Power-law tails from multiplicative noise

Tamás S. Biró

MTA KFKI Research Institute for Particle and Nuclear Physics, H-1525 Budapest Pf. 49, Hungary

Antal Jakovác

Research Group for Theoretical Condensed Matter of HAS and TU Budapest, H-1521 Budapest, Hungary

Abstract We show that the well-known linear Langevin equation, modeling the Brownian motion and leading to a Gaussian stationary distribution of the corresponding Fokker-Planck equation, is changed by the smallest multiplicative noise. This leads to a power-law tail of the distribution for large enough momenta. At finite ratio of the correlation strength for the multiplicative and additive noise the stationary energy distribution becomes exactly the Tsallis distribution.

Power-law tails are present in numerous distributions studied in physics or elsewhere when dealing with complex systems. They are of generic interest, regarded as to signal long range order, non-vanishing correlations or scale invariance in complex systems with strong dynamics, which’s details are mostly unknown. These are often contrasted to the traditional statistical system, showing the Gibbs distribution \((\exp(-E/T))\) in energy, which is Gaussian in the momenta of free, massive particles \((\exp(-p^2/2mT))\). The latter is considered as the generic case for thermal equilibrium of non-correlated or short-range correlated systems. This concept has been carried far beyond of its original field describing monoatomic ideal gas (Maxwell-Boltzmann statistics), by applying the Gibbs distribution in thermal equilibrium to areas such as particle physics and field theory. It serves as starting point of high-temperature field theory calculations both with analytical and numerical methods. Lattice gauge theory is based on the formal similarity between Euclidian path integrals and the canonical partition sum.

A very simple and elegant, microdynamical explanation for the Maxwell-Boltzmann statistics is offered by the Langevin equation, describing a free particle moving under the influence of a deterministic damping force and a stochastic drive. The latter accelerates the particle in a short time, changing its momenta randomly and uncorrelated. The stationary solution of this stochastic equation follows the Gaussian statistics, compatible to Gibbs’ principle. It seems that in many statistical considerations of complex physical models from that on it is tacitly assumed that the presence of this additive noise is a dominant effect: the equilibrium distribution follows Gibbs’ formula (the Gaussian distribution in momentum for a free, massive particle). Since the harmonic oscillator is just the extension of this free motion Langevin equation into the phase space, also for free quantum systems the above picture is generally accepted.

We shared this expectations, thinking that any non-Gaussian (or in the energy non-exponential) distribution, especially the power-law tail observed in many phenomena including particle spectra in high energy physics, may only come from non-thermalized or in an other way non-equilibrium situation. In this note we would like to share our deep astonishment about that this is not so: we found out that treating the damping constant in the Langevin equation also stochastically (considering this way both multiplicative and additive noise) the stationary distribution is in general non-Gaussian. Moreover, above a certain momentum, depending on the strength (i.e. self-correlation width) of the multiplicative noise, the stationary solution goes over into a power-law.

This finding seems to have manifold consequences. Even in equilibrium, even averaging over a huge number of elementary events, measurements on quantum systems, such as particles or fields, are bound to find a power-law tail irrespective if the underlying dynamical system had equilibrated itself long enough in units of a characteristic time. Power-law tails in pion spectra, found experimentally in high energy \(e^+e^-\), \(p\bar{p}\) or heavy-ion collisions, in the view of this simple mathematical result may reflect an already stationary distribution. Power-law tails found in other areas, in particular as properties of average correlations (cf. time series in stock market), may also be a non-temporary, long term effect.

In this paper we derive a generic stationary distribution for the Langevin-type equation with both additive and multiplicative noise. Conform to the original assumptions both noise terms are white (Dirac delta correlated in time), but they may show cross-correlations. This situation may stem naturally from the widespread treatment of field theoretical operator equations, where the fields are split to a large, "classical" expectation value and to certain momentum, depending on the strength (i.e. self-correlation width) of the multiplicative noise, the stationary distribution is in general non-Gaussian. Moreover, above a certain momentum, depending on the strength (i.e. self-correlation width) of the multiplicative noise, the stationary solution goes over into a power-law.

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In this paper we derive a generic stationary distribution for the Langevin-type equation with both additive and multiplicative noise. Conform to the original assumptions both noise terms are white (Dirac delta correlated in time), but they may show cross-correlations. This situation may stem naturally from the widespread treatment of field theoretical operator equations, where the fields are split to a large, "classical" expectation value and to a noisy quantum or thermal fluctuation part: the different noise terms do have a common origin, so it is natural to consider cross-correlations among them. Often the overdamped approximation is applied for studies of plasmas, solving then effectively a differential equation first order in time, instead of second order. We think therefore that the analytically solvable case, we present in this paper, may carry a quite general lesson.

We present the solution of the Langevin equation with both additive and multiplicative noise terms applying the classical method of Wang and Uhlenbeck. Our starting
point is the linear equation:

\[ \dot{p} + \gamma p = \xi, \]  

(1)

where now both \( \xi \) and \( \gamma \) are stochastic variables. They both may have a non-zero mean value (motivated by possible field theory applications),

\[ \langle \xi(t) \rangle = F, \quad \langle \gamma(t) \rangle = G, \]  

(2)

and show white-noise (i.e. extremely short term) correlations:

\[ \langle \xi(t)\xi(t') \rangle - \langle \xi(t) \rangle \langle \xi(t') \rangle = 2D \delta(t-t'), \]
\[ \langle \gamma(t)\gamma(t') \rangle - \langle \gamma(t) \rangle \langle \gamma(t') \rangle = 2C \delta(t-t'), \]
\[ \langle \gamma(t)\xi(t') \rangle - \langle \gamma(t) \rangle \langle \xi(t') \rangle = 2B \delta(t-t'), \]
\[ \langle \xi(t)\gamma(t') \rangle - \langle \xi(t) \rangle \langle \gamma(t') \rangle = 2B \delta(t-t'), \]  

(3)

This problem can be solved analytically. We will determine the time dependence of the distribution of \( p \) values, denoted by \( f(p,t) \). In this notation \( f(p_0,t)dp \) is the probability that after time \( t \) the variable \( p \) has the value in the range \([p_0, p_0 + dp]\). We rewrite (1) as a difference equation

\[ p(t + dt) = p(t) + \int_0^{t+dt} dt' \ (\xi(t') - \gamma(t') \ p(t')) , \]  

(4)

If \( p(t) \) is a smooth function of time we can replace \( p(t') \) either by \( p(t) \) or \( p(t + dt) \), or any value in between. We choose here the Ito prescription, which uses \( p(t) \) under the integral in the \( dt \to 0 \) limit. In order to simplify notation we write the integral term as \( \langle x \rangle \), with \( x \) denoting the general integrand. Now we can write down a Fokker-Planck equation for the distribution: the probability to have the value \( p(t + dt) \) at \( t + dt \) is the probability that we have the value \( p(t) \) at time \( t \) and noise values \( \xi \) and \( \gamma \) that just satisfy (1):

\[ f(p, t + dt) = \int d\xi d\gamma \ \mathcal{P}(\xi, \gamma) \ f(p - dt \ (\xi) + pdt \ (\gamma)). \]  

(5)

Unfortunately this form is not appropriate to directly create differential equation as \( dt \to 0 \). Instead we follow the method of Wang and Uhlenbeck: we have a trial function \( R(p) \) that is smooth enough and we compute the expectation value of \( R \) (averaged over the noise) as a function of time

\[ \langle R(t) \rangle = \int dp \ R(p) \ f(p, t). \]  

(6)

Applying this form to (4) we have

\[ \int dp \ R(p) \ f(p, t + dt) = \]
\[ \int d\xi d\gamma \ \mathcal{P}(\xi, \gamma) \ \int dp \ R(p + dt \ (\xi) - p \ dt \ (\gamma)) \ f(p, t). \]  

(7)

By Taylor expanding \( R(p) \) and integrating over the noise distribution we get

\[ \langle R(p + dt \ (\xi) - p dt \ (\gamma)) \rangle = \langle R(p) + dt \ R'(p) \ K_1(p) + dt \ R''(p) \ K_2(p) + O(dt^2) \rangle \]  

(8)

with

\[ K_1 = F - Gp, \quad K_2 = D - 2Bp + Cp^2 \]  

(9)

in the present case. This leads to the following general Fokker-Planck equation

\[ \frac{\partial f}{\partial t} = - \frac{\partial (K_1 f)}{\partial p} + \frac{\partial^2 (K_2 f)}{\partial p^2}. \]  

(10)

The stationary solution satisfies

\[ \frac{d}{dp} \ (K_2 f) = K_1 f, \]  

(11)

which is analytically solvable. It leads to

\[ f(p) = f(0) \frac{K_2(0)}{K_2(p)} \exp \left( \frac{L(p)}{K_2(p)} \right) \]  

(12)

with

\[ L(p) = \int_0^p dq \ K_1(q) / K_2(q). \]  

(13)

In the case of two noises correlated the way given in eq. (4) we arrive at the following logarithm of the stationary distribution:

\[ \ln \left( \frac{f(p)}{f(0)} \right) = - \left( 1 + \frac{G}{2C} \right) \ln \frac{K_2(p)}{D} - \frac{\alpha}{\vartheta} \text{atan} \left( \frac{\vartheta p}{D - Bp} \right), \]  

(14)

with

\[ \vartheta = \sqrt{CD - B^2} \quad \text{and} \quad \alpha = G \frac{B}{C} - F. \]  

(15)

Here ‘atan’ denotes the inverse tangent function, not always taking the first principle value, but rather continuing at \( p > D/B \) smoothly. Considering physical applications the noise correlation values, \( D, C \) and \( B \) build a positive semi-definite matrix. This ensures that \( C \) and \( D \) are non-negative values, and the determinant \( \vartheta \) is real and also non-negative. The same applies for the function \( K_2(p) \) occurring under the logarithm. In this context zero values are limiting cases and the stability of the stationary solution (14) against choosing a small positive value has to be investigated.

The lesson for physical applications lies in the analysis of different limiting cases. First we consider the traditional case: \( C = B = 0 \), not allowing for any noise (fluctuation) in the multiplicative factor \( \gamma \) (damping constant). This leads back to the familiar Gauss distribution:

\[ f(p) = f(0) e^{-\frac{\gamma p^2}{2}} e^{\frac{\vartheta p}{B}}, \]  

(16)
with an eventual shift in the mean momentum for a non-zero mean driving force $\langle \xi \rangle = F$. Another limiting case is that of the purely multiplicative noise with $D = 0, B = 0$ (G.Wilk has applied it for the heat conduction equation and obtained a Gamma distribution for the inverse temperature $1/T$). Now the stationary solution becomes

$$f(p) = f(0) p^{-2 - G/C} e^{-|p|^C}$$  \hspace{1cm} (17)

a Gamma distribution in $1/p$. For large $p$ this approaches a pure power-law. It is particularly interesting to investigate the mathematically degenerate case of $\vartheta = 0$. Now $K_2(p)$ reaches zero at the critical momentum, $p_c = \sqrt{D/C}$, leading to zero probability in the stationary distribution $f(\sqrt{D/C}) = 0$. This occurs as a "limiting momentum" in the physical distribution.

The above limiting cases rely on an expansion of the generic solution. They are valid only in a limited range of momenta: the Gaussian solution for small, the Gamma distribution for large argument of the inverse tangent function. Correspondingly the widely beloved Gaussian distribution can be a good approximation to the stationary solution only for momenta $p \ll \sqrt{D/C}$. (For $C = 0$ strictly, of course this is the solution for any finite momentum.) Generally the small argument of the inverse tangent is fulfilled for $p \ll \sqrt{D/C}$. This result also means that for the smallest fluctuation in the multiplicative factor the stationary Gauss distribution develops a power-law tail. The power in this tail, $p^{-2v}$, is given by $v = 1 + G/C$. In order to offer a visual insight into the nature of the generic stationary solution we show stationary spectra for different parameters (cf. Fig(1)).

In the case $B = 0$ (no cross correlation between the noises), and $F = 0$ (no drift term due to the additive noise), the exact stationary distribution is the Tsallis distribution. In order to achieve this result one uses the energy of the free particle, $E = p^2/2m$ as the distribution variable and eq.(14). We get

$$f(E) = f_0 \left(1 + (q - 1)K(T)^{2 - q}/T\right)^{q - 1}$$  \hspace{1cm} (18)

where the parameters of the Tsallis distribution are given by

$$T = \frac{D}{mG}, \quad q = 1 + \frac{2C}{G}.$$  \hspace{1cm} (19)

Again for $C = 0$ (only additive noise) $q = 1$ and the Tsallis distribution goes over into the Gibbs distribution,

$$f(E) = f_0 \exp(-E/T).$$  \hspace{1cm} (20)

Tsallis and others have worked out a thermodynamical framework offering the distribution as the canonical distribution. This approach, the non-extensive thermodynamics, however, is based on a non-extensive entropy measure. This eventually unwanted property is not fatal: the distribution can also be obtained based on the extensive Rényi entropy. We note by passing this point that the presence of the two uncorrelated noise and the corresponding Fokker-Planck equation also can be obtained from an inhomogeneous diffusion coefficient. Instead of (1) one may consider

$$\dot{p} + Gp = (D + C p^2)^{1/2} \eta$$  \hspace{1cm} (21)

With a single noise $\eta$, normalized to unity:

$$\langle \eta(t) \rangle = 0, \quad \langle \eta(t) \eta(t') \rangle = 2\delta(t - t').$$  \hspace{1cm} (22)

This is a particular case of a more general field-dependent noise considered by Arnold, Son and Yaffe in the context of non-abelian plasmas. The mechanism outlined in the present article may work for non-abelian gauge theories, as well. The basic gluonic field, described by a vector potential $A$, satisfies the equation

$$\ddot{A} + \sigma \dot{A} = D \times (D \times A),$$  \hspace{1cm} (23)

with $D$ being the gauge-covariant derivative and $\sigma$ the color conductivity factor. We consider an overdamped dynamics, when the second time derivative is ignored. Driving this to the extreme we single out in a Fourier expansion the zero mode, $A_0$ and consider the following effective equation

$$\sigma \dot{A}_0 = \sum_{k, q} (k - gA_k)(q - gA_q) A_{-k-q}.$$  \hspace{1cm} (24)

Treating the the $k = 0$ and $q = 0$ contributions on the right hand side separately we arrive at

$$\dot{A}_0 - \frac{g^2}{\sigma} A_0^3 = -\gamma A_0 + \xi,$$  \hspace{1cm} (25)
\[ G = \frac{g^2}{2\sigma^2} \]
\[ C = \frac{g^4}{2\sigma^3} \]
\[ D = \frac{g^4}{2\sigma^3} \]

**FIG. 2:**

with

\[
\gamma = -2\frac{g}{\sigma} \sum_{k > 0} (k - gA_k)A_{-k},
\]
\[
\xi = \frac{1}{\sigma} \sum_{k > 0, q > 0} (k - gA_k)(q - gA_q)A_{-k-q}. \tag{26}
\]

Ignoring the classical \( A_0^2 \) contribution for the following discussion, one realizes that \( \gamma \) and \( \xi \) do contain hard Fourier component contributions. In the Langevin equation with both additive and multiplicative noise these are regarded as noisy, fast-fluctuating quantities.

The full quantum field theoretical treatment of these factors is rather involved. In order to gain estimates over the parameters of a possible stationary distribution of \( A_0^2 \) values, averages and correlations of \( \gamma \) and \( \xi \) have to be obtained. This process can be facilitated by using a graphical notation: external legs stand for the zero mode and internal lines for the hard modes. (cf. Fig 2). For no spontaneous symmetry breaking, \( F = 0 \) and \( B = 0 \) immediately follows, so the stationary distribution is a Tsallis-distribution for \( A_0^2 \) and eventually in the soft energy \( E \sim \Lambda^2 A_0^2 = A_0^2/(2m) \), where \( \Lambda \) is the underlying energy scale. A qualitative, order of magnitude estimate for the parameters of the Tsallis distribution can be given as: \( T \sim g^2\Lambda^4/(\sigma m^2) \) and \( q - 1 \sim g^2\Lambda^2/\sigma^2 \). The starting point of the power law in eq. (15) is at \( E_c = T/(q - 1) \sim \Lambda^2/m \).

Finally we would like to check whether quantitative estimates give reliable results. In high energy particle physics experiments the transverse momentum distribution has been investigated for long. From Gaussian fits to the parton distribution one conjectures a ratio \( D/G = \langle p_T^2 \rangle \approx 1 - 1.5 \) GeV². On the other hand power-law tails at high transverse momenta make a value of \( v \approx 5.8 \pm 0.5 \) realistic. This fixes the ratio to \( G/C \approx 9.6 \pm 1 \) and the Tsallis index to \( q \approx 1.2 \pm 0.03 \). The critical transverse momentum, beyond which the power-law dominates the familiar Gauss distribution, can be calculated from this to be \( p_c \approx 3 - 4 \) GeV. Compared to experimental spectra this is a quite realistic estimate.

**In conclusion** we have shown that the smallest multiplicative white-noise related to the classically deterministic damping constant in the Langevin equation leads to a stationary distribution of particle momenta, which ends in a power-law tail at high values. This result seems to undermine the well-established thermal approaches to phenomenological and field theoretical studies in particle physics, where the presence of a multiplicative noise is not less probable than the presence of an additive one in any simplification (linearization) of the underlying microdynamical problem. This approach on the other hand offers a new way to deal with the interpretation of power-law tails occurring in experimental findings, as well as it animates to seek new methods in thermal field theory exceeding the traditional thermodynamical approach.

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