Symbolic Optimal Control
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Abstract
We present novel results on the solution of a class of leavable, undiscounted optimal control problems in the minimax sense for nonlinear, continuous-state, discrete-time plants. The problem class includes entry-(exit-)time problems as well as minimum time, pursuit-evasion and reach-avoid games as special cases. We utilize auxiliary optimal control problems ("abstractions") to compute both upper bounds of the value function, i.e., of the achievable closed-loop performance, and symbolic feedback controllers realizing those bounds. The abstractions are obtained from discretizing the problem data, and we prove that the computed bounds and the performance of the symbolic controllers converge to the value function as the discretization parameters approach zero. In particular, if the optimal control problem is solvable on some compact subset of the state space, and if the discretization parameters are sufficiently small, then we obtain a symbolic feedback controller solving the problem on that subset. These results do not assume the continuity of the value function or any problem data, and they fully apply in the presence of hard state and control constraints.

Index Terms
Discrete abstraction, optimal control, difference inclusion, nonlinear system, symbolic control, approximate dynamic programming; MSC: Primary, 49M25; Secondary, 93C10, 93C55, 93C73

I. Introduction
In this paper we present novel results on the solution of optimal control problems, in which we follow a symbolic synthesis approach [1]–[3] and utilize finite, auxiliary problems ("abstractions") obtained from discretizing the original problem data. Our theory provides symbolic feedback controllers, and it culminates in novel convergence and completeness results including the following: If the optimal control problem is solvable on some compact subset of the state space, and if the discretization parameters are sufficiently small, then the obtained controller solves the problem on that subset.

More specifically, we consider discrete-time control systems that are defined by difference inclusions of the form

$$x(t+1) \in F(x(t), u(t)),$$

where $x(t) \in X$ and $u(t) \in U$ represents the state and the input signal, respectively. Typically, the sets $X$ and $U$ are uncountably infinite. We use set-valued transition functions $F: X \times U \rightarrow X$ to account for possible perturbations such as actuator inaccuracies and modeling uncertainties; see e.g. [2]. The problem data also includes non-negative, extended real-valued running and terminal cost functions, $g$ and $G$,

$$g: X \times X \times U \rightarrow \mathbb{R}_+ \cup \{\infty\},$$

$$G: X \rightarrow \mathbb{R}_+ \cup \{\infty\}.$$ (2a) (2b)

As we demonstrate in Section VIII, the infinite costs are useful to represent hard actuation and state space constraints.
Given the aforementioned problem data, we investigate optimal control problems where the evolution of the closed-loop must be stopped at some finite, but not predetermined, time. At that point, the total cost is determined as the sum of the terminal cost and the previously accumulated running costs. We seek to synthesize a feedback controller that minimizes, or approximately minimizes, the total cost in the minimax (worst-case) sense, in which the controller generates both an input signal for the plant (1) and additionally a signal that determines the stopping time. In particular, the considered optimal control problem is leavable as the controller is allowed to stop the evolution of the closed-loop at any time [4]. In contrast to similar settings, in our problem stopping is mandatory and not discretionary, and we penalize non-stopping evolutions with infinite costs. The problem class is formally defined in Section III-A and includes entry-(or exit-)time problems as well as minimum time, pursuit-evasion and reach-avoid games as special cases. Examples are given in Sections III-B and VIII.

Outline of the Proposed Approach. We follow a symbolic synthesis approach [1]–[3]: First, an abstraction, i.e., a finite, auxiliary optimal control problem, is constructed by discretizing the problem data. Second, a controller solving the auxiliary problem is synthesized, and third, the latter controller is refined to obtain a controller for the original problem. In this context, we label quantities and objects that are defined with respect to the original and to the auxiliary optimal control problem as concrete and abstract, respectively. In our theory, abstractions shall be constructed so that the abstract value function, i.e., the best achievable performance of the abstract closed-loop, provides an upper bound of the concrete value function. Conforming to the correct-by-construction paradigm of the symbolic approach, the theory also guarantees that the closed-loop value function associated with the abstract controller, i.e., the worst-case performance of that controller used in the abstract closed loop, provides an upper bound of the closed-loop value function associated with the concrete controller.

Since even rather coarse discretizations of the problem data may very well qualify as abstractions, the abstract value function will provide a rather conservative bound on the concrete value function, in general. To limit the amount of conservatism, we shall introduce a suitable notion of precision for abstractions, which is closely related to the accuracy by which the problem data is discretized. As our main results, we shall establish the convergence of both of the aforementioned upper bounds to the concrete value function as the precision of the abstraction approaches zero. In turn, as we shall also show, our synthesis approach is complete in the following sense: If the original optimal control problem is solvable on a compact subset of the state space, then the obtained controller solves the original problem on that subset whenever a sufficiently precise abstraction is employed.

Our results do not assume the continuity of the value function or any problem data, and they fully apply in the presence of hard state and control constraints. The resulting feedback controllers are memoryless, finitely representable and symbolic, i.e., they require only quantized as opposed to full state information.

Related Work. The symbolic synthesis scheme has been applied to a variety of optimal control problems including minimum time problems [5], [6], entry-time problems [7], [8] and finite horizon problems [9]. Optimality properties in combination with regular language specifications are analyzed in [10]. The results in [5], [7] are based on approximate alternating simulation relations. As discussed in detail in [2, Sec. IV], this leads to overly complex, dynamic controllers which additionally require full state information. The controllers synthesized in [6] also require full state information. Moreover, while the works [8]–[10] lead to arbitrarily close approximations of value functions, the respective convergence results do not account for perturbations [8]–[10], do not apply in the presence of hard constraints and discontinuous value functions [8], [9], or require piecewise linear plant dynamics [10]. Additionally, the approach in [8] relies on the ability to exactly determine first integrals of the plant dynamics, and the one in [10], on the ability to verify a non-trivial property for an exact optimal solution (which is assumed to exist).

Closely related to our approach is the numerical approximation of the value function, which has a rich history and has been a major research focus since the early days of Dynamic Programming
sets of real numbers, non-negative real numbers, integers and non-negative integers, respectively, and
which we announced some of the results in Sections IV and VI appeared in [27].

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introduced [26], to compute abstractions for a class of sampled control systems, and we also present
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VI). Related results do not account for hard constraints or discontinuous value functions, focus on
functions, which additionally imply the completeness of our method in a well-defined sense (Section
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(Section V). Compared to existing methods, we do not impose restrictive assumptions on the system
dynamics, nor is our approach limited to constant cost functions or finite horizon problems, and the
resulting controllers are memoryless and require only quantized state information. Third, we establish
powerful convergence results in the presence of perturbations, hard constraints and discontinuous value
functions, which additionally imply the completeness of our method in a well-defined sense (Section
VI). Related results do not account for hard constraints or discontinuous value functions, focus on
special classes of cost functions or on continuous-time problems, or are not suitable for correct-by-construction controller synthesis.

For the sake of self-consistency of the paper, we present in Section VII a method, which we
introduced [26], to compute abstractions for a class of sampled control systems, and we also present
an algorithm to efficiently solve auxiliary, abstract optimal control problems. In Section VIII, we
illustrate the application of our approach on three examples. A preliminary version of this paper, in
which we announced some of the results in Sections IV and VI appeared in [27].

II. Preliminaries

The relative complement of the set A in the set B is denoted by B \ A. \( \mathbb{R}, \mathbb{R}_+, \mathbb{Z} \) and \( \mathbb{Z}_+ \) denote the
sets of real numbers, non-negative real numbers, integers and non-negative integers, respectively, and
\( \mathbb{N} = \mathbb{Z}_+ \setminus \{0\} \). We adopt the convention that \( -\infty + x = -\infty \) for any \( x \in \mathbb{R} \). \([a, b], [a, b[, [a, b], and \([a, b]\)
 denote closed, open and half-open, respectively, intervals with end points \( a \) and \( b \), e.g., \([0, \infty[ = \mathbb{R}_+,\]
\([a; b], [a; b[, [a; b], and \([a; b]\) stand for discrete intervals, e.g., \([a; b] = [a, b] \cap \mathbb{Z}, [1; 4[ = \{1, 2, 3\}, \) and
\([0; 0[ = \emptyset, \) max \( M \), \min \( M \), \sup \( M \) and \( \inf \ M \) denote the maximum, the minimum, the supremum and
the infimum, respectively, of the nonempty subset \( M \subseteq [-\infty, \infty) \), and we adopt the convention that
\( \sup \emptyset = 0 \).

\( f : A \rightarrow B \) denotes a set-valued map from the set A into the set B, whereas \( f : A \rightarrow B \) denotes an
ordinary map; see [28]. The set of maps \( A \rightarrow B \) is denoted \( B^A \). If \( f \) is set-valued, then \( f \) is strict and
single-valued if \( f(a) \neq \emptyset \) and \( f(a) \) is a singleton, respectively, for every \( a \).

We identify set-valued maps \( f : A \Rightarrow B \) with binary relations on \( A \times B \), i.e., \((a, b) \in f \) iff \( b \in f(a) \).
Moreover, if \( f \) is single-valued, it is identified with an ordinary map \( f : A \rightarrow B \). The restriction of
\[ f \] to a subset \( M \subseteq A \) is denoted \( f|_M \). The inverse mapping \( f^{-1} : B \Rightarrow A \) is defined by \( f^{-1}(b) = \{ a \in A | b \in f(a) \} \), \( f \circ g \) denotes the composition of \( f \) and \( g \), \( (f \circ g)(x) = f(g(x)) \), and the image of a subset \( C \subseteq A \) under \( f \) is denoted \( f(C) \), \( f(C) = \bigcup_{a \in C} f(a) \).

If \( A \) and \( B \) are metric spaces, then \( f \) is upper semi-continuous (u.s.c.) if \( f^{-1}(\Omega) \) is closed for every closed subset \( \Omega \subseteq B \). Alternatively, if \( B = [-\infty, \infty] \), then \( f \) is bounded on the subset \( C \subseteq A \) if \( f(C) \) is a bounded subset of \( \mathbb{R} \).

For maps \( f,g : X \rightarrow [-\infty, \infty] \), the relations \(<, \leq, \geq, > \) are defined point-wise, e.g. \( f < g \) if \( f(x) < g(x) \) for all \( x \in X \). Analogously, the relations are interpreted component-wise for elements of \( [-\infty, \infty]^n \). The set of minimum points of \( f \) in some subset \( Q \subseteq X \) is denoted \( \text{argmin} \{ f(x) | x \in Q \} \). \( \text{hypo} f = \{ (x, \gamma) \in X \times \mathbb{R} | \gamma \leq f(x) \} \) is the hypograph of \( f \), and \( f \) is u.s.c. if \( X \) is a metric space and \( \text{hypo} f \subseteq X \times \mathbb{R} \) is closed [28], [29].

The backward shift operator \( \sigma \) is defined as follows. If the map \( f \) is defined on \([0; T[ \) for some \( T \in \mathbb{N} \cup \{\infty\} \), then \( \sigma f \) is the map defined on \([0; T - 1[ \) and given by \((\sigma f)(t) = f(t + 1)\).

### III. A Leavable Optimal Control Problem

We develop our theory in a rather general setting, and for now we simply assume that \( X \) and \( U \) are nonempty sets. These assumptions already allow for a fixed-point characterization of the value function. As we progress with our analysis we gradually impose stricter assumptions. In particular, we demonstrate the upper semi-continuity of the value function under assumptions including that \( X \) and \( U \) are metric spaces. Here the abstract treatment of \( X \) and \( U \) is crucial. Even if the original system evolves in \( \mathbb{R}^n \), the abstractions we shall construct do not. Similarly, to prove our main results in Section VI, we will need to construct yet another auxiliary problem with a non-euclidean state alphabet. Moreover, our setting covers plants whose states naturally form finite-dimensional manifolds, which is common in e.g. robot dynamics [30].

#### A. Problem definition

Formally, we treat (1) as special case of a more general class of discrete-time dynamical systems given by

\[
\begin{align}
    x(t+1) &\in F(x(t), z(t)), \\
    (y(t), z(t)) &\in H(x(t), u(t)),
\end{align}
\]

where \( x(t) \in X, y(t) \in Y, u(t) \in U, z(t) \in Z \) is the state, output, input and internal input signal, respectively. The use of the internal input signal \( z(t) \) results from the general system definition with set-valued maps \( F \) and \( H \) and is needed to provide a meaningful definition of system composition. See [2, Section III] for a more detailed explanation of this point.

#### III.1 Definition. A system is a septuple

\[(X, X_0, U, Z, Y, F, H),\]

where \( X, X_0, U, Z \) and \( Y \) are nonempty sets, \( X_0 \subseteq X \), and \( H : X \times U \Rightarrow Y \times Z \) and \( F : X \times Z \Rightarrow X \) are strict. Let \( S \) be a system of the form (4). A quadruple \((u, z, x, y) \in U^{Z_+} \times Z^{Z_+} \times X^{Z_+} \times Y^{Z_+}\) is a solution of the system (4) (starting at \( x(0) \)) if (3a) and (3b) hold for all \( t \in Z_+ \), and \( x(0) \in X_0 \).

We call the set \( X, X_0, U, Z, \) and \( Y \) the state, initial state, input, internal variable, and output alphabet, respectively, and the maps \( F \) and \( H \) are the transition function and the output function, respectively.

In our framework, the plant (1) is represented by a system of the form

\[(X, X, U, U, X, F, \text{id}),\]

where \( \text{id} \) denotes the identity function.
whose output function id is the *identity map* and all of whose states are admissible as initial states. A controller, on the other hand, is a general system that accepts output signals of the plant at its input, produces output signals that decompose into input signals for the plant and stopping signals, and so restricts the behavior of the plant within a closed-loop and determines a stopping time. These concepts are formalized in the following definition. See also Fig. 1.

**III.2 Definition.** Let $S$ be a system of the form (5). A system $C$ of the form
\[
(X', X'_0, U', Z', Y' \times \{0, 1\}, F', H')
\]
is *feedback composable* with $S$, denoted by $C \in \mathcal{F}(X, U)$, if $X \subseteq U'$ and $Y' \subseteq U$. In this case, the *behavior* $\mathcal{B}(C \times S) \subseteq (U \times \{0, 1\} \times X)^{2_+}$ of the closed-loop composed of $C$ and $S$ is defined by the requirement that $(u, v, x) \in \mathcal{B}(C \times S)$ iff there exist signals $x': \mathbb{Z}_+ \to X'$ and $z': \mathbb{Z}_+ \to Z'$ such that $(x, z', x', (u, v))$ is a solution of $C$ and $(u, u, x, x)$ is a solution of $S$. In addition, the *behavior initialized at $p \in X$* is denoted by $\mathcal{B}_p(C \times S)$ and defined by $\mathcal{B}_p(C \times S) = \{(u, v, x) \in \mathcal{B}(C \times S) \mid x(0) = p\}$.

A particularly important class of controllers consists of static (or memoryless) controllers, i.e., of systems (6) with $X'$ being a singleton. As we shall see later, that class is sufficient to approximately solve the optimal control problems investigated in the present paper, to arbitrary accuracy.

We proceed with our formulation of optimal control problems: The problem data also includes a *running cost function* $g$ and a *terminal cost function* $G$ as in (2). The total cost to be minimized is then given by the *cost functional* $J: (X \times U \times \{0, 1\})^{2_+} \to [0, \infty]$, which is defined as the sum of the terminal cost and accumulated running costs, i.e.,
\[
J(u, v, x) = G(x(T)) + \sum_{t=0}^{T-1} g(x(t), x(t+1), u(t))
\]
if $v \neq 0$ and $T = \min v^{-1}(1)$, and otherwise we define $J$ by
\[
J(u, v, x) = \infty.
\]

Throughout the paper, we identify the optimal control problem with its problem data and use the following definition.

**III.3 Definition.** An *optimal control problem* is a tuple
\[
(X, U, F, G, g),
\]
where (5) is a system and $G$ and $g$ are non-negative extended real-valued functions as in (2).

The notions of state alphabet, input alphabet and transition function are carried over from the system (5) to the optimal control problem (8) in the obvious way.
As already indicated, solving the problem (8) means to find controllers \( C \in \mathcal{F}(X, U) \) which, for every state \( p \in X \), minimize or approximately minimize the cost (7) for \((u, v, x) \in \mathcal{B}_p(C \times S)\) in a worst-case sense, where \( S \) denotes the plant (5). Here, the stopping signal \( v: \mathbb{Z}_+ \to \{0, 1\} \) determines, by its first 0-1 edge, the instance of time when the optimization process stops and the terminal costs are evaluated, and the worst-case cost is given by the \textit{closed-loop value function} \( L: X \to [0, \infty] \) of (8) associated with \( C \),

\[
L(p) = \sup_{(u, v, x) \in \mathcal{B}_p(C \times S)} J(u, v, x).
\]

It follows that the achievable closed-loop performance is determined by the \textit{value function} \( V: X \to [0, \infty] \) of (8),

\[
V(p) = \inf_{C \in \mathcal{F}(X, U)} \sup_{(u, v, x) \in \mathcal{B}_p(C \times S)} J(u, v, x).
\]

Let us emphasize that our notion of controller subsumes other such notions that are often used in the perturbed optimal control and differential games literature, e.g. the notion of \textit{policy} used in [31, p. 197ff.] and \textit{causal feedback strategies} in [32, Chapter VIII]. Moreover, as we show in Theorem IV.1, the value function satisfies

\[
V(p) = \sup_{\beta \in \Delta(p)} \inf_{u \in \mathbb{Z}_+} \inf_{v \in \{0, 1\}^+} J(u, v, \beta(u))
\]

for all \( p \in X \), where \( \Delta(p) \) is the set of all strictly causal maps \( \beta: U^\mathbb{Z}_+ \to X^{\mathbb{Z}_+} \) for which \((u, v, \beta(u), \beta(u))\) is a solution of \( S \) starting at \( p \), for every \( u \in U^{\mathbb{Z}_+} \). Here, \( \beta \) is causal (resp., strictly causal) if \( \beta(u)|_{[0:T]} = \beta(\tilde{u})|_{[0:T]} \) whenever \( u, \tilde{u} \in U^{\mathbb{Z}_+}, T \in \mathbb{Z}_+ \), and \( u|_{[0:T]} = \tilde{u}|_{[0:T]} \) (resp., \( u|_{[0:T]} = \tilde{u}|_{[0:T]} \)).

It follows that there is no performance gain in using alternative information patterns such as \textit{non-anticipating strategies} [23]. See also [32, Chapter VIII].

### B. Important Special Cases

We briefly discuss special cases of the class of optimal control problems considered in this paper. For further examples, including an entry-(or exit-)time problem and a problem whose underlying dynamics is chaotic, see Section VIII.

#### III.4 Example (Shortest Path Problem)

Given a directed graph, we wish to find shortest paths from a specified source vertex to all other vertices [33]. This problem and its generalizations have numerous applications [23], [33]–[35]. For a formal description, let \( X \) and \( A \subseteq X \times X \) be finite sets of vertices and of arcs, respectively, of a directed graph, and let \( s \in X \) be a distinguished source vertex. Let a non-negative length \( w_{p,q} \) be associated with each arc \((p, q)\), i.e., \( w: A \to \mathbb{R}_+ \), and define the length of a path as the sum of the lengths of its arcs. The distance from \( s \) to \( p \in X \), denoted \( d(p) \), is the minimum length of any (directed) path from \( s \) to \( p \), and is defined to be \( \infty \) if no such path exists. The problem is to determine \( d(p) \), and a path of length \( d(p) \) from \( s \) to \( p \) if \( d(p) < \infty \), for all \( p \in X \).

The problem can be reduced to the following instance of the optimal control problem (8). Define \( U = X \), \( G(s) = 0 \), and \( G(p) = \infty \) for all \( p \in X \setminus \{s\} \), let \( g \) be such that \( g(p, q, u) = w_{q,p} \) whenever \((q, p) \in A \), and let \( F \) be single-valued such that \( F(p, U) = \{p\} \cup \{q \in X \mid (q, p) \in A\} \). Then there exists a static controller \( C \), with single-valued output function, that is feedback composable with the system \( S \) in (5), such that the closed-loop value function of (8) associated with \( C \) equals the value function \( V \) of (8); see e.g. Section VII-B. In turn, as is easily seen, a shortest path from \( s \) to \( p \) can be obtained from the unique element of \( \mathcal{B}_p(C \times S) \), and \( d = V \).

#### III.5 Example (Reach-Avoid Problem)

The problem of steering the state of the plant into a target set while avoiding obstacles appears in many applications, e.g. [30], [36]. For a formal description, let \( S \) be a plant of the form \( (5) \), and let a target set \( D \subseteq X \) and an obstacle set \( M \subseteq X \) be given. The controller \( C \in \mathcal{F}(X, U) \) is successful for the state \( p \in X \) if for every \((u, v, x) \in \mathcal{B}_p(C \times S) \)
IV. Fixed-point characterization and regularity of the value function

In this section, we shall first characterize the value function (10) as the maximal fixed-point of the dynamic programming operator $P$ associated with the optimal control problem (8),

$$P(W)(p) = \min \left\{ G(p), \inf_{u \in U} \sup_{q \in F(p,u)} g(p, q, u) + W(q) \right\},$$

(12)

which maps the space of functions $X \to [0, \infty]$ into itself. This characterization will in turn permit us to represent the value function as the limit of repeated applications of $P$ to the terminal cost function and to prove that this limit is semi-continuous. These results are new, see our discussion at the end of this section. Moreover, they will be useful later, when they facilitate the comparison of value functions in Section V as well as our convergence proofs in Section VI. In addition, as a side product we obtain the identity (11), which shows that in our setting, the value function could equivalently be defined using alternative information patterns, e.g. [23].

IV.1 Theorem. Let (8) be an optimal control problem, and let $V$ and $P$ be the associated value function and dynamic programming operator as defined in (10) and (12), respectively. Then $V$ is the maximal fixed point of $P$, i.e., $PV = V$, and $W \leq PW$ implies $W \leq V$. Moreover, the identity (11) holds for all $p \in X$.

Proof. We first observe that $P$ is monotone, i.e., that $PW \leq PW'$ whenever $W \leq W'$, and that

$$J(u, v, x) = G(x(0)) \quad \text{if } v(0) = 1,$n$$

$$J(u, v, x) = g(x(0), x(1), u(0)) + J(\sigma u, \sigma v, \sigma x), \text{ otherwise},$$

there exists some $s \in \mathbb{Z}_+$ satisfying $x(s) \in D$ and $x(t) \notin M$ for all $t \in [0; s]$. We say that $p$ can be forced into the target set if there exists a controller that is successful for $p$. The problem is to determine the subset $E \subseteq X$ of states that can be forced into the target set, and a controller that is successful for all states in $E$.

The problem can be reduced to the following instance of the optimal control problem (8). Define $G(p) = 0$ if $p \in D \setminus M$, and otherwise $G(p) = \infty$, and define $g(p, q, u) = 0$ if $p \notin M$, and otherwise $g(p, q, u) = \infty$. Then $E$ equals the effective domain $V^{-1}(\mathbb{R}_+)$ of the value function $V$ of (8), and a controller $C$ is successful for all states in $E$ iff the closed-loop value function of (8) associated with $C$ equals $V$.

The problem can be approximately solved using the results in this paper, which, for each compact subset $K \subseteq E$, yield a static controller $C$ that is successful for all states in $K$. See Corollary VI.10. □

III.6 Example (Minimum Time Problem). Various practical problems require solving reach-avoid problems in minimum time, e.g. [21], [30], [37]. For a formal description, let $S$, $D$ and $M$ as in Example III.5, and define the entry time $T_C(p)$ from $p \in X$ under feedback $C \in \mathcal{F}(X, U)$ as the infimum of all $\tau \in \mathbb{Z}_+$ satisfying the following condition: For every $(u, v, x) \in B_p(C \times S)$ there exists some $s \in [0; \tau]$ such that $x(s) \in D$ and $x(t) \notin M$ for all $t \in [0; s]$. The entry time $T(p)$ from $p \in X$ is the infimum of $T_C(p)$ over all controllers $C \in \mathcal{F}(X, U)$. The problem is to determine the value $T(p)$ for all $p \in X$, and a controller $C \in \mathcal{F}(X, U)$ satisfying $T = T_C$.

The problem can be reduced to the following instance of the optimal control problem (8). Define $G(p) = 0$ if $p \in D \setminus M$, and otherwise $G(p) = \infty$, and define $g(p, q, u) = 1$ if $p \notin M$, and otherwise $g(p, q, u) = \infty$. Then the minimum time function $T$ equals the value function $V$ of (8), and for every controller $C \in \mathcal{F}(X, U)$, $T_C$ equals the closed-loop value function of (8) associated with $C$.

The problem can be approximately solved using the results in this paper, which, for each compact subset $K \subseteq X$, yield a static controller $C$ satisfying $\sup T_C(K) = \sup T(K)$. See Corollary VI.10. □
for all \((u, v, x) \in (U \times \{0, 1\} \times X)^{\mathbb{Z}^+}\). Using a controller whose output function maps into \(U \times \{1\}\) we see that \(V \leq G\).

In what follows, we shall denote by \(R(p)\) the right hand side of (11) to show that \(R \leq V \leq PV\) and that \(W \leq PW\) implies \(W \leq R\), which proves the theorem. In particular, the case \(W = PV\) shows that \(PV \leq V\).

To prove that \(R \leq V\) holds, assume that \(V(p) < R(p)\) for some \(p \in X\). Then there exists a controller \(C\) of the form (6) and a map \(\beta \in \Delta(p)\) satisfying \(J(u, v, x) < J(u, v, \beta(u))\) for every \((u, v, x) \in B_p(C \times S)\). We will inductively construct \(u\) and \(v\) such that \((u, v, \beta(u)) \in B_p(C \times S)\), which is a contradiction and so proves \(R \leq V\). To this end, consider the following condition for any \(T \in \mathbb{Z}^+\):

The signals \(u, v\) and \(z'\) have already been defined on \([0; T]\), and the signal \(x'\) has already been defined on \([0; T]\) such that

\[
(u(t), v(t), z'(t)) \in H'(x'(t), \beta(u(t))),
\]

\[
x'(t + 1) \in F'(x'(t), z'(t))
\]

hold for all \(t \in [0; T]\). Here, \(\beta(u(t))\) denotes \(\beta(\tilde{u})(t)\) for any extension \(\tilde{u} : Z_+ \to U\) of \(u\), which is an unambiguous abbreviation as \(\beta\) is causal. Pick any \(x'(0) \in X_0\) to satisfy the condition for \(T = 0\), and assume the condition holds for some \(T \in \mathbb{Z}^+\). To extend the signals \(u, v, x'\) and \(z'\) we first pick \((u(T), v(T), z'(T)) \in H'(x'(T), \beta(u(T)))\), which is feasible as \(\beta\) is strictly causal, and then we pick any \(x'(T + 1) \in F'(x'(T), z'(T))\). Then the condition holds with \(T + 1\) in place of \(T\) as \(\beta\) is causal. Consequently, there exist signals \(u, v, x'\) and \(z'\) defined on \(Z_+\) such that \((\beta(u), z', x', (u, v))\) is a solution of \(\hat{C}\), and so \((u, v, \beta(u)) \in B_p(C \times S)\) as \(\beta \in \Delta(p)\).

To prove that \(V \leq PV\) holds, it suffices to show that

\[
V(p) \leq \sup_{q \in F(p, \xi)} g(p, q, \xi) + V(q)
\]

for all \(p \in X\) and all \(\xi \in U\). To this end, let \(p \in X, \xi \in U\) and \(\varepsilon > 0\). For every \(q \in F(p, \xi)\) there is a controller \(C_q \in \mathcal{F}(X, U)\) such that

\[
\sup_{(u, v, x) \in B_p(C_q \times S)} J(u, v, x) \leq V(q) + \varepsilon.
\]

We may assume without loss of generality that \(C_q\) is of the form \(C_q = (A_q, \{a_{q,0}\}, X, Z_q, U \times \{0, 1\}, F_q, H_q)\), in which the state alphabets \(A_q\) as well as the internal variable alphabets \(Z_q\) are pairwise disjoint. Let \(a_0, a_1 \notin A_q\) for every \(q, a_0 \neq a_1\), define \(A = \{a_0, a_1\} \cup \bigcup_{q \in F(p, \xi)} A_q\) and \(Z = \bigcup_{q \in F(p, \xi)} Z_q\), let \(\mu\) be a system of the form \((A, \{a_{q,0}\}, X, Z, U \times \{0, 1\}, F, H)\) that satisfies the following conditions for every \(q \in F(p, \xi)\): \(H(a_0, p) = \{(\xi, 0)\} \times Z\) and \(\tilde{F}(a_0, z) = \{a_1\}\) for all \(z \in Z\); \(H(a_1, q) = H_q(a_0, q)\) and \(\tilde{F}(a_1, z) = F_q(a_0, q, z)\) whenever \(z \in Z_q\); and \(H(a, \cdot) = H_q(a, \cdot)\) and \(\tilde{F}(a, \cdot) = F_q(a, \cdot)\) whenever \(a \in A_q\). Then \(\mu \in \mathcal{F}(X, U)\), and one easily shows that \((u, v, x) \in B_p(\mu \times S)\) implies \(x(0) = p, v(0) = 0, u(0) = \xi, x(1) \in F(p, \xi)\), and \((\sigma u, \sigma v, \sigma x) \in B_{q(1)}(C_{q(1)} \times S)\). Using the definition of \(V\), the observation at the beginning of this proof, and (14), it then follows that \(V(p) \leq \sup_{q \in F(p, \xi)} g(p, q, \xi) + V(q) + \varepsilon\). This implies (13), and so \(V \leq PV\).

Finally, suppose that \(W \leq PW\) and that \(R(p) + 2\varepsilon < W(p)\) for some \(p \in X\) and some \(\varepsilon > 0\). We claim that there exists a map \(\beta \in \Delta(p)\) such that

\[
R(p) + (1 + 2^{-t})\varepsilon < W(\beta(u(t))) + S(\beta(u), u, t)
\]

holds for all \(t \in \mathbb{Z}^+\) and all \(u : Z_+ \to U\), where \(S\) is defined by \(S(x, u, t) = \sum_{\tau=0}^{t-1} g(x(\tau), x(\tau+1), u(\tau))\). Since \(W \leq PW \leq G\) it then follows that \(R(p) + \varepsilon \leq J(u, v, \beta(u))\) for all \(u\) and \(v\), which contradicts the definition of \(R\), and hence, shows that \(W \leq PW\) implies \(W \leq R\).

To prove our claim, we define \(\beta(u)(0) = p\) for every \(u\), so that the inequality (15) for \(t = 0\) reduces to our assumption on \(\varepsilon\). Next, we assume that for some \(t \in \mathbb{Z}^+\) and all \(\tau \in [0; t]\), the value of \(\beta(u)|_{[0,\tau]}\) has already been defined as a function of \(u|_{[0,\tau]}\) such that (15) holds. Then the inequality
\( W \leq PW \) implies that given \( u|_{[0; t+1]} \), there is some \( q \in F(\beta(u)(t), u(t)) \) such that \( W(\beta(u)(t)) \leq 2^{-(t+1)}\varepsilon + g(\beta(u)(t), q, u(t)) + W(q) \). Hence, the choice \( \beta(u)(t+1) = q \) defines \( \beta(u)|_{[0; t+1]} \) as a function of \( u|_{[0; t+1]} \) such that (15) holds with \( t+1 \) in place of \( t \). This proves our claim, and completes the proof of the theorem.

For our representation of the value function as the semi-continuous limit of value iteration, i.e., of successive applications of the dynamic programming operator \( P \) to the terminal cost function \( G \), we consider the following hypothesis.

\((A_1)\) \( X \) and \( U \) are metric spaces, \( F \) is compact-valued, and \( g, G \) and \( F \) are u.s.c..

**IV.2 Corollary.** In the setting of Theorem IV.1, additionally assume \((A_1)\). Then \( V \) is u.s.c.,

\[
V(p) = \lim_{T \to \infty} P^T(G)(p)
\]

for all \( p \in X \), and \( P^{T+1}G \leq P^T G \) for all \( T \in \mathbb{Z}_+ \).

**Proof.** Obviously, \( 0 \leq PW \leq G \) for every \( W : X \to [0, \infty) \), and \( P \) is monotone. This proves the monotonicity claim and shows that the limit on the right hand side of (16), which we will denote by \( V_{\infty}(p) \), exists in \([0, \infty]\). In addition, the inequality \( V \leq G \) implies \( V \leq P^T G \) for all \( T \in \mathbb{Z}_+ \), hence \( V \leq V_{\infty} \). Next, using Berge’s Maximum Theorem A.3 and the fact that the infimum of u.s.c. maps is u.s.c., we see that \( PW \) is u.s.c. if \( W \) is. Thus, \( P^T G \) is u.s.c. for every \( T \in \mathbb{Z}_+ \). Then \( V_{\infty} \) is the infimum of u.s.c. maps, and hence, is u.s.c..

It remains to show that \( V_{\infty} \leq PV_{\infty} \). Then Theorem IV.1 implies that \( V_{\infty} \leq V \), and so \( V_{\infty} = V \). Indeed, as \( P^T G \) is monotonically decreasing in \( T \), we may apply Proposition A.4 with \( f_k(q) := g(p, q, u) + P^k(G)(q) \) to see that

\[
\lim_{T \to \infty} \sup_q g(p, q, u) + P^T(G)(q) = \sup_q g(p, q, u) + V_{\infty}(q)
\]

for arbitrary \( p \) and \( u \), where the suprema are over \( q \in F(p, u) \). As the limit is an infimum, we have \( \lim_{T \to \infty} P(P^T G)(p) = P(V_{\infty})(p) \), which completes the proof.

We note that while Theorem IV.1 does not assume any regularity of the problem data, the hypothesis \((A_1)\) mandates that e.g. in the Reach-Avoid Problem of Example III.5 the target set and the obstacle set is open and closed, respectively. Moreover, if any one of the assumptions in \((A_1)\) is dropped, then the identity (16) fails to hold, in general. Also our assumptions of semi-continuity and compact-valuedness in \((A_1)\) are automatically satisfied if the state and input alphabets are finite.

We would like to emphasize that while fixed-point characterizations and value iteration methods are well known in the field of Dynamic Programming, e.g. [4], [12], [13], [38], [39], the available results do not apply in our setting. Specifically, the theory in [12] requires that cost functionals are represented as limits of finite horizon costs, which is impossible for the functional in (7). The hypotheses in [38] imply that the dynamic programming operator has a unique fixed-point, and so are not satisfied by e.g. the Reach-Avoid Problem of Example III.5 whenever the transition function \( F \) is single-valued and there exists a state that cannot be forced into the target set. Similarly, for the unconstrained Minimum Time Problem of Example III.6, the hypotheses in [13] imply that the entry time is finite for every state [13, Sect. 3.2.1], or alternatively, that there exists a uniform bound on all finite entry times [13, Sect. 3.2.2]. These assumptions are typically not satisfied if the state alphabet of the plant is infinite, and are not imposed in the present paper. Results on stochastic games, e.g. [4], [39], can be directly interpreted in our setting only if the transition function of the plant is single-valued. In addition, running costs are typically assumed to vanish and terminal costs are required to be real-valued. Moreover, the class of controllers is also restricted, which can be seen from the result [39, Ch. 2.9, Th. 1] which does not hold in our setting: If the state alphabet of the plant is finite and the controller eventually stops every solution of the closed-loop, then the stopping times are uniformly bounded.
V. COMPARISON OF CLOSED-LOOP PERFORMANCES

In this section, we introduce **valuated alternating simulation relations** and **valuated feedback refinement relations** between optimal control problems, which are novel, quantitative variants of known qualitative system relations. As we shall show, the former concept allows for the efficient comparison of value functions of related optimal control problems, while the latter guarantees that the concrete closed-loop value function is upper-bounded, in a well-defined sense, by the abstract closed-loop value function. These results will be needed in the proofs of our main results in Section VI.

A. Comparison of value functions

V.1 Definition. Consider optimal control problems

\[ \Pi_i = (X_i, U_i, F_i, G_i, g_i), \quad (i = 1, 2) \]

and denote the dynamic programming operator associated with the problem \( \Pi_i \) by \( P_i \), \( i \in \{1, 2\} \). The relation \( Q: X_1 \Rightarrow X_2 \) is a **valuated alternating simulation relation** from \( \Pi_1 \) to \( \Pi_2 \), denoted by \( \Pi_1 \leq_{Q} \Pi_2 \), if the following conditions hold for all \( (p_1, p_2) \in Q \) and all \( u_2 \in U_2 \):

(i) \( G_1(p_1) \leq G_2(p_2) \);

(ii) if \( G_1(p_1) > 0 \) and the maps \( g_2(p_2, u_2) \) and \( (P_0) \circ Q^{-1} \) are bounded on the set \( F_2(p_2, u_2) \), where 0 denotes the zero function on \( X_1 \), then for all \( \epsilon > 0 \) we have:

\[ \exists u_1 \in U_1 \mid \forall q_1 \in F_1(p_1, u_1) \exists q_2 \in F_2(p_2, u_2) \cap Q(q_1) \]

\[ g_1(p_1, q_1, u_1) \leq \epsilon + g_2(p_2, q_2, u_2). \] (18)

The notion of valuated alternating simulation relation is related to its well-known qualitative variant in [1] as well as to the quantitative variants employed in [5], [7], [10]. The concepts in [1], [5], [7], [10] require that the first line of condition (18) holds for all \( (p_1, p_2) \in Q \) and all \( u_2 \in U_2 \), which implies, roughly speaking, behavioral inclusion between the two dynamical systems underlying the optimal control problems \( \Pi_1 \) and \( \Pi_2 \). It is the weaker conditions imposed in Definition V.1 that facilitate the application of valuated alternating simulation relations in our convergence proof in Section VI-B, where behavioral inclusion cannot be presumed. Comparison of the value functions associated with two related optimal control problems is still possible using our fixed-point characterization in Theorem IV.1:

V.2 Theorem. Let \( \Pi_1 \) and \( \Pi_2 \) be two optimal control problems with value functions \( V_1 \) and \( V_2 \), respectively. If \( \Pi_1 \leq_{Q} \Pi_2 \), then \( V_1(p_1) \leq V_2(p_2) \) for every \( (p_1, p_2) \in Q \).

Proof. Suppose that \( \Pi_1 \) is of the form (17) and let \( P_i \) be the associated dynamic programming operator, \( i \in \{1, 2\} \). We claim that \( (P_1 V_1)(p_1) \leq (P_2 W)(p_2) \) for all \( (p_1, p_2) \in Q \), where the function \( W: X_2 \to [0, \infty) \) is defined by

\[ W(p_2) = \sup \{ V_1(p_1) \mid (p_1, p_2) \in Q \}. \] (19)

Then, by applying Theorem IV.1 twice, we obtain \( W \leq P_2 W \), and in turn, \( W \leq V_2 \), which proves the assertion.

Let \( (p_1, p_2) \in Q \). Our claim is obvious if \( G_1(p_1) = 0 \), so we may assume throughout that \( G_1(p_1) > 0 \). Moreover, from Definition V.1(i), we see that it suffices to prove that

\[ \inf_{u_1 \in U_1} \sup_{q_1 \in F_1(p_1, u_1)} g_1(p_1, q_1, u_1) + V_1(q_1) \leq \sup_{q_2 \in F_2(p_2, u_2)} g_2(p_2, q_2, u_2) + W(q_2) \] (20)

holds for all \( u_2 \in U_2 \).

Let \( u_2 \in U_2 \), denote the value of the right hand side of (20) by \( R \), and suppose that \( R < \infty \). Then the map \( g_2(p_2, \cdot, u_2) \) is bounded on the set \( F_2(p_2, u_2) \). The same holds for the map \( (P_0) \circ Q^{-1} \) as \( V_1 = P_1 V_1 \geq P_1 0 \). Thus, we may assume that (18) holds. Moreover, the estimate (20) holds if for all \( \epsilon > 0 \) there exists \( u_1 \in U_1 \) such that \( \sup_{q_1 \in F_1(p_1, u_1)} g_1(p_1, q_1, u_1) + V_1(q_1) \leq \epsilon + R \). This, in turn, is guaranteed if for all \( q_1 \in F_1(p_1, u_1) \) there exists \( q_2 \in F_2(p_2, u_2) \) satisfying \( g_1(p_1, q_1, u_1) + V_1(q_1) \leq \epsilon + g_2(p_2, q_2, u_2) + W(q_2) \), and so an application of (18) and (19) completes the proof. \( \square \)
B. Controller refinement and comparison of closed-loop value functions

We have just seen that the existence of a valuated alternating simulation relation between optimal control problems implies a comparison between the respective value functions. We now proceed to introduce the stronger notion of valuated feedback refinement relation to additionally facilitate the refinement of solutions of one of the two problems, to the other problem, which is needed in the proof of one of our main results in Section VI.

V.3 Definition. Consider two optimal control problems \( \Pi_1 \) and \( \Pi_2 \) of the form (17). The relation \( Q : X_1 \Rightarrow X_2 \) is a valuated feedback refinement relation from \( \Pi_1 \) to \( \Pi_2 \), denoted \( \Pi_1 \preccurlyeq_Q \Pi_2 \), if \( Q \) is strict and the following conditions hold for all \( (p_1, p_2), (q_1, q_2) \in Q \) and all \( u \in U_2 \):

(i) \( U_2 \subseteq U_1 \);
(ii) \( G_1(p_1) \leq G_2(p_2) \);
(iii) \( g_1(p_1, q_1, u) \leq g_2(p_2, q_2, u) \);
(iv) \( Q(F_1(p_1, u)) \subseteq F_2(p_2, u) \).

We first note that every valuated feedback refinement relation is also a valuated alternating simulation relation. We state this simple fact as a formal result for later reference:

V.4 Proposition. \( \Pi_1 \preccurlyeq_Q \Pi_2 \) implies \( \Pi_1 \preccurlyeq^o_Q \Pi_2 \).

Apart from conditions (ii) and (iii) in Definition V.3, and in the special case of strict transition functions considered in the present paper, the notion of valuated feedback refinement relation coincides with its qualitative variant introduced in [2], and so we can take advantage of the controller refinement scheme presented in [2]. That is, we refine any abstract controller by serially connecting it with a valuated feedback refinement relation used as an interface; see Fig. 2. We therefore need to formalize the concept of serial composition:

V.5 Definition. Let \( C \) be a system of the form (4), \( U' \) be a non-empty set and \( Q : U' \Rightarrow U \) be a strict map. The serial composition of \( Q \) and \( C \), denoted \( C \circ Q \), is the system \((X, X_0, U', Z, Y, F, H')\) with \( H'(x, u') = H(x, Q(u')) \).

As demonstrated in [2] the proposed controller refinement scheme implies a comparison between closed-loop behaviors. Here we extend that result to guarantee a comparison between closed-loop value functions:

V.6 Theorem. Let \( \Pi_1 \) and \( \Pi_2 \) be optimal control problems of the form (17), and suppose that \( \Pi_1 \preccurlyeq_Q \Pi_2 \) and \( C \in F(X_2, U_2) \). Then \( C \circ Q \in F(X_1, U_1) \) and we have

\[
\forall p_1 \in X_1 \exists p_2 \in Q(p_1) \, L_1(p_1) \leq L_2(p_2),
\]

where \( L_1 \) and \( L_2 \) are the closed-loop value functions of \( \Pi_1 \) and \( \Pi_2 \) associated with \( C \circ Q \) and \( C \), respectively.
Proof. The proof is by reduction to the setting of [2] and exploiting behavioral inclusion [2, Thm. V.4].

Let \( S_1 = (X_1, X_2, U_1, U_0, X_0, F_0, \text{id}) \) with \( U_i = U_i \times \{0, 1\} \) and \( F_0(x, (u, v)) = F_i(x, u) \) for \( i \in \{1, 2\} \). It is straightforward to check that \( \Pi_1 \preceq_Q \Pi_2 \) implies that \( Q \) is a feedback refinement relation (according to [2, Def. V.2]) from \( S_1 \) to \( S_2 \). Moreover, from \( C \in \mathcal{F}(X_2, U_2) \) and Lemma A.1 it follows that \( C \) is feedback composable (according to [2, Def. III.3]) with \( S_2 \). Hence, Theorem V.4 in [2] applies and we observe that \( C \circ Q \) is feedback composable with \( S_1 \), which by Lemma A.1, implies \( C \circ Q \in \mathcal{F}(X_1, U_1) \).

Let \( J_i \) denote the cost functionals associated with \( \Pi_i \). Let \( p_1 \in X_1 \) and \( \varepsilon > 0 \). We pick \( (u, v, x)_1 \in B_{p_1}((C \circ Q) \times S_1) \) so that \( L_1(p_1) - \varepsilon \leq J_1(u, v, x_1) \). By Lemma A.1 we have \((0, ((u, v), x_1)) \in B((C \circ Q) \times S_1) \) and we apply [2, Thm. V.4, (iii)] to ensure the existence of a signal \( x_2 : [0; \infty[ \to X_2 \) such that \( (0, ((u, v), x_2)) \in B(C \circ S_2) \) and \( (x_1(t), x_2(t)) \in Q \) for all \( t \in [0; \infty[ \). A second application of Lemma A.1 shows that \( (u, v, x_2) \in B_{p_2}(C \times S_2) \) for \( p_2 = \tilde{x}_2(0) \). If \( v = 0 \), we obtain \( J_1(u, v, x_1) = J_2(u, v, x_2) = \infty \) which implies \( L_1(p_1) = L_2(p_2) = \infty \). If \( v \neq 0 \), we let \( T = \min v^{-1}(1) \) and use (ii) and (iii) in Definition V.3 to derive the inequalities \( L_1(p_1) - \varepsilon \leq G_1(x_1(T)) + \sum_{t=0}^{T-1} g_1(x_1(t), x_1(t+1), u(t)) \leq G_2(x_2(T)) + \sum_{t=0}^{T-1} g_2(x_2(t), x_2(t+1), u(t)) \leq L_2(p_2) \). As this holds for every \( \varepsilon > 0 \) the assertion follows.

For easier reference in later sections, we reformulate Theorem V.6 in terms of pointwise upper performance bounds:

**V.7 Definition.** Let \( Q : X_1 \Rightarrow X_2 \) be strict and let \( f : X_2 \to [0, \infty[ \). Then the function \( \hat{f}(Q) : X_1 \to [0, \infty[ \) defined by

\[ \hat{f}(Q)(x) = \sup f(Q(x)) \]

is called the pointwise upper bound of \( f \) associated with \( Q \).

**V.8 Corollary.** Under the hypotheses and in the notation of Theorem V.6 we have \( L_1 \leq \hat{f}(Q) \).

## VI. Main results

In this section, we introduce a notion of abstraction of optimal control problems which comes with a non-negative precision parameter. We will then show that the concrete value function can be approximated arbitrarily closely using value functions of sufficiently precise abstractions. Moreover, we shall show that if abstract controllers can be chosen to be optimal, the performance of the closed-loop in Fig. 2 converges to the concrete value function as well. The latter result implies a kind of completeness property of controller synthesis based on abstractions of precision introduced in this paper, an aspect to be discussed at the end of the section.

### A. Abstractions and their precision

To begin with, we first introduce abstractions devoid of any notion of precision. In doing so, we focus on a case where the abstract state space is a cover of the concrete state space, which has turned out to be canonical in the qualitative setting [2, Sec. VII]. Here, a cover of a set \( X \) is a set of subsets of \( X \) whose union equals \( X \).

**VI.1 Definition.** Let \( \Pi_1 \) and \( \Pi_2 \) be optimal control problems of the form (17), where \( X_2 \) is a cover of \( X_1 \) by non-empty subsets. Then \( \Pi_2 \) is an abstraction of \( \Pi_1 \) if \( \Pi_1 \preceq_{\varepsilon} \Pi_2 \) for \( \varepsilon : X_1 \Rightarrow X_2 \) denotes the membership relation.

For later reference, we explicitly state our requirements on abstractions.

**VI.2 Proposition.** Let \( \Pi_1 \) and \( \Pi_2 \) be optimal control problems of the form (17), where \( X_2 \) is a cover of \( X_1 \) by non-empty subsets. Then \( \Pi_1 \preceq_{\varepsilon} \Pi_2 \) iff the following conditions hold whenever \( p \in \Omega \in X_2 \), \( p' \in \Omega' \in X_2 \) and \( u \in U_2 \):

1. \( U_2 \subseteq U_1 \);
(ii) $G_1(p) \leq G_2(\Omega)$;
(iii) $g_1(p, p', u) \leq g_2(\Omega, \Omega', u)$;
(iv) $\Omega \cap F_1(\Omega, u) \neq \emptyset \Rightarrow \Omega' \in F_2(\Omega, u)$.

Proof. Obviously, the relation $\varepsilon$ is strict as $X_2$ is a cover of $X_1$, and if $Q = \varepsilon$, then the conditions (i) through (iii) are equivalent to the respective conditions in Definition VI.3. The equivalence of condition (iv) to the condition (iv) in Definition V.3 is obtained by an application of [2, Prop. VII.1] to the systems $S_i = (X_i, X_i, U_i, U_i, F_i, id), i \in \{1, 2\}$.

As we can see, even rather conservative approximations of the concrete optimal control problem may qualify as abstractions. We aim at limiting the amount of conservatism by introducing a suitable notion of precision. To this end, we first need to introduce some additional notation. For any metric space $(X, d)$ we define

\[
d(x, N) = \inf \{d(x, y) \mid y \in N\},
\]
\[
d(M, N) = \inf \{d(x, y) \mid x \in M, y \in N\}
\]
for all $x \in X$ and all nonempty subsets $M, N \subseteq X$. We use $B(c, r)$ and $\bar{B}(c, r)$ to denote the open, respectively, closed ball with center $c \in X$ and radius $r > 0$, and we adopt the convention that $\bar{B}(c, 0) = \{c\}$. We denote the diameter of a subset $M \subseteq X$ by $\text{diam}(M)$. See [29].

VI.3 Definition. Let $\Pi_2$ be an abstraction of $\Pi_1$ and suppose that $\Pi_1$ and $\Pi_2$ are of the form (17), that $U_1$ and $X_1$ are metric spaces, and that the elements of $X_2$ are closed subsets of $X_1$.

Then $\Pi_2$ is called an abstraction of precision $\infty$ of $\Pi_1$. Moreover, $\Pi_2$ is an abstraction of precision $\rho \in \mathbb{R}_+$ of $\Pi_1$ if the following conditions hold for all $\Omega, \Omega' \in X_2$ and all $u \in U_2$:

(i) $U_1 = \bar{B}(U_2, \rho)$;
(ii) $G_2(\Omega) \leq \rho + \sup G_1(\Omega)$;
(iii) $g_2(\Omega, \Omega', u) \leq \rho + \sup g_1(\Omega, \Omega', u)$.

If $\Omega$ satisfies the condition

\[
G_1(\Omega) \cup g_1(\Omega, X_1, U_1) \neq \{\infty\},
\]
then we additionally require the following:

(iv) $F_2(\Omega, u) \subseteq \{\Omega'' \in X_2 \mid d(\Omega'', F_1(\Omega, u)) \leq \rho\}$, where $d$ denotes the metric on $X_1$;
(v) $\text{diam}(\Omega) \leq \rho$.

As we had announced, Definition VI.3 limits the conservatism of abstractions. Specifically, while the conditions (i) through (iv) in Proposition VI.2 demand that $U_1, G_2(\Omega), g_2(\Omega, \Omega', u)$ and $F_2(\Omega, u)$ merely over-approximate $U_2, \sup G_1(\Omega), \sup g_1(\Omega, \Omega', u)$ and $F_1(\Omega, u)$ respectively, the respective conditions in Definition VI.3 mandate that the approximation error does not exceed the value of the precision parameter $\rho$, and (v) bounds the error by which abstract states over-approximate concrete states. The condition (21) restricts the requirements (iv) and (v) to regions where the concrete value function is possibly finite.

B. Arbitrarily close approximation of concrete value functions

We next need to choose a suitable notion of convergence. On the one hand, pointwise convergence is not powerful enough, e.g. to imply our completeness results in Section VI-C, and similarly for convergence in Lebesgue spaces as employed in [23]. On the other hand, the stronger concept of uniform convergence would require that any discontinuities of the concrete value function are also present, exactly and not only approximately, in the functions to approximate it, which is not realistic to assume. We here rely on a concept that lies in between the aforementioned extremes, and the first main result of our paper shows that the hypographs of pointwise upper bounds of the abstract value functions locally approximate the hypograph of the concrete value function. See Fig. 3. The result
requires tightening the hypothesis (A₁) on the optimal control problem (8) as follows:

\[(A₂)\] \(X\) is a proper metric space, \(U\) is a compact metric space, \(F\) is compact-valued, and \(g, G\) and \(F\) are u.s.c..

Here, a metric space is proper if every closed ball is compact, a requirement satisfied, e.g. by \(\mathbb{R}^n\) and by all of its closed metric subspaces. Hypothesis (A₂) is extended to optimal control problems \(Π_i\) of the form (17) in the obvious way. In the following, we do not mention explicitly the association of pointwise upper bounds on abstract value functions with the respective membership relations.

**VI.4 Theorem.** Let \(Π\) be the optimal control problem (8) and let \(V\) denote the value function of \(Π\). Then the pointwise upper bound of the value function of any abstraction of \(Π\) is an upper bound on \(V\). If (8) additionally satisfies (A₂), then for every \(p \in X\) and every \(\varepsilon > 0\) there exist a neighborhood \(N \subseteq X\) of \(p\) and some \(\rho \in \mathbb{R}_+ \setminus \{0\}\) such that

\[(N \times \mathbb{R}) \cap \text{hypo } W \subseteq B(\text{hypo } V, \varepsilon) \tag{22}\]

holds whenever \(W\) is the pointwise upper bound on the value function of an abstraction of precision \(\rho\) of (8).

To prove the theorem we will introduce an auxiliary optimal control problem \(Π_3\) with the following properties. Firstly, the state space \(X_3\) of \(Π_3\) comprises both a copy of the concrete state space and (almost the whole of) the state spaces of all abstractions, of arbitrary precision. Secondly, the value function \(V_3\) of \(Π_3\) restricted to the concrete state space coincides with the concrete value function \(V\). Thirdly, \(V_3\) is an upper bound on any abstract value function, on the respective subset of \(X_3\). Using the semi-continuity of \(V_3\) on the whole of \(X_3\), we then conclude that the abstract value function arbitrarily closely approximates \(V\) whenever the abstract state space sufficiently closely approximates the concrete one.

In our proof below, the notion of graph of a set-valued map \(f: X \rightrightarrows Y\) refers to the set

\[\{(x, y) \in X \times Y \mid y \in f(x)\},\]

and we also use the space \(K(X)\) of non-empty compact subsets of \(X\) endowed with the Hausdorff metric [28], [29] associated with the metric on \(X\), and its subspaces \(K_\rho(X)\) defined by

\[K_\rho(X) = \{\Omega \in K(X) \mid \text{diam } \Omega \leq \rho\}.\]
VI.5 Lemma. Let $\Pi_1$ be an optimal control problem of the form (17) that satisfies $(A_2)$, and denote the metric on $X_1$ by $d$. Let $\Pi_3 = (X_3, U_3, F_3, G_3, g_3)$ be given by $X_3 = K(X_1) \times \mathbb{R}_+$, $U_3 = U_1$ and

$$F_3((\Omega, \rho), u) = \left\{ \Omega' \in K_p(X_1) \mid d(\Omega', F_1(\Omega, \bar{B}(u, \rho))) \leq \rho \right\} \times \{\rho\},$$

$$G_3((\Omega, \rho), \Omega') = \rho + \sup G_1(\Omega),$$

$$g_3((\Omega, \rho), (\Omega', \rho'), u) = \rho + g_1(\Omega, \Omega', \bar{B}(u, \rho)).$$

Then $\Pi_3$ is an optimal control problem satisfying $(A_2)$.

Proof. $\Pi_3$ is clearly an optimal control problem by our hypotheses, and in particular, $F_3$ is strict. Moreover, $U_3$ is compact, and $K(X_1)$ is proper as $X_1$ is so. In addition, using Proposition A.5 it is easily seen that the maps $\alpha : K(X_1) \ni \Omega \mapsto \beta : U_1 \times \mathbb{R}_+ \ni u_1$ given by $\alpha(\Omega) = \Omega$ and $\beta(u_1, r) = \bar{B}(u_1, r)$ are u.s.c. and compact-valued, or usco for short. Then $G_3$ and $g_3$ are u.s.c. by Theorem A.3.

To show that $F_3$ is usco, define the map $H : K(X_1) \times U_1 \times \mathbb{R}_+ \ni (\Omega, u, \rho) \mapsto X_1$ by

$$H(\Omega, u, \rho) = \beta(F_1(\alpha(\Omega), \beta(\rho_1, u_1)), \rho),$$

let $((\Omega_k, \rho_k), (\Omega_k', \rho_k'))_{k \in \mathbb{N}}$ be a sequence in the graph of $F_3$, and suppose that the sequences $\Omega, \rho$ and $u$ converge to $\Gamma \in K(X_1)$, $r \in \mathbb{R}_+$, and $v \in U_1$, respectively. Then $\Omega_k' \in K_{\rho_k}(X_1)$ for all $k$, and since $F_1$ is usco, we also have $\Omega_k' \cap H(\Omega_k, u_k, \rho_k) \neq \emptyset$ for all $k$. Thus, there exists a sequence $\{p_k\}_{k \in \mathbb{N}}$ satisfying $p_k \in \Omega_k' \cap H(\Omega_k, u_k, \rho_k)$ for all $k$, and by Proposition A.5, a subsequence of $p$ converges to some $p \in H(\Gamma, v, r)$ since $H$ is usco. We may assume that the whole sequence converges. Then the sequence $\Omega'$ is bounded, and so may be assumed to converge to some $\Gamma' \in K(X_1)$ since $K(X_1)$ is proper. Additionally, $\Gamma' \in K_{\rho}(X_1)$ by the continuity of the map diam on $K(X_1)$, and $q \in \Gamma'$. We conclude that $(\Gamma', r) \in F_3((\Gamma, r), v)$, and so $F_3$ is usco by Prop. A.5. 

VI.6 Lemma. Under the hypotheses and in the notation of Lemma VI.5, let $\Pi_2$ be an abstraction of precision $\rho \in \mathbb{R}_+$ of $\Pi_1$, of the form (17). Let $V_i$ denote the value function of $\Pi_i$, $i \in \{1, 2, 3\}$, and let $X_2' \subseteq X_2$ be the subset of cells $\Omega$ that satisfy (21). Then the following holds:

(i) $V_1(p) = V_3(\{p\}, 0)$ for all $p \in X_1$

(ii) $V_2(\Omega) \leq V_3(\Omega, \rho)$ for all $\Omega \in X_2'$

(iii) $V_2(\Omega) \leq V_3(\Omega, \rho)$ whenever $p \in \Omega \in X_2 \setminus X_2'$. 

Proof. We claim that $\Pi_1 \preceq_\rho^3 \Pi_3 \preceq_\rho^1$. $\Pi_1$ holds for the single-valued map $Q : X_1 \ni \Omega \mapsto X_2$ given by $Q(\Omega) = \{p\}$, $p \in X_2$. Indeed, let $p \in X_1$ and $u \in U_3$. Then $G_3(Q(p)) = G_1(p)$ and $g_3(Q(p), Q(u), u) = g_1(p, q, u)$ for all $q \in X_1$. Moreover, $Q(F_1(p, u)) = F_3(\{p\}, 0, u)$ as $F_1$ is compact-valued. Thus, both conditions in Definition V.1 are met with $\Pi_3$ in place of $\Pi_2$, and they are also met with $\Pi_3$ and $\Pi_1$ in place of $\Pi_1$ and $\Pi_2$, respectively. This proves our claim, and (i) follows from Theorem V.2.

To prove (ii) and (iii) we shall show that $\Pi_2 \preceq_\rho^3 \Pi_3$ holds for the relation $Q : X_2 \ni \Omega \mapsto X_3$ given by $Q(\Omega) = \{\Omega, \rho\}$ if $\Omega \in X_2'$, and by $Q(\Omega) = \{\{p\}, \rho\}$, $p \in \Omega$, otherwise.

Let $(\Omega, (\Omega', \rho)) \in Q$ and $u_3 \in U_3$. Then $\Omega' \subseteq \Omega$, and additionally $(\Omega', \rho) \subseteq X_3$ as required since $X_2' \subseteq K_{\rho}(X_1)$. Moreover, the estimate $G_2(\Omega) \leq G_3(\Omega', \rho)$ is immediate from Definition VI.3 if $\Omega \in X_2'$. It also holds if $\Omega \in X_2 \setminus X_2'$, for then (21) is violated, which implies $G_3(\Omega', \rho) = \infty$. Hence, the first requirement in Definition V.3 is met with $\Pi_2$ and $\Pi_3$ in place of $\Pi_1$ and $\Pi_2$, respectively.

In our proof of the second requirement we may assume that the map $g_3((\Omega', \rho), ., u_3)$ is bounded on the set $F_3((\Omega', \rho), u_3)$. Then $g_1(\Omega, X_1, u_3) \neq \{\infty\}$ by the definition of $g_3$, and so $\Omega = \Omega' \subseteq X_2'$. We next pick any $u_2 \in B(u_3, \rho) \cap U_2$, which is possible by condition (i) in Definition VI.3, and any $\Omega'' \in F_2(\Omega, u_2)$. Then the condition (iv) in Def. VI.3 shows that

$$d(\Omega'', F_1(\Omega, \bar{B}(u_3, \rho))) \leq \rho. \quad (23)$$

If $\Omega'' \in X_2'$, then $(\Omega'', \rho) \in F_3((\Omega, \rho), u_3) \cap Q(\Omega')$. Moreover, the condition (iii) in Definition VI.3 with $\Omega''$ and $u_2$ in place of $\Omega'$ and $u$, respectively, shows that $g_2(\Omega, \Omega'', u_2) \leq g_3(\Omega, \rho, (\Omega'', \rho), u_3)$, and we
are done. If, on the other hand, $\Omega'' \notin X'_2$, then $G_1(\Omega'') \cup q_1(\Omega'', X_1, U_1) = \{ \infty \}$, and hence, $G_2(\Omega'') = \infty$ and $g_2(\Omega'', X_2, U_2) = \{ \infty \}$ by Proposition VI.2. This shows that $(P_2(0)(\Omega'') = \infty$. Moreover, $(\{q\}, \rho) \in F_3((\Omega, \rho), u_3)$ for some $q \in \Omega''$ by (23) and a compactness argument. Since additionally $(\{q\}, \rho) \in Q(\Omega')$ it follows that the map $(P_2(0) \circ Q^{-1}$ is not bounded on the set $F_3((\Omega, \rho), u_3)$, which completes our proof.

**Proof of Theorem VI.4.** The first claim of the theorem directly follows from Definition VI.1, Proposition V.4, and Theorem V.2. To prove the second claim, let $\varepsilon > 0$, $p \in X$ and $\rho > 0$, let $\Pi_i$, $V_i$ and $X'_2$ be as in Lemmas VI.5 and VI.6, $i \in \{1, 2, 3\}$, let $N = B(p, \varepsilon) \subseteq X_1$, and let $W$ be the pointwise upper bound of $V_2$. If (22) does not hold with $V_1$ in place of $V$, then there exists $x \in N$ satisfying $V_1(p) + \varepsilon/2 < W(x)$. Then $V_1(p) + \varepsilon/2 < V_2(\Omega')$ for some $\Omega' \in X_2$ containing $x$, by the definition of $W$, and $V_2(\Omega') \leq V_3(\rho, \varepsilon)$ for some $\Omega \in K_{\rho}(X_1)$ containing $x$, by Lemma VI.6; specifically, $\Omega = \Omega'$ if $\Omega' \in X'_2$, and $\Omega = \{x\}$, otherwise.

We conclude that, if the second claim of the theorem does not hold, then there exist $\varepsilon > 0$, $p \in X$ and a sequence $(\Omega_k)_{k \in \mathbb{N}}$ in $K(X_1)$ converging to $\{p\}$ such that $V_1(p) + \varepsilon/2 < V_3(\Omega_k, 1/k)$ for all $k \in \mathbb{N}$. On the other hand, $V_3$ is u.s.c. by Lemma VI.5 and Corollary IV.2, and this together with Lemma VI.6(i) shows that $\limsup_{k \to \infty} V_3(\Omega_k, 1/k) \leq V_1(p)$, which is a contradiction.

C. Convergence of the closed-loop performance to the concrete value function

Finally, we will demonstrate that the performance of the concrete closed-loop in Fig. 2 converges to the concrete value function, in which we use the following notion of convergence; see [28], [29] and Proposition A.2.

**VI.7 Definition.** Let the map $V: X \to \mathbb{R}_+ \cup \{\infty\}$ be u.s.c. on the metric space $X$, and let $L_i: X \to \mathbb{R}_+ \cup \{\infty\}$ satisfy $L_i \geq V$, for all $i \in \mathbb{N}$. Then the sequence $(L_i)_{i \in \mathbb{N}}$ hypo-converges to $V$, denoted $V = \lim_{i \to \infty} L_i$, if the following condition holds. For every $p \in X$ and every $\varepsilon > 0$ there exist a neighborhood $N \subseteq X$ of $p$ such that the inclusion

\[
(N \times \mathbb{R}) \cap \text{hypo } L_i \subseteq B(\text{hypo } V, \varepsilon)
\]

holds for all sufficiently large $i \in \mathbb{N}$.

In addition to hypothesis $(A_2)$, throughout the rest of this section we shall assume the following.

\[(A_3)\]

(i) For every $i \in \mathbb{N}$, $\Pi_i$ is an abstraction of precision $\rho_i \in \mathbb{R}_+ \cup \{\infty\}$ of (8), of the form (17).

$C_i$ is an optimal controller for $\Pi_i$, and $L_i$ is the closed-loop value function of (8) associated with $C_i \circ \varepsilon$, where $\varepsilon: X \Rightarrow X_i$ is the membership relation and $\lim_{i \to \infty} \rho_i = 0$.

(ii) $V$ is the value function of (8).

Here, $C_i \in \mathcal{F}(X_i, U_i)$ is an optimal controller for $\Pi_i$, if the value function of $\Pi_i$ coincides with the closed-loop value function of $\Pi_i$ associated with $C_i$, i.e., if $C_i$ realizes the achievable performance of the abstract closed-loop. As detailed in Section VII, optimal abstract controllers exist whenever abstractions are finite, and finite, arbitrarily precise abstractions can actually be computed in the case of sampled-data control systems.

We are now ready to present the second (and final) main result of our paper.

**VI.8 Theorem.** Assume $(A_2)$ and $(A_3)$. Then $h\lim_{i \to \infty} L_i = V$.

**Proof.** Obviously, $L_i \geq V$ for all $i$, $X$ is a metric space, and $V$ is u.s.c. by Corollary IV.2. Let $W_i$ be the value function of $\Pi_i$, and let $p \in X$ and $\varepsilon > 0$. By Theorem VI.4 there exists a neighborhood $N \subseteq X$ of $p$ such that $(N \times \mathbb{R}) \cap \text{hypo } W_i(\varepsilon) \subseteq B(\text{hypo } V, \varepsilon)$ holds for all sufficiently large $i \in \mathbb{N}$. Then, since $L_i \leq W_i(\varepsilon)$ for all $i$ by Corollary V.8, the requirement in Definition VI.7 is satisfied.
The theorem implies that the concrete value function $V$ is uniformly approximated on compact sets by the actual closed-loop performances $L_i$. Specifically, for every $\varepsilon > 0$ and every compact subset $N \subseteq X$ the inclusion (24) holds for all sufficiently large $i \in \mathbb{N}$. See also Fig. 3. Theorem VI.8 also implies pointwise convergence, and it even implies uniform convergence on any set on which such a strong convergence property can possibly be expected:

**VI.9 Corollary.** Assume $(A_2)$ and $(A_3)$. Then we have

$$V(p) = \lim_{i \to \infty} L_i(p) \quad \text{for all } p \in X,$$

and the following holds for every compact subset $N \subseteq X$.

(i) For every $\varepsilon > 0$ and all sufficiently large $i \in \mathbb{N}$ we have $sup L_i(N) \leq \varepsilon + sup V(N)$.

(ii) If $V$ is real-valued on $N$, then $sup V(N) < \infty$, and if $V$ is additionally continuous on $N$, then the convergence in (25) is uniform with respect to $p \in N$.

**Proof.** If (i) does not hold, then there exist $\varepsilon > 0$, $p \in N$ and a sequence $(x_i)_{i \in \mathbb{N}}$ in $N$ converging to $p$ and satisfying $L_i(x_i) > \varepsilon + sup V(N)$ for infinitely many $i \in \mathbb{N}$. This implies $\limsup_{i \to \infty} L_i(x_i) > V(p)$, which contradicts Proposition A.2. The same argument with the inequality $L_i(x_i) > \varepsilon + V(x_i)$ proves the second claim in (ii), and the first claim follows since $V$ is u.s.c. by Corollary IV.2, and so $V(N) \subseteq \mathbb{R}$ implies $sup V(N) < \infty$. Finally, the identity (25) follows from the estimate $V \leq L_i$ and the special case $N = \{p\}$ of (i).

Assertion (i) in Corollary VI.9 can be seen as a completeness result. Indeed, if for every initial state in a compact subset $N \subseteq X$ the achievable closed-loop performance for (8) is finite, then using sufficiently precise abstractions it is possible to synthesize controllers for (8) whose worst-case performance gaps on $N$ are arbitrarily small. In particular, the closed-loop value functions are finite on $N$. An interesting special case arises when the cost functions (2) map into the discrete set

$$D = \lambda \mathbb{Z}_+ \cup \{\infty\}$$

for some $\lambda \in \mathbb{R}_+$, in which the subcase $\lambda = 0$, or equivalently, $D = \{0, 1\}$, corresponds to qualitative problems. Then, without loss of generality, all abstract cost functions map into the set (26) either. We would like to explicitly spell out this case, which includes, e.g. the Reach-Avoid Problem and the Minimum Time Problem in Examples III.5 and III.6:

**VI.10 Corollary.** Assume $(A_2)$ and $(A_3)$. Suppose that both the concrete cost functions $g$ and $G$ and the abstract cost functions $g_i$ and $G_i$ map into the set (26), for some $\lambda \in \mathbb{R}_+$ and every $i \in \mathbb{N}$.

Then for every compact subset $N \subseteq X$ we have $sup L_i(N) = sup V(N)$ for all sufficiently large $i \in \mathbb{N}$. In particular, if $\lambda = 0$ and $V$ vanishes on some compact subset $N \subseteq X$, so does $L_i$ for all sufficiently large $i \in \mathbb{N}$.

**VII. Algorithmic Solution**

The practical applicability of our main results in Section VI depends on our ability to both compute finite abstractions of arbitrary precision and solve finite optimal control problems. For the sake of self-consistency of the present paper, we shall discuss both issues, where for the former problem we focus on our solution in [26] for a class of optimal control problems arising in the context of sampled-data control systems.

**A. A sampled optimal control problem**

We introduce a class of optimal control problems for which we devised an algorithm in [26] to compute finite abstractions of arbitrary precision. The discrete-time plant represents the sampled
behavior of a continuous-time control system, which we describe by a nonlinear differential equation with additive, bounded disturbances of the form

\[ \dot{x} \in f(x, u) + [-w, w] \]

(27)

where \( f: \mathbb{R}^n \times U \to \mathbb{R}^n \), \( U \subseteq \mathbb{R}^m \), and \( w \in \mathbb{R}^n_+ \). Here, the summation in (27) is interpreted as the Minkowski set addition \([28]\), and \([-w, w]\) denotes a hyper interval in \( \mathbb{R}^n \) given by \([-w, w] = [-w_1, w_1] \times \ldots \times [-w_n, w_n] \). Given an input signal \( u: J \subseteq \mathbb{R} \to U \), a locally absolutely continuous map \( \xi: I \to \mathbb{R}^n \) is a solution of (27) on \( I \) generated by \( u \) if \( I \subseteq J \) is an interval and \( \xi(t) \in f(\xi(t), u(t)) + [-w, w] \) holds for almost every \( t \in I \). Whenever \( u \) is constant on \( I \) with value \( \bar{u} \in U \), we slightly abuse the language and refer to \( \xi \) as a solution of (27) on \( I \) generated by \( \bar{u} \).

We consider the following optimal control problem associated with the sampled behavior of (27).

**VII.1 Definition.** Given a sampling time \( \tau > 0 \) and cost functions

\[ g_1: \mathbb{R}^n \times \mathbb{R}^n \times U \to \mathbb{R}_+ \cup \{\infty\}, \quad G_1: \mathbb{R}^n \to \mathbb{R}_+ \cup \{\infty\}, \]

the tuple \( \Pi_1 = (X_1, U_1, F_1, G_1, g_1) \) is the optimal control problem associated with (27) and \( \tau \), where \( X_1 = \mathbb{R}^n \), \( U_1 = U \), and \( F_1: X_1 \times U_1 \rightrightarrows X_1 \) is implicitly defined by \( x' \in F_1(x, u) \) iff there exists a solution \( \xi \) of (27) on \([0, \tau]\) generated by \( u \in U \) that satisfies \( \xi(0) = x \) and \( \xi(\tau) = x' \).

The following hypothesis ensures that \( \Pi_1 \) is actually an optimal control problem in the sense of Definition III.3 that additionally satisfies Hypothesis (A₂), i.e., a problem to which our results in Section VI apply.

**(A₂)** The input set satisfies \( U = \bigcup_{i \in [1:l]} [\bar{u}_i, \bar{\bar{u}}_i] \), with \( \bar{u}_i, \bar{\bar{u}}_i \in \mathbb{R}^n_+ \), \( \bar{u}_i \leq \bar{\bar{u}}_i \), and \( l \in \mathbb{N} \). The function \( G_1 \) and \( g_1 \) is continuous on the set \( G_1^{-1}(\mathbb{R}) \) and \( g_1^{-1}(\mathbb{R}) \), respectively, and these sets are open. The map \( f \) is continuous, and for all \( i, j \in [1:n] \), the partial derivative \( D_j f_i \) with respect to the \( j \)th component of \( f_i \) exists and is continuous. Every solution \( \xi \) of (27) on \([0, s]\) generated by some \( u \in U \), where \( s < \tau \), can be extended to a solution on \([0, \tau]\) generated by \( u \).

**VII.2 Lemma (Lemma 1 [26]).** Consider an optimal control problem \( \Pi_1 = (X_1, U_1, F_1, G_1, g_1) \) associated with (27) and \( \tau > 0 \) and suppose that (A₁) holds. Then \( \Pi_1 \) is an optimal control problem that satisfies (A₂).

For the actual computation of abstractions, we introduce the domain \( K \) of \((X_1, U_1, F_1, G_1, g_1)\),

\[ K = \{ p \in X_1 | g_1(X_1, p, U_1) \cup g_1(p, X_1, U_1) \cup \{G_1(p)\} \neq \{\infty\} \} \]

(28)

which includes the effective domain of the value function.

In the construction of an abstraction of the optimal control problem associated with (27) various bounds related to the dynamics and the cost functions are used, as detailed below. Here and in Section VIII, \(|x|\) and \(\|x\|\) denote the component-wise absolute value, respectively, the infinity norm of \(x \in \mathbb{R}^n\), and all balls are understood with respect to the infinity norm.

**(A₃)** Let \( K \) be defined by (28). Let \( K' \) be convex and compact and so that for every \( u \in U \) and every solution \( \xi \) of (27) on \([0, \tau]\) generated by \( u \) with \( \xi(0) \in K \) we have \( \xi([0, \tau]) \subseteq K' \). The constants \( A_0 \in \mathbb{R}^n_+ \), \( A_1 \in \mathbb{R}^{n \times n} \), \( A_2, A_3 \geq 0 \) and \( \varepsilon > 0 \) satisfy the inequalities (component-wise)

\[ A_0 \geq |f(p, u)| + w, \]  

(29a)

\[ (A_1)_{i,j} \geq \begin{cases} D_j f_i(x, u), & \text{if } i = j, \\ |D_j f_i(x, u)|, & \text{otherwise} \end{cases} \]  

(29b)
for all \( u \in U \) and all \( p \in \bar{B}(K', \varepsilon) \). Moreover, for all \( p, \bar{p} \in G_{\tau}^{-1}(\mathbb{R}) \) we have
\[
\|p - \bar{p}\| A_2 \geq |G_1(p) - G_1(\bar{p})|,
\]
and for all \( (p, q, u), (\bar{p}, \bar{q}, u) \in g_{\tau}^{-1}(\mathbb{R}) \) we have
\[
(\|p - \bar{p}\| + \|q - \bar{q}\|) A_3 \geq |g_1(p, q, u) - g_1(\bar{p}, \bar{q}, u)|.
\]

We refer the interested reader to [26] for a discussion of the computation of the quantities in \( (A_3) \).

Following [26], an abstraction \( \Pi_2 \) of the optimal control problem \( \Pi_1 \) associated with (27) and \( \tau \) is obtained as follows. Let \( \Pi_1 \) and \( \Pi_2 \) be of the form (17). The state alphabet \( X_2 \) is constructed from a uniform discretization of the domain (28) of \( \Pi_1 \) using the discretization parameter \( \eta \in (\mathbb{R}_+ \setminus \{0\})^n \). Similarly, the input alphabet \( U_2 \) is obtained by a discretization of \( U_1 \) using the discretization parameter \( \mu \in (\mathbb{R}_+ \setminus \{0\})^m \). The transition function \( F_2 \) is obtained from an over-approximation of the attainable set of (27) whose computation is outlined in Algorithm 1 in [26]. To this end, the sampling time \( \tau \) is subdivided in \( k \) inter-sampling times \( t = \tau/k \). At each of those inter-sampling times, the attainable set is over-approximated by a union of hyper-intervals using a growth bound [2, Def. VIII.2], [41] to bound the distance of neighboring trajectories. Here the estimates \( A_0 \) and \( A_1 \) in \( (A_3) \) are instrumental. In order to control the error due to the over-approximation, at each inter-sampling time, each hyper-interval in the approximation can be subdivided in smaller hyper-intervals, whose size is determined by the parameter \( \theta > 0 \). Throughout the computation, several initial value problems have to be solved numerically. The resulting error together with other errors, e.g. rounding errors, can be accounted for using the parameter \( \gamma > 0 \). The cost functions \( G_2, g_2 \) of the abstraction are derived from the values of the cost functions \( G_1, g_1 \) evaluated at the discretized states and inputs. The Lipschitz constants in (29c) and (29d) are used to ensure that the functions \( G_2, g_2 \) indeed are upper bounds in the sense of (ii) and (iii) in Proposition VI.2. The parameters of the construction of the abstraction in [26] are summarized in Tab. I.

We use \( \Pi \) to refer to the optimal control problem associated with (27) and \( \tau \), and we consider sequences of parameters in Tab. I satisfying
\[
\lim_{i \to \infty} \eta_i = 0, \; \lim_{i \to \infty} \mu_i = 0, \; \lim_{i \to \infty} \left( \theta_i \|\eta_i\| + \frac{1}{k_i} + \gamma_i k_i \right) = 0.
\]

Then the method in [26, Sec. V] produces a sequence \( (\Pi_i)_{i \in \mathbb{N}} \) of finite abstractions \( \Pi_i \) of some precision \( \rho_i \in \mathbb{R}_+ \cup \{\infty\} \) of \( \Pi \), satisfying \( \lim_{i \to \infty} \rho_i = 0 \), as required in hypothesis \( (A_3) \) in Section VI-C. See [26, Th. 1, 2].

### B. Solution of finite optimal control problems

We propose Algorithm 1 to efficiently solve the optimal control problem (8) whenever the state and input alphabets are finite; see Theorem VII.3 below. The algorithm can be regarded as an implementation of the high-level algorithm in [34], with improved run time bound and suitable modifications to additionally compute a controller realizing the achievable closed-loop performance. We also present a condition under which the run time is linear in the size of the abstraction of the

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**Table I**

Parameters of the computation of the abstraction in [26].

| \( \eta \in (\mathbb{R}_+ \setminus \{0\})^n \) | state alphabet discretization |
| \( \mu \in (\mathbb{R}_+ \setminus \{0\})^m \) | input alphabet discretization |
| \( k \in \mathbb{N} \) | sample interval discretization |
| \( \theta > 0 \) | subdivision factor |
| \( \gamma > 0 \) | bound on numerical errors |
Algorithm 1 Dijkstra-like algorithm to solve finite problems

**Input:** Optimal control problem \((X, U, F, G, g)\)

**Require:** \(X, U\) finite

1. \(W := G\) // value function
2. \(Q := \{x \in X \mid G(x) < \infty\}\) // priority queue
3. \(E := \emptyset\) // set of settled states
4. for all \(p \in X\) do
   5. \(c(p) := \emptyset\) // controller
6. while \(Q \neq \emptyset\) do
   7. \(q := \arg \min \{W(x) \mid x \in Q\}\)
   8. \(Q := Q \setminus \{q\}\)
   9. \(E := E \cup \{q\}\)
10. for all \((p, u) \in F^{-1}(q)\) do
    11. \(M := \max \{g(p, y, u) + W(y) \mid y \in F(p, u)\}\)
    12. if \(F(p, u) \subseteq E\) and \(W(p) > M\) then
    13. \(W(p) := M\)
    14. \(Q := Q \cup \{p\}\)
    15. \(c(p) := \{u\}\)

**Output:** \(c, W\)

plant. This result applies e.g. to the Reach-Avoid and Minimum Time Problems in Examples III.5 and III.6, and contains the unweighted case of [42] as a special case. In the following, \(\text{card}(M)\) denotes the cardinality of the set \(M\).

**VII.3 Theorem.** If \(X\) and \(U\) are finite, then Algorithm 1 terminates, the controller
\[C = (\{0\}, \{0\}, X, X, U \times \{0, 1\}, F', H')\] defined by the requirement that \(F'(0, p) = \{0\}\) and
\[H'(0, p) = \begin{cases} \{(u_0, 1, p)\}, & \text{if } c(p) = \emptyset, \\ c(p) \times \{0\} \times \{p\}, & \text{otherwise} \end{cases} \tag{30}\]
for all \(p \in X\) and some \(u_0 \in U\), is static and feedback composable with \(S\), and \(L = V = W\), where \(S\), \(L\) and \(V\) denote the system (5), the closed-loop value function of \(S\) associated with \(C\), and the value function of (8).

Moreover, Algorithm 1 can be implemented such that it runs in \(O(m + n \log n)\) time, where \(n = \text{card}(X)\) and \(m = \sum_{p \in X} \sum_{u \in U} \text{card}(F(p, u))\), and in \(O(m)\) time if additionally
\[g(X, X, U) \subseteq \{\gamma, \infty\} \text{ and } G(X) \subseteq \{\Gamma, \gamma + \Gamma, \infty\}\] \tag{31}
for some \(\gamma, \Gamma \in \mathbb{R}_+\).

**Proof (Sketch).** It is easily seen that, throughout the algorithm, the value of \(W(q)\) on lines 8-15 monotonically increases and satisfies \(W(q) \geq \max W(E)\). Thus, each \(q\) is removed from \(Q\) at most once, the while-loop is entered at most \(n\) times, and the algorithm terminates. Moreover, using Theorem IV.1 it is straightforward to show that \(W \geq V\) on lines 6-15 throughout the algorithm, and \(W = V\), upon termination.

To see that \(L = W\) upon termination, note that if \(q \not\in E\), then \(W(q) = \infty\) and \(c(q) = \emptyset\), and so \(L(q) = \infty\) by (30), (7a) and (9), and additionally \(L(q) = W(q)\) on line 9 throughout the algorithm. Indeed, assume that \(L(x) = W(x)\) holds on line 8 for all \(x \in E\). Then \(c(q) \neq \emptyset\) and
\[W(q) = \max \{g(q, y, c(q)) + L(y) \mid y \in F(q, c(q))\}\] \tag{32}
Moreover, for every \((u, v, x) \in B_q(C \times S)\) we have \(v(0) = 0\) by (30), and in turn, \(J(u, v, x) = g(q, x(1), c(q)) + J(\sigma u, \sigma v, \sigma x)\). Then \(L(q) = W(q)\) by (9) and (32).

The data \(G, W, E\) and \(c\) are maintained as arrays, so the respective operations in the algorithm require unit time. Given an adjacency lists representation \([33]\) of \(F\) that also stores the map \(g\), both an analogous representation of \(F^{-1}\) can be obtained and the condition (31) can be verified, in \(O(m)\) time.

Lines 11-15 are executed at most \(m\) times. Using auxiliary counters the conditions \(F(p, u) \subseteq E\) on line 12 can be tested in \(O(m)\) total time \([42]\), and analogously for computing the maximum on line 11. Thus, Algorithm 1 requires \(O(m)\) time, plus the time for executing line 2, executing lines 7, 8 and 14 at most \(n\) times, and for executing line 13 at most \(m\) times. Consequently, the first time bound is met if \(Q\) is maintained as a Fibonacci heap \([33, \text{Th. A.18}]\). If condition (31) holds, then \(M = \gamma + W(q)\) on line 13, and so \(W(Q) \subseteq \{W(q), \gamma + W(q)\}\) on line 8. Thus, \(Q\) can be maintained as a FIFO queue \([33]\), which proves the second bound.

\[\square\]

VIII. Illustrative Examples and Applications

We apply our results to an optimal control problem involving chaotic dynamics and to two classical synthesis problems.

A. A minimum time problem involving chaotic dynamics

To demonstrate the capability of our theory to approximate complex value functions, we first apply it to an instance \(\Pi\) of the Minimum Time Problem in Example III.6 whose underlying dynamics is chaotic. Specifically, \(\Pi = ([0, 1], \{0\}, F, G, g)\), where the transition function \(F\) is the logistic map \([43]\), \(F(p, 0) = \{4p(1 - p)\}\), and the target and obstacle sets are given by \(D = \{0.415, 0.69\}\) and \(M = \emptyset\).

The value function \(V\) of \(\Pi\) is discontinuous and rather irregular, see Fig. 4, but can be determined exactly by rewriting the iteration in Corollary IV.2 into an iteration for sublevel sets, \(V^{-1}(0) = D\) and \(V^{-1}(0; T + 1) = F^{(-1)}(0; T)\).

For every \(N \in \mathbb{N}\) it is straightforward to compute an abstraction \(\Pi_N = (X_N, \{0\}, F_N, G_N, g_N)\) of precision \(1/N\) of \(\Pi\), where \(F_N\) satisfies the conditions in Definition VI.3,

\[
X_N = \{\Omega_0, \ldots, \Omega_N\},
\]

\[
\Omega_i = \left[\frac{i}{N} + \left[-\frac{1}{2N}, \frac{1}{2N}\right]\right] \cap [0, 1],
\]

\[
g_N(\Omega, \Omega', 0) = 1, \quad \text{and}
\]

\[
G_N(\Omega) = \begin{cases} 
0, & \text{if } \Omega \subseteq D, \\
\infty, & \text{otherwise},
\end{cases}
\]

for all \(i \in [0; N]\) and all \(\Omega, \Omega' \in X_N\). The value function \(V_N\) of \(\Pi_N\) is easily computed using Algorithm 1 in Section VII-B. Fig. 4 illustrates the approximation of \(V\) by \(V_N\), for selected values of the precision \(1/N\).

B. An entry-time problem for the inverted pendulum

We consider a variant of the popular inverted pendulum problem with perturbations, where the motion of the cart is not modeled; see e.g. \([44], [45]\). The acceleration \(u\) of the cart, which is constrained to \([-2, 2]\), is the input to the system

\[
\dot{x}_1 = x_2
\]

\[
\dot{x}_2 = \sin(x_1) + u \cos(x_1) - 2\kappa x_2 + [-w, w],
\]

the states \(x_1\) and \(x_2\) correspond to the angle, respectively, the angular velocity of the pole, \(\kappa = 0.01\) is a friction coefficient, and \(w = 0.1\) accounts for any uncertainties.
We restrict the domain of the problem to \( K = [-2\pi, 2\pi] \times [-3, 3] \), i.e., the set \( \mathbb{R}^2 \setminus K \) is an obstacle, and choose a neighborhood \( D \) of the upwards pointing equilibrium \((0, 0)\), \( D = \{ x \in \mathbb{R}^2 \mid 63x_1^2 + 12x_2x_1 + 56x_2^2 < 42 \} \), as the target set. In correspondence with \( K \) and \( D \), we define the terminal and running cost functions \( G \) and \( g \) by 
\[
G(p) = 0 \quad \text{if} \quad p \in D \cap K = D, \\
g(p, q, u) = \begin{cases} 
    u^2, & \text{if} \quad p \in K, \\
    \infty, & \text{otherwise},
\end{cases}
\]
We use \( \Pi \) to refer to the optimal control problem associated with the system (33), the sampling time \( \tau = 0.2 \) and the cost functions \( G \) and \( g \). With \( \Pi \) we aim at minimizing the actuation energy to steer the system into the target \( D \). We pick the constants in \( (A_5) \) to
\[
A_0 := \begin{pmatrix} 4 \\ 2.5 \end{pmatrix}, \quad A_1 := \begin{pmatrix} 0 \\ 2.25 \end{pmatrix}, \quad A_2 := 0, \quad A_3 := 0.
\]
We use \( A_0 \) to verify that \( K' = \overline{B}(\text{cl}K, 0.9) \) contains any solution of (33) originating from \( K \) since \( \overline{B}(K, \tau\|A_0\|) \subseteq K' \). Moreover, (29) is satisfied on \([-8, 8] \times [-4, 4] \supseteq \overline{B}(K', 0.1) \), and we see that \( (A_5) \) holds for \( \varepsilon = 0.1 \).

We conducted several experiments using \( \theta = 1 \) and four parameter tuples \((\eta, \mu, k)\) with values \( p_1 = ((0.08, 0.08), 0.2, 1), p_2 = ((0.04, 0.04), 0.15, 2), p_3 = ((0.02, 0.02), 0.1, 3) \) and \( p_4 = ((0.01, 0.01), 0.05, 4) \). The solution of initial value problems, which are necessary in the construction of the abstraction [26], we use the Taylor series method [46] of order 5 with stepsize \( \tau/(5k) \). We use \( \gamma \) to account for any numerical errors, which we derive from the 6th order remainder term of the Taylor expansion maximized over the appropriate domain. Specifically, for \( k = 1, k = 2, k = 3 \) and \( k = 4 \) we obtain \( \gamma = 6.3 \cdot 10^{-7}, \gamma = 9.9 \cdot 10^{-9}, \gamma = 8.7 \cdot 10^{-10} \) and \( \gamma = 1.6 \cdot 10^{-10} \), respectively. The computation time to
compute the abstraction $\Pi_i$ and the optimal controller $C_i$ (Alg. 1) is 0.5, 8.5, 139 and 4715 seconds, for the parameter tuple $p_i$, $i \in [1; 4]$, respectively. (Here and for the following example, computations are conducted on 3.5 GHz Intel Core i7 CPU with 32GB memory.) The performance of the controllers $C_1 \circ \in$ through $C_4 \circ \in$ is illustrated in Fig. 5.

C. The Homicidal Chauffeur Game

In this pursuit-evasion game, a car with restricted turning radius, traveling at some constant velocity, aims at catching an agile pedestrian as quickly as possible [36]. The problem can be posed as a Minimum Time Problem by choosing the center of the car as origin and directing the $y$ axis along the velocity vector of the car. The dynamics is then described by

$$\dot{x} = -yu + v_1$$
$$\dot{y} = xu - 1 + v_2,$$

where the input $|u| \leq 1$ is the forward velocity of the car, and $v = (v_1, v_2)$ is the velocity vector of the pedestrian [36], which we consider as a perturbation with bound $||v|| \leq 0.3$. Using the sampling time $\tau = 0.1$ and the domain $K = [-5, 5] \times [-5, 5]$, we cast the sampled differential game as Minimum Time Problem with target set $D = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 < 0.9\}$ and the obstacle set $M = \mathbb{R}^2 \setminus K$. The cost functions follow according to Example III.6 and it is straightforward to verify the Hypothesis $(A_5)$ as follows. We fix $\varepsilon = 0.1$, $A_0 = (6.4, 6.4)$, and $(A_1)_{11} = (A_1)_{22} = 0$, $(A_1)_{12} = (A_1)_{21} = 1$, $A_2 = A_3 = 0$ and $K' = [-6, 6] \times [-6, 6]$. The estimates (29) are obvious, and (29a) implies that every solution $\xi$ on $[0, \tau]$ evolves inside $B(K, \tau ||A_0||) \subseteq K'$, and so $(A_5)$ holds.

We approximately solve $\Pi$ using $\theta = 2$ and four parameter tuples $(\eta, \mu, k)$ with values $p_1 = ((0.03, 0.03), 0.2, 1)$, $p_2 = ((0.02, 0.02), 0.1, 2)$, $p_3 = ((0.015, 0.015), 0.1, 3)$ and $p_4 = ((0.01, 0.01), 0.05, 4)$. As the nominal dynamics under constant control inputs can be solved exactly, we neglect the numerical
errors and set $\gamma = 0$. The computation time to compute the abstraction $\Pi_i$ and the optimal controller $C_i$ (Alg. 1) is 3.5, 34, 133 and 1851 seconds, for the parameter tuple $p_i$, $i \in [1; 4]$, respectively. Naturally, with finer discretization parameters the computation times increases. The performance of the controllers $C_1 \circ \in$ through $C_4 \circ \in$ is illustrated in Fig. 6.

**IX. Summary and Conclusions**

We have presented a novel approach to solve a class of leavable, undiscounted optimal control problems in the minimax sense for nonlinear control systems in the presence of perturbations and constraints. The approach is correct-by-construction, i.e., the closed-loop value function of the synthesized controller is upper bounded by the closed-loop value function of the abstract controller, and compared to previously known results, it is applicable to more general cost functions and plant dynamics, and the resulting controllers are memoryless and symbolic. Moreover, as we have shown, the closed-loop value function associated with the concrete controller hypo-converges to the concrete value function as the precision of the abstraction approaches zero. This powerful convergence result distinguishes itself form previously known results in several important aspects. Most notably, it applies to discontinuous value functions and implies that our approach is complete in a well-defined sense.

We have illustrated our results on three optimal control problems, two of which involving discrete-time plants that represent the sampled behavior of continuous-time, nonlinear control systems with additive disturbances. Here, we employed an algorithm that we have proposed in [26], to compute abstractions of arbitrary precision. The algorithm is based on the uniform discretization of the state space, and hence, the computational complexity is expected to growth exponentially with the dimension of the state space of the control system. To increase the computational efficiency of the overall synthesis approach proposed in this paper is a subject of our current research.
APPENDIX

A. Relationship to the theory in reference [2]

The following lemma establishes the exact relationship between our notions of feedback composit-
ability and behavior and the respective notions in [2], to facilitate our use of the theory from the
latter work.

A.1 Lemma. Let $S$ and $C$ be systems of the form (5) and (6), respectively. Consider the system

$$\hat{S} = (X, X, \hat{U}, \hat{U}, X, \hat{F}, \text{id})$$

(34)

with $\hat{U} = U \times \{0, 1\}$ and for all $x \in X$, $u \in U$, $v \in \{0, 1\}$ we have $\hat{F}(x, (u, v)) = F(x, u)$. Then $C \in \mathcal{F}(X, U)$ iff $C$ is feedback composable with $\hat{S}$ in the sense of [2, Def. III.3]. Let $C \in \mathcal{F}(X, U)$, then $(u, v, x) \in \mathcal{B}(C \times S)$ iff $(0, ((u, v), x)) \in \mathcal{B}(C \times \hat{S})$, where $\mathcal{B}(C \times \hat{S})$ denotes the behavior of the closed-loop composed of $C$ and $\hat{S}$ in the sense of [2, Def. V.1].

Proof. Note that $F$ being strict implies that $\hat{F}$ is strict, which in turn implies that $(Z)$ in [2, Def. III.3] is satisfied with $C$ and $\hat{S}$ in place of $S_1$ and $S_2$, respectively. Moreover, $X \subseteq U''$ and $Y' \subseteq U$ holds iff $X \subseteq U'$ and $Y' \times \{0, 1\} \subseteq U$ holds and it follows that $C \in \mathcal{F}(X, U)$ iff $C$ is feedback composable (f.c.) with $\hat{S}$ according to [2, Def. III.3].

Let $C \in \mathcal{F}(X, U)$ and $(x, z', x', (u, v))$ be a solution of $C$. Then $(u, u, x, x)$ is a solution of $S$ iff $((u, v), (u, v), x, x)$ is a solution on $[0; \infty]$ of $\hat{S}$ according to [2, Def. III.1]. Using [2, Prop. III.4], we see that $(0, (z', u), (x', x), ((u, v), x))$ is a solution on $[0; \infty]$ of $C \times \hat{S}$ iff $(x, z', x', (u, v))$ is a solution of $C$ and $(u, u, x, x)$ is a solution of $S$. Since $F$ and $F'$ are strict, it follows from [2, Def. V.1] that $(0, ((u, v), x)) \in \mathcal{B}(C \times \hat{S})$ iff $(x, u, v) \in \mathcal{B}(C \times S)$.

B. The Notion of Hypo-Convergence

The result below shows that, for the special case considered in Definition VI.7, that definition is
equivalent to respective definitions in the literature [28, Ch. 7.B], [29, Cor. VII.5.26].

A.2 Proposition. Let $X$, $V$ and $L$ be as in Definition VI.7. Then $V = \text{h-lim}_{i \to \infty} L_i$ iff

$$\limsup_{i \to \infty} L_i(x_i) \leq V(p) \text{ for every } p \in X \text{ and every sequence } (x_i)_{i \in \mathbb{N}} \text{ converging to } p.$$ 

Proof. For sufficiency, let $p \in X$ and $\varepsilon > 0$, and assume that the condition in Definition VI.7 does not hold. Then there exists a sequence $(x_i)_{i \in \mathbb{N}}$ in $X$ converging to $p$ and satisfying $L_i(x_i) > V(p) + \varepsilon/2$ for infinitely many $i \in \mathbb{N}$. This implies $\limsup_{i \to \infty} L_i(x_i) > V(p)$, which is a contradiction. For necessity, assume that the latter inequality holds for some $p \in X$ and some sequence $(x_i)_{i \in \mathbb{N}}$ in $X$ converging to $p$. Then $L_i(x_i) \geq \lambda > V(p)$ for some $\lambda \in \mathbb{R}$ and infinitely many $i \in \mathbb{N}$. In addition, as $V$ is u.s.c., there exists $\varepsilon > 0$ such that $V(q) < \lambda - \varepsilon$ for all $q \in B(p, 2\varepsilon)$. As $V = \text{h-lim}_{i \to \infty} L_i$ there exists a neighborhood $N \subseteq X$ of $p$ such that (24) holds for all sufficiently large $i \in \mathbb{N}$. Then there exists some $i$ such that $x_i \in B(p, \varepsilon)$ and $(x_i, \lambda) \in B(\text{hypo } V, \varepsilon)$. In turn, there exists $(q, \alpha) \in \text{hypo } V$ such that $\lambda < \alpha + \varepsilon$ and $x_i \in B(q, \varepsilon)$. This implies $q \in B(p, 2\varepsilon)$, hence $\alpha \leq V(q) < \lambda - \varepsilon$, which is a contradiction.

C. Some Results on Semi-Continuous Maps

Throughout, $X$ and $Y$ are metric spaces. See [29], [47].

A.3 Theorem (Berge’s Maximum Theorem). Let $H : X \rightrightarrows Y$ be compact-valued and u.s.c.,
and let $f : X \times Y \rightrightarrows [-\infty, \infty]$ be u.s.c.. Then the map $g : X \rightrightarrows [-\infty, \infty]$ defined by $g(x) = \sup \{f(x, y) \mid y \in H(x)\}$ is u.s.c..

A.4 Proposition. Let $\Omega \subseteq X$ be compact, and suppose that the sequence $(f_k)_{k \in \mathbb{N}}$ of u.s.c. maps $f_k : X \to [-\infty, \infty]$ is monotonically decreasing and converges pointwise to $g : X \to [-\infty, \infty]$. Then $\lim_{k \to \infty} \sup_{x \in \Omega} f_k(x) = \sup_{x \in \Omega} g(x)$, where limits are understood to take values in $[-\infty, \infty]$. 
A.5 Proposition. The map $H : X \to Y$ is compact-valued and u.s.c. iff the following condition holds: If $(x_k, y_k)_{k \in \mathbb{N}}$ is a sequence in the graph of $H$ and $(x_k)_{k \in \mathbb{N}}$ converges to $p \in X$, then there exists a subsequence of $(y_k)_{k \in \mathbb{N}}$ converging to some point in $H(p)$.

A.6 Corollary. If the map $H : X \to Y$ is compact-valued and u.s.c., then $H(\Omega)$ is compact for all compact subsets $\Omega \subseteq X$.

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