Welfare Guarantees in Schelling Segregation

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Martin Bullinger
Segregation on a checkerboard
Segregation on a checkerboard
Segregation on a checkerboard

neighbors define happiness
Segregation on a checkerboard
Segregation on a checkerboard

[Image of a checkerboard with pieces arranged in a pattern indicating segregation.]
Segregation on a checkerboard
Segregation on a checkerboard

![Checkerboard Diagram]
Segregation on a checkerboard
Segregation on a checkerboard
Schelling segregation

- Seeks to explain segregation in metropolitan areas
- First applied to grids (checkers) and lines (Schelling 1969, 1971)
- Surprising convergence under low threshold for movement
- Recently: Game-theoretic approach (Chauhan et al. 2018, Echzell et al. 2019, Elkind et al. 2019)
Formal model

- Set of $n$ agents
- Partitioning into two classes
- Topology graph with at least $n$ vertices
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Utilities and Social welfare

- Output: assignment of agents to nodes
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- Utilities: fractions friends/neighbors (0 if no neighbors)
Utilities and Social welfare

- Output: assignment of agents to nodes
- Utilities: fractions friends/neighbors (0 if no neighbors)
- Social welfare: sum of utilities

Social Welfare $= \frac{75}{6}$
It is NP-complete to maximize social welfare in Schelling instances, even for the class of instances where the number of agents is equal to the number of nodes.

- Very restrictive class of instances
- Previous reductions require auxiliary agent type
- Approximation of social welfare?
Let’s play a game!
Schelling Welfare Game

- Game board: topology graph
Schelling Welfare Game

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- Red and blue pieces placed cooperatively
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- Game board: topology graph
- Red and blue pieces placed cooperatively
- Award: social welfare of assignment

\[ SW = 3.5 + 2.67 = 6.17 \]
Schelling Welfare Game

- Game board: topology graph
- Red and blue pieces placed cooperatively
- Award: social welfare of assignment
- What is a good move?
Schelling Welfare Game

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- Award: social welfare of assignment
- What is a good move?

Position with **high expected welfare under uniform distribution**
Approximation of welfare

Theorem

For any Schelling instance with \( n \) agents, there exists an assignment with social welfare at least \( \frac{n}{2} - 1 \), which can be computed in polynomial time.

- Consider assignment chosen uniformly at random
- Derandomize this selection
- Tight bound slightly larger
Two striking examples
Star topology

\( n - 2 \) blue agents

2 red agents
Star topology

All assignments are Pareto optimal.

Social welfare linear factor apart.

Ordered utility vector of right assignment dominates left one.
Star topology

- All assignments are Pareto optimal
- Social welfare linear factor apart
Star topology

$OUV = (1, \frac{1}{n-1}, 0, \ldots, 0)$

$OUV = (1, \ldots, 1, \frac{n-3}{n-1}, 0, 0)$

- All assignments are Pareto optimal
- Social welfare linear factor apart
- Ordered utility vector of right assignment dominates left one
Star topology

\[
\begin{align*}
(1, 1, 1, \frac{n-3}{n-1}, 0, 0) \\
(1, \frac{1}{n-1}, 0, 0, 0, 0)
\end{align*}
\]

- All assignments are Pareto optimal
- Social welfare linear factor apart
- *Ordered utility vector* of right assignment dominates left one
Complete bipartite topology
Complete bipartite topology
Complete bipartite topology

\[ SW = 4 - \frac{8}{n} \]

\[ SW = \frac{n}{2} \]

- Both assignments are Pareto optimal
- Social welfare linear factor apart
Complete bipartite topology

\[ SW = 4 - \frac{8}{n} \]

\[ SW = \frac{n}{2} \]

\[ OUV = \left( \frac{n-2}{n}, \frac{n-2}{n}, \frac{2}{n}, \ldots, \frac{2}{n} \right) \]

\[ OUV = \left( \frac{1}{2}, \ldots, \frac{1}{2} \right) \]

■ Both assignments are Pareto optimal
■ Social welfare linear factor apart
■ Ordered utility vectors undominated
Complete bipartite topology

\[
\begin{align*}
SW &= 4 - \frac{8}{n} \\
SW_B &= 2 - \frac{4}{n} \\
SW_R &= 2 - \frac{4}{n}
\end{align*}
\]

\[
\begin{align*}
SW &= \frac{n}{2} \\
SW_B &= \frac{n}{4} \\
SW_R &= \frac{n}{4}
\end{align*}
\]

\[
OUV = \left( \frac{n-2}{n}, \frac{n-2}{n}, \frac{2}{n}, \ldots, \frac{2}{n} \right)
\]

\[
OUV = \left( \frac{1}{2}, \ldots, \frac{1}{2} \right)
\]

- Both assignments are Pareto optimal
- Social welfare linear factor apart
- Ordered utility vectors undominated
- Domination in group welfare
Welfare notions

- Maximum welfare
- Group-welfare optimality
- Pareto optimality
- Utility-vector optimality
Welfare guarantees

- Group-welfare optimality guarantees welfare $\frac{n}{n-1}$
- Utility-vector optimality guarantees welfare $1$
- Pareto optimality guarantees welfare $\frac{1}{\sqrt{n}}$

Group-welfare optimality: $\frac{n}{n-1}$

Maximum welfare: $\frac{n}{2} - 1$

Pareto optimality: $\frac{1}{\sqrt{n}}$

Utility-vector optimality: $1$
Welfare guarantees

- Group-welfare optimality guarantees welfare $n/(n-1)$
- Utility-vector optimality guarantees welfare 1
- Pareto optimality guarantees welfare $1/\sqrt{n}$
- Pareto optimality guarantees welfare $n/(n-1)$ for tree topologies

Group-welfare optimality: $n/(n-1)$

Maximum welfare: $n/2 - 1$

Pareto optimality: $1/\sqrt{n}$

Utility-vector optimality: 1
Agents of positive utility

- Not all agents may obtain positive utility
Agents of positive utility

- Not all agents may obtain positive utility
- Minimum degree of 2 allows to give everyone positive utility*

* if number of agents equals number of nodes
Agents of positive utility

- Not all agents may obtain positive utility
- Minimum degree of 2 allows to give everyone positive utility
- Efficient decidability on tree topologies

[Diagram of tree topologies]
Conclusion

- Maximum welfare can be approximated well
- New welfare notions differentiate Pareto-optimal assignments
- Basic happiness of all agents can often be achieved