Twist deformations of Newtonian Schwarzschild-(Anti-)de Sitter classical system

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Abstract – In this article we provide three new twist-deformed Newtonian Schwarzschild-(Anti)-de Sitter models. They are defined on the Lie-algebraically as well as on the canonically non-commutative space-times, respectively. Particularly we find the corresponding Hamiltonian functions and the proper equations of motion. The relations between the models are discussed as well.

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Introduction. – The Schwarzschild-(Anti-)de Sitter metric plays an important role for the most general solution of the vacuum Einstein equation with point-like massive source and nonvanishing cosmological constant Λ. It has been proposed many years ago by Kottler [1] but despite that it still seems to be quite interesting. Since the SdS¹ tensor contains the Λ-terms it can describe the space-time geometry near the heavy object in the Universe of which the expansion is generated by cosmological repulsive force. Recently, for example (see paper [2]) the impact of such a force on the light bending has been studied. Besides this the classical tests² of the Newtonian Schwarzschild-de Sitter space have been performed in article [3].

Regardless of the above considerations there appeared a lot of papers concerning the influence of space-time noncommutativity on the dynamics of physical systems. The proper investigation has been accomplished in the theoretical field (see, e.g., [4–10]), chaos modeling (see, e.g., [11,12]) as well as in the classical and quantum mechanical (see, e.g., [13–22]) context. Consequently, it seems to be quite vital to study the ascendancy of quantum space on the structure of the relativistic and nonrelativistic Schwarzschild-(Anti-)de Sitter systems as well. It should be noted that such a research has been achieved at both velocity scale levels only in the case of canonical³ deformation in articles [34,35].

In this paper, we provide the three noncommutative Newtonian SdS and S(A)dS models. All of them are defined on the twisted⁴ Galilei space-times such as canonically as well as Lie-algebraically deformed spaces. Particularly, we provide the proper Hamiltonian functions and the corresponding equations of motion. Apart from that we dynamically couple the constructed models with the use of the so-called active control synchronization procedure [38,39]⁵. In such a way we introduce and analyze the nonrelativistic systems which formally describe the impact of both trans-Planckian (noncommutativity) and cosmological (presence of parameter Λ) distance scales on the dynamics of the Newtonian S(A)dS models. What seems to be especially interesting are the Lie-algebraically noncommutative systems due to the fact that they provide in a natural way the deformation parameter κ which plays the role of Planck mass [40]. It is commonly belived that studies on such a type of space-time noncommutativity might shed some additional light on, for example,

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¹The acronyms SdS and S(A)dS mean Schwarzschild-de Sitter and Schwarzschild-(Anti-)de Sitter metrics, respectively.
²There have been studied the effects of cosmological constant Λ on Mercury’s perihelion precession and light bending in the context of the Newtonian limit of SdS space-time.
³In accordance with the Hopf-algebraic classification of all deformations of relativistic [23] and nonrelativistic [24] symmetries, one can distinguish three basic types of space-time noncommutativity (see also [25] for details): canonical [26–28], Lie-algebraic [28–31] and quadratic deformation of Minkowski and Galilei spaces [28,31–33].
⁴For details concerning the twist deformation of Hopf algebras see [36] and [37].
⁵By synchronization we mean a dynamical coupling of particles moving in the presence of different (deformed) dynamics such that their phase space trajectories for large times of the evolution become the same.
the properties of the quantum gravity theory [41]. For this reason the models proposed in this article have a chance to give an alternative description of the nonrelativistic quantum gravity effects in the cosmological context [42].

The paper is organized as follows. In the next section we recall the basic facts concerning the Schwarzschild-(Anti-)de Sitter metric and its nonrelativistic limit. The third section is devoted to the canonically and Lie-algebraically twist deformations of Galilei Hopf algebra. In the fourth section we provide the corresponding S(A)dS systems while the relations between models and their synchronizations are discussed in the fifth section. The final remarks close the paper.

Newtonian Schwarzschild-(Anti-)de Sitter classical model. – Let us start with the vacuum Einstein equation for the nonvanishing cosmological constant $\Lambda$,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \Lambda g_{\mu\nu}. \quad (1)$$

Its most general spherically symmetric, so-called Schwarzschild-de Sitter ($\Lambda > 0$) or Schwarzschild-Anti-de Sitter ($\Lambda < 0$) solutions take the form

$$ds^2 = -f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2, \quad (2)$$

with

$$f(r) = 1 - \frac{2GM}{r} - \frac{\Lambda c^2}{3} r^2, \quad (3)$$

where $G$ and $M$ denote the Newton constant and mass of the point-like source, respectively, while $\vec{r} = [x_1, x_2, x_3]$. Besides, one can check that the nonrelativistic limit of the above metric generates the following potential:

$$\phi_{SdS/AdS} = - \frac{GM}{r} - \frac{\Lambda c^2}{3} \vec{r}^2, \quad (4)$$

which leads to the proper Schwarzschild-(\Lambda)de Sitter Hamiltonian function:

$$H_{SdS/AdS} = \frac{\vec{p}^2}{2m} - \frac{GmM}{r} - \frac{m\Lambda c^2}{3} \vec{r}^2, \quad (5)$$

as well as to the canonical equations of motion given by

$$\begin{align*}
\dot{p}_1 &= -G \frac{mM}{r^3} x_1 + \frac{2}{3} m\Lambda c^2 x_1, \\
\dot{p}_2 &= -G \frac{mM}{r^3} x_2 + \frac{2}{3} m\Lambda c^2 x_2, \\
\dot{p}_3 &= -G \frac{mM}{r^3} x_3 + \frac{2}{3} m\Lambda c^2 x_3, \\
\dot{x}_1 &= \frac{p_1}{m}, \\
\dot{x}_2 &= \frac{p_2}{m}, \\
\dot{x}_3 &= \frac{p_3}{m},
\end{align*} \quad (6)$$

Of course, for $M = 0$ and $\Lambda \neq 0$ we get from the above model the attractive or repulsive oscillator system, while for $M \neq 0$ and $\Lambda = 0$ we reproduce the Newtonian model of a particle moving in the central gravitational field.

Twist deformations of Galilei Hopf algebra. –

Canonical twist deformation of Galilei Hopf algebra. The canonically deformed Galilei Hopf algebra $U_{\theta}(G)$ has been provided in article [28] by the proper contraction of its relativistic counterpart. In accordance with the general twist procedure it is given by the classical algebraic sector

$$\begin{align*}
[K_{ij}, K_{kl}] &= i(\delta_{ik} K_{jl} - \delta_{il} K_{jk} + \delta_{jk} K_{il} - \delta_{jl} K_{ik}), \quad (8) \\
[K_{ij}, \Pi_k] &= i(\eta_{jk} \Pi_i - \eta_{ik} \Pi_j), \quad (9) \\
[V_i, \Pi_j] &= [V_i, \Pi_0] = 0, \quad [V_i, \Pi_0] = -i\Pi_i, \quad (10) \\
[\Pi_\mu, \Pi_\nu] &= 0, \quad (11)
\end{align*}$$

where $K_{ij}$, $\Pi_0$ and $V_i$ can be identified as the rotation, time translation and boost operators as well as by the following twisted coproducts:

$$\begin{align*}
\Delta_{(\theta ; i)}(\Pi_\mu) &= \Delta(0)(\Pi_\mu), \\
\Delta_{(\theta ; i)}(V_i) &= \Delta(0)(V_i), \\
\Delta_{(\theta ; i)}(K_{ij}) &= \Delta(0)(K_{ij})
\end{align*} \quad (12)$$

Besides, it should be noted that the corresponding quantum space-time is given by

$$[t, x_i] = 0, \quad [x_i, x_j] = i\theta_{ij}, \quad (13)$$

and for the deformation parameter $\theta_{ij}$ approaching zero it becomes classical.

\footnote{It is described by the classical r-matrix of the form $r_{\theta;ij} = \frac{1}{2} \theta^{1i} \Pi_j \wedge \Pi_i$, where $a \wedge b = a \otimes b - b \otimes a$ and with $\Pi_i$ denoting the momentum generators.}

\footnote{In accordance with the twist procedure [36], the algebraic sector of the deformed Hopf structure remains classical. However, the corresponding coproducts transform in a nontrivial way as follows ($\kappa$ denotes the deformation parameter):

$$\Delta_{(\kappa)}(a) \to \Delta_{(\kappa)}(a) = F_a \cdot \Delta_{(\kappa)}(a) \cdot F_a^{-1},$$

$$\Delta_{(\kappa)}(a) = a \otimes 1 + 1 \otimes a,$$

with twist factor $F_a = \exp i r_a$, where $r_a \in U_{\theta}(A) \otimes U_{\theta}(A)$ denotes so-called classical r-matrix satisfying the classical Yang-Baxter equation (CYBE) of the form: $[r_a, r_b] = 0$; the symbol $[\cdot, \cdot]$ plays a role of so-called Schouten bracket [57].}
Lie-algebraic twist deformations of Galilei Hopf structure. The two Lie-algebraically twist-deformed Galilei Hopf structures $U_{a1}(\mathcal{G})$ and $U_{a2}(\mathcal{G})$ have been introduced in [28] as well and their algebraic sectors remain classical (see formulas (8)–(10)) while the coproducts are given by\textsuperscript{9,10}

$$
\Delta_{(a1)}(\Pi_0) = \Delta_{(0)}(\Pi_0),
\Delta_{(a1)}(\Pi_i) = \Delta_{(0)}(\Pi_i) + \sin\left(\frac{1}{\kappa_1}\Pi_i\right) \wedge (\delta_{kl}\Pi_l - \delta_{il}\Pi_k)
$$

$$
\Delta_{(a1)}(K_{ij}) = \Delta_{(0)}(K_{ij}) + K_{kl} \wedge \left[\cos\left(\frac{1}{\kappa_1}\Pi_l\right) - 1\right]
$$

in the case of the first quantum group, and

$$
\Delta_{(a2)}(\Pi_0) = \Delta_{(0)}(\Pi_0) + \frac{1}{\kappa_3}\Pi_l \wedge \Pi_k,
\Delta_{(a2)}(\Pi_i) = \Delta_{(0)}(\Pi_i),
\Delta_{(a2)}(V_i) = \Delta_{(0)}(V_i),
$$

with $k_{ij} = \cos\left(\frac{1}{\kappa_1}\Pi_l\right) - 1$, and

$$
\Delta_{(a2)}(K_{ij}) = \Delta_{(0)}(K_{ij}) + \frac{i}{\kappa_2} [K_{kl}, V_i] \wedge \left[\cos\left(\frac{1}{\kappa_1}\Pi_l\right) - 1\right],
$$

in the case of the second quantum group, and

$$
\Delta_{(a2)}(V_i) = \Delta_{(0)}(V_i) + [V_i, K_{kl}] \wedge \sin\left(\frac{1}{\kappa_1}\Pi_l\right)
$$

for the second, $U_{a2}(\mathcal{G})$ Hopf structure. One can also check that the corresponding quantum space-times look as follows:

$$
[x_i, x_j] = \frac{i}{\kappa_1} \delta_{ij} (\delta_{lk} x_l - \delta_{il} x_k) + \frac{i}{\kappa_1} \delta_{ij} (\delta_{lk} x_k - \delta_{il} x_l), \quad [t, x_i] = 0,
$$

and

$$
[x_i, x_j] = \frac{i}{\kappa_2} t (\delta_{lk} \delta_{lj} - \delta_{lk} \delta_{ij}), \quad [t, x_i] = 0.
$$

Obviously, for the deformation parameters $\kappa_1$ and $\kappa_2$ running to infinity the above relations become commutative.

Twist deformations of Newtonian Schwarzschild-(Anti-)de Sitter classical system.

Canonical deformation. In the first step of our construction we put in the canonical commutation relations (13) the parameter $\theta_{12}$ equal to $\theta$ and $\theta_{23} = 0 = \theta_{13}$; in such a way we get\textsuperscript{11}

$$
\{x_1, x_2\} = \theta, \quad \{x_1, x_3\} = 0 = \{x_2, x_3\}.
$$

Next, we extend the above structure to the whole phase space as follows [43]\textsuperscript{12}:

$$
\{x_1, \hat{x}_2\} = \theta, \quad \{x_1, \hat{p}_j\} = \delta_{ij},
\{\hat{p}_i, \hat{x}_j\} = 0 = \{\hat{x}_2, \hat{x}_3\},
$$

with $\hat{p}_i$ denoting the canonical momentum conjugated to the $x_i$-variable. By direct calculation one can check that the brackets (24) satisfy the Jacobi identity and for the deformation parameter $\theta$ approaching zero they reproduce the classical ones

$$
\{x_i, p_j\} = \delta_{ij}, \quad \{x_i, x_j\} = 0 = \{p_i, p_j\}.
$$

Besides, it should be noted that the quantum variables $(\hat{x}_i, \hat{p}_i)$ can be represented in terms of commutative ones $(x_i, p_i)$ with the use of the Bopp shift [44],

$$
\hat{x}_1 = x_1 - \frac{\theta}{2} p_2, \quad \hat{x}_2 = x_2 + \frac{\theta}{2} p_1, \quad \hat{x}_3 = x_3, \quad \hat{p}_1 = p_1,
$$

and then, the Hamiltonian (5) takes the form

$$
H_\theta(\hat{x}, \hat{p}) = \frac{\bar{p}^2}{2m} - \frac{G m M}{\sqrt{(x_1 - \frac{\theta}{2} p_2)^2 + (x_2 + \frac{\theta}{2} p_1)^2 + x_3^2 - \frac{m \Delta c^2}{3} \bar{r}^2 - \frac{m \Delta c^2}{12} \bar{q}^2 (\bar{p}_1^2 + \bar{p}_2^2) - \frac{m \Delta c^2}{3} \theta L_\Lambda; \quad L_\Lambda = x_2 p_1 - x_1 p_2,}
$$

\textsuperscript{11}\{,\} = \frac{1}{\kappa_2}\{,\}.
\textsuperscript{12}It is the simplest phase space for canonical space-time noncommutativity which satisfies the Jacobi condition.

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while the corresponding equations of motion are given by

\[
\dot{p}_1 = \frac{2}{3} m \Lambda c^2 x_1 - \frac{1}{2} \Lambda c^2 m \dot{p}_2 \frac{(x_1 - \frac{\theta}{2} p_2)^2 + (x_2 + \frac{\theta}{2} p_1)^2 + x_3^2}{GmM \left[ x_1 - \frac{\theta}{2} p_2 \right]} \]

\[
\dot{p}_2 = \frac{2}{3} m \Lambda c^2 \frac{(x_1 - \frac{\theta}{2} p_2)^2 + (x_2 + \frac{\theta}{2} p_1)^2 + x_3^2}{GmM \left[ x_2 + \frac{\theta}{2} p_1 \right]} \]

\[
\dot{p}_3 = \frac{2}{3} m \Lambda c^2 x_1 \frac{GmM x_3 \left[ (x_1 - \frac{\theta}{2} p_2)^2 + (x_2 + \frac{\theta}{2} p_1)^2 + x_3^2 \right]}{6} \]

\[
\dot{x}_1 = \frac{1}{m} - \frac{m \Lambda c^2}{6} \theta \frac{p_1}{GmM \theta \left[ x_1 + \frac{\theta}{2} p_1 \right]} \]

\[
+ \frac{2}{m} \left[ (x_1 - \frac{\theta}{2} p_2)^2 + (x_2 + \frac{\theta}{2} p_1)^2 + x_3^2 \right] \]

\[
\dot{x}_2 = \frac{1}{m} - \frac{m \Lambda c^2}{6} \theta \frac{p_2}{GmM \theta \left[ x_1 - \frac{\theta}{2} p_2 \right]} \]

\[
- \frac{2}{m} \left[ (x_1 - \frac{\theta}{2} p_2)^2 + (x_2 + \frac{\theta}{2} p_1)^2 + x_3^2 \right] \]

\[
\dot{x}_3 = \frac{p_3}{m} \]

Consequently, the Hamiltonian of the system takes the form

\[
H_\theta (\dot{x}, \dot{p}) = \frac{1}{2m} \dot{p}^2 - \frac{GmM}{\sqrt{\left( x_1 - \frac{1}{\kappa_1} x_3 p_2 + \frac{1}{\kappa_1} x_3 p_3 \right)^2 + x_2^2 + x_3^2}} \]

\[- \frac{m \Lambda c^2}{3 \kappa_1} x_1 L_1 \frac{\left( x_1 - \frac{1}{\kappa_1} x_3 p_2 + \frac{1}{\kappa_1} x_3 p_3 \right)^2 + x_2^2 + x_3^2}{\kappa_1 L_1^2} \]

while the canonical equations of motion are given by

\[
\dot{p}_1 = \frac{2}{3} m \Lambda c^2 x_1 + \frac{2}{3 \kappa_1} m \Lambda c^2 L_1 \]

\[- \frac{GmM}{\sqrt{\left( x_1 - \frac{1}{\kappa_1} x_3 p_2 + \frac{1}{\kappa_1} x_3 p_3 \right)^2 + x_2^2 + x_3^2}} \]

\[- \frac{m \Lambda c^2}{3 \kappa_1} x_1 L_1 \frac{\left( x_1 - \frac{1}{\kappa_1} x_3 p_2 + \frac{1}{\kappa_1} x_3 p_3 \right)^2 + x_2^2 + x_3^2}{\kappa_1 L_1^2} \]

\[
\dot{p}_2 = \frac{2}{3} m \Lambda c^2 x_2 + \frac{2}{3 \kappa_1} m \Lambda c^2 x_1 p_3 + \frac{2}{\kappa_1} m \Lambda c^2 L_1 p_3 \]

\[- \frac{GmM}{\sqrt{\left( x_1 - \frac{1}{\kappa_1} x_3 p_2 + \frac{1}{\kappa_1} x_3 p_3 \right)^2 + x_2^2 + x_3^2}} \]

\[- \frac{m \Lambda c^2}{3 \kappa_1} x_1 L_1 \frac{\left( x_1 - \frac{1}{\kappa_1} x_3 p_2 + \frac{1}{\kappa_1} x_3 p_3 \right)^2 + x_2^2 + x_3^2}{\kappa_1 L_1^2} \]

\[
\dot{p}_3 = \frac{2}{3} m \Lambda c^2 x_3 - \frac{2}{3 \kappa_1} m \Lambda c^2 x_1 p_3 - \frac{2}{\kappa_1} m \Lambda c^2 L_1 p_2 \]

\[- \frac{GmM}{\sqrt{\left( x_1 - \frac{1}{\kappa_1} x_3 p_2 + \frac{1}{\kappa_1} x_3 p_3 \right)^2 + x_2^2 + x_3^2}} \]

\[- \frac{m \Lambda c^2}{3 \kappa_1} x_1 L_1 \frac{\left( x_1 - \frac{1}{\kappa_1} x_3 p_2 + \frac{1}{\kappa_1} x_3 p_3 \right)^2 + x_2^2 + x_3^2}{\kappa_1 L_1^2} \]

\[
\dot{x}_1 = \frac{p_1}{m} \]

\[
\dot{x}_2 = \frac{p_2}{m} + \frac{2}{3 \kappa_1} m \Lambda c^2 x_1 x_3 + \frac{2}{\kappa_1} m \Lambda c^2 L_1 x_3 \]

\[- \frac{GmM x_3}{\sqrt{\left( x_1 - \frac{1}{\kappa_1} x_3 p_2 + \frac{1}{\kappa_1} x_3 p_3 \right)^2 + x_2^2 + x_3^2}} \]

\[- \frac{m \Lambda c^2}{3 \kappa_1} x_1 L_1 \frac{\left( x_1 - \frac{1}{\kappa_1} x_3 p_2 + \frac{1}{\kappa_1} x_3 p_3 \right)^2 + x_2^2 + x_3^2}{\kappa_1 L_1^2} \]

\[
\dot{x}_3 = \frac{p_3}{m} + \frac{2}{3 \kappa_1} m \Lambda c^2 x_1 x_2 - \frac{2}{\kappa_1} m \Lambda c^2 L_1 x_2 \]

\[- \frac{GmM x_2}{\sqrt{\left( x_1 - \frac{1}{\kappa_1} x_3 p_2 + \frac{1}{\kappa_1} x_3 p_3 \right)^2 + x_2^2 + x_3^2}} \]

\[- \frac{m \Lambda c^2}{3 \kappa_1} x_1 L_1 \frac{\left( x_1 - \frac{1}{\kappa_1} x_3 p_2 + \frac{1}{\kappa_1} x_3 p_3 \right)^2 + x_2^2 + x_3^2}{\kappa_1 L_1^2} \]

Of course, for the deformation parameter \( \kappa_1 \) running to infinity the above model becomes commutative.

Second Lie-algebraic twist deformation. For the second Lie-algebraic twist deformation (see formula (22)) we take

\[
\{ \dot{x}_1, \dot{x}_2 \} = \frac{t}{\kappa_2}, \quad \{ \dot{x}_1, \dot{x}_3 \} = 0 = \{ \dot{x}_2, \dot{x}_3 \}, \]

and \([43]^{14}\)

\[
\{ \dot{x}_1, \dot{x}_j \} = 0, \quad \{ \dot{x}_i, \dot{p}_j \} = \delta_{ij}, \quad \{ \dot{p}_i, \dot{x}_j \} = 0. \]

\(^13\)Here we consider the so-called first type of deformed phase space for noncommutative space-time (29) proposed in [43]. It has been obtained as a solution of the proper condition for the Jacobi identity.

\(^{14}\)Here we consider the phase space for relations (36) proposed in [43] which satisfies the Jacobi identity.
where
\[
\begin{align*}
\dot{x}_1 &= x_1 - \frac{t}{2\kappa_2} p_2, \quad \dot{x}_2 = x_2 + \frac{t}{2\kappa_2} p_1, \\
\dot{x}_3 &= x_3, \quad \dot{p}_i = p_i.
\end{align*}
\]
(38)

Then, the corresponding Hamiltonian function as well as the proper canonical equations look as follows:
\[
H_{\kappa_2}(x, p, t) = \frac{p^2}{2m} - \frac{GmM}{\sqrt{(x_1 - \frac{t}{2\kappa_2} p_2)^2 + (x_2 + \frac{t}{2\kappa_2} p_1)^2 + x_3^2}} - \frac{m \Lambda^2}{3} \kappa^2 - \frac{m \Lambda^2}{12\kappa^2} (p_1^2 + p_2^2) - \frac{m \Lambda^2}{3\kappa_2} L_3; \\
L_3 &= x_2 p_1 - x_1 p_2,
\]
(39)

and
\[
\begin{align*}
\dot{p}_1 &= \frac{2}{3} m \Lambda^2 x_1 - \frac{1}{3\kappa_2} \Lambda^2 mtp_2 - \frac{2}{3} m \Lambda^2 x_2 + \frac{1}{3\kappa_2} \Lambda^2 mtp_1 - \frac{2}{3} m \Lambda^2 x_2,
\end{align*}
\]
(40)

respectively. Obviously, for the deformation parameter \(\kappa_2\) approaching infinity the above model becomes classical.

Relations between models. Let us now compare the models (28), (35) and (40) provided in this paper. First of all, one should notice that eqs. (35) are contrary to the remaining two systems highly nonlinear in the \((x_i, p_i)\) variables. Besides, the Hamiltonian function for the third model (39) is not conserved in time. Nevertheless, all the twisted Schwarzschild-(Anti-)de Sitter systems can be directly linked with the use of the so-called active control procedure [38,39]. In its framework one can provide the proper dynamical coupling of the differently deformed particles such that for large times of the evolution their phase-space trajectories become identical, i.e., the systems become synchronized (connected)\textsuperscript{15}. Formally, such an interaction is described by the control functions which in the case of synchronization of the canonically deformed model with the second one look as follows\textsuperscript{16,17,18}:

\[
\begin{align*}
&u_{\theta, \kappa_1, \pi_1} = \frac{2}{3} m \Lambda^2 x_1 - \frac{1}{3\kappa_2} \Lambda^2 mtp_2 - \frac{2}{3} m \Lambda^2 x_2 + \frac{1}{3\kappa_2} \Lambda^2 mtp_1 - \frac{2}{3} m \Lambda^2 x_2 + \frac{1}{3\kappa_2} \Lambda^2 mtp_1 - \frac{2}{3} m \Lambda^2 x_2,
\end{align*}
\]
\[
\begin{align*}
&u_{\theta, \kappa_1, \pi_2} = \frac{2}{3} m \Lambda^2 x_1 - \frac{1}{3\kappa_2} \Lambda^2 mtp_2 - \frac{2}{3} m \Lambda^2 x_2 + \frac{1}{3\kappa_2} \Lambda^2 mtp_1 - \frac{2}{3} m \Lambda^2 x_2 + \frac{1}{3\kappa_2} \Lambda^2 mtp_1 - \frac{2}{3} m \Lambda^2 x_2,
\end{align*}
\]
\[
\begin{align*}
&u_{\theta, \kappa_1, \pi_3} = \frac{2}{3} m \Lambda^2 x_1 - \frac{1}{3\kappa_2} \Lambda^2 mtp_2 - \frac{2}{3} m \Lambda^2 x_2 + \frac{1}{3\kappa_2} \Lambda^2 mtp_1 - \frac{2}{3} m \Lambda^2 x_2 + \frac{1}{3\kappa_2} \Lambda^2 mtp_1 - \frac{2}{3} m \Lambda^2 x_2,
\end{align*}
\]
(41)

\textsuperscript{15}From the physical point of view, the above mentioned synchronization procedure gives an answer on the question: How should interact two cosmological particles moving in the presence of different (deformed) dynamics in order to their trajectories for large times become the same?

\textsuperscript{16}For details of finding the control functions see [38,39].

\textsuperscript{17}The trajectories \((x, p)\) and \((y, \pi)\) correspond to the master canonically (master) and Lie-algebraically (slave) deformed systems respectively while \(K_1 = y_2 \pi_3 - y_1 \pi_2\).

\textsuperscript{18}The controllers (41) (the interaction terms) are added to the equations of motion (40) and due to the Lyapunov theorem the systems are synchronized.
\[
\begin{align*}
\theta_{\kappa, y_1} &= \left[ \frac{1}{m} - \frac{m\Lambda c^2}{6} \theta^2 \right] p_1 \\
&+ \frac{GmM\theta [x_2 + \frac{\theta}{2} p_1]}{2 \left[ (x_1 - \frac{\theta}{2} p_1)^2 + (x_2 + \frac{\theta}{2} p_1)^2 + x_3^2 \right]^{3/2}} \\
&+ x_1 - y_1 - \frac{\pi_1}{m}, \\
\end{align*}
\]

\[
\begin{align*}
\theta_{\kappa, y_2} &= \left[ \frac{1}{m} - \frac{m\Lambda c^2}{6} \theta^2 \right] p_2 \\
&- \frac{GmM\theta [x_1 - \frac{\theta}{2} p_2]}{2 \left[ (x_1 - \frac{\theta}{2} p_2)^2 + (x_2 + \frac{\theta}{2} p_2)^2 + x_3^2 \right]^{3/2}} \\
&+ x_2 - y_2 - \frac{\pi_2}{m} \\
&- \frac{2}{3\kappa_1} \frac{m\Lambda c^2 y_1 y_3}{m} - \frac{2}{\kappa_1} \frac{m\Lambda c^2 K_1 y_3}{m} \\
&+ \frac{GmM y_3 \left[ y_1 - \frac{1}{\kappa_1} y_3 \pi_2 + \frac{1}{\kappa_1} x_2 p_3 \right]}{\kappa_1 \left[ (x_1 - \frac{1}{\kappa_1} x_3 p_2 + \frac{1}{\kappa_1} x_2 p_3)^2 + x_2^2 + x_3^2 \right]^{3/2}}, \\
\end{align*}
\]

Besides, the active controllers \( (u_{\theta,\kappa,\pi}) \) combining the first and third systems as well as the functions \( (u_{\kappa,\kappa_1,\pi}) \) coupling the Lie-algebraically noncommutative models \( 35 \) and \( 40 \) can be found as well. However, due to the complicated form their presentation has been omitted in this paper.

**Final remarks.** – In this article we provide three twist-deformed Newtonian Schwarzschild-(Anti-)de Sitter models. They are defined on the Lie-algebraically as well as on the canonically noncommutative space-times, respectively. Particularly, we find the corresponding Hamiltonian functions and the proper equations of motion. The synchronization of the models is discussed as well.

It should be noted that the presented systems are quite interesting. They formally describe, for example, the impact of two different distance scales such as the trans-Planckian (noncommutativity) and the cosmological (\( \Lambda \)) scale on the dynamics of nonrelativistic particles moving in the central gravitational field. However, a better understanding of such a property of the models requires more investigations which are in progress.

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