Efficient Range Reporting of Convex Hull

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Abstract. We consider the problem of reporting convex hull points in an orthogonal range query in two dimensions. Formally, let $P$ be a set of $n$ points in $\mathbb{R}^2$. A point lies on the convex hull of a point set $S$ if it lies on the boundary of the minimum convex polygon formed by $S$. In this paper, we are interested in finding the points that lie on the boundary of the convex hull of the points in $P$ that also fall within an orthogonal range $[x_l, x_r] \times [y_b, y_t]$. We propose a $O(n \log^2 n)$ space data structure that can support reporting points on a convex hull inside an orthogonal range query, in time $O(\log^3 n + h)$. Here $h$ is the size of the output.

This work improves the result of (Brass et al. 2013) [5] that builds a data structure that uses $O(n \log^2 n)$ space and has a $O(\log^5 n + h)$ query time. Additionally, we show that our data structure can be modified slightly to solve other related problems. For instance, for counting the number of points on the convex hull in an orthogonal query rectangle, we propose an $O(n \log^2 n)$ space data structure that can be queried upon in $O(\log^3 n)$ time. We also propose a $O(n \log^2 n)$ space data structure that can compute the area and perimeter of the convex hull inside an orthogonal range query in $O(\log^4 n)$ time.

Keywords: Convex Hull, Plane, Orthogonal range, Reporting, Counting, Area, Perimeter

1 Introduction

Range searching is one of the most commonly studied topics in spatial databases and computational geometry. Informally, range searching can be stated as follows: Given a point set $P$ we wish to pre-process $P$ into a data structure such that given any rectangular query region $q$, we can efficiently report the points in $P \cap q$ or count their number in $P \cap q$. Range aggregate geometric querying is a variant of range searching where we calculate a geometrical function $f$ inside $P \cap q$. In this work we efficiently compute $f(P \cap q)$ where the function $f$ is to compute the convex hull for the point set $P \cap q$. Range aggregate querying has been studied recently by Kalavagatttu et al. [2], Das et al. [3] and Brodal et al. [4] for maximal points.

However, advances in the creation and archival of digital information have led to an information explosion and therefore the number of objects inside a query range can be huge. In such cases, reporting a sample of the result set is preferred.
In this paper, we use the concept of convex hull query for reporting the boundary points. Reporting of convex points can be useful in situations where we need a shape with minimum area/perimeter for a set of points. Therefore, it may have applications in spatial databases [10], computer graphics and computer vision.

In this work, we propose a data structure to solve the problem of reporting convex hull points in an orthogonal query rectangle. We also study the problem of counting the number of points on the convex hull inside $P \cap q$, and computing its area/perimeter.

1.1 Problem Definitions

In this section, we formally define the problems and related terminology. Given a set of $n$ points $P = \{p_1, \ldots, p_n\}$ in $\mathbb{R}^2$. We use $x(p)$ to represent the $x$ co-ordinate of a point $p$ and $y(p)$ to represent its $y$ co-ordinate. The convex hull of any point set $S$ in the Euclidean plane is the smallest convex polygon that contains $S$. Henceforth, when we refer to convex hull points, we essentially mean the points on the boundary of the convex hull. We are given a set $P$ of $n$ points in $\mathbb{R}^2$ and a query $q = [x_{lt}, x_{rt}] \times [y_b, y_t]$. We find the convex hull on the point set $P \cap q$, denoted by $ch(P \cap q)$. We also denote points with minimum $x$ co-ordinate, maximum $x$ co-ordinate, minimum $y$ co-ordinate and maximum $y$ co-ordinate of point set $P \cap q$ by $x_{\text{min}}$, $x_{\text{max}}$, $y_{\text{min}}$ and $y_{\text{max}}$ respectively.

In this work, we study the following problems: We wish to pre-process $P$ into a data structure such that given an orthogonal query region $q$, we can efficiently

- **Problem 1**: report the points on convex hull of $P \cap q$.
- **Problem 2**: count the number of points on the convex hull of $P \cap q$.
- **Problem 3**: find the area of the convex hull of $P \cap q$.
- **Problem 4**: find the perimeter of the convex hull of $P \cap q$.

1.2 Previous Work

The convex hull for a static data set of two-dimensional points can be computed in $O(n \log h)$ time [1] where $h$ is the number of points on the convex boundary. Gupta et al. discuss non-intersecting queries on aggregated geometric data [9]. Brass et al. gave the first solution for **Problem 1** that takes $O(n \log^2 n)$ space, $O(n \log^3 n)$ preprocessing time and $O(\log^5 n + h)$ query time [5]. For any given orthogonal range query on a standard 2d-range tree they identify $O(\log^2 n)$ disjoint canonical convex hulls. There are $O(\log^4 n)$ pairs of disjoint convex hulls and they compute the tangent for each pair of disjoint convex hulls. Computing tangents between each pair of local convex hulls takes $O(\log n)$ time [7]. They use a method similar to the gift-wrapping algorithm. Therefore, total query time is $O(\log^5 n + h)$.

1.3 Our Results

The following table summarizes the results presented in this paper.
| Query Type | Query Time | Preprocessing Time | Storage Space | Theorems |
|------------|------------|--------------------|---------------|----------|
| Reporting  | $O(\log^3 n + h)$ | $O(n \log^3 n)$ | $O(n \log^2 n)$ | Theorem 1 |
| Counting   | $O(\log n)$ | $O(n \log^3 n)$ | $O(n \log^2 n)$ | Theorem 2 |
| Area       | $O(\log^3 n)$ | $O(n \log^3 n)$ | $O(n \log^2 n)$ | Theorem 3 |
| Perimeter  | $O(\log n)$ | $O(n \log^3 n)$ | $O(n \log^2 n)$ | Theorem 4 |

The rest of the paper is organized as follows. In Section 2 we give the details of the preprocessing and the query algorithm for reporting convex points inside an orthogonal range query. In Section 3 we study the problem of counting convex hull points inside $P \cap q$ and the problem of computing the area/perimeter of the convex hull inside $P \cap q$. We discuss future work and conclude in Section 4.

## 2 Our Solution

In this section we explain our solution to the problem of reporting the convex hull points inside $P \cap q$. In Section 2.1 we describe how to construct the required data structure of size $O(n \log^2 n)$ in time $O(n \log^3 n)$ on a static point set $P$. In Section 2.2 we explain how to report points on the convex hull inside $P \cap q$ for a monotonic chain from $x_{\text{max}}$ to $y_{\text{max}}$. Points on the convex hull can be broken into four monotone chains, namely maximal chain from $x_{\text{max}}$ to $y_{\text{max}}$, maxY-minX chain from $y_{\text{max}}$ to $x_{\text{min}}$, minimal chain from $x_{\text{min}}$ to $y_{\text{min}}$ and minY-maxX chain from $y_{\text{min}}$ to $x_{\text{max}}$ as shown in Figure 1(b). Area of such a convex hull gets divided into four quadrants Q1, Q2, Q3 and Q4 as shown in Figure 1. In this work we present an algorithm to construct the maximal chain from $x_{\text{max}}$ to $y_{\text{max}}$. A similar approach can be applied for the other monotone chains.

### 2.1 Preprocessing

Our solution uses a 2d-range tree as described in [5]. Each internal node of the primary tree $T_x$ represents a horizontal range $[x_i, x_j]$ for $i \neq j$. The set of points
rooted at an internal node $v$ are represented by $S(v)$. For each internal node $v$ of $T_x$ we have a secondary (binary) tree $T_y(v)$ such that leaves of tree $T_y(v)$ store the points in $S(v)$ in non-decreasing order of their $y$-coordinate. Each internal node of the secondary tree $T_y(v)$ represents a vertical range $[y_i, y_j]$ for $i \neq j$.

Given a query $q = [x_l, x_r] \times [y_l, y_r]$, we search $x_l$ and $x_r$ in the primary tree to get canonical nodes $v_1, v_2, \ldots, v_i, \ldots, v_l$. As shown in Figure 2. Here $l = O(\log n)$ is the height of the primary tree. Therefore, the range query $[x_l, x_r]$ is divided into $l = O(\log n)$ canonical subsets $S(v_1), S(v_2), \ldots, S(v_l)$. Similarly, we search $y_l$ and $y_r$ in the secondary tree to get canonical nodes $w_1, w_2, \ldots, w_i, \ldots, w_l$. Here $m = O(\log |S(v_i)|)$ is the height of the secondary tree. Therefore, in every secondary tree, the range query $[y_l, y_r]$ gets divided into $m = O(\log n)$ canonical subsets $S(w(i, 1)), S(w(i, 2)), \ldots, S(w(i, m))$. For each internal node $w$ of the secondary tree, we compute the convex hull over the set $S(w)$. We store points of this convex hull in an array $ch_w$ in the anti-clockwise direction starting from the point with the maximum $y$-coordinate. We also find the point $p_{\text{max}}x_{w(i,j)}$ with maximum $x$-coordinate from the set $S(w(i,j))$ and store it at each internal node $w$. Therefore, any given query $q = [x_l, x_r] \times [y_l, y_r]$ gets divided into $O(\log^2 n)$ disjoint canonical subsets and for each of these subsets we have stored a convex hull $ch_{w(i,j)}$ on the set $S(w(i,j))$.

It takes $O(|S(w)| \log |S(w)|)$ time to compute convex hull $ch_w$ over set $S(w)$ and $O(|S(w)|)$ space to store $ch_w$ at internal node $w$ of secondary tree $T_y(v)$. Therefore, it takes $\sum_{v \in T_x(v)} O(|S(w)| \log |S(w)|) = O(|T_y| \log^2 |T_y|)$ preprocessing time and $\sum_{v \in T_x(v)} O(|S(w)|) = O(|T_y| \log |T_y|)$ space for secondary tree $T_y(v)$.

Fig. 2. (a) Range tree with the canonical nodes highlighted (b) Processing an orthogonal range query for $y_l$ and $y_r$ in the secondary tree $T_y(v_i)$ for $i = 1, 2, 3, \ldots, l$ to get canonical nodes $w(i, 1), w(i, 2), \ldots, w(i,j), \ldots, w(i, m)$. Here $m = O(\log |S(v_i)|)$ is the height of the secondary tree. Therefore, in every secondary tree, the range query $[y_l, y_r]$ gets divided into $m = O(\log n)$ canonical subsets $S(w(i, 1)), S(w(i, 2)), \ldots, S(w(i,m))$. For each internal node $w$ of the secondary tree, we compute the convex hull over the set $S(w)$. We store points of this convex hull in an array $ch_w$ in the anti-clockwise direction starting from the point with the maximum $y$-coordinate. We also find the point $p_{\text{max}}x_{w(i,j)}$ with maximum $x$-coordinate from the set $S(w(i,j))$ and store it at each internal node $w$. Therefore, any given query $q = [x_l, x_r] \times [y_l, y_r]$ gets divided into $O(\log^2 n)$ disjoint canonical subsets and for each of these subsets we have stored a convex hull $ch_{w(i,j)}$ on the set $S(w(i,j))$.

For any given query $[x_l, x_r]$ on the primary tree $T_x$ we have $l = O(\log n)$ secondary trees. Therefore, it takes $\sum_{v \in T_x} O(|T_y(v)| \log^2 |T_y(v)|) = O(n \log^3 n)$ preprocessing time and $\sum_{v \in T_x} O(|T_y(v)| \log |T_y(v)|) = O(n \log^2 n)$ storage space.
Lemma 1. Given a set \( P \) of \( n \) points in \( \mathbb{R}^2 \), we can pre-process \( P \) into a data structure that takes \( O(n \log^2 n) \) storage space and \( O(n \log^3 n) \) pre-processing time (as explained above).

2.2 Query Algorithm

In this algorithm we use two stacks, a hull stack \( HS \) and a tangent stack \( TS \). An element of hull stack \( HS \) is a pointer to some canonical convex hull \( \text{ch}(w(i,j)) \). An element of tangent stack \( TS \) is a tuple \( t = (i_1, i_2) \) where \( i_1 \) and \( i_2 \) are indices of points on two different convex hulls \( C_1 \) and \( C_2 \) as shown in Figure 3. Given an orthogonal range query \( q = [x_{lt}, x_{rt}] \times [y_{lb}, y_{ub}] \), the query algorithm for reporting convex hull points in \( P \cap q \) is as follows:

1. Express the range of \( x \) co-ordinates in \( [x_{lt}, x_{rt}] \) as the disjoint union of \( l = O(\log n) \) canonical subsets. Let the canonical nodes be \( v_1, v_2, \ldots, v_l \) from left to right in that order, as shown in Figure 2(a).
2. Consider a node \( v_i \) of the primary tree \( T_x \). Make a range query \( [y_{lb}, y_{ub}] \) on the associated secondary tree \( T_y(v_i) \) to find canonical nodes \( w(i,1), w(i,2), \ldots, w(i,m) \). Do a linear search on \( x(p_{maxw(i,1)}), x(p_{maxw(i,2)}), \ldots, x(p_{maxw(i,m)}) \) to find point \( X_{maxw} \) with maximum \( x \) co-ordinate. Then traverse the canonical nodes from right to left, starting from \( v_l \) back to \( v_1 \), as shown in Figure 2(a). Initialize \( i \leftarrow l \) and \( y_{low} \leftarrow y(X_{maxw}) \).
3. Consider a node \( v_i \) of the primary tree \( T_x \). For each internal node \( v_i \) of \( T_x \) there is an associated secondary tree \( T_y(v_i) \). Make a range query \( [y_{low}, y_{ub}] \) on secondary tree \( T_y(v_i) \) to find canonical nodes \( w(i,1), w(i,2), \ldots, w(i,m) \), as shown in the Figure 2(b) where \( m = O(\log |T_y(v_i)|) \).
4. Consider the array \( \text{ch}_{w(i,j)} \). If the array \( \text{ch}_{w(i,j)} \) is empty then go to step 5. Otherwise set \( y_{low} \leftarrow y(\text{ch}_{w(i,m)}[1]) \) and then traverse the canonical nodes from bottom to top, starting from \( w(i,1) \) back to \( w(i,m) \) as shown in Figure 2(b) as follows:

   for \( j \leftarrow 1 \) to \( m \) call Algorithm \text{Merge}(\text{ch}_{w(i,j)}, \text{HS}, \text{TS}) \) (see Section 2.2.1).

5. At this point, we have processed the nodes \( v_l, v_{l-1}, \ldots, v_i \). Set \( i \leftarrow i - 1 \) and if \( i \geq 1 \), move to the node \( v_i \) and goto step 3 else exit.
6. Call Algorithm \text{Report}(\text{HS}, \text{TS}) \) (see Section 2.2.2) to report the points on the convex monotone from maximum \( x \) to maximum \( y \) inside \( P \cap q \).

2.2.1 Merge Algorithm

In this section we explain the Algorithm \text{Merge}(\_\_) used for merging canonical convex hulls for any given query. Note that the algorithm proposed is similar to the Graham Scan algorithm \[8\] where we combine points which are sorted on \( x \) co-ordinate. Instead of points we have disjoint convex hulls sorted on non-increasing \( x \) co-ordinate in the stack \( HS \). In line 1 of the algorithm, a while loop checks whether hull stack \( HS \) has sufficient elements to continue. In line 2 we
Algorithm 1: Merge()

Data: \( ch_{w(i,j)} \), \( HS = \emptyset \), \( TS = \emptyset \)

Result: Updated Stacks \( HS \) and \( TS \)

1. while \( \text{size}(HS) \geq 2 \) do
2. \( C_1 \leftarrow \text{secondtop}(HS) \); \( C_2 \leftarrow \text{top}(HS) \); \( C_3 \leftarrow ch_{w(i,j)} \);
3. find points \( P_a \) and \( P_b \) on tangent \( T_1 = (i_1, i_2) \) joining hulls \( C_1 \) and \( C_2 \);
4. find points \( P_c \) and \( P_d \) on tangent \( T_2 = (i_3, i_4) \) joining hulls \( C_2 \) and \( C_3 \);
5. if \( \text{orient}(P_a, P_b, P_d) \leq 0 \) then /* (refer Figure 3(b)) */
6. pop(\( HS \)); pop(\( TS \));
7. else /* (refer Figure 3(a)) */
8. push(\( T_2 \)); break;
9. end
10. end
11. push \( ch_{w(i,j)} \) onto stack \( HS \);

store hulls at \( \text{secondtop}(HS) \), \( \text{top}(HS) \) and \( ch_{w(i,j)} \) in variables \( C_1 \), \( C_2 \) and \( C_3 \). In line 3 we find points \( P_a \leftarrow C_1[i_1] \) and \( P_b \leftarrow C_2[i_2] \) incident with tangent \( T_1 = (i_1, i_2) \) between hulls \( C_1 \) and \( C_2 \) as shown in the Figure 3. Computing such a tangent takes \( O(\log(n_1 + n_2)) \) points where \( n_1 = |C_1| \) and \( n_2 = |C_2| \) are sizes of the convex hulls. Similarly in line 4 we find points \( P_c \leftarrow C_2[i_3] \) and \( P_d \leftarrow C_3[i_4] \) incident with tangent \( T_2 = (i_3, i_4) \) between hulls \( C_2 \) and \( C_3 \) as shown in the Figure 3. In line 5 we compute the orientation of the points \( P_a, P_b \) and \( P_d \) using function \( \text{orient()} \) (a primitive operation in computational Geometry). If the orientation of the points is clockwise (negative) as shown in Figure 3(b) then we pop out an element from the stack \( HS \) and the stack \( TS \) in line 6, else we push tangent \( T_2 \) into stack \( TS \) and break out of the loop in line 8. Every time an element gets pushed onto the stack we iterate the above procedure on the top two elements of stack \( HS \) and \( ch_{w} \) to discard canonical set of convex hulls \( ch_{w(i,j)} \) that do not contribute points to \( ch(P \cap q) \). For every call to Algorithm Merge() there can be

![Figure 3. (a)Positive Orientation (b) Negative Orientation](image-url)
zero or more pop() operations on stack HS. An element once popped out of the
stack HS never gets pushed in again. Therefore, the total number of push and
pop operations is $O(|HS|)$. For each pop and push operation on the stack HS,
the required tangent can be computed in $O(\log n)$ time. Finding the tangent
between any two canonical convex hulls takes $O(\log(n1 + n2)) = O(\log n)$ time
[7]. Therefore, the merging algorithm takes no more than $O(|HS| \cdot \log n)$ time
where $|HS|$ is size of the stack. We end this section with the following lemma.

Lemma 2. Given the stack HS, the stack TS and the hull $ch_{w(i,j)}$ as input to
Algorithm Merge(), it takes $O(|HS| \cdot \log n)$ time where $|HS|$ is size of the
stack HS.

2.2.2 Reporting algorithm In this section we explain the algorithm Report() used for reporting points on the convex hull. This reporting algorithm is the
last step of the query algorithm. We report points from canonical convex hulls

Algorithm 2: Report()

Data: Stack HS, Stack TS
Result: Convex hull points in $P \cap q$
1 $ch \leftarrow \text{pop}(HS)$; index $i \leftarrow \text{size}(ch)$;
2 Report the point $ch[i]$ with maximum y co-ordinate;
3 while size(HS) > 0 do
4 tangent $t \leftarrow \text{pop}(TS)$;
5 index $j \leftarrow t(i_2)$;
6 Report points from $ch[i]$ to $ch[j]$;
7 $i \leftarrow t(i_1)$;
8 array $ch \leftarrow \text{pop}(HS)$;
9 end
10 Report points from $ch[i]$ till $x_{\text{max}}$ in array $ch$;

stored in stack HS using information about tangents stored in the stack TS.
In line 1 of the algorithm we pop out a hull from the stack HS and index $i$
is assigned with the size of array $ch$. In line 2 we report the point $ch[i]$ with
maximum y co-ordinate. In line 3 we check the while loop condition, if the stack
is not empty then we enter the while loop. In line 4 we pop out tangent $t$ from
the stack TS and in line 5 we assign $j$ with the first index $i_2$ of the tangent. In
line 6 we report points from $ch[i]$ to $ch[j]$. In line 7 we assign $i$ with the second
index $i_1$ of tangent $t$ and in line 8 we pop out a canonical convex hull from the
stack TS.

In the algorithm Report() we iterate till the stack HS is empty and in
each iteration of the while loop we report a set of points. Let total number points reported by Algorithm Report() be $h$. It takes $O(|HS|)$ time for the algorithm Report() because while loop gets iterated $|HS|$ times. Therefore, it takes
$O(|HS| + h)$ time for algorithm Report(). We end this section with the following
lemma.
Lemma 3. Given the stack $HS$ and the stack $TS$ as input to Algorithm Report(), it takes $O(|HS| + h_1)$ time where $|HS|$ is size of the stack $HS$ and $h$ is the number of points reported.

2.3 Putting Everything Together

We will now prove the following theorem:

Theorem 1. Given a set $P$ of $n$ points in $\mathbb{R}^2$, we can pre-process $P$ into a data structure of size $O(n \log^2 n)$ in time $O(n \log^3 n)$ such that, given an orthogonal range query $q = [x_l, x_r] \times [y_b, y_t]$, we can report the points of the convex hull inside $P \cap q$ in time $O((\log^3 n + h))$, where $h$ is the number of convex hull points reported.

Proof: Lemma\[1\] for preprocessing indicates the storage space used by our data structure and the preprocessing time to build it. Now we analyze the complexity of our query algorithm explained in Section 2.2. In step 1 it takes $O((\log n) + l)$ time to find $l = O((\log n))$ canonical nodes because the height of primary tree $T_x$ is $O((\log n))$. In step 2 it takes $O((\log |S(v_i)|))$ time to find canonical subset of nodes $w(l, 1), w(l, 2), \ldots, w(l, m)$. It takes $m = O((\log |S(v_i)|))$ time to perform a linear search on $x_{\max}(w_{i, 1}), x_{\max}(w_{i, 2}), \ldots, x_{\max}(w_{i, m})$ to find point $p_{\max}$ with maximum $x$ co-ordinate. In step 3 we consider each $v_i$ from $i \leftarrow 1$ to $l$. In $i^{th}$ iteration of step 3 we spend $O((\log |S(v_i)|)) = O(\log n)$ time finding the canonical subset of nodes $w(i, 1), w(i, 2), \ldots, w(i, m)$ by making an updated query on the associated secondary tree $T_y(v_i)$. Therefore, the total time spent in finding all canonical sets of nodes for any given query is $O((\log |S(v_1)|)) + O((\log |S(v_2)|)) + \ldots + O((\log |S(v_l)|))$ for $i = 1, 2, \ldots, l$ then total time is $O(l \cdot \log n) = O((\log^2 n)$.

In step 4 at most $m = O((\log^2 n)$ calls are made to Algorithm Merge(). Therefore total calls made to algorithm Merge() is $m \cdot l = O((\log^2 n)$). From Lemma\[2\] we know that time taken for Algorithm Merge() is $O(|HS| \cdot \log n)$ Therefore, it takes $O((\log^3 n)$ time to find updated $HS$ that contains convex hulls that contribute at least one point to $ch(P \cap q)$. In step 6 of the query algorithm a call to the algorithm Report($HS, TS$) is made. According to Lemma\[3\] it takes $O(|HS| + h_1)$ time where $|HS|$ is size of the stack $HS$ and $h$ is the number of points reported. Therefore, it takes $O(|HS| + h_1)$ time for step 6 to execute where $|HS| = O((\log^2 n)$. Recall that time taken for step 3 – 5 is $O((\log^3 n)$. Therefore it takes $O((\log^3 n + O((\log^2 n) + h_1) = O((\log^3 n + h_1)$ to report the points of $ch(P \cap q)$ in the first quadrant Q1 (see Figure\[4\]).

3 Related Problems

In this section we study related problems such as counting the number of convex hull points and finding the area/perimeter of the convex hull inside $P \cap q$. One may argue that it is not necessary to study these problems separately once we have reported the points of $ch(P \cap q)$ in query time $O((\log^3 n + h)$ using the algorithm described in Section 2.2 where $h$ is the total number of points on
the convex hull. However, $h$ can be as large as $O(n)$ for some queries. If these problems can be solved independent of $O(h)$ then the running time is unaffected by output size. With a few clever modifications on the data structure explained in Section 2.1 and the algorithm proposed in Section 2.2 we can achieve results that are independent of output size for the above mentioned problems.

3.1 The Counting Problem

Most of the preprocessing of the point set $P$ for the counting problem is the same as the preprocessing described in Section 2.1. In addition to it, at each internal node $w$ of every secondary tree $T_y(.)$ we store three indices $\text{index}_{xmax}$, $\text{index}_{xmin}$ and $\text{index}_{ymin}$ where $ch_w[\text{index}_{xmax}]$, $ch_w[\text{index}_{xmin}]$ and $ch_w[\text{index}_{ymin}]$ are points with maximum $x$ co-ordinate, minimum $x$ coordinate and minimum $y$ co-ordinate. To get the stack $HS$ which contains all the canonical convex hulls that contribute to convex hull $ch(P \cap q)$ we use same Query Algorithm of Section 2.2. Below is the algorithm for counting the number of points on the convex hull of set $S(P \cap q)$. To count the points in $ch(P \cap q)$, algorithm counting() can be used as subroutine in step 6 of the query algorithm. Algorithm Counting() is similar in spirit to the Algorithm Report() of Section 2.2.2. In Step 2 of this algorithm count is initialized to 1 because we count all the points from point $ch[j]$ till the point with maximum $y$ co-ordinate in $P \cap q$. In Step 6 the variable count gets updated with the number of points that the current canonical hull contributes to the output. This is computed as the difference $i - j + 1$ in each iteration of while loop. This process is repeated until the stack $HS$ is empty. In step 10 count gets updated with the difference $\text{index}_{xmax} - i + 1$ after exiting from the while loop, where $\text{index}_{xmax}$ is the index of point with the maximum $x$ co-ordinate in the array $ch$.

It can be seen that Algorithm Counting() runs in $O(|HS|) = O(\log^2 n)$ query time to count points of the convex hull $ch(P \cap q)$ from maximum $x$ to maximum $y$. Similar counting algorithms can be used to find the count of points of the convex hull $ch(P \cap q)$ for the other three monotone chains (see also Figure 1).
Theorem 2. Given a set $P$ of $n$ points in $\mathbb{R}^2$, we can pre-process $P$ into a data structure of size $O(n \log^2 n)$ in time $O(n \log^3 n)$ such that, given an orthogonal range query $q = [x_l, x_r] \times [y_b, y_t]$, we can count the points of the convex hull inside $P \cap q$ in time $O(\log^3 n)$.

3.2 The Perimeter Problem

For computing the perimeter of the convex hull of the points in $P \cap q$ on a two-dimensional point set $P$ and an orthogonal range $q$, we perform a preprocessing phase that is similar to the one described in Section 2.1. In addition, at every internal node $w$ of every secondary tree, we store an auxiliary array $P_w$ that stores the cumulative perimeter. $P_w[i] = \sum_{1 \leq j \leq i} \text{dist}(j, j+1)$ where $\text{dist}(j, j+1)$ is the distance between points $ch_w[j]$ and $ch_w[j+1]$ respectively. Below Algorithm Perimeter() used for finding the perimeter of the convex hull $ch(P \cap q)$ from maximum $x$ to maximum $y$. Algorithm Perimeter() is similar in spirit to the reporting algorithm (Algorithm Report) of Section 3.1. In Step 2 of this algorithm, the variable perimeter is initialized to 0. In Step 6 we store point $ch[i_1]$ of some hull $C_1$ into point $a$. In Step 7 the variable perimeter is updated with difference $P_w[i] - P_w[j]$ in each iteration of while loop. This process is repeated until the stack $HS$ is empty. In Step 13 the variable perimeter is updated with difference $P_w[i] - P_w[\text{index}_{\text{xmax}}]$ after exiting from while loop where $\text{index}_{\text{max}}$ is index of point with the maximum $x$ co-ordinate in array $ch$. All other steps are similar to counting algorithm in section 3.1.

It can be noticed that the Algorithm Perimeter runs in $O(|HS|) = O(\log^2 n)$ time to find perimeter of the convex hull $ch(P \cap q)$ from maximum $x$ to maximum $y$. A similar algorithm can be used to find the perimeter of the convex hull $ch(P \cap q)$ for other three monotone chains. (See also Figure 1.)
Theorem 3. Given a set P of n points in $\mathbb{R}^2$, we can pre-process P into a data structure of size $O(n \log^2 n)$ in time $O(n \log^3 n)$ such that, given an orthogonal range query $q = [x_l, x_r] \times [y_b, y_t]$, we can compute the perimeter of the convex hull inside $P \cap q$ in time $O(\log^3 n)$.

3.3 The Area Problem

During the preprocessing phase, an auxiliary array $A_w$ is stored at each internal node of every secondary tree. Each element of such an array stores the cumulative area $A_w[i] = \sum_{3 \leq j \leq i} aot(1, j - 1, j)$ where $aot(1, j - 1, j)$ is the area of the triangle between points $chw[1], chw[j - 1]$ and $chw[j]$ respectively. $A_w[1] = 0$ because it represents the area of point $chw[1]$ and $A_w[2] = 0$ because it is the area of the line joining points $chw[1]$ and $chw[2]$. Below is an algorithm for computing the area of the convex hull $(P \cap q)$. The algorithm Area() finds the area of quadrant Q1.

Algorithm 5: Area()

Data: Stacks $HS, TS, AR = \phi$
Result: Area of the convex hull of Q1 in Figure 1

1. $area_{Q1} = 0$;
2. point $fixedPoint$ array $ch \leftarrow pop(HS)$; index $i \leftarrow 1$;
3. push($chw[i]$) onto AR;
4. $y(fixedPoint) = y(ch[1])$;
5. while size($HS$) > 0 do
6. tangent $t \leftarrow pop(TS)$;
7. index $j \leftarrow t(i_2)$;
8. push($chw[j]$) onto AR;
9. $area_{Q1} \leftarrow area_{Q1} + (A_w[i] - A_w[j]) - aot(1, i, j)$;
10. $i \leftarrow t(i_1)$;
11. push($chw[i]$) onto AR;
12. ch=pop($HS$);
13. end
14. $j \leftarrow index_{y_{\max}}$;
15. push($chw[j]$) onto AR;
16. $area_{Q1} \leftarrow area_{Q1} + (A_w[i] - A_w[j]) - aot(1, i, j)$;
17. $x(fixedPoint) = x(ch[j])$;
18. point $k1 \leftarrow pop(AR)$;
19. while size($AR$) > 0 do
20. point $k2 \leftarrow pop(AR)$;
21. $area_{Q1} \leftarrow area_{Q1} + aot(fixedPoint, k1, k2)$;
22. $k1 \leftarrow k2$;
23. end

(Figure 1). Similarly we can find areas of the other quadrants Q2, Q3 and Q4. The area of $ch(P \cap q)$ can be obtained using the following formula.

$Area(ch(P \cap q)) = area_{Q1} + area_{Q2} + area_{Q3} + area_{Q4} - (|x(y_{\max}) - x(y_{\min})| *$
\[ |y(x_{\text{max}}) - y(x_{\text{min}})| \] where \( x_{\text{min}}, x_{\text{max}}, y_{\text{min}} \) and \( y_{\text{max}} \) are the points with minimum \( x \), maximum \( x \), minimum \( y \) and maximum \( y \) in \( P \cap q \).

**Theorem 4.** Given a set \( P \) of \( n \) points in \( \mathbb{R}^2 \), we can pre-process \( P \) into a data structure of size \( O(n \log^2 n) \) in time \( O(n \log^3 n) \) such that, given an orthogonal range query \( q = [x_l, x_r] \times [y_b, y_t] \), we can compute the area of the convex hull inside \( P \cap q \) in time \( O(\log^3 n) \).

### 4 Conclusion

In this work, we studied the problem of reporting convex hull points for any given orthogonal range query, in the Pointer Machine Model. We also solved the problem of counting, area and perimeter in \( O(\log^3 n) \) time. We restricted the point set to static two-dimensional points. It will be interesting to see these problems in higher dimensions and dynamic versions of the problems. It will be also interesting to study these problems related to convex hull in the Multi-shot Model. It will be interesting to study these problems in word-RAM model and Cell-probe Model.

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