G-twisted braces and orbifold Landau-Ginzburg Models

Weiqiang He, Si Li and Yifan Li

Abstract

Given an algebra with group $G$-action, we construct brace structures for its $G$-twisted Hochschild cochains. As an application, we construct $G$-Frobenius algebras for orbifold Landau-Ginzburg B-models and present explicit orbifold cup product formula for all invertible polynomials.

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1 Introduction

Consider a triple $(A, W, G)$ (we call $G$-twisted curved algebra), where $A$ is an associative algebra with a compatible $G$-action and $W$ is a $G$-invariant central element of $A$. Associated to $A$, we have Hochschild cochains $C^\bullet(A, A)$ where many homological constructions are captured by brace structures [19, 20, 52]. In this paper, we generalize such brace structures to $G$-twisted Hochschild cochains $C^\bullet(A, A[G])$ (Definition 2.10) and prove a $G$-twisted version of higher pre-Jacobi identities (Proposition 2.22).

Let

$$\partial_H : C^\bullet(A, A[G]) \to C^{\bullet+1}(A, A[G])$$
be the Hochschild differential. Let $\HH^*(A, A[G])$ be the Hochschild cohomology with a natural cup product $\cup$. We show that $\cup$ satisfies a $G$-twisted commutativity relation (see also [45]) and the $G$-invariant subspace $\HH^*(A, A[G])^G$ is a Gerstenhaber algebra (Theorem 2.7). When $G$ is a finite group, a result of $\breve{\text{S}}$tefan (Cor. 3.4 in [48]) implies that $\HH^*(A[G], A[G])$ and $\HH^*(A, A[G])^G$ are isomorphic as graded vector spaces (see also [11]). Using $G$-twisted brace structures, we extend this to an isomorphism as Gerstenhaber algebras (Theorem 2.10).

Let $W$ be a $G$-invariant central element of $A$. It leads to a curving differential (Definition 2.7)

$$d_W : \mathcal{C}^\bullet(A, A[G]) \to \mathcal{C}^{\bullet-1}(A, A[G])$$

such that $(\mathcal{C}^\bullet(A, A[G]), \partial_H, d_W)$ forms a mixed complex. All the above constructions apply to $(\partial_H + d_W)$-cohomologies which are instead $\mathbb{Z}/2\mathbb{Z}$-graded.

Our study of $(A, G, W)$ is motivated by mirror symmetry between two singularity theories: Landau-Ginzburg (LG) A-model and Landau-Ginzburg (LG) B-model. Such LG/LG mirror symmetry is parallel to the well-studied Calabi-Yau/Calabi-Yau and Toric/Landau-Ginzburg mirror symmetry. The basic data for LG model consists of $(A, W, G)$ where $W$ is a holomorphic function (called the superpotential)

$$W : X \to \mathbb{C}$$

on a complex variety $X$ and $G$ is a group (called the orbifold group) acting on $X$ preserving $W$. $A = \mathcal{O}(X)$ is the structure ring of $X$. A version of compact type Hochschild cohomology ([25] [36] [38], see Definition 2.13) turns out to be relevant for the state spaces of orbifold LG models [51, 26].

In this paper we focus on the case when $W : \mathbb{C}^N \to \mathbb{C}$ is a weighted homogeneous polynomial

$$W(\lambda^0 x_i) = \lambda W(x_i), \quad \forall \lambda \in \mathbb{C}^*$$

with an isolated critical point at the origin and contains no monomials of the form $x^i x^j$ for $i \neq j$. Here $q_i \in (0, \frac{1}{2}] \cap \mathbb{Q}$ is called the weight of $x_i$. The orbifold group $G$ will be a subgroup of $G_W$ where

$$G_W = \left\{ (\lambda_1, \ldots, \lambda_N) \in (\mathbb{C}^*)^N \mid W(\lambda_1 x_1, \ldots, \lambda_N x_N) = W(x_1, \ldots, x_N) \right\}.$$

There exists a construction of LG/LG mirror pairs which originates from a physical construction of Berglund-Hübsch [4] and is further completed by Krawitz [31]. In this construction, the polynomial $W$ is required to be invertible [33, 8, 9], i.e., the number of variables must equal to the number of monomials of $W$. By rescaling the variables, we can always write $W$ as

$$W = \sum_{i=1}^N \prod_{j=1}^N x_j^{a_{ij}}.$$

We denote its exponent matrix by $E_W = (a_{ij})_{N \times N}$. The mirror polynomial of $W$ is

$$W^T = \sum_{i=1}^N \prod_{j=1}^N x_j^{a_{ji}},$$

i.e., the exponent matrix $E_{W^T}$ of the mirror polynomial is the transpose matrix of $E_W$. The mirror construction between orbifold groups $G^T \subset G_{W^T}$ and $G \subset G_W$ is more involved but explicitly known. It has the property that bigger $G$ corresponds to smaller $G^T$ and vice versa.

A general mirror theorem is proved (see [24] and references therein for related works) between LG A-model (FJRW theory [15]) of $(W^T, G_{W^T})$ and LG B-model (Saito-Givental theory [39, 21]) of $(W, G = \text{trivial})$. Correlation functions for LG B-model when $G$ is not a trivial group is less known.
As an application of methods developed in this paper, we construct $G$-Frobenius algebraic structure in the sense of Kaufmann [29,30] for orbifold Landau-Ginzburg B-models (Theorem 3.11) (see also [3] for certain axiomatic discussion of Kaufman’s definition in orbifold LG models). Moreover, it is the first time we are able to compute explicit orbifold cup product formula for all invertible polynomials. This computation is based on a construction of explicit homotopy retract (Section 3.1).

\[ \hat{\eta}^* \left( C^*(\overline{A}, A[G]), \partial_H + d_W \right) \xrightarrow{\hat{\phi}^*} (K^*(A, A[G]), \partial_K + \hat{d}_W) \]

between $G$-twisted reduced Hochschild complex $C^*(\overline{A}, A[G])$ and $G$-twisted Koszul complex $K^*(A, A[G])$. This homotopy retract is obtained from a version of homological perturbation from a construction of Shepler and Witherspoon [42] in the case without $W$. The result is summarized as follows.

According to [32], an invertible polynomial is classified as a direct sum of three types of elementary invertible polynomials:

(a) Fermat type: $W = x_1^n$.

(b) Loop type: $W = x_1^{n_1}x_2 + x_2^{n_2}x_3 + \cdots + x_N^{n_N}x_1$.

(c) Chain type: $W = x_1^{n_1}x_2 + x_2^{n_2}x_3 + \cdots + x_N^{n_N}$.

Let $g \in G_W$, $g(x_i) = \lambda_i x_i$ where not all $\lambda_i$’s are 1. We define a version of $g$-twisted Hessian $\text{Hess}^g(W)$ (Definition 3.10). For example, for loop type polynomial, the $g$-twisted Hessian has the form

\[ \text{Hess}^g(W) = \frac{(-1)^{N+1} + n_1 n_2 \cdots n_N}{(1 - \lambda_1)(1 - \lambda_2) \cdots (1 - \lambda_N)} x_1^{n_1-1} x_2^{n_2-1} \cdots x_N^{n_N-1} . \]

Let $1_g$ be the generator of the $g$-twisted sector (see 3.33). Then our orbifold cup product formula reads

\[ 1_g \cup 1_{g^{-1}} = (-1)^{N(N-1)/2} \text{Hess}^g(W) 1_e \]

where $e \in G$ is the identity. This explicitly determines the full orbifold cup product (Theorem 3.20).

Alternatively, there exists categorical approach to orbifold Landau-Ginzburg models in terms of the dg-category $\text{MF}(A[G], W)$ of matrix factorizations. In the case when $G$ is trivial, Hochschild cohomology is computed by Dyckerhoff in terms of compact generators [12]. Alternatively, there is another approach via curved algebra which we follow in this paper. In [46], Polishchuk and Positselski identify Hochschild cohomology and compact type Hochschild cohomology of $\text{MF}(A, W)$. Segal [40] constructs a quasi-isomorphism between compact type Hochschild complex of $\text{MF}(A, W)$ and that of the curved algebra $(A, W)$. Căldăraru and Tu compute the compact type Hochschild (co)-homology of $(A, W)$ explicitly in [6]. In the orbifold case when $G$ is nontrivial, Hochschild cohomology is computed in terms of compact generators [37] or directly in terms of curved algebras [6, 46]. These two approaches are identified in [50]. Based on the result of [46], Shklyarov [46] shows that the categorical cup product of matrix factorizations is identical to the Hochschild cup product of $G$-twisted curved algebras.

On the other hand, it is very difficult to compute orbifold cup product through categories. In [46], Shklyarov deduces a formula of cup product in terms of certain complicated unknown coefficient $\sigma_{g,k}$. In appendix of [46], Basalaev and Shklyarov actually obtain interesting closed formula of cup product in some special cases (two variable chain polynomial and Fukaya category of surface).

In contrast to Shklyarov’s approach, we use an explicit homotopy retract between Hochschild complex and Koszul complex to deduce a combinatorial formula. This allows us to explicitly compute orbifold
cup product for all invertible polynomials. Our result confirms a conjecture by Basalaev and Shklyarov in Appendix A of [46]. It is a very interesting question to compare our results with Shklyarov’s formula.

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II G-twisted Hochschild complexes for curved algebras

In this paper we fix a field \( k \) of characteristic zero. All vector spaces and algebras are defined over \( k \). Let \( V \) be a \( \mathbb{Z} \) or \( \mathbb{Z}/2\mathbb{Z} \)-graded \( k \)-vector space. Let \( V_d = \{ x \in V \mid |x| = d \} \subset V \) denote the subspace of degree \( d \) elements. Let \( V[m] \) denote the degree shifting by \( m \) such that

\[
V[m]_d = V_{d+m}.
\]

When \( V \) is \( \mathbb{Z}/2\mathbb{Z} \)-graded, the above degree shifting is understood modulo \( 2\mathbb{Z} \). For example

\[
V[1]_{\bar{0}} = V_{\bar{1}}, \quad V[1]_{\bar{1}} = V_{\bar{0}}, \quad \text{where} \quad 0, 1 \in \mathbb{Z}/2\mathbb{Z}.
\]

Definition 2.1. A curved algebra is a pair \((A, W)\) where \( A \) is an associative algebra and \( W \) is a central element of \( A \). We will sometimes denote the pair \((A, W)\) for a curved algebra by \( A_W \).

Definition 2.2. Let \( G \) be a group. A \( G \)-twisted curved algebra is a triple \((A, W, G)\) where \( A \) is an associative algebra with \( G \)-equivariant product and \( W \) is a \( G \)-invariant central element of \( A \). \( \forall g \in G \) and \( a \in A \), we denote by \( g a \) the left action of \( a \) by \( g \). The identity element of \( G \) is always denoted by \( e \).

Definition 2.3. Given a \( G \)-twisted curved algebra \((A, W, G)\), we define its \( G \)-orbifolding to be the \( G \)-twisted curved algebra \((A[G], W, G)\) where

(a) \( A[G] = A \otimes_k k[G] \) is the crossed product algebra where \( k[G] \) is the group algebra. The product is

\[
(a_1 g_1) \cdot (a_2 g_2) := (a_1 \cdot g_1 a_2) g_1 g_2, \quad \text{where} \quad a_1, a_2 \in A, g_1, g_2 \in G.
\]

(b) \( G \)-action:

\[
G \times A[G] \to A[G] \quad (g, ag') \mapsto g.(ag') := (g a) g g^{-1}
\]

(c) \( G \)-invariant central element = \( W \).

In this section, we construct \( G \)-twisted brace structures (Definition 2.10) on \( G \)-twisted Hochschild cochains (Definition 2.9). This generalizes the usual brace operations on Hochschild cochains of associative algebras. As an application, we obtain a \( G \)-twisted commutative structure on \( G \)-twisted Hochschild cohomology (Theorem 2.7) and a comparison theorem between two versions of \( G \)-twisted Gerstenhaber algebras (Theorem 2.10 and Theorem 2.11). This will be applied in the next section to establish our main results on \( G \)-Frobenius algebras of orbifold Landau-Ginzburg models.
2.1 Hochschild cochains and brace structure

**Definition 2.4.** Given an associative algebra \( A \), we denote Hochschild cochains and compact type Hochschild cochains of \( A \) by

\[
C^\bullet(A, A) = \bigoplus_{p=0}^{\infty} C^p(A, A), \quad C^\bullet_c(A, A) = \bigoplus_{p=0}^{\infty} C^p(A, A),
\]

where

\[
C^p(A, A) = \text{Hom}_k(A^{\otimes p}, A).
\]

Given \( \phi \in C^p(A, A) \), we write \(|\phi| = p\) for its degree.

Following [19, 20, 17, 7, 49], we can define higher operations on Hochschild cochains as a generalization of Gerstenhaber product introduced in [16]. Given \( \phi \in C^p(A, A) \), \( \phi_i \in C^{p_i}(A, A) \) for \( i = 1, \ldots, k \) and \( a_1, a_2 \cdots \in A \), we define (if \( k \leq p \))

\[
\phi \{ \phi_1, \phi_2, \cdots \phi_k \} (a_1 \otimes a_2 \otimes \cdots) = \sum_{I \in \mathcal{I}} (-1)^{\sum_{j=1}^{k} (i_j - 1)(|\phi_j| - 1)} \phi_1 (a_1 \otimes \cdots \otimes a_{i_1-1} \otimes \phi_1 (a_{i_1} \otimes \cdots) \otimes \cdots \otimes a_{i_j-1} \otimes \phi_j (a_{i_j} \otimes \cdots) \otimes \cdots),
\]

where

\[
\mathcal{I} = \{ I = (i_1 < i_2 < \cdots < i_k) \mid i_1 > 0 \text{ and } \forall 1 \leq j < k, i_j + |\phi_j| \leq i_{j+1} \}.
\]

If \( k > p \), we set \( \phi \{ \phi_1, \phi_2, \cdots \phi_k \} = 0 \). The map

\[
C^\bullet(A, A) \to C^\bullet(C^\bullet(A, A), C^\bullet(A, A))
\]

\[
\phi \mapsto \phi \{ \cdots \}
\]

gives brace structures [19, 20, 52]. It satisfies the following higher pre-Jacobi identity

\[
\phi \{ \phi_1, \phi_2 \cdots \phi_n \} (\psi_1, \psi_2 \cdots \psi_m) = \sum \pm \phi (\psi_1 \cdots \psi_{j_1-1}, \phi_1 (\psi_{j_1} \cdots) \cdots \psi_{j_n-1}, \phi_n (\psi_{j_n} \cdots) \cdots \psi_m). \tag{2.3}
\]

When \( k = 1 \), we get the Gerstenhaber product

\[
\phi \{ \phi_1 \} = \phi \circ \phi_1. \tag{2.4}
\]

Pre-Jacobi identity implies that the Gerstenhaber bracket on \( C^\bullet(A, A) \) defined by

\[
\{ \phi_1, \phi_2 \} = \phi_1 \{ \phi_2 \} - (-1)^{(|\phi_1| - 1)(|\phi_2| - 1)} \phi_2 \{ \phi_1 \}, \quad \forall \phi_1, \phi_2 \in C^\bullet(A, A),
\]

gives a graded Lie algebra structure on \( C^\bullet(A, A)[1] \).

The product \( \cdot \) on \( A \) gives rise to a Hochschild cochain

\[
m_2 \in C^2(A, A), \quad m_2 (a_1, a_2) = a_1 \cdot a_2, \quad a_i \in A.
\]

The associativity of the product is equivalent to

\[
\{ m_2, m_2 \} = 0.
\]
Definition 2.5. The cup product $\cup$ and Hochschild differential $\partial_H$ on $C^\bullet(A, A)$ are defined by
\[
\phi_1 \cup \phi_2 = (-1)^{\phi_1(\phi_2 - 1)} m_2(\phi_1, \phi_2), \quad \partial_H(\phi) := (-1)^{\phi-1} \{m_2, \phi\}. \tag{2.6}
\]

Higher pre-Jacobi identity for brace operations implies the following higher homotopies,
\[
\partial_H(\phi) \{\phi_1, \phi_2, \ldots, \phi_n\} \\
= (-1)^{\xi_n} \partial_H\left(\phi \{\phi_1, \ldots, \phi_n\}\right) - \sum_{k=1}^{n} (-1)^{\xi_k} \phi \{\phi_1, \ldots, \phi_{k-1}, \partial_H(\phi_k), \phi_{k+1}, \ldots, \phi_n\} \\
+ (-1)^{\phi_1} \xi_{\phi_1} \cup \phi \{\phi_2, \phi_3, \ldots, \phi_n\} - (-1)^{\phi_1 + \phi_n + \xi_n - 1} \phi \{\phi_1, \phi_2, \ldots, \phi_{n-1}\} \cup \phi_n \\
+ \sum_{k=1}^{n-1} (-1)^{\xi_k + \phi_1 + \phi_{k+1}} \phi \{\phi_1, \ldots, \phi_k \cup \phi_{k+1}, \ldots, \phi_n\}, \tag{2.7}
\]
with
\[
\begin{align*}
\xi_k &= |\phi_1| + |\phi_2| + \cdots + |\phi_k| - k, \\
\xi_k' &= |\phi_{k+1}| + |\phi_{k+2}| + \cdots + |\phi_n| - (n - k).
\end{align*}
\]

Here are some identities for lower braces

- $n = 1$. We find
\[
(-1)^{\phi_1} \partial_H(\phi_1) + (\partial_H \phi) \{\phi_1\} = \phi_1 \cup \phi - (-1)^{\phi_1} \phi_1 \cup \phi_1. \tag{2.8}
\]
This says that $\cup$ on $C^\bullet(A, A)$ is commutative up to homotopy. Switching the role of $\phi, \phi_1$ and comparing the difference, we find the compatibility of $\partial_H$ with the Gerstenhaber bracket $[16]$,
\[
\partial_H(\phi, \phi_1) = (-1)^{\phi_1 - 1} \{\partial_H \phi, \phi_1\} + \{\phi, \partial_H \phi_1\}. \tag{2.9}
\]

- $n = 2$. We find
\[
\partial_H(\phi_1, \phi_2) = (-1)^{\phi_1 - 1} \{\partial_H \phi_1, \phi_2\} - \phi_1 \cup \phi_2 - (-1)^{\phi_1 + \phi_2} \partial_H(\phi) \{\phi_1, \phi_2\} \\
=(-1)^{\phi_1 - 1} \phi_2 \left(\phi_1 \cup \phi_2 - (-1)^{\phi_1 + \phi_2} \phi_1 \cup \phi_2 - \phi_1 \cup \phi_2\right). \tag{2.10}
\]
Set $\phi = m_2$ and use $m_2 \{m_2\} = 0$, we find
\[
\partial_H(\phi_1 \cup \phi_2) = \partial_H(\phi_1) \cup \phi_2 + (-1)^{\phi_1} \phi_1 \cup \partial_H(\phi_2). \tag{2.11}
\]

This says that the triple
\[
(C^\bullet(A, A), \partial_H, \cup)
\]
defines a differential graded algebra (dga).

Definition 2.6. A Gerstenhaber algebra $(A, \cdot, \{,\})$ is a $\mathbb{Z}$ (or $\mathbb{Z}/2\mathbb{Z}$)-graded commutative algebra with a graded Lie structure on $A[1]$ satisfying the shifted Poisson identities. Thus, for $a, b, c \in A$, we have

- $[a \cdot b] = |a| + |b|$ and $a \cdot b = (-1)^{|a||b|} b \cdot a$.
- $|\{a, b\}| = |a| + |b| - 1$, $\{a, b\} = (-1)^{|a| - 1} (|b| - 1) \{b, a\}$ and $\{a, \{b, c\}\} = \{\{a, b\}, c\} + (-1)^{|a| - 1} (|b| - 1) \{b, \{a, c\}\}$.
- $\{a \cdot b, c\} = \{a, b\} \cdot c + (-1)^{|a||c| - 1} a \cdot \{b, c\}$.

The direct consequence of $2.5$ $2.8$ $2.9$ $2.10$ $2.11$ says that the Hochschild cohomology
\[
HH^\bullet(A) := H^\bullet(C^\bullet(A, A), \partial_H)
\]
together with the cup product and Gerstenhaber bracket form a Gerstenhaber algebra.
2.2 Curved algebras and mixed complex

We consider a curved algebra \((A, W)\) where \(W\) is a central element of \(A\). The curving \(W\) defines

\[ m_0 \in C^0(A, A), \quad m_0(1) = W. \]

**Definition 2.7.** We define the **curving differential** \(d_W\) on Hochschild cochains of \(A\) by

\[ d_W = (-1)^p \{ m_0, - \} : C^p(A, A) \to C^{p-1}(A, A) \]  \hspace{1cm} (2.12)

and the **curved Hochschild differential** by

\[ \partial_W^H = \partial_H + d_W. \]  \hspace{1cm} (2.13)

The following identities hold

\[ \partial_H^2 = d_W^2 = (\partial_W^H)^2 = 0. \]

Therefore the triple \(\{ C^\bullet(A, A), \partial_H, d_W \}\) defines a mixed complex. We are interested in the cohomology for the mixed differential \(\partial_W^H\), which is sensitive to the topology we use. Following [5, 6, 40], the appropriate complex for Landau-Ginzburg models turns out to be the one of compact type above as induced in [25, 36, 38].

**Definition 2.8.** We define the **compact type Hochschild cohomology** for a curved algebra \((A, W)\) by

\[ HH_c(A_W) = \text{Hom}_k(C^\bullet_c(A, A), \partial_W^H). \]  \hspace{1cm} (2.14)

\(HH_c(A_W)\) is \(\mathbb{Z}/2\mathbb{Z}\)-graded in terms of the parity of the degree

\[ HH_c(A_W) = HH^0_c(A_W) \oplus HH^1_c(A_W). \]

**Proposition 2.1.** \(HH_c(A_W)\) is a \(\mathbb{Z}/2\mathbb{Z}\)-graded Gerstenhaber algebra.

**Proof.** Equations (2.8) (2.9) (2.10) (2.11) still hold if \(\partial_H\) is replaced by \(\partial_W^H\). It follows that the cup product \(\cup\) and the Gerstenhaber bracket \(\{ -, - \}\) induce the Gerstenhaber algebra structure on \(HH_c(A_W)\). \bbox

2.3 \(G\)-curved algebras and \(G\)-twisted brace structures

**Definition 2.9.** Let \((A, W, G)\) be a \(G\)-twisted curved algebra. We define the **\(G\)-twisted Hochschild cochains and compact type Hochschild cochains** by

\[ C^\bullet(A, A[G]) = \prod_{p=0}^{\infty} C^p(A, A[G]), \quad C^\bullet_c(A, A[G]) = \bigoplus_{p=0}^{\infty} C^p(A, A[G]), \]  \hspace{1cm} (2.15)

where

\[ C^p(A, A[G]) = \text{Hom}_k(A \otimes^p, A[G]). \]

There is a natural \(G\)-action on \(G\)-twisted Hochschild cochains

\[ G \times C^p(A, A[G]) \to C^p(A, A[G]), \]

\[ (g, \phi) \mapsto g^*(\phi), \]

given by

\[ g^*(\phi)(a_1 \otimes a_2 \otimes \cdots \otimes a_p) = g \cdot \phi(g^{-1}a_1 \otimes g^{-1}a_2 \otimes \cdots \otimes g^{-1}a_p) \cdot g^{-1}. \]  \hspace{1cm} (2.16)
**G-twisted braces**

**Definition 2.10 (G-twisted braces).** Given \( \phi \in \mathcal{C}^p(A, A_g) \), \( \phi_i \in \mathcal{C}^p(A, A_g) \) for \( i = 1, \ldots, k \) and \( a_1, a_2 \cdots \in A \), we define (if \( k \leq p \))

\[
\phi \{ \phi_1, \phi_2, \ldots, \phi_k \} (a_1 \otimes a_2 \otimes \cdots) := \sum_{I \in \mathcal{I}} (-1)^{\sum_{j=1}^{k} (i_j - 1)} (|\phi|_j - 1) \phi^\phi \left( a_1 \otimes \cdots \otimes a_{i_1 - 1} \otimes \phi_1^\phi (a_{i_1} \otimes \cdots \otimes a_{i_1 + |\phi|_1 - 1}) \otimes g_1 a_{i_1 + |\phi|_1} \otimes \cdots \right.
\]

\[
\cdots \otimes g_{j-1} a_{i_j - 1} \otimes \phi_j^\phi (g_{j-1} a_{i_j} \otimes \cdots) \otimes g_j a_{i_j + |\phi|_j} \otimes \cdots \right.
\]

\[
\cdots \otimes \phi_k^\phi (g_{k-1} a_{i_k} \otimes \cdots) \otimes g_k a_{i_k + |\phi|_k} \otimes \cdots \left. \right) gg_k \cdots g_1, \quad (2.17)
\]

where

\( \mathcal{I} := \{ I = (i_1 < i_2 < \cdots < i_k) \mid 0 < i_1 \text{ and } \forall 1 \leq j < k, i_j + |\phi|_j \leq i_{j+1} \} \).

Here \( \phi^\phi \in \mathcal{C}^*(A, A) \) such that \( \phi = \phi^\phi g \in \mathcal{C}^*(A, A_g) \). Similarly for \( \phi^\phi_g \), we will set \( \phi \{ \phi_1, \phi_2, \ldots, \phi_k \} = 0 \), if \( k > p \). This extends linearly to brace structures on \( G \)-twisted Hochschild cochains

\[ C^*(A, A[G]) \to C^*(C^*(A, A[G]), C^*(A, A[G])), \quad \phi \mapsto \phi \{ \cdots \}, \]

which we call \( G \)-twisted braces.

**Remark.** We may use tree graphs to express terms above. For example, we write

\[ (-1)^{(i_1 - 1)(|\phi_1| - 1) + (i_2 - 1)(|\phi_2| - 1)} \]

for

\[ (-1)^{(i_1 - 1)(|\phi_1| - 1) + (i_2 - 1)(|\phi_2| - 1)} \phi^\phi \left( a_1 \otimes \cdots \otimes a_{i_1 - 1} \otimes \phi_1^\phi (a_{i_1} \otimes \cdots) \otimes g_1 a_{i_1 + |\phi|_1} \otimes \cdots \right.
\]

\[
\cdots \otimes g_{i_2 - 1} a_{i_2 - 1} \otimes \phi_2^\phi (g_{i_2} a_{i_2} \otimes \cdots) \otimes g_{i_2} a_{i_2 + |\phi|_2} \otimes \cdots \left. \right) gg_{i_2} g_1,
\]

as a term in \( \phi \{ \phi_1, \phi_2 \} (a_1, a_2, \cdots) \). As another example, the following graph

\[ (-1)^{(i_1 - 1)(|\phi_1| - 1) + (i_2 - 1)(|\phi_2| - 1)} \]

for

\[ (-1)^{(i_1 - 1)(|\phi_1| - 1) + (i_2 - 1)(|\phi_2| - 1)} \phi^\phi \left( a_1 \otimes \cdots \otimes a_{i_1 - 1} \otimes \phi_1^\phi (a_{i_1} \otimes \cdots) \otimes g_1 a_{i_1 + |\phi|_1} \otimes \cdots \right.
\]

\[
\cdots \otimes g_{i_2 - 1} a_{i_2 - 1} \otimes \phi_2^\phi (g_{i_2} a_{i_2} \otimes \cdots) \otimes g_{i_2} a_{i_2 + |\phi|_2} \otimes \cdots \left. \right) gg_{i_2} g_1,
\]

appears as a term in \( \phi \{ \phi_1 \} \{ \phi_2 \} (a_1, a_2, \cdots) \).

**Remark.** This brace structure is a generalization of twisted versions of Gerstenhaber products in \[ \mathcal{C} \mathcal{C} \mathcal{C} ]
Lemma 2.3. The twisted brace structures are $G$-equivariant with respect to the $G$-action defined in (2.16), i.e., $\forall h \in G$ and $\phi, \phi_1, \phi_2, \cdots \in C^*[A, A[G]]$, we have

$$h^* (\phi \{ \phi_1, \phi_2, \cdots \} ) = h^*(\phi) \{ h^*(\phi_1), h^*(\phi_2), \cdots \}. \quad (2.19)$$
Proof. We illustrate for the case $\phi\{\phi_1, \phi_2\}$. Terms in both sides are given by

\[
\begin{align*}
\text{Left side:} & \quad hgg_2g_1h^{-1} = hgh^{-1}hgh^{-1}hgh^{-1}hgh^{-1}h.
\end{align*}
\]

$G$-twisted Hochschild differential and cup product

Now we introduce differentials on $G$-twisted Hochschild cochains. We will always identify

\[
C^\bullet(A, A) \cong C^\bullet(A, Ae) \mapsto C^\bullet(A, A[G])
\]

as the identity sector of $C^\bullet(A, A[G])$. In particular, we identify the product and the curving of $A$

\[
m_2 \in C^2(A, Ae), \quad m_0 \in C^0(A, Ae),
\]

as $G$-invariant cochains in $C^\bullet(A, A[G])$.

**Definition 2.11.** We define the Hochschild differential $\partial_H$ and the curving differential $d_W$ on $G$-twisted Hochschild cochains by

\[
\partial_H(\phi) = (-1)^{|\phi|-1}m_2\{\phi\} - \phi\{m_2\}, \quad d_W(\phi) = \phi\{m_0\}, \quad \phi \in C^\bullet(A, A[G]).
\]

We also denote

\[
\partial_H^W = \partial_H + d_W.
\]

**Definition 2.12.** We define the cup product on $G$-twisted Hochschild cochains by

\[
\phi_1 \cup \phi_2 = (-1)^{|\phi_1|(\phi_2|^{-1})}m_2\{\phi_1, g_1^*\phi_2\}, \quad \phi_i \in C^\bullet(A, A_{g_i}).
\]

Note that this is the usual cup product that arises from the algebra structure on $A[G]$ by Definition 2.10.

**Lemma 2.4.** The following $G$-twisted version of higher homotopy identities holds,

\[
\partial_H(\phi) \{\phi_1, \phi_2, \ldots, \phi_n\} = (-1)^{\xi_n} \partial_H(\phi \{\phi_1, \ldots, \phi_n\}) - \sum_{k=1}^n (-1)^{\xi_k} \phi \{\phi_1, \ldots, \phi_{k-1}, \partial_H(\phi_k), \phi_{k+1}, \ldots, \phi_n\}
\]

\[
+ (-1)^{|\phi_1|+|\phi_2|} \phi_1 \cup (g_1^{-1})^* \phi_2, \phi_3, \ldots, \phi_n\}
\]

\[
+ \sum_{k=1}^{n-1} (-1)^{\xi_k + |\phi_k|} \phi \{\phi_1, \ldots, \phi_k \cup (g_k^{-1})^* \phi_{k+1}, \ldots, \phi_n\}
\]

\[
- (-1)^{|\phi_1|+|\phi_2|+\cdots+|\phi_n|} \phi \{\phi_1, \phi_2, \ldots, \phi_{n-1}\} \cup ((g_{n-1} \cdots g_1)^{-1})^* \phi_n,
\]

with

\[
\xi_k := |\phi_1| + |\phi_2| + \cdots + |\phi_k| - k,
\]

\[
\xi_n := |\phi_{k+1}| + |\phi_{k+2}| + \cdots + |\phi_n| - (n - k).
\]

Here $\phi \in C^\bullet(A, A_{g_i})$, $\phi_i \in C^\bullet(A, A_{g_i})$. The same is true if $\partial_H$ is replaced by $\partial_H^W$. 

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Proof. The lemma follows from the twisted higher pre-Jacobi identities and the $G$-invariance of $m_2, m_0$. □

Corollary 2.5. The cup product $\cup$ on $\mathcal{C}^\bullet(A, A[G])$ satisfies the following twisted commutativity up to homotopy: for any $\phi \in \mathcal{C}^\bullet(A, A_g)$, $\phi_1 \in \mathcal{C}^\bullet(A, A_{g_1})$

\[-1]^{[\phi_1]} \partial_H(\phi_1) + ([\partial_H \phi_1] - [1]^{[\phi_1]} \phi \partial_H \phi_1) = \phi_1 \cup (g_1^{-1})^* \phi - [1]^{[\phi_1]} \phi \cup \phi_1. \quad (2.21)\]

Proof. This follows from Lemma 2.4 in the case $n = 1$. □

Corollary 2.6. The triple $\{\mathcal{C}^\bullet(A, A[G]), \cup, \partial_H\}$ defines a differential graded algebra. If we replace $\partial_H$ with $\partial_H^W$, we get a $\mathbb{Z}/2\mathbb{Z}$-graded differential graded algebra.

Proof. Let $\phi \in \mathcal{C}^\bullet(A, A_g), \phi_1 \in \mathcal{C}^\bullet(A, A_{g_1})$. Lemma 2.4 implies

\[
\partial_H(\phi + \phi_1, \phi_2) - (-1)^{[\phi][1]} \phi_2 \partial_H \phi_1, \phi_2 - \phi_1 \partial_H \phi_2 - (-1)^{[\phi][1]} \partial_H(\phi_1, \phi_2) = (-1)^{[\phi_1][1]} (\phi_1 \cup (g_1^{-1})^* \phi_2) - (-1)^{[\phi][1]} \phi_1 \cup (g_1^{-1})^* \phi_2 - \phi_1 \cup (g_1^{-1})^* \phi_2.
\]

Set $\phi = m_2$ and use $m_2 \{m_2\} = 0$, we find

\[
\partial_H(\phi_1 \cup \phi_2) = \partial_H(\phi_1) \cup \phi_2 + (-1)^{[\phi_1]} \phi_1 \cup \partial_H(\phi_2).
\]

(2.22) The proof for $\partial_H^W$ is similar. □

Definition 2.13. Let

\[
\mathcal{H}(A, A[G]) = H(C^\bullet(A, A[G]), \partial_H), \quad \mathcal{H}_c(A, A[G]) = H(C^\bullet_c(A, A[G]), \partial_H)
\]

denote the $G$-twisted Hochschild cohomologies and similarly for the curved case

\[
\mathcal{H}(A_W, A_W[G]) = H(C^\bullet(A, A[G]), \partial_H + d_W), \quad \mathcal{H}_c(A_W, A_W[G]) = H(C^\bullet_c(A, A[G]), \partial_H + d_W).
\]

All the above cohomologies carry a natural $G$-action induced by $C$. □

Theorem 2.7. The cup product $\cup$ defines $\mathbb{Z}$-graded algebras on $\mathcal{H}(A, A[G]), \mathcal{H}_c(A, A[G])$ and $\mathbb{Z}/2\mathbb{Z}$-graded algebras on $\mathcal{H}(A_W, A_W[G]), \mathcal{H}_c(A_W, A_W[G])$ satisfying the twisted commutativity relation

\[
[\phi_1] \cup [\phi_2] = (-1)^{[\phi_1][1]} [\phi_2] \cup (g_2^{-1})^* [\phi_1], \quad \phi_i \in \mathcal{C}^\bullet(A, A_g).
\]

Moreover, their $G$-invariant subspaces, $\mathcal{H}(A, A[G])^G, \mathcal{H}_c(A, A[G])^G, \mathcal{H}(A_W, A_W[G])^G$ and $\mathcal{H}_c(A_W, A_W[G])^G$, inherit natural Gerstenhaber algebra structures.

Proof. The induced cup product on cohomologies and twisted commutativity follow from (2.21) (2.22). To see Gerstenhaber algebra structures on $G$-invariant cohomologies, we consider for example

\[
\mathcal{H}(A, A[G])^G = H(C^\bullet(A, A[G]), \partial_H),
\]

where $C^\bullet(A, A[G])^G$ are $G$-invariant cochains. On $C^\bullet(A, A[G])^G$, the $G$-twisted higher pre-Jacobi identities (2.18) reduce to the same form as the untwisted one (2.7), from which we deduce the Gerstenhaber algebra structures by (2.5) (2.8) (2.9) (2.10) (2.11). □
Remark. The $G$-twisted commutativity of the cup product is also obtained by Shklyarov in [46].

**Proposition 2.8.** Let $H$ be a subgroup of $G$, then the inclusion

$$C^*(A, A[H]) \hookrightarrow C^*(A, A[G])$$

induce an embedding of $\mathbb{Z}/2\mathbb{Z}$-graded algebras

$$\text{HH}_c(A_W, A_W[H]) \hookrightarrow \text{HH}_c(A_W, A_W[G]).$$

The same is true if we consider $\text{HH}(A, A[H]), \text{HH}(A_W, A_W[H])$.

**Proof.** It is easy to see that $\text{HH}_c(A_W, A_W[H])$ is a $\mathbb{Z}/2\mathbb{Z}$-graded subspace of $\text{HH}_c(A_W, A_W[G])$ and closed under the cup product.

\[ \Box \]

### 2.4 A comparison between Gerstenhaber algebras

In this subsection, $G$ will be a finite group. In this case, Ştefan [48] proved that there is an isomorphism as graded vector spaces between $\text{HH}^*(A[G])$ and $\text{HH}^*(A, A[G])^G$ (see also [11]). Using $G$-twisted brace structures, we extend this to an isomorphism between Gerstenhaber algebras.

Consider the following map $\Psi: C^*(A, A[G])^G \rightarrow C^*(A[G], A[G])$

where for $a_1g_1, a_2g_2, \ldots, a_pg_p \in A[G],$

$$\Psi(\phi)(a_1g_1 \otimes a_2g_2 \otimes \cdots \otimes a_pg_p) = \phi(a_1 \otimes g_1a_2 \otimes \cdots \otimes g_1g_2 \cdots g_{p-1}a_p)g_1g_2 \cdots g_p. \quad (2.23)$$

**Lemma 2.9.** $\Psi$ preserves brace structures: for any $\phi, \phi_1 \in C^*(A, A[G])^G,$

$$\Psi(\phi) \{\Psi(\phi_1), \Psi(\phi_2), \cdots \} = \Psi(\phi \{\phi_1, \phi_2, \cdots \}). \quad (2.24)$$

**Proof.** Let $\phi_k$ be a $G$-invariant $G$-twisted cochain. We write

$$\phi_k = \sum_{g \in G} \phi^g_k g, \quad \phi^g_k \in C^*(A, A).$$

$G$-invariance implies that

$$h^*(\phi^g_k) = \phi^{gh^{-1}}.$$ 

Given input $a_1h_1 \otimes a_2h_2 \otimes \cdots$, the left half side of (2.24) gives a sum of terms like

$$\pm \phi^g \left( a_1 \otimes h_1a_2 \otimes \cdots \otimes h_{i-1} \phi^{h_{i-1}g_1h_1} (a_{i_1} \otimes h_{i_1}a_{i_1+1} \otimes \cdots) \otimes \cdots \right) \ldots \phi_k^{h_{k-1}g_kh_{k-1} \cdots g_{k-1}h_1}(a_{i_k} \otimes \cdots) \otimes \cdots \otimes g \cdots g_2g_1h_1h_2 \cdots$$

$$= \pm \phi^g \left( a_1 \otimes h_1a_2 \otimes \cdots \otimes \phi^g_i (h_{i_1}a_{i_1} \otimes h_{i_1}h_{i_1}a_{i_1+1} \otimes \cdots) \otimes \cdots \right) \phi_k^{h_{k-1}g_kh_{k-1} \cdots g_{k-1}h_1}(a_{i_k} \otimes \cdots) \otimes \cdots \otimes g \cdots g_2g_1h_1h_2 \cdots,$$
where \( h_i := h_1 h_2 \cdots h_i \). The right half side of (2.24) gives a sum of terms like
\[
\pm \phi^g \left( a_1 \otimes h_1 a_2 \otimes \cdots \otimes \phi_{h_1} \left( h_{i_1} \otimes h_{i_2} \otimes \cdots \right) \otimes \cdots \right) 
\]
\[
\cdots \phi_{h_2} \left( g_{i_1-1} g_{i_2-2} \cdots g_1 h_i \otimes \cdots \right) g_2 g_1 h_1 h_2 \cdots .
\]
The Lemma follows.

**Theorem 2.10.** Let \( G \) be a finite group. Then \( \Psi \) defined by (2.23) induces an isomorphism between \( \mathbb{Z} \)-graded Gerstenhaber algebras
\[
\Psi : \text{HH}^\bullet(A, A[G]) \to \text{HH}^\bullet(A[G]).
\]

**Proof.** It is easy to see that \( \Psi \) is compatible with Hochschild differential \( \partial_H \). By (2.2), \( \Psi \) induces a \( \mathbb{Z} \)-graded vector space isomorphism between \( \text{HH}^\bullet(A, A[G])^G \) and \( \text{HH}^\bullet(A[G]) \). The theorem is now a formal consequence of Lemma 2.9.

Now we consider the curved case.

**Theorem 2.11.** Let \( G \) be a finite group. \((A, W, G)\) be a \( G \)-twisted curved algebra. Then \( \Psi \) defined by (2.23) induces an isomorphism between \( \mathbb{Z}/2\mathbb{Z} \)-graded Gerstenhaber algebras
\[
\Psi : \text{HH}_c(A_W, A_W[G]) \to \text{HH}_c(A_W[G]).
\]

**Proof.** By Lemma 2.9 we only need to prove that
\[
\Psi : C_c^\bullet(A, A[G])^G \to C_c^\bullet(A[G])
\]
induce a vector space isomorphism on \( (\partial_H + d_W) \)-cohomology. Introduce a formal variable \( u \) of degree 2 and we extend \( \Psi \) to a cochain map between \( \mathbb{Z} \)-graded complex
\[
\Psi : (C_c^\bullet(A, A[G])^G[u, u^{-1}], \partial_H + u d_W) \to (C_c^\bullet(A[G], A[G])[u, u^{-1}], \partial_H + u d_W). \tag{2.25}
\]

For any \( k \in \mathbb{Z} \), there are natural identifications
\[
\text{HH}_c^k(A_W, A_W[G])^G = H^{2k}(C_c^\bullet(A, A[G])^G[u], \partial_H + u d_W)
\]
\[
\text{HH}_c^k(A_W, A_W[G])^G = H^{2k+1}(C_c^\bullet(A, A[G])^G[u], \partial_H + u d_W)
\]
and similarly for \( \text{HH}_c(A_W[G]) \). It suffices to show that (2.25) is a quasi-isomorphism.

Consider the decreasing filtration \( \cdots \subset F^{p+1} \subset F^p \subset F^{p-1} \subset \cdots \) where
\[
F^p = u^p C_c^\bullet[u].
\]

Here \( C_c^\bullet \) denotes \( C_c^\bullet(A, A[G])^G \) or \( C_c^\bullet(A[G], A[G]) \). Since we work with compact type complex, this filtration is exhaustive (see [53] for details) and there is an associated convergent spectral sequence. \( \Psi \) in (2.25) induces an isomorphism between \( E_1 \)-pages which are computed by \( \partial_H \)-cohomologies. It follows that (2.25) defines a quasi-isomorphism.
III Orbifold Landau-Ginzburg B-models

In this section we study the orbifold Landau-Ginzburg model associated to a $G$-twisted curved algebra

$$(A = \mathbb{C}[x_1, \cdots, x_N], W, G),$$

where $W$ is a weighted homogeneous invertible polynomial and $G$ is a subgroup of $(\mathbb{C}^*)^N$ preserving $W$. We show that the cohomology $HH_* (A_W, A_W [G])$ is a $G$-Frobenius algebra. In particular, the $G$-invariant subspace $HH_*(A_W, A_W [G])^G$ has an induced Frobenius algebra structure. We give closed formulae for the cup product on $HH_*(A_W, A_W [G])^G$ for all Fermat, Loop and Chain types. This is computed via an explicit homotopy between Koszul resolution and bar resolution.

For convenience, we fix the following notations in this section.

**Definition 3.1.** For $g \in G$, we let $\text{Fix}(g) = \{ v \in \mathbb{C}^N \mid g v = v \} \subset \mathbb{C}^N$ be the fixed locus of $g$ and $A_g = \mathbb{C}[\text{Fix}(g)]$ denote polynomial functions on $\text{Fix}(g)$. We write

$$N_g = \dim_{\mathbb{C}} \text{Fix}(g), \quad W_g = W|_{\text{Fix}(g)} \in A_g.$$ 

We also denote $I_g = \{ i_1 < i_2 < \cdots < i_{N - N_g} \}$ where $\{ x_{i_k} \}$’s are variables such that $g x_{i_k} \neq x_{i_k}$. $I_g$ will be called the moving index of $g$.

**Definition 3.2.** Let $x_1^{\gamma_1} \cdots x_N^{\gamma_N}$ be a monomial in $A$, and $g \in G$. We define

$$\rho_i (g) (x_1^{\gamma_1} \cdots x_N^{\gamma_N}) = (g x_1)^{\gamma_1} \cdots (g x_{i - 1})^{\gamma_{i - 1}} x_i^{\gamma_i} \cdots x_N^{\gamma_N}, \quad (3.1)$$

and the quantum differential operator

$$\partial_{x_i}^g (x_1^{\gamma_1} \cdots x_N^{\gamma_N}) = \begin{cases} [\gamma]_{\lambda_i} x_1^{\gamma_1} \cdots x_{i - 1}^{\gamma_{i - 1}} x_i^{\gamma_i} \cdots x_N^{\gamma_N} & \text{if } \gamma_i > 0, \\ 0 & \text{else}. \end{cases} \quad (3.2)$$

Here $\lambda_i$ is the weight: $g x_i = \lambda_i x_i$. $[\gamma]_{\lambda} (\gamma \geq 1)$ is defined by

$$[\gamma]_{\lambda} = 1 + \lambda + \lambda^2 + \cdots + \lambda^{\gamma - 1}. \quad (3.3)$$

Both $\rho_i (g)$ and $\partial_{x_i}^g$ extend linearly to operators on $A$.

### 3.1 Bar resolution vs. Koszul resolution

For simplicity, we work with the reduced bar resolution (or ‘normalized bar resolution’, see [34]),

$$\cdots \longrightarrow b A \otimes \overline{A} \oplus b A \otimes \overline{A} \otimes A \longrightarrow b A \otimes \overline{A} \otimes A \longrightarrow b A \otimes A \longrightarrow b A \otimes \overline{A} \otimes A \longrightarrow A \longrightarrow 0,$$

where $\overline{A} = A / \mathbb{C} = \mathbb{C}[x^\gamma] / \mathbb{C}$. It gives a free resolution of $A$ as $A^e$-modules (equivalently $A$-bimodules) where $A^e = A \otimes A^{op}$. Applying the functor $\text{Hom}_{A^e}(\cdot, A[G])$, we obtain the (reduced) $G$-twisted Hochschild cochain complex

$$\partial_H : C^p(\overline{A}, A[G]) \rightarrow C^{p+1}(\overline{A}, A[G]).$$
Here $C^p(\mathcal{A}, A[G]) = \text{Hom}(\mathcal{A}^G, A[G])$ and $\partial_H$ is the Hochschild differential which is also well-defined on the reduced complex. We will denote

$$C^* (\mathcal{A}, A[G]) = \bigoplus_{p=0}^{\infty} C^p (\mathcal{A}, A[G]), \quad C^*_e (\mathcal{A}, A[G]) = \bigoplus_{p=0}^{\infty} C^p_e (\mathcal{A}, A[G]). \quad (3.4)$$

The curving differential $d_W$ is defined the same as before

$$d_W : C^p (\mathcal{A}, A[G]) \to C^{p-1} (\mathcal{A}, A[G]).$$

Since $A = \mathbb{C}[x_i]$, we have a simpler $A^e$-module resolution of $A$ via the Koszul resolution

$$\cdots \xrightarrow{\cdot b} \bigoplus_{i<j<k} e_i e_j e_k A^e \xrightarrow{\cdot b} \bigoplus_{i<j} e_i e_j A^e \xrightarrow{\cdot b} \bigoplus_{i} e_i A^e \xrightarrow{\cdot b} A^e \xrightarrow{m} A \to 0,$$

where $e^i$'s are odd variables: $\deg e_i = -1$ and $e^i e^j = -e^j e^i$. The Koszul differential $\tilde{b}$ is a derivation of the ring $A^e[e^i]$ generated by $\tilde{b}(e^i) = x_i \otimes 1 - 1 \otimes x_i \in A^e$.

Applying the functor $\text{Hom}_{A^e} (\cdot, A[G])$, we get the $G$-twisted Koszul cochain complex

$$\partial_K : K^* (A, A[G]) = A[e_i][G] \to K^{*+1} (A, A[G]). \quad (3.5)$$

Here $e_i$ is the dual basis of $e^i$ such that $\deg (e_i) = 1$, $e_i e_j = -e_j e_i$. The Koszul differential $\partial_H$ is

$$\partial_K (\phi g) = \sum_{i=1}^{N} (x_i - g x_i) e_i \phi g, \quad \phi \in A[e_i], g \in G. \quad (3.6)$$

**Definition 3.3.** We define the curving differential on $G$-twisted Koszul cochains

$$\tilde{d}_W : K^p (A, A[G]) \to K^{p-1} (A, A[G])$$

by

$$\tilde{d}_W (ae_{i_1} \cdots e_{i_p} g) = \sum_{k=1}^{p} (-1)^{k-1} a \rho_{i_k} (g) (\partial^0_{x_{i_k}} (W)) e_{i_1} \cdots \hat{e}_{i_k} \cdots e_{i_p} g. \quad (3.7)$$

Here $a \in A, g \in G$ and $1 \leq i_1 < \cdots < i_p \leq N$.

In [32], Shepler and Witherspoon introduced chain maps between these two resolutions,

$$\cdots \xrightarrow{\cdot b} \bigoplus_{i<j<k} e_i e_j e_k A^e \xrightarrow{\cdot b} \bigoplus_{i<j} e_i e_j A^e \xrightarrow{\cdot b} \bigoplus_{i} e_i A^e \xrightarrow{\cdot b} A^e \xrightarrow{m} A \to 0,$$

where $\Phi_1 : e^1 e^2 \cdots e^r A \otimes b \mapsto \sum_{\sigma \in S_p} (-1)^{\sigma} a \otimes x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(p)} \otimes b, \quad a, b \in A. \quad (3.8)
Let $\Phi^*_p$ denote the induced map on cochains
\[
\Phi^*_p : C^p(\mathcal{A}, A[G]) \rightarrow K^p(A, A[G]),
\]
(3.9)
\[
\phi \mapsto \sum_{i_1 < \cdots < i_p} \sum_{\sigma \in S_p} (-1)^{\sigma} \phi(x_{i_{\sigma(1)}} \otimes \cdots \otimes x_{i_{\sigma(p)}}) e_{i_1} \cdots e_{i_p}.
\]

- The chain map $\Upsilon$ from bar resolution to Koszul resolution is constructed in the following steps:

**Step 1** Let $a \otimes a_1 \otimes \cdots \otimes a_p \otimes b \in A \otimes \mathcal{A} \otimes \mathcal{A} \otimes A$ where $a_k = \prod_{i=1}^N x_i^{\gamma_i^k}, (\gamma_i^k \geq 0)$.

**Step 2** For each $1 \leq i_1 < i_2 < \cdots < i_p \leq N$ denoted by $I$, let us define $\mathcal{S}(I)$ to be the set of sequence $s = (s_1, s_2, \cdots s_p)$ such that
\[
0 \leq s_k < \gamma_i^k.
\]
Given $s \in \mathcal{S}(I)$, we define a splitting of each $a_k$ into two parts:
\[
\begin{align*}
\{ a_{k,1,I,s}^1 & = x_1^{\gamma_{i_1}} \cdots x_{i_{k-1}}^{\gamma_{i_{k-1}}} x_{i_k}, \\
\{ a_{k,2,I,s}^2 & = x_{i_k}^{\gamma_{i_k}} \cdots x_{i_p}^{\gamma_{i_p}} x_{i_{p+1}}^{\gamma_{i_{p+1}}} \cdots x_N^{\gamma_{i_1}}.
\end{align*}
\]

**Step 3** $\Upsilon$ is defined by
\[
\Upsilon_p(a \otimes a_1 \otimes \cdots \otimes a_p \otimes b) = \sum_{I,s \in \mathcal{S}(I)} (aa_1^1 \cdots a_p^1) \otimes (a_1^2 \cdots a_p^2) e_{i_1} e_{i_2} \cdots e_{i_p},
\]
(3.10)
where $a_k^1$ and $a_k^2$ are short for $a_{k,1,I,s}^1$ and $a_{k,2,I,s}^2$. The proof for $\Upsilon$ being a chain map can be found in the Appendix of [42].

Let $\Upsilon^*$ denote the induced map on cochains
\[
\Upsilon^*_p : K^p(A, A[G]) \rightarrow C^p(\mathcal{A}, A[G]).
\]
(3.11)

For $\psi = ae_{i_1} \cdots e_{i_p}g$ where $1 \leq i_1 < i_2 < \cdots < i_p \leq N$ denoted by $I, a \in A, g \in G$, we have
\[
\Upsilon_p(\psi)(a_1 \otimes \cdots \otimes a_p) = \psi(\Upsilon(1 \otimes a_1 \otimes \cdots \otimes a_p \otimes 1))
\]
\[
= \sum_{s \in \mathcal{S}(I)} a_1^1 a_2^1 \cdots a_p^1 a (\rho_{a_1} a_1^2) (\rho_{a_2} a_2^2) \cdots (\rho_{a_p} a_p^2) g
\]
\[
= \rho_{\psi}(g)(\partial_{x_1} \rho_{a_1}(g) \partial_{x_2} a_2(\partial_{x_{p+1}} a_p) \cdots \rho_{\psi}(g), \rho_{\psi}(g), \partial_{x_{p+1}} a_p) a g.
\]
(3.12)

**Lemma 3.1.** $\Upsilon^*$ is compatible with the curving differential
\[
d_W \circ \Upsilon^* = \Upsilon^* \circ d_W.
\]
(3.13)

**Proof.** It can be checked directly. $\square$

**Theorem 3.2** [43]. The composition
\[
\Upsilon \circ \Phi = \text{id},
\]
(3.14)

is the identity on Koszul resolution.
This theorem implies that the other composition \( \Phi \circ \Upsilon \) will be homotopic to the identity \( \text{id} \) on bar resolution. Let us describe such a homotopy \( H \) explicitly. It will allow us to compute various algebraic structures on Hochschild cohomology and orbifold Landau-Ginzburg models.

**Definition 3.4.** We define \( H_p : A \otimes \overline{A}^p \otimes A \to A \otimes \overline{A}^{(p+1)} \otimes A \) by

\[
H_p(a_0 \otimes a_1 \otimes \cdots \otimes a_p \otimes a_{p+1}) = \sum_{i=1}^{p+1} (-1)^i a_0 \otimes \cdots \otimes a_{i-1} \otimes \Phi_{p-i+1} \circ \Upsilon_{p-i+1}(1 \otimes a_i \otimes \cdots \otimes a_p \otimes a_{p+1}).
\]  

(3.15)

**Proposition 3.3.** \( H \) gives a homotopy between \( \text{id} \) and \( \Phi \circ \Upsilon \) on the bar resolution

\[
\text{id} - \Phi \circ \Upsilon = b \circ H + H \circ b.
\]  

(3.16)

**Proof.** Let \( s \) be the homotopy for bar resolution,

\[
\cdots b \xrightarrow{s} A \otimes \overline{A}^3 \otimes A \xrightarrow{b} A \otimes \overline{A}^2 \otimes A \xrightarrow{b} A \otimes \overline{A} \otimes A \xrightarrow{b} A \otimes A \xrightarrow{0}
\]

Dually, \( H \) induces a homotopy \( H^* \) between \( \text{id} \) and \( \Upsilon^* \circ \Phi^* \) on \( G \)-twisted cochains

\[
\text{id} - \Upsilon^* \circ \Phi^* = H^* \circ \partial_H + \partial_H \circ H^* : C^*(\overline{A}, A[G]) \to C^*(\overline{A}, A[G]).
\]  

(3.17)

**Proof.** Let \( s \) be the homotopy for bar resolution,

\[
\cdots b \xrightarrow{s} A \otimes \overline{A}^3 \otimes A \xrightarrow{b} A \otimes \overline{A}^2 \otimes A \xrightarrow{b} A \otimes \overline{A} \otimes A \xrightarrow{b} A \otimes A \xrightarrow{m_2} A \xrightarrow{0}
\]

where \( s(a_0 \otimes \cdots \otimes a_k) = 1 \otimes a_0 \otimes \cdots \otimes a_k \). It satisfies

\[
b \circ s + s \circ b = \text{id}.
\]

Using the fact that \( \Phi \circ \Upsilon \) commutes with \( b \), we have

\[
(b \circ H + H \circ b)(a_0 \otimes a_1 \otimes \cdots \otimes a_p \otimes a_{p+1}) = -\sum_{i=1}^{p+1} a_0 \otimes \cdots \otimes (a_{i-1} \Phi \circ \Upsilon(1 \otimes a_i \otimes \cdots \otimes a_p \otimes a_{p+1})) + a_0 \otimes \cdots \otimes a_p \otimes m_2(\Phi \circ \Upsilon(1 \otimes a_{p+1})) + \sum_{i=1}^{p} a_0 \otimes \cdots \otimes a_{p+1} - (\text{id} - \Phi \circ \Upsilon)(a_0 \otimes a_1 \otimes \cdots \otimes a_p \otimes a_{p+1}).
\]

The last statement follows from the fact that \( H \) is an \( A \)-bimodule homomorphism. \( \square \)

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Therefore we have a homotopy retraction between two cochains.

\[ H^* \xrightarrow{\Phi^*} (C^*(\overline{A}, A[G]), \partial_H) \xrightarrow{\Phi^*} (K^*(A, A[G]), \partial_K). \]

**Lemma 3.4.** The homotopy \( H^* \) satisfies

\[ H^* \circ \Upsilon^* = 0, \quad \Phi^* \circ H^* = 0, \quad \text{and} \quad H^* \circ \Phi^* = 0. \] (3.18)

In other words, the triple \((\Upsilon^*, \Phi^*, H^*)\) is a special homotopy retraction (see for example [10]).

**Proof.** We have

\[
\Upsilon_{p+1} \circ H_p(1 \otimes a_1 \otimes \cdots \otimes a_p \otimes 1)
= \sum_{i=1}^{p+1} (-1)^{i+\lceil \frac{p}{2} \rceil} \Upsilon_{p+1}(1 \otimes \cdots \otimes x_{\sigma(i-1)} \otimes \Phi_{p-i+1} \circ \Upsilon_{p-i+1}(1 \otimes x_{\sigma(i)} \otimes \cdots \otimes x_{\sigma(p)} \otimes 1))
= 0,
\]

where the last equality follows from (3.8) and the ordering (3.10).

\[
H \circ \Phi(1 \otimes 1 \otimes e^1 \otimes e^2 \cdots e^p)
= \sum_{\sigma \in S_p} \sum_{i=1}^{p+1} (-1)^{i+\lceil \frac{p}{2} \rceil} 1 \otimes \cdots \otimes x_{\sigma(i-1)} \otimes \Phi_{p-i+1} \circ \Upsilon_{p-i+1}(1 \otimes x_{\sigma(i)} \otimes \cdots \otimes x_{\sigma(p)} \otimes 1),
\]

which is zero as a reduced chain since \( \Phi_{p-i+1} \circ \Upsilon_{p-i+1}(1 \otimes x_{\sigma(i)} \otimes \cdots \otimes x_{\sigma(p)} \otimes 1) \) contributes 1 at some middle position. Similarly,

\[
H \circ H(1 \otimes x_{\sigma(i)} \otimes \cdots \otimes x_{\sigma(p)} \otimes 1)
= \sum_{i=1}^{p+1} (-1)^i (1 \otimes \cdots \otimes x_{\sigma(i-1)} \otimes \Phi_{p-i+1} \circ \Upsilon_{p-i+1}(1 \otimes x_{\sigma(i)} \otimes \cdots \otimes x_{\sigma(p)} \otimes 1))
\]

is zero as a reduced chain. \( \square \)

We will be interested in studying the \((\partial_H + d_W)\)-cohomology in terms of Koszul complex. Viewing \( d_W \) as a small perturbation, standard homological perturbation lemma (together with Lemma 3.1 and Lemma 3.4) allows us to construct a new homotopy retract \((\Upsilon^*, \Phi^*, H^*)\)

\[ \tilde{H}^* \xrightarrow{\Phi^*} (C^*(\overline{A}, A[G]), \partial_H + d_W) \xrightarrow{\Phi^*} (K^*(A, A[G]), \partial_K + \tilde{d}_W). \]

Here the perturbed homotopy \( \tilde{H}^* \) and retract \( \tilde{\Phi}^* \) are defined by

\[
\tilde{\Phi}^* = \Phi^* - \Phi^*(\text{id} + d_W H^*)^{-1} d_W H^*, \quad \tilde{H}^* = H^* (\text{id} + d_W H^*)^{-1},
\] (3.19)

where \((\text{id} + d_W H^*)^{-1} = \sum_{m \geq 0} (-1)^m (d_W H^*)^m\). It can be checked directly that

\[
\begin{aligned}
\tilde{\Phi}^* \circ \Upsilon^* &= \text{id} \\
\text{id} - \Upsilon^* \circ \tilde{\Phi}^* &= \tilde{H}^* \circ (\partial_H + d_W) + (\partial_H + d_W) \circ \tilde{H}^*.
\end{aligned}
\] (3.20)

We will need the following property to compute the cup product.
Lemma 3.5. For any $\phi \in C^p(\overline{A}, A[G])$ and $\psi \in C^q(\overline{A}, A[G])$, we have

$$H^*(\phi \cup \psi) = (-1)^p \phi \cup H^*(\psi) + H^*(\psi \cup (\mathcal{T}^* \circ \Phi^*(\psi))).$$

(3.21)

Proof. Let us introduce the notation $\cdots$ for the input such that

$$\phi(a_0 \otimes \cdots \otimes a_p) := \phi(a_0 \otimes \cdots \otimes a_{p-1})a_p, \quad \phi \in C^p(A, A[G]).$$

According to (3.15),

$$(-1)^p \phi \cup H^*(\psi)(a_1 \otimes \cdots \otimes a_p \otimes a_{p+1} \otimes \cdots \otimes a_{p+q-1})$$

$$= \sum_{i=1}^{q-1} (-1)^{p+i} \phi(a_1 \otimes \cdots \otimes a_p) \psi(a_{p+1} \otimes \cdots \otimes a_{p+i-1} \otimes \Phi \circ \mathcal{Y}(1 \otimes a_{p+i} \otimes \cdots \otimes a_{p+q-1} \otimes 1))$$

$$= \sum_{i=1}^{q-1} (-1)^{p+i} (\phi \cup \psi)(a_1 \otimes \cdots \otimes a_{p+i-1} \otimes \Phi_{q-i} \circ \mathcal{Y}_{q-i}(1 \otimes a_{p+i} \otimes \cdots \otimes a_{p+q-1} \otimes 1)),
$$

and

$$H^*(\psi \cup (\mathcal{T}^* \circ \Phi^*(\psi)))(a_1 \otimes \cdots \otimes a_p \otimes a_{p+1} \otimes \cdots \otimes a_{p+q-1})$$

$$= \sum_{i=1}^{p} (-1)^i (\phi \cup (\mathcal{T}^* \circ \Phi^*(\psi)))(a_1 \otimes \cdots \otimes a_{i-1} \otimes \Phi_{p+q-i} \circ \mathcal{Y}_{p+q-i}(1 \otimes a_i \otimes \cdots \otimes a_{p+1} \otimes \cdots \otimes a_{p+q-1} \otimes 1))$$

$$+ \sum_{i=1}^{q-1} (-1)^{p+i} \phi(a_1 \otimes \cdots \otimes a_p) \mathcal{T}^* \circ \Phi^*(\psi)(a_{p+1} \otimes \cdots \otimes a_{p+i-1} \otimes \Phi_{q-i} \circ \mathcal{Y}_{q-i}(1 \otimes a_{p+i} \otimes \cdots \otimes a_{p+q-1} \otimes 1))$$

$$= \sum_{i=1}^{p} (-1)^i (\phi \cup \psi)(a_1 \otimes \cdots \otimes a_{i-1} \otimes \Phi_{p+q-i} \circ \mathcal{Y}_{p+q-i}(1 \otimes a_i \otimes \cdots \otimes a_{p+1} \otimes \cdots)).$$

Here the last equality holds because the second summand vanishes (recall the definition of $\mathcal{T}$, we pick an increasing splitting for terms in a sequence and put all the latter halves into the first term to make a new sequence, in which we cannot find an new increasing splitting), and $\mathcal{T}^* \circ \Phi^*$ will do nothing in the first summand because they will acts on a sequence of terms like $x_{i_1} \otimes x_{i_2} \cdots$ \ with \ $i_1 < i_2 < \cdots$.

Sum them up and we will get $H^*(\phi \cup \psi)(a_1 \otimes \cdots \otimes a_{p+q-1}).$ \qed

Example 3.1. Consider the case $N = 2$ and write $x = x_1, y = x_2$. For $\Phi_k \in C^k(\overline{A}, A[G])$, we have

$$H^0(\phi_1) = 0,$$

$$H^1(\phi_2)(x^a y^b) = -\sum_{0 \leq s < a} \phi_2(x^s y^b \otimes x) x^{a-s-1} - \sum_{0 \leq t < b} \phi_2(y^t \otimes y) x^a y^{b-t-1},$$

$$H^2(\phi_3)(x^a y^b \otimes x^a y^b)$$

$$= -\sum_{0 \leq s < a_1, 0 \leq t < b_1} \phi_3(x_s y^{b_1+t} \otimes (x \otimes y - y \otimes x)) x^{a_1+a_2-s-1} y^{b_2-t-1}$$

$$+ \sum_{0 \leq s < a_2} \phi_3(x^{a_1} y^{b_1} \otimes x^s y^{b_2} \otimes x) x^{a_2-s-1}$$

$$+ \sum_{0 \leq t < b_2} \phi_3(x^{a_1} y^{b_1} \otimes y^t \otimes y) x^{a_2} y^{b_2-t-1}.$$
3.2 \( G \)-twisted cohomology

Let us now compute the cohomology

\[ \text{HH}^*_g(A_W, A_W[G]) \]

which is the state space of orbifold Landau-Ginzburg model.

**Definition 3.5.** Let \( g \in G \), \( \text{Fix}(g) \) be the fixed locus of \( g \). We let

\[ \text{PV}^*(\text{Fix}(g)) = A_g \otimes \wedge^* T\text{Fix}(g) \]  

\[ (3.22) \]

denote algebraic polyvector fields on \( \text{Fix}(g) \). Here \( T\text{Fix}(g) \) are algebraic vector fields on \( \text{Fix}(g) \).

As in the proof of Theorem 2.11 there is a spectral sequence converging to \( \text{HH}_*(A_W, A_W[G]) \) whose \( E_1 \) page is \( \text{HH}_*(A, A[G]) \). This can be computed by the Koszul resolution

\[ \text{HH}^*_g(A, A[G]) \simeq H(K^*(A, A[G]), \partial_K). \]

Using (3.6), we find (see also [35] and [23])

**Lemma 3.6.** There is an natural isomorphism between graded vector spaces:

\[ \Theta_g : \text{PV}^*(\text{Fix}(g))[N - N_g] \rightarrow \text{HH}^*_g(A, A_g) \simeq H(K^*(A, A_g), \partial_K) \]

\[ \partial_i = \partial/\partial x_i \text{ of degree } \deg \partial_i = 1. J_g = \{ i_1 < i_2 < \cdots < i_{N-N_g} \} \text{ is the moving index of } g \text{ and } e_{J_g} := e_{i_1} \cdots e_{i_{N-N_g}}. \]

Our assumption on \( W \) implies that \( W_g \) has an isolated singularity at the origin (see [15] for details). The complex in the \( E_2 \)-page is precisely the Koszul resolution for the critical locus of \( W_g \). Therefore the spectral sequence converges at \( E_2 \)-page whose cohomology in the \( g \)-sector is \( \text{Jac}(W_g)[N - N_g] \).

In conclusion, we have (see also Theorem 4.2. and Theorem 6.3. in [6])

**Theorem 3.7.** \( \Theta_g \) in Lemma 3.6 induces a natural isomorphism between \( \mathbb{Z}/2\mathbb{Z} \)-graded vector spaces

\[ \Theta_g : \text{Jac}(W_g)[N - N_g] \cong \text{HH}^*_g(A_W, A_W g). \]  

\[ (3.23) \]

**Example 3.2** (Fermat type). \( A = \mathbb{C}[x_1], W = x_1^g \). Let \( g \in G_W \) and \( g \neq e \). Then

\[ \text{HH}_*(A_W, A_W g) = \mathbb{C} e_1 g \]

is one-dimensional and has odd parity.

**Example 3.3** (Loop type). \( A = \mathbb{C}[x_1, \ldots, x_N], W = x_1^{n_1} x_2 + x_2^{n_2} x_3 + \cdots + x_N^{n_N} x_1. \) Let \( g \in G_W \) and \( g \neq e \). Assume \( g(x_i) = \lambda_i x_i \). Then \( \lambda_i \neq 1 \) for any \( i \). Therefore

\[ \text{HH}_*(A_W, A_W g) = \mathbb{C} e_1 e_2 \cdots e_N g \]

is one-dimensional and has the same parity as \( N \).

**Example 3.4** (Chain type). \( A = \mathbb{C}[x_1, \ldots, x_N], W = x_1^{n_1} x_2 + x_2^{n_2} x_3 + \cdots + x_N^{n_N} x_1. \) Let \( g \in G_W \) and \( g \neq e \). Assume \( g(x_i) = \lambda_i x_i \). There exists \( l_g \) such that \( \lambda_i \neq 1 \) for \( i \leq l_g \) and \( \lambda_i = 1 \) for \( i > l_g \). Hence

\[ \text{HH}_*(A_W, A_W g) = \text{Jac} \left( x_{l_g+1}^{n_{l_g+1}} x_{l_g+2} + \cdots + x_N^{n_N} x_N + x_N^{n_N} \right) e_1 \cdots e_{l_g} g, \]

whose parity is the same as \( l_g \).
3.3 $G$-Frobenius algebraic structure

**Definition 3.6** ([29][30]). A $G$-Frobenius algebra on $C$ consists of $(G, H, \cup, \eta, \rho, 1_e, \chi)$, where

1. $G$ is a finite group;
2. $H$ is a finite dimensional $\mathbb{Z}$ (or $\mathbb{Z}/2\mathbb{Z}$)-graded vector space with a sector decomposition
   \[ H = \bigoplus_{g \in G} H_g; \]  
   \[ (3.24) \]
3. $\chi \in \text{Hom}(G, C^*)$ is a character of $G$,
4. $\rho: G \rightarrow \text{Aut}(H)$ defines a $G$-action on $H$ satisfying
   i) $\rho(g): H_h \rightarrow H_{gh^{-1}}$;
   ii) $\rho(g): H_g \rightarrow H_g$ is the scalar multiplication by $\chi(g)^{-1}$;
5. $\eta$ is a non-degenerate bilinear form on $H$ satisfying
   i) for $\alpha_g \in H_g, \alpha_h \in H_h$, $\eta(\alpha_g, \alpha_h) = 0$ unless $gh = e$,
   ii) for $g \in G$ and $\alpha, \beta \in H$, 
   \[ \eta(\rho(g)\alpha, \rho(g)\beta) = \chi(g)^{-2}\eta(\alpha, \beta); \]  
   \[ (3.25) \]
6. $(H, \cup, 1_e)$ is an associative algebra with the identity $1_e$ satisfying
   i) $\cup$ is compatible with the sector decomposition,
   \[ \cup: H_g \times H_h \rightarrow H_{gh}; \]  
   ii) $\cup$ is $G$-equivariant and $1_e$ is $G$-invariant,
   iii) $\cup$ is twisted commutative,
   \[ \alpha_g \cup \rho(g^{-1})\alpha_h = (-1)^{||\alpha_g||\alpha_h} \alpha_h \cup \alpha_g, \quad \forall \alpha_g \in H_g, \alpha_h \in H_h, \]  
   \[ (3.26) \]
   iv) $\cup$ is compatible with $\eta$,
   \[ \eta(\alpha \cup \beta, \gamma) = \eta(\alpha, \beta \cup \gamma), \quad \forall \alpha, \beta, \gamma \in H, \]  
   \[ (3.27) \]
   v) the following projective trace axiom holds,
   \[ \chi(h)\text{Tr}_s(L_\alpha \rho(h)|_{H_e}) = \chi(g)^{-1}\text{Tr}_s(\rho(g^{-1})L_\alpha|_{H_h}), \quad \forall g, h \in G, \alpha \in H_{ghg^{-1}h^{-1}}, \]  
   \[ (3.28) \]
   where $L_\alpha$ is an operator on $H$ given by
   \[ L_\alpha(\beta) := \alpha \cup \beta, \]  
   \[ (3.29) \]
   and $\text{Tr}_s$ denotes the trace on $\mathbb{Z}$-graded vector spaces and denotes the super trace on $\mathbb{Z}/2\mathbb{Z}$-graded vector spaces.

**Definition 3.7.** A $G$-Frobenius algebra $H$ is called special if

- there exists generator $1_g \in H_g$ for any $g \in G$ such that $H_g = H_e \cup 1_g$,.
there exists $\rho_{g,h} \in \mathbb{C}^*$ for any $g, h \in G$ such that,

$$\rho(g)(1_h) = \rho_{g,h} 1_{ghg^{-1}}. \quad (3.30)$$

In this subsection we show that the state space of orbifold Landau-Ginzburg model

$$\text{HH}_c(A_W, A_W[G])$$

carries a natural structure of special $G$-Frobenius algebra. This will be constructed in several steps.

Firstly, we have the sector decomposition

$$\text{HH}_c(A_W, A_W[G]) = \bigoplus_{g \in G} H_g,$$

where $H_g = \text{HH}_c(A_W, A_W^g)$, together with compatible $G$-action and twisted commutative cup product $\cup$ by Theorem 2.7.

Define a character $\chi : G \to \mathbb{C}^*$ of $G$ by

$$\chi(g) = \det(g) = \lambda_{g_1} \lambda_{g_2} \cdots \lambda_{g_N}, \quad \text{if} \quad ^g x_i = \lambda_{g_i} x_i. \quad (3.31)$$

**Lemma 3.8.** For any $g \in G$ and $\alpha_g \in H_g$, we have

$$g^*(\alpha_g) = \chi(g)^{-1} \alpha_g. \quad (3.32)$$

**Proof.** This can be checked explicitly using Lemma 3.6.

Let us now construct the generator $1_g \in H_g$ for each $g \in G$. Define

$$1_g = \Theta_g(1) \in H_g \quad (3.33)$$

where $\Theta_g$ is defined in Theorem 3.7 and 1 is the identity in Jac($W_g$). The parity of $1_g$ is the same as $N - N_g$. It is easy to see that $1_e$ is the unit of $\text{HH}_c(A_W, A_W[G])$. For $g, h \in G$, define

$$\rho_{g,h} = \prod_{i \in I_h} (\lambda_{g_i}^q)^{-1}, \quad (3.34)$$

where $I_h$ is the moving index of $h$ and $\lambda_{g_i}^q$ is defined by $^g x_i = \lambda_{g_i}^q x_i$. It is checked that

$$g^*(1_h) = \rho_{g,h} 1_h = \rho_{g,h} 1_{ghg^{-1}}. \quad (3.35)$$

Let

$$\Pi_g : \text{Jac}(W) \to \text{Jac}(W_g) \quad (3.36)$$

denote the natural restriction map. This is well-defined since $W$ is $g$-invariant.

**Lemma 3.9.** For any $f \in \text{Jac}(W)$

$$[\Theta_g(f)] \cup 1_g = [\Theta_g(\Pi_g(f))]. \quad (3.37)$$

In particular, $H_g$ is a cyclic $H_e$-module generated by $1_g$. 

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Proof. Identity (3.37) follows by checking
\[
\Upsilon^*(\Theta_e(f)) \cup \Upsilon^*(\Theta_g(1)) = \Upsilon^*(\Theta_g(f)).
\]
The last statement follows from the surjectivity of \(\Pi_g\). \qed

We now construct a bilinear form \(\eta\) on \(HH_*(A_W, A_W[G])\). On the identity sector
\[
H_e \cong \text{Jac}(W),
\]
the pairing \(\eta_e\) is defined by the residue
\[
\eta_e([\Theta_e(f_1)], [\Theta_e(f_2)]) = \text{Res}_{CN} \left[ f_1 f_2 dx_1 \wedge dx_2 \wedge \cdots \wedge dx_N \right].
\]
(3.38)

Since \(W\) has an isolated singularity at the origin, the residue pairing on \(\text{Jac}(W)\) is non-degenerate. We extend \(\eta_e\) onto twisted sectors by defining
\[
\eta_g : H_g \otimes H_{g^{-1}} \to \mathbb{C}, \quad (\alpha, \beta) \mapsto \eta_e(\alpha \cup \beta, 1_e).
\]
(3.39)

Then the bilinear form \(\eta\) is defined by
\[
\eta = \sum_{g \in G} \eta_g.
\]
(3.40)

Lemma 3.10. The bilinear form \(\eta\) on \(HH_*(A_W, A_W[G])\) is compatible with the cup product
\[
\eta(\alpha \cup \beta, \gamma) = \eta(\alpha, \beta \cup \gamma), \quad \forall \alpha, \beta, \gamma \in H
\]
and satisfies the \(G\)-equivariance condition
\[
\eta(g^*(\alpha), g^*(\beta)) = \chi(g)^{-2} \eta(\alpha, \beta), \quad \forall \alpha, \beta \in H.
\]
(3.42)

Proof. By construction,
\[
\eta(\alpha \cup \beta, \gamma) = \eta(\alpha, \beta \cup \gamma) = \eta(\alpha \cup \beta \cup \gamma, 1_e).
\]
Equation (3.41) follows, Equation (3.42) follows from the \(G\)-equivariance of \(\cup\), the \(G\)-invariance of \(W\) and the property of residue. \qed

The discussion above is summarized as follows.

Theorem 3.11. Let \(W\) be a non-degenerate invertible polynomial. Then the state space \(HH_*(A_W, A_W[G])\) of orbifold Landau-Ginzburg model together with the \(G\)-action (2.16), cup product \(\cup\), bilinear form \(\eta\) (3.40), character \(\chi\) (3.31) and generators \(1_g\) (3.33) form a special \(\mathbb{Z}/2\mathbb{Z}\)-graded \(G\)-Frobenius algebra.

Proof. The proof of (3.28) is via direct calculation by Kaufmann in [29]. Hence, we only need to check the non-degeneracy of \(\eta\) on \(g\)-sectors where \(g \neq e\) for elementary invertible polynomials. If \(W\) is of Fermat type or loop type, \(HH_*(A_W, A_Wg)\) is one-dimensional (Example 3.2 and Example 3.3) and \(\eta(1_g, 1_g^{-1}) \neq 0\) by Theorem 3.20. If \(W\) is of chain type, it can be checked directly that \(\eta\) on \(HH_*(A_W, A_Wg)\) is proportional to the residue pairing on \(\text{Jac}(W_g)\) using Example 3.4, Lemma 3.9 and Theorem 3.20. \(\square\)
Remark. We expect that the pairing $\eta$ on $\text{HH}_c(A_W, A_W g)$ and $\text{HH}_c(A_W, A_W g^{-1})$ is the same as the residue pairing in $\text{Jac}(W_g)$ as long as $W_g$ has an isolated singularity. Alternatively, there is categorical construction of pairing (Mukai pairing) from the dg-category of matrix factorization\cite{5, 44, 45}. The Mukai pairing of G-equivariant case is explicitly computed in \cite{37}. If one uses the trivial volume form to identity Hochschild homology with cohomology, then our pairing here coincides with their result (see Theorem 4.2.1 in \cite{37} and Theorem 3.20 below). The non-degeneracy of Mukai pairing is proved by Shklyarov in \cite{44} for all homological smooth dg-algebra.

3.4 Quantum differential operator and cup product

In this section we establish an explicit formula for cup product of $G$-twisted Hochschild cohomology. This is achieved via the following established quasi-isomorphisms (formula (3.20) and Theorem 3.7)

\[ (C^\bullet(\overline{A}, Ag), \partial_H + d_W) \xrightarrow{\delta^*} (K^\bullet(A, Ag), \partial_{Kc} + \tilde{d}_W) \xrightarrow{p} (\text{Jac}(W_g)[N - N_g], 0). \]

Here $p$ is the natural projection onto the appropriate components of polyvectors.

Given $\alpha_g \in \text{Jac}(W_g)[N - N_g], \beta_h \in \text{Jac}(W_h)[N - N_h]$, their cup product in the $gh$-sector can be computed by

\[ p \Phi^* \left( (\Upsilon^* i(\alpha_g)) \cup (\Upsilon^* i(\beta_h)) \right). \quad (3.43) \]

For invertible polynomials, it turns out that $\cup$ on cohomology is determined by the cup product between $g$-sector and $g^{-1}$-sector. In this case $(\Upsilon^* i(\alpha_g)) \cup (\Upsilon^* i(\beta_h))$ is a cocycle in the identity e-sector. It is easy to see that formula (3.43) is reduced to the following

**Lemma 3.12.** For a $(\partial_H + d_W)$-closed element in the identity e-sector

\[ \phi = \sum_{k=0}^{l} \phi_{2k}, \quad \phi_{2k} \in C^{2k}(\overline{A}, Ae), \]

we have

\[ p \circ \Phi^* (\phi) = \left[ \sum_{k=0}^{l} (-1)^k (d_W H^*)^k \phi_{2k} \right]. \quad (3.44) \]

Here $[ \ ]$ on the right hand side represents its class in $\text{Jac}(W)$.

We first establish some properties of the cup product $\cup$ and the homotopy operator $H^*$.

**Definition 3.8.** Given $g = (q_1, \cdots, q_N) \in (C^*)^N$, we define the decomposition

\[ g = \prod_{i=1}^{N} g^{(i)}, \]

where $g^{(i)} = (1, \cdots, q_i, \cdots, 1)$ is called the $i$'th component of $g$. We define $\tilde{G}$ to be the group generated by $g^{(i)}$'s for all $g \in G$. 

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It would be convenient to extend our Hochschild cochains to $C^\bullet(A, A[\hat{G}])$ and identify $C^\bullet(A, A[G])$ as a sub-algebra under the natural embedding

$$C^\bullet(A, A[G]) \hookrightarrow C^\bullet(A, A[\hat{G}]).$$

We will also simply write $\phi_1 \phi_2$ for the cup product $\phi_1 \cup \phi_2$ in this subsection.

Recall the quantum differential operator $\partial^g_\epsilon$, defined by (3.2).

**Definition 3.9.** For $g \in G$, we define the first order quantum differential operators $\partial^g_\epsilon \in C^1(A, A[\hat{G}])$ by

$$\partial^g_\epsilon : f \rightarrow \partial^g_\epsilon(f)g^{(i)},$$

and the second order quantum differential operators $\partial^{g,h}_{\epsilon,\epsilon} \in C^1(A, A[\hat{G}])$ by

$$\partial^{g,h}_{\epsilon,\epsilon} = \begin{cases} \partial^g_{\epsilon} \{ \partial^h_{\epsilon} \} & i \neq j \text{ (mixed type)}, \\ -H^* (\partial^g_{\epsilon} \partial^h_{\epsilon}) & i = j \text{ (pure type)}. \end{cases}$$

Here $H^*$ is the homotopy on $G$-twisted Hochschild cochains defined in Proposition 3.8.

**Example 3.5.** Let $g_{x_i} = \epsilon_1 x_i$ and $h_{x_i} = \epsilon_2 x_i$. Recall definition (3.3). We have

$$\partial^{g,h}_{\epsilon,\epsilon} (x^n_\epsilon) = \frac{\epsilon_1^{n-1} [n]_{\epsilon_2} - [n]_{\epsilon_1,\epsilon_2}}{\epsilon_1 - 1} x^{n-2}_\epsilon g^{(i)} h^{(j)}.$$  

In particular, if $\epsilon_1 \epsilon_2 = 1$, i.e., $g^{(i)} = (h^{(i)})^{-1}$,

$$\partial^{g,h}_{\epsilon,\epsilon} (x^n_\epsilon) = \frac{[n]_{\epsilon_1} - n}{\epsilon_1 - 1} x^{n-2}_\epsilon e.$$  

Using quantum differential operators, the cochain map (3.12)

$$\Upsilon^* : K^\bullet(A, A[G]) \rightarrow C^\bullet(\hat{A}, A[G])$$

can be expressed in terms of cup product

$$\Upsilon^* (e_{i_1} \cdots e_{i_p} g) = g^{(i_1)} \cdots g^{(i_{p-1})} \partial^{g}_{\epsilon_1} g^{(i_p+1)} \cdots g^{(i_2-1)} \partial^{g}_{\epsilon_2} g^{(i_1+1)} \cdots g^{(i_1-1)} \partial^{g}_{\epsilon_2} g^{(i_2+1)} \cdots g^{(i_p-1)} \partial^{g}_{\epsilon_1} g^{(i_p+1)} \cdots g^{(N)},$$

where $1 \leq i_1 < i_2 < \cdots < i_p \leq N$, and $g^{(i)}$ is viewed naturally as an element in $C^0(\hat{A}, A[g^{(i)}])$.

**Lemma 3.13.**

(a) First order quantum differential operators are $\partial_H$-closed,

$$\partial_H \partial^g_\epsilon = 0.$$  

(b) Second order quantum differential operators are symmetric

$$\partial^{g,h}_{\epsilon_1,\epsilon_2} = \partial^{h,g}_{\epsilon_2,\epsilon_1},$$

and satisfy

$$\partial_H \partial^{g,h}_{\epsilon,\epsilon} = \begin{cases} -\partial^g_{\epsilon} \partial^h_{\epsilon} - \partial^h_{\epsilon} \partial^g_{\epsilon} & i \neq j \\ -\partial^g_{\epsilon} \partial^h_{\epsilon} & i = j. \end{cases}$$
Proof. (a) is equivalent to the twisted Leibniz rule
\[ \partial_i^g(ab) = \partial_i^g(a)b + a\partial_i^g(b) = \partial_i^g(a)(g^i(b)) + a\partial_i^g(b)g^i, \quad \forall a, b \in A. \]

The first equation in (b) is by direct check. The second equation follows from (2.21), (3.50) and (3.17). □

The following lemma gives some useful reorganizing rules for the homotopy $H^*$.

**Lemma 3.14.** Let $I = \{i_1, i_2, \cdots, i_p\}, \quad i_2 < i_3 < \cdots < i_p$ and $i_1 < i_k, k \leq i_k+1$. Then we have the following formula for the homotopy operator
\[ H^*(\partial_{i_1}^{g_1}, \partial_{i_2}^{g_2}, \cdots, \partial_{i_p}^{g_p}) = -\partial_{i_1}^{g_1} \partial_{i_2}^{g_2} \cdots \partial_{i_p}^{g_p} - \partial_{i_2}^{g_2} \partial_{i_1}^{g_1} \partial_{i_3}^{g_3} \cdots \partial_{i_p}^{g_p} \]
\[ = \cdots - \partial_{i_2}^{g_2} \cdots \partial_{i_k-1}^{g_{k-1}} \partial_{i_1}^{g_1} \partial_{i_k}^{g_k} \cdots \partial_{i_p}^{g_p}. \] (3.53)

More generally, if on the left hand side we insert several 0-cochains $h_1^{(j_1)}, \cdots, h_q^{(j_q)}$ into the product $\partial_{i_1}^{g_1} \partial_{i_2}^{g_2} \cdots \partial_{i_p}^{g_p}$ at positions after $\partial_{i_1}^{g_1}$, then on the right hand side each term is modified by inserting $h_1^{(j_1)}, \cdots, h_q^{(j_q)}$ in ascending order as follows: if $i_k < j_s < i_{k+1}(k \geq 2)$, then $h_1^{(j_1)}$ is inserted between quantum differential operators indexed by $i_k$ and $i_{k+1}$; if $i_k = j_s < i_{k+1}$, then we combine $h_s^{(j_s)}$ and $\partial_{i_k}^{g_k}$ into $\partial_{i_k}^{h_s^{(j_s)}}$.

**Proof.** Under the assumption $i_2 < i_3 < \cdots < i_p$, we have only one term survive in (3.15).
\[ H^*(\partial_{i_1}^{g_1}, \partial_{i_2}^{g_2}, \cdots, \partial_{i_p}^{g_p})(a_1 \otimes a_2 \otimes \cdots \otimes a_p-1) = -\partial_{i_1}^{g_1} \partial_{i_2}^{g_2} \cdots \partial_{i_p}^{g_p}(\Phi \circ \Upsilon(1 \otimes a_1 \otimes \cdots \otimes a_p-1)). \]

Here the notation $(\cdots)$ means that
\[ \phi(a_0 \otimes \cdots \otimes a_p) := \phi(a_0 \otimes \cdots \otimes a_p) \phi, \quad \phi \in C^0(A, A[G]). \]

Let $J = \{i_2, \cdots, i_{p-1}\}, g = g_1^{(i_1)} \cdots g_p^{(i_p)}$ and we keep the notations in (3.10). Then
\[ H^*(\partial_{i_1}^{g_1}, \partial_{i_2}^{g_2}, \cdots, \partial_{i_p}^{g_p})(a_1 \otimes a_2 \otimes \cdots \otimes a_p-1) \]
\[ = - \sum_{s \in S(J)} \partial_{x_{i_1}}^{g_1} \partial_{x_{i_2}}^{g_2} \cdots \partial_{x_{i_p}}^{g_p} \left(a_1 \cdots a_{p-1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_p} \otimes a_1^2 \cdots a_p^2 \right) \]
\[ = - \sum_{s \in S(J)} \partial_{x_{i_1}}^{g_1} (a_1 \cdots a_{p-1}) (a_1^2 \cdots a_p^2) g \]
\[ = - \sum_{j=1}^{k-1} \partial_{i_2}^{g_2} \cdots \partial_{i_j}^{g_j} \partial_{i_{j+1}}^{g_{j+1}} \cdots \partial_{i_p}^{g_p} (a_1 \otimes a_2 \otimes \cdots \otimes a_p-1). \]

The general case with 0-cochain insertions is proved similarly. □

The next lemma gives certain vanishing conditions for $H^*$ on twisted cochains.

**Lemma 3.15.** Let $\phi \in C^*(A, A[G])$ be a $G$-twisted cochain expressed in terms of cup product of first and second order quantum differential operators and elements of $G$ (viewed as 0-th cochain). Then $H^*\phi = 0$ in the following cases:

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(a) \( \phi = \cdots \partial_{\theta_j, \theta} \) has a second order quantum differential operator appearing at the last position;

(b) \( \phi \) is of the form of arbitrary insertion of elements of \( \hat{G} \) into the expression

\[ \cdots \partial_{\theta_{i_k}, \theta} \partial_{\theta_{i_{k+1}}} \cdots \partial_{\theta_{i_{p-1}}} \partial_{\theta} \]

with \( i_k < i_{k+1} < \cdots < i_p \) or \( i_k' < i_{k+1} < \cdots < i_p' \);

(c) \( \phi \) is of the form of arbitrary insertion of elements of \( \hat{G} \) into the expression

\[ \partial_{\theta_{i_1}} \cdots \partial_{\theta_{i_{p-1}}} \partial_{\theta} \]

with \( i_1 < i_2 < \cdots < i_p \).

Proof. They are checked directly by (3.10) and (3.15).

We are ready to compute cup product in terms of (3.43) and (3.19). We need to understand how \( \Phi^* = \Phi^* (\text{id} + d_W H^*)^{-1} d_W H^* \) acts on cochains of quantum differential operators.

**Lemma 3.16.** Consider the following cup product of cochains of quantum differential operators

\[ \phi = a_1 g^{(1)} \cdots g^{(i_1-1)} \partial_{\theta} g^{(i_1)} \cdots \partial_{\theta} g^{(i_{p-1})} \cdots g^{(N)} \cup a_2 h^{(1)} \cdots h^{(j_1)} \partial_{\theta} h^{(j_1+1)} \cdots \partial_{\theta} h^{(j_{q-1})} \cdots h^{(N)} \]

where \( i_1 < i_2 < \cdots < i_p \) and \( j_1 < j_2 < \cdots < j_q \). \( a_1, a_2 \in A, g, h \in G \). Then

\[ H^*(\phi) = \sum_{p', j_k} \pm a_1 a_2 g^{(1)} \cdots g^{(i_1-1)} \partial_{\theta} g^{(i_1)} \cdots \partial_{\theta} g^{(i_{p'-1})} \cdots g^{(i_{p'-(p'+1)})-1} \cup \tilde{h}^{(1)} \cdots \partial_{\theta}^{j_k} h^{(j_k+1)} \cdots \partial_{\theta} h^{(j_{q-1})} \cdots h^{(N)} \] (3.54)

Here the summation is over all \( p' < p, j_k \) such that \( i_{(p'+1)} \geq j_k \) and \( i_{(p'+1)} \neq j_k \) for all \( p' \geq p' + 1, 1 \leq l \leq q \).

\[ \tilde{h}^{(j)} = \begin{cases} h^{(j)} & \text{if } 1 \leq j \leq i_{(p'+1)}, \\ h^{(j)} g^{(j)} & \text{if } i_{(p'+1)} < j \leq N. \end{cases} \] (3.55)

Proof. Assume \( j_1 \leq i_\tilde{p} < j_{1+1} \) for some \( 1 \leq l < q \) or \( j_q \leq i_\tilde{p} \). If not, we consider \( i_{p-1} \) and so on. Let

\[ \alpha = a_1 g^{(1)} \cdots g^{(i_1-1)} \partial_{\theta} g^{(i_1)} \cdots g^{(i_{p-1})} \]

\[ \beta = \partial_{\theta}^{i_\tilde{p}} g^{(i_{p'-1})} \cdots g^{(N)} \cup a_2 h^{(1)} \cdots h^{(j_1)} \partial_{\theta} h^{(j_1+1)} \cdots \partial_{\theta} h^{(j_{q-1})} \cdots h^{(N)}. \]

We apply Lemma 3.5 to \( \phi = \alpha \cup \beta \)

\[ H^*(\phi) = \pm \alpha \cup H^*(\beta) + H^*(\alpha \cup (\Upsilon^* \circ \Phi^*(\beta))). \]

The first term is computed by Lemma 3.14. In the second term, \( \Upsilon^* \circ \Phi^*(\beta) = 0 \) if \( i_\tilde{p} = j_1 \); otherwise the role of \( \Upsilon^* \circ \Phi^* \) amounts to change terms in \( \beta \) into the order

\[ \Upsilon^* \circ \Phi^*(\beta) = \pm \cdots \partial_{\theta}^{j_1} \cdots \partial_{\theta} h^{(j_1+1)} \cdots \partial_{\theta} h^{(j_{q-1})} \cdots \partial_{\theta} h^{(N)}. \]

Now we apply the same process to \( H^* (\alpha \cup (\Upsilon^* \circ \Phi^*(\beta))) \) but consider \( i_{p-1} \). Recursively we arrive at the lemma.

\[ \square \]
Theorem 3.17. Consider the cup product between a $q$-sector and $g^{-1}$-sector

$$
\phi = \Upsilon^*(c_{i_1} \cdots c_{i_p}, g) \cup \Upsilon^*(c_{j_1} \cdots c_{j_q}, g^{-1})
$$

where $i_1 < i_2 < \cdots < i_p$ and $j_1 < j_2 < \cdots < j_q$. $a_1, a_2 \in A$. Then $p \circ \tilde{\Phi}^*(\phi)$ is computed as follows.

1. if $p \neq q$, then $p \circ \tilde{\Phi}^*(\phi) = 0$.
2. if $p = q$, consider the following subset of the $p$-th permutation group

$$
V = \{ \sigma \in S_p \mid j_{\sigma(k)} \leq i_k, \forall 1 \leq k \leq p \}.
$$

Let $g^{(i,j)} = g^{(i)}g^{(i+1)} \cdots g^{(j)}$ for $i \leq j$. Then $p \circ \tilde{\Phi}^*(\phi)$ equals

$$
(-1)^{\frac{p(p-1)}{2}} \sum_{\sigma \in V} \text{sgn}(\sigma)g^{(j_{\sigma(1)}, i_1-1)}\partial^{q_1}g^{-1}_{j_1, j_{\sigma(1)}}(W)(g^{-1}(j_{\sigma(1)}+1, j_1) \cdots g^{(j_{\sigma(p)}, i_p-1)}\partial^{q_p}g^{-1}_{i_p, j_{\sigma(p)}}(W)(g^{-1}(j_{\sigma(p)}+1, i_p))
$$

as elements in $\text{Jac}(W)$.

Proof. We consider the case $p = q$. The case for $p \neq q$ follows by the same steps. Explicitly,

$$
\phi = a_1 g^{(1)} \cdots g^{(i-1)}\partial^{q_1}g^{(i+1)} \cdots \partial^{q_p}g^{(i_p+1)} \cdots g^{(N)}
\cup a_2 (g^{-1})^{(1)} \cdots (g^{-1})^{(i-1)}\partial^{q_1}g^{-1}_{j_1} \cdots \partial^{q_p}g^{-1}_{j_p} \cdots (g^{-1})^{(N)}
$$

and by (3.44)

$$
p \circ \tilde{\Phi}^*(\phi) = (-1)^p(d_W)^p \phi.
$$

$H^*(\phi)$ is computed by (3.55). We consider the action of $d_W$ on each term in (3.55). There are two cases:

Case 1 $W$ is acted on by some first order quantum differential operators. In this case, there will be a second order quantum differential operator left in the cochain. If we take a further homotopy $H^*$, it will become zero by Lemma (3.15) (a), (b).

Case 2 $W$ is acted on by some second order quantum differential operators. In this case, a further homotopy $H^*$ will bring the cochain back to the form of (3.53) by Lemma (3.14). The relevant second order differential operator is combined by $\partial^{q}_{i(p+1)}$ and $\partial^{q}_{j_k}$, where $i_{(p+1)} \geq j_k$.

Repeating the above steps we arrive at the theorem.

Example 3.6. For $g^{(1)}\partial^{q_{2}}g^{(3)}\partial^{q_{4}}h_{[2]}\partial^{q_{3}}\partial^{h}h_{[5]}$ with $g = h^{-1}$, there are two relevant index

$$
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
\end{array}
$$

, and

$$
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
\end{array}
$$

The first one contributes

$$
-g^{(1)}\partial^{q_{2}}g^{-1}_{2,1}(W)(g^{-1})^{(2)}g^{(3)}\partial^{q_{4}}g^{-1}_{4,3}(W)\partial^{q_{3}}g^{-1}_{6,4}(W)(g^{-1})^{(5)}.
$$

The second one contributes

$$
g^{(1)}\partial^{q_{2}}g^{-1}_{2,1}(W)(g^{-1})^{(2)}\partial^{q_{4}}g^{-1}_{4,3}(W)g^{(3)}g^{(4)}\partial^{q_{3}}g^{-1}_{6,3}(W)(g^{-1})^{(4)}(g^{-1})^{(5)}.
$$
3.5 Invertible polynomials and cup product formula

As an application of the method we have developed, we give explicit cup product formula for a general invertible polynomial $W$ and a finite group $G \subseteq G_W$. We can write $W = W_1 \oplus W_2 \oplus \cdots \oplus W_r$ and $G = G_1 \times G_2 \times \cdots \times G_r$, where $W_k$ is an elementary invertible polynomial with dimension $N_k$ and $G_k$ is a finite subgroup of $G_{W_k}$. Also $A = A_1 \otimes \cdots \otimes A_r$ is a tensor product of elementary ones.

First of all, we have the following K"unneth type formula.

Proposition 3.18. As $\mathbb{Z}/2\mathbb{Z}$-graded algebras

$$\text{HH}_c(A_W, A_W[G]) = \text{HH}_c((A_1)_{W_1}, (A_1)_{W_1}[G_1]) \otimes \cdots \otimes \text{HH}_c((A_r)_{W_r}, (A_r)_{W_r}[G_r])$$

where the tensor product on the right hand side is the graded tensor product.

Proof. This is a direct consequence of Theorem 3.7 and the fact that

$$\text{Jac}(W) = \text{Jac}(W_1) \otimes \cdots \otimes \text{Jac}(W_r).$$

This proposition reduces the problem to an elementary invertible polynomial only. Let $1_g \in \text{HH}_c(A_W, A_W g)$ be the generator defined in (3.33).

Lemma 3.19. The cup product on $\text{HH}_c^*(A_W, A_W [G])$ is completely determined by the generators

$$1_g \cup 1_h, \quad g, h \in G.$$

Proof. This follows from Lemma 3.9

Definition 3.10. Let $W(x_i)$ be an elementary invertible polynomial. Let $g \neq e \in G_W$, $g(x_i) = \lambda_i x_i$ and $l_g = \sharp \{\lambda_i | \lambda_i \neq 1\}$. We define the $g$-twisted Hessian, denoted by $\text{Hess}^g(W)$, as follows.

(a) If $W = x_1^n$ is of Fermat type, then

$$\text{Hess}^g(W) = \frac{n}{1 - \lambda_1} x_1^{n-2}.$$  

(b) If $W = x_1^{n_1} x_2 + x_2^{n_2} x_3 + \cdots + x_N^{n_N}$ $x_1$ is of loop type, then

$$\text{Hess}^g(W) = \frac{(-1)^{N+1} + n_1 n_2 \cdots n_N}{(1 - \lambda_1)(1 - \lambda_2) \cdots (1 - \lambda_N)} x_1^{n_1-1} x_2^{n_2-1} \cdots x_N^{n_N-1}.$$  

Note that $l_g = N$ for any $g \neq e$ in the loop case.

(c) If $W = x_1^{n_1} x_2 + x_2^{n_2} x_3 + \cdots + x_N^{n_N}$ is of chain type, then

$$\text{Hess}^g(W) = \begin{cases} \frac{n_1 n_2 \cdots n_N}{(1 - \lambda_1)(1 - \lambda_2) \cdots (1 - \lambda_N)} x_1^{n_1-2} x_2^{n_2-1} \cdots x_{l_g}^{n_{l_g}-1} x_{l_g+1} & \text{if } l_g < N \\ \frac{n_1 n_2 \cdots n_N}{(1 - \lambda_1)(1 - \lambda_2) \cdots (1 - \lambda_N)} x_1^{n_1-2} x_2^{n_2-1} \cdots x_{N}^{n_N-1} & \text{if } l_g = N \end{cases}$$

Note that $\lambda_{l_g+1} = \cdots = \lambda_N = 1$ for any $g \neq e$ in the chain case.
The rest of this paper is devoted to prove part (2) of this theorem.

**Remark.** We follow the argument in [3] in the proof of part (1) above.
Fermat type

\[ W = x_1^3 \text{ and } G = \langle \sigma \rangle \cong \mathbb{Z}/n\mathbb{Z}, \text{ where } n \geq 2 \text{ and } \sigma = \exp(\frac{2\pi i}{n}). \]

For \( 1 \leq k < n \), let us first find the representative of \( 1_{g^k} \) in Koszul cochains. This amounts to extend \( e_1 \sigma^k \) into a cochain annihilated by \( \partial_K + d_W \). It is easy to see that \( d_W(e_1 \sigma^k) = 0 \), hence

\[ 1_{g^k} = e_1 \sigma^k \]

already does the job. Since \( \Upsilon^*(e_1 \sigma^k) = \partial^k \), we find

\[ 1_{g^k} \cup 1_{g^{-k}} = p \Phi^*(\partial^k \partial^{-k}) \quad \text{Thm} \quad 3.17 \quad \partial_{q, \sigma^{-k}}(W) \quad \text{Ex} \quad 3.35 \quad n \equiv \sigma^k x_1^{n-2}. \]

This proves part (2) of Theorem 3.20 in the Fermat case.

Loop type

\[ W = x_1^{n_1}x_2 + x_2^{n_2}x_3 + \cdots + x_m^{n_m}x_1. \]

Let us first assume \( N \geq 3 \).

Let \( g = \text{diag}(\lambda_1, \ldots, \lambda_N) \in G_W \), \( \lambda_i = e^{2\pi i / n} \), where

\[ n_i q_i + s_{i+1} \in \mathbb{Z}, \quad \forall 1 \leq i \leq N, \quad \text{where } q_{N+1} \equiv q_i. \]

We require \( g \neq e \), then \( 0 < q_i < 1 \) for any \( i \). Let us first find the representative of \( 1_g \) in Koszul cochains. This amounts to extend \( \kappa_0 = e_1 e_2 \cdots e_N g \) into \( \kappa_0 + \kappa_2 + \cdots \) where \( \kappa_{2k} \in R^{2k}(A, Ag) \) satisfying

\[ \partial_K(\kappa_{2k-2}) + d_W(\kappa_{2k}) = 0. \]

Then \( \lambda_g = \sum_{0 \leq 2k \leq N} \kappa_{2k} \).

For \( I = \{i_1 < i_2 < \cdots < i_n\} \), let us denote \( e_I = e_{i_1} \cdots e_{i_n} \). In the loop case, we have

\[ d_W(\lambda_2 I g) = \sum_{k=1}^{p} (-1)^{k-1} \Delta(p^g_k) e_{I \setminus \{i_k\}} g, \quad (3.59) \]

where

\[ p^g_i = \begin{cases} x_N^{n_i} + [n_i]_1 x_1^{n_1-1} & \text{if } i = 1, \\ \lambda_1^{n_i-1} x_{i-1}^{n_{i-1}} + [n_i]_1 x_i^{n_i-1} & \text{if } 1 < i < N, \\ \lambda_N^{n_N-1} x_{N-1}^{n_{N-1}} + [n_N]_1 x_N^{n_N-1} & \text{if } i = N. \end{cases} \quad (3.60) \]

Let us consider the following index set

\[ B_s := \{i_1 < i_2 < \cdots < i_s \mid i_{k+1} - i_k > 1, \text{ for } 1 \leq k < s, \text{ and } i_1 + N - i_s > 1\}, \]

and introduce the notation

\[ b^g_i := \begin{cases} \lambda_i^{n_i} x_i^{n_i-1} & \text{if } 1 \leq i < N, \\ (-1)^{N-1} & \text{if } i = N. \end{cases} \quad (3.61) \]

For each \( b = \{i_1 < i_2 < \cdots < i_s\} \in B_s \), define

\[ I_b := \begin{cases} \{i_1, i_1 + 1, i_2, i_2 + 1, \cdots, i_s, i_s + 1\} & \text{while } i_s \neq N, \\ \{1, i_1, i_1 + 1, i_2, i_2 + 1, \cdots, i_s\} & \text{while } i_s = N. \end{cases} \quad (3.62) \]
Lemma 3.21. Let $I_g = \{1, 2, \cdots, N\}$ and define
\[
\kappa^g_{2s} := \sum_{b \in B_s} \left( \prod_{i \in b} b^g_i \right) e_{I_b \setminus I_s} g.
\]
Then
\[
\sum_{N \geq 2s \geq 0} \kappa^g_{2s}
\]
gives a Koszul representative for $1_g$.

Proof. For convenience, we will identify index $i$ with $i + N$. For ever $p_i^g$, we denote
\[
p_i^g = (p_i^g)_1 + (p_i^g)_2, \quad \text{where} \quad (p_i^g)_1 \propto x_i, \quad (p_i^g)_2 \propto x_i^{n_{\nu_i}+1}.\]
For any index set $I \subset I_g$ with $|I| = 2s + 1$, denote
\[
\mathcal{H}_I = \{(b, j) \mid b \in B_s, j \notin I_b \text{ and } I_b \cup \{j\} = I\}.
\]
For $I$ such that $\mathcal{H}_I \neq \emptyset$, let $I_0 = \{i_1, i_2 + 1, \cdots, i_s + 2 \ell, I \} \subseteq I$, such that $i_I - 1, i_I + 2 \ell I + 1 \notin I$. Then
\[
\mathcal{H}_I = \{((b(\ell), j(\ell)))_{\ell=0}^{t_I}, \quad \text{where} \quad j(\ell) = i_I + 2 \ell.\}
\]
Note that for $0 \leq \ell \leq \ell - 1$,
\[
(p^g_{j(\ell)})_2 b^g_{j(\ell)+1} + b^g_{j(\ell)} (p^g_{j(\ell+1)})_1 = 0.
\]
Therefore
\[
\tilde{d}_W \left( \kappa^g_{2s} \right) = \sum_{b \in B_s, j_b \notin I_b} \left( \prod_{i \in b} b^g_i \right) e_{I_b \setminus (I_b \cup \{j_b\})} g
\]
\[
= \sum_{|I| = 2s + 1} \sum_{(b, j) \in \mathcal{H}_I} (-1)^{\ell - 1} p^g_{j_I} \left( \prod_{i \in b} b^g_i \right) e_{I_b \setminus I} g
\]
\[
= \sum_{|I| = 2s + 1} \sum_{\ell=0}^{t_I} (-1)^{\ell - 1} p^g_{j(\ell)} \left( (p^g_{j(\ell)})_1 + (p^g_{j(\ell)})_2 \right) \left( \prod_{i \in b(\ell)} b^g_i \right) e_{I_b \setminus I} g.
\]
where $\text{sgn}(I) = \sharp \{(1, 2, \cdots, i_I - 1) \setminus I\}$. Observe
\[
(p^g_{i_I+1})_1 = (1 - \lambda_{i_I}) x_i b^g_i, \quad - (p^g_{i_I})_2 = (1 - \lambda_{i_I+1}) x_i b^g_i,
\]
we have
\[
- \partial_K \left( \kappa^g_{2s-2} \right) = \sum_{|I| = 2s + 1} (-1)^{\text{sgn}(I)} \left( (1 - \lambda_{i-I-1}) x_{i-I-1} \prod_{i \in b(I)^+} b^g_i \right.
\]
\[
- (1 - \lambda_{i_I+2 \ell I + 1}) x_{i_I+2 \ell I + 1} \prod_{i \in b(I)^-} b^g_i \left) e_{I_b \setminus I} g \right.
\]
where $I_b(I)^+ = I \cup \{i_I - 1\}$ and $I_b(I)^- = I \cup \{i_I + 2 \ell I + 1\}$. We find
\[
\partial_K \left( \kappa^g_{2s-2} \right) + \tilde{d}_W \left( \kappa^g_{2s} \right) = 0.
\]
Now we can compute $1_g \cup 1_g^{-1}$ by

$$p \circ \Phi^*\left(\mathcal{T}^* (\kappa_0^g + \kappa_{-2}^g + \cdots) \cup \mathcal{T}^* (\kappa_0^{g^{-1}} + \kappa_{-2}^{g^{-1}} + \cdots)\right).$$

(3.64)

The following Lemma is a direct consequence of Theorem 3.17 and Lemma 3.21.

**Lemma 3.22.**

$$1_g \cup 1_g^{-1} = \sum_{\Gamma} \text{Val}(\Gamma)$$

is a sum of ‘values’ for all graphs $\Gamma$ with

(a) a set of vertices $V(\Gamma)$ indexed by $1, 2, \cdots, N$ (coloured by ‘+’) and $1, 2, \cdots, N$ (coloured by ‘-’);

(b) a set of edges $E(\Gamma)$ of three types such that each vertex has valency 1:

Type ‘++’: edges having sources coloured + and targets coloured +, indexed by $(v_{\text{source}}, v_{\text{target}})$, such that for each edge $(i, j)$, $i \equiv j + 1 \mod N$,

Type ‘−−’: edges having sources coloured − and targets coloured −, indexed by $(v_{\text{source}}, v_{\text{target}})$, such that for each edge $(i, j)$, $j \equiv i + 1 \mod N$,

Type ‘+-’: edges having sources coloured + and targets coloured −, indexed by $(v_{\text{source}}, v_{\text{target}})$, such that for each edge $(i, j)$, $i = j$ or $i = j + 1$ or $(i, j) = (N, 1)$.

Given such a graph $\Gamma$, we assign the value of each edge by

$$\text{Val}(+_i(i, i)_-) = \frac{[n_i]_\lambda - n_k}{\lambda_i - 1} x_i^{n_i-2} x_{i+1},$$

$$\text{Val}(+_i(i, i-1)_-) = \frac{[n_i-1]_\lambda}{\lambda_i - 1} x_i^{n_i-1},$$

$$\text{Val}(+_i(N, 1)_-) = [n_N]_\lambda x_N^{n_N-1},$$

$$\text{Val}(+_i(i, i+1)_+) = b_i^+ = \begin{cases} 
\frac{\lambda_i}{1 - \lambda_i} x_i^{n_i-1} & \text{while } 1 \leq i < N, \\
(-1)^{N-1}/(1 - \lambda_N) x_N^{n_N-1} & \text{while } i = N,
\end{cases}$$

$$\text{Val}(-i(i, i+1)_-)_- = g b_i^{-1} = \begin{cases} 
-1/(1 - \lambda_i) x_i^{n_i-1} & \text{while } 1 \leq i < N, \\
(-1)^N \lambda_N^{n_N} x_N^{n_N-1} & \text{while } i = N,
\end{cases}$$

and the value of a graph $\Gamma$ is defined by

$$\text{Val}(\Gamma) := (-1)^{\text{sgn}(\Gamma)} \prod_{e \in E(\Gamma)} \text{Val}(e).$$

Here $s_\Gamma$ is the number of edges of type ‘+-’ of $\Gamma$. If the type ‘+-’ edges have source index $i_1 < i_2 < \cdots < i_{s_\Gamma}$ and target index $j_1 < j_2 < \cdots < j_{s_\Gamma}$ such that it connects $i_k$ to $j_{\sigma(k)}$, then $(-1)^{\text{sgn}(\Gamma)}$ is the sign of the permutation $\sigma$. 

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Example 3.7. For $N = 3$, we need to consider the following graphs.

There are three types of graphs in Lemma 3.22:

**Type I:** Graphs with all the type ‘$+$’ edges indexed by $+ (i, i)$. Then, the other edges come in pairs;

**Type II:** Graphs with all the type ‘$+$’ edges indexed by $+ (i, i - 1)$, for $i \neq 1$. Then, the other edges come in pairs;

**Type III:** Graphs with all the type ‘$+$’ edges indexed by $+ (N, 1)$ and by $+ (N - 1, N)$. Then, there exists edges $+ (1, 2)$ and $- (N - 1, N)$ and the other edges come in pairs.

We can simplify the graphs by gluing vertices with the same index but different colours, then the three types of graphs look like as follows,

Note that for $1 \leq i < N$, we have

$$\text{Val}(+ (i + 1, i)) = \frac{1 - \lambda_i^n}{1 - \lambda_i} x_i^{n_i - 1} = -\text{Val}(+ (i, i + 1)) - \text{Val}(- (i, i + 1)),$$

which means we can replace all these edges of type ‘$+$’ in graphs of type II by edges of type ‘$+$’ of type ‘$-$’ and multiply by $(-1)^{ir}$. Hence, graphs with

$$i \quad \ldots \quad i + 1$$
for \(2 \leq i \leq N - 1\) will be cancelled and we are left with the graphs

\[
\begin{array}{c}
\begin{tikzpicture}
\node (a) at (0,0) {}; 
\node (b) at (0:1cm) {}; 
\node (c) at (90:1cm) {}; 
\node (d) at (180:1cm) {}; 
\node (e) at (-90:1cm) {}; 
\draw (a) -- (b) -- (c) -- (d) -- (e) -- (a);
\end{tikzpicture}
\end{array}
\quad \text{for } 1 \leq i \leq N - 1,
\begin{array}{c}
\begin{tikzpicture}
\node (a) at (0,0) {}; 
\node (b) at (0:1cm) {}; 
\node (c) at (90:1cm) {}; 
\node (d) at (180:1cm) {}; 
\node (e) at (-90:1cm) {}; 
\draw (a) -- (b) -- (c) -- (d) -- (e) -- (a);
\end{tikzpicture}
\end{array}
\quad \text{for } 2 \leq i \leq N.
\]

multiplied by \((-1)^{(N-1)(N-2)/2}\) (for these \(\Gamma\), \(\text{sgn}(\Gamma) = 0\) and \(s_\Gamma = N - 2\)).

We can do the same thing to graphs of type III. Then the rest graphs are

\[
\begin{array}{c}
\begin{tikzpicture}
\node (a) at (0,0) {}; 
\node (b) at (0:1cm) {}; 
\node (c) at (90:1cm) {}; 
\node (d) at (180:1cm) {}; 
\node (e) at (-90:1cm) {}; 
\draw (a) -- (b) -- (c) -- (d) -- (e) -- (a);
\end{tikzpicture}
\end{array}
\quad \text{for } 2 \leq i \leq N - 1 \text{ and multiplied by } (-1)^{(N-2)(N-3)/2} \text{ (for these } \Gamma, \text{ sgn}(\Gamma) = N - 3 \text{ and } s_\Gamma = N - 2).\n\]

Since \((-1)^N \text{Val}+(N,1)- = -\text{Val}+(N,1)+ - \text{Val}-(N,1)-\),

the sum of graphs of type II and type III gives only two graphs

\[
\begin{array}{c}
\begin{tikzpicture}
\node (a) at (0,0) {}; 
\node (b) at (0:1cm) {}; 
\node (c) at (90:1cm) {}; 
\node (d) at (180:1cm) {}; 
\node (e) at (-90:1cm) {}; 
\draw (a) -- (b) -- (c) -- (d) -- (e) -- (a);
\end{tikzpicture}
\end{array}
\quad \begin{array}{c}
\begin{tikzpicture}
\node (a) at (0,0) {}; 
\node (b) at (0:1cm) {}; 
\node (c) at (90:1cm) {}; 
\node (d) at (180:1cm) {}; 
\node (e) at (-90:1cm) {}; 
\draw (a) -- (b) -- (c) -- (d) -- (e) -- (a);
\end{tikzpicture}
\end{array}
\]

multiplied by \((-1)^{(N-1)(N-2)/2} = (-1)^{(N-1)N}(-1)^{N-1}\). Let us consider the matrix

\[
H^g_W = \begin{pmatrix}
h_{11} & h_{12} & h_{13} & \cdots & h_{1N} \\
h_{21} & h_{22} & h_{23} & \cdots & h_{2N} \\
h_{31} & h_{32} & h_{33} & \cdots & h_{3N} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
h_{N1} & \cdots & \cdots & \cdots & h_{NN}
\end{pmatrix}, \tag{3.65}
\]

where the entries are defined by

\[
h_{ij} := \begin{cases} 
\text{Val}+(i,i) & \text{if } i = j, \\
\text{Val}+(i,i+1) & \text{if } N \geq j = i + 1 \geq 2, \\
\text{Val}-(N,1) & \text{if } (i,j) = (N,1), \\
\text{Val}-(i-1,i) & \text{if } N - 1 \geq j = i - 1 \geq 1, \\
\text{Val}+(N,1) & \text{if } (i,j) = (1,N), \\
0 & \text{otherwise}
\end{cases}
\]
According to the discussion above, the sum of Type I, II, III graphs leads to
\[
1_y \cup 1_y^{-1} = (-1)^{\frac{(N-1)N}{2}} \det H_W^g.
\] (3.66)

Note that \(\det H_W^g\) is valued in \(J\ac(W)\) where
\[
x_i^{n_i-1} + n_i x_i^{n_i-1} x_{i+1} = 0, \quad 1 \leq i \leq N.
\]
Here we always identify indices \(N = 0, N + 1 = 1\).

The two graphs of type II and type III contribute the following two terms in \(\det H_W^g\):
\[
(-1)^{\frac{(N-1)N}{2}} (-1)^{N-1} (h_{12} h_{23} \cdots h_{N1} + h_{21} h_{32} \cdots h_{1N})
\]
\[
= (-1)^{\frac{(N-1)N}{2}} - \frac{n_1 \lambda_1 [n_2 \lambda_2 \cdots [n_N \lambda_N - 1]}{(\lambda_1 - 1)(\lambda_2 - 1) \cdots (\lambda_N - 1)} x_1^{n_1-1} \cdots x_N^{n_N-1}.
\]

To compute the contribution of type I graphs, we observe that \(h_{i(i+1)}\) and \(h_{i(i+1)}\) always come in pairs. The contribution of type I graphs does not change if we replace \(h_{i(i+1)}, h_{i(i+1)}\) by other values \(\tilde{h}_{i(i+1)}, \tilde{h}_{i(i+1)}\) as long as \(h_{i(i+1)} h_{i(i+1)} = \tilde{h}_{i(i+1)} \tilde{h}_{i(i+1)}\) holds for all \(1 \leq i \leq N\). In \(J\ac(W)\), we have
\[
h_{i(i+1)} h_{i(i+1)} = \frac{\lambda_i^{n_i}}{(1 - \lambda_i)^2} x_i^{2n_i-2}
\]
\[
= \frac{(1 - \lambda_i^{n_i}) n_i x_{i+1}^{n_i-2} x_{i+2}^{n_i-1}}{(1 - \lambda_i)^2 (1 - \lambda_{i+1})} x_i^{n_i-2} x_{i+1}^{n_i-1}
\]
\[
= \left( \frac{n_i \lambda_i}{\lambda_i - 1} x_i^{n_i-2} x_{i+1} \right) \left( - \frac{n_i x_{i+1}^{n_i-2} x_{i+2}}{\lambda_i - 1 - \lambda_{i+1}} \right).
\]

Let us replace \(h_{i(i+1)}, h_{i(i+1)}\) by
\[
\begin{cases}
\tilde{h}_{i(i+1)} &= \frac{n_i \lambda_i}{\lambda_i - 1} x_i^{n_i-2} x_{i+1},
\tilde{h}_{i(i+1)} &= - \frac{n_i x_{i+1}^{n_i-2} x_{i+2}}{\lambda_i - 1 - \lambda_{i+1}}.
\end{cases}
\]

Observe that
\[
h_{ii} = \tilde{h}_{i(i+1)} + \tilde{h}_{i(i-1)}.
\]
It leads to cancellation of sum of type I graphs that we are left with only two terms
\[
(-1)^{\frac{(N-1)N}{2}} \left( \tilde{h}_{12} \tilde{h}_{23} \cdots \tilde{h}_{N1} + \tilde{h}_{21} \tilde{h}_{32} \cdots \tilde{h}_{1N} \right)
\]
\[
= (-1)^{\frac{(N-1)N}{2}} \frac{n_1 \lambda_1 [n_2 \lambda_2 \cdots [n_N \lambda_N - 1]}{(\lambda_1 - 1)(\lambda_2 - 1) \cdots (\lambda_N - 1)} x_1^{n_1-1} \cdots x_N^{n_N-1}.
\]

The sum of all the above contributions proves part (2) of Theorem 5.20 in the loop case for \(N \geq 3\).

Remark. In \(N = 2\) case, we can choose
\[
k_{2-2}^g := b_1^g + b_2^g = \frac{\lambda_1^{n_1}}{1 - \lambda_1} x_1^{n_1-1} g - \frac{1}{1 - \lambda_2} x_2^{n_2-1} g,
\]
and define matrix
\[
H_W^g = \begin{pmatrix}
\frac{n_1 \lambda_1}{\lambda_1 - 1} x_1^{n_1-2} x_2 & \frac{\lambda_1^{n_1}}{1 - \lambda_1} x_1^{n_1-1} - \frac{1}{1 - \lambda_2} x_2^{n_2-1} \\
\frac{1}{1 - \lambda_2} x_1^{n_1-1} + \frac{\lambda_2^{n_2}}{1 - \lambda_2} x_2^{n_2-1} & \frac{n_2 \lambda_2}{\lambda_2 - 1} x_1 x_2^{n_2-2}
\end{pmatrix}.
\]

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We leave it to the reader to check directly that

\[ 1_g \cup 1_g^{-1} = \frac{1 - n_1 n_2}{(\lambda_1 - 1)(\lambda_2 - 1)} x_1^{n_1 - 1} x_2^{n_2 - 1}. \]

This proves part (2) of Theorem 3.24 in the loop case for \( N = 2 \).

**Chain type**

\[ W = x_1^{n_1} x_2 + x_2^{n_2} x_3 + \cdots + x_N^{n_N} \] (\( N \geq 2 \)). Let \( g = \text{diag}(\lambda_1, \cdots, \lambda_N) \in G_W \), \( \lambda_i = e^{2\pi \sqrt{-1} g_i} \), where

\[
\begin{align*}
\lambda_i &= 1 \quad \text{if } 1 < i < l_g, \\
\lambda_i &= 1 \quad \text{if } i = l_g, \\
\lambda_i &= 1 \quad \text{if } l_g < i \leq N,
\end{align*}
\]

Here \( \lambda_i \neq 1 \) for \( i \leq l_g \). Let us find a Koszul representative of \( 1_g \).

Let \( I_g = \{1, 2, \ldots, l_g\} \). For \( I = \{i_1 < i_2 < \cdots < i_m\} \), let us denote \( e_I := e_{i_1} \cdots e_{i_k} \). In the chain case,

\[
\tilde{d}_W(a g e_I) = \sum_{k=1}^{p} (-1)^{k-1} a p^g_{i_k} g e_{I \setminus \{i_k\}} ,
\]

where

\[
p^g_{i_k} = \begin{cases} [n_1] \lambda_i x_1^{n_1 - 1} x_2 & \text{if } i = 1, \\ \lambda_i^{-1} x_1^{n_1 - 1} x_2 + [n_1] \lambda_i x_1^{n_1 - 1} x_{i+1} & \text{if } 1 < i < l_g, \\ \lambda_i^{-1} x_1^{n_1 - 1} x_{l_g - 1} & \text{if } i = l_g .
\end{cases}
\]

Let us define the index set

\[
B_s := \{ \{i_1 < i_2 < \cdots < i_s\} \subset I_g \mid i_{k+1} - i_k > 1, \text{ for } 1 \leq k \leq s, \text{ and } i_s < l_g \}\,
\]

and

\[
b^g_i = \frac{\lambda_i^{n_i}}{1 - \lambda_i} x_i^{n_i - 1}, \quad 1 \leq i \leq l_g - 1.
\]

For each \( b = \{i_1 < i_2 < \cdots < i_s\} \in B_s \), let

\[
I_b = \{i_1, i_1 + 1, i_2, i_2 + 1, \cdots, i_s, i_s + 1\}.
\]

Then

\[
\sum_{N \geq 2s \geq 0} \kappa^g_{-2s}
\]

gives a Koszul representative for \( 1_g \) where

\[
\kappa^g_{-2s} = \sum_{b \in B_s} \left( \prod_{i \in b} b^g_i \right) g e_{I_{b} \setminus b} .
\]

The proof is the same as Lemma 3.21. As in the loop case,

\[
1_g \cup 1_g^{-1} = p \circ \Phi^* \left( \Gamma^* (\kappa^g_{0} + \kappa^g_{-2} + \cdots) \cup \Gamma^* (\kappa^g_{0}^{-1} + \kappa^g_{-2}^{-1} + \cdots) \right) ,
\]

is a sum over all the graphs \( \Gamma \) with
(a) a set of vertices $V(\Gamma)$ indexed by $1, 2, \cdots, l_g$;

(b) a set of edges $E(\Gamma)$ of three types such that each vertex has valency 2:

Type $'++'$: edges having sources coloured $+$ and targets coloured $+$, indexed by $\gamma^+(v_{\text{source}}, v_{\text{target}})$, satisfying that for each edge $\gamma^+(i, j)$, $j = i + 1$,

Type $'--'$: edges having sources coloured $-$ and targets coloured $-$, indexed by $\gamma^-(v_{\text{source}}, v_{\text{target}})$, satisfying that for each edge $\gamma^-(i, j)$, $j = i + 1$,

Type $'+-'$: edges having sources coloured $+$ and targets coloured $-$, indexed by $\gamma^+(v_{\text{source}}, v_{\text{target}})$, satisfying that for each edge $\gamma^+(i, j)$, $i = j$ or $i = j + 1$,

Given such a graph $\Gamma$, we assign the value of each edge by

$$\text{Val}(\gamma^+(i, i-1)) = \left\{ \begin{array}{ll}
\frac{[n_1]_{\lambda_i} - n_k}{\lambda_i - 1} x_i^{n_i-2} x_{i+1} & \text{if } 1 \leq i < l_g,
\frac{-n_i}{\lambda_i - 1} x_i^{n_i-2} x_{i+1} & \text{if } i = l_g < N,
\frac{-n_i}{\lambda_i - 1} x_i^{n_i-2} & \text{if } i = l_g = N,
\end{array} \right.$$ $$\text{Val}(\gamma^+(i, i-1)) = \left\{ \begin{array}{ll}
\frac{[n_i]_{\lambda_i-1} x_i^{n_i-1}}{\lambda_i - 1},
\frac{\lambda_i}{1 - \lambda_i} x_i^{n_i-1},
\frac{n_i}{1 - \lambda_i} x_i^{n_i-1},
\end{array} \right.$$ $$\text{Val}(\gamma^+(i, i+1)) = b_i^\gamma = \frac{\lambda_i}{1 - \lambda_i} x_i^{n_i-1},$$ $$\text{Val}(\gamma^+(i, i+1)) = b_i^\gamma = \frac{\lambda_i}{1 - \lambda_i} x_i^{n_i-1},$$

and the value of a graph $\Gamma$ is defined as

$$\text{Val}(\Gamma) = (-1)^{\text{sgn}(\Gamma)} \cdot \prod_{e \in E(\Gamma)} \text{Val}(e).$$

Here $s_{1\Gamma}$ is the number of edges of type $'+-'$ of $\Gamma$. If the type $'+-'$ edges have source index $i_1 < i_2 < \cdots < i_{s_{1\Gamma}}$ and target index $j_1 < j_2 < \cdots < j_{s_{1\Gamma}}$ such that it connects $i_k$ to $j_{\sigma(k)}$, then $(-1)^{\text{sgn}(\Gamma)}$ is the sign of the permutation $\sigma$.

**Example 3.8.** For $l_g = 3$, we need to consider the following graphs.

$$\begin{array}{cccc}
& \circ & \circ & \circ \\
& \circ & \circ & \circ \\
& \circ & \circ & \circ \\
\end{array}$$

Graphs for chain types are those of type I graphs for the loop case.

$$\begin{array}{cccc}
& \circ & \circ & \circ \\
& \circ & \circ & \circ \\
& \circ & \circ & \circ \\
\end{array}$$

Similar to the discussion for the loop case, let us define the matrix

$$H^\gamma_{1\Gamma} = \begin{pmatrix}
h_{11} & h_{12} & h_{13} & \cdots & h_{1(l_g-1)} \\
h_{21} & h_{22} & h_{23} & \cdots & h_{2(l_g-1)} \\
h_{31} & h_{32} & h_{33} & \cdots & h_{3(l_g-1)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
h_{l_g-1,1} & h_{l_g-1,2} & h_{l_g-1,3} & \cdots & h_{l_g-1,l_g} \\
h_{l_g,1} & h_{l_g,2} & h_{l_g,3} & \cdots & h_{l_g,l_g-1}
\end{pmatrix}$$

(3.67)
where the entries are defined by

\[
h_{ij} = \begin{cases} 
\text{Val}( (i, i)- ) & \text{if } i = j, \\
\text{Val}( (i, i+1)+ ) & \text{if } l_y \geq j = i + 1 \geq 2, \\
\text{Val}( (i-1, i)- ) & \text{if } l_y - 1 \geq j = i - 1 \geq 1, \\
0 & \text{otherwise.}
\end{cases}
\]

Then we have the analogue of (3.66) in the chain case

\[
1_g \cup 1_{g-1} = (-1)^{\frac{lg-1}{2}} \det H^g_{W}.
\] (3.68)

The computation of this determinant is also the same as in the loop case for type I graphs. In \(\text{Jac}(W)\),

\[
\begin{cases}
x_i^{n_i-1} + n_i x_i^{n_i-1} x_i+1 = 0 & \text{for } 2 \leq i \leq N - 1, \\
x_N^{n_N-1} + n_N x_N^{n_N-1} = 0.
\end{cases}
\]

For \(l_y < N\), since \((i, i+1)+\) and \((i, i-1)-\) come in pairs and

\[
h_{i(i+1)} \bar{h}_{(i+1)i} = -\frac{\lambda_i^{n_i}}{(1 - \lambda_i)^2} x_i^{2n_i - 2} = \frac{(1 - \lambda_i^{n_i})n_i + 1}{(1 - \lambda_i)^2 (1 - \lambda_i^{n_i} + 1)} x_i^{n_i - 2} x_i+1 \cdot x_i+2
\]

\[
= \left[ \frac{n_i}{\lambda_i} \right]_{\lambda_i} x_i^{n_i - 2} x_i+1 \cdot x_i+2.
\]

Without changing the value of \(\det H^g_{W}\), we can replace \(h_{i(i+1)}\) and \(h_{(i+1)i}\) by

\[
\tilde{h}_{i(i+1)} = \frac{n_i}{\lambda_i} x_i^{n_i - 2} x_i+1,
\]

\[
\tilde{h}_{(i+1)i} = -\frac{n_i}{\lambda_i + 1} x_i^{n_i - 2} x_i+2.
\]

Observe that

\[
h_{ii} = \begin{cases} 
\tilde{h}_{i(i+1)} + \tilde{h}_{(i-1)i} & \text{while } 2 \leq i \leq l_y - 1, \\
\tilde{h}_{y(y-1)} & \text{while } i = l_y.
\end{cases}
\]

It leads to cancellation of sum of graphs that we are left with only two terms

\[
(-1)^{\frac{lg-1}{2}} ( (h_{11} - \tilde{h}_{12}) \tilde{h}_{21} \cdots \tilde{h}_{y(y-1)} )
\]

\[
= (-1)^{\frac{lg-1}{2}} \frac{n_1 n_2 \cdots n_y}{(1 - \lambda_1) \cdots (1 - \lambda_y)} x_1^{n_1 - 2} x_2^{n_2 - 1} \cdots x_y^{n_y - 1} x_{y+1}.
\]

For \(l_y = N\), the computation is similar. This proves part (2) of Theorem 3.20 in the chain case.

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Weiqiang He, YMSC, TSINGHUA UNIVERSITY, BEIJING 100084, CHINA  
E-mail address: wqhe@math.tsinghua.edu.cn

Si Li, YMSC, TSINGHUA UNIVERSITY, BEIJING 100084, CHINA  
E-mail address: sili@mail.tsinghua.edu.cn

Yifan Li, YMSC, TSINGHUA UNIVERSITY, BEIJING 100084, CHINA  
E-mail address: yf-li14@mails.tsinghua.edu.cn