On Quillen’s conjecture for p-solvable groups

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Introduction

D. Quillen, "Homotopy properties of the poset of nontrivial $p$-subgroups of a group," Adv. Math. 28 (1978), no. 2, 101–128.

For a finite group $G$ and a prime $p$, let $A_p(G)$ be the poset of non-trivial elementary abelian $p$-subgroups of $G$. Let $O_p(G)$ be the largest normal $p$-subgroup of $G$.

Theorem (Quillen) $O_p(G) \neq 1 \Rightarrow |A_p(G)| \sim *$.

Conjecture (Quillen) $|A_p(G)| \sim * \Rightarrow O_p(G) \neq 1$. 
INTRODUCTION

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A finite group, $p$ a prime.

$\mathcal{A}_p(G) =$ poset of non-trivial elementary abelian $p$-subgroups of $G$.

$\mathcal{O}_p(G) =$ largest normal $p$-subgroup of $G$.

Theorem (Quillen) $\mathcal{O}_p(G) \neq 1 \Rightarrow |\mathcal{A}_p(G)| \sim \ast$.

Conjecture (Quillen) $|\mathcal{A}_p(G)| \sim \ast \Rightarrow \mathcal{O}_p(G) \neq 1$. 
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- $G$ finite group, $p$ a prime.
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**Theorem (Quillen)**

$O_p(G) \neq 1 \Rightarrow |\mathcal{A}_p(G)| \cong \ast.$
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**Theorem (Quillen)**

$$O_p(G) \neq 1 \Rightarrow |A_p(G)| \cong \ast.$$ 

**Conjecture (Quillen)**

$$|A_p(G)| \cong \ast \Rightarrow O_p(G) \neq 1.$$
Introduction

**Conjecture (QC)**

\[ \text{Op}(G) = 1 \Rightarrow |A_{\text{p}}(G)| \neq \ast. \]

**Known cases** \( r = \text{rank} \mathcal{P}(G) = \dim(A_{\text{p}}(G)) + 1 \):

- \( r = 1 \) (Quillen),
- \( r = 2 \) (Quillen-Serre),
- \( G \) solvable (Quillen, Alperin 80s),
- \( G_p \) solvable (Alperin 80s, Aschbacher-Smith 93, Hawks-Isaacs 87),
- \( p > 5 \) and \( G \) has no component \( U_n(q) \) with \( q \equiv -1 \mod p \), \( q \) odd (Aschbacher-Smith 93).

**Theorem (D 2016)**

Quillen’s conjecture holds if \( G \) is solvable or \( p \)-solvable.

**Highlights**:

- Can build sphere in non-solvable situation,
- Can lift spheres from quotient group.
Introduction

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\[ O_p(G) = 1 \Rightarrow |A_p(G)| \not\approx \ast. \]
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Conjecture (QC)

\[ O_p(G) = 1 \implies |A_p(G)| \not\approx *. \]

Known cases \((r = \text{rank}_p(G) = \text{dim}(A_p(G)) + 1)\):
- \(r = 1\) (Quillen), \(r = 2\) (Quillen-Serre).
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**Theorem (D 2016)**

*Quillen’s conjecture holds if \(G\) is solvable or \(p\)-solvable.*

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- Can build sphere in non-solvable situation.
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Building a homology sphere in top dimension

Alperin's solution:

\[ G \text{ p-solvable} \rightarrow \rightarrow K \rtimes \mathcal{C} \text{r}_{p} \]

\[ \text{CFSG} \downarrow \downarrow \]

\[ G \text{ solvable} \rightarrow \rightarrow K \rtimes \mathcal{C} \text{r}_{p}, \quad K \text{ solvable} \rtimes \text{quotient} \rightarrow \rightarrow (A \times B) \rtimes \mathcal{C} \text{r}_{p} \rightarrow \rightarrow \mathcal{O}_{p}(G) = 1, \quad r = \text{rank } p(G), \quad K_{p}^{' \text{-group}}, \quad A \text{ is an abelian } p^{' \text{-group}}, \quad B \text{ is a direct product of nonabelian simple } p^{' \text{-groups}}, \text{action of } \mathcal{C} \text{r}_{p} \text{ on } K \text{ is faithful.} \]
Alperin’s solution:

\[ G \text{ } p\text{-solvable} \xrightarrow{Q.\text{ reduction}} K \rtimes C_p \]

\[ G \text{ solvable} \xrightarrow{Q.\text{ reduction}} K \rtimes C_p, \text{ } K \text{ solvable} \xrightarrow{r\text{-sphere}} \]
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G \text{ } \text{solvable} \xrightarrow{Q\text{-reduction}} K \rtimes C_p , K \text{ solvable} \xrightarrow{\text{Alperin}} r\text{-sphere}
\]

New solution:

\[
G \text{ } p\text{-solvable} \xrightarrow{Q\text{-reduction}} K \rtimes C_p \xrightarrow{\text{quotient/lift}} (A \times B) \rtimes C_p
\]

\[
G \text{ } \text{solvable} \xrightarrow{Q\text{-reduction}} K \rtimes C_p , K \text{ solvable} \xrightarrow{\text{quotient/lift}} A \rtimes C_p
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Alperin’s solution:

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\[ G \text{ solvable} \xrightarrow{Q\text{-reduction}} K \rtimes C_p \text{, } K \text{ solvable} \xrightarrow{\text{Alperin}} r\text{-sphere} \]

New solution:

\[ G \text{ } p\text{-solvable} \xrightarrow{Q\text{-reduction}} K \rtimes C_p \xrightarrow{\text{quotient/lift}} (A \times B) \rtimes C_p \]
\[ K \text{ solvable} \xrightarrow{\text{quotient/lift}} A \times C_p \]

\[ O_p(G) = 1, \text{ } r = rank_p(G), \text{ } K \text{ } p'\text{-group, } A \text{ is an abelian } p'\text{-group, } B \text{ is a direct product of nonabelian simple } p'\text{-groups, action of } C_p^r \text{ on } K \text{ is faithful.} \]
BUILDING A HOMOLOGY SPHERE IN TOP DIMENSION

\[ H = \mathbb{C}p, a \in \mathbb{Z} \rightarrow \mathbb{Z} \]

\[ H_{[i_1, \ldots, i_l]} = \{ (x_1, \ldots, x_r) \in H | x_{i_1} = \cdots = x_{i_l} = 0 \} \leq H. \]

\[ \sigma_{[i_1, \ldots, i_l]} = H_{[i_1, \ldots, i_l]} < H_{[i_1, i_2]} < H_{[i_1]} < H \in |A_p(H)|. \]

\[ Z_H, a = \sum_{[i_1, \ldots, i_{r-1}]} \cdot a \cdot \sigma_{[i_1, \ldots, i_{r-1}]} \in C^{r-1}(|A_p(H)|). \]

Example

For \( r = 3 \), \( S_3^2 \) = \{ [1, 2], [2, 1], [1, 3], [3, 1], [2, 3], [3, 2] \}, in \(|A_p(C^3_p)|\):

\[ H_{[1]} < H_{[1, 2]} < H_{[1]} < H_{[1, 2]} < H_{[1, 3]} < H_{[1]} < H_{[2, 1]} < H_{[2]} < H_{[2, 3]} < H_{[2]} < H_{[3, 1]} < H_{[3]} < H_{[3, 2]} < H_{[3]} < H_{[2, 3]} < H_{[2, 3]} < H_{[3, 1]} < H_{[3]} \]
$H = C^r_p$
$H = C_p, a \in \mathbb{Z}$
\[ H = C_p, \, a \in \mathbb{Z} \not\sim \mathbb{Z}_H, a \in C_{r-1}(\lvert A_p(H) \rvert). \]
H = C'_p, \ a \in \mathbb{Z} \mapsto Z_{H,a} \in C_{r-1}(|A_p(H)|).

▶ \ H_{[i_1, \ldots, i_l]} = \{(x_1, \ldots, x_r) \in H | x_{i_1} = \ldots = x_{i_l} = 0\} \leq H.
$H = C^r_p, a \in \mathbb{Z} \rightarrow \mathbb{Z}_{H,a} \in C_{r-1}(|A_p(H)|)$.

- $H_{[i_1, \ldots, i_l]} = \{(x_1, \ldots, x_r) \in H|x_{i_1} = \ldots = x_{i_l} = 0\} \leq H$.
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Example

For $r = 3$, $S^3_3 = \{\begin{array}{c}
[1,2] \\
[2,1] \\
[1,3] \\
[3,1] \\
[2,3] \\
[3,2]
\end{array}\}$, in $|A_p(C^3_p)|$:
$H = C^r_p, \ a \in \mathbb{Z} \mapsto Z_{H,a} \in C_{r-1}(|A_p(H)|)$.

- $H_{[i_1, \ldots, i_l]} = \{(x_1, \ldots, x_r) \in H | x_{i_1} = \ldots = x_{i_l} = 0\} \leq H$.
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- $Z_{H,a} = \sum_{[i_1, \ldots, i_{r-1}]} \epsilon_{[i_1, \ldots, i_{r-1}]} \cdot a \cdot \sigma_{[i_1, \ldots, i_{r-1}]} \in C_{r-1}(|A_p(H)|)$. 

Example

For $r = 3$, $S_3^2 = \{[1, 2], [2, 1], [1, 3], [3, 1], [2, 3], [3, 2]\}$, in $|A_p(C_3^p)|$. 

$H_{[1]} < H_{[1, 2]} < H_{[1]} < H_{[1, 2]} < H_{[1, 2]} < H_{[1]} < H_{[2, 1]} < H_{[2, 1]} < H_{[1]} < H_{[2]} < H_{[3]} < H_{[3]}$. 

$H_{[3]} < H_{[3, 1]} < H_{[3]} < H_{[3, 1]} < H_{[3, 1]} < H_{[3]}. $
**Introduction**

**Building a Homology Sphere in Top Dimension**

\[ H = C_p^r, \ a \in \mathbb{Z} \Rightarrow Z_{H,a} \in C_{r-1}(|A_p(H)|). \]

- \( H_{[i_1, \ldots, i_l]} = \{ (x_1, \ldots, x_r) \in H | x_{i_1} = \ldots = x_{i_l} = 0 \} \leq H. \)
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**Example**

For \( r = 3, S_2^3 = \{ [1, 2], [2, 1], [1, 3], [3, 1], [2, 3], [3, 2] \}, \) in \( |A_p(C_p^3)|: \)

```
H[1,2] < H[1] < H
H[2,1] < H[2] < H
H[1,3] < H[1] < H
H[3,1] < H[3] < H
H[2,3] < H[2] < H
H[3,2] < H[3] < H
```
\[ H = C^r_p, a \in \mathbb{Z} \Rightarrow Z_{H,a} \in C_{r-1}(|A_p(H)|). \]

- \( H_{[i_1,...,i_l]} = \{ (x_1, \ldots, x_r) \in H | x_{i_1} = \ldots = x_{i_l} = 0 \} \leq H. \)
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For \( r = 3, S^3_2 = \{ [1, 2], [2, 1], [1, 3], [3, 1], [2, 3], [3, 2] \}, \) in \( |A_p(C^3_p)|: \)
BUILDING A HOMOLOGY SPHERE IN TOP DIMENSION

\[ G = K \rtimes H, \quad H = C_r \cdot a \cdot S \in \text{Syl}_p(G) \xrightarrow{\sim} Z_G, \quad a \cdot S \in C_r - 1(|A_p(G)|) . \]

\[ \forall S \in \text{Syl}_p(G) \Rightarrow S = k_S H \text{ for some } k_S \in K . \]

\[ Z_G, a \cdot S = \sum_{S \in \text{Syl}_p(G)} k_S (Z_H, a_S) \in C_r - 1(|A_p(G)|) . \]

\[ d(Z_G, a \cdot S) = 0 \text{ if and only if: } \sum_{S \in N(I)} k_H[i] a_S = 0 \text{ for all } i \in \{1,...,r\} \text{ and all } K \text{-conjugates } k_H[i] \text{ of } H[i], \]

where:

\[ N(I) = \{ S \in \text{Syl}_p(H) \mid I \leq S \} \text{ and } |N(I)| = \frac{|C_K(I)|}{|C_K(H)|} \text{ for } I \leq H . \]

We have removed a subdivision.

Faithful action implies \(|N(H[i])| > 1\) for \(i = 1, ..., r\).

\[ \tilde{H}_{cr - 1}(|A_p(G)|; Z) \leq \tilde{H}_{r - 1}(|A_p(G)|; Z) . \]
\(G = K \rtimes H, H = C_p\)
$G = K \rtimes H, H = C^r_p, a_\circ = (a_S)_{S \in \text{Syl}_p(G)}$
$G = K \rtimes H, H = C_p, \ a_\cdot = (a_S)_{S \in \text{Syl}_p(G)} \mapsto Z_G.a_\cdot \in C_{r-1}(|A_p(G)|).$
\( G = K \rtimes H, H = \mathbb{C}_p, a_\bullet = (a_S)_{S \in \text{Syl}_p(G)} \mapsto Z_G.a_\bullet \in C_{r-1}(|\mathcal{A}_p(G)|). \)

- \( S \in \text{Syl}_p(G) \Rightarrow S = k_S H \) for some \( k_S \in K \).
$G = K \rtimes H$, $H = C_p^{r}$, $a_\bullet = (a_S)_{S \in \text{Syl}_p(G)} \sim Z_G.a_\bullet \in C_{r-1}(|A_p(G)|)$.

- $S \in \text{Syl}_p(G) \Rightarrow S = k_S H$ for some $k_S \in K$.
- $Z_{G,a_\bullet} = \sum_{S \in \text{Syl}_p(G)} k_S(Z_{H,a_S}) \in C_{r-1}(|A_p(G)|)$. 
\[ G = K \rtimes H, H = C_p^r, a_\bullet = (a_S)_{S \in \text{Syl}_p(G)} \mapsto Z_G.a_\bullet \in C_{r-1}(|A_p(G)|). \]

- \( S \in \text{Syl}_p(G) \Rightarrow S = kS H \) for some \( k_S \in K \).
- \( Z_G.a_\bullet = \sum_{S \in \text{Syl}_p(G)} k_S (Z_H.a_S) \in C_{r-1}(|A_p(G)|). \)
- \( d(Z_G.a_\bullet) = 0 \) if and only if:

\[
\sum_{S \in \mathcal{N}^{(kH[i])}} a_S = 0
\]

for all \( i \in \{1, \ldots, r\} \) and all \( K \)-conjugates \( kH[i] \) of \( H[i] \), where:

\[
\mathcal{N}(I) = \{S \in \text{Syl}_p(H) | I \leq S\} \text{ and } |\mathcal{N}(I)| = |C_K(I)| / |C_K(H)| \text{ for } I \leq H.
\]
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\[ G = K \rtimes H, \quad H = C_p, \quad a \cdot = (a_S)_{S \in \text{Syl}_p(G)} \sim Z_{G,a \cdot} \in C_{r-1}(|A_p(G)|). \]

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- We have removed a subdivision.
\[ G = K \rtimes H, H = C_p^r, a_\bullet = (a_S)_{S \in \text{Syl}_p(G)} \mapsto Z_G, a_\bullet \in C_{r-1}(|A_p(G)|). \]

- \( S \in \text{Syl}_p(G) \Rightarrow S = k_S H \) for some \( k_S \in K \).
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- We have removed a subdivision.
- Faithful action implies \( |\mathcal{N}(H[i])| > 1 \) for \( i = 1, \ldots, r \).
\( G = K \rtimes H, H = C_p^r, a_\bullet = (a_S)_{S \in \text{Syl}_p(G)} \mapsto Z_G.a_\bullet \in C_{r-1}(|A_p(G)|). \)

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- We have removed a subdivision.
- Faithful action implies \( |\mathcal{N}(H[i])| > 1 \) for \( i = 1, \ldots, r \).
- \( \tilde{H}_r^c(|A_p(G)|; \mathbb{Z}) \leq \tilde{H}_{r-1}(|A_p(G)|; \mathbb{Z}). \)
Theorem (Asymptotic $p$-solvable case of Quillen's conjecture)

\[ \tilde{H}_{r-1}(|A_p(G)|; \mathbb{Z}) \neq 0 \quad \text{if} \quad |K| = q^{e_1 \cdot \ldots \cdot e_l} \]

satisfies that $r < q_i$ for all $i = 1, \ldots, l$, $H = C_{r^p}$ acts faithfully, $K = p'$-group.

\[ \text{▶ Proof:} \quad \text{There are more variables than equations and} \quad \mathbb{Z} \text{is a P.I.D.} \]

Theorem (Existence of top dimensional sphere)

$G = K \rtimes H$, $H = C_{r^p}$, $K$ a $p'$-group, elements $c_i \in C_K(H)[i] \setminus C_K(H)$ for $i = 1, \ldots, r$ such that $[c_i, c_j] = 1$ for all $i$ and $j$.

\[ \text{▶ Proof:} \quad \text{Set} \quad a_S = (-1)^{\sum_{i} \delta_i} \quad \text{for} \quad S = c_{\delta_1}^{a_1} \cdot \ldots \cdot c_{\delta_r}^{a_r}H, \quad \delta_i \in \{0, 1\}, \quad \text{and} \quad a_S = 0 \quad \text{otherwise.} \]
Theorem (Asymptotic $p$-solvable case of Quillen’s conjecture)

\[ \tilde{H}_{r-1}(|A_p(K \rtimes H)|; \mathbb{Z}) \neq 0 \text{ if } |K| = q_1^{e_1} \cdots q_l^{e_l} \text{ satisfies that } r < q_i \text{ for all } i = 1, \ldots, l, \text{ } H = C_p \text{ acts faithfully, } K p'-\text{group.} \]
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▶ Proof: There are more variables than equations and $\mathbb{Z}$ is a P.I.D.
Theorem (Asymptotic $p$-solvable case of Quillen’s conjecture)

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- Proof: There are more variables than equations and \( \mathbb{Z} \) is a P.I.D.

Theorem (Existence of top dimensional sphere)

\[ G = K \rtimes H, H = C_p^r, K \text{ a } p'-\text{group, elements } c_i \in C_K(H[i]) \setminus C_K(H) \text{ for } i = 1, \ldots, r \text{ such } [c_i, c_j] = 1 \text{ for all } i \text{ and } j \Rightarrow \tilde{H}_{r-1}(|A_p(G)|; \mathbb{Z}) \neq 0. \]
Theorem (Asymptotic $p$-solvable case of Quillen’s conjecture)

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\[ G = K \rtimes H, \text{ } H = C'_p, \text{ } K \text{ a } p'^{-}\text{-group, elements } c_i \in C_K(H_{[i]}) \setminus C_K(H) \text{ for } i = 1, \ldots, r \text{ such } [c_i, c_j] = 1 \text{ for all } i \text{ and } j \Rightarrow \widetilde{H}_{r-1}(|A_p(G)|; \mathbb{Z}) \neq 0. \]

- Proof: Set \( a_S = (-1)^{\sum_i \delta_i} \) for \( S = c_1^{\delta_1} \cdots c_r^{\delta_r} H, \delta_i \in \{0, 1\}, \text{ and } a_S = 0 \) otherwise.
Theorem (Top dimensional sphere in abelian case)

\[ G = A \rtimes H, \text{A abelian, } H \text{ acts faithfully} \Rightarrow \tilde{H} \cap r^{-1}(|A_2(G)|; \mathbb{Z}) \neq 0. \]

Example

\[ G = C_5 \times C_5 \times C_5 \rtimes C_2 \times C_2 \times C_2, \]

\[ (x_1, x_2, x_3) \mapsto (x_1, x_2, x_3), \]

\[ (x_1, -x_2, x_3) \text{ and } (x_1, x_2, -x_3). \]

\[ c_i \in C_K(H[i]) \setminus C_K(H); i = 1, 2, 3. \]
Theorem (Top dimensional sphere in abelian case)

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**Theorem (Top dimensional sphere in abelian case)**

\[ G = A \rtimes H, \text{ } A \text{ abelian, } H \text{ acts faithfully} \implies \tilde{H}_{r-1}(|\mathcal{A}_p(G)|; \mathbb{Z}) \neq 0. \]

**Example**

\[ G = C_5 \times C_5 \times C_5 \times C_2 \times C_2 \times C_2, \text{ } (x_1, x_2, x_3) \mapsto (-x_1, x_2, x_3), \]

\[(x_1, -x_2, x_3) \text{ and } (x_1, x_2, -x_3). \text{ } c_i \in C_K(H_{[i]}) \setminus C_K(H) \text{ } i = 1, 2, 3. \]
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Graph in \( A_2(G) \).
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Graph in \( A_2(G) \). In \( |A_2(G)| \):
**Theorem (Top dimensional sphere in abelian \times simple case)**

\[ G = (A \times B) \rtimes H, \text{ } A \text{ abelian, } B \text{ product of nonabelian simple } p' \text{-groups, } H \text{ acts faithfully} \Rightarrow \tilde{\text{H}}_r - 1 (|A_p(G)|; \mathbb{Z}) \neq 0. \]

**Theorem**

If \( H = C_r p \) acts transitively on a direct product \( Y \) of \( m \) copies of a nonabelian simple \( p' \)-group \( X \) and \( H_1, \ldots, H_n \) are linearly independent hyperplanes of \( H \) with \( C_Y(H_i) > C_Y(H) \), then there exist elements \( c_i \in C_Y(H_i) \backslash C_Y(H) \) such that

\[ [c_i, c_j] = 1 \text{ for all } i, j \text{ (and } n \leq m). \]

▶ Proof: Uses the following consequence of the CFSG (already used by Aschbacher-Smith):

If \( P \) is a \( p \)-group and \( 1 \neq P \leq \text{Out}(X) \) for a nonabelian simple \( p' \)-group \( X \), then \( X \) is of Lie type and \( P \) is cyclic and consists of field automorphisms.
Theorem (Top dimensional sphere in abelian\times simple case)

\[ G = (A \times B) \rtimes H, \text{ } A \text{ abelian, } B \text{ product of nonabelian simple } p'\text{-groups, } H \text{ acts faithfully} \Rightarrow \tilde{H}_{r-1}(|A_p(G)|; \mathbb{Z}) \neq 0. \]
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Theorem

If \( H = C_p \) acts transitively on a direct product \( Y \) of \( m \) copies of a nonabelian simple \( p' \)-group \( X \) and \( H_1, \ldots, H_n \) are linearly independent hyperplanes of \( H \) with \( C_Y(H_i) > C_Y(H) \), then there exist elements \( c_i \in C_Y(H_i) \setminus C_Y(H) \) such that \([c_i, c_j] = 1\) for all \( i, j \) (and \( n \leq m \)).
Theorem (Top dimensional sphere in abelian×simple case)

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Theorem

If \( H = C_p^r \) acts transitively on a direct product \( Y \) of \( m \) copies of a nonabelian simple \( p'\)-group \( X \) and \( H_1, \ldots, H_n \) are linearly independent hyperplanes of \( H \) with \( C_Y(H_i) > C_Y(H) \), then there exist elements \( c_i \in C_Y(H_i) \setminus C_Y(H) \) such that \([c_i, c_j] = 1\) for all \( i, j \) (and \( n \leq m \)).

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\[ \text{If } P \text{ is a } p\text{-group and } 1 \neq P \leq \text{Out}(X) \text{ for a nonabelian simple } p'\text{-group } X, \text{ then } X \text{ is of Lie type and } P \text{ is cyclic and consists of field automorphisms.} \]
Work in progress with Nadia Mazza

Build spheres in dimension \( \text{rank}_p(G) - 1 \) for \( G \) a finite simple group (only possible for Lie type in non-defining characteristic, exceptional, alternating and sporadic).

**Theorem (D-Mazza, 2017)**

Let \( H = C_r^p \leq G \) with \( r = \text{rank}_p(G) \), elements \( c_i \in C_G(H_i) \setminus N_G(H) \) with \([c_i, c_j] = 1\) for all \( i \) and \( j \). Then \( \tilde{H}_{r-1}(|A_p(G)|; \mathbb{Z}) \neq 0 \).

**Example (D-Mazza, 2017)**

Let \( G = A_n \). Then \( r = \text{rank}_p(G) = \lfloor \frac{n}{p} \rfloor \) and a maximal elementary abelian \( p \)-group of \( A_n \), \( H = C_r^p \), is generated by

\[
(1, \ldots, p), (p + 1, \ldots, 2p), \ldots, (r(p - 1) + 1, \ldots, rp).
\]
Thanks!