ARENAS OF FINITE STATE MACHINES

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Abstract. Finite state machines are widely used as a sound mathematical formalism that appropriately describes large scale, distributed and complex systems. Multiple interactions of finite state machines in complex systems are well captured by the notion of non-flat systems. Non-flat systems are "finite state machines" where each "state" can be either a basic state or an aggregate of finite state machines. By expanding a non-flat system, a flat system is obtained which is an ordinary finite state machine. In this paper we introduce a novel class of non-flat systems called Arena of Finite State Machines (AFSM). AFSMs are collections of finite state machines that interact concurrently through a communication network. We propose a notion of equivalence, termed compositional bisimulation, that allows the complexity reduction of AFSMs without the need of expanding them to the corresponding FSMs. The computational complexity gain obtained from this approach is formally quantified in the paper. An application of the proposed framework to the regulation of gene expression in the bacterium Escherichia coli is also presented.

1. Introduction

Finite state machines (FSMs) are widely used in modeling complex systems ranging from computer and communication networks, automated manufacturing systems, air traffic management systems, distributed software systems, among many others, see e.g. [CL99, CGP99]. The increasing complexity of large scale systems demanded during the years for formal methods that can render their analysis tractable from a computational complexity point of view. Several approaches have been proposed in the literature, which include abstraction, modular verification methods, symmetry and partial order reduction, see e.g. [CGP99]. The common goal of these approaches is to find an FSM that is equivalent to the original one, but with a set of states of smaller size. In this paper we follow the approach by Alur and co-workers (see e.g. [AY01, AKY99]), where a complex system is viewed as a "non-flat" system. A non-flat system is a "finite state machine" where each "state" can be either a basic state or a superstate [Har87] that hides inside an FSM or even a composition of FSMs. By expanding the superstates of a non-flat system to their corresponding FSMs an ordinary FSM is obtained. One of the early non-flat systems that appeared in the literature are hierarchical state machines (HSMs) [AY01]. While HSMs well capture modeling features of many design languages as for example Statecharts [Har87], they only consider sequential interaction among the FSMs involved. Recursive state machines (RSMs) [ABE+05] extend HSMs by allowing recursion in the sequential interaction of FSMs. As such, they well model sequential programming languages with recursive procedure calls. Recursive Game Graphs, a natural adaptation of RSMs to a game theoretic setting, have been studied in [Ete04]; Pushdown Graphs have been studied in [Cac02]. Both HSMs and RSMs do not exhibit concurrent compositional features. Communicating hierarchical state machines (CHSMs) [AKY99] generalize HSMs, by allowing FSMs to interact not only sequentially but also concurrently, through the notion of parallel composition. Reachability problems and checking language and bisimulation equivalences for CHSMs are proven in [AKY99] to fall in the class of exponential time and space complexity problems. This complexity result is in line with the ones further established in [LS00, SJ09] on complexity arising in checking a range of equivalence notions in the linear time–branching time spectrum [vG90] for networks of FSMs, modeled by parallel composition of FSMs. By following the conjecture of Rabinovich in [Rab97], the work in [LS00, SJ09] strongly suggest that there is

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no way to escape the so-called state explosion problem, when checking behavioral relations and in particular bisimulation equivalence, for non-flat systems exhibiting concurrent-types interaction.

In this paper we identify a novel class of concurrent non-flat systems, termed Arenas of Finite State Machines (AFSMs) for which complexity reduction via bisimulation can be performed without incurring in the state explosion problem. AFSMs are collections of FSMs that interact concurrently, through a communication network. For AFSMs we propose a notion of equivalence, termed compositional bisimulation, that is based on the communication network governing the interaction mechanism among the FSMs. The main contribution of the paper resides in showing that compositional bisimulation equivalence between AFSMs implies bisimulation equivalence between the corresponding expanded FSMs. This result is important because it implies that all properties preserved by bisimulation equivalence, e.g. linear temporal logic properties [CGP99], are also preserved by compositional bisimulation. Therefore, it can be of help in the formal verification and control design of complex systems modeled by AFSMs that admit compositional bisimulation. A computational complexity analysis reported in the paper reveals that checking compositional bisimulation between a pair of AFSMs scales as \(O(N_1^2 + N_2^2)\) in space complexity and as \(O((N_1^2 + N_2^2) \ln(N_1 + N_2))\) in time complexity, with the numbers \(N_1\) and \(N_2\) of FSMs composing the AFSMs. A standard approach, based on expanding the AFSMs to the corresponding FSMs exhibits an exponential space and time complexity. An application of the proposed results to the modeling and complexity reduction of the regulation of gene expression in the single-celled bacterium \(E. coli\) is included.

2. Preliminary definitions

2.1. Notation. Given a set \(A\), the symbol \(2^A\) denotes the set of subsets of \(A\) and the symbol \(|A|\) denotes the cardinality of \(A\). If \(|A| = 1\) then \(A\) is said a singleton. A relation \(R \subseteq A \times B\) is said to be total if for any \(a \in A\) there exists \(b \in B\) such that \((a, b) \in R\) and conversely, for any \(b \in B\) there exists \(a \in A\) such that \((a, b) \in R\). Given a relation \(R \subseteq A \times B\), the inverse of \(R\), denoted \(R^{-1}\), is defined as \(\{(b, a) \in B \times A : (a, b) \in R\}\). A relation \(R \subseteq A \times B\) is the identity relation if \(A = B\) and \(a = b\) for all \((a, b) \in R\). A directed graph is a tuple \(G = (V, E)\) where \(V\) is the set of vertices and \(E\) is the set of edges. We denote by \(\mathbb{N}\) the set of positive integers.

2.2. Finite State Machines. In this paper we consider finite state machines in the formulation of Moore machines where states are labeled with outputs and transitions are labeled with inputs.

Definition 2.1. [BG01] A Finite State Machine (FSM) is a tuple
\[
M = (X, x^0, U, Y, H, \Delta),
\]
where \(X\) is a finite set of states, \(x^0 \in X\) is the initial state, \(U\) is a finite set of input symbols, \(Y\) is a finite set of output symbols, \(H : X \to 2^Y\) is an output map, and \(\Delta \subseteq X \times 2^U \times X\) is a transition relation.

When \(x^0\) is skipped from the tuple in (2.1) any state in \(X\) is assumed to be an initial state. We denote a transition \((x, u, x') \in \Delta\) of FSM \(M\) by \(x \xrightarrow{u, \Delta} x'\). By definition of \(\Delta\), a transition of the form \(x \xrightarrow{\emptyset, \Delta} x'\) is allowed. Such a transition is viewed as private or internal to \(M\). Throughout the paper we refer to an input \(u = \emptyset\) as internal, and an input \(u \neq \emptyset\) as external to \(M\). Analogously, for a state \(x \in X\), \(H(x) = \emptyset\) is allowed, meaning that state \(x\) is not visible from the external environment. Despite classical formulations of Moore machines that model the transition relation as \(\Delta \subseteq X \times U \times X\) and the output function as \(H : X \to Y\), we model here \(\Delta\) as a subset of \(X \times 2^U \times X\) and \(H\) as a function from \(X\) to \(2^Y\). By this choice, multiple interactions of FSMs can be considered, as illustrated in Example 3.2 on a simple distributed system.

2.3. Equivalence notions. Several notions of equivalence have been proposed for the class of finite state machines, see e.g. [vG90]. In this paper we focus on the notion of bisimulation equivalence [Mil89, Par81] that is widely used as an effective tool to mitigate complexity of verification and control design of large scale complex systems, see e.g. [CGP99]. Consider a pair of FSMs \(M_i = (X_i, x^0_i, U_i, Y_i, H_i, \Delta_i)\) \((i = 1, 2)\). We start by recalling the notion of isomorphism.
The notion of isomorphism is an equivalence relation on the class of FSMs. The notion of bisimulation equivalence is reported hereafter.

Definition 2.2. The FSMs \( M_1 \) and \( M_2 \) are isomorphic, denoted \( M_1 \cong_{iso} M_2 \), if there exists a bijective function \( T : X_1 \to X_2 \) such that \( x_2^0 = T(x_1^0) \), \( H_1(x_1) = H_2(T(x_1)) \) for any \( x_1 \in X_1 \), and \( x_1 \xrightarrow{u \Delta_i} x_1' \) if and only if \( T(x_1) \xrightarrow{u \Delta_2} T(x_1') \).

The notion of isomorphism is an equivalence relation on the class of FSMs. The notion of bisimulation equivalence is reported hereafter.

Definition 2.3. A set \( R \subseteq X_1 \times X_2 \) is a bisimulation relation between \( M_1 \) to \( M_2 \) if for any \( (x_1, x_2) \in R \),

(i) \( H_1(x_1) = H_2(x_2) \);
(ii) existence of \( x_1 \xrightarrow{\frac{u_1}{\Delta_1}} x_1' \) implies existence of \( x_2 \xrightarrow{\frac{u_2}{\Delta_2}} x_2' \) such that \( u_1 = u_2 \) and \( (x_1', x_2') \in R \);
(iii) existence of \( x_2 \xrightarrow{\frac{u_2}{\Delta_2}} x_2' \) implies existence of \( x_1 \xrightarrow{\frac{u_1}{\Delta_1}} x_1' \) such that \( u_1 = u_2 \) and \( (x_1', x_2') \in R \).

FSMs \( M_1 \) and \( M_2 \) are bisimilar, denoted \( M_1 \cong M_2 \), if

(iv) \( (x_1, x_2) \in R \).

When the initial states \( x_1^0 \) and \( x_2^0 \) are skipped from the tuples \( M_1 \) and \( M_2 \), condition (iv) is replaced by requiring \( R \) to be total. Bisimulation equivalence is an equivalence relation on the class of FSMs. The maximal bisimulation relation between FSMs \( M_1 \) and \( M_2 \) is a bisimulation relation \( R^*(M_1, M_2) \) such that \( R \subseteq R^*(M_1, M_2) \) for any bisimulation relation \( R \) between \( M_1 \) and \( M_2 \). The maximal bisimulation relation exists and is unique. Given an FSM \( M \) the set \( R^*(M, M) \) is an equivalence relation on the set of states of \( M \). The quotient of \( M \) induced by \( R^*(M, M) \), denoted \( M_{min}(M) \), is the FSM bisimilar to \( M \) with the minimal number of states \([CGP99] \). FSM \( M_{min}(M) \) exists and is unique up to isomorphisms.

Lemma 2.4. If \( M_{min}(M_1) \cong M_{min}(M_2) \) then \( M_1 \cong_{iso} M_2 \).

Proof. Let \( X_i \) be the set of states of \( M_i \). Minimality of \( M_{min}(M_1) \) and \( M_{min}(M_2) \) implies that the maximal bisimulation relation \( R^* \) between \( M_{min}(M_1) \) and \( M_{min}(M_2) \) is such that for any \( x_1 \in X_1 \) and \( x_2 \in X_2 \), sets \( R^*(x_1) = \{ x_2 \in X_2 | (x_1, x_2) \in R^* \} \) and \( (R^*)^{-1}(x_2) = \{ x_1 \in X_1 | (x_1, x_2) \in R^* \} \) are singletons. Hence, define function \( T : X_1 \to X_2 \) by \( T(x_1) = x_2 \) when \( R^*(x_1) = \{ x_2 \} \). It is easy to see that function \( T \) satisfies the properties required in Definition 2.2. \( \square \)

We conclude this section by recalling space and time complexity in checking bisimulation equivalence between FSMs.

Proposition 2.5. \([PT87]\) Space complexity in checking \( M_1 \cong M_2 \) is \( O(|X_1| + |\Delta_1| + |X_2| + |\Delta_2|) \).

Proposition 2.6. \([PT87]\) Time complexity in checking \( M_1 \cong M_2 \) is \( O((|\Delta_1| + |\Delta_2|) \ln(|X_1| + |X_2|)) \).

3. Arenas of finite state machines

In this section we introduce a new class of non–flat systems \([AY01] \), called Arenas of Finite State Machines (AFSMs). AFSMs are collections of FSMs that interact concurrently through a communication network. The syntax of an AFSM is specified by a directed graph:

\[ A = (V, E), \]

where \( V \) is a collection of \( N \) FSMs \( M_i \) and \( E \subseteq V \times V \) describes the communication network of the FSMs \( M_i \). In the definition of \( E \) self loops \( (M_i, M_i) \in E \) would model communication of \( M_i \) with itself, which is tautological. For this reason in the sequel we assume \( (M_i, M_i) \notin E \). By expanding each vertex \( M_i \in V \) of \( A \) an ordinary FSM is obtained, which is defined by:

\[ M(A) = (X, x^0, U, Y, H, \Delta), \]
where \( X = X_1 \times X_2 \times \ldots \times X_N \), \( x^0 = (x^0_1, x^0_2, \ldots, x^0_N) \), \( U = \bigcup_{i \in V} U_i \), \( Y = \bigcup_{i \in V} Y_i \), \( H((x_1, x_2, \ldots, x_N)) = \bigcup_{i \in V} H_i(x_i) \), and \( \Delta \subseteq X \times 2^U \times X \) is such that

\[
(x_1, x_2, \ldots, x_N) \xrightarrow{u \Delta} (x'_1, x'_2, \ldots, x'_N),
\]

whenever \( x_i \xrightarrow{u_i \Delta} x'_i \) is a transition of \( M_i \) for some \( u_i \) (\( i = 1, 2, \ldots, N \)) and

\[
u = \bigcup_{i \in V} (u_i \setminus \bigcup_{j \in \text{Pre}(\mathbb{A}, M_i)} H_j(x_j)),
\]

where \( \text{Pre}(\mathbb{A}, M_i) = \{ M_j \in \mathbb{V} | (M_j, M_i) \in \mathbb{E} \} \).

**Proposition 3.1.** Given an AFSM \( \mathbb{A} \), the FSM \( M(\mathbb{A}) \) is unique.

**Proof.** Entities \( X, x^0, U, Y \) and \( H \) in \( M(\mathbb{A}) \) are uniquely determined from \( \mathbb{A} \). For any collection of \( N \) transitions \( x_i \xrightarrow{u_i \Delta} x'_i \) in \( M_i \) there exists one and only one transition in \( M(\mathbb{A}) \) of the form (3.1) with \( u \) uniquely specified by (3.2). \( \Box \)

FSM \( M(\mathbb{A}) \) specifies the semantics of the AFSM \( \mathbb{A} \). Such a semantic is implicitly given through a composition of FSMs that can be regarded as a notion of parallel composition \([\text{CGP99}]\) that respects the topology of the AFMS communication network. The following simple example illustrates syntax and semantics of AFMSs.

**Example 3.2.** Consider a distributed system composed of three computers \( C_1, C_2 \) and \( C_3 \), whose goal is to compute the Euclidean norm \( ||z|| = \sqrt{z_1^2 + z_2^2} \) of a vector \( z = (z_1, z_2) \in \mathbb{R}^2 \) in a distributed fashion. While \( C_1 \) and \( C_2 \) are delegated to compute respectively \( z_1^2 \) and \( z_2^2 \), \( C_3 \) takes as inputs the computations of \( C_1 \) and \( C_2 \) and outputs \( ||z|| \). This simple distributed system can be modeled as the AFMS \( \mathbb{A} = (\mathbb{V}, \mathbb{E}) \) where \( \mathbb{V} = \{ M_1, M_2, M_3 \} \) and \( \mathbb{E} = \{(M_1, M_3), (M_2, M_3)\} \). FSMs \( M_i \), each one modeling computers \( C_i \), are illustrated in Figures 1(a)(b)(c), while AFSM \( \mathbb{A} \), modeling the computers’ network, is depicted in Figure 1(d). By expanding \( \mathbb{A} \), the FSM \( M(\mathbb{A}) \) is obtained, whose accessible part 2 is depicted in Figure 2. Starting from (1, 3, 5), when receiving the input \( \{z_1, z_2\} \), FSM \( M(\mathbb{A}) \) outputs in state (2, 4, 6) the set \( \{z_1^2, z_2^2\} \) and finally in state (1, 3, 7) the requested output \( \{||z||\} \). For illustrating the construction of FSM \( M(\mathbb{A}) \), we describe in detail the construction of the transition (2, 4, 6) \( \xrightarrow{u \Delta} \) (1, 3, 7). By applying the compositional rules defining the semantics of AFMSs, one gets: 2 \( \xrightarrow{\emptyset} \) 1 is in \( M_1 \), 4 \( \xrightarrow{\emptyset} \) 3 is in \( M_2 \), and 6 \( \xrightarrow{\{z_1^2, z_2^2\}} \) 7 is in \( M_3 \). Moreover, one first note that \( \text{Pre}(\mathbb{A}, M_1) = \text{Pre}(\mathbb{A}, M_2) = \emptyset \) and \( \text{Pre}(\mathbb{A}, M_3) = \{M_1, M_2\} \), from which \( u = \emptyset \). The resulting transition (2, 4, 6) \( \xrightarrow{\emptyset} \) (1, 3, 7) is indeed in \( M(\mathbb{A}) \), as shown in Figure 2.

4. **Compositional bisimulation of AFMSs**

A naïve approach to check bisimulation equivalence of two AFMSs \( \mathbb{A}_1 \) and \( \mathbb{A}_2 \) consists in first expanding them to FSMs \( M(\mathbb{A}_1) \) and \( M(\mathbb{A}_2) \) and then apply standard bisimulation algorithms (see e.g. [PT87, DPP04, Hop71]). The main practical limitation of this approach resides in the well–known state explosion problem, see e.g. [LS00, SJ09]. This is the key reason for us to propose an alternative approach to check bisimulation equivalence of AFMSs which is centered on the notion of **compositional bisimulation** that is introduced hereafter.

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1Each circle denotes a state and each edge a transition. In each circle, upper symbol denotes the state and lower symbol the output set associated with the state; symbols labeling edges denote the input sets associated with the transitions.

2The accessible part of the FSM \( M \) in (2.1) is the unique sub–finite state machine extracted from \( M \), containing all and only the states of \( M \) that are reachable (or equivalently, accessible) in a finite number of transitions from its initial state \( x^0 \), see e.g. [CL99].
Definition 4.1. Given a pair of Arenas $\mathcal{A}^j = (\mathcal{V}^j, \mathcal{E}^j)$ of FSMs $M^j_1, M^j_2, ..., M^j_{N^j}$ ($j = 1, 2$), a set $R \subseteq \mathcal{V}^1 \times \mathcal{V}^2$, is a compositional bisimulation relation between $\mathcal{A}^1$ and $\mathcal{A}^2$ if for any $(M^1_i, M^2_j) \in R$ the following conditions are satisfied:

- $M^1_i \equiv M^2_j$;
- existence of $(M^1_i, M^1_1) \in \mathcal{E}^1$ implies existence of $(M^2_j, M^2_1) \in \mathcal{E}^2$ such that $(M^1_i, M^2_j) \in R$;
- existence of $(M^1_j, M^2_j) \in \mathcal{E}^2$ implies existence of $(M^1_j, M^2_j) \in \mathcal{E}^1$ such that $(M^1_i, M^2_j) \in R$.

The FSMs $\mathcal{A}^1$ and $\mathcal{A}^2$ are compositionally bisimilar, denoted $\mathcal{A}^1 \cong \mathcal{A}^2$, if there exists a total compositional bisimulation relation between $\mathcal{A}^1$ and $\mathcal{A}^2$.

The notion of compositional bisimulation is an equivalence relation on the class of AFSMs. The maximal compositional bisimulation relation between AFSMs $\mathcal{A}^1$ and $\mathcal{A}^2$ is a compositional bisimulation relation $R^*(\mathcal{A}^1, \mathcal{A}^2)$ such that $\mathbb{R} \subseteq \mathbb{R}^*(\mathcal{A}^1, \mathcal{A}^2)$ for any compositional bisimulation relation $R$. The maximal compositional bisimulation exists and is unique. The set $\mathbb{R}^*(\mathcal{A}, \mathcal{A})$ is an equivalence relation on the collection of FSMs in $\mathcal{A}$. The quotient of $\mathcal{A}$ induced by $\mathbb{R}^*(\mathcal{A}, \mathcal{A})$ is the minimal (in terms of the number of the FSMs involved) compositionally bisimilar AFSM of $\mathcal{A}$. The minimal AFSM of an AFSM $\mathcal{A}$, denoted $\mathcal{A}_{\text{min}}(\mathcal{A})$, exists and is unique, up to isomorphisms.

Checking compositional bisimulation equivalence of AFSMs is equivalent to checking bisimulation equivalence of appropriate FSMs, as discussed hereafter. Consider a pair of AFSMs $\mathcal{A}^j = (\mathcal{V}^j, \mathcal{E}^j)$ ($j = 1, 2$). Since bisimulation is an equivalence relation on the set $\mathcal{V}_1 \cup \mathcal{V}_2$ of FSMs, it induces a partition of $\mathcal{V}_1 \cup \mathcal{V}_2$ in $K$ equivalence classes $C_1, C_2, ..., C_K$ where $M_i, M_j \in C_k$ if and only if $M_i \cong M_j$. Note that $\{C_k\}_{k \in K}$ is a finite
set. Define the tuple:

\[(4.1) \quad M_{A_j} = (X_{A_j}, U_{A_j}, Y_{A_j}, H_{A_j}, \Delta_{A_j}),\]

where \(X_{A_j} = \emptyset, U_{A_j} = \emptyset, Y_{A_j} = \{C_k\}_{k \in K}, H_{A_j} : X_{A_j} \to 2^{\cup_{j} \Delta_{A_j}}\) is defined by \(H_{A_j}(M_i^j) = \{C_k\}\) if \(M_i^j \in C_k\), and \(\Delta_{A_j} \subseteq X_{A_j} \times \emptyset \times X_{A_j}\) is such that \(M_i^j \circ \Delta_{A_j} = M_i^j\) when \((M_i^j, M_i^j) \in \mathcal{E}_j\). By definition of \(H_{A_j}\), \(H_{A_j}(M_i^j) = H_{A_j}^j(M_i^j)\) if and only if \(M_i^j \equiv M_i^j\). The syntax of the tuple in (4.1) is the same as the one of FSMs from which, the following result holds.

**Proposition 4.2.** \(A^1 \equiv_c A^2\) if and only if \(M_{A^1} \equiv M_{A^2}\).

**Proof.** By Definitions 2.3 and 4.1 it is readily seen that \(A^1 \equiv_c A^2\) if and only if the set \(\mathcal{R}^*(A^1, A^2)\) is a total bisimulation relation between \(M_{A^1}\) and \(M_{A^2}\).

We are now ready to present the main result of the paper, that shows that the notion of compositional bisimulation of AFSMs is consistent with the notion of bisimulation of the corresponding expanded FSMs.

**Theorem 4.3.** If \(A^1 \equiv_c A^2\) then \(M(A^1) \equiv M(A^2)\).

**Proof.** Let be \(A^j = (\mathcal{V}^j, \mathcal{E}^j)\) with \(\mathcal{V}^j = \{M_i^j, M_i^j, ..., M_i^j\}\) and \(M_i^j = (X_i^j, x_i^0, U_i^j, Y_i^j, H_i^j, \Delta_i^j)\) \((i = 1, 2, ..., N_j, j = 1, 2)\). Set \(M(A^j) = (\mathcal{V}^j, \mathcal{E}^j, Y^j, H^j, \Delta^j)\) \((j = 1, 2)\). Since \(A^1 \equiv_c A^2\), relation \(\mathcal{R}^*(A^1, A^2)\) is total. Consider the relation \(R \subseteq X^1 \times X^2\) defined by \((x_1, x_2) \in R\) with \(x_1 = (x_1, x_2, ..., x_N)\) and \(x_2 = (x_1, x_2, ..., x_N)\) if and only if \((x_1, x_2) \in R^*(M_i^1, M_i^2)\) and \((M_i^1, M_i^2) \in \mathcal{R}^*(A^1, A^2)\). Consider \((x_1, x_2) = ((x_1, x_2), ..., x_N), (x_1, x_2, ..., x_N)) \in R\). By definition of \(R\), \(H_1(x_1) = H_2(x_2) = \bigcup_{k \in P(i)} H_k^1(x_k)\) for any \(i = 1, 2, ..., N_1\) and \(j \in I(i) = \{k \in \{1, 2, ..., N_2\} | (M_i^1, M_k^2) \in \mathcal{R}^*(A^1, A^2)\}\). Hence one gets:

\[
H_1(x_1) = \bigcup_{M_i^1 \in \mathcal{V}^1} H_1^1(x_1) = \bigcup_{M_i^1 \in \mathcal{V}^1} \left( \bigcup_{k \in P(i)} H_k^1(x_k) \right) = \bigcup_{M_i^2 \in \mathcal{V}^2} H_2^2(x_2) = H_2^2(x_2).
\]

Note that the third equality in the above chain of equalities holds because \(\mathcal{R}^*(A^1, A^2)\) is total. Hence, condition (i) in Definition 2.3 is satisfied. Consider any transition \((x_1, x_2, ..., x_N) \xrightarrow{u} (z_1, z_2, ..., z_N)\) in \(M(A^1)\). By definition of \(M(A^1)\) there exist transitions \(x_1 \xrightarrow{u_1} z_1\) of \(M_i^1\) \((i = 1, 2, ..., N_1)\), such that:

\[
(4.2) \quad u_1 = \bigcup_{M_i^1 \in \mathcal{V}^1} \left( u_1 \cap \left( \bigcup_{M_i^1 \in \mathcal{V}^1} H_1^1(x_1) \right) \right).
\]

By definition of \(R\), for any \(i = 1, 2, ..., N_1\) there exist transitions \(x_i \xrightarrow{u_2} z_i^2\) of \(M_i^2\), with \((M_i^1, M_i^2) \in \mathcal{R}^*(A^1, A^2)\) and

\[
(4.3) \quad (z_1^1, z_2^2) \in \mathcal{R}^*(M_i^1, M_i^2),
\]

\[
(4.4) \quad u_1 = u_2,
\]

\[
(4.5) \quad H_1^1(x_1) = H_2^2(x_2).
\]

Set:

\[
(4.6) \quad u_1 = \bigcup_{M_i^1 \in \mathcal{V}^1} \left( u_1 \cap \left( \bigcup_{M_i^1 \in \mathcal{V}^1} H_1^1(x_1) \right) \right),
\]

and consider the transition \((x_1, x_2, ..., x_N) \xrightarrow{u_2} (z_1^2, z_2^2, ..., z_N^2)\) in \(M(A^2)\). By definition of the relation \(R\) and by combining (4.2), (4.4), (4.5) and (4.6), one gets \(u_1 = u_2\) from which, together with condition (4.3), one gets \(((z_1^1, z_2^2, ..., z_N^2)) \in R\). Thus, condition (ii) in Definition 2.3 is proved. Condition (iii) can be proven by using similar arguments. Finally condition (iv) holds by definition of \(R\).
The converse implication of the above result, i.e. whether $M(A^1) \cong M(A^2)$ implies $A^1 \cong_c A^2$, is not true in general, as shown in the following counterexample.

**Example 4.4.** Consider four FSMs $M_i = (X_i, X_i^0, U_i, Y_i, H_i, \Delta_i)$, where each $M_i$ is characterized by the unique transition $x_i^0 \xrightarrow{u_i} x_i^+$, where:

| $u_i$ | $M_1$ | $M_2$ | $M_3$ | $M_4$ |
|------|-------|-------|-------|-------|
| $a$  | $\{a\}$ | $\{c\}$ | $\{b, d\}$ | $\{a, d\}$ |
| $b$  | $\{b\}$ | $\{d\}$ | $\{e\}$ | $\{b, c\}$ |
| $c$  | $\{f\}$ | $\{f\}$ | $\{f\}$ | $\{f\}$ |

Consider a pair of FSMs $A^1 = (V^1, E^1)$ and $A^2 = (V^2, E^2)$, depicted in Figure 3, where:

- $V^1 = \{M_1, M_2, M_3\}$, $E^1 = \{(M_1, M_3), (M_2, M_3)\}$,
- $V^2 = \{M_2, M_4\}$, $E^2 = \{(M_2, M_4)\}$.

It is easy to see that FSM $M(M(A^1))$ is composed by the unique transition:

$$(x_1^0, x_2^0, x_3^0) \xrightarrow{(a, c)} (x_1^+, x_2^+, x_3^+),$$

with output function $H^1$ defined by $H^1(x_1^0, x_2^0, x_3^0) = \{b, d, e\}$ and $H^1(x_1^+, x_2^+, x_3^+) = \{f\}$. Moreover, FSM $M(A^2)$ is characterized by the unique transition:

$$(x_2^0, x_4^0) \xrightarrow{(a, c)} (x_2^+, x_4^+),$$

with output function $H^2$ defined by $H^2(x_2^0, x_4^0) = \{b, d, e\}$ and $H^2(x_2^+, x_4^+) = \{f\}$. Hence, FSMs $M(A^1)$ and $M(A^2)$ are bisimilar. On the other hand, it is easy to see that FSM $M_4$ is bisimilar to no FSM $M_i$, $i = 1, 2, 3$. Hence, $A^1$ and $A^2$ are not compositionally bisimilar.

Theorem 4.3 can be used to reduce the size of FSMs through compositional bisimulation, as follows.

**Corollary 4.5.** $M_{\min}(M(A)) \cong_{iso} M_{\min}(M(A_{\min}(A)))$.

**Proof.** By definition of $M_{\min}$, $M_{\min}(M(A)) \cong M(A)$ and $M(A_{\min}(A)) \cong M_{\min}(M(A_{\min}(A)))$. Since $A \cong_c A_{\min}(A)$, by Theorem 4.3 $M(A) \cong M(A_{\min}(A))$. Hence,

$M_{\min}(M(A)) \cong M(A) \cong M(A_{\min}(A)) \cong M_{\min}(M(A_{\min}(A)))$

that, by transitivity implies $M_{\min}(M(A)) \cong M_{\min}(M(A_{\min}(A)))$ which, by Lemma 2.4 concludes the proof. \[\square\]

The above result suggests a method to use compositional bisimulation for complexity reduction of FSMs, as summarized in the following algorithm:

- Compute the relation $R^*(A, A)$.
- Compute the quotient $A_{\min}(A)$.
- Expand the FSM $A_{\min}(A)$ to the FSM $M(A_{\min}(A))$.
- Compute the relation $R^*(M(A_{\min}(A)), M(A_{\min}(A)))$.
- Compute the quotient $M_{\min}(M(A_{\min}(A)))$. 

![Figure 3](image-url)
5. Complexity analysis

In this section we compare computational complexity in checking compositional bisimulation equivalence between AFSMs and bisimulation equivalence between the corresponding expanded FSMs. Consider a pair of AFSMs $A^i = (V^i, E^i)$ composed of $N_i$ FSMs and set $M(A^i) = (X^i, x_0^i, U^i, Y^i, H^i, \Delta^i)$ ($i = 1, 2$). As common practice in the analysis of non–flat systems, e.g. [LS00, SJ09], in the sequel we evaluate how computational complexity scales with the number $N_i$ of FSMs in AFSMs $A^i$. We start by evaluating the computational complexity in checking bisimulation equivalence of the flattened systems $M(A^1)$ and $M(A^2)$. As a direct application of Propositions 2.5 and 2.6, one gets the following results.

**Corollary 5.1.** Space complexity in checking $M(A^1) \cong M(A^2)$ is $O(2^{N_1} + 2^{N_2})$.

**Corollary 5.2.** Time complexity in checking $M(A^1) \cong M(A^2)$ is $O((2^{N_1} + 2^{N_2}) \ln(2^{N_1} + 2^{N_2}))$.

The above result quantifies the aforementioned state explosion problem [LS00, SJ09] in the class of AFSMs. We now discuss computational complexity in checking compositional bisimulation.

**Proposition 5.3.** Space complexity in checking $A^1 \cong_c A^2$ is $O(N_1^2 + N_2^2)$.

**Proof.** Direct consequence of Propositions 2.5 and 4.2. □

**Proposition 5.4.** Time complexity in checking $A^1 \cong_c A^2$ is $O((N_1^2 + N_2^2) \ln(N_1 + N_2))$.

**Proof.** By Proposition 4.2, checking $A^1 \cong_c A^2$ reduces to: (1) construct FSMs $M_{A^1}$ and $M_{A^2}$ and, (2) check if $M_{A^1} \cong M_{A^2}$. Regarding (1), time complexity effort reduces to the one of defining functions $H_{A^1}$ and $H_{A^2}$ which amounts to $O((N_1 + N_2)^2)$. Regarding (2), by Proposition 2.6, time complexity in checking $M_{A^1} \cong M_{A^2}$ is given by $O((N_1^2 + N_2^2) \ln(N_1 + N_2))$. Since the last term is dominant over $O((N_1 + N_2)^2)$, the result follows. □

6. Regulation of gene expression in E. coli

Several mathematical models have been proposed in the control systems’ and computer science literature to model genetic regulatory systems, see e.g. [Jon02], and the references therein. We recall directed and undirected graphs, bayesian, boolean and generalized logical networks in the class of discrete systems, and nonlinear, piecewise–linear, qualitative, partial differential equations in the class of continuous systems. Stochastic hybrid systems have been proposed in [JHS+08]. These models can be broadly classified along two
Figure 5. FSMs modeling proteins involved in the AFSM $A$.

orthogonal mathematical paradigms: (i) discrete, vs. continuous, vs. hybrid models; (ii) deterministic, vs. non-deterministic, vs. stochastic models. AFSMs fall in the class of discrete non-deterministic systems. An exhaustive comparison of AFSMs with the aforementioned models is out of the scope of this section. We only mention that the discrete systems proposed in the literature well capture the network of the genes but lack in modeling the dynamics of each gene. In the following we show that AFSMs are appropriate to describe
Figure 6. AFSM $A$ in the left panel and the minimal AFSM $A_{\text{min}}(A)$ in the right panel.

both the genes’ network and the dynamics of each gene. Moreover, we show that the notion of compositional bisimulation can lead to a sensible reduction in the size of the proposed model.

We consider the genetic regulatory system of the single–celled bacterium Escherichia Coli ($E. \ coli$). In the sequel we only report basic facts about this regulatory system; the interested reader can refer to e.g. [LNC93, Rus02, Alo07] for more details. $E. \ coli$ is a single–celled bacterium with a few million of proteins. During its life $E. \ coli$ encounters situations in which production of proteins is required. Each protein is produced by its gene. Each gene is a double strand of the Deoxyribonucleic acid (DNA) which encodes the information needed for the production of a specific protein. The transcription of a gene is the process by which Ribonucleic acid (RNA) polymerase produces messenger RNA (mRNA) corresponding to the sequence of genetic code. The mRNA is then translated into a protein that is known as gene product. The genes whose activity is controlled in response to the needs of the cell, are called regulated genes. Regulated genes require special proteins called effectors or inducers which implement a kind of induction in the target gene. These proteins can bind to DNA and promote RNA transcription. When extracellular stimuli are perceived, such effectors promote RNA transcription and thus protein translation, as requested. Figure 4 describes a transcription network, representing about 10% of the transcription interactions in $E. \ coli$. Each node represents a gene. An edge $(X,Y)$ indicates that the transcription factor encoded in $X$ regulates operon $Y$.

$E. \ coli$ grows in moist soil containing salts which include a source of nitrogen, and a carbon source as glucose. The energy needed for biochemical reactions in $E. \ coli$ is derived from the metabolism of glucose and other secondary sugars including lactose, galactose and arabinose. For simplicity, in the following we focus on (only) the metabolic regulation of lactose, galactose and arabinose, see e.g. [Rus02, LNC93, Alo07]. When lactose is the sole source of carbon in the soil, three proteins are synthesized, which are necessary to metabolize lactose:

- $\beta$–galactosidase ($LacZ$). This enzyme catalyzes splitting of lactose into glucose and galactose and catalyzes isomerization of lactose to allolactose.
- Lactose permease ($LacY$). It is located in the cytoplasmic membrane of $E. \ coli$ and is needed for the active transport of lactose into the cell.
- Transacetylase ($LacA$). This enzyme modifies toxic molecules of lactose to facilitate their removal from the cell.

When glucose is present, on average, only three molecules of $\beta$–galactosidase are present in the cell. This is because the genes of the three proteins are repressed by a protein encoded by gene $LacI$. After entering into the cell, lactose is converted into allolactose through a biochemical reaction that is catalyzed by one of the few copies of $\beta$–galactosidase. Then, allolactose binds to repressor $LacI$ and after its dissociation, genes $LacZ$, $LacY$ and $LacA$ are expressed, thus producing a 1000–fold increase in the concentration of $\beta$–galactosidase. As already mentioned, for lactose to be metabolized, two conditions are needed: presence of lactose and absence
of glucose. The latter is perceived by the cell via the cyclic adenosine monophosphate (cAMP) that is a molecule whose concentration is inversely proportional to that of glucose. This molecule acts as a coactivator in respect of an activator protein, called cAMP receptor protein (CRP). When glucose is absent, the cAMP–CRP complex binds to a specific site near the promoter of the genes for LacZ, LacY and LacA and increases 50 times the transcription of their mRNA. Metabolic regulation of lactose can be formalized as an AFSM (see Figure 6 (Left Panel)) which involves proteins CRP, LacI, LacZ, LacY and LacA. We start by describing the FSM modeling the protein complex CRP–cAMP (Figure 5(a)). Complex CRP–cAMP switches from the inactive state 1 to the active state 2 when input \{cAMP\}, signaling absence of glucose, is perceived. As soon as glucose is perceived by the cell, the FSM switches to the initial state 1. Similarly we can represent the evolution of protein LacI. The corresponding FSM is depicted in Figure 5(c). It consists of three states: state 7, modeling that the protein is disabled, state 8, modeling activation of the protein, and state 9 modeling high activation of the protein. Transcribed proteins LacZ and LacY are illustrated in Figures 5(b)(d). We do not report the FSM of LacA because the mechanism by which LacA reacts to external stimuli is the same as the one of LacY; hence, we assume \( LacY \cong LacA \). If regulator proteins CRP and LacI are disabled, proteins LacZ and LacY are in their basal states 3 and 10, respectively. Both LacZ and LacY switch from states 3 and 10 to low transcription states 4, 5 and respectively 11, 12 if only one of proteins CRP and LacI are activated or equivalently, if either CRP is in state 2 or LacI is in state 8. Finally, LacZ and LacY switch to high transcription states 6 and 13 if both CRP and LacI are in states 2 and 8, respectively. Moreover, when LacZ is in state 6, it induces a transition in LacI from state 8 (modeling activation of protein LacI) to state 9 (modeling high activation of protein LacI).

The regulatory mechanism for the production of proteins capable of recruiting galactose and arabinose is similar to that of lactose. In particular, the galactose system involves proteins GalS, GalE, GalT, GalK. Figures 5(e)(f) reports FSM modeling of GalE and GalS respectively. FSMs of proteins GalT and GalK are not reported because the mechanism by which proteins GalT and GalK react to external stimuli is the same as the one of GalE, from which we assume \( GalE \cong GalT \cong GalK \). The arabinose system involves proteins AraA, AraB, AraC, AraD, AraE, AraF, AraG and AraK. Figures 5(g)(h)(i) reports FSM modeling of AraB, AraE and AraC respectively. FSMs of proteins AraA, AraC, AraD, AraF and AraG are not reported because the external behavior of proteins AraE, AraF, AraG and AraH can be considered as equivalent, which implies \( AraE \cong AraF \cong AraG \cong AraH \); similarly, the external behavior of proteins AraB, AraA and AraD can be considered as equivalent, which implies \( AraB \cong AraA \cong AraD \). The obtained AFSM \( A \) is reported in Figure 6 (left panel).

We conclude this section by computing the minimal bisimilar FSM of \( M(A) \) through the algorithm illustrated in Section 4.1:

- The relation \( R^*(A, A) \) has been computed and the induced equivalence classes are: \{AraC\}, \{CPR\}, \{GalS\}, \{LacZ\}, \{LacI\}, \{LacY, LacA\}, \{GalE, GalT, GalK\}, \{AraA, AraB, AraD\} and \{AraE, AraF, AraG, AraH\}.
- The quotient \( A_{\text{min}}(A) \) has been computed and is illustrated in Figure 6 (right panel).
- By expanding \( A_{\text{min}}(A) \) the FSM \( M(A_{\text{min}}(A)) \) is obtained, which consists of 55,296 states.
- The relation \( R^*(M(A_{\text{min}}(A)), M(A_{\text{min}}(A))) \) is the identity relation.
- The quotient \( M_{\text{min}}(M(A_{\text{min}}(A))) \) coincides with \( M(A_{\text{min}}(A)) \).

The above computations required to run bisimulation algorithms on the collection of FSMs \( M_i \) composing \( A \), whose sets of states sum up to 35 states, and the FSM \( M_A \) induced by \( A \), whose states are 17. A naive approach to compute \( M_{\text{min}}(M(A_i)) \), would apply bisimulation algorithms directly to \( M(A_i) \), which is composed of 4,831,838,828 states.

### 7. Conclusion

In this paper we introduced the class of Arenas of Finite State Machines. We also proposed the notion of compositional bisimulation that provides a method to assess bisimulation equivalence between AFSMs,
without the need of expanding them to the corresponding FSMs and hence, without incurring in the state explosion problem. Future research direction is two–fold. From the theoretical point of view, we will focus on generalizations of the results presented here to non–flat systems exhibiting more general compositional features, as both parallel and sequential composition. From the point of view of the systems’ biology application that we proposed, we will investigate the use of AFSMs for the formal analysis of such systems.

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