An $L$-function free proof of Hua’s Theorem on sums of five prime squares

CLAUS BAUER

Abstract

We provide a new proof of Hua’s result that every sufficiently large integer $N \equiv 5 \,(mod \,24)$ can be written as the sum of the five prime squares. Hua’s original proof relies on the circle method and uses results from the theory of $L$-functions. Here, we present a proof based on the transference principle first introduced in[5]. Using a sieve theoretic approach similar to ([10]), we do not require any results related to the distributions of zeros of $L$-functions. The main technical difficulty of our approach lies in proving the pseudorandomness of the majorant of the characteristic function of the $W$-tricked primes which requires a precise evaluation of the occurring Gaussian sums and Jacobi symbols.

Mathematics Subject classification: 11(P32), 11(P70), 11(N35).

1 Introduction

In 1938, Hua ([7]) showed the following result in additive prime number theory related to the sum of five prime squares:

Theorem 1.1 Every sufficiently large integer $N \equiv 5 \,(mod \,24)$ can be written as the sum of five prime squares.

Similar to most results in additive prime theory, Hua’s proof uses the circle method and relies on the theory of Dirichlet $L$-functions. Starting with the work of Green ([5], [6]), a different approach to problems in additive prime number theory that relies on the transference principle in additive combinatorics has been applied. In [8] and [9], it has been used to prove a density version of Vinogradov’s three primes theorem. Whereas these proofs rely on results related to the distribution of zeros of $L$-functions, in [10] a sieve theoretic approach not relying on the theory of $L$-functions is used in combination

* Claus Bauer, Dolby Laboratories, Beijing, China, cb@dolby.com
with the transference principle to derive a new proof of Vinogradov’s theo-
rem. The approach in [10] further differs from the methods in [8], [9] by the
fact that the transference principle is applied to the set of positive integers \( \mathbb{Z} \) instead of being applied to the modular group \( \mathbb{Z}/N\mathbb{Z} \). So far, the transference principle has been mostly applied to linear problems in additive prime number theory. In [4], a first application to a non-linear problems is described. In this paper, we will prove Theorem 1.1. Different from [4], we will - as in [10] - not use the theory of \( L \)-functions. A main result and a principal ingredient for the proof of Theorem 1.1 is the following transference principle in \( \mathbb{Z} \):

**Theorem 1.2** (Transference principle in \( \mathbb{Z} \) with majorant equal to one) Let \( 0 < \delta, \kappa < 1, 1/10 < \delta_1 < 1 \) be given. Let \( N \) be a sufficiently large positive integer. Let \( N_1 = N_2 = \lfloor N/6 \rfloor, N_3 = N_4 = \lfloor N/2 \rfloor, \) and \( N_5 = N \). For \( i = 1, 2, 3, 4, 5 \), let \( a_i : [1, N_i] \to [0, 1] \) be arbitrary functions. Let \( \alpha_i \) be the average of \( a_i \) for \( i = 1, \ldots, 5 \). Suppose that the functions \( a_i \) satisfy the following assumptions:

1. (Mean condition) \( \alpha_i \geq \delta_1 (i = 1, 2), \alpha_i \geq \delta (i = 3, 4, 5), \frac{1}{2} \left( \min (1, \alpha_3 + \alpha_4) + \alpha_4 \right) + \alpha_5 \geq 1 + \delta \).
2. (Regularity condition for \( a_4 \)) The function \( a_4 \) is \( (\delta/50, \kappa) \)-regular.

Then

\[
\sum_{n_1 + n_2 + n_3 + n_4 + n_5 = N} a_1(n_1)a_2(n_2)a_3(n_3)a_4(n_4)a_5(n_5) \geq cN^4,
\]

where \( c = c(\delta, \kappa) > 0 \) is a constant depending only on \( \delta \) and \( \kappa \).

The regularity condition (2) in Theorem 1.2 is defined as follows. For \( y > 2 \), let \( P(y) \) be the product of all primes up to \( y \).

**Definition 1.1** Let \( f : [1, N] \to \mathbb{R} \) be an arbitrary function. The function is said to be \((\beta, \kappa)\)-regular if

\[
\sum_{(u, v) \in M} f(u)f(v) \geq \kappa N^2,
\]

where

\[
M = \{(u, v) : u \leq \beta N, v \geq (1 - \beta)N, (v - u, P(\beta^{-1})) = 1\}.
\]

We note that Theorem 1.2 assumes that the functions \( a_i \) are bounded by a constant majorant equal to one. For many applications ([5], [9], [10]), this
requirement is too strict as one requires a transference principle valid for more general majorant functions. Due to the assumed discrete majorant property of the functions \(a_i\), one can derive such transference principles valid for more general majorants that are pseudorandom. Similarly in this paper, for the proof of Theorem 1.1, we will require the following transference principle valid for majorants that are pseudorandom:

**Theorem 1.3 (Transference principle in \(\mathbb{Z}\))** Let \(0 < \delta, \kappa < 1, 1/10 < \delta_1 < 1\) be given. Let \(\eta > 0\) be sufficiently small and \(N\) be a sufficiently large positive integer. Let \(N_1 = N_2 = \lfloor N/6 \rfloor, N_3 = N_4 = \lfloor N/2 \rfloor, \text{ and } N_5 = N\). For \(i = 1, 2, 3, 4, 5\), let \(a_i, v_i : [1, N_i] \to \mathbb{R}\) be arbitrary functions. Let \(\alpha_i\) be the average of \(a_i\) for \(i = 1, \ldots, 5\). Suppose that the functions \(a_i\) and \(v_i\) satisfy the following requirements:

1. **(Majorization condition)** \(0 \leq a_i(n) \leq v_i(n)\) for all \(1 \leq n \leq N_i, i = 1, 2, 3, 4, 5\).

2. **(Mean condition)** \(\alpha_i \geq \delta_1(i = 1, 2), \alpha_i \geq \delta(i = 3, 4, 5), \frac{1}{2} \left( \min(1, \alpha_3 + \alpha_4) + \alpha_4 \right) + \alpha_5 \geq 1 + \delta\).

3. **(Pseudorandom condition)** The functions \(v_i, i = 1, \ldots, 5\) are pseudorandom.

4. **(Discrete majorant property)** The functions \(a_i, i = 1, \ldots, 5\) satisfy the discrete majorant property for some \(4 < q < 5\).

5. **(Regularity condition for \(a_4\))** The function \(a_4\) is \((\delta/50, \kappa)\)-regular.

Then

\[
\sum_{\substack{n_1 + n_2 + n_3 + n_4 + n_5 = N \\ n_i \in N_i}} a_1(n_1)a_2(n_2)a_3(n_3)a_4(n_4)a_5(n_5) \geq cN^4,
\]

where \(c = c(\delta, \kappa) > 0\) is a constant depending only on \(\delta\) and \(\kappa\).

To define the pseudorandom condition (3) and the discrete majorant property (4) in Theorem 1.3, we introduce some more terminology: For a (compactly supported) function \(f : \mathbb{Z} \to \mathbb{R}\), its Fourier transform is defined as

\[
\hat{f}(\theta) = \sum_{n \in \mathbb{Z}} f(n)e(n\theta),
\]
where \( e(n\theta) = \exp(2\pi in\theta) \). The \( L^q \) norm of the Fourier transform is defined by

\[
||\hat{f}||_q = \left( \int_0^1 |\hat{f}(\theta)|^q d\theta \right)^{1/q}.
\]

**Definition 1.2** The function \( f \) is said to be \( \eta \)-pseudorandom if

\[
|\hat{f}(r/N) - \delta_r,0| \ll \eta N \quad \text{for each} \quad r \in \mathbb{Z}/N\mathbb{Z},
\]

where \( \delta_r,0 \) is the Kronecker delta.

**Definition 1.3** The function \( f \) is said to satisfy the discrete majorant property if

\[
||\hat{f}||_q \ll q N^{1-1/q},
\]

where the implied constant depends on \( q \) only.

Our paper is structured as follows. We first prove Theorem 1.2 in section 2. Subsequently, we derive Theorem 1.3 from Theorem 1.2 in section 3. In section 4, we will define the \( W \)-tricked primes squares and their majorant function. In sections 5 - 8, we will show that the characteristic function of the \( W \)-tricked primes squares and its majorant satisfy the assumptions of Theorem 1.3. This will allow us applying Theorem 1.3 to prove Theorem 1.1.

## 2 Proof of Theorem 1.2

We will make use of the following Theorem ([10, Th. 1.4]):

**Theorem 2.1** (Transference principle in \( \mathbb{Z} \) with majorant equal to one) Let \( 0 < \delta, \kappa < 1 \) be given. Then for sufficiently small \( \eta > 0 \) and sufficiently large positive \( N \), the following statement holds: Let \( N_1 = N_2 = [N/2] \), and \( N_3 = N \). For \( i = 1, 2, 3 \) let \( a_i : [1, N_i] \to [0, 1] \) be arbitrary functions. Let \( \alpha_i \) be the average of \( a_i \) for \( i = 1, 2, 3 \). Suppose that the functions \( a_i \) satisfy the following assumptions:

1. (Mean condition) \( \alpha_i \geq \delta (i = 1, 2, 3) \), \( \frac{1}{2} (\min (1, \alpha_1 + \alpha_2) + \alpha_2) + \alpha_3 \geq 1 + \delta \).

2. (Regularity condition for \( a_3 \)) The function \( a_1 \) is \((\delta/50, \kappa)\)-regular.

Then

\[
\sum_{n_1 + n_2 + n_3 = N \atop n_1 \in N_1} a_1(n_1)a_2(n_2)a_3(n_3) \geq cN^2,
\]

where \( c = c(\delta, \kappa) > 0 \) is a constant depending only on \( \delta \) and \( \kappa \).
We now proceed to the proof of Theorem 1.2. For a given positive integer \( m \leq N \) we define \( m_i, i = 1, 2, 3, 4, 5 \), as a function of \( m \) in the same way we have defined \( N_i \) as a function of \( N \) in Theorem 1.2. Further, we for each \( i = 1, 2, 3, 4, 5 \), we define a function \( a_{i,m}(n) \) as follows: \( a_{i,m}(n) = a_i(n) \) for \( n \leq m_i \) and \( a_{i,m}(n) = 0 \) for \( n_i < n \leq N \). Thus \( a_{i,m}(n) \leq a_i(n), \forall n \in [1, N] \).

We note that Theorem 2.1 is valid for all \( N > N_c \), where \( N_c \) is a large positive integer. If we choose \( N \) sufficiently large, we can assume that \( N/100 > N_c \), which we will do in the following. Applying Theorem 2.1 to the functions \( a_{3,m}, a_{4,m} \) and \( a_{5,m} \) and using the mean condition \( \alpha_i \geq \frac{100}{m_i}, i = 1, 2 \), we see

\[
\sum_{n_1 + n_2 + n_3 + n_4 + n_5 = N} a_1(n_1)a_2(n_2)a_3(n_3)a_4(n_4)a_5(n_5)
= \sum_{m \leq N} \sum_{n_1 + n_2 = N - m} a_1(n_1)a_2(n_2) \sum_{n_3 + n_4 + n_5 = m} a_3(n_3)a_4(n_4)a_5(n_5)
\geq \sum_{N/100 < m \leq N} \sum_{n_1 + n_2 = N - m} a_1(n_1)a_2(n_2) \sum_{n_3 + n_4 + n_5 = m} a_3(n_3)a_4(n_4)a_5(n_5)
\geq \frac{cN^2}{10000} \sum_{N/100 < m \leq N} \sum_{n_1 + n_2 = N - m} a_1(n_1)a_2(n_2)
\geq \frac{cN^2}{10000} \sum_{2N/3 < m \leq 9N/100} \sum_{n_1 + n_2 = N - m} a_1(n_1)a_2(n_2)
= \frac{cN^2}{10000} \sum_{n_1, n_2 \in [N/200, N]} a_1(n_1)a_2(n_2)
\geq \frac{cN^4}{10000} \left( \delta_1 - \frac{1}{200} \right)^2,
\]

which proves Theorem 1.2 with \( c := 10^{-8}c \).

3 Proof of Theorem 1.3

We introduce some additional terminology. For notational convenience, we will fix some \( i = 1, 2, 3, 4, 5 \), and set \( a(n) := a_i(n), \alpha_i := \alpha, N := N_i \). Let \( 0 < \epsilon < 1 \) be a small parameter chosen later which depends only on \( \delta \) and \( \kappa \). We set \( T_\epsilon := [0, 1] \). Let

\[
T_\epsilon = \{ \theta \in T : |\hat{a}(\theta)| \geq \epsilon N \}.
\]

Define

\[
B = B_\epsilon = \{ 1 \leq b \leq \epsilon N : \|b\theta\| < \epsilon \text{ for all } \theta \in T_\epsilon \}.
\]
where \(|x|\) denotes the distance from \(x\) to its closest integer. Using these definitions, we define the following functions:

\[
a'(n) = \mathbb{E}_{b_1, b_2 \in B} a(n + b_1 - b_2) = \frac{1}{B^2} \sum_{b_1, b_2 \in B} a(n + b_1 - b_2), \quad a''(n) = a(n) - a'(n).
\]

We will need the following Lemma:

**Lemma 3.1** Suppose that \(\eta\) is sufficiently small depending on \(\epsilon\). The functions \(a'\) and \(a''\) defined above have the following properties:

1. (\(a'\) is set-like) \(0\leq a'(n) \leq 1 + O_\epsilon(\eta)\) for any \(n\). Moreover, \(\mathbb{E}_{1 \leq n \leq N} a'(n) = \alpha + O(\epsilon)\).
2. (\(a''\) is uniform) \(a''(\theta) = O(\epsilon N)\) for all \(\theta\).
3. (\(a'_4\) is regular) \(a'_4(\delta/50, \kappa - O(\epsilon))\) is regular.
4. \(||\hat{a}'||_q \leq ||\hat{a}||_q, ||\hat{a}''||_q \leq ||\hat{a}||_q\).

**Proof:** The proof is word-by-word identical with the proof of [10, Lemma 3.3].

The following Lemma is the main ingredient for the proof of Theorem 1.3.

**Lemma 3.2** With the functions \(a_i\) and \(a'_i\) as defined above, we have

\[
\left| \sum_{n_1 \in N_1, \ldots, n_5 = N} a_1(n_1)a_2(n_2)a_3(n_3)a_4(n_4)a_5(n_5) \right| - \left| \sum_{n_1 \in N_1, \ldots, n_5 = N} a'_1(n_1)a'_2(n_2)a'_3(n_3)a'_4(n_4)a'_5(n_5) \right| \ll \epsilon^{5-q}N^4.
\]

**Proof:** The difference on the left side can be expressed as the sum of several terms of the form

\[
\sum_{n_1 \in N_1, \ldots, n_5 = N} f_1(n_1)f_2(n_2)f_3(n_3)f_4(n_4)f_5(n_5) = \int_{0}^{1} \hat{f}_1(\theta)\hat{f}_2(\theta)\hat{f}_3(\theta)\hat{f}_4(\theta)\hat{f}_5(\theta)e(-N\theta) d\theta.
\]

where \(f_i \in \{a_i, a'_i, a''_i\}\), and \(f_i = a''_i\) for at least one \(i\). Without loss of generality, we assume that \(f_5 = a''_5\). By Hölder’s inequality, for some \(4 < q < 5\), the right-hand side of (3.1) is bounded by

\[
||\hat{f}_5||_\infty^{5-q}||\hat{f}_5||_q^{-q}||\hat{f}_1||_q||\hat{f}_2||_q||\hat{f}_3||_q||\hat{f}_4||_q.
\]
By Lemma \[3.1\] \(|\hat{f}_5|_\infty \ll \epsilon N\). Further, by the discrete majorant property and Lemma \[3.1\] \(|\hat{f}_i|_q \ll N^{1-\frac{1}{4}}, i = 1, 2, 3, 4, 5\). Combining these estimates, we obtain the desired estimate.

**Proof of Theorem** \[1.3\] By Lemma \[3.1\] the functions \(a'_i\) are all bounded above uniformly by \(1 + O(\epsilon)\) with averages \(a_i + O(\epsilon)\) and \(a'_i\) is \((\delta/50, \kappa/2)\) - regular. If \(\epsilon\) and \(\eta\) are chosen small enough, then Theorem \[1.2\] implies

\[
\sum_{n_1 + n_2 + n_3 + n_4 + n_5 = N} a'_1(n_1)a'_2(n_2)a'_3(n_3)a'_4(n_4)a'_5(n_5) \geq c' N^4. \tag{3.2}
\]

Combining this with Lemma \[3.2\] we deduce by choosing \(\epsilon\) small enough that

\[
\sum_{n_1 + n_2 + n_3 + n_4 + n_5 = N} a_1(n_1)a_2(n_2)a_3(n_3)a_4(n_4)a_5(n_5) \geq \frac{c'}{2} N^4.
\]

This completes the proof of Theorem \[1.3\].

**4 Proof of Theorem** \[1.1\]

We first introduce the concept of \(W\)-tricked prime squares and define their majorant function. Subsequently, we prove Theorem \[1.1\].

**4.1 \(W\)-tricked prime squares and Selberg’s majorant**

Let \(W = 8 \prod_{2 < p \leq w} p\) be the product of the number eight and all odd primes not larger than a constant \(w\) and let \(b\) be a reduced residue class modulo \(W\). We set \([W] := \{1, \ldots, W\}\). We say that an integer \(q\) is \(w\)-smooth if none of its prime divisors is larger than \(w\). Otherwise, we call \(q\) \(w\)-rough. We denote the set of primes by \(\mathbb{P}\) and the set of prime squares by \(\mathbb{P}^2\). We fix a small constant \(\delta\) and set \(z_i = N_i^{1/4 - 2\delta}, i = 1, \ldots, 5\), where \(N_i\) is defined as in Theorem \[1.2\]. We assume that \(N\) is a sufficiently large integer depending on \(w\) and \(\delta\). Further, we define \(P := P_i\) as the product of all primes \(p < z_i\) and \((p, W) = 1\). We set

\[
\sigma(b) := \# \{z \in [W] : z^2 - b \equiv 0(\text{mod } W)\}.
\]

We define the \(\sigma(b)\) reduced residue classes \(h_j, 1 \leq j \leq \sigma(b)\), modulo \(W\) via the relation \(h_j^2 \equiv b(\text{mod } W)\). Using ideas from \[3\] and \[10\], we define the \(W\)-shifted prime squares:

\[
a_i(n) = \begin{cases} 
\frac{2\phi(W)\sqrt{Wn+b_i}}{W\sigma(b)} \log z_i, & \text{if } Wn + b_i = x^2 \text{ for some prime number } x \text{ with } z_i \leq x \leq \sqrt{W}N_i + b_i, \\
0, & \text{else.}
\end{cases}
\]

7
Following the approach in [10], we use Selberg’s upper bound sieve to define the majorant of $a_i(n)$ as follows:

$$v_i(n) = \begin{cases} 
\frac{2\phi(W)\sqrt{Wn+b_i}\log z_i}{W\sigma(b)} \left( \sum_{d|Wn+b_i, P} p_d \right)^2, & \text{if } Wn+b_i = x^2 \text{ for some natural integer } x \leq \sqrt{WN_i+b_i}, \\
0, & \text{else.}
\end{cases}$$

(4.1)

The real weights $\rho_d$ are supported on $d < z_i$, $\mu(d) \neq 0$, and satisfy $|\rho_d| \leq 1$, and $\rho_1 = 1$. If $a_i(n) \neq 0$, then $\sqrt{Wn+b_i} \geq z_i$ and therefore $(Wn+b_i, P) = 1$, such that $a_i(n) = v_i(n)$. This shows that $v_i(n)$ is a majorant of $a_i(n)$.

For later usage, we introduce the following notation:

$$J := J_i = \sum_{d|P \, d < z_i} \frac{1}{\phi(d)} = \sum_{d < z_i, d \text{ squarefree}} \frac{1}{\phi(d)}. \quad (4.2)$$

We know from [10] Appendix A, (A.1)] that

$$J_i = \frac{\phi(W)}{W} \left( \log z_i + O_W(1) \right). \quad (4.3)$$

**4.2 Proof of Theorem 1.1**

We will need the following auxiliary Lemmas:

**Lemma 4.1** Let $p$ be a prime number. Let $A, B, C$ be non-empty subsets of $\mathbb{Z}/p\mathbb{Z}$ such that $|A| + |B| + |C| \geq p + 2$. Then,

$$A + B + C = \mathbb{Z}/p\mathbb{Z}.$$

*Proof:* See [11] Proposition 2.6).

**Lemma 4.2** Let $p$ be a prime number $\geq 5$ and define $A_p$ as the set of all quadratic residues modulo $p$. Then ,

$$A_p + A_p + A_p + A_p + A_p = \mathbb{Z}/p\mathbb{Z}.$$

*Proof of Lemma 4.2:* As $|A_p| = (p-1)/2$, we see that for $p \geq 7$, 

$$|A_p + A_p + A_p| + |A_p| + |A_p| \geq |A_p| + |A_p| + |A_p| \geq p + 2.$$ 

For $p \geq 7$, the Lemma now follows from Lemma 4.1. For $p = 5$, the Lemma follows by a case by case analysis.
Lemma 4.3 For any integer $N \equiv 5 \pmod{24}$, there are integers $b_i, 1 \leq b_i \leq W$, $(W, b_i) = 1$, $i = 1, \ldots, 5$ where each $b_i$ is a quadratic residue modulo $W$ such that $N \equiv b_1 + b_2 + b_3 + b_4 + b_5 \pmod{24}$.

Proof: By case-by-case inspection, we see that $b = 1$ is the only quadratic residue modulo 24. Therefore, $N \equiv b_1 + b_2 + b_3 + b_4 + b_5 \pmod{24}$ for any quadratic residues $b_i$ modulo $W$. The Lemma now follows from Lemma 4.2 and the Chinese remainder theorem.

Proof of Theorem 1.1. Let $M$ be a sufficiently large integer with $M \equiv 5 \pmod{24}$. We will show that $M$ can be written as the sum of five prime squares. By Lemma 4.3 we can find integers $b_1, \ldots, b_5$ which are quadratic residues modulo $W$ such that $M \equiv b_1 + b_2 + b_3 + b_4 + b_5 \pmod{W}$. We set $N = (M - b_1 - b_2 - b_3 - b_4 - b_5)/W$. Using the integers $b_i$, we define the functions $a_i$ and $v_i$ as in section 1.1. In section 1.1 we have shown that the functions $v_i$ are majorants for the functions $a_i$. In sections 5-8 we will show that the functions $a_i$ and $v_i$ also satisfy the conditions (2) - (5) of Theorem 1.3. Therefore, by Theorem 1.3 there exist $n_i \in N_i$, $i = 1, \ldots, 5$, such that $a_i(n_i) > 0$, i.e., $Wn_i + b_i \in \mathbb{P}^5$, and $N = n_1 + n_2 + n_3 + n_4 + n_5$. Thus,

$$M = WN + \sum_{i=1}^{5} b_i = \sum_{i=1}^{5} Wn_i + b_i,$$

which proves that $M$ is the sum of five prime squares.

5 Mean condition

Lemma 5.4 For $i = 1, \ldots, 5$, we define by $\alpha_i$ the mean of $a_i(n)$.

i) For $i = 1, \ldots, 5$,

$$\alpha_i \geq 1/2 - 5\delta.$$

ii) For $\delta = 0.001$, the functions $a_i(n), i = 1, \ldots, 5$, satisfy the mean condition (2) of Theorem 1.3.

Proof of Lemma 5.4: To simplify notation, we write $a(n) = a_i(n)$, $b(n) = b_i(n)$, $z = z_i$, and $N = N_i$. We first prove part i).

$$\sum_{n \leq N} a(n) = \frac{2\phi(W)}{W\sigma(b)} \log z \sum_{\substack{n \leq N \leq \sqrt{Wn+b} \text{ prime } x \text{ with } z \leq x \leq \sqrt{Wn+b}}} x = \frac{2\phi(W)}{W\sigma(b)} \log z \sum_{\substack{j=1 \leq x \leq \sqrt{Wn+b} \text{ prime } x \text{ with } z \leq x \leq \sqrt{Wn+b}}} x$$
\[ \frac{2\phi(W)}{W\sigma(b)} \log z \sum_{j=1}^{\sigma(b)} \left( \frac{NW}{2\phi(W)\log(\sqrt{WN} + b)} - 1 \right) \]

\[ = N\log z / \log(\sqrt{WN} + b) - 2\epsilon \log z \geq N(1/4 - 2\delta)2(1 - \epsilon) \]

\[ \geq N(1/2 - 5\delta), \]

for \( \epsilon \leq \delta/100 \) which implies part i) of the Lemma. Part ii) is a direct consequence of i).

6 Pseudorandom condition

In this section, we set \( v(n) = v_i(n) \), \( N = N_i \), \( z = z_i \), and \( J = J_i \), for fixed \( i \in \{1, 2, 3, 4, 5\} \). The main purpose of this section is to prove the following Lemma:

**Lemma 6.1** For any \( r \in \mathbb{Z}/N\mathbb{Z} \),

\[ \hat{v} \left( \frac{r}{N} \right) = \left( \delta_{r, 0} + O_w \left( w^{-1/2 + \epsilon} \right) \right) N, \]

where \( \delta_{r, 0} \) is the Kronecker delta.

To prepare for the proof of Lemma 6.1 we introduce some further terminology. For integers \( q, d_1, d_2, W \), we define \( q_{d_1, d_2} := q / (q, [d_1, d_2]^2) \) and \( a_{d_1, d_2} = a[d_1, d_2]^2 / [q, [d_1, d_2]^2] \). We note that \( (W, q) = (W, q_{d_1, d_2}) \) is independent of \( [d_1, d_2] \) as \( (W, d_1 d_2) = 1 \) which follows from the summation condition \( d_i | \sqrt{W n} + b \) in (4.1). We define,

\[ h := (W, q) = (W, q_{d_1, d_2}), \quad W_1 := W/h, \quad q_{W, d_1, d_2} := q_{d_1, d_2} / h. \] (6.1)

For the proof of Lemma 6.1 we will analyse the following term defined for \( 1 \leq Y \leq N \):

\[ f_{d_1, d_2}(Y, \alpha) := \sum_{n \leq Y} \frac{2\sqrt{Wn + b}}{\sigma(b)} e(\alpha n). \] (6.2)

For the analysis of \( f_{d_1, d_2}(Y, \alpha) \), we divide the integral \( [0, W[ z^2 ]] \) into major arcs \( M \) and minor arcs \( m \) as follows: We set \( Q = [N^{2\delta/5}] \), \( R = [N^{1-\delta/2}] \), and

\[ M = \bigcup_{q_1 \leq Q} \bigcup_{a_1 = 1}^{q_1-1} M(a_1, q_1), \]

\[ M(a_1, q_1) = \left\{ \alpha \in [0, W[ z^2 ]]: \left| \alpha - a_1 \right| / q_1 \leq \frac{1}{q_1 R} \right\}, \]

\[ m = [0, W[ z^2 ]] \setminus M. \] (6.3)
Using the definitions (6.2) and (6.3), we can express \( \hat{v}(r/N) \) as follows:

\[
\hat{v}(r/N) = \hat{v}_1(r/N) + \hat{v}_2(r/N) + \hat{v}_3(r/N),
\]

(6.4)

where

\[
\begin{align*}
\hat{v}_1(r/N) &= \phi(W) \log z W \sum_{d_1, d_2} q_{[d_1, d_2]^2} p_{d_1} p_{d_2} f_{d_1, d_2}(N, r/N), \\
\hat{v}_2(r/N) &= \phi(W) \log z W \sum_{d_1, d_2} \frac{p_{d_1} p_{d_2} f_{d_1, d_2}(N, r/N)}{rW[d_1, d_2]^2/N \in \mathbb{M}} \\
\hat{v}_3(r/N) &= \phi(W) \log z W \sum_{d_1, d_2} \frac{p_{d_1} p_{d_2} f_{d_1, d_2}(N, r/N)}{rW[d_1, d_2]^2/N \in \mathbb{M}}.
\end{align*}
\]

(6.5)

Lemma 6.1 follows from the following three Lemmas:

**Lemma 6.2** For any \( r \in \mathbb{Z}/N\mathbb{Z} \),

\[
\hat{v}_1(r/N) = \left( \delta_{r, 0} + O_w \left( w^{-1/2+\epsilon} \right) \right) N.
\]

**Lemma 6.3** For any \( r \in \mathbb{Z}/N\mathbb{Z} \),

\[
\hat{v}_2(r/N) \ll_w N w^{-1/2+\epsilon}.
\]

**Lemma 6.4** For any \( r \in \mathbb{Z}/N\mathbb{Z} \),

\[
\hat{v}_3(r/N) \ll N^{1-\delta/400}.
\]

We prove the Lemmas 6.2 - 6.4 in the next three sections. We will use ideas from [3, Section 5] and [10, Appendix A].

**6.1 Proof of Lemma 6.2**

We initially assume \( \frac{r}{N} = \frac{a}{q} \) and analyse the term of \( f_{d_1, d_2}(Y, a/q) \) in section 6.1.1. Then, we prove Lemma 6.2 in section 6.1.2.

**6.1.1 The case \( \frac{r}{N} = \frac{a}{q} \)**

We will make use of the following lemma:
Lemma 6.5 For any positive integer $q$ dividing $P$, the sum
\[ T(q) := \sum_{\substack{d_1,d_2|P \\ q|[d_1,d_2]}} \frac{p_{d_1}p_{d_2}}{|d_1,d_2|} \] (6.6)
satisfies
\[ |T(q)| \ll J^{-1}q^{-1+\epsilon}, \]
where $J$ is defined in (4.3). Moreover, $T(1) = J^{-1}$.

Proof of Lemma: See [10, Lemma A.3].

Lemma 6.6 For any positive integer $q,d_1,d_2,W$, with $q|[d_1,d_2]^2$, $([d_1,d_2],W) = 1$, and for two co-prime integers $a$ and $q$ there is:
\[ f_{d_1,d_2}(Y,a/q) = \frac{\epsilon_q Y}{[d_1,d_2]} + O\left(Y^{1/2}W^{1/2}\right), \]
where $\epsilon_q = \epsilon_q(a/q,W,b)$ does not depend on $Y$. Moreover $\epsilon_q = 1$ if $q = 1$, $|\epsilon_q| \leq 1$ if $q > 1$, $(q,W) = 1$, and $\epsilon_q = 0$ if $(q,W) > 1$.

Proof of Lemma 6.6: We write $f_{d_1,d_2}(Y,a/q)$ as follows:
\[ f_{d_1,d_2}(Y,a/q) = \frac{2}{\sigma(b)} \sum_{\substack{x \leq \sqrt{W+Y} \\ x^2 \equiv a \pmod{d_1,d_2}}} xe\left(\frac{a(x^2-b)}{qW}\right) \] (6.7)
\[ = \frac{2e\left(-\frac{ab}{qW}\right)}{\sigma(b)} [d_1,d_2] \sum_{u^2[d_1,d_2]^2 - k \equiv 0 (\bmod W)} ud\left(\frac{a[d_1,d_2]^2u^2}{qW}\right), \] (6.8)

Breaking the sum over $u$ into congruence classes modulo $W$, we see from (6.8) and $q|[d_1,d_2]^2$:
\[ f_{d_1,d_2}(Y,a/q) = \frac{2e\left(-\frac{ab}{qW}\right)}{\sigma(b)} [d_1,d_2] \sum_{z \in [W]} \sum_{\substack{z+YW \leq \sqrt{YW+b}/[d_1,d_2]}} e\left(\frac{a[d_1,d_2]^2(z+yW)^2}{qW}\right) \]
\[ = \frac{2e\left(-\frac{ab}{qW}\right)}{\sigma(b)} [d_1,d_2] \sum_{z \in [W]} \sum_{\substack{z+YW \leq \sqrt{YW+b}/[d_1,d_2]}} e\left(\frac{a[d_1,d_2]^2z^2}{qW}\right) \]
\[ = \left(\frac{Y}{[d_1,d_2]} + O\left(Y^{1/2}W^{1/2}\right)\right) \frac{1}{\sigma(b)} \sum_{z \in [W]} e\left(\frac{a([d_1,d_2]^2z^2-b)}{qW}\right). \] (6.9)
We note that the inner sum over $z$ is equal to $\sigma(b[d_1, d_2]^2) = \sigma(b)$ if $q = 1$. If $q > 1$, and $(q, W) = 1$ the absolute value of the sum over $z$ is $\leq \sigma(b[d_1, d_2]^2) = \sigma(b)$. Further, if $(q, W) > 1$, the assumptions $(W, b) = 1$ and $q|d_1, d_2|^2$, contradict with the summation condition $[d_1, d_2]|\sqrt{Wn + b}$ in (6.2), i.e., $(q, W) > 1 \Rightarrow f_{d_1, d_2}(Y, a/q) = 0$. The Lemma now follows from (6.9).

By the definition of $v(n)$ in (4.1), we can always assume that $d_1$ and $d_2$ are square-free. Thus, if $q|d_1, d_2|^2$, $q$ cannot be divided by the third power of a prime number. Therefore, we can write $q = q_1 q_2^2$, where $(q_1, q_2) = 1$, $\mu(q_1 q_2) \neq 0$. We note that $q|d_1, d_2|^2 \iff q_1 q_2|d_1, d_2|$, (6.10) and $q_1 q_2 \geq q^{1/2}$. (6.11)

**Lemma 6.7** Set $U = N^{1/2-5/100}$. For two co-prime integers $a$ and $q = q_1 q_2^2$, where $(q_1, q_2) = 1$, $\mu(q_1 q_2) \neq 0$, and $Y \leq N$, there is

$$\sum_{d_1, d_2 | \sqrt{W} \leq z} e_1 q_1 q_2 f_{d_1, d_2}(Y, a/q) = \epsilon_1 q Y T(q_1 q_2) + O \left( Y^{1/2} W^{1/2} z^2 \right),$$

where $T(q)$ is defined in (6.6) and $\epsilon_1$ is as defined in Lemma 6.6.

**Proof of Lemma 6.7** From (6.6), Lemma 6.6 (6.10) and the fact that $|p_d| \leq 1$, we see

$$\sum_{d_1, d_2 | \sqrt{W} \leq z} e_1 q_1 q_2 f_{d_1, d_2}(Y, a/q) = \sum_{q_1 q_2 | d_1, d_2} e_1 q_1 q_2 f_{d_1, d_2}(Y, a/q)$$

$$= \epsilon_1 q Y T(q_1 q_2) + O \left( Y^{1/2} W^{1/2} \left( \sum_{d \leq z} |p_d|^\alpha \right)^2 \right) = \epsilon_1 q Y T(q_1 q_2) + O \left( Y^{1/2} W^{1/2} z^2 \right).$$

(6.12)

**6.1.2 Proof of Lemma 6.2**

In addition to the set of major arcs $\mathfrak{M}$ and minor arcs $\mathfrak{m}$ defined in (5.3), we defined a second set of major arcs $\mathfrak{M}_1$ and minor arcs $\mathfrak{m}_1$ as follows: We recall
the definition of $R = \lfloor N^{1-\delta/2} \rfloor$, set $U = N^{1/2-\delta/100}$, and define

$$
\mathcal{M}_1 = \bigcup_{q \leq U} \bigcup_{(a, q) = 1} \mathcal{M}(a_1, q_1),
$$

$$
\mathcal{M}_1(a, q) = \left\{ \alpha \in [0, 1] : \left| \alpha - \frac{a}{q} \right| \leq \frac{1}{qR} \right\},
$$

$$
m_1 = \mathbb{Z}/N\mathbb{Z} \setminus \mathcal{M}_1. \quad (6.13)
$$

We prove Lemma 6.2 separately for $r/N \in \mathcal{M}_1$ and $r/N \in m_1$ in the next two paragraphs.

6.1.2.1 The major arc case: For $r/N \in \mathcal{M}_1$, we write

$$
\frac{r}{N} = \frac{a}{q} + \beta, \quad q \leq U, \quad (a, q) = 1, \quad |\beta| \leq 1/qR. \quad (6.14)
$$

If $\beta = 0$, we see from Lemma 6.7

$$
\frac{\phi(W) \log z}{W} \sum_{d_1, d_2 \mid P, q \mid d_1, d_2} p_{d_1} p_{d_2} f_{d_1, d_2}(Y, a/q) = \frac{\phi(W) \log z}{W} (\epsilon_q YT(q_1 q_2) + E(Y)), \quad (6.15)
$$

where

$$
E(Y) = O \left( Y^{1/2} W^{1/2} z^2 \right).
$$

Applying partial summation, we derive from (6.15) and (6.16),

$$
\hat{v}_1 \left( \frac{r}{N} \right) = \frac{\phi(W) \log z}{W} \int_0^N e(\beta x) d \left( \sum_{d_1, d_2 \mid P} p_{d_1} p_{d_2} f_{d_1, d_2}(x, a/q) \right)
$$

$$
= \frac{\phi(W) \log z}{W} \epsilon_q T(q_1 q_2) \int_0^N e(\beta x) dx + \frac{\phi(W) \log z}{W} \int_0^N e(\beta x) dE(x). \quad (6.16)
$$

Using (6.14), we estimate the error term in (6.16) as follows:

$$
\left| \int_0^N e(\beta x) dE(x) \right| \ll |E(N)| + \int_0^N E(x)(2\pi i \beta) e(\beta x) dx
$$
Combining (6.16) - (6.17), we obtain

\[ \hat{v}_1 \left( \frac{r}{N} \right) = \frac{\phi(W)}{W} \log z \epsilon_q T(q_1,q_2) \int_0^N e(\beta x) dx + O \left( \frac{N^{1-\delta/10}}{z} \right). \]

(6.18)

We now derive Lemma 6.2 from (6.18): If \( w < q \leq U \), then Lemma 6.2 follows from (6.3), Lemma 6.5, and (6.11). If \( 1 < q \leq w \), then \( (q,W) > 1 \), and thus \( \epsilon_q = 0 \). If \( q = 1 \) and \( \beta > 0 \), then \( \beta \) is an integer multiple of \( 1/N \), and thus the integral in (6.18) equals zero. Finally, if \( q = 1 \) and \( \beta = 0 \), then \( \epsilon_q = 1 \). Thus, using (6.3) and Lemma 6.5, we obtain

\[ \hat{v}_1 (0) = \frac{\phi(W)}{W} \log z \left( J^{-1} + O_w \left( w^{-1/2+\delta} \right) \right) N = (1 + O_w(w^{-1/2+\delta})N. \]

This proves Lemma 6.2 for the major arcs case sufficiently large \( N \) and \( z \).

6.1.2.2 The minor arc case: By (6.13) and Dirichlet’s theorem on rational approximation, we can write

\[ \frac{r}{N} = \frac{a}{q} + \beta, \quad U < q \leq R, \quad (a,q) = 1, \quad |\beta| \leq q^{-2}. \]

(6.19)

We argue as in the major arc case and derive (6.16). For the major term in (6.16), we argue in the same way as for the estimate of the major term in (6.18) in the case \( w < q \leq U \), and obtain an upper bound \( \ll NU^{-1/4} \). Estimating the error term as in (6.17) and using (6.19), we see

\[ \left| \int_0^N e(\beta x) dE(x) \right| \ll |E(N)| + \left| \int_0^N E(x)(2\pi i \beta) e(\beta) dx \right| \ll N^{1/2}W^{1/2}z^2(1 + |\beta|N) \]
\[ \ll N^{1/2}W^{1/2}z^2 \frac{N}{U^2} \]
\[ \ll N^{1-\delta/10}. \]

This proves Lemma 6.2 for the minor arcs case for sufficiently large \( N \) and \( z \).
6.2 Proof of Lemma 6.3

We first assume \( r = \frac{a}{q} \) and analyse the term of \( f_{d_1, d_2}(Y, a/q) \). For the analysis, we separately consider the two cases 1) \( q_{d_1, d_2} \) is \( w \)-smooth and/or \( h \parallel 2 \) and 2) \( w \)-rough and \( h|2 \) in sections 6.2.1 and 6.2.2. We then prove Lemma 6.3 in section 6.2.3.

6.2.1 \( q_{d_1, d_2} \) is \( w \)-smooth and/or \( h \parallel 2 \):

We will use the following Lemma:

**Lemma 6.8** For \( z \in [W] \) satisfying \( z^2 - b \equiv 0(\text{mod} W) \), we define

\[
S_q(a, z) = \sum_{r=1}^{q} e \left( \frac{a \left( W r^2 + 2r + \frac{z^2-b}{W} \right)}{q} \right). \tag{6.20}
\]

a) If \((a, q) = 1\), \( q > 1 \), and \( q \) is \( w \)-smooth, then

\[
\sum_{z \in [W]} S_q(a, z) = 0.
\]

b) If \((a, q) = 1\) and \((q, W) \parallel 2\),

\[
S_q(a, z) = 0.
\]

c) For \((a, q) = 1\), there is,

\[
|S_q(a, z)| \leq 2\sqrt{q}.
\]

**Proof:** Part a) and c) are Lemma 5.3 and Lemma 5.2 in [3], respectively. Part b) is stated in the proof of [3] Lemma 5.3.

**Lemma 6.9** If \( q_{d_1, d_2} > 1 \) is \( w \)-smooth and/or \( h \parallel 2 \), then for any two co-prime integers \( a \) and \( q \) there is:

\[
f_{d_1, d_2}(Y, a/q) \ll Y^{1/2} q_{d_1, d_2}^{1/2}.
\]

**Proof of Lemma 6.9** We analyze the right-hand side of (6.7). As by assumption \( q \parallel [d_1 d_2]^2 \), we have \( 1 \leq q_{d_1, d_2} \leq q \). We see

\[
\sum_{z \leq W, z \equiv \frac{x-b}{qW} \pmod{W}} xe \left( \frac{a \left( x^2 - b \right)}{qW} \right) = e \left( \frac{-ab}{qW} \right) \left[ d_1, d_2 \right] \sum_{m \leq \sqrt{W} \frac{h}{\text{gcd}[d_1, d_2]}} m e \left( \frac{q_{d_1, d_2} m^2}{q_{d_1, d_2} W} \right), \tag{6.21}
\]

16
where \([d_1, d_2][d_1, d_2] \equiv 1(\text{mod} \ W)\). Splitting the inner summation over \(m\) into rest classes modulo \(W\), we find
\[
\sum_{m \leq \sqrt{W} Y + b/[d_1, d_2]} m e\left(\frac{a_{d_1, d_2} m^2}{q_{d_1, d_2} W}\right)
= \sum_{z \in [W]} e\left(\frac{a_{d_1, d_2} (z + yW)^2}{q_{d_1, d_2} W}\right) \sum_{z + yW \leq \sqrt{W} Y + b/[d_1, d_2]} (z + WY) e\left(\frac{a_{d_1, d_2} (z + WY)^2}{q_{d_1, d_2} W}\right)
\]

(6.22)

Splitting the summation over \(y\) into rest classes modulo \(q_{d_1, d_2}\), we can write the inner sum in (6.22) as follows:
\[
\sum_{z + WY + b/[d_1, d_2]} (z + WY) e\left(\frac{a_{d_1, d_2} (z + WY)^2}{q_{d_1, d_2} W}\right) = \sum_{r=1}^{q_{d_1, d_2}} e\left(\frac{a_{d_1, d_2} (z + Wr)^2}{q_{d_1, d_2} W}\right) \sum_{z + Wr + Wq_{d_1, d_2} s \leq \sqrt{W} Y + b/[d_1, d_2]} (z + Wr + Wq_{d_1, d_2} s).
\]

(6.23)

We now evaluate the inner sum over \(s\) in (6.23):
\[
\sum_{z + Wr + Wq_{d_1, d_2} s \leq \sqrt{W} Y + b/[d_1, d_2]} (z + Wr + Wq_{d_1, d_2} s)
= Wq_{d_1, d_2} \sum_{s \leq ((\sqrt{W} Y + b/[d_1, d_2]) - z - Wr) / Wq_{d_1, d_2}} s + O\left(\frac{(z + Wr) \sum_{s \leq \sqrt{W} Y + b/[d_1, d_2]} 1}{Wq_{d_1, d_2}[d_1, d_2]}\right)
= Wq_{d_1, d_2} \sum_{s \leq \sqrt{W} Y + b/Wq_{d_1, d_2}[d_1, d_2]} s
+ O\left(Wq_{d_1, d_2} \sum_{((\sqrt{W} Y + b/[d_1, d_2]) - z - Wr) / Wq_{d_1, d_2} s \leq \sqrt{W} Y + b/Wq_{d_1, d_2}[d_1, d_2]} s\right)
+ O\left(Wq_{d_1, d_2} \sum_{s \leq \sqrt{W} Y + b/Wq_{d_1, d_2}[d_1, d_2]} 1\right)
= Wq_{d_1, d_2} \sum_{s \leq \sqrt{W} Y + b/Wq_{d_1, d_2}[d_1, d_2]} s + O\left((\sqrt{W} Y/[d_1, d_2])\right).
\]

(6.24)
Combining (6.21) - (6.24), see see
\[
\sum_{x \leq \sqrt{Wy+\varepsilon} \atop x^2 - b \equiv 0 \mod W} xe \left( \frac{a \left( x^2 - b \right)}{qW} \right)
\]
\[
eq e \left( \frac{-ab}{qW} \right) [d_1, d_2] \sum_{s \leq \sqrt{W}y+b/W} \sum_{r=1}^{q_d, d_2} e \left( \frac{a_{d_1, d_2}(zWr)^2}{q_d, d_2 W} \right)
\]
\[
\times \left( W q_{d_1, d_2} \sum_{s \leq \sqrt{W}y+b/W} s + O \left( \sqrt{W}y/|d_1, d_2| \right) \right)
\]
\[
eq e \left( \frac{-ab}{qW} + \frac{a_{d_1, d_2}b}{q_{d_1, d_2} W} \right) [d_1, d_2] \sum_{s \leq \sqrt{W}y+b/W} s + O \left( \sqrt{W}y/|d_1, d_2| \right)
\]
\[
\times \left( W q_{d_1, d_2} \sum_{s \leq \sqrt{W}y+b/W} s + O \left( \sqrt{W}y/|d_1, d_2| \right) \right)
\]
\[
eq e \left( \frac{-ab}{qW} + \frac{a_{d_1, d_2}b}{q_{d_1, d_2} W} \right) \frac{WY + b}{2W[d_1, d_2]q_{d_1, d_2}} \sum_{z \leq W} S_{q_{d_1, d_2}}(a_{d_1, d_2}, z)
\]
\[
+ O \left( \sqrt{W}y \sigma(b) \max_{z \leq W} \max_{x \leq \sqrt{W}y+b/W} \left| S_{q_d, d_2}(a_{d_1, d_2}, z) \right| \right).
\]

Applying Lemma 6.8(b), we estimate the O-term in (6.25) as follows:
\[
\sqrt{W}y \sigma(b) \max_{z \leq W} \left| S_{q_{d_1, d_2}}(a_{d_1, d_2}, z) \right| \ll Y^{1/2} \left| q_{d_1, d_2} \right|^{1/2}.
\]

Lemma 6.9 now follows from (6.7), (6.23), Lemma 6.8(a) and b), and (6.26).

**Lemma 6.10** For two co-prime integers a and q there is:
\[
\sum_{q \mid d_1, d_2} \sum_{d_1, d_2 \leq Q} p_{d_1, d_2} f_{d_1, d_2}(Y, a/q) \ll z^2 Y^{1/2} Q^{1/2}.
\]

**Proof of Lemma 6.10** Using Lemma 6.9 and noting that \( q_{d_1, d_2} \leq Wq_{W,d_1, d_2} \), we see
\[
\sum_{q \mid d_1, d_2} \sum_{d_1, d_2 \leq Q} p_{d_1, d_2} f_{d_1, d_2}(Y, a/q) \ll \sum_{q \mid d_1, d_2} \sum_{d_1, d_2 \leq Q} p_{d_1, d_2} f_{d_1, d_2}(Y, a/q)
\]

18
\[ \left( \sum_{d \leq z} 1 \right)^2 Y^{1/2} Q^{1/2} \leq z^2 Y^{1/2} Q^{1/2}, \]  

(6.27)

qd, d_2 is w-rough and h|2:

We will first derive the auxiliary Lemmas 6.11 - 6.14. Subsequently we will prove the main Lemmas 6.15 and 6.16 of this paragraph.

We will make use of the generalized Gauss sum \( G(a, b, c) \) defined as follows:

\[ G(a, b, c) = \sum_{n=1}^{c} e \left( \frac{an^2 + bn}{c} \right). \]  

(6.28)

**Lemma 6.11** For co-prime integers \( g \) and \( h \), \( (h, 2) = 1 \), let \( \left( \frac{g}{h} \right) \) denote the Jacobi symbol.

\[ G(a, 0, c) = \begin{cases} \left( \frac{a}{c} \right) \sqrt{c}, & c \equiv 1 (mod 4), \\ \left( \frac{a}{c} \right) i \sqrt{c}, & c \equiv 3 (mod 4), \end{cases} \]

Proof of Lemma 6.11: See [1, Theorem 1.5.2].

**Lemma 6.12** For any integers \( a, b \) and \( c \) with \( (c, 2a) = 1 \), we have

\[ G(a, b, c) = e \left( \frac{-2 \beta \delta^2}{c} \right) \times \begin{cases} \left( \frac{a}{c} \right) \sqrt{c}, & c \equiv 1 (mod 4), \\ \left( \frac{a}{c} \right) i \sqrt{c}, & c \equiv 3 (mod 4), \end{cases}, \]

where the integers \( \beta \) and \( \delta \) modulo \( c \) are defined through the relations \( \beta^2 \equiv a \beta \equiv 1 (mod c) \).

Proof of Lemma 6.12:

\[
G(a, b, c) = e \left( \frac{-2 \beta \delta^2}{c} \right) \sum_{n=1}^{c} e \left( \frac{a (n^2 + \beta n + (2 \beta \delta)^2)}{c} \right) \\
= e \left( \frac{-2 \beta \delta^2}{c} \right) \sum_{n=1}^{c} e \left( \frac{a (n^2 + 2 \beta \delta)^2}{c} \right) = e \left( \frac{-2 \beta \delta^2}{c} \right) \sum_{n=1}^{c} e \left( \frac{a n^2}{c} \right) \\
= e \left( \frac{-2 \beta \delta^2}{c} \right) G(a, 0, c). \]  

(6.29)

Applying Lemma 6.11 to (6.29), we derive Lemma 6.12.
Lemma 6.13 For fixed \( k | q, k < q \), there is

\[
\sum_{d_1, d_2 | P_{(q,d_1,d_2)^2}} P_{d_1} P_{d_2} \left\lfloor \frac{d_1 d_2}{d_1, d_2} \right\rfloor \ll d(q)J^{-1}k^{-1/2+\epsilon}.
\]

Proof of Lemma 6.13 We write \( \frac{q}{k} = \prod_{i \leq M} p_i^\alpha_i \), where the \( p_i \) are different prime numbers, \( \alpha_i \in \mathbb{Z}^+ \), and \( M \leq d(p/k) \) is an integer depending on the prime decomposition of \( \frac{q}{k} \). Applying the inclusion-exclusion principle, we find

\[
\sum_{d_1, d_2 | P_{(q,d_1,d_2)^2}} P_{d_1} P_{d_2} \left\lfloor \frac{d_1 d_2}{d_1, d_2} \right\rfloor = \left( \sum_{d_1, d_2 | P_{(q,d_1,d_2)^2}} P_{d_1} P_{d_2} \frac{d_1 d_2}{d_1, d_2} \right) + \sum_{1 \leq j \leq M} (-1)^j \sum_{\beta_1 \leq 1} \cdots \sum_{\beta_M \leq 1} \sum_{\beta_1 + \cdots + \beta_M = j} \prod_{i \leq M} p_i^\beta_i \left\lfloor \frac{d_1 d_2}{d_1, d_2} \right\rfloor \right)
\]

(6.30)

From (6.30), we see

\[
\left| \sum_{d_1, d_2 | P_{(q,d_1,d_2)^2}} P_{d_1} P_{d_2} \frac{d_1 d_2}{d_1, d_2} \right| \leq d(q) \max_{j:k|j, j|q} \sum_{d_1, d_2 | P_{(q,d_1,d_2)^2}} P_{d_1} P_{d_2} \frac{d_1 d_2}{d_1, d_2} \right) .
\]

(6.31)

From (6.31), we see

\[
\left| \sum_{d_1, d_2 | P_{(q,d_1,d_2)^2}} P_{d_1} P_{d_2} \frac{d_1 d_2}{d_1, d_2} \right| \leq d(q) \max_{j:k|j, j|q} \sum_{d_1, d_2 | P_{(q,d_1,d_2)^2}} P_{d_1} P_{d_2} \frac{d_1 d_2}{d_1, d_2} \right) .
\]

From (6.31), we see

\[
\left| \sum_{d_1, d_2 | P_{(q,d_1,d_2)^2}} P_{d_1} P_{d_2} \frac{d_1 d_2}{d_1, d_2} \right| \leq d(q) \max_{j:k|j, j|q} \sum_{d_1, d_2 | P_{(q,d_1,d_2)^2}} P_{d_1} P_{d_2} \frac{d_1 d_2}{d_1, d_2} \right) .
\]

(6.31)

Defining \( j = j_1 j_2 \) in the same way we have defined \( q = q_1 q_2 \) in (6.10) and applying Lemma 6.2, we obtain from (6.11) and (6.31):

\[
\left| \sum_{d_1, d_2 | P_{(q,d_1,d_2)^2}} P_{d_1} P_{d_2} \frac{d_1 d_2}{d_1, d_2} \right| \ll d(q)J^{-1} \max_{j:k|j, j|q} j^{-1/2+\epsilon} \leq d(q)J^{-1}k^{-1/2+\epsilon}.
\]

qed.

Lemma 6.14 Let \( \left( \frac{a}{b} \right) \) denote the Jacobi symbol. Let \( a, b \) and \( v \) be three strictly positive integers satisfying \( (a, b) = 1, v | b, \) and \( \left( \frac{b}{2}, 2 \right) = 1 \). Then, for any strictly positive integer \( c \) with \( (c^2, b) = v \), there is

\[
\left( \frac{a^{c^2}}{b^v} \right) = S_{a,b,v},
\]

where \( S_{a,b,v} \in \{-1, 1\} \) depends on \( a, b, v \), but is independent of \( c \).
Proof of Lemma 6.14: We write the prime decomposition of \( v \) as 
\[
v = \prod_{i \leq U} p_i^{\gamma_i}, \quad \gamma_i \geq 1.
\]
Further, we write the prime decomposition of \( c^2 \) as 
\[
c^2 = \prod_{i \leq U} p_i^{2\beta_i}, \quad \beta_i \geq \max(\gamma_i, 2), \quad (d, b) = 1, \quad p_i | b.
\]
Thus, \( c^2 / v = \prod_{i \leq U} p_i^{2\beta_i - \gamma_i} \). We see,
\[
\left( \frac{a^2}{v} \right) = \left( \frac{d}{v} \right)^2 \prod_{i \leq U} \left( \frac{p_i}{v} \right)^{2\beta_i - \gamma_i}
\]
\[
= \left( \frac{a}{v} \right) \prod_{\gamma_i \text{odd}} \left( \frac{p_i}{v} \right) = \left( \frac{a}{v} \right) \prod_{\gamma_i \text{odd}} \left( \frac{p_i}{v} \right), \quad (6.32)
\]
The last term in (6.32) does not depend on \( c \) which proves the Lemma.

Lemma 6.15 If \( q_{d_1, d_2} \) is \( w \)-rough and \( h | 2 \), then for any two co-prime integers \( a \) and \( q \) there is:
\[
f_{d_1, d_2}(Y, a/q) = e \left( \frac{-ab}{qW} \right) e \left( \frac{g(q, [d_1, d_2]^2)}{W_1} \right) \frac{Y}{\sqrt{q_{d_1, d_2}h}} V_{qW, d_1, d_2} \left( \frac{W_1 a_{d_1, d_2}}{qW, d_1, d_2} \right)
\]
\[\quad + \quad O \left( Y^{1/2} q_{d_1, d_2}^{1/2} \right),\]
where \( h, W_1, \) and \( q_{W, d_1, d_2} \) are as defined in (6.1), \( g(q, [d_1, d_2]^2) \) is an integer modulo \( W \) which for fixed \( a, q, b, \) and \( W \) depends on \( \{q, [d_1, d_2]^2\} \) only, and
\[
V_{qW, d_1, d_2} = \begin{cases} 1, & qW, d_1, d_2 \equiv 1 \text{ (mod 4)}; \\ i, & qW, d_1, d_2 \equiv 3 \text{ (mod 4)}. \end{cases}
\]
(6.33)

Proof of Lemma 6.15: By the definitions (6.20) and (6.28) and the assumption \( h | 2 \),
\[
S_{q_{d_1, d_2}}(a_{d_1, d_2}, z) = e \left( \frac{(z^2 - b)a_{d_1, d_2}}{q_{d_1, d_2}W} \right) G(W_1 a_{d_1, d_2}, 2za_{d_1, d_2} / h, q_{W, d_1, d_2}).
\]
Inserting (6.34) in (6.25) and using (6.20), we find
\[
\sum_{x \leq \sqrt{W + |z|}} x e \left( \frac{a (x^2 - b)}{QW} \right)
\]
\[
= e \left( \frac{-ab}{QW} \right) \frac{WY + b}{2W [d_1, d_2] q_{d_1, d_2}} \sum_{x \leq \sqrt{W}} e \left( \frac{a_{d_1, d_2} x^2}{q_{d_1, d_2} W} \right) G(W_1 a_{d_1, d_2}, 2za_{d_1, d_2} / h, q_{W, d_1, d_2})
\]
\[\quad + \quad O \left( Y^{1/2} q_{d_1, d_2}^{1/2} \right), \quad (6.35)
\]
We recall that we assume $h \in \{1, 2\}$ which implies that $(q_{W,d_1,d_2}, 2) = 1$. Applying Lemma 6.12 to (6.35), we derive

$$
\sum_{x \leq \sqrt{q_{W,d_1,d_2}}} xe \left( \frac{a (x^2 - b)}{q W} \right) = e \left( \frac{-ab}{q W} \right) \frac{W Y + b}{2 W [d_1,d_2]} \sum_{z \in [W]} e \left( \frac{a_{d_1,d_2} z^2}{q_{d_1,d_2} W} \right) e \left( \frac{- (\overline{a})^2 W_1 a_{d_1,d_2} (2z a_{d_1,d_2}/h)^2}{q W, d_1, d_2} \right)
$$

$$
= e \left( \frac{-ab}{q W} \right) \frac{W Y + b}{2 W [d_1,d_2]} \sum_{z \in [W]} e \left( \frac{a_{d_1,d_2} z^2}{q_{d_1,d_2} W} \right) e \left( \frac{- (\overline{a})^2 W_1 a_{d_1,d_2} (2z a_{d_1,d_2}/h)^2}{q W, d_1, d_2} \right)
$$

$$
= e \left( \frac{-ab}{q W} \right) \frac{W Y + b}{2 W [d_1,d_2]} \sum_{z \in [W]} e \left( \frac{a_{d_1,d_2} z^2}{q_{d_1,d_2} W} \right) e \left( \frac{- (\overline{a})^2 W_1 a_{d_1,d_2} (2z a_{d_1,d_2}/h)^2}{q W, d_1, d_2} \right)
$$

$$
\times V_{q_{W,d_1,d_2}} \left( \frac{W_1 a_{d_1,d_2}}{q_{W,d_1,d_2}} \right) + O \left( y^{1/2} q_{d_1,d_2} \right)
$$

$$
= e \left( \frac{-ab}{q W} \right) \frac{W Y + b}{2 W [d_1,d_2]} \sum_{z \in [W]} e \left( \frac{a_{d_1,d_2} z^2}{q_{d_1,d_2} W} \right) e \left( \frac{- (\overline{a})^2 W_1 a_{d_1,d_2} (2z a_{d_1,d_2}/h)^2}{q W, d_1, d_2} \right)
$$

$$
= e \left( \frac{-ab}{q W} \right) \frac{W Y + b}{2 W [d_1,d_2]} \sum_{z \in [W]} e \left( \frac{a_{d_1,d_2} z^2}{q_{d_1,d_2} W} \right) e \left( \frac{- (\overline{a})^2 W_1 a_{d_1,d_2} (2z a_{d_1,d_2}/h)^2}{q W, d_1, d_2} \right)
$$

$$
\times V_{q_{W,d_1,d_2}} \left( \frac{W_1 a_{d_1,d_2}}{q_{W,d_1,d_2}} \right) + O \left( y^{1/2} q_{d_1,d_2} \right).
$$

We recall that in (6.36) for any integer $c$ prime to $q_{W,d_1,d_2}$, $\overline{a}$ is defined via the relation $c \equiv 1 (mod q_{W,d_1,d_2})$. Thus, in particular

$$
a_{d_1,d_2} d_{d_1,d_2} \equiv 1 (mod q_{W,d_1,d_2}).
$$

(6.37)

Further, we set

$$
W_1 W_1 = 1 + s_{q, [d_1, d_2]^2} q_{W,d_1,d_2}.
$$

(6.38)

In view of the definitions in (6.1), we see that - for fixed $q$ and $W$ - $s_{q, [d_1, d_2]^2}$ depends on $(q, [d_1, d_2]^2)$ only. Similarly, we notice that for fixed values of $q$ and $W$, the value of the integer $\overline{a}$ only depends on $(q, [d_1, d_2]^2)$, i.e.,

$$
\overline{a} := u_{q, [d_1, d_2]^2}.
$$

(6.39)

Finally, we consider a fixed $z$ satisfying the congruence condition in (6.36), i.e., $z^2 [d_1, d_2]^2 \equiv b (mod W)$. This implies that

$$
a_{d_1,d_2} z^2 = \frac{a_{d_1,d_2} z^2}{(q, [d_1, d_2]^2)^2} \equiv ab(q, [d_1, d_2]^2)(mod W),
$$

(6.40)
where \((q, [d_1, d_2]^2) = 1 (\text{mod } W)\). Subsequently applying (6.37) - (6.40), we can calculate the product of the exponential terms in (6.36) as follows:

\[
\begin{align*}
&\ e \left( \frac{a_{d_1, d_2} z^2}{q d_1, d_2 W} \right) e \left( -\left( \frac{2}{2} \right)^2 W_1 a_{d_1, d_2} \left( 2 \pi \theta_{d_1, d_2} / h \right)^2 \right) \\
= &\ e \left( \frac{a_{d_1, d_2} z^2}{q d_1, d_2 W} \right) e \left( -\left( \frac{2}{2} \right)^2 W_1 a_{d_1, d_2} \left( 2 \pi / h \right)^2 \right) \\
= &\ e \left( \frac{a_{d_1, d_2} z^2}{q d_1, d_2 W} \right) e \left( -s_{q, d_1, d_2} \left( 2 \pi / h \right)^2 \right) \left( \frac{W_1}{W_1} \right) \\
= &\ e \left( \frac{a_{d_1, d_2} z^2}{q d_1, d_2 W} \right) e \left( -s_{q, d_1, d_2} \left( 2 \pi / h \right)^2 \right) \left( \frac{W_1}{W_1} \right) \\
:= &\ e \left( \frac{g_{q, d_1, d_2} z^2}{W_1} \right), \quad (6.41)
\end{align*}
\]

where - for fixed \(a, q, b,\) and \(W = g_{q, d_1, d_2} \) is an integer modulo \(W\) that depends on \((q, [d_1, d_2]^2)\) only. Inserting (6.41) into (6.36), we obtain

\[
\sum_{x \leq \sqrt{W}} \sum_{x^2 - k h (0 \text{mod } W)} x e \left( \frac{a (x^2 - b)}{q W} \right) = \frac{\sigma(b)}{2} e \left( \frac{-ab}{q W} \right) e \left( \frac{g_{q, d_1, d_2} z^2}{W_1} \right) \left( \frac{Y}{d_1, d_2} \right) + O \left( \left( \frac{Y^{1/2} q^{1/2}}{d_1, d_2} \right) \right). \quad (6.42)
\]

Now the Lemma follows from (6.37) and (6.42).

**Lemma 6.16** For two co-prime integers \(a\) and \(q\) there is:

\[
\begin{align*}
&\sum_{q, d_1, d_2 \in \mathbb{Z}} p_{d_1, d_2} f_{d_1, d_2} (Y, a/q) \\
= &\sum_{k \leq \delta} \frac{t_k}{q/k} \sum_{d_1, d_2 \in \mathbb{Z}} p_{d_1, d_2} (Y, a/q) + O \left( \frac{Z^{1/2} Q^{1/2}}{W} \right),
\end{align*}
\]

where \(t_k\) is a complex number that for fixed \(a, q, b,\) and \(W\) depends on \(k\) only and \(|t_k| = 1\).
Proof of Lemma 6.16: Using Lemma 6.15, we see

\[
\sum_{q, d_1, d_2 \leq Q, qd_1 \cdot d_2 > w, w \text{-rough and } h} p_{d_1} p_{d_2} f_{d_1, d_2} (Y, a/q)
\]

\[
= e \left( -ab \right) \frac{Y}{\sqrt{h}} \cdot \sum_{q, d_1, d_2 \leq Q, qd_1 \cdot d_2 > w, w \text{-rough and } h} \left( \frac{g(q, [d_1, d_2]^2)}{W_1} \right) V_{q, d_1, d_2} \left( \frac{W_1 a [d_1, d_2]^2}{q/h^k} \right)
\]

\[
+ O \left( z^2 Y^{1/2} Q^{1/2} \right),
\]

where we have estimated the O-term arguing similarly as in (6.27). If \( qd_1, d_2 \) is \( w \)-rough, then \( qd_1, d_2 > w \), which implies that \( (q, [d_1, d_2]^2) < q/w \). Thus, we can rewrite the right-hand side of (6.43) as follows:

\[
\sum_{q, d_1, d_2 \leq Q, qd_1 \cdot d_2 > w, w \text{-rough and } h} p_{d_1} p_{d_2} f_{d_1, d_2} (Y, a/q)
\]

\[
= e \left( -ab \right) \frac{Y}{\sqrt{h}} \cdot \sum_{k < q/h, k|q} \left( \frac{g(k/W_1) V_{W_1}}{\sqrt{k}} \right) \sum_{d_1, d_2 | \frac{q, [d_1, d_2]^2}{k}} p_{d_1} p_{d_2} \left( \frac{W_1 a [d_1, d_2]^2}{q/h^k} \right)
\]

\[
+ O \left( z^2 Y^{1/2} Q^{1/2} \right).
\]

Applying Lemma 6.14 with \( a = W_1 a, b = q/h, v = k, \) and \( c = [d_1, d_2] \), we see that \( \left( \frac{W_1 a [d_1, d_2]^2}{q/h^k} \right) = S_{W_1 a, q/h, k} \). Thus, for fixed \( a, q \) and \( W_1 \), \( \left( \frac{W_1 a [d_1, d_2]^2}{q/h^k} \right) \) depends on \( k \) only. We now define the complex number \( t_k \) as

\[
t_k := e \left( \frac{-ab}{q/h} \right) e \left( \frac{bk/W_1}{q/h} \right) \sum_{d_1, d_2 | \frac{q, [d_1, d_2]^2}{k}} \left( \frac{W_1 a [d_1, d_2]^2}{q/h^k} \right).
\]

By the foregoing discussion and the definition of \( g(k) \) and \( V_{W_1} \), we see that for fixed \( a, q, b, \) and \( W_1 \), \( t_k \) depends on \( k \) only and \( |t_k| = 1 \). Thus, we can rewrite (6.44) as

\[
\sum_{q, d_1, d_2 \leq Q, qd_1 \cdot d_2 > w, w \text{-rough and } h} p_{d_1} p_{d_2} f_{d_1, d_2} (Y, a/q)
\]

\[
= \frac{Y}{\sqrt{h}} \cdot \sum_{k < q/h, k|q} t_k \sum_{d_1, d_2 | \frac{q, [d_1, d_2]^2}{k}} p_{d_1} p_{d_2} \left( \frac{W_1 a [d_1, d_2]^2}{q/h^k} \right) + O \left( z^2 Y^{1/2} Q^{1/2} \right),
\]

\[
\text{qed.}
\]
6.2.3 Proof of Lemma 6.3

By (6.3) and (6.5), for fixed \( r \), we only need to consider those pairs \( d := (d_1, d_2) \) for which

\[
\frac{r W[d_1, d_2]^2}{N} - \frac{a_{1d}}{q_{1d}} \leq \frac{1}{q_{1d} R}, \quad (a_{1d}, q_{1d}) = 1, \quad q_{1d} \leq Q,
\]

which implies

\[
\frac{r}{N} = \frac{a_d}{q_d} + \beta_d, \quad q_d \leq q_{1d} W[d_1, d_2]^2 \leq Q W z^2 \leq N^{1/2 - \delta/100}, \quad |\beta_d| \leq 1/R.
\]

(6.45)

We now show that for different pairs \( d := (d_1, d_2) \) and \( d^* := (d_1^*, d_2^*) \) which satisfy (6.45), there is \( a_d = a_d^* \) and \( q_d = q_d^* \) which implies that \( \beta_d = \beta_d^* \). This follows from (6.45) and the relation

\[
\left| \frac{a_d}{q_d} - \frac{a_d^*}{q_d^*} \right| \geq \frac{1}{q_d q_d^*} > \frac{1}{q_d R} + \frac{1}{q_d^* R},
\]

which holds because of \( q_d, q_{d^*} \leq N^{1/2 - \delta/100} \). Thus, we rewrite (6.45) as

\[
\frac{r}{N} = \frac{a}{q} + \beta, \quad (a, q) = 1, \quad q \leq N^{1/2 - \delta/100}, \quad |\beta| \leq 1/R. \tag{6.46}
\]

In view of (6.46), we first consider the case \( \beta = 0 \) i.e., \( \frac{r}{N} = \frac{a}{q} \). By (6.3) and (6.5), for fixed \( r \), we see that \( r W[d_1, d_2]^2 / N \in \mathbb{N} \) if and only if

\[
\frac{r W[d_1, d_2]^2}{N} = \frac{a W[d_1, d_2]^2}{q W[d_1, d_2] q W[d_1, d_2]}, \quad q W[d_1, d_2] \leq Q. \tag{6.47}
\]

From (6.5), Lemma 6.10, Lemma 6.16, and (6.47), we see

\[
\hat{v}_2 \left( \frac{a}{q} \right) = \phi(W) \log \frac{z}{W} \sum_{\substack{d_1, d_2 \mid P \quad q \mid d_1, d_2 \quad q W[d_1, d_2] \leq Q}} p_{d_1} p_{d_2} f_{d_1, d_2}(Y, a/q)
\]

\[
= \phi(W) \log \frac{z}{W} \left( Y M(q, a) + E_1(Y) \right), \tag{6.48}
\]

where

\[
M(a, q) = \frac{1}{\sqrt{\kappa}} \sum_{k < \frac{z^2}{\sqrt{q/k}} \leq Q} \frac{t_k}{\sqrt{q/k}} \sum_{\substack{d_1, d_2 \mid P \quad (q, d_1, d_2)^2 = k \quad \# W \text{-rough and } (W, \#)^2 \leq k}} p_{d_1} p_{d_2} \frac{d_1, d_2}{[d_1, d_2]^2},
\]

\[
E_1(Y) = O \left( \varepsilon^2 Y^{1/2} Q^{1/2} \right).
\]

25
We notice from the definition of $M(a, q)$ that
\[ M(a, q) \neq 0 \Rightarrow q > w. \]  
(6.49)

We now consider the general case $\frac{r}{N} = \frac{a}{q} + \beta$, Applying partial summation, we derive from (6.5), (6.46), and (6.48),
\[ \hat{v}_2 \left( \frac{r}{N} \right) = \frac{\phi(W) \log z}{W} \frac{M(q, a)}{e^x} \int_0^N e(\beta x) dx \]
\[ + \frac{\phi(W) \log z}{W} \int_0^N e(\beta x) dE_1(x). \]  
(6.50)

We first estimate the main term in (6.50). Applying Lemma 6.13 and using (4.3), we find:
\[ \phi(W) \log z \frac{M(q, a)}{M(q, a)} \ll \log z d(q) \max_{d_1, d_2} \left| \frac{p_{d_1} p_{d_2}}{d_1 d_2} \right| \]
\[ \ll \log J^{-1} q^{-1/2} k^{-1/2 + \epsilon} q^{-1/2 + 2\epsilon} \ll q^{-1/2 + 2\epsilon}. \]  
(6.51)

We recall the well-known estimate
\[ \int_0^N e(\beta x) dx \ll \min \left( N, \frac{1}{|\beta|} \right). \]  
(6.52)

Combining (6.49), (6.51), and (6.52), we find
\[ \phi(W) \log z \frac{M(q, a)}{M(q, a)} \int_0^N e(\beta x) dx \ll Nw^{1-2+2\epsilon}. \]  
(6.53)

Using (6.46), we estimate the error term integral in (6.50) as follows:
\[ \left| \int_0^N e(\beta x) dE_1(x) \right| \ll |E_1(N)| + \int_0^N E_1(x) (2\pi i \beta) e(\beta x) dx \]
\[ \ll \left( z^2 N^{1/2} Q^{1/2} \right) (1 + |\beta| N) \]
\[ \ll \left( z^2 N^{1/2} Q^{1/2} \right) \frac{N}{R} \]
\[ \ll N^{-\delta/10}. \]  
(6.54)

Lemma 6.3 now follows from (6.50), (6.53), and (6.54).
6.3 Proof of Lemma 6.4

Using (6.7) with \( \frac{r}{q} \) instead of \( \frac{a}{q} \), we see

\[
|f_{d_1, d_2}(N, r/N)| = \left| \frac{2}{\sigma(b)} \sum_{\substack{x \leq \sqrt{WN+1} \\{d_1, d_2]\ \mid x^2 - b \equiv 0 (\mod W) \\{d_1, d_2\}}} xe \left( \frac{r(x^2 - b)}{NW} \right) \right|
\]

\[
\ll \sum_{j=1}^{\sigma(b)} \sum_{x \equiv g_j (\mod W[d_1, d_2])} xe \left( \frac{rx^2}{NW} \right),
\]

(6.55)

where \( g_j = g_j(h_j, W[d_1, d_2]) \). Here we have used the fact that due to \( (W, P) = 1 \) and \( d_1, d_2 \mid P \), there is \((W, [d_1, d_2]) = 1\). Using partial summation, we estimate the inner sum over \( x \) in (6.55) for \( Y \leq N \) as follows:

\[
\sum_{x \equiv g_j (\mod W[d_1, d_2])} xe \left( \frac{rx^2}{NW} \right) \ll \sqrt{WN} \max_{K \leq \sqrt{WN+6}} |U(K, r/N)|,
\]

(6.56)

where

\[
U(K, r/N) := \sum_{x \equiv g_j (\mod W[d_1, d_2])} e \left( \frac{rx^2}{NW} \right)
\]

\[
= \sum_{x \leq K - g_j/W[d_1, d_2]} e \left( \frac{rW[d_1, d_2]^2 s^2}{N} + \frac{2rg_2 [d_1, d_2] s}{N} + \frac{rg_2^2}{NW} \right).
\]

(6.57)

To estimate the right-hand side of (6.57), we will make use of Weyl’s Lemma [12, Lemma 2.4]:

**Lemma 6.17** Let \( \alpha, \alpha_1 \) and \( \alpha_2 \) be real numbers. If \( \left| \alpha - \frac{a}{q} \right| \leq q^{-2} \), for two integers \( a \) and \( q \) with \((a, q) = 1\), then

\[
\sum_{n \leq N} e \left( an^2 + \alpha_1 n + \alpha_2 \right) \ll N^{1+\epsilon} \left( q^{-1} + N^{-1} + qN^{-2} \right)^{1/2}.
\]
By the definition of the minor arcs (6.3) and Dirichlet’s theorem on rational approximation, we know that there exists integers $a$ and $q$ with $(a, q) = 1$, and $Q < q \leq R$ such that $\left| \frac{rW[d_2, d_2]}{N} - \frac{a}{q} \right| \leq 1/q^2$. Thus, applying Lemma 6.17 to (6.57) with $\alpha = rW[d_2, d_2]/N$, we find

$$U(K, r/N) \ll \frac{K^{1+\epsilon}}{W^{1+\epsilon}[d_1, d_2]^{1+\epsilon}} (Q^{-1} + K^{-1} + RK^{-2})^{1/2}. \quad (6.58)$$

The right-hand side of (6.58) is an increasing function in $K$. Therefore,

$$\max_{K \leq \sqrt{W/N+5}} |U(K, r/N)| \ll \frac{N^{1/2+\epsilon}}{[d_1, d_2]^{1+\epsilon}} (Q^{-1} + N^{-1/2} + RN^{-1})^{1/2}.$$

$$\ll \frac{N^{1/2+\epsilon}}{[d_1, d_2]^{1+\epsilon}} \left( N^{-2\delta/5} + N^{-1/2} + N^{-\delta/2} \right)^{1/2} \ll \frac{N^{\frac{\delta}{100}}}{[d_1, d_2].} \quad (6.59)$$

From (6.56) and (6.59), we see

$$\left| \sum_{\substack{r \equiv \eta_j (\text{mod } W[d_1, d_2])}} xe \left( \frac{\alpha x^2}{W} \right) \right| \ll \frac{N^{1-\frac{\delta}{100}}}{[d_1, d_2]}. \quad (6.60)$$

From (6.5), (6.55), (6.60) and using $|\rho_d| \leq 1$, we see

$$\hat{v}_3 \left( \frac{r}{N} \right) \ll N^{1-\frac{\delta}{100}} \sum_{d_1, d_2 \mid P} \frac{1}{[d_1, d_2]} \ll N^{1-\frac{\delta}{100}} \sum_{d_1, d_2 \leq z} \frac{1}{[d_1, d_2]}$$

$$\ll N^{1-\frac{\delta}{100}} \sum_{d \leq z^2} \sum_{(d, W) = 1} \left( \sum_{d_1, d_2 \leq z} \frac{1}{[d_1, d_2]^{1/d}} \right) \frac{1}{d}. \quad (6.61)$$

For any fixed squarefree $d \leq z^2$, there are at most $3^{\omega(d)} \leq d^{5/900} \leq N^{\delta/900}$ pairs $[d_1, d_2] = d$. Thus, we see from (6.61)

$$\hat{v}_3 \left( \frac{r}{N} \right) \ll N^{1-\frac{\delta}{300}} \sum_{d \leq z^2} d^{-1} \ll \ll N^{1-\frac{\delta}{400}},$$

which proves Lemma 6.4.
7 Restriction estimate

In this section, we set \( a(n) = a_i(n) \), \( b = b_i \), \( z = z_i \), \( v(n) = v_i(n) \), and \( N = N_i \). The main purpose of this section is to show the following Lemma 7.1. Our proof follows the argument in [3, Section 6] with some minor modifications.

**Lemma 7.1** For any real number \( p > 4 \) there exists an absolute constant \( C_p \) such that

\[
\int_T |\hat{a}(\theta)|^p \, d\theta \leq C_p N^{p-1}.
\]

For the proof of Lemma 7.1 we will make use of the following Lemma:

**Lemma 7.2** There exists an absolute constant \( C \) such that

\[
\int_T |\hat{a}(\theta)|^4 \, d\theta \leq N^{3+C/\log \log N}.
\]

**Proof of Lemma 7.2:** We note that

\[
\int_T |\hat{a}(\theta)|^4 \, d\theta = \left\| \sum_{n,m \in \mathbb{Z}} a(m)\bar{a}(n)\bar{e}(\theta(m-n)) \right\|_2^2
\]

\[
\leq \sum_{|k| \leq N} \sum_{n,m \in \mathbb{Z}, n-m=k} a(m)\bar{a}(n)^2.
\]

The contribution to the sum over \( k \) for \( k = 0 \) is

\[
\leq \sum_{n \in \mathbb{Z}} |a(n)|^2 \ll N^3(\log N)^2 = N^{3+O(1/\log \log N)}.
\]

If \( k \neq 0 \), we see from the definition of \( a(n) \),

\[
\sum_{n,m \in \mathbb{Z}, n-m=k} a(m)\bar{a}(n) = \left( \frac{2\phi(W)\log z}{\sigma(b)W} \right)^2 \sum_{z \leq x, y \leq \sqrt{W+b} \text{, } x,y \text{ prime and } x^2 \equiv y^2 \equiv b (\text{mod } W)} xy.
\]

Using the Cauchy-Schwarz inequality, we estimate the sum on the right hand side in (7.3) as follows:

\[
\sum_{z \leq x, y \leq \sqrt{W+b} \text{, } x,y \text{ prime}} xy \ll d^{1/2}(k) \left( \sum_{z \leq x, y \leq \sqrt{W+b} \text{, } x,y \text{ prime and } x^2 \equiv y^2 \equiv b (\text{mod } W)} x^2 y^2 \right)^{1/2}.
\]
Combining (7.1) - (7.4), we see
\[
\int_{\mathbb{T}} |\hat{a}(\theta)|^4 \, d\theta \ll N^2 (\log N)^4 \sum_{0 < |k| \leq N} d(k) \sum_{\substack{x, y \leq \sqrt{W N} + b \mod W \, \text{prime}}} 1 + N^{3 + O(1/\log \log N)}.
\]
(7.5)

Recalling the standard estimate for the divisor function \( d(k) \ll N^{O(1/\log \log N)} \) for \( 0 \leq |k| \leq N \), we see from (7.5):
\[
\int_{\mathbb{T}} |\hat{a}(\theta)|^4 \, d\theta \ll N^{O(1/\log \log N)} N^2 (\log N)^4 \sum_{0 < |k| \leq N} \sum_{\substack{x, y \leq \sqrt{W N} + b \mod W \, \text{prime}}} 1
\]
\[
+ N^{3 + O(1/\log \log N)}
\]
\[
\ll N^{2 + O(1/\log \log N)} \left( \sum_{x \leq \sqrt{W N} + b \mod W \, \text{prime}} 1 \right)^2 \ll N^{3 + O(1/\log \log N)}
\]
(7.6)

which proves Lemma 7.2.

**Lemma 7.3** Define the region
\[
\mathbb{R}_{\delta_1} = \{ \theta \in \mathbb{T} : |\hat{a}(\theta)| > \delta_1 N \}.
\]
for any \( \delta_1 \in (0, 1) \). For any \( \delta_1 \in (0, 1) \) and any \( \epsilon > 0 \) there exists a constant \( C_\epsilon \) depending only on \( \epsilon \) such that
\[
\text{meas}(\mathbb{R}_{\delta_1}) \leq \frac{C_\epsilon}{\delta_1^{4 + \epsilon}} N.
\]

As shown in [3], Lemma 7.4 is a direct consequence of Lemma 7.3.

**Proof of Lemma 7.3.** Using Lemma 7.2 instead of [3, Lemma 6.2], we derive in the same way as in [3, Proof Lemma 6.3] that we only need to consider the case
\[
\delta_1 > N^{-C(\epsilon \log \log N)^{-1}}.
\]
(7.7)

As in [3, Section 6], we let \( \theta_1, \ldots, \theta_R \) be \( 1/N \) spaced points in \( \mathbb{T} \) such that \( |\hat{a}(\theta_r)| \geq \delta_1 N \), for \( 1 \leq r \leq R \). We known from [3, Proof Lemma 6.3] that in order to prove Lemma 7.3 it is sufficient to show that
\[
R \ll \frac{C_\epsilon}{\delta_1^{4 + \epsilon}}.
\]
(7.8)
In order to prove (7.8), we let \( f_n \in \mathbb{R} \) be such that \(|f_n| \leq 1\) and \( a(n) = f_n v(n) \) for integers \( 1 \leq n \leq N \). Furthermore, define \( c_r \in \mathbb{C} \) with \(|c_r| = 1\) such that \( c_r \hat{a}(\theta_r) = |\hat{a}(\theta_r)|^2 \) for \( 1 \leq r \leq R \). Then it follows from the Cauchy-Schwarz inequality and the prime number theorem in arithmetic progressions with the constant module \( W \),

\[
\delta_1^2 N^2 R^2 \leq \left( \sum_{1 \leq r \leq R} |\hat{a}(\theta_r)| \right)^2
\]

\[
= \left( \sum_{1 \leq r \leq R} c_r \sum_{n \leq N} a(n) e(n \theta_r) \right)^2
\]

\[
= \left( \sum_{n \leq N} a(n)^{1/2} (f_n v(n))^{1/2} \sum_{1 \leq r \leq R} c_r e(n \theta_r) \right)^2
\]

\[
\ll \sum_{n \leq N} a(n) \sum_{n \leq N} v(n) \left| \sum_{1 \leq r \leq R} c_r e(n \theta_r) \right|^2
\]

\[
\ll N \sum_{n \leq N} v(n) \left| \sum_{1 \leq r \leq R} c_r e(n \theta_r) \right|^2,
\]

which implies

\[
\delta_1^2 N R^2 \ll \sum_{1 \leq r, r' \leq R} |\hat{a}(\theta_r - \theta_{r'})|.
\] (7.9)

Assume \( \gamma > 2 \) be fixed. Applying Hölder’s inequality to (7.9), we find

\[
\delta_1^2 \gamma^\gamma N^\gamma R^2 \ll \sum_{1 \leq r, r' \leq R} |\hat{a}(\theta_r - \theta_{r'})|^\gamma.
\] (7.10)

We put \( \theta = \theta_r - \theta_{r'} \) for given \( r \neq r' \). With \( \theta \) in place of \( \frac{\theta}{N} \), we see from (6.4), (6.18), (6.50), (6.51), (6.54), and Lemma 6.4,

\[
|\hat{v}(\theta)| \ll \frac{\phi(W)}{W} \log z \left( |T(q_1 q_2)| + q^{-1/2+\epsilon} \right) \left| \int_0^N e \left( \left( \theta - \frac{a}{q} \right) x \right) dx \right| + N^{1-\delta/400} g.
\] (7.11)

Using (4.3), Lemma 6.5, (6.11), and (6.52), we see from (7.11),

\[
|\hat{v}(\theta)| \ll \left( \log N |T(q_1 q_2)| + q^{-1/2+\epsilon} \right) \min \left\{ N, \left| \theta - \frac{a}{q} \right|^{-1} \right\} + N^{1-\delta/400}
\]
\[ \ll N \left( q^{-1/2+\epsilon} \left( 1 + N \left\| \theta - \frac{\alpha}{q} \right\| \right)^{-1} + N^{-\delta/400} \right). \]  

(7.12)

Let \( Q_1 = \delta_1^{-d} \) for some \( d \in [5, 6] \) which we will fix later. If \( q > Q_1 \), then by (7.7) and (7.12),

\[ |\hat{v}(\theta)| \ll Q_1^{-1/2+\epsilon} N \leq \delta_1^{(5/2+5\epsilon)} N. \]  

(7.13)

We derive from (7.12) and (7.13) that the contribution of all \( \theta = \theta_r - \theta_{r'} \in M(a, q) \) with \( q \geq Q_1 \) in the sum over \( r, r' \) in the right-hand side of (7.10) is \( \ll R^2 \delta_1^{(5/2+5\epsilon)} N^\gamma \), which implies that it is smaller than the left-hand side of (7.10) and we can therefore neglect it. Summarizing the above, we derive from (7.7), (7.10), and (7.12),

\[ \delta_1^{2\gamma} R^2 \ll \sum_{q \leq Q_1} \sum_{\substack{a \mod q \equiv 1 \ \forall (a, q) \equiv 1 \ \forall r, r' \leq R \left( 1 + N \left\| \theta_r - \theta_{r'} - \frac{\alpha}{q} \right\| \right)^\gamma,} \]

which implies

\[ \delta_1^{2\gamma + \epsilon d} R^2 \leq \sum_{q \leq Q_1} \sum_{\substack{a \mod q \equiv 1 \ \forall (a, q) \equiv 1 \ \forall r, r' \leq R \left( 1 + N \left\| \theta_r - \theta_{r'} - \frac{\alpha}{q} \right\| \right)^{\gamma/2}}. \]  

(7.14)

We assume \( \epsilon \leq 10^{-9} \). We set \( \delta_2 = \delta_1^{2\gamma + \epsilon d} \). Further we choose \( d \) such that \( 5 = d \delta_2 \epsilon \delta_1^{2\gamma + \epsilon d} \). This implies that \( d = \frac{10\epsilon}{2\gamma - 9\epsilon} \in [5, 6] \). Thus, we see from (7.14):

\[ \delta_2^{2\gamma} R^2 \leq \sum_{q \leq \delta_2^{-5}} \sum_{\substack{a \mod q \equiv 1 \ \forall (a, q) \equiv 1 \ \forall r, r' \leq R \left( 1 + N \left\| \theta_r - \theta_{r'} - \frac{\alpha}{q} \right\| \right)^{\gamma/2}}. \]  

(7.15)

This is exactly the same expression that was found in the proof of [3, Lemma 6.3]. It is stated in [3, end of section 6] that, using the argument in [2], (7.15) implies (7.8) with \( \delta_2 \) instead of \( \delta_1 \). As \( \epsilon \leq 10^{-9} \) and \( \gamma > 2 \), this in turn implies

\[ R \ll \frac{C_\epsilon}{\delta_1^{(4+\epsilon)\delta_1^{2\gamma + \epsilon d}}} \leq \frac{C_\epsilon}{\delta_1^{4+8\epsilon}}. \]

This proves (7.8) with \( \epsilon := 8\epsilon \).

### 8 Regularity condition

In this section, we prove that the function \( a_4(n) \) satisfies the regularity condition (5) of Theorem 1.2. Following the argument in [10, Section 5], we write
\[ \beta = \delta/50, \quad Y = \prod_{p \leq \beta^{-1}} p, \quad \text{and} \quad \beta \text{ is chosen such that } Y \mid W. \]

We set
\[ U = \{ 1 \leq u \leq \beta N_1 : Wu + b_4 \in \mathbb{P}^2 \}, \]
\[ V = \{ (1 - \beta) N_4 \leq v \leq N_4 : Wv + b_4 \in \mathbb{P}^2 \}. \]

We see,
\[
\sum_{(u,v) \in M} a_4(u) a_4(v) = \left( \frac{2 \phi(W)}{\sigma(b) W} \log z_4 \right)^2 \sum_{(u,v) \in M} \sqrt{Wu + b \sqrt{Wv + b}}
\]
\[
\gg NW \left( \frac{\phi(W)}{\sigma(b) W} \log z_4 \right)^2 \sum_{u \in U, v \in V} 1
\]
\[
\gg NW \left( \frac{\phi(W)}{\sigma(b) W} \log z_4 \right)^2 \sum_{s_1, s_2 (\text{mod} W)} |U \cap YZ + s_1| |V \cap YZ + s_2|. \tag{8.1}
\]

We now consider the set \( U \cap YZ + s_1 \). For any \( u \in U \), there exists a unique pair \( h_j \) and \( u_1, h_j \in [W] \), \( j \in [1, \ldots, \sigma(b_4)] \), \( h_j^2 \equiv b_4 (\text{mod} W) \), \( u_1 \in [1, U_1] \), \( U_1 := (\sqrt{W} \beta N_4 + b_4 - h_j) / W \), such that \( W u_1 + h_j \in \mathbb{P} \) and
\[
(W u_1 + h_j)^2 = Wu + b_4. \tag{8.2}
\]

Conversely, for each such pair \( u_1 \) and \( h_j \), there exists exactly one \( u \in U \) such that (8.2) holds. We write \( h_j^2 = c_j W + b_4 \) such that \( Wu + b_4 = W(W u_1^2 + 2h_j u_1 + c_j) + b_4 \). As \( Y \mid W \), the congruence condition \( u \equiv s_1 (\text{mod} Y) \) in (8.1) implies that
\[
2h_j u_1 \equiv s_1 - c_j (\text{mod} Y). \tag{8.3}
\]

In the same way, considering the set \( V \cap YZ + s_2 \), we find integers \( v_1 \in [V_1, V_2] \), \( V_1 = \left( \sqrt{W} (1 - \beta) N_4 + b_4 - h_k \right) / W \), \( V_2 = \left( \sqrt{W} N_4 + b_4 - h_k \right) / W \), \( h_k \), and \( c_k \) such that \( h_k^2 \equiv b_4 (\text{mod} W) \), and
\[
2h_k v_1 \equiv s_2 - c_k (\text{mod} Y). \tag{8.4}
\]

Combining (8.1) - (8.4), we obtain
\[
\sum_{(u,v) \in M} a_4(u) a_4(v)
\]
\[
\gg NW \left( \frac{\phi(W)}{\sigma(b) W} \log z_4 \right)^2 \sum_{s_1, s_2 (\text{mod} Y)} \sum_{j=1}^{\sigma(b)} \sum_{h=1}^{\sigma(b)} \sum_{1 \leq u_1 \leq U_1} \sum_{W u_1 + h_j \in \mathbb{P}} \sum_{2h_j u_1 \equiv s_1 - c_j (\text{mod} Y)} 1.
\]

(8.5)

33
For a fixed pair \( s_1 \) and \( s_2 \) with \((s_1 - s_2, Y) = 1\), note that the summation condition \((s_1 - s_2, Y) = 1\) implies that \((s_1, 2) = 1\) and \((s_2, 2) = 2\) or vice versa. Assuming - without loss of generality - that \((s_1, 2) = 1\) and \((s_2, 2) = 2\), the congruence conditions in the summations over \( u_1 \) and \( v_1 \) imply that \((c_{j_0}, 2) = 1\) and \((c_{k_0}, 2) = 2\). Assuming that for each pair \((s_1, s_2)\) we can find at least one pair \( h_{j_0} \) and \( h_{k_0} \) such that \( c_{j_0} \) and \( c_{k_0} \) satisfy these conditions, we see from [Ref.]:

\[
\sum_{(u,v) \in M} a_4(u)a_4(v) \geq NW \left( \frac{\phi(W)}{W} \log z_4 \right)^2 \sum_{1 \leq u_1 \leq U_1} \sum_{W^{u_1} + a_{j_0} \in \mathcal{P}} \sum_{u_1 \equiv h_{j_0}(s_1 - c_{j_0})/2(\text{mod } Y)} 1
\]

\[
\geq NW \left( \frac{\phi(W)}{W} \log z_4 \right)^2 Y \phi(Y) \left( \frac{\sqrt{\beta N_4}}{\log N_4} \frac{\sqrt{W}}{\phi(W)Y} \right)^2
\]

\[
\gg N^2 \kappa,
\]

Here, we have used the prime number theorem in arithmetic progressions of modulus \( WY \) which is a constant. To conclude the proof, we show that for a fixed pair \( s_1 \) and \( s_2 \) we can indeed find at least one pair \( h_{j_0} \) and \( h_{k_0} \) such that \( h_{j_0} = h_{k_0} = b_4(\text{mod } W) \) such that \((c_{j_0}, 2) = 1\) and \((c_{k_0}, 2) = 2\). Assume a given \( h_{j_0} \) with \( h_{j_0}^2 = c_{j_0}W + b_4 \) such that \((c_{j_0}, 2) = 2\). We define \( h_{j_1} := h_{j_0} + \frac{W}{2} \). We see \( h_{j_1}^2 = Wc_{j_1} + b_4 \), where \( c_{j_1} = c_{j_0} + h_{j_0} + \frac{W}{2} \). As \((c_{j_0} + h_{j_0} + \frac{W}{2}, 2) = 1\), we have found \( h_{j_1} \) such that \((c_{j_1}, 2) = 1\). In the same way, we can show that there exists \( h_{k_0} \) such that \((c_{k_0}, 2) = 1\).

References

[1] B. Berndt, R.J. Evans, K.S. Williams, *Gauss and Jacobi Sums*, Canadian Mathematical Sociey Series Monographs and Advanced texts, John Wiley and Sons (1998).

[2] J. Bourgain, *On \( \Lambda(p) - \) subsets of squares*, Israel J. Math. 67 (1989), 291-311.

[3] T.D. Browning and S.M. Prendiville, *A transference approach to a Roth-type theorem in the squares*, International Mathematics Research Notices 2016, p. 1-30.

[4] S. Chow, *Roth - Waring - Goldbach*, International Mathematics Research Notices 2017.
[5] B. Green, *Roths theorem in the primes*, Ann. of Math. (2), 161(3), 2005.

[6] B. Green and T. Tao., *Restriction theory of the Selberg sieve with applications*, Journal de Théorie des Nombres Bordeaux, 18(1), 2006.

[7] L. K. Hua, *Some results in the additive prime number theory*, Quart. J. Math., 9 (1938), 68 - 80.

[8] H. Li and H. Pan, *A density version of Vinogradovs three primes theorem*, Forum Math. 22(4), 2010

[9] X. Shao, *A density version of the Vinogradov three primes theorem*, Duke Math. J. 163(3) (2014), 489512.

[10] X. Shao, *An L-function free proof of Vinogradov’s three primes theorem*, Forum of Mathematics, Sigma (2014), Vol. 2, e27.

[11] T. Tao, *Lecture Notes 1 for 254A*,
https://www.math.cmu.edu/~af1p/Teaching/AdditiveCombinatorics/Tao.pdf

[12] R.C. Vaughan, *The Hardy-Littlewood method*, 2nd ed., Cambridge Tracts in Mathematics, vol. 125, cambridge University Press, Cambridge, 1997.