Simple model of 1/f noise

B. Kaulakys
Institute of Theoretical Physics and Astronomy, A. Goštauto 12, 2600
Vilnius, Lithuania
(Received 1 March 1998)

Abstract

Simple analytically solvable model of 1/f noise is proposed. The model consists of one or few particles moving in the closed contour. The drift period of the particle round the contour fluctuates about some average value, e.g. due to the external random perturbations of the system’s parameters. The model contains only one relaxation rate, however, the power spectral density of the current of particles reveals an exact 1/f spectrum in any desirable wide range of frequency and can be expressed by the Hooge formula. It is likely that the analysis and the generalizations of the model can strongly influence on the understanding of the origin of 1/f noise.
PACS number(s): 05.40.+j, 02.50.-r, 72.70.+m
The puzzle of the origin and omnipresence of 1/f noise – also known as 'flicker' or 'pink' noise – is one of the oldest unsolved problem of the contemporary physics. Since the first observation of the flicker noise in the current of electron tube more than 70 years ago by Johnson [1], fluctuations of signals and physical variables exhibiting behavior characterized by a power spectral density $S(f)$ diverging at low frequencies like $1/f^\delta$ ($\delta \sim 1$) have been discovered in large diversity of uncorrelated systems. We can mention here processes in condensed matter, traffic flow, quasar emissions, music, biological, evolution and artificial systems, human cognition and even distribution of prime numbers (see [2–5] and references herein).

1/f noise is an intermediate between the well understood white noise with no correlation in time and the random walk (Brownian motion) noise with no correlation between increments. Widespread occurrence of signals exhibiting power spectral density with 1/f behavior suggests that a general mathematical explanation of such an effect might exist. However, except some formal mathematical speculations like ”fractional Brownian motion” or half-integral of a white noise signal [1] no generally recognized mathematical explanation of the ubiquity of 1/f noise is still proposed. Physical models of 1/f noise in some physical systems are usually very specialized, complicated (see [2–5] and references herein) and they do not explain the omnipresence of the processes with $1/f^\delta$ spectrum [1–4].

Note also some mathematical algorithms and models of the generation of the processes with 1/f noise [10–12]. These models also expose some shortcomings: they are very specific, formal or unphysical, they can not, as a rule, be solved analytically and they do not reveal the origin as well as the necessary and sufficient conditions for the appearance of 1/f type fluctuations.

History of the progress in different areas of physics indicates to the crucial influence of simple models on the advancement of the understanding of the main points of the new phenomena. We note here only the decisive influence of the Bohr model of hydrogen atom on the development of the quantum theory, the role of the Lorenz model as well as the logistic and standard (Chirikov) maps for understanding of the deterministic chaos and the quantum kicked rotator for the revealing the quantum localization of classical chaos. On the contrary, the simple model of 1/f noise, to the best of our knowledge, is absent until now. This is one of the reasons of the incomprehensibility of the origin and nature of this phenomenon.

It is the purpose of this letter to present the simplest analytically solvable model of 1/f noise which can essentially influence on the understanding of the origin, main properties and parameter dependences of the intensity and domination of the flicker noise. Our model is a result of the search of necessary and sufficient conditions for the appearance of 1/f fluctuations in simple systems affected by the random external perturbations initiated in [13] and originated from the observation of the transition from chaotic to nonchaotic behavior in the ensemble of randomly driven systems [14]. Contrary to the Mc Whorter model [15] based on the superposition of large number of Lorentzian spectra and requiring a very wide distribution of relaxation times, our model contains only one relaxation rate and can have an exact 1/f spectrum in any desirable wide range of frequency.

The simplest version of our model consists of one particle moving in the closed contour. The period of the drift of the particle round the contour fluctuates (due to the external random perturbations of the system’s parameters) about some average value $\tau$. So, a sequence
of the transit times \( t_k \) when the particle crosses some point of the contour \( L_m \) is described by the recurrent equations

\[
\begin{align*}
    t_k &= t_{k-1} + \tau_k, \\
    \tau_k &= \tau_{k-1} - \gamma (\tau_{k-1} - \tau) + \sigma \varepsilon_k.
\end{align*}
\]  (1)

Here \( \gamma \ll 1 \) is the period’s relaxation rate, \( \{\varepsilon_k\} \) denotes the sequence of uncorrelated normally distributed random variables with zero expectation and unit variance (the white noise source) and \( \sigma \) is the standard deviation of white noise.

The intensity of the current of the particle through the section of the contour \( L_m \) is expressed as

\[
I(t) = \sum_k \delta(t - t_k)
\]  (2)

where \( \delta(t) \) is the Dirac delta function. The power spectral density of the current (2) is

\[
S(f) = \lim_{T \to \infty} \langle \frac{2}{T} \sum_{k=k_{\text{min}}}^{k_{\text{max}}} e^{-i2\pi ft_k} \rangle^2 = \lim_{T \to \infty} \langle \frac{2}{T} \sum_k \sum_{q=k_{\text{min}}-k}^{k_{\text{max}}-k} e^{i2\pi f(t_{k+q} - t_k)} \rangle
\]  (3)

where \( T \) is the whole observation time interval, \( k_{\text{min}} \) and \( k_{\text{max}} \) are minimal and maximal values of index \( k \) in the interval of observation and the brackets \( \langle \ldots \rangle \) denote the averaging over realizations of the process.

From Eqs. (1) it follows an expression for the period

\[
\tau_k = \tau + (\tau_0 - \tau)(1 - \gamma)^k + \sigma \sum_{j=1}^k (1 - \gamma)^{k-j} \varepsilon_j
\]  (4)

where \( \tau_0 \) is the initial period.

After some algebra we also can easily obtain an explicit expression for the transit times \( t_k \),

\[
t_k = t'_0 + k\tau + \frac{\sigma}{\gamma} \sum_{l=1}^k \left[ 1 - (1 - \gamma)^{k+1-l} \right] \varepsilon_l.
\]  (5)

Here \( t'_0 \) is some constant for \( k \gg \gamma^{-1} \) or \( \tau_0 = \tau \). In the later case \( t'_0 \) is the initial time \( t_0 \).

At \( k \gg \gamma^{-1} \) Eq. (5) generates the stationary time series and the transit times \( t_{k+q} \) and \( t_k \) difference in Eq. (3) is

\[
t_{k+q} - t_k = q\tau + \frac{\sigma}{\gamma} \left\{ [1 - (1 - \gamma)^q] \sum_{l=1}^k (1 - \gamma)^{k+1-l} \varepsilon_l + \sum_{l=k+1}^{k+q} [1 - (1 - \gamma)^{k+q+1-l}] \varepsilon_l \right\}, \quad q \geq 0.
\]  (6)

Substitution of Eq. (6) into Eq. (3) and averaging over realizations of the process or over the normal distribution of the random variables \( \varepsilon_l \) yield

\[
S(f) = \lim_{T \to \infty} \frac{2}{T} \sum_k \sum_{q} e^{i2\pi f\tau q - \pi^2 \sigma^2 f^2 g(q)}
\]  (7)

where

\[
g(q) = \frac{2}{\gamma^2} \left\{ [1 - (1 - \gamma)^q]^2 \sum_{l=1}^k (1 - \gamma)^{2l} + \sum_{l=1}^q [1 - (1 - \gamma)^l]^2 \right\}, \quad q \geq 0.
\]  (8)
Summations in Eq. (8) for \( k \gg \gamma^{-1} \) result in

\[
g(q) = \frac{2}{\gamma^2} \left\{ q - 2 \left( 1 - \gamma \right) \left[ 1 - (1 - \gamma)^q \right] \right\}.
\]

(9)

Expansion of expression (9) in powers of \( \gamma q \ll 1 \) is

\[
g(q) = \frac{1}{\gamma} q^2 - \frac{1}{3} q^3 + \frac{1}{2} q^2 + \ldots \quad q \geq 0.
\]

(10)

Note the parity of the function \( g(q), g(-q) = g(q) \), following from Eqs. (6)–(8) at \( k - |q| \gg \gamma^{-1} \).

For \( f \ll f_\tau = (2\pi\tau)^{-1} \) and \( f < f_2 = 2\sqrt{\gamma}/\pi\sigma \) we can replace the summation in Eq. (7) by the integration

\[
S(f) = 2\bar{I} \int_{-\infty}^{+\infty} e^{i2\pi f q - \pi^2 f^2 g(q)} dq.
\]

(11)

Here \( \bar{I} = \lim_{T \to \infty} \frac{1}{T} (k_{\text{max}} - k_{\text{min}} + 1) \) is the averaged current. Furthermore, at \( f \gg f_1 = \gamma^{3/2}/\pi\sigma \) it is sufficient to take into account only the first term of expansion (10). Integration in Eq. (11) hence yields to 1/f spectrum

\[
S(f) = \frac{\bar{I}^2}{f} \alpha_H, \quad f_1 < f < f_2, f_\tau
\]

(12)

where \( \alpha_H \) is a dimensionless constant (the Hooge parameter)

\[
\alpha_H = \frac{2}{\sqrt{\pi}} K e^{-K^2}, \quad K = \frac{\tau \sqrt{\gamma}}{\sigma}.
\]

(13)

The model containing only one relaxation time \( \gamma^{-1} \) for sufficiently small parameter \( \gamma \) can, therefore, produce an exact 1/f-like spectrum in any desirable wide range of frequency. Furthermore, due to the contribution to the transit times \( t_k \) of the large number of the random variables \( \epsilon_l \) \( (l = 1, 2, \ldots, k) \), our model represents a 'long-memory' random process. As a result of the nonzero relaxation rate \( (\gamma \neq 0) \) and, consequently, due to the finite dispersion of the \( \tau \) period, \( \sigma_\tau^2 \equiv \langle \tau_k^2 \rangle - \langle \tau_k \rangle^2 = \sigma^2/2\gamma (1 - \gamma/2) \) \( (2k\gamma \gg 1) \), the model, however, is free from the unphysical divergency of the spectrum at \( f \to 0 \). So, using an expansion of expression (9) at \( \gamma q \gg 1 \), \( g(q) = 2q/\gamma^2 \), we obtain from Eq. (11) the spectrum density \( S(f) = \bar{I}^2 (2\sigma^2/\tau \gamma^2) \) for \( f \ll f_1, f_0 = \tau \gamma^2/\pi\sigma^2 \).

Eqs. (11)–(13) describe quite well the power spectrum of the random process (1). As an illustrative example in Fig. 1 the numerically calculated power spectral density averaged over five realizations of the process (1) is compared with the analytical calculations according to Eqs. (9)–(13). Note, that for the used in calculations parameters \( \tau, \sigma \), and \( \gamma \) according to Eqs. (11)–(13) we have \( K = 2, \alpha_H = 0.04, f_1 = 5 \times 10^{-4}, f_2 = 0.06, f_\tau = 0.08, f_0 = 1 \times 10^{-3} \), while spectrum for \( f_1 < f < f_2 \) according to Eq. (12) is \( S(f) = 0.01/f \) and for \( f \ll f_1, f_0 \) tends to the constant \( S(f \ll f_1) = 156 \). These results are in excellent agreement with the numerical analysis.

This simple, consistent and exactly solvable model can easily be generalized in different directions: for large number of particles moving in similar contours with the coherent (identical for all particles) or independent (uncorrelated for different particles) fluctuations of the
periods, for the non-Gaussian and for the continuous perturbations of the systems’ parameters and for some spatially extended systems. So, when an ensemble of $N$ particles moves in the closed contours and the period of each particle fluctuates independently (due to the perturbations by uncorrelated, different for each $\nu$ particle, sequences of random variables $\{\varepsilon_k^\nu\}$) the power spectral density of the collective current $I$ of all particles can be calculated by the above method too and is expressed as the Hooge formula \[ 2 \]

$$S(f) = \bar{I}^2 \frac{\alpha_H}{N f}.$$  \hspace{1cm} (14)

Summarizing, simple analytically solvable model of $1/f$ noise revealing main features and parameter dependences of the power spectral density of the noise is proposed. The model and its generalizations may essentially influence on the understanding of the origin, the main properties and parameter dependences of the intensity and domination of the flicker noise.

The author acknowledges stimulating discussions with Prof. R. Katilius, Prof. A. Matulionis, and Dr. A. Bastys as well as numerical simulation data provided by Mr. T. Meškauskas. The research of this publication was made possible in part by support of the Alexander von Humboldt Foundation and Lithuanian State Science and Studies Foundation.
REFERENCES

[1] J. B. Johnson, Phys. Rev. 26, 71 (1925).
[2] F. N. Hooge, T. G. M. Kleinpenning, and L. K. J. Vadamme, Rep. Prog. Phys. 44, 479 (1981); P. Dutta and P. M. Horn, Rev, Mod. Phys. 53, 497 (1981); M. B. Weissman, Rev. Mod. Phys. 60, 537 (1988); G. P. Zhigal’skii, Usp. Fiz. Nauk 167, 623 (1997) [Phys.-Usp. 40, 599 (1997)].
[3] T. Musha and Higuchi, Jap. J. Appl. Phys. 15, 1271 (1976); X. Zhang and G. Hu, Phys. Rev. E 52, 4664 (1995).
[4] W. H. Press, Comments Astrophys. 7, 103 (1978); A. Lawrence, M. G. Watson, K. A. Pounds, and M. Elvis, Nature (London) 325, 694 (1987); M. Gartner, Sci. Am. 238, 16 (1978); S. Maslov, M. Paczuski, and P. Bak, Phys. Rev. Lett. 73, 2162 (1994); M. Usher and M. Stemmler, ibid 74, 326 (1995); M. Wolf, Physica A 241, 493 (1997).
[5] M. Kobayashi and T. Musha, IEEE Trans. Biomed. Eng. BME-29, 456 (1982); D. L. Gilden, T. Thornton, and M. W. Mallon, Science 267, 1837 (1995).
[6] B. B. Mandelbrot and J. W. Van Ness, SIAM Rev. 10, 422 (1968); E. Masry, IEEE Trans. Inform. Theory 37, 1173 (1991); B. Ninness, ibid 44, 32 (1998).
[7] Proc. 13th Int. Conf. on Noise in Physical Systems and 1/f Fluctuations, Palanga, Lithuania, 29 May-3 June 1995. Eds. V. Bareikis and R. Katilius (World Scientific, Singapore, 1995).
[8] Proc. 14th Int. Conf. on Noise in Physical Systems and 1/f Fluctuations, Leuven, Belgium, 14-18 July 1997. Eds. C. Claey ys and E. Simoen (World Scientific, Singapore, 1997).
[9] Proc. 1th Int. Conf. on Unsolved Problems of Noise, Szeged, Hungary, 3-7 Sept. 1996. Eds. Ch. Doering, L. B. Kiss, and M. F. Shlesinger (World Scientific, Singapore, 1997).
[10] H. J. Jensen, Physica Scripta 43, 593 (1991); H. F. Quyang, Z. Q. Huang, and E. J. Ding, Phys. Rev. E 50, 2491 (1994); E. Millotti, ibid 51, 3087 (1995).
[11] J. Kumičak, Ann. Physik 3, 207, (1994); J. Kumi čak, in Ref [8], p. 93; H. Akbane and M. Agu, in Ref [8], p. 601 ; J. Timmer and M. König, Astron. Astrophys. 300, 707 (1995).
[12] S. Sinha, Phys. Rev. E 53, 4509 (1996); T. Ikeguchi and K. Aihara, ibid 55, 2530 (1997).
[13] B. Kaulakys and G. Vektaris, in Ref [4], p. 677; B. Kaulakys and T. Meškauskas, in Ref [8], p. 126.
[14] B. Kaulakys and G. Vektaris, Phys. Rev. E 52, 2091 (1995).
[15] J. Bernamont, Ann. Physik 7, 71 (1937); M. Surdin, J. Phys. Radium (Serie 7) 10, 188 (1939); F. K. du Pré, Phys. Rev. 78, 615 (1950); A. Van der Ziel, Physica 16, 359 (1950); A. L. McWhorter, Techn. Rept. 80, MIT Lincoln Lab. (1955); F. N. Hooge, in Ref [8], p. 3.
Captions to the figure of the paper

B. Kaulakys

“Simple model of 1/f noise”

Fig. 1. Power spectral density vs frequency of the current generated by Eqs. (1) with $\tau = 2$, $\sigma = 0.04$, and $\gamma = 0.016$. Sinuous fine curve is the averaged over five realizations result of numerical simulations, the heavy line corresponds to the numerical integration of Eq. (11) with $g(q)$ from Eq. (9), and the thin straight line represents the analytical spectrum according to Eqs. (12) and (13).
Fig. 1