A NOTE ON A RESULT OF SAKS AND ZYGMUND ON ADDITIVE
FUNCTIONS OF RECTANGLES

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Abstract. We modify the proof of the basic lemma of a paper of Saks and Zygmund on additive
functions of rectangles.

1. Introduction.
In the paper [1], Saks and Zygmund present a theorem on additive functions on rectangles that
implies a slightly more general form of two theorems of Besicovitch about the set of removable
singularities of a continuous or bounded analytic function on a simply connected domain.
The main result (Theorem 5.1) is based on a lemma (Lemma 5.1) whose proof needs to be modified.

2. Discussion on the proof of Saks-Zygmund’s lemma.
First of all let us introduce some definitions following [1].
We shall only consider rectangles and squares in the plane with sides parallel to the axis. A function
$F$ of rectangles is said to be additive if
$F(I_1 \cup I_2) = F(I_1) + F(I_2)$
for any pair of adjacent rectangles $I_1$ and $I_2$.
For a rectangle $I$, we will denote by $\delta(I)$ its diameter and by $|I|$ its measure. The function of
rectangles $F$ is said to be continuous if $F(I) \to 0$ as $\delta(I) \to 0$.
We consider the set of dyadic squares of the plane (meshes in [1]). The dyadic squares of order $n$ of
size $2^{-n}$ are the set of squares into which the plane is divided by the two systems of parallel lines
$x = k \cdot 2^{-n} \quad y = m \cdot 2^{-n}, \quad k, m \in \mathbb{Z}$.
Let $F$ be an additive and continuous function of rectangles, let $I_0$ be a fixed rectangle and $0 \leq \alpha \leq 2$.
Lemma 5.1 in [1] states that if $F$ satisfies the conditions
$$\liminf_{\delta(Q) \to 0} \frac{F(Q)}{|Q|^\alpha} \geq 0,$$
where the limit is taken over arbitrary squares containing any point $x \in I_0$, and
$$\liminf_{\delta(Q) \to 0} \frac{F(Q)}{|Q|} \geq 0,$$
where, again, the limit is taken over arbitrary squares containing any point $x \in I_0$ except for a set
of $\sigma$–finite $H^\alpha$ Hausdorff measure, then $F(I_0) \geq 0$.
Taking into account the continuity of $F$, the argument is based on the reduction to the case in which
$I_0$ is a dyadic square. In other words, one assumes that if an additive and continuous function of
rectangles $F$ satisfies $F(Q_d) \geq 0$ for any dyadic square $Q_d$, then $F(I_0) \geq 0$ for every rectangle $I_0$.
But this is false as the following example shows.

Proposition 1. There is an additive and continuous function of rectangles $F$ such that $F(Q_d) \geq 0$
for every dyadic square $Q_d$ but $F(I_0) < 0$ for some rectangle $I_0$.

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Proof.
A standard way to define an additive function $F$ of rectangles is the following one. Take a function $f$ defined on $\mathbb{R}^2$ and then, for any rectangle

$$I = [x_1, x_2] \times [y_1, y_2], \quad x_j, y_j \in \mathbb{R},$$

set $F(I) = f(x_2, y_2) + f(x_1, y_1) - f(x_1, y_2) - f(x_2, y_1)$, which is clearly additive.

Consider now the function $f$ defined as

$$f(x, y) = \begin{cases} 1 & \text{if } y \notin \mathbb{Q}, \\ xy & \text{if } y \in \mathbb{Q}. \end{cases}$$

Then, for any dyadic square $Q_d = [x_1, x_2] \times [y_1, y_2]$ one gets easily $F(Q_d) = (x_2 - x_1)(y_2 - y_1) = |Q_d| > 0$, but taking $I = [x_1, x_2] \times [y_1, y_2]$ with $y_1 \in \mathbb{Q}, y_1 > 0$ and $y_2 \notin \mathbb{Q}$, one gets $F(I_0) = (x_1 - x_2)y_1 < 0$.

Note that this function $F$ is continuous since $\delta(I) \to 0$ implies that $x_2 - x_1 \to 0$ and $y_2 - y_1 \to 0$ if $I = [x_1, x_2] \times [y_1, y_2]$ and consequently, $F(I) \to 0$ in any case. \hfill $\square$

Remark.
One could define another type of continuity for an additive function of rectangles $F$, namely that $F$ is continuous if $F(I) \to 0$ when $|I| \to 0$. This is a stronger notion of continuity than the one used in the quoted paper.

If one assumes this kind of continuity for $F$ then it is true that $F(Q_d) \geq 0$ for any dyadic square implies that $F(I_0) \geq 0$ for any rectangle $I_0$. This can be checked just by approximating $I_0$ by dyadic squares inside.

Note that the function $F$ exhibited in the proof of Proposition 1 is not continuous in this stronger sense.

3. Modification of the proof of the lemma.

In order to provide a proof of Lemma 5.1 in [1] we first remark that the hypothesis in this lemma imply that $F(Q) \geq 0$ for any square $Q \subset I_0$.

To see this, fix a square $Q$ of sidelenlength $\ell$ and consider as dyadic squares the squares obtained by doing a dyadic division of $Q$. Consequently, at the n-th step one obtains $2^{2n}$ squares of sidelenlength $\ell/2^n$.

Now Lemma 4.1 in [1] for this familiy of dyadic squares holds (going to the unit square by a dilation and a translation) with assertion (i) with a different constant from the constant 32 appearing there.

Then one can follow the proof of Lemma 5.1 to obtain $F(Q) \geq 0$. Therefore Lemma 5.1 will be proved if we stablilize the following fact.

Proposition 2. Let $F$ be an additive and continuous function of rectangles such that $F(Q) \geq 0$ for any square $Q \subset I_0$, $I_0$ some rectangle. The $F(I_0) \geq 0$.

Proof.
Assume that $I_0$ has sides of lengths $\ell_0 > \ell_1$. Put inside $I_0$ adjacent squares of sidelenlength $\ell_1$ as many times as possible and let $R_1$ the remaining rectangle.

Following this process with the rectangle $R_1$, after a number of iterations we will obtain a decomposition

$$I_0 = Q_1 \cup Q_2 \cup \cdots \cup Q_n \cup R_n$$

with $Q_1, Q_2, \cdots, Q_n$ squares and $R_n$ a rectangle with largest side $\ell_n$. If we show that $\ell_n \simeq \delta(R_n) \to 0$ as $n \to \infty$ we are done.
If $\ell_{n+1} \leq \ell_n/2$ for any $n$, the result is clear. However it can occur that for some $n$ one has $\ell_{n+1} > \ell_n/2$, but in this case, $\ell_{n+2} \leq \ell_n/2$. In fact, $\ell_{n+1}$ only fits once in $\ell_n$ and one has

$$\ell_{n+2} = \ell_n - \ell_{n+1} \leq \ell_n - \frac{\ell_n}{2} = \frac{\ell_n}{2}. $$

So, at every step or at every two steps, the sidelength is reduced at least by a factor $1/2$ and, consequently $\ell_n \to 0$, since $(\ell_n)$ is decreasing.

\[\square\]

REFERENCES

[1] S. Saks; A. Zygmund. *On functions of rectangles and their application to analytic functions.* Ann. Scuola Norm. Super. Pisa Cl. Sci. (2) **3** (1934) 27–32.

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