Upper bounds for the uniform simultaneous Diophantine exponents

Dzmitry Badziahin

School of Mathematics and Statistics, The University of Sydney, Camperdown, NSW, Australia

Correspondence
Dzmitry Badziahin, The University of Sydney, Camperdown, NSW, 2006, Australia.
Email: dzmitry.badziahin@sydney.edu.au

Abstract

We give several upper bounds for the uniform simultaneous Diophantine exponent \( \hat{\lambda}_n(\xi) \) of a transcendental number \( \xi \in \mathbb{R} \). The most important one relates \( \hat{\lambda}_n(\xi) \) and the ordinary simultaneous exponent \( \omega_k(\xi) \) in the case when \( k \) is substantially smaller than \( n \). In particular, in the generic case \( \omega_k(\xi) = k \) with a properly chosen \( k \), the upper bound for \( \hat{\lambda}_n(\xi) \) becomes as small as \( \frac{3}{2n} + O(n^{-2}) \) which is substantially better than the best currently known unconditional bound of \( \frac{2}{n} + O(n^{-2}) \). We also improve an unconditional upper bound on \( \hat{\lambda}_n(\xi) \) for even values of \( n \).

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1 INTRODUCTION

Given a real number \( \xi \), its \( n' \)-th (uniform) simultaneous Diophantine exponent \( \lambda_n(\xi) \) (respectively, \( \hat{\lambda}_n(\xi) \)) is defined as the supremum of all \( \lambda \) such that the system of inequalities

\[
\max_{1 \leq i \leq n} |x_0^\xi x_i - x_i| < Q^{-\lambda}, \quad 1 \leq |x_0| \leq Q;
\]

has a solution \( x = (x_0, x_1, \ldots, x_n) \in \mathbb{Z}^{n+1} \) for an increasing unbounded sequence of values \( Q \) (respectively, for all large enough values of \( Q \)).

The \( n' \)-th (uniform) dual Diophantine exponent \( \omega_n(\xi) \) (respectively, \( \hat{\omega}_n(\xi) \)) is defined as the supremum of \( \omega \) such that the system of inequalities

\[
0 < |P(\xi)| \leq Q^{-\omega}, \quad \deg(P) \leq n, \quad H(P) \leq Q
\]

The most important one relates \( \hat{\lambda}_n(\xi) \) and the ordinary simultaneous exponent \( \omega_k(\xi) \) in the case when \( k \) is substantially smaller than \( n \). In particular, in the generic case \( \omega_k(\xi) = k \) with a properly chosen \( k \), the upper bound for \( \hat{\lambda}_n(\xi) \) becomes as small as \( \frac{3}{2n} + O(n^{-2}) \) which is substantially better than the best currently known unconditional bound of \( \frac{2}{n} + O(n^{-2}) \). We also improve an unconditional upper bound on \( \hat{\lambda}_n(\xi) \) for even values of \( n \).

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has infinitely many solutions \( P \in \mathbb{Z}[x] \) for an unbounded sequence of values \( Q \) (respectively, for all large values \( Q \)). Here, \( H(P) \) is the trivial height of the polynomial \( P \), that is, it is the supremum norm of the vector of its coefficients.

The Diophantine exponents are extensively studied in the literature. They help us understand the approximational properties of real numbers. The notion of \( \omega_n(\xi) \) was first introduced by Mahler in 1932 [8] where he defined his classification of real numbers. To the best of my knowledge, the simultaneous Diophantine exponents were formally defined much later (perhaps, in [4]) but were studied long before that without having an explicit name.

The classical Dirichlet and Minkowski theorems imply that for transcendental \( \xi \), \( \lambda_n(\xi) \geq \hat{\lambda}_n(\xi) \geq 1/n \) and \( \omega_n(\xi) \geq \hat{\omega}_n(\xi) \geq n \) for all \( \xi \in \mathbb{R} \). It was conjectured by Mahler and later established by Sprindžuk [15] that these bounds are in fact equalities for almost all values of \( \xi \) in terms of Lebesgue measure. The precise values of all Diophantine exponents are also known for algebraic numbers, due to Schmidt Subspace Theorem [14]. If \( \xi \) is a real algebraic number of degree \( d \), then

\[
\omega_n(\xi) = \hat{\omega}_n(\xi) = \min\{n, d-1\}; \quad \lambda_n(\xi) = \hat{\lambda}_n(\xi) = \frac{1}{\min\{n, d-1\}}.
\]

Further in this paper, we will always assume that \( \xi \) is transcendental.

Diophantine exponents play an important role in understanding how well a given real number \( \xi \) can be approximated by algebraic numbers of bounded degree. Heuristically, it is easy to see for \( \omega_n(\xi) \). Indeed, large values of \( \omega_n(\xi) \) imply that there exist polynomials that take very small values at \( \xi \), and that in turn implies that one of the roots of those polynomials is near \( \xi \). This idea was used by Wirsing [17] in 1961 where he showed that for any transcendental \( \xi \in \mathbb{R} \) and \( \omega^* < \frac{\omega_n(\xi)+3}{2} \) there exist infinitely many algebraic real numbers \( \alpha \) of degree at most \( n \) such that

\[
|\xi - \alpha| < H(\alpha)^{-\omega^*}.
\]

Later, this inequality was strengthened in [2, 5, 16]. In 1969, Davenport and Schmidt [6] discovered a relation between \( \omega^* \) and \( \hat{\lambda}_n(\xi) \). Wirsing himself conjectured that for all transcendental \( \xi \) the degree \( \omega^* \) can be made arbitrarily close to \( n + 1 \). This problem remains open for all dimensions \( n \geq 3 \). The best known bounds on \( \omega^* \) can be found in [1]. For more properties and relations between the Diophantine exponents of \( \xi \) and the notion of \( \omega^* \), we refer to an extensive overview [3].

In this paper, we will closely look at the exponents \( \hat{\lambda}_n \). We do not know much about what values they can take. It is known that for \( n = 2 \), \( \hat{\lambda}_2(\xi) \) can change between \( \frac{1}{2} \) and \( \frac{\sqrt{5}-1}{2} \) where both upper and lower bounds are sharp [11]. Recent developments about the spectrum of values \( \hat{\lambda}_2(\xi) \) and \( \hat{\omega}_2(\xi) \) can be found in [4, 9]. However, for \( n \geq 3 \) and transcendental \( \xi \), it is not even known if \( \hat{\lambda}_n(\xi) \) can take any other value than \( 1/n \).

In 1969, Davenport and Schmidt [6] verified that \( \hat{\lambda}_n(\xi) \leq 1/\lfloor \frac{n}{2} \rfloor \) which, in view of the lower bound \( 1/n \leq \hat{\lambda}_n(\xi) \), leaves a small window for the values of the exponent \( \hat{\lambda}_n \). Later, their bound was slightly improved by several authors, most recent results belong to Laurent [7] and Schleischitz [13]. However, all these upper bounds are still of the form \( \frac{2}{n} + O(n^{-2}) \).

**Theorem LS.** For any transcendental \( \xi \in \mathbb{R} \) and integer \( n \geq 2 \), one has

\[
\hat{\lambda}_n(\xi) \leq \begin{cases} 
\frac{2}{n+1} & \text{for odd } n \\
\frac{1}{\tau_n} & \text{for even } n,
\end{cases}
\]

where \( \tau_n \) is the Lucas number.
where \( \tau_n \) is the solution in the interval \( \left[ \frac{2}{n + 2}, \frac{2}{n} \right) \) of the equation

\[
\left( \frac{n}{2} \right)^n x^{n+1} - \left( \frac{n}{2} + 1 \right) x + 1 = 0.
\]

In this paper, we improve the upper bound of \( \hat{\lambda}_n(\xi) \) for even \( n \), while asymptotically it is still in the range \( \frac{2}{n} + O(n^{-2}) \).

**Theorem 1.** For any positive integer \( n \), one has

\[
\hat{\lambda}_{2n}(\xi) \leq t_n = \frac{\sqrt{n^2 + 4n - n}}{2n},
\]

where \( t_n \) is the positive root of the equation \( nx^2 + nx - 1 = 0 \).

Observe that for \( n = 1 \) the bound in Theorem 3 is the same as in the result of Davenport and Schmidt [6]. Therefore, it is sufficient to prove this theorem for \( n \geq 2 \). For larger values of \( n \), we get \( t_2 \approx 0.366 < 0.371 \approx \tau_4 \); \( t_3 \approx 0.264 < 0.268 \approx \tau_6 \), where \( \tau_{2n} \) are defined in Theorem LS. In general, one can check that \( t_n < \tau_{2n} \) for all \( n \geq 2 \).

**Remark.** After this paper was initially submitted for publication, Poëls and Roy [10] announced stronger upper bounds for \( \hat{\lambda}_n(\xi) \) than in this paper. In particular, they got \( \hat{\lambda}_4(\xi) \leq 0.3370 \ldots \) and \( \hat{\lambda}_6(\xi) \leq 0.2444 \ldots \)

The Diophantine exponent \( \hat{\lambda}_3(\xi) \) was studied in more detail by Roy [12] where he got a better upper bound \( \hat{\lambda}_3(\xi) \leq \lambda_0 \approx 0.4245 \) than for general \( n \).

The most important results of this paper show that the upper bounds on \( \hat{\lambda}_n(\xi) \) can be made much tighter if some of the Diophantine exponents \( \omega_k(\xi) \) with \( 2k + 1 \leq n \) are close to their generic values \( k \).

**Theorem 2.** Let \( \xi \in \mathbb{R} \) be a transcendental number. Assume that for a given \( k \in \mathbb{N} \),

\[
\delta_k := \frac{k}{\omega_k(\xi) + 1 - k} \geq 1. \tag{1}
\]

Then one has

\[
\hat{\lambda}_n(\xi) \leq \begin{cases} 
\frac{1}{n - k} & \text{for } 2k + 1 \leq n < 2k + 1 + \delta_k \\
[2ex] \min \left\{ \frac{1}{n - \left\lfloor \frac{n - \delta_k - 1}{2} \right\rfloor}, \frac{1}{\left\lfloor \frac{n - \delta_k - 1}{2} \right\rfloor + 1 + \delta_k} \right\} & \text{for } n \geq 2k + 1 + \delta_k.
\end{cases}
\]

While for fixed \( k \) and \( n \) tending to infinity, the upper bound in Theorem 2 is still of the form \( \frac{2}{n} + O(n^{-2}) \), it may become as small as \( \frac{3}{2n} + O(n^{-2}) \) if \( n = 3k + O(1) \) and \( \omega_k(\xi) = k \). We manage to slightly improve the bound in Theorem 2 for some values of \( n \) and \( k \). However, due to the complicated nature of those bounds, we do not provide them in the introduction, but leave them in § 6, see Theorem 4.
Remark. In fact, the notion $\delta_k$ in Theorem 2 can be replaced by in some cases smaller notion $\hat{\omega}_{k,k+1}(\xi)$ which is defined as the supremum of $\omega$ such that the inequalities

$$0 < |P(\xi)| \leq Q^{-\omega}, \quad \deg(P) \leq k, \quad H(P) \leq Q$$

have $k+1$ linearly independent solutions $P \in \mathbb{Z}[x]$ for all large values of $Q$. While it is not hard to adapt the proof of Theorem 2 to this new notion, we leave the details of the proof to the interested reader.

The last result of this paper gives a slight improvement of Theorem 2 for the case $n = 3$. It provides a better bound on $\hat{\lambda}_3(\xi)$ than that of Roy in the case when $\omega_1(\xi)$ is close to 1.

**Theorem 3.** For a transcendental number $\xi \in \mathbb{R}$, the Diophantine exponents $\hat{\lambda}_3(\xi)$ and $\omega_1(\xi)$ satisfy the inequality

$$2\hat{\lambda}_3(\xi)^3 \omega_1(\xi) < \hat{\lambda}_3(\xi)^3 (\omega_1(\xi) - 1) + 2\hat{\lambda}_3(\xi) \leq 1.$$  

(2)

For $\omega_1(\xi) = 1$ (which is satisfied for almost all $\xi$), this inequality is equivalent to $\hat{\lambda}_3(\xi) \leq \lambda_3 = 0.42385...$ which is better than the best-known bound $0.4245...$ from [12]. In fact, condition (2) gives a better estimate on $\hat{\lambda}_3(\xi)$ than the currently known one for $\omega_1(\xi) \leq 1.07...$

2 | NOTATION

For real numbers $A$ and $B$, we write $A \ll B$ if $A \leq cB$ for some absolute constant $c > 0$. The notation for $A \gg B$ and $A \asymp B$ is defined similarly. In the further discussion, we fix a real number $\xi$ and assume that $\xi \approx 1$, that is, $\xi$ is bounded from above and below by some absolute positive constants.

We will borrow much of the notation from [12]. Given a vector $\mathbf{x} = (x_0, \ldots, x_n) \in \mathbb{R}^{n+1}$, we define a function $L(\mathbf{x})$ as follows:

$$L(\mathbf{x}) := \max_{1 \leq i \leq n} |x_0 \xi^i - x_i|.$$  

(3)

This definition implies that for all $k, l \in \mathbb{Z}_{\geq 0}$ with $l + k \leq n$, one has

$$|x_k \xi^l - x_{k+l}| = |(x_k - x_0 \xi^k) \xi^l - (x_{k+l} - x_0 \xi^{k+l})| \ll L(\mathbf{x}).$$  

(4)

By $(\mathbf{x}_i)_{i \in \mathbb{N}}$ we denote the sequence of minimal points for $(\xi, \xi^2, \ldots, \xi^n)$. This is a sequence of points $\mathbf{x}_i = (x_{i,0}, x_{i,1}, \ldots, x_{i,n})$ in $\mathbb{Z}^{n+1}$ such that the following conditions are satisfied.

• The positive integers $X_i := ||x_i||_\infty$ form a strictly increasing sequence with $X_1 = 1$.
• The positive real numbers $L_i := L(\mathbf{x}_i)$ form a strictly decreasing sequence.
• If some nonzero point $\mathbf{x} \in \mathbb{Z}^{n+1}$ satisfies $L(\mathbf{x}) < L_i$, then $||\mathbf{x}||_\infty \geq X_{i+1}$.

Throughout the paper, we always deal with the supremum norm $|| \cdot ||_\infty$. Therefore, to make the notation shorter, we will omit the subscript and write $||\mathbf{x}||$ for the supremum norm of $\mathbf{x}$.

One can verify that for transcendental $\xi$ the sequence $(\mathbf{x}_i)_{i \in \mathbb{N}}$ is uniquely defined, apart from probably the first term $\mathbf{x}_1$. In further discussion, we always assume that $\xi$ is transcendental. Note that since $\xi \approx 1$, then for large enough $i$ we have $x_{i,0} \approx x_{i,1} \approx \cdots \approx x_{i,n} \approx X_i$. 
We fix a positive real number $\lambda$ for which there exists a constant $c > 0$ such that for all $X \geq 1$ the inequalities

$$||x|| \leq X; \quad L(x) \leq cX^{-\lambda}$$

have a solution $x \in \mathbb{Z}^{n+1} \setminus \{0\}$. Clearly, any number smaller than $\lambda_n(\xi)$ satisfies this condition, while $\lambda_n(\xi)$ itself may or may not satisfy it.

By choosing $X = X_{i+1} - \frac{1}{2}$ we immediately verify that $L_i \leq c(X_{i+1} - \frac{1}{2})^{-\lambda}$ or for simplicity we will write it in a shorter form:

$$L_i \ll X_{i+1}^{-\lambda}.$$ (5)

We fix positive real numbers $\omega_k$, $k \in \{1, \ldots, n\}$ for which the inequalities

$$||x|| \leq X; \quad |x_0 + x_1\xi + x_2\xi^2 + \cdots + x_k\xi^k| \ll X^{-\omega_k}$$ (6)

may have solutions $x \in \mathbb{Z}^{k+1} \setminus \{0\}$ only for a bounded set of values $X$. It is straightforward to check that any number $\omega_k$ bigger than $\omega_k(\xi)$ satisfies this condition, while $\omega_k(\xi)$ itself may or may not satisfy it.

One can easily verify that any two consecutive vectors $x_{i-1}$ and $x_i$ are linearly independent. By $W_i$, we denote the span of these two vectors:

$$W_i := \langle x_{i-1}, x_i \rangle.$$ 

Now we note that if for all $i \geq i_0$ the vectors $x_i$ belong to a proper subspace of $\mathbb{R}^{n+1}$, then $\xi$ must be an algebraic number of degree at most $n$. Indeed, in that case without loss of generality we may assume that the points lie in a hyperplane $a \cdot x_i = 0$ for some vector $a$. Since all $x_i$ are integer vectors, $a$ may be chosen to be integer too. Then the points $\frac{x_i}{x_i,0}$ also lie in the same hyperplane. Finally, we observe that their limit

$$\lim_{i \to \infty} \frac{x_i}{x_i,0} = (1, \xi, \xi^2, \ldots, \xi^n)$$

is also in that hyperplane and hence $\xi$ is algebraic.

The observation above implies that for $n \geq 2$ there exists infinitely many indices $i$ such that $x_{i-1}, x_i$ and $x_{i+1}$ are linearly independent. By $I$, we denote the set of indices which satisfy this property. Formally,

$$I := \{i \in \mathbb{N} : W_i \neq W_{i+1}\}.$$ (7)

For $i \in I$, by $U_i$ we denote the span of three consecutive vectors

$$U_i := \langle x_{i-1}, x_i, x_{i+1} \rangle = W_i + W_{i+1}.$$ 

For $n \geq 3$ and transcendental $\xi$, we must have that the sequence of subspaces $U_i$ is not eventually constant. In other words, there must be infinitely many indices $j \in I$ whose successor $i \in I$ satisfies $U_j \neq U_i$. We denote such a set of indices $j$ by $J$. 

For larger values of $n$, one can continue defining systems of higher dimensional subspaces, analogously to $W_i$ and $U_j$, but we will not use them in this paper.

Given a vector $x \in \mathbb{R}^{n+1} = (x_0, x_1, ..., x_n)$, we define vectors $x^{(k,l)} \in \mathbb{R}^{l+1}$, $k \geq 0$, $k + l \leq n$ as follows:

$$x^{(k,l)} := (x_k, x_{k+1}, ..., x_{k+l}).$$

(8)

### 3 | PREPARATORY RESULTS

One of the most important ideas introduced by Davenport and Schmidt [6] is to consider vectors of the form $x^{(j,m)}_i$ for a fixed $m < n$ and all $j \in \{0, ..., n - m\} \in \mathbb{N}$. They are all almost parallel to the vector $(1, \xi, \xi^2, ..., \xi^m)$, therefore the wedge product of such vectors must have a very small norm.

**Proposition 1.** Let $m, d \in \mathbb{Z}$ satisfy $0 \leq m \leq n$ and $0 \leq d \leq m + 1$. Let $(l_1, k_1), ..., (l_d, k_d)$ be a sequence of integer pairs such that $0 < l_1 \leq l_2 \leq \cdots \leq l_d$ and $0 \leq k_i \leq n - m$ for all $i \in \{1, ..., d\}$.

Then

$$|\bigwedge_{i=1}^{d} x^{(k_i,m)}_{l_i}| \ll X_{l_d} \prod_{i=1}^{d-1} L_{l_i}. \tag{9}$$

**Proof.** Consider any $d \times d$ minor of the matrix which is composed of $x^{(k_1,m)}_{l_1}, ..., x^{(k_d,m)}_{l_d}$ as rows:

$$M := \begin{pmatrix}
x_{l_1,k_1+j_1} & x_{l_1,k_1+j_2} & \cdots & x_{l_1,k_1+j_d} \\
x_{l_2,k_2+j_1} & x_{l_2,k_2+j_2} & \cdots & x_{l_2,k_2+j_d} \\
\vdots & \vdots & \ddots & \vdots \\
x_{l_d,k_d+j_1} & x_{l_d,k_d+j_2} & \cdots & x_{l_d,k_d+j_d}
\end{pmatrix}.$$

In order to prove the proposition, we need to verify that $|\det M| \ll X_{l_d} \prod_{i=1}^{d-1} L_{l_i}$.

For each $i$ between 2 and $d$, we multiply the first column of $M$ by $\xi^{l_1-j_1}$ and subtract it from $i$’th row of $M$. As a result, we get a matrix $M^*$ such that, by (4), its entries in row $i$ and columns 2 to $d$ are $\ll L_{l_i}$. Now we expand the determinant of $M^*$ by the first column to get $|\det M| \ll X_{l_d} \prod_{i=1}^{d-1} L_{l_i}$.

The idea of this proposition is that for appropriately chosen vectors $x^{(k_i,m)}_{l_i}$ and for $\lambda$ large enough, the right-hand side of (9) becomes smaller than 1. Since on the left-hand side there is an integer multivector, this implies that $x^{(k_i,m)}_{l_i}$ are linearly dependent. The remaining part of the arguments is then to verify that linear dependence of those vectors implies that $\xi$ is algebraic.

The following lemma is proven in [1, Lemma 3.1] and is based on the ideas of Laurent [7].

**Lemma 1.** Let $m$ be an integer such that $1 \leq m \leq n/2$. Assume that

$$\lambda > \frac{1}{n - m + 1}. \tag{10}$$

Then for any large $i$ the vectors $x^{(0,n-m)}_i, x^{(1,n-m)}_i, ..., x^{(m,n-m)}_i$ are linearly independent.
The next result states that sometimes we can find even more vectors that are linearly independent. It may be of independent interest.

**Proposition 2.** Let \( m \) and \( k \) be nonnegative integers such that either \( k \leq 2 \) and \( 1 \leq m + k \leq n/2 \) or \( k \geq 3 \) and \( 1 \leq m + 2k - 2 \leq n/2 \). Assume that

\[
\lambda > \begin{cases} 
1 & \text{for } k \leq 2 \\
\left\lceil \frac{n - m - k + 1}{k} \right\rceil & \text{for } k \geq 3.
\end{cases}
\]

Let \( y_0, y_1, \ldots, y_k \) be large enough linearly independent minimal points of \( \xi \), not necessarily consecutive. Then the dimension of the span of \( \left\{ y^{(j,m-n)}_{i} \right\}_{0 \leq i \leq k; 0 \leq j \leq m} \) is at least \( m + 1 + k \).

Note that Lemma 1 is a particular case of this proposition for \( k = 0 \). We believe that the proposition can be strengthened so that \( \lambda > (n - m - k + 1)^{-1} \) should be sufficient for \( k \geq 2 \) too but we do not see an easy way of proving it.

**Proof.** We prove by induction on \( k \). Lemma 1 immediately verifies the base of induction \( k = 0 \): all the vectors \( y^{(0,n-m)}_0, \ldots, y^{(m,n-m)}_0 \) are linearly independent. For convenience, denote by \( T_k \) the span of \( \left\{ y^{(j,m-n)}_{i} \right\}_{0 \leq i \leq k; 0 \leq j \leq m} \). Assume that the proposition is true for \( k - 1 \), that is, that for all triples \((m, k, \lambda)\) which satisfy the conditions of the proposition, \( \dim T_{k-1} \geq m + k \). Now we prove the statement for \( k \).

Note that if the triple \((m, k, \lambda)\) satisfies the conditions of the propositions, then the triples \((m, k - 1, \lambda)\) and \((m + 1, k - 1, \lambda)\) also do. In particular, that implies that, by the inductive assumption, \( \dim T_k \geq \dim T_{k-1} \geq m + k \). If it is at least \( m + k + 1 \), then we are done. Assume now that \( \dim T_{k-1} = m + k \).

If \( 1 \leq k \leq 2 \), we consider the pair \((m + 1, k - 1)\) in place of \( m \) and \( k \) together with vectors \( y_0, \ldots, y_{k-1} \) and apply the proposition. By inductive assumption, there exist indices \((j_1, \ldots, j_{k-1})\) and \((i_1, \ldots, i_{k-1})\) such that

\[
\left\{ y^{(i,m-n)}_0, y^{(j_1,n-m-1)}_{i_1}, \ldots, y^{(j_{k-1},n-m-1)}_{i_{k-1}} \right\}
\]

are linearly independent. For \( k = 2 \), there always exist either a vector \( y^{(j_1-1,n-m)}_{1} \) or a vector \( y^{(j_1,n-m)}_{1} \). We assume that the last case is satisfied, the first case can be dealt analogously. For \( k = 1 \), no cases need to be considered because the only vectors in the system are of the form \( y^{(i,m-n-1)}_0 \). Then, by adding one extra component to each of the above vectors, and throwing away \( y^{(i,m+1,n-m-1)}_0 \), we get that the system of vectors

\[
S := \left\{ y^{(i,m-n)}_0, \ldots, y^{(j_{k-1},n-m)}_{i_{k-1}} \right\} = \{ s_1, \ldots, s_{k+m} \}
\]

is also linearly independent.

If \( k \geq 3 \), we consider the pair \((m + 1, k - 1)\) in place of \( m \) and \( k \) together with vectors \( y^{(0,n-1)}_0, \ldots, y^{(0,n-1)}_{k-1} \). Then the conditions of Proposition 2 are satisfied for these new parameters
and by inductive assumption there exist indices \( 0 \leq j_1, \ldots, j_{k-1} < m \) and \((i_1, \ldots, i_{k-1})\) such that

\[
\begin{pmatrix}
(y_0^{(i,n-m-1)})_{0 \leq i \leq m}, & y_{i_1}^{(j_1,n-m-1)}, & \ldots, & y_{i_{k-1}}^{(j_{k-1},n-m-1)}
\end{pmatrix}
\]

are linearly independent. Then, as before, we add one extra component to each of the above vectors from the right and get that the system of vectors

\[
S := \left\{ \begin{pmatrix}
(y_0^{(i,n-m)})_{0 \leq i \leq m}, & y_{i_1}^{(j_1,n-m)}, & \ldots, & y_{i_{k-1}}^{(j_{k-1},n-m)}
\end{pmatrix} \right\} = \{s_1, \ldots, s_{k+m}\}
\]

is also linearly independent.

Observe that \( \dim \text{span } S = \dim \text{span } \mathcal{T}_{k-1} \) and therefore \( \text{span } S = \text{span } \mathcal{T}_{k-1} \). Consider the set \( \mathbf{R} \) of vectors \( z \in \mathbb{R}^{n+1} \) such that all vectors \( z^{(i,n-m)}, i \in \{0, \ldots, m\} \) belong to the span of \( S \). In view of the previous remark, here the span of \( S \) can be interchanged with the span of \( \mathcal{T}_{k-1} \). It is straightforwardly verified that this set is a vector subspace of \( \mathbb{R}^{n+1} \). We will bound its dimension.

For each vector \( z \in \mathbf{R} \), we get

\[
z^{(i,n-m)} = \sum_{j=1}^{k+m} a_{i,j} s_j.
\]

By the construction of vectors \( z^{(i,n-m)} \), we get relations for all \( i \in \{0, \ldots, m-1\}, l \in \{1, \ldots, n-m\} \):

\[
\sum_{j=1}^{k+m} a_{i,j} s_j,l = \sum_{j=1}^{k+m} a_{i+1,j} s_{j,l-1}.
\]

(11)

Compose a vector \( a \) by arranging the coefficients \( a_{i,j} \) in the following way

\[
a := (a_{0,1}, a_{0,2}, \ldots, a_{0,m+1}, a_{1,1}, \ldots, a_{1,m+1}, \ldots, a_{m,m+1}, a_{0,m+2}, \ldots, a_{0,m+k}, \ldots, a_{1,m+2}, \ldots, a_{1,m+k}, \ldots, a_{m,m+k}).
\]

Then the conditions (11) can be written in a matrix form \( Ma = 0 \) where \( M = (A | B) \). The part \( A \) of \( M \) is written in the block form as

\[
A := \begin{pmatrix}
H^+ & -H^- & 0 & 0 & 0 & \cdots & 0 \\
0 & H^+ & -H^- & 0 & 0 & \cdots & 0 \\
0 & 0 & H^+ & -H^- & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \cdots & H^+ & -H^- 
\end{pmatrix},
\]

where

\[
H^+ := \begin{pmatrix}
y_0^{(1,n-m-1)} & y_0^{(2,n-m-1)} & \cdots & y_0^{(m+1,n-m-1)}
\end{pmatrix},
\]

\[
H^- := \begin{pmatrix}
y_0^{(0,n-m-1)} & y_0^{(1,n-m-1)} & \cdots & y_0^{(m,n-m-1)}
\end{pmatrix},
\]
and vectors $y_0^{(i,n-m-1)}$ are considered as columns. The part $B$ is composed similarly

$$B := \begin{pmatrix} K^+ & -K^- & 0 & 0 & \ldots & 0 \\ 0 & K^+ & -K^- & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & \ldots & K^+ & -K^- \end{pmatrix},$$

$$K^+ := \begin{pmatrix} y_{i_1}^{(j_1+1,n-m-1)} & y_{i_2}^{(j_2+1,n-m-1)} & \ldots & y_{i_{k-1}}^{(j_{k-1}+1,n-m-1)} \end{pmatrix},$$

$$K^- := \begin{pmatrix} y_{i_1}^{(j_1,n-m-1)} & y_{i_2}^{(j_2,n-m-1)} & \ldots & y_{i_{k-1}}^{(j_{k-1},n-m-1)} \end{pmatrix}.$$

Perform the following column operation for $A$ which do not change the rank of this matrix: starting from $i = 1$ till $i = m$, add columns $(m+1)(i-1)+i, (m+1)(i-1)+i+1, \ldots, (m+1)(i-1)+m$ to columns $(m+1)i+i+1, \ldots, (m+1)(i+1)$, respectively. Then going backwards from $i = m$ till $i = 2$, add columns $(m+1)i+2, (m+1)i+3, \ldots, (m+1)i+i$ to columns $(m+1)(i-1)+1, \ldots, m(i-1)$. After that matrix $A$ transforms to

$$A^* = \begin{pmatrix} H_1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & H_2 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & H_m & 0 \end{pmatrix},$$

where

$$H_i := \begin{pmatrix} y_0^{(i,n-m-1)} & y_0^{(i+1,n-m-1)} & \ldots & y_0^{(m+1,n-m-1)} & -y_0^{(0,n-m-1)} & \ldots & y_0^{(i-1,n-m-1)} \end{pmatrix}.$$

By assumption, all column vectors of $H_i$ together with column vectors of $K^-$ are linearly independent. This fact implies that the left $(m+2)m$ columns of $A^*$ together with $(k-1)m$ right columns of $B$ are linearly independent, hence $\text{rk}(M) \geq m(m+k+1)$. Finally, $\dim R \leq (k+m)(m+1) - m(m+k+1) = k$.

We finish the proof by observing that the span of $y_0, y_1, \ldots, y_{k-1}$ belongs to $R$ and its dimension is $k$ therefore these two subspaces coincide. The vector $y_k$ which is linearly independent with $y_0, y_1, \ldots, y_{k-1}$ cannot belong to $R$ and therefore one of its components $y_k^{(j,n-m)}$ is linearly independent together with $S$. \hfill $\square$

In this paper, we will apply Proposition 2 for the case $k = 1$ and the pairs of consecutive minimal points. We formulate it in the corollary.

**Corollary 1.** Let $m$ be an integer such that $1 \leq m \leq n/2 - 1$. Assume that $\lambda > (n-m)^{-1}$. Then for any two consecutive minimal points $x_{i-1}, x_i$ of $\xi$ the vectors $x_i^{(0,n-m)}, \ldots, x_i^{(m,n-m)}$ are linearly independent. On top of that, at least one of the vectors $x_i^{(0,n-m)}$, $\ldots$, $x_i^{(m,n-m)}$ is linearly independent with $x_i^{(0,n-m)}, \ldots, x_i^{(m,n-m)}$. 

\hfill
The next lemma uses the ideas of Laurent [7] and allows to transfer the linear relation \( a \cdot x_i = 0 \) from a minimal point \( x_i \) to the next (or previous) one \( x_{i \pm 1} \).

**Proposition 3.** Let \( a \in \langle x_i \rangle^\perp \cap \mathbb{Z}^{n+1} \). There exists an absolute constant \( c > 0 \) which satisfies the following conditions.

- If \( ||a|| \leq \frac{cX_i}{X_{i+1}L_i} \), then \( a \in \langle x_{i+1} \rangle^\perp \).
- If \( ||a|| \leq cL_{i-1}^{-1} \), then \( a \in \langle x_{i-1} \rangle^\perp \).

**Proof.** For \( a = (a_0, a_1, \ldots, a_n) \), consider the polynomial \( P_a(\xi) := \sum_{j=0}^n a_j \xi^j \). In view of \( a \cdot x_i = 0 \), one computes

\[
|x_{i,0}P_a(\xi)| = |a \cdot x_i + a_1(x_{i,0}\xi - x_{i,1}) + a_2(x_{i,0}\xi^2 - x_{i,2}) + \cdots + a_n(x_{i,0}\xi^n - x_{i,n})| \ll ||a|||L_i. \tag{12}
\]

Recall that we have \( x_{i,0} \asymp X_i \). Therefore, \( |P_a(\xi)| \ll ||a||X_i \).

To prove the first statement, we do similar computations for \( x_{i+1,0}P_a(\xi) \):

\[
|a \cdot x_{i+1}| \leq |x_{i+1,0}P_a(\xi)| + |a_1(x_{i+1,0}\xi - x_{i+1,1}) + \cdots + a_n(x_{i+1,0}\xi^n - x_{i+1,n})| \ll ||a||X_{i+1}L_i \ll ||a||L_{i+1} \frac{|X_{i+1}L_i|}{X_i}. \tag{13}
\]

Hence, there exists an absolute constant \( c > 0 \) such that if \( ||a|| \leq \frac{cX_i}{X_{i+1}L_i} \), then \( a \cdot x_{i+1} \leq 1 \) which in turn implies \( a \in \langle x_{i+1} \rangle^\perp \).

For the second statement, we do the same computation but with \( a \cdot x_{i-1} \):

\[
|a \cdot x_{i-1}| \leq |x_{i-1,0}P_a(\xi)| + |a_1(x_{i-1,0}\xi - x_{i-1,1}) + \cdots + a_n(x_{i-1,0}\xi^n - x_{i-1,n})| \ll ||a||X_{i-1}L_i \ll ||a||L_{i-1} \frac{|X_{i-1}L_i|}{X_i}. \]

Therefore, for small enough absolute constant \( c > 0 \), if \( ||a|| \leq cL_{i-1}^{-1} \), then \( a \cdot x_{i-1} = 0 \) and \( a \in \langle x_{i-1} \rangle^\perp \). \( \square \)

We end this section with the result for the case \( n = 3 \). It is essentially an adaptation of Proposition 5.2 from [12].

**Lemma 2.** Assume that \( n = 3 \), \( \xi \) is transcendental and \( \lambda > \sqrt{2} - 1 \). Then for all large enough \( i \in \mathbb{N} \), at least one of the numbers \( |x_{i-1,0} \wedge x_{i,0} \wedge x_{i,1}| \) or \( |x_{i,1} \wedge x_{i,0} \wedge x_{i,2}| \) is nonzero. That immediately implies \( X_i \gg L_i^{\frac{1}{\lambda-1}} \gg X_i^{\frac{1}{\lambda-2}} \).

**Proof.** Proposition 5.2 from [12] implies that for \( \lambda > \sqrt{2} - 1 \), the subspaces \( \langle x_{i,0}, x_{i,1} \rangle \) and \( \langle x_{i,0}^2, x_{i,1}^2 \rangle \) do not coincide for all large \( i \). By Lemma 2.3 from [12], we have that \( x_{i,0} \) and \( x_{i,1} \) are linearly independent for all large \( i \). Therefore one of the numbers \( |x_{i,0}^2 \wedge x_{i,0} \wedge x_{i,1}| \) and
\[ |x_i^{(1,2)} \land x_i^{(0,2)} \land x_i^{(1,2)}| \text{ is nonzero. By Proposition 1, both numbers are estimated as} \]

\[
1 \leq \max\{|x_i^{(0,2)} \land x_i^{(0,2)} \land x_i^{(1,2)}|, \{x_i^{(1,2)} \land x_i^{(0,2)} \land x_i^{(1,2)}|\} \ll X_i L_{i-1} \ll X_i^{1-\lambda} L_i.
\]

Thus we have \( X_i \gg L_i^{\frac{1}{\lambda-1}} \). The second inequality follows from \( L_i \ll X_i^{1-\lambda} \).

\[ \square \]

\section*{4 \ UNCONDITIONAL IMPROVEMENT OF AN UPPER BOUND FOR \( \lambda \)\]

We now prove Theorem 1. Let \( m \in \mathbb{N}, n = 2m + 2 \) and \( \lambda > (m + 2)^{-1} \). Then Lemma 1 implies that the vectors \( (x_i^{(j,n-m-1)})_{0 \leq j \leq m+1} \) are linearly independent for all large values of \( i \). That gives us the estimate

\[
1 \leq \left\| \bigwedge_{j=0}^{m+1} x_i^{(j,n-m-1)} \right\| (9) \ll X_i L_i^{m+1} \ll (5) X_i \gg X_i^{(m+1)\lambda}.
\]  

\[
\text{(14)}
\]

Moreover, it follows from Corollary 1 that at least one of the vectors \( x_i^{(l,n-m)} \) for \( 0 \leq l \leq m \) is linearly independent with the system of vectors \( V_i := (x_i^{(j,n-m)})_{0 \leq j \leq m} \).

Suppose that there exist arbitrarily large \( i \) such that there are two vectors \( x_i^{(l_1,n-m)}, x_i^{(l_2,n-m)} \) which are linearly independent together with \( V_i \). In this case, we have a stronger condition

\[
1 \leq \left\| \bigwedge_{j=0}^{m} x_i^{(j,n-m)} \land x_i^{(l_1,n-m)} \land x_i^{(l_2,n-m)} \right\| (9) \ll X_i L_i^m L_i^{2}.
\]

and therefore, by (5),

\[
X_i \gg X_i^{2\lambda} X_i^{m\lambda}.
\]

Since \( \lambda > (m + 2)^{-1} \), for large enough \( i \) we get \( X_i > X_i^{1+1} \) which is impossible. We conclude that the assumption is not satisfied and therefore the dimension of any space \( \langle V_i, V_i+1 \rangle \) equals \( m + 2 \).

Note that there exist arbitrarily large values of \( i \) such that two spaces \( \langle V_{i-1}, V_i \rangle \) and \( \langle V_{i+1}, V_i+1 \rangle \) do not coincide. Indeed, otherwise for large enough values of \( i \), all \( x_i \) lie in a proper subspace of \( \mathbb{Z}^{n+1} \) and hence \( \xi \) is algebraic. Therefore, there exist arbitrarily large \( i \) and the values \( 0 \leq k, l \leq m \) such that \( x_i^{(k,n-m)}, x_i^{(l,n-m)} \) together with \( V_{i+1} \) are linearly independent. Proposition 1 then implies

\[
1 \leq \left\| \bigwedge_{j=0}^{m} x_i^{(j,n-m)} \land x_i^{(k,n-m)} \land x_i^{(l,n-m)} \right\| (9) \ll X_i^{1+1} L_{i+1}^m L_{i-1} \ll X_i^{1-\lambda} X_i^{m\lambda} X_i^{1-\lambda}.
\]

By (14), we further bound it from above by

\[
X_i^{1-(m+1)\lambda-(m+1)\lambda^2}.
\]
Therefore, if $\lambda$ is bigger than the positive root $\lambda_0$ of $(m + 1)x^2 + (m + 1)x - 1 = 0$, then this expression becomes arbitrarily close to zero which is a contradiction. Observe that $(m + 2)^{-1} < \lambda_0 < (m + 1)^{-1}$ which finally gives that $\lambda < \lambda_0$. The proof of Theorem 1 is finished.

5 RELATIONS BETWEEN $\lambda$ AND $\omega_k$

Given a positive parameter $Y$ and a positive integer $k$, consider a centrally-symmetric convex figure $F(Y) \subset \mathbb{R}^{k+1}$ defined by the inequalities

$$F(Y) := \left\{ y \in \mathbb{R}^{k+1} : \|y^{(1,k-1)}\| \leq Y; \ |y_0 + y_1 \zeta + \cdots + y_k \zeta^k| \leq Y^{-k} \right\}.$$

We first establish an estimate on the last successive minimum of $F(Y)$, based on $\omega_k$.

**Lemma 3.** For all $Y \geq 1$, the $(k + 1)'th$ successive minimum $\tau_{k+1}$ of $F(Y)$ is bounded from above by

$$\tau_{k+1} \ll Y^{\frac{k-\omega_k}{1+\omega_k}}.$$  \hspace{1cm} (15)

**Proof.** We first note that the first successive minimum $\tau_1$ of $F(Y)$ is bigger than $cY^{\frac{k-\omega_k}{1+\omega_k}}$ for some small absolute constant. Indeed, if the inequalities

$$||y|| \leq Y^{\frac{k-\omega_k}{1+\omega_k}} \cdot Y = Y^{\frac{k+1}{1+\omega_k}}; \ |y_0 + y_1 \zeta + \cdots + y_k \zeta^k| \leq Y^{\frac{k-\omega_k}{1+\omega_k}} Y^{-k} = Y^{\frac{k-(k+1)\omega_k}{1+\omega_k}}$$

are satisfied then we get that the inequalities (6) have an integer solution $y$ for $X = Y^{\frac{k+1}{1+\omega_k}}$. If $X$ lies outside a bounded region from (6) then we immediately get a contradiction. For smaller values of $X$ we may have a finite number of solutions $y$ of the inequalities above. All of them can be ruled out by taking the constant $c$ in the expression $\tau_1 \geq cY^{\frac{k-\omega_k}{1+\omega_k}}$ small enough.

Secondly, the volume of $F(Y)$ is $2^{k+1}$ and hence by the Minkowski’s theorem on successive minima, we have $\tau_1 \tau_2 \cdots \tau_{k+1} \asymp 1$. That gives an upper bound

$$\tau_{k+1} \ll \tau_1^{-k} \ll Y^{\frac{k-\omega_k}{1+\omega_k}}.$$ \hspace{1cm} \square

For a given vector $a = (a_0, a_1, \ldots, a_k) \in \mathbb{R}^{k+1}$ we denote by $P_a(x)$ a polynomial $a_0 + a_1 x + \cdots + a_k x^k$. And we also define an inverse operation: for a given a polynomial $P \in \mathbb{R}[x]$ of degree $k$ we denote by $a(P)$ a vector in $\mathbb{R}^{k+1}$ composed from the polynomials coefficients.

**Lemma 4.** let $P \in \mathbb{R}[x]$ be a polynomial of degree $m$ and $x \in \mathbb{R}^{m+k+1}$. Suppose there exist $k + 1$ linearly independent polynomials $Q_i \in \mathbb{R}[x], i \in \{0, \ldots, k\}$ of degree at most $k$ such that for all $0 \leq i \leq k$, $a(Q_i P) \in (x)_{\perp}$. Then one has $a(P) \in \langle x^{(0,m)}, x^{(1,m)}, \ldots, x^{(k,m)} \rangle_{\perp}$.

**Proof.** The space of polynomials of degree at most $k$ has dimension $k + 1$, therefore the polynomials $Q_i$ form its basis. That in turn implies that for each $j \in \{0, \ldots, k\}$ one can write $x^j$ as a linear
combination

\[ x^i = \sum_{i=0}^{k} c_{j,i} Q_i(x). \]

We then have \( a(x^i \cdot P(x)) \in \langle x \rangle^\perp \) which is equivalent to \( a(P) \in \langle x^{(j,m)} \rangle^\perp \). To finish the proof, we observe that

\[ \left\langle x^{(0,m)}, x^{(1,m)}, \ldots, x^{(k,m)} \right\rangle^\perp = \bigcap_{i=0}^{k} \left\langle x^{(i,m)} \right\rangle^\perp. \]

Fix a positive integer \( k \). The key idea of the proof below is to consider the shortest nonzero vector \( a \) from \( \langle x^{(0,n-m)}, \ldots, x^{(m,n-m)} \rangle^\perp \cap \mathbb{Z}^{n-m+1} \) where \( m \geq k \) and construct \( k + 1 \) linearly independent polynomials \( Q_j, 0 \leq j \leq k \) such that \( a(Q_j P) \) belongs to \( \langle x^{i+1} \rangle^\perp \). That will imply that \( a \) belongs to \( \langle x^{(0,n-m)}, \ldots, x^{(m,n-m)} \rangle^\perp \) and by repeating this process again and again we infer that vectors \( x_i \) for all large \( i \) belong to a proper subspace of \( \mathbb{R}^{n+1} \) which contradicts to that \( \xi \) is transcendental.

Denote \( \Lambda_i := \langle x^{(0,n-m)}, \ldots, x^{(m,n-m)} \rangle^\perp \cap \mathbb{Z}^{n-m+1} \). For the rest of this section, let \( a = a_i \) be the shortest nonzero vector in \( \Lambda_i \). The latter set is a sublattice in \( \mathbb{R}^{n-m+1} \). By Lemma 1, if \( m \leq \frac{n}{2} \) and \( \lambda > (n - m + 1)^{-1} \), then the vectors \( x^{(j,n-m)}_i \) are linearly independent and the dimension of \( \Lambda_i \) is \( n - 2m \). Therefore \( \Lambda_i \) is indeed a nontrivial sublattice and not just a single zero point if \( n \geq 2m + 1 \) or \( m \leq \lfloor \frac{n-1}{2} \rfloor \).

The covolume of \( \Lambda_i \) in the subspace \( \langle x^{(0,n-m)}, \ldots, x^{(m,n-m)} \rangle^\perp \) equals the length of the primitive multi-vector which is a scalar multiple of

\[ \bigwedge_{j=0}^{m} x^{(j,n-m)}_i. \]

Therefore by the Minkowski’s theorem and Proposition 1, the shortest nonzero vector of \( \Lambda_i \) satisfies

\[ ||a|| \ll \left| \bigwedge_{j=0}^{m} x^{(j,n-m)}_i \right|^{\frac{1}{n-2m}} \ll (X_i L_i^m)^{\frac{1}{n-2m}}. \]  \hspace{1cm} (16)

The linear independence of vectors \( x^{(j,n-m)}_i \) then gives us that \( X_i L_i^m \gg 1 \). Hence by (5), we get that for all large \( i \),

\[ X_i \gg X_i^{m\lambda}. \]  \hspace{1cm} (17)

Corollary 1 may give us a better estimate on \( ||a|| \) in some cases. If \( \lambda > (n - m)^{-1} \), then one of the vectors \( x^{(i,n-m)}_i, 0 \leq i \leq m \) is linearly independent together with \( (x^{(j,n-m)}_i)_{0 \leq j \leq m} \). Therefore, the shortest nonzero vector of \( \Lambda_i \) is not longer than the shortest nonzero vector of \( \Lambda_i \cap \langle x^{(i,n-m)}_i \rangle^\perp \). The latter is a sublattice of dimension \( n - 2m - 1 \) which contains nonzero vectors for \( m \leq \lfloor \frac{n-2}{2} \rfloor \).
The application of Minkowski theorem and Proposition 1 leads to

$$||a|| \ll \left( n-2m-1 \right) \left( X_i L_i \right)^{1/2}.$$  \hspace{1cm} (18)

Conditions $\lambda > (n - m)^{-1}$ together with $n \geq 2m + 2$ imply that the vectors $x_i^{(j,n-m)}$, $j \in \{0, ..., m+1\}$ are linearly independent which, analogously to (17), implies

$$X_i \gg X_i^{(m+1)^{\lambda}}.$$ \hspace{1cm} (19)

We denote the length of $a$ by $A$ and keep in mind that both bounds (16) and (18), with slightly different conditions on $m$ and $\lambda$, are satisfied.

Note that

$$|x_{i,0}P_a(\xi)| = |a_1(x_{i,0}^k - x_{i,1}) + \cdots + a_m(x_{i,0}^{k^m} - x_{i,m})| \ll AL_i.$$

Therefore, $|P_a(\xi)| \ll \frac{AL_i}{X_i}$.

By Lemma 3, for any large enough $Y$, there exist $k+1$ linearly independent polynomials $Q_j \in \mathbb{Z}[x]$, $0 \leq j \leq k$ such that

$$\text{deg } Q_j \leq k, \quad ||a(Q_j)|| \ll Y^{\frac{k(\omega_k - k)}{1+\omega_k}} + |Q_j(\xi)| \ll Y^{\frac{k(\omega_k - k)}{1+\omega_k}-k}.$$ \hspace{1cm} (20)

Choose $Y$ such that for all $j \in \{0, ..., k\}$, one has $|x_{i+1,0}Q_j(\xi)P_a(\xi)| < \frac{1}{2}$. We have that

$$|x_{i+1,0}Q_j(\xi)P_a(\xi)| \ll \frac{X_i AL_i Y^{\frac{k(\omega_k - k)}{1+\omega_k}}}{X_i^{1+k}}.$$ \hspace{1cm} (21)

Hence,

$$Y = C \left( \frac{X_i AL_i}{X_i^{1+k}} \right)^{\frac{1+\omega_k}{1+\omega_k}} \ll \left( \frac{X_i AL_i}{X_i^{1+k}} \right)^{\frac{1+\omega_k}{1+\omega_k}}.$$ \hspace{1cm} (22)

does the job, assuming that $C$ is a large enough absolute constant.

Let $b = b(j) := a(Q_j P_a) \in \mathbb{Z}^{n-m+k+1}$. Similarly to (13), we compute

$$|b \cdot x_{i+1,0}^{(0,n-m+k)}| \leq |x_{i+1,0}Q_j P_a(\xi)| + |b_1(x_{i+1,0}^k - x_{i+1,1}) + \cdots + b_{n-m+k}(x_{i+1,0}^{k^m} - x_{i+1,n-m+k})|.$$ \hspace{1cm} (23)

The second term on the right-hand side is $\ll ||b||L_{i+1}$ which, by (20), is bounded from above by

$$Y^{\frac{(k+1)\omega_k-k+1}{1+\omega_k}} AL_{i+1} \ll \left( \frac{X_i AL_i}{X_i^{1+k}} \right)^{\frac{\omega_k-k+1}{k}} \cdot AL_{i+1} = \left( X_i AL_i \right)^{\frac{\omega_k-k+1}{k}} A^{\frac{\omega_k+1}{k}} L_{i+1}.$$ \hspace{1cm} (24)
Now, if the last expression is smaller than some small enough absolute constant, so that the second term in (22) is smaller than $\frac{1}{2}$, then we derive that $b(j) \cdot x_{i+1}^{(0,n-m+k)} = 0$ for all $j \in \{0, \ldots, k\}$. By analogy, under the same conditions we derive that $b(j) \cdot x_{i+1}^{(n-m+k)} = 0$ for all $j \in \{0, \ldots, k\}$ and $l \in \{0, \ldots, m - k\}$. Then Lemma 4 implies that $a \in \Lambda_{i+1}$.

We apply the upper bound (16) for $A$. Recall that it is satisfied in the case $n \geq 2m + 1$ and $\lambda > (n - m + 1)^{-1}$. Then we continue estimating (23):

$$
\left( \frac{X_{i+1}L_i}{X_i} \right)^{\frac{\omega_k + 1}{k}} A^{\frac{\omega_k - k + 1}{k}} L_{i+1} \ll X_{i+1} \frac{\omega_k + 1}{k(n-2m)} X_i \frac{\omega_k - k + 1}{k} L_i \frac{m(\omega_k + 1)}{k(n-2m)} L_{i+1}.
$$

We consider two cases.

**Case 1.** The degree of $X_i$ is positive. That is equivalent to

$$
1 > \frac{(\omega_k + 1)(n - 2m - 1)}{k(n - 2m)} \iff n < 2m + 1 + \frac{k}{\omega_k + 1 - k}.
$$

In this case, we can use a straightforward bound $X_i < X_{i+1}$. We also use $L_{i+1} < L_i$ to finally get an upper bound for (24):

$$
\frac{\omega_k - k + 1}{k} \frac{\omega_k + 1}{k(n-2m)} \frac{\omega_k + 1}{k(n-2m)} L_{i+1} = X_{i+1} \frac{\omega_k + 1}{k(n-2m)} \frac{\omega_k + 1}{k(n-2m)} L_i.
$$

Note that for $\lambda > (n - m)^{-1}$ the last expression becomes arbitrary close to zero as $X_{i+1} \to \infty$. Therefore, for large enough $i$ the shortest vector of $\Lambda_i$ also belongs to $\Lambda_{i+1}$. This implies that the sequence of lengths $||a_i||$ of the shortest nonzero vectors of $\Lambda_i$ is monotonically nonincreasing. But this sequence cannot decrease infinitely often, therefore there exists $i_0 \in \mathbb{N}$ such that for all $i > i_0$ the sequence $||a_i||$ is constant, hence all the vectors $x_i$ belong to a finite number of proper rational subspaces and thus $\xi$ is algebraic.

We derive that for $n$ in the range

$$
2m + 1 \leq n < 2m + 1 + \frac{k}{\omega_k + 1 - k} = 2m + 1 + \delta_k,
$$

the parameter $\lambda$ must satisfy $\lambda \leq (n - m)^{-1}$.

The expression $(n - m)^{-1}$ grows together with $m$, therefore to make an upper bound on $\lambda$ as small as possible, we need to take the smallest $m$ such that the condition (25) is satisfied. In view of the condition $m \geq k$, if $2k + 1 \leq n < 2k + 1 + \delta_k$, then the smallest possible $m$ is $m = k$ and $\lambda \leq (n - k)^{-1}$. Since $\omega_k$ can be taken arbitrary close to $\omega_k(\xi)$, the last assertion implies the first statement of Theorem 2.

If $n \geq 2k + 1 + \delta_k$, then $m \geq n - \delta_k - 1 \geq k$, that is, the smallest possible $m$ is $\lfloor \frac{n - \delta_k - 1}{2} \rfloor$. Because of the assumption (1) that $\delta_k \geq 1$, we always get $2m + 1 \leq n$ and the condition (25) is satisfied. Now the inequality $\lambda \leq (n - m)^{-1}$ implies the first part of the second assertion of Theorem 2.
Case 2. The degree of $X_i$ is not positive which is equivalent to
\[ n \geq 2m + 1 + \frac{k}{\omega_k + 1 - k}. \]
In this case, we use the bound (17) to further estimate (24):
\[
X_{i+1} = X_i \frac{\omega_k - k + 1}{k(\omega_k - k + 1)m + \omega_k + 1} = X_i \frac{\omega_k - k + 1}{k} \frac{m\lambda - \omega_k + 1}{\lambda}.
\]
This expression becomes arbitrarily small for $i$ large enough if
\[
\lambda > \frac{\omega_k - k + 1}{(\omega_k - k + 1)m + \omega_k + 1} = \frac{1}{m + 1 + \delta_k}.
\]
As in the case 1, for such $\lambda$ we get that the sequence $||a_i||$ for the shortest vectors $a_i$ of lattices $\Lambda_i$ is eventually constant and hence $\xi$ is algebraic.

The last bound on $\lambda$ decays as $m$ grows, therefore the smallest possible upper bound on $\lambda$ will be when $m$ is the largest possible such that $2m + 1 + \delta_k \leq n$ or $m = \left\lfloor \frac{n - \delta_k - 1}{2} \right\rfloor$. Taking into account that all the arguments only work for $\lambda > (n - m + 1)^{-1}$, we finally derive the upper bound:
\[
\lambda \leq \max \left\{ \frac{1}{n - m + 1}, \frac{1}{\left\lfloor \frac{n - \delta_k - 1}{2} \right\rfloor + 1 + \delta_k} \right\} = \frac{1}{\left\lfloor \frac{n - \delta_k - 1}{2} \right\rfloor + 1 + \delta_k},
\]
which is the second term in the second assertion of Theorem 2.

Finally, we need to check the condition $m \geq k$. It is only satisfied when $n \geq 2k + 1 + \delta_k$. That finishes the proof of Theorem 2.

6 | LIGHT IMPROVEMENT OF THEOREM 2

For some values of $n$ and $\delta_k$, the upper bound on $\hat{\lambda}_n(\xi)$ in Theorem 2 can be slightly improved if we use estimates (18) and (19) instead of (16) and (17), respectively. However, the corresponding expressions become much more technical. Also, to use those estimates, we need to make stronger assumptions on $m$ and $\lambda$: $n \geq 2m + 2$ and $\lambda > (n - m)^{-1}$.

Theorem 4. Let $\xi \in \mathbb{R}$ be a transcendental number, $k \in \mathbb{N}$ and $\delta_k$ be defined by (1). Then
\[
\hat{\lambda}_n(\xi) \leq \frac{1}{n - m}
\]
for the minimal value of $m$ such that $m \geq k$ and
\[ 2m + 2 \leq n < 2m + 1 + \left(1 - \frac{1}{n - m}\right)(1 + \delta_k). \]
For a given \( m \in \mathbb{N} \), consider the positive root \( x = x(m) \) of the quadratic equation
\[
\frac{1}{\delta_k} = \frac{(n-m-1)x}{\delta_k(n-2m-1)} - \frac{(n-m-1)x}{n-2m-1} + \left( \frac{1-x}{n-2m-1} - \frac{n-2m-2+x}{\delta_k(n-2m-1)} \right)(m+1)x.
\] (27)

Then for any \( m \) such that \( 2m + 2 \leq n, 2m + 1 + (1 - x(m))(1 + \delta_k) \leq n \) and \( k \leq m \), one has \( \hat{\lambda}_n(\xi) \leq x(m) \).

Proof. We apply an upper bound (18) on \( A \) in estimate (23):
\[
\left( \frac{X_{i+1}L_i}{X_i} \right)^{\omega_k+k+1 \atop k} \ll X_i^{\omega_k+k+1 \atop k} L_i^{\omega_k+k+1 \atop k} \ll X_i^{\omega_k+k+1 \atop k} L_i^{\omega_k+k+1 \atop k}.
\] (28)

As in the proof of Theorem 2, we consider two cases.

**Case 1*. The degree of \( X_i \) is nonnegative. That is equivalent to
\[
1 \geq \frac{(\omega_k + 1)(n-2m-2+\lambda)}{k(n-2m-1)} \iff n \leq 2m + 1 + (1-\lambda)(1+\delta_k).
\]

Then the inequalities \( X_i < X_{i+1} \) and \( L_{i+1} < L_i \) give an upper bound for (28):
\[
X_i^{\omega_k+1 \atop k} L_i^{\omega_k+1 \atop k} \ll X_i^{\omega_k+1 \atop k} L_i^{\omega_k+1 \atop k}.
\]

If \( \lambda > (n-m)^{-1} \), we immediately get that the last expression becomes arbitrary close to zero as \( X_{i+1} \to \infty \). By analogy with Case 1, that implies that the sequence of \( ||a_i|| \) is constant for \( i > i_0 \) and \( \xi \) is algebraic.

If \( n \) is such that
\[
2m + 2 \leq n < 2m + 1 + \left( 1 - \frac{1}{n-m} \right)(1+\delta_k),
\] (29)

then there exists \( \lambda = (n-m)^{-1} + \varepsilon \) for small enough \( \varepsilon \) such that \( n \leq 2m + 1 + (1-\lambda)(1+\delta_k) \) and we are in Case 1*. However, if the upper bound (29) on \( n \) is not satisfied, then the degree of \( X_i \) in (28) is never positive for \( \lambda > (n-m)^{-1} \) and hence Case 1* does not take place. This gives us the first assertion of Theorem 4.

**Case 2*. The degree of \( X_i \) is negative which is equivalent to \( n > 2m + 1 + (1-\lambda)(1+\delta_k) \). Then we use (19) to further estimate (28):
\[
\ll X_i^{\omega_k+1 \atop k} L_i^{\omega_k+1 \atop k} \ll X_i^{\omega_k+1 \atop k} L_i^{\omega_k+1 \atop k}.
\]
We slightly simplify the degree of $X_{i+1}$:

$$\frac{\omega_k - k + 1}{k} - \frac{(\omega_k + 1)(n - m - 1)}{k(n - 2m - 1)} \lambda + \left(1 - \frac{(\omega_k + 1)(n - 2m - 2)}{k(n - 2m - 1)} - \frac{(\omega_k + 1)\lambda}{k(n - 2m - 1)}\right)(m + 1)\lambda$$

$$= \frac{1}{\delta_k} - \frac{(n - m - 1)\lambda}{\delta_k(n - 2m - 1)} - \frac{(n - m - 1)\lambda}{n - 2m - 1} + \left(1 - \frac{\lambda(n - 2m - 2)}{n - 2m - 1} - \frac{n - 2m - 2 + \lambda}{\delta_k(n - 2m - 1)}\right)(m + 1)\lambda. \quad (30)$$

Note that this expression is exactly (27) with $\lambda$ instead of $x(m)$. For $\lambda = (n - m)^{-1}$, one has $(m + 1)\lambda < 1$ and hence the expression (30), as in Case $1^*$, is bounded from below by

$$\frac{(\omega_k + 1)(1 - (n - m)\lambda)}{k(n - 2m - 1)} = 0.$$

Therefore, we have $x(m) > (n - m)^{-1}$ and any $\lambda > x(m)$ also satisfies the condition $\lambda > (n - m)^{-1}$. While it is not needed for the proof itself, it is useful to observe that by the same method we can check that $x(m)$ lies between $(n - m)^{-1}$ and $(m + 1)^{-1}$.

For $\lambda > x(m)$, the expression (30) becomes negative and the upper bound in (28) becomes arbitrary close to zero as $X_{i+1} \to \infty$ and, as before, this implies that $\xi$ is algebraic. Therefore, we must have $\lambda \leq x(m)$.

To demonstrate that Theorem 4 indeed gives slightly better estimates in some cases, we consider a couple of examples. Observe that, by Dirichlet theorem, $\omega_k$ can take values not smaller than $k$ and therefore $\delta_k$ takes values between $0$ and $k$.

**Example 1.** $k = 1$ and $\delta_k = 1$. That is, $\xi$ is not very well approximable.

We emphasize that the conditions of Example 1 are satisfied for almost all real numbers $\xi$ in terms of the Lebesgue measure.

In this case, Theorem 2 gives us that for any $n \geq 2$,

$$\hat{\lambda}_{2n}(\xi) \leq \frac{1}{n + 1}, \quad \hat{\lambda}_{2n-1}(\xi) \leq \frac{1}{n}.$$

The achieved upper bound on $\hat{\lambda}_{2n}(\xi)$ is currently best known (of course, it depends on the condition $\delta_1 = 1$). However, the bound on $\hat{\lambda}_{2n-1}(\xi)$ coincides with that of Laurent [7].

Now we apply Theorem 4. Note that for odd $n$ the condition (26) is not satisfied for all integer values of $m$, hence the first assertion of this theorem cannot be applied. On the other hand, for $n \geq 2$ this assertion gives

$$\hat{\lambda}_{2n}(\xi) \leq \frac{1}{n + 1}.$$

That is exactly the same bound as in Theorem 2. Now we apply the second assertion of Theorem 4. Equation (27) in this case simplifies to

$$2(m + 1)x^2 + (m(n - 2m) + 3n - 7m - 5)x - (n - 2m - 1) = 0.$$
If $n = 2m + 3$, then the equation simplifies even further to $(m + 1)x^2 + (m + 2)x - 1 = 0$. One can check that its positive root $x(m)$ belongs to $(\frac{1}{m+3}, \frac{1}{m+2})$. Simple calculations then verify that all the conditions of Theorem 4 are satisfied and

$$\hat{\lambda}_{2m+3} \leq x(m) = \sqrt{\frac{m^2 + 8m + 8 - (m + 2)}{2(m + 1)}}.$$ 

This upper bound is better than $\frac{1}{m+2}$ derived from Theorem 2. For example,

$$\hat{\lambda}_5(\xi) \leq \frac{\sqrt{17} - 3}{4} \approx 0.2808 < \frac{1}{3}; \quad \hat{\lambda}_7(\xi) \leq \frac{\sqrt{7} - 2}{3} \approx 0.2153 < \frac{1}{4}.$$ 

**Example 2.** $k = 2$ and $\delta_k = 3/2$ which in turn means that $\omega_2(\xi) = 7/3$.

Theorem 2 gives

$$\hat{\lambda}_5(\xi) \leq \frac{1}{3}, \quad \hat{\lambda}_{2n}(\xi) \leq \frac{1}{n+1}, \quad \hat{\lambda}_{2n+1}(\xi) \leq \frac{2}{2n+3} \quad \forall n \geq 3.$$ 

All these upper bounds are smaller than the best currently known bounds on $\hat{\lambda}_n(\xi)$.

The first assertion of Theorem 4 gives the same upper bounds on $\hat{\lambda}_{2n}(\xi)$, $n \geq 3$ as Theorem 2. Also, the conditions of the first assertion are never satisfied for $n = 5$ and $n = 7$. However, for $m \geq 3$, it gives

$$\hat{\lambda}_{2m+3}(\xi) \leq \frac{1}{m + 3}, \quad (31)$$

which is better than in Theorem 2.

Equation (27) simplifies to

$$5(m + 1)x^2 + (2m(n - 2m) + 7n - 16m - 12)x - 2(n - 2m - 1) = 0$$

For $n = 7, m = 2$ we get the solution $x(m) = \frac{1}{5}$ and the second assertion of Theorem 4 infers $\hat{\lambda}_7(\xi) \leq \frac{1}{5}$ which complements (31) for $m = 2$. If $n$ and $m$ are related by $n = 2m + 4$, the equation further simplifies to

$$5(m + 1)x^2 + (6m + 16)x - 6 = 0.$$ 

By substituting $x = (m + 3)^{-1}$ and $x = (m + 4)^{-1}$ into the quadratic polynomial above and checking that the result is positive in the first case and is negative in the second one, one concludes that $x(m) \in (\frac{1}{m+4}, \frac{1}{m+3})$ which is smaller than the value $\frac{1}{m+3}$ provided by Theorem 2. In particular,

$$\hat{\lambda}_8(\xi) \leq \frac{\sqrt{286} - 14}{15} \approx 0.1941 < \frac{1}{5}; \quad \hat{\lambda}_{10}(\xi) \leq \frac{\sqrt{409} - 17}{20} \approx 0.1612 < \frac{1}{6}.$$ 

We can see that in Example 2, Theorem 4 gives better upper bounds for $\hat{\lambda}_n(\xi)$ for all values of $n \geq 7$. 


Remark. In light of [10] which appeared after this paper was written, some of the upper bounds from both examples are now superseded. However, the upper bound on $\hat{\lambda}_4(\xi)$ from Example 1 and on $\hat{\lambda}_{2m+3}(\xi)$ from Example 2 are still best currently known.

7 BETTER RELATION BETWEEN $\hat{\delta}$ AND $\omega_1$ FOR $n = 3$

Here we prove Theorem 3. Denote by $c = c_i$ the shortest vector in the subspace $\langle x_i \rangle \cap \mathbb{Z}^d$. One can easily check that for transcendental $\xi$ there must exist arbitrarily large values of $i$ such that $c_i \notin \langle x_{i+1} \rangle \cap \mathbb{Z}^d$. Fix one such value of $i$. By Proposition 3, we have that

$$||c|| \gg \frac{X_i}{X_{i+1} L_i} \gg \frac{X_i}{X_{i+1}^{1-\lambda}}.$$  (32)

By the Minkowski’s theorem, $||c|| \leq X_{1/3}^j$ which is in turn smaller than $L_{-1}^{-1}$. Therefore Proposition 3 implies that $c_i$ also belongs to $\langle x_{i-1} \rangle \cap \mathbb{Z}^d$. Since $c_i \notin \langle x_{i+1} \rangle \cap \mathbb{Z}^d$, we get that $x_{i-1}, x_i$ and $x_{i+1}$ are linearly independent or equivalently, $i \in I$.

Let $j$ be the predecessor of $i$ in the set $I$. Since $I$ is an infinite set, we can choose $i$ large enough so that $j$ exists. Then we have $\langle x_j, x_{j+1} \rangle = \langle x_{i-1}, x_i \rangle$ and any vector from $\langle x_{j-1}, x_j, x_{j+1} \rangle$ is orthogonal to $x_i$. Therefore, by Proposition 1 and (5), we get

$$||c_i|| \leq ||x_{j-1} \wedge x_j \wedge x_{j+1}|| \ll X_{j-1}^{-\lambda} X_j^{-\lambda}.$$  (33)

Combining two inequalities (32) and (33) together gives us

$$X_{j+1}^{-\lambda} X_{i+1}^{-\lambda} \gg X_j^{\lambda} X_i.$$  (34)

From now on, we assume that $\lambda > \sqrt{2} - 1$ which allows us to apply Lemma 2. Since $x_i \in \langle x_j, x_{j+1} \rangle$, we have $x_i = ux_j + vx_{j+1}$ for some real $u$ and $v$. Also, since $x_i$ is not a scalar multiple of $x_j$, we have that $v$ is nonzero.

The following arguments are adapted from the proof of Lemma 4.2 in [12]. By Lemma 3.1 from [12], the coefficients $u$ and $v$ are integer. If $X_i > 3|u|X_{j+1}$, we have $|u|X_j = |x_i - vx_{j+1}| > 2|v|X_{j+1}$ and so $|u| > 2|v|$. But then we find $L_i \geq |u|L_j - |v|L_{j+1} > L_{j+1}$ which is impossible. This contradiction shows that $|v| \geq X_j/(3X_{j+1}) \gg \frac{X_i}{X_{j+1}}$.

Now for $l \in \{0, 1\}$, we have that

$$\left| x_j^{(0,2)} \wedge x_j^{(1,2)} \wedge x_j^{(l,2)} \right| = v \left| x_j^{(0,2)} \wedge x_j^{(1,2)} \wedge x_j^{(l,2)} \right|.$$  

By Lemma 2, we have that at least one of $x_j^{(0,2)}$ or $x_j^{(1,2)}$ is linearly independent with $(x_j^{(0,2)}, x_j^{(1,2)})$ and hence the same is true for at least one of the vectors $x_i^{(0,2)}$ or $x_i^{(1,2)}$.

Now we will closely follow the proof of Theorem 2 with $k = m = 1$. Let $a = a_j$ be the shortest nonzero vector in $\Lambda_j$. Then (16) gives us $A := ||a|| \ll X_j L_j$. Observe that $|x_{j,0} P_a(\xi)| \ll AL_j$ and hence $|P_a(\xi)| \ll \frac{AL_j}{X_j}$. 
Choose two linearly independent polynomials $Q_0, Q_1 \in \mathbb{Z}[x]$ such that (20) is satisfied for $k = 1$. By Lemma 3, we can do that for any positive $Y$. Choose $Y$ such that for both $l \in \{0, 1\}$, one has $|x_{i,0}Q_l(\xi)P_\alpha(\xi)| < \frac{1}{2}|v|$. Then for a small enough absolute constant $c$, the value

$$Y = c\left(\frac{X_iAL_j}{|v|X_j}\right)^{\frac{1+\omega_1}{2}}$$

(35)
does the job.

Let $b = b(l) := a(Q_lP_\alpha) \in \mathbb{Z}^{n+1}$. We then have $b \cdot x_j = 0$. Indeed, if $Q_l(\xi) = c_1\xi + c_0$, then $b \cdot x_j = (0, c_1a) \cdot x_j + (c_0a, 0) \cdot x_j$ and the last expression equals 0 since $a \in \Lambda_j$. From (20), we have that $||b|| \ll Y^{\frac{2\omega_1}{1+\omega_1}}A$. Next, we compute

$$|b \cdot x_i| \leq |x_{i,0}Q_l(\xi)P_\alpha(\xi)| + |b_1(x_{i,0}\xi - x_{i,1}) + \cdots + b_3(x_{i,0}\xi^3 - x_{i,3})|.$$ 

The left-hand side is equal to $|b \cdot x_i| = |b \cdot (ux_j + vx_{j+1})| = |v||b \cdot x_{j+1}|$ and we derive that it is a multiple of $v$. If the second term on the right-hand side is smaller than $\frac{1}{2}|v|$, then we get $b(0) \cdot x_i = b(1) \cdot x_i = 0$ which by Lemma 4 implies that $a_j \in \Lambda_i$. Finally, that means that the systems $x_j^{(0,2)}, x_j^{(1,2)}, x_i^{(0,2)}$ and $x_j^{(0,2)}, x_j^{(1,2)}, x_i^{(1,2)}$ are linearly dependent which is impossible, as we showed above.

We conclude that the second term must be at least $\frac{1}{2}|v|$ for at least one of the vectors $b(0), b(1)$. This fact, together with (35) and $A \ll X_jL_j$ infers

$$\frac{X_i}{X_{j+1}} \ll |v| \ll Y^{\frac{2\omega_1}{1+\omega_1}}AL_i \ll \left(\frac{X_iL_j}{|v|X_j}\right)^{\omega_1} (X_jL_j)^{\omega_1+1}L_i,$$

$$\ll X_j^{\omega_1}X_jL_j^{2\omega_1+1}L_i \ll X_j^{\omega_1}X_j^{-(2\omega_1+1)}X_{j+1}^{-\omega_1}X_j,$$

or

$$X_j \gg X_iX_j^{\frac{1}{1+\omega_1-(2\omega_1+1)\lambda}}.$$ 

Now we substitute this estimate for $X_j$ into (34) to get

$$X_{j+1}^{1-\lambda}X_{i+1}^{1-\lambda} \gg X_i^2X_{j+1}^{1-\lambda}X_{j+1}^{-(1-\omega_1+(2\omega_1+1)\lambda)\lambda}X_i$$

$$\implies X_{i+1}^{1-\lambda-\lambda^2} \gg X_i^{1+\lambda}X_{j+1}^{-1+(2\omega_1+1)\lambda-\omega_1\lambda}.$$ 

One can easily check that for $0 < \lambda < \frac{1}{2}$, the degree of $X_{j+1}$ in the last expression is negative. Hence, we can use the inequality $X_{j+1} \leq X_i$ and Lemma 2 to get

$$X_{i+1}^{1-\lambda-\lambda^2} \gg X_i^{((2\omega_1+1)\lambda-\omega_1+1)\lambda} \gg X_i^{\frac{((2\omega_1+1)\lambda-\omega_1+1)\lambda^2}{1-\lambda^2}}.$$
That implies

$$(1 - \lambda - \lambda^2)(1 - \lambda) \geq ((2\omega_1 + 1)\lambda - \omega_1 + 1)\lambda^2.$$ 

By rearranging terms, one can easily check that the last inequality is equivalent to (2) with $\lambda$ and $\omega_1$ in place of $\hat{\lambda}_3(\xi)$ and $\omega_1(\xi)$, respectively. Since $\omega_1$ can be taken arbitrarily close to $\omega_1(\xi)$ and $\lambda$ can be taken arbitrarily close to $\hat{\lambda}_3(\xi)$, we finish the proof of Theorem 3.

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