ON SEMI-CONTINUITY PROBLEMS FOR MINIMAL LOG DISCREPANCIES

YUSUKE NAKAMURA

ABSTRACT. We show the semi-continuity property of minimal log discrepancies for varieties which have a crepant resolution in the category of Deligne-Mumford stacks. Using this property, we also prove the ideal-adic semi-continuity problem for toric pairs.

1. Introduction

The minimal log discrepancy (mld for short) was introduced by Shokurov, in order to reduce the conjecture of terminations of flips to a local problem about singularities. Recently, this has been a fundamental invariant in the minimal model program. There are two related conjectures about mld’s, the ACC (ascending chain condition) conjecture and the LSC (lower semi-continuity) conjecture. Shokurov showed that these two conjectures imply the conjecture of terminations of flips \[19\].

In the first half of this paper, we consider the LSC conjecture, proposed by Ambro \[1\].

**Conjecture 1.1 (LSC conjecture).** Let \((X, \Delta)\) be a log pair. Then, the function

\[ |X| \to \mathbb{R}_{\geq 0} \cup \{-\infty\}; \quad x \mapsto \text{mld}_x(X, \Delta) \]

is lower semi-continuous, where we denote by \(|X|\) the set of all closed points of \(X\).

In this conjecture, \(\text{mld}_x(X, \Delta)\) denotes the minimal log discrepancy, and will be defined in section 2.1. The value \(\text{mld}_x(X, \Delta)\) reflects the singularity of \(X\) and \(\Delta\) around \(x\).

The LSC conjecture is known under some conditions. If \(X\) is toric or \(\dim X \leq 3\), this conjecture is known by work of Ambro \[2\]. If \(X\) is smooth, this conjecture was proved by Ein, Mustață, and Yasuda \[7\]. They used the description of mld’s by the language of jet schemes. Ein and Mustață extended this result to locally complete intersection varieties \[6\].

The first purpose of this paper is to prove the LSC conjecture on varieties with quotient singularities.

Let \(X\) be a \(\mathbb{Q}\)-Gorenstein normal variety, and assume that \(X\) has a crepant resolution in the category of Deligne-Mumford stacks. Here, a proper birational morphism \(f : \mathcal{X} \to X\) from a smooth Deligne-Mumford stack \(\mathcal{X}\) is crepant if \(K_X = f^* K_{\mathcal{X}}\) holds.

2010 Mathematics Subject Classification. Primary 14B05; Secondary 14E18.

Key words and phrases. minimal log discrepancies, ACC conjecture, semi-continuity problem, twisted jet stacks.
Theorem 1.2. Let $X$ be a $\mathbb{Q}$-Gorenstein normal variety. Assume $X$ has a crepant resolution in the category of Deligne-Mumford stacks. Then, the followings hold.

(i) Let $\Delta$ be an effective $\mathbb{R}$-Cartier divisor on $X$, and $a$ an $\mathbb{R}$-ideal sheaf on $X$, that is, a formal product $\prod a_i^{r_i}$ of ideals $a_i$ with $r_i$ positive real numbers. Then the function $|X| \to \mathbb{R}_{\geq 0} \cup \{-\infty\}; \quad x \mapsto \operatorname{mld}_x(X, \Delta, a)$ is lower semi-continuous.

(ii) Let $T$ be a variety, $\Delta$ an effective $\mathbb{R}$-Cartier divisor on $X$, and $x$ a closed point of $X$. For an $\mathbb{R}$-ideal sheaf $\mathfrak{A}$ on $X \times T$, the function $|T| \to \mathbb{R}_{\geq 0} \cup \{-\infty\}; \quad p \mapsto \operatorname{mld}_x(X, \Delta, \mathfrak{A}_p)$ is lower semi-continuous, where $\mathfrak{A}_p$ is the restriction of $\mathfrak{A}$ to $X \times \{p\}$.

Especially, Conjecture 1.1 holds for the variety $X$.

Assume that a finite group $G$ acts on a smooth variety $M$, and that this action is free in codimension 1. Then the quotient variety $M/G$ and the quotient stack $[M/G]$ are isomorphic in codimension 1. Hence, $M/G$ has a crepant resolution $[M/G] \to M/G$. Therefore we get the following corollary.

Corollary 1.3. Conjecture 1.1 holds if $X$ has only quotient singularities.

To prove Theorem 1.2, we employ Yasuda’s theory of twisted jet stacks [20, 21]. We take a crepant resolution $f : \mathcal{X} \to X$, and describe $\operatorname{mld}_x$ by the dimensions of twisted jet stacks on $X$.

As a corollary, we have an application to terminations. We can show that there is no infinite sequence of flips if we start from a symplectic variety with only quotient singularities. This is an extension of Matsushita’s result [10, Proposition 2.1, Lemma 2.1]. In the statement below, we consider log minimal model programs. For a log pair $(X, D)$, a $(K_X + D)$-MMP is a log minimal model program which contracts a $(K_X + D)$-negative extremal ray repeatedly. The $(K_{X_i} + D_i)$-flip is the flip corresponding to a flipping contraction of a $(K_{X_i} + D_i)$-negative extremal ray (See [14] for more details). Note that $K_X$ is trivial if $X$ is a symplectic variety.

Corollary 1.4. Let $X$ be a symplectic variety with only terminal and quotient singularities, and $D_0$ an effective $\mathbb{R}$-Cartier divisor on $X$. Then, there is no infinite sequence of flips

$$X = X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots$$

such that $f_i$ is a $(K_{X_i} + D_i)$-flip, where $D_i$ is the $\mathbb{R}$-Cartier divisor defined as $D_{i+1} := f_i(D_i)$ inductively.

Especially, if the linear system $|D_0|$ has no fixed divisor, any $(K_X + D_0)$-MMP terminate.

In the latter half of this paper, we consider the following related conjecture proposed by Mustaţă.

Conjecture 1.5 (Mustaţă). Let $(X, \Delta)$ be a log pair, $Z$ a closed subset of $X$, and $I_Z$ its ideal sheaf. Let $a = \prod_{i=1}^s a_i^{r_i}$ be an $\mathbb{R}$-ideal sheaf with ideal sheaves $a_1, \ldots, a_s$ on $X$ and positive real numbers $r_1, \ldots, r_s$.
Then there exists a positive integer $l$ such that the following holds: if ideal sheaves $b_1, \ldots, b_s$ satisfy $a_i + I_{Z}^l = b_i + I_{Z}^l$, then

$$\text{mld}_{Z}(X, \Delta, a) = \text{mld}_{Z}(X, \Delta, b)$$

holds, where we put $b := \prod_{i=1}^{s} b_{i}^{r_{i}}$.

This conjecture is related to Shokurov’s ACC conjecture on mld’s by generic limits of ideals [9, Remark 2.5.1]. Generic limits were introduced by Kollár [13], and using this method, de Fernex, Ein, and Mustață proved Shokurov’s ACC conjecture for log canonical thresholds [3]. The generic limit is one of constructions of a limit of a sequence of ideals. This idea of considering the limit of ideals originates in de Fernex and Mustață [4], and they constructed a limit using non-standard analysis.

Mustață’s conjecture is known under some conditions. If $\text{mld}_{Z}(X, \Delta, a) = 0$ holds, the conjecture is known by work of de Fernex, Ein, and Mustață [3]. If the triple $(X, \Delta, a)$ is Kawamata log terminal around $Z$, then the conjecture is known by work of Kawakita [9]. The remaining case is when $(X, \Delta, a)$ is log canonical around $Z$ and satisfies $\text{mld}_{Z}(X, \Delta, a) > 0$. The conjecture in dimension 2 was proved by Kawakita [11].

The second purpose of this paper is to prove Mustață’s conjecture on varieties with a $\mathbb{C}^*$-action. Let $X = \text{Spec} A$ be an affine variety with a $\mathbb{C}^*$-action, and $A = \bigoplus_{m \in \mathbb{Z}} A^{(m)}$ the induced graded ring structure. Then, the action of $\mathbb{C}^*$ on $X = \text{Spec} A$ is said to be of ray type if $A^{(m)} = 0$ holds for all $m < 0$ or $A^{(m)} = 0$ holds for all $m > 0$. In the following proposition, we assume that the action is of ray type.

**Proposition 1.6.** Let $X$ be a $\mathbb{Q}$-Gorenstein normal affine variety, $\Delta$ an effective $\mathbb{R}$-Cartier divisor, $a = \prod_{i=1}^{s} a_{i}^{r_{i}}$ be an $\mathbb{R}$-ideal sheaf on $X$, and $Z$ a closed subset of $X$. Suppose that $\mathbb{C}^*$ acts on $X$ and assume the following conditions:

- The variety $X$ has a $\mathbb{C}^*$-equivariant crepant resolution in the category of Deligne-Mumford stacks.
- The $\mathbb{C}^*$-action on $X$ is of ray type.
- All the components of $\Delta$ and $a_i$ are $\mathbb{C}^*$-invariant.
- $Z$ is the set of all $\mathbb{C}^*$-fixed points in $X$.

Then, Conjecture 1.5 holds for $(X, \Delta, a)$ and $Z$.

**Remark 1.7.** In the above proposition, we assume that the ideal $a_i$ is $\mathbb{C}^*$-invariant, but the ideal $b_i$ is not necessarily $\mathbb{C}^*$-invariant.

As an application, we can prove Mustață’s conjecture for toric varieties.

**Theorem 1.8.** Let $X$ be a normal toric variety, $(X, \Delta)$ a log pair, and $Z$ a closed subset of $X$. Assume that $\Delta$ and $Z$ are torus invariant. Then, for an $\mathbb{R}$-ideal sheaf $a = \prod_{i=1}^{s} a_{i}^{r_{i}}$ with torus invariant ideal sheaves $a_i$, Conjecture 1.5 holds for $(X, \Delta, a)$ and $Z$.

**Remark 1.9.** As Remark 1.7, in the above theorem, we assume that the ideal $a_i$ is torus invariant, but the ideal $b_i$ in Conjecture 1.5 is not necessarily torus invariant. Therefore the theorem cannot follow directly from a combinatorial description of mld’s for toric triples.
In the proof of Proposition 1.6, showing the inequality
\[ \text{mld}_Z(X, \Delta, a) \leq \text{mld}_Z(X, \Delta, b) \]
is essential (see Remark 1.2). To prove this inequality, we consider a degeneration of the ideal \( b \). We explain the idea of the proof below. For the sake of simplicity, we assume \( s = 1 \) and denote \( a' := a_1, b' := b_1 \), and \( r := r_1 \). When \( l \) is sufficiently large, we can degenerate \( b' \) to some ideal \( b'_0 \) which contains \( a' \). Hence we get \( \text{mld}_Z(X, \Delta, (a')^r) \leq \text{mld}_Z(X, \Delta, (b'_0)^r) \), and the proposition reduces to the inequality \( \text{mld}_Z(X, \Delta, (b'_0)^r) \leq \text{mld}_Z(X, \Delta, (b')^r) \). This inequality follows from the semi-continuity property like Theorem 1.2 (ii). However, the above inequality does not directly follow from Theorem 1.2 (ii). It is because \( Z \) has possibly positive dimension. Hence, we need to look at the construction of the above degeneration.

The paper is organized as follows. In section 2, we review the definition of mld’s and the theory of Yasuda’s twisted jet stacks. Some lemmas necessary for the proof of Theorem 1.2 are also proved in section 2. In section 3, we prove Theorem 1.2. In section 4, we prove Proposition 1.6 and Theorem 1.8.

**Notation and convention.** Throughout this paper, we work over the field of complex numbers \( \mathbb{C} \).

- We denote by \( \mathbb{N} \) the set of all non-negative integers.
- Every Deligne-Mumford stack in this paper is separated.
- For a Deligne-Mumford stack \( X \), we denote by \( |X| \) the set of all \( \mathbb{C} \)-valued points.
- For a morphism \( f : X \to T \) from a Deligne-Mumford stack to a variety \( T \), and a closed point \( p \in |T| \), we denote by \( X_p \) the fiber of \( f \) over \( p \). For an ideal sheaf \( a \) on \( X \), we denote by \( a_p \) the restriction to the fiber \( X_p \). In addition, for a morphism \( g : X \to Y \) between Deligne-Mumford stacks over \( T \), we denote by \( g_p : X_p \to Y_p \) the induced morphism between the fibers.

## 2. Preliminaries

### 2.1. Minimal log discrepancies

A *log pair* \( (X, \Delta) \) is a normal variety \( X \) and an effective \( \mathbb{R} \)-divisor \( \Delta \) such that \( K_X + \Delta \) is \( \mathbb{R} \)-Cartier. An *\( \mathbb{R} \)-ideal sheaf* of \( X \) is a formal product \( a_1^{r_1} \cdots a_s^{r_s} \), where \( a_1, \ldots, a_s \) are ideal sheaves on \( X \) and \( r_1, \ldots, r_s \) are positive real numbers. For a log pair \( (X, \Delta) \) and an \( \mathbb{R} \)-ideal sheaf \( a \), we call \( (X, \Delta, a) \) a *log triple*. When \( \Delta = 0 \), we sometimes drop \( \Delta \) and write \( (X, a) \). If \( Y_i \) is the closed subscheme of \( X \) corresponding to \( a_i \), we sometimes identify the triple \( (X, \Delta, \sum_{i=1}^s r_i Y_i) \) with the triple \( (X, \Delta, \prod_{i=1}^s a_i^{r_i}) \).

For a proper birational morphism \( f : X' \to X \) from a normal variety \( X' \) and a prime divisor \( E \) on \( X' \), the *log discrepancy* of \( (X, \Delta, a) \) at \( E \) is defined as
\[
a_E(X, \Delta, a) := 1 + \text{ord}_E(K_{X'} - f^*(K_X + \Delta)) - \text{ord}_E a,
\]
where \( \text{ord}_E a := \sum_{i=1}^s r_i \text{ord}_E a_i \). The image \( f(E) \) is called the *center* of \( E \) on \( X \), and we denote it by \( c_X(E) \). For a closed subset \( Z \) of \( X \), the *minimal
log discrepancy (mld for short) over $Z$ is defined as

$$\text{mld}_Z(X, \Delta, a) := \inf_{c_X(E) \subset Z} a_E(X, \Delta, a).$$

In the above definition, the infimum is taken over all prime divisors $E$ on $X'$ with the center $c_X(E) \subset Z$, where $X'$ is a higher birational model of $X$, that is, $X'$ is the source of some proper birational morphism $X' \to X$.

Remark 2.1. It is known that $\text{mld}_Z(X, \Delta, a)$ is in $\mathbb{R}_{\geq 0} \cup \{-\infty\}$ and that if $\text{mld}_Z(X, \Delta, a) \geq 0$, then the infimum on right hand side in the definition is actually the minimum.

Remark 2.2. Let $(X, \Delta, a)$ and $Z$ be as above. Mld’s have the following properties.

(i) If $Z_1, \ldots, Z_c$ are the irreducible components of $Z$, then

$$\text{mld}_Z(X, \Delta, a) = \min_{1 \leq i \leq c} \text{mld}_{Z_i}(X, \Delta, a).$$

(ii) If $U_1, \ldots, U_c$ are open subsets of $X$ such that $Z \subset \bigcup_{j=1}^c U_j$, then

$$\text{mld}_Z(X, \Delta, a) = \min_{1 \leq j \leq c} \text{mld}_{Z \cap U_j}(U_j, \Delta|_{U_j}, a|_{U_j}),$$

where $a|_{U_j} := \prod_{i=1}^c (a|_{U_i})^{r_i}$. These properties easily follow from the definition.

2.2. Notation and remarks on Deligne-Mumford stacks. In this section, we review some properties of Deligne-Mumford stacks (DM stacks for short). In this paper, we are mainly interested in separated DM stacks of finite type over complex number $\mathbb{C}$.

Let $\mathcal{X}$ be a DM stack of finite type over $\mathbb{C}$. We can consider the set of $\mathbb{C}$-points $|\mathcal{X}|$ of $\mathcal{X}$, and it admits a Zariski topology [15]. Keel and Mori [12] proved that the coarse moduli space $X$ exists for $\mathcal{X}$. Then, the induced map $|\mathcal{X}| \to |X|$ is a homeomorphism.

It is known that a DM stack is étale locally isomorphic to a quotient stack (see for instance [12]). Let $X$ be the coarse moduli space of $\mathcal{X}$. Then, there exists an étale covering $\{X_i \to X\}_i$, such that for every $i$, the étale pullback $\mathcal{X} \times_X X_i$ is isomorphic to a quotient stack $[M_i/G_i]$, where $M_i$ is a variety over $\mathbb{C}$ and $G_i$ is a finite group.

We prove two lemmas about DM stacks. The first one is about families of DM stacks.

Lemma 2.3. Let $M$ be a variety and $G$ a finite group. Suppose $G$ acts on $M$. Let $T$ be a variety and $f : [M/G] \to T$ be a morphism of stacks. Then, for a closed point $p \in |T|$, the fiber $[M/G]_p$ is isomorphic to the quotient stack $[M_p/G]$. 

Proof. First, we remark that $M_p$ is $G$-invariant because the morphism $M \to T$ factors through the quotient stack $[M/G]$.

Let $n$ be the order of the group $G$. It follows that both of the morphisms $M_p \to [M/G]_p$ and $M_p \to [M_p/G]$ are surjective finite étale morphisms of degree $n$. Since $M_p \to [M/G]_p$ factors through $M_p \to [M_p/G]$, we can conclude that $[M/G]_p$ is isomorphic to $[M_p/G]$. $\square$
The second lemma is about semi-continuity of the dimensions of fibers on a family of DM stacks.

**Lemma 2.4.** Let $\mathcal{X}$ be a DM stack of finite type over $\mathbb{C}$, and $f : \mathcal{X} \to T$ a morphism from $\mathcal{X}$ to a variety $T$. Then, for every $n \in \mathbb{N}$, the set

$$|\mathcal{X}|_{\geq n} := \{ x \in |\mathcal{X}| \mid \dim_x f^{-1}(f(x)) \geq n \}$$

is closed in $|\mathcal{X}|$. In the above equation, we denote by $\dim_x$ the dimension around $x$.

**Proof.** If $\mathcal{X}$ is a variety, then this lemma is well-known (see for instance [3]).

Let $X$ be the coarse moduli space of $\mathcal{X}$. By the universality of the coarse moduli space, $f : \mathcal{X} \to T$ factors through $X \to T$. Since the induced map $|\mathcal{X}| \to |X|$ is a homeomorphism, the assertion follows from the case of varieties. □

2.3. **Yasuda’s twisted jet stacks.** Twisted jet stacks were introduced by Yasuda [20], [21]. First, we recall the definition.

Let $X$ be a DM stack over $\mathbb{C}$. For an affine scheme $S = \text{Spec} R$ over $\mathbb{C}$ and a non-negative integer $n$, we denote $D_{n,S} := \text{Spec} R[t]/(t^{n+1})$.

For a positive integer $l$, we denote by $\mu_l$ the cyclic group of $l$-th roots of the unity. We consider the natural group action $\mu_l$ on $D_{n,S}$ induced by the following ring homomorphism:

$$R[t]/(t^{nl+1}) \to R[t]/(t^{nl+1}) \otimes \mathbb{C}[x]/(x^l - 1); \quad t \mapsto t \otimes x.$$

Then, we denote

$$\mathcal{D}_{n,S} := [D_{n,S}/\mu_l].$$

A twisted $n$-jet of order $l$ of $\mathcal{X}$ over $S$ is a representable morphism $\mathcal{D}_{n,S} \to \mathcal{X}$. Yasuda defined the stack of twisted $n$-jets of order $l$, and we denote it by $\mathcal{J}_{n,\mathcal{X}}$. An object of $\mathcal{J}_{n,\mathcal{X}}$ over $S$ is a twisted $n$-jet of order $l$. For a morphism $f : S \to T$ over $\mathbb{C}$, we have an induced morphism $f' : \mathcal{D}_{n,S} \to \mathcal{D}_{n,T}$. A morphism in $\mathcal{J}_{n,\mathcal{X}}$ from $\gamma : \mathcal{D}_{l,n,S} \to \mathcal{X}$ to $\gamma' : \mathcal{D}_{l,n,T} \to \mathcal{X}$ is a 2-morphism from $\gamma$ to $\gamma' \circ f'$. The stack $\mathcal{J}_{n,\mathcal{X}}$ of twisted $n$-jets is the disjoint union of the stacks $\bigsqcup_{l \geq 1} \mathcal{J}_{n,l,\mathcal{X}}$. Both $\mathcal{J}_{n,\mathcal{X}}$ and $\mathcal{J}_{n,\mathcal{X}}$ are actually DM stacks [21 Theorem 18]. When $X$ is a scheme, $\mathcal{J}_{n,X} = \emptyset$ holds for $l \geq 2$, and $\mathcal{J}_{n,X}$ can be identified with the usual $n$-th jet scheme $J_n X$.

For $0 \leq n_1 \leq n_2$, we have the truncation morphism $\varphi_{n_2,n_1} : \mathcal{J}_{n_2,X} \to \mathcal{J}_{n_1,X}$. This corresponds to the surjective ring homomorphism $R[t]/(t^{n_2l+1}) \to R[t]/(t^{n_1l+1})$. Since $\varphi_{n_2,n_1}$ is an affine morphism, we have a projective limit and projections

$$\mathcal{J}_\infty \mathcal{X} := \lim_{\leftarrow} \mathcal{J}_n \mathcal{X}, \quad \varphi_{\infty,n} : \mathcal{J}_\infty \mathcal{X} \to \mathcal{J}_n \mathcal{X}.$$

The stack $\mathcal{J}_0 \mathcal{X}$ can be identified with the inertia stack and the projection $\varphi_{\infty,0} : \mathcal{J}_0 \mathcal{X} \to \mathcal{X}$.
is a finite morphism. Here, we use “b” in the index as an abbreviation of the word “base”. We denote by $\varphi_{n,b}$ the composite map

$$J_nX \xrightarrow{\varphi_{n,b}} J_0X \xrightarrow{\varphi_{0,b}} X$$

for $n \in \mathbb{N} \cup \{\infty\}$.

In the theory of jet schemes, the $C^*$-action and the relative jet schemes can be defined [17, Section 2]. In the following part of this section, we generalize these concepts to twisted jet stacks.

We already defined the truncation morphism $\varphi_{n,0}^X : J_nX \to J_0X$. On the other hand, we also have the zero section $\sigma_n^X : J_0X \to J_nX$. This is defined by the composition with the stack morphism $D_{n,S}^l \to D_{0,S}^l$ induced by the ring inclusion $R \to R[t]/(t^{nl+1})$. We will write simply $\varphi_{m,n}$ (resp. $\sigma_n$) instead of $\varphi_{m,n}^X$ (resp. $\sigma_n^X$) when no confusion can arise.

The twisted jet stack $J_nX$ admits the $C^*$-action which is induced by the following $C^*$-action on $R[t]/(t^{nl+1})$:

$$R[t]/(t^{nl+1}) \to R[t]/(t^{nl+1}) \otimes \mathbb{C}[s, s^{-1}]; \ t \to t \otimes s.$$

The following lemma is used in the proof of Theorem 1.2.

**Lemma 2.5.** Let $W$ be a $C^*$-invariant closed subset of $|J_nX|$. Then $\varphi_{n,0}(W)$ is a closed subset of $|J_0X|$.

**Proof.** The $C^*$-action on $J_nX$ is induced by the action $C^* \times D_{n,S}^l \to D_{n,S}^l$. By the definition above, the morphism $C^* \times D_{n,S}^l \to D_{n,S}^l$ is uniquely extended to the morphism $C \times D_{n,S}^l \to D_{n,S}^l$. Therefore, the $C^*$-action $C^* \times J_nX \to J_nX$ is extended to the morphism

$$\psi : C \times J_nX \to J_0X.$$

By definition, for any $\alpha \in |J_nX|$, we have $\psi(0, \alpha) = \sigma_n(\varphi_{n,0}(\alpha))$.

It is sufficient to show the equality $\varphi_{n,0}(W) = \sigma_n^{-1}(W)$ because $\sigma_n^{-1}(W)$ is closed. Fix an element $\alpha \in W$. Since $W$ is $C^*$-invariant, $\psi(\gamma, \alpha) \in W$ for any $\gamma \in C^*$. Because $W$ is closed, $\psi(0, \alpha) \in W$ holds. Since $\psi(0, \alpha) = \sigma_n(\varphi_{n,0}(\alpha))$, we have the equality $\varphi_{n,0}(W) = \sigma_n^{-1}(W)$. \hfill $\Box$

Next, we construct the relative twisted jet stacks.

**Lemma 2.6.** Let $f : Y \to T$ be a morphism from a DM stack $Y$ to a variety $T$. Then there exist a DM stack $J_n(Y/T)$ and a morphism $g : J_n(Y/T) \to T$ such that for any point $p \in |T|$, $J_n(Y/T)_p \cong J_n(Y_p)$ holds, where $J_n(Y/T)_p$ is the fiber of $g$ over $p$, and $Y_p$ is the fiber of $f$ over $p$.

For the proof of Lemma 2.6 we use an étale local description of twisted jet stacks [21, Proposition 16].

Let $[M/G]$ be a quotient stack with scheme $M$ and finite group $G$. Fix a positive integer $l$ and an embedding $a : \mu_l \hookrightarrow G$. We denote by $J_nM$ the $n$-th jet scheme of $M$, then $\mu_l$ acts on $J_nM$ in two ways. First, the action $\mu_l \curvearrowright D_{n,S}$ induces an action $\mu_l \curvearrowright J_nM$. On the other hand, we have the action $G \curvearrowright J_nM$, and this action induces an action $\mu_l \curvearrowright J_nM$ by the
embedding $a : \mu_l \hookrightarrow G$. We define $J_n^{(a)} M$ to be the closed subscheme of $J_n M$ where the above two actions are identical. Then we have a concrete description of twisted jet stacks:

$$\mathcal{J}_n^l \mathcal{X} \cong \bigsqcup_{a \in \text{Conj}(\mu_l, G)} [J_n^{(a)} M/C_a],$$

where $\text{Conj}(\mu_l, G)$ is the conjugacy classes of embeddings $\mu_l \hookrightarrow G$, and $C_a$ is the centralizer of $a$.

**Proof of Lemma 2.6.** When $\mathcal{Y} = Y$ is a scheme, the relative jet scheme $J_n(Y/T)$ exists and

$$J_n(Y/T)_p \cong J_n(Y_p)$$

holds [17, Section 2]. Besides, $J_n(Y/T)$ can be characterized by the following representability:

$$\text{Hom}_{\mathcal{Sch}/T}(Z, J_n(Y/T)) \cong \text{Hom}_{\mathcal{Sch}/T} \left(Z \times_{\mathcal{C}} \text{Spec}(\mathbb{C}[t]/(t^{n+1})), Y\right)$$

for every scheme $Z$ over $T$.

First, we consider the case when $\mathcal{Y}$ is a quotient stack $M/G$. Fix a positive integer $l$ and an embedding $a : \mu_l \hookrightarrow G$. Then, two actions $\mu_l \actson J_n(M/T)$ are induced by the actions $\mu_l \actson \text{Spec}(\mathbb{C}[t]/(t^{n+1}))$ and $\mu_l \actson M$. These actions are compatible with above two $\mu_l$-actions on $J_n(M_p)$ when we restrict them to the fibers $J_n(M/T)_p$. Set $J_n^{(a)}(M/T)$ to be the closed subscheme of $J_n(M/T)$ where the above two actions are identical.

We put

$$Z^l := \bigsqcup_{a \in \text{Conj}(\mu_l, G)} [J_n^{(a)} (M/T)/C_a].$$

Restricting this to the fiber over $p \in |T|$, we have

$$Z^l_p \cong \bigsqcup_{a \in \text{Conj}(\mu_l, G)} [J_n^{(a)} (M_p)/C_a] \cong \mathcal{J}_n^l([M/G]_p)$$

by Lemma 2.3. Hence, $\bigsqcup_{l \geq 1} Z^l$ satisfies the property in the statement.

For the general case, we introduce the relative twisted jet stack as a category. We define $\mathcal{J}_n^l(Y/T)$ as follows. For a scheme $S$ over $T$, an object of $\mathcal{J}_n^l(Y/T)$ over $S$ is a representable morphism $\mathcal{D}_{n,S} \to \mathcal{Y}$ over $T$. For a morphism $f : S_1 \to S_2$ over $T$, we have an induced morphism $f' : \mathcal{D}_{n,S_1} \to \mathcal{D}_{n,S_2}$. A morphism in $\mathcal{J}_n^l(Y/T)$ from $\gamma : \mathcal{D}_{n,S_1} \to \mathcal{Y}$ to $\gamma' : \mathcal{D}_{n,S_2} \to \mathcal{Y}$ is a 2-morphism from $\gamma$ to $\gamma' \circ f'$. Then, the category $\mathcal{J}_n^l(Y/T)$ is actually a stack by the same reason as in the absolute case [21, Lemma 12]. We define $\mathcal{J}_n(Y/T) = \bigsqcup_{l \geq 1} \mathcal{J}_n^l(Y/T)$.

To complete the proof, it is sufficient to show that for a quotient stack $\mathcal{Y} = [M/G]$,

$$\mathcal{J}_n^l([M/G]/T) = \bigsqcup_{a \in \text{Conj}(\mu_l, G)} [J_n^{(a)} (M/T)/C_a]$$

holds. This equation can be confirmed by following the proof for the absolute case [21, Proposition 16]. Here, we use the property of the relative jet schemes about the representability as remarked above. \qed
For the relative twisted jet stacks, we can also define the truncation morphism \( \varphi_{r,0}^{Y/T} : J_n(\mathcal{Y}/\mathcal{T}) \to J_n(\mathcal{X}/\mathcal{T}) \), the zero section \( \sigma_{n}^{Y/T} : \mathcal{J}_0(\mathcal{Y}/\mathcal{T}) \to \mathcal{J}_n(\mathcal{Y}/\mathcal{T}) \), and \( \mathbb{C}^*\)-action on \( \mathcal{J}_n(\mathcal{Y}/\mathcal{T}) \). If we restrict them to the fiber over a closed point \( p \in T \), these definitions are compatible with the definitions in the absolute case. We also have the following lemma.

**Lemma 2.7.** Let \( W \) be a \( \mathbb{C}^*\)-invariant closed subset of \( |\mathcal{J}_n(\mathcal{Y}/\mathcal{T})| \). Then \( \varphi_{r,0}^{Y/T}(W) \) is a closed subset of \( |\mathcal{J}_0(\mathcal{Y}/\mathcal{T})| \).

**Proof.** We have the equality \( \varphi_{r,0}^{Y/T}(W) = (\sigma_{n}^{Y/T})^{-1}(W) \) by the proof of Lemma 2.5. The right hand side is clearly closed in \( |\mathcal{J}_0(\mathcal{Y}/\mathcal{T})| \). \( \square \)

### 2.4 Motivic integration.

First, we define the space in which motivic integration takes value. We introduce the notion of the Grothendieck semiring, following Yasuda [21, Section 3].

In this section, we fix a positive integer \( r \). A *convergent stack* is the pair \((\mathcal{X}, \alpha)\) of a DM stack \( \mathcal{X} \) and a function \( \alpha : \{ \text{connected component of } \mathcal{X} \} \to \frac{1}{r}\mathbb{Z} \) satisfying the following two conditions:

- \( \mathcal{X} \) has at most countably many connected components and the all connected components are of finite type, and
- for every integer \( n \), there are at most finitely many connected components \( \mathcal{V} \) such that \( \dim \mathcal{V} + \alpha(\mathcal{V}) > n \).

A DM stack \( \mathcal{X} \) of finite type is identified with the convergent stack \((\mathcal{X}, 0)\). For two convergent stacks \((\mathcal{X}, \alpha)\) and \((\mathcal{Y}, \beta)\), a morphism \( f : (\mathcal{X}, \alpha) \to (\mathcal{Y}, \beta) \) of convergent stacks is a morphism \( g : \mathcal{X} \to \mathcal{Y} \) of stacks satisfying \( \beta = \alpha \circ f \). We denote by \( (\mathcal{R}^1/r)' \) the set of the isomorphism classes of convergent stacks. \( (\mathcal{R}^1/r)' \) admits a semiring structure by the disjoint union \( \sqcup \) and the product \( \times \). For \((\mathcal{X}, \alpha) \in (\mathcal{R}^1/r)', \) we can define the dimension \( \dim(\mathcal{X}, \alpha) \) by

\[
\max\{ \dim \mathcal{V} + \alpha(\mathcal{V}) \mid \mathcal{V} \text{ is a connected component of } \mathcal{X} \}.
\]

For empty set \( \emptyset \), we define \( \dim \emptyset := -\infty \).

For each \( n \in \mathbb{Z} \), we can define a relation \( \sim_n \) on \((\mathcal{R}^1/r)'\) to be the strongest equivalence relation satisfying the following three relations:

- If \((\mathcal{X}, \alpha)\) and \((\mathcal{Y}, \beta)\) are convergent stacks with \( \dim(\mathcal{Y}, \beta) < n \), then \((\mathcal{X}, \alpha) \sim_n (\mathcal{X}, \alpha) \sqcup (\mathcal{Y}, \beta) \).
- If \((\mathcal{X}, \alpha)\) is a convergent stack and \( \mathcal{Y} \) is a closed substack of \( \mathcal{X} \), then \((\mathcal{X}, \alpha) \sim_n (\mathcal{Y}, \alpha) \sqcup (\mathcal{X} \setminus \mathcal{Y}, \alpha) \).
- Let \((\mathcal{X}, \alpha)\) and \((\mathcal{Y}, \beta)\) be convergent stacks. Assume there exists a representable morphism of stacks \( f : \mathcal{X} \to \mathcal{Y} \) such that \( f^{-1}(x) \cong \mathcal{A}^{\beta(x) - \alpha(f^{-1}(x))} \) holds for any point \( x \in \mathcal{Y} \). Then \((\mathcal{X}, \alpha) \sim_n (\mathcal{Y}, \beta) \).

Then we can define new relation \( \sim \) on \((\mathcal{R}^1/r)'\) as follows: For any \( a, b \in (\mathcal{R}^1/r)' \), \( a \sim b \) if and only if \( a \sim_n b \) for any integer \( n \). For a convergent stack \((\mathcal{X}, \alpha) \in (\mathcal{R}^1/r)'\), we denote by \( \{ (\mathcal{X}, \alpha) \} \) the equivalence classes of \( \mathcal{X} \). Besides, we denote by \( \mathcal{R}^1/r \) the set of all the equivalence classes. By
definition, the map
\[ \dim : \mathfrak{A}^{1/r} \to \frac{1}{r} \mathbb{Z} \cup \{-\infty\}, \quad \{(\mathcal{X}, \alpha)\} \mapsto \dim(\mathcal{X}, \alpha) \]
is well defined, and a semiring structure on \(\mathfrak{A}^{1/r}\) is induced by the semiring structure on \((\mathfrak{A}^{1/r})'\).

Next, we introduce the motivic measure on a smooth DM stack \[21\] Section 2.

Let \(\mathcal{X}\) be a smooth DM stack of finite type and pure dimension \(d\). For a non-negative integer \(n\), a subset \(A \subset |\mathcal{J}_n\mathcal{X}|\) is called an \(n\)-cylinder if \(A = \varphi_{\infty,n}^{-1}(\varphi_{\infty,n}(A))\) and \(\varphi_{\infty,n}(A)\) is a constructible subset of \(|\mathcal{J}_n\mathcal{X}|\). For an \(n\)-cylinder \(A \subset |\mathcal{J}_n\mathcal{X}|\) we define
\[ \mu_\mathcal{X}(A) := \{\varphi_{\infty,n}(A)\}_L^{-nd} \in \mathfrak{A}^{1/r}, \]
where we denote \(\{A^1\}\) by \(L\). This definition is independent of \(n\) \[21\] Lemma 43. A subset \(A \subset |\mathcal{J}_n\mathcal{X}|\) is called negligible if there exist cylinders \(C_i\) for \(i \geq 1\) such that \(A = \bigcap_{i \geq 1} C_i\) and \(\lim_{i \to \infty} \text{codim} C_i = \infty\) hold, where we denote \(\text{codim} C := \text{codim}_{|\mathcal{J}_n\mathcal{X}|} \varphi_{\infty,n}(C)\) for an \(n\)-cylinder \(C\).

Let \(A \subset |\mathcal{J}_n\mathcal{X}|\) be a subset. A function \(F : A \to \mathfrak{A}^{1/r}\) is called measurable if there exist a negligible subset \(B\) and countably many cylinders \(A_i\) such that \(A = B \cup \bigcup_{i \geq 1} A_i\) and \(F\) is constant on \(A_i\). For such \(F\), we define the motivic integration as follows:
\[ \int_A Fd\mu_\mathcal{X} := \sum_{i \geq 1} F(A_i) \cdot \mu_\mathcal{X}(A_i) \in \mathfrak{A}^{1/r} \cup \{\infty\}, \]
where it takes value \(\infty\) if there exist infinitely many \(i\) with \(\text{dim} F(A_i) \cdot \mu_\mathcal{X}(A_i) > m\) for some integer \(m\).

Next, we define the motivic measure on singular varieties \[5, 21\].

Let \(X\) be a variety of dimension \(d\). For a non-negative integer \(n\), a subset \(A \subset |\mathcal{J}_nX|\) is called stable at level \(n\) if
- \(A = \varphi_{\infty,n}^{-1}(\varphi_{\infty,n}(A))\),
- \(\varphi_{\infty,n}(A)\) is a constructible subset of \(|\mathcal{J}_nX|\), and
- for any \(m \geq n\) the truncation morphism \(\varphi_{m,n} : \varphi_{\infty,m}(A) \to \varphi_{\infty,n}(A)\) is a piecewise trivial fibration with fibers \(\mathbb{A}\{m-n\}^d\).

For such a subset \(A \subset |\mathcal{J}_nX|\), we define
\[ \mu_X(A) := \{\varphi_{\infty,n}(A)\}_L^{-nd} \in \mathfrak{A}^\frac{d}{r}. \]
A subset \(A \subset |\mathcal{J}_nX|\) is called negligible if there exist constructible subsets \(C_i \subset \varphi_{\infty,i}(\mathcal{J}_nX)\) for \(i \geq 0\) such that \(A = \bigcap_{i \geq 0} C_i\) and \(\lim_{i \to \infty} \text{dim} C_i - di = -\infty\) hold. As in the case of smooth DM stacks, we can define the motivic integration on singular varieties, replacing \(n\)-cylinders by stable subsets at level \(n\) in the above definition.

2.5. The transformation rule. Yasuda proved the transformation rule for a proper birational morphism from a smooth DM stack to a variety.

First, we define the shift function \[21\].

Let \(\mathcal{X}\) be a smooth DM stack of dimension \(d\), \(x \in |\mathcal{X}|\) a \(\mathbb{C}\)-valued point, and \(a : \mu_1 \hookrightarrow \text{Aut}(x)\) an embedding. Then the cyclic group \(\mu_1\) acts on the tangent space \(T_x\mathcal{X}\) by the embedding \(a\), and induces a decomposition
$T_x \mathcal{X} = \bigoplus_{i=1}^l T_{x,i}$, where $T_{x,i}$ is the eigenspace on which $\xi_i := e^{2\pi \sqrt{-1}/l} \in \mu_l$ acts by the multiplication of $\xi_i$. Then we define

$$\text{sht}(a) := d - \frac{1}{l} \sum_{i=1}^l i \cdot \dim T_{x,i} \in \mathbb{Q}.$$  

Since the space $|\mathcal{J}_0 \mathcal{X}|$ is set theoretically equal to $\bigsqcup_{i \geq 1} \bigsqcup_{x \in |\mathcal{X}|} \text{Conj} \left( \mu_l, \text{Aut}(x) \right)$, we can write a point of $|\mathcal{J}_0 \mathcal{X}|$ by a pair $(x, a)$ with $x \in |\mathcal{X}|$ and $a \in \text{Conj}(\mu_l, \text{Aut}(x))$. Then we can define the shift function on $|\mathcal{J}_0 \mathcal{X}|$ by followings

$$\text{sht} : |\mathcal{J}_0 \mathcal{X}| \rightarrow \mathbb{Q} : (x, a) \rightarrow \text{sht}(a).$$

This function is constant on each connected component $\mathcal{Y}$ of $|\mathcal{J}_0 \mathcal{X}|$. We define the shift function $\text{sht} \mathcal{X}$ on $|\mathcal{J}_0 \mathcal{X}|$ by the composite map

$$|\mathcal{J}_0 \mathcal{X}| \xrightarrow{\varphi_{\infty, 0}} |\mathcal{J}_0 \mathcal{X}| \xrightarrow{\text{sht}} \mathbb{Q}.$$  

So we have a measurable function

$$\mathbb{L}_{\infty, \mathcal{X}} : |\mathcal{J}_0 \mathcal{X}| \rightarrow \mathcal{R}^{1/r}$$

for some sufficiently divisible positive integer $r$.

Next, we define the order function along an ideal. Let $\mathcal{X}$ be a DM stack, $\mathcal{Y}$ a closed substack of $\mathcal{X}$, and $I$ its ideal sheaf on $\mathcal{X}$. For $\alpha \in |\mathcal{J}_0 \mathcal{Y}| \setminus |\mathcal{J}_0 \mathcal{X}|$, we define a rational number $\text{ord}_I \alpha$ as follows. The twisted jet $\alpha$ can be written by a morphism $\alpha : [\text{Spec} \mathbb{C}[\![t]\!] / \mu_l] \rightarrow \mathcal{X}$ for some $l \geq 1$. Hence we have the lift $\overline{\alpha} : [\text{Spec} \mathbb{C}[\![t]\!] \rightarrow \mathcal{X}$ and have $\overline{\alpha}^{-1} : \mathcal{O}_X \rightarrow \mathbb{C}[\![t]\!]$. Let $m$ be the integer satisfying $\overline{\alpha}^{-1}I = (t^m)$. Then we define $\text{ord}_I \alpha := \frac{m}{l}$.

Let $f : \mathcal{X} \rightarrow X$ be a morphism from a DM stack to a variety. Then $f$ induces a map $f_\infty : |\mathcal{J}_0 \mathcal{X}| \rightarrow |J_\infty X|$ as follows: For a twisted jet $D_{n, S}^l \rightarrow \mathcal{X}$, we have the composition $D_{n, S}^l \rightarrow \mathcal{X} \rightarrow X$. Since $D_{n, S}$ is the coarse moduli space of $D_{n, S}^l$, the above map uniquely factors through $D_{n, S}^l \rightarrow D_{n, S}$, and we have an $n$-th jet $D_{n, S}^l \rightarrow D_{n, S}$ to $X$. Since the map $D_{n, S}^l \rightarrow D_{n, S}$ is defined by $t \mapsto t^l$, the order functions on $\mathcal{X}$ and $X$ are compatible with $f_\infty$. That is, for an ideal sheaf $I$ on $X$, $\text{ord}_I \mathcal{O}_X = \text{ord}_I f_\infty$ holds.

Here, we can state Yasuda’s transformation rule for a proper birational morphism $f : \mathcal{X} \rightarrow X$ from a smooth DM stack to a variety. The Jacobian ideal sheaf of $f$ is defined to be the 0-th Fitting ideal sheaf of $\Omega_{\mathcal{X}/X}$, and we denote by $\text{Jac} f$.

**Theorem 2.8 (21).** Let $X$ be a variety, $\mathcal{X}$ a smooth DM stack, and $f : \mathcal{X} \rightarrow X$ a proper birational morphism. Let $A$ be a subset of $|\mathcal{J}_0 \mathcal{X}|$ and $F : A \rightarrow \mathcal{R}^{1/r}$ a measurable function. Then $F \circ f_\infty : f_\infty^{-1}(A) \rightarrow \mathcal{R}^{1/r}$ is measurable, and

$$\int_A F d\mu_X = \int_{f_\infty^{-1}(A)} (F \circ f_\infty) \mathbb{L}^{-\text{ord}_{\text{Jac} f} + \text{sht}} d\mu_X \in \mathcal{R}^{1/r} \cup \{\infty\}$$

holds.
2.6. Minimal log discrepancies and jet schemes. In [7], Ein, Mustaţi, and Yasuda showed that mld’s can be described by the language of jet schemes. Suppose $X$ is a $\mathbb{Q}$-Gorenstein variety with Gorenstein index $r$ and set $n := \dim X$. Then, we have a natural map

$$(\Omega^n_X)^{\otimes r} \longrightarrow \mathcal{O}_X(rK_X).$$

Since $\mathcal{O}_X(rK_X)$ is an invertible sheaf, we have an ideal sheaf $J_{r,X}$ such that the image of this map is $J_{r,X} \mathcal{O}_X(rK_X)$.

**Theorem 2.9** ([7]). Let $X$ be a normal, $d$-dimensional $\mathbb{Q}$-Gorenstein variety with Gorenstein index $r$, $Y_1, \ldots, Y_s$ proper closed subschemes of $X$, and $W$ a proper closed subset of $X$. If $q_1, \ldots, q_s$ are positive real numbers, then

$$\text{mld}_W(X, \sum_{i=1}^s q_i Y_i) = \inf_{m \in \mathbb{N}^s} \left\{ d - \sum_{i=1}^s q_i m_i - \dim \int_{A_{m}} \mathbb{L}^{\frac{1}{2}} \text{ord}_{J_{r,X}} d \mu_X \right\},$$

where $A_m := \varphi_{-1,0}^{-1}(W) \cap \bigcap_{1 \leq i \leq s} \text{ord}_{\mathcal{I}_{Y_i}}^{-1}(\geq m_i) \subset J_{\infty,X}$.

Moreover, if the mld is finite, then the infimum on the right-hand side is actually a minimum, and the minimum is attained by some $m \in S$, where $S$ is a finite subset of $\mathbb{N}^s$ and only depends on the numerical data of a log resolution of $(X, \sum_{i=1}^s q_i Y_i)$ and $W$. If the mld is infinite, then a negative value is attained by some $m \in S$.

**Remark 2.10.** The finite set $S$ can be constructed as follows. Take a log resolution $\pi : X' \to X$ of the log pair $(X, \sum_{i=1}^s q_i Y_i)$ and $W$. Let $D_1, \ldots, D_c$ be the divisors on $X'$ satisfying $\pi(D_j) \subset W$. Then we can take $S$ as

$$S = \{ m = (\text{ord}_{D_1} Y_1, \ldots, \text{ord}_{D_j} Y_s) \in \mathbb{N}^s \mid 1 \leq j \leq c \}.$$

3. Lower semi-continuity problems on varieties with quotient singularities

3.1. Minimal log discrepancies and twisted jet stacks. Let $X$ be a $d$-dimensional $\mathbb{Q}$-Gorenstein variety with Gorenstein index $r$, $\mathcal{a}$ an $\mathbb{R}$-ideal sheaf, and $W$ a closed subset of $X$. Assume that there exists a crepant resolution $f : \mathcal{X} \to X$ in the category of DM stacks. In this setting, we can describe $\text{mld}_W(X, \mathcal{a})$ by a motivic integration on $\mathcal{X}$.

The $\mathbb{R}$-ideal sheaf $\mathcal{a}$ can be written by $\mathcal{a} = \prod_{i=1}^s a_i^{q_i}$ with ideal sheaves $\mathcal{a}_i$ and positive real numbers $q_i$. By Theorem 2.9

$$\text{mld}_W(X, \mathcal{a}) = \inf_{m \in \mathbb{N}^s} \left\{ d - \sum_{i=1}^s q_i m_i - \dim \int_{A_{W,m}} \mathbb{L}^{\frac{1}{2}} \text{ord}_{J_{r,X}} d \mu_{\mathcal{X}} \right\},$$

where $A_{W,m} := \varphi_{-1,0}^{-1}(W) \cap \bigcap_{1 \leq i \leq s} \text{ord}_{\mathcal{I}_{\mathcal{a}_i}}^{-1}(\geq m_i) \subset |J_{\infty,\mathcal{X}}|$.

We apply Theorem 2.9 to the resolution $f$. We write $\mathcal{a}_i' := \mathcal{a}_i \mathcal{O}_X$.

**Lemma 3.1.** Let $X, \mathcal{X}, \mathcal{a}, \mathcal{a}'$ and $W$ be as above. Then we have

$$\text{mld}_W(X, \mathcal{a}) = \inf_{m \in \mathbb{N}^s} \left\{ d - \sum_{i=1}^s q_i m_i - \dim \int_{A'_{W,m}} \mathbb{L}^{\frac{1}{2}} d \mu_{\mathcal{X}} \right\},$$

where we denote

$$A'_{W,m} := (f \circ \varphi_{-1,0})^{-1}(W) \cap \bigcap_{i=1}^s \text{ord}_{\mathcal{a}_i'}^{-1}(\geq m_i) \subset |J_{\infty,\mathcal{X}}|. $$
Proof. First, we easily have $f_{\infty}^{-1}(A_{W,m}) = A'_{W,m}$.

By the definition of the Jacobian ideal $\text{Jac}_f$, the image of the canonical morphism $f^*\Omega^2_X \to \Omega^2_X$ is equal to $\text{Jac}_f \otimes \Omega^2_X$. Let $r$ be the Gorenstein index of $X$. By the definition of $J_{r,X}$, the image of the canonical morphism $(\Omega^d_X)^r \to \mathcal{O}_X(rK_X)$ is equal to $J_{r,X} \otimes \mathcal{O}_X(rK_X)$. Therefore we have an equation $\text{Jac}_{r,X} = J_{r,X} \mathcal{O}_X(-rK_X/X)$. Since $f$ is crepant in this case, we have
\[
\frac{1}{r}\text{ord}_{J_{r,X}} \circ f_{\infty} = \text{ord}_{\text{Jac}_f}.
\]

By Theorem 2.8, we can conclude
\[
\int_{A_{W,m}} L^1 d\text{ord}_{J_{r,X}} \mu_X = \int_{A'_{W,m}} L^s d\mu_X,
\]
which completes the proof.

3.2. Proof of Theorem 1.2. We prove a more general statement.

Theorem 3.2. Let $f : X \to T$ be a flat surjective morphism between varieties, $\mathcal{A}$ an $\mathbb{R}$-ideal sheaf on $X$, and $W$ a closed subset of $X$. Assume the following three conditions:

- $W$ is proper over $T$ and each $W_p$ is a proper subset of $X_p$,
- each $X_p$ is irreducible, reduced, normal, and $\mathbb{Q}$-Gorenstein, and
- there exist a DM stack $\mathcal{X}$ and a morphism $g : \mathcal{X} \to X$ such that for each closed point $p \in T$, the induced morphism $g_p : X_p \to X_p$ is a crepant resolution.

Then, the function
\[
|T| \to \mathbb{R} \cup \{ \pm \infty \}; \quad p \mapsto \text{mld}_{W_p}(X_p, \mathcal{A}_p)
\]
is lower semi-continuous.

To prove this theorem, we borrow an idea from the argument in [17, Proposition 2.3]. We need two lemmas. First one is about the set $S$ in Remark 2.10.

Lemma 3.3. Let $f : X \to T$, $\mathcal{A}$, and $W$ be as in Theorem 3.2. In addition, suppose that $\mathcal{A}_p \neq 0$ for any $p \in |T|$. Then, for each $p \in |T|$, we can take a finite set $S_p \subset \mathbb{N}^s$ in Theorem 2.4 for $(X_p, \mathcal{A}_p)$ and $W_p$ such that $\bigcup_{p \in T} S_p$ is also a finite set.

Proof. Take a log resolution $g : X' \to X$ of $(X, \mathcal{A})$ and $W$. Let $D = \bigcup_{i \in I} D_i$ be the union of the exceptional divisors and the supports of $\mathcal{A}\mathcal{O}_X$, and $g^{-1}(W)$. By generic smoothness, we can take a non-empty open set $U \subset T$ such that $X'$ is smooth over $U$ and $\bigcap_{i \in I} D_i$ is also smooth over $U$ for any subset $I' \subset I$. Take $V \subset X$ be a non-empty open set over which $g$ is isomorphic. Since $f$ is flat and surjective, $f(V)$ is also open. Then, for any $p \in U \cap f(V)$, the restriction to the fiber $g_p : X'_p \to X_p$ is a log resolution of $(X_p, \mathcal{A}_p)$ and $W_p$. Hence, we can take $S_p$ uniformly for $p \in U \cap f(V)$. Since $U \cap f(V)$ is a non-empty open subset of $T$, the assertion follows from the induction on $\dim T$. \qed
Second lemma is about the shift function on a family. For a smooth morphism \( f : \mathcal{X} \to T \) from a DM stack to a variety, the relative 0-th twisted jet stack \( \mathcal{J}_0(\mathcal{X}/T) \) can be defined. An element of \( |\mathcal{J}_0(\mathcal{X}/T)| \) can be written by \((p, \alpha)\) with \( p \in |T| \) and \( \alpha \in \mathcal{J}_0(\mathcal{X}_p) \). On \( |\mathcal{J}_0(\mathcal{X}/T)| \), the shift function \( \text{sht} \) can be defined by \( \text{sht}(p, \alpha) = \text{sht}(\alpha) \).

**Lemma 3.4.** Let \( f : \mathcal{X} \to T \) be as above. For a connected component \( \mathcal{V} \) of \( |\mathcal{J}_0(\mathcal{X}/T)| \), \( \text{sht} \) is constant on \( \mathcal{V} \).

**Proof.** We may assume that \( \mathcal{X} \) is a quotient stack \([M/G]\). Then the relative twisted jet stack can be written by

\[
\mathcal{J}_n([M/G]/T) = \bigsqcup_{l \geq 1} \bigsqcup_{a \in \text{Conj}(\mu_l, G)} [\mathcal{J}^{(a)}_n(M/T)/C_a].
\]

Take \( l \geq 1 \) and \( a : \mu_l \hookrightarrow G \) such that \( \mathcal{V} \subset [\mathcal{J}^{(a)}_n(M/T)/C_a] \). Then, for every \( x \in |\mathcal{V}|, a : \mu_l \to G \) factors through \( \text{Aut} x \subset G \).

Let \( T_{X/T} \) be the relative tangent bundle on \( \mathcal{X} \) over \( T \), and \( i : \mathcal{V} \to \mathcal{X} \) the projection. Then \( \mu_l \) acts on \( i^*T_{X/T} \) over \( T \), and this action is fiberwise compatible with the \( \mu_l \)-action on \( T_{X_u} \) in Section 2.3. This action induces a decomposition \( i^*T_{X/T} = \bigoplus_{i=1}^l T_i \), where \( T_i \) is the vector bundle on which \( \xi_i \in \mu_l \) acts by the multiplication of \( \xi_i \). It follows that for each \( i \), \( \dim T_{X_i} \) is constant on \( |\mathcal{V}| \), which completes the proof.

**Proof of Theorem 3.2.** If \( W_p = \emptyset \), then \( \text{mld}_{W_p}(X_p, \mathfrak{A}_p) = \infty \). If \( W_p \neq \emptyset \) and \( \mathfrak{A}_p = 0 \), then \( \text{mld}_{W_p}(X_p, \mathfrak{A}_p) = -\infty \). Since \( W \) is proper over \( T \),

\[
T' := f(W) \cap \{ p \in |T| \mid \mathfrak{A}_p = 0 \}
\]

is closed in \(|T|\). Replacing \( T \) by \( T \setminus T' \), we may assume that \( \mathfrak{A}_p \neq 0 \) for any \( p \in |T| \).

Since \( \mathfrak{A} \) is an \( \mathbb{R} \)-ideal sheaf, it can be written by \( \mathfrak{A} = \prod_{i=1}^s \mathfrak{A}_i \) with ideal sheaves \( \mathfrak{A}_i \) and positive real numbers \( q_i \). We denote \( \mathfrak{A}_i' := \mathfrak{A}_i \mathcal{O}_{\mathcal{X}} \) and denote by \( (\mathfrak{A}_i')_p \) the restriction of ideal \( \mathfrak{A}_i' \) to the fiber \( X_p \).

For \( p \in |T| \) and \( m \in \mathbb{N}^s \), we put

\[
B_{p,m} := (g_p \circ \varphi_{\infty, p})^{-1}(W_p) \cap \bigcap_{1 \leq i \leq s} \text{ord}_{(\mathfrak{A}_i')_p}^{-1}(m_i).
\]

Take a finite set \( S_p \subset \mathbb{N}^s \) for each closed point \( p \in T \) as in Lemma 3.3. Fix a multi-index \( m \in \bigcup_{p \in |T|} S_p \). By Lemma 3.1, it is sufficient to prove that the function

\[
|T| \to \mathbb{Q}; \quad p \mapsto \dim \int_{B_{p,m}} \mathbb{L}^2 x_p d\mu_{x_p}
\]

is upper semi-continuous.

Let \( \mathcal{Y}_i \) be the closed substack of \( \mathcal{X} \) corresponding to the ideal sheaf \( \mathfrak{A}_i' \). We identify the relative twisted jet stack \( \mathcal{J}_m(\mathcal{Y}_i/T) \) with a closed substack of \( \mathcal{J}_m(\mathcal{X}/T) \), and we also identify \( B_{p,m} \) with a closed subset of \( \mathcal{J}_\infty(\mathcal{X}/T)_p \).

For a connected component \( \mathcal{V} \) of \( |\mathcal{J}_0(\mathcal{X}/T)| \), the shift function is constant on \( \mathcal{V} \) by Lemma 3.4. Since \( |\mathcal{J}_0(\mathcal{X}/T)| \) has only finitely many connected components, it is sufficient to prove that the function

\[
|T| \to \mathbb{Z}; \quad p \mapsto \dim \mu_{x}((\varphi_{\infty, 0}^{-1}(\mathcal{V}) \cap B_{p,m})
\]
is upper semi-continuous for each \( Y \). Take a positive integer \( m' \) such that \( m' \geq m_i \) for any \( i \). Since \( B_{p,m} \) is an \( m' \)-cylinder, we have

\[
\mu_X((\varphi_{X,T}^{-1}(\mathcal{V}) \cap B_{p,m}) = \{S \cap (f \circ g \circ \varphi_{m',b})^{-1}(p)\} \mathbb{A}^{-\mu'},
\]

where we put

\[
S := (\varphi_{m',0}^{-1}(\mathcal{V}) \cap (\varphi_{m',b})^{-1}(g^{-1}(W)) \cap \bigcap_{1 \leq i \leq s} \varphi_{m',m_i}^{-1}((\mathcal{J}_{m_i}(Y_i/T))).
\]

Let \( F \) be the composite map

\[
S \hookrightarrow |\mathcal{J}_{m'}(X/T)| \overset{\varphi_{m',0}/T}{\longrightarrow} |\mathcal{J}_0(X/T)| \overset{\varphi_{0,b}}{\longrightarrow} |X| \overset{f}{\longrightarrow} |T|.
\]

Then this theorem reduces to the upper semi-continuity property of the function

\[
|T| \rightarrow \mathbb{Z}; \quad p \mapsto \dim F^{-1}(p).
\]

For each integer \( n \), we set

\[
S_{\geq n} := \{s \in S \mid \dim_s F^{-1}(f(s)) \geq n\},
\]

\[
|T|_{\geq n} := \{p \in |T| \mid \dim F^{-1}(p) \geq n\}.
\]

Then we have \( F(S_{\geq n}) = |T|_{\geq n} \). We need to show that \( |T|_{\geq n} \) is closed in \( |T| \), but we know only that \( S_{\geq n} \) is a closed subset of \( S \) by Lemma 2.4. The space \( |\mathcal{J}_{m'}(X/T)| \) admits a \( C^* \)-action (see Section 2.3). Each \( |\mathcal{J}_{m_i}(Y_i/T)| \) is a \( C^* \)-invariant closed subset of \( |\mathcal{J}_{m'}(X/T)| \), and each fibre of the morphism \( \varphi_{X,T}^1 \) is invariant on this action. Hence \( S_{\geq n} \) is a \( C^* \)-invariant closed subset of \( |\mathcal{J}_{m'}(X/T)| \). By Lemma 2.7, \( \varphi_{X,T}^1|_{\mathcal{J}_{m'}(X/T)}(S_{\geq n}) \) is closed in \( |\mathcal{J}_0(X/T)| \). Since both of the maps \( \varphi_{0,b}^1 \) and \( g \) are closed, the set

\[
|X|_{\geq n} := (g \circ \varphi_{0,b}^1)(\varphi_{m',0}^{-1}(S_{\geq n}))
\]

is also closed in \( |X| \). Because \( |X|_{\geq n} \) is a subset of \( |W| \) and \( W \) is proper over \( T \), we can conclude the set

\[
|T|_{\geq n} = f(|X|_{\geq n})
\]

is also closed in \( |T| \), which completes the proof.

Proof of Theorem 1.3. We prove (i) first. Since \( X \) is \( \mathbb{Q} \)-Gorenstein, we may assume \( \Delta = 0 \) by forcing \( \Delta \) to \( \mathfrak{a} \). For \( i = 1, 2 \), we denote by \( p_i : X \times X \rightarrow X \) the \( i \)-th projection. Since \( \mathfrak{a} \) is an \( \mathbb{R} \)-ideal sheaf, it can be written by

\[
\mathfrak{a} = \prod_{i=1}^n \mathfrak{a}_i^{q_i}
\]

with \( \mathbb{R} \)-ideal sheaves \( \mathfrak{a}_i \) and positive real numbers \( q_i \). Set \( \mathfrak{A} = \prod_{i=1}^n (p_i^*\mathfrak{a}_i)^{q_i} \), and \( W \) be the diagonal set of \( X \times X \). Applying Theorem 3.2 to the morphism \( p_2 : X \times X \rightarrow X \), the ideal \( \mathfrak{A} \), and the closed set \( W \), we have the assertion in (i).

Next we prove (ii). Since \( X \) is \( \mathbb{Q} \)-Gorenstein, we may assume \( \Delta = 0 \) by forcing \( \Delta \times T \) to \( \mathfrak{A} \). Applying Theorem 3.2 to the projection \( X \times T \rightarrow T \), the ideal \( \mathfrak{A} \), and the closed set \( \{x\} \times T \), we have the assertion in (ii).
Now, we can prove Corollary 1.4. Matsushita proved the same statement when $X$ is a smooth symplectic variety and $D_0$ is a $\mathbb{Q}$-divisor [16 Proposition 2.1, Lemma 2.1]. First, by Theorem 1.2 (1), this can be extended to the case when $X$ is a symplectic variety with only quotient singularities. In addition, Kawakita’s result about the ACC conjecture [10] can extend it to the case when $D_0$ is an $\mathbb{R}$-divisor.

Proof of Corollary 1.4 Let $Y$ be a normal variety and $\Gamma$ be a finite subset of $[0, 1]$. Then, we define a set $A(X, \Gamma)$ as

$$A(X, \Gamma) := \{ \text{mld}_x(X, \Delta) \mid (X, \Delta) \text{ is a log canonical pair, and } \Delta \in \Gamma \},$$

here we write $\Delta \in \Gamma$ if all coefficients of $\Delta$ are in $\Gamma$.

Let $\Gamma$ be the set of all coefficients of $D_0$. By the proof of [19 Theorem], it is sufficient to show that the set $\bigcup_{t \geq 0} A(X_t, \Gamma)$ satisfies the ascending chain condition and that the LSC conjecture holds for each $X_t$.

By [18 Main Theorem, Proposition 1], since $X_0$ has only terminal singularities, $X_0$ and $X_1$ can deform to a variety $X'$ by locally trivial deformations at the same time. So, inductively, we can say that $X_i$ has only quotient singularities and that $A(X_i, \Gamma) = A(X, \Gamma)$. By [10 Theorem 1.2], we already know that $A(X, \Gamma)$ is finite. Hence $\bigcup_{t \geq 0} A(X_t, \Gamma)$ is also finite. On the other hand, since $X_i$ has only quotient singularities, the LSC conjecture holds for each $X_i$ by Theorem 1.2. Thus, we are done. \hfill $\square$

4. The ideal-adic semi-continuity problem on varieties with a $\mathbb{C}^*$-action

4.1. Proof of Proposition 1.6 Before we start to prove Proposition 1.6 we provide some general remarks on Conjecture 1.5.

**Remark 4.1.** Let $(X, \Delta, a)$ and $Z$ be as in Conjecture 1.5 and $f : X' \to X$ a proper birational morphism from a $\mathbb{Q}$-Gorenstein variety $X'$. We denote $a_{\mathcal{O}_{X'}} := \prod_{i=1}^s (a_i \mathcal{O}_{X'})^{\nu_i}$. Suppose $\Delta' := f^*(K_X + \Delta) - K_{X'}$ is effective. Then, Conjecture 1.5 holds for $(X, \Delta, a)$ and $Z$ if the conjecture holds for $(X', \Delta', a_{\mathcal{O}_{X'}})$ and $f^{-1}(Z)$.

This follows from that $\text{mld}_Z(X, \Delta, c) = \text{mld}_{f^{-1}(Z)}(X', \Delta', c_{\mathcal{O}_{X'}})$ holds for any $\mathbb{R}$-ideal sheaf $c$ and that $a_i + I_Z = b_i + I_{f^{-1}(Z)}$ implies $a_i c_{\mathcal{O}_{X'}} + I_{f^{-1}(Z)} = b_i c_{\mathcal{O}_{X'}} + I_{f^{-1}(Z)}$.

**Remark 4.2.** Let $\prod_{i=1}^s a_i$ be an $\mathbb{R}$-ideal sheaf, $f : X' \to X$ a proper birational morphism, and $E$ a divisor on $X'$. If $c_X(E) \subset Z$, there exists a positive integer $l_E$ such that: if ideal sheaves $b_1, \ldots, b_s$ satisfy $a_i + I_{l_E} = b_i + I_{l_E}$, then

$$a_E(X, \Delta, \prod_{i=1}^s a_i) = a_E(X, \Delta, \prod_{i=1}^s b_i)$$

holds. In fact, if we take $l_E$ such that $\text{ord}_E f_{l_E} > \text{ord}_E a_i$ for each $i$, then $\text{ord}_E a_i = \text{ord}_E b_i$ holds, and this implies the above equation.

By this remark and Remark 2.1 we can conclude the inequality

$$\text{mld}_Z(X, \Delta, \prod_{i=1}^s a_i) \geq \text{mld}_Z(X, \Delta, \prod_{i=1}^s b_i)$$
In Conjecture 1.5. In fact, by Remark 2.1
\[
\text{mld}_Z(X, \Delta, \prod_{i=1}^s a_i^{r_i}) = a_E(X, \Delta, \prod_{i=1}^s a_i^{r_i})
\]
holds for some \(E\) when \(\text{mld}_Z(X, \Delta, \prod_{i=1}^s a_i^{r_i})\) is non-negative. Then, by the above remark and the definition of mld’s,
\[
a_E(X, \Delta, \prod_{i=1}^s a_i^{r_i}) = a_E(X, \Delta, \prod_{i=1}^s b_i^{r_i}) \geq \text{mld}_Z(X, \Delta, \prod_{i=1}^s b_i^{r_i})
\]
holds for any \(l > l_E\). When \(\text{mld}_Z(X, \Delta, \prod_{i=1}^s a_i^{r_i})\) is negative, by Remark 2.1, \(a_E(X, \Delta, \prod_{i=1}^s a_i^{r_i})\) is negative for some \(E\), and we can continue the same argument.

Let \(X = \text{Spec} A\) be an affine variety with a \(\mathbb{C}^*\)-action, and \(A = \bigoplus_{m \in \mathbb{Z}} A^{(m)}\) be the induced graded ring structure, where
\[
A^{(m)} := \{f \in A \mid \gamma \cdot f = \gamma^m f \text{ for any } \gamma \in \mathbb{C}^*\}.
\]
In what follows, we assume that the \(\mathbb{C}^*\)-action is of ray type, and \(A = \bigoplus_{m \geq 0} A^{(m)}\).

For an element \(f \in A\), we get the unique expression \(f = \sum_{m \geq 0} f^{(m)}\) with \(f^{(m)} \in A^{(m)}\). Then we define \(\deg f\) and \(f \circ t\) as follows:
\[
\deg f := \min\{m \mid f^{(m)} \neq 0\}, \quad f \circ t := \sum m f^{(m)} \in A[t].
\]
For an ideal \(I \subset A\), we define \(\widetilde{I} \subset A[t]\) as
\[
\widetilde{I} := \text{ideal} \left( \frac{f \circ t}{\deg f} \mid f \in A \setminus \{0\} \right).
\]
For \(\gamma \in \mathbb{C}\), the restriction \(\widetilde{I}_\gamma\) is defined as the ideal
\[
\widetilde{I}_\gamma := \{f(\gamma) \mid f(t) \in \widetilde{I} \subset A\}.
\]
It is clear that \(\widetilde{I}_1 = I\). For \(\gamma \in \mathbb{C}^*\), we have the ring automorphism
\[
\phi_\gamma : A \rightarrow A; \quad \sum m f^{(m)} \mapsto \sum m \gamma^m f^{(m)},
\]
and \(\phi_\gamma(\widetilde{I}_1) = \widetilde{I}_\gamma\) holds.

**Proof of Proposition 1.7.** Since \(X\) is \(\mathbb{Q}\)-Gorenstein, we may assume \(\Delta = 0\) by forcing \(\Delta\) to \(\prod_{i=1}^s a_i^{r_i}\). By exchanging \(t\) with \(t^{-1}\) in \(\mathbb{C}^* = \text{Spec} \mathbb{C}[t, t^{-1}]\), we may assume \(A = \bigoplus_{m \geq 0} A^{(m)}\). Since \(Z\) is the set of all \(\mathbb{C}^*\)-fixed points, it follows that \(I_Z = \bigoplus_{m \geq 1} A^{(m)}\). Since ideals \(a_1, \ldots, a_s\) are \(\mathbb{C}^*\)-invariant, they are homogeneous. Let \(f_{i1}, \ldots, f_{ik_i}\) be homogeneous elements in \(A\) such that they generate \(a_i\). Set \(l_i := 1 + \max 1 \leq j \leq k_i \{\deg f_{ij}\}\).

By Remark 2.2, it is sufficient to show that if ideal sheaves \(b_1, \ldots, b_s\) satisfy \(a_i + I_Z^{(l_i)} = b_i + I_Z^{(l_i)}\), then
\[
\text{mld}_Z(X, \prod_{i=1}^s a_i^{r_i}) \leq \text{mld}_Z(X, \prod_{i=1}^s b_i^{r_i})
\]
holds.
We first prove that \(a_i \subset (\overline{b}_i)_0\). Since \(a_i = \text{ideal}(f_{i1}, \ldots, f_{ik_i}) \subset b_i + \mathcal{I}_Z^1\), there exists \(h_{ij}\) for each \(1 \leq j \leq k_i\) such that

\[
f_{ij} + h_{ij} \in b_i, \quad h_{ij} \in \mathcal{I}_Z^1.
\]

In respect to the degrees, we have \(\deg f_{ij} \leq l_i - 1 < \deg h_{ij}\). Since \(f_{ij}\) is homogeneous, we have \(f_{ij} \in (\overline{b}_i)_0\). We thus get \(a_i \subset (\overline{b}_i)_0\).

This inclusion implies the inequality

\[
\text{mld}_Z(X, \prod_{i=1}^s a_i^{r_i}) \leq \text{mld}_Z(X, \prod_{i=1}^s (\overline{b}_i)_0^{r_i}).
\]

Because \((\overline{b}_i)_1 = b_i\), it is sufficient to show that

\[
\text{mld}_Z(X, \prod_{i=1}^s (\overline{b}_i)_0^{r_i}) \leq \text{mld}_Z(X, \prod_{i=1}^s (\overline{b}_i)_1^{r_i}).
\]

On the other hand, we have the ring automorphism \(\phi_\gamma : A \to A\) for \(\gamma \in \mathbb{C}^\ast\). By the definition of \(\overline{b}_i\), we have \(\phi_\gamma((\overline{b}_i)_1) = ((\overline{b}_i)_1)\). Hence we have

\[
\text{mld}_Z(X, \prod_{i=1}^s (\overline{b}_i)_1^{r_i}) = \text{mld}_Z(X, \prod_{i=1}^s (\overline{b}_i)_1^{r_i})
\]

for every \(\gamma \in \mathbb{C}^\ast\). Therefore, the proof can be completed by showing that the function

\[
|A^1| \to \mathbb{R} \cup \{-\infty\}; \quad p \mapsto \text{mld}_Z(X, \prod_{i=1}^s (\overline{b}_i)_1^{r_i})
\]

is lower semi-continuous.

In order to prove the semi-continuity, we consider the relative twisted jet stacks. Take a \(\mathbb{C}^\ast\)-equivariant crepant resolution \(X \to X\) and fix a positive integer \(l\) and \(m \in \mathbb{N}^s\). Let \(\mathcal{Y}_i \subset X \times \mathbb{A}^1\) be the closed substack corresponding to \(\mathfrak{b}_i\mathfrak{o}_{X \times \mathbb{A}^1}\). Take a positive integer \(m'\) such that \(m' \geq m_i\) holds for any \(i\). In addition, fix a connected component \(\mathcal{V}\) of \([\mathcal{J}_0(X \times \mathbb{A}^1/\mathbb{A}^1)]\). Consider the following twisted jet stacks and morphisms

\[
|\mathcal{J}_{m'}(X \times \mathbb{A}^1/\mathbb{A}^1)| \xrightarrow{\phi_{m',0}} |\mathcal{J}_0(X \times \mathbb{A}^1/\mathbb{A}^1)| \xrightarrow{\phi_{0,b}} |X \times \mathbb{A}^1| \xrightarrow{\mathcal{V}} |X \times \mathbb{A}^1| \xrightarrow{g} |\mathbb{A}^1|,
\]

where \(f\) is the second projection and \(g\) is the morphism induced by the resolution \(X \to X\). Then, we set

\[
S := \phi_{m',0}(\mathcal{V}) \cap \phi_{m',0}(g^{-1}(Z \times \mathbb{A}^1)) \cap \bigcap_{1 \leq i \leq s} \phi_{m',m_i}^{-1}(|\mathcal{J}_{m_i}(\mathcal{Y}_i/\mathbb{A}^1)|).
\]

Let \(F\) be the composite map \(S \to |\mathcal{J}_{m'}(X \times \mathbb{A}^1/\mathbb{A}^1)| \to |\mathbb{A}^1|\). Then, as in the proof of Theorem 3.2, the assertion can reduce to the upper semi-continuity property of the function

\[
|\mathbb{A}^1| \to \mathbb{Z}; \quad p \mapsto \dim F^{-1}(p).
\]

For each integer \(n\), we set

\[
S_{\geq n} := \{s \in S \mid \dim_n F^{-1}(F(s)) \geq n\},
\]

\[
|\mathbb{A}^1|_{\geq n} := \{p \in |\mathbb{A}^1| \mid \dim F^{-1}(p) \geq n\}.
\]

Then we have \(F(S_{\geq n}) = |\mathbb{A}^1|_{\geq n}\).
By the same reason as in the proof of Theorem 3.2 it follows that \((g \circ \psi_{m',b})(S_{\geq n})\) is closed in \(|X \times A^1|\). By definition, \(\tilde{b}_1\) is trivial over \(A^1 \setminus \{0\} = \mathbb{C}^*\). In addition, we note that \((g \circ \psi_{m',b})(S_{\geq n})\) is contained in \(|Z \times A^1|\) and that \(Z \subset X\) consists of \(\mathbb{C}^*\)-fixed points. Hence, if \((p, \gamma) \in (g \circ \psi_{m',b})(S_{\geq n})\) for some \(p \in |X|\) and \(\gamma \in A^1 \setminus \{0\}\), then \((p, \gamma') \in (g \circ \psi_{m',b})(S_{\geq n})\) for any \(\gamma' \in A^1 \setminus \{0\}\). Since \((g \circ \psi_{m',b})(S_{\geq n})\) is closed, the latter condition implies \(\langle p, 0 \rangle \in (g \circ \psi_{m',b})(S_{\geq n})\). Therefore we can conclude that \(F(S_{\geq n})\) is one of \(\emptyset, \{0\}\) or \(A^1\), which completes the proof.

\[\square\]

4.2. \textbf{The toric case.} If \((X, \Delta, \prod_{i=1}^s a_i\})\) is a toric triple and \(Z\) is a torus invariant closed set, then we can construct a \(\mathbb{C}^*\)-action on \(X\) as in Proposition 4.6.

\textit{Proof of Theorem 4.8.} By Remark 4.2, we may assume that \(X\) is an affine toric variety and \(Z\) is irreducible. In addition, by Remark 4.1, we may assume that \(X\) is an affine \(\mathbb{Q}\)-factorial toric variety. Note that a \(\mathbb{Q}\)-factorial toric variety has a crepant resolution in the category of smooth DM toric stacks. As in the proof of Proposition 4.6 we may assume \(\Delta = 0\).

Let \(n\) be the dimension of \(X\), \(M\) a free \(\mathbb{Z}\)-module of rank \(n\), and set \(M_{\mathbb{R}} := M \otimes \mathbb{R}\). Since \(X\) is an affine normal toric variety, we may assume \(X = \text{Spec} \mathbb{C}[\sigma \cap M]\) for some rational convex cone \(\sigma \subset M_{\mathbb{R}}\). For \(m \in \sigma \cap M\), we denote by \(\chi^m\) the corresponding function in \(\mathbb{C}[\sigma \cap M]\). We call such an element a \textit{monomial}. Since \(a_i\) and \(I_Z\) are torus invariant ideals, they are generated by monomials in \(\mathbb{C}[\sigma \cap M]\).

If \(Z = \emptyset\), then the assertion is trivial. In what follows, we assume \(Z \neq \emptyset\). Since \(Z\) is torus invariant, there exists a face \(F\) of \(\sigma\) such that \(I_Z\) is generated by the monomials \(\chi^m\) for \(m \in (\sigma \setminus F) \cap M\). Since \(Z \neq \emptyset\), we can take a hyperplane \(H\) such that \(\sigma \cap H = F\). Hence there exists \(w \in M^\vee\) such that \(\langle m, w \rangle \geq 1\) for every \(m \in (\sigma \setminus F) \cap M\) and that \(\langle m, w \rangle = 0\) for every \(m \in F\). We fix such \(w \in M^\vee\).

The element \(w \in M^\vee\) provides the \(\mathbb{C}^*\)-action on the ring \(\mathbb{C}[\sigma \cap M]\) as follows: for \(\gamma \in \mathbb{C}^*\)

\[\mathbb{C}[\sigma \cap M] \rightarrow \mathbb{C}[\sigma \cap M]; \quad \chi^m \mapsto \gamma^{\langle m, w \rangle} \chi^m.\]

Then, this \(\mathbb{C}^*\)-action is of ray type because \(\langle m, w \rangle \geq 0\) for every \(m \in \sigma \cap M\).

In addition, \(Z\) is the set of all \(\mathbb{C}^*\)-fixed points because \(I_Z = \bigoplus_{m \geq 1} A^{(m)}\).

Since \(a_1, \ldots, a_s\) are torus invariant, \(a_1, \ldots, a_s\) are \(\mathbb{C}^*\)-invariant. Therefore we can apply Proposition 4.6 to the toric pair and complete the proof. \[\square\]

\textbf{Acknowledgments}

The author expresses his gratitude to his advisor Professor Yujiro Kawamata for his encouragement and valuable advice. He is grateful to Atsushi Ito and Professors Shihoko Ishii, Daisuke Matsushita, Shin-nosuke Okawa, Shunsuke Takagi, and Yoshinori Gongyo for useful comments and suggestions. He is supported by the Grant-in-Aid for Scientific Research (KAKENHI No. 25-3003) and the Grant-in-Aid for JSPS fellows.
References

[1] F. Ambro, *The Adjunction Conjecture and its applications*, available at arXiv:9903060v3

[2] ——, *On minimal log discrepancies*, Math. Res. Lett. 6 (1999), no. 5-6, 573–580.

[3] T. de Fernex, L. Ein, and M. Mustaţă, *Shokurov’s ACC conjecture for log canonical thresholds on smooth varieties*, Duke Math. J. 152 (2010), no. 1, 93–114.

[4] T. de Fernex and M. Mustaţă, *Limits of log canonical thresholds*, Ann. Sci. Éc. Norm. Supér. (4) 42 (2009), no. 3, 491–515.

[5] J. Denef and F. Loeser, *Germs of arcs on singular algebraic varieties and motivic integration*, Invent. Math. 135 (1999), no. 1, 201–232.

[6] L. Ein and M. Mustaţă, *Inversion of adjunction for local complete intersection varieties*, Amer. J. Math. 126 (2004), no. 6, 1355–1365.

[7] L. Ein, M. Mustaţă, and T. Yasuda, *Jet schemes, log discrepancies and inversion of adjunction*, Invent. Math. 153 (2003), no. 3, 519–535.

[8] U. Görtz and T. Wedhorn, *Algebraic geometry I*, Advanced Lectures in Mathematics, Vieweg + Teubner, Wiesbaden, 2010.

[9] M. Kawakita, *Ideal-adic semi-continuity problem for minimal log discrepancies*, Math. Ann. 356 (2013), no. 4, 1359–1377.

[10] ——, *Discreteness of log discrepancies over log canonical triples on a fixed pair*, available at arXiv:1204.5248v1

[11] ——, *Ideal-adic semi-continuity of minimal log discrepancies on surfaces*, Michigan Math. J. 62 (2013), no. 2, 443–447.

[12] S. Keel and S. Mori, *Quotients by groupoids*, Ann. of Math. (2) 145 (1997), no. 1, 193–213.

[13] J. Kollár, *Which powers of holomorphic functions are integrable?*, available at arXiv:0805.0756v1

[14] J. Kollár and S. Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics, vol. 134, Cambridge University Press, 1998.

[15] G. Laumon and L. Moret-Bailly, *Champs algébriques*, Vol. 39, Springer-Verlag, Berlin, 2000.

[16] D. Matsushita, *On almost holomorphic Lagrangian fibrations*, available at arXiv:1209.1194v1

[17] M. Mustaţă, *Singularities of pairs via jet schemes*, J. Amer. Math. Soc. 15 (2002), no. 3, 599–615.

[18] Y. Namikawa, *On deformations of Q-factorial symplectic varieties*, J. Reine Angew. Math. 599 (2006), 97–110.

[19] V. V. Shokurov, *Letters of a bi-rationalist. V. Minimal log discrepancies and termination of log flips*, Tr. Mat. Inst. Steklova 246 (2004), no. Algebr. Geom. Metody, Svyazi i Prilozh., 328–351; English transl., Proc. Steklov Inst. Math. 3 (246) (2004), 315–336.

[20] T. Yasuda, *Twisted jets, motivic measures and orbifold cohomology*, Compos. Math. 140 (2004), no. 2, 396–422.

[21] ——, *Motivic integration over Deligne-Mumford stacks*, Adv. Math. 207 (2006), no. 2, 797–861.

Graduate School of Mathematical Sciences, the University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo 153-8914, Japan.
E-mail address: nakamura@ms.u-tokyo.ac.jp