Kinematic dynamos in multiple scale flows

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Abstract. A 2D periodic flow is investigated in kinematic dynamo simulations as a model for a turbulent flow. The flow is a superposition of periodic arrays of right handed helices. The arrays have different periodicity lengths but contain eddies of identical helicity. These model flows reproduce the multiple scale character of helical turbulence. Numerical calculations are used to investigate whether adding small scale turbulence to a large scale flow improves or deteriorates the ability of the flow to generate magnetic fields. Two different, sometimes competing effects are revealed. The small scale eddies can in suitable geometries be more efficient at producing magnetic fields than the large scale eddies, but the distortion of the magnetic field caused by the small eddies is detrimental to the dynamo action of the large eddies.

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1. Introduction

It is commonly accepted that the magnetic fields of most celestial bodies are created by the dynamo effect which converts kinetic into magnetic energy. The main ingredient of the dynamo effect is a liquid conductor in motion such that a situation with zero magnetic field is unstable. Small magnetic seed fields are amplified in a dynamo and grow until the magnetic forces acting on the liquid conductor modify its motion to prevent further growth. The kinematic dynamo problem ignores the back reaction of the magnetic field on the flow and only inquires whether a given flow yields growing magnetic fields or not.

Not every arbitrary flow leads to magnetic field generation. Most importantly, the flow needs to be fast enough. This is quantified by a magnetic Reynolds number \( R_m \) which needs to exceed a critical value. The critical value depends on the form of the velocity field. In addition, anti-dynamo theorems tell us that flow fields in dynamos have to meet some structural requirements. For example, streamlines may not be confined to planes.

In most cases, the dynamo problem has to be solved numerically. This implies that the investigated flows are either laminar or only modestly turbulent. Flows in nature or in laboratory experiments supporting magnetic fields are on the contrary very turbulent, so that the validity of simulations aimed at geo- or astrophysical applications or even laboratory experiments is always uncertain. We therefore have to ask what can be learned about fully developed dynamos from laminar or weakly chaotic dynamos investigated numerically. The most crucial question is whether a flow predicted to amplify magnetic fields according to numerical simulations is unable to do so if small scale turbulence or other fluctuations are added. In short, we want to know whether turbulence is helpful or deleterious to the dynamo effect. Small scale turbulence can contribute to the generation of magnetic fields, but it also increases mixing and diffusion, which causes magnetic fields to dissipate.

The literature contains examples of both: turbulence helping and destroying the dynamo. Turbulence introduces two aspects: temporal fluctuations and a velocity field with eddies of many different length scales. Pédrélis and Fauve [1] shows that temporal fluctuations increase the critical magnetic Reynolds number in the example of a 2D periodic flow. At the same time, it is well known that a whole class of fast dynamos relies on time variations in order to work as a dynamo at all. In as far as different length scales are concerned, [2] gives an example of a dynamo which is suppressed if small scale turbulence is added. Ponty \textit{et al} [3] studies a flow with a critical magnetic Reynolds number almost insensitive to turbulence. Laval \textit{et al} [4] presents a problem in which the critical magnetic Reynolds number decreases if turbulence at small scales is added but increases if turbulence is added at large scales. In the framework of mean field magnetohydrodynamics, [5] finds that turbulence increases the critical magnetic Reynolds in a screw dynamo. Finally, [6] shows in an analytical model that the spectrum of kinetic energy must decay fast enough towards small scales for dynamo action.

Clearly, turbulence can have a variety of effects which remains to be understood in general. The purpose of this paper is to investigate the issue within the kinematic dynamo problem so that we can choose a class of flows which allows simple numerical calculations but also reasonable analytic approximations. The obvious choice is to use the 2D periodic flow introduced in [7]. This flow is a prototypical \( \alpha^2 \)-dynamo and functions according to the same principle as the one believed to be responsible for planetary dynamos. Turbulence is readily mimicked by adding periodic velocity fields with different periodicity lengths. The velocity field is stationary in time.
It will be shown that the effects of turbulence observed in a variety of flows in the references above already occur in this simple flow.

2. Mathematical formulation

Let us consider the non-dimensional induction equation for the magnetic field \( B(r, t) \) and a prescribed velocity field \( v(r, t) \) as functions of position \( r \) and time \( t \) in the form

\[
\partial_t B + \nabla \times (B \times v) = \frac{1}{Rm} \nabla^2 B, \quad \nabla \cdot B = 0, \tag{1}
\]

\( Rm \) is the magnetic Reynolds number. In all applications, the induction equation is solved in a finite volume surrounded by vacuum. In order to simplify numerical and analytical treatments, the induction equation will be solved for the present study in a cartesian box with periodic boundary conditions along the axes \( x, y, z \) and size \( 0 \leq x, y \leq a, 0 \leq z \leq d \). The velocity \( v \) is chosen to be a sum

\[
v = \sum_n c_n v_n, \tag{2}
\]

where the \( c_n \) are real coefficients and each term \( v_n \) represents a 2D periodic flow independent of \( z \) and periodicity length \( a_n = a/n \) along the \( x \)- and \( y \)-axes

\[
v_n = \begin{pmatrix}
\sqrt{2} \sin \left( \frac{2\pi}{a_n} x \right) \cos \left( \frac{2\pi}{a_n} y \right) \\
-\sqrt{2} \cos \left( \frac{2\pi}{a_n} x \right) \sin \left( \frac{2\pi}{a_n} y \right) \\
2 \sin \left( \frac{2\pi}{a_n} x \right) \sin \left( \frac{2\pi}{a_n} y \right)
\end{pmatrix}.
\tag{3}
\]

The case \( v = v_1 \) was first investigated by Roberts [7] and is sketched in figure 1. The other \( v_n \) have a different periodicity length but otherwise have the same structure. In particular, all \( v_n \) have the same helicity. In helical turbulence, helicity is a quantity with a cascade. It is therefore appropriate to choose terms with helicities of the same sign in (2) if the total velocity field is to mimic a turbulent helical flow.

For any choice of the coefficients \( c_n \), the flow (2) is independent of \( z \) and time, so that the solutions of (1) must be of the form

\[
B(r, t) = B_0(x, y) \cdot \exp (pt) \cdot \exp (ik_z z), \tag{4}
\]

with a growth rate \( p \) and a vertical wavenumber \( k_z \). Depending on the chosen \( c_n \), the flow has more or less energy so that it will be interesting to introduce an effective magnetic Reynolds number defined by

\[
Rm_{\text{eff}} = Rm \sqrt{2 E_{\text{kin}} / V_c} \tag{5}
\]
Figure 1. Sketch of the flow $v_1$. Helical eddies are arranged in a square grid. The sense of the swirling motion in the $(x, y)$-plane is indicated by arrows. Signs show the direction of the $z$-component of $v_1$.

where $E_{\text{kin}}$ is the kinetic energy, $E_{\text{kin}} = \int \frac{1}{2} v^2 dV$. The volume integral extends over one periodicity cell, $\int dV \ldots = \int_0^a dx \int_0^a dy \int_0^d dz \ldots$, and $V_c = a^2 d$ is the volume of the periodicity cell.

It will be important to know the contribution of different $v_n$ to the production of mean magnetic field, i.e., the magnetic field averaged over the $(x, y)$-plane. If this average is denoted by an overbar, so that

$$\bar{B}(z, t) = \frac{1}{a^2} \int_0^a dx \int_0^a dy \bar{B}(r, t),$$

the induction equation becomes

$$\partial_t \bar{B} + \nabla \times \bar{B} \times v = \frac{1}{Rm} \nabla^2 \bar{B}. \quad (7)$$

Multiplying this equation by $\bar{B}$, integrating over $z$, and identifying the second term with the production of mean magnetic field, we obtain the following expression for the production $P$ and its spectral decomposition $P_i$,

$$P = \left( \int \bar{B} \cdot \nabla \times \bar{v} \times \bar{B} dV \right) \left( \int \frac{1}{2} \bar{B}^2 dV \right)^{-1} = - \left( \int \bar{v} \cdot [(\nabla \times \bar{B}) \times \bar{B}] dV \right) \left( \int \frac{1}{2} \bar{B}^2 dV \right)^{-1} = \sum_i P_i. \quad (8)$$

The production in (8) is normalized with the energy of the mean magnetic field because the amplitude of the magnetic field is arbitrary in kinematic calculations. Throughout this paper we
will look at $Rm$ small enough so that not every single eddy is a dynamo and a large scale field with nonzero mean in the $(x, y)$-plane is generated. That is why it is reasonable to normalize all quantities relevant for the dynamo process by the energy contained in the mean field.

For the interpretation of the numerical results below, it will prove useful to spectrally decompose $P$ and various other quantities in the $(x, y)$-plane. Assume two arbitrary vector functions, $F(r, t)$ and $G(r, t)$, and their Fourier decompositions in the periodic box

$$F = \sum_k \hat{F}_k \exp(ikr), \quad G = \sum_k \hat{G}_k \exp(ikr).$$

We will be interested in quantities $Q$ of the form

$$Q = \int F \cdot G dV = V c \sum_k \hat{F}_k^* \hat{G}_k = \sum_k \hat{Q}_k.$$  
(10)

For easier graphical representation, the $\hat{Q}_k$ will be binned

$$\sum_k \hat{Q}_k = \sum_i Q_i \quad \text{with} \quad Q_i = \sum_{k \in \text{bin } i} \hat{Q}_k,$$

and

$$k \in \text{bin } i \quad \text{if} \quad i - 1 \leq \left( \frac{k_1^2 + k_2^2}{2} \right)^{1/2} \frac{a}{2\pi} < i.$$  
(12)

This procedure is applied in (8) to obtain $P = \sum_i P_i$. A positive $P_i$ indicates that Fourier components of the velocity field with wavevectors $k$ in bin $i$ contribute to the creation of a mean field, whereas a negative $P_i$ indicates that the same components destroy the mean field.

Another quantity of the same form is the magnetic helicity

$$h = \left( \int (\nabla \times B) \cdot B dV \right) \left( \int \frac{1}{2} B^2 dV \right)^{-1} = \sum_i h_i.$$  
(13)

3. The mean field approach

In order to interpret the numerical results, it will be useful to have a theoretical guideline. Mean field magnetohydrodynamics offer a convenient approach for small $Rm$. It will be sufficient to consider the case where the velocity field is characterized by a single length scale, $v = v_1$. We decompose the magnetic field into its mean $\bar{B}$ and the remainder $b$, $B = \bar{B} + b$. The averaged induction equation is

$$\partial_i \bar{B} + \nabla \times b \times v_1 = \frac{1}{Rm} \nabla^2 \bar{B},$$  
(14)

which we can subtract from the full induction equation to obtain

$$\partial_i b + (v_1 \cdot \nabla) \bar{B} - (\bar{B} \cdot \nabla)v_1 = \frac{1}{Rm} \nabla^2 b - \{ \nabla \times (b \times v_1) - \nabla \times b \times v_1 \}.$$  
(15)
Let us now focus on the case $Rm$ close to its critical value, $Rm_c$, so that the magnetic field varies slowly and $\partial_t b$ can be neglected in (15). If $Rm_c$ is small, then $|b| \ll |\bar{B}|$ and the curly bracket is negligible, so that (15) reduces to

$$(v_1 \cdot \nabla)\bar{B} - (\bar{B} \cdot \nabla)v_1 = \frac{1}{Rm} \nabla^2 b.$$ (16)

This equation is solved by

$$b = -\frac{1}{2} \left(\frac{a}{2\pi}\right)^2 Rm((v_1 \cdot \nabla)\bar{B} - (\bar{B} \cdot \nabla)v_1).$$ (17)

because $\nabla^2 v_1 = -2(2\pi/a)^2 v_1$ (note that in some previous papers in which almost identical equations are derived [8, 9], $a$ was used for the eddy size which is half the periodicity length of $v_1$). Inserting this expression for $b$ into (15) and evaluating the averages yields

$$\partial_t \bar{B} = \nabla \times \alpha \bar{B} + \frac{1}{Rm} \nabla^2 \bar{B} + Rm \frac{1}{2} \frac{a^2}{(2\pi)^2} \partial_{zz} \bar{B},$$ (18)

where all entries of the tensor $\alpha$ are zero except $\alpha_{xx} = \alpha_{yy} = -Rm \cdot a/(\sqrt{8\pi})$. The first term on the right hand side is responsible for magnetic field generation if indeed there is a growing field. The second term is the diffusion, and the last term on the right hand side is a diffusion term which describes the eddy diffusion or enhanced dissipation introduced by the velocity field.

One can deduce from (18) that $Rm_c$ satisfies

$$Rm_c^2 = \frac{8\pi^2}{\sqrt{2ad} - a^2},$$ (19)

and the corresponding mean field is of the form

$$\bar{B} \propto \left(\cos \left(\frac{2\pi}{d} z\right), \sin \left(\frac{2\pi}{d} z\right), 0\right).$$ (20)

The expression for $Rm_c$ has a minimum for $a = d/\sqrt{2}$. This means that in a superposition like (2), one of the $v_n$ will be the most efficient in generating the magnetic field, and it won’t necessarily be the component with the largest periodicity length. The minimum is determined by a trade-off between the induction, which depends linearly on $a$ in (18), and the eddy diffusion caused by $v_1$, which depends quadratically on $a$ in (18). However, the formula for $Rm_c$ is obviously wrong for large $a$ at constant $d$ since $Rm_c^2$ becomes negative. This is due to the assumption of small $Rm$ made during its derivation, so that the existence of that minimum in $Rm_c$ will have to be verified numerically.

Let us now turn to helicities. The helicity of $v_1$ is positive everywhere, $(\nabla \times v_1) \cdot v_1 > 0$, because $v_1$ satisfies $\nabla \times v_1 = \sqrt{2}2\pi/a \cdot v_1$. For the mean magnetic field, one has $\nabla \times \bar{B} = -2\pi/d \cdot \bar{B}$, so that $(\nabla \times \bar{B}) \cdot \bar{B} < 0$ everywhere. The mean field has the opposite helicity from the velocity field. For the small scale magnetic field, it can be shown in the limit $a \ll d$ that
Numerical simulations will show that \( \bar{B} \) and \( b \) have opposite helicities even if \( a \ll d \) does not hold.

What if the velocity field is the multiscale flow (2)? In this case, (16) is solved by

\[
b = \sum_n c_n b_n
\]

with

\[
b_n = -\frac{1}{2} \left( \frac{a_n}{2\pi} \right)^2 Rm[(v_n \cdot \nabla)\bar{B} - (\bar{B} \cdot \nabla)v_n].
\]

(21)

\( b_n \) has the same spatial periodicity as \( v_n \). When computing \( b \times v \) in (14), only terms of the form \( b_n \times v_n \) remain because \( b_n \times v_m = 0 \) for \( n \neq m \). As a result, the coefficients in front of \( \partial_z \bar{B} \) in (18) and the \( \alpha \)-tensors contributed by the different \( v_n \) simply add. At the level of approximation used in this section, velocity components of different length scales generate magnetic fields independently of each other. The numerical simulations in the next section explore whether this is still true if the condition \( a \ll d \) is violated or if \( Rm \) is large. It will be shown that in general, significant interaction can occur, to the extent that the superposition (2) is not a dynamo even though individual constituents of that velocity field are capable of sustaining a magnetic field.

4. Numerical simulations

The induction equation (1) is solved with a spectral method which uses Fourier modes to discretize in space the three cartesian components of the magnetic field. Time stepping is done with a second order Adams–Bashforth scheme for the induction term coupled to a Crank–Nicolson step for the diffusion term. The induction term is computed in direct space. The method is described in greater detail in [10]. The velocity field is independent of \( z \) and the magnetic field populates a single vertical wavenumber so that four grid points in direct space are enough to discretize in \( z \). The resolution in \( x \) and \( y \) was typically 32 points.

After some transient, the magnetic energy grows or decays exponentially if \( v \) is independent of time. The growth rate \( p \) in (4) is numerically determined from the computed time dependence of the magnetic energy.

An automatic root search finds \( Rm_c \), as the value of \( Rm \) for which \( p = 0 \). All results reported below are for \( Rm = Rm_c \). The critical field is the most interesting one because the solution of the full dynamo problem with a saturated magnetic field also has a velocity field which, if introduced into the kinematic problem (1), yields zero growth rate. The kinematic dynamo problem thus is the most relevant to the full problem if \( Rm = Rm_c \).

From now on, the periodicity length along \( z \) is set to \( d = 1 \).

Figure 2 confirms the prediction made in section 3 that there is an optimal periodicity length \( a \) for which the critical \( Rm \) is minimal if \( v = v_1 \). The minimum is located near \( a/d \approx 0.5 \) and not exactly at \( a/d = 1/\sqrt{2} \) as predicted by mean field theory at the level of approximation of section 3 because \( Rm \) is around 20 in figure 1 and not small. The same optimum at \( a/d \approx 0.5 \) has been found before in spherical geometry [8].

Before studying velocity fields with many length scales, we will consider in guise of preparation the case of only two components with different periodicity. For the purpose of demonstration, let us take \( v = v_1 + c_4 v_4 \), so that the two components have periodicities \( a \) and \( a/4 \). For simplicity, \( a \) will be called the ‘large scale’ and \( a/4 \) the ‘small scale’. The total kinetic energy changes as \( c_4 \) is varied. In order to compute critical \( Rm \) which can be compared with each other, one can either normalize \( v \) to unit kinetic energy, or report the critical value of the
The critical magnetic Reynolds number $R_m$ as a function of $a$ for the velocity field $v_1$ and $d = 1$.

$R_{m,c,\text{eff}}$ for the flow $v_1 + c_4 v_4$ as a function of $c_4$ for $d = 1$ and $a = 0.75$ (long dashed, red), $a = 1$ (solid, black), $a = 1.25$ (dot dashed, blue) and $a = 1.5$ (short dashed, green).

effective magnetic Reynolds number defined in (5), $R_{m,c,\text{eff}}$, which will be done here. Figure 3 shows $R_{m,c,\text{eff}}$ as a function of $c_4$ for various $a$. Depending on $a$, increasing $c_4$ may improve or deteriorate the dynamo.

One contribution to this effect is readily explained with the help of figure 2. If $v_4$ is less efficient than $v_1$ in generating magnetic fields (meaning that the critical $Rm$ for $a/4$ is larger than for $a$ in figure 2), increasing $c_4$ increases the proportion of inefficient flow in the total flow and $R_{m,c,\text{eff}}$ increases. For small $a$, we thus expect $R_{m,c,\text{eff}}$ to increase with $c_4$. For large $a$ on
the contrary, $v_4$ is more efficient than $v_1$ and $Rm_{c, eff}$ should decrease with increasing $c_4$. This behaviour is qualitatively observed in figure 3, but the transition from increasing to decreasing behaviour already occurs at $a$ around 1, which shows that yet another effect must be important.

The effect of the velocity field on the magnetic field can be quantified by the work flow performed against the Lorentz force. In general, arbitrary velocity fields deform magnetic field lines by introducing small scales into the magnetic field which enhances diffusion and is detrimental for the dynamo. Only the energy put into the least damped magnetic mode is useful for the dynamo. In the cases studied here, dynamo fields with a nonzero average over $(x, y)$-planes are generated, i.e., fields with $B \neq 0$. We are thus led to look at the production of mean field defined in (8). With the binning introduced in section 2, $v_1$ contributes to $P_2$ and $v_4$ contributes to $P_5$. Figure 4 shows $P_2$ as a function of $c_4$, again for different $a$. As $c_4$ is increased, the production by the large scale decreases in all cases. This is partly due to the normalization with the energy of the mean magnetic energy which increases if there is also a productive small scale. However, $P_2$ can become negative, in which case the large scale is destructive for dynamo action due to the presence of the small scale! The opposite does not happen: the small scale always makes a positive contribution to mean field generation, no matter how much of the large velocity component is present, as shown in figure 5.

Essentially the same behaviour is observed if $v$ contains more terms. The following velocity fields were investigated with algebraically decaying coefficients

$$v = \sum_{n=1}^{8} c_n v_n, \quad c_n = n^{-\gamma}. \quad (22)$$

Figure 6 shows $Rm_{c, eff}$ for different $\gamma$ and $a$. If $\gamma$ is small, the spectrum is broad and many scales contribute significantly. If $\gamma$ is large, only $v_1$ makes an important contribution. If $a < d/2$,
Figure 5. $P_5$, the production of mean magnetic field contributed by $v_4$, as a function of $c_4$ for $d = 1$ and $a = 0.75$ (long dashed, red), $a = 1$ (solid, black), $a = 1.25$ (dot dashed, blue) and $a = 1.5$ (short dashed, green).

Figure 6. $Rm_{c,\text{eff}}$ as a function of $\gamma$ for $d = 1$ and $a = 0.5$ (dot dashed, black), $a = 0.75$ (long dashed, red), $a = 1$ (solid, black), $a = 1.25$ (dot dashed, blue) and $a = 1.5$ (short dashed, green).

decreasing $\gamma$ adds more inefficient components of velocity to $v_1$, so that $Rm_{c,\text{eff}}$ increases with decreasing $\gamma$. But if $a$ is equal to $d$ or larger, two effects compete: a broad spectrum adds more efficient scales, at the same time, the presence of the small eddies reduces the efficiency of the large scales. We therefore find non monotonous behaviour in figure 6. Figure 7 shows $P_l$ for $\gamma = 0.5$ and $a = 1.25$. The largest scales indeed have negative $P_l$, which means that they destroy the magnetic field.
We finally rationalize why the $P_i$ can become negative. There cannot be any formally rigorous argument in the framework of section 3 because different length scales are independent within the approximations made there, but we can gain some intuition from that section. The mean induction contributed by the velocity component $\mathbf{v}_n$ is $\mathbf{b}_n \times \mathbf{v}_n$, because the averages $\mathbf{b}_n \times \mathbf{v}_n$ vanish for $n \neq m$. The magnetic field $\mathbf{b}_n$ of the same periodicity as the velocity component $\mathbf{v}_n$ is given by (21) in the mean field approach. If we go beyond the approximation of section 3, the field on a large scale, say $\mathbf{b}_n$, is modified by the presence of small eddies, say $\mathbf{v}_m$. In the simplest picture, $\mathbf{b}_n$ becomes a superposition of expression (21) and the field distortion created by $\mathbf{v}_m$. This qualitatively changes the structure of $\mathbf{b}_n$ if $\mathbf{v}_m$ is a dynamo on its own: it was stated in section 3 that in the case of a single scale velocity field, the helicity of the magnetic field is opposite at large and small scales. If the small scale component of $\mathbf{v}$, $\mathbf{v}_m$, generates a magnetic field, it will have negative helicity at large scales and positive helicity at small scales. If the field induced by $\mathbf{v}_m$ is strong enough, the large scale component of $\mathbf{v}$, $\mathbf{v}_n$, operates on a magnetic field $\mathbf{b}_n$ of the opposite helicity it would have in the absence of small eddies. On the contrary, magnetic field generated by the large scale component $\mathbf{v}_n$ creates a magnetic field of negative helicity at small length scales, which is the same helicity the magnetic field would have at small scales in the absence of large scale eddies. The presence of small scale eddies can therefore qualitatively modify the structure of the magnetic field at the scale of larger eddies but not the other way round. Assuming that in the case of a single scale velocity field the magnetic field chooses a structure which optimizes dynamo action at that scale, one is led to conclude that adding small scale eddies can hurt the dynamo production by the large scales. This assumption also implies that only those scales produce mean magnetic field at which the helicity of the magnetic field is opposite to the helicity of the mean magnetic field. Figure 8 shows the helicity spectrum for the same parameters as in figure 7. At the scales where $P_i$ is negative, the helicity is negative.

The assumption just made is purely heuristic and there is no rigorous connection between $P_i$ and $h_i$ because $\mathbf{b} \times \mathbf{v}$ is not formally linked to helicity. In fact, combinations of $a$ and $\gamma$ exist.

Figure 7. Mean field production $P_i$ as a function of wavenumber $k$ in the $(x, y)$-plane, $k = (k_x^2 + k_y^2)^{1/2}$, for $a = 1.25$ and $\gamma = 0.5$. 

![Figure 7](http://www.njp.org/)
for which $P_i$ and $h_i$ do not change sign at exactly the same $k$. Nonetheless, the discussion of
the previous paragraph and figures 4, 5, 7 and 8 are neatly summarized by the following picture: the
magnetic field generated by eddies of a given size has a structure at length scales larger than the
eddy size qualitatively different from the structure of the same field at length scales smaller than
the eddy size. One indicator of that structural difference is the change of sign of helicity in going
from large to small scales. Heuristic reasoning suggests, and numerical simulation demonstrates,
that this magnetic field structure has the property of decreasing the magnetic field production
of eddies of larger size, but does not deteriorate the dynamo action of eddies of smaller scale.

5. Conclusion

The 2D periodic helical flow (3) has already been of great pedagogical value in the past because
it demonstrates important effects of dynamo theory in the simplest possible setting. This is
exploited here to study the interaction between different length scales in the dynamo process in
a well controlled flow and to expose two mechanisms by which this interaction may occur.

In the examples investigated here, adding small scales to a large scale velocity field can be
both helpful and harmful to dynamo action. The first relevant effect is that for a given system
size, a certain eddy size is most efficient at generating the magnetic field, and this eddy size need
not be the largest possible length scale. There is an optimal eddy size because a velocity field
generating a magnetic field also increases dissipation of magnetic field through eddy diffusion.
The second important effect has to do with the structure of the magnetic field generated by the
dynamo process which is such that the presence of small scales in the velocity field deteriorates
the efficiency of the large scales, whereas the presence of large scales in the flow does not disrupt
the ability of small scales to generate a mean magnetic field. This second effect is independent
of enhanced ohmic dissipation due to small scale turbulence, which is the effect usually invoked.
when small scales turn out to be destructive for a dynamo. In the picture delineated here, the eddy diffusion is already accounted for by the existence of an optimal eddy size.

Are there any dynamos of current interest in which small scales might be helpful? Let us consider first numerical simulations of planetary dynamos. At resolutions routinely used nowadays, the unresolved eddies are smaller by a factor of 30 or so than the radius of the spherical shell filled by conductor. If the calculations above are representative of helical flows in general, the unresolved eddies are too small to be near the optimum size and cannot significantly help the dynamo effect. In reasonably resolved simulations, the unresolved eddies have a small magnetic Reynolds number and contribute little to the induction anyway.

Next, there are laboratory experiments. The Karlsruhe experiment [11] was built as a helical dynamo with optimal eddy size [8], so that additional small scales do not help in lowering the critical magnetic Reynolds number. Similar optimization has been done for the experiment in Riga [12]. Currently, several experiments are under construction which reproduce simple helical flows in spheres [13]. These flows are related to a single scale velocity field (3) like $v_1$ when the axis of each eddy in $v_1$ is wrapped into a ring and the resulting torus-shaped eddy is inserted into a sphere. The proposed experiments excite only two eddies in a sphere or a cylinder. It is conceivable from the calculations presented here that these flows profit from additional smaller helical eddies. This issue requires further numerical study to be settled.

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