Complementarity between Position and Momentum as a Consequence of Kochen-Specker Arguments

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We give two simple Kochen-Specker arguments for complementarity between the position and momentum components of spinless particles, arguments that are identical in structure to those given by Peres and Mermin for spin-1/2 particles.

PACS numbers: 03.65.Bz

I. INTRODUCTION

Complementarity is the idea that mutually exclusive pictures are needed for a complete description of quantum-mechanical reality. The paradigm example is the complementarity between particle and wave (or ‘spacetime’ and ‘causal’) pictures, which Bohr took to be reflected in the uncertainty relation $\Delta x \Delta p \geq \hbar$. Bohr saw this relation as defining the latitude of applicability of the concepts of position and momentum to a single system, not just as putting a limit on our ability to predict the values of both position and momentum within an ensemble of identically prepared systems. Furthermore, right at the start of his celebrated reply to the Einstein-Podolsky-Rosen (EPR) argument against the completeness of quantum theory [4], Bohr confidently asserted: “it is never possible, in the description of the state of a mechanical system, to attach definite values to both of two canonically conjugate variables” [2]. Critics have often pointed out that complementarity does not logically follow from the uncertainty relation without making the positivistic assumption that position and momentum can only be simultaneously defined if their values can be simultaneously measured or predicted [3]. However, we shall show here how direct Kochen-Specker arguments for complementarity between position and momentum can be given that are entirely independent of the uncertainty relation and its interpretation.

The aim of a Kochen-Specker argument is to establish that a certain set of observables of a quantum system cannot have simultaneously definite values that respect the functional relations between compatible observables within the set [4]. Let $\mathcal{O}$ be a collection of bounded self-adjoint operators (acting on some Hilbert space) containing the identity $I$ and both $AB$ and $\lambda A + \mu B$ ($\lambda, \mu \in \mathbb{R}$), whenever $A, B \in \mathcal{O}$ and $[A, B] = 0$. Kochen and Specker called such a structure a partial algebra because there is no requirement that $\mathcal{O}$ contain arbitrary self-adjoint functions of its members (such as $i[A, B]$ or $A + B$, when $[A, B] \neq 0$). They then assumed that an assignment of values $[\cdot]: \mathcal{O} \to \mathbb{R}$ to the observables in $\mathcal{O}$ should at least be a partial homomorphism, respecting linear combinations and products of compatible observables in $\mathcal{O}$. That is, whenever $A, B \in \mathcal{O}$ and $[A, B] = 0$,

$$[AB] = [A][B], \quad [\lambda A + \mu B] = \lambda[A] + \mu[B], \quad [I] = 1. \quad (1)$$

Clearly these constraints are motivated by analogy with classical physics, in which all physical magnitudes (functions on phase space) trivially commute, and possess values (determined by points of phase space) that respect their functional relations. (The requirement that $[I] = 1$ is only needed to avoid triviality; for $[F^2] = [F]^2$ implies $[I] = 0$ or 1, and if we took $[I] = 0$, it would follow that $[A] = [AF] = [A][F] = 0$ for all $A \in \mathcal{O}$.)

Constraints [3] are not entirely out of place in quantum theory. For example, any common eigenstate $\Psi$ of a collection of observables $\mathcal{O}$ automatically defines a partial homomorphism, given by assigning to each $A \in \mathcal{O}$ the eigenvalue $A$ has in state $\Psi$. Difficulties — called Kochen-Specker contradictions or obstructions — arise when not all observables in $\mathcal{O}$ share a common eigenstate. In that case, there is no guarantee that value assignments on all the commutative subalgebras of $\mathcal{O}$ can be extended to a partial homomorphism on $\mathcal{O}$ as a whole. Should such an extension exist, one could be justified in thinking of the noncommuting observables in $\mathcal{O}$ as having simultaneously definite values, notwithstanding that a quantum state may not permit all their values to be predicted with certainty. But should some particular collection of observables $\mathcal{O}$ not possess any partial homomorphisms, the natural response would be to concede to Bohr that the observables in $\mathcal{O}$ “transcend the scope of classical physical explanation” and cannot be discussed using “ambiguous language with suitable application of the terminology of classical physics” [4]. That is, one would have strong reasons for taking the noncommuting observables in $\mathcal{O}$ to be mutually complementary.

Bell [3] has emphasized one other way to escape an obstruction with respect to some set of observables $\mathcal{O}$. One could still take all $\mathcal{O}$’s observables to have definite values by allowing the value of a particular $A \in \mathcal{O}$ to be a function of the context in which $A$ is measured. Thus,
suppose $\mathcal{O}_1, \mathcal{O}_2 \subseteq \mathcal{O}$ are two different commuting subalgebras both containing $A$, where $[\mathcal{O}_1, \mathcal{O}_2] \neq 0$. Then if $[,][,*]$ are homomorphisms on these subalgebras such that $[A][,] \neq [A][,*]$, one could interpret this difference in values (the obstruction) as signifying that the measured result for $A$ has to depend on whether it is measured along with the observables in $\mathcal{O}_1$ or those in $\mathcal{O}_2$. Such value assignments to the observables in $\mathcal{O}$ are called contextual, because the context in which an observable is measured is allowed to influence what outcome is obtained. For example, Bohm’s theory is contextual in exactly this sense. On the other hand (as Bell himself was quick to observe), complementarity also demands a kind of contextualism: in some contexts it is appropriate to assign a system a definite position, and in other contexts, a definite momentum. The difference from Bohm is that Bohr takes the definiteness of the values of observables itself to be a function of context. And this makes all the difference in cases where value contextualism can only be enforced by making the measured value of an observable nonlocally depend on whether an observable of another spacelike-separated system is measured. We shall see below that complementarity between position and momentum can only be avoided by embracing such nonlocality.

Numerous Kochen-Specker obstructions have been identified in the literature, and their practical and theoretical implications continue to be analyzed. While obstructions cannot occur for observables sharing a common eigenstate, failure to possess a common eigenstate does not suffice for an obstruction. As Kochen and Specker themselves showed, the partial algebra generated by all components of a spin-1/2 particle possesses plenty of partial homomorphisms. But for particles with higher spin, or collections of more than one spin-1/2 particle, obstructions can occur, perhaps the simplest being those identified by Peres in the case of two spin-1/2 particles, and Mermin in the case of three. Obstructions for sets that contain functions of position and momentum observables have been identified, but additional observables need to be invoked that weaken the case for complementarity between position and momentum alone. In the arguments below, we shall only need simple continuous functions of the individual position and momentum components of a system. Though all our observables have purely continuous spectra, obstructions arise in exactly the same way that they do in the arguments given by Peres and Mermin for the spin-1/2 case. And because our obstructions depend only on the structure of the Weyl algebra, they immediately extend to relativistic quantum field theories, which are constructed out of representations of the Weyl algebra.

II. THE WEYL ALGEBRA

Let $\vec{x} = (x_1, x_2, x_3)$ and $\vec{p} = (p_1, p_2, p_3)$ be the unbounded position and momentum operators for three degrees of freedom. We cannot extract a Kochen-Specker contradiction directly out of these operators, since domain questions prevent them from defining a simple algebraic structure. However, we may just as well consider the collection of all bounded, continuous, self-adjoint functions of $x_1$, and, similarly, the same set of functions in each of the variables $x_2, x_3, p_1, p_2, p_3$. Taking $\mathcal{O}$ to be the partial algebra of observables generated by all these functions (obtained by taking compatible products and linear combinations thereof), we shall show that $\mathcal{O}$ does not possess any partial homomorphisms.

Our arguments are greatly simplified by employing the following method, analogous to simplifying a problem in real analysis by passing to the complex plane. Assuming that $\mathcal{O}$ does possess a partial homomorphism $[\cdot] : \mathcal{O} \rightarrow \mathbb{R}$ (an assumption we shall eventually have to discharge), we can extend this mapping to the set $\mathcal{O}_C = \mathcal{O} + i\mathcal{O}$ in a well-defined manner, by taking $[X] \equiv [\mathbb{R}(X)] + i[\Im(X)] \in \mathbb{C}$, where $\mathbb{R}(X)$ and $\Im(X)$ are the unique real and imaginary parts of $X$. Now, if we consider any pair of commuting unitary operators $U, U' \in \mathcal{O}_C$, then since $U, U^*, U', U'^* \allowbreak$ pairwise commute, the four self-adjoint operators

$$[UU'] = [\mathbb{R}(U)\mathbb{R}(U') - \mathbb{R}(U)\Im(U')]$$

$$+ i[\mathbb{R}(U)\Im(U') + \mathbb{R}(U')\Im(U)],$$

using the fact that $\mathcal{O}$ is a partial algebra. In addition,

$$[UU'] = [\mathbb{R}(U)\mathbb{R}(U') - \mathbb{R}(U)\Im(U')]$$

$$+ i[\mathbb{R}(U)\Im(U') + \mathbb{R}(U')\Im(U)],$$

$$= ([\mathbb{R}(U)] [\mathbb{R}(U')] - [\mathbb{R}(U)] [\Im(U')]$$

$$+ i([\mathbb{R}(U)] [\Im(U')] + [\Im(U')] [\mathbb{R}(U')]),$$

$$= ([\mathbb{R}(U)] + i[\Im(U')]) ([\mathbb{R}(U')] + i[\Im(U')])$$

$$= [U][U'],$$

using the fact that $[\cdot]$ is a partial homomorphism in step. So we have established that the following product rule must hold in $\mathcal{O}_C$:

$$U, U' \in \mathcal{O}_C \ \& \ \ [U, U'] = 0$$

$$\Rightarrow \ UU' \in \mathcal{O}_C \ \& \ \ [UU'] = [U][U].$$

Henceforth, we shall only this need this simple product rule, together with $|[\pm I]| = \pm 1$. Our obstructions will manifest themselves as contradictions obtained by applying the product rule to compatible unitary operators in $\mathcal{O}_C$. 

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To see what operators those are, we first recall the definition of the Weyl algebra for three degrees of freedom. Consider the two families of unitary operators given by

\[ U_\alpha = e^{-i\alpha \cdot \vec{x}/\hbar}, \quad V_\beta = e^{-i\beta \cdot \vec{p}/\hbar}, \quad \alpha, \beta \in \mathbb{R}^3. \]

(10)

These operators act on any wavefunction \( \Psi \in L_2(\mathbb{R}^3) \) as

\[ (U_\alpha \Psi)(\vec{x}) = e^{-i\alpha \cdot \vec{x}/\hbar} \Psi(\vec{x}), \quad (V_\beta \Psi)(\vec{x}) = \Psi(\vec{x} - \vec{b}), \]

(11)

and satisfy the Weyl form of the canonical commutations relations \([x_j, p_k] = \delta_{jk} \hbar I\),

\[ U_\alpha V_\beta = e^{-i\alpha \cdot \vec{b}/\hbar} V_\beta U_\alpha. \]

(12)

The Weyl algebra (which is independent of the representation in (11)) is just the C*-algebra generated by the two families of unitary operators \([U_\alpha, V_\beta]\) subject to the commutation relation (12).

\[ O_C \]

is properly contained in the Weyl algebra. Indeed, writing \( U_{a_j} \) \((= e^{-ia_j x_j/\hbar})\) for the \( j \)th component of the operator \( U_\alpha \), and similarly \( V_{b_j} \), \((= e^{-ib_j p_j/\hbar})\), all nine of these component generators of the Weyl algebra lie in \( O_C \), because their real and imaginary parts, cosine and sine functions of the \( x_j \)'s and \( p_j \)'s, lie in \( O \). By the product rule, \( O_C \) also contains the products of compatible unitary operators for different degrees of freedom, as well as compatible products of those products. But, unlike the full Weyl algebra, \( O_C \) does not contain incompatible products, like \( U_{a_j} V_{b_j} \) when \( a_j b_j \neq 2n\pi \hbar \) \((n \in \mathbb{Z})\). Nevertheless, \( O_C \) is all we need to exhibit obstructions. The key is that we can choose values for the components of \( \alpha, \beta \) so that, for \( j = 1 \) to \( 3 \), \( a_j b_j = (2n + 1)\pi \hbar \). In that case, we immediately obtain from (12) the anti-commutation rule

\[ [U_{b_1 + b_2}, V_{\pm b_1}] = 0 = [U_{\mp a_1}, V_{\pm b_2}], \]

(13)

which, together with the product rule, will generate the required obstructions.

### III. OBSTRUCTIONS FOR TWO AND THREE DEGREES OF FREEDOM

We first limit ourselves to continuous functions of the four observables \( x_1, x_2, p_1, p_2 \), extracting a contradiction in exactly the way Peres \[9\] does for a pair of spin-1/2 particles. A first application of the product rule in \( O_C \) yields

\[ [U_{-a_1} U_{a_2}] = [U_{-a_1}] [U_{a_2}], \]

(14)

\[ [U_{a_1} V_{b_2}] = [U_{a_1}] [V_{b_2}], \]

(15)

\[ [V_{b_1} U_{-a_2}] = [V_{b_1}] [U_{-a_2}], \]

(16)

\[ [V_{-b_1} V_{-b_2}] = [V_{-b_1}] [V_{-b_2}], \]

(17)

Multiplying equations (14)–(17) together, and using one further (trivial) application of the product rule

\[ [U_{a_j}] [U_{-a_j}] = [I] = 1 = [V_{b_k}] [V_{-b_k}], \]

(18)

one obtains

\[ [U_{-a_1} U_{a_2}] [V_{-b_1} V_{-b_2}] [U_{a_1} V_{b_2}] [V_{b_1} U_{-a_2}] = 1. \]

(19)

However, because of the anti-commutation rule (13), the first pair of product operators occurring in (19) actually commute, as do the second pair of product operators. Hence we may make a further application of the product rule to (19), to get

\[ [U_{-a_1} U_{a_2} V_{-b_1} V_{-b_2}] [U_{a_1} V_{b_2} V_{b_1} U_{-a_2}] = 1. \]

(20)

Again, due to the anti-commutation rule, the two remaining (four-fold) product operators occurring in (20) commute, and their product is \( -I \). Thus, a final application of the product rule to (20) yields the contradiction \([-I] = -I + 1 = 1 \).

Notice that this obstruction remains for any given nonzero values for \( a_1 \) and \( a_2 \), provided only that we choose \( b_{1,2} = (2n + 1)\pi \hbar /a_{1,2} \). The obstruction would vanish if, instead, we chose any of the numbers \( a_1, a_2, b_1, b_2 \) to be zero. When \( a_1 = a_2 = 0 \) or \( b_1 = b_2 = 0 \), this is to be expected, since one would then no longer be attempting to assign values to nontrivial functions of both the positions and momenta. However, the breakdown of the argument when either \( a_2 \) or \( b_2 \) is zero does not necessarily mean that a more complicated argument could not be given for position-momentum complementarity by invoking only a single degree of freedom.

As Mermin \[10\] has emphasized (for the spin-1/2 analogue of the above argument), one can get by without independently assuming the existence of values for the two commuting unitary operators occurring in (20), and thereby strengthen the argument. For we can suppose that the quantum state of the system is an eigenstate of these operators, with eigenvalues that necessarily multiply to -1. Using (13), a wavefunction \( \Psi \) will be an eigenstate of both products in (20) just in case

\[ e^{i(a_1 x_1 - a_2 x_2)/\hbar} \Psi(x_1 + b_1, x_2 + b_2) = c \Psi(x_1, x_2), \]

(21)

\[ -e^{-i(a_1 x_1 - a_2 x_2)/\hbar} \Psi(x_1 - b_1, x_2 - b_2) = c' \Psi(x_1, x_2), \]

(22)

for some \( c, c' \in \mathbb{C} \). We should not expect there to be a normalizable wavefunction satisfying (21) and (22), because the commuting product operators in (20) have purely continuous spectra. But if we allow ourselves the idealization of using Dirac states (which can be approximated arbitrarily closely by elements of \( L_2(\mathbb{R}^2) \)), and just choose \( a_1 = a_2 \) for simplicity, then the two-dimensional delta function \( \delta(x_1 - x_2 - x_0) \) — an improper eigenstate of the relative position operator \( x_1 - x_2 \) with ‘eigenvalue’ \( x_0 \in \mathbb{R} \) — provides a simple solution to the above equations. However, this state cannot also be used to independently justify the assignment of values to the operators \( U_{a_1} V_{b_2} \) and \( V_{b_1} U_{-a_2} \) occurring in (13), which do not have \( \delta(x_1 - x_2 - x_0) \) as an eigenstate.
It is ironic that \( \delta(x_1 - x_2 - x_0) = \delta(p_1 + p_2) \) is exactly the state of two spacelike-separated particles that EPR invoked to argue against position-momentum complementarity. So in a sense the EPR argument carries the seeds of its own destruction. For suppose we follow their reasoning by invoking locality and the strict correlations entailed by the EPR state between \( x_1 \) and \( x_2 \), and between \( p_1 \) and \( p_2 \), to argue for the existence of noncontextual values for all four positions and momenta. Then all eight component unitary operators we employed above must have definite noncontextual values, since their real and imaginary parts are simple functions of those \( x \)'s and \( p \)'s. It is then a simple step to conclude that the four product operators in (19) should also have definite noncontextual values satisfying the product rule, and from there contradiction follows. This final step cannot itself be justified by appeal to locality, for the product observables in (19) do not pertain to either particle on its own and, hence, a measurement context for any one of these operators (i.e., their self-adjoint real and imaginary parts) necessarily requires a joint measurement undertaken on both particles [1]. Still, the above argument sheds an entirely new light on the nonclassical features of the original EPR state, which have hitherto only been discussed from a statistical point of view [13].

Our second argument employs all three degrees of freedom, extracting a contradiction in exactly the way Mermin [10] does for three spin-1/2 particles. Again, a first application of the product rule in (2) yields

\[
\begin{align*}
[U_{a_1} V_{b_2} V_{b_3}] & = [U_{a_1}] [V_{b_2}] [V_{b_3}] , \quad (23) \\
[V_{b_1} U_{a_2} V_{b_3}] & = [V_{b_1}] [U_{a_2}] [V_{b_3}] , \quad (24) \\
[V_{b_1} V_{b_2} U_{a_3}] & = [V_{b_1}] [V_{b_2}] [U_{a_3}] , \quad (25) \\
[U_{-a_1} U_{-a_2} U_{-a_3}] & = [U_{-a_1}] [U_{-a_2}] [U_{-a_3}] . \quad (26)
\end{align*}
\]

Multiplying (23)–(26) together, again using (18), yields

\[
\begin{align*}
[U_{a_1} V_{b_2} V_{b_3}] [V_{b_1} U_{a_2} V_{b_3}] [V_{b_1} V_{b_2} U_{a_3}] [U_{-a_1} U_{-a_2} U_{-a_3}] & = 1 . \quad (27)
\end{align*}
\]

But now, exploiting the anti-commutation rule once again, the four product operators occurring in square brackets in (27) pairwise commute, and their product is easily seen to be \(-I\). So one final application of the product rule to (27) once more yields the contradiction \([-I] = -1 = 1\).

As before, we may interpret the \( x \)'s and \( p \)'s as the positions and momenta of three spacelike-separated particles. And we can avoid independently assuming values for the four products in (27) by taking the state of the particles to be a simultaneous (improper) eigenstate of these operators — exploiting that state’s strict correlations and EPR-type reasoning from locality to motivate values for all the component operators. (The reader is invited to use (11) to determine the set of all such common eigenstates, which are new position-momentum analogues of the Greenberger-Horne-Zeilinger state [10].) This time, the only way to prevent contradiction is to introduce contextualism to distinguish, for example, the value of \( U_{a_1} \) as it occurs in (23) from the value this operator (or rather its inverse) receives in (26) in the context of different operators for particles 2 and 3 — forcing the values of \( \sin a_1 x_1 \) and \( \cos a_1 x_1 \) to nonlocally depend on whether position or momentum observables for 1 and 2 are measured. Bohr of course denied that there could be any such nonlocal “mechanical” influence, but only “an influence on the very conditions which define” which of the two mutually complementary pictures available for each system can be unambiguously employed [4].

The author would like to thank All Souls College, Oxford for support, and Paul Busch, Jeremy Butterfield, and Hans Halvorson for helpful discussions.

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