Optimal Control Of Dynamic IS-LM Bussiness Cycle Model With Two Time Delay

Airin Nur Hidayati¹,*, Erna Apriliani¹ and I Gst Ngr Rai Usadha¹
¹Department of Mathematics, Institut Teknologi Sepuluh Nopember, Sukolilo-Surabaya 60111, Indonesia

Email: airinnurhidayati@gmail.com

Abstract. One of the business cycle model in the dynamics economy is the IS-LM business cycle model with time delay. This model talks about stability in the micro-economic system. Meanwhile, the time delay in the IS-LM business cycle model involve a change in stability at the equilibrium point so that a bifurcation is occurs. In this study, analysis of stability and optimal control on the IS-LM business cycle model with time delay. Based on simulation with numerical computation, show that there is a change in the stability when the delay value was given exceed the critical delay value. The stability change occur when the delay value arose a pure eigen value so that there was a limit cycle that show a Hopf bifurcation. Furthermore, optimal control in the IS-LM business cycle model given when the system changes to be unstable, i.e. when the delay value passed the critical delay value. Variable control use in the interest rate function. While the objective function maximize the total money supply from the country, the optimal solution is obtained by using the Pontryagin Maximum Principle. The results of this simulations with numerical computation show that maximizing the rate of income, the rate of interest rates, and the rate of capital stock lead to the stability point at thirtieth time.

1. Introduction

Dynamic systems can be developed in various fields of life, such as engineering [1], biology [2], economics [3], robotic [4], encryption [5] and other social sciences [6]. Generally dynamic systems can be modeled in the form of differential equations. In general there are two kinds of differential equations, namely ordinary differential equations and partial differential equations. Ordinary differential equations are differential equations which only have one independent variable and contain ordinary derivatives, while partial differential equations are differential equations which have more than one independent variable and contain partial derivatives[7].

At present, many delay differential equations are developed or commonly called time delays. One model that contains a delay differential equation in the economic field is the IS-LM business cycle (Investment Saving-Liquidity Money) model. This model involves the investment function (I), the deposit function (S), the money demand function (L) and consists of three non-independent variables that represent income, interest rates, and capital stock. In the IS-LM business cycle model, with two delays, the value
of delay is given assuming that investment depends on income when the investment is first made and in the stock of capital when the time to invest ends [8], [9],[10], [11], [12]. The IS-LM business cycle model has been developed in many researches. One of them is the research conducted by Cai [13]. This study examines the IS-LM business cycle model with an additional lag time in capital accumulation, which is assumed that the invested profits and capital growth are due to past investment decisions. Furthermore, Zhou and Li examined the IS-LM business cycle model by assuming that investment depends on past income and accumulated capital stock, and also on different preparation periods. Zhou and Li used the Hopf bifurcation theorem to show that the delay in the process of capital accumulation can lead to loss of balance or stability of profits and cycles in dynamic macroeconomics [14]. Whereas in 2016, research conducted by Zhou and Li was continued by Rosmely by investigating the occurrence of oscillations in time delay parameters in the IS-LM business cycle model with two time delays. The results show that when the Hopf bifurcation occurs, graph in the solution field shows a constant oscillation movement from the level of income, interest rates and capital stock. If the value of the delay time given is less than the critical value of delay, then the system has a controlled solution towards a balanced condition. Then when the value of the delay time is greater than the critical value of delay, then the system solution will continue to fluctuate, causing unstable system conditions [15].

In this paper, we discuss the optimal stability and control analysis in the IS-LM business cycle model that contains time delay. Stability analysis on the surface is to determine the effect of the value of delay given to the system [16], [17]. Then, the control is given to the interest rate variable. Given the control provided, it is expected to help reduce interest rates when interest rates begin to increases.

2. The IS-LM Business Cycle Model With Two Time Delays
Research by Zhou and Li (2009) studied the IS-LM business cycle model of time delay in the equation of capital accumulation, which is assumed that investment depends on past income and capital accumulation, and also on different preparation periods. The IS-LM Zhou and Li business cycle models are given by the following equation:

\[
\begin{aligned}
\dot{Y}(t) &= \alpha \left[ I_1(Y(t), r(t)) + \beta_1 K(t) - S(Y(t), r(t)) \right] \\
\dot{r}(t) &= \beta \left[ L(Y(t), r(t)) - \bar{M} \right] \\
\dot{K}(t) &= I_1(Y(t - \tau_1), r(t)) - (\delta - \beta_2) K(t - \tau_2)
\end{aligned}
\]

In 2005, De Casare and Sportelli formulated functions I, S and L in the following forms [18]:

\[
\begin{aligned}
I_1(Y(t), r(t)) &= A \frac{[Y(t)]^a}{[r(t)]^b} \\
S(Y(t), r(t)) &= s_1[Y(t)]^a [r(t)]^b \\
L(Y(t), r(t)) &= L_1(Y(t)) + L_2(r(t)) = gY(t) + \frac{h}{r(t) - \bar{r}_1}
\end{aligned}
\]

Thus the IS-LM Zhou and Li (2009) business cycle models and De Casare and Sportelli (2005) can be stated as follows:

\[
\begin{aligned}
\dot{Y}(t) &= \alpha \left[ A \frac{[Y(t)]^a}{[r(t)]^b} + \beta_1 K(t) - s_1[Y(t)]^a [r(t)]^b \right] \\
\dot{r}(t) &= \beta [gY(t) + \frac{h}{r(t) - \bar{r}_1} - \bar{M}] \\
\dot{K}(t) &= A \frac{[Y(t-\tau_1)]^a}{[r(t)]^b} - (\delta - \beta_2) K(t - \tau_2)
\end{aligned}
\]
with $\dot{Y}(t)$ is the rate of income that is affected by the function of income, function of interest rates, function of capital stock, and parameters in goods market index ($\alpha$), technology productivity ($A$), level of decline in investment in capital stock ($\beta_1$), level growth of savings against income ($s_1$). Whereas $\dot{r}(t)$ is the interest rate that is influenced by the income function, interest rate function and the money market index parameter ($\beta$), the amount of money demand for income ($g$), the amount of money demand from the interest rate ($h$), the lowest fixed rate of the interest rate ($\delta$), the money supply constant ($M$). Similarly, $K(t)$ is the rate of capital stock affected by the function of income, function of interest rates, function of capital stock, and parameters of technological productivity, capital depreciation constant ($\delta$) rate of decline in investment in capital stock.

3. Analysis Of The Stability Of The IS-LM Business Cycle Model With Two Time Delays

Stability analysis is obtained by determining the equilibrium point or the critical point, linearizing the model if the model used is nonlinear and determining the characteristic value of the model. The IS-LM business cycle model in Equation (1) contains a delay differential equation, so to obtain the equilibrium point, the case is considered without delay ($\tau_1 = \tau_2 = 0$). The equilibrium point is obtained from $\dot{Y}(t) = \dot{r}(t) = K(t) = 0$ so that the equilibrium point is obtained

$$E_1 = \left(0, \frac{h + M\ddot{r}_1}{M}, 0\right)$$

and $E_2 = \left(\frac{1}{g} \left(\tilde{M} - \frac{h}{r(t) - \ddot{r}_1}\right), \frac{A\delta}{s_1(\delta - \beta_1)} \frac{1}{e^{\frac{\delta}{(\delta - \beta_1)|r(t)|^b}} \right)$.

Furthermore, The IS-LM business cycle model is a non-linear model, for it to be linearized or to obtain the characteristic values of the model. Equation (1) contains a delay parameter, so that the linearization of the IS-LM business cycle model is as follows:

$$\begin{pmatrix} \dot{Y} \\ \dot{r} \\ \dot{K} \end{pmatrix} = J_1 \begin{pmatrix} Y(t) \\ r(t) \\ K(t) \end{pmatrix} + J_2 \begin{pmatrix} Y(t - \tau_1) \\ r(t - \tau_1) \\ K(t - \tau_1) \end{pmatrix} + J_3 \begin{pmatrix} Y(t - \tau_2) \\ r(t - \tau_2) \\ K(t - \tau_2) \end{pmatrix}$$

where the matrix $J_1, J_2$ and $J_3$ are the Jacobi matrix as follows:

$$J_1 = \begin{pmatrix} ad_1 & ad_2 & -a\beta_1 \\ \beta g & \beta d_3 & 0 \\ 0 & 0 & 0 \end{pmatrix}, J_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ d_5 & 0 & 0 \end{pmatrix}, J_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\delta + \beta \end{pmatrix}$$

with

$$d_1 = \frac{Aa[Y(t)]^{a-1}}{[r(t)]^b} - a s_1 [Y(t)]^{a-1} [r(t)]^b$$
$$d_2 = \frac{Aa[Y(t)]^{a}}{[r(t)]^{b+1}} - b s_1 [Y(t)]^{a} [r(t)]^{b-1}$$
$$d_3 = \frac{h}{(r(t) - \ddot{r}_1)^2}$$
$$d_4 = \frac{Aa[Y(t - \tau_1)]^{a}}{[r(t)]^{b+1}}$$
$$d_5 = \frac{Aa[Y(t - \tau_1)]^{a}}{[r(t)]^b}$$

Furthermore, the characteristic equation of the IS-LM business cycle model will be determined by completing

$$\det(\lambda I - J_1 - J_2 e^{-\lambda \tau_1} - J_3 e^{-\lambda \tau_2}) = 0$$

So that the following results are obtained:
Equation (2) contains $e^{-\lambda t_1}$ and $e^{-\lambda t_2}$ so that it cannot be solved exactly. For that, a numerical approach is needed to solve it.

Analysis of the stability of the IS-LM business cycle model can be seen from the characteristics of the model. If the characteristic value is less than zero then the model is stable. Stability analysis is carried out in several cases, i.e:

1. Case 1. When $\tau_1 = \tau_2 = 0$ (without delay)
   In the IS-LM business cycle model without delay, the following characteristics are obtained:
   \[ \lambda^3 + (p_2 + r_2)\lambda^2 + (p_1 + q_1 + r_1)\lambda + (p_0 + q_0 + r_0) = 0 \]
   - For fixed points $E_1 = \left(0, \frac{h + Mp_2}{M}, 0\right)$ obtained eigenvalue
     \[ \lambda_1 = 0, \lambda_2 = -\frac{M^2 \beta}{\delta}, \lambda_3 = -\delta + \beta \]
   - For fixed points $E_2 = \left(\frac{1}{\delta} \left(\frac{M}{h - r(1) - r_1}, \left(\frac{\delta \hat{\beta}}{\delta - \beta_1} \right)^{\frac{1}{\delta}}, \hat{A}^{[\varphi]}(0)^{\frac{1}{\delta}} \right), \left(\frac{\delta - \beta_1}{\delta - \beta} \right)^{\frac{1}{\delta}} \right)
     \]
     obtained the following characteristic equation:
     \[ \lambda^3 + (p_2 + r_2)\lambda^2 + (p_1 + q_1 + r_1)\lambda + (p_0 + q_0 + r_0) = 0 \]
     Based on Routh-Hurwitz criteria for the third degree characteristic equation, so that the local asymptotic stable $E_2$ equilibrium point must meet the following criteria:
     \[ p_2 + r_2 > 0, \quad p_0 + q_0 + r_0 > 0 \]
     and
     \[ (p_2 + r_2)(p_1 + q_1 + r_1) - (p_0 + q_0 + r_0) > 0 \]

2. Case 2. When $\tau_1 \neq 0$ and $\tau_2 = 0$
   In the IS-LM business cycle model at the time $\tau_1 \neq 0$ and $\tau_2 = 0$, obtained the following characteristic equation:
   \[ \lambda^3 + (p_2 + r_2)\lambda^2 + (p_1 + r_1)\lambda + p_0 + r_0 - (q_1\lambda + q_0)e^{-\lambda t_1} = 0 \] (3)
   Let $\omega = i\omega$, $\omega > 0$, $\omega \in R$, then Equation (2) can be written in the form of:
   \[ (i\omega)^3 + (p_2 + r_2)(i\omega)^2 + (p_1 + r_1)(i\omega) + p_0 + r_0 + (q_1(i\omega) + q_0) \]
   \[ (\cos(\omega \tau_1) - i \sin(\omega \tau_1)) = 0 \] (4)
   So that the real part is obtained in Equation (4)
   \[ -(p_2 + r_2)\omega^2 + (p_0 + r_0) + q_1 \omega \sin(\omega \tau_1) + q_0 \cos(\omega \tau_1) = 0 \]
   \[ (p_2 + r_2)\omega^2 - (p_0 + r_0) = q_1 \omega \sin(\omega \tau_1) + q_0 \cos(\omega \tau_1) \] (5)
   while the imaginary part is
   \[ -\omega^3 + (p_1 + r_1)\omega + q_1 \omega \cos(\omega \tau_1) - q_0 \sin(\omega \tau_1) = 0 \]
   \[ \omega^3 - (p_1 + r_1)\omega = q_1 \omega \cos(\omega \tau_1) - q_0 \sin(\omega \tau_1) \] (6)
   Next, Equation (5) and Equation (6) can be simplified to become
\[ \omega^6 + (p_2^2 + 2p_2r_2 + r_2^2 - 2(p_1 + r_1))\omega^4 + ((p_1 + r_1)^2 - 2(p_2 + r_2)(p_0 + r_0) - q_1^2)\omega^2 + ((p_0 + r_0)^2 - q_0^2) = 0 \]

Or it can be rewritten to

\[ \omega^6 + G\omega^4 + H\omega^2 + I = 0 \]

with

\[
G = p_2^2 + 2p_2r_2 + r_2^2 - 2(p_1 + r_1) \\
H = (p_1 + r_1)^2 - 2(p_2 + r_2)(p_0 + r_0) - q_1^2 \\
I = (p_0 + r_0)^2 - q_0^2
\]

For example \( \omega^2 = z \), then

\[ z^3 + Gz^2 + H + I = 0 \]

Suppose Equation (8) has positive roots \( z_1, z_2 \) and \( z_3 \), then the roots of Equation (7) are

\[ \omega_1 = \sqrt{z_1}, \omega_2 = \sqrt{z_2}, \text{dan} \omega_3 = \sqrt{z_3} \]

Review Equation (5) and Equation (6)

\[
(p_2 + r_2)\omega^2 - (p_0 + r_0) - q_1\omega \sin(\omega \tau_1) = q_0 \cos(\omega \tau_1)
\]

so that it is obtained

\[ \cos(\omega \tau_1) = \frac{(p_2 + r_2)\omega^2 - (p_0 + r_0) - q_1\omega \sin(\omega \tau_1)}{q_0} \]

Next, substitute Equation (9) in Equation (6)

\[
\omega^3 - (p_1 + r_1)\omega = q_1\omega \left( \frac{(p_2 + r_2)\omega^2 - (p_0 + r_0) - q_1\omega \sin(\omega \tau_1)}{q_0} \right) - q_0 \sin(\omega \tau_1)
\]

Thus obtained

\[ \sin(\omega \tau_1) = \frac{q_1\omega^3(p_2 + r_2) - q_1\omega(p_0 + r_0) - (\omega^3 - (p_1 + r_1)\omega)q_0}{(q_1\omega)^2 - q_0^2} \]

Next, substitute Equation (9) in Equation (8)

\[ \cos(\omega \tau_1) = \frac{(p_2 + r_2)\omega^2 - (p_0 + r_0) - q_1\omega \left( \frac{q_1\omega^3(p_2 + r_2) - q_1\omega(p_0 + r_0) - (\omega^3 - (p_1 + r_1)\omega)q_0}{(q_1\omega)^2 - q_0^2} \right)}{q_0} \]

Thus obtained

\[ \tau_1 = \frac{1}{\omega} \cos^{-1} \left( \frac{q_0\omega^2(p_2 + r_2) - q_0(p_0 + r_0) + q_1\omega(\omega^3 - (p_1 + r_1)\omega)}{(q_1\omega)^2 - q_0^2} \right) \]

Changes in characteristic values occur because \( \tau_1 \) increases, so a bounded value is needed called a delay critical value or denoted by \( \tau_{k_{\text{cr}}} \) which is

\[ \tau_{k_{\text{cr}}}^{i,j} = \frac{1}{\omega_j} \arccos \left( \frac{-(p_2 + r_2)q_0\omega_j^2 + (p_0 + r_0)q_0 + (\omega_j^2 - (p_1 + r_1)\omega_j)q_1\omega_j}{(q_1\omega_j)^2 - q_0^2} \right) + \frac{2\pi}{\omega_j} \]

With \( i = 0,1,2, \ldots \) and \( j = 1,2,3 \).
Because $\tau_{i,j}$ has an infinite value, the critical value of delay is $\tau_{1,\text{bif}} = \min_{i=0,1,2,...} \tau_{1,i}$ and $\omega_j$ are selected from the three positive roots of Equation (3).

To prove the occurrence of the Hopf bifurcation at $\tau_1 = \tau_{1,\text{bif}}$, it is necessary to point out that $\text{Re}\left(\frac{d}{d\tau_1}\left(\lambda(\tau_1)\right)\right) \neq 0$.

Suppose Equation (3) can be rewritten with

$$G(\lambda(\tau_1), \tau_1) = \lambda^3 + A\lambda^2 + B\lambda + C + D\lambda e^{-\lambda \tau_1} + E e^{-\lambda \tau_1}$$

Then, differential $G(\lambda(\tau_1), \tau_1)$ produces

$$\frac{dG}{d\lambda} d\lambda + \frac{dG}{d\tau_1} d\tau_1 = 0$$

$$\frac{d\lambda}{d\tau_1} = -\frac{\frac{dG}{d\lambda}}{\frac{dG}{d\tau_1}} = \frac{(D\lambda^2 + E\lambda)e^{-\lambda \tau_1}}{3\lambda^3 + 2A\lambda + B + (D - \tau_1 D\lambda - \tau_1 E)e^{-\lambda \tau_1}}$$

Next, the derivative of $G$ is calculated at the time $\tau_1 = \tau_{1,\text{bif}}$ dan $\lambda = i \omega_{\text{bif}}$

$$\frac{d}{d\tau_1}(\lambda(\tau_1)) = \frac{D(i\omega_{\text{bif}})^2 + E(i\omega_{\text{bif}})}{3(i\omega_{\text{bif}})^2 + 2A(i\omega_{\text{bif}}) + B + (D - i\tau_{1,\text{bif}} D\omega_{\text{bif}} - \tau_{1,\text{bif}} E)e^{-i\omega_{\text{bif}} \tau_1}}$$

Substitute $e^{-i\omega_{\text{bif}} \tau_1} = \cos(\omega_{\text{bif}} \tau_{1,\text{bif}}) - i \sin(\omega_{\text{bif}} \tau_{1,\text{bif}})$, and with simplification obtained

$$\text{Re}\left(\frac{d}{d\tau_1}(\lambda(\tau_1))\right) = 3\omega_{\text{bif}}^4(D\omega_{\text{bif}} \cos(\omega_{\text{bif}} \tau_{1,\text{bif}}) - E \sin(\omega_{\text{bif}} \tau_{1,\text{bif}})) - B\omega_{\text{bif}}(D\omega_{\text{bif}} \cos(\omega_{\text{bif}} \tau_{1,\text{bif}}) - E \sin(\omega_{\text{bif}} \tau_{1,\text{bif}})) - D^2\omega_{\text{bif}}^2 + 2A\omega_{\text{bif}}^2(D\omega_{\text{bif}} \sin(\omega_{\text{bif}} \tau_{1,\text{bif}}) + E \cos(\omega_{\text{bif}} \tau_{1,\text{bif}}))$$

4. Optimal Control of the IS-LM Business Cycle Model with Two Delays

The problem of optimal control of the IS-LM business cycle model in Equation (2) is to balance or stabilize the supply of money and demand for money in the community. This is expected to stabilize the economy of a country. In this study, what is used as a controller is the supply of money $\bar{M}$.

Optimal control of the IS-LM business cycle model to get the optimal shape so as to minimize the objective function with the $\bar{M}$ control are:

$$J(\bar{M}) = \int_0^T (r(t) + c_1 \bar{M}^2) \, dt$$
Completion of Optimal Control:

1. Determine the Hamiltonian function

\[ \mathcal{H} = r(t) + \bar{M}^2 + \lambda_1 \left( \alpha \left[ A \frac{[Y(t)]^a}{[r(t)]^b} + \beta_1 K(t) - s_1 [Y(t)]^a [r(t)]^b \right] \right) \]

\[ + \lambda_2 \left( \beta \left[ gY(t) + \frac{h}{r(t) - \bar{r}_1} - \bar{M} \right] \right) + \lambda_3 \left( A \frac{[Y(t) - \tau_1]^a}{[r(t)]^b} - (\delta - \beta_1) K(t - \tau_2) \right) \]

2. Stationary conditions

\[ \frac{\partial \mathcal{H}}{\partial \bar{M}} = 0 \text{ then } \bar{M}^* = \frac{\beta}{2c_1} \lambda_2 \]

So that it is obtained:

\[ \mathcal{H}^* = r(t) + \left( \frac{\beta}{2c_1} \lambda_2 \right)^2 + \lambda_1 \left( \alpha \left[ A \frac{[Y(t)]^a}{[r(t)]^b} + \beta_1 K(t) - s_1 [Y(t)]^a [r(t)]^b \right] \right) \]

\[ + \lambda_2 \left( \beta \left[ gY(t) + \frac{h}{r(t) - \bar{r}_1} - \frac{\beta}{2c_1} \lambda_2 \right] \right) + \lambda_3 \left( A \frac{[Y(t) - \tau_1]^a}{[r(t)]^b} - (\delta - \beta_1) K(t - \tau_2) \right) \]

3. Equation State

\[ \dot{Y}^*(t) = \frac{\partial \mathcal{H}^*}{\partial \lambda_1} \]

\[ = \alpha \left[ A \frac{[Y(t)]^a}{[r(t)]^b} + \beta_1 K(t) - s_1 [Y(t)]^a [r(t)]^b \right] \]

\[ \dot{r}^*(t) = \frac{\partial \mathcal{H}^*}{\partial \lambda_2} \]

\[ = \beta \left[ gY(t) + \frac{h}{r(t) - \bar{r}_1} - \bar{M} \right] \]

\[ \dot{K}^*(t) = \frac{\partial \mathcal{H}^*}{\partial \lambda_3} \]

\[ = A \frac{[Y(t) - \tau_1]^a}{[r(t)]^b} - (\delta - \beta_1) K(t - \tau_2) \]

4. Costate equation

\[ \dot{\lambda}_1^*(t) = -\frac{\partial \mathcal{H}^*}{\partial Y^*(t)} - \frac{\partial \mathcal{H}^*}{\partial Y^*(t - \tau_1)} \]

\[ = -\lambda_1 \left( \alpha aA \frac{[Y^*(t)]^{a-1}}{[r^*(t)]^b} - \alpha a s_1 [Y^*(t)]^{a-1} [r^*(t)]^b \right) - \lambda_2 \beta g - \lambda_3 aA \frac{[Y^*(t - \tau_1)]^{a-1}}{[r^*(t)]^b} \]

\[ -\lambda_1 \left( -a b A \frac{[Y^*(t)]^a}{[r^*(t)]^{b+1}} - b s_1 [Y^*(t)]^a [r^*(t)]^{b-1} \right) - \lambda_2 \beta \frac{h}{(r(t) - \bar{r}_1)^2} \]

\[ \dot{\lambda}_2^*(t) = -\frac{\partial \mathcal{H}^*}{\partial r^*(t)} \]

\[ = -1 - \lambda_1 \left( -a b A \frac{[Y^*(t)]^a}{[r^*(t)]^{b+1}} - b s_1 [Y^*(t)]^a [r^*(t)]^{b-1} \right) - \lambda_2 \beta \frac{h}{(r(t) - \bar{r}_1)^2} \]

\[ -\lambda_3 b A \frac{[Y^*(t - \tau_1)]^a}{[r^*(t)]^{b+1}} \]

\[ \dot{\lambda}_3^*(t) = -\frac{\partial \mathcal{H}^*}{\partial K^*(t - \tau_2)} \]

\[ = -\lambda_1 a \beta_1 + \lambda_3 (\delta - \beta_1) \]
5. **Numerical Simulation**

In the next section a simulation will be performed to show changes in system stability and optimal control in the IS-LM business cycle model. Model simulations are shown in graphical form, by substituting parameter values in Table 1.

| Parameter | Description                                      | Values |
|-----------|--------------------------------------------------|--------|
| $A$       | technology productivity                         | 0.1    |
| $\alpha$ | index in the goods market                       | 1      |
| $\beta$  | index in the money market                        | 1      |
| $a$       | coefficient of adjustment in the goods market   | 1.03   |
| $b$       | coefficient of adjustment on the money market   | 0.6    |
| $\beta_1$| the level of decline in investment in the capital stock | $-0.1$ |
| $\delta$ | capital depreciation constant                   | 0.1    |
| $s_1$     | growth rate of deposits against income          | 0.08   |
| $g$       | amount of money demand for income               | 0.05   |
| $h$       | the amount of money demand for interest rates   | 0.01   |
| $\bar{r}_1$ | lowest fixed rate of interest rates          | 0.001  |
| $\bar{M}$ | money supply constant                           | 0.1    |

**5.1 Stability on IS-LM Business Cycle Model**

Analysis of the stability of the IS-LM business cycle model can be seen from the characteristics of the model. If the characteristic value is less than zero then the model is stable. Stability analysis is carried out in several cases, i.e: Case 1. When $\tau_1 = \tau_2 = 0$ (without delay) and Case 2: for $\tau_1 \neq 0$ and $\tau_2 = 0$

**5.1.1 Case 1. When $\tau_1 = \tau_2 = 0$ (without delay)**

Based on the parameter values in Table 1, obtained a fixed point $E_1 = (0, 0.101, 0)$ and $E_2 = (1.7037, 0.6759, 1.0949)$. For fixed points $E_1 = (0, 0.101, 0)$, the eigenvalue is $\lambda_i = (-0.1, 0, -0.2)$ with $i = 1, 2, 3$. Because there is an eigen value of 0, then at a fixed point $E_1$ does not oscillate. Whereas the fixed point $E_2 = (1.7037, 0.6759, 1.0949)$ is asymptotically stable because it meets the Routh-Hurwitz criteria. Then simulations are performed for different $s_1$ parameter values, where the parameter $s_1$ is a parameter of the growth rate of deposits against income. Selected parameters $s_1 = 0.08$ and $s_1 = 0.01$ with initial values $\bar{Y} = 1.5, r = 0.5$, and $K = 1$. 


In Figure 1 it can be seen that when the parameter value $s_1 = 0.08$, the graph solution goes to the fixed point so that the system is stable.

In Figure 2 it can be seen that when the parameter value $s_1 = 0.08$, the phase field solution goes to the fixed point so that the system is stable.

In Figure 3 it can be seen that when the parameter value $s_1 = 0.1$, the graph solution has oscillation and is further away from the fixed point $E_2$ so that the system is unstable.

In Figure 4 it can be seen that the phase field solution has oscillation and is further away from the fixed point $E_2$ so that the system is unstable.
In Figure 4, it can be seen that when the parameter value $s_1 = 0.01$, the phase field is further away from the fixed point $E_2$ so that the system is unstable.

Thus, the simulation results above show that at the fixed point $E_2$, there is a bifurcation with a change in the value of the parameter $s_1$.

5.1.2 Case 2: for $\tau_1 \neq 0$ and $\tau_2 = 0$

Based on the parameter values in Table 1, obtained a fixed point $E_2 = (1.7037, 0.6759, 1.0949)$ and a delay critical value $\tau_1 = 2.7076$. Parameter values meet the Routh-Hurwitz criteria so that at a fixed point $E_2$ is stable. Simulations were carried out at $\tau_1 = 1$ and $\tau_1 = 3.8$ with initial values $Y = 1.5, r = 0.5$ and $K = 1$.

In Figure 5, it can be seen that when the delay time is $\tau_1 = 1$, it is explained that the time delay on income causes the rate of income, interest rate, and rate of capital stock to decrease, consequently the system will move towards $E_2$ so that the system is stable.

In Figure 6, it can be seen that when the delay time is equal $\tau_1 = 1$, the phase field solution goes to the fixed point $E_2$ that the system is stable.
In Figure 7, can be seen when the delay time $\tau_1 = 2.7076$ system shows a constant oscillation movement. The rate of income, interest rates, and the rate of capital stock do not seem to move towards a certain value. This is because there is an eigenvalue into a pure imaginary form, so that the stability of the system changes. Eigenvalues in the form of pure imaginary indicate a limit cycle. This limit cycle illustrates that when the delay value is $1 = 2.7076$, the system will not go to a certain value but only oscillates over time which will only change when the time value of the delay gets bigger.

In Figure 8, it can be seen when $\tau_1 = 2.7076$ phase fields form a constant spiral, so the system experiences oscillation.
In Figure 9, it can be seen that when $\tau_1 = 3.8$ shows a very large change in oscillation, especially in the income variable, as a result the income rate, interest rate, and the rate of capital stock will increase because the delay time will increase, to the point of stability.

In Figure 10, it can be seen that when $\tau_1 = 3.8$ phase fields form a spiral that gets bigger and does not lead to $E_2$ equilibrium point, so the system is unstable.

The simulation results show that there is a stability change when $\tau_1 = 3.8$ and $\tau_2 = 3.8$ so that according to the theorem occurs bifurcation. Stability changes occur when the value $\tau_1 = 2.7076$ is when the eigenvalues in the form of pure imaginary arise so that there is a limit cycle that reveals that there is a Hopf bifurcation. Hopf bifurcation at the equilibrium point describes the existence of a boundary value called the delay critical point value. The income rate, interest rate, and capital stock rate have a stable system solution when given a value of delay that is less than the value of the delay critical point $0 < \tau_1 < 2.7076$. If the value of the delay given is more than the value of the delay critical point ($\tau_1 > 2.7076$), the system solution shows the rate of income, the interest rate, and the rate of capital stock will continue to fluctuate, causing unstable economic conditions.

5.2 Optimal Control of the IS-LM Business Cycle Model

In this section numerical simulations optimal control in the IS-LM business cycle model is performed. To solve the state equation, the forward of 4th order Runge Kutta method is used, while the completion of the co-state equation is used as the backward of 4th order runge kutta method. In this simulation, parameter values are based on Table 1 with a value of $\tau_1 = 3.5$ and $\tau_2 = 6.5$. The simulation for optimal control $\bar{M}$ in the interest rate equation can be seen in the following figure.

In Figure 11, it can be seen that when the values $\tau_1 = 3.5$ and $\tau_2 = 6.5$ pass the critical delay value, the changes in oscillations are very large, especially in the capital stock variable. As a result, the rate of income, interest rates, and the rate of capital stock will increase because the delay time is getting bigger, so the system is not unstable. Furthermore optimal control is carried out so that the system becomes stable.
In Figure 12, it can be seen that when giving money supply control to the interest rate variable, the income rate decreases and starts towards the stability point in the 50th time. While the interest rate starts towards the stability point at the 40th time. Likewise with the rate of capital stock starting towards the stability point at the 30th time. Thus it can be concluded that with the control given, the rate of income, the interest rate and the rate of capital stock will go to the point of stability.

6. Conclusion
Based on the discussion above it can be concluded that based on the analysis of the stability of the IS-LM business cycle model, the simulation results show that there is a stability change when the delay value given exceeds the critical value of delay. Stability changes occur when the delay value given arises a pure eigenvalue in the form of pure imaginary so that there is a limit cycle that reveals that there is a Hopf bifurcation. While based on optimal control provided by maximizing the total money supply, the rate of income, interest rate, and the rate of capital stock decrease. As a result, the system will lead to stability.

7. References
[1] Sambas A, Vaidyanathan S, Mamat M, Mohamed M A and Sanjaya W S M 2018 International Journal of Electrical and Computer Engineering 8 4951-4958.
[2] Vaidyanathan S, Feki M, Sambas A and Lien C H 2018 International Journal of Simulation and Process Modelling 13 419-432.
[3] Vaidyanathan S, Sambas A, Kacar S and Cavusoglu U 2019 Nonlinear Engineering 8 193-205.
[4] Vaidyanathan S, Sambas A, Mamat M and Sanjaya W M 2017 Archives of Control Sciences 27 541-554.
[5] Mobayen S, Vaidyanathan S, Sambas, Kaçar S and Çavuşoğlu Ü 2019 Iranian Journal of Science and Technology, Transactions of Electrical Engineering 43, 1-12.
[6] Vespignani A 2012 Nature physics 8, 32.
[7] Tastrawati N K T 2012 Mathematical Journal 2 1.
[8] Neamţu M, Opreş D and Chilarescu C 2007 Chaos, Solitons & Fractals 34, 519-530.
[9] Kaddar A and Alaoui H T 2008 Electronic Journal of Differential Equations 134, 1-9.
[10] Sportelli M, De Cesare L and Binetti M T 2014 Applied Mathematics and Computation 243, 728-739.
[11] Ma J and Ren W 2016 International Journal of Bifurcation and Chaos 26, 1650181.
[12] Liu X, Cai W, Lu J and Wang Y 2015 Communications in Nonlinear Science and Numerical Simulation 25, 149-161.
[13] Cai J 1986 Electronic Journal of Differential Equations 15, 1-6.
[14] Zou L and Li Y 2009 Journal of Computational and Applied Mathematics 228 182-187.
[15] Rosmely, Nugrahani E H and Sianturi P 2016 Mathematical Journal 15 1-10.
[16] Kaddar A and Alaoui H T 2008 Applied Mathematical Sciences 2 1529-1539.
[17] Neri U and Venturi B 2007 International Review of Economics 54 53-65.
[18] De Cesare L and Sportelli M 2005 Chaos, Solutions and Fractals 25 233-244.