Plane Formation by Synchronous Mobile Robots without Chirality

Yusaku Tomita, Yukiko Yamauchi, Shuji Kijima, and Masafumi Yamashita

Graduate School of Information Science and Electrical Engineering,
Kyushu University, Japan
tomita@tcslab.csce.kyushu-u.ac.jp,
{yamauchi, kijima, mak}@inf.kyushu-u.ac.jp

Abstract. We consider a distributed system consisting of autonomous mobile computing entities, called robots, moving in a specified space. The robots are anonymous, oblivious, and have neither any access to the global coordinate system nor any explicit communication medium. Each robot observes the positions of other robots and moves in terms of its local coordinate system. To investigate the self-organization power of robot systems, formation problems in the two dimensional space (2D-space) have been extensively studied. Yamauchi et al. (DISC 2015) introduced robot systems in the three dimensional space (3D-space). While existing results for 3D-space assume that the robots agree on the handedness of their local coordinate systems, we remove the assumption and consider the robots without chirality. One of the most fundamental agreement problems in 3D-space is the plane formation problem that requires the robots to land on a common plane, that is not predefined. It has been shown that the solvability of the plane formation problem by robots with chirality is determined by the rotation symmetry of their initial local coordinate systems because the robots cannot break it. We show that when the robots lack chirality, the combination of rotation symmetry and reflection symmetry determines the solvability of the plane formation problem because a set of symmetric local coordinate systems without chirality is obtained by rotations and reflections. This richer symmetry results in the increase of unsolvable instances compared with robots with chirality and a flaw of existing plane formation algorithm. In this paper, we give a characterization of initial configurations from which the robots without chirality can form a plane and a new plane formation algorithm for solvable instances.

1 Introduction

Distributed coordination of mobile computing entities has been gaining increasing attention from many areas such as robotics, transportation, construction, material engineering, DNA computing, and so on. Though these wide areas of applications require complicated operations, they can be classified into fundamental tasks, for example, gathering, formation, exploration, surveillance,
flocking, and partitioning. The underlying goals of these distributed coordination tasks are agreement and self-organization. We focus on a theoretical aspect of one of such mobile computing entity models, called autonomous mobile robots [2,7,9,11,12,13,15,16,17]. A mobile robot system consists of a set of robots each of which autonomously moves in a specified space. Each robot is an anonymous (indistinguishable) point, and it executes a common distributed algorithm. Each robot repeats a Look-Compute-Move cycle, where it takes a snapshot of the positions of other robots in a Look phase, computes its next position in the Compute phase, and moves to the next position in the Move phase. A configuration of such a system is the set of positions of the robots observed in the global coordinate system, in other words, a set of points. The robots have neither any access to the global coordinate system nor any explicit communication medium. Each robot observes and moves in terms of its local coordinate system. Though observation is the only way for the robots to cooperate with each other, they have to tolerate inconsistency among observations. A robot is oblivious if in a Compute phase, it does not remember the past observations and the past computations, and can use the observation obtained in the Look phase of the current cycle. Otherwise, a robot is non-oblivious, which means it is equipped with local memory. Existing literature introduces the following three asynchrony models: In the fully-synchronous (FSYNC) model, the robots execute the ith Look-Compute-Move cycle at the same time. Thus the robots execute a cycle at each time step \( t = 0, 1, 2, \ldots \). In the semi-synchronous (SSYNC) model, the robots follow discrete time steps, but some robots may skip cycles. In the asynchronous (ASYNC) model, no assumption is made except that the length of each cycle is finite.

The self-organization power of mobile robot systems has been studied for robots in a discrete space (e.g., graphs) [5,6], in the two-dimensional space (2D-space or plane) [2,4,7,9,11,12,13,15], and in the three-dimensional space (3D-space) [14,16,17]. The formation problem requires the robots to form a specified pattern from a given initial configuration. The set of formable patterns indicates the self-organization power of a robot system. Depending on the specified pattern, the formation problem is classified into the following problems; the point formation problem, which is the simplest form of the agreement problem among the robots [2,8], the circle formation problem [7,12], and the pattern formation problem for arbitrary target pattern [9,11,13,15]. Since real systems work in 3D-space and applications such as drones become widely available, robot systems in 3D-space form an important and promising field. Yamauchi et al. proposed the plane formation problem that requires the robots to land on a common plane without making any multiplicity [6]. The plane formation problem is one of the simplest agreement problems in 3D-space and it bridges between the robots in 3D-space and the robots in 2D-space, so that existing techniques in 2D-space can be used in 3D-space.

---

1 As the plane formation problem does not allow multiplicity, point formation is not a solution.
In this paper, we consider the plane formation problem by mobile robots that lack chirality. A robot system does not have chirality when the robots may not agree on the handedness (right-handed or left-handed) of their local coordinate systems. On the other hand, a robot system has chirality when the handedness of all local coordinate systems are identical. Lack of chirality introduces heterogeneity among the robots, and the model is expected to reveal the self-organization power of the weakest robot model. For example, Flocchini et al. and Mamino et al. showed that more than four oblivious ASYNC robots can form a circle without chirality \cite{7,12}.

Existing studies show that the set of formable patterns in 2D-space is determined by the initial symmetry among the robots. Consider an initial configuration of the four robots in 2D-space, where they form a square and their local coordinate systems are symmetric regarding the center of the square (Fig. 1). Since the robots execute a common algorithm, from this initial configuration, they keep square positions forever if they execute cycles synchronously. Yamashita et al. introduced the notion of symmetricity that gives formal explanation for such situation \cite{13,15}. We consider the decomposition of a set of points $P$ into regular $m$-gons centered at one point. We consider that one point is a regular 1-gon with an arbitrary center and two points form a regular 2-gon with the center being the midpoint. Then the maximum value of such $m$ is the symmetricity $\rho(P)$ of $P$ in 2D-space. When $\rho(P)$ is greater than one, the common center is the center of the smallest enclosing circle of $P$, denoted by $c(P)$, and $\rho(P)$ is generally the order of the cyclic group that acts on $P$. However, when $c(P) \in P$, this definition allows $\rho(P) = 1$, which means the symmetry of $P$ can be broken. This is achieved by the robot on $c(P)$ leaving its current position. It has been shown that irrespective of obliviousness and asynchrony, the robots with chirality in 2D-space can form a target pattern $F$ from an initial configuration $P$ if and only if $\rho(P)$ divides $\rho(F)$ except the case where $F$ is a point with multiplicity two \cite{11,13,15}. The exception is called the rendezvous problem, which is trivially solvable by FSYNC robots while not solvable by SSYNC (thus ASYNC) robots.

The notion of symmetricity is later extended to the robots in 3D-space \cite{17}. In 3D-space, a set of symmetric local coordinate systems with chirality is obtained by rotations on the global coordinate systems, and there are five types of rotation symmetry: the cyclic groups, the dihedral groups, the tetrahedral group, the octahedral group, and the icosahedral group. Each rotation symmetry forms
a group that can be recognized as the set of symmetric rotation operations on
a prism, a pyramid, a regular tetrahedron, a regular octahedron, and a regular
icosahedron, respectively. In other words, each rotation group is determined by
the arrangement of rotation axes and their foldings. A rotation axis is a k-fold
axis if it admits rotations by $2\pi i/k$ for $i = 1, 2, \ldots, k$. Yamauchi et al. intro-
duced rotation group and symmetricity for a set of points $P$ in 3D-space. The rotation group $\gamma(P)$ of a set of points $P$ is the rotation group that acts on $P$ and
none of its supergroup in the set of rotation groups acts on $P$. The symmetricity
$\rho(P)$ of $P$ is the set of rotation groups $G$ such that the group action of $G$ on $P$
divides $P$ into $|G|$-sets where $|G|$ is the order of $G$. In the same way as 2D-space,
the definition of symmetricity implies symmetry breaking by movement of the
robots because when some robots are on the rotation axes of $\gamma(P)$, the robots
do not allow the specified decomposition regarding the rotation axis. In other
words, $\rho(P)$ consists of the rotation groups formed by “unoccupied” rotation
axes of $\gamma(P)$. Actually, the robots on rotation axes can remove the rotation axes
by leaving their current positions. Yamauchi et al. showed that irrespective of
obliviousness, the FSYNC robots with chirality can form a target pattern $F$
from an initial configuration $P$ if and only if $\rho(P)$ is a subset of $\rho(F)$ [17].

However, all these results assume chirality among the robots. After Ya-
mashita et al. present pattern formation algorithms for the oblivious SSYNC
robots with chirality in 2D-space [13,15], Fujinaga et al. investigate the embedded pattern formation problem, where a target pattern is given as a set of land-
marks on the plane [10]. They showed that oblivious ASYNC robots can form
any embedded target pattern by presenting an algorithm that is based on the
“clockwise” minimum-weight perfect matching between the robots and the land-
marks. Based on this clockwise matching algorithm, Fujinaga et al. presented a pattern formation algorithm for oblivious ASYNC robots with chirality [11].
Later Cicerone et al. pointed out that the clockwise matching algorithm does not
work when the robots lack chirality, and showed a new embedded target pattern
formation algorithm [1]. They also pointed out that robots without chirality may
forever move symmetrically regarding an axis of symmetry.

Our contribution. The goal of our study is to formalize the degree of symmetry
among the robots without chirality in 3D-space and investigate their formation
power. The contribution of this paper is twofold. First, we give a definition of
symmetricity among the robots without chirality in 3D-space. We consider both
rotation symmetry and reflection symmetry because when the robots lack chi-
rality, a local coordinate system is obtained by a uniform scaling, a translation,
a rotation, a reflection by a mirror plane, or a combination of them on the global
coordinate system. The combination of rotation symmetry and reflection symmetry
introduces seventeen types of symmetry groups, which is well studied in
group theory and crystal symmetry [3]. We extend the notion of symmetricity in
[17] to these seventeen symmetry groups. We validate the definition by showing
that the robots cannot resolve their symmetricity forever. Then, we give a nec-

\footnote{When the robots have chirality, reflection is not necessary since reflection changes the handedness of local coordinate system.}
Fig. 2: Examples of plane formation procedure. (a) Robots directly land on a plane. (b) Robots with chirality follow a “right-screw rule”. (c) Robots without chirality cannot avoid multiplicity. (d) Robots on the horizontal mirror plane can break the reflection symmetry.

necessary and sufficient condition for FSYNC robots without chirality to solve the plane formation problem. To show the sufficiency, we present a new plane formation algorithm since the existing plane formation algorithms for robots with chirality [14,16] do not work in our model.

We focus on the FSYNC robots with rigid movement, that is, all robots synchronously execute a cycle and reach their next positions in each cycle. If a robot stops en route, its movement is non-rigid. While most existing results assume non-rigid movement, the worst case is when the robots cannot resolve their symmetry. Thus the worst case is determined by synchrony and rigid movement. Formally, any execution of the FSYNC robots with rigid movement appears in the SSYNC (thus ASYNC) model with non-rigid movement.

In [16], the cyclic groups and the dihedral groups are called 2D rotation groups because one rotation axis is recognized, and when the rotation group of the current configuration is a 2D rotation group, the robots with chirality can easily land on a “horizontal” plane perpendicular to this rotation axis (Fig. 2a and Fig. 2b). On the other hand, the remaining three rotation groups do not act on a set of points on a plane, and they are called 3D rotation groups. The necessary and sufficient condition in [16] is rephrased as follows: The FSYNC robots with chirality can form a plane from an initial configuration $P$ if and only if $\gamma(P)$ consists of 2D rotation groups. This characterization implies that the FSYNC robots with chirality can form a plane from an initial configuration where they form a regular polyhedron (except a regular icosahedron) or an icosidodecahedron, while they cannot form a plane from the remaining (convex) uniform polyhedra. Clearly, the necessity of this result holds for the robots without chirality.

When the robots lack chirality, even when $\gamma(P)$ is a 2D rotation group, the “horizontal” plane can be a mirror plane and the robots cannot resolve the symmetry regarding this mirror plane (Fig. 2c). Actually, the only plane that the robots can agree is this mirror plane, but they cannot avoid multiplicity on it. As a result, a cube is removed from the set of solvable instances, when compared with the robots with chirality. However, we will show that when there is at least one robot on the horizontal mirror plane, the robots can remove the mirror plane.
and can form a plane. Intuitively, our necessary and sufficient condition for the FSYNC robots without chirality requires that an initial symmetricity contains neither any 3D rotation group nor any combination of a 2D rotation group and an “empty” horizontal mirror plane. Our current results are preliminary in the sense of symmetry by rotation axes and mirror planes in 3D-space. For example, we defined the symmetricity among the robots, and prove that the robots cannot resolve it, but the symmetry breaking is not fully explored as we will address in the conclusion section.

**Organization.** In Section 2 we define our robot model and introduce rotation symmetry and reflection symmetry in 3D-space. We present a necessary and sufficient condition for plane formation by FSYNC robots without chirality. We show the necessity of the condition in Section 3, and we prove the sufficiency by presenting a plane formation algorithm in Section 4. We conclude this paper with Section 5.

2 Preliminary

2.1 Robot Model

Let \( R = \{r_1, r_2, \ldots, r_n\} \) be a set of \( n \) anonymous robots, each of which is a point in 3D-space. We use \( r_i \) just for description. We consider discrete time \( t = 0, 1, 2, \ldots \) and let \( p_i(t) = (x_i(t), y_i(t), z_i(t)) \in \mathbb{R}^3 \) be the position of \( r_i \) at time \( t \) in the global \( x-y-z \) coordinate system \( Z_0 \), where \( \mathbb{R} \) is the set of real numbers. The configuration of \( R \) at time \( t \) is \( P(t) = \{p_1(t), p_2(t), \ldots, p_n(t)\} \). We denote the set of all possible configurations of \( R \) by \( P^n \). We assume that the initial positions of robots are distinct, i.e., \( p_i(0) \neq p_j(0) \) for \( r_i \neq r_j \) and \( |P(0)| = n \). We also assume that \( n \geq 4 \) since any three robots are on one plane.

Each robot \( r_i \) has no access to the global coordinate system, and it uses its local \( x-y-z \) coordinate system \( Z_i \). The origin of \( Z_i \) is the current position of \( r_i \) while the unit distance, the directions, and the orientations of the \( x, y, \) and \( z \) axes of \( Z_i \) are arbitrary and never change. Hence, it is appropriate to denote \( Z_i(t) \), but we use a shorter description. Each \( Z_i \) is either right-handed or left-handed. Thus the robots do not have chirality. We denote the coordinates of a point \( p \) in \( Z_i \) by \( Z_i(p) \).

We consider the fully-synchronous (FSYNC) model, where the robots start the \( t \)th Look-Compute-Move cycle at the beginning of time \( (t-1) \) and finishes it before time \( t \) \( (t = 1, 2, \ldots) \). Each of the Look phase, the Compute phase, and the Move phase of a cycle is completely synchronized at each time step. At time \( t \), each robot \( r_i \) obtains a set \( Z_i(P(t)) = \{Z_i(p_1(t)), Z_i(p_2(t)), \ldots, Z_i(p_n(t))\} \) in the Look phase. Then \( r_i \) computes its next position by using a common algorithm \( \psi \) in the Compute phase. A robot is oblivious if it does not remember the past observations and the past computations, thus the input to \( \psi \) is \( Z_i(P(t)) \). Otherwise, it is non-oblivious and the input to \( \psi \) contains the past observations.

---

3 When more than one robots are at one point, it is impossible to separate them by a deterministic algorithm.
and the past computations. Finally, \( r_i \) moves to the next point in the Move phase. We assume that each robot always reaches its next position in a move phase and we do not care for the route to reach there. Thus we consider rigid movement.

An execution of an algorithm \( \psi \) from an initial configuration \( P(0) \) is a sequence of configurations \( P(0), P(1), P(2), \ldots \). When the initial local coordinate systems of \( P(0) \), the algorithm \( \psi \), and initial local memory content (if any) are fixed, the FSYNC execution is uniquely determined.

The plane formation problem requires that the robots land on a plane, which is not predefined, without making any multiplicity. Hence point formation is not a solution for the plane formation problem. We say that an algorithm \( \psi \) forms a plane from an initial configuration \( P(0) \), if, regardless of the choice of initial local coordinate systems \( Z_i \) for each \( r_i \in R \), any execution \( P(0), P(1), \ldots \) there exists a finite \( t \geq 0 \) such that (i) \( P(t) \) is contained in a plane, (ii) \( |P(t)| = n \), i.e., all robots occupy distinct positions, and (iii) once the system reaches \( P(t) \), the robots do not move anymore.

For a set of points \( P \), we denote the smallest enclosing ball (SEB) of \( P \) by \( B(P) \) and its center by \( b(P) \). A point on the sphere of a ball is said to be on the ball, and we assume that the interior or the exterior of a ball does not include its sphere. The innermost empty ball \( I(P) \) is the ball whose center is \( b(P) \), that contains no point of \( P \) in its interior and contains at least one point of \( P \) on its sphere. When all points of \( P \) are on \( B(P) \), we say \( P \) is spherical.

### 2.2 Symmetry by Rotations and Reflections

We consider symmetry among the robots which is caused by not only symmetric positions of the robots but also symmetric local coordinate system of them. Since any local coordinate system is obtained by a uniform scaling, a translation, a rotation, a reflection by a mirror plane, or a combination of them on the global coordinate system, we focus on symmetry operations by rotation axes and mirror planes.

A \( k \)-fold axis admits rotations by \( 2\pi i/k \) (\( i = 1, 2, \ldots, k \)). These \( k \) operations form the cyclic group \( C_k \) of order \( k \). When there are more than one rotation axes, they also form a group, and there are five kinds of rotation groups in 3D-space, each of which is determined by the types of rotation axes and the arrangement of them.\(^4\) Clearly, these multiple rotation axes intersect at one point. The dihedral group \( D_\ell \) consists of a single \( \ell \)-fold axis called the principal axis and \( \ell \) 2-fold axes perpendicular to the principal axis, and its order is \( 2\ell \).\(^4\) We can recognize \( D_\ell \) by the rotations on a prism with regular \( \ell \)-gon bases. We abuse the term “principal axis” for the single rotation axis of a cyclic group.

The remaining three rotation groups are the tetrahedral group, the octahedral group, and the icosahedral group, and we can recognize them by the rotations on

\(^4\) The rotation group \( D_2 \) consists of three 2-fold rotation axes, and it has been shown that the principal axis of \( D_2 \) can be also recognized.\(^6\) This is because we do not consider \( D_2 \) only, but a set of points and a rotation (or symmetry) group that acts on the points.
the corresponding regular polyhedra. The *tetrahedral group* $T$ consists of three 2-fold axes and four 3-fold axes, and its order is 12. The *octahedral group* $O$ consists of six 2-fold axes, four 3-fold axes, and three 4-fold axes, and its order is 24. The *icosahedral group* $I$ consists of fifteen 2-fold axes, ten 3-fold axes, and six 5-fold axes, and its order is 60. We call the cyclic groups and the dihedral groups 2D rotation groups, and we call the remaining three rotation groups $T$, $O$, and $I$ 3D rotation groups because a 3D rotation group does not act on a point on a plane.

A mirror plane changes the handedness and a mirror image of an object has a different handedness from the original object. This is the reason why we need to consider reflection symmetry when we consider the robots without chirality. The *bilateral symmetry* $C_s$ consists of one mirror plane and its order is 2. When there are more than one mirror planes, an intersection of mirror planes introduces a rotation axis. We consider the compositions of rotation symmetry and mirror planes. Each symmetry type also forms a group. Clearly, the rotation axes and mirror planes of the symmetry type intersect at one point. The composition of $C_k$ ($k > 1$) and a mirror plane perpendicular to the principal axis is denoted by $C_{kh}$, where $h$ represents the “horizontal” mirror plane. The order of $C_{kh}$ is $2k$. The composition of $C_k$ ($k > 1$) and $k$ mirror planes containing the principal axis is denoted by $C_{kv}$, where $v$ represents the “vertical” mirror planes. The order of $C_{kv}$ is $2k$. The composition of $D_\ell$ ($\ell \geq 2$) and a horizontal mirror plane regarding the principal axis is denoted by $D_{th}$. However, this horizontal mirror plane together with rotation axes of $D_\ell$ forces $\ell$ vertical mirror planes each of which contains two 2-fold axes and the principal axis. The order of $D_{th}$ is $4\ell$. The composition of $D_\ell$ ($\ell \geq 2$) and $\ell$ vertical mirror planes is denoted by $D_{tv}$. The vertical mirror planes do not contain any 2-fold axes, otherwise the rotation axes of $D_\ell$ forces a horizontal mirror plane. The order of $D_{tv}$ is $4\ell$.

The composition of $T$ and three mutually perpendicular mirror planes, each of which contains two 2-fold axes is denoted by $T_h$. The order of $T_h$ is 24. The composition of $T$ and six mirror planes, each of which contains two 3-fold axes is denoted by $T_d$. The order of $T_d$ is 24. The composition of $O$ and nine mirror planes is denoted by $O_h$. Three of the mirror planes are mutually perpendicular and each of them contains two 4-fold axes. Each of the remaining six mirror planes contains two 3-fold axes. The order of $O_h$ is 48. The composition of $I$ and fifteen mirror planes, each of which contains two 5-fold axes is denoted by $I_h$. The order of $I_h$ is 120.

Another type of composite symmetry is *rotation reflection*, where a rotation regarding a single rotation axis and taking a mirror image regarding a mirror plane perpendicular to the rotation axis are alternated. This type of symmetry is denoted by $S_m$. Because of the alternation, the folding of the rotation axis is even. $S_2$ corresponds to the central inversion, which is denoted by $C_1$. See Appendix A for more detail.

Let $S = \{C_1, C_s, C_k, C_{kh}, C_{kv}, D_\ell, D_{th}, D_{tv}, S_m, T, T_h, T_d, O, O_h, I, I_h \mid k = 2, 3, \ldots, \ell = 2, 3, \ldots \}$ where $C_1$ consists of only the identity element. We call the elements of $S$ symmetry groups. These seventeen types of symmetry
groups describe all symmetry in 3D-space \[\mathbb{S}\]. In this paper we consider rotation symmetry separately. We call the elements of \(\{C_k, D\ell, T, O, I \mid k = 1, 2, \ldots, \ell = 2, 3, \ldots\}\) the rotation groups.

We denote the order of \(G \in \mathbb{S}\) with \(|G|\). When \(G'\) is a subgroup of \(G\) (\(G, G' \in \mathbb{S}\)), we denote it by \(G' \leq G\). If \(G'\) is a proper subgroup of \(G\) (i.e., \(G \neq G'\)), we denote it by \(G' \prec G\). For example, we have \(D_2 \prec T\), \(T \prec O, I\), but \(O \not\prec I\). If \(G \in \mathbb{S}\) has a \(k\)-fold axis, \(C_k' \leq G\) if \(k'\) divides \(k\). For symmetry groups containing mirror planes, \(T \prec T_h \prec O_h\) but \(T_h \not\prec O\). For \(S_m\), we have \(C_{m/2} \prec S_m \prec C_{mh}\).

### 2.3 Rotation Group, Symmetry Group, and Symmetricity

Let \(P \in \mathcal{P}^n\) be a set of \(n\) points. The rotation group \(\gamma(P)\) of \(P\) is the rotation group that acts on \(P\) and none of its proper supergroup in \(\{C_k, D\ell, T, O, I \mid k = 1, 2, \ldots, \ell = 2, 3, \ldots\}\) acts on \(P\). The symmetry group \(\theta(P)\) of \(P\) is the symmetry group that acts on \(P\) and none of its proper supergroup in \(\mathbb{S}\) acts on \(P\). Clearly, \(\gamma(P)\) is a subgroup of \(\theta(P)\) (\(\gamma(P) \leq \theta(P)\)), and they are uniquely determined.\(^5\) By the definition, when \(\theta(P)\) is either \(C_1, C_1, C_s\), \(\gamma(P) = C_1\) because such configuration \(P\) does not have any rotation axis. Table 1 shows the rotation group of a set of vertices of each regular polyhedron.

The group action of \(\theta(P)\) decomposes \(P\) into disjoint subsets. Let \(Orb(p) = \{g \ast p \mid g \in \theta(P)\}\) be the orbit of \(p \in P\) where \(\ast\) denotes the action of \(g\) on \(s\), and the orbit space \(\{Orb(p) \mid p \in P\} = \{P_1, P_2, \ldots, P_m\}\) is called the \(\theta(P)\)-decomposition of \(P\). Each element \(P_i\) is transitive because it is one orbit regarding \(\theta(P)\).

Yamauchi et al. showed that in configuration \(P\) without any multiplicity, the robots with chirality can agree on the \(\theta(P)\)-decomposition \(\{P_1, P_2, \ldots, P_m\}\) of \(P\) and a total ordering among the elements so that (i) \(P_1\) is on \(I(P)\), (ii) \(P_m\) is on \(B(P)\), and (iii) \(P_{i+1}\) is not in the interior of the ball centered at \(b(P)\) and containing \(P_i\) on its sphere.\(^6\) Though their technique relies on chirality, we can extend it to robots without chirality. In \([16]\), each robot translates its local observations to a “celestial map” by considering \(I(P)\) as the earth and its current position is on the half line from \(b(P)\) containing the north pole. Then, the

---

\(^5\) See for example \([3]\), that shows an algorithm to uniquely determine the symmetry group of a polyhedra. The algorithm checks rotation axes, mirror planes, and a point of inversion. Since we consider a set of points and their convex-hulls, we can use the same algorithm.

---

| Polyhedron          | Rotation group | Symmetry group | Symmetricity                        |
|---------------------|----------------|----------------|-------------------------------------|
| Regular tetrahedron | \(T\)          | \(T_d\)        | \(\{D_2, S_4\}\)                   |
| Regular octahedron  | \(O\)          | \(O_h\)        | \(\{D_4, S_4\}\)                   |
| Cube                | \(O\)          | \(O_h\)        | \(\{D_4, D_{2h}, D_{2w}, C_{4h}, S_4\}\) |
| Regular dodecahedron| \(I\)          | \(I_h\)        | \(\{D_5, D_2, S_{10}\}\)          |
| Regular icosahedron | \(I\)          | \(I_h\)        | \(\{I, D_3, S_6\}\)               |
robot selects an appropriate robot to define the prime meridian and translates the position of each robot to a triple consisting of its altitude, latitude, and longitude. The ordered sequence of these triples is the local observation of the robots. However, the lack of chirality does not allow the robots to agree on the direction of longitude. Then we make a robot consider both directions and select the direction that produces the smallest sequence. In the same way as [10], we have the following property.

**Lemma 1.** Let \( P \in \mathcal{P}_n^3 \) and \( \{P_1, P_2, \ldots, P_m\} \) be a configuration of robots represented as a set of points and its \( \theta(P) \)-decomposition, respectively. Then we have the following two properties:

1. For each \( P_i \) \( (i = 1, 2, \ldots, m) \), all robots in \( P_i \) have the same local view.
2. Any two robots, one in \( P_i \) and the other in \( P_j \), have different local views, for all \( i \neq j \).

**Proof.** The first property is obvious by the definitions of \( \theta(P) \)-decomposition and local view, since for any \( p, q \in P_i \) there is an element \( g \in \theta(P) \) such that \( q = g \ast p \).

As for the second property, to derive a contradiction, suppose that there are distinct integers \( i \) and \( j \), such that robots \( r_k \in P_i \) and \( r_\ell \in P_j \) have the same local view. Let \( V_\ell^* \) and \( V_i^* \) be the local view of \( r_k \) and \( r_\ell \). Thus, we have \( V_\ell^* = V_i^* \). Let us consider a function \( f \) that maps the \( d \)th element of \( V_\ell^* \) to that of \( V_i^* \). More formally, letting the \( d \)th element of \( V_\ell^* \) (resp., \( V_i^* \)) be \( p_x^\ast \) (resp., \( p_y^\ast \)), \( f \) maps \( p_x \) to \( p_y \). Then \( f \) is a transformation that keeps \( b(P) \) unchanged by the definition of local view, i.e., \( f \) is a rotation or an reflection in \( \theta(P) \), which contradicts to the definition of \( \theta(P) \)-decomposition. \( \square \)

By Lemma 1, the robots can agree on a total ordering of the elements of the \( \theta(P) \)-decomposition of \( P \). In the following, we assume that \( \{P_1, P_2, \ldots, P_m\} \) is sorted by this ordering.

We denote the set of local coordinate systems for configuration \( P \) with \( Q = \{(o_i, x_i, y_i, z_i \mid p_i \in P)\} \) where \( o_i \) is the position of \( p_i \in P \) (i.e., the origin of \( Z_i \)) and \( x_i, y_i, \) and \( z_i \) are the \((1, 0, 0), (0, 1, 0), \) and \((0, 0, 1)\) of \( Z_i \) observed in the global coordinate system \( Z_0 \). We use \((P, Q)\) to explicitly show the set of local coordinate systems for \( P \) though \( Q \) contains \( P \) as \( \{o_1, o_2, \ldots, o_n\} \). We define the symmetry group \( \sigma(P, Q) \) of \((P, Q)\) as the symmetry group of that acts on \((P, Q)\) and none of its proper supergroup in \( S \) acts on it. Clearly, we have \( \sigma(P, Q) \leq \theta(P, Q) \). We define the \( \sigma(P, Q) \)-decomposition of \((P, Q)\) in the same way as the \( \theta(P) \)-decomposition of \( P \). We note that the robots of \( P \) cannot obtain \( Q \) or \( \sigma(P, Q) \) because they can observe only the positions of themselves.

Given a set \( P \) of points, \( \theta(P) \) determines the arrangement of its rotation axes and mirror planes in \( P \). We thus use \( \theta(P) \) and the arrangement of its rotation axes and mirror planes in \( P \) interchangeably. For two groups \( G, H \in S \), an embedding of \( G \) to \( H \) is an embedding of each rotation axis and each mirror plane of \( G \) to one of the rotation axes and one of the mirror planes of \( H \) with keeping their arrangement in \( G \). Any \( k \)-fold axis of \( G \) is embedded so that it
overlaps a $k'$-fold axis of $H$, where $k$ divides $k'$, and any mirror plane of $G$ is embedded to a mirror plane of $H$. However we need careful treatment for $S_m$. When $G = S_m$, its mirror plane is embedded to a mirror plane of $H$, and when $H = S_m$, its mirror plane cannot grant any mirror plane of $G$. For example, we can embed $T$ to $O$. There are three embeddings of $C_4$ to $O$ depending on the choice of the 4-fold axis. We can embed $C_3$ to $S_6$, and $S_6$ to $C_{6h}$. We can embed $D_{ℓv}$ to $D_{2ℓh}$ but cannot to $D_{ℓh}$. Observe that we can embed $G$ to $H$ if and only if $G \preceq H$.

We can also consider a $G$-decomposition of a set of points $P$ for some $G \prec θ(P)$ for an embedding of $G$ in $θ(P)$. However, for such $G$-decomposition, the robots cannot agree the ordering among the elements since Lemma 1 does not hold.

We now define symmetricity of a set of points in 3D-space. Intuitively, symmetricity shows all possible symmetry groups to which the robots may forever subject. As the symmetry groups are partially ordered, we consider a set of such rotation groups.

**Definition 1.** Let $P ∈ P_n^3$ be a set of points. The symmetricity $ℓ(P)$ of $P$ is the set of symmetry groups $G ∈ S$ that acts on $P$ (thus $G \preceq θ(P)$) and there exists an embedding of $G$ to $θ(P)$ such that each element of the $G$-decomposition of $P$ is a $|G|$-set.

We define $ℓ(P)$ as a set because the “maximal” symmetry group that satisfies the definition is not uniquely determined. Maximality means that there is no proper supergroup in $S$ that satisfies the condition of Definition 1. When it is clear from the context, we denote $ℓ(P)$ by the set of such maximal elements. For example, if $P$ forms a cube, $ℓ(P) = \{C_1, C_4, C_2, C_{4h}, C_{2v}, D_2, D_4, D_{2h}, D_{2v}, S_4\}$, and we denote it by $ℓ(P) = \{D_4, D_{2h}, D_{2v}, C_{4h}, S_4\}$. The set $ℓ(P)$ does not contain $O$ itself since $O$-decomposition of $P$ consists of one 8-set, while $|O| = 24$.

From the definition, $ℓ(P)$ contains every element of $S$ that is a subgroup of $G$ if $G ∈ ℓ(P)$. See Table 1 as an example.

We can rephrase the definition of symmetricity of a set of points $P$ as a set of symmetry groups formed by rotation axes and mirror planes of $θ(P)$ that do not contain any point of $P$. This is because a point on a rotation axis (a mirror plane, respectively) does not allow a decomposition into $|G|$-sets for any $G$ containing the rotation axis.

We conclude this section with the following two lemmas, that validate the definition of symmetricity. Lemma 2 shows that there exists an arrangement of local coordinate systems $Q$ for any initial configuration $P$ and $G ∈ ℓ(P)$ such that $σ(P,Q) = G$. Then, Lemma 3 shows that the robots are caught in this initial symmetry.

---

6 We assume that a set of points $P$ does not contain any multiplicity. In other words, we consider an initial configuration $P$. 

11
In the proofs, we take a new view of the positions of the robots. In the definition of symmetricity \( \phi(P) \) for a set of points \( P \), we consider an arrangement of a symmetry group in \( P \). On the other hand, to show that the initial symmetry cannot be broken, we consider the cases where the positions of robots are caught in an arrangement of \( G \in \phi(P) \).

**Lemma 2.** For an arbitrary initial configuration \( P \in \mathcal{P}^n \) and any \( G \in \phi(P) \), there exists a set of local coordinate systems \( Q \) such that \( \sigma(P,Q) = G \).

**Proof.** We show a construction of \( Q \) for \( P \) and \( G \). Let \( \{P_1, P_2, \ldots, P_m\} \) be the \( G \)-decomposition of \( P \) for some embedding of \( G \) to \( \theta(P) \). Clearly, such embedding exists since \( G \leq \theta(P) \). From the definition, \( |P_j| = |G| \) for \( j = 1, 2, \ldots, m \). For each \( P_j \), we arbitrary fix a local coordinate system of one robot \( p_i \in P_j \). Then for each \( p_k \in P_j \), there exists a unique element of \( G \) such that \( p_k = g_k * p_i \) and \( g_k \neq g_\ell \) if \( p_k \neq p_\ell \) for any \( p_\ell \in P_j \). Then we fix the local coordinate system of \( p_k \) by applying \( g_k \) to the local coordinate system of \( p_i \). The local coordinate systems \( Q \) obtained by this procedure satisfies the property. \( \square \)

**Lemma 3.** Irrespective of obliviousness, for an arbitrary initial configuration \( P \in \mathcal{P}^n \), any \( G \in \phi(P) \), and any algorithm \( \psi \), there exists an execution \( P(0) = P, P(1), P(2), \ldots \) such that \( \theta(P(t)) \geq G \).

**Proof.** Let \( Q \) be initial local coordinate systems for \( P \) such that \( \sigma(Q,P) = G \) for arbitrary \( G \in \phi(P) \). By Lemma 2 such \( Q \) always exists. Let \( \{P_1, P_2, \ldots, P_m\} \) be the \( G \)-decomposition of \( P \).

From this arrangement of initial local coordinate systems, the robots forming \( P_j \) keeps their symmetry group \( G \) forever for any algorithm \( \psi \). We first show an induction for the oblivious FSYNC robots. For any \( p_i, p_k \in P_j \), when \( \psi(Z_i[P(0)]) = x \) holds, we have \( \psi(Z_k[P(0)]) = x \) and \( Z_0[Z_k(\psi(Z_k[P(0)]))] = g_k * Z_0[Z_k(\psi(Z_i[P(0)]))] \). Let \( P_j(1) \subseteq P(1) \) be the positions of robots of \( P_j \) in \( P(1) \). Thus \( \theta(P_j(1)) = G \) and \( \theta(P(1)) \geq G \). By an easy induction for \( t = 1, 2, \ldots \), we have the property for any \( P(t) \).

Non-obliviousness does not improve the situation. When the initial memory contents are identical (for example, empty), the above discussion holds for the transition from \( P(0) \) to \( P(1) \). During this transition, the robots in the same element \( P_j \) of the \( G \)-decomposition of \( P \) obtain the same local observation, performs the same computation, and exhibits the same movement. Thus, their local memory content are still the same in \( P(1) \), and they continue symmetric movement during the transition from \( P(1) \) to \( P(2) \). \( \square \)

### 3 Impossibility of Plane Formation

The following theorem shows a necessary condition for the FSYNC robots without chirality to form a plane, that will be shown to be a sufficient condition in Section 4. The condition means that to solve the plane formation from an initial configuration \( P \), \( \phi(P) \) should not contain any of the following symmetry groups: \( T, T_d, T_h, O, O_h, I, I_h, C_{kh} (k \geq 3), \) and \( D_{th} (\ell \geq 2) \).
**Theorem 1.** Irrespective of obliviousness, the FSYNC robots without chirality can form a plane from an initial configuration \( P \) only if \( \rho(P) \) consists of \( C_1, C_s, C_i, C_k, C_{kv}, C_{2h}, D_t, D_{kv}, \) and \( S_m \).

**Proof.** Let \( \psi \) be an arbitrary plane formation algorithm for an initial configuration \( P \) such that \( \rho(P) \) contains a 3D rotation group, \( C_{kh} \) \((k \geq 3)\), or \( D_{\ell h} \) \((\ell \geq 2)\). We have the following three cases.

**Case A:** \( g(P) \) contains \( C_{kh} \) for some \( k \geq 3 \).

Let \( Q \) be a set of initial local coordinate systems for \( P \) such that \( \sigma(P,Q) = C_{kh} \in g(P) \) \((k \geq 3)\). From Lemma \( \text{Lemma 3} \) irrespective of obliviousness, for any algorithm \( \psi \), there exists an execution \( P = P(0), P(1), P(2), \ldots \) such that \( \theta(P(t)) \supseteq C_{kh} \) for any \( t \geq 0 \). Assume that \( P(t') \) is a terminal configuration. Then \( \theta(P(t')) \) is a supergroup of \( C_{kh} \), and \( \theta(P(t')) \) has the mirror plane of \( \theta(P) \). The robots are on the mirror plane, otherwise the robots are not on one plane because of their symmetry. Let \( \{P_1, P_2, \ldots, P_m\} \) be the \( \sigma(P,Q) \)-decomposition of \( P(= P(0)) \). For each \( P_i \) \((1 \leq i \leq m)\) and \( p \in P_i \), there exists \( q \in P_i \) such that \( p \) and \( q \) are at symmetric positions regarding the mirror plane of \( C_{kh} \).

By Lemma \( \text{Lemma 3} \) the robots of \( P_i \) move with keeping the rotation axis and the mirror plane of the embedding of \( C_{kh} \) in \( P \). Thus \( p \) and \( q \) occupy the same point on the mirror plane of \( C_{kh} \) in \( P(t') \). Hence the robots cannot avoid multiplicity and \( P(t') \) is not a terminal configuration of the plane formation problem.

**Case B:** \( g(P) \) contains \( D_{\ell h} \) for some \( h \geq 2 \).

Let \( Q \) be a set of initial local coordinate systems for \( P \) such that \( \sigma(P,Q) = D_{\ell h} \in g(P) \) \((\ell \geq 2)\). By Lemma \( \text{Lemma 3} \) irrespective of obliviousness, for any algorithm \( \psi \), there exists an execution \( P = P(0), P(1), P(2), \ldots \) such that \( \theta(P(t)) \supseteq D_{\ell h} \) for any \( t \geq 0 \). We have the same discussion as Case A. If there exists a terminal configuration, the robots are on the initial horizontal mirror plane of \( D_{\ell h} \). Hence, the robots cannot avoid multiplicity and \( P(t') \) is not a terminal configuration of the plane formation problem.

**Case C:** \( g(P) \) contains a 3D-rotation group.

The impossibility for this case has been shown for robots with chirality in [16] and the result holds for our robots because our model allows the robots with chirality. We note that when \( g(P) \) contains \( T_d, T_h, O_h \) or \( I_h \), then it contains the corresponding rotation group because it is a subgroup without any mirror plane.

\( \square \)

### 4 Plane Formation Algorithm

In this section, we show a plane formation algorithm for oblivious FSYNC robots without chirality and prove our main theorem.

**Theorem 2.** Irrespective of obliviousness, the FSYNC robots without chirality can form a plane from an initial configuration \( P \) if and only if \( g(P) \) consists of \( C_1, C_i, C_s, C_k, C_{kv}, C_{2h}, D_t, D_{kv}, \) and \( S_m \).
The necessity is clear from Theorem 1. We prove the sufficiency by presenting a plane formation algorithm for solvable instances (i.e., initial configurations). Because of the condition of the theorem, solvable instances are classified into the following three types.

**Type 1:** Initial configurations with 3D rotation groups. From the condition of Theorem 2, any initial configuration $P$ of this type contains one of the following polyhedra as an element of its $\theta(P)$-decomposition because some robots are on some rotation axes: a regular tetrahedron, a regular octahedron, a regular dodecahedron, and an icosidodecahedron.\(^7\)

**Type 2:** Initial configurations with 2D rotation groups with at least one rotation axis. From the condition of Theorem 2, any initial configuration $P$ of this type satisfies that either $\theta(P)$ does not have the horizontal mirror plane or there are some robots on a horizontal mirror plane.

**Type 3:** Initial configurations without any rotation axis. This case is further divided into asymmetric initial configurations, a symmetric initial configurations with a single mirror plane, and a symmetric initial configurations with point of inversion.

The proposed algorithm handles these three types separately. The robots can agree on the type of the current configuration and they execute the corresponding algorithm. For the first case, the robots first break their symmetry and translates an initial Type 1 configuration to another Type 2 or Type 3 configuration. For the second case, the robots agree on a plane perpendicular to the principal axis and containing the center of their smallest enclosing ball, and land on the plane. For the third case, if the initial configuration is asymmetric, the robots agree on a plane by using the total ordering among themselves. Otherwise, the robots agree on a plane other than the mirror plane by using two elements of their decomposition and lands on it.

Before we go into the detailed description of the proposed algorithm, we show preparation steps for an initial configuration $P$. These steps can be realized very easily in the FSYNC model because the set of robots to move is easy to recognize, and the movement neither makes collisions nor changes the symmetry group and the symmetricity of the robots. In the following, we use a point and a robot at the point interchangeably. For example, the position $p_i \in P$ of robot $r_i$ means $r_i$, and $P' \subseteq P$ means the set of robots at positions of $P'$.

First, when $b(P) \in P$, the preparation phase sends the robot on $b(P)$ to an arbitrary point in the interior of $I(P)$, so that a resulting configuration will be asymmetric. The next position of the robot at $b(P)$ is, for example, a point neither on any rotation axis nor on any mirror plane of $\theta(P \setminus \{b(P)\})$.

Second, for Type 1 cases, the preparation phase moves an element $P_i$ of the $\theta(P)$-decomposition of $P$ forming one of the specified polyhedra to the interior of $I(P)$.

\(^7\) Points on rotation axes of a 3D rotation group also forms a cube, a cuboctahedron, and a regular icosahedron. However, these polyhedra allow $T(\prec O, I)$ to join their symmetricity.
Finally, for Type 2 cases, the preparation phase moves an element $P_j$ of the $\theta(P)$-decomposition of $P$ on the horizontal mirror plane of $\theta(P)$ to the interior of $I(P)$. At the same time, if there exists another element $P_k$ of the $\theta(P)$-decomposition of $P$ that is on neither the horizontal mirror plane nor the principal rotation axis, the preparation phase makes the element slightly “shrink” so that $\theta(P)$ is kept by $P_k$. For example, if $P$ with $\theta(P) = D_{4h}$ consists of a cube and a square on the horizontal mirror plane, the robots forming a cube shrink to translate the cube to a long square prism.

For the second and the third cases, we select the minimum index among the elements satisfying the condition and move each $p \in P_j$ ($P_j$, respectively) along the line $pb(P)$. Since we select the minimum index, this movement introduces no collision. Additionally, in the third case, the robots of $P_j$ move on the horizontal mirror plane toward $b(P)$.

4.1 Symmetry Breaking

We consider a Type 1 configuration $P$. Let $\{P_1, P_2, \ldots, P_m\}$ the the $\theta(P)$-decomposition of $P$. The preparation step guarantees that $P_1$ forms one of the following four polyhedra: a regular tetrahedron, a regular octahedron, a regular dodecahedron, and an icosidodecahedron. Then the proposed plane formation algorithm first makes the robots execute the go-to-center algorithm (Algorithm 1) proposed in [16]. Each robot of $P_1$ selects an adjacent face of the polyhedron and moves to the center of the selected face. But it stops $\epsilon$ before the center to avoid collisions.

Algorithm 1 does not depend on chirality among the robots and it has been shown that the rotation group of a resulting configuration is always a 2D rotation group [16]. Since our robots lack chirality, we should consider the combination of such 2D rotation groups and mirror planes. The following lemma guarantees that any resulting configuration does not have any horizontal mirror plane (except $C_{2h}$). We note that we do not have to care for rotation reflections in this phase because such configurations are Type 3 configurations.

**Lemma 4.** Let $P$ be a configuration such that $\gamma(P)$ is a 3D rotation group and $\varphi(P)$ contains neither any 3D rotation group nor any symmetry group with a horizontal mirror plane (except $C_{2h}$). Let $\{P_1, P_2, \ldots, P_m\}$ be the $\theta(P)$-decomposition of $P$. Then there exists one element of $P_i$ ($1 \leq i \leq m$) that forms one of the following polyhedra: a regular tetrahedron, a regular octahedron, a regular dodecahedron, or an icosidodecahedron. We further assume that $P_1$ forms one of the above polyhedra. Then one step execution of Algorithm 1 translates $P$ into another configuration $P'$ that satisfies (i) $\gamma(P')$ is a 2D rotation group, and (ii) if $\gamma(P') \neq C_1, C_2$, $\theta(P')$ does not have any horizontal mirror plane.

**Proof.** To show the first property of the lemma, we introduce another technique to check transitive set of points regarding a symmetry group. Given an arrangement of $G \in S$ and a seed point $s$ in an arrangement of $G$, by applying all
Algorithm 1 Symmetry breaking algorithm for robot \( r_i \) \[16\]

Notation
- \( P \): Current configuration observed in \( Z_i \).
- \( p_i \in P \): The position of \( r_i \) (i.e., the origin of \( Z_i \)).
- \( \{P_1, P_2, \ldots, P_m\} \): \( \theta(P) \)-decomposition of \( P \), where \( P_1 \) forms one of the four polyhedra.
- \( \epsilon = \ell/100 \), where \( \ell \) is the length of an edge of the polyhedron that \( P_1 \) forms.

Algorithm
If \( p_i \in P_1 \) then
  If \( P_1 \) is an icosidodecahedron then
    Select an adjacent regular pentagon face of \( P_1 \).
    Destination \( d \) is the point \( \epsilon \) before the center of the face on the line from \( p_i \) to the center.
  Else
    // \( P_1 \) is a tetrahedron, an octahedron, or a dodecahedron.
    Select an adjacent face of \( P_1 \).
    Destination \( d \) is the point \( \epsilon \) before the center of the face on the line from \( p_i \) to the center.
  Endif
  Move to \( d \).
Endif

elements of \( G \) to \( s \), we obtain an orbit of \( S = \{g \ast s \mid g \in G\} \). Clearly, \( S \) is transitive regarding \( G \) and the location of \( s \) determines the size of \( S \). For example, when \( s \) is on a \( k \)-fold rotation axis of \( G \), \( |S| = |G|/k \), when \( s \) is on a rotation axis of \( G \) but not on a rotation axis of \( G \), \( |S| = |G|/2 \), and when \( s \) is neither on any mirror plane nor on any rotation axis, \( |S| = |G| \).

Since \( \gamma(P) \) is a 3D rotation group and \( g(P) \) does not contain any 3D rotation group, there are some robots on the rotation axes of \( \gamma(P) \). A seed point on a rotation axis of a 3D rotation group produces a regular tetrahedron, a regular octahedron, a cube, a cuboctahedron, a regular dodecahedron, a regular icosahedron, or an icosidodecahedron. However, since \( T \prec O \) and \( T \prec I \), a cuboctahedron and an icosahedron allow \( T \) to remain in symmetricity. Additionally, a cube allows \( C_{4h} \) to remain in symmetricity, and the plane formation is not possible from a cubic initial configuration. We have the first property of the lemma.

Assume that the robots of \( P_1 \) occupy the points \( P'_1 \subseteq P' \). It suffices to show that \( \theta(P'_1) \) does not have any horizontal mirror plane, since \( \theta(P') \) is a subgroup of \( \theta(P'_1) \).

We first check the rotation group of any resulting configuration of Algorithm \[16\] and then we proceed to the combinations of rotation axes and mirror planes. In \[16\], it has been shown that after one step execution of Algorithm \[16\] the rotation group of any resulting configuration is one of the rotation groups
Table 2: Symmetricity and rotation group of the four polyhedra and rotation groups after the execution of Algorithm 1

| Polyhedron of $P$ | $|P| \gamma(P) \varrho(P)$ | Candidates of $\gamma'(P'')$ |
|-------------------|-----------------------------|-----------------------------|
| Regular tetrahedron | 4 $T$ $\{D_2, S_4\}$ | $D_2, C_2, C_1$ |
| Regular octahedron | 6 $O$ $\{D_3, S_6\}$ | $D_3, C_3, C_1$ |
| Regular dodecahedron | 20 $I$ $\{D_5, D_2, S_{10}\}$ | $D_5, D_2, C_5, C_2, C_1$ |
| Icosidodecahedron | 30 $I$ $\{S_{10}, S_6\}$ | $C_5, C_3, C_1$ |

Fig. 3: The set of candidate destinations. (a) an $\epsilon$-expanded tetrahedron, (b) an $\epsilon$-expanded cube, (c) an $\epsilon$-expanded icosahedron, and (d) an $\epsilon$-truncated icosahedron.

shown in Table 2. Hence $\theta(P')$ is a composition of these rotation symmetry and reflection symmetry.

The set of candidate destinations of Algorithm 1 forms the polyhedra shown in Fig. 3. For example, when $P$ forms a regular octahedron, possible next positions of the six robots are around the centers of the faces, thus, around the vertices of the dual cube. Since the robots do not move to the center, the candidate destinations form an $\epsilon$-expanded cube, which is obtained by expanding the faces of a cube. The rotation group of an $\epsilon$-expanded cube is the same as its original polyhedra, i.e., a cube, and it is $O$. Additionally, its vertices form a transitive set regarding $O$. The six robots select a subset of the vertices of this $\epsilon$-expanded cube in Algorithm 1. In the same way, when $P$ forms a regular tetrahedron, the candidate destinations form an $\epsilon$-expanded tetrahedron, when $P$ forms a regular dodecahedron, the candidate destinations form an $\epsilon$-expanded
icosahedron, and when $P$ forms an icosidodecahedron, the candidate destinations form an $\epsilon$-truncated icosahedron. Each of these polyhedra is also transitive (hence spherical) regarding the rotation group of its original polyhedron.

We check the symmetry group of $P_1'$ and depending on $P_1$, we have the following four cases.

**Case A: When $P_1$ forms a regular tetrahedron.** The set of candidate destinations form an $\epsilon$-expanded tetrahedron and $|P_1'| = 4$. By Table 2 we check whether $\theta(P_1')$ is $D_{2h}$.

Assume that $\theta(P_1') = D_{2h}$. When the points of $P_1'$ are on the principal axis (secondary axes, respectively), $P_1'$ is on one plane. When the points of $P_1'$ are on mirror planes but not on any rotation axes, still $P_1'$ is on one plane. Otherwise, we have $|P_1'| = |D_{2h}| = 8$, and we do not have this case.

However, since the four robots select one face of a regular tetrahedron, $P_1'$ is not on a plane. There are following four cases: (i) three robots select the same face, (ii) two robots select the same face and the remaining two robots select another face, (iii) robots are divided into a 2-set and two 1-sets and the three groups select different faces, and (iv) each robot selects different face. In any of the four cases, the four robots are not on one plane. Hence, we do not have the case where $\theta(P_1') = D_{2h}$.

**Case B: When $P_1$ forms a regular octahedron.** The set of candidate destinations form an $\epsilon$-expanded cube and $|P_1'| = 6$. By Table 2 we check whether $\theta(P_1')$ is $D_{3h}$ or $C_{3h}$.

We first show that $P_1'$ is not on a plane. The candidate destinations of one $p \in P_1$ forms a square face of an $\epsilon$-expanded cube. If $P_1'$ is on one plane, say $H$, $H$ contains at least one vertex of each square face of an $\epsilon$-expanded cube. Clearly, such $H$ does not exist.

Assume that $\theta(P_1') = D_{3h}$. Thus the points of $P_1'$ are on some mirror planes, otherwise we have $|P_1'| = 12$. Since $P_1'$ is not on one plane, $P_1'$ forms a triangular prism. As Fig. 4 shows, any regular triangle in an $\epsilon$-expanded cube is centered at a point on a 3-fold rotation axis. Additionally, no combination of these triangles form a triangular prism. Hence, we have $\theta(P_1') \neq D_{3h}$.

Assume that $\theta(P_1') = C_{3h}$. Since $P_1'$ is not on one plane, $P_1'$ is not on the horizontal mirror plane of $C_{3h}$ and it forms a triangular prism. In the same way as the above discussion, we do not have this case.
Case C: When $P_1$ forms a regular dodecahedron. The set of candidate destinations form an $\epsilon$-expanded icosahedron and $|P'_1| = 20$. By Table 2, we check whether $\theta(P'_1)$ is $D_{5h}$, $D_{2h}$, or $C_{5h}$.

We first show that $P'_1$ is not on a plane. The candidate destinations of one $p \in P_1$ forms a regular triangle face of an $\epsilon$-expanded icosahedron. If $P'_1$ is on one plane, say $H$, $H$ contains at least one vertex of each regular triangle face of an $\epsilon$-expanded icosahedron. Clearly, such $H$ does not exist.

Assume that $\theta(P'_1) = D_{5h}$. Since $P'_1$ is not on one plane, we have the following two cases: (a) $P'_1$ contains a pentagonal prism (size 10), and (b) $P'_1$ contains a transitive 20-set regarding $D_{5h}$, (c) $P'_1$ contains a set of points on the principal rotation axis of $D_{5h}$. As Fig. 5 shows, any regular pentagon in an $\epsilon$-expanded icosahedron is centered at a point on a 5-fold rotation axis. Additionally, no combination of these pentagons form a pentagonal prism. Hence, we do not have case (a).

Any transitive 20-set regarding $D_{5h}$ consists of two pentagonal prisms. From the above discussion, we do not have case (b).

As shown in Fig. 5 any pentagon in an $\epsilon$-expanded icosahedron has no point above its rotation axis because there is no point of an $\epsilon$-expanded icosahedron on its 5-fold rotation axis. Thus we do not have case (c) and we have $\theta(P'_1) \neq D_{5h}$.

Assume that $\theta(P'_1) = C_{5h}$. Since $P'_1$ is not on one plane, we have the following two cases: (d) $P'_1$ contains a pentagonal prism (size 10), and (e) $P'_1$ contains a set of points on the principal axis. In the same way as above discussion, we have $\theta(P'_1) \neq C_{5h}$.

Assume that $\theta(P'_1) = D_{2h}$. The size of a transitive set of points regarding $D_{2h}$ is either 8, 4 (on a mirror plane), or 2 (on a rotation axis). Since $P'_1$ is not on one plane, $P'_1$ does not consist of transitive 4-sets. When $P'_1$ contains a transitive 2-set, the number of transitive 2-sets is greater than one because $|P'_1| = 20$. However, since an $\epsilon$-expanded icosahedron is spherical, at most two points of it are on a line. Thus $P'_1$ should contain a transitive 4-set because of its size.

Fig. 6 shows all possible cuboids in an $\epsilon$-expanded icosahedron. Then their mirror planes contain a 2-fold axis and two 5-fold axes, in other words, four vertices of the original icosahedron. Since a vertex of a regular icosahedron is broken.
Fig. 6: Cuboids in an $\epsilon$-expanded icosahedron. The figures show two parallel faces of each possible cuboid.

Fig. 7: Pentagons in an $\epsilon$-truncated icosahedron

into five points in an $\epsilon$-expanded icosahedron, there is no rectangle containing the mirror plane of any of such cuboids. Hence, we have $\theta(P'_1) \neq D_{2h}$.

**Case D: When $P_1$ forms a regular icosidodecahedron.** The set of candidate destinations form an $\epsilon$-truncated icosahedron and $|P'_1| = 30$. By Table 2 we check whether $\theta(P'_1)$ is $C_{5h}$ or $C_{3h}$.

We first show that $P'_1$ is not on a plane. The candidate destinations of one $p \in P_1$ forms an edge of an $\epsilon$-truncated icosahedron. If $P'_1$ is on one plane, say $H$, $H$ contains at least one endpoint of each edge of an $\epsilon$-expanded icosahedron. Clearly, such $H$ does not exist.

Assume that $\theta(P'_1) = C_{5h}$. Since $P'_1$ is not on one plane, we have one of the following two cases: (a) $P'_1$ contains a pentagonal prism, or (b) $P'_1$ contains a set of points on the rotation axis of $C_{5h}$. As Fig. 7 shows, any regular pentagon in an $\epsilon$-truncated icosahedron is centered at a point on a 5-fold rotation axis. Additionally, no combination of these pentagons form a pentagonal prism. Hence, we do not have case (a).

As sown in Fig. 7 any pentagon in an $\epsilon$-truncated icosahedron has no point above its rotation axis because there is no point of an $\epsilon$-truncated icosahedron on its 5-fold rotation axis. Thus we do not have case (b) and we have $\theta(P'_1) \neq C_{5h}$.

Assume that $\theta(P'_1) = C_{3h}$. Since $P'_1$ is not on one plane, we have one of the following two cases: (c) $P'_1$ contains a triangular prism, or (d) $P'_1$ contains a set of points on the rotation axis of $C_{3h}$. As Fig. 8 shows, any regular triangle in an $\epsilon$-truncated icosahedron is centered at a point on a 3-fold rotation axis. Additionally, no combination of these triangles form a triangular prism. Hence, we do not have case (c).
As seen in Fig. 8, any regular triangle in an $\epsilon$-truncated icosahedron has no point above its rotation axis because there is no point of an $\epsilon$-truncated icosahedron on its 3-fold rotation axis. Thus we do not have case (d) and we have $\theta(P') \neq C_{3h}$.

From the above four cases, we conclude that $\theta(P')$ is a 2D rotation group and if $\theta(P')$ have at least one rotation axis, it does not have a horizontal mirror plane (except $C_{2h}$). Since $\theta(P')$ is a subgroup of $\theta(P')$, we have the lemma. □

We finally note that the robots cannot remove vertical mirror planes of $P$ with the go-to-center algorithm. For example, consider an initial configuration $P$ where the robots form a regular octahedron. Thus $\theta(P) = O_h$. Consider an embedding of $D_{3v}$ in $\theta(P)$. The 3-fold rotation axis of $D_{3v}$ overlaps a 3-fold rotation axis of $O_h$ and each vertical mirror plane of $D_{3v}$ contains two trajectories of the go-to-center algorithm. Actually, the six robots can take these trajectories and the resulting configuration $P'$ forms a triangular anti-prism (thus $\theta(P') = D_{3v}$).

4.2 Landing Algorithm

In this section, we show a plane formation algorithm for Type 2 and Type 3 initial configurations. When $\gamma(P)$ of a current configuration is a cyclic group or a dihedral group, our basic strategy is to make the robots agree on the plane perpendicular to the principal axis and containing $b(P)$ and then we send the robots to the plane. Each robot moves along a perpendicular to the plane. To avoid multiplicities, we need some tricks for the following two cases: First, when $\theta(P)$ has a horizontal mirror plane, the condition of Theorem 2 guarantees that there is at least one element of the $\theta(P)$-decomposition of $P$ on it. To remove this mirror plane, we first make the robots of such an element leave their current positions (Fig. 9a). The other case is when $\gamma(P)$ is a dihedral group and some element, say $P_i$, of the $\theta(P)$-decomposition of $P$ has a horizontal mirror plane. Since the target plane is this mirror plane, the final destination of any symmetric two robots are the same. However, Theorem 2 guarantees that there exists at least one element, say $P_j$, of the $\theta(P)$-decomposition of $P$ that does not have a horizontal mirror plane. The robots of $P_i$ use $P_j$ to break their symmetric landing points (Fig. 9b).

The proposed algorithm consists of five phases. The first three phases break the mirror plane of $\theta(P)$ and resolves the collisions on the target plane. The
Fig. 9: Removing horizontal mirror plane and resolving collisions. (a) The robots on the horizontal mirror plane move vertically, and removes the plane. (b) The robots forming a prism rotates by using an anti-prism.

The fourth phase makes the robots agree on the target plane and in the fifth phase each robot computes the destinations of all robots to avoid any collision. The fourth and the fifth phases are done in local computation at each robot. Finally, the robots move to their final destinations in the same cycle. In any configuration $P$, the robots execute the algorithm for the smallest phase number. The robots can easily agree on which phase to execute because the condition for each of the five phases divide the set of all configurations with 2D rotation groups into disjoint subsets. Depending on the execution, some phases may be skipped. Since the proposed algorithm is designed for the oblivious robots, it is always described for a current configuration.

**First Phase: Removing the Mirror Plane** By the condition of Theorem 2 when $\gamma(P)$ is a cyclic group or a dihedral group and has a horizontal mirror plane, the mirror plane contains some robots. The preparation step guarantees that this element is $P_1$ of the $\theta(P)$-decomposition of $P$. Let $k$ be the folding of the principal axis of $\theta(P)$. Intuitively, the first phase makes the robots select the upward direction or the downward direction regarding this mirror plane and the robots move to the selected directions. Any resulting configuration does not have the horizontal mirror plane any more because for each new positions of the robots, there is no corresponding point regarding the horizontal mirror plane. However, for the simplicity of the correctness proof, these next positions are selected more carefully.

The robots consider a fictitious prism with a regular $k$-gon base inscribed in $I(P)$, that share the horizontal mirror plane (Fig. 10). However, the size is selected so that the length of the edge of the regular $k$-gon base is one tenth of the length of its side edge, and its arrangement is determined so that the plane formed by $b(P)$ and a side edge contains a point of $P_1$. Then, each $p \in P_1$ moves toward one of the nearest vertex of this fictitious prism.
Lemma 5. Let $P$ be a configuration such that $\gamma(P)$ is a 2D rotation group ($\neq C_1, C_2$) and some robots are on the horizontal mirror plane of $\theta(P)$. In this configuration, the robots execute the first phase, and a new configuration $P'$ yields. Then $P'$ satisfies one of the following conditions: (a) $\gamma(P') = C_1$, (b) $\gamma(P') = C_2$, or (c) $\theta(P')$ does not have any horizontal mirror plane.

Proof. Let $\{P_1, P_2, \ldots, P_m\}$ and $k$ be the $\theta(P)$-decomposition of $P$ and the folding of the principal axis of $\theta(P)$. We denote the vertices of the fictitious prism by $C$. Clearly, $\gamma(C) = D_k$. During the transition from $P$ to $P'$, the robots of $P_1$ select a subset $C' \subset C$ and moves to the selected vertices. Other robots of $P \setminus P_1$ do not move. Thus clearly the mirror plane of $\theta(P)$ is not a mirror plane of $\theta(P')$ (even $\theta(C')$) because each $c \in C'$ does not have the corresponding point regarding this initial mirror plane.

We separately consider the following two cases. First, when $\theta(P \setminus P_1) = \theta(P)$, $\theta(P' \setminus C') = \theta(P)$ and $\theta(P')$ is a subgroup of $\theta(P)$. In other words, the symmetry group of $P$ is kept by the robots of $P \setminus P_1$ because these robots do not move during the transition from $P$ to $P'$, the symmetry group that acts on them does not change. Since $\theta(P')$ is a subgroup of $\theta(C')$ and $\theta(P' \setminus C') = \theta(P)$, we check $\theta(C')$.

We first consider rotation axes of $\theta(C')$. We have the following four cases:

Case A: If the principal axis of $\theta(C)$ remains as some rotation axis of $\gamma(C')$, its folding is a divisor of $k$ because of the movement of the robots of $P_1$. Clearly, $\gamma(C')$ has no horizontal mirror plane from the above discussion.

Case B: If a 2-fold axis of $\theta(C)$ remains as some rotation axis of $\gamma(C')$, its folding remains two because any subset of $C$ that is on a plane perpendicular to this 2-fold axis forms a line or a rectangle because the prism is long. Actually, we do not have the case of a square because $C'$ is not symmetric regarding the horizontal mirror plane of $C$.

Case C: If a new rotation axis appears and it has an intersection with the top or the base of $C$, it is a 2-fold axis because any subset of $C$ that is on a plane perpendicular to this axis forms a line.

Case D: If a new rotation axis appears and it has an intersection with the side face of $C$, it is a 2-fold axis because any subset of $C$ that is on a plane perpendicular to this axis forms a line.
From the above four cases, \( \theta(C') \) is neither \( D_{\ell h} \) nor \( C_{\ell h} \) for any \( \ell > 2 \) because the possible principal axis do not have any mirror plane by Case A. The remaining case is \( D_{2h} \). The rotation axes of Case A and Case B do not form \( D_{2h} \) because there is no horizontal mirror plane. Even when the rotation axes of Case C and Case D form \( D_{2h} \), this \( D_{2h} \) does not act on \( P' \setminus C' \) since \( \theta(P' \setminus C') = \theta(P) \). Hence, \( \theta(P') \) is \( C_{2h} \) or does not have any horizontal mirror planes.

Second, we consider the case where \( \theta(P \setminus P_1) \neq \theta(P) \). From the preparation phase, all points of \( P \) are not on the horizontal mirror plane (thus the plane formation is finished), or all points of \( P \) are on the principal axis. In the first case, the assumption \( \theta(P \setminus P_1) = \theta(P) \) is used in the combination of the rotation axes of Case C and Case D. Thus what we should check is this case. When the rotation axes of Case C and Case D form \( D_{2h} \), it should act on the points on the original principal axis of \( \theta(P) \). However, they do not act on these points since they do not overlap these points. Hence, \( \theta(P') \) is \( C_{2h} \) or does not have any horizontal mirror planes.

In the following, we assume that when the current configuration \( P \) has a rotation axis, it does not have any horizontal mirror plane for the principal axis or \( \theta(P) = C_{2h} \).

**Second Phase: Collision Avoidance for Dihedral Groups** When \( \gamma(P) \) is a dihedral group, say \( D_{\ell} \) or \( D_{\ell v} \), and the \( \theta(P) \)-decomposition of \( P \) contains a prism, our basic strategy makes multiplicities. Let \( P_i \) be the element of the \( \theta(P) \)-decomposition of \( P \) that forms a prism (Fig. 9b). Since \( \theta(P) \) is \( D_{\ell} \) or \( D_{\ell v} \), any side edge of \( P_i \)'s prism contains a 2-fold axis. Remember that the vertical mirror planes of \( D_{\ell v} \) do not contain any 2-fold axis. Since \( \theta(P) \) does not have a horizontal mirror plane, there exists at least one element \( P_j \) that does not form such a prism. Let \( j \) be the minimum index among such elements. Then for each robot \( p \in P_i \), the nearest point of \( P_j \) is uniquely determined. If there are nearest two points of \( P_j \) for \( p \in P_i \), then \( P_j \) is not transitive regarding \( \theta(P) \) as shown in Fig. 11. When \( \theta(P) = D_{\ell v} \), if \( P_j \) is not on any mirror plane of \( \theta(P) \) and each \( p \in P_i \) has two nearest points of \( P_j \), then the mirror planes produce \( 8\ell \) points, which means \( P_j \) is not transitive (Fig. 11a). Otherwise, the nearest points are on the mirror planes of \( \theta(P) \) and \( |P_j| \) should be \( 2\ell \), but the 2-fold axes produces \( 4f \) points, a contradiction (Fig. 11b). We can show the property for the case of \( \theta(P) = D_{\ell} \) in the same argument (Fig. 11c and 11d).

This selection gives an agreement of direction to the robots of \( P_i \). The robots of \( P_i \) circulates toward the nearest point of \( P_j \) and twist their prism. This movement resolves the collisions among the perpendiculars from the robots of \( P_i \) to the target plane.

**Third Phase: Collision Avoidance on the Principal Axis** When \( \gamma(P) \) is a dihedral group and some robots are on the principal axis, we need another trick to resolve the collisions of these robots. Clearly, these robots form element(s) of the \( \theta(P) \)-decomposition of \( P \) and the size of each of such elements \( P_k \) is two.
Fig. 11: Examples of multiple nearest points. The figures show the top (or the base) of a prism. Lines show 2-fold axes and broken lines show vertical mirror planes. The principal axis passes the center of the circle and the black circles are the point forming a prism. The white circles are examples of two nearest points for each black circle. (a) When a white circle is not on a mirror plane, $D_{3v}$ produces 24 points. (b) When a white circle is on a mirror plane, $D_{3v}$ produces 12 points. (c) When a white circle is on a mirror plane, $D_3$ produces 12 points. (d) When a white circle is on a rotation axes, $D_3$ produces a prism.

We also use an element of $P_j$ that forms a “twisted” prism in the same way as the previous case. Each point of $p \in P_k$ selects the nearest point of $P_j$, however in this case, the robots of $P_k$ do not move. Other robots consider the vertices of the twisted prism as possible destinations of this fictitious move.

**Fourth Phase: Agreement of the Target Plane**

The robots agree on the target plane. Depending on $\theta(P)$ of the current configuration $P$, we have the following five cases: When $\gamma(P)$ is a cyclic group or a dihedral group, the robots agree on the plane perpendicular to the principal axis and containing $b(P)$. The exceptional case is when $\theta(P) = C_{2h}$. In this case, since the robots are not on one plane, there exists at least one element $P_i$ of the $\theta(P)$-decomposition of $P$ such that $|P_i| = 4$. Let $i$ be the minimum index among such elements. Then, the robots agree on the plane formed by $P_i$. We note that $P_i$ forms a rectangle perpendicular to the horizontal mirror plane and the agreed plane is not a mirror plane for $P$.

When $\theta(P)$ is a rotation-reflection, the robots agree on the mirror plane of $\theta(P)$.

When $\theta(P)$ is a bilateral symmetry, the size of each element of the $\theta(P)$-decomposition of $P$ is one or two. Since the robots are not on one plane, there exists at least one element $P_i$ such that $|P_i| = 2$ and forms a perpendicular line regarding the mirror plane. If there is just one such element $P_i$, the robots agree on the plane defined by $P_i$ (a line) and $P_1$ (a point). If $i = 1$, then the algorithm uses $P_2$ instead of $P_1$. Otherwise, let $P_i$ and $P_j$ be the elements with the minimum and second-minimum index of such elements. Then the robots agree on the plane defined by these two elements. The selected plane is not a mirror plane for $P$.

When $\theta(P)$ is a central inversion, the size of each element of the $\theta(P)$-decomposition of $P$ is one or two. Since the robots are not one one plane, there are more than one elements of size two. Each of these elements form a line and
they all intersect at the center of inversion. Let \( P_i \) and \( P_j \) be such elements with the minimum and second-minimum index. Then the robots agree on the plane defined by these two elements.

When \( \theta(P) \) is \( C_1 \), by Lemma \( \text{[1]} \) the robots can agree on the total ordering of themselves and agree on the plane defined by \( P_1, P_2, \) and \( P_3 \).

**Fifth Phase: Computation of Final Positions** Let \( P \) and \( \{P_1, P_2, \ldots, P_m\} \) be the current configuration and its \( \theta(P) \)-decomposition. Through the previous four phases, for each element \( P_i \) (\( i = 1, 2, \ldots, m \)), the foots of perpendiculars from the points of \( P_i \) to the target plane do not overlap.

The robots land on the agreed plane along a perpendicular to it. However, to avoid collisions among different elements of the \( \theta(P) \)-decomposition of \( P \), each robot computes the destinations of all other robots. The computation of destinations starts from \( P_1 \) and proceeds to \( P_2, P_3, \ldots \). When the destination of robot \( p \in P_i \) is already a destination of another robot \( q \in P_j \) (\( j < i \)), \( p \) first computes the largest circle \( C \) that is centered at the destination and does not contain any other destinations of the robots in \( P_1, P_2, \ldots, P_{i-1} \) except the center. Then, let \( C' \) be the concentric circle of \( C \) whose radius is \( 1/2 \). Robot \( p \) selects an arbitrary point of \( C' \) as its destination and \( C' \) is considered as (possible) destination of \( p \) in the subsequent computation. Finally, the robots move to their destinations in the same cycle.

As the non-oblivious robots can execute the proposed algorithm, we have the following theorem, that together with Theorem \( \text{[1]} \) proves our main theorem.

**Theorem 3.** Irrespective of obliviousness, the FSYNC robots without chirality can form a plane from an initial configuration \( P \) if \( \theta(P) \) consists of \( C_1, C_s, C_i, C_k, C_{kv}, C_{2h}, D_\ell, D_{\ell v}, \) and \( S_m \).

5 Conclusion and Discussion

We considered the plane formation problem by FSYNC robots without chirality. We extended the notion of symmetricity in \( \text{[17]} \) to the composition of rotation symmetry and reflection symmetry. We gave a characterization of initial configurations from which the FSYNC robots without chirality can form a plane. We then showed a plane formation algorithm for oblivious FSYNC robots without chirality because existing plane formation algorithm does not work correctly for our robots.

References

1. Serafino Cicerone, Gabriele Di Stefano, and Alfredo Navarra. Asynchronous embedded pattern formation without orientation. In Proceedings of the 30th International Symposium on Distributed Computing (DISC 2016), pages 85–98. Springer Berlin Heidelberg, 2016.
2. Mark Cieliebak, Paola Flocchini, Giuseppe Prencipe, and Nicola Santoro. Distributed computing by mobile robots: gathering. *SIAM J. Comput.*, 41(4):829–879, 2012.

3. Peter R. Cromwell. *Polyhedra*. University Press, 1997.

4. Shantanu Das, Paola Flocchini, Nicola Santoro, and Masafumi Yamashita. Forming sequence of geometric patterns with oblivious mobile robots. *Distrib. Comput.*, 28:131–145, 2015.

5. Stéphane Devisnes, Franck Petit, and Sébastien Tixeuil. Optimal probabilistic ring exploration by semi-synchronous oblivious robots. *Theor. Comput. Sci.*, 498:10–27, 2013.

6. Paola Flocchini, David Ilcinkas, Andrzej Pelc, and Nicola Santoro. Computing without communicating: Ring exploration by asynchronous oblivious robots. *Algorithmica*, 65(3):562–583, 2013.

7. Paola Flocchini, Giuseppe Prencipe, Nicola Santoro, and Giovanni Viglietta. Distributed computing by mobile robots: Solving the uniform circle formation problem. In *Proceedings of the 18th International Conference on Principles of Distributed Systems (OPODIS 2014)*, pages 217–232. Springer International Publishing, 2014.

8. Paola Flocchini, Giuseppe Prencipe, Nicola Santoro, and Peter Widmayer. Gathering of asynchronous robots with limited visibility. *Theor. Comput. Sci.*, 337:147–168, 2005.

9. Paola Flocchini, Giuseppe Prencipe, Nicola Santoro, and Peter Widmayer. Arbitrary pattern formation by asynchronous, anonymous, oblivious robots. *Theor. Comput. Sci.*, 407:412–447, 2008.

10. Nao Fujinaga, Hirotaka Ono, Shuji Kijima, and Masafumi Yamashita. Pattern formation through optimum matching by oblivious cord robots. In *Proceedings of the 14th International Conference On Principles Of Distributed Systems (OPODIS 2010)*, pages 1–15. Springer, Berlin, Heidelberg, 2010.

11. Nao Fujinaga, Yukiko Yamauchi, Hirotaka Ono, Shuji Kijima, and Masafumi Yamashita. Pattern formation by oblivious asynchronous mobile robots. *SIAM J. Comput.*, 44(3):740–785, 2015.

12. Marcello Mamino and Giovanni Viglietta. Square formation by asynchronous oblivious robots. In *Proceedings of the 28th Canadian Conference on Computational Geometry (CCCG16)*, pages 1–6, 2016.

13. Ichiro Suzuki and Masafumi Yamashita. Distributed anonymous mobile robots: Formation of geometric patterns. *SIAM J. Comput.*, 28(4):1347–1363, 1999.

14. Taichi Uehara, Yukiko Yamauchi, Shuji Kijima, and Masafumi Yamashita. Plane formation by semi-synchronous robots in the three dimensional euclidean space. In *Proceedings of the 18th International Symposium on Stabilization, Safety, and Security of Distributed Systems (SSS 2016)*, pages 383–398. Springer International Publishing, 2016.

15. Masafumi Yamashita and Ichiro Suzuki. Characterizing geometric patterns formable by oblivious anonymous mobile robots. *Theor. Comput. Sci.*, 411:2433–2453, 2010.

16. Yukiko Yamauchi, Taichi Uehara, Shuji Kijima, and Masafumi Yamashita. Plane formation by synchronous mobile robots in the three dimensional euclidean space. *Journal of the ACM*, (to appear).

17. Yukiko Yamauchi, Taichi Uehara, and Masafumi Yamashita. Brief announcement: Pattern formation problem for synchronous mobile robots in the three dimensional euclidean space. In *Proceedings of the 35th ACM SIGACT-SIGOPS Symposium*.
on Principles of Distributed Computing (PODC 2016), pages 447–449. ACM New York, 2016.
A Seventeen Symmetry Groups in 3D-space

We summarize the seventeen symmetry groups in 3D-space. Each rotation group is determined by rotation axes, mirror planes, and their arrangement. Our results also heavily rely on the order of each symmetry group and horizontal mirror planes. The following three tables show these properties together with typical polyhedra obtained by a seed point in each symmetry group.

Table 3: Symmetry groups without rotation axis

| Symmetry | Order |
|----------|-------|
| C1       | 1     |
| C1i      | 2     |
| C1s      | 2     |

Table 4: Symmetry groups with 2D rotation groups

| Principal axis | Other axes | Horizontal mirror | Order |
|----------------|------------|--------------------|-------|
| C_k            | k-fold     | -                  | N     |
| C_kh           | k-fold     | -                  | Y     |
| C_kv           | k-fold     | -                  | N     |
| D_l            | t-fold     | 2-fold axes        | N     |
| D_lh           | t-fold     | 2-fold axes        | Y     |
| D_lv           | t-fold     | 2-fold axes        | N     |
| S_m            | m-fold     | -                  | Y     |

Example polyhedra:
- Pyramid with regular k-gon base
- Hexagonal prism for D_6
- Hexagonal anti-prism for D_{10}
- Hexagonal anti-prism for S_{12}

Table 5: Symmetry groups with 3D rotation groups

| Rotation axes (folding) | Mirror planes | Order |
|-------------------------|---------------|-------|
| T 3 4 - 5               | 0             | 12    |
| T_d 3 4 - 6             | 6             | 24    |
| T_h 3 4 - 3             | 3             | 24    |
| O 6 4 3 - 0             | 24            |
| O_{15} 6 4 3 - 9        | 48            |
| I 15 10 - 6             | 60            |
| I_{15} 15 10 - 6        | 120           |

Example polyhedra:
- Snub tetrahedron
- Regular tetrahedron
- Snub cube
- Cube, regular octahedron
- Snub icosahedron
- Regular dodecahedron
Fig. 12: Mirrors in composite symmetry. The rotation axes (bold lines) and mirror planes of (a) \(D_{3h}\) and (b) \(D_{3v}\). Since \(T_d, T_h \preceq O_h\), the mirror planes show the arrangement of rotation axes and mirror planes of (c) \(T_h\) and (d) \(T_d\). For example, the three mirror planes of \(T_h\) and the 3-fold axes connecting the opposite vertices of a cube also show the arrangement of mirror planes and rotation axis of \(T_h\).

Fig. 12 shows the arrangement of the rotation axes and the mirror planes of \(D_{3h}, D_{3v}, O_h,\) and \(O_v\). The arrangement of the mirrors of \(O_h\) is related to that of \(T_h\) and \(T_d\).