Breakdown of strong solutions of the Thermal Quasi-Geostrophic equation

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Abstract. The thermal Quasi-Geostrophic equation is a coupled system of equations that governs the evolution of the buoyancy and the potential vorticity of a fluid. It has a local in time solution as proved in [3]. In this paper, we give criterion for the breakdown of solutions to the Thermal Quasi-Geostrophic (TQG) equations, in the spirit of the classical Beale–Kato–Majda blowup criterion (cf. [2]) for the solution of the Euler equation.

Keywords: Blowup criterion, Thermal quasigeostrophic equation, Modified Helmholtz operator.

1 Introduction

The Thermal Quasi-Geostrophic (TQG) equations is a coupled system of equations governed by the evolution of the buoyancy $b$ and the potential vorticity $q$ in the following way:

$$
\begin{align*}
\partial_t b + (u \cdot \nabla) b &= 0, \\
\partial_t q + (u \cdot \nabla)(q - b) &= -(u_h \cdot \nabla)b, \\
b(0, x) &= b_0(x), \\
q(0, x) &= q_0(x),
\end{align*}
$$

where

$$
\begin{align*}
\mathbf{u} = \nabla \perp \psi, \\
u_h = \frac{1}{2} \nabla \perp h, \\
q = (\Delta - 1) \psi + f.
\end{align*}
$$

Here, $\psi$ is the streamfunction and $h$ is the spatial variation around a constant bathymetry profile. Since we are working on the whole space, we can supplement our system with the far-field condition

$$
\lim_{|x| \to \infty} (b(x), u(x)) = 0.
$$

Our given set of data is $(\mathbf{u}_h, f, b_0, q_0)$ with regularity class:

$$
\begin{align*}
\mathbf{u}_h \in W^{3,2}(\mathbb{R}^2), \\
f \in W^{2,2}(\mathbb{R}^2), \\
b_0 \in W^{3,2}(\mathbb{R}^2), \\
q_0 \in W^{3,2}(\mathbb{R}^2).
\end{align*}
$$
The TQG equation models the dynamics of a submesoscale geophysical fluid in thermal geostrophic balance, for which the Rossby number, the Froude number and the stratification parameter are all of the same asymptotic order.

In the following, we are interested in strong solutions of the system (1.1)–(1.4). To be precise, we state the notion of strong solution below:

**Definition 1 (Local strong solution).** Let \((u_h, f, b_0, q_0)\) be of regularity class (1.5). For some \(T > 0\), we call the triple \((b, q, T)\) a strong solution to the system (1.1)–(1.4) if the following holds:

- The buoyancy \(b\) satisfies \(b \in C([0, T]; W^{3,2}(\mathbb{R}^2))\) and the equation
  \[ b(t) = b_0 - \int_0^t \text{div}(bu) \, d\tau, \]
  holds for all \(t \in [0, T]\);
- the potential vorticity \(q\) satisfies \(q \in C([0, T]; W^{2,2}(\mathbb{R}^2))\) and the equation
  \[ q(t) = q_0 - \int_0^t \left[ \text{div}((q - b)u) + \text{div}(bu_h) \right] \, d\tau \]
  holds for all \(t \in [0, T]\).

Such local strong solutions exist on a maximal time interval. We define this as follows.

**Definition 2 (Maximal solution).** We call \((b, q, T_{\text{max}})\) a maximal solution to the system (1.1)–(1.4) if:

- there exists an increasing sequence of time steps \((T_n)_{n \in \mathbb{N}}\) whose limit is \(T_{\text{max}} \in (0, \infty]\);
- for each \(n \in \mathbb{N}\), the triple \((b, q, T_n)\) is a local strong solution to the system (1.1)–(1.4) with initial condition \((b_0, q_0)\);
- if \(T_{\text{max}} < \infty\), then
  \[ \limsup_{T_n \to T_{\text{max}}} \|b(T_n)\|^2_{W^{3,2}(\mathbb{R}^2)} + \|q(T_n)\|^2_{W^{2,2}(\mathbb{R}^2)} = \infty. \] (1.6)

We shall call \(T_{\text{max}} > 0\) the maximal time.

The existence of a unique strong solution of (1.1)–(1.4) has recently been shown in [3, Theorem 2.10] on the torus. A unique maximal solutions also exist [3, Theorem 2.14] and the result also applies to the whole space [3, Remark 2.1]. We state the result here for completeness.

**Theorem 1.** For \((u_h, f, b_0, q_0)\) of regularity class (1.5), there exist a unique maximal solution \((b, q, T)\) of the system (1.1)–(1.4).

Before we state our main result, let us first present some notations used throughout this work.
1.1 Notations

In the following, we write $F \lesssim G$ if there exists a generic constant $c > 0$ such that $F \leq cG$. For $k \in \mathbb{N} \cup \{0\}$ and $p \in [1, \infty]$, $W^{k,p}(\mathbb{R}^2)$ is the usual Sobolev space. For $p = 2$, $W^{k,2}(\mathbb{R}^2)$ is a Hilbert space with inner product $\langle u, v \rangle_{W^{k,2}(\mathbb{R}^2)} = \sum_{|\beta| \leq k} \langle \partial^\beta u, \partial^\beta v \rangle$, where $(\cdot, \cdot)$ denotes the standard $L^2$-inner product. For general $s \in \mathbb{R}$, we use the norm

$$\|v\|_{W^{s,2}(\mathbb{R}^2)} \equiv \left( \int_{\mathbb{R}^2} (1 + |\xi|^2)^s |\hat{v}(\xi)|^2 \, d\xi \right)^{\frac{1}{2}} \tag{1.7}$$

defined in frequency space. Here, $\hat{v}(\xi)$ denotes the Fourier coefficients of $v$. For simplicity, we write $\| \cdot \|_{s,2}$ for $\| \cdot \|_{W^{s,2}(\mathbb{R}^2)}$. When $k = s = 0$, we get the usual $L^2(\mathbb{R}^2)$ space whose norm we will simply denote by $\| \cdot \|_2$. A similar notation will be used for norms $\| \cdot \|_p$ of general $L^p(\mathbb{R}^2)$ spaces for any $p \in [1, \infty]$ as well as for the inner product $\langle \cdot, \cdot \rangle_{k,2} := \langle \cdot, \cdot \rangle_{W^{k,2}(\mathbb{R}^2)}$ when $k \in \mathbb{N}$. Additionally, $W^{k,p}_{\text{div}}(\mathbb{R}^2)$ represents the space of divergence-free vector-valued functions in $W^{k,p}(\mathbb{R}^2)$.

With respect to differential operators, we let $\nabla_0 := (\partial_{x_1}, \partial_{x_2}, 0)^T$ and $\nabla_0^\perp := (-\partial_{x_2}, \partial_{x_1}, 0)$ be the three-dimensional extensions of the two-dimensional differential operators $\nabla = (\partial_{x_1}, \partial_{x_2})^T$ and $\nabla^\perp := (-\partial_{x_2}, \partial_{x_1})$ by zero respectively. The Laplacian $\Delta = \text{div}\nabla = \partial_{x_1} \partial_{x_1} + \partial_{x_2} \partial_{x_2}$ remains two-dimensional.

1.2 Main result

Our main result is to give a blowup criteria, of Baele–Kato–Majda-type [2], for a strong solution $(b, q, T)$ of (1.1)–(1.4). In particular, we show the following result.

**Theorem 2.** Suppose that $(b, q, T)$ is a local strong solution of (1.1)–(1.4). If

$$\int_0^T \left( \|q(t)\|_\infty + \|\nabla b(t)\|_\infty \right) \, dt \equiv K < \infty, \tag{1.8}$$

then there exists a solution $(b', q', T')$ with $T' > T$, such that $(b', q') = (b, q)$ on $[0, T]$. Moreover, for all $t \in [0, T]$,

$$\|b(t)\|_{3,2} + \|q(t)\|_{2,2} \leq \left[ e + \|b_0\|_{3,2} + \|q_0\|_{2,2} \right] \exp(\text{cK}) \exp(cT \exp(cK)).$$

An immediate consequence of the above theorem is the following:

**Corollary 1.** Assume that $(b, q, T)$ is a maximal solution. If $T < \infty$, then

$$\int_0^T \left( \|q(t)\|_\infty + \|\nabla b(t)\|_\infty \right) \, dt = \infty$$

and in particular,

$$\sup_{t \leq T} \left( \|q(t)\|_\infty + \|\nabla b(t)\|_\infty \right) = \infty.$$
2 Blowup

We devote the entirety of this section to the proof of Theorem 2. In order to achieve our goal, we first derive a suitable exact solution for what is referred to as the modified Helmholtz equation. Some authors also call it the Screened Poisson equation while others rather mistakenly call it the Helmholtz equation.

2.1 Estimate for the 2D modified Helmholtz equation or the screened Poisson equation

In the following, we want to find an exact solution $\psi : \mathbb{R}^2 \to \mathbb{R}$ of

$$(\Delta - 1)\psi(x) = w(x), \quad \lim_{|x| \to \infty} \psi(x) = 0 \quad (2.1)$$

for a given function $w \in W^{2,2}(\mathbb{R}^2)$ that is sufficiently smooth. The corresponding two-dimensional free space Green’s function $G_{\text{free}}(x)$ for (2.1) must therefore solve

$$(\Delta - 1)G_{\text{free}}(x - y) = \delta(x - y), \quad \lim_{|x| \to \infty} G_{\text{free}}(x - y)(x) = 0 \quad (2.2)$$

Indeed, one can verify that the Green’s function is given by

$$G_{\text{free}}(x - y) = \frac{1}{2\pi} K_0(|x - y|) \quad (2.3)$$

see [1, Table 9.5], where

$$K_0(z) = \int_0^\infty \frac{e^{-\sqrt{z^2 + r^2}}}{\sqrt{z^2 + r^2}} \, dr$$

is the modified Bessel function of the second kind, see equation (8.432-9), page 917 of [4] with $\nu = 0$ and $x = 1$. Note that $\Gamma(1/2) = \sqrt{\pi}$. However, since the integral above is an even function, it follows that

$$G_{\text{free}}(x - y) = \frac{i}{4} H_{(1)}^{(1)}(i|x - y|) = \frac{1}{4\pi} \int_{\mathbb{R}} \frac{e^{-\sqrt{|x - y|^2 + r^2}}}{\sqrt{|x - y|^2 + r^2}} \, dr \quad (2.4)$$

which is the zeroth-order Hankel function of the first kind, see equation (11.117) in [1] and equation (8.421-9) of [4] on page 915. Therefore,

$$\psi(x) = \frac{1}{4\pi} \int_{\mathbb{R}^2} \frac{e^{-|x - y - r|}}{|(x - y, -r)|} w((y, 0)) \, dy \, dr$$

$$=: \psi((x, 0))$$

where we have used the identity $\sqrt{|x - y|^2 + r^2} = |(x, 0) - (y, r)| = |(x - y, -r)|$. We can therefore view the argument of the streamfunction $\psi$ as a 3D-vector with zero vertical component.
2.2 Log-Sobolev estimate for velocity gradient

Our goal now is to find a suitable estimate for the Lipschitz norm of $\mathbf{u}$ that solves

$$
\mathbf{u} = \nabla^\perp \psi, \quad (\Delta - 1) \psi = w
$$

(2.5)

where $w \in W^{2,2}(\mathbb{R}^2)$ is given. In particular, inspired by [2], we aim to show the following result.

**Proposition 1.** For a given $w \in W^{2,2}(\mathbb{R}^2)$, any $\mathbf{u}$ solving (2.5) satisfies

$$
\|\mathbf{u}\|_{1,\infty} \lesssim 1 + (1 + 2 \ln^+ (\|w\|_{2,2})) \|w\|_{\infty}
$$

(2.6)

where $\ln^+ a = \ln a$ if $a \geq 1$ and $\ln^+ a = 0$ otherwise.

**Proof.** To show (2.6), we fix $L \in (0,1]$ and for $\mathbf{z} \in \mathbb{R}^3$, we let $\zeta_L(\mathbf{z})$ be a smooth cut-off function satisfying

$$
\zeta_L(\mathbf{z}) = \begin{cases}
1 & : |\mathbf{z}| < L, \\
0 & : |\mathbf{z}| > 2L
\end{cases}
$$

and $0 \leq |\nabla_0 \nabla^\perp \zeta_L(\mathbf{z})| \lesssim L^{-2}$. This latter requirement ensures that the point of inflection of the graph of the cut-off, the portion that is constant, concave upwards and concave downwards are all captured. We now define the following

$$
B_1 := \{(y, r) \in \mathbb{R}^3 : |(x, 0) - (y, r)| = |(x - y, -r)| < 2L\},
$$

$$
B_2 := \{(y, r) \in \mathbb{R}^3 : L \leq |(x - y, -r)| \leq 1\},
$$

$$
B_3 := \{(y, r) \in \mathbb{R}^3 : |(x - y, -r)| > 1\},
$$

and let

$$
|\nabla \mathbf{u}(x)| = |\nabla_0 \nabla^\perp \psi((x, 0))| \leq |\nabla_0 (\mathbf{u}_1((x, 0), 0))| + |\nabla_0 (\mathbf{u}_2((x, 0), 0))|$

$$
+ |\nabla_0 (\mathbf{u}_3((x, 0), 0))| + |\nabla_0 \mathbf{u}_4| + |\nabla_0 \mathbf{u}_5| + |\nabla_0 \mathbf{u}_6|
$$

where

$$
\nabla_0 \mathbf{u}_1 := \frac{1}{4\pi} \int_{B_1} \zeta_L((x - y, -r)) \frac{\exp[-(|x-x-y,-r|)]}{(|x-x-y,-r|)^{3/2}} \nabla_0 \nabla^\perp w((y, 0)) \, dydr,
$$

$$
\nabla_0 \mathbf{u}_2 := \frac{1}{4\pi} \int_{B_2} [1 - \zeta_L((x - y, -r))] \nabla_0 \nabla^\perp \frac{\exp[-(|x-x-y,-r|)]}{(|x-x-y,-r|)} w((y, 0)) \, dydr,
$$

$$
\nabla_0 \mathbf{u}_3 := \frac{1}{4\pi} \int_{B_3} \nabla_0 \nabla^\perp [1 - \zeta_L((x - y, -r))] \frac{\exp[-(|x-x-y,-r|)]}{(|x-x-y,-r|)} w((y, 0)) \, dydr,
$$

$$
\nabla_0 \mathbf{u}_4 := \frac{1}{4\pi} \int_{B_1} \zeta_L((x - y, -r)) \frac{\exp[-(|x-x-y,-r|)]}{(|x-x-y,-r|)^{3/2}} \nabla_0 \nabla^\perp w((y, 0)) \, dydr,
$$

$$
\nabla_0 \mathbf{u}_5 := \frac{1}{4\pi} \int_{B_2} [1 - \zeta_L((x - y, -r))] \nabla_0 \nabla^\perp \frac{\exp[-(|x-x-y,-r|)]}{(|x-x-y,-r|)} w((y, 0)) \, dydr,
$$

$$
\nabla_0 \mathbf{u}_6 := \frac{1}{4\pi} \int_{B_3} \nabla_0 \nabla^\perp [1 - \zeta_L((x - y, -r))] \frac{\exp[-(|x-x-y,-r|)]}{(|x-x-y,-r|)} w((y, 0)) \, dydr.
$$
For \( L \in (0, 1) \), we have that

\[
|\nabla_0 u_1| \lesssim \left( \int_{B_1} \frac{e^{-2|\theta(x,0)-(y,r)|}}{|\theta(x,0)-(y,r)|^2} \, dy \, dr \right)^{1 \over 2} \| \nabla_0 \nabla_0^\perp w((y,0)) \|_2
\]

\[
\lesssim \left( \int_0^{2L} \frac{e^{-2s}}{s^2} \, ds \right)^{1 \over 2} \| u \|_{2,2} \lesssim (1-e^{-4L})^{1 \over 2} \| u \|_{2,2} \lesssim \pi L^{1 \over 2} \| u \|_{2,2}.
\]

Now note that

\[
\nabla_0 u_2^1 := \frac{1}{4\pi} \int_{B_2} \left[ 1 - \zeta_L((x-y,-r)) \right] \left\{ \frac{2(x-y)^T(x-y)}{|x-y|^2 + r^2} + \frac{3(x-y)^T(x-y)}{|x-y|^2 + r^2} \right\}
\]

\[
- \frac{1}{|x-y|^2 + r^2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \frac{1}{|x-y|^2 + r^2} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]

\[
+ \frac{(x-y)^T(x-y)}{|x-y|^2 + r^2} + \frac{(x-y)^T(x-y)}{|x-y|^2 + r^2} e^{-|x-y,-r|} w(y) \, dy \, dr
\]

\[
=: \sum_{i=1}^6 K_i((x-y,-r)).
\]

Clearly, \(|x-y|^2 \leq |x-y|^2 + r^2 = |(x-y,-r)|^2\) and for any \( L \in (0, 1) \), the inequalities

\[
(e^{-L} - e^{-1}) \lesssim (1-e^{-1}) \lesssim (1-e^{-1})(1 - \ln(L)) \lesssim (1 - \ln(L))
\]

holds independent of \( L \). Therefore, for \( L \in (0, 1) \), it follows that

\[
|K_1((x-y,-r))| + |K_3((x-y,-r))| + |K_6((x-y,-r))|
\]

\[
\lesssim \| u \|_{\infty} \int_{B_2} \frac{e^{-|x-y,-r|}}{|x-y,-r|^2} \, dy \, dr
\]

\[
\lesssim \| u \|_{\infty} \int_0^1 \frac{e^{-s}}{s^2} \, ds
\]

\[
\lesssim \| u \|_{\infty} (1 - \ln(L)).
\]

Again, we can use \(|x-y|^2 \leq |x-y|^2 + r^2\) and the fact that the inequalities

\[
(e^{-L}(L+1) - 2e^{-1}) \lesssim (1-2e^{-1}) \lesssim (1-2e^{-1})(1 - \ln(L)) \lesssim (1 - \ln(L))
\]

holds independent of any \( L \in (0, 1) \) to obtain

\[
|K_5((x-y,-r))| \lesssim \| u \|_{\infty} \int_{B_2} \frac{e^{-|x-y,-r|}}{|x-y,-r|^2} \, dy \, dr
\]

\[
\lesssim \| u \|_{\infty} \int_0^1 \frac{e^{-s}}{s} \, ds
\]

\[
\lesssim \| u \|_{\infty} (1 - \ln(L)).
\]
Finally, for $\mathbb{K}_2$ and $\mathbb{K}_4$, we also obtain
\[
\|\mathbb{K}_2((x - y, -r))\| + \|\mathbb{K}_4((x - y, -r))\| \lesssim \|w\|_\infty \int_{B_2} \frac{e^{-|(x-y,-r)|}}{|(x-y,-r)|^3} \, d\mathbf{y} \lesssim \|w\|_\infty \int_1^L \frac{1}{s^{3/2}} \, ds \\
\lesssim \|w\|_\infty e^{-L} \int_1^L \frac{1}{s} \, ds \\
\lesssim \|w\|_\infty e^{-0} (1 - \ln(L)).
\]

We have shown that
\[
|\nabla_0 \mathbf{u}_1| \lesssim \|w\|_\infty (1 - \ln(L)) \quad (2.7)
\]
for $L \in (0, 1]$. Also, the quantity $(1/L^2)[e^{-L}(L+1) - e^{-2L}(2L+1)]$ is uniformly bounded for any $L \in (0, 1]$ and as such,
\[
|\nabla_0 \mathbf{u}_2| \lesssim \left(\int_1^{2L} \frac{e^{-\frac{1}{sL^2}}}{s^{2/3}} \, ds\right) \|w\|_\infty \lesssim \|w\|_\infty. \quad (2.8)
\]

Similar to the estimate for $\nabla \mathbf{u}_1$, we have that
\[
|\nabla_0 \mathbf{u}_3| \lesssim \|w\|_\infty. \quad (2.9)
\]

It follows by summing up the various estimates above that
\[
\|\nabla \mathbf{u}\| \lesssim L^{1/2} \|w\|_{2,2} + (1 - \ln(L)) \|w\|_\infty. \quad (2.10)
\]

It remains to show that the estimate (2.10) also holds for $\mathbf{u}$. For this, we let
\[
B := \{(y, r) \in \mathbb{R}^3 : |(x-y,-r)| < d\}
\]
be a ball where $d > 1$ is chosen large enough so that on $\mathbb{R}^2 \setminus B$, we have that
\[
|x-y| \leq |x-y|^2. \quad (2.11)
\]

Now write
\[
\mathbf{u}(x) = \frac{1}{4\pi} \left( \int_{\mathbb{R}^3 \setminus B} + \int_B \right) \nabla_0^+ \left[ \frac{e^{-|(x-y,-r)|}}{|(x-y,-r)|} \right] w((y,0)) \, d\mathbf{y}dr \\
=: I_1 + I_2. \quad (2.12)
\]

For $I_1$, we use Hölder’s inequality, the fact that
\[
|\nabla_0^+ \frac{e^{-|(x-y,-r)|}}{|(x-y,-r)|}| \lesssim \left[ \frac{|x-y|}{(|x-y|^2 + r^2)^{1/2}} + \frac{|x-y|}{|x-y|^2 + r^2} \right] e^{-|(x-y,-r)|}
\]
and (2.11) to obtain
\[
|I_1| \lesssim \|w\|_{\infty} \int_{\mathbb{R}^3 \setminus B} \frac{1}{\left(|x| + r \right)^{\frac{d}{2}}} \frac{1}{\left(|x - y| + r \right)^{\frac{d}{2}}} e^{-|x - y - r|} dy dr
\]
\[
\lesssim \|w\|_{\infty} \int_0^{\infty} e^{-s} s^2 ds + \|w\|_{\infty} \int_0^{\infty} e^{-s} s^2 ds
\]
\[
\lesssim \|w\|_{\infty}.
\]
(2.13)

For \(I_2\), we use the fact that
\[
|x - y| \leq (|x| + r^2)^{\frac{d}{2}}
\]
(2.14)
to obtain
\[
|I_2| \lesssim \|w\|_{\infty} \int_{B} \frac{1}{\left(|x| + r \right)^{\frac{d}{2}}} \frac{1}{\left(|x - y| + r \right)^{\frac{d}{2}}} e^{-|x - y - r|} dy dr
\]
\[
\lesssim \|w\|_{\infty} \int_0^{d} e^{-s} s^2 ds + \|w\|_{\infty} \int_0^{d} e^{-s} s^2 ds
\]
\[
\lesssim \|w\|_{\infty}.
\]
(2.15)

Therefore, it follows from (2.10) and the estimates for \(I_1\) and \(I_2\) that
\[
\|u\|_{1, \infty} \lesssim L^{\frac{d}{2}} \|w\|_{2,2} + (1 - \ln(L)) \|w\|_{\infty}.
\]
(2.16)

If \(\|w\|_{2,2} \leq 1\), we choose \(L = 1\) and if \(\|w\|_{2,2} > 1\), we take \(L = \|w\|_{2,2}^{-2}\) so that (2.16) holds. This finishes the proof.

Before we end the subsection, we also note that a direct computation using the definition of Sobolev norms in frequency space (1.7) immediately yield
\[
\|u\|_{k+1, 2} \lesssim \|w\|_{k, 2}
\]
(2.17)
for any \(k \in \mathbb{N} \cup \{0\}\) where \(w \in W^{k, 2}(\mathbb{R}^2)\) is a given function in (2.5).

### 2.3 A priori estimate

In order to proof Theorem 2, we first need some preliminary estimates for \((b, q)\).

In the following, we define
\[
\|(b, q)\| := \|b\|_{3, 2} + \|q\|_{2, 2}.
\]

\textbf{Lemma 1.} A strong solution of (1.1) – (1.4) satisfies the bound
\[
\frac{d}{dt} \|(b, q)\|^2 \lesssim (1 + \|u\|_{1, \infty} + \|\nabla b\|_{\infty} + \|q\|_{\infty}) (1 + \|(b, q)\|^2).
\]
(2.18)
Proof. Since the space of smooth functions is dense in the space $W^{3,2}(\mathbb{R}^2) \times W^{2,2}(\mathbb{R}^2)$ of existence, in the following, we work with a smooth solution pair $(b, q)$. To achieve our desired estimate, we apply $\partial^\beta$ to (1.1) for $|\beta| \leq 3$ to obtain

$$
\partial_t \partial^\beta b + \mathbf{u} \cdot \nabla \partial^\beta b = R_1
$$

(2.19)

where

$$
R_1 := \mathbf{u} \cdot \partial^\beta \nabla b - \partial^\beta (\mathbf{u} \cdot \nabla b).
$$

Now since $\text{div}\mathbf{u} = 0$, if we multiply (2.19) by $\partial^\beta b$ and sum over the multiindex $\beta$ so that $|\beta| \leq 3$, we obtain

$$
\frac{d}{dt} \|b\|_{3,2}^2 \lesssim (\|\nabla \mathbf{u}\|_{\infty} \|b\|_{3,2} + \|\nabla b\|_{\infty} \|\mathbf{u}\|_{3,2}) \|b\|_{3,2}
$$

(2.20)

where we have used (2.17) for $w = q - f$ and $k = 2$.

Next, we find a bound for $\|q\|_{3,2}^2$. For this, we apply $\partial^\beta$ to (1.2) for $|\beta| \leq 2$ and we obtain

$$
\partial_t \partial^\beta q + \mathbf{u} \cdot \nabla \partial^\beta (q - b) + \mathbf{u}_h \cdot \nabla \partial^\beta b = R_2 + R_3 + R_4
$$

(2.21)

where

$$
R_2 := \mathbf{u} \cdot \partial^\beta \nabla q - \partial^\beta (\mathbf{u} \cdot \nabla q),
$$

$$
R_3 := -\mathbf{u} \cdot \partial^\beta \nabla b + \partial^\beta (\mathbf{u} \cdot \nabla b),
$$

$$
R_4 := \mathbf{u}_h \cdot \partial^\beta \nabla b - \partial^\beta (\mathbf{u}_h \cdot \nabla b).
$$

Now notice that for $\nabla \mathbf{u} := \nabla \mathbf{u}$, it follows from interpolation that

$$
\|\nabla \mathbf{u}\|_4 \lesssim \|\nabla \mathbf{u}\|_{\infty}^{\frac{2}{3}} \|\nabla^2 \mathbf{u}\|_{2}^{\frac{1}{3}}
$$

and so,

$$
\|\nabla^2 \mathbf{u}\|_4 \lesssim \|\nabla \mathbf{u}\|_{\infty}^{\frac{2}{3}} \|\mathbf{u}\|_{3,2}^{\frac{1}{3}}.
$$

Similarly

$$
\|\nabla q\|_4 \lesssim \|\nabla q\|_{\infty}^{\frac{2}{3}} \|q\|_{2,2}^{\frac{1}{3}}.
$$

Therefore,

$$
\|\nabla q\|_4 \|\nabla^2 \mathbf{u}\|_4 \lesssim \|q\|_{\infty} \|\nabla \mathbf{u}\|_{3,2} + \|\nabla \mathbf{u}\|_{\infty} \|q\|_{2,2}.
$$

By using this estimate, we deduce from (2.17) and commutator estimates that

$$
\|R_2\|_2 \lesssim \|\nabla \mathbf{u}\|_{\infty} \|q\|_{2,2} + \|q\|_{\infty} (1 + \|q\|_{2,2}).
$$

(2.22)
The commutators $R_3$ and $R_4$ are easy to estimate and are given by
\begin{align}
\| R_3 \|_2 & \lesssim \| \nabla u \|_\infty \| b \|_{3,2} + \| \nabla b \|_\infty (1 + \| q \|_{2,2}). \\
\| R_4 \|_2 & \lesssim \| b \|_{3,2} + \| \nabla b \|_\infty,
\end{align}
respectively, for a given $u_h \in W^{3,2}(\mathbb{R}^2)$. Next, by using $\text{div} u = 0$, we obtain
\[ \langle (u \cdot \nabla \partial^\beta q), \partial^\beta q \rangle = 0. \tag{2.25} \]
Additionally, the following estimates holds true
\begin{align}
\left| \langle (u \cdot \nabla \partial^\beta b), \partial^\beta q \rangle \right| & \lesssim \| u \|_\infty \| b \|_{3,2}^2 + \| u \|_\infty \| q \|_{3,2}^2, \\
\left| \langle u_h \cdot \nabla \partial^\beta b, \partial^\beta q \rangle \right| & \lesssim \| b \|_{3,2}^2 + \| q \|_{2,2}^2,
\end{align}
since $u_h \in W^{3,2}(\mathbb{R}^2)$. If we now collect the estimates above (keeping in mind that $f \in W^{2,2}(\mathbb{R}^2)$ and $u_h \in W^{3,2}(\mathbb{R}^2)$), we obtain by multiplying (1.2) by $\partial^\beta q$ and then summing over $|\beta| \leq 2$, the following
\[ \frac{d}{dt} \| q \|_{2,2}^2 \lesssim (1 + \| u \|_{1,\infty} + \| \nabla b \|_\infty + \| q \|_{\infty})(1 + (b, q)^2). \tag{2.28} \]
Summing up (2.20) and (2.28) yields the desired result.

We now have all in hand to proof our main theorem, Theorem 2.

**Proof (Proof of Theorem 2).** In the following, we define the time-dependent function $g$ as
\[ g(t) := e + \|(b, q)(t)\|, \quad \text{for} \quad t \in [0, T]. \tag{2.29} \]
Next, without loss of generality, we assume that $f = 0$ so that from Proposition 4 we obtain
\[ \| u(t) \|_{1,\infty} \lesssim 1 + (1 + \ln \| q(t) \|_{2,2})(\| \nabla b(t) \|_\infty + \| q(t) \|_{\infty}) \tag{2.30} \]
for $t \in [0, T]$. Using the monotonic properties of logarithms, it follows from the above that
\[ \| u(t) \|_{1,\infty} \lesssim 1 + \ln(g(t))(\| \nabla b(t) \|_\infty + \| q(t) \|_{\infty}). \tag{2.31} \]
Furthermore, since $1 \leq \ln(e + |x|)$ for any $x \in \mathbb{R}$, we can deduce from the inequality above that
\[ \| u(t) \|_{1,\infty} + \| \nabla b(t) \|_\infty + \| q(t) \|_{\infty} \lesssim 1 + \ln(g(t))(\| \nabla b(t) \|_\infty + \| q(t) \|_{\infty}). \tag{2.32} \]
On the other hand, it follows from Lemma 4 that
\[ g(t) \leq g(0) \exp \left( c \int_0^t (1 + \| u(s) \|_{1,\infty} + \| \nabla b(s) \|_\infty + \| q(s) \|_{\infty}) \, ds \right) \tag{2.33} \]
for any \( t \in [0, T] \). Combining (2.32) and (2.33) yields
\[
g(t) \leq g(0) \exp \left( c \int_0^t \left( 1 + \ln[g(s)] \right) \left( \|\nabla b(s)\|_\infty + \|q(s)\|_\infty \right) ds \right).
\] (2.34)

We can now take logarithm of both sides and apply Grönwall’s lemma to the resulting inequality to obtain
\[
\ln[g(t)] \leq \left( \ln[g(0)] + cT \right) \exp \left( c \int_0^t \left( \|\nabla b(s)\|_\infty + \|q(s)\|_\infty \right) ds \right).
\] (2.35)

At this point, we can now utilize (1.8), take exponentials in (2.35) and obtain
\[
\|b(t)\| + \|q(t)\| \leq [g(0)]^{\exp(cK)} \exp[cT \exp(cK)]
\] (2.36)
for any \( t \in [0, T] \). Since the right-hand side is finite, it follows that the solution \((b, q)\) can be continued on some interval \([0, T')\) for some \( T' > T \). This finishes the proof.

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