Power behavior of test based on the moving sums (MOSUM) of spatial least squares residuals

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Abstract. The optimality of a test can be studied by investigating the behavior of the corresponding power function under alternatives. In this paper we study the rate of decay of power functions obtained from asymptotic model validity tests in linear regression analysis based on the moving sums (MOSUM) process of ordinary least squares (OLS) residuals of univariate spatial observations. The powers under study appear as the evaluation of the probability of events generated by the Kolmogorov-Smirnov (KS) and Cramer-von Mises functionals of the Slepian field plus a deterministic trend. It is shown under mild conditions that the rate of the power to the size of test is bounded by a constant which is proportional to the length of the trend. It is also shown that the rate of decays of the powers of the KS and CvM MOSUM tests coincide with the Neyman-Pearson test which is most powerful (MP) test for simple hypotheses. The performance of the power functions are also discussed by simulation.

1. Introduction
The application of MOSUM technique in statistical modelling of phenomenon using linear regression has been studied in many literatures. For instance, Chu and et al. [1] investigated the problem of testing for regression constancy over time based on the MOSUM process of recursive residuals. The limit process was established by applying the functional central limit theorems proposed by Kramer and et al. [2] and Sen [3]. Next, Somayasa and et al. [4, 5, 6] proposed more general method based on the set-indexed partial sum of ordinary least squares (OLS) residual for detecting valid linear regression model for spatial observations over a compact region in which the limit process was derived by generalizing the method of Bischoff and Somayasa [7] and Xie and MacNeill [8] and the invariance principle of Alexander and Pyke [9].

The purpose of the present paper is to study the rate of decays of the power function of Kolmogorov-Smirnov (KS) and Cramér-von Mises (CvM) type tests appeared in testing for the validity of a linear regression model based on the MOSUM process of the spatial OLS residuals. As stated in Lehmann and Romano [10], p. 423, power function of a test takes important role for evaluating the optimality of the test.

To see the problem in detail, let \( R_{n_1 \times n_2} := (r_{k\ell})_{k=1,\ell=1}^{n_2,n_1} \) be the matrix of the OLS residuals obtained from the matrix of independent spatial observations of the following regression model

\[
Y(t, s) = g(t, s) + \varepsilon(t, s), \quad (t, s) \in D := [a_1, b_1] \times [a_2, b_2] \subset \mathbb{R}^2
\]

sampled according to a probability measure \( P_0 \) defined on \( (D, \mathcal{B}(D)) \) with \( E(\varepsilon(t, s)) = 0 \) and \( \text{var}(\varepsilon(t, s)) = \sigma^2 \), see Somayasa and Budiman [11] and Somayasa [12] for the sampling schema.
according to $P_0$. The corresponding probability distribution function $F_0$ of $P_0$ is assumed to be factorable as $F_0 = F_{01} \times F_{02}$, such that $F_0(t, s) = F_{01}(t)F_{02}(s)$, for $(t, s) \in \mathbf{D}$. Let $\mathbf{W}$ be a subspace in $L_2(P_0, \mathbf{D})$ generated by known regression functions $\{f_1, \ldots, f_g\}$. Let $h_1$ and $h_2$ be fixed positive real numbers such that $0 \leq h_1 \leq b_1 - a_1$ and $0 \leq h_2 \leq b_2 - a_2$. By extending the ideas of [1, 8], whether or not the unknown regression function $g$ belongs to $\mathbf{W}$ can be detected by applying the KS and CvM functionals of the MOSUM of the OLS residuals defined respectively by

$$
\mathcal{KS}_{n_1 \times n_2} := \sup_{(t, s) \in \mathbf{D}_{h_1h_2}} \left| \frac{1}{\sigma} M S_{h_1h_2;P_0}(\mathbf{R}_{n_1 \times n_2})(t, s) \right|
$$

$$
\mathcal{CM}_{n_1 \times n_2} := \int_{\mathbf{D}_{h_1h_2}} \left( \frac{1}{\sigma} M S_{h_1h_2;P_0}(\mathbf{R}_{n_1 \times n_2})(t, s) \right)^2 P_0(dt, ds),
$$

where for every $(t, s) \in \mathbf{D}_{h_1h_2} := [a_1, b_1 - h_1] \times [a_2, b_2 - h_2],

$$
M S_{h_1h_2;P_0}(\mathbf{R}_{n_1 \times n_2})(t, s) := \frac{1}{\sqrt{n_1n_2}} \left[ \frac{n_2F_{02}(s+h_2)}{[n_2F_{02}(s+h_2)]} \right] \left[ \frac{n_1F_{01}(t+h_1)}{[n_1F_{01}(t+h_1)]} \right] \sum_{k=1}^{n_2} \sum_{\ell=1}^{n_1} r_{k\ell},
$$

to thereby $[x] := \max \{z \in \mathbb{Z}_{\geq 0} \mid z \leq x\}$. As a special case, when $P_0$ is given by the uniform probability measure $\lambda_D^*$ having the probability distribution function

$$
F_0^*(t, s) := \frac{(t-a_1)(s-a_2)}{(b_1-a_1)(b_2-a_2)}, \quad (t, s) \in \mathbf{D},
$$

we have

$$
M S_{h_1h_2;\lambda_D^*}(\mathbf{R}_{n_1 \times n_2})(t, s) = \frac{1}{\sqrt{n_1n_2}} \left[ \frac{n_2(s+h_2-a_2)}{[n_2(s+h_2-a_2)]} \right] \left[ \frac{n_1(t+h_1-a_1)}{[n_1(t+h_1-a_1)]} \right] \sum_{k=1}^{n_2} \sum_{\ell=1}^{n_1} r_{k\ell}.
$$

Suppose the matrix of independent observations are generated based on the localized model

$$
Y(t, s) = \frac{1}{\sqrt{n_1n_2}} g(t, s) + \varepsilon(t, s), \quad (t, s) \in \mathbf{D}.
$$

If the regression functions $\{f_1, \ldots, f_g\}$ have bounded variation on $\mathbf{D}$ and build an orthonormal basis in $L_2(\mathbf{D}, P_0)$, then by the limit theorems in [4, 7] and by the well-known continuous mapping theorem (cf. Billingsley [13]), it holds for $g \not\in \mathbf{W}$ and $n_1, n_2 \to \infty$,

$$
\frac{1}{\sigma} M S_{h_1h_2;P_0}(\mathbf{R}_{n_1 \times n_2})(\cdot) \Rightarrow \mathcal{R}(\cdot) := \mathcal{S}_{h_1h_2;P_0}(\cdot), \quad (1)
$$

where for any $(t, s) \in \mathbf{D}_{h_1h_2},$

$$
\mathcal{S}_{h_1h_2;P_0}(t, s) := \Delta_{[t,t+h_1] \times [s,s+h_2]} S_g - \sum_{i=1}^{g} \int_{\mathbf{D}} f_i(x, y)g(x, y)P_0(dx, dy)\Delta_{[t,t+h_1] \times [s,s+h_2]}S_f_i,
$$

$$
\varphi_g(t, s) := \Delta_{[t,t+h_1] \times [s,s+h_2]} S_g - \sum_{i=1}^{g} \int_{\mathbf{D}} f_i(x, y)g(x, y)P_0(dx, dy)\Delta_{[t,t+h_1] \times [s,s+h_2]}S_f_i.
$$
Thereby for every real-valued function \( w \) on \( D \), \( S_w \) is an absolutely continuous real function defined by \( S_w(t, s) := \int_{[a_1, t] \times [a_2, s]} w(x, y) P_0(dx, dy) \),

\[
\Delta_{[t,t+h_1] \times [s, s+h_2]} w := w(t + h_1, s + h_2) - w(t, s + h_2) - w(t + h_1, s) + w(t, s)
\]

and \( Z_{P_0} \) is the Gaussian white noise with the control measure \( P_0 \), see Bass [14] and Lifshits [15] for the definition of \( Z_{P_0} \). Let us called \( R(\cdot) \) throughout the paper as the residual CUSUM limit process. It is noticed that, when \( g \) is actually in \( W \), the term \( \varphi_g \) in (1) vanishes uniformly living \( S_{h_1h_2,P_0}(\cdot) \) as the limit process. In that case if \( W = [f_1] \), with \( f_1 \equiv 1 \), the limit reduces to \( S_{h_1h_2,P_0}^0 = \Delta_{[t,t+h_1] \times [s, s+h_2]} Z_{P_0} \) which is a Gaussian process with the covariance function given by

\[
K_{S_{h_1h_2,P_0}}(t_1, s_1; t_2, s_2) = P_0([t_1, t_1 + h_1] \times [s_1, s_1 + h_2] \cap [t_2, t_2 + h_1] \times [s_2, s_2 + h_2]).
\]

We call the process \( S_{h_1h_2,P_0}^0 \) as the generalized \((h_1h_2)\)-Slepian field with the control measure \( P_0 \). In this case, when \( P_0 = \lambda \mathbb{D} \), the limit process coincides with the \((h_1h_2)\)-Slepian field having the covariance function given by

\[
K_{S_l}(t_1, s_1; t_2, s_2) = (h_1 - |t_1 - t_2|)(h_2 - |s_1 - s_2|),
\]

where \( x^+ := \max\{x, 0\} \), see Gao and Li [16]. Another well-known Gaussian process is obtained under constant model, that is, when \( W = [f_1] \), with \( f_1 \equiv 1 \). Under this condition we have

\[
S_{h_1h_2,P_0}^1(t, s) = \Delta_{[t,t+h_1] \times [s, s+h_2]} Z_{P_0} - Z_{P_0}(\mathbb{D}) P_0([t, t + h_1] \times [s, s + h_2]), (t, s) \in D_{h_1 h_2},
\]

whose covariance function is given by

\[
K_{S_{h_1h_2,P_0}}^1(t_1, s_1; t_2, s_2) = P_0([t_1, t_1 + h_1] \times [s_1, s_1 + h_2] \cap [t_2, t_2 + h_1] \times [s_2, s_2 + h_2])
- P_0([t_1, t_1 + h_1] \times [s_1, s_1 + h_2]) P_0([t_2, t_2 + h_1] \times [s_2, s_2 + h_2]).
\]

It is worth mentioning that by the definitions, the probability distributions of \( S_{h_1h_2,P_0}^0 \) and \( S_{h_1h_2,P_0}^1 \) clearly depend not only on the design \( P_0 \) but also on the choice of \( h_1 \) and \( h_2 \).

The limiting power functions of the size \( \alpha \) KS and CvM CUSUM tests for \( H_0 : g \in W \) against \( H_1 : g \notin W \) are given respectively by

\[
\mathcal{K}_{S_{h_1h_2,P_0}}(\tilde{t}_{1-\alpha}, \varphi_g) := P \left\{ \sup_{(t,s) \in D_{h_1,h_2}} |\varphi_g(t, s) + S_{h_1h_2,P_0}(t, s)| \geq \tilde{t}_{1-\alpha} \right\}
\]

\[
\mathcal{C}_{S_{h_1h_2,P_0}}(\tilde{q}_{1-\alpha}, \varphi_g) := P \left\{ \int_{D_{h_1,h_2}} (\varphi_g(t, s) + S_{h_1h_2,P_0}(t, s))^2 P_0(dt, ds) \geq \tilde{q}_{1-\alpha} \right\}
\]

where \( \tilde{t}_{1-\alpha} \) and \( \tilde{q}_{1-\alpha} \) are the constants that satisfy \( \mathcal{K}_{S_{h_1h_2,P_0}}(\tilde{t}_{1-\alpha}, 0) = \alpha \) and \( \mathcal{C}_{S_{h_1h_2,P_0}}(\tilde{q}_{1-\alpha}, 0) = \alpha \). These conditions are fulfilled when \( g \) varies in \( W \), otherwise they will move away from \( \alpha \). Unfortunately, it is impossible to conduct analytic computation to \( \mathcal{K}_{S_{h_1h_2,P_0}}(\tilde{t}_{1-\alpha}, \varphi_g) \) as well as \( \mathcal{C}_{S_{h_1h_2,P_0}}(\tilde{q}_{1-\alpha}, \varphi_g) \). Therefore, it becomes the purpose of the present paper to establish approximation procedure for the rate of decays of the power functions of the tests.

We organize the rest of the present paper as follows. Some preliminary results regarding the properties of the residual MOSUM limit process which are important as the auxiliaries for deriving the main results are discussed in Section 2. The rate of decays of the power functions of the KS and CvM MOSUM tests are investigated analytically in Section 3. The finite sample size behavior of the rate of decay of the power is also studied by Simulation, see Section 4. We close the paper in Section 5 with some conclusions and remarks for future researches.
2. Preliminary results

In this section we present some preliminary results regarding the properties of the process \( S_{h_1,h_2:P_0} \) useful for deriving the approximation to the rate of decays of the power functions of both KS and CvM MOSUM tests.

In general the process \( S_{h_1,h_2:P_0} \) does not have independent increments (see Definition A.1 in the Appendix for the definition of the increment of \( S_{h_1,h_2:P_0} \)). It can be shown that for any two disjoint adjacent rectangles \( J_{w_1,w_2} = [t_{w_1-1}, t_{w_1}] \times [s_{w_2-1}, s_{w_2}] \) and \( J_{w_1+1,w_2} = [t_{w_1}, t_{w_1+1}] \times [s_{w_2-1}, s_{w_2}] \), it holds

\[
\text{Cov} \left( \Delta_{w_1w_2} S^0_{h_1,h_2:P_0}, \Delta_{w_1+1w_2} S^0_{h_1,h_2:P_0} \right) = -2P_0 \left( [t_{w_1}, t_{w_1} + h_1] \times [s_{w_2}, s_{w_2} + h_2] \right) + 2P_0 \left( [t_{w_1}, t_{w_1} + h_1] \times [s_{w_2-1}, s_{w_2-1} + h_2] \right) \neq 0,
\]

unless \( P_0 \) is restricted to the uniform measure \( \lambda^p_0 \). Hence, the \((h_1h_2)\)-Slepian field \( S^0_{h_1,h_2:P_0} \) is a spatial process with independent increments, see also Proposition A.1 in the Appendix.

The notion of the reproducing kernel Hilbert space (RKHS) of \( S_{h_1,h_2:P_0} \), denoted by \( \mathcal{H}_{S_{h_1,h_2:P_0}} \) is decisive for our result. It can be derived by utilizing factorization theorem (Theorem 4.1 in [15]) as follows. Let \( \{ m(t,s) \in L_2(P_0, D) \mid (t,s) \in \mathcal{D}_{h_1h_2} \} \) be the family of functions in \( L_2(P_0, D) \) that satisfies the condition

\[
\mathcal{K}_{S_{h_1,h_2:P_0}}(t_1, s_1; t_2, s_2) = \langle m(t_1, s_1), m(t_2, s_2) \rangle_{L_2(P_0, D)}.
\]

Then by Theorem 4.1 in [15], the RKHS of \( S_{h_1,h_2:P_0} \) is given by

\[
\mathcal{H}_{S_{h_1,h_2:P_0}} \triangleq \left\{ u \mid \exists \ell \in L_2(P_0, D), \ u(t,s) = \int_D m(t,s)(x,y)\ell(x,y)P_0(dx,dy), (t,s) \in \mathcal{D}_{h_1h_2} \right\}.
\]

For every \( u \in \mathcal{H}_{S_{h_1,h_2:P_0}} \) that satisfies \( u(t,s) = \int_D m(t,s)(x,y)\ell(x,y)P_0(dx,dy) \), for some \( \ell \in L_2(P_0, D) \), the norm of \( u \) is defined by

\[
\|u\|_{\mathcal{H}_{S_{h_1,h_2:P_0}}}^2 \triangleq \inf \{ \|\ell\|_{L_2(P_0, D)}^2 \mid \ell : u(t,s) = \int_D m(t,s)(x,y)\ell(x,y)P_0(dx,dy), (t,s) \in \mathcal{D}_{h_1h_2} \}.
\]

If \( \ell \) is unique, then the norm is simply defined by

\[
\|u\|_{\mathcal{H}_{S_{h_1,h_2:P_0}}}^2 \triangleq \|\ell\|_{L_2(P_0, D)}^2.
\]

**Corollary 2.1** For the generalized \((h_1h_2)\)-Slepian field \( S^0_{h_1,h_2:P_0} \), we have

\[
\mathcal{H}_{S^0_{h_1,h_2:P_0}} \triangleq \left\{ u \mid \exists \ell \in L_2(P_0, D), \ u(t,s) = \int_{[t,t+h_1] \times [s,s+h_2]} \ell(x,y)P_0(dx,dy), (t,s) \in \mathcal{D}_{h_1h_2} \right\}
\]

by the reason there exists a family of indicator functions \( \{ 1_{[t,t+h_1] \times [s,s+h_2]} \mid (t,s) \in \mathcal{D}_{h_1h_2} \} \), such that for \( (t_1, s_1), (t_2, s_2) \in \mathcal{D}_{h_1h_2} \),

\[
\mathcal{K}_{S^0_{h_1,h_2:P_0}}(t_1, s_1; t_2, s_2) = P_0 \left( [t_1, t_1 + h_1] \times [s_1, s_1 + h_2] \cap [t_2, t_2 + h_1] \times [s_2, s_2 + h_2] \right) = \langle 1_{[t_1,t_1+h_1] \times [s_1,s_1+h_2]}, 1_{[t_2,t_2+h_1] \times [s_2,s_2+h_2]} \rangle_{L_2(P_0, D)}.
\]
Corollary 2.2 For the more general residual MOSUM process $S_{h_1 h_2}^0$, it holds

$$
\mathcal{H}_{S_{h_1 h_2}^0} := \left\{ u \mid \ell \in L_2(P_0, D), \ u(t, s) = \int_{[t, t+h_1] \times [s, s+h_2]} \ell(x, y) P_0(dx, dy) - P_0([t, t+h_1] \times [s, s+h_2]) \int_{D_{h_1 h_2}} \ell(x, y) P_0(dx, dy) \right\},
$$

by the fact the family

$$
\{m_{(t,s)}^1 := 1_{[t, t+h_1] \times [s, s+h_2]} - P_0([t, t+h_1] \times [s, s+h_2]) 1_{D_{h_1 h_2}} \mid (t, s) \in D_{h_1 h_2}\}
$$
can be shown to satisfy the following equation:

$$
K_{S_{h_1 h_2}^0} (t_1, s_1; t_2, s_2) = \langle m^{1}_{(t_1, s_1)}, m^{1}_{(t_2, s_2)} \rangle_{L_2(P_0, D)}, \ (t_1, s_1), (t_2, s_2) \in D_{h_1 h_2}.
$$

The following corollary provides the Cameron-Martin formula for the density function of the shifted $(h_1 h_2)$ Slepian field $\varphi_g + S_{h_1 h_2}^0$, with $\varphi_g \in \mathcal{H}_{S_{h_1 h_2}^0}$, where for every $(t, s) \in D_{h_1 h_2}$,

$$
\varphi_g(t, s) = \int_{[t, t+h_1] \times [s, s+h_2]} g(x, y) \lambda_D^*(dx, dy).
$$
The result is due to Theorem A.4 in the Appendix.

Corollary 2.3 Let $P$ be the probability distribution of $S_{h_1 h_2}^0$ on $(C(D_{h_1 h_2}), B(C(D_{h_1 h_2})))$ and $P_{\varphi_g}$ be defined as $P_{\varphi_g}(A) = P(A - \varphi_g)$, for $A \in B(C(D_{h_1 h_2}))$. Then the Cameron-Martin density of $\varphi_g + S_{h_1 h_2}^0 \lambda_D^*$ is given by

$$
\frac{dP_{\varphi_g}}{dP} = \exp \left\{ \frac{1}{2} \int_{D_{h_1 h_2}} g(x, y) dS_{h_1 h_2}^0 \lambda_D^* (x, y) - \frac{1}{2} \| \varphi_g \|_{\mathcal{H}_{S_{h_1 h_2}^0}}^2 \right\},
$$

with

$$
L(\varphi_g, S_{h_1 h_2}^0 \lambda_D^*) = \frac{1}{2} \int_{D_{h_1 h_2}} g(x, y) dS_{h_1 h_2}^0 \lambda_D^* (x, y).
$$

Furthermore, by applying Proposition A.1 and Proposition A.2 in the Appendix, we get

$$
Cov \left( \int_{D_{h_1 h_2}} g_1(x, y) dS_{h_1 h_2}^0 \lambda_D^* (x, y), \int_{D_{h_1 h_2}} g_2(x, y) dS_{h_1 h_2}^0 \lambda_D^* (x, y) \right)
= 4 \int_{D_{h_1 h_2}} g_1(x, y) g_2(x, y) \lambda_D^* (dx, dy) = 4 (g_1, g_2)_{L_2(\lambda_D^*, D_{h_1 h_2})}.
$$

Hence, for every $\varphi_g \in \mathcal{H}_{S_{h_1 h_2}^0}$, it holds

$$
L(\varphi_g, S_{h_1 h_2}^0 \lambda_D^*) \sim N \left( 0, \int_{D_{h_1 h_2}} g^2(x, y) \lambda_D^* (dx, dy) \right), \quad (2)
$$

where

$$
\| \varphi_g \|_{\mathcal{H}_{S_{h_1 h_2}^0}}^2 := \int_{D_{h_1 h_2}} g^2(x, y) \lambda_D^* (dx, dy).
$$

(3)
3. Main results

In this section we state the main results of our investigation regarding the rate of decays of the power function of the KS and CvM MOSUM test. However by technical reason the consideration is restricted to the \((h_1 h_2)\)-Slepian field only. This is due to the fact that \(S_{h_1 h_2; \lambda^*}^0\) has stationary and independent increments. We need more effort and further researches to get similar results for the more general process. Therefore discussion for the more general residual MOSUM limit process is beyond the scope of this paper.

The discussion is started with the case of the power of the KS type MOSUM test. Treatment for the CvM test will be presented in the second part of this section.

**Theorem 3.1** Let \(\varphi_g\) be a function defined by

\[
\varphi_g(t, s) := \int_{t}^{t+h_1|\lambda|} g(x, y) \lambda_D^*(dy, dx).
\]

For \(\alpha \in (0, 1)\), let \(\tilde{t}_{1-\alpha}\) be a constant defined by

\[
P\left\{ \sup_{(t,s) \in D_{h_1 h_2}} \left| S_{h_1 h_2; \lambda^*}^0(t, s) \right| \geq \tilde{t}_{1-\alpha} \right\} = \alpha.
\]

Then, it holds

\[
\left| KS_{h_1 h_2; \lambda^*}^0(\tilde{t}_{1-\alpha}, \varphi_g) - \alpha \right| \leq \frac{1}{2\sqrt{2\pi}} \|\varphi_g\|_{\mathcal{H}_{h_1 h_2}^0}. \]

**Proof:** Let \(A := \left\{ w \in \mathcal{C}(D_{h_1 h_2}) : \sup_{(t,s) \in D_{h_1 h_2}} |w(t, s)| \geq \tilde{t}_{1-\alpha} \right\}\), where \(\mathcal{C}(D_{h_1 h_2})\) is the space of continuous functions on \(D_{h_1 h_2}\). Since \(\varphi_g \in \mathcal{H}_{h_1 h_2; \lambda^*}^0 \subset \mathcal{C}(D_{h_1 h_2})\), then

\[
A - \varphi_g = \left\{ w - \varphi_g \in \mathcal{C}(D_{h_1 h_2}) : \sup_{(t,s) \in D_{h_1 h_2}} |w(t, s)| \geq \tilde{t}_{1-\alpha} \right\}
\]

Thus, we have \(KS_{h_1 h_2; \lambda^*}^0(\tilde{t}_{1-\alpha}, \varphi_g) = P_{\varphi_g}(A)\), where \(P\) is the probability distribution of \(S_{h_1 h_2; \lambda^*}^0\). Next, let \(\Phi(\cdot)\) be the distribution function of the standard normal distribution. Since \(\varphi_g \in \mathcal{H}_{h_1 h_2; \lambda^*}^0\) and \(P(A) = \alpha\), then by applying Li-Kuelbs inequality (cf. [15, 17]), it holds

\[
\Phi \left( \Phi^{-1}(\alpha) - \frac{L(\varphi_g, \varphi_g)}{\|\varphi_g\|_{\mathcal{H}_{h_1 h_2; \lambda^*}^0}} \right) \leq KS_{h_1 h_2; \lambda^*}^0(\tilde{t}_{1-\alpha}, \varphi_g) \leq \Phi \left( \Phi^{-1}(\alpha) + \frac{L(\varphi_g, \varphi_g)}{\|\varphi_g\|_{\mathcal{H}_{h_1 h_2; \lambda^*}^0}} \right)
\]

Since \(\frac{L(\varphi_g, \varphi_g)}{\|\varphi_g\|_{\mathcal{H}_{h_1 h_2; \lambda^*}^0}} = \frac{1}{2}\|\varphi_g\|_{\mathcal{H}_{h_1 h_2; \lambda^*}^0}\), then the last inequality becomes

\[
\Phi \left( \Phi^{-1}(\alpha) - \frac{1}{2}\|\varphi_g\|_{\mathcal{H}_{h_1 h_2; \lambda^*}^0} \right) \leq KS_{h_1 h_2; \lambda^*}^0(\tilde{t}_{1-\alpha}, \varphi_g) \leq \Phi \left( \Phi^{-1}(\alpha) + \frac{1}{2}\|\varphi_g\|_{\mathcal{H}_{h_1 h_2; \lambda^*}^0} \right). \tag{4}
\]

The application of the well-known mean value theorem to (4), implies

\[
KS_{h_1 h_2; \lambda^*}^0(\tilde{t}_{1-\alpha}, \varphi_g) - \alpha \leq \Phi \left( \Phi^{-1}(\alpha) + \frac{1}{2}\|\varphi_g\|_{\mathcal{H}_{h_1 h_2; \lambda^*}^0} \right) - \Phi(\Phi^{-1}(\alpha)) \leq \frac{1}{2}\|\varphi_g\|_{\mathcal{H}_{h_1 h_2; \lambda^*}^0} \phi(\eta)
\]

for some \(\eta \in \left( \Phi^{-1}(\alpha), \Phi^{-1}(\alpha) + \frac{1}{2}\|\varphi_g\|_{\mathcal{H}_{h_1 h_2; \lambda^*}^0} \right)\), where \(\phi\) is the density of \(N(0, 1)\). Since \(\phi\) attains the maximum value \(1/\sqrt{2\pi}\), then the last inequality reduces to

\[
KS_{h_1 h_2; \lambda^*}^0(\tilde{t}_{1-\alpha}, \varphi_g) - \alpha \leq \frac{1}{2\sqrt{2\pi}} \|\varphi_g\|_{\mathcal{H}_{h_1 h_2; \lambda^*}^0}. \tag{5}
\]
Similarly, we get

\[ -\left( \Phi \left( \Phi^{-1}(\alpha) \right) \right) - \Phi \left( \Phi^{-1}(\alpha) - \frac{1}{2} \| \varphi_g \|_{H_{\alpha h_2;\lambda_2}^0} \right) \leq K S_{\alpha h_2;\lambda_2}^0 (\tilde{t}_{1-\alpha}, \varphi_g) - \alpha \]

\[ \iff -\frac{1}{2\sqrt{2\pi}} \| \varphi_g \|_{H_{\alpha h_2;\lambda_2}^0} \leq K S_{\alpha h_2;\lambda_2}^0 (\tilde{t}_{1-\alpha}, \varphi_g) - \alpha \quad (6) \]

The assertion follows by combining (5) and (6). This is finishing the proof.

By Theorem 3.1 it is seen that the convergence of the power to the size of the test is bounded point wise by the norm of the regression function. However, when the power is evaluated on any compact subset of \( H_{\alpha h_2;\lambda_2}^0 \), we get a uniformly bound by the continuity of the norm.

The following theorem presents the rate of decay of the power of the CvM MOSUM test. Although the result can be proved similarly as in the case of KS test, for the sake of brevity we insist to present the proof.

**Theorem 3.2** Let \( q_{1-\alpha} \) be a constant defined by

\[ P \left\{ \int_{D_{h_1\lambda_1}} \left( S_{h_2;\lambda_2}^0 (t, s) \right)^2 \geq q_{1-\alpha} \right\} = \alpha, \]

for \( \alpha \in (0, 1) \). Suppose that \( \varphi_g \) is defined on \( D_{h_1\lambda_1} \) by \( \varphi_g(t, s) := \int_{[t,t+h_2] \times [s,s+h_2]} g(x, y) \lambda_D^2 (dx, dy) \). Then it holds

\[ |C M_{h_2;\lambda_2} \left( q_{1-\alpha}, \varphi_g \right) - \alpha | \leq \frac{1}{2\sqrt{2\pi}} \| \varphi_g \|_{H_{\alpha h_2;\lambda_2}^0}. \]

**Proof**: The proof of this theorem can be established analogously as that of Theorem 3.1. Let \( B \) be a subset of \( C(D_{h_1\lambda_1}) \), defined by

\[ B := \left\{ w \in C(D_{h_1\lambda_1}) : \int_{D_{h_1\lambda_1}} w^2(x, y) \lambda_D^2 (dx, dy) \geq q_{1-\alpha} \right\}. \]

Since \( \varphi_g \in C(D_{h_1\lambda_1}) \), then we have

\[ B - \varphi_g = \left\{ w - \varphi_g \in C(D_{h_1\lambda_1}) : \int_{D_{h_1\lambda_1}} (w(x, y) - \varphi_g(x, y))^2 \lambda_D^2 (dx, dy) \geq q_{1-\alpha} \right\} \]

\[ = \left\{ v \in C(D_{h_1\lambda_1}) : \int_{D_{h_1\lambda_1}} (\varphi_g(x, y) + v(x, y))^2 \lambda_D^2 (dx, dy) \geq q_{1-\alpha} \right\}. \]

Hence, the power function \( C M_{\alpha h_2;\lambda_2} (q_{1-\alpha}, \varphi_g) \) coincides with \( P(B - \varphi_g) = P\phi_{\varphi_g}(B) \), provided \( P \) is the probability distribution of \( S_{h_2;\lambda_2}^0 \). Furthermore, since \( \varphi_g \in H_{\alpha h_2;\lambda_2} \), then by applying Li-Kuelbs inequality documented in [15, 17], the following inequality holds true

\[ \Phi \left( \Phi^{-1}(\alpha) - \frac{1}{2} \| \varphi_g \|_{H_{\alpha h_2;\lambda_2}^0} \right) \leq C M_{\alpha h_2;\lambda_2} (q_{1-\alpha}, \varphi_g) \leq \Phi \left( \Phi^{-1}(\alpha) + \frac{1}{2} \| \varphi_g \|_{H_{\alpha h_2;\lambda_2}^0} \right). \]

Next, by applying the similar steps as in the proof of Theorem 3.1, the assertion follows. We are done.

Theorem 3.1 and Theorem 3.2 provide information that the rate of decays of both power functions to the size of the tests depend on the length of the regression function. The smaller the norm the faster the rate of decay. Conversely, the larger the norm the slower the decay.
We show below that the rate of decays of the power functions presented above coincide with the rate of decay of a most powerful (PM) test for simple hypotheses. In fact, by recalling (1), testing \( H_0 : g \in \mathbf{W} \) against \( H_1 : g \notin \mathbf{W} \) based on the \text{MOSUM process} \( MS_{h_1h_2;P_0}(\mathbf{R}_{n_1 \times n_2}) \) is asymptotically equivalent to the problem of testing

\[
H_0 : \varphi_g \equiv 0 \text { against } H_1 : \varphi_g \equiv \varphi_{w_0}, \text { for some } \varphi_{w_0} \in \mathcal{H}_{S_{h_1h_2,P_0}}
\]  

(7)

when the the residual \text{MOSUM process} \( \mathcal{R}(\cdot) = \varphi_{w_0}(\cdot) + \mathcal{S}_{h_1,h_2;P_0}(\cdot) \) is observed. It is well known that \text{Neyman-Pearson (N-P)} test for testing simple hypotheses coincides with an MP test, see either Theorem 3.2.1 in \cite{10} or Theorem III.1.1 in \cite{18}. By this reason we consider the rate of decay of the power of N-P test by observing the process \( \{\mathcal{R}(t,s) : (t,s) \in D_{h_1h_2}\} \).

**Theorem 3.3** Let \( \mathcal{NP}_{S_{h_1h_2;\lambda_D}^0}^0 : \mathcal{H}_{S_{h_1h_2;P_0}} \to (0,1) \) be the power function of the size \( \alpha \) N-P test for testing (7). Then it holds true

\[
\mathcal{NP}_{S_{h_1h_2;\lambda_D}^0}(\varphi_w) = \Phi \left( \frac{L(\varphi_{w_0}, \varphi_w)}{\|\varphi_{w_0}\|_{S_{h_1h_2;\lambda_D}^0}} + \Phi^{-1}(\alpha) \right), \quad \varphi_w \in \mathcal{H}_{S_{h_1h_2;P_0}},
\]

Furthermore, for any \( \alpha \in (0,1) \) and \( \varphi_w \in \mathcal{H}_{S_{h_1h_2;P_0}} \) if \( L(\varphi_{w_0}, \varphi_w) > 0 \), then

\[
\left| \mathcal{NP}_{S_{h_1h_2;\lambda_D}^0}(\varphi_w) - \alpha \right| \leq \frac{1}{2\pi} \frac{L(\varphi_{w_0}, \varphi_w)}{\|\varphi_{w_0}\|_{S_{h_1h_2;\lambda_D}^0}},
\]

(8)

where

\[
L(\varphi_{w_0}, \varphi_w) = \frac{1}{2} \int_{D_{h_1h_2}} w_0(x,y) d\varphi_w(x,y).
\]

**Proof:** Let \( \gamma_0(\cdot) \) and \( \gamma_1(\cdot) \) be the density of the probability distribution of \( \mathcal{R}(\cdot) \) under \( H_0 \) and under \( H_1 \), respectively. By recalling Corollary 2.3 and the definition of the N-P test (cf. \cite{10} and Pestman \cite{18}), for any \( \alpha \in (0,1) \), we have

\[
P \left\{ \frac{\gamma_0(\mathcal{R})}{\gamma_1(\mathcal{R})} \leq k |H_0\right\} = \alpha \iff P \left\{ \exp \left\{ -L(\varphi_{w_0}, \mathcal{R}) + \frac{1}{2} \|\varphi_{w_0}\|_{S_{h_1h_2;\lambda_D}^0}^2 \right\} \leq k |H_0\right\} = \alpha
\]

\[
\iff P \left\{ L(\varphi_{w_0}, \mathcal{S}_{h_1h_2;\lambda_D}^0) \geq - \ln k + \frac{1}{2} \|\varphi_{w_0}\|_{S_{h_1h_2;\lambda_D}^0}^2 |H_0\right\} = \alpha
\]

\[
\iff P \left\{ L(\varphi_{w_0}, \mathcal{S}_{h_1h_2;\lambda_D}^0) \geq - \ln k + \frac{1}{2} \|\varphi_{w_0}\|_{S_{h_1h_2;\lambda_D}^0}^2 \right\} = \alpha.
\]

Hence, by applying (2), it can be shown that the size \( \alpha \) rejection region of the test is given by

\[
\mathcal{NP}_\alpha := \left\{ \mathcal{R} : L(\varphi_{w_0}, \mathcal{R}) \geq \Phi^{-1}(1 - \alpha) \|\varphi_{w_0}\|_{S_{h_1h_2;\lambda_D}^0} \right\}.
\]

The power of the size \( \alpha \) N-P test at any \( \varphi_w \in \mathcal{H}_{S_{h_1h_2;\lambda_D}^0} \) can be derived as follows

\[
\mathcal{NP}_{S_{h_1h_2;\lambda_D}^0}(\varphi_w) = P \left\{ L(\varphi_{w_0}, \mathcal{R}) \geq \Phi^{-1}(1 - \alpha) \|\varphi_{w_0}\|_{S_{h_1h_2;\lambda_D}^0} |\varphi_g \equiv \varphi_w \right\}
\]
\[ P \left( \varphi_{w_0}, \varphi_w + S_{h_1, h_2, \lambda^*_D}^0 \right) \geq \Phi^{-1}(1 - \alpha) \| \varphi_{w_0} \| S_{h_1, h_2, \lambda^*_D}^0 \right) \]

\[ = P \left( L(\varphi_{w_0}, \varphi_w, S_{h_1, h_2, \lambda^*_D}^0) \geq \Phi^{-1}(1 - \alpha) \| \varphi_{w_0} \| S_{h_1, h_2, \lambda^*_D}^0 - L(\varphi_{w_0}, \varphi_w) \right) \]

where the last equality follows by the symmetry of \( \Phi \). This is establishing the first result. By the mean value theorem and the symmetry of \( \Phi \), for every \( \alpha \in (0, 1) \), we have

\[ \mathcal{NP}_{S_{h_1, h_2, \lambda^*_D}^0}(\varphi_w) - \alpha = \Phi \left( \frac{L(\varphi_{w_0}, \varphi_w)}{\| \varphi_{w_0} \| S_{h_1, h_2, \lambda^*_D}^0} + \Phi^{-1}(\alpha) \right) - \Phi(\Phi^{-1}(\alpha)) \]

\[ \leq \frac{1}{\sqrt{2\pi}} \frac{L(\varphi_{w_0}, \varphi_w)}{\| \varphi_{w_0} \| S_{h_1, h_2, \lambda^*_D}^0} \]

Conversely, since \( L(\varphi_{w_0}, \varphi_w) \) is assumed to be positive real number, it is always true that

\[ \mathcal{NP}_{S_{h_1, h_2, \lambda^*_D}^0}(\varphi_w) - \alpha \geq \frac{-1}{\sqrt{2\pi}} \frac{L(\varphi_{w_0}, \varphi_w)}{\| \varphi_{w_0} \| S_{h_1, h_2, \lambda^*_D}^0} \]

completing the proof of the theorem.

We notice that, if the power \( \mathcal{NP}_{S_{h_1, h_2, \lambda^*_D}^0}(\cdot) \) is evaluated at \( \varphi_{w_0} \), then by the definition of \( L(\varphi_{w_0}, \varphi_w) \), the upper bound in (8) reduces to \( \frac{1}{\sqrt{2\pi}} \frac{L(\varphi_{w_0}, \varphi_w)}{\| \varphi_{w_0} \| S_{h_1, h_2, \lambda^*_D}^0} \), which is equal to the rate of decay of the KS and CvM MOSUM tests exhibited in Theorem 3.1 and Theorem 3.2, respectively. This means that the rate of both KS as well as CvM MOSUM tests can be interpreted as the rate of an MP test.

4. Simulation

In this section we give an investigation to the power of the tests by simulation. The observations are sampled according to a regular lattice of size \( n_1 \times n_2 \) on \( I = [0, 1] \times [0, 1] \), where for every \( \ell \) and \( k \), with \( 1 \leq \ell \leq n_1 \) and \( 1 \leq k \leq n_2 \), the observation \( Y_{\ell k} \) is generated independently from the \( N(\theta_0(\ell, k), \sigma^2) \) distributions, with \( g(\ell, k) = 1 + \ell/n_1 + k/n_2 \), where \( \theta \) varies in some interval including zero. The limit process of the MOSUM of the OLS residuals is given by \( \mathcal{R}(\cdot) = \varphi_{\theta g}(\cdot) + S_{h_1, h_2, \lambda^*_D}^0(\cdot) \), where \( \varphi_{\theta g}(t, s) = \theta \int_{[t-t+1]} \cdot [s, s+h_2] (1 + x + y)dxdy \). We simulate two cases: in the first case we choose \( h_1 = 0.02 \) and \( h_2 = 0.02 \), so that by recalling (3) the norm of \( \varphi_{\theta g}(\cdot) \) is given by

\[ \| \varphi_{\theta g} \| S_{h_1, h_2, \lambda^*_D}^0 = \sqrt{\theta^2 \int_{[0,0.88] \times [0,0.88]} (1 + x + y)^2dxdy = 1.684|\theta|} \]

Hence, by Theorem 3.1 and Theorem 3.2, for the size \( \alpha \) KS and CvM tests, \( \mathcal{L} \) and \( \mathcal{U} \) are computed by the formulas \( \mathcal{L} = \alpha - \frac{1.684|\theta|}{2\sqrt{2\pi}} \) and \( \mathcal{U} = \alpha + \frac{1.684|\theta|}{2\sqrt{2\pi}} \). In the second scenario we consider the
Table 1. Empirical upper and lower bounds for the rate of decays of the size $\alpha$ power functions of the KS and CvM MOSUM tests evaluated at $\theta g(t, s) = \theta(1 + t + s)$, $(t, s) \in [0, 1] \times [0, 1]$, for several chosen values of $\theta$ and for $\alpha = 0.01$ and 0.05. The size of the window are $h_1 = 0.02$ and $h_2 = 0.02$.

| $\alpha$ | $\theta$ | $h_1$ | $h_2$ | $\mathcal{L}$ | $\mathcal{K}_S^{\theta h_1 h_2, \lambda_D}$ | $\mathcal{C}_M^{\theta h_1 h_2, \lambda_D}$ | $U$ |
|----------|----------|-------|-------|--------------|--------------------------------|--------------------------------|------|
| 0.01     | 0.0      | 0.02  | 0.02  | 0.0100       | 0.0095                         | 0.0118                         | 0.0100 |
| 0.2      | -0.0572  |       |       | 0.0085       | 0.0124                         | 0.0772                         |
| 0.4      | -0.1244  |       |       | 0.0099       | 0.0103                         | 0.1444                         |
| 0.6      | -0.1916  |       |       | 0.0115       | 0.0100                         | 0.2116                         |
| 0.9      | -0.2923  |       |       | 0.0110       | 0.0125                         | 0.3123                         |
| 2.0      | -0.6618  |       |       | 0.0098       | 0.0104                         | 0.6818                         |
| 4.0      | -1.3336  |       |       | 0.0124       | 0.0309                         | 1.3536                         |
| 6.0      | -2.0055  |       |       | 0.0125       | 0.0984                         | 2.0255                         |
| 0.05     | 0.0      | 0.02  | 0.02  | 0.0500       | 0.0473                         | 0.0509                         | 0.0500 |
| 1.0      | -0.2859  |       |       | 0.0435       | 0.0523                         | 0.3859                         |
| 5.0      | -1.6296  |       |       | 0.0572       | 0.1796                         | 1.7296                         |
| 7.0      | -2.3014  |       |       | 0.0613       | 0.4134                         | 2.4014                         |
| 8.5      | -2.8052  |       |       | 0.0739       | 0.6810                         | 2.9052                         |
| 10.0     | -3.3091  |       |       | 0.0664       | 0.8924                         | 3.4091                         |
| 12.0     | -3.9809  |       |       | 0.0909       | 0.9932                         | 4.0809                         |
| 15.0     | -4.9886  |       |       | 0.1127       | 1.0000                         | 5.0886                         |

Table 2. Empirical upper and lower bounds for the rate of decays of the size $\alpha$ power functions of the KS and CvM MOSUM tests evaluated at $\theta g(t, s) = \theta(1 + t + s)$, $(t, s) \in [0, 1] \times [0, 1]$, for several chosen values of $\theta$ and $\alpha = 0.01$ and 0.05. Window size is $h_1 = 0.30$ and $h_2 = 0.40$.

| $\alpha$ | $\theta$ | $h_1$ | $h_2$ | $\mathcal{L}$ | $\mathcal{K}_S^{\theta h_1 h_2, \lambda_D}$ | $\mathcal{C}_M^{\theta h_1 h_2, \lambda_D}$ | $U$ |
|----------|----------|-------|-------|--------------|--------------------------------|--------------------------------|------|
| 0.01     | 0.0      | 0.30  | 0.40  | 0.0100       | 0.0080                         | 0.0103                         | 0.0100 |
| 2.0      | -0.4221  |       |       | 0.0093       | 0.0134                         | 0.4421                         |
| 5.0      | -1.0701  |       |       | 0.0098       | 0.0443                         | 1.0901                         |
| 10.0     | -2.1503  |       |       | 0.0153       | 0.5755                         | 2.1703                         |
| 15.0     | -3.2304  |       |       | 0.0233       | 0.9990                         | 3.2504                         |
| 20.0     | -4.3105  |       |       | 0.0417       | 1.0000                         | 4.3305                         |
| 0.05     | 0.0      | 0.30  | 0.40  | 0.0500       | 0.0487                         | 0.0546                         | 0.0500 |
| 3.0      | -0.5981  |       |       | 0.0561       | 0.0793                         | 0.6981                         |
| 5.0      | -1.0301  |       |       | 0.0594       | 0.1633                         | 1.1301                         |
| 6.0      | -1.2462  |       |       | 0.0621       | 0.2445                         | 1.3462                         |
| 9.0      | -1.8945  |       |       | 0.0779       | 0.6596                         | 1.9943                         |
| 11.0     | -2.3263  |       |       | 0.0931       | 0.9215                         | 2.4263                         |
| 12.0     | -2.5923  |       |       | 0.0944       | 0.9751                         | 2.5923                         |

case $h_1 = 0.30$ and $h_2 = 0.40$, so we have

$$
\| \varphi_g \|_{\mathcal{H}_S^{\theta h_1 h_2, \lambda_D}} = \sqrt{\theta^2 \int_{[0,0.70] \times [0,0.60]} (1 + x + y)^2 \, dx \, dy} = 1.083 \| \theta \|.
$$
Similarly, by Theorem 3.1 and Theorem 3.2 the lower and upper bounds for the size \( \alpha \) power functions of the KS and CvM MOSUM tests are computed by as follows \( \mathcal{L} = \alpha - \frac{1.083\sqrt{\theta}}{2\sqrt{2\pi}} \) and \( \mathcal{U} = \alpha + \frac{1.083\sqrt{\theta}}{2\sqrt{2\pi}} \).

The simulation results for these two cases are presented respectively in Table 1 and Table 2 under 10000 runs and 100 \( \times \) \( \alpha \). Similarly, by Theorem 3.1 and Theorem 3.2 the lower and upper bounds for the size \( \theta \) power functions of the KS and CvM MOSUM tests are computed by as follows \( \mathcal{L} = \alpha - \frac{1.083\sqrt{\theta}}{2\sqrt{2\pi}} \) and \( \mathcal{U} = \alpha + \frac{1.083\sqrt{\theta}}{2\sqrt{2\pi}} \).

5. Concluding remarks
Mathematical formulas for the rate of decays or the rate of convergence of the power of the KS and CvM MOSUM tests in regression have been established by utilizing the Cameron-Martin translation formula for Gaussian process. However, due to the property of independent increments, the results are derived only for the KS and CvM functionals of the \((h_1 h_2)\) Slepian random field with deterministic trends. Advanced research is needed for establishing results for the more general residual MOSUM processes. In the forthcoming paper we derive the Cameron-Martin translation formula for the multidimensional \((h_1 h_2)\) Slepian filed for obtaining the rate of decays of KS and CvM MOSUM type tests in multivariate linear regression.

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Appendix

Definition A.1 Let \( \mathcal{P}_1 := \{[t_0, t_1], [t_1, t_2], \ldots, [t_{(m_1 - 1)}, t_{m_1}]\} \) be a set of \( m_1 \) closed intervals on \([a_1, b_1]\), such that \( a_1 = t_0 < t_1 < t_2 < \cdots < t_{m_1} = b_1 \). Let \( \mathcal{P}_2 := \{[s_0, s_1], [s_1, s_2], \ldots, [s_{(m_2 - 1)}, s_{m_2}]\} \) be a set of \( m_2 \) closed intervals on \([a_2, b_2]\), such that \( a_2 = s_0 < s_1 < s_2 < \cdots < s_{m_2} = b_2 \). The Cartesian product between \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \) denoted by \( \mathcal{K} := \mathcal{P}_1 \times \mathcal{P}_2 \) is a partition on \( \mathcal{D} \) which consists of \( m_1 m_2 \) closed rectangles in \( \mathcal{D} \). The points generating the rectangles are determined by the probability measure \( P_0 \). For fixed \( w_1 \) and \( w_2 \), such that \( 1 \leq w_i \leq m_i \), with \( i = 1, 2 \), let \( J_{w_1 w_2} \) be the element of \( \mathcal{K} \) defined by \( J_{w_1 w_2} := \{t_{w_1 - 1}, t_{w_1}\} \times \{s_{w_2 - 1}, s_{w_2}\} \). The increment of \( S_{h_1 h_2; P_0} \) on \( J_{w_1 w_2} \) is denoted by \( \Delta_{w_1 w_2} S_{h_1 h_2; P_0} \), given by

\[
\Delta_{w_1 w_2} S_{h_1 h_2; P_0} := S_{h_1 h_2; P_0}(t_{w_1}, s_{w_2}) - S_{h_1 h_2; P_0}(t_{w_1 - 1}, s_{w_2}) - S_{h_1 h_2; P_0}(t_{w_1}, s_{w_2 - 1}) + S_{h_1 h_2; P_0}(t_{w_1 - 1}, s_{w_2 - 1}).
\]

Proposition A.2 The \((h_1 h_2)\)-Slepian field \( S_{h_1 h_2}^0 := S_{h_1 h_2; P_0}^0 \) is a centered Gaussian process with independent increments.

Proof: It is sufficient to show that \( \text{Cov} \left( \Delta_{w_1 w_2} S_{h_1 h_2}^0, \Delta_{w'_1 w'_2} S_{h_1 h_2}^0 \right) = 0 \), for arbitrary and disjoint rectangles \( [t_{w_1}, t_{w_1 + 1}] \times [s_{w_2}, s_{w_2 + 1}] \) and \( [t_{w'_1}, t_{w'_1 + 1}] \times [s_{w'_2}, s_{w'_2 + 1}] \) on \( \mathcal{I} \), such that \( t_{w_1 + 1} \leq t_{w'_1}, s_{w_2 + 1} \leq s_{w'_2} \), \( 0 \leq t_{w_1}, t_{w'_1} \leq m_1 - 1 \) and \( |t_{w_1 + 1} - t_{w'_1}| < h_1 \) and \( |s_{w_2 + 1} - s_{w'_2}| < h_2 \).
By recalling the definition of the increments of $S_{h_1,b_2}^0$, we get the following result:

$$
Cov \left( \Delta_{w_1+1,w_2+1} h^1, \Delta_{w_1',w_2'+1} h^1 \right) = Cov \left( S_{h_1,b_2}^0 (t_{w_1+1}, s_{w_2+1}), S_{h_1,b_2}^0 (t_{w_1',w_2'+1}) \right) - Cov \left( S_{h_1,b_2}^0 (t_{w_1+1}, s_{w_2+1}), S_{h_1,b_2}^0 (t_{w_1',s_{w_2'+1}}) \right) - Cov \left( S_{h_1,b_2}^0 (t_{w_1',s_{w_2'+1}}), S_{h_1,b_2}^0 (t_{w_1',s_{w_2'+1}}) \right)
$$

By recalling the normality of the increment of $S_{h_1,b_2}^0$, the last equation establishes the proof.

**Proposition A.3** For any rectangle $I_{kl} := [t_{n_1} \ell, t_{n_1} \ell + 1] \times [s_{n_2} k, s_{n_2} k + 1]$ in $D_{h_1,b_2}$, such that $h_1 < |t_{n_1} \ell + 1 - t_{n_1} \ell|$ and $h_2 < |s_{n_2} k + 1 - s_{n_2} k|$, it holds $Var \left( \Delta_{t_{n_1} \ell + 1 + s_{n_2} k + 1} h^1 \right) = 4h_1 h_2$. Conversely, we have $Var \left( \Delta_{t_{n_1} \ell + 1 + s_{n_2} k + 1} h^1 \right) = 4h_1 h_2$. 

**Proof:** Let $K_{ij}$, $i,j = 1,2,3,4$ be constants defined by

$$
K_{11} = K_{22} = K_{33} = K_{44} = h_1 h_2, \quad K_{12} = h_2 (h_1 - t_{n_1} \ell + 1) + s_{n_2} k, \quad K_{14} = h_1 (h_2 - s_{n_2} k + 1 + s_{n_2} k), \\
K_{23} = h_2 (h_1 - t_{n_1} \ell + 1), \quad K_{24} = h_1 (h_2 - s_{n_2} k + 1 + s_{n_2} k),
$$

Let $h$ be defined as

$$
h = (S_{h_1,b_2}^0 (t_{n_1} \ell, s_{n_2} k), S_{h_1,b_2}^0 (t_{n_1} \ell, s_{n_2} k + 1), S_{h_1,b_2}^0 (t_{n_1} \ell + 1, s_{n_2} k), S_{h_1,b_2}^0 (t_{n_1} \ell + 1, s_{n_2} k + 1))^{\top}.
$$

By the definition of the covariance function of $K_{S_{h_1,b_2}^0}$, we have $E(h h^\top) = (K_{ij})_{i=1,j=1}^{4,4}$. By a standard result in multivariate analysis, cf. Johnson and Wichern [19], we get
Let \( \mathcal{W} \) be a centered Gaussian process with sample paths in a metric space \( \mathcal{C}(\Omega) \) and \( T \) is a deterministic function defined on \( \Omega \). Let \( \mathbf{P} \) and \( \mathbf{P}^T \) be the distribution of \( \mathcal{W} \) and \( T + \mathcal{W} \), respectively, defined by

\[
\mathbf{P}^T(B) := \mathbf{P}(B - T), \quad \forall B \in \mathcal{B}(\mathcal{C}(\Omega)).
\]

Let \( \mathcal{H}_W \) be the RKHS of \( \mathcal{W} \). Then \( T \in \mathcal{H}_W \), if and only if \( \mathbf{P}^T \) is absolutely continuous with respect to \( \mathbf{P} \). Moreover, if \( T \in \mathcal{H}_W \), then the density of \( \mathbf{P}^T \) with respect to \( \mathbf{P} \) is given by

\[
d\mathbf{P}^T(w) = \exp \left\{ L(T, w) - \frac{1}{2} \| T \|^2_{\mathcal{H}_W} \right\}, \quad \text{for almost all } w \in \mathcal{C}(\Omega),
\]

where \( L(T, w) \) is a bilinear form satisfies the condition

\[
\text{Cov} \left( L(T_1, \mathcal{W}), L(T_2, \mathcal{W}) \right) = \langle T_1, T_2 \rangle_{\mathcal{H}_W}.
\]

**Proof:** The proof is referred to Theorem 5.1 in [15]. See also Bass [14] and Janssen A and Ünlü [20] for further references.

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