A COUPLED $p$-LAPLACIAN ELLIPTIC SYSTEM: EXISTENCE, UNIQUENESS AND ASYMPTOTIC BEHAVIOR

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Abstract. We prove uniqueness, existence and asymptotic behavior of positive solutions to the system coupled by $p$-Laplacian elliptic equations

$$\begin{cases}
-\Delta_p z_1 = \lambda_1 g_1 (z_2) \text{ in } \Omega, \\
-\Delta_p z_2 = \lambda_2 g_2 (z_1) \text{ in } \Omega, \\
z_1 = z_2 = 0 \text{ on } \partial \Omega,
\end{cases}$$

where $\Delta_p u = \text{div}(|\nabla u|^{p-2} \nabla u)$, $1 < p < \infty$, $\lambda_1$ and $\lambda_2$ are positive parameters, $\Omega$ is the open unit ball in $\mathbb{R}^N$, $N \geq 2$.

1. Introduction. It is well known that $p$-Laplace equations are quasilinear equations when $p \neq 2$(see [22], [12]), and there are many important applications in physics, game theory and image processing (see [14], [5]). In the past few decades, a good many of results have been developed for single $p$-Laplace equations by different methods, for instance, see [21, 23, 19, 18] and the references cited therein. Specially, in [24], Zhang and Li considered the following $p$-Laplacian equation

$$\begin{cases}
-\Delta_p u = g(u) \text{ in } \Omega, \\
u = 0 \text{ on } \partial \Omega,
\end{cases}$$

(1)

where $\Delta_p u = \text{div}(|\nabla u|^{p-2} \nabla u)$ is the $p$-Laplacian operator, $N < p < \infty$, $\Omega$ is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 1$. The authors applied differential equations theory in Banach spaces and dynamics theory to study problem (1), and obtained excellent multiple solutions and sign-changing solutions theorems of $p$-Laplacian.

At the same time, we notice that many authors have paid more attention to existence and uniqueness problems, for example, see Castro, Sankar and Shivaji [2], Lin [13], Hai [10], and Guo [6]. Specially, Guo and Webb [7] considered the following $p$-Laplacian equation

$$\begin{cases}
\Delta_p u = -\lambda f(u) \text{ in } \Omega, \\
u = 0 \text{ on } \partial \Omega.
\end{cases}$$

(2)

They obtained existence and uniqueness results to problem (2) for large $\lambda$ if $f \geq 0$, $(f(x)/x^{p-1})' < 0$ for $x > 0$ and $f$ satisfies some $p$-sublinearity conditions at $\infty$.

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and 0. In [11], using sub-supersolution method together with sharp estimates near the boundary, Hai and Shivaji improved the results of (2) in a unit ball under much weaker assumptions than in [7]. Recently, Shivaji, Sim and Son [20], and Chu, Hai and Shivaji [3] generalize the study in [7] from a bounded domain to the exterior domains and obtained some excellent results.

Moreover, we notice that various of system problems have become an important area of investigation in recent years. To identify a few, we refer the reader to [4, 15, 16, 17]. In [10], Hai considered the existence and uniqueness of positive area of investigation in recent years. To identify a few, we refer the reader to

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Weaker assumptions than in [7]. Recently, Shivaji, Sim and Son [20], and Chu, Hai and Son [3] generalize the study in [7] from a bounded domain to the exterior domains and obtained some excellent results.
Lemma 2.1. Let $p > 1, q > 1$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Then $\varphi_p(s) = |s|^{p-2}s$ is odd, and $s\varphi_p(s) > 0$ if $s \neq 0$, $\varphi_p(st) = \varphi_p(s)\varphi_p(t)$, $\varphi_p(0) = 0$, $\varphi_p(1) = 1$, $\varphi_p(-1) = -1$,

$$
\varphi_p(s + t) = \begin{cases} 
2^{p-1}(\varphi_p(s) + \varphi_p(t)), & \text{if } p \geq 2, \ s, t > 0, \\
\varphi_p(s) + \varphi_p(t), & \text{if } 1 < p < 2, \ s, t > 0.
\end{cases}
$$

On the other hand, $\varphi_p(s)$ is increasing on $[0, \infty)$, and for $a \geq 0$, $\varphi_p(s^a) = \varphi_p^a(s)$ on $[0, \infty)$.

Next we mainly analyze the existence of positive solutions for system (4). In order to get our theorems, we let $\mathbb{R}_+ = [0, +\infty)$ and $g_i$ satisfy (C_0) $g_1$ and $g_2 : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous.

(C_1) $g_1$ and $g_2 : \mathbb{R}_+ \to \mathbb{R}_+$ are nondecreasing, $C^1$ on $(0, \infty)$ and

$$
\lim_{x \to 0^+} \sup \, s g_1'(x) < \infty, \quad \lim_{x \to 0^+} \sup \, s^2 g_2'(x) < \infty.
$$

(C_2) There exist nonnegative numbers $a, b, A, D$, where $ab < 1$ and $A, D > 0$ such that

$$
\lim_{x \to 0^+} \inf \frac{g_1(x)}{\varphi_p(x^a)} > 0, \quad \lim_{x \to 0^+} \inf \frac{g_2(x)}{\varphi_p(x^b)} > 0,
$$

$$
\lim_{x \to \infty} \frac{g_1(x)}{\varphi_p(x^a)} = A, \quad \lim_{x \to \infty} \frac{g_2(x)}{\varphi_p(x^b)} = D
$$

and for $a_1 > a$ and $b_1 > b$,

$$
\frac{g_1(x)}{\varphi_p(x^{a_1})} \quad \text{and} \quad \frac{g_2(x)}{\varphi_p(x^{b_1})}
$$

are nonincreasing for $x$ large.

A pair of functions $u, v \in C[0, 1] \cap C^1(0, 1)$ with $\phi_p(u'), \phi_p(v') \in C^1(0, 1)$ is called to be a positive solution of (4) if $u(t), v(t) > 0$ for all $t \in (0, 1)$, and $u$ and $v$ satisfy (4).

Let

$$
E = C([0, 1], \mathbb{R}) \times C([0, 1], \mathbb{R}).
$$

Then $E$ is a Banach space with the norm $\|(u, v)\| = \max\{\|u\|_\infty, \|v\|_\infty\}$, where $\|u\|_\infty = \max_{t \in [0, 1]} |u(t)|$.

Define a cone $P$ by

$$
P = \{(u, v) \in E : u \geq 0, v \geq 0\}.
$$

Define an operator $F : E \to E$ by

$$
F(u, v)(t) = (A(u, v)(t), B(u, v)(t)), \quad t \in [0, 1],
$$

where

$$
A(u, v)(t) = \int_t^1 \varphi_q\left(\frac{1}{s^{N-1}}\int_0^s \lambda_1 \tau^{N-1} g_1(v(\tau))d\tau\right)ds,
$$

and

$$
B(u, v)(t) = \int_t^1 \varphi_q\left(\frac{1}{s^{N-1}}\int_0^s \lambda_2 \tau^{N-1} g_2(u(\tau))d\tau\right)ds.
$$

It is easy to check that $F : P \to P$ is completely continuous and the solution of system (4) is equivalent to the fixed point equation

$$
F(u, v) = (u, v).
$$

Therefore, the task of the present paper is to search nonzero fixed points of $F$.

The following well-known results are crucial in the proofs of our results.
Lemma 2.2. (See Lemma 2.4 of [9], on page 131) Let $P$ be a cone in a Banach space $E$ and $T: P \to P$ be a completely continuous mapping satisfying

(a) There exist $k \in K$, $\|k\| = 1$, and a number $r > 0$ such that all solutions $y \in P$ of

$$y = Ty + \theta k, \quad 0 < \theta < \infty$$

satisfy $\|y\| \neq r$.

(b) There exists $R > r$ such that all solutions $z \in P$ of

$$z = \theta Tz, \quad 0 < \theta < 1.$$ 

Then $T$ has a fixed point $x \in P$, $r \leq \|x\| \leq R$.

3. Uniqueness of positive solution. In this section, we analyze the uniqueness of fixed point of $F$ for $\lambda_1 \lambda_2^a$ and $\lambda_1^b \lambda_2$ sufficiently large.

Lemma 3.1. Let $h$ be continuous on $\mathbb{R}_+$ and $C^1$ on $(0, \infty)$ such that

$$\lim_{x \to 0^+} \sup_{x \in [0, \infty)} x h'(x) < \infty.$$ 

Let $M$, $\varepsilon$, $r$ be positive numbers with $\varepsilon < 1$. Then there is a positive constant $C$ such that

$$|h(\gamma x) - \varphi_p(\gamma^r)h(x)| \leq C(1 - \gamma)$$

for $\varepsilon \leq \gamma < 1$ and $0 \leq x \leq M$.

Proof. Let $0 \leq x \leq M$. Define $H(\gamma) = h(\gamma x) - \varphi_p(\gamma^r)h(x)$, $\varepsilon \leq \gamma < 1$. Using the mean value theorem, there is a $c \in (\gamma, 1)$ such that

$$|H(\gamma)| = |H(\gamma) - H(1)| = (1 - \gamma)|xh'(cx) - r(p - 1)c^{p - 1}(p - 1) - 1)h(x)| \leq C(1 - \gamma),$$

where

$$C = \sup\{xyh'(y) : 0 < y \leq M\} + r(p - 1)\max(\varepsilon^{p - 1}, 1)\sup\{|h(y) : y \leq M\}.$$ 

Next we will check the upper and lower estimates for possible positive solutions of system (4).

Lemma 3.2. Let $(u, v)$ be a positive solution of (4). Then there exist positive constants $M_i$, $i \in \{1, 2, 3, 4\}$ and $M$ independent of $u, v$ such that

$$M_1(\varphi_q(\lambda_1 \lambda_2^a))^{\frac{1}{\gamma_1 \gamma_2}} (1 - t) \leq u(t) \leq M_2(\varphi_q(\lambda_1 \lambda_2^a))^{\frac{1}{\gamma_1 \gamma_2}} (1 - t), \quad 0 < t < 1,$$

$$M_3(\varphi_q(\lambda_2 \lambda_1^b))^{\frac{1}{\gamma_1 \gamma_2}} (1 - t) \leq v(t) \leq M_4(\varphi_q(\lambda_2 \lambda_1^b))^{\frac{1}{\gamma_1 \gamma_2}} (1 - t), \quad 0 < t < 1$$

for $\min\{\lambda_1 \lambda_2^a, \lambda_2 \lambda_1^b\} \geq M$.

Proof. Suppose that $u$ and $v$ are a pair of positive solutions for system (4). By integrating, we get

$$u(t) = \int_t^1 \varphi_q\left(\frac{1}{s^{N-1}} \int_0^s \lambda_1 \tau^{N-1} g_1(v(\tau))d\tau\right)ds,$$

$$v(t) = \int_t^1 \varphi_q\left(\frac{1}{s^{N-1}} \int_0^s \lambda_2 \tau^{N-1} g_2(u(\tau))d\tau\right)ds.$$
Next, we can denote by $c_i$, $i = 1, 2, \ldots$, positive constants independent of $u, v, \lambda_1, \lambda_2$. Since $v$ is decreasing, we have
\[
u(\frac{1}{2}) \geq \int_{\frac{1}{2}}^{1} \varphi_q\left(\frac{1}{s^{N-1}} \int_{0}^{\frac{1}{2}} \lambda_1 s^{N-1} g_1(v(\tau)) d\tau\right) ds
\]

Similarly we can get
\[
u(\frac{1}{2}) \geq \frac{1}{2} \varphi_q\left(\frac{\lambda_2}{N^{2N}}\right) \varphi_q\left(g_2(\frac{1}{2})\right). \tag{6}
\]

By (C2), there are two positive constants $K_1$ and $K_2$ such that
\[
g_1(x) \geq \varphi_p(K_1 x^a), \quad g_2(x) \geq \varphi_p(K_2 x^b).
\]

This together with (5) and (6), shows that
\[
u(\frac{1}{2}) \geq \frac{1}{2} \varphi_q\left(\frac{\lambda_2}{N^{2N}}\right) K_1 \left[\frac{1}{2} \varphi_q\left(\frac{\lambda_2}{N^{2N}}\right)\right]^a K_2^b (u(\frac{1}{2}))^a = c_1 \varphi_q(\lambda_1 \lambda_2) (u(\frac{1}{2}))^a.
\]

Thus,
\[
u(\frac{1}{2}) \geq (c_1) \frac{1}{1+a} \varphi_q(\lambda_1 \lambda_2) \frac{1}{1+a} = c_2 \varphi_q(\lambda_1 \lambda_2) \frac{1}{1+a}. \tag{7}
\]

Similarly, we can get
\[
u(\frac{1}{2}) \geq (c_2) \frac{1}{1+b} \varphi_q(\lambda_2 \lambda_1) \frac{1}{1+b} = c_3 \varphi_q(\lambda_1 \lambda_2) \frac{1}{1+b}. \tag{8}
\]

It follows from (7), (8) and (9) that for $t \geq \frac{1}{2}$
\[
-u'(t) = \varphi_q\left(\frac{\lambda_1}{t^{N-1}} \int_{0}^{t} s^{N-1} g_1(v) ds\right)
\]

Then by integrating, for $t \geq \frac{1}{2}$, we get
\[
u(t) \geq c_3 \varphi_q(\lambda_1 \lambda_2) \frac{1}{1+a} (1-t). \tag{10}
\]

Similarly,
\[
u(t) \geq c_4 \varphi_q(\lambda_2 \lambda_1) \frac{1}{1+b} (1-t). \tag{11}
\]
Because of $u,v$ being decreasing, this shows the left-side inequalities for $u,v$ in Lemma 3.2.

According to the formulas for $u,v$, we can see that

$$u(t) \leq |u|_{\infty}$$

$$\leq \int_{0}^{1} \varphi_{\psi}(\frac{1}{s^{\lambda_{2}-1}} \int_{0}^{1} \int_{0}^{s} \lambda_{1} \tau^{N-1} g_{1}(|v|_{\infty})d\tau)ds$$

$$\leq \int_{0}^{1} \varphi_{\psi}(\frac{1}{s^{\lambda_{2}-1}} \int_{0}^{1} \lambda_{1} \tau^{N-1} g_{1}(|v|_{\infty})d\tau)ds$$

$$\leq \varphi_{\psi}(\lambda_{1} g_{1}(|v|_{\infty}))$$

(12)

and

$$|v|_{\infty} \leq \varphi_{\psi}(\lambda_{2} g_{2}(|u|_{\infty})).$$

(13)

By (7) and (11), for large $\lambda_{1} \lambda_{2}^{a}$ and $\lambda_{2} \lambda_{1}^{a}$, we get

$$|v(t)| \geq c_{4}(\varphi_{\psi}(\lambda_{1} \lambda_{2}^{b}))^{\frac{1}{1-a}}(1-t) \gg 1 \quad (i.e. \; |v|_{\infty} \text{ is large})$$

(14)

and

$$\lambda_{1} g_{1}(|v|_{\infty}) \geq \lambda_{1} \varphi_{p}(K_{1} |v|_{\infty}^{a})$$

$$\geq \lambda_{1} \varphi_{p}(K_{1} [c_{4}(\varphi_{\psi}(\lambda_{2} \lambda_{1}^{b}))^{\frac{1}{1-a}}]^{a})$$

$$\geq \lambda_{1} \varphi_{p}(K_{1} [c_{4}(\varphi_{\psi}(\lambda_{1} \lambda_{2}^{a}))^{\frac{1}{1-a}}])$$

$$\gg 1.$$

Note that from (C_{2}) it follows that

$$|v|_{\infty} \leq \varphi_{\psi}(\lambda_{2} g_{2}(|u|_{\infty}))$$

$$\leq \varphi_{\psi}(\lambda_{2} g_{2}(\varphi_{\psi}(\lambda_{1} g_{1}(|v|_{\infty}))))$$

$$\leq \varphi_{\psi}(\lambda_{2} g_{2}(\varphi_{\psi}(\lambda_{1} K_{1} |v|_{\infty}^{a}))$$

$$\leq \varphi_{\psi}(\lambda_{2} K_{2} \varphi_{p}((\varphi_{\psi}(\lambda_{1} K_{1}))^{b}(|v|_{\infty})^{ab}))$$

$$= \varphi_{\psi}(K_{1}^{b} K_{2} \varphi_{p}(\lambda_{2} \lambda_{1}^{b})(|v|_{\infty})^{ab})$$

$$= c_{5} \varphi_{\psi}(\lambda_{2} \lambda_{1}^{b})(|v|_{\infty})^{ab},$$

or

$$|v|_{\infty} \leq c_{5}^{\frac{1}{1-\alpha}}(\varphi_{\psi}(\lambda_{2} \lambda_{1}^{b}))^{\frac{1}{1-\alpha}}.$$  

(15)

By combining the equation of $u'$ and (15), we can get

$$-u'(t) \leq \varphi_{\psi}(\frac{\lambda_{1}}{t^{N-1}} \int_{0}^{t} s^{N-1} g_{1}(|v|_{\infty})ds)$$

$$\leq \varphi_{\psi}(\lambda_{1} g_{1}(|v|_{\infty}))$$

$$\leq \varphi_{\psi}(\lambda_{1} f(c_{5}^{\frac{1}{1-a}}(\varphi_{\psi}(\lambda_{2} \lambda_{1}^{b}))^{\frac{1}{1-\alpha}}))$$

$$\leq \varphi_{\psi}(\lambda_{1} K_{1} c_{5}^{\frac{1}{1-\alpha}}(\varphi_{\psi}(\lambda_{1} \lambda_{2}^{a}))^{\frac{1}{1-a}})$$

$$= c_{6}(\varphi_{\psi}(\lambda_{1} \lambda_{2}^{a}))^{\frac{1}{1-a}},$$

(16)

and then it follows from integrating that

$$u(t) \leq c_{6}(\varphi_{\psi}(\lambda_{1} \lambda_{2}^{a}))^{\frac{1}{1-a}}(1-t), \quad 0 < t < 1.$$
Similarly, we can get the upper estimate for \( v(t) \). This completes the proof. \( \square \)

**Theorem 3.3.** Assume \( (C_0) \) -- \( (C_2) \) hold. Then there is a constant \( \beta > 0 \) such that the system (4) admits a unique positive solution for \( \min(\lambda_1\lambda_2^2, \lambda_2\lambda_1^2) \geq \beta \).

**Proof.** We shall check the conditions of Lemma 2.2 to prove the existence of solution. Let \((u, v) \in P\) satisfy
\[
(u, v) = F(u, v) + \theta(1, 1)
\]
for some \( \theta > 0 \). Because of \( u, v \) are nonincreasing and \( u, v > 0 \) on \((0, 1)\), the proof is analogous to that of Lemma 3.2 that
\[
\varphi(\frac{1}{2}) \geq c_2(\varphi_2(\lambda_1\lambda_2^3))^{\frac{1}{1-\alpha}}.
\]
So, \( \| (u, v) \| \neq r \), where \( 0 < r < c_2(\varphi_2(\lambda_1\lambda_2^3))^{\frac{1}{1-\alpha}} \).

Next, set \((u, v) \in P\) with
\[
(u, v) = \theta F(u, v)
\]
for some \( \theta \in (0, 1) \). Then, by (12) and (13) we get
\[
|v| \leq \varphi(\lambda_2^2g_2(\varphi_2(\lambda_1g_1(|v|_{\infty})))) \quad |u| \leq \varphi(\lambda_1g_1(\varphi_2(\lambda_2g_2(|u|_{\infty}))))
\]
In addition, if \( |v|_{\infty} \to \infty \), by \( (C_2) \) and \( ab < 1 \) we can see that
\[
1 \leq \frac{\varphi(\lambda_2g_2(\varphi_2(\lambda_1g_1(|v|_{\infty}))))}{|v|_{\infty}} \leq c_2\varphi_2(\lambda_1\lambda_2^3)\lambda^{ab} = 0,
\]
which is impossible. So, there is a number \( R > r \) such that \( \| (u, v) \| \neq R \). Therefore, Lemma 2.2 shows that \( F \) has a fixed point \((u, v) \) with \( r \leq \| (u, v) \| \leq R \). Thus, it follows that (4) admits one positive solution, and the existence is proved.

Next, we shall show that the solution is unique. Suppose that \((u, v)\) and \((u_1, v_2)\) are positive solutions of system (4) and let \( \min\{\lambda_1\lambda_2^2, \lambda_2\lambda_1^2\} \) be large enough such that Lemma 3.2 holds. It follows from Lemma 3.2 that
\[
\frac{M_1}{M_2}u_1 \leq u \leq \frac{M_2}{M_1}u_1 \quad \text{on} \quad (0, 1).
\]
Let \( \alpha = \sup\{d > 0 : u \geq du_1 \in (0, 1)\} \). Then obviously \( d_0 \leq \alpha \leq \infty \) and \( u \geq \alpha u_1 \in (0, 1) \), where \( \alpha_0 = \frac{M_1}{M_2} \). We assert that \( \alpha \geq 1 \). In fact, we can assume by contradiction that \( \alpha < 1 \). Since \( u, v \) are decreasing and
\[
(t^{N-1}\varphi_p(u'))' = -\lambda_1t^{N-1}g_1(\int_t^1\varphi_\lambda(\frac{1}{s^{N-1}}\int_0^s\varphi_\lambda(\lambda_2s^{N-1-1}g_2(\varphi_\lambda(\lambda_1g_1(|u|_{\infty}))))d\tau)ds),
\]
\[
(t^{N-1}\varphi_p(\alpha u_1'))' = -\lambda_1t^{N-1}\varphi_p(\alpha)(\int_t^1\varphi_\lambda(\frac{1}{s^{N-1}}\int_0^s\varphi_\lambda(\lambda_2s^{N-1-1}g_2(\varphi_\lambda(\lambda_1g_1(|u|_{\infty}))))d\tau)ds),
\]
we can get
\[
[t^{N-1}(\varphi_p(u') - \varphi_p(\alpha u_1'))]' \leq -\lambda_1t^{N-1}|g_1(\int_t^1\varphi_\lambda(\frac{\lambda_2}{s^{N-1}}\int_0^s\varphi_\lambda(\lambda_2s^{N-1-1}g_2(\varphi_\lambda(\lambda_1g_1(|u|_{\infty}))))d\tau)ds)\varphi_p(\alpha)g_1(\int_t^1\varphi_\lambda(\frac{\lambda_2}{s^{N-1}}\int_0^s\varphi_\lambda(\lambda_2s^{N-1-1}g_2(\varphi_\lambda(\lambda_1g_1(|u|_{\infty}))))d\tau)ds)].
\]
Let \( b_1 > b_2 > b \), \( a_1 > a \) and \( a_1b_1 < 1 \). Then we assert that
\[
\int_0^s\tau^{N-1}g_2(\varphi_\lambda(\lambda_1g_1(|u|_{\infty}))))d\tau \geq \varphi_p(a)\int_0^s\tau^{N-1}g_2(\varphi_\lambda(\lambda_1g_1(|u_1|_{\infty}))))d\tau.
\]
According to $\frac{g_2(x)}{\varphi_p(x^{b_2})}$ is nonincreasing for $x \gg 1$ and $\alpha \geq \alpha_0$, we can get

$$\frac{g_2(\alpha x)}{\varphi_p((\alpha x)^{b_2})} \geq \frac{g_2(x)}{\varphi_p(x^{b_2})},$$

or

$$g_2(\alpha x) \geq \varphi_p(\alpha^{b_2})g_2(x).$$

By Lemma 3.2, we can get

$$u(r) \geq M_1(\varphi_q(\lambda_1^{b_2}))^{-\frac{1}{\lambda_1-\alpha}}(1-T) \gg 1,$$

where $T \in (\frac{1}{T}, 1)$.

Since $\alpha < 1$, then for $s \leq T$, we have

$$\int_0^s \tau^{N-1}(g_2(\alpha u) - \varphi_p(\alpha^{b_1})g_2(u))d\tau \geq (\varphi_p(\alpha^{b_2}) - \varphi_p(\alpha^{b_1})) \int_0^s \tau^{N-1}g_2(u)\,d\tau \geq 0.$$

For $s > T$, combined with Lemma 3.1, we have

$$\int_0^T \tau^{N-1}(g_2(\alpha u) - \varphi_p(\alpha^{b_1})g_2(u))d\tau = \int_0^T \tau^{N-1}(g_2(\alpha u) - \varphi_p(\alpha^{b_1})g_2(u))d\tau + \int_T^s \tau^{N-1}(g_2(\alpha u) - \varphi_p(\alpha^{b_1})g_2(u))d\tau
\geq (\varphi_p(\alpha^{b_2}) - \varphi_p(\alpha^{b_1})) \int_0^T \tau^{N-1}g_2(u)\,d\tau - C(1-T)(1-\alpha).$$

Because

$$\int_0^T \tau^{N-1}g_2(u)\,d\tau \geq \int_0^{\frac{1}{2}} \tau^{N-1}g_2(u)\,d\tau \geq \frac{g_2(u(\frac{1}{2}))}{N2^N} \geq \frac{K_2}{N2^N}$$

and since there is a positive number $l > 0$ such that

$$(\varphi_p(\alpha^{b_2}) - \varphi_p(\alpha^{b_1})) \geq l(1-\alpha^{p-1}) \text{ for } \alpha_0 \leq \alpha \leq 1.$$
4. New existence results. In this section, we will establish some new existence results of positive solutions for system (4). To achieve this goal, we will define a new cone $\tilde{P}$ and a composite operator $T$.

Lemma 4.1. (See Theorem 2.3.6 of [8], on page 99) Suppose that $D$ is an open subset of the infinite-dimensional real Banach space $E$, $\theta \in D$, and $P$ is a cone of $E$. If the operator $\Gamma : P \cap D \to P$ is completely continuous with $\Gamma \theta = \theta$ and satisfies

$$\inf_{x \in P \cap \partial D} \Gamma x > 0,$$

then all conditions of Theorem 2.3.6 are satisfied and $\tilde{P}$ is a cone in $E$.

Theorem 4.2. Suppose that $\Gamma$ is a completely continuous operator and $\tilde{P}$ is a cone in $E$. If $\Gamma \theta = \theta$ and $\tilde{P}$ is a cone, then $\Gamma \tilde{P}$ is a cone for any $\theta \in \tilde{P}$.

Proof. It follows from the definition of $\Gamma$ and the properties of $\tilde{P}$ that $\Gamma \tilde{P} = \tilde{P}$. Therefore, $\Gamma \tilde{P}$ is a cone.

Finally, we prove the existence of positive solutions for system (4). To achieve this, we will define a new cone $\tilde{P}$ and a composite operator $T$. This shows that there is a constant $\alpha > 0$ in $(0, 1)$ such that $u \geq \alpha u_1$, which is a contradiction. Thus $\alpha \geq 1$ and hence $u = u_1$ in $(0, 1)$. Similarly, we can verify $v = v_1$ in $(0, 1)$ and so we finish the proof of Theorem 3.3.
then $\Gamma$ has a proper element on $P \cap \partial D$ associated with a positive eigenvalue. That is, there exist $x_0 \in P \cap \partial D$ and $\mu_0 > 0$ such that $\Gamma x_0 = \mu_0 x_0$.

Let $E = C[0,1]$. Then $E$ is a real Banach space with the norm $\| \cdot \|$ defined by

$$\|x\| = \max_{t \in J} |x(t)|.$$ 

Let $J = [0,1]$ and $P$ be the cone

$$P := \left\{ v \in E : v(t) \geq 0, \ t \in J, \ v(t) \geq \frac{1}{4} \| v \|, \ t \in \left[ \frac{1}{4}, \frac{3}{4} \right] \right\}. \quad (20)$$

It is easy to see that $P$ is a normal cone of $E$.

For $v \in P$, define $T_i : P \to E (i = 1, 2)$ as

$$(T_1 v)(t) = \int_t^1 \varphi_t \left( \frac{1}{N - 1} \int_0^s g_1(v(s))ds \right) d\tau, \quad (21)$$

$$(T_2 v)(t) = \varphi_t(\lambda_2) \int_t^1 \varphi_t \left( \frac{1}{N - 1} \int_0^s g_2(v(s))ds \right) d\tau. \quad (22)$$

It follows from Lemma 3 in [1] that $T_i (i = 1, 2)$ maps $P$ into itself. Moreover, $T_1$ and $T_2$ are completely continuous by standard arguments.

Define a composite operator $T = T_1 T_2$, which is also completely continuous from $P$ to itself. So the operator $T$ also maps $P$ into $P$. Therefore the next task of this paper is to search nonzero fixed points of operator $T$.

Let

$$g_1^\infty := \lim_{v \to \infty} \frac{g_1(v)}{\varphi_p(v)}, \quad g_0^1 := \lim_{v \to 0} \frac{g_1(v)}{\varphi_p(v)};$$

$$g_2^\infty := \lim_{v \to \infty} \frac{g_2(v)}{\varphi_p(v)}, \quad g_0^2 := \lim_{v \to 0} \frac{g_2(v)}{\varphi_p(v)},$$

and

$$A = \int_{\frac{3}{4}}^1 s^{-1} ds = \frac{3N - 1}{N 4^N}, \quad B = \int_{\frac{3}{4}}^1 \varphi_t \left( \frac{1}{N - 1} \right) d\tau, \quad B^* = \int_0^1 \varphi_t \left( \frac{1}{N - 1} \right) d\tau. \quad (23)$$

**Theorem 4.2.** Suppose that (C3) holds. If $0 < g_i^\infty < +\infty (i = 1, 2)$, then there exists $\beta_0 > 0$ such that, for every $R > \beta_0$, system (4) admits a pair of positive solutions $u_R, v_R$ satisfying $\| u_R \| = R$ for any

$$\lambda_1 R \lambda_2 \in [\lambda_R, \tilde{\lambda}_R],$$

where $\lambda_R$ and $\tilde{\lambda}_R$ are positive finite numbers.

**Proof.** Since $0 < g_i^\infty < +\infty$, there exist $0 < l_1 < l_2, \mu > 0$ so that

$$l_1 \varphi_p(v) < g_1(v) < l_2 \varphi_p(v), \quad \forall v \geq \mu;$$

$$l_1 \varphi_p(u) < g_2(u) < l_2 \varphi_p(u), \quad \forall u \geq \mu.$$

Next, we verify that $\beta_0 = 4\mu$ is required. Letting

$$\Omega_R = \{ x \in E : \| x \| < R \},$$

then $0 \in \Omega_R$ and $\Omega_R$ is a bounded open subset of Banach space $E$.

Since $R > \beta_0$, for any $u, v \in P \cap \partial \Omega_R$, we get

$$u(t) \geq \frac{1}{4} \| u \| = \frac{1}{4} R, \quad v(t) \geq \frac{1}{4} \| v \| = \frac{1}{4} R, \quad t \in \left[ \frac{1}{4}, \frac{3}{4} \right].$$
and
\[ u(t) \geq \frac{1}{4} \|u\| > \frac{1}{4} \beta_0 = \mu, \ v(t) \geq \frac{1}{4} \|v\| > \frac{1}{4} \beta_0 = \mu, \ t \in \left[ \frac{1}{4}, \frac{3}{4} \right]. \]

So, for any \( v \in P \cap \partial \Omega_R \), we have
\[
(T_1v)(t) \geq \int_0^1 \phi_q \left( \frac{1}{\tau^{N-1}} \right) \left( \frac{1}{4} \right)^{N-1} g_1(v(s)) ds d\tau \]
\[
\geq \int_0^1 \phi_q \left( \frac{1}{\tau^{N-1}} \right) \left( \frac{1}{4} \right)^{N-1} g_2(u(s)) ds d\tau \]
\[
\geq \int_0^1 \phi_q \left( \frac{1}{\tau^{N-1}} \right) \left( \frac{1}{4} \right)^{N-1} l_1 \varphi_p(u(s)) ds d\tau \]
\[
\geq \int_0^1 \phi_q \left( \frac{1}{\tau^{N-1}} \right) \left( \frac{1}{4} \right)^{N-1} l_1 \varphi_p \left( \frac{1}{4} \|v\| \right) ds d\tau \]
\[
= \frac{1}{4} \|v\| \phi_q(l_1 A) B, \ \forall t \in J. \]

Analogously, for \( u \in P \cap \partial \Omega_R \), we obtain
\[
(T_2u)(t) \geq \int_0^1 \phi_q \left( \frac{1}{\tau^{N-1}} \right) \left( \frac{1}{4} \right)^{N-1} g_2(u(s)) ds d\tau \]
\[
\geq \int_0^1 \phi_q \left( \frac{1}{\tau^{N-1}} \right) \left( \frac{1}{4} \right)^{N-1} g_2(u(s)) ds d\tau \]
\[
\geq \int_0^1 \phi_q \left( \frac{1}{\tau^{N-1}} \right) \left( \frac{1}{4} \right)^{N-1} l_1 \varphi_p(u(s)) ds d\tau \]
\[
\geq \int_0^1 \phi_q \left( \frac{1}{\tau^{N-1}} \right) \left( \frac{1}{4} \right)^{N-1} l_1 \varphi_p \left( \frac{1}{4} \|u\| \right) ds d\tau \]
\[
= \frac{1}{4} \|u\| \phi_q(l_1 A) B, \ \forall t \in J. \]

Therefore, we get
\[
(Tu)(t) = (T_1 T_2u)(t) \]
\[
\geq \frac{1}{4} \|T_2u\| \phi_q(l_1 A) B \]
\[
\geq \frac{1}{16} \|u\| \left( \phi_q(l_1 A) B \right)^2. \]

This gives that
\[
\inf_{u \in P \cap \partial \Omega_R} Tu \geq \frac{1}{16} \|u\| \left( \phi_q(l_1 A) B \right)^2 > 0. \]

For any \( R > \beta_0 \), Lemma 4.1 yields that operator \( T \) admits a proper element \( u_R \in P \) associated with the eigenvalue \( \mu_{1R} > 0 \), and \( u_R \) satisfies \( \|u_R\| = R \).

For operator \( T \), we can denote \( v_R = T_2u_R \), then \( u_R \) and \( v_R \) are the solutions of system (4).

Let \( \lambda_{1R} = \frac{1}{\varphi_p(l_{1R})} \). Then we get
\[
Tu_R = \mu_{1R} u_R = \frac{1}{\varphi_q(l_{1R})} u_R. \quad (24) \]
It follows from the proof above that, for any $R > \beta_0$, system \((4)\) has a pair of positive solutions $u_R$ and $v_R$ with $u_R \in P \cap \partial \Omega_R$ associated with $\lambda_1 = \lambda_{1R} > 0$. Thus, by (24) we get
\[
   u_R(t) = \varphi_q(\lambda_{1R}) Tu_R,
\]
and so
\[
   u_R(t) = \varphi_q(\lambda_{1R}) \int_t^1 \varphi_q\left(\frac{1}{\tau^{N-1}}\right) \int_0^\tau s^{N-1} g_1(v_R(s)) ds \, d\tau,
\]
\[
   v_R(t) = \varphi_q(\lambda_{2R}) \int_t^1 \varphi_q\left(\frac{1}{\tau^{N-1}}\right) \int_0^\tau s^{N-1} g_2(u_R(s)) ds \, d\tau
\]
with $\|u_R\| = R$.

On the one hand,
\[
   u_R(t) \geq \varphi_q(\lambda_{1R}) \int_t^1 \varphi_q\left(\frac{1}{\tau^{N-1}}\right) \int_0^\tau s^{N-1} g_1(v_R(s)) ds \, d\tau
\]
\[
   \geq \varphi_q(\lambda_{1R}) \int_t^1 \varphi_q\left(\frac{1}{\tau^{N-1}}\right) \int_0^\tau s^{N-1} g_1(1) ds \, d\tau
\]
\[
   \geq \varphi_q(\lambda_{1R}) \int_t^1 \varphi_q\left(\frac{1}{\tau^{N-1}}\right) \int_0^\tau s^{N-1} - \lambda_1 \varphi_p(v_R(s)) ds \, d\tau
\]
\[
   \geq \varphi_q(\lambda_{1R}) \int_t^1 \varphi_q\left(\frac{1}{\tau^{N-1}}\right) \int_0^\tau s^{N-1} - \lambda_1 \varphi_p(\frac{4}{3} v_R(t)) ds \, d\tau
\]
\[
   = \frac{1}{4} \varphi_q(\lambda_{1R} \lambda_{1} A) B \|v_R\|, \ \forall t \in J.
\]

Analogously, we can show that
\[
   (v_R(t) \geq \frac{1}{4} \varphi_q(\lambda_{2R} A) B \|u_R\|, \ \forall t \in J.
\]

Therefore, we get
\[
   \|u_R\| \geq \frac{1}{16} \varphi_q(\lambda_{1R} \lambda_{1}^2 B^2 A^2) \|u_R\|,
\]
Suppose that

Theorem 4.3.

and so,

\[ \lambda_1 R \lambda_2 \leq \frac{\varphi_p(16)}{l_1^2 A^2 \varphi_p(B^2)} = \bar{\lambda}_R. \]  \tag{25}

We hence get \( \lambda_1 R \lambda_2 \in [\lambda_R, \bar{\lambda}_R] \). This gives the proof. \( \square \)

If we define another composite operator \( T^* = T^*_2 T^*_1 \), where

\[
(T^*_1 v)(t) = \varphi_q(\lambda_1) \int_t^1 \varphi_q(\frac{1}{t^{N-1}}) \int_0^t s^{N-1} g_1(v(s)) ds d\tau, 
\tag{26}
\]

\[
(T^*_2 v)(t) = \int_t^1 \varphi_q(\frac{1}{t^{N-1}}) \int_0^t s^{N-1} g_2(v(s)) ds d\tau. \tag{27}\]

Corollary 1. Let \( T^* = T^*_2 T^*_1 \). Suppose that \((C_0)\) holds. If \(0 < g_i^* < +\infty (i = 1, 2)\), then there exists \( \beta_0 > 0 \) such that, for every \( R > \beta_0 \), system \((4)\) admits a pair of positive solutions \( u_R, v_R \) satisfying \( \|v_R\| = R \) for any

\[ \lambda_1 \lambda_2 R \in [\lambda_R, \bar{\lambda}_R], \]  \tag{28}\n
where \( \lambda_R \) and \( \bar{\lambda}_R \) are positive finite numbers.

Proof. Similar to the proof of Theorem 4.2, we can prove Corollary 1. \( \square \)

Theorem 4.3. Suppose that \((C_0)\) holds. If \(0 < g_i^* < +\infty (i = 1, 2)\), then there exists \( \beta_0 > 0 \) such that, for every \( 0 < r < \beta_0 \), system \((4)\) admits a pair of positive solutions \( u_r, v_r \) satisfying \( \|u_r\| = r \) for any

\[ \lambda_1 r \lambda_2 \in [\lambda_r, \bar{\lambda}_r], \]  \tag{29}\n
where \( \lambda_r \) and \( \bar{\lambda}_r \) are positive finite numbers.

Proof. Similar to the proof of Theorem 4.2, we can prove Theorem 4.3. \( \square \)

Theorem 4.4. Suppose that \((C_0)\) holds. If \( g_i^* = +\infty (i = 1, 2) \), then there exists \( \beta_0 > 0 \) such that, for every \( R > \beta_0 \), system \((4)\) admits a pair of positive solutions \( u_{R*}, v_{R*} \) satisfying \( \|u_{R*}\| = R_* \) for any

\[ \lambda_1 R_* \lambda_2 \in (0, \lambda_{R*}], \]  \tag{29}\n
where \( \lambda_{R*} \) is a positive finite number.

Proof. Since \( g_i^* = +\infty \), there exist \( l_* > 0, \) \( \mu^* > 0 \) so that

\[ g_1(v) > l_* \varphi_p(v), \forall v \geq \mu^*; \]

\[ g_2(u) > l_* \varphi_p(u), \forall u \geq \mu^*. \]

Now, we show that \( \beta_0 = 4 \mu^* \) is required. Set

\[ \Omega_{R*} = \{ x \in E : \|x\| < R_* \}. \]

Since \( R_* > \beta_0 \), for any \( u, v \in P \cap \partial \Omega_{R*} \), we get

\[ u(t) \geq \frac{1}{4} \|u\| = \frac{1}{4} R_* \quad v(t) \geq \frac{1}{4} \|v\| = \frac{1}{4} R_* \quad t \in \left[ \frac{1}{4}, \frac{3}{4} \right], \]

and

\[ u(t) \geq \frac{1}{4} \|u\| > \frac{1}{4} \beta_0 = \mu^* \quad v(t) \geq \frac{1}{4} \|v\| > \frac{1}{4} \beta_0 = \mu^* \quad t \in \left[ \frac{1}{4}, \frac{3}{4} \right]. \]
So, for any \( v \in P \cap \partial \Omega_{R_*} \), we have
\[
(T_1 v)(t) \geq \int_0^1 \varphi_q \left( \frac{1}{\tau^{N-1}} \right) \int_0^{\frac{1}{4}} s^{N-1} g_1(v(s)) ds \, d\tau
\]
\[
\geq \int_0^1 \varphi_q \left( \frac{1}{\tau^{N-1}} \right) \int_0^{\frac{1}{4}} s^{N-1} g_2(u(s)) ds \, d\tau
\]
\[
\geq \int_0^1 \varphi_q \left( \frac{1}{\tau^{N-1}} \right) \int_0^{\frac{1}{4}} s^{N-1} l_* \varphi_p (u(s)) ds \, d\tau
\]
\[
\geq \int_0^1 \varphi_q \left( \frac{1}{\tau^{N-1}} \right) \int_0^{\frac{1}{4}} s^{N-1} l_* \varphi_p \left( \frac{1}{4} \| u \| \right) ds \, d\tau
\]
\[
= \frac{1}{4} \| u \| \varphi_q (l_* A) B, \forall t \in J.
\]

Analogously, for \( u \in P \cap \partial \Omega_{R_*} \), we obtain
\[
(T_2 u)(t) \geq \int_0^1 \varphi_q \left( \frac{1}{\tau^{N-1}} \right) \int_0^{\frac{1}{4}} s^{N-1} g_2(u(s)) ds \, d\tau
\]
\[
\geq \int_0^1 \varphi_q \left( \frac{1}{\tau^{N-1}} \right) \int_0^{\frac{1}{4}} s^{N-1} g_2(u(s)) ds \, d\tau
\]
\[
\geq \int_0^1 \varphi_q \left( \frac{1}{\tau^{N-1}} \right) \int_0^{\frac{1}{4}} s^{N-1} l_* \varphi_p (u(s)) ds \, d\tau
\]
\[
\geq \int_0^1 \varphi_q \left( \frac{1}{\tau^{N-1}} \right) \int_0^{\frac{1}{4}} s^{N-1} l_* \varphi_p \left( \frac{1}{4} \| u \| \right) ds \, d\tau
\]
\[
= \frac{1}{4} \| u \| \varphi_q (l_* A) B, \forall t \in J.
\]

Therefore, we get
\[
(T u)(t) = (T_1 T_2 u)(t)
\]
\[
\geq \frac{1}{4} \| T_2 u \| \varphi_q (l_* A) B
\]
\[
\geq \frac{1}{16} \| u \| (\varphi_q (l_* A) B)^2.
\]

This gives that
\[
\inf_{u \in P \cap \partial \Omega_{R_*}} Tu \geq \frac{1}{16} \| u \| (\varphi_q (l_* A) B)^2 > 0.
\]

For any \( R_* > \bar{\beta}_0 \), Lemma 4.1 yields that operator \( T \) admits a proper element \( u_{R_*} \in P \) associated with the eigenvalue \( \mu_{R_*} > 0 \), and \( u_{R_*} \) satisfies \( \| u_{R_*} \| = R_* \).

For operator \( T \), we denote \( v_{R_*} = T_2 u_{R_*} \), then \( u_{R_*} \) and \( v_{R_*} \) are the solutions of system (4).

Let \( \lambda_{1 R_*} = \frac{1}{\varphi_q (l_* A) B} \). Next, similar to the proof of (25), we can verify that (29) holds. This finishes the proof of Theorem 4.4.

**Theorem 4.5.** Suppose that (f) holds. If \( g_i^0 = +\infty (i = 1, 2) \), then there exists \( \beta_1 > 0 \) such that, for every \( 0 < \rho^* < \beta_1 \), system (4) admits a nontrivial radial
solution \( u_{r^*} = (u_{1r^*}, u_{2r^*}) \) satisfying \( \|u_{1r^*}\| = r^* \) for any 
\[
\lambda_{1r^*} \lambda_2 \in (0, \lambda^{**}],
\]
where \( \lambda^{**} \) is a positive finite number.

**Proof.** Similar to the proof of Theorem 4.4, we can prove Theorem 4.5. \( \square \)

5. **Asymptotic behavior of positive solutions.** In this section, we study the asymptotic behavior of positive solutions for system (4).

Let \( P \) be defined as (20), and \( T_1^* \) and \( T_2 \) be respectively defined in (26) and (22). Define a composite operator \( \widetilde{T}_1 = T_1^* T_2 \), which is completely continuous from \( P \) to itself. So the operator \( \widetilde{T}_1 \) also maps \( P \) into \( P \). We also define another composite operator
\[
\widetilde{T}_2 = T_2 T_1^*,
\]
which has the same meaning as \( \widetilde{T}_1 \).

**Theorem 5.1.** Suppose that \( (C_0) \) holds. For \( i \in \{1, 2\} \), then we have the following two conclusions.

\( (C_3) \) If \( g_i^0 = 0 \) and \( g_i^\infty = \infty \), then for every \( \lambda_i > 0 \) system (4) admits a pair of positive solutions \( u_{\lambda_i}, v_{\lambda_i} \) with 
\[
\lim_{\lambda_i \to 0^+} \|u_{\lambda_i}\| = \infty, \quad \lim_{\lambda_i \to 0^+} \|v_{\lambda_i}\| = \infty;
\]

\( (C_4) \) If \( g_i^0 = \infty \) and \( g_i^\infty = 0 \), then for every \( \lambda_i > 0 \) system (4) admits a pair of positive solutions \( u_{\lambda_i}, v_{\lambda_i} \) with 
\[
\lim_{\lambda_i \to 0^+} \|u_{\lambda_i}\| = 0, \quad \lim_{\lambda_i \to 0^+} \|v_{\lambda_i}\| = 0.
\]

**Proof.** We need only verify this theorem under condition \( (C_3) \) because the proof is similar when \( (C_4) \) is satisfied. For \( i \in \{1, 2\} \), let \( \lambda_i > 0 \). Since \( g_i^0 = 0 \), there exists \( r > 0 \) such that 
\[
g_1(v) \leq \frac{1}{\lambda_1 \varphi_p(B^*)} \varphi_p(v), \quad 0 \leq v \leq r,
\]
\[
g_2(u) \leq \frac{1}{\lambda_2 \varphi_p(B^*)} \varphi_p(u), \quad 0 \leq u \leq r,
\]
where \( B^* \) is defined in (23).

Thus, for \( i = \{1, 2\} \) and \( u, v \in P \cap \partial \Omega_r \), we get
\[
(T_1^* v)(t) = \varphi_q(\lambda_1) \int_t^1 \varphi_q(\frac{1}{\tau^{N-1}}) \int_0^1 s^{N-1} g_1(v(s)) ds d\tau 
\leq \varphi_q(\lambda_1) \int_0^1 \varphi_q(\frac{1}{\tau^{N-1}}) \int_0^1 s^{N-1} g_1(v(s)) ds d\tau 
\leq \varphi_q(\lambda_1) \int_0^1 \varphi_q(\frac{1}{\tau^{N-1}}) \int_0^1 s^{N-1} \frac{1}{\lambda_1 \varphi_p(B^*)} \varphi_p(v(s)) ds d\tau 
\leq \|v\|, \quad \forall t \in J,
\]
and
\[
(T_2 u)(t) = \varphi_q(\lambda_2) \int_{\tau}^{1} \varphi_q(\frac{1}{\tau^{N-1}}) \int_{0}^{\tau} s^{N-1} g_2(u(s)) ds d\tau
\]
\[
\leq \varphi_q(\lambda_2) \int_{0}^{\tau} \varphi_q(\frac{1}{\tau^{N-1}}) \int_{0}^{\tau} s^{N-1} g_2(u(s)) ds d\tau
\]
\[
\leq \varphi_q(\lambda_2) \int_{0}^{\tau} \varphi_q(\frac{1}{\tau^{N-1}}) \int_{0}^{\tau} s^{N-1} \frac{1}{\lambda_2 \varphi_p(B^*) \varphi_p(u(s))} ds d\tau
\]
\[
\leq \varphi_q(\lambda_2) \int_{0}^{\tau} \varphi_q(\frac{1}{\tau^{N-1}}) \int_{0}^{\tau} s^{N-1} \frac{1}{\lambda_2 \varphi_p(B^*) \varphi_p(\|u\|)} ds d\tau
\]
\[
\leq \|u\|, \forall t \in J.
\]

So
\[
\|\tilde{T}_1 u\| = \|T_1^* T_2 u\|
\]
\[
\leq \|T_2 u\|
\]
\[
\leq \|u\|. \tag{30}
\]

Next, for \( i = \{1, 2\} \), considering \( g_i^\infty = \infty \), there exists \( \hat{R} \) satisfying \( 0 < r < \hat{R} \) so that
\[
g_1(v) \geq \varepsilon \varphi_p(v), \quad \forall v \geq \hat{R},
\]
\[
g_2(u) \geq \varepsilon \varphi_p(u), \quad \forall u \geq \hat{R},
\]
where \( \varepsilon > 0 \) satisfies
\[
\varphi_q(\lambda_1 \lambda_2 A^2 \varepsilon^2) B^2 \geq 1, \tag{31}
\]
where \( A \) and \( B \) are respectively defined in (23).

Let \( R > 4\hat{R} \). Then, for \( u, v \in P \cap \partial \Omega_R \), we get
\[
u(t) \geq \frac{1}{4} \|u\| \geq \hat{R}, \quad v(t) \geq \frac{1}{4} \|v\| \geq \hat{R}, \quad t \in \left[ \frac{1}{4}, \frac{3}{4} \right],
\]
and then
\[
(T_1^* v)(t) = \varphi_q(\lambda_1) \int_{\tau}^{1} \varphi_q(\frac{1}{\tau^{N-1}}) \int_{0}^{\tau} s^{N-1} g_1(v(s)) ds d\tau
\]
\[
\geq \varphi_q(\lambda_1) \int_{\frac{1}{4}}^{1} \varphi_q(\frac{1}{\tau^{N-1}}) \int_{\frac{1}{4}}^{\frac{3}{4}} s^{N-1} g_1(v(s)) ds d\tau
\]
\[
\geq \varphi_q(\lambda_1) \int_{\frac{1}{4}}^{1} \varphi_q(\frac{1}{\tau^{N-1}}) \int_{\frac{1}{4}}^{\frac{3}{4}} s^{N-1} \varepsilon \varphi_p(v(s)) ds d\tau
\]
\[
\geq \varphi_q(\lambda_1) \int_{\frac{1}{4}}^{1} \varphi_q(\frac{1}{\tau^{N-1}}) \int_{\frac{1}{4}}^{\frac{3}{4}} s^{N-1} \varepsilon \varphi_p(\frac{1}{4} \|v\|) ds d\tau
\]
\[
= \varphi_q(\lambda_1 A \varepsilon) B \|v\|, \quad \forall t \in J.
\]
Similarly, we get
\[
(T_2 u)(t) \geq \varphi_q(\lambda_2 A \varepsilon) B \|u\|, \quad \forall t \in J.
\]
So, by (31), we have
\[
(T_1v_1)(t) = (T_1^*T_2u)(t)
\]
\[
\geq \varphi_q(\lambda_1 \varepsilon) B \|T_2u\|
\]
\[
\geq \varphi_q(\lambda_1 \lambda_2 A^2 \varepsilon^2) B^2 \|u\|
\]
\[
\geq \|u\|.
\]

\[\text{(32)}\]

From the above estimate and the fixed point theorem of cone expansion and compression of norm type, we deduce that operator \(T_1\) has a fixed point \(u \in P \cap (\Omega_R \setminus \Omega_r)\). Denote \(v = T_2u\), then \(u\) and \(v\) are the desired solution of system (4).

Similarly, we can prove that \(T_2\) has a fixed point \(v \in P \cap (\Omega_R \setminus \Omega_r)\).

Next, for \(i \in \{1, 2\}\), we prove that \(\|u_{\lambda_i}\| \to +\infty\), \(\|v_{\lambda_i}\| \to +\infty\) as \(\lambda_i \to 0^+\). In deed, if not, there are a number \(\xi_i > 0\) and a sequence \(\lambda_{im} \to +\infty\) such that
\[
\|u_{\lambda_{im}}\| \leq \xi_i, \quad \|v_{\lambda_{im}}\| \leq \xi_2 \quad (m = 1, 2, 3, \ldots).
\]

Moreover, the sequence \(\{\|u_{\lambda_{im}}\|\}\) and \(\{\|v_{\lambda_{im}}\|\}\) respectively contain a subsequence that converges to a number \(\eta_i(0 \leq \eta_i \leq \xi_i)\). For simplicity, we suppose that \(\{\|u_{\lambda_{1m}}\|\}\) itself converges to \(\eta_1\), and \(\{\|v_{\lambda_{2m}}\|\}\) itself converges to \(\eta_2\).

If \(\eta_1 > 0, \eta_2 > 0\), then \(\|u_{\lambda_{1m}}\| > \frac{\eta_1^2}{2}\), \(\|v_{\lambda_{2m}}\| > \frac{\eta_2^2}{2}\) for sufficiently large \(m > M\) (\(m \geq M\), \(M\) denotes a natural number), and so
\[
\frac{1}{\varphi_q(\lambda_{1m})} = \|\int_{\frac{1}{\tau - r}}^1 \varphi_q(\frac{1}{\tau - r}) \int_0^\tau s^{N-1} g_1(v(s))ds d\tau\| \quad \frac{\|u_{\lambda_{1m}}\|}{\|u_{\lambda_{1m}}\|}
\]
\[
\leq \frac{\|\int_{\frac{1}{\tau - r}}^1 \varphi_q(\frac{1}{\tau - r}) \int_0^\tau s^{N-1} g_1(v(s))ds d\tau\|}{\|u_{\lambda_{1m}}\|}
\]
\[
\leq \frac{\varphi_q(D_1) B^*}{\|u_{\lambda_{1m}}\|}
\]
\[
< \frac{2 \varphi_q(D_1) B^*}{\eta_1} \quad (m > M),
\]

and
\[
\frac{1}{\varphi_q(\lambda_{2m})} = \|\int_{\frac{1}{\tau - r}}^1 \varphi_q(\frac{1}{\tau - r}) \int_0^\tau s^{N-1} g_2(u(s))ds d\tau\| \quad \frac{\|v_{\lambda_{2m}}\|}{\|v_{\lambda_{2m}}\|}
\]
\[
\leq \frac{\|\int_{\frac{1}{\tau - r}}^1 \varphi_q(\frac{1}{\tau - r}) \int_0^\tau s^{N-1} g_2(u(s))ds d\tau\|}{\|v_{\lambda_{2m}}\|}
\]
\[
\leq \frac{\varphi_q(D_2) B^*}{\|v_{\lambda_{2m}}\|}
\]
\[
< \frac{2 \varphi_q(D_1) B^*}{\eta_2} \quad (m > M),
\]

where,
\[
D_1 = \max \left\{ g_1(v), \ r \leq \|v\| \leq R \right\},
\]
\[
D_2 = \max \left\{ g_2(u), \ r \leq \|u\| \leq R \right\}.
\]

This gives a contradiction as \(\lambda_{im} \to 0^+\) for \(i \in \{1, 2\}\).
If \( \eta_1 = 0 \) and \( \eta_2 = 0 \), then \( \|u_{\lambda_1 m}\| \to 0, \|v_{\lambda_2 m}\| \to 0 \) for sufficiently large \( m \) \((m > M)\), and so it follows from \((C_3)\) that for any \( \varepsilon > 0 \) there is \( r^* > 0 \) so that
\[
g_1(v_{\lambda_2 m}) \leq \varepsilon \varphi_p(v), \quad \forall \, 0 \leq v_{\lambda_2 m} \leq r^*,
\]
\[
g_2(u_{\lambda_1 m}) \leq \varepsilon \varphi_p(u), \quad \forall \, 0 \leq u_{\lambda_1 m} \leq r^*.
\]
Then, for \( u_{\lambda_1 m}, v_{\lambda_2 m} \in P \cap \partial \Omega_{r^*} \), we have
\[
\frac{1}{\varphi_q(\lambda_1 m)} = \frac{\| \int_0^1 \varphi_q(\frac{1}{\tau}) \int_0^\tau s^{N-1} g_1(v(s)) ds d\tau \|}{\|u_{\lambda_1 m}\|} \leq \frac{\| \int_0^1 \varphi_q(\frac{1}{\tau}) \int_0^\tau s^{N-1} \varepsilon \varphi_p(v(s)) ds d\tau \|}{\|u_{\lambda_1 m}\|} \leq \frac{\varphi_q(\varepsilon) B^\ast \|v\|}{\|u_{\lambda_1 m}\|}.
\]
and
\[
\frac{1}{\varphi_q(\lambda_2 m)} = \frac{\| \int_0^1 \varphi_q(\frac{1}{\tau}) \int_0^\tau s^{N-1} g_2(u(s)) ds d\tau \|}{\|v_{\lambda_2 m}\|} \leq \frac{\| \int_0^1 \varphi_q(\frac{1}{\tau}) \int_0^\tau s^{N-1} \varepsilon \varphi_p(v(s)) ds d\tau \|}{\|v_{\lambda_2 m}\|} \leq \frac{\varphi_q(\varepsilon) B^\ast \|v\|}{\|v_{\lambda_2 m}\|},
\]
where \( B^\ast \) is defined in \((23)\). Because \( \varepsilon \) is arbitrary, for \( i \in \{1, 2\} \), we get \( \lambda_{im} \to +\infty \) \((m \to +\infty)\), which contradicts \( \lambda_{im} \to 0^+ \). The proof of Theorem 5.1 is finished.

6. Some remarks. In this section, we offer some remarks and applications on the associated system \((4)\).

**Remark 6.1.** The present research extends the study in Hai \([10]\) from Laplacian system to \(p\)-Laplacian system. Meanwhile, we obtain some new existence results by defining composite operators and using the eigenvalue theory in cones. Moreover, we also analyze the asymptotic behavior of positive solutions to system \((4)\).

**Remark 6.2.** In this paper, we also generalize the study in Guo \([6]\), Guo and Webb \([7]\), Hai and Shivaji \([11]\), Shivaji, Sim and Son \([20]\), and Chu, Hai and Shivaji \([3]\) from single \(p\)-Laplacian equation to coupled \(p\)-Laplacian system. Here, we not
only get the uniqueness results, but also we obtain some existence results, and we consider the asymptotic behavior of positive solutions.

**Remark 6.3.** The approaches to prove Theorem 3.3, Theorem 4.2-Theorem 4.5 and Theorem 5.1 can be applied to the single equation case

\[
\begin{aligned}
-\Delta_p z &= \lambda g(z) \quad \text{in } \Omega, \\
z &= 0 \quad \text{on } \partial\Omega,
\end{aligned}
\]

where \(\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u)\), \(1 < p < N\), \(\lambda\) is a positive parameter, \(\Omega\) is the open unit ball in \(\mathbb{R}^N\).

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