Anticipated mean-field backward stochastic differential equations with jumps

Tao Hao

School of Statistics, Shandong University of Finance and Economics, Jinan 250014, China
(e-mail: taohao@sdufe.edu.cn)

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Abstract. In this paper, we prove an existence and uniqueness theorem and a comparison theorem for a class of anticipated mean-field backward stochastic differential equations with jumps.

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1 Introduction

Stochastic delay differential equations (SDDEs) can be met frequently in the fields of finance, economics, and physics. Recently, stochastic optimal control problems for SDDEs have attracted an increasing attention. We refer to [7,8,9,13] for the maximum principle and to [10,11,12] for the dynamic programming principle and the probability interpretation of the related Hamilton–Jacobi–Bellman equations. As we know, when adopting the dual method to investigate necessary conditions of optimality for control systems with delay, the dual equations of the first-order variational equations are anticipated backward stochastic differential equations (BSDEs), which were first considered by Peng and Yang [18] in 2009. Later, many scholars dedicated themselves to studying this kind of equations, such as Yang and Elliot [19] and Lu and Ren [17].

On the other hand, mean-field BSDEs as the limit states of characterizing the asymptotic behavior of large stochastic particle systems with mean-field interaction when the size of the system becomes very large have also received a lot of attention; see, for example, [2, 3]. In particular, a recent series of works of Lions [16] (also see Cardaliaguet [5]) gave a huge impulse to investigate the general mean-field BSDEs with coefficients depending on the law of the solution, not the expectation; see [1,4,14,15].

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In this paper, we are interested in the following general anticipated mean-field BSDE with jumps:

\[-dY_t = f\left(t, Y_t, Z_t, \int_G K_t(e) l(e) \lambda(de), A_t, B_t, C_t, \overline{A}_t, \overline{B}_t, \overline{C}_t, \mathbf{P}_{II}\right) dt \]

\[- Z_t dW_t - \int_G K_t(e) N_\lambda(de, dt), \quad Y_T = \varphi_T, \quad Z_T = \phi_T, \quad K_t = \psi_t(\cdot), \quad t \in [T, T+M], \]

where \(A_t = Y_{t+\delta_1}, B_t = Z_{t+\delta_2}, C_t = \int G K_{t+\delta_2} l(e) \lambda(de), \overline{A}_t = \int_0^{\delta_1} e^{-\rho s} Y_{t+s} ds, \overline{B}_t = \int_0^{\delta_2} e^{-\rho s} Z_{t+s} ds, \overline{C}_t = \int_0^{\delta_1} e^{-\rho s} \int G K_{t+s} l(e) \lambda(de) ds, \Pi_t = (Y_t, Z_t, \int G K_t l(e) \lambda(de)), \varphi, \phi, \psi\) are given functions on \([T, T+M], W) is a d-dimensional Brownian motion, \(N_\lambda\) is a Poisson martingale measure, \(\mathbf{P}_\xi = \mathbf{P} \circ \xi^{-1}\) is the law (distribution) of a random variable \(\xi \in L^2(\Omega, \mathcal{F}, \mathbf{P})\), the mapping \(L : G \to \mathbb{R}\) satisfies \(0 < l(e) \leq C(1 + |e|)\) for some constant \(C > 0\). Here \(A_t, \overline{A}_t\) can be regarded as the counterparts of the one point delay and the average delay in the corresponding mean-field SDE with jumps. We call them the one point anticipated term and the average anticipated term, respectively; \(B_t, \overline{B}_t, C_t, \overline{C}_t\) can be understood similarly. We prove an existence and uniqueness theorem and a comparison theorem for one-dimensional anticipated mean-field BSDEs with jumps.

The motivation comes, on the one hand, from the rapid development of the theory of mean-field BSDEs and, on the other hand, from the necessity of studying the optimal control problems driven by mean-field SDEs with jumps or anticipated mean-field BSDEs with jumps.

Compared with the spermic work of Peng and Yang [18], the potential obstacle of this paper lies in involving the mean-field term and jump term, which means that we need more subtle calculation; see the proof of Lemma 1.

This paper is organized as follows. In Section 2, we recall the notion of the derivative in the Wasserstein space and some usual functional spaces. Section 3 is devoted to showing the existence and uniqueness theorem. The comparison theorem is supplied in Section 4.

## 2 Preliminaries

In this section, we introduce the differentiability of a function defined on \(\mathcal{P}_2(\mathbb{R}^d)\) and some usual spaces.

### 2.1 Derivative in the Wasserstein space

Let \(\mathcal{P}_2(\mathbb{R}^d)\) be the space of all probability measures over \((\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))\) with finite second-order moment endowed with the 2-Wasserstein’s distance

\[W_2(\nu_1, \nu_2) := \inf \left\{ \left( \int_{\mathbb{R}^{2d}} |a_1 - a_2|^2 \pi(a_1, a_2) \right)^{1/2}, \pi \in \mathcal{P}_2(\mathbb{R}^{2d}) \text{ with marginals } \nu_1 \text{ and } \nu_2 \right\}.\]

By \(\langle \cdot, \cdot \rangle\) we denote the “dual product” on \(L^2(\mathcal{F}; \mathbb{R}^d)\), and by \(\delta_\theta\) the Dirac measure at \(\theta\).

Let us now recall the notion of the differentiability of a function \(\varphi : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}\) in \(\nu \in \mathcal{P}_2(\mathbb{R}^d)\). Let \((\Omega, \mathcal{F}, \mathbf{P})\) be a “rich enough” space, which means that for each \(\nu \in \mathcal{P}_2(\mathbb{R}^d)\), there exists a random variable \(\xi \in L^2(\mathcal{F}; \mathbb{R}^d)\) such that \(\nu = \mathbf{P}_\xi\).

**Definition 1.** (See [16].) For \(\xi_0 \in L^2(\mathcal{F}; \mathbb{R}^d)\), we call the function \(\varphi : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}\) differentiable at \(\mathbf{P}_{\xi_0}\) if for all \(\xi \in L^2(\mathcal{F}; \mathbb{R}^d)\), the “lifted” function \(\varphi^#(\xi) := \varphi(\mathbf{P}_\xi)\) is differentiable at \(\xi_0\) in the Fréchet sense.
This means that there exists a continuous linear mapping \( D\varphi^\#(\xi_0) : L^2(\mathcal{F}; \mathbb{R}^d) \to \mathbb{R} \) such that for \( \zeta \in L^2(\mathcal{F}; \mathbb{R}^d) \),
\[
\varphi^\#(\xi_0 + \zeta) - \varphi^\#(\xi_0) = D\varphi^\#(\xi_0)(\zeta) + o(\|\zeta\|_{L^2}).
\] (2.1)
Riesz’ representation theorem allows us to show that there exists \( \eta_0 \in L^2(\mathcal{F}; \mathbb{R}^d) \) such that
\[
D\varphi^\#(\xi_0)(\zeta) = \langle \eta_0, \zeta \rangle.
\]
Cardaliaguet [5] proved that there exists a Borel-measurable function \( h \) depending on the law of \( \xi_0 \), not on the random variable \( \xi_0 \) itself, such that \( \eta_0 = h(\xi_0) \). Consequently, (2.1) can be described as
\[
\varphi(\mathbf{P}_{\xi_0 + \zeta}) - \varphi(\mathbf{P}_{\xi_0}) = \langle h(\xi_0), \zeta \rangle + o(\|\zeta\|_{L^2}), \quad \zeta \in L^2(\mathcal{F}; \mathbb{R}^d).
\]
We call \( \partial_\nu \varphi(\mathbf{P}_{\xi_0}; a) = h(a), a \in \mathbb{R}^d \), the derivative of \( \varphi : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R} \) at \( \mathbf{P}_{\xi_0} \). It is easy to see \( D\varphi^\#(\xi_0) = h(\xi_0) = \partial_\nu \varphi(\mathbf{P}_{\xi_0}; \xi_0) \).

In this paper, for convenience, we assume that all the functions \( \varphi^\# : L^2(\mathcal{F}; \mathbb{R}^d) \to \mathbb{R} \) are Fréchet differentiable over the whole space \( L^2(\mathcal{F}; \mathbb{R}^d) \), which naturally guarantees that the corresponding functions \( \varphi : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R} \) are differentiable in all probability measures from \( \mathcal{P}_2(\mathbb{R}^d) \). Note that in this situation, \( \partial_\nu \varphi(\mathbf{P}_\xi; a), \xi \in L^2(\mathcal{F}; \mathbb{R}^d), a \in \mathbb{R}^d \), exists \( \mathbf{P}_\xi(d\alpha) \)-a.e., and moreover, by [6, Lemma 3.2], if there exists a constant \( K > 0 \) such that for \( \xi_1, \xi_2 \in L^2(\mathcal{F}; \mathbb{R}^d) \),
\[
\mathbb{E}|\partial_\nu \varphi(\mathbf{P}_{\xi_1}; \xi_1) - \partial_\nu \varphi(\mathbf{P}_{\xi_2}; \xi_2)|^2 \leq K^2\mathbb{E}|\xi_1 - \xi_2|^2,
\]
then for all \( \xi \in L^2(\mathcal{F}; \mathbb{R}^d) \), there is a \( \mathbf{P}_\xi \)-version of \( \partial_\nu \varphi(\mathbf{P}_\xi; \cdot) : \mathbb{R}^d \to \mathbb{R}^d \) such that
\[
|\partial_\nu \varphi(\mathbf{P}_{\xi}; a) - \partial_\nu \varphi(\mathbf{P}_{\xi}; a')| \leq K|a - a'| \quad \text{for } a, a' \in \mathbb{R}^d.
\]

2.2 Functional spaces

Let \( T \) be a given time horizon, and let \( (\Omega, \mathcal{F}, \mathbf{P}; \mathbb{F} = \{\mathcal{F}_t\}_{t \in [0, T]} \) be a complete filtered probability space on which a \( d \)-dimensional Brownian motion \( W \) is defined. Let \( G \subseteq \mathbb{R} \) be a nonempty open set, equipped with its Borel \( \sigma \)-algebra \( \mathcal{B}(G) \), and let \( \lambda \) be a \( \sigma \)-finite Lévy measure on \( (G, \mathcal{B}(G)) \), that is, \( \int_G (1 \land |e|) \lambda(de) < +\infty \).

Let \( N \) be a Poisson random measure on \([0, T] \times G\), independent of the Brownian motion \( W \), with compensator \( \mu(de, dt) = \lambda(de) dt \) such that \( \{N_\lambda((s, t] \times B) = (N - \mu)((s, t] \times B), s \leq t, B \in \mathcal{B}(G) \) with \( \lambda(B) < +\infty \)\} is a martingale. By \( \mathcal{P} \) we denote the \( \sigma \)-field of \( \mathbb{R} \)-predictable subsets of \( \Omega \times [0, T] \).

We suppose that \( \mathbb{F} = \{\mathcal{F}_t\}_{t \in [0, T]} \) is the natural filtration generated by the Brownian motion \( W \) and Poisson random measure \( N \), augmented with an independent \( \sigma \)-algebra \( \mathcal{G}^0 \subset \mathcal{F} \), that is,
\[
\mathcal{F}_t = \sigma \{W_s, N([0, s] \times A) \mid s \leq t, A \in \mathcal{B}(G)\},
\]
\[
\mathcal{F}_t := \bigcap_{s \leq t} \mathcal{F}_s \vee \mathcal{G} \vee \mathcal{N}_\mathbf{P}, \quad t \in [0, T],
\]
where \( \mathcal{N}_\mathbf{P} \) is the set of all \( \mathbf{P} \)-null subsets, and \( \mathcal{G}^0 \subset \mathcal{F} \) has the following properties:

(i) \( \mathcal{G}_0 \) is independent of the Brownian motion \( W \) and the Poisson random measure \( N \);
(ii) \( \mathcal{G}_0 \) is “rich enough”, that is, \( \mathcal{P}_2(\mathbb{R}^d) = \{\mathbf{P}_\xi, \xi \in L^2(\mathcal{G}_0; \mathbb{R}^d)\} \).
We frequently use the following spaces:

\[ S^2_F(s, t; \mathbb{R}^d) := \left\{ \psi \mid \psi : \Omega \times [s, t] \to \mathbb{R}^d \text{ is an } \mathbb{F}\text{-adapted càdlàg process} \right\}, \]

with \( \mathbb{E}\left[ \sup_{s \leq r \leq t} |\psi_r|^2 \right] < +\infty \),

\[ \mathcal{H}^2_F(s, t; \mathbb{R}^d) := \left\{ \psi \mid \psi : \Omega \times [s, t] \to \mathbb{R}^d \text{ is an } \mathbb{F}\text{-predictable process} \right\}, \]

with \( \|\psi\|^2 := \mathbb{E}\left[ \int_s^t |\psi_r|^2 \, dr \right] < +\infty \),

\[ K^2_\lambda(s, t; \mathbb{R}^d) := \left\{ K \mid K : \Omega \times [s, t] \times G \to \mathbb{R}^d \text{ is } \mathbb{P} \otimes \mathcal{B}(G)\text{-measurable} \right\}, \]

with \( \|K\|^2 := \mathbb{E}\left[ \int_s^t \int_G |K_r(e)|^2 \lambda(de) \, dr \right] < +\infty \).

In what follows, by \( S^2_F(s, T + M), \mathcal{H}^2_F(s, T + M), \text{ and } K^2_\lambda(s, T + M) \) we shortly denote \( S^2_F(s, T + M; \mathbb{R}), \mathcal{H}^2_F(s, T + M; \mathbb{R}), \text{ and } K^2_\lambda(s, T + M; \mathbb{R}) \).

### 3 Existence and uniqueness theorem

In this section, we prove the existence and uniqueness theorem for Eq. (1.1).

Let \( \delta_i : [0, T] \to \mathbb{R}^+, i = 1, 2, 3 \), satisfy:

(H1) (i) There exists a constant \( M > 0 \), such that for \( t \in [0, T] \),

\[ t + h(t) \leq T + M, \quad h = \delta_1, \delta_2, \delta_3; \]

(ii) There exists a constant \( L > 0 \), such that for \( t \in [0, T] \) and all nonnegative integrable functions \( \Gamma(\cdot) \) and \( \overline{\Gamma}(\cdot, \cdot) \),

\[
\int_t^T \Gamma(s + \delta_i(s)) \, ds \leq L \int_t^{T+M} \Gamma(s) \, ds, \quad i = 1, 2, \tag{3.1}
\]

\[
\int_t^T \int_G \overline{\Gamma}(s + \delta_3(s), e) \lambda(de) \, ds \leq L \int_t^{T+M} \int_G \overline{\Gamma}(s, e) \lambda(de) \, ds.
\]

Let the mapping

\[ f(\omega, s, y, z, k, \xi, \eta, \zeta, \xi, \eta, \zeta, \nu) : \]

\[ \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \times \left( L^2(\mathcal{F}_{r_1}; \mathbb{R}) \times L^2(\mathcal{F}_{r_2}; \mathbb{R}^d) \times L^2(\mathcal{F}_{r_3}; \mathbb{R}) \right)^2 \times \mathcal{P}_2(\mathbb{R}^{(1+d+1)}) \]

\[ \to L^2(\mathcal{F}_s; \mathbb{R}), \]

\[ r_1, r_2, r_3 \in [s, T + M], \] satisfy:
\(\text{(H2)}\) (i) There exists a constant \(C > 0\) such that for all \(s \in [0, T]\), \(y, y' \in \mathbb{R}, z, z' \in \mathbb{R}^d, k, k' \in \mathbb{R}, \xi, \xi', \zeta, \zeta', \zeta'' \in \mathcal{H}_F^2(s, T + M), \eta, \eta', \eta'' \in \mathcal{H}_F^2(s, T + M; \mathbb{R}^d), \zeta, \zeta', \zeta'' \in \mathcal{H}_F^2(s, T + M), \) \(r_1, r_2, r_3 \in [s, T + M]\), and \(\nu, \nu' \in \mathcal{P}_2(\mathbb{R}^{1+d+1}), \mathcal{P}\) \(\)-a.s.,

\[
\left| f(s, y, z, k, \xi_{r_1}, \eta_{r_1}, \zeta, \zeta'_{r_1}, \xi', \eta'_{r_1}, \zeta, \zeta''_{r_1}, \nu) - f(s, y', z', k', \xi'_{r_1}, \eta'_{r_1}, \zeta, \zeta''_{r_1}, \xi', \eta'_{r_1}, \zeta, \zeta''_{r_1}, \nu') \right|
\leq C \left[ |y - y'| + |z - z'| + |k - k'| + E^F \left[ |\xi_{r_1} - \xi'_{r_1}| + |\eta_{r_1} - \eta'_{r_1}| + |\zeta - \zeta'| \right]
+ |\xi'_{r_1} - \xi''_{r_1}| + |\eta'_{r_1} - \eta''_{r_1}| + |\zeta - \zeta''| \right] + W_2(\nu, \nu');
\]

(ii) \(\mathbb{E} \left[ \int_0^T |f(s, 0, 0, 0, 0, 0, 0, 0, 0, \delta_0)|^2 \, ds \right] < +\infty\), where \(\delta_0\) denotes the Dirac measure at the \((1 + d + 1)\)-dimensional zero vector;

(iii) There exists a constant \(C > 0\) such that the mapping \(l : G \rightarrow \mathbb{R}\) satisfies \(0 < l(e) \leq C(1 \wedge |e|)\).

**Theorem 1.** Let assumptions \((\text{H1})\) and \((\text{H2})\) be satisfied, let \(\varphi, \in \mathcal{S}_F^2(T, T + M), \phi, \in \mathcal{H}_F^2(T, T + M; \mathbb{R}^d), \) and \(\psi, \in K^2_\lambda(T, T + M), \) and let the anticipated mean-field BSDE \((1.1)\) possess a unique adapted solution 

\[(Y, Z, K) \in \mathcal{S}_F^2(0, T + M) \times \mathcal{H}_F^2(0, T + M; \mathbb{R}^d) \times K^2_\lambda(0, T + M).\]

**Proof.** For \((y, z, k) \in \mathcal{H}_F^2(0, T + M) \times \mathcal{H}_F^2(0, T + M; \mathbb{R}^d) \times K^2_\lambda(0, T + M),\) we define the following norms:

\[
\| (y, z, k) \|_\beta := \mathbb{E} \int_0^{T+M} e^{\beta s} \left( |y_s|^2 + |z_s|^2 + \int_G |k_s(e)|^2 \lambda(de) \right) \, ds, \quad \beta > 0,
\]

under which the contractive mapping theorem can be applied more expediently.

Let us consider the equation

\[
-dY_t = f \left( t, y_t, z_t, \int_G k_t(e) l(e) \lambda(de), a_t, b_t, c_t, \bar{a}_t, \bar{b}_t, \bar{c}_t, \Pi_t, P, \pi_t \right) \, dt
- Z_t \, dW_t - \int_G K_t(e) N_\lambda(de, dt), \quad t \in [0, T],
\]

\[
Y_t = \varphi_t, \quad Z_t = \phi_t, \quad K_t = \psi_t(\cdot), \quad t \in [T, T + M],
\]

where \((a_t, b_t, c_t, \bar{a}_t, \bar{b}_t, \bar{c}_t, \pi_t)\) are defined similar to \((A_t, B_t, C_t, \overline{A}_t, \overline{B}_t, \overline{C}_t, \Pi_t)\) in \((1.1)\), but with \((y, z, k)\) instead of \((Y, Z, K)\).

It is easy to check that Eq. \((3.2)\) has a unique solution

\[(Y, Z, K) \in \mathcal{S}_F^2(0, T + M) \times \mathcal{H}_F^2(0, T + M; \mathbb{R}^d) \times K^2_\lambda(0, T + M),\]

from which we can define a mapping \(\Phi : \mathcal{H}_F^2(0, T + M) \times \mathcal{H}_F^2(0, T + M; \mathbb{R}^d) \times K^2_\lambda(0, T + M) \rightarrow \mathcal{H}_F^2(0, T + M) \times \mathcal{H}_F^2(0, T + M; \mathbb{R}^d) \times K^2_\lambda(0, T + M)\) such that

\[\Phi(y, z, k) = (Y, Z, K).\]

Let us now show that \(\Phi\) is a strictly contractive mapping for some suitable \(\beta > 0\). To this end, let \((y^i, z^i, k^i) \in \mathcal{H}_F^2(0, T + M) \times \mathcal{H}_F^2(0, T + M; \mathbb{R}^d) \times K^2_\lambda(0, T + M)\) and \((Y^i, Z^i, K^i) = \Phi(y^i, z^i, k^i), i = 1, 2.\) By \(\Delta Y\) we denote the difference of \(Y^1\) and \(Y^2\), with \(\Delta Z, \Delta K, \Delta y, \Delta z, \Delta k\) having the similar meanings.

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Applying Itô’s formula to \(e^{\beta s}|\Delta Y_s|^2\), we have
\[
de^{\beta s}|\Delta Y_s|^2 = \beta e^{\beta s}|\Delta Y_s|^2 \, ds + e^{\beta s}2 \Delta Y_s \Delta Z_s + e^{\beta s}|\Delta Z_s|^2 \, ds
\]
\[
+ \int_G e^{\beta s}|\Delta K_s(e)|^2 \lambda(\text{de}) \, ds + \int_G e^{\beta s}|\Delta K_s(e)|^2 N_\lambda(\text{de}, ds)
\]
\[
e^{\beta s} \left( \beta |\Delta Y_s|^2 + |\Delta Z_s|^2 + \int |\Delta K_s(e)|^2 \lambda(\text{de}) \right) \, ds
\]
\[
- e^{\beta s}2 \Delta Y_s \Delta f(s) \, ds + e^{\beta s}2 \Delta Y_s \Delta Z_s \, dW_s
\]
\[
+ e^{\beta s}2 \Delta Y_s - \int_G \Delta K_s(e) N_\lambda(\text{de}, ds) + \int_G e^{\beta s}\Delta K_s(e)^2 N_\lambda(\text{de}, ds),
\]
where
\[
\Delta f(s) = f \left( t, y^1_t, z^1_t, \int G k^1(e) l(e) \lambda(\text{de}), a^1_t, b^1_t, c^1_t, \bar{a}^1_t, \bar{b}^1_t, \bar{c}^1_t, P_{\pi^1} \right)
\]
\[
- f \left( t, y^2_t, z^2_t, \int G k^2(e) l(e) \lambda(\text{de}), a^2_t, b^2_t, c^2_t, \bar{a}^2_t, \bar{b}^2_t, \bar{c}^2_t, P_{\pi^2} \right).
\]
\[
\pi^i_t = \left( y^i_t, z^i_t, \int G k^i(e) l(e) \lambda(\text{de}) \right), \quad i = 1, 2.
\]
Integrating from \(t\) to \(T\) and then taking conditional expectation, we get
\[
e^{\beta t}|\Delta Y_t|^2 + \mathbb{E}^{F_t} \left[ \int_t^T e^{\beta s} \left( \beta |\Delta Y_s|^2 + |\Delta Z_s|^2 + \int G |\Delta K_s(e)|^2 \lambda(\text{de}) \right) \, ds \right]
\]
\[
= \mathbb{E}^{F_t} \left[ \int_t^T e^{\beta s}2 \Delta Y_s \Delta f(s) \, ds \right]. \tag{3.3}
\]
In particular, as \(t = 0\),
\[
\mathbb{E} \left[ \int_0^T e^{\beta s} \left( \frac{\beta}{2} |\Delta Y_s|^2 + |\Delta Z_s|^2 + \int G |\Delta K_s(e)|^2 \lambda(\text{de}) \right) \, ds \right]
\]
\[
\leq \frac{2C^2}{\beta} \mathbb{E} \left[ \int_0^T e^{\beta s} \left( |\Delta y_s| + |\Delta z_s| + \int G |\Delta k_s(e)^l(e) \lambda(\text{de})| + \mathbb{E}^{F_s} \left[ |\Delta y_{s+\delta_1(s)}| + |\Delta z_{s+\delta_2(s)}| \right]
\right.
\]
\[
+ \int G |\Delta k_{s+\delta_1(s)}(e)^l(e) \lambda(\text{de})| + \int_0^{\delta_1(s)} e^{-\rho r} |\Delta y_{s+r}| \, dr + \int_0^{\delta_2(s)} e^{-\rho r} |\Delta z_{s+r}| \, dr
\]
\[
+ \left. \int_0^{\delta_3(s)} e^{-\rho r} |\Delta k_{s+r}(e)^l(e) \lambda(\text{de})| \, dr \right] + W_2(P_{\pi^1}, P_{\pi^2})^2 \, ds \right].
\]
By the Hölder inequality, since $W_2(P_\xi, P_\eta) \leq \{E[|\xi - \eta|^2]\}^{1/2}$, we have

$$\begin{align*}
\mathbb{E} \left[ \int_0^T e^{\beta s} \left( \frac{\beta}{2} |\Delta Y_s|^2 + |\Delta Z_s|^2 + \int_G |\Delta k_s(e)|^2 \lambda(de) \right) ds \right] \\
\leq \frac{20C^2}{\beta} \mathbb{E} \left[ \int_0^T e^{\beta s} \left( |\Delta y_s|^2 + |\Delta z_s|^2 + \int_G |\Delta k_s(e) l(e)\lambda(de)|^2 + |\Delta y_{s+\delta_2(s)}|^2 \\
+ |\Delta z_{s+\delta_2(s)}|^2 + \int_G |\Delta k_{s+\delta_2(s)}(e) l(e)\lambda(de)|^2 + \int_0^s e^{-\rho r} |\Delta y_{s+r}|^2 dr \\
+ \int_0^s e^{-\rho r} |\Delta z_{s+r}|^2 dr + \int_0^s e^{-\rho r} \int_G |\Delta k_{s+r}(e) l(e)\lambda(de)| dr \right)^2 \\
+ \mathbb{E} \left[ |\Delta y_s|^2 + |\Delta z_s|^2 + \int_G |\Delta k_s(e) l(e)\lambda(de)|^2 \right) ds \right]. \quad (3.4)
\end{align*}$$

From (3.1) and the Hölder inequality it is clear that

$$\begin{align*}
\int_0^T e^{\beta s} \int_G |\Delta k_{s+\delta_3(s)}(e) l(e)\lambda(de)|^2 ds \\
\leq \int_G |l(e)|^2 \lambda(de) \cdot \int_0^T \int_G e^{\beta s} |\Delta k_{s+\delta_3(s)}(e)|^2 \lambda(de) ds \\
\leq L \int_G |l(e)|^2 \lambda(de) \int_0^{T+M} e^{\beta s} |\Delta k_s(e)|^2 \lambda(de) ds. \quad (3.5)
\end{align*}$$

Moreover, noting that $\delta_3(s) \leq T + M$, by the Hölder inequality we get

$$\begin{align*}
\int_0^T e^{\beta s} \int_0^s e^{-\rho r} \int_G |\Delta k_{s+r}(e) l(e)\lambda(de)| dr \frac{\delta_3(s)}{ds} \\
\leq \int_0^T e^{\beta s} \int_0^s e^{-\rho r} dr \int_0^s \left( \int_G |\Delta k_{s+r}(e) l(e)\lambda(de)| \right) dr ds \\
\leq \frac{1}{2\rho} \left( 1 - e^{-2\rho(T+M)} \right) \int_G |l(e)|^2 \lambda(de) \cdot \int_0^{T+M} \int_0^s e^{\beta s} |\Delta k_{s+r}(e)|^2 \lambda(de) dr ds. \quad (3.6)
\end{align*}$$
Denote $C_{T,M} = (1 - e^{-2\rho(T+M)})/(2\rho)$. Since $s \leq T \leq T + r$, we have

$$
\int_0^T e^{\beta s} \left| \int_0^T e^{-\rho r} \int_G \Delta k_{s+r}(e) l(e) \lambda(de) \, dr \right|^2 ds \leq C_{T,M} \int_G |l(e)|^2 \lambda(de) \int_0^T e^{\beta T} \int_0^T e^{\beta(u-s)} \int_G |\Delta k_u(e)|^2 \lambda(de) \, du \, ds.
$$

Letting $u = s + r$, from $\delta_\beta(s) \leq T + M$ we have

$$
\int_0^T e^{\beta s} \left| \int_0^T e^{-\rho r} \int_G \Delta k_{s+r}(e) l(e) \lambda(de) \, dr \right|^2 ds \leq C_{T,M} e^{\beta T} \int_G |l(e)|^2 \lambda(de) \int_0^T e^{\beta(u-s)} \int_0^{T+M} e^{\beta u} |\Delta k_u(e)|^2 \lambda(de) \, du \, ds.
$$

Hence from (3.4), (3.5), and (3.8), by the argument of calculating (3.5) and (3.8) it follows that

$$
\mathbb{E} \left[ \int_0^{T+M} e^{\beta s} \left( \frac{\beta}{2} |\Delta Y_s|^2 + |\Delta Z_s|^2 + \int_G |\Delta K_s(e)|^2 \lambda(de) \right) \, ds \right] \leq \frac{20C^2}{\beta} \left( 2 + \left( 1 + \int_G |l(e)|^2 \lambda(de) \right) (L + C_{T,M} e^{\beta T}) \right) \times \mathbb{E} \left[ \int_0^{T+M} e^{\beta s} \left( |\Delta y_s|^2 + |\Delta z_s|^2 + \int_G |\Delta k_s(e)|^2 \lambda(de) \right) \, ds \right].
$$

Then choosing $\beta = 40C^2 (2 + (1 + \int_G |l(e)|^2 \lambda(de))(L + C_{T,M} e^{\beta T}) + 2$, we obtain

$$
\mathbb{E} \left[ \int_0^{T+M} e^{\beta s} \left( |\Delta Y_s|^2 + |\Delta Z_s|^2 + \int_G |\Delta K_s(e)|^2 \lambda(de) \right) \, ds \right] \leq \frac{1}{2} \mathbb{E} \left[ \int_0^{T+M} e^{\beta s} \left( |\Delta y_s|^2 + |\Delta z_s|^2 + \int_G |\Delta k_s(e)|^2 \lambda(de) \right) \, ds \right],
$$

which means that $\Phi$ is a strictly contractive mapping. Hence Eq. (1.1) has a unique solution $(Y, Z, K) \in \mathcal{H}_2^2(0, T + M) \times \mathcal{H}_2^2(0, T + M; \mathbb{R}^d) \times \mathcal{K}_2^2(0, T + M)$. Moreover, observing (3.3), by a similar argument and the Burkholder–Davis–Gundy inequality we can check that $Y \in \mathcal{S}_2^2(0, T + M)$. The proof is completed. \qed
Proposition 1. Let assumptions (H1) and (H2) be satisfied. Then there exists a constant \( L_0 \) depending on \( L, C, \) and \( T, \) such that for \( (\varphi, \phi, \psi) \in S_2^\delta(T, T + M) \times H_2^\delta(T, T + M; \mathbb{R}^d) \times K_\lambda^2(T, T + M), \) and \( t \in [0, T], \)

\[
E \left[ \sup_{t \leq s \leq T} |Y_s|^2 + \int_t^T |Z_s|^2 \, ds + \int_t^T \int G |K_s(e)|^2 \lambda(de) \, ds \right]
\]

\[
\leq L_0 E \left[ |\varphi_T|^2 + \int_T^{T+M} \left( |\varphi_s|^2 + |\phi_s|^2 + \int_G |\psi_s(e)|^2 \lambda(de) \right) ds \right.
\]

\[
+ \int_t^T \left| f(s, 0, 0, 0, 0, 0, 0, 0, 0, 0, \delta_0) \right|^2 ds ,
\]

where \( \delta_0 \) is given in (H2).

Proof. The proof is standard; refer to [18, Prop. 4.4]. \( \square \)

4 Comparison theorem

Let us now analyze the comparison theorem for Eq. (1.1). First, Peng and Yang [18] have stated with two examples that the comparison theorem of anticipated BSDEs does not hold when the coefficient \( f \) is decreasing in the anticipated term of \( Y \) and depends on the anticipated term of \( Z. \) Second, if the coefficient \( f \) depends on the mean-field term of \( Z \) or \( f \) is decreasing with respect to the mean-field term of \( Y, \) then the comparison theorem of mean-field BSDEs also becomes invalid; see a counterexample in [3]. Therefore here we just consider the comparison theorem of a class of anticipated mean-field BSDEs with jumps. Let us introduce it in detail.

We assume that for \( r_1 \in [s, T + M], \) the mapping

\[
f(\omega, s, y, z, k, \xi, \nu) : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \times L^2(\mathcal{F}_{r_1}; \mathbb{R}) \times L^2(\mathcal{F}_{r_1}; \mathbb{R}) \times \mathcal{P}_2(\mathbb{R}) \to L^2(\mathcal{F}_s; \mathbb{R})
\]

satisfies (H2).

Let us consider the following anticipated mean-field BSDE:

\[
-dY_t = f(t, Y_t, Z_t, \int_G K_t(e) l(e) \lambda(de), A_t, \overline{A}_t, \mathbf{P}_Y) \, dt - Z_t \, dW_t - \int_G K_t(e) N_\lambda(de, dt),
\]

\[
Y_t = \varphi_t, \quad Z_t = \phi_t, \quad K_t = \psi_t(\cdot), \quad t \in [T, T + M],
\]

(4.1)

where \( A_t \) and \( \overline{A}_t \) are given in (1.1).

To prove the comparison theorem for (4.1), let us first investigate the comparison theorem for general mean-field BSDEs with jumps.

Lemma 1. Let \( f_i : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}, i = 1, 2, \) be the drivers. Suppose that there exists a constant \( C > 0 \) such that the derivatives of \( f_1 \) with respect to \( \nu \) and \( k \) are positive and bounded by \( C > 0, \) that is, \( 0 < \partial_k f_1 \leq C \) and \( 0 < \partial_\nu f_1 \leq C. \) Let \( (Y^i, Z^i, K^i), i = 1, 2, \) be the solutions of the following
mean-field BSDEs with jumps:

\[-dY_t^i = f_i\left(t, Y_t^i, Z_t^i, \int G K_i^j(e) l(e) \lambda(\text{d}e), \mathbb{P}_{Y_t^i}\right) \text{d}t - Z_t^i \text{d}W_t - \int G K_i^j(e) N_\lambda(\text{d}e, \text{d}t),\]

\[Y_T^i = \varphi_T^i.\]

If \(f_1 \geq f_2\) and \(\varphi_T^1 \geq \varphi_T^2\), then \(Y_T^1 \geq Y_T^2\) for a.e. \(t \in [0, T]\), a.s.

**Proof.**

We denote \(\Delta Y = Y^2 - Y^1, \Delta Z = Z^2 - Z^1, \Delta K = K^2 - K^1, \Delta \varphi = \varphi^2 - \varphi^1\). Then

\[-d\Delta Y_s = \left(\delta f(s) + \alpha_y(s) \Delta Y_s + \alpha_z(s) \Delta Z_s + \int G \alpha_k(s) \Delta K_s l(e) \lambda(\text{d}e) + \widehat{E}[\alpha_\nu(s) \Delta \tilde{Y}_s]\right) \text{d}s\]

\[\quad - \Delta Z_s \text{d}W_s - \int G \Delta K_s(e) N_\lambda(\text{d}e, \text{d}s), \quad s \in [0, T],\]

\[\Delta Y_T = \Delta \varphi_T,\]

where for \(\ell = y, z, k,\)

\[\delta f(s) := f_2\left(s, Y_s^2, Z_s^2, \int G K_s^2(e) l(e) \lambda(\text{d}e), \mathbb{P}_{Y_s^2}\right) - f_1\left(s, Y_s^1, Z_s^1, \int G K_s^1(e) l(e) \lambda(\text{d}e), \mathbb{P}_{Y_s^1}\right),\]

\[\alpha_\ell(s) := \int_0^1 \frac{\partial f_1}{\partial \ell}\left(s, Y_s^1 + \rho(Y_s^2 - Y_s^1), Z_s^1 + \rho(Z_s^2 - Z_s^1), \int G K_s^1(e) + \rho(K_s^2(e) - K_s^1(e)) l(e) \lambda(\text{d}e), \mathbb{P}_{Y_s^1 + \rho(Y_s^2 - Y_s^1)}\right) \text{d}\rho,\]

\[\tilde{\alpha}_\nu(s) := \int_0^1 \frac{\partial f_1}{\partial \nu}\left(s, Y_s^1 + \rho(Y_s^2 - Y_s^1), Z_s^1 + \rho(Z_s^2 - Z_s^1), \int G K_s^1(e) + \rho(K_s^2(e) - K_s^1(e)) l(e) \lambda(\text{d}e), \mathbb{P}_{Y_s^1 + \rho(Y_s^2 - Y_s^1)}; \tilde{Y}_s^1 + \rho(\tilde{Y}_s^2 - \tilde{Y}_s^1)\right) \text{d}\rho.\]

Obviously, from (H2) it follows that \(|\alpha_\ell(s)| \leq C, s \in [0, T]|.

Applying Itô’s formula to \(((\Delta Y_t)^+)^2\), we obtain

\[\left((\Delta Y_t)^+\right)^2 + \int_t^T 1_{\{\Delta Z_s > 0\}} |\Delta Z_s|^2 \text{d}s\]

\[+ \int_t^T \int G \left(((\Delta Y_{s-} + \Delta K_s(e))^+)^2 - ((\Delta Y_{s-})^+)^2 - 21_{\{\Delta Y_s > 0\}} \Delta Y_{s-} \Delta K_s(e)\right) N(\text{d}e, \text{d}s)\]
\[ = (\Delta Y_T)^+)^2 + \int_t^T 2\mathbf{1}_{\{\Delta Y_s > 0\}} \alpha_y(s) \Delta Y_s + \alpha_z(s) \Delta Z_s + \int_G \alpha_k(s) \Delta K_s(e) l(e) \lambda(de) \]

\[ + \hat{E}[\hat{\alpha}_\nu(s) \Delta Y_s] + \delta f(s) \} ds \]

\[- \int_t^T 2\mathbf{1}_{\{\Delta Y_s > 0\}} \Delta Y_s \Delta Z_s ds - \int_t^T \int_G 2\mathbf{1}_{\{\Delta Y_s > 0\}} \Delta Y_s \Delta K_s(e) N\lambda(de, ds). \] (4.2)

Taking expectation on both sides of (4.2) and noting that \( \delta f(s) \leq 0 \) and \( \Delta Y_T \leq 0 \), we have

\[ \mathbb{E}\left[ (\Delta Y_t)^+)^2 + \int_t^T 1_{\{\Delta Y_s > 0\}} |\Delta Z_s|^2 ds \right] \]

\[ + \int_t^T \int_G ((\Delta Y_s + \Delta K_s(e))^+)^2 - ((\Delta Y_s)^+)^2 - 2(\Delta Y_s^+ \Delta K_s(e)) \lambda(de) ds \]

\[ \leq \mathbb{E}\left[ \int_t^T 2\mathbf{1}_{\{\Delta Y_s > 0\}} \Delta Y_s \left\{ \alpha_y(s) \Delta Y_s + \alpha_z(s) \Delta Z_s \right\} \right. \]

\[ + \left. \int_G \alpha_k(s) \Delta K_s(e) l(e) \lambda(de) + \hat{E}[\hat{\alpha}_\nu(s) \Delta Y_s] \} ds \right]. \] (4.3)

On the other hand, from the boundedness of \( \alpha_z(s) \) and the inequality \( 2ab \leq 2a^2 + b^2/2 \) it follows that

\[ \mathbb{E}\left[ \int_t^T 2\mathbf{1}_{\{\Delta Y_s > 0\}} \Delta Y_s \alpha_z(s) \Delta Z_s ds \right] \]

\[ \leq 2C^2 \mathbb{E}\left[ \int_t^T (\Delta Y_s)^+)^2 ds \right] + \frac{1}{2} \mathbb{E}\left[ \int_t^T 1_{\{\Delta Y_s > 0\}} |\Delta Z_s|^2 ds \right]. \] (4.4)

Moreover, from Jensen’s inequality and the assumption \( 0 < \partial_\nu f_1 \leq C \) we obtain

\[ \mathbb{E}\left[ \int_t^T 2\mathbf{1}_{\{\Delta Y_s > 0\}} \Delta Y_s \hat{\alpha}_\nu(s) \Delta Y_s ds \right] \]

\[ \leq 2CE \int_t^T \mathbf{1}_{\{\Delta Y_s > 0\}} \Delta Y_s \mathbb{E}[(\Delta Y_s)^+] \leq 2CE \left[ \int_t^T (\Delta Y_s)^+)^2 ds \right]. \] (4.5)

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Combining (4.3), (4.4), and (4.5), we can rewrite (4.3) as

\[
E \left[ (\Delta Y_t^+)^2 \right] + \frac{1}{2} \int_t^T 1_{\{\Delta Y_s^+ > 0\}} |\Delta Z_s|^2 \, ds
\]

\[
+ \int_t^T \int_G \left( \left( (\Delta Y_s^+ + \Delta K_s(e))^2 - (\Delta Y_s^+)^2 - 2(\Delta Y_s^+ \Delta K_s(e)) \lambda(de) \right) \right) \, ds
\]

\[
\leq (4C + 2C^2) E \left[ \int_t^T (\Delta Y_s^+)^2 \, ds \right]
\]

\[
+ E \left[ \int_t^T \int_G 21_{\{\Delta Y_s^+ > 0\}} \Delta Y_s \Delta K_s(e) \alpha_k(s) l(e) \lambda(de) \, ds \right].
\] (4.6)

For convenience, we denote

\[
A = \{(s, \omega) \mid \Delta Y_s > 0\} \quad \text{and} \quad B = \{(s, \omega) \mid \Delta Y_s + \Delta K_s(e) > 0\}.
\]

It is easy to check that

\[
E \left[ \int_t^T \int_G \left( \left( (\Delta Y_s^+ + \Delta K_s(e))^2 - (\Delta Y_s^+)^2 - 2(\Delta Y_s^+ \Delta K_s(e)) \lambda(de) \right) \right) \, ds \right]
\]

\[
= E \left[ \int_t^T \int_G (1_{AB} + 1_{A^c}) (\Delta Y_s^+ + \Delta K_s(e))^2
\]

\[
- (1_{AB} + 1_{A^c})( (\Delta Y_s^2 + 2\Delta Y_s \Delta K_s(e)) \lambda(de) \, ds
\]

\[
\geq E \left[ \int_t^T \int_G 1_{AB} (\Delta K_s(e))^2 - 1_{AB^c} ( (\Delta Y_s^2 + 2\Delta Y_s \Delta K_s(e)) \lambda(de) \, ds. \right. \] (4.7)

Combining (4.6) with (4.7), we have

\[
E \left[ (\Delta Y_t^+)^2 \right] + \frac{1}{2} E \left[ \int_t^T 1_A |\Delta Z_s|^2 \, ds \right] + E \left[ \int_t^T 1_B (\Delta K_s(e))^2 \lambda(de) \, ds \right]
\]

\[
+ E \left[ \int_t^T \int_G 1_{AB^c} \left( - (\Delta Y_s)^2 - 2\Delta Y_s \Delta K_s(e) \right) (1 + \alpha_k(s) l(e)) \lambda(de) \, ds \right]
\]

\[
\leq (4C + 2C^2) E \left[ \int_t^T (\Delta Y_s^+)^2 \, ds \right] + E \left[ \int_t^T \int_G 21_{AB} \Delta Y_s \Delta K_s(e) \alpha_k(s) l(e) \lambda(de) \, ds \right].
\]
By the boundedness assumption of $\alpha_k(s)$, the Hölder inequality, and the inequality $2ab \leq 2a^2 + b^2/2$, we have

$$E\left[ \left( (\Delta Y^i_t)^+ \right)^2 \right] + \frac{1}{2} E \left[ \int_t^T 1_A |\Delta Z_s|^2 \, ds \right] + \frac{1}{2} E \left[ \int_t^T 1_{AB} |\Delta K_s(e)|^2 \lambda(de) \, ds \right]$$

$$+ E \left[ \int_t^T \int_{G} 1_{AB'} \Delta Y_s \Delta K_s(e) \left( 1 + \alpha_k(s)l(e) \right) \lambda(de) \, ds \right]$$

$$\leq \left( 4C + 2C^2 + 2C^2 \int_{G} |l(e)|^2 \lambda(de) \right) E \left[ \int_t^T \left( (\Delta Y^i_t)^+ \right)^2 \, ds \right].$$

We claim that $\Gamma := E \left[ \int_t^T \int_{G} 1_{AB'} \Delta Y_s \Delta K_s(e) \left( 1 + \alpha_k(s)l(e) \right) \lambda(de) \, ds \right] \geq 0$. Indeed, for all $e \in G$ and $(s, \omega) \in AB'$, we have $0 < \Delta Y_s \leq -\Delta K_s(e)$. The nonnegativity assumptions on $\partial_k f_1$ and $l$ easily imply that $\Gamma \geq 0$. Hence

$$E \left[ \left( (\Delta Y^i_t)^+ \right)^2 \right] \leq \left( 4C + 2C^2 + 2C^2 \int_{G} |l(e)|^2 \lambda(de) \right) E \left[ \int_t^T \left( (\Delta Y^i_t)^+ \right)^2 \, ds \right],$$

which, together with the Gronwall lemma, shows the desired result. \(\square\)

Let us state the second main result of this paper, a comparison theorem. We make an extra assumption:

(H3) Let $f_i$, $i = 1, 2$, be two drivers of (4.1) that satisfy:

(i) $f_2(t, y, z, k, \xi_r, \zeta_r, \nu) \geq f_2(t, y, z, k, \xi'_r, \zeta'_r, \nu)$, $(t, y, z, k, \nu) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R})$, if $\xi_r, \zeta_r, \xi'_r, \zeta'_r \in \mathcal{H}_2^d(t, T + M)$;

(ii) There exists a constant $C > 0$ such that the derivatives of $f_1$ with respect to $\nu$ and $k$ are positive and bounded by $C > 0$, that is, $0 < \partial_k f_1 \leq C$ and $0 < \partial_{\nu} f_1 \leq C$.

**Theorem 2 [Comparison theorem].** Let assumptions (H1)–(H3) be satisfied, and let $\varphi^i \in S_{p}^2(T, T + K)$, $i = 1, 2$. By $(Y^i, Z^i, K^i)$ we denote the solution of Eq. (4.1) with data $(f_1, \varphi^i)$. If $\varphi^3 \geq \varphi^2$, $s \in [T, T + K]$, and $f_1(s, y, z, k, \theta_r, \nu) \geq f_2(s, y, z, k, \theta_r, \theta_r, \nu)$ for $s \in [0, T]$, $y \in \mathbb{R}$, $z \in \mathbb{R}^d$, $k \in \mathbb{R}$, $\theta_r, \theta_r \in \mathcal{H}_2^d(s, T + M)$, $\nu \in \mathcal{P}_2(\mathbb{R})$, and $r \in [t, T + M]$, then $Y^1_t \geq Y^2_t$ for a.e. $t \in [0, T]$, a.s.

**Proof.** For $i = 1, 2, 3, \ldots$, we set $A^i_s = Y^i_{s+\delta_i(s)}$ and $\overline{A}^i_s = \int_0^{\delta_i(s)} e^{-\rho u} Y^i_{s+u} \, du$. Let $(Y^3, Z^3, K^3) \in S_{p}^2(0, T) \times \mathcal{H}_2^d(0, T; \mathbb{R}^d) \times \mathcal{K}_\lambda^2(0, T)$ be a solution of the following mean-field BSDE with jumps:

$$Y^3_t = \varphi^3_T + \int_t^T \left[ f_2 \left( s, Y^3_s, Z^3_s, K^3_s(e) \lambda(de), A^3_s, \overline{A}^3_s, \mathbf{P}, Y^3_s \right) \right] \, ds$$

$$- \int_t^T Z^3_s \, dW_s - \int_t^T K^3_s(e) \lambda(de, dt), \quad t \in [0, T],$$

$$Y^3_T = \varphi^3_T, \quad t \in [T, T + M].$$

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From Lemma 1 it follows that $Y^1_t \geq Y^3_t$ for a.e. $t \in [0, T]$, a.s.

We now consider

\[
Y^4_t = \varphi^2_T + \int_0^T f_2(s, Y^4_s, Z^4_s, \int_G K^4_s(e) \lambda(de), A^3_s, \overline{A}^3_s, P_{Y^4_s}) \, ds
\]

\[
- \int_0^T Z^4_s \, dW_s - \int_0^T \int_G K^4_s(e) N \lambda(de, dt), \quad t \in [0, T],
\]

\[
Y^3_t = \varphi^2_t, \quad t \in [T, T + M].
\]

Since $f_2(s, y, z, k, \cdot, \nu)$ is increasing, we can check that $Y^3_t \geq Y^4_t$ for a.e. $t \in [0, T]$, a.s. Repeating the above argument, we obtain $Y^3_t \geq Y^n_t$ for a.e. $t \in [0, T]$, a.s., where $(Y^n, Z^n, K^n)$ is the solution of the mean-field BSDE with jumps: for $n \geq 4$,

\[
Y^n_t = \varphi^2_T + \int_0^T f_2(s, Y^n_s, Z^n_s, \int_G K^n_s(e) \lambda(de), A^{n-1}_s, \overline{A}^{n-1}_s, P_{Y^n_s}) \, ds
\]

\[
- \int_0^T Z^n_s \, dW_s - \int_0^T \int_G K^n_s(e) N \lambda(de, dt), \quad t \in [0, T],
\]

\[
Y^n_t = \varphi^2_t, \quad t \in [T, T + M].
\]

We will prove the existence of the limit of $(Y^n, Z^n, K^n)$ and show that it is just $(Y^2, Z^2, K^2)$. For this purpose, we define for $n \geq 1$,

\[
(\tilde{Y}^n, \tilde{Z}^n, \tilde{K}^n, \tilde{A}^n, \tilde{\overline{A}}^n)
\]

\[
:= (Y^n - Y^{n-1}, Z^n - Z^{n-1}, K^n - K^{n-1}, A^n - A^{n-1}, \overline{A}^n - \overline{A}^{n-1}).
\]

By the Lipschitz property of $f$ and the inequality $W_2(P_\xi, P_\eta) \leq \{E[\xi - \eta]^2\}^{1/2}$ the Itô formula enables us to show that, for $\beta > 0$,

\[
E \left[ \int_0^T e^{\beta s} \left( \frac{\beta}{2} |\tilde{Y}^n_s|^2 + |\tilde{Z}^n_s|^2 + \int G |\tilde{K}^n_s(e)|^2 \lambda(de) \right) \, ds \right]
\]

\[
\leq \frac{2}{\beta} E \left[ \int_0^T e^{\beta s} f_2 \left( s, Y^n_s, Z^n_s, \int_G K^n_s(e) \lambda(de), A^{n-1}_s, \overline{A}^{n-1}_s, P_{Y^n_s} \right)
\]

\[
- f_2 \left( s, Y^{n-1}_s, Z^{n-1}_s, \int_G K^{n-1}_s(e) \lambda(de), A^{n-2}_s, \overline{A}^{n-2}_s, P_{Y^{n-1}_s} \right) \right|^2 \, ds \right]
\]

\[
\leq \frac{2C^2}{\beta} E \left[ \int_0^T e^{\beta s} \left( |\tilde{Y}^n_s| + |\tilde{Z}^n_s| + \int G |\tilde{K}^n_s(e)| \lambda(de) \right) + E^{F_s} [||A^{n-1}_s|| + ||\overline{A}^{n-1}_s||] \right]
\]
Consequently, we have

\[
\begin{align*}
&\int_0^T e^{\beta s} \left[ \int_0^s e^{\beta u} \left| \tilde{Y}^{n-1}_{s+u} \right|^2 du \right] ds \\
&\leq \frac{12C^2}{\beta} \mathbb{E} \left[ \int_0^T e^{\beta s} \left( \left| \tilde{Y}^{n}_{s} \right|^2 + \left| \tilde{Z}^{n}_{s} \right|^2 + \int_G \left| \tilde{K}^n(s) \right|^2 \lambda(de) \right)^2 \\
&\quad + \mathbb{E} \left[ \left| \tilde{Y}^{n}_{s} \right|^2 \right] \right] ds \\
&\leq \frac{12C^2}{\beta} \left( 2 + \int_G (1 \wedge |e|^2) \lambda(de) \right) \mathbb{E} \left[ \int_0^T e^{\beta s} \left( \left| \tilde{Y}^{n}_{s} \right|^2 ds + \left| \tilde{Z}^{n}_{s} \right|^2 + \int_G \left| \tilde{K}^n(s) \right|^2 \lambda(de) \right) ight] ds \\
&\quad + \frac{12C^2}{\beta} (L + \kappa_0) \mathbb{E} \int_0^T e^{\beta s} \left| \tilde{Y}^{n-1}_{s} \right|^2 ds,
\end{align*}
\]

where \( \kappa_0 = (1/(2\rho))(1 - e^{-2\rho(T+M)})Te^{\beta T} \).

Choosing \( \beta = 36C^2 (2 + \int_G (1 \wedge |e|^2) \lambda(de) + L + \kappa_0) + 3 \), we have

\[
\begin{align*}
\mathbb{E} \left[ \int_0^T e^{\beta s} \left( \left| \tilde{Y}^{n}_{s} \right|^2 + \left| \tilde{Z}^{n}_{s} \right|^2 + \int_G \left| \tilde{K}^n(s) \right|^2 \lambda(de) \right) ds \\
&\leq \frac{1}{2} \mathbb{E} \left[ \int_0^T e^{\beta s} \left( \left| \tilde{Y}^{n-1}_{s} \right|^2 + \left| \tilde{Z}^{n-1}_{s} \right|^2 + \int_G \left| \tilde{K}^{n-1}(s) \right|^2 \lambda(de) \right) ds \right].
\end{align*}
\]
Therefore
\[
E \left[ \int_0^T e^{\beta s} \left( \left| \tilde{Y}^n_s \right|^2 + \left| \tilde{Z}^n_s \right|^2 + \int_G \left| K^n_s(e) \right|^2 \lambda(de) \right) ds \right]
\]
\leq \frac{1}{2n-4} E \left[ \int_0^T e^{\beta s} \left( \left| \tilde{Y}^4_s \right|^2 + \left| \tilde{Z}^4_s \right|^2 + \int_G \left| K^4_s(e) \right|^2 \lambda(de) \right) ds \right],
\]

which implies that \((Y^n, Z^n, K^n)_{n \geq 4}\) is a Cauchy sequence. By \((Y, Z, K)\) we denote its limit. It is easy to get that \((Y, Z) \in \mathcal{H}^2_{\mathcal{F}}(0, T; \mathbb{R}^{1+d}), K \in \mathcal{K}^2_\lambda(0, T)\), and, moreover, \(Y^n_t \geq Y_t\) for a.e. \(t \in [0, T]\), a.s.

Taking the limit in (4.9), we have
\[
Y_t = \varphi_T^2 + \int_0^T f_2(s, Y_s, Z_s, \int_G K_s(e)l_s(e) \lambda(de), \Lambda_s, \mathcal{A}_s, \mathcal{P}_{Y_s}) ds
\]
\[
- \int_0^T Z_s dW_s - \int_0^T \int_G K_s(e)N\lambda(de, dt), \quad t \in [0, T],
\]
\[
Y_t = \varphi_T^2, \quad t \in [T, T + M].
\]

The existence and uniqueness of the solution of the anticipated mean-field BSDEs (see Theorem 1) allows us to show \(Y_t = Y_T^2\) for a.e. \(t \in [0, T]\), a.s. Then the desired result follows from the fact \(Y_t^1 \geq Y_t^3 \geq \cdots \geq Y_t = Y_T^2\) for a.e. \(t \in [0, T]\), a.s. \(\Box\)

**Example 1.** Let \(f_1(t) = |E^{\mathcal{F}_t}[\eta_{t+\delta}]| + |\int_{\mathbb{R}} k(e) (1 \wedge |e|) \lambda(de)| + |E[\zeta_t]|, f_2(t) = 0, t \in [0, T]\), where \(\delta > 0\) is a given constant, and let \(\eta_{t+\delta} \in L^2(\mathcal{F}_{t+\delta}; \mathbb{R}), \zeta_t \in L^2(\mathcal{F}_t; \mathbb{R}); k(\cdot) \in L^2(\mathcal{B}(G); \mathbb{R})\). Obviously, \(f_1\) and \(f_2\) satisfy assumptions (H1)–(H3). Let \(\xi \in L^2(\mathcal{F}_T; \mathbb{R})\) and \(\varphi_1(t) = |\xi|, \varphi_2(t) = 0, t \in [T, T + K]\). By \((Y^1, Z^1, K^1)\) we denote the solution of (4.1) with data \((f_1, \varphi^1)\). Then from Theorem 2 we have \(Y_t^1 \geq 0\) for a.e. \(t \in [0, T]\), a.s.

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