Black hole with primary scalar hair in Einstein-Weyl-Maxwell-conformal scalar theory

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Abstract

We obtain a non-charged BBMB (Bocharova-Bronnikov-Melnikov-Bekenstein) black hole solution from the Einstein-Weyl-Maxwell-conformal (conformally coupled) scalar theory with a positive Weyl coupling parameter $m_2^2$ numerically. This solution has a primary scalar hair, compared to a secondary scalar hair in the charged BBMB black hole solution. The limiting case of $m_2^2 \to \infty$ leads to the series form of the charged BBMB black hole solution. However, for $m_2^2 < 0$, we could not obtain any asymptotically flat black hole solution with scalar hair.

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1 Introduction

An extension of Einstein gravity including $R^2$ and $C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}$ was renormalizable in Minkowski spacetimes [1], but it is still suffering from the loss of unitarity due to a massive spin-2 ghost with $m_2^2 > 0$. One usually excludes the case of $m_2^2 < 0$ because it corresponds to the tachyonic mass. Recently, the Einstein-Weyl gravity ($R - C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}/2m_2^2$) without having $R^2$ has attracted considerable interest in obtaining asymptotically flat black hole solutions numerically. It was known that the condition of vanishing Ricci scalar has simplified the search for any static black hole solutions of a Ricci quadratic gravity. As a famous example, a non-Schwarzschild black hole solution found in [2] has a vanishing Ricci scalar and non-vanishing Ricci tensor.

In this direction, it is meaningful to note that the instability of Schwarzschild black hole (vanishing Ricci tensor) is closely related to the appearance of non-Schwarzschild black hole with non-vanishing Ricci tensor [3, 4, 5, 6]. Explicitly, the small Schwarzschild black hole is unstable against the s-mode of Ricci-tensor perturbation (equivalently, massive spin-2 mode) under the condition of $0 < m_2 \leq m_{th} = 0.876/r_+$ with $m_{th}$ and $r_+$ representing the threshold mass for instability and the black hole radius. For $0 < m_2 \leq m_{th}$, one finds any non-Schwarzschild black hole solution, implying a single branch of non-Schwarzschild black holes. However, the general problem of connecting the strong field regime to the weak one is not completely understood because a shooting method was used to determine the weak field region. In other words, one might successively integrate out further and further approaching asymptotically flat space by tuning the initial conditions near the horizon, resulting in the lack for a systematically asymptotic expansion to the numerical solution [7]. More recently, a multiple shooting approach was introduced to handle this problem [8].

On the other hand, considering the Einstein-Maxwell-conformally coupled scalar theory has admitted the charged BBMB black hole whose scalar hair is secondary [9]. This might be regarded as the second counterexample to the no-hair theorem on black holes even though the scalar hair blows up on the horizon. Incorporating the Weyl-squared term into the Einstein-conformally coupled scalar theory [10], one could find asymptotic expansions as well as a non-BBMB black hole solution with primary scalar hair. We would like to stress that the conformally coupled scalar played an important role in obtaining asymptotic forms of metric functions.

In this work, we wish to derive a non-charged BBMB black hole solution with pri-
mary scalar hair numerically by considering the Einstein-Weyl-Maxwell-conformally coupled scalar (EWMCS) action. If one excludes the Einstein-Hilbert term, this WMCS theory corresponds to a conformally invariant scalar-vector-tensor theory. However, the Einstein-Hilbert term will be necessary to have a constraint of vanishing Ricci scalar which plays a crucial role in obtaining numerical black hole solutions because the EWMCS theory belongs to a fourth-order gravity theory.

Our paper is organised as follows. We start with introducing the EWMCS theory which is not a conformally invariant scalar-vector-tensor theory in section 2. Section 3 focuses on obtaining non-charged BBMB black holes numerically by considering three cases of $m_2^2 > 0$, $= \infty$, $< 0$. Finally, we summarize our main results in section 4.

## 2 EWMCS theory

We start with the Einstein-Weyl-Maxwell-conformally coupled scalar (EWMCS) action given by

$$S_{\text{EWMCS}} = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left[ R - \frac{1}{2m_2^2} C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} - F_{\mu\nu} F^{\mu\nu} - \beta \left( \phi^2 R + 6 \partial_\mu \phi \partial^\mu \phi \right) \right],$$

(1)

The second term denotes the Weyl-squared term, called the conformal gravity solely. The conformal coupling parameter $\beta$ is chosen to be $\beta = 1/3$ for simplicity. In limit of $m_2^2 \to \infty$, the above action reduces to the Einstein-Maxwell-conformally coupled scalar theory which provided a charged BBMB black hole. Furthermore, excluding the Einstein-Hilbert term from (1) leads to the action that is invariant under the conformal transformation

$$g_{\mu\nu} \to \Omega^2(x), \quad \phi \to \frac{\phi}{\Omega},$$

(2)

where $\Omega(x)$ is an arbitrary positive smooth function of $x$. This implies that adding the Einstein-Hilbert term to the action breaks the conformal symmetry. On later, this will gives us an important condition of $R = 0$ for obtaining a numerical black hole solution.

The Einstein equation could be derived from (1) as

$$E_{\mu\nu} \equiv G_{\mu\nu} - \frac{2}{m_2^2} B_{\mu\nu} - 2 T^M_{\mu\nu} - T^\phi_{\mu\nu} = 0,$$

(3)

where the Einstein tensor is given by $G_{\mu\nu} = R_{\mu\nu} - R g_{\mu\nu}/2$ and the Bach tensor $B_{\mu\nu}$ describes
fourth-order terms

\[ B_{\mu \nu} = \left( R_{\mu \nu \rho \sigma} R^{\rho \sigma} - \frac{1}{4} R^{\rho \sigma} R_{\rho \sigma} g_{\mu \nu} \right) - \frac{1}{3} R \left( R_{\mu \nu} - \frac{1}{4} R g_{\mu \nu} \right) + \frac{1}{2} \left( \nabla^2 R_{\mu \nu} - \frac{1}{6} \nabla^2 R g_{\mu \nu} - \frac{1}{3} \nabla_{\mu} \nabla_{\nu} R \right). \]  

(4)

We note here that its trace is zero \((B^\mu_{\mu} = 0)\). The energy-momentum tensors for Maxwell theory and conformally coupled scalar theory are defined by, respectively

\[ T_{\mu \nu}^M = F_{\mu \rho} F^{\rho \nu} - \frac{F^2}{4} g_{\mu \nu}, \]  

(5)

\[ T_{\mu \nu}^\phi = \frac{1}{3} \left[ \phi^2 G_{\mu \nu} + g_{\mu \nu} \nabla^2 (\phi^2) - \nabla_{\mu} \nabla_{\nu} (\phi^2) + 6 \partial_{\mu} \phi \partial_{\nu} \phi - 3 (\partial \phi)^2 g_{\mu \nu} \right]. \]  

Note the traceless condition of \(T_{\mu \mu}^M = 0\). Also, the Maxwell equation is given by

\[ \nabla^\mu F_{\mu \nu} = 0. \]  

(6)

Apparently, the scalar equation seems to be a conformally coupled scalar equation

\[ \nabla^2 \phi - \frac{1}{6} R \phi = 0. \]  

(7)

However, taking the trace of \( (5) \) together with \( (7) \) leads to the vanishing Ricci scalar

\[ R = 0, \]  

(8)

which will play an important role in reducing the third-order Einstein equation to the second-order equation. Taking into account \( (8) \), one finds a minimally coupled scalar equation

\[ \nabla^2 \phi = 0, \]  

(9)

which is found due to the Einstein-Hilbert term \((R)\), breaking the conformal symmetry. Considering \( (9) \), it is important to note that the scalar hair has nothing to do with the spontaneous scalarization where acquiring of a scalar hair by the black hole arises from the nonminimal coupling of a scalar to the Gauss-Bonnet term \( f(\phi)G \) \([11, 12, 13]\) and the Maxwell term \( f(\phi) F^2 \) \([14]\). Imposing \( (8) \) and \( (9) \) on \( (3) \) leads to the reduced Einstein equation

\[ R_{\mu \nu} = \frac{2}{m^2} \left( R_{\mu \nu \rho \sigma} R^{\rho \sigma} - \frac{1}{4} R^{\rho \sigma} R_{\rho \sigma} g_{\mu \nu} + \frac{1}{2} \nabla^2 R_{\mu \nu} \right) + 2 T_{\mu \nu}^M + \frac{1}{3} [\phi^2 R_{\mu \nu} - (\partial \phi)^2 g_{\mu \nu} + 4 \partial_{\mu} \phi \partial_{\nu} \phi - 2 \phi \nabla_{\mu} \nabla_{\nu} \phi]. \]  

(10)
Finally, a conformally invariant scalar-vector-tensor (WMCS) theory predicts equations of motion
\[ \frac{2}{m_2^2} B_{\mu\nu} = 2T^M_{\mu\nu} + T^\phi_{\mu\nu}, \quad \nabla^\mu F_{\mu\nu} = 0, \quad \nabla^2 \phi - \frac{1}{6} R \phi = 0, \] (11)
which might not be suitable for admitting a numerical black hole solution with scalar hair because of \( R \neq 0 \).

3 Scalar hairy black holes

3.1 Charged BBMB black hole solution

In limit of \( m_2^2 \to \infty \), the action (11) reduces to the one admitting the charged BBMB black hole [9]
\[
\begin{align*}
 ds^2_{\text{BBMB}} &= -\tilde{\mathcal{N}}(r)e^{-2\tilde{\delta}(r)}dt^2 + \frac{dr^2}{\tilde{\mathcal{N}}(r)} + r^2 d\Omega_2^2, \\
 \tilde{\mathcal{N}}(r) &= \left(1 - \frac{m}{r}\right)^2, \quad \tilde{\delta}(r) = 0, \quad \tilde{\phi}(r) = \sqrt{3(m^2 - Q^2)} \frac{r}{r - m}, \quad \tilde{A}_t = \tilde{v}(r) = \frac{Q}{r} - \frac{Q}{r_+},
\end{align*}
\] (12)
where \( m \) is the mass of the black hole and \( Q \) denotes the Maxwell charge. This line element takes the same form as in the extremal Reissner-Nordström black hole, but the scalar hair blows up at the horizon \( r = m \) and \( \tilde{\phi}(r) \) belongs to the secondary hair because it does not have an independent scalar charge \( Q_s \). It was shown forty years ago that this black hole is unstable against the scalar perturbation because it belongs to an extremal RN black hole [15]. We would like to mention that the charged BBMB solution (12) was obtained under the conditions of vanishing Ricci scalar and non-vanishing Ricci tensor, in addition to non-vanishing scalar. Also, the extremality of (12) is closely related to the conformal scalar coupling to the Einstein-Hilbert term [16, 17].

On the other hand, it is worth noting that a non-extremal black hole with constant scalar hair could be found when taking the limit of \( m_2^2 \to \infty \) in the EWMCS theory [18].

3.2 Non-charged BBMB black holes

Now, let us derive a non-charged BBMB black hole solution which will carry a primary scalar hair by including the Weyl-squared term. In order to find scalarized charged black
holes, one proposes the metric ansatz
\[
d s^2_{\text{schb}} = - N(r)e^{-2\delta(r)}d t^2 + \frac{d r^2}{N(r)} + r^2 d \Omega^2, \quad \phi(r) \neq 0, \quad A_t = v(r),
\]
where \(\delta(r) \neq 0\) will describe a non-charged BBMB black hole. Substituting Eq. (13) into Eqs. (3), (6), and (9), one finds four equations for \(N(r)\), \(\delta(r)\), \(\phi(r)\), and \(v(r)\)

\[
N''r^2[rN' - 2N(1 + r\delta')] + N'r[4r\delta'(rN' - 2) - 2N[2 + rN'(4r\delta' + 2r^2\delta^2 - 1)]
+ N^2(4 + 6r^2\delta'^2 + 4r^3\delta''^2) + \frac{2m^2}{3}[(\phi^2(rN' - 1) - 3(rN' + e^{2\delta}r^2v^2 - 1)
+ r^2N'\phi\phi' + N(6r\delta' - 3 + \phi^2(1 - 2r\delta') - 2r\phi(r\delta' - 2)\phi' + 3r^2\phi'^2)] = 0, \quad (14)
\]
and
\[
2 + rN'(3r\delta' - 4) - r^2N'' + N(2r^2\delta'' - 2r^2\delta'^2 + 4r\delta' - 2) = 0, \quad (15)
\]
\[
rN'\phi' + N[(2 - r\delta')\phi' + r\phi''] = 0, \quad (16)
Q + e^{\delta}r^2v' = 0. \quad (17)
\]

In deriving (14), we wish to mention the Einstein equation (3) that \(E_{tt} = 0\): fourth-order equation; \(E_{rr} = 0\): third-order equation; \(E_{\theta\theta} = 0\): fourth-order equation. We obtain one third-order \(N'''(r) + \cdots = 0\) by making use of two fourth-order equations. Together \(E_{rr} = 0\) with \(N'''(r) + \cdots = 0\) leads to the second-order equation (14). During this process, we use \(R = 0\) (15) to eliminate \(\delta''(r)\) in (14). At this stage, we have to mention that one arrives at the same equation as (14) when using the reduced Einstein equation (10).

First of all, we introduce the near-horizon expansion by considering the charged BBMB solution (12)

\[
N(r) = \sum_{i=2}^{\infty} N_i(r - r_+)^i, \quad \delta(r) = \sum_{i=0}^{\infty} \delta_i(r - r_+)^i,
\]
\[
v(r) = \sum_{i=1}^{\infty} v_i(r - r_+)^i, \quad \psi(r) = \sum_{i=1}^{\infty} \psi_i(r - r_+)^i, \quad (18)
\]
where ‘\(i = 2\)’ in \(N(r)\) reflects the conformally coupled scalar background solution. Also it is worth noting that \(\psi = 1/\phi\) is introduced in the near-horizon for technical reason. The
first few coefficients are computed as

\[
N_2 = \frac{1}{r_+}, \quad N_3 = -\frac{6(r_+^2 - Q^2)\psi_1^2}{3r_+^4} + 4, \quad N_4 = \frac{5}{3r_+^4} + \frac{5(r_+^2 - Q^2)\psi_2^2}{r_+^4} - \frac{3(r_+^2 - Q^2)^2\psi_1^4}{r_+^4},
\]

\[
N_5 = -\frac{86}{45r_+^4} - \frac{(37r_+^2 - 46Q^2)\psi_1^2}{5r_+^4} - \frac{27(r_+^2 - Q^2)(2Q^2 - r_+^2 - \frac{6}{m_2^2})\psi_1^4}{5r_+^4} - \frac{6(r_+^2 - Q^2)^3\psi_1^6}{r_+^4},
\]

\[
\delta_0, \quad \delta_1 = \frac{2 + 6(Q^2 - r_+^2)\psi_1^2}{3r_+}, \quad \delta_2 = -\frac{1 + 15(Q^2 - r_+^2)\psi_1^2 + 36(Q^2 - r_+^2)^2\psi_1^4}{9r_+^4},
\]

\[
\delta_3 = \frac{34}{405r_+^4} + \frac{2(38Q^2 - 11r_+^2)\psi_2^2}{45r_+^3} + \frac{2(Q^2 - r_+^2)(59Q^2 - 32r_+^2 - \frac{162}{m_2^2})\psi_1^4}{15r_+^3} + \frac{32(Q^2 - r_+^2)^3\psi_1^6}{3r_+^3},
\]

\[
v_1 = -\frac{e^{-\delta_0}Q}{r_+^4}, \quad v_2 = \frac{e^{-\delta_0}Q}{r_+^4} \left[ \frac{4}{3} - (r_+^2 - Q^2)\psi_1^2 \right],
\]

\[
v_3 = -\frac{e^{-\delta_0}Q}{r_+^4} \left[ \frac{14}{9} - \frac{7}{3}(r_+^2 - Q^2)\psi_1^2 + 2(r_+^2 - Q^2)^2\psi_1^4 \right],
\]

\[
v_4 = \frac{e^{-\delta_0}Q}{r_+^4} \left[ \frac{232}{153} \frac{(107r_+^2 - 116Q^2)\psi_2^2}{30} + \frac{(r_+^2 - Q^2)(59r_+^2 - 68Q^2 + \frac{54}{m_2^2})\psi_1^4}{10} + 5(r_+^2 - Q^2)^3\psi_1^6 \right],
\]

\[
\phi_1 = \frac{\psi_2}{3r_+} - \frac{3(r_+^2 - Q^2)}{9r_+^4} \psi_1^2, \quad \psi_3 = \frac{2\psi_1[1 - 3(r_+^2 - Q^2)\psi_1^2]^2}{9r_+^4},
\]

\[
\psi_4 = \frac{-22\psi_1}{135r_+^4} + \frac{(37r_+^2 - 46Q^2)\psi_2^2}{30r_+^4} - \frac{3(r_+^2 - Q^2)(13r_+^2 - 16Q^2 + \frac{18}{m_2^2})\psi_3^5}{10r_+^4} - \frac{5(r_+^2 - Q^2)^3\psi_1^7}{r_+^4},
\]

where \( Q \) denotes the \( U(1) \) charge. We note that \( \{N_5, \delta_3, v_4, \psi_4\} \) contain the Weyl term of \( 1/m_2^2 \). In our theory, two free parameters \( \delta_0 \) and \( \psi_1 \) will be determined when matching [8] with the following asymptotic expansions for \( r \gg r_+ \):

\[
N(r) = 1 - \frac{2m}{r} + \frac{Q^2 + Q_2^2}{r^2} + \frac{2}{3r^4} \left[ (Q^2 + Q_2^2)(Q_2^2 + \frac{6}{m_2^2}) - m^2Q_2^2 \right] + \ldots,
\]

\[
\delta(r) = \frac{1}{6r^2} \left[ (Q^2 + Q_2^2)(Q_2^2 + \frac{6}{m_2^2}) - m^2Q_2^2 \right] - \frac{4mQ_2^2(m^2 - Q^2 - Q_2^2)}{5r^5} + \ldots,
\]

\[
v(r) = -\frac{Q}{r_+} + \frac{Q}{r} - \frac{Q}{30r^5} \left[ (Q^2 + Q^2)(Q_2^2 + \frac{6}{m_2^2}) - m^2Q_2^2 \right] + \ldots,
\]

\[
\phi(r) = \frac{\sqrt{3}Q_8}{r} + \frac{\sqrt{3}Q_8m}{r^2} + \frac{\sqrt{3}Q_8(4m^2 - Q^2 - Q_2^2)}{3r^3} + \frac{\sqrt{3}Q_8m}{r^2} (2m^2 - Q^2 - Q_2^2)
\]

\[
+ \frac{\sqrt{3}Q_8}{10r^5} \left[ 3m^4 - m^2(24Q^2 + 23Q_2^2) + (Q^2 + Q_2^2)(2Q^2 + Q_2^2 - \frac{6}{m_2^2}) \right] + \ldots
\]

where \( m \) is the ADM (Arnowitt-Deser-Misner) mass and \( Q_8 \) represents the scalar charge.
Here, we observe that $O(r^{-3})$ is missed in $N(r)$, $\delta(r)$, and $v(r)$, which implies the feature of a conformally coupled scalar in the EWMCS theory. In addition, we note that asymptotic expansions are not available if the Maxwell and scalar fields are turned off. This explains why the asymptotic expansion is not available in the Einstein-Weyl gravity [2].

At this stage, it is important to check whether the limit of $m^2_2 \to \infty$ recovers the charged BBMB black hole solution. In this case, we obtain three relations from numerical computation as [19]

$\tilde{r} \approx m, \quad m \approx \sqrt{Q^2 + Q_s^2}, \quad \psi_1 \approx \frac{1}{\sqrt{3(r_+^2 - Q^2)}}, \quad \delta_0 = 0.0001.$

Using (21), we recover the near-horizon expansion for the charged BBMB solution from (19)

$\tilde{N}(r) = \frac{(r - r_+)^2}{r_+^2} - 2 \frac{(r - r_+)^3}{r_+^3} + 3 \frac{(r - r_+)^4}{r_+^4} + \ldots \to \left[\left(1 - \frac{r_+}{r}\right)^2\right]_{r=r_+},$

$\tilde{v}(r) = -Q \frac{r - r_+}{r_+^2} + Q \frac{(r - r_+)^2}{r_+^3} - Q \frac{(r - r_+)^3}{r_+^4} + \ldots \to \left[Q \frac{r}{r_+} - \frac{Q}{r_+}\right]_{r=r_+},$

$\tilde{\delta}(r) = 0,$

$\tilde{\psi}(r) = \frac{r - r_+}{\sqrt{3(r_+^2 - Q^2)}}.$

On the other hand, making use of (21), the asymptotic expansion for the charged BBMB solution could be obtained from (20) as

$\tilde{N}(r) = 1 - \frac{2m}{r} + \frac{m^2}{r^2} = \left(1 - \frac{m}{r}\right)^2,$

$\tilde{v}(r) = \frac{Q}{r} - \frac{Q}{r_+},$

$\tilde{\delta}(r) = 0$

$\tilde{\phi}(r) = \sqrt{3(m^2 - Q^2)}\left[\frac{1}{r} + \frac{m}{r^2} + \frac{m^2}{r^3} + \frac{m^3}{r^4} + \frac{m^4}{r^5} \cdots \right] \to \left[\frac{\sqrt{3(m^2 - Q^2)}}{r - m}\right]_{r \gg m}.$

We observe that (23) does not contain scalar charge $Q_s$. This implies that the scalar hair is secondary in the charged BBMB solution.

Let us consider the $m^2_2 > 0$ case. Explicitly, we may choose the horizon radius $r_+ = 0.75$, $U(1)$ charge $Q = 0.5$ together with $m^2_2 = 2$ to construct a non-charged BBMB black hole with the ADM mass $m = 0.7974$ and scalar charge $Q_s = 0.5766$. This implies a relation of $m^2 \neq Q^2 + Q_s^2$, implying a non-extremal black hole. As are shown in Figs. 1 and 2, the
Figure 1: Plot of metric function $N(r)$ for a non-charged BBMB solution with $m_2^2 = 2$, compared to $\bar{N}(r)$ for the charged BBMB solution with $r_+ = m = 0.75$ and $Q_s = 0.5590$. Here, the horizon radius is located at $r_+ = 0.75$ (implying $\ln r_+ = -0.274$) together with charge $Q = 0.5$. $\delta(r)$ starts with $\delta_0 = 0.215$, while it is always zero [$\bar{\delta}(r) = 0$] for the charged BBMB solution.

Figure 2: (Left) Plot of scalar hair $\phi(r)$ for a non-charged BBMB solution with $\psi_1 = 1.3803$ compared to $\bar{\phi}(r)$ for the charged BBMB solution with $\psi_1 = 1.0327$. (Right) Plot of vector potential $A_t = v(r)$ for a non-charged BBMB solution, comparing with $\bar{A}_t(r) = \bar{v}(r)$ for the charged BBMB solution.

metric functions, scalar hair, and vector potential are compared to those for the charged BBMB black hole without Weyl-squared term. The appearance of non-zero $\delta(r)$ and scalar charge $Q_s$ indicates clearly that the non-charged BBMB solution has a primary scalar hair. This represents a non-charged BBMB black hole in the single branch of $m_2^2 > 0$.

However, for $m_2^2 < 0$, we employ $m_2^2 = -1.5$, $\psi_1 = 0.5972$, and $\delta_0 = -1.5$ to derive a numerical solution \{${N}_n(r), \delta_n(r), \nu_n(r), \phi_n(r)$\} shown in Fig. 3. We stress that one could not obtain any asymptotically flat numerical black hole solutions for $m_2^2 < 0$ because $N(r)$ diverges as $r$ increases. In Table 1, we list the cases that are used for finding divergent
Figure 3: (Left) Plot of divergent metric function $N_n(r)$ for a non-charged BBMB solution with $m_2^2 = -1.5$ and $\psi_1 = 0.5972$. Here, the horizon radius is located at $r_+ = 0.75$ together with $U(1)$ charge $Q = 0.5$. $\delta_n(r)$ starts with $\delta_0 = -1.5$. (Right) Plot for scalar hair $\phi_n(r)$ with divergent behavior near the horizon.

metric functions $N(r)$. A set of $\{m_2^2, \psi_1, \delta_0\}$ determines the near-horizon expansion. Even though these do not represent a complete set for $m_2^2 < 0$, it is reasonable to say that any asymptotically flat numerical black hole solutions are not allowed for $m_2^2 < 0$ because it corresponds to tachyonic mass.

| $m_2^2$  | $\psi_1$ | $\delta_0$ |
|----------|----------|------------|
| -0.0357  | 0.2676   | -0.0114    |
| -0.0555  | 0.2712   | -0.0167    |
| -0.0833  | 0.3207   | -0.0263    |
| -0.125   | 0.3423   | -0.0357    |
| -0.2     | 0.4027   | -0.0556    |
| -0.75    | 0.6254   | -0.25      |
| -2.0     | 0.7686   | -0.5       |
| -12      | 0.8838   | -1.0       |

Table 1: List of finding divergent metric functions $N(r)$ for $m_2^2 < 0$.

4 Summary

We have obtained a non-charged BBMB black hole solution from the Einstein-Weyl-Maxwell-conformally coupled scalar theory which is not a conformally invariant scalar-vector-tensor theory, for $m_2^2 > 0$ numerically. This solution includes a primary scalar hair, compared to
a secondary scalar hair in the charged BBMB black hole solution found from the Einstein-Maxwell-conformally coupled scalar theory. Interestingly, the limit of $m_2^2 \to \infty$ leads to the series form of the charged BBMB black hole solution. However, for $m_2^2 < 0$, we could not obtain any asymptotically flat black hole solution with scalar hair. This may be so because the case of $m_2^2 < 0$ corresponds to tachyonic mass for a spin-2 mode when perturbing around the Minkowski background.

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References

[1] K. S. Stelle, Phys. Rev. D 16, 953 (1977). doi:10.1103/PhysRevD.16.953

[2] H. Lu, A. Perkins, C. N. Pope and K. S. Stelle, Phys. Rev. Lett. 114, no. 17, 171601 (2015) doi:10.1103/PhysRevLett.114.171601 [arXiv:1502.01028 [hep-th]].

[3] H. Lu, A. Perkins, C. N. Pope and K. S. Stelle, Phys. Rev. D 96, no. 4, 046006 (2017) doi:10.1103/PhysRevD.96.046006 [arXiv:1704.05493 [hep-th]].

[4] K. S. Stelle, Int. J. Mod. Phys. A 32, no. 09, 1741012 (2017). doi:10.1142/S0217751X17410123

[5] Y. S. Myung, Phys. Rev. D 88, no. 2, 024039 (2013) doi:10.1103/PhysRevD.88.024039 [arXiv:1306.3725 [gr-qc]].

[6] B. Whitt, Phys. Rev. D 32, 379 (1985). doi:10.1103/PhysRevD.32.379

[7] K. Goldstein and J. J. Mashiyane, Phys. Rev. D 97, no. 2, 024015 (2018) doi:10.1103/PhysRevD.97.024015 [arXiv:1703.02803 [hep-th]].

[8] A. Bonanno and S. Silveravalle, Phys. Rev. D 99, no. 10, 101501 (2019) doi:10.1103/PhysRevD.99.101501 [arXiv:1903.08759 [gr-qc]].

[9] J. D. Bekenstein, Annals Phys. 82, 535 (1974). doi:10.1016/0003-4916(74)90124-9

[10] Y. S. Myung and D. C. Zou, Phys. Rev. D 100, no. 6, 064057 (2019) doi:10.1103/PhysRevD.100.064057 [arXiv:1907.09676 [gr-qc]].

[11] D. D. Doneva and S. S. Yazadjiev, Phys. Rev. Lett. 120, no. 13, 131103 (2018) doi:10.1103/PhysRevLett.120.131103 [arXiv:1711.01187 [gr-qc]].

[12] H. O. Silva, J. Sakstein, L. Gualtieri, T. P. Sotiriou and E. Berti, Phys. Rev. Lett. 120, no. 13, 131104 (2018) doi:10.1103/PhysRevLett.120.131104 [arXiv:1711.02080 [gr-qc]].

[13] G. Antoniou, A. Bakopoulos and P. Kanti, Phys. Rev. Lett. 120, no. 13, 131102 (2018) doi:10.1103/PhysRevLett.120.131102 [arXiv:1711.03390 [hep-th]].

[14] C. A. R. Herdeiro, E. Radu, N. Sanchis-Gual and J. A. Font, Phys. Rev. Lett. 121, no. 10, 101102 (2018) doi:10.1103/PhysRevLett.121.101102 [arXiv:1806.05190 [gr-qc]].
[15] K. A. Bronnikov and Y. N. Kireev, Phys. Lett. A 67, 95 (1978). doi:10.1016/0375-9601(78)90030-0

[16] B. C. Xanthopoulos and T. Zannias, J. Math. Phys. 32, 1875 (1991).

[17] Y. Tomikawa, T. Shiromizu and K. Izumi, Class. Quant. Grav. 34, no. 15, 155004 (2017) doi:10.1088/1361-6382/aa7906 [arXiv:1702.05682 [gr-qc]].

[18] M. Astorino, Phys. Rev. D 88, no. 10, 104027 (2013) doi:10.1103/PhysRevD.88.104027 [arXiv:1307.4021 [gr-qc]].

[19] D. C. Zou and Y. S. Myung, arXiv:1911.08062 [gr-qc].