Noncommutativity in open string: New results in a
gauge independent analysis

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Abstract

Noncommutativity in an open string moving in a background Neveu-Schwarz field is inves-
tigated in a gauge independent Hamiltonian approach, leading to new results. The noncom-
mutativity is shown to be a direct consequence of the non-trivial boundary conditions, which,
contrary to several approaches, are not treated as constraints. We find that the noncommu-
tativity persists for all string points. In the conformal gauge our results reduce to the usual
noncommutativity at the boundaries only.

Keywords: Hamiltonian analysis, Noncommutativity, Strings

PACS 11.15.-q, 10.11.Ef, 11.25.-w

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1 Introduction

The study of open string, in the presence of a background Neveu-Schwarz two-form field $B_{\mu\nu}$, leading to a noncommutative structure has recently evoked considerable interest [1, 2]. This structure manifests in the noncommutativity in the spacetime coordinates of D-branes, where the end points of the string are attached. Different approaches have been adopted to obtain this result. A Hamiltonian operator treatment was provided in [3] and a world sheet approach in [4]. Also, an alternative Hamiltonian (Dirac [5]) approach based on regarding the Boundary Conditions (BC) as constraints was given in [6]; the corresponding Lagrangian (symplectic) version being done in [7]. The interpretation of the BC as primary constraints usually led to an infinite tower of second class constraints [8], in contrast to the usual Dirac formulation of constrained systems [5, 9]. Some other approaches to this problem have been discussed in [10, 11]. As has been stressed in [1], it is very important to understand this noncommutativity from different perspectives.

In the present work, we provide an exhaustive analysis of the noncommutativity in open string theory moving in the presence of a constant Neveu-Schwarz field, in the conventional Hamiltonian framework. In contrast to the usual studies, our model of string theory is very general in the sense that no gauge is fixed at the beginning. Let us recall that all computations of noncommutativity, mentioned before, were done in the conformal gauge. Our gauge independent analysis yields a new noncommutative structure, which correctly reduces to the usual one in conformal gauge. This shows the compatibility of the present analysis with the existing literature. In the general case, the noncommutativity is manifested at all points of the string, in contrast to conformal gauge results where it appears only at the boundaries. Indeed, in this gauge independent scheme, one finds a noncommutative algebra among the coordinates, even for a free string, a fact that was not observed before. Expectedly, this noncommutativity vanishes in the conformal gauge. Note however, that there is no gauge for which noncommutativity vanishes in the interacting theory. To gain further insight, both the Polyakov and Nambu-Goto (NG) formalisms of string theory have been studied.

At the outset, let us point out the crucial difference between existing Hamiltonian analysis [6] and our approach. This is precisely in the interpretation of the BC arising in the string theory. The general consensus has been to consider the BCs as primary constraints of the theory and attempt a conventional Dirac constraint analysis [5]. The aim is to induce the noncommutativity in the form of Dirac Brackets between coordinates. The subsequent analysis turns out to be ambiguous since it involves the presence of $\delta(0)$-like factors, (see Chu and Ho in [6]). Different results are obtained depending on the interpretation of these factors.

We, on the other hand, do not treat the BCs as constraints, but show that they can be systematically implemented by modifying the canonical Poisson Bracket (PB) structure. In this sense our approach is quite similar in spirit to that of Hanson, Regge and Teitelboim [4], where modified PBs were obtained for the free NG string, in the orthonormal gauge, which is the counterpart of the conformal gauge in the free Polyakov string.

The paper is organized as follows: In section 2, the gauge independent analysis of free Polyakov string is discussed. This also helps to fix the notations. The free NG string is
developed in section 3 for a comparison. A new structure, in the form of an interpolating action is presented in section 4, which connects the Polyakov and NG actions in a smooth way. It also highlights the role of the boundary conditions in the present context. The noncommutativity is revealed in a gauge independent analysis, in free Polyakov model in section 5, which incidentally is a new result. Section 6 discusses the noncommutativity in the interacting theory in the Polyakov formulation and section 7 does the same in the NG formalism. The paper ends with a conclusion in section 8.

2 The free Polyakov string

In order to study the various ramifications of different formulations of string theory, let us first consider the free Polyakov string action,

\[ S_P = -\frac{1}{2} \int_{-\infty}^{+\infty} d\tau \int_{0}^{\pi} d\sigma \sqrt{-g} g^{ab} \partial_a X^\mu \partial_b X_\mu \]  

(1)

where \( \tau \) and \( \sigma \) are the usual world-sheet parameters and \( g_{ab} \), up to a Weyl factor, is the induced metric on the world-sheet. \( X^\mu(\xi) \) are the string coordinates in the D-dimensional Minkowskian target space with metric \( G_{\mu\nu} = \text{diag}(-1, 1, 1, \ldots, 1) \). This action has the usual Poincare, Weyl and diffeomorphism invariances. Contrary to the usual approach of working in the reduced space by choosing the conformal gauge at the very beginning, we prefer to carry out the analysis in the complete space by regarding both \( X^\mu \) and \( g_{ab} \) as independent dynamical variables [12]. The canonical momenta are,

\[ \Pi_\mu = \frac{\delta L_P}{\delta (\partial_0 X^\mu)} = -\sqrt{-g} \partial^0 X_\mu \]

\[ \pi_{ab} = \frac{\delta L_P}{\delta (\partial_0 g^{ab})} = 0 \]

(2)

It is clear that while \( \Pi_\mu \) is a genuine momenta, \( \pi_{ab} \approx 0 \) are the primary constraints of the theory. To determine the secondary constraints, one can either follow the traditional Dirac’s Hamiltonian approach, or just read it off from the equation obtained by varying \( g_{ab} \) since this is basically a Lagrange multiplier. This imposes the vanishing of the symmetric energy-momentum tensor,

\[ T_{ab} = \frac{2}{\sqrt{-g}} \frac{\delta S_P}{\delta g^{ab}} = -\partial_a X^\mu \partial_b X_\mu + \frac{1}{2} g_{ab} g^{cd} \partial_c X^\mu \partial_d X_\mu = 0 \]  

(3)

Because of the Weyl invariance, the energy-momentum tensor is traceless,

\[ T^{a}_a = g^{ab} T_{ab} = 0 \]  

(4)

so that only two components of \( T_{ab} \) are independent. These components, which are the constraints of the theory, are given by,

\[ \chi_1 = g T^{00} = -T_{11} = \frac{1}{2} (\Pi^2 + (\partial_1 X)^2) = 0 \]
\[ \chi_2 = \sqrt{-g}T^0_1 = \Pi \partial_1 X = 0 \]  
\[ H = \int d\sigma \sqrt{-g}T^0_0 = \int d\sigma \sqrt{-g}(\frac{1}{2g_{11}}\chi_1 + \frac{g_{01}}{\sqrt{-gg_{11}}}\chi_2) \]  

Expectedly, the Hamiltonian turns out to be a linear combination of the constraints.

Just as variation of \( g_{ab} \) yields the constraints, variation of \( X^\mu \) gives the equation of motion,

\[ \partial_a(\sqrt{-gg^{ab}\partial_b X^\mu}) = 0 \]  

Finally, there is a mixed BC,

\[ \partial^1 X^\mu(\tau, \sigma)\big|_{\sigma=0,\pi} = 0 \]

where the string parameters are in the region \(-\infty \leq \tau \leq +\infty, 0 \leq \sigma \leq \pi\). In the covariant form involving phase space variables, this is given by

\[ (\partial X^\mu + \sqrt{-gg^{01}\Pi^\mu})\big|_{\sigma=0,\pi} = 0. \]

It is quite clear that the above boundary conditions are incompatible with the first of the basic Poisson brackets (PB),

\[ \{X^\mu(\tau, \sigma), \Pi^\nu(\tau, \sigma')\} = \delta^\mu_\nu \delta(\sigma - \sigma') \]

\[ \{g_{ab}(\tau, \sigma), \pi^{cd}(\tau, \sigma')\} = \frac{1}{2}(\delta^c_a \delta^d_b + \delta^d_a \delta^c_b)\delta(\sigma - \sigma') \]

where \( \delta(\sigma - \sigma') \) is the usual one-dimensional Dirac delta function. We would also like to mention that there is an apparent contradiction of the constraint \( \pi_{ab} \approx 0 \) with the PB (9). However this equality is valid in Dirac’s “weak” sense only, so that it can be set equal to zero only after the relevant brackets have been computed. These weak equalities will be designated by \( \approx \), rather than an equality, which is reserved only for a strong equality. In this sense, therefore, there is no clash between this constraint and the relevant PB. Indeed, we can even ignore the canonical pair \((g_{ab}, \pi^{cd})\). From the basic PB, it is easy to generate a first class (involutive) algebra,

\[ \{\chi_1(\sigma), \chi_1(\sigma')\} = 4(\chi_2(\sigma) + \chi_2(\sigma'))\partial_\sigma \delta(\sigma - \sigma'), \]

\[ \{\chi_2(\sigma), \chi_1(\sigma')\} = (\chi_1(\sigma) + \chi_1(\sigma'))\partial_\sigma \delta(\sigma - \sigma'), \]

\[ \{\chi_2(\sigma), \chi_2(\sigma')\} = (\chi_2(\sigma) + \chi_2(\sigma'))\partial_\sigma \delta(\sigma - \sigma'). \]

The situation is quite similar to usual electrodynamics. There the Lagrange multiplier is \( A_0 \), which corresponds to \( g_{ab} \) in the string theory. The multiplier \( A_0 \) enforces the Gauss constraint just as \( g_{ab} \) enforces the constraints \( \chi_1 \) and \( \chi_2 \). Furthermore, the Gauss constraint generates the time independent gauge transformations, while \( \chi_1, \chi_2 \) generate the diffeomorphism transformations.

\[ ^4\text{It is a mixed boundary condition in the sense that } \partial^1 X^\mu = g^{11}\partial_1 X^\mu + g^{10}\partial_0 X^\mu \text{ will consist of both } \tau \text{ and } \sigma \text{ derivatives.} \]
The BC (8), on the other hand, is not a constraint in the Dirac sense [5], since it is applicable only at the boundary [5]. Thus, there has to be an appropriate modification in the PB, to incorporate this condition. This is not unexpected and occurs, for instance, in the example of a free scalar field \( \phi(x) \) in 1 + 1 dimension, subjected to periodic BC of period, say, \( 2\pi \) \( (\phi(t, x + 2\pi) = \phi(t, x)) \). There the PB between the field \( \phi(t, x) \) and its conjugate momentum \( \pi(t, x) \) are given by,

\[
\{ \phi(t, x), \pi(t, y) \} = \delta_P(x - y) \tag{11a}
\]

where,

\[
\delta_P(x - y) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} e^{in(x-y)} \tag{11b}
\]

is the periodic delta function of period \( 2\pi \) and occurs in the closure properties of the basis functions \( e^{inx} \) for the space of square integrable functions, defined on the unit circle \( S^1 \). In fact, one can easily show that this PB algebra is obtained automatically if one starts with the canonical harmonic oscillator algebra for each mode in the Fourier space.

Before actually computing the modifications in the usual PB, let us take a look at the free NG action.

### 3 The free Nambu-Goto action

The NG action is given by,

\[
S_{NG} = - \int d\tau d\sigma [(\dot{X} \cdot X')^2 - \dot{X}^2 X'^2]^{\frac{1}{2}} \tag{12}
\]

where \( X'^\mu = \frac{\partial X^\mu}{\partial \sigma} = \partial_1 X^\mu \) and \( \dot{X}^\mu = \frac{\partial X^\mu}{\partial \tau} = \partial_0 X^\mu \) have been introduced for notational convenience. Note that, here the induced metric on the world-sheet has not been introduced, as we are exclusively working with \( \tau, \sigma \) variables. A systematic constrained analysis of this action has already been carried out in [9] and here we just give the results. This will also put the analysis of the Polyakov string formulation in a proper perspective. The Euler equations are,

\[
\partial_0 \Pi^\mu + \partial_1 K^\mu = 0 \tag{13a}
\]

where,

\[
\Pi_\mu = \frac{\partial L_{NG}}{\partial \dot{X}^\mu} = \frac{(X' \cdot \ddot{X})X'_\mu - X'^2 \ddot{X}_\mu)}{[(X' \cdot \dot{X})^2 - \dot{X}^2 X'^2]^{\frac{1}{2}}} \tag{13b}
\]

and

\[
K_\mu = \frac{\partial L_{NG}}{\partial X'^\mu} = \frac{(X' \cdot \ddot{X})X'_\mu - \dot{X}^2 X'_\mu}{[(X' \cdot \dot{X})^2 - \dot{X}^2 X'^2]^{\frac{1}{2}}} \tag{13c}
\]

---

\[5\] We are therefore differing from recent approaches [6] which regard the BCs as Dirac constraint. Our views are similar to those of [9], who discuss the free NG string.
The definition of the momenta $\Pi_\mu$ immediately leads to two primary constraints,

$$\Pi^2 + X'^2 \approx 0 \quad (14a)$$

$$\Pi.X' \approx 0 \quad (14b)$$

And the BCs are,

$$K_\mu(\tau,0) = K_\mu(\tau,\pi) = 0 \quad (15)$$

A simple comparison shows that although the constraints in the Polyakov (5) and NG formulations (14) have the same functional form, the BCs do not share this property (8, 15).

If one wants to match the BCs also, it is necessary to choose a particular gauge. In the NG formulation one can take the orthonormal gauge conditions [9],

$$\lambda_\mu(X^\mu(\tau,\sigma) - \frac{\tau}{\pi}P_\mu) \approx 0,$$

$$\lambda_\mu(\Pi_\mu(\tau,\sigma) - \frac{1}{\pi}P_\mu) \approx 0 \quad (16)$$

where $\lambda_\mu$ is an arbitrary constant D-vector and $P_\mu = \int_0^\pi d\sigma \Pi_\mu$ denotes the conserved momentum, following from the equations of motion.

With these conditions the NG action weakly (i.e. on the constraint surface) reduces to,

$$S_{NG} \approx \frac{1}{2} \int d\tau d\sigma (\dot{X}^2 - X'^2) \quad (17)$$

while the BCs become the usual Neumann type:

$$X'^\mu|_{\sigma=0,\pi} \approx 0. \quad (18)$$

The orthonormal gauge corresponds to the conformal gauge in the Polyakov formulation, so that the induced metric $g_{ab} = \eta_{ab} = \text{diag}(1,1)$. Then the Polyakov action (1) and the BC (8) exactly match with the corresponding expressions for the NG case.

## 4 The interpolating free string action

From our analysis in the previous sections, we saw that the NG and Polyakov actions, along with their BCs, agreed in the orthonormal and conformal gauge respectively. Here we discuss a new form of the action that interpolates between the two forms, without the need of any gauge fixing.

The starting point is to rewrite the free NG action in a first order form [12], incorporating the constraints,

$$L_I = \Pi_\mu \dot{X}^\mu - \mathcal{H} = \Pi_\mu \dot{X}^\mu + \frac{1}{2} \lambda(\Pi_\mu^2 + X'^2) + \rho \Pi_\mu X'^\mu \quad (19)$$

Note that there is no contribution from the canonical Hamiltonian, obtained by a Legendre transformation, as it vanishes identically-a typical feature of a reparametrisation invariant
theory like this. So the expression of the Hamiltonian $H$ appearing here is just a linear combination of the constraints, with $\lambda$ and $\rho$ playing the roles of Lagrange multipliers enforcing the respective constraints. Consequently, the time evolution of the system here is given by a gauge transformation.

Coming back to (19), we observe that $\Pi_\mu$ appears here as an auxiliary variable. It is thus possible to eliminate it using its equation of motion. We find,

$$\mathcal{L}_I = -\frac{1}{2\lambda} (\dot{X}_\mu^2 + 2\rho \dot{X}_\mu X'^\mu + (\rho^2 - \lambda^2) X'^2)$$

This is the cherished form of our interpolating Lagrangian.

If $\rho$ and $\lambda$ are eliminated by their respective equations of motion,

$$\rho = -\frac{\dot{X}_\mu X'^\mu}{X'^2}$$
$$\lambda^2 = -\frac{\hbar}{X'^2 X'^2}$$

then the above Lagrangian (20) reduces to the NG form (12).

If, on the other hand, we identify $\rho$ and $\lambda$ with the following contravariant components of the world-sheet metric,

$$g^{ab} = (-g)^{-\frac{1}{4}} \left( \frac{1}{\lambda} \begin{pmatrix} \frac{1}{\lambda} & \rho \\ \rho & \lambda^2 \end{pmatrix} \right)^{\frac{1}{4}}$$

then the action reduces to the Polyakov form (1). In this sense, therefore, the Lagrangian in (20) is referred to as an interpolating Lagrangian [13]. Also, note that with this mapping, the Hamiltonian read-off from (19) just reproduces the result (6).

Next, the BC is analysed. In general the BC of an open string is given by,

$$K^\mu = \frac{\partial L}{\partial X'_\mu}|_{\sigma=0,\pi} = 0.$$  

From the interpolating Lagrangian (20), we find,

$$K^\mu = (\rho \dot{X}_\mu + \rho^2 - \lambda^2 X'^\mu)|_{\sigma=0,\pi} = 0$$

at $\sigma = 0, \pi$. Now using the expressions (21) for $\rho$ and $\lambda$, we recover the usual BC (15) for NG string.

To get the BC for Polyakov string, it is useful to rewrite (23) in terms of phase space variables, $X^\mu$ and $\Pi_\mu$, as

$$K^\mu = (\rho \Pi_\mu + \lambda X'^\mu)|_{\sigma=0,\pi} = 0$$

where

$$\Pi_\mu = \frac{\partial \mathcal{L}_I}{\partial \dot{X}_\mu} = -\frac{1}{\lambda} (\dot{X}_\mu + \rho X'^\mu)$$
Now identifying $\rho$ and $\lambda$ with the metric components, it is easy to check that the Polyakov form of BC (8) is reproduced. Hence it is possible to interpret either of (23) or (24) as an interpolating BC.

It is noteworthy that although the Polyakov BC can be expressed in terms of pure phase space variables, the Nambu-Goto BC cannot be done so, because of the presence of velocities in $\rho$ (see (21)). This is an important distinction when it comes to the study of the modification in the basic algebra, as will become evident in the next section.

5 Boundary Conditions and modified brackets for a free theory

Before discussing the mixed type condition, that emerged in a completely gauge independent formulation of the Polyakov action, consider the simpler Neumann type condition (8) that leads to $(\partial_\tau X^\mu)|_{\sigma=0,\pi} = 0$ in an orthonormal (conformal) gauge.

Since the string coordinates $X^\mu(\tau, \sigma)$ transform as a world-sheet scalar under its reparametrisation, it will be even more convenient to get back to our scalar field $\phi(t, x)$ defined on 1 + 1 dimensional space-time, but with the periodic BC of $2\pi$ replaced by Neumann BC

$$\partial_x \phi|_{\sigma=0,\pi} = 0 \quad (26)$$

at the end points of a 1-dimensional box of compact size, i.e. of length $\pi$. Correspondingly, the $\delta_P(x)$ appearing there in the PB (11)-consistent with periodic BC- have to be replaced now with a suitable “delta function” incorporating Neumann BC, rather than periodic BC. Interestingly, such a “delta function” is not difficult to construct from purely algebraic arguments.

One starts by noting that the usual properties of a delta function is also satisfied by $\delta_P(x)$:

$$\int_{-\pi}^{\pi} dx' \delta_P(x' - x)f(x') = f(x) \quad (27)$$

for any periodic function $f(x) = f(x + 2\pi)$ defined in the interval $[-\pi, +\pi]$. Let us now restrict to the case of even (odd) functions $f_\pm(-x) = \pm f_\pm(x)$. Then it can be easily seen that the above integral (27) reduces to,

$$\int_0^{\pi} dx' \Delta_\pm(x', x)f_\pm(x') = f_\pm(x) \quad (28)$$

where,

$$\Delta_\pm(x', x) = \delta_P(x' - x) \pm \delta_P(x' + x) \quad (29a)$$

Using (11b), the explicit form of $\Delta_+(\sigma', \sigma)$, in particular, can be given as,

$$\Delta_+(\sigma, \sigma') = \frac{1}{\pi} + \frac{1}{\pi} \sum_{n\neq 0} cos(n\sigma')cos(n\sigma) \quad (29b)$$
We will not have to deal with $\Delta_- (\sigma', \sigma)$ henceforth in our paper, for reasons explained below.

Since any function $\phi(x)$ defined in the interval $[0, \pi]$ can be regarded as a part of an even/odd function $f_{\pm}(x)$ defined in the interval $[-\pi, \pi]$, both $\Delta_{\pm}(\sigma', \sigma)$ act as delta functions defined in half of the interval at the right i.e. $[0, \pi]$ (28). It is still not clear which of these $\Delta(x', x)$ functions should replace $\delta_P(x' - x)$ in the PB relation. We can invoke the Neumann BC, at this stage, to fix the matter. To see this, consider the Fourier decomposition of an arbitrary function $f(x)$ satisfying periodic BC, $(f(x) = f(x + 2\pi))$

$$f(x) = \sum_{n \in \mathbb{Z}} f_n e^{inx}. \quad (30)$$

Clearly,

$$f'(0) = i \sum_{n > 0} n(f_n - f_{-n})$$

$$f'{}(\pi) = i \sum_{n > 0} (-1)^n n(f_n - f_{-n}) \quad (31)$$

Now for even(odd) functions, the Fourier coefficients are related as,

$$f_{-n} = \pm f_n \quad (32)$$

so that Neumann’s BC

$$f'(0) = f'(\pi) = 0 \quad (33)$$

are satisfied if and only if $f(x)$ is even. Therefore, one has to regard the scalar field $\phi(x)$ defined in the interval $[0, \pi]$ and subjected to Neumann BC (26) as a part of an even periodic function $f_{+}(x)$ defined in the extended interval $[-\pi, +\pi]$. It thus follows that the appropriate PB for the scalar theory is given by,

$$\{ \phi(t, x), \pi(t, x') \} = \Delta_{+}(\sigma, \sigma')$$

It is clearly consistent with Neumann BC as $\partial_\sigma \Delta_{+}(\sigma, \sigma')|_{\sigma=0,\pi} = \partial_{\sigma'} \Delta_{+}(\sigma, \sigma')|_{\sigma=0,\pi} = 0$ is automatically satisfied. It is straightforward to generalise it to the string case, where it is given by,

$$\{ X^\mu(\tau, \sigma), \Pi^\nu(\tau, \sigma') \} = \delta^\mu_\nu \Delta_{+}(\sigma, \sigma') \quad (34a)$$

and the Lorentz indices are playing the role of “isospin” indices, as viewed from the world-sheet. This form first appeared in [9]. Observe also that the other brackets

$$\{ X^\mu(\tau, \sigma), X'^\nu(\tau, \sigma') \} = 0 \quad (34b)$$

and

$$\{ \Pi^\mu(\tau, \sigma), \Pi'^\nu(\tau, \sigma') \} = 0 \quad (34c)$$

are consistent with the BCs and hence remain unchanged.

For a gauge independent analysis, the Nambu-Goto BC poses problems since it cannot be expressed in phase space variables. To overcome this problem it is necessary to fix a gauge and
this was elaborated in section 3. The generalisation of this in the interacting NG string will be given later in section 7. Here we take recourse to the mixed condition (8) that occurs in the Polyakov string. A simple inspection shows that this is also compatible with the modified brackets (34a, 34c), but not with (34b). Hence the bracket among the coordinates should be altered suitably. We therefore make an ansatz,

\[ \{X^\mu(\tau, \sigma), X^\nu(\tau, \sigma')\} = C^{\mu\nu}(\sigma, \sigma') \]  

(35a)

where,

\[ C^{\mu\nu}(\sigma, \sigma') = -C^{\nu\mu}(\sigma', \sigma). \]  

(35b)

Imposing the BC (8) on this algebra, we get,

\[ \partial_\sigma C^{\mu\nu}(\sigma, \sigma')|_{\sigma = 0, \pi} = \partial_\sigma C^{\mu\nu}(\sigma, \sigma')|_{\sigma = 0, \pi} = -\sqrt{-g} g^{01} \{\Pi^\mu(\tau, \sigma), X^\nu(\tau, \sigma')\} \]

\[ = \sqrt{-g} g^{01} G^{\mu\nu} \Delta_+(\sigma, \sigma') \]  

(36)

For an arbitrary form of the metric tensor, it might be technically problematic to find a solution for \( C^{\mu\nu}(\sigma, \sigma') \). However, for a restricted class of metric \( g^{ab} \) that satisfy,

\[ \partial_1 g_{ab} = 0 \]  

(37)

it is possible to give a quick solution of \( C^{\mu\nu}(\sigma, \sigma') \) as,

\[ C^{\mu\nu}(\sigma, \sigma') = \sqrt{-g} g^{01} G^{\mu\nu}[\Theta(\sigma, \sigma') - \Theta(\sigma', \sigma)] \]  

(38)

where the generalised step function \( \Theta(\sigma, \sigma') \) satisfies,

\[ \partial_\sigma \Theta(\sigma, \sigma') = \Delta_+(\sigma, \sigma') \]  

(39)

An explicit form of \( \Theta \) is given by [3],

\[ \Theta(\sigma, \sigma') = \frac{\sigma}{\pi} + \frac{1}{\pi} \sum_{n \neq 0} \frac{1}{n} \sin(n\sigma) \cos(n\sigma') \]  

(40a)

having the properties,

\[ \Theta(\sigma, \sigma') = 1 \quad \text{for} \quad \sigma > \sigma', \]

and

\[ \Theta(\sigma, \sigma') = 0 \quad \text{for} \quad \sigma < \sigma'. \]  

(40b)

Using these relations, the simplified structure of noncommutative algebra follows,

\[ \{X^\mu(\tau, \sigma), X^\nu(\tau, \sigma')\} = 0 \quad \text{for} \quad \sigma = \sigma' \]

\[ \{X^\mu(\tau, \sigma), X^\nu(\tau, \sigma')\} = \pm \sqrt{-g} g^{01} G^{\mu\nu} \quad \text{for} \quad \sigma > \sigma' \quad \text{and} \quad \sigma < \sigma' \]  

(41)

\[ ^6\text{Such conditions also follow from a standard treatment of the light-cone gauge [14]} \]
respectively. Thus a noncommutative algebra for distinct coordinates \( \sigma \neq \sigma' \) of the string emerges automatically in a free string theory if a gauge independent analysis is carried out like this. But this non-commutativity can be made to vanish in gauges like conformal gauge, where \( g^{01} = 0 \), thereby restoring the usual commutative structure. However, the non-commutativity among the string coordinates cannot be made to vanish in any gauge if the string is coupled to a constant external B-field, as we show in the next section.

Before we conclude this section, we would like to mention that the essential structure of the involutive algebra (10) is still preserved, only that \( \delta(\sigma - \sigma') \) has to be replaced by \( \Delta_+(\sigma, \sigma') \). And this is despite the fact that the original basic brackets (9) have now been modified to (34a,34c,41). Indeed, using these relations, one can show that

\[
\{ \chi_1(\sigma), \chi_1(\sigma') \} = 4(\chi_2(\sigma) + \chi_2(\sigma')) \partial_\sigma \Delta_+(\sigma, \sigma'), \\
\{ \chi_2(\sigma), \chi_1(\sigma') \} = (\chi_1(\sigma) + \chi_1(\sigma')) \partial_\sigma \Delta_+(\sigma, \sigma') \\
\{ \chi_2(\sigma), \chi_2(\sigma') \} = (\chi_2(\sigma) + \chi_2(\sigma')) \partial_\sigma \Delta_+(\sigma, \sigma').
\]

(42)

Note that, contrary to the usual case, the right hand side vanishes identically on the boundary. A crucial intermediate step in this derivation is to use the relation,

\[
\{ X^\mu(\sigma), X^\nu(\sigma') \} = 0
\]

which follows from the basic bracket (41).

6 The interacting theory: Polyakov formulation

The Polyakov action for a bosonic string moving in the presence of a constant background Neveu-Schwarz two-form field \( B_{\mu\nu} \) is given by,

\[
S_P = \int d\tau d\sigma (-\frac{1}{2} \sqrt{-g} g^{\alpha\beta} \partial_\alpha X_\mu \partial_\beta X_\nu + \epsilon^{\alpha\beta} B_{\mu\nu} \partial_\alpha X_\mu \partial_\beta X_\nu)
\]

(43)

where we have introduced a ‘coupling constant’ \( \epsilon \) and \( \epsilon^{01} = -\epsilon^{10} = +1 \). Here too we shall carry out a gauge independent analysis at the beginning, rather than making use of any gauge fixing condition (like conformal gauge) right at this stage. A usual canonical analysis leads to the following set of primary first class constraints,

\[
g T^{00} = \frac{1}{2} (\Pi_\mu + \epsilon B_{\mu\nu} \partial_1 X^\nu)^2 + (\partial_1 X)^2 \approx 0
\]

(44)

\[
\sqrt{-g} T^{01} = \Pi_\mu \partial_1 X \approx 0
\]

(45)

where

\[
\Pi_\mu = -\sqrt{-g} \partial_0 X_\mu + \epsilon B_{\mu\nu} \partial_1 X^\nu
\]

(46)

is the momentum conjugate to \( X^\mu \).
Likewise, the BC is given by,

$$\left. \left( \partial_1^1 X^\mu + \frac{1}{\sqrt{-g}} e B^{\mu \nu} \partial_0 X_\nu \right) \right|_{\sigma=0, \pi} = 0 \quad (47)$$

Using phase space variables, this can be written in a completely covariant form as,

$$\left. \left( \partial_1^1 X^\rho M^\rho_\mu + \Pi^\nu N_{\nu \mu} \right) \right|_{\sigma=0, \pi} = 0 \quad (48)$$

where,

$$M^\rho_\mu = \frac{1}{g_{11}} \left[ \delta^\rho_\mu - \frac{2e}{\sqrt{-g}} g_{01} B^\rho_\mu + e^2 B^{\rho \nu} B_{\nu \mu} \right] \quad (49a)$$

$$N_{\nu \mu} = -\frac{g^{01}}{g^{00} \sqrt{-g}} G_{\nu \mu} - \frac{1}{g_{11}} e B_{\nu \mu} \quad (49b)$$

are two matrices. This nontrivial BC leads to a modification in the original (naive) canonical PBs.

Now the BC (48) is recast as,

$$\left. \left( \partial_1^1 X^\mu + \Pi^\rho (NM^{-1})_{\rho \mu} \right) \right|_{\sigma=0, \pi} = 0 \quad (50)$$

The \{X^\mu(\sigma), \Pi_\nu(\sigma')\} PB is the same as that of the free string (34a). Considering the general structure (35), we obtain,

$$\{ \partial_\sigma X^\mu(\sigma), X^\nu(\sigma') \} = \partial_\sigma C^{\mu \nu}(\sigma, \sigma') \quad (51)$$

Putting the BC and exploiting (34a), we get

$$\partial_\sigma C_{\mu \nu}(\sigma, \sigma') \left|_{\sigma=0, \pi} \right. = \left( NM^{-1} \right)_{\nu \mu} \Delta_+(\sigma, \sigma') \left|_{\sigma=0, \pi} \right. . \quad (52)$$

As we did in the free case, we restrict to the class of metrics defined by (37). Taking a cue from the free theory, the solution for \( C_{\mu \nu}(\sigma, \sigma') \) must involve the generalised \( \Theta \) function, introduced in (40). Splitting \( (NM^{-1})_{\nu \mu} \) into its symmetric \( (NM^{-1})_{\nu \mu} \) and antisymmetric \( (NM^{-1})_{[\nu \mu]} \) components, a general solution for \( C_{\mu \nu} \) is given by,

$$C_{\mu \nu}(\sigma, \sigma') = \frac{1}{2} (NM^{-1})_{\nu \mu} [\Theta(\sigma, \sigma') - \Theta(\sigma', \sigma)] + \frac{1}{2} (NM^{-1})_{[\nu \mu]} [\Theta(\sigma, \sigma') + \Theta(\sigma', \sigma) - 1]. \quad (53)$$

Observe that, by demanding (35b), \( (NM^{-1})_{[\nu \mu]} \) must be multiplied by an antisymmetric combination of \( \Theta \)'s, which is precisely \( [\Theta(\sigma, \sigma') - \Theta(\sigma', \sigma)] \) Likewise, the other factor \( (NM^{-1})_{\nu \mu} \) must be multiplied by a symmetric combination \( [\Theta(\sigma, \sigma') + \Theta(\sigma', \sigma)] \), plus an undetermined constant. We fix this constant to \(-1\) by requiring that the vanishing result (41) in the free case is retained for all \( \sigma = \sigma' \) away from the boundary (using \( \Theta(\sigma, \sigma) = \frac{1}{2} \)). An advantage of this normalisation is that by passing to the conformal gauge, where \( g = -1 \) and \( g^{01} = 0 \), one obtains,

$$C_{\mu \nu}(\sigma, \sigma') = \tilde{B}_{\mu \nu} [\Theta(\sigma, \sigma') + \Theta(\sigma', \sigma) - 1] \quad (54a)$$
where,
\[ \tilde{B}_{\mu
u} = -e[B(1 + e^2B^2)^{-1}]_{\mu
u} \tag{54b} \]
which reproduces the standard non-commutative algebra in the presence of a background field.

It is evident that the modified algebra is gauge dependent, depending on the choice of the metric. However, there is no choice, for which the non-commutativity vanishes. To show this, note that the origin of the non-commutativity is the presence of non-vanishing \( \Pi^\mu \) term in the BC (48). If this can be eliminated, then the usual commutative algebra is obtained. This requires \( N_{\nu\mu} = 0 \). From (49b) this implies \( B_{\mu\nu} \) and \( G_{\mu\nu} \) have to be proportional which obviously cannot happen, as the former is an antisymmetric and the latter is a symmetric tensor. Hence non-commutativity will persist for any choice of world-sheet metric \( g_{ab} \). Specially interesting are the expressions for noncommutativity (53) at the boundaries,
\[ C_{\mu\nu}(0, 0) = -C_{\mu\nu}(\pi, \pi) = \frac{1}{2}(NM^{-1})_{[\nu\mu]} \quad C_{\mu\nu}(0, \pi) = -C_{\mu\nu}(\pi, 0) = -\frac{1}{2}(NM^{-1})_{(\nu\mu)} \tag{55} \]
It should be pointed out that in the conformal gauge, \( (NM^{-1}) \) does not have a symmetric component, so that
\[ C_{\mu\nu}(0, \pi) = C_{\mu\nu}(\pi, 0) = 0. \]

7 The interacting theory: Nambu-Goto formulation

Although the Polyakov and NG formulations for free strings are regarded to be classically equivalent, there are some subtle issues. Indeed the structures of BC’s in the two formulations are different as was also illuminated by our interpolating action. Also more complications are expected in the presence of interactions. Since the occurrence of noncommutativity is directly connected with the BC’s, it is therefore useful to study this feature in the NG formulation. This motivates us to carry out an exhaustive analysis of the classical relativistic string interacting with a constant, second rank, antisymmetric tensor \( B_{\mu\nu} \) in the NG formulation in this subsection. Here we present a generalisation of the analysis of Hanson, Regge and Teitelboim [9] for the free string, to show that the noncommutativity appears directly from taking proper account of the boundary conditions. The analysis for the free theory [9] has already been reproduced briefly in section 3.

We start with the action,
\[ S = \int_{-\infty}^{+\infty} d\tau \int_{0}^{\pi} d\sigma [\mathcal{L}_0 + eB_{\mu\nu}\dot{X}^\mu X^{\nu}] \tag{56} \]
Here \( \mathcal{L}_0 \) denotes the free string Lagrangian density appearing in (12). From the variation of the action, we obtain the following equations of motion and the BC’s,
\[ \dot{\Pi}^\mu + K^{\mu} = 0, \tag{57} \]
\[ K^{\mu}|_{\sigma=0,\pi} = 0, \tag{58} \]
where
\[ \Pi^\mu = \frac{\partial L}{\partial \dot{X}_\mu} = \mathcal{L}_0^{-1}(-X'^2 \hat{X}^\mu + (\hat{X} \cdot X') X'^\mu) + eB^{\mu\nu} \dot{X}_\nu, \] (59)
\[ K^\mu = \frac{\partial L}{\partial X'_\mu} = \mathcal{L}_0^{-1}(-\dot{X}^2 X'^\mu + (\dot{X} \cdot X') \dot{X}^\mu) - eB^{\mu\nu} \ddot{X}_\nu. \] (60)

The Primary constraints of the theory are,
\[ \chi_1 = (\Pi^\mu - eB^{\mu\nu} X'_\nu)^2 + X'^2 \approx 0, \chi_2 = \Pi \cdot X' \approx 0. \] (61)

which are similar to those obtained in the Polyakov version (see (44,45)). Using the standard canonical PB, it is straightforward to verify the diffeomorphism algebra (10).

A gauge independent analysis, as was done for the Polyakov formulation, is not feasible here, since the BC involves time derivatives that are not eliminatable in terms of the momenta. To properly account for the BCs, a gauge choice becomes necessary. This is equally valid for a free theory. As was shown in [9] and discussed in section 3, the free theory becomes most tractable in the orthonormal gauge. Inspired by their choice, we consider the following gauge conditions,
\[ \lambda^\mu (X^\mu - P^\mu_\pi) \approx 0, \lambda^\mu (\Pi^\mu - \frac{\mathcal{P}^\mu_\pi}{\pi}) \approx 0, \] (62)
which for \( e = 0 \) reduce to the orthonormal gauge in free theory. Here \( \lambda^\mu \) is a constant D-vector.

For our present analysis there arises no need to fix \( \lambda^\mu \). Same notations as in section 3 are used here.

Let us study the consequences of the gauge choice. From the gauge conditions (62), we obtain \( \partial_0(\lambda \Pi) = \frac{\partial_0(\lambda \mathcal{P})}{\pi} = 0 \) and together with the equations of motion (57) this leads to \( \partial_1(\lambda K) = 0 \). Compatibility with the BC (58) then ensures that \( \lambda K = 0 \) for all \( \sigma \). Again, from the gauge choice (62), we find \( \lambda \cdot X' = 0 \) and \( \lambda \cdot X = (\lambda \Pi) \). In short, the following three exact relations are valid,
\[ \lambda \cdot K = 0, \lambda \cdot X' = 0, \lambda \cdot \dot{X} = (\lambda \Pi). \] (63)

From the defining equations (59),(60) and (62), we find,
\[ (\lambda \Pi) = -\mathcal{L}_0^{-1}X'^2(\lambda \cdot \hat{X}) + eB^{\mu\nu} \lambda^\mu X'_\nu, \] (64)
\[ \lambda \cdot K = \mathcal{L}_0^{-1}(\hat{X} \cdot X')(\lambda \cdot \dot{X}) - eB^{\mu\nu} \lambda^\mu \dot{X}_\nu = 0. \] (65)

Using (62) once again we obtain,
\[ \mathcal{L}_0^{-1}X'^2 = -1 + e\mathcal{A}, \mathcal{L}_0^{-1}(\hat{X} \cdot X') = e\mathcal{B} \] (66a)
where,
\[ \mathcal{A} = \frac{B^{\mu\nu} \lambda^\mu X'_\nu}{\lambda \Pi}, \mathcal{B} = \frac{B^{\mu\nu} \lambda^\mu \dot{X}_\nu}{\lambda \Pi}. \] (66b)
From now on we will work in the lowest nontrivial order in the coupling $e$. The explicit expressions for $A$ and $B$ are not needed for the $O(e)$ results presented here. Recalling the explicit form of $L_0$ from (56), we find,

$$L_0 \approx -\left(-\dot{X}^2 X'^2\right)^{\frac{1}{2}}.$$  \hspace{1cm} (67)

Using (66) we obtain,

$$X'^2 = -\dot{X}^2(1 - 2eA) \rightarrow \dot{X}^2 + X'^2 \approx 2eA\dot{X}^2.$$  \hspace{1cm} (67)

The $O(e)$ correction to the orthonormality vanishes in the free theory ($e = 0$), where $\dot{X}.X' = 0$, and $\dot{X}^2 + X'^2 = 0$ \[\square\]. Now $L_0$ is simplified to,

$$L_0 \approx -\left(-\dot{X}^2 X'^2\right)^{\frac{1}{2}} \approx \dot{X}^2(1 - eA)$$ \hspace{1cm} (68)

Finally we recover the Lagrangian of the string coupled to $B_{\mu\nu}$, in this particular gauge, to lowest order in the coupling $e$ as,

$$L = \frac{1}{2}(\dot{X}^2 - X'^2) + eB_{\mu\nu}\dot{X}^{\mu}X'^{\nu} + O(e^2)$$  \hspace{1cm} (69)

The equation of motion in this gauge is that of a free theory,

$$(\partial_0^2 - \partial_1^2)X^\mu = 0,$$ \hspace{1cm} (70)

but crucial modifications have appeared in the BC,

$$(X'^{\mu} + \frac{e}{N}B_{\mu\nu}\dot{X}^{\nu})|_{\sigma=0,\pi} = 0.$$ \hspace{1cm} (71)

In fact, the interaction have changed the BC from Neumann type in free theory to a mixed one. Elimination of $\dot{X}_\mu$ from (58) reproduces the BC in phase space,

$$X'^{\mu} + e(M^{-1}B)^{\mu\nu}\Pi_\nu = 0,$$ \hspace{1cm} (72)

where $M^{\mu\lambda} = G^{\mu\lambda} - \frac{e^2}{N^2}B^{\mu\nu}B_\nu^\lambda$. It is amusing to note that this BC is identical to the one used in the Polyakov model in the conformal gauge \[\square\] \[\square\] but in our case we should consider $M^{\mu\lambda} \approx G^{\mu\lambda}$, since our results are of $O(e)$ only.

It is worthwhile to make a comparison with Polyakov formulation at this stage. The Lagrangian (69) is identical to the Polyakov one (43) in the conformal gauge. There is a similar mapping between BCs (72) and (48) again in the conformal gauge. Consequently, we shall be reproducing the same set of modified brackets (34a) and (54), displaying noncommutativity among various coordinates. It should be emphasised, however, that this agreement is only upto $O(e)$ in the coupling parameter in the specific gauge (62).
8 Conclusion

In this paper we have derived expressions for a noncommutative algebra that are more general than the standard results found in the conformal gauge. Indeed, our results reproduce the standard ones, once the conformal gauge is implemented.

The origin of any modification in the usual Poisson algebra is the presence of boundary conditions. This phenomenon is quite well known for a free scalar field subjected to periodic boundary conditions. We showed that its exact analogue is the conformal gauge fixed free string, where the boundary condition is of Neuman-type. This led to a modification only in the \( \{X^\mu(\sigma), \Pi_\nu(\sigma')\} \) algebra, where the usual Dirac delta function got replaced by \( \Delta_+(\sigma, \sigma') \). A more general type of boundary condition occurs in the gauge independent formulation of a free Polyakov string. Using certain algebraic consistency requirements, we showed that the boundary conditions in the free theory naturally led to a noncommutative structure among the coordinates. This non-commutativity vanishes in the conformal gauge, as expected.

The same technique was adopted for the interacting string. A more involved boundary condition led to a more general type of noncommutativity than has been observed before. Contrary to the standard conformal gauge expressions, this noncommutative algebra survives at all points of the string and not just at the boundaries. Furthermore, in contrast to the free theory, this noncommutativity cannot be removed in any gauge. We have also shown that, the noncommutativity does not affect the usual diffeomorphism algebra among the gauge generators. In the conformal gauge, our results reduce to the standard noncommutativity found only at the string end points.

A perturbative analysis of the noncommutativity has also been performed in the interacting Nambu-Goto string. Surprisingly, the conformal gauge result in the Polyakov formulation is reproduced in the Nambu-Goto scheme in the lowest nontrivial order in the Neveu-Schwarz coupling, in an orthonormal-like gauge. It would be interesting to see if there is an alternative gauge condition in which the above equivalence can be shown exactly.
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