A REFINEMENT OF THE OZSVÁTH-SZABÓ LARGE INTEGER SURGERY FORMULA AND KNOT CONCORDANCE

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Abstract. We compute the knot Floer filtration induced by the \((n, 1)\)-cable of the meridian of a knot in the manifold obtained by large integer surgery along the knot. We give a formula in terms of the original knot Floer complex of the knot in \(S^3\). As an application, we show that the concordance invariant \(a_1(K)\) of \(\text{Hom}\) can equivalently be defined in terms of filtered maps on the Heegaard Floer homology groups induced by the two-handle attachment cobordism of surgery along a knot in \(S^3\).

1. Introduction

Let \(S^3_t(K)\) denote the manifold constructed as Dehn surgery along \(K \subset S^3\) with surgery coefficient \(t\). In [OS04] Ozsváth and Szabó construct a chain homotopy equivalence between certain subquotient complexes of the full knot Floer chain complex \(\text{CFK}^\infty(S^3, K)\) and Heegaard Floer chain complexes \(\widehat{\text{CF}}(S^3_t(K), s_m)\) for sufficiently large integers \(t\) for each spin\(^c\) structure \(s_m\). This equivalence is known as the large integer surgery formula.

The meridian \(\mu\) of \(K\) naturally lies inside of the knot complement \(S^3 \setminus K\) and the surgered manifold \(S^3_t(K)\). The meridian \(\mu\) induces a filtration on \(\widehat{\text{CF}}(S^3_t(K), s_m)\) for each spin\(^c\) structure \(s_m\). In [Hed07] Hedden gives a formula for the filtered complex \(\text{CFK}(S^3_t(K), \mu, s_m)\) in terms of \(\text{CFK}^\infty(S^3, K)\) for sufficiently large \(t\). As an application of this formula, Hedden computes the knot Floer homology of Whitehead doubles and the Ozsváth-Szabó concordance invariant \(\tau\) of Whitehead doubles. In [HKL16] Hedden, Kim, and Livingston generalize Hedden’s formula by computing the full knot Floer complex \(\text{CFK}^\infty(S^3_t(K), \mu, s_m)\) in terms of \(\text{CFK}^\infty(S^3, K)\) for sufficiently large \(t\). As an application to knot concordance, they show that the subgroup of topologically slice knots of the concordance group contains a \(\mathbb{Z}^\infty_2\) subgroup.

![Figure 1. The two-component link \(\mu_n\) and \(K\) for \(n = 5\)](image)

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We refine the theorems of Ozsváth-Szabó, Hedden and Hedden-Kim-Livingston to determine the filtered chain homotopy type of $\text{CFK}^\infty(S^3_t(K), \mu_n)$, where $\mu_n$ denotes the $(n, 1)$–cable of the meridian of $K$, viewed as a knot in $S^3_t(K)$. See Figure 1. For each spin$^c$ structure $s_m$, we show that the complex $\text{CFK}^\infty(S^3_t(K), \mu_n, s_m)$ is isomorphic to $\text{CFK}^\infty(S^3, K)$, but endowed with a different $\mathbb{Z} \oplus \mathbb{Z}$ filtration and an overall shift in the homological grading.

**Theorem 1.1.** Let $K$ be a knot in $S^3$ and fix $m, n \in \mathbb{Z}$. Then there exists $T = T(m, n) > 0$ such that for all $t > T$, the complex $\text{CFK}^\infty(S^3_t(K), \mu_n, s_m)$ is isomorphic to $\text{CFK}^\infty(S^3, K)[\epsilon]$ as an unfiltered complex, where $[\epsilon]$ denotes a grading shift that depends only on $m$ and $t$. Given a generator $[x, i, j]$ for $\text{CFK}^\infty(S^3, K)$, the $\mathbb{Z} \oplus \mathbb{Z}$ filtration level of the same generator, viewed as a chain in $\text{CFK}^\infty(S^3_t(K), \mu_n, s_m)$, is given by:

$$\mathcal{F}([x, i, j]) = \begin{cases} [i, i] & \text{if } j \leq m + i \\ [j - m, j - m - k] & \text{if } j = m + i + k, \text{ where } 1 \leq k < n \\ [j - m, j - m - n] & \text{if } j \geq m + i + n \end{cases}$$

As a corollary, the $\mathbb{Z}$-filtered complex $\widehat{\text{CFK}}(S^3_t(K), \mu_n, s_m)$ is isomorphic to a subquotient complex of $\text{CFK}^\infty(S^3, K)$, endowed with an $(n + 1)$ step filtration $\mathcal{F}$:

$$0 \subseteq C_{\{i < -n+1, j = m\}} \subseteq \cdots \subseteq C_{\{i < 0, j = m\}} \subseteq C_{\{\max(i, j - m) = 0\}}$$

This filtration is illustrated in Figure 2 in the case $n = 3$.

**Corollary 1.2.** Let $K \subset S^3$ be a knot, and fix $m, n \in \mathbb{Z}$. Then there exists $T = T(m, n) > 0$ such that for all $t > T$, the $\mathbb{Z}$–filtration on $\widehat{\text{CF}}(S^3_t(K), s_m)$ induced by $\mu_n \subset S^3_t(K)$ is isomorphic to the filtered chain homotopy type of the $(n + 1)$ step filtration on $C_{\{\max(i, j - m) = 0\}}$ described above.

![Figure 2](image)

**Figure 2.** $C_{\{\max(i, j - m) = 0\}}$ is the shaded region. The sub-regions bounded by the colored dots represent subcomplexes of the filtration $\mathcal{F}$ in the case $n = 3$.

As an application, we show that the concordance invariant $a_1(K)$ of $\text{Hom}[\text{Hom}14b]$ can equivalently be defined in terms of filtered maps on the Heegaard Floer homology groups induced by the two-handle attachment cobordism of surgery along a knot $K$ in $S^3$. The rationally null-homologous knot $\mu_n \subset S^3_t(K)$ induces a $\mathbb{Z}$-filtration of $\widehat{\text{CF}}(S^3_t(K), s_\tau)$ and $\widehat{\text{CF}}(S^3_{t-1}(K), s_\tau)$, that is, a sequence of subcomplexes:

$$0 \subset \mathcal{F}_{\text{bottom}} \subset \mathcal{F}_{\text{bottom}+1} \subset \cdots \subset \mathcal{F}_{\text{top} - 1} \subset \mathcal{F}_{\text{top}} = \widehat{\text{CF}}(S^3_t(K), s_\tau).$$
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Using the knot filtrations, an equivalent definition of $a_1(K)$ can be formulated in terms of the filtration $\mathcal{F}$ and $\mathcal{F}'$ induced by $\mu_n$ as a knot inside $S^3(K)$ and $S^3_{t}(K).

**Theorem 1.3.** Let $n > 2g(K)$. For sufficiently large surgery coefficient $t$, the concordance invariant $a_1(K)$ is equal to:

$$
a_1(K) = \begin{cases} \max \left\{ m \mid \hat{\mathcal{CF}}(S^3_t K, s_\tau) / \mathcal{F}_{t-1-m} \to \hat{\mathcal{CF}}(S^3) \right\}, & \text{if } \varepsilon(K) = -1, \\
0, & \text{if } \varepsilon(K) = 0, \\
\min \left\{ m \mid \hat{\mathcal{CF}}(S^3) \to \mathcal{F}_{t+m} \subseteq \hat{\mathcal{CF}}(S^3_{t} K, s_\tau) \right\}, & \text{if } \varepsilon(K) = 1. \end{cases}
$$

This interpretation of the invariant $a_1(K)$ offers a topological perspective that complements the original algebraic definition of $a_1(K)$. We will also include properties of the invariant $a_1(K)$ as well as computations of $a_1(K)$ for homologically thin knots and $L$–space knots.

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2. **The knot Floer filtration of cables of the meridian in Dehn surgery along a knot**

In this section we will refine the theorem of Ozsváth-Szabó to determine the filtered chain homotopy type of the knot Floer complex of $(S^3(K), \mu_n)$.

We begin by recalling the large integer surgery formula from Ozsváth and Szabó [OS04]. Let $(\Sigma, \alpha_1, \ldots, \alpha_g, \gamma_1, \ldots, \gamma_g, w, z)$ be a doubly-pointed Heegaard diagram for $\text{CFK}^\infty(S^3, K)$, where

- the curve $\gamma_g = \mu$ is a meridian of the knot $K$
- the curve $\alpha_g$ is a longitude for $K$
- there is a single intersection point in $\alpha_g \cap \gamma_g = x_0$
- the basepoints $w$ and $z$ lie on either side of $\gamma_g$

Let $\beta = \{\gamma_1, \ldots, \gamma_{g-1}, \lambda_t\}$ be the set of curves in $\gamma$, with $\gamma_g$ replaced by a longitude $\beta_g = \lambda_t$ winding $t$ times around $\mu$. Label the unique intersection point $\gamma_g \cap \beta_g = \theta$.

The Heegaard triple diagram $(\Sigma, \alpha, \beta, \gamma, w, z)$ represents a cobordism between $S^3$ and $S^3_{t}(K)$. See Figure 3.

Let $C_{\{\max(i, j - m)\}} = 0$ denote the subquotient complex of $\text{CFK}^\infty(S^3, K)$ generated by triples $[x, i, j]$ with the $i$ and $j$ filtration levels satisfying the specified constraints.
Theorem 2.1 ([OS04]). Let $K \subset S^3$ be a knot, and fix $m \in \mathbb{Z}$. Then there exists $T = T(m) > 0$ such that for all $t > T$, the chain map

$$
\Phi_m : \hat{CF}(S^3_t(K), s_m) \to C\{\max(i, j - m) = 0\}
$$

defined by

$$
\Phi_m([x]) = \sum_{y \in T_\alpha \cap T_\beta} \sum_{\psi \in \pi_2(x, \beta, y)} \{y, -n_z(\psi), m - n_z(\psi)\}
$$

induces an isomorphism of chain complexes.

Remark 2.2. Here, as usual, the labeling of the spin$^c$ structures is determined by the condition that $s_m$ can be extended over the cobordism $-W_t$ from $-S^3_t(K)$ to $-S^3$ associated to the two-handle addition along $K$ with framing $t$, yielding a spin$^c$ structure $\tau_m$ satisfying

$$
\langle c_1(\tau_m, [S]) \rangle + t = 2m.
$$

Above, $S$ denotes a surface in $W_t$ obtained from closing off a Seifert surface for $K$ in $S^3$ to produce a surface $S$ of square $t$.

We refine the theorem of Ozsváth-Szabó to determine the filtered chain homotopy type of the knot Floer complex of $(S^3_t(K), \mu)$. Consider the meridian $\mu = \mu_K$ of a knot $K$. The meridian $\mu$ naturally lies inside of the knot complement $S^3 \setminus K$ and the surgered manifold $S^3_t(K)$. For $n \in \mathbb{N}$, $\mu_n$ denotes the $(n, 1)$–cable of $\mu_K$, and also lies inside $S^3 \setminus K$ and the surgered manifold $S^3_t(K)$. The knot $\mu_n$ is homologically equivalent to $n \cdot [\mu]$ in $H_1(S^3_t(K))$. When $n = 1$, $\mu_1 = \mu$. See Figure 1 for a picture of the two-component link $K \cup \mu_n$.

For all $n \geq 1$ there is a natural $(n + 1)$–step algebraic filtration $\mathcal{F}$ on the subquotient complex $C_{\{\max(i, j - m) = 0\}}$ of $\text{CFK}^\infty(S^3, K)$:

$$
0 \subseteq C_{\{i < -n + 1, j = m\}} \subseteq \cdots \subseteq C_{\{i < 0, j = m\}} \subseteq C_{\{\max(i, j - m) = 0\}}.
$$

This filtration is illustrated in the case $n = 3$ in Figure 2.

Theorem 2.3 says that this algebraic filtration $\mathcal{F}$ corresponds to a relative $\mathbb{Z}$–filtration on $\hat{CF}(S^3_t(K), s_m)$ induced by $\mu_n \in S^3_t(K)$. This generalizes work of Hedden [Hed07] who studied the $n = 1$ case of the filtered complex $\hat{CF}(S^3_t(K), \mu, s_m)$.

Theorem 2.3. Let $K \subset S^3$ be a knot, and fix $m, n \in \mathbb{Z}$. Then there exists $T = T(m, n) > 0$ such that for all $t > T$, the following holds: The filtered chain homotopy type of the $(n+1)$–step filtration $\mathcal{F}$ on $C_{\{\max(i, j - m) = 0\}}$ described above is filtered chain homotopy equivalent to that of the filtration on $\hat{CF}(S^3_t(K), s_m)$ induced by $\mu_n \subset S^3_t(K)$. 
Proof. The key observation will be that the triple diagram \((\Sigma, \alpha, \beta, \gamma, w, z)\) used to define \(\Phi_m\) not only specifies a Heegaard diagram for the knot \((S^3, K)\), but also a Heegaard diagram for the knot \((S^3(K), \mu_n)\) with the addition of a basepoint \(z'\). Place an extra basepoint \(z' = z_n\) so that it is \(n\) regions away from the basepoint \(w\) in the Heegaard triple diagram representing the cobordism between \(S^3\) and \(S^3(K)\) as in Figure 4. (This can be accomplished if \(t\) is sufficiently large, e.g., if \(t > 2n\)). The knot represented by the doubly-pointed Heegaard diagram \((\Sigma, \alpha, \beta, w, z_n)\) is \(\mu_n\) in \(S^3(K)\).

An intersection point \(x' \in T_\alpha \cap T_\beta\) is said to be supported in the winding region if the component of \(x'\) in \(\alpha_g\) lies in the local picture of Figure 4. Intersection points in the winding region are in \(t\) to 1 correspondence with intersection points \(x\) in \(T_\alpha \cap T_\gamma\).

Fix a \(\text{Spin}^c\) structure \(s_m\) where \(m \in \mathbb{Z}\). For \(t\) (the surgery coefficient) sufficiently large, any generator \(x' \in T_\alpha \cap T_\beta\) representing \(\text{Spin}^c\) structure \(s_m\) is supported in the winding region. In this case, there is a uniquely determined \(x \in T_\alpha \cap T_\gamma\) and a canonical small triangle \(\psi \in \pi_2(x, \theta, x')\).

Suppose \(\psi \in \pi_2(x, \theta, x')\) is the canonical small triangle and \(x' \in T_\alpha \cap T_\beta\) is a generator representing \(\text{Spin}^c\) structure \(s_m\). If \(k = n_x(\psi) \geq 0\) (so \(n_w(\psi) = 0\)), then the \(\alpha_g\) component of \(x'\) is \(x_k\) (and lies \(k\) units to the left of \(x_0\)) in Figure 4. In this case, \(\Phi_m\) maps \(x'\) to \(C\{i = 0, j \leq m\}\). On the other hand, if \(x'\) is a generator with \(n_x(\psi) = 0\) and \(l = n_w(\psi) > 0\), then the \(\alpha_g\) component of \(x'\) is \(x_{-l}\) (and lies \(l\) steps to the right of \(x_0\)) in Figure 4. In this case, \(\Phi_m\) maps \(x'\) to the subcomplex \(C\{i \leq -l, j = m\} \subset C\{i < 0, j = m\}\).

Figure 4. Local picture of the winding region of the Heegaard triple diagram \((\Sigma, \alpha, \beta, \gamma, w, z_n)\) for the cobordism between \(S^3(K)\) and \(S^3\). The basepoint \(z_n\) is located \(n\) regions away from the basepoint \(w\) in the Heegaard diagram \((\Sigma, \alpha, \beta, w, z_n)\). Here we depict the basepoint \(z_n\) for \(n = 3\).

The following lemma (which generalizes Lemma 4.2 of [Hed07]) will be used to finish the proof.

**Lemma 2.4.** Let \(p \in \widehat{\text{CFK}}(S^3(K), \mu_n, s_m)\) be a generator supported in the winding region, and let \(x_i\) denote the \(\alpha_g\) component of the corresponding intersection point in \(T_\alpha \cap T_\beta\), where the \(x_i\) are labeled as in Figure 4. Then

\[
F(p) = \begin{cases} 
F_{\text{top}} & i > 0; \\
F_{\text{top} + i} & -n < i < 0; \\
F_{\text{bottom}} & i \leq -n.
\end{cases}
\]

Here, \(F_{\text{top}}\) (respectively, \(F_{\text{bottom}}\)) denotes the top (respectively, bottom) filtration level of \(\widehat{\text{CFK}}(S^3(K), \mu_n, s_m)\). \(F_{\text{top} + i}\) denotes the filtration level that is \(i\) lower than \(F_{\text{top}}\). In addition \(F_{\text{bottom}} = F_{\text{top} - n}\), so this is an \((n + 1)\)-step filtration.
Proof. The $\mathbb{Z}$-filtration $F$ is defined by the relative Alexander grading $A'_n$ induced by $\mu_n$ on $\text{CF}^\infty(S^3_t K, s_m)$. That is,

$$F(p) - F(q) = n_w(\phi) - n_w(\phi)$$

where $\phi \in \pi_2(p, q)$ is a Whitney disk connecting $p, q \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$.

Let $p, q \in \overline{\text{CFK}}(S^3_t K, \mu_n, s_m)$ be generators supported in the winding region, and let $x_i, x_j$ denote the $\alpha_q$ components of the corresponding intersection points $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$. Assume without loss of generality that $i < j$ (so that $x_i$ lies to the right of $x_j$).

We will define a set of $n$ arcs $\delta_1, \ldots, \delta_n$ on $\beta$ as follows. Let $\delta_1$ denote the arc on $\beta$ connecting $x_1$ to $x_{-1}$. Let $\delta_k$ denote on the arc on $\beta$ connecting $x_{-(k-1)}$ to $x_{-k}$, for $k \in \{2, \ldots, n\}$.

We will construct a Whitney disk $\phi_{p,q} \in \pi_2(p, q)$ with the following properties:

- If $i > 0$ and $j > 0$, (that is, $x_i, x_j$ both lie on the left of $x_0$), then $\partial \phi_{p,q}$ doesn’t contain any arc $\delta_k$. Therefore,

$$F(p) - F(q) = 0.$$

- If $i \leq -n$ and $j \leq -n$, (that is, $x_i, x_j$ both lie $n$ steps to the right of $x_0$), then $\partial \phi_{p,q}$ doesn’t contain any arc $\delta_k$. Therefore,

$$F(p) - F(q) = 0.$$

- If $i < -n$ and $j > 0$, (that is, $x_j$ lies to the left of $x_0$ and $x_i$ lies $i$ steps to the right of $x_0$), then $\partial \phi_{p,q}$ contains the $n$ arcs $\delta_1, \ldots, \delta_n$, each with multiplicity one. Therefore,

$$F(p) - F(q) = -n.$$

- If $-n \leq i < 0$ and $j > 0$, (that is, $x_j$ lies to the left of $x_0$ and $x_i$ lies $i$ steps to the right of $x_0$), then $\partial \phi_{p,q}$ contains the $i$ arcs $\delta_1, \ldots, \delta_i$, each with multiplicity one. Moreover, $\partial \phi_{p,q}$ doesn’t contain the arcs $\delta_k$ for $k > i$. Therefore,

$$F(p) - F(q) = -i.$$

- If $-n < j < 0$ and $i \leq -n$, (that is, $x_i$ lies $n$ steps to the right of $x_0$ and $x_j$ lies $j$ steps to the right of $x_0$), then $\partial \phi_{p,q}$ contains the $n + j$ arcs $\delta_{|\gamma|+1}, \ldots, \delta_n$, each with multiplicity one. Moreover, $\partial \phi_{p,q}$ doesn’t contain the arcs $\delta_k$ for $k \leq |\gamma|$. Therefore,

$$F(p) - F(q) = -n - j.$$

- If $-n < i < 0$ and $-n < j < 0$, (that is, $x_j$ lies $j$ steps to the right of $x_0$ and $x_i$ lies $i$ steps to the right of $x_0$), then $\partial \phi_{p,q}$ contains the $j - i$ arcs $\delta_{|\gamma|+1}, \ldots, \delta_{|\gamma|}$, each with multiplicity one. Therefore,

$$F(p) - F(q) = i - j.$$

Assuming the existence of such $\phi_{p,q}$, the lemma follows immediately.

In [Hed07, Lemma 4.2] Hedden constructs a Whitney disk $\phi_{p,q} \in \pi_2(p, q)$. The above enumerated properties of $\partial \phi_{p,q}$ will be immediate from the construction. We restate his construction here. Note first since $p, q$ lie in the winding region, they correspond uniquely to intersection points $\hat{p}, \hat{q} \in \mathbb{T}_\alpha \cap \mathbb{T}_\gamma$. These intersection points $\hat{p}, \hat{q}$ can be connected by a Whitney disk $\phi \in \pi_2(\hat{p}, \hat{q})$ with $n_w(\phi) = 0$ and $n_z(\phi) = k$ for some $k \in \mathbb{Z}_{\geq 0}$. This means that $\partial \phi$ contains $\gamma_{\phi}$ with multiplicity $k$, which further implies that the distance between $x_i$ and $x_j$ is $k$, that is, $i - j = k$. The domain
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of \( \phi_{p,q} \) can then be obtained from the domain of \( \phi \) by a simple modification in the winding region as described in [Hed07]. This modification is shown in Figure 5. It replaces the boundary component \( k \cdot \gamma_g \) by a simple closed curve from an arc connecting \( x_i \) and \( x_j \) along \( \alpha_g \) followed by an arc connecting \( x_j \) to \( x_i \) along \( \beta_g \), and which wraps \( k \) times around the neck of the winding region. □

This completes the description of the knot Floer complex \( \widehat{CFK}(S^3_t(K), \mu_n) \) in terms of the complex \( CFK^\infty(S^3, K) \). □

\( (\Lambda) \) The domain of a disk \( \phi \in \pi_2(\tilde{p}, \tilde{q}) \).

\( (\text{B}) \) \( \phi_{p,q} \in \pi_2(p, q) \) where \( p, q \) have \( \alpha_g \) components \( x_{-3}, x_{-1} \). \( \partial \phi_{p,q} \) contains arcs \( \delta_2 \) and \( \delta_3 \) on \( \beta \) drawn in violet.

\( (\text{C}) \) \( \phi_{p,q} \in \pi_2(p, q) \) where \( p, q \) have \( \alpha_g \) components \( x_{-2}, x_{-1} \).

\( (\text{D}) \) \( \phi_{p,q} \in \pi_2(p, q) \) where \( p, q \) have \( \alpha_g \) components \( x_1, x_2 \).

\( (\text{E}) \) \( \phi_{p,q} \in \pi_2(p, q) \) where \( p, q \) have \( \alpha_g \) components \( x_{-2}, x_1 \).

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\textbf{Figure 5.} The domain of a disk \( \phi_{p,q} \in \pi_2(p, q) \), for \( p, q \in T_\alpha \cap T_\beta \) in the winding region can be identified with the domain of a disk \( \phi \in \pi_2(\tilde{p}, \tilde{q}) \).

Theorem 2.3 described the \( \mathbb{Z} \)-filtered chain homotopy type of knot Floer chain complex \( \widehat{CFK}(S^3_t(K), \mu_n, s_m) \) for \( t \) large with respect to \( m \) and \( n \). In Theorem 1.1 we describe the \( \mathbb{Z} \oplus \mathbb{Z} \)-filtered chain homotopy type of \( CFK^\infty(S^3, K, \mu_n, s_m) \). This generalizes Theorem 4.2 of Hedden-Kim-Livingston [HKL16] which studies the \( n = 1 \) case.
Proof of Theorem 1.1. The isomorphism of chain complexes induced by the map (defined in [OS04])

\[ \Phi_m : \text{CF}^\infty(S^3_t(K), s_m) \rightarrow \text{CFK}^\infty(S^3, K) \]

respects the \( \mathbb{F}[U, U^{-1}] \)-module structure of both complexes, and hence determines one of the \( \mathbb{Z} \)-filtrations (called the \( U \)-filtration) of \( \text{CFK}^\infty(S^3_t(K), \mu_n, s_m) \).

The knot \( \mu_n \subset S^3_t(K) \) induces an additional \( \mathbb{Z} \)-filtration (the Alexander filtration) on \( \hat{\text{CF}}(S^3_t(K), s_m) \) and on \( \text{CFK}^\infty(Y_t(K), s_m) \). The additional \( \mathbb{Z} \)-filtration on \( \text{CFK}^\infty(Y_t(K), \mu_n, s_m) \) can be determined in exactly the same way as it was determined for the case of \( \text{CF}(S^3_t(K), s_m) \). Lemma 2.4 identifies the \( \mathbb{Z} \)-filtration induced on any given \( i = \text{constant} \) slice in \( \text{CF}^\infty(S^3_t, s_m) \) with a \( (n+1) \)-step filtration as above. This yields the statement of the theorem.

Alternatively, the additional (Alexander) \( \mathbb{Z} \)-filtration on \( \text{CFK}^\infty(Y_t(K), \mu_n, s_m) \) can be obtained from the Alexander filtration on \( \hat{\text{CFK}}(Y_t(K), \mu_n, s_m) \) by the fact that the \( U \) variable decreases Alexander grading by one, i.e. we have the relation \( A(U \cdot x) = A(x) - 1 \).

□

Corollary 2.5. Let \( K \) be a knot in \( S^3 \) and fix \( m, n \in \mathbb{Z} \). Then there exists \( T = T(m, n) > 0 \) such that for all \( t > T \) the following holds: Up to a grading shift,
the $p$th filtration level of $\CFK^\infty(S^3_t(K), \mu_n, s_m)$ is described in terms of the original $\mathbb{Z} \oplus \mathbb{Z}$–filtered knot Floer homology $\CFK^\infty(S^3, K)$

$$\max(i, j - m - n) = p.$$ 

That is, each Alexander filtration level $p$ of $\CFK^\infty(S^3_t(K), \mu_n, s_m)$ is a “hook” shaped region in $\CFK^\infty(S^3, K)$.

Proof. This follows from Theorem I.1 □

**Proposition 2.6.** Let $m \in \mathbb{Z}$ with $|m| \leq g(K)$ and let $n > 2g(K)$. For sufficiently large surgery coefficient $t$, the Alexander filtration induced by $\mu_n$ on $\CFK^\infty(S^3_t(K), s_m)$ coincides with the algebraic $i$–filtration on $\CFK^\infty(S^3, K)$ under the correspondence given by $\Phi_m$.

Proof. Since $\hat{\CF}(Y, K)$ has degree equal to the Seifert genus of the knot, $\CFK^\infty(Y, K)$ is supported along a thick diagonal of width $2g(K) + 1$. By the hypothesis, we have $m + n > g(K)$.

Therefore the corner $(p, m + n + p)$ of the hook region $C\{\max(i, j - m - n) = p\}$ of each constant Alexander filtration level $p$ of $\CFK^\infty(S^3_t(K), \mu_n, s_m)$ lies above the thick diagonal along which $\CFK^\infty(Y, K)$ is supported. See Figure 6. For spin$^c$ structures $s_m$ where $|m| \leq g(K)$, this means that the Alexander filtration induced by $\mu_n$ on $\CFK^\infty(S^3_t(K), \mu_n, s_m)$ coincides with the algebraic $i$–filtration on $\CFK^\infty(S^3, K)$ under the correspondence given by $\Phi_m$. □

Because the algebraic $i$–filtration is used to define concordance invariants (such as $a_1(K)$), which can be interpreted as an integer lift of the Hom $\varepsilon$ invariant [Hom14a], the filtration induced by $\mu_n$ on $\CFK^\infty(S^3_t(K), s_m)$ can be used to study the concordance class of a knot $K$. We will see that we can extract concordance invariants of $K$ from $\CFK^\infty(S^3_t(K), \mu_n, s_m)$.

3. A knot concordance invariant

As an application for the results in the previous section on the $\mathbb{Z}$–filtration induced on $\hat{\CF}(S^3_t(K), s_m)$ by the $(n, 1)$–cable of the meridian $\mu_n$, our main result in this section (Theorem 3.3) shows that the concordance invariant $a_1(K)$ of Hom [Hom14b], which has an algebraic definition in terms of maps on subquotient complexes of $\CFK^\infty(K)$, can be equivalently defined by studying filtered maps on the (hat version of the) Heegaard Floer homology groups induced by the two-handle attachment cobordism of large integer surgery along a knot $K$ in $S^3$ and the filtration induced by the knot $\mu_n$ inside of the surgered manifold.

Our result is analogous to the statement that the concordance invariants $\nu(K)$ of Ozsváth-Szabó [OS11] and $\varepsilon(K)$ of Hom [Hom14a] can be defined algebraically or in terms of maps on the (hat version of the) Heegaard Floer homology groups induced by the two-handle attachment cobordism of large integer surgery along a knot $K$ in $S^3$. Definition 3.1 gives an algebraic definition of $\varepsilon(K)$ in terms of certain chain maps on the subquotient complexes of the knot Floer chain complex $\CFK^\infty(K)$. Due to the Ozsváth-Szabó large integer surgery formula [OS04], $\varepsilon(K)$ can equivalently be defined in terms of maps on the Heegaard Floer chain complexes induced by the two-handle attachment cobordism of (large integer) surgery.
We begin by recalling the definition of the concordance invariants $\varepsilon(K)$. Let $N$ be a sufficiently large integer relative to the genus of a knot $K$. Consider the map 

$$F_s : \widehat{HF}(S^3) \to \widehat{HF}(S^3_{\infty}(K), [s]),$$

induced by the two-handle cobordism $W^4_{-N}$. Here, $[s]$ denotes the restriction to $S^3_{\infty}(K)$ of the Spin$^c$ structure $s$ over $W^4_{-N}$ with the property that

$$\langle c_1(s), [\widehat{F}] \rangle - N = 2s,$$

where $|s| \leq \frac{N}{2}$ and $\widehat{F}$ denotes the capped off Seifert surface in the four-manifold.

We also consider the map

$$G_s : \widehat{HF}(S^3_{\infty}(K), [s]) \to \widehat{HF}(S^3),$$

induced by the two-handle cobordism $-W^4_{3}$. The maps $F_s$ and $G_s$ can be defined algebraically by studying certain natural maps on subquotient complexes of $\text{CFK}^\infty(K)$, as in [OS04]. The map $F_s$ is induced by the chain map

$$C\{i = 0\} \to C\{\min(i, j - s) = 0\}$$

consisting of quotienting by $C\{i = 0, j < s\}$ followed by the inclusion. Similarly, the map $G_s$ is induced by the chain map

$$C\{\max(i, j - s) = 0\} \to C\{i = 0\}$$

consisting of quotienting by $C\{i < 0, j = s\}$ followed by the inclusion.

**Definition 3.1** ([Hom14a], [Hom14b]). Let $\tau = \tau(K)$ be the Ozsváth-Szabó concordance invariant. The invariant $\varepsilon(K)$ is defined as follows:

- $\varepsilon(K) = 1$ if $F_\tau$ is trivial (in which case $G_\tau$ is necessarily non-trivial).
- $\varepsilon(K) = -1$ if $G_\tau$ is trivial (in which case $F_\tau$ is necessarily non-trivial).
- $\varepsilon(K) = 0$ if $F_\tau$ and $G_\tau$ are both non-trivial.

In [Hom14b], Hom defines a concordance invariant $a_1(K)$ for knots with $\varepsilon(K) = 1$ that is a refinement of $\varepsilon(K)$.

**Definition 3.2** ([Hom14b]). If $\varepsilon(K) = 1$ ($F_\tau$ is trivial), define

$$a_1(K) = \min\{s \mid H_s : H_s(C\{i = 0\}) \to H_s(C\{\min(i, j - \tau) = 0, i \leq s\}) \text{ is trivial}\}.$$ 

We extend this definition of $a_1(K)$ to all knots (to include knots with $\varepsilon(K) \neq 1$). Consider the maps

$$G_{-s,\tau} : C\{\max(i, j - \tau) = 0, i \geq -s\} \to C\{i = 0\}$$

$$F_{s,\tau} : C\{i = 0\} \to C\{\min(i, j - \tau) = 0, i \leq s\}$$

**Definition 3.3.** Given a knot $K$ inside $S^3$, define:

$$a_1(K) = \begin{cases} 
\max\{-s \mid G_{-s,\tau} \text{ is trivial on homology}\}, & \text{if } \varepsilon(K) = -1; \\
0, & \text{if } \varepsilon(K) = 0; \\
\min\{s \mid F_{s,\tau} \text{ is trivial on homology}\}, & \text{if } \varepsilon(K) = 1.
\end{cases}$$

Note that $a_1(K)$ only depends on the doubly-filtered chain homotopy type of the knot Floer chain complex $\text{CFK}^\infty(K)$, so it is a knot invariant.
Remark 3.4. When $\varepsilon(K) = 1$, the definition of $a_1(K)$ agrees with the invariant $a_1(K)$ defined in Lemma 6.1 in [Hom14b]. As remarked in [Hom14b], $a_1(K)$ measures the “length” of the horizontal differential hitting the special class generating the vertical homology of $\hat{CF}(S^3)$. Similarly, when $\varepsilon(K) = -1$, $a_1(K)$ measures the “length” of the horizontal differential coming out of the special class generating the vertical homology of $\hat{CF}(S^3)$.

Recall that the rationally null-homologous knot $\mu_n \subset S^3_t(K)$ induces a $\mathbb{Z}$-filtration of $\hat{CF}(S^3_t(K), s_\tau)$ and $\hat{CF}(S^3_t(K), s_\tau)$, that is, a sequence of subcomplexes:

$$0 \subset F_{bottom} \subset F_{bottom+1} \subset \cdots \subset F_{top-1} \subset F_{top} = \hat{CF}(S^3_t(K), s_\tau).$$

Using Theorem 2.3 and Proposition 2.6, an equivalent definition of $a_1(K)$ can be formulated in terms of the filtration $F$ and $F'$ induced by $\mu_n$ as a knot inside $S^3_t(K)$ and $S^3_t(K)$. This interpretation of the invariant $a_1(K)$ offers a topological perspective that complements the original algebraic definition of $a_1(K)$.

**Theorem 3.5.** Let $n > 2g(K)$. For sufficiently large surgery coefficient $t$, the concordance invariant $a_1(K)$ is equal to

$$a_1(K) = \begin{cases} 
\max \left\{ m \mid \hat{CF}(S^3_tK, s_\tau)/F_{top-1-m} \rightarrow \hat{CF}(S^3) \right. 
\text{induces a trivial map on homology} \bigg\} & \text{if } \varepsilon(K) = -1, \\
0 & \text{if } \varepsilon(K) = 0, \\
\min \left\{ m \mid \hat{CF}(S^3) \rightarrow F_{bottom+m} \subset \hat{CF}(S^3_tK, s_\tau) \right. 
\text{induces a trivial map on homology} \bigg\} & \text{if } \varepsilon(K) = 1.
\end{cases}$$

**Proof.** Since $|\tau| \leq g_4(K) \leq g(K)$, we can apply Proposition 2.6 which states that in the spin$^c$ structure $s_\tau$, the algebraic $i$-filtration on $CFK^\infty(S^3_tK)$ coincides with the filtration induced by $\mu_n$ on $\hat{CF}(S^3_t(K), s_\tau)$ under the identification of the two filtered chain complexes in Theorem 2.3.

**Remark 3.6.** Recall that $a_1(K)$ is a concordance invariant (see Proposition 3.7) that fits into a family of concordance invariants studied by Dai, Hom, Stoffregen and the author in [DIHST19]. It would be interesting to see if an analogue of Theorem 3.5 exists for this entire family of algebraically defined invariants corresponding to the standard local representative (over $\mathbb{F}[U, V]/(UV)$) of the knot.

**Proposition 3.7 ([Hom14b]).** The invariant $a_1(K)$ is a concordance invariant.

**Proof.** Suppose $K_1$ and $K_2$ are concordant knots, i.e. $K_1 \# K_2$ is slice. Then $\varepsilon(K_1 \# K_2) = 0$. By Proposition 3.11 in [Hom15], we may find a basis for $CFK^\infty(K_1 \# K_2)$ with a distinguished element $x$ that generates the homology $HF^\infty(K_1 \# K_2)$ and splits off as a direct summand of $CFK^\infty(K_1 \# K_2)$. Similarly, we can find a basis for $CFK^\infty(K_2 \# K_2)$ with a distinguished element $y$ with the same properties. Then to compute $a_1(K_2 \# K_1 \# K_2)$, by the Kunneth principle we can consider either chain complex:

$$CFK^\infty(K_1 \# K_2) \otimes_{\mathbb{Z}[U,U^{-1}]} CFK^\infty(K_2) \text{ or } CFK^\infty(K_1) \otimes_{\mathbb{Z}[U,U^{-1}]} CFK^\infty(K_2 \# K_2).$$
Using the special bases from above, the relevant summands to $a_1$ are

$$\{x\} \otimes \text{CFK}^\infty(K_2) \quad \text{or} \quad \text{CFK}^\infty(K_1) \otimes \{y\}.$$ 

Thus, $a_1(K_2) = a_1(K_2 \# K_1 \# \overline{K}_2) = a_1(K_1)$. \hfill \square

**Example 3.8** (Homologically thin knots). Model complexes for CFK\(^\infty\) of homologically thin knots are studied in [Pet13]. Petkova shows that if $\tau(K) = n$, the model complex contains a direct summand isomorphic to

$$\text{CFK}^\infty(T_{2,2n+1}) \text{ if } n > 0 \quad \text{and} \quad \text{CFK}^\infty(T_{2,2n-1}) \text{ if } n < 0.$$ 

This summand supports $H_4(\text{CFK}^\infty(K))$ and thus determines the value of $a_1(K)$. It is easy to see from the complex that $a_1(K) = \text{sgn}(\tau(K))$.

**Proposition 3.9.** The following are properties of $a_1(K)$:

1. If $K$ is smoothly slice, then $a_1(K) = 0$.
2. $\text{sgn}(a_1(K)) = \varepsilon(K)$.
3. $a_1(K) = -a_1(\overline{K})$.
4. If $a_1(K) = 0$, then $a_1(K \# K') = a_1(K')$.

**Proof of (1).** If $K$ is smoothly slice, then $\varepsilon(K) = 0$; therefore, $a_1(K) = 0$. \hfill \square

**Proof of (2).** By construction, if $a_1(K) > 0$, then $\varepsilon(K) = 1$; if $a_1(K) < 0$, then $\varepsilon(K) = -1$.

If $a_1(K) = 0$, we show that $\varepsilon(K) = 0$. Suppose $\varepsilon(K) = -1$. Then the vanishing of

$$a_1(K) = \max\{n \mid G_{n,\tau} \text{ is trivial on homology}\}$$ 

implies that the map $G_{0,\tau} : C\{i = 0, j \leq \tau\} \rightarrow C\{i = 0\}$ is trivial on homology, which contradicts the definition of $\tau$. Similarly, $\varepsilon(K) \neq 1$ if $a_1(K) = 0$.

Finally, according to [Hom14a], $\varepsilon(K) = 0$ implies that $\tau(K) = 0$. \hfill \square

**Proof of (3).** The symmetry properties of CFK\(^\infty\) of Section 3.5 in [OS04] imply that $a_1(K) = -a_1(\overline{K})$. \hfill \square

**Proof of (4).** If $a_1(K) = 0$, $\varepsilon(K) = 0$. By Lemma 3.3 from [Hom14a], we may find a basis for CFK\(^\infty\)(K) with a distinguished element $x$ which is the generator of both vertical and horizontal homology. Then $a_1(K \# K')$ can be computed from $\{x\} \otimes \text{CFK}^\infty(K')$. \hfill \square

In fact, we can extend Proposition [3.9]{#3} to describe the behavior of $a_1$ under connect sum in many (but not all) cases.

**Proposition 3.10.**

1. If $a_1(K_1) > 0$ and $a_1(K_2) < 0$ and $a_1(K_1) + a_1(K_2) < 0$, then $a_1(K_1 \# K_2) = a_1(K_1)$.
2. If $a_1(K_1) > 0$ and $a_1(K_2) < 0$ and $a_1(K_1) + a_1(K_2) > 0$, then $a_1(K_1 \# K_2) = a_1(K_2)$.
3. If $a_1(K_1) > 0$ and $a_1(K_2) > 0$, then $a_1(K_1 \# K_2) = \text{min}(a_1(K_1), a_1(K_2))$.
4. If $a_1(K_1) < 0$ and $a_1(K_2) < 0$, then $a_1(K_1 \# K_2) = \text{max}(a_1(K_1), a_1(K_2))$.

**Proof.** Note that we use $-K$ to denote the mirror of a knot $K$.

(1) See the proof of Lemma 6.3 of [Hom14b].
(2) The mirrors $-K_1$ and $-K_2$ satisfy the hypothesis of (1), so

$$a_1(-K_1\# - K_2) = a_1(-K_2).$$

Apply the symmetry property of $a_1$ under mirroring \([3.9]\):

$$-a_1(K_1\#K_2) = -a_1(K_2)$$

as desired.

(3) By Lemma 6.2 of \([Hom14b]\), there exists a basis \(\{x_i\}\) over \(\mathbb{F}[U,U^{-1}]\) for \(\text{CFK}^\infty(K_1)\) with basis elements \(x_0\) and \(x_1\) with the property that

1. There is a horizontal arrow of length \(a_1\) from \(x_1\) to \(x_0\).
2. There are no other horizontal arrows or vertical arrows to or from \(x_0\).
3. There are no other horizontal arrows to or from \(x_1\).

Similarly, we may find a basis \(\{y_i\}\) over \(\mathbb{F}[U,U^{-1}]\) for \(\text{CFK}^\infty(K_2)\) with basis elements \(y_0\) and \(y_1\) with the above properties. Without loss of generality, assume that \(a_1(K_1) \leq a_1(K_2)\).

Notice \(x_0y_0\) generates the vertical homology \(H_*(C(\{i = 0\}))\) of \(\text{CFK}^\infty(K_1\#K_2)\). Let \(\tau = \tau(K_1\#K_2)\). Consider the subquotient complex

$$A = C\{\min(i, j - \tau) = 0\}.$$

There is a direct summand of \(A\) consisting of the generators \(x_0y_0, x_0y_1, x_1y_0, \) and \(x_1y_1\), and four horizontal arrows as shown in Figure 7. The arrow \(x_1y_0\) to \(x_0y_0\) has length \(a_1(K_1)\). Clearly, \(\varepsilon(K_1\#K_2) = 1\) and \(a_1(K_1\#K_2) = a_1(K_1)\).

![Figure 7](image)

This is the summand that is relevant for computing \(a_1\), as it contains the generator \(x_0y_0\) of vertical homology \(H_*(C(\{i = 0\}))\).

(4) The mirrors $-K_1$ and $-K_2$ satisfy the hypothesis of (3). So

$$-a_1(K_1\#K_2) = a_1(-K_1\# - K_2) = \min(a_1(-K_1), a_1(-K_2)) = \min(-a_1(K_1), -a_1(K_2)) = -\max(a_1(K_1), a_1(K_2)) = 0.$$  

Proposition 3.10 can be rewritten as the following.

**Proposition 3.11.** If \(a_1(K_1) \neq 0\) and \(a_1(K_2) \neq 0:\)

1. If \(a_1(K_1) + a_1(K_2) < 0\), then \(a_1(K_1\#K_2) = \max(a_1(K_1), a_1(K_2))\).
2. If \(a_1(K_1) + a_1(K_2) > 0\), then \(a_1(K_1\#K_2) = \min(a_1(K_1), a_1(K_2))\).

**Remark 3.12.** If \(a_1(K) \neq 0\) and \(a_1(K') \neq 0\), and \(a_1(K)+a_1(K') = 0\), then \(a_1(K\#K')\) is indeterminate. The next two examples illustrate this case.

**Example 3.13.** The connect sum of any knot \(K\) with the reverse of its mirror \(-K\), i.e. the inverse of \(K\) in the concordance group \(C\), has vanishing \(a_1(K\# - K) = 0\).
Example 3.14. The full knot Floer chain complexes $CFK^\infty$ of the mirror $-T_{2,3;2,5}$ of the $(2,5)$-cable of the torus knot $T_{2,3}$, the torus knot $T_{2,9}$, and the connect sum $-T_{2,3;2,5}\#T_{2,9}$ are described in [HW14]. It is easy to see that $a_1(-T_{2,3;2,5}) = -1$, $a_1(T_{2,9}) = 1$, and $a_1(-T_{2,3;2,5}\#T_{2,9}) = -1$.

We conclude with some computations of the $a_1$–invariant.

Example 3.15. In [Hom16] Hom produces the relevant summand of $CFK^\infty$ for computing $\varepsilon$ and hence $a_1$ for the knot $T_{4,5}\#-T_{2,3;2,5}$. It is easy to see that $a_1(T_{4,5}\#-T_{2,3;2,5}) = 2$.

Example 3.16. The Conway knot $C_{2,1}$ has $a_1(C_{2,1}) = 0$. According to [Pet10], the knot Floer chain complex $CFK^\infty(C_{2,1})$ is generated as a $\mathbb{F}[U,U^{-1}]$–module by a single isolated $\mathbb{F}$ at the origin plus a collection of null-homologous “boxes”.

Example 3.17. The knot Floer chain complex of an L-space knot is given by Theorem 2.1 in [OSS14]. If $K$ is an L-space knot, with Alexander polynomial

$$\Delta_K(t) = \sum_{i=0}^{k} (-1)^i t^{n_i},$$

where $n_0 > n_1 > \cdots > n_k$, then $a_1(K) = n_0 - n_1$ by Lemma 6.5 [Hom14b].

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