Fractal Effect of Random Disturbance on Reaction-diffusion Equation

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Abstract. Fractal dimension is the main feature of many nonlinear phenomena like the coastline, stock indexes and surface growth. Reaction-diffusion equation which is used mainly in different fields such as physics and medicine has complicated characteristics as its components are nonlinear. In this paper, the fractal effect of the external disturbance on the reaction-diffusion equation which is a partial differential equation is studied. The relationship of the disturbance between the fractal dimension of the output variable has been obtained by finite difference method simultanously. Even there exists random term in the disturbance, the fractal dimension is also stable characteristics of the output variable of the reaction-diffusion equation.

Keywords: Disturbance; Fractal; Diffusion equation.

1. Introduction
Fractal theory has been a hot topic in describing the nonlinear features of many natural phenomena like the shapes of clouds, the length of the ocean line and the urban growth process since its born from 1980s\cite{1-5}. For example, Diffusion Limited Aggregated (DLA) model as a stochastic fractal growth model was first used in urban growth in 1989\cite{6,7}. The Dielectric Breakdown (DB) model comprises deterministic and stochastic parts to get the similar fractal features like DLA and related similar features with urban sprawl\cite{8,9}.

Growth process related a reaction-diffusion equation as a partial differential equation has fractal features in reality such as in black dendritic growth of copper, the soil structures and movements of population under local environment\cite{10-12}.

Actually, there are many disturbance influencing the dynamical behavior of the growth process. However, few researches focus on the relationship between the fractal feature and the reaction-diffusion equation with random disturbance. In addition, there still exist many questions to be answered: Can we find out an appropriate model to combine the random disturbance describing real growth process? Is there any inner relationship between the model and the fractal dimension? Is there any effect of the random term in the disturbance and initial condition on the variation of the fractal dimension?

In order to address the above questions, a reaction-diffusion model introducing a disturbance function related with the distance from the growth center is set up. The relationship between other growth density models are also presented. Moreover, numerical results that the fractal dimension of the output variable are stable are shown illustrately.

The organization of this paper is as follows. A partial differential equation is presented based on the
empirical results about population diffusion process under external disturbance. The connection between the disturbance function and the other five growth functions used in empirical study mainly is also obtained. Simulations illustrate the stable and variable connection between the fractal dimension of the output variable and the disturbance function.

2. Preliminaries

The following reaction-diffusion equation has been used in many growth process like in gene study, turbulent flows and the population density[13-15]:

$$ \frac{\partial \rho(x,y,t)}{\partial t} = B \Delta \rho(x,y,t) + k \rho(x,y,t)(\sigma - \rho(x,y,t)), $$  \hspace{1cm} (1)

where \( B \) is the diffusion parameter, \( k \) the intrinsic growth rate, \( \sigma \) the carrying capacity and the Laplacian of \( \rho \) is

$$ \Delta \rho(x,y,t) := \frac{\partial^2 \rho(x,y,t)}{\partial x^2} + \frac{\partial^2 \rho(x,y,t)}{\partial y^2}. $$

According to the similar introduction in Ref.[16], the following dynamical system is presented:

$$ \frac{\partial \rho(x,y,t)}{\partial t} = B \Delta \rho(x,y,t) + k \rho(x,y,t)(\sigma - \rho(x,y,t)) $$

$$ -\beta \rho(x,y,t) \iint_{A(x,y)} \rho(x',y',t)\psi(x',y',t)dx'dy', $$

where \( \psi(x,y,t) \) is a generalized disturbance factor related to location and time, \( A(x,y) \) spans the growth location influencing the decision making,

$$ A(x,y) = \{ (x',y') | r_0 \leq \sqrt{(x')^2 + (y')^2} \leq \sqrt{x^2 + y^2} \}, $$

where the origin of coordinates is placed at the center of the growth location, and \( r_0 > 0 \) is a constant representing the initial distance from the center which has population density.

For the simplicity computationally, Eq.(2) is rewritten as the following form by Cartesian coordinate computation:

$$ \rho_r(r,t) = B(\rho_r(r,t) + \frac{\rho_r(r,t)}{r}) + k \rho(r,t)(\sigma - \rho(r,t)) $$

$$ -2\pi\beta \rho(r,t) \int_0^r r' \psi(r',t)\rho(r',t)dr'. $$

3. Methods and Main Results

The existence and uniqueness of solutions of Eq. (3) can be proved easily by the Lipschitz property similarly in Ref.[17]. After calculation, the different growth models are obtained as the steady solutions for Eq. (3) shown in Table 1.

When the random term exists in \( \psi(r) \) shown in Table 1 as: \( c_0(t) := \cos(w_n(t)), b_0(t) := \sin(w_n(t)), \) where \( w_n(t) \) is a random value at each time step, the initial condition is listed based on the empirical study[23]:

$$ \rho(r,0) = \rho(r_0) e^{\frac{r-r_0}{w_n(r)}}. $$
As it is impossible to solve Eq. (3) analytically for general disturbance function $\psi(r, t)$, Eq.(3) can be studied by finite differencing method with central difference representation as following:

$$P_{i,j,t+1} = P_{i,j,t} \left(1 + \frac{B \Delta t}{4x(t)^2} - \frac{2B \Delta t}{(\Delta t)^2} + k \sigma \Delta t\right) - \frac{k \Delta t}{\sqrt{x(i)}} P_{i,j}^2 + \frac{B \Delta t}{(\Delta t)^2} P_{i+1,j}^t$$

$$+ \frac{B \Delta t}{(\Delta t)^2} P_{i-1,j} - \pi \beta \sigma \Delta x P_{i,j} \sum_{k=1}^{l} (P_{i+1,j} + P_{i,j}) \psi(k, j) \sqrt{x(k)},$$

where $P(r, t) := \rho(r, t) \sqrt{r}, 2B \Delta t \leq (\Delta t)^2$.

Table 1. The relationship between $\psi(r)$ and the steady solutions of Eq.(3) as $B=0$ and $r0=0$.

| $\psi(r)$ | Steady solution $\rho(r)$ | Other growth models |
| --- | --- | --- |
| $\frac{1}{2 \pi \beta} \frac{k}{c_1r}$ | $\rho_0(0)e^{-\frac{r}{c_0}}$ | Clark model[18] |
| $\frac{1}{2 \pi \beta} \frac{k}{c_0^2}$ | $\rho_0(0)e^{-\frac{r^2}{2c_0^2}}$ | Sherrat model[19] |
| $\frac{1}{2 \pi \beta} \frac{kc_0}{r}$ | $\rho_0(0)r^{-c_0}$ | Smeed model[20] |
| $\frac{1}{2 \pi \beta} \frac{k(2c_0 - b_0)}{r}$ | $\rho_0(0)e^{b_0r-c_0r^2}$ | Newling model[21] |
| $\frac{1}{2 \pi \beta} \frac{k(2c_0 + b_0)}{r}$ | $\rho_0(0)r^{-b_0}e^{-c_0r}$ | Tanner model[22] |

The output variable of Eq.(3) Q(r, t) as defined by: $Q(r, t) := 2\pi \int_0^t r^r \rho(r', t) dr'$ has the following scaling relationship:

$$Q(r, t) \propto r^{D(t)},$$

which can be illustrated in Figure 1(a) even though the random term in the disturbance function and in the initial condition. For each time, it is possible to calculate the power of $r$ in this way; this is the fractal dimension at each time, and is shown in Figure 1(b). It is clear that the fractal dimension decreases sharply at early times, and then displays mild variation which is concave down.

![Figure 1](image-url)
However, there still exists an important problem: what is the effect of the random terms on the fractal dimension? So in order to illustrate the different effects, the simulated process goes on 30 times and the distribution of the fractal dimension $D$ and its adjusted $R^2$ are illustrated in Figure 2(a) and 2(b). It is significant as the adjusted $R^2$ value shown in Fig 2(b), demonstrating that the confidence in the fractal dimension computations is high. Figure 2(c) shows that $\psi_4(r)$ has the steepest with the highest kurtosis value and the greatest left deviation as the smallest skewness value for the distribution of the fractal dimension while $\psi_1(r)$ has the opposite results. Averagely, $\psi_4(r)$ would get the largest fractal dimension while $\psi_4(r)$ has the smallest one. From the point of the divergence as variance, stand derivation, range and coefficient of variation, $\psi_4(r)$ has the largest variation for the calculation of the fractal dimension and $\psi_1(r)$ has the smallest one.

![Figure 2](image-url)

Figure 2. Variation of the fractal dimension $D$ and its boxplot distribution for disturbance $\psi_i$.

4. Conclusions

Here we showed that the reaction-diffusion equation can give insight of the significant role that a disturbance function with random term can play an important role in the fractal features related with the diffusion equation. The disturbance function systematically generalized many classical growth models present in the literature. For the purpose of comparing the power of the disturbance models, the behavior of the fractal dimension was significantly impacted by the choice of the disturbance model. The fractal feature under the two random terms in initial condition and in the five different disturbance functions, were significant with high confidence. And more than 50% of the value of the fractal dimensions were in the range of $[1.6, 1.8]$ statistically. The fractal dimension increased in short time and then decreased to converge to some value $1 < D < 2$ after fixing the initial condition and still keeping the random term in
the disturbance functions.

In addition, the obtained simulations indicate that the disturbance function (relating cost to the distance from the centre area) had more important role in driving population into other locations than other parameters. Similarly, the fractal dimension displayed sensitivity to the choice of the disturbance term, indicating that this can be an important---though less considered---factor in population growth modeling. The results presented here is a basic step toward the further understanding and applications of the fractal theory and the reaction-diffusion equations deeply.

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