RESPONSE SOLUTIONS TO HARMONIC OSCILLATORS
BEYOND MULTI–DIMENSIONAL BRJUNO FREQUENCY

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Abstract. This paper focuses on the quasi–periodically forced nonlinear harmonic oscillators
\[ \ddot{x} + \lambda^2 x = \epsilon f(\omega t, x), \]
where \( \lambda \in \Omega \), a closed interval not containing zero, the forcing term \( f \) is real analytic, and the frequency vector \( \omega \in \mathbb{R}^d \) is beyond Brjuno frequency, which we call as Liouvillean frequency. For the given class of the frequency \( \omega \in \mathbb{R}^d \), which will be given later, we prove the existence of real analytic response solutions (the response solution is the quasi–periodic solution with the same frequency as the forcing) for the above equation. The proof is based on a modified KAM (Kolmogorov–Arnold–Moser) theorem for finite–dimensional harmonic oscillator systems with Liouvillean frequency.

1. Introduction and main result. In physics and mechanics, there is a variety of nonlinear vibration problems [7, 26, 28] which can be characterized by the following forced harmonic oscillators
\[ \ddot{x} + \lambda^2 x = \epsilon f(\omega t, x), \quad (1.1) \]
where the parameters \( 0 < \epsilon \ll 1 \) and the forcing term \( f \) is quasi–periodic in \( t \) with frequency vector \( \omega = (\omega_1, \ldots, \omega_d) \in \mathbb{R}^d \). For the system (1.1), one of the classical questions is to find suitable conditions (spectral, non–degeneracy, smoothness, etc) that guarantee the persistence of the quasi–periodic solutions.

In dynamical systems, the KAM theory is a very powerful tool to construct the quasi–periodic solutions. For more details, one may refer to [3, 11, 12, 13, 14, 18, 19, 20, 21, 22, 23] and see also [8, 9] for further developments. J. Moser [25], M. Friedman [10] and B. Braaksma and H. Broer [4] considered the problem for general nonlinear oscillators and obtained the existence of response solutions by KAM and normal form method. It is well known that the resonance, small divisors, is a notorious problem in KAM theory. To overcome this problem, the forcing
frequency $\omega \in \mathbb{R}^d$ is required to satisfy some non-degeneracy conditions, such as Diophantine condition, i.e., there exist $\tau > d - 1$ and $\gamma > 0$ such that

$$|\langle k, \omega \rangle| \geq \frac{\gamma}{|k|^\tau}, \quad \forall k \in \mathbb{Z}^d \setminus \{0\},$$

or Brjuno condition which is slightly weaker than the Diophantine condition and defined by

$$\sum_{n \geq 0} \frac{1}{2^n} \max_{0 < |k| \leq 2^n, k \in \mathbb{Z}^d} \ln \frac{1}{|\langle k, \omega \rangle|} < \infty,$$

where $\langle k, \omega \rangle := \sum_{i=1}^d k_i \omega_i$, $|k| := \sum_{i=1}^d |k_i|$. If the frequency $\omega$ is not Brjuno, we call it Liouvillean.

A natural question is whether the results for Diophantine or Brjuno frequency are possible to be extended to Liouvillean frequency? Recently, some results in reducibility theory show that it is possible to develop KAM theory for some systems with Liouvillean frequency. We say that a linear quasi-periodic system is reducible if there is a nonsingular quasi-periodic change of variables such that it conjugates the system to a linear constant system. The small divisor obstructions related to the frequency are present in the problem of reducibility. In 2011, A. Avila, B. Fayad and R. Krikorian [1] developed a non-standard KAM scheme and, by using algebra-conjugacy trick, proved that most of the $SL(2, \mathbb{R})$ cocycles with one Liouvillean frequency is rotations reducible provided that cocycle is homotopic to identity. X. Hou and J. You [15] considered the real analytic quasi-periodically forced continuous cocycle in $sl(2, \mathbb{R})$ with two frequencies $\omega = (\alpha, 1)$ and proved that it is almost reducible or reducible if the fibered rotation number is Diophantine with $\omega$.

Subsequently, Q. Zhou and J. Wang [35] generalized the $SL(2, \mathbb{R})$ cocycles in [1] to the $GL(d, \mathbb{R})$ cocycle. They used the method of periodic approximation and KAM schemes to study the reducibility problems for quasi-periodic cocycles $GL(d, \mathbb{R})$ with Liouvillean frequency. Since the linear quasi-periodic system can be reviewed as a linear Hamiltonian system and the reducibility shows there exists quasi-periodic solution to the system, it provides a trick to study the nonlinear Hamiltonian systems.

Recently, based on KAM scheme in [1, 15, 35], J. Wang, J. You and Q. Zhou [30] generalized the work [1] to finite dimensional nonlinear Hamiltonian system and proved the existence of response solutions for the quasi-periodically forced harmonic oscillators (1.1) with $\lambda$ in a large measure set and the frequency $\omega = (1, \alpha), \forall \alpha \in \mathbb{R} \setminus \mathbb{Q}$. Z. Lou and J. Geng [24] developed a similar KAM scheme for the reversible system and proved the existence of response solutions for reversible forced harmonic oscillators (1.1) with two basic frequencies $\omega = (1, \alpha), \forall \alpha \in \mathbb{R} \setminus \mathbb{Q}$. Moreover, R. Krikorian, J. Wang, J. You and Q. Zhou [17] proved that an analytic quasi-periodically forced circle flow, which is close enough to a constant rotation, with a non-super-Liouvillean base frequency is rotations reducible provided its fibered rotation number is Diophantine with respect to the base frequency. J. Wang and J. You [29] used the similar approach and technique with the one in [17] to investigate the boundedness of solutions for nonlinear quasi-periodic Hamiltonian systems with a non-super-Liouvillean frequency.

For the infinite-dimensional dynamical system, there are some results about the persistence of invariant torus with Liouvillean frequency. By applying a similar trick given in [1], X. Xu, J. You and Q. Zhou [32] constructed the quasi-periodic solutions.
of forced nonlinear Schrödinger equations with the frequency vector \( \omega = \xi (\omega_1, \omega_2) \) and \( \omega_1 = (1, \alpha) \) satisfying 
\[
\left\{ \begin{array}{l}
\beta(\alpha) := \limsup_{n > 0} \frac{\ln \ln q_{n+1}}{\ln q_n} < \infty, \\
\langle (k, \omega_1) + (l, \omega_2) \rangle \geq \frac{\gamma}{1 + \|k + l\|}, \quad \text{for } k \in \mathbb{Z}^2, l \in \mathbb{Z}^d \setminus \{0\},
\end{array} \right.
\]
where \( p_n/q_n \) is the continued fraction approximates to \( \alpha \). H. Cheng, W. Si and J. Si [5] constructed whiskered tori for forced beam equations with multi-dimensional Liouvillean frequency \( \omega = \xi \bar{\omega} \) where \( \xi \in [1, 2] \) and fixed \( \bar{\omega} \in \mathbb{R}^d \) satisfying
\[
\max_{0 < |k| \leq K, k \in \mathbb{Z}^d} \frac{1}{\langle (k, \bar{\omega}) \rangle} \leq K (\ln |K|)^{-a},
\]
for \( a \in (0, 1) \) and any \( K > 1 \).

Motivated by the works above, in this paper, we generalize, by using some technique in [5], the result in [30] to the existence of response solutions with multi-dimensional Liouvillean frequency. We construct the quasi–periodic response solution to (1.1) with the forcing frequency vector \( \omega \in \mathbb{R}^d \) \((d \geq 2)\) satisfying: for the fixed \( n_* \in \mathbb{N}^+ \), \( a \in (0, 1] \), and \( K > 1 \),
\[
\max_{0 < |k| \leq K, k \in \mathbb{Z}^d} \frac{1}{\langle (k, \omega) \rangle} \leq K \alpha(K),
\]
where \( \alpha(K) := \left( \ln \cdots \ln K \right)^{-a} \).

The two parameters \( n_* \) and \( a \) in (1.3) measure the non–degeneracy of the frequency \( \omega \). The smaller \( a \in (0, 1] \) is or the bigger \( n_* \in \mathbb{N}^+ \) is, the weaker the frequency \( \omega \) is. It is showed that, in [5], some of the frequencies satisfying (1.2) are Liouvillean. Note that the frequency defined by (1.3) is more weaker than the one defined by (1.2) (the frequency in [5]) since the function \( \alpha(K) \) decreases more slowly than \( (\ln |K|)^{-a} \). Thus, the frequency \( \omega \) given by (1.3) also includes some (actually, more) Liouvillean frequencies.

Note that, in [32], the authors only allow \( \omega_1 \) to be Liouvillean, not to all \( \omega \). Then if we allow \( d > 2 \), our frequency is much weaker than the frequency in [32]. However, if we restrict \( d = 2 \), the frequency in [30] is much weaker than the one in our paper. The reason why we put these assumptions on the frequency is that the \( d \)-dimensional (\( d > 2 \)) frequency is much more complicated than the 2–dimensional frequency \( (1, \alpha) \), \( \alpha \in \mathbb{R} \setminus \mathbb{Q} \). For example, there is no CD–bridge in this work.

In the Appendix, we construct another function that decreases more slowly to zero than the function \( \alpha(K) \) in (1.3). Moreover, we prove that the two inequalities in Lemma 3.1 also hold if we replace \( \alpha(K) \) by the function we construct. That is the KAM theorem we construct can be also applied to another kind of frequency which includes more Liouvillean frequencies than the one defined by (1.3). Please see the Appendix for details.

Now we are ready to state our main result.

**Theorem 1.1.** Assume that \( f : \mathbb{T}^d \times \mathbb{R} \to \mathbb{R}, d \geq 2, \) is a real analytic function, \( \lambda \in \mathcal{O} \subset \mathbb{R} \setminus \{0\} \), where \( \mathcal{O} \) is a closed interval, and \( \omega \) satisfies the hypothesis (1.3). Then for any given \( 0 < \gamma < 1 \), there exist \( \epsilon_* > 0 \) (depending on \( f, \gamma \) and the frequency \( \omega \)), and a Cantor subset \( \mathcal{O}_\gamma \subset \mathcal{O} \) with \( \text{meas} \mathcal{O}_\gamma \geq (1 - \epsilon \gamma) \text{meas} \mathcal{O} \), such that for any \( \lambda \in \mathcal{O}_\gamma \) the equation (1.1) possesses a quasi–periodic response solution provided that \( \epsilon < \epsilon_* \).
Note that, in our work, we allow the frequency to be Liouvillean. If $\omega$ is ‘too close’ to rational vector, in general, the persistence of the invariant torus can be destroyed since the small divisor $|(k, \omega)|$ is not controllable. Thus, the resonant terms in the perturbation (i.e., $\sum_{|a| \leq 1} \sum_{0 \neq k \in \mathbb{Z}^d} \hat{P}_{a, \alpha}(k; \xi)e^{i(k, \theta)z^\alpha z^\alpha}$) whose small divisor is $|(k, \omega)|$ can not be eliminated and we have to put these resonant terms into the normal form. Therefore, the new normal form depends on the angle variable $\theta$. This, consequently, leads to solving the homological equation with variable coefficients,

$$\partial_\omega u + i(\mu(\xi) + B(\theta; \xi) + b(\theta; \xi))u = f(\theta; \xi), \quad \partial_\omega = \sum_{j=1}^d \omega_j \partial_{\theta_j}.$$

We will eliminate the resonant terms of function $B(\theta; \xi)$ by solving the equation

$$\partial_{\xi_j}B(\theta; \xi) = -B(\theta; \xi) + [B(\theta; \xi)]_{\theta_j},$$

where $[B(\theta; \xi)]_{\theta_j}$ is the average of $B(\theta; \xi)$ on $\mathbb{T}^d$. In [30], the authors fixed the frequency as $\omega = (1, \alpha)$ and controlled the small divisor problem by the theory of continued fractions to ensure that the solution $B(\theta; \xi)$ is controllable. However, in the case $d > 2$, the frequency satisfying (1.3) is more complicated than $(1, \alpha)$ even though the theory of continued fractions still exists, but the estimates are more complicated than the two dimensional ones. For example, the estimate $|q_n, \alpha - p_n|$ just depends on $q_n$, while $|q_n\omega - p_n|(|q_n \in \mathbb{Z}, p_n \in \mathbb{Z}^d)$ depends not only $q_n$ but also $m_n \in \mathbb{Z}^d$ with $m_{n,j} = p_{n,j}q_{j+1} - p_{j+1,i}q_j$. See [6] for more details.

We will use the new technique in [5] to make $B(\theta; \xi)$ controllable. In section 3.3, we use the special structure of $B(\theta; \xi)$ to obtain a nice estimate of the solution $B(\theta; \xi)$ by carefully choosing the iterative parameters in the iteration and taking advantage of the bound in (1.3). Moreover, to overcome the difficulty caused by the variable coefficients, one step of KAM iteration, from $n$-th step to $(n + 1)$-th step, for example, will be finished by an iterative process including $\mathcal{N}_n$ steps, where $\mathcal{N}_n$ depends on $n$ and goes to $\infty$ as $n$ goes to $\infty$.

If the smallness of the perturbation does not depend on the Diophantine constants of the frequency $\omega$, we say the result is non–perturbative. We stress that the smallness of the perturbation in our result depends on the forcing frequency $\omega$. Thus, our result is perturbative unlike the result in [30]. For example, one of the hypotheses on $\epsilon_*$ is $\ln \epsilon_* \geq 3 \exp \cdots \exp \left\{(3^{-1} e^{-4} s_0(n_* + 3)^{-2})^{1/2} \right\}$, where $n_*$ and $a$ are the constants in (1.3) and $0 < s_0 < 1$.

The paper is organized as follows. In Section 2, we give some notations, definitions and a finite dimensional KAM theorem. In Section 3, we prove the KAM theorem in detail, such as we give the technique to solve the homological equation with variable coefficients. In Section 4, we prove Theorem 1.1 by the KAM theorem. In the Appendix, we construct another function which can give a weaker condition than (1.3).

2. Some notations and a KAM theorem.

2.1. Some notations. Let $\mathbb{T}^d := \mathbb{R}^d / 2\pi\mathbb{Z}^d$ (or $\mathbb{C}^d / 2\pi\mathbb{Z}^d$) be the standard $d$-dimensional real (complex) torus. For $s > 0$, denote the complex neighborhood of $\mathbb{T}^d$ by

$$D(s) = \{ \theta \in \mathbb{T}^d : \| \text{Im} \theta \| < s \},$$

where $\| \cdot \|$ is the sup-norm of complex vectors.
For the function \( f(\theta; \xi) \) defined on \( D(s) \times \mathcal{O} \), where the parameter set \( \mathcal{O} \subset \mathbb{R} \) is a bounded closed interval, with the Fourier expansion

\[
f(\theta; \xi) = \sum_{k \in \mathbb{Z}^d} \hat{f}(k; \xi)e^{i(k, \theta)},
\]

we define the norms \( \|f\|_{s,\mathcal{O}}^* \) and \( \|f\|_{s,\mathcal{O}}^L \) as

\[
\|f\|_{s,\mathcal{O}}^* = \sum_{k \in \mathbb{Z}^d} \|\hat{f}(k\|_{\mathcal{O}} \| e^{i|k|s}, \quad \|f\|_{s,\mathcal{O}}^L = \sum_{k \in \mathbb{Z}^d} \|\hat{f}(k)\|_{\mathcal{O}} \| e^{i|k|s},
\]

with

\[
\|\hat{f}(k)\|_{\mathcal{O}} = \sup_{\xi \in \mathcal{O}} |\hat{f}(k; \xi)|, \quad \|\hat{f}(k)\|_{\mathcal{O}} = \sup_{\xi \in \mathcal{O}} |\frac{\hat{f}(k; \xi_1) - \hat{f}(k; \xi_2)}{\|\xi_1 - \xi_2\|}|
\]

And we also define the norms \( \|\hat{f}(k)\|_{\mathcal{O}} \) and \( \|f\|_{s,\mathcal{O}} \) as

\[
\|\hat{f}(k)\|_{\mathcal{O}} = \|\hat{f}(k)\|_{\mathcal{O}}^* + \|\hat{f}(k)\|_{\mathcal{O}}^L, \quad \|f\|_{s,\mathcal{O}} = \|f\|_{s,\mathcal{O}}^* + \|f\|_{s,\mathcal{O}}^L.
\]

For any \( K \geq 1 \), we define the truncation operator \( T_K \) and projection operator \( R_K \) as

\[
T_K f(\theta; \xi) = \sum_{k \in \mathbb{Z}^d, |k| \leq K} \hat{f}(k; \xi)e^{i(k, \theta)}, \quad R_K f(\theta; \xi) = \sum_{k \in \mathbb{Z}^d, |k| > K} \hat{f}(k; \xi)e^{i(k, \theta)}.
\]

We denote the average of \( f(\theta; \xi) \) on \( \mathbb{T}^d \) by

\[
[f(\theta; \xi)]_0 = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(\theta; \xi) \, d\theta = \hat{f}(0; \xi).
\]

For \( s, r > 0 \), we denote the complex neighborhood of \( \mathbb{T}^d \times \{0\} \times \{0\} \) by

\[
D(s, r) = \{ (\theta, z, \bar{z}) : \|\text{Im}\theta\| < s, \|z\| < r, \|\bar{z}\| < r \} \subset \mathbb{T}_e^d \times \mathbb{C}^m \times \mathbb{C}^m := \mathcal{P},
\]

where \( d \geq 2 \), \( m \geq 1 \). Denote \( \delta = (\delta_1, \ldots, \delta_m) \), \( \beta = (\beta_1, \ldots, \beta_m) \), where \( \delta_n, \beta_n \in \mathbb{N}, 1 \leq n \leq m \). For the function \( P : D(s, r) \times \mathcal{O} \to \mathbb{C} \), which is real analytic in variables \( (\theta, z, \bar{z}) \) and Lipschitz with respect to the parameter \( \xi \in \mathcal{O} \), we take the following Taylor–Fourier expansion

\[
P(\theta, z, \bar{z}; \xi) = \sum_{\delta, \beta \in \mathbb{N}^m} P_{\delta, \beta}(\theta; \xi) z^\delta \bar{z}^\beta = \sum_{\delta, \beta \in \mathbb{N}^m} \sum_{k \in \mathbb{Z}^d} \widehat{P}_{\delta, \beta}(k; \xi)e^{i(k, \theta)} z^\delta \bar{z}^\beta,
\]

where \( z^\delta \bar{z}^\beta = \prod_{j=1}^m z_j^{\delta_j} \bar{z}_j^{\beta_j} \). Moreover, we define the norms \( \|P\|_{s,r,\mathcal{O}}^* \) and \( \|P\|_{s,r,\mathcal{O}}^L \) as follows

\[
\|P\|_{s,r,\mathcal{O}}^* = \sup_{\|z\| \leq r, \|\bar{z}\| \leq r} \sum_{\delta, \beta \in \mathbb{N}^m} \|P_{\delta, \beta}\|_{s,\mathcal{O}}^* \|z^\delta\| \|\bar{z}^\beta\|,
\]

\[
\|P\|_{s,r,\mathcal{O}}^L = \sup_{\|z\| \leq r, \|\bar{z}\| \leq r} \sum_{\delta, \beta \in \mathbb{N}^m} \|P_{\delta, \beta}\|_{s,\mathcal{O}}^L \|z^\delta\| \|\bar{z}^\beta\|,
\]

and the norm \( \|P\|_{s,r,\mathcal{O}} \) as

\[
\|P\|_{s,r,\mathcal{O}} = \|P\|_{s,r,\mathcal{O}}^* + \|P\|_{s,r,\mathcal{O}}^L.
\]
2.2. A finite–dimensional KAM Theorem. In this section, we develop an abstract KAM theorem for a general finite–dimensional system with tangent frequency satisfying (1.3). As an immediate application of the theorem, we can prove Theorem 1.1.

We consider a real analytic Hamiltonian $H$ defined on $T^d \times \mathbb{C}^d \times \mathbb{C}^d \times \mathcal{O}$ with the form

$$H(\theta, I, z, \bar{z}; \xi) = \langle \omega, I \rangle + (\Omega(\xi)z, \bar{z}) + P(\theta, z, \bar{z}; \xi)$$

(2.1)

and the symplectic form is $dI \wedge d\theta + idz \wedge d\bar{z}$, where

$$\Omega(\xi) = \text{diag}(\Omega_1(\xi), \cdots, \Omega_m(\xi))$$

(2.2)

is Lipschitz with respect to the parameter $\xi \in \mathcal{O}$. For the diagonal matrix $\Omega(\xi)$, we also identify it as the vector $\Omega(\xi) = (\Omega_1(\xi), \cdots, \Omega_m(\xi))^T$. Similarly, for the vectors

$$B(\theta; \xi) = (B_1(\theta; \xi), \cdots, B_m(\theta; \xi))^T, \quad b(\theta; \xi) = (b_1(\theta; \xi), \cdots, b_m(\theta; \xi))^T$$

defined later, we also identify them as the diagonal matrices.

The proof of Theorem 1.1 is based on the following KAM theorem.

**Theorem 2.1.** Assume that $\omega \in \mathbb{R}^d$ satisfies (1.3), and $0 < s, r < 1, \tau > d$. Consider the real analytic Hamiltonian $H$ given by (2.1) satisfying the non–degeneracy condition: for $l \in \mathbb{Z}^d, 0 < |l| \leq 2$,

$$\inf_{\xi_1, \xi_2 \in \mathcal{O}} \frac{|(l, \Omega(\xi_1) - \Omega(\xi_2))|}{|\xi_1 - \xi_2|} \geq \mu > 0, \quad \|\Omega(\xi)\|_{\mathcal{O}} \leq \rho.$$  

(2.3)

Then for any $0 < \gamma \ll 1$, there exists a constant $\epsilon_*(\omega, \gamma, s, r, \tau, \mu, \rho) > 0$, such that for every real analytic perturbation $P$ with

$$\epsilon := \|P\|_{s, r, \mathcal{O}} \leq \epsilon_*(\omega, \gamma, s, r, \tau, \mu, \rho),$$

there exists a nonempty subset $\mathcal{O}_\gamma \subset \mathcal{O}$ with $\text{meas} \mathcal{O}_\gamma \geq (1 - c\gamma) \text{meas} \mathcal{O}$, and for every $\xi \in \mathcal{O}_\gamma$, there is a real analytic symplectic map $\Phi$ of the form

$$(\theta, I, z, \bar{z}) \mapsto (\theta, U(\theta, I, z, \bar{z}; \xi), W(\theta, z, \bar{z}; \xi), \overline{W}(\theta, z, \bar{z}; \xi)),$$

where $W, \overline{W} : D(s, \frac{\tau}{2}) \times \mathcal{O}_\gamma \rightarrow D(s, r)$ are affine in $z, \bar{z}$, such that the transformation $\Phi$ casts the Hamiltonian $H$ into $H \circ \Phi = N_\ast + P_\ast$, where

$$N_\ast(\theta, I, z, \bar{z}; \xi) = e_*(\theta; \xi) + \langle \omega, I \rangle + \langle (\Omega(\xi) + B_\ast(\theta; \xi))z, \bar{z} \rangle,$$

$$P_\ast(\theta, z, \bar{z}; \xi) = \sum_{|\delta + \beta| \geq 3} P_{\delta, \beta}^*(\theta; \xi) e^\delta \bar{z}^\beta$$

with

$$\|e_\ast\|_{s, \gamma, \mathcal{O}_\gamma} < 4\epsilon, \quad \|B_\ast\|_{s, \gamma, \mathcal{O}_\gamma} \leq 4\epsilon.$$  

(2.4)

3. Proof of Theorem 2.1.

3.1. The strategy of the proof. The essential idea to prove Theorem 2.1 is to construct a symplectic transformation by a modified KAM scheme, consisting of infinitely many successive steps (called as KAM steps) of iteration. At each step of KAM iteration, we will construct a symplectic transformation to change the system into a new one such that the new perturbation of the transformed system is smaller. As all KAM steps can be carried out inductively, we only describe one step of KAM iteration in more details.

The key, in each step of iteration, to obtaining the symplectic transformation is to solve the homological equation. Since we assume the frequency $\omega$ just satisfies
(1.3), there will be new difficulties compared with the classical KAM theory under Diophantine or Brjuno conditions. The main difficulty is that we can not kill the resonant terms \( \sum_{|\alpha| \leq 1} \sum_{0 \neq k \in \mathbb{Z}} P_{\alpha,\sigma}(k; \xi) z^\alpha \bar{z}^\alpha \) in the perturbation since the small divisor \(|(k, \omega)|\) is not controllable. We overcome this problem by putting these resonant terms \( \sum_{|\alpha| \leq 1} P_{\alpha,\sigma}(\theta; \xi) z^\alpha \bar{z}^\alpha \) into the normal from.

More concretely, let us start from the normal form of system (2.1), which is constant. After first step, there will be functions \( \epsilon_i(\theta; \xi) \) and \( B_i(\theta; \xi) \), which are from the perturbation \( P(\theta, z, \bar{z}; \xi) \), in the normal formal of the new system. Inductively, we get the normal form

\[
N_n = \epsilon_n(\theta; \xi) + \langle \omega, I \rangle + \langle (\Omega(\xi) + B_n(\theta; \xi)) z, \bar{z} \rangle,
\]

at the \( n \)-th step with \( \epsilon_0(\theta; \xi) = 0 \) and \( B_0(\theta; \xi) = 0 \). As a consequence, the homological equation in this paper depends on the angle variable \( \theta \). What we want to stress is that the function \( B_n(\theta; \xi) \) owns the special structure (see Remark 2 for details) and this special structure plays an important role when we try to eliminate the effect of \( B_n(\theta; \xi) \).

In this paper, since the effect of relatively large \( B_n(\theta; \xi) \), we can not obtain a strong enough estimate on the solution of the homological equation. We only obtain an approximate solution to the homological equation with nice estimate which can permit an iterative construction to prove the iterative lemma in the KAM scheme. Therefore, one KAM step (the proof of Iterative Lemma 3.3) will be finished by a finite iterative procedure (Proposition 2), which is different from the general KAM theory under the Diophantine or Brjuno non–degeneracy conditions. In the following, we give the detail of \( n \)-th step of KAM iteration.

### 3.2. Iterative sequences

For any given \( 0 < \epsilon_0 < 1, \tau > d \), we denote the parameter \( K_{-1} \) by

\[
K_{-1}^{\frac{1}{2}} = \ln(\frac{1}{\min(2d+1, 2\tau+1)}), \text{ i.e., } \epsilon_0 = \exp\{-40(2\tau + 1)K_{-1}^{\frac{1}{2}}\}.
\]

For \( i \geq 0 \), let \( \eta_i = (i + 2)^{-2} \), and we assume that \( \epsilon_0 \) is small enough such that

\[
\ln \epsilon_0^{-\frac{1}{40(2\tau+1)}} > 3 \exp \prod_{n_s} \exp \left\{ \left(3^{-1}e^{-4}s_0\eta_{n_s+1}^{-1}\right)^{-\frac{1}{2}} \right\},
\]

where \( 0 < s_0 < 1 \).

**Remark 1.** Note that \( \epsilon_0 \), the smallness of perturbation, depends on the parameters \( n_s \) and \( a \), which depict the weakness of the Liouvillean frequency \( \omega \). Thus \( \epsilon_0 \) is related to the forcing frequency \( \omega \).

For above \( \epsilon_0, K_{-1} \) and \( s_0 \), we denote the iterative sequence: for \( i \geq 0 \),

\[
K_i = \exp\{K_{i-1}^{\frac{1}{2}}\}, \quad \epsilon_{i+1} = \exp\{-40(2\tau + 1)K_i^{\frac{1}{2}}\}, \quad \bar{\epsilon}_{i,j} = \epsilon_i^{\left(\frac{15}{2}\right)^j},
\]

\[
s_{i+1} = s_0 \prod_{j=0}^{i}(1 - 2\eta_j), \quad \sigma_{i,j} = 4^{-1}\eta_j s_i, \quad \bar{T}_{i,j} = \sigma_{i,j}^{-1} \ln \bar{\epsilon}_{i,j}, \quad (3.1)
\]

where \( j = 0, \ldots, N_i - 1 \) and \( N_i \) is the smallest integer number such that \( \bar{\epsilon}_{i,N_i} \leq \epsilon_{i+1} \), that is

\[
\bar{\epsilon}_{i,N_i} \leq \epsilon_{i+1} < \bar{\epsilon}_{i,N_i-1}.
\]
Obviously, for any $n \geq 1$,
\[
s_n = s_0 \prod_{i=0}^{n-1} (1 - 2\eta_i) \geq s_0 \prod_{i=0}^{\infty} [1 - 2(i + 2)^{-2}] = s_0 \exp \left\{ \sum_{i=0}^{\infty} \ln [1 - 2(i + 2)^{-2}] \right\} \geq s_0 \exp \left\{ \sum_{i=0}^{\infty} -4(i + 2)^{-2} \right\} > s_0 e^{-4}.
\]
Hence, we know that
\[
e^{-4}s_0 < s_n \leq s_0, \quad \forall n \geq 0.
\]

The above two inequalities are used in many places, and we will not emphasize the references about them. Moreover, we will also assume $\epsilon_0$ is small enough such that for the sequence \{$K_j \}_{j \geq -1}$ defined above, the following inequalities hold:
\[
K_{n+1} > K_n^3, \quad K_{n+1} > 20K_n, \quad \ln K_n < K_n^\frac{1}{\tau}, \quad \forall n \geq 0.
\]

For the sequences in (3.1), we will give two important inequalities in the following lemma, which are used to solve the homological equations in Lemma 3.2 and Proposition 1.

**Lemma 3.1.** (1) For the sequences defined in (3.1), we have
\[
\exp \left\{ T_{n-3,N_{n-3}} \right\} < \ln \epsilon_n^{-1}, \quad \forall n \geq 3. \tag{3.2}
\]
(2) For the function $\alpha(T) = (\ln \cdots \ln |T|)^{-n}$, we also have
\[
\alpha(\ln K_n^\frac{1}{\tau}) < 3^{-n} e^{-4}s_0\eta_{n+1}, \quad \forall n \geq 0. \tag{3.3}
\]

**Proof.** (1) Since $\tilde{\epsilon}_{n,N_n} \leq \epsilon_n < \epsilon_n N_n$, we know that
\[
\epsilon_n^{-\frac{15}{14}} N_n^{-1} < \epsilon_{n+1}^{-1} \leq \epsilon_n^{-\frac{15}{14}} N_n^{-1},
\]
which implies
\[
\left( \frac{15}{14} \right)^{N_n^{-1}} < \frac{\log \epsilon_{n+1}}{\log \epsilon_n} = K_n^\frac{1}{\tau} K_{n+1}^{-\frac{1}{\tau}} \leq \left( \frac{15}{14} \right)^{N_n^{-1}}. \tag{3.4}
\]
Note that $(\frac{15}{14})^{14} < e < (\frac{15}{14})^{15}$, then by the above inequalities, we have
\[
K_n^\frac{1}{\tau} K_{n+1}^{-\frac{1}{\tau}} \leq \left( \frac{15}{14} \right)^{N_n^{-1}} < e^{\frac{15}{14}}, \quad e^{\frac{15}{14}} < \left( \frac{15}{14} \right)^{N_n^{-1}} < K_n^\frac{1}{\tau} K_{n+1}^{-\frac{1}{\tau}} < K_n^\frac{1}{\tau}.
\]
Thus
\[
\frac{14}{3} \ln K_n < N_n < \frac{15}{2} \ln K_n + 1. \tag{3.5}
\]

Then by (3.1), we have
\[
T_{n,N_{n-1}} = \sigma_{n,N_{n-1}}^{-1} \ln \tilde{\epsilon}_{n,N_{n-1}}^{-1} < \sigma_{n,N_{n-1}}^{-1} \ln \epsilon_{n+1}^{-1}
= 160(n + 2)^2(N_n + 1)^2 s_n^{-1}(2\tau + 1) K_n^\frac{2}{3}
< 10240(2\tau + 1)e^8 s_0^{-1}(n + 2)^2(\ln K_n)^2 K_n^\frac{2}{3}
< K_n^\frac{2}{3} \left( = \exp \left( \frac{2}{3} K_n^\frac{1}{\tau} \right) \right).
\]
Hence, we obtain
\[
\ln \epsilon_n^{-1} = 40(2\tau + 1)K_{n-1}^{\frac{2}{3}} = 40(2\tau + 1) \exp\left\{\frac{1}{2}K_{n-2}^{\frac{2}{3}}\right\}
\]
> \exp\{K_{n-3}^{\frac{2}{3}}\} > \exp\{\bar{T}_{n-3,N_{n-3}}\}, \quad \forall n \geq 3.

The last inequality is derived from (3.5) with \((n - 3)\) in place of \(n\). This is the proof of (3.2).

(2) To prove (3.3), we distinguish \(n\) according to \(n_*\).

\textit{Case I:} \(0 \leq n \leq n_*\).

Note that \(K_0^{\frac{1}{3}} > 3 \exp \cdots \exp((3^{-1}e^{-4s_0\eta_{n+1}})^{-\frac{1}{3}})\), i.e.,
\[
K_0^{\frac{1}{3}} > \exp \cdots \exp((3^{-1}e^{-4s_0\eta_{n+1}})^{-\frac{1}{3}}).
\]

Thus
\[
\ln \cdots \ln K_{n_*+1}^{\frac{1}{3}} \geq \ln \cdots \ln K_{n_*+1}^{\frac{1}{3}} > (3^{-1}e^{-4s_0\eta_{n+1}})^{-\frac{1}{3}} \geq (3^{-1}e^{-4s_0\eta_{n+1}})^{-\frac{1}{3}}.
\]

Hence by the above inequality, we obtain
\[
\alpha(\ln K_{n_*}^{\frac{1}{3}}) = (\ln \cdots \ln K_{n_*}^{\frac{1}{3}})^{-a} < 3^{-1}e^{-4s_0\eta_{n+1}}, \quad \forall 0 \leq n \leq n_*.
\]

\textit{Case II:} \(n \geq n_* + 1\).

We will use induction to prove the inequality (3.3) in this case. Suppose that for \(j \geq n_*\), the following inequality holds:
\[
\left(\ln \cdots \ln K_{j+1}^{\frac{1}{3}}\right)^{-a} < 3^{-1}e^{-4s_0\eta_{j+1}},
\]

which implies
\[
\ln \cdots \ln K_{n_*+1}^{\frac{1}{3}} > (3^{-1}e^{-4s_0\eta_{j+1}})^{-\frac{1}{3}}.
\]

Note that \(K_{j+1} = \exp\{K_j^{\frac{1}{3}}\}\), then we get
\[
\ln \cdots \ln K_{j+1}^{\frac{1}{3}} = \ln \cdots \ln \exp\left\{\frac{1}{3}K_j^{\frac{1}{3}}\right\} \geq \ln \cdots \ln \exp\{K_j^{\frac{1}{3}}\} = \ln \cdots \ln K_j^{\frac{1}{3}}
\]
> \exp\left\{(3^{-1}e^{-4s_0\eta_{j+1}})^{-\frac{1}{3}}\right\} \geq \frac{1}{2} (3^{-1}e^{-4s_0\eta_{j+1}})^{-\frac{1}{3}}
\]
> \left(3^{-1}e^{-4s_0\eta_{j+1}}\right)^{-\frac{3}{2}} > \left(3^{-1}e^{-4s_0\eta_{j+1}}\right)^{-\frac{3}{2}}.

From the case I, we know that
\[
\left(\ln \cdots \ln K_{n_*}^{\frac{1}{3}}\right)^{-a} < 3^{-1}e^{-4s_0\eta_{n_*+1}}.
\]

Hence, we obtain
\[
\alpha(\ln K_{n_*}^{\frac{1}{3}}) = (\ln \cdots \ln K_{n_*}^{\frac{1}{3}})^{-a} < 3^{-1}e^{-4s_0\eta_{n+1}}, \quad \forall n \geq n_* + 1.
\]

Finally, combining (3.6) with (3.7), we finish the proof of (3.3).
3.3. The homological equation and its approximate solution. The main idea of proving Theorem 2.1 is to construct a series of symplectic coordinate transformations, and the key point in one KAM step is to solve the homological equation to obtain a transformation. In this section, we will show how to solve the variable coefficient homological equation with $\omega$ be the one defined by (1.3).

Recall that $\{G, H\}$ is the Poisson bracket of smooth functions $G(\theta, I, z, \bar{z}; \xi)$ and $H(\theta, I, z, \bar{z}; \xi)$:

$$\{G, H\} = \left\langle \frac{\partial G}{\partial \theta}, \frac{\partial H}{\partial \theta} \right\rangle - \left\langle \frac{\partial G}{\partial I}, \frac{\partial H}{\partial I} \right\rangle + i \left\langle \frac{\partial G}{\partial z}, \frac{\partial H}{\partial \bar{z}} \right\rangle - i \left\langle \frac{\partial G}{\partial \bar{z}}, \frac{\partial H}{\partial z} \right\rangle.$$ 

Assume that, for any $n \in \mathbb{N}$, $B_n(\theta; \xi) = (B^1_n(\theta; \xi), \cdots, B^m_n(\theta; \xi))^T$ is the real analytic vector valued function defined on $D(s_n) \times \mathcal{O}$. Moreover, it can be written in the following form

$$B_n(\theta; \xi) = \sum_{i=0}^{n} b_i(\theta; \xi) = \sum_{i=0}^{n} \sum_{j=0}^{N_{i-1}} b_{i,j}(\theta; \xi), \quad \forall n \geq 0, \quad (3.8)$$

where $b_{i,j}(\theta; \xi) = (b_{i,j}^1(\theta; \xi), \cdots, b_{i,j}^m(\theta; \xi))^T$ are real analytic vector valued functions and for $i = 1, \cdots, n$, $j = 1, \cdots, N_{i-1}$,

$$b_{i,j}(\theta; \xi) = \sum_{k \in \mathbb{Z}^d, |k| \leq \tilde{T}_{i-1,j-1}} \tilde{b}_{i,j}(k; \xi) e^{i(k, \theta)}, \quad \|b_{i,j}\|_{s_n, \mathcal{O}} \leq \tilde{T}_{i-1,j-1}. \quad (3.9)$$

Note that

$$b_{0,j}(\theta; \xi) = 0, \quad \forall j = 0, \cdots, N_{-1},$$

$$b_{i,0}(\theta; \xi) = 0, \quad \forall i = 0, \cdots, n.$$ 

Because there are no such functions in the original systems and $N_{-1}, \tilde{T}_{-1,j}, \bar{c}_{-1,j}$ ($j = 0, \cdots, N_{-1} - 1$) are virtual, we just give them formally.

**Remark 2.** As in our work, one step of KAM iteration will be finished by a finite induction. One can find the detail in the proof of Lemma 3.3. Taking, for example, the $i$-th step of KAM iteration and $j$-th step of the finite induction, there is a term, which we denote as $b_{i,j+1}z$, will be put into the normal form. Thus the function $B_n(\theta; \xi)$ enjoys such a special form (3.8).

For any $n \geq 3$, we define a sequence $\{Q^n_{i,j}\}$ ($i = n - 1, n, j = 0, \cdots, N_{i-1}$). We let $Q_{i,j}^n$ be the smallest integer number such that $\exp\{3^{-Q_{i,j}^n} \tilde{T}_{i-1,j-1}\} \leq \ln \epsilon_n^{-1}$, i.e.,

$$\exp\{3^{-Q_{i,j}^n} \tilde{T}_{i-1,j-1}\} \leq \ln \epsilon_n^{-1} < \exp\{3^{-Q_{i,j}^n-1} \tilde{T}_{i-1,j-1}\}. \quad (3.10)$$

For $i = n - 1, n$ and $j = 0, \cdots, N_{i-1}$, we denote, $\forall l = 0, \cdots, Q^n_{i,j} - 1$,

$$\bar{B}_{i,j}^{(l)}(\theta; \xi) = \sum_{3^{-l}\tilde{T}_{i-1,j-1} < |k| 
 \leq 3^{-l}\tilde{T}_{i-1,j-1}} \tilde{b}_{i,j}(k; \xi) e^{i(k, \theta)},$$

and

$$\bar{B}_{i,j}^{(Q^n_{i,j})}(\theta; \xi) = \sum_{|k| \leq 3^{-Q^n_{i,j}} \tilde{T}_{i-1,j-1}} \tilde{b}_{i,j}(k; \xi) e^{i(k, \theta)}.$$ 

By the above discussion, we can rewrite $B_n(\theta; \xi)$ as

$$B_n(\theta; \xi) = \sum_{i=n-1}^{n} \sum_{j=0}^{N_{i-1}} Q^n_{i,j} \bar{B}_{i,j}^{(l)}(\theta; \xi) + \sum_{i=0}^{n-2} \sum_{j=0}^{N_{i-1}} \bar{B}_{i,j}(\theta; \xi), \quad (3.11)$$
with $\tilde{B}_{i,j}(\theta;\xi) = b_{i,j}(\theta;\xi)$ for all $i = 0, \ldots, n-2$, $j = 0, \ldots, N_{i-1}$.

**Remark 3.** The reason we do not rewrite $b_{i,j}(\theta;\xi)$ ($j = 0, \ldots, N_{i-1}, i = 0, \ldots, n-2$) as the sum of a sequence functions like we did with $b_{i,j}(\theta;\xi)$ ($j = 0, \ldots, N_{i-1}, i = n-1, n$) is that the inequality in (3.2) guarantees that the solutions to the equations about these $\tilde{B}_{i,j}(\theta;\xi)$ are controllable. See the discussion in Lemma 3.2 for the detail. Moreover, if $n \leq 2$, we know that there is no $\tilde{B}_{i,j}(\theta;\xi)$ ($\tilde{B}_{0,j}(\theta;\xi) = 0$) for $j = 0, \ldots, N_{i-1}, i \leq n-2$. So when we consider these terms, we mean $n \geq 3$.

Set

$$N = e_n(\theta;\xi) + \langle \omega, I \rangle + (\langle \Omega(\xi) + B_n(\theta;\xi) + b(\theta;\xi) \rangle z, \tilde{z}).$$

Moreover, for a real analytic function $R(\theta, z, \tilde{z};\xi)$ with Taylor–Fourier expansion

$$R(\theta, z, \tilde{z};\xi) = \sum_{\delta - \beta = 1, |\delta| + |\beta| \leq 2} R_l(\theta;\xi) z^\delta \tilde{z}^\beta = \sum_{\delta - \beta = 1, |\delta| + |\beta| \leq 2} \sum_{k \in \mathbb{Z}^d} \tilde{R}_{\delta,\beta}(k;\xi)e^{i(k,\theta)} z^\delta \tilde{z}^\beta,$$

which is defined on $D(s_n, r_n) \times \mathcal{O}$, we consider the homological equation on the unknown function $F$

$$(3.12) \quad \{F, N\} = R.$$

If we set

$$F(\theta, z, \tilde{z};\xi) = \sum_{\delta - \beta = 1, |\delta| + |\beta| \leq 2} F_l(\theta;\xi) z^\delta \tilde{z}^\beta = \sum_{\delta - \beta = 1, |\delta| + |\beta| \leq 2} \sum_{k \in \mathbb{Z}^d} \tilde{F}_{\delta,\beta}(k;\xi)e^{i(k,\theta)} z^\delta \tilde{z}^\beta,$$

then direct calculation shows that (3.12) is equivalent to the variable coefficients homological equations as follows

$$\partial_\omega F_l(\theta;\xi) + i\langle l, \Omega(\xi) + B_n(\theta;\xi) + b(\theta;\xi) \rangle F_l(\theta;\xi) = R_l(\theta;\xi), \quad \forall 0 < |l| \leq 2.$$

In order to solve the above homological equations, we need to eliminate the effect of the relatively large $B_n(\theta;\xi)$. We remove the non–resonant terms of $B_n(\theta;\xi)$ by solving the equation

$$\partial_\omega B(\theta;\xi) = -B_n(\theta;\xi) + [B_n(\theta;\xi)]_\theta.$$  \hspace{1cm} (3.13)

We will give a good control over the solution of (3.13) in the following lemma.

**Lemma 3.2.** Assume that $B_n(\theta;\xi)$ is the one defined in (3.8) with the estimate (3.9). Then the equation (3.13) has a unique solution satisfying the estimate

$$||B||_{\tilde{z},\mathcal{O}} < (480)^{-1} \ln \epsilon_n^{-1},$$  \hspace{1cm} (3.14)

where $\tilde{s} := s_n(1 - \eta_n) > e^{-4}s_0$.

**Proof.** Rewrite the function $B_n(\theta;\xi)$ as the one in (3.11). Assume that the function $\mathcal{B}^{(l)}_{i,j}(\theta;\xi)$ solves

$$\partial_\omega \mathcal{B}^{(l)}_{i,j}(\theta;\xi) = -\tilde{B}^{(l)}_{i,j}(\theta;\xi) + [\tilde{B}^{(l)}_{i,j}(\theta;\xi)]_\theta$$  \hspace{1cm} (3.15)

for $l = 0, \ldots, Q^n_{i,j}$, $j = 0, \ldots, N_{i-1}$, $i = n-1, n$, and function $\mathcal{B}_{i,j}(\theta;\xi)$ solves

$$\partial_\omega \mathcal{B}_{i,j}(\theta;\xi) = -\tilde{B}_{i,j}(\theta;\xi) + [\tilde{B}_{i,j}(\theta;\xi)]_\theta$$  \hspace{1cm} (3.16)

for $j = 0, \ldots, N_{i-1}$, $i = 0, \ldots, n-2$. Then the function

$$\mathcal{B}(\theta;\xi) = \sum_{j=0}^{n-2} \sum_{i=0}^{N_{i-1}} \mathcal{B}^{(j)}_{i,j}(\theta;\xi) + \sum_{i=0}^{n-2} \sum_{j=0}^{N_{i-1}} \mathcal{B}_{i,j}(\theta;\xi)$$
solves (3.13). By comparing the Fourier coefficients of (3.13), we have

$$\hat{B}(k; \xi) = \frac{i\hat{B}(k; \xi)}{\langle k, \omega \rangle}, \quad \forall k \in \mathbb{Z}^d.$$  

In order to derive the estimate of $B(\theta; \xi)$, we need the following inequality. For any $l = 0, \cdots , Q^n_{i,j}, j = 0, \cdots , N_i, i = n - 1, n$, we know that

$$3^{-l}\bar{T}_{i-1,j-1} \geq 3^{-Q^n_{i,j}}\bar{T}_{i-1,j-1} = 3^{-13}Q^n_{i,j}^{-1}\bar{T}_{i-1,j-1}$$

$$> 3^{-1}\ln\ln e_n^{-1} = 3^{-1}\ln\{40(2\tau + 1)K_n^{-1}\}$$

$$> 3^{-1}\ln\{K_n\} > 3^{-1}\exp\cdots\exp\{(3^{-1}e^{-4}s_0\eta_n + 1)^{-\frac{1}{2}}\}$$

$$> \exp\cdots\exp\{(e^{-4}s_0\eta_n + 1)^{-\frac{1}{2}}\},$$  

(3.17)

where the second inequality is due to the right side of (3.10) and the penultimate inequality is from $K_n \geq \exp\cdots\exp\{(3^{-1}e^{-4}s_0\eta_n + 1)^{-\frac{1}{2}}\}$.

Note that the function

$$\alpha(T) = (\ln \cdots \ln |T|)^{-\alpha}$$

is monotonically decreasing on the interval $[\exp\cdots\exp\{(e^{-4}s_0\eta_n + 1)^{-\frac{1}{2}}\}, +\infty)$. So we know that $\alpha(3^{-l}\bar{T}_{i-1,j-1})$ is well defined due to (3.17) and for each $l \leq Q^n_{i,j}, j \leq N_i, i = n - 1, n$, we have

$$0 < \alpha(3^{-l}\bar{T}_{i-1,j-1}) < e^{-4}s_0\eta_n + 1 < 1.$$  

Let us consider the equation (3.15) first.

Case I: $l = 0, \cdots , Q^n_{i,j} - 1$. By the assumption (1.3) and $\hat{s} = s_n(1 - \eta_n)$, we obtain

$$\|B_{i,j}\|_{\mathcal{S}, \mathcal{O}} = \sum_{3^{-l+1}\bar{T} < k < 3^{-l}\bar{T}} \|\hat{B}_{i,j}(k; \xi)\|_{\mathcal{O}}e^{k|s_n(1-\eta_n)|}$$

$$\leq \exp(3^{-l}\bar{T}_\alpha(3^{-l}\bar{T})) \exp\{-3^{-l+1}\bar{T}s_n\eta_n\}$$

$$\sum_{3^{-l+1}\bar{T} < k < 3^{-l}\bar{T}} \|\hat{B}_{i,j}(k; \xi)\|_{\mathcal{O}}e^{k|s_n|}$$

$$= \exp(3^{-l}\bar{T}_\alpha(3^{-l}\bar{T})) \exp\{-3^{-l+1}\bar{T}s_n\eta_n\} \|\hat{B}_{i,j}\|_{\mathcal{S}_n, \mathcal{O}}$$

$$\leq \exp(3^{-l}\bar{T}_\alpha(3^{-l}\bar{T})) \exp\{-3^{-l+1}\bar{T}3^{-1}e^{-4}s_0\eta_n\} \|\hat{B}_{i,j}\|_{\mathcal{S}_n, \mathcal{O}}$$

$$\leq \|\hat{B}_{i,j}\|_{\mathcal{S}_n, \mathcal{O}} < \ln e_n^{-1}\|\hat{B}_{i,j}\|_{\mathcal{S}_n, \mathcal{O}},$$

where $\bar{T} := \bar{T}_{i-1,j-1}$ in this formula for short and the third inequality follows from the following:

First, using the same calculations in (3.17), for $l \leq Q^n_{i,j} - 1, j \leq N_i, i = n - 1, n$, we obtain

$$3^{-l}\bar{T}_{i-1,j-1} \geq 3^{-Q^n_{i,j} - 1}\bar{T}_{i-1,j-1} > \ln \ln e_n^{-1} = \ln\{40(2\tau + 1)K_n^{-2}\} > \ln K_n^{-2},$$

which implies
\[ \alpha(3^{-l} \overline{T}_{n-1,j-1}) < \alpha(\ln K_{n-1}^2). \]  

(3.18)

Moreover, from (3.3) and (3.18), we have
\[ \alpha(3^{-l} \overline{T}_{n-1,j-1}) < \alpha(\ln K_{n-1}^2) < 3^{-1}e^{-4}s_0\eta_n, \quad \forall l \leq Q_{n-1}. \]

Thus
\[ \exp\{3^{-l} \overline{T}_{n-1,j-1}\alpha(3^{-l} \overline{T}_{n-1,j-1})\} < \exp\{3^{-l} \overline{T}_{n-1,j-1}3^{-1}e^{-4}s_0\eta_n\}, \]

i.e.,
\[ \exp\{3^{-l} \overline{T}_{n-1,j-1}\alpha(3^{-l} \overline{T}_{n-1,j-1})\} \exp\{-3^{-l} \overline{T}_{n-1,j-1}3^{-1}e^{-4}s_0\eta_n\} < 1. \]

Case II: \( l = Q_{n-1}. \) Due to \( \alpha(3^{-l} \overline{T}_{n-1,j-1}) < 1, \) we have the following estimate:
\[ \|B_{i,j}\|_{\tilde{s},O} = \sum_{0<|k|\leq 3^{-l}Q_{n-1}} \|\tilde{B}_{i,j}^{(Q_{n-1})}(k;\xi)\|_{O} e^{k|s_n(1-n\varphi)|} \]
\[ \leq \sum_{0<|k|\leq 3^{-l}Q_{n-1}} \exp\{3^{-l} \overline{T}_{n-1,j-1}\} \|\tilde{B}_{i,j}^{(Q_{n-1})}(k;\xi)\|_{O} e^{k|s_n(1-n\varphi)|} \]
\[ = \exp\{3^{-l} \overline{T}_{n-1,j-1}\} \|\tilde{B}_{i,j}^{(Q_{n-1})}\|_{\tilde{s},O} \leq \ln \epsilon^{-1}_n \|\tilde{B}_{i,j}^{(Q_{n-1})}\|_{s_n,O}, \]

where the last inequality is derived from the inequality on the left side of (3.10).

Now we consider the equation (3.16). Note that \( \overline{T}_{n-1,j-1} \leq \overline{T}_{n-3,n-3-1}, \forall j = 0, \cdots , N_{i-1}, \) \( i \leq n - 2, \) and due to (3.2), we have for \( j = 0, \cdots , N_{i-1}, i \leq n - 2, \)
\[ \exp\{\overline{T}_{i-1,j-1}\} \leq \exp\{\overline{T}_{n-3,n-3-1}\} < \ln \epsilon^{-1}_n. \]

Moreover, by the similar calculations in (3.5), we also have
\[ \overline{T}_{n-3,n-3-1} = \sigma_{n,N_{i-1}}^{-1} \ln \epsilon_{n,N_{i-1}}^{-1} = \frac{14}{15} \sigma_{n,N_{i-1}}^{-1} \ln \epsilon_{n,N_{i-1}}^{-1} > \ln \epsilon^{-1}_n > K_n^2. \]

(3.19)

Therefore, by (3.3) and (3.19), we also have, for \( n \geq 3, \)
\[ \alpha(\overline{T}_{n-3,n-3-1}) < \alpha(\overline{T}_{n,N_{i-1}}) < \alpha(\ln K_n^2) < 3^{-1}e^{-4}s_0\eta_n < 1. \]

Then with the similar discussions in the case II, we obtain, for \( i \leq n - 2, j = 0, \cdots , N_{i-1}, \)
\[ \|B_{i,j}\|_{\tilde{s},O} \leq \ln \epsilon^{-1}_n \|\tilde{B}_{i,j}\|_{s_n,O}. \]

Due to the above discussion, the function \( B(\theta; \xi) \), the solution to (3.13), satisfies
\[ \|B\|_{\tilde{s},O} = \left\| \sum_{i=0}^{n-1} \sum_{j=0}^{N_{i-1}} Q_{n-1}^{(j)} B_{i,j}^{(j)} + \sum_{i=0}^{n-2} \sum_{j=0}^{N_{i-1}} B_{i,j} \right\|_{\tilde{s},O} \]
\[ \leq \sum_{i=0}^{n-1} \sum_{j=0}^{N_{i-1}} \left\| B_{i,j}^{(j)} \right\|_{\tilde{s},O} + \sum_{i=0}^{n-2} \sum_{j=0}^{N_{i-1}} \left\| B_{i,j} \right\|_{\tilde{s},O} \]
\[ \leq \ln \epsilon^{-1}_n \left\{ \sum_{i=0}^{n-1} \sum_{j=0}^{N_{i-1}} \left\| B_{i,j}^{(j)} \right\|_{s_n,O} + \sum_{i=0}^{n-2} \sum_{j=0}^{N_{i-1}} \left\| B_{i,j} \right\|_{s_n,O} \right\} \]
\[ = \ln \epsilon^{-1}_n \sum_{i=0}^{n-1} \sum_{j=0}^{N_{i-1}} \|b_{i,j}\|_{s_n,O} < 2\epsilon_0 \ln \epsilon^{-1}_n < (480)^{-1} \ln \epsilon^{-1}_n. \]
Now we consider the homological equation (3.12), and we have the following proposition.

**Proposition 1.** Assume that $B_n(\theta; \xi)$ is given by (3.8) and $b(\theta; \xi)$ is defined on $D(s) \times \mathcal{O}$ $(e^{-4}s_0 < s < s_n(1 - \eta_n))$ satisfying $\|b\|_{s, \mathcal{O}} \leq \epsilon_n$. If for every $0 < \gamma < 1, \tau > d$, and $\xi \in \mathcal{O}$ such that $\bar{\Omega}(\xi) := \Omega(\xi) + [B_n(\theta; \xi)]_\theta$ satisfies the Melnikov’s condition

$$\|\langle k, \omega \rangle + \langle l, \bar{\Omega}(\xi) \rangle \| \geq \gamma \langle k \rangle^{-\tau}, \quad k \in \mathbb{Z}^d, \quad 0 < |l| \leq 2,$$

(3.20)

where $\langle k \rangle = \max\{|k|, 1\}$, then for the real analytic function $R$ defined on $D(s, r) \times \mathcal{O}$, the homological equation (3.12) has a real analytic approximate solution $F$ of the same form as $R$ satisfying

$$\|F\|_{s, r, \mathcal{O}} \leq 32\gamma^{-2} T_{n, j}^{2\tau} \epsilon_n^{-\frac{1}{2}} \|R\|_{s, r, \mathcal{O}}.$$

Moreover, the error term is

$$P^{(c)}(\theta, z, \bar{z}; \xi) = \sum_{|\delta - \beta| = 1, 1 \leq |\delta + \beta| \leq 2, \delta \neq \beta} P^{(c)}_l(\theta; \xi) z^\delta \bar{z}^\beta$$

with

$$P^{(c)}_l(\theta; \xi) = e^{i\langle l, B(\theta; \xi) \rangle} R_l(\theta; \xi) - i\langle l, b(\theta; \xi) \rangle e^{-i\langle l, B(\theta; \xi) \rangle} F_l(\theta; \xi),$$

and for $\sigma < s$, the following estimate holds:

$$\|P^{(c)}\|_{s-\sigma, r, \mathcal{O}} \leq e^{-T_{n, j}^{-\frac{1}{2}} \epsilon_n} \|R\|_{s, r, \mathcal{O}} + 2\epsilon_n \|F\|_{s, r, \mathcal{O}}.$$

**Proof.** Firstly, we consider the case $n \geq 1$. For the functions $R(\theta, z, \bar{z}; \xi)$ and $F(\theta, z, \bar{z}; \xi)$ with the Taylor expansions

$$R(\theta, z, \bar{z}; \xi) = \sum_{|\delta - \beta| = 1, 1 \leq |\delta + \beta| \leq 2} R_l(\theta; \xi) z^\delta \bar{z}^\beta, \quad F(\theta, z, \bar{z}; \xi) = \sum_{|\delta - \beta| = 1, 1 \leq |\delta + \beta| \leq 2} F_l(\theta; \xi) z^\delta \bar{z}^\beta,$$

we denote $\tilde{R}_l(\theta; \xi) = e^{-i\langle l, B(\theta; \xi) \rangle} R_l(\theta; \xi)$ and $\tilde{F}_l(\theta; \xi) = e^{-i\langle l, B(\theta; \xi) \rangle} F_l(\theta; \xi)$, where $B(\theta; \xi)$ is the solution of (3.13) in Lemma 3.2. Then from (3.12), we obtain

$$\partial_{\omega} \tilde{F}_l(\theta; \xi) + i\langle l, \bar{\Omega}(\xi) + b(\theta; \xi) \rangle \tilde{F}_l(\theta; \xi) = \tilde{R}_l(\theta; \xi),$$

where $\bar{\Omega}(\xi) = \Omega(\xi) + [B_n(\theta; \xi)]_\theta$. We first solve

$$T_{\bar{\tau}, n, j}[\partial_{\omega} \tilde{F}_l(\theta; \xi) + i\langle l, \bar{\Omega}(\xi) + b(\theta; \xi) \rangle \tilde{F}_l(\theta; \xi)] = T_{\bar{\tau}, n, j} \tilde{R}_l(\theta; \xi), \quad T_{\bar{\tau}, n, j} \tilde{F}_l(\theta; \xi) = \tilde{F}_l(\theta; \xi).$$

By comparing the Fourier coefficients, for each $0 < |l| \leq 2$ and $|k| \leq \bar{\tau}_{n, j}$, we have

$$i(\langle k, \omega \rangle + \langle l, \bar{\Omega}(\xi) \rangle) \tilde{F}_l(k; \xi) + i \sum_{|k_1| \leq \bar{\tau}_{n, j}} \langle l, \bar{b}(k - k; \xi) \rangle \tilde{F}_l(k_1; \xi) = \tilde{R}_l(k; \xi).$$

(3.23)

Rewrite (3.23) as a matrix equation

$$(\tilde{E} + \Xi \tilde{D} \Xi^{-1}) \Xi_{n, j} F_l = \Xi_{n, j} R_l,$$

where

$$\tilde{E} = \text{diag}(\cdots, i(\langle k, \omega \rangle + \langle l, \bar{\Omega}(\xi) \rangle), \cdots)_{|k| \leq \bar{\tau}_{n, j}},$$

$$\tilde{D} = i(\langle l, \bar{b}(k - k; \xi) \rangle)_{|k_1|, |k| \leq \bar{\tau}_{n, j}}, \quad \Xi = \text{diag}(\cdots, e^{i|k|}, \cdots)_{|k| \leq \bar{\tau}_{n, j}},$$

$$\tilde{F}_l = F_l(\xi) = (\tilde{F}_l(k; \xi))_{|k| \leq \bar{\tau}_{n, j}}, \quad R_l = R_l(\xi) = (\tilde{R}_l(k; \xi))_{|k| \leq \bar{\tau}_{n, j}}.$$
Since $\omega$ and $\hat{\Omega}(\xi)$ satisfy (3.20), we have
\[
\|\hat{E}^{-1}\|_{op(l^1)} \leq \gamma^{-1}\hat{T}_{n,j}^\tau,
\]
where $op(l^1)$ denotes the operator norm associated with the $l^1$-norm, which is defined by, for the vector $u = (u(k))^T [k] \leq \hat{T}_{n,j}$, $\|u\|_1 = \sum_{[k] \leq \hat{T}_{n,j}} |u(k)|$. Due to the inequality (3.5), we obtain
\[
\hat{T}_{n,j} \leq \hat{T}_{n,Nn-1} < \exp\left\{ \frac{2}{3} \hat{K}_{n-1}^\tau \right\} \leq \exp\{K_{n-1}^\tau\} = \epsilon_n^{-\frac{1}{2m(\lambda+\nu)}}.
\]
Therefore,
\[
\|\hat{E}^{-1}\|_{op(l^1)} \leq \gamma^{-1}\hat{T}_{n,j}^\tau < 4^{-1}\epsilon_n^{-\frac{1}{2m}}.
\]
By direct calculation, we have
\[
\|\hat{\Xi}_{s} \hat{D}\hat{\Xi}_{s}^{-1}\|_{op(l^1)} \leq 2\|b\|_{s,0} = 2\epsilon_n.
\]
The above two inequalities yield
\[
\|\hat{E}^{-1}\hat{\Xi}_{s} \hat{D}\hat{\Xi}_{s}^{-1}\|_{op(l^1)} \leq \frac{1}{2},
\]
which implies that $\hat{E} + \hat{\Xi}_{s} \hat{D}\hat{\Xi}_{s}^{-1}$ has a bounded inverse. Therefore, we obtain
\[
\|(\hat{E} + \hat{\Xi}_{s} \hat{D}\hat{\Xi}_{s}^{-1})^{-1}\|_{op(l^1)} \leq \|(I + \hat{E}^{-1}\hat{\Xi}_{s} \hat{D}\hat{\Xi}_{s}^{-1})^{-1}\|_{op(l^1)} \leq \|\hat{E}^{-1}\|_{op(l^1)} \leq 2\gamma^{-1}\hat{T}_{n,j}^\tau.
\]
It follows that
\[
\|\hat{F}_l|_{s,0}^* = \sum_{[k] \leq \hat{T}_{n,j}} \|\hat{F}_l(k)|_{s,0}^* = \|\hat{\Xi}_s \hat{F}_l|_{s,0}^* \leq \|\hat{E} + \hat{\Xi}_s\hat{D}\hat{\Xi}_s^{-1}\|\|\hat{R}_l|_{s,0}^* \leq 2\gamma^{-1}\hat{T}_{n,j}^\tau \|\hat{R}_l|_{s,0}^*.
\]
Therefore,
\[
\|F_l|_{s,0}^* \leq \epsilon^2\|B|_{s,0} \|\hat{F}_l|_{s,0}^* \leq 2\gamma^{-1}\epsilon^2\|B|_{s,0} \hat{T}_{n,j}^\tau \|\hat{R}_l|_{s,0}^* \leq 2\gamma^{-1}\hat{T}_{n,j}^\tau \epsilon_n^{-1\frac{1}{2m}} \|\hat{R}_l|_{s,0}^*.
\]
where the last inequality is from (3.14) in Lemma 3.2. Since $F$ and $R$ are polynomials in variables $\xi$ and $\hat{z}$ of order 2, we obtain
\[
\|F|_{s,r,0}^* \leq 2\gamma^{-1}\hat{T}_{n,j}^\tau \epsilon_n^{-1\frac{1}{2m}} \|R|_{s,r,0}^*.
\]
Next, we give the estimate about $\|F|_{s,r,0}^L$. Denote $\Delta\xi, \xi W = W(\cdot; \xi) - W(\cdot; \xi_2)$. For $[k] \leq T_{n,j}$, from (3.23), we have
\[
i(\langle k, \omega \rangle + \langle l, \hat{\Omega}(\xi) \rangle)\Delta\xi, \xi \hat{F}_l(k) + i \sum_{[k_1] \leq T_{n,j}} \langle l, \hat{b}(k-k_1; \xi) \rangle \Delta\xi, \xi \hat{F}_l(k_1)
\]
\[
+ i\Delta\xi, \xi_2 \langle \langle k, \omega \rangle + \langle l, \hat{\Omega} \rangle \rangle \hat{F}_l(k; \xi_2) + i \sum_{[k_1] \leq T_{n,j}} \langle l, \Delta\xi, \xi_2 \hat{b}(k-k_1) \rangle \hat{F}_l(k_1; \xi_2)
\]
\[
= \Delta\xi, \xi \hat{R}_l(k).
\]
In the similar way to get (3.24), we obtain
\[
\|\Delta\xi, \xi F_l|_{s,0}^* \leq 2\gamma^{-1}\hat{T}_{n,j}^\tau \epsilon_n^{-1\frac{1}{2m}} \left\{ \|\Delta\xi, \xi(\hat{E} + \hat{D})\|_{op(l^1)} \|F_l|_{s,0}^* + \|\Delta\xi, \xi \hat{R}_l|_{s,0}^* \right\}.
\]
Dividing $|\xi_1 - \xi_2|$ and taking supreme over $\xi_1 \neq \xi_2 \in \mathcal{O}$, and by the similar way to get (3.25), we obtain
\[
\|F\|_{s,r,\mathcal{O}}^L \leq 2\gamma^{-1}T_{n,j}^*\epsilon_n^{-\frac{1}{10}}\left\{ (5\epsilon_0 + 2\rho)\|F\|_{s,r,\mathcal{O}}^* + \|R\|_{s,r,\mathcal{O}}^L \right\} \\
\leq 2\gamma^{-1}T_{n,j}^*\epsilon_n^{-\frac{1}{10}}\left\{ (10\epsilon_0 + 4\rho)\gamma^{-1}T_{n,j}^*\epsilon_n^{-\frac{1}{10}}\|R\|_{s,r,\mathcal{O}}^* + \|R\|_{s,r,\mathcal{O}}^L \right\} \\
\leq 30\gamma^{-2}T_{n,j}^*\epsilon_n^{-\frac{1}{10}}\|R\|_{s,r,\mathcal{O}},
\]
provided that $\epsilon_0^\frac{1}{10} \rho \leq 1$ and $\epsilon_0$ is small enough.

From (3.25) and the above inequality, we obtain
\[
\|F\|_{s,r,\mathcal{O}} \leq 32\gamma^{-2}T_{n,j}^*\epsilon_n^{-\frac{1}{10}}\|R\|_{s,r,\mathcal{O}}.
\]

Obviously, the error term $P(\epsilon)$ can be shown by the formula in (3.21) and the following estimate holds:
\[
\|P(\epsilon)\|_{s,\mathcal{O}} \leq e^{-\frac{1}{10}T_{n,j}^*}\epsilon_n^{-\frac{1}{10}} (\|R\|_{s,\mathcal{O}} + 2\|b\|_{s,\mathcal{O}} \|F\|_{s,\mathcal{O}}).
\]

Then the estimate in (3.22) follows straightforwardly.

In the case $n = 0$, $B_0(\theta; \xi) = 0$, we do not need to make the transformation $R_l(\theta; \xi) = e^{-i(\mathcal{B}(\theta; \xi))} R_l(\theta; \xi)$ and $F_l(\theta; \xi) = e^{-i(\mathcal{B}(\theta; \xi))} F_l(\theta; \xi)$, but directly deal with the equations with respect to $F_l$ and $R_l$. In this case,
\[
P_l(\epsilon)(\theta; \xi) = \mathcal{R}_{T_{n,j}} [R_l(\theta; \xi) - i(l, b(\theta; \xi)) F_l(\theta; \xi)],
\]
and the estimates about $F$ and $P(\epsilon)$ also hold.

\[\square\]

3.4. Iterative lemma. Beside the parameters defined in (3.1), for $0 < \gamma < 1$, $\epsilon_0^\frac{13}{20} < r < 1$, we will also define the iterative sequences $(\gamma_n)_{n \geq 0}$, $(r_n)_{n \geq 0}$, and a sequence of domain $(D_n)_{n \geq 0}$ in the following manner:
\[
r_0 = r, \quad r_{n+1} = \epsilon_n^\gamma r_n, \quad \gamma_0 = \gamma, \quad \gamma_n = \gamma_0 \eta_n, \quad D_n = D(s_n, r_n).
\]

For the sequences $\{\epsilon_n\}_{n \geq 0}$ defined in (3.1) and $\{r_n\}_{n \geq 0}$, we know that $\epsilon_n$ decreases much more quickly than $r_n$. Actually,
\[
\epsilon_{n+1} = \exp\{-40(2r + 1)K_n\} < \exp\{-800(2r + 1)K_{n-1}\} = \epsilon_n^{20},
\]
since $K_n > 20K_{n-1}$. That is $\epsilon_n > \epsilon_n^{\frac{20}{21}}$. Therefore,
\[
\epsilon_{n+1} = \epsilon_n \epsilon_n^{-1} \epsilon_{n+1} < \epsilon_n \epsilon_n^{\frac{19}{21}}.
\]

Obviously, $\epsilon_n^\frac{19}{21} \epsilon_{n+1} < \epsilon_n^{\frac{19}{21}}$.

According to the preceding analysis, we can obtain the following Iterative Lemma.

Lemma 3.3 (Iterative Lemma). Suppose that the real analytic Hamiltonian $H_n = N_n + P_n$ is defined on $D_n \times \mathcal{O}_n$, where
\[
N_n = e_n(\theta; \xi) + (\omega, I) + ((\Omega(\xi) + B_n(\theta; \xi))z, \bar{z}), \quad P_n = P_n(\theta, z, \bar{z}; \xi),
\]
satisfying
\[
\|e_n - e_{n-1}\|_{s_n, \mathcal{O}_n} \leq 2\epsilon_{n-1},
\]
\[
\|P_n\|_{s_n, r_n, \mathcal{O}_n} \leq \epsilon_n.
\]
B_n(\theta; \xi) is defined in (3.8) satisfying the estimates (3.9), and the parameter set \( \mathcal{O}_n \) is defined as

\[
\mathcal{O}_n = \left\{ \xi \in \mathcal{O}_{n-1} : |\langle k, \omega \rangle + \langle l, \Omega(\xi) + [B_n(\theta; \xi)]_\phi \rangle| \geq \frac{\gamma_n}{(k)^{\gamma}}, \ \forall 0 < |l| \leq 2, k \in \mathbb{Z}^d \right\}.
\]

Then there exist a real analytic symplectic transformation of variables

\[
\Phi_{n+1} : D_{n+1} \times \mathcal{O}_n \to D_n
\]

of form

\[
(\theta, I, z, \tilde{z}) \mapsto (\theta, U_n(\theta, I, z, \tilde{z}; \xi), W_n(\theta, z, \tilde{z}; \xi), \overline{W}_n(\theta, z, \tilde{z}; \xi)),
\]

where \( W_n, \overline{W}_n : D_{n+1} \times \mathcal{O}_n \to D_n \) are affine in \( z \) and \( \tilde{z} \), satisfying

\[
\|\Phi_{n+1} - id\|_{s_{n+1}, r_{n+1}; \mathcal{O}_n} \leq 2\varepsilon_n^2,
\]

and a subset \( \mathcal{O}_{n+1} \subset \mathcal{O}_n \),

\[
\mathcal{O}_{n+1} = \left\{ \xi \in \mathcal{O}_n : |\langle k, \omega \rangle + \langle l, \Omega(\xi) + [B_{n+1}(\theta; \xi)]_\phi \rangle| \geq \frac{\gamma_{n+1}}{(k)^{\gamma}}, \ \forall 0 < |l| \leq 2, k \in \mathbb{Z}^d \right\},
\]

such that \( H_{n+1} = H_n \circ \Phi_{n+1} \) has the analogous form of \( H_n \) and satisfies the conditions (3.26) and (3.27), and \( B_{n+1}(\theta; \xi) \) is given by the formula in (3.8) and satisfies the estimates (3.9) with \( (n+1) \) in place of \( n \).

3.5. Proof of Lemma 3.3. Because we don’t impose any Diophantine restriction on \( \omega \), we need a finite induction to construct a bounded transformation \( \Phi_{n+1} \) at the \( n \)-th step of KAM iteration.

In order to prove Lemma 3.3, we let

\[
\tilde{\tau}_{n,0} = r_n, \quad \tilde{s}_{n,0} = s_n(1 - \eta_n), \quad \gamma = \gamma_n, \quad \mathcal{O} = \mathcal{O}_n.
\]

For \( j \geq 0 \), we define the following sequences:

\[
\delta_{n,j} = e_{n,j}^+ - e_{n,j}^-, \quad \tau_{n,j+1} = \delta_{n,j} \tilde{r}_{n,j}, \quad 
\tilde{s}_{n,j+1} = \tilde{s}_{n,j} - 4\sigma_{n,j}, \quad D_j = D(\tilde{s}_{n,j}, \tilde{r}_{n,j}).
\]

Consider the real analytic Hamiltonian \( H_n = N_n + P_n \) defined on \( D_n \times \mathcal{O}_n \), and we rewrite it as

\[
\tilde{H}_0 = \tilde{N}_0 + \tilde{P}_0 = \tilde{e}_{n,0}(\theta; \xi) + \langle \omega, I \rangle + \langle \Omega(\xi) + B(\theta; \xi), z, \tilde{z} \rangle + \tilde{P}_0(\theta, z, \tilde{z}; \xi),
\]

which is defined on \( \tilde{D}_0 \times \mathcal{O} \), where \( \tilde{e}_{n,0}(\theta; \xi) := e_n(\theta; \xi), B(\theta; \xi) := B_n(\theta; \xi), \) and \( \tilde{P}_0 = P_n \). Obviously,

\[
\|\tilde{P}_0\|_{\tilde{D}_0, \mathcal{O}} \leq \tilde{\epsilon}_{n,0}.
\]

Proposition 2. Suppose that assumptions in Lemma 3.3 hold, then for each \( \xi \in \mathcal{O} \), the following results hold: For \( 0 \leq j \leq N_n - 1 \), there exists a real analytic Hamiltonian \( F_{j+1} \) such that

\[
\tilde{H}_{j+1} = \tilde{H}_j \circ X_{F_{j+1}}^1 = \tilde{N}_{j+1} + \tilde{P}_{j+1}
\]

\[
= \tilde{e}_{n,j+1}(\theta; \xi) + \langle \omega, I \rangle + \langle \Omega(\xi) + (B + \sum_{l=0}^{j+1} b_{n+1,l}) \rangle(\theta; \xi) z, \tilde{z} \rangle + \tilde{P}_{j+1}(\theta, z, \tilde{z}; \xi),
\]

with

\[
\|\tilde{P}_{j+1}\|_{\tilde{s}_{n,j+1}, \tilde{r}_{n,j+1}, \mathcal{O}} < \tilde{\epsilon}_{n,j+1}, \quad (3.30)
\]

\[
\|\tilde{e}_{n,j+1} - \tilde{e}_{n,j}\|_{\tilde{s}_{n,j}, \mathcal{O}} < \tilde{\epsilon}_{n,j}, \quad (3.31)
\]
where \( b_{n+1,l}(\theta; \xi) = \text{diag}(b_{n+1,l}^1(\theta; \xi), \ldots, b_{n+1,l}^m(\theta; \xi)) \), \( b_{n+1,0}(\theta; \xi) = 0 \), and
\[
\begin{align*}
\forall \nu \geq 1, \quad & b_{n+1,\nu}(\theta; \xi) = \sum_{|k| \leq \tilde{T}_{n,l-1}} \tilde{b}_{n+1,\nu}(k; \xi)e^{i\langle k, \theta \rangle}, \\
& \|b_{n+1,\nu}\|_{\tilde{X}_{\nu,0}} \leq \tilde{c}_{n,l-1}.
\end{align*}
\tag{3.32}
\]
on domain \( \tilde{D}_{j+1} \times \mathcal{O} \).

Moreover, the real analytic symplectic map \( X_{F_{j+1}}^1 \) satisfies
\[
\begin{align*}
\|X_{F_{j+1}}^1 - i\theta\|_{\tilde{n}_{j+1}, \tilde{r}_{j+1}, \mathcal{O}} & < \epsilon_{n,j}^{\frac{4}{3}}, \tag{3.33} \\
\|DX_{F_{j+1}}^1 - I\|_{\tilde{n}_{j+1}, \tilde{r}_{j+1}, \mathcal{O}} & < \epsilon_{n,j}^{\frac{4}{5}}. \tag{3.34}
\end{align*}
\]

Proof. Suppose that there exists a real analytic Hamiltonian \( F_{\nu} \) such that
\[
H_{\nu} = H_{\nu - 1} \circ X_{F_{\nu}}^1 = \tilde{N}_{\nu} + \tilde{P}_{\nu}
\]
satisfies (3.30) and (3.31) with \((\nu - 1)\) in place of \( j \). Then our goal is to find \( F_{\nu + 1} \) defined on \( \tilde{D}_{\nu + 1} \times \mathcal{O} \) such that \( H_{\nu + 1} = H_{\nu} \circ X_{F_{\nu + 1}}^1 = \tilde{N}_{\nu + 1} + \tilde{P}_{\nu + 1} \) satisfies the corresponding estimates. Moreover, the symplectic map \( X_{F_{\nu + 1}}^1 \) satisfies the estimates (3.33) and (3.34) with \( \nu \) in place of \( j \) on the domain \( \tilde{D}_{\nu + 1} \times \mathcal{O} \).

In the following, we will construct such function \( F_{\nu + 1} \). By using the Taylor–Fourier expansion, we rewrite \( \tilde{P}_{\nu} \) as
\[
\tilde{P}_{\nu}(\theta, z, \bar{z}; \xi) = \sum_{\delta, \beta \in \mathbb{C}^n} \sum_{k \in \mathbb{Z}^d} \tilde{P}_{\nu, \delta, \beta}(k; \xi)e^{i\langle k, \theta \rangle}z^\delta \bar{z}^\beta,
\]
and split it into three different parts:
\[
\tilde{P}_{\nu} = R_{\nu}^{(el)} + R_{\nu}^{(nf)} + R_{\nu}^{(pe)},
\]
where
\[
\begin{align*}
R_{\nu}^{(el)}(\theta, z, \bar{z}; \xi) &= \sum_{|\delta + \beta| \leq 2, \delta \neq \beta, k} \tilde{P}_{\nu, \delta, \beta}(k; \xi)e^{i\langle k, \theta \rangle}z^\delta \bar{z}^\beta, \\
R_{\nu}^{(nf)}(\theta, z, \bar{z}; \xi) &= \sum_{|\delta + \beta| \leq 2, \delta = \beta, |k| \leq \tilde{T}_{n,\nu}} \tilde{P}_{\nu, \delta, \beta}(k; \xi)e^{i\langle k, \theta \rangle}z^\delta \bar{z}^\beta
\quad + \sum_{|\delta + \beta| \geq 3, k} \tilde{P}_{\nu, \delta, \beta}(k; \xi)e^{i\langle k, \theta \rangle}z^\delta \bar{z}^\beta \\
&= R_{\nu}^{(pe1)} + R_{\nu}^{(pe2)}.
\end{align*}
\]
Shorten the notations \( \tilde{r}_{n,\nu}, \tilde{s}_{n,\nu}, \sigma_{n,\nu}, \delta_{n,\nu}, \tilde{T}_{n,\nu} \) as \( \tilde{r}, \tilde{s}, \sigma, \delta \) and \( T \) respectively. Obviously,
\[
\|R_{\nu}^{(el)}\|_{\tilde{s}, \tilde{r}, \mathcal{O}}, \quad \|R_{\nu}^{(nf)}\|_{\tilde{s}, \tilde{r}, \mathcal{O}} \leq \|\tilde{P}_{\nu}\|_{\tilde{s}, \tilde{r}, \mathcal{O}} \leq \epsilon_{n,\nu}.
\tag{3.35}
\]

For \( R_{\nu}^{(pe1)} \), we can rewrite it as
\[
R_{\nu}^{(pe1)}(\theta, z, \bar{z}; \xi) = \int \sum_{|\delta + \beta| = 3} \frac{\partial|\delta + \beta|}{\partial \rho^\delta \partial \bar{p}^\beta} \tilde{P}_{\nu}(\theta, \rho, \bar{p}; \xi) \, d\rho^\delta \, d\bar{p}^\beta,
\]
where the symbol $\int$ stands for $\int_0^{z_1} \cdots \int_0^{z_1} \cdots \int_0^{z_m} \int_0^{b_{m+1}}$, $\delta = (\delta_1, \cdots, \delta_m)$ and
\[ \beta = (\beta_1, \cdots, \beta_m) \] with $|\delta + \beta| = 3$. Then by Cauchy estimate, we obtain
\[ \|R^{(pc)}_{\nu}\|_{\tilde{z}, 2\delta, \tilde{\nu}, \mathcal{O}} \leq \frac{c(\delta f)^3}{\tilde{f}^3} \left\| \tilde{P}_\nu \right\|_{\tilde{z}, \tilde{\nu}, \mathcal{O}} = c\delta^3 \left\| \tilde{P}_\nu \right\|_{\tilde{z}, \tilde{\nu}, \mathcal{O}} \leq c\frac{\epsilon_{n,\nu} + 1}{16}, \]
where the constant $c$ depends only on $d$ and $m$. Moreover, for $R^{(pe)}_{\nu}$, we have
\[ \|R^{(pe)}_{\nu}\|_{\tilde{z}, \sigma, 2\delta, \tilde{\nu}, \mathcal{O}} \leq e^{-T\sigma} \|\tilde{P}_\nu\|_{\tilde{z}, \tilde{\nu}, \mathcal{O}} \leq \epsilon_{n,\nu} \|\tilde{P}_\nu\|_{\tilde{z}, \tilde{\nu}, \mathcal{O}} < \frac{\epsilon_{n,\nu} + 1}{16}. \]
That is
\[ \|R^{(pe)}_{\nu}\|_{\tilde{z}, \sigma, 2\delta, \tilde{\nu}, \mathcal{O}} < \frac{\epsilon_{n,\nu} + 1}{8}. \] (3.36)
We rewrite $\tilde{H}_\nu$ as $\tilde{H}_\nu = \tilde{N}_{\nu+1} + R^{(cl)}_{\nu} + R^{(pe)}_{\nu}$, where
\[ \tilde{N}_{\nu+1} := \tilde{N}_\nu + R^{(nf)}_{\nu} = \epsilon_{n,\nu+1}(\theta; \xi) + \langle \omega, \nu \rangle + \langle \Omega(\xi) + (B + \sum_{l=0}^{\nu+1} b_{n+1,l}) \nu \rangle \tilde{z}, \tilde{z} \]
with $\epsilon_{n,\nu+1}(\theta; \xi) = \epsilon_{n,\nu}(\theta; \xi) + \epsilon_{n,\nu+1}(\theta; \xi)$. Obviously, $\tilde{N}_{\nu+1}$ is real analytic since $\tilde{N}_\nu$ and $\tilde{P}_\nu$ are real analytic.

The transformation of variables we need is the time-1-map of the flow $X^1_{F_{\nu+1}}$. Using Taylor formula to expand $\tilde{H}_\nu \circ X^1_{F_{\nu+1}}$, we obtain
\[ \begin{align*}
\tilde{H}_\nu \circ X^1_{F_{\nu+1}} &= \tilde{N}_{\nu+1} \circ X^1_{F_{\nu+1}} + R^{(cl)}_{\nu} \circ X^1_{F_{\nu+1}} + R^{(pe)}_{\nu} \circ X^1_{F_{\nu+1}} \\
&= \tilde{N}_{\nu+1} + \{ \tilde{N}_{\nu+1}, F_{\nu+1} \} + R^{(cl)}_{\nu} + \int_0^1 \{ R^{(cl)}_{\nu}, F_{\nu+1} \} \circ X^1_{F_{\nu+1}} dt \\
&\quad + \int_0^1 (1 - t) \{ \{ \tilde{N}_{\nu+1}, F_{\nu+1} \}, F_{\nu+1} \} \circ X^1_{F_{\nu+1}} dt + R^{(pe)}_{\nu} \circ X^1_{F_{\nu+1}}.
\end{align*} \]
We want to find $F_{\nu+1}$ such that
\[ \{ F_{\nu+1}, \tilde{N}_{\nu+1} \} = R^{(cl)}_{\nu}. \] (3.37)
From Proposition 1, we know that the homological equation (3.37) has a real analytic approximate solution satisfying
\[ \|F_{\nu+1}\|_{\tilde{z}, \tilde{\nu}, \mathcal{O}} \leq 32\gamma^{-2} T^{2\gamma} \epsilon_n^{-\frac{1}{8}} \|R^{(cl)}_{\nu}\|_{\tilde{z}, \tilde{\nu}, \mathcal{O}}. \] (3.38)
Moreover, the error term $P^{(e)}_{\nu+1}$ satisfies
\[ \begin{align*}
\|P^{(e)}_{\nu+1}\|_{\tilde{z}, \sigma, \tilde{\nu}, \mathcal{O}} &\leq e^{-T\sigma} \epsilon_n^{-\frac{1}{8}} \left( \|R^{(cl)}_{\nu}\|_{\tilde{z}, \tilde{\nu}, \mathcal{O}} + 2\epsilon_n \|F_{\nu+1}\|_{\tilde{z}, \tilde{\nu}, \mathcal{O}} \right) \\
&\leq \epsilon_n^{-\frac{1}{8}} \epsilon_n^{-\frac{2}{2\gamma}} + 64\epsilon_n \epsilon_n^{-2} T^{2\gamma} \epsilon_n^{-\frac{1}{8}} \|R^{(cl)}_{\nu}\|_{\tilde{z}, \tilde{\nu}, \mathcal{O}} \\
&\leq \epsilon_n^{-\frac{1}{8}} \epsilon_n^{-\frac{2}{2\gamma}} + 64\epsilon_n \epsilon_n^{-2} \epsilon_n^{-\frac{1}{8}} - \epsilon_n^{-\frac{1}{8}} \epsilon_n^{-\frac{2}{2\gamma}} \epsilon_n^{-\frac{1}{8}} < \frac{\epsilon_{n,\nu+1}}{8}, \quad (3.39)
\end{align*} \]
where the third inequality is derived from the following (the second inequality below is from (3.5))
\[ T^{2\gamma} \leq T^{2\gamma} < \exp \left\{ \frac{2(2\gamma + 1)}{3} K_{n-1}^{-\frac{1}{2}} \right\} = \exp \left\{ \frac{40(2\gamma + 1)}{60} K_{n-1}^{-\frac{1}{2}} \right\} = \epsilon_n^{-\frac{1}{8}}. \]
In the following, we give the estimate for coordinate transformation $X^i_{\nu+1}$. Denote
\[ ||D^i F_{\nu+1}||_{s,r,\bar{O}} = \max \left\{ \left\| \frac{\partial^{\left| \alpha \right|} F}{\partial r^{\alpha}} \right\|_{s,r,\bar{O}} : |\alpha| + |\beta| = l \right\}. \]
Note that $r_0 > \epsilon_{i}^{13}$, and we assume that $r_i > \epsilon_{i}^{13}$ for $i \geq 0$. Then we have
\[ r_{i+1} = \epsilon_{i+1} \nu \epsilon_i > \epsilon_{i+1} \nu \epsilon_i > \epsilon_{i+1} \nu \epsilon_{i+1} > \epsilon_{i+1} \nu, \]
where the second inequality is derived from $\epsilon_i > \epsilon_{i+1} \nu$. That is $r_i > \epsilon_{i+1} \nu$, $\forall i \geq 0$, which implies
\[ \tilde{r}_{i,0} = r_i > \epsilon_{i+1} \nu = \epsilon_{i+1} \nu, \quad \forall i \geq 0. \]
For the fixed $i \geq 0$, we assume that $\tilde{r}_{i,j} > \epsilon_{i}^{13}$ for $j \leq N_i$. Then
\[ \tilde{r}_{i,j} = \tilde{r}_{i,j+1} \epsilon_{i,j} > \epsilon_{i,j} \epsilon_{i,j} = \left( \epsilon_{i,j} \right) \epsilon_{i,j} > \left( \epsilon_{i,j} \right) \epsilon_{i,j} = \epsilon_{i,j+1}. \]
By the above discussion, we know that
\[ \tilde{r}_{i,j} > \epsilon_{i,j}, \quad \forall j = 0, \ldots, N_i - 1, \quad (3.40) \]
for fixed $i \geq 0$.

Since $F_{\nu+1}$ is a polynomial in $z, \tilde{z}$ of order 2, by (3.38),(3.40) and the Cauchy estimate, we obtain
\[ \|DF_{\nu+1}\|_{\tilde{z},\sigma,2-1\tilde{r},\bar{O}} \leq \frac{c}{\min\{\sigma,2-1\tilde{r}\}} < 2\sigma - 132 \gamma^{-2} T^{-2} \epsilon_n \left( \epsilon_{\nu + 1} \right) \|R_{\nu}^{(\epsilon)}\|_{\tilde{z},\bar{O}}, \]
\[ \leq 64 \gamma^{-2} \epsilon_{n,\nu} \epsilon_n \left( \epsilon_{\nu + 1} \right) \epsilon_n \epsilon_{n,\nu} < \epsilon_{\nu}^{-1}, \quad (3.41) \]
where $\sigma > \tilde{r}$ and $c$ is a constant depending only on $d$ and $m$. Similarly, by (3.41), we also have
\[ \|D^2 F_{\nu+1}\|_{\tilde{z},2\sigma,4-1\tilde{r},\bar{O}} \leq \frac{c}{\min\{\sigma,4-1\tilde{r}\}} < 4\sigma - 164 \gamma^{-2} \epsilon_{n,\nu} \epsilon_n \left( \epsilon_{\nu + 1} \right) \epsilon_n \epsilon_{n,\nu} \]
\[ \leq 256 \gamma^{-2} \epsilon_{n,\nu} \epsilon_n \left( \epsilon_{\nu + 1} \right) \epsilon_n \epsilon_{n,\nu} < \epsilon_{\nu}^{-1}, \quad (3.42) \]
where $\tilde{c}$ is a constant depending only on $d$ and $m$.

Then the flow $X^i_{\nu+1}$ of the vector field $F_{\nu+1}$ exists on $D(\tilde{s} - 3\sigma, 8^{-1}\tilde{r})$ for $0 \leq t \leq 1$ and takes this domain into $D(\tilde{s} - 2\sigma, 4^{-1}\tilde{r})$. Similarly, it takes $D(\tilde{s} - 4\sigma, 16^{-1}\tilde{r})$ into $D(\tilde{s} - 3\sigma, 8^{-1}\tilde{r})$. It is easy to see that for $0 \leq t \leq 1$
\[ \|X^i_{\nu+1} - tD\|_{\tilde{z},3\sigma,8^{-1}\tilde{r},\bar{O}} \leq c \|DF_{\nu+1}\|_{\tilde{z},2\sigma,4^{-1}\tilde{r},\bar{O}} < \epsilon_{\nu}^{-1}, \]
and
\[ \|DX^i_{\nu+1} - tD\|_{\tilde{z},4\sigma,16^{-1}\tilde{r},\bar{O}} \leq c \|D^2 F_{\nu+1}\|_{\tilde{z},2\sigma,4^{-1}\tilde{r},\bar{O}} < \epsilon_{\nu}^{-1}. \]
Moreover, $X^i_{\nu+1}$ is of form
\[ \left\{ \begin{array}{l}
\theta = \theta_+ \\
I = G_{\nu+1}(\theta_+, I_+, z_+, \bar{z}_+; \xi), \\
z = V_{\nu+1}(\theta_+, z_+, \bar{z}_+; \xi), \\
\bar{z} = \bar{V}_{\nu+1}(\theta_+, z_+, \bar{z}_+; \xi),
\end{array} \right. \]
where $V_{\nu+1}$ and $\bar{V}_{\nu+1}$ are affine in $z$ and $\bar{z}$.
By the definition of $X^1_{F_{\nu+1}}$ and (3.37), we know that

$$\tilde{H}_\nu \circ X^1_{F_{\nu+1}} = \tilde{N}_{\nu+1} + \tilde{P}_{\nu+1}$$

$$= \tilde{N}_{\nu+1} + \int_0^1 t \{ R^{(c)}_{\nu+1}, F_{\nu+1} \} \circ X^t_{F_{\nu+1}} dt + R^{(pe)}_{\nu+1} \circ X^1_{F_{\nu+1}}$$

$$+ P^{(c)}_{\nu+1} + \int_0^1 (1-t) \{ P^{(c)}_{\nu+1}, F_{\nu+1} \} \circ X^t_{F_{\nu+1}} dt$$

is well defined on domain $D(\bar{s}_{\nu+1}, \bar{r}_{\nu+1}) \times \mathcal{O}$. In view of (3.35), (3.38) and by Cauchy estimate, we obtain

$$\| \{ tR^{(c)}_{\nu+1}, F_{\nu+1} \} \|_{\bar{s}_{\nu+1}, \bar{r}_{\nu+1}, \mathcal{O}} \leq 4 \| D^2 R^{(c)}_{\nu+1} \|_{\bar{s}_{\nu+1}, \bar{r}_{\nu+1}, \mathcal{O}} \| DF_{\nu+1} \|_{\bar{s}_{\nu+1}, \bar{r}_{\nu+1}, \mathcal{O}}$$

$$\leq 4(c^2 \bar{r}^{-2}) \| R^{(c)}_{\nu+1} \|_{\bar{s}_{\nu+1}, \bar{r}_{\nu+1}, \mathcal{O}} \| F_{\nu+1} \|_{\bar{s}_{\nu+1}, \bar{r}_{\nu+1}, \mathcal{O}} \leq \frac{14}{e_{\nu+1}^\nu} < \frac{\bar{\epsilon}_{\nu+1}}{4},$$

with the similar calculations in (3.42), where the constant $c$ depends only on $d$ and $m$. With the estimates (3.36) and (3.39), we can also obtain the same bound for the rest three terms of $\bar{P}_{\nu+1}$. So we omit the details. Then we arrive at the estimate

$$\| \bar{P}_{\nu+1} \|_{\bar{s}_{\nu+1}, \bar{r}_{\nu+1}, \mathcal{O}} < \bar{\epsilon}_{\nu+1}. \quad \square$$

We can use Proposition 2 iteratively to prove Lemma 3.3. Once we reach the $N_n$-th step, we terminate the iteration. Let

$$H_{n+1} := \tilde{H}_{N_n} = \tilde{N}_{N_n} + \tilde{P}_{N_n} =: N_{n+1} + P_{n+1},$$

and

$$\Phi_{n+1} = X^1_{F_1} \circ X^1_{F_2} \circ \cdots \circ X^1_{F_{N_n}}.$$ 

Recall that

$$\bar{s}_{n,j+1} = \frac{1}{\epsilon_{n,j}^{\frac{1}{n}}} \bar{s}_{n,j}, \quad \bar{r}_{n,j+1} = \bar{s}_{n,j} - 4\epsilon_{n,j}.$$ 

Since

$$\prod_{j=0}^{N_n-1} \frac{1}{\epsilon_{n,j}^{\frac{1}{n}}} \prod_{j=0}^{N_n-1} \frac{1}{\epsilon_{n,j}^{\frac{1}{n}}} \bar{s}_{n,j}^{\frac{1}{n}} = \frac{1}{\epsilon_{n,0}^{\frac{1}{n}} \epsilon_{n,N_n-1}^{\frac{1}{n}}} > \frac{1}{\epsilon_{n,0}^{\frac{1}{n}} \epsilon_{n,N_n-1}^{\frac{1}{n}}} = \frac{1}{\epsilon_{n,0}^{\frac{1}{n}} \epsilon_{n,N_n-1}^{\frac{1}{n}}} > \frac{1}{\epsilon_{n+1}^{\frac{1}{n}}}$$

where the last inequality is derived from $\bar{s}_{n,N_n-1} > \epsilon_{n+1}$, we obtain

$$\bar{r}_{n,N_n} = r_n \prod_{j=0}^{N_n-1} \frac{1}{\epsilon_{n,j}^{\frac{1}{n}}} > r_n \epsilon_{n+1}^{\frac{1}{n}} = r_{n+1}.$$ 

Moreover,

$$\bar{s}_{n,N_n} = \bar{s}_{n,0} - 4 \sum_{j=0}^{N_n-1} \epsilon_{n,j} \geq \bar{s}_{n,0} - 4 \epsilon_{n+1} = \bar{s}_{n+1} + 1.$$ 

The above estimates imply that $H_{n+1}$ is well defined on $D(r_{n+1}, \sigma_{n+1}) \times \mathcal{O}_{n+1}$. Moreover, $B_{n+1}$ is of the similar form in (3.8) and satisfies the estimates in (3.9). The functions $c_{n+1}$ and $P_{n+1}$ satisfy the estimates in (3.26) and (3.27) respectively with $(n+1)$ in place of $n$. From (3.33) and (3.34), we know that $\Phi_{n+1}$ satisfies (3.28) and (3.29).
3.6. Convergence and measure estimate. Now we use Lemma 3.3 to prove Theorem 2.1. First, we consider the Hamiltonian $H$ given by (2.1) on $D(s_0, r_0) \times O_0$ with $\Omega$ being the ones given by (2.2) and $\epsilon_0 := \epsilon_0$, that is

$$\|P\|_{s_0, r_0, c_0} \leq \epsilon_0.$$ 

Moreover, the parameter set $O_0$ is the following

$$O_0 = \left\{ \xi \in O : |(k, \omega) + (l, \Omega(\xi))| \geq \frac{\gamma_0}{\langle k \rangle^{1/2}}, \forall k \in \mathbb{Z}^d, 0 < |l| \leq 2 \right\}.$$ 

Obviously, $\epsilon_0 (\theta; \xi) = 0$ and $B_0 (\theta; \xi) = 0$ in this case, and it is easy to check that Hamiltonian (2.1) satisfies all hypotheses of Lemma 3.3 with $n = 0$. Note that

$$s_* := e^{-4} s_0 < s_n \leq s_0, \quad \forall n \geq 0.$$ 

Moreover,

$$r_\infty = r_0 \prod_{n=0}^{\infty} \epsilon_{n+1}^n = 0 =: r_*.$$ 

Then we have a decreasing sequence of domains

$$D(s_0, r_0) \supset D(s_1, r_1) \supset \cdots \supset D(s_\infty, r_\infty) \supset D(s_*, r_*).$$

Applying the Lemma 3.3 iteratively, we can obtain a sequence of transformations $\Phi^n := \Phi_1 \circ \Phi_2 \circ \cdots \circ \Phi_n : D(s_n, r_n) \times O_{n+1} \to D(s_0, r_0)$. Then

$$H_n = H \circ \Phi^n = N_n + P_n$$

is the one in Lemma 3.3.

From Lemma 3.3, we know that $H_n, N_n, P_n, \Phi^n$ and $D\Phi^n$ converge uniformly on $D(s_*, r_*) \times O_\gamma$, where $O_\gamma = \bigcap_{l=0}^{\infty} O_l$. Let the limits be $H_\gamma, N_\gamma, P_\gamma, \Phi$ and $D\Phi$ respectively. By (3.28) and (3.29), we know that

$$\|\Phi - Id\|_{s_*, r_*, \gamma} \leq 4 \epsilon_0, \quad \|D\Phi - Id\|_{s_*, r_*, \gamma} \leq 4 \epsilon_0^{1/2}, \quad (3.43)$$

and

$$N_* = e_* (\theta; \xi) + \langle \omega, I \rangle + \langle [\Omega (\xi) + B_* (\theta; \xi)] z, \bar{z} \rangle,$$

$$P_* = \sum_{|\delta + \beta| \geq 3} P_{\delta, \beta}^* (\theta; \xi) z^\delta \bar{z}^\beta.$$ 

Moreover, from Section 3.5, we know that the inequalities in (2.4) hold. Because our symplectic transformation $\Phi$ is affine in $z$ and $\bar{z}$, we obtain that $\Phi$ is defined on the domain $D(s_*, \bar{z}) \times O_\gamma$ following the analysis in [26].

Measure estimate. Finally, we give the estimate of Lebesgue measure of the parameter set $O_\gamma$. According to Lemma 3.3, we know that

$$O_\gamma = O \setminus \bigcup_{n=0}^{\infty} \bigcup_{k \in \mathbb{Z}^d} \Gamma_k^n (\gamma_n),$$

where

$$\Gamma_k^n (\gamma_n) = \left\{ \xi \in O_{n-1} : |(k, \omega) + (l, \Omega (\xi) + [B_n (\theta; \xi)] \theta | < \frac{\gamma_n}{\langle k \rangle^{1/2}}, \quad 0 < |l| \leq 2 \right\}.$$ 

with $O_{-1} = O$.

In view of the non–degeneracy condition (2.3), we have $\forall n \geq 0$

$$|\Delta_{\xi_1, \xi_2} (l, \Omega (\xi) + [B_n (\theta; \xi)] \theta | \geq |\Delta_{\xi_1, \xi_2} (l, \Omega (\xi))| + |\Delta_{\xi_1, \xi_2} (l, [B_n (\theta; \xi)] \theta |).$$
\[ \geq (\mu - 2)(B_n(\theta; \xi)|a|^2|c_{n-1}|)\xi_1 - \xi_2 > \frac{\mu}{2} |\xi_1 - \xi_2| \]

for any \( \xi_1, \xi_2 \in \mathcal{O}_{n-1} \) provided that \( \epsilon_0 < \frac{\mu}{2} \). Therefore, following the analysis in [16] we get

\[ \text{meas } \Gamma_k^n(\gamma_n) \leq \frac{2^{m+1} \gamma_n}{\mu(k)\tau}. \]

Hence,

\[ \text{meas}(\mathcal{O} \setminus \mathcal{O}_\gamma) = \sum_{n=0}^{\infty} \sum_{k \in \mathbb{Z}^d} \text{meas } \Gamma_k^n(\gamma_n) \leq \frac{2^{d+m+1} \gamma}{\mu} \sum_{n=0}^{\infty} (n + 2)^{-2} \sum_{\nu=0}^{\infty} \frac{1}{\nu^{\tau-d+1}} = O(\gamma) \]

provided that \( \tau > d \) and \( \gamma \) is sufficiently small. Then we obtain the measure estimate

\[ \text{meas } \mathcal{O}_\gamma \geq (1 - c\gamma) \text{meas } \mathcal{O}, \]

where \( c \) is a constant depending on \( \mu, \tau, d \) and \( m \).

4. **Proof of Theorem 1.1.** Denote \( \dot{x} = \lambda y \), then equation (1.1) becomes the following system

\[ \begin{cases} 
\dot{\theta} = \omega, \\
\dot{x} = \lambda y, \\
\dot{y} = -\lambda x + \epsilon \lambda^{-1} f(\theta, x).
\end{cases} \tag{4.1} \]

Moreover, make the transformation of variables

\[ z = \frac{x - iy}{\sqrt{2}}, \quad \bar{z} = \frac{x + iy}{\sqrt{2}}. \]

Then (4.1) is changed into

\[ \begin{cases} 
\dot{\theta} = \omega, \\
\dot{z} = i\lambda z - i\epsilon(\sqrt{2}\lambda)^{-1} f(\theta, \frac{z + \bar{z}}{\sqrt{2}}), \\
\dot{\bar{z}} = -i\lambda \bar{z} + i\epsilon(\sqrt{2}\lambda)^{-1} f(\theta, \frac{z + \bar{z}}{\sqrt{2}}).
\end{cases} \tag{4.2} \]

The equations of motion (4.2) is a Hamiltonian system with the Hamiltonian

\[ H = \langle \omega, I \rangle + \lambda z\bar{z} + P(\theta, z, \bar{z}; \lambda) \tag{4.3} \]

and symplectic structure \( dI \wedge d\theta + idz \wedge d\bar{z} \), where the added variable \( I \in \mathbb{C}^d \) is canonically conjugate to \( \theta \in \mathbb{T}_c^d \). Moreover, the perturbation \( P \) is

\[ P(\theta, z, \bar{z}; \lambda) = -\epsilon \lambda^{-1} \int_0^{\frac{z + \bar{z}}{\sqrt{2}}} f(\theta, s) \, ds. \]

Without loss of generality, we suppose that \( \lambda \in \mathcal{O} := [u, v] \), where \( v > u > 0 \) are constants. For \( ||\text{Im}\theta|| < s, ||z||, ||\bar{z}|| \leq 1 \), the perturbation \( P \) satisfies the smallness condition:

\[ ||P||_{s,1,\mathcal{O}} \leq \sqrt{2}\epsilon(u^{-1} + u^{-2}) ||f||_{s,2,2,\mathcal{O}} = O(\epsilon). \]

It is obvious that \( \frac{\Delta_{\lambda_1,\lambda_2}(l\lambda)}{\lambda_1 - \lambda_2} = |l| \geq 1 \) for \( l \in \mathbb{Z}, 0 < |l| \leq 2 \). That implies the non–degeneracy condition (2.3) holds for this Hamiltonian. From Theorem 2.1, we know that for sufficiently small \( \gamma > 0 \) and \( \tau > d \), there exits \( \epsilon_* \) depending on \( \gamma, f, \mathcal{O}, \tau, \omega \).
such that if \( \epsilon \leq \epsilon_\ast \), there exists a symplectic transformation of variables \( \Phi \) defined on \( D(s_\ast, r/2) \times O_\ast \) casts the Hamiltonian \( H \) given by (4.3) into
\[
H_s = H \circ \Phi = e, (\theta; \lambda) + (\omega, I) + (\lambda + B_\ast(\theta; \lambda)) z \bar{z} + P_s(\theta, z, \bar{z}; \lambda),
\]
where \( P_s(\theta, z, \bar{z}; \lambda) = \sum_{|\delta| + |\beta| \geq 3} P_{s, \delta, \beta}(\theta; \lambda) z^\delta \bar{z}^\beta. \)

Moreover, the equations of motion determined by \( H_s \) is (without the added variable \( I \))
\[
\begin{align*}
\dot{\theta} &= \omega, \\
\dot{z} &= i(\lambda + B_\ast) z + i\partial_\bar{z} P_s, \\
\dot{\bar{z}} &= -i(\lambda + B_\ast) \bar{z} - i\partial_z P_s.
\end{align*}
\]
System (4.4) possesses the invariant tori \( \theta = \theta_0 + \omega t, \quad z = \bar{z} = 0. \)

Let \( (\theta(t), z(t), \bar{z}(t)) = \Phi(\theta_0 + \omega t, 0, 0; \lambda). \) Then
\[
(\theta(t), z(t), \bar{z}(t)) = (\theta_0 + \omega t, X(\theta_0 + \omega t), \bar{X}(\theta_0 + \omega t))
\]
is a solution to (4.2), where \( \bar{X}(\theta_0 + \omega t) \) is the complex conjugate of \( X(\theta_0 + \omega t). \) Then
\[
x(t) = \frac{X(\theta_0 + \omega t) + \bar{X}(\theta_0 + \omega t)}{\sqrt{2}}
\]
is the solution to (1.1), which means that, for the parameter \( \lambda \in O_\gamma, \) the equation (1.1) has a quasi–periodic solution with the frequency \( \omega. \)

5. Appendix. In this section, we give another function which decreases more slowly than \( \alpha(x) \) does. We define the function
\[
F(x) = \left( \ln \cdots \ln x \right)^{-1}, \quad n_{**} > n_\ast + 2.
\]

Obviously,
\[
\ln F(x) = -\left( \ln \cdots \ln x \right)^{-1} \to -\infty, \quad x \to +\infty,
\]

since \( n_{**} > n_\ast + 2. \) Thus \( F(x) \to 0 \) as \( x \to +\infty. \)

Note that \( \alpha(x) \leq F(x), \quad x \geq M := \exp \cdots \exp \left\{ \frac{1}{\delta} \right\}. \) Then the frequency defined by the following
\[
\max_{0 < |k| \leq K, \, k \in \mathbb{Z}^d} \ln \left( \frac{1}{|k, \omega|} \right) \leq K F(K),
\]
is much weaker than the one defined by (1.3).

Now we will prove that for the sequences defined by (3.1) with \( K_{-1} \) satisfying
\[
(F(\frac{1}{3}K_{-1}^{1/2}z_1^{1/2}))^{-1} = \left\{ \ln \cdots \ln \left( \frac{1}{3}K_{-1}^{1/2}z_1^{1/2} \right) \right\}^{-1} \to (3^{-1}e^{4s_0 \eta_{n_{**}}})^{-1}, \quad (5.1)
\]
the two inequalities in Lemma 3.1 also hold with \( F(x) \) in place of \( \alpha(x). \) Moreover, we also prove that the function \( F \) is decreasing to zero as \( x \) goes to +\infty (
Note that $K_{-1}^\frac{1}{n} := \ln \epsilon_0^{-\frac{1}{\eta n}}$, then we can set $\epsilon_0$ small enough such that $K_{-1}^\frac{1}{n} \geq 3(\exp \cdots \exp 1)$ is sufficiently big satisfying the inequality in (5.1)).

Note that the proof of Theorem 2.1 depends heavily on the two inequalities in Lemma 3.1. Thus, once we prove these two inequalities hold for the frequency defined above, the proof of Theorem 2.1 with this frequency will be the same with the proof given in this work with just some small modifications. We omit the detail. Actually, we just need to prove the second inequality.

**Lemma 5.1.** The function $F(T) = (\ln \cdots \ln |T|)^{-1}$ with $n_{**} > n_*$ is monotone decreasing to zero in the interval $[\exp \cdots \exp 1, +\infty)$. Moreover, for the sequences defined in (3.1) with $K_{-1}$ satisfying (5.1), we have

$$F(\ln K_{-1}^\frac{1}{n}) < 3^{-1}e^{-4}s_0n_{h+1}, \quad \forall n \geq 0.$$  

**Proof.** Set $Z(T) = \ln \cdots \ln |T|$, then $F(T)$ can be rewritten as

$$F(T) = G \circ Z(T), \quad G(x) = \{\exp \cdots \exp x\}^{-\frac{1}{2}}.$$  

For the function $G$, we have

$$\ln G(x) = -x^{-1}\exp \cdots \exp x.$$  

Thus

$$(G(x))^{-1}\frac{d}{dx}G(x) = x^{-2}\exp \cdots \exp x - x^{-1}\exp \cdots \exp x \cdot \exp x$$

$$= x^{-2}\exp \cdots \exp x(1 - x \exp \cdots \exp x) < 0.$$  

That is $\frac{d}{dx}G(x) < 0$ since $(G(x))^{-1} > 0$. Thus function $G$ is monotone decreasing in $[1, +\infty)$. Obviously, the function $Z$ is monotone increasing in $[\exp \cdots \exp 1, +\infty)$ and the range of $Z$ is $[1, +\infty)$. Thus, the composition function $F(T) = G \circ Z(T)$ is monotone decreasing in $[\exp \cdots \exp 1, +\infty)$.

Moreover, note

$$\lim_{x \to +\infty} \ln G(x) = \lim_{x \to +\infty} -x^{-1}\exp \cdots \exp x = -\infty$$

and

$$\lim_{T \to +\infty} Z(T) = \lim_{T \to +\infty} \ln \cdots \ln T = +\infty,$$

thus

$$\lim_{T \to +\infty} F(T) = \lim_{T \to +\infty} G \circ Z(T) = \lim_{x \to +\infty} G(x) = 0.$$  

Now we turn to the inequality (5.2). Similar with the proof of (3.3), we distinguish $n$ according to $n_*$. 

Case I: $0 \leq n \leq n_*$. By (5.1) and note that $K_0 = \exp\{K_1^{\frac{1}{n}}\}$, we obtain
\[
\frac{1}{n_*+1} \ln \cdots \ln K_0^{\frac{1}{n_*+1}} > (3^{-1}e^{-4}s_0\eta_{n_*+1})^{-1} \geq (3^{-1}e^{-4}s_0\eta_{n+1})^{-1}.
\]
Hence by the above inequality, we obtain
\[
F(\ln K_n^{\frac{1}{n}}) \leq F(\ln K_0^{\frac{1}{n_*+1}}) = (\ln \cdots \ln K_0^{\frac{1}{n_*+1}})_{n_*+1} < 3^{-1}e^{-4}s_0\eta_{n+1}, \quad \forall 0 \leq n \leq n_*.
\]

Case II: $n \geq n_*+1$. We will use induction technique to prove the inequality (5.2) in this case. Suppose that for $n \geq n_*$, the following inequality holds:
\[
\frac{1}{n_*+1} \ln \cdots \ln K_n^{\frac{1}{n_*+1}} > (3^{-1}e^{-4}s_0\eta_{n+1})^{-1},
\]
which implies
\[
\Gamma(\ln K_n^{\frac{1}{n_*+1}}) := \frac{\ln \cdots \ln K_n^{\frac{1}{n_*+1}}}{n_*+1} > \ln(3^{-1}e^{-4}s_0\eta_{n+1})^{-1}. \tag{5.3}
\]
Denote
\[
\Gamma(\ln K_n^{\frac{1}{n_*+1}}) = \frac{\ln \cdots \ln K_n^{\frac{1}{n_*+1}}}{n_*+1} = \frac{\exp \cdots \exp X}{X},
\]
\[
\Gamma(\ln K_{n+1}^{\frac{1}{n_*+1}}) = \frac{\ln \cdots \ln K_{n+1}^{\frac{1}{n_*+1}}}{n_*+1} = \frac{\exp \cdots \exp Y}{Y},
\]
where $X := \ln \cdots \ln K_n^{\frac{1}{n_*+1}}$ and $Y := \ln \cdots \ln K_{n+1}^{\frac{1}{n_*+1}}$. Note that $K_{n+1} = \exp\{K_n^{\frac{1}{n}}\}$, then we get
\[
Y = \frac{\ln \cdots \ln K_{n+1}^{\frac{1}{n_*+1}}}{n_*+1} = \ln \cdots \ln \exp\left\{\frac{1}{n_*+1}K_n^{\frac{1}{n_*+1}}\right\} \geq \exp\left\{\ln \cdots \ln K_n^{\frac{1}{n_*+1}}\right\} = e^X > 4X
\]
since $K_n \gg 1$. Hence,
\[
\Gamma(\ln K_{n+1}^{\frac{1}{n_*+1}}) = \frac{1}{Y} \exp \cdots \exp Y \geq \exp \cdots \exp \left\{\frac{Y}{2}\right\} \geq \exp \cdots \exp \{2X\}
\]
\[
\geq \exp \cdots \exp \{2\exp\{X\}\} \quad \text{(since } e^{2X} > 2e^X)\]
\[
\geq \exp \left\{ \frac{2 \exp \cdots \exp \{X\}}{n_*-(n_*+2)} \right\} \quad \text{(inductively)}
\]
\[
\begin{align*}
&= \left( \exp \cdots \exp \{X\} \right)^2 \geq \left\{ \frac{1}{X} \exp \cdots \exp X \right\}^2 = \left( \Gamma \left( \ln K_{n_*}^{\frac{1}{3}} \right) \right)^2 \\
&> \left\{ \ln \left( (3^{-1} e^{-4 s_0 \eta n+1})^{-1} \right) \right\}^2 \\
&> 2 \ln \left( (3^{-1} e^{-4 s_0 \eta n+1})^{-1} \right) = \ln(3^{-1} e^{-4 s_0 \eta n+1})^{-2} \\
&> \ln(3^{-1} e^{-4 s_0 \eta n+2})^{-1},
\end{align*}
\]

where the sixth inequality is by (5.3). The above inequality implies

\[
\ln \cdots \ln K_{n_*+1}^{\frac{1}{3}} \geq \ln(3^{-1} e^{-4 s_0 \eta n+2})^{-1}.
\]

From case I we know that

\[
\ln \cdots \ln K_{n_*}^{\frac{1}{3}} \geq \ln(3^{-1} e^{-4 s_0 \eta n+1})^{-1}.
\]

Thus (5.4) holds for all \( n \geq n_* + 1 \), which implies

\[
F(\ln K_{n_*}^{\frac{1}{3}}) = \left( \ln \cdots \ln K_{n_*}^{\frac{1}{3}} \right)^{-1} < 3^{-1} e^{-4 s_0 \eta n+1}, \quad \forall n \geq n_* + 1. \quad \square
\]

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REFERENCES

[1] A. Avila, B. Fayad and R. Krikorian, A KAM scheme for SL(2, \mathbb{R}) cocycles with Liouvillean frequencies, Geom. Funct. Anal., 21 (2011), 1001–1019.
[2] A. Avila, J. You and Q. Zhou, Sharp phase transitions for the almost Mathieu operator, Duke Math. J., 166 (2017), 2697–2718.
[3] M. Berti, KAM theory for partial differential equations, Anal. Theory Appl., 35 (2019), 235–267.
[4] B. L. J. Braaksma and H. W. Broer, On a quasiperiodic Hopf bifurcation, Ann. Inst. Henri Poincare Anal. Non Lineaire, 4 (1987), 115–168.
[5] H. Cheng, W. Si and J. Si, Whiskered tori for forced beam equations with multi-dimensional liouvillean frequency, J. Dyn. Differ. Equ., 32 (2020), 705–739.
[6] Y. Cheung, Hausdorff dimension of the set of singular pairs, Ann. Math., 173 (2011), 127–167.
[7] L. H. Eliasson, Perturbations of stable invariant tori for Hamiltonian systems, Ann. Scuola Norm. Super. Pisa-Cl. Sci., 15 (1988), 115–147.
[8] L. H. Eliasson, B. Grébert and S. B. Kuksin, KAM for the nonlinear beam equation, Geom. Funct. Anal., 26 (2016), 1588–1715.
[9] L. H. Eliasson and S. B. Kuksin, KAM for the nonlinear Schrödinger equation, Ann. Math., 172 (2010), 371–435.
[10] M. Friedman, Quasi-periodic solutions of nonlinear ordinary differential equations with small damping, *Bull. Amer. Math. Soc.*, **73** (1967), 460–464.

[11] J. Geng and X. Ren, Lower dimensional invariant tori with prescribed frequency for nonlinear wave equation, *J. Differ. Equ.*, **249** (2010), 2796–2821.

[12] J. Geng, X. Xu and J. You, An infinite dimensional KAM theorem and its application to the two dimensional cubic Schrödinger equation, *Adv. Math.*, **226** (2011), 5361–5402.

[13] J. Geng and J. You, A KAM theorem for Hamiltonian partial differential equations in higher dimensional spaces, *Commun. Math. Phys.*, **262** (2006), 343–372.

[14] Y. Han, Y. Li and Y. Yi, Degenerate lower-dimensional tori in Hamiltonian systems, *J. Differ. Equ.*, **227** (2006), 670–691.

[15] X. Hou and J. You, Almost reducibility and non-perturbative reducibility of quasi-periodic linear systems, *Invent. Math.*, **190** (2012), 209–260.

[16] T. Kappeler and J. Pöschel, *KdV & KAM*, Springer-Verlag, Berlin, 2003.

[17] R. Krikorian, J. Wang, J. You and Q. Zhou, Linearization of quasiperiodically forced circle flows beyond brjuno condition, *Commun. Math. Phys.*, **358** (2018), 81–100.

[18] S. B. Kuksin, A KAM-theorem for equations of the Korteweg-de Vries type, *Rev. Math. Math. Phys.*, **10** (1998), 1–64.

[19] S. B. Kuksin, Analysis of Hamiltonian PDEs, Oxford University Press, Oxford, 2000.

[20] S. Kuksin and J. Pöschel, Invariant Cantor manifolds of quasi-periodic oscillations for a nonlinear Schrödinger equation, *Ann. Math.*, **143** (1996), 149–179.

[21] Y. Li and Y. Yi, Persistence of lower dimensional tori of general types in Hamiltonian systems, *T. Am. Math. Soc.*, **357** (2005), 1565–1600.

[22] J. Liu and X. Yuan, Spectrum for quantum Duffing oscillator and small-divisor equation with large-variable coefficient, *Commun. Pure Appl. Math.*, **63** (2010), 1145–1172.

[23] J. Liu and X. Yuan, A KAM theorem for Hamiltonian partial differential equations with unbounded perturbations, *Commun. Math. Phys.*, **307** (2011), 629–673.

[24] Z. Lou and J. Geng, Quasi-periodic response solutions in forced reversible systems with liouvillean frequencies, *J. Differ. Equ.*, **263** (2017), 3894–3927.

[25] J. Moser, Combination tones for Duffing's equation, *Commun. Pure Appl. Math.*, **18** (1965), 167–181.

[26] J. Pöschel, On elliptic lower-dimensional tori in Hamiltonian systems, *Math. Z.*, **202** (1989), 559–608.

[27] W. Si and J. Si, Response solutions and quasi-periodic degenerate bifurcations for quasi-periodically forced systems, *Nonlinearity*, **31** (2018), 2361–2418.

[28] J. J. Stoker, *Nonlinear Vibrations in Mechanical and Electrical Systems*, Interscience Publishers, Inc., New York, N.Y., 1950.

[29] J. Wang and J. You, Boundedness of solutions for non-linear quasi-periodic differential equations with Liouvillean frequency, *J. Dyn. Differ. Equ.*, **261** (2016), 1068–1098.

[30] J. Wang, J. You and Q. Zhou, Response solutions for quasi-periodically forced harmonic oscillators, *T. Am. Math. Soc.*, **369** (2017), 4251–4274.

[31] J. Xu, J. You and Q. Qiu, Invariant tori for nearly integrable Hamiltonian systems with degeneracy, *Math. Z.*, **226** (1997), 375–387.

[32] X. Xu, J. You and Q. Zhou, Quasi-periodic solutions of NLS with Liouvillean frequency, preprint, *arXiv*:1707.04048

[33] J. You and Q. Zhou, Phase transition and semi-global reducibility, *Commun. Math. Phys.*, **330** (2014), 1095–1113.

[34] D. Zhang, J. Xu and X. Xu, Reducibility of three dimensional skew symmetric system with Liouvillean basic frequencies, *Discrete Contin. Dyn. Syst.*, **38** (2018), 2851–2877.

[35] Q. Zhou and J. Wang, Reducibility results for quasiperiodic cocycles with liouvillean frequency, *J. Dyn. Differ. Equ.*, **24** (2012), 61–83.

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