The effect of coordinate and momentum uncertainties on collision of coherent electrons

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Abstract.
We have examined the head-on collision of two electrons in approximation of coherent states. We have shown that the character of collision depends mainly on ratio of initial relative electron’s momentum to momentum uncertainty of electrons. When this ratio becomes greater than 1, the Coulomb interaction does not practically influence the scattering.

1. Introduction

It is well known that solution of the electron-electron scattering problem, both classical and quantum, is generally reduced to finding the dependence of differential cross-section on the particles deflection angle. At that, the case of head-on collision is principally ignored despite the fact that it is in these scatterings quantum properties of electrons reveal themselves maximally. This is related first of all to their extreme mutual closing in, and therefore the manifestations of anti-symmetry of two-particle wave function become maximal. Moreover, the influence of Coulomb repulsion energy on two-particle system’s behavior must be most significant. In this paper we have used the coherent states as one-particle states and have examined the head-on collision of coherent electrons in triplet state in center-of-mass system.

2. Kinetic and interaction energy of two coherent electrons

Let us assume that at an instant $t = 0$ the first electron is in coherent state $|\tilde{\alpha}\rangle$ and the second one is in coherent state $|-\tilde{\alpha}\rangle$. States $|\tilde{\alpha}\rangle$ and $|\tilde{\alpha}\rangle$ are eigenstates for operator $\tilde{a}$ with eigenvalues $\tilde{\alpha}$ and $-\tilde{\alpha}$ correspondingly:

$$\tilde{a} |\pm\tilde{\alpha}\rangle = \pm \tilde{\alpha} |\pm\tilde{\alpha}\rangle, \quad \tilde{a} = \frac{\sigma}{\hbar} \tilde{r} + i \tilde{p}/2\sigma, \quad \tilde{\alpha} = \frac{\sigma}{\hbar} \langle \tilde{r}_{\alpha} \rangle + i \langle \tilde{p}_{\alpha} \rangle/2\sigma,$$

and may be written as [2]:

$$|\pm\tilde{\alpha}\rangle = \frac{1}{\big(\sqrt{\sigma} \sqrt{2\pi}\big)^3} \exp \left( -\frac{(\tilde{p} + \langle \tilde{p}_{\alpha} \rangle)^2}{4\sigma^2} \mp i \frac{\tilde{p} \langle \tilde{r}_{\alpha} \rangle}{\hbar} \right).$$
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where $\pm \langle \vec{p}_\alpha \rangle$ and $\pm \langle \vec{r}_\alpha \rangle$ are mean momentum and mean position of electrons at the instant $t = 0$, $\sigma$ is uncertainty of every component of particle's momentum in coherent states $|\pm \vec{\alpha} \rangle$:

$$\sigma^2 = \langle p_i^2 \rangle - \langle p_i \rangle^2.$$ 

Spatial part of two-particle state must be anti-symmetrical for triplet:

$$|\Psi \rangle = \frac{|\vec{\alpha} \rangle |-\vec{\alpha} \rangle - |\vec{\alpha} \rangle |-\vec{\alpha} \rangle}{\sqrt{2 \left(1 - |N|^2\right)}} \uparrow\uparrow,$$

where $N = \langle \vec{\alpha} | -\vec{\alpha} \rangle$ is overlapping integral, $|N|^2 = \exp \left(-4 |\vec{\alpha}|^2\right)$. At a given instant $t$:

$$|\Psi, t \rangle = \exp \left(-i \frac{H_0}{\hbar} t\right) |\Psi \rangle.$$ 

This state satisfies Schroedinger equation:

$$i\hbar \frac{d}{dt} |\Psi, t \rangle = H_0 |\Psi, t \rangle,$$

where $H_0 = \frac{\vec{p}_1^2}{2m} + \frac{\vec{p}_2^2}{2m}$. 

Average value of kinetic energy in the state $|\Psi, t \rangle$ comprises three terms:

$$\langle T \rangle = T_{cl} + T_{conf} + T_{corr} = \frac{\langle \vec{p}_\alpha \rangle^2}{m} + \frac{3\sigma^2}{m} + \frac{4\sigma^2 |\vec{\alpha}|^2 |N|^2}{m \left(1 - |N|^2\right)}.$$ 

The first term corresponds to free motion of classical particles, the second (energy of configuration) appears to be the contribution of momentum uncertainty; the third one (energy of correlation) appears due to overlapping of two non-orthogonal one-particle states.

Average value of Coulomb repulsion energy in the triplet state $|\Psi, t \rangle$

$$\langle V \rangle = \langle \Psi, t | \frac{e_0^2}{|\vec{r}_1 - \vec{r}_2|} |\Psi, t \rangle,$$

is equal to:

$$\langle V \rangle = e_0^2 \frac{1}{\bar{R}} \text{erf} \left(\frac{\sigma |\bar{R}|}{\hbar \sqrt{1 + \omega^2 t^2}}\right) + \frac{i}{|\bar{L}|} \text{erf} \left(\frac{i\sigma |\bar{L}|}{\hbar \sqrt{1 + \omega^2 t^2}}\right) |N|^2,$$

where

$$\bar{R} = 2 \left(\langle \vec{r}_\alpha \rangle + \frac{\langle \vec{p}_\alpha \rangle}{m} t\right), \quad \bar{L} = \frac{\hbar}{\sigma^2} \langle \vec{p}_\alpha \rangle - 2 \langle \vec{r}_\alpha \rangle \omega t, \quad \omega = \frac{2\sigma^2}{\hbar m}.$$
3. Head-on collision of two coherent electrons

Let us assume, that at the instant $t = 0$ the first electron is in coherent state $|\alpha\rangle$ and has mean momentum $\vec{p}_1(t = 0) = \langle \vec{p}_\alpha \rangle$, and mean position $\vec{r}_1(t = 0) = \langle \vec{r}_\alpha \rangle$; the second electron is in coherent state $|\alpha\rangle$ with mean momentum and mean position $\vec{p}_2(t = 0) = - \langle \vec{p}_\alpha \rangle$, $\vec{r}_2(t = 0) = - \langle \vec{r}_\alpha \rangle$, respectively. As far as we examine the case of collision, the initial momentum is negative for the first electron and positive for the second.

As the solution of Schroedinger equation in zero-order approximation we have used the state \[ |\psi\rangle = |\alpha\rangle \] in which $\langle \vec{p}_\alpha \rangle$ and $\langle \vec{r}_\alpha \rangle$ depend upon time according to Hamilton equations:
\[
\frac{dr}{dt} = \frac{1}{2} \frac{\partial H}{\partial p}, \quad \frac{dp}{dt} = - \frac{1}{2} \frac{\partial H}{\partial r},
\]
where
\[
H = \langle T \rangle + \langle V \rangle, \quad p = \langle \vec{p}_\alpha \rangle, \quad r = \langle \vec{r}_\alpha \rangle.
\]

In [3] it is shown that $\langle V \rangle$ can be considered as perturbation in wide enough range of $\langle \vec{p}_\alpha \rangle$ and $\langle \vec{r}_\alpha \rangle$. Since analytical solution of the system of equations (4) is not found we use its numerical solution.

In Fig.1 the two typical phase trajectories are shown for head-on collision of two coherent electrons. The distance between electrons and their momenta are in atomic units, initial distance is 5 a.u. One can see that according to the ratio of initial relative momentum to momentum uncertainty of each electron, two principally different kinds of behavior of phase trajectories are possible. As long as the initial momentum of electrons is less than some critical momentum $p_{cr}$ the effective deflection occurs. When initial momentum of electrons exceeds $p_{cr}$ the effective penetration is the case. Let us note that these two cases are experimentally indistinguishable due to identity of electrons.

Concerning the identity of particles the electron’s return time $t$ may be used as an observable feature of collision process. This term implies time span necessary to revert the distance between electrons to initial value. For calculation of $t$ the plots of dependence of electron’s coordinate upon time are used. The example of such dependence is shown in Fig.2 at the initial distance between electrons 5 a.u. It should be noticed that a significant delay near the origin is possible both in the case of deflection and penetration.

The dependence of return time $t$ on initial momentum at the initial distance between particles 5 a.u. is shown in Fig.3 with boxes. For comparison, analogous dependences for non-interacting particles (dashed line) and classical electrons (solid line) are given, and they almost coincide for large relative momenta. Fig.3 shows that dependence of return time $t$ on the initial momentum of electrons $p_0$ coincides with that for classical electrons when $p_0/\sigma < 0.1$ only. In the range $0.2 < p_0/\sigma < 1$ the Coulomb repulsion may be neglected, though as Fig.3 shows, this range corresponds to effective deflection. In the area $p_0/\sigma \approx 1$ a delay occurs. When initial momentum $p_0$ is large, $p_0/\sigma > 1$, interaction may be ignored and phase trajectories accord with effective penetration.
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4. Conclusions

The analysis of head-on collision of two coherent electrons has shown that the Coulomb repulsion is considerable in the area of small values of relative momenta only. For almost all values of initial relative momenta the system’s behavior practically does not differ from classical. Essentially quantum scattering can be observed only in the area \( p_0/\sigma \approx 1 \), when the distance between particles at the instant of stop is less than position uncertainty of electrons.

References

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[2] V.O. Gnatovskyy, C.V. Usenko, Ukr. J. Phys. 46,999-1006 (2001)
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Figure 1. Phase trajectories for head-on collision. Initial distance between electrons is 5 a.u., momentum uncertainty of particles is \( \sigma = 1 \) a.u.
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Figure 2. Dependence of first electron’s position on time. Initial coordinate of the first electron is equal to 2.5 a.u., $\sigma = 1$ a.u.

Figure 3. Dependence of return time of first electron on initial momentum $p_0$. Momentum uncertainty of electrons is $\sigma = 1$ a.u. Initial distance between particles is 5 a.u.