ANOTHER ALMOST ZERO-DIMENSIONAL SPACE OF EXACT MULTIPLICATIVE CLASS 3

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Abstract. We show that the escaping set for $f(z) = \exp(z) - 1$ is nowhere $\sigma$-complete. This establishes that the escaping endpoint set $\dot{E}(f)$ is a first category almost zero-dimensional space which is $F_{\sigma\delta}$ and nowhere $G_{\delta\sigma}$. Only two other elementary spaces with those properties are known: $Q^\omega$ and Erdős space $\mathfrak{E}$. Previous work has shown that $\dot{E}(f)$ is homeomorphic to neither of those.

1. Introduction

In the 1980’s van Engelen proved that $Q^\omega$ is the unique zero-dimensional space which is first category, $F_{\sigma\delta}$, and nowhere $G_{\delta\sigma}$. In the larger class of almost zero-dimensional spaces there is another such example, namely the Erdős space $\mathfrak{E} = \{x \in \ell^2 : x_n \in \mathbb{Q} \text{ for all } n < \omega\}$. The goal of this paper is to demonstrate a third fundamental example: The set of endpoints of the Julia set of $f(z) = \exp(z) - 1$ which escape to infinity under iteration of $f$. In symbols, $\dot{E}(f) = E(f) \cap I(f)$ where $E(f)$ is the set of endpoints of the Cantor bouquet Julia set $J(f)$, and $I(f) = \{z \in \mathbb{C} : f^n(z) \to \infty\}$. That $\dot{E}(f)$ is a first category almost zero-dimensional $F_{\sigma\delta}$-space was observed in [5]; see also [1]. Here we prove:

**Theorem 1.** $\dot{E}(f)$ is nowhere $\sigma$-complete.

Rempe [1] recently proved that $I(f)$ is nowhere $\sigma$-compact. Since $\dot{E}(f)$ is a dense $G_{\delta\sigma}$-subset of $I(f)$, Theorem 1 implies the stronger result:

**Corollary 2.** $I(f)$ is nowhere $\sigma$-complete.

Corollary 2 implies that $J(f) \setminus I(f)$ is nowhere $F_{\sigma\delta}$. Since $J(f) \setminus I(f)$ is zero-dimensional [6] and $\mathbb{P}^\omega \setminus (\mathbb{Q} + \sqrt{2})^\omega$ is the unique zero-dimensional $\sigma$-complete Baire space which is nowhere $F_{\sigma\delta}$ [11, Theorem A.2.6], we get:

**Corollary 3.** $J(f) \setminus I(f) \cong \mathbb{P}^\omega \setminus (\mathbb{Q} + \sqrt{2})^\omega$.

The following summarizes what is currently known about the topology of $\dot{E}(f)$:

- almost zero-dimensional and cohesive [5, 1]
- first category and $F_{\sigma\delta}$ [5, 10]
- nowhere $G_{\delta\sigma}$
- rim-complete and nowhere rim-$\sigma$-compact [6, 3]
- contains a dense copy of $\mathfrak{E}$ [7]
- each point is contained in a closed copy of $\mathfrak{E}_c := \{x \in \ell^2 : x_n \notin \mathbb{Q} \text{ for all } n < \omega\}$ [1, 6]
- cannot be written as a countable union of nowhere dense C-sets. [8]
The last property distinguishes $\hat{E}(f)$ from $\mathbb{Q}^\omega$ and $\mathcal{E}$. It is unknown whether $\hat{E}(f)$ is a topological group, or is at least homogeneous.

**Question 1.** Is $\hat{E}(f)$ homogeneous?

2. Preliminaries

2.1. Terminology. An intersection of clopen subsets of a space $X$ is called a C-set in $X$. A space $X$ is almost zero-dimensional if each point $x \in X$ has a neighborhood basis of C-sets in $X$. Equivalently, $X$ is almost zero-dimensional if $X$ is homeomorphic to the graph of an upper semi-continuous function $\varphi : Z \to [0, 1]$ with zero-dimensional domain $Z$ [2, Lemma 4.11].

A space $X$ is first category if $X$ can be written as a countable union of (closed) nowhere dense subsets. Completely metrizable spaces are precisely the absolute $G_\delta$ spaces, and by Baire’s theorem these spaces are not first category.

Absolute $F_{\sigma\delta}$ spaces are said to be of multiplicative class three; they are the spaces in level $\Pi^0_3$ of the Borel hierarchy. A space $X$ is $\sigma$-complete if $X$ can be written as a countable union of completely metrizable subspaces. These are precisely the absolute $G_{\delta\sigma}$-spaces (the members of $\Sigma^0_3$). A space $X$ is of exact multiplicative class three if $X$ is $F_{\sigma\delta}$ but not $G_{\delta\sigma}$, i.e. if $X \in \Pi^0_3 \setminus \Sigma^0_3$.

2.2. Straight brush model of $J(f)$. To prove Theorem 1 we will use the following model of $J(f)$ described in [1] and [9]. Let $\mathbb{Z}^\omega$ denote the space of integer sequences

$$\mathbb{Z} = \{s_0s_1s_2\ldots\}.$$ 

Define $\mathcal{F} : [0, \infty) \times \mathbb{Z}^\omega \to \mathbb{R} \times \mathbb{Z}^\omega$ by

$$\langle t, \mathbb{Z} \rangle \mapsto \langle F(t) - |s_1|, \sigma(\mathbb{Z}) \rangle,$$

where $F(t) = e^t - 1$ and $\sigma(s_0s_1s_2\ldots) = s_1s_2s_3\ldots$ is the shift on $\mathbb{Z}^\omega$. For each $x = \langle t, \mathbb{Z} \rangle \in [0, \infty) \times \mathbb{Z}^\omega$ put $T(x) = t$ and $\mathbb{Z}(x) = \mathbb{Z}$. Let

$$J(\mathcal{F}) = \{x \in [0, \infty) \times \mathbb{Z}^\omega : T(\mathcal{F}(x)) \geq 0 \text{ for all } n \geq 0\}.$$ 

The action of $\mathcal{F}$ on $J(\mathcal{F})$ models the action of $f$ on $J(f)$.

2.3. Endpoints of $J(\mathcal{F})$. Let

$$\mathbb{S} = \{\mathbb{Z} \in \mathbb{Z}^\omega : \text{there exists } t \geq 0 \text{ such that } \langle t, \mathbb{Z} \rangle \in J(\mathcal{F})\}.$$ 

For each $\mathbb{Z} \in \mathbb{S}$ put

$$t_{\mathbb{Z}} = \min\{t \geq 0 : \langle t, \mathbb{Z} \rangle \in J(\mathcal{F})\}.$$ 

Let $E(\mathcal{F}) = \{\langle t_{\mathbb{Z}}, \mathbb{Z} \rangle : \mathbb{Z} \in \mathbb{S}\}$. Note that $E(\mathcal{F})$ is a $G_\delta$-subset of $J(\mathcal{F})$ which is closed in $\mathbb{Z}^\omega \times [0, \infty)$. Therefore $E(\mathcal{F})$ is completely metrizable. Let

$$\tilde{E}(\mathcal{F}) = \{x \in E(\mathcal{F}) : t_{\sigma^n(\mathbb{Z})} \to \infty\}.$$ 

The conjugacy [9, Theorem 9.1] shows that $\tilde{E}(\mathcal{F}) \simeq \hat{E}(f)$.

2.4. Approximating $t_{\mathbb{Z}}$. The following estimates of $t_{\mathbb{Z}}$ will be useful. Let $F^{-1}$ denote the inverse of $F$. So $F^{-1}(t) = \ln(t + 1)$ for all $t \geq 0$. For each $n \geq 1$, the $n$-fold composition of $F^{-1}$ is denoted $F^{-n}$. Let

$$t_{\mathbb{Z}}^* = \sup_{k \geq 1} F^{-k}|s_k|.$$ 

Note that $t_{\sigma^n(\mathbb{Z})}^* = \sup_{k \geq 1} F^{-k}|s_{n+k}|$.

**Proposition 1** ([1, Lemma 3.8]). $t_{\mathbb{Z}}^* \leq t_{\mathbb{Z}} \leq t_{\mathbb{Z}}^* + 1$.  

Proposition 2 ([1, Lemma 3.8]). \( \langle t_\omega, s \rangle \in \tilde{E}(F) \iff t_\omega^* < \infty \) and \( t_{\sigma^M(\omega)}^* \rightarrow \infty \).

Proposition 3 ([1, Observation 3.7]). If \( s^0, s \in S \) and \( |s_n| \geq |s^0_n| \) for all \( n < \omega \), then \( t_\omega \geq t_\omega^0 \).

Proposition 4 ([1, Observation 3.9]). Let \( s^0, s \in S \) and suppose \( |s^0_j| = |s_j| \) for \( j = 1, \ldots, M \) and that \( \delta := t_{\sigma^M(\omega)} - t_{\sigma^M(\omega)} > 0 \). Then \( t_\omega \leq t_\omega^0 \leq t_\omega + F^{-M}(\delta) \).

3. A stratification of \( \tilde{E}(F) \)

Here we will describe a system \( (X_\alpha) \) of subsets of \( \tilde{E}(F) \) which will be used in the proof of Theorem 1. For any subset \( X \) of \( \tilde{E}(F) \) we will let \( \overline{X} \) denote the closure of \( X \) in \( E(F) \).

The sets \( X_\alpha \) are defined recursively for increasing elements \( \alpha \in \mathbb{N}^{<\omega} \). Let \( X_0 = \tilde{E}(F) \). For each \( N \in \mathbb{N} \) define \( X_N = \{ x \in \tilde{E}(F) : t_{\sigma^n(x)} > 1 \) for all \( n \geq N \} \). If \( \alpha = (N_0, N_1, \ldots, N_{\text{dom}(\alpha)-1}) \in \mathbb{N}^{<\omega} \) is an increasing sequence of integers, \( X_\alpha \) has been defined, and \( N > N_{\text{dom}(\alpha)-1} \), then define

\[
X_\alpha \setminus N = \{ x \in X_\alpha : t_{\sigma^n(x)} > 3 \text{dom}(\alpha) + 1 \) for all \( n \geq N \} \).
\]

Claim 1. \( \overline{X_\alpha \setminus N} = \{ x \in \overline{X_\alpha} : t_{\sigma^n(x)} \geq 3 \text{dom}(\alpha) + 1 \) for all \( n \geq N \} \).

Proof. Note that \( t_{\sigma^n(x)} = T(F^n(x)) \). So the set on the right is closed by continuity of \( T \circ F^n \). Clearly it contains \( X_\alpha \setminus N \), so it contains \( \overline{X_\alpha \setminus N} \). Conversely suppose \( x \in \overline{X_\alpha} \) and \( t_{\sigma^n(x)} \geq 3 \text{dom}(\alpha) + 1 \) for all \( n \geq N \). Let \( U \) be an open set containing \( x \). We will show that \( U \) contains a point of \( \overline{X_\alpha \setminus N} \).

Let \( y \in U \cap X_\alpha \), and put \( s = s(y) \).

For each \( M > N \) we will define \( s^M \in \mathbb{Z}^\omega \) as follows. Note that

\[
t_{\sigma^n(x)} \leq t_{\sigma^n(x)}^* \geq 3 \text{dom}(\alpha)
\]

for all \( n \geq N \) by Proposition 1. There is a least \( l \in [N, M] \) such that \( t_{\sigma^n(x)} \leq t_{\sigma^n(x)}^* \geq 3 \text{dom}(\alpha) + 1 \) such that \( l + k > M \) and \( F^{-k}|s_{l+k}| > 3 \text{dom}(\alpha) - 1 \). Let \( n(0) = l \), and let \( n(0) \geq 1 \) such that \( F^{-k}(s_{n(0)+k(0)}) > 3 \text{dom}(\alpha) - 1 \). Assuming \( n(i) \) and \( k(i) \) have been defined, let \( n(i+1) = n(i) + k(i) \), and choose \( k(i+1) \geq 1 \) such that \( F^{-k(i+1)}|s_{n(i+1)+k(i+1)}| > 3 \text{dom}(\alpha) - 1 \). In this manner we recursively define integers \( n(0) + k(0) < n(1) + k(1) < \ldots \) so that

\[
F^{-k(i)}|s_{n(i)+k(i)}| > 3 \text{dom}(\alpha) - 1.
\]

For each \( i \) there exists an integer \( p_i \) such that

\[
3 \text{dom}(\alpha) + 1 < F^{-k(i)}|s_{n(i)+k(i)}| + p_i < F^{-k(n)}|s_{n(i)+k(i)}| + 3.
\]

Define \( s^M \) by

\[
s^M_j = \begin{cases} 
  s^M_j & \text{if } j \neq n(i) + k(i) \text{ for any } i \\
  |s^M_j| + p_i & \text{if } j = n(i) + k(i).
\end{cases}
\]

Note that \( t_\omega^* \leq t_\omega^{*M} < t_\omega^* + 3 < \infty \), so \( \langle t_\omega^*, s^M \rangle \in \tilde{E}(F) \) by Proposition 2. Further, \( \langle t_\omega^*, s^M \rangle \in X_\alpha \setminus N \). By Propositions 1 and 3, \( t_{\sigma^M(\omega)}^* \leq t_{\sigma^M(\omega)}^* + 3 \) implies that

\[
0 \leq t_{\sigma^M(\omega)}^* - t_{\sigma^M(\omega)}^* \leq 4.
\]

By Proposition 4 we have \( t_\omega \leq t_{\sigma^M(\omega)} \leq t_{\sigma^M(\omega)} + F^{-M}(4) \), hence \( t_{\sigma^M(\omega)} \rightarrow t_\omega \). Thus \( \langle t_{\sigma^M(\omega)}, s^M \rangle \rightarrow y \) and so eventually \( \langle t_{\sigma^M(\omega)}, s^M \rangle \in U \). This shows \( x \in \overline{X_\alpha \setminus N} \).

Claim 2. \( \overline{X_\alpha \cap \tilde{E}(F)} = \bigcup \{ \overline{X_\alpha \setminus N} \cap \tilde{E}(F) : N \in \mathbb{N} \} \)
Proof. Clearly \( \bigcup (X_{\alpha - N} \cap \bar{E}(\mathcal{F}) : N \in \mathbb{N}) \subset X_{\alpha} \cap \bar{E}(\mathcal{F}) \). Now suppose \( x \in X_{\alpha} \cap \bar{E}(\mathcal{F}) \). Then there exists \( N \) such that \( t_{\sigma^n(x)} \geq 3 \text{dom}(\alpha) + 1 \) for all \( n \geq N \). By Claim 1 we have \( x \in X_{\alpha - N} \cap \bar{E}(\mathcal{F}) \).

Our goal now is to use Claim 2 to show that each \( X_{\alpha} \cap \bar{E}(\mathcal{F}) \) is first category in itself. For \( \alpha = \emptyset \) this is already known (i.e. \( \bar{E}(\mathcal{F}) \) is first category). Assume now that \( \text{dom}(\alpha) > 0 \). For each \( s \in S \) and \( m < \omega \) define \( \tilde{s}^m \) by

\[
\tilde{s}^m_n = \begin{cases} 
 s_n & \text{if } n \leq m \\
 \min\{\lfloor s_n \rfloor, \lfloor \text{dom}(\alpha) \rfloor - 1\} & \text{if } n > m.
\end{cases}
\]

Note that \( \tilde{s}^m \in S \) because \( t^m \leq t^\omega < \infty \).

Claim 3. For any \( s \in \{s(x) : x \in X_{\alpha}\} \) and integer \( M \) there exists \( m \geq M \) such that \( (t_{s^m}, \tilde{s}^m) \in X_{\alpha - N} \).

Proof. Let \( s \in \{s(x) : x \in X_{\alpha}\} \) and \( M \) be given. Put \( N_{\text{dom}(\alpha)} = N \). For each \( i < \text{dom}(\alpha) \) and \( n \in [N_i, N_{i+1}) \) there exists \( k_n \geq 1 \) such that

\[
F^{-k_n}|s_n + k_n| > 3i - 2.
\]

Assume \( M > N + \max\{k_n : n < N\} \). For each \( n \in [N, M] \) there exists \( k_n \) such that

\[
F^{-k_n}|s_n + k_n| > 3 \text{dom}(\alpha) - 2.
\]

Let \( m = \max\{n + k_n : n \in [N, M]\} \). Observe that \( (t_{s^m}, \tilde{s}^m) \in \bar{E}(\mathcal{F}) \) by Proposition 2 and the assumptions \( x \in X_{\alpha} \) with \( \text{dom}(\alpha) > 0 \). Also, by Proposition 1 we have

\[
3 \text{dom}(\alpha) - 2 < F^{-1}[|F(3 \text{dom}(\alpha) - 1)|] \leq t_{\sigma^n(s)} \leq t_{\sigma^n(s)}
\]

for each \( n \geq m \), and

\[
t_{\sigma^n(s)} \leq t_{\sigma^n(s)} + 1 \leq \text{dom}(\alpha).
\]

We conclude that \( m \geq M \) and \( (t_{s^m}, \tilde{s}^m) \in X_{\alpha - N} \). \( \square \)

Claim 4. If \( U \) is open and \( U \cap X_{\alpha} \neq \emptyset \) then \( U \cap X_{\alpha - N} \neq \emptyset \).

Proof. Let \( (t_{s^m}, \tilde{s}^m) \in U \cap X_{\alpha} \). Clearly \( |s^m_n| \leq |s_n| \) for every \( n < \omega \). So \( t_{s^m} \leq t_{s} \) by Proposition 3. Also \( s^m \to s \). Since \( J(\mathcal{F}) = \bigcup \{[t, \infty) \times \{s \} : s \in S\} \) is closed in \( \mathbb{Z}^\omega \times [0, \infty) \), we get \( (t_{s^m}, \tilde{s}^m) \to (t_s, s) \). It now follows from Claim 3 that

\[
(t_{s^m}, \tilde{s}^m) \in U \cap X_{\alpha - N}
\]

for some \( m \). \( \square \)

Claim 5. Every \( X_{\alpha - N} \cap \bar{E}(\mathcal{F}) \) is first category in itself.

Proof. Each \( X_{\alpha - N} \) is nowhere dense in \( X_{\alpha} \) by Claim 4. Therefore \( X_{\alpha - N} \cap \bar{E}(\mathcal{F}) \) is nowhere dense in \( X_{\alpha} \cap \bar{E}(\mathcal{F}) \). Thus by Claim 2 \( X_{\alpha} \cap \bar{E}(\mathcal{F}) \) is a first category space. \( \square \)

4. Proof of Theorem 1

We need to show that \( \bar{E}(\mathcal{F}) \) has no \( \sigma \)-complete neighborhood. To that end, suppose that \( A := \{A_n : n < \omega\} \) is a collection of completely metrizable subspaces of \( \bar{E}(\mathcal{F}) \). Let \( U \) be any non-empty open subset of \( \bar{E}(\mathcal{F}) \). We will show that \( A \) does not cover \( U \).

Let \( d \) be a complete metric for \( E(\mathcal{F}) \). Since \( U \) is first category we have that \( U \cap X_0 \) is a non-empty open set. Let \( U_0 \) be a non-empty open subset of \( E(\mathcal{F}) \) such that \( U_0 \cap X_0 \subset U \) and \( \text{diam}(U_0) < 1 \), and \( \overline{U_0} \cap A_0 = \emptyset \). There exists \( N_0 \) such that \( X_{\langle N_0 \rangle} \cap U_0 \neq \emptyset \).
Since \( U_0 \cap \bar{X}(N_0) \) is first category (Claim 5), it does not have a dense completely metrizable subspace. Note that \( U_0 \cap \bar{X}(N_0) \cap A_1 \) is complete because it is a closed subset of \( A_1 \). Therefore

\[
U_0 \cap \bar{X}(N_0) \setminus U_0 \cap \bar{X}(N_0) \cap A_1 \neq \emptyset.
\]

So there is a non-empty relatively open subset \( U_1 \) of \( U_0 \cap \bar{X}(N_0) \) such that \( \text{diam}(U_1) < 1/2 \) and \( U_1 \cap \bar{X}(N_0) \cap A_1 = \emptyset \). Choose \( N_1 > N_0 \) such that \( X(N_0, N_1) \cap U_1 \neq \emptyset \).

This process can be continued to produce an increasing sequence

\[ \lambda = (N_0, N_1, \ldots) \in \mathbb{N}^\omega \]

and non-empty sets \( U_0 \supset U_1 \supset U_2 \supset \ldots \) such that:

- \( U_n \) is a relatively open subset of \( \bar{X}_\lambda|n \cap \bar{E}(\mathcal{F}) \),
- \( \text{diam}(U_n) < 1/(n + 1) \) in the metric \( d \), and
- \( U_n \cap \bar{X}_\lambda|n \cap A_n = \emptyset \).

By completeness of the metric space \( (E(\mathcal{F}), d) \) there exists

\[
x \in \bigcap \{X_\lambda|n \cap U_n : n < \omega \} \subset V \cap E(\mathcal{F}) \setminus \bigcup A.
\]

Note that \( t_{\sigma^n}(x) \to \infty \) by the easy direction in Claim 1, so \( x \in \bar{E}(\mathcal{F}) \) as desired.

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