The pigeonhole principle is not violated by quantum mechanics

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ABSTRACT
Bell’s theorem has long been considered to establish local realism as the fundamental principle that contradicts quantum mechanics. It is therefore surprising that the quantum pigeonhole effect points to the pigeonhole principle as yet another source of contradiction [Aharonov et al., Proc. Natl. Acad. Sci. USA, 113, 532-535 (2016)]. Here we construct two new forms of Bell’s inequality with the pigeonhole principle, then reconstruct Aharonov et al.’s weak measurement on a bipartite system. We show that in both cases it is counterfactual reasoning by the assumption of realism rather than the pigeonhole principle being violated by quantum mechanics. We further show that the quantum pigeonhole effect is in fact a new version of Bell’s theorem without inequality. With the pigeonhole principle as the same conduit, a comparison between two versions of Bell’s theorem becomes straightforward as it only relies on the orthonormality of the Bell states.

KEYWORDS
Bell’s theorem; pigeonhole principle; local realism; counterfactual reasoning; separability; weak measurement

1. Introduction

On the quantum theory of black holes, Stephen Hawking once said, “Not only does God play dice, …he sometimes confuses us by throwing them where they can’t be seen” [1]. We find this statement most fitting to the quantum pigeonhole effect discovered recently by Aharonov et al. [2]: When three quantum particles are put in two boxes, there are instances where no two particles are in the same box. This effect appears to be a violation of the pigeonhole principle, also known as Dirichlet’s drawer principle [3], which has been a fundamental tool in combinatorics and number theory [4]. Considering the legendary EPR paradox of 1935 [5], Bell’s theorem of 1964 [6], and numerous debates that follow, the quantum pigeonhole effect presents yet another

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challenge to not only our understanding of the reality of quantum states, but also our most basic intuition about counting numbers.

However, here we show that the pigeonhole principle does not contradict quantum mechanics, and the contradictions revealed by the quantum pigeonhole effect can be accounted for by counterfactual reasoning with the assumption of realism. We make our arguments by first constructing two new forms of Bell’s inequality with the pigeonhole principle, then reconstructing the weak measurement used by Aharonov et al. on a bipartite system with a generalized class of observables. These analyses show that the quantum pigeonhole effect is in fact a new version of Bell’s theorem without inequality. With the pigeonhole principle as the same conduit for both violation of Bell’s inequality and the quantum pigeonhole effect, we can directly compare these two versions of Bell’s theorem, and show that they are opposite aspects of the same phenomenon as they only differ in the directions of detecting entanglement.

2. Bell’s Inequality by the Pigeonhole Principle

In the following, we show that applying the pigeonhole principle to a bipartite system can directly give rise to Bell’s inequality. We first consider the case of perfect correlation. Given three binary random variables, $a, b, c = \pm 1$, we have three pigeons $a, b,$ and $c$, and two pigeonholes $+1$ and $-1$. The pigeonhole principle dictates that at least two of the variables take the same sign so that at least one of the three products $ab$, $ac$, and $bc$ takes the value $+1$, therefore,

$$ab + ac + bc \geq -1.$$  

(1)

Define the correlation $\rho(x, y) = \langle xy \rangle$ as the expected value of the product $xy$, where $x$ and $y$ represent the observables that respectively measure $x$ and $y$. If Eq. 1 holds in all possible conditions, then

$$\rho(a, b) + \rho(a, c) + \rho(b, c) \geq -1.$$  

(2)

In the case of anti-correlation, suppose that the three random variables are $a, b, c = \pm i$ (here $\pm i$ represent two orthogonal states so that there exists a Hermitian operator with corresponding eigenstates that we can measure). Then, the pigeonhole principle dictates that at least one of the three products is $-1$ so that

$$ab + ac + bc \leq 1,$$  

(3)

which leads to

$$\rho(a, b) + \rho(a, c) + \rho(b, c) \leq 1.$$  

(4)
Figure 1. Violation of Bell’s inequality derived by the pigeonhole principle (Eq. 2 and Eq. 4). (a): An experimental setting with three spin measurements in the XZ-plane (the $e_1$-$e_3$ plane). Measurement $a$ is set to vertical, and measurements $b$ and $c$ are set respectively to $\theta$ and $-\theta$ off vertical, $\theta \in [0, \pi]$. (b): When measuring the singlet $|\beta_{11}\rangle$, the total correlation $\rho(a, b) + \rho(a, c) + \rho(b, c)$ (solid red) and its components along $Z \otimes Z$ (dashed blue) and $X \otimes X$ (dotted green) are plotted as functions of $\theta$ (Eq. 20). A violation is observed when $\theta > 90^\circ$ and reaches the maximum at $\theta = 120^\circ$.

Apparantly, Eq. 2 and Eq. 4 belong to the family of Bell’s inequality. To see this, let $a, b, a', b' = \pm 1$ or $\pm i$, so that one of $(b + b')$ and $(b - b')$ is zero and the other is $\pm 2$ or $\pm 2i$, then

$$a(b + b') + a'(b - b') = \pm 2,$$

which leads to the CHSH inequality [7],

$$-2 \leq \rho(a, b) + \rho(a, b') + \rho(a', b) - \rho(a', b') \leq 2.$$  \hspace{1cm} (6)

Let $a' = -b$ and $b' = c$, Eq. 6 reduces to Eq. 2 if $b = \pm 1$ and reduces to Eq. 4 if $b = \pm i$.

The following experiment shows that the inequalities in Eq. 2 and Eq. 4 are violated by quantum mechanics. There are three directions of spin measurements, $a$, $b$, and $c$, in the same plane at equal intervals $\theta = 120^\circ$ (Fig. 1 illustrates a more general setting, including an eigen-analysis that will be presented in a later section). Given a sequence of pairs of particles prepared in the same Bell state, each time we choose two of the directions to measure the two particles, respectively. If the two particles are in the Bell state $|\beta_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, then quantum mechanics predicts that the probability of agreement between settings $x$ and $y$ is $P_{x,y} = \cos^2 \theta$ where $\theta$ is the angle between the two settings [8], so that the correlation is $\rho(x, y) = 2P_{x,y} - 1 = \cos 2\theta$, and

$$\rho(a, b) + \rho(a, c) + \rho(b, c) = 3 \cos 2\theta = -\frac{3}{2},$$

which violates Eq. 2 by a factor of 1.5. If the two particles are in the Bell state $|\beta_{11}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$, then quantum mechanics predicts that $\rho(x, y) = -\cos \theta$ so
that

\[ \rho(a, b) + \rho(a, c) + \rho(b, c) = -3 \cos \theta = \frac{3}{2}, \]  

which violates Eq. 4 by a factor of 1.5 as well.

The experimental setting we just described here is similar to the setting presented by Feynman [8], and essentially the same as the setting presented by Mermin [9] and Penrose [10] (they use the same three measurement directions at equal 120° intervals for each of the two observation parties). A major difference is that all of those accounts rely on a local realistic model without an explicit form of Bell’s inequality. This probably is due to the fact that the original Bell’s inequality [6] was given as

\[ 1 + \rho(b, c) \geq |\rho(a, b) - \rho(a, c)|, \]  

which can be rearranged as

\[ \rho(a, b) - \rho(a, c) - \rho(b, c) \leq 1, \]  

and neither of these two forms contradicts quantum mechanics at the setting of equal 120° intervals. Nevertheless, Eq. 2 and Eq. 4 are merely one rotation away from Eq. 10 (a proof is given later by Eq. 21). Therefore the accounts by Feynman, Mermin and Penrose on violation of Bell’s inequality apply to Eq. 2 and Eq. 4 as well, and they all lead to the rejection of local realism. Here we particularly emphasize that these two new forms of Bell’s inequality, albeit being derived by the pigeonhole principle, do not put the principle in contradiction of quantum mechanics. Instead, contradiction arises from counterfactual reasoning, a notion considered to be equivalent to the assumption of realism that physical properties have definite values without being measured [11].

A reexamination of Eq. 1 shows that the pigeonhole principle starts with (rather than making) the assumption that \(a, b, c = \pm 1\), but in the actual experiment only two of the settings are measured from one EPR pair at a time. For the Bell’s inequality in Eq. 2 to hold, the unmeasured setting must be assigned with a definite outcome +1 or −1. Therefore, it is counterfactual reasoning rather than the pigeonhole principle being violated by quantum mechanics.

### 3. The Quantum Pigeonhole Effect

We now show that the quantum pigeonhole effect discovered by Aharonov et al. [2] can also be accounted for by counterfactual reasoning. To do so, we reconstruct the weak measurement by Aharonov et al. on a system with only two particles.

Consider two particles to be put in two boxes labeled by 0 (left) and 1 (right). We first prepare each particle in an even superposition \( |+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \) so that the
Given that $\Pi$ finds two particles in the same box, $\Pi$ finds them in different boxes. Given that $\Pi + \Pi = I$ and $\langle \psi | \Pi \psi \rangle = \langle \psi | \Pi \psi \rangle = \frac{1}{2}$, the states immediately after the projective measurements are two maximally entangled Bell states,

$$\frac{\Pi_{\text{same}}}{\sqrt{\langle \psi | \Pi_{\text{same}} | \psi \rangle}} = \frac{1}{\sqrt{2}} ( |00 \rangle + |11 \rangle ) = | \beta_{00} \rangle,$$

$$\frac{\Pi_{\text{diff}}}{\sqrt{\langle \psi | \Pi_{\text{diff}} | \psi \rangle}} = \frac{1}{\sqrt{2}} ( |01 \rangle + |10 \rangle ) = | \beta_{01} \rangle.$$

Lastly, construct all four possible postselected states $| \phi_k \rangle$ as direct products of the orthogonal states $|+\rangle = \frac{1}{\sqrt{2}} ( |0 \rangle + |1 \rangle )$ and $|-\rangle = \frac{1}{\sqrt{2}} ( |0 \rangle - |1 \rangle )$,

$$| \phi_1 \rangle = |+\rangle |+\rangle,$$

$$| \phi_2 \rangle = |+\rangle |-\rangle,$$

$$| \phi_3 \rangle = |-\rangle |+\rangle,$$

$$| \phi_4 \rangle = |-\rangle |-\rangle,$$

where $\sum_k | \phi_k \rangle \langle \phi_k | = I$. Then, the probabilities of finding two particles in the same box or in different boxes with each $| \phi_k \rangle$ are respectively,

$$| \langle \phi_1 | \beta_{00} \rangle |^2 = | \langle \phi_4 | \beta_{00} \rangle |^2 = | \langle \phi_2 | \beta_{01} \rangle |^2 = | \langle \phi_3 | \beta_{01} \rangle |^2 = 0,$$

$$| \langle \phi_2 | \beta_{00} \rangle |^2 = | \langle \phi_3 | \beta_{00} \rangle |^2 = | \langle \phi_1 | \beta_{01} \rangle |^2 = | \langle \phi_4 | \beta_{01} \rangle |^2 = \frac{1}{2}.$$

That is, we can find $| \beta_{00} \rangle$ (two particles in the same box) with $| \phi_2 \rangle$ and $| \phi_3 \rangle$ but not with $| \phi_1 \rangle$ and $| \phi_4 \rangle$, and find $| \beta_{01} \rangle$ (two particles in different boxes) with $| \phi_1 \rangle$ and $| \phi_4 \rangle$ but not with $| \phi_2 \rangle$ and $| \phi_3 \rangle$.

A contradiction will arise if we extend Eq. 14 to a system of three or more particles with counterfactual reasoning. Suppose that the preselected state for a three-particle system is $| \Psi \rangle = |+\rangle_1 |+\rangle_2 |+\rangle_3$. Apply the intermediate measurement $\Pi_{\text{same}}$ in Eq. 11 on particles 1 and 2, then the postselected state $| \Phi \rangle = |+\rangle_1 |+i\rangle_2 |+i\rangle_3$ finds that

$$\langle \Phi | (\Pi_{\text{same}} \otimes I) | \Psi \rangle = \left( |+i\rangle_1 |+i\rangle_2 \Pi_{\text{same}} |+\rangle_1 |+\rangle_2 \right) \langle +\rangle_3$$

$$= \frac{1}{\sqrt{2}} \langle \phi_1 | \beta_{00} \rangle \langle +| +\rangle_3 = 0,$$

which means that we cannot find particles 1 and 2 in the same box. Let $\lambda_i$ be a local binary variable denoting the final location of particle $i$, we have $\lambda_1 \neq \lambda_2$. Extend this result to every particle pair, we have the quantum pigeonhole effect, $\lambda_1 \neq \lambda_2$, $\lambda_2 \neq \lambda_3$ and $\lambda_1 \neq \lambda_3$, which is an apparent contradiction to the pigeonhole principle. However,
this contradiction is a consequence of counterfactual reasoning. Eq. 15 makes clear that in each measurement only two particles have been measured through the intermediate state \( |\beta_{00} \rangle \), and to the best of our knowledge, the unmeasured third particle remains in the superposition \( |+ \rangle = \frac{1}{\sqrt{2}}(|0 \rangle + |1 \rangle) \). That is, contradiction arises when we attempt to compare all three \( \lambda \) values at the same time by the assumption that any particle must have a \( \lambda \) value without being measured, an assumption made by counterfactual reasoning not by the pigeonhole principle.

4. Bell’s Theorem without Inequality

The argument above indicates that the quantum pigeonhole effect is in fact a new version of Bell’s theorem without inequality, similar to the GHZ theorem [12] and Hardy’s proof of nonlocality [13] but in a simpler construction. By our formulation, it only relies on the orthonormality of the Bell states. Crucially, with the pigeonhole principle as the same conduit, we can now directly compare Bell’s theorem without inequality with the version that exhibits violation of Bell’s inequality. To see this clearly, we observe that all versions of Bell’s theorem without inequality hinge on some zero-probability events, and that the quantum pigeonhole effect hinges on the vanishing trace when a certain postselected state \( |\phi_k \rangle \) is orthogonal to a Bell state (Eq. 14). Combine those postselected states into a pair of density operators,

\[
\rho_Y^+ = \frac{1}{2} (|\phi_1 \rangle \langle \phi_1| + |\phi_4 \rangle \langle \phi_4|) = \frac{1}{2} (|\beta_{01} \rangle \langle \beta_{01}| + |\beta_{10} \rangle \langle \beta_{10}|),
\rho_Y^- = \frac{1}{2} (|\phi_2 \rangle \langle \phi_2| + |\phi_3 \rangle \langle \phi_3|) = \frac{1}{2} (|\beta_{00} \rangle \langle \beta_{00}| + |\beta_{11} \rangle \langle \beta_{11}|),
\]

where \( |\beta_{00} \rangle = \frac{1}{\sqrt{2}}(|00 \rangle + |11 \rangle) \), \( |\beta_{01} \rangle = \frac{1}{\sqrt{2}}(|01 \rangle + |10 \rangle) \), \( |\beta_{10} \rangle = \frac{1}{\sqrt{2}}(|00 \rangle - |11 \rangle) \), and \( |\beta_{11} \rangle = \frac{1}{\sqrt{2}}(|01 \rangle - |10 \rangle) \). This shows that each of \( \rho_Y^\pm \) is an equal mixture of product states \( |\phi_k \rangle \langle \phi_k| \) therefore a separable state that contains no quantum correlation, yet at the same time it is an equal mixture of two maximally entangled Bell states. By the orthonormality of the Bell states, \( \rho_Y^\pm \) produce exactly the same vanishing effect as the individual post-selected states \( |\phi_k \rangle \) in Eq. 14,

\[
\text{tr} (\rho_Y^+ |\beta_{00} \rangle \langle \beta_{00}|) = \text{tr} (\rho_Y^- |\beta_{01} \rangle \langle \beta_{01}|) = 0,
\text{tr} (\rho_Y^- |\beta_{00} \rangle \langle \beta_{00}|) = \text{tr} (\rho_Y^+ |\beta_{01} \rangle \langle \beta_{01}|) = \frac{1}{2}.
\]

It can be shown that \( \rho_Y^\pm \) belong to a family whose members are separable states
and yet are equal mixtures of two maximally entangled Bell states,

\[
\begin{align*}
    \rho_Y^+ &= \frac{1}{2} (|\beta_{00}\rangle \langle \beta_{00}| + |\beta_{10}\rangle \langle \beta_{10}|) = \frac{1}{4} (I \otimes I + Z \otimes Z), \\
    \rho_X^- &= \frac{1}{2} (|\beta_{00}\rangle \langle \beta_{00}| + |\beta_{01}\rangle \langle \beta_{01}|) = \frac{1}{4} (I \otimes I + X \otimes X), \\
    \rho_Y^- &= \frac{1}{2} (|\beta_{00}\rangle \langle \beta_{00}| + |\beta_{11}\rangle \langle \beta_{11}|) = \frac{1}{4} (I \otimes I - Y \otimes Y), \\
    \rho_X^+ &= \frac{1}{2} (|\beta_{01}\rangle \langle \beta_{01}| + |\beta_{10}\rangle \langle \beta_{10}|) = \frac{1}{4} (I \otimes I + Y \otimes Y), \\
    \rho_Z^- &= \frac{1}{2} (|\beta_{01}\rangle \langle \beta_{01}| + |\beta_{11}\rangle \langle \beta_{11}|) = \frac{1}{4} (I \otimes I - Z \otimes Z),
\end{align*}
\]  

(17)

where \(X, Y, Z\) are the Pauli matrices. There exists a unitary transformation between each pair of these states, except that the transformation between \(\rho_Y^+\) and \(\rho_Y^-\) is a partial transposition that ensures the separability of both states \([14,15]\). Then, all zero-probability events when measuring a Bell state can be summarized as an effect of the vanishing trace

\[
\frac{1}{4} \text{tr} \left( (\sigma_i \otimes \sigma_i)(\sigma_j \otimes \sigma_j) \right) = \delta_{ij},
\]

(18)

where \(\sigma_0 = I, \sigma_1 = X, \sigma_2 = Y, \sigma_3 = Z\), \((\sigma_i \otimes \sigma_i)\) are components of \(\rho_i^\pm\), and \((\sigma_j \otimes \sigma_j)\) are components of a Bell state by the Pauli representation,

\[
\begin{align*}
    |\beta_{00}\rangle \langle \beta_{00}| &= \frac{1}{2} (I \otimes I + Z \otimes Z + X \otimes X - Y \otimes Y), \\
    |\beta_{10}\rangle \langle \beta_{10}| &= \frac{1}{2} (I \otimes I + Z \otimes Z - X \otimes X + Y \otimes Y), \\
    |\beta_{01}\rangle \langle \beta_{01}| &= \frac{1}{2} (I \otimes I - Z \otimes Z + X \otimes X + Y \otimes Y), \\
    |\beta_{11}\rangle \langle \beta_{11}| &= \frac{1}{2} (I \otimes I - Z \otimes Z - X \otimes X - Y \otimes Y).
\end{align*}
\]

Eq. 18 is the essence underlying both Bell’s theorem without inequality and violation of Bell’s inequality. Specifically, the entanglement in each Bell state is represented by the maximal correlation or anti-correlation in the directions of \(\pm \sigma_j \otimes \sigma_j, j = 1, 2, 3\). A zero-probability event can be defined when a member of Eq. 17 annihilates entanglement in a single direction. Then, the quantum pigeonhole effect can be produced when \(|\beta_{00}\rangle \langle \beta_{00}|\) is measured by any pair of \(\rho_i^\pm\) so that two particles in the same box may vanish. For example, the \(Y \otimes Y\) component of \(\rho_Y^+\) couples with the \(-Y \otimes Y\) component of \(|\beta_{00}\rangle \langle \beta_{00}|\) to annihilate the trace in \(I \otimes I\), whereas cross terms such as \((Y \otimes Y)(X \otimes X)\) remain traceless. Similarly, \(|\beta_{11}\rangle \langle \beta_{11}|\) vanishes with \(\rho_X^+\) in the direction of \(X \otimes X\), \(|\beta_{01}\rangle \langle \beta_{01}|\) vanishes with \(\rho_Z^-\) in the direction of \(Z \otimes Z\), and so on.

In contrast, violation of Bell inequality requires a measurement that collects correlations in multiple directions. To see this, we can define a Bell operator for the inequalities in Eq. 2 and Eq. 4,

\[
\mathcal{B} = \sigma \cdot a \otimes \sigma \cdot b + \sigma \cdot a \otimes \sigma \cdot c + \sigma \cdot b \otimes \sigma \cdot c,
\]

(19)
where \( \sigma \cdot a = \sum a_i \sigma_i \), \( \sum_i a_i^2 = 1 \), \( \rho(a, b) = \langle \sigma \cdot a \otimes \sigma \cdot b \rangle \), and so on. Suppose that all three vectors are in the \( XZ \)-plane (because components of \( \mathcal{B} \) in different planes would vanish in the trace, cf., Eq. 18), where \( a = (0, 0, 1) \), \( b = (\sin \alpha, 0, \cos \alpha) \), \( c = (-\sin \beta, 0, \cos \beta) \), and \( \alpha, \beta \in [0, \pi] \) (cf., Fig. 1). We can omit cross terms such as \( Z \otimes X \) and \( X \otimes Z \) because they remain traceless when measuring a Bell state or any mixture of the Bell states. Then, \( \mathcal{B} \) is reduced to

\[
\mathcal{B}_R = (\cos \alpha + \cos \beta + \cos \alpha \cos \beta)(Z \otimes Z) - \sin \alpha \sin \beta \, (X \otimes X).
\]

(20)

When measuring \(|\beta_00\rangle\) or \(|\beta_{11}\rangle\), the function

\[
F = \cos \alpha + \cos \beta + \cos \alpha \cos \beta - \sin \alpha \sin \beta
\]

is minimized to the value \(-\frac{3}{2}\) when \( \alpha = \beta = 120^\circ \) (Fig. 1 shows \(-F\) as a function of \( \alpha = \beta = \theta \)). This gives us a Bell operator that maximally violates Bell’s inequality

\[
\mathcal{B}_M = -\frac{3}{4} (Z \otimes Z + X \otimes X),
\]

(21)

with eigenpairs \((-\frac{3}{2}, |\beta_{00}\rangle\), \((0, |\beta_{01}\rangle\), \((0, |\beta_{10}\rangle\) \) and \((\frac{3}{2}, |\beta_{11}\rangle\) \), so that the bounds of the maximal violation are \(\pm \frac{3}{2}\). The same bound has been derived by [16], but our method here is much simpler. Replacing \( c \) in Eq. 19 with \( c' = -c \) gives us a Bell operator corresponding to the original Bell inequality in Eq. 10, which exhibits the maximal violation at the same bounds when \( \theta_{a,c'} = 60^\circ \), \( \theta_{b,c'} = 60^\circ \), and \( \theta_{a,b} = 120^\circ \), but no violation at equal 120° intervals. By a similar method, the Bell operator that maximally violates the CHSH inequality (Eq. 6) can be written in the same form but with a different coefficient, \( \mathcal{B}^{CHSH}_M = \sqrt{2}(Z \otimes Z + X \otimes X) \), whose largest and smallest eigenvalues are \(\pm 2\sqrt{2}\), known as the Tsirelson bound [17].

A comparison of Eq. 17 and Eq. 21 makes clear that the vanishing trace in the quantum pigeonhole effect and violation of Bell’s inequality are two sides of the same coin as they only differ in the directions of detecting entanglement. Another advantage of this formulation is that given the SU(2)-SO(3) homomorphism \( U \sigma_j U^\dagger = \sigma \cdot (P e_j) \), where \( j = 1, 2, 3 \), \( U \in SU(2) \) and \( P \in SO(3) \), the observables in Eq. 17 and Eq. 21 have an intuitive geometric representation on the Euclidean sphere (e.g., Fig. 1). Furthermore, these observables can be easily combined into an entanglement witness operator for more complex mixtures of the Bell states. For example, the entanglement witness operator for the Werner state by [18,19] can be rewritten as

\[
W = \frac{1}{3} (I \otimes I + Z \otimes Z + X \otimes X + Y \otimes Y).
\]
5. Conclusion

A signature of quantum entanglement is that it often suggests a conflict between our intuitions of the classical world and the quantum world. The quantum pigeonhole effect provides yet another example of such conflict, particularly about what can be counted as an event and what a quantum state represents. In the present paper, we argue that much of the conflict can be resolved if we give up counterfactual reasoning, rather than questioning the classical pigeonhole principle. In the case where Bell’s inequality is violated, an event of agreement or disagreement between two experimental settings occurs only when we make a measurement on them. In the case of the quantum pigeonhole effect, the superposition $|\pm\rangle$ represents our best knowledge of the unmeasured particle, which cannot be updated without a measurement. From an operational point of view, giving up counterfactual reasoning in both cases simply means not to equate a quantum state with the eigenvalues of the observable.

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References

[1] Hawking SW, Penrose R. The nature of space and time. Princeton (NJ): Princeton University Press; 1996.
[2] Aharonov Y, Colombo F, Popescu S, et al. Quantum violation of the pigeonhole principle and the nature of quantum correlations. Proc Natl Acad Sci USA. 2016;113(3):532–535.
[3] Dirichlet PGL, Dedekind R. Vorlesungen über zahlentheorie [Lectures on number theory]. Braunschweig (Germany): Vieweg; 1863. German.
[4] Allenby R, Slomson A. How to count: An introduction to combinatorics. 2nd ed. Boca Raton (FL): Chapman & Hall / CRC; 2011. (Discrete mathematics and its applications).
[5] Einstein A, Podolsky B, Rosen N. Can quantum-mechanical description of physical reality be considered complete? Phys Rev. 1935;47(10):777–780.
[6] Bell JS. On the Einstein-Podolsky-Rosen paradox. Physics. 1964;1(3):195–200.
[7] Clauser JF, Horne MA, Shimony A, et al. Proposed experiment to test local hidden-variable theories. Phys Rev Lett. 1969;23(15):880–884.
[8] Feynman RP. Simulating physics with computers. Int J Theor Phys. 1982;21(6):467–488.
[9] Mermin ND. Is the Moon there when nobody looks? Reality and the quantum theory. Phys Today. 1985;38(4):38–47.
[10] Penrose R. The emperor's new mind. New York: Penguin Books; 1991.
[11] Gill RD. Statistics, causality and Bell's theorem. Stat Sci. 2014;29(4):512–528.
[12] Greenberger DM, Horne MA, Shimony A, et al. Bells theorem without inequalities. Am J Phys. 1990;58(12):1131–1143.
[13] Hardy L. Nonlocality for two particles without inequalities for almost all entangled states. Phys Rev Lett. 1993;71(11):1665–1668.
[14] Peres A. Separability criterion for density matrices. Phys Rev Lett. 1996;77(8):1413–1415.
[15] Horodecki M, Horodecki P, Horodecki R. Separability of mixed states: Necessary and sufficient conditions. Phys Lett A. 1996;223(1):1–8.
[16] Ardehali M. Clauser-Horne-Shimony-Holt correlation and Clauser-Horne correlation do not lead to the largest violations of Bell's inequality. Phys Rev A. 1998;57(1):114–119.
[17] Cirel’son BS. Quantum generalizations of Bell’s inequality. Lett Math Phys. 1980;4(2):93–100.
[18] Branciard C, Rosset D, Liang YC, et al. Measurement-device-independent entanglement witnesses for all entangled quantum states. Phys Rev Lett. 2013;110(6):060405.
[19] Werner RF. Quantum states with Einstein-Podolsky-Rosen correlations admitting a hidden-variable model. Phys Rev A. 1989;40(8):4277–4281.