Distributed Nash equilibrium seeking for aggregative games with coupled constraints  *

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Abstract

In this paper, we study a distributed continuous-time design for aggregative games with coupled constraints in order to seek the generalized Nash equilibrium by a group of agents via simple local information exchange. To solve the problem, we propose a distributed algorithm based on projected dynamics and non-smooth tracking dynamics, even for the case when the interaction topology of the multi-agent network is time-varying. Moreover, we prove the convergence of the non-smooth algorithm for the distributed game by taking advantage of its special structure and also combining the techniques of the variational inequality and Lyapunov function.

Key words: Distributed algorithms; aggregative games; generalized Nash equilibrium; projected dynamics; coupled constraints.

1 Introduction

The seek of generalized Nash equilibrium for non-cooperative games with coupled constraints has been widely investigated due to various applications in natural/social science and engineering (such as telecommunication power allocation and cloud computation (Pang, Scutari, Facchinei \& Wang 2008, Ardagna, Paniucci \& Passacantando 2013)). Significant theoretic and algorithmic achievement has been done, referring to (Pavel 2007, Altman \& Solan 2009, Arslan, Demirkol \& Yueksel 2015) and (Facchinei \& Kanzow 2010).

Distributed equilibrium seeking algorithms guide a group of players or agents to cooperatively achieve the Nash equilibrium (NE), based on players’ local information and information exchange between their neighbors in a network. The NE seeking may be viewed as an extension of distributed optimization problems, which have been widely studied recently (see (Nedić \& Ozdaglar 2009, Shi, Johansson \& Hong 2013, Kia, Cortés \& Martínez 2015)), and on the other hand, distributed optimization problems can be handled with a game-theoretic approach (Li \& Marden 2013). In fact, in the study of complicated behaviors of strategic-interacted players in large-scale networks, it is quite natural to investigate game theory in a distributed way. For example, distributed convergence to NE of zero-sum games over two subnetworks was obtained in (Lou, Hong, Xie, Shi \& Johansson 2016). Moreover, a distributed fictitious play algorithm was proposed in (Swenson, Kar \& Xavier 2015), while a gossip-based approach was employed for seeking an NE of noncooperative games in (Salehisadaghiani \& Pavel 2016).

Aggregative games have become an important type of games since the well-known Cournot model was proposed, and have recently been studied in the literature, referring to (Jensen 2010, Corne \& Hartley 2012), for its broad application in public environmental models (Corne 2016), congestion control of communication networks (Barrera \& García 2015), and demand response management of power systems (Ye \& Hu 2017). Usually, linear aggregation functions and quadratic cost functions in such games were considered, for example, in (Parise, Gentile, Grammatico \& Lygeros 2015, Paccagnan, Gentile, Parise, Kamgarpour \& Lygeros 2016, Ye \& Hu 2017). Also, a recent result was given for distributed discrete-time algorithms to seek the NE of an aggregative game with time-varying topologies in (Koshal, Nedić \& Shanbhag 2016).

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The objective of this paper is to develop a novel distributed continuous-time algorithm for nonlinear aggregative games with linear coupled constraints and time-varying topologies. In recent years, continuous-time algorithms for distributed optimization become more and more popular (Shi et al. 2013, Kia et al. 2015, Yi, Hong & Liu 2016), partially because they may be easily implemented in continuous-time or hybrid physical systems. However, ideas and approaches for continuous-time design may not be the same as those for the discrete-time one. Thanks to various well-developed continuous-time methods, distributed continuous-time algorithms or protocols keep being constructed, but the (convergence) conditions may be different from those in discrete-time cases.

In our problem setup, every player tries to optimize its local cost function by updating its local decision variable. The cost function depends on not only the local variable but also a nonlinear aggregation. Moreover, feasible decision variables of players are coupled by linear constraints. Existing distributed algorithms for aggregative games (Koshal et al. 2016, Ye & Hu 2017) cannot solve our problems since they did not consider coupled constraints. The contribution of this paper can be summarized as follows:

- The aggregative game model in this paper generalizes the previous ones (Paccagnan et al. 2016, Ye & Hu 2017) by allowing nonlinear aggregation term and non-quadratic cost functions, and also those in (Koshal et al. 2016) by considering coupled constraints. In addition, the considered game can be non-potential.
- Inspired from distributed average tracking dynamics and projected primal-dual dynamics, we take advantage of continuous-time techniques to solve the distributed problem. With the new idea, our algorithm is described as a non-smooth multi-agent system with two interconnected dynamics: a projected gradient one for the equilibrium seeking, and a consensus one for the synchronization of the aggregation and the dual variables. In addition, our algorithm need not solve the best response subproblems, different from those in (Parise et al. 2015), and can keep private some information about the cost functions, local decisions, and constraint coefficients.
- We provide a method to prove the correctness and convergence of the continuous-time algorithm by combining the techniques from variational inequality theory and Lyapunov stability theory.

Notations: Denote \( \mathbb{R}^n \) as the \( n \)-dimensional real vector space; denote \( \mathbf{1}_n = (1, \ldots, 1)^T \in \mathbb{R}^n \), and \( \mathbf{0}_n = (0, \ldots, 0)^T \in \mathbb{R}^n \). Denote \( \text{col}(x_1, \ldots, x_n) = (x_1^T, \ldots, x_n^T)^T \) as the column vector stacked with column vectors \( x_1, \ldots, x_n \), \( \|\cdot\| \) as the Euclidean norm, and \( I_n \in \mathbb{R}^{n \times n} \) as the identity matrix. Denote \( \nabla f \) as the gradient vector of a function \( f \) and \( JF \) as the Jacobian matrix of a map \( F \). Let \( C_1 \pm C_2 = \{z_1 \pm z_2 | z_1 \in C_1, z_2 \in C_2\} \) be the Minkowski sum/minus of sets \( C_1 \) and \( C_2 \), and \( \text{rint}(C) \) be the relative interior of a convex set \( C \) (Rockafellar & Wets 1998, page 25 and page 64).

2 Preliminaries

In this section, we give some preliminary knowledge related to convex analysis, variational inequality, and graph theory.

A set \( C \subseteq \mathbb{R}^n \) is convex if \( \lambda z_1 + (1 - \lambda) z_2 \in C \) for any \( z_1, z_2 \in C \) and \( 0 \leq \lambda \leq 1 \). For a closed convex set \( C \), the projection map \( P_C : \mathbb{R}^n \to C \) is defined as

\[
P_C(x) \triangleq \text{argmin}_{y \in C} \|x - y\|.
\]

The following two basic properties hold:

\[
(x - P_C(x))^T (P_C(x) - y) \geq 0, \quad \forall y \in C, \quad (1)
\]

\[
\|P_C(x) - P_C(y)\| \leq \|x - y\|, \quad \forall x, y \in \mathbb{R}^n. \quad (2)
\]

For \( x \in C \), the tangent cone to \( C \) at \( x \) is

\[
\mathcal{T}_C(x) \triangleq \left\{ \lim_{k \to \infty} \frac{x_k - x}{t_k} | x_k \in C, t_k > 0, \quad \text{and} \quad x_k \to x, t_k \to 0 \right\}.
\]

and the normal cone to \( C \) at \( x \) is

\[
\mathcal{N}_C(x) \triangleq \{ v \in \mathbb{R}^n | v^T (y - x) \leq 0, \text{ for all } y \in C \}.
\]

Lemma 1 (Rockafellar & Wets 1998, Theorem 6.42) Let \( C_1 \) and \( C_2 \) be two closed convex subsets of \( \mathbb{R}^n \). If \( 0 \in \text{rint}(C_1 - C_2) \), then

\[
\mathcal{T}_{C_1 \cap C_2}(x) = \mathcal{T}_{C_1}(x) \cap \mathcal{T}_{C_2}(x), \quad \forall x \in C_1 \cap C_2.
\]

A function \( f : \mathbb{R}^n \to \mathbb{R} \) is convex if \( f(\lambda z_1 + (1 - \lambda) z_2) \leq \lambda f(z_1) + (1 - \lambda) f(z_2) \) for any \( z_1, z_2 \in C \) and \( 0 \leq \lambda \leq 1 \). A map \( F : \mathbb{R}^n \to \mathbb{R}^n \) is said to be monotone (strictly monotone) on a set \( \Omega \) if \( (x - y)^T (F(x) - F(y)) \geq 0 \) (or \( > 0 \)) for all \( x, y \in \Omega \) and \( x \neq y \). A differentiable map \( F \) is monotone if and only if the Jacobian matrix \( JF(x) \) (not necessarily symmetric) is positive semidefinite for each \( x \) (Rockafellar & Wets 1998, Theorem 12.3).

Given a subset \( \Omega \subseteq \mathbb{R}^n \) and a map \( F : \Omega \to \mathbb{R}^n \), the variational inequality, denoted by VI(\( \Omega, F \)), is to find a vector \( x \in \Omega \) such that

\[
(y - x)^T F(x) \geq 0, \quad \forall y \in \Omega,
\]

and the set of solutions to this problem is denoted by \( \text{SOL}(\Omega, F) \) (Facchinei & Pang 2003). When \( \Omega \) is closed and convex, the solution of VI(\( \Omega, F \)) can be equivalently reformulated via projection as follows:

\[
x \in \text{SOL}(\Omega, F) \iff x = P_{\Omega}(x - F(x)). \quad (3)
\]
Lemma 2 (Facchinei & Pang 2003, Corollary 2.2.5, and Theorem 2.2.3) Consider $\text{VI}(\Omega, F)$, where the set $\Omega \subset \mathbb{R}^n$ is convex and the map $F : \Omega \to \mathbb{R}^n$ is continuous. The following two statements hold:

1) if $\Omega$ is compact, then $\text{SOL}(\Omega, F)$ is nonempty and compact;

2) if $\Omega$ is closed and $F(x)$ is strictly monotone, then $\text{VI}(\Omega, F)$ has at most one solution.

The following lemma about a regularized gap function is important for our results.

Lemma 3 (Fukushima 1992) Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a differentiable map and $H(x) = P_{\Omega}(x - F(x))$. Define $g : \mathbb{R}^n \to \mathbb{R}$ as

$$g(x) = (x - H(x))^T F(x) - \frac{1}{2} \|x - H(x)\|^2.$$

Then $g(x) \geq 0$ is differentiable and its gradient is

$$\nabla g(x) = F(x) + (JF(x) - I_n)(x - H(x)).$$

Furthermore, it is known that the information exchange among agents can be described by a graph. A graph with node set $V$ and edge set $E$ is written as $G = (V, E)$ (Godsil & Royle 2001). If agent $i \in V$ can receive information from agent $j \in V$, then $(j, i) \in E$ and agent $j$ belongs to agent $i$’s neighbor set $N_i = \{j \mid (j, i) \in E\}$. $G$ is said to be undirected if $(i, j) \in E \iff (j, i) \in E$, and $G$ is said to be connected if any two nodes in $V$ are connected by a path (a sequence of distinct nodes in which any consecutive pair of nodes share an edge).

3 Problem Formulation

Consider an $N$-player aggregative game with coupled constraints as follows. For $i \in V = \{1, ..., N\}$, the $i$th player aims to minimize its cost function $J_i(x_i, x_{-i}) : \Omega \to \mathbb{R}$ by choosing the local decision variable $x_i$ from a local strategy set $\Omega_i \subset \mathbb{R}^n$, where $x_{-i} \triangleq \text{col}(x_1, ..., x_{i-1}, x_{i+1}, ..., x_N)$, $\Omega \triangleq \Omega_1 \times \cdots \times \Omega_N \subset \mathbb{R}^n$, and $n = \sum_{i \in V} n_i$. The strategy profile of this game is $x \triangleq \text{col}(x_1, ..., x_N) \in \Omega$. The aggregation map $\sigma : \mathbb{R}^n \to \mathbb{R}^m$, to specify the cost function as $J_i(x_i, x_{-i}) = \vartheta_i(x_i, \sigma(x))$ with a function $\vartheta_i : \mathbb{R}^{n_i} \to \mathbb{R}$, is defined as

$$\sigma(x) \triangleq \frac{1}{N} \sum_{i=1}^N \varphi_i(x_i),$$

where $\varphi_i : \mathbb{R}^{n_i} \to \mathbb{R}$ is a (nonlinear) map for the local contribution to the aggregation. In addition, the feasible strategy set of this game is $\mathcal{K} = \Omega \cap \mathcal{X}$, where $\mathcal{X}$ is a set for linear coupled constraints, defined as

$$\mathcal{X} \triangleq \{x \in \mathbb{R}^n \mid \sum_{i \in V} A_i x_i = \sum_{i \in V} b_i\} = \{x \in \mathbb{R}^n \mid Ax - b = 0\},$$

for some $A_i \in \mathbb{R}^{l \times n_i}$, $b_i \in \mathbb{R}^l$, $A = [A_1, ..., A_N]$, and $b = \sum_{i \in V} b_i$.

For such games with coupled constraints, the following concept of generalized Nash equilibrium is considered.

Definition 1 A strategy profile $x^*$ is said to be a generalized Nash equilibrium (GNE) of the game if

$$J_i(x_i^*, x_{-i}^*) \leq J_i(y, x_{-i}^*), \forall y : (y, x_{-i}^*) \in \mathcal{K}, i \in V.$$

Condition (6) means that all players simultaneously take their own best (feasible) responses at $x^*$, where no player can further decrease its cost function by changing its decision variable unilaterally.

Moreover, a strategy profile is said to be a variational equilibrium, or variational GNE, if it is a solution of $\text{VI}(\mathcal{K}, F)$, where the map $F(x) : \mathbb{R}^n \to \mathbb{R}^n$ is defined as

$$F(x) \triangleq \text{col}\{\nabla x_i J_1(\cdot, x_{-1}), ..., \nabla x_N J_N(\cdot, x_{-N})\}.$$ (7)

The variational GNEs are well-defined due to the following result.

Lemma 4 (Facchinei & Kanzow 2010, Theorem 3.9) If $\mathcal{K}$ is convex, every solution of the $\text{VI}(\mathcal{K}, F)$ is also a GNE.

The following assumption and theorem are associated with the game and the variational GNEs.

Assumption 1

- Smoothness: $\forall i \in V$, the cost function $J_i(x_i, x_{-i})$ is twice continuously differentiable.
- Monotonicity: $F(x)$ in (7) is strictly monotone.
- Feasibility: $\Omega$ is compact and convex.
- Constraint qualification: $0 \in \text{rint}(\Omega - \mathcal{X})$.

Theorem 1 Under Assumption 1, the considered game admits a unique variational GNE.

Proof. According to Lemma 2, the smoothness and feasibility in Assumption 1 guarantee the existence of a variational GNE, and the monotonicity in Assumption 1 guarantees the uniqueness. \qed

The constraint qualification in Assumption 1 is quite mild and can be easily verified. In fact, it suffices to check whether the set $\mathcal{K}$ has nonempty relative interior, i.e., $\text{rint}(\mathcal{K}) \neq \emptyset$, because in this case,

$$0 \in \text{rint}(\mathcal{K} - \mathcal{K}) \subseteq \text{rint}(\Omega - \mathcal{X}).$$

In the distributed design for our aggregative game, the communication topology for each player to exchange information is assumed as follows.
Assumption 2 The (time-varying) graph \(G(t)\) is undirected and connected.

Then we formulate our problem.

**Problem 1:** Design a distributed algorithm to seek the variational GNE for the considered aggregative game with coupled constraints.

Here are some remarks about our formulation:

- Our formulation is quite similar to that in (Koshal et al. 2016), but we further consider linear coupled constraints and study its distributed continuous-time GNE seeking algorithm.
- Players may not measure the values of the aggregation directly, in contrast to (Parise et al. 2015).
- Our aggregation function can be nonlinear and the cost functions can be non-quadratic, which are different from (Paccagnan et al. 2016) and (Ye & Hu 2017).

In addition, \(JF(x)\) can be asymmetric\(^1\) which was also discussed in (Paccagnan et al. 2016).

4 Main Results

In this section, we first propose our distributed algorithm and then analyze its correctness and convergence.

4.1 Distributed Algorithm

Let \(\alpha, \beta, \gamma > 0\) be some constants satisfying \(\alpha > (N - 1) \bar{f}_1\) and \(\beta > \gamma(N - 1) \bar{f}_2\), where

\[
\bar{f}_1 = \sup_{i \in V} \left( \sup_{x_i \in \Omega_i} \| \nabla \varphi_i(x_i) \| \sup_{y, z \in \Omega} \| y - z \| \right),
\]

\[
\bar{f}_2 = \sup_{i \in V} \left( \sup_{x_i \in \Omega_i} \| A_i x_i - b_i \| \right).
\]

For \(i \in V\), define the map \(G_i(\cdot) : \mathbb{R}^{n_i+m} \rightarrow \mathbb{R}^{n_i}\) as

\[
G_i(x_i, y_i) \triangleq \nabla x_i J_i(\cdot, x_i - \sigma)|_{\sigma(x)=y_i} = (\nabla x_i \theta_i(\cdot, \sigma) + \frac{1}{N} \nabla \sigma \theta_i(x_i, \cdot) T \nabla \varphi_i)|_{\sigma=y_i}.
\]

Then the distributed continuous-time algorithm to solve Problem 1 is designed as follows:

\[
\begin{aligned}
\dot{x}_i &= P_{\Omega_i}(x_i - G_i(x_i, \eta_i) - \frac{\gamma}{N} A_i^T \lambda_i) - x_i \\
\dot{\lambda}_i &= \beta \sum_{j \in N_i} \text{sgn}(\lambda_j - \lambda_i) + \gamma(A_i x_i - b_i) \\
\dot{\zeta}_i &= \alpha \sum_{j \in N_i} \text{sgn}(\eta_j - \eta_i) \\
\dot{\eta}_i &= \zeta_i + \varphi_i(x_i)
\end{aligned}
\]

where \(\text{sgn}(\cdot)\) is the sign function, \(\lambda_i(t) \in \mathbb{R}^l, \zeta_i(t) \in \mathbb{R}^m\). The initial conditions of algorithm (11) are provided as follows:

\[
x_i(0) \in \Omega_i, \quad \lambda_i(0) = A_i x_i(0) - b_i, \quad \zeta_i(0) = 0, \quad \eta_i(0) = 0.
\]

To set the parameters \(\alpha, \beta, \gamma\) in algorithm (11) needs the values of \(\bar{f}_1, \bar{f}_2\) in (8) and (9), which involves additional distributed calculation as follows: take variables \(z_i(0) = w_i\) for \(i \in V\), and update them by \(z_i(k+1) = \sup \{z_i(k), z_j(k), j \in N_i\}\); in this way, one can obtain \(\sup \{w_i, i \in V\}\) within \(N - 1\) steps.

**Remark 1** The design idea for this continuous-time algorithm (11) is totally different from that given in (Koshal et al. 2016) for discrete-time algorithms. Note that \(\eta_i\) in our algorithm is to estimate the value of aggregation \(\sigma(x)\), while \(\lambda_i\) is to estimate a dual variable associated with the coupled constraints. Moreover, our algorithm is fully distributed and also preserves some privacy because the information, such as the local cost functions, decision variables, and coefficients of the coupled constraints, need not be shared.

The solution of (11) (with a discontinuous righthand side) can be well defined in the Filippov sense, which is unique and absolutely continuous. For convenience, we can make \(\lim_{t \rightarrow +\infty} \nu_i(t) = 0\) for any \(i \in V\) with an exponential convergence rate.

4.2 Convergence Analysis

Here we give some correctness and convergence analysis for our proposed algorithm.

First of all, we get the following result by extending (Chen, Cao & Ren 2012, Theorem 3).

**Lemma 5** Under Assumption 2, if

\[
\alpha > (N - 1) \bar{f}, \quad \bar{f} \geq \sup_{t \in [0, \infty)} \| r_i(t) \|, \quad \forall i \in V
\]

then the following system

\[
\begin{aligned}
\nu_i(t) &= \alpha \sum_{j \in N_i(t)} \text{sgn}(\nu_j(t) - \nu_i(t)) \\
\nu_i(t) &= \mu_i(t) + r_i(t), \quad \mu_i(0) = 0
\end{aligned}
\]

1 A game is a potential game if there is a function \(P(x)\) such that \(\nabla P(x) = F(x)\) (Ye & Hu 2017). This equation holds if and only if the Jacobian matrix \(JF(x)\) is symmetric (Facchinei & Pang 2003, Theorem 1.3.1).
• $\sum_{i=1}^{N} |\nu_i - \frac{1}{N} 1^T \nu| \leq \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1,j\neq i}^{N} |\nu_i - \nu_j| \leq \frac{N-1}{2} \sum_{(i,j)\in E(t)} |\nu_i - \nu_j|$.  
Define $c(t) \triangleq \max_{i,j \in \mathcal{V}} |\nu_i(t) - \nu_j(t)|$ and

$$V(t) \triangleq \frac{1}{2} \| \nu(t) - \frac{1}{N} 11^T \nu(t) \|^2.$$  
Clearly, $c(t) \leq \frac{1}{2} \sum_{(i,j)\in E(t)} |\nu_i(t) - \nu_j(t)|$ and $V(t) = \frac{1}{2} \sum_{i=1}^{N} |\nu_i(t) - \frac{1}{N} \sum_{k=1}^{N} \nu_k(t)|^2 \geq 0$. Since $V(t)$ is absolutely continuous, we have that, for almost all $t > 0$,

$$\dot{V}(t) \leq -\frac{\alpha - (N-1)}{2} \sum_{(i,j)\in E(t)} |\nu_i(t) - \nu_j(t)| \leq -(\alpha - (N-1)) c(t) \leq 0.$$

Because $c(t) \geq 0$ is absolutely continuous and $(\alpha - (N-1)) c(t) \leq V(0) < +\infty$, $c(t) \to 0$ as $t \to +\infty$. Therefore, $V(t) \leq c(t)$ and $V(t) \leq (\alpha - (N-1)) c(t)$, $\forall n$, $\mathcal{V}$, $\mathbb{N}$, and $\nu$. Thus, for almost all $t \in [\tilde{t}, +\infty)$, with a sufficient large $\tilde{t}$, which implies the conclusion. \hfill $\square$

Next, we give the following result, whose proof is quite straightforward by Lemma 5.

**Lemma 6** Consider algorithm (11) under Assumption 2. Let $\sigma(x)$ be in (4) and define

$$\tilde{\lambda}(t) = \frac{\gamma}{N} \left( Ax(0) - b + \int_0^t (Ax(\tau) - b) d\tau \right).$$

Then, for any $i \in \mathcal{V}$, $|\nu_i(t) - \sigma(x(t))| \to 0$ and $|\lambda_i(t) - \tilde{\lambda}(t)| \to 0$ exponentially.

Let $X^* \times \Lambda^*$ be the solution set of the following equation with respect to $(x, \lambda)$

$$\begin{cases} 0 = P_{\Omega}(x - F(x) - \frac{\gamma}{N} A^T \tilde{\lambda}) - x, \\
0 = \frac{\gamma}{N} (Ax - b) 
\end{cases}$$

and let $X^* \times \Lambda^*$ be the positive limit set of $(x(t), \tilde{\lambda}(t))$, where $x(t)$ is in (11) and $\lambda(t)$ is in (14).

**Lemma 7** If $\lim_{t \to +\infty} \dot{x}(t) = 0$ and $\lim_{t \to +\infty} \dot{\lambda}(t) = 0$, then $X^* \times \Lambda^* \subseteq X^* \times \Lambda^*$.

Proof. If $X^* \times \Lambda^* \neq \emptyset$, then, for any $(x^*, \tilde{\lambda}^*) \in X^* \times \Lambda^*$, there exists $(t_q)_{q=1}^{+\infty}$ such that $\lim_{q \to +\infty} x(t_q) = x^*$ and $\lim_{q \to +\infty} \tilde{\lambda}(t_q) = \tilde{\lambda}^*$. By taking the limit of $t_q$ to (11) and (14), one obtains according to Lemma 6 that $(x^*, \tilde{\lambda}^*)$ is a solution of equation (15). Thus, the conclusion follows. \hfill $\square$

The next result reveals another property of $X^* \times \Lambda^*$, related to the variational GNE.

**Theorem 2** Under Assumption 1, $x^*$ is the variational GNE if and only if there exists some $\tilde{\lambda}^* \in \mathbb{R}^1$ such that $(x^*, \tilde{\lambda}^*) \in X^* \times \Lambda^*$.

Proof. Sufficiency. Suppose $(x^*, \tilde{\lambda}^*) \in X^* \times \Lambda^*$. Then it follows from the definition of projection operator that $x^* \in \Omega$. Moreover, $x^* \in \mathcal{X}$ since $Ax^* - b = 0$. By (3), $x^* \in \text{SOL}(\Omega, F(x) + \frac{\gamma}{N} A^T \tilde{\lambda}^*)$, i.e.,

$$(x - x^*)^T F(x^*) + \frac{\gamma}{N} A^T \tilde{\lambda}^* \geq 0, \forall x \in \Omega.$$  

Because $\mathcal{K} \subseteq \Omega$ and $(x^*)^T A^T \tilde{\lambda}^* = b^T \tilde{\lambda}^*$, we have

$$(x - x^*)^T F(x^*) \geq -\frac{\gamma}{N} (x - x^*)^T A^T \tilde{\lambda}^* = -\frac{\gamma}{N} (\tilde{\lambda}^*)^T (Ax - b) = 0, \forall x \in \mathcal{K}.$$  

Thus, $(x^*) = \text{SOL}(\mathcal{K}, F)$, i.e., $x^*$ is the variational GNE.

Necessity. Suppose that $x^*$ is the variational GNE. We claim that there exists some $\tilde{\lambda}^*$ such that

$$-F(x^*) \in \frac{\gamma}{N} A^T \tilde{\lambda}^* + \mathcal{N}_\Omega(x^*).$$

Otherwise, there is a hyperplane separating the point $-F(x^*)$ and the set $\{A^T \tilde{\lambda} : \tilde{\lambda} \in \mathbb{R}^1 \} + \mathcal{N}_\Omega(x^*)$. Namely, there is a vector $\omega \in \mathbb{R}^n$ such that

$$\omega^T F(x^*) < 0,$$

$$\omega^T A \omega = 0,$$

$$d^T \omega \leq 0, \forall d \in \mathcal{N}_\Omega(x^*).$$

From (18b) and (18c), $\omega \in \mathcal{N}_{\Omega}(x^*) \cap \mathcal{N}_\mathcal{K}(x^*)$. Moreover, with Assumption 1, $\omega \in \mathcal{T}_\mathcal{K}(x^*) = \mathcal{T}_\mathcal{K}(x^*)$ according to Lemma 1. Recalling the definition of the tangent cone, there exist $x_k \in \mathcal{K}$ and $t_k > 0$ such that

$$x_k \to x^*, t_k \to 0, \text{ and } \lim_{k \to +\infty} \frac{x_k - x^*}{t_k} = \omega.$$  

Consequently,

$$\omega^T F(x^*) = \lim_{k \to +\infty} \frac{(x_k - x^*)^T F(x^*)}{t_k} \geq 0,$$

which contradicts to (18a). It completes the proof. \hfill $\square$

Clearly, $x^* = \{x^*\}$ from Theorems 1 and 2, where $x^*$ is the unique variational GNE under Assumption 1.

Finally, it is time to give our convergence result.

**Theorem 3** Under Assumptions 1 and 2, system (11) is stable and converges to the set $X^* \times \Lambda^*$, that is,  

$$\begin{cases} \lim_{t \to +\infty} |\lambda_i(t) - \tilde{\lambda}(t)| = 0, \forall i = 1, ..., N \\
\lim_{t \to +\infty} ||(x(t), \tilde{\lambda}(t)) - P_{X^* \times \Lambda^*}(x(t), \tilde{\lambda}(t))|| = 0 
\end{cases}$$

with $\tilde{\lambda}(t)$ defined in (14).
Proof. Since \( \dot{x} \in T_{t}(x), x(t) \in \Omega \) for all \( t \geq 0 \). Also, we have
\[
\dot{x}(t) = P_{\Omega}(x_{i} - G_{i}(x_{i}, \sigma(x)) - \frac{\gamma}{N} A_{i}^{T} \tilde{\lambda}(t)) - x_{i}(t) + e_{i}(t),
\]
where \( \|e_{i}(t)\| = \|P_{\Omega}(x_{i} - G_{i}(x_{i}, \sigma(x)) - \frac{\gamma}{N} A_{i}^{T} \tilde{\lambda}(t)) - P_{\Omega}(x_{i} - G_{i}(x_{i}, \eta_{i}) - \frac{\gamma}{N} A_{i}^{T} \lambda_{i}(t))\| \leq \|G_{i}(x_{i}, \sigma(x) - G_{i}(x_{i}, \eta_{i}) + \frac{\gamma}{N} A_{i}^{T}(\lambda(t) - \lambda_{i}(t)))\|, \) for some \( k > 0 \). Let \( e(t) = col \{e_{1}(t), ..., e_{N}(t)\} \). Hence, \( e(t) \) vanishes exponentially according to Lemma 6.

Define
\[
\theta(t) = \begin{bmatrix} x(t) \\ \tilde{\lambda}(t) \end{bmatrix}, \quad \hat{\theta}(t) = \begin{bmatrix} F(x) + \frac{\gamma}{N} A^{T} \tilde{\lambda} \\ -\frac{\gamma}{N}(Ax - b) \end{bmatrix}, \quad \Theta \triangleq \Omega \times \mathbb{R}^{t}, \quad \hat{H}(\hat{\theta}) \triangleq P_{\Theta}(\hat{\theta} - \hat{\theta}(\hat{\theta})),
\]
and let \( \theta^{*} \in X^{*} \times \Lambda^{*} \). Consider the following Lyapunov function
\[
V(t) \triangleq (\theta - \hat{H}(\theta))^{T} \hat{F}(\hat{\theta}) - \frac{1}{2} \|\theta - \hat{H}(\theta)\|^{2} - \frac{1}{2} \|\theta - \theta^{*}\|^{2}.
\]
It follows from Lemma 3 that \( V(t) \geq 0 \) and
\[
\dot{V}(t) = (\nabla_{\theta} V)^{T} \dot{\theta}(t) = (\nabla_{\theta} V)^{T} (\hat{H}(\theta) - \theta) + \dot{e}(t), \quad (23)
\]
where
\[
\dot{e}(t) = [e(t), 0]^{T} \nabla_{\theta} V = e(t)^{T} [F(x) + \frac{\gamma}{N} A^{T} \tilde{\lambda}(t)]
\]
\[
+ (\nabla F(x) - I_{n})(x - P_{t}(x - F(x)) - \frac{\gamma}{N} A^{T} \tilde{\lambda}(t))
\]
\[
+ \frac{\gamma}{N} A^{T}(Ax - b) + x(t) - x^{\ast}].
\]

Since \( x(t) \in \Omega \) is bounded, \( \tilde{\lambda}(t) \) in (14) satisfies \( \|	ilde{\lambda}(t)\| \leq K_{1} + K_{2}t \) for \( t \geq 0 \) with some constants \( K_{1}, K_{2} > 0 \). Also, since \( e(t) \) is exponentially convergent, \( \|e(t)\| \leq (K_{3} + K_{4})e^{-K_{5}t} \), for some constants \( K_{3}, K_{4}, K_{5} > 0 \), which further implies
\[
\int_{0}^{+\infty} \|\dot{e}(t)\|dt < +\infty. \quad (24)
\]

On the other hand, after calculations, we have
\[
(\nabla_{\theta} V)^{T}(\hat{H}(\theta) - \theta) = -W_{1}(\theta) - W_{2}(\theta) - W_{3}(\theta) - W_{4}(\theta),
\]
where
\[
W_{1}(\theta) = (\theta^{*} - \hat{H}(\theta))^{T} (\hat{F}(\theta) + \hat{H}(\theta) - \theta), \quad (25a)
\]
\[
W_{2}(\theta) = (\theta - \theta^{*})^{T} \hat{F}(\theta^{*}), \quad (25b)
\]
\[
W_{3}(\theta) = (\theta - \theta^{*})^{T} (\hat{F}(\theta) - \hat{F}(\theta^{*})), \quad (25c)
\]
\[
W_{4}(\theta) = (\hat{H}(\theta) - \theta)^{T} \mathcal{J} \hat{F}(\theta)(\hat{H}(\theta) - \theta). \quad (25d)
\]

It follows from (1) that \( W_{1}(\theta) \geq 0 \). Moreover,
\[
W_{2}(\theta) \in (x - x^{*})^{T} (-\mathcal{N}_{\Omega}(x^{*})) \subseteq \mathbb{R}_{+}, \quad \forall \theta \in \Theta. \quad (26)
\]
Furthermore, for any \( \theta, \theta^{*} \in \Theta \) with \( x \neq x^{*} \), we obtain
\[
(\theta - \theta^{*})^{T}(\hat{F}(\theta) - \hat{F}(\theta^{*})) = (x - x^{*})^{T}(F(x) - F(x^{*})) > 0 \quad (27)
\]
since \( F(x) \) is strictly monotone. Then \( W_{3}(\theta) \geq 0 \). Also, because \( \mathcal{J} \hat{F}(\theta) \) is positive semidefinite, \( W_{4}(\theta) \geq 0 \). Therefore,
\[
\dot{V}(t) \leq -W_{3}(t) + \dot{e}(t). \quad (28)
\]
Then it follows from (24) that
\[
0 \leq \int_{0}^{+\infty} W_{3}(t)dt \leq V(0) - \lim_{t \to +\infty} sup V(t)
\]
\[
+ \int_{0}^{+\infty} \dot{e}(t)dt < +\infty,
\]
Consequently, \( W_{3}(t) \to 0 \) as \( t \to +\infty \), which implies \( \lim_{t \to +\infty} x(t) = x^{*} \). Moreover, \( \lim_{t \to +\infty} V(t) < +\infty \), which implies that \( \tilde{\lambda}(t) \) is bounded. Therefore, the positive limit set \( X^{+} \times \Lambda^{+} \neq \emptyset \). It follows from (14) that \( \lim_{t \to +\infty} \tilde{\lambda}(t) = Ax^{*} - b = 0 \). Furthermore, \( \dot{x}(t) \) is uniformly continuous because the trajectory of (11) is absolutely continuous and the right-hand side of the differential equation with respect to \( x(t) \) in (11) is uniformly continuous in \( t \). Since \( x(t) \) is convergent, \( \lim_{t \to +\infty} \dot{x}(t) = 0 \) by the well-known Barbalat’s lemma. Thus, (19) holds according to Lemmas 6 and 7. \( \square \)

**Remark 2** Our algorithm can be viewed as a distributed perturbed projected dynamics with the exponentially vanishing perturbation term \( e_{i}(t) \) in (20). Although some projected dynamics without any perturbation was studied, e.g., in (Yi et al. 2016), the analysis for the perturbed one is novel.

## 5 Numerical Examples

Two numerical examples are given in this section.

### 5.1 Nash-Cournot Game

Consider a Nash-Cournot game played by \( N \) competitive firms to produce a kind of commodity. For \( i \in \mathcal{V} = \{1, ..., N\} \), firm \( i \) chooses \( x_{i} \in \Omega_{i} \) as the quantity of the commodity to produce and has the cost function as \( \vartheta_{i}(x_{i}, \sigma) = (c_{i} - p(\sigma))x_{i} \), where \( c_{i} \) is the production price.
of firm $i$, and $p = d - N\sigma(x)$ is the market price determined by the aggregation function $\sigma(x) = \frac{1}{N} \sum_{j \in V} x_j^2$. Our numerical setting is as follows.

(i) $N = 20$ and $V = \{1, \ldots, 20\}$.
(ii) For each firm $i \in V$, $\Omega_i = [0, 20]$, $c_i = 10 + 20(i - 1)$ and $d = 1200$.
(iii) Firms from $i = 1$ to $i = 10$ share a scare resource as $\sum_{i=1}^{10} x_i = 20$.
(iv) the communication graph $\mathcal{G}(t)$ is time-varying and randomly generated.
(v) Parameter setting of our algorithm is $\alpha = 20$, $\beta = 400$ and $\gamma = 20$.

Figure 1 shows the convergence to the NE (the upper one) and GNE (the lower one), which illustrates the effectiveness of our algorithm.

5.2 Demand Response Management

Consider $N$ electricity users with the demand of energy consumption. For each $i \in V$, $x_i \in [\underline{r}_i, \bar{r}_i]$ is the energy consumption of the $i$th user and $C_i(x_i, \sigma(x))$ is the cost function in the following form

$$C_i(x_i, \sigma(x)) = k_i(x_i - \chi_i)^2 + P(\sigma(x))x_i,$$

where $k_i$ is constant and $\chi_i$ is the nominal value of energy consumption for $i = 1, \ldots, N$, with $P(\sigma(x)) = aN\sigma(x) + p_0$ and $\sigma(x) = \frac{1}{N} \sum_{i \in V} x_i$. We adopt the same numerical setting as given in (Ye & Hu 2017) as follows:

(i) $N = 5$, $k_i = 1$, $a = 0.04$, $p_0 = 5$.
(ii) $\chi_1 = 50$, $[\underline{r}_1, \bar{r}_1] = [45, 55]$; $\chi_2 = 55$, $[\underline{r}_2, \bar{r}_2] = [44, 66]$; $\chi_3 = 60$, $[\underline{r}_3, \bar{r}_3] = [46, 72]$; $\chi_4 = [52, 78]$; $\chi_5 = 70$, $[\underline{r}_5, \bar{r}_5] = [56, 84]$.

The NE of the game has been calculated in (Ye & Hu 2017) as $x^* = [45, 46.4, 51.3, 56.2, 61.1]^T$. Note that the total energy consumption at this NE may be too far from the normal value because $\sum_{i=1}^{5} x_i = \sum_{i=1}^{5} \chi_i - 25$. The communication graph is randomly generated and the parameters in our algorithm are given as $\alpha = 30$, $\beta = 100$, and $\gamma = 2$. Then the GNE is obtained as $x^* = [45.2, 50.1, 55, 59.9, 64.8]^T$. Figure 2 shows the convergence to the NE (the upper one) and GNE (the lower one), which again illustrates our algorithm.

6 Conclusions

In this paper, aggregative games with linear coupled constraints were considered and a distributed continuous-time projection-based algorithm was proposed for the GNE seeking. The correctness and convergence of the proposed non-smooth algorithm were proved by virtue of variational inequalities and Lyapunov functions, and moreover, two numerical examples were given for illustration.

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