Erdős-Rényi phase transition in the Axelrod model on complete graphs

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Abstract
The Axelrod model has been widely studied since its proposal for social influence and cultural dissemination. In particular, the community of statistical physics focused on the presence of a phase transition as a function of its two main parameters, \(F\) and \(Q\). In this work, we show that the Axelrod model undergoes a second order phase transition in the limit of \(F \to \infty\) on a complete graph. This transition is equivalent to the Erdős-Rényi phase transition in random networks when it is described in terms of the probability of interaction at the initial state, which depends on a scaling relation between \(F\) and \(Q\). We also found that this probability plays a key role in sparse topologies by collapsing the transition curves for different values of the parameter \(F\). We explore the extent of this collapse and the dynamical mechanisms that lead to this.

1 Introduction
The Axelrod model, originally proposed for cultural dissemination [1], is grounded in two key dynamical features: Social influence, through which people become more similar when they interact; and homophily, which is the tendency of individuals to interact preferentially with similar ones. Specifically, the agents are described by a vector of \(F\) components called cultural features, which can take one of \(Q\) integer values called cultural traits. The dynamics of the model is based on an imitation rule: A random agent adopts a cultural trait of another one with a probability proportional to the number of shared features.

Despite its simplicity, the Axelrod model attracted the attention of the statistical physics community due to the emergency of a phase transition from a monocultural to a multicultural state [2,3]. The phase transition takes place by varying the number of cultural traits \(Q\) for a given fixed \(F\). If the number of cultural traits is low, the probability of interaction is high, leading the system

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to a monocultural state. If $Q$ is high, the mentioned probability is low and, after few interactions, the system evolves to a stationary multicultural state.

This phase transition is usually studied by taking the size of the biggest fragment as the order parameter. The transition was reported to be continuous for one-dimensional networks and discontinuous for two dimensions when $F > 2$ [4], although a continuous transition is recovered when the topology becomes small-world [5]. On the other hand, for $F = 2$, the type of the transition is the opposite: Continuous for 2-D, and discontinuous for small-world networks [5]. The case of $F = 2$ is important due to the possibility of taking an analytical approach to study the model [6].

Several scaling relationships have been found in the Axelrod model. For instance, in [7] and [8] a finite-size scaling analysis is performed for $F = 2$ in square-lattices and small-world networks. A scaling relation between $Q$ and size $N$ can be found in scale-free networks for $F = 10$ [3], a scaling relationship between the density of active bonds and time in one-dimensional network in [9], and the finding of an effective noise rate is explored in [10].

Among the reported scaling relations, there is a particular one reported in [11] and [12] where the transition curves in one-dimensional networks collapse when the control parameter is $F/Q$. This ratio has an immediate interpretation as the mean value of shared features given two agents in the initial state, suggesting that the initial distribution contains key information about the final outcome of the Axelrod model.

In this work, we review the Axelrod model in terms of the initial interaction probability between agents. In particular, we found that the second order phase transition in the limit of $F \to \infty$ on a complete graph is equivalent to the phase transition observed in Erdős-Rényi random networks [13]. In this model, a set of $N$ initially disconnected nodes are linked with a probability $p$, and for $p > p_c$, a fragment which scales with the size of the system emerges [13,14].

Notoriously, the initial interaction probability between agents plays also a key role in describing the Axelrod model on sparse topologies, by making the transition curves collapse for different values of $F$. We end our work by discussing the mechanisms which lead to this collapse in the transition curves.

2 Results

We studied the transition of the Axelrod model for a wide range of $F$ and $Q$ values. We explore the transition between mono and multicultural states on two different topologies: Complete graphs and the classical two-dimensional lattice.

2.1 Axelrod model in complete graphs

We analyze the transition in the Axelrod model for a $N = 1024$ complete graph. In Fig. 1 we show the relative size of the biggest fragment in the initial and final state, for different values of $F$. A fragment is defined as a group of topologically connected agents by active links, which are defined as a link who connect two agents with at least one feature in common. We can observe that the biggest fragment in the final state is equal or smaller than in the initial state for low values of $F$. When $F$ increases, this difference approaches to zero. This result
suggests that in the limit of $F \to \infty$, the size of the stationary biggest fragment is fully determined at the initial condition.

The importance of the initial condition is reflected in the fact that two agents who initially do not share any feature cannot interact (at least, until other interactions take place and eventually change their cultural states). Given two agents, the parameters $F$ and $Q$ set their initial number of shared features, by sampling this quantity from a binomial distribution with parameters $F$ and $1/Q$. If we define the interaction probability $p_{\text{int}}$ as the probability of having an active link between them:

$$p_{\text{int}} = 1 - (1 - \frac{1}{Q})^F,$$

the initial state of the Axelrod model on a complete graph is equivalent to the Erdős-Rényi model with parameter $p_{\text{int}}$.

In the limit of $F \to \infty$ the stationary sizes of the biggest fragment converge to their initial state, as can be seen in Fig. 1. This suggests that the transition in this limit of the Axelrod model is similar to the Erdős-Rényi one. In fact, Fig. 2 shows that this happens by taking $p_{\text{int}}$ as the control parameter. This can be observed both for the biggest fragment (panel (a)) and the average finite-fragment size (panel (b)).

The definition of $p_{\text{int}}$ (Eq. (1)) allows to estimate the critical value of the transition, $Q_c$, in the limit of $F \to \infty$. Since the biggest fragment emerges in an Erdős-Rényi network when $Np_{\text{int}} = 1$ [14], it gives:

$$Q_c = (1 - (1 - \frac{1}{N})^{\frac{1}{F}})^{-1},$$
in the Axelrod model. This analogy provides a good estimation of $Q_c$ for large values of $F$. For instance, if $N = 1024$ and $F = 100$, $Q_c \sim 10^5$, as can be seen in Fig. 1.

The equivalence between both phase transitions can be completed by the calculation of the critical exponents. We estimate them for the case of $F = 100$, given the closeness of its curves to the Erdős-Rényi transition (see Fig. 2). The critical exponents [14] are introduced following the usual relationships:

$$S_{\text{max}} \sim (p_{\text{int}} - p_{\text{int}}^c)^\beta,$$

$$\langle s \rangle \sim |p_{\text{int}} - p_{\text{int}}^c|^{-\gamma},$$

where $p_{\text{int}}^c$ is the critical probability, $S_{\text{max}}$ is the biggest fragment and $\langle s \rangle$ is the average finite-fragment size, which is a measure of the fluctuations of the order parameter. For finite systems, the critical exponents can be calculated by performing finite-size scaling following [15]. Here, the authors propose the following scaling relationships:

$$S_{\text{max}} = N^{-\frac{\beta}{\nu}} F_1[(p_{\text{int}} - p_{\text{int}}^c)N^{\frac{\beta}{\nu}}]$$

$$(s) = N^{\frac{\gamma}{\nu}} F_2[(p_{\text{int}} - p_{\text{int}}^c)N^{\frac{\gamma}{\nu}}],$$

where $F_1$ and $F_2$ are unknown scaling functions, but with the property that $F_{1(2)}(x) \rightarrow \text{constant}$, when $x \rightarrow 0$ (which means, near the critical value). This
implies that exactly at the critical point, $S_{\text{max}} \sim N^{-\frac{\beta}{\nu}}$ and $\langle s \rangle \sim N^\gamma$. These expressions can be used to estimate the relation between exponents without knowledge about $F_1$ and $F_2$. The argument of these scaling functions defines another relationship followed by the exponent $\nu$ and the critical value $p_{\text{int}}^c$, which can be read in the following equation:

$$p_{\text{int}}^c(N) = p_{\text{int}}^c - bN^{-\frac{\beta}{\nu}},$$

where $p_{\text{int}}^c(N)$ is the pseudo-critical value in which $\langle s \rangle$ takes its maximum values as was shown in panel (b) of Fig. 2 for different values of $F$. Finally, the fragments size distribution $f(s)$ near the critical point follows a power-law distribution with parameter $\tau$, i.e. $f(s) \sim s^{-\tau}$. This relation defines this last critical exponent.

Fig. 3 shows the scaling relationships for $F = 100$ as a function of $N$. Panel (a) shows the scaling relationship derived from Eq. (4). Panel (b) and (c) show the scaling relationships at the critical point for both the biggest fragment and the average fragment size. Finally, panel (d) shows that the fragments size distribution follows a power-law distribution at the critical point. The estimated exponents are pointed out both in Fig. 3 and Table 1. The exponent $\tau$ was calculated following the methodology sketched in [16].

Table 1 shows that the estimations are consistent with the theoretical values predicted for the Erdős-Rényi model [14]. Given that the equivalence is in the limit of $F \to \infty$, we expect that the matching between both set of exponents improves for larger values of $F$.

Figure 3: Finite-size scaling and critical behaviour for $F = 100$ on a complete network. Full lines point out the fitted curves. The estimation of the critical exponents are also shown. Panel (d) belongs to the fragment distribution at the critical point with $N = 1024$. 

\begin{align*}
\nu^{-1} &= -1.12 \pm 0.01 \\
-\beta/\nu &= -0.35 \pm 0.05 \\
\gamma/\nu &= 0.34 \pm 0.02 \\
\tau &= -2.4 \pm 0.2
\end{align*}
Table 1: Critical exponents. Estimation from the Axelrod model with $F = 100$ and the predicted theoretical values for the Erdős-Rényi phase transition.

| Exponent | Estimated     | Theoretical |
|----------|---------------|-------------|
| $p_{int}$ | $(1 \pm 0.7) \times 10^{-4}$ | 0           |
| $\nu$    | $0.89 \pm 0.02$ | 1           |
| $\beta/\nu$ | $0.35 \pm 0.05$ | $1/3$       |
| $\gamma/\nu$ | $0.34 \pm 0.02$ | $1/3$       |
| $\tau$   | $2.4 \pm 0.2$  | 2.5         |

It should be noticed that the equivalence between both phase transitions (described by similar critical exponents) is not present in other topological features because of the dynamical evolution of the Axelrod model. This could be understood as follows: Given an active link between two agents, the Axelrod model always tends to increase their similarity. An active link can only become inactive by third party interactions. When the value of $F$ increases, the probability that an active link becomes inactive decreases and goes to zero when $F \to \infty$. Then, the initial active links define the sizes of the connected components (as in Erdős-Rényi model) and the only effect of the dynamics is to transform all the connected components in cliques of the same size.

2.2 Axelrod model in 2D lattices

Let’s explore now the Axelrod transition in classical 2D lattices as a function of the new control parameter $p_{int}$. Fig. 4 shows the transition curves for different $F$ as a function of $p_{int}$, in addition to the initial size of the biggest fragment. The inset of this figure shows the same curves as function of $Q$, where it can be seen that the transition shifts to larger values of $Q$ when $F$ increases. Fig. 4 also shows that all transition curves collapse to one. This collapse is also found in other sparse topologies, as random regular networks with equivalent mean degree (not shown). However, in contrast to the observed behavior in complete networks (see Fig. 1), the collapsed curve does not match the corresponding to the initial state.

The collapse as a function of $p_{int}$ is essentially the same pointed out by [11] for one dimensional networks. As was mentioned above, $F$ and $Q$ set the number of shared features for a pair of agents, by sampling this quantity from a binomial distribution with parameters $F$ and $1/Q$. These quantities also set the value of $p_{int}$. In the limit of large $F$ and $Q$, this binomial distribution can be well approximated by a Poisson one with parameter $F/Q$, which is the mean shared features by two random agents, and it is the control parameter introduced in [11] for one-dimensional networks.

To understand the mechanisms underlying the collapsing of the curves as a function of $p_{int}$, we look for a dynamical observable who tracks the activation and deactivation of links during the dynamics. We define the fraction of deactivated links $f_{dl}$:

$$f_{dl} = \frac{d}{d + c},$$

where $d$ is the number of active links which became inactive during the
dynamics and $c$ the number of links who were inactive and become active. This quantity can be measured both, in complete networks and lattices. In lattices, we also discriminate between any pairs of agents (homophilic links) and pairs of connected agents (physical links). This distinction is unnecessary for complete networks.

Fig. 5 shows, in panel (a) and (b), the value of $f_{dl}$ as a function of $p_{int}$ near the critical point for both homophilic and physical links in a lattice. Panel (a) shows that the homophilic links have the same rate of deactivation, independently of the value of $F$. Panel (b) shows this same behaviour for larger values of $p_{int}$, but the curves differ when $p_{int}$ goes to zero. However, in this case there is few events associated to an activation or deactivation of a physical link, and the fluctuations of $f_{dl}$ makes unclear the differences among curves. Within the error bars, we conclude that the dynamic makes a similar effect on the initial state no matter the value absolute of $F$, leading to the collapse of the transition curves observed in Fig. 4.

In contrast, panel (c) of Fig. 5 shows that $f_{dl}$ on a complete network is systematically lower for $F = 100$ respect to other values of $F$, over a range of values of $p_{int}$. A low value of $f_{dl}$ means that there are more links that become active than inactive, and in particular leads to preserve most of the initial active links in the final state. This preservation is consistent with the observed equivalence between phase transitions sketched in previous section.
Figure 5: **Fraction of deactivated links.** Panel (a) and (b) shows the homophilic and physical links on a 2D lattice. The largest error bars are due to the occurrence of few events that activate or deactivate links. Panel (c) shows homophilic links for a complete network (same as physical links in this case). These results correspond to a system with \( N = 100 \).

### 3 Conclusions

In this work, we show that the Axelrod model on complete graphs displays a phase transition equivalent to the one observed in the Erdős-Rényi model when the order parameter is plotted as a function of \( p_{\text{int}} \), which is the probability that two agents share at least one feature in the initial state. This happens in the limit of \( F \rightarrow \infty \). This claim is supported by the calculation of critical exponents following the approach of finite size scaling sketched in [15]. Despite this similarity, the Axelrod dynamics leads to a stationary state where the connected components are also cliques, in contrast to what happens in the Erdős-Rényi model.

When same scaling relationship is analyzed in Axelrod model on sparse graphs, we found the collapse of transition curves for different values of \( F \). These collapsed curves do not coincide with the initial state, as did in complete networks for \( F \rightarrow \infty \). Both behaviors can be understood in terms of the fraction of links that become inactive during the dynamics. We have observed that this quantity is the same for different values of \( F \) in a lattice and tends to zero when \( F \rightarrow \infty \) in a complete network.

Summarizing, the dynamics of the Axelrod model produces different stationary states depending on the underlying connectivity. For complete network, all agents with homophily different from zero (active links) are able to interact. In
the particular case of \( F \to \infty \), the active links have low probability to become inactive and therefore are preserved in the final state (see Fig. 5). Here, the main effect of the dynamics is to transform the connected components (fixed by the set of initial active links) in cliques of the same size. On the other hand, the lack of this kind of physical connectivity in sparse networks changes dramatically the effect of the dynamics. Most of the agents which share at least one feature are not able to interact and the final state cannot be obtained by simply fulfilling cliques. However, we observed that the rate of activation and deactivation of links is the same for different values of \( F \), leading to the curve collapse observed in Fig. 3.

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