Abstract

We study restricted chain-order polytopes associated to Young diagrams using combinatorial mutations. These polytopes are obtained by intersecting chain-order polytopes with certain hyperplanes. The family of chain-order polytopes associated to a poset interpolate between the order and chain polytopes of the poset. Each such polytope retains properties of the order and chain polytope; for example its Ehrhart polynomial. For a fixed Young diagram, we show that all restricted chain-order polytopes are related by a sequence of combinatorial mutations. Since the property of giving rise to the period collapse phenomenon is invariant under combinatorial mutations, we provide a large class of rational polytopes that give rise to period collapse.

1 Introduction

Let \( P \subset \mathbb{R}^N \) be a \( d \)-dimensional rational polytope. The lattice point counting function \( L_P(n) := |nP \cap \mathbb{Z}^N| \) is a quasi-polynomial in \( n \) of degree \( d \), that is, a polynomial \( L_P(n) = c_d(n)n^d + c_{d-1}(n)n^{d-1} + \cdots + c_1(n)n + c_0(n) \) whose coefficients \( c_i(n) \) are periodic in \( n \) [4]. The least common multiple of the periods of \( c_i \) is called the period of \( P \). Generically, the period of \( P \) is equal to its denominator, which is the smallest positive integer \( m \) such that all vertices of the \( m \)th dilate of \( P \) lie in \( \mathbb{Z}^N \). By a well-known theorem of Ehrhart, the period divides the denominator. So all lattice polytopes are polytopes with period one, however the converse is false. We say that \( P \) has period collapse if its period is not equal to its denominator. It was proved in [15, Theorem 2.2] that, for all positive integers \( d, D, \) and \( s \) with \( d \geq 2 \) such that \( s \) divides \( D \), there exists a \( d \)-dimensional rational polytope with its denominator \( D \) and period \( s \). Since then, period collapse has become one of the main topics in Ehrhart theory. See, e.g, [5, 12, 15].

Our method to study polytopes involves combinatorial mutations [1, 13], which were originally defined in the study of mirror symmetry for Fano varieties. There are two complementary perspectives on combinatorial mutations that are related by taking the dual. Our perspective is derived from the so-called \( M \)-lattice. That is, we consider a combinatorial mutation to be a piece-wise linear map. In this setting, polytopes that are related by a combinatorial mutation have the same Ehrhart polynomial. See [1, Proposition 4] and Proposition 2.2. In particular, if \( P \) has period collapse, then all polytopes that are mutation equivalent to \( P \) also have period collapse.
Despite preserving their Ehrhart polynomials, many other salient features of polytopes are not invariant under combinatorial mutations. For example, the denominator and the number of vertices may change after a combinatorial mutation.

**Example 1.1.** Let $P \subset \mathbb{R}^2$ be the convex hull of the rational points $(1, 0), (0, -1), (-1, 0)$ and $(0, \frac{1}{2})$. Let $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ be the piece-wise linear map given by

$$
\varphi(x, y) = \begin{cases} 
(x, y) & \text{if } x \leq 0, \\
(x, x + y) & \text{if } x \geq 0.
\end{cases}
$$

The map $\varphi$ is an example of a *tropical map* $\varphi_{w,F}$ with respect to the data $w = (0, 1)$ and $F = \text{conv}\{(0,0),(-1,0)\}$. See Section 2.1. The image of $P$ under $\varphi$ is the lattice simplex $Q$ with vertices $(-1,0), (0, -1)$ and $(1, 1)$. See Figure 1. Since $P$ and $Q$ are mutation equivalent, they have the same Ehrhart polynomial. Moreover, the polytope $Q$ is a lattice polytope, so we deduce that $P$ has period collapse.

![Figure 1: The combinatorial mutation in Example 1.1.](image)

In this paper we focus on a class of polytopes related to poset, Gelfand-Tsetlin, and Birkhoff polytopes [2, 3, 11, 18]. The Gelfand-Tsetlin polytopes $\text{GT}_{\alpha,\beta}$ arise from representation theory, where their lattice points form a basis for the corresponding irreducible representation of the Lie algebra $\mathfrak{sl}_n$. Using the perspective from representation theory, it is known that the Ehrhart quasi-polynomials of such Gelfand-Tsetlin polytopes are indeed polynomials [14]. The Gelfand-Tsetlin polytope is an example of the *order polytope* of a poset intersected with certain affine-hyperplanes. The order and chain polytopes, originally defined by Stanley [18], are polytopes associated to a poset. The chain-order polytopes are polytopes that interpolate between the order and chain polytopes via a sequence of piece-wise linear maps called *transfer maps* [18, Definition 3.1]. It was shown that these piece-wise linear maps, similarly to combinatorial mutations, preserve the Ehrhart polynomial. Note that the chain-order polytopes were originally defined for *marked* posets (see [10]), which is a vast generalisation. In this paper, we specialise this definition so that our notions are concurrent with usual poset polytopes.
Recently, in [2], it was shown that the restricted order and restricted chain polytopes that correspond to certain Gelfand-Tsetlin and Birkhoff polytopes, respectively, are related by a sequence of piece-wise linear maps. These piece-wise linear maps were originally defined by Pak [16], and is almost equivalent to the Robinson–Schensted–Knuth (RSK) correspondence [17]. However, as noted in [2, Section 5], the transfer maps do not commute with taking intersections with the hyperplanes that define the restricted order polytope and restricted chain polytope. In this paper we resolve this problem by decomposing Pak’s piece-wise linear map into smaller pieces, which are combinatorial mutations. See Proposition 3.2. We define the restricted analogues of chain-order polytopes in Definition 3.3 and prove that these polytopes are mutation equivalent, providing a complete answer to [2, Remark 6.4].

Outline. In Section 2 we introduce the necessary preliminaries to state our main results. In particular, we define combinatorial mutations in Section 2.1, recall the definitions of polytopes from posets and Young diagrams in Section 2.2, and fix our main setup (Setup 2.7) in Section 2.3. In Section 3, we describe our main results. Our main theorem (Theorem 3.1) shows mutation equivalence of the restricted order and chain polytopes. We describe a sequence of mutations that relate these polytopes and define our main tool for proving Theorem 3.1: the restricted chain-order polytopes in Definition 3.3. In Section 4, we prove our main results in two steps using Lemmas 4.1 and 4.2. In Section 5 we provide some corollaries of our results, together with some computations and further questions.

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2 Preliminaries

In this section, we recall the main definitions and fix our notation. See Setup 2.7. We use this notation to explain our main results in Section 3.

2.1 Combinatorial mutations

Fix a natural number $N$. For each $x$ and $y$ in $\mathbb{R}^N$, denote by $x \cdot y$ the usual dot product of $x$ and $y$. We recall the definition of combinatorial mutation of a polytope $P \subseteq \mathbb{R}^N$ in terms of a piece-wise linear map. Let $w \in \mathbb{Z}^N$ be a primitive lattice point and $F \subseteq w^\perp \subseteq \mathbb{R}^N$ a lattice polytope, where $w^\perp = \{x \in \mathbb{R}^N \mid x \cdot w = 0\}$. We define the tropical map

$$\varphi_{w,F} : \mathbb{R}^N \to \mathbb{R}^N, \quad x \mapsto x - x_{\min}w,$$

where $x_{\min} = \min\{x \cdot f \mid f \in F\}$. Each vertex $v \in F$ defines a region of linearity of $\varphi_{w,F}$ given by $U = \{x \in \mathbb{R}^N \mid x_{\min} = x \cdot v\}$. The restriction $\varphi_{w,F}|_U : x \mapsto x - (x \cdot v)w$ is a unimodular map given by a shear.
Let \( P \subseteq \mathbb{R}^N \) be a polytope. If \( \varphi_{w,F}(P) \) is convex, we say that \( \varphi_{w,F}(P) \) is a combinatorial mutation of \( P \). Two polytopes \( P \) and \( Q \) in \( \mathbb{R}^N \) are said to be combinatorial-mutation equivalent, or simply mutation equivalent, if there exists a sequence of combinatorial mutations

\[
P_1 = \varphi_{w_1,F_1}(P_0), \ P_2 = \varphi_{w_2,F_2}(P_1), \ldots, \ P_k = \varphi_{w_k,F_k}(P_{k-1})
\]

such that \( P_0 = P \) and \( P_k = Q \). We refer to the polytopes \( P_2, P_3, \ldots, P_{k-1} \) as the intermediate polytopes of the sequence of combinatorial mutations.

**Remark 2.1.** Combinatorial mutations arise in the context of mirror symmetry in the study of the classification of Fano varieties [1]. They have also been shown to connect families of Newton-Okounkov bodies for partial flag varieties [6, 7, 8] and adjacent tropical cones [9].

The Ehrhart series is the generating function of the number of lattice points \( |nP \cap \mathbb{Z}^N| \) in the \( n \)th dilate of \( P \). We denote it

\[
E_P(t) := \sum_{n \geq 0} |nP \cap \mathbb{Z}^N| \ t^n.
\]

**Proposition 2.2** ([1, Proposition 4]). Mutation equivalent polytopes have the same Ehrhart series.

For clarity we provide our own proof of this proposition. Note that we do not assume that the dual polytopes are lattice polytopes.

**Proof.** Suppose that \( \varphi_{w,F}(P) \) is a combinatorial mutation of \( P \). The regions of linearity of \( \varphi_{w,F} \) are the maximal cones of a polyhedral fan \( \Sigma \). Let \( \Sigma^\circ = \{ \sigma^\circ \mid \sigma \in \Sigma \} \) be the collection of relative interiors of cones in \( \Sigma \). Note that \( \Sigma \) is a complete fan, so \( \mathbb{R}^N = \bigcup_{\sigma^\circ \in \Sigma} \sigma^\circ \). Since \( \varphi_{w,F} \) is piece-wise unimodular, for each \( \sigma^\circ \in \Sigma^\circ \), we have that \( \varphi_{w,F}(P \cap \sigma^\circ) \) and \( P \cap \sigma^\circ \) have the same Ehrhart series. So, we have

\[
E_{\varphi_{w,F}(P)}(t) = \sum_{\sigma^\circ \in \Sigma^\circ} E_{\varphi_{w,F}(P \cap \sigma^\circ)}(t) = \sum_{\sigma^\circ \in \Sigma^\circ} E_{P \cap \sigma^\circ}(t) = E_P(t).
\]

\[ \square \]

### 2.2 Posets and polytopes

Let \((\mathcal{P}, \lt)\) be a partially ordered set, which we usually denote as \( \mathcal{P} \). A subset \( U \subseteq \mathcal{P} \) is called an up-set if for all \( x \in U \) and \( y \in \mathcal{P} \), we have that \( x \lt y \) implies that \( y \in U \). A subset \( D \subseteq \mathcal{P} \) is a down-set if its complement \( \mathcal{P}\setminus D \) is an up-set. A subset \( S \subseteq \mathcal{P} \) is called a chain if each pair of elements in \( S \) is comparable. A subset \( S \subseteq \mathcal{P} \) is an antichain if no pair of elements in \( S \) is comparable. Given a subset \( S \subseteq \mathcal{P} \), the collections of minimal and maximal elements are respectively \( \min(S) = \{ s \in S \mid \ x \lt s \text{ for all } x \in S \} \) and \( \max(S) = \{ s \in S \mid \ x \not\lt s \text{ for all } x \in S \} \).

We denote by \( \mathbb{N} = \{1, 2, \ldots, \} \) the set of natural numbers and the set \([n] = \{1, 2, \ldots, n\} \) for any \( n \in \mathbb{N} \). We equip \( \mathbb{N}^2 \) with the component-wise partial order: \( (a, b) \leq (c, d) \) if \( a \leq c \) and \( b \leq d \). A Young diagram \( \lambda \subseteq \mathbb{N}^2 \) is a finite down-set, which we take to be a sub-poset of \( \mathbb{N}^2 \). Young diagrams...
are often defined as partitions of natural numbers. Explicitly, a partition \((\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k)\) of \(n \in \mathbb{N}\) is naturally associated to the down-set
\[
\lambda = \{(1,1), (1,2), \ldots, (1,\lambda_1), (2,1), (2,2), \ldots, (2,\lambda_2), \ldots, (k,1), \ldots, (k,\lambda_k)\} \subseteq \mathbb{N}^2.
\]

Typically, a Young diagram \(\lambda\) is depicted as a collection of boxes, with \((1,1)\) located in the top-left, \((1,2)\) to the immediate right of \((1,1)\), \((2,1)\) immediately below \((1,1)\), and so on. We say that a box \(r \in \lambda\) is a corner if \(r \in \text{max}(\lambda)\) is a maximal element.

**Example 2.3.** The Young diagram given by the partition \((4,4,3)\), is represented via the following collection of boxes.

![Young diagram](image)

The Young diagram has two corners: \((2,4)\) and \((3,3)\), which are the shaded boxes above.

We write \(\mathbb{R}^\lambda\) for the real vector space with distinguished basis \(\{e_p \mid p \in \lambda\}\). Given \(x, y \in \mathbb{R}^\lambda\) we write \(x \cdot y = \sum_{p \in \lambda} x_p y_p\) for the standard dot-product. For all \(x \in \mathbb{R}^\lambda\) and \(p \notin \lambda\), we take the convention that \(x_p = 0 \in \mathbb{R}\) and \(e_p = 0 \in \mathbb{R}^\lambda\).

We recall the definition of two polytopes classically associated to a poset. The **order polytope** of the Young diagram \(\lambda\) is the polytope
\[
O(\lambda) = \{x \in \mathbb{R}^\lambda \mid 0 \leq x_p \leq x_q \leq 1 \text{ for all } p \leq q \text{ in } \lambda\}.
\]

The **chain polytope** of \(\lambda\) is the polytope
\[
C(\lambda) = \{x \in \mathbb{R}^\lambda \mid 0 \leq x_{p_1} + x_{p_2} + \cdots + x_{p_k} \leq 1 \text{ for all } p_1 < p_2 < \cdots < p_k \text{ in } \lambda\}.
\]

Each point \(x \in \mathbb{R}^\lambda_{\geq 0}\) can be thought of as a non-negative filling of the Young diagram, i.e. writing the value \(x_p\) in the box \(p \in \lambda\). A point \(x \in \mathbb{R}^\lambda_{\geq 0}\) lies in the \(k\)th dilate \(kO(\lambda)\) if and only if the values in each box do not exceed \(k\) and increase when moving down and to the right. Similarly, a point \(x \in \mathbb{R}^\lambda_{\geq 0}\) lies in \(kC(\lambda)\) if and only if the sum of values along any path in \(x\) that starts at \((1,1)\) and moves down to the right is at most \(k\). The vertices of these polytopes have the following descriptions.

**Proposition 2.4** ([18, Corollary 1.3 and Theorem 2.2]). Fix a Young diagram \(\lambda\) and for each subset \(S \subseteq \lambda\) define the characteristic vector \(\chi(S) \in \mathbb{R}^\lambda\) by \(\chi(S)_p = 1\) if \(p \in S\) and \(\chi(S)_p = 0\) if \(p \notin S\). The vertices of \(O(\lambda)\) and \(C(\lambda)\) are
\[
V(O(\lambda)) = \{\chi(S) \mid S \text{ an up-set of } \lambda\} \quad \text{and} \quad V(C(\lambda)) = \{\chi(S) \mid S \text{ an antichain of } \lambda\}.
\]

The order polytope and chain polytope are mutation equivalent [13]. The sequence of mutations can be realised by a decomposition of a piece-wise linear map called the transfer map, introduced in [18], which interpolates between the polytopes. Intermediate polytopes of this sequence of mutations are given by the so-called chain-order polytope.
Definition 2.5. Let $\lambda$ be a Young diagram and let $C \subseteq \lambda$ be a proper up-set. The chain-order polytope of $\lambda$ with respect to $C$ is the polytope

$$O_C(\lambda) = \left\{ x \in \mathbb{R}^\lambda : \begin{cases} 0 \leq x_p \leq 1 & \text{for all } p \in \lambda, \\ x_p \leq x_q & \text{for all } p \leq q \text{ in } \lambda \setminus C, \\ x_p + x_{q_1} + \cdots + x_{q_n} \leq 1 & \text{for all } p \in \lambda \setminus C \text{ and } q_1, \ldots, q_n \in C \\ \text{such that } p < q_1 < \cdots < q_n. \end{cases} \right\}.$$ 

If $C = \emptyset$, then $O_C(\lambda) = O(\lambda)$ coincides with the order polytope. If $C = \lambda \setminus \{(1, 1)\}$, then $O_C(\lambda) = C(\lambda)$ coincides with the chain polytope. For $C = \lambda$, we define $O_C(\lambda) = C(\lambda)$ to be the chain polytope.

Example 2.6. Consider the Young diagram $\lambda$ given by the partition $(3, 2)$ and the up-set $C = \{(1, 2), (1, 3), (2, 2)\}$ corresponding to the highlighted boxes below. The chain-order polytope of $\lambda$ with respect to $C$ has vertices:

$$\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
\end{array} \text{ and } \begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}$$

2.3 Restricted chain and order polytopes

We now consider restricted versions of the order and chain polytopes; that is, the intersection of these polytopes with certain hyperplanes. To define them, we fix the following setup that will be used throughout the rest of the paper.

Setup 2.7. Fix a Young diagram $\lambda$. Let $m_1, m_2 \in \mathbb{N}$ be smallest natural numbers such that $\lambda \subseteq [m_1] \times [m_2]$. For example, if $\lambda$ is given by the partition $\{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n\}$, then $(m_1, m_2) = (s, \lambda_1)$. Fix a vector $d = (d_{m_1-1}, d_{m_1-2}, \ldots, d_0, d_{-1}, \ldots, d_{1-m_2}) \in \mathbb{N}^{m_1+m_2-1}$ and a natural number $k \in \mathbb{N}$. Define the set of integers $\text{Diag}(\lambda) := \{1 - m_2, 2 - m_2, \ldots, m_1 - 1\}$, which index the diagonals of $\lambda$. For each $\ell \in \text{Diag}(\lambda)$, denote by $r_\ell$ the maximal element in the $\ell$th diagonal of $\lambda$: $\{(i, j) \in \lambda \mid i - j = \ell\}$. Visually, the box $r_\ell$ is the bottom-right-most box of the Young diagram on the $\ell$th diagonal. See Figure 2.

Definition 2.8. Fix Setup 2.7. The restricted order polytope of $\lambda$ with respect to $d$ and $k$ is the restriction of the $k$th dilate of the order polytope of $\lambda$ defined by:

$$O(\lambda)_d^k = \left\{ x \in kO(\lambda) : \sum_{i-j=\ell} x_{(i,j)} = d_\ell \text{ for all } \ell \in \text{Diag}(\lambda) \right\}.$$ 

The restricted chain polytope of $\lambda$ with respect to $d$ and $k$ is the restriction of the $k$th dilate of the chain polytope of $\lambda$ defined by:

$$C(\lambda)_d^k = \left\{ x \in kC(\lambda) : \sum_{p \leq r_\ell} x_p = d_\ell \text{ for all } \ell \in \text{Diag}(\lambda) \right\}.$$
Figure 2: Highlighted in blue, the 1st diagonal given by the boxes $(2, 1)$ and $r_1 = (3, 2)$. Highlighted in purple, the box $r_{-2} = (2, 4)$.

Remark 2.9. If $\lambda = \{(1, 1), \ldots, (n, n)\}$ is square and $d = (1, 2, \ldots, n - 1, n, n - 1, \ldots, 1)$, then the restricted order polytope is equal to the restricted Gelfand-Tsetlin polytope and the restricted chain polytope is equal to the restricted Birkhoff polytope. The period-collapse phenomenon of these polytopes is studied in [2]. In Section 5, we show that period-collapse also occurs for some other families of polytopes.

3 Main results

In this section we explain our main results, which use combinatorial mutations to connect restricted order polytopes to restricted chain polytopes. In particular, by Proposition 2.2, if a polytope exhibits period collapse, then all mutation equivalent polytopes simultaneously exhibit period collapse.

Theorem 3.1. Fix Setup 2.7. The restricted order polytope $O(\lambda)_{\frac{d}{2}}$ and the restricted chain polytope $C(\lambda)_{\frac{d}{2}}$ are mutation equivalent.

We construct a sequence of mutation equivalent polytopes that interpolate between the restricted order and restricted chain polytopes. We do this by defining a restricted analogue of the chain-order polytope. We show that all piece-wise linear maps that connect these polytopes are combinatorial mutations. The foundation for our construction is the piece-wise linear map introduced by Pak [16].

Pak’s piece-wise linear map. Fix Setup 2.7. We recall, from [16, Section 4], the piece-wise linear map $\xi_\lambda : \mathbb{R}^\lambda \to \mathbb{R}^\lambda$ which bjectively maps $O(\lambda)_{\frac{d}{2}}$ to $C(\lambda)_{\frac{d}{2}}$. The map is defined inductively on $|\lambda|$. If $|\lambda| = 1$, then $\xi_\lambda = \text{id}$. Otherwise, if $|\lambda| > 1$, then let $r = r_\ell$ be a corner of $\lambda$. We recall the convention that for any $x \in \mathbb{R}^\lambda$, we have $x_p = 0$ if $p \notin \lambda$. Define the map $\chi_r : \mathbb{R}^\lambda \to \mathbb{R}^\lambda$ as follows:

$$
\chi_r(x)_{(i,j)} = \begin{cases} 
  x_{(i,j)} & \text{if } i - j \neq \ell, \\
  x_{(i,j)} - \max\{x_{(i-1,j)}, x_{(i,j-1)}\} & \text{if } (i, j) = r, \\
  \max\{x_{(i-1,j)}, x_{(i,j-1)}\} & \text{if } i - j = \ell \text{ and } (i, j) \neq r.
\end{cases}
$$

By induction, the map $\xi_{\lambda \setminus \{r\}}$ is already defined, so we define $\xi_\lambda = (\xi_{\lambda \setminus \{r\}} \times \text{id}) \circ \chi_r$. By [16, Theorem 4], the map $\xi_\lambda$ is well-defined, in particular it does not depend on the choice of corner $r$ above.
We naturally extend the definition of the map $\chi_r$ to non-corners $r \in \lambda$. This is done by defining $\chi_r(x)_{(i,j)} = x_{(i,j)}$ for all $(i,j) \neq r$ and taking the above definition for all other coordinates $(i,j) \leq r$. So we may express Pak’s map as a composition:

$$\xi_\lambda = \chi_{r_1} \circ \chi_{r_2} \circ \cdots \circ \chi_{r_n},$$

(1)

for any permutation $(r_1, \ldots, r_n)$ of the elements of $\lambda$ that satisfies $r_i < r_j \implies i < j$.

**A decomposition of Pak’s map.** Fix $r = (a,b) \in \lambda$ in the $\ell$th diagonal of $\lambda$. We express the map $\chi_r$ as the composition of tropical maps and a unimodular map. In fact, we will show that this gives a sequence of combinatorial mutations. We define the tropical maps and unimodular map as follows:

- For each $0 \leq i \leq i_{\text{max}}(r) := \min\{a, b\} - 1$, let

$$w_i = e_{(a-i-1, b-i-1)} - e_{(a-i, b-i)} \quad \text{and} \quad F_i = \text{conv} \left\{ e_{(a-i-1, b-i)}, e_{(a-i, b-i-1)} \right\}.$$

Define the tropical map $\varphi_i := \varphi_{w_i,F_i}$.

- Let $\psi$ be the unimodular map:

$$\psi(x)_{(i,j)} = \begin{cases} x_{(i,j)} & \text{if } i - j \neq \ell, \\ x_{(i,j)} - (x_{i-1,j} + x_{i,j-1}) & \text{if } (i, j) = r, \\ -x_{(i,j)} + (x_{i-1,j} + x_{i,j-1}) & \text{if } i - j = \ell \text{ and } (i, j) \neq r. \end{cases}$$

**Proposition 3.2.** *With the notation above, $\chi_r = \psi \circ \varphi_{i_{\text{max}}(r)} \circ \cdots \circ \varphi_0$.***

**Proof.** It is straightforward to show that the composition of the tropical maps is

$$(\varphi_{i_{\text{max}}(r)} \circ \cdots \circ \varphi_0(x))_{(i,j)} = \begin{cases} x_{(i,j)} & \text{if } i - j \neq \ell, \\ x_{(i,j)} + \min\{x_{i-1,j}, x_{i,j-1}\} & \text{if } (i,j) = r, \\ x_{(i,j)} + \min\{x_{i-1,j}, x_{i,j-1}\} - \min\{x_{i+1,j}, x_{i,j+1}\} & \text{if } i - j = \ell \text{ and } (i,j) < r. \end{cases}$$

For all $u, v \in \mathbb{R}$ we have $\min\{u, v\} - u - v = -\max\{u, v\}$, so it follows that

$$\psi \circ \varphi_{i_{\text{max}}(r)} \circ \cdots \circ \varphi_0(x) = \chi_r(x)$$

for every $x \in \mathbb{R}^\lambda$. \qed

To prove that these tropical maps are combinatorial mutations, we proceed inductively. For each $0 \leq i \leq i_{\text{max}}(r)$, we show that the image of the polytope under $\varphi_i$ is convex. As a result, we obtain a collection of mutation equivalent polytopes. Recall that Pak’s map $\xi_\lambda$ can be expressed as a composition of maps $\chi_r$, as in Equation (1). Then, for each $1 \leq s \leq n$, the polytope

$$\mathcal{O}_C(\lambda)^k = \chi_{r_s} \circ \chi_{r_{s+1}} \circ \cdots \circ \chi_{r_n}(\mathcal{O}(\lambda)^k)$$

can be regarded as the restricted analogue of the chain-order polytope $\mathcal{O}_C(\lambda)$ with respect to the up-set $C = \{r_s, r_{s+1}, \ldots, r_n\}$. We give an abstract definition of the polytope $\mathcal{O}_C(\lambda)^k$ and prove that the above equality holds in Section 4.
Definition 3.3. Fix Setup 2.7 and let $C \subseteq \lambda$ be an up-set. For each $\ell \in \text{Diag}(\lambda)$, we define:

- $R_\ell = \{(a, b) \in \lambda \mid (a, b) \leq r_\ell\}$ the rectangular sub-Young diagram of $\lambda$,
- $S_\ell = \max((\lambda \setminus C) \cap R_\ell)$ the corners of $(\lambda \setminus C) \cap R_\ell$,
- $\overline{S}_\ell = \{(a, b) \in R_\ell \mid (a + i, b + i) \in S_\ell$ for some $i \geq 0\}$ the diagonals of $R_\ell$ ending in $S_\ell$,
- $\overline{T}_\ell = \{(a, b) \in R_\ell \mid (a + i, b + i) \in T_\ell$ for some $i \geq 1\}$ the diagonals in $R_\ell$ that end one step before $T_\ell$.

The restricted chain-order polytope with respect to $d \in \mathbb{N}^{m_1 + m_2 - 1}$ and $k \in \mathbb{N}$ is

$$O_C(\lambda)_d^k = \left\{ x \in kO_C(\lambda) \mid \sum_{s \in \overline{S}_\ell} x_s - \sum_{t \in \overline{T}_\ell} x_t + \sum_{u \in C \cap R_\ell} x_u = d_\ell$ for each $\ell \in \text{Diag}(\lambda) \right\}.$$  

Example 3.4. Let $\lambda = \{(1, 1), \ldots, (5, 6)\}$ be the rectangular Young diagram with 4 rows and 5 columns. Let $C \subseteq \lambda$ be the up-set with minimal elements $(4, 3)$ and $(3, 5)$. Fix $\ell = -1$, then $r_\ell = (5, 6)$ and the set $R_\ell = \lambda$ is the entire poset. The sets $S_\ell$, $\overline{S}_\ell$, $T_\ell$ and $\overline{T}_\ell$ are shown in Figure 3.

Figure 3: Depiction of the Young diagram in Example 3.4. Shaded boxes represent the up-set $C$. Fix $\ell = -1$. The set $S_\ell$ is given by the boxes labelled with $s$, the set $\overline{S}_\ell$ are the boxes $s$ and $\overline{s}$, the set $T_\ell$ are the boxes $t$, and the set $\overline{T}_\ell$ are the boxes $\overline{t}$.

4 Proofs

In this section we prove all the results from Section 3, in particular Theorem 3.1, which follow from Lemmas 4.1 and 4.2. We begin by fixing the notation for the affine hyperplanes that define the restricted chain-order polytopes. We then proceed to prove the two lemmas followed by the main theorem.
Notation. Following the notation from Definition 3.3, given a restricted chain-order polytope $O_C(\lambda)_d^k$ and $\ell \in \text{Diag}(\lambda)$, we define the hyperplane

$$H_\ell = \left\{ x \in \mathbb{R}^\lambda \left| \sum_{s \in S_\ell} x_s - \sum_{t \in T_\ell} x_t + \sum_{u \in C \cap R_\ell} x_u = d_\ell \right. \right\}.$$  

In particular we have that $O_C(\lambda)_d^k = kO_C(\lambda) \cap \bigcap_{\ell} H_\ell$, where the intersection runs over $\text{Diag}(\lambda)$. Whenever we work with hyperplanes $H_\ell$ with respect to different up-sets $C$, we write $H_\ell^C$ for $H_\ell$ to avoid ambiguity. Similarly, we write $S_\ell^C$, $\overline{S_\ell}^C$, $T_\ell^C$ and $\overline{T_\ell}^C$ for the sets in Definition 3.3.

Lemma 4.1. Fix Setup 2.7 and let $C \subseteq \lambda$ be an up-set. Let $r = (a, b)$ be a corner of $\lambda \setminus C$ and $\ell = a - b$. We denote by $\chi_r : \mathbb{R}^\lambda \rightarrow \mathbb{R}^\lambda$ the extension of Pak’s map defined on $\mathbb{R}^{\lambda \setminus C}$. Then $\chi_r : O_C(\lambda) \rightarrow O_{C \cup \{r\}}(\lambda)$ is a bijection. Recall the decomposition $\chi_r = \psi \circ \varphi_{i_{\max}(r)} \circ \cdots \circ \varphi_0$, from Proposition 3.2. For all $i \in \{0, \ldots, i_{\max}(r)\}$ and $\ell \in \text{Diag}(\lambda)$, we have that $\varphi_i(H_\ell^C) = H_\ell^C$ and $\psi(H_\ell^C) = H_{\ell \cup \{r\}}^C$. Hence $\chi_r : O_C(\lambda)_d^k \rightarrow O_{C \cup \{r\}}(\lambda)_d^k$ is a bijection.

Proof. By [16], we have that $\chi_r$ admits an inverse given by:

$$\chi_r^{-1}(x)_{(i,j)} = \begin{cases} x_{(i,j)} + \max\{x_{(i-1,j)}, x_{(i,j-1)}\} & \text{if } (i, j) = r, \\ \max\{x_{(i-1,j)}, x_{(i,j-1)}\} & \text{if } i - j = \ell \text{ and } (i, j) < r, \\ + \min\{x_{(i+1,j)}, x_{(i,j+1)}\} - x_{(i,j)} & \text{otherwise}. \end{cases}$$

Hence, to show that $\chi_r : O_C(\lambda) \rightarrow O_{C \cup \{r\}}(\lambda)$ is a bijection, it suffices to prove that $\chi_r(O_C(\lambda)) = O_{C \cup \{r\}}(\lambda)$. Fix $x \in O_C(\lambda)$. We show that $\chi_r(x) \in O_{C \cup \{r\}}(\lambda)$ by showing that $\chi_r(x)$ satisfies all the defining inequalities of $O_{C \cup \{r\}}(\lambda)$:

- If $i - j \neq \ell$ or both $i - j = \ell$ and $(i, j) > r$, then $0 \leq \chi_r(x)_{(i,j)} = x_{(i,j)} \leq 1$. Since $r$ is a corner of $\lambda \setminus C$, for every $(i, j) \leq r$ with $i - j = \ell$, it follows that $\max\{x_{(i-1,j)}, x_{(i,j-1)}\} \leq x_{(i,j)} \leq \min\{x_{(i+1,j)}, x_{(i,j+1)}\}$, hence $0 \leq \chi_r(x)_{(i,j)} \leq 1$.

- Let $(i_1, j_1) \leq (i_2, j_2) \in \lambda \setminus (C \cup \{r\})$. Since, for every $(i, j) \in \lambda \setminus (C \cup \{r\})$ with $i - j = \ell$, we have that $\max\{x_{(i-1,j)}, x_{(i,j-1)}\} \leq \chi_r(x)_{(i,j)} \leq \min\{x_{(i+1,j)}, x_{(i,j+1)}\}$, it follows that $\chi_r(x)_{(i_1,j_1)} \leq \chi_r(x)_{(i_2,j_2)}$.

- It remains to show that if $p \in \lambda \setminus (C \cup \{r\})$ and $q_1, \ldots, q_n \in C \cup \{r\}$ with $p < q_1 < \cdots < q_n$, then $\chi_r(x)_p + \chi_r(x)_{q_1} + \cdots + \chi_r(x)_{q_n} \leq 1$. First assume that $q_i \neq r$ for every $i \in \{1, \ldots, n\}$. If $p$ does not lie on the $\ell$th diagonal, then $\chi_r(x)_p + \chi_r(x)_{q_1} + \cdots + \chi_r(x)_{q_n} = x_p + x_{q_1} + \cdots + x_{q_n} \leq 1$. Otherwise, if $p$ lies on the $\ell$th diagonal, then $\chi_r(x)_p + \chi_r(x)_{q_1} + \cdots + \chi_r(x)_{q_n} \leq x_p + x_{q_1} + \cdots + x_{q_n} \leq 1$.

Suppose that $r \in \{q_1, \ldots, q_n\}$. It follows that $r = q_1$. Since $p < q_1$ and $x_p \leq \max\{x_{(a-1,b)}, x_{(a,b-1)}\}$, we have $\chi_r(x)_p + \chi_r(x)_{q_1} \leq \chi_r(x)_p + x_{(a,b)} - \max\{x_{(a-1,b)}, x_{(a,b-1)}\} \leq x_r$. Hence $\chi_r(x)_p + \chi_r(x)_{q_1} + \cdots + \chi_r(x)_{q_n} \leq x_r + x_{q_2} + \cdots + x_{q_n} \leq 1$. 

So, we have shown that $\chi_r(\mathcal{C}\lambda) \subseteq \mathcal{O}_{\mathcal{C}\cup\{r\}}(\lambda)$. Using the explicit formula for $\chi_r^{-1}$, the reverse inclusion $\chi_r^{-1}(\mathcal{O}_{\mathcal{C}\cup\{r\}}(\lambda)) \subseteq \mathcal{O}_C(\lambda)$ follows similarly.

Recall the decomposition $\chi_r = \psi \circ \varphi_{\text{max}(r)} \circ \cdots \circ \varphi_0$ from Proposition 3.2 and the defining equation of the hyperplane $H^C_\ell$ given by

$$H^C_\ell : \sum_{s \in S^C_\ell} x_s - \sum_{t \in T^C_\ell} x_t + \sum_{u \in R^C_\ell} x_u = d_\ell.$$ 

Consider the tropical map $\varphi_i = \varphi_{\omega_i, R_i}$. We have that $w_i = e_\alpha - e_\beta$ for some $\alpha$ and $\beta$ in $\lambda$ that lie on the same diagonal of $\lambda \setminus C$. Since the sets $S^C_\ell$ and $T^C_\ell$ are unions of diagonals of $\lambda \setminus C$, it follows that $\varphi_i(H^C_\ell) = H^C_\ell$ for each $i \in \{0, \ldots, i_{\text{max}}(r)\}$ and $\ell \in \text{Diag}(\lambda)$.

It remains to show that $\psi(H^C_\ell) = H^C_{\mathcal{C}\cup\{r\}}$. Let $x \in H^C_\ell$, then we have

$$\sum_{i-j=\ell \atop (i,j) \leq r} x(i,j) = \psi(x)_{(a,b)} + \psi(x)_{(a-1,b)} + \psi(x)_{(a,b-1)} + \sum_{i-j=\ell \atop (i,j) < r} (-\psi(x)_{(i,j)} + \psi(x)_{(i-1,j)} + \psi(x)_{(i,j-1)}).$$

Since $\psi(x)_{(i,j)} = x(i,j)$ for every $(i,j)$ with $i - j \neq \ell$, using the previous equality, the following holds:

$$d_\ell = \sum_{s \in S^C_\ell} x_s - \sum_{t \in T^C_\ell} x_t + \sum_{u \in R^C_\ell} x_u$$

$$= \sum_{s \in S^C_\ell \atop i-j \neq \ell} x_s - \sum_{t \in T^C_\ell \atop i-j \neq \ell} x_t + \sum_{u \in R^C_\ell} x_u + \sum_{i-j=\ell \atop (i,j) \leq r} x(i,j)$$

$$= \sum_{s \in S^C_\ell \atop i-j \neq \ell} \psi(x)_s - \sum_{t \in T^C_\ell \atop i-j \neq \ell} \psi(x)_t + \sum_{u \in R^C_\ell} \psi(x)_u$$

$$+ \psi(x)_{(a,b)} - \sum_{i-j=\ell \atop (i,j) < r} \psi(x)_{(i,j)} + \sum_{i-j=\ell \atop (i,j) \leq r} \psi(x)_{(i,j-1)} + \psi(x)_{(i,j-1)}.$$

Note that $r \in T^C_{\mathcal{C}\cup\{r\}}$. There are four possible behaviours of $C$ near the corner $r$ of $\lambda \setminus C$ as shown below, where the highlighted boxes lie in $C$:

\begin{center}
\begin{tabular}{cccc}
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
\end{tabular}
\end{center}

\begin{center}
\begin{tabular}{cccc}
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
\end{tabular}
\end{center}

\begin{center}
\begin{tabular}{cccc}
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
\end{tabular}
\end{center}

\begin{center}
\begin{tabular}{cccc}
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
\end{tabular}
\end{center}

or

\begin{center}
\begin{tabular}{cccc}
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
\end{tabular}
\end{center}

In particular, since $r$ is a corner of $\lambda \setminus C$, we have that $(a-1, b)$ and $(a, b-1)$ are not in $S^C_\ell$. Moreover, $(a+1, b) \in T^C_\ell$ if and only if $(a, b-1) \not\in S^C_{\mathcal{C}\cup\{r\}}$. Similarly, $(a, b+1) \in T^C_\ell$ if and only if $(a-1, b) \not\in S^C_{\mathcal{C}\cup\{r\}}$. So, the previous expression can be written as:

$$\sum_{s \in S^C_{\mathcal{C}\cup\{r\}}} \psi(x)_s - \sum_{t \in T^C_{\mathcal{C}\cup\{r\}}} \psi(x)_t + \sum_{u \in (\mathcal{C}\cup\{r\}) \cap R_\ell} \psi(x)_u = d_\ell$$

that is, $\psi(H^C_\ell) = H^C_{\mathcal{C}\cup\{r\}}$. \qed
Lemma 4.2. Fix Setup 2.7 and let $C \subseteq \lambda$ be an up-set. Let $r = (a, b)$ be a corner of $\lambda \setminus C$ and recall the decomposition $\chi_r = \psi \circ \varphi_{\max(r)} \circ \cdots \circ \varphi_0$ from Proposition 3.2. For every $i \in \{0, \ldots, i_{\max}(r)\}$, the tropical map $\varphi_i = \varphi_{w_i, F_i}$ gives a combinatorial mutation of the polytope $(\varphi_{i-1} \circ \cdots \circ \varphi_0)(O_C(\lambda)^\delta_2)$.

Proof. By Lemma 4.1, for each $i \in \{0, \ldots, i_{\max}(r)\}$ and $\ell \in \{1-m_2, \ldots, m_1-1\}$, the hyperplane $H_{\ell}^C$ is invariant under $\varphi_i$. Hence, it is enough to prove that $\varphi_i$ gives a combinatorial mutation of the polytope $P_i := (\varphi_{i-1} \circ \cdots \circ \varphi_0)(O_C(\lambda))$.

We introduce the following notation for readability:

$$A = (a - i - 1, b - i - 1), \quad B = (a - i - 1, b - i), \quad C = (a - i, b - i - 1), \quad D = (a - i, b - i).$$

Recall that $F_i = \text{conv}\{e_B, e_C\}$. We may assume that both $B$ and $C$ lie in $\lambda$, otherwise $\varphi_i$ is a unimodular map and the result follows immediately. So, the regions of linearity of the tropical map $\varphi_i$ are given by

$$U_\geq := \{x \in \mathbb{R}^\lambda \mid x_B \geq x_C\} \quad \text{and} \quad U_\leq := \{x \in \mathbb{R}^\lambda \mid x_B \leq x_C\}.$$ 

Similarly, we define $U_>$ and $U<_C$ to be the interiors of $U_\geq$ and $U_\leq$ respectively. Let $U_\leq := U_\geq \cap U_\leq$ be the intersection of the regions of linearity.

Let $p \in U_\geq \cap P_i$ and $q \in U_\leq \cap P_i$ be two points that lie in the interiors of the regions of linearity. To show that $P_{i+1} := \varphi_i(P_i)$ is convex, it suffices to show that the line segment between $\varphi_i(p)$ and $\varphi_i(q)$ is contained in $P_{i+1}$. Without loss of generality, we may assume that $p$ and $q$ are vertices of $P_i$. For each $i \in \{0, \ldots, i_{\max}(r)\}$, we will show that the line segment $[\varphi_i(p), \varphi_i(q)]$ is contained in $P_{i+1}$. Moreover, we will show that $\varphi_i$ gives a bijection between the vertices of $P_i$ and $P_{i+1}$. We proceed by induction on $i$.

Fix $i = 0$. Since $p_B > p_C, q_B > q_C$, and $p$ and $q$ are vertices of $O_C(\lambda)$, we have that

$$\begin{cases} p_A = p_C = 0 \\ p_B = p_D = 1 \end{cases} \quad \text{and} \quad \begin{cases} q_A = q_B = 0 \\ q_C = q_D = 1. \end{cases}$$

Note that, by the definition of the chain-order polytope, we have that $p_s = q_s = 0$ for all $s \leq A$. We define the points $\alpha, \beta, \gamma, \delta \in \mathbb{R}^\lambda$ as follows:

- $\alpha_s = 0$ for all $s \leq D$ and $\alpha_s = \min\{p_s, q_s\}$ for all $s \not\leq D$, in particular $\alpha_A = \alpha_B = \alpha_C = \alpha_D = 0$,
- $\beta_s = 0$ for all $s < D$ and $\beta_s = \min\{p_s, q_s\}$ for all $s \not< D$, in particular $\beta_A = \beta_B = \beta_C = 0$ and $\beta_D = 1$,
- $\gamma_s = 0$ for all $s \leq A$ and $\gamma_s = \max\{p_s, q_s\}$ for all $s \not\leq A$, in particular $\gamma_A = 0$ and $\gamma_B = \gamma_C = \gamma_D = 1$,
- $\delta_s = 0$ for all $s < A$, $\delta_A = 1$, and $\delta_s = \max\{p_s, q_s\}$ for all $s \not< A$, in particular $\delta_A = \delta_B = \delta_C = \delta_D = 1$. 

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It is straightforward to check that the points $\alpha, \beta, \gamma$ and $\delta$ lie in $\mathcal{O}_C(\lambda)$. Observe that $\alpha, \beta, \gamma, \delta \in U_=$ and $p + q = \beta + \gamma$. In particular, the line segment $[p, q]$ is not an edge of $\mathcal{O}_C(\lambda)$. By the definition of the tropical map $\varphi_0$, we have that $\varphi_0(p) + \varphi_0(q) = \varphi_0(\alpha) + \varphi_0(\delta)$. To see this, note that $\varphi_0$ fixes all coordinates except $A$ and $D$, which vary based only on the values of $B$ and $C$. Writing out the coordinates $A, B, C, D$ of these points we have:

$$\varphi_0(p) + \varphi(q) = \varphi_0 \left( \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array} \right) + \varphi_0 \left( \begin{array}{c} 0 \\ 1 \\ 0 \\ 1 \end{array} \right) = \left( \begin{array}{c} 0 \\ 1 \\ 1 \\ 1 \end{array} \right) + \left( \begin{array}{c} 0 \\ 1 \\ 1 \\ 1 \end{array} \right) = \left( \begin{array}{c} 0 \\ 1 \\ 1 \\ 2 \end{array} \right)$$

where order of the coordinates is given by $\left( \begin{array}{cc} A & B \\ C & D \end{array} \right)$. For the remaining coordinates, the equality follows immediately from the definition of $\alpha$ and $\delta$ above. Therefore, the line segment $[\varphi_0(p), \varphi(q)]$ is contained in the convex hull $\text{conv} \left\{ \varphi_0(p), \varphi(q), \varphi_0(\alpha), \varphi_0(\delta) \right\}$. Hence $P_1$ is convex. Observe that the line segment $[\varphi_0(p), \varphi(q)]$ is not an edge of $P_1$.

We now show that $\varphi_0$ gives a bijection between the vertices of $\mathcal{O}_C(\lambda)$ and $P_1$. Clearly, $\varphi_0$ is a bijection between the vertices of $P_1$ and $\mathcal{O}_C(\lambda)$ that lie in the interiors $U_>$ and $U_\prec$. Let $v$ be any vertex of $\mathcal{O}_C(\lambda)$ that lies in $U_\succ$. Assume by contradiction that $\varphi_0(v)$ is not a vertex of $P_1$, then $\varphi_0(v)$ lies in strict interior of an edge of $P_1$. Therefore, there exist vertices $p \in P_1 \cap U_>$ and $q \in P_0 \cap U_\prec$ such that $\varphi_0(v)$ lies on the line segment between them, which is an edge of $P_1$. However, we have already observed that all such line segments are not edges of $P_1$, a contradiction. Therefore $\varphi_0(v)$ is a vertex of $P_0$.

Now, let $v$ be a non-vertex of $\mathcal{O}_C(\lambda)$ and assume by contradiction that $\varphi_0(v)$ is a vertex of $P_1$. Since $\varphi_0(v)$ is a vertex, it follows that $v$ lies on an edge of $\mathcal{O}_C(\lambda)$ that intersects the interiors of both regions of linearity $U_>$ and $U_\prec$. Hence, there exist vertices $p \in U_\succ \cap \mathcal{O}_C(\lambda)$ and $q \in U_\prec \cap \mathcal{O}_C(\lambda)$ such that $v \in [p, q]$. However, we have already observed that there are no such edges, a contradiction. Therefore, $\varphi_0(v)$ is not a vertex of $P_1$. And so we have shown that $v$ is a vertex of $\mathcal{O}_C(\lambda)$ if and only if $\varphi_0(v)$ is a vertex of $P_0$.

The inductive step, for $i > 0$, follows almost identically to the case $i = 0$. The only difference is that, for each $x \in \mathbb{R}^\lambda$, we do not work with the value $x_{(a-i, b-i)}$. Instead, we use the value $x_{(a-i, b-i+1)} + \min \{x_{(a-i+1, b-i)}, x_{(a-i, b-i+1)}\}$.

□

**Proof of Theorem 3.1.** Fix a maximal flag of up-sets of $\lambda$: $\emptyset = C_0 \subset C_1 \subset C_2 \subset \cdots \subset C_u = \lambda$. Hence, for each $i \in \{1, \ldots, u\}$, we have that $C_{i+1} \setminus C_i \subset \{r\}$ such that $r$ is a corner of $\lambda \setminus C_{i-1}$. By Lemma 4.1, the map $\chi_r$ is a bijection between the restricted chain-order polytopes $\chi_r : \mathcal{O}_{C_{i-1}}(\lambda)_{\downarrow}^k \to \mathcal{O}_{C_i}(\lambda)_{\downarrow}^k$. By Proposition 3.2 we have that $\chi_r = \psi \circ \varphi_{\text{max}(r)} \circ \cdots \circ \varphi_0$ is the composition tropical maps $\varphi_r$ and a unimodular map $\psi$. By Lemma 4.2, we have that for each $j \in \{0, \ldots, i_{\text{max}}(r)\}$, the tropical map $\varphi_j$ gives a combinatorial mutation of the polytope $\varphi_{j-1} \circ \cdots \circ \varphi_0(\mathcal{O}_{C_{i-1}}(\lambda)_{\downarrow}^k)$. So $\mathcal{O}_{C_{i-1}}(\lambda)_{\downarrow}^k$ and $\mathcal{O}_{C_{i}}(\lambda)_{\downarrow}^k$ are mutation equivalent for each $i$. Hence $\mathcal{O}(\lambda)_{\downarrow}^k = \mathcal{O}_{C_0}(\lambda)_{\downarrow}^k$ and $\mathcal{O}(\lambda)_{\downarrow}^k = \mathcal{O}_{C_u}(\lambda)_{\downarrow}^k$ are mutation equivalent. □
Remark 4.3. The proof of Lemma 4.2 shows that all chain-order polytopes $O_C(\lambda)$ have the same number of vertices. However, this does not hold for the restricted polytopes. See [2, Examples 2.1 and 2.3].

5 Observations about period collapse

In this section we focus on the phenomenon of period collapse, as studied in [2]. We provide some small computations, straightforward corollaries of our main results, and further questions. We begin with the following observation.

Proposition 5.1 ([2, Theorem 2.1 and Lemma 2.2]). If $\lambda$ is rectangular then the Ehrhart quasi-polynomial of $O(\lambda)^k_d$ is a polynomial.

Proof. The proof follows identically to [2, Theorem 2.1 and Lemma 2.2] with a slight modification for the rectangular case. Explicitly, if $\lambda = [m_1] \times [m_2]$, then $O(\lambda)^k_d$ is integrally equivalent to the Gelfand-Tsetlin polytope $\text{GT}_{\alpha,\beta}$ of shape $\alpha$ and content $\beta$ where $\alpha = (k^{m_1}, 0^{m_2})$ and

$$\beta = (d_{m_1-1}, d_{m_1-2} - d_{m_1-1}, d_{m_1-3} - d_{m_1-2}, \ldots, d_{m_1-m_2} - d_{m_1-m_2+1},$$

$$k + d_{m_1-m_2-1} - d_{m_1-m_2}, k + d_{m_1-m_2-2} - d_{m_1-m_2-1}, \ldots, k + d_{-m_2+1} - d_{-m_2+2}).$$

The notation $k^{m_1}$ above means $k$ repeated $n$ times.

By Proposition 2.2, combinatorial mutations preserve the Ehrhart (quasi)-polynomial, so we immediately obtain the following corollary of Theorem 3.1.

Corollary 5.2. If $\lambda$ is a rectangular Young diagram then the Ehrhart quasi-polynomial of the restricted chain-order polytope $O_C(\lambda)^k_d$ is a polynomial, for any $d$, $k$ and up-set $C \subseteq \lambda$.

We observe that some non-rectangular Young diagrams give rise to restricted chain-order polytopes with period collapse. For the next proposition, we require the following definition. Let $\lambda$ be a Young diagram and fix a pair of adjacent diagonals $\ell$ and $\ell + 1$ where $\ell \geq 0$ (resp. $\ell$ and $\ell - 1$ where $\ell \leq 0$). Assume the diagonals have the same length. We define the Young diagram $\lambda \setminus \ell$ with diagonal $\ell$ removed as follows:

$$\lambda \setminus \ell := \begin{cases} 
\{ (i, j) \in \lambda \mid i - j < \ell \} \cup \{ (i - 1, j) \mid (i, j) \in \lambda \text{ and } i - j > \ell \} & \text{if } \ell \geq 0, \\
\{ (i, j) \in \lambda \mid i - j > \ell \} \cup \{ (i, j - 1) \mid (i, j) \in \lambda \text{ and } i - j < \ell \} & \text{resp. if } \ell \leq 0.
\end{cases}$$

Given a vector $d$, as in Setup 2.7 for the Young diagram $\lambda \subseteq [m_1] \times [m_2]$, we also define $d \setminus \ell = (d_{m_1-1}, \ldots, d_{\ell+1}, d_{\ell-1}, \ldots, d_{1-m_2})$. We think of $d \setminus \ell$ as the corresponding vector for the Young diagram $\lambda \setminus \ell$. See Example 5.6.

Proposition 5.3. If $\lambda$ contains two adjacent diagonals of the same length, say $\ell$ and $\ell + 1$ with $\ell \geq 0$ (resp. $\ell$ and $\ell - 1$ with $\ell \leq 0$), and $d_\ell = d_{\ell+1}$ (resp. $d_\ell = d_{\ell-1}$) then the polytopes $O(\lambda)^k_d$ and $O(\lambda \setminus \ell)^k_{d \setminus \ell}$ are integrally equivalent. Also if $d_{\ell+1} < d_\ell$ (resp. $d_{\ell-1} < d_\ell$) then $O(\lambda)^k_d$ is empty.
Proof. Assume $\ell \geq 0$ and take any point $x \in \mathcal{O}(\lambda)^k_d$. Write $\alpha_1, \ldots, \alpha_t$ for the boxes along the $\ell$th diagonal in $\lambda$ and $\beta_1, \ldots, \beta_t$ for the coordinates along the $(\ell + 1)$th diagonal. By definition of the restricted order polytope, we have that $x_{\alpha_1} + \cdots + x_{\alpha_t} = d_\ell = d_{\ell+1} = x_{\beta_1} + \cdots + x_{\beta_t}$ and for each $i \in [t]$ we have $x_{\alpha_i} \leq x_{\beta_i}$. It follows immediately that $x_{\alpha_i} = x_{\beta_i}$ for all $i \in [t]$. Therefore the projection from $\mathcal{O}(\lambda)^k_d$ to $\mathcal{O}(\lambda|\ell)^k_{d|\ell}$ which removes the $\ell$th diagonal of $\lambda$ gives an integral equivalence of polytopes.

Since $x_{\alpha_i} \leq x_{\beta_i}$ for all $i \in [t]$, it follows that $d_\ell \leq d_{\ell+1}$. So, if $d_{\ell+1} < d_\ell$ then we have that $\mathcal{O}(\lambda)^k_d$ is empty. A similar argument proves the result when $\ell \leq 0$. \qed

So for any Young diagram $\lambda$, we may apply Propositions 5.1 and 5.3 to construct restricted order polytopes that exhibit period collapse. We obtain the following corollary of Theorem 3.1.

**Corollary 5.4.** Let $d$ be the vector given by $(d)_\ell = |\{\text{boxes of the $\ell$th diagonal of } \lambda\}|$. Then the Ehrhart quasi-polynomial of $\mathcal{O}_C(\lambda)^k_d$ is a polynomial.

**Remark 5.5.** We note that Proposition 5.1 does not immediately generalise to the non-rectangular cases. Unless the vector $d$ contains repeated entries, as in Proposition 5.3, it is not clear whether restricted order polytopes $\mathcal{O}(\lambda)^k_d$ for non-rectangular posets $\lambda$ are related to the Gelfand-Tsetlin polytope $\text{GT}_{\alpha, \beta}$. Restricted order polytopes can be realised as the intersection of a GT-polytope with certain hyperplanes. However, it is not clear why such an intersection would give a polytope with period collapse.

**Example 5.6.** Consider the partition of 11 given by $(4,4,3)$ and let $\lambda$ be the corresponding Young diagram. Let $k = 2$ and $d = (1,2,3,2,2,1)$. Then the vertices of restricted order $\mathcal{O}(\lambda)^k_d$ are

\[
\begin{array}{ccccccc}
1 & 1 & 1 & 1 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 & 1 & 1 \\
1 & 1 & 1 & 0 & 1 & 1 & 1 \\
1 & 1 & 2 & 1 & 1 & 2 & 2 \\
1 & 2 & 2 & 1 & 1 & 2 & 2 \\
\end{array}
\]

and

\[
\begin{array}{cccc}
0 & \frac{1}{2} & \frac{1}{2} & 1 \\
\frac{1}{2} & \frac{3}{2} & \frac{3}{2} & \frac{3}{2} \\
\frac{1}{2} & \frac{3}{2} & \frac{3}{2} & 2 \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 \\
\frac{1}{2} & \frac{3}{2} & \frac{3}{2} & \frac{3}{2} \\
1 & \frac{3}{2} & 2 & 1 \\
\end{array}
\]

The shaded adjacent diagonals $\ell = -1$ (in blue) and $\ell - 1 = -2$ (in purple) have a same values since $d_{-1} = d_{-2} = 2$. Let $\lambda' = \lambda|\ell$ be the square Young diagram given by the partition $(3,3,3)$ and $d' = d|\ell = (1,2,3,2,1)$. By Proposition 5.3, the vertices of the restricted order polytope $\mathcal{O}(\lambda')^{k}_{d'}$ can be obtained from the above vertices by applying the projection:

\[
\begin{array}{ccc}
a & b & c \\
d & e & f \\
g & h & i \\
\end{array} \rightarrow \begin{array}{ccc}
a & b & c \\
d & e & f \\
g & h & i \\
\end{array}
\]
This projection gives an integral equivalence between the two restricted order polytopes. The Ehrhart polynomial of this polytope is given by:

$$\frac{1}{12}t^4 + \frac{1}{2}t^3 + \frac{17}{12}t^2 + 2t + 1$$

and its $h^*$-vector is $(1, 0, 1, 0, 0)$.

**Example 5.7.** We give an example of a restricted order polytope for some non-rectangular Young diagram $\lambda$ that does not satisfy the assumptions Propositions 2.2 or 5.3. Consider the partition $\lambda = (4, 4, 3), k = 3$ and $d = (1, 2, 3, 2, 3, 1)$. We index the coordinates of $\mathbb{R}^\lambda$ as follows:

$$(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11)$$

and let $V = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \end{pmatrix} \in (\mathbb{R}^\lambda)^{11}$

The vertices of the restricted chain-order polytope $O(\lambda)^R$ are the columns of $V$. The Ehrhart quasi-polynomial of this polytope is the polynomial given by

$$\frac{7}{120}t^5 + \frac{1}{2}t^4 + \frac{41}{24}t^3 + 3t^2 + \frac{41}{15}t + 1.$$

The $h^*$-vector for this polytope is $(1, 3, 3, 0, 0, 0)$.

**Example 5.8.** Let $\lambda$ be the Young diagram associated to the partition $(4, 4, 3, 2)$. Let $k = 3$ and $d = (1, 3, 2, 3, 2, 3, 1)$. Label the coordinates of $\mathbb{R}^\lambda$ as follows:

$$(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11)$$

and let $V = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 & 1 & 2 & 2 & 1 & 2 & 2 & 2 \\ 2 & 2 & 2 & 3 & 2 & 2 & 3 & 2 & 3 & 2 & 3 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 3 & 3 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 \end{pmatrix} \in (\mathbb{R}^\lambda)^{18}$

The restricted order polytope $O(\lambda)^R$ has 18 vertices which are given by the columns of $V$. Similarly, to the previous examples, the polytope exhibits period collapse. Its Ehrhart polynomial is

$$\frac{7}{240}t^6 + \frac{77}{240}t^5 + \frac{23}{16}t^4 + \frac{163}{48}t^3 + \frac{68}{15}t^2 + \frac{197}{60}t + 1$$

and its $h^*$-vector is $(1, 7, 11, 2, 0, 0, 0)$. 

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**Question 5.9.** Is the Ehrhart quasi-polynomial of the restricted chain-order polytope $O_C(\lambda)^k_d$ a polynomial for any up-set $C \subseteq \lambda$, vector $d$ and positive integer $k$?

By Theorem 3.1, it suffices to consider only the restricted order polytopes. Also, using the combinatorial mutations defined in Section 3, the question is simultaneously answered for all other intermediate polytopes that appear in the sequence of combinatorial mutations between restricted chain-order polytopes.

**Question 5.10.** What is the degree of the $h^*$-polynomial of a restricted chain-order polytope?

The examples above seem to suggest that the degree of the $h^*$-polynomial is bounded above by half the dimension of the polytope. Given a Young diagram with $n$ boxes and $\ell$ diagonals, for sufficiently generic $k$ and $d$, the dimension of the the restricted order polytope $O(\lambda)^k_d$ is $n - \ell$.

**Question 5.11.** Which non-lattice restricted order polytopes are mutation equivalent to lattice polytopes?

Consider the restricted order polytope in Example 5.6. It turns out that all intermediate polytopes, which appear in the sequence of mutations constructed in Section 3, are non-lattice polytopes. For other non-lattice restricted order polytopes, we ask whether any of the intermediate polytopes are lattice polytopes. We note that if a restricted order polytope is mutation equivalent to a lattice polytope, then this gives an alternative proof that the Ehrhart quasi-polynomial is a polynomial.

**References**

[1] M. Akhtar, T. Coates, S. Galkin, and A. M. Kasprzyk. Minkowski polynomials and mutations. *SIGMA. Symmetry, Integrability and Geometry: Methods and Applications*, 8:paper 094, 17, 2012.

[2] P. Alexandersson, S. Hopkins, and G. Zaimi. Restricted Birkhoff polytopes and Ehrhart period collapse. *arXiv preprint arXiv:2206.02276*, 2022.

[3] F. Ardila, T. Bliem, and D. Salazar. Gelfand–Tsetlin polytopes and Feigin–Fourier–Littelmann–Vinberg polytopes as marked poset polytopes. *Journal of Combinatorial Theory, Series A*, 118(8):2454–2462, 2011.

[4] M. Beck and S. Robins. *Computing the continuous discretely*. Undergraduate Texts in Mathematics. Springer New York, 2007.

[5] M. Beck, S. V. Sam, and K. M. Woods. Maximal periods of (Ehrhart) quasi-polynomials. *J. Combin. Theory Ser. A*, 115(3):517–525, 2008.

[6] O. Clarke, A. Higashitani, and F. Mohammadi. Combinatorial mutations and block diagonal polytopes. *Collectanea Mathematica*, pages 1–31, 2021.
[7] O. Clarke, A. Higashitani, and F. Mohammadi. Combinatorial mutations of Gelfand-Tsetlin polytopes, Feigin-Fourier-Littelmann-Vinberg polytopes, and block diagonal matching field polytopes. *arXiv preprint arXiv:2208.04521*, 2022.

[8] O. Clarke, F. Mohammadi, and F. Zaffalon. Toric degenerations of partial flag varieties and combinatorial mutations of matching field polytopes. *arXiv preprint arXiv:2206.13975*, 2022.

[9] L. Escobar and M. Harada. Wall-crossing for Newton-Okounkov bodies and the tropical Grassmannian. *International Mathematics Research Notices, rnaa230*, 2020.

[10] X. Fang and G. Fourier. Marked chain-order polytopes. *European J. Combin.*, 58:267–282, 2016.

[11] I. M. Gelfand and M. L. Tsetlin. Finite-dimensional representations of the group of unimodular matrices. *Dokl. Akad. Nauk SSSR*, 71(5):825–828, 1950.

[12] C. Haase and T. B. McAllister. Quasi-period collapse and $GL_n(\mathbb{Z})$-scissors congruence in rational polytopes. In *Integer points in polyhedra—geometry, number theory, representation theory, algebra, optimization, statistics*, volume 452 of *Contemp. Math.*, pages 115–122. Amer. Math. Soc., Providence, RI, 2008.

[13] A. Higashitani. Two poset polytopes are mutation-equivalent. *arXiv:2002.01364*, 2020.

[14] A. N. Kirillov. Ubiquity of Kostka polynomials. *Physics and Combinatorics*, pages 85–200, 2001.

[15] T. B. McAllister and K. M. Woods. The minimum period of the Ehrhart quasi-polynomial of a rational polytope. *J. Combin. Theory Ser. A*, 109(2):345–352, 2005.

[16] I. Pak. Hook length formula and geometric combinatorics. *Séminaire Lotharingien de Combinatoire*, 46:B46f, 13 p., 2001.

[17] C. Schensted. Longest increasing and decreasing subsequences. *Canadian Journal of Mathematics*, 13:179–191, 1961.

[18] R. P. Stanley. Two poset polytopes. *Discrete Comput. Geom.*, 1(1):9–23, 1986.

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