Tracy-Widom Distributions for the Gaussian Orthogonal and Symplectic Ensembles Revisited: A Skew-Orthogonal Polynomials Approach

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Abstract
We study the distribution of the largest eigenvalue in the “Pfaffian” classical ensembles of random matrix theory, namely in the Gaussian orthogonal (GOE) and Gaussian symplectic (GSE) ensembles, using semi-classical skew-orthogonal polynomials, in analogy with the approach of Nadal and Majumdar (NM) for the Gaussian unitary ensemble (GUE). Generalizing the techniques of Adler, Forrester, Nagao and van Moerbeke, and using “overlapping Pfaffian” or “compound Pfaffian” identities, we explicitly construct these semi-classical skew-orthogonal polynomials in terms of the semi-classical orthogonal polynomials studied by NM in the case of the GUE. With these polynomials we obtain expressions for the cumulative distribution functions of the largest eigenvalue in the GOE and the GSE. Further, by performing asymptotic analysis of these skew-orthogonal polynomials in the limit of large matrix size, we obtain an alternative derivation of the Tracy-Widom distributions for GOE and GSE. This asymptotic analysis relies on a certain Pfaffian identity, the proof of which employs the characterization of Pfaffians in terms of perfect matchings and link diagrams.

Keywords Random matrices · Extreme value statistics · Tracy-Widom distributions · Skew-orthogonal polynomials

1 Introduction

Since their discovery more than 25 years ago, the Tracy-Widom (TW) distributions [32,81,82] have become cornerstones of extreme value statistics of strongly correlated variables [49]. While they were initially found as the limiting distributions describing the typical fluctua-
tions of the largest eigenvalues of large random matrices belonging to the classical Gaussian ensembles of random matrix theory (RMT), namely the Gaussian orthogonal, unitary and symplectic ensembles (respectively denoted as GOE, GUE and GSE), they have since found a large number of applications (for a review see [46]). Indeed, TW distributions have emerged in a variety of problems at the interface between statistical mechanics and mathematics, including the longest increasing subsequence of random permutations [10], directed polymers [9,10,12] and related growth models [40,47,69], in the Kardar-Parisi-Zhang (KPZ) universality class in (1+1) dimensions as well as for the continuum (1+1)-dimensional KPZ equation [6,13,21,28,38,44,71], sequence alignment problems [48], height fluctuations of non-intersecting Brownian motions over a fixed time interval [34,45,64], height fluctuations of non-intersecting interfaces in the presence of a long-range interaction induced by a substrate [61], or more recently in the context of trapped fermions [25–27,78], as well as in finance [15]. Remarkably, the TW distributions have been recently observed in experiments on nematic liquid crystals [79,80] (for the GOE and GUE) as well as in experiments involving coupled fibre lasers (for the GOE), and in dissipative self-assembled systems [51] (for the GUE).

In the pioneering works on the largest eigenvalue in the classical ensembles of RMT [32,81,82], the authors used the powerful tools of determinantal (for GUE) or Pfaffian (for GOE and GSE) point processes. This naturally led to the expression of these distributions in terms of a Fredholm determinant (for GUE) or a Fredholm Pfaffian (for GOE and GSE). Using rather involved “operator theoretic” techniques [81,82], it was further shown how to relate these Fredholm determinants and Pfaffians to sets of partial differential equations. In the limit of large matrix size $N$, this eventually led to a fairly explicit expression of these distributions for GOE, GUE and GSE in terms of a special solution of a Painlevé II equation (the so called Hastings-McLeod solution, see also below).

More recently, an alternative derivation of the TW distribution for the GUE was proposed by Nadal and Majumdar in Ref. [62] using (semi-classical) orthogonal polynomials. The idea of the method is rather simple and also quite instructive since one sees how the Painlevé II equation emerges from the asymptotic analysis of the three-term recurrence relation satisfied by these orthogonal polynomials, which are some deformations of the standard Hermite polynomials, in the limit of large matrix size $N$. Furthermore, this approach was further extended in Ref. [66] to compute the distribution of the first gap (between the first two eigenvalues), and more generally the statistics of near extreme eigenvalues in the GUE, which could be expressed in a rather compact form in terms of Painlevé transcendents, from which very precise asymptotics could be derived (see also [84] for yet another derivation of the statistics of the first gap in the GUE). It would thus be very useful to obtain such an alternative derivation of the TW distributions in the other classical ensembles, namely the GOE and the GSE. This would be particularly interesting in the case of GOE, since this would provide a very efficient method to compute the statistics of near-extreme eigenvalues for this ensemble, which is directly relevant to the description of static [58] and dynamical [36] properties of a well known mean-field spin-glass model, namely the spherical Sherrington-Kirkpatrick model. Up to now, the statistics of near-extreme eigenvalues in these ensembles have only been studied numerically [67]. The goal of this paper is precisely to extend the method of Ref. [62] and provide an alternative derivation of the TW distributions in the GOE and the GSE, by developing an approach based on (semi-classical) skew-orthogonal polynomials. This is a first important step towards a precise and useful description of the statistics of near-extreme eigenvalues, e.g. the first gap between the two largest eigenvalues, in terms of Painlevé transcendents in these ensembles [54].
2 Summary of Main Results

In the following we consider Gaussian random matrices $M = [m_{ij}]$ belonging to the aforementioned classical ensembles of random matrices with real symmetric (GOE), complex Hermitian (GUE) or real quaternionic self-dual (GSE) entries respectively [33,55] (see also Appendix A), characterised by a Dyson index $\beta = 1, 2$ and 4 respectively. In these three cases, the probability measure associated to the matrix ensemble is given by

$$\Pr(M) \propto e^{-\beta(\text{Tr}M^2)/2}. \quad (1)$$

In what follows we denote by $G\beta E$ these ensembles with $\beta = 1$ for the GOE, $\beta = 2$ for the GUE and $\beta = 4$ for the GSE. By performing a change of variables from the matrix entries $m_{ij}$ to the eigenvalues and eigenvectors of $M$, one obtains the joint probability density function (JPDF) of the (real) eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_N$ in the $G\beta E$ ensembles as (see [33,55])

$$P_\beta(\lambda_1, \ldots, \lambda_N) = \frac{1}{Z_{\beta,N}} \prod_{j=1}^{N} e^{-\beta \lambda_j^2/2} \prod_{j<k} |\lambda_k - \lambda_j|^\beta, \quad (2)$$

where $Z_{\beta,N}$ is a normalization constant such that

$$\int_{-\infty}^{\infty} d\lambda_1 \cdots \int_{-\infty}^{\infty} d\lambda_N P_\beta(\lambda_1, \ldots, \lambda_N) = 1 \quad (3)$$

and is given explicitly by

$$Z_{\beta,N} = \beta^{-N/2} \frac{N^\beta}{N!} (N-1) \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1+\beta}{2}\right)} \prod_{j=0}^{N-1} \frac{\Gamma\left(1 + (j+1)\frac{\beta}{2}\right)}{\Gamma\left(1 + \frac{\beta}{2}\right)}. \quad (4)$$

where $\Gamma(z)$ is the gamma function. We will compute the cumulative distribution function (CDF) of the largest eigenvalue, i.e. $F_{\beta,N}(y) \equiv \Pr(\lambda^{(\beta)}_{\text{max}} < y)$, or equivalently, the probability that all eigenvalues are less than some upper bound $y$

$$F_{\beta,N}(y) \equiv \Pr(\lambda^{(\beta)}_{\text{max}} < y) = N! \int_{-\infty}^{y} d\lambda_1 \int_{\lambda_1}^{y} d\lambda_2 \cdots \int_{\lambda_{N-1}}^{y} d\lambda_N P_\beta(\lambda_1, \ldots, \lambda_N), \quad (5)$$

where the factorial $N!$ comes from the fact that in Eq. (5), the eigenvalues are ordered, i.e. $\lambda_1 < \lambda_2 < \cdots < \lambda_N \leq y$. (Note that this ordering is not required here, however it will be convenient later to work with ordered eigenvalues and therefore we impose the ordering from the beginning.) It is well known that the JPDF in Eq. (2) can be interpreted as the Boltzmann weight of a one-dimensional gas of $N$ charged particles where $\lambda_i$ denotes the position of the $i$-th particle and $\beta$ the inverse temperature [29]. These particles interact via a repulsive logarithmic interaction while they are subjected to an external quadratic potential: this is the so-called log-gas. Hence the CDF $F_{\beta,N}(y)$ in Eq. (5) is the partition function of this log-gas in the presence of a hard wall at position $y$ [50]—such partition functions are called “restricted partition functions” in the following. (Note that they are normalized such that they become unity when $y \to \infty$.)

\[1\] Note that, for $\beta = 4$, the $\text{Tr}$ function needs to be interpreted as a quaternion trace, which can be understood as one half of the regular trace (for more information on the quaternionic definitions in RMT see, for example, [20,29,30,53,55]).
To compute $F_{\beta,N}(y)$, it is useful to introduce sets of orthogonal and skew-orthogonal polynomials. Specifically, we define the $y$-dependent inner (or scalar) product for $\beta = 2$

$$\langle f, g \rangle^y_2 = \int_{-\infty}^{y} e^{-x^2} f(x)g(x)dx,$$

and the skew-inner products for $\beta = 4$

$$\langle f, g \rangle^y_4 = \frac{1}{2} \int_{-\infty}^{y} dx \ e^{-2x^2} \left[ f(x)g'(x) - g(x)f'(x) \right]$$

$$= \frac{1}{2} \int_{-\infty}^{y} dx \ e^{-x^2} \left[ f(x) \frac{d}{dx} \left( e^{-x^2} g(x) \right) - g(x) \frac{d}{dx} \left( e^{-x^2} f(x) \right) \right],$$

and for $\beta = 1$

$$\langle f, g \rangle^y_1 = \frac{1}{2} \int_{-\infty}^{y} dx \ e^{-x^2/2} f(x) \int_{-\infty}^{y} dz \ e^{-z^2/2} g(z) \text{sgn}(z - x)$$

$$= \frac{1}{2} \int_{-\infty}^{y} dx \ e^{-x^2/2} f(x) \int_{-\infty}^{y} dz \ e^{-z^2/2} g(z)$$

$$- \frac{1}{2} \int_{-\infty}^{y} dx \ e^{-x^2/2} f(x) \int_{-\infty}^{x} dz \ e^{-z^2/2} g(z).$$

Then for each ensemble, we seek a set of (monic) polynomials $\{P_j(x,y)\}$ for $\beta = 2$, $\{Q_j(x,y)\}$ for $\beta = 4$, and $\{R_j(x,y)\}$ for $\beta = 1$ (by increasing order of complexity, as we will see) with the orthogonality/skew-orthogonality properties

$$\langle P_j, P_k \rangle^y_2 = p_j(y) \delta_{j,k},$$

$$\langle Q_{2j}, Q_{2k} \rangle^y_4 = \langle Q_{2j+1}, Q_{2k+1} \rangle^y_4 = 0,$$

$$\langle Q_{2j}, Q_{2k+1} \rangle^y_4 = -\langle Q_{2k+1}, Q_{2j} \rangle^y_4 = q_j(y) \delta_{j,k},$$

and

$$\langle R_{2j}, R_{2k} \rangle^y_1 = \langle R_{2j+1}, R_{2k+1} \rangle^y_1 = 0,$$

$$\langle R_{2j}, R_{2k+1} \rangle^y_1 = -\langle R_{2k+1}, R_{2j} \rangle^y_1 = r_j(y) \delta_{j,k},$$

where, to be explicit, the respective normalizations are

$$p_j(y) := \langle P_j, P_j \rangle^y_2,$$

$$q_j(y) := \langle Q_{2j}, Q_{2j+1} \rangle^y_4,$$

$$r_j(y) := \langle R_{2j}, R_{2j+1} \rangle^y_1.$$

(Note that the orthogonal and skew-orthogonal polynomials depend on the parameter $y$, although we will often suppress the explicit notation of that dependence for brevity.) In fact, the CDF $F_{\beta,N}(y)$ can be expressed only in terms of the norms $p_j(y), q_j(y)$ and $r_j(y)$ for $\beta = 2, 4$ and $1$ respectively. For $\beta = 2$, it was indeed shown in [62] that

$$F_{2,N}(y) = \frac{2^{N(N-1)/2}}{\pi^{N/2}} \prod_{j=0}^{N-1} \frac{p_j(y)}{j!}.$$

In the present paper we apply the techniques of [2,3] to write the monic $\beta = 1$ and $\beta = 4$ polynomials in the basis of these (monic) $\beta = 2$ polynomials, and we find the coefficients
\[ \alpha_{j,k} \] in the linear combination
\[ \eta_j = P_j + \alpha_{j,j-1} P_{j-1} + \cdots + \alpha_{j,0} P_0, \quad (\eta_j = R_j \text{ for } \beta = 1, \eta_j = Q_j \text{ for } \beta = 4), \]
(16)
to be a ratio of Pfaffians of \( \beta = 2 \) quantities (see Sect. 4). (Note that a “monic polynomial” is defined as having leading coefficient equal to 1, hence \( \alpha_{j,j} = 1 \).) Then for \( \beta = 1 \) (and where \( N \) is restricted to be even for simplicity), with the polynomials \( R_j \) in this basis, we have
\[ F_{1,N}(y) = \frac{2^{N/4}}{\pi^{N/4}} \prod_{j=0}^{N/2 - 1} \frac{1}{(2j)!} \text{Pf} \, V_{N-1}, \]
(17)
where the matrix \( V_m \) is given by
\[
V_m = \begin{bmatrix}
0 & p_0 - X_{0,1} - \Phi_{0,1} & X_{2,0} + \Phi_{2,0} & X_{3,0} + \Phi_{3,0} & \ldots & X_{m,0} + \Phi_{m,0} \\
-p_0 + X_{0,1} + \Phi_{0,1} & 0 & p_1 - X_{1,2} - \Phi_{1,2} & X_{3,1} + \Phi_{3,1} & \ldots & X_{m,1} + \Phi_{m,1} \\
-X_{2,0} - \Phi_{2,0} & -p_1 + X_{1,2} + \Phi_{1,2} & 0 & p_2 - X_{2,3} - \Phi_{2,3} & \ldots & X_{m,2} + \Phi_{m,2} \\
-X_{3,0} - \Phi_{3,0} & -X_{3,1} - \Phi_{3,1} & -p_2 + X_{2,3} + \Phi_{2,3} & 0 & \ldots & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots \\
-X_{m,0} - \Phi_{m,0} & -X_{m,1} - \Phi_{m,1} & -X_{m,2} + \Phi_{m,2} & 0 & \ldots & \ldots \\
\end{bmatrix}
\]
(18)
and
\[
\Phi_{j,k} := \frac{1}{2} \int_{-\infty}^{y} e^{-z^2/2} P_j(z, y) dz \int_{y}^{\infty} e^{-z^2/2} P_k(z, y) dz, \]
(19)
\[
X_{j,k} := \frac{1}{2} \left( \int_{-\infty}^{y} P_j(x, y) e^{-x^2/2} \text{erf} \left( \frac{x}{\sqrt{2}} \right) dx \right) \int_{-\infty}^{\infty} e^{-z^2/2} P_k(z, y) dz. \]
(20)
For \( \beta = 4 \) we require a slightly modified (by a simple rescaling) skew-inner product with associated modified skew-orthogonal polynomials \( \tilde{Q}_j \) and normalizations \( \tilde{q}_j \) [see Eqs. (59) and (60)]. But we can again use the basis (16) to give
\[ F_{4,N}(y) = \frac{2^{N^2}}{\pi^{N/2}} \prod_{j=0}^{N-1} \frac{1}{(2j + 1)!} \text{Pf} \, W_{2N-1} \bigg|_{y = \sqrt{2y}'}, \]
(21)
\[ \text{erf}(x) := \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-t^2} dt \quad \text{erfc}(x) := 1 - \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_{x}^{\infty} e^{-t^2} dt, \]
and they are related by the expressions \( \text{erfc}(-x) = 2 - \text{erfc}(x) = 1 + \text{erf}(x) \).

\[ \text{This modification of the skew-inner product allows us to define a single operator } A \text{ which can be used to construct both the } \beta = 1 \text{ and } \beta = 4 \text{ cases [2]. See Sect. 4.1 for the details.} \]
Fig. 1 The histograms correspond to a numerical evaluation of the CDF of the largest eigenvalue sampled from 50,000 matrices in the $\beta = 1$ ensemble for $N = 8$ (left panel) and in the $\beta = 4$ ensemble for $N = 4$ (right panel). The solid red line represents the exact result given, in the left panel, by Eq. (17) and, in the right panel, by Eq. (21).

where the matrix $W_m$ is given by

$$W_m = \begin{bmatrix}
0 & p_1 + \frac{\Omega_{0,1}}{2} & \frac{\Omega_{0,2}}{2} & \cdots & \frac{\Omega_{0,m}}{2} \\
-p_1 - \frac{\Omega_{0,1}}{2} & 0 & p_2 + \frac{\Omega_{1,1}}{2} & \cdots & \frac{\Omega_{1,m}}{2} \\
-p_2 - \frac{\Omega_{1,2}}{2} & -p_1 - \frac{\Omega_{0,2}}{2} & 0 & \cdots & \frac{\Omega_{2,m}}{2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-p_m - \frac{\Omega_{0,m}}{2} & -p_{m-1} - \frac{\Omega_{1,m}}{2} & -p_{m-2} - \frac{\Omega_{2,m}}{2} & \cdots & 0
\end{bmatrix}. \tag{22}$$

and

$$\Omega_{j,k} := e^{-y^2} P_j(y,y) P_k(y,y). \tag{23}$$

As a test of our expressions for $V_m$ and $W_m$ (and consequently, of our expressions for the $\beta = 1$ and $\beta = 4$ skew-orthogonal polynomials, which are given in terms of these matrices) we present in Fig. 1 a comparison between a numerical evaluation of the formulae (17) and (21) and a direct numerical computation of these CDFs by sampling GOE and GSE random matrices, showing very good agreement. This also demonstrates that our expressions are useful for high-precision numerical computations, for small $N$. However once $N$ starts to become large ($N \gtrsim 10$), the exact expressions for the required $\beta = 2$ polynomials $P_j(x, y)$ and their normalizations $p_j(y)$ become unmanageably long, and consequently we have not computed the $\beta = 1$ or $\beta = 4$ CDFs nor the skew-orthogonal polynomials for larger $N$ as yet. For numerical calculations for $\beta = 1$ with larger $N$ (up to $N \approx 500$) the reader is referred to [23], where a very neat recursive formula for the CDF is presented. We would like to note that our expressions for the skew-orthogonal polynomials will also allow high-precision computation of quantities other than the $\beta = 1$ CDF, such as the eigenvalue gap distributions and correlation functions (for small $N$). They are also quite useful in providing expressions for $F_{1,N}(y)$ and $F_{4,N}(y)$ that are amenable to asymptotic analysis, which is the main application we put them to in the present work.

Indeed, we emphasize that, because of the decomposition (16), the expressions on the right hand side of Eqs. (17) and (21) depend only on the $\beta = 2$ orthogonal polynomials, and do not depend on the skew-orthogonal polynomials at all. In Sect. 4 we provide explicit expressions for the coefficients $\alpha_{j,k}$ for the $\beta = 1$ and $\beta = 4$ polynomials, and these are
then amenable to asymptotic analysis. Thus, for an analysis of the large $N$ asymptotics of the GOE and GSE, we only need the asymptotic behaviour of the basis polynomials.

These $\beta = 2$ polynomials [the orthogonal polynomials $P_k$ for the inner product in (6)] have already been studied, first in [62] and later in [66]. Here we call these polynomials the Nadal–Majumdar (NM) polynomials. Interestingly, these NM polynomials naturally arise also in the study of the so called level curvature distribution at the soft edge of random Hermitian matrices [35]. Although they do not have a known closed formula, they satisfy the three-term recurrence relation [since they are orthogonal with respect to the inner product in (6)]

$$
\lambda P_k(\lambda, y) = P_{k+1}(\lambda, y) + \hat{S}_k(y) P_k(\lambda, y) + \hat{R}_k(y) P_{k-1}(\lambda, y) \quad (24)
$$

$$
\hat{R}_k(y) = \frac{p_k(y)}{p_{k-1}(y)} \quad (25)
$$

$$
\hat{S}_k(y) \neq 0, \quad (26)
$$

where the last expression follows because the domain of integration in the inner product (6) is not symmetric. (Note that we have used the “check” and sans serif font to distinguish $\hat{R}_k$ from the $\beta = 1$ polynomials $R_j$.) We use the same style for $\hat{S}_k$ for consistency.) In the limit $y \to \infty$ the NM polynomials become the (monic, “physicist’s”) Hermite polynomials, i.e. [62]

$$
P_j(\lambda, y) = \frac{1}{2^j} H_j(\lambda) + O\left(e^{-y^2}\right), \quad (27)
$$

where the Hermite polynomials of index $j$, $H_j(x)$, are orthogonal with respect to the weight function $e^{-x^2}$, and the division by $2^j$ is here to ensure monicity. In fact, in this limit the inner product (6) and skew-inner products (7)–(8) all reduce to their classical Gaussian counterparts, with norms [33,55]

$$
p_j(\infty) = \frac{\pi^{1/2}}{2^j} \Gamma(j + 1), \quad q_j(\infty) = \frac{\pi^{1/2}}{2^{4j+2}} \Gamma(2j + 2), \quad r_j(\infty) = \frac{\pi^{1/2}}{2^{2j}} \Gamma(2j + 1). \quad (28)
$$

The corresponding classical skew-orthogonal polynomials are known, and recalled in Appendices E.4 and F.2.

However, for finite $y$, there are no known statements analogous to (24)–(26) for $\beta = 1$ and 4 polynomials. Yet, as a first approach, we can iteratively use the skew-inner products (7) and (8) to construct these polynomials. An important property is that these polynomials are not unique, since skew-inner products are invariant under the polynomial transformation

$$
\eta_{2j+1} \mapsto \eta_{2j+1} + d \eta_{2j} \quad (29)
$$

where $d$ is any constant (and $\eta_k = Q_k$ or $R_k$), and therefore a set of skew-orthogonal polynomials is unique only up to this symmetry in the odd degree polynomials [33]. By specifying this constant $d$ we employ this iterative process to construct the skew-orthogonal polynomials defined in Eqs. (10) and (11) in Appendix D. However this method is not convenient for the asymptotic analysis of the quantities in (5). Instead, in [62,66], it was shown that the recurrence relations (24)–(26) can be exploited to obtain the asymptotic

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6 Note that this invariance follows from the anti-symmetry of skew-inner products, and also that the invariance does not hold for the even degree polynomials; that is the mapping $\eta_{2j} \mapsto \eta_{2j} + d \eta_{2j-1}$ is not invariant — we leave the confirmation of this as a short exercise for the reader.
behaviors of the norms $p_j(y)$ and the polynomials $P_j(\lambda, y)$ themselves in the limit of large $N$ and large $y$. So, as mentioned above, we extend the approach developed in [2,3] to obtain explicit expressions for the sets of semi-classical skew-orthogonal polynomials $\{Q_j\}$ and $\{R_j\}$ in the basis of the orthogonal polynomials $\{P_j\}$ (the NM polynomials) as in (16). This is the content of Proposition 4 (for the GSE) and Proposition 5 (for the GOE). Interestingly, the proofs of these results are achieved by using results on compound Pfaffians [65], also called overlapping Pfaffians by Knuth in Ref. [43] — more specifically, we obtain some sums over differences of Pfaffians [in (E.26) and (E.47)], which have the structure of overlapping Pfaffians. (See the proofs of Propositions 4 and 5 in Appendix E for the details.) Propositions 4 and 5 are the first main technical contributions of this work. As a byproduct of our analysis, we also recover the classical skew-orthogonal polynomials as the $y \to \infty$ limit of our results here (see Appendices E and F).

We will then use this explicit construction, together with the asymptotic analysis of the polynomials $P_j(x, y)$, to compute the large $N$ asymptotic limit of $F_{1,N}$ and $F_{4,N}$. Indeed for the case of the GSE, we show that

$$\lim_{N \to \infty} F_{4,N} \left( y = \sqrt{2N + \frac{s}{2^{7/6}N^{1/6}}} \right) = \exp \left( -\frac{1}{2} \int_s^\infty (x - s)q^2(x)dx \right) \cosh \left( \frac{1}{2} \int_s^\infty q(x)dx \right),$$

where $q(x)$ is the Hastings-McLeod solution of the Painlevé II equation, i.e.

$$q''(x) = xq(x) + 2q^3(x), \quad \text{with} \quad q(x) \sim \text{Ai}(x),$$

and $\text{Ai}(x)$ is the standard Airy function. On the other hand, for the case of the GOE, we show that

$$\lim_{N \to \infty} F_{1,N} \left( y = \sqrt{2N + \frac{s}{\sqrt{2N^{1/6}}} \right) = \exp \left( -\frac{1}{2} \int_s^\infty (x - s)q(x)^2dx \right) \exp \left( \frac{1}{2} \int_s^\infty q(x)dx \right),$$

with, again, $q(x)$ given in Eq. (31). We thus recover the known expressions of the TW distributions for GSE and GOE [82], by using here a completely different method. This is the second main achievement of the present paper. The key result used to obtain the TW distributions is an identity proved in Lemma 1 [see Eq. (94)] that allows us to obtain explicit expressions of the Pfaffians entering the expressions in Eqs. (17) and (21), which are then conveniently amenable to an asymptotic analysis in the limit of large $N$. The proof of this identity (94) relies on the expression of a Pfaffian as a sum over perfect matchings recalled in (B.7) of the Appendices — this representation is used extensively throughout the present paper.

The paper is organized as follows. In Sect. 3 we use the polynomials $Q_j$ and $R_j$, defined in (10) and (11) respectively, to find Pfaffian expressions for restricted partition functions

7 Such relations on compound Pfaffians have been obtained in the physics literature as a consequence of a general Wick theorem for fermions [65], which turned out to be useful in the study of the 2d-Ising model, see e.g. [8].

8 Note that the factor $2^{-7/6}$ differs by a factor $2^{-1/6}$ from the result obtained in the original paper [82]. This mistake was actually noticed in [60, p.47] — see also [19]. There, this factor was corrected by matching with known asymptotic results for large (positive and negative) arguments. Here, we obtain this correct factor $2^{-7/6}$ by a direct computation.

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such as the CDFs $F_{1,N}(y)$ and $F_{4,N}(y)$ using standard techniques. In Sect. 4 we construct explicitly these skew-orthogonal polynomials in the basis of the NM polynomials $P_j(x, y)$ and their normalizations $p_j(y)$ [as in (16)], finding, in particular, Pfaffian expressions for the coefficients $\alpha_{j,k}$ and the normalizations $q_j(y)$ and $r_j(y)$. In Sects. 5 and 6 we present the asymptotic analysis of $F_{4,N}(y)$ and $F_{1,N}(y)$ respectively, leading to the expressions given in Eqs. (30) and (32). Finally, Sect. 7 contains our conclusions and perspectives. Several technical details about the results presented in this paper have been left to the Appendices.

3 Restricted Partition Functions and Generalizations

In this section, we show how to compute restricted partition functions such as the CDFs $F_{\beta,N}(y)$ in Eq. (5). We actually consider slightly more general quantities defined as the following averages over the eigenvalue JPDFs for the GOE ($\beta = 1$), GUE ($\beta = 2$) and the GSE ($\beta = 4$):

$$
\hat{Z}_{\beta,N}[a, y] = \left\langle \prod_{j=1}^{N} a(\lambda_j) \right\rangle_{P_{\beta}}^y = \frac{1}{Z_{\beta,N}} \int_{-\infty}^{y} d\lambda_1 \cdots \int_{-\infty}^{y} d\lambda_N \prod_{j=1}^{N} a(\lambda_j) e^{-\beta \lambda_j^2/2} \prod_{j<k} |\lambda_k - \lambda_j|^\beta. \tag{33}
$$

Each of the $\hat{Z}_{\beta,N}[a, y]$ will be put into determinant/Pfaffian form — the construction of the associated matrices will depend on its own set of monic polynomials. While these polynomials are in principle arbitrary, it is convenient to specify them to be the respective orthogonal/skew-orthogonal polynomials. If we think of the integral in (33) as an average over a truncated version of the density (2), i.e.

$$
P_{\beta}(\lambda_1, \ldots, \lambda_N; y) := P_{\beta}(\lambda_1, \ldots, \lambda_N) \chi_{(-\infty, y)}(\lambda_1, \ldots, \lambda_N), \tag{34}
$$

where $\chi_A(x) = 1$ if $x \in A$ and zero otherwise, then we are in the realm of Janossy densities [41] (see [39] as well as [76] for a clear introduction to the topic and references). In [18] the authors discussed “determinantal” Janossy densities (where the particle JPDF and $n$-point correlation functions can be written in terms of a determinant) and computed the corresponding kernel. In [75] these results were extended to “Pfaffian” Janossy densities, that is, the author found the matrix kernel for Janossy JPDFs and $n$-point correlation functions that are expressed as Pfaffians. Our eigenvalue JPDFs (2) have this determinantal ($\beta = 2$) or Pfaffian ($\beta = 1, 4$) structure, and so the $n$-point correlations will also have determinantal/Pfaffian structure. We will explicitly construct these correlation functions in a future work, and use them to calculate gap probabilities and the density of states near the largest eigenvalue [54]. Here, however, we restrict ourselves to the calculation of the averages (33), which gives us the CDF of the largest eigenvalue (5) via

$$
F_{\beta,N}(y) = \hat{Z}_{\beta,N}[1, y]. \tag{35}
$$

Below we treat the case $\beta = 2$, $\beta = 4$ and $\beta = 1$, again by increasing order of complexity. Note that these results are known, and the techniques we use to derive them are standard (see [33,55]), but we include them to show how our results fit inside the larger theory of Gaussian random matrices, and, in particular, why we must specify that $N$ is even or odd when $\beta = 1$.
3.1 $\beta = 2$

Although this is not needed for the $\beta = 1, 4$ cases, for completeness we also include the $\beta = 2$ result, which can be obtained using the Vandermonde determinant identity (the procedure is a slight modification to that in [33, §5.2.1])

$$\hat{Z}_{2,N}[a, y] = \frac{N!}{Z_{2,N}} \det \left[ \gamma_{j,k}^{(2)}[a, y] \right]_{j,k=0,\ldots,N-1},$$

where $Z_{2,N}$ is given in (4) and

$$\gamma_{j,k}^{(2)}[a, y] := \int_{-\infty}^{y} a(\lambda) e^{-\lambda^2} P_j(\lambda, y) P_k(\lambda, y) d\lambda.$$ (37)

The polynomials $P_j$ in (37) are the NM polynomials, i.e. the monic polynomials of degree $j$ that are orthogonal with respect to the inner product (6). A consequence of this (in the limit $y \to \infty$) is the known result [33,55]

$$Z_{2,N} = N! \prod_{j=0}^{N-1} p_j(\infty),$$ (38)

where $p_j(\infty)$ is given in (28), which agrees with (4).

With $a(x) = 1$ the integral $\gamma_{j,k}^{(2)}$ becomes the inner product (6), so with the orthogonal polynomials $P_j$ we use (35) to obtain the known result

$$F_{2,N}(y) = \prod_{j=0}^{N-1} \frac{p_j(y)}{p_j(\infty)},$$ (39)

which gives (15).

3.2 $\beta = 4$

Proposition 1 The average (33) for $\beta = 4$ is

$$\hat{Z}_{4,N}[a, y] = \frac{N! 2^N}{Z_{4,N}} \text{Pf} \left[ \gamma_{j,k}^{(4)}[a, y] \right]_{j,k=0,\ldots,2N-1},$$ (40)

where

$$\gamma_{j,k}^{(4)}[a, y] := \frac{1}{2} \int_{-\infty}^{y} d\lambda \, a(\lambda) e^{-\lambda^2}$$

$$\times \left[ Q_j(\lambda, y) \frac{d}{d\lambda} \left( e^{-\lambda^2} Q_k(\lambda, y) \right) - Q_k(\lambda, y) \frac{d}{d\lambda} \left( e^{-\lambda^2} Q_j(\lambda, y) \right) \right]$$ (41)

and the $Q_j$ are monic polynomials of degree $j$ that are skew-orthogonal with respect to the skew-inner product (7).

Using the theory of Sect. 4 below, we can make a quick check of (40) by noting that when $a(x) = 1$ the matrix in (40) is of the form (52), and so from (53) the Pfaffian is given by the product $q_0(y)q_1(y) \cdots q_{N-1}(y)$. Then, in the limit $y \to \infty$, we recover the result analogous
to (38) \cite{33,55}
\[
\hat{Z}_{4,N}[1, y] \bigg|_{y \to \infty} = 1 \quad \Rightarrow \quad Z_{4,N} = N!2^N \prod_{j=0}^{N-1} q_j(\infty),
\]
where $q_j(\infty)$ is given in (28), and this agrees with (4).

To obtain our results we let $a(x) = 1$ in (40), and then $\gamma_{j,k}^{(4)}[1, y]$ reduces to the skew-inner product (7), giving us the CDF
\[
F_{4,N}(y) = \prod_{j=0}^{N-1} \frac{q_j(y)}{q_j(\infty)},
\]
However, clearly we will need the explicit forms of the normalizations $q_j$ before we can obtain the expression in (21), which is a new result — this will be achieved below in Sect. 4.

### 3.3 $\beta = 1$, with $N$ even

Recall that in the statement of the CDF (17) for the largest eigenvalue of the GOE we restricted $N$ to be even. The parity of $N$ plays an important role since for the $\beta = 1$ case we have the difficulty of the absolute value of the Vandermonde determinant in (2). To deal with it, we apply the method of integration over alternate variables, which was introduced by de Bruijn \cite{24} and applied to integrals similar to (33) by Mehta \cite{55}. However, this method is dependent on the parity of $N$, which can be seen in the proof (Appendix C.2) when one pairs up the rows in (C.7) — an odd number of rows obviously cannot be perfectly paired. We discuss the odd case in the following subsection.

**Proposition 2** With $N$ even the average (33) for $\beta = 1$ is
\[
\hat{Z}_{1,N}[a, y] = \frac{2^{N/2}N!}{Z_{1,N}} \text{Pf} \left[ \gamma_{j,k}^{(1)}[a, y] \right]_{j,k=0,\ldots,N-1},
\]
where
\[
\gamma_{j,k}^{(1)}[a, y] := \frac{1}{2} \int_{-\infty}^{y} dx \ a(x) e^{-x^2/2} R_j(x, y) \int_{-\infty}^{y} dz \ a(z) e^{-z^2/2} R_k(z, y) \text{sgn}(z-x)
\]
and the $R_j$ are monic polynomials of degree $j$ that are skew-orthogonal with respect to the skew-inner product (8).

We can recover the known result (4) \cite{33,55} with $a(x) = 1, y \to \infty$
\[
\hat{Z}_{1,N}[1, y] \bigg|_{y \to \infty} = 1 \quad \Rightarrow \quad Z_{1,N} = N!2^{N/2} \prod_{j=0}^{N/2-1} r_j(\infty),
\]
where $r_j(\infty)$ is given in (28).

As for $\beta = 4$ above, $\gamma_{j,k}^{(1)}[1, y]$ is the skew-inner product (8) and so
\[
F_{1,N}(y) = \prod_{j=0}^{N/2-1} \frac{r_j(y)}{r_j(\infty)},
\]
yet we still need explicit expressions for the normalizations $r_j$ before we can obtain the expression for the CDF $F_{1,N}$ in (17). We will find these expressions in Sect. 4.
3.4 $\beta = 1$, with $N$ odd

The technique that we use to prove the $N$ odd case in Appendix C.3 can be found in [33,55], but there are variations on this technique, and entirely different techniques for dealing with odd dimensional real random matrix ensembles elsewhere in the literature — see for example [31,52,72–74].

**Proposition 3** With $N$ odd the average (33) for $\beta = 1$ is

$$
\hat{Z}_{1,N_{\text{odd}}}[a, y] = \frac{2^{(N-1)/2}N!}{Z_{1,N}} \text{Pf} \left[ \begin{array}{c|c} [Y_{j,k}(a, y)] & [v_j[a, y]] \\ \hline -v_k[a, y] & 0 \end{array} \right]_{j,k=0,...,N-1},
$$

where $Y_{j,k}(a, y)$ and the polynomials $R_j$ are as in Proposition 2 and

$$
v_k[a, y] := \int_{-\infty}^{y} e^{-x^2/2} a(x) R_k(x) \, dx.
$$

With $a(x) = 1$ we get

$$
F_{1,N_{\text{odd}}}(y) = \frac{v_{N-1}[1, y]}{v_{N-1}[1, \infty]} \prod_{j=0}^{(N-1)/2-1} \frac{r_j(y)}{r_j(\infty)},
$$

and using the matrix $\mathbf{V}_m$ in (18) we can therefore write

$$
F_{1,N_{\text{odd}}}(y) = \frac{2^{(2N-4N-3)/4}}{\pi^{(N-1)/4} \Gamma(N/2)} \left( \prod_{j=0}^{(N-1)/2-1} \frac{1}{(2j)!} \right) \frac{v_{N-1}[1, y]}{v_{N-1}[1, \infty]} \prod_{j=0}^{(N-3)/2} r_j(y)
$$

$$
= \frac{2^{(2N-4N-3)/4}}{\pi^{(N-1)/4} \Gamma(N/2)} \left( \prod_{j=0}^{(N-1)/2-1} \frac{1}{(2j)!} \right) \frac{v_{N-1}[1, y]}{v_{N-1}[1, \infty]} \text{Pf} \mathbf{V}_{N-2},
$$

which is an explicit expression for the $N$ odd CDF, however it still relies on the skew-orthogonal polynomial $R_{N-1}$, or at least its representation as a sum of NM polynomials via the coefficients $\alpha_{j,k}$ in (16). (Recall that the $N$ even expression in (17) for $N$ had no reference to the skew-orthogonal polynomials at all.)

4 Explicit Construction of the Skew-Orthogonal Polynomials

The averages (40) and (44) in Sect. 3 above contain integrals over the respective skew-orthogonal polynomials $Q_j$ and $R_j$. The major advantage of using these polynomials can be seen if we first consider the case of $\beta = 2$, from the expression (36): we see that when $a(x) = 1$ the matrix in the determinant becomes $[(P_j, P_k)^Y]_{j,k=0,...,N-1}$, and so the determinant will be simply calculated if the polynomials $P_j$ are orthogonal with respect to the inner product (6) since the resulting matrix is diagonal. Indeed, this was the approach taken in [62,66], where the orthogonal polynomials are the NM polynomials, which obey the relations (24)–(26).
We will use the same approach for the $\beta = 4$ and $\beta = 1$ cases; that is we will construct the polynomials $Q_j$ and $R_j$ such that the matrices in (40) and (44) are of skew-diagonal form\(^9\)

\[
S = \begin{bmatrix}
0 & s_1 & 0 & 0 & \cdots & 0 & 0 \\
-s_1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & s_2 & \cdots & 0 & 0 \\
0 & 0 & -s_2 & 0 & \vdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \ddots \\
0 & 0 & 0 & \cdots & 0 & s_N \\
0 & 0 & 0 & \cdots & -s_N & 0
\end{bmatrix}.
\]

(52)

The only non-zero elements of $S$ are in $2 \times 2$ blocks $\begin{bmatrix} 0 & s_j \\ -s_j & 0 \end{bmatrix}$ on the diagonal, and we then have the simple result

\[
Pf S = \prod_{j=1}^{N} s_j.
\]

(53)

In other words, we are looking for two sets of monic polynomials $\{Q_j\}, \{R_j\}$ that satisfy the conditions in (10) and (11) respectively. Such polynomials are called skew-orthogonal polynomials. Recall that these polynomials are only unique up to the transformation (29), where $\eta_j = Q_j (\beta = 4)$ and $\eta_j = R_j (\beta = 1)$.

As discussed in Introduction we can, in principle, construct the polynomials iteratively using the conditions (10) and (11), but this technique does not yield expressions that are amenable to asymptotic analysis. Nor is there a known closed form or recursive expression for these polynomials. Rather, we apply the method of [2,3] where we aim to express our skew-orthogonal polynomials in the basis of the NM polynomials $\{P_j\}$ from the analogous $\beta = 2$ problem as in (16). If we can find the coefficients $\alpha_{j,k}$ in (16) then we can use the properties of the polynomials $\{P_j\}$ to obtain asymptotic results for the problems that we consider here. Note that (29) implies that we have freedom in the choice of the $\alpha_{2j+1,2j}$ (that is, the second term in the odd-degree polynomials) in equation (16), and we will typically choose $\alpha_{2j+1,2j} = 0$. [We will see that this choice is quite natural once we have the general formula for the coefficients, see (E.36).] We find that the coefficients and the polynomial normalizations $q_j(y)$ and $r_j(y)$ are given as ratios of Pfaffians. To contrast this with the classical Gaussian case (in the limit $y \to \infty$) we have included the skew-orthogonal polynomials for $\beta = 4$ in Appendix E.4 and for $\beta = 1$ in Appendix F.2.

The key to the method of [2] is an operator $A$, which acts thusly

\[
Af[x] = e^{x^2/2} \frac{d}{dx} \left( e^{-x^2/2} f(x) \right).
\]

(54)

We will also need the inverse operator

\[\text{---}
\]

\(^9\) The term **skew-diagonal** is used here in analogy with the term **diagonal**, that is, the (non-trivial) skew-symmetric ($M = -M^T$) analogue of a diagonal matrix.
\[ A^{-1} f[x] = \frac{e^{x^2/2}}{2} \int_{-\infty}^{\infty} \text{sgn}(x - z) e^{-z^2/2} f(z) dz \]
\[ = \frac{e^{x^2/2}}{2} \left( \int_{-\infty}^{x} e^{-z^2/2} f(z) dz - \int_{x}^{\infty} e^{-z^2/2} f(z) dz \right). \tag{55} \]

That this is the inverse can be checked by explicitly calculating \( AA^{-1} f[x] \) and \( A^{-1}Af[x] \), and using the identity \( \frac{d}{dx} \text{sgn}(x - z) = 2\delta(x - z) \) (where care is taken to use the distributional derivative). The use of these operators will allow us to find relations between the \( \beta = 2 \) inner product (6) and the \( \beta = 4, 1 \) skew-inner products (7) and (8), and then to find the sought relations between the polynomials themselves.

### 4.1 \( \beta = 4 \)

Recall that the goal is to find the coefficients \( \alpha_{j,k} \) in (16) so that we can express the \( Q_j \) in the basis of the \( P_j \), which are the \( \beta = 2 \) orthogonal polynomials [orthogonal with respect to the integral of \( f(x)g(x) \) (56)], and we will use the operator \( A \) from (54). This operator will allow us to develop both the \( \beta = 1 \) and \( \beta = 4 \) cases in the same framework, however, we find ourselves arriving at a \( \beta = 4 \) skew-inner product that is slightly modified from (7). But the associated skew-orthogonal polynomials \( \tilde{Q}_j \) are related to the \( \tilde{Q}_j \) by a simple rescaling and so the cost of the technique seems outweighed by its simplicity and uniformity. The modified \( \beta = 4 \) skew inner product is defined as

\[
\langle f, g \rangle_4^\beta := \frac{1}{2} \int_{-\infty}^{\infty} dx \ e^{-x^2} \left[ f(x)g'(x) - g(x)f'(x) \right]
\]
\[
= \frac{1}{2} \int_{-\infty}^{\infty} dx \ e^{-x^2/2} \left[ f(x) \frac{d}{dx} \left( e^{-x^2/2} g(x) \right) - g(x) \frac{d}{dx} \left( e^{-x^2/2} f(x) \right) \right], \tag{56} \]

which is the same as (7), except that we have replaced \( e^{-x^2} \mapsto e^{-x^2}/2 \). We also define the associated monic skew-orthogonal polynomials \( \{ \tilde{Q}_j \}_{j=0,1,\ldots} \) and normalizations \( \{ \tilde{q}_j \}_{j=0,1,\ldots} \):

\[
\langle \tilde{Q}_j, \tilde{Q}_k \rangle_4^\beta = \langle \tilde{Q}_{j+1}, \tilde{Q}_{k+1} \rangle_4^\beta = 0,
\]
\[
\langle \tilde{Q}_j, \tilde{Q}_{k+1} \rangle_4^\beta = -\langle \tilde{Q}_{k+1}, \tilde{Q}_j \rangle_4^\beta = \tilde{q}_j(y)\delta_{j,k}. \tag{57} \]

Note that the use of the tilde \(^ \sim \) here and elsewhere in this paper (which matches the notation in [33]) denotes that the quantity is related to this modified \( \beta = 4 \) skew-inner product (56), rather than the standard skew-inner product (7).

By performing a change of variables we have the relations

\[
q_j(y) = 2^{-2j-\frac{1}{2}} \tilde{q}_j \left( \sqrt{2} y \right), \tag{59} \]
\[
Q_k(x, y) = 2^{-k/2} \tilde{Q}_k \left( \sqrt{2} x, \sqrt{2} y \right), \tag{60} \]

and so we can recover the polynomials that we are searching for. (Note that the factor of \( 2^{-k/2} \) ensures that the polynomials remain monic.) We can check these relations by generating the first few polynomials \( \tilde{Q}_j \), as done for the \( Q_j \) in Appendix D,

\[
\tilde{Q}_0(\lambda, y) = 1, \quad \tilde{Q}_1(\lambda, y) = \lambda, \quad \tilde{Q}_2(\lambda, y) = \lambda^2 + 2\lambda + \frac{1}{2} \left( 1 - \frac{y}{2} \right), \tag{61} \]
\[
\tilde{Q}_3(\lambda, y) = \lambda^3 - 3 \frac{1 - y\tilde{b}}{2} \lambda - \tilde{b}(1 + y^2)
\]  
(62)

[where we used (29) for \(\tilde{Q}_3(\lambda)\)] and the corresponding normalizations

\[
\tilde{q}_0(y) := \langle \tilde{Q}_0, \tilde{Q}_1 \rangle_4^y = \frac{\sqrt{\pi}}{4} \text{erfc}(-y) = \frac{e^{-y^2}}{2\tilde{b}},
\]  
(63)
\[
\tilde{q}_1(y) := \langle \tilde{Q}_2, \tilde{Q}_3 \rangle_4^y = \frac{1}{8} \left(3\sqrt{\pi} \text{erfc}(-y) - e^{-y^2} y(9 + 2y^2) - e^{-y^2}(4 + y^2)\tilde{b}\right),
\]  
(64)

with the parameter \(\tilde{b}\) given by

\[
\tilde{b} = \frac{2e^{-y^2}}{\sqrt{\pi}(1 + \text{erf}(y))} = \frac{2e^{-y^2}}{\sqrt{\pi} \text{erfc}(-y)}.
\]  
(65)

To use this modified skew-inner product we will introduce the operator \(A\) from (54) into the \(\beta = 2\) inner product (6) and we have the properties (by integrating by parts)

\[
(f, Ag)_2^y = -(g, Af)_2^y + \Omega(f, g; y),
\]  
(66)
\[
(f, Af)_2^y = \frac{\Omega(f, f; y)}{2},
\]  
(67)

where

\[
\Omega(f, g; y) := \left[e^{-x^2} f(x)g(x)\right]_{-\infty}^y = \lim_{x \to y} \left(e^{-x^2} f(x)g(x)\right) - \lim_{x \to -\infty} \left(e^{-x^2} f(x)g(x)\right).
\]  
(68)

Then we can write

\[
\langle f, g \rangle_4^y = \frac{1}{2} \left((f, Ag)_2^y - (g, Af)_2^y\right) = (f, Ag)_2^y - \frac{\Omega(f, g; y)}{2}.
\]  
(69)

So we are searching for coefficients \(\tilde{a}_{j,k}\)

\[
\tilde{Q}_j = P_j + \tilde{a}_{j,j-1} P_{j-1} + \cdots + \tilde{a}_{j,1} P_1 + \tilde{a}_{j,0}
\]  
(70)

such that the relations (57)–(58) hold, and we will use (69)–(70) to recast the problem in terms of the \(\beta = 2\) inner product and associated polynomials. Note that the coefficients \(\tilde{a}_{j,k}\) depend on \(y\). Here we only present the results, with the detailed derivations in Appendix E.

With \(W_m\) from (22) we have \(W_m = \left[\langle P_j, P_k \rangle_4^y\right]_{j,k=0,...,m}\), and so we can use the fact that \(\tilde{Q}_j\) is a linear combination of the \(P_j\)’s to obtain information about the coefficients \(\tilde{a}_{j,k}\) in (70). Then we use a result of Knuth on overlapping Pfaffians [43, (5.0)–(5.1)] (see also Appendix E for more details) to obtain the following.

**Proposition 4** Assuming

\[
\tilde{a}_{j,j-1}(y) = 0, \quad j \text{ odd},
\]  
(71)
then for \( j \geq 2 \)

\[
\tilde{q}_{j,k}(y) = \begin{cases} 
- \frac{\text{Pf } W_{j-1}^{(k\rightarrow j)}}{\text{Pf } W_{j-1}}, & \text{if } j \text{ even, } k \leq j - 1, \\
- \frac{\text{Pf } W_{j-2}^{(k\rightarrow j)}}{\text{Pf } W_{j-2}}, & \text{if } j \text{ odd, } k \leq j - 2,
\end{cases}
\]

(72)

where \( W_m^{(\eta\rightarrow v)} \) is the matrix \( W_m \) from (22) with all occurrences of the index \( \eta \) replaced by the index \( v \).

The normalizations are

\[
\tilde{q}_j(y) = \frac{\text{Pf } W_{2j+1}}{\text{Pf } W_{2j-1}},
\]

(73)

with the convention

\[
\text{Pf } W_{-1} = 1.
\]

(74)

The proofs of these results are contained in Appendix E.

With Proposition 4, we can obtain the skew-orthogonal polynomials for the skew-inner product (7) via (60)

\[
Q_j(\lambda, y) = 2^{-j/2} \left[ P_j(\sqrt{2}\lambda, \sqrt{2}y) + \tilde{a}_{j,j-1}(\sqrt{2}y) P_{j-1}(\sqrt{2}\lambda, \sqrt{2}y) + \cdots + \tilde{a}_{j,1}(\sqrt{2}y) P_1(\sqrt{2}\lambda, \sqrt{2}y) + \tilde{a}_{j,0}(\sqrt{2}y) \right],
\]

(75)

where the \( P_j \) are the NM polynomials, and the normalizations are obtained from (73) via (59)

\[
q_j(y) = 2^{-j-1} \left. \frac{\text{Pf } W_{2j+1}}{\text{Pf } W_{2j-1}} \right|_{y\rightarrow\sqrt{2}y}.
\]

(76)

From (35) we know that the CDF of the largest eigenvalue is expressed in terms of the average (33), with the function \( a(x) = 1 \). Also, with \( a(x) = 1 \) we have \( \gamma_j^{(4)}[1, y] = \langle Q_j, Q_k \rangle_{4}^{y} \).

So using the skew-orthogonal polynomials (75), the relations (10) tell us that the matrix in \( \tilde{Z}_{N,4}[1, y] \) is of the form (52), and so its Pfaffian is given by (53) with \( s_j = q_j \), yielding (43). Then, substitution of the normalization (76) into (43) gives the result in (21).

The expression for \( \tilde{a}_{j,k} \) can be seen to recover the classical Gaussian case (with \( y \rightarrow \infty \)), since in this limit the polynomials \( P_j \) are the Hermite polynomials [from (27)] and also that \( \Omega_{j,k} = 0 \), so the matrix (22) is then the same as that in [33, Prop 6.2.1]. The derivation of the \( \alpha_{j,k} \) in (E.56) then proceeds identically.

### 4.2 \( \beta = 1 \)

As above, we want to express the skew-orthogonal polynomials \( \{ R_j \} \) from (11) in terms of the polynomials \( \{ P_j \} \) from (9). So we look for coefficients \( \alpha_{j,k} \) such that

\[
R_j = P_j + \alpha_{j,j-1} P_{j-1} + \cdots + \alpha_{j,1} P_1 + P_0,
\]

(77)

and again these coefficients will depend on \( y \). To make further progress, we use the operator \( A^{-1} \) from (55). First we note from (66) that

\[
(f, A^{-1} g)_2^y = (AA^{-1} f, A^{-1} g)_2 = -(A^{-1} f, g)_2^y + \Omega(A^{-1} f, A^{-1} g; y)
\]
\[ = -(g, A^{-1} f)_2^y - \Phi(f, g) - \Phi(g, f), \]  
where
\[ \Phi(f, g) := \frac{1}{2} \int_{-\infty}^{\infty} e^{-z^2/2} f(z) dz \int_{\infty}^{\infty} e^{-z^2/2} g(z) dz. \]

Now we can re-write the skew-inner product (8) as
\[ (f, g)_1^y = -\frac{1}{2} \left( (f, A^{-1} g)_2^y - (g, A^{-1} f)_2^y + \Phi(f, g) - \Phi(g, f) \right) \]
\[ = -(f, A^{-1} g)_2^y - \Phi(f, g). \]  
(80)

From here we follow the same procedure as for \( \beta = 4 \), but replacing the matrix \( W_m \) in (22) with the more complicated matrix \( V_m \) in (18) (where we have defined \( \Phi_{j,k} := \Phi(P_j, P_k) \) for simplicity).

Note that we have the equality
\[ V_m = \left[ (P_j, P_k)_1^y \right]_{j,k=0, \ldots, m} = -\left[ (P_j, A^{-1} P_k)_2^y + \Phi_{j,k} \right]_{j,k=0, \ldots, m}. \]  
(81)

We now give expressions for the coefficients \( \alpha_{j,k}(y) \) and normalizations \( r_j(y) \) in terms of the matrix \( V_m \). (The details of the derivation are provided in Appendix F.)

**Proposition 5** Assuming
\[ \alpha_{j,j-1}(y) = 0, \quad j \text{ odd,} \]  
(82)
then for \( j \geq 2 \)
\[ \alpha_{j,k}(y) = \begin{cases} 
\text{Pf } V_{j-1}^{(k=0)} & j \text{ even, } k \leq j - 1, \\
-\frac{\text{Pf } V_{j-1}^{(k=1)}}{\text{Pf } V_{j-1}} & j \text{ odd, } k \leq j - 1, \\
-\frac{\text{Pf } V_{j-2}^{(k=0)}}{\text{Pf } V_{j-2}} & j \text{ odd, } k \leq j - 2, \\
-\frac{\text{Pf } V_{j-2}^{(k=1)}}{\text{Pf } V_{j-2}} & j \text{ even, } k \leq j - 2, 
\end{cases} \]  
(83)

where \( V_m^{(\eta \rightarrow \nu)} \) is the matrix \( V_m \) with all occurrences of the index \( \eta \) replaced by the index \( \nu \).

The normalizations are
\[ r_j(y) = \frac{\text{Pf } V_{2j+1}}{\text{Pf } V_{2j-1}} \]  
(84)
with the convention
\[ \text{Pf } V_{-1} = 1. \]  
(85)

Note that the coefficients \( \alpha_{j,k} \) depend on \( y \). We give the proof of Proposition 5 in Appendix F.

The coherence of the \( \alpha_{j,k} \) in Proposition 5 with the \( y \to \infty \) classical Gaussian result in (F.21) is not as straightforward as in the \( \beta = 4 \) case above, and we go through the details in Appendix F.2. The extra complications are because the technique of [2] did not use an exact analogue of our matrix \( V_m \) in (18); they instead used some shrewd linear algebra to express the matrix \( [(P_j, A^{-1} P_k)] \) in terms of the matrix \( [(P_j, A P_k)] \) and some other matrices containing the polynomial normalizations \( p_j \). This approach worked as it relied on inverting matrices that are (almost) diagonal, however the analogous step in our case (with finite \( y \)) involves inverting a full \( N \times N \) matrix, and so it is infeasible here. At any rate, setting \( y = \infty \), we
see from (19) that the matrix $\Phi(P_j, P_k) = 0$, and from (27) that the polynomials $P_j$ become the Hermite polynomials leading us to the simple expression (F.24) for the elements of $X_{j,k}$. Using these facts we recover the classical case in (F.21), where $\alpha_{2j+1,2j-1} = -j$ and is zero otherwise — see Appendix F.2 for the details.

As with the $\beta = 4$ case above, by use of the polynomials $R_j$ the matrix in $\hat{Z}_{1,N}[1, y]$ of (44) has the skew-diagonal structure in (52) and so its Pfaffian is given by (53) with $s_j = r_j$, and we get (47). Substitution of (84) into (47) then gives the expression for the CDF of the largest eigenvalue in (17).

4.3 Skew-Orthogonal Polynomials for More General Weight Functions

As mentioned after Eq. (33), the density functions in this paper are of the form (34), which are a type of Janossy density, so a natural question is to ask if our methods can be applied more generally. We see from (69) and (80) that the key step involved in calculating the polynomial coefficients in Proposition 4 (for the GSE) and Proposition 5 (for the GOE) is writing the corresponding skew-inner product in terms of the GUE inner product. The quantity separating the procedure here from the classical case in [2] is $\Omega$ in (68), which [via (78)] also determines the quantity $\Phi$ in (79). This $\Omega$ function is particular to the Gaussian weight and the eigenvalue domain $(-\infty, y)$, however, from following the matrix manipulations in Appendices E and F, we can conclude that our method will work for Janossy densities over more general domains and for the other classical weight functions. Indeed, let

$$w_\beta(x) := \begin{cases} e^{-\beta x^2/2}, & \text{(Gaussian)}, \\ x^{a\beta/2}e^{-\beta x^2/2}, & \text{(Laguerre)}, \\ (1-x)^{a\beta/2}(1+x)^{b\beta/2}, & \text{(Jacobi)}, \\ (1+x^2)^{-\beta(N-1)/2+1}, & \text{(Cauchy)}, \end{cases}$$

and define the inner product

$$(f, g)_Y^2 := \int_Y w_2(x)f(x)g(x)dx$$

and the skew-inner products

$$\langle f, g \rangle_Y^4 := \frac{1}{2} \int_Y w_4(x)\left[f(x)g'(x) - g(x)f'(x)\right]dx,$$  

$$\langle f, g \rangle_Y^1 := \frac{1}{2} \int_Y w_1(x)f(x)\int_Y w_1(z)g(z)\text{sgn}(z-x)dzdx,$$

where $Y \subset U_w$, with $U_w$ the maximal domain for the weight function $w_\beta$. Then we can define a new $\Omega$ and apply the procedures in Appendices E and F to obtain the skew-orthogonal polynomials in terms of the orthogonal polynomials. (Of course, explicitly calculating the $\beta = 1$ and $\beta = 4$ polynomials using this method relies on knowing the orthogonal polynomials for the corresponding $\beta = 2$ problem; a non-trivial hurdle.) Note that in the case of unitary, Hermitian matrix models (i.e. with $\beta = 2$) such orthogonal polynomials have been studied in the context of the counting statistics of eigenvalues in these ensembles (see e.g. [22,85]), including at the hard edge of the Laguerre ensemble [14,68]. In fact, this method relying on orthogonal polynomials was even extended to study the edge statistics of non-classical random matrix ensembles, such as multi-critical matrix models [4,7].
For the purposes of illustration assume we have the interval \( Y = (y_1, y_2) \subset \mathbb{R} \), then we replace (68) and (79) by

\[
\Omega(f, g; Y) = \lim_{x \to y_2} \left( w_2(x) f(x) g(x) \right) - \lim_{x \to y_1} \left( w_2(x) f(x) g(x) \right)
\]

and then Propositions 4 and 5 hold, with the matrices \( W_m \) and \( V_m \) modified accordingly.

5 Asymptotic Analysis of the CDF of the Largest Eigenvalue for \( F_{4,N} \) for Large \( N \)

In this section we show that our formula for the CDF \( F_{4,N}(y) \) is amenable to an asymptotic analysis in the large \( N \) limit, which allows us to obtain an alternative derivation of the Tracy-Widom formula for \( \beta = 4 \) [82]. Indeed, we will show below that, from the expression in (21), we can obtain the scaled limit (30). To show this result, starting from our expression in (21), we will first provide an explicit expression for \( \text{Pf} \) \( W_{2N-1} \), where the matrix \( W_m \) is defined in (22), and for that we will need the following lemma.

**Lemma 1** For all anti-symmetric \( 2N \times 2N \) matrices of the form

\[
A = T + B,
\]

where \( T \) and \( B \) are each anti-symmetric with upper triangular elements

\[
T_{j,k} = t_k \delta_{j+1,k} \quad \text{and} \quad B_{j,k} = f_j g_k \quad (j, k = 0, 1, \ldots, 2N - 1)
\]

for some functions \( f \) and \( g \), then

\[
\text{Pf} A = \left( \prod_{j=0}^{N-1} t_{2j+1} \right) \left( 1 + \sum_{p=1}^{N} \sum_{I_{2p}} L_{i_1,i_2} L_{i_3,i_4} \cdots L_{i_{2p-1},i_{2p}} \right),
\]

where

\[
L_{i_1,i_2} = \frac{\prod_{m=i_1}^{i_2-1} t_{2m+1} f_{2i_1} g_{2i_2+1}}{\prod_{m=i_1}^{i_2} t_{2m+1}}, \quad i_2 \geq i_1,
\]

and the summation indices \( I_{2p} \) are defined by

\[
I_{2p} : 0 \leq i_1 \leq i_2 < i_3 \leq i_4 \leq i_5 \leq \cdots < i_{2p-1} \leq i_{2p} \leq N - 1.
\]

**Proof** From [77, Lemma 4.2] we have

\[
\text{Pf} \left( A \right) = \text{Pf} \left( T + B \right) = \sum_{|I| \text{ is even}} (-1)^{d(I)} \text{Pf} \left( T_I \right) \text{Pf} \left( B_{I^c} \right),
\]

where \( T_I \) denotes that we are restricting \( T \) to just the rows and columns indexed by \( I \) (or its complement \( I^c = \{0, 1, \ldots, 2N - 1\} \backslash I \) for \( B_{I^c} \)), and

\[
d(I) := -\frac{|I|}{2} + \sum_{j \in I} j.
\]
Using the expression for the Pfaffian in (B.7) we have

\[
Pf \left( T_I \right) = \sum_{\mu \in M_I} \varepsilon(\mu) T_{j_1,k_1} T_{j_2,k_2} \cdots T_{j_{|I|/2},k_{|I|/2}},
\]

where the perfect matchings \( \mu = \{(j_1, k_1), \ldots, (j_{|I|/2}, k_{|I|/2})\} \in M_I \) are on the index set \( I \) and are represented by link diagrams as described in Appendix B.1. From the structure of \( T \), we see that Pf \( (T_I) \) is only non-zero if the associated link diagram contains only “little links” from site \( i \) to \( i+1 \) (see Fig. 2); in other words \( I = \{a_1, a_1+1, a_2, a_2+1, \ldots, a_{|I|/2}, a_{|I|/2}+1\} \).

Since little links never have crossings we must have \( \varepsilon(\mu) = +1 \). As noted below (B.7), the left vertex of each link gives us the row index and the right edge gives us the column index, so we have

\[
Pf \left( T_I \right) = \prod_{s=1}^{|I|/2} i_{a_s+1}.
\]

The sum of any neighbouring pair of integers is odd, and the number of pairs is given by \(|I|/2\) so we have \((-1)^{d(I)} = +1\).

With \( I \) given as above, \( I^c \) is of the form

\[ I^c = \{2i_1, 2i_2 + 1, 2i_3, 2i_4 + 1, \ldots, 2i_{2p-1}, 2i_{2p} + 1\}, \]

where \( i_1, \ldots, i_{2p} \) obey the restrictions in (96), and \( p = |I^c|/2 \). Note for later reference that \( I^c \) contains alternating even-odd pairs, and always begins with an even number.

With the perfect matchings on \( I^c \) denoted as \( \hat{\mu} = \{(j_1, k_1), (j_2, k_2) \cdots (j_p, k_p)\} \), we have

\[
Pf \left( B_{I^c} \right) = \sum_{\hat{\mu} \in M_{I^c}} \varepsilon(\hat{\mu}) B_{j_1,k_1} B_{j_2,k_2} \cdots B_{j_p,k_p}
\]

\[
= \sum_{\hat{\mu} \in M_{I^c}} \varepsilon(\hat{\mu}) f_{j_1} g_{k_1} f_{j_2} g_{k_2} \cdots f_{j_p} g_{k_p}.
\]

This can be simplified in the following way. Consider the following three link patterns that match everywhere except at the four sites \( \hat{j}_s < \hat{j}_{s+1} < \hat{k}_s < \hat{k}_{s+1} \):

\[
\hat{\mu} = \{\cdots, (\hat{j}_s, \hat{k}_s), (\hat{j}_{s+1}, \hat{k}_{s+1}), \cdots\} = \begin{array}{c}
\cdots \hat{j}_s \cdots \hat{j}_{s+1} \cdots \hat{k}_s \cdots \hat{k}_{s+1} \cdots
\end{array}
\]

\[
\hat{\mu}' = \{\cdots, (\hat{j}_s, \hat{k}_{s+1}), (\hat{j}_{s+1}, \hat{k}_s), \cdots\} = \begin{array}{c}
\cdots \hat{j}_s \cdots \hat{j}_{s+1} \cdots \hat{k}_s \cdots \hat{k}_{s+1} \cdots
\end{array}
\]

\[
\hat{\mu}'' = \{\cdots, (\hat{j}_s, \hat{j}_{s+1}), (\hat{k}_s, \hat{k}_{s+1}), \cdots\} = \begin{array}{c}
\cdots \hat{j}_s \cdots \hat{j}_{s+1} \cdots \hat{k}_s \cdots \hat{k}_{s+1} \cdots
\end{array}
\]
In the expansion of the Pfaffian, these three terms can be combined into a single term containing a factor of
\[
( - \hat{f}_{j_1} \hat{g}_{k_1} \hat{f}_{j_1+1} \hat{g}_{k_1+1} + \hat{f}_{j_1} \hat{g}_{k_1+1} \hat{f}_{j_1+1} \hat{g}_{k_1} + \hat{f}_{j_1} \hat{g}_{k_1} \hat{f}_{j_1+1} \hat{g}_{k_1+1}),
\] (105)
where the negative in the first term is the contribution to \( \varepsilon(\hat{\mu}) \) of the crossing in \( \hat{\mu} \). The first two terms here cancel, leaving only the contribution from \( \hat{\mu}'' \), which is a variation on the observation by Muir [59, §168] that in the expansion of a Pfaffian there is exactly one more positive term than negative.

Every perfect matching except one will have some set of four sites paired as in \( \hat{\mu} \) or \( \hat{\mu}' \), and will thus have its contribution to \( \text{Pf} (\mathcal{B}_{I^c}) \) cancelled by the corresponding \( \hat{\mu}' \) or \( \hat{\mu} \). The only term in the sum that remains will be that corresponding to the perfect matching \( \{2i_1, 2i_2 + 1\}, \{2i_3, 2i_4 + 1\}, \cdots, \{2i_{2p-1}, 2i_{2p} + 1\} \), where every even site is connected to the next odd site to the right in \( I^c \). Thus (101) becomes
\[
\text{Pf} (\mathcal{B}_{I^c}) = \prod_{s=1}^{p} f_{2i_{2s-1}} g_{2i_{2s} + 1}.
\] (106)
(As discussed above, the signature \( \varepsilon(\hat{\mu}) = +1 \) for the non-cancelling contributions.) So we have
\[
\text{Pf} (A) = \sum_{I \subseteq \{0, 1, \ldots, 2N-1\}} \frac{|I|}{2} \prod_{s=1}^{|I|/2} t_{a_{s}+1} \prod_{s=1}^{|I^c|/2} f_{2i_{2s-1}} g_{2i_{2s} + 1}
\]
\[
= t_1 t_3 \cdots t_{2N-1} + \sum_{I \subseteq \{0, 1, \ldots, 2N-1\}} \frac{|I|}{2} \prod_{s=1}^{|I|/2} t_{a_{s}+1} \prod_{s=1}^{|I^c|/2} f_{2i_{2s-1}} g_{2i_{2s} + 1},
\] (107)
where in the second equality we have restricted the sum to only those \( I \) that are strict subsets.

Recall from above that \( I^c \) contains alternating even-odd pairs (in that order), and that the remaining vertices (in \( I \)) are all connected by little links. So for each pair \( f_{2i_{2s-1}} g_{2i_{2s} + 1} \) we have all the even-indexed \( i \)'s in-between \( 2i_{2s-1} \) and \( 2i_{2s} + 1 \), and for each pair of pairs, we have all the odd-indexed \( i \)'s in-between \( 2i_{2s} + 1 \) and \( 2i_{2s+1} \). For example, let \( I = \{0, 1, 3, 4, 6, 7, 10, 11\} \) and \( I^c = \{2, 5, 8, 9\} \) then
\[
\prod_{s=1}^{|I|/2} t_{a_{s}+1} \prod_{s=1}^{|I^c|/2} f_{2i_{2s-1}} g_{2i_{2s} + 1} = t_1 f_2 t_4 g_5 t_7 f_8 g_9 t_{11}.
\] (108)
Factoring out \( t_1 t_3 \cdots t_{2N-1} \) we then obtain all the odd-indexed \( i \)'s between \( 2i_{2s-1} \) and \( 2i_{2s} + 1 \) (including \( 2i_{2s+1} \)) in the denominator, and we cancel all the \( i \)'s between pairs of pairs. So for each pair \( f_{2i_{2s-1}} g_{2i_{2s} + 1} \) we have a factor \( L_{i_{2s-1}, i_{2s}} \). There will be \( p = |I^c|/2 \) of these factors for each \( I \), and summing over \( p \) gives the result in (94).

To use Lemma 1 we first define
\[
M_{i_1, i_2} (y) = \frac{1}{2} \prod_{m=1}^{i_2-1} p_{2m+2} (y) P_{2i_1} (y, y) P_{2i_2+1} (y, y) e^{-y^2}, \quad i_2 \geq i_1,
\] (109)
where we recall that the \( P_k \)'s are the NM orthogonal polynomials (24)–(26) while the \( p_k \)'s are their corresponding norms (12). (We will often suppress the explicit dependence on \( y \) for
We can further use (113) to obtain
\[ M_{i_1, i_2}(y) = \frac{1}{2} \frac{1}{\sqrt{\hat{R}_{2i_1+1}(y)}} \prod_{m=i_1}^{i_2-1} \frac{\sqrt{\hat{R}_{2m+2}(y)}}{\hat{R}_{2m+1}(y)} \psi_{2i_1}(y, y) \psi_{2i_2}(y, y), \quad i_2 \geq i_1 \] (110)
in terms of \( \hat{R}_m = p_m/p_{m-1} \) from (25) and the so-called “wave functions” \( \psi_k(x, y) \) given by
\[ \psi_k(x, y) = \frac{P_k(x, y)}{\sqrt{p_k(y)}} e^{-\frac{x^2}{2}}. \] (111)

Use of Lemma 1 with the Pfaffian in (21) gives us
\[ F_{4,N}(y) = \frac{2N^2}{\pi^{N/2}} \prod_{j=0}^{N-1} \frac{1}{(2j+1)!} \times \left( \prod_{j=0}^{N-1} p_{2j+1} \right) \left( 1 + \sum_{p=1}^{N} \sum_{I_{2p}} M_{i_1, i_2} M_{i_3, i_4} \cdots M_{i_{2p-1}, i_{2p}} \right) \bigg|_{y \rightarrow \sqrt{2y}} \] (112)
an expression which is quite convenient for the large \( N \) analysis of \( F_{4,N}(y) \). It is first important to check that this formula for the CDF of the largest eigenvalue is correctly normalized, i.e., \( \lim_{y \rightarrow \infty} F_{4,N}(y) = 1 \). For this purpose, we first notice that when \( y \rightarrow \infty \), the norms \( p_j \) converge to the norms of the Hermite polynomial of degree \( j \) [see Eq. (28)], i.e.
\[ p_j(\infty) = \lim_{y \rightarrow \infty} p_j(y) = \sqrt{\pi} \frac{j!}{2^j}. \] (113)

Combined with the expression of \( M_{i_1, i_2}(y) \) given in Eq. (109), this implies that \( M_{i_1, i_2}(y) \rightarrow 0 \) as \( y \rightarrow \infty \) for all pairs of indices \((i_1, i_2)\), because of the Gaussian factor \( e^{-\frac{y^2}{2}} \) in that expression (109). Therefore
\[ \lim_{y \rightarrow \infty} \left( 1 + \sum_{p=1}^{N} \sum_{I_{2p}} M_{i_1, i_2} M_{i_3, i_4} \cdots M_{i_{2p-1}, i_{2p}} \right) = 1. \] (114)

We can further use (113) to obtain
\[ \lim_{y \rightarrow \infty} \prod_{j=0}^{N-1} p_{2j+1}(y) = \pi^{N/2} \frac{\prod_{j=0}^{N-1} (2j+1)!}{2^{\sum_{j=0}^{N-1} (2j+1)}} = \pi^{N/2} \frac{\prod_{j=0}^{N-1} (2j+1)!}{2^{N^2}}, \] (115)
which implies, by combining (112), (114) and (115), that
\[ \lim_{y \rightarrow \infty} F_{4,N}(y) = 1, \] (116)
as it should.

We now proceed to obtain the scaled limit (30). As we will see below, we will make a crucial use of the asymptotic results for the norms \( p_k(y) \) obtained in Ref. [62] and for the wave functions \( \psi_k(x, y) \) derived in Ref. [66] in the limit of \( N \) and large \( y \). Let us start by analyzing the first factors of \( F_{4,N}(y) \) in (112) and define
\[ Z_N(y) = \frac{2N^2}{\pi^{N/2}} \prod_{j=0}^{N-1} \frac{1}{(2j+1)!} \left( \prod_{j=0}^{N-1} p_{2j+1}(y) \right). \] (117)
It is easy to check that

\[
\frac{Z_{N-1}(y)Z_{N+1}(y)}{Z_N(y)^2} = \frac{1}{N(N+1/2)} \frac{p_{2N+1}(y)}{p_{2N-1}(y)} = \frac{\tilde{R}_{2N+1}(y)\tilde{R}_{2N}(y)}{N(N+1/2)} .
\]  

(118)

From the result of Tracy and Widom [82] [i.e. Eq. (30)], one anticipates that, in the large \(N\) limit, \(F_{4,N}(y)\) becomes a function of the scaling variable \(s = 2^{7/6}N^{1/6}(y - \sqrt{2N})\). Given the expression of \(F_{4,N}\) in Eq. (112), it is thus natural to assume that the first factor, namely \(Z_N(y)\) given in Eq. (117), also becomes a function of the same scaling variable \(s\). Therefore we assume the asymptotic scaling behavior

\[
\ln Z_N(y) \sim f \left( 2^{7/6}N^{1/6}(y - \sqrt{2N}) \right) ,
\]

(119)

with some function \(f\), independent of \(N\), yet to be determined. Assuming this scaling behavior (119), and setting \(y = \sqrt{2N} + (s/2^{7/6})N^{-1/6}\), the left hand side of Eq. (118) becomes

\[
\ln Z_{N-1}(y) + \ln Z_{N+1}(y) - 2 \ln Z_N(y) = 2^{4/3} f''(s) N^{-2/3} + o(N^{-2/3}) .
\]

(120)

Let us now analyse the right hand side of (118) in the large \(N\) limit, where, from [62], we have the asymptotic behavior

\[
\tilde{R}_N \left( \sqrt{2N} + \frac{x}{\sqrt{2}} N^{-1/6} \right) = \frac{N}{2} \left( 1 - N^{-2/3} q^2(x) + o(N^{-2/3}) \right) ,
\]

(121)

where \(q(s)\) is defined in (31). This implies, setting again \(y = \sqrt{2N} + (s/2^{7/6})N^{-1/6}\), that

\[
\tilde{R}_{2N} \left( \sqrt{2} y \right) = N \left( 1 - (2N)^{-2/3} q^2(s) + o(N^{-2/3}) \right) .
\]

(122)

Hence the logarithm of the right hand side of Eq. (118) reads

\[
\ln \left( \frac{\tilde{R}_{2N+1}(\sqrt{2} y)\tilde{R}_{2N}(\sqrt{2} y)}{N(N+1/2)} \right) = -2^{1/3} N^{-2/3} q^2(s) + o(N^{-2/3}) .
\]

(123)

Taking the logarithm of the relation in (118) and equating the leading terms, of order \(O(N^{-2/3})\) on both sides, one finds

\[
f''(s) = -\frac{1}{2} q^2(s) .
\]

(124)

Integrating twice this relation (124), using that \(\lim_{s \to \infty} f'(s) = 0\) [since the probability density function \(F_{4,N}(y) \to 0\) as \(y \to \infty\)] as well as \(\lim_{s \to \infty} f(s) = 0\) [since \(F_{4,N}(y) \to 1\) as \(y \to \infty\), see Eq. (116)], one obtains

\[
f(s) = -\frac{1}{2} \int_s^\infty (x-s)q^2(x)dx .
\]

(125)

Therefore, recalling (119) one obtains

\[
\lim_{N \to \infty} Z_N \left( y = \sqrt{2N} + (s/2^{7/6})N^{-1/6} \right) \bigg|_{y \to \sqrt{2}y} = \exp \left[ -\frac{1}{2} \int_s^\infty (x-s)q^2(x)dx \right] ,
\]

(126)

which gives the first factor of the Tracy-Widom distribution for \(\beta = 4\) [see Eq. (30)].

We now analyse the large \(N\) behavior of the right-most factor in the expression of the Pfaffian in Eq. (112). For this purpose, we will take advantage of the analysis performed in [66]. In fact, one can show that, in the large \(N\) limit, the multiple sums in Eq. (112) are
dominated by the region where \( i_1, i_2, \cdots, i_{2p} \) are close to \( N \). For later convenience, we reverse the order of the indices in the product of \( M_{i_1, i_2} \) by looking for \( M_{N-k_1, N-k_2} \), then from the results obtained in [66] for the asymptotic forms of the “wave functions” in (111)

\[
\psi_N(y, y) \sim N^{-1/12} q \left( \frac{\sqrt{2} N^{1/6} (y - \sqrt{2} N)}{s} \right)
\]

and using \( y \mapsto \sqrt{2} y = 2 \sqrt{N} + \frac{s}{2^{2/3} N^{1/6}} \) we have

\[
\psi_{2(N-k)}(y, y) \sim N^{-1/12} q \left( s + 2^{2/3} \frac{k}{N^{1/3}} \right).
\]

Using (122) for the pre-factors in (110) one gets

\[
M_{N-k_1, N-k_2} \left( y = 2 \sqrt{N} + \frac{s}{2^{2/3} N^{1/6}} \right) \sim \frac{1}{(2N)^{2/3}} q \left( s + 2^{2/3} \frac{k_1}{N^{1/3}} \right) q \left( s + 2^{2/3} \frac{k_2}{N^{1/3}} \right),
\]

which will be the useful form in the following. Indeed, performing first the change of variables \( i_j = N - k_j \) in the second factor of Eq. (112) and then using (129) one finds, at leading order for large \( N \), setting again \( y = \sqrt{2} N + (s/2^{7/6}) N^{-1/6} \),

\[
\sum_{\mathcal{I}_{2p}} M_{i_1, i_2} M_{i_3, i_4} \cdots M_{i_{2p-1}, i_{2p}} \bigg|_{y \mapsto \sqrt{2} y} \sim \frac{1}{(2N)^{2p}} \sum_{K_{2p}} q \left( s + 2^{2/3} \frac{k_1}{N^{1/3}} \right) \cdots \left( s + 2^{2/3} \frac{k_{2p}}{N^{1/3}} \right),
\]

where, similar to (96), we denote

\[
K_{2p} : N \geq k_1 \geq k_2 \geq k_3 \geq k_4 \geq k_5 \geq k_6 \geq \cdots \geq k_{2p-1} \geq k_{2p} \geq 1.
\]

In the limit \( N \to \infty \) the discrete sums over the \( k_j \)'s become integrals. Performing the change of variables \( v_j = 2^{2/3} k_j / N^{1/3} \) one finds

\[
\sum_{\mathcal{I}_{2p}} M_{i_1, i_2} M_{i_3, i_4} \cdots M_{i_{2p-1}, i_{2p}} \sim \frac{1}{2^{2p}} \int_0^\infty dv_2 \int_0^\infty dv_2 \cdots \int_0^\infty dv_1 q(s + v_1) \cdots q(s + v_{2p-1}) q(s + v_{2p}).
\]

Since the integrand in (132) is completely symmetric under the permutation of the variables \( v_i \)'s, the nested integral can simply be written as

\[
\sum_{\mathcal{I}_{2p}} M_{i_1, i_2} M_{i_3, i_4} \cdots M_{i_{2p-1}, i_{2p}} \sim \frac{1}{(2p)!} \left( \frac{1}{2} \int_0^\infty dx q(x) \right)^{2p}.
\]

Finally, summing over \( p \) in Eq. (112), one obtains

\[
\lim_{N \to \infty} \left[ 1 + \sum_{p=1}^N \sum_{\mathcal{I}_{2p}} M_{i_1, i_2} M_{i_3, i_4} \cdots M_{i_{2p-1}, i_{2p}} \right] = \sum_{p=0}^\infty \frac{1}{(2p)!} \left( \frac{1}{2} \int_0^\infty dx q(x) \right)^{2p} = \cosh \left( \frac{1}{2} \int_0^\infty dx q(x) \right).
\]

Combining Eqs. (112), (117), (126) and (134), one obtains the desired expression given in (30) for the \( \beta = 4 \) Tracy-Widom distribution.
6 Asymptotic Analysis of the CDF of the Largest Eigenvalue for $F_{1,N}$ for Large $N$

We now show that starting with (17), we can obtain the limiting formula for $\beta = 1$ in (32) [82], where we proceed in much the same way as in Sect. 5 above for $\beta = 4$.

From the definitions in (19) and (20) we have

\[
X_{j,k} + \Phi_{j,k} = \frac{1}{2} \left( \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} P_j(x, y) \text{erf} \left( \frac{x}{\sqrt{2}} \right) dx \right) \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} P_k(x, y) dx + \frac{1}{2} \int_{-\infty}^{y} e^{-\frac{x^2}{2}} P_j(x, y) dx \int_{y}^{\infty} e^{-\frac{x^2}{2}} P_k(x, y) dx
\]

\[
= \frac{1}{2} \left( \int_{-\infty}^{y} e^{-\frac{x^2}{2}} P_j(x, y) \text{erfc} \left( -\frac{x}{\sqrt{2}} \right) dx \right) \Xi_{k}(\infty, y) - \frac{1}{2} \Xi_{j}(y, y) \Xi_{k}(y, y),
\]

(135)

where we have introduced the notation

\[
\Xi_{j}(x, y) := \int_{-\infty}^{x} e^{-\frac{z^2}{2}} P_j(z, y) dz, \quad \Xi_{j}(\infty, y) := \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} P_j(z, y) dz.
\]

(136)

For later use we also similarly define

\[
\Psi_{j}(\infty, y) := \int_{-\infty}^{\infty} \psi_{j}(z, y) dz.
\]

(137)

Using the identities (F.13) and (F.14) to perform the integrals in (135) gives

\[
\Xi_{j}(x, y) = \frac{1}{2} \text{erfc} \left( -\frac{x}{\sqrt{2}} \right) \Xi_{j}(\infty, y) - e^{-x^2/2} P_{j-1}(x, y)
\]

\[
+ e^{-x^2/2} \text{LoP}_{j-2}
\]

(138)

\[
\int_{-\infty}^{x} e^{-\frac{z^2}{2}} P_j(z, y) \text{erfc} \left( -\frac{z}{\sqrt{2}} \right) dz = -e^{-\frac{x^2}{2}} \text{erfc} \left( -\frac{x}{\sqrt{2}} \right) P_{j-1}(x, y)
\]

\[
+ \frac{1}{4} \text{erfc} \left( -\frac{x}{\sqrt{2}} \right)^2 \Xi_{j}(\infty, y)
\]

\[
+ e^{-\frac{x^2}{2}} \text{LoP}_{j-2} + e^{-x^2} \text{LoP}_{j-2}
\]

(139)

where the notation LoP$_{j-2}$ denotes “lower-order polynomials” up to degree $j - 2$, that is, some linear combination of $P_0(x, y)$, $P_1(x, y)$, ..., $P_{j-2}(x, y)$.

Noting that $\text{erfc}(x) \in (0, 2)$ (so it is bounded) and recalling the $O \left( e^{-y^2} \right)$ corrections in (27), we substitute (138) and (139) into (135) to give (at leading order for large $y$)

\[
X_{j,k} + \Phi_{j,k} \sim \frac{1}{2} e^{-\frac{x^2}{2}} \left( P_{k-1}(y, y) \Xi_{f}(\infty, y) - P_{j-1}(y, y) \Xi_{k}(\infty, y) \right).
\]

(140)
Keeping just the leading order polynomial (meaning that we use only the larger of \( j \) or \( k \)), we substitute (140) into (18) to obtain

\[
\begin{align*}
&\zeta_j = \zeta_j(\infty, y) \quad \text{and we have suppressed all the function arguments to save space.}\nonumber
&\text{Looking at the matrix in (141), we see that it has the structure of (92) and so using (94) with (17) we conclude that for large } y \text{ (recalling that } N \text{ is even)}
\end{align*}
\]

\[
F_{1,N}(y) \sim \frac{2^{N}((\frac{N}{2} - 1)}{\pi^{N/4}} \prod_{j=0}^{N/2 - 1} \left( \frac{N}{2} \right)! \left( \prod_{j=0}^{N/2 - 1} p_{2j}(y) \right) \left( 1 + \sum_{p=1}^{N/2} \sum_{i_2p} T_{i_1,i_2} T_{i_3,i_4} \cdots T_{i_{2p-1},i_{2p}} \right),
\]

\[
T_{i_2} : 0 \leq i_1 \leq i_2 \leq i_3 \leq i_4 \leq i_5 \leq i_6 \leq \cdots \leq i_{2p-1} \leq i_{2p} \leq \frac{N}{2} - 1
\]

with

\[
T_{i_1,i_2}(y) = -\frac{1}{2} \prod_{m=i_1}^{i_2-1} p_{2m+1}(y) \zeta_{2i_1}(\infty, y) p_{2i_2}(y, y) e^{-y^2/2}, \quad i_2 \geq i_1,
\]

\[
= -\frac{1}{2} \prod_{m=i_1+1}^{i_2} \sqrt{\frac{R_{2m-1}(y)}{R_{2m}(y)}} \Psi_{2i_1}(\infty, y) \Psi_{2i_2}(y, y), \quad i_2 \geq i_1,
\]

where \( \Psi_j(\infty, y) \) is from (137) and \( \tilde{R}_k(y) = p_k(y)/p_{k-1}(y) \) is from (25).

Denoting the prefactor in (142) by

\[
\zeta_N(y) := \frac{2^N((\frac{N}{2} - 1)}{\pi^{N/4}} \prod_{j=0}^{N/2 - 1} \left( \frac{N}{2} \right)! \left( \prod_{j=0}^{N/2 - 1} p_{2j}(y) \right)
\]

then

\[
\frac{\zeta_{N-2}(y)\zeta_{N+2}(y)}{\zeta_N(y)^2} = \frac{4}{N(N-1)} \frac{p_N(y)}{p_{N-2}(y)} = \frac{4\tilde{R}_N(y)\tilde{R}_{N-1}(y)}{N(N-1)}
\]

and so using (121)

\[
\ln \left( \frac{4\tilde{R}_N\tilde{R}_{N-1}}{N(N-1)} \right) = -2\frac{q(s)^2}{N^{2/3}} + o(N^{-2/3}).
\]

Now, following the same line of reasoning that led us to assume the scaling form given in (119) in the case \( \beta = 4 \), we assume here [see Eq. (32)],

\[
\ln \zeta_N(y) \sim f \left( \sqrt{2N^{1/6}}(y - \sqrt{2N}) \right),
\]

\( \zeta \) Springer
and then with \( y = \sqrt{2N} + \frac{s}{\sqrt{2N^{1/6}}} \) we have

\[
\ln Z_{N-2}(y) + \ln Z_{N+2}(y) - 2 \ln Z_N(y) = \frac{4}{N^{2/3}} f''(s) + o\left(N^{-2/3}\right) .
\]  
(149)

Equating (147) and (149) we have (to leading order)

\[
f''(s) = -\frac{1}{2} q^2(s)
\]
(150)

identically with the \( \beta = 4 \) case in (125). Therefore, we have

\[
\lim_{N \to \infty} Z_N\left(y = \sqrt{2N} + \frac{s}{\sqrt{2N^{1/6}}}\right) = \exp\left[ -\frac{1}{2} \int_s^\infty (x-s)q^2(x)dx \right],
\]
(151)

which is the first factor in (32).

For the right-most factor in (142) we use the known asymptotic behaviour (127), with \( y = \sqrt{2N} + \frac{s}{\sqrt{2N^{1/6}}} \), to find

\[
\psi_{N-2k}(y, y) \sim 2^{1/4}N^{-1/12}q\left(s + 2\frac{k}{N^{1/3}}\right).
\]
(152)

For \( \psi_{N-2k}(\infty, y) \) in (137), we notice that the integral over \( z \) runs over the whole real line \( z \in (-\infty, +\infty) \), and in practice only over the “bulk” region \( z \in [-\sqrt{2N}, +\sqrt{2N}] \) since, for \( |z| > \sqrt{2N} \), \( \psi_{N-2k}(z, y) \) is exponentially small (with \( N \)), for \( y = \sqrt{2N} + \frac{s}{\sqrt{2N^{1/6}}} \). In fact, for \( z \) inside the bulk, the wave function \( \psi_{N-2k}(z, y) \) behaves as if the “wall” at \( y \) were sent to \( y \to +\infty \) [66]. Therefore, in the integral in (137), \( \psi_{N-2k}(z, y) \) can be replaced by its asymptotic behaviour for large \( y \), namely [66]

\[
\psi_{N-2k}(z, y) \sim \frac{H_{N-2k}(z)}{\pi^{1/4}2^{(N-2k)/2}N^{-1/2k+1}}e^{-z^2/2}.
\]
(153)

Using the well known recursion relations for Hermite polynomials [1, Chapter 22]

\[
H_{j+1}(z) = 2zH_j(z) - H'_j(z), \quad H'_j(z) = 2jH_{j-1}(z)
\]
(154)

we can show that

\[
\int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} H_j(z)dz = \begin{cases} 2^{j+1/2} \Gamma\left(\frac{j+1}{2}\right), & \text{j even}, \\
0, & \text{j odd}, \end{cases}
\]
(155)

where the second line follows since Hermite polynomials of odd degree are odd functions. Therefore we have (recalling that \( N \) is even)

\[
\psi_{N-2k}(\infty, y) \sim \frac{2^{3/4}}{N^{1/4}},
\]
(156)

and with (121) we obtain

\[
T_{N/2-k_1, N/2-k_2}(y = \sqrt{2N} + \frac{s}{\sqrt{2N^{1/6}}}) \sim -\frac{1}{N^{1/3}}q\left(s + 2\frac{k_2}{N^{1/3}}\right).
\]
(157)

Then we find ourselves at the analogue of (130)

\[
\sum_{I_{2p}} T_{i_{1,2}, i_{3,4}} \cdots T_{i_{2p-1,2p}} \sim \left(-\frac{1}{N^{p}}\right)^p N^{p} \sum_{K_p} q\left(s + 2\frac{k_2}{N^{1/3}}\right)q\left(s + 2\frac{k_4}{N^{2/3}}\right) \cdots q\left(s + 2\frac{k_{2p}}{N^{p/3}}\right).
\]
(158)
where we denote

\[ K_p : \frac{N}{2} \geq k_2 > k_4 > k_6 > \cdots > k_{2p-2} > k_{2p} \geq 1. \]  \hspace{1cm} (159)

[Note that, in contrast to \( K_{2p} \) in (131), \( K_p \) contains only the even indices. In the \( \beta = 4 \) case, both even and odd indices contributed factors of \( q \), as can be seen in (129). But here the odd indices are attached to the integrals \( \Psi_{N-2k_{odd}} \), and only contribute factors of \( N \) and 2 as per (156).] Changing variables \( v_j = 2k_j/N^{1/3} \) gives

\[
\sum_{II_p} T_{i_1,i_2} T_{i_3,i_4} \cdots T_{i_{2p-1},i_{2p}} \\
\sim \frac{(-1)^p}{2^p} \int_0^\infty q(s + v_{2p})dv_{2p} \int_{v_{2p}}^\infty q(s + v_{2p-2})dv_{2p-2} \\
\cdots \int_{v_6}^\infty q(s + v_4)dv_4 \int_{v_4}^\infty q(s + v_2)dv_2 \\
= \frac{(-1)^p}{p!} \left( \frac{1}{2} \int_s^\infty dx q(x) \right)^p,
\]

where, in the final line, we have removed the ordering from the integration variables since the integrand is symmetric in the \( v_j \)'s. Summing over \( p \) and taking the limit we have

\[
\lim_{N \to \infty} \left[ 1 + \sum_{p=1}^{N/2} \sum_{II_p} T_{i_1,i_2} T_{i_3,i_4} \cdots T_{i_{2p-1},i_{2p}} \right] = \sum_{p=0}^\infty \frac{1}{p!} \left( \frac{1}{2} \int_s^\infty dx q(x) \right)^p \\
= \exp \left( \frac{1}{2} \int_s^\infty dx q(x) \right),
\]

which is the second factor in (32).

7 Conclusions and Perspectives

In this paper, we have revisited the computation of the cumulative distribution functions of the largest eigenvalue in the classical Pfaffian ensembles of RMT, namely the GOE and the GSE, using the techniques of skew-orthogonal polynomials, thus extending the approach of Nadal and Majumdar [62] developed for the GUE. By adapting the method of Refs. [2,3], we have explicitly constructed these (semi-classical) skew-orthogonal polynomials in terms of the so-called “Nadal-Majumdar” orthogonal polynomials introduced in the case of the GUE. This construction involves some non-trivial Pfaffians, which we have related to “overlapping Pfaffians”, studied by Knuth [43]. We were then able to carry out the asymptotic analysis of these skew-orthogonal polynomials and of their norms to obtain the well known Tracy-Widom distributions, using a method that is quite different from the original one [82], and also different from the more recent one obtained via the so-called stochastic Airy operator [16]. Our method relied on a certain Pfaffian identity, which is the content of Lemma 1. Here, we have obtained the limiting form of the (appropriately centered and scaled) CDF’s \( F_{4,N}(y) \) and \( F_{1,N}(y) \) in the limit \( N \to \infty \) [see Eqs. (30) and (32) for \( \beta = 4 \) and \( \beta = 1 \) respectively]. For finite but large \( N \), there will be corrections to these limiting behaviors described by the Tracy-Widom distributions. These corrections have been studied in the case of the GUE and the GOE and in both cases it was shown that the leading corrections are of order \( O(N^{-1/6}) \).
Although, to our knowledge, there are no such results available for the GSE, it is rather reasonable to expect that the leading corrections are also of the same order $O(N^{-1/6})$ (see e.g. Ref. [11] for a discussion in a more general context). In principle, it should be possible to compute these finite $N$ corrections using the present method — and in fact a very similar method has been used to compute these corrections at the hard edge of the Laguerre Unitary Ensemble [68]. However, this goes beyond the scope of the present work.

As discussed in Sect. 3, it is known that “Pfaffian” Janossy densities (of which our $\beta = 1$ and $\beta = 4$ densities are examples) have $n$-point correlation functions given by Pfaffians. These correlation functions can be calculated via standard techniques (see [33,55]) — these calculations will be presented in a follow-up work [54]. It would be interesting to see how one can use the skew-orthogonal polynomials constructed in the present work to compute various linear statistics at the edge of the GOE and the GSE, as studied in [56,57] using different techniques. In fact, these skew-orthogonal polynomials will also allow us to analyze other extreme observables, such as the density of eigenvalues near the largest one and the statistics of the gap between the two largest eigenvalues in the GSE and the GOE. These quantities are particularly interesting in the challenging case of GOE since they naturally enter into the computation of physical observables in the spherical Sherrington-Kirkpatrick model of mean-field spin glasses [36].

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**Appendices**

**A Reminder on the Classical Ensembles of RMT: GOE, GUE and GSE**

For self-consistency, we recall here the definition of the classical ensembles of RMT studied in this paper:

- The **Gaussian Orthogonal Ensemble** (GOE) is the set of $N \times N$ real symmetric matrices

  $$M = \frac{Y + Y^T}{2}, \quad (A.1)$$

  where $Y$ contains standard normally distributed elements $y_{j,k} \sim \mathcal{N}[0, 1]$ resulting in a matrix PDF proportional to $e^{-(\text{Tr}M^2)/2}$ which is invariant under orthogonal conjugation $M \mapsto O^T MO$.

- The **Gaussian Unitary Ensemble** (GUE) is the set of complex Hermitian matrices

  $$M = \frac{Y + Y^\dagger}{2} \quad (A.2)$$

  with real independent Gaussian components $y_{j,k} \sim \mathcal{N}[0, \frac{1}{\sqrt{2}}] + i\mathcal{N}[0, \frac{1}{\sqrt{2}}]$ giving a matrix PDF proportional to $e^{-\text{Tr}M^2}$ which is invariant under unitary conjugation $M \mapsto U^\dagger MU$. 

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$$M = \frac{Y + Y^T}{2}, \quad (A.1)$$

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The Gaussian Unitary Ensemble (GUE) is the set of complex Hermitian matrices

$$M = \frac{Y + Y^\dagger}{2} \quad (A.2)$$

with real independent Gaussian components $y_{j,k} \sim \mathcal{N}[0, \frac{1}{\sqrt{2}}] + i\mathcal{N}[0, \frac{1}{\sqrt{2}}]$ giving a matrix PDF proportional to $e^{-\text{Tr}M^2}$ which is invariant under unitary conjugation $M \mapsto U^\dagger MU$. 

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The Gaussian Symplectic Ensemble (GSE) is defined similarly for normally distributed quaternionic entries. The reader can find detailed treatments about the quaternions in random matrix theory in several places (see, for example, [29, 53, 55]), however this will not be required for understanding the current work, as we use the equivalent $2 \times 2$ representation of quaternions

$$
\begin{bmatrix}
    a_1 + ib_1 & a_2 + ib_2 \\
    -a_2 + ib_2 & a_1 - ib_1
\end{bmatrix} \quad (a_1, a_2, b_1, b_2 \in \mathbb{R}).
$$

(A.3)

The ensemble is then the set of $2N \times 2N$ matrices,

$$
M = \frac{Y + Y^\dagger}{2},
$$

(A.4)

where each $2 \times 2$ block of $Y$ is of the form (A.3) with each independent real component normally distributed $a_1, b_1, a_2, b_2 \sim \mathcal{N}[0, \frac{1}{2}]$. The matrix PDF is then proportional to $e^{-\text{Tr}M^2}$, which is invariant under symplectic conjugation, that is conjugation by a unitary matrix $M \mapsto U^\dagger MU$, with the restriction that

$$
UZ_NU^T = \pm Z_N,
$$

(A.5)

where

$$
Z_N := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \otimes I_N = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},
$$

(A.6)

and $I_N$ is the $N \times N$ identity matrix.

**B Pfaffians**

Pfaffians are a key tool in our work here, so for the non-specialist we provide some background information. The statements in this section are classical results — references include [24, 59, 77]. A brief historical survey on the topic is provided in [43, §6]

**Definition B.1 (Pfaffian)** Let $M = [m_{j,k}]_{j,k=1,\ldots,2N}$, where $m_{j,k} = -m_{k,j}$, so that $M$ is an anti-symmetric matrix of even size. Then the Pfaffian of $M$ is defined by

$$
Pf M = \sum_{\pi \in S_{2N \times 2N}^{\pi(2j) > \pi(2j-1)}} \epsilon(\pi)m_{\pi(1),\pi(2)}m_{\pi(3),\pi(4)} \cdots m_{\pi(2N-1),\pi(2N)}
$$

$$
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad = \frac{1}{N!} \sum_{\pi \in S_{2N}^{\pi(2j) > \pi(2j-1)}} \epsilon(\pi)m_{\pi(1),\pi(2)}m_{\pi(3),\pi(4)} \cdots m_{\pi(2N-1),\pi(2N)}
$$

$$
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad = \frac{1}{2^N N!} \sum_{\pi \in S_{2N}} \epsilon(\pi)m_{\pi(1),\pi(2)}m_{\pi(3),\pi(4)} \cdots m_{\pi(2N-1),\pi(2N)},
$$

(B.1)
where $S_{2N}$ is the group of permutations of $2N$ letters and $\varepsilon(\pi)$ is the signature of the permutation $\pi$. The * above the first sum indicates that the sum is over distinct terms only (that is, all permutations of the pairs of indices are regarded as identical).

Note that in the second equality of (B.1) the factors of 2 are associated with the restriction $\pi(2j) > \pi(2j - 1)$ while the factorial is associated with counting only distinct terms. $[N!]$ is the number of ways of arranging the $N$ pairs of indices $\pi(2l - 1)$, $\pi(2l)]$. Pfaffians can be calculated via a version of Laplace expansion, however the Pfaffian minors $M^{j,k}$ that one needs to calculate are obtained by blocking out both the $j$th and $k$th row and the $j$th and $k$th column.

The definition of a Pfaffian is very close to that of a determinant, and for the matrix $\mathbf{M}$ (antisymmetric of size $2N \times 2N$), they are related by

$$\text{Pf} \mathbf{M}^2 = \det \mathbf{M},$$

We will also have need of the identity [24]

$$\text{Pf} (\mathbf{BMB}^T) = \det(\mathbf{B}) \text{Pf} (\mathbf{M}),$$

where $\mathbf{B}$ is a general $2N \times 2N$ matrix.

### B.1 Pfaffians and Perfect Matchings

In several places of this work (e.g. Lemma 1 and Appendix F.2) we use an expression equivalent to (B.1) in terms of perfect matchings and link patterns. Expressions for Pfaffians in terms of perfect matchings have been known for a long time, and they are discussed in many places — we refer to [43,70,77].

A perfect matching $\mu$ is a set of links between $2N$ sites, where each site is connected to exactly one other site. Diagrammatically, this is expressed as a link diagram, and most easily seen via an example: let

$$\mu = \{(2, 3), (5, 1), (4, 6)\} \quad (B.4)$$

and the link diagram is given in Fig. 3. The signature $\varepsilon(\mu)$ of the perfect matching is given by $(-1)^\chi$, where $\chi$ is the number of crossings in the link pattern — for the example in (B.4) we have $\varepsilon(\mu) = (-1)^1$. We denote the set of all perfect matchings on $2N$ sites by $M_{2N}$, and the number of perfect matchings is

$$|M_{2N}| = (2N - 1)!! = (2N - 1) \cdot (2N - 3) \cdot \ldots \cdot (3) \cdot (1),$$

since there are $2N - 1$ sites for the first site to pair with, then $2N - 3$ sites for the second site to pair with, etc. (Note that usually a perfect matching is defined as a set of edges on a graph such that every vertex is included exactly once. However this characterization will not be useful for us, and for a complete graph it is equivalent to the definition we use in terms of link patterns.)

The connection to Pfaffians comes from the fact that there is a bijection from $M_{2N}$ to a subset of $S_{2N}$, the set of permutations of $\{1, \ldots, 2N\}$. The bijection is found by taking a perfect matching $\mu$ and ordering the components of each pair such that $\mu = \{(\mu_{j,L}, \mu_{j,R})\}_{j=1,\ldots,N}$.
where \( \mu_{j,L} \) and \( \mu_{j,R} \) are respectively the left and right terminals of link \( j \). (In graph parlance, this creates a **directed link pattern**, where all links point from, say, left to right.) Then we define an ordering between the pairs according to some scheme (say, that \( \mu_{j,L} < \mu_{j+1,L} \)), which results in a unique representative ordered set of pairs for each perfect matching. Then, by removing the pairing, we obtain a unique \( s \in S_{2N} \). For the example in (B.4) we find

\[
M_6 \ni \{(2, 3), (5, 1), (4, 6)\} \mapsto (1, 5, 2, 3, 4, 6) \in S_6.
\] (B.6)

The reverse mapping \( S_{2N} \supset \hat{S}_{2N} \rightarrow M_{2N} \) is clear: \( s \in \hat{S}_{2N} \) is a permutation of \( 1, \ldots, 2N \) such that \( s(2j - 1) < s(2j) \) and \( s(2j - 1) < s(2k - 1) \) for \( j < k \).

In order for this mapping to make sense, we need \( \varepsilon(\mu) = \varepsilon(s) \), that is the number of crossings in the perfect matching \( \mu \) must be the same as the signature of the permutation \( s \), given by \( (-1)\tau \) where \( \tau \) is the number of transpositions required to return \( s \) to the identity permutation. This can be shown by first noting that the identity permutation gives a link pattern with no crossings, and then that a crossing can always be removed by a single transposition, while a link pattern with no crossings can be transformed to the identity by an even number of transpositions.

The conditions defining \( \hat{S}_{2N} \) are the same restrictions on \( S_{2N} \) as those implied by the first line of (B.1), and so we have the following equivalent expression for the Pfaffian

\[
Pf M = \sum_{\mu \in M_{2N}} \varepsilon(\mu) \cdot m_{i_1,j_1} m_{i_2,j_2} \cdots m_{i_N,j_N},
\] (B.7)

where \( M_{2N} \) is the set of all perfect matchings \( \mu = \{(i_1, j_1), \ldots, (i_N, j_N)\} \) on \( 2N \) sites, and \( \varepsilon(\mu) \) is the signature of the perfect matching, or equivalently, the signature of the corresponding permutation. Note that in this representation, since we are in the upper half-triangle of the matrix (where the row index is less than the column index) the left vertex \( \mu_{j,L} \) of each edge corresponds to the row index of the matrix, while the right vertex \( \mu_{j,R} \) corresponds to the column.

## C Proofs of Restricted Partition Functions

Although the statements in Propositions 1 and 2 are straightforward variations on the \( y = \infty \) results (see [33,55,83]), we provide the proofs for our specific case here to keep this work self-contained for the non-specialist.

### C.1 Proof of Proposition 1 (\( \beta = 4 \))

We start with the identity [55]

\[
\prod_{1 \leq j < k \leq N} (\lambda_j - \lambda_k)^4 = \det \left[ \lambda_j^{k-1} \right]_{j=1, \ldots, N}^{k=1, \ldots, 2N}, \quad (C.1)
\]

and note that each even row is the derivative of the odd row immediately above it. Then in this matrix, for each column, by adding linear combinations of the columns to the left of that column (starting from the left-most column) we can create arbitrary monic polynomials, while preserving the derivative relationship between the even and odd rows. So for our purpose, we choose the polynomials to be the \( Q_j \), which are skew-orthogonal with respect
to the skew-inner product (7), giving

$$\hat{Z}_{4,N}[a, y] = \frac{1}{Z_{4,N}} \int_{-\infty}^{y} d\lambda_1 \cdots \int_{-\infty}^{y} d\lambda_N \prod_{j=1}^{N} a(\lambda_j) e^{-2\lambda_j^2} \det \begin{bmatrix} Q_{2k-2}(\lambda_j) & Q_{2k-1}(\lambda_j) \\ Q_{2k-2}'(\lambda_j) & Q_{2k-1}'(\lambda_j) \end{bmatrix}_{j,k=1,\ldots,N}$$

$$= \frac{1}{Z_{4,N}} \sum_{\pi \in S_{2N}} \varepsilon(\pi) \prod_{j=1}^{N} \int_{-\infty}^{\pi(\lambda_j)} d\lambda \ a(\lambda) e^{-2\lambda^2} Q_{\pi(2j-1)-1}(\lambda) Q_{\pi(2j)-1}'(\lambda), \quad (C.2)$$

where the second line follows from Laplace expansion of the determinant, and we apply the integrals to each matched pair of $Q$ and $Q'$. (Note that we suppress the dependence on $y$ for brevity.)

For each pair of indices on the $Q$ and $Q'$ in (C.2), we then match up each permutation with the corresponding permutation where that index pair is interchanged, hence picking up a $(-1)$, giving

$$\hat{Z}_{4,N}[a, y] = \frac{1}{Z_{4,N}} \times \sum_{\pi \in S_{2N}} \varepsilon(\pi) \prod_{j=1}^{N} \int_{-\infty}^{\pi(\lambda_j)} d\lambda \ a(\lambda) e^{-2\lambda^2} \left( Q_{\pi(2j-1)-1}(\lambda) Q_{\pi(2j)-1}'(\lambda) - Q_{\pi(2j)-1}(\lambda) Q_{\pi(2j-1)-1}'(\lambda) \right), \quad (C.3)$$

where we need to restrict the sum to just those permutations obeying the rule $\pi(2j) > \pi(2j-1)$ for all $j$. Introducing a factor of $\frac{1}{N!}$ for each integral (incurring a pre-factor of $2^N$), then using the definition of the Pfaffian recalled in (B.1) we obtain

$$\hat{Z}_{4,N}[a, y] = \frac{2^N N!}{Z_{4,N}} \text{Pf} \left[ \frac{1}{2} \int_{-\infty}^{y} d\lambda \ a(\lambda) e^{-2\lambda^2} \left( Q_{\lambda}(\lambda) Q_{\lambda}'(\lambda) - Q_{\lambda}(\lambda) Q_{\lambda}'(\lambda) \right) \right]_{j,k=0,\ldots,2N-1}. \quad (C.4)$$

The equality between the first and second lines in (7) gives the result in (40).

### C.2 Proof of Proposition 2 ($\beta = 1$, $N$ even)

We start by ordering the eigenvalues $-\infty < \lambda_1 < \cdots < \lambda_N < y$ (incurring a factor of $N!$) in (33) so that we can remove the absolute value from the product of differences. Then we use the Vandermonde determinant expression (suppressing the polynomial dependence on $y$)

$$\hat{Z}_{1,N}[a, y] = \frac{N!}{Z_{1,N}} \int_{-\infty}^{y} d\lambda_N \int_{-\infty}^{\lambda_N} d\lambda_{N-1}$$

$$\cdots \int_{-\infty}^{\lambda_2} d\lambda_1 \prod_{j=1}^{N} e^{-\lambda_j^2/2} a(\lambda_j) \prod_{1 \leq j < k \leq N} (\lambda_k - \lambda_j)$$

$$= \frac{N!}{Z_{1,N}} \int_{-\infty}^{y} d\lambda_N \int_{-\infty}^{\lambda_N} d\lambda_{N-1}$$

$$\cdots \int_{-\infty}^{\lambda_2} d\lambda_1 \det \left[ e^{-\lambda_j^2/2} a(\lambda_j) \lambda_j^{k-1} \right]_{j,k=1,\ldots,N}.$$
where the third equality follows from elementary column operations. This is the same procedure that was applied to (C.1) in the \( \beta = 4 \) case above, and it allows us to obtain any set of monic polynomials in the columns; for our purposes we specify the polynomials to be the \( \{R_j\} \), which are skew-orthogonal with respect to the skew-inner product (8).

Now we wish to apply the method of integration over alternate variables (mentioned above), and to prepare for that we change the order of the integrals, with even integrals on the left and odd integrals on the right

\[
\hat{Z}_{1,N}[a, y] = \frac{N!}{Z_{1,N}} \int_{-\infty}^{y} d\lambda_N \int_{-\infty}^{\lambda_N} d\lambda_{N-2} \cdots \int_{-\infty}^{\lambda_2} d\lambda_N \det \left[ e^{-\lambda_j^2/2} a(\lambda_j) R_{k-1}(\lambda_j) \right]_{j,k=1,\ldots,N},
\]

where we have added the first row to the third row, and the first and third rows to the fifth row, and so on, so all the integrals have lower terminal \(-\infty\). (This sequence of steps is the method of integration over alternate variables.)

We see that the determinant in (C.7) is now symmetric in the variables \( \lambda_2, \lambda_4, \ldots, \lambda_N \), and so we can remove the ordering \( \lambda_2 < \lambda_4 < \ldots < \lambda_N \) at the cost of dividing by \( (N/2)! \). Taking the Laplace expansion of the determinant we find

\[
\hat{Z}_{1,N}[a, y] = \frac{1}{Z_{1,N}} \frac{N!}{(N/2)!} \sum_{\pi \in S_N} \varepsilon(\pi) \prod_{j=1}^{N/2} \mu_{\pi(2j-1), \pi(2j)},
\]

where

\[
\mu_{j,k} := \int_{-\infty}^{y} dx \ e^{-x^2/2} a(x) \ R_{k-1}(x) \int_{-\infty}^{x} dz \ e^{-z^2/2} a(z) \ R_{j-1}(z),
\]

and \( \varepsilon(\pi) \) is the signature of the permutation \( \pi \). By defining

\[
y_{j,k}^{(1)} := \frac{1}{2} (\mu_{j,k} - \mu_{k,j}),
\]
incurred a factor of $2^{N/2}$, then we can restrict the sum to terms with $\pi(2j) > \pi(2j - 1)$ for all $j$, giving

$$
\hat{Z}_{1,N}[a, y] = \frac{1}{Z_{1,N}} \frac{2^{N/2} \cdot N!}{(N/2)!} \sum_{\pi \in \mathcal{S}_N} \varepsilon(\pi) \prod_{j=1}^{N/2} \gamma_{(2j-1), (2j)}^{(1)}.
$$

(C.11)

Now using (B.1) we have the result in (44)–(45) [where we cancel the factor of $(N/2)!$ to account for summing over distinct terms only].

### C.3 Proof of Proposition 3 ($\beta = 1$, $N$ odd)

As for the $N$ even case we follow the same technique that led to (C.5) and (C.6), but now with an odd number of $\lambda_j$, so (C.6) becomes

$$
\hat{Z}_{1,N_{odd}}[a, y] = \frac{N!}{Z_{1,N}} \int_{-\infty}^{y} d\lambda_{N-1} \cdots \int_{-\infty}^{\lambda_4} d\lambda_2 \int_{\lambda_N-1}^{\lambda_2} d\lambda_N \int_{\lambda_{N-3}}^{\lambda_2} d\lambda_{N-2} 
$$

$$
\cdots \int_{\lambda_2}^{\lambda_4} d\lambda_3 \int_{-\infty}^{\lambda_1} d\lambda_1 \det \left[ e^{-\lambda_j^2/2} a(\lambda_j) R_{k-1}(\lambda_j) \right]_{j=1,\ldots,(N-1)/2}^{k=1,\ldots,N}.
$$

(C.12)

We depart from the $N$ even case when we pair up the even and odd rows ("integration over alternate variables"), and we must have one unpaired row. So we replace (C.7) with

$$
\hat{Z}_{1,N_{odd}}[a, y] = \frac{N!}{Z_{1,N}} \int_{-\infty}^{y} d\lambda_{N-1} \int_{-\infty}^{\lambda_{N-3}} d\lambda_{N-3} 
$$

$$
\cdots \int_{-\infty}^{\lambda_4} d\lambda_2 \det \left[ e^{-\lambda_j^2/2} a(\lambda_j) R_{k-1}(\lambda_j) \right]_{j=1,\ldots,(N-1)/2}^{k=1,\ldots,N}.
$$

(C.13)

where, in the second equality we have removed the ordering on the $(N - 1)/2$ even variables. So now with $\mu_{j,k}$ from (C.9) and $\nu_j$ from (49) we expand this determinant to obtain

$$
\hat{Z}_{1,N_{odd}}[a, y] = \frac{1}{Z_{1,N}} \frac{N!}{((N - 1)/2)!} \prod_{\pi \in \mathcal{S}_N} \varepsilon(\pi) \nu_{\pi(N)-1}^{(N-1)/2} \prod_{j=1}^{(N-1)/2} \mu_{\pi(2j-1), \pi(2j)}.
$$

(C.14)
Introducing $\gamma_{j,k}^{(1)}$ via (C.10) and restricting the sum to those terms with $\pi(2j) > \pi(2j-1)$ we have

$$\tilde{Z}_{1,N_{\text{odd}}}[a, y] = \frac{2^{(N-1)/2}}{Z_{1,N}} \frac{N!}{((N-1)/2)!} \sum_{\pi \in S_N} \varepsilon(\pi) \nu_{\pi(N)-1}[a, y] \prod_{j=1}^{(N-1)/2} \gamma_{\pi(2j-1), \pi(2j)}^{(1)},$$

(C.15)

Further restricting the sum to only distinct terms [giving a factor of $((N-1)/2)!$] and identifying $\nu_{\pi(N)-1,N} := \nu_{\pi(N)-1}$ we use (B.1) with the permutations $\pi \in S_{N-1}$ over just the first $N-1$ elements and we have the result in Proposition 3.

**D Iterative Construction of the First Few Skew-Orthogonal Polynomials**

In this Appendix, we iteratively construct the first few skew-orthogonal polynomials defined in Eqs. (10) and (11).

First for $\beta = 4$, by monicity, we must have $Q_0(\lambda) = 1$ and by (29) we can assume that $Q_1(\lambda) = \lambda$, then we use the skew-inner product relations (7) to iteratively solve for the higher degree polynomials, so the first four skew-orthogonal polynomials are

$$Q_0(\lambda, y) = 1, \quad Q_1(\lambda, y) = \lambda, \quad Q_2(\lambda, y) = \lambda^2 - \lambda b + \frac{1 - 2yb}{4},$$

(D.1)

$$Q_3(\lambda, y) = \lambda^3 - 3 \frac{1 - 2yb}{4} \lambda - b \frac{1 + 2y^2}{2}$$

(D.2)

[where we used (29) for $Q_3(\lambda, y)$] with normalizations

$$q_0(y) := \langle Q_0, Q_1 \rangle_4^y = \frac{\sqrt{\pi}}{4\sqrt{2}} \text{erfc}(-\sqrt{2}y) = \frac{e^{-2y^2}}{4b},$$

(D.3)

$$q_1(y) := \langle Q_2, Q_3 \rangle_4^y = \frac{1}{64} \left(3\sqrt{2\pi} \text{erfc}(-\sqrt{2}y) - 2e^{-2y^2} y(9 + 4y^2) - 4e^{-2y^2}(2 + y^2)b\right),$$

(D.4)

where

$$b = \sqrt{2e^{-2y^2}} \frac{\sqrt{\pi}(1 + \text{erf}(-\sqrt{2}y))}{\sqrt{\pi} \text{erfc}(-\sqrt{2}y)}.$$ 

(D.5)

For $\beta = 1$, again by monicity and (29) we have, $R_0(\lambda) = 1$, $R_1(\lambda) = \lambda$ and using the relations (8) we can obtain the first four polynomials

$$R_0(\lambda, y) = 1, \quad R_1(\lambda) = \lambda,$$

(D.6)

$$R_2(\lambda, y) = \lambda^2 + \lambda \frac{c}{\sqrt{\pi}} \left(2e^{-y^2/2} + \sqrt{2\pi} \text{erfc}(-y/\sqrt{2})\right) + ce^{y^2/2} \text{erfc}(-y) - 1,$$

(D.7)

$$R_3(\lambda, y) = \lambda^3 + \lambda c \left(\frac{2ye^{-y^2/2}}{\sqrt{\pi}} - \frac{2}{c} e^{y^2/2} \text{erfc}(-y) + y^2 \sqrt{2} \text{erfc}(-y/\sqrt{2})\right) - \frac{2ce^{-y^2/2}}{\sqrt{\pi}}$$

(D.8)
with
\[
c = \left( 2e^{y^2/2} \text{erfc}(-y) - \sqrt{2} \text{erfc}(-y/\sqrt{2}) \right)^{-1}
\]  
(D.9) 

and
\[
r_0(y) = \frac{\sqrt{\pi}}{2} \left( \text{erfc}(-y) - \frac{e^{-y^2/2}}{\sqrt{2}} \text{erfc}(-y/\sqrt{2}) \right) = \frac{\sqrt{\pi} e^{-y^2/2}}{4c}
\]  
(D.10) 

\[
r_1(y) = \frac{\sqrt{\pi}}{8} \text{erfc}(-y) - \frac{ye^{-y^2}}{4} - c \left( \frac{e^{-3y^2/2}}{\sqrt{\pi}} + \frac{y^2 \sqrt{\pi}}{2\sqrt{2}} \text{erfc}(-y) \text{erfc}(-y/\sqrt{2}) \right)
\]
\[
+ \frac{ye^{-y^2}}{2} \text{erfc}(-y) + \frac{ye^{-y^2}}{\sqrt{2}} \text{erfc}(-y/\sqrt{2}) - \frac{\sqrt{\pi} e^{-y^2/2}}{4} \text{erfc}(-y)^2 \right).
\]  
(D.11) 

### E Skew-Orthogonal Polynomials for \( \beta = 4 \)

For the ease of the reader, we try to use the same notation as in \([33, \S 6.2 \& \S 6.4]\), where the case \( y = \infty \) is discussed in detail. Also note that all the quantities in this section depend on \( y \), however we will suppress the explicit notation of such, to save space.

The goal is to write the \( \beta = 4 \) skew-orthogonal polynomials \( \{Q_j\} \), defined by (7) and (10), in terms of the polynomials orthogonal with respect to the inner product (6), the NM polynomials \( P_j \) which obey the relations (24)–(26). However, as discussed in Sect. 4.1 we will instead use the modified skew-inner product (56), and look for polynomials \( \{\tilde{Q}_j\} \) that obey the relations (57) and (58), up to the invariance (29). Since the orthogonal polynomials form a complete set we can find coefficients \( \tilde{\alpha}_{j,k} \) such that
\[
\tilde{Q}_j = \tilde{\alpha}_{j,j} P_j + \tilde{\alpha}_{j,j-1} P_{j-1} + \cdots + \tilde{\alpha}_{j,1} P_1 + \tilde{\alpha}_{j,0} P_0, \quad \tilde{\alpha}_{j,k} \in \mathbb{C}.
\]  
(E.1) 

Recall that the tilde \( \tilde{\ } \) means that the quantity is associated with this modified skew-inner product. From monicity and (29) we have
\[
\tilde{\alpha}_{j,j} = 1, \quad \tilde{\alpha}_{2j+1,2j} = 0.
\]  
(E.2) 

We can write (E.1) in the matrix form
\[
\tilde{Q} = \tilde{X} P
\]  
(E.3) 

where
\[
\tilde{Q} = \begin{bmatrix} \tilde{Q}_0 \\ \tilde{Q}_1 \\ \vdots \end{bmatrix}, \quad P = \begin{bmatrix} P_0 \\ P_1 \\ \vdots \end{bmatrix}
\]  
(E.4) 

\[
\tilde{X} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ \tilde{\alpha}_{2,0} & \tilde{\alpha}_{2,1} & 1 & 0 & 0 & \cdots \\ \tilde{\alpha}_{3,0} & \tilde{\alpha}_{3,1} & 0 & 1 & 0 & \cdots \\ \tilde{\alpha}_{4,0} & \tilde{\alpha}_{4,1} & \tilde{\alpha}_{4,2} & \tilde{\alpha}_{4,3} & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}.
\]  
(E.5)
For the calculation, we will find it more convenient to work with the equation

\[ P = \tilde{X}^{-1} \tilde{Q}. \] (E.6)

Since the skew-orthogonal polynomials will also form a complete set, we know that \( \tilde{X} \) is invertible and we denote

\[
\tilde{X}^{-1} := \left[ \tilde{\beta}_{s,t} \right]_{s,t=1,...,j} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & \cdots \\
0 & 1 & 0 & 0 & 0 & \cdots \\
\tilde{\beta}_{2,0} & \tilde{\beta}_{2,1} & 1 & 0 & 0 & \cdots \\
\tilde{\beta}_{3,0} & \tilde{\beta}_{3,1} & 0 & 1 & 0 & \cdots \\
\tilde{\beta}_{4,0} & \tilde{\beta}_{4,1} & \tilde{\beta}_{4,2} & \tilde{\beta}_{4,3} & 1 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots \\
\end{bmatrix}
\] (E.7)

where we have used the assumptions analogous to (E.2)

\[ \tilde{\beta}_{j,j} = 1, \quad \tilde{\beta}_{2j+1,2j} = 0 \] (E.8)

(recalling that the first equality is the definition of monicity). So instead of looking for the coefficients in (E.1) we will solve for the coefficients \( \tilde{\beta}_{j,k} \) in

\[ P_j = \tilde{Q}_j + \tilde{\beta}_{j,j-1} \tilde{Q}_{j-1} + \cdots + \tilde{\beta}_{j,1} \tilde{Q}_1 + \tilde{\beta}_{j,0} \tilde{Q}_0, \quad \tilde{\beta}_{j,k} \in \mathbb{C}, \] (E.9)

and then hope to invert the relations to recover the \( \tilde{\alpha}_{j,k} \). We also define the matrix of inner products

\[
\tilde{q} := \left[ \langle \tilde{Q}_j, \tilde{Q}_k \rangle \right]_{j,k} = \begin{bmatrix}
0 & \tilde{q}_0 & 0 & 0 & 0 & \cdots \\
-\tilde{q}_0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & \tilde{q}_1 & 0 & 0 & \cdots \\
0 & 0 & 0 & \tilde{q}_1 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots \\
\end{bmatrix}
\] (E.10)

Using (69) we can write the modified \( \beta = 4 \) skew-inner product in terms of the \( \beta = 2 \) inner product, with the inclusion of the operator \( A \) defined in (54). To make use of this we first note that if \( f_k \) is any monic polynomial of degree \( k \) then we have

\[
A f_k[x] = -\left( x f_k(x) - f'_k(x) \right) = -\left( P_{k+1}(x) + \sum_{j=0}^{k} c_j P_j(x) \right),
\] (E.11)

where we have decomposed \( x f_k(x) - f'_k(x) \) into a sum over the (monic) orthogonal polynomials \( P_j \), with coefficients \( c_j \). Combining this fact with (66), (67) and the normalization of the \( P_j \) from (12) we have the matrix

\[
A := \left[ \langle P_j, A P_k \rangle \right]_{j,k=0,...,N-1} = \begin{bmatrix}
\frac{\Omega_{0,0}}{2} & p_1 + \Omega_{0,1} & \Omega_{0,2} & \Omega_{0,3} & \Omega_{0,4} & \Omega_{0,5} & \cdots \\
-p_1 & \frac{\Omega_{1,1}}{2} & p_2 + \Omega_{1,2} & \Omega_{1,3} & \Omega_{1,4} & \Omega_{1,5} & \cdots \\
0 & -p_2 & \frac{\Omega_{2,2}}{2} & p_3 + \Omega_{2,3} & \Omega_{2,4} & \Omega_{2,5} & \cdots \\
0 & 0 & -p_3 & \frac{\Omega_{3,3}}{2} & p_4 + \Omega_{3,4} & \Omega_{3,5} & \cdots \\
0 & 0 & 0 & -p_4 & \frac{\Omega_{4,4}}{2} & p_5 + \Omega_{4,5} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \\
\end{bmatrix}. \] (E.12)
So now we can write
\[
\tilde{q} = \left\langle \tilde{Q}_j, \tilde{Q}_k \right\rangle^4 = \left\langle \tilde{Q}^T \right\rangle^4_4 = \left\langle \tilde{X} P P^T \tilde{X}^T \right\rangle^4
\]
\[
= \tilde{X} \left\langle P P^T \right\rangle^4 \tilde{X}^T = \tilde{X} \left( A - \frac{1}{2} [\Omega_{j,k}] \right) \tilde{X}^T
\]
\[
= \tilde{X} W \tilde{X}^T,
\]
(E.13)
where \( W = A - \frac{1}{2} [\Omega_{j,k}] \) is the anti-symmetric matrix in (22). (Note that for a matrix \( M \) the notation \( \langle\langle M \rangle\rangle^4 \) implies that the average is applied elementwise to the matrix.) Rearranging (E.13)
\[
\tilde{X}^{-1} \tilde{q} (\tilde{X}^{-1})^T = W,
\]
(E.14)
and expanding out the left hand side we get
\[
\left[ \tilde{X}^{-1} \tilde{q} (\tilde{X}^{-1})^T \right]_{j,k} = \sum_{m=0,1,\ldots,j} \tilde{q}_{m,n} \tilde{p}_{k,n}
\]
\[
= \sum_{m \text{ even}} \tilde{p}_{j,m} \tilde{q}_{m,m+1} + \sum_{m \text{ odd}} \tilde{p}_{j,m} \tilde{q}_{m-1,m+1} + \tilde{p}_{j,m} \tilde{q}_{2m/2} - \tilde{p}_{j,m} \tilde{q}_{(m-1)/2} - \tilde{p}_{j,m+1} \tilde{q}_{2m+1/2} - \tilde{p}_{j,m+1} \tilde{q}_{(m+1)/2},
\]
(E.15)
noting that this is a finite sum since all \( \tilde{p}_{\mu,\nu} \) are zero when \( \nu > \mu \). So we have the set of equations
\[
0 = \sum_{m \text{ even}} \tilde{q}_{m/2} \left( \tilde{p}_{j,m} \tilde{p}_{k,m+1} - \tilde{p}_{j,m+1} \tilde{p}_{k,m} \right) - w_{j,k}
\]
(E.16)
(where we denote the elements of \( W \) by \( w_{j,k} \)) and we are now in a position to solve for the normalizations \( \tilde{q}_j \) and the coefficients \( \tilde{p}_{j,k} \).

**E.1 Expressions for \( \tilde{q}_j \)**

Let the matrices in (E.14) be of size \( 2n \times 2n \). Then, taking the Pfaffian we get
\[
Pf W = Pf (\tilde{X}^{-1} \tilde{q} (\tilde{X}^{-1})^T)^{-1} = det(\tilde{X}^{-1}) Pf \tilde{q} = Pf \tilde{q} ,
\]
(E.17)
where we used the Pfaffian identity (B.3) for the second equality, and the fact that \( \tilde{X}^{-1} \) is a triangular matrix with 1s on the diagonal for the third equality.

Because \( \tilde{q} \) is a skew-diagonal matrix, as in (52), we have
\[
Pf \tilde{q} = \prod_{j=0}^{n-1} \tilde{q}_j = Pf W_{2n-1}.
\]
(E.18)
Beginning with \( n = 1 \) and iterating, we obtain (73), with the convention (74).
E.2 Expressions for $\tilde{\beta}_{j,k}$

Let $k$ be even, then the last term in the sum of (E.16) is $-\tilde{\beta}_{j,k+1}\tilde{q}/2$ (when $m = k$), and so solving for this $\tilde{\beta}$ we obtain

$$
\tilde{\beta}_{j,k+1} = \frac{1}{\tilde{q}/2} \left[ \sum_{m=0, \text{m even}}^{k-2} \tilde{q}m/2 \left( \tilde{\beta}_{j,m}\tilde{\beta}_{k,m+1} - \tilde{\beta}_{j,m+1}\tilde{\beta}_{k,m} \right) - w_{j,k} \right], \quad k \text{ even. (E.19)}
$$

For $k$ odd the last term (when $m = k - 1$) is $\tilde{q}((k-1)/2(\tilde{\beta}_{j,k-1} - \tilde{\beta}_{j,k}\tilde{\beta}_{k,k-1})$, but recall from (E.8) that (when $k$ is odd) we have set $\tilde{\beta}_{k,k-1} = 0$ [using (29)], so we obtain

$$
\tilde{\beta}_{j,k-1} = -\frac{1}{\tilde{q}(k-1)/2} \left[ \sum_{m=0, \text{m even}}^{k-3} \tilde{q}m/2 \left( \tilde{\beta}_{j,m}\tilde{\beta}_{k,m+1} - \tilde{\beta}_{j,m+1}\tilde{\beta}_{k,m} \right) - w_{j,k} \right], \quad k \text{ odd. (E.20)}
$$

From these two expressions we see that each $\tilde{\beta}_{j,2k}$ and $\tilde{\beta}_{j,2k+1}$ only depends on the $\tilde{\beta}$s in the same row, and in columns $0, 1, \ldots, 2k - 1$. This allows us to inductively solve for the $\tilde{\beta}$: first we solve for $\tilde{\beta}_{j,0}$, $\tilde{\beta}_{j,1}$, then $\tilde{\beta}_{j,2}$, $\tilde{\beta}_{j,3}$, etc.

It is this decoupling of the $\tilde{\beta}$ equations that is the reason for working with $\tilde{X}^{-1}$ instead of $\bar{X}$.

Proposition E.1

$$
\tilde{\beta}_{j,k} = \begin{cases} 
\frac{\text{Pf } W_{k+j}^{(k+j)}}{\text{Pf } W_{k+1}} & \text{, } k \text{ even}, \\
\frac{\text{Pf } W_{k+j}^{(k+j)}}{\text{Pf } W_{k}} & \text{, } k \text{ odd},
\end{cases}
$$

(E.21)

where $W_{\mu}^{(\eta \rightarrow \nu)}$ is the matrix $W_\mu$ from (22) with all occurrences of the index $\eta$ replaced by the index $\nu$.

Proof As mentioned above, we will employ an inductive proof. We need both even and odd base cases. Expanding out (E.16) with $k = 0$ we have

$$
0 = -\tilde{\beta}_{j,1}\tilde{q}_0\tilde{\beta}_{0,0} - w_{j,0}
$$

(E.22)

$$
\Rightarrow \tilde{\beta}_{j,1} = -\frac{w_{j,0}}{\tilde{q}_0} = \frac{w_{0,j}}{w_{0,1}} = \frac{\text{Pf } W_{1}^{(1 \rightarrow j)}}{\text{Pf } W_{1}}.
$$

(E.23)

Similarly, [recalling that $\tilde{\beta}_{1,0} = 0$ from (E.8)] with $k = 1$, we get

$$
0 = \tilde{\beta}_{j,0}\tilde{q}_0\tilde{\beta}_{1,1} - \tilde{\beta}_{j,1}\tilde{q}_0\tilde{\beta}_{1,0} - w_{j,1}
$$

(E.24)

$$
\Rightarrow \tilde{\beta}_{j,0} = \frac{w_{j,1}}{\tilde{q}_0} = \frac{\text{Pf } W_{1}^{(0 \rightarrow j)}}{\text{Pf } W_{1}}.
$$

(E.25)
Now we move to the inductive step. For convenience, here we restrict to \( k \) even. Assume that we have (E.21) for all \( \tilde{\beta}_{j,0}, \tilde{\beta}_{j,1}, \ldots, \tilde{\beta}_{j,k-1} \) and we substitute (E.21) and (73) into (E.19) to get

\[
\tilde{\beta}_{j,k+1} = \frac{\text{Pf} \, W_{k-1}}{\text{Pf} \, W_{k+1}} \left[ \sum_{m=0, m \text{ even}}^{k-2} \frac{\text{Pf} \, W_{m+1}^{(m \rightarrow j)}}{\text{Pf} \, W_{m+1}^{(m+1 \rightarrow k)}} - \frac{\text{Pf} \, W_{m+1}^{(m+1 \rightarrow j)}}{\text{Pf} \, W_{m+1}^{(m \rightarrow k)}} - w_{j,k} \right].
\]  

(E.26)

Using [43, (1.1)] we obtain

\[
\text{Pf} \, W_{m+1}^{(m \rightarrow j)} - \text{Pf} \, W_{m+1}^{(m+1 \rightarrow k)} - \text{Pf} \, W_{m+1}^{(m-k)} \]

\[
= \text{Pf} \, W_{m-1} \text{Pf} \, W_{m+3}^{(m+2 \rightarrow k, m+3 \rightarrow j)} - \text{Pf} \, W_{m+1} \text{Pf} \, W_{m+1}^{(m \rightarrow j, m+1 \rightarrow k)}.
\]  

(E.27)

We note that these “overlapping Pfaffians” have appeared earlier than in [43] (eg. [65], and even as far back as [37]), however we will use Knuth’s formulation simply because of our familiarity with it. The notation in [43] is quite different to that used here, so we briefly outline how (E.27) follows from [43, (1.1)], which we quote here, rearranged for convenience

\[
-f[\alpha x z]f[\alpha w y] + f[\alpha w z]f[\alpha x y] = f[\alpha]f[\alpha w xyz] - f[\alpha w x]f[\alpha y z]
\]  

(E.28)

where \( w, x, y, z \in \mathbb{Z} \) are matrix indices and \( \alpha \in \mathbb{Z}^p \) is an ordered set of indices. For index sets \( \alpha_1 \in \mathbb{Z}^p, \alpha_2 \in \mathbb{Z}^q \) the product \( \alpha_1 \alpha_2 \in \mathbb{Z}^{p+q} \) is the concatenation of the index sets. The function \( f[\alpha] \) is then the Pfaffian of the matrix \( f[jk] \) with index set \( \alpha \), i.e.

\[
\begin{align*}
\{f[\alpha]\} = & \text{Pf} \left[ f[jk] \right], j, k \in \alpha \\
\end{align*}
\]  

(E.29)

defined recursively, where for a pair of indices \( f[jk] \) is the matrix element, and

\[
f[jk] = - f[kj]
\]  

(E.30)

since Pfaffian matrices are anti-symmetric. So then to match (E.28) with (E.27) we take

\[
\alpha = \{0, 1, \ldots, m - 1\}, \quad w = \{m\}, \quad x = \{m + 1\}, \quad y = \{k\}, \quad z = \{j\},
\]  

(E.31)

and apply (E.30) to rearrange the indices as needed.

Substituting (E.27) into (E.26) we obtain

\[
\tilde{\beta}_{j,k+1} = \frac{\text{Pf} \, W_{k-1}}{\text{Pf} \, W_{k+1}} \left[ \sum_{m=0, m \text{ even}}^{k-2} \frac{\text{Pf} \, W_{m+1}^{(m+2 \rightarrow k, m+3 \rightarrow j)}}{\text{Pf} \, W_{m+1}^{(m+1 \rightarrow k, m+1 \rightarrow j)}} - w_{j,k} \right],
\]  

(E.32)

which is a telescoping sum, leaving

\[
\tilde{\beta}_{j,k+1} = \frac{\text{Pf} \, W_{k-1}}{\text{Pf} \, W_{k+1}} \left[ \frac{\text{Pf} \, W_{k+1}^{(k+1 \rightarrow j)}}{\text{Pf} \, W_{k+1}^{(k+1 \rightarrow j)}} - \frac{\text{Pf} \, W_{1}^{(0 \rightarrow k, 1 \rightarrow j)}}{\text{Pf} \, W_{1}^{(0 \rightarrow k, 1 \rightarrow j)}} - w_{j,k} \right]
\]  

(E.33)

\[
= \frac{\text{Pf} \, W_{k+1}^{(k+1 \rightarrow j)}}{\text{Pf} \, W_{k+1}^{(k+1 \rightarrow j)}} - w_{j,k}
\]  

(E.34)

since \( \text{Pf} \, W_{1}^{(0 \rightarrow k, 1 \rightarrow j)} = w_{k,j} = - w_{j,k} \), and we also used the convention (74).

For the \( k \) odd case, one proceeds from (E.20) in a similar fashion. □
Note from (E.21) that
\[ \tilde{\beta}_{2n+1,2n} = \frac{\text{Pf } W_{2n+1}^{(2n \rightarrow 2n+1)}}{\text{Pf } W_{2n+1}}, \quad \text{(E.35)} \]
and since \(w_{n,n} = 0\) for all \(n\), then the Pfaffian in the numerator has two identical columns (the right-most) and two identical rows (the bottom-most), which implies
\[ \text{Pf } W_{2n+1}^{(2n \rightarrow 2n+1)} = 0 \quad \Rightarrow \quad \tilde{\beta}_{2n+1,2n} = 0. \quad \text{(E.36)} \]
Also, we clearly have
\[ \tilde{\beta}_{j,j} = \begin{cases} \frac{\text{Pf } W_{j+1}^{(j \rightarrow j)}}{\text{Pf } W_{j+1}}, & j \text{ even}, \\ \frac{\text{Pf } W_{j}^{(j \rightarrow j)}}{\text{Pf } W_{j}}, & j \text{ odd}, \end{cases} \quad \text{(E.37)} \]
so, even though we assumed \(\tilde{\beta}_{j,j} = 1\) in (E.8), the result (E.21) naturally extends to cover this case.

### E.3 Expressions for \(\tilde{\alpha}_{j,k}\) in Proposition 4

From the matrix product
\[ \tilde{X}^{-1}\tilde{X} = I \quad \text{(E.38)} \]
we have
\[ \tilde{\alpha}_{j,k} = - \sum_{m=k}^{j-1} \tilde{\beta}_{j,m} \tilde{\alpha}_{m,k} \quad \text{(E.39)} \]
for \(j > k\). Using this and the expressions for the \(\tilde{\beta}_{j,k}\) in (E.21) we can find expressions for the \(\tilde{\alpha}_{j,k}\).

**Proof of Proposition 4:** From (E.9) we have
\[ \tilde{Q}_j = P_j - \sum_{k=0}^{j-1} \tilde{\beta}_{j,k} \tilde{Q}_k \]
\[ = P_j - \tilde{\beta}_{j,j-1} \tilde{Q}_{j-1} - \sum_{k=0}^{j-2} \tilde{\beta}_{j,k} \tilde{Q}_k, \quad \text{(E.40)} \]
so, with (70), this implies
\[ \tilde{\beta}_{j,j-1} \tilde{Q}_{j-1} = \tilde{\beta}_{j,j-1} (P_{j-1} + \tilde{\alpha}_{j-1,j-2} P_{j-2} + \ldots), \quad \text{(E.41)} \]
and thus
\[ \tilde{Q}_j = P_j - \tilde{\beta}_{j,j-1} P_{j-1} - \text{ lower degree polynomials}. \quad \text{(E.42)} \]
So we have that
\[ \tilde{\alpha}_{j,j-1} = -\tilde{\beta}_{j,j-1}, \quad \text{(E.43)} \]
which is equal to zero when \(j\) is odd by (E.8), and we have consistency with (71).
For (72) we will use an inductive proof similar to that used in Proposition E.1. We see from (E.39) that each \( \tilde{\alpha}_{j,k} \) only depends on the \( \tilde{\beta} \)'s (which are known) and the \( \tilde{\alpha} \)'s above it in the same column of the matrix \( \tilde{X} \) [in (E.5)]. From (E.43) we have

\[
\tilde{\alpha}_{j,j-1} = -\tilde{\beta}_{j,j-1} = \begin{cases} \frac{\text{Pf } W_{j-1}^{(j-1\to j)}}{\text{Pf } W_{j-1}}, & j \text{ even}, \\ 0, & j \text{ odd}, \end{cases} \tag{E.44}
\]

and from (E.39)

\[
\tilde{\alpha}_{j,j-2} = -\tilde{\beta}_{j,j-2}\tilde{\alpha}_{j-2,j-2} - \tilde{\beta}_{j,j-1}\tilde{\alpha}_{j-1,j-2} = -\tilde{\beta}_{j,j-2} = \begin{cases} \frac{\text{Pf } W_{j-1}^{(j-1\to j)}}{\text{Pf } W_{j-1}}, & j \text{ even}, \\ \frac{\text{Pf } W_{j-2}^{(j-2\to j)}}{\text{Pf } W_{j-2}}, & j \text{ odd}, \end{cases} \tag{E.45}
\]

since one of \( \tilde{\beta}_{j,j-1} \) or \( \tilde{\alpha}_{j-1,j-2} \) must be zero by (E.8) or (E.44). The equations (E.44) and (E.45) give us expressions for all \( \tilde{\alpha} \)'s on the first and second lower diagonals of \( \tilde{X} \). So for any column \( k \), there is a row \( j \) for which all the \( \tilde{\alpha}_{j-m,k} \) above it are known, so we have our base cases.

Now for the inductive step, we expand (E.39) to obtain

\[
\tilde{\alpha}_{j,k} = \begin{cases} -\tilde{\beta}_{j,k} - \sum_{m=k+2}^{j-1} \tilde{\beta}_{j,m} \tilde{\alpha}_{m,k}, & j \text{ even}, k \leq j-1, k \text{ even}, \\ -\tilde{\beta}_{j,k} - \sum_{m=k+1}^{j-2} \tilde{\beta}_{j,m} \tilde{\alpha}_{m,k}, & j \text{ even}, k \leq j-1, k \text{ odd}, \\ -\tilde{\beta}_{j,k} - \sum_{m=k+2}^{j-2} \tilde{\beta}_{j,m} \tilde{\alpha}_{m,k}, & j \text{ odd}, k \leq j-1, k \text{ even}, \\ -\tilde{\beta}_{j,k} - \sum_{m=k+1}^{j-2} \tilde{\beta}_{j,m} \tilde{\alpha}_{m,k}, & j \text{ odd}, k \leq j-1, k \text{ odd}. \end{cases} \tag{E.46}
\]

We assume that \( \tilde{\alpha}_{m,k} \) is given by (72) for all \( m \leq j-1 \) (j even) or \( m \leq j-2 \) (j odd), while all \( \tilde{\beta} \)'s are given by (E.21). Taking \( j, k \) both even (the other cases follow similarly), we substitute these known \( \tilde{\alpha} \)'s and \( \tilde{\beta} \)'s into the first row of (E.46) to give

\[
\tilde{\alpha}_{j,k} = \frac{\text{Pf } W_{k+1}^{(k\to j)}}{\text{Pf } W_{k+1}} + \sum_{m=k+2}^{j-2} \frac{\text{Pf } W_{m+1}^{(m\to j)}}{\text{Pf } W_{m+1}} \frac{\text{Pf } W_{m}^{(k\to m)}}{\text{Pf } W_{m}} + \sum_{m=k+3}^{j-1} \frac{\text{Pf } W_{m}^{(m\to j)}}{\text{Pf } W_{m}} \frac{\text{Pf } W_{m-2}^{(k\to m)}}{\text{Pf } W_{m-2}} \\
= \frac{\text{Pf } W_{k+1}^{(k\to j)}}{\text{Pf } W_{k+1}} + \sum_{m=k+2}^{j-2} \frac{\text{Pf } W_{m+1}^{(m\to j)}}{\text{Pf } W_{m+1}} \frac{\text{Pf } W_{m}^{(k\to m)}}{\text{Pf } W_{m}} + \frac{\text{Pf } W_{m}^{(m+1\to j)}}{\text{Pf } W_{m+1}} \frac{\text{Pf } W_{m-1}^{(k\to m+1)}}{\text{Pf } W_{m-1}}, \tag{E.47}
\]

keeping in mind that we have the convention that \( \text{Pf } W_{-1} = 1 \).

We now use [43, (5.1)] (again quoted here and rearranged for convenience)
\[ f[\alpha xuv]f[\alpha yz] - f[\alpha xuv]f[\alpha wyz] = -f[\alpha uvw]f[\alpha xy] + f[\alpha uyz]f[\alpha xuw] + f[\alpha z]f[\alpha uvwxy] - f[\alpha y]f[\alpha uvwxz] \] (E.48)

with
\[ \alpha = \{0, 1, \ldots, k-1, k+1, \ldots, m-1\}, \ x = \{j\}, \ u = \{k\}, \ v = \{m\}, \ w = \{m+1\} \] (E.49)
\[ y = z = \emptyset \quad \text{(the empty set).} \] (E.50)

Rearranging indices according to (E.30), the equality (E.48) gives
\[ \text{Pf } W_{m+1}^{(m_j \to j)} \text{Pf } W_{m-1}^{(k_i \to m_j)} + \text{Pf } W_{m+1}^{(m_j \to j)} \text{Pf } W_{m-1}^{(k_i \to m_j)} \] (E.51)
\[ = \text{Pf } W_{m+1}^{(k_i \to j)} - \text{Pf } W_{m-1}^{(k_i \to j)}, \] (E.52)
and substituting into (E.47) we get
\[ \tilde{\alpha}_{j,k} = -\frac{\text{Pf } W_{k+1}^{(k_i \to j)}}{\text{Pf } W_{k+1}} + \sum_{m=k+2}^{j-2} \frac{\text{Pf } W_{m-1}^{(k_i \to j)}}{\text{Pf } W_{m-1}} - \frac{\text{Pf } W_{m+1}^{(k_i \to j)}}{\text{Pf } W_{m+1}}. \] (E.53)

This is a telescoping sum, which reduces to (72). The other cases in (E.46) are calculated similarly. \[ \Box \]

### E.4 \( \beta = 4 \) Polynomials in the Classical Limit

In the classical limit \((y \to \infty)\) the skew inner product (7) becomes
\[ \langle f, g \rangle_4 := \frac{1}{2} \int_{-\infty}^{\infty} dx \ e^{-2x^2} \left[ f(x)g'(x) - g(x)f'(x) \right], \] (E.54)
and the associated skew-orthogonal polynomials obeying
\[ \langle Q_{2j}, Q_{2k} \rangle_4 = \langle Q_{2j+1}, Q_{2k+1} \rangle_4 = 0 \]
\[ \langle Q_{2j}, Q_{2k+1} \rangle_4 = -\langle Q_{2k+1}, Q_{2j} \rangle_4 = q_j \delta_{j,k} \] (E.55)
are given by \([2,63]\)
\[ Q_{2j+1}(x) = P_{2j+1}(\sqrt{2}x), \quad Q_{2j}(x) = \sum_{t=0}^{j} \left( \prod_{s=t+1}^{j} \frac{p_{2s}}{p_{2s-1}} \right) P_{2t}(\sqrt{2}x) \]
\[ = \sum_{t=0}^{j} \frac{j!}{t!} P_{2t}(\sqrt{2}x) \] (E.56)

[up to the invariance (29)], where the polynomials
\[ P_j(x) = \frac{1}{2^j} H_j(x) \] (E.57)
are the (monic, “physicist’s”) Hermite polynomials in (27) and \( p_j = p_j(\infty) \) from (28). The corresponding normalizations \( q_j = q_j(\infty) \) are also from (28).

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As mentioned after Proposition 4, it can be seen that the results of that Proposition reduce to the classical polynomials (E.56), since in the limit \( y \to \infty \) the matrix \([\Omega_{j,k}] = 0\) in (E.13), and we then follow exactly the steps in [2] to obtain (E.56).

**F Skew-Orthogonal Polynomials for \( \beta = 1 \)**

We again suppress the explicit dependence on \( y \) to save space, although all the quantities here depend on \( y \).

We can follow the same steps as for the \( \beta = 4 \) case in Appendix E to find the coefficients \( \alpha_{j,k} \) in (77). With \( p \) from (E.4) we first rewrite equation (77) as

\[
R = XP \quad \Rightarrow \quad P = X^{-1}R,
\]

where

\[
R = \begin{bmatrix}
R_0 \\
R_1 \\
\vdots
\end{bmatrix},
\]

and \( X \) and \( X^{-1} \) are the same as in (E.5) and (E.7), but without the tildes. Also define the matrices

\[
r := [(R_{j}, R_k)]_{j,k=0,1,...,N-1},
\]

\[
B := [(P_j, A^{-1}P_k)]_{j,k=0,1,...,N-1},
\]

\[
\Phi := [\Phi_{j,k}]_{j,k=0,1,...,N-1},
\]

where \( r \) is of skew-diagonal form (52). Then

\[
r = [(R_{j}, R_k)]_{1} = (RR^T)_1 = (Xpp^T X^T)_1
\]

\[
= X (PP^T)_1 X^T
\]

\[
= -X (B + \Phi) X^T
\]

\[
= XVX^T,
\]

where the anti-symmetric matrix \( V \) is defined in (18) — we will discuss the derivation of the specific structure of the elements of \( V \) in Appendix F.1 below. (As above, the averages over matrix arguments imply that the average is applied elementwise to the matrix.)

We now follow the same steps as in (E.14)–(E.16) to get

\[
X^{-1} r (X^{-1})^T = V
\]

\[
\Rightarrow \quad \sum_{m \text{ even}} r_{m/2} (\beta_{j,m} \beta_{k,m+1} - \beta_{j,m+1} \beta_{k,m}) - v_{j,k} = 0
\]

with \( V = V_m = [v_{j,k}]_{j,k=0,...,m} \) from (18). Assuming \( m = 2n \), taking the Pfaffian of (F.7) we have

\[
Pf r = \prod_{j=0}^{n-1} r_j = Pf V_{2n-1}
\]
and we obtain (84), with the convention (85).

Then, since the equations in (F.8) are of the same form as (E.16), we apply the same reasoning as that in Proposition E.1 to obtain solutions for the $\beta_{j,k}$

$$\beta_{j,k} = \begin{cases} 
\text{Pf} V_{k+1}^{(k\rightarrow j)}, & k \text{ even,} \\
\text{Pf} V_{k}^{(k\rightarrow j)} / \text{Pf} V_{k}, & k \text{ odd,}
\end{cases} \quad (F.10)$$

where again, $V_{\mu}^{(\eta\rightarrow \nu)}$ is the matrix $V_{\mu}$ with all occurrences of the index $\eta$ replaced by the index $\nu$. Now using the equations

$$X^{-1}X = I \quad \Rightarrow \quad \alpha_{j,k} = -\sum_{m=k}^{j-1} \beta_{j,m} \alpha_{m,k}, \quad (F.11)$$

and by following the same steps as in Appendix E.3 and we establish the remaining statements in Proposition 5.

**F.1 Entries of the Matrix $V_m$**

For a general polynomial

$$g_j(x) = c_{j,j}x^j + c_{j,j-1}x^{j-1} + \cdots + c_{j,1}x + c_{j,0} \quad (F.12)$$

we use the identities (calculated via repeated integration by parts)

$$\int_a^b e^{-u^2/2}u^{2k+1}du = (2k)!! \left( \sum_{m=0}^{k} \frac{e^{-a^2/2}a^{2m} - e^{-b^2/2}b^{2m}}{(2m)!!} \right) \quad (F.13)$$

$$\int_a^b e^{-u^2/2}u^{2k}du = (2k - 1)!! \left( \sum_{m=1}^{k} \frac{e^{-a^2/2}a^{2m-1} - e^{-b^2/2}b^{2m-1}}{(2m - 1)!!} \right)$$

$$\quad \quad \quad \quad \quad + (2k - 1)!! \sqrt{\frac{\pi}{2}} \left( \text{erf} \left( \frac{b}{\sqrt{2}} \right) - \text{erf} \left( \frac{a}{\sqrt{2}} \right) \right) \quad (F.14)$$

to obtain

$$\int_{-\infty}^{\infty} e^{-z^2/2}g_k(z)dz = \sqrt{2\pi} \sum_{t=0}^{[k/2]} c_{k,2t}(2t - 1)!! \quad (F.15)$$

and

$$A^{-1}g_k[z] = \left( \frac{e^{x^2/2}}{2} \text{erf} \left( \frac{x}{\sqrt{2}} \right) \int_{-\infty}^{\infty} e^{-z^2/2}g_k(z)dz \right) - g_{k-1}(x) - (\text{lower order polynomials}). \quad (F.16)$$
So then, with \( \mathbf{B} \) defined in (F.4), and using the orthogonality of the NM polynomials \( P_j \) we have

\[
\mathbf{B} = \begin{bmatrix}
-\Phi_{0,0} - p_0 + X_{0,1} & b_{0,2} & b_{0,3} \\
X_{1,0} - \Phi_{1,1} & -p_1 + X_{1,2} & b_{1,3} & \cdots \\
X_{2,0} - \Phi_{2,2} & 0 & b_{2,3} & \cdots \\
\vdots & \vdots & \ddots & \ddots \\
p_0 - X_{0,1} - \Phi_{0,1} & -\Phi_{1,1} & -p_1 + X_{1,2} & b_{1,3} & \cdots \\
p_0 - X_{0,1} - \Phi_{0,1} & 0 & -p_1 + X_{1,2} & \Phi_{1,2} & -b_{2,3} & \cdots \\
p_0 - X_{0,1} - \Phi_{0,1} & X_{2,0} + \Phi_{2,0} & p_1 - X_{1,2} & -p_2 + X_{2,3} & \cdots \\
p_0 - X_{0,1} - \Phi_{0,1} & X_{3,0} + \Phi_{3,0} & X_{3,1} + \Phi_{3,1} & p_2 - X_{2,3} & \cdots \\
\vdots & \vdots & \ddots & \ddots & \\
\end{bmatrix},
\]

(F.17)

where the \( b_{j,k} \) represent the currently unknown elements of \( \mathbf{B} \), and the second equality comes from the use of (78). Adding the matrix \( \Phi \) from (F.5) gives the (negative of the) anti-symmetric matrix \( \mathbf{V} \) from (18), allowing us to specify the remaining elements of \( \mathbf{B} \) as so

\[
\mathbf{B} + \Phi = -\mathbf{V}
\]

\[
\begin{bmatrix}
0 & -p_0 + X_{0,1} + \Phi_{0,1} & -X_{2,0} - \Phi_{2,0} & -X_{3,0} - \Phi_{3,0} \\
p_0 - X_{0,1} - \Phi_{0,1} & 0 & -p_1 + X_{1,2} + \Phi_{1,2} & -X_{3,1} - \Phi_{3,1} & \cdots \\
p_0 - X_{0,1} - \Phi_{0,1} & X_{2,0} + \Phi_{2,0} & p_1 - X_{1,2} - \Phi_{1,2} & 0 & -p_2 + X_{2,3} + \Phi_{2,3} & \cdots \\
p_0 - X_{0,1} - \Phi_{0,1} & X_{3,0} + \Phi_{3,0} & X_{3,1} + \Phi_{3,1} & p_2 - X_{2,3} - \Phi_{2,3} & 0 & \cdots \\
\vdots & \vdots & \ddots & \ddots & \ddots \\
\end{bmatrix},
\]

(F.18)

\[\text{F.2 } \beta = 1 \text{ Polynomials in the Classical Limit}\]

Similar to Appendix E.4 above we have the \( y \to \infty \) limit of the skew-inner product (8) as

\[
\langle f, g \rangle_1 = \frac{1}{2} \int_{-\infty}^{\infty} dx \ e^{-x^2/2} f(x) \int_{-\infty}^{\infty} dz \ e^{-z^2/2} g(z) \text{sgn}(z - x),
\]

(F.19)

with the associated skew-orthogonal polynomials obeying the equations

\[
\langle R_{2j}, R_{2k} \rangle_1 = \langle R_{2j+1}, R_{2k+1} \rangle_1 = 0
\]

and

\[
\langle R_{2j}, R_{2k+1} \rangle_1 = -\langle R_{2k+1}, R_{2j} \rangle_1 = r_j \delta_{j,k}.
\]

(F.20)

These polynomials are given explicitly [up to the invariance (29)] by [2,63]

\[
R_{2j}(x) = P_{2j}(x), \quad R_{2j+1}(x) = P_{2j+1}(x) - \frac{P_{2j}}{p_{2j-1}} P_{2j-1}(x)
\]

\[
= P_{2j+1}(x) - j P_{2j-1}(x),
\]

(F.21)

where the polynomials \( P_j(x) \) are the Hermite polynomials in (E.57) and \( p_j = p_j(\infty) \). The normalizations \( r_j = r_j(\infty) \) are from (28).
To check coherence between (83) and (F.21) we can use integration by parts, the identities (154) and
d\frac{d}{dx}\text{erf} \left( \frac{x}{\sqrt{2}} \right) = \sqrt{\frac{\pi}{2}} e^{-\frac{x^2}{2}} \tag{F.22}
to give us
\int_{-\infty}^{\infty} e^{-x^2/2} H_j(x) \text{erf} \left( \frac{x}{\sqrt{2}} \right) dx = \begin{cases} 2^{(j+2)/2} (j - 1)!!, & j \text{ odd}, \\ 0, & j \text{ even}. \end{cases} \tag{F.23}
Substitution into (20) yields
\left. X_{j,k} \right|_{y \to \infty} = \frac{1}{2j+k+1} \left( \int_{-\infty}^{\infty} H_j(x) e^{-x^2/2} \text{erf} \left( \frac{x}{\sqrt{2}} \right) dx \right) \int_{-\infty}^{\infty} e^{-z^2/2} H_k(z) dz
= \begin{cases} \Gamma \left( \frac{j+1}{2} \right) \Gamma \left( \frac{k+1}{2} \right) = p_k(\infty) \frac{\Gamma \left( \frac{j+1}{2} \right)}{\Gamma \left( \frac{k+2}{2} \right)}, & j \text{ odd } \land k \text{ even}, \\ 0, & \text{otherwise}, \end{cases} \tag{F.24}
where we used (155) for the integral over $H_k$. The second line (equalling zero) follows easily from the fact that the error function is an odd function and that $H_j(x)$ is an even or odd function depending on the parity of $j$. We will also make use of the formula
\[ p_j(\infty) = \Gamma \left( \frac{j+1}{2} \right) \Gamma \left( \frac{j+2}{2} \right), \tag{F.25} \]
which can be shown via Legendre’s duplication formula for gamma functions.
In the case that $y = \infty$ then from (19) the function $\Phi_{j,k} = 0$ and we also use (F.24) to find that the matrix $V_m$ in (18) has entries
\[ V_m = \begin{bmatrix} 0 & p_0 & 0 & X_{3,0} & 0 & X_{5,0} \\ -p_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & p_2 & 0 & X_{5,2} \\ -X_{3,0} & 0 & -p_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & p_4 \\ -X_{5,0} & 0 & -X_{5,2} & 0 & -p_4 & 0 \\ \vdots & \ddots \end{bmatrix} \tag{F.26} \]
meaning
\[ v_{j,k} = \begin{cases} p_j, & j \text{ even } \land k = j + 1, \\ 0, & j \text{ odd } \lor k \text{ even}, \\ X_{k,j}, & j \text{ even } \land k \text{ odd } \land j < k - 1, \end{cases} \tag{F.27} \]
with the anti-symmetry condition
\[ v_{j,k} = -v_{k,j}. \tag{F.28} \]
So $V_m$ is a sparse checkerboard matrix and it is known that Pfaffians of checkerboard matrices are equivalent to determinants of a condensed matrix (see, for example, [5] and [17]). Explicitly, for a matrix $\hat{A}_1 = [\alpha_{i,j}]_{i,j=1,\ldots,2N}$ we have
\[ \text{Pf} \hat{A}_1 = \det[\alpha_{2i-1,2j}]_{i,j=1,\ldots,N}, \tag{F.29} \]
and applying this to the matrix in (F.26) we obtain a determinant with zeroes everywhere below the diagonal, giving

\[ \text{Pf} V_{2j-1} = p_0 p_2 \cdots p_{2j-2}. \]  

(F.30)

For the numerator of \( \alpha_{j,k} \) we have four cases to consider, being the four possibilities given by the parities of \( j \) and \( k \).

\( \alpha_{2j,2k} \):

In the \( 2k \)-th column we have the matrix entries

\[ v_{s,2k} \mapsto v_{s,2j} = X_{2j,s} = 0 \quad (s < 2k) \]  

(F.31)

while in the \( 2k \)-th row we have

\[ v_{2k,t} \mapsto v_{2j,t} = -X_{2j,t} = 0 \quad (2k < t) \]  

(F.32)

so we have zeros above and to the right of the \((2k, 2k)\) entry (in the same column and row), which gives us

\[ \text{Pf} V_{2j-1}^{(2k\mapsto2j)} = 0, \]  

(F.33)

since at least one of these zero factors must appear in each term of the Pfaffian.

\( \alpha_{2j,2k+1} \):

Similar to the above, we have

\[ v_{s,2k+1} \mapsto v_{s,2j} = X_{2j,s} = 0 \quad (s < 2k + 1) \]  

(F.34)

and

\[ v_{2k+1,t} \mapsto v_{2j,t} = -X_{2j,t} = 0 \quad (2k + 1 < t). \]  

(F.35)

So now we have zeros above and to the right of the \((2k + 1, 2k + 1)\) entry, which gives us

\[ \text{Pf} V_{2j-1}^{(2k+1\mapsto2j)} = 0. \]  

(F.36)

\( \alpha_{2j+1,2k} \):

Now we have

\[ v_{s,2k} \mapsto v_{s,2j+1} = X_{2j+1,s} = 0 \quad (s < 2k \land s \text{ odd}) \]  

(F.37)

so we still have every odd row containing only zeros (in the upper triangle). Thus, as in (F.30), the only term in the Laplace expansion that could be non-zero is \( p_0 p_2 \cdots p_{2j-3} \). However,

\[ p_{2k} = v_{2k,2k+1} \mapsto v_{2j+1,2k+1} = -X_{2j+1,2k+1} = 0, \]  

(F.38)

and so

\[ \text{Pf} V_{2j-1}^{(2k\mapsto2j+1)} = 0. \]  

(F.39)

\( \alpha_{2j+1,2k+1} \):

Using the expressions (F.24) and (F.25) we have the identity

\[ X_{2m+1,2r} X_{2r+1,2n} = p_{2r} X_{2m+1,2n}, \quad (m > t > n), \]  

(F.40)

which we will make use of below. First we recall from (F.26) that in the upper triangle \( v_{j,k} \neq 0 \) only when \( j \) is even and when \( k \) is odd, which implies that all the even sites in the corresponding diagram connect to the right, and all the odd sites connect to the left. However, we will have an exception to this when we make the replacement \( 2k + 1 \mapsto 2j + 1 \).
Fig. 4 A general link diagram in the case that the vertex $2j + 1$ connects to the right. The left-pointing arrow on this link indicates that the corresponding matrix entry has row index larger than the column index (which is different to the convention on all other links). From (F.28) we see that this left-pointing arrow will introduce a negative sign.

Fig. 5 An example of the type of link diagrams possible with the restrictions in Fig. 4. The ellipses “...” denote identity links. The labels “e” and “o” denote generic even and odd vertices respectively.

Specifically, in terms of link diagrams there are two possibilities for the links involving site $2j + 1$: either $(2s, 2j + 1)$ or $(2j + 1, 2t)$ (so $2j + 1$ is either the right or left vertex of the link). We note that the other vertex must be even, since any odd-odd or even-even link results in $X_{\text{odd}, \text{odd}} = 0 = X_{\text{even}, \text{even}}$. It is easiest to consider the two cases separately:

(i) Assume $2j + 1$ connects to the left, that is we have a link $(2s, 2j + 1)$. Since all other odd sites connect left and all other even sites connect right, this must be the identity link diagram.

(ii) Assume $2j + 1$ connects to the right, that is we have a link $(2j + 1, 2t)$, then we must have identity links at sites to the left of $2k$ and to the right of $2t + 1$, as depicted in Fig. 4. [The left-pointing arrow on the edge $(2j + 1, 2t)$ indicates that the left vertex is greater than the right vertex, which is the opposite convention to all the other links, and this introduces a negative sign from (F.28).] In this case, we see from the diagram that there are 2 possible connections for $2t - 2$, and then another 2 possible connections for $2t - 4$, and so on. Thus there are $2^{t-k-1}$ link diagrams corresponding to Fig. 4.

Summing over the possible values of $t = k + 1, \ldots, j - 1$ in (ii), and adding the identity link pattern from (i), we have the number of valid link patterns on $N$ sites $L(N)$ given by

$$L(N) = 1 + \sum_{t=k+1}^{j-1} 2^{t-k-1} = 2^{j-k-1}. \quad (F.41)$$

So for $2k + 1 < 2j - 1$ we have an even number of terms in the Pfaffian, and it turns out that they all cancel.

To show this, note that the restriction that all odd vertices connect to the left and all even vertices connect to the right (except for $2j + 1$ and $2t$) means that a general link diagram must look like that in Fig. 5. That is, big interconnected links, with a large rainbow link $(2j + 1, \text{even})$, and interspersed with little links. The big links must interconnect at neighbouring sites, since otherwise we would have two neighbouring vertices pointing in the
The same direction, violating the even/right–odd/left rule. We can construct every diagram of the type in Fig. 5 by application of the equality (F.40), by recasting that equation into the link diagram equalities in Fig. 6, for the particular case when \( m = j \). In Fig. 6a note the link diagram on the right has a left-pointing arrow (implying that the row index is larger than the column index), and so from (F.28) we introduce a negative sign on the corresponding matrix entry. In Fig. 6b we have left-pointing arrows on both sides of the equality, but we have an additional sign introduced since the diagrams differ by an odd number of crossings.

In Fig. 7 we give the example of constructing the link diagram in Fig. 5 from the identity diagram by repeated application of equalities in Fig. 6—starting from the left at the link \((2k, 2j + 1)\) we first apply equality (a), and then, moving to the right, we repeatedly apply (b) until we have the final diagram. Each application of the equalities (a) and (b) introduces a negative sign.

In the identity diagram there are \( j - k - 1 \) little links to the right of site \( 2k + 1 \), so there are \( \binom{j-k-1}{p} \) link diagrams obtained from \( p \) uses of the equalities in Fig. 6, which gives us that

\[
P^f_{2j-1}^{(2k+1\rightarrow 2j+1)} = (p_0p_2\cdots p_{2k-2})v_{2k,2j+1}(p_{2k+2}\cdots p_{2j-2}) \sum_{p=0}^{j-k-1} (-1)^p \binom{j-k-1}{p} = 0 \quad \text{for } k < j - 1,
\]

where \((p_0p_2\cdots p_{2k-2})v_{2k,2j+1}(p_{2k+2}\cdots p_{2j-2})\) is the term from the identity link diagram (i.e. the top diagram in Fig. 7). The second equality follows since the sum of alternating binomial coefficients is equal to zero, which can be seen from the binomial expansion of \((x - y)^{j-k-1}\), with \( x, y \rightarrow 1 \). Thus \( a_{2j+1,2k+1} = 0 \) when \( k < j - 1 \).

From (F.41) we see the only scenario where we do not have an even number of cancelling link diagrams is when \( k = j - 1 \), and we have only the identity link pattern. In this case,
Fig. 7 Constructing the link diagram in Fig. 5 using the diagram equalities in Fig. 6. The labels to the left of each link diagram refer to which of the equalities in Fig. 6 was applied, and the sign of the corresponding term in the Pfaffian equation (F.42) becomes

\[ \text{Pf} \begin{pmatrix} 2j & 2j+1 \\ 2j-1 & 2j+2 \end{pmatrix} = p_0 p_2 \cdots p_{2j-4} X_{2j+1,2j-2} \]  

(F.43)

since \( v_{2j-2,2j-1} \mapsto v_{2j-2,2j+1} = X_{2j+1,2j-2} \). Substituting (F.43) and (F.30) (with \( m = 2j - 1 \)) into (83) we have

\[ a_{2j+1,2j-1} = -\frac{X_{2j+1,2j-2}}{p_{2j-2}} = -\frac{p_{2j}}{p_{2j-1}} = -\frac{\Gamma(j+1)}{\Gamma(j)} = -j, \]  

(F.44)

where we used (F.24) for the second equality. Combining this result with (F.33), (F.36), (F.39) and (F.42) we recover (F.21).

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