Common origin of $\theta_{13}$ and $\Delta m^2_{12}$ in a model of neutrino mass with quaternion symmetry

Michele Frigerio$^1$ and Ernest Ma$^2$

$^1$ Service de Physique Théorique, CEA-Saclay, 91191 Gif-sur-Yvette Cedex, France
$^2$ Department of Physics and Astronomy, University of California, Riverside, California 92521, USA

(Dated: August 6, 2007)

The smallness of the $1 - 3$ lepton mixing angle $\theta_{13}$ and of the neutrino mass-squared-difference ratio $\Delta m^2_{12}/\Delta m^2_{23}$ can be understood as the departure from a common limit where they both vanish. We discuss in general the conditions for realizing the mass degeneracy of a pair of neutrinos and show that the vanishing of a CP violating phase is needed. We find that the discrete quaternion group $Q$ of eight elements is the simplest family symmetry which correlates the smallness of $\Delta m^2_{12}$ to the value of $\theta_{13}$. In such a model we predict $0.12 \lesssim \sin \theta_{13} \lesssim 0.2$ if the ordering of the neutrino mass spectrum is normal, and $\sin \theta_{13} \lesssim 0.12$ if it is inverted.

PACS numbers: 11.30.Hv, 14.60.Pq

Introduction. Contrary to quarks and charged leptons, the three neutrinos are known to have a mass spectrum with a weak hierarchy, possibly quasi-degenerate. In particular, the two mass eigenstates $\nu_1$ and $\nu_2$ in the “solar pair” are very close in mass in the case of inverted ordering and also in the case of normal ordering, as long as the yet unknown absolute mass scale is larger than $\sim 0.02$ eV. Indeed, a global fit of neutrino oscillation data gives

$$
\begin{align*}
\Delta m^2_{12} &\equiv m_2^2 - m_1^2 = (7.9_{-0.8}^{+1.0}) \cdot 10^{-5} \text{eV}^2, \\
\Delta m^2_{23} &\equiv m_3^2 - m_2^2 = \pm (2.6 \pm 0.6) \cdot 10^{-3} \text{eV}^2, \\
\sin^2 \theta_{12} &\approx 0.30_{-0.06}^{+0.10}, \\
\sin^2 \theta_{23} &\approx 0.50_{-0.18}^{+0.05}, \\
\sin^2 \theta_{13} &\lesssim 0.040,
\end{align*}
$$

where we took the best fit values and the $3\sigma$ intervals from the last update of Ref.\[1\] (v5).

The smallness of the $1 - 2$ mass splitting compared to the $2 - 3$ “atmospheric” splitting may be explained as the departure from a symmetric limit where $1 - 2$ mass degeneracy holds. Other small parameters of the lepton flavor sector may also be interpreted as deviations from the same limit, such as the $1 - 3$ mixing angle, the deviation of the $1 - 2$ and possibly $2 - 3$ angles from the maximal value $\pi/4$, and the mass ratios $(m_e/m_\mu, m_\mu/m_\tau)$ of charged leptons.

In this paper we analyze the neutrino mass structures corresponding to a quasi-degenerate pair of states and the phenomenological correlations with other lepton flavor parameters, in particular a non-vanishing $1 - 3$ mixing, whose measure is the objective of an extensive experimental program \[2\]. We also search for the simplest flavor symmetries which can be used to realize such a mass degeneracy. The discrete quaternion group of eight elements $Q$ is identified as the most suitable for this purpose and a complete model is constructed.

The group $Q$ (sometimes called $Q_8$ or $Q_4$) was introduced in \[3\] to build a model of quark and lepton masses and mixing. Discrete subgroups of quaternions with unit norm, i.e. $SU(2)$, were already used in \[4\] to suppress the neutrino mass while allowing for large neutrino magnetic moments. They were also discussed as flavor symmetries in a series of papers by Frampton and collaborators \[5\]. A specific model based on the quaternion group of 12 elements $Q_6$ was proposed as well \[6\]. Models were built \[7\] using the binary tetrahedral group $T'$ (double covering of the tetrahedral group $A_4$), which is also a discrete subgroup of quaternions with 24 elements. The group $T'$ was recently employed to accommodate tri-bi-maximal mixing \[8\]. Note that $A_4$ is not a subgroup of $T'$, but $Q$ is. Geometrically, $SU(2)$ is isomorphic to the hypersphere in four dimensions, the 8 elements of $Q$ form the 8 vertices of the perfect hyperoctahedron (dual of the hypercube), whereas the 24 elements of $T'$ form the 24 vertices of the hyperdiamond (which is self-dual).

Mass matrix of 2 degenerate neutrinos. Let us begin considering the Majorana mass matrix $m_\nu$ for two neutrino states. Under the requirement of mass degeneracy ($m_1 = m_2 = m$), $m_\nu$ can be written in full generality as

$$
D_\phi m_\nu D_\phi = \begin{pmatrix} a & b \\ b & -a \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix} m,
$$

where $D_\phi = \text{diag}(e^{i\phi_1}, e^{i\phi_2})$ is a diagonal matrix of phases, $a$ and $b$ are real and positive, $m = \sqrt{a^2 + b^2}$ and $\tan 2\theta = b/a$. In terms of the $m_\nu$ matrix elements, the requirement of mass degeneracy is equivalent to 2 conditions, $|m_{11}| = |m_{22}|$ and $\arg(m_{11}m_{22}/m^2_{12}) = \pi$.

The simplest cases $b = 0$ or $a = 0$ have often been discussed, while the general case was studied in just a few interesting papers, as a prototypical example of a pseudo-Dirac neutrino mass matrix \[9\] \[10\] \[11\]. Here we analyze in detail how this matrix structure relates to the physical observables, with special attention to the effects of possible CP violating phases.

The diagonalization of the matrix in eq.\[2\] presents some subtleties, which turn out to be important to understand the effect of small perturbations responsible for $m_1 \neq m_2$. Notice first that the two neutrino masses $m_{1,2}$,
and the moduli of the mass matrix elements, \(a\) and \(b\), are physically well-defined quantities (they can be measured, at least in principle). This determines uniquely the parameter \(\theta\), which one is tempted to identify with a physical mixing angle \(\theta_{12}\) between \(\nu_1\) and \(\nu_2\). However, in the degenerate limit there is no mixing angle responsible for neutrino oscillations: \(m_\nu m_\nu^\dagger = \text{diag}(m^2, m^2)\). To settle this apparent contradiction let us rewrite \(m_\nu\) in the standard parameterization,

\[
m_\nu = U^{\dagger} m_\nu^{\text{diag}} U^\dagger
\]

\[
= D_\phi' \begin{pmatrix} c_{12} & s_{12} \\ -s_{12} & c_{12} \end{pmatrix} \begin{pmatrix} m e^{-2i\phi} & 0 \\ 0 & m \end{pmatrix} \begin{pmatrix} c_{12} & -s_{12} \\ s_{12} & c_{12} \end{pmatrix} D_\phi, \tag{3}
\]

where \(D_\phi'\) is a diagonal matrix of phases, \(c_{12} \equiv \cos \theta_{12}\), \(s_{12} \equiv \sin \theta_{12}\) and \(\rho\) is the relative Majorana phase between the two mass eigenstates, varying between 0 and \(\pi\). With some easy algebra one finds \(\sin 2\theta = \sin \rho \sin 2\theta_{12}\). Only this combination has physical meaning (it is measurable), while the mixing angle \(\theta_{12}\) and the Majorana phase \(\rho\) cannot be determined uniquely (a similar discussion can be found in [12]).

The important consequence is that different small perturbations which generate \(\Delta m^2_{12}\) may select very different values for \(\theta_{12}\), which is the crucial parameter to determine the oscillation probability. Let us consider a positive mass \(\epsilon \ll a, b\). If \(m_\nu\) diagonal entries are corrected as \(a \rightarrow (a - \epsilon)\) and \(-a \rightarrow (-a - \epsilon)\), then \(\Delta m^2_{12} = 4\epsilon a^2 + b^2\) but the mixing angle is a free parameter, \(\tan 2\theta_{12} = b/a\), and the two neutrinos have opposite CP-parity, \(\rho = \pi/2\). If instead \(m_\nu\) off-diagonal entries are corrected as \(b \rightarrow (b + i\epsilon)\), then \(\Delta m^2_{12} = 4\epsilon a\) and the mixing angle is maximal, \(\theta_{12} = \pi/4\), but the Majorana phase is a free parameter, \(\sin \rho \approx b/\sqrt{a^2 + b^2}\).

**Quaternion model.** We now search for family symmetries that can lead to the structure in eq. (2). The equality \(|m_{11}| = |m_{22}|\) cannot be explained by an Abelian symmetry, since in this case each lepton family would transform independently under the action of the symmetry group, so that equalities among independent mass matrix elements cannot be justified. Hence the two quasi-degenerate neutrino families should sit in a two-dimensional irreducible representation (2-dim irrep) of a non-Abelian group. Let us assign, therefore, the three Standard Model (SM) lepton doublets as follows:

\[
\begin{pmatrix} L_1 \\ L_2 \end{pmatrix} \sim 2, \quad L_3 \sim 1, \tag{4}
\]

where the 1-dim irrep is not necessarily the singlet invariant under the symmetry group. To realize the structure in eq. (2), one needs two invariants, \(a(L_1 L_1^\dagger - L_2 L_2^\dagger)\) and \(2b L_1 L_2^\dagger\), and at the same time the combination \(L_1 L_1^\dagger + L_2 L_2^\dagger\) should not contribute. We found that this \(2 \times 2\) pattern may be obtained by using the 2-dim irrep of any of the three smallest non-Abelian groups, \(S_3\), \(D_4\) and \(Q\). However, in the \(S_3\) case it is not possible to maintain this pattern in a complete model with three families. In the \(D_4\) case the combination \(L_1 L_1^\dagger + L_2 L_2^\dagger\) is a group invariant and cannot be discarded without extra assumptions.

We therefore focus on the smallest quaternion group \(Q\). All the details on the group structure, the character table and our conventions for the irreps and their tensor products can be found in [3]. For our purposes here it is sufficient to recall that \(Q\) has four 1-dim irreps, \(1^{++}, 1^{--}, 1^{++} - 1^{--}\), with tensor product rules made obvious by the superscripts, and one 2-dim irrep \(2\). The product of two \(Q\)-doublets \((\psi_1, \psi_2)^T, (\chi_1, \chi_2)^T\) \(\sim 2\) goes as follows:

\[
(\psi_1 \chi_2 - \psi_2 \chi_1) \sim 1^{++}, \quad (\psi_1 \chi_1 - \psi_2 \chi_2) \sim 1^{++}, \quad (\psi_1 \chi_2 + \psi_2 \chi_1) \sim 1^{--}, \quad (\psi_1 \chi_1 + \psi_2 \chi_2) \sim 1^{--}. \tag{5}
\]

The charged lepton masses arise from the Yukawa coupling \(y_{ijk} L_i \psi_j^c q_k\) where \(\phi_k = (\phi^0, \phi^-)_k\) are Higgs doublets with vacuum expectation values (VEVs) \(\langle \phi^0_k \rangle \equiv v_k\). The neutrino masses arise from the Majorana-type Yukawa coupling \(f_{ijk} L_i L_j \Delta_k\) where \(\Delta_k = (\Delta^{++}, \Delta^+, \Delta^0)_k\) are Higgs triplets with \(\langle \Delta^0_k \rangle = u_k\). These \(\Delta_k\) are naturally small when the triplets are super-heavy, by virtue of the type II seesaw mechanism.

Let us consider the following \(Q\) assignments:

\[
\begin{pmatrix} L_1 \\ L_2 \end{pmatrix} \sim 2, \quad L_3 \sim 1^{++}, \quad e^c_1 \sim 1^{--}, \quad 1^{+-}, \quad 1^{--}, \quad \phi_1 \sim 2, \quad \phi_2 \sim 1^{--}, \quad \Delta_1 \sim 1^{--}, \quad 1^{+-}, \quad 1^{++}. \tag{6}
\]

Then the charged lepton and neutrino mass matrices have the following structure:

\[
m_l = \begin{pmatrix} y_1 v_2 & -y_2 v_1 & y_3 v_1 \\ y_1 v_1 & y_2 v_2 & y_3 v_3 \\ 0 & 0 & y_4 v_3 \end{pmatrix}, \tag{7}
\]

\[
m_\nu = \begin{pmatrix} f_2 v_2 & f_1 u_1 & 0 \\ f_2 v_2 & f_1 u_2 & 0 \\ 0 & 0 & f_3 u_3 \end{pmatrix}.
\]

The \(1 \sim 2\) sector of \(m_\nu\) has the desired form of eq. (2). Therefore, in the CP conserving case, the \(1 \sim 2\) mass degeneracy is realized. Moreover, when \(Q\) is broken by the Higgs doublet VEVs in the direction \(v_1 = 0\), the unique off-diagonal element in the charged lepton mass matrix is \(m_{12}^{(23)}\). Therefore the \(1 \sim 2\) mixing comes entirely from \(m_\nu\), the \(2 \sim 3\) mixing entirely from \(m_l\) and one predicts \(\theta_{13} = 0\). The scalar potential, the generation of VEVs and their alignment are discussed in the Appendix.

A possible extension of the \(Q\) symmetry to the quark sector is discussed in [3], where the phenomenological constraints on the extra Higgs doublets are also estimated. Notice that, since charged leptons mix only in the \(2 \sim 3\) sector, there are no flavor-changing neutral currents (FCNCs) involving the electron, which are strongly constrained experimentally. Several possible tests of FCNCs in the \(\mu - \tau\) sector are discussed e.g. in [14].
Before performing a detailed analysis of lepton masses and mixing angles associated with the matrices in eq. (7) and their small perturbations, we would like to stress that the same pattern is maintained in many possible variants of the $Q$ model, with a different symmetry breaking sector, different field assignments and/or a different type of seesaw.

First, one may dislike the presence of multiple Higgs doublets at electroweak scale, because of sizable FCNC effects, a harsher hierarchy problem (more than one fine-tuning), worsened gauge coupling unification, etc. Of course all such worries are based on some amount of theoretical prejudice. In any case, one can rephrase the flavor model above in terms of a unique Higgs doublet $\phi$ invariant under the family symmetry, then adding SM singlets $\varphi_i$ (flavons) charged under $Q$ as $(2, 1^{--})$. They enter charged lepton Yukawa couplings as $y_{ij}L_i\varphi_j/\Lambda$ where $\Lambda > \langle \varphi \rangle$ is some cutoff scale. In this way one can reproduce the same mass matrix structure as before, while maintaining the SM particle content only at electroweak scale. In this context, superheavy triplets may also be eliminated. One may think of neutrino masses originating from the effective operator 

$$ m_{ij} \propto y_{ij}L_i\varphi_j/\Lambda, $$

with flavons $\varphi_k$ charged under $Q$. The neutrino singlet $\nu_c$ (flavons) charged under $Q$ as $1^{--}$ and $1^{+-}$. Second, a different structure of $m_\nu$ leading to the same mixing pattern as in eq. (7) can be obtained also when charged lepton singlets $\nu_i$ and lepton doublets $L_i$ both transform as $(2, 1^{++})$, which is required in left-right symmetric extensions of the SM. In this case, adding $\phi_4 \sim 1^{+-}$, one finds

$$ m_\nu = \begin{pmatrix} y_{33}v_3 - y_{44}v_4 & 0 & -y_2v_2 \\ 0 & y_{33}v_3 + y_{44}v_4 & y_2v_1 \\ -y_1v_2 & y_1v_1 & 0 \end{pmatrix}. \tag{8} $$

When $Q$ is broken by Higgs doublet VEVs in the direction $v_2 = 0$, one can accommodate $m_{c,\mu,\tau}$ and, at the same time, large (maximal) $2 - 3$ mixing.

Third, the neutrino mass matrix in eq. (7) can be derived with just a little bit more effort even in the context of type I seesaw, by introducing neutrino singlets $\nu_i^c \sim (2, 1^{--})$. The neutrino Dirac mass matrix $m_D$ comes from the Yukawa coupling $y_{ij}L_i\nu^c_j\phi$, where $\phi = (\phi^+, \phi_0)$ is a Higgs doublet transforming as $1^{--}$, so that $m_D = diag(x, x, y)$. The neutrino singlet mass matrix $M_R$ comes from the coupling $f_{ijk}\nu^c_i\nu^c_jS_k$, where $S_k$ are Higgs singlets superheavy VEVs, transforming as $(1^{+-}, 1^{--}, 1^{++})$. In this way one finds that $M_R$, $M_R^{-1}$ and $m_\nu \equiv -m_DM_R^{-1}m_D$ all have the same structure as $m_\nu$ in eq. (7). A similar model with type I seesaw which realizes the same form of $m_\nu$ by means of a SU($2$) x U($1$) family symmetry can be found in [16].

Correlations among observables. The neutrino mass matrix in eq. (7) represents an interesting limit: $\theta_{13} = 0$ and either $\Delta m_{12}^2 = 0$, in the case of no CP violation (that is, when $(f_{2u2}/f_{1u1})$ is real), or $\theta_{12} = \pi/4$, when a nontrivial CP violating phase is present. In both cases solar oscillation data (see eq. (1)) call for a perturbation to this matrix structure. In the CP-conserving case, the perturbation leads in general to a correlation between the values of $\Delta m_{12}^2$ and $\theta_{13}$. In the CP-violating case, the correlation will be between the deviation of $1 - 2$ mixing from maximal and nonzero $1 - 3$ mixing. Such correlations will be probed in future searches of $\theta_{13}$ [2].

What may be the origin of such perturbations? A first possibility is that the structures in eq. (7) are modified by radiative corrections from some large scale down to the electroweak scale. This scenario can be justified assuming that the family symmetry is broken at the large scale (by the VEVs of flavon fields) and that $m_D$ runs below that scale as in the SM (or the MSSM). The radiative corrections to $m_\nu$ for matrix structures similar to those considered here have been studied in great detail in [12, 17, 18]. It is found that it is possible to generate radiatively both $\Delta m_{12}^2$ and $\theta_{13}$, but it is problematic to obtain the present values of parameters in the Large Mixing Angle MSW region. We will not consider this radiative possibility in the following.

A more straightforward way to introduce a perturbation is to add extra Higgs multiplets, with different $Q$ assignments, which may provide a sub-dominant contribution to $m_\nu$. First, consider a Higgs triplet $\Delta_4 \sim 1^{+-}$. Its VEV gives an equal contribution to the 11- and 22-entry of $m_\nu$. In the CP conserving (violating) case, a small perturbation of this type generates $\Delta m_{12}^2 \neq 0 (\theta_{12} = \pi/4)$ but does not affect $\theta_{13} = 0$. Each observable is reproduced by a different parameter and therefore no correlations are predicted. In particular one cannot tell this scenario from any other model with $\theta_{13} = 0$.

The most interesting scenario is obtained adding, instead, $(\Delta_4, \Delta_5) \sim 2$, with $u_4 = 0$ (same direction in group space as for the Higgs doublets $\phi_1, \phi_2 \sim 2$ with $v_1 = 0$, see the Appendix). Then the neutrino mass matrix has the form

$$ m_\nu = \begin{pmatrix} a & b & d \\ b & -a & 0 \\ d & 0 & c \end{pmatrix}, \tag{9} $$

where $d$ is proportional to $u_5$. Let us recall that we are working in a basis where $m_2$ contains an arbitrary $2 - 3$ mixing $\theta_{23}$, so the $2 - 3$ mixing $\theta_{23}$ in $m_\nu$ is not required to match the observed value of $\theta_{23} = \theta_{23}^0 + \theta_{23}^\nu$. Neutrino mass matrices with one zero element and two independent nonzero elements equal to each other are a typical outcome of models based on the family symmetry $Q$ [3]. All possible matrices with this feature in the basis where $m_1$ is diagonal were analyzed in [19].

CP-conserving case. Let us diagonalize $m_\nu$ in eq. (9) in the case where all matrix elements are real (without loss of generality one can take $a, b$ and $d$ positive). In the limit $d \equiv c \ll a, b, |c|$, defining $m \equiv \sqrt{a^2 + b^2}$ and

$$ a \to a + mc, \quad b \to b - mc, \quad d \to d - mc, \quad c \to 0, \tag{10} $$

...
expanding in $\epsilon$, the three neutrino masses are given by

$$m_1 \approx m + \frac{\epsilon^2 (m + a)}{2m(m - c)}, \quad m_2 \approx -m + \frac{\epsilon^2 (a - m)}{2m(m + c)},$$

$$m_3 \approx c + \frac{\epsilon^2 (c + a)}{c^2 - m^2}.$$  \hspace{1cm} (10)

(strictly speaking these equations hold only as long as $\epsilon \ll |m \pm c|$). The mass squared differences are then easily derived:

$$\Delta m_{12}^2 \equiv m_1^2 - m_2^2 \approx \frac{2m(a + c)}{c^2 - m^2} \epsilon^2,$$

$$\Delta m_{23}^2 \equiv m_2^2 - m_3^2 \approx c^2 - m^2.$$  \hspace{1cm} (11)

The mixing angles are

$$\sin \theta_{13} \approx \frac{\epsilon (c + a)}{c^2 - m^2}, \quad \tan^2 \theta_{12} \approx \frac{m - a}{m + a}, \quad \sin \theta_{23} \approx \frac{\epsilon b}{c^2 - m^2}.$$  \hspace{1cm} (12)

The almost maximal $2 - 3$ mixing arises from the charged lepton sector. The ordering of the mass spectrum is normal (inverted) for $c^2 - m^2 > 0$ ($< 0$). Since $\tan^2 \theta_{12} < 1$, solar neutrino data require $\Delta m_{12}^2 > 0$, that is $a + c > 0$ ($< 0$) for the case of normal (inverted) ordering. Finally and most importantly, the value of $\theta_{13}$ is correlated with the other observables:

$$\sin^2 \theta_{13} \approx \frac{1}{2} \frac{\Delta m_{12}^2}{\Delta m_{23}^2} \cos 2\theta_{12} + \frac{m_3}{m}.$$  \hspace{1cm} (13)

In the case of normal ordering, $m_3 > m$ and $\theta_{13}$ decreases by increasing the absolute neutrino mass scale, with a lower bound

$$\sin \theta_{13}|_{\text{normal}} > \cos \theta_{12} \sqrt{\frac{\Delta m_{12}^2}{\Delta m_{23}^2}} \approx 0.15.$$  \hspace{1cm} (14)

In the case of inverted ordering $m_3 < -m \cos 2\theta_{12}$ and $\theta_{13}$ increases by increasing the absolute mass scale, with an upper bound

$$\sin \theta_{13}|_{\text{inverted}} < \sin \theta_{12} \sqrt{\frac{\Delta m_{12}^2}{\Delta m_{23}^2}} \approx 0.10.$$  \hspace{1cm} (15)

These correlations are the main predictions of our model, which links the smallness of $\Delta m_{12}^2$ and $\theta_{13}$.

We performed a numerical analysis for the mass matrix in eq.\((9)\), scanning over the values of $a$, $b$, $c$ and $d$. The prediction for $\sin \theta_{13}$ is shown in Fig.\((1)\) as a function of the neutrino mass-squared-difference ratio $\Delta m_{12}^2/\Delta m_{23}^2$. The bounds \((14)\) and \((15)\), which assume best fit values of the measured parameters, are slightly relaxed but hold qualitatively: we find $\sin \theta_{13}|_{\text{normal}} > 0.12$ and $\sin \theta_{13}|_{\text{inverted}} < 0.12$.

The prediction for $\sin \theta_{13}$ is shown in Fig.\((2)\) as a function of the absolute neutrino mass scale. The lower density of allowed points for larger absolute mass scales indicates that a quasi-degenerate spectrum requires some fine-tuning in the input parameters $a$, $b$, $c$ and $d$. In the normal ordering case the experimental constraint $\sin \theta_{13} < 0.2$ implies a lower bound $m_1 \gtrsim 0.02$ eV. In the inverted ordering case $\sin \theta_{13}$ vanishes for $m_3 \sim 0.02$ eV. A lower bound on $\sin \theta_{13}$ holds also in the case of inverted ordering if $m_3$ is sufficiently large. This can be probed in neutrinoless $2\beta$ decay searches, which can measure the effective mass parameter $m_{ee} \equiv |m_{11}| = a$. In the limit of small $d$ one has $m_{ee} \approx m \cos 2\theta_{12} \sim m/2$. If $m_{ee} \gtrsim 0.04$ eV is found, our model predicts a lower bound $\sin \theta_{13} \gtrsim 0.05$.

One should notice that the matrix in eq.\((9)\) may accommodate data even when $d$ is not much smaller that the other parameters, as shown by the black region in Figs.\((1)\) and \((2)\). Indeed, $\Delta m_{12}^2$ and $\theta_{13}$ both vanish not only in the limit $d \to 0$, that was studied above, but also in the limit $c + a \to 0$. When $a + c = 0$ one has

$$m_{1,2} = \pm m = \pm \sqrt{a^2 + b^2 + c^2}, \quad m_3 = -a,$$

$$\tan \theta'_{23} = -\frac{d}{b}, \quad \tan 2\theta_{12} = \frac{\sqrt{d^2 + b^2}}{a}.$$  \hspace{1cm} (16)

Therefore the ordering of the mass spectrum is inverted.
and both large angles can be accommodated (there is no need of large 2–3 mixing in $m_\nu$). Defining $\epsilon \equiv a + c$ and taking the limit $|\epsilon| \ll a, b, d$, we find

$$\Delta m_{12}^2 \sim \frac{2d^2mc}{b^2 + d^2},$$
$$\sin \theta_{13} \approx \frac{1}{2} \frac{\Delta m_{12}^2 \sin 2\theta_{12}}{\Delta m_{23}^2 \tan \theta_{23}}. \tag{17}$$

When $b \rightarrow 0$, $\theta_{13}$ vanishes but at the same time $\Delta m_{12}^2$ is nonzero. This is why in the case of inverted ordering there is no lower bound on $\sin \theta_{13}$ (see lower panel in Figs. 1 and 2).

**CP-violating case.** Let us consider the neutrino mass matrix in eq. (6) in the general case of complex matrix elements. In the limit $d = 0$ one has $\theta_{12} = \pi/4$, $\theta_{13} = 0$, $\Delta m_{12}^2 = 4\text{Im}(ab^*)$ and $\Delta m_{23}^2 = |c|^2 - |a|^2 - |b|^2$. Notice that, in the presence of non-trivial phases, the smallness of $\Delta m_{12}^2/\Delta m_{23}^2$ is accidental.

When $d \equiv \epsilon$ is small, that is, $|\epsilon| \ll |a|, |b|, |c|$, a small $\theta_{13}$ is generated and its value is correlated to the deviation from maximal 1–2 mixing. One should diagonalize

$$m_\nu m_\nu^{-1} = \begin{pmatrix} |a|^2 + |b|^2 & ab^* - a^*b & ae^* + c^*c \\ ab^* - a^*b & |a|^2 + |b|^2 & c^*b \\ ae^* + c^*c & c^*b & |c|^2 + |\epsilon|^2 \end{pmatrix}. \tag{18}$$

At leading order, we find

$$\sin \theta_{13} \approx \frac{|ae^* + c^*c|}{\Delta m_{23}^2},$$
$$\sin^2 \theta_{12} \approx \frac{1}{2} \left[ 1 - \frac{|\epsilon|^2}{\Delta m_{12}^2} \frac{|P_2(a, b, c)|}{\Delta m_{23}^2} \right], \tag{19}$$

where $P_2$ is a lengthy expression quadratic in $a, b$ and $c$.

We performed a numerical analysis for the most general choice of complex phases. The predicted correlation between $\sin^2 \theta_{12}$ and $\sin \theta_{13}$ is shown in Fig. 3. We find that, in the normal ordering case, $\sin \theta_{13} \lesssim 0.12$ is required to accommodate the non-maximal 1–2 mixing. Besides the opportunity to measure this value

![Image](image-url)
of $\theta_{13}$ already in the Double Chooz experiment, this scenario is also promising for future searches of leptonic CP violation \cite{2}. In the inverted ordering case, instead, there is no lower nor upper bound on $\sin \theta_{13}$. However, when $\epsilon$ is small we find $0.05 \lesssim \sin \theta_{13} \lesssim 0.12$ (gray region in the lower panel of Fig[3]. We checked that this is the case when $m_3 \gtrsim 0.05$ eV, in analogy with the CP-conserving case (gray region in the lower panel of Fig[2].

Conclusions. We studied the most general mass matrix for two mass-degenerate neutrinos. This is possibly a good limit to understand the smallness of the ‘solar’ mass splitting $\Delta m^2_{12}$ relative to $\Delta m^2_{23}$. We have shown that such a mass degeneracy requires the equality of the 11 and 22 matrix elements, as well as the vanishing of one CP violation. The first requirement points to a non-Abelian family symmetry, the second indicates that CP violation in the lepton sector should be not generic, if present at all. The matrix structure leading to the mass degeneracy is most easily accommodated, in the framework of three families, if neutrinos only mix in the 1–2 sector and, therefore, the large 2–3 lepton mixing comes from the charged lepton sector.

We have shown that all these features can be explained by the simplest quaternion family symmetry $Q$, together with the requirement of no CP violation. We discussed several realizations of our $Q$ model, either employing several Higgs doublets or heavy flavon fields, and realizing the seesaw by either Higgs triplets or right-handed neutrinos. In the limit where $\Delta m^2_{12} = 0$, the model predicts also $\theta_{13} = 0$.

We discussed the possible perturbations generating nonzero $\Delta m^2_{12}$ and the consequent correlation with the nonzero value of $\theta_{13}$. We studied in detail the predictive case where $m_e$ depends only on three real parameters plus one small perturbation. Both normal and inverted ordering of the mass spectrum can be realized. In the normal case, $\sin \theta_{13} \gtrsim 0.12$ should be found, close to the present upper bound. In the inverted case, $\sin \theta_{13}$ is smaller than about 0.1. A lower bound ($\sin \theta_{13} \gtrsim 0.05$) holds in the inverted case when the absolute neutrino mass scale $m_3$ (or equivalently the neutrinoless $2\beta$ effective mass $m_{ee}$) is larger than about 0.05 eV.

If CP violating phases are present, the $Q$ model predicts a correlation between nonzero $\theta_{13}$ and the deviation of $\theta_{12}$ from the maximal value. Also in this scenario a lower bound $\sin \theta_{13} \gtrsim 0.12$ holds in the normal ordering case.

Acknowledgments. We thank the Aspen Center for Physics for hosting the 2007 workshop on “Neutrino Physics: Looking Forward” where we began our discussions. MF also thanks the Department of Physics and Astronomy, University of California, Riverside, for hospitality during a subsequent visit. MF was supported in part by the CNRS/USA exchange grant 3503 and by the RTN European Program MRTN-CT-2004-503369. EM was supported in part by the U.S. Department of Energy under Grant No. DE-FG03-94ER40837.

Appendix: Alignment of VEVs

The $Q$ model defined by eq.(6) contains three Higgs doublets and three Higgs triplets. Here we will discuss some features of the scalar potential responsible for generating their VEVs.

Let us consider the most general $Q$-invariant potential for $(\phi_1, \phi_2) \sim 2$ (notice that the conjugate $Q$-doublet is $(\phi'_2, -\phi'_1)$):

$$V = m^2(\phi_1^+ \phi_1 + \phi_2^+ \phi_2) + \frac{1}{2} \lambda_1[(\phi_1^+ \phi_1)^2 + (\phi_2^+ \phi_2)^2] + \lambda_3(\phi_1^+ \phi_1)(\phi_2^+ \phi_2) + \lambda_4(\phi_1^+ \phi_1)^2 + \lambda_4(\phi_2^+ \phi_2)^2$$

(20)

It is bounded from below as long as $\lambda_1 > |\lambda_2| + |\lambda_3|$. For $m^2 < 0$, the electroweak symmetry is broken in a minimum where $|v_1|^2 = |v_2|^2 = -m^2/(\lambda_1 + \lambda_3 + \lambda_4 - |\lambda_2|)$. The vacuum with $v_1 = 0$ and $v_2 = -m^2/\lambda_1$ is a saddle. In order to make it the minimum, one needs to break softly Q to Z4, by adding

$$V_{soft} = \mu^2(\phi_2^+ \phi_2 - \phi_1^+ \phi_1)$$

(21)

Notice that no other soft terms are allowed by the residual Z4 symmetry. Then the electroweak symmetry is broken if $\mu^2 < |\mu|^2$ and the vacuum with $v_1 = 0$ and $v_2 = -(m^2 + \mu^2)/\lambda_1$ is the absolute minimum as long as $\mu^2(\lambda_1 + \lambda_3 + \lambda_4 - |\lambda_2|) > m^2(\lambda_1 - \lambda_3 - \lambda_4 + |\lambda_2|)$. This alignment of VEVs is the one analyzed in the text.

In fact, the analysis would be pretty much the same, even if one does not want to introduce $V_{soft}$. In this case the minimum is given by $v_1 = v_2 = v$ (strictly speaking the relative phase between $v_1$ and $v_2$ can be $\pm i$, but for the sake of brevity we illustrate only one possibility). Then, the charged lepton mass matrix in eq.(7) can be rewritten as

$$m_l = \begin{pmatrix}
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\
0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
0 & -\sqrt{2} y_{12} v & 0 \\
\sqrt{2} y_{12} v & 0 & \sqrt{2} y_{13} v \\
0 & 0 & y_{14} v_3
\end{pmatrix}$$

(22)

Therefore, up to a maximal 1–2 rotation on the left and a harmless swap of $e_1^c$ and $e_2^c$ on the right, one recovers the same structure of $m_l$ as in the case $v_1 = 0$. Notice that the maximal 1–2 rotation does not modify the structure of the neutrino mass matrix $m_\nu$, in eq.(7), so that all predictions for the lepton mixing angles are unchanged. However, such rotation introduces FCNCs involving the electron, so that the non-standard Higgs bosons should be heavier to evade the experimental constraints.

The addition of the third Higgs doublet $\phi_3 \sim 1^{−}$ does not change the qualitative features of the scalar potential discussed above.
Let us briefly discuss the origin of the VEVs of the three Higgs triplets in eq.\((\ref{eq:VEV})\), as well as of the two triplets \((\Delta_4, \Delta_5) \sim 2\), needed to generate \(m_\nu\) in eq.\((\ref{eq:mass})\). The \(Q\)-invariant scalar potential reads

\[
V = \sum_{i=1}^{3} M_i^2 \Delta_i^4 + M_4^2 (\Delta_1^4 \Delta_4 + \Delta_2^4 \Delta_5) + \mu_1 \Delta_1 \phi_1 \phi_2 + \mu_2 \Delta_2 (\phi_1 \phi_1 - \phi_2 \phi_2) + \mu_3 \Delta_3 \phi_3 \phi_3 + \mu_4 (\Delta_1 \phi_1 \phi_3 + \Delta_5 \phi_2 \phi_3) + \text{h.c.} + V_{\text{quartic}}^\Delta.
\]

In the hypothesis that \(M_1\) is much larger than the electroweak scale, one can integrate out \(\Delta_i\) using its equation of motion. The triplet VEVs are thus determined as follows:

\[
u_i \equiv -\frac{\lambda_{u3}}{M_4^2} v_1 v_2, \quad v_2 = -\frac{\lambda_{u3}}{M_4^2} (v_1^2 - v_2^2), \quad u_3 = -\frac{\lambda_{u3}}{M_4^2} v_{1,2} v_3.
\]

In the case \(v_1 = 0\), one has automatically \(u_4 = 0\) as assumed in eq.\((\ref{eq:mass})\), however also \(u_1\) vanishes, which is not acceptable. There are two ways to cure this problem. The first is once again to break \(Q\) softly, e.g. \(V_{\text{soft}}^\Delta = M_4^2 (\Delta_1^4 \Delta_2 + \Delta_2^4 \Delta_1)\), that allows to induce both \(u_1\) and \(u_2\) from \(v_2\) only.

A second possibility without soft breaking is to resort to \(Q\)-invariant quartic couplings among Higgs doublets and triplets. Consider, in particular,

\[
\lambda (\phi_1 \phi_1 - \phi_2 \phi_2) (\Delta_1^4 \Delta_1 + \Delta_2^4 \Delta_3) \subset V_{\text{quartic}}^\Delta.
\]

Then the equation of motion for \(\Delta_1\) leads to \(\nu_1 = \lambda u_3^2 v_2^2 / M_4^2\). This contribution is tiny if all triplet masses are of the same order, but when \(M_1\) is instead close to the electroweak scale one can obtain \(u_1 \sim u_3\).

---

[1] M. Maltoni, T. Schwetz, M. A. Tortola and J. W. F. Valle, New J. Phys. 6 (2004) 122 [hep-ph/0401172].
[2] D. Duchesneau [OPERA Collaboration], eConf C0209101 (2002) TH09 [Nucl. Phys. Proc. Suppl. 123 (2003) 279 [hep-ex/0209026]; F. Ardelli et al. [Double Chooz Collaboration], hep-ex/0609025; Y. Itow et al. [The T2K Collaboration], hep-ex/0903055; P. Huber, M. Lindner, M. Rolines, T. Schwetz and W. Winter, Phys. Rev. D 70 (2004) 073014 [hep-ph/0403068].
[3] M. Frigerio, S. Kaneko, E. Ma and M. Tanimoto, Phys. Rev. D 71 (2005) 011901 [hep-ph/0409187]; M. Frigerio, hep-ph/0505144.
[4] D. Chang, W. Y. Keung and G. Senjanovic, Phys. Rev. D 42 (1990) 1599; D. Chang, W. Y. Keung, S. Lipovaca and G. Senjanovic, Phys. Rev. Lett. 67 (1991) 953.
[5] P. H. Frampton and T. W. Kephart, Int. J. Mod. Phys. A 10 (1995) 4689 [hep-ph/9409330]; P. H. Frampton and O. C. W. Kong, Phys. Rev. Lett. 75 (1995) 781 [hep-ph/9502395]; P. H. Frampton and A. Rasin, Phys. Lett. B 478 (2000) 424 [hep-ph/9910522].
[6] K. S. Babu and J. Kubo, Phys. Rev. D 71 (2005) 056006 [hep-ph/0411226]; Y. Kajiyama, E. Itou and J. Kubo, Nucl. Phys. B 743 (2006) 74 [hep-ph/0511268].
[7] A. Aranda, C. D. Carone and R. F. Lebed, Phys. Lett. B 474 (2000) 170 [hep-ph/9910392]; A. Aranda, C. D. Carone and R. F. Lebed, Phys. Rev. D 62 (2000) 016009 [hep-ph/0002044]; A. Aranda, 0707.3661 [hep-ph].
[8] P. D. Carr and P. H. Frampton, hep-ph/0701034; F. Fergedon, C. Hagedorn, Y. Lin and L. Merlo, Nucl. Phys. B 775 (2007) 120 [hep-ph/0702194]; M. C. Chen and K. T. Mahanthappa, 0705.0714 [hep-ph]; P. H. Frampton and T. W. Kephart, 0706.1186 [hep-ph].
[9] L. Wolfenstein, Nucl. Phys. B 186 (1981) 147.
[10] S. T. Petcov, Phys. Lett. B 110 (1982) 245.
[11] A. S. Joshipura and S. D. Rindani, Phys. Lett. B 494 (2000) 114 [hep-ph/0007334].
[12] A. S. Joshipura, S. D. Rindani and N. N. Singh, Nucl. Phys. B 660 (2003) 362 [hep-ph/0211378].

---

An extensive list of references on 

---

[13] An extensive list of references on 

---

[14] W. Grimus and L. Lavoura, Phys. Lett. B 572 (2003) 189 [hep-ph/0305046]; W. Grimus, A. S. Joshipura, S. Kaneko, L. Lavoura and M. Tanimoto, JHEP 0407 (2004) 078 [hep-ph/0407112]; T. Kobayashi, S. Raby and R. J. Zhang, Nucl. Phys. B 704 (2005) 3 [hep-ph/0409098]; T. Kobayashi, H. P. Nilles, F. Ploeg, S. Raby and M. Ratcliffe, Nucl. Phys. B 768 (2003) 155 [hep-ph/0611029].
[15] S. L. Chen, M. Frigerio and E. Ma, Phys. Lett. B 612 (2005) 29 [hep-ph/0412018].
[16] A. S. Joshipura and S. D. Rindani, Phys. Rev. D 67 (2003) 073009 [hep-ph/0211404].
[17] A. S. Joshipura, Phys. Lett. B 543 (2002) 276 [hep-ph/0205038].
[18] A. S. Joshipura and S. Mohanty, Phys. Rev. D 660 (2003) 091302 [hep-ph/0302181].
[19] S. Kaneko, H. Sawanaka and M. Tanimoto, JHEP 0508 (2005) 073 [hep-ph/0504074].