Inward and Outward Network Influence Analysis

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ABSTRACT

Measuring heterogeneous influence across nodes in a network is critical in network analysis. This article proposes an inward and outward network influence (IONI) model to assess nodal heterogeneity. Specifically, we allow for two types of influence parameters; one measures the magnitude of influence that each node exerts on others (outward influence), while we introduce a new parameter to quantify the receptivity of each node to being influenced by others (inward influence). Accordingly, these two types of influence measures naturally classify all nodes into four quadrants (high inward and high outward, low inward and high outward, low inward and low outward, and high inward and low outward). To demonstrate our four-quadrant clustering method in practice, we apply the quasi-maximum likelihood approach to estimate the influence parameters, and we show the asymptotic properties of the resulting estimators. In addition, score tests are proposed to examine the homogeneity of the two types of influence parameters. To improve the accuracy of inferences about nodal influences, we introduce a Bayesian information criterion that selects the optimal influence model. The usefulness of the IONI model and the four-quadrant clustering method is illustrated via simulation studies and an empirical example involving customer segmentation.

1. Introduction

Rapid digital technology advancement has produced large amounts of data that can be used for network analyses in various disciplines and professions such as business, engineering, health care, and science. For example, Valente (2010) studied social network applications to medical and public health; Katona, Zubcsek, and Sarvary (2011) investigated network effects and personal influences in marketing; Banerjee et al. (2013) examined the diffusion of people’s participation in the microfinance program through social networks; Banerjee et al. (2014) further identified highly central individuals via the diffusion path of the network; Lawyer (2015) assessed the network influence of spreading processes as seen in epidemic diseases; Maggio et al. (2019) demonstrated that the network relationships between brokers and institutional investors can influence market outcomes. Recently, Tabassum et al. (2018) provided a succinct overview of social network analysis.

In practice, one of the important tasks in network analysis is to identify influential nodes (or actors). For example, evaluating the effect of each user’s behavior on other users or determining the extent of spread from each COVID-19 carrier to other people. Hence, it is useful to measure the magnitude of the outward influence each node exerts on others (outward influence). Examples include analyzing how the behavior of social media influencers is itself affected by the other users or measuring the receptivity of each person who is exposed to COVID-19 carriers. While much research has focused on outward influence, such studies are limited in their predictive power as they neglect the compounding effect of heterogeneity across nodes in the level of receptivity to influence from others (i.e., inward influence). Our article improves on prior literature by introducing heterogeneity in inward influence, in addition to heterogeneity of outward influence. We propose a model for working with these two influence factors simultaneously. Many existing models that study outward influence alone can be seen as a special case of our model.

To study both types of influences, we construct a network with \( n \) nodes by defining the adjacency matrix \( A = (a_{ij})_{n \times n} \in \mathbb{R}^{n \times n} \) to characterize the relationship between any two adjacent nodes in the network (see, e.g., Carrington, Scott, and Wasserman 2005; Scott 2013). Specifically, we define \( a_{ij} = 1 \) if there is a direct connection from node \( i \) to node \( j \) and \( a_{ij} = 0 \) otherwise.

In addition, we denote \( a_{ii} = 0 \) for any \( 1 \leq i \leq n \). For example, we define \( a_{ij} = 1 \) if user \( i \) in the Sina Weibo network follows user \( j \) and \( a_{ij} = 0 \) otherwise; see the real data analysis in Section 3.2. Subsequently, we normalize \( a_{ij} \) and obtain \( W = (w_{ij}) \in \mathbb{R}^{n \times n} \) with \( w_{ij} = a_{ij} / \sum_{j=1}^{n} a_{ij} \), which is the weighted adjacency matrix.

After constructing the network structure, we then define the response variable \( Y = (Y_1, \ldots, Y_n) \) that usually depends on the target of the study. For example, suppose that our aim is to identify the specific users who can most influence others’ activity in the Sina Weibo network given in Section 3.2. The \( Y_i \) \((i = 1, \ldots, n)\) could be the number of posts made by the \( i \)th...
user as a measure of user i's activity. Although the response of the ith node, $Y_i$, can be influenced by the other nodes' responses, it can also depend on its own attributes, such as gender and duration in the Sina Weibo network, and we name this vector of attributes the nodal covariate $X_i$. Therefore, to identify influential nodes, for each node $1 \leq i \leq n$, we consider the continuous response variable $Y_i \in \mathbb{R}^1$ and its associated $p$-dimensional nodal covariate $X_i = (X_{i1}, \ldots, X_{ip})^T \in \mathbb{R}^p$. By integrating this nodal information with the network structure induced by the adjacency matrix $A$, we propose the following model:

$$
Y_i = \sum_{j=1}^{n} b_{ij} Y_j + X_i^T \eta + \varepsilon_i, \quad (1.1)
$$

where $b_{ij}$ is the function of $w_{ij}$ that characterizes the influence effect between nodes $i$ and $j$ for $i,j = 1, \ldots, n$, $\eta = (\eta_1, \ldots, \eta_p)^T \in \mathbb{R}^p$ is the $p$-dimensional unknown regression coefficient vector, and $\varepsilon_i$ is the random error for $i = 1, \ldots, n$. If we parameterize $b_{ij} = \lambda_{w_{ij}}$ by a single influence parameter $\lambda$, then the resulting autoregressive model has been studied for a long history in the field of economics (see, e.g., Anselin 1988; Manski 1993a, 1993b; Bramoullé, Djebarri, and Fortin 2009; LeSage and Pace 2009; Liu and Lee 2010; Gupta and Robinson 2015; Zhou et al. 2017; Zhang and Yu 2018; Gao et al. 2019; Kwok 2019; Zhu et al. 2019). It is of interest to note that this type of model was labeled as “mixed-regression,” “spatial autoregressive,” “spatial correlation,” and “spatial autoregressive model (SAR)” by Anselin (1988), Manski (1993a), and LeSage and Pace (2009), respectively, although the term SAR is most commonly used in the literature.

As Gupta and Robinson (2015) and Lam and Souza (2019) noted, using the single parameter $\lambda$ to characterize the influence effect among the $n$ nodes may not capture all of their heterogeneity. To alleviate this limitation, Zhu et al. (2019) and Zou et al. (2021) proposed the network influence model by parameterizing $b_{ij}$ in Equation (1.1) as $\lambda_j w_{ij}$. Thus, the influence effect of node $j$ on node $i$ not only relies on the network structure $w_{ij}$, but also depends on $\lambda_j$. We name $\lambda_j$ the outward influence index of node $j$. It is worth noting that the influence of node $j$ on node $i$ does not take into account the magnitude of the response of node $i$. Hence, we cannot measure the true influence of node $j$ accurately. For example, consider a Sina Weibo network with three nodes, $j_1, j_2, j_3$. Suppose that $j_1$ is a famous star and both nodes $j_2$ and $j_3$ follow node $j_1$, while node $j_2$ is quite active and node $j_3$ is inactive. Under this scenario, we expect that $j_1$ has a larger influence on $j_2$ than on $j_3$ since $j_2$ renders more responses to $j_1$. This situation can also occur in the infectious diseases network. For instance, let $j_1$ be a super spreader, $j_2$ be an older person with chronic susceptibility, and $j_3$ be a younger person with a strong immune system. It is expected that $j_1$ will have more apparent influence on the health of $j_2$ than on the health of $j_3$ since $j_2$ is more likely than $j_3$ to yield an immune response to $j_1$. Thus, we cannot detect potential collective outcomes accurately by considering the outward influence index alone. This motivates us to introduce an inward influence measure that can characterize each individual’s own response to others’ influence.

To determine the real influence, we introduce the inward and outward network influence (IONI) model given below by assuming $b_{ij}$ in Equation (1.1) to be $\gamma_i w_{ij} \lambda_j$.

$$
Y_i = \sum_{j=1}^{n} (\gamma_i w_{ij} \lambda_j) Y_j + X_i^T \eta + \varepsilon_i, \quad (1.2)
$$

where $\gamma_i$ measures the degree of node i's receptivity to being influenced by others and $\lambda_j$ determines the influence magnitude of node $j$ on others. Hence, we name $\gamma_i$ the inward influence index, and $\lambda_j$ is the outward influence index mentioned earlier. In addition, we assume that $0 \leq \gamma_i \leq 1$ and $0 \leq \lambda_j \leq 1$. It is worth noting that the influence indices $\gamma_i$ and $\lambda_j$ are usually related to the nodes' attributes (see, e.g., Trusov, Bodapati, and Bucklin 2010; Zou et al. 2021), which we denote them $Z_i$ and $V_j$, respectively. For instance, in the Sina Weibo example, $Z_i$ and $V_j$ can include the number of each user's followers and the number of each user's followees, that is, other people that a given user is following. Accordingly, we parameterize $\gamma_i$ and $\lambda_j$ as the known functions of the nodes' attributes $Z_i$ and $V_j$, respectively, in Section 2.1.

To illustrate the usefulness of the $\gamma$ and $\lambda$ indices, we consider customer segmentation in marketing as an application. Figure 1 depicts $\gamma$ and $\lambda$, which increase in the directions of the $x$ and $y$ arrows, respectively, with the origin $(\gamma_0, \lambda_0)$. Then we classify the nodes of the network into four groups according to the values of $\gamma$ and $\lambda$. The first quadrant of Figure 1 is group I, which consists of customers with high $\gamma$ and $\lambda$. Accordingly, those customers not only have strong influence over others, but also can be influenced greatly by others. This type of customer shares similar properties to “early adopters” in marketing; see, for example, Wolf and Seebauer (2014), Catalini and Tucker (2017). Hence, identifying this type of customer is essential for promoting new products. The second quadrant is group II, in which the customers have low $\gamma$ and high $\lambda$. This group's customers tend to influence others, but not vice versa. They can be named “opinion leaders” or “social influencers;” see, for example, Li et al. (2013), Ma and Liu (2014), and Bamakan, Nurgaliyev, and Qu (2019). Group III is located in the third quadrant, in which the customers have both low $\gamma$ and $\lambda$. This type of customer neither exerts influence nor is influenced by others. Hence, these customers do not create awareness or show interest in products, and they can be considered “inactive customers;” see Kumar and Reinartz (2018). Finally, group IV is located in the fourth quadrant, in which the customers have high $\gamma$ and $\lambda$.
low λ. This type of customer seems to be very susceptible but has little influence on others, and they can be treated as the “early majority,” see, for example, Ram and Jung (1994) and Mattila, Karjaluoto, and Pento (2003).

Based on the above four-quadrant clustering, practitioners can segment Sina Weibo’s users into the four groups to improve product promotion or opinion dissemination. For example, if practitioners need to promote a new product on Sina Weibo, they could target the “early adopters” (i.e., the users with high γ and high λ). In contrast, if practitioners would like to extend the life cycle of mature products, they should focus on the “early majority” (i.e., the users with high γ and low λ). To effectively make use of the above two types of customers, decision makers should focus on the “opinion leaders” (i.e., the users with low γ and high λ) for promotions. Finally, removing or ignoring prolonged “inactive users” (i.e., the users with low γ and low λ) can cut down on promoting costs. We provide detailed illustrations in Section 3.2.

The aim of this article is to propose the IONI model, which allows us to identify four types of influencers in network analysis. To estimate the model and influence indices, we study identification conditions and parameterize λ, γ as known functions of the nodes’ attributes, respectively. Then, we employ Lees’ (2004) quasi-maximum likelihood approach to obtain parameter estimators. In addition, we demonstrate their asymptotic properties without imposing any specific error distributions. A Bayesian information criterion is also proposed for selecting optimal models, and its consistency is demonstrated. Moreover, score tests are provided to examine the homogeneity of λ and γ. Simulation studies and an empirical example for differentiating the four different types of users in the Sina Weibo network are presented to demonstrate the utility of the proposed model. The rest of the article is organized as follows. Section 2 describes the model structure in the matrix form, parameter estimators, four-quadrant clustering, selection criterion and test statistics. Section 3 presents numerical studies. We provide a discussion with some concluding remarks in Section 4. All technical details are relegated to the supplementary material.

2. Methodology and Theoretical Results

2.1. Model Setting and Identification Condition

To effectively examine the IONI model, we express the proposed model (1.2) in the matrix form as follows:

$$Y = \Gamma W \Lambda Y + X \eta + \epsilon, \quad (2.1)$$

where $Y = (Y_1, \ldots, Y_n)^T \in \mathbb{R}^n$, $X = (X_1, \ldots, X_n)^T \in \mathbb{R}^{n \times p}$, $\Gamma = \text{diag}(\gamma_1, \ldots, \gamma_n)$, $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$, $\epsilon = (\epsilon_1, \ldots, \epsilon_n)^T \in \mathbb{R}^n$, and $\epsilon_i$ are independent and identically distributed with mean 0 and variance $\sigma^2$ for $i = 1, \ldots, n$. In addition, $\gamma_i$ (and $\lambda_i$) are the inward (and outward) influence parameters or indices, respectively.

It is worth noting that there are $2n$ unknown parameters in (2.1), which cannot be estimated via the $n$ observations. However, $\gamma_i$ and $\lambda_i$ can each be considered a function of the associated attributes of nodes $i$ and $j$, respectively. To this end, we parameterize $\gamma_i$ with $\gamma_i(\alpha) = F(Z_i \alpha)$ and $\lambda_i$ with $\lambda_i(\beta) = F(V_i \Gamma \beta)$, where $F(\cdot)$ is a strictly monotone and known function, and $Z_i$ and $V_j$ are the corresponding attributes associated with the inward and outward influence indices, $\gamma_i$ and $\lambda_j$, respectively.

In addition, $Z_i = (z_{i1}, \ldots, z_{id})^T \in \mathbb{R}^{d \times 1}$ with $z_{i1} = 1$ and $V_j = (v_{j1}, \ldots, v_{jd})^T \in \mathbb{R}^{d \times 1}$ with $v_{j1} = 1$, and the corresponding unknown regression coefficient vectors are $\alpha = (\alpha_1, \ldots, \alpha_d)^T \in \mathbb{R}^{d \times 1}$ and $\beta = (\beta_1, \ldots, \beta_d)^T \in \mathbb{R}^{d \times 1}$. Moreover, define $Z_{1-1} = (z_{12}, \ldots, z_{1d})^T$ and $V_{1-1} = (v_{21}, \ldots, v_{jd})^T$, and assume that $Z_{1-1} = (Z_{1-1}, \ldots, Z_{1-n})^T$ and $V_{1-1} = (V_{1-1}, \ldots, V_{1-1})^T$ are of full rank. Since both $\Lambda$ and $\Gamma$ can be expressed as a function of $\alpha$ and $\beta$, respectively, we express (2.1) as follows:

$$Y = \Gamma(\alpha) W \Lambda(\beta) Y + X \eta + \epsilon. \quad (2.2)$$

To make the proposed model (2.2) practically useful, one needs to specify the function $F(\cdot)$ that links attributes to influence parameters. We first adopt the sigmoid function, which is often used in machine learning (see, e.g., Hastie, Tibshirani, and Friedman 2009). That is, $F(Z_i \alpha) = e^{\gamma_i(\alpha)}/(1 + e^{\gamma_i(\alpha)})$. We next consider two related link functions: the inverse of log-log (i.e., $F(Z_i \alpha) = 1 - e^{-\gamma_i(\alpha)}$) and the inverse of probit (i.e., $F(Z_i \alpha) = \Phi(Z_i \alpha)$), where $\Phi(\cdot)$ is the distribution function of the standard normal distribution.

In order to estimate the unknown parameters in model (2.2), one needs to consider the identification problem of $\gamma_i$ and $\lambda_j$.

That is, for any $\gamma = (\gamma_1, \ldots, \gamma_n)$, $\lambda = (\lambda_1, \ldots, \lambda_n)$, $\Gamma = \text{diag}([\gamma_1, \ldots, \gamma_n])$ and $\Lambda = \text{diag}([\lambda_1, \ldots, \lambda_n])$, $\Gamma W \Lambda$ leads to $\Gamma = \Gamma \Lambda = \Lambda$, where $\gamma_i = \gamma_i(\alpha)$ and $F(Z_i \alpha) = F(V_i \Gamma \beta)$, $\gamma_i = \gamma_i(\alpha) = F(Z_i \alpha)$, $\lambda_j = \lambda_j(\beta) = F(V_j \Gamma \beta)$, $\gamma_i = \gamma_i(\alpha) = F(Z_i \alpha)$. Similarly, if $\Gamma W \Lambda = \Gamma W \Lambda$ and $w_{ij} > 0$ for some $i \neq j$, then $F(Z_i \alpha)/F(Z_i \alpha) = F(V_j \Gamma \beta)/F(V_j \Gamma \beta) = c$ for some constant $c$. To this end, we impose the constraint that either $\alpha_i = 0$ or $\beta_j = 0$, to ensure identifiability. To illustrate the necessity for this constraint, we define $\alpha_{i-1} = (\alpha_2, \ldots, \alpha_d)^T \in \mathbb{R}^{d-1}$ and $\beta_{j-1} = (\beta_2, \ldots, \beta_d)^T \in \mathbb{R}^{d-1}$. Suppose that $\alpha_{i-1} = 0$ and $\beta_{j-1} = 0$. Then $F(Z_i \alpha)/F(Z_i \alpha) = c$ implies that $\alpha_{i-1} = 0$ and $\beta_{j-1} = 0$. Similarly, we obtain $\beta_{j-1} = 0$ and $F(V_j \Gamma \beta) = c$. Thus, $F(\alpha_{i-1})F(\beta_{j-1}) = F(\alpha_{i-1})F(\beta_{j-1})$. Using the fact that $F(\cdot)$ is strictly monotone, the model cannot be identified unless we impose the constraint that either $\alpha_i = 0$ or $\beta_j = 0$.

Without loss of generality, we assume $\alpha_1 = 0$, or equivalently $Z_i \alpha = Z_{i-1} \alpha$. For the sake of simplification, however, we slightly abuse notation and still use $\alpha = (\alpha_1, \ldots, \alpha_d)^T \in \mathbb{R}^{d \times 1}$ and $Z_i = (z_{i1}, \ldots, z_{id})^T \in \mathbb{R}^{d \times 1}$ hereafter, rather than $\alpha_{i-1}$ and $Z_{i-1}$, as the parameter vector and its associated attributes, respectively.

Remark 1: In network analysis, most existing studies have focused on outward influence. To construct spatio-temporal models for panel data, however, Dou, Parrella, and Yao (2016) introduced a pure spatial effect measure. Without considering the temporal structure, we can consider their measure to be inward influence by parameterizing $b_{ij}$ as $\gamma_i w_{ij}$. Dou, Parrella, and Yao (2016) allowed the temporal observations to tend toward infinity and showed that their parameters are estimable without introducing a link function. However, our model does...
2.2. Parameter Estimation

In model (2.2), we do not assume that the random errors are normally distributed. To obtain nice properties of the estimators, we adopt Lee's (2004) approach by employing the quasi-maximum likelihood estimation (QMLE) method to estimate unknown parameters. Define $S(\alpha, \beta) = I_n - \Gamma(\alpha)W\Lambda(\beta)$. We then have $e = e(\eta, \alpha, \beta) = S(\alpha, \beta)^{-1}Y_n$, and the normal log-likelihood function of Equation (2.2) is

$$
\ell(\theta) = -\frac{n}{2} \log 2\pi - \frac{n}{2} \log \sigma^2 - \frac{1}{2\sigma^2}(S(\alpha, \beta)^{-1}Y_n - \eta)^\top S(\alpha, \beta)^{-1}(S(\alpha, \beta)^{-1}Y_n - \eta) + \log |\det(S(\alpha, \beta))|,
$$

where $\theta = (\eta^\top, \alpha^\top, \beta^\top)^\top$. We next apply the concentrated QMLE method to estimate the parameters. Specifically, given $\alpha$ and $\beta$, we maximize $\ell(\theta)$ with respect to $\eta$ and $\sigma^2$, which leads to

$$
\hat{\eta}(\alpha, \beta) = (X^\top X)^{-1}X^\top S(\alpha, \beta)^{-1}Y_n \quad \text{and} \quad \hat{\sigma}^2(\hat{\eta}(\alpha, \beta), \alpha, \beta) = n^{-1}e(\hat{\eta}(\alpha, \beta), \alpha, \beta) \in \mathbb{R}^n.
$$

To further obtain the theoretical properties of the estimates, we introduce some notation and equations below.

Let $\Lambda_{ij}(\alpha) = \partial^2 \Gamma(\alpha)/\partial \alpha_i \partial \alpha_j = \text{diag}(z_{ik}F'(Z_i^\top \alpha), \ldots, z_{nk}F'(Z_n^\top \alpha))$ for $k = 1, \ldots, d_1$, and $\Lambda_{ij}(\beta) := \partial^2 \Lambda(\beta)/\partial \beta_l \partial \beta_l = \text{diag}(v_{il}F'(V_i^\top \beta), \ldots, v_{il}F'(V_n^\top \beta))$ for $l = 1, \ldots, d_2$, where $F'(\cdot)$ is the first derivative of $F$. In addition, we denote the Fisher information matrix of the quasi-maximum likelihood $l(\theta)$ by $I_n(\theta) = -\frac{1}{n}E[\partial^2 \ell(\theta)/\partial \theta \partial \theta^\top]$. To show the asymptotic distribution of estimated parameters, we introduce some notation and equations below.

We next define the necessary norms and introduce conditions used in the theoretical proofs. Denote $\| \cdot \|_s$ the $s$-norm of a vector (or a matrix) for $1 \leq s \leq \infty$. Specifically, for any generic vector $x = (x_1, \ldots, x_q)^\top \in \mathbb{R}^q$, $\|x\|_s = (\sum_{i=1}^q |x_i|^s)^{1/s}$, and, for any generic matrix $G \in \mathbb{R}^{m \times q}$,

$$
\|G\|_s = \sup \left\{ \frac{|Gx|_s}{\|x\|_s} : x \in \mathbb{R}^{q \times 1} \text{ and } x \neq 0 \right\}.
$$

In addition, define the element-wise $\ell_s$ norm for any generic matrix $G$ as $|G|_s = \|\text{vec}(G)\|_s$, where vec($G$) denotes the vectorization for any generic matrix $G$. We subsequently present five technical conditions, which are required to ensure the theoretical properties of the estimates.

(C1) Assume that the random errors $e_i$ are independent and identically distributed with mean 0, and $E|e_i|^{4 + \xi} < \infty$ for some $\xi > 0$.

(C2) Assume $\sup_{n \geq 1} ||W||_1 < \infty$ and $\sup_{n \geq 1} ||W||_\infty < \infty$.

(C3) Assume that $S(\alpha, \beta) = I_n - \Gamma(\alpha)W\Lambda(\beta)$ is nonsingular uniformly over $\alpha$ and $\beta$ in the compact parameter space $\mathcal{P}$, and the true values of $\alpha$ and $\beta$ are also in the interior of $\mathcal{P}$. In addition, assume that $\sup_{(\alpha, \beta) \in \mathcal{P}} \sup_{n \geq 1} ||S^{-1}(\alpha, \beta)||_1 < \infty$ and $\sup_{(\alpha, \beta) \in \mathcal{P}} \sup_{n \geq 1} ||S^{-1}(\alpha, \beta)||_\infty < \infty$ hold.

(C4) Assume $\sup_{n \geq 1} ||x_i||_\infty < \infty$. In addition, for the true parameters $\alpha$ and $\beta$, assume that

$$
\sup_{n \geq 1} \max_{1 \leq i \leq n} |z_{ik}F'(Z_i^\top \alpha)| < \infty,
$$

$$
\sup_{n \geq 1} \max_{1 \leq i \leq n} |v_{il}F'(V_i^\top \beta)| < \infty,
$$

$$
\sup_{n \geq 1} \max_{1 \leq i \leq n} |z_{ik_1}z_{ik_2}F''(Z_i^\top \alpha)| < \infty,
$$

$$
\sup_{n \geq 1} \max_{1 \leq i \leq n} |v_{il_1}v_{il_2}F''(V_i^\top \beta)| < \infty,
$$

$$
\sup_{n \geq 1} \max_{1 \leq i \leq n} |z_{ik_1}z_{ik_2}z_{ik_3}F''(Z_i^\top \alpha)| < \infty,
$$

$$
\sup_{n \geq 1} \max_{1 \leq i \leq n} |v_{il_1}v_{il_2}v_{il_3}F''(V_i^\top \beta)| < \infty,
$$

for any $k_1, k_2, k_3 \in \{1, \ldots, d_1\}$ and $l_1, l_2, l_3 \in \{1, \ldots, d_2\}$, where the link function $F$ is assumed to be three times differentiable.

(C5) Assume $I_n(\theta) \to I(\theta)$ and $J_n(\theta, \mu_3, \mu_4) \to J(\theta, \mu_3, \mu_4)$ as $n \to \infty$. We further assume that $I(\theta)$ and $I(\theta) + J(\theta, \mu_3, \mu_4)$ are finite and positive definite, where $I_n(\theta)$ and $J_n(\theta, \mu_3, \mu_4)$ are defined in Section 1 of the supplementary material.

Condition (C1) is a moment condition, which means it is weaker than a distribution assumption, such as the normal distribution considered in Zhou et al. (2017) and sub-Gaussian assumption considered in Zhu et al. (2019). Conditions (C2) and (C3) are standard regularity conditions used in the literature (Lee 2004; Yu, de Jong, and Lee 2008). We can verify that these two conditions are satisfied as long as the network is sparse so that each individual node is connected only to a finite number of other nodes (i.e., $\sum_{j=1}^n a_{ij} < \infty$ for any $i = 1, \ldots, n$). Accordingly, $\sup_{n \geq 1} ||W||_\infty < \infty$ and $\sup_{n \geq 1} ||W||_1 < \infty$ are satisfied. In addition, $1 \leq \sum_{j=1}^n |\gamma_j \lambda_j| |w_j| + 1 < \infty$ for any $i$. 

not consider the temporal structure. Hence, we introduce link functions for influence parameter estimation.
This implies that $1 \leq \inf \| S^{-1}(\alpha, \beta) \|_1 < \infty$. By the definition of $\| S^{-1}(\alpha, \beta) \|_1 = 1/\inf \{ ||S(\alpha, \beta)|| : x \in \mathbb{R}^n \}$, we immediately obtain $\sup_{(\alpha, \beta) \in \mathcal{P}} \sup_{n \geq 1} || S^{-1}(\alpha, \beta) ||_1 < \infty$. Employing a similar approach, $\sup_{(\alpha, \beta) \in \mathcal{P}} \sup_{n \geq 1} || S^{-1}(\alpha, \beta) ||_1 < \infty$ can be established.

Condition (C4) imposes conditions on the link function and attributes, and it can be satisfied as long as the link function $F(\cdot)$ has bounded first three derivatives and the elements of attributes $X_i, Z_i$ and $V_i$ are uniformly bounded constants for all $i$ (see, e.g., Assumption 6 in Lee 2004). It is worth noting that the first three derivatives of the three link functions (i.e., sigmoid function, inverse of log-log, and inverse of probit) mentioned in Section 2.1 are bounded and they satisfy this condition. Condition (C5) is a type of law of large numbers assumption for ensuring the convergence of the Hessian matrix and the variance of the score function, in order to establish the asymptotic normality of $\hat{\gamma}$. Similar conditions can be found in Fan and Li (2001), Lee (2004), and Zhu et al. (2017). Under Condition (C5), the concentrated quasi-log-likelihood function in Equation (2.3) is concave and has a global maximizer. Based on the above notations and conditions, we then have the following results.

**Theorem 1.** Under Conditions (C1)–(C5), we then have (i) $\sqrt{n}(\hat{\gamma} - \gamma)$ is asymptotically distributed as $N(0, I^{-1}(\theta) J(\theta, \mu_3, \mu_4) I^{-1}(\theta))$; and (ii) for any fixed $i \geq 1$, $\sqrt{n}(\hat{\lambda}_i - \lambda_i)$ $\xrightarrow{d}$ $N(0, \{F'(V_i)\}^2 V_i' D_{\beta}(\theta, \mu_3, \mu_4)V_i)$, where $I(\theta)$ and $J(\theta, \mu_3, \mu_4)$ are defined in Condition (C5), and $D_{\beta}(\theta, \mu_3, \mu_4)$ and $D_{\mu}(\theta, \mu_3, \mu_4)$ are the corresponding convergent matrices of $D_{\alpha,n}(\theta, \mu_3, \mu_4)$ and $D_{\beta,n}(\theta, \mu_3, \mu_4)$ defined above.

In practice, we can estimate the unknown quantities $D_{\alpha,n}(\theta, \mu_3, \mu_4)$ and $D_{\beta,n}(\theta, \mu_3, \mu_4)$ consistently by their corresponding estimators $\hat{D}_{\alpha,n}(\hat{\theta}, \hat{\mu_3}, \hat{\mu_4})$ and $\hat{D}_{\beta,n}(\hat{\theta}, \hat{\mu_3}, \hat{\mu_4})$, where $\hat{\mu_3} = n^{-1} \sum \hat{\gamma}_i, \hat{\mu_4} = n^{-1} \sum \hat{\lambda}_i$, and $\hat{\gamma}_i = v_i(\hat{\gamma}, \hat{\beta}) = (\hat{\eta}_1, \ldots, \hat{\eta}_n)$ $\in \mathbb{R}^n$. Using the continuous mapping theorem, we can also estimate $F'(Z_i' \alpha)$ and $F'(V_i' \beta)$ consistently by the estimators $\hat{F}'(Z_i' \hat{\alpha})$ and $\hat{F}'(V_i' \hat{\beta})$, respectively.

### 2.3. Four-Quadrant Clustering

To make use of the inward and outward influence indices, we can apply the four-quadrant clustering approach to partition all nodes into four groups with known threshold values $\gamma_0$ and $\lambda_0$ as depicted in Figure 1. To achieve this end, a heuristic approach can be applied to select the threshold values $\gamma_0$ and $\lambda_0$ via the empirical quantiles and then classify nodes by comparing their estimated inward and outward influence indices with the threshold values, respectively. However, this simple heuristic approach does not take into account the variation in the estimated inward and outward influence indices. To this end, we consider the following two hypotheses for $i = 1, \ldots, n$: $H_{0i,\gamma}$: $\gamma_i \leq \gamma_0$ vs. $H_{1i,\gamma}$: $\gamma_i > \gamma_0$ and $H_{0i,\lambda}$: $\lambda_i \leq \lambda_0$ vs. $H_{1i,\lambda}$: $\lambda_i > \lambda_0$. Their corresponding test statistics are $T_{i,\gamma} = \frac{\sqrt{n}(\gamma_i - \gamma_0)}{\sqrt{\{F'(Z_i' \hat{\alpha})\}^2 Z_i' D_{\alpha,n}(\hat{\theta}, \hat{\mu_3}, \hat{\mu_4})Z_i}}$ and $T_{i,\lambda} = \frac{\sqrt{n}(\lambda_i - \lambda_0)}{\sqrt{\{F'(V_i' \hat{\beta})\}^2 V_i' D_{\beta,n}(\hat{\theta}, \hat{\mu_3}, \hat{\mu_4})V_i}}$, where $D_{\alpha,n}(\hat{\theta}, \hat{\mu_3}, \hat{\mu_4})$ and $D_{\beta,n}(\hat{\theta}, \hat{\mu_3}, \hat{\mu_4})$ are given above in the discussion of Theorem 1. By Theorem 1, we can immediately obtain that $T_{i,\gamma}$ and $T_{i,\lambda}$ are asymptotically distributed as $N(0,1)$, which yields their respective $p$-values.

It is worth noting that the above testing procedure implicitly involves $n$ multiple tests. This motivates us to control the false discovery rate (FDR) by selecting the threshold of the $p$-values obtained from the multiple tests and then identifying significant hypotheses and determining the rejection regions (see, e.g., Storey, Taylor, and Siegmund 2004).

In sum, Theorem 1 not only provides the asymptotic properties of parameter estimators, but also allows us to classify existing nodes into adequate categories via inward and outward influence indices. Thus, the four-quadrant clustering method can play an important role in network applications.

### 2.4. Variable Selections

The influence parameters $\gamma_i$ and $\lambda_i$ in model (2.2) are known functions of attributes. To make better inferences and predictions, we employ Schwarz’s Bayesian information criterion to select attributes in the correct model consistently. Define the full model $S_T = \{S_{F}, S_{P} \}$, where $S_{F} = \{1, \ldots, d_1 \}$ and $S_{P} = \{1, \ldots, d_2 \}$. In addition, denote true model as $S_T = \{S_{F}, S_{P} \}$, where $S_{F} = \{i \geq 1, \alpha_i \neq 0 \}$ and $S_{P} = \{j \geq 1, \beta_j \neq 0 \}$.

Accordingly, the full model includes all possible attributes (i.e., covariates), while the true model contains relevant covariates. For the sake of convenience, we use generic notation $S \subseteq S_T$ to represent an arbitrary candidate model with $S = \{S_{F}, S_{P} \}$, and then denote $a_S = (\alpha_k : k \in S_{F})$ and $b_S = (\beta_l : l \in S_{P})$.

Based on Equation (2.3), the Bayesian information criterion is

$$
BIC(S) = -2\ell_c(\hat{\alpha}_S, \hat{\beta}_S) + df \times \log n,
$$

where $\ell_c(\hat{\alpha}_S, \hat{\beta}_S)$ represents the concentrated quasi-log-likelihood function for any subset model $S$, $\hat{\alpha}_S = (\hat{\alpha}_{k,S_{F}} : k \in S_{F})$ and $\hat{\beta}_S = (\hat{\beta}_{l,S_{P}} : l \in S_{P})$ are the maximum likelihood estimates of $\alpha_S$ and $\beta_S$, respectively, and $df = |S|$ is the size of $S$.

Removing the irrelevant constants involved in the concentrated log-likelihood function, we have

$$
BIC(S) = n \log \hat{\sigma}^2(\hat{\eta}(\hat{\alpha}_S, \hat{\beta}_S), \hat{\alpha}_S, \hat{\beta}_S) - 2 \log \left| \det(S(\hat{\alpha}_S, \hat{\beta}_S)) \right| + |S| \times \log n.
$$

Then the optimal model selected by the Bayesian information criterion is $S_{BIC} = \arg\min_{S \subseteq S_T} BIC(S)$. To obtain the theoretical property of $BIC$, we introduce an additional condition here.

(C6) For any underfitted model, which means $S \subset S_T$ but $S \supset S_T$, assume that there exists a positive constant $c_{min} > 0$ such that $\min \left\{ \inf_{(\alpha_S, \beta_S)} E\left[n^{-1} \ell_c(\alpha_S, \beta_S) \right] - \inf_{(\alpha_S, \beta_S)} E\left[n^{-1} \ell_c(\alpha_S, \beta_S) \right] > c_{min} \right\} > c_{min}$.

Condition (C6) indicates that the mean of the concentrated quasi-log-likelihood function of any underfitted model is inferior to that of the true model (i.e., no underfitted model can fit
better than the true model). Similar conditions can exist in linear regression models; see, for example, (Shi and Tsai 2002, assump. 3) and (Wang, Li, and Tsai 2007, condit. 2). We now present the following result.

**Theorem 2.** Under Conditions (C1)–(C6), we have $P(S_{\text{BIC}} = S_T) \to 1$ as $n \to \infty$.

The above theorem implies that the Bayesian information criterion can determine the true model consistently as long as $n$ tends to infinity. To obtain the solution of $S_{\text{BIC}}$, we apply the backward elimination method (see, e.g., Zhang and Wang 2011), which can reduce the computational complexity from $O(2^{d_1+d_2})$ to $O((d_1 + d_2)^2)$. Thus, $S_{\text{BIC}}$ is computable when $d_1 + d_2$ is not very large.

### 2.5. **Hypothesis Testing**

After obtaining the parameter estimator of $\theta = (\eta^T, \alpha^T, \beta^T, \sigma^2)^T$, we perform the following three tests to examine the homogeneity of $\lambda$ and $\gamma$.

**Test I.** We first test the homogeneity of $\lambda$ and $\gamma$ by considering the null and alternative hypotheses:

$$H_0: \gamma_1 = \gamma_2 = \cdots = \gamma_n \quad \text{and} \quad \lambda_1 = \lambda_2 = \cdots = \lambda_n$$

vs.

$$H_1: \gamma_{i_1} \neq \gamma_{i_2}, \quad \text{or} \quad \lambda_{j_1} \neq \lambda_{j_2}, \quad \text{for some} \quad i_1 \neq i_2 \quad \text{and} \quad j_1 \neq j_2.$$  

Assuming that $\gamma(\alpha) = F(Z_1^T \alpha), \lambda(\beta) = F(V_1^T \beta), F(\cdot)$ is strictly monotone, no intercept is included in $\alpha$, and $Z$ and $V$ - 1 are of full rank as defined in Subsection 2.1, the above hypotheses are equivalent to $H_0$: $\alpha_1 = \cdots = \alpha_{d_1} = 0$ and $\beta_2 = \cdots = \beta_{d_2} = 0$ vs. $H_1$: $\alpha_{i_1} \neq 0$ or $\beta_{i_2} \neq 0$ for some $i \geq 1$ or $j > 1$.

Recall that $\theta = (\eta^T, \alpha^T, \beta^T, \sigma^2)^T$ and $\beta_1$ is an intercept. To test the null hypothesis, we let $\theta_1 = (\eta^T, \beta_1^T, \sigma^2)^T$ and $\theta_2 = (\alpha_1, \beta_2, \cdots, \beta_{d_2})^T$. With a slight abuse of notation, we reset $\theta = (\theta_1^T, \theta_2^T)^T$. The notation, functions and equations used in the following theorem and propositions are based on this new setting. Let $\hat{\gamma}(\cdot)$ be the constrained QMLE under the null hypothesis. Then, the resulting quasi-Lagrange-multiplier test statistic is

$$T_\gamma = \frac{1}{n} \left\{ \frac{\partial \hat{\epsilon}(\hat{\theta}(\cdot))}{\partial \theta} \right\}^T I^{-1}(\hat{\theta}(\cdot)) \left\{ \frac{\partial \hat{\epsilon}(\hat{\theta}(\cdot))}{\partial \theta} \right\},$$

where $\tilde{\epsilon}(\hat{\theta}(\cdot))$ and $I^{-1}(\hat{\theta}(\cdot))$ are, respectively, the score function and the Fisher information matrix, both evaluated at $\hat{\theta}(\cdot)$. Subsequently, the asymptotic distribution of $T_\gamma$ is given below.

**Theorem 3.** Under Conditions (C1)–(C5) and the null hypothesis of $H_0$, $T_\gamma$ is asymptotically distributed as $\sum_{l=1}^{p+d_1+d_2+1} \lambda_l(\theta, \mu_3, \mu_4) X_l^2(1)$ as $n \to \infty$, where $\lambda_l(\theta, \mu_3, \mu_4)$ is the $l$th largest eigenvalue of $K^{1/2}(\theta, \mu_3, \mu_4)(I^{-1}(\theta) - J_0(\theta))K^{1/2}(\theta, \mu_3, \mu_4), K^{1/2}(\theta, \mu_3, \mu_4)$ and $J_0(\theta)$ are defined in Section 2 of the supplementary material, and $X_l^2(1)$ are independent chi-squared random variables with 1 degree of freedom for $l = 1, \ldots, (p + d_1 + d_2 + 1)$. Furthermore, $T_\gamma$ is asymptotically $\chi^2(d_1 + d_2 + 1)$ when $e$ is normally distributed.

If the null hypothesis is not rejected, then we can consider the classical spatial autoregressive model. Otherwise, there exists heterogeneity either among inward or outward influence indices, or both, which motivates us to conduct the next two tests.

**Test II.** To test the homogeneity of $\gamma$, we consider the null and alternative hypotheses:

$$H_{0,\gamma}: \gamma_1 = \gamma_2 = \cdots = \gamma_n \quad \text{vs.} \quad H_{1,\gamma}: \gamma_{i_1} \neq \gamma_{i_2}, \quad \text{for some} \quad i_1 \neq i_2.$$  

Assuming that $\gamma(\alpha) = F(Z_1^T \alpha), F(\cdot)$ is strictly monotone, no intercept is included in $\alpha$, and $Z$ is of full rank defined in Section 2.1, the above hypotheses are equivalent to

$$H_{0,\alpha}: \alpha_1 = \alpha_2 = \cdots = \alpha_{d_1} = 0 \quad \text{vs.} \quad H_{1,\alpha}: \alpha_i \neq 0, \quad \text{for some} \quad i.$$  

(2.5)

Under the null hypothesis of $H_{0,\alpha}$, we can obtain the constrained QMLE $\hat{\theta}_{\alpha}(\cdot)$, whose associated quasi-log-likelihood function is $\ell(\hat{\theta}_{\alpha}(\cdot))$. Accordingly, the quasi-Lagrange multiplier test statistic of $\alpha$ is

$$T_{\alpha} = \frac{1}{n} \left\{ \frac{\partial \hat{\epsilon}(\hat{\theta}_{\alpha}(\cdot))}{\partial \theta} \right\}^T I^{-1}(\hat{\theta}_{\alpha}(\cdot)) \left\{ \frac{\partial \hat{\epsilon}(\hat{\theta}_{\alpha}(\cdot))}{\partial \theta} \right\},$$

and its theoretical property is given below.

**Proposition 1.** Under Conditions (C1)–(C5) and the null hypothesis of $H_{0,\alpha}$, $T_{\alpha}$ is asymptotically distributed as $\sum_{l=1}^{p+d_1+d_2+1} \lambda_l(\theta, \mu_3, \mu_4) X_l^2(1)$ as $n \to \infty$, where $\lambda_l(\theta, \mu_3, \mu_4)$ is the $l$th largest eigenvalue of $K^{1/2}(\theta, \mu_3, \mu_4)(I^{-1}(\theta) - J_0(\theta))K^{1/2}(\theta, \mu_3, \mu_4), \infty$ is defined in Section 2 of the supplementary material, and $X_l^2(1)$ are independent chi-squared random variables with 1 degree of freedom for $l = 1, \ldots, (p + d_1 + d_2 + 1)$. Furthermore, under the normality assumption of $e$, $T_{\alpha}$ is asymptotically $\chi^2(d_1)$.

If the null hypothesis is rejected, then there exists heterogeneity among inward influence indices. Otherwise, we can consider the network influence model proposed by Zou et al. (2021).

**Test III.** To test the homogeneity of $\lambda$, we consider the null and alternative hypotheses:

$$H_{0,\lambda}: \lambda_1 = \lambda_2 = \cdots = \lambda_n \quad \text{vs.} \quad H_{1,\lambda}: \lambda_{j_1} \neq \lambda_{j_2}, \quad \text{for some} \quad j_1 \neq j_2.$$  

Assuming that $\lambda(\beta) = F(V_1^T \beta), F(\cdot)$ is strictly monotone, and all $V$ - 1 is of full rank, the above hypotheses are equal to

$$H_{0,\beta}: \beta_2 = \cdots = \beta_{d_2} = 0 \quad \text{vs.} \quad H_{1,\beta}: \beta_{j_1} \neq 0, \quad \text{for some} \quad j > 1.$$  

Under the null hypothesis of $H_{0,\beta}$, we can obtain the constrained QMLE $\hat{\theta}_{\beta}(\cdot)$ and its associated quasi-log-likelihood function $\ell(\hat{\theta}_{\beta}(\cdot))$. Accordingly, the quasi-Lagrange multiplier test statistic of $\beta$ is

$$T_{\beta} = \frac{1}{n} \left\{ \frac{\partial \hat{\epsilon}(\hat{\theta}_{\beta}(\cdot))}{\partial \theta} \right\}^T I^{-1}(\hat{\theta}_{\beta}(\cdot)) \left\{ \frac{\partial \hat{\epsilon}(\hat{\theta}_{\beta}(\cdot))}{\partial \theta} \right\},$$

and its theoretical property is given below.

**Proposition 2.** Under Conditions (C1)–(C5) and the null hypothesis of $H_{0,\beta}$, $T_{\beta}$ is asymptotically distributed as $\sum_{l=1}^{p+d_1+d_2+1} \lambda_l(\theta, \mu_3, \mu_4) X_l^2(1)$ as $n \to \infty$, where $\lambda_l(\theta, \mu_3, \mu_4)$ is the $l$th largest eigenvalue of $K^{1/2}(\theta, \mu_3, \mu_4)(I^{-1}(\theta) - J_0(\theta))K^{1/2}(\theta, \mu_3, \mu_4), \infty$ is defined in Section 2 of the supplementary material, and $X_l^2(1)$ are independent chi-squared random variables with 1 degree of freedom for $l = 1, \ldots, (p + d_1 + d_2 + 1)$. Furthermore, under the normality assumption of $e$, $T_{\beta}$ is asymptotically $\chi^2(d_1)$.
in Section 2 of the supplementary material, and $\chi^2(1)$ are
independent chi-squared random variables with 1 degree of
freedom for $l = 1, \ldots, (p + d_1 + d_2 + 1)$. Furthermore, $T_{\hat{y}}$
is asymptotically $\chi^2(d_2 - 1)$ under the normality assumption
of $\epsilon$.

If the null hypothesis is rejected, then there exists heterogeneity
among outward influence indices.

It is worth noting that the asymptotic results in The-
orem 3 and Propositions 1 and 2 depend on unknown
parameters. In practice, we can replace them by their cor-
responding consistent estimators. For example, in the test
statistic $T_\lambda$, $\lambda(\theta, \mu_3, \mu_4)$ is unknown. We can replace it by
its consistent estimator $\lambda_{n,\hat{y}}(\hat{\theta}(\cdot), \hat{\mu}(\cdot, \cdot), \hat{\mu}(\cdot, \cdot))$, which is
the lth largest eigenvalue of $K_{n,\beta}^1(\hat{\theta}(\cdot), \hat{\mu}(\cdot, \cdot), \hat{\mu}(\cdot, \cdot))$(I$n^{-1}(\hat{\theta}(\cdot)) - I_n(\hat{\theta}(\cdot)))K_{n,\beta}^1(\hat{\theta}(\cdot), \hat{\mu}(\cdot, \cdot), \hat{\mu}(\cdot, \cdot))$. Here, $K_{n,\beta}^1(\hat{\theta}(\cdot), \hat{\mu}(\cdot, \cdot), \hat{\mu}(\cdot, \cdot)) = I_n(\hat{\theta}(\cdot)) + J_n(\hat{\theta}(\cdot), \hat{\mu}(\cdot, \cdot), \hat{\mu}(\cdot, \cdot))$ is a consistent estimator of
$K(\theta, \mu_3, \mu_4)$, where $\hat{\mu}(\cdot, \cdot) = n^{-1} \sum_{i=1}^n \hat{\mu}(\cdot, \cdot)$, and $\hat{\mu}(\cdot, \cdot)$ is the r-th element of $\epsilon(\hat{\theta}(\cdot), \hat{\mu}(\cdot, \cdot), \hat{\mu}(\cdot, \cdot))$. In addition, $T_{\hat{y}}(\hat{\theta}(\cdot))$
is a consistent estimator of $T_\lambda(\theta)$. Analogous replacements
are made for the asymptotic distributions of $T_{\hat{y}}$ and $T_{\hat{y}}$, which
are omitted here.

3. Numerical Studies

3.1. Simulation Studies

In this section, we conduct simulation studies to investigate the
finite sample performance of parameter estimator, variable
selection, and test statistics. Let the off-diagonal elements,
$a_{ij}$, of the adjacency matrix $A$ be independent and identically
generated from the Bernoulli distribution with probability $5/n$,
and let the diagonal elements, $a_{ii}$, of $A$ be zeros. We then
define the weighted adjacency matrix $W$ as $W = (w_{ij})_{n \times n} \in \mathbb{R}^{n \times n}$ with $w_{ij} = a_{ij}/\sum_{i=1}^n a_{ij}$ for $i, j = 1, \ldots, n$. Next,
the elements of $Z_i = (z_{i1}, z_{i2}, z_{i3})^T$, $V_i = (v_{i1}, v_{i2}, v_{i3}, v_{i4})^T$ and $X_i = (x_{i1}, x_{i2}, x_{i3})$ are
independent and identically generated from the standard normal
distribution $N(0, 1)$ except $v_{i1} = x_{i1} = 1$. Their corresponding
coefficients are $\alpha = (\alpha_1, \alpha_2, \alpha_3)^T = (0, 0.4, 0.6, 0.7)^T$, $\beta = (\beta_1, \beta_2, \beta_3, \beta_4)^T = (1.2, 1.4, 1.3, 1.6)^T$, and $\eta = (\eta_1, \eta_2, \eta_3)^T = (1, 2, 3)^T$. Subsequently, we have $\Gamma = \text{diag}(F(Z_1^\top \alpha), \ldots, F(Z_n^\top \alpha))$ and $\Lambda = \text{diag}(F(V_1^\top \beta), \ldots, F(V_n^\top \beta))$. Because $Z_i$ does not include
the intercept, the above covariate parameter setting ensures the
model is identifiable; see Section 2.1. In addition, the random
errors $e_i$ ($i = 1, \ldots, n$) are independent and identically gen-
erated from two different distributions: the standard normal
distribution, $N(0, 1)$, and the mixture normal distribution, $0.9N(0, 5/9) + 0.1N(0, 5)$, respectively. Finally, the response vector $Y$
is generated from model (2.2) with $Y = (I_n - \Gamma A W)^{-1}(X_n + \epsilon)$ and parameter vector $\theta = (\eta^T, \alpha^T, \beta^T, \sigma^2)^T$. All of these
settings satisfy Conditions (C1)-(C6).

In simulation studies, all results are based on 500 realizations
with $n=500, 1000$, and $2000$. Let $\hat{\theta}^{(k)} = (\hat{\theta}_1^{(k)}, \hat{\theta}_2^{(k)}, \hat{\gamma}_1^{(k)}, \hat{\gamma}_2^{(k)}, \hat{\xi}_1^{(k)}, \hat{\xi}_2^{(k)}, \hat{\xi}_3^{(k)}, \hat{\beta}_1^{(k)}, \hat{\beta}_2^{(k)}, \hat{\beta}_3^{(k)}, \hat{\beta}_4^{(k)}, \hat{\sigma}_2^{(k)})^T \in \mathbb{R}^{11}$ be the
QMLE of $\theta$ at the kth realization and let $\hat{\theta}_j^{(k)}$ be its jth component.
To evaluate the performance of the parameter estimates,
we consider the measurement $\text{BIAS}(\hat{\theta}^{(k)}) = 500^{-1} \sum_{k=1}^{500} (\hat{\theta}^{(k)} - \theta)$, which is the averaged bias of $\hat{\theta}^{(k)}$. Based on $\hat{\theta}^{(k)}$, we can further obtain the estimates $\hat{\gamma}^{(k)}$ and $\hat{\lambda}^{(k)}$ of $\gamma$ and $\lambda$, respectively. Their corresponding bias measurements are
$\text{BIAS}(\hat{\gamma}) = n^{-1} \sum_{i=1}^n \{500^{-1} \sum_{k=1}^{500} (\hat{\gamma}_i^{(k)} - \gamma_i)\}$ and $\text{BIAS}(\hat{\lambda}) = n^{-1} \sum_{i=1}^n \{500^{-1} \sum_{k=1}^{500} (\hat{\lambda}_i^{(k)} - \lambda_i)\}$. By Theorem 1, for each component $\hat{\theta}_j^{(k)}$, we can obtain the estimated standard error $SD_j^{(k)}$. Thus, the average of the estimated standard errors for each $\theta_j = SD_j = 500^{-1} \sum_k SD_j^{(k)}$. Subsequently, the averages of the estimated standard errors of the $n$ nodes via 500 realizations
for $\gamma$ and $\lambda$ are $SD_\gamma = n^{-1} \sum_{i=1}^n \{500^{-1} \sum_k SD_{\gamma}^{(k)}\}$ and $SD_\lambda = n^{-1} \sum_{i=1}^n \{500^{-1} \sum_k SD_{\lambda}^{(k)}\}$, respectively. To obtain an
overall measurement, we also consider the root mean squared
error, $\text{RMSE} = \sqrt{SD^2 + \text{BIAS}^2}$, for $\theta_j, \gamma$, and $\lambda$.

Table 1 presents the BIAS, SD, and RMSE of $\hat{\theta}$, $\hat{\gamma}$, and $\hat{\lambda}$
via 500 realizations under normal random errors with three
sample sizes. The results indicate that, for all three link func-
tions, the absolute value of BIAS and the SD of the parameter
estimates decrease when $n$ gets large. In addition, the biases
of $\gamma$ and $\lambda$ approach 0, and the average estimated errors
are quite small. Hence, it is not surprising that RMSE reveals
the same pattern. Notably, the accurate estimates of $\gamma$ and $\lambda$ allow
us to subsequently conduct four-quadrant clustering, which is not
reported here to save space. Moreover, we find that both the
estimated Hessian matrix and the information matrix are
positive definite at each iterative step across all realizations.
Accordingly, $\hat{\theta}$ is the global maximizer of the log-likelihood
function, and $\hat{\theta}$ is close to its true value, as shown in Table 1.
Under mixture normal errors, the results are qualitatively similar
to those in Table 1, and we relegate them to Table S1 in the
supplementary material. As suggested by an anonymous
reviewer, we consider the simulation setting with an over-
lapping design. The results are qualitatively similar to those in
Table 1, and we present them in Table S3 of the supplementary
material.

We next study the finite sample performance of the BIC variable
selection criterion. To this end, we set up the full model size
$|S_{TA}| = |S_{TP}| = 6$ and the true model size $|S_{TA}| = 3$, $|S_{TP}| = 4$. In addition, three measures are used to assess the
selection performance given below: (i) the average percentage of
correct fit (CF), that is, $I(S\hat{\theta} = S_{TA}) + I(S\hat{\beta} = S_{TB})$; (ii) the average true positive rate (TPR),
that is, $|S\hat{\theta} \cap S_{TA}|/|S_{TA}|$ and $|S\hat{\beta} \cap S_{TB}|/|S_{TB}|$; and (iii) the average false positive rate (FPR),
that is, $|S\hat{\theta} \cap S_{TP}|/|S_{TP}|$ and $|S\hat{\beta} \cap S_{TP}|/|S_{TP}|$, where $I$ is an
indicator function. $\hat{S}_{TA}$ and $\hat{S}_{TB}$ are the selected models via BIC,
$S_{TA} = S_{TA} \setminus S_{TA}$ and $S_{TB} = S_{TB} \setminus S_{TB}$. Table 2 reports CF, TPR, and FPR under normal random
errors via 500 realizations. The results indicate that, as the
sample size increases, the average percentage of CF and the
average TPR approach 1, while the average FPR is close to 0.
Accordingly, the optimal model selected by the BIC is consistently
approaching the true model, which supports Theorem 2. For
the mixture normal errors, the results are qualitatively similar
to those in Table 2, and we present them in Table S2 of the
supplementary material.
We finally investigate the performance of the three quasi Lagrange-multiplier tests by testing the homogeneity of the inward and outward influence indices under normal random errors. To this end, we set up the three hypotheses: (i) $\alpha = (0.4, 0.6, 0.7, 0.7)\top$ and $\beta = (1.2, 1.4, 1.3, 1.6)\top$, (ii) $\alpha = (0.4, 0.6, 0.7, 0.7)\top$ and $\beta = (1.2, 1.4, 1.3, 1.6)\top$, and (iii) $\alpha = (0.4, 0.6, 0.7, 0.7)\top$ and $\beta = (1.2, 1.4, 1.3, 1.6)\top$ for Tests I, II, and III, respectively with $\delta = 0, 0.1, 0.15, 0.2, 0.25, 0.3, 0.4, 0.7$. Here, $\delta = 0$ evaluates the empirical sizes of the three tests, while the other values of $\delta$ are used to assess the empirical powers. Then, we calculate the empirical sizes and powers of $T_{\alpha i}$, $T_{\alpha j}$, and $T_{\beta i j}$ under the three link functions with the significance level of 0.05 via 500 realizations. Figures 2–4 demonstrate that the empirical sizes of the three tests approach the predetermined significance level, 0.05, as the sample size $n$ grows large, which demonstrates the validity of the three testing procedures mentioned in Section 2.4. In addition, the powers of the three tests increase and tend to 100\% when the sample size $n$ becomes large. In sum, our proposed three tests perform well for assessing the heterogeneity of influence indices.

### 3.2. Real Data Analysis

To illustrate the practical usefulness of the IONI model and four-quadrant clustering method, we collect social network data from Sina Weibo, which is the biggest social media platform in China. Our Sina Weibo dataset is a snapshot taken on November 13, 2013, and contains 2,580 online users’ activity and their corresponding attribute variables between January 1, 2013 and November 13, 2013, where all users in our dataset had registered before January 1, 2013. Accordingly, we define the adjacency matrix $A = (a_{ij}) \in \mathbb{R}^{2,580 \times 2,580}$ as follows: $a_{ij} = 1$ if user $i$ is following user $j$ and $a_{ij} = 0$ otherwise; a similarly defined adjacency matrix can be found in Bramoulle, Djebbari, and Fortin (2009), Zhou et al. (2017), and Zhu et al. (2019). Since the network density is approximately equal to 1.3\%, this network is extremely sparse. Thus, Conditions (C2) and (C3) are satisfied. The response variable $Y_i$ is the $i$th user's activity level, which is proxied by the number of Sina Weibo posts made by user $i$. However, on any given day, we find that most users have an activity level (i.e., number of posts) of zero. Thus, our response variable is the total number of posts over the entire study period. In this way, our proposed model allows us to assess the overall inward and outward network influence across the network during this period.

To characterize users’ inward and outward influence, we consider the following five attributes: (i) In-degree: the number of each user's followers in the network; (ii) Out-degree: the number of each user's followees in the network; (iii) Duration:
Figure 2. The empirical sizes and powers of Test I under the three link functions with significance level 0.05. The signal strengths are $\delta = 0, 0.1, 0.15, 0.2, 0.25, 0.3, 0.4, \text{ and } 0.7$, which correspond to the settings of $\alpha = (0.4\delta, 0.6\delta, 0.7\delta)^\top$ and $\beta = (1.2, 1.4\delta, 1.3, 1.6\delta)^\top$. The random errors are independently and identically simulated from the normal distribution $N(0, 1)$.

Figure 3. The empirical sizes and powers of Test II under the three link functions with significance level 0.05. The signal strengths are $\delta = 0, 0.1, 0.15, 0.2, 0.25, 0.3, 0.4, \text{ and } 0.7$, which correspond to the settings of $\alpha = (0.4\delta, 0.6\delta, 0.7\delta)^\top$ and $\beta = (1.2, 1.4, 1.3, 1.6)^\top$. The random errors are independently and identically simulated from the normal distribution $N(0, 1)$.

Figure 4. The empirical sizes and powers of Test III under the three link functions with significance level 0.05. The signal strengths are $\delta = 0, 0.1, 0.15, 0.2, 0.25, 0.3, 0.4, \text{ and } 0.7$, which correspond to the settings of $\alpha = (0.4, 0.6, 0.7)^\top$ and $\beta = (1.2, 1.4, 1.3, 1.6\delta)^\top$. The random errors are independently and identically simulated from the normal distribution $N(0, 1)$. 
the time elapsed since registering on the Sina Weibo platform; (iv) Self-Introduction: the length of a user’s description about
himself/herself; and (v) Gender: male=1 and female=0. We then assign all five attributes to each of the three covariates, Z, V and X. Afterwards, we fit them with the model (2.2), and find that none of their associated coefficients α, β, and η are significant at the 5% significance level. Thus, we iteratively eliminated the attribute with the largest p-value across Z, V and X until one of the attributes becomes significant. As a result, the remaining attributes in Z, V, and X are (“Out-degree”, “Duration” and “Self-Introduction”), (“In-degree,” “Duration,” and “Self-Introduction”), and (“Gender”), respectively.

In this empirical example, the sample size is large and the full model is finite. This motivates us to further employ the BIC consistent criterion to select the best model for each of the three link functions: sigmoid, inverse of log-log and inverse of probit. Inspired by Vuong (1989), we further choose the link function with the smallest BIC value, which is the inverse of probit. Table 3 reports the QMLEs of the parameters and their corresponding t-statistics and p-values. For Z covariates, the attribute “Out-degree” is selected, and it is significant at the 5% level with a positive coefficient. This finding is sensible since users with more out-degree may have broad interests, outgoing personality, and strong curiosity to accept new things and are thus more likely to be influenced by others. As for V covariates, the attribute “Duration” is selected, and it is significant at the 5% level with a positive coefficient. This result is also reasonable because the users with longer duration are generally called “veteran users,” who tend to have greater influence in the network. Finally, the p-value of the X variable “gender” is 0.1857. Thus, the users’ activity level is not strongly related to gender.

It is worth noting that, without employing an initial variable elimination procedure, none of the selected variables are significant at the 5% significance level when using BIC to select relevant variables from the full model with fifteen candidate variables (five attributes in each of three covariates Z, V, and X); the results are omitted to save space. In addition, we find that both the estimated Hessian matrix and information matrix are positive definite at each iterative step, which ensures that the QMLEs listed in Table 3 are global maximizers.

We next examine the heterogeneity of inward and outward influence among individuals via the three test statistics in Section 2.5. The p-values of all three tests are close to 0, which indicate high heterogeneity in both inward and outward influence among the 2580 users. To evaluate these phenomena, we calculate the estimated influence indices  \( \hat{\gamma}_i = \Phi(Z_i^\top \hat{a}) \) and  \( \hat{\lambda}_j = \Phi(V_j^\top \hat{\beta}) \). We then sort them and obtain  \( \hat{\gamma}_1 > \hat{\gamma}_2 > \ldots > \hat{\gamma}_n \) and  \( \hat{\lambda}_1 > \hat{\lambda}_2 > \ldots > \hat{\lambda}_n \). This allows us to conduct user segmentation. To this end, as shown in Figure 1, we designate the origin as  \( y_0 = \hat{\gamma}_1 \) and  \( \lambda_0 = \hat{\lambda}_1 \), where  \( k_1 \) and  \( k_2 \) satisfy  \( \sum_{i=1}^{k_1} \hat{\gamma}_i = 12.5\% \) and  \( \sum_{i=1}^{k_2} \hat{\lambda}_i = 12.5\% \), respectively. To illustrate the usefulness of our proposed clustering method, we choose the 12.5% quantile as the threshold for our user segmentation. In practice, decision makers can balance the benefits and costs of choosing a different threshold.

With the given threshold values  \( y_0 \) and  \( \lambda_0 \), we now employ the four quadrant method of classifying users, following two different approaches. First, we conduct the “indices-comparison (IC)” by comparing  \( \hat{\gamma}_i \) and  \( \hat{\lambda}_i \) with the corresponding threshold values  \( y_0 \) and  \( \lambda_0 \) for  \( i = 1, \ldots, n \). Second, we proceed with the “indices-test (IT)” by calculating the test statistics  \( T_{iY} \) and  \( T_{i\lambda} \) for  \( i = 1, \ldots, n \), and obtain the corresponding p-values  \( p_{iY} \) and  \( p_{i\lambda} \) by controlling the FDR as mentioned in Section 2.3. Figure 5 depicts the percentage of users being classified into the four quadrants by the aforementioned two approaches. Because the results are similar, we focus only on the “indices-test” approach.

Based on user segmentation, practitioners can design marketing strategies to improve a product promotion or opinion dissemination. For example, if practitioners need to promote a new product on Weibo, they could focus on not just “influencers” but specifically the 4.0% “early adopters.” In contrast, if practitioners want to conduct a promotion for a mature product, they should focus on the 7.7% “early majority.” To effectively use the above two types of users, decision makers can adopt the “fan economy” strategy to increase their conversion rate. With this strategy, they should first focus on the 15.2% “opinion leaders” to make such promotions. Finally, removing or ignoring the 73.1% who are prolonged “inactive users” can reduce promoting costs and improve marketing efficiency. In sum, this example demonstrates that obtaining the nodes’ inward and outward influence indices can play an important role in social network applications.

| Attribute-types | Variables | Estimates | Standard error | t-statistic | p-value |
|-----------------|-----------|-----------|----------------|-------------|---------|
| Z               | Out-degree | 0.1263    | 0.0181         | 6.9747      | <0.0001 |
|                 | Duration  | 0.6705    | 0.1356         | 4.9463      | <0.0001 |
| X               | Intercept | 2.7460    | 0.1137         | 24.1494     | <0.0001 |
|                 | Gender    | 0.0611    | 0.0684         | 0.8939      | 0.1857  |

Table 3. Variable selection results from the Sina Weibo data with the inverse of the probit link function.

Figure 5. The proportion (%) of each user segment in the Sina Weibo data as classified by the IC and IT methods. The Q1, Q2, Q3, and Q4 represent the first quadrant (“early adopters”), the second quadrant (“opinion leaders”), the third quadrant (“inactive users”), and the fourth quadrant (“early majority”), respectively.
4. Concluding Remarks

In large scale networks, we propose an IONI model that not only captures the heterogeneity among nodes, but is also applicable to user segmentation. Without imposing any specific error distribution, we apply the quasi-maximum likelihood approach to obtain the estimators of the inward and outward influence indices. This allows us to employ the four-quadrant clustering method to segment users. The theoretical properties of parameter estimates are established, and simulation studies as well as an empirical analysis are presented to demonstrate the usefulness of the IONI model.

To broaden the usefulness of IONI and four-quadrant clustering, we identify the following four possible future research avenues. First, the adjacency matrix of the network is observed in our study. For an unobserved adjacency matrix, we could allow it to follow a probability distribution and estimate the adjacency matrix. Then we could employ IONI for studying influential power within networks. It is not surprising that indirectly connected nodes can influence each other in a network; see Lan et al. (2018). Hence, the second avenue is to take into account higher-order adjacency matrices to enlarge the application of the IONI model. Third, motivated by an anonymous reviewer’s comment, one can consider observations and network structures that vary with time. Hence, it is useful to extend IONI to dynamic network models; see, for example, Dou, Parrella, and Yao (2016) and Gao et al. (2019). Last, based on an anonymous reviewer’s suggestion, it is of great interest to generalize IONI to accommodate data with discrete responses (see Zhang et al. 2020). We believe each of the above efforts would increase the value of IONI considerably.

Supplementary Materials

The online supplementary material includes six sections. Section 1 presents detailed expressions of the Fisher information matrix \( I_\theta(\theta) \) and the quantity \( J_\theta(\mu_3, \mu_4) \) used in estimating the asymptotic covariance matrix of parameter estimators. Additional notations used for proving Theorem 3 and Propositions 1-2 are given in Section 2. Section 3 provides five technical lemmas. Section 4 presents the proofs of theorems and propositions. Simulation studies for mixture normal errors and for an overlapping design are provided in Sections 5 and 6, respectively. Both additional studies are used to demonstrate the robustness of our proposed estimators.

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