ARITHMETIC PHASE TRANSITIONS FOR MOSAIC MARYLAND MODEL

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Abstract. We give a precise description of spectral types of the Mosaic Maryland model with any irrational frequency, which provides a quasi-periodic unbounded model with non-monotone potential has arithmetic phase transition.

1. Introduction
In this paper, we study the spectral property of almost-periodic unbounded Schrödinger operator

\((Hu)_n = u_{n+1} + u_{n-1} + \lambda v_n u_n,\)

where \(v_n\) is a unbounded almost-periodic sequence. An important example is

\((H_{\lambda \tan \alpha \theta} u)_n = u_{n+1} + u_{n-1} + \lambda \tan(\theta + n\alpha) u_n,\)

where \(\lambda \in \mathbb{R}\) is the coupling constant, \(\alpha \in \mathbb{R} \setminus \mathbb{Q}\) is the frequency, \(\theta \notin \frac{1}{2} + \alpha \mathbb{Z} + \mathbb{Z}\) is the phase. In the following, we just denote \(\Theta \triangleq \frac{1}{2} + \alpha \mathbb{Z} + \mathbb{Z}\), and we say “all \(\theta\), if \(\theta \notin \Theta\).

This model was first proposed by Grempel, Fishman, and Prange in 1982 [15] as a model stemming from the study of quantum chaos, and later dubbed the Maryland model by B. Simon [37]. In recent years, Maryland model has received extensive research due to the rich backgrounds in quantum physics [10, 14, 17, 34]. The richness of its spectral theory, abundance of unusual features, and amenability to analysis also make it a crucial component of general conjectures and counterexamples in mathematics.

It is worth noting that if the potential is unbounded, there is no absolutely continuous spectrum for all \(\theta\) [38]. As a result, it is natural to expect localized eigenfunctions. Grempel-Fishman-Prange [15] obtained in an essentially rigorous way, a dense set of explicitly determined eigenvalues, corresponding to exponentially decaying eigenfunctions, for Diophantine frequencies. Here \(\alpha\) is Diophantine, if there exist \(\gamma, \sigma > 0\), such that

\[\inf_{j \in \mathbb{Z}} |n\alpha - j| \geq \frac{\gamma}{|n|^{\sigma}} \forall n \neq 0.\]

Indeed for Diophantine frequencies \(\alpha\), Maryland model has Anderson localization: pure point spectrum with exponentially decaying eigenfunctions, for all \(\theta\) [13, 37]. Motivated by the Maryland model, Anderson localization for Maryland-type model have recently sparked tremendous interest in spectral theory of Schrödinger operator. In general, \(V\) is called Maryland-type potential, if \(V\) is a function

\[f : (-1/2, 1/2) \to (-\infty, +\infty), \quad f(-1/2 \pm 0) = \mp \infty,\]

and can be extended into \(\mathbb{R} \setminus \mathbb{Z} + 1/2\) by 1-periodicity. We call \(V\) is Lipschitz monotone if there exists \(\gamma > 0\) such that \(f(y) - f(x) \geq \gamma(y - x)\) for all \(0 < x < y < 1\). Using a KAM-type procedure, Béllissard, Lima, and Scoppola [9] obtained Anderson localization for a class of meromorphic functions \(V\) whose restrictions onto \(\mathbb{R}\) are also 1-periodic and Lipschitz monotone with Diophantine frequencies. This result [9] is perturbative, that is, once \(V\) is fixed, one can only obtain localization for the potential \(AV\) with \(\lambda \geq \lambda_0(\alpha)\), where \(\lambda_0\) depends on the Diophantine constant of \(\alpha\) and does not have a uniform lower bound on a full measure set of frequencies. Additionally, a large family of 1D quasiperiodic operators with unbounded monotone potentials and Diophantine frequencies were constructed in [30] as the non-perturbative outcome of Anderson localization. Another example of Maryland-type potential in a closely related setting is given in [31], where the authors show that Anderson localization can still be explored in operators with unbounded monotone potential, which are not required to be strictly monotone and can have flat segments under certain geometric conditions. As we can see, all the preceding results all assume that the potential is monotonic, a natural question is whether or not a non-monotone \(V\) exhibits Anderson localization. That is the primary motivation for our paper.
Our second motivation stems from the Maryland model’s phase transition. Before we go any further, we consider another more famous quasi-periodic model:

\[(H_{λ, α, θ}u)_n = u_{n+1} + u_{n-1} + 2λ\cos2π(nα + θ)u_n,\]

This model is known as almost-Mathieu operator (or Aubry-Andre model in physical literature), which is a bounded self-adjoint operator on \(ℓ^2(ℤ)\). The Almost-Mathieu operator (AMO) was first proposed by Peierls [35], as a model for an electron on a 2D lattice, acted on by a homogeneous magnetic field [20, 36]. AMO undergoes a phase transition at \(λ = 1\), where the Lyapunov exponent changes from zero everywhere on the spectrum [11] to positive everywhere on the spectrum [21]. Aubry-Andre conjectured [1], that at \(λ = 1\) the spectrum changes from absolutely continuous for \(λ < 1\) to pure point for \(λ > 1\). This has since been proved, for all \(α, θ\) for \(λ < 1\) [2, 4, 7, 32] and for Diophantine \(α, θ\) (so a.e.) for \(λ > 1\) [22]. Indeed, there exists second transition line from singular continuous spectrum to pure point spectrum. Let \(p_n/q_n\) be the continued fraction approximates of \(α\). The index \(β(α)\) that measures Liouvillessness of the frequency is defined as follows:

\[
β(α) = \limsup_{n→∞} \frac{\ln q_{n+1}}{q_n}.
\]

If \(1 < λ < e^β\), then \(H_{λ, α, θ}\) has purely singular continuous spectrum for all \(θ\) [8], and if \(λ > e^β\), then \(H_{λ, α, θ}\) has Anderson localization for \(γ(α, θ) = 0\) [8, 24], where we recall that

\[
γ(α, θ) = -\limsup_{k→∞} \frac{\ln ||2θ + kα||_R}{|k|}.
\]

Moreover, if \(\ln |λ| < γ(α, θ)\), \(H_{λ, α, θ}\) has purely singular continuous spectrum [25], and if \(\ln |λ| > γ(α, θ)\), then \(H_{λ, α, θ}\) has Anderson localization for \(β(α) = 0\) [25]. To summarize, AMO has two different types of resonances: frequency resonances and phase resonances, where \(β(α)\) measures exponential strength of the frequency resonances, and \(γ(α, θ)\) measure exponential strength of the phase resonances. The second transition line claims that the operator displays localization when the Lyapunov exponent beats frequency/phase resonances.

Let us return to the Maryland model. As previously stated, if the frequency \(α\) is Diophantine, the Maryland model has Anderson localization for all \(θ\) [13, 37]. Indeed it was shown by Jitomirskaya-Liu [23] that \(σ_{pp}(H_{λ \tan, α, θ})\) can be characterized arithmetically: by defining an index

\[
(1.1) \quad δ(α, θ) := \limsup_{n→∞} \frac{\ln q_{n+1} + \ln \|q_n(θ - \frac{1}{2})\|_R}{q_n},
\]

Jitomirskaya-Liu [23] show that

\[
(1.2) \quad σ_{pp}(H_{λ \tan, α, θ}) = \{E : L(E) ≥ δ(α, θ)\},
\]

while we have

\[
σ_{sc}(H_{λ \tan, α, θ}) = \{E : L(E) < δ(α, θ)\}
\]

where \(L(E)\) is the Lyapunov exponent. This makes the Maryland the first model where arithmetic spectral transition is described without any parameter exclusion. It should be noted that the proofs of localization in [23], as well as the original physics paper [15], is based on a Cayley transform [37] that reduced the eigenvalue problem to solving certain explicit cohomological equation. In [29], the authors provided a constructive proof for the localization component by expanding Jitomirskaya’s localization approach [22], obtaining Anderson localization for all \(θ\) and Diophantine \(α\). Quite recently, Han-Jitomirskaya-Yang [19] extended [29], gave a constructive proof of (1.2) for any irrational \(α\). More importantly, they investigated that, different from AMO, the Maryland model has another resonance: anti-resonance; this type of observation is critical in proving the arithmetic transition. The natural question is whether there are other quasi-periodic unbounded models that exhibit arithmetic phase transitions, and whether the monotonicity is an essential assumption.

To answer these questions, we study the following unbounded Schrödinger operator:

\[
(1.3) \quad (H_{V, α, θ}u)_n = u_{n+1} + u_{n-1} + λV_1(θ + \frac{nα}{2}, n)u_n = E u_n,
\]

where

\[
V_1(θ, n) = \begin{cases} \tan πθ, & n ∈ 2ℤ, \\ 0, & \text{else} \end{cases}
\]
Be aware that this potential exhibits strong oscillations, we refer to it as the Mosaic Maryland operator. The name of the operator was inspired by a recently researched quasi-periodic mosaic model\cite{40, 41}:

\[(H_{V_2, \alpha, \theta} u)_n = u_{n+1} + u_{n-1} + V_\theta(n)u_n,\]

where

\[V_\theta(n) = \begin{cases} 2\lambda \cos 2\pi(\theta + n\alpha), & n \in \kappa \mathbb{Z}, \\ 0, & \text{else}, \end{cases} \]

and the authors demonstrate the existence of exact mobility edges \cite{40}, which are energies separating absolutely continuous spectrum from pure point spectrum. For the mosaic Maryland model, we show the following phase transition result:

**Theorem 1.1.** Let \(\alpha \in \mathbb{R} \setminus \mathbb{Q}\), then Lyapunov exponent of \(H_{V_2, \alpha, \theta}\) satisfy

\[L(E) = \operatorname{arccosh} \left( \frac{\sqrt{(E^2 - 4)^2 + (\lambda E)^2} + \sqrt{E^4 + (\lambda E)^2}}{4} \right).\]

Moreover, \(H_{V_1, \alpha, \theta}\) has purely singular continuous spectrum on \(\{E : 0 < L(E) < \delta(\alpha, \theta)/2\}\), and pure point spectrum on \(\{E : L(E) > \delta(\alpha, \theta)/2\}\).

Let’s give some comments why Theorem 1.1 is interesting. While Cayley transform \cite{37} can be used to prove pure point part of the Maryland model, it doesn’t work the mosaic Maryland model, thus to prove the Anderson localization part of Theorem 1.1, we have to adopt the constructive proof \cite{19, 29}. Note that for quasi-periodic unbounded models, if the potential is monotonic and the frequency is Diophantine, there are essentially no resonances, making localization proof relatively simple, this can be seen either from the KAM side \cite{9} or from the Green’s function estimation side\cite{29}. In our non-monotonic model, our proof follows from \cite{19}, and we will further explore the anti-resonances lead to Anderson localization. From the singular continuous side, the proof will be based on sharp Gordon’s argument \cite{8, 23}. To the best knowledge of the authors, we present the first quasi-periodic unbounded model with non-monotonic model, that has arithmetic phase transition.

We also note the Mosaic Maryland operator is generated by product systems, which corresponds to a periodic multiplicative modulation of Maryland potential. Clearly then, \(V_1(\theta, n)\) admits a description in terms of the product system \(X = \mathbb{T} \times \mathbb{Z}_2, T : X \to X, (\theta, n) \mapsto (\theta + n + 1)\). In particular,

\[V_1(n, \theta) = V_\omega(n) = f(T^n \omega),\]

where \(\omega = (n, \theta)\) and

\[f(n, \theta) = \tan(\pi(\theta))f_2(n),\]

with \(f_2(n) = \delta_{n \mod 2, \theta}\). Indeed, it is a special case of ergodic Schrödinger operators defined over product dynamical systems in which one factor is periodic and the other factor is either a subshift over a finite alphabet or an irrational rotation of the circle. We point the reader to \cite{12} for a thorough account of spectral properties of dynamically defined Schrödinger operators.

2. Preliminaries

2.1. Rational approximations. Let \(\alpha \in (0, 1) \setminus \mathbb{Q}, a_0 = 0\), and let \(a_0 = \alpha\). Inductively for \(k \geq 1\),

\[a_k = \left[\frac{\alpha_{k-1}}{\alpha_{k-2}}\right], \quad a_k = \frac{1}{\alpha_{k-1}} - a_k = G(\alpha_{k-1}) = \left\{ \frac{1}{\alpha_{k-1}} \right\}.\]

Let \(p_0 = 0, p_1 = 1, q_0 = 1, q_1 = a_1\), then we define inductively \(p_k = a_k p_{k-1} + p_{k-2}, q_k = a_k q_{k-1} + q_{k-2}\). The sequence \((q_n)\) are the denominators of the best rational approximations of \(\alpha\), since we have

\[\forall 1 \leq k < q_n, \quad \|k\alpha\|_\mathbb{T} \geq \|q_{n-1}\alpha\|_\mathbb{T},\]

and

\[\frac{1}{2q_{n+1}} \leq \|q_n\alpha\| \leq \frac{1}{q_{n+1}},\]

\[\|q_{n+1}\alpha\| = a_{n+1} \|q_n\alpha\| + \|q_{n+1}\alpha\|.\]
2.2. Cocycles and growth of the cocycle. Let $X$ be a compact metric space, $(X, \nu, T)$ be ergodic. A cocycle $(\alpha, A) \in \mathbb{R} \times C^\omega(X, M(2, \mathbb{R}))$ is a linear skew product:

$$(T, A) : \quad X \times \mathbb{R}^2 \to X \times \mathbb{R}^2
\quad (x, \phi) \mapsto (Tx, A(x) \cdot \phi).$$

For $n \in \mathbb{Z}$, $A_n$ is defined by $(T, A)^n = (T^n, A_n)$. Thus $A_0(x) = id$,

$$A_n(x) = \prod_{j=n-1}^{0} A(T^j x) = A(T^{n-1}x) \cdots A(Tx)A(x), \text{ for } n \geq 1.$$ and $A_{-n}(x) = A_n(T^{-n}x)^{-1}$. $A_n$ is called the $n$-step transfer matrix. For this kind of cocycles, the Lyapunov exponent

$$L(\alpha, A) = \lim_{n \to \infty} \frac{1}{n} \int \ln \|A_n(\theta)\| \ d\theta$$
is well defined. In this paper, we will consider the following cocycle:

$$X$$


Thus, any (formal) solution $\phi$ of (2.4) can be reconstructed via the following relation

$$S^V_E(\cdot) = \left( \begin{array}{cc} E - V(\cdot) & -1 \\ 1 & 0 \end{array} \right), \quad E \in \mathbb{R}.$$}

2.3. Trigonometric product. The following lemma from [5] gives a useful estimate of products appearing in our analysis.

**Lemma 2.1.** For $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, $\theta \in \mathbb{R}$ and $0 \leq j_0 \leq q_n - 1$ be such that

$$|\cos \pi (\theta + j_0\alpha)| = \inf_{0 \leq j \leq q_n - 1} |\cos \pi (\theta + j\alpha)|,$$

then for some absolute constant $C > 0$

$$-C \ln q_n \leq \sum_{j=0, j \neq j_0}^{q_n-1} \ln |\cos \pi (\theta + j\alpha)| + (q_n - 1) \ln 2 \leq C \ln q_n.$$

3. Lyapunov Exponents

To exactly calculate the Lyapunov exponent, we need to consider $L(\alpha, A(\cdot + i\varepsilon))$ with complex phase $\varepsilon$. The basic idea is to reduce the non-trival problem of computing the Lyapunov exponent of a given non-constant cocycle to an "almost constant" cocycle by taking $\varepsilon \to \infty$. This approach was first developed by Avila.

Let us make a short review of Avila’s global theory of one-frequency quasi-periodic cocycles [3]. Suppose that $D \in C^\omega(T, M(2, \mathbb{C}))$ admits a holomorphic extension to $\{|\Im \theta| < h\}$. Then for $|\varepsilon| < h$, we define $D_\varepsilon \in C^\omega(T, M(2, \mathbb{C}))$ by $D_\varepsilon(\cdot) = S^V_E(\cdot + i\varepsilon)$, and define the the acceleration of $(\alpha, D_\varepsilon)$ as follows

$$\omega(\alpha, D_\varepsilon) = \frac{1}{2\pi} \lim_{h \to 0^+} \frac{L(\alpha, D_{\varepsilon+h}) - L(\alpha, D_\varepsilon)}{h}.$$

The acceleration was first introduced by Avila for analytic $SL(2, \mathbb{C})$ cocycles [3], and extended to analytic $M(2, \mathbb{C})$ cocycles by Jitomirskaya and Marx [26, 27]. It follows from the convexity and continuity of the Lyapunov exponent that the acceleration is an upper semicontinuous function in parameter $\varepsilon$. The key property of the acceleration is that it is quantized:

**Theorem 3.1.** Suppose that $(\alpha, D) \in (\mathbb{R} \setminus \mathbb{Q}) \times C^\omega(T, M_2(\mathbb{C}))$ with $\det D(\theta)$ bound away from 0 on the strip $\{|\Im \theta| < h\}$, then $\omega(\alpha, D_\varepsilon) \in \frac{1}{2}\mathbb{Z}$ in the strip. Moreover, if $D \in C^\omega(T, SL(2, \mathbb{C}))$, then $\omega(\alpha, D_\varepsilon) \in \mathbb{Z}$. 
Now, we consider the Lyapunov exponent of the model defined in (1.3). $V_1$ is defined on $\mathbb{T} \times \mathbb{Z}_2$, consequently (1.3) induces an almost-periodic Schrödinger cocycle $\left( T_\alpha, S_{E}^{V_1} \right)$ where $T_\alpha(\theta, n) = (\theta + \alpha/2, n + 1)$. Although $\left( T_\alpha, S_{E}^{V_1} \right)$ is not a quasi-periodic cocycle in the strict sense, its iterate

\[
\left( \alpha, D_{E}^{V_1} \right) =: \left( \alpha, S_{E}^{V_1}(\theta, 1) \times S_{E}^{V_1}(\theta, 0) \right),
\]

indeed defines an analytic quasi-periodic cocycle. By simple calculation,

\[
D_{E}^{V_1}(\theta) = \begin{pmatrix}
E & -1 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
E - \lambda \tan \pi \theta & -1 \\
1 & 0
\end{pmatrix}
= \begin{pmatrix}
E^2 - \lambda E \tan \pi \theta - 1 & -E \\
E - \lambda \tan \pi \theta & -1
\end{pmatrix}.
\]

It is easy to see that $L \left( T_\alpha, S_{E}^{V_1} \right) = \frac{1}{2} L \left( \alpha, D_{E}^{V_1} \right)$. The latter can be explicitly computed by Avila’s global theory, thus we have the following result:

**Lemma 3.2.** For $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $\lambda \in \mathbb{R}$, we have

\[
4 \cosh(L \left( T_\alpha, S_{E}^{V_1} \right)) = \sqrt{(E^2 - 4)^2 + (\lambda E)^2} + \sqrt{(E^4 + (\lambda E)^2)}.
\]

**Proof.** For simplicity, denote $L(E) = L \left( T_\alpha, S_{E}^{V_1} \right)$. It suffices for us to prove that for any $E \in \Sigma(H_{V_1, \alpha, \theta})$, we have

\[
4 \cosh \left( \frac{1}{2} L \left( \alpha, D_{E}^{V_1} \right) \right) = \sqrt{(E^2 - 4)^2 + (\lambda E)^2} + \sqrt{(E^4 + (\lambda E)^2)}.
\]

First we rewrite the matrix $D_{E}^{V_1}(\theta)$ as

\[
D_{E}^{V_1}(\theta) = \begin{pmatrix}
E^2 + i\lambda \frac{e^{i2\pi(\theta + i)}/(e^{i2\pi(\theta + i) + 1})}{e^{i2\pi(\theta + i) + 1}} E - 1 & -E \\
E + i\lambda \frac{e^{i2\pi(\theta + i)}/(e^{i2\pi(\theta + i) + 1})}{e^{i2\pi(\theta + i) + 1}} & -1
\end{pmatrix},
\]

then we complexify the phase

\[
D_{E}^{V_1}(\theta + i\epsilon) = \begin{pmatrix}
E^2 + i\lambda \frac{e^{i2\pi(\theta + i)}/(e^{i2\pi(\theta + i) + 1})}{e^{i2\pi(\theta + i) + 1}} E - 1 & -E \\
E + i\lambda \frac{e^{i2\pi(\theta + i)}/(e^{i2\pi(\theta + i) + 1})}{e^{i2\pi(\theta + i) + 1}} & -1
\end{pmatrix}.
\]

Let $\epsilon$ goes to infinity, then

\[
D_{E}^{V_1}(\theta + i\epsilon) = D_\infty + o(1),
\]

where

\[
D_\infty = \begin{pmatrix}
E & -1 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
E - i\lambda & -1 \\
1 & 0
\end{pmatrix}.
\]

By the continuity of the LE [11, 27]

\[
L(\alpha, D_{E}^{V_1}(\theta + i\epsilon)) = L(\alpha, D_\infty) + o(1).
\]

The quantization of acceleration yields

\[
L(\alpha, D_{E}^{V_1}(\theta + i\epsilon)) = L(\alpha, D_\infty).
\]

for all $\epsilon > 0$ sufficiently large. In addition the convexity, continuity and symmetry of $L(\alpha, D_{E}^{V_1}(\theta + i\epsilon))$ with respected to $\epsilon$, gives

\[
L(\alpha, D_{E}^{V_1}(\theta + i\epsilon)) = L(\alpha, D_\infty),
\]

for all $\epsilon > 0$. Note that symmetry means $L(\alpha, D_{E}^{V_1}(\theta + i\epsilon)) = L(\alpha, D_{E}^{V_1}(\theta - i\epsilon))$, this implies

\[
L(E) = L(\alpha, D_\infty)/2.
\]

Then Lemma 3.2 follows from solving for the eigenvalue of $D_\infty$ (a constant matrix) directly. \qed

It is obviously that $L(E) = 0$ if and only if $E = 0$. Now, we will prove:

**Lemma 3.3.** We have, $0 \in \Sigma(H_{V_1, \alpha, \theta})$
Proposition 4.1. Let
\[
\begin{align*}
   u_n = \begin{cases} 
   1 & n = 4k + 1 \\
   -1 & n = 4k + 3 \\
   0 & \text{else}
   \end{cases}
\end{align*}
\]
where \( k \in \mathbb{Z} \), direct computation shows the sequence \((u_n)_{n \in \mathbb{Z}}\) satisfy the equation in (1.3) when \( E = 0 \).

By Schnol’s theorem[18], \( 0 \in \Sigma(H_{V_1,\alpha,\theta}) \).

\[ \square \]

Remark 3.4. In the remaining of the paper, we only consider the case energy \( E \in \Sigma(H_{V_1,\alpha,\theta}) \) with positive Lyapunov exponent.

4. SINGULAR CONTINUOUS SPECTRUM

Denote \( A(\theta) = D_E^V(\theta) \times \cos(\theta) \) and
\[
A_{m}(\theta) = A(\theta + (m-1)\alpha) \cdots A(\theta + \alpha)A(\theta),
\]
\[
= A^m(\theta) \cdots A^2(\theta)A^1(\theta).
\]
Let \( B(\theta) = S_E^V(\theta) \) and
\[
B_m(\theta) = B(\theta + (m-1)\hat{\alpha}) \cdots B(\theta + \hat{\alpha})B(\theta),
\]
\[
= B^m(\theta) \cdots B^2(\theta)B^1(\theta).
\]
for \( m \geq 1 \) and \( \hat{\alpha} = \alpha/2 \). We also denote \( B_{-m}(\theta) = B_m(\theta - m\hat{\alpha})^{-1} \).

Then, we have the following

**Proposition 4.1.** If \( E \in \{ E : 0 < 2L(E) < \delta(\alpha, \epsilon) \} \), there exists \( N = N(E, \lambda, \epsilon) > 0 \) such that if \( q_{n_i} > N \), let \( \varphi(k) \) be a normalized solution of (1.3), \( \tilde{u}_E^q = \left( \begin{array}{c} \varphi(0) \\ \varphi(-1) \end{array} \right) \), then we have
\[
\| (B_{2q_{n_i}}(\theta + 2q_{n_i}\hat{\alpha}) - B_{2q_{n_i}}(\theta))\tilde{u}_E^q \| \leq e^{(2L-\delta(\alpha)+4\epsilon)q_{n_i}},
\]
\[
\| (B_{-2q_{n_i}}(\theta + 2q_{n_i}\hat{\alpha}) - B_{-2q_{n_i}}(\theta))\tilde{u}_E^q \| \leq e^{(2L-\delta(\alpha)+4\epsilon)q_{n_i}}.
\]

**Proof.** We only give the proof of (4.2), the proof of (4.3) is similar. Note \( B_{2q_{n_i}}(\theta) = \prod_{j=0}^{q_{n_i}-1} A_{q_{n_i}+j}(\theta) = \prod_{j=0}^{q_{n_i}-1} A^j(\theta) \), where \( A^j(\theta) = A(\theta + j\alpha), c_j = \cos(\pi(\theta + j\alpha)) \). By telescoping argument (One can consult [28] for details), we have
\[
\| (B_{2q_{n_i}}(\theta + 2q_{n_i}\hat{\alpha}) - B_{2q_{n_i}}(\theta))\tilde{u}_E^q \| 
\leq \sum_{j=0}^{q_{n_i}-1} \left\| \prod_{l=0}^{j-1} \frac{A^{q_{n_i}+l}}{c_{q_{n_i}+l}} - \prod_{l=0}^{j-1} \frac{A^{q_{n_i}+j}}{c_{q_{n_i}+j}} \left( \begin{array}{c} \varphi_{j-1} \\ \varphi_{j-2} \end{array} \right) - \frac{c_{q_{n_i}+j}}{c_{q_{n_i}+j}} \left( \begin{array}{c} \varphi_j \\ \varphi_{j-1} \end{array} \right) \right\|.
\]
Since \( \varphi \in \ell^2 \) is decaying solution, there exists a constant \( C > 0 \) such that
\[
\left\| \begin{array}{c} \varphi_k \\ \varphi_{k-1} \end{array} \right\| \leq C.
\]
And we need to estimate the norms \( A_{q_{n_i}+j-1} \). The following control of the norm of the transfer matrix of a uniquely ergodic continuous cocycle by the Lyapunov exponent is well known.

**Theorem 4.2.** ([16, 40]) Let \( (\alpha, M) \) be a continuous cocycle, then for any \( \varepsilon > 0 \), for \( |n| \) large enough,
\[
\| M_n(\theta) \| \leq e^{|n|(L(\alpha, M)+\varepsilon)} \text{ for any } \theta \in \mathbb{T}.
\]
Since \( A(\theta) = D_E^V(\theta) \times \cos(\theta) \) is analytic, we have that \( \ln \| A_n(\theta) \| \) is a continuous subadditive cocycle, by Theorem 4.2, we have
\[
\| A_n(\theta) \| \leq e^{|n|(L(\alpha, A)+\varepsilon)} \text{ for any } \theta \in \mathbb{T},
\]
for any \( \varepsilon > 0 \), for \( |n| \) large. And by the fact that \( \int_\mathbb{T} \ln |\cos \pi \theta|d\theta = -\ln 2 \), we have
\[
L(\alpha, A) = 2L(E) - \ln 2.
\]
Considering 1-dimensional continuous cocycles, by Theorem 4.2, we have the following corollary.
Corollary 4.3 ([19]). Let $I = [\ell_1, \ell_2] \subset \mathbb{Z}$, we have
$$\prod_{\ell=\ell_1}^{\ell_2} |\cos(\pi(\theta + \ell \alpha))| \leq C(\varepsilon)e^{(\ell_2-\ell_1)(-\ln 2+\varepsilon)} \inf_{j=\ell_1}^{\ell_2} |\cos(\pi(\theta + j \alpha))|,$$
where $C(\varepsilon)$ is a constant that depends only on $\varepsilon$.

As for the lower bound of $\prod_j c_j$, we will use the following Lemma.

Lemma 4.4 (Theorem 2.3, [23]). For any $\varepsilon > 0$, there exists a subsequence $q_{n_i}$ of $q_n$ such that the following estimate holds
\begin{equation}
\prod_{j=0}^{q_{n_i}-1} |c_j| \geq e^{(\delta(\alpha, \theta)-\ln 2-\varepsilon)q_{n_i}}.
\end{equation}

Observe that $\sup_{\theta \in \mathbb{T}} \|A_{\pm q_{n_i}}(\theta + 2q_{n_i}\hat{\alpha}) - A_{\pm q_n}(\theta)\| \leq \frac{C}{q_{n_i}+1}$, combining (4.4), Corollary 4.3 with Lemma 4.4, we have
\begin{align*}
&\left\| (B_{2q_{n_i}}(\theta + 2q_{n_i}\hat{\alpha}) - B_{2q_n}(\theta)) \begin{pmatrix} \varphi_0 \\ \varphi_{-1} \end{pmatrix} \right\| \\
&\leq Cq_{n_i}e^{q_{n_i}(2L(E)-\ln 2+\varepsilon)} \cdot e^{n_{q_{n_i}}(2L(E)-\ln 2-\varepsilon)} \\
&\leq Cq_{n_i}(2L(E)-\delta(\alpha)+4\varepsilon).
\end{align*}
\hfill \square

As a result of Proposition 4.1, we have the following:

Corollary 4.5. Let $\varphi(k)$ be a normalized solution of (1.3), $\tilde{u}_E^\theta = \begin{pmatrix} \varphi(0) \\ \varphi(-1) \end{pmatrix}$, then we have
\begin{equation}
\max \left\{ \|B_{2q_{n_i}}(E,\theta)\tilde{u}_E^\theta\|, \|B_{-2q_{n_i}}(E,\theta)\tilde{u}_E^\theta\|, \|B_{2q_{n_i}}(E,\theta)\tilde{u}_E^\theta\| \right\} \geq \frac{1}{4}.
\end{equation}

Proof. The proof is essentially contained in Lemma 3.2 of [6]. We remark that this result is only valid in the subsequence $n_i$. \hfill \square

Now as a result of Corollary 4.5, one can conclude that $H_{V_1,\alpha,\theta}$ has purely singular continuous spectrum on $\{E : 0 < L(E) < \delta(\alpha,\theta)/2\}$.

5. Pure Point Spectrum

In this section, we are devote to prove Anderson localization in the regime $\{E : L(E) > \delta(\alpha,\theta)/2\}$. We first introduce some notations and recall the key framework, modified from the one developed in [19, 24] also with adaptions from [29, 40]. For any generalized eigenvalue $E$, assume $\phi$ is the corresponding generalized eigenfunction of $H_{V_1,\alpha,\theta}$, without loss of generality assume
\begin{equation}
|\phi(0)| \geq 1,
\end{equation}
and
\begin{equation}
|\phi(k)| \leq C_0|k|.
\end{equation}
We shall write $\delta(\alpha,\theta)$ as $\delta$ and $\beta(\alpha)$ as $\beta$ for simplicity. Define
\begin{equation}
\beta_n := \frac{\ln q_{n+1}}{q_n},
\end{equation}
and
\begin{equation}
\delta_n := \frac{\ln \|q_n \left( \theta - \frac{1}{2} \right) \| - \ln \|q_n \alpha\|}{q_n},
\end{equation}
then one can check that

Lemma 5.1. [19] We have $0 \leq \delta \leq \beta$ for all $\alpha, \theta$, and $\delta = \lim \sup \max (0, \delta_n)$. 

Fix a small $\varepsilon > 0$ such that
\begin{equation}
2L(E) > \delta + 700\varepsilon.
\end{equation}

Since $\limsup_{n \to \infty} \delta_n = \delta$, we have that for $n > N(\varepsilon)$ large enough,
\begin{equation}
2L(E) > \delta_n + 680\varepsilon.
\end{equation}

Then we have the following:

**Theorem 5.2.** If $E \in \{ E: 2L(E) > \delta(\alpha, \theta) \}$, let $\phi$ be an generalized eigenfunction satisfying $|\phi(0)| \geq 1$ and (5.2). Then for $n > N(\alpha, E, \theta, \varepsilon, C_0)$ large enough and $\frac{1}{n} q_n \leq |k| < \frac{1}{n} q_{n+1}$, we have
\begin{equation}
|\phi(k)| \leq e^{-(L - \delta(\alpha)/2 - 330\varepsilon)|k|}.
\end{equation}

Before giving the proof, we first introduce some useful notations and concepts. Denote by $M_k(\theta)$ the $k-$ step transfer-matrix of $H_{V,\alpha,\theta} u = Eu$, and denote
\begin{align*}
Q_k(\theta) &= \det \left[ (H_{V,\alpha,\theta} - E)_{[0,k-1]} \right], \\
P_k(\theta) &= \det \left[ (H_{V,\alpha,\theta} - E)_{[1,k]} \right],
\end{align*}
for $k \geq 1$, then the $k$-step transfer-matrix can be written as
\begin{equation}
M_k(\theta) = (-1)^k \begin{pmatrix}
Q_k(\theta) & P_{k-1}(\theta) \\
-Q_{k-1}(\theta) & -P_{k-2}(\theta)
\end{pmatrix}.
\end{equation}

Let $\tilde{Q}_k(\theta) : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ be defined as $\tilde{Q}_{2k}(\theta) = \prod_{j=0}^{k-1} \cos \pi(\theta + j\alpha) \cdot Q_{2k}(\theta)$ and $\tilde{Q}_{2k+1}(\theta) = \prod_{j=0}^{k} \cos \pi(\theta + j\alpha) \cdot Q_{2k+1}(\theta)$. Respectively, $\tilde{P}_k(\theta)$ can be also defined as $\tilde{P}_{2k}(\theta) = \prod_{j=1}^{k} \cos \pi(\theta + j\alpha) \cdot P_{2k}(\theta)$ and $\tilde{P}_{2k+1}(\theta) = \prod_{j=1}^{k} \cos \pi(\theta + j\alpha) \cdot P_{2k+1}(\theta)$. Then clearly, it turns out $A_k(\theta)$ defined in (4.1) has the following expression
\begin{equation}
A_k(\theta) = M_{2k}(\theta) \prod_{j=0}^{k-1} \cos \pi(\theta + j\alpha) = \begin{pmatrix}
\tilde{Q}_{2k}(\theta) & -\tilde{P}_{2k-1}(\theta + \alpha) \cos \pi\theta \\
-\tilde{Q}_{2k-1}(\theta) & -\tilde{P}_{2k-2}(\theta + \alpha) \cos \pi\theta
\end{pmatrix}.
\end{equation}

Then, we have the following upper bound of $\tilde{P}_k$ and $\tilde{Q}_k$.

**Lemma 5.3.** For any $\varepsilon > 0$, for $|k|$ large enough,
\begin{equation}
|\tilde{P}_k(\theta)| \leq e^{(L(E)+\varepsilon)|k|} \text{ for any } \theta \in T,
\end{equation}
and
\begin{equation}
|\tilde{Q}_k(\theta)| \leq e^{(L(E)+\varepsilon)|k|} \text{ for any } \theta \in T,
\end{equation}
where $\tilde{L}(E) = L(E) - \frac{\ln 2}{2}$.

**Proof.** It follows from (4.4) and (5.7). \hfill \Box

We can also have the average lower bound of $\tilde{P}_k$.

**Lemma 5.4.** By Herman’s subharmonic trick, one has
\begin{equation}
\frac{1}{k} \int_0^1 \ln |\tilde{P}_k(\theta)| \, d\theta = \frac{1}{k} \int_0^1 \ln |\tilde{P}_k(2\theta)| \, d\theta \geq \tilde{L}
\end{equation}

The proof of this lemma is modification of that of Lemma 3.1 in [29]. We will leave it in the appendix. An important observation that makes our analysis possible is

**Lemma 5.5.** $\frac{\tilde{P}_{2k-1}(\theta)}{\cos^{k-1}(\pi\theta)}$ and $\frac{\tilde{P}_{2k}(\theta)}{\cos^k(\pi\theta)}$ can be expressed as a polynomial of degree $k-1$ and $k$ respectively in $\tan \pi \theta$, namely,
\begin{equation}
\frac{\tilde{P}_{2k-1}(\theta)}{(\cos \pi \theta)^k} \triangleq g_{k-1}(\tan \pi \theta),
\end{equation}
\begin{equation}
\frac{\tilde{P}_{2k}(\theta)}{(\cos \pi \theta)^{k+1}} \triangleq f_k(\tan \pi \theta),
\end{equation}
where $g_{k-1}$ is a polynomial of degree $k - 1$, respectively, $f_k$ is a polynomial of degree $k$. 

Proof. Note that $V_1(\theta, 2n+1) = 0$ and $V_1(\theta, n+2) = V_1(\theta, n, n)$. Then if we expand the determinant \[
\det ((H_{V,\alpha,\theta} - E)[_{1,2k-1}]) \text{ and } \det ((H_{V,\alpha,\theta} - E)[_{1,2k}]) \text{ by the last column, we have}
\]
\[
P_{2k-1}(\theta) = -EP_{2k-2}(\theta) - P_{2k-3}(\theta),
\]
\[
P_{2k}(\theta) = (\tan(\pi(\theta + k\alpha) - E)P_{2k-1}(\theta) - P_{2k-2}(\theta).
\]
Recall the definition of $\tilde{P}_k$, we have
\[
\tilde{P}_{2k-1}(\theta) = -E\tilde{P}_{2k-2}(\theta) - \tilde{P}_{2k-3}(\theta)\cos(\theta + (k-1)\alpha),
\]
\[
\tilde{P}_{2k}(\theta) = (\tan(\pi(\theta + k\alpha)) - E)\cos(\pi(\theta + k\alpha)\tilde{P}_{2k-1}(\theta) - \tilde{P}_{2k-2}(\theta)\cos(\pi(\theta + k\alpha)),
\]
then (5.11) and (5.12) follow from an induction, by using (5.13). \qed

By the Lagrange interpolation formula, for any set of $k+1$ distinct $\theta_i$'s in $(-1/2, 1/2)$, we have the following convenient representation
\[
\tilde{P}_{2k}(\theta) = (\cos \pi \theta)^k g_k(\tan \pi \theta) = \sum_{i=0}^{k} \tilde{P}_{2k}(\theta_i) \prod_{l \neq i} \tan \pi \theta - \tan \pi \theta_i \cdot \cos^k \pi \theta_i \frac{\cos \pi \theta}{\cos \pi \theta_i},
\]
(5.14)
\[
= \sum_{i=0}^{k} \tilde{P}_{2k}(\theta_i) \prod_{l \neq i} \sin \pi (\theta - \theta_i) \sin \pi (\theta_i - \theta_l),
\]
also
\[
\tilde{P}_{2k+1}(\theta) = (\cos \pi \theta)^k g_k(\tan \pi \theta) = \sum_{i=0}^{k} \tilde{P}_{2k+1}(\theta_i) \prod_{l \neq i} \tan \pi \theta - \tan \pi \theta_i \cdot \cos^k \pi \theta_i \frac{\cos \pi \theta}{\cos \pi \theta_i},
\]
(5.15)
\[
= \sum_{i=0}^{k} \tilde{P}_{2k+1}(\theta_i) \prod_{l \neq i} \sin \pi (\theta - \theta_i) \sin \pi (\theta_i - \theta_l).
\]

In this regard, we also recall the following useful concept:

Definition 5.6. [5] We say that the set $\{\theta_1, \ldots, \theta_{k+1}\}$ is $\epsilon$ -uniform if
\[
\max_{\theta \in [0,1]} \max_{\prod_{l \neq i} \sin \pi (\theta - \theta_i) \sin \pi (\theta_i - \theta_l) < e^{k\epsilon},
\]
(5.16)

We use $G_{[x_1,x_2]}(E)(x, y)$ for the Green function of the operator $H$ restricted to the interval $[x_1, x_2]$ with zero boundary conditions at $x_1 - 1$ and $x_2 + 1$. We will omit $E$ when it is fixed throughout the argument. A useful definition about Green’s function is the following:

Definition 5.7. [22] A point $y \in \mathbb{Z}$ will be called $(m, h)$-regular if there exists an interval $[x_1, x_2]$, $x_2 = x_1 + h - 1$, containing $y$, such that
\[
|G_{[x_1,x_2]}(x_i, y)| < e^{-m|y-x_i|}, |y-x_i| \geq \frac{1}{4} h, \text{ for } i = 1, 2.
\]
 Otherwise, $y \in \mathbb{Z}$ will be called $(m, h)$-singular.

Let $\phi(x)$ be a solution of $H\phi(x) = E\phi(x)$ and let $[x_1, x_2]$ be an interval containing $y$. We have
\[
\phi(y) = -G_{[x_1,x_2]}(x_1, y) \phi(x_1 - 1) - G_{[x_1,x_2]}(x_2, y) \phi(x_2 + 1).\]
(5.17)

In general, if $I = [a, b]$, let $\partial I := \{a, b\}$ and $a' := a - 1, b' := b + 1$. If we denote
\[
\Delta_{m,n}(\theta) = \det \left[(H_{V,\alpha,\theta} - E)[_{m,n}]\right].
\]

By Cramer’s rule, we have the following connection between the determinants $P_k$ and Green function:
\[
|G_{[x_1,x_2]}(x_1, y)| = \frac{\Delta_{y+1,x_2}(\theta)}{|\Delta_{x_1,x_2}(\theta)|},
\]
(5.18)
\[
|G_{[x_1,x_2]}(y, x_2)| = \frac{\Delta_{y-1,x_2}(\theta)}{|\Delta_{x_1,x_2}(\theta)|}.
\]
Furthermore, if \( y = 2n \) and \( x_i = 2n_i + 1 \) with \( n, n_i \in \mathbb{N} \) for \( i = 1, 2 \), we have
\[
(5.19) \quad |\phi(y)| \leq \left| \sum_{i=1}^{n-1} \frac{P_{x_i-y} \left( \theta_n \right)}{P_{x_{i-1}-y} \left( \theta_n \right)} \right| \prod_{k=n}^{n_2} |\cos(\pi \theta_k)| \cdot |\phi(x_1 - 1)| + \left| \sum_{i=1}^{n_2} \frac{P_{y-x_i} \left( \theta_n \right)}{P_{x_{i+1}-y} \left( \theta_n \right)} \right| \prod_{k=n}^{n_2} |\cos(\pi \theta_k)| \cdot |\phi(x_2 + 1)|,
\]
and if \( y = 2n \) and \( x_i = 2n_i \) with \( n, n_i \in \mathbb{N} \) for \( i = 1, 2 \), we have
\[
(5.20) \quad |\phi(y)| \leq \left| \sum_{i=1}^{n-1} \frac{Q_{x_i-y} \left( \theta_n \right)}{Q_{x_{i-1}-y} \left( \theta_n \right)} \right| \prod_{k=n}^{n_2} |\cos(\pi \theta_k)| \cdot |\phi(x_1 - 1)| + \left| \sum_{i=1}^{n_2} \frac{Q_{y-x_i} \left( \theta_n \right)}{Q_{x_{i+1}-y} \left( \theta_n \right)} \right| \prod_{k=n}^{n_2} |\cos(\pi \theta_k)| \cdot |\phi(x_2 + 1)|.
\]

where \( \theta_k = \theta + k\alpha \). One should be mentioned that if we use (5.19) to expand even point \( y \) with odd endpoints \( x_1 \) and \( x_2 \), then we can also expand even point \( x'_1 \) in an interval with odd endpoints. The parity of these points will contribute to keep the the numerators and denominators of Green’s functions from being replaced by \( Q_k \).

### 5.1 Key technical Lemmas.
In the remaining of this paper, We want to prove the generalized eigenfunction \( \phi \) decays exponentially (Theorem 5.2). To do so, first we need to obtain good bounds of \( P_k \) and the product (indeed the minimum) of cosines in (5.19). Before giving these bounds of \( P_k \) and the product of cosines, we need some concepts, which were first introduced in [19].

**Definition 5.8.** [19] We call \( (m, \ell) \in \mathbb{Z}^2 \) is \( \theta \) minimal on scale \( q_n \) if the following holds
- (1) \( m \in [-q_n/2, q_n/2] \),
- (2) \( \| \ell \| \leq \frac{1}{q_n} \left( e^{\delta \sqrt{q_n}} + q_n + 1 \right) \),
- (3) \( \| \theta - \frac{1}{2} + (m + \ell q_n) \| < \left( \frac{1}{2} + \frac{1}{2q_n} \right) \| q_n \alpha \| \),
- (4) (i). For \( a_{n+1} \geq 4 \), we have
  \[
  \left\| \theta - \frac{1}{2} + (m + jq_n) \alpha \right\| \leq 20 \min_{|k| < q_n} \left\| \theta - \frac{1}{2} + (m + jq_n + k) \alpha \right\|,
  \]
  holds for any \( |j| \leq a_{n+1}/6 \).
- (ii). For \( a_{n+1} \leq 3 \), we have
  \[
  \left\| \theta - \frac{1}{2} + m \alpha \right\| \leq 20 \min_{-q_n/2 \leq k < q_n/2} \left\| \theta - \frac{1}{2} + k\alpha \right\|.
  \]

The following Lemma show the existence of \( \theta \)-minimal \( (m, \ell) \).

**Lemma 5.9.** [19] For any \( q_n \) sufficiently large, there exists \( \theta \)-minimal \( (m_n, \ell_n) \) at scale \( q_n \).

Define
\[
(5.21) \quad c_{n,\ell} := |\cos(\pi \theta_{m_n+\ell q_n})|.
\]
As a corollary of Lemma 5.9, we have the following Lemma.

**Lemma 5.10.** Let \( I = [\ell_1, \ell_2] \subset \mathbb{Z} \) be such that there exists \( j \in \mathbb{Z}, |j| < q_{n+1}/(6q_n) \), that satisfies
\[
I \subset [m_n + jq_n + 1, m_n + (j + 1)q_n - 1].
\]
Then for \( n > N(\varepsilon) \) large enough, we have
\[
\prod_{\ell \in I} |\cos(\pi \theta_\ell)| \geq e^{-c(2q_n-|I|)}e^{-(\ln 2)|I|}.
\]
Furthermore, for \( \beta_n \geq 200\varepsilon \), for \( |\ell| \leq q_{n+1}/(6q_n) \) and some absolute constant \( 0 < C < 8 \)
\[
(5.22) \quad c_{n,\ell} \leq C \max \{|\ell|, e^{\delta q_n}, 1\} e^{-\beta q_n}.
\]

**Proof.** One can consult Corollary 5.5 and Corollary 5.6 of [19] for details. \( \square \)

Now, it is time to estimating \( \hat{P}_k \).

**Lemma 5.11.** Let \( I_1, I_2 \) be two disjoint intervals in \( \mathbb{Z} \) such that \( |I_1 \cup I_2| = k \) and \( \{ \theta + \ell \alpha \}_{\ell \in I_1 \cup I_2} \) is \( \varepsilon_k \)-uniform, then exists \( x_1 \in I_1 \cup I_2 \) such that
\[
|\hat{P}_{2k-1}(\theta + x_1 \alpha)| \geq e^{2k(L - 2\varepsilon_k)}.\]
Proof. The result is direct consequence of the Lagrange interpolation formula and (5.10). We omit the details.

Usually, the numerators of Green’s functions can be bounded uniformly by (5.8). Using the strategy in [24] and the inequality above one can prove \( \phi(y) \) exponential decay. However this does not work for \( \delta < 2L(E) \beta \), so one has to look for an additional decay, which was first introduced in [19]. The following lemmas on \( \hat{P}_k \) are essential for proving Anderson localization in the sharp regime \( \{E : L(E) > \delta(\alpha, \theta) / 2\} \).

**Lemma 5.12.** (Corollary 7.4, [19]) For \( |\ell| < 2q_{n+1} / (3q_n) \), assume \( k < 2q_n \) and

\[
y \leq \ell q_n + m_n, \quad \text{and} \quad y + k - 1 \geq (\ell + 1) q_n + m_n - 1,
\]

we then have

\[
|\hat{P}_{2k-1}(\theta_y)| \leq g_{k,\ell} e^{(2k-1) L z}.
\]

where

\[
g_{k,\ell} := \max \left( e^{\beta_n q_n}, e^{\beta_n / 2} \right) e^{-(\beta_n - 6\varepsilon) q_n}, \quad \text{if} \quad \beta_n \geq \delta_n + 200\varepsilon,
\]

\[
e^{\beta_n q_n - 6\varepsilon}, \quad \text{if} \quad \beta_n < \delta_n + 200\varepsilon.
\]

5.2. **Some useful Propositions.** Choose a value (from multiple possible values) of \( \tau_n \) such that

\[
\tau_n \in \left( \frac{\varepsilon}{2 \max(L, 1)}, \frac{\varepsilon}{\max(L, 1)} \right)
\]

and \( \tau_n q_n \in \mathbb{Z} \). Define \( b_n = \tau_n q_n \). For any \( m_n \in \mathbb{Z} \) we call \( m_n \) resonant (at the scale of \( q_n \)) if dist \( (m_n, q_n \mathbb{Z}) \leq b_n \), otherwise we call \( y \) non-resonant. We call \( y \) even-resonant (at the scale of \( q_n \)) if dist \( (y, 2q_n \mathbb{Z}) \leq b_n \), otherwise we call \( y \) is not even-resonant. We introduce some notations:

\[
I^- := [2\ell q_n + 2b_n, 2(\ell q_n + m_n) - 1],
\]

\[
I^+ := [2(\ell q_n + m_n) + 1, 2(\ell + 1) q_n - 2b_n],
\]

\[
\phi(x_0) := \max_{x \in I^-} |\phi(y)|,
\]

\[
\phi(x_0) := \max_{x \in I^+} |\phi(y)|,
\]

\[
R_\ell := [2\ell q_n - 2b_n, 2\ell q_n + 2b_n],
\]

\[
r_\ell := \max_{k \in R_\ell} |\phi(k)|.
\]

In the following, we distinguish the proof according to \( m_n \) is resonant or not. And each part can be divided into two cases depending on \( y \) is even-resonant or not.

**Proposition 5.13.** Assume \( \text{dist} (m_n, q_n \mathbb{Z}) > b_n \),

(1) If \( y \) is not even resonance, we have: for \( y = 2(\ell q_n + m_n) \),

\[
|\phi(y)| \leq e^{29\varepsilon q_n} c_{n,\ell} e^{-(\beta_n q_n) L r_\ell} e^{-(2(\ell + 1) q_n - y) L r_{\ell+1}}.
\]

For any \( y \in I^- \),

\[
|\phi(y)| \leq e^{29\varepsilon q_n} \max \left( e^{-(y - 2q_n) L r_\ell}, e^{-(2(\ell + 1) q_n - y) L r_{\ell+1}} \right).
\]

For any \( y \in I^+ \),

\[
|\phi(y)| \leq e^{29\varepsilon q_n} \max \left( e^{-(y - 2q_n) L r_\ell}, e^{-(2(\ell + 1) q_n - y) L r_{\ell+1}} \right).
\]

(2) If \( y \) is even resonance, we have: for any \( \ell \neq 0, |\ell| \leq q_n + 1 / (6q_n) \),

\[
r_\ell \leq e^{-(2L - 5\varepsilon) q_n} \max(|\ell|, \max_{\ell \in \mathbb{Z}} e^{\beta_n q_n}, e^{\beta_n q_n}) e^{-2L z} e^{-(2(\ell + 1) q_n - y) L r_{\ell+1}}.
\]

**Proposition 5.14.** Assume \( \text{dist} (m_n, q_n \mathbb{Z}) \leq b_n \),

(1) If \( y \) is not even resonance, we have:

\[
|\phi(y)| \leq e^{40\varepsilon q_n} \max \left( e^{-(y - 2q_n) L r_\ell}, e^{-(2(\ell + 1) q_n - y) L r_{\ell+1}} \right).
\]

where

\[
R_\ell^+ := [2\ell q_n + 2m_n + 1, 2\ell q_n + 2b_n], \quad \text{and} \quad R_\ell^- := [2\ell q_n - 2b_n, 2\ell q_n + 2m_n - 1],
\]

and

\[
r_\ell^+ := \max_{y \in R_\ell^+} |\phi(y)| \quad \text{and} \quad r_\ell^- := \max_{y \in R_\ell^-} |\phi(y)|.
\]
(2) If \( y \) is even resonance, for any \( \ell \neq 0 \) such that \(|\ell| < q_{n+1}/(6q_n)\), we have
\[
(5.25) \quad r_{\ell} \leq e^{-((2L-70c)q_n)} \max(|\ell|, 1) \max(r_{\ell-1}, r_{\ell+1}) \times \left\{ \begin{array}{ll}
\max(|\ell|, e^{\beta_n q_n}), & \text{if } \beta_n \geq \delta_n + 200\varepsilon \\
e^{\beta_n q_n}, & \text{if } \beta_n < \delta_n + 200\varepsilon
\end{array} \right.
\]

The above two propositions will be proved in Section 6. They will be used to prove Theorem 5.2 in the case \( \beta_n \) is not too small. As for relevant Diophantine case, in other words, \( 0 \leq \beta_n \leq 300\varepsilon \), we have the following:

**Proposition 5.15.** For \( n \) large enough,
1. If \( \frac{2\delta_n}{\beta_n} < k < q_n, \ k \in 2\mathbb{N} \), we have \( |\phi(k)| \leq e^{-k(L-24\varepsilon)} \).
2. If \( q_n < k < \frac{4\delta_n}{\beta_n} \), \( k \in 2\mathbb{N} \), we have \( |\phi(k)| \leq e^{-((L-330\varepsilon)k)} \).

It is a variant of case 1 of Lemma 6.1 in [19]. We only need to replace \( k \) by \( k/2 \) in the argument.

5.3. Proofs of Theorem 5.2. The remaining of this paper will be devoted to the proof of Theorem 5.2, dividing into the following three cases.

Case 1. \( \beta_n \geq \delta_n + 200\varepsilon \);
Case 2. \( 300\varepsilon \leq \beta_n \leq \delta_n + 200\varepsilon \);
Case 3. \( 0 \leq \delta_n \leq \beta_n \leq 300\varepsilon \).

Case 1 require some key estimates presented in Subsection 5.1. It is the most technical part in this paper as it showed in [19]. In case 2, we have \( 2L > \beta_n + 200\varepsilon \), we will use the strategy in [24] to handle this case. Compared to the Case 1, Case 2 has a lot of simplifications. Case 3 is similarly to the Diophantine case that is handled in [29].

**Case 1** Assume \( \beta_n \geq \delta_n + 200\varepsilon \). Let \( y \in (2q_{n+1} + 2b_n, 2(\ell + 1)q_n - 2b_n) \) for some \(|\ell| \leq \frac{q_{n+1}}{6q_n} \). Without loss of generality, we assume \( \ell \geq 0 \).

If \( \ell \neq 0, -1 \), we need the following Lemma:

**Lemma 5.16.** For any \( \ell_0 \) such that \( 1 \leq |\ell_0| \leq q_{n+1}/(6q_n) \), we have
\[
(5.26) \quad r_{\ell_0} \leq e^{2(\delta_n/2-L+54\varepsilon)|\ell_0|q_n}
\]

**Proof.** In view of (5.24) and (5.25), for any \( 0 < |\ell_0| \leq q_{n+1}/(6q_n) \), we have
\[
(5.26) \quad r_{\ell_0} \leq e^{(\delta_n/2-L+50\varepsilon)2q_n} \max_{\ell_1=\ell_0\pm 1} r_{\ell_1}.
\]

One can iterate (5.26) until one reaches \( \ell_t \) (and stops the iteration once reaches such a \( \ell_t \)):
1. \( t = 0 \),
2. \( t = 2\ell_0 \),
3. the iterating number reaches \( [q_{n+1}/(12q_n)] \).

Hence one obtains
\[
r_{\ell_0} \leq \max_{(\ell_0, \ell_1, \ldots, \ell_t) \in \mathcal{G}} e^{(\delta_n/2-L+50\varepsilon)2q_n r_{\ell_t}}
\]
where \( \mathcal{G} = \{ (\ell_0, \ldots, \ell_t) : |\ell_i - \ell_{i-1}| = 1 \} \).

Then Lemma 5.16 follows from bounding \( \ell_t \) by (5.2).

Combing Proposition 5.13 (if \( m_n \) is non-resonant) and 5.14 (if \( m_n \) is resonant) with Lemma 5.16, we have
\[
(5.27) \quad |\phi(y)| \leq e^{10c q_n} \max \left( e^{-(y-2\ell q_n)L r_{\ell}}, e^{-(2(\ell+1)q_n-y)\ell r_{\ell+1}} \right)
\]

By (5.2), we have
\[
(5.28) \quad r_0 \leq 2C_0 r_n q_n.
\]

Using (5.28) and Lemma 5.16 to bound \( r_{\ell_t} \), by (5.27), we have
\[
(5.29) \quad |\phi(y)| \leq e^{10c q_n} \max \left( e^{-(y-2\ell q_n)\ell r_{\ell}}, e^{-(2(\ell+1)q_n-y)\ell r_{\ell+1}} \right),
\]

**Case 2 of Theorem 5.2.** The proofs of Case 1 and 2 of Theorem 5.2 are completely analogous. We only give a brief proof. Compared to the Case 1, Case 2 has a lot of simplifications. We don’t need to care about the minimum values of (the absolute values of) cosines.

Assume \( \beta_n \leq \delta_n + 200\varepsilon \), by Proposition 5.13 and 5.14, bound \( c_{n,\ell} \) by 1, we have
\[
(5.29) \quad |\phi(y)| \leq e^{10c q_n} \max \left( e^{-(y-2\ell q_n)\ell r_{\ell}}, e^{-(2(\ell+1)q_n-y)\ell r_{\ell+1}} \right),
\]
if \(2\ell q_n + 2b_n < y < 2(\ell + 1)q_n - 2b_n\), for some \(|\ell| \leq q_{n+1}/(6q_n)\). And for any \(\ell \neq 0, |\ell| \leq q_{n+1}/(6q_n)\), we have
\[
(5.30) \quad r_\ell \leq e^{-(2L-70\varepsilon-7\beta)q_n} \frac{e^{-2|\ell|q_n}}{\max(|\ell|, 1)} \max(r_{\ell-1}, r_{\ell+1}).
\]

Then similarly to Lemma 5.16, we have

**Lemma 5.17.** For any \(\ell_0\) such that \(1 \leq |\ell_0| \leq q_{n+1}/(6q_n)\), we have
\[
r_{\ell_0} \leq e^{2(\beta/2-L+54\varepsilon)|\ell_0|q_n}.
\]

**Proof.** It follows from (5.29), (5.30) and by arguments similar to those in Lemma 5.16. In order to avoid repetition, we omit the details. \(\square\)

Combing (5.28), (5.29) with Lemma 5.17, thus we have proved Cases 2 of Theorem 5.2.

**Case 3 of Theorem 5.2** For \(k = 2n + 1, n \in \mathbb{Z}\),
\[
(6.31) \quad \phi(2n + 2) + \phi(2n) = E\phi(2k + 1),
\]
then we have
\[
(6.32) \quad |\phi(2n + 1)| \leq C \max(|\phi(2n)|, |\phi(2n + 2)|),
\]
where \(C = C(E)\).

Combing (5.32) with Proposition 5.15, thus we have proved Cases 3 of Theorem 5.2.

6. Proofs of Some useful Propositions.

6.1. Proofs of Proposition 5.13. We will first prove not even-resonant \(y\)'s can be dominated by even resonances, and then study the relation between adjacent even resonant regions. In the remaining of this paper, by (5.32), we will only consider the case \(k \in 2\mathbb{N}\) without additional statement.

6.1.1. \(y\) is not even resonance.

**Lemma 6.1.** Assume \(2\ell q_n + 2b_n \leq y \leq 2(\ell + 1)q_n - 2b_n\), we have, for \(y = 2(\ell q_n + m_n)\),
\[
(6.1) \quad |\phi(y)| \leq e^{29\varepsilon q_n} c_{n, \ell} \max\left(e^{-(y-2\ell q_n)Lr_\ell}, e^{-(2(\ell+1)q_n-y)Lr_{\ell+1}}\right).
\]

For any \(y \in I^\prime\),
\[
(6.2) \quad |\phi(y)| \leq e^{29\varepsilon q_n} \max\left(e^{-(y-2\ell q_n)Lr_\ell}, c_{n, \ell} e^{-2(\ell+1)q_n-y)Lr_{\ell+1}}\right).
\]

For any \(y \in I^\prime\),
\[
(6.3) \quad |\phi(y)| \leq e^{29\varepsilon q_n} \max\left(c_{n, \ell} e^{-(y-2\ell q_n)Lr_\ell}, e^{-2(\ell+1)q_n-y)Lr_{\ell+1}}\right).
\]

We will give the proof of this lemma in the end of the subsection.

For a not even resonant \(y\) and \(y \in 2\mathbb{N}\), let \(n_0\) be the least positive integer so that
\[
4q_{n-n_0} \leq \text{dist} (y, 2q_n \mathbb{Z}).
\]

Once \(n_0\) is chosen, we can fix \(s\) be the greatest positive integer such that
\[
4sq_{n-n_0} \leq \text{dist} (y, 2q_n \mathbb{Z}).
\]

Clearly, Let
\[
\tilde{I}_0 = [-s[q_{n-n_0}/2] - s[q_{n-n_0}], -s[q_{n-n_0}/2]] \cap \mathbb{Z},
\]
\[
\tilde{I}_y = [y/2 - s[q_{n-n_0}/2] - s[q_{n-n_0}], y/2 - [s[q_{n-n_0}/2] - 1]] \cap \mathbb{Z}.
\]

Clearly \(\tilde{I}_0 \cup \tilde{I}_y\) contains \(2sq_{n-n_0} + 1\) distinct numbers. The choice of \(n_0\) was first introduced in [33]. It is actually a very useful technical improvement (simplify one case appearing in [5]) and now everyone in this field is using it as a standard technique. It should be noted that by the choice of \(n_0\), we have
\[
2b_n < \text{dist} (y, 2q_n \mathbb{Z}) < 4q_{n-n_0+1}.
\]

and also
\[
sq_{n-n_0} < q_{n-n_0+1}.
\]
then we have
Lemma 6.2. For a not even-resonant $y$, for $n > N(\varepsilon)$ large enough, we have $\{\theta_{t}\}_{t \in \tilde{I}_0 \cup \tilde{I}_y}$ are $\varepsilon$-uniform.

This is essentially Lemma 4.1 in [19], we thus omit the proof. Then we have the following:

Corollary 6.3. There exists $x_1 \in \tilde{I}_0 \cup \tilde{I}_y$ such that

$$\left| \tilde{P}_{4\sigma_{q_{n-\eta}-1}}(\theta_{x_1}) \right| \geq e^{(\tilde{L} - 2\varepsilon)(4\sigma_{q_{n-\eta}-1})}.$$  

Proof. It follows from Lemma 5.4 and Lemma 6.2. \qed

By a standard argument, we have the following:

Lemma 6.4. For $n > N(\varepsilon)$ large enough, there exists $x_1 \in \tilde{I}_y$ so that

$$\left| \tilde{P}_{4\sigma_{q_{n-\eta}-1}}(\theta_{x_1}) \right| \geq e^{(\tilde{L} - 2\varepsilon)(4\sigma_{q_{n-\eta}-1})}.$$  

Proof. Suppose otherwise, by Corollary 6.3, we have that for some $x_1 \in \tilde{I}_0$,

$$\begin{align*}
\left| \tilde{P}_{4\sigma_{q_{n-\eta}-1}}(\theta_{x_1}) \right| \geq e^{(\tilde{L} - 2\varepsilon)(4\sigma_{q_{n-\eta}-1})}.
\end{align*}$$

Denoting $x_2 := x_1 + 2\sigma_{q_{n-\eta}-1} - 1$ and $I := [2x_1 + 1, 2x_2 + 1]$. By the Green’s formula, we have

$$\begin{align*}
0 & \leq |G_I(2x_1, 0)| \cdot |\phi(2x_1)| + |G_I(2x_2 + 1, 0)| \cdot |\phi(2x_2 + 2)|
\end{align*}$$

where we used $\partial I(y) = \{z_1, z_2\}$. By Green’s function expansion, we have

$$\begin{align*}
\phi(y) &= \sum_{z \in \partial I(y)} G_I(y, z) \phi(z').
\end{align*}$$

If $z_1 = 2x_1 - 2 > 2\sigma_{q_{n-\eta}-1} + 2b_n$ or $z_2 = 2x_2 < 2(\ell + 1)q_n - 2b_n$, we could expand $\phi(x_1 - 2)$ or $\phi(x_2)$.

We will continue this process until we arrive at a $z$ so that $z \leq 2\ell q_n + 2b_n$ or $z \geq (2\ell + 1)q_n - 2b_n$, or the iterating number reaches $t_0' := \left\lceil \frac{24}{\pi n} \right\rceil + 1$. We obtain, after a series of expansions, the following

$$\begin{align*}
\phi(2k) &= \sum_{z_{i+1}; z_{i+1} \in I(z_i')} G_I(y, z_{i+1}) G_I(z_{i+1}; z_2) \cdots G_I(z_{i+1}; z_{i+1}) \phi(z_{i+1}).
\end{align*}$$

where $z_{i+1}'$ either satisfies

- Case 1: $2\ell q_n \leq z_{i+1}' \leq 2\ell q_n + 2b_n$ and $t < t_0$ or,
- Case 2: $2(\ell + 1)q_n \geq z_{i+1}' \geq 2(\ell + 1)q_n - 2b_n$ and $t < t_0$ or,
- Case 3: $t = t_0$.

For simplicity, let us denote $y = z_0'$. If $z_{i+1}'$ satisfies Case 1. For each $z_j'$, $0 \leq j \leq t$, denote $\partial I(z_j') = \{z_{j+1}, y_{j+1}\}$. Combing with corollary 4.3, Lemma 6.4 and Lemma 4.2, we have

$$\begin{align*}
\left| G_I(z_j') \right| & \leq C(\varepsilon) e^{-|z_j'-z_{j+1}|(L - 12\varepsilon)},
\end{align*}$$

Furthermore

$$\begin{align*}
\sum_{z_{i+1} \in I(z_i')} G_I(y, z_{i+1}) G_I(z_{i+1}; z_2) \cdots G_I(z_{i+1}; z_{i+1}) \phi(z_{i+1}) & \leq (C(\varepsilon))^{t_0} e^{-(y - 2\ell q_n - 2b_n)(L - 12\varepsilon)} \langle t \rangle.
\end{align*}$$
If \( z_{t+1}' \) satisfies Case 2., there must be a \( z_j' \) such that \( aq_n + m_n \in I'(z_j') \), we estimate similarly to Case 1, only modifying the estimate of the cosine product, we have

\[
G_{I(y)}(y,z_1)\ldots G_{I(z'_t)}(z'_t,z_{t+1}) \phi(z'_{t+1}) \leq (C(\varepsilon))^{t+1} e^{\varepsilon q_n e^{-(2(\ell+1)q_n-y)(L-12c)}}c_{n,\ell}r_{\ell+1}.
\]

If \( z_{t+1}' \) satisfies Case 3, we bound \( |\phi(z'_{t+1})| \) by

\[
|\phi(z'_{t+1})| \leq \begin{cases} 
|\phi(x_0^-)|, & \text{if } z'_{t+1} \in I^- \\
|\phi(2(\ell q_n+m_n))|, & \text{if } z'_{t+1} = 2\ell q_n + 2m_n \\
|\phi(x_0^+)|, & \text{if } z'_{t+1} \in I^+ 
\end{cases}
\]

Using the Green’s function estimate (6.6), we have

\[
\left| G_{I(y)}(y,z_1)\ldots G_{I(z'_t)}(z'_t,z_{t+1}) \phi(z'_{t+1}) \right| 
\leq (C(\varepsilon))^{t+1} e^{-\frac{1}{2}r_{\ell,\ell+1}} \max \{|\phi(x_0^-)|, |\phi(2\ell q_n + 2m_n)|, c_{n,\ell} |\phi(x_0^+)|\}
\leq e^{-6q_n(L-12c)} \max \{|\phi(x_0^-)|, |\phi(2\ell q_n + 2m_n)|, c_{n,\ell} |\phi(x_0^+)|\}.
\]

Taking into account all the three cases (6.7), (6.8) and (6.10), we have proved that for even point \( y \in I^- \),

\[
|\phi(y)| \leq (C(\varepsilon))^{t+1} \max \{ e^{\varepsilon q_n e^{-(y-2q_n)(L-12c)}r_{\ell,\ell}, e^{\varepsilon q_n e^{-(2(\ell+1)q_n-y)(L-12c)}c_{n,\ell}r_{\ell+1}}, e^{\varepsilon q_n e^{-(2(\ell+1)q_n-y)(L-12c)}c_{n,\ell}r_{\ell+1}}, e^{-3q_n(L-12c)} \max \{|\phi(x_0^-)|, |\phi(2\ell q_n + 2m_n)|, |\phi(x_0^+)|\} \}.
\]

Letting \( y = x_0^- \), we have \(|\phi(x_0^-)| \leq (C(\varepsilon))^{t+1} \max (r_{\ell,\ell+1}) \).

Similarly, one can show that for \( y \in I^+ \),

\[
|\phi(y)| \leq (C(\varepsilon))^{t+1} \max \{ e^{\varepsilon q_n e^{-(y-2q_n)(L-12c)}c_{n,\ell}r_{\ell,\ell}, e^{\varepsilon q_n e^{-(2(\ell+1)q_n-y)(L-12c)}c_{n,\ell}r_{\ell+1}}, e^{-3q_n(L-12c)} \max \{|\phi(x_0^-)|, |\phi(2\ell q_n + 2m_n)|, |\phi(x_0^+)|\} \}.
\]

and

\[
|\phi(2\ell q_n + 2m_n)| \leq (C(\varepsilon))^{t+1} c_{n,\ell} \max \{ e^{\varepsilon q_n e^{-(y-2q_n)(L-12c)}r_{\ell,\ell}, e^{\varepsilon q_n e^{-(2(\ell+1)q_n-y)(L-12c)}r_{\ell+1}}, e^{-3q_n(L-12c)} \max \{|\phi(x_0^-)|, |\phi(2\ell q_n + 2m_n)|, |\phi(x_0^+)|\} \}.
\]

Letting \( y = x_0^+ \) in (6.12), together with (6.13), we have

\[
\max \{|\phi(x_0^-)|, |\phi(x_0^+)|, |\phi(2\ell q_n + 2m_n)|\} \leq (C(\varepsilon))^{t+1} \max (r_{\ell,\ell+1}) \).
\]

Plugging this back into (6.11), (6.12), (6.13), we obtain the claimed result for all even points. Combining these with (5.32), we obtain the claimed result for all \( y \in \mathbb{N} \).

6.1.2. \( y \) is even-resonance.

\textbf{Lemma 6.5.} For any \( \ell \neq 0, |\ell| \leq q_{n+1}/(6q_n), \)

\[
r_{\ell,\ell+1} \leq e^{-(2(\ell-55)q_n)} \frac{1}{\max(|\ell|, 1)} \max (r_{\ell-1,\ell+1}) \times \begin{cases} 
\max (|\ell|, e^{\delta_n q_n}), & \text{if } \beta_n \geq \delta_n + 200\varepsilon \\
\max (|\ell|, e^{\delta_n q_n}), & \text{if } \beta_n < \delta_n + 200\varepsilon
\end{cases}
\]

\textbf{Proof.} For \( \ell \in \mathbb{Z} \), let \( I_{\ell} \) be defined below

\[
I_{\ell} := [\ell - 1, q_n - q_n/2, \ell q_n - q_n/2 - 1] \cap \mathbb{Z}.
\]

for \( \ell > 0 \) and

\[
I_0 := [-q_n - q_n/2, q_n - q_n/2] \cap \mathbb{Z}.
\]

\textbf{Lemma 6.6} (Lemma 4.3, [19]). For \( \ell \) such that \( 0 < |\ell| \leq 2q_{n+1}/(3q_n) \), there exists \( x_1 \in I_0 \cup I_{\ell} \) such that

\[
\hat{P}_{4q_n-1}(\theta_{x_1}) \geq \frac{|\ell| e^{(L-2c)(4q_n-1)}}{q_n+1}.
\]

More precisely,
Lemma 6.8. For any $\ell \neq 0, |\ell| \leq q_{n+1}/(6q_n)$, there exists $x_1 \in I_{\ell}$ such that

$$\left|\tilde{P}_{4q_n-1}(\theta_{x_1})\right| \geq \max(|\ell|, 1)e^{-\beta_n q_n e^{(L-2\delta)(4q_n-1)}}.$$

Proof. Similar to the argument in the proof of Lemma 6.4, we have that for any $x_1 \in I_0$ so that

$$\left|\tilde{P}_{4sq_n-1}(\theta_{x_1})\right| \leq \frac{|\ell|}{q_{n+1}}e^{(L-2\delta)(4q_n-1)}.$$

Therefore Lemma 6.8 holds.

Lemma 6.9. Assume that there exists $x_1 \in I_{\ell}$ such that

$$\left|\tilde{P}_{4q_n-1}(\theta_{x_1})\right| \geq \max(|\ell|, 1)e^{-\beta_n q_n e^{(L-2\delta)(4q_n-1)}}. \tag{6.14}$$

Then we have

$$r_{\ell} \leq \frac{e^{-(2L-\beta_n-55\delta)q_n}}{\max(|\ell|, 1)} \max\{c_{n,\ell-1}r_{\ell-1}, c_{n,\ell}r_{\ell+1}\}. \tag{6.15}$$

This is a variant of Lemma 8.3 in [19]. If $\beta_n \geq \delta_n + 200\varepsilon$, bound the $c_{n,j}$’s by $c_{n,j}=0$. Otherwise trivially bound the $c_{n,j}$’s by 1, then Lemma 6.5 follows from combining Lemma (6.9) with Lemma 6.8.

Proposition 5.13 follows directly by Lemma 6.1 and Lemma 6.5.

6.2. Proofs of Proposition 5.14.

6.2.1. $y$ is not even-resonance.

Lemma 6.10. If $2\ell q_n + 2b_n < y < 2(\ell + 1)q_n - 2b_n$, for some $|\ell| \leq q_{n+1}/(6q_n)$. Then

$$|\phi(y)| \leq c_{40q_n} \max\{e^{-y-2\ell q_n}L_{r_{\ell}^+}, e^{-(2(\ell+1)q_n-y)\ell q_{n+1}}\}.$$ 

Proof. The proof of this lemma is almost identical to that of Lemma 6.1. We only give a brief proof. By Green’s function expansion, we have

$$\phi(y) = \sum_{z \in \partial I(y)} G_{I(y)}(z,y)\phi(z').$$

If $2x_1 - 2 > 2\ell q_n + 2b_n$ or $2x_2 < 2(\ell + 1)q_n - 2b_n$, we continue to expand $\phi(2x_1 - 2)$ or $\phi(2x_2)$. We repeat this process until we arrive at a $z$ so that $z \leq 2\ell q_n + 2b_n$ or $z \geq 2(\ell + 1)q_n - 2b_n$, or the iterating number reaches $t_0 := [24/\tau_n] + 1$. We obtain, after a series of expansions, the following

$$\phi(y) = \sum_{z : z_{t+1} \in \ell(z')} G_{I(y)}(y,z_1)G_{I(z'_1)}(z'_1,z_2) \cdots G_{I(z'_{t_0})}(z'_{t_0},z_{t+1}) \phi(z'_{t_0}). \tag{6.15}$$

where $z'_{t+1}$ either satisfies

- Case 1: $z'_{t+1} \in R_{\ell}^+ \cup \{2\ell q_n + 2m_n\}$ and $t < t_0$ or,
- Case 2: $z'_{t+1} \in R_{\ell}^- \cup \{2\ell q_n + 2m_n\}$ and $t < t_0$ or,
- Case 3: $z'_{t+1} \in R_{\ell}^- \cup \{2\ell q_n + 2m_n\}$ and $t < t_0$ or,
- Case 4: $t = t_0$.

Therefore, we have

$$|\phi(y)| \leq (C\varepsilon)^{t_0}e^{18\varepsilon q_n} \max\{e^{-y-2\ell q_n}L_{r_{\ell}^+}, c_{n,\ell}e^{-y-2\ell q_n}L_{r_{\ell}^-}, e^{-(2(\ell+1)q_n-y)\ell q_{n+1}}\}.$$ 

Then, we will use the following lemmas to study the relation of $r_{\ell}^-$ and $r_{\ell}^+$.

Lemma 6.11. (Corollary 5.8, [19]) Let $I = [2\ell_1, 2\ell_2] \subset Z$ be such that $2\ell_1 \in [2(j-1)q_n + 2m_n - 2, 2jq_n + 2m_n - 2]$ and $2\ell_2 \in [2jq_n + 2m_n + 2, 2(j+1)q_n + 2m_n - 2]$, for some $j \in \mathbb{Z}, |j| < q_{n+1}/(6q_n)$. For $n > N(\varepsilon)$ large enough we have

$$\|A_{I/2}(\theta_{\ell_1})\| \leq e^{7q_n} \frac{1}{c_{n,j}} e^{L|I|}.$$
Thus we have
\[
    r_{\ell}^- \leq e^{18 \varepsilon q_n} \frac{1}{c_{n,\ell}} r_{\ell}^+.
\]
Hence (6.16) yields
\[
    |\phi(y)| \leq e^{40 \varepsilon q_n} \max \left( e^{-(y - 2\ell q_n) L_{r_{\ell}^+}}, e^{-(2\ell + 1) q_n - y} L_{r_{\ell}^+} \right).
\]
This proves the claimed result.

6.2.2. *y is even-resonance.* Assume without loss of generality that $0 < m_n \leq b_n$. The main lemma of this section is the following.

**Lemma 6.12.** For any $\ell \neq 0$ such that $|\ell| < q_n + 1 / (6 q_n)$, we have
\[
    r_{\ell} \leq \frac{e^{-(2L - 70\varepsilon) q_n}}{\max(|\ell|, 1)} \\max (r_{\ell-1}^- r_{\ell+1}^- + 1, r_{\ell-1}^+ r_{\ell+1}^+),
\]
where
\[
    r_{\ell}^- \leq \frac{e^{-(2L - 70\varepsilon) q_n}}{\max(|\ell|, 1)} \\max (c, r_{\ell-1}^- r_{\ell+1}^+ + 1, c, r_{\ell-1}^+ r_{\ell+1}^-),
\]
and
\[
    r_{\ell}^+ \leq \frac{e^{-(2L - 70\varepsilon) q_n}}{\max(|\ell|, 1)} \\max (c, r_{\ell-1}^- r_{\ell+1}^+ + 1, c, r_{\ell-1}^+ r_{\ell+1}^-).\]

**Proof.** This argument is very similar to that of Lemma 6.5. Firstly, we need the following Lemma:

**Lemma 6.13.** Assume that there exists $x_1 \in I_{\ell}$, for some $|\ell| < q_n + 1 / (6 q_n)$, such that
\[
    \left| \tilde{P}_{4q_n - 1}(\theta_{x_1}) \right| \geq \max(|\ell|, 1) e^{-\beta q_n} e^{(L - 2\varepsilon)(4q_n - 1)}.
\]
We have
\[
    r_{\ell}^+ \leq \frac{e^{-(2L - 70\varepsilon) q_n}}{\max(|\ell|, 1)} \\max (c, r_{\ell-1}^- r_{\ell+1}^+ + 1, c, r_{\ell-1}^+ r_{\ell+1}^-),
\]
and
\[
    r_{\ell}^- \leq \frac{e^{-(2L - 70\varepsilon) q_n}}{\max(|\ell|, 1)} \\max (c, r_{\ell-1}^+ r_{\ell+1}^- + 1, c, r_{\ell-1}^- r_{\ell+1}^+).\]

This is a variant of Lemma 9.3 in [19]. If $\beta_n \geq \delta_n + 200\varepsilon$, bound the $c_n, j$'s by (5.22). Otherwise trivially bound the $c_n, j$'s by 1, then combining Corollary Lemma 6.8 with Lemma 6.13, for any $\ell \neq 0$ such that $|\ell| \leq q_n + 1 / (6 q_n)$, the following hold
\[
    r_{\ell} \leq \frac{e^{-(2L - 70\varepsilon) q_n}}{\max(|\ell|, 1)} \\max (r_{\ell-1}^- r_{\ell+1}^- + 1, r_{\ell-1}^+ r_{\ell+1}^+),
\]
It should be noted that
\[
    \frac{e^{-(2L - 70\varepsilon) q_n}}{\max(|\ell|, 1)} \\max (r_{\ell-1}^+ r_{\ell+1}^- + 1, r_{\ell-1}^- r_{\ell+1}^+) \leq e^{-(2L - 70\varepsilon) q_n} \leq 1,
\]
so the $r_{\ell}$ terms on the right-hand-side of the equation above can be dropped. This proves Lemma 6.12.

Proposition 5.14 is a consequence of Lemma 6.10 and Lemma 6.12.

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We have
\[
\tilde{P}_{2k}(2\theta) = \det \begin{bmatrix}
t_1 & c_1 & \cdots & c_{2k-1} \\
\tilde{c}_2 & t_2 & \cdots & \tilde{c}_2 \\
\vdots & \vdots & \ddots & \vdots \\
\tilde{c}_{2k} & \tilde{c}_2 & \cdots & t_{2k}
\end{bmatrix}_{2k \times 2k}
\]
where \( t_{2j} \triangleq E \cos 2\pi (\theta + j\alpha) - \lambda \sin 2\pi (\theta + j\alpha), \) \( t_{2j+1} \triangleq E, \) \( c_{2j} \triangleq -\cos 2\pi (\theta + j\alpha) \) and \( c_{2j+1} \triangleq -1. \) Then
\[
\begin{cases}
\tilde{t}_{2j}(z) \triangleq e^{2\pi i j\alpha} \cdot t_{2j}(z) = \frac{E + \lambda}{2} e^{4\pi i j\alpha} z^2 + \frac{E - \lambda}{2}, \\
\tilde{c}_{2j}(z) \triangleq e^{2\pi i j\alpha} \cdot c_{2j}(z) = \frac{1}{2} e^{4\pi i j\alpha} z^2 - \frac{1}{2},
\end{cases}
\]
and
\[
\begin{cases}
\tilde{t}_{2j+1}(z) \triangleq t_{2j+1}(z), \\
\tilde{c}_{2j+1}(z) \triangleq c_{2j+1}(z).
\end{cases}
\]
Since \(|z| = 1\), we have
\[
(A.1) \quad \left| \tilde{P}_{2k}(2\theta) \right| = |f_k(z)| = \det \begin{bmatrix}
\tilde{t}_1(z) & \tilde{c}_1(z) & \cdots & \tilde{c}_{k-2}(z) \\
\tilde{c}_2(z) & \tilde{t}_2(z) & \cdots & \tilde{c}_{k-2}(z) \\
\cdots & \cdots & \ddots & \cdots \\
\tilde{t}_{2k}(z) & \tilde{c}_{2k}(z) & \cdots & \tilde{c}_{2k}(z)
\end{bmatrix}_{2k \times 2k}.
\]
Clearly, \( \ln |f_k(z)| \) is a subharmonic function, therefore
\[
\frac{1}{k} \int_T \ln \tilde{P}_k(2\theta) \, d\theta = \frac{1}{k} \int_T \ln |f(e^{2\pi i \theta})| \, d\theta \geq \frac{1}{k} \ln |f_k(0)|.
\]
\[
f_{2k}(0) = \det \begin{bmatrix}
E & -1 & \cdots & -1 \\
-1/2 & (E - i\lambda)/2 & \cdots & -1/2 \\
-1 & \cdots & \cdots & \cdots \\
-1/2 & (E - i\lambda)/2 & \cdots & -1/2 \\
-1/2 & \cdots & \cdots & \cdots \\
\end{bmatrix}_{2k \times 2k}
\]
\[
= \frac{1}{(2)^k} \det \begin{bmatrix}
-E & 1 & \cdots & 1 \\
1 & i\lambda - E & \cdots & 1 \\
\cdots & \cdots & \cdots & \cdots \\
1 & i\lambda - E & \cdots & 1 \\
\end{bmatrix}_{2k \times 2k}
\triangleq \frac{1}{(2)^k} a_{2k}.
\]
Similarly, we can denote \( f_{2k+1}(0) = \frac{1}{(2)^k} b_{2k+1} \). Obviously \( b_{2k+1} = (i\lambda - E)a_{2k} - a_{2k-1} \) and \( a_{2k} = -Eb_{2k-1} - b_{2k-2} \). Thus
\[
|a_{2k}| \sim |b_{2k+1}| \sim C |x_2|^k \quad \text{as} \quad k \to \infty,
\]
where \(|x_1| < 1 < |x_2|\) are solutions of the characteristic equation
\[
x^2 - (E^2 - i\lambda E - 2)x + 1 = 0.
\]
Therefore we have
\[
\lim_{k \to \infty} \frac{1}{k} \int_0^1 \ln \left| \tilde{P}_k(\theta) \right| \, d\theta \geq \ln |x_2| - \frac{\ln 2}{2}.
\]
Then the result follows from Lemma 3.2.
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