quad (6)$$

Let $\ell_{s,f}$ and $\ell_{t,g}$ be two $\ell$-maps on $\Delta \times \Gamma$. If $(k, i) \in \Delta \times \Gamma$, then

$$(\ell_{s,f}\ell_{t,g})(k, i) = \ell_{s,f}(t, g(i)) = (s, f(g(i))) = \ell_{s,f,g}(k, i)$$

by (6). Hence, we have

$$\ell_{s,f}\ell_{t,g} = \ell_{s,f,g} \quad \text{for } (s, f), (t, g) \in \Delta \times Sym\Gamma. \quad (7)$$

Proposition 2.1 If $G$ is a subgroup of $Sym\Gamma$, then the set

$$\ell_{\Delta \times G} := \{ \ell_{s,f} \mid (s, f) \in \Delta \times G \}$$

is a subsemigroup of $T(\Delta \times Sym\Gamma)$ having the following two properties:
1. \( \ell_{\Delta \times G} \) has a right unit \( e := \ell_{0,1} \), where \( 0 \) is a fixed element of \( \Delta \).

2. Every element \( \ell_{s,f} \) of \( \ell_{\Delta \times G} \) has a left inverse \( \ell_{s,f}^{-1} := \ell_{0,f^{-1}} \) in \( \ell_{\Delta \times G} \) with respect to the right unit \( e = \ell_{0,1} \).

**Proof** It is clear by (7).

**Proposition 2.2** Let \( G \) be a subgroup of \( \text{Sym}\Gamma \), and let \( \theta : G \to \text{Sym}\Delta \) be a group homomorphism.

(i) The map

\[
\ell_f \mapsto \tilde{\ell}_f \quad \text{for } f \in G
\]

is a group homomorphism from \( G \) to \( \text{Sym}(\Delta \times \Gamma) \), where \( \tilde{\ell}_f : \Delta \times \Gamma \to \Delta \times \Gamma \)

is defined by

\[
\tilde{\ell}_f(k, i) := (\theta(f)(k), f(i)) \quad \text{for } (k, i) \in \Delta \times \Gamma .
\]

(8)

\( \tilde{\ell}_f \) will be called a \( \theta \)-permutation on \( \Delta \times \Gamma \).

(ii) If \((s, f), (t, g) \in \Delta \times \Gamma \), then

\[
\ell_s \ell_{t,g} = \ell_{\theta(f)(t),fg},
\]

(9)

\[
\ell_{s,f} \ell_g = \ell_{s,fg}.
\]

(10)

**Proof** (i) It is clear that \( \tilde{\ell}_f \in \text{Sym}(\Delta \times \Gamma) \) by (8). For \( f, g \in G \) and \((k, i) \in \Delta \times \Gamma \), we have

\[\tilde{\ell}_f(k, i) = \tilde{\ell}_f (\theta(g)(k), g(i)) = (\theta(f)\theta(g)(k), f(g(i))) = (\theta(fg)(k), (fg)(i)) = \tilde{\ell}_g(k, i),\]

which implies that

\[\tilde{\ell}_f \tilde{\ell}_g = \tilde{\ell}_{fg} \quad \text{for } f, g \in G .\]

This proves that the map \( f \mapsto \tilde{\ell}_f \) is a group homomorphism.

(ii) For \((k, i) \in \Delta \times \Gamma \), we have

\[
(\ell_{t,g} \ell_{s,f})(k, i) = \ell_f(t, g(i)) = (\theta(f)(t), f(g(i))) = \ell_{\theta(f)(t),fg}(k, i)
\]

and

\[
(\ell_{s,f} \ell_g)(k, i) = \ell_s, f(\theta(g)(k), g(i)) = (s, (fg)(i)) = \ell_{s,fg}(k, i).
\]

Hence, (ii) is true.

Motivated by the facts in [5], we introduce the construction of a transformation digroup in the next proposition.
Proposition 2.3 If $\mathcal{G}$ is a subgroup of $\text{Sym}\Gamma$ and $\theta : \mathcal{G} \rightarrow \text{Sym}\Delta$ is a group homomorphism, then $\ell_{\Delta \times \mathcal{G}}$ is a digroup under the following two binary operations:

\[
\ell_{s,f} \cdot \ell_{t,g} := \ell_{s,f} \ell_{t,g} = \ell_{s,t}g,
\]

\[
\ell_{s,f} \cdot \ell_{t,g} := \ell_{s,f} \ell_{t,g} = \ell_{\theta(f)(t),g},
\]

where $(s,f), (t,g) \in \Delta \times \mathcal{G}$. $e := \ell_{0,1}$ is a bar-unit, and the left inverse and the right inverse of an element with respect to the bar-unit $\ell_{0,1}$ are given by

\[
(\ell_{s,f})^{-1} = \ell_{0,f}^{-1}, \quad (\ell_{s,f})^{-1} = \ell_{\theta(f^{-1})(0),f}^{-1}.
\]

Proof For $(s,f), (t,g)$ and $(v,h) \in \Delta \times \mathcal{G}$, we have

\[
\ell_{s,f} \cdot (\ell_{t,g} \cdot \ell_{v,h}) = (\ell_{s,f} \cdot \ell_{t,g}) \cdot \ell_{v,h} = \ell_{s,f} \cdot (\ell_{t,g} \cdot \ell_{v,h}) = \ell_{s,fg},
\]

\[
\ell_{s,f} \cdot \ell_{t,g} \cdot \ell_{v,h} = \ell_{s,f} \cdot \ell_{t,g} = \ell_{s,f} \cdot \ell_{\theta(f)(g),h} = \ell_{s,fg},
\]

\[
(\ell_{s,f} \cdot \ell_{t,g}) \cdot \ell_{v,h} = (\ell_{s,f} \cdot \ell_{t,g}) \cdot \ell_{v,h} = \ell_{s,f} \cdot (\ell_{t,g} \cdot \ell_{v,h}) = \ell_{\theta(f)(g),fg}.
\]

This proves that the two binary operations $\cdot$ and $\cdot$ are diassociative. The remaining parts are clear by (12) and (13).

A subdigroup of $\ell_{\Delta \times \mathcal{G}}$ is called the transformation digroup on $\Delta \times \Gamma$ induced by $(\mathcal{G},\theta)$. In particular, the digroup $\ell_{\Delta \times \text{Sym}\Gamma}$ is called the symmetric digroup on $\Delta \times \Gamma$ induced by the group homomorphism $\theta : \text{Sym}\Gamma \rightarrow \text{Sym}\Delta$. It is clear that the symmetric digroup becomes the symmetric group on $\Gamma$ if $|\Delta| = 1$.

We finish this section with the description of the halo and subdigroups of $\ell_{\Delta \times \mathcal{G}}$.

Proposition 2.4 Let $\ell_{\Delta \times \mathcal{G}}$ be the transformation digroup on $\Delta \times \Gamma$ induced by $(\mathcal{G},\theta)$. Let $\mathcal{Z}(\mathcal{G})$ be the center of the group $\mathcal{G}$. Then

(i) $\mathcal{Z}(\ell_{\Delta \times \mathcal{G}}) = \{ \ell_{s,1} \mid s \in \Delta \}$.

(ii) $\ell_{s,1}$ is an identity of $\ell_{\Delta \times \mathcal{G}}$ if and only if $\text{Im}\theta \subseteq (\text{Sym}\Delta)_s$, where

\[
(S\text{ym}\Delta)_s := \{ x \in \text{Sym}\Delta \mid x(s) = s \}
\]

is the stabilizer of $s \in \Delta$ in $\text{Sym}\Delta$.

(iii) $H$ is a subdigroup of $\ell_{\Delta \times \mathcal{G}}$ if and only if there exist a subgroup $\mathcal{H}$ of $\mathcal{G}$ and a fixed block $\Omega$ of $\text{Im}\mathcal{H}$ such that $H = \ell_{\Omega \times \mathcal{H}}$.

(iv) The target center $\mathcal{Z}^t(\ell_{\Delta \times \mathcal{G}})$ is given by

\[
\mathcal{Z}^t(\ell_{\Delta \times \mathcal{G}}) := \{ \ell_{s,f} \mid s \in \Delta \text{ and } f \in \text{Ker}\theta \cap \mathcal{Z}(\mathcal{G}) \}.
\]
The source center $Z^*(\ell_{\Delta \times \mathcal{G}})$ is given by

$$Z^*(\ell_{\Delta \times \mathcal{G}}) := \{ \ell_{s,f} \mid s \in \Delta, \operatorname{Im} \theta \subseteq \operatorname{Sym}\Delta \} \text{ and } f \in \mathcal{Z}(\mathcal{G}).$$

**Proof** The results follow from (12) and (13).

3 The Counterpart of Cayley’s Theorem

In this section we prove that every digroup is isomorphic to a transformation digroup. We begin with the following property of left translations.

**Proposition 3.1** Let $e$ be a bar-unit of a digroup $G$ and let $\phi : G \to \mathcal{G}$ be a map defined by

$$\phi(g) := \bar{g},$$

where $g \in G$, $\bar{g} := e \cdot g$ and

$$\mathcal{G} := \{ \bar{g}_i \mid i \in \bar{\Gamma} \text{ and } \bar{g}_i \neq \bar{g}_j \text{ for } i, j \in \bar{\Gamma} \text{ and } i \neq j \}.$$ If $f \in G$, then $\mathcal{L}_f \in \operatorname{Sym}(\Gamma)$, where

$$\Gamma := \{ \phi^{-1}(\bar{g}_i) \mid i \in \bar{\Gamma} \}.$$ 

**Proof** It is clear that $\phi$ is surjective. Hence, $\phi^{-1}(\bar{g}_i)$ is not empty for all $i \in \bar{\Gamma}$. Since

$$e \cdot \mathcal{L}_f (\phi^{-1}(\bar{g}_i)) = e \cdot (f \cdot \phi^{-1}(\bar{g}_i)) = (e \cdot f) \cdot (\phi^{-1}(\bar{g}_i)) = e \cdot f \cdot \bar{g}_i = \bar{f} \cdot \bar{g}_i,$$

we have

$$\mathcal{L}_f (\phi^{-1}(\bar{g}_i)) \subseteq \phi^{-1}(\bar{f} \cdot \bar{g}_i).$$

Using the properties of left translations and (13), we have

$$\phi^{-1}(\bar{f} \cdot \bar{g}_i) = \mathcal{L}_e (\phi^{-1}(\bar{f} \cdot \bar{g}_i)) = \mathcal{L}_{f^{-1}f} (\phi^{-1}(\bar{f} \cdot \bar{g}_i))$$

or

$$\mathcal{L}_f (\phi^{-1}(\bar{g}_i)) \supseteq \phi^{-1}(\bar{f} \cdot \bar{g}_i).$$

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1The basic properties of left translations were given in Chapter 2 of [6].
It follow from (15) and (16) that
\[ L_f (\phi^{-1} (\vec{g}_i)) = \phi^{-1} \left( \vec{f} \cdot \vec{g}_i \right) \quad \text{for } i \in \bar{\Gamma}. \] (17)

Since \( L_f \) is a permutation on \( G \), \( L_f \) can be regarded as a permutation on \( \Gamma \) by (17).

For an element \( f \) of a digroup \( G \), we define a map \( \Psi_f : G \to G \) by
\[ \Psi_f(x) := f \cdot \vec{e} \cdot \vec{x} \cdot f^{-1} = f \cdot \vec{e} \cdot \vec{x} \cdot f^{-1} \quad \text{for } x \in G, \] (18)
where \( e \) is a bar-unit of \( G \). Since the definition of \( \Psi_f \) is independent of the choice of the bar-unit \( e \), (18) is also written as
\[ \Psi_f(x) := f \cdot \vec{e} \cdot \vec{x} \cdot f^{-1} = f \cdot \vec{e} \cdot \vec{x} \cdot f^{-1} \quad \text{for } x \in G, \]
where \( f^{-1} \) and \( f^{-1} \) denote the left inverse and the right inverse of \( f \) with respect to any bar-unit, respectively.

It is clear that \( \Psi_f \) can be regarded as a permutation on the halo of \( G \).

**Proposition 3.2** If \( G \) is a digroup, then
\[ \theta : L_f \mapsto \Psi_f| \bar{h}(G) \quad \text{for } f \in G \] (19)
is a group homomorphism from the subgroup \( \bar{L}_G \) of \( \text{Sym}\Gamma \) to \( \text{Sym}\bar{h}(G) \), where
\[ \bar{L}_G := \{ \bar{L}_f \mid f \in G \} \quad \text{and} \quad \Gamma := \{ \phi^{-1}(\vec{g}_i) \mid i \in \bar{\Gamma} \} \]

**Proof** \( \theta \) is well-defined. In fact, if \( f, g \in G \) and \( \bar{L}_f = \bar{L}_g \), then \( f \cdot \vec{e} = g \cdot \vec{e} \) for a bar-unit \( e \) of \( G \). For all \( x \in G \), we have
\[ \Psi_f(x) = f \cdot \vec{e} \cdot \vec{x} \cdot f^{-1} \]
\[ = f \cdot \vec{e} \cdot \vec{x} \cdot \vec{e} \cdot f^{-1} = (f \cdot \vec{e}) \cdot \vec{x} \cdot (f \cdot \vec{e})^{-1} \]
\[ = (g \cdot \vec{e}) \cdot \vec{x} \cdot (g \cdot \vec{e})^{-1} = \Psi_g(x). \]

Hence, \( \Psi_f = \Psi_g \).

Since
\[ \Psi_{fg} = \Psi_f \Psi_g \quad \text{for } f, g \in G, \]
\( \theta \) is a group homomorphism.
Let $G$ be a digroup. By Proposition 3.1 and Proposition 3.2, $\ell_{\Delta \times L_G}$ is a transformation digroup on $\Delta \times \Gamma$ induced by $(G, \theta)$, where

$$\Delta := \bar{h}(G) \quad \text{and} \quad \Gamma := \{ \phi^{-1}(\tilde{g}_i) \mid i \in \tilde{\Gamma} \}.$$

We now define a map $\lambda$ from $G$ to $\ell_{\Delta \times L_G}$ by

$$\lambda : \alpha \twoheadrightarrow f \mapsto \ell_{\alpha, L_f} \quad \text{for} \ \alpha \in \bar{h}(G) \text{ and } f \in G. \quad (20)$$

First, we prove that $\lambda$ is well-defined. Every element of $G$ is of the form $\alpha \twoheadrightarrow f$ for some $\alpha \in \bar{h}(G)$ and $f \in G$. If $\alpha \twoheadrightarrow f = \beta \twoheadrightarrow g$, where $\alpha, \beta \in \bar{h}(G)$ and $f, g \in G$, then $\alpha = \beta$ and $\alpha \twoheadrightarrow f = \alpha \twoheadrightarrow g$ or $f \twoheadrightarrow \alpha = g \twoheadrightarrow \alpha$. It follows that $\ell_{L_f} = L_g$. This proves that $\lambda$ is well-defined.

Next, we prove that $\lambda$ is injective. For $\alpha, \beta \in \bar{h}(G)$ and $f, g \in G$, we have

$$\lambda \left( \left( \alpha \twoheadrightarrow f \right) \twoheadrightarrow \left( \beta \twoheadrightarrow g \right) \right) = \lambda \left( \beta \twoheadrightarrow g \right)$$

$$\Rightarrow \ell_{\alpha, L_f} = \ell_{\beta, L_g}$$

$$\Rightarrow \alpha = \beta \quad \text{and} \quad L_f = L_g$$

$$\Rightarrow \alpha = \beta \quad \text{and} \quad f \twoheadrightarrow \alpha = g \twoheadrightarrow \alpha$$

$$\Rightarrow \alpha = \beta \quad \text{and} \quad \alpha \twoheadrightarrow f = \beta \twoheadrightarrow g.$$

Finally, we prove that $\lambda$ preserves both the left product and the right product on $G$. For $\alpha, \beta \in \bar{h}(G)$ and $f, g \in G$, we have

$$\lambda \left( \left( \alpha \twoheadrightarrow f \right) \twoheadrightarrow \left( \beta \twoheadrightarrow g \right) \right) = \lambda \left( \beta \twoheadrightarrow g \right)$$

$$= \lambda \left( \alpha \twoheadrightarrow f \twoheadrightarrow g \right) = \ell_{\alpha, L_f} \twoheadrightarrow \ell_{\beta, L_g}$$

$$= \ell_{\alpha, L_f} \twoheadrightarrow \ell_{\beta, L_g} = L_f \twoheadrightarrow L_g$$

$$= \lambda \left( \alpha \twoheadrightarrow f \right) \twoheadrightarrow \lambda \left( \beta \twoheadrightarrow g \right)$$

and

$$\lambda \left( \left( \alpha \twoheadrightarrow f \right) \twoheadrightarrow \left( \beta \twoheadrightarrow g \right) \right) = \lambda \left( \beta \twoheadrightarrow g \right)$$

$$= \lambda \left( \alpha \twoheadrightarrow f \twoheadrightarrow \beta \twoheadrightarrow g \right).$$
Thus we get the following counterpart of Cayley’s Theorem.

**Proposition 3.3** Any digroup is isomorphic to a transformation digroup.

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