ATTRACTOR MECHANISMS OF MODULI SPACES OF CALABI–YAU 3-FOLDS

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ABSTRACT. We investigate the complex and Kähler attractor mechanisms of moduli spaces of Calabi–Yau 3-folds. The complex attractor mechanism was previously studied by Ferrara–Kallosh–Strominger, Moore and others in string theory. It is concerned with the minimizing problems of the normalized central charges of 3-cycles and defines a new interesting class of Calabi–Yau 3-folds called, the complex attractor varieties. In light of mirror symmetry, we introduce the Kähler attractor mechanism and define the Kähler attractor varieties. The complex and Kähler attractor varieties are expected to possess very rich structures, in particular certain complex and Kähler rigidities.

1. INTRODUCTION

Let $X$ be a projective Calabi–Yau 3-fold. Let $\pi : \widetilde{\mathcal{M}}_{\text{Cpx}} \to \mathcal{M}_{\text{Cpx}}$ be the universal covering of the complex moduli space $\mathcal{M}_{\text{Cpx}}$ of $X$. The normalized central charge of a 3-cycle $\gamma \in H_3(X,\mathbb{Z})$ is defined by

$$Z(\Omega_{X_z}, \gamma) = e^{K_B(z) \int_{\gamma} \Omega_{X_z}}$$

where $\Omega_{X_z}$ is a holomorphic volume form of $X_z$ and $K_B(z)$ is the Weil–Petersson potential on $\widetilde{\mathcal{M}}_{\text{Cpx}}$. It induces a function $|Z(\cdot, \gamma)| : \widetilde{\mathcal{M}}_{\text{Cpx}} \to \mathbb{R}_{\geq 0}$, called the mass function of $\gamma$, and we are interested in its stationary points, called the attractors. Our investigation is motivated by the study of black holes in string theory (Ferrara–Kallosh–Strominger [9]), where it is of great interest to find a 3-cycle $\gamma \in H_3(X,\mathbb{Z})$ which supports a BPS state. Finding stationary points of the mass function $|Z(\cdot, \gamma)|$ is a purely mathematical problem and can be answered in parts by using the attractor mechanism investigated by Moore in his unpublished article [23]. The Calabi–Yau 3-folds corresponding to the attractors are called the attractor varieties. This new class of Calabi–Yau 3-folds are conjectured to possess very rich structures.

Moore’s article [23] is full of beautiful insights and he posed many interesting mathematical questions (the attractor conjectures) pertaining to the arithmetic nature of the attractor varieties. In fact, the attractor varieties can be considered as a vast generalization of the rigid Calabi–Yau 3-folds.

In light of mirror symmetry, which is a duality between complex and Kähler (symplectic) geometries of distinct Calabi–Yau manifolds, a natural question is, what is the mirror of the attractor mechanism? In this article,
we introduce the Kähler attractor mechanism of the Kähler moduli space and develop parallel theories to the complex side. Moreover, the Kähler attractor mechanism leads us to the idea of rigid Kähler structures, which should be mirror to the rigid complex structures. This direction of research is further carried out from the viewpoint of generalized Calabi–Yau geometry in a separate article [15].

The present work is based on our previous work [8]. It investigates the A-model Weil–Petersson geometry on the Kähler moduli space (or more precisely the space of Bridgeland stability conditions), which is supposed to be mirror to the classical Weil–Petersson geometry on the complex moduli spaces [26, 27].

The objective of this article is twofold. The first is to provide mathematical foundations of the complex attractor mechanism (Section 2). The second is to introduce the Kähler attractor mechanism inspired by mirror symmetry (Section 4).

| (A-side) | (B-side) |
|----------|----------|
| Kähler moduli $\mathcal{M}_{\text{Kah}}$ | complex moduli $\mathcal{M}_{\text{Cpx}}$ |
| Weil–Petersson metric $g^A$ | Weil–Petersson metric $g^B$ |
| (Fan–Kanazawa–Yau [8]) | (Tian [26], Todorov [27]) |
| Kähler attractor mechanism | complex attractor mechanism |
| (Fan–Kanazawa [present work]) | (Moore [23]) |

**Structure of article.** Section 2 provides mathematical foundations of the complex attractor mechanism based on [23]. Section 3 is a brief review of our previous work [8] on the Weil–Petersson geometry by means of the Bridgeland stability conditions. Section 4 introduces the Kähler attractor mechanism and develops parallel theories to the complex side. We finally compare the complex and Kähler attractor mechanisms from the viewpoint of mirror symmetry.

**Notation and conventions.** Throughout the article, we work over complex numbers $\mathbb{C}$. A Calabi–Yau $n$-fold is an $n$-dimensional Kähler manifold whose canonical bundle is trivial. $\text{ch}(\cdot)$ denotes the Chern character and $\text{Td}_X$ denotes the Todd class of $X$. For $R = \mathbb{Z}, \mathbb{Q}, \mathbb{R}$, $H^{i,i}(X,R)$ denotes the intersection $H^{i,i}(X) \cap H^{2i}(X,R)$. $\mathfrak{H}_g$ denotes the Siegel upper half-space of degree $g$.

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2. Complex attractor mechanism

2.1. Foundation of complex attractor mechanism. Let $X$ be a projective Calabi–Yau 3-fold and $\mathcal{M}_{\text{Cpx}}$ the complex moduli space of $X$. We consider the vector bundle $\mathcal{H} = R^3\pi_*\mathcal{L} \to \mathcal{M}_{\text{Cpx}}$ equipped with a natural Hodge filtration $F^3\mathcal{H} \subset \cdots \subset F^0\mathcal{H}$ of weight 3. By the Calabi–Yau condition, the first piece of the filtration defines a holomorphic line bundle $L = F^3\mathcal{H} \to \mathcal{M}_{\text{Cpx}}$, which we call the Hodge bundle. It is classically known that $\mathcal{M}_{\text{Cpx}}$ carries a natural Kähler metric, called the Weil–Petersson metric $g^B$, whose Kähler potential is given by

$$K^B(z) = -\log(\sqrt{-1} \int_{X_z} \Omega_{X_z} \wedge \overline{\Omega_{X_z}}),$$

where $\{\Omega_{X_z}\}_z$ is a nowhere-zero (holomorphic) local section of the Hodge bundle $\mathcal{L}$. We call $K^B$ the Weil–Petersson potential. For later use, we introduce the following useful formula for computing $K^B$.

**Proposition 2.1** ([8]). Assume that there exist formal sums of Lagrangian submanifolds $\{L_i\}$ representing a basis of $H_3(X,\mathbb{Z})/\text{tor}(H_3(X,\mathbb{Z}))$. Then

$$K^B(z) = -\log(\sqrt{-1} \sum_{i,j} \chi_{Fuk}^{ij} \int_{L_i} \Omega_{X_z} \int_{L_j} \overline{\Omega_{X_z}}),$$

where $(\chi_{Fuk}^{ij}) = (\chi_{Fuk}(L_i, L_j))^{-1}$ is the inverse matrix for the Euler paring $\chi_{Fuk}$ of the Fukaya category $\mathcal{D}^b\text{Fuk}(X)$.

We will develop the $A$-model Weil–Petersson geometry based upon this new expression (Equation (2.2)).

**Definition 2.2.** Let $\gamma \in H_3(X,\mathbb{Z})$ be a non-trivial 3-cycle. Given an identification $H_3(X,\mathbb{Z}) \cong H_3(X_z,\mathbb{Z})$, we define the normalized central charge of $\gamma$ by

$$Z(\Omega_{X_z}, \gamma) = e^{K^B(z)} \int_{\gamma} \Omega_{X_z},$$

where by abuse of notation $K^B(z)$ is given by the Equation (2.1) (it depends not only on $z$ but also on $\Omega_{X_z}$).

Hereafter we always assume $\gamma \neq 0$. Let $\pi : \widetilde{\mathcal{M}}_{\text{Cpx}} \to \mathcal{M}_{\text{Cpx}}$ be the universal covering. Then $Z(-, \gamma)$ is a smooth function on the total space of the pullback $\pi^*\mathcal{L}$ of the Hodge bundle $\mathcal{L}$

$$Z(-, \gamma) : \pi^*\mathcal{L} \to \mathbb{C}.$$

We observe that the absolute value $|Z(\Omega_{X_z}, \gamma)|$ is independent of a choice of $\Omega_{X_z} \neq 0$. Therefore we obtain a function

$$|Z(-, \gamma)| : \widetilde{\mathcal{M}}_{\text{Cpx}} \to \mathbb{R}_{\geq 0}.$$

We call it the mass function of $\gamma \in H_3(X,\mathbb{Z})$. 
Theorem 2.3. A stationary point \( z \in \widetilde{\mathcal{M}}_{\text{Cpx}} \) of the mass function \( |Z(\cdot, \gamma)| \): 
\( \widetilde{\mathcal{M}}_{\text{Cpx}} \to \mathbb{R}_{\geq 0} \) with \( Z(\Omega_{X_z}, \gamma) \neq 0 \) is characterized by the equation 
(2.3) \( \gamma^P \Omega = \text{Re}(C\Omega_{X_z}), \quad (\exists C \in \mathbb{C}) \)
in \( H^3(X, \mathbb{Z}) \), where \( \gamma^P \) denotes the Poincaré dual of \( \gamma \). We call the Equation (2.3) the complex attractor equation. It is equivalent to the condition 
\( \gamma^P \in H^{3,0}(X) \oplus H^{0,3}(X) \).

Proof. By the Bogomolov–Tian–Todorov Theorem, the Kodaira–Spencer map provides an identification between an open neighborhood \( U \) of \( z \in \widetilde{\mathcal{M}}_{\text{Cpx}} \) and an open neighborhood \( U' \) of 0 \( \in H^1(X, TX) \cong H^{2,1}(X) \). Therefore, for a basis \( \Omega_1, \ldots, \Omega_k \) of \( H^{2,1}(X) \), the variation \( \Omega_{X_z} = \Omega_{X_z} + \sum_i \varepsilon_i \Omega_i \) gives a local coordinate \( \varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \) of \( U \). Then a straightforward calculation shows
\[
\frac{\partial}{\partial \varepsilon_i} \bigg|_{\varepsilon=0} |Z(\Omega_{X_z}, \gamma)|^2 = \frac{\partial^2}{\partial \varepsilon_i \partial \varepsilon_j} \bigg|_{\varepsilon=0} \frac{|\int_\gamma \Omega_{X_z}|^2}{\sqrt{-1} \int_{X_z} \Omega_{X_z} \wedge \overline{\Omega_{X_z}}} = e^{K_B(z)} \int_{X_z} \Omega_i \int_\gamma \varepsilon_i
\]
By the assumption that \( Z(\Omega_{X_z}, \gamma) = e^{\frac{K_B(z)}{2}} \int_\gamma \Omega_{X_z} \neq 0 \), \( z \in \widetilde{\mathcal{M}}_{\text{Cpx}} \) is a stationary point if and only if \( \int_\gamma \Omega_i = 0 \) (\( 1 \leq i \leq k \)). Since \( \Omega_1, \ldots, \Omega_k \) form a basis of \( H^{2,1}(X) \), this condition is equivalent to 
\( \gamma^P \in H^{3,0}(X) \oplus H^{0,3}(X) \).
Therefore, since \( \gamma^P \in H^3(X, \mathbb{Z}) \), 
\( \gamma^P = (C\Omega_{X_z} + \overline{C\Omega_{X_z}}) = \text{Re}(C\Omega_{X_z}) \)
for some \( C \in \mathbb{C} \). \( \square \)

Theorem 2.4. A stationary point \( z \in \widetilde{\mathcal{M}}_{\text{Cpx}} \) of the mass function \( |Z(\cdot, \gamma)| \) with \( Z(\Omega_{X_z}, \gamma) \neq 0 \) is a local minimizer. Moreover, such points are discrete.

Proof. For a stationary point \( z \in \widetilde{\mathcal{M}}_{\text{Cpx}} \), a straightforward but tedious calculation shows
\[
\frac{\partial^2}{\partial \varepsilon_i \partial \varepsilon_j} \bigg|_{\varepsilon=0} |Z(\Omega_{X_z}, \gamma)|^2 = \frac{\partial^2}{\partial \varepsilon_i \partial \varepsilon_j} \bigg|_{\varepsilon=0} \frac{|\int_\gamma \Omega_{X_z}|^2}{\sqrt{-1} \int_{X_z} \Omega_{X_z} \wedge \overline{\Omega_{X_z}}} = 2e^{K_B(z)} \int_{X_z} \Omega_i | \int_\gamma \Omega_{X_z} |^2 g_{ij}^B(z).
\]
Therefore the complex Hessian of the function \( |Z(\cdot, \gamma)|^2 \) at \( z \) is identified with the Weil–Petersson metric \( g_{ij}^B(z) \), rescaled by a positive constant, and hence is positive definite. \( \square \)
Theorem 2.4 asserts that there are 3 different types of the behavior of the mass function \(|Z(\cdot, \gamma)|\) depending on the nature of \(\gamma \in H_3(X, \mathbb{Z})\).

1. There exists no stationary point.
2. There exists a stationary point \(z \in \mathcal{M}_{\text{Cpx}}\) with \(Z(\Omega_{X_z}, \gamma) = 0\). In this case the equation \(\int_\gamma \Omega_{X_z} = 0\) defines a divisor on \(\mathcal{M}_{\text{Cpx}}\).
3. There exists a stationary point \(z \in \mathcal{M}_{\text{Cpx}}\) with \(Z(\Omega_{X_z}, \gamma) \neq 0\).

**Definition 2.5.** A stationary point \(z \in \mathcal{M}_{\text{Cpx}}\) with \(Z(\Omega_{X_z}, \gamma) \neq 0\) is called a complex attractor for \(\gamma\). The corresponding Calabi–Yau 3-fold \(X_z\) is called a complex attractor variety for \(\gamma\). We denote by \(\text{Attr}_{\text{Cpx}}(\gamma) \subset \mathcal{M}_{\text{Cpx}}\) the set of complex attractors for \(\gamma\). Then we define

\[
\text{Attr}_{\text{Cpx}} = \pi(\cup_{\gamma} \text{Attr}_{\text{Cpx}}(\gamma)) \subset \mathcal{M}_{\text{Cpx}},
\]

where \(\gamma\) runs over \(H_3(X, \mathbb{Z})\), and call it the complex attractor constellation of \(X\).

**Remark 2.6.** The complex attractor equation is concerned with \(\Omega_{X_z}\). It in general does not determine a complex attractor \(z \in \mathcal{M}_{\text{Cpx}}\) because a Torelli type theorem fails in 3-dimensions.

It is natural to ask whether or not a complex attractor gives the global minimum, but the situation is rather complicated partly due to the non-compactness of the complex moduli space \(\mathcal{M}_{\text{Cpx}}\). In fact, it is claimed in [23, Section 9.2] that there exists a Calabi–Yau 3-fold for which a single \(\gamma\) leads to several distinct complex attractors with different values of local minima.

Another important problem is to investigate the distribution of the complex attractor constellation \(\text{Attr}_{\text{Cpx}} \subset \mathcal{M}_{\text{Cpx}}\). Note that it is not an intrinsic property of the complex manifold \(\mathcal{M}_{\text{Cpx}}\). For example, many 1-parameter families of Calabi–Yau 3-folds share their complex moduli \(\mathbb{P}^1 \setminus \{0, 1, \infty\}\), but the complex constellations should be different because they depend on the Calabi–Yau 3-folds they parametrize (to be precise, the variation of Hodge structures).

**Conjecture 2.7.** Let \(\mathcal{M}_{\text{Cpx}} \subset \overline{\mathcal{M}}_{\text{Cpx}}\) be a (partial) compactification. The complex attractor constellation \(\text{Attr}_{\text{Cpx}}\) is dense near a large complex structure limit \(z \in \overline{\mathcal{M}}_{\text{Cpx}}\).

This conjecture is inspired by the observation that there seem infinitely many Kähler attractor points (to be introduced in Section 4) near the large volume limit of a Calabi–Yau 3-folds.

**Remark 2.8.** A Calabi–Yau 3-fold \(X\) is called rigid if \(H^{2,1}(X) = 0\). A rigid Calabi–Yau 3-fold \(X\) is by definition a complex attractor variety for any \(\gamma \in H_3(X, \mathbb{Z})\). From this perspective, the complex attractor varieties are a vast generalization of the rigid Calabi–Yau 3-folds, whose arithmetic properties are of considerable interest. Indeed, in [23, Section 8.2], Moore
posed several interesting questions (the attractor conjectures) pertaining to the arithmetic nature of the complex attractor varieties. This direction of research has recently been carried out by Lam and Tripathy [19, 20].

We now take a closer look at the complex attractors. The plane
\[ V(z) = H^{3,0}(X_z) \oplus H^{0,3}(X_z) \subset H^3(X, \mathbb{C}) \]
vary as \( z \) moves in \( \mathcal{M}_{\text{Cpx}} \), where we have a natural identification
\[ H^3(X_z, \mathbb{C}) \cong H^3(X, \mathbb{C}) \]
for a reference \( X \). The intersection with the real \((2h^{3,1} + 2)\)-dimensional space \( H^3(X, \mathbb{R}) \) is the 2-plane spanned over \( \mathbb{R} \) by \( \text{Re}(\Omega_{X_z}) \) and \( \text{Im}(\Omega_{X_z}) \). For a generic \( z \in \mathcal{M}_{\text{Cpx}} \), the plane \( V_\mathbb{R}(z) \) intersects \( H^3(X, \mathbb{Z}) \subset H^3(X, \mathbb{R}) \) only in 0.

**Definition 2.9.** Let \( z \in \mathcal{M}_{\text{Cpx}} \) be a complex attractor for some \( \gamma' \in H_3(X, \mathbb{Z}) \). There are two cases:

1. The intersection \( V_\mathbb{R}(z) \cap H^3(X, \mathbb{Z}) \) is a lattice line. The point \( z \) is a complex attractor for any non-zero \( \gamma^{PD} \in V_\mathbb{R}(z) \cap H^3(X, \mathbb{Z}) \). In this case \( z \) is called a complex attractor of rank 1.

2. The intersection \( V_\mathbb{R}(z) \cap H^3(X, \mathbb{Z}) \) is a lattice plane. Then there exist \( \gamma_1, \gamma_2 \in H_3(X, \mathbb{Z}) \) such that the intersection \( \gamma_1 \cap \gamma_2 \not= 0 \) and \( V(z) \) is the complexification of the lattice \( \mathbb{Z}\gamma_1^{PD} + \mathbb{Z}\gamma_2^{PD} \). Therefore \( \gamma_1, \gamma_2 \) simultaneously satisfy the complex attractor equations. In this case, \( z \) is called a complex attractor of rank 2.

**Proposition 2.10.** Let \( z \in \mathcal{M}_{\text{Cpx}} \) be a complex attractor of rank 2 such that \( \gamma_1, \gamma_2 \in H_3(X, \mathbb{Z}) \) with \( \gamma_1 \cap \gamma_2 \not= 0 \) simultaneously satisfy the complex attractor equations
\[ \gamma_1^{PD} = \text{Re}(C_1\Omega_{X_z}), \quad \gamma_2^{PD} = \text{Re}(C_2\Omega_{X_z}). \]
for some \( C_1, C_2 \in \mathbb{C} \). Then
\[ \Omega_{X_z} = \frac{\sqrt{-1}}{\text{Im}(C_1C_2)} (C_1\gamma_2^{PD} - C_2\gamma_1^{PD}). \]

**Proof.** Since \( \Omega_{X_z} \) lies in \( V(z) \), we can write \( \Omega_{X_z} = a_1\gamma_1^{PD} + a_2\gamma_2^{PD} \) for some \( a_1, a_2 \in \mathbb{C} \). By plugging this in the complex attractor equations (2.4), we determine the coefficients \( a_1, a_2 \). \( \square \)

While the complex attractors of rank 1 are expected to be dense in the moduli space, those of rank 2 are expected to be rare, as the underlying Calabi–Yau 3-fold in general need to satisfy very stringent conditions. In fact, Moore showed by using mirror symmetry the complex attractors of rank 1 are dense near a large complex structure limit (Conjecture 2.7) [23].

### 2.2. Complex attractor mechanism for torus

Let us consider a real 6-dimensional torus \( X = \mathbb{C}^3/(\mathbb{Z}^3 + \sqrt{-1}\mathbb{Z}^3) \). We introduce a complex structure...
on $X$ in such a way that

$$dz_i = dx_i + \sum_{j=1}^{3} T^{ij} dy_j \quad (1 \leq i \leq 3)$$

are holomorphic 1-forms for a period matrix $T = (T^{ij}) \in \mathfrak{h}_3$. Such a complex torus is denoted by $X_T$. Then $X_T$ is biholomorphic to $\mathbb{C}^3/(\mathbb{Z}^3 + T \mathbb{Z}^3)$ equipped with the natural complex structure by the map

$$\phi : \mathbb{C}^3/(\mathbb{Z}^3 \oplus \sqrt{-1} \mathbb{Z}^3) \rightarrow \mathbb{C}^3/(\mathbb{Z}^3 + T \mathbb{Z}^3), \quad z = x + \sqrt{-1} y \mapsto z' = x + Ty.$$

We see $\phi$ is holomorphic because $\phi^*(dz') = d\phi^*(z') = dx + T dy$. We vary the complex structure of $X_T$ by, not varying the lattice as usual, but by varying the holomorphic volume form

$$\Omega_{X_T} = dz_1 \wedge dz_2 \wedge dz_3.$$

We fix a symplectic basis of $H^3(X_T, \mathbb{Z})$ as follows

$$\alpha_0 = dx_1 \wedge dx_2 \wedge dx_3,$$

$$\alpha_{ij} = \frac{1}{2} \sum_{l,m=1}^{3} \epsilon_{ilm} dx_l \wedge dx_m \wedge dy_j \quad (1 \leq i, j \leq 3)$$

$$\beta^0 = -dy_1 \wedge dy_2 \wedge dy_3,$$

$$\beta^{ij} = \frac{1}{2} \sum_{l,m=1}^{3} \epsilon_{jlm} dx_i \wedge dy_l \wedge dy_m \quad (1 \leq i, j \leq 3).$$

where $\epsilon_{ilm}$ denotes the Levi–Civita symbol. With respect to this basis, $\Omega_{X_T}$ has an expansion

$$\Omega_{X_T} = \alpha_0 + \sum_{i,j=1}^{3} T^{ij} \alpha_{ij} + \sum_{i,j=1}^{3} (\text{Cof}(T)_{ij}) \beta^{ij} - (\det(T)) \beta^0$$

where $\text{Cof}(T) = (\text{Cof}(T)_{ij})$ denotes the cofactor matrix of $T$. We fix a 3-cycle $\gamma \in H_3(X_T, \mathbb{Z})$ and write it as

$$\gamma = q_0 A_0 + \sum_{i,j=1}^{3} Q_{ij} A_{ij} + \sum_{i,j=1}^{3} P^{ij} B^{ij} + p^0 B^0$$

where $A_0, A_{ij}, B^{ij}, B^0$ form a basis of $H_3(X_T, \mathbb{Z})$ dual to the symplectic basis $\alpha_0, \alpha_{ij}, \beta^{ij}, \beta^0$. Then the normalized central charge of $\gamma$ reads

$$Z(\Omega_{X_T}, \gamma) = e^{-\frac{k B(T)}{2}} \int_{\gamma} \Omega_{X_T}$$

$$= e^{-\frac{k B(T)}{2}} (q_0 + \sum_{i,j=1}^{3} Q_{ij} T^{ij} + \sum_{i,j=1}^{3} P^{ij} (\text{Cof}(T)_{ij}) - p^0 \det(T)).$$
Therefore the complex attractor equation \( \text{Re}(C \Omega_{X_T}) = \gamma^{PD} \) is equivalent to the following system of equations:

\[
\begin{align*}
\text{Re}(C) &= p^0 \\
\text{Re}(CT^{ij}) &= P^{ij} \\
\text{Re}(C\text{Cof}(T)_{ij}) &= -Q_{ij} \\
\text{Re}(C \text{det}(T)) &= q_0.
\end{align*}
\]

**Theorem 2.11** (Moore [23]). A complex attractor for \( \gamma \in H_3(X, \mathbb{Z}) \) exists if and only if the coefficient matrices \( P = (P^{ij}), Q = (Q_{ij}) \in M_3(\mathbb{Z}) \) are symmetric. Then a complex attractor is unique and given by

\[
T = ((2PQ - (p^0q_0 + \text{tr}(PQ)E_3)) + \sqrt{-DE_3})(2R)^{-1} \in S_3
\]

where

\[
\begin{align*}
R &= \text{Cof}(P) + p^0Q, \\
D &= ((\text{tr}(PQ))^2 - \text{tr}((PQ)^2)) - (p^0q_0 + \text{tr}(PQ))^2 + 4(p^0 \text{det}(Q) - q_0 \text{det}(P)).
\end{align*}
\]

**Proof.** We provide in Appendix A a rigorous and accessible proof based on Moore’s original argument. One of our contributions is to show that there is no complex attractor if \( P, Q \) are not symmetric. \( \Box \)

The complex attractor variety \( X_T \cong \mathbb{C}^3/(\mathbb{Z}^3 + T\mathbb{Z}^3) \) has the following interesting property. The lattice embedding

\[
\mathbb{Z}^3 + T(2R)\mathbb{Z}^3 \hookrightarrow \mathbb{Z}^3 + T\mathbb{Z}^3
\]

induces an isogeny

\[
\phi : (E_{\sqrt{-D}})^3 \cong \mathbb{C}^3/(\mathbb{Z}^3 + T(2R)\mathbb{Z}^3) \twoheadrightarrow X_T.
\]

In other words, the complex attractor variety \( X_T \) is isogenous to the self-product \( (E_{\sqrt{-D}})^3 \) of the elliptic curves \( E_{\sqrt{-D}} \) with complex multiplication by the covering map \( \phi \) of degree \( 8 \text{det}(R) \). In particular, \( X_T \) is defined over a finite extension of the field \( \mathbb{Q}(\sqrt{-D}) \).

For a projective complex manifold \( X \) the Lefschetz (1,1)-theorem asserts that the Néron–Severi group \( NS(X) = H^2(X, \mathbb{Z}) \cap H^{1,1}(X) \). The rank \( \rho(X) \) of the Néron–Severi group, the so-called Picard number, satisfies the inequality \( 1 \leq \rho(X) \leq h^{1,1}(X) \).

**Theorem 2.12** ([13, Theorem 2.1]). Let \( A \) be an abelian variety of dimension \( g \). The following are equivalent.

1. The Picard number is maximal, i.e. \( \rho(A) = g^2 \);
2. \( A \) is isogenous to the self-product of an elliptic curve \( E \) with complex multiplication, i.e. \( A \sim E^g \);
3. \( A \) is isomorphic to the product of some pairwise isogenous elliptic curves \( E_1, \ldots, E_g \) with complex multiplication, i.e. \( A \cong E_1 \times \cdots \times E_g \).
Theorem 2.12 points out how the Picard number forces the structure of an abelian variety to be rigid. It is classically known that the algebraic varieties with the maximum Picard number possible often possess interesting arithmetic and geometric properties.

**Corollary 2.13.** The complex attractor variety $X_T$ has the maximal Picard number $\rho(X_T) = 9$, and hence is of rank 2. Moreover, $X_T \cong E_1 \times E_2 \times E_3$ for some pairwise isogenous elliptic curves $E_1, E_2, E_3$ with complex multiplication.

In fact, the converse is also true and we have the following.

**Theorem 2.14.** The complex constellation $\text{Attr}_{Cpx}$ bijectively corresponds to the abelian 3-folds with Picard number 9.

**Proof.** It suffices to show that an abelian 3-fold $X_T = \mathbb{C}^3/(\mathbb{Z}^3 + T\mathbb{Z}^3)$ with $\rho(X_T) = 9$ is a complex attractor variety for some $\gamma \in H_3(X_T, \mathbb{Z})$. A proof is based on a straightforward but tedious computation, and we leave it in Appendix B. □

### 2.3. Complex attractor mechanism for $E \times S$.

Let $E = \mathbb{C}/(\mathbb{Z} + \sqrt{-1}\mathbb{Z})$ be a real 2-dimensional torus. We put a complex structure on $E$ in such a way that $dz = dx + \tau dy$ is holomorphic for $\tau \in \mathbb{H}$ so that $E \cong \mathbb{C}/(\mathbb{Z} + \tau \mathbb{Z})$ as a complex manifold. Let $S$ be a K3 surface equipped with a holomorphic volume form $\Omega_S \in H^{2,0}(S)$. We consider the product Calabi–Yau 3-fold $X = E \times S$, which carries a natural holomorphic volume form

$$\Omega_X = dz \wedge \Omega_S.$$

Note that $dx, dy$ form a symplectic basis of $H^1(E, \mathbb{Z})$. By the Künneth theorem, we have the identification $H^3(X, \mathbb{Z}) \cong H^1(E, \mathbb{Z}) \otimes \mathbb{Z} \oplus H^2(S, \mathbb{Z})$. Therefore the Poincaré dual of a 3-cycle $\gamma \in H_3(X, \mathbb{Z})$ can be expressed as

$$\gamma^{PD} = dx \otimes u_1 + dy \otimes u_2,$$

for some $u_1, u_2 \in H^2(S, \mathbb{Z})$. We define $D_{u_1,u_2} = u_1^2 - u_2^2 \in \mathbb{Z}$.

Then the complex attractor equation $\text{Re}(C\Omega_X) = \gamma^{PD}$ is equivalent to the following system of equations

$$\text{Re}(C\Omega_S) = u_1$$

$$\text{Re}(C\tau \Omega_S) = u_2$$

Before solving the complex attractor equation, we introduce some notations. The Néron–Severi lattice of $S$ is $NS(S) = H^2(X, \mathbb{Z}) \cap H^{1,1}(S)$ equipped with the cup product. It is of signature $(1, \rho(S) - 1)$. The transcendental lattice is its complement $T(S) = NS(S)^\perp \subset H^2(S, \mathbb{Z})$. It is of signature $(2, 20 - \rho(S))$ and characterized as the minimal sublattice of $H^2(S, \mathbb{Z})$ whose complexification contains $\Omega_S$. 


Theorem 2.15 (Moore [23]). A complex attractor for \( \gamma \in H_3(X, \mathbb{Z}) \) exists if and only if the lattice \( \mathbb{Z}u_1 + \mathbb{Z}u_2 \) is positive definite. Moreover, if it exists, it is uniquely determined by the following periods

\[
\tau = \left( \frac{u_1}{u_1^2}, u_2 + \sqrt{-D_{u_1, u_2}} \right), \quad \Omega_S = -\sqrt{-1}(\tau u_1 - u_2).
\]

In particular, it is of rank 2.

Proof. By Proposition 2.10, we obtain

\[
\Omega_S = -\sqrt{-1}(\tau u_1 - u_2).
\]

The global Torelli theorem asserts that the K3 surface \( S \) is uniquely determined by \( \Omega_S \in H^{2,0}(S) \), up to isomorphism. On the other hand, the Hodge–Riemann bilinear relation \( \Omega_S \wedge \Omega_S = 0 \) implies

\[
u_2^2 \tau^2 - 2(u_1, u_2)\tau + u_2^2 = 0.
\]

If \( u_1^2 = (u_1, u_2) = 0 \), then the transcendental lattice \( T(S) \subset \mathbb{Z}u_1 + \mathbb{Z}u_2 \) is degenerate and this is a contradiction. If \( u_1^2 = 0 \) and \( (u_1, u_2) \neq 0 \), then \( \tau = \frac{u_2}{2(u_1, u_2)} \notin \mathbb{H} \) and this is a contradiction. Hence \( u_1^2 \neq 0 \) and we get

\[
\tau = \left( \frac{u_1}{u_1^2}, u_2 + \sqrt{-D_{u_1, u_2}} \right).
\]

We need \( D_{u_1, u_2} > 0 \) in order for \( \tau \) to lie in \( \mathbb{H} \). If \( u_1^2 < 0 \), \( T(S) \subset \mathbb{Z}u_1 + \mathbb{Z}u_2 \) is negative definite and this is a contradiction. Therefore we conclude that \( \mathbb{Z}u_1 + \mathbb{Z}u_2 \) is positive definite. In this case, we have

\[
\tau = \left( \frac{u_1}{u_1^2}, u_2 + \sqrt{-D_{u_1, u_2}} \right).
\]

Note that a change of the symplectic basis \( dx, dy \) by an element of \( \text{SL}(2, \mathbb{Z}) \) does not change \( \Omega_S \) and \( \tau \).

At the complex attractor point, the Néron–Severi lattice is \( NS(S) = (\mathbb{Z}u_1 + \mathbb{Z}u_2)^\perp \) and the Picard number \( \rho(S) \) obtains its maximal possible value 20. A K3 surface with \( \rho(S) = 20 \) is known as a singular K3 surface. It admits a rational map of degree 2 to a Kummer surface constructed from the product of two isogenous elliptic curves which have complex multiplications (a Shioda–Inose structure) [24]. Theorem 2.15 shows that any singular K3 surface appears as the second factor of some complex attractor variety \( X = E \times S \).

Remark 2.16. A singular K3 surface is known as a rigid K3 surface as it does not admit any complex deformation keeping the property \( \rho(S) = 20 \).

Remark 2.17. In the foundational article [7], Dolgachev formulated mirror symmetry for lattice polarized K3 surfaces. Although his formulation works beautifully in many cases, a singular K3 surface has been an exception. The long-standing problem of mirror symmetry for singular K3 surfaces is recently settled in [15] inspired by the results in this article (cf. Section 4.5).
Theorem 2.18. There is a bijective correspondence between the isomorphism classes of singular K3 surfaces and the isomorphism classes of positive definite even lattices of rank 2.

The bijective correspondence is given by associating a singular K3 surface $S$ with its transcendental lattice $T(S)$.

Lemma 2.19. Let $L$ be a positive definite even lattices of rank 2. Then

$$Q_L = \left\{ \frac{(u_1, u_2) + \sqrt{-D_{u_1,u_2}}}{u_1^2} \right\}_{u_1,u_2} \subset \mathbb{H}$$

is dense, where $u_1, u_2$ run over $\mathbb{Z}$-linearly independent vectors in $L$.

Proof. Let us write $\tau_{u_1,u_2} = \frac{(u_1, u_2) + \sqrt{-D_{u_1,u_2}}}{u_1^2}$. Then for $k, l \in \mathbb{Z}$ we have the elementary identities:

$$\tau_{ku_1,lu_2} = \frac{l}{k} \tau_{u_1,u_2}, \quad \tau_{u_1,ku_1+u_2} = k + \tau_{u_1,u_2}.$$  

They show that $Q_L \subset \mathbb{H}$ is dense. □

Theorem 2.20. For $X = E \times S$, the complex constellation $\text{Attr}_{\text{Cpx}}$ is dense in the complex moduli space $\mathbb{M}_{\text{Cpx}}$.

Proof. By a result of Beauville [2], we have a canonical isomorphism $\text{Aut}(E \times S) = \text{Aut}(E) \times \text{Aut}(S)$. Then there is a natural fibration $p : \mathbb{M}_{\text{Cpx}} \to \mathbb{M}_{K3}^{\text{Cpx}}$, where $\mathbb{M}_{K3}^{\text{Cpx}}$ denotes the complex moduli space of K3 surfaces. The image $p(\text{Attr}_{\text{Cpx}}) \subset \mathbb{M}_{K3}^{\text{Cpx}}$ corresponds to the isomorphism classes of singular K3 surfaces and is hence dense. Therefore it suffices to show that $p^{-1}(s) \cap \text{Attr}_{\text{Cpx}} \subset p^{-1}(s) \cong \text{SL}(2,\mathbb{Z}) \backslash \mathbb{H}$ is dense where $s$ corresponds to a singular K3 surface $S$. This assertion follows from the fact that $Q_{T(S)} \subset \mathbb{H}$ is dense (Lemma 2.19). □

The results presented in Sections 2.2 and 2.3 imply that the complex attractors have maximal Picard numbers possible. This kind of results do not hold for the Calabi–Yau 3-folds with $h^{2,1}(X) = 0$ as their Picard numbers are topological. Nevertheless, the complex attractors are discrete and the complex attractor varieties possess complex rigidity.

3. Weil–Petersson geometry on Bridgeland stability space

We provide a brief review of our previous work [8]. It introduced a provisional mirror Weil–Petersson geometry on the space of Bridgeland stability conditions $\text{Stab}(\mathcal{D}_X)$, which can be thought of as an approximation of the Kähler moduli space $\mathbb{M}_{\text{Kah}}$. 


3.1. Bridgeland stability conditions. Let $X$ be a smooth projective variety of dimension $n$. We define $D_X = D^b\text{Coh}(X)$ to be the bounded derived category of coherent sheaves on $X$.

The numerical Grothendieck group $\mathcal{N}(D_X) = K(D_X)/K(D_X)^\perp$ is the quotient group of the Grothendieck group $K(D_X)$ by the null group $K(D_X)^\perp$ of the Euler form $\chi$. It is a free abelian group of rank $\sum_{i=0}^n h^{i,i}(X)$.

**Definition 3.1 ([4]).** A (Bridgeland) stability condition $\sigma = (Z, P)$ on $D_X$ consists of a group homomorphism $Z : \mathcal{N}(D_X) \to \mathbb{C}$ and a collection $P = \{P(\phi)\}_{\phi \in \mathbb{R}}$ of full additive subcategories of $D_X$ parametrized by $\phi \in \mathbb{R}$ such that:

1. If $0 \neq F \in P(\phi)$, then $Z(F) \in \mathbb{R}_{>0} e^{\sqrt{-1} \pi \phi}.
2. $P(\phi + 1) = P(\phi)[1]$.
3. If $\phi_1 > \phi_2$ and $A_i \in P(\phi_i)$, then $\text{Hom}_{D_X}(A_1, A_2) = 0$.
4. For every $0 \neq F \in D_X$, there exists a sequence of exact triangles

$$
\begin{array}{cccccccc}
0 & \rightarrow & F_0 & \rightarrow & F_1 & \rightarrow & F_2 & \rightarrow & \cdots & \rightarrow & F_{k-1} & \rightarrow & F_k & \rightarrow & F \\
& & & & & & & & \downarrow & & & \downarrow & & & \\
& & & & & & & & & A_1 & & & A_2 & & & \\
& & & & & & & & & A_1 & & & A_2 & & & \\
& & & & & & & & & & \vdots & & & \vdots & & & \\
& & & & & & & & & & A_{k-1} & & & A_k & & & \\
& & & & & & & & & & & \vdots & & & \vdots & & & \\
& & & & & & & & & & & A_{k-1} & & & A_k & & & \\
\end{array}
$$

such that $A_i \in P(\phi_i)$ and $\phi_1 > \phi_2 > \cdots > \phi_k$.
5. (Support property [18]) There exist a constant $C > 0$ and a norm $\| \cdot \|$ on $\mathcal{N}(D_X) \otimes \mathbb{R}$ such that $\|F\| \leq C|Z(F)|$ for any semistable object $E$.

$Z$ is called a central charge and an element $A \in P(\phi)$ is called a semistable object of phase $\phi$.

We denote by $\text{Stab}(D_X)$ the set of stability conditions on $D_X$. There is a nice topology on it such that the forgetful map

$$
\text{Stab}(D_X) \rightarrow \text{Hom}(\mathcal{N}(D_X), \mathbb{C}), \quad \sigma = (Z, P) \mapsto Z
$$

is a local homeomorphism [4]. In other words, the deformations of the central charge lift uniquely to deformations of the stability condition. Therefore $\text{Stab}(D_X)$ naturally becomes a complex manifold, locally modelled on the $\mathbb{C}$-vector space $\text{Hom}(\mathcal{N}(D_X), \mathbb{C}) \cong \mathbb{C}^{\sum_{i=0}^n h^{i,i}(X)}$.

Moreover, $\text{Stab}(D_X)$ naturally carries a right action of the group $\widetilde{\text{GL}}^+(2, \mathbb{R})$, the universal covering of the group of orientation-preserving linear transformations $\text{GL}^+(2, \mathbb{R})$, as well as a left action of the group $\text{Aut}(D_X)$ of autoequivalences of $D_X$. The $\text{GL}^+(2, \mathbb{R})$-action is given by post-composition on the central charge $Z : \mathcal{N}(D_X) \rightarrow \mathbb{C} \cong \mathbb{R}^2$ with a suitable relabelling of the phases $\phi$. We often restrict this action to the subgroup $\mathbb{C} \subset \text{GL}^+(2, \mathbb{R})$ which acts freely.
3.2. **Central charge via twisted Mukai pairing.** The Mukai pairing on $H^*(X, \mathbb{C})$ is defined, for $v, w \in H^*(X, \mathbb{C})$,

$$
\langle v, w \rangle = \int_X e^{c_1(X)/2} v^\vee w,
$$

where $v = \sum v_j \in \bigoplus_j H_j(X, \mathbb{C})$ and its Mukai dual $v^\vee = \sum \sqrt{-1}^j v_j$. Note it differs from the usual Mukai pairing for K3 surfaces by a sign. We define a twisted Mukai vector of $F \in D_X$ by

$$
v_\Lambda(F) = \text{ch}(F) \sqrt{\text{Td}_X} e^{\sqrt{-1} \Lambda}
$$

for any $\Lambda \in H^*(X, \mathbb{C})$ such that $\Lambda^\vee = -\Lambda$. A twisted Mukai pairing is compatible with the Euler pairing; by the Hirzebruch–Riemann–Roch theorem,

$$
\chi(E, F) = \int_X \text{ch}(E) \text{ch}(F) \text{Td}_X = \langle v_\Lambda(E), v_\Lambda(F) \rangle.
$$

A geometric twisting $\Lambda_X$ compatible with the integral structure on the quantum cohomology was introduced by Iritani [14] and Katzarkov–Kontsevich–Pantev [17].

It is called the the log Gamma class and, in the Calabi–Yau case, we can explicitly write it as

$$
\Lambda_X = -\frac{\zeta(3)}{(2\pi)^3} c_3(X) + \frac{\zeta(5)}{(2\pi)^5} (c_5(X) - c_2(X)c_3(X)) + \ldots
$$

For K3 and abelian surfaces, there is no effect of twisting as $\Lambda_X = 0$. For Calabi–Yau 3-folds, the modification is given by the first term, which is familiar in the B-model period computations.

**Definition 3.2.** We define $v_X(F)$ to be the twisted Mukai vector of $F \in \mathcal{N}(D_X)$ associated to the log Gamma class $\Lambda_X$, namely

$$
v_X(F) = \text{ch}(F) \sqrt{\text{Td}_X} e^{\sqrt{-1} \Lambda_X}
$$

Let $X$ be a projective Calabi–Yau manifold equipped with a complexified Kähler parameter

$$
\omega = B + \sqrt{-1} \kappa \in H^{1,1}(X, \mathbb{C}),
$$

where $\kappa$ is a Kähler class. The set of such classes is the complexified Kähler cone $\mathcal{K}_X^\mathbb{C}$. We set $q = e^{2\pi \sqrt{-1} \omega}$.

**Conjecture 3.3** (Bridgeland [14]). Let $\mathcal{M}_{\text{Cpx}}$ be the Kähler moduli space of a projective Calabi–Yau manifold $X$. Then there exists an embedding

$$
\text{per}^\vee : \mathcal{M}_{\text{Kah}} \hookrightarrow \text{Aut}(D_X) \backslash \text{Stab}(D_X)/\mathbb{C}.
$$

The complexified Kähler cone $\mathcal{K}_X^\mathbb{C}$ gives a local chart of $\mathcal{M}_{\text{Kah}}$ and, near the large volume limit, there exists a stability condition $\sigma_\omega$ with central charge of the form

$$
Z_{\sigma_\omega}(F) = -\left\langle (2\pi \sqrt{-1})^{-\deg J(-2\pi \sqrt{-1} \omega)}, v_X(F) \right\rangle.
$$
Here $J(\tau) = J(\tau, 1)$ denotes the $J$-function of $X$ evaluated at the spectral parameter $z = 1$ and deg is the degree operator defined by $\deg(\alpha) = 2p\alpha$ for $\alpha \in H^{p,p}(X)$. This expression was introduced by Iritani in his study of integral structures of quantum cohomology [14]. $\mathcal{Z}_{\sigma}$ is called the quantum cohomology central charge (cf. Hosono [11]). The embedding $\mathrm{per}^{\vee}$ is comparable with a period map in mirror symmetry.

The asymptotic behavior of the quantum cohomology central charge $\mathcal{Z}_{\sigma}(F)$ is given by

$$\mathcal{Z}_{\sigma}(F) \sim -\int_X e^{-\omega}v_X(F) + O(q).$$

The existence of a stability condition with the asymptotic central charge given by the leading term has been proven for various examples including K3 surfaces, abelian surfaces [5], and abelian 3-folds [1, 22].

### 3.3. Weil–Petersson geometry.

In this subsection we assume $X$ is a projective Calabi–Yau $n$-fold. Then the Serre duality implies that for $E, F \in \mathcal{D}_X$, there is a natural functorial isomorphism

$$\operatorname{Hom}^*_\mathcal{D}_X(E, F) \cong \operatorname{Hom}^*_\mathcal{D}_X(F, E)[n]^{\vee}.$$ 

An important consequence of the Calabi–Yau condition is that the Euler form on $\mathcal{N}(\mathcal{D}_X)$ is (skew-) symmetric if $n$ is even (odd). The following bilinear form is inspired by Proposition 2.1.

**Definition 3.4.** Let $\{F_i\}$ be a basis of $\mathcal{N}(\mathcal{D}_X)$. We define a bilinear form $\mathfrak{b} : \operatorname{Hom}(\mathcal{N}(\mathcal{D}_X), \mathbb{C})^2 \to \mathbb{C}$ by

$$(Z_1, Z_2) \mapsto \sum_{i,j} \chi^{i,j}Z_1(F_i)Z_2(F_j),$$

where $(\chi^{i,j}) = (\chi(F_i, F_j))^{-1}$. Then $\mathfrak{b}$ is independent of the choice of a basis.

We think of $\operatorname{Hom}(\mathcal{N}(\mathcal{D}_X), \mathbb{C})$ as the tangent space of $\mathrm{Stab}(\mathcal{D}_X)$ at a point. Therefore $\mathfrak{b}$ defines a holomorphic symplectic structure on $\mathrm{Stab}(\mathcal{D}_X)$ for odd $n$.

**Definition 3.5.** We define $\mathrm{Stab}^+(\mathcal{D}_X) \subset \mathrm{Stab}(\mathcal{D}_X)$ by

$$\mathrm{Stab}^+(\mathcal{D}_X) = \{\sigma = (Z, P) \mid \mathfrak{b}(Z, Z) = 0, (\sqrt{-1})^{-n}\mathfrak{b}(Z, Z) > 0\}.$$ 

If $n$ is odd, the first condition is vacuous as $\mathfrak{b}$ is skew-symmetric.

**Remark 3.6.** It is worth mentioning that $\mathrm{Stab}^+(\mathcal{D}_X)$ is an analogue of a period domain in the Hodge theory and the defining equations are an analogue of the Hodge–Riemann bilinear relations. The natural free $\mathbb{C}$-action on $\mathrm{Stab}(\mathcal{D}_X)$ preserves $\mathrm{Stab}^+(\mathcal{D}_X)$.

**Definition 3.7.** Let $s = (Z_{\sigma}, P_{\sigma})$ be a holomorphic local section of the $\mathbb{C}$-torsor $\mathrm{Stab}^+(\mathcal{D}_X) \to \mathrm{Stab}^+(\mathcal{D}_X)/\mathbb{C}$. Then

$$(3.3) \quad K^A(\bar{\sigma}) = -\log ((\sqrt{-1})^{-n}\mathfrak{b}(Z_{\sigma}, \overline{Z_{\sigma}})).$$
defines a local smooth function on $\text{Stab}^+(\mathcal{D}_X)/\mathbb{C}$. We call $K^A$ the A-model Weil–Petersson potential.

**Proposition 3.8** ([8 Proposition 3.5]). The complex Hessian $g^A = \sqrt{-1} \partial \bar{\partial} K^A$ of the A-model Weil–Petersson potential $K^A$ is independent of the choice of a local section $s$. Moreover, it descends to the quotient

$$\text{Aut}(\mathcal{D}) \backslash \text{Stab}^+(\mathcal{D}_X)/\mathbb{C}$$

away from the singular loci.

We call $g^A$ the A-model Weil–Petersson metric on $\text{Aut}(\mathcal{D}) \backslash \text{Stab}^+(\mathcal{D}_X)/\mathbb{C}$. Note that $g^A$ is in general a degenerate metric. The following examples are discussed in [8].

**Example 3.9.** Let $X$ be an elliptic curve. Since $\widetilde{\text{GL}}^+(2, \mathbb{R})$-action on $\text{Stab}(\mathcal{D}_X)$ is free and transitive, we observe

$$\text{Stab}^+(\mathcal{D}_X) = \text{Stab}(\mathcal{D}_X) \cong \widetilde{\text{GL}}^+(2, \mathbb{R}) \cong \mathbb{C} \times \mathbb{H}.$$ 

Therefore we conclude

$$\text{Aut}(\mathcal{D}) \backslash \text{Stab}^+(\mathcal{D}_X)/\mathbb{C} \cong \text{SL}(2, \mathbb{Z}) \backslash \mathbb{H}.$$ 

This is indeed the expected Kähler moduli space of $X$. Up to the $\mathbb{C}$-action, the central charge at $\tau \in \mathbb{H}$ is given by

$$Z(F) = -\deg(F) + \tau \text{rank}(F).$$

Since $K(\mathcal{D}_X) = \mathbb{Z}O_X \oplus \mathbb{Z}O_p$, the A-model Weil–Petersson potential is

$$K^A(\tau) = -\log \left((\sqrt{-1})^{-1}(Z(O_p)\overline{Z}(O_X) - Z(O_X)\overline{Z}(O_p))\right)$$

$$= -\log(\text{Im}(\tau)) - \log 2.$$ 

This is the Poincaré potential on $\mathbb{H}$ and descends to $\text{SL}(2, \mathbb{Z}) \backslash \mathbb{H}$. Therefore $g^A$ is the Poincaré metric. This computation is compatible with the fact that a mirror of an elliptic curve is an elliptic curve.

**Example 3.10.** Let $X$ be the self-product $E_\tau \times E_\tau$ of an elliptic curve $E_\tau = \mathbb{C}/(\mathbb{Z} + \tau \mathbb{Z})$. Then there is an identification

$$\overline{\text{Aut}}_{\text{CY}}(\mathcal{D}_X) \backslash \text{Stab}^+(\mathcal{D}_X)/\mathbb{C}^\times \cong \text{Sp}(4, \mathbb{Z}) \backslash \mathbb{H}_2$$

where $\overline{\text{Aut}}_{\text{CY}}(\mathcal{D}_X)$ is an appropriate group induced by the group of autoequivalences. Moreover, the A-model Weil–Petersson metric on the LHS is identified with the Bergman metric on the RHS (a Siegel modular variety). This result is compatible with the mirror symmetry between $X$ and a principally polarized abelian surface. The complex moduli space of the latter is given by $\text{Sp}(4, \mathbb{Z}) \backslash \mathbb{H}_2$.

**Example 3.11.** Let $X \subset \mathbb{P}^4$ be a quintic Calabi–Yau 3-fold, for which the existence of a stability condition is recently proven by Li [21]. Let
\( \tau H \in H^2(X, \mathbb{C}) \) be the complexified Kähler class, where \( H \) is the hyperplane class and \( \tau \in \mathbb{H} \). Then we have

\[
J(2\pi \sqrt{-1} \tau H, z) = e^{2\pi \sqrt{-1} \tau H} (1 + \frac{1}{2^2} \sum_{d \geq 0} N_d^X q^d dH^2 - \frac{2}{2^3} \sum_{d \geq 0} N_d^X q^d dH^3)
\]

where \( q = e^{2\pi \sqrt{-1} \tau} \) and \( N_d^X \) denotes the genus 0 Gromov–Witten invariant of \( X \) of degree \( d \) (cf. [6]). Then we have

\[
J(2\pi \sqrt{-1} \tau H) = e^{2\pi \sqrt{-1} \tau H} (1 + \frac{1}{5} \sum_{d \geq 0} N_d^X q^d dH^2 - \frac{2}{5} \sum_{d \geq 0} N_d^X q^d dH^3)
\]

Hence, near the large volume limit, the A-model Weil–Petersson potential is given by

\[
K^A(\tau) = -\log\left(\frac{2^3 \cdot 5}{3!} \text{Im}(\tau)^3\right) + O(q).
\]

and hence \( g^A \) is a quantum deformation of the Poincaré metric. In particular, for sufficiently small \( q \), it is non-degenerate and the Weil–Petersson distance to the large volume limit is infinite.

4. Kähler attractor mechanism

We will introduce the provisional definition of the Kähler attractor mechanism mirror to the complex attractor mechanism. Throughout this section, \( X \) is a projective Calabi–Yau 3-fold. Let \( k = \dim H^{1,1}(X) \) be the expected dimension of the complexified Kähler moduli space. We define \( H^{ev}(X, \mathbb{C}) = \bigoplus_{i=0}^{3} H^{i,i}(X, \mathbb{C}) \).

4.1. Kähler attractor mechanism of stability space. We first investigate some fundamental structures on \( \text{Stab}^+(D_X) \).
Definition 4.1. For $\sigma = (Z, P) \in \text{Stab}^+(\mathcal{D}_X)$, we define the normalized Kähler central charge of $F \in \mathcal{N}(\mathcal{D}_X)$ by

$$V(\sigma, F) = e^{\frac{K^A(\sigma)}{2}} \mathcal{Z}(F),$$

where by abuse of notation $K^A(\sigma)$ is given by the Equation (3.3) ($K^A(\sigma)$ really depends on $\sigma$, not on the class $\bar{\sigma}$).

Then $V(-, F)$ defines a smooth function

$$V(-, F) : \text{Stab}^+(\mathcal{D}_X) \longrightarrow \mathbb{C}.$$ 

Since the absolute value $W(\sigma, F)$ is invariant under the $\mathbb{C}$-action, we obtain a function

$$|V(-, F)| : \text{Stab}^+(\mathcal{D}_X)/\mathbb{C} \longrightarrow \mathbb{R}_{\geq 0}.$$ 

We call it the Kähler mass function of $F$.

Let us recall some basic facts about $\text{Stab}^+(\mathcal{D}_X)/\mathbb{C}$, a projectivization of a holomorphic symplectic manifold $\text{Stab}^+(\mathcal{D}_X)$. The tangent space of $\text{Stab}^+(\mathcal{D}_X)/\mathbb{C}$ at $\sigma = (Z, P)$ is naturally identified with $\text{Hom}(\mathcal{N}(\mathcal{D}_X), \mathbb{C})/\mathbb{C}Z$, and hence $\text{Stab}^+(\mathcal{D}_X)/\mathbb{C}$ is a holomorphic contact manifold with the canonical contact form

$$\theta = b(dZ, Z).$$

There is a precise relation between Legendrian and Lagrangian submanifolds: the lift of a Legendrian submanifold in $\text{Stab}^+(\mathcal{D}_X)/\mathbb{C}$ is a Lagrangian submanifold in $\text{Stab}^+(\mathcal{D}_X)$. Moreover, a Legendrian subspace $L$ defines a polarized Hodge structure of weight 3 on $\text{Hom}(\mathcal{N}(\mathcal{D}_X), \mathbb{C})$ equipped with $b$, namely

$$H^{3,0} = \mathbb{C}Z, \quad H^{3,0} \oplus H^{2,1} = \pi^{-1}(L),$$

where $\pi : \text{Stab}^+(\mathcal{D}_X) \rightarrow \text{Stab}^+(\mathcal{D}_X)/\mathbb{C}$ is the quotient map.

Theorem 4.2. Let $F \in \mathcal{N}(\mathcal{D}_X)$ and $\sigma = (Z, P) \in \text{Stab}^+(\mathcal{D}_X)/\mathbb{C}$ such that $\mathcal{Z}(F) \neq 0$. Given a Legendrian subspace $L \subset \text{Hom}(\mathcal{N}(\mathcal{D}_X), \mathbb{C})/\mathbb{C}Z$, then $\sigma$ is a stationary point of the Kähler mass function $|V(-, F)|$ along the $L$-direction if and only if

$$\chi(F, -) = \text{Re}(CZ(-)) \quad (\exists C \in \mathbb{C})$$

holds in $\text{Hom}(\mathcal{N}(\mathcal{D}_X), \mathbb{C})$.

Proof. The proof is parallel to that of Theorem [2,3]. Let $L_1, \ldots, L_k \in \text{Hom}(\mathcal{N}(\mathcal{D}_X), \mathbb{C})$ be lifts of a basis of $L$. By Bridgeland’s result [4], the deformation of the central charge

$$\mathcal{Z}_\epsilon = \mathcal{Z} + \sum_{i=1}^{k} \epsilon_i L_i \in \text{Hom}(\mathcal{N}(\mathcal{D}_X), \mathbb{C})$$

for small $\epsilon = (\epsilon_i)_{i=1}^{k} \in \mathbb{C}^k$ induces a unique deformation $\sigma_\epsilon$ of the stability condition $\sigma$. Then a straightforward calculation shows

$$\left. \frac{\partial}{\partial \epsilon_i} \right|_{\epsilon=0} |V(\sigma_\epsilon, F)|^2 = e^{K^A(\sigma)} \mathcal{Z}(F)L_i(F).$$
By the assumption that $Z(F) \neq 0$, $\sigma$ is a stationary point of $|V(-, E)|^2$ along the $L$-direction if and only if $L_i(F) = 0$ for $1 \leq i \leq k$. Then the Legendrian property of $L$ implies that $\chi(F, -) \in \mathbb{C}Z \oplus \overline{\mathbb{C}Z}$. Since $\chi(E, -) \in \text{Hom}(\mathcal{N}(D_X), Z)$, we have

$$\chi(F, -) = (CZ + \overline{CZ}) = \text{Re}(CZ(-))$$

for some $C \in \mathbb{C}$.

\[\Box\]

4.2. Kähler attractor mechanism of complexified Kähler cone. We defined the normalized Kähler central charges on $\text{Stab}^+(D_X)/\mathbb{C}$. However, $\text{Stab}^+(D_X)/\mathbb{C}$ is in general conjectured to be much larger than (the universal covering of) the Kähler moduli space $\mathfrak{M}_{\text{Kah}}$ (Conjecture 3.3). Therefore it is reasonable to restrict ourselves to a submanifold of the expected dimension $k$. In this section, we will consider the complexified Kähler cone $\mathcal{K}_X^\mathbb{C}$, a natural candidate of such a submanifold.

Let $\omega = B + \sqrt{-1}K \in \mathcal{K}_X^\mathbb{C}$ be a complexified Kähler class of $X$. Henceforth we consider the quantum cohomology central charge (Equation (3.2))

$$Z_{\sigma, \omega}(F) = -\left\langle \tilde{J}(\omega), v_X(F) \right\rangle$$

where we write $\tilde{J}(\omega) = (2\pi \sqrt{-1})^{-\deg J}(-2\pi \sqrt{-1}\omega)$ for the sake of shorthand.

Let $\phi_1, \ldots, \phi_k \in H^{1,1}(X)$ be a basis and $t_1, \ldots, t_k$ the linear coordinate system of $H^{1,1}(X)$ dual to the basis, i.e. we may write $\omega = \sum_{i=1}^k t_i \phi_i$. Let $L(\omega)$ be the fundamental solution of the quantum differential equation, that is the $\text{End}(H^{ev}(X, \mathbb{C}))$-valued function satisfying

$$\nabla^A L(\omega) = 0, \quad L(\omega) = \text{id} + O(\omega),$$

where $\nabla^A = d + \sum_{i=1}^k (\phi_i^*)dt_i$ denotes the Dubrovin connection on $H^{ev}(X, \mathbb{C})$. Then the J-function is obtained by applying the fundamental solution $L(\omega)$ to $1 \in H^0(X, Z)$, i.e. $J(\omega) = L(\omega)1$.

Proposition 4.3. Near the large volume limit (i.e. for sufficiently small $q$), $\sqrt{-1}\beta(Z_{\sigma, \omega}, \overline{Z_{\sigma, \omega}}) > 0$ holds.

Proof. This is a quantum corrected version of [8, Proposition 4.6], where $\tilde{J}(\omega)$ is replaced by $e^\omega$. Since $\tilde{J}(\omega) = e^\omega + O(q)$, the assertion follows. \[\Box\]

Theorem 4.4. The conjectural embedding (Conjecture 3.3)

$$\iota: \mathcal{K}_X^\mathbb{C} \longrightarrow \text{Stab}^+(D_X)/\mathbb{C}, \quad \omega \mapsto \sigma_\omega = (Z_{\sigma, \omega}, P_{\sigma, \omega})$$

is Legendrian (if it exists).

Proof. It suffices to check that

$$\mathbb{C}(\frac{\partial}{\partial t_1}Z_{\sigma, \omega}, \ldots, \frac{\partial}{\partial t_k}Z_{\sigma, \omega}) \subset \text{Hom}(\mathcal{N}(D_X), \mathbb{C})/\mathbb{C}Z_{\sigma, \omega}$$
is a Legendrian subspace. This follows from the following calculation.

\[
\mathfrak{b}(\frac{\partial}{\partial t}Z_{\sigma}, \frac{\partial}{\partial t}Z_{\sigma}) = -\langle \frac{\partial}{\partial t}J(\omega), \frac{\partial}{\partial t}J(\omega) \rangle \\
= \frac{1}{(-2\pi\sqrt{-1})^3}(\frac{\partial}{\partial t}J(-2\pi\sqrt{-1}\omega), \frac{\partial}{\partial t}J(-2\pi\sqrt{-1}\omega)) \\
= \frac{1}{2\pi\sqrt{-1}}(L(-2\pi\sqrt{-1}\omega)\phi_i, L(-2\pi\sqrt{-1}\omega)\phi_j) \\
= \frac{1}{2\pi\sqrt{-1}}(\phi_i, \phi_j) \\
= 0
\]

The first equality is due to \([8, \text{Lemma 4.4}]\). Although \([8, \text{Lemma 4.4}]\) is classical (no quantum correction), an identical proof works. The third equality follows from the definition of the \(J\)-function. The fourth equality follows from \([14, \text{Proposition 4.2}]\) (essentially the Frobenius property of the quantum product). \(\square\)

Motivated by the Theorem 4.4, we define the normalized Kähler central charge of \(F \in \mathcal{N}(\mathcal{D}_X)\) on \(\mathcal{K}^C_X\) by

\[
W(\omega, F) = e^{K^A(\omega)}Z_{\sigma}(F),
\]

where \(K^A\) denotes the A-model Weil–Petersson potential. Then \(W(-, F)\) is a smooth function

\[
W(-, F) : \mathcal{K}^C_X \rightarrow \mathbb{C}
\]

and the Kähler mass function of \(F \in \mathcal{N}(\mathcal{D}_X)\) is defined by

\[
|W(-, F)| : \mathcal{K}^C_X \rightarrow \mathbb{R}_{\geq 0}.
\]

To summarize, we have the following commutative diagram

\[
\begin{array}{ccc}
\mathcal{K}^C_X & \xrightarrow{\iota} & \text{Stab}^+(\mathcal{D}_X)/\mathbb{C} \\
\downarrow{\scriptstyle |W(-, F)|} & & \downarrow{\scriptstyle |V(-, F)|} \\
\mathbb{R}_{\geq 0} & & \mathbb{R}_{\geq 0}
\end{array}
\]

where \(\iota\) is in general hypothetical, but the Kähler mass functions are well-defined.

**Theorem 4.5.** For \(F \in \mathcal{N}(\mathcal{D}_X)\), a stationary point \(\omega \in \mathcal{K}^C_X\) of the Kähler mass function such that \(W(\omega, F) \neq 0\) is characterized by the equation

\[
\chi(F, -) = \text{Re}(CZ_{\sigma}(\omega)), \quad (\exists C \in \mathbb{C})
\]

in \(\text{Hom}(\mathcal{N}(\mathcal{D}_X), \mathbb{C})\). We call Equation (4.1) the Kähler attractor equation.

**Proof.** The assertion follows from Theorem 4.2 and Theorem 4.4. Note that we do not used \(P_{\sigma}\) but only \(Z_{\sigma}\) in Theorem 4.4. \(\square\)
**Definition 4.6.** A stationary point \( \omega \in K_{\mathcal{X}}^{\mathbb{C}} \) with \( W(\omega, F) \neq 0 \) is called a Kähler attractor for \( F \). The corresponding Calabi–Yau 3-fold \( (X, \omega) \) is called a Kähler attractor variety for \( F \).

**Remark 4.7.** Near the large volume limit, we have an asymptotic expansion \( K^A(\omega) = -\log \text{Im}(\omega)^3 + O(q) \) (up to constant term). Then the A-model Weil–Petersson metric \( g^A \) is positive definite and we can show that the Kähler attractors are discrete by an almost identical argument to the complex side (Theorem 2.4). Note that this sort of asymptotic metric has previously been investigated by Trenner and Wilson [28]. Our work [8] can be considered as a globalization of their pioneering work.

### 4.3. Kähler attractor mechanism for torus

Let us consider a complex 3-torus \( Y = \mathbb{C}^3/\mathbb{Z}^3 + \sqrt{-1}\mathbb{Z}^3 \). We choose a symplectic basis of \( H^{ev}(Y, \mathbb{Z}) = \oplus_{i=0}^3 H^{i,i}(Y, \mathbb{Z}) \) as follows

\[
\delta_0 = 1 \in H^{0,0}(Y, \mathbb{Z}),
\]

\[
\delta_{ij} = \frac{\sqrt{-1}}{2} dz_i \wedge d\bar{z}_j \in H^{1,1}(Y, \mathbb{Z}) \quad (1 \leq i, j \leq 3)
\]

\[
\epsilon^0 = \left( \frac{\sqrt{-1}}{2} \right)^3 dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2 \wedge dz_3 \wedge d\bar{z}_3 \in H^{3,3}(Y, \mathbb{Z})
\]

\[
\epsilon^{ij} = \left( \frac{\sqrt{-1}}{2} \right)^{-1} \frac{\partial}{\partial z_i} \left( \frac{\partial}{\partial \bar{z}_j} \epsilon^0 \right) \in H^{2,2}(Y, \mathbb{Z}) \quad (1 \leq i, j \leq 3).
\]

We introduce a complexified Kähler structure on \( Y \) by

\[
\omega = B + \sqrt{-1}\kappa = \sum_{i,j=1}^3 \omega^{ij} \delta_{ij} \in H^{1,1}(Y, \mathbb{C}).
\]

and identify \( \omega \) with the matrix \( \Omega = (\omega^{ij}) \in \mathfrak{h}_3 \) for the sake of convenience.

The twisted Mukai vector of \( F \in N(D_Y) \) has an expansion

\[
v_Y(F) = v^0 \delta_0 + \sum_{i,j} v^{ij} \delta_{ij} + \sum_{i,j} u_{ij} \epsilon^{ij} + u_0 \epsilon^0.
\]

There is no quantum correction \( (Z_{\sigma_\omega}(F) = -\langle e^\omega, F \rangle) \), and hence the Kähler attractor equation \( \chi(F, -) = \text{Re}(CZ_{\sigma_\omega}(-)) \) is equivalent to the following system of equations

\[
\text{Re}(C) = -v^0
\]

\[
\text{Re}(C \omega^{ij}) = -v^{ij}
\]

\[
\text{Re}(C \text{Cof}(\Omega)_{ij}) = u_{ij}
\]

\[
\text{Re}(C \det(\Omega)) = -u_0.
\]

They are parallel to the complex attractor equation for \( T^6 \). We are able to solve the equations to obtain the solutions in an explicit form.
Theorem 4.8. Assume that the coefficient matrices \( V = (v_{ij}), U = (u_{ij}) \in M_3(\mathbb{Z}) \) are symmetric. There exists a unique Kähler attractor
\[
\Omega = ((2Vu - (v^0u_0 + \text{tr}(VU)E_3)) + \sqrt{-D}E_3)(2R)^{-1} \in \mathbb{H}_3
\]
where
\[
R = \text{Cof}(V) + v^0U,
\]
\[
D = ((\text{tr}(VU))^2 - \text{tr}((VU)^2)) - (v^0u_0 + \text{tr}(VU))^2 + 4(v^0 \det(U) - u_0 \det(V)).
\]

Proof. The proof is based on a step-by-step explicit calculation given in Appendix A and is parallel to the complex attractor case. \(\square\)

In light of the B-model side, we introduce a covering of the Kähler attractor variety \((Y, \omega)\). The lattice embedding
\[
\mathbb{Z}^3 + \sqrt{-1}(2R)\mathbb{Z}^3 \hookrightarrow \mathbb{Z}^3 + \sqrt{-1}\mathbb{Z}^3
\]
induces a covering map
\[
\phi^\vee : Y' = \mathbb{C}^3/(\mathbb{Z}^3 + \sqrt{-1}(2R)\mathbb{Z}^3) \rightarrow Y = \mathbb{C}^3(\mathbb{Z}^3 + \sqrt{-1}\mathbb{Z}^3).
\]
Then \(Y'\) carries a natural complexified Kähler structure \(\omega'\) given by the pullback
\[
\Omega' = (\phi^\vee)^* \Omega = (2Vu - (v^0u_0 + \text{tr}(VU)E_3)) + \frac{\sqrt{-D}}{2}E_3
\]
If we regard the complexified Kähler structures as elements of \(H^2(Y', \mathbb{C})/H^2(Y', \mathbb{Z})\), then the B-field \(\text{Re}(\omega')\) becomes trivial and the Kähler structure reads
\[
\text{Im}(\omega') = \frac{\sqrt{-D}}{2} \sum_{1 \leq i,j \leq 3} \delta_{ij}dz_i \wedge d\bar{z}_j = \sqrt{D} \sum_{1 \leq i \leq 3} dx_i \wedge dy_i
\]
Hence \(Y'\) is a principally polarized abelian 3-fold with the Kähler structure \(\sqrt{D}\). This computation is compatible with the fact that the product \(X' = (E_{\sqrt{-D}})^3\) of an elliptic curve \(E_{\sqrt{-D}}\) is mirror symmetric to a principally polarized abelian 3-fold \(Y'\) (c.f. [16]).

We conclude that a complex attractor variety \(X\) (resp. \(X')\) is mirror symmetric to a Kähler attractor variety \(Y\) (resp. \(Y'\)) provided that \(\gamma \in H_3(X, \mathbb{Z})\) and \(F \in \mathcal{N}(D_Y)\) are mirror cycles.

\[
(E_{\sqrt{-D}})^3 \xrightarrow{\text{mirror}} (\mathbb{C}^3/(\mathbb{Z}^3 + \sqrt{-1}(2R)\mathbb{Z}^3), \omega') \xrightarrow{\phi^\vee} (\mathbb{C}^3/(\mathbb{Z}^3 + \sqrt{-1}\mathbb{Z}^3), \omega)
\]

Remark 4.9. It is expected that if \(\Omega\) is a complex attractor of a charge \(\gamma\), then \(\gamma\) supports BPS states with respect to \(\Omega\) [23, Section 2.6]. In terms of Kähler attractors and stability conditions on the mirror side, one expects that if \(\sigma\) is a Kähler attractor of a class \(v\), then \(v\) should support a Bridgeland semistable object with respect to \(\sigma\). The space of stability conditions
on abelian threefolds has been studied in [1]. It would be interesting to verify that the Kähler attractors found in Theorem 4.8 do indeed support semistable objects.

4.4. Kähler attractor mechanism for $E \times S$. For an elliptic curve $E = \mathbb{C}/(\mathbb{Z} + \tau \mathbb{Z})$ and a K3 surface $S$, we consider the product Calabi–Yau 3-fold $Y = S \times E$. In the following, we use $\langle -, - \rangle$ for the Mukai pairing on $H^*(Y, \mathbb{Z})$, $\langle -, - \rangle^S$ for the Mukai pairing on $H^*(S, \mathbb{Z})$, and $(-, -)$ for the cup pairing on $H^2(S, \mathbb{Z})$. We also define the algebraic lattice by

$$NS'(S) = H^0(S, \mathbb{Z}) \oplus NS(S) \oplus H^4(S, \mathbb{Z}).$$

The twisted Mukai vector $v_Y(F)$ of $F \in \mathcal{N}(D_Y)$ can be written as

$$v_Y(F) = v_1 + v_2 \frac{-1}{2} dz \wedge d\bar{z} \in H^{ev}(Y, \mathbb{Z})$$

where for $i = 1, 2$

$$v_i = (r_i, D_i, s_i) \in NS'(S).$$

We would like to find $\omega_S \in NS(S)_\mathbb{C}$ and $\omega_E \in \mathbb{H}$ such that

$$\omega = \omega_S + \omega_E \frac{-1}{2} dz \wedge d\bar{z} \in H^{1,1}(Y)$$

satisfies the Kähler attractor equation

$$\text{Re}(CZ \omega(-)) = \langle v_Y(F), - \rangle$$

for some $C \in \mathbb{C}$, where we rewrite the equation by the Hirzebruch–Riemann–Roch theorem (Equation (3.1)).

**Lemma 4.10.** The Kähler attractor equation is equivalent to the following system of equations

$$\text{Re}(C \langle \delta, - \rangle^S) = \langle v_1, - \rangle^S,$$

$$\text{Re}(C \omega_E \langle \delta, - \rangle^S) = \langle v_2, - \rangle^S,$$

where $\delta = e^{\omega_S}$.

**Proof.** Recall first that there is no quantum correction for $Y$, and hence we have $\tilde{J}(\omega) = e^{\omega}$. We plug $\alpha \in \oplus_{i=0}^2 H^{i,i}(S)$ in the Kähler attractor equation to obtain

$$\langle v_2, \alpha \rangle^S = \langle v_Y(F), \alpha \rangle = \text{Re}(C \langle e^{\omega}, \alpha \rangle) = \text{Re}(C \omega_E \langle \delta, \alpha \rangle).$$

Similarly for $\beta dz \wedge d\bar{z}$ where $\beta \in \oplus_{i=0}^2 H^{i,i}(S)$ we obtain obtain

$$\langle v_1, \beta \rangle^S = \langle v_Y(F), \beta \rangle = \text{Re}(C \langle e^{\omega}, \beta \rangle) = \text{Re}(C \langle \delta, \beta \rangle).$$

\[ \square \]

Let us write $D_{v_1, v_2} = v_1^2 v_2^2 - \langle v_1, v_2 \rangle^S S \in \mathbb{Z}$, where $v_1^2 = \langle v_1, v_1 \rangle^S$. 
Proposition 4.11. There exist \( \omega_S \in H^{1,1}(S) \) and \( \omega_E \in \mathbb{H} \) satisfying the Kähler attractor equation for \( F \) if and only if the lattice \( \mathbb{Z}v_1 + \mathbb{Z}v_2 \) is positive definite. Moreover, they are unique and given by
\[
\delta = \frac{-\sqrt{-1}}{C \text{Im}(\omega_E)}(v_2 - \overline{\omega_E}v_1), \quad \omega_E = \frac{(v_1, v_2)_S + \sqrt{-D_{v_1, v_2}}}{v_1^2},
\]
where the constant \( C \) is chosen so that the degree 0 part of \( \delta \) is \( 1 \in H^0(S, \mathbb{C}) \).

Proof. First, by Proposition 2.10 we can solve the Kähler attractor equation to obtain
\[
\delta = \frac{-\sqrt{-1}}{\text{Im}([C]^{1,2}\omega_E)}(Cv_2 - C\omega_1v_1) = \frac{-\sqrt{-1}}{C \text{Im}(\omega_E)}(v_2 - \overline{\omega_E}v_1).
\]
Assume that there exist \( \omega_S \in H^{1,1}(S) \) and \( \omega_E \in \mathbb{H} \) satisfying the Kähler attractor equation. The condition that \( \delta \) is of the form \( e^{\omega_S} \) implies that \( \langle \delta, \delta \rangle = 0 \) and hence
\[
\omega_E^2v_1^2 - 2\omega_E(v_1, v_2)_S + v_2^2 = 0.
\]
By a parallel argument to the proof of Proposition 2.13 we conclude that the lattice \( \mathbb{Z}v_1 + \mathbb{Z}v_2 \) is positive definite and
\[
\omega_E = \frac{(v_1, v_2)_S + \sqrt{-D_{v_1, v_2}}}{v_1^2} \in \mathbb{H}.
\]
On the other hand, assume that \( \mathbb{Z}v_1 + \mathbb{Z}v_2 \) is positive definite, then \( (r_1, r_2) \neq (0, 0) \). Moreover, we can choose a constant \( C \) so that the degree 0 part of
\[
\delta = \frac{-\sqrt{-1}}{C \text{Im}(\omega_E)}(v_2 - \overline{\omega_E}v_1)
\]
is 1, and then \( \delta = e^{\omega_S} \) can be solved for \( \omega_S = \log \delta \in H^{1,1}(S) \). \( \omega_E \) is uniquely determined in a similar manner. \( \square \)

Note that \( \omega_S \in H^{1,1}(S) \) in Proposition 4.11 is not necessarily a complexified Kähler class. The best we could prove is the following (cf. Remark 4.16).

Proposition 4.12. Assume the lattice \( \mathbb{Z}v_1 + \mathbb{Z}v_2 \) is positive definite. For \( \omega_S = \log \delta \in H^{1,1}(S) \), we have \( \text{Im}(\omega_S)^2 > 0 \).

Proof. Recall that we write \( v_i = (r_i, D_i, s_i) \in NS'(S) \). A straightforward computation of shows that \( \text{Im}(\omega_S)^2 = (r_2D_1 - r_1D_2)^2 \). Since \( \mathbb{Z}v_1 + \mathbb{Z}v_2 \) is positive definite, \( (r_1, r_2) \neq (0, 0) \). Assume first that \( r_1r_2 \neq 0 \). Then, since \( \mathbb{Z}v_1 + \mathbb{Z}v_2 \) is positive definite
\[
0 < \left( \frac{1}{r_1}v_1 - \frac{1}{r_2}v_2 \right)^2 = 0, \quad \frac{1}{r_1}D_1 - \frac{1}{r_2}D_2, \quad \frac{s_1}{r_1} - \frac{s_2}{r_2} = \left( \frac{1}{r_1}D_1 - \frac{1}{r_2}D_2 \right)^2.
\]
Therefore we have
\[
\text{Im}(\omega_S)^2 = (r_1r_2)^2\left( \frac{1}{r_1}D_1 - \frac{1}{r_2}D_2 \right)^2 > 0.
\]
Assume next that \( r_1 = 0 \) and \( r_2 \neq 0 \). Then we have
\[
\text{Im}(\omega_S)^2 = r_2^2 D_1^2 = r_2^2 v_1^2 > 0.
\]
Similarly for \( r_1 \neq 0 \) and \( r_2 = 0 \). \( \square \)

**Example 4.13.** Let \( S \) be a K3 surface such that \( NS(S) = \mathbb{Z}H \), where \( H \) is ample with \( H^2 = 2n > 0 \). It is known that for \( v \in NS'(S) \) with \( v^2 > 0 \) there is a sheaf \( \mathcal{E} \) such that \( v_S(\mathcal{E}) = v \). In particular there are sheaves \( \mathcal{E}_1, \mathcal{E}_2 \) whose twisted Mukai vectors are
\[
v_1 = (1, 0, -n), \quad v_2 = (0, -H, 0) \in NS'(S).
\]
For \( Y = S \times E \), let us consider
\[
F = \mathcal{E}_1 \boxtimes \mathcal{O}_E \oplus \mathcal{E}_2 \boxtimes \mathcal{O}_p \in \mathcal{N}(Y).
\]
Then
\[
e^{\sqrt{-1}H} \in (\mathbb{Z}v_1 + \mathbb{Z}v_2)_C, \quad \omega_E = \sqrt{-1} \in \mathbb{H}
\]
satisfy the Kähler attractor equation for \( F \).

Let us take a close look at Example 4.13
\[
e^{\sqrt{-1}H} = (1, \sqrt{-1}H, -n)
\]
\[
e^{\sqrt{-1}H} = v_1 + \sqrt{-1}v_2 \in (\mathbb{Z}v_1 + \mathbb{Z}v_2)_C \subset NS'(S)_C
\]
On the other hand, for \( \epsilon^2 \notin \mathbb{Q} \),
\[
e^{\sqrt{-1}H} = (1, \sqrt{-1}\epsilon H, -\epsilon^2 n)
\]
\[
e^{\sqrt{-1}H} = (1, 0, -\epsilon^2 n) + \sqrt{-1}\epsilon(0, H, 0)
\]
\[
e^{\sqrt{-1}H} = (1, 0, 0) - \epsilon^2(0, 0, n) + \sqrt{-1}\epsilon(0, H, 0) \in NS'(S)_C.
\]
Hence there is no proper sublattice \( L \subset NS'(S)_C \) such that \( e^{\sqrt{-1}H} \in L_C \). Therefore the Kähler structure \( H \) is not deformable in such a way that \( e^{\omega_S} \in L_C \) for some lattice \( L \subset NS'(S)_C \) of rank 2. This calculation illustrates that \( e^{B+\sqrt{-1}\omega} \) is able to detect a fine integral structure of the Kähler moduli space.

**Definition 4.14.** A complexified Kähler structure \( \omega_S \) is called Kähler rigid if there exists a rank 2 lattice \( L \subset NS'(S)_C \) such that \( e^{\omega_S} \in L_C \).

**Theorem 4.15** (cf. [15]). A complexified Kähler structure \( B + \sqrt{-1}\kappa \in H^{1,1}(S) \) is Kähler rigid if and only if \( B \in H^{1,1}(S, \mathbb{Q}) \) and \( \kappa^2 \in H^4(S, \mathbb{Q}) \)

**Proof.** We consider an existence condition of a rank 2 sublattice \( L \subset H^*(M, \mathbb{Z}) \) such that
\[
e^{B+\sqrt{-1}\kappa} = 1 + B + \frac{1}{2}(B^2 - \kappa^2) + \sqrt{-1}(\kappa + B \wedge \kappa) \in L_C.
\]
First, $B$ needs to be rational, and hence so is $\kappa^2$. Then we may write $\kappa = kH$ for $k^2 \in \mathbb{Q}$ and $H \in H^2(S, \mathbb{Z})$ with $H^2 > 0$. Indeed, in this case, there exist $m, n \in \mathbb{N}$ such that

$$m \text{Re}(e^{B + \sqrt{-1}kH}), \ n \text{Im}(e^{B + \sqrt{-1}kH}) \in H^*(S, \mathbb{Z}).$$

and the complexification $L_C$ of the lattice

$$L = \mathbb{Z}m \text{Re}(e^{B + \sqrt{-1}kH}) + \mathbb{Z}n \text{Im}(e^{B + \sqrt{-1}kH}) \subset H^*(S, \mathbb{Z})$$

contains $e^{B + \sqrt{-1}kH}$. \qed

It is natural to expect that a Kähler rigid K3 surface is mirror to a singular K3 surface. However, there is an obvious puzzle. The dimension of the Kähler moduli space of a singular K3 surface is 20, while the dimension of the complex moduli space of a Kähler rigid K3 surface is at most 19. It turns out that the correct framework of mirror symmetry for K3 surfaces is the generalized Calabi–Yau structures developed by Hitchin [10] and Huybrechts [12]. To solve the above puzzle, we need to incorporate deformations as a generalized K3 surface (namely a Kähler rigid K3 surface should be defined as a generalized K3 surface). A recent article [15] investigates mirror symmetry for generalized K3 surfaces with particular emphasis on complex and Kähler rigid structures.

**Remark 4.16.** As to Proposition 4.12, the reason why the imaginary part of $\omega_S = \log \delta \in H^{1,1}(S)$ is not necessarily Kähler but merely positive is also well-explained from the viewpoint of the generalized Calabi–Yau structures.

**4.5. Kähler constellation.** From the perspective of homological mirror symmetry, the derived equivalent Calabi–Yau 3-folds share the same mirror Calabi–Yau 3-fold (remember that birationality implies derived equivalence for Calabi–Yau 3-folds [3]). It is a folklore conjecture that the complexified Kähler cones of the derived equivalent Calabi–Yau 3-folds give local charts of the Kähler moduli space $\mathcal{M}_{\text{Kah}}$ (Conjecture 3.3). Indeed, the union of the Kähler cones of birational Calabi–Yau 3-folds form a cone, known as the movable cone, and has been extensively studied in birational geometry. From this point of view, the Kähler constellation $\text{Attr}_{\text{Kah}}$ of a Calabi–Yau 3-fold $Y$ is defined as the union of the Kähler attractors of Calabi–Yau 3-folds derived equivalent to $Y$.

In light of mirror symmetry, if $X$ and $Y$ are mirror Calabi–Yau 3-folds, then the mirror map should induce a bijective correspondence between the complex constellation $\text{Attr}_{\text{Cpx}}^X$ of $X$ and the Kähler constellation $\text{Attr}_{\text{Kah}}^Y$ of $Y$.

$$\mathcal{M}^X_{\text{Cpx}} \cong \mathcal{M}^Y_{\text{Kah}}$$

$$\cup$$

$$\text{Attr}^X_{\text{Cpx}} \cong \text{Attr}^Y_{\text{Kah}}$$
The mirror correspondence of the complex rigid and Kähler rigid K3 surfaces is one occurrence of such \[15\] (cf. Section 2.3 and Section 4.4).

**Appendix A.**

Let \(p_0, q_0 \in \mathbb{R}\) and \(P = (P^{ij}), Q = (Q_{ij}) \in M_3(\mathbb{R})\). We investigate conditions on \(p_0, q_0, P, Q\) under which there exist \(C \in \mathbb{C}\) and \(A = (A^{ij}) \in M_3(\mathbb{C})\) such that the following system of equations hold.

\[
\begin{align*}
(A.1) \quad & \quad \text{Re}(C) = p_0, \\
(A.2) \quad & \quad \text{Re}(CA^{ij}) = P^{ij}, \\
(A.3) \quad & \quad \text{Re}(CC^{(A)}_{ij}) = -Q_{ij}, \\
(A.4) \quad & \quad \text{Re}(C \det(A)) = q_0.
\end{align*}
\]

Here \(C^{(A)}_{ij}\) denotes the \((i,j)\)-th entry of the cofactor matrix of \(A\).

We first define

\[
\begin{align*}
R & := \text{Cof}(P) + p_0^0 Q, \\
M & := 2 \text{det}(P) + (p_0^0)^2 q_0 + p_0^0 \text{tr}(P^T Q), \\
D & := 2 \left((\text{tr}(P^T Q))^2 - \text{tr}((P^T Q)^2)\right) - (p_0^0 q_0 + \text{tr}(P^T Q))^2 \\
& \quad + 4(p_0^0 \text{det}(Q) - q_0 \text{det}(P)).
\end{align*}
\]

**Lemma A.1.** We have the identity

\[
4 \text{det}(R) - M^2 = (p_0^0)^2 D.
\]

**Proof.** We first assume that \(P\) is invertible. Then \(\text{det}(R)\) can be expressed as

\[
\begin{align*}
\text{det}(C \text{of}(P) + p_0^0 Q) & = \text{det}(P)^{-1} \text{det} \left( P^T \text{of}(P) + p_0^0 P^T Q \right) \\
& = \text{det}(P)^2 \text{det} \left( E_3 + \frac{p_0^0}{\text{det}(P)} P^T Q \right)
\end{align*}
\]

where \(E_3 \in M_3(\mathbb{C})\) denotes the identity matrix. Using the standard formula

\[
\text{det}(E_3 + B) = 1 + \text{tr}(B) + \frac{1}{2} \left((\text{tr}B)^2 - \text{tr}(B^2)\right) + \text{det}(B)
\]

for \(B \in M_3(\mathbb{C})\), we can check by direct computation that the Equation (A.5) holds.

Since the Equation (A.5) is an algebraic identity for \((p_0^0, q_0, P, Q) \in \mathbb{R}^2 \times M_3(\mathbb{R})^2\) which holds in the open dense subset \(\{\text{det}(P) \neq 0\} \subset \mathbb{R}^2 \times M_3(\mathbb{R})^2\), it remain valid in \(\mathbb{R}^2 \times M_3(\mathbb{R})^2\).

**Theorem A.2.** Suppose that \(p_0^0, q_0, P, Q\) are not all zero. The following two statements are equivalent:

1. There exists \((C, A)\) satisfying (A.1)-(A.4) in which \(\text{Im}(A)\) is invertible;
2. \(\text{det}(R) > 0\) and \(D > 0\).
In this case, there are exactly two solutions \((C, A)\), given by

\[
C = p^0 \pm \sqrt{-1} \frac{M}{\sqrt{D}}, \\
A = \left(2PQ - (p^0q_0 + \text{tr}(P^TQ))E_3 \mp \sqrt{-D}\right) (2R)^{-1}. 
\]

Moreover, the following two statements are equivalent:

(1) There exists \((C, A)\) satisfying (A.1)-(A.4) in which \(A\) is symmetric and \(\text{Im}(A)\) is positive definite;

(2) \(P\) and \(Q\) are symmetric, \(R\) is positive definite, and \(D > 0\).

In this case, the unique solution is given by

\[
C = p^0 - \sqrt{-1} \frac{M}{\sqrt{D}}, \\
A = \left(2PQ - (p^0q_0 + \text{tr}(PQ))E_3 + \sqrt{-D}\right) (2R)^{-1}. 
\]

**Proof.** We first prove the theorem under the assumption that \(p^0 \neq 0\) and \(M \neq 0\). Suppose that \((C, A)\) satisfies the Equation (A.1)-(A.4) and \(\text{Im}(A)\) is invertible. By the Equation (A.1),

\[
C \in \mathbb{C}
\]

can be written as

\[
C = p^0 + i\zeta^0
\]

for some \(\zeta^0 \in \mathbb{R}\). We denote

\[
X := \text{Im}(A) \in M_3(\mathbb{R}).
\]

The Equation (A.2) gives

\[
P^{ij} = \text{Re} \left( (p^0 + i\zeta^0)(\text{Re}(A^{ij}) + \sqrt{-1}\text{Im}(A^{ij})) \right)
= p^0\text{Re}(A^{ij}) - \zeta^0\text{Im}(A^{ij}).
\]

Since we assume \(p^0 \neq 0\), therefore

\[
\text{Re}(A^{ij}) = \frac{1}{p^0}(P^{ij} + \zeta^0X^{ij}).
\]

Using these expressions of the real and imaginary parts of \(A\), we have

\[
\text{Re}(CA^{ij}A^{kl}) = \text{Re} \left( (p^0 + i\zeta^0)(\text{Re}(A^{ij}) + \sqrt{-1}\text{Im}(A^{ij}))(\text{Re}(A^{kl}) + \sqrt{-1}\text{Im}(A^{kl})) \right)
= \frac{1}{p^0}(P^{ij}p^{kl} - |C|^2X^{ij}X^{kl})
\]

for any \(1 \leq i, j, k, l \leq 3\). Together with the Equation (A.3), we have

\[
-Q = \text{Re}(C\text{Cof}(A)) = \frac{1}{p^0}(\text{Cof}(P) - |C|^2\text{Cof}(X)).
\]

Hence

\[
\text{(A.6) } \text{Cof}(X) = \frac{1}{|C|^2}(\text{Cof}(P) + p^0Q) = \frac{1}{|C|^2}R.
\]

Since \(X = \text{Im}(A)\) is invertible, we have

\[
\det(\text{Cof}(X)) = \det(X)^2 > 0
\]
therefore $\det(R) > 0$.

Using again the expressions of $\text{Re}(A)$ and $\text{Im}(A)$, we have

$$\text{Re}(CA^i A^{kl} A^{mn}) = \frac{1}{(p^0)^2} \left(-2\zeta^0|C|^2 X^{ij} X^{kl} X^{mn} + P^{ij} P^{kl} P^{mn} - |C|^2 \left(\frac{P^{ij} X^{kl} X^{mn} + P^{kl} X^{ij} X^{mn} + P^{mn} X^{ij} X^{kl}}{2}\right)\right)$$

Together with the Equation (A.4) and (A.6), we obtain

$$q_0 = \text{Re}(C \det(A)) = \frac{1}{(p^0)^2} \left(-2\zeta^0|C|^2 \det(X) + \det(P) - |C|^2 \text{tr}(P^T \text{Cof}(X)) + \det(\text{Cof}(X)) = \frac{1}{|C|^6} \det(R).\right)$$

By Lemma (A.1)

$$\langle p^0\rangle^2 D = 4 \det(R) - M^2 = M^2 \left(\frac{|C|^2}{(\zeta^0)^2} - 1\right) = \frac{M^2(p^0)^2}{(\zeta^0)^2}.$$}

Hence $D > 0$ and

$$\zeta^0 = \pm \frac{M}{\sqrt{D}}.$$}

By Equation (A.6) and (A.7), we have

$$X = |C|^2 \det(X) R^{-1,T} = \pm \sqrt{D}(2R)^{-1,T}.$$}

Hence

$$\text{Re}(A) = \frac{1}{p^0} (P + \zeta^0 X) = \frac{1}{p^0} (P - \frac{M}{2} R^{-1,T}) = \frac{1}{p^0} (PR^T - \frac{M}{2} E_3) R^{-1,T} = \left(PQ^T - \frac{1}{2} (p^0 q_0 + \text{tr}(P^T Q)) E_3\right) R^{-1,T}.$$}

Therefore

$$A = \left(2PQ^T - (p^0 q_0 + \text{tr}(P^T Q)) E_3 \mp \sqrt{D}\right) (2R)^{-1,T}.$$}

This proves the statement under the assumption that $p^0 \neq 0$ and $M \neq 0$. 

Next we assume that $p^0 \neq 0$ and $M = 0$. Then the above argument up to the Equation A.7 is still valid. By Lemma A.1 and (A.6), we have
\[
D = \frac{4 \det(R)}{(p^0)^2} > 0.
\]

Now assume that $p^0 = 0$. First we observe that the solutions are still valid in this case. Hence it suffices to show that they are the only two solutions. Since $p^0, q_0, P, Q$ are not all zero, we have $c = i\zeta^0 \neq 0$. Then Equation (A.2) gives
\[
X := \text{Im}(A) = -\frac{1}{\zeta^0} P.
\]

Note that $P$ is invertible since $X = \text{Im}(A)$ is assumed to be invertible. We denote
\[
Y := \text{Re}(A).
\]

By Equation (A.3), we have
\[
\begin{pmatrix}
0 & 0 & 0 & 0 & P^{33} & -P^{32} & 0 & -P^{23} & P^{22} \\
0 & 0 & 0 & 0 & -P^{33} & 0 & P^{31} & p^{23} & 0 \\
0 & 0 & 0 & 0 & p^{32} & -P^{31} & 0 & -P^{22} & P^{21} \\
0 & -P^{33} & p^{32} & 0 & 0 & 0 & 0 & P^{13} & -P^{12} \\
P^{33} & 0 & -P^{31} & 0 & 0 & 0 & -P^{13} & 0 & P^{11} \\
-P^{32} & P^{31} & 0 & 0 & 0 & 0 & P^{12} & -P^{11} & 0 \\
0 & p^{23} & -P^{22} & 0 & -P^{13} & P^{12} & 0 & 0 & 0 \\
-P^{23} & 0 & P^{21} & P^{13} & 0 & -P^{11} & 0 & 0 & 0 \\
-P^{22} & -P^{21} & 0 & P^{12} & P^{11} & 0 & 0 & 0 & 0
\end{pmatrix}
= \begin{pmatrix}
Y^{11} \\
Y^{12} \\
Y^{13} \\
Y^{21} \\
Y^{22} \\
Y^{23} \\
Y^{31} \\
Y^{32} \\
Y^{33}
\end{pmatrix}
= \begin{pmatrix}
Q_{11} \\
Q_{12} \\
Q_{13} \\
Q_{21} \\
Q_{22} \\
Q_{23} \\
Q_{31} \\
Q_{32} \\
Q_{33}
\end{pmatrix}
\]
The $9 \times 9$ matrix on the left hand side has determinant $-2(\det(P))^3 \neq 0$, hence $Y$ can solved uniquely.

\[
\text{Re}(CA^{ij}A^{kl}) = (p^{ij}Y^{kl}Y^{mn} + P^{kl}Y^{ij}Y^{mn} + P^{mn}Y^{ij}Y^{kl}) - \frac{1}{\zeta^0)^2} p^{ij} P^{kl} P^{mn}.
\]

By Equation (A.4),
\[
q_0 = \text{tr}(P^T \text{Cof}(Y)) - \frac{1}{\zeta^0)^2} \det(P).
\]

This solves $\zeta^0$ up to sign. This proves that there are no other solutions. \qed

APPENDIX B.

**Theorem B.1** (Theorem 2.13). The complex constellation Attr$^{\text{Cpx}}$ bijectively corresponds to the abelian 3-folds with Picard number 9.
Proof. It suffices to show that an abelian 3-fold \(X_T = \mathbb{C}^3/(\mathbb{Z}^3 + T\mathbb{Z}^3)\) with \(\rho(X_T) = 9\) is a complex attractor variety. By Theorem 2.12, \(X_T\) is isogenous to \((E\sqrt{-D})^3\) for some \(D \in \mathbb{N}\). Therefore there exists a lattice embedding \(\mathbb{Z}^3 + \sqrt{-D}\mathbb{Z}^3 \hookrightarrow \mathbb{Z}^3 + T\mathbb{Z}^3\) induced by \(R \in M_3(\mathbb{Z})\) such that

\[TR \in M_3(\mathbb{Z}) + \sqrt{-DE_3}.\]

Then \(R = \sqrt{D}\text{Im}(T)^{-1}\) is symmetric and positive definite. We will find a 3-cycle \(\gamma \in H_3(X_T, \mathbb{Z})\) for which \(T\) is a complex attractor point. We write \(\gamma\) as

\[\gamma = q_0A_0 + \sum_{i,j=1}^{3} Q_{ij}A_{ij} + \sum_{i,j=1}^{3} P^{ij}B^{ij} + p^0B^0\]

as in Section 2.2. We introduce

- \(n := (D + 1)\det(R) \in \mathbb{N}\).
- \(M := 2n\det(R) \in \mathbb{N}\).
- \(S := 2n(TR - \sqrt{-DE_3}) = 2n\text{Re}(T)R \in M_3(\mathbb{Z})\).
- \(p^0 := \det(R) \in \mathbb{Z}\).
- \(P := (p^0S + ME_3)(2nR)^{-1} \in M_3(\mathbb{Q})\).
- \(Q := \frac{1}{p^0}(nR - \text{Cof}(P)) \in M_3(\mathbb{Q})\).
- \(q_0 := \frac{1}{(p^0)^2}(2n\det(R) - 2\det(P) - p^0\text{tr}(PQ)) \in \mathbb{Q}\).

Now we show that \(T\) is an attractor of \(\gamma\) given by \((p^0, P, Q, q_0)\). Define as in Appendix A:

\[
\tilde{R} := \text{Cof}(P) + p^0Q, \\
\tilde{M} := 2\det(P) + (p^0)^2q_0 + p^0\text{tr}(PQ), \\
\tilde{D} := 2(\text{tr}(PQ)^2 - \text{tr}((PQ)^2)) - (p^0q_0 + \text{tr}(PQ))^2 \\
+ 4(p^0\det(Q) - q_0\det(P)).
\]

Then we have

\[
\tilde{R} = nR, \quad \tilde{M} = 2n\det(R) = M.
\]

Moreover, by Lemma A.1

\[
\tilde{D} = \frac{1}{(p^0)^2}(4\det(\tilde{R}) - \tilde{M}^2) \\
= \frac{1}{\det(R)^2}(4n^3\det(R) - 4n^2\det(R)^2) \\
= 4(D + 1)^2\det(R)(n - \det(R)) \\
= 4D(D + 1)^2\det(R)^2 \\
= 4n^2D > 0.
\]

By Theorem A.2 an attractor is given by

\[
\tilde{T} = \left(2PQ - (p^0q_0 + \text{tr}(PQ))E_3 + \sqrt{-\tilde{D}}\right)(2\tilde{R})^{-1}.
\]
Hence
\[ \text{Im}(\tilde{T}) = \sqrt{D(2\tilde{R})^{-1}} = \sqrt{DR^{-1}} = \text{Im}(T), \]
and
\[ 2n\text{Re}(\tilde{T})R = 2PQ - (p^0q_0 + \text{tr}(PQ))E_3 \]
\[ = \frac{2}{p^0}(P\tilde{R} - \text{det}(P)E_3) - \frac{1}{p^0}(\tilde{M} - 2\text{det}(P))E_3 \]
\[ = \frac{1}{p^0}(2P\tilde{R} - \tilde{M}E_3) \]
\[ = \frac{1}{p^0}(2(p^0S + ME_3)(2nR)^{-1}(nR) - ME_3) \]
\[ = S = 2n\text{Re}(T)R \]
since \( P\tilde{R} = \text{det}(P)E_3 + p^0PQ \). Therefore we have \( \tilde{T} = T \), so \( T \) is an attractor. \( \square \)

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