Application of Runge-Kutta method for finding multiple numerical solutions to intuitionistic fuzzy differential equations

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Abstract. The present study is aimed to discuss multiple numerical solutions to first order ordinary differential equation which is intuitionistic fuzzy in nature, under the concept of generalised differentiability. The first order intuitionistic fuzzy differential equation which has been taken for the present study, is changed into four systems of ordinary differential equations by the \((\alpha, \beta)\)-cut representation of an intuitionistic fuzzy set. After the transformation, each system contains two pair of equations; one is for membership function and the other for non-membership function. Then, the fourth order Runge-Kutta method is applied in each pair and the competence of the method over Euler method and Modified Euler method are shown by solving a real time problem.

1. Introduction

Many real world problems can be solved by converting them into ordinary differential equations. But in most of the cases, the exact solution is not possible and hence it is necessary to study its numerical solutions. In general most of the real world problems do not contain crisp or precise data. Therefore, to accommodate the impreciseness, the concept of fuzzy set was introduced. However, there are situations in which even the fuzzy sets are also not enough. For, the refinement of fuzzy set, namely, intuitionistic fuzzy set was introduced. In the literature, it is found that a very few investigations on the study of numerical solutions of ordinary differential equations which are intuitionistic fuzzy in nature. An ordinary differential equation with intuitionistic fuzzy number as its initial value, was studied and solved numerically using Runge-Kutta method by Abbasbandy and Allahviranloo[1]. A study has been made on an \(N\)th-order intuitionistic fuzzy linear differential equation which is time dependent, by Lata and Kumar [2]. In recent times, strong and weak solutions of first order homogeneous intuitionistic fuzzy differential equation have been discussed by Mondal, et al. [3] and the authors have studied an application of a system of differential equations with triangular intuitionistic numbers as its initial value [4]. Nirmala and Chenthur Pandian [5] have used Euler method for the discussion of the numerical solution of intuitionistic fuzzy differential equation (IFDE)
Let be considered a first order ordinary differential equation

\[ \dot{z} = f(w,z), w \in I = [a, b] \]  

where (i) \( z \) is a function which is intuitionistic fuzzy in nature
(ii) \( z \) is a function of the crisp variable \( w \)
(iii) \( f(w, z) \) is an intuitionistic fuzzy function and
(iv) \( \dot{z} \) is the intuitionistic fuzzy derivative.

An intuitionistic fuzzy Cauchy problem of first order is obtained by taking the initial value \( z(w_0) = w_0 \) as an intuitionistic fuzzy number.

Since each intuitionistic fuzzy number is a conjunction two fuzzy numbers [12], Eqn (2.1) is substituted by a corresponding system of equations as follows:

\[ \begin{align*}
\dot{z}^+(w) &= \min \{ f(w, z^+, z^+) \} = F(w, z^+, z^+), z^+(w_0) = z_0^+ \\
\dot{z}^-(w) &= \max \{ f(w, z^+, z^-) \} = G(w, z^+, z^-), z^-(w_0) = z_0^- \\
\dot{z}^+(w) &= \min \{ f(w, z^-, z^-) \} = F(w, z^-, z^-), z^-(w_0) = z_0^- \\
\dot{z}^-(w) &= \max \{ f(w, z^-, z^-) \} = G(w, z^-, z^-), z^-(w_0) = z_0^- 
\end{align*} \]  

The unique solution of the system of equations given in (2.2) and (2.3) is given by \( [z^+, z^-] \in B = \bar{C}(0,1)X\bar{C}(0,1) \) and the unique solution of the system of equations given in (2.4) and (2.5) is \( [z^-, z^-] \in B = \bar{C}(0,1)X\bar{C}(0,1) \).

Therefore the system equations given from (2.2) to (2.5) possesses the unique solution \( z(w) = \{ [z^+(w), z^-(w)] \} \in BXB \) which is an intuitionistic fuzzy function.

(i.e) for each \( w, z(w; (\alpha, \beta)) = \{ [z^+(w; \alpha), z^+(w; \alpha)], [z^-(w; \beta), z^-(w; \beta)] \} \), where \( 0 \leq \alpha + \beta \leq 1 \) and \( \alpha, \beta \in [0,1] \), is an intuitionistic fuzzy number.

The system of equations given from (2.2) to (2.5) is represented in the parametric form as follows:

\[ \begin{align*}
\dot{z}^+(w; \alpha) &= F(w, z^+(w; \alpha), z^+(w; \alpha)), z^+(w_0; \alpha) = z_0^+(\alpha) \\
\dot{z}^+(w; \beta) &= G(w, z^+(w; \alpha), z^+(w; \beta)), z^+(w_0; \beta) = z_0^+(\beta) \\
\dot{z}^-(w; \alpha) &= F(w, z^-(w; \alpha), z^-(w; \alpha)), z^-(w_0; \alpha) = z_0^-(\alpha) \\
\dot{z}^-(w; \beta) &= G(w, z^-(w; \beta), z^-(w; \beta)), z^-(w_0; \beta) = z_0^-(\beta)
\end{align*} \]

for \( \alpha, \beta \in [0,1] \) and \( 0 \leq \alpha + \beta \leq 1 \).

3. The runge-kutta method of order four

In this section, a numerical scheme is presented to solve IFDEs by Runge-Kutta method of order 4.

To solve the system of IFDEs in \([w_0, w_1], [w_1, w_2], \ldots, [w_k, w_{k+1}], \ldots\) for \( \alpha, \beta \in [0,1] \), each interval
[w_k, w_{k+1}] is substituted by a set of N_k + 1 regularly spaced points. The grid points on [w_k, w_{k+1}] will be w_{k,n} = w_k + n h_k, where h_k = \frac{w_{k+1} - w_k}{N_k} and 0 \leq n \leq N_k.

**Algorithm:** To find approximate solution of the fuzzy initial value problem given in Eqs (2.6) to (2.9).

**Case 1:** (1, 1)-Differentiability

**Step 1:** Let h_k = \frac{w_{k+1} - w_k}{N_k}, \quad z_a^+(w_k, 0) = a_0, \quad z_a^-(w_k, 0) = a_1, \quad z_a^+(w_k, 0) = b_0 and \quad z_a^-(w_k, 0) = b_1

**Step 2:** Let i = 1

**Step 3:** Let

\[ z_a^+(w_k, i) = \frac{1}{4} \left[ k_{1a}^+ + 2 k_{2a}^+ + 2 k_{3a}^+ + k_{4a}^+ \right] \]

\[ z_a^-(w_k, i) = \frac{1}{4} \left[ k_{1a}^- + 2 k_{2a}^- + 2 k_{3a}^- + k_{4a}^- \right] \]

Where, \( k_{1a}^+ = h_k F_a[w_{k,i-1}, z_a^+(w_{k,i-1})] \), \( k_{1a}^- = h_k G_a[w_{k,i-1}, z_a^+(w_{k,i-1})] \), \( k_{2a}^- = h_k F_a[w_{k,i-1} + \frac{h_k}{2}, z_a^+(w_{k,i-1}) + \frac{u_k}{2}] \), \( k_{2a}^- = h_k G_a[w_{k,i-1} + \frac{h_k}{2}, z_a^-(w_{k,i-1}) + \frac{u_k}{2}] \), \( k_{3a}^+ = h_k F_a[w_{k,i-1} + \frac{h_k}{2}, z_a^+(w_{k,i-1}) + \frac{u_k}{2}] \), \( k_{3a}^- = h_k G_a[w_{k,i-1} + \frac{h_k}{2}, z_a^-(w_{k,i-1}) + \frac{u_k}{2}] \), \( k_{4a}^+ = h_k G_a[w_{k,i-1} + h_k, z_a^+(w_{k,i-1}) + k_{3a}^+] \), \( k_{4a}^- = h_k G_a[w_{k,i-1} + h_k, z_a^-(w_{k,i-1}) + k_{3a}^-] \), \( z_a^+(w_{k,i}) \in [z_a^+(w_{k,i-1}), z_a^+(w_{k,i-1})], \quad z_a^-(w_{k,i}) \in [z_a^-(w_{k,i-1}), z_a^-(w_{k,i-1})] \)

\[ u_k^+(w_{k,i}) \in [k_{1a}^+ + k_{2a}^+ + k_{3a}^+ + k_{4a}^+], \quad u_k^-(w_{k,i}) \in [k_{1a}^- + k_{2a}^- + k_{3a}^- + k_{4a}^-] \]

\[ z_a^+(w_k, i) = \frac{1}{6} \left[ k_{1a}^+ - 2 k_{2a}^- + 2 k_{3a}^- + k_{4a}^- \right] \]

\[ z_a^- (w_k, i) = \frac{1}{6} \left[ k_{1a}^- - 2 k_{2a}^+ + 2 k_{3a}^+ + k_{4a}^+ \right] \]

Where, \( k_{1a}^- = h_k H_a[w_{k,i-1}, z_a^+(w_{k,i-1})] \), \( k_{1a}^- = h_k H_a[w_{k,i-1}, z_a^-(w_{k,i-1})] \), \( k_{2a}^- = h_k F_a[w_{k,i-1} + \frac{h_k}{2}, z_a^+(w_{k,i-1}) + \frac{v_k}{2}] \), \( k_{2a}^- = h_k G_a[w_{k,i-1} + \frac{h_k}{2}, z_a^-(w_{k,i-1}) + \frac{v_k}{2}] \), \( k_{3a}^- = h_k F_a[w_{k,i-1} + \frac{h_k}{2}, z_a^+(w_{k,i-1}) + \frac{v_k}{2}] \), \( k_{3a}^- = h_k G_a[w_{k,i-1} + \frac{h_k}{2}, z_a^-(w_{k,i-1}) + \frac{v_k}{2}] \), \( k_{4a}^+ = h_k G_a[w_{k,i-1} + h_k, z_a^+(w_{k,i-1}) + k_{3a}^-] \), \( k_{4a}^- = h_k G_a[w_{k,i-1} + h_k, z_a^-(w_{k,i-1}) + k_{3a}^-] \)

\[ z_a^+(w_{k,i}) \in [z_a^+(w_{k,i-1}), z_a^-(w_{k,i-1})], \quad z_a^-(w_{k,i}) \in [z_a^-(w_{k,i-1}), z_a^+(w_{k,i-1})] \)

**Step 4:** \( w_{k,i+1} = w_{k,i} + (i + 1) h_k \)

**Step 5:** Let \( i = i + 1 \)

**Step 6:** If \( i \leq N_k \), go to Step 3

**Step 7:** The algorithm ends and \([z_a^+(w_{k+1}), z_a^-(w_{k+1})]\) approximates the value of the exact solution of the fuzzy initial value problem given in Eqs (2.6) to (2.9). The membership function and \([z_a^+(w_{k+1}), z_a^-(w_{k+1})]\) approximates the value of the exact solution \([z_a^+(w_{k+1}), z_a^-(w_{k+1})]\) of the non-membership function.

Similar algorithms can be given for (1, 2)-Differentiability, (2, 1)-Differentiability and (2, 2)-Differentiability.

**4. Illustration**

**Example 4.1:** Let be considered a first order ordinary differential equation

\[ y'(w) = -\gamma y(w), \quad y(w_0) = y_0, \quad w \in I = [w_0, a] \]

(4.1)

where, \( y(w) \) denotes the number of radio nuclides present in a given radioactive material, \( \gamma \) represents the decay constant and \( y_0 \) is the number of radio nuclides at the initial time. Since nuclear
disintegration is a stochastic process, vagueness can be introduced on the number of radio nuclides. In this problem, let the uncertainty be introduced on the initial value \( y_0 \). However, there are some circumstances in which there may be hesitation over the number of radio nuclides present in the radioactive material under study.

Here, the initial value \( y_0 \) is considered as an intuitionistic fuzzy number which is triangular in shape.

Let \( \chi = 1 \) and \( I = [0,1] \) and \( y_0 = (5,7,9;3,7,11) \). \((\alpha, \beta)\) - cut of \( y(w_0) = y_0 \) is given by \( y(w_0, r) = y_0(r) = \left\{ \left[ y_{\alpha}^+, y_{\alpha}^- \right], \left[ y_{\beta}^+, y_{\beta}^- \right] \right\} \), \( r \in [0,1] \) and \( 0 \leq r = \alpha + \beta \leq 1 \).

(i.e) \( y(w_0, r) = y_0(r) = \left\{ [5 + 2\alpha, 9 - 2\alpha], [3 + 4\beta, 11 - 4\beta] \right\}, r \in [0,1] \) and \( 0 \leq r = \alpha + \beta \leq 1 \).

In all the four cases, Java (Version 1.5) is used to obtain the numerical solutions of Eqn(4.1).

Approximate solutions for both membership and non-membership functions are obtained by (1) Euler Method, (2) Modified Euler method and (3) Runge-Kutta method.

**Case 1: \((1,2)\)-Differentiability**

Applying the concept of \((1,2)\)-Differentiability to equation (4.1), the exact solutions of membership function are given by

\[
y_{\alpha}^+(w) = (2\alpha - 2)e^w + 7e^{-w}; y_{\alpha}^-(w) = -(2\alpha - 2)e^w + 7e^{-w};
\]

and the exact solutions of non-membership function are given by

\[
y_{\beta}^-(w) = (3 + 4\beta)e^w; y_{\beta}^- (w) = (11 - 4\beta)e^{-w}.
\]

The total error between exact and the approximate solutions for membership function at different \( r \)-levels are shown in Table 1.

| \( r \) | Error by Euler Method | Error by Modified Euler Method | Error by Runge-Kutta Method |
|-------|-----------------------|-------------------------------|----------------------------|
| 0     | 0.498157473           | 0.016803928                   | 8.338E-06                   |
| 0.2   | 0.398525979           | 0.013443142                   | 6.669E-06                   |
| 0.4   | 0.298894484           | 0.010082356                   | 5.002E-06                   |
| 0.6   | 0.268814014           | 0.009261612                   | 4.666E-06                   |
| 0.8   | 0.268814015           | 0.009261612                   | 4.665E-06                   |
| 1     | 0.268814014           | 0.009261612                   | 4.665E-06                   |

The total error between exact and the approximate solutions for non-membership function at different \( r \)-levels are shown in Table 2.

| \( r \) | Error by Euler Method | Error by Modified Euler Method | Error by Runge-Kutta Method |
|-------|-----------------------|-------------------------------|----------------------------|
| 0     | 0.268814016           | 0.30475884                    | 3.09790333                  |
| 0.2   | 0.268814015           | 0.243807073                   | 0.24783268                  |
| 0.4   | 0.268814015           | 0.182855305                   | 0.185874201                 |
| 0.6   | 0.268814014           | 0.121903536                   | 0.123916134                 |
| 0.8   | 0.268814014           | 0.06951768                    | 0.061958067                 |
| 1     | 0.268814014           | 0.009261612                   | 4.666E-06                   |

**Case 2: \((2,1)\)-Differentiability**

Applying the concept of \((2,1)\)-Differentiability to equation (4.1), the exact solutions of membership function are given by

\[
y_{\alpha}^+(w) = (5 + 2\alpha)e^{-w}; y_{\alpha}^-(w) = (9 - 2\alpha)e^{-w};
\]

and the exact solutions of non-membership function are given by
The total error between exact and the approximate solutions for membership function at different $r$-levels are shown in Table 3.

**Table 3. Absolute Error for Membership function**

| $r$ | Error by Euler Method | Error by Modified Euler Method | Error by Runge-Kutta Method |
|-----|------------------------|-------------------------------|----------------------------|
| 0   | 0.268814016            | 0.152379421                  | 0.154895167                |
| 0.2 | 0.268814021            | 0.121903536                  | 0.123916134                |
| 0.4 | 0.268814015            | 0.091427653                  | 0.092937101                |
| 0.6 | 0.268814014            | 0.060951768                  | 0.061958067                |
| 0.8 | 0.268814015            | 0.030475884                  | 0.030979033                |
| 1   | 0.268814014            | 0.009261612                  | 4.666E-06                  |

The total error between exact and the approximate solutions for non-membership function at different $r$-levels are shown in Table 1.

**Table 4. Absolute Error for Non-Membership function**

| $r$ | Error by Euler Method | Error by Modified Euler Method | Error by Runge-Kutta Method |
|-----|------------------------|-------------------------------|----------------------------|
| 0   | 0.996314947            | 0.033607855                  | 1.6675E-05                  |
| 0.2 | 0.797051957            | 0.026886284                  | 1.334E-05                  |
| 0.4 | 0.597788968            | 0.020164713                  | 1.000E-05                  |
| 0.6 | 0.398525979            | 0.013443142                  | 6.669E-06                  |
| 0.8 | 0.268814014            | 0.009261612                  | 4.666E-06                  |
| 1   | 0.268814014            | 0.009261612                  | 4.666E-06                  |

**Case 3: (1, 1)-Differentiability (Differentiability-1)**

Applying the concept of $(1, 1)$-Differentiability to equation (4.1), the exact solutions of membership function are given by

$$y_{\alpha}^{-}(w) = (2\alpha - 2)e^w + 7e^{-w}; \quad \overline{y}_{\alpha}^{-}(w) = -(2\alpha - 2)e^w + 7e^{-w};$$

and the exact solutions of non-membership function are given by

$$y_{\beta}^{-}(w) = (4\beta - 2)e^w + 7e^{-w}; \quad \overline{y}_{\beta}^{-}(w) = -(4\beta - 2)e^w + 7e^{-w};$$

A comparison between the errors by the numerical methods discussed in this paper is given in the following figure and the figure is obtained by importing data into MATLAB (Version R2012b).
Case 4: (2, 2)-Differentiability (Differentiability-2)
Applying the concept of (2, 2)-Differentiability to equation (4.1), the exact solutions of membership function are given by
\[ y_a^+(w) = (5 + 2\alpha)e^{-w}; \quad y_a^-(w) = (9 - 2\alpha)e^{-w}; \]
and the exact solutions of non-membership function are given by
\[ y_b^+(w) = (3 + 4\beta)e^{-w}; \quad y_b^-(w) = (11 - 4\beta)e^{-w}; \]
Comparison of errors between exact and approximate solutions for membership and non-membership functions respectively is depicted in Figure 2.
It is obvious that the lengths of the supports of the solutions of equation (4.1) under (1,1)-Differentiability, (1,2)-Differentiability and (2,1)-Differentiability will be increasing as the independent variable \( t \) increases. This says that the radio activity of the system increases as time increases and the number of radio nuclides \( y(t) \) may be negative. However, it is known that the radio activity of a material constantly decreases with time and it cannot go beyond zero. Therefore, it is evident that (2, 2)-Differentiability is appropriate for these types of problems. In all the four cases, the numerical solutions of equation (4.1) by Runge-Kutta Method improve to a great extent, than those by the other two methods. However, minimum step size could reduce the error.
5. Conclusion
This paper proposes an orderly approach for finding numerical solutions of intuitionistic fuzzy Cauchy problems when they are expressed in \((\alpha, \beta)\text{-cut}\) representation. In this paper, a real time problem is taken and is written as an ordinary differential equation in an intuitionistic fuzzy environment. The problem which is taken for the study is replaced by two pair of ordinary differential equations in the parametric form. Each pair is solved by the methods of (1) Euler, (2) Modified Euler and (3) fourth order Runge-Kutta. In all the four cases the numerical solutions by Runge-Kutta method improves to a great extent, than those by the Euler method and Modified Euler’s Method. However, it is found that among the four cases, (2, 2)-Differentiability is suitable for the problem under study. So it is evident from the study that the selection of the differentiability completely depends on the temperament of the problem. Further studies, in future can be done on the multiple numerical solutions of IFDEs by higher order methods.

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