ULTRADIFFERENTIABLE CR MANIFOLDS

STEFAN FÜRDŐS

Abstract. In this article the notion of ultradifferentiable CR manifold is introduced and an ultradifferentiable regularity result for finitely nondegenerate CR mappings is proven. Here ultradifferentiable means with respect to Denjoy-Carleman classes defined by weight sequences. Furthermore the regularity of infinitesimal CR automorphisms on ultradifferentiable abstract CR manifolds is investigated.

1. Introduction

The primary focus of this article is the study of the regularity of CR mappings. Looking at the literature concerning this problems, one observes that most theorems about the regularity of CR mappings are of a similar form which can be summarized as follows: We consider a CR mapping $H$ between two CR submanifolds $M$ and $M'$ with some a-priori regularity that extends to a holomorphic mapping defined on a wedge with edge $M$. If the mapping and/or the manifolds satisfy certain nondegeneracy conditions at some point then it is proven that $H$ is actually of optimal regularity near this point, that is smooth if $M$ and $M'$ are smooth, or real-analytic if the manifolds are real-analytic. We should mention that the nondegeneracy assumptions are heavily tailored towards the methods applied in the various different proofs. In particular, it is worth noting that in most instances the conditions in the smooth setting differ sharply from those used in the analytic category. In the case of smooth CR manifolds the fundamental contributions are the pioneering works of Fefferman [15] and Nirenberg-Webster-Yang [32]. We should mention that in the analytic setting surprisingly weak assumptions often suffice, c.f. e.g. the classical results of Baouendi-Jacobowitz-Treves [32], Huang [23] and Pinčuk [35].

One of the rare cases, where under the identical assumptions it has been possible to show that $H$ is smooth if the manifolds are smooth and analytic if $M$ and $M'$ are both analytic manifolds, have been the results of Bernhard Lamel [27, 28]. He proved that every finitely nondegenerate CR mapping between two generic submanifolds that extends holomorphically is smooth and even analytic if both manifolds are real-analytic.

Recently Berhanu-Xiao [4] were able to strengthen this result in the smooth case by relaxing partially its assumptions. They require only the target manifold to be an embedded CR manifold, the source manifold could be only an abstract CR manifold. The finitely nondegenerate condition on the mapping remains unchanged but the holomorphic extension obviously makes no sense in this situation. It is replaced in the theorem of Berhanu-Xiao with the assumption that the fibers of the wavefront set of $H$ do not include opposite directions.

This microlocal assumption is automatically satisfied in the embedded setting if extension to a wedge is assumed since Baouendi-Chang-Treves [2] showed that for CR distributions on CR submanifolds of $\mathbb{C}^N$ the holomorphic extension into wedges is in fact a microlocal condition, which they used to define the hypoanalytic wavefront set of CR distributions. It coincides with the analytic wavefront set if the manifold is analytic. If the manifold is only smooth then the hypoanalytic wavefront set includes the smooth wavefront set.

Since the results of Lamel and Berhanu-Xiao suggest that finite nondegeneracy preserves regularity quite well, the following question arises naturally. Given a subsheaf $\mathcal{A}$ of the sheaf of smooth functions we may ask that if in the formulation of the theorem of Lamel the manifolds are assumed to be of class $\mathcal{A}$, does it follow that the CR mapping has to be of class $\mathcal{A}$ as well?

Of course we have to assume that $\mathcal{A}$ satisfies certain properties. First of all, in order for the conjecture above to make sense, $\mathcal{A}$ must be closed under composition and the implicit function theorem must hold in the category of mappings of class $\mathcal{A}$. Furthermore if we try to modify the existing proofs in the smooth category then we need some version of $\mathcal{A}$-wavefront set or more precisely a definition of $\mathcal{A}$-microlocal regularity. We should note at this point that in both Lamel’s proof and that of Berhanu-Xiao the

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characterization of the smooth wavefront set by almost-analytic extensions was heavily used as both relied on an almost-analytic version of the implicit function theorem.

We are mainly interested in subsheaves of smooth functions that contains strictly the sheaf of real-analytic functions. We shall call the elements of such sheafs ultradifferentiable functions. Generally ultradifferentiable functions are determined either by estimates on its derivatives or its Fourier transform. The most well-known examples of ultradifferentiable classes are the Gevrey classes, see e.g. [37].

Here we consider the category of so-called Denjoy-Carleman classes, which are defined in the following way. If $\mathcal{M} = (m_j)_j$ is a sequence of positive real numbers then the Denjoy-Carleman class associated with $\mathcal{M}$ consists of those smooth functions that satisfy the following generalized Cauchy estimate

$$|\partial^\alpha f(x)| \leq C h^{[\alpha]} |\alpha|! \quad (1.1)$$

on compact sets, where $C$ and $h$ are constants independent of $\alpha$. We will also say that a smooth function $f$ obeying (1.1) is of class $\{\mathcal{M}\}$. In particular, if $\mathcal{M} = (j!)_j$ then the associated Denjoy-Carleman class to $\mathcal{M}$ is the Gevrey class of order $s + 1$.

Examining the literature concerning the Denjoy-Carleman classes and their properties one can observe that stability conditions of the associated class correlate with properties of the weight sequence. For example, we know that, if $\mathcal{M}$ is a regular weight sequence in the sense of [12], then the Denjoy-Carleman class associated to $\mathcal{M}$ is closed under composition, solving ordinary differential equations and the implicit function theorem holds in the class, c.f. e.g. [36]. Hence for regular sequences $\mathcal{M}$ we can consider manifolds of Denjoy-Carleman type. We shall say such a manifold is an ultradifferentiable manifold of class $\{\mathcal{M}\}$.

On the other hand, Hörmander [21] introduced the ultradifferentiable wavefront set for distributions defined on open subsets of the euclidean space. But since he worked under comparatively weak conditions on the weight sequence Hörmander was only able to define the ultradifferentiable wavefront set $WF_{\mathcal{M}} u$ of distributions $u$ on real-analytic manifolds but not distributions defined on ultradifferentiable manifolds.

However using Dyn’kims characterization of ultradifferentiable functions by almost analytic extensions [12] [11] we were able in [16] to develop a geometric theory for the ultradifferentiable wavefront set. In particular, if the weight sequence is regular, the ultradifferentiable wavefront set of a distribution on an ultradifferentiable manifold is shown to be well defined.

With these results at hand and an $\mathcal{M}$-almost analytic version of the almost-analytic implicit function theorem used in Lamel [28] and Berhanu-Xiao [4] it is possible to prove the ultradifferentiable version of the regularity result of Lamel:

**Theorem 1.1.** Let $M \subseteq \mathbb{C}^N$ and $M' \subseteq \mathbb{C}^{N'}$ be two generic ultradifferentiable submanifolds of class $\{\mathcal{M}\}$, $p_0 \in M$, $p'_0 \in M'$ and $H : (M, p_0) \to (M', p'_0)$ a $C^{k_0}$-CR mapping that is $k_0$-nondegenerate at $p_0$. Suppose furthermore that $H$ extends continuously to a holomorphic map in a wedge $W$ with edge $M$. Then $H$ is ultradifferentiable of class $\{\mathcal{M}\}$ in a neighbourhood of $p_0$.

For the definition of finite nondegeneracy of a CR mapping we refer to the beginning of section 5.

More precisely this paper is structured as follows. In section 2 the results on regular Denjoy-Carleman classes and ultradifferentiable manifolds that are needed are discussed. In section 3 we first recall the results from Dyn’kin [12] [11] on the almost analytic extension of ultradifferentiable functions. Furthermore we give the definition and basic results on the ultradifferentiable wavefront set according to Hörmander [22] and close the section by presenting the geometric theory for the ultradifferentiable wavefront set given in [16].

In section 4 basic definitions and first results on ultradifferentiable CR manifolds are given, whereas the proofs of Theorem [14] and of ultradifferentiable versions of other regularity results of Lamel and Berhanu-Xiao are presented in section 5. The last section is devoted to present essentially the generalization of [17] concerning the smoothness of infinitesimal CR automorphisms to regular Denjoy-Carleman classes. We end by examining smooth infinitesimal CR automorphisms on formally holomorphic nondegenerate quasianalytic CR submanifolds.

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## 2. Regular Denjoy-Carleman classes

In this section we summarize the results for Denjoy-Carleman classes that we need in the following. For a more detailed presentation see [16]. Note that, unless stated otherwise, $\Omega \subseteq \mathbb{R}^n$ will be an open set.
**Definition 2.1.** A sequence \( M = (m_k)_k \) is a regular weight sequence iff it satisfies the following conditions.

\[
\begin{align*}
m_0 &= m_1 = 1 & \quad (M1) \\
\sup_k \sqrt[k]{m_{k+1}} &= \infty & \quad (M2) \\
m_k^2 &\leq m_{k-1}m_{k+1} & k \in \mathbb{N} & \quad (M3) \\
\lim_{k \to \infty} \sqrt[k]{m_k} &= \infty & \quad (M4)
\end{align*}
\]

**Definition 2.2.** Let \( M \) be a regular weight sequence. Then we say that a smooth function \( f \in \mathcal{E}(\Omega) \) is ultradifferentiable of class \( \{M\} \) iff for all compact sets \( K \subseteq \Omega \) there are constants \( C, h > 0 \) such that

\[
|\partial^\alpha f(x)| \leq Ch^{|\alpha|}m_{|\alpha|}|\alpha|!
\]

for all \( x \in K \). The space of all ultradifferentiable functions of class \( \{M\} \) is denoted by \( \mathcal{E}_M(\Omega) \). It is sometimes also called the Denjoy-Carleman class associated to \( M \).

**Example 2.3.** If \( s > 0 \) consider the regular weight sequence \( M^s = (ks)_k \). Its associated Denjoy-Carleman class is the Gevrey class \( G^{s+1}(\Omega) = \mathcal{E}_M(\Omega) \) of order \( s+1 \) on \( \Omega \), c.f. \cite{37}.

On the other hand, the constant sequence \( M^0 = (1)_k \) gives the space \( \mathcal{O}(\Omega) \) of real-analytic functions on \( \Omega \). Note that \( M^0 \) is not regular in the sense of Definition 2.1.

We consider here only regular weight sequences but might occasionally omit the word “regular”.

If \( M \) and \( N = (n_k)_k \) are two weight sequences then we write \( M \prec N \) iff there is a constant \( Q \) such that \( m_k \leq Q^n n_k \). It holds that \( E_M \subseteq E_N \) if and only if \( M \preceq N \). Thus we see that (M1) means that \( \mathcal{O} \subseteq E_M \) and (M2) implies that \( E_M \) is closed under derivation, i.e. if \( f \in \mathcal{E}_M(\Omega) \) then \( \partial^s f \in \mathcal{E}_M(\Omega) \) for all \( s \). Furthermore we have

**Lemma 2.4** (c.f. Remark 2.5 in \cite{19}). Let the Denjoy-Carleman class \( \mathcal{E}_M(\Omega) \) be closed under derivation closed and suppose that \( f \in \mathcal{E}_M(\Omega) \) and \( f(x_1, \ldots, x_{j-1}, a, x_{j+1}, \ldots, x_n) = 0 \) for some fixed \( a \in \mathbb{R} \) and all \( x_k, k \neq j \), with the property that \( (x_1, \ldots, x_{j-1}, a, x_{j+1}, \ldots, x_n) \in \Omega \). Then there exists some \( g \in \mathcal{E}_M(\Omega) \) such that

\[
f(x) = (x_j - a)g(x).
\]

In fact, if \( M \) is a regular weight sequence then the associated Denjoy-Carleman class satisfies also the following stability properties.

**Theorem 2.5.** Let \( M \) be a regular weight sequence and \( \Omega_1 \subseteq \mathbb{R}^m \) and \( \Omega_2 \subseteq \mathbb{R}^n \) open sets. Then the following holds:

1. The algebra \( \mathcal{E}_M(\Omega) \) is inverse closed, i.e. if \( f \in \mathcal{E}_M(\Omega) \) does not vanish at any point of \( \Omega \) then \( 1/f \in \mathcal{E}_M(\Omega) \) (c.f. \cite{26} and the remarks therein).
2. The class \( \mathcal{E}_M \) is closed under composition \( (39) \) (see also \cite{5}) i.e. let \( F : \Omega_1 \to \Omega_2 \) be an \( \mathcal{E}_M \)-mapping, that is each component \( F_j \) of \( F \) is ultradifferentiable of class \( \{M\} \) in \( \Omega_1 \), and \( g \in \mathcal{E}_M(\Omega_2) \). Then also \( g \circ F \in \mathcal{E}_M(\Omega_1) \).
3. The inverse function theorem holds in the Denjoy-Carleman class \( \mathcal{E}_M \) \cite{25}:
   Let \( F : \Omega_1 \to \Omega_2 \) be an \( \mathcal{E}_M \)-mapping and \( p_0 \in \Omega_1 \) such that the Jacobian \( F'(p_0) \) is invertible. Then there exist neighbourhoods \( U \) of \( p_0 \) in \( \Omega_1 \) and \( V \) of \( q_0 = F(p_0) \) in \( \Omega_2 \) and an \( \mathcal{E}_M \)-mapping \( G : V \to U \) such that \( G(q_0) = p_0 \) and \( F \circ G = \text{id}_V \).
4. The implicit function theorem is valid in \( \mathcal{E}_M \) \cite{25}:
   Let \( F : \mathbb{R}^{n+d} \supseteq \Omega \to \mathbb{R}^d \) be a \( \mathcal{E}_M \)-mapping and \( (x_0, y_0) \in \Omega \) such that \( F(x_0, y_0) = 0 \) and \( \frac{\partial F}{\partial y}(x_0, y_0) \) is invertible. Then there exist open sets \( U \subseteq \mathbb{R}^n \) and \( V \subseteq \mathbb{R}^d \) with \( (x_0, y_0) \in U \times V \subseteq \Omega \) and an \( \mathcal{E}_M \)-mapping \( G : U \to V \) such that \( G(x_0) = y_0 \) and \( F(x, G(x)) = 0 \) for all \( x \in V \).

In particular we note that \( \mathcal{E}_M(\Omega) \) is closed under solving ODEs. More precisely we have the following result.

**Theorem 2.6** \cite{14}, see also \cite{26}. Let \( M \) be a regular weight sequence, \( 0 \in I \subseteq \mathbb{R} \) an open interval, \( U \subseteq \mathbb{R}^n \), \( V \subseteq \mathbb{R}^d \) open and \( F \in \mathcal{E}_M(I \times U \times V) \).

Then the initial value problem

\[
\begin{align*}
x'(t) &= F(t, x(t), \lambda) & t \in I, \lambda \in V \\
x(0) &= x_0 & x_0 \in U
\end{align*}
\]
has locally a unique solution $x$ that is ultradifferentiable near $0$.

More precisely, there is an open set $\Omega \subseteq I \times U \times V$ that contains the point $(0, x_0, \lambda)$ and an $\mathcal{E}_M$-mapping $x = x(t, y, \lambda) : \Omega \to U$ such that the function $t \mapsto x(t, y_0, \lambda_0)$ is the solution of the initial value problem

$$
x'(t) = F(t, x(t), \lambda_0) \quad x(0) = y_0.
$$

Using Theorem 2.6 we are able to define

**Definition 2.7.** Let $M$ be a smooth manifold and $\mathcal{M}$ a weight sequence. We say that $M$ is an ultradifferentiable manifold of class $\{\mathcal{M}\}$ iff there is an atlas $\mathcal{A}$ of $M$ that consists of charts such that

$$
\phi' \circ \phi^{-1} \in \mathcal{E}_M
$$

for all $\phi, \phi' \in \mathcal{A}$.

If $M \subseteq \mathbb{R}^N$ is an ultradifferentiable submanifold of class $\{\mathcal{M}\}$ then the following characterization is proven exactly as the analogous result in the smooth setting.

**Proposition 2.8.** Let $M \subset \mathbb{R}^N$ be a smooth manifold of dimension $n$ and $p \in \mathcal{M}$ and $\mathcal{M}$ be a weight sequence. The following statements are equivalent:

1. The manifold $M$ is ultradifferentiable of class $\{M\}$ near $p$.
2. There are an open neighbourhood $U \subseteq \mathbb{R}^N$ of $p$ and an $\mathcal{E}_M$-mapping $\rho : U \to \mathbb{R}^{N-n}$ such that $dp$ has rank $N-n$ on $W$ and $\rho^{-1}(0) = M \cap U$.

A mapping $F : M \to N$ between two manifolds of class $\{M\}$ is ultradifferentiable of class $\{M\}$ iff $\psi \circ F \circ \phi^{-1} \in \mathcal{E}_M$ for any charts $\phi$ and $\psi$ of $M$ and $N$, respectively. Thus it is possible to consider the category of ultradifferentiable manifolds with all the usual constructions like vector fields, differential forms and so on.

**Definition 2.9.** Let $M$ be an ultradifferentiable manifold of class $\{M\}$. We say that a smooth vector bundle $\pi : E \to M$ is an ultradifferentiable vector bundle of class $\{M\}$ iff for any point $p \in M$ there is a neighbourhood $U$ of $p$ and a local trivialization $\chi$ of class $\{M\}$ on $U$.

**Remark 2.10.** Let $E$ be an ultradifferentiable vector bundle of class $\{M\}$. Then $E$ can also be considered as a smooth vector bundle or as a vector bundle of class $N$ for any weight sequence $N \succ M$. We observe in particular that a local basis of $\mathcal{E}_M(M, E)$ is also a local basis of $\mathcal{E}_N(M, E)$ and $\mathcal{E}(M, E)$, respectively.

We denote by $\mathfrak{X}_M(M) = \mathcal{E}_M(M, TM)$ the Lie algebra of ultradifferentiable vector fields on $M$. Note that, if $M$ is a regular weight sequence, an integral curve of an ultradifferentiable vector field of class $\{M\}$ is an $\mathcal{E}_M$-curve by Theorem 2.6.

The next result is an ultradifferentiable version of Sussmann’s Theorem 40.

**Theorem 2.11.** Let $p_0 \in \Omega$ and a collection $\mathcal{D}$ of ultradifferentiable vector fields of class $\{M\}$. Then there exists an ultradifferentiable submanifold $W$ of $\Omega$ through $p_0$ such that all vector fields in $\mathcal{D}$ are tangent to $W$ at all points of $W$ and such that the following holds:

1. The germ of $W$ at $p_0$ is unique, i.e. if $W'$ is an ultradifferentiable submanifold of $\Omega$ containing $p_0$ and to which all vector fields of $\mathcal{D}$ are tangent at every point of $W'$ then there is a neighbourhood $V \subseteq \Omega$ of $p_0$ such that $W \cap V \subseteq W' \cap V$.
2. For every open set $U \subseteq \Omega$ containing $p_0$ there exists $J \in \mathbb{N}$ and open neighbourhoods $V_1 \subseteq V_2 \subseteq U$ of $p_0$ such that every point $p \in W \cap V_1$ can be reached from $p_0$ by a polygonal path of $J$ integral curves of vector fields in $\mathcal{D}$ contained in $W \cap V_2$.

The proof of Theorem 2.11 is essentially the same as in the smooth setting, c.f. e.g. [3], due to Theorem 2.6.

The (unique) germ of the manifold $W$ will be denoted as the local Sussmann orbit of $p_0$ relative to $\mathcal{D}$. The local Sussman orbit does not depend on $\Omega$.

One of the main differences between the space of smooth functions and the space of real analytic functions is that in the smooth case there exist nontrivial test functions $\varphi \in \mathcal{D}(\Omega)$ whereas $\mathcal{D} \cap \mathcal{O} = \{0\}$. Since the existence of functions of nontrivial test functions is equivalent to the existence of nonzero flat functions, it makes sense to give the following definition in the ultradifferentiable setting.
Definition 2.12. Let $E \subseteq \mathcal{E}(\Omega)$ be a subalgebra. We say that $E$ is quasianalytic iff for $f \in E$ the fact that $D^\alpha f(p) = 0$ for some $p \in \Omega$ and all $\alpha \in \mathbb{N}_0^n$ implies that $f \equiv 0$ in the connected component of $\Omega$ that contains $p$.

In the case of Denjoy-Carleman classes quasianalyticity is characterized by the following theorem.

Theorem 2.13 (Denjoy\[10\]-Carleman\[8, 7\]). The space $\mathcal{E}_M(\Omega)$ is quasianalytic if and only if

$$\sum_{k=1}^\infty \frac{m_{k-1}}{km_k} = \infty.$$  (2.2)

We say that a weight sequence is quasianalytic if it satisfies (2.2) and non-quasianalytic if not.

Example 2.14. Let $\sigma > 0$ be a parameter. We define a family $N^\sigma$ of weight sequences by

$$n_k^\sigma = (\log(k + e))^\sigma.$$  

The weight sequence $N^\sigma$ is quasianalytic if and only if $0 < \sigma \leq 1$, see [11].

If $\mathcal{M}$ is a quasianalytic regular weight sequence then it is possible to show a quasianalytic version of Nagano’s theorem [31], c.f. [16]. As in the case of the ultradifferentiable version of Sussmann’s theorem the proof is just a straightforward adaptation of the proof of the classical result, see e.g. [3].

Theorem 2.15. Let $U$ be an open neighbourhood of $p_0 \in \mathbb{R}^n$ and $\mathcal{M}$ a quasianalytic regular weight sequence. Furthermore let $\mathfrak{g}$ be a Lie subalgebra of $\mathfrak{X}_\mathcal{M}(U)$ that is also an $\mathcal{E}_\mathcal{M}$-module, i.e. if $X \in \mathfrak{g}$ and $f \in \mathcal{E}_\mathcal{M}(U)$ then $fX \in \mathfrak{g}$.

Then there exists an ultradifferentiable submanifold $W$ of class $\{\mathcal{M}\}$ in $U$, such that

$$T_pW = \mathfrak{g}(p) \quad \forall p \in W.$$  (2.3)

Moreover, the germ of $W$ at $p_0$ is uniquely defined by this property.

As in the analytic category, c.f. [3], we have the following result.

Corollary 2.16. Let $\mathcal{M}$ be quasianalytic and $\mathcal{D} \subseteq \mathfrak{X}_\mathcal{M}(\Omega)$ a collection of ultradifferentiable vector fields. If $\mathfrak{g} = \mathfrak{g}_\mathcal{D}$ is the Lie algebra generated by $\mathcal{D}$ and $p_0 \in \Omega$ then the local Sussman orbit of $p_0$, relative to $\mathcal{D}$, coincides with the local Nagano leaf of $\mathfrak{g}$.

Proof. Let $W_N$ be a representative of the local Nagano leaf of $\mathfrak{g}$ at $p_0$ and $W_S$ a representative of the local Sussman orbit of $p_0$, relative to $\mathcal{D}$. By Theorem 2.14 (1) there exists an open neighbourhood $V$ of $p_0$ such that $W_S \cap V \subseteq W_N \cap V$. On the other hand $\mathfrak{g}(p) = T_pW_N$ for all $p \in W_N$ and $\mathfrak{g}(p) \subseteq T_pW_S$ at every $p \in W_S$, hence $\mathfrak{g}(p) = T_pW_S$ for $p \in W_S \cap V$. The uniqueness part of Theorem 2.15 gives the equality of the local Nagano leaf and the local Sussman orbit.

We want to close this section by showing how the results pertaining the division of smooth functions in [17], section 4] transfer to the category of ultradifferentiable functions of class $\{\mathcal{M}\}$. This is possible because these classes are closed under division by a coordinate, i.e. Lemma 2.17.

Lemma 2.17. Let $\lambda$ be an ultradifferentiable function of class $\{\mathcal{M}\}$ defined near $0 \in \mathbb{R}$ that is non-flat at the origin, i.e. there is a positive integer $k \in \mathbb{N}$ such that $\lambda^{(j)}(0) = 0$ for all integers $0 \leq j \leq k - 1$ and $\lambda^{(k)}(0) \neq 0$. Further assume that there is a locally integrable function $u$ defined near $0$ such that the product $f = \lambda u$ is of class $\{\mathcal{M}\}$ in some neighbourhood of the origin.

Then $u$ is ultradifferentiable of class $\{\mathcal{M}\}$ near the origin.

Proof. First, we note that the zero of $\lambda$ at 0 is isolated. Therefore we restrict ourselves to an open interval $I$ that contains the origin and such that 0 is the only zero of $\lambda$ on $I$. Iterating Lemma 2.17 we see that there is a function $\lambda$ of class $\{\mathcal{M}\}$ defined near 0 such that $\lambda(0) \neq 0$ and $\lambda(x) = x^k\tilde{\lambda}(x)$.

In order to proceed we want a similar decomposition of $f$. But, since we are not able to say anything apriori about the values of the derivatives of $f$ at the origin, we can only find an ultradifferentiable function $f_1$ such that $f(x) = xf_1(x)$ in a neighbourhood of 0. If $k > 1$ then we would have that $u(x) = x^{1-k}f_1(x)$ $\tilde{\lambda}(x)$.
in a punctured neighbourhood of 0. Hence, if \( f_1(0) \neq 0 \) then \( u \sim x^{1-k} \) for \( x \to 0 \). This is a contradiction to the assumption that \( u \) is locally integrable. Therefore \( f_1(0) = 0 \) and there has to be a function \( f_2 \) of class \( \{\mathcal{M}\} \) such that \( f(x) = x^2f_2(x) \) near 0. Repeating this argument if necessary, we obtain that there is a function \( f_k \) ultradifferentiable of class \( \{\mathcal{M}\} \) defined near the origin such that

\[
 f(x) = x^k f_k(x).
\]

It follows that

\[
 u(x) = \frac{f_k(x)}{\lambda(x)}
\]

in some neighbourhood of 0.

\[\square\]

**Proposition 2.18.** Let \( p_0 \in \mathbb{R}^n \) and \( \lambda \) an ultradifferentiable function of class \( \{\mathcal{M}\} \) defined in a neighbourhood of \( p_0 \) and \( \lambda(p_0) = 0 \). Suppose that \( \lambda^{-1}(0) \) is a hypersurface of class \( \{\mathcal{M}\} \) near \( p_0 \) and that there are \( v \in \mathbb{R}^n \) and \( k \in \mathbb{N} \) such that \( \partial_v^j(p) = 0 \) for \( 0 \leq j < k \) and \( \partial_v^k(p) \neq 0 \) for all \( p \in \lambda^{-1}(0) \cap U \) where \( U \) is a neighbourhood of \( p_0 \).

If \( u \) is a locally integrable function defined near the origin in \( \mathbb{R}^n \) such that \( \lambda \cdot u = f \) is ultradifferentiable of class \( \{\mathcal{M}\} \) near \( p_0 \) then \( u \) has also to be of class \( \{\mathcal{M}\} \) in some neighbourhood of \( p_0 \).

**Proof.** We can choose ultradifferentiable coordinates \((x_1, \ldots, x_{n-1}, x_n) = (x', x_n)\) in a neighbourhood \( V \) of \( p_0 \) in \( \mathbb{R}^n \) such that \( p_0 = 0, \lambda^{-1}(0) \cap V = \{(x', x_n) \in V \mid x_n = 0\} \) and

\[
 \frac{\partial^j \lambda}{\partial x_n}(0) = 0, \quad 0 \leq j < k,
\]

\[
 \frac{\partial^k \lambda}{\partial x_n}(0) \neq 0.
\]

Similarly to above, using Lemma 2.14, we conclude, if we shrink \( V \), that there is \( \tilde{\lambda} \in \mathcal{E}\mathcal{M}(V) \) with the following properties: \( \tilde{\lambda}(x) \neq 0 \) and \( \lambda(x) = x_n^k \tilde{\lambda}(x) \) for all points \( x \in V \). There is also a Denjoy-Carleman function \( f_1 \) on \( V \) such that \( f(x', x_n) = x_n f_1(x', x_n) \). We want to show, as in the 1-dimensional case, that \( f_1(x', 0) = 0 \) for \((x', 0) \in V \) if \( k > 1 \): Suppose that there exists some \( y \in \mathbb{R}^{n-1} \) with \((y, 0) \in V \) and \( f_1(y, 0) \neq 0 \). Then there is a neighbourhood \( W' \) of \((y, 0) \) such that \( f_1(x) \neq 0 \) and also \( \tilde{\lambda}(x) \neq 0 \) for \( x \in W'. \) W.l.o.g. the open set \( W \) is of the form \( W = W' \times I \subseteq \mathbb{R}^{n-1} \times \mathbb{R} \) and set

\[
 F(x_n) := \int_{W'} \left| \frac{f_1(x)}{\tilde{\lambda}(x)} \right| \, dx
\]

for \( x_n \in I \). We conclude that

\[
 \int_W |u(x)| \, dx = \int_I |x_n|^{1-k} F(x_n) \, dx = \infty
\]

and hence \( u \) cannot be locally integrable near \((y, 0)\) which contradicts our assumption. Therefore we obtain by iteration a function \( \tilde{f} \) of class \( \{\mathcal{M}\} \) defined near the origin in \( \mathbb{R}^n \) such that \( f(x', x_n) = x_n^k \tilde{f}(x', x_n) \). Hence \( u = \tilde{f}/\tilde{\lambda} \) is also of class \( \{\mathcal{M}\} \) in a neighbourhood of 0.

\[\square\]

**Corollary 2.19.** Let \( U \subseteq \mathbb{R}^n \) a neighbourhood of 0, \( \lambda \in \mathcal{E}\mathcal{M}(U) \) and suppose that \( \lambda \) is of the form \( \lambda(x) = x^n \tilde{\lambda}(x) \) where \( \alpha \in \mathbb{N}_0^n \) and \( \tilde{\lambda} \in \mathcal{E}\mathcal{M}(U) \) with \( \tilde{\lambda}(0) \neq 0 \).

If \( u \) is a locally integrable function near 0 with the property that the product \( f := \lambda \cdot u \) is of class \( \{\mathcal{M}\} \) near the origin, then \( u \) is also ultradifferentiable near 0.

**Proof.** Note first that, if \( \alpha = \alpha_j e_j \) then the statement is just Proposition 2.18. In the general case we argue as follows: Set \( \tilde{f} = f/\tilde{\lambda} \) and

\[
 u_k(x) = \prod_{j=k+1}^{n} x_j^{\alpha_j} u(x)
\]

for all \( k \in \{1, \ldots, n-1\} \). The function \( \tilde{f} \) is of class \( \{\mathcal{M}\} \) whereas the functions \( u_k \) are locally integrable near 0. Furthermore we define \( u_n = u \) and obtain

\[
 x_k^{\alpha_k+1} u_1(x) = \tilde{f}(x)\]

\[
 x_k^{\alpha_k+1} u_{k+1}(x) = u_k(x) \quad 1 \leq k \leq n-1.
\]

Hence repeated application of Proposition 2.18 finishes the proof.
In the literature the focus regarding questions of divisibility of functions seems to be more on the problem if it is possible to show that functions that are formally divisible, i.e. their Taylor series are divisible, are actually divisible. Indeed, the Weierstrass division theorem for example implies that two real-analytic functions that are formally divisible are also divisible as functions.

However, the equivalent of the Weierstrass division theorem does not hold for general quasianalytic Denjoy-Carleman classes \[1\], \[34\], c.f. also \[14\]. In general the algebraic structure of quasianalytic Denjoy-Carleman classes is far more complicated than that of the space of real-analytic functions, c.f. the survey \[11\].

Despite this there are some positive results known for quasianalytic regular classes, e.g. \[5\] showed that certain desingularization theorems hold in these classes whereas \[35\] proved that quasianalytic regular Denjoy-Carleman classes define \(\alpha\)-minimal structures. Both of these approaches can be used to prove division theorems. Especially the following result was shown by \[33\].

**Theorem 2.20.** Let \(p \in \mathbb{R}^n\), \(\mathcal{M}\) quasianalytic and \(f, g \in \mathcal{E}_\mathcal{M}\) are defined near \(p\) with power series expansions \(\hat{f}\) and \(\hat{g}\) at \(p\). If \(\hat{f} \in \hat{g} \cdot C[[x]]\) then \(f \in g \cdot \mathcal{E}_\mathcal{M}\) near \(p\).

3. ALMOST ANALYTIC EXTENSIONS AND THE WAVEFRONT SET IN THE ULTRADIFFERENTIABLE SETTING

In this section we recall the almost analytic extension of ultradifferentiable functions given by Dyn’kin in \[12\] \[11\] and its connection with the ultradifferentiable wavefront set introduced by Hörmander in \[20\] that was proven in \[16\].

We recall (see e.g. \[12\]) that a smooth function \(F\) given on an open subset \(\hat{\Omega} \subseteq \mathbb{C}^n\) is almost analytic iff

\[
\partial_j F = \frac{\partial}{\partial x_j} F = \frac{1}{2} \left( \frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j} \right) F
\]

is flat on \(\hat{\Omega} \cap \mathbb{R}^n\). The motivation to consider almost analytic function in the ultradifferentiable setting is the well-known fact that a function \(f\) is smooth on \(\Omega\) if and only if there is an almost analytic function \(F\) on some open set \(\hat{\Omega} \subseteq \mathbb{C}^n\) with \(\Omega \cap \mathbb{R}^n = \Omega\) such that \(F|_\Omega = f\). In the ultradifferentiable category the idea is now that if \(f\) is ultradifferentiable of class \(\{\mathcal{M}\}\) then it should be possible to construct an almost analytic extension \(F\) of \(f\) such that the decrease of \(\partial_j F\) can be measured in terms of the weight sequence \(\mathcal{M}\). (c.f. \[13\]).

In order to specify this decay we introduce for a regular weight sequence \(\mathcal{M}\) its associated weight given by

\[
h_{\mathcal{M}}(t) = \inf_k t^k m_k \quad \text{if } t > 0 \quad \& \quad h_{\mathcal{M}}(0) = 0. \tag{3.1}
\]

Conversely it is possible to extract the weight sequence from its weight:

\[
m_k = \sup_t \left( \frac{h_{\mathcal{M}}(t)}{t^k} \right)
\]

The weight \(h_{\mathcal{M}}\) is continuous with values in \([0,1]\), equals 1 on \([1,\infty)\) and goes more rapidly to 0 than \(t^p\) for any \(p > 0\) for \(t \to 0\), c.f. \[16\]. Before we are able to state the Theorem of Dyn’kin alluded above, we have to note that his result gives not the existence of a global extension as in the smooth case but only a semiglobal statement. This corresponds with the fact that real-analytic functions have generally only local holomorphic extensions. In order to state the precise form of Dyn’kin’s result we recall the following definition from e.g. \[24\]. If \(K \subseteq \Omega\) is compact then \(\mathcal{E}_\mathcal{M}(K)\) is the space of smooth functions which are defined on some neighbourhood of \(K\) and on \(K\) they satisfy \[21\] for some constants \(C,h > 0\).

**Theorem 3.1.** Let \(\mathcal{M}\) be a regular weight sequence, \(K \subseteq \mathbb{R}^n\) a compact and convex set with \(K = \overline{K}\). Then \(f \in \mathcal{E}_\mathcal{M}(K)\) if and only if there exists a test function \(F \in \mathcal{D}^{\mathbb{C}^n}\) with \(F|_K = f\) and if there are constants \(C,Q > 0\) such that

\[
|\partial_j F(z, \bar{z})| \leq C h_{\mathcal{M}}(Q d_K(z)) \tag{3.2}
\]

where \(1 \leq j \leq n\) and \(d_K\) is the distance function with respect to \(K\) on \(\mathbb{C}^n\setminus K\).

The local form of Theorem \[3,1\] is

**Corollary 3.2.** If \(f\) is ultradifferentiable of class \(\{\mathcal{M}\}\) near \(p\), then there are an open neighbourhood \(W \subseteq \Omega\), a constant \(\rho > 0\) and a function \(F \in \mathcal{E}(W + iB(0,\rho))\) such that \(F|_W = f|_W\) and

\[
|\partial_j F(x + iy)| \leq C h_{\mathcal{M}}(Q |y|) \tag{3.3}
\]

for some positive constants \(C,Q\) and all \(1 \leq j \leq n\) and \(x + iy \in W + iB(0,\rho)\).
We call such function $F$ an $M$-almost analytic extension of $f$.

The following theorem is the $M$-almost analytic version of the almost-holomorphic implicit function theorem proven in [28].

**Theorem 3.3.** Let $M$ be a regular weight sequence, $U \subseteq \mathbb{C}^N$ a neighbourhood of the origin, $A \in \mathbb{C}^p$ and $F : U \times \mathbb{C}^p \to \mathbb{C}^N$ of class $\{M\}$ on $U$ and polynomial in the last variable with $F(0, A) = 0$ and $F_2(0, A)$ invertible. Then there exists a neighbourhood $U' \times V'$ of $(0, A)$ and a smooth mapping $\phi = (\phi_1, \ldots, \phi_N) : U' \times V' \to \mathbb{C}^N$ with $\phi(0, A) = 0$ with the property that if $F(Z, \bar{Z}, W) = 0$ for some $(Z, W) \in U' \times V'$, then $Z = \phi(Z, \bar{Z}, W)$. Furthermore, there are constants $C, \gamma > 0$ such that

$$\left| \frac{\partial \phi_j}{\partial Z_k}(Z, \bar{Z}, W) \right| \leq Ch_M(\gamma |\phi(Z, \bar{Z}, W) - Z|)$$

(3.4)

for all $1 \leq j, k \leq N$ and $\phi$ is holomorphic in $W$.

**Proof.** We write $F(Z, \bar{Z}, W) = F(x, y, W)$, where $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N$ are the underlying real coordinates of $\mathbb{C}^N$, i.e. $Z_j = x_j + iy_j$ for $1 \leq j \leq N$. Let $U_0 \subseteq \mathbb{R}^N$ be a convex neighbourhood of 0 such that $U_0 \times U_0 \subseteq U$. Using Theorem 3.1 we find a smooth mapping

$$\tilde{F} = F_0 \times \mathbb{R}^N \to \mathbb{R}^N \times \mathbb{C}^p \to \mathbb{C}^N$$

such that $\tilde{F}(x, x', y, y', W)|_{x' = y' = 0} = F(x, y, W)$ and if we write $\xi_k = x_k + iy_k$, $\eta_k = y_k + iy_k$ for $k = 1, \ldots, N$ and set $\zeta = (\xi, \eta)$, then for each compact subset $K \subset \subset \mathbb{C}^p$ there are constants $C, \gamma > 0$ such that

$$\left| \frac{\partial \tilde{F}}{\partial \xi_k}(\zeta, \zeta, W) \right| \leq Ch_M(\gamma |\zeta|)$$

(3.5a)

$$\left| \frac{\partial \tilde{F}}{\partial \eta_k}(\zeta, \zeta, W) \right| \leq Ch_M(\gamma |\zeta|)$$

(3.5b)

for $(\zeta, W) \in (U_0 + i\mathbb{R}^N)^2 \times K$ and $1 \leq j, k \leq N$. Note also that $\tilde{F}$ is still polynomial in the variable $W$.

We introduce new variables $\chi = (\chi_1, \ldots, \chi_N) \in \mathbb{C}^N$ by

$$\xi_k = \frac{Z_k + \chi_k}{2} \quad \eta_k = \frac{Z_k - \chi_k}{2i} \quad 1 \leq k \leq N$$

and note that

$$x_k = \frac{Z_k + \chi_k}{2} \bigg|_{\chi_k = Z_k} \quad y_k = \frac{Z_k - \chi_k}{2i} \bigg|_{\chi_k = Z_k}.$$

We also set $G(Z, \bar{Z}, \chi, \bar{\chi}, W) = \tilde{F}(\xi, \eta, \bar{\eta}, W)$. The function $G$ is therefore smooth in the first $2N$ variables in some neighbourhood of the origin and polynomial in the last $p$ variables. Due to the definition of $G$ we have

$$\frac{\partial G}{\partial Z}(0, A) = \frac{\partial F}{\partial Z}(0, A)$$

and

$$\frac{\partial G}{\partial Z}(0, A) = \frac{1}{2} \left( \frac{\partial F}{\partial \xi}(0, A) - i \frac{\partial F}{\partial \eta}(0, A) \right) = 0$$

and thus

$$\det \left( \frac{\partial G}{\partial Z} \frac{\partial G}{\partial \bar{Z}} \right)(0, A) = \left| \det \frac{\partial F}{\partial Z}(0, A) \right|^2 \neq 0$$

by assumption. Hence, by the smooth implicit function theorem, there is a smooth mapping $\psi$ defined in some open neighbourhood of $(0, A)$, valued in $\mathbb{C}^N$ and holomorphic in the variable $W$ such that $Z = \psi(\chi, \bar{\chi}, W)$ solves the equation $G(Z, \bar{Z}, \chi, \bar{\chi}, W) = 0$ uniquely. Since $G(Z, \bar{Z}, \chi, \bar{\chi}, W) = F(Z, \bar{Z}, W)$, this fact implies that if $F(Z, \bar{Z}, W) = 0$ then $Z = \psi(Z, \bar{Z}, W)$. We set $\phi(Z, \bar{Z}, W) = \psi(Z, \bar{Z}, W)$ and claim that $\phi$ satisfies (3.4).
In fact, if we differentiate the implicit equation $G(\psi(\chi, \bar{\chi}, W), \psi(\chi, \bar{\chi}, W), \chi, \bar{\chi}, W) = 0$ then we obtain

\begin{align*}
G_Z \psi_\chi + G_\bar{Z} \bar{\psi}_\chi + G_\chi &= 0 \\
G_Z \bar{\psi}_\chi + G_\bar{Z} \psi_\chi + G_\bar{\chi} &= 0.
\end{align*}

If we multiply the last line with $G_\bar{Z} G_Z^{-1}$ and subtract the result from the first line then

\[(G_Z - G_\bar{Z} G_Z^{-1} G_{\bar{Z}}) \psi_\chi = G_\bar{Z} G_Z^{-1} G_\chi - G_\bar{\chi}.\]

Hence we have in a small neighbourhood of $(0, A)$ that

\[
\phi(Z, \bar{Z}, W) = \psi_\chi(Z, \bar{Z}, W) = \left( \frac{G_\bar{Z} G_Z^{-1} G_{\bar{Z}} - G_\chi}{G_Z - G_\bar{Z} G_Z^{-1} G_{\bar{Z}}} \right) \left( \psi(Z, \bar{Z}, W), \overline{\psi(Z, \bar{Z}, W)}, Z, W \right).
\]

This formula shows that any function $\partial_{Z_i} \phi_j$ is a sum of products each of which contains a factor of the form $G_Z$, or $G_\bar{Z}$, for some $\ell$. Note also that by definition $\text{Im} \xi = \frac{1}{2} (\text{Im} Z + \text{Im} \chi)$ and $\text{Im} \eta = -\frac{1}{2} (\text{Re} Z - \text{Re} \chi)$.

Hence (3.5) implies on some compact neighbourhood of $(0, A)$, where $\det G_Z^{-1}$ is bounded,

\[
|\phi(Z, \bar{Z}, W)| \leq \text{Ch}_M \left( \frac{1}{2} (|\text{Im} \phi(Z, \bar{Z}, W)| - |\text{Im} Z|^2 + |\text{Re} Z - \text{Re} \phi(Z, \bar{Z}, W)|^2)^{\frac{1}{2}} \right)
\]

for some positive constants $C$ and $\gamma$.

In the following we recall the results on the ultradifferentiable wavefront set that we need in this paper. We start with the definition given in [20].

**Definition 3.4.** Let $u \in D'(\Omega)$ and $(x_0, \xi_0) \in T^*\Omega \setminus \{0\}$. We say that $u$ is **microlocally ultradifferentiable of class $\{M\}$** at $(x_0, \xi_0)$ iff there is a bounded sequence $(u_N) \subseteq E'(\Omega)$ such that $u_N|_{V} \equiv u|_{V}$, where $V \in \mathcal{U}(x_0)$ and a conic neighbourhood $\Gamma$ of $\xi_0$ such that for some constant $Q > 0$,

\[
\sup_{\xi \in \Gamma} \frac{\langle \xi \rangle^N |\hat{u}_N|}{Q^N m_N N!} < \infty.
\]

The ultradifferentiable wavefront set $WF_M u$ is then defined as

\[
WF_M u := \{(x, \xi) \in T^*\Omega \setminus \{0\} \mid u \text{ is not microlocally of class } \{M\} \text{ at } (x, \xi)\}.
\]

The basic properties $WF_M u$ shown by Hörmander in [22] are the following.

**Theorem 3.5 (22) Theorem 8.4.5-8.4.7.** Let $u \in D'(\Omega)$ and $M, N$ weight sequences. Then we have

1. $WF_M u$ is a closed conic subset of $\Omega \times \mathbb{R}^n \setminus \{0\}$.
2. The projection of $WF_M u$ in $\Omega$ is

\[
\pi_1(WF_M u) = \text{sing supp}_u = \{x \in \Omega \mid \exists V \in \mathcal{U}(x) : u|_V \in \mathcal{E}_M(V)\}.
\]
3. $WF u \subseteq WF_N u \subseteq WF_M u$ if $M \ll N$.
4. If $P = \sum \rho_j D^{m_j}$ is a partial differential operator with ultradifferentiable coefficients of class $\{M\}$ then $WF_M Pu \subseteq WF_M u$.

Additionally we note that $WF_M u$ satisfies the following **microlocal reflection property**:

\[
(x, \xi) \notin WF_M u \iff (x, -\xi) \notin WF_M \hat{u} (3.7)
\]

In particular, if $u$ is a real-valued distribution, i.e. $\hat{u} = u$, then $WF_M u|_{\mathbb{R}^n} := \{\xi \in \mathbb{R}^n \mid (x, \xi) \in WF_M u\}$ is symmetric at the origin.

It is a classic fact that the analytic wavefront set can not only characterized by the Fourier transform but also holomorphic extension in certain directions, see [6]. Likewise, the smooth wavefront set can be characterized by almost-analytic extensions, c.f. [30]. We present now the basic results on the connection between almost-analytic extensions and the ultradifferentiable wavefront set that we proved in [16]. In order to do so we need first to recall some notations used in [16]: A subset $\Gamma \subseteq \mathbb{R}^d$ is a cone iff for all $\lambda > 0$ and $y \in \Gamma$ we have $\lambda y \in \Gamma$. If $r > 0$ then

\[
\Gamma_r := \{y \in \Gamma \mid |y| < r\}.
\]

\footnote{We use in the following the notation $D_j = -i \partial_j$.}
If $\Gamma' \subseteq \Gamma$ is also a cone we write $\Gamma' \subset \subset \Gamma$ if $\big(\Gamma' \cap S^{d-1}\big) \subset \subset \big(\Gamma \cap S^{d-1}\big)$.

If $\mathcal{M}$ is a weight sequence with associated weight $h_\mathcal{M}$ then a function $F \in \mathcal{E}(\Omega \times U \times \Gamma_r)$, $U \subseteq \mathbb{R}^d$ open, is said to be $\mathcal{M}$-almost analytic in the variables $(x, y) \in U \times \Gamma_r$ with parameter $x' \in \Omega$ iif for all $K \subset \subset \Omega$, $L \subset \subset U$ and cones $\Gamma' \subset \subset \Gamma$ there are constants $C, Q > 0$ such that for some $r'$ we have

$$\frac{\partial F}{\partial x_j}(x', x, y) \leq Ch_\mathcal{M}(Q|y|) \quad (x', x, y) \in K \times L \times \Gamma'_r, \quad j = 1, \ldots, d$$

(3.8)

where $\frac{\partial}{\partial x_j} = \frac{1}{2}(\partial_{x_j} + i\partial_{y_j})$.

We may also say generally that a function $g \in C(\Omega \times U \times \Gamma_r)$ is of slow growth in $y \in \Gamma_r$ if for all $K \subset \subset \Omega$, $L \subset \subset U$ and $\Gamma' \subset \subset \Gamma$ there are constants $c, k > 0$ such that

$$|g(x', x, y)| \leq c|y|^{-k} \quad (x', x, y) \in K \times L \times \Gamma'_r.$$  (3.9)

**Theorem 3.6.** Let $F \in \mathcal{E}(\Omega \times U \times \Gamma_r)$ be $\mathcal{M}$-almost analytic in the variables $(x, y) \in U \times \Gamma_r$ and of slow growth in the variable $y \in \Gamma_r$. Then the distributional limit $u$ of the sequence $u_\varepsilon = F(\ldots, \varepsilon) \in \mathcal{E}(\Omega \times U)$ exists. We say that $u = br(F) \in \mathcal{D}'(\Omega \times U)$ is the boundary value of $F$. Furthermore, we have

$$WF_{\mathcal{M}} u \subseteq (\Omega \times U) \times (\mathbb{R}^n \times \Gamma^\circ)$$

where $\Gamma^\circ = \{y \in \mathbb{R}^d \mid \langle y, y \rangle \geq 0 \quad \forall y \in \Gamma\}$ is the dual cone of $\Gamma$ in $\mathbb{R}^d$.

**Theorem 3.7.** Let $\Gamma \subseteq \mathbb{R}^n$ be an open convex cone and $u \in \mathcal{D}'(\Omega)$ with $WF_{\mathcal{M}} u \subseteq \Omega \times \Gamma^\circ$. If $V \subset \subset \Omega$ and $\Gamma'$ is an open convex cone with $\Gamma' \subseteq \Gamma \cup \{0\}$ then there is an $\mathcal{M}$-almost analytic function $F$ on $V + i\Gamma'_r$ of slow growth for some $r > 0$ such that $u|_V = br(F)$.

Using Theorem 3.6 and Theorem 3.7 we were able in [16] to show the characterization of the ultradifferentiable wavefront set by $\mathcal{M}$-almost analytic extensions.

**Corollary 3.8.** Let $u \in \mathcal{D}'(\Omega)$ and $(x_0, \xi_0) \in \Omega \times \mathbb{R}^n \setminus \{0\}$. Then $(x_0, \xi_0) \notin WF_{\mathcal{M}} u$ if and only if there are a neighbourhood $V$ of $x_0$, open convex cones $\Gamma_1, \ldots, \Gamma_N$ with the properties $\xi_0 \Gamma_j < 0$, $j = 1, \ldots, N$ and $\Gamma_j \cap \Gamma_k = \emptyset$ for $j \neq k$, and $\mathcal{M}$-almost analytic functions $h_j$ on $V + i\Gamma'_r$, $r_j > 0$, of slow growth such that

$$u|_V = \sum_{j=1}^N b_{\Gamma_j}(h_j)$$

In [16] Corollary 3.8 is then applied to show the following Theorem.

**Theorem 3.9.** Let $F$ be an $\mathcal{E}_{\mathcal{M}}$-diffeomorphism then

$$WF_{\mathcal{M}}(F^* u) = F^* \left(WF_{\mathcal{M}} u\right).$$

Hence if $M$ is an $\mathcal{E}_{\mathcal{M}}$-manifold and $u \in \mathcal{D}'(M)$ we can define $WF_{\mathcal{M}} u$ invariantly as a subset of $T^*M \setminus \{0\}$. We refer to [18] or [9] for the definition of distributions on manifolds, either scalar or with values in vector bundles. Let $u$ be a distribution on an ultradifferentiable manifold $M$ of class $\{M\}$ with values in an $\mathcal{E}_{\mathcal{M}}$-vector bundle over $M$. In particular we can write locally $u|_V = \sum_{j=1}^N u_j \omega^j$, where $V \subseteq M$ is an open subset of $M$, scalar-valued distributions $u_j \in \mathcal{D}'(V)$ and the sections $\omega^1, \ldots, \omega^N \in \mathcal{E}_{\mathcal{M}}(V, E|_V)$ constitute a local basis of $\mathcal{E}_{\mathcal{M}}(M, E)$. The ultradifferentiable wavefront set of $u$ is then defined locally by

$$WF_{\mathcal{M}} u = \bigcup_{j=1}^N WF_{\mathcal{M}} u_j.$$
fiber dimension $\nu$. An ultradifferentiable differential operator $P: \mathcal{E}_M(M, E) \to \mathcal{E}_M(M, F)$ of class $\{ M \}$ and order $\leq m$ is given locally in some trivialization by

$$Pu = \begin{pmatrix} P_{11} & \cdots & P_{1\nu} \\ \vdots & \ddots & \vdots \\ P_{\nu1} & \cdots & P_{\nu\nu} \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_\nu \end{pmatrix},$$

where the $P_{jk}$ are partial differential operators with ultradifferentiable coefficients of order $\leq m$ defined on some chart neighbourhood. The operator $P$ is of order $m$ if it is not of order $\leq m - 1$. The principal symbol $p$ of $P$ is an ultradifferentiable mapping on $T^*M$ with values in the fiber-linear maps from $E$ to $F$, that is given locally by

$$p(x, \xi) = \begin{pmatrix} p_{11}(x, \xi) & \cdots & p_{1\nu}(x, \xi) \\ \vdots & \ddots & \vdots \\ p_{\nu1}(x, \xi) & \cdots & p_{\nu\nu}(x, \xi) \end{pmatrix}$$

where $p_{jk}$ is the principal symbol of $P_{jk}$. The operator $P$ is not characteristic (or non-characteristic) at a point $(x, \xi) \in T^*M \setminus \{0\}$ if $p(x, \xi)$ is an invertible linear mapping. The set of all characteristic points is defined by

$$\text{Char } P = \{(x, \xi) \in T^*M \setminus \{0\} : P \text{ is characteristic at } (x, \xi)\}.$$

After this lengthy preparation we are able to state the elliptic regularity theorem for partial differential operators between ultradifferentiable vector bundles.

**Theorem 3.10.** Let $M$ be an $\mathcal{E}_M$-manifold and $E, F$ two ultradifferentiable vector bundles on $M$ of the same fiber dimension. If $P(x, D)$ is a differential operator between $E$ and $F$ with $\mathcal{E}_M$-coefficients and $p$ its principal symbol, then

$$\text{WF}_M u \subseteq \text{WF}_M(Pu) \cup \text{Char } P \quad u \in \mathcal{D}'(M, E).$$

4. **CR Manifolds of Denjoy Carleman Type**

In this section we rapidly recall the basic definitions of CR geometry, for more details see [3]. We begin with the embedded case. Let $M \subseteq \mathbb{C}^N$ be a real submanifold of $\mathbb{C}^N$, then $T_pM \subset T_p\mathbb{C}^N$ ($p \in M$) as real vector spaces, but $T_p\mathbb{C}^N = \mathbb{R}^{2N} \cong \mathbb{C}^N$ inherits a complex structure from $\mathbb{C}^N$. Hence there is a maximal complex subspace $T_p^\mathbb{C}M$ of $T_p\mathbb{C}^N$ such that $T_p^\mathbb{C}M \subseteq T_pM \subseteq T_p\mathbb{C}^N$.

**Definition 4.1.** A submanifold $M \subseteq \mathbb{C}^N$ is said to be CR if the mapping

$$M \ni p \mapsto \dim_\mathbb{C} T_p^\mathbb{C}M$$

is constant. The CR dimension of $M$ is then defined as $\dim_{\text{CR}} M := \dim_{\mathbb{C}} T_p^\mathbb{C}M$.

Note that any real hypersurface $M \subseteq \mathbb{C}^N$ is CR. An arbitrary submanifold $M \subseteq \mathbb{C}^N$ of codimension $d$ is said to be generic iff it can be realized as the intersection of $d$ real hypersurfaces whose complex tangent spaces are in general position as complex vector spaces. The manifold $M$ is said to be generic at a point $p \in M$ iff there is a neighbourhood $U$ of $p$ in $\mathbb{C}^N$ such that $M \cap U$ is generic. We recall that if $M \subseteq \mathbb{C}^N$ is a generic submanifold of CR dimension $n$ and real codimension $d$ then $n + d = N$.

It is easy to see that for a CR manifold $M$ we can consider the complex tangential bundle $T^\mathbb{C}M \subseteq TM$. However the complex tangential bundle, although being a vector bundle over $\mathbb{C}$, is realized as a subbundle of the real bundle $TM$. Often it is more convenient to take a different approach for the definition of CR manifolds. For this end consider the complexified tangent bundle $\mathbb{C}TM = \mathbb{C} \oplus TM$ of a manifold $M \subseteq \mathbb{C}^N$. Furthermore let $p \in M$ and set $\mathbb{C}T_p\mathbb{C}^N = T_p^10\mathbb{C}^N \oplus T_p^{01}\mathbb{C}^N$. If $z_j = x_j + iy_j$, $j = 1, \ldots, N$ denote the coordinates of $\mathbb{C}^N$ then the spaces $T_p^{1,0}\mathbb{C}^N$ and $T_p^{0,1}\mathbb{C}^N$ are generated by $\partial/\partial z_j|_p$ and $\partial/\partial \bar{z}_j|_p$, $j = 1, \ldots, N$, respectively. If we set $V_p = \mathbb{C}T_p\mathbb{C}^N \cap T_p^{1,0}\mathbb{C}^N$ then $\dim_{\mathbb{C}} V_p = \dim_{\mathbb{C}} T_p^\mathbb{C}M$. If $M$ is a CR submanifold, then $\mathcal{V} = \bigsqcup V_p$ is said to be the CR bundle associated to $M$. It is easy to see that $\mathcal{V}$ is involutive, i.e. $[\mathcal{V}, \mathcal{V}] \subseteq \mathcal{V}$, and $\mathcal{V} \cap \mathcal{V} = \{0\}$. Using this it is possible to generalize the notion of CR manifold.

**Definition 4.2.** Let $M$ be a manifold (not necessarily embedded) and $\mathcal{V} \subseteq \mathbb{C}TM$ a subbundle. We say that $(M, \mathcal{V})$ is an abstract CR manifold iff $\mathcal{V}$ is an involutive bundle and $\mathcal{V} \cap \mathcal{V} = \{0\}$. The CR dimension of $M$ is defined as $\dim_{\text{CR}} M = \dim \mathcal{V}$. If $\dim_{\mathbb{R}} M = m + n$ then the CR codimension is given by $d = m - n$. 

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If \( M \) is a CR manifold of class \( \{M\} \) then a CR vector field \( L \) is an ultradifferentiable section of \( V \), i.e. \( L \in \mathcal{E}_{\mu}(M,V) \). If \( p \in M \) and \( n = \dim_{\text{CR}} M \) then a local basis of CR vector fields near \( p \) consists of \( n \) CR vector fields \( L_1, \ldots, L_n \) defined near \( p \) that are linearly independent. We also set \( L^n = L_1^n \cdots L_n^n \) for \( a \in \mathbb{N}_0^n \).

A CR function or CR distribution is a function or distribution on \( M \) that is annihilated by all CR vector fields. We refer to \( T^0M \):= \( V^\perp \) as the holomorphic cotangent bundle. \( T^0M \) is a complex vector bundle on \( M \) with fiber dimension \( N = n + d \). Its ultradifferentiable sections are called holomorphic forms. The real subbundle \( T^0\mathcal{M} \subseteq T^0M \) that consists of the real dual vectors that vanish on \( V \oplus \overline{V} \) is called the characteristic bundle of \( M \) and its sections of class \( \{M\} \) are the characteristic forms on \( M \).

Note that if \( L \) is a CR vector field, we have generally that \( \text{Char} L \subseteq T^0M \), hence we obtain for any CR distribution \( u \) that \( \text{WF}_M u \subseteq T^0M \).

A \( C^1 \)-mapping \( H \) between two CR manifolds \( (M, V) \) and \( (M', V') \) is CR iff for all \( p \in M \) we have \( H_*(V_p) \subseteq V'_H(p) \). Here \( H_* \) denotes the tangent map of \( H \). If \( M' \subseteq \mathbb{C}^{N'} \) is an embedded CR submanifold and \( Z' = (Z'_1, \ldots, Z'_n) \) some set of local holomorphic coordinates in \( \mathbb{C}^{N'} \) then \( H_j = Z'_j \circ H \), \( 1 \leq j \leq N' \) is a CR function on the CR manifold \( M \) for all \( 1 \leq j \leq N' \).

We continue with a first look at specific results about ultradifferentiable CR manifolds.

**Proposition 4.3.** Let \( M \subseteq \mathbb{C}^N \) be a generic manifold of class \( \{M\} \) of codimension \( d \) and \( p_0 \in M \). If \( n \) denotes the CR dimension of \( M \) then there are holomorphic coordinates \( (z, w) \) in \( \mathbb{C}^n \times \mathbb{C}^d \) defined near \( p_0 \) that vanish at \( p_0 \) and a function \( \varphi \in \mathcal{E}_{\mu}(U \times V, \mathbb{R}^d) \) defined on a neighbourhood \( U \times V \) of the origin in \( \mathbb{R}^{2n} \times \mathbb{R}^d \) with \( \varphi(0) = 0 \) and \( \nabla \varphi(0) = 0 \), such that near \( p_0 \) the manifold \( M \) is given by

\[
\text{Im} \ w = \varphi(z, \bar{z}, \text{Re} \ w).
\]  

(4.1)

**Proof.** We follow the proof in [3] for the result in the smooth category.

After an affine transformation we may assume that \( p_0 = 0 \). Let \( \rho = (\rho_1, \ldots, \rho_d) \) be a defining function for \( M \) near 0. The complex differentials \( \partial \rho_1, \ldots, \partial \rho_d \) are linearly independent over \( \mathbb{C} \) near 0 since \( M \) is generic. For each \( k \in \{1, \ldots, d\} \) we write

\[
\rho_k(Z, \bar{Z}) = \sum_{r=1}^{N} \left( a_{kr} x_r + b_{kr} y_r \right) + O(2)
\]

where \( O(2) \) denotes terms that vanish at least of quadratic order at 0. Since \( \rho_k \) is real-valued, the coefficients \( a_{kr} \) and \( b_{kr} \) have to be real numbers. We define a linear form \( \ell_k \) on \( \mathbb{C}^N \) by

\[
\ell_k(Z) = \sum_{r=1}^{N} \left( b_{kr} + ia_{kr} \right) Z_r
\]

and thus the above equation becomes

\[
\rho_k(Z, \bar{Z}) = \text{Im} \ \ell_k(Z) + O(2).
\]

The linear forms \( \ell_k, k = 1, \ldots, d \) are linearly independent over \( \mathbb{C} \) since the differentials \( \partial \rho_k, k = 1, \ldots, d \), are \( \mathbb{C} \)-linearly independent. After renumbering the coordinates \( Z_j \) we can assume that

\[
Z_1, \ldots, Z_n, \ell_1, \ldots, \ell_k
\]

are linearly independent as linear forms over \( \mathbb{C} \).

We define new holomorphic coordinates \( (z, w) \) near \( (0, 0) \in \mathbb{C}^{n+d} \) by

\[
z_j = Z_j, \quad 1 \leq j \leq n
\]

\[
w_k = \ell_k(Z), \quad n + 1 \leq k \leq N = n + d.
\]

In these new coordinates we have, if we set \( \tilde{\rho}(z, \bar{z}, w, \bar{w}) = \rho(Z(z, w), \overline{Z(z, w)}) \),

\[
\tilde{\rho}(z, \bar{z}, w, \bar{w}) = \text{Im} \ w + O(2)
\]  

(4.2)

and therefore we can locally near 0 solve the equation

\[
\tilde{\rho}(z, \bar{z}, w, \bar{w}) = 0
\]  

(4.3)

with respect to \( t = \text{Im} \ w \) according to Theorem [25]. We obtain an ultradifferentiable solution \( \varphi \) of class \( \{M\} \) defined near \( 0 \in \mathbb{R}^{2n+d} \subseteq \mathbb{C}^n \times \mathbb{R}^d \) and valued in \( \mathbb{R}^d \). The properties \( \varphi(0) = 0 \) and \( \nabla \varphi(0) = 0 \) are easy consequences of (4.2) and (4.3). We also see that in view of (4.2) and

\[
\tilde{\rho}(z, \bar{z}, s + i \varphi(z, \bar{z}, s), s - i \varphi(z, \bar{z}, s)) = 0
\]
the function $\psi(z, \bar{z}, s, t) = t - \varphi(z, \bar{z}, s)$ is also a defining function for $M$ near 0. This finishes the proof. \hfill $\Box$

**Remark 4.4.** We note that Proposition 4.3 can be used to give a special local basis of CR vector fields. Indeed, let $M \subseteq \mathbb{C}^N$ be a generic submanifold of codimension $d$ that is given locally near a point $p_0 \in M$ by a defining function $\rho = (\rho_1, \ldots, \rho_d)$. If we use the coordinates $(z, w) \in \mathbb{C}^{n+d}$ from above then we can formally view $\rho$ as a function on the variables $(z, \bar{z}, w, \bar{w})$. Let $\rho_z$, $\rho_{\bar{z}}$, $\rho_w$ and $\rho_{\bar{w}}$ the Jacobi matrices of $\rho$ with respect to $z, \bar{z}, w$ and $\bar{w}$ respectively. We can assume that $\rho_w$ and $\rho_{\bar{w}}$ are invertible in a neighbourhood of $p_0$. According to [3] §1.6 a local basis of CR vector fields near $p_0$ is given by

$$(L) = (\partial_z - \tau_{\rho_w} \rho_{\bar{w}}^{-1} \partial_{\bar{w}})$$

where we have used the following notation

$$\begin{pmatrix} L_1 \\ \vdots \\ L_n \end{pmatrix}, \quad \begin{pmatrix} \partial_z \\ \vdots \\ \partial_{\bar{w}} \end{pmatrix} = \begin{pmatrix} \partial_{z_1} \\ \vdots \\ \partial_{z_n} \end{pmatrix}, \quad \begin{pmatrix} \partial_{\bar{w}} \\ \vdots \\ \partial_{\bar{w}} \end{pmatrix}.$$

If we use the defining function $\rho = t - \varphi$ induced by (4.1) then this local basis is of the following form

$$L_j = \frac{\partial}{\partial z_j} - \sum_{\mu=1}^d 2b_{\mu}^{i_j} \frac{\partial}{\partial w_{i\mu}}$$

with

$$b_{\mu}^{i_j} = \frac{i}{\det \Phi} \frac{\det B_{i_j}^\mu}{\det \Phi}.$$

Here we used

$$\Phi = \rho_w = \begin{pmatrix} 1 + i(\varphi_1)_{s_1} & \cdots & i(\varphi_1)_{s_d} \\ \vdots & \ddots & \vdots \\ i(\varphi_d)_{s_1} & \cdots & 1 + i(\varphi_d)_{s_d} \end{pmatrix}$$

and $B_{i_j}^\mu$ is the following matrix. Let $\delta_{\mu \nu}$ be the Kronecker delta defined by $\delta_{\mu \nu} = 1$ and $\delta_{\mu \nu} = 0$ otherwise and set

$$(\varphi)_{s_\nu} = \begin{pmatrix} \delta_{1\nu} + i(\varphi_1)_{s_\nu} \\ \vdots \\ \delta_{d\nu} + i(\varphi_d)_{s_\nu} \end{pmatrix} \quad \text{and} \quad (\varphi)_{s_j} = \begin{pmatrix} (\varphi_1)_{s_j} \\ \vdots \\ (\varphi_d)_{s_j} \end{pmatrix}.$$

Then

$$B_{i_j}^\mu = \begin{pmatrix} (\varphi)_{s_1} & \cdots & (\varphi)_{s_{\nu-1}} & (\varphi)_{s_j} & (\varphi)_{s_{\nu+1}} & \cdots & (\varphi)_{s_d} \end{pmatrix}. $$

In particular, if $M \subseteq \mathbb{C}^{n+1}$ is a real hypersurface of class $\{M\}$ locally given by the equation $\text{Im} \ w = \varphi(z, \bar{z}, \text{Re} \ w)$ where $\varphi \in \mathcal{E}_M$ then the vector fields

$$L_j = \frac{\partial}{\partial z_j} - 2i \frac{\varphi_{s_j}}{1 + i\varphi_\nu} \frac{\partial}{\partial w} \quad j = 1, \ldots, n$$

form a local basis of the CR vector fields of $M$. When we use the local coordinates $(z, \bar{z}, s)$ of $M$ induced by (4.1) then this basis takes the form

$$L_j = \frac{\partial}{\partial z_j} - i \frac{\varphi_{s_j}}{1 + i\varphi_\nu} \frac{\partial}{\partial s} \quad j = 1, \ldots, n.$$

Next we give a first result on the structure of ultradifferentiable CR manifolds.

**Definition 4.5.** Let $M \subseteq \mathbb{C}^N$ a CR submanifold. The CR orbit $\text{Orb}_p$ of $p \in M$ is the local Sussman orbit of $p$ in $M$ relative to the set of ultradifferentiable sections of $T^c M$.

Note that if $p_0 \in M$ then by construction $T^c_p \text{Orb}_{p_0} = T^c_p M$ for all $p \in \text{Orb}_{p_0}$ thence $\text{Orb}_{p_0}$ is the germ of a CR submanifold of $\mathbb{C}^N$ of CR dimension $n$.

**Definition 4.6.** Let $M \subseteq \mathbb{C}^N$ a CR manifold and $p_0 \in M$.

(1) We say that $M$ is minimal at $p_0$ if there is no submanifold $S \subseteq M$ through $p_0$ such that $T^c_p S \subseteq T^c_p S$ for all $p \in S$ and $\dim_{\mathbb{R}} S < \dim_{\mathbb{R}} M$. 

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We set
$$\{M\}$$
also a local basis of
$$E$$
and thus also
$$\text{dim}_g \text{Orb}_{p_0} = \text{dim}_g \gamma_{p_0}(g) = \text{dim}_g g(p_0)$$.

On the other hand, if
$$\text{dim}_g \text{Orb}_{p_0} < \text{dim}_g M$$
then any representative
$$W$$
of
$$\text{Orb}_{p_0}$$
is by the remark below Definition 4.5 a submanifold of
$$M$$
and
$$T_p^j W = T_p^j M$$
for all
$$p \in W$$. It remains to prove that (2) implies (3).

By Corollary 2.10 we have that
$$\text{Orb}_{p_0} = \gamma_{p_0}(g)$$,
where
$$g$$
is the Lie algebra generated by the ultradifferentiable sections of
$$T^* U$$
with
$$U$$
being a sufficiently small neighbourhood of
$$p_0$$
and
$$\gamma_{p_0}(g)$$
the local Nagano leaf of
$$g$$
at
$$p_0$$. Hence
$$\text{dim}_g \text{Orb}_{p_0} = \text{dim}_g \gamma_{p_0}(g) = \text{dim}_g g(p_0)$$.

We shall note we could have shown the equivalence of (1) and (2) by citing the corresponding proof in [3, Theorem 4.1.3.]. Indeed, let
$$M \subseteq \mathbb{C}^N$$
be an ultradifferentiable CR submanifold of class \( \{M\} \) and
$$p_0 \in M$$. Then we can consider
$$M$$
as also an smooth CR manifold and define similar to [3, \text{Orb}_{p_0}]$$
as the Sussman Orbit relative to the smooth sections of
$$T^* M$$
near
$$p_0$$.

However, if
$$X_1, \ldots, X_n$$
is a local basis of
$$\mathcal{E}_M(M, T^* M)$$
near
$$p_0$$
then we have that
$$\text{Orb}_{p_0}$$
is generated by
$$\mathcal{D} = \langle X_1, \ldots, X_n \rangle$$, c.f. Theorem 2.11. On the other hand, since the vector fields
$$X_1, \ldots, X_n$$
constitute also a local basis of
$$\mathcal{E}(M, T^* M)$$
near
$$p_0$$
we obtain also that
$$\text{Orb}_{p_0}$$
is generated by
$$\mathcal{D}$$. It follows that
$$\text{Orb}_{p_0} = \text{Orb}_{p_0}$$
as germs of manifolds at
$$p_0$$.

The next example is a straightforward generalization of [3, Example 1.5.16.].

Example 4.8. Let
$$\mathcal{M}$$
be a non-quasianalytic weight sequence and
$$\psi \in \mathcal{E}_M(\mathbb{R})$$
a real valued function such that
$$\psi(y) = 0$$
for
$$y \leq 0$$
and
$$\psi(y) > 0$$
for
$$y > 0$$. We define a real hypersurface in \( \mathbb{C}^2 \) by
$$M = \{(z, w) \in \mathbb{C}^2 \mid \text{Im } w = \varphi(\text{Im } z)\}$$.

Then
$$M$$
is minimal at the origin but not of finite type at 0. Indeed, if
$$M$$
is non-minimal at 0 then according to [3, Theorem 1.5.15] there is a holomorphic hypersurface
$$S \subseteq M$$
through the origin. Since
$$\partial / \partial z$$
is tangent to
$$S$$
at 0 it follows that
$$S$$
is given near the origin by the defining equation
$$w = h(z)$$
where
$$h$$
is a holomorphic function defined in some neighbourhood of
$$0 \in \mathbb{C}$$
with
$$h(0) = 0$$. We conclude that due to
$$S \subseteq M$$
we necessarily have that
$$h(z) = \overline{h(z)} = 2i \varphi(\text{Re } z)$$
in some neighbourhood of 0. It follows that
$$\psi$$
has to be real-analytic near 0 which contradicts the definition of
$$\psi$$.

Since
$$\psi$$
is flat at the origin, it follows that
$$M$$
cannot be of finite type at 0.

We close this section by recalling the space of multipliers for an ultradifferentiable abstract CR manifold \((M, \mathbb{V})\), which was introduced by [17] in the smooth setting. To begin with consider the following sequence of spaces of sections
$$E_k = \langle \mathcal{L}_{K_1}, \ldots, \mathcal{L}_{K_j}; j \leq k, K_q \in \mathcal{E}_M(M, \mathbb{V}), \theta \in \mathcal{E}_M(M, T^0 M) \rangle.$$ We note that
$$E_0 = \mathcal{E}_M(M, T^0 M)$$,
and
$$E_j \subseteq \mathcal{E}_M(M, T^j M)$$
for all
$$j \in \mathbb{N}_0$$,
and set
$$E = \bigcup_{j \in \mathbb{N}_0} E_j$$.

We associate to the increasing chain
$$E_k$$
the increasing sequence of ideals
$$S^k \subseteq \mathcal{E}_M(M, \mathbb{C})$$,
where
$$S^k = \bigwedge E_k = \left\{ \begin{array}{c} V^1(\mathbb{V}^1) \cdots V^1(\mathbb{V}^N) V^1(\mathbb{V}^1) \cdots V^1(\mathbb{V}^N) \cdots V^N(\mathbb{V}^1) \cdots V^N(\mathbb{V}^N) \end{array} \right\} : V^j \in E_k, \mathbb{V}^j \in \mathcal{E}_M(M, (T^0 M)^*)$$.

We set
$$S = S(M) = \bigcup_{k \in \mathbb{N}_0} S^k$$
and call it the space of multipliers of
$$M$$. In fact each
$$S^k$$
and thus also
$$S$$
can be considered actually as ideal sheaves, if we define
$$E^k(U)$$
and
$$S^k(U)$$
accordingly.
Note that locally one can find smaller sets of generators: Let \( U \subset M \) be open, and assume that \( L_1, \ldots, L_n \) is a local basis for \( \Gamma(U, \mathcal{V}) \), that \( \theta^1, \ldots, \theta^d \) is a local basis for \( \Gamma(U, T^0 M) \), and that \( \omega^1, \ldots, \omega^N \) is a local basis of \( T^1 M \). We write \( L_j = L_{*j} \) for \( j = 1, \ldots, n \) and \( L^\alpha = L_1^{\alpha_1} \cdots L_n^{\alpha_n} \) for any multi-index \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n \). We note that, since \( \mathcal{V} \) is formally integrable, the \( L^\alpha \), where \( |\alpha| = k \), generate all \( k \)-th order homogeneous differential operators in the \( L_j \), and we thus have

\[
E_k|_U = \langle L^\alpha \theta^\mu : 1 \leq \mu \leq d, |\alpha| \leq k \rangle.
\]

We can expand

\[
L^\alpha \theta^\mu = \sum_{\ell=1}^N A^\alpha_\ell \theta^\ell
\]

and for any choice \( \alpha = (\alpha^1, \ldots, \alpha^N) \) of multi-indices \( \alpha^1, \ldots, \alpha^N \in \mathbb{N}^n \) and \( r = (r_1, \ldots, r_N) \in \{1, \ldots, d\}^N \) we define the functions

\[
D(\alpha, r) = \det \begin{pmatrix} A_1^{\alpha_1 r_1} & \cdots & A_N^{\alpha_1 r_1} \\ \vdots & \ddots & \vdots \\ A_1^{\alpha_N r_N} & \cdots & A_N^{\alpha_N r_N} \end{pmatrix}.
\]

With this notation, we have

\[
S^k|_U = \langle D(\alpha, r) : |\alpha| \leq k \rangle;
\]

we shall denote the stalk of \( S^k \) at \( p \) by \( S^k_p \).

The space of multipliers of a CR manifold \( M \) clearly encodes the nondegeneracy properties of \( M \). We close this section by taking a closer look at the connection of \( S \) with finite nondegeneracy. We recall from [3] the definition of finite nondegeneracy for abstract CR manifolds.

**Definition 4.9.** Let \( M \) be an abstract CR manifold and

\[
E_k(p) = \langle L_{K_1} \cdots L_{K_j} \theta(p) : j \leq k, K_q \in \mathcal{E}(M, \mathcal{V}), \theta \in \mathcal{E}(M, T^0 M) \rangle.
\]

for \( p \in M \) and \( k \in \mathbb{N} \). Then \( M \) is \( k_0 \)-nondegenerate at \( p_0 \in M \) if \( E_{k_0-1} \subseteq E_{k_0} = T^1_{p_0} M \). We say that \( M \) is finite nondegenerate iff \( M \) is finite nondegenerate at every point.

**Remark 4.10.** This definition is in fact local, since by [3] Proposition 11.1.10. if \( L_1, \ldots, L_n \) is a local basis of CR vector fields and \( \theta^1, \ldots, \theta^d \) is a local basis of characteristic forms near \( p_0 \) then \( M \) is \( k_0 \)-nondegenerate if and only if

\[
T^1_{p_0} M = \operatorname{span}_C \{ L^\mu \theta^\mu(p_0) : |\alpha| \leq k_0, \mu \in \{1, \ldots, d\} \}.
\]

Hence we may replace \( M \) with any open neighbourhood \( U \subseteq M \) of \( p_0 \) in (4.6). Thus we observe that a CR submanifold \( M \) is \( k_0 \)-nondegenerate at \( p_0 \in M \) if and only if \( S^k_{p_0} = \langle E_{k_0}(p) \rangle_{p_0} \).

More precisely, let \( U \subseteq M \) be an open subset and \( q \in U \). Then \( M \) is \( k_0 \)-nondegenerate at \( q \) if and only if there is a multiplier \( f \in S^{k_0}(U) \) that does not vanish at \( q \), i.e. \( f(q) \neq 0 \).

Indeed, if \( f(q) \neq 0 \) then obviously \( E_{k_0}(q) = T_q^1 M \). On the other hand, if \( g(q) = 0 \) for all multipliers \( g \in S^{k_0}(U) \) then necessarily \( E_{k_0}(q) \neq T_q^1 M \).

5. Ultradifferentiable regularity of CR mappings

The main goal of this section is to present the proof of Theorem [11]. Furthermore we show also ultradifferentiable versions of further regularity results of [28] and [4]. However, first we need to recall the definition of finite nondegeneracy of a CR mapping.

**Definition 5.1.** Let \( M \) be an abstract CR manifold and \( M' \subseteq C^{N'} \) a generic submanifold. Furthermore let \( \rho' = (\rho'_{1}, \ldots, \rho'_{d'}) \) be a defining function of \( M' \) near a point \( q_0 \in M' \), \( L_1, \ldots, L_n \) a local basis of CR vector fields on \( M \) near \( p_0 \in M \) and \( H : M \to M' \) an \( \mathcal{C}^{\infty} \)-CR mapping with \( H(p_0) = q_0 \).

For \( 0 \leq k \leq m \) define an increasing sequence of subspaces \( E_k(p_0) \subseteq C^{N'} \) by

\[
E_k(p_0) := \operatorname{span}_C \left\{ L^\alpha \partial_{\rho'}(H(Z))(H(Z)|_{Z=p_0} : 0 \leq |\alpha| \leq k, 1 \leq l \leq d' \right\}.
\]

We say that \( H \) is \( k_0 \)-nondegenerate at \( p_0 \) \((0 \leq k_0 \leq m)\) if \( E_{k_0-1}(p_0) \subseteq E_{k_0}(p_0) = C^{N'} \).

**Remark 5.2.** Comparing Definition 5.1 with Definition 4.9 we observe that a CR submanifold \( M \in C^{N} \) is \( k_0 \)-nondegenerate if and only if \( \text{id} : M \to M \) is \( k_0 \)-nondegenerate. We note also the fact that any CR diffeomorphism between two \( k_0 \)-nondegenerate CR submanifolds is \( k_0 \)-nondegenerate.
Finally we need to recall that if $\rho$ is a local defining function of $M$, $\Gamma \subseteq \mathbb{R}^d$ an open convex cone, $p_0 \in M$ and $U \subseteq \mathbb{C}^n$ an open neighbourhood of $p_0$, then a wedge $W$ with edge $M$ centered at $p_0$ is an open subset of the form $W := \{ Z \in U \mid \rho(Z, \bar{Z}) \in \Gamma \}$.

**Proof of Theorem 1.1.** Since the assertion of the theorem is local, we are going to work on a neighbourhood $\Omega \subseteq \mathbb{C}^n$ of $p_0$. If $\Omega$ is small enough then by Proposition 3.3 there are open neighbourhoods $U \subseteq \mathbb{C}^n$ and $V \subseteq \mathbb{R}^d$ of the origin and a function $\varphi \in \mathcal{E}_M(U \times V, \mathbb{R}^d)$ with $\varphi(0,0) = 0$ and $\nabla \varphi(0,0) = 0$ such that

$$M \cap \Omega = \{(z, w) \in \Omega \mid \text{Im } w = \varphi(z, \bar{z}, \text{Re } w)\}.$$  

From now we denote $M \cap \Omega$ by $M$. If we choose $U$ and $V$ to be small enough we can consider the diffeomorphism

$$\Psi : U \times V \longrightarrow M$$

$$(z, s) \longmapsto (z, s + i\varphi(z, \bar{z}, s)).$$

If we shrink the neighbourhoods $U, V$ a little bit (such that $\varphi \in \mathcal{E}_M(U \times V, \mathbb{R}^d)$) and assume that w.l.o.g. both sets are convex we can extend the mapping $\Psi$, $\mathcal{M}$-almost analytically in the $s$-variables, i.e. there exists a smooth function $\tilde{\Psi} : U \times V \times \mathbb{R}^d$ and we can extend the mapping $\Psi$, $\mathcal{M}$-almost analytically in the $s$-variables, i.e. there exists a smooth function $\tilde{\Psi} : U \times V \times \mathbb{R}^d$ and for each component $\Psi_k$, $k = 1, \ldots, N$, of $\Psi$ we have

$$\left| \frac{\partial \tilde{\Psi}_k}{\partial \bar{w}_j}(z, \bar{z}, s, t) \right| \leq Ch_{\mathcal{M}}(\gamma |t|) \quad j = 1, \ldots, d,$$

for some constants $C, \gamma > 0$. Here $w' = s + it \in V + i\mathbb{R}^d$. We see that there is some $r > 0$ such that $\tilde{\Psi}_{U \times V \times \mathbb{R}^d(0)}$ is a diffeomorphism.

By assumption $H = (H_1, \ldots, H_N)$ extends continuously to a holomorphic mapping on a wedge $W$ near $0$. If we shrink $W$ we may assume that $\partial H_j, j = 1, \ldots, N'$, is bounded on $W$. By definition

$$W = \{ Z \in \Omega_0 \mid |\rho(Z, \bar{Z})| \leq 1 \}$$

for a neighbourhood $\Omega_0$ of the origin in $\mathbb{C}^n$ and an open acute cone $\Gamma \subseteq \mathbb{R}^d$. If we shrink $U, V$, when necessary, and choose a suitable open and acute cone $\Gamma$, we achieve that

$$\tilde{\Psi}(U \times V \times \Gamma) \subseteq W$$

for some $r \geq \delta > 0$. Note that $\tilde{\Psi}(U \times V \times \Gamma)$ is open in $\mathbb{C}^n$. For each $j = 1, \ldots, N'$ set $h_j = H_j \circ \tilde{\Psi}$ and $u_j = h_j \circ \Psi$. Since

$$\frac{\partial h_j}{\partial u_k} = \sum_{\ell=1}^N \frac{\partial H_j}{\partial \bar{w}_\ell} \frac{\partial \tilde{\Psi}_\ell}{\partial \bar{w}_k} \quad j = 1, \ldots, N', \ k = 1, \ldots, d,$$

and $\partial H_j$ is bounded, each function $h_j$ is $\mathcal{M}$-almost analytic on $U \times V \times \Gamma$ due to (5.1) and extends $u_j \in \mathcal{C}^{k_0}(U \times V)$. Hence Theorem 3.3 implies

$$WF_{\mathcal{M}} u_j \subseteq (U \times V) \times (\mathbb{R}^{2n} \times \Gamma) \setminus \{0\}. \quad (5.2)$$

If $L_j, j = 1, \ldots, n$, is a basis of the CR vector fields on $M = M \cap \Omega$, then $\Lambda_j \circ u_k = 0$ for $j = 1, \ldots, n$ and $k = 1, \ldots, N'$.

Let $\rho'$ be a defining function of $M'$ near $p_0' = 0 \in \mathbb{C}^{N'}$. Then there are ultradifferentiable functions $\Phi_{\ell, \alpha}(Z', \bar{Z}', W)$ for $|\alpha| \leq k_0$, $\ell = 1, \ldots, d'$, defined in a neighbourhood of $\{0\} \times \mathbb{C}^{k_0} \subseteq \mathbb{C}^{N'} \times \mathbb{C}^{k_0}$ and polynomial in the last $K_\mathcal{O} = N' \cdot |\{ \alpha \in \mathbb{N}_0^d \mid |\alpha| \leq k_0 \} | \{ |\beta| \leq k_0 \}$ variables such that

$$\Lambda_\alpha(\rho' \circ u)(z, \bar{z}, s) = \Phi_{\ell, \alpha}(u(z, \bar{z}, s), \bar{u}(z, \bar{z}, s), (\Lambda^\beta \bar{u}(z, \bar{z}, s))|_{\beta \leq k_0}) = 0 \quad (5.3)$$

and

$$\Lambda_\alpha(\rho' \circ u)(0, 0, 0) = \Phi_{\ell, \alpha, Z'}(0, 0, (\Lambda^\beta \bar{u}(0, 0, 0))|_{\beta \leq k_0}).$$

Since $H$ is $k_0$-nondegenerate there are multi-indices $\alpha_1, \ldots, \alpha_{N'}$ and $\ell_1, \ldots, \ell_{N'} \in \{1, \ldots, d'\}$ such that if we set

$$\Phi = (\Phi_{\ell_1, \alpha_1}, \ldots, \Phi_{\ell_{N'}, \alpha_{N'}})$$

the matrix $\Phi_{Z'}$ is invertible. Hence by Theorem 3.3 there is a smooth function $\phi = (\phi_1, \ldots, \phi_{N'})$ defined in a neighbourhood of $(0, (\Lambda^\beta \bar{u}(0, 0, 0))|_{\beta \leq k_0})$ in $\mathbb{C}^{N'} \times \mathbb{C}^{k_0}$ such that, if we shrink $U \times V$ accordingly,

$$u_j(z, \bar{z}, s) = \phi_j(u(z, \bar{z}, s), \bar{u}(z, \bar{z}, s), (\Lambda^\beta \bar{u}(z, \bar{z}, s))|_{\beta \leq k_0}) \quad (z, s) \in U \times V, \ j = 1, \ldots, N'$$

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and (5.3) holds. If we further shrink $U \times V$ and $\delta$ and choose $\Gamma' \subset \Gamma$ appropriately we see that
\[
g_j(z, \bar{z}, s, t) = \phi_j(h(z, \bar{z}, s, t), \bar{h}(z, \bar{z}, s, t)), (\bar{h}_{k, \beta}(z, \bar{z}, s, t))_{k, \beta \in \Gamma_0} \tag{5.4}
\]
is well defined for $t \in -\Gamma'_s$. Here $\bar{h}_{j, \beta}$ is the $M$-almost analytic extension of $\Lambda^d \bar{u}_j$ on $U \times V \times (-\Gamma'_s)$, which exists due to (5.2), (5.10), Proposition 5.3 and Theorem 5.7. It is also easy to see that $h(z, \bar{z}, s, -t)$ is $M$-almost analytic on $U \times V \times (-\Gamma'_s)$. We have that
\[
\frac{\partial g_j}{\partial w'_k} = \sum_{k=1}^{N'} \frac{\partial \phi_j}{\partial w'_k} + \sum_{k=1}^{N'} \frac{\partial \phi_j}{\partial \bar{w}'_k} + \sum_{k=1}^{N'} \sum_{|\beta| \leq \kappa_0} \frac{\partial \phi_j}{\partial W_{k, \beta}} \frac{\partial \bar{h}_{k, \beta}}{\partial w'_k} \tag{5.5}
\]
for $j = 1, \ldots, N'$ and $\ell = 1, \ldots, d$. Note that we can choose $U \times V$ and $\Gamma'_s$ so small that all functions appearing on the right-hand side are uniformly bounded. Hence, since $\partial w'_j = \partial \bar{w}'_k$, the last two terms on the right hand side of (5.5) are $M$-almost analytic. The estimates (5.4) and the arguments in Section 3.3 give that the first sum on the right hand side of (5.5) is also $M$-almost analytic. We conclude that $g_j$ is an $M$-almost analytic extension on $U \times V \times (-\Gamma'_s)$ of $u_j$ and thus
\[
WF_M u_j \subseteq (U \times V) \times (\mathbb{R}^n \times (\Gamma' \cup -\Gamma'^c)) \setminus \{0\} = (U \times V) \times (\mathbb{R}^n \setminus \{0\} \times \{0\}).
\]

On the other hand, since each $u_j$ is CR we have that $WF_M u_j|_{\{0\}} \subseteq \{0\} \times \mathbb{R}^d \setminus \{0\}$ by (5.10) and we deduce that in fact $WF_M u_j|_{\{0\}} = \emptyset$ for all $j = 1, \ldots, N'$. Hence the mapping $H$ is ultradifferentiable of class $\{M\}$ near $p_0$. $\Box$

If we recall the well-known result of Tumanov [13] which states that any CR function on a minimal CR submanifold $M$ extends to a holomorphic function on a wedge with edge $M$, then we obtain the following corollary.

**Corollary 5.3.** Let $M \subseteq \mathbb{C}^N$ and $M' \subseteq \mathbb{C}^{N'}$ generic submanifolds of class $\{M\}$, $p_0 \in M$, $p'_0 \in M'$, $M$ minimal at $p_0$ and $H : (M, p_0) \to (M', p'_0)$ a $C^{k_0}$-CR mapping that is $k_0$-nondegenerate at $p_0$. Then $H$ is ultradifferentiable of class $\{M\}$ in some neighbourhood of $p_0$.

This leads to the following result.

**Corollary 5.4.** Let $M \subseteq \mathbb{C}^N$ and $M' \subseteq \mathbb{C}^{N'}$ generic submanifolds of class $\{M\}$ that are $k_0$-nondegenerate at $p_0 \in M$ and $p'_0 \in M'$, respectively. Furthermore assume that $M$ is minimal at $p_0$ and let $H : M \to M'$ a CR diffeomorphism that is $C^{k_0}$ near $p_0$ and satisfies $H(p_0) = p'_0$. Then $H$ has to be ultradifferentiable of class $\{M\}$ near $p_0$.

Recently Berhanu-Xiao [3] showed that it is possible to slightly weaken the prerequisites of the smooth reflection principle of Lamel. In particular, the source manifold $M$ can be chosen to be an abstract CR manifold. Using the methods developed previously we can also generalize this result to the ultradifferentiable category.

**Theorem 5.5.** Let $(M, V)$ be an abstract CR manifold and $M' \subseteq \mathbb{C}^{N'}$ be a generic submanifold, both of class $\{M\}$. Furthermore let $p_0 \in M$, $H : M \to M'$ a $C^{k_0}$-CR mapping that is $k_0$-nondegenerate at $p_0$ and there is a closed acute cone $\Gamma \subseteq \mathbb{R}^d$ such that $WF_M H|_{p_0} \subseteq \{0\} \times \Gamma$. Then $H$ is ultradifferentiable of class $\{M\}$ near $p_0$.

**Proof.** Since the assertion is local we will work on a small chart neighbourhood $\Omega = U \times V \times W \subseteq \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^d$ of $M$ of $p_0 = 0$. Here $n$ denotes the CR-dimension of $M$ whereas $d$ is the CR-codimension of $M$. We use coordinates $(x, y, s)$ on $\Omega$ and write $z = x + iy$. In these coordinates a local basis of the CR vector fields of $M$ is given by
\[
L_j = \frac{\partial}{\partial z_j} + \sum_{k=1}^n a_{jk} \frac{\partial}{\partial z_k} + \sum_{\alpha=1}^d b_{j\alpha} \frac{\partial}{\partial s_\alpha} \quad j = 1, \ldots, n.
\]

From the assumptions we conclude that if $\Omega$ is small enough that there is an open, convex cone $\Gamma_1 \subseteq \mathbb{R}^N \setminus \{0\}$ such that
\[
WF_M H = \bigcup_{j=1}^{N'} WF_M H_j \subseteq \Omega \times \Gamma_1^c \tag{5.6}
\]
due to the closedness of $WF_M H$ in $T^* M \setminus \{0\}$. If we further shrink $\Omega$ (resp. $U$, $V$ and $W$) and choose an open convex cone $\Gamma_2 \subseteq \mathbb{R}^N \setminus \{0\}$ such that $\overline{\Gamma_2} \subseteq \Gamma_1 \cup \{0\}$ we have by Theorem 5.7 that there is
an $\mathcal{M}$-almost extension $\tilde{F}$ with slow growth of $H$ onto $\Omega \times \Gamma_2$. If we now choose an open convex cone $\Gamma_3 \subseteq \mathbb{R}^d \setminus \{0\}$ with $\{0\} \times \Gamma_3 \subseteq \Gamma_2$ we infer that

$$F := \tilde{F}|_{\Omega \times \{0\} \times \Gamma_3}$$

is an $\mathcal{M}$-almost analytic function on $U \times V \times W \times \Gamma_3$ with values in $\mathbb{C}^{N'}$ and

$$\lim_{\gamma_3 \rightarrow t \rightarrow 0} F(\cdot, \ldots, t) = H$$

in the sense of distributions.

Let $\rho' = (\rho'_1, \ldots, \rho'_n)$ be an ultradifferentiable defining function of $M'$ near $\rho'_0 = H(p_0)$. As before in the proof of Theorem 1.1 we conclude that there are ultradifferentiable functions $\Phi_{t,\alpha}(Z', Z', W)$ for $|\alpha| \leq k_0$, $\ell = 1, \ldots, d'$, defined in a neighbourhood of $\{0\} \times \mathbb{C}^{K_0} \subset \mathbb{C}^{N'} \times \mathbb{C}^{K_0}$ and polynomial in the last $K_0 = N'[\alpha \in \mathbb{N}_0^{n'} \mid |\alpha| \leq k_0]$ variables. From now on we can follow the proof of Theorem 1.1 verbatim.

6. Ultradifferentiable regularity of infinitesimal CR automorphisms

In this section we show how the results in [17] concerning the smoothness of infinitesimal CR automorphisms transfer to the ultradifferentiable setting. Since our presentation here differs in some details from that given in [17] we first recall the framework we are going to work in. In this section $(M, V)$ is always an ultradifferentiable abstract CR manifold of class $[M]$.

**Definition 6.1.** Let $U \subseteq M$ an open subset and $X : U \rightarrow TM$ a vector field of class $C^1$. We say that $X$ is an infinitesimal CR automorphism iff its flow $H^\tau$, defined for small $\tau$, has the property, that there is $\varepsilon > 0$ such that $H^\tau$ is a CR mapping provided that $|\tau| \leq \varepsilon$.

We need for the proofs of the regularity results a more suitable characterization of infinitesimal CR automorphisms. We call a section $\mathfrak{V} \in \Gamma(M, (T'M)^*)$ a holomorphic vector field on $M$.

Apparently every vector field $X \in \Gamma(M, TM)$ gives rise to a holomorphic vector field by first extending $X$ to $\mathbb{C}TM$ and then restricting the extension to $T^*M$. For a partial converse, we recall from [17] the following purely algebraic result.

**Lemma 6.2.** Let $\mathfrak{V} \in \Gamma(M, (T'M)^*)$. Then there exists a unique vector field $X \in \Gamma(M, TM)$ such that $\mathfrak{V}$ is induced by $X$ if and only if $\mathfrak{V}(\tau) = \mathfrak{V}(\tau)$ for all characteristic forms $\tau$.

From now on we shall not distinguish between $X$ being a real vector field or a holomorphic vector field.

We recall the well-known identity, see e.g. [19],

$$\mathcal{L}_X \alpha(Y) = d\alpha(X, Y) + Y\alpha(X) = X\alpha(Y) - \alpha([X, Y]),$$

which holds for arbitrary complex vector fields $X, Y$ and complex forms $\alpha$ on smooth manifolds.

We conclude that accordingly the Lie derivative

$$\mathcal{L}_L \omega(\cdot) = d\omega(L, \cdot)$$

of a holomorphic form $\omega$ with respect to a CR vector field $L$ is again a holomorphic form. It is now possible to make the following definition. We shall say that a holomorphic vector field $\mathfrak{V} \in \Gamma(M, (T'M)^*)$ is CR iff

$$L_\omega(\mathfrak{V}) = d\omega(L, \mathfrak{V})$$

for every CR vector field $L$ and holomorphic form $\omega$. In particular a real vector field $X$ is CR if and only if

$$\omega([L, X]) = 0$$

for all CR vector fields $L$ and holomorphic forms $\omega$. We recall from [17] the following fact.

**Proposition 6.3.** If $X$ is an infinitesimal CR automorphism on $M$, then $X$ considered as a holomorphic vector field, i.e. $X \in \mathcal{C}^1(M, (T'M)^*)$ is CR.

We are now able to generalize the notion of infinitesimal CR automorphism. To this end consider the space $\mathcal{D}'(M, (T'M)^*)$ of distributions with values in $(T'M)^*$.

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Definition 6.4. An infinitesimal CR diffeomorphism with distributional coefficients on $M$ is a generalized holomorphic vector field $\mathcal{Y} \in \mathcal{D}^\prime(M,(T^\prime M)^\ast)$ that satisfies
\[ L_\omega(\mathcal{Y}) = (L_\omega(L)) \] (6.1)
for every CR vector field $L$ and holomorphic form $\omega$ and
\[ \mathcal{Y}(\tau) = \mathcal{Y}(\tau) \] (6.2)
for all characteristic forms $\tau$.

Note that (6.1) is in fact a CR equation for $\mathcal{Y}$. If $U \subseteq M$ is an open subset of $M$ then we say that $\mathcal{Y} \in \mathcal{D}^\prime(M,(T^\prime M)^\ast)$ is an infinitesimal CR automorphism on $U$ iff (6.1) and (6.2) hold for all local sections $L \in \mathcal{E}_M(U,V|U)$ and $\theta \in \mathcal{E}_M(U,T^0M|U)$, respectively. Let the subset $U \subset M$ be small enough such that there is a local basis $L_1, \ldots, L_n$ of CR vector fields and also a local basis $\{\omega^1, \ldots, \omega^N\}$ of the space of holomorphic forms. We recall that locally a distribution $\mathcal{Y} \in \mathcal{D}^\prime(M,(T^\prime M)^\ast)$ is of the form
\[ \mathcal{Y}|_U = \sum_{j=1}^N X_j \omega^j \] (6.3)
with $X_j \in \mathcal{D}^\prime(U)$. We introduce also the following operators on $U$
\[ L_j = L_j \cdot \text{Id}_N = \begin{pmatrix} L_j & 0 \\ \vdots & \ddots \\ 0 & L_j \end{pmatrix} \]
and note that since $d\omega^k(L_j, \cdot)$ is again a holomorphic form we have
\[ d\omega^k(L_j, \cdot) = \sum_{\ell=1}^N B^k_{j,\ell} \omega^\ell \]
with $B^k_{j,\ell} \in \mathcal{E}_M(U)$. We observe that $\mathcal{Y}$ is CR on $U$ if and only if
\[ L_j X_k = L_j (\omega^k(\mathcal{Y})) = d\omega^k(L_j, \mathcal{Y})) = \sum_{\ell=1}^N B^k_{j,\ell} X_\ell \]
for all $1 \leq j \leq n$ and $0 \leq k \leq N$. We set
\[ B_j = \begin{pmatrix} B^1_{j,1} & \cdots & B^1_{j,N} \\ \vdots & \ddots & \vdots \\ B^N_{j,1} & \cdots & B^N_{j,N} \end{pmatrix}. \]

Furthermore, using its local representation (6.3), we can identify $\mathcal{Y}$ with the vector $X = (X_1, \ldots, X_N)$. Hence (6.1) turns into
\[ L_j X = B_j \cdot X \]
or
\[ P_j X = 0 \]
respectively, where
\[ P_j = L_j - B_j \]
In particular we infer from above and Theorem 3.10 that
\[ \text{WF}_M(\mathcal{Y}) \subseteq T^0M. \] (6.4)

Definition 6.5. Let $(M,\mathcal{V})$ be an ultradifferentiable abstract CR manifold of class $\{M\}$, and $\mathcal{Y}$ an infinitesimal CR diffeomorphism with distributional coefficients of $M$.

We say that $\mathcal{Y}$ extends microlocally to a wedge with edge $M$ if there exists a set $\Gamma \subseteq T^0M$ such that for each $p \in M$, the fiber $\Gamma_p \subseteq T^0M \setminus \{0\}$ is a closed, convex cone, and
\[ \text{WF}_M(\omega(\mathcal{Y})) \subseteq \Gamma \]
for every holomorphic form $\omega \in \mathcal{E}_M(M,T^\prime M)$.

Note that the condition $\Gamma \subseteq T^0M$ is not as strict as it seems, because $\text{WF}_M(\omega(\mathcal{Y})) \subseteq T^0M$ by (6.3).
Theorem 6.6. Let \((M,\mathcal{V})\) be an ultradifferentiable abstract CR structure of class \(\{\mathcal{M}\}\), and \(\mathcal{Q}\) an infinitesimal CR diffeomorphism of \(M\) with distributional coefficients which extends microlocally to a wedge with edge \(M\).

Then, for any \(\omega \in E\), the evaluation \(\omega(\mathcal{Q})\) is ultradifferentiable, and for any \(\lambda \in \mathcal{S}\), the vector field \(\lambda \mathcal{Q}\) is also of class \(\{\mathcal{M}\}\).

Proof. Since the assertion is local we will work in a suitable small open set \(U \subseteq M\) such that there are local bases \(L_1, \ldots, L_n\) of \(E_M(U,\mathcal{Y})\) and \(\omega^1, \ldots, \omega^N\) of \(E_M(U,T^0M)\), respectively. We recall that we can represent \(\mathcal{Q}\) on \(U\) by \((\mathcal{L},\mathcal{X}) = (X_1, \ldots, X_N) \in \mathcal{D}'(U,\mathcal{C}^N)\). By assumption we know that there is a closed convex cone \(\Gamma \subseteq T^0M\setminus \{0\}\) such that \(WF_{M_j}X_j \subseteq \Gamma\) for each \(j = 1, \ldots, N\). If we set \(W^+ = (\Gamma)^c \subseteq T^0M\setminus \{0\}\), then \(WF_{M_j}X_j \cap W^+ = \emptyset\) for all \(j = 1, \ldots, N\). We may refer to this fact by saying that \(X\) extends above. On the other hand, if we analogously put \(W^- = (-\Gamma)^c \subseteq T^0M\setminus \{0\}\) then \(WF_{M_j}X_j \cap W^- = \emptyset\) by \((6.3)\); we say that \(X\) extends below.

Furthermore let \(\{\theta^1, \ldots, \theta^d\}\) be a generating set of \(E_M(U,T^0M)\) and recall \((6.4)\), i.e.

\[
L^\alpha \theta^\nu = \sum_{\ell=1}^N A^\alpha_\ell \omega^\ell
\]

with \(A^\alpha_\ell \in E_M(U)\) for \(\alpha \in \mathbb{N}_0^n\) and \(\nu = 1, \ldots, d\). In particular, \((6.2)\), i.e. \(\theta(\mathcal{Q}) = \theta(\mathcal{Q})\), turns into

\[
\sum_{\ell=1}^N A^\alpha_\ell X_\ell = \sum_{\ell=1}^N A^\alpha_\ell X_\ell
\]

and applying \(L^\alpha\) to \((6.2)\) yields

\[
\sum_{\ell=1}^N A^\alpha_\ell X_\ell = \sum_{\ell=1}^N \sum_{|\alpha| \leq |\alpha|} C^\beta_\ell L^\beta X_\ell,
\]

where \(C^\beta_\ell \in E_M(U)\). Note that in both equations above the left hand side extends above, while the right hand side extends below.

Now choose any \(N\)-tuple \(\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{N}_0^n\) of multi-indices with \(|\alpha| \leq k\) for all \(j = 1, \ldots, N\) and \(r = (r_1, \ldots, r_N) \in \{1, \ldots, d\}^N\). Then we have

\[
\begin{pmatrix}
A_{1,r}^{1} & \cdots & A_{1,r}^{N}\n\vdots & \ddots & \vdots
\end{pmatrix}
\begin{pmatrix}
X_1 \\
\vdots \\
X_N
\end{pmatrix} = \begin{pmatrix}
\sum C_{\beta}^{1,r} L^\beta X_1 \\
\vdots \\
\sum C_{\beta}^{N,r} L^\beta X_N
\end{pmatrix}.
\]

If we multiply the equation with the classic adjoint of the matrix

\[
\begin{pmatrix}
A_{1,r}^{1} & \cdots & A_{1,r}^{N}\n\vdots & \ddots & \vdots
\end{pmatrix}
\begin{pmatrix}
A_{1,r}^{1} & \cdots & A_{1,r}^{N}\n\vdots & \ddots & \vdots
\end{pmatrix}
\]

then we obtain

\[
D(\alpha, r)X_j = \sum_{|\beta| \leq k} D^{\beta}_{\ell,j} L^\beta X_j
\]

for each \(j = 1, \ldots, N\) where the \(D^{\beta}_{\ell,j}\) are ultradifferentiable functions on \(U\). It follows that the right hand side of this equation extends below, whereas the left hand side obviously extends above. Hence \(WF_{M_j}D(\alpha, r)X = \emptyset\). We conclude that \(\lambda X \in E_M(U)\) for any \(\lambda \in \mathcal{S}^k(U)\) since \(\mathcal{S}^k(U)\) is generated by the functions \(D(\alpha, r)\).

The next statement is an obvious corollary of Theorem 6.6.

Corollary 6.7. Let \((M,\mathcal{Y})\) be finitely nondegenerate and \(X\) an infinitesimal CR diffeomorphism of \(M\) with distributional coefficients which extends microlocally to a wedge with edge \(M\). Then \(X\) is ultradifferentiable of class \(\{\mathcal{M}\}\).

However, the condition that \(M\) is actually finitely nondegenerate is far too restrictive. We shall say that \((M,\mathcal{Y})\) is CR-regular if for every \(p \in M\) there exists a multiplier \(\lambda \in \mathcal{S}\) with the property that near \(p\), the zero set of \(\lambda\) is a finite intersection of real hypersurfaces in \(M\), and such that \(\lambda\) does not vanish to infinite order at \(p\). Hence we can apply Proposition 2.13 or Corollary 2.14 respectively.
Theorem 6.8. Let $(M, V)$ be an abstract CR structure, $p \in M$, and assume that $M$ is CR-regular near $p$. Then any locally integrable infinitesimal CR diffeomorphism $X$ of $M$ which extends microlocally to a wedge with edge $M$ is of class $\{M\}$ near $p$.

In general it might be difficult to determine if a certain CR manifold is CR-regular. In the forthcoming we want to present some instances of CR-regular manifolds. But first we take a closer look at the Lie derivatives of characteristic forms.

Suppose that $M$ is a CR manifold and near a point $p_0 \in M$ there are local coordinates $(x, y, s)$ of $M$ such that the vector fields

$$L_j = \frac{\partial}{\partial x_j} - \sum_{\tau=1}^d b^\tau_j \frac{\partial}{\partial s^\tau}, \quad j = 1, \ldots, n, \quad z_j = x_j + y_j,$$

(6.5)

where $b^\tau_j \in \mathcal{E}_M$, are a local basis of CR vector fields near $p_0$. In this setting (c.f. Remark 4.4) the characteristic bundle is spanned by the forms

$$\theta^\tau = ds^\tau + \sum_{j=1}^n b^\tau_j dz_j + \sum_{j=1}^n b^\tau_j dz_j, \quad \tau = 1, \ldots, d.$$

Furthermore, the forms $\theta^\tau, \tau = 1, \ldots, d,$ and $\omega^j = dz_j, j = 1, \ldots, n,$ constitute a local basis of holomorphic forms on $M$ near $p_0$. We also define the functions

$$\lambda^j_{\mu} := L_k b^k_{\mu} - \tilde{L}_j b^k_{\mu}$$

for $j, k = 1, \ldots, n$ and $\mu = 1, \ldots, d$.

Consider a general holomorphic form

$$\eta = \sum_{\mu=1}^d \sigma_\mu \theta^\mu + \sum_{j=1}^n \rho_j \omega^j.$$

The Lie derivative of $\eta$ with respect to the CR vector field $L_k$ is

$$L_k \eta = d\eta(L_k, \cdot) = \sum_{\mu=1}^d \left( L_k \sigma_\mu - \sum_{\nu=1}^d \sigma_\nu (b^k_\nu)_s \right) \theta^\mu + \sum_{j=1}^n (L_k \rho_j + \sum_{\mu=1}^d \sigma_\mu \lambda^j_{\mu}) \omega^j.$$

(6.6)

Let $\alpha \in N_0^n$ a multi-index of length $|\alpha| = m$. We introduce the finite sequence $m_j := \sum_{\ell \leq j} \alpha_\ell, j = 1, \ldots, n$, and set $m_0 := 0$ and associate to $\alpha$ the function $p_\alpha : \{0, 1, \ldots, m\} \to \{0, 1, \ldots, n\}$ which is defined by

$$p_\alpha(\ell) = j \quad \text{if} \quad \ell \in (m_{j-1}, m_j)$$

for $\ell = 1, \ldots, m$ and $p_\alpha(0) = 0$. We also associate the following sequences of multi-indices to $\alpha$

$$\alpha(\ell) := \sum_{q \leq \ell} c_{p_\alpha(q)} \quad \ell = 0, 1, \ldots, m,$$

$$\hat{\alpha}(\ell) := \sum_{q > \ell} c_{p(q)},$$

where $c_j$ is the $j$-th standard unit vector in $\mathbb{R}^n$.

With this notation and (6.6) we can now state what the Lie derivative of the characteristic form $\theta^\mu$ ($\mu = 1, \ldots, d$) is:

$$L^\alpha \theta^\mu = \sum_{\tau=1}^d T^\alpha_{\tau} \theta^\tau + \sum_{j=1}^n A_j^{\alpha, \mu} \omega^j$$

(6.7)

The functions $T^\alpha_{\tau}$ and $A_j^{\alpha, \mu}$ are defined iteratively by

$$T^\alpha_{\tau} = \delta^\alpha_{\tau},$$

(6.8a)

$$T^\alpha_{\tau} = L_{p_\alpha(1)} T^\alpha_{\tau} - \sum_{\nu=1}^d (b^\nu(1))_s T^\alpha_{\tau}$$

and

$$A_j^{\alpha, \mu} = \sum_{k=1}^m \sum_{\nu=1}^d L^\alpha(\nu, k) \left( T^{\alpha, \mu, \nu} \lambda^j_{\nu, p_{\alpha}(k)} \right).$$

(6.8b)

We are now able to give the first example of a CR regular submanifold of $\mathbb{C}^N$. 

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Definition 6.9. We say that a real hypersurface $M \subseteq \mathbb{C}^N$ is weakly nondegenerate at $p_0$ iff there exist coordinates $(z, w) \in \mathbb{C}^n \times \mathbb{C}$ near $p_0$ and numbers $k, m \in \mathbb{N}$ such that $p_0 = 0$ in these coordinates and near $p_0$ $M$ is given by an equation of the form

$$\text{Im } w = (\text{Re } w)^m \varphi(z, \bar{z}, \text{Re } w),$$

where

$$\frac{\partial^{(\alpha)} \varphi}{\partial z^{\alpha}}(0, 0, 0) = 0, \quad |\alpha| \leq k,$n

and

$$\overline{\text{span}}_{\mathbb{C}} \{ \varphi_{z^{\alpha}}(0, 0, 0) : |\alpha| \leq k \} = \mathbb{C}^n.$$

If $k_0$ is the smallest $k$ for which the preceding condition holds, we say that $M$ is weakly $k_0$-nondegenerate at $p_0$.

Proposition 6.10. Let $M \subseteq \mathbb{C}^N$ be an ultradifferentiable real hypersurface, $p_0 \in M$, and assume that $M$ is weakly $k_0$-nondegenerate at $p_0$. Then $M$ is CR regular near $p_0$. In particular, any locally integrable infinitesimal CR diffeomorphism of $M$ which extends microlocally to a wedge with edge $M$ near $p_0$ is ultradifferentiable near $p_0$.

Proof. In order to show that $M$ is CR regular we are going to construct a multiplier $\lambda \in S$ of the form

$$\lambda(z, \bar{z}, s) = s^k \psi(z, \bar{z}, s)$$

in suitable local coordinates and with $\psi \in \mathcal{E}_M$ not vanishing at $s = 0$ and $\ell \in \mathbb{N}$.

Recall that by assumption there are coordinates $(z, w) \in \mathbb{C}^n \times \mathbb{C}$ such that $p_0 = 0$ and $M$ is given locally by

$$\text{Im } w = (\text{Re } w)^m \varphi(z, \bar{z}, \text{Re } w)$$

where $m \in \mathbb{N}$ and $\varphi$ is an ultradifferentiable real-valued function defined near $0$ with the property that $\varphi_{z^{\alpha}}(0) = \varphi_{\bar{z}^{\alpha}}(0) = 0$ for $|\alpha| \leq k_0$ and

$$\overline{\text{span}}_{\mathbb{C}} \{ \varphi_{z^{\alpha}}(0, 0, 0) : 0 < |\alpha| \leq k_0 \} = \mathbb{C}^n.$$

In these coordinates a local basis of the CR vector fields on $M$ is given by

$$L_j = \frac{\partial}{\partial z_j} - b_j \frac{\partial}{\partial s}, \quad 1 \leq j \leq n,$$

with

$$b_j = i \frac{s^m \varphi_{z_j}}{1 + i(s^m \varphi)_s},$$

whereas the characteristic bundle is spanned near the origin by

$$\theta = ds + \sum_{j=1}^{n} b_j d\bar{z}_j + \sum_{j=1}^{n} b_j dz_j$$

and $\theta$ together with the forms $\omega^j = dz_j$ constitute a local basis of $TM$ near the origin.

We observe that for $1 \leq j, \ell \leq n$

$$\chi^j_\ell := L_j b^\ell - \bar{L}_\ell b^j$$

$$= s^m \left( i \varphi_{z_j z_\ell}(1 + i(s^m \varphi)_s) + \varphi_{z_j}(s^m \varphi_{z_\ell})_s \right) \left( 1 + i(s^m \varphi)_s \right)^2$$

$$+ \varphi_{z_j}(s^m \varphi_{z_\ell})_s (1 + i(s^m \varphi)_s) - is^m \varphi_{z_j}(s^m \varphi)_{ss}$$

$$+ \frac{i \varphi_{z_j z_\ell}(1 + i(s^m \varphi)_s) + \varphi_{z_\ell}(s^m \varphi_{z_j})_s}{1 + i(s^m \varphi)_s}$$

$$- \frac{\varphi_{z_j}(s^m \varphi_{z_\ell})_s (1 + i(s^m \varphi)_s) - s^m \varphi_{z_j}(s^m \varphi)_{ss}}{1 + i(s^m \varphi)_s^3}$$

$$= s^m \chi^j_\ell$$

and $\chi^j_\ell(0) = 2i \varphi_{z_j z_\ell}(0)$ by the assumptions on $\varphi$. 

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In this setting (6.7) takes the form

\[ \mathcal{L}^\alpha \theta = T^\alpha \theta + \sum_{j=1}^n A^\alpha_j \omega^j \]

and (6.8) implies that

\[ T^\alpha = L_{p(1)} T^{\alpha(1)} - (b^{(1)})_s T^{\alpha(1)}, \quad T^0 = 1, \]

\[ A^\alpha_j = \sum_{k=1}^{[\alpha]} L^{\alpha(k-1)} \left(T^{\alpha(k)}\right)_{\nu(k)}. \]

If we use the two simple facts for smooth functions \( f, g \), namely \( (s^q f)_s = s^{q-1} f + s q f_s \) for \( q \in \mathbb{N} \) we see that \( T^\beta = s^{m-1} C^\beta \) for \( |\beta| \geq 1. \) Hence, if \( m \geq 2 \) we have

\[ A^\alpha_j(z, \bar{z}, s) = s^m \frac{2i \varphi_{zz\alpha_j}(z, \bar{z}, s)}{1 + (s^m \varphi(z, \bar{z}, s))^2} + s^{2m-1} R^\alpha_j(z, \bar{z}, s) = s^m B^\alpha_j(z, \bar{z}, s). \]

On the other hand we obtain for \( m = 1 \) the following representation

\[ A^\alpha_j(z, \bar{z}, s) = s \frac{2i \varphi_{zz\alpha_j}(z, \bar{z}, s)}{1 + (\varphi(z, \bar{z}, s))^2} + s S^\alpha_j(z, \bar{z}, s) + s^2 R^\alpha_j(z, \bar{z}, s) = s B^\alpha_j(z, \bar{z}, s), \]

where \( S^\alpha \) is a sum of products of rational functions with respect to \( \varphi \) and its derivatives. Each of these summands contains at least one factor of the form \( \varphi_{z^\beta} \) or \( \varphi_{z^\beta} \) with \( |\beta| \leq |\alpha| \leq k_0 \) and therefore \( S^\alpha(0) = 0. \)

By assumption there have to be multi-indices \( \alpha^1, \ldots, \alpha^n \neq 0 \) of length shorter than \( k_0 \) such that

\[ \{ \varphi_{zz^{\alpha^1}}(0), \ldots, \varphi_{zz^{\alpha^n}}(0) \} \]

is a basis for \( \mathbb{C}^n. \) Now we choose \( \alpha = (0, \alpha^1, \ldots, \alpha^n) \) and calculate according to (16.5) the multiplier \( D(\alpha) = D(\alpha, 1) \) (note that \( d = 1) \):

\[ D(\alpha) = \det \begin{pmatrix} 1 & 0 & \ldots & 0 \\ A_0^1 & A_1^1 & \ldots & A_n^1 \\ \vdots & \vdots & \ddots & \vdots \\ A_0^n & A_1^n & \ldots & A_n^n \end{pmatrix} = s^{n-m} \det \begin{pmatrix} 1 & 0 & \ldots & 0 \\ A_0^1 & B_1^1 & \ldots & B_n^1 \\ \vdots & \vdots & \ddots & \vdots \\ A_0^n & B_1^n & \ldots & B_n^n \end{pmatrix} = s^{n-m} Q(\alpha) \]

where

\[ Q(\alpha) = \det \begin{pmatrix} 1 & 0 & \ldots & 0 \\ A_0^1 & B_1^1 & \ldots & B_n^1 \\ \vdots & \vdots & \ddots & \vdots \\ A_0^n & B_1^n & \ldots & B_n^n \end{pmatrix} = \det \begin{pmatrix} B_1^1 & \ldots & B_n^1 \\ \vdots & \ddots & \vdots \\ B_1^n & \ldots & B_n^n \end{pmatrix}. \]

hence

\[ Q(\alpha)(0) = (2i)^n \det \begin{pmatrix} \varphi_{zz^{\alpha^1}}(0) \\ \vdots \\ \varphi_{zz^{\alpha^n}}(0) \end{pmatrix} \neq 0. \]

We conclude that \( M \) is CR-regular. \( \square \)

Obviously, a similar approach as in the hypersurface case above can be used to find manifolds of higher codimension that are CR-regular.

**Definition 6.11.** We say that a CR manifold \( M \subseteq \mathbb{C}^N \) of codimension \( d \) is weakly nondegenerate at \( p_0 \in M \) (in the first codimension) iff there are local coordinates \( (z, w) \in \mathbb{C}^{n+d} \) near \( p_0 \) such that \( M \) is given by the equations

\[ \text{Im} w_\mu = (\text{Re} w)^p \varphi_\mu(z, \bar{z}, \text{Re} w), \quad \mu = 1, \ldots, d, \]
with $\gamma^1 < \gamma^\nu$, $\nu = 2, \ldots, d$, and $|\gamma^1| \geq 2$. Furthermore the function $\varphi_1$ satisfies for some $k$

$$\text{span}_{\mathbb{C}} \{ (\varphi_1)_{\mathbb{Z}^n} (0,0,0) : |\alpha| \leq k \} = \mathbb{C}^n.$$  

If $k_0$ is the smallest integer $k$ for which the above condition holds, we say that $M$ is weakly $k_0$-nondegenerate at $p_0$.

**Proposition 6.12.** Let $M \subseteq \mathbb{C}^N$ be a generic ultradifferentiable CR submanifold of codimension $d$, $p_0 \in M$, and assume that $M$ is weakly nondegenerate at $p_0$. Then any locally integrable infinitesimal CR diffeomorphism of $M$ which extends microlocally to a wedge with edge $M$ near $p_0$ is ultradifferentiable near $p_0$.

**Proof.** Similar to before we have to construct a multiplier $\lambda \in \mathcal{S}$ of the form $\lambda(z, \bar{z}, s) = s^\beta \psi(z, \bar{z}, s)$ where $\psi \in \mathcal{E}_\mathcal{M}$ and $\psi(0) \neq 0$. By assumption there are coordinates $(z, w) \in \mathbb{C}^{n+d}$ near $p_0 = 0$ such that $M$ is given by

$$\text{Im } w_{\mu} = (\text{Re } w)^{\gamma^\mu} \varphi_\mu(z, \bar{z}, \text{Re } w), \quad \mu = 1, \ldots, d.$$  

In particular note that $\alpha^1 \leq \alpha^\mu$ for $\mu = 2, \ldots, d$.

We deduce from Remark 4.4 that the vector fields

$$L_j = \frac{\partial}{\partial z_j} - \sum_{\mu=1}^d b^j_\mu \frac{\partial}{\partial s_\mu}$$

are a local basis of the CR vector fields near the origin. The coefficients $b^j_\mu$ are of the form

$$b^j_\mu = i \left( \det (\text{Id}_d + i \Phi) \right)^{-1} \cdot \det B^j_\mu$$

where $\Phi$ denotes the Jacobi matrix of the map $(s^{\gamma^\mu} \varphi_\mu)_\mu$ with respect to the variables $s = (s_1, \ldots, s_d)$ and

$$B^j_\mu = \begin{pmatrix}
1 + i(s^{\gamma^1} \varphi_1)_{s_1} & \ldots & \ldots & \ldots & i(s^{\gamma^1} \varphi_1)_{s_d} \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
i(s^{\gamma^\mu} \varphi_\mu)_{s_1} & \ldots & \ldots & \ldots & i(s^{\gamma^\mu} \varphi_\mu)_{s_d} \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
i(s^{\gamma^d} \varphi_d)_{s_1} & \ldots & \ldots & \ldots & 1 + i(s^{\gamma^d} \varphi_d)_{s_d}
\end{pmatrix}.$$  

Hence for all $j = 1, \ldots, n$ and $\mu = 1, \ldots, d$ we have

$$b^j_\mu = i s^{\gamma^1} \left( \det (\text{Id}_d + i \Phi) \right)^{-1} \det C^j_\mu$$

with

$$C^j_\mu = \begin{pmatrix}
1 + i(s^{\gamma^1} \varphi_1)_{s_1} & \ldots & \ldots & \ldots & i(s^{\gamma^1} \varphi_1)_{s_d} \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
i(s^{\gamma^\mu} \varphi_\mu)_{s_1} & \ldots & \ldots & \ldots & i(s^{\gamma^\mu} \varphi_\mu)_{s_d} \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
i(s^{\gamma^d} \varphi_d)_{s_1} & \ldots & \ldots & \ldots & 1 + i(s^{\gamma^d} \varphi_d)_{s_d}
\end{pmatrix}$$

and $\bar{\gamma}^\mu = \gamma^\mu - \gamma^1 > 0$. We observe that

$$\det C^j_\mu \big|_{s_\mu = 0} = (\varphi_1)_{z_j} (z, \bar{z}, 0)$$

$$\det C^j_\mu = 0, \quad \mu = 2, \ldots, d,$$

since $|\gamma^\mu| \geq |\gamma^1| \geq 2$.

Furthermore the forms

$$\theta^\mu = ds_\mu + \sum_{j=1}^d b^j_\mu dz_j + \sum_{j=1}^d \bar{b}^j_\mu d\bar{z}_j, \quad \mu = 1, \ldots, d,$$

span the characteristic bundle near 0 and $\theta^\mu, \mu = 1, \ldots, d$ and $\omega^j = dz_j, j = 1, \ldots, n$, form a local basis of the holomorphic forms on $M$. From (6.7) we recall for $\alpha \in \mathbb{N}_0^d$ and $\mu = 1, \ldots, d$ that

$$L^\alpha_{\theta^\mu} = \sum_{\tau=1}^d T^{\alpha^\mu}_{\theta^\mu} + \sum_{j=1}^n A^{\alpha^\mu}_{\omega^j}.$$
We recall that and note that (6.9) and (6.10) imply that
\[ T_{\alpha}^{\gamma,\mu} = \sum_{k=1}^{d} L^{\alpha(k-1)}_{\nu} \beta_{\nu}^{\gamma} A_{\nu}^{\alpha(j)}(k) \]

where
\[ \lambda^{j,k}_{\nu} = L_{\nu}^k b_{\nu}^k - L_{\nu}^j b_{\nu}^j \]
\[ = (b_{\nu}^k)_{z_k} - \sum_{\mu=1}^{d} b_{\mu}^k (b_{\nu}^j)_{s_{\mu}} - (b_{\nu}^j)_{z_j} + \sum_{\mu=1}^{d} b_{\mu}^j (b_{\nu}^j)_{s_{\mu}} \]

and note that (6.9) and (6.10) imply that
\[ \lambda^{j,k}_{\nu} = 2is^\gamma R^{j,k}_{\nu} \]
where
\[ R^{j,k}_{\nu}|_{s=0} = (\phi_1)_{z_k z_j}|_{s=0} \]
\[ R^{j,k}_{\nu}|_{s=0} = 0 \]
\[ \nu = 1, \ldots, d. \]

It is easy to see that also \( T_{\alpha}^{\gamma,\mu}|_{s=0} = 0 \) for \( \alpha \neq 0 \). We conclude that for all \( \alpha \neq 0 \), and \( j = 1, \ldots, n \)
\[ A_{\alpha}^{\gamma,\mu} = 2is^\gamma \tilde{A}_{\gamma}^{\alpha,\mu} \]
where
\[ \tilde{A}_{\gamma}^{\alpha,1}|_{s=0} = (\phi_1)_{z_{\alpha} z_j}|_{s=0} \]
\[ \tilde{A}_{\gamma}^{\alpha,\mu}|_{s=0} = 0 \]
\[ \mu = 2, \ldots, d. \]

By assumption there are multi-indices \( \alpha, \ldots, \alpha^n \in \mathbb{N}_0^n \) of length at most \( k_0 \) such that the vectors
\[ (\phi_1)_{z_{\alpha} z_j}(0), \quad j = 1, \ldots, n, \]
form a basis of \( \mathbb{C}^n \).

We compute the multiplier \( D(\overline{\alpha},r) \) for \( \overline{\alpha} = (0, \ldots, 0, \alpha^1, \ldots, \alpha^n) \) and \( r = (1, 2, \ldots, d, 1, \ldots, 1) \). By (6.5) we have

\[
D(\overline{\alpha},r) = \det \begin{pmatrix}
1 & \ldots & 0 & 0 & \ldots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \ldots & 1 & 0 & \ldots & 0 \\
T_{1}^{\alpha} & \ldots & T_{d}^{\alpha} & A_{1}^{\alpha} & \ldots & A_{n}^{\alpha} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
T_{1}^{\alpha^n} & \ldots & T_{d}^{\alpha^n} & A_{1}^{\alpha^n} & \ldots & A_{n}^{\alpha^n}
\end{pmatrix}
\]

\[
= \det \begin{pmatrix}
A_{1}^{\alpha} & \ldots & A_{n}^{\alpha} \\
\vdots & \ddots & \vdots \\
A_{1}^{\alpha^n} & \ldots & A_{n}^{\alpha^n}
\end{pmatrix}
\]

\[
= \det \begin{pmatrix}
2is^\gamma \tilde{A}_{\gamma}^{\alpha} & \ldots & 2is^\gamma \tilde{A}_{\gamma}^{\alpha^n} \\
\vdots & \ddots & \vdots \\
2is^\gamma \tilde{A}_{\gamma}^{\alpha^n} & \ldots & 2is^\gamma \tilde{A}_{\gamma}^{\alpha^n}
\end{pmatrix}
\]

\[
= (2i)^{n} s^\gamma \det \begin{pmatrix}
\tilde{A}_{\gamma}^{\alpha} & \ldots & \tilde{A}_{\gamma}^{\alpha^n} \\
\vdots & \ddots & \vdots \\
\tilde{A}_{\gamma}^{\alpha^n} & \ldots & \tilde{A}_{\gamma}^{\alpha^n}
\end{pmatrix}
\]

\[
= (2i)^{n} s^\gamma \det \begin{pmatrix}
\tilde{A}_{\gamma}^{\alpha} & \ldots & \tilde{A}_{\gamma}^{\alpha^n} \\
\vdots & \ddots & \vdots \\
\tilde{A}_{\gamma}^{\alpha^n} & \ldots & \tilde{A}_{\gamma}^{\alpha^n}
\end{pmatrix}
\]
\[ = (2i)^n s^\omega \Lambda(\underline{\alpha}, r). \]

We conclude

\[ \Lambda(\underline{\alpha}, r)(0) = \det \begin{pmatrix} (\varphi_1)_{z_{\alpha_1}}(0) \\ \vdots \\ (\varphi_1)_{z_{\alpha_n}}(0) \end{pmatrix} \neq 0. \]

\[ \square \]

In the preceding results we required the involved manifolds to have a special form in order to simplify the necessary calculations, but of course there are many more CR regular manifolds. The next example gives a CR manifold that is not weakly nondegenerate at 0 in the sense of Definition 6.11 but is still CR regular.

**Example 6.13.** Let \( M \subseteq \mathbb{C}^3 \) be the CR manifold given by

\[ \text{Im } w_1 = \text{Re } w_1 |z|^2 \]
\[ \text{Im } w_2 = \text{Re } w_2 |z|^2. \]

The CR bundle \( V \) of \( M \) is spanned by

\[ L = \frac{\partial}{\partial z} - i \frac{s_1 z}{1 + i|z|^2} \frac{\partial}{\partial s_1} - i \frac{s_2 z}{1 + i|z|^2} \frac{\partial}{\partial s_2} \]

Thus a basis of the characteristic form is given by

\[ \theta^1 = ds_1 + i \frac{s_1 z}{1 + i|z|^2} dz - i \frac{s_1 \bar{z}}{1 - i|z|^2} d\bar{z} \]
\[ \theta^2 = ds_2 + i \frac{s_2 z}{1 + i|z|^2} dz - i \frac{s_2 \bar{z}}{1 - i|z|^2} d\bar{z}. \]

We know that \( \theta^1, \theta^2 \) and \( \omega = dz \) form a basis of \( T'M \). If \( \alpha = e_3 \) we recall from (6.7) that

\[ \mathcal{L}^\alpha \theta^1 = T^\alpha_1 \theta^1 + T^\alpha_2 \theta^2 + A^\alpha \omega. \]

Using (6.8) we observe that

\[ T^\alpha_1 = -i \frac{z}{1 + i|z|^2} \]
\[ T^\alpha_2 = 0 \]
\[ A^\alpha = -2is_1 \frac{1 - |z|^4}{(1 + |z|^4)^2}. \]

Hence, if we set \( \underline{\alpha} = (0, 0, \alpha) \) and \( r = (1, 2, 1) \) then the multiplier \( D(\underline{\alpha}, r) \) of \( M \) given by (6.9) is

\[ D(\underline{\alpha}, r) = \det \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -i \frac{z}{1 + i|z|^2} & 0 & -2is_1 \frac{1 - |z|^4}{(1 + |z|^4)^2} \end{pmatrix} = -2is_1 \frac{1 - |z|^4}{(1 + |z|^4)^2} \]

and thus \( M \) is CR regular.

We could now give an ultradifferentiable version of the example given in section 7 of [17] in order to show that in the previous statements the requirement on the infinitesimal automorphisms to be locally integrable is essential for the assertions to hold. However, to do this it would be enough to replace everywhere in section 7 of [17] the word *smooth* with the term *ultradifferentiable of class \( \{M\} \).*

Instead we take a closer look into the case of quasianalytic manifolds. We begin with recalling the following definition from [3] §11.7. Let \( M \subseteq \mathbb{C}^N \) be a CR submanifold with defining functions \( \rho = (\rho_1, \ldots, \rho_d) \) near \( p_0 \in M \). A *formal holomorphic vector field* at \( p_0 \) is a vector field of the form

\[ X = \sum_{j=1}^N a_j(Z) \frac{\partial}{\partial Z_j} \]

with the coefficients \( a_j \) being formal power series in \( Z - \bar{p}_0 \) with complex coefficients. The formal vector field \( X \) is said to be tangent to \( M \) at \( p_0 \) if there exists a \( d \times d \) matrix \( c(Z, \bar{Z}) \) consisting of formal power series in the variables \( Z - p_0 \) and \( \bar{Z} - \bar{p}_0 \) such that

\[ X \rho(Z, \bar{Z}) \sim c(Z, \bar{Z}) \rho(Z, \bar{Z}), \]
where \( \sim \) denotes equality as formal power series in \( Z - p_0 \) and \( \bar{Z} - \bar{p}_0 \). Note that the existence of nontrivial holomorphic vector fields at \( p_0 \) tangent to \( M \) does not depend on the choice of holomorphic coordinates and defining equations near \( p_0 \).

**Definition 6.14.** A generic submanifold \( M \subseteq \mathbb{C}^N \) is formally holomorphically nondegenerate at \( p_0 \in M \) if there is no nontrivial formal holomorphic vector field at \( p_0 \) that is tangent to \( M \).

**Remark 6.15.** If \( M \) is formally holomorphically nondegenerate at \( p_0 \) then \( M \) is formally holomorphically nondegenerate at every point of some neighbourhood \( U \) of \( p_0 \). Furthermore if \( M \) is formally holomorphically nondegenerate on an open set \( U \subseteq M \) then \( M \) is finitely nondegenerate on an open and dense subset \( V \subseteq U \), c.f. [3, Theorem 11.7.5].

**Theorem 6.16.** Let \( M \) be a quasianalytic regular weight sequence and \( M \subseteq \mathbb{C}^N \) a generic submanifold of class \( \{M\} \) that is formally holomorphically nondegenerate.

Every smooth CR diffeomorphism \( \Psi \) that extends microlocally to a wedge with edge \( M \) is ultradifferentiable of class \( \{M\} \).

**Proof.** As usual we argue locally near a point \( p_0 \). After a choice of local bases of CR vector fields and holomorphic forms and selecting a generating set for the characteristic forms we can use the representation [6,3] near \( p_0 \). By Theorem 6.16 we know that for any multiplier \( \lambda \) the product \( A_j = \lambda \cdot X_j \) is ultradifferentiable for \( j = 1, \ldots, N \). Since \( X_j \) is smooth by assumption we have that the equality holds also for the formal power series at \( p_0 \) of \( A_j \), \( \lambda \) and \( X_j \). Since \( M \) is formally holomorphically nondegenerate at \( p_0 \) there has to be a multiplier \( \lambda \in S \) with nontrivial formal power series at \( p_0 \). Indeed, if the power series of \( \lambda \) at \( p_0 \) equals \( 0 \) then \( \lambda \) itself has to vanish in a neighbourhood of \( p_0 \) by the quasianalyticity of \( M \). On the other hand in every neighbourhood of \( p_0 \) there is a point \( q \) at which \( M \) is finitely nondegenerate by [3, Theorem 11.7.5]. Hence by Remark 6.15 there has to be a nontrivial multiplier \( \lambda' \) defined on some neighbourhood \( U \) of \( p_0 \).

We conclude that the formal power series of \( A_j' = \lambda' \cdot X_j \) at \( p_0 \) is divisible by the Taylor series of \( \lambda' \) at \( p_0 \). Hence Theorem 2.20 gives that \( X_j \) is ultradifferentiable of class \( \{M\} \) near \( p_0 \). \( \square \)

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Faculty of Mathematics, University of Vienna, Oskar-Morgenstern-Platz 1, 1090 Vienna, Austria

Current address: Department of Mathematics and Statistics, Masaryk University, Kotlarska 2, 611 37 Brno, Czech Republic

E-mail address: stefan.fuerdoes@univie.ac.at