HEAT TRACE ASYMPTOTICS WITH SINGULAR WEIGHT FUNCTIONS II

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Abstract. We study the weighted heat trace asymptotics of an operator of Laplace type with mixed boundary conditions where the weight function exhibits radial blowup. We give formulas for the first three boundary terms in the expansion in terms of geometrical data.

1. Introduction

An important issue for several decades has been to obtain explicitly the coefficients of the short-time asymptotic expansion of the heat kernel associated with a Laplace type operator on a $m$-dimensional Riemannian manifold $M$ [15, 19]. In mathematics this interest stems in particular from the link between the spectrum of the operator and the underlying geometry of $M$ [18], but it extends to basically all of Geometric Analysis [15]. In physics the heat kernel asymptotic expansion has been realized to be a particularly useful tool to determine various approximations of effective actions and the Casimir energy [2, 9, 13].

Instead of simply analyzing the integrated heat trace one often puts a weight in the evaluation of the trace, sometimes called the localizing or smearing function. This function is introduced for various reasons. First it allows one to obtain local information from the integrated one, therefore, most importantly it is possible to recover the local behavior near the boundary. Furthermore, it is this smeared coefficient that appears in the integration of conformal anomalies relevant for several physical applications, see, e.g., [6, 11, 19]. For smooth localizing functions the results for the first few heat kernel coefficients are available for several years now [16, 19]. A detailed analysis of what happens for singular weighting functions has only been started recently. In the context of the heat content asymptotics the weighting function plays the role of an initial temperature distribution. In the context of black hole physics singular conformal transformations play an important role when mapping black holes to their Penrose diagrams [5].

The heat content asymptotics of an operator of Laplace type with singular initial temperature distribution and with Dirichlet or Robin boundary conditions were investigated in [4]. A similar study of the heat trace asymptotics with singular weighting function and Dirichlet boundary conditions was performed in [3]. In this paper, we conclude this line of investigation by extending the results of [3] concerning heat trace asymptotics to Robin, and more generally, to mixed boundary conditions. We anticipate that also this singular setting will find its applications in physics.

1.1. Operators of Laplace type. Let $M$ be a compact Riemannian manifold of dimension $m$ with smooth non-empty boundary $\partial M$. Let $V$ be a smooth vector
bundle over $M$ and let $D$ be an operator of Laplace type on the space of smooth sections $C^\infty(V)$. This means that locally we may express $D$ in the form
\begin{equation}
(1.a) \quad D = -(g^{\mu\nu} \partial_{x^\mu} \partial_{x^\nu} \text{Id} + \Lambda^r \partial_{x^r} + B)
\end{equation}
for suitably chosen matrices $\Lambda^r$ and $B$ where we adopt the Einstein convention and sum over repeated indices and where $g^{\mu\nu}$ denotes the inverse matrix. It is possible to express $D$ invariantly [15] using a Bochner formalism. There exists a unique connection $\nabla$ on $V$ and a unique endomorphism $E$ of $V$ so that
\begin{equation}
D\phi = -(g^{\mu\nu} \phi_{,\mu\nu} + E\phi),
\end{equation}
where we use ‘,’ to denote the components of multiple covariant differentiation. Let $\Gamma$ be the Christoffel symbol of the Levi-Civita connection. We then have
\begin{equation}
(1.b) \quad \omega_\delta = \frac{1}{2}g_{\delta\delta}(A^{\nu} + g^{\mu\sigma} \Gamma_{\mu\nu}^{\sigma} \text{Id}),
\end{equation}
\begin{equation}
E = B - g^{\mu\nu}(\partial_{x^\mu} \omega_{\nu} + \omega_{\nu} \partial_{x^\mu} - \omega_{\sigma} \Gamma_{\nu}^{\sigma\mu}).
\end{equation}

1.2. Boundary conditions. We recall the formalism of Branson and Gilkey [7]. Let $\varepsilon > 0$ be the injectivity radius of the boundary $\partial M$ in $M$. Use the geodesic flow defined by the unit inward normal vector field $\partial_r$ to define a diffeomorphism between the collar $C_\varepsilon := \partial M \times [0, \varepsilon]$ and a neighborhood of the boundary in $M$ which identifies $\partial M \times \{0\}$ with $\partial M$. The curves $r \mapsto (y_0, r)$ for $r \in [0, \varepsilon]$ are unit speed geodesics perpendicular to the boundary and $r$ is the geodesic distance to the boundary.

Let $\chi \in C^\infty(\text{End}(V|_{\partial M}))$ satisfy $\chi^2 = 1$. Extend $\chi$ to the collar $C_\varepsilon$ so that $\nabla_{\partial_r} \chi = 0$. Let $\Pi_{\pm} := \frac{1}{2}(1 \pm \chi)$ be projections on the $\pm 1$ eigenbundles $V_{\pm}$ of $\chi$. Let $S \in \text{End}(V|_{\partial M})$ be an auxiliary endomorphism with $\Pi_+ S = S \Pi_+ = S$. If $\phi \in C^\infty(V)$, let $B = B(\chi, S)$ be the mixed boundary operator:
\begin{equation}
(1.c) \quad B\phi := \{\Pi_+ \phi\}|_{\partial M} \oplus \{\Pi_+(\nabla_{\partial_r} + S)\Pi_+ \phi\}|_{\partial M}.
\end{equation}

Let $D_B$ be the realization of $D$ with this boundary condition. We set $\Pi_+ = 0$ to define the Dirichlet boundary operator $D_D$ and we set $\Pi_- = 0$ and $S = 0$ to define the Neumann boundary operator $D_N$.

Operators of this type arise when studying the Gauss-Bonnet theorem for manifolds with boundary [15] and will play an important role in the analysis of Section 4. Let $\Delta = d\delta + \delta d$ be the Laplace-Beltrami operator on the space of smooth differential forms. Let $(y, r)$ be coordinates on the collar $C_\varepsilon$. Set
\begin{equation}
\Pi_+(dy_I) = dy_I \quad \text{and} \quad \Pi_-(dy_I \wedge dr) = dy_I \wedge dr
\end{equation}
and define the absolute boundary operator $B_a$ by taking
\begin{equation}
B_a\{\phi_I dy_I + \psi_J dy_J \wedge dr\} = \{(\partial_r \phi_I)dy_I\}|_{\partial M} \oplus \{\psi_J dy_J\}|_{\partial M}.
\end{equation}

Extend the second fundamental form $L$ (see Section 1.6 below) to act as a derivation on the space of differential forms. Then:
\begin{equation}
\nabla_{\partial_r} (f_I dy_I) = (\partial_r + L)(f_I dy_I) \quad \text{so} \quad B_a = B(\chi, -L).
\end{equation}

Let $\Delta_p$ be Laplacian on the space of smooth $p$-forms. If $M$ is a closed manifold, then $\ker(\Delta_p, B_a)$ is naturally isomorphic to the topological cohomology groups $H^p(M; \mathbb{C})$. Relative boundary conditions $B_r$ are defined similarly using the Hodge $*$ operator and one may identify $\ker(\Delta_p, B_r)$ with the relative cohomology groups $H^p(M, \partial M; \mathbb{C})$. 


1.3. **The heat equation.** For $t > 0$ and $\phi \in L^2(V)$, let $u = e^{-tD_B}\phi$ be the solution of the heat equation:

$$(\partial_t + D_B)u(x; t) = 0, \quad Bu = 0, \quad \lim_{t \downarrow 0} u(\cdot; t) = \phi(\cdot) \text{ in } L^2(V).$$

Let $dvol_M$ (resp. $dvol_{\partial M}$) be the Riemannian measure on $M$ (resp. $\partial M$). There is a smooth kernel $p_{D_B}(x, \tilde{x}; t)$ which gives the fundamental solution of the heat equation:

$$u(x; t) = \int_M p_{D_B}(x, \tilde{x}; t)\phi(\tilde{x}) \, dvol_M(\tilde{x}).$$

If $D_B$ is formally self-adjoint with respect to a fiber metric, we can take a spectral resolution $\{\lambda_{\nu}, \theta_{\nu}\}$ for $D_B$ where $\{\theta_{\nu}\}$ is a complete orthonormal basis for $L^2$ with $B\theta_{\nu} = 0$ and $D\theta_{\nu} = \lambda_{\nu}\theta_{\nu}$. We then have

$$p_{D_B}(x, \tilde{x}, t) = \sum_{\nu} e^{-t\lambda_{\nu}}\theta_{\nu}(x)\theta_{\nu}(\tilde{x}).$$

This series converges in the $C^\infty$ topology for $t > 0$. (There are some additional notational complexities in the bundle valued case we suppress in the interests of simplicity).

1.4. **Weighting functions.** We study the weighted heat trace $\text{Tr}_{L^2}(F e^{-tD_B})$. Previous work has concentrated on the smooth section - we review that work presently in Section 1.5. However, in this paper, we shall concentrate on a more general setting and consider the following class of smearing or weighting functions. Let $\alpha < 1$. Let $F$ be a smooth function on the interior of $M$. We assume that $r^\alpha F$ is smooth on the collar $C_r := \partial M \times [0, \varepsilon]$; the parameter $\alpha$ controls the growth (if $\alpha > 0$) or decay (if $\alpha < 0$) of $F$ near the boundary. We expand $F$ in a modified Taylor series near the boundary:

$$F(y, r) \sim r^{-\alpha}(F_0(y) + rF_1(y) + r^2F_2(y) + ...) \quad \text{where}$$

$$F_1(y) = \frac{1}{m!}(\partial_r)^m\{r^\alpha F\}|_{r=0}.$$

We remark that the assumption that $\alpha < 1$ ensures that $F \in L^1(M)$. With Dirichlet boundary conditions, the fundamental solution of the heat equation vanished to second order on the boundary and it was possible to consider the region $\alpha < 3$; logarithmic singularities then appeared when $\alpha = 1, 2$. This is not possible in the more general situation since the fundamental solution of the heat equation $p_{D_B}$ need not vanish on $\partial M$ and we must restrict to $\alpha < 1$ to ensure convergence.

1.5. **Heat trace asymptotics in the smooth setting.** Suppose $\alpha = 0$ so that $F$ is smooth on all of $M$; this is the case considered classically. Work of Greiner [17] and of Seeley [21] shows:

**Theorem 1.1.** Let $D$ be an operator of Laplace type on a compact Riemannian manifold $M$ with smooth boundary. Let $D_B$ be the realization of $D$ with respect to the mixed boundary conditions $B$ given in Equation (1.c). There is a full asymptotic series as $t \downarrow 0$ of the form:

$$\text{Tr}_{L^2}(F e^{-tD_B}) \sim (4\pi)^{-m/2}t^{-m/2} \sum_{n=0}^\infty t^n a_n(F, D)$$

$$+ (4\pi)^{-m/2}t^{-(m-1)/2} \sum_{\ell=0}^\infty t^{\ell/2} \nu_{\ell}^{bd}(F, D, B).$$
There are local invariants defined on $M$ and on $\partial M$ so that

$$a_n(F, D) = \int_M F(x) a_n(x, D) \, d\text{vol}_M(x),$$

$$a_{\ell}^{bd}(F, D, B) = \sum_{i=0}^{\ell} \int_{\partial M} F_i(y) a_{\ell,i}^{bd}(y, D, B) \, d\text{vol}_{\partial M}(y).$$

These invariants play an important role in index theory; they are also important in regularization results for mathematical physics [15, 19]. We remark that we have used a different indexing convention than is sometimes used in the literature and that we have handled the normalizing constants involving $4\pi$ slightly differently.

1.6. Local formulas. One has explicit combinatorial formulas for these invariants in the smooth setting; the interior invariants are known for $n \leq 5$ [1, 14, 22] and the boundary invariants are known for $\ell \leq 5$ [8, 16, 19, 23]. We introduce the requisite notation as follows.

Let $R_{ijkl}$ be the components of the curvature tensor of the Riemannian manifold; with our sign convention, $R_{1221} = +1$ on the unit sphere in $\mathbb{R}^2$. Let $\tau := R_{ijkl}$ be the scalar curvature of the manifold. Let $\rho_{mm} = R_{imm} i^n$ be the normal component of the Ricci tensor. Let $L_{ab}$ be the components of the second fundamental form on the boundary relative to an orthonormal frame $\{e_1, \ldots, e_{m-1}\}$ for the tangent bundle of $\partial M$; $L_{ab} = g(e_m, \nabla_{e_a} e_b)$. Relative to the coordinate frame, we have

$$L(\partial_{\mu}, \partial_{\nu}) = \Gamma_{\mu
u}^{\tau} = -\frac{1}{2} \partial_{\mu} g_{\nu} \quad \text{for} \quad 1 \leq \mu, \nu \leq m - 1.$$

Thus $L_{11} = 1$ for the unit disk in $\mathbb{R}^2$. Express $D = -(g^{\mu\nu} \nabla_{\nu} \nabla_{\mu} + E)$ where $\nabla$ and $E$ are as in Equation (1.b). Then:

**Theorem 1.2.**

1. $a_0(x, F, D) = \text{Tr}\{F \text{Id}\}$.
2. $a_1(x, F, D) = \text{Tr}\{6F E + F \tau \text{Id}\}$.

**Theorem 1.3.**

1. $a_0^{bd}(y, F, D, B) = \frac{\sqrt{\pi}}{4} \text{Tr}\{F_0 \chi\}$,
2. $a_1^{bd}(y, F, D, B) = \frac{\sqrt{\pi}}{6} \text{Tr}\{2F_0 L_{aa} \text{Id} + 3F_1 \chi + 12F_0 S\}$,
3. $a_2^{bd}(y, F, D, B) = \frac{\sqrt{\pi}}{8\pi} \text{Tr}\{F_0 [96\chi E + 16\tau \chi - 8\rho_{mm} \chi + L_{aa} L_{bb}(13\Pi_+ - 7\Pi_-) + L_{ab} L_{ab}(2\Pi_+ + 10\Pi_-) + 96L_{aa} S + 192S^2 - 12\chi_{a} \chi_{aa}] + F_1 [L_{aa} (6 \Pi_+ + 30 \Pi_-) + 96S] + 48F_2 \chi\}$.

1.7. The shifted asymptotic series. If $\alpha \neq 0$, there is a shift in the power of $t$ for the boundary invariants but the interior series discussed in Section 1.5 is unchanged. In [3], we used the calculus of pseudo-differential operators to establish the existence of an asymptotic series with Dirichlet boundary conditions. The same approach extends directly to the situation at hand to yield the following generalization of Theorem 1.1:

**Theorem 1.4.** Let $D$ be an operator of Laplace type on a compact Riemannian manifold $M$ with smooth boundary. Let $D_B$ be the realization of $D$ with respect to the mixed boundary conditions $B$ given in Equation (1.c). Let $\alpha < 1$. Let $F$ be smooth on the interior of $M$ and let $r^\alpha F$ be smooth near the boundary. There is a full asymptotic series as $t \downarrow 0$ of the form:

$$\text{Tr}_{L^2}(F e^{-tD_B}) \sim (4\pi)^{-m/2} t^{-m/2} \sum_{n=0}^{\infty} t^n a_n(F, D)$$

$$+ (4\pi)^{-m/2} t^{-(m-1)/2} \sum_{\ell=0}^{\infty} t^{(\ell-\alpha)/2} a_{\ell, \alpha}^{bd}(F, D, B).$$
Theorem 1.5. Computed computations and forms an essential starting point for the study of the general case: Dirichlet boundary conditions in [3]; the following result is a consequence of those.

1.8. Dirichlet Boundary Conditions. We computed the boundary invariants for Dirichlet boundary conditions in [3]; the following result is a consequence of those computations and forms an essential starting point for the study of the general case:

Theorem 1.6. Let $B$ define Dirichlet boundary conditions. Let $\kappa_\alpha := \frac{1}{2} \Gamma \left( 1 - \frac{\alpha}{2} \right)$. 

1.9. Heat trace asymptotics for mixed boundary invariants. The following is the main result of this paper. It generalizes Theorem 1.5 to general mixed boundary conditions:

Theorem 1.7. Let $\ell, \alpha$ be the invariants of Theorem 1.4.

Here is a brief outline to the paper. In Section 2, we express the invariants $a_{0,\alpha}^{bd}, a_{1,\alpha}^{bd},$ and $a_{2,\alpha}^{bd}$ in terms of geometrical quantities with 8 undetermined universal coefficients $\vartheta_{\alpha}$; we refer to Lemma 2.8 for details. In Section 3, we determine the...
Lemma 2.3. Let \( IS \) be a Riemannian manifold with smooth boundary. Let \( D \) be a product formula. The desired result then follows by equating terms in the asymptotic series.

2. The method of universal coefficients

2.1. Weighted homogeneity and dimensional analysis. We assign weight \( k \) to the \( k \)th derivative of the metric, weight \( k + 1 \) to the \( k \)th derivative of the connection form \( \omega \) of Equation (1.1b), and weight \( k + 2 \) to the \( k \)th derivative of the endomorphism \( E \) of Equation (1.1b). We also assign weight \( k \) to the \( k \)th tangential derivative of \( \chi \), weight \( k \) to \( F \), and weight \( k + 1 \) to the \( k \)th tangential derivative of \( S \). Thus, in particular, the components \( R_{ijkl} \) of the curvature tensor have weight 2 and the components \( L_{ab} \) of the second fundamental form have weight 1. Standard arguments using dimensional analysis shows establishes the following result; we omit details in the interests of brevity and instead refer to [3, 15, 19] where similar results were established:

Lemma 2.1. The local invariants \( a_{i,j,a,b}^{\ell,i} \) of Theorem 1.4 are weighted homogeneous of degree \( \ell - i \).

2.2. Orthogonal invariants. Weyl’s theory of orthogonal invariants [24] may be used to construct a spanning set for the space of invariants which are homogeneous of weight \( k \). One uses the metric to contract indices in pairs. We let \( \chi_{a} \) denote the components of tangential covariant differentiation of the tensor \( \chi \). Lemma 2.2 then leads to the following result; again, we omit details as by now the arguments are standard:

Lemma 2.2. There exist universal constants \( \{ \varrho_{a}^{i,\pm} \} \) so that:

1. \( a_{0}^{i,j,a}(y,F,D,\mathcal{B}) = \text{Tr}\{ F_{0}[\varrho_{a}^{0,+}\Pi_{+} + \varrho_{a}^{0,-}\Pi_{-}] \} \).
2. \( a_{1}^{i,j,a}(y,F,D,\mathcal{B}) = \text{Tr}\{ F_{1}[\varrho_{a}^{1,+}\Pi_{+} + \varrho_{a}^{1,-}\Pi_{-}] + F_{0}[L_{aa}(\varrho_{a}^{2,+}\Pi_{+} + \varrho_{a}^{2,-}\Pi_{-}) + \varrho_{a}^{3}S] \} \).
3. \( a_{2}^{i,j,a}(y,F,D,\mathcal{B}) = \text{Tr}\{ F_{2}[\varrho_{a}^{4,+}\Pi_{+} + \varrho_{a}^{4,-}\Pi_{-}] + F_{1}[L_{aa}(\varrho_{a}^{5,+}\Pi_{+} + \varrho_{a}^{5,-}\Pi_{-}) + \varrho_{a}^{6}S] + F_{0}[\rho_{mm}(\varrho_{a}^{7,+}\Pi_{+} + \varrho_{a}^{7,-}\Pi_{-}) + L_{aa}L_{bb}(\varrho_{a}^{8,+}\Pi_{+} + \varrho_{a}^{8,-}\Pi_{-}) + L_{ab}L_{ab}(\varrho_{a}^{9,+}\Pi_{+} + \varrho_{a}^{9,-}\Pi_{-}) + E(\varrho_{a}^{10,\pm}\Pi_{\pm} + \varrho_{a}^{10}\Pi_{\pm}) + \tau(\varrho_{a}^{11,\pm}\Pi_{\pm} + \varrho_{a}^{11,-}\Pi_{-} + \varrho_{a}^{12}S + \varrho_{a}^{13}S^{2} + \varrho_{a}^{14}S^{2} + \chi_{a}\chi_{a})] \} \).

2.3. Product formulas. The following is a useful observation.

Lemma 2.3. Let \( M = M_{1} \times M_{2} \) where \( M_{1} \) is a closed Riemannian manifold and \( M_{2} \) is a Riemannian manifold with smooth boundary. Let \( D_{1} \) be operators of Laplace type on \( M_{1} \) and let \( \mathcal{B} \) be a mixed boundary operator on \( M_{2} \) which we extend to \( M \). Then:

\[
a_{i,j,a,b}(x_{1},x_{2},D,\mathcal{B}) = \sum_{2k+j=t} a_{k}(x_{1},D_{1})a_{j,a,b}^{i}(x_{2},D_{2},\mathcal{B}) .
\]

Proof. Because the structures decouple, we have that \( e^{-tD_{a}} = e^{-tD_{1}}e^{-tD_{2}} \). Let \( F(x_{1},x_{2}) = F_{1}(x_{1})F_{2}(x_{2}) \). Then:

\[
\text{Tr}_{L^{2}(M)}\{ Fe^{-tD_{a}} \} = \text{Tr}_{L^{2}(M_{1})}\{ F_{1}e^{-tD_{1}} \} \text{Tr}_{L^{2}(M_{2})}\{ F_{2}e^{-tD_{2}} \}.
\]

The desired result then follows by equating terms in the asymptotic series.
2.4. Dimension shifting. A-priori, the constants in Lemma 2.2 depend on the dimension. Fortunately, that is not the case.

Lemma 2.4. The constants $\rho^{i,\pm}_\alpha$ are independent of the dimension $m$.

Proof. Let $M_1 := S^1$ and $D_1 := -\partial_\theta^2$ where $\theta$ is the usual periodic parameter on the circle. Since $M_1$ is a closed manifold, there are no boundary invariants. Since the structures are flat, $a_n(\theta, D_1) = 0$ for $n > 0$. By Theorem 1.2, $a_0(\theta, D_1) = 1$. Consequently, by Lemma 2.3,

$$a_{\ell,0,0(i\theta, y_2), D, B} = a_{\ell,0,0(i\theta, y_2), D, B}.$$ 

It now follows that $\rho^{i,\pm}_\alpha$ in dimension $m$ is equal to $\rho^{i,\pm}_\alpha$ in dimension $m + 1$. \qed

2.5. The coefficients of $E$ and of $\tau$. In the proof of Lemma 2.4, we applied Lemma 2.3 with $M_1 = S^1$. We take a product with $S^2$ to establish:

Lemma 2.5. We have the relations $\rho^{10,\pm}_\alpha = \rho^{0,\pm}_\alpha$ and $\rho^{11,\pm}_\alpha = \frac{1}{6} \rho^{0,\pm}_\alpha$.

Proof. Let $M_1 = S^2$ be the sphere of radius $\varepsilon$ in $\mathbb{R}^3$, let $\Delta$ be the scalar Laplacian on $S^2$, and let $D_1 := \Delta - \delta$. We have $\tau = \varepsilon^{-2}$ and $E = \delta$. We apply Theorem 1.2 to see

$$a_0(x, D) = 1 \quad \text{and} \quad a_1(x, D) = \delta + \frac{1}{6} \varepsilon^{-2}.$$ 

Let $D_2$ be an operator of Laplace type on $M_2$ and let $B$ be mixed boundary conditions. We form $M := M_1 \times M_2$ and $D = D_1 + D_2$. We omit terms which involve neither $\delta$ nor $\varepsilon$ and use Lemma 2.2 and Lemma 2.3 to see

$$a_{\ell,0,0(i\theta, y_2), D, B} = \text{Tr} \{ \delta(g^{10,\pm}_\alpha \Pi_+ + g^{10,\pm}_\alpha \Pi_-) + \varepsilon^{-2}(g^{11,\pm}_\alpha \Pi_+ + g^{11,\pm}_\alpha \Pi_-) + ... \}$$ 

$$= a_0(x_1, D_1)a_{\ell,0,0(i\theta, y_2), D_2, B} + a_1(x_1, D_1)a_{\ell,0,0(i\theta, y_2), D_2, B}$$ 

$$= \text{Tr} \{ (\delta + \frac{1}{6} \varepsilon^{-2})(g^{0,\pm}_\alpha \Pi_+ + g^{0,\pm}_\alpha \Pi_-) + ... \}.$$ 

Equating coefficients of $\delta$ and $\varepsilon^{-2}$ yields the desired identity. \qed

2.6. Degree shifting. In Lemma 2.5, we related $\rho^{10,\pm}_\alpha$ and $\rho^{11,\pm}_\alpha$ to $\rho^{0,\pm}_\alpha$. There are other relations of this form which are available:

Lemma 2.6.

1. If $1 \leq i \leq \ell$, then $a_{\ell,0,i} = a_{\ell-1,0,i-1}.$
2. $g^{1,\pm}_\alpha = g^{0,\pm}_\alpha.$
3. $g^{2,\pm}_\alpha = g^{0,\pm}_\alpha$ and $g^{3,\pm}_\alpha = g^{0,\pm}_\alpha.$
4. $g^{4,\pm}_\alpha = g^{0,\pm}_\alpha.$

Proof. Choose $\chi(r)$ to be a smooth cut-off function which is identically 0 near $r = \varepsilon$ and which is identically 1 near $r = 0$. Let $F^i_\alpha(y, r) := \chi(r) r^{-\alpha} r^i f(y)$; then we have $(F^i_\alpha)_j(y) = \delta^i_j f(y)$. We suppress the interior terms to express:

$$\text{Tr}_{L^2}(F^i_\alpha e^{-tD^2}) \sim (4\pi)^{-m/2} t^{-(m-1)/2} \sum_{\ell=0}^{\infty} t^{(\ell-\alpha)/2} \int_{\partial M} f(y)a_{\ell,0,i}^{bd}(y) \, d\text{vol}_{\partial M}(y).$$ 

Let $1 \leq i \leq \ell$. We have $F^i_\alpha = F^{i-1}_\alpha$. Equating powers of $t$ in the asymptotic expansions then yields the relation of Assertion (1). We apply Assertion (1) with $\ell = 1$ and $i = 1$ to derive Assertion (2); we apply Assertion (1) with $\ell = 2$ and $i = 1$ to derive Assertion (3); we apply Assertion (1) with $\ell = 2$ and $i = 2$ to see $\rho^{1,\pm}_\alpha = \rho^{0,\pm}_\alpha$ and then apply Assertion (2) (after replacing $\alpha$ by $\alpha - 1$) to establish Assertion (4). \qed
2.7. Relating pure Neumann and pure Dirichlet boundary conditions.

The following Lemma gives some relationships between the coefficients defining pure Neumann and pure Dirichlet boundary conditions.

Lemma 2.7. \( \rho^\alpha_{0,+} + \rho^\alpha_{0,-} = \rho^\alpha_{0,+} + \rho^\alpha_{0,-} = 0. \)

Proof. Let \( M \) be the upper hemisphere of the unit sphere of \( \mathbb{R}^{m+1} \). Let

\[
T(x_1, \ldots, x_{m+1}) = (x_1, \ldots, x_m, -x_{m+1})
\]

be an isometric involution of \( S^m \) whose fixed point set is the boundary of \( M \). Let

\[
E(\lambda, \Delta_{S^m}) = \{ \phi \in C^\infty(S^m) : \Delta_{S^m} \phi = \lambda \phi \}
\]

be the eigenspaces of the spherical Laplacian on \( S^m \). Let \( (T^* \phi)(x) := \phi(Tx) \). Since \( T \) is an isometry of \( S^m \), it commutes with the Laplacian and we may decompose

\[
E(\lambda, \Delta_{S^m}) = E^+(\lambda, \Delta_{S^m}) \oplus E^-(\lambda, \Delta_{S^m})
\]

into the \( \pm 1 \) eigenvalues of \( T^* \). It is then immediate that the elements of \( E^- \) satisfy Dirichlet boundary conditions while the elements of \( E^+ \) satisfy Neumann boundary conditions. If \( p_D \) and \( p_N \) are the fundamental solutions of the heat equation of the Laplacians \( \Delta_D \) and \( \Delta_N \) on \( M \) for Dirichlet and Neumann boundary conditions, respectively, we may then conclude, after allowing for the renormalization of the \( L^2 \) norms of the eigenvectors, that:

\[
p_D(x, x; t) + p_N(x, x; t) = 2p_{S^m}(x, x; t) \quad \text{for} \quad x \in M.
\]

Since \( S^m \) is a homogeneous space, there are constants \( \tilde{a}_n \) so that

\[
p_{S^m}(x, x; t) = p_{S^m}(t) \sim (4\pi t)^{-m/2} \sum_{n} t^n \tilde{a}_n.
\]

Let \( F \in C^\infty(S^m) \) satisfy \( T^* F = F \). Then

\[
\text{Tr}_{L^2}(Fe^{-t\Delta_D}) + \text{Tr}_{L^2}(Fe^{-t\Delta_N}) \sim (4\pi t)^{-m/2} \int_{S^m} F(x) \text{dvol}(x) \cdot \sum_{n=0}^{\infty} t^n \tilde{a}_n.
\]

We may suppose \( \alpha \notin \mathbb{Z} \). Since there are no \( t^{-(m-1)/2}(t-\alpha)/2 \) terms in the asymptotic expansion of the right hand side of the above display, the boundary terms must vanish. The two relations of the Lemma now follow. \( \square \)

The coefficients \( \rho^\alpha_{0,-} \) may be evaluated using Theorem 1.5. After changing notation appropriately to simplify the relevant formulas, we summarize the results of this section in the following result:

Lemma 2.8. There exist universal constants \( \{ \vartheta^\alpha_n \} \) so that:

1. \( \alpha_{0,\alpha}(y, F, D, B) = \kappa_{\alpha-1} \text{Tr} \{ F_0[\Pi_+ - \Pi_-] \} dy. \)

2. \( \alpha_{1,\alpha}(y, F, D, B) = \kappa_{\alpha-1} \text{Tr}[F_1[\Pi_+ - \Pi_-] + F_0 LA_\alpha[\vartheta^1_{0,1} + \frac{\alpha-4}{2(\alpha-2)} \Pi_- + \vartheta^2_{0,0} F_0 S] dy. \)

3. \( \alpha_{2,\alpha}(y, F, D, B) = \kappa_{\alpha-2} \text{Tr}[F_2[\Pi_+ - \Pi_-] + F_1[L_\alpha[\vartheta^1_{0,1} + \frac{\alpha-5}{2(\alpha-4)} \Pi_- + \vartheta^2_{0,1} S] + F_0[-\frac{1}{6} \rho_{mm}(\Pi_+ - \Pi_-) + L_\alpha L_{ab}[\vartheta^a_{0,0} \Pi_+ - \frac{(\alpha-7)}{2(\alpha-6)} \Pi_-] + L_{ab} L_{ab}[\vartheta^2_{0,0} \Pi_+ + \frac{\alpha-3}{2(\alpha-4)} \Pi_-] + \vartheta^0_{0,0} L_{aa} S + \vartheta^0_{0,1} S^2 + \frac{1}{3(1-\alpha)}(\tau + 6E)(\Pi_+ - \Pi_-) + \vartheta^0_{0,0} \chi_{\alpha,\alpha,\alpha}] \} dy. \)
3. Special case computations on the interval

We note that \( s\Gamma(s) = \Gamma(s + 1) \). Since \( \kappa_\alpha := \frac{1}{\Gamma(\frac{N-2}{2})} \), we have the following identities which we note for future reference:

\[
(3.a) \quad \frac{\kappa_\alpha}{\kappa_{\alpha-2}} = \frac{2}{1 - \alpha}, \quad \frac{\kappa_{\alpha+1}}{\kappa_{\alpha-1}} = -\frac{2}{\alpha}.
\]

**Lemma 3.1.** We have that \( \vartheta^{2}_{\alpha} = \frac{4}{1 - \alpha} \) and that \( \vartheta^{7}_{\alpha} = \frac{8}{(2 - \alpha)(1 - \alpha)} \).

**Proof.** Let \( M = [0, 1] \). Let \( F_a = r^{-\alpha} \) near \( x = 0 \) and \( F_a = 0 \) near \( x = 1 \). Let \( 0 \neq b \in \mathbb{R} \). We form

\[
A := \partial_x + b, \quad A^* := -\partial_x + b, \quad D := A^*A = AA^* = -(\partial_{xx}^2 - b^2).
\]

Let \( D_D \) and \( D_R \) be the realizations, respectively, of \( D \) with respect to Dirichlet boundary conditions and Robin boundary conditions with \( S(0) = b \) and \( S(1) = -b \). Thus we may identify \( B_R \phi = A\phi|_{\partial M} \). We integrate by parts to derive the Green’s formula:

\[
\int_0^1 \left\{ (A^*Au, v) - (u, A^*Av) \right\}(x)dx = \left\{ -(Au, v) + (u, Av) \right\}|_0^1.
\]

This vanishes if \( u = v = 0 \) on \( \partial M \) or if \( Au = Av = 0 \) on \( \partial M \). Consequently both \( D_D \) and \( D_R \) are self-adjoint. If \( D\phi = 0 \), then \( \phi'' = b^2\phi \) so \( \phi = a_0 e^{bx} + a_1 e^{-bx} \). Thus \( \phi \) satisfies Dirichlet boundary conditions means \( \phi = 0 \) and thus \( \ker(D_D) = \{0\} \). Let

\[
\{\theta_{\nu}, \lambda_{\nu}\} = \{\sin(\pi \nu x), 2\pi^2 \nu^2 + b^2\} \quad \text{for} \quad \nu = 1, 2, ...
\]

be a spectral resolution for \( D_D \). We have similarly that \( \ker(D_R) = e^{-bx} \cdot \mathbb{R} \) and that \( \{A^*\theta_{\nu}/\sqrt{\lambda_{\nu}}, \lambda_{\nu}\} \) is a spectral resolution of \( D_R \) on \( \ker(D_R)^\perp \). Let \( p_D \) and \( p_R \) be the fundamental solutions of the heat equation for \( D_D \) and \( D_R \), respectively. We compute:

\[
\partial_t p_D(x, x; t) = -\sum_{\nu} \lambda_{\nu} e^{-t\lambda_{\nu}} \theta_{\nu}(x)^2 = -\sum_{\nu} e^{-t\lambda_{\nu}} D\theta_{\nu} \cdot \theta_{\nu},
\]

\[
\partial_t p_R(x, x; t) = -\sum_{\nu} e^{-t\lambda_{\nu}} A^*\theta_{\nu} \cdot A^*\theta_{\nu}.
\]

This then yields the identity:

\[
2\partial_t \{p_D(x, x; t) - p_R(x, x; t)\} = -2\sum_{\nu} e^{-t\lambda_{\nu}} \{D\theta_{\nu} \cdot \theta_{\nu} - A^*\theta_{\nu} \cdot A^*\theta_{\nu}\}
\]

\[
= 2\sum_{\nu} e^{-t\lambda_{\nu}} \{(\theta_{\nu}'' \theta_{\nu} - b^2 \theta_{\nu}^2) + (\theta_{\nu}' \theta_{\nu}' - 2b \theta_{\nu}' \theta_{\nu} + b^2 \theta_{\nu}^2)\}
\]

\[
= \partial_x(\partial_x - 2b)p_D(t; x, x).
\]

We suppose \( \alpha << 0 \) and \( \alpha \neq \mathbb{Z} \) to ensure convergence and to ensure that the interior and the boundary terms do not interact; the general case then follows by analytic continuation. We integrate by parts to see:

\[
2\partial_t \{\text{Tr}_{L^2}\{F_\alpha e^{-tD_D}\} - \text{Tr}_{L^2}\{F_\alpha e^{-tD_R}\}\}
\]

\[
= \int_M 2F_\alpha \partial_t(p_D(x, x; t) - p_R(x, x; t)) \text{dvol}(x)
\]

\[
= \int_M F_\alpha \partial_x(\partial_x - 2b)p_D(x, x; t) \text{dvol}(x)
\]

\[
= \int_M (F_\alpha'' + 2bF_\alpha')p_D(x, x; t) \text{dvol}(x)
\]

\[
= \text{Tr}_{L^2}\{\{(F_\alpha'' + 2bF_\alpha')e^{-tD_D}\} \}.
\]
Notice that $\partial_\tau F_\alpha = -\alpha F_{\alpha+1}$ and $\partial_\tau^2 F_\alpha = \alpha(\alpha+1)F_{\alpha+2}$ near the boundary of $M$. Since the underlying operator is the same, the difference of the interior terms cancel and we have:

$$2\partial_\tau [\text{Tr}_{L^2}(F_\alpha e^{-tD_\alpha}) - \text{Tr}_{L^2}(F_\alpha e^{-tD_\beta})]$$

$$\sim \sum_\ell (\ell - \alpha)t^{(\ell - \alpha - 2)/2}\{a_{\ell,\alpha}(F_\alpha, D, B_\alpha) - a_{\ell,\alpha}(F_\alpha, D, B_\beta)\}$$

$$= \text{Tr}_{L^2}\{(F''_\alpha + 2bF'_\alpha)e^{-tD_\alpha}\}$$

$$\sim \alpha(\alpha+1)\sum_j t^{(j-\alpha-2)/2}a_{j,\alpha+2}(F_{\alpha+2}, D, B_\beta)$$

$$-2ab\sum_k t^{(k-\alpha-1)/2}a_{k,\alpha+1}(F_{\alpha+1}, D, B_\beta).$$

We equate the coefficients in the asymptotic expansions to see

$$(\ell - \alpha)\{a_{\ell,\alpha}^b(F_\alpha, D, B_\beta) - a_{\ell,\alpha}^b(F_\alpha, D, B_\beta)\}$$

$$= \alpha(\alpha+1)a_{1,\alpha+2}(F_{\alpha+2}, D, B_\beta) - 2ab(a_{0,\alpha+1}(F_{\alpha+1}, D, B_\beta).$$

This leads to the identity:

$$(1 - \alpha)\{-\kappa_{\alpha-1}^b\}b = 2a\{\kappa_{\alpha+1}\}b.$$

We apply Equation (3.a) to see:

$$\vartheta_\alpha^2 = -2a\frac{\kappa_{\alpha+1}}{1 - \alpha} = \frac{-2a}{1 - \alpha} = \frac{4}{1 - \alpha}.$$

Finally, we take $\ell = 2$ to see:

$$(2 - \alpha)\{a_{2,\alpha}^b(F_\alpha, D, B_\beta) - a_{2,\alpha}^b(F_\alpha, D, B_\beta)\}$$

$$= \alpha(\alpha+1)a_{2,\alpha+2}(F_{\alpha+2}, D, B_\beta) - 2ab(a_{1,\alpha+1}(F_{\alpha+1}, D, B_\beta).$$

This gives rise to the identity:

$$(2 - \alpha)\kappa_{\alpha-2}b^2\left\{\frac{4}{1 - \alpha} - \vartheta_\alpha^2\right\} = \kappa_{\alpha}b^2\left(\alpha(\alpha+1)\frac{6}{3(1 - \alpha + 2)}\right).$$

We use Equation (3.a) to solve this identity for $\vartheta_\alpha^7$ to see:

$$\vartheta_\alpha^7 = \frac{4}{1 - \alpha} - \frac{1}{2 - \alpha} - \frac{2}{\alpha(\alpha+1)} - \frac{2}{1 - \alpha} = \frac{8}{(1 - \alpha)(2 - \alpha)}.$$

4. Absolute and relative boundary conditions

We establish the following result by generalizing the 1-dimensional construction of Lemma 3.1 to the 2-dimensional setting.

**Lemma 4.1.** We have that

1. $\vartheta_\alpha^6 = \frac{a_\alpha^2}{2a_\alpha(1 - (\alpha - 1))}$.
2. $(1 - \alpha)^2\{\vartheta_\alpha^6 - 8\vartheta_\alpha^8\} - 2\alpha(\alpha+1)\{\vartheta_\alpha^4 + \vartheta_\alpha^5\} = \frac{4(\alpha-1)}{\alpha-2} + \frac{\alpha(\alpha+1)(\alpha-1)}{4(\alpha-4)}.$
Proof. We modify an argument from McKean and Singer [20]. Let \( M \) be a Riemann surface. Let \( \Delta_a \) denote the Laplacian with absolute boundary conditions \( B_a \) as discussed above. Let \( \Lambda^0 := \Lambda^0 \oplus \Lambda^2 \) and let \( \Lambda^a := \Lambda^1 \). Let \( \{ \theta_\nu, \lambda_\nu \} \) be a spectral resolution of \( \Delta_a^{ev} \) on \( \ker(\Delta_a^{ev}) \). Then

\[
\{(d + \delta_\nu)\theta_\nu / \sqrt{\lambda_\nu}, \lambda_\nu \}
\]

is a spectral resolution of \( \Delta_a^{ad} \) on \( \ker(\Delta_a^{ad}) \). One has:

\[
2\partial_t \left\{ p_{\Delta_a^{ev}}(x,x,t) - p_{\Delta_a^{ad}}(x,x,t) \right\}
= 2\partial_t \sum_{\nu : \lambda_\nu > 0} e^{-t\lambda_\nu} \left\{ (\theta_\nu, \theta_\nu) - ((d + \delta)\theta_\nu), (d + \delta)\theta_\nu / \lambda_\nu \right\}
\]

\[
= -2 \sum_{\nu : \lambda_\nu > 0} e^{-t\lambda_\nu} \left\{ \lambda_\nu (\theta_\nu, \theta_\nu) - ((d + \delta)\theta_\nu), (d + \delta)\theta_\nu \right\}
\]

\[
= -2 \sum_{\nu : \lambda_\nu > 0} e^{-t\lambda_\nu} \left\{ (\Delta_a^{ev}\theta_\nu, \theta_\nu) - ((d + \delta)\theta_\nu, (d + \delta)\theta_\nu) \right\}.
\]

Now comes a crucial point. The Laplacian \( \Delta_a \) decomposes as the direct sum of two scalar operators on \( \Lambda^a \). Let \( \Theta \) be a function. Then

\[
2\Delta_a \Theta = \Theta e^1 \wedge e^2,
\]

\[
2(\delta + \delta) \Theta = 2\delta \Theta = -2\theta_1 e^2 + 2\theta_2 e^1,
\]

\[
2\{(\Delta_a \Theta, \Theta) - ((d + \delta)\Theta, (d + \delta)\Theta)\} = -2\theta_1 \Theta - 2\theta_2 \Theta = \Delta_a (\Theta^2)
\]

\[
= \Delta_0 (\Theta, \Theta).
\]

Next, let \( \Theta = \theta e^1 \wedge e^2 \) be a 2-form. We compute:

\[
2\Delta_a \Theta = -2\Theta e^1 \wedge e^2,
\]

\[
2(\delta + \delta) \Theta = 2\delta \Theta = -2\theta_1 e^2 + 2\theta_2 e^1,
\]

\[
2\{(\Delta_a \Theta, \Theta) - ((d + \delta)\Theta, (d + \delta)\Theta)\} = -2\theta_1 \Theta - 2\theta_2 \Theta = \Delta_a (\Theta^2)
\]

\[
= \Delta_0 (\Theta, \Theta).
\]

Consequently we have

\[
-2 \sum_{\nu : \lambda_\nu > 0} e^{-t\lambda_\nu} \left\{ (\Delta_a^{ev}\theta_\nu, \theta_\nu) - ((d + \delta)\theta_\nu, (d + \delta)\theta_\nu) \right\} = - \sum_{\nu : \lambda_\nu > 0} e^{-t\lambda_\nu} \Delta_0 \left\{ (\theta_\nu, \theta_\nu) \right\}.
\]

We suppose \( \alpha < 0 \) and \( \alpha \neq \mathbb{Z} \). We may then integrate by parts to see:

\[
\sum_{\ell} (\ell - \alpha - 1) t^{(\ell - \alpha - 1)/2 - 1} \left\{ a^{bd}_{\ell, \alpha}(F_\alpha, \Delta^{ev}, B_a) - a^{bd}_{\ell, \alpha}(F_\alpha, \Delta^{ad}, B_a) \right\}
\]

\[
\sim - \sum_k a^{bd}_{\ell, \alpha + 2}(\Delta F_\alpha, \Delta^{ev}, B_a) t^{(k - \alpha - 2)/2}.
\]

Equating terms in the asymptotic expansion then yields

\[
(\ell - \alpha - 1) \left\{ a^{bd}_{\ell, \alpha}(F_\alpha, \Delta^{ev}, B_a) - a^{bd}_{\ell, \alpha}(F_\alpha, \Delta^{ad}, B_a) \right\}
\]

\[
= -a^{bd}_{\ell, \alpha + 2}(\Delta F_\alpha, \Delta^{ev}, B_a).
\]

We specialize to the case \( M \) is the disk of radius 1 in \( \mathbb{R}^2 \). Introduce the usual coordinates \((R, \theta)\) so that \( x = R \cos \theta \) and \( y = R \sin \theta \); the distance to the boundary is then given by \( r = 1 - R \). We have

\[
F_\alpha = (1 - R)^{-\alpha},
\]

\[
\Delta = -\partial_R^2 - R^{-1} \partial_R - R^{-2} \partial_\theta^2,
\]

\[
\Delta F_\alpha = -\alpha(\alpha + 1) F_{\alpha + 2} - \alpha R^{-1} F_{\alpha + 1}.
\]
We have that $R^{-1} = (1 - r)^{-1} = 1 + r + \ldots$. Since only the first 3 terms in the Taylor series expansion of $\Delta F_\alpha$ play a role for $\ell = 1, 2$, we obtain the identity:

$$
(\ell - \alpha - 1)\{a_{\ell,\alpha}^{bd}(F_\alpha, \Delta^{ev}, B_\alpha) - a_{1,\alpha}^{bd}(F_\alpha, \Delta^{od}, B_\alpha)\} = \alpha a_{\ell,\alpha+2}\{(\alpha + 1)F_{\alpha+2} + F_{\alpha+1} + F_\alpha, \Delta^{ev}, B_\alpha\} \quad \text{for} \quad \ell = 1, 2.
$$

We first set $\ell = 1$. After canceling the factors of $-\alpha$ from both sides of the equation we get the relation

$$
a_{1,\alpha}^{bd}(F_\alpha, \Delta^{ev}, B_\alpha) - a_{1,\alpha}^{bd}(F_\alpha, \Delta^{od}, B_\alpha) = -a_{1,\alpha+2}\{(\alpha + 1)F_{\alpha+2} + F_{\alpha+1} + F_\alpha, \Delta^{ev}, B_\alpha\}.
$$

The desired result now follows.

In this section, we will use the pseudo-differential calculus to complete the calculation. Only the invariant $\chi \alpha \chi \alpha$ genuinely involves a vector valued context; it will be determined by Lemma 4.1 once the remaining coefficients are determined. Thus we will restrict our attention to the case in which

$$
D = \Delta_M = -g^{\mu\nu}\partial_\mu \partial_\nu + b^\mu \partial_\mu,
$$
is the scalar Laplacian. We shall work with Robin boundary conditions.

We begin by reviewing some fairly standard material. Let \( \tilde{\alpha} = (\alpha_1, ..., \alpha_m) \) be a multi-index. We set:

\[
|\tilde{\alpha}| = \alpha_1 + ... + \alpha_m, \quad \tilde{\alpha}! = \alpha_1! \times ... \times \alpha_m!,
\]

\[
x^\tilde{\alpha} = x_1^{\alpha_1} \times ... \times x_m^{\alpha_m}, \quad D^\tilde{\alpha}_x = \left( \frac{\partial}{\partial x_1} \right)^{\alpha_1} \times ... \times \left( \frac{\partial}{\partial x_m} \right)^{\alpha_m},
\]

We apologize in advance for the slight notational confusion involved with using \( \alpha \) to control the growth of \( F \) and also to using \( \tilde{\alpha} \) as a multi-index. We use the metric to raise and lower indices; \(^a\)_ will denote partial differentiation.

We refer to [10, 12, 15, 17, 21] for additional material about pseudo-differential operators. We wish to construct the resolvent \( (\Delta - \lambda)^{-1} \) for large \( \lambda \). We first suppose \( M \) is a closed manifold. In the evaluation of the heat equation asymptotics homogeneity properties of symbols are relevant and it turns out that collecting terms according to homogeneity is useful; the complex parameter \( \lambda \) has weight 2.

Expand the symbol of \( \Delta_M - \lambda \) in the form \( a_2(x, \xi, \lambda) + a_1(x, \xi) + a_0(x, \xi) \) where:

\[
a_2(x, \xi, \lambda) = g^{\mu \nu} \xi_\mu \xi_\nu - \lambda \equiv |\xi|^2 - \lambda,
\]

\[
a_1(x, \xi, \lambda) = \sqrt{-1} b^\sigma \xi_\sigma, \quad \text{and} \quad a_0(x, \xi, \lambda) = 0.
\]

We formally expand the symbol of the resolvent in an asymptotic series:

\[
\sigma((\Delta_M - \lambda)^{-1})(x, \xi, \lambda) \sim \sum_{l=0}^{\infty} q_{-2-l}(x, \xi, \lambda).
\]

The \( q_k \) are then determined by the recursive relations:

\[
1 = a_2(x, \xi, \lambda) q_{-2}(x, \xi, \lambda),
\]

\[
0 = \sum_{k=2+1+|\tilde{\alpha}|} \frac{1}{\tilde{\alpha}!} [d^\tilde{\alpha}_{x}(a_{2}(x, \xi, \lambda))] \cdot [D^\tilde{\alpha}_x q_{-2-l}(x, \xi, \lambda)] \quad \text{for} \ k \geq 1.
\]

To complete the proof of Theorem 1.6, we must examine \( q_{-2}, \ q_{-3}, \) and \( q_{-4} \). We summarize the facts we shall need and omit details in the interests of brevity – the fact that \( \Delta_M \) is scalar plays an essential role. Let greek indices range from 1 through \( m \). We have:

\[
q_{-2} = a_2^{-1},
\]

\[
q_{-3} = -a_1 q_{-2}^2 + c_{-3,3} q_{-2},
\]

\[
q_{-4} = -a_0 q_{-2}^3 + c_{-4,3} q_{-2}^2 + c_{-4,4} q_{-2} + c_{-4,5} q_{-2}^5,
\]

where

\[
c_{-3,3} = -\sqrt{-1} (\partial_\xi^\nu a_2)(\partial_\mu^\nu a_2),
\]

\[
c_{-4,3} = a_1^2 - \sqrt{-1} (\partial_\xi^\nu a_2)(\partial_\mu^\nu a_2) - \sqrt{-1} (\partial_\xi^\nu a_2)(\partial_\nu^\xi a_2) - \frac{1}{4} (\partial_\xi^\nu a_2)(\partial_\nu^\mu a_2)(\partial_\mu^\sigma a_2),
\]

\[
c_{-4,4} = -3a_1 c_{-3,3} + \sqrt{-1} (\partial_\xi^\nu a_2)(\partial_\xi^\sigma c_{-3,3}) + (\partial_\xi^\nu a_2)(\partial_\xi^\nu a_2)(\partial_\mu^\nu a_2),
\]

\[
c_{-4,5} = 3c_{-3,3}^2.
\]
One has that:

\[
q_j(x, \xi, \lambda) = \frac{1}{(1+|\xi|^2-\lambda)^j} \left( -b^\mu b^\sigma \xi^\mu \xi^\nu - b^\nu b^\sigma \xi^\mu \xi^\nu + 2b^\mu g^\nu_\sigma \xi^\nu \xi_\sigma - g^\mu_\nu g^\nu_\sigma \xi^\mu \xi_\sigma \right) \\
+ \frac{1}{(1+|\xi|^2-\lambda)^{(j+1)}} \left( -6b^\mu g^\nu_\sigma g^\nu_\beta \xi^\mu \xi^\nu \xi^\beta \xi^\sigma + 4g^\nu_\sigma g^\nu_\beta g^\nu_\gamma \xi^\mu \xi^\nu \xi^\beta \xi^\gamma \\
+ 4g^\nu_\sigma g^\nu_\beta g^\nu_\gamma \xi^\mu \xi^\nu \xi^\beta \xi^\gamma \right) \\
+ \frac{1}{(1+|\xi|^2-\lambda)^{(j+2)}} \left( -12g^\nu_\sigma g^\nu_\beta g^\nu_\gamma \xi^\mu \xi^\nu \xi^\beta \xi^\gamma \right). 
\]

If the manifold has a boundary the expansion (5.5) has to be augmented by a boundary correction. To formulate the conditions to be satisfied by the boundary correction we expand about \( r = 0 \). One may express the metric on the collar \( C_r \) in the form

\[
d\tilde{s}^2 = g_{\sigma\rho}(y, r) dy^\sigma \circ dy^\rho + dr^2.
\]

The coordinate \( y \) locally parametrizes the boundary, and \( r \) is the geodesic distance to the boundary, so \( x = (y, r) \). A tilde above any quantity will indicate that it is to be evaluated at the boundary, that is at \( r = 0 \). Furthermore, we use \( \xi = (\omega, \tau) \).

We find

\[
\Delta_M - \lambda = \sum_{k=0}^{\infty} \frac{1}{k!} r^k \left( \sum_{|\alpha| \leq 2} \frac{\partial^{k}}{\partial r^{k}} a_{\alpha}(y, r) \right) D_{y, r}^\alpha |_{r=0} D_{\tilde{y}, \tilde{r}}^\alpha,
\]

with the notation

\[
D_{y, r}^\alpha = \left( \prod_{i=1}^{\alpha} D_{y, r}^{a_i} \right) D_{\tilde{r}}^{p_m}.
\]

Introducing

\[
a_j(y, r, \omega, D_r, \lambda) = \begin{cases} 
\sum_{|\alpha|=2} a_{\alpha}(y, r) \left( \prod_{i=1}^{\alpha} \omega_i^{a_i} \right) D_{r}^{p_m} & \text{for } j = 0, 1, \\
\sum_{|\alpha|=2} a_{\alpha}(y, r) \left( \prod_{i=1}^{\alpha} \omega_i^{a_i} \right) D_{r}^{p_m} - \lambda & \text{for } j = 2
\end{cases}
\]

we define the partial symbol

\[
\sigma'(\Delta_M - \lambda) = \sum_{k=0}^{\infty} \frac{1}{k!} r^k \sum_{j=0}^{2} \frac{\partial^k}{\partial r^k} a_j(y, r, \omega, D_r, \lambda) |_{r=0}.
\]

As it turns out, the symbols

\[
a_j(y, r, \omega, D_r, \lambda) = \sum_{l=0}^{2} \sum_{k=0}^{\infty} \frac{1}{k!} r^k \frac{\partial^k}{\partial r^k} a_l(y, r, \omega, D_r, \lambda) |_{r=0}
\]

have suitable homogeneity properties and using these symbols we write

\[
\sigma'(\Delta_M - \lambda) = \sum_{j=-\infty}^{2} a_j(y, r, \omega, D_r, \lambda).
\]

We write the symbol of the resolvent as

\[
\sigma((\Delta_M - \lambda)^{-1})(y, r, \omega, \tau, \lambda) = \sum_{j=0}^{\infty} q_{-j}(y, r, \omega, \tau, \lambda) - e^{-\sqrt{-1}r \tau} \sum_{j=0}^{\infty} h_{-j}(y, r, \omega, \tau, \lambda),
\]

(5.5)
where the second term is the boundary correction. The factor $e^{-\sqrt{-\tau r}}$ appears because the operator constructed from these terms is the $Op'(h)$ in [21], and $Op'(h) = Op(h e^{-\sqrt{-\tau r}})$. This shows

$$\sigma'(\Delta_M - \lambda) \circ \sum_{j=0}^{\infty} h_{-j-2}(y, r, \omega, \lambda) = 0.$$ 

Here $\circ$ denotes the symbol product on $\mathbb{R}^{m-1}$. Analogously to Equation (5.5.b) this equation leads to the differential equations

$$0 = a^{(2)}(y, r, \omega, D_r, \lambda) h_{-2}(y, r, \omega, \tau, \lambda),$$

$$0 = a^{(2)}(y, r, \omega, D_r, \lambda) h_{-2-j}(y, r, \omega, \tau, \lambda) + \sum_{j=1}^{m} \frac{1}{\alpha^j} [D^2_{\omega} a^{(2)}(y, r, \omega, D_r, \lambda)] \left[ (\sqrt{-\tau D^y})_{a} h_{-2-j}(y, r, \omega, \tau, \lambda) \right].$$

For the present considerations we need $h_{-2-j}$ for $j = 0, 1, 2$, and we have more explicitly (repeated letters $a, b, c, \ldots$ run over tangential coordinates $\{1, 2, \ldots, m-1\}$)

$$0 = a^{(2)}(y, r, \omega, D_r, \lambda) h_{-2}(y, r, \omega, \tau, \lambda),$$

$$0 = a^{(2)}(y, r, \omega, D_r, \lambda) h_{-4}(y, r, \omega, \tau, \lambda) + a^{(0)}(y, r, \omega, D_r, \lambda) h_{-2}(y, r, \omega, \tau, \lambda) + a^{(1)}(y, r, \omega, D_r, \lambda) h_{-2-j}(y, r, \omega, \tau, \lambda) + \sum_{j=1}^{m} \frac{1}{\alpha^j} [D^2_{\omega} a^{(2)}(y, r, \omega, D_r, \lambda)] \left[ (\sqrt{-\tau D^y})_{a} h_{-2-j}(y, r, \omega, \tau, \lambda) \right].$$

The relevant equations for $a^{(i)}(y, r, \omega, D_r, \lambda)$, $i = 0, 1, 2$ are

$$a^{(2)}(y, r, \omega, D_r, \lambda) = a_2(y, r, \omega, D_r, \lambda)|_{r=0} + \sum_{j=1}^{m} \frac{1}{\alpha^j} a^{(2)}(y, r, \omega, D_r, \lambda)|_{r=0}$$

$$0 = a^{(1)}(y, r, \omega, D_r, \lambda) = r(\partial_r a_1(y, r, \omega, D_r, \lambda)|_{r=0} + a_1(y, r, \omega, D_r, \lambda)|_{r=0})$$

$$0 = a^{(0)}(y, r, \omega, D_r, \lambda) = \frac{1}{2} r^2 (\partial^2_r a_2(y, r, \omega, D_r, \lambda)|_{r=0} + r(\partial_r a_1(y, r, \omega, D_r, \lambda)|_{r=0})$$

The differential equations have to be augmented by a growth condition

$$(5.d) \quad h_{-2-j}(y, r, \omega, \tau, \lambda) \to 0 \quad \text{as} \quad r \to \infty,$$

and an initial condition corresponding to the Robin boundary condition

$$B \phi = (\partial_r + S) \phi$$
considered here. The first few boundary symbols satisfy
\[ \partial_t h_{-2}(y, r, \omega, \tau, \lambda)|_{r=0} = \sqrt{-1} \tau q_{-2}(y, r, \omega, \tau, \lambda)|_{r=0} , \]
\[ \partial_t h_{-3}(y, r, \omega, \tau, \lambda)|_{r=0} = -Sh_{-2}(y, r, \omega, \tau, \lambda)|_{r=0} + Sq_{-2}(y, r, \omega, \tau, \lambda)|_{r=0} + \partial_t q_{-2}(y, r, \omega, \tau, \lambda)|_{r=0} , \]
\[ \partial_t h_{-4}(y, r, \omega, \tau, \lambda)|_{r=0} = -Sh_{-3}(y, r, \omega, \tau, \lambda)|_{r=0} + Sq_{-3}(y, r, \omega, \tau, \lambda)|_{r=0} + \partial_t q_{-3}(y, r, \omega, \tau, \lambda)|_{r=0} . \]
(5.e)

Once the symbols \( h_{-2-j} \) have been determined, their contribution to the asymptotics of the trace of the heat kernel follows from multiple integration. As before, we suppose \( r^\alpha F \in C^\infty(C_\varepsilon) \). The contribution reads
\[ \sum_{l=0}^{\infty} t^{\frac{\Lambda - m}{2}} \int_{\partial M} \eta^{+}_{\frac{1}{2}}(y, F, \Delta M) dy \]
with
\[ \eta^{+}_{\frac{1}{2}}(y, F, \Delta M) = \frac{1}{(2\pi)^{m+1}} \sum_{j+k=l} \int_{\mathbb{R}^{m-1}} d\omega \int_{-\infty}^{\infty} ds \]
(5.f) \times \int_{\gamma} d\tau e^{\sqrt{-1} \tau \bar{\varepsilon}} \left( - \int_{\gamma} d\tau e^{-\sqrt{-1} \tau \bar{\varepsilon}} h_{-2-j}(y, \bar{\varepsilon}, \omega, \tau, -\sqrt{-1} s) \right)^{r^{k-\alpha}} F_k(y) ,
where \( \gamma \) is anticlockwise enclosing the poles of \( h_{-2-j} \) in the lower half-plane. The integral with respect to \( s \) is the contour integral transforming the resolvent to the heat kernel. Note that from (5.c) the contribution to the heat kernel is minus the above.

As will become clear in the following, with \( \Lambda = \sqrt{|\omega|^2 + \sqrt{-1} s} \), we need integrals of the type
\[ T^{k\ell jn}_{ab...} \equiv \int_{\mathbb{R}^{m-1}} d\omega \int_{-\infty}^{\infty} ds \int_{\gamma} d\tau e^{\sqrt{-1} \tau \bar{\varepsilon}} \left( - \int_{\gamma} d\tau e^{-\sqrt{-1} \tau \bar{\varepsilon}} \right)^{r^{k} \bar{\varepsilon}^{l-\alpha}} \frac{\omega_\ell \omega_j \omega_b \ldots}{\Lambda^{l} (\tau^2 + \Lambda^2)^{n}} e^{-\bar{\varepsilon} \Lambda} . \]
The \( \tau \) integration can be done using
\[ \frac{d\tau}{(\tau^2 + \Lambda^2)^{n}} = \frac{1}{(n-1)!} \left( \frac{\sqrt{-1}}{\Lambda} \right)^{n} \int_{\mathbb{R}^{m-1}} d\omega \int_{-\infty}^{\infty} ds \]
(5.f) \times \int_{\gamma} d\tau e^{\sqrt{-1} \tau \bar{\varepsilon} s^{l-\alpha}} \frac{\omega_\ell \omega_j \omega_b \ldots}{\Lambda^{l} (\tau^2 + \Lambda^2)^{n-1}} e^{-\bar{\varepsilon} \Lambda} \left( \frac{1}{\Lambda} \right)^{n-1} \Lambda^{k-1} \bar{\varepsilon} .
Performing the \( \Lambda \)-differentiation, different \( \bar{\varepsilon} \)-dependent functions would occur. It is therefore desirable to first perform the \( \bar{\varepsilon} \)-integration before performing the \( \Lambda \)-derivatives explicitly. This is achieved by noting that \( z = \Lambda \) has to be put after the \( \Lambda \) differentiation has been performed.
\[ T^{k\ell jn}_{ab...} = \frac{1}{(n-1)!} \int_{\mathbb{R}^{m-1}} d\omega \int_{-\infty}^{\infty} ds \int_{\gamma} d\tau e^{\sqrt{-1} \tau \bar{\varepsilon} s \omega_\ell \omega_j \omega_b \ldots} \frac{\Lambda^{k-1}}{\Lambda^{l}} \left( \frac{1}{2 \Lambda \bar{\Lambda}} \right)^{n-1} \Lambda^{k-1} \bar{\varepsilon} \right|_{z=\Lambda} .
\]
We can proceed in general by introducing numerical multipliers $c_{nkl}$ according to

$$
\left( \frac{1}{2\Lambda} \frac{d}{d\Lambda} \right)^{n-1} \left. \frac{\Lambda^{k-1}}{(\Lambda + z)^{l+1-\alpha}} \right|_{z=\Lambda} = c_{nkl} \frac{1}{\Lambda^{l+2n-k-\alpha}}.
$$

The $s$-integration is then performed using

$$
\int_{-\infty}^{\infty} ds \left( \frac{e^{\sqrt{-1}s}}{|s|^2 + \sqrt{-1}s} \right)^2 = \frac{2\pi}{\Gamma(\beta)} \frac{e^{-|s|^2}}{|s|^\beta}.
$$

The final $\omega$-integrations follow from

$$
C(y) \equiv \int_{\mathbb{R}^{m-1}} d\omega e^{-\bar{g}_{ab} \omega_a \omega_b + \sqrt{-1} y^a \omega_a} = \pi^{\frac{m+k}{2}} \sqrt{g} e^{-\bar{g}_{ab} y^a y^b},
$$

by observing that

$$
\int_{\mathbb{R}^{m-1}} d\omega \omega_a \omega_b \cdots \omega_n e^{-\bar{g}_{ab} \omega_a \omega_b} = \left( \frac{1}{\sqrt{-1}} \right)^r \partial_{y^{a_1}} \cdots \partial_{y^{a_r}} C(y) \big|_{y=0}.
$$

In particular

$$
\int_{\mathbb{R}^{m-1}} d\omega e^{-|\omega|^2} = \frac{\pi^{\frac{m}{2}}}{\sqrt{g}},
$$

$$
\int_{\mathbb{R}^{m-1}} d\omega \omega_a \omega_b e^{-|\omega|^2} = \frac{\pi^{\frac{m+k}{2}}}{\sqrt{g}} \bar{g}_{ab},
$$

$$
\int_{\mathbb{R}^{m-1}} d\omega \omega_a \omega_b \omega_c e^{-|\omega|^2} = \frac{\pi^{\frac{m+k-1}{2}}}{\sqrt{g}} \bar{g}_{ab} \bar{g}_{cd} + \bar{g}_{ac} \bar{g}_{bd} + \bar{g}_{ad} \bar{g}_{bc}.
$$

Introducing the numerical multipliers $d_{kljn}$ according to

$$
d_{kljn} = \frac{2(\sqrt{-1})^k (-1)^{n+k+1} \pi^2 \Gamma(l+1-\alpha)c_{nkl}}{(n-1)! \Gamma\left(\frac{l+k-\alpha}{2} + n\right)},
$$

we obtain the compact-looking answers

$$
T_{ab\cdots} = d_{kljn} \int_{\mathbb{R}^{m-1}} d\omega \omega_a \omega_b \cdots e^{-|\omega|^2},
$$

where the last $\omega$-integration is performed with the above results.

Note that the numerical multipliers $d_{kljn}$ are easily determined using an algebraic computer program. Therefore, all appearing integrals can be very easily obtained.

Let us apply this formalism explicitly to the leading orders, and we start with $h_{-2}(y, r, \omega, \tau, \lambda)$. The relevant differential equation reads

$$
(\partial_r^2 - \Lambda^2) h_{-2}(y, r, \omega, \tau, \lambda) = 0,
$$

which has the general solution

$$
h_{-2}(y, r, \omega, \tau, \lambda) = g_1 e^{-r\Lambda} + g_2 e^{r\Lambda}.
$$

The asymptotic condition (5.4) on the symbol as $r \to \infty$ imposes $g_2 = 0$. The initial condition $\partial_r h_{-2}|_{r=0} = \sqrt{-1} \tau q_{-2}|_{r=0}$ gives $g_1 = -\sqrt{-1} (\Lambda (r^2 + \Lambda^2))^{-1} r$. Putting the information together we have obtained

$$
h_{-2}(y, r, \omega, \tau, \lambda) = -\frac{\sqrt{-1} r}{\Lambda (r^2 + \Lambda^2)} e^{-r\Lambda}.
$$

Performing the relevant integrals, with the notation

$$
\int dI = \int_{\mathbb{R}^{m-1}} d\omega \int_{-\infty}^{\infty} ds \int_{0}^{\infty} d\bar{r} e^{\sqrt{-1}s} \left( -\int_{0}^{\gamma} d\bar{r} e^{\sqrt{-1}\bar{r}\bar{r}} \right) \bar{r}^{-\alpha},
$$
produces
\[
\int dI h_{-2}(y, r, \omega, \tau, -\sqrt{-1} s) = -\sqrt{-1} d_{1011} \pi^{\frac{m-1}{2}} \sqrt{g}
\]
\[
= -\frac{2\alpha \pi^2 \Gamma(1 - \alpha)}{\Gamma(1 - \frac{\alpha}{2})} \pi^{\frac{m-1}{2}} \sqrt{g} = -\pi \Gamma\left(1 - \frac{\alpha}{2}\right) \pi^{m/2} \sqrt{g}.
\]
Taking into account the prefactor in (5.6) and the change of sign, this agrees with Assertion (1) of Theorem 1.6.

In the next order we obtain
\[
(\partial^2_r - \Lambda^2) h_{-3}(y, r, \omega, \tau, \lambda) = (E + U_1)e^{-\tau^2} + (F + U_2)re^{-\tau^2}.
\]
where
\[
E = \frac{\sqrt{-1} b^\tau}{\tau^2 + \Lambda^2}, \quad F = -\frac{\sqrt{-1} r \tilde{g}^b \omega_a \omega_b}{\Lambda(\tau^2 + \Lambda^2)},
\]
\[
U_1(\omega) = \frac{\tilde{b}^a \omega_a \tau}{\Lambda(\tau^2 + \Lambda^2)} + \frac{\tilde{g}^{a\omega_a c} \omega_a \omega_c}{\Lambda^3(\tau^2 + \Lambda^2)^2} + \frac{2 \tilde{g}^b \omega_a \omega_b \omega_c \tau}{\Lambda(\tau^2 + \Lambda^2)^2},
\]
\[
U_2(\omega) = \frac{\tilde{g}^{a\omega_a c} \omega_a \omega_c}{\Lambda^2(\tau^2 + \Lambda^2)^2}.
\]
Note, for later arguments, that \(U_1(\omega)\) and \(U_2(\omega)\) are odd functions in \(\omega\). Furthermore, for the scalar Laplacian at hand \(b^a = g^{bc} \Gamma_{bc} a\); thus they contain only tangential derivatives of the metric.

Using for example the annihilator method, we write down the general form of the solution to this differential equation as
\[
h_{-3}(y, r, \omega, \tau, \lambda) = c_1 e^{-\tau^2} + c_2 r e^{-\tau^2} + c_3 r^2 e^{-\tau^2} + c_4 r^3.
\]
From the asymptotic condition (5.12) we conclude \(c_4 = 0\). From the initial condition given in Equation (5.13) we obtain
\[
c_1 = -\frac{1}{4 \Lambda^2} (F + U_2) - \frac{1}{2 \Lambda^2} (E + U_1)
\]
\[
-\frac{\sqrt{-1} S \tau}{\Lambda^2(\tau^2 + \Lambda^2)} - \frac{S}{\Lambda(\tau^2 + \Lambda^2)^2} - \frac{\tilde{b}^a \omega_a \tau}{\Lambda(\tau^2 + \Lambda^2)^2} + \frac{2 \tilde{g}^{a\omega_a c} \omega_a \omega_c}{\Lambda^3(\tau^2 + \Lambda^2)^3} + \frac{\tilde{g}^{a\omega_a c} \omega_a \omega_c}{\Lambda(\tau^2 + \Lambda^2)^2}.\]
From the differential equation we derive
\[
c_2 = -\frac{1}{4 \Lambda^2} (F + U_2) - \frac{1}{2 \Lambda} (E + U_1),
\]
\[
c_3 = -\frac{1}{4 \Lambda^2} (F + U_2).
\]
Collecting the available information, we see
\[
h_{-3}(y, r, \omega, \tau, \lambda) = De^{-\tau^2} + Bre^{-\tau^2} + Cr^2 e^{-\tau^2} + O(\omega),
\]
with
\[
D = \frac{\sqrt{-1} \tilde{g}^{a\omega_a c} \omega_a \omega_c}{4 \Lambda^3(\tau^2 + \Lambda^2)} - \frac{\sqrt{-1} b^\tau}{2 \Lambda^2(\tau^2 + \Lambda^2)} - \frac{\sqrt{-1} S \tau}{\Lambda^2(\tau^2 + \Lambda^2)^2} - \frac{\tilde{b}^a \omega_a \tau}{\Lambda(\tau^2 + \Lambda^2)^2} + \frac{2 \tilde{g}^{a\omega_a c} \omega_a \omega_c}{\Lambda^3(\tau^2 + \Lambda^2)^3} + \frac{\tilde{g}^{a\omega_a c} \omega_a \omega_c}{\Lambda(\tau^2 + \Lambda^2)^2},
\]
\[
B = \frac{\sqrt{-1} \tilde{g}^{a\omega_a c} \omega_a \omega_c}{4 \Lambda^2(\tau^2 + \Lambda^2)} - \frac{\sqrt{-1} b^\tau}{2 \Lambda(\tau^2 + \Lambda^2)^2},
\]
\[
C = \frac{\sqrt{-1} \tilde{g}^{a\omega_a c} \omega_a \omega_c}{4 \Lambda(\tau^2 + \Lambda^2)^2},
\]
and where $O(\omega)$ is an odd function in $\omega$. Furthermore, $O(\omega)$ contains only tangential derivatives of the metric. We next perform the multiple integrals; note, odd functions in $\omega$ do not contribute. We obtain

$$
\int dh_{-3}(y, \tilde{r}, \omega, \tau, -\sqrt{-1}s) = \frac{4\pi^{-m/2}}{\Gamma (1 - \frac{\alpha}{4})} \frac{\alpha}{\pi} \frac{m + 2}{2} S \left(1 - \frac{\alpha}{4}\right) \left(\frac{\pi^2}{\lambda_0} - \frac{1}{\lambda_0} + \frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right).
$$

This confirms the value of $\vartheta^a_0$ in Lemma 4.1 and of $\vartheta^2_0$ in Lemma 3.1 after taking into account the prefactor in (5.1) and the fact that $\tilde{g}_{ab}^{\rho}g_{ab} = -\tilde{g}^{ab}g_{ab, r} = 2g_{ab}L_{ab}$. Up to this point the calculation can be considered a warm up for the next order. Leaving aside the $S$-terms for the moment, we would like to determine the universal coefficients of the geometric invariants $L_{aa}L_{bb}$ and $L_{ab}L_{ab}$. In terms of the metric these are determined by

$$L_{ab} = -\frac{1}{2} \tilde{g}_{ab, r}.$$

Using the Christoffel symbols

$$\Gamma_{jk}^l = \frac{1}{2} g^{il} \left(g_{lj,k} + g_{lk,j} - g_{jk,l}\right),$$

and taking into account that with our sign convention the scalar curvature is given by the contraction $g^{jk}R_{ijk}$, we may expand the Riemann curvature tensor in the form:

$$R_{ijk} = \Gamma_{jk}^l \Gamma_{ik}^l - \Gamma_{ik}^j \Gamma_{jk}^l + \Gamma_{jl}^n \Gamma_{jk}^n - \Gamma_{jk}^l \Gamma_{ln}^l.$$

The normal projection of the Riemann curvature tensor reads

$$R_{\alpha \beta \gamma \delta} = -\frac{1}{2} \tilde{g}^{ac} \tilde{g}_{ac, \gamma \delta} + \frac{1}{2} \tilde{g}^{ac} \tilde{g}_{ac, \gamma \delta} + \frac{1}{2} \tilde{g}^{ac} \tilde{g}_{ac, \gamma \delta} + \frac{1}{2} \tilde{g}^{ac} \tilde{g}_{ac, \gamma \delta} + \frac{1}{2} \tilde{g}^{ac} \tilde{g}_{ac, \gamma \delta} + \frac{1}{2} \tilde{g}^{ac} \tilde{g}_{ac, \gamma \delta}.$$

The above results suggest a strategy for the calculation. It suffices to consider the special case where the metric is independent of $y$. As a consequence, our answer will have the form

$$(4\pi)^{-m/2} \left\{ H\tilde{g}^{ac}\tilde{g}_{ac, \gamma \delta} + K\tilde{g}^{ab}\tilde{g}_{ab, \gamma \delta} + L\tilde{g}^{ac}\tilde{g}_{ab, \gamma \delta} + M\tilde{g}^{ab}\tilde{g}_{ab, \gamma \delta}\right\}$$

plus terms involving $S$. This has to be compared with the terms in $a_{2, \alpha}^d(F, \Delta_M)$ that possibly contribute to these geometric invariants. In detail one can show these terms are (mod terms with tangential derivatives of the metric)

$$\left(\frac{-1}{3(1-\alpha)}\right) - \frac{1}{\lambda_0} \vartheta_{mm} + \frac{1}{\lambda_0} \vartheta_{ab}L_{ab}L_{ab} + \vartheta_{ab}L_{ab}L_{ab}$$

$$= \left(\frac{1}{12} - \frac{1}{3(1-\alpha)}\right)\tilde{g}^{ac}\tilde{g}_{ac, \gamma \delta} + \frac{1}{\lambda_0} \vartheta_{ab}g_{ab, \gamma \delta} + \frac{1}{\lambda_0} \vartheta_{ab}g_{ab, \gamma \delta}$$

$$+ \left(\frac{-3}{3(1-\alpha)}\right) - \frac{1}{\lambda_0} \vartheta_{ab}g_{ab, \gamma \delta} + \frac{1}{\lambda_0} \vartheta_{ab}g_{ab, \gamma \delta}.$$

So once we know $H, K, L$, as a check we can verify that

$$H = \left(\frac{1}{12} - \frac{1}{3(1-\alpha)}\right)\kappa_{\alpha - 2},$$

and we can deduce

$$\vartheta_{ab} = \frac{4K}{\kappa_{\alpha - 2}} + \frac{1}{\kappa_{\alpha - 2}}, \quad \vartheta_{ab} = \frac{4L}{\kappa_{\alpha - 2}} - \frac{1}{\kappa_{\alpha - 2}} + \frac{1}{\alpha}. $$

In summary, when writing down the differential equation for $h_{-4}(y, r, \omega, \tau, \lambda)$, we can neglect all terms that are odd in $\omega$ as well as all terms that contain tangential derivatives of the metric. We obtain (up to irrelevant terms)

$$(\vartheta^2 - \Lambda^2)h_{-4}(y, r, \omega, \tau, \lambda) = Me^{-r\Lambda} + Ne^{-r\Lambda} + Pr^2e^{-r\Lambda} + Qr^3e^{-r\Lambda},$$
where
\[ M = -\Lambda \hat{b} r D + \hat{b} r B, \]
\[ N = \frac{\sqrt{r} b r}{\tau^2 + \Delta^2} + \hat{g}_{r r}^{ab} \omega_{a} \omega_{b} D - \Lambda \hat{b} r B + 2 \hat{b} r C, \]
\[ P = -\frac{\sqrt{r} g_{r r}^{ab} \omega_{a} \omega_{b} \tau}{2 \Lambda (\tau^2 + \Delta^2)} + \hat{g}_{r r}^{ab} \omega_{a} \omega_{b} B - \Lambda \hat{b} r C, \]
\[ Q = \hat{g}_{r r}^{ab} \omega_{a} \omega_{b} C, \]
with \( B, C, D \) given above.

So the solution has the form, taking into account the asymptotic behavior (5.d),
\[ h_{-4}(y, r, \omega, \tau, \lambda) = -\hat{a} e^{-\tau \Lambda} + \beta r e^{-\tau \Lambda} + \gamma r^2 e^{-\tau \Lambda} + \delta r^3 e^{-\tau \Lambda} + \epsilon r^4 e^{-\tau \Lambda}. \]

To simplify the notation, let \( \Xi := \tau^2 + \Delta^2 \). From the initial condition we obtain, up to irrelevant terms,
\[ \hat{a} = \frac{\beta}{\Lambda} + \frac{\alpha D \tau}{\Lambda} + \frac{\sqrt{r} b r}{\Lambda^2} + \frac{2 \sqrt{r} g_{r r}^{ab} \omega_{a} \omega_{b} \tau}{\Lambda^2} \]
\[ + \frac{\sqrt{r} g_{r r}^{ab} \omega_{a} \omega_{b} \tau}{\Lambda^3} - \frac{2 \sqrt{r} b r \hat{g}_{r r}^{ab} \omega_{a} \omega_{b} \tau}{\Lambda^3} + \frac{2 \sqrt{r} b r \hat{g}_{r r}^{ab} \omega_{a} \omega_{b} \tau}{\Lambda^4} - \frac{6 \sqrt{r} g_{r r}^{ab} \omega_{a} \omega_{b} \tau}{\Lambda^4}. \]

From the differential equation we obtain the conditions
\[ M = -2 \Lambda \delta + 2 \gamma, \quad N = -4 \Lambda \gamma + 6 \delta, \]
\[ P = -6 \Lambda \delta + 12 \epsilon, \quad Q = -8 \epsilon \Lambda. \]

This determines the numerical multipliers \( \beta, \gamma, \delta \) and \( \epsilon \) to be
\[ \beta = \frac{3}{4} \frac{\tau}{\Delta} - \frac{1}{4} \frac{\tau}{\tau} - \frac{1}{4} \frac{\tau}{\Xi} - \frac{1}{2} \frac{\tau}{\Xi}, \quad \gamma = \frac{1}{2} \frac{\tau}{\Xi}, \quad \delta = -\frac{1}{2} \frac{\tau}{\Xi}, \quad \epsilon = -\frac{1}{8} \frac{\tau}{\Xi}. \]

For \( \Delta M \), we have:
\[ \hat{b} r = -\frac{1}{2} \hat{g}_{r r}^{ab} \hat{g}_{ab, r}, \]
\[ \hat{b} r \hat{b} r = \frac{1}{4} \hat{g}_{r r}^{ab} \hat{g}_{cd}^{ab} \hat{g}_{ab, r} \hat{g}_{cd, r}, \]
\[ \hat{b} r \hat{g}_{r r}^{ab} \hat{g}_{ab, r} = \frac{1}{2} \hat{g}_{r r}^{ab} \hat{g}_{r r}^{cd} \hat{g}_{ab, r} \hat{g}_{cd, r}, \]
\[ \hat{b} r \hat{g}_{r r}^{ab} \hat{g}_{ab, r} \hat{g}_{cd, r} = \frac{1}{4} \hat{g}_{r r}^{ab} \hat{g}_{cd}^{ab} \hat{g}_{ab, r} \hat{g}_{cd, r} - \frac{1}{2} \hat{g}_{r r}^{ab} \hat{g}_{ab, r} \hat{g}_{cd, r}, \]
\[ \hat{g}_{r r}^{ab} \hat{g}_{cd}^{ab} \hat{g}_{ab, r} \hat{g}_{cd, r} = \frac{1}{2} \hat{g}_{r r}^{ab} \hat{g}_{cd}^{ab} \hat{g}_{ab, r} \hat{g}_{cd, r} + \frac{1}{2} \hat{g}_{r r}^{ab} \hat{g}_{cd}^{ab} \hat{g}_{ab, r} \hat{g}_{cd, r}, \]

Performing the integrations we obtain, modulo normalizing constants of \( \pi^{(m-1)/2} \sqrt{g} \), one obtains:
\[ \hat{a}_I = \hat{g}_{r r}^{ab} \hat{g}_{ab, rr} \left[ -\frac{\sqrt{r} d_{1051}}{16} + \frac{\sqrt{r} d_{1031}}{4} + \frac{\sqrt{r} d_{3013}}{2} - \frac{\sqrt{r} d_{1013}^2}{4} \right] + 2 \sqrt{r} d_{3014} - \frac{\sqrt{r} d_{1014}^2}{2} - \frac{\sqrt{r} d_{1014}^3}{4} \]
\[ + \frac{\sqrt{r} d_{1014}^4}{8} - \frac{\sqrt{r} d_{1014}}{16} + \frac{\sqrt{r} d_{1014}^2}{4} - \frac{\sqrt{r} d_{1014}^3}{8} - \frac{\sqrt{r} d_{1014}^4}{16} - \frac{\sqrt{r} d_{1014}^5}{32}. \]
Adding up all terms and simplifying using the functional equation and the doubling formula for the $\Gamma$-function, the contribution to the heat kernel coefficient reads

$$\beta_I = \tilde{g}^{ab} \tilde{g}_{ab,rr} \left[ -\frac{\sqrt{\pi}}{16} d_{1141} + \frac{\sqrt{\pi}}{8} d_{1211} \right]$$

$$\gamma_I = \tilde{g}^{ab} \tilde{g}_{ab,rr} \left[ -\frac{\sqrt{\pi}}{16} d_{1231} + \frac{\sqrt{\pi}}{8} d_{1211} \right]$$

$$\delta_I = \tilde{g}^{ab} \tilde{g}_{ab,rr} \left[ -\frac{\sqrt{\pi}}{24} d_{1321} + \frac{\sqrt{\pi}}{8} d_{1321} \right]$$

$$\epsilon_I = \tilde{g}^{ab} \tilde{g}_{ab,rr} \left[ -\frac{\sqrt{\pi}}{8} d_{1431} + \frac{\sqrt{\pi}}{8} d_{1431} \right].$$

Adding up all terms and simplifying using the functional equation and the doubling formula for the $\Gamma$-function, the contribution to the heat kernel coefficient reads

$$(4\pi)^{-m/2} \kappa_{\alpha}^{-2} \sqrt{g} \left\{ \frac{8}{(1-\alpha)(2-\alpha)} S^2 + \frac{2(\alpha^2-\alpha-1)}{(\alpha-1)(\alpha-2)(\alpha-3)} S \right\}$$

$$\left( \frac{1}{12} - \frac{1}{3(1-\alpha)} \right) \tilde{g}^{ab} \tilde{g}_{ab,rr} + \frac{1}{4} \tilde{g}^{ab} \tilde{g}_{ab,rr} \left( \frac{1}{2\alpha} - \frac{1}{2} - \frac{\alpha^2-10\alpha^2+2\alpha+44}{4(\alpha-6)(\alpha-4)(\alpha-1)} \right)$$

$$\frac{1}{4} \tilde{g}^{ab} \tilde{g}_{ab,rr} \left( -\frac{1}{3(1-\alpha)} + \frac{\alpha^2-6\alpha^2-2\alpha+104}{8(\alpha-6)(\alpha-4)(\alpha-2)(\alpha-1)} \right).$$

This allows us conclude:

$$\vartheta_4^\alpha = \frac{\alpha^4-6\alpha^3-\alpha^2-2\alpha+104}{8(\alpha-6)(\alpha-4)(\alpha-2)(\alpha-1)},$$

$$\vartheta_5^\alpha = -\frac{\alpha^4-10\alpha^2+2\alpha+44}{4(\alpha-6)(\alpha-4)(\alpha-1)},$$

$$\vartheta_6^\alpha = \frac{2(\alpha^2-\alpha-8)}{(\alpha-1)(\alpha-2)(\alpha-4)}.$$

The values for $\alpha = 0$ reproduce the result for the smooth setting. We also confirm the result for $\vartheta_6^\alpha$. We can now employ Lemma 4.1 (2) to determine $\vartheta_6^\alpha$. We find

$$\vartheta_6^\alpha = -\frac{1}{(\alpha-1)(\alpha-4)},$$

which in the limit $\alpha \to 0$ reproduces the correct answer.
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