Thermodynamic Bethe Ansatz
and
Threefold Triangulations

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Abstract

In the Thermodynamic Bethe Ansatz approach to 2D integrable, ADE-related quantum field theories one derives a set of algebraic functional equations (a Y-system) which play a prominent role. This set of equations is mapped into the problem of finding finite triangulations of certain 3D manifolds. This mapping allows us to find a general explanation of the periodicity of the Y-system. For the $A_N$ related theories and more generally for the various restrictions of the fractionally-supersymmetric sine-Gordon models, we find an explicit, surprisingly simple solution of such functional equations in terms of a single unknown function of the rapidity. The recently-found dilogarithm functional equations associated to the Y-system simply express the invariance of the volume of a manifold for deformations of its triangulations.

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1 Introduction

In the Thermodynamic Bethe Ansatz (TBA) approach [1], the renormalization group behaviour of a two-dimensional, integrable quantum field theory is described by the ground state energy $E(R)$ of the system on an infinitely long cylinder of radius $R$. The equations known (or conjectured) to give $E(R)$ are of the form

$$E(R) = -\frac{1}{2\pi} \sum_{a=0}^{a=N} \int_{-\infty}^{\infty} d\theta \nu_a(\theta) \log (1 + Y_a(\theta)) ,$$  \hspace{1cm} (1.1)

where the $Y_a(\theta)$ are $R$ dependent functions determined by a set of coupled integral equations known as TBA equations; the $\nu_a(\theta)$ are known functions describing the asymptotic behaviour of the solutions: $Y_a(\theta) \to \nu_a(\theta)$ for $\theta \to \pm \infty$. One of the main results of ref. [2] was that any solution $\{Y_a(\theta)\}$ of the TBA equations satisfies a set of simple functional algebraic equations, called the Y-system. Conversely it is easy to show that a set of entire functions satisfying the Y-system with a suitable asymptotic behaviour is a solution of the TBA equations, thus the Y-system encodes all the dynamical properties of the model.

In [2]-[7] a large class of TBA systems classified according to the ADET Dynkin diagrams was proposed to describe integrable perturbed coset theories. The Y-systems associated to all these models can be written in terms of an ordered pair $G \times H$ of ADET Dynkin diagrams in the following form

$$Y_a^b \left( \theta + \frac{i\pi}{\tilde{g}} \right) Y_a^b \left( \theta - \frac{i\pi}{\tilde{g}} \right) = \prod_{c=1}^{r_G} \left( 1 + Y_c^b(\theta) \right) G_{a,c} \prod_{d=1}^{r_H} \left( 1 + \frac{1}{Y_d^a(\theta)} \right)^{-H_{b,d}}$$ \hspace{1cm} (1.2)

where $G_{a,c}$ and $H_{b,d}$ are the adjacency matrices of the corresponding ADET Dynkin diagram, $\tilde{g}$ is the dual Coxeter number of $G$, $r_G$ and $r_H$ are the ranks of the corresponding algebras. These ADE related Y-systems do not exhaust at all the set of such systems. For instance, recently a new class of Y-systems associated to sorts of bent $D_n$ diagrams describing the sine-Gordon models at rational points has been described [8]. The functional equations (1.2) are universal in the sense that using different infra-red boundary conditions, they describe different theories or different regimes of the same theory.

The $Y$ functions solving whatever Y-system with arbitrary boundary conditions have two general, intriguing properties.

One is that the $Y$’s are periodic functions, as first pointed out by Zamolodchikov [2]. In particular, for the set of Y-systems described in Eq.(1.2),
denoting with \( \tilde{h} \) the dual Coxeter number of \( H \) one can verify by direct successive substitutions or, in the high rank cases, by numerical computations that

\[
Y^b_a \left( \theta + i\pi \frac{\tilde{h} + \tilde{g}}{\tilde{g}} \right) = Y^\tilde{b}_{\tilde{a}}(\theta),
\]

(1.3)

where \( \tilde{a} \) and \( \tilde{b} \) denote the nodes of the Dynkin diagram conjugate to \( a \) and \( b \). Although this periodicity has many important consequences and is in relation with the conformal dimension of the perturbing operator in the ultra-violet region, it has not been proven in a general way until now.

The other intriguing property, recently pointed out in ref. \[9\], is that the \( Y \) functions solving Eq.(1.2) are the arguments of a new infinite family of functional equations for the Rogers dilogarithm \( L(x) \) of the from

\[
\sum_{a=1}^{r_G} \sum_{b=1}^{r_H} \left( \sum_{n=0}^{\tilde{h} + \tilde{g} - 1} L \left( \frac{\Upsilon^b_a(n)}{1 + \Upsilon^b_a(n)} \right) \right) = \frac{\pi^2}{6} r_G r_H \tilde{g},
\]

(1.4)

where

\[
\Upsilon^b_a(n) = Y^b_a \left( \theta + i\frac{n\pi}{\tilde{g}} \right) \quad n = 0, 1, 2, \ldots
\]

(1.5)

They were verified by extensive numerical checks and proven only for low rank cases. Thus they are (until now) at the same conjectural level of the periodicity discussed above and are strictly related to it: if the \( \Upsilon \) variables were not periodic Eq.(1.4) would be meaningless.

From a physical point of view, these two properties play an essential role in finding the deformed conformal theory associated to the integrable model, indeed form these functional equations one can easily derive the whole set of the dilogarithm sum-rules yielding the effective central charge of the UV fixed point \[10-17\], while the period of the \( Y \) function fixes, as already mentioned, the conformal dimension of the perturbing field.

In this paper we describe a simple geometrical interpretation of these two properties. In particular, for the infinite family of models related to \( A_N \) and, more generally, for the new set of Y-systems \[8\] describing the \( \phi_{1,1,3} \)-perturbations of \( SU(2) \)-coset models with fractional supersymmetry, we work out explicitly the solution for arbitrary boundary conditions. For the \( A_N \) systems we also find a general proof of the dilogarithm functional identities.
2 Threefold Triangulations

Volume calculations in the three-dimensional hyperbolic space involve the dilogarithm function of complex argument, as is well known in the mathematical literature since the time of Lobachevskij. Similarly, that the Rogers dilogarithm of real argument is related to volume calculation of other threefolds. In particular it has been shown \cite{18} that there is at least a manifold (a compactification of the universal covering group of the projective SL(2,R) group) where there are special or "ideal" tetrahedra whose four vertices are parametrized by four real numbers. The volume of the ideal tetrahedron of vertices \(a, b, c, d\) depends only on the cross-ratio \(x = (abcd)\) and is given in terms of the Rogers dilogarithm by

\[
\text{vol}(abcd) = L(x) - \frac{\pi^2}{6} .
\]

(2.6)

Notice that the dilogarithm \(L(z)\) is a multivalued function, then the above definition has some ambiguity unless it is implemented by a consistent choice of the sheet of the Riemann surface associated to \(L(z)\). We choose the sheet in which \(|\text{vol}(abcd)|\) has its minimum value.

With an abuse of notation we shall use in the following the same symbol \((abcd)\) to denote both the cross-ratio as well as an ideal tetrahedron of vertices \(a, b, c, d\) with the proper orientation, whenever it does not generate confusion; when instead it is important to distinguish between a tetrahedron and its cross-ratio we shall use square brackets.

We do not need to commit ourselves on the nature of the threefold \(\mathcal{M}\) where Eq.(2.6) holds, and we consider Eq.(2.6) as the definition of volume of an ideal tetrahedron of a suitable \(\mathcal{M}\) threefold.

Using the properties of the cross-ratio and of \(L(x)\) it is immediate to see that this volume is independent of even permutations of the vertices, as it should be; notice also that there are always two vertex renumberings, corresponding to two different orientations, where the value of the cross-ratio \(x\) belongs to the interval \(0 \leq x \leq 1\). Indeed if the ideal tetrahedron \(T\) is associated to the cross-ratio \((abcd) = x\) then the the tetrahedron \(U\) of reversed orientation is associated to the cross-ratio \((acbd) = 1 - x\). Note that, according to Eq.(2.6), these two tetrahedra do not have in general the same volume, although the (negative) sign of the volume is the same. However, owing to the Euler relation

\[
L(x) + L(1-x) = L(1) = \frac{\pi^2}{6} ,
\]

(2.7)
we get \( \text{vol}(abcd) + \text{vol}(acbd) = -\frac{\pi^2}{6} \). This has a simple geometric interpretation. First, let us define the boundary \( \partial T \) of the tetrahedron \( T = [abcd] \) in the following standard way

\[
\partial [abcd] = [abc] - [bcd] + [cda] - [dab],
\]

where \([ijk]\) denotes an oriented triangle. An odd permutation of the vertices gives the reversed orientation, then, for instance \([ijk] = -[jik]\). It follows that the chain \( T + U = [abcd] + [acbd] \) satisfies the 3-cycle condition

\[
\partial(U + T) = 0.
\]

Thus we can view \( U \) as the complement \( U = \mathcal{M} \setminus T \) of \( T \) and the 3-cycle \( T + U \) is a triangulation of the threefold \( \mathcal{M} \). We can define the volume of \( \mathcal{M} \) as the sum of the volumes of the tetrahedra of the triangulation:

\[
\text{vol}(T) + \text{vol}(U) = \text{vol}(abcd) + \text{vol}(acbd) = \text{vol}(\mathcal{M}),
\]

and \( \text{vol}(\mathcal{M}) = -\frac{\pi^2}{6} \). The negative value of this volume is simply a consequence of the sign chosen in the definition of volume in Eq. (2.4).

An elementary analog of the above special triangulation can be found for instance in the sphere \( S^2 \): a spherical triangle \( t \) plus its complement \( u = S^2 \setminus t \) form a triangulation of \( S^2 \) and the area of \( S^2 \) is simply the sum of the areas of these two triangles. One can get other less trivial triangulations by replacing one (or both) triangle(s), say \( t \), with a chain \( \sum_i t_i \) of triangles with the same orientation whose boundary is that of \( t \): \( \partial \sum_i t_i = \partial t \). We shall see that a similar procedure can be adopted for the ideal tetrahedra of \( \mathcal{M} \).

Consider indeed a domain formed by two adjacent tetrahedra \( U_1 \leftrightarrow (adbc) \) and \( U_2 \leftrightarrow (bece) \) like in Fig. 1.

It can also be considered as the union of the three tetrahedra \( T_3 \leftrightarrow (cdae) \), \( T_4 \leftrightarrow (deba) \) and \( T_5 \leftrightarrow (each) \):

\[
U_1 \cup U_2 = T_3 \cup T_4 \cup T_5,
\]

or, using a more appropriate homology language, the chain \( C = U_1 + U_2 - T_3 - T_4 - T_5 \) is a 3-cycle, i.e. the boundary of \( U_1 + U_2 \) coincides with the boundary of \( T_1 + T_2 + T_3 \) so that \( \partial C = 0 \).

Since the volume is additive with respect to division of the domain into a finite number of pieces, we have

\[
\text{vol}(U_1) + \text{vol}(U_2) = \text{vol}(T_3) + \text{vol}(T_4) + \text{vol}(T_5).
\]
Figure 1: Two adjacent tetrahedra can be divided into three.

If the above tetrahedra are ideal tetrahedra of $\mathcal{M}$ we get through Eq.(2.6) the five term Abel functional identity for the Rogers dilogarithm (see for instance ref.[18]) which coincides with the $A_1 \times A_2$ case of Eq. (1.4). However in this way the hidden $\mathbb{Z}_5$ symmetry observed in ref.[9], which played a crucial role in the construction of the new dilogarithm identities does not have a clear geometric meaning. For a more symmetric approach, we combine Eq.(2.12) with Eq.(2.10), which is the geometric version of the Euler equation. In particular, let us consider the complementary tetrahedra $T_1 = \mathcal{M} \setminus U_1 \leftrightarrow (abcd)$ and $T_2 = \mathcal{M} \setminus U_2 \leftrightarrow (beed)$. Clearly the chain $C = T_1 + U_1 + T_2 + U_2$ is a 3-cycle, and Eq.(2.10) gives

$$vol(T_1) + vol(U_1) + vol(T_2) + vol(U_2) = 2 vol(\mathcal{M}) \quad . \quad (2.13)$$

Note that the volume of these four tetrahedra have the same sign, hence $C$ can be considered as a triangulation of a 3D manifold $\mathcal{M}_2$ which covers $\mathcal{M}$ twice. Now, if we replace the sum $U_1 + U_2$ with $T_3 + T_4 + T_5$ in $C$, we get a

\[1\] The Abel and the Euler equations in the form Eqs. (2.10) and (2.12) give us the rules of passing from a tetrahedron to its complement and for the decomposition of two adjacent tetrahedra in to three. These are the only two ingredients we need in order to prove the more general $A_1 \times A_N$ ($N>2$) equations.
new triangulation of $M_2$
\[ T = \sum_{i=1}^{5} T_i \] (2.14)

which is regular, in the sense that the group of the automorphisms of the triangulation, denoted by $G_T$, is transitive: in other terms, given two arbitrary, distinct tetrahedra $T_i$ and $T_j$ there is at least a permutation of the tetrahedra preserving all the adjacency relations which carries $T_i$ into $T_j$. Performing the cyclic permutation of the five vertices $a \rightarrow b \rightarrow c \rightarrow d \rightarrow e \rightarrow a$ in the triangulation $T$ yields $T_i \rightarrow T_{i+1}$ (the indices are taken modulo 5). This shows that $T$ is regular and that $G_T \supset \mathbb{Z}_5$. Note also that, taking the vertices ordered as $a < b < c < d < e$, the cross-ratios associated to the 5 ideal tetrahedra belong to the interval $[0, 1]$ and the Abel identity (2.12) can be written in a more symmetric way as
\[ \sum_{i=1}^{5} \text{vol}(T_i) = 2 \text{vol}(M) \] (2.15)

It is now clear how generalize the above geometrical construction to the other Y-systems: we conjecture that the Y variables are associated to the ideal tetrahedra of a triangulation $T$ of a manifold which is a multiple covering of $M$. This should imply that writing the Y’s as cross-ratios of a suitable set of points solves the Y-system equations. The $\mathbb{Z}_P$ symmetry of the Y-system ($P$ is the periodicity) is mapped into the automorphism group $G_T$ of the triangulation. In the more symmetric cases $G_T$ is transitive, yielding a regular triangulation. Finally, the geometrical meaning behind the Rogers dilogarithm functional identities of Eq.(1.4) is simply that the volume of a manifold is independent of its triangulations.

It is important to stress that, although we use a geometric language to express our results, the whole set of equations we write is independent of this geometrical interpretation: all our derivations are simply algebraic consequences of the Euler and Abel identities given in Eq.s (2.10) and (2.15); nevertheless the geometric language is extremely useful to guide our intuition.

We worked out explicitly the construction described above for the infinite family of the kind $A_1 \times A_N$, and more generally for the new set of Y-systems \[8\] describing the various restrictions of the fractionally-supersymmetric sine-Gordon models. The solution is surprisingly simple.
We start by describing the case $A_1 \times A_3$ and then we will generalize to a generic $A_N$.

Consider an ordered set of six points $x_1 < x_2 < \ldots < x_6$ of the real line which will form the vertices of a regular triangulation of a suitable threefold. We give to the indices of $x_i$ a cyclic order by putting

$$x_{i+6} = x_i \quad .$$

(2.16)

Consider now the following three ideal tetrahedra

$$T_i \leftrightarrow (x_{i+2} x_{i+3} x_{i+1} x_i) \quad i = 1, 2, 3 \quad .$$

(2.17)

They have the same adjacency relations of the nodes of the $A_3$ diagram. We now associate to each $T_i$ its complement

$$U_i = \mathcal{M} \setminus T_i \leftrightarrow (x_{i+2} x_{i+1} x_{i+3} x_i) \quad i = 1, 2, 3 \quad .$$

(2.18)

According to Eq. (2.10) we have

$$\sum_{i=1,2,3} [vol(T_i) + vol(U_i)] = 3vol(\mathcal{M}) \quad .$$

(2.19)

Then the 3-cycle $\sum_{i=1}^3 [T_i + U_i]$ can be considered as a triangulation of a manifold $\mathcal{M}_3$ covering $\mathcal{M}$ three times. We now apply few times the five term relation (2.12) to pairs of adjacent tetrahedra of the kind $U_i$, in order to obtain a regular triangulation. In the following chain of relations the braces select the pair of such adjacent tetrahedra.

$$\sum_{i=1}^3 vol(U_i) = \{vol(3241) + vol(4352)\} + vol(5463) =$$

$$vol(1254) + vol(2315) + \{vol(5143) + vol(5463)\} =$$

$$vol(1254) + vol(3416) + \{vol(2315) + vol(1365)\} + vol(6154) =$$

$$+ vol(1254) + vol(3416) + vol(5632) +$$

$$+ vol(1265) + vol(2316) + vol(6154) \quad .$$

(2.20)

Note that, as a consequence of the ordering $x_1 < x_2 < \ldots$ of the vertices, all the cross-ratios belong to the interval $[0, 1]$ and all the volumes of the tetrahedra listed in the above identities have the same sign. If we add to the three
Figure 2: Adjacency relations for the $A_3$ triangulation.

$T_i$’s of Eq.(2.18) these six tetrahedra which replace the $U_i$’s we get a regular triangulation $C$ of $M_3$. Indeed these nine ideal tetrahedra are associated to the cross-ratios of the form $(x_j x_{j+1} x_{i+1} x_i) \leftrightarrow T_{ij}$ ($T_{i,j} = T_{j,i}$), where $i$ and $j$ are two non-consecutive, cyclic indices ($i + 6 = i$, $j + 6 = j$).

The initial $T_i$ tetrahedra correspond, in this notation, to $T_{i,i+2}$. In order to study the adjacency relations of these nine tetrahedra it is useful to draw a graph like in Fig.2 where the nodes are associated to the tetrahedra $T_{ij}$ and adjacent nodes represent adjacent tetrahedra. It is easy to verify that this graph, besides the obvious $\mathbb{Z}_3$ cyclic symmetry $T_{ij} \to T_{i+2,j+2}$, fulfills another more hidden $\mathbb{Z}_3$ symmetry generated by $(T_{13}T_{14}T_{24})(T_{35}T_{36}T_{46})(T_{15}T_{25}T_{26})$, where $(xyz)$ denotes the cyclic permutation $x \to y \to z \to x$. Combining these two symmetries it is evident that each $T_{ij}$ can be mapped in any other $T_{i'j'}$ as required in a regular triangulation.

The generalization of the above construction to a generic $A_N$ is straightforward: the starting object is an ordered set of $N + 3$ points $x_1 < x_2 < \ldots < x_{N+3}$ which will form the vertices of a regular triangulation. Being the cross-ratio invariant under the projective group, one can perform a projective transformation carrying three arbitrary distinct points of this set into 0, 1, $\infty$. As a consequence this object depends on $N$ real parameters. Consider now the following set of $N$ ideal tetrahedra $T_i \leftrightarrow (x_{i+2} x_{i+3} x_{i+1} x_i)$.
having the same adjacency relations of the nodes of the $A_N$ diagram. These are the seeds of the wanted triangulation. Indeed these with their complements $U_i = M \setminus T_i$ form a 3-cycle where, using again Eq. (2.10), we get $\sum_{i=1}^{N} [\text{vol}(T_i) + \text{vol}(U_i)] = N \text{vol}(M)$, then it can be thought as a triangulation of a manifold $M_N$ covering $M$ $N$ times. One can try again to transform this triangulation into a regular one by using the decomposition of Eq. (2.12) in suitable places. Actually a good strategy is to eliminate all the tetrahedra of kind $U_i$, because pair of complementary tetrahedra as $\{U_i, T_i\}$ are easily seen to be obstructions to the transitivity of the automorphisms.

For the general $A_N$ case one ends up with a chain $C$ of $N(N+3)/2$ ideal tetrahedra $T_{ij}$ associated to the cross-ratio $(x_j x_{j+1} x_{i+1} x_i)$, labelled by two non-consecutive indices, with $1 < i < i+1 < j \leq N+3$ or $2 < j < N+3$ if $i = 1$, where the indices are understood to be cyclically ordered by defining $x_{i+N+3} = x_i$. Notice that all these cross-ratios belong to the interval $[0, 1]$ as a consequence of the ordering of the vertices.

Now it is easy to show directly that $C$ is a regular triangulation of $M_N$: Note that the chain $C$ satisfies the 3-cycle condition because, as is immediate to verify using Eq. (2.8), each face of the $T_{ij}$’s is shared by two tetrahedra and appear with the two opposite orientations, then $\partial C = 0$. If we subtract from $C$ the chain of seed tetrahedra, we get a chain $C' = C - \sum_{i=1}^{N} T_i$ with the same vertices of of $C$. Now $C'$ is no longer a cycle, but has a boundary which coincides with the boundary of $\sum_{i=1}^{N} U_i$. It follows that $C'$ is a triangulation of $\sum_{i=1}^{N} U_i$ and $C$ is a triangulation of $M$. Looking at the adjacency relations it is not difficult to see that it is regular.

As a first consequence, the volume of $M_N$ can be written as
\[
\sum_{T_{ij} \in C} \text{vol}(T_{ij}) = N \text{vol}(M),
\] (2.21)
which is nothing but the Rogers dilogarithm identity for the $A_N \times A_1$ system in a slightly disguised form. Indeed, rewriting Eq. (1.2) for this family of diagrams as
\[
Y_a \left( \theta + i \frac{\pi}{N+1} \right) Y_a \left( \theta - i \frac{\pi}{N+1} \right) = \prod_{c=1}^{N} (1 + Y_c(\theta))^{G_{a,c}},
\] (2.22)
the identity (1.4) in its \[\] reduced form becomes
\[
\sum_{\alpha=1}^{N} \sum_{n=0}^{N+2} \delta(\alpha + n) L \left( \frac{\Upsilon_a(n)}{1 + \Upsilon_a(n)} \right) = \frac{\pi^2}{6} N(N+1)
\] (2.23)
where the projector $\delta(j) = (1 + (-1)^j)/2$ constrains the double sum to the subset in which $a + n$ is an even number and

$$\Upsilon_a(n) = Y_a \left( \theta + i \frac{n \pi}{N + 1} \right) \quad n = 0, 1, 2, \ldots N + 2 \quad .$$  \hspace{1cm} (2.24)

Then we have to map the two indices $i, j$ labelling the ideal tetrahedra into the two indices $a = 1, \ldots N$ and $n = 1, \ldots N + 3$. It is convenient to choose

$$i = \left\lfloor \frac{n - a}{2} \right\rfloor - 1 \quad , \quad j = \left\lfloor \frac{n + a}{2} \right\rfloor \quad ,$$  \hspace{1cm} (2.25)

where $[x]$ denotes the integer part, modulo $N + 3$. An obvious permutation of the indices in the cross-ratio yields

$$\Upsilon_a(n) = -(x_j x_i x_{i+1} x_{j+1}) \quad .$$  \hspace{1cm} (2.26)

With this choice Eq.(2.24) is transformed, using Eq.(2.6), into the Eq.(2.23) and the Y-system (2.22) is identically fulfilled as a consequence of the following trivial identity among cross -ratios involving six different points

$$(ebcf)(dabc) = (ebaf)(dcbe) \quad .$$  \hspace{1cm} (2.27)

In particular, for variables associated to the two end nodes of the $A_N$ diagram the number of different points in the above identity is reduced to five and one of the two cross-ratios in the r.h.s is equal to 1 as it should.

An important feature of the map (2.25) is that the indices $a$ and $n$, which in the Y-system have a completely different role (the former labels the nodes of $A_N$ the latter is the recursion index) here appear summed together to form the indices labelling the vertices $x_i, x_j$ of the triangulation. Since $n$ in Eq.(2.24) enters in the imaginary part of the rapidity $\theta$, these vertices can be thought as the different values of a single periodic function $x(\theta)$ for different values of the imaginary part of the argument. Thus, combining Eq.s(2.24), (2.26) and (2.25) we may put

$$x_j = x \left( \theta + i \frac{j}{N + 1} \right) \quad$$  \hspace{1cm} (2.28)

with the periodicity condition

$$x \left( \theta + i \frac{N + 3}{N + 1} \right) = x(\theta) \quad .$$  \hspace{1cm} (2.29)
In conclusion, inserting Eq.(2.28) in Eq.(2.26) we get a solution of the Y-system functional equations for the $A_1 \times A_N$ case. It is immediate also to see that such a solution is the most general one, because it is known that this Y-system can be considered from the algebraic point of view as a recursion relation depending on $N$ free parameters, which is just the number of free parameters in our construction. According to Eq.(2.29) the general solution of the Y-system is periodic, so we get, as a by-product, a simple proof of the periodicity conjecture.

Finally, notice that the whole set of $Y$-functions is parametrized through Eq.(2.26) by a single function $x(\theta)$ which becomes the only unknown of the TBA equations. It is suggestive that this result goes in the direction of [19] where an alternative version for the XXZ vacuum energy, depending only from one unknown function, has been derived.

3 Triangulations for minimal models

Among the integrable theories, the sine-Gordon model and its quantum-reduced versions (RSG) corresponding to the $\phi_{13}$-thermal perturbations of the minimal $c < 1$ conformal field theories, are certainly the more studied in the literature. The thermodynamic equations for the vacuum energy of these models has been determined in the repulsive phase [20] as well in the attractive [8] one. In particular in [8] the algebraic RSG-functional equations, obtained using the S-matrix of the sine-Gordon model at arbitrary rational coupling constant, have been presented. It is important to stress [21, 22, 8] that these set of equations describe not only the thermal perturbation of the $c < 1$ minimal models but indeed the whole set of reduced models associated to the fractionally-supersymmetric sine-Gordon theories [23]. In the following we will report the general solutions for the Y-systems, proposed in that paper, and the reader should consult directly that work for a correct interpretation of our results. Let us remember that the RSG Y-systems are uniquely defined by the simple-continued fractions representation for the "dressed " coupling constant $\xi$ of the SG model

\[
\xi = \frac{p}{q-p} = \hat{\xi}(n_1, n_2, \ldots, n_F) := \frac{1}{n_1 + \frac{\ddots}{n_2 + \frac{1}{n_3 + \frac{\ddots}{\ddots + \frac{1}{n_{F-1} + \frac{1}{n_F}}}}}},
\]  

and that the TBA equation can be written in term of a set of unknown functions $Y_k$ with $k = 1, 2, \ldots, n_T - 3$ and $n_T = \sum_{i=1}^{F} n_i$. One of the main differences between the RSG system and the $ADE$ is that the shifts involved
in the Y-equations are now index-dependent and in general they are linear combinations of
\[ s_1 = i\pi \frac{\xi_1}{2}, \quad s_2 = i\pi \frac{\xi_1\xi_2}{2}, \quad \ldots, \quad s_F = i\pi \frac{\xi_1\xi_2\ldots\xi_F}{2} = \frac{i\pi}{2q - 2p}, \quad (3.31) \]
with
\[ \xi_a = \hat{\xi}(n_a, n_{a+1}, \ldots, n_F). \quad (3.32) \]
Defining the matrix
\[ c_{j,k} = \begin{cases} 0 & \text{for } j \neq k \pm 1, \\ (-1)^{a-1} & \text{for } j = k \pm 1, \sum_{i=1}^{a-1} n_i < j, k \leq 1 + \sum_{i=1}^a n_i \end{cases} \quad (3.33) \]
j, k = 1, 2, \ldots, n_T - 3. The Y’s are the solutions of the following system of coupled equations (\(\hat{c}_k = c_{k,k+1}\)).
\[ Y_k (\theta + S_k) \ Y_k (\theta - S_k) = (1 + Y_{k-1}(\theta)^{c_{k,k-1}})^{c_{k,k-1}} (1 + Y_{k+n_{a+1}+1}(\theta)^{\hat{c}_k})^{\hat{c}_k} \]
\[ \prod_{j=k+1}^{k+n_{a+1}} (1 + Y_j(\theta + (k + n_{a+1} - j)S_j + S_{k+n_{a+1}+1})^{\hat{c}_k})^{\hat{c}_k} \]
\[ \prod_{j=k+1}^{k+n_{a+1}} (1 + Y_j(\theta - (k + n_{a+1} - j)S_j - S_{k+n_{a+1}+1})^{\hat{c}_k})^{\hat{c}_k}, \quad (3.34) \]
for \(k = \sum_{i=1}^a n_i\) and \(a < F - 1\), with
\[ S_k = s_a \sum_{i=1}^{a-1} n_i < k \leq \sum_{i=1}^a n_i, \quad (3.35) \]
\[ Y_k (\theta + S_k) \ Y_k (\theta - S_k) = (1 + Y_{k-1}(\theta)^{c_{k,k-1}})^{c_{k,k-1}} (1 + Y_{n_T-3}(\theta)^{-\hat{c}_k})^{-\hat{c}_k} \]
\[ \prod_{j=k+1}^{n_T-3} (1 + Y_j(\theta + (n_T - 1 - j)S_j)^{\hat{c}_k})^{\hat{c}_k} \]
\[ \prod_{j=k+1}^{n_T-3} (1 + Y_j(\theta - (n_T - 1 - j)S_j)^{\hat{c}_k})^{\hat{c}_k}, \quad (3.36) \]
for \(k = n_T - n_F\), and finally
\[ Y_k (\theta + S_k) \ Y_k (\theta - S_k) = \prod_j (1 + Y_j^{c_j,k}(\theta))^{c_j,k}, \quad (3.37) \]
for all the other values of $k$. For our purposes it is convenient to introduce a new set of positive integers $\{\tilde{n}_a\}$ defined in terms of the $\{n_a\}$ in Eq. (3.30) as $\tilde{n}_1 = n_1 + 1$, $\tilde{n}_F = n_F - 1$, $\tilde{n}_a = n_a$, $a = 2, \ldots, F - 1$ and the integer shifts $\tilde{S}_a = 2(q - p)s_a/\pi$. With the choice $\tilde{S}_0 = q$ they satisfy the relation
\begin{equation}
\tilde{S}_{a+1} = \tilde{n}_a \tilde{S}_a + \tilde{S}_{a-1}.
\end{equation}

Let us also define a new set of functions labelled by two indices $a$ and $i$
\begin{equation}
Z^a_i(n)^{(-1)^{a+1}} = \tilde{Y}_k(n) \equiv Y_k \left( \theta + \imath \pi \frac{n}{2(q - p)} \right),
\end{equation}
with $n = 0, 1, \ldots, 2q - 1$. The indices $a$ and $i$ are defined through the relations
\begin{equation}
k = \sum_{j=1}^{a-1} \tilde{n}_j + i,
\end{equation}
\begin{equation}
1 \leq i \leq \tilde{n}_a - 1 \quad \text{for } a = 1
\end{equation}
\begin{equation}
0 \leq i \leq \tilde{n}_a - 1 \quad \text{for } a = 2, \ldots, F - 1
\end{equation}
\begin{equation}
0 \leq i \leq \tilde{n}_a - 3 \quad \text{for } a = F.
\end{equation}

Our starting object is now a ordered set of $q$ points $x_1 < x_2 < \ldots < x_q$, as in the $A_N$ case three of these can be fixed to be $0, 1, \infty$ for the cross-ratio invariance under projective transformations. Using the experience on the $A_N \times A_1$ systems we can again try to solve explicitly the $Y$-system of these RSG models by expressing quantities (3.39) as suitable cross-ratios of the above $q$ points. Actually it is straightforward to verify by direct substitution that the general solution of Eqs. (3.34)-(3.39) is
\begin{equation}
Z^a_i(n) = -(x_b \ x_c \ x_d \ x_e)
\end{equation}
with $x_{i+q} = x_i$ and
\begin{equation}
b = \left[ \frac{n - \tilde{S}_{a-1} + i\tilde{S}_a}{2} \right], \quad c = \left[ \frac{n + \tilde{S}_{a-1} - (i + 2)\tilde{S}_a}{2} \right],
\end{equation}
\begin{equation}
d = \left[ \frac{n + \tilde{S}_{a-1} - i\tilde{S}_a}{2} \right], \quad e = \left[ \frac{n - \tilde{S}_{a-1} + (i + 2)\tilde{S}_a}{2} \right],
\end{equation}
where $[x]$ denotes the integer part, modulo $q$. $Z^1_0$ is identically equal to 0 and for $\xi = 1/(N + 2)$ Eq. (3.43), up to an irrelevant common shift $q$, reduces to Eq. (2.26).
As a simple, illustrative example, let us consider the theory at $\xi = 3/4$. We have $\tilde{n}_1 = 2$, $\tilde{n}_2 = 3$, $\tilde{S}_1 = 3$, $\tilde{S}_2 = 1$, from Eq.s (3.42,3.43) we find the fourteen tetrahedra of this triangulation

$$T_n = \begin{cases} ((n - 4)/2) [((n - 2)/2) [(n + 4)/2] [(n + 2)/2]) & \text{for } n \text{ even} \\ ((n - 3)/2) [((n - 1)/2) [(n + 1)/2] [(n + 3)/2]) & \text{for } n \text{ odd} \end{cases}$$

and $n = 0, 1, \ldots 13$. Drawing the adjacency diagram (see Fig.3), it is easy to see that, although the diagram has an evident $\mathbb{Z}_7$ symmetry, the triangulation, due to the asymmetry under the exchange $T_{n=odd} \leftrightarrow T_{n=even}$, is not regular.

4 Conclusions

The dilogarithm functions play an important role in various branches of mathematics and physics [24]-[28]. In particular, all the properties studied in this paper for the Rogers dilogarithm have an ($m = 2$) Bloch-Wigner counterpart [25]. The Bloch-Wigner function is used for volume calculation on hyperbolic manifold, and obviously some of the geometrical concepts introduced in this article have been already used in different contexts by
various authors. However we only use the geometry as guide to our proofs, which are ultimately purely algebraic. We would also like to stress that, although we learned that the $\mathbb{Z}_5$ symmetry in the Abel equation noticed in [9] was already known by the author of [26], to our knowledge the main results presented in this paper and in [9, 8] are new to the mathematics as well to the physics literature.

It was known for long time that a three-dimensional viewpoint about 2D conformal field theories yields a better unifying understanding of them: many different aspects of rational conformal field theories have emerged as natural consequences of the topology of three dimensional manifolds. Our paper indicates that it is possible to enlarge this point of view also to the perturbed conformal field theories: different properties of the renormalization group evolution of a 2D integrable theory perturbed with a relevant operator, which are encoded in apparently mysterious properties of the Y-systems, like their link with the Dynkin diagrams of simply-laced Lie algebras, the $\mathbb{Z}_p$ symmetry of the recursion relations, the periodicity of the $Y$ variables and the functional identities of the dilogarithm, find again a unifying three-dimensional viewpoint through the ideal triangulations of suitable threefolds.

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