Quantifying continuous-variable realism

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The debate instigated by the seminal works of Einstein, Podolsky, Rosen, and Bell, put the notions of realism and nonlocality at the core of almost all philosophical and physical discussions underlying the foundations of quantum mechanics. However, while experimental criteria and quantifiers are by now well established for nonlocality, there is no clear quantitative measure for the degree of reality associated with continuous variables such as position and momentum. This work aims at filling this gap. Considering position and momentum as effectively discrete observables, we implement an operational notion of projective measurement and, from that, a criterion of reality for these quantities. Then, we introduce a quantifier for the degree of irreality of a discretized continuous variable which, when applied to the conjugated pair position-momentum, is shown to obey an uncertainty relation, this meaning that quantum mechanics prevents classical realism for conjugated quantities. As an application of our formalism, we study the emergence of elements of reality in an instance where a Gaussian state is submitted to the dissipative dynamics implied by the Caldirola-Kanai Hamiltonian. In particular, at the equilibrium, we make some links with the measurement problem and identify aspects that can be taken as the quantum counterpart for the notion of rest.

I. INTRODUCTION

Until the beginning of the twentieth century, classical physics nourished the belief that the physical properties of a system are all well defined at every instant of time, even when the system is ideally isolated from its surroundings. Quantum mechanics, however, tells us that this intuition about the physical reality actually emerges because we, observers, “look” at the systems through a vast number of projective measurements performed one after another almost instantaneously. This is what happens, for instance, when we collect the photons that are scattered by an object under observation. The resulting sequence of collapses keeps the values of physical quantities well defined and produces the preconception of an observer-independent physical reality. This feeling is further strengthened by the fact the macroscopic objects are barely disturbed by the measurement act and thus do not significantly deviate from their Newtonian trajectories.

As has repeatedly been shown by experiments with isolated microscopic systems, such a classical notion of reality cannot be generally maintained. Perhaps the most emblematic experiment in this regard be the double-slit setup for matter (see, e.g., [1, 2]), where massive particles are put in a coherent superposition of different locations and then produce an interference pattern. Here comes the conceptual difficulty: What can one say about the positional elements of reality of each particle in this experiment? Does a particle pass through both slits simultaneously, as a wave, or is its position in a state of fundamental indefiniteness?

The implications of the superposition principle for the physical reality soon bothers the founding fathers of quantum theory. To emphasize the conundrum, Schrödinger showed that, governed by quantum laws, nature admits exotic scenarios where complex beings can be set in a sort of suspended reality which interpolates between states of “being dead” and “being alive” [3]. Approaching the issue from a different perspective, Einstein, Podolsky, and Rosen (EPR) put forward in 1935 a rationale defending that quantum mechanics could not be our ultimate description of nature [4]. Taking locality as a cornerstone of physics an introducing a criterion of reality, they conceived superposition states for which, they claimed, incompatible observables would have simultaneous elements of reality, even though such elements cannot be predicted by quantum mechanics. Quantum mechanics was then regarded as incomplete. Ironically, Bell showed later on that any theory aiming at completing quantum mechanics would be unavoidably nonlocal [5] (Bohmian mechanics [6, 7] being a prominent illustration of that). In light of the substantial empirical evidence obtained from accurate loophole-free Bell tests [8–13], it is fair to say nowadays that the fundamental premise of EPR, namely, locality, is unsustainable.

Discussions about the physical reality implied by the wave function were recently polarized in two main classes, both supported by substantial amount of theoretical work. While on the one hand ψ-ontic models aims at attaching to the wave function some realistic substance, on the other hand ψ-epistemic models suggest that it actually represents the knowledge one has about an underlying reality. Specialized literature has by now cumulated a number of contributions in favour of both ψ-ontic [14, 25] and ψ-epistemic [24, 30] models, with a recent work arguing that quantum mechanics can be viewed as classical statistical mechanics with an ontic extension and an epistemic restriction [31]. Within the decoherence paradigm, where environmental models are provided to account for the disappearance of quantum superpositions, considerable progress has been made towards a deep understanding of the quantum measurement problem and the emergence of objective classicality in the framework of the quantum Darwinism [32, 36]. Also noteworthy is the framework according to which quantum physics is an elementary theory of information and, as such, some ontological status is to be given to the very notion of information [37, 39].

Besides inciting a number of developments around the notions of entanglement and Bell nonlocality, EPR’s criterion of reality also led to heated debates about the physical realism. Bohr’s reply to EPR was given in terms of the complemen-
...arity principle [40], which defends that elements of reality of incompatible observables cannot be established in the same experiment, but only through mutually exclusive experimental arrangements. In Bohr’s view, one cannot speak of the nature of microscopic systems before making a measurement, that is, physical reality only emerges upon interaction with a macroscopic apparatus. This perspective refutes EPR’s rationale and elects the correlations generated in the experimental setup as an important mechanism responsible for the establishment of physical reality (see Ref. [41] for a related discussion). Published in the same year as EPR’s and Bohr’s articles, Rauk’s work [42] pointed out that EPR’s conclusion derived from a criterion that “is directly opposed to the view held by many theoreticians, that a physical property of a given system has reality only when it is actually measured”. Later on, Redhead [43] defended a subtle reformulation of EPR’s criterion: “If we can predict with certainty, or at any rate with probability, the result of measuring a physical quantity at time t, then, at time t, there exists an element of reality corresponding to this physical quantity and having value equal to the predicted measurement result”. This proposal aimed at softening the condition on the relativistic causality hypothesis. In 1996, Vaidman introduced a slightly different perspective to the issue. Realizing that a point common to many criteria of reality is the link with actual results of projective measurements, he proposed that “for any definite result of a measurement there is a corresponding element of reality” [44]. Vaidman regarded as “definitive result” the definite shift of the probability distribution of the pointer variable and thus suggested the following definition of elements of reality: “If we are certain that a procedure for measuring a certain variable will lead to a definite shift of the unchanged probability distribution of the pointer, then there is an element of reality: the variable equal to this shift”. With that, different shades of reality were attached to physical realism. A few years ago, Bilobran and Angelo introduced a measurement-based criterion of reality which allowed for the quantification of the degree of reality of a discrete-spectrum observable for a given multipartite quantum state [45]. This approach led to further foundational developments, as for instance the definition of the realism-based nonlocality [46, 47], which captures nonlocal aspects that are dramatically different from Bell nonlocality, and the derivation of a information-reality complementarity relation [48], which is shown to apply even to scenarios of weak disturbances. Interestingly, this framework received experimental corroboration via weak measurements implemented in a photonic platform [49].

With regards to the emergence of reality, a crucial aspect is recognized in Ref. [48], namely, that in all measurements, the degree of freedom that is intended to be measured is effectively discarded. In the Stern-Gerlach setting, for instance, the information about the spin is encoded in the spatial degree of freedom of the silver atom. After interacting with the magnet, the atom is set in the entangled state $|\psi_+\rangle + |\psi_-\rangle$ and then its position is registered (via some ionizing process) in a screen. The spin is then inferred from this position measurement. The discard of the spin subspace reduces the accessible state to $|\psi_+\rangle |\psi_+\rangle + |\psi_-\rangle |\psi_-\rangle$. If $\langle \psi_+ | \psi_- \rangle = 0$, then this state corresponds to a classical statistical mixture (with no interference terms in the position basis). In this case, the spin can be inferred with certainty, the measurement is ideally selective, and an element of reality emerges for the spin (even when nobody looks at the screen). In terms of the information-reality complementarity [48], the reality of the spin increases because information about it flows to the spatial degree of freedom. In fact, the narrative goes the other way around: because information about the atom position flows to the spin degree of freedom, which is inevitably discarded, an element of reality emerges for the position. This interpretation is consistent with the fact that no interference pattern is observed in the screen. In addition, it suggests, in agreement with the framework delineated in Ref. [41], that the wave-particle duality, widely accepted as a fundamental principle of quantum theory, to which both matter and radiation are submitted, can be phrased in terms of the dichotomy “absence versus presence of quantum correlations”.

As illustrated in the aforementioned paradigmatic experiment, measurements of arbitrary physical quantities can always be reduced to a position measurement. Indeed, at the very last stage of any measurement, we always look at a “pointer”, that is, we invariably receive, from a physical carrier, information about the occurrence of a given event in a well-defined point in spacetime. That is what happens, for instance, when a photon (or a sound wave) reaches an observer after being generated by a phosphorescent mark in a screen (or a click in a detector array) upon arrival of a particle. Thus, within the context of the Bilobran-Angelo elements of reality, the question naturally arises as to whether one can quantify the extent to which the position of a system can be regarded as an element of physical reality. In spite of its acute foundational relevance, this problem has attracted little attention from the physical community, possibly because of the many technical and conceptual difficulties underlying this task. This work is devoted to solve this problem. Specifically, we want to extend the Bilobran-Angelo approach to the domain of continuous variables and then look at the consequences implied by the derived measure for the elements of reality of canonically conjugated observables, such as position and momentum. As an application, we investigate the emergence of reality in the dissipative dynamics implied by the Caldirola-Kanai model, whose classical analog allows for the achievement of rest. Additionally, looking for a dynamical description of a position measurement, we expand the model in a way to allow for the analysis of an entanglement dynamics of a particle and a pointer.

II. BASIC CONCEPTS AND UNCERTAINTY RELATION

Recently, Bilobran and Angelo (BA) put forward a proposal for quantifying elements of reality of discrete-spectrum observables [45]. Here we present a brief review of the BA approach since it will be the main platform for the present work. The whole idea is based on the single premise that upon the completion of a measurement of a given observable, say $A$, there exists an element of reality for $A$. The rationale goes as
follows.

After the measurement of an observable \( A = \sum_a a A_a \), with discrete spectrum \( \{ a \} \) and projectors \( A_a = |a\rangle\langle a| \) on \( \mathcal{H}_A \), the preparation \( \rho \) on \( \mathcal{H}_A \otimes \mathcal{H}_B \) (with \( \mathcal{H}_B = \bigotimes_{i=1}^n \mathcal{H}_i \) and \( d_{\mathcal{H}_B} = \dim(\mathcal{H}_A) \)) collapses to \( \rho^{(a)} = (A_a \otimes \mathbb{1}_B) \rho (A_a \otimes \mathbb{1}_B) / \rho_a \), where \( \rho_a = \text{Tr}[A_a \otimes \mathbb{1}_B \rho (A_a \otimes \mathbb{1}_B)] \). According to the aforementioned premise, an element of reality then emerges for \( A \). Mathematically, this is explicitly revealed by the projector appearing in \( \rho^{(a)} = A_a \otimes \langle a|a\rangle \). Consider, however, as a more general instance, that the outcome \( a \) obtained by the experimentalist is not revealed to some external observer. This mere omission can by no means alter the state of reality of \( A \). The best description the ignorant observer can give to the post-measurement state is

\[
\sum_a \rho_a \rho^{(a)} = \sum_a (A_a \otimes \mathbb{1}_B) \rho (A_a \otimes \mathbb{1}_B) =: \Phi_A(\rho),
\]

where \( \Phi_A(\rho) \) is a completely positive trace-preserving (CPTP) map written in terms of the operator-sum representation, with Kraus operators \( A_a \) satisfying \( \sum_a A_a^\dagger A_a = \mathbb{1}_A \). In this respect, \( \Phi_A \) constitutes a particular form of quantum operation \([50]\).

The unrevealed-measurement procedure thus led to the state \( \Phi_A(\rho) \) which, by virtue of the premise adopted, is to be interpreted as a state of reality for \( A \). Notice, in particular, that further unrevealed measurements of \( A \) cannot change a state of reality \( \rho' = \Phi_A(\rho) \), since \( \Phi_A \) is a CPTP map. This allows us to take

\[
\Phi_A(\rho) = \rho \quad \text{(BA criterion of realism) \quad (2)}
\]

as a formal statement of a scenario where \( A \) is real for a given preparation \( \rho \). With that, it is possible to employ the relative entropy \( S(\rho||\sigma) := \text{Tr}[\rho \log \rho - \rho \log \sigma] \) of states \( \rho \) and \( \sigma \) as an “entropic metric” to evaluate the so-called irreality \( \Im(\rho) \) of \( A \) given \( \rho \), that is, the amount by which the BA realism is violated for the context defined by \( A \) and \( \rho \):

\[
\Im(\rho) := S(\rho||\Phi_A(\rho)) = S(\Phi_A(\rho)||\rho) - S(\rho). \quad (3)
\]

Since \( S(\rho||\sigma) \) is always nonnegative and equals zero if and only if \( \rho = \sigma \), then its is guaranteed that the irreality will vanish if only if the BA realism occurs. Indeed, it is straightforward to check that \( \Im(\rho) = 0 \) for any \( \rho \). On the other hand, this is not so if the preparation is a state of reality for an incompatible observable. That is, for \( \{ A, A' \} \neq 0 \) one has that \( \Im(\Phi_{A'}(\rho)) \neq 0 \), with the equality holding only for \( \rho \) being already a state of reality for \( A \). This suggests that we cannot make the irrealties of incompatible observables vanish simultaneously. Interestingly, now we prove that this is indeed the case. Using definition \([3]\) along with the result \( S(\rho) + S(\Phi_{A'}(\rho)) \leq S(\Phi_A(\rho)) + S(\Phi_{A'}(\rho)) \), proved in Ref. \([48]\), we can immediately demonstrate, for any \( A \) and \( A' \) acting on \( \mathcal{H}_A \), that

\[
\Im(\rho) + \Im(A'(\rho)) \geq I_{\mathcal{A}B}(\rho), \quad (4)
\]

where \( I_{\mathcal{A}B}(\rho) = \ln d_A - S_{\mathcal{A}B}(\rho) \) with \( S_{\mathcal{A}B}(\rho) = S(\rho) - S(\rho_B) \) and \( \rho_B = \text{Tr}_B(\rho) \). Since \( S_{\mathcal{A}B} \) is the conditional entropy, the lower bound \( I_{\mathcal{A}B} \) can be interpreted as the information available about the subsystem \( \mathcal{A} \) from knowledge about the subsystem \( \mathcal{B} \). This term can yet be written as

\[
I_{\mathcal{A}B}(\rho) = I(\rho_A) + I_{\mathcal{A}B}(\rho_B) = S(\rho || \frac{1}{d_A} \otimes \rho_B), \quad (5)
\]

with \( I_{\mathcal{A}B}(\rho) = S(\rho || \rho_A \otimes \rho_B) \) being the mutual information of \( \rho \) and \( I(\rho_A) = \ln d_A - S(\rho_A) \) the information associated with \( \rho_A \), for reduced states \( \rho_{\mathcal{A}B} = \text{Tr}_{\mathcal{B}}(\rho) \). Inequality \([4]\) defines an uncertainty relation for the irrealties of arbitrary observables \( A \) and \( A' \) on \( \mathcal{H}_A \). The equality can be seen to hold when \( \rho = \Phi_A(\rho_A) = \frac{1}{d_A} \otimes \rho_B \) (a state of simultaneous reality for maximally incompatible observables \( A \) and \( A' \)). This instance, however, brings no interesting insight, since \( \Im(\rho_A) = \Im(A'(\rho_A)) = I_{\mathcal{A}B}(\rho) = 0 \). On the other hand, consider the pure state \( \rho = |\psi\rangle\langle\psi| \), for two equidimensional subsystems \((d_A = d_B = d)\), with Schmidt decomposition \( |\psi\rangle = \sum_i |i\rangle \sqrt{\lambda_i} \). In this case:\( I(\rho_A) = 0 \) and \( I_{\mathcal{A}B}(\rho) = 2S(\rho_A) = 2 \ln d \) (twice as much the entanglement of \( |\psi\rangle \)), showing that entanglement prevents any two observables of having simultaneous reality.

The BA criterion of realism \([2]\) and the quantification of its violation \([3]\) counts by now with further developments and applications \([46–49]\). All these works are, however, strictly connected with discrete-spectrum observables. So far, this has been a mandatory specialization, as the BA approach fundamentally relies on projectors, whose definition is tricky for continuous variables. In what follows, we develop a scheme to suitably treat elements of reality associated with position and momentum.

### III. DISCRETIZATION

Here we show that one can directly apply the BA approach to the case of continuous-spectrum observables by making small adaptations. The main idea consists of keeping the definitions \([2,3]\) intact and treating position and momentum as discrete quantities. This is, in fact, rather convenient for eventual confronts with experimental contexts, wherein finite-resolution detectors operationally prevent the observation of genuine continuous behavior for whatever physical quantities. Let us then consider the eigenvalue relation \( Q|q\rangle = q|q\rangle \) for the operator \( Q \) (denoting position or any continuous generalized coordinate), where \( |q\rangle \) is the eigenvector associated with the eigenvalue \( q = k \delta q \), with \( k \) being an integer and \( \delta q \) an (operational) resolution of the position space. Within this model, \( \delta q > 0 \in \mathbb{R} \) is a (small) free parameter with dimension of position. We then introduce

\[
|q\rangle \langle q'\rangle = \frac{\delta_{qq'}}{\delta q} \quad (6)
\]

for the scalar product of the space, where \( \delta_{qq'} \) is the Kronecker delta function. Clearly, this relation is dimensionally correct and leads to the expected result in the continuous variable limit \((\delta q \to 0)\). As projectors we propose

\[
\Pi_k = |q_k\rangle \langle q_k| \quad (7)
\]


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1 This uncertainty relation can also be derived from some results of Ref. \([51]\) as long as we require, in addition, the restriction that \( A \) and \( A' \) be maximally incompatible observables. This means that the eigenbasis \(|u\rangle\) and \(|u'\rangle\) of \( A \) and \( A' \), respectively, must form mutually unbiased bases respecting \(|u\langle u'||^2 = d_A^{-1} \).
satisfying
\[ \Pi_k \Pi_{k'} = \delta_{kk'} \Pi_k \quad \text{and} \quad \sum_{k=-L_p}^{L_p} \Pi_k = I. \quad (8) \]

The parameter \( L_q \), to be posteriorly fixed as an integer function of \( \delta q \), defines the dimension \( 2L_q + 1 \) of the discretized space. Its introduction will prove relevant for the consistency of the method. The completeness relation given above allows one to expand any vector \( |\psi\rangle \) as
\[ |\psi\rangle = \sum_{k=-L_p}^{L_p} \delta q \, \psi(q_k) |q_k\rangle, \quad (9) \]
with amplitude \( \psi(q_k) = \langle q_k | \psi \rangle \) and probability \( |\psi(q_k)|^2 \delta q \). An analog discretization scheme can be postulated for the momentum space. To this end, we introduce
\[ \langle p|p_t \rangle = \frac{\delta p}{\delta p} \quad \text{and} \quad \Xi_I = \delta p |p_t \rangle \langle p|, \quad (10) \]
such that
\[ \Xi_I \Xi_{I'} = \delta_{II'} \Xi_I \quad \text{and} \quad \sum_{l=-L_p}^{L_p} \Xi_{l} = I. \quad (11) \]
\( \delta p \) and \( L_p \) play the roles of momentum resolution and momentum space dimension, respectively. Using the \( (|p\rangle) \) representation we can expand any \( |\psi\rangle \) as
\[ |\psi\rangle = \sum_{l=-L_p}^{L_p} \delta p \, \psi(p_l) |p_l\rangle, \quad (12) \]
with amplitude \( \psi(p_l) \) and probability \( |\psi(p_l)|^2 \delta p \).

In order to obtain interrelations for the various free parameters of the model, we require the validity of the usual algebra \[ \{ q, p \} \] associated with the canonical couple \( \{ Q, P \} \). Let \( T(\delta) := I - i \delta \hat{P}/\hbar \) be the standard translation operator associated with an infinitesimal displacement \( \delta \gg \delta q \) and generator \( P \). By demanding that \( T(\delta q)|q_k\rangle = |q_k + \delta q\rangle = |q_{k+1}\rangle \) we directly find \( \{ Q, T(\delta q) \} |q_k\rangle = \delta q |q_k\rangle \). Using the explicit form of \( T(\delta q) \) we obtain \( \{ Q, P \} = i \hbar \), which confirms that the translation generator \( P \) is indeed the momentum operator satisfying \( P |p_l\rangle = p_l |p_l\rangle \), with \( p_l = l \delta p \). Since \( P \) is Hermitian, all the properties expected for \( T \) are (up to the first-order approximation with respect to \( \delta \)) validated, in particular one shows that \( T(-\delta) T(\delta) = T(0) \) and \( T(\delta) T(\delta') = T(\delta + \delta') \). Also, from \( \langle q_k|T(\delta q)\psi\rangle = \langle q_k + \delta q|\psi\rangle = \psi(q_k + \delta q) \) we have
\[ \langle q_k|P|\psi\rangle = \frac{\hbar}{i} \psi'(q_k) = \frac{\hbar}{i} \left[ \psi(q_k + \delta q) - \psi(q_k) \right]/\delta q, \quad (13) \]
which identifies momentum in the position representation with a discrete derivative with respect to position, as expected.

In the continuous regime \( (\delta q \to 0) \), specializing \( |\psi\rangle = |p\rangle \) above gives, as solution, the plane wave \( \langle q|p\rangle = e^{i qp}/\sqrt{2\pi} \). In the discrete regime, we still have (up to the first order in \( \delta q \))
\[ \langle q_k|p_l\rangle = \frac{e^{i k q_k p_l} / \hbar}{\sqrt{2\pi} \hbar} = \frac{e^{i k q_k p_l} / \hbar}{\sqrt{2\pi} \hbar}. \quad (14) \]

Now, using this relation along with \[ (9) \) and \[ (11) \) yields
\[ \delta q \langle q_k|q_{k'} \rangle = \frac{\delta q \delta p}{2\pi \hbar} \sum_{l=-L_p}^{L_p} e^{i k (q_k - q_{k'})} \delta q / \delta p = \delta_{kk'}, \quad (15) \]
which can be compared with the identity
\[ \frac{1}{\xi} \sum_{k=1}^{L_p} e^{2\pi i (k-k')/\xi} = \delta_{kk'}, \quad (16) \]
to produce \( L_p = \lfloor (\xi - 1)/2 \rfloor \) and
\[ \xi = \frac{2\pi \hbar}{\delta q \delta p}. \quad (17) \]
The symbol \[ \lfloor x \rfloor \] denotes the ceiling function, which maps the real number \( x \) to the least integer greater than or equal to \( x \).

A. Application for a Gaussian state

The importance of Gaussian states for physics does not need emphasis. In the present work, these states will also play a distinctive role as will be shown latter, especially because of their analytical properties. It is, therefore, instructive to discuss the implications and limitations of the discretization method to this context. The standard continuous-variable minimum-uncertainty Gaussian state can be written as
\[ |\psi\rangle = \int_{-\infty}^{\infty} dq \, \psi(q - \bar{q}) |q\rangle, \quad \psi(q) = e^{-q^2 / 2} e^{i q \bar{q} / \hbar}, \quad (18) \]
for which one shows that
\[ \langle Q \rangle = \bar{q}, \quad \Delta Q = \sqrt{\langle Q^2 \rangle - \langle Q \rangle^2} = \Delta q, \quad (19) \]
\[ \langle P \rangle = \bar{p}, \quad \Delta P = \sqrt{\langle P^2 \rangle - \langle P \rangle^2} = \hbar / \Delta q. \]
To migrate to the discrete model, we introduce
\[ |\psi\rangle = \sum_k \sqrt{\delta q} \, \psi_{k-\bar{k}} |k\rangle, \quad \psi_k = e^{-\bar{q}^2 / 2} e^{i \bar{q} k / \hbar}, \quad (20) \]
with \( |k\rangle \equiv |q_k\rangle, \Delta q \equiv \Delta q / \delta q, \bar{q} = \bar{k} \delta q, \) and \( \bar{p} = \bar{\bar{k}} \delta p \). We opted for dealing with a dimensionless amplitude \( \psi_k \), which explains the appearance of the term \( \sqrt{\delta q} \) instead of \( \delta q \). The
normalization term, which is fixed through \( \langle \psi | \psi \rangle = 1 \), can be accurately approximated as

\[
N = \sum_{k} |\psi_{k} - \bar{k}|^2 \approx \sum_{k=\infty}^{\infty} e^{-\frac{k^2}{2\Delta_q}} = \theta_3(0, e^{-\frac{1}{2\Delta_q}}) =: N_{\Delta_q},
\]

where \( \theta_3(z, q) = \sum_{k=\infty}^{\infty} q^{k} e^{2zk} \) with \( z \in \mathbb{C} \) stands for the Jacobi theta function. Further analyses allow one to obtain the following analytical approximation for \( N_{\Delta_q} \):

\[
N_{\Delta_q} \approx \begin{cases} 1 + 2 \left( e^{-\frac{1}{2\Delta_q}} + e^{-\frac{1}{2\Delta_q^2}} + e^{-\frac{1}{4\Delta_q^2}} \right) & \text{if } 0 \leq \Delta_q < 1, \\ 2\pi \Delta_q^2 & \text{if } \Delta_q > 1, \end{cases}
\]

which never implies an error greater than 0.0271% with respect to \( \theta_3(0, \exp(-1/2\Delta_q^2)) \) for all \( \Delta_q \gg 0 \). The quality of these results increases as the continuous limit \( \delta q \to 0 \) (\( \xi \to \infty \)) is approached.

Having calculated the normalization, we can use the discretized wave function \( | \psi \rangle \) to assess the quality of the resulting statistics. Employing the above approximations and the discrete derivative \( \delta q \) and preserving only leading terms with respect to \( \delta q \), one can check that

\[
\langle Q \rangle = \sum_{k} \psi_{k}^* (k \delta q) \psi_{k} \equiv \delta q \sum_{k=\infty}^{\infty} \psi_{k}^* (k + \Delta_q) \psi_{k} = \hat{k} \delta q,
\]

\[
\langle P \rangle = \sum_{k} \psi_{k}^* (-i\hbar) \psi_{k} = \Delta_q + \frac{\hbar}{2\Delta_q},
\]

\[
\langle P^2 \rangle = \sum_{k} \psi_{k}^* (-i\hbar) \psi_{k} = (\Delta_q)^2 + \frac{\hbar}{2\Delta_q},
\]

which can always be identified to their continuous-case counterparts within certain approximations. Taking the identity \( \Delta_q \exp[-\kappa^2/(2\Delta_q^2)] = -2\Delta_q \lim_{\kappa \to 1} \frac{d}{d\kappa} \exp[-\kappa^2/(2\Delta_q^2)] \) it is possible to advance the computation for \( \langle Q^2 \rangle \) and for the uncertainties \( \Delta Q \) and \( \Delta P \), whose product can be written as

\[
\left( \Delta Q \Delta P / \hbar / 2 \right)^2 = \sum_{k=\infty}^{\infty} k^2 |\psi_{k}|^2 = -\frac{2}{N_{\Delta_q}} \lim_{\kappa \to 1} \frac{d}{d\kappa} N_{\Delta_q} =: \eta_{\Delta_q}^2.
\]

With the approximation (22), the function \( \eta_{\Delta_q} \) can be analytically computed. Its behavior is presented in Fig. 1 as a function of \( \Delta_q = \Delta q / \delta q \), showing that the discretized model is in full agreement with the continuous one as long as the state width \( \Delta q \) is not appreciably smaller than the resolution \( \delta q \).

IV. QUANTIFYING REALISM VIOLATIONS

Having developed the discretized model for position and momentum, we can now apply the BA formalism to quantify the degree of irreality of these quantities for a given preparation \( \rho \). The unrevealed-measurement map [10] is now written as \( \Phi_\rho(\rho) = \sum_k \Pi_k \rho \Pi_k \), with projectors \( \Pi_k \) defined by Eq. (7) and \( k \) running from \(-L\) to \( L = [(\xi - 1)/2] \). The construction of \( \Phi_\rho(\rho) \) is made in complete analogy by use of the projectors [10]. It can be checked that \( \Phi_\rho(\rho) \), with \( R \in \{ Q, P \} \), preserves all the desired properties; in particular: it is a CPTP map and \( \Phi_\rho^2(\rho) = \Phi_\rho(\rho) \). The BA criterion of realism [2] expresses itself for position and momentum in the form \( \Phi_\rho(\rho) = \Phi_R \). The degree of irreality of \( \hat{R} \) can be diagnosed as

\[
\mathcal{J}(R|\rho) = S(\Phi_R(\rho)) - S(\rho) \quad (R \in \{ Q, P \}).
\]

The form of the von Neumann entropy, \( S(\rho) = -\text{Tr}(\rho \ln \rho) \), is preserved in the discretized model provided we write the trace operation as \( \text{Tr}(O) = \sum_k \delta q \langle q | O | q \rangle = \sum_k \delta p \langle p | O | p \rangle \), for a generic operator \( O \). In addition, one shows that \( S = 0 \) and \( S = \ln D \) (with \( D = 2L + 1 \)) for pure and maximally mixed states, respectively, in agreement with the continuous-variable formulation.

A. Examples

It is instructive to compute the irreality for some simple cases. Consider as a first example a uniform state given by

\[
| \psi \rangle = \sum_{k=\bar{k}}^{\bar{k}} \sqrt{\delta q} \psi_k |k\rangle, \quad \psi_k = \frac{1}{\sqrt{\Delta_q}},
\]

where \( \bar{k} = (\Delta_q - 1)/2 \) and \( \Delta_q = 2n + 1 \) with \( n \in \mathbb{N} \). It follows that

\[
\Phi_Q(\rho) = \sum_k \Pi_k |\psi \rangle \langle \psi | \Pi_k = \sum_k |\psi_k|^2 \Pi_k = \frac{1}{\Delta_q}.
\]

Since \( \Phi_Q(\rho) \) is a statistical mixture with eigenvalues \( 1/\Delta_q \) and \( S(|\psi \rangle) = 0 \), we obtain

\[
\mathcal{J}(\rho |\psi \rangle) = \ln \Delta_q \quad \text{(uniform state)}.
\]
This shows that the irrelity increases with the width of the superposition, being, in this case, also a direct measure of quantum coherence [45].

For our second example, we come back to the Gaussian state [20]. In this case, we find

$$\Phi_Q(\rho) = \sum_k |\psi_k|^2 \Pi_k \approx \frac{1}{N_{\Delta_\rho}} \sum_{k=-\infty}^{\infty} e^{\frac{-q^2}{2\Delta_\rho^2}} \Pi_k,$$

(28)

whose eigenvalues read $\exp\left[-k^2/(2\Delta_\rho^2)\right]/N_{\Delta_\rho}$. This leads to the following irrelity:

$$\Im(Q\rho) = -\sum_{k=-\infty}^{\infty} e^{\frac{-q^2}{2\Delta_\rho^2}} \ln\left(\frac{e^{\frac{-q^2}{2\Delta_\rho^2}}}{N_{\Delta_\rho}}\right) = \ln N_{\Delta_\rho} + \frac{\eta_{\Delta_\rho}^2}{2},$$

(29)

where the relations (21) and (23) have been used. With the relations (22) and (23) derived for $N_{\Delta_\rho}$ and $\eta_{\Delta_\rho}$, respectively, one verifies that $\Im(Q\rho) \to 0$ as $\Delta_\rho \to 0$, always preserving the positivity of the irrelity. However, as pointed out above, in order not to violate the uncertainty principle we have to confine ourselves to $\Delta_\rho > 1$, domain in which $\eta_{\Delta_\rho} = 1$ and $N_{\Delta_\rho} = \sqrt{2\pi} \Delta_\rho$. With that, we finally obtain

$$\Im(Q\rho) = \ln(\sqrt{2\pi} \Delta_\rho) \quad \text{(Gaussian state).}$$

(30)

It is interesting to note that the probability of finding the particle in the range $(Q - \frac{d}{2}, Q + \frac{d}{2})$ for a Gaussian state with root mean square $\Delta q$ results $0.383$ for $d = \Delta q$ and $0.961$ for $d = \sqrt{2\pi} \Delta q$. In this sense, $\sqrt{2\pi} \Delta q$ can be viewed as a better candidate for discriminating the “effective width” of the wave-packet with respect to the resolution $\Delta q$. This observation unifies the results (27) and (30).

**B. Position-momentum uncertainty relation**

Since the wave-packet considered above manifests itself as a Gaussian distribution also in the momentum representation, it is clear that we should have $\Im(P\rho) = \ln(\sqrt{2\pi} \Delta p)$. It follows that $\Im(Q\rho) + \Im(P\rho) = \ln(2\pi \Delta q \Delta p)$, which is in agreement with results reported in conceptually different contexts [42].

As previously mentioned, in order to keep the discretized model in consistency with Heisenberg’s uncertainty, we should demand that $\Delta q > 1$ and $\Delta p > 1$, which implies that

$$\Im(Q\rho) + \Im(P\rho) \geq \ln(2\pi e).$$

(31)

Once this lower bound has been derived for the minimum uncertainty state, we expect that such inequality be valid in general. This tell us that we can never prepare a state $\rho$ for which both position and momentum are simultaneous elements of reality. We see, therefore, that Heisenberg’s uncertainty relation imposes severe restrictions to the classical notion of realism, which here is written as $\Im(Q\rho) = \Im(P\rho) = 0$.

**V. QUANTUM-MECHANICAL REST**

One of the tenets of quantum measurement theory prescribes that the realization of sequential measurements of a given observable must produce always the same outcome revealed by the first of these measurements, as long as the system is not allowed to dynamically evolve between two contiguous measurements. If the observable under consideration is the position of a quantum particle, then we might conclude that such a sequential protocol would be able to confine the particle to some state of rest. Of course, due to Heisenberg’s uncertainty principle, which implies full indefiniteness for the canonical momentum after each position measurement, the resulting state might not be strictly compatible with our classical notion of rest. However, as far as position and velocity are the figure of merit, it is still possible to find classical rest—with emergent elements of reality for both quantities simultaneously—even departing from a strictly quantum sub-stratum. The aim of this section is to make this point through a perspective according rest can be achieved through an overdamped quantum dynamics.

Our argument is constructed by use of the Caldirola-Kanai (CK) model [54, 55], which effectively implements the dissipative dynamics of a block of mass $m$ attached to a spring with elastic constant $k$. The classical time-dependent Hamiltonian reads

$$H_t = \frac{p^2}{2m} - \frac{kq^2}{2} + \frac{kq^2}{2} e^{2\tau},$$

(32)

where $\tau = \lambda t$ is a dimensionless time, $t$ is the physical time, and $\lambda$ is a frequency that determines the dissipation rate. Hamilton’s equation of motion lead to $\dot{q} + 2\lambda q + \omega^2 q = 0$, with $\omega^2 = k/m$, where one can recognize a velocity-dependent term typical of damped motion. Direct integration of the equations of motion, with initial conditions $q_0$ and $p_0$, yield

$$q_t = q_0 e^{-\tau} \left[\cosh(\zeta \tau) + \left(1 + \frac{p_0^2}{m e_0}\right) \frac{\sinh(\zeta \tau)}{\zeta}\right],$$

(33a)

$$p_t = p_0 e^{\tau} \left[\cosh(\zeta \tau) - \left(1 + \frac{kq_0^2}{e_0}\right) \frac{\sinh(\zeta \tau)}{\zeta}\right],$$

(33b)

where $\zeta = (1 - \omega^2 / \lambda^2)^{1/2}$, and $e_0 = \lambda q_0 p_0$. Despite the notably divergent form of the canonical momentum, we may identify scenarios typical of mechanical rest if we look at the velocity $v_t = dq_t / dt$. In Table 1, the asymptotic behaviors ($t \to \infty$) of some physical quantities are presented as a function of the dimensionless time $\tau$ for distinct regimes of damping. It is clear that the block will eventually come to a fixed position with no

---

2 Since $S(\rho) = 0$ and $\Delta q \Delta p = \hbar/2$ for the state under consideration, the present result can be written in the form $H_q + H_p = \ln(2\pi e)$, which has been found in Ref. [53], where $H_q = S(\Phi_q(\rho))$ and $H_p = S(\Phi_p(\rho))$ are the Shannon entropies associated with probability distributions for the classical variables $q$ and $p$. Consistency with the result [44] is checked by writing $\Im(Q\rho) + \Im(P\rho) = \ln(e/2) > \ln \xi = i\rho$. Recall that there are no correlations in this case and that $\xi$ accounts for the dimension of the space.
velocity and no kinetic energy. The potential energy accumulated by the spring, \( V_t = \frac{kq_t^2}{2} \), will be fully suppressed as well. It is then evident that \( h_t \), which may increase with time, is not to be taken as the energy of the oscillator; this is the case only if \( \lambda = 0 \). Instead, it should be interpreted as the total energy of a system composed of the oscillator (block + spring) and an external environment that drains the mechanical energy of the oscillator. Also worth noticing is the dramatic difference between velocity and canonical momentum. Indeed, from Hamilton’s equation we have \( m \dot{v} = p e^{-2\tau} \). It is immediate seen that mechanical rest can be generally claimed to occur for the block in the CK model since

\[
\lim_{t \to \infty} (q_t, v_t) = 0. \tag{34}
\]

The direct quantization of the classical model gives

\[
H_t = \frac{p^2}{2m} e^{-2\tau} + \frac{kQ^2}{2} e^{2\tau}, \tag{35}
\]

with \([Q, P] = ih\). Using Heisenberg’s picture we can show that \( \dot{Q}_t + 2i\dot{Q}_t + \omega^2 Q_t = 0 \), with \( \omega^2 = k/m \), where \( Q_t = U_t^\dagger Q U_t \), and \( U_t = \exp \left[ -i \int_0^t dt' H_{t'} \right] \). This shows that, for a well-localized initial state \( \rho_0 \) and \( \lambda \) sufficiently large, the mean value \( \langle Q_t \rangle = \text{Tr}[\rho_0 Q U_t] \) evolves in time as a typical trajectory of a damped motion, so that the previously studied classical behavior will approximately apply (Ehrenfest’s theorem). However, since we are interested in analyze if and how the elements of reality emerge from the quantum dynamics, the study of the centroid does not suffice. In particular, because the irreality quantifier is a state variable, the Schrödinger picture should be preferred.

The quantum CK model [58] has a long history of conceptual discussions [56,59], applications [60,63], and derivations of analytical solutions [64,70]. Here we adopt the method developed in Ref. [71], which is of particular convenience for our purposes because it offers a formal solution for \( i\hbar \partial_t U_t = H_t U_t \) in terms of the time-evolution operator \( U_t \) in cases where the Hamiltonian can be written as

\[
H_t = a_+^\dagger J_+ + a_0^\dagger J_0 + a_-^\dagger J_- \tag{36}
\]

where \( J_{a,0} \) form the SU(2) Lie algebra characterized by \([J_+, J_-] = 2 J_0 \) and \([J_0, J_a] = \pm J_a \), and \( a_{\pm,0} \) are arbitrary functions of time. The identification of the Hamiltonians and \([Q, P] = \mathcal{O} P + P Q \). The method then allows one to write

\[
U_t = e^{i\eta[t] J_0} e^{i\eta[0] J_0} e^{i\eta[t] J_0}, \tag{38}
\]

with time-dependent coefficients given by

\[
c_t^+ = \mu_t \quad c_t^0 = -2 \ln \left( \frac{u_t}{u_0} \right) \quad c_t^- = -u_0^2 \int_0^t \frac{dt'}{\mu_t u_t'} \tag{39}
\]

with \( \epsilon_0 = 0 \) and \( \dot{u}_t + \frac{\mu_t}{m} u_t + \frac{k}{m} u_t = 0 \), such that \( u_0 \neq 0 \) and \( u_0 = 0 \). Without the explicit expressions of \( \mu_t \) and \( \kappa_t \) we find \( \dot{u}_t + 2\dot{u}_t + \omega^2 u_t = 0 \), which shows that the quantum solution encapsulates the classical trajectory. Taking \( u_0 = 1 \) and \( u_0 = 0 \) as initial conditions, we obtain the following particular solution

\[
u_t = e^{-\tau} \left[ \cosh \left( \frac{\zeta \tau}{\lambda} \right) + \frac{\sinh \left( \frac{\zeta \tau}{\lambda} \right)}{\zeta} \right], \tag{40}
\]

which leads to

\[
c_t^+ = -k \left[ \frac{e^{\xi \tau}}{1 + \zeta \coth \left( \frac{\zeta \tau}{\lambda} \right)} \right], \tag{41a}
\]

\[
c_t^0 = 2 \ln \left[ \frac{e^\tau}{\cosh \left( \frac{\zeta \tau}{\lambda} \right) + \frac{\sinh \left( \frac{\zeta \tau}{\lambda} \right)}{\zeta}} \right], \tag{41b}
\]

\[
c_t^- = -\frac{1}{m \alpha} \left[ \frac{1}{1 + \zeta \coth \left( \frac{\zeta \tau}{\lambda} \right)} \right] \tag{41c}
\]

Notice that such solutions do not hold for \( \lambda = 0 \). Having constructed the solution for the evolution operator [58], we can proceed with the calculation of the wave function. Since \( J_0 = \frac{1}{3} + \frac{v P}{2 \hbar} \) and \( \langle \phi \rangle \exp (i c^+ \tau / 2 \hbar) Q P | \Theta(q) = e^{i \tau / 2 \hbar} \phi(q T), \Theta(q) \rangle = \Theta(e^{i \tau / 2 \hbar}) \), we find

\[
\psi_t(q) = \langle q | U_t | \psi_0 \rangle = e^{i \tau / 2 \hbar} e^{i(\tau / 2 \hbar)} \phi(q T), \Theta(q) \rangle, \tag{42}
\]

where \( \phi(q, T) := \langle q | e^{i \tau / 2} \langle \psi_0 \rangle \). From \( i \dot{c}_t^\tau J_- = -i P^2 T_t / (2m \hbar) \) with

\[
\lambda T_t := \frac{1}{1 + \zeta \coth \left( \frac{\zeta \tau}{\lambda} \right)} \tag{43}
\]

we see that the solution for \( \phi(q, T) \) can be obtained from the problem of a free particle evolving during a time-interval \( T_t \) (which is a monotonically increasing function of the dimensionless time \( \tau \)). Thus, taking the standard free-particle solution for a Gaussian probability density, we return to Eq. [42]...
to finally obtain

\[ |\psi(q)|^2 = \frac{\exp\left[-\frac{(q-e^{-\tau/2}q_0)^2}{2\sigma_0^2}\right]}{\sqrt{2\pi\sigma_0^2}}, \quad \Delta q_i = \sigma_0 e^{-\tau/2}, \quad (44) \]

\[ \alpha_i = [1 + (T_i/T_E)^2]^{1/2}, q_{T_i} = q_0 + p_0 T_i/m, \text{ and } T_E = 2m\sigma_0^2/h \] (the Ehrenfest time). This solution presumes that at \( t = 0 \) one has a Gaussian wave packet with mean values \((q_0, p_0)\) and uncertainties \((\sigma_0, \frac{h}{2\pi})\). The centroid evolves in time according to \( \langle Q \rangle_t = \langle 0 \rangle + c_0 \tau + \lambda \chi q_0 e^{-(1+\zeta)t} \) for \( t \rightarrow \infty \) and \( \zeta \in (0, 1) \), one has \( \langle Q \rangle_t \approx q_0 e^{-t/\sigma_0^2} \) and \( \langle V \rangle_t \approx -\lambda q_0 e^{-(1+\zeta)t} \), both vanishing for long times. Therefore, as time increases, the center of the wave packet starts to move as a free particle but inevitably goes to the origin of the coordinate system while the width rapidly diminishes. This dynamics is illustrated in Fig. 2.

\[ |\bar{\psi}(p)|^2 = \frac{\exp\left[-\frac{p^2}{2\Delta p_i^2}\right]}{\sqrt{2\pi\Delta p_i^2}}, \quad \Delta p_i = \frac{\hbar}{2\sigma_0} e^{\tau/2}, \quad (45) \]

\[ \text{with } \beta_i = [1 + 4 e^{-\tau/2} f_i^{1/2}, f_i = \chi_0 \left( T_i/T_E + \chi (\Delta q_i/\sigma_0)^2 \right), \text{ and } \chi_0 = c_i^2 \sigma_0^2/\hbar. \]

In Fig. 2, a simulation is presented for the time evolution of this probability distribution. It is immediately seen from the solutions (44) and (45) that Heisenberg’s principle is always satisfied and that the asymptotic behaviors \( \Delta q_i \approx e^{-\tau/2} \) and \( \Delta p_i \approx e^{\tau/2} \) point out to a full localization of the particle at \( q = 0 \) and a complete delocalization of its momentum. All these results are in qualitative consonance with the classical results presented in Table I.

We are now ready to assess the irrelevancies associated with position, momentum, and velocity. As previously discussed, the adequacy of formula (50) is conditioned to the relations \( \Delta q \rho \gg 1 \), which require \( \delta q \leq \sigma_0 e^{-\tau/2} \) and \( \delta p \geq \frac{\hbar}{\sigma_0} \beta_i e^{\tau/2} \) to hold simultaneously, and validate the uncertainty relation (51). These inequalities impose an upper bound for time as a function of \( \sigma_0 \), in such a way that the greater \( \sigma_0 \) the greater the time domain within which the irrelevancies are valid nonnegative quantities and the discretized model applies. Once such inequalities are respected, we then have, via Eq. (50), the following results:

\[ \Im(Q|\rho_i) = \ln \sqrt{2\pi e} \frac{\sigma_0}{8} \alpha_i e^{-\tau/2}, \quad (46a) \]

\[ \Im(P|\rho_i) = \ln \sqrt{2\pi e} \frac{\hbar}{2\sigma_0} \beta_i e^{\tau/2}, \quad (46b) \]

which show that the irrelevancy of position (momentum) is a monotonically decreasing (increasing) function of time for \( \rho_i = |\bar{\psi}(i)|^2 \), with \( |\bar{\psi}(i)|^2 \) being the Gaussian state whose wave function is given by Eq. (54). Even though the above results depend on the resolutions \( \delta q \) and \( \delta p \), the irrelevancy variation \( \Delta \Im^Q = \Im(\rho_{i+1}) - \Im(\rho_i) \) of the observable \( R \in \{Q, P\} \) does not. One has \( \Delta \Im^Q = \ln(\alpha_i e^{-\tau/2}) \) and \( \Delta \Im^P = \ln(\beta_i e^{\tau/2}) \), which yield

\[ \Delta \Im^Q + \Delta \Im^P = \ln(\alpha_i \beta_i) \rightarrow 2 \zeta \tau. \quad (47) \]

Therefore, while the position irrelevancy is rapidly suppressed, the mean production rate \( \Delta \Im^Q/\Delta t \) of momentum irrelevancy equals the constant \( 2\zeta \tau \) already for times of the order of \( T_E \).

Although the behavior \( \Im(Q|\rho_{i\rightarrow}) \rightarrow 0 \) is consistent with the notion of rest, the fact that \( \Im(P|\rho_{i\rightarrow}) \rightarrow \infty \) might, in principle, not be. However, as previously realized for the classical CK model, the correct observable to look at is velocity. Using Heisenberg’s equations, we find

\[ V_H := \frac{dQ_H}{dt} = \frac{[Q_H, H]}{i\hbar} = \frac{P_H}{m} e^{-2\tau}. \quad (48) \]
It follows that $\langle V^m \rangle = \text{Tr}[\rho_0 V^m] = \text{Tr}[\rho_0 P^m] \gamma^2 t^{-2m} = \langle P^m \rangle t^{-2m}$ for $n \geq 1 \in \mathbb{N}$. We then find
\[
\Delta q_i := \sqrt{\langle V^2 \rangle_i - \langle V \rangle_i^2} = \Delta \rho_i \beta_i \frac{\hbar}{2 \sigma_0 \beta_i} \delta v^i e^{-\gamma^2 t_i^2 / 2}. \tag{49}
\]

Since $c_i^0 \approx 2 \tau_i(1 - \zeta)$ and $\beta_i \approx \epsilon \gamma^2 t_i \rightarrow \infty$, in the overdamped regime ($0 < \xi < 1 \in \mathbb{R}$), it is clear that $\Delta q_i \approx \epsilon \gamma^2 t^{-1 / 2} \rightarrow 0$, which rapidly goes to zero with time. Because the probability distribution for velocity is also Gaussian, we have
\[
\text{As we learned from the discussion associated with Eq. (44), as time passes $|\varphi(x, x_p)|$ approaches the Dirac delta function $\delta(x - x_p)$. It follows that $|\Psi_{\rho_0}| \approx \int dx \varphi_{\rho_0}(x)|x\rangle|\xi\rangle$, which is a nonnormalizable estimate indicating that the asymptotic state is maximally entangled. A detailed computation of the degree of entanglement $E$ in the joint state can be obtained through the linear entropy of the reduced state, $E_I = 1 - \Pi[\rho_i^0]$, where $\Pi[\rho_i^0] := \text{Tr}(\rho_i^0)$ is the purity of $\varphi$ and $\rho_i^0 = \text{Tr}_{x_p}|\Psi_i\rangle\langle\Psi_i|$. Direct calculation yields
\[
\Pi[\rho_i^0] = \frac{\gamma_i^2}{(1 + m^2 \gamma_i^2)} \left(1 + m^2 \gamma_i^2\right), \tag{53}
\]

meaning that not only the average position and velocity are consistent with the classical notion of rest, but also their respective elements of reality. It is worth noticing that the generalization of the relation $\Delta \xi = \Delta \rho / m$ straightforwardly implies that $\Delta q \Delta \rho = m \Delta \rho \Delta \xi \approx \hbar / 2$, which allows for the simultaneous emergence of reality for position and velocity ($\Delta q = \Delta \rho \rightarrow 0$) for heavy particles ($m \rightarrow \infty$), in harmonic coexistence with Heisenberg’s uncertainty principle.

In trying to apply the present model to dynamically describe a position measurement, an important drawback is found: the asymptotic position of the particle, here to be interpreted as representative of the measurement outcome, is always zero. This does not reproduce the random aspect of measurement outcomes for a particle prepared in a Gaussian state. A possible way out of this difficulty is to consider that the above description actually refers to the position of the particle relative to a pointer, not to the laboratory reference frame. That is, let us hereafter assume that $q = x - x_p$ (a relative coordinate), where $x$ and $x_p$ denote the positions of the particle and the pointer $\varphi$ relative to the laboratory. If we consider that the particle and the pointer form a perennial closed system, then we can assume that the center of mass (cm) remains uncorrelated with the relative coordinate for all times. Thus, the joint state of the system can be written as $|\Psi_i\rangle = \varphi_{\langle\rho\rangle} \otimes |\varphi_i\rangle$, where $|\varphi_{\langle\rho\rangle}\rangle$ is the solution we just obtained above, adapted with the replacement $m \rightarrow m = m_{\rho} / (m + m_p)$, and $|\varphi_i\rangle$ is the state associated with the center of mass. For simplicity, in what follows we assume that $\varphi_{\langle\rho\rangle}(x_{cm})$ is the Gaussian solution of a free particle with $\langle x_{cm}^m \rangle = \langle p_{cm}^m \rangle = 0$, $\Delta x_{cm} = \sigma_m 1 + (t^m_{\gamma} \hbar)^{1 / 2}$, and $\Delta p_{cm} = \hbar / (2 \sigma_m)$, where $t^m_{\gamma} = 2 \hbar \sigma_m / \gamma^2$. We then have $|\Psi_i\rangle = \int \int d\varphi_{\langle\rho\rangle} d\varphi_{\langle\rho\rangle} \varphi_{\langle\rho\rangle}(x_{cm}) \varphi_i(q)(\varphi_{\langle\rho\rangle}(x_{cm})(\varphi_i(q))$, with $\varphi_i(q)$ given by Eq. (12). Employing the traditional transformations from laboratory coordinates to center of mass plus relative coordinates, $x_{cm} = \frac{m + m_p}{m} x$ $q = x - x_p$, we conceive the inverse map
\[
|\varphi_{\langle\rho\rangle}(q)| \mapsto |x_{cm} = \frac{m + m_p}{m} q| (x_{cm} - \frac{m + m_p}{m} q) \tag{51}
\]
to link every state in $H_{cm} \otimes H_q$ with one in $H_x \otimes H_{x_p}$. Using this map and performing a change of dummy variables, we rewrite the joint state as
\[
|\Psi_i\rangle = \int \int d\varphi_{\langle\rho\rangle} \varphi_i \left(\frac{m + m_p}{m}\right) \psi_i(x - x_p)|\xi\rangle|x_{\varphi} \rangle. \tag{52}
\]

Therefore, we have found a framework where it is easy to conclude that $\Delta q_i$ will rapidly vanish as well. Clearly, the asymptotic reduced state of the pointer is a fully incoherent statistical mixture, whose discretized version satisfies $\Phi_{x_p}(\rho_{cm}^{\ast}) = \rho_{cm}^{\ast}$, which then implies an element of reality for the pointer position, that is $\mathcal{S}(x_{cm} | \rho_{cm}) = 0$. Therefore, given the inescapable discard of the particle position, the pointer position is certain to be in a real state. In this sense, the measurement problem dissipates.

VI. SUMMARY

By explicitly presenting a formalism through which one can quantify the degree of unreality associated with a continuous variable for a given quantum state, this work extends the approach recently put forward by Bilobran and Angelo [45], which allows one to make inferences about realism in a quantitative fashion. As a first contribution, we derived the uncertainty relation [4], which indicates a lower bound for the total amount of unreality one can simultaneously set to arbitrary observables $A$ and $A'$ acting on $H_{cm}$ by preparing a quantum state. We then showed how to consistently discretize the position and momentum representations in terms of operational resolutions $\delta q$ and $\delta p$, which define the space dimension $L = \left[\xi^{-1} / 2\right]$ of the discretized model, with $\xi = 2 \gamma \hbar (\delta q \delta p)$. As expected, the continuous-variable formalism is fully retrieved at $\xi \rightarrow \infty$. With this strategy, we succeeded in explicitly computing the irreality of position and momentum for Gaussian states and, in agreement with inequality [4], deriving a position-momentum uncertainty relation [see inequality (51)]. This result points out that the classical notion of
a simultaneous position-momentum realism is forbidden by quantum mechanics.

As an application of the presently developed formalism, we demonstrated by example how the classical notion of rest can emerge from quantum mechanics. Using the Caldirola-Kanai model, we studied an effective dynamics whereby the mechanical energy of a harmonic oscillator is entirely dissipated into a reservoir while the system state remains pure. Even though the uncertainty and the irreality of the canonical momentum exponentially increase with time, it is shown that both the mean position and the mean velocity of the particle simultaneously go to zero along with their respective irrealities [see Eq. (50)]. This is the expression of quantum rest, which occurs in full consistency with Heisenberg’s uncertainty principle. Finally, applying the Caldirola-Kanai model to a two-body system, we showed that the dynamical emergence of reality for the particle from the perspective of the pointer manifests itself, in the laboratory reference frame, as the creation of maximum entanglement between the parts. Following Ref. [48], this result points out to a solution for the measurement problem.

The techniques developed here, along with the quantifier introduced in Ref. [45], constitute well defined tools for one to characterize realism in a quantitative way. This may eventually be useful in several contexts involving spatial degrees of freedom, as for instance in foundational and applied studies related to arenas such as optomechanics, Stern-Gerlach experiments, Bell tests, quantum random walks, double-slit experiments, quantum gravity, among others.

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