Dispersive Model for Undular Hydraulic Jump Behind a Weir

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Abstract. In this paper, we investigate a suitable weir design which yields in an undular jump phenomenon using both hydrostatic and non-hydrostatic models. We solved the hydrostatic model using numerical and analytical methods. After that, we validated the output from both methods using the experimental data. In order to simulate the undular jump better, modification towards the hydrostatic model is needed. Hence, for the non-hydrostatic model, we consider the existence of hydrodynamic pressure and solved the model numerically. The outcome of this paper can be beneficial for hydraulic structures design and the management of water resources.

Keyword: undular jump, weir, dispersive, shallow water equation, non-hydrostatic

1. Introduction

This research provides a model for hydraulic infrastructure for water gate in a weir. The model must be able to control the water level and it has three main functions; sustain flood safety, improving water quality, and increasing revenue by optimizing the use of the weir, while also decreasing cost for its construction. Shallow water equations are a set of hyperbolic partial differential equations that can be used to describe water flow below a pressure. In this research, we will use Shallow Water Equations (SWE) to investigate a suitable weir design to produce certain water level and undular jump. It will produce various range of Froude number that can be noticed in the appearance of undulations behind a weir.

The important issues regarding this phenomenon are; under which flow condition they occur, and what the characteristics these undulations are. A well-known equation that is frequently used to simulate free surface flows and the appearance of wave propagation is Non-Linear Shallow Water Equation (NSWE), because it is easy to be solved numerically compared to other equations. However, it failed to simulate this phenomenon because it does not take into account the dispersive effect. Therefore, we modify the NSWE which is called the Non-Hydrostatic Model as seen in Stelling and Zijlema [4], Magdalena [7], Cui [9], Magdalena [10], and Pudjaprasetya et al. [12]. We solve the equation numerically using finite volume on the staggered grid and simulate hydraulic jump behind a weir. Subsequently, we use the energy conservation law and continuity law to obtain the analytical solution result in the height of the water level behind the weir. To validate our numerical model, we compare the results with the analytical solution and the published data.

Numerically, we will solve the equation using finite volume on the staggered grid and simulate hydraulic jump behind a weir. The energy conservation law and continuity law will obtain analytical...
solution result in the height of the water level behind the weir. To validate the numerical model, we’ll compare the result with numerical solution and published data.

The outline of this paper is, first we will discuss about the hydrostatic part that consist of governing equation, solving analytical and numerical solution, also the comparison between analytical and numerical results. After that, the next thing we want to examine is the non-hydrostatic part starting from modifying the equation from hydrostatic part, solving the mathematical model numerically, and then validating the results.

2. Hydrostatic Model

2.1. Governing Equation
Let \( \eta(x, t) \) denotes surface elevation and \( u(x, t) \) denotes average horizontal velocity with \( g \) is gravitational acceleration constant. \( h = \eta + d \) denotes the water thickness and \( d \) is the bottom topography.

\[
\eta_t + (hu)_x = 0, \\
u_t + uu_x + g\eta_x = 0.
\]

Here, the first equation is the mass conservation and the second equation is the momentum conservation in horizontal direction.

2.2. Analytical Solution
In this section, we solved the Shallow Water Equation analytically. Consider the following problem sketch:
\[(H + Y) + \frac{v_1^2}{2g} + p_1 = (H + y_c) + \frac{v_c^2}{2g} + p_c\]  \hspace{1cm} (3)

Assuming that the atmospheric pressure is constant, \(p_1 = p_c\). Then, the equation can be written as:

\[(H + Y) + \frac{v_1^2}{2g} = (H + y_c) + \frac{v_c^2}{2g}\]  \hspace{1cm} (4)

By eliminating \(H\) in both sides and multiplying both sides with \(y_c^2\), we obtain:

\[\frac{v_1^2}{2g} y_c^2 - \frac{v_c^2 y_c^2}{2g} = y_c^3 - y_c^2 Y\]  \hspace{1cm} (5)

Apply the simplified continuity equation, \(v_1 y_1 = v_c y_c\) where \(y_1 = Y + H\)

\[\frac{v_1^2}{2g} y_c^2 - \frac{v_1^2 (Y + H)^2}{2g} = y_c^3 - y_c^2 Y\]  \hspace{1cm} (6)

Lastly, simplify the equation into third degree polynomial:

\[y_3^3 - \left(\frac{v_1^2}{2g} + y_1 - H\right) y_c^2 + \frac{v_1^2 y_1^2}{2g} = 0\]  \hspace{1cm} (7)

The root of this equation is the value of \(y_c\). Next, determine the value of \(y_c\) using the previous equation:

\[(H + Y) + \frac{v_1^2}{2g} = (H + y_c) + \frac{v_c^2}{2g}\]  \hspace{1cm} (8)

We now have:

\[y_c = \frac{v_1^2}{3g} + \frac{2}{3} Y\]  \hspace{1cm} (9)

Note that \(Q_{weir} = v_c y_c\) and because section 2 is located above the weir, we can assume the Froude number in this section equals to 1. Therefore, \(v_c^2 = g y_c\). Thus, we have:

\[Q_{weir} = (g y_c^3)^{\frac{1}{2}}\]  \hspace{1cm} (10)

Substitute the equation above, we obtain:

\[Q_{weir} = \frac{2}{3} H_0 \sqrt{\frac{2g}{3} H_0}, \text{ where } H_0 = \frac{v_1^2}{2g} + Y\]  \hspace{1cm} (11)

Then, we can determine the value of \(v_c\) by using

\[v_c = \frac{Q_{weir}}{y_c}\]  \hspace{1cm} (12)

After managed to determine the value of \(y_c\) and \(v_c\), we can obtain \(y_3\). We use the momentum balance equation between section 2 and section 3 yields:

\[\frac{1}{2} \rho g y_c^2 + \rho v_c^2 y_c = \frac{1}{2} \rho g y_3^2 + \rho v_3^2 y_3\]  \hspace{1cm} (13)

Then, by multiplying both sides by 2 and dividing both sides by \(\rho\), we gained:

\[g y_c^2 + 2v_c^2 y_c = g y_3^2 + 2v_3^2 y_3\]  \hspace{1cm} (14)

We know the simplified continuity equation:
\[ g v_c y_c = v_3 y_3 \]  

So, we can substitute \( y_3^3 = \frac{v_3 y_c}{v_c} \) into the equation and obtain third degree polynomials:

\[ y_3^3 - ((y_c + H)^2 + \frac{2v_y^2}{g} y_c) y_3 + \frac{2}{g} (v_c y_c)^2 = 0 \]  

Finally, the root of this equation is the value of \( y_3 \).

### 2.3. Numerical Solution

Here, we implement the finite volume method on a staggered grid to solve equation (1) and equation (2) numerically. First, we consider a spatial domain \([0, L]\) with a staggered grid partition \( x_1 = 0, x_\frac{3}{2}, x_2, \ldots, x_{Nx}, x_{Nx+\frac{1}{2}} = L \). Using the staggered formulation, the values of \( \eta \) are computed at every full grid points \( x_i = i \Delta x \), with \( i = 1, 2, \ldots, Nx \) using equation (1). Whereas velocities \( u \) are computed at every staggered grid points \( x_{i+\frac{1}{2}} = (i + 1/2) \Delta x \), with \( i = 0, 1, 2, \ldots, Nx \) using equation (2), see Figure 3.

![Illustration of the staggered grid with configuration of the calculated variables \( \eta \) and \( u \)](image)

Then, the approximation for equation (1) and equation (2) are:

\[ \frac{\eta_i^{n+1} - \eta_i^n}{\Delta t} + \frac{hu_i|_{i+1/2} - hu_i|_{i-1/2}}{\Delta x} = 0, \]  

\[ \frac{u_i^{n+1} - u_i^n}{\Delta t} + g(\eta_i^{n+1} - \eta_i^{n+1}) \frac{\Delta x}{\Delta x} + (uu_x)^n_{i+1/2} = 0. \]  

In equation (17), the values of total water depth \( h_{i+1/2} \), at half grid points are unknown, and we approximate their values using the first order upwind method:

\[ *h_{i+1/2} = \begin{cases} h_i, & \text{if } u_{i+1/2} \geq 0, \\ h_{i+1}, & \text{if } u_{i+1/2} < 0. \end{cases} \]  

The upwinding method depends on the direction of the flow. If the flow is going to the right we take the information from the left and vice versa. The most difficult discretization in Shallow Water Equation is the advection term. Its approximation \( uu_x \) relates with momentum variable \( q = hu \) is given by
\[ u u_x = \frac{1}{h} \left( \frac{\partial(qu)}{\partial x} - u \frac{\partial q}{\partial x} \right) \]  

(20)

and the discretization for the advection term in equation (20) reads as

\[ (u u_x)_{i+\frac{1}{2}} = \frac{1}{h_{i+\frac{1}{2}}} \left( \frac{q_{i+1}^- u_{i+1}^- - q_{i}^- u_i}{\Delta x} - u_{i+\frac{1}{2}}^\frac{q_{i+1}^- - q_i^-}{\Delta x} \right), \]

(21)

whereas

\[ h_{i+\frac{1}{2}} = \frac{1}{2} (h_i + h_{i+1}), \quad q_i = \frac{1}{2} (q_{i+1}^- + q_{i}^-), \quad q_{i+\frac{1}{2}} = *h_{i+\frac{1}{2}} u_{i+\frac{1}{2}}, \]

with the following upwind approximation of \( u_i \)

\[ *u_i = \begin{cases} u_{i-\frac{1}{2}}, & \text{if } q_i \geq 0, \\ u_{i+\frac{1}{2}}, & \text{if } q_i < 0. \end{cases} \]

(22)

Hence, the staggered conservative scheme for the equations are equation (17), equation (19) for mass conservation, and equation (18), equation (21), equation (22) for momentum balance. In Stelling and Duinmeijer [3], this approximation is called the momentum conservative scheme. Some validations and verifications of this momentum conservative scheme have been shown in Magdalena, et al. [1], Pudjaprasetya and Magdalena [2], and Magdalena et al. [11]. We showed that the momentum conservative approach is suitable for dam break simulation for various bottom topographies and transcritical flow simulation with a shock and dam break over a rectangular bed. Moreover, this method also works for rapidly varied flow with a large range of Froude numbers.

2.4. Result and Discussion

In this section, we will apply the numerical scheme to simulate undular hydraulic jump phenomenon under two bottom topography conditions. The result from the numerical scheme will be compared with experimental data and results from the analytical solution.

2.4.1. Undular Hydraulic Jump over a Flat Bottom Topography

In this simulation, we want to examine the phenomenon that occurred on a flat bottom topography. The result will be compared with the corresponding analytical solution recorded in [6]. The correct dam break simulation over a dry bed deserves a non standard numerical solver, and it has been considered as a benchmark test. Here, consider an observation domain \([-10, 10]\], and the initial condition \(h(x, 0) = 1\) for \(x < 0\) and \(h(x, 0) = 0\) elsewhere, with initial velocity \(u(x, 0) = 0\). For this simulation we use \(\Delta x = 0.01\), \(\Delta t = 0.003\), and gravity \(g = 9.81\) m/s². For simulating dam break on a dry bed, the wet-dry procedure is necessarily incorporated. For flat bottom case, channel with no slope, we compute surface elevation and horizontal velocity for two cases: no friction and with friction. Figure 4 displays numerical results for the no friction case, in which the wave front propagates in the form of rarefaction wave. The result of momentum conservative scheme confirms the exact solution. We note that, as described by Stelling & Duinmeijer in [3], other conservative schemes, such as energy-head scheme, will produce a shock wave front, which is incorrect. Hence, momentum conservative scheme is necessary for this dry bed case. Figure 4 shows that in the presence of frictional force with \(C_f = 0.1\), the wave front location and velocity are reduced.
Figure 4. Dam-break solution at time $t = 1.8$ s without friction and with friction $C_f = 0.1$. (Left) Water level $h$. (Right) Horizontal velocity $u$.

From the agreement between numerical results and the exact solutions we conclude that our numerical scheme is suitable for this case.

2.4.2. Undular Hydraulic Jump behind a Weir

For this simulation, we use the same numerical solution as the Dam Break Simulation and add a rectangular weir in the bottom topography, considering the conditions where $T = 10$ s, $L = 10$ m, initial surface elevation = 0 m, initial water velocity = 0.5 m/s.

Figure 5. Numerical simulation for $h = 5$ m

The result from the numerical scheme are compared with the analytical solution for various weir height, denoted by $h$. From Figure 6, we can see that the numerical and analytical result have the same trend and they are in a good agreement. In addition, the trend shows that as the height of weir increases, the wave amplitude resulted from the undular hydraulic jump decreases.

Figure 6. Comparison between numerical and analytical solution

3. Non-Hydrostatic Model

Since the undular hydraulic jump is a dispersive phenomenon, we need to modify the Shallow Water Equation to understand the phenomenon better. Thus, we improved the equation by adding hydrodynamic pressure, denoted by $P$. 
3.1. Governing Equation

Here, we modify the governing equation in section 2.1 by considering the existence of hydrodynamic pressure, $P$, in the momentum conservation equation.

\[ \eta_t + (hu)_x = 0, \]  
\[ u_t + uu_x + g\eta_x = -\frac{1}{h}\int_{-d}^{\eta} \frac{\partial P}{\partial x} dz, \]

We can assume that $w(x,z,t)$ as vertical velocity that vertically homogeneous. The relation between hydrodynamic pressure $P$ and vertical velocity $w(x,z,t)$ can be written as below:

\[ w_t = -P_z. \]

3.2. Numerical Solution

In this subsection, we will discuss about how to handle the hydrodynamic pressure term $P$. First, the right hand side of equation (24) is simplified by applying the Leibniz’s rule as follows:

\[ \int_{-d}^{\eta} \frac{\partial P}{\partial x} dz = \frac{\partial}{\partial x} \int_{-d}^{\eta} P dz - P_{z=-d} \frac{\partial d}{\partial x} P_{z=\eta} \frac{\partial \eta}{\partial x}. \]

where subscripts $z = \eta$ denotes that the quantity evaluated at the free surface and $z = -d$ at the bottom. The integral on the right-hand side is approximated using trapezoidal approximation:

\[ \int_{-d}^{\eta} P dz = \frac{1}{2} h (P_{z=\eta} + P_{z=-d}). \]

Along the surface, dynamic pressure is zero, hence $P_{z=\eta} = 0$, and the term in equation (26) is simplified to

\[ \int_{-d}^{\eta} \frac{\partial P}{\partial x} dz = \frac{1}{2} h \frac{\partial P_{z=-d}}{\partial x} + \frac{1}{2} P_{z=-d} \frac{\partial (\eta-d)}{\partial x}. \]

Hence, the one layer approximation of the momentum equation reads

\[ u_t + uu_x + g\eta_x = -\frac{1}{h} \frac{\partial P_{z=-d}}{\partial x} - \frac{1}{2h} P_{z=-d} \frac{\partial (\eta-d)}{\partial x}. \]

Next, the linearized momentum equation in the $z$-coordinate reads

\[ w_t = -P_z. \]

and its discrete form, obtained by applying the Keller-box scheme, read as

\[ \frac{w_{z=\eta}^{n+1} - w_{z=\eta}^{n} + w_{z=-d}^{n+1} - w_{z=-d}^{n}}{2\Delta t} = -\frac{P_{z=\eta}^{n+1} - P_{z=-d}^{n+1}}{h^{n}}. \]

Along the bottom the vertical velocity component is zero or $w_{z=-d} = 0$, and again along the free surface, hydrodynamic pressure is zero or $P_{z=\eta} = 0$, then equation (30) is simplified to

\[ \frac{w_{z=\eta}^{n+1} - w_{z=\eta}^{n}}{2\Delta t} = \frac{P_{z=-d}^{n+1}}{h^{n}}. \]

Finally, a closure relation is needed to calculate $P_{z=-d}$, and it comes from the discrete Laplace equation
\[ u_x + \frac{w_x}{h} \eta - \frac{w}{h} z = 0, \]

or

\[ u_x + \frac{w_x}{h} \eta = 0, \]

(32)

Recapitulating, the full set of non-hydrostatic SWE with one layer approximation are equation (23), equation (28), equation (31), equation (33). To handle the non-hydrostatic terms numerically, we use predictor-corrector method. First, we calculate \( \eta, u \) without non-hydrostatic terms as explained in the hydrostatic part. We consider the values of \( \eta \) and \( u \) from hydrostatic model as the predictor values denoted by superscript *. Now, we do correction by including hydrodynamic pressure terms with vertical momentum equation. In the correction step, the pressure variables \( P \) and vertical velocities \( w \) are calculated at the full grid points, see Figure 7.

**Figure 7.** Configuration of the calculated variables in the staggered grid of non-hydrostatic model.

For computation, in all equations we use notations \( P_l^{n+1} = P_l^{n+1} \) and \( w_i^{n+1} = w_i^{n+1} \). And the corrections for \( u \) and \( w \) are given by:

\[ u_{i+1/2}^{n+1} = u_{i+1/2}^* - \Delta t \frac{P_{i+1}^{n+1} - P_i^{n+1}}{\Delta x}, \]

\[ w_i^{n+1} = w_i^* + \Delta t \frac{P_{i-1}^{n+1} - P_{i+1}^{n+1}}{h_i^{n+1}}. \]

(34)

(35)

In the corrector steps above, pressure values \( P \) are needed, therefore it should be calculated first from Laplace equation (33) with its discrete counter part

\[ w_i^{n+1} + \Delta t \frac{u_i^{n+1} - u_{i+1}^{n+1}}{\Delta x} = 0. \]

(36)

Next, the computational procedure is divided into three steps. First, substitute equation (34), equation (35) into equation (36) to get the tridiagonal system of equations for \( P_l \). Second, we use the Thomas Algorithm to solve the tridiagonal system. Third, we corrected \( u_i^{n+1} \) and \( w_i^{n+1} \) a by substituting \( P_l \) into equation (34) and equation (35), respectively.

### 3.3. Result and Discussion

In this section, we take the numerical scheme from the non-hydrostatic model to simulate the undular jump phenomenon over two types of bottom topography. We first simulate the phenomenon over the flat bottom topography and compared the result with experiment from Frazao. Lastly, we will simulate the confirmed numerical scheme to observe the undular jump phenomenon over a weir structure.

#### 3.3.1. Undular Hydraulic Jump over a Flat Bottom Topography

For numerical simulation, we use the computational domain [-50, 50] and depth \( d = 1 \) m with hard wall as the left boundary and absorbing boundary on the right side. The initial condition is as follows:
\[ u_1(x, 0) = \frac{1}{2} u_0 \sqrt{g d (1 - \tanh(\frac{x}{a}))} \]  \hspace{1cm} (37)

\[ \eta(x, 0) = \frac{1}{g} \left( u_1(x, 0) \sqrt{g d} + \frac{1}{4} u_1(x, 0)^2 \right) \]  \hspace{1cm} (38)

with \( u_0 = 0.1 \) m/s, \( g = 9.81 \) m/s\(^2\), \( a = 5 \) m, and \( q_k(1, t) = u_k(1, t) h_k(1) \). For the numerical computation, we choose \( t = 0.01 \) s, \( x = 0.2 \) m. We varied the Froude number from 1 to 1.7 and obtain the result as shown in Figure 8.

We compared our model with the free surface undulation model from Frazao and Guinot [5]. Figure 9 shows a comparison between the two models at four different time. Our numerical dispersive model produces the undular phenomenon comes with the wave train. For this flat bottom topography, our numerical dispersive results are in a good agreement with the experimental model by Frazao.

**Figure 8.** Result plot obtained by varying the Froude number.

**Figure 9.** Numerical waves model compared to MUSCL4-Frazao model at time \( t = 0, 6.36, 12.77, 15.96 \) s
3.3.2. Undular Hydraulic Jump behind a Weir

Now, after we have validated the numerical scheme for our non-hydrostatic model, we simulated the interaction with a weir.

![Figure 10. Illustration of the undular jump caused by a weir.](image)

Here, we used the same initial condition and boundary, with the computational domain of [0,2500] and depth \(d = 0.25\) m.

![Figure 11. Wave interaction with the weir on the bottom topography](image)

Results are given in Figure 11 with the comparison to the hydrostatic model. It is shown that the hydrodynamic pressure performs better simulation of the undular hydraulic jump, which confirms the results reported by Wols [8] and Cui [9].

4. Conclusion

In this research, we have successfully developed hydrostatic and non-hydrostatic models to assess the design of a weir. First, for the hydrostatic model, we solved the mathematical model using the analytical and numerical method. The comparison between experimental data, numerical solution, and analytical solution shows a good agreement. Second, we solved the non-hydrostatic model numerically and validated the result with experimental data in the case of flat bottom topography. In addition, from the numerical simulation of the non-hydrostatic model, it is implied that the non-hydrostatic model can simulate the undular jump phenomenon better than the hydrostatic model.

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