Bochner-Weitzenböck formulas associated with the Rarita-Schwinger operator

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1 Introduction

In this paper, we establish basic material for future investigations of the analysis and geometry of the twistor bundle, and of differential operators with the twistor bundle as source and/or target, especially the Rarita-Schwinger operator, a first order differential operator taking twistors to twistors. Some of the material that we shall present generalizes to arbitrary irreducible tensor-spinor bundles \([7]\). In addition, some material which does not have a clear generalization of this breadth should nevertheless extend to statements about spinor-forms (see \([4]\)), or about bundles contained in the tensor product of the spinors with the trace-free symmetric tensors. There are some nice complementary results in more analytic directions for flat structures; see for example \([16]\) and \([14]\). One direct inspiration for our investigations is the success of the spinor program \([2, 3]\), and we have been guided by a desire to obtain analogues of the most important results of this field.

Some of the results we state here are undoubtedly not the most refined or extensive possible. However, the relevant identities and decompositions do not seem to be in general circulation. Given this, it seems timely to put some of this material into print, together with sufficient concrete results to indicate the motivation and effectiveness of the method. One of our main themes is that to “work on twistors and the Rarita-Schwinger operator,” one needs to also consider several other related bundles and operators.

2 Familiar vector bundles and first-order differential operators

Let \((M, g, E, \gamma)\) be an \(n\)-dimensional Riemannian spin manifold. That is, we have a Riemannian manifold \((M, g)\) which admits spin structure, and thus has a volume \(n\)-form \(E\), a spinor bundle \(\Sigma\), and a fundamental tensor-spinor \(\gamma\); this is a smooth section of the bundle \(TM \otimes \text{End}(\Sigma)\) with

\[
\gamma^\alpha \gamma^\beta + \gamma^\beta \gamma^\alpha = -2g^{\alpha\beta} \quad \text{and} \quad \nabla \gamma = 0.
\]

The connection \(\nabla\) is the natural extension of the Levi-Civita connection on \(TM\) to tensor-spinors of arbitrary type. Here and below, we use an abstract tensor index notation, but do not write spinor indices explicitly. “Abstract” is meant in the sense of Penrose (\([22], \S2\)): the indices do not refer to a choice of local frame, but rather are placeholders; indicating, among other things, how to compute the expression locally should choices be made. The allowable manipulations may then be described by a finite number of axioms. The dimension, but not the signature of the metric, is detectable via such manipulations. As usual in tensor calculus, an expression like
\( \nabla_\alpha \omega_\beta \) denotes \((\nabla \omega)_{\alpha\beta}\). An index which appear twice in a term, once up and once down, indicates a contraction, and indices may be lowered and raised using the metric tensor and its inverse.

Given a vector bundle \( V \), we denote by \( \Gamma(V) \) its smooth section space. The Dirac operator is, up to normalization, the operator

\[
\nabla : \Gamma(\Sigma) \to \Gamma(\Sigma),
\]

\[
\psi \mapsto \gamma^\alpha \nabla_\alpha \psi.
\]

Let \( T \) be the twistor bundle; that is, the subbundle of \( T^*M \otimes \Sigma \) determined by the (pointwise) equation

\[
\gamma^\alpha \varphi_\alpha = 0.
\]

The twistor operator is the operator

\[
\mathcal{T} : \Gamma(\Sigma) \to \Gamma(T),
\]

\[
\psi \mapsto \nabla_\alpha \psi + \frac{1}{n} \gamma_\alpha \nabla \psi.
\]

The formal adjoint of the twistor operator is

\[
\mathcal{T}^* : \Gamma(T) \to \Gamma(\Sigma),
\]

\[
\varphi \mapsto -\nabla^\alpha \varphi_\alpha,
\]

as one sees via the calculation

\[
\left\langle \left( \nabla_\alpha + \frac{1}{n} \gamma_\alpha \nabla \right) \psi, \varphi^\alpha \right\rangle = - \left\langle \psi, \nabla_\alpha \varphi^\alpha + \frac{1}{n} \nabla (\gamma_\alpha \varphi^\alpha) \right\rangle + \text{(exact divergence)}
\]

\[
= - \left\langle \psi, \nabla^\alpha \varphi_\alpha \right\rangle + \text{(exact divergence)}
\]

\[
= \left\langle \psi, \nabla^* \varphi \right\rangle + \text{(exact divergence)}.
\]

Here we have used the covariant constancy of the spin metric \( \langle \cdot, \cdot \rangle \), the skew-adjointness of \( \gamma^\alpha \) (and the consequent formal self-adjointness of \( \nabla \)), and the fact that \( \gamma^\alpha \varphi_\alpha = 0 \) for a section of \( T \). By “exact divergence”, we mean an expression of the form \( \nabla^\alpha \omega_\alpha \), where \( \omega \in \Gamma(T^*M) \). We shall often make calculations like this, without explicitly noting all the steps.

The operator \( T \) may be described as \( P \circ \nabla \), where \( \nabla \) is the covariant derivative \( \Gamma(\Sigma) \to \Gamma(T^*M \otimes \Sigma) \), and \( P \) is the orthogonal projection of \( T^*M \otimes \Sigma \) onto \( T \). Since \( T \) is a \( \text{Spin}(n) \)-subbundle of \( T^*M \otimes \Sigma \), the projection \( P \) is \( \text{Spin}(n) \)-equivariant. Given this, the formula (2.2) is not surprising, and is an example of a more general phenomenon: since orthogonal projections are self-adjoint, \( (P \circ \nabla)^* = \nabla^* \circ P = \nabla^* \) on \( \Gamma(T) \).

The \( \text{Spin}(n) \)-bundle complementary to \( T \) is the image of \( \Sigma \) under the injection \( I : \psi \mapsto \gamma_\alpha \psi \). This map is clearly \( \text{Spin}(n) \)-equivariant. Injectivity is guaranteed by the calculations

\[
I^* \varphi = -\gamma_\alpha \varphi^\alpha, \quad I^* I = n \text{Id}_\Sigma,
\]
which show that $n^{-1/2}I$ is an isometric injection.

In view of (2.1), the corresponding decomposition of $\nabla \psi$ is

$$\nabla \psi = T\psi - \frac{1}{n} I \nabla \psi.$$ 

The Rarita-Schwinger operator (see (1.2)) on $\Gamma(T)$ is, up to normalization, the operator

$$S^0 : \Gamma(T) \to \Gamma(T),$$

$$\varphi \mapsto \gamma^\lambda \nabla_\lambda \varphi - \frac{2}{n} \gamma_\alpha \nabla^\alpha \varphi.$$ 

(2.3)

The operator $S^0$, like $\nabla$, is formally self-adjoint. It may be described (and will be below) as the orthogonal projection of the operator $\gamma^\lambda \nabla_\lambda$ on $\Gamma(T)$ to the (unique) subbundle $W$ of $T^* M \otimes T$ which is isomorphic to $T$, followed by a bundle isomorphism $W \to T$.

## 3 Further relevant bundles and operators

Let $\text{TFS}^2$ be the bundle of trace-free symmetric two-tensors, and let $Z$ be the subbundle of $\text{TFS}^2 \otimes \Sigma$ determined by the pointwise condition

$$\gamma^\beta \Phi_{\alpha \beta} = 0.$$  

(3.1)

With (3.1) in place, the trace-free condition is actually redundant, since by the Clifford relations, $0 = \gamma^\beta \gamma^\alpha \Phi_{\alpha \beta} = -g^{\alpha \beta} \Phi_{\alpha \beta}$. Note that (3.1) requires $\Phi$ to also be a section of $T^* M \otimes T$, so that

$$Z = (\text{TFS}^2 \otimes \Sigma) \cap (T^* M \otimes T).$$

Similarly, let $Y$ be the subbundle of $\Lambda^2 \otimes \Sigma$ determined by the condition (3.1); that is,

$$Y = (\Lambda^2 \otimes \Sigma) \cap (T^* M \otimes T).$$

Two more subbundles of $T^* M \otimes T$ may be defined by injecting $\Sigma$ into $T^* M \otimes T$ using the map

$$\psi \mapsto (I_{\Sigma} \psi)_{\alpha \beta} = \left\{ \gamma_\alpha \gamma_\beta + \frac{n - 2}{n} \gamma_\beta \gamma_\alpha \right\} \psi,$$

and by injecting $T$ into $T^* M \otimes T$ using the map

$$\varphi_\alpha \mapsto (I_{T} \varphi)_{\alpha \beta} = \gamma_\alpha \varphi_\beta - \frac{2}{n} \gamma_\beta \varphi_\alpha.$$ 

The maps $I_{\Sigma}$ and $I_{T}$ are clearly Spin($n$)-equivariant. Injectivity is guaranteed by the calculations

$$I_{\Sigma} \Phi = -2 \Phi^\alpha _\alpha, \quad I_{\Sigma}^* I_{\Sigma} = 4(n - 1) \text{Id}_{\Sigma},$$

$$I_{T}^* \Phi = -\gamma^\beta \Phi_{\beta \alpha}, \quad I_{T}^* I_{T} = \frac{(n + 2)(n - 2)}{n} \text{Id}_{T},$$ 

(3.2)
which also show that \(\{4(n-1)\}^{-1/2}I_{\Sigma}\) and \(\{(n+2)(n-2)/n\}^{-1/2}I_{T}\) are isometric injections.

For use in some of the following formulas, we introduce the *antisymmetric Clifford symbols*

\[
\gamma_{\alpha\beta} := \frac{1}{2}(\gamma_\alpha \gamma_\beta - \gamma_\beta \gamma_\alpha),
\]

so that

\[
\gamma_\alpha \gamma_\beta = \gamma_{\alpha\beta} - g_{\alpha\beta}.
\]

The four subbundles of \(T^*M \otimes T\) given above are clearly orthogonal. Moreover, the maps \(P_{\Sigma}, P_T, P_Y, P_Z\) given by

\[
(P_{\Sigma} \Phi)_{\alpha\beta} = -\frac{1}{2(n-1)} \left\{ \frac{n-2}{n} \gamma_\alpha \gamma_\beta + \gamma_\alpha \gamma_\beta \right\} \Phi^\lambda
\]

\[
= \frac{1}{n} \left\{ g_{\alpha\beta} - \frac{1}{n} \gamma_{\alpha\beta} \right\} \Phi^\lambda,
\]

\[
(P_T \Phi)_{\alpha\beta} = \frac{1}{(n+2)(n-2)} \left\{ -n \gamma_\alpha \gamma^\lambda \Phi_{\lambda\beta} + 2 \gamma_\beta \gamma^\lambda \Phi_{\lambda\alpha} + 2 \gamma_\alpha \gamma_\beta \Phi^\lambda \right\}
\]

\[
- \frac{4}{n} \gamma_\beta \gamma_\alpha \Phi^\lambda
\]

\[
= \frac{1}{(n+2)(n-2)} \left\{ -n \gamma_\alpha \gamma^\lambda \Phi_{\lambda\beta} + 2 \gamma_\beta \gamma^\lambda \Phi_{\lambda\alpha} + n \Phi_{\alpha\beta} - 2 \Phi_{\beta\alpha} \right\}
\]

\[
+ \left( \frac{2}{n(n-2)} \gamma_{\alpha\beta} - \frac{2}{n+2} g_{\alpha\beta} \right) \Phi^\lambda,
\]

\[
(P_Y \Phi)_{\alpha\beta} = \frac{1}{2} \left\{ \Phi_{\alpha\beta} - \Phi_{\beta\alpha} \right\} + \frac{1}{2(n-2)} \left\{ \gamma_\alpha \gamma^\lambda \Phi_{\lambda\beta} - \gamma_\beta \gamma^\lambda \Phi_{\lambda\alpha} \right\}
\]

\[
- \frac{1}{2(n-1)(n-2)} \left\{ \gamma_\alpha \gamma^\lambda \Phi_{\lambda\beta} - \gamma_\beta \gamma^\lambda \Phi_{\lambda\alpha} \right\} \Phi^\lambda
\]

\[
= \frac{n-3}{2(n-2)} \left\{ \Phi_{\alpha\beta} - \Phi_{\beta\alpha} \right\} + \frac{1}{2(n-2)} \left\{ \gamma_\alpha \gamma^\lambda \Phi_{\lambda\beta} - \gamma_\beta \gamma^\lambda \Phi_{\lambda\alpha} \right\}
\]

\[
- \frac{1}{n-1(n-2)} \gamma_{\alpha\beta} \Phi^\lambda,
\]

\[
(P_Z \Phi)_{\alpha\beta} = \frac{1}{2} \left\{ \Phi_{\alpha\beta} + \Phi_{\beta\alpha} \right\} + \frac{1}{2(n+2)} \left\{ \gamma_\alpha \gamma^\lambda \Phi_{\lambda\beta} + \gamma_\beta \gamma^\lambda \Phi_{\lambda\alpha} - 2 g_{\alpha\beta} \Phi^\lambda \right\}
\]

\[
= \frac{1}{2(n+2)} \left\{ (n+1)(\Phi_{\alpha\beta} + \Phi_{\beta\alpha}) + \gamma_\alpha \gamma^\lambda \Phi_{\lambda\beta} + \gamma_\beta \gamma^\lambda \Phi_{\lambda\alpha} - 2 g_{\alpha\beta} \Phi^\lambda \right\},
\]

are complementary projections: one has

\[
P_{\Sigma} + P_T + P_Y + P_Z = \text{Id}_{T^*M \otimes T},
\]

\[
P_u^2 = P_u,
\]

\[
P_u P_v = 0, \; u \neq v,
\]

where \(u, v\) run through the labels \(\Sigma, \; T, \; Y; \) and \(Z\). The projections \(P_{\Sigma}, \; P_T, \; P_Y, \) and \(P_Z\) are valued in \(I_{\Sigma} \Sigma, \; I_T T, \; Y, \) and \(Z\) respectively. In particular,

\[
T^* M \otimes T = I_{\Sigma} \Sigma \oplus I_T T \oplus Y \oplus Z.
\]
We define the first-order differential operators $\mathbf{G}_u$ by applying the above projections to $\nabla \varphi$ for $\varphi \in \Gamma(\mathbf{T})$:

$$
(\mathbf{G}_\Sigma \varphi)_{\alpha\beta} = -\frac{1}{2(n-1)} \left\{ \gamma_\alpha \gamma_\beta + \frac{n-2}{n} \gamma_\beta \gamma_\alpha \right\} \nabla^\lambda \varphi_\lambda
$$

$$
= \frac{1}{n} \left\{ g_{\alpha\beta} - \frac{1}{n-1} \gamma_{\alpha\beta} \right\} \nabla^\lambda \varphi_\lambda,
$$

(3.3)

$$
(\mathbf{G}_T \varphi)_{\alpha\beta} = \frac{1}{(n+2)(n-2)} \left\{ - n \gamma_\alpha \gamma_\beta \nabla_\lambda \varphi_\beta + 2 \gamma_\beta \gamma_\lambda \nabla_\lambda \varphi_\alpha + 2 \gamma_\alpha \gamma_\lambda \nabla_\lambda \varphi_\beta
$$

$$
- \frac{4}{n} \gamma_\beta \gamma_\alpha \nabla^\lambda \varphi_\lambda \right\}
$$

$$
= \frac{1}{(n+2)(n-2)} \left\{ - n \gamma_\alpha \gamma_\beta \nabla_\lambda \varphi_\beta + 2 \gamma_\beta \gamma_\lambda \nabla_\lambda \varphi_\alpha + n \nabla_\alpha \varphi_\beta - 2 \nabla_\beta \varphi_\alpha \right\}
$$

$$
+ \left( \frac{2}{n(n-2)} \gamma_{\alpha\beta} - \frac{2}{n(n+2)} g_{\alpha\beta} \right) \nabla^\lambda \varphi_\lambda,
$$

(3.4)

$$
(\mathbf{G}_Y \varphi)_{\alpha\beta} = \frac{1}{2} \left\{ \nabla_\alpha \varphi_\beta - \nabla_\beta \varphi_\alpha \right\} + \frac{1}{(n-2)} \left\{ \gamma_\alpha \gamma_\beta \nabla_\lambda \varphi_\beta - \gamma_\beta \gamma_\lambda \nabla_\lambda \varphi_\alpha \right\}
$$

$$
- \frac{1}{2(n-1)(n-2)} \left\{ \gamma_\alpha \gamma_\beta - \gamma_\beta \gamma_\alpha \right\} \nabla^\lambda \varphi_\lambda
$$

$$
= \frac{n-3}{2(n-2)} \left\{ \nabla_\alpha \varphi_\beta - \nabla_\beta \varphi_\alpha \right\} + \frac{1}{2(n-2)} \left\{ \gamma_\alpha \gamma_\beta \nabla_\lambda \varphi_\beta - \gamma_\beta \gamma_\lambda \nabla_\lambda \varphi_\alpha \right\}
$$

$$
- \frac{1}{(n-1)(n-2)} \gamma_{\alpha\beta} \nabla^\lambda \varphi_\lambda,
$$

(3.5)

$$
(\mathbf{G}_Z \varphi)_{\alpha\beta} = \frac{1}{2} \left\{ \nabla_\alpha \varphi_\beta + \nabla_\beta \varphi_\alpha \right\} + \frac{1}{2(n+2)} \left\{ \gamma_\alpha \gamma_\beta \nabla_\lambda \varphi_\beta
$$

$$
+ \gamma_\beta \gamma_\lambda \nabla_\lambda \varphi_\alpha - 2 g_{\alpha\beta} \nabla^\lambda \varphi_\lambda \right\}
$$

$$
= \frac{1}{2(n+2)} \left\{ (n+1) \left( \nabla_\alpha \varphi_\beta + \nabla_\beta \varphi_\alpha \right) + \gamma_\alpha \gamma_\beta \nabla_\lambda \varphi_\beta
$$

$$
+ \gamma_\beta \gamma_\lambda \nabla_\lambda \varphi_\alpha - 2 g_{\alpha\beta} \nabla^\lambda \varphi_\lambda \right\}.
$$

(3.6)

By the above remarks on formal adjoints, $\mathbf{G}_u^* \mathbf{G}_u = \nabla^* \mathbf{G}_u$:

$$
(\mathbf{G}_\Sigma^* \mathbf{G}_\Sigma \varphi)_{\beta} = \frac{1}{2(n-1)} \left\{ \gamma_\alpha \gamma_\beta + \frac{n-2}{n} \gamma_\beta \gamma_\alpha \right\} \nabla^\alpha \nabla^\lambda \varphi_\lambda
$$

$$
= -\frac{1}{n} \left\{ g_{\alpha\beta} - \frac{1}{n-1} \gamma_{\alpha\beta} \right\} \nabla^\alpha \nabla^\lambda \varphi_\lambda,
$$

$$
(\mathbf{G}_T^* \mathbf{G}_T \varphi)_{\beta} = \frac{1}{(n+2)(n-2)} \left\{ n \gamma_\alpha \gamma_\beta \nabla^\alpha \nabla_\lambda \varphi_\beta - 2 \gamma_\beta \gamma_\lambda \nabla^\alpha \nabla_\lambda \varphi_\alpha
$$

$$
- 2 \gamma_\alpha \gamma_\beta \nabla^\alpha \nabla^\lambda \varphi_\lambda + \frac{4}{n} \gamma_\beta \gamma_\alpha \nabla^\alpha \nabla^\lambda \varphi_\lambda \right\}
$$

$$
= \frac{1}{(n+2)(n-2)} \left\{ n \gamma_\alpha \gamma_\beta \nabla^\alpha \nabla_\lambda \varphi_\beta - 2 \gamma_\beta \gamma_\lambda \nabla^\alpha \nabla_\lambda \varphi_\alpha
$$

$$
- n \nabla^\alpha \nabla_\alpha \varphi_\beta + 2 \nabla^\alpha \nabla_\beta \varphi_\alpha \right\}
$$

$$
- \left( \frac{2}{n(n-2)} \gamma_{\alpha\beta} - \frac{2}{n(n+2)} g_{\alpha\beta} \right) \nabla^\alpha \nabla^\lambda \varphi_\lambda,
$$

(3.7)
\[(G_Y^*G_Y \varphi)_\beta = -\frac{1}{2} \{ \nabla^\alpha \nabla_\alpha \varphi_\beta - \nabla^\alpha \nabla_\beta \varphi_\alpha \} \]
\[+ \frac{1}{2(n-2)} \{ \gamma_\alpha \gamma^\lambda \nabla^\alpha \nabla_\lambda \varphi_\beta - \gamma_\beta \gamma^\lambda \nabla^\alpha \nabla_\lambda \varphi_\alpha \} \]
\[+ \frac{1}{2(n-1)(n-2)} \{ \gamma_\alpha \gamma_\beta - \gamma_\beta \gamma_\alpha \} \nabla^\alpha \nabla^\lambda \varphi_\beta \]
\[+ \frac{1}{2(n-2)} \{ \gamma_\alpha \gamma^\lambda \nabla^\alpha \nabla_\lambda \varphi_\beta - \gamma_\beta \gamma^\lambda \nabla^\alpha \nabla_\lambda \varphi_\alpha \} \]
\[= -\frac{n-3}{2(n-2)} \{ \nabla^\alpha \nabla_\alpha \varphi_\beta - \nabla^\alpha \nabla_\beta \varphi_\alpha \} \]
\[+ \frac{1}{2(n-2)} \{ \gamma_\alpha \gamma^\lambda \nabla^\alpha \nabla_\lambda \varphi_\beta - \gamma_\beta \gamma^\lambda \nabla^\alpha \nabla_\lambda \varphi_\alpha \} \]
\[+ \frac{n-1}{n-2} \gamma_\alpha \gamma^\lambda \nabla^\alpha \nabla_\lambda \varphi_\beta , \]

\[(G_Z^*G_Z \varphi)_\beta = -\frac{1}{2} \{ \nabla^\alpha \nabla_\alpha \varphi_\beta + \nabla^\alpha \nabla_\beta \varphi_\alpha \} - \frac{1}{2(n+2)} \{ \gamma_\alpha \gamma^\lambda \nabla^\alpha \nabla_\lambda \varphi_\beta \]
\[+ \gamma_\beta \gamma^\lambda \nabla^\alpha \nabla_\lambda \varphi_\alpha - 2g_\alpha \beta \nabla^\alpha \nabla^\lambda \varphi_\lambda \} \]
\[= -\frac{1}{2(n+2)} \{ (n+1) \{ \nabla^\alpha \nabla_\alpha \varphi_\beta + \nabla^\alpha \nabla_\beta \varphi_\alpha \} \]
\[+ \gamma_\alpha \gamma^\lambda \nabla^\alpha \nabla_\lambda \varphi_\beta + \gamma_\beta \gamma^\lambda \nabla^\alpha \nabla_\lambda \varphi_\alpha - 2g_\alpha \beta \nabla^\alpha \nabla^\lambda \varphi_\lambda \} \].

The following is a consequence of the general elliptic classification scheme of [7], Theorem 4.10. We also supply an elementary proof below.

**Lemma 1** The operator \(G_T^*G_T\) is strongly elliptic, in the sense that its leading symbol is bounded below by a positive constant times the leading symbol of \(\nabla^* \nabla\).

**Proof.** One computes that if
\[(T(\xi) \varphi)_\beta := (n-1)\xi^\alpha \xi_\beta \varphi_\alpha + \gamma_\beta \alpha \xi^\alpha \xi^\lambda \varphi_\lambda , \quad (3.7)\]
then \(T(\xi)^2 = (n-1)|\xi|^2 T(\xi)\), and
\[\sigma_2(G_T^*G_T)(\xi) = -\frac{1}{(n+2)(n-2)} \left( -n|\xi|^2 + \frac{4}{n} T(\xi) \right) . \]
As a result, if
\[E_T := -(n+2)^2 G_T^*G_T + \frac{2(n+2)(n^2 - 2n + 2)}{n(n-2)} \nabla^* \nabla , \]
then
\[\sigma_2(E_T)(\xi) \sigma_2(G_T^*G_T)(\xi) = |\xi|^4 . \]
Thus \(\sigma_2(G_T^*G_T)(\xi)\) is invertible for nonzero \(\xi\), so \(G_T^*G_T\) is elliptic. Hence, since \(\sigma_2(G_T^*G_T)(\xi) = \sigma_1(G_T)(\xi)^* \sigma_1(G_T)(\xi)\) is clearly positive semidefinite, it is positive definite for \(\xi \neq 0\). By equivariance, the eigenvalues of \(\sigma_2(G_T^*G_T)(\xi)\) have the form
µ_i|ξ|^2, where the list of µ_i is independent of x ∈ M (and in fact independent of the manifold M). By positive definiteness, 0 < µ := min{µ_i}, and we have

$$\sigma_2(G^*_T G_T) \geq \mu \sigma_2(\nabla^* \nabla).$$  \hspace{1cm} (3.8)

In fact, the proof of Lemma 1 gives us more precise information:

**Corollary 2**  For a, b > 0, the operators

$$G^*_Z G_Z \text{ and } aG^*_\Sigma G_\Sigma + bG^*_Y G_Y$$

are strongly elliptic, and

$$\sigma_2(G^*_T G_T)(ξ) \geq \frac{n-2}{n(n+2)}|ξ|^2; \hspace{1cm} (3.9)$$

$$\sigma_2(G^*_Z G_Z)(ξ) \geq \frac{n+1}{2(n+2)}|ξ|^2; \hspace{1cm} (3.10)$$

$$\sigma_2(aG^*_Z G_\Sigma + bG^*_Y G_Y)(ξ) \geq \min \left( \frac{a}{n}, \frac{b(n-3)}{2(n-2)} \right). \hspace{1cm} (3.11)$$

**Proof.** By the proof of Lemma 1, the eigenvalues µ_i of $\sigma_2(G^*_T G_T)(ξ)$ for $|ξ|^2 = 1$ must be roots of the quadratic

$$-(n+2)^2\mu_i^2 + \frac{2(n+2)(n^2-2n+2)}{n(n-2)}\mu_i - 1 = -(n+2)^2 \left( \mu_i - \frac{n-2}{n(n+2)} \right) \left( \mu_i - \frac{n}{(n+2)(n-2)} \right).$$

In particular, we have the estimate (3.9). We may also compute that, in the notation of (3.7) above,

$$\sigma_2(G^*_Z G_\Sigma)(ξ) = \frac{1}{n(n-1)}T(ξ) = \frac{n|ξ|^2 - (n+2)(n-2)\sigma_2(G^*_T G_T)(ξ)}{4(n-1)},$$

$$\sigma_2(G^*_Y G_Y)(ξ) = \frac{n-3}{8(n-1)} \left\{ -(n-2)|ξ|^2 + n(n+2)\sigma_2(G^*_T G_T)(ξ) \right\},$$

$$\sigma_2(G^*_Z G_Z)(ξ) = \frac{1}{8} \left\{ (n+2)|ξ|^2 - n(n-2)\sigma_2(G^*_T G_T)(ξ) \right\}. $$

Thus, with respect to the block diagonalization in which

$$\sigma_2(G^*_T G_T)(ξ) = \text{diag} \left( \frac{n}{(n+2)(n-2)}, \frac{n-2}{n(n+2)} \right) |ξ|^2, \hspace{1cm} (3.12)$$
we also have
\[
\sigma_2(G^*_\Sigma G_\Sigma)(\xi) = \text{diag} \left( \frac{1}{n}, \frac{n-3}{2(n-2)}, 0 \right) |\xi|^2, \\
\sigma_2(G^*_\Upsilon G_\Upsilon)(\xi) = \text{diag} \left( \frac{n-3}{2(n-2)}, 0 \right) |\xi|^2, \\
\sigma_2(G^*_Z G_Z)(\xi) = \text{diag} \left( \frac{n+1}{2(n+2)}, \frac{n}{n+2} \right) |\xi|^2.
\]
(3.13)

The estimates (3.10,3.11) follow.

The following is a provisional form of Theorem 4 below:

**Corollary 3** The following operators have order zero:
\[
Z_1 := \frac{(n-3)(n-2)}{2n} G^*_\Sigma G_\Sigma - \frac{(n-3)(n+2)}{2n} G^*_T G_T + G^*_\Upsilon G_\Upsilon, \\
Z_2 := -\frac{(n-1)(n+2)}{2n} G^*_\Sigma G_\Sigma - \frac{(n-2)(n+1)}{2n} G^*_T G_T + G^*_Z G_Z.
\]

**Proof.** Equations (3.12,3.13) show that the \(Z_i\) have order at most 1. But invariant theory shows that any equivariant operator of homogeneity 2 and order < 2 is an action of the Riemann curvature.

\[\square\]

4 Bundles associated to representations of the spin group

Here we would like to provide some background and motivation for the decompositions above. Strictly speaking, this material is not needed to follow the arguments of this paper. Accordingly, we do not fill in the details of, for example, the process of matching dominant weight labels to tensor symmetry types of bundles; see [23, 24] more information along these lines. We believe, however, that an understanding of the representation theoretic thinking behind this work will be valuable in further investigations.

Irreducible representations of Spin\((n), \ n \geq 2,\) are parameterized by *dominant weights* \((\lambda_1, \ldots, \lambda_\ell) \in \mathbb{Z}^\ell \cup (\frac{1}{2} + \mathbb{Z})^\ell, \ \ell = [n/2],\) satisfying the inequality constraint
\[
\lambda_1 \geq \ldots \geq \lambda_\ell \geq 0, \quad \text{\(n\) odd,} \\
\lambda_1 \geq \ldots \geq \lambda_{\ell-1} \geq |\lambda_\ell|, \quad \text{\(n\) even.}
\]

The dominant weight \(\lambda\) is the highest weight of the corresponding representation. The representations which factor through SO\((n)\) are exactly those with \(\lambda \in \mathbb{Z}^\ell\). We shall denote by \(V(\lambda)\) the representation with highest weight \(\lambda\). If \(M\) is an \(n\)-dimensional
smooth manifold with $\text{Spin}(n)$ structure and $\mathcal{F}$ is the bundle of spin frames, we denote by $\mathbb{V}(\lambda)$ the vector bundle $\mathcal{F} \times_{\lambda} V(\lambda)$. When spin structure is not involved (i.e. when $\lambda$ is integral), we may use the orthonormal frame bundle in constructing $\mathbb{V}(\lambda)$.

One important highest weight is that of the defining representation $V(1,0,\ldots,0)$ of $\text{SO}(n)$. The classical selection rule describes the $\text{Spin}(n)$ decomposition of $V(1,0,\ldots,0) \otimes V(\lambda)$ for an arbitrary dominant $\lambda$:

$$V(1,0,\ldots,0) \otimes V(\lambda) \cong_{\text{Spin}(n)} V(\sigma_1) \oplus \cdots \oplus V(\sigma_{N(\lambda)}),$$

where the $\sigma_u$ are distinct: $\sigma_u \cong_{\text{Spin}(n)} \sigma_v \Rightarrow u = v$. A given $\sigma$ appears if and only if $\sigma$ is a dominant weight and

$$\sigma = \lambda \pm e_a, \text{ some } a \in \{1,\ldots,\ell\}, \quad \text{or} \quad n \text{ is odd, } \lambda\ell \neq 0, \quad \sigma = \lambda. \quad (4.1)$$

Here $e_a$ is the $a^{\text{th}}$-standard basis vector in $\mathbb{R}^\ell$. Note that $N(\lambda)$, the number of selection rule “targets” of $V(\lambda)$, depends on $\lambda$. We shall use the notation

$$\lambda \leftrightarrow \sigma$$

for the selection rule: $\lambda \leftrightarrow \sigma$ if and only if $V(\sigma)$ appears as a summand in $V(1,0,\ldots,0) \otimes V(\lambda)$. The notation $\leftrightarrow$ is justified because the relation is symmetric. In fact, one can see a priori that the relation must be symmetric: the defining representation of $\text{SO}(n)$ is real, and thus self-contragredient.

An interesting concept related to the selection rule is that of generalized gradients, or Stein-Weiss operators [23]. The covariant derivative $\nabla$ carries sections of $\mathbb{V}(\lambda)$ to sections of

$$T^*M \otimes \mathbb{V}(\lambda) \cong_{\text{Spin}(n)} \mathbb{V}(1,0,\ldots,0) \otimes \mathbb{V}(\lambda) \cong_{\text{Spin}(n)} \mathbb{V}(\sigma_1) \oplus \cdots \oplus \mathbb{V}(\sigma_{N(\lambda)}).$$

Since the selection rule is multiplicity free, we may project onto the unique $\sigma_u$ summand; the result is our gradient:

$$G_u = G_{\lambda \sigma_u} = P_{u^o} \nabla.$$

Up to normalization and isomorphic realization of bundles, some examples of gradients, or direct sums of gradients, are the exterior derivative $d$, its formal adjoint $\delta$, the conformal Killing operator $S$, the Dirac operator, the twistor operator, and the Rarita-Schwinger operator. In fact, every first-order $\text{Spin}(n)$-equivariant differential operator is a direct sum of gradients [17].

In even dimensions, if $\lambda\ell \neq 0$, the bundles $\mathbb{V}(\lambda)$ and $\mathbb{V}(\bar{\lambda})$, where

$$\bar{\lambda} = (\lambda_1, \ldots, \lambda_{\ell-1}, -\lambda_\ell),$$

...
are distinguished by the action of the volume element, which commutes with the action of SO($n$) or Spin($n$), but not with that of O($n$) or Pin($n$). This shows up in \textit{duality} and \textit{chirality} considerations. For example, we get the split between the two eigenbundles of the pointwise operator $E_{\alpha_1 \ldots \alpha_n} \gamma^{\alpha_1} \ldots \gamma^{\alpha_n}$ on spinors, or between the two eigenbundles of the Hodge $\star$ operator on (complexified) $n/2$-forms. In the situation where duality and/or chirality are not in play, it is convenient to define, for $\lambda_\ell \geq 0$,

$$\mathbb{U}(\lambda) := \begin{cases} V(\lambda) \oplus V(\bar{\lambda}), & n \text{ even and } \lambda_\ell > 0, \\ V(\lambda), & \text{otherwise.} \end{cases}$$

The bundle $\mathbb{U}(\lambda)$ is defined when

$$\lambda_1 \geq \ldots \geq \lambda_\ell \geq 0$$

for both even and odd $n$. When $\lambda_\ell = 1/2$ (a case of much interest in the present work), $\mathbb{U}(\lambda)$ is always a direct summand in $T^* M \otimes \mathbb{U}(\lambda)$.

The gradient concept is the motivation behind the definitions of the operators $G_{\Sigma}$, $G_T$, $G_Y$, and $G_Z$ above. The spinor bundle is

$$\Sigma \cong_{\text{Spin}(n)} \mathbb{U}(\frac{1}{2}, \ldots, \frac{1}{2}).$$

Similarly, the twistor bundle is

$$T \cong_{\text{Spin}(n)} \mathbb{U}(\frac{3}{2}, \frac{1}{2}, \ldots, \frac{1}{2}).$$

The other bundles introduced in the last section are

$$Y \cong_{\text{Spin}(n)} \begin{cases} \mathbb{U}(\frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \ldots, \frac{1}{2}), & n \geq 4, \\ \mathbb{U}(\frac{1}{2}), & n = 3, \\ 0, & n = 2, \end{cases}$$

and

$$Z \cong_{\text{Spin}(n)} \mathbb{U}(\frac{5}{2}, \frac{1}{2}, \ldots, \frac{1}{2}).$$

The selection rule (4.1) shows that there are gradients, or direct sums of gradients, acting between copies of the following bundles. Up to isomorphic realizations of bundles and constant factors, the corresponding gradients or direct sums of gradients are also listed:

- $\Sigma \rightarrow \Sigma$   \quad $\nabla$
- $\Sigma \rightarrow T$   \quad $\nabla$
- $T \rightarrow \Sigma$   \quad $T$ or $G_{\Sigma}$
- $T \rightarrow T \ (n \geq 3)$   \quad $S^0$ or $G_T$
- $T \rightarrow Y \ (n \geq 4)$   \quad $G_Y$
- $T \rightarrow Z$   \quad $G_Z$
(Recall the definition of $S^0$ in (2.3).) The operators $T^*$ and $G_\Sigma$ are targeted at different realizations of $\Sigma$; thus they are not the same operator, but each is a constant factor times the composition of the other with a Spin($n$)-bundle isomorphism. A similar statement holds for $S^0$ and $G_T$ with $T$. More precisely, using (3.2), (3.3), and (3.4), one has

$$
T^* = \frac{1}{2} I_\Sigma^* G_\Sigma
$$

$$
S^0 = -I_T^* G_T.
$$

(4.2)

Certain natural first-order operators are especially interesting in that they can be realized so that the source and target bundle are the same. In particular, these operators have spectra. Examples are the Dirac and Rarita-Schwinger operators, and the operator $\star d$ on $\Gamma(\Lambda^{(n-1)/2})$ for odd $n$. Such operators arise as follows. In odd dimensions, we take the gradient corresponding to the exceptional case of the selection rule (the second line of (4.1)). In this case, $G_{\lambda\lambda}$ carries sections of $V(\lambda)$ to sections of a copy of $V(\lambda)$ which lives as a subbundle in $T^*M \otimes V(\lambda)$. If we would like to use the source realization of $V(\lambda)$ as both source and target for a realization $S_\lambda$ of $G_{\lambda\lambda}$, we need a choice of normalization. First, normalize the Hermitian inner product on $T^*M \otimes V$ so that

$$
|\xi \otimes v|^2 = |\xi|^2 |v|^2;
$$

then normalize $S_\lambda$ so that

$$
S_\lambda^2 = G_{\lambda\lambda}^* G_{\lambda\lambda}.
$$

(4.4)

This determines $S_\lambda$ up to multiplication by $\pm 1$.

In even dimensions, take a dominant weight $\lambda$ with $\lambda_\ell = 1/2$. Then there are gradients $G_{\lambda\lambda}$ and $G_{\lambda\lambda}$, giving rise to a first-order operator

$$
\begin{pmatrix}
0 & G_{\lambda\lambda} \\
G_{\lambda\lambda} & 0
\end{pmatrix}
$$

carrying sections of $U(\lambda) = V(\lambda) \oplus V(\bar{\lambda})$ to sections of an isomorphic copy of this bundle, realized in its tensor product with $T^*M$. Remarks similar to those above, on normalization and realization, then hold, and we obtain a first-order differential operator $S_{\lambda\bar{\lambda}\lambda}$ on $\Gamma(V(\lambda) \oplus V(\bar{\lambda}))$, again determined up to a factor of $\pm 1$, normalized so that

$$
(S_{\lambda\bar{\lambda}\lambda})^2 = G_{\lambda\lambda}^* G_{\lambda\bar{\lambda} \lambda} \oplus G_{\lambda\lambda}^* G_{\lambda\lambda}.
$$

(4.5)

The sign ambiguity in the $S$ operators is in the nature of things: it is analogous to the ambiguity in the naming of the complex units $\pm \sqrt{-1}$. Indeed, this is more than an analogy: gradients generalize the Cauchy-Riemann equations [23], which are sensitive to the renaming of $\pm \sqrt{-1}$. In our examples, the ambiguity may be viewed as residing in a choice of fundamental tensor-spinor (or more generally, a Clifford structure) $\gamma$. The Clifford relations and spin connection (in particular the relation $\nabla\gamma = 0$) are
invariant under interchange of $\gamma$ and $-\gamma$, but the Dirac operator $\gamma^a \nabla_a$ undergoes a sign change.

The principal examples of such self-gradients of interest to us are those which act on $\Gamma(\Sigma)$ and $\Gamma(T)$, namely

$$S_{\Sigma} = \frac{1}{\sqrt{n}} \nabla$$

and

$$S_T : \varphi \mapsto \sqrt{\frac{n}{(n+2)(n-2)}} \left( \gamma^\lambda \nabla_{\lambda} \varphi - \frac{2}{n} \gamma^\alpha \nabla_{\lambda} \varphi^\alpha \right),$$

i.e.,

$$S_T = \sqrt{\frac{n}{(n+2)(n-2)}} S^{0} = - \sqrt{\frac{n}{(n+2)(n-2)}} I^*_T G_T.$$

The normalizations are computed from (4.4) and (4.5). The normalized Rarita-Schwinger operator $S_T$ will appear in formulas below. We shall also have use for self-gradients on $Y$ and $Z$; see (6.13) and (6.23) below.

An important point is that there are distinguished normalizations for $G^*_\Sigma G_\Sigma$, $G^*_T G_T$, $G^*_Y G_Y$, $G^*_Z G_Z$, and all similarly defined operators. In fact, the issue is exactly that of normalizing the formal adjoint by getting a relative normalization for the source and target bundles of a gradient (or a suitable direct sum of gradients), say $\mathcal{V}$ and $\mathcal{W}$ respectively. This is provided by taking the realization of $\mathcal{W}$ in $T^*M \otimes \mathcal{V}$, and normalizing its metric according to (4.3). Allowing the metric on $\mathcal{V}$ to determine that on $\mathcal{W}$ in this way, our operators $G^*G$ remain invariant under rescalings of the metric on $\mathcal{V}$.

5 Bochner-Weitzenböck formulas

A Bochner-Weitzenböck formula (henceforth a BW formula) may be described in general as an equation

$$D_1 = D_2 + Z,$$

where $D_1$ and $D_2$ are natural, nonnegative definite second-order differential operators with the same leading symbol, on sections of a vector bundle $\mathcal{V}$, and $Z$ is a natural bundle endomorphism of $\mathcal{V}$; in particular, a differential operator of order zero. The importance of such formulas derives from the elementary observation that if $Z \geq c \cdot \text{Id}_{\mathcal{V}}$ pointwise, for some constant $c > 0$, then $D_1 \geq c \cdot \text{Id}_{\Gamma(\mathcal{V})}$; if $-Z \geq k \cdot \text{Id}_{\mathcal{V}}$, then $D_2 \geq k \cdot \text{Id}_{\Gamma(\mathcal{V})}$. This observation usually appears as part of a longer argument in which devices particular to the situation, notably variations of the underlying geometric structure, are also employed.
There is an essentially unique BW formula on $\Sigma$. One way to express this is the Schrödinger-Lichnerowicz formula\[20\]

\[
\nabla^2 = \nabla^* \nabla + \frac{K}{4},
\]

\[\text{(5.1)}\]

where $K$ is the scalar curvature. By the discussion at the end of the last section,

\[
\nabla^* \nabla = S_\Sigma^2 + \mathcal{T}^* \mathcal{T} = \frac{1}{n} \nabla^2 + \mathcal{T}^* \mathcal{T},
\]

so (5.1) is equivalent to

\[
\nabla^2 = \frac{n}{n-1} \mathcal{T}^* \mathcal{T} + \frac{nK}{4(n-1)}; \tag{5.2}
\]

this is sometimes known as the Lichnerowicz identity\[21\]. It is, in a certain sense, an optimal way to write (5.1), since it brings us into contact with an orthogonal decomposition of $\nabla \psi$.

Both (5.1) and (5.2) are manifestations of the same BW formula: each computes the same linear combination of $\nabla^2$ and $\mathcal{T}^* \mathcal{T}$; the unique such combination that has vanishing leading symbol. As Corollary 3 makes clear, the operators $G_\Sigma^* G_\Sigma$, $G_T^* G_T$, $G_Y^* G_Y$, and $G_Z^* G_Z$ give rise to two essentially different BW formulas. To state these, let us standardize some notation. Let $R$ be the Riemann curvature tensor, with the convention on index placement that gives $[\nabla_\alpha, \nabla_\beta] X^\lambda = R^\lambda_{\mu\alpha\beta} X^\mu$ for $X$ a vector field. Then $r_{\mu\beta} = R^\alpha_{\mu\alpha\beta}$ is the Ricci tensor, and $K = r_{\beta\beta}$ is the scalar curvature. Define the Einstein (trace-free Ricci) tensor $b$ by

\[
b_{\alpha\beta} = r_{\alpha\beta} - \frac{K}{n} g_{\alpha\beta}, \tag{5.3}
\]

and define the Weyl tensor $C$ to be the totally trace-free part of $R$. Explicitly, if

\[
J := \frac{K}{2(n-1)} \quad \text{and} \quad V := \frac{r - J g}{n-2},
\]

then

\[
C^\alpha_{\beta\kappa\lambda} = R^\alpha_{\beta\kappa\lambda} + V^\alpha_{\beta\lambda} \delta^\alpha_{\kappa} - V^\beta_{\beta\lambda} \delta^\alpha_{\kappa} + V^\alpha_{\lambda\kappa} g_{\beta\lambda} - V^\alpha_{\kappa} g_{\beta\lambda}. \tag{5.4}
\]

If $\varphi \in \Gamma(T)$, let

\[
(b \cdot \varphi)_\mu := \frac{1}{n} \left( (n - 2) b_{\mu\lambda} \varphi^\lambda - b_{\alpha\lambda} \gamma^\alpha_{\gamma\mu} \varphi^\lambda \right)
\]

\[
= \frac{1}{n} \left( (n - 1) b_{\mu\lambda} \varphi^\lambda - b_{\alpha\lambda} \gamma^\alpha_{\mu} \varphi^\lambda \right),
\]

\[
(C \circ \varphi)_\mu := C_{\alpha\beta}^\lambda \gamma^\alpha_{\mu} \gamma^\beta_{\varphi\lambda} = C_{\alpha\beta}^\lambda \gamma^\alpha_{\mu} \gamma^\beta_{\varphi\lambda}.
\]

By the Clifford relations, the Bianchi identity, and the trace-free nature of $C$,

\[
\gamma^\mu (b \cdot \varphi)_\mu = \gamma^\mu (C \circ \varphi)_\mu = 0.
\]
Thus both \( b \) and \( C \) are Spin(\( n \))-bundle endomorphisms of \( T \). In fact, the maps \( \beta \otimes \varphi \mapsto \beta \cdot \varphi \) and \( \zeta \otimes \varphi \mapsto \zeta \diamond \varphi \) describe Spin(\( n \))-bundle homomorphisms \( \text{TFS}^2 \otimes T \to T \) and \( \mathcal{W} \otimes T \to T \) respectively, where \( \mathcal{W} \) is the bundle of algebraic Weyl tensors. Note that by \([24]\),

\[
\text{TFS}^2 \cong_{\text{Spin}(n)} \mathbb{V}(2, 0, \ldots, 0),
\]

and

\[
\mathcal{W} \cong_{\text{Spin}(n)} \mathbb{V}(2, 2, 0, \ldots, 0) = \begin{cases} 
\mathbb{V}(2, 2, 0, \ldots, 0), & n \geq 5, \\
\mathbb{V}(2, 2) \oplus \mathbb{V}(2, -2), & n = 4, \\
0, & n < 4.
\end{cases}
\]

By the Clifford relations, the skew-adjointness of \( \gamma^\mu \), and the twistor condition \( \gamma^\mu \varphi_\mu = 0 \), we have

\[
\langle (b \cdot \varphi)_\mu, \varphi^\mu \rangle = b_\mu^\lambda \langle \varphi_\lambda, \varphi^\mu \rangle. \tag{5.5}
\]

**Theorem 4**

\[
Z_1 : = \frac{(n - 3)(n - 2)}{2n} G_\Sigma^* G_\Sigma - \frac{(n + 2)(n - 3)}{2n} G_T^* G_T + G_Y^* G_Y
\]

\[
= \frac{1}{8} C \diamond + \frac{n - 3}{2(n - 2)} b \cdot \frac{(n - 2)(n - 3)}{8n(n - 1)} K,
\]

\[
Z_2 : = \frac{(n - 1)(n + 2)}{2n} G_\Sigma^* G_\Sigma - \frac{(n + 1)(n - 2)}{2n} G_T^* G_T + G_Z^* G_Z
\]

\[
= \frac{3}{8} C \diamond - \frac{n + 1}{2(n - 2)} b \cdot \frac{(n + 2)(n + 1)}{8n(n - 1)} K.
\]

*Proof.* We promote the leading symbol calculation of Lemma [\( \| \) and Corollaries [\( \| \) and [\( \| \) to operator calculations, keeping track of curvature terms. The following identities are used. Let \( \mathcal{R} \) be the Riemannian spin curvature; that is, \( \mathcal{R}_{\alpha\beta} = [\nabla_\alpha, \nabla_\beta] \), the precise effect of which depends on what sort of index expression appears to its right. If \( \psi \) is a spinor, then

\[
\mathcal{R}_{\lambda\mu} \psi := W_{\lambda\mu} \psi = -\frac{1}{4} R_{\kappa\nu\lambda\mu} \gamma^\kappa \gamma^\nu \psi,
\]

where \( W \) is the spin curvature. If \( \varphi \) is a spinor-one-form, then

\[
\mathcal{R}_{\lambda\mu} \varphi_\alpha = W_{\lambda\mu} \varphi_\alpha - R_{\alpha\lambda\mu} \varphi_\nu.
\]

The classical Lichnerowicz calculation, which combines the Clifford relations and the Bianchi identity, shows that

\[
\gamma^\mu W_{\lambda\mu} = -\frac{1}{2} \gamma^\mu.
\]

In particular,

\[
\gamma^\lambda \gamma^\mu W_{\lambda\mu} = \frac{1}{2} K. 
\]
This leads to the formula
\[ \gamma^\lambda \gamma^\mu R_{\lambda\mu} \varphi_\alpha = 4 W^\lambda_\alpha \varphi_\lambda + \frac{1}{2} K \varphi_\alpha. \]
We also have
\[ R^\alpha_{\mu \varphi} = (W^\alpha_{\mu} + r^\alpha_{\mu}) \varphi_\alpha, \]
so that
\[ \gamma^\mu R^\alpha_{\mu \varphi} = \frac{1}{2} r^\alpha_{\mu \varphi}. \]
We then write the curvature terms in terms of \( K, b, \) and \( C \) using (5.3, 5.4).

**Remark 5** In [7], Theorem 5.10, it is shown that, for the \( N(\lambda) \) gradients \( G_u \) emanating from a given irreducible \( \text{Spin}(n) \)-bundle \( \mathbb{V}(\lambda) \),
\[ \dim \text{span} \{ \sigma_2(G_u^*G_u) \mid u = 1, \ldots, N(\lambda) \} = \left\lfloor \frac{N(\lambda) + 1}{2} \right\rfloor. \]
That is, only about half of the leading symbols of the \( G_u^*G_u \) are linearly independent. This means that there is an \( \left\lfloor N(\lambda)/2 \right\rfloor \)-parameter family of BW formulas relating these operators (since \( \left\lfloor N(\lambda)/2 \right\rfloor = N(\lambda) - [(N(\lambda) + 1)/2] \)). In fact, there are numbers \( \tilde{c}_u \) and \( s_u \) such that
\[ \sigma_2 \left( \sum_{u=1}^{N(\lambda)} b_u G_u^*G_u \right) = 0 \iff \sum_{u=1}^{N(\lambda)} b_u \tilde{c}_u s_u^{2j} = 0, \quad j = 0, 1, \ldots, \left\lfloor \frac{N(\lambda) + 1}{2} \right\rfloor - 1. \quad (5.6) \]
Given such a “null” linear combination of operators \( G_u^*G_u \), the corresponding BW formula takes the form
\[ \sum_{b_u > 0} b_u G_u^*G_u = \sum_{-b_u > 0} (-b_u) G_u^*G_u + Z, \]
where \( Z \) has order zero.

**Remark 6** Among all the BW formulas described in the last remark are some distinguished ones. First, for any \( \mathbb{V}(\lambda) \), there is a formula, studied by Gauduchon in [18], Appendix B, in which the order zero operator \( Z \) is controlled by the curvature operator, in an appropriate sense. The value of the Casimir operator of the Lie algebra \( \mathfrak{so}(n) \) in the representation \( V(\lambda) \) is, in the notation of the last section,
\[ \lambda(\text{Cas}_{\mathfrak{so}(n)}) = \sum_{a=1}^{t} \left\{ \left( \lambda + \frac{n - 2a}{2} \right)^2 - \left( \frac{n - 2a}{2} \right)^2 \right\}. \quad (5.7) \]
Let
\[ s_u := \frac{1}{2} \left( \lambda(C_{\text{so}(n)}) - \sigma_u(C_{\text{so}(n)}) \right); \quad (5.8) \]
this is in fact the same quantity \( s_u \) that appears in \( (5.6) \). Gauduchon showed that if \( n \geq 3 \) and \( \lambda \) is integral,
\[ P_\lambda := \sum_{u=1}^{N(\lambda)} \left( s_u + \frac{n-1}{2} \right) G^*_u G_u = -\sum I_{\lambda}(X_I) \lambda(R^{op}(X_I)), \quad (5.9) \]
where \((X_I)\) is any local orthonormal frame for the bundle \( \Lambda^2 M \), and \( R^{op} \) is the curvature operator on sections \( \eta \) of \( \Lambda^2 M \):
\[ (R^{op}\eta)_{\alpha\beta} = \frac{1}{2} R_{\alpha\beta}^{\mu\nu} \eta_{\mu\nu}. \]
(The fact that this particular linear combination of \( G^*_u G_u \) must produce a BW formula was actually noted earlier in [5]; see Remark 7 below.) In [12], the present authors show, among other things, that this result extends to half-integral \( \lambda \). For any \( \lambda \), the significance of the result is that it enforces pointwise bounds on \( P_\lambda \), which has order zero as a differential operator and thus is actually a section of \( \text{End} V(\lambda) \). These bounds are multiples of the bottom and top pointwise eigenvalues of the curvature operator \( R^{op} \), which is a section of \( \text{End} \Lambda^2 \). If, at a point \( x \in M \),
\[ q_x \text{Id}_{\Lambda^2_x M} \leq R^{op}_x \leq Q_x \text{Id}_{\Lambda^2_x M} \quad (5.10) \]
for some constants \( q_x \), \( Q_x \), in the sense of ordering of endomorphisms \( A \leq B \) iff \( B - A \) is positive semidefinite), then
\[ q_x \lambda(C_{\text{so}(n)}) \text{Id}_V(\lambda)_x \leq (P_\lambda)_x \leq Q_x \lambda(C_{\text{so}(n)}) \text{Id}_V(\lambda)_x, \quad (5.11) \]
also in the sense of endomorphism ordering. (Note that both endomorphisms, \( R^{op} \) and \( (P_\lambda)_x \), are symmetric.)

Another important combination of the \( G^*_u G_u \) is the one involved in the conformally covariant operator [1, 3]
\[ D_\lambda = \frac{K}{2(n-1)} - \sum_{u=1}^{N(\lambda)} \left( s_u + \frac{1}{2} \right)^{-1} G^*_u G_u. \]
This is well-defined as long as no \( s_u + \frac{1}{2} \) vanishes; i.e., provided it is not the case that
\[ n \text{ is even and } \lambda_\ell = 0 \neq \lambda_{\ell-1}. \quad (5.12) \]
When \( (5.12) \) does not hold,
\[ \text{ord } D_\lambda = \begin{cases} 2, & N(\lambda) \text{ odd}, \\ 0, & N(\lambda) \text{ even}. \end{cases} \]
In case $\text{ord} \mathcal{D}_\lambda = 0$, we have a BW formula; by the conformal covariance of $\mathcal{D}_\lambda$ and some invariant theory,

$$\mathcal{D}_\lambda \varphi = \alpha(C, \varphi)$$

for all sections $\varphi$ of $\mathcal{V}(\lambda)$, where $\alpha$ is some section of the bundle $\text{Hom}(\mathcal{W} \otimes \mathcal{V}(\lambda), \mathcal{V}(\lambda))$. In other words, the BW formula that we get from the combination of $G^* G_u$ appearing in $\mathcal{D}_\lambda$ omits the Einstein tensor, in the sense that its right-hand side depends only on $K$ and $C$. This formula was exploited in [11] to set up a systematic approach for obtaining vanishing theorems, and in some cases eigenvalue estimates, based on the relative size of the bottom eigenvalue of the Yamabe operator $\Delta + (n-2)K/(4(n-1))$ on scalars, and the pointwise eigenvalues of the Weyl tensor $C$. To be precise, the relevant Weyl tensor data are bounds of the form (5.10) on the operator $C^{\text{op}}$ defined in analogy with the curvature operator: $(C^{\text{op}} \eta)_{\alpha\beta} = \frac{1}{2} C_{\alpha\beta}^{\mu\nu} \eta_{\mu\nu}$.

When $N(\lambda) \geq 4$ is even and (5.12) does not hold, the BW formulas associated to the $\mathcal{P}_\lambda$ and $\mathcal{D}_\lambda$ are different; that is, the coefficient arrays

$$\left( s_u + \frac{n-1}{2} \right)_{u=1}^{N(\lambda)} \quad \text{and} \quad \left( \left( s_u + \frac{1}{2} \right)^{-1} \right)_{u=1}^{N(\lambda)} \quad (5.13)$$

are linearly independent ([12], Lemma 2.1). When $N(\lambda) = 2$, they must be proportional by the above-described result counting the linearly independent BW formulas. In fact, when $N(\lambda) = 2$, the first array in (5.13) is $2\lambda(C_{\text{ase}(n)})/n$ times the second ([12], Remark 2.2).

Remark 7 In [5], p.46, equation (3.30), an interesting relation between $\mathcal{P}_\lambda$ and $\mathcal{D}_\lambda$ is noted. This holds in all cases except (5.12); i.e., even if $\mathcal{D}_\lambda$ has order 2. Up to a nonzero constant multiple, the second conformal variation of $\mathcal{D}_\lambda$ in the direction of conformal factors $e^{2\varepsilon \omega}$, where $\omega \in C^\infty(M)$, and $\varepsilon \in \mathbb{R}$ is a variational parameter, is the second-order symbol of $\mathcal{P}_\lambda$ evaluated at the covector field $\xi = d\omega$. Since $\mathcal{D}_\lambda$ is conformally covariant and $(d\omega)_x$ may be arbitrarily prescribed at any point $x \in M$, this establishes that $\mathcal{P}_\lambda$ has order less than 2, and thus (by invariant theory and homogeneity) order 0. This argument, however, does not provide the precise right-hand side of (5.9).

For $n \geq 4$, the twistor bundle $\mathbf{T}$ is either a Spin($n$)-irreducible bundle with $N(\lambda) = 4$ ($n$ odd), or a direct sum of two Spin($n$)-irreducibles, each having $N(\lambda) = 4$. Thus the list of BW formulas in Theorem [4] above is complete. Since there are exactly two BW formulas, the last remark shows that each is a linear combination of the formulas associated to the $\mathcal{P}_\lambda$ and the $\mathcal{D}_\lambda$ combinations. In fact, the conformally covariant
operator (or direct sum of such for \( n \) even) is

\[
\mathcal{D} = \frac{K}{2(n-1)} - \frac{2}{n+1} G^{*}_\Sigma G_\Sigma - 2 G^{*}_T G_T
\]

\[
+ \frac{n-3}{K} \cdot \frac{2}{n+1} G^{*}_\Sigma G_Y + \frac{2}{n+1} G^{*}_Z G_Z
\]

\[
= \frac{2(n-1)}{n-2} Z_1 + \frac{2}{n+1} Z_2
\]

\[
= \frac{(n-3)(n+1)}{n-2} C \diamond .
\]

The Gauduchon operator (or direct sum of such if \( n \) is even) is

\[
\mathcal{P} = \frac{2n-1}{2} G^{*}_\Sigma G_\Sigma + \frac{n-1}{2} G^{*}_T G_T
\]

\[
+ \frac{1}{n-2} G^{*}_\Sigma G_Y - \frac{3}{2} G^{*}_Z G_Z
\]

\[
= \frac{1}{2} Z_1 - \frac{3}{2} Z_2
\]

\[
= -\frac{1}{2} C \diamond + \frac{n}{n-2} b \cdot \frac{n+7}{8(n-1)} K
\]

\[
=: R\diamond .
\]

The \( R\diamond \) notation will be useful below. One can show control over the curvature action \( R\diamond \) by the curvature operator by an elementary argument, without using representation theory. Let \( A \) be the following action of two-forms on twistors:

\[
(A(\eta) \varphi)_\mu = \eta_{\alpha \beta} \gamma^\alpha \gamma^\beta \varphi_\mu - 4 \eta_{\mu} \lambda \varphi_\lambda .
\]

A short calculation shows that \( \gamma^\mu (A(\eta) \varphi)_\mu = 0 \), so \( A \in \text{Hom}(\Lambda^2 M \otimes T, T) \). Another short calculation shows that \( A(\eta) \) is skew-adjoint as a bundle endomorphism of \( T \). Thus if \( (X_I) \) is an orthonormal basis of \( \Lambda^2_x M \) diagonalizing the symmetric operator \( R^{op}_x \) with eigenvalues \( a_I \), we have

\[
- \sum_I a_I A(X_I)^2
\]

nonnegative (or nonpositive) if \( R^{op} \) is nonnegative (or nonpositive). However, direct calculation shows that

\[
- \sum_I a_I A(X_I)^2 = 8 R\diamond .
\]

(One can organize the calculation as follows: compute \( -A(\eta)^2 \varphi \), and then replace each occurrence of \( \eta_{\alpha \beta} \eta_{\lambda \mu} \) by \( \frac{1}{2} R_{\alpha \beta \lambda \mu} \); this is the value of \( (5.16) \). In particular,

\[
R^{op} \geq 0 \quad \Rightarrow \quad R\diamond \geq 0,
\]

\[
R^{op} \leq 0 \quad \Rightarrow \quad R\diamond \leq 0.
\]
To get precise control by the curvature operator, let \([\{K\}], [b],\) and \(C\) be the (orthogonal) contributions of the scalar curvature \(K\), the Einstein tensor \(b\), and the Weyl tensor \(C\) to the Riemann curvature \(R\):

\[
R = C + [b] + [[K]].
\]

In detail,

\[
[[K]]_{\alpha\beta\lambda\mu} = \frac{K}{n(n-1)} (g_{\alpha\lambda}g_{\beta\mu} - g_{\alpha\mu}g_{\beta\lambda}),
\]

\[
[b]_{\alpha\beta\lambda\mu} = \frac{1}{n-2} (b_{\beta\mu}g_{\alpha\lambda} - b_{\beta\lambda}g_{\alpha\mu} - b_{\alpha\mu}g_{\beta\lambda} + b_{\alpha\lambda}g_{\beta\mu}).
\]

The curvature operator is

\[
(R_{\text{op}} \eta)_{\alpha\beta} = \frac{1}{n(n-1)} K \eta_{\alpha\beta} + \frac{1}{n-2} (b^\lambda \alpha_{\alpha\eta\lambda\beta} + b^\lambda \beta_{\eta\alpha\lambda\beta}) + \frac{1}{2} C^\lambda_{\alpha\beta\lambda\mu} \eta_{\lambda\mu}.
\]

Let \(q\) and \(Q\) be smooth real-valued functions on \(M\). The condition that \(R_{\text{op}} \geq q \text{Id}_{\Lambda^2 M}\) is thus equivalent to the condition that the algebraic curvature tensor

\[
R_q := [[K - n(n-1)q]] + [b] + C
\]  

has its curvature operator \(R_{q_{\text{op}}}^{\text{op}}\) nonnegative. Similarly, the condition \(R_{\text{op}} \leq Q \text{Id}_{\Lambda^2 M}\) is equivalent to the condition that \(R_{Q_{\text{op}}}^{\text{op}}\) is nonpositive. We have:

\[
q \leq R_{\text{op}} \leq Q \iff R_{Q_{\text{op}}}^{\text{op}} \leq 0 \iff R_{Q_{\text{op}}}^{\text{op}} \leq 0 \iff \frac{n(n+7)}{8} Q \leq \frac{n(n+7)}{8} q \leq \frac{n(n+7)}{8} R_{Q_{\text{op}}}^{\text{op}} \leq 0 \iff n(n+7) q \leq P \leq n(n+7) Q.
\]  

(5.18)

Here we have used the fact that, by (5.15) and (5.17),

\[
R_{f_{\text{op}}} = R_{\text{op}} - \frac{n(n+7)}{8} f.
\]

To compare with (5.11), note that the value of the Casimir operator in the twistor representation(s) is \(n(n+7)/8\).

The same reasoning may be applied to get precise control over the quantity in (5.14). First notice that

\[
C_{\text{op}} = -2C_{\text{op}}.
\]

If for some smooth functions \(t, T\) we have

\[
t \leq C_{\text{op}} \leq T,
\]

then

\[
\frac{n(n+7)}{8} t \leq C_{\text{op}} \leq \frac{n(n+7)}{8} T,
\]
whence
\[-\frac{n(n+7)}{4} T \leq C^\circ \leq \frac{n(n+7)}{4} t,\]
and
\[
\frac{(n-2)n(n+7)}{4(n-3)(n+1)} t + \frac{K}{2(n-1)} \leq D_0 \leq \frac{(n-2)n(n+7)}{4(n-3)(n+1)} T + \frac{K}{2(n-1)},
\]
where
\[
D_0 : = \frac{2}{n+1} G_*^\Sigma G_\Sigma + 2G_*^T G_T - \frac{2}{n-3} G_*^Y G_Y - \frac{2}{n-2} G_*^Z G_Z - \frac{2}{2(n-1)} C^\circ - \frac{n(n+7)}{8} q,
\]
\[
= \frac{2}{n+1} G_*^\Sigma G_\Sigma + 2G_*^T G_T \geq D_0 \geq \frac{(n-2)n(n+7)}{4(n-3)(n+1)} t + \frac{K}{2(n-1)},
\]
\[
\frac{2}{n-3} G_*^Y G_Y + \frac{2}{n+1} G_*^Z G_Z \geq -D_0 \geq -\frac{(n-2)n(n+7)}{4(n-3)(n+1)} T - \frac{K}{2(n-1)},
\]
as operator inequalities. We immediately get the following vanishing results:

**Theorem 8** Let \( n \geq 4 \). If \( R^\text{op} > 0 \), then \( \mathcal{N}(G_\Sigma) \cap \mathcal{N}(G_T) \cap \mathcal{N}(G_Y) = 0 \). If \( R^\text{op} < 0 \), then \( \mathcal{N}(G_Z) = 0 \). If
\[
C^\text{op} > -\frac{2(n-3)(n+1)K}{(n-1)(n-2)n(n+7)},
\]
(in particular, if \( g \) is conformally flat and \( K > 0 \)), then \( \mathcal{N}(G_\Sigma) \cap \mathcal{N}(G_T) = 0 \). If
\[
C^\text{op} < -\frac{2(n-3)(n+1)K}{(n-1)(n-2)n(n+7)},
\]
(in particular, if \( g \) is conformally flat and \( K < 0 \)), then \( \mathcal{N}(G_Y) \cap \mathcal{N}(G_Z) = 0 \).

The BW formula (5.19) is potentially interesting because it eliminates the Einstein tensor term. We may similarly compute BW formulas that eliminate other elements: the Weyl tensor, the scalar curvature, or any given Stein-Weiss operator \( G_*^\ast G \). Taking
\[
3 \cdot Z_1 - Z_2, \text{ we have}
\]
\[
2 \left\{ \frac{(n-7)^2}{4} + \frac{15}{16} \right\} G_*^\Sigma G_\Sigma - \frac{(n-1)^2}{4} G_*^T G_T + 3G_*^Y G_Y - G_*^Z G_Z = 2b \cdot \frac{n-8}{4n} K.
\]
This eliminates the Weyl tensor $C$. In particular, for $n = 8$, one has
\[ 10G^*_{2}G_{\Sigma} - 6G^*_{T}G_{T} + 3G^*_{Y}G_{Y} - G^*_{Z}G_{Z} = 2b ; \] (5.21)
this eliminates both $C$ and $K$. We eliminate $K$ by looking at
\[ (n + 1)(n + 2)Z_{1} - (n - 3)(n - 2)Z_{2} . \]
The result is
\[ (n - 3)(n - 2)(n + 2)G^*_{2}G_{\Sigma} - 4(n - 3)(n + 1)G^*_{T}G_{T} \\
+ (n + 1)(n + 2)G^*_{Y}G_{Y} - (n - 3)(n - 2)G^*_{Z}G_{Z} \\
= - \frac{(n - 8)(n - 1)}{4} C \circ + \frac{(n - 3)n(n + 1)}{n - 2} b . \]
We eliminate $G^*_{2}G_{\Sigma}$ by looking at
\[ (n - 3)n(n - 2)(n + 2)Z_{1} + (n - 3)(n - 2)Z_{2} . \]
This gives
\[ -(n - 3)n^{2}G^*_{T}G_{T} + (n - 1)(n + 2)G^*_{Y}G_{Y} + (n - 3)(n - 2)G^*_{Z}G_{Z} \\
= \frac{1}{2} \left\{ (n - \frac{1}{4})^{2} + \frac{15}{16} \right\} C \circ + \frac{(n - 3)n(n + 1)}{n - 2} b - \frac{(n - 3)(n - 2)(n + 2)}{4(n - 1)} K . \] (5.22)
We eliminate $G^*_{T}G_{T}$ by looking at
\[ (n + 1)(n - 2)Z_{1} - (n + 2)(n - 3)Z_{2} . \]
This gives
\[ n^{2}(n - 3)G^*_{2}G_{\Sigma} + (n + 1)(n - 2)G^*_{Y}G_{Y} - (n + 2)(n - 3)G^*_{Z}G_{Z} \\
= - \frac{1}{4} \left\{ (n - \frac{1}{2})^{2} - \frac{33}{4} \right\} C \circ + \frac{(n - 3)n(n + 1)}{n - 2} b + \frac{(n + 1)(n - 3)}{n - 1} K . \]

**Proposition 9** Suppose the Einstein tensor satisfies $p \leq b \leq P$ in the sense of endomorphisms, for some constants $p, P$. (Here we view $b$ as residing in $\text{End} TM$). If
\[ p \geq \frac{n - 8}{8n} K , \]
with strict inequality at some point, then $\mathcal{N}(G_{\Sigma}) \cap \mathcal{N}(G_{Y}) = 0$. In particular, if $g$ is Einstein and $(n - 8)K \leq 0$, with strict inequality at some point, then $\mathcal{N}(G_{\Sigma}) \cap \mathcal{N}(G_{Y}) = 0$. If
\[ P \leq \frac{n - 8}{8n} K , \]
with strict inequality at some point, then $\mathcal{N}(G_{T}) \cap \mathcal{N}(G_{Z}) = 0$. In particular, if $g$ is Einstein and $(n - 8)K \geq 0$, with strict inequality at some point, then $\mathcal{N}(G_{T}) \cap \mathcal{N}(G_{Z}) = 0$. 
Proof. By (5.3),
\[ p \leq b \leq P \Rightarrow p \cdot b \leq P. \]
The statements are now immediate from (5.20). □

Proposition 10 If \( n = 8 \) and \( g \) is Einstein, any twistor in \( \mathcal{N}(G_{\Sigma}) \cap \mathcal{N}(G_{Y}) \) or \( \mathcal{N}(G_{T}) \cap \mathcal{N}(G_{Z}) \) is parallel.

Proof. This is immediate from (5.21), together with the fact that \( \nabla = G_{\Sigma} + G_{T} + G_{Y} + G_{Z} \). □

6 Mixed Bochner-Weitzenböck formulas

In this section, we compute compositions of different gradients which will play a role in studying necessary conditions for the existence of special sections of the twistor bundle. Consider the possible compositions

\[ V(\lambda) \xrightarrow{G_{\lambda}} V(\tau) \xrightarrow{G_{\mu}} V(\mu), \]  

(6.1)

acting between irreducible Spin(\( n \))-bundles \( V(\lambda) \) and \( V(\mu) \) with \( \lambda \neq \mu \). By the selection rule (4.1), there are either 0, 1, or 2 compositions (6.1) for a given pair \((\lambda, \mu)\).

In fact, the number of compositions (6.1) is

\[ r(\lambda, \mu) := \dim \text{Hom}_{\text{Spin}(n)}(T^* \otimes T^* \otimes V(\lambda), V(\mu)). \]

Since \( T^* \otimes T^* \cong_{\text{Spin}(n)} \text{TFS}^2 \oplus \Lambda^0 \oplus \Lambda^2 \), the number \( r(\lambda, \mu) \) breaks up into summands attributable to \( \text{TFS}^2, \Lambda^0, \) and \( \Lambda^2 \):

\[ r(\lambda, \mu) = r_{\text{TFS}^2}(\lambda, \mu) + r_{\Lambda^0}(\lambda, \mu) + r_{\Lambda^2}(\lambda, \mu). \]

By \[ \text{[8], Lemma 2.2}, \]

\[ r(\lambda, \mu) = 2 \Rightarrow r_{\text{TFS}^2}(\lambda, \mu) = r_{\Lambda^2}(\lambda, \mu) = 1. \]

When \( r(\lambda, \mu) = 1 \), the contribution may come from \( \text{TFS}^2 \) or \( \Lambda^2 \), depending on the pair \((\lambda, \mu)\). (See \[ \text{[8], Lemma 2.2 for a precise classification.}, \]

Of course, in the complementary case \( \lambda = \mu \), we have already described these numbers:

\[ r_{\Lambda^0}(\lambda, \lambda) = 1, \quad r_{\text{TFS}^2}(\lambda, \lambda) = [(N(\lambda) - 1)/2], \quad r_{\Lambda^2}(\lambda, \lambda) = [N(\lambda)/2]. \]

Since equivariant second-order leading symbols of differential operators are identified with elements of \( \text{Hom}_{\text{Spin}(n)}(\text{Sym}^2 \otimes V(\lambda), V(\mu)) \), the space of such leading symbols
has dimension $r_{TFS^2}(\lambda, \mu)$. (Note that since $\lambda \neq \mu$, we have $r_{\lambda^0}(\lambda, \mu) = 0$.) Thus in the case $r(\lambda, \mu) = 2$, there is a nontrivial linear relation

$$c_1 \sigma_2 (G_{\tau_1 \mu} G_{\lambda \tau_1}) + c_2 \sigma_2 (G_{\tau_2 \mu} G_{\lambda \tau_2}) = 0,$$

where $\tau_1, \tau_2$ are the intermediate weights in (6.1):

As a result, the operator $c_1 G_{\tau_1 \mu} G_{\lambda \tau_1} + c_2 G_{\tau_2 \mu} G_{\lambda \tau_2}$ is a curvature action, and since $r_{\lambda^0}(\lambda, \mu) = 0$, the scalar curvature does not contribute to this: there are actions $\alpha(C)$ and $\beta(b)$ of the Weyl and Einstein tensors such that

$$c_1 G_{\tau_1 \mu} G_{\lambda \tau_1} + c_2 G_{\tau_2 \mu} G_{\lambda \tau_2} = \alpha(C) + \beta(b).$$

(6.2)

When $r(\lambda, \mu) = r_{\lambda^2}(\lambda, \mu) = 1$, there are no second-order symbols, since then $r_{TFS^2}(\lambda, \mu) = 0$. In addition, since $b$ is a section of $TFS^2$, the Einstein tensor cannot act from $\nabla(\lambda)$ to $\nabla(\mu)$. Thus in this case, $G_{\tau \mu} G_{\lambda \tau}$ is an action of the Weyl tensor ($\tau$ being the unique intermediate weight from (5.1)); say

$$G_{\tau \mu} G_{\lambda \tau} = \alpha(C).$$

(6.3)

When $\nabla(\lambda)$ is a twistor bundle, these remarks apply for several values of $\nabla(\mu)$. Suppose that $n \geq 4$, and consider the following diagram.
This contains a new bundle and some new operators, all of which will be defined and/or computed in the following sections. Corresponding to each of the four simple closed loops, we have an equation of the form (6.2).

Strictly speaking, we only have this immediately for $n$ odd, since each bundle in the diagram is reducible in the even-dimensional case; they are $U(\lambda)$ rather than $V(\lambda)$ bundles. However, applying the same arguments to their irreducible summands, we get the result. At any rate, we shall compute each relation explicitly, so we do not really rely on the general principle (6.2).

We also have equations of the form (6.3) corresponding to the compositions

\[ \Sigma \xrightarrow{T} T \xrightarrow{G_Y} Y \]
\[ T \xrightarrow{G_Y} Y \xrightarrow{G_X} Y \]  

(6.4)
6.1 Mixed BW formulas targeted at spinor-form bundles

In this subsection we compute the composition (6.4) and the loops $1$ and $2$. This brings us into contact with the theory of spinor-forms developed in $[4]$; see also $[16]$. The starting point of $[4]$ is the introduction of variants of the exterior and interior operators (both differential and multiplicative) familiar from the de Rham complex. On a spinor-$k$-form,

$$
(\tilde{d}\varphi)_{a_0 \ldots a_k} = \sum_{s=0}^{k}(-1)^s \nabla_{a_s} \varphi_{a_0 \ldots \hat{a}_s \ldots a_k},
$$

$$
(\delta \varphi)_{a_2 \ldots a_k} = -\nabla_{a_2} \varphi_{a_0 \ldots a_k},
$$

$$
(\varepsilon(\gamma) \varphi)_{a_0 \ldots a_k} = \sum_{s=0}^{k}(-1)^s \gamma_{a_s} \varphi_{a_0 \ldots \hat{a}_s \ldots a_k},
$$

$$
(\iota(\gamma) \varphi)_{a_2 \ldots a_k} = \gamma^\lambda \varphi_{a_0 \ldots a_k}.
$$

It is convenient to have a compact notation for the operator

$$
\mathbb{D} = \iota(\gamma) \tilde{d} + \tilde{d} \iota(\gamma) = -(\tilde{\delta} \varepsilon(\gamma) + \varepsilon(\gamma) \tilde{\delta}),
$$

which, in index notation, appears as

$$(\mathbb{D} \varphi)_{a_1 \ldots a_k} = \gamma^\lambda \nabla_{\lambda} \varphi_{a_1 \ldots a_k}.
$$

Let $\Sigma \Lambda^k$ be the bundle of spinor-$k$-forms. The following are identities that can be computed immediately, and which are used repeatedly:

$$
\iota(\gamma) \varepsilon(\gamma) - \varepsilon(\gamma) \iota(\gamma) = -(n - 2k),
$$

$$
\iota(\gamma) \tilde{\delta} = -\tilde{d} \iota(\gamma),
$$

$$
\varepsilon(\gamma) \tilde{\delta} = -\tilde{d} \varepsilon(\gamma),
$$

$$
\iota(\gamma) \mathbb{D} + \mathbb{D} \iota(\gamma) = 2\tilde{\delta},
$$

$$
\varepsilon(\gamma) \mathbb{D} + \mathbb{D} \varepsilon(\gamma) = -2\tilde{d},
$$

and

$$
(\tilde{d} \tilde{d} \varphi)_{a_1 \ldots a_{k+2}} = \sum_{1 \leq s < t \leq k+2} (-1)^{s+t-1} W_{a_s a_t} \varphi_{a_1 \ldots \hat{a}_s \ldots \hat{a}_t \ldots a_k},
$$

$$
(\tilde{\delta} \tilde{\delta} - \nabla^* \nabla) \varphi_{a_1 \ldots a_k} = -\sum_{s=1}^{k}(-1)^s R^a_{a_0} \varphi_{a_1 \ldots \hat{a}_s \ldots a_k},
$$

$$
(\tilde{d} \mathbb{D} - \mathbb{D} \tilde{d}) \varphi_{a_2 \ldots a_k} = -\gamma^\lambda R^{\alpha_1 \lambda} \varphi_{a_0 \ldots a_2 \ldots a_k},
$$

$$
(\tilde{d} \mathbb{D} - \mathbb{D} \tilde{d}) \varphi_{a_0 \ldots a_k} = \sum_{s=0}^{k}(-1)^s \gamma^\lambda R_{a_0 \ldots a_s \ldots \hat{a}_s \ldots a_k},
$$

$$
\mathbb{D}^2 - \nabla^* \nabla = \frac{1}{2} \gamma^i \gamma^j R_{ij}.
$$

Here $W$ is the spin curvature, and $R$ is the “all-purpose” curvature, whose meaning depends on the valence of what sits to its right. In particular, in the formula for $\tilde{d} \tilde{d}$, it is $W$ rather than $R$ that appears – the terms giving the tensorial part of the curvature cancel, for the same reason that $d d$ vanishes on differential forms.
In particular, for \( \psi \in \Gamma(\Sigma) \), we have \( \tilde{\delta}^2 \psi = 0 \) and

\[
(\tilde{d}^2 \psi)_{\alpha \beta} = W_{\alpha \beta} \psi,
\]

\[
(\tilde{\delta} \tilde{d} + \tilde{d} \tilde{\delta} - \nabla^* \nabla) \psi = (\tilde{\delta} \tilde{d} - \nabla^* \nabla) \psi = 0,
\]

\[
(\tilde{\delta} \tilde{D} - \tilde{D} \tilde{\delta} \tilde{d}) \psi = \gamma^\lambda W_{\alpha \lambda} \psi = -\frac{1}{2} r_{\alpha \lambda} \gamma^\lambda \psi
\]

\[
(\tilde{D}^2 - \nabla^* \nabla) \psi = \frac{K}{4} \psi.
\]

For \( \varphi \in \Gamma(\Sigma \Lambda^1) \), we get

\[
(\tilde{d} \tilde{d} \varphi)_{\alpha \beta \lambda} = W_{\alpha \beta} \varphi_{\lambda} - W_{\alpha \lambda} \varphi_{\beta} + W_{\beta \lambda} \varphi_{\alpha},
\]

\[
(\tilde{\delta} \tilde{d} + \tilde{d} \tilde{\delta} - \nabla^* \nabla) \varphi = \mathcal{R}^\beta_{\alpha \beta} \varphi_{\beta} = (W^\beta_{\alpha} + r^\beta_{\alpha}) \varphi_{\beta},
\]

\[
(\tilde{\delta} \tilde{D} - \tilde{D} \tilde{\delta} \tilde{d}) \varphi = -\frac{1}{2} r^\alpha_{\beta \gamma} \gamma^\beta \varphi_{\alpha},
\]

\[
(\tilde{d} \tilde{D} - \tilde{D} \tilde{d}) \varphi = \gamma^\beta \left( -\frac{1}{2} r_{\alpha \beta} \varphi_{\alpha 1} - R^\mu_{\alpha 1 \alpha 0} \varphi_{\mu} + \frac{1}{2} r_{\alpha 1 \beta} \varphi_{\alpha 0} + R^\mu_{\alpha 1 \alpha 0 \beta} \varphi_{\mu} \right). 
\]

Let \( \Sigma \Lambda^1_{\text{top}} \) be the subbundle of \( \Sigma \Lambda^1 \) annihilated by \( \iota(\gamma) \). For example, \( \Sigma \Lambda^0_{\text{top}} = \Sigma \), \( \Sigma \Lambda^1_{\text{top}} = \mathbf{T} \), \( \Sigma \Lambda^2_{\text{top}} = \mathbf{Y} \). The bundle \( \mathbf{Y} \) from (6.4) can now be defined as \( \Sigma \Lambda^3_{\text{top}} \). For \( (n - 2k)(n - 2k + 1) \neq 0 \), when we compress \( \tilde{d} \) to an operator going from \( \Sigma \Lambda^k_{\text{top}} \) to \( \Sigma \Lambda^{k+1}_{\text{top}} \), we get

\[
\tilde{d}^1_{k_{\text{top}}} = \tilde{d} + \frac{1}{n - 2k} \varepsilon(\gamma) \mathbb{D} + \frac{1}{(n - 2k)(n - 2k + 1)} \varepsilon(\gamma)^2 \tilde{\delta}.
\]

For \( k = 0, 1, 2 \), these operators are related to those used in diagram (6) by

\[
\tilde{d}^0_{\text{top}} = \mathcal{T} = \frac{1}{2} G^*_\Sigma I \Sigma, \quad \tilde{d}^1_{\text{top}} = 2G_\Sigma, \quad \tilde{d}^2_{\text{top}} := G_\mathbf{Y}.
\]

On weight theoretic grounds, we can predict that \( \tilde{d}^0_{k+1} \tilde{d}^0_{k} \) is an action of the Weyl tensor, at least provided we stay safely below the middle order of form. Indeed, using the above identities, we readily obtain the result that \( \tilde{d}^0_{k+1} \tilde{d}^0_{k} \) is a curvature action. But by (6.3), there is no action of symmetric two-tensors carrying \( \Sigma \Lambda^k_{\text{top}} \) to \( \Sigma \Lambda^{k+2}_{\text{top}} \), thus the Ricci tensor cannot act. Thus the action depends the Riemann curvature only through the Weyl tensor. For \( k = 0 \) or \( k = 1 \), the above identities yield to the following Weyl curvature actions

\[
(\tilde{d}_1^0 \tilde{d}_0^0 \psi)_{\alpha \beta} = -\frac{1}{4} C_{\lambda \mu \alpha \beta} \gamma^\lambda \gamma^\mu \psi,
\]

\[
(\tilde{d}_2^0 \tilde{d}_1^0 \varphi)_{\alpha 0 \alpha 1} = 6 \left\{ -\frac{1}{8} C_{\mu \alpha 0 \alpha 1} \gamma^\lambda \gamma^\mu \varphi_{\alpha 2} + \frac{1}{n - 4} C_{\alpha 2 \alpha 1 \mu} \gamma_{\alpha 0} \gamma^\mu \varphi_{\lambda} + \frac{1}{4(n - 4)(n - 3)} C_{\lambda \mu} \beta \gamma_{\alpha 0} \gamma_{\alpha 1} \gamma^\lambda \gamma^\mu \varphi_{\beta} \right\}
\]

\[
:= (\alpha(C) \varphi)_{\alpha 0 \alpha 1}.
\]
where square brackets denote antisymmetrization. (For the second identity, assume \( n > 4 \).) The extreme right-hand side is in fact (up to constant multiples) the unique action of the Weyl tensor from \( \Sigma \Lambda^1_{\text{top}} \) to \( \Sigma \Lambda^3_{\text{top}} \). In particular, if we take any of its three terms, each of which is valued in \( \Sigma \Lambda^3 \), and project to \( \Sigma \Lambda^3_{\text{top}} \), the whole expression will emerge. With the notation of diagram (6), the above relations translate to

**Proposition 11** For \( n \geq 4 \), the compositions (6.4) are given by

\[
(G_Y T^\psi)_{\alpha\beta} = -\frac{1}{8} C^\lambda_{\mu\alpha \beta} \gamma^\lambda \gamma^\mu \psi, \\
G_Y G_Y \psi = \frac{1}{2} \alpha(C) \phi,
\]

(6.9)

where the action of the Weyl tensor \( \alpha(C) \) is given by (6.8).

We are also interested in the self-gradient on \( \Sigma \Lambda^k_{\text{top}} \), since the case \( k = 2 \) enters in our considerations. We can get this by compressing the conformally covariant operator \( P_k \) of [4] from \( \Sigma \Lambda^k \) to \( \Sigma \Lambda^k_{\text{top}} \). Since

\[
P_k = \frac{n - 2k + 4}{2} \iota(\gamma) \tilde{d} + \frac{n - 2k}{2} \left( \tilde{d}(\gamma) - \tilde{\delta} \varepsilon(\gamma) \right) - \frac{n - 2k - 4}{2} \varepsilon(\gamma) \tilde{d},
\]

we have

\[
P_k|_{\Sigma \Lambda^k_{\text{top}}} = \left. \left( \frac{n - 2k + 4}{2} \mathbb{D} - \frac{n - 2k}{2} \tilde{\delta} \varepsilon(\gamma) - \frac{n - 2k - 4}{2} \varepsilon(\gamma) \tilde{d} \right) \right|_{\Sigma \Lambda^k_{\text{top}}},
\]

(6.10)

since \( \iota(\gamma) \) annihilates \( \Sigma \Lambda^k_{\text{top}} \). Some weight theory (including the conformal weights of the operators involved) predicts that the restriction of \( P_k \) to \( \Sigma \Lambda^k_{\text{top}} \) will also be the compression; that is, that \( P_k \) carries \( \Sigma \Lambda^k_{\text{top}} \) to itself. We can check this by computing

\[
\iota(\gamma) \left( (n - 2k + 2) \mathbb{D} + 2 \varepsilon(\gamma) \tilde{d} \right) \big|_{\Sigma \Lambda^k_{\text{top}}};
\]

(6.11)

this should vanish. Using our list of identities to move \( \iota(\gamma) \) to the right, we get

\[
\iota(\gamma) \mathbb{D} \big|_{\Sigma \Lambda^k_{\text{top}}} = 2 \tilde{d} \big|_{\Sigma \Lambda^k_{\text{top}}}, \\
\iota(\gamma) \varepsilon(\gamma) \tilde{d} \big|_{\Sigma \Lambda^k_{\text{top}}} = -(n - 2k + 2) \tilde{d} \big|_{\Sigma \Lambda^k_{\text{top}}},
\]

so (6.11) vanishes as predicted.

Let \( \tilde{S}_k = P_k|_{\Sigma \Lambda^k_{\text{top}}} \). Then, for \( k = 0 \) in (6.10), one gets

\[
\tilde{S}_0 = (n + 2) \nabla.
\]

(6.12)
For \( k = 1 \), and for any \( \varphi \in \Gamma(\mathbf{T}) \), we obtain
\[
(\tilde{S}_1 \varphi)_\alpha = \left( (n \mathcal{D} + 2 \varepsilon(\gamma) \tilde{\delta}) \varphi \right)_\alpha = n \gamma^\lambda \nabla_{\lambda} \varphi_\alpha - 2 \gamma_\alpha \nabla^\lambda \varphi_\lambda .
\] (6.13)

Thus by (2.3),
\[
\tilde{S}_1 = n \mathcal{S}^0.
\] (6.14)

Letting \( k = 2 \) in (6.10), we denote the self-gradient \( \tilde{S}_2 \) by \( S_Y \). For any \( \varphi \in \Gamma(Y) \), we get
\[
(S_Y \varphi)_{\alpha \beta} = (n - 2) \gamma^\lambda \nabla_{\lambda} \varphi_{\alpha \beta} - 2 \left( \gamma_\alpha \nabla^\lambda \varphi_{\lambda \beta} - \gamma_\beta \nabla^\lambda \varphi_{\lambda \alpha} \right).
\] (6.15)

By (6.6) and (6.10), there is a linear relation between the leading symbols of \( \tilde{S}_{k+1} d^\text{top}_k \) and \( d^\text{top}_k \tilde{S}_k \). We may calculate this directly as follows: let \( s = n - 2k \), and let \( \sim \) be equality modulo a curvature action. Then
\[
\tilde{S}_{k+1} d^\text{top}_k = (s - 2) \mathcal{D} - s \varepsilon(\gamma) \mathcal{D}^2 - \frac{s - 2}{s + 1} \varepsilon(\gamma) \tilde{\delta} \mathcal{D} + \frac{4}{s} \varepsilon(\gamma) \tilde{\delta} \mathcal{D} + 2 \varepsilon(\gamma) \tilde{\delta} d, \\
d^\text{top}_k \tilde{S}_k = (s + 2) \mathcal{D} + s \varepsilon(\gamma) \mathcal{D}^2 - \frac{s + 2}{s + 1} \varepsilon(\gamma) \tilde{\delta} \mathcal{D} - \frac{2(s + 2)}{s} \varepsilon(\gamma) \tilde{\delta} d. 
\] (6.16)

As a result,
\[
\tilde{S}_{k+1} d^\text{top}_k - \frac{s - 2}{s + 2} d^\text{top}_k \tilde{S}_k \sim 2 \varepsilon(\gamma) \left( - \mathcal{D}^2 + \tilde{\delta} d + d \tilde{\delta} \right) \sim 0;
\]
this is the desired linear relation.

Keeping track of lower-order terms, we use the above identities to compute that
\[
\tilde{S}_{k+1} d^\text{top}_k = \frac{s - 2}{s + 2} d^\text{top}_k \tilde{S}_k = s \left[ \mathcal{D}, \tilde{d} \right] + \frac{3}{s + 1} \varepsilon(\gamma)^2 \left[ \mathcal{D}, \tilde{\delta} \right] \\
+ \frac{\varepsilon(\gamma)^3 \tilde{\delta} \tilde{\delta} + 2 \varepsilon(\gamma) \left( \tilde{\delta} \tilde{d} + d \tilde{\delta} - \mathcal{D}^2 \right)}{(s + 1)(s + 2)}.
\] (6.17)

In particular, for \( k = 0 \), using (6.3) to compute the right side of (6.17), we get
\[
\tilde{S}_1 d^\text{top}_0 = \frac{n - 2}{n + 2} d^\text{top}_0 \tilde{S}_0 = \frac{n}{2} b_{T \rightarrow \Sigma}(b)^*,
\] (6.18)

where the action of the Einstein tensor \( b \) taking a twistor \( \varphi \) to a spinor is given by
\[
b_{T \rightarrow \Sigma}(b) \varphi := -b_{\alpha \beta} \gamma^\alpha \varphi^\beta,
\]
and its adjoint, taking a spinor field \( \psi \) to a twistor, is
\[
(b_{T \rightarrow \Sigma}(b)^*)^\alpha \psi := b_{\alpha \beta} \gamma^\beta \psi. 
\] (6.19)

For \( k = 1 \), we get
\[
\tilde{S}_2 d^\text{top}_1 = \frac{n - 4}{n} \tilde{d}_1 \tilde{S}_1 = C(C) + (n - 4) b(b),
\] (6.20)
where
\[
(C(C)\varphi)_{\alpha\beta} := (n - 2) C^\lambda_{\mu\alpha\beta} \gamma^\mu \varphi_\lambda - C_{\lambda\mu} \gamma_{[\alpha\gamma\beta]} \gamma^{\lambda\mu} \varphi_{\nu} \quad (6.21)
\]
and
\[
(b(b)\varphi)_{\alpha\beta} := \gamma^\lambda b_{\lambda\alpha\beta} - \frac{2}{n - 2} b_{[\alpha\gamma\beta]} \varphi_\lambda + \frac{1}{(n - 1)(n - 2)} \gamma_{\alpha\beta} b^\lambda b^{\mu} \gamma^\mu \varphi_\lambda. \quad (6.22)
\]

Note that \(b\) is the unique action of the Einstein tensor carrying \(\Sigma \Lambda_1\) to \(\Sigma \Lambda_2\). Up to a constant factor, it could already have been predicted by looking at either operator in (6.16): the leading symbol of each must exhibit the unique TFS\(^2\) action: replacing \(b\) by \(\xi \otimes \xi - |\xi|^2 g/n\) in (6.22) produces a constant multiple of the leading symbol, evaluated at the covector \(\xi\).

With the identifications (6.7), (6.12), (6.14) and (6.15), formulas (6.18) and (6.20) yield

**Proposition 12** The identities of (6.2) corresponding to the adjoint of loop 1 and to loop 2 are realized by

\[
S^0 T - \frac{n - 2}{n} \nabla T = \frac{1}{2} b_{\Gamma - \Sigma}(b)^* \quad \text{for} \quad n \geq 2,
\]

\[
2S_G G_Y - 2(n - 4) G_Y S^0 = C(C) + (n - 4) b(C) \quad \text{for} \quad n \geq 4,
\]

where the curvature actions are given by (6.19), (6.21) and (6.22).

Note that the first formula in Proposition 12 is proved in [25]. The second formula, in the case \(n = 4\), is actually an additional realization of the second formula in Proposition 11 above: their abstract targets are realized within both spinor-3-forms and spinor-2-forms.

### 6.2 Mixed BW formulas targeted at other tensor-spinor bundles

In this section, we compute the instance of formula (6.2) corresponding to loop 3. For this, we need to compute the self-gradient \(S_Z\) that acts in the target bundle \(Z\) for \(G_Z\). This target consists of spinor-2-tensors \(\varphi = (\varphi_{\alpha\beta})\) which are trace free and symmetric in the two tensor arguments, and which are annihilated by interior Clifford multiplication in the sense that \(\gamma^\alpha \varphi_{\alpha\beta} = 0\). As in the remark after (3.1), the trace-free condition is actually redundant.

It is not difficult to compute that the operator

\[
(S^0_Z \varphi)_{\alpha\beta} = (\nabla \varphi)_{\alpha\beta} - \frac{2}{n + 2} (\gamma_\alpha \nabla^\lambda \varphi_{\lambda\beta} + \gamma_\beta \nabla^\lambda \varphi_{\lambda\alpha}) \quad (6.23)
\]
has its range in \( \mathbb{Z} \). Being manifestly equivariant, it must be a realization of the self-gradient if \( n \) is odd, and of the gradients

\[
\mathcal{V}(\frac{5}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, \pm \frac{1}{2}) \rightarrow \mathcal{V}(\frac{5}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, \mp \frac{1}{2})
\]

for \( n \) even. Here and below, just as in the spinor-form case, \( \mathbb{D} \) is the Dirac expression \( \gamma^\lambda \nabla_\lambda \), which can act on any bundle \( \Sigma \otimes T \) for which \( T \) is a tensor bundle. This particular normalization of the operator has no special meaning, but it seems convenient to have coefficient 1 on the \( \mathbb{D} \) part. Recall that we adopted the same convention to normalize the Rarita-Schwinger operator \( \mathcal{S}^0 \):

\[
(\mathcal{S}^0 \varphi)_\beta = (\mathbb{D} \varphi)_\beta - \frac{2}{n} \gamma_\beta \text{div} \varphi,
\]

where

\[
\text{div} \varphi = \nabla^\lambda \varphi_\lambda.
\]

Using these expressions and the explicit expression (3.10) for \( \mathcal{G}_Z \), we find a relation between \( \mathcal{S}_Z \mathcal{G}_Z \) and \( \mathcal{G}_Z \mathcal{S}^0 \) on the leading symbol level:

\[
\sigma_2(\mathcal{S}_Z^0 \mathcal{G}_Z) = \frac{n}{n + 2} \sigma_2(\mathcal{G}_Z \mathcal{S}^0).
\]

In fact, each side just above has the same second-order symbol as

\[
\frac{1}{2} \left( \nabla_\alpha (\mathbb{D} \varphi)_\beta + \nabla_\beta (\mathbb{D} \varphi)_\alpha \right) - \frac{1}{n + 2} \left( \gamma_\beta \nabla_\alpha \text{div} \varphi + \gamma_\alpha \nabla_\beta \text{div} \varphi \right)
\]

\[
+ \frac{1}{2(n + 2)} \left( \gamma_\alpha (\mathbb{D}^2 \varphi)_\beta + \gamma_\beta (\mathbb{D}^2 \varphi)_\alpha \right) - \frac{1}{n + 2} g_{\alpha \beta} \mathbb{D} \text{div} \varphi.
\]

That there should be such a relation between leading symbols is expected, by (6.2). The difference

\[
\frac{n}{n + 2} \mathbb{G}_Z \mathcal{S}^0 - \mathcal{S}_Z^0 \mathbb{G}_Z
\]

is some curvature action \( \mathbb{T} \rightarrow \mathbb{Z} \), and computing a little more, we can find it. Note that this curvature action cannot involve the scalar curvature (since \( \mathbb{Z} \not\cong \text{Spin}(n) \mathbb{T} \)). Up to constant multiples, there is just one action of the Einstein tensor which can appear, since \( \text{TF}\mathbb{S}^2 \otimes \mathbb{T} \) contains just one copy of \( \mathbb{Z} \); this is the same fact used to obtain (6.2). This action must already be visible in the leading symbol of the operator (6.24): replacing each \( \nabla \nabla \) in this formula by \( b \) (noting that \( \nabla \) is implicit in \( \mathbb{D} \) and \( \text{div} \)), we get \( \frac{1}{2} \mathbb{b}(b) \), where

\[
(\mathbb{b}(b) \varphi)_{\alpha \beta} = \gamma^\mu (b_{\alpha \nu} \varphi_\beta + b_{\beta \nu} \varphi_\alpha) - \frac{2}{n + 2} \left\{ b_\lambda \gamma_\beta + b_\beta \gamma_\lambda + g_{\alpha \beta} b_\lambda \gamma^\mu \right\} \varphi_\lambda.
\]

It is not immediately clear how many Weyl tensor actions carry \( \mathbb{T} \) to \( \mathbb{Z} \), but in fact, trying all the combinatorial possibilities, it is straightforward to show that there is just one:

\[
(\mathbb{C}(C) \varphi)_{\alpha \beta} = \gamma^\lambda \left( C_{\beta \lambda \alpha} + C_{\alpha \lambda \beta} \right) \varphi_\nu - \frac{3}{2(n + 2)} \left( \gamma_\alpha C_{\beta \lambda \nu} + \gamma_\beta C_{\alpha \lambda \nu} \right) \gamma^\kappa \gamma^\nu \varphi_\lambda.
\]
Computing the difference (6.25) explicitly, one obtains:

**Proposition 13** A realization of (6.2) for loop $\mathcal{B}$ in diagram (6) is given by

$$\frac{n}{n+2} G_Z S^0 - S Z G_Z = -\frac{n}{4(n-2)} b(b) + \frac{1}{2} C(C),$$

where the Einstein and Weyl actions are given by (6.26) and (6.24).

### 6.3 Mixed BW formulas with target in higher tensor-spinor bundles

The new objects in diagram (6) and (6.4) are defined as follows. The bundle $Z$ is a tensor-spinor realization of $U(\frac{5}{2}, \frac{3}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$.

There are two competing realizations of this bundle, of approximately the same complexity. To describe these, it is convenient to first describe a corresponding pair of competing realizations of $U(2, 1, 0, \ldots, 0)$. As always, the tensor realizations and differential operator formulas speak for themselves, and an understanding of the representation-theoretic background is not strictly required.

Consider tensors $\varphi_{\lambda\alpha\beta}$ in $T^* \otimes \Lambda^2$; that is, tensors with the symmetry $\varphi_{\lambda\alpha\beta} = -\varphi_{\lambda\beta\alpha}$.

Under the action of $O(n)$, there are three projections of such tensors. The $\Lambda^3$ part is

$$(P_{\Lambda^3} \varphi)_{\lambda\alpha\beta} = \frac{1}{3} (\varphi_{\lambda\alpha\beta} + \varphi_{\alpha\beta\lambda} + \varphi_{\beta\lambda\alpha}).$$

The remaining parts, being orthogonal to this, must satisfy the Bianchi-like identity

$$\kappa_{\lambda\alpha\beta} + \kappa_{\alpha\beta\lambda} + \kappa_{\beta\lambda\alpha} = 0. \quad (6.28)$$

The $\Lambda^1$ part is

$$(P_{\Lambda^1} \varphi)_{\lambda\alpha\beta} = \frac{1}{n-1} (g_{\lambda\alpha} \varphi^\mu_{\mu\beta} + g_{\lambda\beta} \varphi^\mu_{\alpha\mu}).$$

(Up to a constant multiple, this is the only “pure trace” that is antisymmetric in the second and third arguments. The constant $1/(n-1)$ is determined by the projection condition.) Note that $\kappa = P_{\Lambda^1} \varphi$ satisfies the Bianchi-like identity (6.28). The remaining part is

$$(P \varphi)_{\lambda\alpha\beta} = \frac{2}{3} \varphi_{\lambda\alpha\beta} - \frac{1}{3} \varphi_{\alpha\beta\lambda} - \frac{1}{3} \varphi_{\beta\lambda\alpha} - \frac{1}{n-1} (g_{\lambda\alpha} \varphi^\mu_{\mu\beta} + g_{\lambda\beta} \varphi^\mu_{\alpha\mu}).$$

One may check that

$$(P \varphi)^\alpha_{\alpha\beta} = 0,$$
and that $\kappa = P\varphi$ satisfies (6.28). The bundle we have reached must be isomorphic to $V(2, 1, 0, \ldots, 0)$ by the selection rule (4.11); its symmetry type is: (1) antisymmetric in the last two arguments; (2) totally trace-free; (3) Bianchi-like in the full three arguments.

Now consider a tensor in $\psi \in T^* \otimes \text{TFS}^2$. The projection onto the symmetric 3-tensors $\text{Sym}^3$ is

$$\frac{1}{3} (\psi_{\lambda\alpha\beta} + \psi_{\alpha\beta\lambda} + \psi_{\beta\lambda\alpha}).$$

But $\text{Sym}^3$ splits under $O(n)$, into the direct sum of $\text{TFS}^3$ and $\Lambda^1 = \text{TFS}^1$. The projection of $\psi$ onto $\text{TFS}^3$ will take the form

$$(Q_{\text{TFS}^3}\psi)_{\lambda\alpha\beta} = \frac{1}{3} (\psi_{\lambda\alpha\beta} + \psi_{\alpha\beta\lambda} + \psi_{\beta\lambda\alpha}) - a (g_{\beta\lambda} \psi^\mu_{\mu\alpha} + g_{\alpha\lambda} \psi^\mu_{\mu\beta} + g_{\alpha\beta} \psi^\mu_{\mu\lambda}),$$

where $a$ is some constant. The requirement that the $\alpha\beta$-trace (and thus all traces) vanish gives $a = \frac{2}{3(n + 2)}$:

$$(Q_{\text{TFS}^3}\psi)_{\lambda\alpha\beta} = \frac{1}{3} (\psi_{\lambda\alpha\beta} + \psi_{\alpha\beta\lambda} + \psi_{\beta\lambda\alpha}) - \frac{2}{3(n + 2)} (g_{\beta\lambda} \psi^\mu_{\mu\alpha} + g_{\alpha\lambda} \psi^\mu_{\mu\beta} + g_{\alpha\beta} \psi^\mu_{\mu\lambda}).$$

The $\Lambda^1$ projection will have the form

$$(Q_{\Lambda^1}\psi)_{\lambda\alpha\beta} = c_1 (g_{\lambda\alpha} \psi^\mu_{\mu\beta} + g_{\lambda\beta} \psi^\mu_{\mu\alpha}) + c_2 g_{\alpha\beta} \psi^\mu_{\mu\lambda},$$

(6.29)

where $c_1$ and $c_2$ are constants. The projection condition leads to the system

$$c_2 = c_2 \{ c_2 + (n + 1) c_1 \}, \quad c_1 = c_1 \{ c_2 + (n + 1) c_1 \}.$$  

Thus the projection is trivial unless

$$c_2 + (n + 1) c_1 = 1.$$  

(6.30)

The trace-free condition in $\alpha\beta$ gives

$$2c_1 + nc_2 = 0,$$

and the last two equations force

$$c_2 = -\frac{2}{(n + 2)(n - 1)}, \quad c_1 = \frac{n}{(n + 2)(n - 1)}.$$  

(Note that (6.30) implies that the $\lambda\alpha$ trace of (6.29) is $\psi^\mu_{\mu\beta}$.) Collecting this information, we have

$$(Q_{\Lambda^1}\psi)_{\lambda\alpha\beta} = \frac{n}{(n + 2)(n - 1)} (g_{\lambda\alpha} \psi^\mu_{\mu\beta} + g_{\lambda\beta} \psi^\mu_{\mu\alpha}) - \frac{2}{(n + 2)(n - 1)} g_{\alpha\beta} \psi^\mu_{\mu\lambda}.$$
The remaining projection is
\[
(Q\psi)_{\lambda\alpha\beta} = \frac{2}{3}\psi_{\lambda\alpha\beta} - \frac{1}{3}\psi_{\alpha\beta\lambda} - \frac{1}{3}\psi_{\beta\lambda\alpha} + \frac{2}{3(n-1)}g_{\alpha\beta}\psi^\mu{}_{\mu\lambda} - \frac{1}{3(n-1)}(g_{\lambda\alpha}\psi^\mu{}_{\mu\beta} + g_{\lambda\beta}\psi^\mu{}_{\mu\alpha}).
\]

Note that \(\kappa = Q\psi\) satisfies (6.28), and that \((Q\psi)^{\alpha}_{\alpha\beta} = 0\). By the selection rule (4.1), we have landed in a copy of \(\mathbb{V}(2, 1, 0, \ldots, 0)\), and by the above, we have landed in the following symmetry type: (1) symmetric in the last two arguments; (2) totally trace-free; (3) Bianchi-like in the full three arguments.

The isometry between the two competing realizations of \(\mathbb{V}(2, 1, 0, \ldots, 0)\) is
\[
\psi'_\lambda{}_{\alpha\beta} = -\frac{1}{\sqrt{3}}(\varphi_{\alpha\beta\lambda} + \varphi_{\beta\alpha\lambda}),
\]
\[
\varphi'_\lambda{}_{\alpha\beta} = \frac{1}{\sqrt{3}}(\psi_{\alpha\beta\lambda} - \psi_{\beta\alpha\lambda}).
\] (6.31)

That is, denoting the two tensor bundles by \(\mathbb{V}_P\) and \(\mathbb{V}_Q\), the maps
\[
\mathbb{V}_P \leftrightarrow \mathbb{V}_Q, \quad \varphi \mapsto \psi',
\]
\[
\varphi' \leftarrow \psi
\]
are isometries. One could also reverse the roles of \(\pm 1/\sqrt{3}\) in (6.31).

This material on \(\mathbb{V}(2, 1)\) is significant because \(\mathbb{V}(\frac{5}{2}, \frac{3}{2})\) is the Cartan product (highest weight direct summand) of \(\Sigma \otimes \mathbb{V}(2, 1)\). We may thus realize \(\mathbb{V}(\frac{5}{2}, \frac{3}{2})\) as the bundle of tensor-spinors in \(\Sigma \otimes \mathbb{V}_P\), or in \(\Sigma \otimes \mathbb{V}_Q\), satisfying the interior multiplication conditions
\[
\gamma^\lambda{}_{\kappa\lambda\alpha\beta} = 0, \quad \gamma^{\alpha}_{\kappa\lambda\alpha\beta} = 0.
\] (6.32)

We denote by \((\Sigma \otimes \mathbb{V}_P)_{\text{top}}\) and \((\Sigma \otimes \mathbb{V}_Q)_{\text{top}}\) the subbundles cut out by this condition. This allows us to compute realizations of the gradients
\[
\mathbb{U}(\frac{3}{2}, \frac{3}{2}) \to \mathbb{U}(\frac{5}{2}, \frac{3}{2}) \quad \text{and} \quad \mathbb{U}(\frac{5}{2}, \frac{3}{2}) \to \mathbb{U}(\frac{7}{2}, \frac{3}{2})
\]
(where, for convenience, we have omitted terminal strings of \(\frac{1}{2}\)'s) as differential operators carrying
\[
\mathbf{Y} \to (\Sigma \otimes \mathbb{V}_P)_{\text{top}},
\]
\[
\mathbf{Z} \to (\Sigma \otimes \mathbb{V}_Q)_{\text{top}} \to (\Sigma \otimes \mathbb{V}_P)_{\text{top}},
\]
the very last arrow by the isometry between \(\mathbb{V}_P\) and \(\mathbb{V}_Q\). That is, we agree on one of the competing realizations for \(\mathbb{V}(\frac{5}{2}, \frac{3}{2})\), namely
\[
\mathbf{Z} := (\Sigma \otimes \mathbb{V}_P)_{\text{top}},
\]
for purposes of comparing the operators \(\mathcal{G}_\mathbf{Z}\) and \(\mathcal{H}_\mathbf{Z}\) in loop 3 of diagram (8).
To compute the gradient $G_Z$, we first compute the projection $\Pi$ of $T^* \otimes Y$ onto $Z$. We will then have

$$G_Z \eta = \Pi(\nabla \eta).$$

$\Pi \varphi$ should have the form

$$(\Pi \varphi)_{\lambda\alpha\beta} = \frac{2}{3} \varphi_{\lambda\alpha\beta} - \frac{1}{3} \varphi_{\alpha\beta\lambda} - \frac{1}{3} \varphi_{\beta\lambda\alpha} + a_1 \gamma_\lambda \gamma^\mu \varphi_{\mu\alpha\beta} + a_2 (\gamma_\alpha \gamma^\mu \varphi_{\mu\lambda\beta} - \gamma_\beta \gamma^\mu \varphi_{\mu\lambda\alpha}) + a_3 (g_{\lambda\alpha} \varphi^\mu_{\mu\beta} - g_{\lambda\beta} \varphi^\mu_{\mu\alpha}) + a_4 (\gamma_\lambda \gamma_\alpha \varphi_{\mu\beta} - \gamma_\beta \gamma_\alpha \varphi_{\mu\beta}) + a_5 (\gamma_\alpha \gamma_\beta - \gamma_\beta \gamma_\alpha) \varphi^\mu_{\mu\lambda}$$

for some constants $a_i$.

We now impose the interior multiplication conditions (6.32); after some calculation, we obtain

$$a_1 = \frac{2}{3(n+2)}, \quad a_2 = \frac{1}{3(n+2)}, \quad a_3 = -\frac{3n+4}{3n(n+2)},$$

$$a_4 = -\frac{1}{3n(n+2)}, \quad a_5 = \frac{3n(n+2)}{3}. $$

(Note, in this connection, that the trace-free conditions actually follow from the interior multiplication conditions and the Clifford relations, as in the remark after (3.1).) The Bianchi-like identity (6.28) now holds automatically for $\kappa = \Pi \varphi$; in fact, this just depends on the conditions

$$a_1 - 2a_2 = a_4 + a_5 = 0.$$

To get the operator $H_Z$, we first compute the projection onto $(\Sigma \otimes Z)_{\text{top}}$ of $\psi \in T^* \otimes Z$; this must have the form

$$(\Xi \psi)_{\lambda\alpha\beta} = \frac{2}{3} \psi_{\lambda\alpha\beta} - \frac{1}{3} \psi_{\alpha\beta\lambda} - \frac{1}{3} \psi_{\beta\lambda\alpha} + b_1 \gamma_\lambda \gamma^\mu \psi_{\mu\alpha\beta} + b_2 (\gamma_\alpha \gamma^\mu \psi_{\mu\lambda\beta} + \gamma_\beta \gamma^\mu \psi_{\mu\lambda\alpha}) + b_3 (g_{\lambda\alpha} \psi^\mu_{\mu\beta} + g_{\lambda\beta} \psi^\mu_{\mu\alpha}) + b_4 g_{\alpha\beta} \psi^\mu_{\mu\lambda} + b_5 (\gamma_\lambda \gamma_\alpha \psi^\mu_{\mu\beta} + \gamma_\beta \gamma_\alpha \psi^\mu_{\mu\alpha})$$

for some constants $b_i$. The interior multiplication conditions give, after some calculation,

$$b_1 = \frac{2}{3(n-2)}, \quad b_2 = b_3 = -\frac{1}{3(n-2)}, \quad b_4 = \frac{2(n-3)}{3n(n-2)}, \quad b_5 = -\frac{1}{n(n-2)}.$$ 

$\Xi \psi$ then automatically satisfies (6.28); this just depends on the relations

$$b_1 + 2b_2 = 2b_3 + b_4 - 2b_5 = 0.$$ 

To reach the $Z$ realization, we now apply the isometry (6.31) in the tensorial factor:

$$\sqrt{3} (\ddot{\Xi} \psi)_{\lambda\alpha\beta} = (\Xi \psi)_{\alpha\beta\lambda} - (\Xi \psi)_{\beta\alpha\lambda} = \psi_{\alpha\beta\lambda} - \psi_{\beta\alpha\lambda} + \frac{1}{n-2} (\gamma_\alpha \gamma^\mu \psi_{\mu\beta\lambda} - \gamma_\beta \gamma^\mu \psi_{\mu\alpha\lambda})$$

$$-\frac{1}{n} (g_{\lambda\alpha} \psi^\mu_{\mu\beta} - g_{\beta\lambda} \psi^\mu_{\mu\alpha})$$

$$-\frac{1}{n(n-2)} \left\{ (\gamma_\alpha \gamma_\beta - \gamma_\beta \gamma_\alpha) \psi^\mu_{\mu\lambda} + \gamma_\alpha \gamma_\lambda \psi^\mu_{\mu\beta} - \gamma_\beta \gamma_\lambda \psi^\mu_{\mu\alpha} \right\}.$$
By the above, our two gradient realizations are

\[(G_Z \varphi)_{\lambda\alpha\beta} := \frac{2}{3} \nabla_\lambda \varphi_{\alpha\beta} - \frac{1}{3} \nabla_\alpha \varphi_{\beta\lambda} - \frac{1}{3} \nabla_\beta \varphi_{\lambda\alpha} + \frac{2}{3} \nabla^\mu \varphi_{\mu\alpha} - \frac{1}{3(n+2)} \left( \gamma_\alpha \gamma^\mu \nabla^\mu \varphi_{\beta\lambda} - \gamma_\beta \gamma^\mu \nabla^\mu \varphi_{\lambda\alpha} \right) \]

and \(1/\sqrt{3}\) times

\[(H_Z \psi)_{\lambda\alpha\beta} := \nabla_\alpha \psi_{\beta\lambda} - \nabla_\beta \psi_{\lambda\alpha} + \frac{1}{n-2} \left( \gamma_\alpha \gamma^\mu \nabla^\mu \psi_{\beta\lambda} - \gamma_\beta \gamma^\mu \nabla^\mu \psi_{\lambda\alpha} \right) \]

What will be important are the compositions \(G_Z \, G_Y\) and \(H_Z \, G_Z\), each of which carries \(T\) to \(Z\). The principal part of each composition turns out to be the following: it carries a section \(\Phi\) of \(T\) to

\[
- \frac{n^2 - 2n - 2}{2(n+2)(n-2)} \left\{ \Phi_{\alpha|\beta\lambda} - \Phi_{\beta|\alpha\lambda} \right\} \\
+ \frac{n^2 - n - 3}{2n(n+2)(n-2)} \left\{ g_{\beta\lambda} \Phi_{a|\mu\lambda} - g_{a\lambda} \Phi_{\beta|\mu\alpha} \right\} \\
- \frac{1}{2(n+2)} \left\{ \gamma_\mu \left( \Phi_{\alpha|\beta\mu} - \Phi_{\beta|\alpha\mu} \right) - \gamma_\beta \Phi_{a|\mu\alpha} + \gamma_\alpha \Phi_{\beta|\mu\alpha} \right\} \\
+ \frac{1}{2(n+2)(n-2)} \left\{ \gamma_\beta \Phi_{a|\mu\lambda} - \gamma_\alpha \Phi_{\beta|\mu\lambda} \right\} \\
- \frac{1}{2(n+2)(n-2)} \left\{ \gamma_\alpha \Phi_{a|\mu\lambda} - \gamma_\beta \Phi_{\alpha|\mu\lambda} \right\} \\
+ \frac{1}{n+2} \left\{ \gamma_\alpha \Phi_{a|\mu\lambda} + \gamma_\beta \Phi_{\alpha|\mu\lambda} - \gamma_{a\mu} \Phi_{\beta|\mu\alpha} \right\} \\
- \frac{1}{n+2} \left\{ \gamma_{a\beta} \Phi_{\beta|\mu\lambda} + \gamma_{\beta\mu} \Phi_{\alpha|\mu\lambda} - \gamma_{a\mu} \Phi_{\beta|\mu\alpha} \right\} \\
- \frac{1}{n+2} \left\{ \gamma_{a\beta} \Phi_{\lambda|\mu\lambda} + \gamma_{\beta\mu} \Phi_{\lambda|\alpha\mu} - \gamma_{a\mu} \Phi_{\lambda|\beta\mu} \right\} \\
- \frac{1}{n+2} \gamma_{a\beta} \Phi_{\lambda|\mu\mu},
\]

where we have used the notation

\(\Phi_{a|\beta\lambda} := \nabla_\lambda \nabla_\beta \Phi_a\).
This computation tells us, as a bonus, what the single (up to a constant multiple) action of TFS\(^2\) carrying \(T\) to \(Z\) must be. In particular, the single action of the Einstein tensor is

\[
(b(b)\Phi)_{\lambda\alpha\beta} = - \left( \frac{n^2 - 2n - 2}{2(n+2)(n-2)} \right) \{ \Phi_\alpha b_\beta \lambda - \Phi_\beta b_\alpha \lambda \} \\
+ \frac{1}{2(n+2)} \left\{ \gamma_\mu \lambda (\Phi_\alpha b_\beta \mu - \Phi_\beta b_\alpha \mu) \right. \\
\left. - g_{\beta\lambda} \Phi_\mu b_\alpha \mu + g_{\alpha\lambda} \Phi_\mu b_\beta \mu \right\} \\
- \frac{1}{2(n+2)(n-2)} \frac{1}{n} \left\{ \gamma_\mu \lambda \Phi_\alpha b_\beta \mu - \gamma_\alpha \lambda \Phi_\mu b_\beta \mu \right\} \\
+ \frac{1}{2(n+2)(n-2)} \left\{ \gamma_\beta \lambda \Phi_\mu b_\alpha \mu - \gamma_\alpha \lambda \Phi_\mu b_\beta \mu \right\} \\
+ \frac{1}{2(n+2)(n-2)} \left\{ \gamma_\alpha \beta \Phi_\mu b_\mu \lambda + \gamma_\beta \mu \Phi_\lambda b_\alpha \mu - \gamma_\alpha \mu \Phi_\beta b_\beta \mu \right\}.
\]

(6.33)

Direct computation, now keeping track of curvature terms, shows that

**Proposition 14** A realization of the relation \((bZ)\) corresponding to loop \(4\) is given by

\[
\mathcal{H}_Z G_Z - G_Z G_Y = \frac{2}{n-2} b(b) + C(C),
\]

where the action \(b(b)\) of the Einstein tensor is given by (6.33), and the action of the Weyl tensor is

\[
(C(C)\Phi)_{\lambda\alpha\beta} := \frac{1}{6} \gamma_{\mu\nu} \left( - \Phi_{\lambda C_{\alpha\beta}}^{\mu\nu} + \Phi_{[\alpha C_{\beta]}^{\mu\nu}} \right) \\
+ \frac{1}{3n(n+2)(n-2)} \left( \frac{3n^3 - 4n^2 - 7n + 8}{2n^3 - 4n^2 - 7n + 8} \right) \gamma_{\mu\nu} \Phi_\nu C_{\alpha\beta}^{\mu\nu} - \gamma_{\mu[\alpha C_{\beta]}^{\mu\nu}} \Phi_\nu \\
- \frac{1}{2n^2 - 9n - 6} \gamma_{\mu\lambda} \Phi_\nu C_{\alpha\beta}^{\nu\mu \lambda} \\
+ \frac{6n(n+2)(n-2)}{4n^2 - 6n - 15} \gamma_{\mu\nu} \Phi_\nu C_{\alpha\beta}^{\mu\nu \lambda} \\
+ \frac{3n(n+2)(n-2)}{n+3}(2n-3) \Phi_\nu \gamma_{\mu[\alpha C_{\beta]}^{\mu\nu \lambda}} \\
+ \frac{3n(n+2)(n-2)}{5n+8} \gamma_{\mu\nu} \Phi_\nu g_{\lambda[\beta C_{\alpha]}^{\mu\nu \rho}} \\
+ \frac{6n(n+2)(n-2)}{n^2 - 8n - 8} \gamma_{\mu\nu} \Phi_\nu g_{\lambda[\beta C_{\alpha]}^{\mu\nu \rho}}.
\]

Here square brackets denote antisymmetrization, and we have employed the fourth-degree antisymmetric Clifford symbols

\[
\gamma_{\alpha\beta\lambda\mu} = \gamma_{[\alpha \gamma \lambda \gamma \mu]}.
\]
7 Overdetermined systems

Consider the following systems of differential equations on twistors $\varphi$:

\begin{align*}
G^*_T G_T \varphi &= \tau^2 \varphi, & G^*_Y G_Y \varphi &= 0, & G^*_Z G_Z \varphi &= 0, \\
G^*_T G_T \varphi &= \tau^2 \varphi, & G^*_Y G_Y \varphi &= 0, & G^*_Z G_Z \varphi &= 0,
\end{align*}

(7.1) \quad (7.2)

where $\tau$ is a fixed but arbitrary smooth real function. By virtue of Lemma 1, each of these systems is overdetermined. However, each has nontrivial solutions on the sphere $S^n$, as we shall show presently. It is thus reasonable to ask whether a given one of these systems characterizes the sphere, in the sense that no other manifold supports solutions. If (7.1) or (7.2) fails to characterize the sphere, one might ask for a classification of the manifolds that are capable of supporting a solution.

Lemma 15 Let $n \geq 4$. If (7.1) holds for some smooth real function $\tau$, then the pointwise equation

\begin{equation}
\frac{1}{2} \left\{ \left( \frac{n-3}{4} \right)^2 + \frac{15}{16} \right\} C \circ \varphi + \frac{(n-3)n}{n-2} b \cdot \varphi \\
+ (n-3) \left\{ n^2 \tau^2 - \frac{(n-2)(n+2)}{4(n-1)} K \right\} \varphi = 0
\end{equation}

(7.3)

holds. In particular, if

\begin{align*}
S_T \varphi &= \tau \varphi, & G^*_Y G_Y \varphi &= 0, & G^*_Z G_Z \varphi &= 0
\end{align*}

(7.4)

then (7.3) holds. On a given manifold, the system (7.4) with $\tau$ constant can have a nonzero solution $\varphi$ only for finitely many values of $\tau$.

Proof. The identity (7.3) is an immediate consequence of (5.22). It shows that the possible values of $\tau^2$ are bounded by a constant times $\max_x \| R^\varphi \|$. But since $G^*_T G_T$ is strongly elliptic (Lemma 1), its eigenvalues $\tau^2_0 \leq \tau^2_1 \leq \cdots$ have Weyl asymptotics $\tau^2_j \sim \text{const} \cdot j^{2/n}$ as $j \to \infty$. Thus only finitely many $\tau^2_j$ can satisfy the curvature operator bound. \qed

8 Spectra on the sphere

Despite being badly overdetermined, the system (7.1) does have solutions, for a certain constant $\tau^2$, on the sphere $S^n$. By the branching rule and Frobenius reciprocity, the Spin($n+1$)-types of sections of $T$ for $n \geq 5$ odd have highest weight labels

$$
\alpha_{j,k,\pm} = \left( \frac{3}{2} + j, \frac{1}{2} + k, \frac{1}{2}, \cdots, \frac{1}{2}, \pm \frac{1}{2} \right),
$$
where \( j \) runs over the natural numbers, and \( k \) runs over \( \{0, 1\} \). Each type occurs with multiplicity one. By [7], Theorem 4.1,

\[
\mathcal{N}(G^*_Z G_Z) = \bigoplus_{j=0}^{\infty} \alpha_{j,k,\pm}
\]

\[
\mathcal{N}(G^*_Y G_Y) = \bigoplus_{k=0}^{\infty} \alpha_{j,k,\pm}
\]

where we have abused notation slightly by writing the highest weight to represent the Spin\((n + 1)\)-type which it labels. Each \( \alpha_{j,k,\pm} \) consists of eigensections of \( S_T \), and thus of \( G^*_T G_T \), since \( S_T \) is Spin\((n + 1)\)-invariant: by Schur’s Lemma and the fact the Spin\((n + 1)\)-types occur with multiplicity one, \( S_T \) must act on each Spin\((n + 1)\) as multiplication by a constant. Thus the Spin\((n + 1)\)-types \((\frac{3}{2}, \frac{1}{2}, \cdots, \frac{1}{2}, \pm \frac{1}{2})\) consist of solutions of (7.1), and choosing just one of these two types, one gets solutions of (7.4). When \( n \) is even, the section space of each (positive and negative) twistor bundle is a (multiplicity one) direct sum of Spin\((n + 1)\)-modules with labels

\[
\alpha_{j,k} = \left( \frac{3}{2} + j, \frac{1}{2} + k, \frac{1}{2}, \cdots, \frac{1}{2} \right),
\]

where \( j \) runs over the natural numbers, and \( k \) runs over \( \{0, 1\} \). Again, the solutions of (7.1) are the summands with \( j = k = 0 \).

The spectra on the sphere of all the operators we study here, and in fact of any operator of the form \( G_{\sigma_u}^* G_{\lambda_u} \) on any irreducible Spin\((n)\)-bundle \( V(\lambda) \), are given in [7], Theorems 4.1 and 5.1. (The first of these theorems gives the spectrum to within an overall normalizing constant, and the second computes the normalizing constant.) The branching rule and Frobenius reciprocity show that the Spin\((n + 1)\) types \( \alpha \) occurring in the space of sections of \( V(\lambda) \) occur with multiplicity one, and are exactly those satisfying the interlacing rule

\[
\begin{align*}
\alpha_1 &\geq \lambda_1 \geq \cdots \geq \lambda_\ell \geq |\alpha_{\ell+1}|, & n \text{ odd}, \\
\alpha_1 &\geq \lambda_1 \geq \cdots \geq \alpha_\ell \geq |\lambda|, & n \text{ even}.
\end{align*}
\] (8.1)

For the operator \( G_{\lambda_u}^* G_{\lambda_u} \), the eigenvalue on the \( \alpha \) summand is

\[
c_{\lambda_u} \prod_{a=1}^{L} (\tilde{\alpha}_a^2 - s_u^2) = \tilde{c}_{\lambda_u} \prod_{a \in \mathcal{T}(\lambda)} (\tilde{\alpha}_a^2 - s_u^2),
\] (8.2)

where \( L = [(n + 1)/2] \), \( s_u \) is the quantity defined in (5.7, 5.8),

\[
\tilde{\alpha}_a = \alpha_a + \frac{n + 1 - 2a}{2},
\]

\( \mathcal{T}(\lambda) \) is the set of all \( a \) in \( \{1, \ldots, L\} \) for which \( \tilde{\alpha}_a^2 \) is allowed only one value by the interlacing rule (8.1), and \( c_{\lambda_u}, \tilde{c}_{\lambda_u} \) are certain normalizing constants. Given \( \lambda \) and
\( \tilde{c}_{\lambda \sigma u} \), one may compute \( c_{\lambda \sigma u} \), so it suffices to describe \( \tilde{c}_{\lambda \sigma u} \). Let \( t(\lambda) \) be the cardinality of \( T(\lambda) \). By [7], Theorem 5.2,

\[
\tilde{c}_{\lambda \sigma u} = \begin{cases} 
( -1 )^{t(\lambda)+1} & \text{N(\( \lambda \)) odd}, \\
\prod_{1 \leq v \leq N(\lambda), v \neq u} (s_v - s_u), & N(\( \lambda \)) even, \ \lambda_{\ell} = 0 \neq \lambda_{\ell-1}, \ |(\sigma_u)_{\ell}| = 1, \\
( -1 )^{t(\lambda)(s_u + \frac{1}{2})} & \text{otherwise}.
\end{cases}
\]

(A unified formula handling all cases is given in [7], Remark 5.6.)

In the present situation, let

\[ s_{\Sigma} = \frac{n}{2}, \quad \tilde{c}_{\Sigma} = -\frac{1}{n(n-1)}, \]
\[ s_{T} = 0, \quad \tilde{c}_{T} = \frac{4}{n(n+2)(n-2)}, \]
\[ s_{Y} = -\frac{n-2}{2}, \quad \tilde{c}_{Y} = \frac{2(n-1)(n-2)}{n-3}, \]
\[ s_{Z} = -\frac{n+2}{2}, \quad \tilde{c}_{Z} = -\frac{1}{2(n+2)}. \]

In odd dimensions, these describe the above constants corresponding to the gradient targets \( \Sigma, T, Y, \) and \( Z \) in that order. In even dimensions, starting with the bundle

\[ T^\pm \cong_{\text{Spin}(n)} \bigvee \left( \frac{3}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, \frac{\pm 1}{2} \right), \]

we get the constants corresponding to the targets \( \Sigma^\pm, T^\mp, Y^\pm, \) and \( Z^\pm \) in that order. As a consequence, we have the following spectra. On the Spin\((n+1)\)-type

\[ \alpha(j, k, \varepsilon) := \left( \frac{3}{2} + j, \frac{1}{2} + k, \frac{1}{2}, \ldots, \frac{1}{2}, \varepsilon \frac{1}{2} \right), \]

where \( j \in \mathbb{N}, k \in \{0, 1\}, \) and

\[ \varepsilon \begin{cases} 
\in \{-1, 1\}, & n \text{ odd}, \\
= 1, & n \text{ even},
\end{cases} \]

the eigenvalues are:

\[
\begin{array}{ll}
\text{operator} & \text{eigenvalue} \\
G^*_\Sigma G_\Sigma & \frac{(j + n + 1)(j+1)(1-k)}{n(n+2)(n-2)(j+n+1)^2 (k+n/2-1)^2} \\
G^*_T G_T & \frac{4}{n(n+2)(n-2)} \left( j + \frac{n}{2} + 1 \right)^2 \left( k + \frac{n}{2} - 1 \right)^2 \\
G^*_Y G_Y & \frac{(n-3)(j+n)(j+2)k}{2(n-2)} \\
G^*_Z G_Z & \frac{j(j+n+2)(2n-(n-1)k)}{2(n+2)}
\end{array}
\]
Note that (8.2) gives formulas for the eigenvalues that are quadratic in \( k \); but since \( k^2 = k \), we may reduce them to linear-in-\( k \) expressions if we like. Note also that when \( n \) is even, the section space of \( T^+ \) contains one copy of the \( \text{Spin}(n+1) \)-type \( \alpha(j,k,1) \), and the section space of \( T^- \) also contains one copy.

The leading asymptotics in \( j \) as \( j \to \infty \) in the above list coincide with the arrays (3.12) and (3.13), of eigenvalues of the leading symbols on arbitrary manifolds. This is an example of a more general phenomenon explored in detail in \( \Pi \).

In particular, the above eigenvalue list shows that

\[
\begin{align*}
G^*_\Sigma G_\Sigma & \text{ annihilates the } k = 1 \text{ summands,} \\
G^*_Y G_Y & \text{ annihilates the } k = 0 \text{ summands,} \\
G^*_Z G_Z & \text{ annihilates the } j = 0 \text{ summands.}
\end{align*}
\]

In fact, (8.3) is predictable from the fact that the sections spaces of \( \Sigma, Y, Z \) do not contain copies of \( \alpha(j,1,\varepsilon) \), \( \alpha(j,0,\varepsilon) \), \( \alpha(0,k,\varepsilon) \) respectively; the operators \( G_\Sigma, G_Y, G_Z \), which are targeted in these bundles, must already annihilate the relevant summands.

The summands \( \alpha(0,0,\varepsilon) \) in the \( \text{Spin}(n+1) \)-decomposition of the \( \text{Spin}(n+1) \)-finite section space thus satisfy the system (7.1), with

\[
\tau^2 = \frac{4}{n(n+2)(n-2)} \left( \frac{n}{2} + 1 \right)^2 \left( \frac{n}{2} - 1 \right)^2.
\]

Similarly, the \( \alpha(0,1,\varepsilon) \) summands satisfy the system (7.2), with

\[
\tau^2 = \frac{4}{n(n+2)(n-2)} \left( \frac{n}{2} + 1 \right)^2 \left( \frac{n}{2} \right)^2.
\]

### 9 Interaction with sharp Kato estimates

Suppose we have a natural irreducible bundle \( V(\lambda) \), with gradients

\[ G_u : V(\lambda) \to V(\sigma_u), \quad u = 1, \ldots, N(\lambda). \]

Partition \( \{1, \ldots, N(\lambda)\} \) into two sets \( A \) and \( A^c \), and suppose that \( \varphi \) is a smooth section of \( V(\lambda) \) on a compact manifold \( M \) satisfying

\[ G_u \varphi = 0, \quad \text{all } u \in A. \quad (9.1) \]

(We assume that \( M \) has whatever structure necessary to support the bundles – \( \text{SO}(n) \) or \( \text{Spin}(n) \).) Choose a BW formula

\[
\sum_{u=1}^{N(\lambda)} t_u G^*_u G_u = \text{Curv},
\]
where “Curv” is an action of the Riemann curvature on $\nabla(\lambda)$. Then

$$
\int_M \langle \text{Curv} \varphi, \varphi \rangle = \int_M \sum_{u \in A^c} t_u \langle G_u^* G_u \varphi, \varphi \rangle \\
= \int_M \sum_{u \in A^c} t_u |G_u \varphi|^2_{N(\lambda)} \\
\geq (\min_{u \in A^c} t_u) \int_M \sum_{u=1}^N |G_u \varphi|^2. 
$$

(9.2)

Associated to the set $A$ is a sharp Kato constant $k_A$ \cite{10, 15}. This is the best universal constant with the (local) property

$$
G_u \varphi = 0, \ \text{all } u \in A \Rightarrow \left| d|\varphi| \right|^2 \leq k_A |\nabla \varphi|^2 \text{ off } \{\varphi_x = 0\}. 
$$

If the system (9.1) is injectively elliptic, then $k_A < 1$; otherwise $k_A = 1$. In fact, $k_A$ is $1 - \varepsilon_A$, where $\varepsilon_A$ is the best ellipticity constant for $\sum_{u \in A} G_u^* G_u$:

$$
\sigma_2 \left( \sum_{u \in A} G_u^* G_u \right)(\xi) \geq \varepsilon_A |\xi|^2. 
$$

(See \cite{10}, Theorems 4 and 7.) Ellipticity constants like this can be computed from arrays like (3.12) and (3.13). This and (9.2) give

$$
\int_M \langle \text{Curv} \varphi, \varphi \rangle \geq \frac{m_A}{k_A} \int_M \left| d|\varphi| \right|^2, 
$$

(9.3)

where

$$
m_A = \min_{u \in A^c} t_u. 
$$

By a standard argument, the restriction “off $\{\varphi_x = 0\}$” disappears upon integration; this depends on the fact that for any smooth section $\varphi$, the scalar quantity $|\varphi|$ is a distribution in the Sobolev space $L^2_1$.

Now assume all integrals are taken with respect to normalized measure. (The foregoing statements about integrals are insensitive to normalization of the measure.) Consider the orthogonal (Hodge) decomposition of $|\varphi|$ as

$$
|\varphi| = |\varphi|_{\text{const}} + |\varphi|_{\text{div}},
$$

where $|\varphi|_{\text{const}}$ is a constant function, and $|\varphi|_{\text{div}}$ is in the $L^2$-span of the eigenfunctions of the scalar Laplacian $\Delta$ which have positive eigenvalues. If $0 = \mu_0 < \mu_1 \leq \cdots$ are the eigenvalues of $\Delta$, with corresponding orthonormal eigenfunctions $\psi_j$, and

$$
|\varphi| = \sum_{j=0}^\infty a_j \psi_j,
$$
then $\psi_0 = 1$ and

$$|\varphi|_{\text{const}} = \int |\varphi| = \|f\|_1$$

is the $L^1$ norm of $\varphi$ in normalized measure. Integrating by parts on the right in (9.3), we have

$$\int_M (\text{Curv} \varphi, \varphi) \geq \frac{m_A}{k_A} \int_M |\text{div} \varphi|_\text{div}$$

$$\geq \frac{\mu_1 m_A}{k_A} \int_M |\varphi|^2_\text{div}$$

$$= \frac{\mu_1 m_A}{k_A} \left\| |\varphi|_{\text{div}} \right\|_2^2.$$ 

As a result, if we have an assumption

$$\text{Curv} \leq C \text{Id}_{V(\lambda)}$$

in the sense of endomorphisms, then we may conclude that

$$\left\| |\varphi|_{\text{div}} \right\|_2^2 \leq \frac{k_A C}{\mu_1 m_A} \|\varphi\|_2^2.$$ 

Note that this estimate is scale invariant (as it must be): if the metric $g$ is rescaled to $A^2 g$, where $A$ is a positive constant, then both $C$ and $\mu_1$ scale by factors of $A^{-2}$, while the other constants remain fixed. Thus the part of $|\varphi|$ which is orthogonal to the constants cannot be too large; in this sense, $|\varphi|$ is approximately constant.

Since

$$\left\| |\varphi|_{\text{div}} \right\|_2^2 = \|\varphi\|_2^2 - \left\| |\varphi|_{\text{const}} \right\|_2^2,$$

we could also write this as

$$\left\| |\varphi|_{\text{const}} \right\|_2^2 \geq \left( 1 - \frac{k_A C}{\mu_1 m_A} \right) \|\varphi\|_2^2.$$ 

Since $|\varphi|_{\text{const}} = \|\varphi\|_1$, this relates the $L^2$ and $L^1$ norms of $\varphi$:

$$\|\varphi\|_1^2 \geq \left( 1 - \frac{k_A C}{\mu_1 m_A} \right) \|\varphi\|_2^2.$$ 

In general, for any section, since the measure has been normalized, $\|\varphi\|_1 \leq \|\varphi\|_2$ (by the convexity of $x \mapsto x^2$), so we have

$$\left( 1 - \frac{k_A C}{\mu_1 m_A} \right) \|\varphi\|_2^2 \leq \|\varphi\|_1^2 \leq \|\varphi\|_2^2.$$ 

(9.4)

This is a statement of the approximate constancy of $|\varphi|$, which becomes stronger as $C$ gets smaller. (If $C \leq 0$, $|\varphi|$ must be constant; in fact (12) implies the stronger statement that $\varphi$ is parallel.) The use of the improved Kato inequality (resulting in the appearance above of $k_A$ in place of 1) also improves things somewhat. If we have
a choice of BW formulas (i.e., if $N(\lambda) \geq 4$), then the interplay between $m$ and $C$ can be somewhat complicated.

If there is a self-gradient and its index $u_0$ lies in $A$, we may replace the condition $G_{u_0}\varphi = 0$ with the eigensection equation $D_{\text{self}}\varphi = \eta\varphi$ (see below for definitions) without disturbing the improved Kato inequality. Recall that for a self-gradient, we need a summand of $T^*M \otimes \mathbb{V}(\lambda)$ to be isomorphic to $\mathbb{V}(\lambda)$ itself; that is, there must be a bundle map in

$$0 \neq \zeta \in \text{Hom}_H(\mathbb{V}, T^*M \otimes \mathbb{V}).$$

This results in a natural first-order differential operator on the original realization of $\mathbb{V}$, namely

$$D_{\text{self}} := -\zeta^* \circ \nabla = -\zeta^* \circ \text{Proj}_{\mathbb{V}} \circ \nabla = -\zeta^* \circ G_s; \quad (9.5)$$

this is the self-gradient. Since the difference between two Spin$(n)$-connections on $\mathbb{V}$ is an element of $\text{Hom}_{\text{Spin}(n)}(\mathbb{V}, T^*M \otimes \mathbb{V})$, we also have the family of modified $H$-connections

$$\tilde{\nabla} := \nabla + a\zeta, \quad a \in \mathbb{R}.$$  

In the spinor case, these are known as Friedrich connections. Using $\tilde{\nabla}$ instead of $\nabla$ in the formula (9.5) for the self-gradient results in

$$\tilde{D} = D - a\zeta^*\zeta.$$  

Though $\mathbb{V}$ need not have a distinguished real form, there is a distinguished real form of $\mathbb{V} \otimes \mathbb{V}^* \cong \text{End}(\mathbb{V})$, namely the self-adjoint endomorphisms. Since $T^*M$ has a distinguished real form, it makes sense to demand that $\zeta$ be imaginary; this is what is required to make the self-gradient $D$ formally self-adjoint. For example, in the spinor case, we construct the Clifford multiplication $\gamma$ so that each $\gamma(\xi)$ is skew-adjoint, with the result that the Dirac operator is formally self-adjoint. This fixes the normalization of $\zeta$ up to a constant factor in $\mathbb{R}^*$. If $G_s$ is the gradient valued in the summand of $T^*M \otimes \mathbb{V}$ which is $H$-isomorphic to $\mathbb{V}$, then the requirement that

$$D^2 = G_s^* G_s \quad (9.6)$$

fixes the normalization of $\zeta$ up to a factor of $\pm 1$. This is in fact the best one can do; see [4]. For example, replacing $\gamma$ by $-\gamma$ has no effect on spinor theory. Ultimately, all these ambiguities are rooted in the fact that $\sqrt{-1} \mapsto -\sqrt{-1}$ is an automorphism of $\mathbb{C}$.

By (9.4) and (9.5),

$$G_s^* \zeta^* G_s = G_s^* G_s,$$

so $\zeta^*$, being an $H$-map on $T^*M \otimes \mathbb{V}$, is the projection on the $\mathbb{V}(\lambda)$-isomorphic summand, and $\zeta^* \zeta$ is the identity on $\mathbb{V}$.

Choosing $\zeta$ to be imaginary also has the effect of making the natural metrics on $\mathbb{V}(\lambda)$ and $T^*M \otimes \mathbb{V}(\lambda)$ compatible with the modified connections $\tilde{\nabla}$; that is, $\tilde{\nabla} h = 0$.
whenever $h$ is one of these metrics. As noted in [13], the sharp Kato constants remain unchanged upon passage from $\nabla$ to a new compatible connection. (One can also easily observe this by examining the argument of [10].)

In the case of twistors, the resulting statements build on, for example, Theorems 8 and 9, giving a weaker conclusion under a relaxed assumption. There are clearly many results along these lines that could be stated; lacking an immediate application, we shall content ourselves here with just a few. Using data from [10] or computing directly from (3.12) and (3.13) to get Kato constants, we have the following relatives of Theorem 8 and Proposition 10:

**Theorem 16** Let $Q$ be the maximum eigenvalue over $M$ of the curvature operator on $\Lambda^2$. With normalized measure, for a section $\varphi$ of $\mathcal{T}$ with $G_{\mathbf{Z}}\varphi = 0$,

$$
\left(1 - \frac{n(n+1)(n+7)Q}{8(n+2)\mu_1}\right)\|\varphi\|_2^2 \leq \|\varphi\|_1^1 \leq \|\varphi\|_2^2.
$$

**Proof.** This is the basic estimate (9.4), based on the BW formula of (5.9). by (5.15), $m_A = 1/2$, and by (3.13), $k_A = (n+1)/(2(n+2))$. By (5.17) and (5.18), the quantity $C$ in (9.4) may be taken to be $n(n+7)Q/8$. \hfill \Box

**Theorem 17** Let $n = 8$, and let $\underline{B}$, $\overline{B}$ be the minimum and maximum eigenvalues of the trace-free Ricci tensor $b$ over $M$. For an eigensection of the Rarita-Schwinger operator with $G_{\mathbf{Z}}\varphi = 0$,

$$
\left(1 - \frac{7\overline{B}}{18\mu_1}\right)\|\varphi\|_2^2 \leq \|\varphi\|_1^1 \leq \|\varphi\|_2^2.
$$

For an eigensection of the Rarita-Schwinger operator with $G_{\mathbf{X}}\varphi = 0$, $G_{\mathbf{Y}}\varphi = 0$,

$$
\left(1 + \frac{2\overline{B}}{5\mu_1}\right)\|\varphi\|_2^2 \leq \|\varphi\|_1^1 \leq \|\varphi\|_2^2.
$$

**Proof.** Both assertions are based on the BW formula (5.21). For the first statement, we may take $m_A = 3$, $C = 2\overline{B}$ (by (5.3)), and $k_A = 7/12$ (by (3.12) and (3.13)). For the second statement, we may take $m_A = 1$, $C = -2\overline{B}$, and $k_A = 1/5$. \hfill \Box

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