Coextension of scalars in operad theory

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Abstract

The functor between operadic algebras given by restriction along an operad map generally has a left adjoint. We give a necessary and sufficient condition for the restriction functor to admit a right adjoint. The condition is a factorization axiom which roughly says that operations in the codomain operad can be written essentially uniquely as operations in arity one followed by operations in the domain operad.

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1 Introduction

A map between (potentially colored) operads yields an associated restriction functor between their respective categories of algebras. The restriction functor is right adjoint to a functor which is a free extension along the operad map. Usually the restriction functor is not a left adjoint. However, in some interesting examples, it is. Here we provide a necessary and sufficient condition for the existence of a right adjoint, and give an explicit construction of the right adjoint in the case that the criterion is satisfied.

The criterion is the existence and uniqueness of a certain kind of factorization. The main theorems below are stated in considerably more generality, but for the purposes of the introduction, we restrict to the monochrome version. For the time being, assume that the ground category is a standard one, for instance abelian groups, topological spaces, spectra, or sets. We use the evocative notation \(\ll\) for the usual composition product of collections (see Definition 2.5 and Notation 2.22).

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Definition 1.1 Let $\phi : \mathcal{P} \to \mathcal{Q}$ be a map of (monochrome) operads. We say $\phi$ is a monoidal extension if the composition

$$\mathcal{P} \odot_{\mathcal{P}(1)} \mathcal{Q}(1) \to \mathcal{Q} \odot_{\mathcal{Q}} \mathcal{Q} \cong \mathcal{Q}$$

is an isomorphism.

Theorem 1.2 Let $\phi : \mathcal{P} \to \mathcal{Q}$ be a map of (monochrome) operads. The restriction functor $\phi^*$ from $\mathcal{Q}$-algebras to $\mathcal{P}$-algebras is a left adjoint if and only if $\phi$ is a monoidal extension.

This result is, in fact, constructive. See Theorem 3.8 and Sect. 4.

An illustrative special case (see Example 4.2) is when $\phi$ is a homomorphism $R \to S$ of rings. It is well known that the restriction from $S$-modules to $R$-modules admits both a left adjoint (extension of scalars) and a right adjoint (coextension of scalars). A ring homomorphism is automatically a monoidal extension, as the map $R \otimes_R S \to S \otimes_S S \cong S$ is an isomorphism of abelian groups.

This paper is organized as follows. In Sect. 2 we fix notation for and describe the structure of colored collections and colored operads. Once these necessities are out of the way, in Sect. 3 we state the main theorem, including the construction of the right adjoint. Section 4 contains several examples, including the inclusion of the Gerstenhaber operad into the Batalin–Vilkovisky operad, the inclusion of the colored operad governing operads into the colored operad governing cyclic operads, and the inclusion of the associative operad into the colored operad governing those operads which are concentrated in positive arity. In Sects. 5 and 6, we prove necessity and sufficiency of our condition.

This paper is weakly inspired by an example in our paper [8] which gives a sufficient condition for the existence of a right-induced model structure in the special case where the right adjoint is also a left adjoint. There, we gave the example of the forgetful functor from cyclic operads to operads (see Sect. 4.7 below). The existence of a right adjoint to this forgetful functor, originally due to Templeton [25], surprised us as well as a number of experts with whom we discussed it. This forgetful functor is in fact a restriction functor along a map of colored operads, and we became curious about what features of the governing operad map enabled the existence of the right adjoint.

Ward [26, Proposition 7.9], gave a sufficient condition for the existence of this kind of right adjoint (assuming some restrictions on the ground category). Ward’s motivations are different; his conditions are significantly more restrictive but they ensure not only the existence of the adjoint but its coincidence with a different functor which exists independently. Our condition here is both necessary and sufficient and applies in more general ground categories. See Theorems 3.4 and 3.8 for the full details.

The analogous question for Lawvere theories was studied by Wraith in [27, §9], but the characterization there is quite different to ours. It would be interesting to compare them in the area of overlap, namely when considering maps of monochrome operads in sets.

We are not aware of other references in the literature that study this question.

Conventions

We will use the notation $(\mathcal{E}, \otimes, 1)$ for a bicomplete symmetric monoidal category, which we abbreviate as $\mathcal{E}$. All such $\mathcal{E}$ will be assumed closed and we write $[-, -]$ for the internal hom. We use $\mathbb{N}$ to denote the set of nonnegative integers. For $n \in \mathbb{N}$, we write $[n]$ for the set $\{1, 2, \ldots, n\}$ and $\Sigma_n$ for the symmetric group on $n$ letters, $\Sigma_n = \text{Aut}([n])$. 
2 Operads and collections

In this section we establish conventions for colored operads. The parts for fixed colors are fairly standard. Some of the machinery related to the change of colors is less standard, although not much harder.

2.1 Collections

Definition 2.2 Suppose that $A$ is a set of colors, and let $S_A^{op}$ denote the groupoid whose objects are lists $a := a_1, \ldots, a_n$ (where $n$ varies and $a_i \in A$) and whose morphisms are

$$a = a_1, \ldots, a_n \xrightarrow{\sigma} a_{\sigma(1)}, \ldots, a_{\sigma(n)} =: a\sigma$$

where $\sigma \in \Sigma_n$ is a permutation.

- If $B$ is another color set, then an $(A, B)$-collection $X$ is an object in the functor category $\mathcal{E}_{A}^{op} \times B$.

- The maps of $(A, B)$-collections are natural transformations between the functors.

- If $A = \emptyset$, we call an $(\emptyset, B)$-collection a $B$-object, and write $X_b$ for $X(\emptyset; b)$.

- We call an object $(a_1, \ldots, a_n; b)$ in $S_A^{op} \times B$ an $(A, B)$-profile or just a profile if $A$ and $B$ are clear from context. We will alternately write such a profile as $(a_1; b) \lor (\emptyset; b)$, so if $X$ is an $(A, B)$-collection we will write $X((a_1; b))$ or $X(a_1; b)$ for the value of $X$ at the indicated profile.

Concretely, an $(A, B)$-collection $X$ consists of:

1. for each (possibly empty) list $(a_1, \ldots, a_n)$ of colors in the color set $A$ and each color $b$ in $B$, an object $X(a_1, \ldots, a_n; b)$ of $\mathcal{E}$, and
2. for each element $\sigma$ of the symmetric group $\Sigma_n$, color $b$ in $B$, and tuple $(a_1, \ldots, a_n)$ in $A$, a morphism $\sigma^*$ in $\mathcal{E}$ of the form

$$X(a_1, \ldots, a_n; b) \rightarrow X(a_{\sigma(1)}, \ldots, a_{\sigma(n)}; b)$$

such that $\id^* = \id$ and $\sigma^* \tau^* = (\tau \sigma)^*$ for all $\sigma$ and $\tau$ in $\Sigma_n$.

A map $X \rightarrow Y$ consists of, for each color $b$ in $B$ and each list $(a_1, \ldots, a_n)$ of colors in $A$, a morphism in $\mathcal{E}$ from $X(a_1, \ldots, a_n; b)$ to $Y(a_1, \ldots, a_n; b)$ such that the evident $\Sigma_n$ equivariance conditions are satisfied.

Notation 2.3 Let $f : A \rightarrow B$ be a map between color sets. We can build two collections out of $f$ as follows. We build an $(A, B)$-collection also called $f$:

$$f(a_1, \ldots, a_n; b) = \begin{cases} 1 & \text{if } n = 1 \text{ and } f(a_1) = b, \\ \emptyset & \text{otherwise}, \end{cases}$$

where $\emptyset$ is the initial object of $\mathcal{E}$. We also have a $(B, A)$-collection $\bar{f}$

$$\bar{f}(b_1, \ldots, b_n; a) = \begin{cases} 1 & \text{if } n = 1 \text{ and } f(a) = b_1, \\ \emptyset & \text{otherwise}. \end{cases}$$

When $f$ is invertible, the collections $f^{-1}$ and $\bar{f}$ are canonically isomorphic.

By convention we use the notation $1_A$ for the $(A, A)$-collection $\id_A$. 
More generally, if \( p \) is any span of sets from \( A \) to \( B \), then there is an associated \((A, B)\)-collection concentrated in arity one. Writing \( p : U \to A \times B \), this collection is given in profile \((a; b)\) by \( \prod_{p^{-1}(a, b)} 1 = p^{-1}(a, b) \cdot 1 \) (where the \( \cdot \) denotes copower). Then \( f \) comes from the span \((\text{id}, f) : A \to A \times B \) while \( \tilde{f} \) comes from the span \((f, \text{id}) : A \to B \times A \).

Above we used the groupoid \( S^\text{op}_A \) of lists of colors. We also have the groupoid \( S_A \) whose morphisms are \( \sigma : a \to \sigma a := a_{\sigma^{-1}(1)} \ldots a_{\sigma^{-1}(n)} \). Notice there are functors from \( S_A \) to the discrete category \( \mathbb{N} = \{0, 1, 2, \ldots\} \) that take a list \( a \) to its length. Both categories \( S^\text{op}_A \) and \( S_A \) are strict monoidal categories (in fact, free strict symmetric monoidal categories). The following definition is essentially adapted from [17, §2], and appears in the colored case when \( A = B \) in [28, §3.1].

**Definition 2.4** (Day powers) Let \( Y \) be an \((A, B)\)-collection.

- For each \( b \in S_B \) of length \( m \) and each \( a \in S^\text{op}_A \), there is a functor

\[
F : \left\{ \prod_{j=1}^m S^\text{op}_A \right\} \times \left\{ \prod_{j=1}^m S_A \right\} \to \mathcal{E}
\]

having value

\[
F([u_j], [v_j]) = S^\text{op}_A (u_1 u_2 \ldots u_m; a) \cdot \bigotimes_{j=1}^m Y(v_{b_j}).
\]

Here the \( \cdot \) denotes copower and \( u_1 u_2 \ldots u_m \) is the concatenation of the lists.

- If \( a \) is a list of elements in \( A \) and \( b \) is a list of elements in \( B \), we define an object

\[
Y^b(a) := \int_{[w_j] \in \prod_{j=1}^m S^\text{op}_A} S^\text{op}_A (w_1 w_2 \ldots w_m; a) \cdot \bigotimes_{j=1}^m Y(w_{b_j})
\]

of \( \mathcal{E} \) as the coend of the bifunctor \( F \).

1 There is an evident naturality in the \( a \) variable given by postcomposition; there is also a naturality in the \( b \) variable which makes this a functor \( S^\text{op}_A \times S_B \to \mathcal{E} \).

**Definition 2.5** (Kelly) Suppose that \( X \) is a \((B, C)\)-collection and \( Y \) is an \((A, B)\)-collection. The **composition product** of \( X \) and \( Y \) is defined to be the \((A, C)\)-collection

\[
(X \triangleleft Y)(\frac{e}{C}) = \int_{b \in S_B} X(\frac{b}{C}) \otimes Y^b(a).
\]

**Remark 2.6** The functor \( (-) \triangleleft Y \) goes from \((B, C)\)-collections to \((A, C)\)-collections and has a right adjoint, temporarily denoted \( \{Y, -\} \). If \( Z \) is an \((A, C)\)-collection, then the \((B, C)\)-collection \( \{Y, Z\} \) is given by the end

\[
\{Y, Z\}(\frac{e}{C}) = \int_{a \in S^\text{op}_A} \left[ Y^b(a), Z(\frac{a}{C}) \right]
\]

where square brackets denote the internal hom of \( \mathcal{E} \). We will frequently need that \( (-) \triangleleft Y \) is a left adjoint in what follows, but we will never explicitly use the formula for \( \{Y, Z\} \).

In general \( X \triangleleft (-) \) is not a left adjoint functor (see [17, p.7]), but is when \( X \) is concentrated in arity one (see Lemma 2.9).

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1 This superscript notation matches with the \( S^m \) and \( (S^m)k \) appearing in [17].
Lemma 2.7 Suppose $Y$ is concentrated in arity one. Then $Y^b(a)$ is naturally isomorphic to

$$\bigsqcup_{\sigma \in \Sigma_m} \bigotimes_{j=1}^m Y(a_{\sigma^{-1}(j)}; b_j).$$

**Proof** By definition,

$$Y^b(a) \cong \int \{ w_j \in \prod \mathbb{S}^\text{op} A(w_1 w_2 \ldots w_m; a) \cdot \bigotimes_{j=1}^m Y(w_j) \},$$

and the indexing category is discrete so this is isomorphic to

$$\bigsqcup_{w \in A^m} \bigsqcup_{\sigma \in \Sigma_m} \mathbb{S}(w; w) \bigotimes_{j=1}^m Y(w_j).$$

We can identify $\mathbb{S}(w; w)$ with the set of $\sigma \in \Sigma_m$ so that $w_j = a_{\sigma^{-1}(j)}$. The set of pairs $(w, \sigma) \in A^m \times \Sigma_m$ satisfying $w_j = a_{\sigma^{-1}(j)}$ is just in bijection with $\Sigma_m$. \qed

On the other hand, if $X$ is concentrated in arity one and we are investigating $X \triangleleft (\_)$, then we are only concerned with the case of $m = 1$. In general we have

$$Y^b(a) = \int \mathbb{S}^\text{op} A(w; a) \cdot Y^b(a) \cong Y^b(a)$$

and we see that

$$(X \triangleleft Y)(\underline{a}) = \int_{b \in B} X(b) \otimes Y^b(a) = \bigsqcup_{b \in B} X(b) \otimes Y^b(a).$$

**Notation 2.8** Suppose that $X$ is a $(B, C)$-collection concentrated in arity one and let $Z$ be an $(A, C)$-collection. Define $(X, Z)$ to be the $(A, B)$-collection given by

$$(X, Z)(\underline{a}) = \prod_{c \in C} X(b) \otimes Y^b(a) = \prod_{c \in C} \left[ X(b), Z(b) \right].$$

Lemma 2.9 Suppose that $X$ is a $(B, C)$-collection concentrated in arity one and let $A$ be a set of colors. The functor $(X, \_)$ from $(A, C)$-collections to $(A, B)$-collections is right adjoint to $X \triangleleft (\_)$.

When $A = B = C$, Definition 2.5 agrees with the colored circle product $\circ$ from [13, 2.2.3], the colored symmetric circle product $\circ^S$ from [28, §3.1], and the $\square$-product from [4, 7.2]. Of course the category of $(A, A)$-collections equipped with this product is a monoidal category (see [17, §3], [28, Proposition 2.1.8], or the sources above). Essentially the same proof of that fact shows that the collection of $(A, B)$-collections as $A$ and $B$ varies forms a bicategory. Indeed, there are natural associator and unitor isomorphisms for $\triangleleft$ and $1_A$ and we have:

**Definition 2.10** The bicategory of colored collections $\mathbf{CoCo}$ has data defined as follows.

- The 0-cells are sets of colors.
- The 1-cells from color $A$ to color $B$ are $(A, B)$-collections.
- The 2-cells from an $(A, B)$-collection $X$ to an $(A, B)$-collection $Y$ are maps of collections from $X$ to $Y$. 

\[ Springer \]
The horizontal composition is \(<\).

The identity morphism for color \(A\) is \(1_A\).

The associators and unitors will not be described explicitly.

Recall from Notation 2.3 that every function determines a collection in two different ways. When considered as 1-cells of the bicategory \(\mathcal{CoCo}\) these two collections are adjoint.

**Example 2.11** Let \(f : A \rightarrow B\) be a function between color sets. Then the compositions of the collections \(f\) and \(\tilde{f}\) are as follows:

\[
(f \triangleright \tilde{f}) (b_1, \ldots, b_n; b) \cong \begin{cases} \bigcup_{a \in f^{-1}(b)} 1 & \text{if } n = 1 \text{ and } b_1 = b, \\ \emptyset & \text{otherwise.} \end{cases}
\]

\[
(\tilde{f} \triangleright f) (a_1, \ldots, a_n; a) \cong \begin{cases} 1 & \text{if } n = 1 \text{ and } f(a_1) = f(a), \\ \emptyset & \text{otherwise.} \end{cases}
\]

Notice that these collections may also be obtained by first composing the span \((id, f)\) with its reverse \((f, id)\) and then taking the corresponding collection.

**Definition 2.12** The canonical counit \(\epsilon_f\) of \(f\) is the map of collections \(f \triangleright \tilde{f} \rightarrow 1_B\) induced in each profile by the fold map \(\bigcup 1 \rightarrow 1\). The canonical unit \(\eta_f\) of \(f\) is the inclusion of collections \(1_A \rightarrow \tilde{f} \triangleright f\).

**Lemma 2.13** If \(f\) is a map of colors \(A \rightarrow B\), then the one-cell \(f\) in the bicategory \(\mathcal{CoCo}\) is left adjoint to the one-cell \(\tilde{f}\). As a consequence, we have the following induced adjunctions:

- The functor \(f \triangleright -\) from \((C, A)\)-collections to \((C, B)\)-collections is left adjoint to the functor \(\tilde{f} \triangleright -\) in the opposite direction.
- The functor \(-\triangleright \tilde{f}\) from \((A, C)\)-collections to \((B, C)\)-collections is left adjoint to the functor \(-\triangleright f\) in the opposite direction.

These functors are equivalences of categories if and only if \(f\) is invertible.

**Proof**  The canonical unit and counit are compatible: the compositions

\[
f \xrightarrow{f \triangleright \eta_f} \tilde{f} \triangleright f \xrightarrow{\epsilon_f \triangleright f} f
\]

and

\[
\tilde{f} \xrightarrow{\eta_f \triangleright \tilde{f}} f \triangleright \tilde{f} \xrightarrow{\tilde{f} \triangleright \epsilon_f} \tilde{f}
\]

are identity maps. The canonical unit and counit are isomorphisms if and only if \(f\) is invertible.

\(\square\)

### 2.14 Operads

We now give a definition of (colored) operad which is convenient for our purposes, as well as several descriptions of maps of such.

**Definition 2.15** A colored operad is a monad (as in [3, 5.4.1]) in the bicategory of colored collections (see also [24], which we will use below for morphisms of monads). In other words, it is a choice of color set \(A\) and a monoid in the monoidal category consisting of \((A, A)\)-collections along with \(<\) and \(1_A\). More explicitly, the data is given by an \(A\)-colored...
collection \( P \) and maps of collections \( \mu_P : P \triangleleft P \to P \) (called composition) and \( \eta_P : 1_A \to P \) (called the unit) which satisfy associativity and unit constraints. This definition essentially appeared in the monochrome case in [17,23], while the colored case appeared in [4, §7.3].

A map of colored operads \((A, P) \to (B, Q)\) is a pair \((f, \phi)\) where \( f \) is a function from \( A \) to \( B \) and \( \phi \) is a 2-cell in \( \text{CoCo} \) from \( f \triangleleft P \) to \( Q \triangleleft f \) so that the following two diagrams commute (up to suppressed associators and unitors).

In other words, it is a pair \((f, \phi)\) such that \((f, \phi)\) is a monad opfunctor [24, §4] or colax map of monads [19, §6.1].

**Remark 2.16** (Warning) Not all monad opfunctors are maps of operads, because the one-cell \( f \) must be of a particular form, i.e., must come from a map of color sets. As pointed out to us by Rune Haugseng, there is a double categorical framework enhancing \( \text{CoCo} \) which takes a little more setup in which the presentation is more uniform.

Unpacking the definition further using the adjunctions of Lemma 2.13, the data of \( \phi \) consists of the following. For each profile \((a_1, \ldots, a_k; a)\) of colors in \( A \), we are given a map in \( \mathcal{E} \) from \( P(a_1, \ldots, a_k; a) \) to \( Q(f(a_1), \ldots, f(a_k); f(a)) \). This map is called the component of \( \phi \) at \((a_1, \ldots, a_k; a)\). Commutativity of the triangle says that the component of \( \phi \) at \((a; a)\) should take the unit at \( a \) to the unit at \( f(a) \). Commutativity of the pentagon says that this collection of maps should intertwine the composition of \( P \) and the composition of \( Q \).

**Example 2.17** If \( A \) is any set of colors, then the \( A \)-colored collection \( 1_A \) is an operad with \( \eta \) the identity morphism and \( \mu \) an instance of the unitor isomorphism.

**Remark 2.18** (Alternative presentations of maps of colored operads) Let’s give two equivalent definitions of a map from \((A, P)\) to \((B, Q)\). In both cases we will have pair consisting of a function \( f : A \to B \) and also a 2-cell in \( \text{CoCo} \). That these are equivalent to the original definition is an exercise using the fact that the 1-cell \( f \) is left adjoint to the 1-cell \( \bar{f} \) in the bicategory \( \text{CoCo} \) by Lemma 2.13.

1. The 2-cell \( \chi \) goes from \( P \) to \( \bar{f} \triangleleft Q \triangleleft f \) and the following two diagrams commute.
2. The 2-cell $\psi$ goes from $P \triangleleft \bar{f}$ to $\bar{f} \triangleleft Q$ and the following two diagrams commute.

![Diagram]

The first of these redefinitions is just saying that a colored operad map $(A, P) \to (B, Q)$ is the same thing as a colored operad map $(A, P) \to (A, \bar{f} \triangleleft Q \triangleleft f)$ which is the identity on color sets.

Note that there is little distinction between the three versions of colored operad map when $f$ is the identity function on $A$.

### 2.19 Modules and algebras

Suppose that $(A, P)$ and $(B, Q)$ are two colored operads. As with monads in any bicategory, there are good notions of left $P$-modules, right $Q$-modules, and $P$-$Q$ bimodules.

**Definition 2.20** Let $(A, P)$ and $(B, Q)$ be colored operads.

- A left $P$-module consists of a $(C, A)$-collection $X$ for some color set $C$ together with a map of $(C, A)$-collections $\lambda : P \triangleleft X \to X$ so that the diagrams

  \[ \begin{array}{ccc}
  P \triangleleft P \triangleleft X & \xrightarrow{\mu_{P \triangleleft X}} & P \triangleleft X \\
  \downarrow \mu_{P \triangleleft X} & & \downarrow \lambda \\
  P \triangleleft X & \xrightarrow{\lambda} & X
  \end{array} \]

  commute. We also call such a $\lambda$ a left $P$-action.

- Likewise, a right $Q$-module consists of a $(B, C)$-collection $X$ for some color set $C$ together with a map of $(B, C)$-collections $\rho : X \triangleleft Q \to X$ so that the diagrams

  \[ \begin{array}{ccc}
  X \triangleleft Q \triangleleft X & \xrightarrow{X \triangleleft \eta_{Q \triangleleft X}} & X \triangleleft Q \\
  \downarrow \rho \triangleleft Q & & \downarrow \rho \\
  X \triangleleft Q & \xrightarrow{\rho} & X
  \end{array} \]

  commute. We also call such a $\rho$ a right $Q$-action.

- A $P$-$Q$ bimodule is a $(B, A)$-collection which is both a left $P$-module and a right $Q$-module so that the square

  \[ \begin{array}{ccc}
  P \triangleleft X \triangleleft Q & \xrightarrow{P \triangleleft \eta_{Q \triangleleft X}} & P \triangleleft X \\
  \downarrow \lambda \triangleleft Q & & \downarrow \lambda \\
  X \triangleleft Q & \xrightarrow{\rho} & X
  \end{array} \]

  commutes. Write $P$-$\text{mod}$-$Q$ for the category of bimodules.

- An algebra over $P$ is an $(\emptyset, A)$-collection $\mathcal{A}$ (that is, an $A$-object), equipped with the structure of a left $P$-module. We write $P$-$\text{alg}$ for the category of $P$-algebras.
Remark 2.21 1. Concretely, the data of an algebra \( A \) is given by an \( A \)-indexed family of \( E \)-objects \( A_a := A(\_; a) \) along with maps
\[
P(a_1, \ldots, a_n; a) \otimes A_{a_1} \otimes \cdots \otimes A_{a_n} \to A_a
\]
which satisfy associativity, unitality, and equivariance constraints.

2. Note that \( P \)-alg is nothing but \( P \)-mod-\( Q \) for \( Q \) the initial colored operad, that is, the unique operad with an empty color set. Many of our results are stated for \( P \)-alg, but hold more generally for \( P \)-mod-\( R \) where \( R \) is some operad which stays fixed throughout the discussion.

3. Every \((C, A)\)-collection \( X \) is automatically a right \( 1_C \)-module, in a way that is compatible with any left \( P \)-module structure on \( X \).

Notation 2.22 Suppose given three colored operads \( P, Q, \) and \( R \). Then there is an induced functor
\[
\llangle_Q : P\text{-mod}-Q \times Q\text{-mod}-R \to P\text{-mod}-R
\]
given at the level of collections as a reflexive coequalizer
\[
X \llangle Q Y \xrightarrow{X \llangle \lambda Y} X \llangle Y \xrightarrow{Y \llangle \rho} X \llangle Q Y.
\]
The special case when \( Q = 1_B \) gives \( X \llangle 1_B Y \cong X \llangle Y \).

Proposition 2.23 If \( Q \) and \( R \) are two colored operads, then
\[
(X \llangle Q Y) \llangle_R Z \cong X \llangle_Q (Y \llangle_R Z).
\]

Proof This essentially follows from the fact that \( W \llangle (-) \) preserves reflexive coequalizers while \((-) \llangle W \) preserves all colimits (in particular, reflexive coequalizers).

2.24 Restriction and extension of algebras

A map of operads induces a well-known adjunction between their categories of algebras. In this section we review this procedure, describing it explicitly and categorically.

Lemma 2.25 Let \((f, \phi)\) be a map of operads from \( P \) to \( Q \). There is an induced left \( P \)-action \( \lambda_\phi \) on \( \tilde{f} \llangle Q \).

Proof We utilize the second adjoint definition from Remark 2.18 and assume we have a 2-cell \( \psi : P \llangle f \to \tilde{f} \llangle Q \) so that the two diagrams

\[
\begin{array}{ccc}
P \llangle f & \xrightarrow{\mu_{P\llangle f}} & P \llangle Q \\
\downarrow \psi & & \downarrow \psi \\
\tilde{f} \llangle Q & \rightarrow & \tilde{f} \llangle Q
\end{array}
\]

commute. Define \( \lambda \) to be the composite
\[
P \llangle f \llangle Q \xrightarrow{\psi \llangle Q} \tilde{f} \llangle Q \llangle Q \xrightarrow{\tilde{f} \llangle \mu_Q} \tilde{f} \llangle Q.
\]
We see that \( \lambda(P \triangleleft \psi \triangleleft Q) = \lambda(\mu \triangleleft \tilde{f} \triangleleft Q) \) holds because of commutativity of the following diagram.

\[
\begin{array}{cccccc}
P \triangleleft P \triangleleft \tilde{f} \triangleleft Q & \xrightarrow{\rho \triangleleft \psi \triangleleft Q} & P \triangleleft \tilde{f} \triangleleft Q \triangleleft Q & \xrightarrow{\rho \triangleleft \psi \triangleleft \mu \triangleleft Q} & P \triangleleft \tilde{f} \triangleleft Q \\
\downarrow & & \downarrow \psi \triangleleft \psi \triangleleft Q & & \downarrow \psi \triangleleft Q \\
P \triangleleft \tilde{f} \triangleleft Q & \xrightarrow{\psi \triangleleft Q} & \tilde{f} \triangleleft Q \triangleleft Q & \xrightarrow{\tilde{f} \triangleleft \psi \triangleleft \mu \triangleleft Q} & \tilde{f} \triangleleft Q \triangleleft Q \\
\end{array}
\]

Here the left-hand rectangle commutes by our assumption on \( \psi \), while the two small squares commute by naturality.

Compatibility with the unit holds by the diagram

\[
\begin{array}{cccccc}
P \triangleleft \tilde{f} \triangleleft Q & \xrightarrow{\eta \triangleleft \tilde{f} \triangleleft Q} & \tilde{f} \triangleleft Q \triangleleft Q & \xrightarrow{\tilde{f} \triangleleft \psi \triangleleft \eta \triangleleft Q} & \tilde{f} \triangleleft Q \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\tilde{f} \triangleleft Q & \xrightarrow{\psi \triangleleft Q} & \tilde{f} \triangleleft Q \triangleleft Q & \xrightarrow{\tilde{f} \triangleleft \psi \triangleleft Q} & \tilde{f} \triangleleft Q \\
\end{array}
\]

\( \square \)

**Remark 2.26** (Induced right module structure) Similarly, given an operad map \( P \to Q \) there is an induced right \( P \)-action \( \rho \phi \) on \( Q \triangleleft f \). The proof is akin to the proof of Lemma 2.25, except one should use the original definition of operad map rather than its adjoint version.

**Remark 2.27** (The usual adjunction between algebras) Suppose that \( (f, \phi) : P \to Q \) is a map of operads. In light of Lemma 2.25 there is a functor \( \phi^* \) from \( Q \)-algebras to \( P \)-algebras given on underlying collections by

\[
B \mapsto (\tilde{f} \triangleleft Q) \triangleleft Q B \cong \tilde{f} \triangleleft B,
\]

with action induced by the left \( P \)-module structure on \( \tilde{f} \triangleleft Q \).

Similarly, in light of Remark 2.26, there is a functor \( \phi_! \) from \( P \)-algebras to \( Q \)-algebras given on underlying collections by

\[
A \mapsto (Q \triangleleft f) \triangleleft P A.
\]

The outcome of this process retains the evident left \( Q \)-action.

Using the canonical unit and counit of \( f \), it can be shown that \( \phi_! \) is left adjoint to \( \phi^* \).

We are interested in the question of when \( \phi^* \) admits an exceptional \textit{right} adjoint \( \phi_* \) in addition to the left adjoint \( \phi_! \). We next explore one situation where this occurs, namely when \( P \) and \( Q \) are actually categories and \( \phi_* \) is right Kan extension.

### 2.28 Categories as operads

**Definition 2.29** Let \( P \) be an \( A \)-colored operad. The \textit{underlying category} \( |P| \) is the \( \mathcal{E} \)-enriched category whose

- objects are the elements of the color set \( A \),

\( \square \) Springer
– morphism object between \( c \) and \( d \) is \( \mathcal{P}(c; d) \), and
– whose unit and composition are given by the unit and the restriction of the composition of \( \mathcal{P} \).

In the same spirit, if \( X \) is an \((A, B)\)-collection, we will write \(|X|\) for the corresponding collection that is concentrated in arity one. That is, \(|X|\) is the collection with

\[
|X|(a_1, \ldots, a_n; b) = \begin{cases} 
X(a_1; b) & \text{if } n = 1 \\
\emptyset & \text{otherwise.}
\end{cases}
\]

The underlying category assignment \(|−|\) is a functor, and is right adjoint to the inclusion of categories into operads. We can also think of the underlying category \(|\mathcal{P}|\) as an \(A\)-colored suboperad of \(\mathcal{P}\) via the counit of the adjunction, which we will do without comment or change of notation.

Let \(\mathcal{M}\) and \(\mathcal{N}\) be categories enriched in \(\mathcal{E}\) with object sets \(A\) and \(B\) respectively. Suppose that \(X\) is a \((B, A)\)-collection concentrated in arity one, which moreover comes with the structure of an \(\mathcal{M} \cdot \mathcal{N}\) bimodule. In this case, we can regard \(X\) as an \(\mathcal{E}\)-functor

\[
\mathcal{N}^{\text{op}} \times \mathcal{M} \rightarrow \mathcal{E}
\]

which on objects sends \((b, a)\) to \(X(b; a) = X^{(b)}_{(a)}\).

Suppose that \(Y\) is a \((C, B)\)-collection which is a left \(\mathcal{N}\)-module. Concretely, \(X \triangleright_{\mathcal{N}} Y\) (from Notation 2.22) is given by the coend

\[
(X \triangleright_{\mathcal{N}} Y)(^a_c) = \int^{b \in \mathcal{N}} X^{(b)}_{(a)} \otimes Y^{(a)}_{(b)}
\]

and the left \(\mathcal{M}\)-action is given by

\[
\mathcal{M}(^a_{a'}) \otimes \int^{b \in \mathcal{N}} X^{(b)}_{(a)} \otimes Y^{(a')}_{(b)} \cong \int^{b \in \mathcal{N}} \mathcal{M}(^a_{a'}) \otimes X^{(b)}_{(a)} \otimes Y^{(a')}_{(b)} \rightarrow \int^{b \in \mathcal{N}} X^{(b)}_{(a')} \otimes Y^{(a')}_{(b)}
\]

Our primary focus going forward will be the case where \(C\) is the empty set.

**Lemma 2.30**  Let \(\mathcal{M}\) and \(\mathcal{N}\) be \(\mathcal{E}\)-enriched categories with objects \(A\) and \(B\) respectively. Let \(X\) be a \((B, A)\)-collection concentrated in arity one, equipped with the structure of an \(\mathcal{M} \cdot \mathcal{N}\) bimodule. Then the functor

\[
X \triangleright_{\mathcal{N}} (−) : \mathcal{N} \cdot \text{alg} \rightarrow \mathcal{M} \cdot \text{alg}
\]

admits a right adjoint.

**Notation 2.31**  The right adjoint is called \([[X, −]]^{\mathcal{M}}\). As the category \(\mathcal{M}\) will always be clear from context, we simply use the shorthand \([[X, −]]\) for this functor. That is, the adjunction from Lemma 2.30 takes the form

\[
\begin{array}{ccc}
\mathcal{N} \cdot \text{alg} & \overset{\scriptscriptstyle{\perp}}{\rightarrow} & \mathcal{M} \cdot \text{alg} \\
\downarrow & & \downarrow \scriptscriptstyle{[[X, −]]} \\
\end{array}
\]
Proof of Lemma 2.30 This is a categorification of a standard base-change adjunction and follows from the adjunction of the closed monoidal structure on \( \mathcal{E} \). In more detail, suppose that \( Z \) is an \( \mathcal{M} \)-algebra. Define the underlying \( \mathcal{B} \)-object of \([X, Z]\) to be the equalizer of the diagram

\[
\begin{align*}
(X, \langle \mathcal{M}, Z \rangle) & \cong \langle X, Z \rangle \\
\langle X, Z \rangle & \longrightarrow \langle \mathcal{M} \triangleleft X, Z \rangle
\end{align*}
\]

where \( Z \rightarrow \langle \mathcal{M}, Z \rangle \) is the adjoint, from Lemma 2.9, of the structure map \( \mathcal{M} \triangleleft Z \rightarrow Z \).

Let us now turn to the \( \mathcal{N} \)-action this \( \mathcal{B} \)-object. To produce the map \( \mathcal{N} \triangleleft v[X, Z] \rightarrow v[X, Z] \) we use the fact that \( \mathcal{N} \triangleleft (\cdot) \) is left adjoint to \( \langle \mathcal{N}, (\cdot) \rangle \). The right adjoint \( \langle \mathcal{N}, (\cdot) \rangle \) commutes with the equalizer, and by adjunction we see that it is equivalent to provide a map

\[
\langle X, Z \rangle \rightarrow \langle \mathcal{N}, v[X, Z] \rangle \cong \langle X \triangleleft \mathcal{N}, Z \rangle
\]

which is then supplied by pulling back along the right action \( X \triangleleft \mathcal{N} \rightarrow X \).

\[\Box\]

Remark 2.32 If \((\mathcal{C}, \mathcal{R})\) is an operad, then the proof of Lemma 2.30 can be readily adapted to provide an adjunction between \( \mathcal{N} \)-mod-\( \mathcal{R} \) and \( \mathcal{M} \)-mod-\( \mathcal{R} \). The lemma we have stated is just the special case when \( \mathcal{R} \) is the initial colored operad (that is, when \( \mathcal{C} = \emptyset \)). Unraveling the equalizer description from the proof reveals that the underlying \((\mathcal{C}, \mathcal{B})\)-collection of \([X, Z]\) may be written in end notation as

\[
[X, Z](\xi) = \int_{a \in \mathcal{M}} [X^{(b)}(a), Z(\xi a)]
\]

which compares favorably with the formula from Notation 2.8.

Corollary 2.33 Let \((f, \phi)\) be a map of colored operads from \( \mathcal{P} \) to \( \mathcal{Q} \). There is an adjunction between \( \vert\mathcal{Q}\vert\)-algebras and \( \vert\mathcal{P}\vert\)-algebras with left adjoint

\[
[\mathcal{X}, \mathcal{Z}] \rightarrow \langle \mathcal{N}, [\mathcal{X}, \mathcal{Z}] \rangle \cong [\mathcal{X} \triangleleft \mathcal{N}, \mathcal{Z}]
\]

and right adjoint

\[
[\mathcal{X}, \mathcal{Z}] \rightarrow \langle \mathcal{N}, [\mathcal{X}, \mathcal{Z}] \rangle \cong [\mathcal{X} \triangleleft \mathcal{N}, \mathcal{Z}]
\]

Proof This is an application of Lemma 2.30 and follows by taking \( \mathcal{M} = \vert\mathcal{P}\vert \), \( \mathcal{N} = \vert\mathcal{Q}\vert \), and \( X = f \triangleleft \vert\mathcal{Q}\vert \).

\[\Box\]

Remark 2.34 Suppose \((f, \phi) : \mathcal{P} \rightarrow \mathcal{Q}\) is a map of operads and that \( \mathcal{B} \) is a \( \mathcal{Q}\)-algebra. We can either restrict the action, considering \( \mathcal{B} \) as a \( \vert\mathcal{Q}\vert\)-algebra, and then apply the functor \( \phi^* \), or we can apply \( \phi^* \) to \( \mathcal{B} \) as in Remark 2.27 and then restrict to get a \( \vert\mathcal{P}\vert\)-algebra. These two procedures coincide.

Remark 2.35 (The adjunctions \( \phi_! \dashv \phi^* \dashv \phi_* \) in the categorical situation) Let \((f, \phi)\) be a map of categories from \( \mathcal{P} \) to \( \mathcal{Q} \), where \( \mathcal{Q} \) has color set \( \mathcal{B} \).

By Remark 2.26, \( \mathcal{Q} \triangleleft f \) has a \( \mathcal{Q}\P \)-bimodule structure. The functor \( [\mathcal{Q} \triangleleft f, (\cdot)] \) is naturally isomorphic to \( \phi^* \). This can be seen at the level of underlying collections by the following formal manipulation which uses Lemma 2.13:

\[
[\mathcal{Q} \triangleleft f, \mathcal{B}] \cong [\mathcal{Q}, f \triangleleft \mathcal{B}] \\
\cong [1_{\mathcal{B}}, f \triangleleft \mathcal{B}]
\]
A diagram chase verifies that the $\mathcal{P}$-actions coincide. This implies that $\phi^*$ has a left adjoint $\phi_!$ (of course, we already knew this by Remark 2.27).

By Lemma 2.25 $\bar{f} \triangleleft Q$ has a $\mathcal{P}$-$Q$ bimodule structure. As we have seen, the functor $(\bar{f} \triangleleft Q) \triangleleft Q (\cdot)$ from $Q$-algebras to $\mathcal{P}$-algebras also has underlying functor $\bar{f} \triangleleft (\cdot)$ and in fact is also naturally isomorphic to $\phi^*$. This shows via Lemma 2.30 that $\phi^*$ also has a right adjoint $\phi_*$.

This flexibility in presentation allows us to choose whether to present $\phi^*$ as manifestly a left adjoint or manifestly a right adjoint and justifies the existence of adjoints to $\phi^*$ on both sides in the categorical context.

In the operadic context, we have already presented $\phi^*$ as ‘manifestly a right adjoint’ (in the sense of the preceding remark) in Remark 2.27. In the coming sections, we will attempt to present $\phi^*$ as ‘manifestly a left adjoint’, which will lead us to a criterion which permits us to upgrade the relevant presentation from the categorical context.

### 3 Main theorem

In this section we state our main results in generality. These appear as Theorem 3.4 and Theorem 3.8 which are, respectively, generalizations of the necessity and sufficiency portions of Theorem 1.2. Several examples are given in Sect. 4.

**Lemma 3.1** Suppose $(f, \phi) : \mathcal{P} \to Q$ is a map of colored operads and let $\lambda$ be the induced left action of $\mathcal{P}$ on $\bar{f} \triangleleft Q$ from Lemma 2.25. The composition

$$\mathcal{P} \triangleleft \bar{f} \triangleleft |Q| \to \mathcal{P} \triangleleft \bar{f} \triangleleft Q \overset{\lambda}{\to} \bar{f} \triangleleft Q$$

descends to a morphism

$$\mathcal{P} \triangleleft |\mathcal{P}| (\bar{f} \triangleleft |Q|) \to \bar{f} \triangleleft Q.$$

**Proof** The domain of the desired morphism, as in Notation 2.22, is the coequalizer of the diagram

$$\mathcal{P} \triangleleft |\mathcal{P}| (\bar{f} \triangleleft |Q|) \xrightarrow{\mu_{\mathcal{P} \triangleleft |\mathcal{P}|} \bar{f} \triangleleft |Q|} \mathcal{P} \triangleleft \bar{f} \triangleleft |Q|.$$

Descent to the coequalizer is implied by the $\mathcal{P}$-module relation on $\lambda_{\phi}$ and the fact that $\lambda_{|\mathcal{P}|}$ is the restriction of $\lambda_{\phi}$. □

**Definition 3.2** Let $(f, \phi) : \mathcal{P} \to Q$ be a map of colored operads. We call the morphism

$$\mathcal{P} \triangleleft |\mathcal{P}| (\bar{f} \triangleleft |Q|) \to \bar{f} \triangleleft Q$$

guaranteed by Lemma 3.1 the extension morphism of $(f, \phi)$. We say $(f, \phi)$ is a categorical extension if the extension morphism is an isomorphism. In this case, we call the inverse isomorphism the factorization isomorphism.

The factorization isomorphism has components (under adjunction)

$$\phi(b_1, \ldots, b_n; f(a)) : Q(b_1, \ldots, b_n; f(a)) \to (f \triangleleft |\mathcal{P}|) (b_1, \ldots, b_n; a).$$

It is a direct verification that the notion of monoidal extension from Definition 1.1 is just the special case of categorical extension when $f$ is the identity function on the point.
Remark 3.3 (Assumptions on the ground category) So far in this paper, we have been working with a bicomplete symmetric monoidal closed category $E$ (that is, $E$ is a Bénabou cosmos). This is strong enough to establish sufficiency in Theorem 1.2. Our method of proof to establish necessity goes as follows: for each profile $(b_a)$, we use the fact that $\phi^*$ preserves colimits to show that a certain map in $E$ is an isomorphism, and then exhibit the extension morphism of $\phi$ at $(b_a)$ as a summand of this isomorphism. We cannot always use this to deduce that the extension morphism is an isomorphism, for instance if $E = \text{Set} \times \text{Set}^{\text{op}}$ (equipped with a Chu-type tensor product). For questions of necessity, we thus assume that the binary coproduct functor $E \times E \to E$ $X, Y \mapsto X \sqcup Y$ is a conservative functor. That is, if $f \sqcup g$ is an isomorphism, then $f$ and $g$ are also isomorphisms. We will simply say that coproducts are conservative whenever this is the case. See [5, Proposition 3.2] for some equivalent characterizations of this assumption.

Coproducts are conservative in many commonly used ground categories, including sets, topological spaces, $R$-modules, spectra, and presheaf categories.

Theorem 3.4 Suppose that coproducts are conservative in $E$. Let $(f, \phi)$ be a map of operads $P \to Q$. If the restriction functor $\phi^*$ from $Q$-algebras to $P$-algebras is a left adjoint, then $(f, \phi)$ is a categorical extension.

This theorem will be proven in Sect. 5. For now, we turn to sufficiency of this criterion, giving a construction of what will turn out to be the right adjoint to $\phi^*$.

Construction 3.5 Let $(f, \phi)$ be a map of colored operads $P \to Q$. Recall that $\tilde{f} \triangleleft |Q|$ is a $|P|-|Q|$ bimodule. Then $[\tilde{f} \triangleleft |Q|, -]$ is a functor from $P$-algebras (or $|P|$-algebras) to $|Q|$-algebras. Now suppose $(f, \phi)$ is a categorical extension. In this case, we equip the functor $[\tilde{f} \triangleleft |Q|, -]$ with a natural transformation $\alpha : (Q \triangleleft (-)) \circ [\tilde{f} \triangleleft |Q|, -] \to [\tilde{f} \triangleleft |Q|, -]$ as follows.

We emphasize that it is not immediate that $[\tilde{f} \triangleleft |Q|, \mathcal{A}]$ is even a $Q$-algebra when equipped with this structure. We will return to this in Sect. 6.

2 In general, a functor is conservative if the only morphisms it sends to isomorphisms are themselves isomorphisms.
Remark 3.6  By the adjunction of Corollary 2.33, specifying a morphism from the $Q$-algebra $Q \triangleleft \mathbb{F} \triangleleft |Q|, \mathcal{A}$ to the enriched morphism object $\mathbb{F} \triangleleft |Q|, \mathcal{A}$ is equivalent to giving a $|P|$-algebra morphism

$$\bar{f} \triangleleft |Q| \triangleleft |Q|, \mathcal{A} \to \mathcal{A}.$$  

The formula for the adjoint of $\alpha$ is somewhat cleaner:

$$\bar{f} \triangleleft |Q| \triangleleft |Q|, \mathcal{A} \cong \alpha \circ \mathbb{F} \triangleleft |Q| \triangleleft |Q|, \mathcal{A}$$

where $\alpha$ is a factorization isomorphism.

$$\mathcal{P} \triangleleft |P| \triangleleft |Q| \triangleleft |Q|, \mathcal{A} \cong \mathcal{P} \triangleleft |P| \triangleleft |Q| \triangleleft |Q|, \mathcal{A}$$

Theorem 3.8  Suppose $(f, \phi)$ is a categorical extension between operads $P$ and $Q$. Then the adjunction $|\phi|^* \dashv |\phi|_*$ of Corollary 2.33 underlies an adjunction $\phi^* \dashv \phi_*$ between $P$-algebras and $Q$-algebras.

Lemma 3.7 and Theorem 3.8 will be proven in Sect. 6.

4 Examples

In this section we provide a number of examples of maps of operads that satisfy our criterion, along with some non-examples. We arrange these in order of complexity of the colors involved, not independent interest or importance.

When discussing non-existence of right adjoints, we always restrict to ground categories $\mathcal{E}$ with conservative coproducts so that Theorem 3.4 holds.
4.1 One-colored operads

The simplest examples are given by monochrome operads. In this case, as in Theorem 1.2, the criterion on an operad map \( \phi : P \to Q \) is that

\[
P \circ_{P(1)} Q(1) \to Q
\]

is an isomorphism.

**Example 4.2** The most trivial case occurs when \( Q(n) \) and \( P(n) \) are the initial object for \( n \neq 1 \), that is, the case of monoids. Then the criterion is automatically satisfied and the right adjoint from \( Q(1) \)-modules to \( P(1) \)-modules is the classical base-change functor \([Q(1), -]\) with \( Q(1) \) viewed as a \( P(1) \)-\( Q(1) \) bimodule. This includes, for example, change of ring functors between modules over rings related by a ring homomorphism. We have already seen a slightly more general version of this example as Remark 2.35.

Here is a less trivial example in monochrome operads.

**Example 4.3** The Gerstenhaber and Batalin–Vilkovisky operads (hereafter \( G \) and \( BV \)) are algebraic models for the \( E_2 \) and framed \( E_2 \) operads in chain complexes over a field \( k \) of characteristic zero [6, 10]. There is an inclusion \( \iota : G \to BV \).

It is well-known [10, Proof of Proposition 4.8] that the Batalin–Vilkovisky operad \( BV \) is isomorphic as a collection to \( G \circ (k[\Delta]/\Delta^2) \), where \( \Delta \) is a unary, degree 1 operator. Here \( k[\Delta]/\Delta^2 \) is precisely \( BV(1) \), while \( G(1) \) is just \( k \). The isomorphism between the collection underlying \( BV \) and the collection \( G \circ (k[\Delta]/\Delta^2) \) is precisely our factorization isomorphism.

What is the form of the right adjoint? According to our formula, for \( X \) a Gerstenhaber algebra, the right adjoint \( \iota_\ast(X) \) is \([k[\Delta]/\Delta^2, X]\), with the internal hom computed in vector spaces. So the elements of degree \( j \) in \( \iota_\ast(X) \) are pairs \((x, y)\) where \( x \) is in degree \( j \) and \( y \) in degree \( j + 1 \). The product is given by

\[
(x_1, y_1)(x_2, y_2) = (x_1 x_2, [x_1, x_2] + y_1 x_2 + (-1)^{|x_1|} x_1 y_2)
\]

(for some appropriately chosen convention for the Gerstenhaber bracket \((-,-)\)) and the Batalin–Vilkovisky operator is

\[
\Delta(x, y) = (y, 0).
\]

This works in other dimensions and in other ground categories. For instance, the inclusion \( \iota \) of the little \( k \)-balls operad into the framed little \( k \)-balls operad (as operads in a convenient, i.e., bicomplete and Cartesian closed, category of topological spaces) is a monoidal extension by \( SO(k) \), so the right adjoint of \( \iota \) applied to an algebra \( X \) over the little \( k \)-balls operad yields the space of continuous functions \( \text{Map}(SO(k), X) \). According to our general setup, the action of an element \( \rho \) of \( SO(k) \) on a map \( g : SO(k) \to X \) is then

\[
\rho(g)(\sigma) = g(\sigma \cdot \rho).
\]

The action of \( E_k^n \) on a tuple of functions \( (g_1, \ldots, g_n) \in \text{Map}(SO(k), X)^\times n \) is given by the formula in Remark 3.6.

This example works in various models, for example the classical model where \( E_k(1) \) consists of pairs \((c, r)\) in \((\mathbb{R}^k, \mathbb{R}_+)\) with \( |c| + r \leq 1 \) with product

\[
(c, r) \cdot (c', r') = (c + r c', r r')
\]

and \( E_k^\frak{fr}(1) \) is triples \((c, r, \rho)\) in \((\mathbb{R}^k, \mathbb{R}_+, SO(k))\) with \( |c| + r \leq 1 \) and product

\[
(c, r, \rho) \cdot (c', r', \rho') = (c + \rho(r c'), r r', \rho \rho').
\]
It is arguably easier to see the monoidal extension property if we instead choose a point-set model where $E_k(1)$ is a single point and $E_k^{fr}(1)$ is the group $SO(k)$. See Fig. 1 for an illustration of this case.

In particular, for $k = 2$, the functor $\iota_*$ is the free loop space functor.

**Example 4.4** Let $\mathcal{P}$ be an operad in graded $R$-modules. For convenience assume that $\mathcal{P}$ is concentrated in arity at least two (with some change of notation the example works more generally). There is an operad $\mathcal{P}_{dg}$ in graded $R$-modules whose algebras are differential graded $\mathcal{P}$-algebras. One way to construct the operad $\mathcal{P}_{dg}$ is by adjoining a new arity one operation $d$ to $\mathcal{P}$ and imposing the relations that $d$ squares to zero and that $d$ is a graded derivation of every homogeneous operation of $\mathcal{P}$.

The ground category here is not chain complexes of $R$-modules and the differential $d$ is an operation in the operad, not part of the structure of an object in the ground category.

We can then recover $\mathcal{P}$ from $\mathcal{P}_{dg}$ by quotienting away the operation $d$. This can be described as a map of operads $(\text{id}, \phi): \mathcal{P}_{dg} \to \mathcal{P}$. The extension map is of the form

$$\mathcal{P}_{dg} \triangleleft |\mathcal{P}_{dg}| |\mathcal{P}| \to \mathcal{P},$$

that is:

$$\mathcal{P}_{dg} \triangleleft_{R[d]/d^2} R \to \mathcal{P},$$

where the action of $d$ in $R[d]/d^2$ on $R$ is trivial. Then the left hand side here is the quotient by $d$ which we have already said is isomorphic to $\mathcal{P}$. Thus $\phi$ is a monoidal extension and $\phi^*$ has both adjoints.

The pullback $\phi^*$ acts on a $\mathcal{P}$-algebra $A$ by adjoining the zero differential. The left adjoint $\phi_!$ acts on a differential graded $\mathcal{P}$-algebra $B$ by quotienting $B$ (viewed as a $\mathcal{P}$-algebra) by the ideal generated by the differential. The right adjoint $\phi_*$ takes a differential graded $\mathcal{P}$-algebra $B$ to the kernel of $d$, which is still a $\mathcal{P}$-algebra essentially by the derivation property.

**Remark 4.5** Examples 4.3 and 4.4 do not meet Ward’s more restrictive criteria [26, Proposition 7.9]. One of his criteria in the fixed color situation is that $|\mathcal{P}| \to |\mathcal{Q}|$ is an extension by a group action in each color. In both of these examples the extension is by non-invertible elements.

**Example 4.6** A class of non-examples consists of maps of operads $\phi$ where $\phi(1): \mathcal{P}(1) \to \mathcal{Q}(1)$ is an isomorphism. For such operad maps $\phi$, the product $\mathcal{P} \triangleleft_{\mathcal{P}(1)} \mathcal{Q}(1)$ is isomorphic to
\(\mathcal{P}\) and the canonical map induced by \(\phi\) is just \(\phi\) under this identification. So a right adjoint to \(\phi^*\) exists if and only if \(\phi\) is an isomorphism.

Thus if we restrict ourselves to operads whose arity one part consists of just the tensor unit, then the right adjoint \(\phi_e\) nearly never exists. This can be observed directly in many instances, for example there is no right adjoint to the functor from commutative algebras to associative algebras.

### 4.7 Bijections on colors

Now we turn to colored operads, and maps of such which are bijections (or the identity) on color sets. Our main examples of interest involve colored operads whose algebras are various variants of operads, cyclic operads, and modular operads. We have placed explanations of the \(\text{Set}\)-operads \(\mathcal{O}, \mathcal{C},\) and \(\mathcal{M}\) and their variants in Appendix A. All of the positive examples work for a general \(\mathcal{E}\), by base-changing the colored operads in question along the cocontinuous functor \(\text{Set} \to \mathcal{E}\) which sends the point to \(\mathbf{1}\).

Suppose we are given a map of operads \((f, \phi)\) from \(\mathcal{P}\) to \(\mathcal{Q}\) where \(f\) is the identity map on some color set \(A\). Then the collections \(f\) and \(\bar{f}\) are both canonically isomorphic to \(\mathbf{1}_A\) so \(\bar{f} \circ (-)\) is canonically isomorphic to the identity. Then suppressing it in the notation, to be a categorical extension means that the extension morphism is an isomorphism:

\[
\mathcal{P} \xymatrix{
|\mathcal{P}| \ar[r]^\sim & |\mathcal{Q}| \ar[r] & \mathcal{Q}
}
\]

We conclude that Example 4.6 holds just as well in this setting, that is, if \(\phi: \mathcal{P} \to \mathcal{Q}\) is a bijection-on-colors operad map so that \(|\phi|: |\mathcal{P}| \to |\mathcal{Q}|\) is an isomorphism, then \(\phi\) is a categorical extension if and only if \(\phi\) is an isomorphism.

We turn now to a motivating example for this paper, the relation between operads and cyclic operads. Recall the colored operads \(\mathcal{O}\) and \(\mathcal{C}_{\text{GK}}\) from Example A.4 and Example A.3 which have as their respective algebras operads and the cyclic operads of Getzler and Kapranov [11]. Both operads have as color set the positive integers, and the operations in each are certain isomorphism classes of ordered trees with at least one boundary component. The difference is that \(\mathcal{C}_{\text{GK}}\) includes all such ordered trees, whereas \(\mathcal{O}\) only includes the rooted trees (those with a downward flow).

**Lemma 4.8** The map \(\mathcal{O} \to \mathcal{C}_{\text{GK}}\) is a categorical extension.

**Proof** Suppose that \(G\) is an ordered tree with at least one boundary element. A preimage of \(G\) under the extension morphism will be an ordered rooted tree with the same underlying graph and the same orderings on \(B(G)\) and \(V(G)\). Moreover, for each \(v\), the two orderings on \(\text{nb}(v)\) will differ by a cyclic permutation. There is exactly one possibility for such an ordered rooted tree in the preimage of \(G\), hence the extension morphism is an isomorphism. \(\square\)

This gives another construction of the following adjoint.

**Corollary 4.9** (Templeton [25]) The forgetful functor from cyclic operads to operads has a right adjoint, with formula given in Construction 3.5.

This adjoint was independently constructed by Ward [26, §9.11] in the ground category of vector spaces over a field of characteristic zero. See [8, Lemma 2.16] for general ground categories. A number of variations, such as for non-unital operads and cyclic operads, or for non-symmetric operads and cyclic operads, are also possible. These are left to the reader.
Table 1  Non-categorical extensions related to generalized operads

\begin{tabular}{|c|c|c|}
\hline
\(\mathcal{P}\)-algebras & \(\mathcal{Q}\)-algebras & \(\phi^*\) \\
\hline
\N \hspace{1em} Non-symmetric operads & Operads & Forget symmetric group actions \\
\N \hspace{1em} Cyclic operads & Modular operads (without genus) & Forget contraction operations \\
\N \times \N \hspace{1em} Dioperads & Properads & Forget higher genus composition \\
\N \times \N \hspace{1em} Properads & Wheeled properads & Forget wheel contractions \\
\hline
\end{tabular}

Fig. 2  No element in the image of \(\mathcal{O}_{\text{ns}} \rightarrow \O\) has the indicated boundary ordering.

**Example 4.10** (Non-examples) Some negative results are included in Table 1, where the map of operads \(\phi : \mathcal{P} \rightarrow \mathcal{Q}\) is not a categorical extension and the exceptional right adjoint to \(\phi^*\) does not exist. We do not give full details but in every case it is possible to find a simple example demonstrating that the extension morphism is not an isomorphism. As one example for operads and non-symmetric operads, in Fig. 2 there is no compatible choice of order for the edges adjacent to the bottom vertex which will make this element appear in the image of the extension morphism of \(\mathcal{O}_{\text{ns}} \rightarrow \O\). As another example, no element of \(\mathcal{M}\) indexed by a graph containing a cycle of length greater than one is in the image of the extension morphism of \(\mathcal{O} \rightarrow \mathcal{M}\).

These non-examples have both unital and non-unital variants and extend to any ground category \(\mathcal{E}\) with conservative coproducts as in Remark 3.3 which is not equivalent to the terminal category. All of the algebras from Table 1 are about monochrome objects, but these work identically when working over a fixed color set (or in the case of the second line, over a fixed involutive color set), as in Example A.5. For instance, a colored version of the first line would indicate that, for a non-empty set \(C\), the forgetful functor from \(C\)-colored operads to non-symmetric \(C\)-colored operads does not have a right adjoint.

### 4.11 The general case

Now we pass to the general case, where colors are allowed to change. Let us first consider the cases of the initial and terminal operads. We already saw in Example 4.6 that the unique map from a colored operad to the terminal colored operad (the commutative operad) is often not a categorical extension.

**Example 4.12** (Trivial example) Suppose that \(\mathcal{P}\) is the initial operad, that is, the unique operad with \(A = \emptyset\). Then for any other operad \(\mathcal{Q}\), the unique map \(\mathcal{P} \rightarrow \mathcal{Q}\) is a categorical extension, as the condition is vacuously satisfied. The exceptional right adjoint takes the unique object of \(\mathcal{P}\)-alg to the terminal object of \(\mathcal{Q}\)-alg.

Our next example is likely not of independent interest, but addresses a necessary complication of our method of proof (see Remark 4.14).
Example 4.13 Let $Q$ be the $\{1, 2\}$-colored operad which is freely generated by a single morphism in $Q(1, 2; 1)$. Let $f : \{1\} \to \{1, 2\}$ be the inclusion, and let $P = \tilde{f} \triangleleft Q \triangleleft f$ be the associated $\{1\}$-colored operad (see Remark 2.18). That is, $P$ is the terminal category. The extension morphism has the form

$$\tilde{f} \triangleleft |Q| \cong P \triangleleft |P| \tilde{f} \triangleleft |Q| \to \tilde{f} \triangleleft Q$$

since $P = |P|$. As the right-hand side has entries outside of arity one, this map is not an isomorphism.

Remark 4.14 Every operad map $(\phi) : P \to Q$ factors as a composite of operad maps

$$P \to \tilde{f} \triangleleft Q \triangleleft f \to Q$$

whose first map fixes colors. If it were the case, for every function $f$ with target $B$, that $R = \tilde{f} \triangleleft Q \triangleleft f \to Q$ happened to be a categorical extension, we would be able to reduce all considerations in the proof of the main theorem to the fixed-color setting. Alas, Example 4.13 tells us that this is not the case. Notice, though, that if we apply $(\_ \triangleleft f)$ to the extension morphism of $R \to Q$, we have that

$$R \triangleleft |R| \tilde{f} \triangleleft |Q| \triangleright f \to \tilde{f} \triangleleft Q \triangleright f$$

is always an isomorphism since $\tilde{f} \triangleleft |Q| \triangleright f = |R|$. But the functor $(\_ \triangleleft f)$ is conservative just when $f$ is a surjective function, so we cannot generally deduce in this setting that the extension morphism is an isomorphism.

Remark 4.14 yields a reduction to the fixed color case when $f$ is a surjective function. There is also a situation which arises in examples in which $f$ is an injective function and $\phi$ is a categorical extension. We state it more generally.

Proposition 4.15 Let $(f, \phi) : P \to Q$ be a map of operads such that the comparison map $P \triangleleft \tilde{f} \to \tilde{f} \triangleleft Q$ is an isomorphism. Then $\phi$ is a categorical extension.

Proof The hypothesis implies that $|P| \triangleleft \tilde{f} \to \tilde{f} \triangleleft |Q|$ is also an isomorphism. Consequently

$$P \triangleleft |P| \tilde{f} \triangleleft |Q| \cong P \triangleleft |P| |P| \triangleleft \tilde{f} \cong P \triangleleft \tilde{f} \cong \tilde{f} \triangleleft Q$$

and this isomorphism coincides with the extension morphism. \qed

Remark 4.16 When $f : A \to B$ is an injection of color sets, the condition that $P \triangleleft \tilde{f} \to \tilde{f} \triangleleft Q$ is an isomorphism reduces to the condition that $P$ is a maximal sieve of $Q$:

1. (fully faithful condition) for any profile in $A$, $P(a; a) \cong Q(f(a); f(a))$ and all structure maps in $P$ are induced from $Q$,
2. (ideal condition) for any profile $(b; a)$ in $S_B^{op} \times A$, if $b$ contains a color not in $A$ then $Q(b; f(a))$ is an initial object of $E$.

This is very close to [26, Prop. 7.9]. Now we give a few examples of this phenomenon.

Example 4.17 There is a suboperad $\mathcal{O}_+$ of $\mathcal{O}$ without the color 1 but with all the same operations for defined profiles. Algebras over $\mathcal{O}_+$ correspond to operads with no arity zero operations.

There is in turn a $\{2\}$-colored suboperad $\mathcal{A}$ of $\mathcal{O}_+$ consisting of those rooted trees where every vertex has valence 2. That is, elements of $\mathcal{A}$ are linear rooted trees, including the tree with no vertices. Algebras over $\mathcal{A}$ are monoids. The underlying category of $\mathcal{A}$ has only the identity morphism.
The inclusion of $A$ into $O_+$ is a categorical extension.

**Proof** The color map $f$ is the inclusion of $\{2\}$ into $\{2, 3, \ldots\}$. We use Remark 4.16. The inclusion of $A$ into $O_+$ satisfies the fully faithful condition by definition. Any rooted tree with overall valence two and vertices of valence at least two can only have vertices of valence two, so $A$ satisfies the ideal condition.

Therefore there is a right adjoint $\phi_*$ to the forgetful functor from operads without 0-ary operations to monoids. By the formula of Construction 3.5, we have that $\phi_*(A)(n)$ is $A$ if $n$ is 1 and the terminal object if $n \geq 2$. This is correct by inspection and is a nice dual result to the shape of the left adjoint, which is the initial object in all arities other than 1.

Note that the inclusion of $A$ into the colored operad governing operads with arity zero operations, $T$, is not a categorical extension. In this case $\tilde{f} \triangleleft T$ contains isomorphism classes of labelled trees with vertices of arbitrary valence and the extension morphism fails to be surjective. Thus there is no right adjoint to the inclusion of monoids into ordinary operads.

**Remark 4.19** The previous example has an extension to the colored setting. If $A$ is a set, the operad $O^A$ for $A$-colored operads (from Example A.5), has analogous suboperads $A^A \subseteq O^A \subseteq O^A$. The inclusion $A^A \rightarrow O^A$ will be a categorical extension, but $A^A \rightarrow O^A$ will not be (unless $A = \emptyset$). Objects of $A^A$-alg are categories with object set $A$, and morphisms are identity-on-object functors between such. Similarly, $O^A$-alg is the category of $A$-colored ‘positive’ operads, which play a role in Sect. 5.

**Example 4.20** ($\ell$-Truncated operads) For $1 \leq \ell \leq \infty$, let $O^\ell$ denote the full suboperad of $O_+$ with color set $\{2, 3, \ldots, \ell + 1\}$. Note that $O^1_+ = A$ and $O^\infty_+ = O_+$. Algebras over $O^\ell_+$ are ‘$\ell$-truncated operads.’ If $\ell \leq m$, then the inclusion $O^\ell_+ \rightarrow O^m_+$ is a maximal sieve. Full faithfulness is automatic, and the ideal condition holds because $O_+(k_1, \ldots, k_n; p)$ is inhabited if and only if

$$p = 2 + \sum_{i=1}^n (k_i - 2).$$

By Remark 4.16, $O^\ell_+ \rightarrow O^m_+$ is a categorical extension. The right adjoint to restriction places the terminal object in arities $\ell < n \leq m$. When $m = \infty$, this right adjoint $O^\ell_+\text{-}alg \rightarrow O_+\text{-}alg$ appeared as the first part of [12, Proposition 4.2.2].

**Example 4.21** In Lemma 4.8 we showed that $T \rightarrow G$ was a categorical extension, giving a right adjoint to the restriction from Getzler–Kapranov cyclic operads to operads. We could just as easily have worked with $T$, the $\mathbb{N}$-colored operad governing all cyclic operads (see Example A.3 for details), as is justified by the following lemma.

**Lemma 4.22** The inclusion of operads $G \rightarrow T$ is a categorical extension.

**Proof** Again we follow Remark 4.16. Both full faithfulness and the ideal condition follow from the fact that a simply connected graph with non-empty boundary, each vertex $v$ of the graph must have non-empty $\text{nb}(v)$.

Therefore the forgetful functor “forget constants and pairings of elements to constants” from cyclic operads to Getzler–Kapranov cyclic operads has a right adjoint which puts in a terminal object for the constants and the unique pairing of elements to the terminal object. In the colored setting, a version of this exceptional right adjoint appears, more or less, as the subcategory inclusion in [9, Lemma 4.2].
Example 4.23 We return to the case of cyclic and modular operads. In Example 4.10, we noted that \( C \to M \) is not a categorical extension. This time we will use the genus-aware model \( M^g \) for modular operads from Example A.2. There is a map \((f, \phi)\) of colored operads from \( C \to M^g \) where \( f \) takes the color \( n \) to the color \((n,0)\) and \( \phi \) takes an ordered tree to itself. Pullback along this map of colored operads is a forgetful functor from genus-aware modular operads to cyclic operads which forgets all operations and gluings of higher genus.

Lemma 4.24 The map \((f, \phi)\) is a categorical extension.

Proof In \( M^g \), an operation has genus zero inputs and output if and only if the corresponding graph is a tree and the genus of each vertex is zero; these operations are precisely the image of the inclusion of \( C \), giving the full faithfulness criterion of Remark 4.16. On the other hand, any operation with genus zero output necessarily only has genus zero inputs, giving the ideal criterion. \( \square \)

We conclude that the “forget higher genus” forgetful functor has a right adjoint. Similarly to the previous examples, the adjoint is given by

\[
\phi^*(C)(n, g) = \begin{cases} C(n) & g = 0 \\ * & \text{otherwise.} \end{cases}
\]

This adjoint was described by Ward [26, §9.1].

In Remark 2.18 and Remark 4.14 we encountered a standard construction that allows one to pull back a \( B \)-colored operad \( Q \) to an \( A \)-colored operad along a function \( A \to B \). This can be reinterpreted in terms of the operads appearing in Example A.5.

Example 4.25 Let \( h : A \to B \) be a function. There is a map of colored operads \((f, \phi) : \mathcal{O}^A \to \mathcal{O}^B \) where \( f = \prod_{n \geq 1} h \times n \) and \( \phi \) takes a rooted tree with coloring function \( E \to A \) to the same rooted tree but with coloring function \( E \to A \to B \). If \( A \) is empty, then \( \mathcal{O}^A \) is the initial operad so Example 4.12 applies to show that \((f, \phi)\) is a categorical extension. With the assumption that \( A \) is nonempty, we claim that \((f, \phi)\) is a categorical extension if and only if \( h \) is a bijection. One can see the nontrivial direction by examining the extension morphism in profiles of the form \((b_1, b_2), (b_2); (a_1))\) where \( b_1 = h(a_1) \). There is a unique rooted ordered tree \( \bullet \cdots \bullet \) containing two vertices of the appropriate valences. The target of the extension morphism at this profile is \( \mathcal{O}^B((b_1, b_2), (b_2); (b_1)) \), which has exactly one element. On the other hand, the preimage of this element under the extension morphism may be identified with \( h^{-1}(b_2) \), the coloring of the internal edge. Assuming the extension morphism is an isomorphism, we then have that \( h^{-1}(b_2) \) is a single point for every \( b_2 \in B \), hence \( h \) is a bijection.

5 Necessity of the criterion

In this section we will prove Theorem 3.4. Let \((f, \phi) : \mathcal{P} \to \mathcal{Q}\) be a map of colored operads. The theorem states that if the restriction functor \( \phi^* \) from \( \mathcal{Q}\)-algebras to \( \mathcal{P}\)-algebras is a left adjoint, then \((f, \phi)\) is a categorical extension. Our strategy is to show that preservation of certain colimits implies the factorization condition. We begin with initial objects.

Notation 5.1 Let \((A, \mathcal{P})\) be an operad. We write \( \hat{\mathcal{P}} \) for the initial \( \mathcal{P}\)-algebra.

The initial \( \mathcal{P}\)-algebra \( \hat{\mathcal{P}} \) has \( \hat{\mathcal{P}}_a = \mathcal{P}(\ ; a) \). This implies the following lemma, whose conclusion is the arity zero part of the condition for a map of operads to be a categorical extension.
Lemma 5.2 Suppose that \((f, \phi) : (A, \mathcal{P}) \to (B, \mathcal{Q})\) is a map of colored operads and that \(\phi^*\) preserves initial objects. Then for every color \(a\) of \(A\), the extension morphism
\[
\mathcal{P} \triangleleft |\mathcal{P}| (\tilde{f} \triangleleft |\mathcal{Q}|) \to \tilde{f} \triangleleft \mathcal{Q}
\]
is an isomorphism in the profile \((; a)\).

**Proof** At profile \((; a)\), the source of the extension morphism is \(\mathcal{P}(; a)\) while the target is \(\mathcal{Q}(; f(a))\). By assumption, \(\phi^*\mathcal{P} \cong \mathcal{Q}\), so we have
\[
\mathcal{P}(; a) = \mathcal{P}_a = (\phi^*\mathcal{P})_f(a) \cong \mathcal{Q}_f(a) = \mathcal{Q}(; f(a)).
\]
\[\square\]

Now we will argue that preservation of binary coproducts suffices to show that all of the other components of the extension morphism are isomorphisms. It is more convenient to assume that we are dealing with positive operads.

**Definition 5.3** Let \(X\) be an \((A, B)\)-collection. We call \(X\) **positive** if \(X(; b)\) is the initial object of \(\mathcal{E}\) for every color \(b\) in \(B\). Likewise, an operad \((A, \mathcal{P})\) is positive just when its underlying collection is positive, that is, just when the initial \(\mathcal{P}\)-algebra coincides with the initial \(A\)-object.

**Notation 5.4** Let \(\mathcal{P}\) be a colored operad. Write \(\mathcal{P}_+\) for the positive operad obtained from \(\mathcal{P}\) by replacing the components in profiles with empty input list with the initial object of \(\mathcal{E}\). This procedure is functorial, and if \((f, \phi)\) is a map of colored operads from \(\mathcal{P}\) to \(\mathcal{Q}\), then we write \((f, \phi_+)\) for the corresponding map of colored operads from \(\mathcal{P}_+\) to \(\mathcal{Q}_+\). Further, the natural transformation \(\iota\) from \((-)_+\) to the identity functor gives, for each operad \((A, \mathcal{P})\), a map
\[
(id_A, \iota_\mathcal{P}) : \mathcal{P}_+ \to \mathcal{P}.
\]

**Remark 5.5** It is reasonable to try to reduce from \(\mathcal{P}\) to \(\mathcal{P}_+\) because the components of their extension morphisms agree on any profile with non-empty input colors. That is, the following diagram, with vertical maps extension morphisms and horizontal maps induced by \(\iota\), commutes more or less by naturality. Moreover, the horizontal maps are isomorphisms in any profile with non-empty input colors:
\[
\begin{array}{ccc}
\mathcal{P}_+ \triangleleft |\mathcal{P}_+| (\tilde{f} \triangleleft |\mathcal{Q}_+|) & \longrightarrow & \mathcal{P} \triangleleft |\mathcal{P}| (\tilde{f} \triangleleft |\mathcal{Q}|) \\
\downarrow & & \downarrow \\
\tilde{f} \triangleleft \mathcal{Q}_+ & \longrightarrow & \tilde{f} \triangleleft \mathcal{Q}.
\end{array}
\]

Coproducts of \(\mathcal{P}\)-algebras and coproducts of \(\mathcal{P}_+\)-algebras differ. However, we have the following implication.

**Lemma 5.6** Suppose that coproducts are conservative in \(\mathcal{E}\) as in Remark 3.3. Let \((f, \phi) : (A, \mathcal{P}) \to (B, \mathcal{Q})\) be a map of colored operads. If \(\phi^*\) preserves initial objects and binary coproducts, then \(\phi^*_+\) preserves binary coproducts.

We will prove this lemma in Sect. 5.8.

Now Theorem 3.4 is implied by the following lemma and Remark 5.5.

**Lemma 5.7** Suppose that coproducts are conservative in \(\mathcal{E}\). Let \((f, \phi) : (A, \mathcal{P}) \to (B, \mathcal{Q})\) be a map of positive colored operads. If \(\phi^*\) preserves binary coproducts, then for every positive length list \(b = (b_1, \ldots, b_n)\) of \(B\) and color \(a\) of \(A\), the component of the extension morphism
\[
\mathcal{P} \triangleleft |\mathcal{P}| (\tilde{f} \triangleleft |\mathcal{Q}|)(b_1, \ldots, b_n; a) \to \tilde{f} \triangleleft \mathcal{Q}(b_1, \ldots, b_n; a)
\]
is an isomorphism.
The proof of this lemma is the topic of Sect. 5.18. This proof rests on an analysis of decompositions of collections of the form $X \prec (\bigsqcup Y \prec Z)$, which we introduce in Sect. 5.10.

5.8 Discarding arity zero

In this section our goal is to prove Lemma 5.6.

First we observe that restriction along the canonical maps $i_P$ and $i_Q$ is well-behaved with respect to $\phi$.

**Lemma 5.9** Suppose $P \xrightarrow{(f, \phi)} Q$ is a map of operads such that $\phi^*$ preserves initial objects. Then restriction along $\phi$ and $i_+$ commutes with induction along $i$ in the sense that the following diagram commutes up to natural isomorphism:

$$
\begin{array}{ccc}
Q_+ \text{-alg} & \xrightarrow{(\iota_Q)_!} & Q \text{-alg} \\
\phi^* \downarrow & \cong & \downarrow \phi^* \\
P_+ \text{-alg} & \xrightarrow{(\iota_P)_!} & P \text{-alg}
\end{array}
$$

**Proof** There is a natural transformation

$$
\begin{aligned}
& ((\iota_P)_! \phi_+^*; \text{unit}) \rightarrow ((\iota_P)_! \phi_+^* \iota_Q^*(\iota_Q)_!); \cong ((\iota_P)_! \iota_P^* \phi^*(\iota_Q)_!); \xrightarrow{\text{counit}} \phi^*(\iota_Q)_!; \\
& (1)
\end{aligned}
$$

where the isomorphism in the middle follows from $i_Q \phi_+ = \phi i_P$. We will argue that this composite natural transformation is an isomorphism, which can be done at the level of underlying collections. For the moment, we write $U$ for any of the functors which take an algebra to its underlying collection. At this level, the functors involved have the following form:

$$
\begin{align*}
U \phi_+^* & \cong \bar{f} \prec U(-) & U \phi^* & \cong \bar{f} \prec U(-) \\
U i_P^* & \cong U & U i_Q^* & \cong U \\
U (\iota_P)_! & \cong U \hat{P} \sqcup U(-) & U (\iota_Q)_! & \cong U \hat{Q} \sqcup U(-).
\end{align*}
$$

Then the morphism underlying the unit natural transformation of $Q_+$-algebras, $\text{id} \rightarrow (\iota_Q)_!$, of the form

$$
U(-) \rightarrow U \hat{Q} \sqcup U(-),
$$

is the universal inclusion of the coproduct. The morphism underlying the counit of $P$-algebras, $(\iota_P)_! \iota_P^* \rightarrow \text{id}$, of the form

$$
U \hat{P} \sqcup U(-) \rightarrow U(-),
$$

is given on the first factor by the $P$-algebra structure and on the second factor by the identity. That is, for a $P$-algebra $A$, the map $U \hat{P} \rightarrow U A$ underlies the unique $P$-algebra morphism. Then the underlying natural transformation of the composite (1) described above, at a $Q_+$-
algebra \( \mathcal{B} \), is

\[
U^\mathcal{P} \sqcup (\tilde{f} \triangleleft UB)
\]

\[
\xrightarrow{\text{inclusion}}
U^\mathcal{P} \sqcup (\tilde{f} \triangleleft (U^\mathcal{Q} \sqcup UB)) \xrightarrow{\cong} U^\mathcal{P} \sqcup (\tilde{f} \triangleleft U\hat{Q}) \sqcup (\tilde{f} \triangleleft UB)
\]

\[
(\tilde{f} \triangleleft U\hat{Q}) \sqcup (\tilde{f} \triangleleft UB).
\]

Again, the leftmost summand of the second vertical map underlies the unique map of \( \mathcal{P} \)-algebras \( U^\mathcal{P} \to U\hat{Q} = \phi^*\hat{Q} \), which is an isomorphism by assumption. Thus the natural transformation (1) is an isomorphism. \( \square \)

The functor which takes an algebra to its underlying collection does not preserve coproducts. At the end of the following proof, and anywhere later in this section where it seems potentially confusing, we will distinguish between coproducts in categories of algebras (over \( \mathcal{P}, \mathcal{P}_+, \mathcal{Q}, \text{or } \mathcal{Q}_+ \)) and coproducts in the categories of objects (that is, \( A \) or \( B \)-objects) using the notation \( \sqcup^\text{alg} \) and \( \sqcup^\text{obj} \).

**Proof of Lemma 5.6** Let \( B_1 \) and \( B_2 \) be \( \mathcal{Q}_+ \)-algebras. We want to show that the comparison map

\[
\phi^*_+ B_1 \sqcup \phi^*_+ B_2 \to \phi^*_+(B_1 \sqcup B_2)
\]

(2)

is an isomorphism of \( \mathcal{P}_+ \)-algebras. First, since by hypothesis \( \phi^* \) preserves finite coproducts, we know that the following map is an isomorphism of \( \mathcal{P} \)-algebras:

\[
\phi^* (\iota_\mathcal{Q})! B_1 \sqcup \phi^* (\iota_\mathcal{Q})! B_2 \xrightarrow{\cong} \phi^* (\iota_\mathcal{Q})! (B_1 \sqcup B_2).
\]

By Lemma 5.9, we can replace \( \phi^* (\iota_\mathcal{Q})! \) with \( (\iota_{\mathcal{P}})! \phi^*_+ \). Then commuting the left adjoint \( (\iota_{\mathcal{P}})! \) past the coproduct, we get

\[
(\iota_{\mathcal{P}})! (\phi^*_+ B_1 \sqcup \phi^*_+ B_2) \cong (\iota_{\mathcal{P}})! \phi^*_+ B_1 \sqcup (\iota_{\mathcal{P}})! \phi^*_+ B_2 \xrightarrow{\cong} (\iota_{\mathcal{P}})! \phi^*_+(B_1 \sqcup B_2),
\]

and by inspection this composition is \( (\iota_{\mathcal{P}})! \), applied to (2).

Using the fact that the functor of collections underlying \( (\iota_{\mathcal{P}})! \) is the coproduct with \( U^\mathcal{P} \), the isomorphism of collections underlying our isomorphism of \( \mathcal{P} \)-algebras is

\[
U^\mathcal{P} \sqcup^\text{obj} \xrightarrow{\cong} U^\mathcal{P} \sqcup^\text{obj} U \phi^*_+(B_1 \sqcup^\text{alg} B_2).
\]

Then since both coproducts and the forgetful functor to collections are conservative, (2) is an isomorphism as well. \( \square \)

**5.10 The multilinear summand**

We will need to be able to refine the description of the composition product of Definition 2.5 by keeping track of special summands. In this section, we introduce a decomposition for composition products of a very particular form. The key results Lemma 5.15, Corollary 5.16, and Lemma 5.17 will all be needed in Sect. 5.18.
For this section, fix color sets A, B, and C. Let X be a (B, C)-collection, let Y be a positive
(A, B)-collection, and for i ∈ [k] = {1, . . . , k}, let Zi be an A-object. Assuming that Y is
positive and that the Zi are objects rather than collections are not logically necessary, but
this simplification allows us to streamline the definitions and proofs below.

We are interested in summands of the iterated composition products

\[ X \triangleleft \left( \bigotimes_{i=1}^{k} Y \triangleleft Z_i \right) \]

in which each factor Z_i “appears precisely once”. In this section, we will formalize what we
mean by this, defining the multilinear decomposition of X \triangleleft(\bigotimes_{i=1}^{k} Y \triangleleft Z_i) into multilinear
and nonlinear summands.

Using the point of view of Day powers from Definitions 2.4 and 2.5, we will first focus our
attention on Y \triangleleft Z_i, identifying a coproduct decomposition of the Day power (\bigotimes_{i=1}^{k} Y \triangleleft Z_i)^b,
and then define a coproduct decomposition of the full composition product X \triangleleft(\bigotimes_{i=1}^{k} Y \triangleleft Z_i).

The collection Y splits as \( Y = Y^{(1)} \sqcup Y^{(2)} \) where \( Y^{(1)} = |Y| \) and \( Y^{(2)} \) has nothing
in arity zero or one. The decomposition can exclude the 0-ary part of Y because Y is positive.
Let S be the two element set \{1, 2\}. Since \((-) \triangleleft Z_i\) is a left adjoint, we have

\[ \bigotimes_{i=1}^{k} Y \triangleleft Z_i = \bigotimes_{[k] \times S} Y^{(s)} \triangleleft Z_i. \]  

(3)

**Definition 5.11** Let Y be a positive (A, B)-collection, and for i ∈ [k] = {1, . . . , k}, let Zi be
an A-object. Write V for the B-object from (3), and let b be a length m list in B. We have that
the power (see Definition 2.4) is given by

\[ V^b = \bigotimes_{j=1}^{m} V_{b_j} = \bigotimes_{j=1}^{m} \bigotimes_{[k] \times S} (Y^{(s)} \triangleleft Z_i)_{b_j} \]

\[ \cong \bigotimes_{g \times h: [m] \rightarrow [k] \times S} \bigotimes_{j=1}^{m} (Y^{(h(j))} \triangleleft Z_{g(j)})_{b_j}. \]

The multilinear summand of \( V^b_{\text{mul}} \), denoted by \( V^b_{\text{mul}} \), is the subsum
indexed by those g × h : [m] → [k] × S satisfying

- \( h(j) = 1 \) for all \( j \), and
- the map g : [m] → [k] is a bijection.

In other words, we have

\[ V^b_{\text{mul}} \cong \bigotimes_{g: [m] \cong [k]} \bigotimes_{j=1}^{m} (|Y| \triangleleft Z_{g(j)})_{b_j}. \]

The nonlinear summand \( V^b_{\text{non}} \) consists of the subsum indexed by the remaining choices
of g × h. We have \( V^b = V^b_{\text{mul}} \sqcup V^b_{\text{non}} \) and both \( V^b_{\text{mul}} \) and \( V^b_{\text{non}} \) are functors from \( S_B \) to \( E \).

**Remark 5.12** Suppose that \( Y \rightarrow Y' \) is a morphism of positive (A, B)-collections and, for
i ∈ [k], \( Z_i \rightarrow Z'_i \) is a map of A-objects. Further, let V = \( \bigotimes_{i=1}^{k} Y \triangleleft Z_i \) and \( V' = \bigotimes_{i=1}^{k} Y' \triangleleft Z'_i \).
Then for every list \( b \) of elements of B, the induced morphism \( V^b \rightarrow (V')^b \) splits as a sum
of \( V^b_{\text{mul}} \rightarrow (V')^b_{\text{mul}} \) and \( V^b_{\text{non}} \rightarrow (V')^b_{\text{non}} \). Further, if each \( Z_i \rightarrow Z'_i \) is an identity and
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$Y \to Y'$ is an isomorphism in arity one, then the multilinear summand $V_{\text{mul}}^b \to (V')_{\text{mul}}^b$ is an isomorphism.

This last condition of the remark includes, of course, the inclusion of $|Y|$ into $Y$.

**Definition 5.13** Given the multilinear decomposition of the Day powers, we extend to a multilinear decomposition of the product $X \triangleleft \bigsqcup_{i=1}^k (Y \triangleleft Z_i)$ as follows

$$
\left( X \triangleleft \bigsqcup_{i=1}^k (Y \triangleleft Z_i) \right) = \left( X \triangleleft \bigsqcup_{i=1}^k (Y \triangleleft Z_i) \right)_{\text{mul}} \sqcup \left( X \triangleleft \bigsqcup_{i=1}^k (Y \triangleleft Z_i) \right)_{\text{non}}
$$

into multilinear and nonlinear summands. Using the $V$ notation from above, the multilinear summand is given by

$$
\left( X \triangleleft \bigsqcup_{i=1}^k (Y \triangleleft Z_i) \right)_{\text{mul},c} := \int_{b \in \mathcal{S}_B} X^{(\ell)} \otimes V^b_{\text{mul}}
$$

and similarly for the nonlinear summand.

**Definition 5.14** Given two collections equipped with multilinear decompositions, we call a map between them which respects the decompositions homogeneous.

Unraveling the definitions of the summands yields the following lemma.

**Lemma 5.15** Let $X \to X'$ be a map of $(\mathcal{B}, \mathcal{C})$-collections, $Y \to Y'$ be a map of positive $(\mathcal{A}, \mathcal{B})$-collections, and, for $i \in [k] = \{1, \ldots, k\}$, let $Z_i \to Z'_i$ be a map of $\mathcal{A}$-objects. Then the map

$$
X \triangleleft \bigsqcup_{i=1}^k (Y \triangleleft Z_i) \to X' \triangleleft \bigsqcup_{i=1}^k (Y' \triangleleft Z'_i)
$$

is homogeneous. \qed

In particular, there are induced maps

$$
\left( X \triangleleft \bigsqcup_{i=1}^k (Y \triangleleft Z_i) \right)_{\text{mul}} \to \left( X' \triangleleft \bigsqcup_{i=1}^k (Y' \triangleleft Z'_i) \right)_{\text{mul}}
$$

$$
\left( X \triangleleft \bigsqcup_{i=1}^k (Y \triangleleft Z_i) \right)_{\text{non}} \to \left( X' \triangleleft \bigsqcup_{i=1}^k (Y' \triangleleft Z'_i) \right)_{\text{non}}
$$

which commute with inclusions of summands.

Combining this lemma with Remark 5.12 gives the following.

**Corollary 5.16** Suppose that $X$ is a $(\mathcal{B}, \mathcal{C})$-collection, $Y$ is a positive $(\mathcal{A}, \mathcal{B})$-collection, and, for $i \in [k] = \{1, \ldots, k\}$, $Z_i$ is an $\mathcal{A}$-object. Then the inclusion $|Y| \to Y$ induces an isomorphism

$$
\left( X \triangleleft \bigsqcup_{i=1}^k (|Y| \triangleleft Z_i) \right)_{\text{mul}} \cong \left( X \triangleleft \bigsqcup_{i=1}^k (Y \triangleleft Z_i) \right)_{\text{mul}}
$$

of multilinear summands. \qed
Lemma 5.17 Let $W$ be a $(C, D)$-collection. Let $X$ be a positive $(B, C)$-collection. Let $Y$ be a positive $(A, B)$-collection. For $i \in [k] = \{1, \ldots, k\}$, let $Z_i$ be $A$-objects. The $D$-object map

$$W \triangleleft \bigoplus_{i=1}^{k} ((X \triangleleft Y) \triangleleft Z_i) \to W \triangleleft X \triangleleft \bigoplus_{i=1}^{k} (Y \triangleleft Z_i)$$

induced by the maps $(X \triangleleft Y) \triangleleft Z_i \cong X \triangleleft (Y \triangleleft Z_i) \to X \triangleleft \bigoplus_{i=1}^{k} (Y \triangleleft Z_i)$ is homogeneous.

Proof (Sketch proof) On the left side, by Remark 5.12, taking multilinear summands of the relevant Day powers commutes with passing to the arity one part of $X \triangleleft Y$. That is,

$$\left(\bigoplus_{i=1}^{k} (X \triangleleft Y) \triangleleft Z_i\right)_{\text{mul}} \cong \left(\bigoplus_{i=1}^{k} |X \triangleleft Y| \triangleleft Z_i\right)_{\text{mul}} \cong \left(\bigoplus_{i=1}^{k} (|X| \triangleleft |Y|) \triangleleft Z_i\right)_{\text{mul}},$$

where the condition to be in the multilinear summand is merely that the function $g$ from Definition 5.11 is a bijection. Since $|X|$ is concentrated in arity one, $|X| \triangleleft (\cdot)$ distributes over the coproduct by Lemma 2.9, and this distribution only changes the function $g$ by reindexing. This shows that the multilinear summand is preserved by the map in the lemma. This also shows that if we are in a summand where all $X \triangleleft Y$ factors are in arity one (i.e., we are working with $|X \triangleleft Y|$) but $g$ is not a bijection, it cannot become a bijection after distribution. This is part of the proof that the nonlinear summand is preserved by the map in the lemma.

There is a second way for a summand to be nonlinear, namely if it contains a factor of the form $(X \triangleleft Y)^{(\geq 2)}$. Because $X$ and $Y$ are both positive, we have

$$(X \triangleleft Y)^{(\geq 2)} = |X| \triangleleft Y^{(\geq 2)} \cup X^{(\geq 2)} \triangleleft Y,$$

so we must have either a factor of $X^{(\geq 2)}$ or a factor of $Y^{(\geq 2)}$. In the former case, the $g$ on the right side of the distribution cannot be a bijection because some index $i$ in its codomain has multiple preimages. In the latter case, the factor from $Y^{(\geq 2)}$ still lives in arity bigger than one after distributing the $X$ factor on the right side and thus will still be nonlinear. \hfill \Box

5.18 Preservation of coproducts

Now that we have the description of the multilinear summands, we can prove Lemma 5.7.

We will use a description of colimits in algebras over operads using reflexive coequalizers in the ground category due to Rezk [22]. The only case we will use is the finite coproduct of algebras which is realized by the following reflexive coequalizer.

Proposition 5.19 For a finite coproduct of $P$-algebras $A_1$ through $A_n$, the coproduct $\bigoplus_{i=1}^{\text{alg}} A_i$ has as its underlying object the reflexive coequalizer

$$P \triangleleft \left(\bigoplus_i \text{obj} \ P \triangleleft A_i\right) \rightrightarrows P \triangleleft \left(\bigoplus_i \text{obj} A_i\right)$$

where the $\bigoplus_i \text{obj}$ coproducts are taken in the ground category and the two maps are

1. composition in the operad $P$ and
2. the action by the operad $P$ on the algebras $A_1$ through $A_n$. \hfill \Box
Now we will realize the extension morphism
\[ \mathcal{P} \triangleleft \mathcal{P} (\bar{f} \triangleleft |\mathcal{Q}|)(b_1, \ldots, b_n; -) \rightarrow \bar{f} \triangleleft \mathcal{Q}(b_1, \ldots, b_n; -) \]
as a summand of the comparison isomorphism between a coproduct of restrictions and a restriction of a coproduct \( \coprod_i \mathcal{P}^{\text{alg}} \mathcal{A}_i \rightarrow \phi^* \coprod_i \mathcal{A}_i \).

**Notation 5.20** Let \((f, \phi) : (A, \mathcal{P}) \rightarrow (B, \mathcal{Q})\) be a map of positive colored operads. Fix a positive length list \(b\) of \(B\) and a color \(a\) of \(A\). By abuse of notation write \(b_i\) for the \((\emptyset, B)\)-collection which is the unit in the ground category concentrated in profile \((; b_i)\). Write \(\mathcal{F}_\mathcal{Q}\) from \(B\)-objects to \(\mathcal{Q}\)-algebras for the left adjoint to the forgetful functor.

**Lemma 5.21** (Homogeneity of the comparison map) Let \((f, \phi) : \mathcal{P} \rightarrow \mathcal{Q}\) be a map of positive colored operads, and let \(b\) be a positive length list in \(B\).

1. The object underlying the coproduct of pulled back algebras \(\coprod_i \mathcal{P}^{\text{alg}} \phi^* \mathcal{F}_\mathcal{Q}(b_i)\) can be realized as the coequalizer of a homogeneous diagram
   \[ \mathcal{P} \triangleleft \left( \coprod_i \mathcal{P}^{\text{obj}} (\mathcal{P} \triangleleft \bar{f} \triangleleft \mathcal{Q}) \triangleleft b_i \right) \Rightarrow \mathcal{P} \triangleleft \left( \coprod_i \mathcal{P}^{\text{obj}} (\bar{f} \triangleleft \mathcal{Q}) \triangleleft b_i \right), \]
   with maps induced by
   - the action of \(\mathcal{P}\) on \(\bar{f} \triangleleft \mathcal{Q}\) using \(\phi\) and \(\mu_\mathcal{Q}\) (Lemma 2.25) and
   - the distributor for the coproduct followed by the composition \(\mu_\mathcal{P}\).

2. The object underlying the pullback of the coproduct algebra \(\phi^* \coprod_i \mathcal{F}_\mathcal{Q}(b_i)\) can be realized as the coequalizer of a homogeneous diagram
   \[ \bar{f} \triangleleft \mathcal{Q} \triangleleft \left( \coprod_i \mathcal{Q}^{\text{obj}} (\mathcal{Q} \triangleleft \mathcal{Q}) \triangleleft b_i \right) \Rightarrow \bar{f} \triangleleft \mathcal{Q} \triangleleft \left( \coprod_i \mathcal{Q}^{\text{obj}} b_i \right), \]
   with maps induced by
   - the operadic composition map \(\mu_\mathcal{Q}\) and
   - the distributor for the coproduct followed by the composition \(\mu_\mathcal{Q}\).

3. given these presentations, the map underlying the comparison map
   \[ \coprod_i \mathcal{P}^{\text{alg}} \phi^* \mathcal{F}_\mathcal{Q}(b_i) \rightarrow \phi^* \coprod_i \mathcal{F}_\mathcal{Q}(b_i) \]
is induced by a homogeneous map of coequalizer diagrams.

**Proof** The presentation as a coequalizer in part 1 is a direct application of Proposition 5.19. Since \(\bar{f} \triangleleft -\), the functor underlying \(\phi^*\), commutes with colimits, the same proposition yields the presentation as a coequalizer in part 2 as well.

In both cases, homogeneity follows from Lemma 5.15 for the first of the two parallel maps. For the second parallel map it follows from Lemma 5.17 coupled with another application of Lemma 5.15.

Given these presentations, the comparison map in part 3 is induced by a map of coequalizer diagrams whose components are in turn induced by repeated use of \(\phi : \mathcal{P} \triangleleft f \rightarrow \bar{f} \triangleleft \mathcal{Q}\) and distributors \(\coprod((\bar{f} \triangleleft -) \rightarrow \bar{f} \triangleleft \coprod -)\). Every map involved is homogeneous by Lemma 5.15 or 5.17. 

\(\square\)
The preceding lemma identifies a splitting of \( \bigoplus \text{alg} \phi^* F_Q(b_i) \) into what would be reasonable to call ‘multilinear’ and ‘nonlinear’ parts, despite being of a different form than our general framework from Definition 5.13. We will use this terminology without further comment, and also refer to the map from Lemma 5.21(3) as being homogeneous (the codomain of the comparison map is isomorphic, as a collection, to \( \bar{f} \triangleleft Q \triangleleft \bigoplus \text{obj} b_i \), and so already has a multilinear decomposition).

The following lemma will provide an identification of certain multilinear summands.

**Lemma 5.22** Suppose that \( Y \) is a positive \((B, C)\)-collection and \( X \) is a \((C, D)\)-collection. Then for each \( d \in D \) and each list \( b \) of elements of \( B \), there is an isomorphism

\[
X \triangleleft \left( \bigoplus_{i=1}^k Y \triangleleft b_i \right)_\text{mul} \cong \left( X \triangleleft |Y| \right)(\frac{b}{d}),
\]

natural in \( X \) and \( Y \).

**Proof** By Corollary 5.16, we may replace \( Y \) by \( |Y| \) on the left-hand side, so for the remainder of the proof we assume \( Y = |Y| \) is concentrated in arity one. By definition the two sides of the purported isomorphism are

\[
\int_{c \in SC} X(c d) \otimes V^c_{\text{mul}} \text{ and } \int_{c \in SC} X(c d) \otimes Y^c(b),
\]

respectively (taking \( Z_i = b_i \) in the specification of \( V^c_{\text{mul}} \) in Definition 5.11), so it suffices to show that \( V^c_{\text{mul}} \cong Y^c(b) \). We have

\[
V^c_{\text{mul}} \cong \prod_{g:|m| \rightarrow [k]} \bigotimes_{j=1}^m (Y \triangleleft b_{g(j)})_{c_j} \cong \prod_{g:|m| \rightarrow [k]} \bigotimes_{j=1}^m Y(b_{g(j)})
\]

which is isomorphic to \( Y^c(b) \) by Lemma 2.7. \( \square \)

This lemma enables the following identification.

**Lemma 5.23** Let \((f, \phi) : P \rightarrow Q\) be a map of positive colored operads, and let \( b \) be a positive length list in \( B \) (the colors of \( Q \)).

Via the identifications of Lemmas 5.21 and 5.22, the value of the multilinear summand of the comparison map

\[
\bigoplus \text{alg} \phi^* F_Q(b_i) \rightarrow \phi^* \bigoplus \text{alg} F_Q(b_i)
\]

in profile \( \binom{a}{b} \) is naturally isomorphic to the component of the extension morphism

\[
P \triangleleft |P| (\bar{f} \triangleleft |Q|) \rightarrow \bar{f} \triangleleft Q
\]

in profile \( \binom{b}{a} \).

**Proof** For the domain, since the coequalizer diagram of Lemma 5.21(1) is homogeneous, the multilinear decomposition distributes through the coequalizer. Then by Lemma 5.22, the multilinear summand of the domain in profile \( \binom{a}{b} \) is the coequalizer of the diagram

\[
(P \triangleleft |P| \triangleleft \bar{f} \triangleleft |Q|)(\frac{b}{a}) \Rightarrow (P \triangleleft \bar{f} \triangleleft Q)(\frac{b}{a}),
\]

with maps induced by the action of \( |P| \) on \( \bar{f} \triangleleft |Q| \) and the composition \( \mu_P \) of the operad \( P \).

This is the domain of the extension morphism.
We can do a similar computation for the codomain using the coequalizer diagram from Lemma 5.21(2). By Lemma 5.22 the multilinear summand in profile \((a)\) is the coequalizer of the diagram

\[ \bar{f} \triangleleft Q \triangleleft |Q| \triangleleft |Q|_a \triangleleft \bar{f} \triangleleft Q \triangleleft |Q|, \]

with maps induced by \(\mu_Q\). This coequalizer collapses to \(\bar{f} \triangleleft Q \triangleleft |Q|\), the codomain of the extension morphism.

By the naturality condition of Lemma 5.22, the map between components of \(P \triangleleft |P| (\bar{f} \triangleleft |Q|)\) and \(\bar{f} \triangleleft Q\) is induced by

\[ P \triangleleft \bar{f} \triangleleft |Q| \overset{\phi}{\to} \bar{f} \triangleleft Q \triangleleft |Q| \]

followed by the collapse to the coequalizer, which is induced by \(\mu_Q\). This is the extension morphism.

\[ \Box \]

**Proof of Lemma 5.7** Since the presentation of the comparison morphism

\[ \coprod^{\text{alg}} \phi^* \mathcal{F}_Q(b_i) \to \phi^* \coprod^{\text{alg}} \mathcal{F}_Q(b_i) \]

of Lemma 5.21 is homogeneous, it respects the multilinear decomposition of that lemma and induces a map between the multilinear summands which was identified in Lemma 5.23 as the extension morphism.

By the hypotheses of Lemma 5.7, the comparison morphism is an isomorphism. Since coproducts are conservative, its multilinear summand, the extension morphism, is also an isomorphism.

\[ \Box \]

### 6 Sufficiency of the criterion

Recall that if \(\mathcal{P} \overset{(f, \phi)}{\rightarrow} Q\) is any map of operads and \(B\) is a \(Q\)-algebra, then by Remark 2.27 we know that \(|\phi|^* B \cong \bar{f} \triangleleft B\) is not just a \(|\mathcal{P}|\)-algebra, but is actually a \(\mathcal{P}\)-algebra. By Corollary 2.33, \(|\phi|^*\) has a right adjoint \(|\phi|_*\). It is not always the case that there is a meaningful \(Q\)-algebra structure on \(|\phi|_* A\) when \(A\) is a \(\mathcal{P}\)-algebra. Throughout this section, let

\[ \mathcal{P} \overset{(f, \phi)}{\rightarrow} Q \]

be a map of operads which is a categorical extension.

Our current task is to prove

- Lemma 3.7, which says that Construction 3.5 gives a functor \(\phi_*\) from \(\mathcal{P}\)-algebras to \(Q\)-algebras, lying over the functor \(|\phi|_* = [\bar{f} \triangleleft |Q|, -]|\), and
- Theorem 3.8, which says that the adjunction

\[ |\phi|^* = \bar{f} \triangleleft |Q| \triangleleft |Q|_\phi \triangleleft [\bar{f} \triangleleft |Q|, -]| = |\phi|_*. \]

of Corollary 2.33 between \(|\mathcal{P}|\) and \(|\mathcal{Q}|\)-algebras lifts to an adjunction \(\phi^* \dashv \phi_*\) between \(\mathcal{P}\) and \(Q\)-algebras.

**Remark 6.1** Let

\[ L = |\phi|^* = \bar{f} \triangleleft |Q| \triangleleft |Q|_\phi \triangleleft [\bar{f} \triangleleft |Q|, -]| \cong \bar{f} \triangleleft (-) \]
be the functor from $|Q|$-algebras to $|P|$-algebras, which is left adjoint to

$$R = |\phi|_* = [[\tilde{f} \circ |Q|, -]].$$

We have a natural transformation

$$\mathcal{P} \triangleleft_{|P|} L(\_ \Rightarrow) L(Q \triangleleft_{|Q|} (\_)) \quad (4)$$

which is given by the extension morphism

$$\mathcal{P} \triangleleft_{|P|} \tilde{f} \circ |Q| \triangleleft |Q| B \rightarrow \tilde{f} \circ Q \triangleleft |Q| B.$$

If $(f, \phi)$ is a categorical extension, then (4) is an isomorphism of functors from $|Q|$-algebras to $|P|$-algebras. In particular, these have the same underlying objects.

**Notation 6.2** To make diagrams in the proofs in this section less busy, we omit the symbol $\triangleleft$ (e.g., $\mathcal{P}(\_)$ means $\mathcal{P} \triangleleft (\_)$), write $\mathcal{P}(\_)$ for $\mathcal{P} \triangleleft_{|P|} (\_)$, and write $Q(\_)$ for $Q \triangleleft_{|Q|} (\_).$ We also follow the notation in Remark 6.1, writing $R$ for the functor $[\tilde{f} \circ |Q|, -] = |\phi|_*$ and similarly for its left adjoint $L = |\phi|^*.$ Further, all functors should be interpreted as being evaluated on everything to the right, which we use to omit all parenthesization.

For example, (4) would be written, at a $Q$-algebra $B,$ as $\mathcal{P} \cdot L B \rightarrow L Q \cdot B.$

**Remark 6.3** We omit the detailed verification that the maps we write down descend to the coequalizers made using $\mathcal{P}(\_)$ and $Q(\_).$ This essentially follows from the fact that everything in sight at least respects $|P|$-algebra and $|Q|$-algebra structures. This includes, in particular, the unit $\eta,$ the counit $\epsilon,$ the operad composition maps $\mu_P$ and $\mu_Q,$ and the $P$-algebra structure map $\lambda.$

**Remark 6.4** Let us recast certain induced structures using the notation now available to us.

- Suppose that $B$ is a $Q$-algebra. The $P$-algebra structure on $L B$ from Remark 2.27 takes the form

$$\mathcal{P} L B \longrightarrow \mathcal{P} \cdot L B \longrightarrow L Q \cdot B \xrightarrow{L \lambda} L B$$

where the middle arrow comes from Remark 6.1.

- Suppose that $A$ is a $P$-algebra. The proposed $Q$-algebra structure on $R A$ comes in two pieces.

  - First, we have the composite from Remark 3.6, which we write as $\hat{\alpha}$ below.

$$L Q R A \longrightarrow L Q \cdot R A \leftarrow \mathcal{P} \cdot L R A \xrightarrow{\mathcal{P} \cdot \epsilon} \mathcal{P} \cdot A \xrightarrow{\lambda} A.$$

  The dashed red arrow is (4) from Remark 6.1, which is an isomorphism by assumption.

  - Second, we have the adjunct of $\hat{\alpha},$ which we call $\alpha,$ which is given by

$$Q R A \xrightarrow{\eta} R L Q R A \xrightarrow{R \hat{\alpha}} R A.$$

This is our proposed action of $Q$ on $R A$ from Construction 3.5.

We start by showing this indeed gives a lift of the functor $R = |\phi|_*$ to algebras.
Proof of Lemma 3.7  We first address objects. Let $A$ be a $P$-algebra. We wish to show that $\alpha$ constitutes a $Q$-algebra structure on $RA$. The aim is to show that the diagram
\[
\begin{array}{c}
\xymatrix{QQRA \ar[r]^{Q\alpha} & QRA \\
QRA \ar[r]^\alpha & RA }
\end{array}
\]
commutes, or by adjointness that the diagram
\[
\begin{array}{c}
\xymatrix{LQRA \ar[r]^{LQ\alpha} & LQRA \\
LQRA \ar[r]^{L\alpha} & LRA }
\end{array}
\]
commutes. We of course only need to show that the inner chamber of this latter diagram commutes.

The maps $\alpha$ and $\hat{\alpha}$ depend upon the inverse of the extension isomorphism $P \triangleleft P | Q | \rightarrow Q$. As in Remark 6.4, we will use dashed red arrows $\xrightarrow{\hspace*{5cm}}$ for maps coming from the natural isomorphism in Remark 6.1. For any $|Q|$-algebra $B$, the diagram
\[
\begin{array}{c}
\xymatrix{\mathcal{P} \cdot \mathcal{P} \cdot LB \ar[r]^{\mu_{\mathcal{P} \cdot L \mathcal{B}}} & \mathcal{P} \cdot L \mathcal{B} \\
\mathcal{P} \cdot L \mathcal{Q} \cdot B \ar[r] & \mathcal{L} \mathcal{Q} \cdot B \ar[r]^{L_{\mu \mathcal{Q} \cdot L \mathcal{B}}} & \mathcal{L} \mathcal{Q} \cdot B }
\end{array}
\]
commutes, essentially by the second adjoint form of colored operad maps from Remark 2.18. When $B = RA$, this forms the middle chamber in the following commutative diagram
\[
\begin{array}{c}
\xymatrix{LQRA \ar[r]^{L\mu RA} & LQRA \\
LQ \cdot RA \ar[r]_{L \mu \mathcal{Q} \cdot RA} & \mathcal{L} \mathcal{Q} \cdot RA \\
\mathcal{P} \cdot \mathcal{P} \cdot LA \ar[u]_{\mathcal{P} \cdot \mathcal{P} \cdot L \mathcal{A}} & \mathcal{P} \cdot \mathcal{P} \cdot LA \ar[u]_{\mathcal{P} \cdot \mathcal{P} \cdot L \mathcal{A}} & \mathcal{P} \cdot \mathcal{P} \cdot LA \ar[u]_{\mathcal{P} \cdot \mathcal{P} \cdot L \mathcal{A}} & \mathcal{P} \cdot \mathcal{P} \cdot LA \ar[u]_{\mathcal{P} \cdot \mathcal{P} \cdot L \mathcal{A}} & \mathcal{P} \cdot \mathcal{P} \cdot LA \ar[u]_{\mathcal{P} \cdot \mathcal{P} \cdot L \mathcal{A}} & \mathcal{P} \cdot \mathcal{P} \cdot LA \ar[u]_{\mathcal{P} \cdot \mathcal{P} \cdot L \mathcal{A}} \\
\mathcal{P} \cdot \mathcal{P} \cdot LA \ar[u]_{\mathcal{P} \cdot \mathcal{P} \cdot L \mathcal{A}} & \mathcal{P} \cdot \mathcal{P} \cdot LA \ar[u]_{\mathcal{P} \cdot \mathcal{P} \cdot L \mathcal{A}} & \mathcal{P} \cdot \mathcal{P} \cdot LA \ar[u]_{\mathcal{P} \cdot \mathcal{P} \cdot L \mathcal{A}} & \mathcal{P} \cdot \mathcal{P} \cdot LA \ar[u]_{\mathcal{P} \cdot \mathcal{P} \cdot L \mathcal{A}} & \mathcal{P} \cdot \mathcal{P} \cdot LA \ar[u]_{\mathcal{P} \cdot \mathcal{P} \cdot L \mathcal{A}} & \mathcal{P} \cdot \mathcal{P} \cdot LA \ar[u]_{\mathcal{P} \cdot \mathcal{P} \cdot L \mathcal{A}} \\
\mathcal{P} \cdot \mathcal{P} \cdot LA \ar[u]_{\mathcal{P} \cdot \mathcal{P} \cdot L \mathcal{A}} & \mathcal{P} \cdot \mathcal{P} \cdot LA \ar[u]_{\mathcal{P} \cdot \mathcal{P} \cdot L \mathcal{A}} & \mathcal{P} \cdot \mathcal{P} \cdot LA \ar[u]_{\mathcal{P} \cdot \mathcal{P} \cdot L \mathcal{A}} & \mathcal{P} \cdot \mathcal{P} \cdot LA \ar[u]_{\mathcal{P} \cdot \mathcal{P} \cdot L \mathcal{A}} & \mathcal{P} \cdot \mathcal{P} \cdot LA \ar[u]_{\mathcal{P} \cdot \mathcal{P} \cdot L \mathcal{A}} & \mathcal{P} \cdot \mathcal{P} \cdot LA \ar[u]_{\mathcal{P} \cdot \mathcal{P} \cdot L \mathcal{A}} \\
\mathcal{P} \cdot \mathcal{P} \cdot LA \ar[u]_{\mathcal{P} \cdot \mathcal{P} \cdot L \mathcal{A}} & \mathcal{P} \cdot \mathcal{P} \cdot LA \ar[u]_{\mathcal{P} \cdot \mathcal{P} \cdot L \mathcal{A}} & \mathcal{P} \cdot \mathcal{P} \cdot LA \ar[u]_{\mathcal{P} \cdot \mathcal{P} \cdot L \mathcal{A}} & \mathcal{P} \cdot \mathcal{P} \cdot LA \ar[u]_{\mathcal{P} \cdot \mathcal{P} \cdot L \mathcal{A}} & \mathcal{P} \cdot \mathcal{P} \cdot LA \ar[u]_{\mathcal{P} \cdot \mathcal{P} \cdot L \mathcal{A}} & \mathcal{P} \cdot \mathcal{P} \cdot LA \ar[u]_{\mathcal{P} \cdot \mathcal{P} \cdot L \mathcal{A}} \\
\mathcal{P} \cdot \mathcal{P} \cdot LA \ar[u]_{\mathcal{P} \cdot \mathcal{P} \cdot L \mathcal{A}} & \mathcal{P} \cdot \mathcal{P} \cdot LA \ar[u]_{\mathcal{P} \cdot \mathcal{P} \cdot L \mathcal{A}} & \mathcal{P} \cdot \mathcal{P} \cdot LA \ar[u]_{\mathcal{P} \cdot \mathcal{P} \cdot L \mathcal{A}} & \mathcal{P} \cdot \mathcal{P} \cdot LA \ar[u]_{\mathcal{P} \cdot \mathcal{P} \cdot L \mathcal{A}} & \mathcal{P} \cdot \mathcal{P} \cdot LA \ar[u]_{\mathcal{P} \cdot \mathcal{P} \cdot L \mathcal{A}} & \mathcal{P} \cdot \mathcal{P} \cdot LA \ar[u]_{\mathcal{P} \cdot \mathcal{P} \cdot L \mathcal{A}} \\
\mathcal{P} \cdot \mathcal{P} \cdot LA \ar[u]_{\mathcal{P} \cdot \mathcal{P} \cdot L \mathcal{A}} & \mathcal{P} \cdot \mathcal{P} \cdot LA \ar[u]_{\mathcal{P} \cdot \mathcal{P} \cdot L \mathcal{A}} & \mathcal{P} \cdot \mathcal{P} \cdot LA \ar[u]_{\mathcal{P} \cdot \mathcal{P} \cdot L \mathcal{A}} & \mathcal{P} \cdot \mathcal{P} \cdot LA \ar[u]_{\mathcal{P} \cdot \mathcal{P} \cdot L \mathcal{A}} & \mathcal{P} \cdot \mathcal{P} \cdot LA \ar[u]_{\mathcal{P} \cdot \mathcal{P} \cdot L \mathcal{A}} & \mathcal{P} \cdot \mathcal{P} \cdot LA \ar[u]_{\mathcal{P} \cdot \mathcal{P} \cdot L \mathcal{A}} }
\end{array}
\]
whose top composite $LQRA \rightarrow A$ is the left-bottom composite of the inner chamber of (5). Here, all unlabeled solid arrows are induced by the relevant structural maps to coequalizer objects. On the bottom row, two of the squares commute by naturality, while the one on the right commutes since $\lambda$ is an action.

On the other hand, we have a commutative diagram in Fig. 3 in which the composite through the top right corner agrees with the composite through the bottom left corner in the previous diagram. Most squares commute by naturality, whereas the upper right triangle is a triangular identity for the pair of adjoint functors.

The left-bottom composite of this diagram is the top-right composite of the inner chamber of (5). Thus (5) commutes.

Suppose that $g : A \rightarrow A'$ is a morphism of $\mathcal{P}$-algebras. We know that $RG$ is a morphism of $|Q|$-algebras, and we wish to show that this is a morphism of $Q$-algebras. That is, the
Fig. 3  Part of the proof of Lemma 3.7

diagram on the left below should commute.

\[ \begin{array}{c}
QRA \xrightarrow{QRg} QRA' \\
\downarrow^{\alpha} \downarrow^{\alpha'} \\
RA \xrightarrow{Rg} RA'
\end{array} \quad \begin{array}{c}
LQRA \xrightarrow{LQRg} LQRA' \\
\downarrow^{\hat{\alpha}} \downarrow^{\hat{\alpha}'} \\
LRA \xrightarrow{g} LA'
\end{array} \]

Of course the diagram on the left is adjoint to the diagram on the right. Expanding out the definitions of \( \hat{\alpha} \) and \( \hat{\alpha}' \), we have the following diagram.

\[ \begin{array}{c}
LQRA \xrightarrow{LQRg} LQRA' \\
\downarrow^{\hat{\alpha}} \downarrow^{\hat{\alpha}'} \\
P.LRA \xrightarrow{PLRg} P.LRA'
\end{array} \quad \begin{array}{c}
P.A \xrightarrow{g} P.A' \\
\downarrow^{\hat{\lambda}_A} \downarrow^{\hat{\lambda}_A'} \\
A \xrightarrow{g} A'
\end{array} \]

The top three squares commute by naturality, and the bottom square commutes since \( g \) is a map of \( P \)-algebras.

Suppose that \( A \) is a \( P \)-algebra and \( B \) is a \( Q \)-algebra. We have now established that the maps in Remark 6.4 indeed give a \( Q \)-algebra structure on \( RA = |\phi|_s A \) and a \( P \)-algebra structure on \( LB = |\phi|^* B \). We have a pair of functors

\[ \begin{array}{c}
Q\text{-alg} \xrightarrow{\phi^*} P\text{-alg} \\
\phi_s
\end{array} \]
and our goal is to show that they are adjoint. In order to prove Theorem 3.8, it is enough to show that the unit and counit of the adjunction

\[
\text{\text{\textit{Q}}}-\text{alg} \xrightarrow{\phi^*} \text{\textit{P}}\text{-\textit{alg}}.
\]

from Corollary 2.33 are compatible with this additional structure.

**Lemma 6.5** Let \( A \) be a \( \text{\textit{P}} \)-algebra, viewed by restriction as a \( \text{\textit{Q}}\)-algebra. The counit of the adjunction \( \phi^* \dashv \phi_* \) at \( A \) is a map of \( \text{\textit{P}} \)-algebras from \( \phi^* \phi_* A \) to \( A \).

**Proof** We must show that the diagram

\[
\begin{array}{c}
\text{\textit{P}} \text{\textit{L}} \text{\textit{R}} A \\
\downarrow \ \\
\text{\textit{L}} \text{\textit{R}} A
\end{array}
\xrightarrow{\text{\textit{P}} \text{\textit{L}} \text{\textit{R}} A} \begin{array}{c}
\text{\textit{P}} A \\
\downarrow \ \\
A
\end{array}
\]

(6)

commutes. The left-hand map utilizes the formula for the action \( \text{\textit{P}} \text{\textit{L}} B \to B \), where \( B \) is a \( \text{\textit{Q}} \)-algebra. This, in turn, relies on the \( \text{\textit{Q}} \)-action, called \( \alpha \), on \( R A \). Refer to Remark 6.4 for both of these.

The bottom two squares of the following diagram commute by naturality, the rightmost cell commutes because the coequalizers are well-behaved, the triangle commutes by a triangular identity, and the top map is defined so that the odd-shaped upper chamber commutes.

The composite (upper-right to lower-left corners) from \( \text{\textit{P}} \text{\textit{L}} \text{\textit{R}} A \) to \( \text{\textit{L}} \text{\textit{R}} A \) is the left-hand map from (6). Thus commutativity of (6). follows from this commutative diagram. \( \square \)

**Lemma 6.6** Let \( B \) be a \( \text{\textit{Q}} \)-algebra, viewed by restriction as a \( \text{\textit{Q}}\)-algebra. The unit of the adjunction \( \phi^* \dashv \phi_* \) at \( B \) is a map of \( \text{\textit{Q}} \)-algebras from \( \phi^* \phi_* B \) to \( B \).

**Proof** As we already know that the unit is a map of \( \text{\textit{Q}} \)-algebras, it is sufficient to show that the diagram

\[
\begin{array}{c}
\text{\textit{Q}} B \\
\downarrow \text{\textit{L}}
\end{array}
\xrightarrow{\text{\textit{Q}} \text{\textit{L}} \text{\textit{R}} B} \begin{array}{c}
\text{\textit{Q}} \text{\textit{R}} B \\
\downarrow \alpha
\end{array}
\xrightarrow{\text{\textit{R}} B} \begin{array}{c}
\text{\textit{R}} B
\end{array}
\]
commutes, where the map on the right is the induced action coming from the $P$-action on $LB$. By adjointness, this is equivalent to showing that the triangle

$$L(Q·B) \xrightarrow{L(Q·η)} L(Q·RLB) \xrightarrow{Lλ_B} LB$$

commutes, where the diagonal map is induced from $\hat{α}$. Expanding this slightly, we have the following:

$$L(Q·B) \xrightarrow{L(Q·η)} L(Q·RLB)$$

$$\xrightarrow{Lλ_B} \xrightarrow{P·LB} \xrightarrow{P·LRLB} \xrightarrow{P·Lη} \xrightarrow{P·LRLB}$$

The bottom left chamber is just the definition of the $P$-action on $LB$, while the other two chambers commute automatically. The composition from upper right to bottom left is induced from $\hat{α}$, so we have shown that the triangle we want to commute does commute. □

**Proof of Theorem 3.8** In light of Lemma 6.5 and Lemma 6.6, we know that the unit and counit of the adjunction $|φ|^* \dashv |φ|^*$ lift to $Q$-alg and $P$-alg. Since the functors $Q$-alg $→ |Q|$-alg and $P$-alg $→ |P|$-alg are faithful, naturality and the triangle identities follow from the corresponding properties for $|φ|^* \dashv |φ|^*$. □

**Remark 6.7** This proof, at the current (colored and categorical) level of generality, is a bit abstract. It is a diverting exercise to verify the triangle identities by hand in, say, the case of monochrome operads in sets.

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**A Examples of colored operads**

Colored operads which describe various types of generalized operads are built on the notion of **graphs with loose ends**, which we will just call **graphs** in what follows. In such graphs, edges need not be attached to anything at one or both edges, or may be attached to themselves, forming a circle. A picture is instructive, and we have included one in Fig. 4. Any such graph consists of

- A finite set of vertices $V$.
- A finite set of edges $E$.
- For each vertex $v$, a set $nb(v)$ of germs of edges at the vertex; there is a function $\bigcup_{v \in V} nb(v) → E$ whose fibers have cardinality less than three.
The only additional condition to be a graph is that each fiber of $B \sqcup \bigsqcup_{v \in V} \text{nb}(v) \to E$ has cardinality either zero or two. If $e$ is an edge such that this fiber is empty, we regard $e$ as being like a circle, while if two elements of $B$ map to $e$, we regard $e$ as being like an interval. Alternative presentations of such graphs may be found in [2, Definition 13.1], [14, Definition 1.1], and [29, Sect. 1.2]. There is a realization functor from graphs to topological spaces whose details we omit; when we refer to topological properties of a graph we always implicitly use this functor. In what follows we always restrict to connected graphs.

An ordered graph is a graph $G$ with the following additional structure:

- A total ordering on the set of vertices $V$.
- A total ordering on each $\text{nb}(v)$.
- A total ordering on the boundary $B$.

The following is a special case of [21, §4.5], when $C$ is a point. Raynor’s term ‘CSM’ refers to what we call ‘modular operad’ in this paper, whereas what Raynor calls a ‘modular operad’ we would call a ‘non-unital modular operad’.

**Example A.1** (Modular operads) Let $\mathcal{M}$ be the $\mathbb{N}$-colored collection whose elements are isomorphism classes of ordered graphs. Specifically, an element of $\mathcal{M}(k_1, \ldots, k_n; p)$ will be represented by an ordered graph $G$ with $n$ vertices $\{v_1, \ldots, v_n\}$ so that $\text{nb}(v_j)$ has cardinality $k_j$ and $B(G)$ has cardinality $p$. There is a function

$$\mathcal{M}(k_1, \ldots, k_n; p) \times \prod_{j=1}^{n} \mathcal{M}(\ell_{j,1}, \ldots, \ell_{j,m_j}; k_j) \to \mathcal{M}(\ell_{1,1}, \ldots, \ell_{n,m_n}; p)$$

which replaces the $v_j \in V(G)$ with a graph $H_j$ with the gluing specified by the unique ordered bijection $B(H_j) \to \text{nb}(v_j)$. This type of graph substitution is both unital (with respect to corollas) by [29, Lemma 5.31] and associative by [29, Theorem 5.32], hence $\mathcal{M}$ is a colored operad. Algebras over $\mathcal{M}$ are a kind of modular operad.

The underlying category of $\mathcal{M}$ consists of ordered graphs with precisely one vertex. Each edge will either be loop at the vertex, or be connected at one end to $v$. Then an equivalent presentation of this category has objects $\mathbb{N}$ and morphisms from $k$ to $p$ consisting of the data:

1. an involution $\iota$ on $[k]$ with precisely $p$ fixed points (the boundary edges) and $\frac{k-p}{2}$ free orbits (the loops), and
2. a bijection between the fixed points of $\iota$ with $[p]$.

In particular the set of morphisms is nonempty if and only if $k \geq p$ and $k \equiv p \pmod{2}$.
The following example can be recovered from the Feynman category of \[16, \S 2.3.3\] using the biequivalence of \[1, \text{Theorem 5.16}\].

**Example A.2** (A genus-aware version) We also have \(\mathcal{M}^g\), which is a \(\mathbb{N}^2\)-colored collection defined as follows. Let \(\mathcal{M}^g((k_1, g_1), \ldots, (k_n, g_n); (p, g))\) be the set of isomorphism classes of ordered graphs as before but where the graph \(G\) is restricted to have first betti number \(g - \sum g_j\). That is, operations of \(\mathcal{M}^g\) are genus-decorated graphs. The composition map defined for \(\mathcal{M}\) respects genus appropriately, making \(\mathcal{M}^g\) an operad. There is a map of operads \(\mathcal{M}^g\) to \(\mathcal{M}\) which on color sets is projection on the first factor, \((k, g) \mapsto k\). In the underlying category of \(\mathcal{M}^g\), the set of morphisms from \((k, g)\) to \((p, h)\) is given by \(|\mathcal{M}|(k; p)\) when \(k - p = 2(h - g)\), while in other cases it is empty.

**Example A.3** (Cyclic operads) Let \(\mathcal{C}\) be the suboperad of \(\mathcal{M}\) with the same color set consisting of ordered graphs which are simply-connected. Algebras over \(\mathcal{C}\) are a variant of cyclic operads. They are slightly more general than the cyclic operads of \[11\] because they contain “constants” in level 0 and two “elements” in level 1 can be paired to give a constant.

We can define a further suboperad \(\mathcal{C}_{\text{GK}}\) recovering Getzler and Kapranov’s cyclic operads precisely. This suboperad has colors the positive integers, and its elements have the additional restriction that the boundary set \(B\) is nonempty. A version of \(\mathcal{C}_{\text{GK}}\) appeared in \[20, \S 1.6.4\].

In both cases, the underlying category is a disjoint union of symmetric groups. That is, the morphisms between different colors are empty, while the endomorphisms of \(n\) are the symmetric group \(\Sigma_n\).

**Example A.4** (Operads) We further restrict \(\mathcal{C}_{\text{GK}} \subseteq \mathcal{M}\) to give an operad \(\mathcal{O}\) governing monochrome operads. This is a variant of the description of \[4, \S 1.5.6\]. Our presentation is slightly more complicated but has the virtue of having a direct relationship with \(\mathcal{C}\) and \(\mathcal{M}\). See also \[7, \S 1.2\] and \[29, \S 14.1\].

Suppose that \(G\) is an ordered graph in \(\mathcal{C}_{\text{GK}}\), that is, suppose that \(G\) is a tree with at least one boundary element. There is a unique edge flow in the direction of the first element of \(B(G)\), which we call the root. That is, we have a partial order with the root as the minimal element. This allows us to declare that the root of a vertex \(v\) is the element of \(\text{nb}(v)\) that is nearest to the global root. We call \(G\) a rooted tree just when, for each \(v\), the root of \(v\) is also the minimal element of \(\text{nb}(v)\). We declare that \(\mathcal{O} \subseteq \mathcal{C}_{\text{GK}}\) is the collection of all rooted trees. Algebras over \(\mathcal{O}\) are operads.

The underlying category is again a disjoint union of symmetric groups. But in the underlying category of \(\mathcal{O}\), the endomorphisms of \(n\) are the symmetric group \(\Sigma_{n-1}\) (the root remains fixed).

Let us give a derived example. There is a suboperad \(\mathcal{O}_{\text{ns}}\) with colors again the positive integers, but fewer operations in most profiles. Namely, for a rooted tree to be in \(\mathcal{O}_{\text{ns}}(k_1, \ldots, k_n; p)\), the orderings on \(B\) must be compatible with the orderings on each \(\text{nb}(v)\). Precisely, suppose that \(a_1\) and \(a_2\) are elements of \(\text{nb}(v)\) and \(b_1\) and \(b_2\) are elements of \(B\) so that the image of \(b_i\) in \(E\) is greater than or equal to the image of \(a_i\) in the partial order on \(E\). Compatibility means that if \(a_1 < a_2\) in the total ordering on \(\text{nb}(v)\), then \(b_1 < b_2\) in the total ordering on \(B\). Algebras over \(\mathcal{O}_{\text{ns}}\) are nonsymmetric operads.

The underlying category of \(\mathcal{O}_{\text{ns}}\) has only the identity in each color because, for a graph with one vertex, the compatibility condition forces the orders on \(B \cong E\) and \(\text{nb}(v) \cong E\) to coincide. This version was studied by van der Laan \[18\].

Our convention for the colors of \(\mathcal{O}\) are shifted by one from all conventions in the literature. We make this nonstandard choice because we are interested in the comparison with \(\mathcal{C}\) and \(\mathcal{M}\) where this shift is natural.
The operad $O$ has operations given by rooted trees. One can imagine analogous operads whose operations are other kinds of directed graphs and whose algebras are dioperads, properads, wheeled operads, and so on. A general construction of such operads is included in Sect. 14.1 of [29], so we will omit further details here.

**Example A.5** (Colored variants) Given a set $A$ of colors, one can form an operad $O^A$ whose set of colors is $\coprod_{n \geq 1} A \times n$ and whose operations are isomorphism classes of ordered rooted trees equipped with a function from the set of edges to the set $A$. Algebras over this $\coprod_{n \geq 1} A \times n$-colored operad are precisely $A$-colored operads. When $A$ is a point, one recovers the operad $O$. The underlying category of $O^A$ is a groupoid of positive length lists of elements of $A$.

This same pattern extends in a straightforward way to other types of directed graphs, and actually falls under the general construction of [29, §14.1]. For operadic structures built on undirected graphs, like cyclic operads and modular operads, one has the flexibility to work with an involutive set of colors $A$. The main difference is that the coloring function $E \to A$ should be replaced with an involutive function from the involutive set of oriented edges to $A$. See, e.g., [9, §2], [15], [21, §4.5], and [14, §2] for implementations of this involutive perspective.

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