Parabolic Kazhdan-Lusztig polynomials, plethysm and generalized Hall-Littlewood functions for classical types

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Abstract

We use power sums plethysm operators to introduce $H$ functions which interpolate between the Weyl characters and the Hall-Littlewood functions $Q'$ corresponding to classical Lie groups. The coefficients of these functions on the basis of Weyl characters are parabolic Kazhdan-Lusztig polynomials and thus, by works of Kashiwara and Tanisaki, are nonnegative. We prove that they can be regarded as quantizations of branching coefficients obtained by restriction to certain subgroups of Levi type. The $H$ functions associated to linear groups coincide with the polynomials introduced by Lascoux Leclerc and Thibon in [J. Math. Phys 38 (1996), 1041-1068].

1 Introduction

Given $\mu$ a partition with at most $n$ parts, the Hall Littlewood function $Q'_\mu$ can be defined by

$$Q'_\mu = \sum \lambda K_{\lambda,\mu}(q)s_\lambda$$

where the sum runs over the partitions of length at most $n$, $K_{\lambda,\mu}(q)$ is the Lusztig $q$-analogue of weight multiplicity associated with $(\lambda, \mu)$ and $s_\lambda$ the Schur function indexed by $\lambda$, that is the Weyl character of the irreducible finite dimensional $GL_n$-module $V(\lambda)$. Since $K_{\lambda,\mu}(1)$ is equal to the dimension of the weight space $\mu$ in $V(\lambda)$, $Q'_\mu$ can be regarded as a quantization of the homogeneous function $h_\mu$. In [8], Lascoux, Leclerc and Thibon have introduced a new family of symmetric functions $H^\ell_\mu$ depending on a fixed nonnegative integer $\ell$ which interpolate between the Schur functions $s_\mu$ and the Hall-Littlewood functions $Q'_\mu$. The polynomials $H^\ell_\mu$ can be combinatorially described in terms of the spin statistic on certain generalized Young tableaux called $\ell$-ribbon tableaux. These ribbon tableaux naturally appear in the description of the action of the power sum plethysm $\psi_\ell$ on symmetric functions. Recall that for any symmetric function $f$, $\psi_\ell(f)$ is obtained by replacing in $f$ each variable $x_i$ by $x_i^\ell$. In particular $\psi_\ell$ multiplies the degrees by $\ell$. The space of symmetric functions is endowed with an inner product $<\cdot, \cdot>$ which makes the basis of Schur functions orthonormal. Then $\varphi_\ell$, the adjoint operator of $\psi_\ell$, divides the degrees by $\ell$. It is well known that $\varphi_\ell(s_\mu)$ can be computed from the Jacobi-Trudi determinantal identity. Namely, we have

$$\varphi_\ell(s_\mu) = 0 \text{ or } \varphi_\ell(s_\mu) = \varepsilon(\sigma_0)s_{\mu(0)}s_{\mu(\ell-1)}$$

(1)

where $\varepsilon(\sigma_0) = \pm 1$ is the signature of a permutation $\sigma_0 \in S_n$ and $(\mu(0), \ldots, \mu(\ell-1))$ a $\ell$-tuple of partitions defined by $\ell$ and $\mu$. By expanding $\varphi_\ell(s_\mu)$ on the basis of Schur functions we obtain then

$$\varphi_\ell(s_\mu) = \varepsilon(\sigma_0)\sum_{\lambda} c^\lambda_{\mu(0),\ldots,\mu(\ell-1)} s_\lambda$$

(2)
where $c_{\lambda(0),...,(\ell-1)}^\lambda$ is the Littlewood-Richardson coefficient giving the multiplicity of $V(\lambda)$ in the tensor product $V(\mu(0)) \otimes \cdots \otimes V(\mu(\ell-1))$. When $\ell = 1$, one has $\varphi_\ell(s_{\mu}) = s_{\mu}$ and when $\ell > n$ one can prove that $\varphi_\ell(s_{\mu}) = h_{\mu}$. Thus the functions $h_{\mu}^{(\ell)} = \varepsilon(\sigma_0)\varphi_\ell(s_{\mu})$ interpolate between the functions $s_{\mu}$ and $h_{\mu}$ and have nonnegative coefficients on the basis of Schur functions.

In [3], the authors have interpreted the algebra of symmetric functions as the bosonic Fock space representation of the quantum affine Lie algebra $U_q(\widehat{sln})$. This permits to introduce a natural quantization $\psi_{q,\ell}$ of the power sum plethysm $\psi_\ell$. Let $\varphi_{q,\ell}$ be the adjoint operator of $\psi_{q,\ell}$ with respect to $<.,.>$. The function $H_\mu^\ell$ is then defined as a simple renormalization of $\varphi_{q,\ell}(s_{\mu})$. This gives an identity of the form

$$H_\mu^\ell = \sum_{\lambda} c_{\lambda(0),...,(\ell-1)}^\lambda q^{|\lambda|} \quad \text{where} \quad c_{\lambda}^\lambda = \prod_{i=0}^{\ell-1} c_{\mu(0),...,(\ell-1)}^\lambda \quad \text{(3)}$$

where the polynomial $c_{\lambda(0),...,(\ell-1)}^\lambda(q)$ is a $q$-analogue of $c_{\mu(0),...,(\ell-1)}^\lambda$.

Lusztig’s $q$-analogues $K_{\lambda,\mu}(q)$ are particular affine Kazhdan-Lusztig polynomials. These polynomials arise in affine Hecke algebra theory as the entries of the transition matrix between the natural basis and a special basis defined by Lusztig. By replacing the affine Hecke algebra $\tilde{H}$ by one of its parabolic modules $\tilde{H}\nu$ ($\nu$ being a weight of the affine root system under consideration), Deodhar has introduced analogues of the Kazhdan-Lusztig polynomials in [3], it is shown that the family constituted by these parabolic Kazhdan-Lusztig polynomials contains in particular the $q$-analogues $c_{\lambda(0),...,(\ell-1)}^\lambda(q)$. By a result of Kashiwara and Tanisaki [2], this implies notably that the coefficients of the polynomial $c_{\lambda(0),...,(\ell-1)}^\lambda(q)$ are nonnegative integers.

The aim of the paper is to introduce analogues of the polynomials $H_\mu^\ell$ for the classical Lie groups $G = SO_{2n+1}, Sp_{2n}$ and $SO_{2n}$ which interpolate between the Weyl characters and the Hall-Littlewood functions associated with $G$. Write also $s_\lambda$ for the Weyl character of the irreducible $G$-module $V(\lambda)$ of highest weight $\lambda$. We define the plethysm operator $\varphi_\ell$ and its dual $\hat{\psi}_\ell$ on the $\mathbb{Z}$-algebra generated by these Weyl characters. By a subgroup $L \subset G$ of Levi type, we mean a subgroup of $G$ isomorphic to the Levi subgroup of one of its parabolic subgroups. Given a highest weight of $L$, we denote by $[V(\lambda) : V_L(\gamma)]$ the multiplicity of the irreducible $L$-module $V_L(\gamma)$ of highest weight $\gamma$ in the restriction of $V(\lambda)$ to $L$. Then, provided that $\ell$ is odd when $G = Sp_{2n}$ or $SO_{2n}$, we establish for any Weyl character $s_\mu$ such that $\varphi_\ell(s_\mu) \neq 0$, a formula of the type

$$\varphi_\ell(s_\mu) = \varepsilon(w_0) \sum_{\lambda} [V(\lambda) : V_L(\mu)] s_\lambda \quad \text{(3)}$$

where $\varepsilon(w_0)$ is the signature of an element $w_0 \in W$ the Weyl group of $G$, $L$ a subgroup of Levi type of $G$ and $\mu$ a dominant weight associated with $L$. The procedure which yields $w_0, L$ and $\mu$ from $\ell$ and $\mu$ can be regarded as an analogue of the algorithm computing the $\ell$-quotient of a partition which implicitly appears in [1]. The identity (2) can also be rewritten as in (3). Indeed, take $L = GL_{r_0} \times \cdots \times GL_{r_{\ell-1}}$ where for any $k = 1, ..., \ell-1$, $r_k$ is the length of $\mu^{(k)}$. Then $(\mu^{(k)}) = (\mu(0), ..., \mu^{(\ell-1)})$ can be interpreted as a dominant weight for the subgroup of Levi type $L$ of $GL_n$ and we have the duality $c_{\mu(0),...,(\ell-1)}^\lambda = [V(\lambda) : V_L(\mu^{(k)})]$. The surprising constraint $\ell$ odd when $G = Sp_{2n}$ or $SO_{2n}$ appearing in (3) follows from the fact that the procedure giving $w_0, L$ and $\mu$ mentioned above depends not only on the Lie group under consideration but also on the parity of the integer $\ell$. For $G = SO_{2n+1}$ the coefficients of $\varepsilon(w_0)\varphi_\ell(s_\mu)$ on the basis of Weyl characters are always branching coefficients corresponding to restriction to $L$. For $G = Sp_{2n}$ or $SO_{2n}$ this is only true when $\ell$ is odd. Note that this difficulty disappears in large rank, that is for $n \geq \ell |\mu|$ (but see Section 6.4).
To define the functions $H_\mu^\ell$ in type $B, C$ or $D$, we prove the equalities

$$|<\psi_\ell(s_\lambda), s_\mu>| = |<s_\lambda, \varphi_\ell(s_\mu)>| = P_{\mu+\rho, \ell\lambda+\rho}^{-}(1)$$

which show that the coefficients of the expansion of $\psi_\ell(s_\lambda)$ on the basis of Weyl characters are, up to a sign, parabolic Kazhdan-Lusztig polynomials specialized at $q = 1$. By using (3) this gives, providing $\ell$ is odd for $G = Sp_{2n}$ or $SO_{2n}$

$$[V(\lambda) : V_L(\mu_\ell)] = P_{\mu+\rho, \ell\lambda+\rho}(1).$$

We then introduce the functions

$$G_\mu^\ell = \sum_\lambda [V(\lambda) : V_L(\mu_\ell)]q^{s_\lambda}$$

where $[V(\lambda) : V_L(\mu_\ell)]_q = P_{\mu+\rho, \ell\lambda+\rho}(q)$. This yields nonnegative $q$-analogues of the branching coefficients $[V(\lambda) : V_L(\mu_\ell)]$. The functions $H_\mu^\ell$ are then defined by setting $H_\mu^\ell = G_{\ell\mu}^\ell$. We obtain the identities $H_\mu^1 = s_\mu$ and $H_\mu^\ell = Q_\mu^{\ell'}$ when $\ell$ is sufficiently large. Thus the functions $H_\mu^\ell$ interpolate between the Weyl characters and the Hall-Littlewood functions associated with $G$.

The paper is organized as follows. In Section 2 we recall the necessary background on classical root systems, Weyl characters, subgroups of Levi type and their corresponding branching coefficients. In Section 3, we define the plethysm operators $\psi_\ell$ and their dual operators $\varphi_\ell$. By abuse of notation, we also denote by $\varphi_\ell$ the linear operator on the group algebra $\mathbb{Z}[\mathbb{Z}^n]$ with basis the formal exponentials $(e^\beta)$ such that

$$\varphi_\ell(e^\beta) = \begin{cases} e^{\beta/\ell} & \text{if } \beta \in (\ell\mathbb{Z})^n \\ 0 & \text{otherwise} \end{cases} \quad \text{for any } \beta \in \mathbb{Z}^n.$$

We then show how the determination of $\varphi_\ell(s_\mu)$ can be reduced to the computation of the polynomial

$$\varphi_\ell(e_\mu \prod_{\alpha \in R_+} (1 - e_\alpha))$$

where $R_+$ is the set of positive roots corresponding to the Lie group $G$. This permits to establish formulas (3) providing $\ell$ is odd when $G = Sp_{2n}$ or $SO_{2n}$. For completion we have also included the case $G = GL_n$ and shown why (3) cannot hold when $\ell$ is even and $G = Sp_{2n}$ or $SO_{2n}$. To make the paper self-contained, we have summarized in Section 4 some necessary results on affine Hecke algebras and parabolic Kazhdan-Lusztig polynomials. Section 5 is devoted to the definition of the polynomials $G_\mu^\ell$ and $H_\mu^\ell$ and to their links with the Weyl characters and the Hall-Littlewood functions. Finally we briefly discuss in Section 6 the problem of defining nonnegative $q$-analogues of tensor product multiplicities when $G \neq GL_n$. We add also a few remarks concerning the exceptional root systems.

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2 Background

2.1 Classical root systems

In the sequel $G$ is one of the complex Lie groups $GL_n, Sp_{2n}, SO_{2n+1}$ or $SO_{2n}$ and $\mathfrak{g}$ its Lie algebra. We follow the convention of [7] to realize $G$ as a subgroup of $GL_N$ and $\mathfrak{g}$ as a subalgebra of $\mathfrak{gl}_N$ where

$$N = \begin{cases} n & \text{when } G = GL_n \\ 2n & \text{when } G = Sp_{2n} \text{ or } SO_{2n} \\ 2n + 1 & \text{when } G = SO_{2n+1} \end{cases}$$

With this convention the maximal torus $T$ of $G$ and the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ coincide respectively with the subgroup and the subalgebra of diagonal matrices of $G$ and $\mathfrak{g}$. Similarly the Borel subgroup $B$ of $G$ and the Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ coincide respectively with the subgroup and subalgebra of upper triangular matrices of $G$ and $\mathfrak{g}$.

Let $d_N$ be the linear subspace of $\mathfrak{gl}_N$ consisting of the diagonal matrices. For any $i \in I_n = \{1, \ldots, n\}$, write $\varepsilon_i$ for the linear map $\varepsilon_i : d_N \to \mathbb{C}$ such that $\varepsilon_i(D) = \delta_{n-i+1}$ for any diagonal matrix $D$ whose $(i,i)$-coefficient is $\delta_i$. Then $(\varepsilon_1, \ldots, \varepsilon_n)$ is an orthonormal basis of the Euclidean space $h^*_R$ (the real part of $h^*$). Let $\langle \cdot, \cdot \rangle$ be the corresponding nondegenerate symmetric bilinear form defined on $h^*_R$. Write $R$ for the root system associated with $G$. For any $\alpha \in R$ we set $\alpha^\vee = \frac{\alpha}{\langle \alpha, \alpha \rangle}$. The Lie algebra $\mathfrak{g}$ admits the diagonal decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$. We take for the set of positive roots:

$$R^+ = \{ \varepsilon_j - \varepsilon_i \text{ with } 1 \leq i < j \leq n \}$$

for the root system $A_{n-1}$

$$R^+ = \{ \varepsilon_j - \varepsilon_i, \varepsilon_j + \varepsilon_i \text{ with } 1 \leq i < j \leq n \} \cup \{ \varepsilon_i \text{ with } 1 \leq i \leq n \}$$

for the root system $B_n$

$$R^+ = \{ \varepsilon_j - \varepsilon_i, \varepsilon_j + \varepsilon_i \text{ with } 1 \leq i < j \leq n \} \cup \{ 2\varepsilon_i \text{ with } 1 \leq i \leq n \}$$

for the root system $C_n$

$$R^+ = \{ \varepsilon_j - \varepsilon_i, \varepsilon_j + \varepsilon_i \text{ with } 1 \leq i < j \leq n \}$$

for the root system $D_n$

Let $\rho$ be the half sum of positive roots. Set $J_n = \{ \overline{1} < \cdots < \overline{i} < 1 < \cdots < n \}$ where, for each integer $i = 1, \ldots, n$, we have written $\overline{i}$ for the negative integer $-i$. For any $x \in J_n$ we have $\overline{x} = x$ and we set $|x| = x$ if $x > 0$, $|x| = \overline{x}$ otherwise. Given a subset $U \subset J_n$, we define $|U| = \{|x| \mid x \in U\}$ and $\overline{U} = \{\overline{x} \mid x \in U\}$.

The Weyl group of $GL_n$ is the symmetric group $S_n$ and for $G = SO_{2n+1}, Sp_{2n}$ or $SO_{2n}$, the Weyl group $W$ of the Lie group $G$ is the subgroup of the permutation group of $J_n$ generated by the permutations

$$s_i = (i, i + 1)(\overline{i}, \overline{i + 1}), \quad i = 1, \ldots, n - 1$$

and

$$s_n = (n, \overline{1})(n, \overline{n})$$

for the root systems $B_n$ and $C_n$

$$s_i = (i, i + 1)(\overline{i}, \overline{i + 1}), \quad i = 1, \ldots, n - 1$$

and $s_n = (n, \overline{n - 1})(n - 1, \overline{1})$ for the root system $D_n$

where for $a \neq b (a, b)$ is the simple transposition which switches $a$ and $b$. For types $B_n$ and $C_n$, $W$ is the group of signed permutations. It is the subgroup of the permutation group of $J_n$ consisting of the permutations $w$ such that $w(\overline{i}) = \overline{w(i)}$. For type $D_n$, the elements of $W$ verify the additional constraint $\{ i \in I_n \mid w(i) < 0 \} \in 2N$. We identify the subgroup of $W$ generated by $s_i = (i, i + 1)(\overline{i}, \overline{i + 1})$, $i = 1, \ldots, n - 1$ with the symmetric group $S_n$. The signature $\varepsilon$ of $w \in W$ is defined by $\varepsilon(w) = (-1)^{\ell(w)}$ where $\ell$ is the length function corresponding to the above sets of generators. Consider the increasing sequence $K = (\overline{1}_p, \ldots, \overline{i}_1, i_1, \ldots, i_p) \subset J_n$. For $X = B, D$ set

$$W_{X,K} = \{w \in W \mid \text{type } X_n \mid w(x) = x \text{ for any } x \notin K\}$$

Then, $W_{X,K}$ is isomorphic to the Weyl group of type $X_p$. Let $\varepsilon_{X,K}$ be the corresponding signature.

**Lemma 2.1.1** Consider $X = B, D$ and $w \in W_{X,K}$. Then we have $\varepsilon_{X,K}(w) = \varepsilon(w)$.
Proof. Suppose \( X = B \). The generators of the Weyl group \( W_{X,K} \) are the \( t_k = (i_k, i_{k+1}) \), \( k = 1, \ldots, p - 1 \) and \( s_n = (i_p, i_p) \). One verifies easily that, considered as elements of \( W \), they have an odd length. We proceed similarly when \( X = D \). ■

The action of \( w \in W \) on \( \beta = (\beta_1, \ldots, \beta_n) \in \mathfrak{h}_R^* \) is defined by

\[
w \cdot (\beta_1, \ldots, \beta_n) = (\beta_1^{w^{-1}}, \ldots, \beta_n^{w^{-1}})
\]

where \( \beta_i^w = \beta_{w(i)} \) if \( w(i) \in I_n \) and \( \beta_i^w = -\beta_{w(\overline{i})} \) otherwise. The dot action of \( W \) on \( \beta = (\beta_1, \ldots, \beta_n) \in \mathfrak{h}_R^* \) is defined by

\[
w \circ \beta = w \cdot (\beta + \rho) - \rho.
\]

The fundamental weights of \( \mathfrak{g} \) belong to \( \left( \frac{Z}{2} \right)^n \). More precisely we have \( \omega_i = (0, \ldots, 1^i, \ldots, 0) \in \mathbb{N}^n \) for \( i < n - 1 \) and also \( i = n - 1 \) for \( \mathfrak{g} \neq \mathfrak{so}_{2n} \), \( \omega_n^C = (1^n) \), \( \omega_n^B = \omega_n^D = (\frac{1}{2}n) \) and \( \omega_n^A = (\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, 1^{n-1}) \). The weight lattice \( P \) of \( \mathfrak{g} \) can be considered as the \( \mathbb{Z} \)-sublattice of \( \left( \frac{Z}{2} \right)^n \) generated by \( \omega_i \), \( i \in I \). For any \( \beta = (\beta_1, \ldots, \beta_n) \in P \), we set \( |\beta| = \beta_1 + \cdots + \beta_n \). Write \( P^+ \) for the cone of dominant weights of \( G \). With our convention, a partition of length \( m \) is a weakly \textit{increasing} sequence of \( m \) nonnegative integers. Denote by \( P_n \) the set of partitions with at most \( n \) parts. Each partition \( \lambda = (\lambda_1, \ldots, \lambda_m) \in P_n \) will be identified with the dominant weight \( \sum_{i=1}^m \lambda_i \varepsilon_i \). Then the irreducible finite dimensional polynomial representations of \( G \) are parametrized by the partitions of \( P_n \). For any \( \lambda \in P_n \), denote by \( V(\lambda) \) the irreducible finite dimensional representation of \( G \) of highest weight \( \lambda \). We will also need the irreducible rational representations of \( GL_n \). They are indexed by the \( n \)-tuples

\[
(\gamma^+, \gamma^-) = (-\gamma_1, \ldots, -\gamma_1, \gamma_1^+, \gamma_2^+, \ldots, \gamma_p^+)
\]

where \( \gamma^+ = (\gamma_1^+, \gamma_2^+, \ldots, \gamma_p^+) \) and \( \gamma^- = (\gamma_1^-, \ldots, \gamma_q^-) \) are partitions of length \( p \) and \( q \) such that \( p + q = n \). Write \( \overline{P}_n \) for the set of such \( n \)-tuples and denote also by \( V(\gamma) \) the irreducible rational representation of \( GL_n \) of highest weight \( \gamma = (\gamma^-, \gamma^+) \in \overline{P}_n \).

In the sequel, our computations will also make appear root subsystems of the root systems \( R \) described above. Suppose that \( G \) is of type \( X_n \) with \( X_n = A_{n-1} \), \( B_n \), \( C_n \) or \( D_n \). Let \( I = (i_1, \ldots, i_r) \) be an increasing sequence of integers belonging to \( I_n \), that is \( i_k \in I_n \) for any \( k = 1, \ldots, r \) and \( i_1 < \cdots < i_r \). Then

\[
R_I = \{ \alpha \in R \cap \oplus_{i \in I} \mathbb{Z} \varepsilon_i \}
\]

is a root subsystem of \( R \) of type \( X_r \). Write \( R_I^+ \) for the set of positive roots in \( R_I \). Then we have \( R_I^+ = R_I \cap R^+ \). The dominant weights associated with \( R_I \) have the form \( (\lambda_{i_1}, \ldots, \lambda_{i_r}) \) where \( \lambda_i \neq 0 \) only if \( i \in I \) and \( \lambda(I) = (\lambda_{i_1}, \ldots, \lambda_{i_r}) \in P_r \). We slightly abuse the notation by identifying \( \lambda \) with \( \lambda(I) \). Consider an increasing sequence \( X = (x_1, \ldots, x_r) \) of integers belonging to \( J_n \) such that \( |x_k| = |x_k'| \) if and only if \( k = k' \). For any integer \( i = 1, \ldots, n \), set \( \varepsilon_i^* = -\varepsilon_i \). Then

\[
R_{A,X} = \{ \pm(\varepsilon_{x_j} - \varepsilon_{x_i}) \mid 1 \leq i < j \leq r \}
\]

is a root subsystem of \( R \) of type \( A_{r-1} \). To see this, consider the linear map \( \theta_X : \mathbb{Z}^r \to \mathbb{Z}^n \) such that \( \theta_X(\varepsilon_i) = \varepsilon_{x_i} \). The map \( \theta \) is injective and preserves the scalar product in \( \mathbb{Z}^r \) and \( \mathbb{Z}^n \). Moreover the root system \( \{ \pm(\varepsilon_j - \varepsilon_i) \mid 1 \leq i < j \leq r \} \subset \mathbb{Z}^r \) of type \( A_r \) is sent on \( R_{A,X} \) by \( \theta_X \). The set of positive roots in \( R_{A,X} \) is equal to \( R_{A,X}^+ = R_{A,X} \cap R^+ \). Denote by \( s \in \{1, \ldots, r\} \) the maximal integer such that \( x_s < 0 \). We associate to \( X \), the increasing sequence of indices \( I \subset I_n \) defined by

\[
I = (\overline{x}_s, \ldots, \overline{x}_1, x_{s+1}, \ldots, x_r).
\]
It will be useful to consider the weights corresponding to $R_{A,X}$ as the $r$-tuples $\beta = (\beta_{x_1}, ..., \beta_{x_r})$ with coordinates indexed by $X$. The coordinates $(\beta'_1, ..., \beta'_n)$ of $\beta$ on the initial basis $(\varepsilon_1, ..., \varepsilon_n)$ are such that $\beta'_i = \beta_{x_i}$ if $i = x_a \in X$, $\beta'_i = -\beta_{x_a}$ if $i = x_a \in X$ and $\beta'_i = 0$ otherwise. With this convention the dominant weights for $R_{A,X}$ have the form

$$\lambda(X) = (\lambda_{x_1}, ..., \lambda_{x_r}) \in \tilde{P}_r.$$  

This simply means that we have chosen to expand the weights of $R_{A,X}$ on the basis $\{\varepsilon_x | x \in X\}$ rather than on the basis $\{\varepsilon_i | i \in I\}$ to preserve the identification of the dominant weights with the nondecreasing $r$-tuples of integers.

**Example 2.1.2** Take $G = Sp_{10}$.

- For $I = (2, 4, 5)$ we have

  $$R^+_I = \{\varepsilon_5 \pm \varepsilon_4, \varepsilon_5 \pm \varepsilon_2, \varepsilon_4 \pm \varepsilon_2, 2\varepsilon_2, 2\varepsilon_4, 2\varepsilon_5\}$$

  which is the set of positive roots of a root system of type $C_3$. The weight $\lambda = (1, 2, 2)$ is dominant for $G_I$. Considered as a weight of $Sp_{10}$, we have $\lambda = (0, 1, 0, 2, 2)$.

- For $X = (5, 3, 1, 4)$ we have

  $$R^+_A,X = \{\varepsilon_4 - \varepsilon_1, \varepsilon_5 - \varepsilon_2, \varepsilon_1 + \varepsilon_2, \varepsilon_1 + \varepsilon_5, \varepsilon_4 + \varepsilon_2, \varepsilon_4 + \varepsilon_5\}$$

  which is the set of positive roots of a root system of type $A_3$. The weight $\gamma = (-3, -1, 4, 5)$ is dominant for $G_X$. Considered as a weight of $Sp_{10}$, we have $\gamma = (4, 1, 0, 5, 3)$.

### 2.2 Subgroups of Levi type

Suppose $G$ is a classical Lie group and consider $R$ the corresponding root system. We shall need Lie subgroups of $G$ associated with particular sub-root systems of $R$. Each of these subgroups will be of Levi type, that is, will be isomorphic to the Levi subgroup of one of the parabolic subgroups of $G$.

Consider $p \geq 1$ an integer. Let $I^{(0)} = (i^{(0)}_1, ..., i^{(0)}_r)$ be an increasing sequence of integers in $I_n$. For $k = 1, ..., p$, consider increasing sequences $X^{(k)} = (x^{(k)}_1, ..., x^{(k)}_{r_k}) \subset J_n$ such that $\text{card}(X^{(k)}) = r_k$. Let $s_k$ be maximal in $\{1, ..., r_k\}$ such that $x^{(k)}_{s_k} < 0$. Set

$$I^{(k)} = (i^{(k)}_1, ..., i^{(k)}_r), x^{(k)}_{s_k + 1}, ..., x^{(k)}_{r_k}) \subset I_n.$$  

We suppose that the sets $I^{(k)}$, $k = 0, ..., p$ are pairwise disjoint and verify $\bigcup_{k=0}^{p} I^{(k)} = I_n$. Set $I = \{I^{(0)}, X^{(1)}, ..., X^{(p)}\}$ and

$$R_I = R_{I^{(0)}} \cup \bigcup_{k=1}^{p} R_{A,X^{(k)}}$$

Then $g_I = h \oplus \bigoplus_{\alpha \in R_I} g_\alpha$ is a Lie subalgebra of $g$. Its corresponding Lie group $G_I$ is a subgroup of $G$ of Levi type and we have

$$G_I \simeq \begin{cases} GL_{r_0} \times GL_{r_1} \times \cdots \times GL_{r_p} & \text{for } G = GL_n \\ SO_{2r_0+1} \times GL_{r_1} \times \cdots \times GL_{r_p} & \text{for } G = SO_{2n+1} \\ Sp_{2r_0} \times GL_{r_1} \times \cdots \times GL_{r_p} & \text{for } G = Sp_{2n} \\ SO_{2r_0} \times GL_{r_1} \times \cdots \times GL_{r_p} & \text{for } G = SO_{2n} \end{cases}.$$
The root system associated with $G_I$ is $R_I$. Denote by $P_I^+$ its cone of dominant weights. The weight lattice of $G_I$ coincides with that of $G$ since the Lie algebras $g_T$ and $g$ have the same Cartan subalgebra. The elements of $P_I^+$ are the $(p + 1)$-tuples $\lambda = (\lambda(0), \lambda(1), \ldots, \lambda(p))$ where $\lambda(0) = (\lambda_i \mid i \in I^{(0)})$ is a dominant weight of $R_{G,I^{(0)}}$ and for any $k = 1, \ldots, p$, $\lambda(k) = (\lambda_i \mid i \in X^{(k)})$ is a dominant weight of $R_{G,X^{(k)}}$. For any $\lambda \in P_I^+$, we denote by $V_I(\lambda)$ the irreducible finite dimensional $G_I$-module of highest weight $\lambda$. Each weight $\beta = (\beta(0), \beta(1), \ldots, \beta(p)) \in P_I$ can be considered as a weight $\beta = (\beta_1', \ldots, \beta_n')$ of $P$. With the convention $[8]$ we have then $\beta_i = \beta_{i_a}$ if $i = i_a^{(0)} \in I^{(0)}$ and for any $k = 1, \ldots, p$, $\beta_i' = \beta'_{i_a}$ if $i = i_a^{(k)} \in X^{(k)}$, $\beta_i' = -\beta'_{i_a}$ if $\overline{i} = i_a^{(k)} \in X^{(k)}$. In the sequel we identify the two expressions

$$
\beta = (\beta(0), \beta(1), \ldots, \beta(p)) \quad \text{and} \quad \beta = (\beta_1', \ldots, \beta_n')
$$

of the weights of $P_I$.

### 2.3 Weyl characters and dual bases

We refer the reader to [13] and [15] for a detailed exposition of the results used in this paragraph. We use as a basis of the group algebra $\mathbb{Z}[\mathbb{Z}^n]$, the formal exponentials $(e^\beta)_{\beta \in \mathbb{Z}^n}$ satisfying the relations $e^{\beta_1} e^{\beta_2} = e^{\beta_1 + \beta_2}$. We furthermore introduce $n$ independent indeterminates $x_1, \ldots, x_n$ in order to identify $\mathbb{Z}[\mathbb{Z}^n]$ with the ring of polynomials $\mathbb{Z}[x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}]$ by writing $e^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n} = x^\beta$ for any $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{Z}^n$. Define the action of the Weyl group $W$ on $\mathbb{Z}[\mathbb{Z}^n]$ by $w \cdot x^\beta = x^{w(\beta)}$. The Weyl character $s_\beta$ is defined by

$$s_\beta = \frac{a_\beta + \rho}{a_\rho} \quad \text{where} \quad a_\beta = \sum_{w \in W} \varepsilon(\sigma)(w \cdot x^\beta)
$$

For any $\beta \in \mathbb{Z}^n$ we have

$$s_\beta = \begin{cases} 
\varepsilon(w)s_\lambda & \text{if there exists } w \in W \text{ and } \lambda \in \mathcal{P}_n \text{ such that } \lambda = w \circ \beta \\
0 & \text{otherwise}
\end{cases}
$$

Let $A$ be the $\mathbb{Z}$-algebra generated by the characters $s_\lambda, \lambda \in \mathcal{P}_n$. For any $\beta \in \mathbb{Z}^n$, denote by $W_\beta$ the stabilizer of $\beta$ under the action of the Weyl group $W$ and by $W_\beta$ a set of representatives in $W/W_\beta$ with minimal length. Then the functions

$$m_\beta = \sum_{w \in W_\beta} w \cdot x^\beta
$$

belong to $A$. Moreover $\{m_\lambda \mid \lambda \in \mathcal{P}_n\}$ is a basis of $A$. We have the decomposition

$$s_\lambda = \sum_{\mu \in \mathcal{P}_n} K_{\lambda, \mu} m_\mu
$$

where $K_{\lambda, \mu}$ is equal to the dimension of the weight space $\mu$ in the irreducible representation $V(\lambda)$. There exists an inner product $< \cdot, \cdot >$ on $A$ which makes the characters $s_\lambda$ orthonormal. We denote by $\{h_\mu \mid \mu \in \mathcal{P}_n\}$ the dual basis of $\{m_\lambda \mid \lambda \in \mathcal{P}_n\}$ with respect to $< \cdot, \cdot >$. The homogeneous functions $h_\mu$ are given in terms of the Weyl characters by the decomposition

$$h_\mu = \sum_{\lambda \in \mathcal{P}_n} K_{\lambda, \mu} s_\lambda.
$$

This decomposition is infinite in general when $G \neq GL_n$. Nevertheless, by embedding $A$ in the ring $\hat{A}$ of universal characters defined by Koike and Terada [7], it makes sense to consider formal series
in the characters $s_{\lambda}, \lambda \in \mathcal{P}_n$. Note that the function $h_\mu$ is not the character of the representation $V(\mu_1\omega_1) \otimes \cdots \otimes V(\mu_n\omega_1)$ when $G \neq GL_n$. For any $\beta \in \mathbb{Z}^n$ we define the function $h_\beta$ by

$$h_\beta = h_\mu,$$  \hspace{1cm} (14)

where $\mu$ is the unique dominant weight contained in the orbit $W \cdot \beta$.

2.4 Jacobi-Trudi identities

Denote by $\mathcal{L}_n = \mathbb{K}[[x^\beta]]$ the vector space of formal series in the monomials $x^\beta$ with $\beta \in \mathbb{Z}$. We identify the ring of polynomials $\mathcal{F}_n = \mathbb{K}[x^\beta]$ with the subspace of $\mathcal{L}_n$ containing the finite formal series. The vector space $\mathcal{L}_n$ is not a ring since $\beta \in \mathbb{Z}$. More precisely, the product $F_1 \cdots F_r$ of the formal series $F_i = \sum_{\beta \in E_i} x^{\beta(i)}$ is defined if and only if, for any $\gamma \in \mathbb{Z}^n$, the number $N_\gamma$ of decompositions $\gamma = \beta^{(1)} + \cdots + \beta^{(r)}$ such that $\beta^{(i)} \in E_i$ is finite and in this case we have

$$F_1 \cdots F_r = \sum_{\gamma \in \mathbb{Z}^n} N_\gamma x^\gamma.$$ 

In particular the product $P \cdot F$ with $P \in \mathcal{P}_n$ and $F \in \mathcal{L}_n$ is well defined.

Set 

$$\nabla = \prod_{\alpha \in R_+} \frac{1}{1 - x^\alpha} \text{ and } \Delta = \prod_{\alpha \in R_+} (1 - x^\alpha).$$

Then $\Delta \in \mathcal{F}_n$ and $\nabla \in \mathcal{L}_n$. We define two linear maps

$$S : \{ \mathcal{L}_n \to \hat{A} \text{ and } H : \{ \mathcal{L}_n \to \hat{A} \text{ by setting } x^\beta \mapsto s_\beta \text{ and } \nabla \mapsto h_\beta \}.$$

From Theorem 2.14 of [15] we obtain

**Proposition 2.4.1** For any $\beta \in \mathbb{Z}^n$, $s_\beta = \sum_{w \in W} \varepsilon(w)h_{\beta + \rho - w \cdot \rho}$.

By using the identity

$$\Delta = x^\beta \sum_{w \in W} \varepsilon(w)x^{-w \cdot \rho} \hspace{1cm} (15)$$

the previous proposition is equivalent to the following identity:

$$S(x^\beta) = H(\Delta \times x^\beta). \hspace{1cm} (16)$$

**Proposition 2.4.2** For any $\beta \in \mathbb{Z}^n$ we have $H(x^\beta) = S(\nabla \times x^\beta)$.

**Proof.** Denote by $\chi_\Delta$ and $\chi_\nabla$ the linear maps defined on $\mathcal{L}_n$ by setting $\chi_\Delta(x^\beta) = \Delta \times x^\beta$ and $\chi_\nabla(x^\beta) = \nabla \times x^\beta$ respectively. By [16] we have $S = H \circ \chi_\Delta$. Moreover for any $\beta \in \mathbb{Z}^n$, $\chi_\Delta \circ \chi_\nabla(x^\beta) = x^\beta$. This gives $S(\nabla \times x^\beta) = S \circ \chi_\nabla(x^\beta) = H \circ \chi_\Delta \circ \chi_\nabla(x^\beta) = H(x^\beta)$. \qed
2.5 Branching coefficients for the restriction to subgroups of Levi type

Consider \( \mathcal{I} = \{ I_0, X_1, \ldots, X_p \} \) as in \[.22\] The set \( \mathcal{I} \) characterizes a subgroup \( G_{\mathcal{I}} \subset G \) of Levi type. Set

\[
\Delta_{\mathcal{I}} = \prod_{\alpha \in R^+_{\mathcal{I}}} (1 - x^\alpha) \quad \text{and} \quad \nabla_{\mathcal{I}} = \prod_{\alpha \in R^+ - R^+_{\mathcal{I}}} \frac{1}{(1 - x^\alpha)}.
\]

Then \( \Delta_{\mathcal{I}} \in \mathcal{F}_n \) and \( \nabla_{\mathcal{I}} \in \mathcal{L}_n \). Note that \( \nabla_{\mathcal{I}} = \nabla \times \Delta_{\mathcal{I}} \).

As a formal series, \( \nabla_{\mathcal{I}} \) can be expanded on the form

\[
\nabla_{\mathcal{I}} = \sum_{\gamma \in \mathbb{Z}^n} \mathcal{P}_{\mathcal{I}}(\gamma) x^\gamma.
\]

Consider \( \lambda \in \mathcal{P}_n \) and \( \mu = (\mu^{(0)}, \ldots, \mu^{(p)}) \) a dominant weight associated with \( G_{\mathcal{I}} \). We denote by \([V(\lambda) : V_{\mathcal{I}}(\mu)]\) the multiplicity of the irreducible representation \( V_{\mathcal{I}}(\mu) \) in the restriction of \( V(\lambda) \) from \( G \) to \( G_{\mathcal{I}} \). The proposition below follows from Theorem 8.2.1 in [11]:

**Proposition 2.5.1** Consider \( \lambda \in \mathcal{P}_n \) and \( \mu = (\mu^{(0)}, \ldots, \mu^{(p)}) \) a dominant weight of \( P^+_{\mathcal{I}} \). Then

\[
[V(\lambda) : V_{\mathcal{I}}(\mu)] = \sum_{w \in W} \varepsilon(w) \mathcal{P}_{\mathcal{I}}(w \circ \lambda - \mu).
\]

Define the linear map

\[
\left\{ \begin{array}{c}
S_{\mathcal{I}} : \mathcal{L}_n \rightarrow \hat{\Lambda} \\
x^\beta \mapsto H(\Delta_{\mathcal{I}} \times x^\beta)
\end{array} \right.
\]

For any dominant weight \( \mu \in P^+_{\mathcal{I}} \), set

\[
S_{\mu,\mathcal{I}} = H(\Delta_{\mathcal{I}} \times x^\mu) = S_{\mathcal{I}}(x^\mu).
\]

**Proposition 2.5.2** With the above notations we have

\[
S_{\mu,\mathcal{I}} = \sum_{\lambda \in \mathcal{P}_n} [V(\lambda) : V_{\mathcal{I}}(\mu)] s_{\lambda}.
\]

**Proof.** For any \( \beta \in \mathbb{Z}^n \), we have obtained in the proof of Proposition \[.24.2\] the identity \( H(x^\beta) = S \circ \chi_{\mathcal{I}}(x^\beta) \). Denote by \( \chi_{\Delta_{\mathcal{I}}} \) the linear map defined on \( \mathcal{L}_n \) by setting \( \chi_{\Delta_{\mathcal{I}}}(x^\beta) = \Delta_{\mathcal{I}} \times x^\beta \). We obtain

\[
S_{\mathcal{I}}(x^\beta) = H(\Delta_{\mathcal{I}} \times x^\beta) = S \circ \chi_{\mathcal{I}} \circ \chi_{\Delta_{\mathcal{I}}}(x^\beta) = S(\nabla_{\mathcal{I}} \times x^\beta) \text{since} \ \nabla_{\mathcal{I}} = \nabla \times \Delta_{\mathcal{I}}. \text{Thus by} \ (17) \text{this yields} \ S_{\mathcal{I}}(x^\beta) = \sum_{\gamma \in \mathbb{Z}^n} \mathcal{P}_{\mathcal{I}}(\gamma) s_{\beta + \gamma}. \text{For any} \ \gamma, \text{we know by} \ (11) \text{that} \ s_{\beta + \gamma} = 0 \text{or there exists} \ \lambda \in \mathcal{P}_n, w \in W \text{such that} \ \lambda = w \circ (\beta + \gamma) \text{and} \ s_{\beta + \gamma} = \varepsilon(w)s_{\lambda}. \text{This permits to write}
\]

\[
S_{\mathcal{I}}(x^\beta) = \sum_{\lambda \in \mathcal{P}_n} \sum_{w \in W} \varepsilon(w) \mathcal{P}_{\mathcal{I}}(w \circ \lambda - \beta)s_{\lambda}.
\]

When \( \beta = \mu \) is a dominant weight of \( P^+_{\mathcal{I}} \), we obtain the desired identity by using Proposition \[.25.1\].

**Remarks:**

(i): When \( G = G_{\mathcal{I}} \) that is, when \( r_0 = n \) and \( r_1 = \cdots = r_p = 0 \), we have \( \mu = \mu^{(0)} \), \( \Delta_{\mathcal{I}} = \Delta \) and \( H_{\mathcal{I}} = H \). Thus \( S_{\mu,\mathcal{I}} = s_{\mu^{(0)}} \). This can be recovered by using \[.19\] since in this case \([V(\lambda) : V_{\mathcal{I}}(\mu)] = 0 \text{ except when} \ \lambda = \mu^{(0)} \).
(ii) When $G_T = H$ the maximal torus of $G$, that is when $n = p + 1$ and $r_k = 1$ for any $k = 0, \ldots, p$, we have $\mu_i = \mu^{(i-1)}$ for any $i = 1, \ldots, n$, $\Delta_T = 1$ and $H_\ell(x^\beta) = h_\beta$ for any $\beta \in \mathbb{Z}^n$. Hence $S_{\mu, \ell} = h_\mu$. In this case $[V(\lambda) : V_\ell(\mu)] = K_{\lambda, \mu}$ for any $\lambda \in \mathcal{P}_n$. Thus (19) reduces to (13).

(iii) By (i) and (ii) the functions $S_{\mu, \ell}$ interpolates between the Weyl characters $s_\mu$ and the homogeneous functions $h_\mu$.

(iv) When $G = GL_n$, we have the duality

$$[V(\lambda) : V_\ell(\mu)] = c_{\ell(0)}^{\lambda(0)} \cdots c_{\ell(p)}^{\mu(p)}$$

where $c_{\ell(0)}^{\lambda(0)} \cdots c_{\ell(p)}^{\mu(p)}$ is the Littlewood-Richardson coefficient associated with the multiplicity of $V(\lambda)$ in the tensor product $V_\mu = (\mu(0)) \otimes \cdots \otimes (\mu(p))$. Thus we can write $S_{\mu, \ell} = \sum_{\lambda \in \mathcal{P}_n} c_{\ell(0)}^{\lambda(0)} \cdots c_{\ell(p)}^{\mu(p)} s_\lambda$. This means that $S_{\mu, \ell}$ is the character of $V_\mu$. Such a duality does not exist for $G = Sp_{2n}$, $SO_{2n+1}$ or $SO_{2n}$, (but see Section 6).

3 Plethysm on Weyl characters

3.1 The operators $\Psi_\ell$ and $\varphi_\ell$

Consider $\ell$ a positive integer. The power sum plethysm operator $\Psi_\ell$ is defined on $A$ be setting $\Psi_\ell(m_\beta) = m_{\beta, \ell}$ for any $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{Z}^n$ where $\ell = (\ell_1, \ldots, \ell_n)$. Since $\{m_\lambda \mid \lambda \in \mathcal{P}_n\}$ and $\{h_\lambda \mid \lambda \in \mathcal{P}_n\}$ are dual bases for the inner product $\langle \cdot, \cdot \rangle$, the adjoint operator $\varphi_\ell$ of $\Psi_\ell$ verifies

$$\varphi_\ell(h_\beta) = \begin{cases} h_{\beta/\ell} & \text{if } \beta \in (\mathbb{Z}^*)^n \\ 0 & \text{otherwise} \end{cases}$$

(20)

where $\beta/\ell = (\beta_1/\ell, \ldots, \beta_n/\ell)$ when $\beta \in (\mathbb{Z}^*)^n$.

By abuse of notation, we also denote by $\Psi_\ell$ and $\varphi_\ell$ the linear operators respectively defined on $L_n$ by setting

$$\Psi_\ell(x^\beta) = x^{\ell \beta} \text{ and } \varphi_\ell(x^\beta) = \begin{cases} x^{\beta/\ell} & \text{if } \beta \in (\mathbb{Z}^*)^n \\ 0 & \text{otherwise} \end{cases}$$

for any $\beta \in \mathbb{Z}^n$. (21)

**Remark:** Since $\Psi_\ell(x^\beta \times x^{\beta'}) = \Psi_\ell(x^\beta) \times \Psi_\ell(x^{\beta'})$ for any $\beta, \beta' \in \mathbb{Z}^n$ the map $\Psi_\ell$ is a morphism of algebra. This is not true for $\varphi_\ell$. Nevertheless, if $\{i_1, \ldots, i_r\}$ and $\{j_1, \ldots, j_s\}$ are disjoint subsets of $I_n$, $i = (i_1, \ldots, i_r) \in \mathbb{Z}^*$ and $\gamma = (\gamma_1, \ldots, \gamma_s) \in \mathbb{Z}^s$ we have

$$\varphi_\ell(x_{i_1}^{\gamma_1} \cdots x_{i_r}^{\gamma_r} \times x_{j_1}^{\gamma_1} \cdots x_{j_r}^{\gamma_r}) = \varphi_\ell(x_{i_1}^{\gamma_1} \cdots x_{i_r}^{\gamma_r}) \times \varphi_\ell(x_{j_1}^{\gamma_1} \cdots x_{j_r}^{\gamma_r}).$$

(22)

For any $\lambda \in \mathcal{P}_n$, $\Psi_\ell(s_\lambda)$ belongs to $A$, thus decomposes on the basis $\{s_\mu \mid \mu \in \mathcal{P}_n\}$. Let us write

$$\Psi_\ell(s_\lambda) = \sum_{\mu \in \mathcal{P}_n} n_{\lambda, \mu} s_\mu.$$

Since $\Psi_\ell$ and $\varphi_\ell$ are dual operators with respect to the scalar product $\langle \cdot, \cdot \rangle$, we can write $n_{\lambda, \mu} = \langle \Psi_\ell(s_\lambda), s_\mu \rangle = \langle s_\lambda, \varphi_\ell(s_\mu) \rangle$. So we have

$$\varphi_\ell(s_\mu) = \sum_{\lambda \in \mathcal{P}_n} n_{\lambda, \mu} s_\lambda.$$

By (16) and Proposition 2.4.1 we obtain the identity

$$s_\mu = \sum_{w \in W} \epsilon(w) h_{\mu + p - w, \rho} = H(\Delta \times x^\mu).$$

Thus from (20) and (21) we derive $\varphi_\ell(s_\mu) = H(\varphi_\ell(\Delta \times x^\mu))$. Set $P_\mu = \Delta \times x^\mu$. From the previous arguments the coefficients $n_{\lambda, \mu}$ are determined by the computation of $\varphi_\ell(P_\mu)$.
3.2 Computation of $\varphi_\ell(P_\mu)$

For any $i \in \{\overline{n}, \ldots, 1\}$ we set $x_i = \frac{1}{i}$. This permits to consider also variables indexed by negative integers. Given $X = (i_1, \ldots, i_r)$ an increasing sequence contained in $J_n$ and $\beta = (\beta_1, \ldots, \beta_r) \in \mathbb{Z}^r$, we set $x_{i}^\beta = x_{i_1}^{\beta_1} \cdots x_{i_r}^{\beta_r}$. We also denote by $S_X$ the group of permutations of the set $X$. Each $\sigma \in S_X$ determines a unique permutation $\sigma^*$ of the set $\{1, \ldots, r\}$ defined by

$$\sigma(i_p) = i_{\sigma^*(p)} \text{ for any } p = 1, \ldots, r. \quad (23)$$

In the sequel, we identify for short $\sigma$ and $\sigma^*$. Similarly, given $Z = (\overline{n}, \ldots, \overline{u_1}, u_1, \ldots, u_r)$ an increasing sequence such that $\{u_1, \ldots, u_r\} \subseteq I_n$, each signed permutation $w$ defined on $Z$ will be identified with the signed permutation $w^*$ defined on $J_r$ by $w(u_p) = u_{w^*(p)}$ for any $p = 1, \ldots, r$.

Set $\rho_n = (1, 2, \ldots, n)$. For any $w \in W$ we have $w \cdot \rho_n = (w(1), \ldots, w(n))$. This permits to write

$$\sum_{w \in W} \varepsilon(w) x^{-w \cdot \rho_n} = \sum_{w \in W} \varepsilon(w) x^{w(1)} \cdots x^{w(n)}. \quad (24)$$

3.2.1 For $G = GL_n$

Set $\kappa_n = (1, \ldots, 1) \in \mathbb{Z}^n$. Since $\sigma(\kappa_n) = \kappa_n$ for any $\sigma \in S_n$, one can replace $\rho$ by $\rho_n = (1, 2, \ldots, n)$ in (15). By using (24) we can write

$$P_\mu = x_1^{\mu_1+1} \cdots x_n^{\mu_n+1} \sum_{\sigma \in S_n} \varepsilon(\sigma) x_1^{-\sigma(1)} \cdots x_n^{-\sigma(n)}$$

where $\mu = (\mu_1, \ldots, \mu_n)$. For any $k \in \{0, \ldots, \ell - 1\}$ consider the ordering sequences

$$I^{(k)} = (i \in I_n \mid \mu_i + i \equiv k \text{ mod } \ell) \text{ and } J^{(k)} = (i \in I_n \mid i \equiv k \text{ mod } \ell). \quad (25)$$

Set $r_k = \text{card}(I^{(k)})$ and write $I^{(k)} = (i_1^{(k)}, \ldots, i_{r_k}^{(k)})$. Then

$$\mu^{(k)}(i) = \left(\frac{\mu_i + i + \ell - k}{\ell} \mid i \in I^{(k)}\right) \in \mathbb{Z}^{r_k}$$

We derive

$$P_\mu = x_{I^{(0)}}^{\mu^{(0)}} x_{I^{(1)}}^{\mu^{(1)}} \cdots x_{I^{(\ell-1)}}^{\mu^{(\ell-1)}} \sum_{\sigma \in S_n} \varepsilon(\sigma) x_1^{-\sigma(1)} \cdots x_n^{-\sigma(n)} \times \prod_{k=0}^{\ell-1} \prod_{a=1}^{r_k} x_{i_a^{(k)}}^{-\sigma(i_a^{(k)})-(\ell-k)}. \quad (26)$$

This gives

$$\varphi_\ell(P_\mu) = x_{I^{(0)}}^{\mu^{(0)}} x_{I^{(1)}}^{\mu^{(1)}} \cdots x_{I^{(\ell-1)}}^{\mu^{(\ell-1)}} \sum_{\sigma \in S_n} \varepsilon(\sigma) \varphi_\ell \left( \prod_{k=0}^{\ell-1} \prod_{a=1}^{r_k} x_{i_a^{(k)}}^{-\sigma(i_a^{(k)})-(\ell-k)} \right).$$

The contribution of a fixed permutation $\sigma \in S_n$ in the above sum is nonzero if and only if for any $k = 0, \ldots, \ell - 1$

$$i \in I^{(k)} \implies \sigma(i) \equiv k \text{ mod } \ell.$$ 

Thus we must have $\sigma(I^{(k)}) \subset J^{(k)}$ for any $k = 0, \ldots, \ell - 1$. Since $\sigma$ is a bijection, $I^{(k)} \cap I^{(k')} = J^{(k)} \cap J^{(k')} = \emptyset$ if $k \neq k'$ and $\cup_{0 \leq k \leq \ell - 1} I^{(k)} = \cup_{0 \leq k \leq \ell - 1} J^{(k)} = I_n$, the restriction of $\sigma$ on $I_k$ is a bijection from $I^{(k)}$ to $J^{(k)}$. In particular card($I^{(k)}$) = card($I^{(k)}$) = $r_k$. This means that we have the equivalences

$$\varphi_\ell \left( \prod_{k=0}^{\ell-1} \prod_{a=1}^{r_k} x_{i_a^{(k)}}^{-\sigma(i_a^{(k)})-(\ell-k)} \right) \neq 0 \iff \sigma(I^{(k)}) = J^{(k)} \text{ for any } k = 0, \ldots, \ell - 1. \quad (27)$$
Write $J^{(k)} = (k, k + \ell, \ldots, k + (r_k - 1)\ell)$. Denote by $\sigma_0 \in S_n$ the permutation verifying
\[ \sigma_0(i_a^{(k)}) = k + (a - 1)\ell \] (28)
for any $k = 0, \ldots, \ell - 1$ and any $a = 1, \ldots, r_k$. Let $S_{I^{(k)}}$ be the permutation group of the set $I^{(k)}$. The permutations $\sigma$ which verify the right-hand side of (27) can be written $\sigma = \sigma_0 \tau$ where $\tau = (\tau(0), \ldots, \tau(\ell-1))$ belongs to the direct product $S_{I^{(0)}} \times \cdots \times S_{I^{(\ell-1)}}$. We have then $\varepsilon(\sigma) = \varepsilon(\sigma_0)(-1)^{l(\tau(0))} \times \cdots \times (-1)^{l(\tau(\ell-1))}$.

For any $k \in \{0, \ldots, \ell - 1\}$ set
\[ P_k = \sum_{\tau(k) \in S_{I^{(k)}}} (-1)^{l(\tau)} \varphi_\ell \left( \prod_{a=1}^{r_k} x_{i_a^{(k)}} - \sigma_0 \tau(k)(i_a^{(k)}) - (\ell - k) \right), \]
From (22) and (26) we derive
\[ \varphi_\ell(P_\mu) = \varepsilon(\sigma_0) \prod_{k=0}^{\ell-1} x_{I^{(k)}}^\mu P_k. \]
Since $\sigma_0(i_a^{(k)}) = k + (a - 1)\ell$, we can write by (23) $\sigma_0 \tau(k)(i_a^{(k)}) = k + (\tau(k)(a) - 1)\ell$. Thus we obtain
\[ P_k = \sum_{\tau(k) \in S_{I^{(k)}}} (-1)^{l(\tau)} x_{i_1^{(k)}}^{-\tau(k)(1)} \cdots x_{i_{r_k}^{(k)}}^{-\tau(k)(r_k)} = x_{I^{(k)}}^{-\rho_k} \Delta_{I^{(k)}} \]
where $\rho_k = (1, 2, \ldots, r_k)$ and $\Delta_{I^{(k)}} = \prod_{i < j, i, j \in I^{(k)}} (1 - x_j/x_i)$. Finally, this gives
\[ \varphi_\ell(P_\mu) = \varepsilon(\sigma_0) \prod_{k=0}^{\ell-1} x_{I^{(k)}}^{\mu(k)-\rho_k} \Delta_{I^{(k)}} = \varepsilon(\sigma_0) \prod_{k=0}^{\ell-1} x_{I^{(k)}}^{\mu(k)} \Delta_{I^{(k)}} \]
where for any $k = 0, \ldots, \ell - 1$,
\[ \mu(k) = \left( \frac{\mu_i + i + \ell - k}{\ell} \right)_{i \in I^{(k)}} - (1, 2, \ldots, r_k) \in \mathbb{Z}^{r_k}. \] (29)

**Theorem 3.2.1** Consider a partition $\mu$ of length $n$ and $\ell$ a positive integer. For any $k = 0, \ldots, \ell - 1$ define the sets $I^{(k)}$ and $J^{(k)}$ as in (27).

- If there exists $k \in \{0, \ldots, \ell - 1\}$ such that $\text{card}(I^{(k)}) \neq \text{card}(J^{(k)})$ then $\varphi_\ell(s_\mu) = 0$.
- Otherwise, for any $k = 0, \ldots, \ell - 1$, set $r_k = \text{card}(I^{(k)}) = \text{card}(J^{(k)})$ and define $\sigma_0$ as in (28). Then each $r_k$-tuple defined by (29) is a partition and we have
  \[ \varphi_\ell(s_\mu) = \varepsilon(\sigma_0) S_\mu(I), \] where $I = \{I^{(0)}, \ldots, I^{(\ell-1)}\}$, $s_\mu = (\mu^{(0)}, \ldots, \mu^{(\ell-1)})$ and $\char(V_\mu)$ is the character of the $GL_n$-module $V_\mu = V(\mu^{(0)}) \otimes \cdots \otimes V(\mu^{(\ell-1)})$.

**Proof.** One verifies easily from (29) that each $\mu(k)$ is a partition. By the previous computation, we obtain
\[ \varphi_\ell(\Delta \times x^\mu) = \varepsilon(\sigma_0) \Delta_\ell \times x_\ell^{(\mu)} \]
(with the notation of [2.5]). By definition of \( \varphi_\ell \) we have also

\[
\varphi_\ell(s_\mu) = \varepsilon(\sigma_0)H \circ \varphi_\ell(\Delta \times x^\mu) = \varepsilon(\sigma_0)H(\Delta_\mathcal{I} \times x^\mu) = \varepsilon(\sigma_0)S^i_{(\mu)}K
\]

where the last equality follows from \([18]\). ■

**Remark:** The subgroup \( G_\mathcal{I} \) appearing in Theorem [3.2.1] is characterized by \( \mathcal{I} = \{I^{(0)}, ..., I^{(\ell-1)}\} \). This means that for type \( A \), we have \( X^{(k)} = I^{(k)} \) for any \( k > 0 \) with the notation of [2.2] that is the sets \( X^{(k)} \) contain only positive indices.

**Example 3.2.2**

Consider \( \mu = (1, 2, 3, 4, 4, 6, 6) \) and take \( \ell = 3 \). We have \( \mu + \rho_8 = (2, 4, 6, 8, 9, 10, 13, 14) \). Thus \( I^{(0)} = \{3, 5\}, I^{(1)} = \{2, 6, 7\}, I^{(2)} = \{1, 4, 8\} \) and \( J^{(0)} = \{3, 6\}, J^{(1)} = \{1, 4, 7\}, J^{(2)} = \{2, 5, 8\} \). Then \( \mu^{(0)} = (1, 1), \mu^{(1)} = (1, 2, 2) \) and \( \mu^{(2)} = (0, 1, 2) \). We have \( G_\mathcal{I} \simeq GL_2 \times GL_3 \times GL_3 \).

**3.2.2 For \( G = Sp_{2n} \)**

We have \( \rho = \rho_n = (1, 2, ..., n) \). By using \([24]\) we deduce the identity:

\[
P_\mu = x_1^{(\mu_1+1)} \cdots x_n^{(\mu_n+n)} \sum_{w \in W} \varepsilon(w)x_1^{-w(1)} \cdots x_n^{-w(n)}
\]

(30)

where \( W \) is the group of signed permutations defined on \( J_n = \{\pi, ..., \bar{\pi}, 1, ..., n\} \), that is the subgroup of permutations \( w \in S_{J_n} \) verifying \( w(\pi) = w(x) \) for any \( x \in J_n \). Given \( k \in \{0, ..., \ell - 1\} \) consider the ordering sequences

\[
I^{(k)} = (i \in I_n | \mu_i + i \equiv k \text{mod}\ell) \quad \text{and} \quad J^{(k)} = (x \in J_n | x \equiv k \text{mod}\ell).
\]

(31)

Set \( p = \frac{\ell}{2} \) if \( \ell \) is even and \( p = \frac{\ell-1}{2} \) otherwise.

**The odd case \( \ell = 2p - 1 \)**

Set \( r_0 = \text{card}(I^{(0)}) \) and for any \( k = 1, ..., p - 1, s_k = \text{card}(I^{(k)}), r_k = \text{card}(I^{(\ell-k)}) \) and \( r_{\ell-k} = \text{card}(I^{(\ell-k)}) \). Write \( X^{(k)}, k = 1, ..., p \) for the increasing reordering of \( I^{(k)} \cup I^{(\ell-k)} \). Set \( I^{(0)} = (i^{(0)}_1, ..., i^{(0)}_{r_0}) \) and for \( k > 0 \)

\[
X^{(k)} = (i^{(k)}_1, ..., i^{(k)}_{r_k}).
\]

(32)

This means that \( I^{(k)} = (i^{(k)}_{s_k+1}, ..., i^{(k)}_{r_k}) \) and \( I^{(\ell-k)} = (i^{(\ell-k)}_{s_{\ell-k}+1}, ..., i^{(\ell-k)}_{r_{\ell-k}}) \). To simplify the computation, we are going to use the indices and the variables \( x_i, i \in X^{(k)} \) rather than the variables \( x_i, i \in I^{(k)} \cup I^{(\ell-k)} \) when \( k \in \{1, ..., p - 1\} \).

Consider

\[
\mu^{(0)} = \left( \frac{\mu_i + i}{\ell} \mid i \in I^{(0)} \right) \in \mathbb{Z}^{r_0} 
\text{and for } k > 0, \mu^{(k)} = \left( \frac{\text{sign}(i) \mu_i + |i| + \text{sign}(i)k}{\ell} \right) \in \mathbb{Z}^{r_k}
\]

where for any \( i \in J_n \), \( \text{sign}(i) = 1 \) if \( i > 0 \) and \(-1\) otherwise. For any \( i \in I \), we have \( x_i^{-w(i)} = x_{\tau(i)}^{-w(\tau(i))} \). Thus

\[
\prod_{i \in X^{(k)}} x_i^{-w(i)} = \prod_{i \in I^{(k)}} x_i^{-w(i)} \prod_{i \in I^{(\ell-k)}} x_i^{-w(i)} \quad \text{and} \quad x_1^{(\mu_1+1)} \cdots x_n^{(\mu_n+n)} = x_1^{\mu^{(0)}_0} \prod_{k=1}^{p-1} x_i^{\mu^{(k)}_i} \prod_{i \in X^{(k)}} x_i^{-k}
\]

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by definition of the $\mu^{(k)}$'s. Then (30) can be rewritten
\[
P_\mu = a_\ell^\mu \prod_{k=1}^{p-1} a_{\ell,k}^{(k)} X^{(k)} \times \sum_{w \in W} \varepsilon(w) \prod_{i \in I(0)} x_{i}^{-w(i)} \prod_{i \in I(0)} \prod_{k=1}^{p-1} x_{i}^{-w(i)} \prod_{k=1}^{p-1} x_{i}^{-w(i) - k}.
\]
This gives
\[
\varphi_\ell(P_\mu) = x_{\ell}^{\mu} \prod_{k=1}^{p-1} x_{\ell,k}^{(k)} X^{(k)} \times \sum_{w \in W} \varepsilon(w) \varphi_\ell \left( \prod_{i \in I(0)} x_{i}^{-w(i)} \prod_{i \in I(0)} \prod_{k=1}^{p-1} x_{i}^{-w(i) - k} \right)
\]
The contribution of a fixed $w \in W$ in the above sum is nonzero if and only if
\[
\begin{align*}
i \in I(0) & \implies w(i) \equiv 0 \mod \ell \\
i \in X^{(k)} & \implies w(i) \equiv -k \mod \ell \text{ for any } k = 1, \ldots, p - 1
\end{align*}
\]
Thus we must have $w(\overline{T(0) \cup I(0)}) = J(0)$, and for any $k = 1, \ldots, p - 1$, $w(X^{(k)}) \subset J^{(\ell-k)}$. Recall that $\overline{T(0)} = J(0)$ and $\overline{T^{(\ell-k)}} = J^{(k)}$ for $k = 1, \ldots, p - 1$. Moreover
\[
I(0) \cup \overline{T(0)} \prod_{k=1}^{p-1} X^{(k)} \cup X^{(k)} = J_n \text{ and } I(0) \cup \bigcup_{k=1}^{p-1} J^{(k)} \cup J^{(\ell-k)} = J_n.
\]
Since the sets appearing in the left hand side of these two equalities are pairwise disjoint, we must have $w(\overline{T(0) \cup I(0)}) = J(0)$, and for $k = 1, \ldots, p - 1$, $w(X^{(k)}) = J^{(\ell-k)}$. In particular $\text{card}(J(0)) = 2\text{card}(I(0)) = 2r_0$ and $\text{card}(J^{(\ell-k)}) = \text{card}(X^{(k)}) = r_k$ for any $k = 1, \ldots, p - 1$. We have the equivalences
\[
\varphi_\ell \left( \prod_{i \in I(0)} x_{i}^{-w(i)} \prod_{k=1}^{p-1} \prod_{i \in X^{(k)}} x_{i}^{-w(i) + k} \right) \neq 0 \iff \begin{cases} (i) : w(\overline{T(0) \cup I(0)}) = J(0) \\ (ii) : w(X^{(k)}) = J^{(\ell-k)} \text{ for any } k = 1, \ldots, p - 1 \end{cases}
\]
Note that condition (ii) can be rewritten: $w(\overline{X^{(k)}}) = J^{(k)}$ for any $k = 1, \ldots, p - 1$. We can set $J(0) = (-r_0\ell, \ldots, r_0\ell)$ and for $k = 1, \ldots, p - 1$,
\[
J^{(\ell-k)} = (-k - \alpha_k \ell, \ldots, -k + \beta_k \ell), \quad J^{(k)} = (k - \beta_k \ell, \ldots, k + \alpha_k \ell)
\]
with $\alpha_k + \beta_k + 1 = r_k$. Consider $w_0 \in W$ defined by
\[
\begin{align*}
w_0(i_a^{(0)}) &= a \ell \text{ for } a \in \{1, \ldots, r_0\} \\
w_0(i_a^{(k)}) &= -k - \alpha_k \ell + (a - 1)\ell \text{ for } a \in \{1, \ldots, p - 1\}
\end{align*}
\]
Denote by $W$ the set of signed permutations $w$ which verify (i) and (ii) in (31). We have $w_0 \in W$. Each $w \in W$ can be written $w = w_0 v$ where $v = (v^{(0)}, \tau^{(1)}, \ldots, \tau^{(p-1)})$ belongs to the direct product $W_{I(0)} \times S_{X^{(1)}} \times \cdots \times S_{X^{(p-1)}}$. Here $W_{I(0)}$ is the group of signed permutations defined on $\overline{T(0) \cup I(0)}$ and for $k = 1, \ldots, p - 1$, $S_{X^{(k)}}$ is the group of signed permutations $\tau^{(k)}$ defined on $\overline{X^{(k)}} \cup X^{(k)}$ and verifying $\tau^{(k)}(X^{(k)}) = X^{(k)}$. Indeed if $\tau^{(k)}(x) \in \overline{X^{(k)}}$ and $x \in X^{(k)}$, we would have $w(x) \in J^{(k)}$ and $x \in X^{(k)}$ which contradicts (ii). This means that $S_{X^{(k)}}$ is in fact isomorphic to the symmetric group $S_{r_k}$. Since the sets $I(0)$ and $X^{(k)}$, $k = 1, \ldots, p - 1$ are increasing subsequences of $J_n$, we have by Lemma 2.1.1 $\varepsilon(w) = \varepsilon(w_0)(-1)^{l(v^{(0)})}(-1)^{l(\tau^{(1)})} \times \cdots \times (-1)^{l(\tau^{(p-1)})}$.  

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Set

\[ P_0 = \sum_{\nu^{(0)} \in W_{I^{(0)}}} (-1)^{l(\nu^{(0)})} \varphi_{\ell} \left( \prod_{i \in I^{(0)}} x_i^{-w_0 \nu^{(0)}(i)} \right) \] and

\[ P_k = \sum_{\tau^{(k)} \in S_{X^{(k)}}} (-1)^{l(\tau^{(k)})} \varphi_{\ell} \left( \prod_{i \in X^{(k)}} x_i^{-w_0 \tau^{(k)}(i)-k} \right), k \in \{1, ..., p-1\}. \]

We obtain

\[ \varphi_{\ell}(P_\mu) = \varepsilon(w_0) x_{I^{(0)}}^{\mu^{(0)}} P_0 \prod_{k=1}^{p-1} x_{X^{(k)}}^{\mu^{(k)}} P_k. \]

From (23) and (35), we have \( w_0 \nu^{(0)}(i_a^{(0)}) = \nu^{(0)}(a) \ell \) for any \( a = 1, ..., r_0 \) and

\[ w_0 \tau^{(k)}(i_a^{(k)}) = -k - \alpha_k \ell + (\tau^{(k)}(a) - 1) \ell \text{ for any } a = 1, ..., r_k. \]

This yields

\[ P_0 = \sum_{\nu^{(0)} \in W_{I^{(0)}}} (-1)^{l(\nu^{(0)})} \prod_{a=1}^{r_0} x_{i_a^{(0)}}^{-\nu^{(0)}(a)} = x_{I^{(0)}}^{-B_0} \Delta I^{(0)} \text{ and} \]

\[ P_k = \sum_{\tau^{(k)} \in S_{X^{(k)}}} (-1)^{l(\tau^{(k)})} \prod_{a=1}^{r_k} x_{i_a^{(k)}}^{-\tau^{(k)}(a)+1} \Delta X^{(k)} \]

where for any \( k = 1, ..., p-1, \eta_{r_k} = -\rho_{r_k} + (\alpha_k + 1, ..., \alpha_k + 1) \in Z^{r_k}, \]

\[ \Delta I^{(0)} = \prod_{i<j \in I^{(0)}} (1 - \frac{x_j}{x_i}) \prod_{r \leq s \in I^{(0)}} (1 - x_r x_s) \]

and \( \Delta X^{(k)} = \prod_{i<j \in X^{(k)}} (1 - \frac{x_j}{x_i}) \) for any \( k = 1, ..., p-1. \)

Finally, this gives

\[ \varphi_{\ell}(P_\mu) = \varepsilon(w_0) x_{I^{(0)}}^{\mu^{(0)}-\rho_0} \Delta I^{(0)} \prod_{k=1}^{p-1} x_{X^{(k)}}^{\mu^{(k)}-\eta_{r_k}} \Delta X^{(k)} = \varepsilon(w_0) x_{I^{(0)}}^{\mu^{(0)}} \Delta I^{(0)} \prod_{k=1}^{p-1} x_{X^{(k)}}^{\mu^{(k)}} \Delta X^{(k)} \]

where

\[ \mu^{(0)} = \left( \frac{\mu_i + i}{\ell} \right) | i \in I^{(0)} \right) - (1, ..., r_0) \in Z^{r_0} \]

and for any \( k = 1, ..., p-1 \)

\[ \mu^{(k)} = \left( \frac{\text{sign}(i) \mu_i + |i| + \text{sign}(i) k}{\ell} \right) | i \in X^{(k)} \right) - (1, ..., r_k) + (\alpha_k + 1, ..., \alpha_k + 1) \in Z^{r_k}. \]

Recall that the weights corresponding to the subgroup of Levi type \( G_T \) are written following the convention \([39]\).

**Theorem 3.2.3** Consider a partition \( \mu \) of length \( n \) and \( \ell = 2p - 1 \) a positive integer. Let \( I^{(0)} \) and \( J^{(0)} \) be as in \([37]\). For any \( k = 1, ..., p-1 \) define the sets \( X^{(k)} \) and \( J^{(k)} \) by \([37]\) and \([32]\).
• If \( \text{card}(I^{(0)}) \neq \frac{1}{2} \text{card}(J^{(0)}) \) or if there exists \( k \in \{1, \ldots, p-1\} \) such that \( \text{card}(X^{(k)}) \neq \text{card}(J^{(k)}) \) then \( \varphi_{(s_\mu)} = 0 \).

• Otherwise, set \( r_0 = \text{card}(I^{(0)}) \) and for any \( k = 1, \ldots, p-1 \), \( r_k = \text{card}(X^{(k)}) \). Let \( w_0 \in W \) be as in (32). Consider \( (\mu') = (\mu^{(0)}, \mu^{(1)}, \ldots, \mu^{(p-1)}) \) where the \( \mu^{(k)} \)'s are defined by (32) and (37). Then \( (\mu') \) is a dominant weight of \( P^I_\mu \) with \( I = \{ I^{(0)}, X^{(1)}, \ldots, X^{(p-1)} \} \) and we have

\[
\varphi_{(s_\mu)} = \varepsilon(w_0)S_{(\mu'), I}.
\]

**Proof.** The proof is essentially the same as in Theorem 3.2.1. We obtain

\[
\varphi_{(s_\mu)} = \varepsilon(w_0)H \circ \varphi_{(\Delta \times x^6)} = \varepsilon(w_0)H(\Delta \times x^6) = \varepsilon(w_0)S_{(\mu'), I}.
\]

Example 3.2.4
Consider \( \mu = (1, 2, 3, 4, 4, 6, 6) \) and take \( \ell = 3 \). We have \( \mu + \rho_8 = (2, 4, 6, 8, 9, 10, 13, 14) \). Thus \( I^{(0)} = \{3, 5\}, X^{(1)} = \{7, 6, 2, 1, 4, 8\} \) and \( J^{(0)} = \{6, 3, 3, 6\}, J^{(1)} = \{8, 5, 2, 1, 4, 7\}, J^{(2)} = \{7, 4, 1, 2, 5, 8\} \). In particular \( \alpha_1 = 2 \). Then \( \mu^{(0)} = (1, 1) \) and \( \mu^{(1)} = \frac{1}{2}(1, 1, 2, 2, 2, 2, 3, 3) \) with the convention (33). We have \( G_I \simeq \text{Sp}_4 \times \text{GL}_6 \).

The even case \( \ell = 2p \)
With the same notation as in the odd case, (30) can be rewritten

\[
P_{\mu} = x_{I^{(0)}}^{\mu^{(0)}} x_{I^{(p)}}^{\mu^{(p)}} \prod_{k=1}^{p-1} x_{X^{(k)}}^{\mu^{(k)}} \sum_{w \in W} \varepsilon(w) \prod_{i \in I^{(0)}} x_i^{-w(i)} \prod_{i \in I^{(p)}} x_i^{-w(i)-p} \prod_{k=1}^{p-1} \prod_{i \in X^{(k)}} x_i^{-w(i)-k}.
\]

where

\[
\mu^{(p)} = \left( \frac{\mu_i + i + p}{\ell} \right) \quad i \in I^{(p)}.
\]

This gives

\[
\varphi_{(s_\mu)}(P_{\mu}) = x_{I^{(0)}}^{\mu^{(0)}} x_{I^{(p)}}^{\mu^{(p)}} \prod_{k=1}^{p-1} x_{X^{(k)}}^{\mu^{(k)}} \sum_{w \in W} \varepsilon(w) \varphi_{(s_\mu)} \left( \prod_{i \in I^{(0)}} x_i^{-w(i)} \prod_{i \in I^{(p)}} x_i^{-w(i)-p} \prod_{k=1}^{p-1} \prod_{i \in X^{(k)}} x_i^{-w(i)-k} \right)
\]

The contribution of a fixed \( w \in W \) in the above sum is nonzero if conditions (34) are verified and

\[
i \in I^{(p)} \implies w(i) \in J^{(p)}.
\]

Since \( p \equiv -p \mod \ell \) we have \( J^{(p)} = J^{(p)} = \{-p - \alpha_p \ell, \ldots, -p, p, \ldots, p + \alpha_p \ell\} \). This implies that \( w(I^{(p)} \cup J^{(p)}) = J^{(p)} \) and thus \( \text{card}(I^{(p)}) = \frac{1}{2} \text{card}(J^{(p)}) \). We then define \( w_0 \) by requiring (35) and \( w_0(i^{(p)}_a) = \left( \frac{w(a) + a + p}{\ell} \right) \)
Proposition 3.2.5

Suppose $G = SP_{2n}$ and $\ell = 2p$.

- If $\text{card}(I(0)) \neq \frac{1}{2} \text{card}(J(0))$, $\text{card}(I(p)) \neq \frac{1}{2} \text{card}(J(p))$ or there exists $k \in \{1, ..., p-1\}$ such that $\text{card}(X(k)) \neq \text{card}(J(k))$ then $\varphi_{\ell}(s_{\mu}) = 0$.

- Otherwise, the coefficients appearing in the decomposition of $\varphi_{\ell}(s_{\mu})$ on the basis of Weyl characters cannot be interpreted as branching coefficients and have signs alternatively positive and negative.
3.2.3 For $G = SO_{2n}$

As for $G = Sp_{2n}$, the coefficients appearing in the decomposition of $\varphi_\ell(s_\mu)$ with $\ell = 2p$ on the basis of Weyl characters cannot be interpreted as branching coefficients. Note that there is an additional difficulty in this case. Indeed, $\varphi_\ell(P_\mu)$ cannot be factorized as a product of polynomials $(1 - x^\beta)$ where $\beta \in \mathbb{Z}^n$. For example, we have for $SO_4$

$$\varphi_2(P_{(0,0)}) = \varphi_2 \left( (1 - \frac{x_2}{x_1})(1 - x_1 x_2) \right) = 1 + x_2.$$

This is due to the incompatibility between the signatures defined on the Weyl groups of types $B$ and $D$ when they are realized as subgroups of the permutation group $S_n$.

So we will suppose $\ell = 2p - 1$ in this paragraph. Recall that the elements of $W$ are the signed permutations $w$ defined on $J_n = \{\overline{1}, \ldots, \overline{1}, 1, \ldots, n\}$ such that $\text{card}(\{i \in I_n \mid w(i) < 0\})$ is even. Set $K_n = \{n - 1, \ldots, \overline{1}, 0, 1, \ldots, n - 1\}$. Each $w \in W$ can be written $w = \zeta \sigma$ according to the decomposition of $W$ as the semidirect product $(\mathbb{Z}/2\mathbb{Z})^{n-1} \rtimes S_n$. For any $x \in J_n$, we have then $\xi(x) = 1$ if $w(x) > 0$ and $\xi(x) = -1$ otherwise. Given $w \in W$ we define $\hat{w} : J_n \to K_n$ such that $\hat{w}(x) = w(x) - \xi(x)$ for any $x \in J_n$. Then $\hat{w}(\overline{w}) = \hat{w}(w)$.

For type $D_n$, we have $\rho = \rho'_n = (0, 1, \ldots, n - 1) = \rho_n - (1, \ldots, 1)$. Hence

$$w \cdot \rho' = w \cdot \rho_n - (\xi(1), \ldots, \xi(n)) = (\hat{w}(1), \ldots, \hat{w}(n)) = \hat{w} \cdot \rho_n.$$

Then we obtain

$$P_\mu = x_1^{(\mu_1 + 0)} \cdots x_n^{(\mu_n + n - 1)} \sum_{w \in W} \varepsilon(w) x_1^{-\hat{w}(1)} \cdots x_n^{-\hat{w}(n)}.$$  

For any $k = 0, \ldots, \ell - 1$ set

$$I^{(k)} = (i \in I_n \mid \mu_i + i - 1 \equiv k \text{mod } \ell) \text{ and } J^{(k)} = (x \in K_n \mid x \equiv k \text{mod } \ell).$$  

We then proceed essentially as in 3.2.2 by using $\hat{w}$ instead of $w$ and $\rho'_n = (0, 1, \ldots, n - 1)$ instead of $\rho_n = (1, \ldots, n)$. We only sketch below the main steps of the computation.

Set $r_0 = \text{card}(I^{(0)})$ and for any $k = 1, \ldots, p - 1$, $s_k = \text{card}(I^{(k)})$, $r_k = \text{card}(J^{(k)})$. For $k = 1, \ldots, p - 1$, $X^{(k)}$ is defined as the increasing reordering of $\overline{I^{(k)}} \cup J^{(k-1)}$. Consider

$$\mu^{(0)} = \left( \frac{\mu_i - 1}{\ell} \mid i \in I^{(0)} \right) \in \mathbb{Z}^{r_0} \text{ and for } k > 0,$$

$$\mu^{(k)} = \left( \frac{\text{sign}(\mu_i) \mu_i + |i| - 1 + \text{sign}(\mu_i)k}{\ell} \mid i \in X^{(k)} \right) \in \mathbb{Z}^{r_k}.$$

We obtain

$$\varphi_\ell(P_\mu) = x_{I^{(0)}}^{\mu^{(0)}} \prod_{k=1}^{p-1} x_{X^{(k)}}^{\mu^{(k)}} \times \sum_{w \in W} \varepsilon(w) \varphi_\ell \left( \prod_{i \in I^{(0)}} x_i^{-\hat{w}(i)} \prod_{k=1}^{p-1} \prod_{i \in X^{(k)}} x_i^{-\hat{w}(i)-k} \right).$$

We also have the equivalences

$$\varphi_\ell \left( \prod_{i \in I^{(0)}} x_i^{-w(i)} \prod_{k=1}^{p-1} \prod_{i \in X^{(k)}} x_i^{-\hat{w}(i)+k} \right) \neq 0 \iff \begin{cases} (i) : \hat{w}(I^{(0)} \cup \overline{T^{(0)})} = J^{(0)} \\ (ii) : \hat{w}(X^{(k)}) = J^{(k-1)} \text{ for any } k = 1, \ldots, p - 1 \end{cases}.$$  

(39)
We can write $J^{(0)} = -(r_0 - 1)\ell, \ldots, 0, \ldots, (r_0 - 1)\ell$ and for $k = 1, \ldots, p$,

$$J^{(k-1)} = (-k - \alpha_k\ell, \ldots, -k + \beta_k\ell), \ \ J^{(k)} = (k - \beta_k\ell, \ldots, k + \alpha_k\ell)$$

with $\alpha_k + \beta_k + 1 = r_k$. Consider $w_0 \in W$ defined by

$$\tilde{w}_0(i_a^{(0)}) = (a - 1)\ell \text{ for } a \in \{1, \ldots, r_0\}$$

$$\tilde{w}_0(i_a^{(k)}) = -k - \alpha_k\ell + (a - 1)\ell \text{ for any } k = 1, \ldots, p - 1.$$

Denote by $W$ the set of signed permutations $w \in W$ which verify (i) and (ii) in (39). We have $w_0 \in W$. Each $w \in W$ can be written $w = w_0v$ where $v = (v^{(0)}, \tau^{(1)}, \ldots, \tau^{(p-1)})$ belongs to the direct product $W_I^{(0)} \times S_X^{(1)} \times \cdots \times S_X^{(p-1)}$ with $W_I^{(0)}$ the Weyl group of type $D_{r_0}$ defined on $T^{(0)} \cup I^{(0)}$. We have by Lemma 2.1.1 $\varepsilon(w) = \varepsilon(w_0)(-1)^l(v^{(0)})(-1)^l(\tau^{(1)}) \cdots (-1)^l(\tau^{(p-1)})$.

Set

$$P_0 = \sum_{v^{(0)} \in W_I^{(0)}} (-1)^l(v^{(0)}) \varphi_\ell \left( \prod_{i \in X^{(0)}} x_i^{-\tilde{w}_0v^{(0)}(i)} \right)$$

$$P_k = \sum_{\tau^{(k)} \in S_X^{(k)}} (-1)^l(\tau^{(k)}) \varphi_\ell \left( \prod_{i \in X^{(k)}} x_i^{-\tilde{w}_0v^{(k)}(i) - k} \right) \text{ for any } k \in \{1, \ldots, p - 1\}.$$

We obtain

$$\varphi_\ell(P_\mu) = \varepsilon(w_0)x^{\mu}_{I^{(0)}} P_0 \prod_{k=1}^{p-1} x^{\mu}_{X^{(k)}} P_k.$$

Given $v^{(0)} \in W_I^{(0)}$, we define $\tilde{v}^{(0)} = v^{(0)} - \xi_a$ where $\xi_a(i_a) = 1$ if $v(i_a) > 0$ and $-1$ otherwise. By (40), we have for any $a = 1, \ldots, r_0$, $\tilde{w}_0v^{(0)}(i_a^{(0)}) = \tilde{v}^{(0)}(a)\ell$. Moreover since $\tau^{(k)} \in S_X^{(k)}$

$$\tilde{w}_0\tau^{(k)}(i_a^{(k)}) = -k - \alpha_k\ell + (\tau^{(k)}(a) - 1)\ell \text{ for any } a = 1, \ldots, r_k.$$

This yields

$$P_0 = \sum_{v^{(0)} \in W_I^{(0)}} (-1)^l(v^{(0)}) \prod_{a=1}^{r_0} x_{i_a^{(0)}}^{-\tilde{v}^{(0)}(a)} = x^{\rho_{r_0}}_{I^{(0)}} \Delta_{I^{(0)}}$$

and

$$P_k = \sum_{\tau^{(k)} \in S_X^{(k)}} (-1)^l(\tau^{(k)}) \prod_{a=1}^{r_k} x_{i_a^{(k)}}^{-\tau^{(k)}(a) + (\alpha_k + 1)} = x^{\eta_k}_{X^{(k)}} \Delta_{X^{(k)}}$$

where for any $k = 1, \ldots, p - 1$, $\eta_{r_k} = -\rho_{r_k} + (\alpha_k, \ldots, \alpha_k) = -\rho_{r_k} + (\alpha_k + 1, \ldots, \alpha_k + 1) \in \mathbb{Z}^{r_k}$,

$$\Delta_{I^{(0)}} = \prod_{i<j, i,j \in I^{(0)}} (1 - \frac{x_j}{x_i}) \prod_{r<s, r,s \in I^{(0)}} (1 - x_r x_s) \text{ and}$$

$$\Delta_{X^{(k)}} = \prod_{i<j, i,j \in X^{(k)}} (1 - \frac{x_j}{x_i}) \text{ for any } k = 1, \ldots, p - 1.$$

This gives

$$\varphi_\ell(P_\mu) = \varepsilon(w_0)x^{\mu}_{I^{(0)}} - \rho_{r_0} \Delta_{I^{(0)}} \prod_{k=1}^{p-1} x^{\mu}_{X^{(k)}} - \eta_k \Delta_{X^{(k)}} = \varepsilon(w_0)x^{\mu}_{I^{(0)}} \Delta_{I^{(0)}} \prod_{k=1}^{p-1} x^{\mu}_{X^{(k)}} \Delta_{X^{(k)}} \text{ (40).}$$
where
\[
\mu^{(0)} = \left( \frac{\mu_i + i - 1}{\ell} \right)_{i \in I^{(0)}} - (0, ..., r_0 - 1) \in \mathbb{Z}^{r_0},
\] (41)

and for any \( k = 1, ..., p - 1, \)
\[
\mu^{(k)} = \{ i \in X^{(k)} \bigg| \frac{\text{sign}(i)\mu_i + |i| - 1 + \text{sign}(i)k}{\ell} \bigg| i \in X^{(k)} \} - (0, ..., r_k - 1) + (\alpha_k, ..., \alpha_k) \in \mathbb{Z}^{r_k}. \] (42)

Note that these formulas are essentially the same as for \( G = Sp_{2n}, \) except that we use \( \rho'_n = (0, ..., n-1) \) instead of \( \rho_n = (1, ..., n) \) for the half sum of positive roots. This gives the following theorem whose proof is identical to that of Theorem 3.2.3.

**Theorem 3.2.6** Consider a partition \( \mu \) of length \( n \) and \( \ell = 2p - 1 \) a positive integer. Let \( I^{(0)} \) and \( J^{(0)} \) be as in (35). For any \( k = 0, ..., p - 1 \) define the sets \( X^{(k)} \) and \( J^{(k)} \) by (41) and (35).

- If \( \text{card}(I^{(0)}) \neq \frac{1}{2}(\text{card}(J^{(0)}) + 1) \) or if there exists \( k \in \{ 1, ..., p - 1 \} \) such that \( \text{card}(X^{(k)}) \neq \text{card}(J^{(k)}) \) then \( \varphi_\ell(s_\mu) = 0. \)
- Otherwise, set \( r_0 = \text{card}(I^{(0)}) \) and for any \( k = 0, ..., p - 1, \) \( r_k = \text{card}(X^{(k)}). \) Let \( \omega_0 \in W \) be as in (40). Consider \( \left( \mu^{(k)} \right) = (\mu^{(0)}, ..., \mu^{(p-1)}) \) where the \( \mu^{(k)} \)'s are defined by (41) and (42). Then \( \left( \nu^{(k)} \right) \) is a dominant weight of \( P^+_I \) with \( I = \{ I^{(0)}, X^{(1)}, ..., X^{(p-1)} \} \) and we have
\[
\varphi_\ell(s_\mu) = \varepsilon(\omega_0)S^{(\nu^{(0)})}_I.
\]

**Example 3.2.7**
Consider \( \mu = (1, 2, 3, 4, 4, 6, 6) \) and take \( \ell = 3. \) We have \( \mu + \rho'_3 = (1, 3, 5, 7, 8, 9, 12, 13). \) Thus \( I^{(0)} = \{ 2, 6, 7 \}, X^{(1)} = \{ 5, 7, 3, 0, 3, 6 \}, J^{(0)} = \{ 5, 7, 2, 1, 4, 7 \} \) and \( J^{(2)} = \{ 7, 4, 7, 2, 5 \}. \) In particular \( \alpha_2 = 1. \) Then \( \mu^{(2)} = (1, 2, 2) \) and
\[
\mu^{(1)} = \left( -\frac{13 - 1}{3} - 1 + 3, -\frac{7 - 1}{3}, -2 + 3, -\frac{1 - 1}{3}, -3 + 3, \frac{5 + 1}{3}, -4 + 3, \frac{8 + 1}{3}, -5 + 3 \right)
\]
\[
= (-2, -1, 0, 1, 1)
\]
We have \( G_I \simeq SO_6 \times GL_5. \)

**3.2.4** For \( G = SO_{2n+1} \)
Set \( L_n = \{ n-1, ..., 1, 0, 1, ..., n \}. \) Each \( w \in W \) can be written \( w = \zeta \sigma \) according to the decomposition of \( W \) as the semidirect product \( (\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n. \) Given \( w \in W \) we define \( \tilde{w} : J_n \rightarrow L_n \) such that \( \tilde{w}(x) = w(x) + \frac{1}{2}(1 - \xi(x)) \) for any \( x \in J_n. \) For any \( y \in L_n, \) set \( y^* = y + 1. \) We have then \( \tilde{w}(\sigma) = (w(x))^* \) \( = w(x) + 1. \)

Observe that \( \rho = \rho''_n = \left( \frac{1}{2}, \frac{3}{2}, ..., n - \frac{1}{2} \right) = \rho_n - \left( \frac{1}{2}, ..., \frac{1}{2} \right). \) Thus
\[
w \cdot \rho''_n = w \cdot \rho_n - \frac{1}{2}(\xi(1), ..., \xi(n)) = (\tilde{w}(1), ..., \tilde{w}(n)) - \frac{1}{2}(1, ..., 1).
\]
This permits to write
\[
P_{\mu} = x_1^{(\mu_1 + 1)} \cdots x_n^{(\mu_n + n)} \sum_{w \in W} \varepsilon(w)x_1^{-\tilde{w}(1)} \cdots x_n^{-\tilde{w}(n)}.
\] (43)

For any \( k = 1, ..., \ell \) (observe that \( k \) does not run over \( \{ 0, ..., \ell - 1 \} \) as for \( G = Sp_{2n} \) or \( SO_{2n} \)) set
\[
I^{(k)} = \{ i \in I_n \mid \mu_i + i \equiv k \bmod \ell \} \text{ and } J^{(k)} = \{ x \in L_n \mid x \equiv k \bmod \ell \}.
\] (44)
Note that \( (J^{(k)})^* = J^{(\ell - k + 1)}. \) We then proceed essentially as in 3.2.2 by using \( \tilde{w} \) instead of \( w. \) We are going to see that for \( G = SO_{2n+1}, \) there exists an analogue of Theorem 3.2.3 whatever the parity of \( \ell. \)
The even case $\ell = 2p$. For any $k=1,...,p$, set $s_k = \text{card}(I^{(k)})$, $r_k = \text{card}(I^{(k)}) + \text{card}(I^{(\ell-k+1)})$ and define $X^{(k)}$ as the increasing reordering of $I^{(k)} \cup I^{(\ell-k+1)}$. Set

$$X^{(k)} = (i_1^{(k)},...,i_{r_k}^{(k)}). \quad (45)$$

For $k=1,...,p$ consider the $r_k$-tuple $\mu^{(k)}$ such that

$$\mu^{(k)} = \begin{pmatrix}
\text{sign}(i)\frac{\mu_{|i|} + |i| + \text{sign}(i)k - \frac{1+\text{sign}(i)}{2}}{\ell} | \ i \in X^{(k)} \end{pmatrix} \in \mathbb{Z}^{r_k}.$$

For any $i \in I^{(k)}$ with $k=1,...,p$, we have $x_i^{-\bar{w}(i)-1} = x_i^w$. Thus

$$\prod_{i \in X^{(k)}} x_i^{-\bar{w}(i)} = \prod_{i \in I^{(k)}} x_i^{-\bar{w}(i)} \prod_{i \in I^{(\ell-k+1)}} x_i^{-\bar{w}(i)} = \prod_{i \in I^{(k)}} x_i^{-\bar{w}(i)-1} \prod_{i \in I^{(\ell-k+1)}} x_i^{-\bar{w}(i)}$$

and by definition of the $\mu^{(k)}$'s, (43) can be rewritten

$$P_\mu = \prod_{k=1}^{p} x_{X^{(k)}}^{\mu^{(k)}} \times \sum_{w \in W} \varepsilon(w) \prod_{k=1}^{p} \prod_{i \in X^{(k)}} x_i^{-\bar{w}(i)} \times \prod_{k=1}^{p} \prod_{i \in X^{(k)}} x_i^{-k+1}.$$  

We obtain

$$\varphi_\ell(P_\mu) = \prod_{k=1}^{p} x_{X^{(k)}}^{\mu^{(k)}} \times \sum_{w \in W} \varepsilon(w) \varphi_\ell \left( \prod_{k=1}^{p} \prod_{i \in X^{(k)}} x_i^{-\bar{w}(i)-k+1} \right)$$

We deduce the equivalences

$$\varphi_\ell \left( \prod_{k=0}^{p} \prod_{i \in X^{(k)}} x_i^{-w(i)+k-1} \right) \neq 0 \iff \bar{w}(X^{(k)}) = J^{(\ell-k+1)} \text{ for any } k = 1,...,p. \quad (46)$$

In particular we must have $\text{card}(J^{(\ell-k+1)}) = \text{card}(J^{(k)}) = r_k$. We can write

$$J^{(\ell-k+1)} = (-k + 1 - \alpha_k \ell, ..., -k + 1 + \beta_k \ell) \text{ and } J^{(k)} = (k - \beta_k \ell, ..., k + \alpha_k \ell)$$

with $\alpha_k + \beta_k + 1 = r_k$. Consider $w_0 \in W$ defined by

$$\bar{w}_0(i^{(k)}_a) = -k + 1 - \alpha_k \ell + (a-1)\ell \text{ for any } k = 1,...,p.$$  

Denote by $W$ the set of signed permutations $w \in W$ which verify the right-hand side of (46). We have $w_0 \in W$. Each $w \in W$ can be written $w = u_0\tau$ where $\tau = (\tau^{(1)},...,\tau^{(p)})$ belongs to the direct product $S_{X^{(1)}} \times \cdots \times S_{X^{(p)}}$. We have also by Lemma [2.1.1] $\varepsilon(w) = \varepsilon(w_0)(-1)^{(l(\tau^{(1)}) \times \cdots \times (-1)^{(l(\tau^{(p)})})}$.

For any $k=1,...,p$, set

$$P_k = \sum_{\tau^{(k)} \in S_{X^{(k)}}} (-1)^{(l(\tau^{(k)}))} \varphi_\ell \left( \prod_{i \in X^{(k)}} x_i^{-\bar{w}_0\tau^{(k)}(i)-k+1} \right).$$

We obtain

$$\varphi_\ell(P_\mu) = \varepsilon(w_0) \prod_{k=1}^{p} x_{X^{(k)}}^{\mu^{(k)}} P_k.$$
By (47) we have
\[ w_0^T (k)^{(1)} (i^{(k)}) = -k + 1 - \alpha_k \ell + (\tau^{(k)}(a) - 1) \ell \] for any \( a = 1, \ldots, r_k \).
This yields
\[ P_k = \sum_{\tau(k) \in S_X^{(k)}} (-1)^{\ell \tau(k)} \prod_{a=1}^{r_k} x_{\tau(k)}^{\tau(k)(a) + (\alpha_k + 1)} = x_{X^{(k)}}^{\eta_{r_k}} \Delta_{X^{(k)}} \]
where for any \( k = 1, \ldots, p, \eta_{r_k} = -\rho_{r_k} + (\alpha_k + 1, \ldots, \alpha_k + 1) \in \mathbb{Z}^{r_k} \) and
\[ \Delta_{X^{(k)}} = \prod_{i<j \in X^{(k)}} (1 - \frac{x_i}{x_j}). \]

Note that the computation only makes root systems of type A appear in this case. This gives
\[ \varphi_{\ell}(P_{\mu}) = \varepsilon(w_0) \prod_{k=1}^{p} x_{X^{(k)}}^{\mu^{(k)} - \eta_{r_k}} \Delta_{X^{(k)}} = \varepsilon(w_0) \prod_{k=1}^{p} x_{X^{(k)}}^{\mu^{(k)}} \Delta_{X^{(k)}} \]
where for any \( k = 1, \ldots, p, \)
\[ \mu^{(k)} = \left( \text{sign}(i) \mu_{[i]} + |i| + \text{sign}(i) \frac{1 + \text{sign}(i)}{2} \right) \quad | i \in X^{(k)} \) - (1, \ldots, r_k) + (\alpha_k + 1, \ldots, \alpha_k + 1) \in \mathbb{Z}^{r_k}. \] (48)

Similarly to Theorem 3.2.3 we obtain:

**Theorem 3.2.8** Consider a partition \( \mu \) of length \( n \) and \( \ell = 2p \) a positive integer. For any \( k = 1, \ldots, p \) define the sets \( X^{(k)}, J^{(k)} \) by (44) and (45).

- If there exists \( k \in \{1, \ldots, p\} \) such that \( \text{card}(X^{(k)}) \neq \text{card}(J^{(k)}) \) then \( \varphi_{\ell}(s_{\mu}) = 0 \).

- Otherwise, for any \( k = 1, \ldots, p \), set \( r_k = \text{card}(X^{(k)}) \). Let \( w_0 \in W \) be as in (47). Consider \( (\mu^k) = (\mu^{(1)}, \ldots, \mu^{(p)}) \) where the \( \mu^{(k)} \)'s are defined by (48). Then \( (\mu^k) \) is a dominant weight of \( P_{\mathcal{I}}^+ \) with \( \mathcal{I} = \{X^{(1)}, \ldots, X^{(p)}\} \) and we have
\[ \varphi_{\ell}(s_{\mu}) = \varepsilon(w_0) S_{(\mu^k)}^{(\mathcal{I})}, \]

**Example 3.2.9** Consider \( \mu = (2, 5, 5, 6, 7, 9) \) and \( \ell = 2 \). Then \( \mu + \rho_6 = (3, 7, 8, 10, 12, 15) \). Hence \( I_1 = \{1, 2, 6\} \) and \( I_2 = \{3, 4, 5\} \). Moreover \( J_2 = \{4, 2, 0, 2, 4, 6\} \) and \( J_1 = \{5, 3, 1, 1, 3, 5\} \). Then \( \bar{w}_0 \) sends \( X_1 = \{5, 2, 3, 4, 5, 6\} \) into \( J_2 \). This gives
\[ \bar{w}_0 = \begin{pmatrix} \bar{6} & \bar{5} & \bar{4} & \bar{3} & \bar{2} & \bar{1} & 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix} \]
by using (??). Hence
\[ w_0 = \begin{pmatrix} 6 & 5 & 4 & 3 & 2 & 1 & 0 & 1 & 3 & 2 & 4 & 6 & 5 \end{pmatrix}. \]
We have \( \varepsilon(\mu) = 1, \alpha_1 = 2 \) and
\[ \mu^{(1)} = (-7, -3, -1, 4, 5, 6) - (1, 2, 3, 4, 5, 6) + (3, 3, 3, 3, 3, 3) = (-5, -2, -1, 3, 3, 3). \]
We have then \( G_{\mathcal{I}} \simeq GL_6 \).
The case \( \ell = 2p+1 \) In addition to the sets \( X^{(k)}, k = 1, ..., p \) defined in (45), we have also to consider \( I^{(p+1)} = \{i_1^{(p+1)}, ..., i_r^{(p+1)}\} \). This yields to define

\[
\mu^{(p+1)} = \left( \frac{\mu_i + i + p}{\ell} \right)_{i \in I^{(p+1)}},
\]

We have

\[
\varphi_{\ell}(P_\mu) = x_{I^{(p+1)}}^{\mu} \prod_{k=1}^{p} x_{I^{(k)}}^{w_k} \times \sum_{w \in W} \varepsilon(w) \varphi_{\ell} \left( \prod_{i \in I^{(p+1)}} x_{i-\tilde{w}(i)-p} \prod_{k=1}^{p} \prod_{i \in X^{(k)}} x_{i-\tilde{w}(i)-k-1} \right)
\]

and the equivalence

\[
\varphi_{\ell} \left( \prod_{i \in I^{(p+1)}} x_{i-\tilde{w}(i)-p} \prod_{k=1}^{p} \prod_{i \in X^{(k)}} x_{i-\tilde{w}(i)-k-1} \right) \neq 0 \iff \begin{cases} \tilde{w}(X^{(k)}) = J^{(\ell-k+1)} \text{ for any } k = 1, ..., p \vspace{1em} \\ \tilde{w}(I^{(p+1)} \cup T^{(p+1)}) = J^{(\ell-p)} = J^{(p+1)} \end{cases}
\]

Indeed \((J^{(p+1)})^* = J^{(p+1)}\). In particular we must have \( \text{card}(J^{(p+1)}) = 2 \text{card}(I^{(p+1)}) = 2r_{p+1} \). Thus we can set \( J^{(p+1)} = (-p - (r_{p+1} - 1)\ell, ..., -p + r_{p+1}\ell) \). Consider \( w_0 \in W \) defined by (47) and

\[
\tilde{w}_0(i_a^{(p+1)}) = -p + a\ell \text{ for any } a = 1, ..., r_{p+1}.
\]

Denote by \( W \) the set of signed permutations \( w \in W \) which verify the right-hand side of (49). We have \( w_0 \in W \). Each \( w \in W \) can be written \( w = w_0v \) where \( v = (\tau^{(1)}, ..., \tau^{(p)}, v^{(p+1)}) \) belongs to the direct product \( S_{X^{(1)}} \times \cdots \times S_{X^{(p)}} \times W_{I^{(p+1)}} \). We have also \( \varepsilon(w) = \varepsilon(w_0)(-1)^{(\tau^{(1)})} \times \cdots \times (-1)^{(\tau^{(p)})}(-1)^{(v^{(p+1)})} \). This permits to write

\[
\varphi_{\ell}(P_\mu) = \varepsilon(w_0)x_{I^{(p+1)}}^{\mu} P_{p+1} \prod_{k=1}^{p} x_{I^{(k)}}^{w_k} P_k \text{ where }
\]

\[
P_{p+1} = \sum_{v^{(p+1)} \in W_{I^{(p+1)}}} (-1)^{(v^{(p+1)})} \varphi_{\ell} \left( \prod_{i \in I^{(p+1)}} x_{i-\tilde{w}_0v(i^{(p+1)})} \right)
\]

The functions \( P_k, k = 1, ..., p \) can be computed as in the even case. For \( P_{p+1} \), observe that each \( v^{(p+1)} \in W_{I^{(p+1)}} \) can be written \( v^{(p+1)} = \zeta \sigma \) with \( \sigma \in S_{I^{(p+1)}} \). According to this decomposition we have for any \( a = 1, ..., r_{p+1} \),

\[
\tilde{w}_0v^{(p+1)}(i_a^{(p+1)}) = \xi(a)(-p + \sigma(a)\ell).
\]

\[
P_{p+1} = \sum_{v^{(p+1)} \in W_{I^{(p+1)}}} (-1)^{(v^{(p+1)})} \varphi_{\ell} \left( \prod_{a=1}^{r_{p+1}} x_{i_a^{(p+1)}}^{-\xi(a)(-p + \sigma(a)\ell)-p} \right) =
\]

\[
\sum_{v^{(p+1)} \in W_{I^{(p+1)}}} (-1)^{(v^{(p+1)})} \prod_{a=1}^{r_{p+1}} x_{i_a^{(p+1)}}^{-\frac{1-\xi(a)}{\ell} - \frac{\xi(a)}{\sigma(a)}}.
\]

Thus

\[
P_{p+1} = \prod_{i \in I^{(p+1)}} x_i^{-1/2} \sum_{v^{(p+1)} \in W_{I^{(p+1)}}} (-1)^{(v^{(p+1)})} \left( \prod_{a=1}^{r_{p+1}} x_{i_a^{(p+1)}}^{-\frac{a - 1}{\ell}} \right) = x_{I^{(p+1)}}^{-r_{p+1}} \Delta_{I^{(p+1)}}
\]
Thus, 

\[ \Delta_{I^{(p+1)}} = \prod_{i<j, i,j \in I^{(p+1)}} (1 - \frac{x_j}{x_i}) \prod_{r<s, r,s \in I^{(p+1)}} (1 - x_r x_s) \prod_{i \in I^{(p+1)}} (1 - x_i). \]

This means that when \( \ell \) is odd

\[ \varphi_{\ell}(P_\mu) = \varepsilon(w_0) \prod_{k=1}^p x_{I^{(k)}} x_{J^{(k)}} \Delta_{X^{(k)}} \times x_{I^{(p+1)}} \Delta_{I^{(p+1)}} \]

where

\[ \mu^{(p+1)} = \left( \frac{\mu_i + i + p}{\ell} \right), \quad i \in I^{(p+1)} \] - (1, ..., \( r_{p+1} \)) \in \mathbb{Z}^r_{p+1}. \] (51)

This gives the following theorem:

**Theorem 3.2.10** Consider a partition \( \mu \) of length \( n \) and \( \ell = 2p+1 \) a positive integer. Define \( X^{(k)}, J^{(k)} \) \( k = 1, ..., p \) and \( I^{(p+1)}, J^{(p+1)} \) by (44) and (45).

- If \( \text{card}(I^{(p+1)}) \neq \frac{1}{2} \text{card}(J^{(p+1)}) \) or if there exists \( k \in \{ 1, ..., p \} \) such that \( \text{card}(X^{(k)}) \neq \text{card}(J^{(k)}) \) then \( \varphi_{\ell}(s_\mu) = 0 \).

- Otherwise, set \( r_{p+1} = \text{card}(I^{(p+1)}) \) and for any \( k = 1, ..., p \), \( r_k = \text{card}(X^{(k)}) \). Let \( w_0 \in W \) verifying (47) and (50). Consider \( (\rho_\mu) = (\mu^{(p+1)}, \mu^{(1)}, ..., \mu^{(p)}) \) where the \( \mu^{(k)} \)'s are defined by (48) and (51). Then \( (\rho_\mu) \) is a dominant weight of \( P_\mu^+ \) with \( I = \{ I^{(p+1)}, X^{(1)}, ..., X^{(p)} \} \) and we have

\[ \varphi_{\ell}(s_\mu) = \varepsilon(w_0) \sigma^{(\rho_\mu)}. \]

**Example 3.2.11** Consider \( \mu = (1, 5, 5, 6, 7, 9) \) and take \( \ell = 3 \). We have \( \mu + \rho_6 = (2, 7, 8, 10, 12, 15) \).

Thus \( X^{(1)} = \{ 4, 2, 5, 6 \}, I^{(2)} = \{ 1, 3 \} \) and \( J^{(1)} = \{ 5, 4, 7, 1 \}, J^{(2)} = \{ 4, 5, 2, 1 \} \). In particular \( \alpha_2 = 1 \). Then

\[ \mu^{(1)} = \left( \frac{-10 - 1}{3} - 1 + 2, -\frac{7 - 1}{3} - 2 + 2, \frac{12}{3} - 3 + 2, \frac{15}{3} - 4 + 2 \right) = (-2, -2, 3, 3) \]

and \( \mu^{(2)} = (\frac{2+1}{3} - 1, \frac{8+1}{3} - 2) = (0, 1) \). Moreover, one has by using (47)

\[ \tilde{w}_0 = \left( \begin{array}{cccccccc} 6 & 5 & 4 & 3 & 2 & 1 & 2 & 3 \\ 3 & 0 & 5 & 4 & 2 & 1 & 2 & 3 \\ 6 & 5 & 4 & 3 & 2 & 1 & 2 & 3 \\ 4 & 1 & 6 & 5 & 3 & 2 & 1 & 2 \\ 6 & 5 & 4 & 3 & 2 & 1 & 2 & 3 \\ 4 & 1 & 6 & 5 & 3 & 2 & 1 & 2 \\ 6 & 5 & 4 & 3 & 2 & 1 & 2 & 3 \\ 4 & 1 & 6 & 5 & 3 & 2 & 1 & 2 \end{array} \right). \]

Hence

\[ w_0 = \left( \begin{array}{cccccccc} 6 & 5 & 4 & 3 & 2 & 1 & 2 & 3 \\ 4 & 1 & 6 & 5 & 3 & 2 & 1 & 2 \\ 6 & 5 & 4 & 3 & 2 & 1 & 2 & 3 \\ 4 & 1 & 6 & 5 & 3 & 2 & 1 & 2 \\ 6 & 5 & 4 & 3 & 2 & 1 & 2 & 3 \\ 4 & 1 & 6 & 5 & 3 & 2 & 1 & 2 \\ 6 & 5 & 4 & 3 & 2 & 1 & 2 & 3 \\ 4 & 1 & 6 & 5 & 3 & 2 & 1 & 2 \end{array} \right). \]

and \( \varepsilon(\mu) = 1 \). We have moreover \( G_I \simeq SO_5 \times GL_4 \).

## 4 Parabolic Kazhdan-Lusztig polynomials

We recall briefly in this section some basics on Affine Hecke algebras and parabolic Kazhdan-Lusztig polynomials associated with classical root systems. The reader is referred to [14] and [16] for detailed expositions. Note that the definition of the Hecke algebra used in [14] coincides with that used in [9] and [16] (with generators \( H_w \)) up to the change \( q \rightarrow q^{-1} \).
4.1 Extended affine Weyl group

Consider a root system of type $A_{n-1}, B_n, C_n$ or $D_n$. For any $\beta \in P \cap \mathbb{Z}^n$, we denote by $t_\beta$ the translation defined in $\mathfrak{h}_R^*$ by $\gamma \mapsto \gamma + \beta$. The extended affine Weyl group $\hat{W}$ is the group

$$\hat{W} = \{wt_\beta \mid w \in W, \beta \in P\}$$

with multiplication determined by the relations $t_\beta t_\gamma = t_{\beta + \gamma}$ and $wt_\beta = t_{w \cdot \beta}w$. The group $\hat{W}$ is not a Coxeter group but contains the affine Weyl group $\widetilde{W}$ generated by reflections through the affine hyperplanes $H_{\alpha,k} = \{\beta \in \mathfrak{h}_R^* \mid \langle \beta, \alpha^\vee \rangle = k\}$. It makes sense to define a length function on $\hat{W}$ verifying

$$l(wt_\beta) = \sum_{\alpha \in R_+} |\langle \beta, \alpha^\vee \rangle + 1_{R_-}(w \cdot \alpha)|$$

(52)

where for any $w \in W$, $1_{R_-}(w \cdot \alpha) = 0$ if $w \cdot \alpha \in R_+$ and $1_{R_-}(w \cdot \alpha) = 1$ if $w \cdot \alpha \in -R_+ = R_-$. Write $n_\beta$ for the element of maximal length in $Wt_\beta W$. It follows from (52) that for any $\lambda \in P_+$ we have $l(wt_\lambda) = l(w) + l(t_\lambda)$. This gives

$$n_\lambda = w_0 t_\lambda$$

(53)

where $w_0$ denotes the longest element of $W$. There exists a unique element $\eta \in R_+$ such that the fundamental alcove

$$A = \{\beta \in \mathfrak{h}_R^* \mid (\beta, \alpha^\vee) \geq 0 \forall \alpha \in R_+ \text{ and } (\beta, \eta^\vee) < 1\}$$

is a fundamental region for the action of $\hat{W}$ on $\mathfrak{h}_R^*$. This means that, for any $\beta \in \mathfrak{h}_R^*$, the orbit $\hat{W} \cdot \beta$ intersects $A$ in a unique point. Each $w \in \hat{W}$ can be written on the form $w = w_A w_\text{aff}$ where $w_\text{aff} \in \hat{W}$ and $w_A$ belongs to the stabilizer of $A$ under the action of $\hat{W}$. This implies that $A$ is also a fundamental domain for the action of $\hat{W}$ on $\mathfrak{h}_R^*$. The Bruhat ordering on $\hat{W}$ is defined by taking the transitive closure of the relations

$$w < sw \text{ whenever } l(w) < l(sw)$$

for all $w \in \hat{W}$ and all (affine) reflections $s \in \hat{W}$.

In fact the natural action of $\hat{W}$ on the weight lattice $P$ obtained by considering $P$ as a sublattice of $\mathfrak{h}_R^*$ is not that which is relevant for our purpose. For any integer $m \in \mathbb{Z}^*$ we obtain a faithful representation $\pi_m$ of $\hat{W}$ on $P$ by setting for any $\beta, \gamma \in P, w \in W$

$$\pi_m(w) \cdot \gamma = w \cdot \gamma \text{ and } \pi_m(t_\beta) \cdot \gamma = \gamma + m \beta.$$  

**Warning:** In the sequel, the extended affine Weyl group $\hat{W}$ acts on the weight lattice $P$ via $\pi_{-\ell}$ where $\ell$ is a fixed nonnegative integer.

We write for simplicity $wt_\beta \cdot \gamma$ rather that $\pi_{-\ell}(wt_\beta) \cdot \gamma$. Hence for any $w \in W$ and any $\beta \in P$, we have $wt_\beta \cdot \gamma = w \cdot \gamma - \ell w \cdot \beta$. The fundamental region for this new action of $\hat{W}$ on $P$ is the alcove $A_\ell$ obtained by expanding $A$ with the factor $-\ell$. This gives

$$A_\ell = \left\{ \nu = (\nu_1, ..., \nu_n) \mid 0 \geq \nu_1 \geq \cdots \geq \nu_n > -\ell \right\} \text{ for types } A,$$
$$\left\{ \nu = (\nu_1, ..., \nu_n) \mid 0 \geq \nu_1 \geq \cdots \geq \nu_n \geq -\ell/2 \right\} \cup \left\{ \nu = (\nu_1, ..., \nu_n) \mid 0 \geq \nu_1 \geq \nu_2 \geq \cdots \geq \nu_n \geq -\ell/2 \right\} \text{ for types } B, C,$$
$$\left\{ \nu = (\nu_1, ..., \nu_n) \mid 0 \geq \nu_1 \geq \nu_2 \geq \cdots \geq \nu_n \geq -\ell/2 \right\} \text{ for type } D.$$

Consider a weight $\beta \in P$. Then its orbit intersects $A_\ell$ in a unique weight $\nu$. Then there is a unique $w(\beta) \in \hat{W}$ of minimal length such that $w(\beta) \cdot \nu = \beta$. We denote by $W_\nu$ the stabilizer of $\nu \in A_\ell$ in $\hat{W}$. Since $\nu \in A_\ell$, $W_\nu$ is in fact a subgroup of $W$.  

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Lemma 4.1.1 Consider $\lambda \in P^+$ and suppose $\ell > n$. Then

1. $w(\ell \lambda + \rho) = n_\lambda \tau^{-n+1}$ with $\lambda^* = -w_0(\lambda)$ and $\tau = s_1s_2 \cdots s_{r-1}t_{e_1}$ for type $A$.

2. $w(\ell \lambda + \rho) = n_\lambda$ for types $B$, $C$ and $D$.

Proof. 1: See Lemma 2.3 in [6].

2: Observe first that $w_0 \cdot \rho = -\rho$ belongs to $A_\ell$ for types $B$, $C$, $D$ since $\ell > n$. We have

$$\ell \lambda + \rho = t_{-\lambda} \cdot \rho = t_{-\lambda}w_0 \cdot (w_0 \cdot \rho) = t_{-\lambda}w_0 \cdot (-\rho).$$

Moreover $W_{-\rho} = \{1\}$. Since $-\rho \in A_\ell$ this means that $w(\ell \lambda + \rho) = t_{-\lambda}w_0 = w_0t_{w_0(-\lambda)} = w_0t_\lambda = n_\lambda$ where the last equality follows from (53). 

4.2 Affine Hecke algebra and K-L polynomials

The Hecke algebra associated with the root system $R$ of type $A_n$, $B_n$, $C_n$ or $D_n$ is the $\mathbb{Z}[q, q^{-1}]$-algebra defined by the generators $T_w$, $w \in \hat{W}$ and relations

$$T_{w_1}T_{w_2} = T_{w_1T_{w_2}} \text{ if } l(w_1w_2) = l(w_1) + l(w_2),$$

$$T_wT_s = (q^{-1} - q)T_w + T_{sw} \text{ if } l(sw) < l(w) \text{ and } i \in I_n.$$ 

In particular we have $T_i^2 = (q^{-1} - q)T_i + 1$ for any $i \in I_n$. The bar involution on $\hat{H}$ is the $\mathbb{Z}$-linear automorphism defined by 

$$\bar{q} = q^{-1} \text{ and } \overline{T_w} = T_{w^{-1}}^{-1} \text{ for any } w \in \hat{W}.$$ 

Kazhdan and Lusztig have proved that there exists a unique basis $\{C'_w \mid w \in \hat{W}\}$ of $\hat{H}$ such that 

$$\overline{C'_w} = C'_w \text{ and } C'_w = \sum_{y \leq w} p_{y,w}T_y$$

where $p_{w,w} = 1$ and $p_{y,w} \in q\mathbb{Z}[q]$ for any $y < w$. We will refer to the polynomials $p_{y,w}(q)$ as Kazhdan-Lusztig polynomials. They are renormalizations of the polynomials $P_{y,w}$ originally introduced by Kazhdan and Lusztig in [6]. Namely we have $p_{y,w} = q^{l(w) - l(y)}P_{y,w}$.

Let us define the $q$-partition function $P_q$ by

$$\prod_{\alpha \in R_+} \frac{1}{1 - qx^\alpha} = \sum_{\beta \in \mathbb{Z}^n} P_q(\beta)x^\beta.$$ 

Given $\lambda$ and $\mu$ in $P$, the Lusztig $q$-analogue $K_{\lambda,\mu}(q)$ is defined by 

$$K_{\lambda,\mu}(q) = \sum_{w \in W} \varepsilon(w)P_q(w \circ \lambda - \mu).$$ 

Then one has the following theorem due to Lusztig:

Theorem 4.2.1 Suppose $\lambda, \mu$ are dominant weights. Then $K_{\lambda,\mu}(q) = p_{n_\mu, n_\lambda}(q)$. 

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One defines the action of the bar involution on the parabolic module $P_{\nu} = \bar{\mathcal{H}} \cdot \nu$, $\nu \in \mathcal{A}_t$, by setting $\bar{q} = q^{-1}$ and $\bar{h} \cdot \nu = \bar{h} \cdot \nu$ for any $h \in \mathcal{H}$. Deodhar has proved that there exist two bases $\{C^+_\lambda \ | \ \lambda \in \hat{W} \cdot \nu\}$ and $\{C^-_\lambda \ | \ \lambda \in \hat{W} \cdot \nu\}$ of $P_{\nu}$ belonging respectively to

$$L^+_\nu = \bigoplus_{\lambda \in \hat{W} \cdot \nu} \mathbb{Z}[q] \lambda \text{ and } L^-_\nu = \bigoplus_{\lambda \in \hat{W} \cdot \nu} \mathbb{Z}[q^{-1}] \lambda$$

characterized by

$$\begin{align*}
\overline{C^+_\lambda} &= C^+_\lambda \\
C^+_\lambda &\equiv \lambda \mod q L^+_\nu \\
\overline{C^-_\lambda} &= C^-_\lambda \\
C^-_\lambda &\equiv \lambda \mod q^{-1} L^-_\nu
\end{align*}$$

We will only need the basis $\{C^-_\lambda \ | \ \lambda \in \hat{W} \cdot \nu\}$ in the sequel. The parabolic Kazhdan-Lusztig polynomials $P^-_{\lambda,\mu}$ are then defined by the expansion

$$C^-_\lambda = \sum_{\mu \in \hat{W} \cdot \lambda} (-1)^{(l(\lambda)) + l(\mu)} P^-_{\mu,\lambda}(q^{-1})$$

(see [16, Theorem 3.5]). In particular they belong to $\mathbb{Z}[q]$. Their expansion in terms of the ordinary Kazhdan-Lusztig polynomials is given by the following theorem due to Deodhar:

**Theorem 4.2.2** Consider $\nu \in \mathcal{A}_t$ and $\lambda \in \hat{W} \cdot \nu$. Then for any $\mu \in \hat{W} \cdot \lambda$ we have

$$P^-_{\lambda,\mu}(q) = \sum_{z \in W_\nu} (-q)^{l(z)} p_{w(\mu)z, w(\lambda)}(q) \quad (54)$$

with the notation of [4.1]

**Remark:** When $\nu$ is regular, that is $W_\nu = \{1\}$, we have $P^-_{\lambda,\mu}(q) = p_{w(\mu), w(\lambda)}(q)$.

## 5 Generalized Hall-Littlewood functions

### 5.1 Plethysm and parabolic K-L polynomials

Consider $t$ a nonnegative integer and $\zeta \in \mathbb{C}$ such that $\zeta^2$ is a primitive $t$-th root of 1. We briefly recall in this paragraph the arguments of [11] which establish that the coefficients of the plethysm $\psi_t(s_\lambda)$ on the basis of Weyl characters are, up to a sign, parabolic Kazhdan-Lusztig polynomials specialized at $q = 1$.

For any $\lambda \in P_+$, denote by $V_q(\lambda)$ the finite dimensional $U_q(\mathfrak{g})$-module of highest weight $\lambda$. Its character is also the Weyl character $s_\lambda$. Let $U_{q,z}(\mathfrak{g})$ be the $\mathbb{Z}[q,q^{-1}]$-subalgebra of $U_q(\mathfrak{g})$ generated by the elements

$$E_i^{(k)} \equiv \frac{E_i^{(k)}}{[k]_i!}, F_i^{(k)} \equiv \frac{F_i^{(k)}}{[k]_i!} \text{ and } K_i^{\pm 1}$$

where $E_i, F_i, K_i^{\pm 1}$, $i \in I_o$ are the generators of $U_q(\mathfrak{g})$. The indeterminate $q$ can be specialized at $\zeta$ in $U_{q,z}(\mathfrak{g})$. Thus it makes sense to define $U_{q,z}(\mathfrak{g}) = U_{q,z}(\mathfrak{g}) \otimes \mathbb{Z}[q,q^{-1}]$ where $\mathbb{Z}[q,q^{-1}]$ acts on $\mathbb{C}$ by $q \mapsto \zeta$.

Fix a highest weight vector $v_\lambda$ in $V_q(\lambda)$. We have $V_q(\lambda) = U_q(\mathfrak{g}) \cdot v_\lambda$. Similarly $V_\zeta(\lambda) = U_{q,z}(\mathfrak{g}) \cdot v_\lambda$ is a $U_{q,z}(\mathfrak{g})$ module called a Weyl module and one has $\text{char}(V_\zeta(\lambda)) = s_\lambda$. The module $V_\zeta(\lambda)$ is not simple but admits a unique simple quotient denoted by $L(\lambda)$.

From results due to Kazhdan-Lusztig and Kashiwara-Tanisaki one obtains the following decomposition of $\text{char}(L(\lambda))$ on the basis of Weyl characters:
Theorem 5.1.1 Consider \( \lambda \in P_+ \).

1. For \( \ell \) sufficiently large, the character of \( L(\lambda) \) decomposes on the form

\[
\text{char}(L(\lambda)) = \sum_{\mu} (-1)^{l(w(\lambda+\rho)) - l(w(\mu+\rho))} P^-_{\mu+\rho,\lambda+\rho}(1)s_\mu
\]

where the sum runs over the dominant weights \( \mu \in P_+ \) such that \( \mu + \rho \in \hat{W} \cdot (\lambda + \rho) \).

2. The parabolic Kazhdan-Lusztig polynomials \( P^-_{\mu+\rho,\lambda+\rho}(q) \) have nonnegative integer coefficients.

Remarks:

(i) The decomposition has been conjectured by Kazhdan-Lusztig and proved by Kashiwara-Tanisaki. In \([5]\), Kashiwara and Tanisaki have also obtained that the parabolic Kazhdan-Lusztig polynomials have nonnegative integer coefficients as soon as the Coxeter system under consideration corresponds to the Weyl group of a Kac-Moody Lie algebra, thus in the particular context of this paper.

(ii) In Assertion 1 of the above theorem, the integer \( \ell \) is only explicitly known in the simply laced case (see Theorem 7.1 in \([17]\)).

Consider a nonnegative integer \( \ell \). The Frobenius map \( \text{Fr}_\ell \) is the algebra homomorphism defined from \( U_\zeta(\mathfrak{g}) \) to \( U(\mathfrak{g}) \) by \( \text{Fr}_\ell(K_i) = 1 \) and

\[
\text{Fr}_\ell(e_i^{(k)}) = \begin{cases} e_i^{(k/\ell)} & \text{if } \ell \text{ divides } k \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \text{Fr}_\ell(f_i^{(k)}) = \begin{cases} f_i^{(k/\ell)} & \text{if } \ell \text{ divides } k \\ 0 & \text{otherwise} \end{cases}
\]

where \( e_i, f_i, i \in I_n \) are the Chevalley generators of the enveloping algebra \( U(\mathfrak{g}) \). This permits to endow each \( U(\mathfrak{g}) \)-module \( M \) with the structure of a \( U_\zeta(\mathfrak{g}) \)-module \( M^{\text{Fr}_\ell} \). Then we have

\[
\text{char}(M^{\text{Fr}_\ell}) = \psi_\ell(\text{char}(M))
\]

in particular for any \( \lambda \in P_+ \), \( \text{char}(V(\lambda)^{\text{Fr}_\ell}) = \psi_\ell(s_\lambda) \).

Each dominant weight \( \lambda \in P_+ \), can be uniquely decomposed on the form \( \lambda = \lambda' + q \lambda'' \) where \( \lambda, \lambda' \in P_+ \) and \( \lambda'' = (\lambda_1, ..., \lambda_n) \) verifies \( 0 \leq \lambda_{i+1} - \lambda_i < \ell \) for any \( i \in I_n \).

Theorem 5.1.2 (Lusztig) The simple \( U_\zeta(\mathfrak{g}) \)-module \( L(\lambda) \) is isomorphic to the tensor product

\[
L(\lambda) \simeq L(\lambda') \otimes V(\lambda'')^{\text{Fr}_\ell}.
\]

By replacing \( \lambda \) by \( \ell \lambda \) in the previous theorem, we have \( \lambda = 0 \) and \( \lambda' = \lambda' \). Thus \( L(\ell \lambda) \simeq V(\lambda')^{\text{Fr}_\ell} \). Then one deduces from \([55]\) the equality

\[
\psi_\ell(s_\lambda) = \text{char}(L(\ell \lambda)) = \sum_{\mu+\rho \in \hat{W} \cdot (\ell \lambda+\rho)} (-1)^{l(w(\lambda+\rho)) - l(w(\mu+\rho))} P^-_{\mu+\rho,\ell \lambda+\rho}(1)s_\mu
\]

which shows that the coefficients of the expansion of \( \psi_\ell(s_\lambda) \) on the basis of Weyl characters are, up to a sign, parabolic Kazhdan-Lusztig polynomials specialized at \( q = 1 \). This gives

\[
| < \psi_\ell(s_\lambda), s_\mu > | = | < s_\lambda, \varphi(s_\mu) > | = P^-_{\mu+\rho,\ell \lambda+\rho}(1).
\]

By definition of the action of \( \hat{W} \) on \( P \) we have \( \hat{W} \cdot (\ell \lambda + \rho) = \hat{W} \cdot \rho \). This implies the
Corollary 5.1.3 (of Theorems 4.2.2 and 5.1.2). For $\ell$ as in 1 of Theorem 4.2.2, we have

$$\psi_{\ell}(s_\lambda) = \sum_{\mu + \rho \in \widehat{W} \cdot \rho} (-1)^{l(w(\lambda + \rho)) - l(w(\mu + \rho))} P^{-}_{\mu + \rho, \ell \lambda + \rho}(1) s_{\mu}.$$  

In particular $\varphi(s_\mu) \neq 0$ if and only if $\mu + \rho \in \widehat{W} \cdot \rho$, that is $\mu + \rho = w \cdot \rho - \ell \beta$ with $w \in W$ and $\beta \in P$.

Remark: The equivalence $\varphi(s_\mu) \neq 0 \iff \mu + \rho \in \widehat{W} \cdot \rho$ can also be obtained more elementary from algorithms described in 3.2.

5.2 Parabolic K-L polynomials and branching coefficients

Warning: In the sequel of the paper we will suppose $\ell$ sufficiently large so that assertion 1 of Theorem 4.2.2 holds. Moreover $\ell$ is assumed odd when the Lie groups under consideration are of type $C$ or $D$.

Under these hypotheses we have for any $\mu \in \mathcal{P}_n \varphi_{\ell}(s_\mu) = 0$ or

$$\varphi_{\ell}(s_\mu) = \varepsilon(w_0) S_{(\mu^\ell)} \mathcal{I}$$  

according to the results of 3.2.

Remark: According to the algorithms described in 3.2 when $\varphi_{\ell}(s_\mu) \neq 0$, the cardinalities of the sets $I^{(k)}$ or $X^{(k)}$ contained in $\mathcal{I}$ are determined by those of the sets $J^{(k)}$. In particular they depend only on $n$ and $\ell$ and not on the partition $\mu$ considered. Thus in (56), the underlying subgroup of Levi type $G_\mathcal{I}$ is, up to isomorphism, independent on $\mu$.

By using Proposition 2.5.2 and Theorems 3.2.1, 3.2.3, 3.2.6, 3.2.8, 3.2.10 we deduce from Corollary 5.1.3 the

Theorem 5.2.1 For any $\lambda, \mu \in \mathcal{P}_n$ such that $\mu + \rho \in \widehat{W} \cdot \rho$

$$P^{-}_{\mu + \rho, \ell \lambda + \rho}(1) = [V(\lambda) : V_\mathcal{I} \left(\begin{array}{c} \mu \\ \ell \end{array}\right)]$$

where $(\mu^\ell)$ and $\mathcal{I}$ are obtained from $\mu$ and $\ell$ by applying the algorithms described in 3.2.

5.3 The functions $H^\ell_\mu$

For any $\mu \in \mathcal{P}_n$, we define the function $G^\ell_\mu$ by setting

$$G^\ell_\mu = \sum_{\lambda \in \mathcal{P}_n} [V(\lambda) : V_\mathcal{I} \left(\begin{array}{c} \mu \\ \ell \end{array}\right)] q s_\lambda$$  

where for any $\lambda \in \mathcal{P}_n$, $[V(\lambda) : V_\mathcal{I} \left(\begin{array}{c} \mu \\ \ell \end{array}\right)] q = P^{-}_{\mu + \rho, \ell \lambda + \rho}(q)$. We also consider the function $H^\ell_\mu$ such that

$$H^\ell_\mu = G^\ell_\mu.$$  

Theorem 5.3.1 Consider a partition $\mu \in \mathcal{P}_n$.

1. The coefficients of $G^\ell_\mu$ and $H^\ell_\mu$ on the basis of Weyl characters are polynomials in $q$ with non-negative integer coefficients.
2. We have $H^1_{\mu} = s_\mu$.

3. For $\ell$ sufficiently large $H^\ell_{\mu} = Q^{'\ell}_{\mu}$, that is $H^\ell_{\mu}$ coincide with the Hall-Littlewood function associated with $\mu$.

To prove our theorem we need the following Lemma:

**Lemma 5.3.2** Consider $\beta \in \mathbb{Z}^n$.

- In type $A_{n-1}$, suppose $\ell > n$. Then the weight $\ell \beta + \rho$ is regular.
- In type $B_n, C_n$ or $D_n$, suppose $\ell > 2n$. Then the weight $\ell \beta + \rho$ is regular.

**Proof.** Consider $w \in W$ and $t_\gamma$ such that $t_\gamma w \cdot (\ell \beta + \rho) = \ell \beta + \rho$. Then $\delta = \ell \beta + \rho - w \cdot (\ell \beta + \rho) \in (\ell \mathbb{Z})^\ell$. Set $\beta = (\beta_1, \ldots, \beta_n)$. For any $i = 1, \ldots, n$, the $i$-th coordinate of $\delta$ is $\delta_i = \ell \beta_i + \rho_i - \ell \beta_{w(i)} - \rho_{w(i)}$. Since $\delta_i \in \ell \mathbb{Z}$, we must have $|\rho_i - \rho_{w(i)}| \in \ell \mathbb{Z}$. One verifies easily that for type $A_{n-1}$, $|\rho_i - \rho_{w(i)}| \leq n - 1$ and for types $B_n, C_n, D_n$ $|\rho_i - \rho_{w(i)}| \leq 2n$. Hence when the conditions of the lemma are verified, $|\rho_i - \rho_{w(i)}| = 0$ for any $i = 1, \ldots, n$. This gives $w = 1$. The equality $t_\gamma w \cdot (\ell \beta + \rho) = \ell \beta + \rho$ implies then that $\gamma = 0$. Thus the stabilizer of $\ell \beta + \rho$ is reduced to $\{1\}$, that is $\ell \beta + \rho$ is regular. ■

**Proof.** (of Theorem 5.3.1)

1. Follows from Theorem 5.1.1 and (57).

2. When $\ell = 1$, we have seen that $G = G_T$ and $\binom{\mu}{1} = \mu$. Thus $[V(\lambda) : V_T(\binom{\mu}{1})_q \neq 0$ only if $\lambda = \mu$. In this case $H^1_{\mu} = s_\mu$ for $[V(\lambda) : V_T(\binom{\mu}{1})_q = [V(\lambda) : V(\lambda)]_q = 1$.

3. Suppose $\ell$ as in the previous lemma. We have $[V(\lambda) : V_T(\binom{\mu}{\ell})_q = P_{\ell \mu + \rho, \ell \lambda + \rho}(q)$. Since $\ell \lambda + \rho$ is regular for the action of $\hat{W}$, we obtain by Theorem 4.2.2, $P_{\ell \mu + \rho, \ell \lambda + \rho}(q) = p_{n\mu, n\lambda}(q)$. By using Lemma 4.1.1 we deduce $P_{\ell \mu + \rho, \ell \lambda + \rho}(q) = p_{n\mu, n\lambda}(q)$. Now by Theorem 4.2.1 this gives $P_{\ell \mu + \rho, \ell \lambda + \rho}(q) = K_{\lambda, \mu}(q)$. Finally

$$H^\ell_{\mu} = \sum_{\lambda \in \mathcal{P}_n} [V(\lambda) : V_T(\binom{\ell \mu}{\ell})_q s_{\lambda} = \sum_{\lambda \in \mathcal{P}_n} K_{\lambda, \mu}(q) s_{\lambda} = Q^{'}_{\mu}.$$  \hspace{2cm} \hspace{2cm} (59)

**Remarks:**

(i) : By the previous theorem the functions $H^\ell_{\mu}$ interpolate between the Weyl characters and the Hall-Littlewood functions.

(ii) : When $\ell$ is even for types $C$ and $D$, one can also define the functions $G^\ell_{\mu}$ and $H^\ell_{\mu}$ by setting $G^\ell_{\mu} = \sum_{\lambda \in \mathcal{P}_n} P_{\mu + \rho, \ell \lambda + \rho}(q) s_{\lambda}$ and $H^\ell_{\mu} = G^\ell_{\ell \mu}$, respectively. When $\ell > 2n$ we have yet $H^\ell_{\mu} = Q^{'}_{\mu}$, but the polynomials $P_{\mu + \rho, \ell \lambda + \rho}(q)$ cannot be interpreted as quantizations of branching coefficients.

(iii) : The conditions $\ell > n$ for type $A_{n-1}$ and $\ell > 2n$ for types $B_n, C_n, D_n$ appear also naturally in the algorithms of 5.2. When they are fulfilled, one has $\varphi(\ell s_{\mu}) = 0$, or $J_k = I_k$ for any $k = 1, \ldots, n$ and $J_k = I_k = 0$ for $k \notin \{1, \ldots, n\}$. Then $[(\ell \mu)/\ell] = \mu$ and $G_T = H$. Hence $[V(\lambda) : V_T(\binom{\ell \mu}{\ell})] = K_{\lambda, \mu}$ for any $\lambda \in \mathcal{P}_n$. This yields equality (59) specialized at $q = 1$.  

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6 Further remarks

6.1 Quantization of tensor product coefficients

Consider \( \mu \in \mathcal{P}_n \) and set \( \mu = (\mu(0), ..., \mu(\ell-1)) \) as in Theorem 3.2.1. For \( G = GL_n \), the duality \( c^\lambda_{(\mu(0), ..., \mu(\ell-1))} = [V(\lambda) : V_T(\mu)] \) yields a \( q \)-analogue of the Littlewood-Richardson coefficient \( c^\lambda_{(\mu(0), ..., \mu(\ell-1))} \) defined by setting

\[
c^\lambda_{(\mu(0), ..., \mu(\ell-1))}(q) = [V(\lambda) : V_T(\mu)]_q = D_{\mu+p, \ell \lambda+p}(q). \tag{60}
\]

By Theorem 5.1.1, \( c^\lambda_{(\mu(0), ..., \mu(\ell-1))}(q) \) have then nonnegative integer coefficients.

In [10], we have shown that there also exists a duality between tensor product coefficients for types \( B, C, D \) defined as the analogues of the Littlewood-Richardson coefficients by counting the multiplicities of the isomorphic irreducible components in a tensor product of irreducible representations and branching coefficients. These branching coefficients correspond to the restriction of \( SO_{2n} \) to subgroups of the form \( SO_{2r_0} \times \cdots \times SO_{2r_p} \) where the \( r_i \)'s are positive integers summing \( n \). These subgroups are not subgroups of Levi type, thus the Littlewood-Richardson coefficients for types \( B, C, D \) cannot be quantified as in (60) by using parabolic Kazhdan-Lusztig polynomials.

For \( G = SO_{2n+1}, Sp_{2n} \) or \( SO_{2n} \) and \( \lambda \in \mathcal{P}_n \), denote by \( \mathcal{U}(\lambda) \) the restriction of the irreducible finite dimensional \( GL_N \)-module of highest weight \( \lambda \) to \( G \). Consider a \( p \)-tuple \( (\mu(0), ..., \mu(p-1)) \) of partitions such that \( \mu(k) \in \mathcal{P}_{r_k} \) for any \( k = 0, ..., p-1 \). One can define the coefficients \( D^\lambda_{(\mu(0), ..., \mu(p-1))} \) as the multiplicity of \( V(\lambda) \) in \( \mathcal{U}(\mu(0)) \otimes \cdots \otimes \mathcal{U}(\mu(p-1)) \), that is such that

\[
\mathcal{U}(\mu(0)) \otimes \cdots \otimes \mathcal{U}(\mu(p-1)) \simeq \bigoplus_{\lambda \in \mathcal{P}_n} V(\lambda) \oplus D^\lambda_{(\mu(0), ..., \mu(p-1))}.
\]

We have also obtained in [10] a duality result between the coefficients \( D^\lambda_{(\mu(0), ..., \mu(p-1))} \) and branching coefficients corresponding to the restriction of \( G \) to the subgroup of Levi type \( GL_{r_0} \times \cdots \times GL_{r_p} \). The coefficients \( D^\lambda_{(\mu(0), ..., \mu(p-1))} \) can be expressed by using a partition function similarly to Proposition 2.5.1. By quantifying this partition function, one shows that they admit nonnegative \( q \)-analogues. It is conjectured that stable one-dimensional sums defined in [1] from affine crystals obtained by considering the affinizations of the classical root systems are special cases of the \( q \)-analogues obtained in this way.

Recall that the subgroups of Levi type \( G_T \) obtained in the theorems of 3.2 are, up to isomorphism, determined only by \( G \) and \( \ell \). This implies that there exist subgroups of Levi type \( L \) in \( G \) which are not isomorphic to a subgroup \( G_T \). This is for instance the case when \( G = Sp_{2n} \) for the subgroups of Levi type \( G_T \simeq GL_{r_0} \times \cdots \times GL_{r_{p-1}} \) such that \( r_k > 1 \) for any \( k = 0, ..., p-1 \). Indeed, by Theorem 3.2.3 when \( r_0 = \text{card}(T) > 1 \), \( G_T \) is isomorphic to

\[
Sp_{2r_0} \times GL_{r_1} \times \cdots \times GL_{r_{p-1}}.
\]

This implies that one cannot obtain in general a quantization of the tensor product coefficients \( D^\lambda_{(\mu(0), ..., \mu(p-1))} \) by using parabolic Kazhdan-Lusztig polynomials as in (60).

6.2 Combinatorial description of the functions \( G^\ell_{\mu} \)

When \( G = GL_n \), the functions \( G^\ell_{\mu} \) defined in [77] admit the following combinatorial description

\[
G^\ell_{\mu} = \sum_{T \in \operatorname{Tab}_\ell(\mu)} q^{s(T)} x^T
\]
where Tab(µ) is the set of ℓ-ribbon tableaux of shape µ on In and s the spin statistic defined on ribbon tableaux (see [8] page 1057). Recently, Haglund, Haiman and Loehr have obtained the expansion of the Macdonald polynomials in terms of simple renormalizations of the LLT polynomials Gµ,ℓ. This expansion yields a combinatorial formula for the Macdonald polynomials [3].

This suggests to investigate the following combinatorial problem:

**Problem 6.2.1** Find a combinatorial description of the polynomials Gµ,ℓ and the q-analogues [V(λ) : VT(ℓ)]q related to the roots systems of type B, C or D.

### 6.3 Exceptional root systems

It is also possible to define the plethysm ψℓ and the dual plethysm ϕℓ for exceptional root systems. Consider such an exceptional root system R and µ a dominant weight for R. Denote also by sµ the Weyl character of the irreducible finite dimensional module of highest weight λ. When ℓ is sufficiently large (the bound depends on R), we have ϕℓ(sµ) = sµ. For the other values of ℓ, one shows that the polynomial ϕℓ(eµ \prod_{α∈R_+}(1 − eα)) do not factorize in general as a product of factors (1 − xβ) where β is a positive root. This implies that one cannot define generalized Hall-Littlewood functions for exceptional types by proceeding as in [58].

### 6.4 Stabilized plethysms

When G = Sp2n or SO2n and ℓ is even, we have seen that the combinatorial methods of Section 3 do not permit to obtain the coefficients of the plethysms ϕ(λ) on the basis of the Weyl characters. In [12], we show that this difficulty can be overcome by considering stabilized power sum plethysms, i.e. by assuming n ≥ ℓ |λ|. Under this hypothesis, one can indeed prove that the coefficients in the expansion of ϕ(λ) coincide for G = SO2n+1, Sp2n and SO2n. So it suffices to compute them in type Bn for which we have a relevant combinatorial procedure in both cases ℓ even and ℓ odd.

**Note:** While revising a previous version of this work [11], I was informed that Grojnowski and Haiman [2] also define, in a paper in preparation, generalized Hall-Littlewood polynomials for reductive Lie groups. Their polynomials are introduced as formal q-characters depending on a subgroup of Levi type. The coefficients of the corresponding expansion on the basis of the Weyl characters are also affine parabolic Kazhdan-Lusztig polynomials. As far as the author can see, the generalization of the Hall-Littlewood polynomials presented in the present paper satisfies the general definition given in [2] (see Definition 5.12). Nevertheless, our combinatorial results based on the study of the power sum plethysms on Weyl characters are completely independent of the approach of Grojnowski and Haiman. It also naturally yields the family of polynomials \{Gℓ,µ | ℓ ∈ N\} in the spirit of the original work by Lascoux, Leclerc and Thibon [8].

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