Polyhedral programming and optimization of discrete control processes with polyhedral performance criterions

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Abstract. The elements of polyhedral programming theory considering the extreme tasks with polyhedral both goal and limiting functions are presented. The authors discuss the application of polyhedral programming device for optimal control tasks by discrete dynamic objects with polyhedral performance criterions. The decision of two optimization tasks of discrete control processes is reduced. They are the control task by finite state and the task of terminal stabilization.

1. Introduction
One of the main mathematical instruments widely used in modern scientific and engineering developments is mathematical programming (MP). It’s the division of applied mathematics, having inquired into the extremal problems and the elaboration of their solution methods. By classical determined finite-dimensional MP problem we mean the following definition problem of extremal (maximal or minimal) value of function \( f(x) \) of vector variable \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \) [1]:

\[
\text{extr}\left\{ f(x) : g_k(x) \leq 0, k = 1, r \right\}
\]

where components \( \{x_i, i = 1, n\} \), of vector \( x \) are unknown, \( f(x) \) and \( g_k(x) \) – are the given scalar functions; \( \mathbb{R} \) – is material numbers field. In doing so the functions \( f(x) \) and \( g_k(x) \) are called goal function and bounding functions respectively.

According to functions properties \( f(x) \), \( g_k(x) \) in MP are distinguished the special divisions investigating the corresponding problems. Thus in linear programming (LP) problems all the indicated functions are linear functions, and in the nonlinear programming (NP) problems at least one of them is nonlinear function. So called piecewise-linear programming (PLP) problems where goal and bounding functions are piece-wise-linear functions (“glued” from the “pieces” of linear functions) occupy the “intermediate” position between the problems of LP and NP.
The bases of theory and methods of solution of PLP problems are founded in the papers of Golshtein and Judin [2, ch. 7] and also Zuhovitsky and Avdeeva [3, ch. V]. The received results made possible to form PLP and to give them the independent division MP status [4].

The wide class of control design and planning problems and also the different applied optimization problems in the very different areas of human activities belongs to PLP problems. So a number of economic and technical problems beginning with the problems of scheduling production planning, some transport problem and ending with choice problems of rational tolerances system in machine-building, the problems of structure of reliable schemes from unreliable elements, the problems of synthesis of formal neurons, automatic systems design problems, the optimal control problems solutions is reduced to PLP problems. Besides it is worth to reformulate many LP problems in PLP terms for decrease of variables number and boundaries.

Unfortunately the development and more fruitful PLP application is restrained itself from that circumstance that in fact it isn’t elucidated in modern educational and monographic literature.

In the present paper are explained the main positions of polyhedral programming (PP) [5] that is PLP division, on the base of which the polyhedral designs are situated: they are polyhedral set, polyhedral function, polyhedral norm and polyhedral metric.

PP encloses the problems on maximum and minimum of goal polyhedral functions, moreover the first problems in contrast to the second one are not concerned to so called convex programming (CP) problems, where the goal and bounding functions are convex one. Let’s note that in this connection PP is not the special case of CP. The principle singularity of PP consists in the possibility of optimization problems reduction to LP problems. That is the reason that PP methods have great applied opportunities and it’s illustrated with the cited examples of two optimization tasks of discrete control processes formalization in PP terms.

2. Main polyhedral designs

Polyhedral sets. Let $E$ be Euclidean space, that is the material linear space provided with scalar product of two vectors $x, x' \in E$, which we’ll be denote by $\langle x, x' \rangle$. In particular if $E = \mathbb{R}^n$ is $n$-dimensional arithmetical space, $x = (x_1, x_2, \ldots, x_n) \in E$, $x' = (x'_1, x'_2, \ldots, x'_n) \in E$, and scalar product is introduced across natural basis then

$$\langle x, x' \rangle = \sum_{i=1}^{n} x_i x'_i.$$

Further for some set $\Omega \subset E$ we’ll be use the following norm notation: $\text{int} \; \Omega$ is the interior, $\partial \Omega$ is bound, $\text{conv} \; \Omega$ - is convex hull, $\text{dim} \; \Omega$ is dimension set $\Omega$.

The orthogonal projection of vector $x \in L$ on some linear subspace $L \subset E$ of Euclidean space $E$ will be denote by $\text{pr}_L (x)$.

Nonempty set $\Pi \subset E$, formed by intersection of closed half-space final number in $E$, we’ll be call polyhedral set or convex polyhedron, and also empty set $\emptyset$ we’ll be consider as polyhedral set by definition.

The following proposal [6] explains the structure of polyhedral sets.

Proposal 1. Every polyhedral set $\Pi \subset E$ is convex hull of final points’ number and final number of closed rays in $E$.

The following proposals show the ways of new polyhedral sets from the input given [7].

Proposal 2. The enumeration of two polyhedral sets is the polyhedral set.

Proposal 3. The convex hull of two polyhedral sets is the polyhedral set.

We’ll call set $\Omega \subset E$ a convex body, if it is convex and contains at least one inner point: $\text{int} \; \Omega \neq \emptyset$.

Theorem 1. Let space $E$ be decomposed into finite number of closed convex bodies $\Omega_1, \Omega_2, \ldots, \Omega_m$, the interiors of which are not intersected:
\[ E = \Omega_1 \cap \Omega_2 \cap \ldots \cap \Omega_m , \quad \text{int} \, \Omega_i \cup \text{int} \, \Omega_j = \emptyset , \quad i \neq j , \quad i, j = \overline{1, m} . \]

Then \( \Omega_1, \Omega_2, \ldots, \Omega_m \) are polyhedral sets.

**Proof.** As it is known two nonempty convex sets \( \Omega' \) and \( \Omega'' \), the interiors of which are not intersected are separated with the help of some hyperplane. Hence if the given set has general boundary points \( \partial \Omega' \cup \partial \Omega'' \neq \emptyset \) then the last one are situated on some separating hyperplane \( \Gamma \), which is also hyperplane of support for sets \( \Omega' \) and \( \Omega'' \). Thus every set \( \Omega' \in \{ \Omega_1, \Omega_2, \ldots, \Omega_m \} \) is the intersection not more then \( m-1 \) of closed half-spaces containing \( \Omega' \) and bounded by hyperplanes, separating the given set from the other sets \( \Omega'' \in \{ \Omega_1, \Omega_2, \ldots, \Omega_m \} \).

**Convex functions.** We’ll consider the continuous material functions for \( X = R^n \). Let us connect the following set with every function \( f : X \to R \):

1) the area of values: \( \text{im}(f) \equiv f(X) \);
2) the graph: \( \text{gr}(f) \equiv \{(x, y) \in X \times R : y = f(x)\} \);
3) the epigraph: \( \text{epi}(f) \equiv \{(x, \mu) \in X \times R : \mu \geq f(x)\} \);
4) set level: \( C_\alpha(f) \equiv \{x \in X : f(x) \leq \alpha, \alpha \in R\} \).

Then we will use the following result [7].

**Proposal 4.** Every function is determined by its epigraph:

\[ f(x) = \inf \{ \mu : (x, \mu) \in \text{epi}(f) \} . \]

The function \( f \) we’ll call the convex (on \( X \)) if for arbitrary \( x_0, x_1 \in X \) and \( \lambda \in [0, 1] \) is realized Jensen inequality:

\[ f((1 - \lambda)x_0 + \lambda x_1) \leq (1 - \lambda)f(x_0) + \lambda f(x_1) . \]

If this inequality is realized strictly for \( x_0 \neq x_1 \) and \( 0 < \lambda < 1 \), then \( f \) is called strictly convex.

The following properties of functions take place:

**Proposal 5.** If \( f \) is convex then for any \( \alpha \in R \) set \( C_\alpha(f) \) is either empty or convex.

**Proposal 6.** The function \( f \) is convex if and only if set \( \text{epi}(f) \) is convex.

**Proposal 7.** The convex functions sum with nonnegative coefficients is the convex function again. For continuous functions class \( f_1(x), f_2(x), \ldots, f_m(x) \) let us define maximum operation [8]:

\[ \bigvee_{i=1}^{m} f_i(x) = (f_1 \lor f_2 \lor \ldots \lor f_m)(x) = \max\{f_1(x), f_2(x), \ldots, f_m(x)\} . \]

The following proposal explains the geometric meaning of the given operation.

**Proposal 8.** If \( f_1(x), f_2(x), \ldots, f_m(x) \) are continuous functions for \( X \), then

\[ f(x) = \bigvee_{i=1}^{m} f_i(x) , \]

if and only if when

\[ \text{epi}(f) = \bigcap_{i=1}^{m} \text{epi}(f_i) . \quad (1) \]

Let us note then the correctness of the given proposal follows from the equitation

\[ f(x) \leq \mu \iff f_i(x) \leq \mu, \quad i = \overline{1, m} . \]

**Polyhedral functions.** Let us consider the final system of linear functions \( \varphi_i : X \to R \) :
\[ \varphi_i(x) = \gamma_i + \langle d_i, x \rangle, \]

where \( \gamma_i \in \mathbb{R}, \ d_i \in \mathbb{X}, \ i = 1, m. \)

Every linear function \( \varphi : \mathbb{X} \to \mathbb{R} \) one may present in the form of may

\[ \varphi(x) = \gamma + \langle d, x \rangle, \]

where \( \gamma = \varphi(0) \) and \( d = \text{grad} \varphi(x) \).

It’s clear that half-space \( P = \text{epi}(\varphi) \) is epigraph of the given linear function.

**Theorem 2.** Let \( P \subset \mathbb{X} \times \mathbb{R} \) be closed half-space and for some \( x' \in \mathbb{X} \)

\[ \inf \{ \mu : (x', \mu) \in P \} > -\infty. \]

Then \( P \) defines the linear function

\[ \varphi(x) = \inf \{ \mu : (x, \mu) \in P \} \]

and coincides with its epigraph: \( P = \text{epi}(\varphi) \).

**Proof.** Half-space \( P \) answers some linear inequality in the form of

\[ \psi(x) - \beta \cdot \mu \leq 0, \]

where \( \psi(x) \) is linear function, and \( \beta = \text{const.} \) The condition (3) means

\[ \inf \{ \mu : \psi(x') - \beta \cdot \mu \leq 0 \} > -\infty, \]

what’s possible only for \( \beta > 0 \). The unknown function is determined by equality

\[ \varphi(x) = \psi(x)/\beta. \]

The function we will call **polyhedral** one [9] if its epigraph is convex polyhedron.

**Proposal 9.** For any \( \alpha \in \mathbb{V} \) level set \( \mathcal{C}_\alpha(f) \) of polyhedral function \( f \) is convex polyhedron.

**Theorem 3.** The function \( f(x) \) is the polyhedral one if and only if it’s maximum function of final number of some nonlinear functions \( \varphi_1(x), \varphi_2(x), \ldots, \varphi_m(x) \):

\[ f(x) = \bigvee_{i=1}^{m} \varphi_i(x). \]

**Proof.** Let us assume that equality (4) takes place. Then according with the supposition 8

\[ \text{epi}(f) = \bigcap_{i=1}^{m} \text{epi}(\varphi_i), \]

that is \( \text{epi}(f) \) coincides with intersection of half-spaces \( \text{epi}(\varphi_i) \), \( i = 1, m \) and so it’s convex polyhedron. Hence \( f \) is polyhedral function.

Let’s take the inverse supposition: \( f \) is polyhedral function. Then its epigraph \( \text{epi}(f) \) is convex polyhedron \( \Pi \). Let \( G_1, G_2, \ldots, G_m \) be \( n \)-dimensional bounds \( \Pi \). It’s clear, that

\[ \text{gr}(f) = G_1 \cap G_2 \cap \ldots \cap G_m. \]

Let’s connect with bound \( G_i \) the closed half-space \( P_i \subset \mathbb{X} \times \mathbb{R} \), having the following properties: it contains \( \Pi \), moreover hyperplane \( \partial P_i \) is hyperplane of support for \( \Pi \) and intersects its by bound \( G_i \). But then polyhedron \( \Pi \) is formed by intersection of the given half-space.

Let’s choose some point \( (x'_i, f(x'_i)) \in G_i \). It follows from (4), that

\[ \inf \{ \mu : (x'_i, \mu) \in P_i \} = f(x'_i) > -\infty. \]
On the strength of theorem 2 half-space \( P_i \) defines the linear function
\[
\varphi_i(x) = \inf \{ \mu : (x, \mu) \in P_i \}, \ i = 1, m.
\]
Thus
\[
\text{epi}(f) = \inf \{ (x, \mu) \in X \times \mathbb{R} : \varphi_i(x) \leq \mu, i = 1, m \}.
\]

Hence according with proposal 4 we’ll find
\[
f(x) = \inf \{ \mu : \varphi_i(x) \leq \mu, i = 1, m \} = \max \{ \varphi_1(x), \varphi_2(x), \ldots, \varphi_m(x) \}.
\]

The following theorem shows that polyhedral functions class coincides with class of convex piecewise-linear functions.

**Theorem 4.** Let \( f : X \rightarrow \mathbb{R} \) be convex function, and let’s assume that space \( X \) one may decomposed into final number of closed convex bodies \( \Omega_1, \Omega_2, \ldots, \Omega_m \)
\[
X = \Omega_1 \cap \Omega_2 \cap \ldots \cap \Omega_m,
\]
the interiors of which are not intersected, and also the function \( f \) is linear in every open area \( \text{int} \Omega_i, i = 1, m \). Then \( f \) is polyhedral function.

**Proof.** First of all let’s note that from the function’s convexity follows its continuity [7].

Next according to theorem 1, each from sets \( \Omega_1, \Omega_2, \ldots, \Omega_m \) is polyhedral. Let
\[
G_i = \{ (x, f(x)) : x \in \Omega_i \}, \ i = 1, m.
\]

As far as \( f(x) \) is linear on \( \Omega_1, \Omega_2, \ldots, \Omega_m \), then \( G_1, G_2, \ldots, G_m \) are convex polyhedrons, moreover
\[
\text{gr}(f) = G_1 \cap G_2 \cap \ldots \cap G_m.
\]
As \( f \) is convex function, then \( \text{epi}(f) \) is convex polyhedron.

**Theorem 5.** If \( f_1(x), f_2(x), \ldots, f_m(x) \) are polyhedral functions, and \( f(x) \) is maximum function:
\[
f(x) = \bigvee_{i=1}^{m} f_i(x),
\]
then \( f \) is polyhedral function.

**Proof.** According to the proposal 8 the equality \( (1) \) takes place, so \( \text{epi}(f) \) is convex polyhedron and hence \( f \) is polyhedral function.

Next the representation of polyhedral function in the form of \( (4) \) we’ll call its *disjunctive expansion*.

Proposal 7 and theorem 5 point to the possibility of new polyhedral functions structure from available functions by means of three operations: multiplication by positive scalar, addition and pointwise maximum.

As example we’ll cite the following polyhedral functions:

1) \( n = 1; \ f_1(x) = |x| = \max \{-x, x\}; \)
2) \( n = 2; \ f_2(x) = \max \{-x_1, -x_2, x_1 + x_2\}; \)
3) \( n = 3; \ f_3(x) = \max \{|x_1|, |x_2|, |x_3|\}; \ f_4(x) = |x_1| + |x_2| + |x_3|.

The function \( f_1(x) \) is the simplest nonlinear polyhedral function. Let’s note, that set level \( C_\alpha \) of the given functions for \( \alpha > 0 \) is not empty, and also for functions \( f_2(x), f_3(x) \) and \( f_4(x) \) it’s triangular, cube and octahedron respectively.
The following proposal being proved simply gives one more method of polyhedral functions formation.

**Proposal 10.** If \( f : \mathbb{R}^k \to \mathbb{R} \) is polyhedral function, then for every matrix \( H \in \mathbb{R}^{k \times n} \) the complex function \( g : \mathbb{R}^n \to \mathbb{R} \) defined by equality

\[
g(x) = f(Hx),
\]

is also polyhedral one.

Thus by means of linear transformation of input polyhedral function argument we succeeded in getting a lot of other polyhedral functions.

**Polyhedral inequality.** Inequalities in the form of

\[
g(x) \leq C,
\]

where \( g(x) \) is some polyhedral function, and \( C = \text{const} \) we’ll call **polyhedral inequalities**.

According to proposal 9 many solutions of inequality (5) is convex polyhedron.

**Proposal 11.** Each final system of polyhedral inequalities

\[
g_k(x) \leq 0, \quad k = \overline{1, r},
\]

one may represent as unique polyhedral inequality

\[
g(x) \leq 0,
\]

where

\[
g(x) = \bigvee_{k=1}^{r} g_k(x).
\]

This proposal follows from the definition of pointwise maximum operation. Let’s emphasize its methodological sense: each totality of polyhedral boundaries one may represent in aggregative form (5).

Two follows proposals being proved easily reveal technology reduction of polyhedral inequalities to linear inequalities system.

**Proposal 12.** Let

\[
g(x) = \bigvee_{j=1}^{q} \psi_j(x),
\]

where \( \psi_j(x) \) are linear functions. Then polyhedral inequality (5) one may imagine as equivalent system of linear inequalities

\[
\psi_j(x) \leq C, \quad j = \overline{1, q}.
\]

**Proposal 13.** Let

\[
g(x) = g_1(x) + g_2(x) + \ldots + g_m(x),
\]

where \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{X} = \mathbb{R}^n \), and \( g_1(x), g_2(x), \ldots, g_m(x) \) are polyhedral functions. Then polyhedral inequality (5) is decomposed in the following inequalities system.

\[
x_{n+1} + x_{n+2} + \ldots + x_{m+n} \leq C,
\]

(6)

\[
g_i(x) \leq x_{n+i}, \quad i = \overline{1, m}.
\]

(7)

Let’s note the transfer from inequality (5) to inequality system (6), (7) in produced via introducing further variables \( x_{n+1}, x_{n+2}, \ldots, x_{m+n} \) which are **majorants** of polyhedral summands \( g_1(x), g_2(x), \ldots, g_m(x) \). Thus solution set totality is extended and now agree with \( \mathbb{R}^{n+m} \).
The application meaning of the last proposal consists in decomposition possibility in linear inequalities of each from polyhedral inequalities (7). So, for example the inequality

$$|x_1| + |x_2| + ... + |x_m| \leq C$$

one may decompose in linear inequalities system

$$x_{n+1} + x_{n+2} + ... + x_{n+m} \leq C,$$

$$-x_{n+1} \leq x_i \leq x_{n+1}, \quad i = 1, m.$$  

3. Polyhedral programming problems

By polyhedral programming (PP) problems we mean class MP problems with polyhedral goal and bounding functions:

$$\text{extr} \left\{ f(x) : g_i(x) \leq 0, i = 1, r \right\},$$  \hspace{1cm} (8)

where $x \in \mathbb{X}$, and $f(x), g_1(x), g_2(x), ..., g_r(x)$ are polyhedral functions. In doing so set

$$D = \left\{ x : g_i(x) \leq 0, i = 1, r \right\}$$

we’ll call the area of admissible solutions (or admissible area for short), and vector $x \in D$ we’ll call the admissible solution of the given problem.

It’s important to emphasize that according to proposal 11 without bound of generality in statement of PP problems the polyhedral boundaries one may formalize by one bounding function:

$$g(x) \leq 0.$$  \hspace{1cm} (9)

All solutions totality of polyhedral inequality (9) determines set $D$ and it’s clear that it’s polyhedral set.

The problem (8) we’ll call solvable if $D \neq \emptyset$ and extremum exists. In this case number

$$f^* = \text{extr} \left\{ f(x) : x \in D \right\}$$

will be named the optimal meaning of goal function; set $D^* = \left\{ x \in D : f(x) = f^* \right\}$ will be named the optimal solutions set, and vector

$$x^* = \arg \text{extr} \left\{ f(x) : g(x) \leq 0 \right\}$$

we’ll call the optimal solution of given problem.

Let’s note the principle singularity of PP problems distinguished them from sequence of other MP problems. So if the problem of goal function maximum search in MP problems (including LP problems) is automatically reduced to minimization problem of inverse goal function then between PP problems on maximum and minimum of goal function are essential distinctions determined by the polyhedral goal functions are convex one. As far as in the future we may be in need of PP problem solution on minimum then let’s consider only this problem.

By PP problems on minimum we means the problems in the form of

$$\mathbf{3}_0 : \quad \min \left\{ f(x) : g(x) \leq 0 \right\},$$

where $x \in \mathbb{X}$; $f(x), g(x) : \mathbb{X} \to \mathbb{R}$ are polyhedral functions.

The area of tolerable solutions $D$ of the problem $\mathbf{3}_0$ is all solution totality of polyhedral inequality (9).

Let’s consider along with input problem $\mathbf{3}_0$ the auxiliary optimization problem in extended space $\bar{\mathbb{R}} = \mathbb{X} \times \mathbb{R}$:

$$\mathbf{3}_0 : \quad \bar{f}(\bar{x}) = x_{n+1} \to \min,$$

$$\bar{g}_i(\bar{x}) = f(x_1, x_2, ..., x_n) - x_{n+1} \leq 0,$$
\[ \mathcal{g}_2(\bar{x}) = g(x_1, x_2, \ldots, x_n) \leq 0. \]

where \( \bar{x} = (x_1, x_2, \ldots, x_n, x_{n+1}) \in \bar{\mathbf{R}} \). Thus this problem is \((n+1)\)-dimensional, moreover the goal function \( \bar{f} \) and bounding functions \( \mathcal{g}_1, \mathcal{g}_2 \) are polyhedral one.

**Theorem 7.** Input problem \( \mathcal{P}_0 \) and auxiliary problem \( \mathcal{P}_0^* \) are equivalent:
1) each of them is solvable when the other one is solvable;
2) the goal functions optimal meanings of both problems are coincided:
\[ f^* = \bar{f}^*; \]
3) if \( x^* \in X \) is optimal solution of input problem then \( \bar{x}^* = (x^*, f^*) \) is optimal solution of auxiliary problem and conversely.

**Proof.** Let’s note, that the admissible area of auxiliary problem \( \bar{D} \) is the epigraph subset of input problem goal function and is determined by admissible area of input problem \( D \):
\[ \bar{D} = \{(x, \mu) \in \text{epi}(f) : x \in D\}. \]

Hence it’s not difficult to make sure of the correctness of theorem 1 conclusion.
Let’s assume that both problems are solvable. Then taking into account the function \( f(x) \) continuity and proposal 4 we have
\[ f^* = \min_{x \in D} f(x) = \min \min_{x \in D} \mu \in \mathbb{R} \{(x, \mu) \in \text{epi}(f)\} = \min_{x \in \bar{D}} \bar{f}(\bar{x}) = \bar{f}^*, \]

that corresponds to the conclusion of theorem 2.

At last for proof of theorem 3 conclusion it’s sufficient to check that
\[ x^* \in D^* \iff (x^*, f^*) \in \bar{D}^*. \]

**Theorem 8.** Each PP problem on minimum is reduced to some LP problem.

**Proof.** Let’s turn to PP problem on minimum \( \mathcal{P}_0 \).

According to theorem 3 the polyhedral functions \( f(x) \) and \( g(x) \) are maximum functions of some linear functions. Let
\[ f(x) = \bigvee_{i=1}^{p} \varphi_i(x), \quad g(x) = \bigvee_{j=1}^{q} \psi_j(x), \quad (10) \]

where \( \varphi_i(x), i = 1, p \) and \( \psi_j(x), j = 1, q \) are linear functions.

Let’s transfer from input problem \( \mathcal{P}_0 \) to auxiliary optimization problem \( \bar{\mathcal{P}}_0 \). The last one on the strength of (10) and definition of pointwise maximum operation one may note in the form of the general LP problem:
\[ \bar{\mathcal{P}}_0: \]
\[ \bar{f}(x_1, x_2, \ldots, x_n, x_{n+1}) = x_{n+1} \rightarrow \min, \]
\[ \varphi_i(x_1, x_2, \ldots, x_n) - x_{n+1} \leq 0, i = 1, p, \]
\[ \psi_j(x_1, x_2, \ldots, x_n) \leq 0, j = 1, q. \]

The following solution method of input problem \( \mathcal{P}_0 \) that is PP problem on minimum corresponds to the stated structures in proof of theorem 8:
- it’s introduced the further variable \( x_{n+1} \) that is goal function majorant of the problem \( \mathcal{P}_0 \);
- the input problem \( \mathcal{P}_0 \) is changing to the extended problem \( \bar{\mathcal{P}}_0 \) with linear goal function \( x_{n+1} \).
• from the problem $\mathcal{H}_0$ by means of decomposition of its polyhedral bounds in linear bounds LP problem $\mathcal{H}_1$ is got;
• the received optimal solution of the last problem gives the input problem solution.

Let’s note what PP effectiveness is specifically determined by the reducibility of PP problems, methods and software solutions of which are well-known. Thus, for example for the last one solution it’s possible to use LP88 program or Matlab system.

As bibliographical commentary for PP let’s note that the best approximation problem of PL Chebyshev functions, which is the classical object of many MP methods application, must be taken as the historically first optimization problem using the polyhedral goal function. The papers of the beginning of the fifties years of G.Sh. Rubinshtejn and S.I. Zuhovitsky connected with the Chebyshev approximation problem solution are the first papers where he optimization problems with polyhedral goal function by MP methods had been solved. It should be noted that the idea of input optimization problems reduction to LP problems by means of the additional variable introduction majorized the goal function used as the basis for PP at first was realized in the paper [10], moreover it was often used by the authors under the different maxmin and minmax solution.

4. Polyhedral norms and metrics
The mathematical structure of many applied optimization problems is determined by means of choice of vectors norms in linear space solutions $X$. For norm $\|x\|$ of vector $x \in X$ the following axioms are to be realized:

1) $\|x\| > 0$, if $x \neq 0$; $\|x\| = 0 \iff x = 0$;
2) $\|\lambda x\| = |\lambda| \|x\|$ $\forall \lambda \in \mathbb{R}$ and $x' \in X$ (absolute homogeneity);
3) $\|x + x'\| \leq \|x\| + \|x'\|$ (triangle inequality).

Norm space is become metric if for distance $\rho(x, x')$ between points $x, x' \in X$ we assume

$$\rho(x, x') = \|x - x'\|.$$ 

In arithmetical linear space $X = \mathbb{R}^n$ Euclidean vectors norm $x = (x_1, x_2, \ldots, x_n) \in X$: $\|x\|_E = \sqrt{x \cdot x}$, which generates the corresponding Euclidean metric is used. The choice of the given norm is very traditional, but not always justified because complicates the optimization problem solution. The introduction of polyhedral norm and polyhedral metric may gives the essential advantages, making possible to formalize the applied optimization problem as PP problem.

**Polyhedral metric** Norm being polyhedral components function we’ll call polyhedral one. The wide extension has the following polyhedral norms.

$$\|x\| = \sum_{i=1}^{n} |x_i|, \|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$$

They are known as octahedronic and cubic (uniform or chebyshev) norms respectively. The given polyhedral norms determine the following two polyhedral metrics:

• the distance of Minkovsky $\rho_1(x, x') = \|x - x'\|_1$;
• the distance of Chebyshev $\rho_\infty(x, x') = \|x - x'\|_\infty$.

Let’s consider the linear forms:

$$\varphi_i(x) = \langle d_i, x \rangle,$$  \hspace{1cm} (11)

where $d_i \in X$, $i = 1, m$, and also the we’ll consider function of their maximum
Vectors totality \( \{ \mathbf{d}_i, i = 1, \ldots, m \} \) we’ll call comprehensive [11] if for any \( x \neq 0, x \in \mathbf{X} \), it will be found index \( j \in \{ 1, 2, \ldots, m \} \) so, that \( \varphi_j(x) > 0 \).

Many-valued mapping \( I^+(x) : \mathbf{X} \to \{ 1, 2, \ldots, m \} \), meanings of which are index sets
\[
I^+(x) = \{ i \in \{ 1, 2, \ldots, m \} : \varphi_i(x) = h(x) \}
\]

Let’s connect with polyhedral function \( h(x) \).

**Theorem 9.** Polyhedral function \( h(x) \) answers axioms of norm if and only if it’s maximum function (12) of some linear forms (11) where vectors totality \( \{ \mathbf{d}_i, i = 1, \ldots, m \} \) is comprehensive.

**Proof.** It’s easy to check that for polyhedral function \( h(x) \) satisfying hypotheses of the theorem axioms of norm are reduced. Really axiom 1 follows from (11), (12), and also from the property of comprehensive vectors totality \( \{ \mathbf{d}_i, i = 1, \ldots, m \} \); the realization of axiom 2 is stipulated by homogeneity of functions (11); axiom 3 follows from function’s convexity \( h(x) \).

Let’s prove that if for polyhedral function axioms of norm are fulfilled then it answers hypotheses of the theorem.

According to theorem 3 the function \( h(x) \) is reduced to the form of (12), where \( \{ \varphi_i(x), i = 1, \ldots, m \} \) are linear functions in the form of (2).

The requirement of absolute homogeneity means that for \( \lambda > 0 \)
\[
h(x) = \lim_{\lambda \to \infty} \frac{1}{\lambda} h(\lambda x),
\]
from where it follows for \( i \in I^+(x) : \)
\[
\varphi_i(x) = \lim_{\lambda \to \infty} \frac{1}{\lambda} \varphi_i(\lambda x) = \langle \mathbf{d}_i, x \rangle,
\]
that is linear functions \( \varphi_i(x) \) are linear forms.

From (11) and (12) it follows that \( h(0) = 0 \). Then if \( x \neq 0 \), then for \( i \in I^+(x) \)
\[
h(x) = \varphi_i(x) > 0,
\]
that is vectors totality \( \{ \mathbf{d}_i, i = 1, \ldots, m \} \) is comprehensive one.

Let’s consider the examples of applied problems formalization in PP terms on the base of polyhedral norms and metrics application.

5. Optimization of discrete control process by mathematical programming methods

Many control processes by social, economic and technical objects are discrete by nature. In these processes the check of state object and control by it is realized at discrete instant time. The given processes obtained main important meaning in connection with the introduction in control practice of computing machinery means.

In modern control theory for optimization of control processes the MP method being powerful and universal mean of finite-dimensional optimization problems solution discover more and more application. In doing so the particular function has the LP methods. This is the most mature and developed MP division. The point is that the LP methods stably were contained in engineering practice for different applied optimization problems solution. They are fully completed by an algorithm and quit effective software are created for them.
At first LP was used to optimal control problems in the papers of Manne [12], Zadeh and Whalen [13], Gnoensky and Movshovich [14]. At present a great many publications both native and foreign authors dedicated to use of LP apparatus for optimization of control processes (for example [15-24]). In all these papers the input optimal control problem in either case is reformulated for some MP problem and then is used any method of reduction of the last one to LP problems.

But in spite of most LP popularity in scientific and engineering developments the linear structure of LP problems considerably narrows the area of its application for optimization problems of control processes by dynamic objects. The polyhedral structure of PP essentially extends the area of its application to the given optimization problems.

Then the PP apparatus application for optimization of discrete control processes with polyhedral qualities criterion is discussed. The formalization of statement in PP terms is given and the solution of two optimal control problems by discrete dynamic objects is reduced. They are the control problem by final state and terminal stabilization problem. The possibility of the \textit{direct phase and resource boundary discount} for control processes occupy a peculiar position. The given results are the natural development and generalization of the papers [5, 25–44].

6. Optimal control problems by final state of discrete objects

Let the dynamic of discrete control object be described by linear difference equation in the form of

$$x(t + 1) = Ax(t) + Bu(t),$$

where \( t \in \mathbb{Z}_+ = \{0, 1, 2, \ldots\} \) is discrete time; \( x \in X = \mathbb{R}^n \) is state vector and \( u \in U \subseteq \mathbb{R}^r \) is control vector of object respectively; \( X \) and \( U \) are state space and control area respectively, moreover \( U \) is \( r \)-dimensional polyhedron; \( A \in \mathbb{R}^{n \times n} \) and \( B \in \mathbb{R}^{n \times r} \) are coefficient matrix.

One of the central places in modern theory and control practice occupy the control problems by final object state where the control purpose consists in transfer of its state at final time to goal set of states \( X^* \subset X \) with regard for the given resource and phase bounds for control process. The application area of control systems by final state is quit wide and actual: beginning with automation problems of controlled programming technological equipment and ending with control by moving objects of air-rocket-cosmic techniques. Let’s assume that \textit{control area} \( U \), characterizing \textit{resource bounds of control process} is polyhedral set and is determined by polyhedral inequality

$$\mu(u) \leq M \ (M = \text{const}).$$

Here \( \mu(u) \) is some polyhedral function:

$$\mu(u) = \bigvee_{i=1}^{p} \mu_i(u),$$

where \( \mu_i(u), \ i = 1, \ldots, p \) are linear functions.

Let’s assume that \textit{goal set} \( X^* \subset X \) is nonempty polyhedral set and is determined by polyhedral inequality

$$\gamma(x) \leq \Gamma \ (\Gamma = \text{const}),$$

where \( \gamma(x) \) is some polyhedral function, given by linear functions \( \gamma_j(x), \ j = 1, \ldots, q \):

$$\gamma(x) = \bigvee_{j=1}^{q} \gamma_j(x).$$
Let’s consider the following *time-optimal control problem by final state* (3): it’s required to transfer the control object (12) from the given initial state \( x(0) = x_0 \) (\( x_0 \not\in X^* \)) to the goal set \( X^* \) for the shortest time:

\[
x(T) \in X^*, \quad T \rightarrow \min .
\]

For the given problem solution let us take the idea of N.N. Krasovsky of the time-optimal problem reduction to class of terminal control problems [45].

Let’s consider the following *auxiliary terminal control problem* (3_3_1): it’s required to transfer the control object (12) from the initial state \( x_0 \) for the given (terminal) instant time \( t = T \) to terminal state \( x(T) \), being extremal in sense of quality criterion:

\[
\gamma(x(T)) \rightarrow \min \quad (13)
\]

Then solving class of the auxiliary problems \( 3_{3_1}, T = 1, 2, \ldots \), it’s possible to find the optimal control time by object \( T^* \) as the smallest from numbers \( T \), satisfying condition

\[
T^* = \min \left\{ T : \gamma(x(T)) \leq T \right\},
\]

Besides the corresponding auxiliary problem \( 3_{3_1} \) solution will give the input problem 3 solution.

Let’s turn to the auxiliary problem solution \( 3_{3_1} \). It’s clear that terminal object’s state \( x(T) \) is determined by control programming in the form of

\[
x(T) = A^T x_0 + \sum_{i=0}^{T-1} A^{T-i-1} B u(t), \quad (14)
\]

where

\[
\mu(u(t)) \leq M, \quad t = 0, T - 1. \quad (15)
\]

According to (14) terminal criterion in (13) is polyhedral functional of control program:

\[
\gamma = \gamma(u(t)), \quad 0 \leq t \leq T - 1,
\]

and so on the strength of polyhedral of resource bounds (15) the considered optimization problem of terminal control \( 3_{3_1} \) is the PP problem on minimum. Taking advantage of solution method of the given problem stated above let us reduce it to the following general LP problem with space of dimension solutions \( N = r \times T + n + 1 \):

\[
3_{3_1}: \quad z_N \rightarrow \min .
\]

\[
z_{T+1} = A^T x_0 + \sum_{i=0}^{T-1} A^{T-i-1} B z_{t+1},
\]

\[
\mu_i(z_i) \leq M, \quad t = 1, T, \quad i = 1, p,
\]

\[
\gamma_j(z_{T+1}) \leq z_N, \quad j = 1, q,
\]

as relating to variables: \( z_i \in \mathbb{R}^r, \quad i = 1, T, \quad z_{T+1} \in \mathbb{R}^n, \quad z_N \in \mathbb{R} \).

The extremal solution of the given problem \( z_{3_1} = (z_1, z_2, \ldots, z_T, z_{T+1}, z_N) \) determines the unknown optimal program of terminal control by object, it is optimal terminal state and meaning of optimal quality criteria:

\[
u(t) = z_{t+1}, \quad 0 \leq t \leq T - 1,
\]
\[ x(T) = z_{T+1}, \]
\[ \gamma(x(T)) = z_N. \]

The problem \( 3_T \) is the base one for input optimization problem \( 3 \) and the search of the last problem solution it’s possible to consider as multistep process on each step of which the base problem \( 3_T \) is solved and if \( z_N > \Gamma \) then terminal instant time \( T \) is increased on unit: \( T = T + 1 \). As a result the first base problem solution \( 3_T \) satisfying inequality \( z_N \leq \Gamma \) will be conform to the first hit of controlled object terminal state \( x(T) \) to goal set \( X^* \). The given solution just will defines the unknown shortest time of control by object \( T^* \), and the corresponding control program \( \{ u(t), t = 0, T - 1 \} \) will give the unknown solution of input \( 3 \).

Thus the application of PP apparatus makes possible to reduce solution of time-optimal control problem by final object state to solution of class LP problems.

It is necessary to note that considered solution method of optimal control problems makes possible to take into account the given phase bounds on admissible trajectory of controlled object movement: for this it’s necessary to present the given bounds in the form of polyhedral inequalities and include them in statement of base problem.

Let’s separately discuss two-point control problem, assuming the goal state \( x^* \) is given, that is \( X^* = \{ x^* \} \).

Let’s choose in \( X \) some polyhedral metric \( \rho(x, x') \), where \( x, x' \in X \), and will be estimate the remoteness of current object state \( x \) from its goal state \( x^* \) by means of this metric: then the process of control by object consists in the direction of its movement to goal state \( x^* \).

If the polyhedral metric \( \rho(x, x') \) we’ll take as goal function \( \gamma(x) = \rho(x, x^*) \) then the solution of considered optimization problem is based on stated above scheme of time-optimal control problem solution: it’s introduced into consideration the auxiliary terminal control problem \( 3_T \) with optimal criterion

\[ \gamma(x(T)) = \rho(x(T), x^*), \]

which is PP problem on minimum and hence is reduced to class of LP problems \( 3_T \), moreover the iterated solution procedure of the last one are being continued till the realization of condition

\[ \rho(x(T), x^*) = 0. \]

Let’s note that in case of equilibrium goal object’s state: \( x^* = 0 \) the given problem is the problem of stabilization and consists in formation of control strategy, enveloping damping of perturbed object movement for the shortest time.

7. Optimal terminal stabilization problem of discrete objects

Let discrete non-stationary control object be described by linear difference equation in the form of (12), where \( t = 0, T \) is discrete time, \( t, T \in \mathbb{Z}_+ \), \( \mathbb{Z}_+ \) is nonnegative numbers set; \( x \in \mathbb{R}^n \) is state vector, \( u \in \mathbb{R}^r \) is control vector of object respectively; \( A(t) \) and \( B(t) \) are variable matrices of object’s state and control, \( A : [0, T] \rightarrow \mathbb{R}^{m \times n} \) and \( B : [0, T] \rightarrow \mathbb{R}^{n \times r} \).

Let’s assume that an equilibrium (non-perturbed) object’s state \( x^* = 0 \) which it’s necessary to stabilize by linear feedback in the form of

\[ u(t) = -K(t)x(t), \text{ } t = 0, T - 1, \]
where $K(t)$ is feedback variable matrix, $K : [0, T - 1] \rightarrow \mathbb{R}^{n}$, is the goal state besides for any (perturbed) initial state $x(0) = x_0 \neq 0$ the object are to be damped at the final (terminal) instant time $t = T$:

$$x(T) = 0.\]

In very general formalization the problem of optimal synthesis of stabilizing terminal feedback (17) consists in choice of finite sequence

$$K = (K(0), K(1), \ldots, K(T - 1)),\]$$

being optimal by some criterion $J(K)$:

$$K^* = \arg \min J(K). \quad (18)$$

The quality of control process by object is determined by the quality of its output reactions.

Let $u(t) = u(t, x_0, K)$ and $x(t) = x(t, x_0, K)$ be current meanings of control and state in closed system (12), (17) at its movement from the state $x_0$, and $\Delta u(t)$ and $\Delta x(t)$ are rates of change respectively:

$$\Delta u(t) = u(t + 1) - u(t), \quad \Delta x(t) = x(t + 1) - x(t).$$

For quality estimation of control process by object it’s possible to use the polyhedral quality criteria in the form of the following loss functionals taking into account the dynamic structure of phase trajectories and controlling actions:

$$J(x_0, K) = \sum_{t=0}^{T-1} Q(x(t), \Delta x(t), u(t), \Delta u(t));$$

$$J(x_0, K) = \max_{0 \leq t \leq T - 1} Q(x(t), \Delta x(t), u(t), \Delta u(t)). \quad (19)$$

Here $Q(x(t), \Delta x(t), u(t), \Delta u(t))$ is goal polyhedral function in the form of

$$Q(x(t), \Delta x(t), u(t), \Delta u(t)) = \lambda_1(t) q_1(x(t)) + \lambda_2(t) q_2(\Delta x(t)) + \lambda_3(t) q_3(u(t)) + \lambda_4(t) q_4(\Delta u(t)),$$ \quad (21)

where $q_1 : \mathbb{R}^n \rightarrow \mathbb{R}, \ q_2 : \mathbb{R}^n \rightarrow \mathbb{R}, \ q_3 : \mathbb{R}^r \rightarrow \mathbb{R}, \ q_4 : \mathbb{R}^r \rightarrow \mathbb{R}$ are some positively homogeneous polyhedral functions; $\lambda_i(t) \geq 0, \ t = 0, T - 1, \ t = 1, 4$, are weight coefficient besides

$$\lambda_1(t) + \lambda_2(t) + \lambda_3(t) + \lambda_4(t) > 0, \ t = 0, T - 1.$$ 

In particular one may consider:

$$q_1(x) = |x_1| + |x_2| + \ldots + |x_n|, \ q_2(\Delta x) = 0,$$

$$q_3(u) = |u_1| + |u_2| + \ldots + |u_r|, \ q_4(\Delta u) = 0,$$

$$\lambda_i(t) = v^r, \text{where} \ v \in \mathbb{Z}_+, \ \lambda_4(t) = 1.$$ 

Let’s assume that it’s given the polyhedron $X_0 \subset X$, presenting the indeterminacy of initial state of object $x_0$. Then the quality of synthesized system (12), (17) it’s possible to estimate by loss functionals (19), (20), determined on class of its transfer processes $x(t) = x(t, x_0, K), \ x_0 \in X_0$:

$$J(K) = \max_{x_0 \in X_0} J(x_0, K). \quad (22)$$
In doing so the initial state \( \mathbf{x}_0^* \), on which maximum of the given functionals is achieved and the corresponding transient process in closed we’ll call stabilization system the worst one.

As a result the input synthesis problem of terminal stabilization (18) it’s possible to set up as the parametric optimization problem of system in sense minimax:

\[
J(K^*) = \min_k \max_{\mathbf{x}_0 = \mathbf{x}_0^*} J(\mathbf{x}_0, K).
\]

(23)

Here the choice of control strategy (23) answers the so called guaranteed result principle.

Let \([\mathbf{X}_0]\) be set of polyhedron tops \( \mathbf{x}_0 \).

**Theorem 7.** For every feedback (17) the worst in sense criterion (23) the initial object’s state (12) coincides with one of the polyhedron tops \( \mathbf{x}_0^* \in [\mathbf{X}_0] \).

**Proof.** Denote by \( \Phi \) transfer matrix of closed system (12), (17). Then it’s evident

\[
\mathbf{x}(t) = \Phi(t, 0)\mathbf{x}_0^*,
\]

(24)

and according with (17),

\[
\mathbf{u}(t) = -K(t)\Phi(t, 0)\mathbf{x}_0^*.
\]

(25)

The permutation of the given expression for criterion (19) gives the polyhedral goal function as relating to \( \mathbf{x}_0 \). The same result takes place and for criterion (20) on the strength of theorem 3.

It is possible to show [5] that for fixed \( K \) the extremal PP problem on maximum

\[
\max_{\mathbf{x}_0 = \mathbf{x}_0^*} J(\mathbf{x}_0, K)
\]

is also reduced to LP problem and so if it’s solved, then extremum is achieved at some top of polyhedron \( \mathbf{X}_0 \).

Thus, according to theorem 7 the polyhedral goal functions (19) and (20) achieve maximum meanings at one of the polyhedron top \( \mathbf{x}_0 \). The given result permits to simplify the optimization problem (23): to change the continual set \( \mathbf{X}_0 \) into final set \( [\mathbf{X}_0] \) that is to reduce it to more simple problem on minimax:

\[
J(K^*) = \min_k \max_{\mathbf{x}_0 = \mathbf{x}_0^*} J(\mathbf{x}_0, K),
\]

(26)

Let’s state the problem solution method (26). Let’s write the solution of the equation (12) in the form of

\[
\mathbf{x}(t) = \Phi_0(t, 0)\mathbf{x}_0 + \sum_{\tau = 0}^{t-1} \Omega_0(t, \tau)\mathbf{u}(\tau),
\]

(27)

where \( \Phi_0 \) and \( \Omega_0 \) are the transfer and impulse transfer matrices of object.

Let’s represent the optimal control in the form of program linearly parametrized by initial object’s state \( \mathbf{x}_0^* \):

\[
\mathbf{u}(t) = \mathbf{P}(t)\mathbf{x}_0^*,
\]

(28)

where \( \mathbf{P}(t) \), \( t = 0, T - 1 \), are the unknown matrix.

Let’s introduce finite sequence \( \mathbf{P} = (\mathbf{P}(0), \mathbf{P}(1), \ldots, \mathbf{P}(T - 1)) \). Then the permutation (28) in (27) gives (24), where

\[
\Phi(t, 0) = \Phi_0(t, 0) + \sum_{\tau = 0}^{t-1} \Omega_0(t, \tau)\mathbf{P}(\tau).
\]

(29)
The relations (28), (24) and (29) permit to represent the considered quality criterions (19), (20) as polyhedral functionals relative to functional matrix \( P(t) \):

\[
J(x_0, K) = \text{\( \tilde{J}(x_0, P) \)},
\]

as a result of this instead of extremal problem (26) we get the following problem on minimax:

\[
\text{\( \tilde{J}(P^*) = \min_P \max_{x_0 \in \{x_0\}} \text{\( J(x_0, P) \)} \)).
\]

The given problem is PP problem on minimum 3 and consequently as stated above may be reduced to class of equivalent general LP problems 3. The key meaning here has the majorants introduction of goal and bounding polyhedral functions for formation of vector problem solution 1.

Let’s assume that as a result of the given LP problem solution are defined matrices \( P(t), t = 0, T - 1 \). Then calculating \( \Phi(t, 0) \) according to formula (29) and comparing (25) and (28), we’ll find the unknown parameters \( K(0), K(1), \ldots, K(T - 1) \) of feedback (17):

\[
K(t) = P(t)\Phi^{-1}(t, 0), \quad t = 0, T - 1.
\]

8. Numerical example of optimal control problem solution by polyhedral programming method

As illustration let us consider the time-optimal discrete control problem by linear object of the second order with transfer function on “input – output” channel in the form of [18]:

\[
W(s) = \frac{1}{s(s + 1)}.
\]

Let us assume that it’s realized the impulse control by object with period of digitization which is equal to 1 second, in addition to this in object’s channel fixer of order zero is used. Then let us take as variables of the object state the output and the rate of output change, the discrete object’s model one may represent via the following vector difference equation:

\[
x(t + 1) = Ax(t) + bu(t),
\]

where \( t \in \mathbb{Z}_+ = \{0, 1, 2, \ldots\} \) is discrete time; \( x = \text{col}(x_1, x_2) \) is vector of object’s state; \( u \) is the scalar controlling object’s input, and \( A \) and \( b \) are numerical matrices in the form of

\[
A = \begin{bmatrix}
1 & 1 - e^{-1} \\
0 & e^{-1}
\end{bmatrix}, \quad b = \begin{bmatrix}
e^{-1} \\
1 - e^{-1}
\end{bmatrix}.
\]

It sets up the following optimal-time control problem 3: it’s required to transfer the control object from zero initial state \( x(0) = 0 \) to goal state \( x^* = \text{col}(x'^*_1, x'^*_2) \) with components \( x'^*_1 = 5, \quad x'^*_2 = 0 \), for the least numbers of steps

\[
x(T) = x^*, \quad T \rightarrow \min,
\]

for the following resource and phase boundaries:

\[
|u| \leq 2, \quad |x_2| \leq 1.
\]

Let’s take as the goal function characterizing the remoteness of current state of object \( x \) from the goal \( x^* \), the polyhedral metric in the form of distances of Minkovksy:

\[
\gamma(x) = \rho_1(x, x^*) = \|x - x^*\| = |x_1 - x'^*_1| + |x_2 - x'^*_2|.
\]
The auxiliary terminal control problem \( 3_T \) consists in object’s transfer to terminal instant time \( t = T \) in terminal state \( x(T) \) with optimal criterion:

\[
\gamma(x(T)) = \left| x_1(T) - x_1^* \right| + \left| x_2(T) - x_2^* \right| \to \min .
\]

Let’s denote by

\[
z_i = u(t + 1), \quad i = 1, T ;
\]

\[
z_{T+1} = \left| x_1(T) - x_1^* \right|, \quad z_{T+2} = \left| x_2(T) - x_2^* \right| .
\]

Then base problem \( 3_T \) for input problem \( 3 \) may be represented in the form of the following LP problem with space of solutions of \( N = T + 2 \)-dimension:

\[
(z_{T+1} + z_{T+2}) \to \min ;
\]

\[
- \begin{bmatrix} z_{T+1} \\ z_{T+2} \end{bmatrix} \leq \begin{bmatrix} \sum_{i=1}^{T} A^T \cdot b \cdot z_i \\ x_1^* \\ x_2^* \end{bmatrix} \leq \begin{bmatrix} z_{T+1} \\ z_{T+2} \end{bmatrix} ;
\]

\[
-1 \leq [0, 1] \begin{bmatrix} \sum_{i=1}^{T} A^T \cdot b \cdot z_i \end{bmatrix} \leq 1, \quad \tau = 1, T ;
\]

\[
-2 \leq z_i \leq 2, \quad i = 1, T ;
\]

containing the variables \( z_i, \quad i = 1, T + 2 \).

Solution \( z = (z_1, z_2, \ldots, z_{T+2}) \) of the given LP problem for some \( T \in Z_+ \) defines the optimal \( T \)-step terminal control program by object and the meaning of quality criterion, corresponding to it:

\[
u(t) = z_{t+1}, \quad 0 \leq t \leq T - 1,
\]

\[
\gamma(x(T)) = z_{T+1} + z_{T+2} .
\]

Minimal number of steps \( T^* = T \), providing the realization of terminal condition in the form of

\[
\gamma(x(T)) = 0,
\]

will define the shortest time of achievement by goal state object \( x^* \).

Thus the solution of the input problem \( 3 \) is \( T^* \)-step control program \( \{ u(t), t = 0, T^* - 1 \} \), determined by solution \( z = (z_1, z_2, \ldots, z_{T+2}) \) of the represented LP problem for

\[
T^* = \min \{ T \in Z_+ : z_{T+1} = 0; z_{T+2} = 0 \} .
\]

For considered problem \( T^* = 6 \). Phase trajectories of controlled object getting as the result of sequential LP problem solution for \( T = 1, 2, \ldots, 6 \) are represented on figure 1.

Let’s represent the corresponding solutions of base LP problems:

- \( T = 1: u(0) = 0; \quad \gamma(x(1)) = 5 \);
- \( T = 2: u(0) = 1.582, \quad u(1) = -0.582; \quad \gamma(x(2)) = 4 \);
- \( T = 3: u(0) = 1.582, \quad u(1) = 1, \quad u(2) = -0.582; \quad \gamma(x(3)) = 3 \);
- \( T = 4: u(0) = 1.582, \quad u(1) = 1, \quad u(2) = 1, \quad u(3) = -0.582; \quad \gamma(x(4)) = 2 \);
- \( T = 5: u(0) = 1.582, \quad u(1) = 1, \quad u(2) = 1, \quad u(3) = 1, \quad u(4) = -0.582; \quad \gamma(x(5)) = 2 \);
- \( T = 6: u(0) = 1.582, \quad u(1) = 1, \quad u(2) = 1, \quad u(3) = 1, \quad u(4) = 1, \quad u(5) = -0.582; \quad \gamma(x(6)) = 0 \).
Figure 1. Phase trajectories of controlled object.

The optimal control program and object movement are represented on figures 2, 3.

Figure 2. The optimal control program.

Figure 3. Object movement characteristics.
Using as a goal function of Chebyshev polyhedral metric:

$$\gamma(x) = \rho_{\infty}(x, x^*) = \|x - x^*\|_{\infty} = \max \left| x_1 - x_1^* \right|, \left| x_2 - x_2^* \right|,$$

the auxiliary terminal control problem $\mathbf{3}_T$ is minimax:

$$\gamma(x(T)) = \max \left\{ \left| x_1(T) - x_1^* \right|, \left| x_2(T) - x_2^* \right| \right\} \rightarrow \min.$$ 

In this case the base problem $\mathbf{3}_T$ analogously is reduced to LP problem with space of solution $N = T + 1$-dimensional. Here phase trajectories of control object, obtained as a result of sequential solution of LP problem for $T = 1, 2, \ldots, 6$ are represented on figure 4. In spite of the fact that they rather differ from phase trajectories obtained in preceding case (see figure 1), the optimal control program and object’s movement in the both cases are completely coincided (see figures 2, 3).

9. Conclusion

Polyhedral methodology opens the real perspectives for wide application in theory and practice of automatic systems of creation of quality of chebyshev type.

Really the given methodology is directed to the solution of two cardinal problems of modern automatics connected with the problem of the direct exponents’ quality of control processes, in the first place, and secondly the problem of direct resource and phase on control processes. The application of the polyhedral criteria of quality leads to the new class of optimization problems that is the problems of polyhedral programming, algorithmization of which is based on powerful device and calculating methods of linear programming.

Polyhedral methodology of formalization of control problems is based on the wide spectrum of criteria of quality of polyhedral structure and also polyhedral phase and resource limits for control processes. The polyhedral approach is effective for the decision of some classical and modern problems of discrete control and observation in conditions full definiteness and uncertainty, in regular, critical and conflict situations, including problem.

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