On the Hölder regularity for solutions of integro-differential equations like the anisotropic fractional Laplacian

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Abstract

In this paper we study integro-differential equations like the anisotropic fractional Laplacian. As in Silvestre (Indiana Univ Math J 55:1155–1174, 2006), we adapt the De Giorgi technique to achieve the $C^\gamma$-regularity for solutions of class $C^2$ and use the geometry found in Caffarelli et al. (Math Ann 360(3–4): 681–714, 2014) to get an ABP estimate, a Harnack inequality and the interior $C^{1,\gamma}$ regularity for viscosity solutions.

Keywords Fractional Laplacian · Integro-differential equations · Regularity theory · Anisotropy

Mathematics Subject Classification 26A33 · 35J70 · 47G20 · 35J60 · 35D35 · 35D40 · 35B65

1 Introduction

In [14], the second author presents the anisotropic fractional Laplacian

$$(-\Delta)^{\beta,s} f(x) = C_{\beta,s} \int_{\mathbb{R}^n} \frac{f(x) - f(z)}{\left(\sum_{i=1}^n |x_i - z_i|^{b_i}\right)^{\frac{\beta}{s}}} \, dz,$$

(1.1)

where $\beta = (b_1, \ldots, b_n) \in \mathbb{R}^n$ represents the different homogeneities in different directions, $b_i > 0$, $0<s<2$, $c = \sum_{i=1}^n \frac{2}{b_i}$ and $C_{\beta,s} > 0$ is a normalization constant. In this work we develop a regularity theory for integro-differential equations like the anisotropic fractional Laplacian.

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where

\[ Lu(x) := \int_{\mathbb{R}^n} (u(x+y) - u(x) - \chi_{B_1}(y) \nabla u(x) \cdot y) K(y) dy, \]

\[ 0 < s < \frac{4}{b_{\max}}, \]

and the kernel \( K \) is symmetric, \( K(y) = K(-y) \), and satisfy the anisotropic bounds

\[ \lambda, \frac{q_{\max,s}}{\|y\|^{c+s}} \leq K(y) \leq \Lambda, \frac{q_{\max,s}}{\|y\|^{c+s}}, \quad \forall y \in \mathbb{R}^n \setminus \{0\}, \]

where \( 0 < \lambda \leq \Lambda \) and we denote \( b_{\max} = \max\{b_1, \ldots, b_n\} \).

\[ \|y\|^2 = \sum_{i=1}^n |y_i|^{b_i} \quad \text{and} \quad q_{\max,s} = \frac{4}{b_{\max}} - s. \]

Integro-differential equations appear in the context of discontinuous stochastic processes. For example, competitive stochastic games with two or more players, which are allowed to choose from different strategies at every step in order to maximize the expected value of some function at the first exit point of a domain. Integral operators like (1.1) correspond to purely jump processes when diffusion and drift are neglected. The anisotropic setting we consider also appears in the context of magnetic resonance imaging (MRI) of the human brain (cf. [12, 15]), anomalous diffusion (cf. [16]), biological tissues (cf. [11, 16]), financial mathematics (see [5, 17]). More concrete examples where the operator \((-\Delta)^{\beta,s}\) appears can be obtained if we consider payoff models and long jump random walks in which the probability for the jumps are governed by kernel

\[ K_0(y) = \frac{1}{\|y\|^{c+s}}, \]

see [1]. The main difference between the fractional Laplacian \((-\Delta)^s\) and the anisotropic fractional Laplacian \((-\Delta)^{\beta,s}\) is the geometry determined by the kernel \( K_0 \). In the seminal work [5], this anisotropic geometry required a refinement of the techniques presented in [6]: for example, a new covering lemma and a suitable scaling. In this paper, the novelty from kernel \( K_0 \) is the geometric exponent \( c \). In [14], motivated by the kernel studied in [5], that is,

\[ K_0(y) = \frac{1}{\|y\|^{c+s}}, \quad \text{where} \ b_i = n + \sigma_i \quad \text{and} \quad s = 2 - c, \]

for \( \sigma_i \in (0, 2) \), the second author studied an extension problem related to the anisotropic fractional laplacian and the constant \( c \) arose when it was necessary to find an equation such that the fundamental solution at the origin has the following form

\[ \frac{1}{\|y\|^\kappa} \]

for a suitable constant \( \kappa > 0 \). In this sense, \( c \) reveals the local version of the anisotropic fractional laplacian, namely, we have
\[ \text{div}(A_\beta(x) \nabla v) = 0, \quad \text{in} \quad \mathbb{R}^n \setminus S, \]  
(1.5)

where \( S = \bigcup_{i=1}^n \{ x_i = 0 \} \) and \( A_\beta = (a_{ij}) \) is the diagonal matrix given by

\[
a_{ij} := \begin{cases} 
\frac{4}{b_i} |x_i|^2 - b_i, & \text{if } i = j, \\
0, & \text{if } i \neq j.
\end{cases}
\]

Naturally, in order to get an anisotropic version of the Almgren’s frequency formula obtained in [7], the second author realized that the divergent equation

\[ \text{div}(A_\beta(x) \nabla u) = 0 \]

required a riemannian metric \( g \). We would also like to remark that the results obtained here will be necessary in the study of the free boundary of obstacle problems to the case anisotropic, see [8, 19].

The paper is divided into two parts. In the sequel, we comment on the strategies to achieve our results:

1. (Smooth solution). In the first part of the paper, we will show that the De Giorgi’s approach, see [10, 13], allows us to reach the \( C^r \)-regularity for smooth solutions \( u \) of (1.2), where the estimates do not depend on the norm of any derivative or modulus of continuity of \( u \). As in [18], we will control the behavior of a solution \( u \) of (1.2) away from the origin to obtain a Growth Lemma and use an iterate argument to get the desired regularity. In this analysis, two tools are crucial: barrier function and suitable scaling. In fact, in order to find an appropriate way to control the behavior of \( u \) away from the origin in the isotropic case [18], Silvestre established an interesting inequality involving radial barriers \( \eta \) and the kernel \( K \):

**Silvestre inequality** Given a \( \delta > 0 \), there exist \( \kappa > 0 \) and \( \tau > 0 \) only depending on \( \beta \), dimension \( n \), \( s \) and \( \delta \) such that for all \( r > 0 \) and \( x \in \mathbb{R}^n \):

\[
\kappa L_r \eta(x) + 2 \int_{\mathbb{R}^n \setminus B_\frac{r}{2}} (|8y|^2 - 1)K(ry)r^\tau dy < \frac{1}{2} \inf_{B \subset B_{2r}, |B| < \delta} \int_B K(ry)r^\tau dy, \quad (1.6)
\]

where

\[
L_r v(x) := \int_{\mathbb{R}^n} (v(x + y) - v(x) - \chi_{B_1}(ry)\nabla v(x) \cdot y)K(ry)r^\tau dy.
\]

The Silvestre inequality reveals the appropriate scaling for our analysis: the scaling determined by the kernel \( K \). Furthermore, the barrier functions \( \eta \) should satisfy the bounds:

\[
-C \leq L_r \eta(x) \leq C, \quad (1.7)
\]

for some positive constant \( C \) depending on \( \beta \), dimension \( n \), and \( s \). In our case, we will use radial functions as barrier functions and the anisotropic scaling \( T_{\beta,r} : \mathbb{R}^n \to \mathbb{R}^n \) defined by

\[
T_{\beta,r}e_i = r^\frac{s}{b_i}e_i, \quad (1.8)
\]

where \( e_i \) is the \( i \)th canonical vector, to get the anisotropic Silvestre inequality and access to the \( C^r \)-regularity.

As in [18], we could reach the \( C^r \)-regularity for more general kernels \( K \). For example,
\[ L = \sum_{j=1}^{k} \lambda_j (-\Delta)^{\beta_j, s_j}, \quad \lambda_j > 0, \quad s_j \in (0, 2), \]
or \( \mathcal{K} = \mathcal{K}(x, y) \) satisfying the following conditions:

\[ \mathcal{K}(x, y) = \mathcal{K}(x, -y) \quad \text{and} \quad \frac{\lambda_j}{\|y\|^{c+s_j(y)}} \leq \mathcal{K}(x, y) \leq \frac{\Lambda}{\|y\|^{c+s_j(x)}}, \]

where \( 0 < s_1 \leq s(x) \leq s_2 < 2 \). Moreover, we could consider a source \( f \) in our equation, that is,

\[ Lu = f. \]

2. (Viscosity solution). In the second part of the paper, we get the regularity theory established in \([5, 6]\) for viscosity solutions of non-local Isaac’s equation like the anisotropic fractional Laplacian

\[ Tu(x) := \inf_{\beta} \sup_{x} L_{x, \beta} u(x) = 0, \quad (1.9) \]

where \( L_{x, \beta} \) is as in (1.2). In \([5, 6]\), the key that gives access to the regularity theory to viscosity solutions \( u \) of the Eq. (1.9) is a non-local ABP estimate. In \([5]\), the correct geometry to reach a non-local ABP estimate for integro-differential equation governed by anisotropic kernels \( \mathcal{K}_{x, \beta} \) was discovered. More precisely, the geometry determined by the level sets of the kernels \( \mathcal{K}_{x, \beta} \):

\[ \Theta_r(x) := \{(y_1, \ldots, y_n) \in \mathbb{R}^n : \|y - x\| < r\}. \]

With this geometry at hand, three steps are fundamental to obtain a non-local ABP estimate, a Harnack Inequality and the desired regularity:

1. \textit{u stays quadratically close} to the tangent plane to concave envelope \( \Gamma \) of \( u \) in a (large) portion of the neighbourhoods of the contact points and such that, in smaller neighbourhoods (with the same geometry), the concave envelope \( \Gamma \) has quadratic growth: here, our neighbourhoods are ellipses \( E_{r, 1} \) with the same geometry of \( \Theta_r \).

2. \textit{Covering lemma} Since our neighbourhoods will be ellipses \( E_{r, 1} \), our covering is naturally made of \( n \)-dimensional rectangles \( \mathcal{R}_r \) and we invoke a covering lemma from \([2]\). (Viscosity solution). In the second part of the paper, we get the regularity theory established in \([5, 6]\) for viscosity solutions of non-local Isaac’s equation like the anisotropic fractional Laplacian

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With this geometry at hand, three steps are fundamental to obtain a non-local ABP estimate, a Harnack Inequality and the desired regularity:
Finally, we emphasize that the restriction $0 < s < 4/b_{\text{max}}$ in our results comes from the class of solutions $u$ we are studying: solutions of class $C^2$ or viscosity solutions ($u$ is touched by a $C^2$ function). However, we believe that the results obtained here can naturally be extended for $0 < s < 2$ if we consider an appropriate class of solutions $u$ and change the metric of $\mathbb{R}^n$, a namely, $(\mathbb{R}^n, g)$, where $g$ is the metric determined by kernel $\| \cdot \|$, see [14]. We plan to address this issue in a forthcoming paper. Furthermore, the Lemma [3] allows the homogeneity degrees $b_i$ depend on $x$, see [9]. We would also like to mention that in [4] an important regularity theory for integro-differential equations was developed, where the kernels are singular, and only charge the coordinate axes for the jumps, and each axis may charge jumps with a different exponent.

The paper is organized as follows. In Sect. 2 we gather all the necessary tools for our analysis: fundamental geometry, Silvestre inequality, the notion of viscosity solution for the problem (1.9), the extremal operators of Pucci type associated with the family of kernels $K_{ab}$ and some notation. In Sect. 3 we present the proof of $C^2$-regularity of smooth solutions and as a corollary we get a result type Liouville. The Sect. 4 is divided in three Sect. 4.1, where the nonlocal ABP estimate for a solution $u$ of Eq. (1.9) is obtained, is the most important of the paper. Sections 4.2 and 4.3 are devoted to the proof of the Harnack inequality and its consequences.

2 Preliminaries

In this section we gather anisotropic versions of some results obtained in [5, 18]. We begin with geometric informations that we will systematically use along the work.

Given $r, l > 0$ and $x \in \mathbb{R}^n$, we will denote

$$E_{r,l}(x) := \left\{ (y_1, \ldots, y_n) \in \mathbb{R}^n : \sum_{i=1}^n \frac{(y_i - x_i)^2}{r^{2b_i}} < l^2 \right\}.$$

If $b_{\text{min}} = \min\{b_1, \ldots, b_n\}$ and $b_{\text{max}} = \max\{b_1, \ldots, b_n\}$ we define

$$R_{r,l}(x) := \left\{ (y_1, \ldots, y_n) \in \mathbb{R}^n : |y_i - x_i| < l b_{\text{min}}^\frac{1}{2} r^\frac{1}{2} \right\},$$

and

$$E_{r,l}^{\text{max}}(x) := \left\{ (y_1, \ldots, y_n) \in \mathbb{R}^n : \sum_{i=1}^n \frac{(y_i - x_i)^2}{r^{2b_{\text{max}}}} < l^2 \right\}.$$

Furthermore, if $C = C > 0$ is a natural number and the $n$-dimensional rectangle

$$R(x) := \left\{ (y_1, \ldots, y_n) \in \mathbb{R}^n : |y_i - x_i| < l_i \right\}$$

satisfies

$$R(x) \subset \left\{ (y_1, \ldots, y_n) \in \mathbb{R}^n : |y_i - x_i| < 2^{-C(k+1)} r^\frac{k}{2} \right\},$$

for some number natural $k$, we define the corresponding $n$-dimensional rectangle $\tilde{R}(x)$ by
\[ \tilde{R}(x) := \left\{ (y_1, \ldots, y_n) \in \mathbb{R}^n : |y_i - x_i| < \left[ 2 - e^{\left(\frac{\max}{n} r\right)} \right] r_{i,j} \right\}. \]

We will also consider the notation

\[ B_r = B_r(0), \quad \Theta_r = \Theta_r(0) \quad \text{and} \quad E_{r,l}^{\max} = E_{r,l}^{\max}(0). \]

The geometric properties of the sets defined above will be crucial in our analysis. We collect them in the following Lemma.

**Lemma 2.1** (Fundamental Geometry) Let \( r > 0 \) and \( l > 0 \). Then, given \( x \in \mathbb{R}^n \), we have the following relations:

1. \( E_{r,1}(x) \subset \Theta_r \subset E_{r,1}(x) \) and \( E_{2^{-l}r}(x) \subset E_{r,1}(x) \), for some natural number \( C = C(n, b_{\max}) > 0 \).
2. If \( R \) is a \( n \)-dimensional rectangle, then \( R(x) \subset \tilde{R}(x) \). Moreover, \( R_{r,l}(x) \subset E_{r,1}(x) \), where \( e_{\max} = n b_{\max} \), if \( r, l \in (0, 1) \).
3. \( E_{r,1}^{\max}(x) \subset E_{r,1/2}(x) \) and \( E_{r,l}^{\max}(x) \subset E_{r,1}(x) \), if \( l \geq 1 \).
4. If \( \tau_1 \) is the topology generated by Euclidean balls \( B_r(z) \) and \( \tau_2 \) is the topology generated by anisotropic balls \( \Theta_r(z) \), then \( \tau_1 = \tau_2 \).
5. If \( T_{\beta,r} : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is defined by

\[ T_{\beta,r} e_i = r_{\beta,i} e_i \quad \text{or} \quad T_{\max,r} e_i = r_{\max,i} e_i, \quad (2.1) \]

where \( e_i \) is the \( i \)-th canonical vector, then \( T_{\beta,r}(B_1) = E_{r,1} \) or \( T_{\max,r}(B_1) = E_{r,1}^{\max} \).

Next we will divide this section into two subsections: Smooth solutions and Viscosity solutions and extremal operators.

**2.1 Smooth solutions**

Without loss of generality, we consider \( L = (-\Delta)^{\beta,s} \). In this subsection, we establish the tools to get the regularity \( C^\beta \) for \( \Delta^{\beta,s} \)-harmonic smooth functions. Precisely, we show that the operator \( \Delta^{\beta,s} \) applied to radial functions \( \eta \) is bounded for \( s \in (0, 4/b_{\max}) \) and we get the Silvestre inequality for \( \Delta^{\beta,s} \eta \).

**Lemma 2.2** (Barrier function) Let \( \eta : \mathbb{R}^n \rightarrow \mathbb{R} \) defined by

\[ \eta(y) = \begin{cases} (1 - |y|^2)^2, & \text{if } y \in B_1, \\ 0, & \text{if } y \in (\mathbb{R}^n \setminus B_1). \end{cases} \quad (2.2) \]

There exist \( C > 0 \) only depending on \( \beta \), dimension \( n \) and \( s \) such that

\[ |(-\Delta)^{\beta,s} \eta(x)| \leq C \quad \text{for all } x \in B_{3/4}. \quad (2.3) \]

**Proof** Choose \( r_0 = r_0(n, \beta) \in (0, 1) \) such that
where $C$ is a positive constant only depending on $\beta$, and dimension $n$. Denote $T_{r_k} := T_{\beta,r_k}$, where $r_k = r_0 2^{-k}$. Then, we get

$$\int_{E_{r_0,1}} \frac{|\eta(x) - \eta(x+y) + \langle \nabla \eta(x), y \rangle|}{\|y\|^{c+s}} dy \leq C \int_{E_{r_0,1}} \frac{|y|^2}{\|y\|^{c+s}} dy$$

and we can estimate

$$\sum_{k=0}^{\infty} r_k^{-s} \int_{B_1 \setminus E_{r_1,1}} \frac{|T_{r_k} y|^2}{\|y\|^{c+s}} dy \leq C \sum_{i=0}^{q_{\max,s}} \int_{B_1 \setminus E_{r_1,1}} \frac{|y|^2}{\|y\|^{c+s}} dy$$

$$= C(n, \beta, s) \frac{1}{1 - 2^{-q_{\max,s}}}$$

where $C(n, \beta, s) = \int_{B_1 \setminus E_{r_1,1}} \frac{|y|^2}{\|y\|^{c+s}} dy$. On the other hand, if $r_1 = r_1(r_0) > 0$ is such that $\Theta_{r_1} \subset E_{r_0,1}$, we obtain

$$\int_{\mathbb{R}^n \setminus E_{r_0,1}} \frac{|\eta(x) - \eta(x+y)|}{\|y\|^{c+s}} dy \leq 2\|\eta\|_{\infty} \int_{\mathbb{R}^n \setminus \Theta_{r_1}} \frac{1}{\|y\|^{c+s}} dy$$

$$= r_1^s \int_{\mathbb{R}^n \setminus \Theta_{r_1}} \frac{1}{\|y\|^{c+s}} dy$$

(2.5)

$$= C(n, \beta, s)$$

Then, we find

$$|(-\Delta)^{\beta,s} \eta(x)| \leq C.$$  (2.6)

Taking into account (2.6) we get the Silvestre inequality for $\Delta^{\beta,s}$:

**Lemma 2.3** (Silvestre inequality) Given a $\delta > 0$, there exist $0 < \kappa < \frac{1}{4}$ and $\tau > 0$ only depending on $\beta$, dimension $n$, $s$ and $\delta$ such that

$$\kappa(-\Delta)^{\beta,s} \eta(x) + 2 \int_{\mathbb{R}^n \setminus B_{\frac{3}{4}}} (|y|^r - 1) K_0(y) dy < \frac{1}{2} \inf_{B < B_2; |B| < \delta} \int_B K_0(y) dy,$$  (2.7)

for all $x \in B_{3/4}$, where $K_0(y) := \frac{1}{|y|^{n+r}}$ for all $y \in \mathbb{R}^n \setminus \{0\}$.  

\[\square\]
2.2 Viscosity solutions and extremal operators

In this subsection we collect the technical properties of the operator $I$ that we will use throughout the paper. Since $K_{\alpha\beta}$ is symmetric and positive, we obtain

$$L_{\alpha\beta}u(x) = \text{PV} \int_{\mathbb{R}^n} (u(x + y) - u(x))K_{\alpha\beta}(y)dy$$

and

$$L_{\alpha\beta}u(x) = \frac{1}{2} \int_{\mathbb{R}^n} (u(x + y) - u(x) - 2(x))K_{\alpha\beta}(y)dy.$$ 

For convenience of notation, we denote

$$\delta(u, x, y) := u(x + y) + u(x - y) - 2u(x)$$

and we can write

$$L_{\alpha\beta}u(x) = \int_{\mathbb{R}^n} \delta(u, x, y)K_{\alpha\beta}(y)dy,$$

for some kernel $K_{\alpha\beta}$.

We now define the adequate class of test functions for our operators.

**Definition 2.4** A function $\varphi$ is said to be $C^{1,1}$ at the point $x$, and we write $\varphi \in C^{1,1}(x)$, if there is a vector $v \in \mathbb{R}^n$ and numbers $M, \eta_0 > 0$ such that

$$|\varphi(x + y) - \varphi(x) - v \cdot y| \leq M|y|^2,$$

for $|x| < \eta_0$. We say that a function $\varphi$ is $C^{1,1}$ in a set $\Omega$, and we denote $\varphi \in C^{1,1}(\Omega)$, if the previous holds at every point, with a uniform constant $M$.

**Remark 2.5** Let $u \in C^{1,1}(x) \cap L^\infty(\mathbb{R}^n)$ and $M > 0$ and $\eta_0 > 0$ be as in Definition 2.4. Then, by Lemma 2.2, we find

$$L_{\alpha\beta}u(x) = \text{PV} \int_{\mathbb{R}^n} \delta(u, x, y)K_{\alpha\beta}(y)dy \leq C(n, \Lambda, b_{\min}, b_{\max}, \eta_0, s).$$

We now introduce the notion of viscosity subsolution (and supersolution) $u$ in a domain $\Omega$, with $C^2$ test functions that touch $u$ from above or from below. We stress that $u$ is allowed to have arbitrary discontinuities outside of $\Omega$.

**Definition 2.6** Let $f$ be a bounded and continuous function in $\mathbb{R}^n$. A function $u : \mathbb{R}^n \to \mathbb{R}$, upper (lower) semicontinuous in $\overline{\Omega}$, is said to be a subsolution (supersolution) to equation $Iu = f$, and we write $Iu \geq f$ ($Iu \leq f$), if whenever the following happen:

1. $x_0 \in \Omega$ is any point in $\Omega$;
2. $B_r(x_0) \subset \Omega$, for some $r > 0$;
3. $\varphi \in C^2(\overline{B_r(x_0)})$;
4. \( \varphi(x_0) = u(x_0); \)
5. \( \varphi(y) > u(y) \) (\( \varphi(y) < u(y) \)) for every \( y \in B_r(x_0) \setminus \{x_0\}; \)

then, if we let

\[
\nu := \begin{cases} 
\varphi, & \text{in } B_r(x_0) \\
u, & \text{in } \mathbb{R}^n \setminus B_r(x_0),
\end{cases}
\]

we have \( \mathcal{I}\nu(x_0) \geq f(x_0) \) (\( \mathcal{I}\nu(x_0) \leq f(x_0) \)).

**Remark 2.7** Functions which are \( C^{1,1} \) at a contact point \( x \) can be used as test functions in the definition of viscosity solution (see Lemma 4.3 in [6]).

Next, we define the class of linear integro-differential operators that will be a fundamental tool for the regularity analysis.

**Definition 2.8** Let \( \mathfrak{L}_0 \) be the collection of linear operators \( L_{xy}\). We define the maximal and minimal operator with respect to \( \mathfrak{L}_0 \) as

\[
\mathcal{M}^+ u(x) := \sup_{L \in \mathfrak{L}_0} Lu(x)
\]

and

\[
\mathcal{M}^- u(x) := \inf_{L \in \mathfrak{L}_0} Lu(x).
\]

By definition, if \( \mathcal{M}^+ u(x) < \infty \) and \( \mathcal{M}^- u(x) < \infty \), we get

\[
\mathcal{M}^+ u(x) = q_{\max,s} \int_{\mathbb{R}^n} \frac{\Lambda \delta^+ - \lambda \delta^-}{\|y\|^{c+s}} dy
\]

and

\[
\mathcal{M}^- u(x) = q_{\max,s} \int_{\mathbb{R}^n} \frac{\lambda \delta^+ - \Lambda \delta^-}{\|y\|^{c+s}} dy.
\]

The proofs of the results that we now present can be found in the sections 3, 4 and 5 of [6]. The first result ensures that if \( u \) can be touched from above, at a point \( x \), with a paraboloid then \( \mathcal{I}u(x) \) can be evaluated classically.

**Lemma 2.9** If we have a subsolution, \( \mathcal{I}u \geq f \) in \( \Omega \), and \( \varphi \) is a \( C^2 \) function that touches \( u \) from above at a point \( x \in \Omega \), then \( \mathcal{I}u(x) \) is defined in the classical sense and \( \mathcal{I}u(x) \geq f(x) \).

Another important property of \( \mathcal{I} \) is the continuity of \( \mathcal{I}\varphi \) in \( \Omega \) if \( \varphi \in C^{1,1}(\Omega) \).

**Lemma 2.10** Let \( v \) be a bounded function in \( \mathbb{R}^n \) and \( C^{1,1} \) in some open set \( \Omega \). Then \( \mathcal{I}v \) is continuous in \( \Omega \).
The next lemma allows us to conclude that the difference between a subsolution of the maximal operator $M^+$ and a supersolution of the minimal operator $M^-$ is a subsolution of the maximal operator.

**Lemma 2.11** Let $\Omega$ be a bounded open set and $u$ and $v$ be two bounded functions in $\mathbb{R}^n$ such that

1. $u$ is upper-semicontinuous and $v$ is lower-semicontinuous in $\Omega$;
2. $Iu \geq f$ and $ Iv \leq g$ in the viscosity sense in $\Omega$ for two continuous functions $f$ and $g$.

Then

$$M^+(u-v) \geq f-g \quad \text{in} \quad \Omega$$

in the viscosity sense.

### 3 Hölder regularity: smooth solutions

As in [18] we will use the De Giorgi’s approach to achieve the $C^\gamma$-regularity for $\Delta^{\beta,s}$-harmonic smooth functions. We begin with a Growth lemma.

**Lemma 3.1** (Growth lemma) If $u$ is a function that satisfies:

1. $(−Δ)^{\beta,s}u \leq 0$ in $B_1$;
2. $u \leq 1$ in $B_1$;
3. $u(x) \leq 2|2x|^\gamma \leq 1$ for all $x \in \mathbb{R}^n \setminus B_1$;
4. $|\{x \in B_1 : u(x) \leq 0\}| > \delta$.

Then, there exists a constant $\mu = \mu(n,s,\beta,\delta) > 0$ such that $u \leq 1 - \mu$ in $B_{1/2}$.

**Proof** Consider $\mu = \kappa(\eta(1/2) - \eta(3/4))$. Suppose, for the purpose of contradiction, that there exists $x_0 \in B_2$ such that

$$u(x_0) > 1 - \mu = 1 - \kappa\eta(1/2) + \kappa\eta(3/4). \quad (3.1)$$

Thus, since $\eta$ is decreasing in any ray from the origin and $u \leq 1$ in $B_1$, we have

$$v(x_0) > v(x), \quad \text{for all} \quad x \in B_1 \setminus B_{3/4}, \quad (3.2)$$

where $v(x) = u(x) + \kappa \eta(x)$. Then, we conclude that

$$1 < \sup_{x \in B_4} v(x) = v(x_1) \quad (3.3)$$

for some $x_1 \in B_{3/4}$. If we define

$$\mathcal{B} = \{y \in \mathbb{R}^n : x_1 + y \in B_1\} \quad \text{and} \quad \mathcal{B}_0 = \{y \in \mathbb{R}^n : x_1 + y \in B_1, \ u(x_1 + y) \leq 0\}$$

we can write

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\[
(-\Delta)^{\beta,s} v(x_1) = \int_{\mathbb{R}^n} (v(x_1) - v(x_1 + y)) K_0(y) dy = I_1 + I_2,
\]

where we denote
\[
I_1 = \int_{B} (v(x_1) - v(x_1 + y)) K_0(y) dy \quad \text{and} \quad I_2 = \int_{\mathbb{R}^n \setminus B} (v(x_1) - v(x_1 + y)) K_0(y) dy.
\]

Since \(v\) has a maximum at \(x_1\) and \(v(x_1) \geq 1\) we estimate
\[
I_1 = \int_{B_0} (v(x_1) - v(x_1 + y)) K_0(y) dy + \int_{B \setminus B_0} (v(x_1) - v(x_1 + y)) K_0(y) dy
\geq \int_{B_0} (v(x_1) - v(x_1 + y)) K_0(y) dy
\geq \int_{B_0} (1 - \kappa \eta(x_1 + y)) K_0(y) dy
\geq \frac{1}{2} \int_{B_0} K_0(y) dy.
\]

Using the conditions 2 and 3 we find
\[
I_2 = \int_{\mathbb{R}^n \setminus B} (v(x_1) - v(x_1 + y)) K_0(y) dy
\geq \int_{\mathbb{R}^n \setminus B} [1 - (2|2(x_1 + y)|^\tau - 1)] K_0(y) dy
= \int_{\mathbb{R}^n \setminus B} [2 - 2^{\tau+1}|x_1 + y|^\tau] K_0(y) dy
\geq \int_{\mathbb{R}^n \setminus B} [2 - 2^{\tau+1}(3/4 + |y|)^\tau] K_0(y) dy.
\]

Moreover, since \((\mathbb{R}^n \setminus B) \subset (\mathbb{R}^n \setminus B_{1/4})\) we obtain
\[
I_2 \geq \int_{\mathbb{R}^n \setminus B} [2 - 2^{\tau+1}|3/4 + y|^\tau] K_0(y) dy
= \int_{\mathbb{R}^n \setminus B_{1/4}} [2 - 2^{\tau+1}|3/4 + y|^\tau] K_0(y) dy - \int_{(\mathbb{R}^n \setminus B_{1/4}) \setminus B} [2 - 2^{\tau+1}(3/4 + |y|)^\tau] K_0(y) dy
\geq \int_{\mathbb{R}^n \setminus B_{1/4}} [2 - 2^{\tau+1}(3/4 + |y|)^\tau] K_0(y) dy.
\]

From condition 1 we have
\[
(-\Delta)^{\beta,s} v(x_1) = (-\Delta)^{\beta,s} (u(x_1) + \kappa \eta(x_1)) \leq \kappa (-\Delta)^{\beta,d} \eta(x_1)
\]
and using the condition 4 we obtain
\[ \kappa(-\Delta)^{\beta,s} \eta(x_1) \geq 2 \int_{\mathbb{R}^n \setminus B_1} (1 - |y|^\gamma) K_0(y) dy + \frac{1}{2} \inf_{B \subset B_2 \subset \mathbb{R}^n \setminus B_{\delta}} \int_B K_0(y) \, dy, \]

which contradicts (2.7).

Using the anisotropic scaling \( T_{\max,r} \) and Lemma 3.1 we get the following scaled version.

**Lemma 3.2** (Growth lemma-anisotropic) If \( u \) is a function that satisfies

\( (-\Delta)^{\beta,s} u \leq 0 \) in \( E_{r,1}^{\max} (x_0) \);
\( u \leq C \) in \( E_{r,1}^{\max} (x_0) \);
\( u(x) \leq C \left( 2 |2 T_{\max,r} (x-x_0)|^\gamma - 1 \right) \) for all \( x \in \mathbb{R}^n \setminus E_{r,1}^{\max} (x_0) \);
\( \max_{r_{\max}^2} \frac{|\{ x \in E_{r,1}^{\max} (x_0) : u(x) \leq 0 \}|}{r_{\max}} \geq \delta. \)

Then, there exists a constant \( \mu = \mu(n,s,\beta,\delta) > 0 \) such that \( u \leq C(1 - \mu) \) in \( E_{x,1}^{\max} \).

**Proof** Define

\[ v(x) = \frac{u(T_{\max,r} x + x_0)}{C}, \]

for all \( x \in \mathbb{R}^n \). Since \( T_{\max,r} (B_1) = E_{r,1}^{\max} \) we conclude that \( v \) satisfies 2 and 3. Furthermore, we find

\[ (-\Delta)^{\beta,s} v(x) \leq 0 \quad \text{and} \quad |\{ x \in B_1 : v(x) \leq 0 \}| > \delta. \]

By Lemma 3.1 there exists a constant \( \mu = \mu(n,s,\beta) > 0 \) such that \( v \leq 1 - \mu \) in \( B_{1/2} \). Thus, we find \( u \leq C(1 - \mu) \) in \( E_{r,1/2}^{\max} \). Finally, by Lemma 2.1 we have \( E_{x,1}^{\max} \subset E_{x,1/2}^{\max} \) and the Lemma 3.2 is concluded.

**Theorem 3.3** If \( u \) is a bounded function that satisfies \( (-\Delta)^{\beta,s} u = 0 \) in \( E_{2r,1}^{\max} \), then for \( \delta = |B_{1/2}| \) there exist constants \( \gamma = \gamma(n,s,\beta) \in (0,1) \) and \( C = C(n,s,\beta) > 0 \) such that

\[ \sup_{x,y \in E_{r,1}^{\max}} \frac{|u(x) - u(y)|}{\|x - y\|^\gamma} \leq \frac{C}{r_{\max}} \|u\|_\infty. \]

In particular, \( u \in C^{\frac{\beta}{1-\beta}}_{\text{loc}} (E_{r,1}^{\max}) \).

**Proof** By considering the anisotropic scaling \( v(x) = u(T_{\max,r} x)/2\|u\|_\infty \) we can suppose that \( \text{osc}_{B_1} u = 1 \) and \( r = 1 \). As in [18], given \( x_0 \in B_1 \) we will construct a nondecreasing sequence \( c_k \) and a nonincreasing sequence \( d_k \) such that \( d_k - c_k = 2^{-kx} \)

\[ d_k - c_k = 2^{-kx} \quad \text{and} \quad c_k \leq u \leq d_k \quad \text{in} \ E_{r,1}^{\max}(x_0), \]

where \( r_k = r_{0}^k \) for any integer number \( k \) and \( 0 < x < 1 \) will be chosen appropriately. Now we consider two cases:
Case 1: $k \leq 0$.

Since $\text{osc}_{\mathbb{R}^n} u = 1$, we can write

$$c_k = \inf_{\mathbb{R}^n} u \quad \text{and} \quad d_k = c_k + r_k^2,$$

for $k \leq 0$ and for all $\epsilon \in (0, 1)$.

Case 2: $k \geq 1$.

Suppose that we already have $c_j$ and $d_j$ for $j = 1, \ldots, k$. We will find $c_{k+1}$ and $d_{k+1}$ satisfying (3.8). In fact, if

$$m = \frac{c_k + d_k}{2},$$

then by (3.8) we find

$$|u - m| \leq \frac{2^{-k\epsilon}}{2} \quad \text{in} \quad E_{r_k, 1}^{\max}(x_0).$$

Now define

$$v(x) = \frac{2\left(u(x) - m\right)}{r_k^2},$$

for all $x \in E_{r_k, 1}^{\max}(x_0)$. Clearly, we have

$$|v| \leq 1 \quad \text{in} \quad E_{r_k, 1}^{\max}(x_0)$$

and

$$(-\Delta)^{\beta/2} v \leq 0 \quad \text{in} \quad E_{r_k, 1}^{\max}(x_0).$$

Next, we will analysis two cases:

(i) Assume that

$$\left\{ x \in E_{r_k, 1}^{\max}(x_0) : v(x) \leq 0 \right\} \geq \frac{|B_1|}{2r_k^2}.$$

Taking into account that

$$x \in \mathbb{R}^n \setminus E_{r_k, 1}^{\max}(x_0) = T_{\text{max}, r_k}^{-1}(\mathbb{R}^n \setminus B_1(x_0))$$

we obtain

$$T_{\text{max}, r_k}^{-1}(x - x_0) \in \mathbb{R}^n \setminus B_1.$$

Thus, there exists $j \in \mathbb{N}$ such that

$$2^j \leq |T_{\text{max}, r_k}^{-1}(x - x_0)| \leq 2^{j+1}$$

Hence, we find

$$T_{\text{max}, r_k}^{-1}(x - x_0) \in B_{2^{j+1}}$$

and from Lemma 2.1
\[ x - x_0 \in E_{r_k,2(j+1)}^{\max} \subseteq E_{2^{-k(j+1)+1},1}^{\max} = E_{r_{k-j-1},1}^{\max}. \] (3.20)

Thus, by inductive hypothesis we estimate
\[
v(x) = 2 \left( \frac{u(x) - m}{r_k^2} \right) \leq 2 \left( \frac{a_{k-j-1} - m}{2^{-kx}} \right) \]
and since \( c_k \) is a nondecreasing sequence we obtain
\[
v(x) \leq 2 \left( \frac{a_{k-j-1} - c_{k-j-1} + c_{k-j-1} - m}{r_k^2} \right) \leq 2 \left( \frac{a_{k-j-1} - c_{k-j-1} + c_k - m}{r_k^2} \right) \]
\[
\leq 2 \left( \frac{2^{-(k-j-1)x} - 1}{2} \right) = 2^{2^{-(k-j-1)x}} - 1,
\]
for all \( x \in \mathbb{R}^n \setminus E_{r_k,1}^{\max}(x_0) \). If we take \( \tau \in (0, \tau] \) we get
\[
v(x) \leq \left( 2 |2T_{\max,r_k}^{-1}(x-x_0)|^\tau - 1 \right) \quad \text{for all } x \in \mathbb{R}^n \setminus E_{r_k,1}^{\max}(x_0).
\]
(3.23)

Then, we can apply the Lemma 3.2 to obtain \( v \leq 1 - \mu \) in \( E_{r_{k+1}}^{\max}(x_0) = E_{r_{k+1},1}^{\max}(x_0) \). We then scale back to \( u \) to find
\[
u \leq c_k + \left( \frac{2 - \mu}{2} \right) r_k^2 \quad \text{in } E_{r_{k+1},1}^{\max}(x_0).
\]
(3.24)

Now we define \( c_{k+1} = c_k \) and \( d_k = c_k + r_{k+1}^2 \). Clearly, \( c_{k+1} \leq u \) in \( E_{r_{k+1},1}^{\max}(x_0) \). Finally, if we choose \( \tau = \min\left\{ \tau, \frac{\ln(1-\mu/2)}{\ln 2} \right\} \) we obtain
\[
u \leq d_{k+1} \quad \text{in } E_{r_{k+1},1}^{\max}(x_0).
\]
(3.25)

(ii) In the case
\[
\left| \left\{ x \in E_{r_k,1}(x_0) : v(x) \leq 0 \right\} \right| \leq \frac{|B_1|}{2}.
\]
(3.26)

we consider \( v = -u \) to obtain
\[
u \geq d_k - \left( \frac{2 - \mu}{2} \right) r_k^2 \quad \text{in } E_{r_{k+1},1}^{\max}(x_0).
\]
(3.27)

Now we define \( d_{k+1} = d_k \) and \( c_{k+1} = d_k - \left( \frac{2 - \mu}{2} \right) r_k^2. \)
Finally, given $x_0 \in B_1$ and $y \in \mathbb{R}^n$ we can choose an integer $k$ such that $x_0 - y \in (E_{\gamma,1}^{\text{max}} \setminus E_{\gamma,1}^{\text{max}})$. Thus, by Lemma 2.1 we can conclude

$$|u(x_0) - u(y)| \leq r_{k-1}^\gamma \leq C|x_0 - y|^\gamma,$$

where $C = C(n, \gamma, b_{\text{min}}, b_{\text{max}}) > 1$ and $\gamma = \frac{2\gamma}{b_{\text{max}}}$. \hfill \Box

**Corollary 3.4** (Liouville property) Let $u$ be a bounded function that satisfies $(-\Delta)^{\beta,s} u = 0$ in $\mathbb{R}^n$. Then, $u$ is constant.

**Proof** Given $x, y \in \mathbb{R}^n$, choose $R > 0$ such that $x, y \in E_{R,1}^{\text{max}}$. By Theorem 3.3 we have

$$\frac{|u(x) - u(y)|}{|x - y|^\gamma} \leq \frac{C}{R^\gamma} \|u\|_{\infty}.$$

Taking $R > 0$ large enough, we get $u(x) = u(y)$. Hence, $u$ is constant. \hfill \Box

4 Hölder regularity: viscosity solutions

In this section, we obtain the ingredients necessary to reach the interior $C^\gamma$ and $C^{1,\gamma}$ regularity for viscosity solutions of $\mathcal{I}u = 0$.

4.1 Nonlocal anisotropic ABP estimate

In this subsection we get an ABP estimate for integro-differential equations like anisotropic fractional Laplacian.

Let $u$ be a non positive function outside the ball $B_1$. We define the concave envelope of $u$ by

$$\Gamma(x) := \begin{cases} \min\{p(x) : \text{for all planes } p \geq u^+ \text{ in } B_3\}, & \text{in } B_3 \\ 0 & \text{in } \mathbb{R}^n \setminus B_3. \end{cases}$$

**Lemma 4.1** Let $u \leq 0$ in $\mathbb{R}^n \setminus B_1$ and $\Gamma$ be its concave envelope. Suppose $f \in L^\infty$ and $\mathcal{M}^+ u(x) \geq -f(x)$ in $B_1$. Let $\rho_0 = \rho_0(n) > 0$,

$$r_k := \rho_0 2^{-\left(\frac{1}{q_{\text{min},s}}\right)} 2^{-\left(\frac{q_{\text{max},s}}{q_{\text{min},s}}\right)k},$$

where $C = C(b_{\text{min}}, b_{\text{max}})$ is a natural number such that

$$E_{lr,1} \subset E_{r,1/2},$$

with $l = 2^{-\left(\frac{q_{\text{min},s}}{q_{\text{min},s}}\right)}$ for all $r > 0$ and $q_{\text{min},s} = \frac{4}{b_{\text{min}}} - s$. Given $M > 0$, we define

$$W_k(x) := E_{r_k,1} \setminus E_{r_{k-1},1} \cap \left\{ y : u(x + y) < u(x) + \langle y, \nabla \Gamma(x) \rangle - M\left(q_{\text{min},s}, q_{\text{max},s}\right) r_k^{q_{\text{min},s}} \right\}.$$
\[ |W_k(x)| \leq C_0 \frac{f(x)}{M} |E_{r_1,1} \setminus E_{r_2,1}|. \tag{4.1} \]

**Proof**  Notice that \( u \) is touched by the plane 
\[ \Gamma(x) + \langle y - x, \nabla \Gamma(x) \rangle \]
from above at \( x \). From Lemma 2.9, \( \mathcal{M}^+ u(x) \) is defined classically and we get
\[ \mathcal{M}^+ u(x) = q_{\max,s} \int_{\mathbb{R}^n} \frac{\Lambda \delta^+ - \lambda \delta^-}{\|y\|^{c+s}} \, dy. \tag{4.2} \]

We will show that
\[ \delta(y) := \delta(u, x, y) = u(x + y) + u(x - y) - 2u(x) \leq 0. \tag{4.3} \]

In fact, if both \( x - y \in B_3 \) and \( x + y \in B_3 \) then we conclude that \( \delta(y) \leq 0 \), since \( u(x) = \Gamma(x) = p(x) \), for some plane \( p \) that remains above \( u \) in the whole ball \( B_3 \). Moreover, if either \( x - y \notin B_3 \) or \( x + y \notin B_3 \), then both \( x - y \) and \( x + y \) are not in \( B_1 \), and thus \( u(x + y) \leq 0 \) and \( u(x - y) \leq 0 \). Therefore, in any case the inequality (4.3) is proved. Combining (4.2) and (4.3), we find
\[ -f(x) \leq \mathcal{M}^+ u(x) \]
\[ = q_{\max,s} \int_{E_{r_1,1}} \frac{-\lambda \delta^-}{\|y\|^{c+s}} \, dy, \tag{4.4} \]

where \( r_0 = \rho_0 2^{-\frac{1}{q_{\min,s}}} \). Since \( x \in \{ u = \Gamma \} \), we would like to emphasize that \( y \in W_k(x) \) implies \( -y \in W_k(x) \). Hence, we find
\[ W_k(x) \subset E_{r_1,1} \setminus E_{r_2,1,1} \cap \left\{ y : -\delta(y) > 2M \left( \frac{q_{\min,s}}{q_{\max,s}} \right)^{\frac{1}{r_k}} \right\}. \tag{4.5} \]

Using (4.4), we estimate
\[ f(x) \geq c(n, \lambda) \left[ q_{\max,s} \sum_{k=0}^{\infty} \int_{E_{r_1,1} \setminus E_{r_2,1,1}} \frac{\delta^-}{\|y\|^{c+s}} \, dy \right] \]
\[ \geq c(n, \lambda) \sum_{k=0}^{\infty} q_{\max,s} (n^{\frac{c+s}{2}} r_k^{-(c+s)}) \int_{W_k} \delta^- \, dy \tag{4.6} \]

Moreover, we have
\[ |E_{r_1,1} \setminus E_{r_2,1,1}| = \left( \prod_{j=1}^{n} r_{j,1}^{p_{j,1}} \right) |B_1 \setminus E_{l,1}| = r_k^l |B_1 \setminus E_{l,1}|, \]

where \( l = 2^{-c_{\min,s}} \). Therefore, we find
Let us assume by contradiction that (4.1) is not valid. Then, from (4.5), (4.6) and ( ), we obtain

\[ f(x) \geq c_1(n, \lambda, b_{\min}, b_{\max}) \left[ q_{\min,s} \sum_{k=0}^{\infty} \left( 2Mr_k^{q_{\min,s}} f(x) \frac{C_0}{M} \right) \right] \]

\[ = c_2(n, \lambda, b_{\min}, b_{\max}) f(x)C_0 q_{\min,s} \left[ \sum_{k=0}^{\infty} (2r_k^{q_{\min,s}}) \right] \]

\[ \geq c_3(n, \lambda, b_{\min}, b_{\max}) f(x)C_0 \rho_0^{q_{\min,s}} q_{\min,s} \left[ \sum_{k=0}^{\infty} 2^{-k} \right] \]

\[ \geq c_2(n, \lambda, b_{\min}, b_{\max}) f(x)C_0 \rho_0^{q_{\min,s}} q_{\min,s} \left[ \sum_{k=0}^{\infty} 2^{-k} \right] \]

Then, we get

\[ f(x) \geq c_4(n, \lambda, b_{\min}, b_{\max}) f(x) \]

Finally, since \( q_{\min,s} \frac{\rho_0}{1 - 2^{-q_{\min,s}}} \) is bounded away from zero, for all \( s \in \left( 0, \frac{4}{b_{\max}} \right) \), we find

\[ f(x) \geq c_4(n, \lambda, b_{\min}, b_{\max}) f(x) \]

which is a contradiction if \( C_0 \) is chosen large enough.

As in [5], the following result is a direct consequence of the arguments used in the proof of [6, Lemma 8.4].

**Lemma 4.2** Let \( r > 0 \) and \( \Gamma \) be a concave function in \( E_{r,1} \). There exists \( e_0 > 0 \) such that if

\[ \left| E_{r,1} \setminus E_{r,1,1} \right| \geq c(b_{\min}, b_{\max}) r_k^e. \]

for \( 0 < \varepsilon \leq e_0 \), then

\[ \Gamma(y) \geq \Gamma(0) + \langle y, \nabla \Gamma(0) \rangle - h \]

in the whole set \( E_{r,1} \).

**Corollary 4.3** Let \( e_0 > 0 \) be as in Lemma 4.2. Given \( 0 < \varepsilon \leq e_0 \), there exists a constant \( C(n, \lambda, b_{\min}, b_{\max}, \varepsilon) > 0 \) such that for any function \( u \) satisfying the same hypothesis as in Lemma 4.1, there exist \( r \in \left( 0, \rho_0 2^{-\frac{1}{q_{\min,s}}} \right) \) and \( k = k(x) \) such that
\[ E_{r,1} \setminus E_{r,1}^2 \cap \left\{ y : u(x + y) < u(x) + \langle y, \nabla \Gamma(x) \rangle - C \left( \frac{q_{\min,s}}{q_{\max,s}} \right) f(x) \sum_{i=1}^{n} r_i^2 \right\} \]

(4.7)

and

\[ \left| \nabla \Gamma \left( E_{r,1}^2(x) \right) \right| \leq C \left( \frac{q_{\min,s}}{q_{\max,s}} \right)^{n} f(x)^{n} |E_{r,1}^2(x)| , \]

where \( r = \rho_0 2^{-q_{\min,s}} 2^{-q_{\max,s} k} \) and \( l = 2^{-q_{\max,s}} \).

**Proof**  Taking \( M = \frac{C_0}{s C_1} f(x) \) in Lemma 4.1, we obtain (4.7) with \( C_2 := \frac{C_0}{s C_1} \), where

\[ C_1 := \frac{|B_1|}{|B_1 \setminus B_{1/2}|} > 1. \]

Consider the sets

\[ W_{1,r} := E_{r,1} \setminus E_{r,1}^2 \cap \left\{ y : \Gamma(x + y) < u(x) + \langle y, \nabla \Gamma(x) \rangle - C_2 \left( \frac{q_{\min,s}}{q_{\max,s}} \right) f(x) r_{\min}^2 \right\} \]

and

\[ W_{2,r}(x) := E_{r,1} \setminus E_{r,1}^2 \cap \left\{ y : u(x + y) < u(x) + \langle y, \nabla \Gamma(x) \rangle - C_2 \left( \frac{q_{\min,s}}{q_{\max,s}} \right) f(x) r_{\min}^2 \right\} . \]

Then, since

\[ E_{r,1} \setminus E_{r,1}^2 \subset E_{r,1} \setminus E_{r,1}, \quad u(x) = \Gamma(x), \quad \text{and} \quad u(x + y) \leq \Gamma(x + y), \]

for \( y \in E_{r,1} \), we have \( W_{1,r} \subset W_{2,r} \subset W_r(x) \). Thus, from (4.7) we obtain

\[ \left| W_{1,r}(x) \right| \leq \left| W_{2,r}(x) \right| \leq \frac{\varepsilon}{C_1} |E_{r,1} \setminus E_{r,1}| . \]

(4.8)

Moreover, we estimate

\[ \frac{\varepsilon}{C_1} |E_{r,1} \setminus E_{r,1}| = \frac{\varepsilon}{C_1} r^c \left| B_1 \setminus B_{1/2} \right| \left| B_1 \setminus B_{1/2} \right| \]

\[ \leq \frac{\varepsilon}{C_1} r^c C_1 \left| B_1 \setminus B_{1/2} \right| \]

\[ \leq c_0 \left| E_{r,1} \setminus E_{r,1/2} \right| . \]

(4.9)

Then, from Lemma 4.2 and the concavity of \( \Gamma \), we find

\[ 0 \leq F(y) \leq 2C_2 \left( \frac{q_{\min,s}}{q_{\max,s}} \right) f(x) r_{\min}^2 \quad \text{in} \quad E_{r,1}^2, \]

where
\[ F(y) := \Gamma(x+y) - \Gamma(x) - \langle y, \nabla \Gamma(x) \rangle + C_2 \left( \frac{q_{\min,s}}{q_{\max,s}} \right) f(x) r_{\min}^\frac{1}{q_{\min}}. \]

Notice that
\[ \nabla F(x+y) = \nabla \Gamma(x+y) - \nabla \Gamma(x). \]

Then, since \( F \) is concave, we find
\[
|\nabla \Gamma(x+y) - \nabla \Gamma(x)| \leq \frac{\|F\|_{L^\infty}(E_{r,\frac{1}{2}})}{\text{dist}(\partial E_{r,\frac{1}{2}}, E_{r,\frac{1}{2}})} \\
\leq C_2 f(x) \left( \frac{q_{\min,s}}{q_{\max,s}} \right) r_{\min}^\frac{1}{q_{\min}} \\
\leq C_3 \left( \frac{q_{\min,s}}{q_{\max,s}} \right) f(x) r_{\min}^\frac{1}{q_{\min}}.
\]

Thus, we have
\[ \nabla \Gamma \left( E_{r,\frac{1}{2}} \right) \subset B \left( \frac{q_{\min,s}}{q_{\max,s}} \right) f(x) r_{\min}^\frac{1}{q_{\min}} \left( \nabla \Gamma(x) \right) \]
and obtain
\[
|\nabla \Gamma \left( E_{r,\frac{1}{2}} \right)| \leq C_4 \left( \frac{q_{\min,s}}{q_{\max,s}} \right)^n f(x)^n \left| E_{r,\frac{1}{2}} \right|.
\]

Finally, taking \( C = \max\{C_2, C_4\} \), the lemma is proven. \( \square \)

The following covering lemma is a fundamental tool in our analysis.

**Lemma 4.4** (Covering Lemma, [2, Lemma 3]) Let \( S \) be a bounded subset of \( \mathbb{R}^n \) such that for each \( x \in S \) there exists an \( n \)-dimensional rectangle \( R(x) \), centered at \( x \), such that:

- the edges of \( R(x) \) are parallel to the coordinate axes;
- the length of the edge of \( R(x) \) corresponding to the \( i \)-th axis is given by \( h_i(t) \), where \( t = t(x) \), \( h_i(t) \) is an increasing function of the parameter \( t \geq 0 \), continuous at \( t = 0 \), and \( h_i(0) = 0 \).

Then there exist points \( \{x_k\} \) in \( S \) such that

1. \( S \subset \bigcup_{k=1}^\infty R(x_k) \);
2. each \( x \in S \) belongs to at most \( C = C(n) > 0 \) different rectangles.

The Corollary 4.3 and the Covering Lemma 4.4 allow us to obtain a lower bound on the volume of the union of the level sets \( E_{r,1} \) where \( \Gamma \) and \( u \) detach quadratically from the corresponding tangent planes to \( \Gamma \) by the volume of the image of the gradient map, as in the standard ABP estimate.
**Corollary 4.5** For each $x \in \Sigma = \{u = \Gamma\} \cap B_1$, let $E_{r,1}(x)$ be the level set obtained in Corollary 4.3. Then, we have

$$C(\sup u)^n \leq \left| \bigcup_{x \in \Sigma} E_{r,1}(x) \right|.$$  

The nonlocal anisotropic version of the ABP estimate now reads as follows.

**Theorem 4.6** Let $u$ and $\Gamma$ be as in Lemma 4.1. There is a finite family of open rectangles $\{R_j\}_{j \in \{1,\ldots,m\}}$ with diameters $d_j$ such that the following hold:

1. Any two rectangles $R_i$ and $R_j$ in the family do not intersect.
2. $\{u = \Gamma\} \subset \bigcup_{j=1}^m R_j$.
3. $\{u = \Gamma\} \cap R_j \neq \emptyset$ for any $R_j$.
4. $d_j \leq \left( \sum_{i=1}^n \left( \rho_0 2^{-\frac{1}{q_{\min,s}}} \right)^2 \right)^{\frac{1}{2}}$.
5. $|\nabla \Gamma(R_j)| \leq C \left( \max_{R_j} f^+ \right)^n |\tilde{R}_j|$.
6. $\left\{ y \in C\tilde{R}_j : u(y) \geq \Gamma(y) - C \left( \max_{R_j} f \right) \left( \tilde{d}_j \right)^2 \right\} \geq \varsigma |\tilde{R}_j|$,

where $\tilde{d}_j$ is the diameter of the rectangle $\tilde{R}_j$ corresponding to $R_j$. The constants $\varsigma > 0$ and $C > 0$ depend only on $n$, $\lambda$, $\Lambda$, $b_{\min}$, $b_{\max}$, and $s$.

**Proof** We cover the ball $B_1$ with a tiling of rectangles of edges

$$\left( \rho_0 2^{-\frac{1}{q_{\min,s}}} \right)^\frac{1}{2}. $$

We discard all those that do not intersect $\{u = \Gamma\}$. Whenever a rectangle does not satisfy (5) and (6), we split its edges by $2^{n\varsigma}$ and discard those whose closure does not intersect $\{u = \Gamma\}$. Now we prove that all remaining rectangles satisfy (5) and (6) and that this process stops after a finite number of steps.

As in [5] we will argue by contradiction. Suppose the process is infinite. Then, there is a sequence of nested rectangles $R_j$ such that the intersection of their closures will be a point $x_0$. Moreover, since

$$\{u = \Gamma\} \cap \overline{R}_j \neq \emptyset$$

and $\{u = \Gamma\}$ is closed, we have $x_0 \in \{u = \Gamma\}$. Let $0 < \varepsilon_1 < \varepsilon_0$, where $\varepsilon_0$ is as in Corollary 4.3. Thus, there exist

$$r \in \left( 0, \rho_0 2^{-\frac{1}{q_{\min,s}}} \right)$$

and $k_0 = k_0(x_0)$ such that
\[ E_r \setminus E_{r,1} \cap \left\{ y : u(x + y) < u(x) + \langle y, \nabla \Gamma(x) \rangle - Cf(x) \sum_{i=1}^{n} r_i^2 \right\} \leq \varepsilon_1 |E_{r,1} \setminus E_{r,1}| \] \hspace{2cm} (4.10)

and

\[ |\nabla \Gamma(E_{r/4}(x_0))| \leq Cf(x_0)^n |E_{r,1/4}(x_0)|, \] \hspace{2cm} (4.11)

where

\[ r = \rho_0 2^{-\epsilon_{b_{\min}} - \epsilon_{b_{\max}} k_0}. \]

Let \( \mathcal{R}_j \) be the largest rectangle in the family containing \( x_0 \) such that

\[ 2^{-\epsilon_{b_{\min}}(k_0+2)} \left( \rho_0 2^{-\epsilon_{b_{\min}} k_0} \right)^{\frac{2}{3}} \leq I_j < 2^{-\epsilon_{b_{\min}}(k_0+1)} \left( \rho_0 2^{-\epsilon_{b_{\min}} k_0} \right)^{\frac{2}{3}}. \]

Thus, from Lemma 2.1 we obtain

\[ \mathcal{R}_j \subset E_{r,1/4} \quad \text{and} \quad E_{r,1} \subset C \mathcal{R}_j, \]

for some \( C = C(n, b_{\min}, b_{\max}) > 1 \). Furthermore, since \( \Gamma \) is concave in \( B_2 \), we find

\[ \Gamma(y) \leq u(x_0) + \langle y - x_0, \nabla \Gamma(x_0) \rangle \]

in \( B_2 \). Thus, denoting

\[ A_j := \left\{ y \in \mathcal{R}_j : u(y) \geq \Gamma(y) - C \left( \max_{\mathcal{R}_j} f \right) (\tilde{d}_j)^2 \right\}, \]

using (4.10), (4.11), we obtain

\[ |A_j| \geq \left| \{ y \in \mathcal{R}_j : u(y) \geq u(x_0) + \langle y - x_0, \nabla \Gamma(x_0) \rangle \right. \]

\[ -Cf(x_0) \sum_{i=1}^{n} r_i^2 \left\} \right| \]

\[ \geq (1 - \varepsilon_1) |E_{r,1} \setminus E_{r,1/2}| \]

\[ = (1 - \varepsilon_1) r^{\frac{2}{3}} |B_1 \setminus E_{l,1}| \]

\[ = \zeta |\mathcal{R}_j| \]

and

\[ |\nabla \Gamma(\mathcal{R}_j)| \leq |\nabla \Gamma(E_{r,1/4}(x_0))| \]

\[ \leq Cf(x_0)^n |E_{r,1/4}(x_0)| \]

\[ = Cf(x_0)^n r^{\frac{2}{3}} |B_{1/4}(x_0)| \]

\[ = C_2 f(x_0)^n |\mathcal{R}_j|. \]

Then \( \mathcal{R}_j \) would not be split and the process must stop, which is a contradiction. \( \square \)
Remark 4.7 We emphasize that if \( b_{\text{max}} = b_{\text{min}} = 2 \) we recover the ABP estimate obtained in [6]. Furthermore, for \( b_{\text{max}} = n + \sigma_{\text{max}} \) and \( b_{\text{min}} = n + \sigma_{\text{min}} \) with \( \sigma_{\text{max}}, \sigma_{\text{min}} \in (0, 2) \) we have the ABP estimate reached in [5].

4.2 A barrier function

In order to locate the contact set of a solution \( u \) of the maximal equation, as in Lemma 4.1, we build a barrier function which is a supersolution of the minimal equation outside a small ellipse and is positive outside a large ellipse.

Lemma 4.8 Given \( R > 1 \), there exist \( p > 0 \) and \( s_0 \in \left( 0, \frac{4}{b_{\text{max}}} \right) \) such that the function

\[
f(x) = \min(2^p, |x|^{-p})
\]

satisfies

\[
\mathcal{M}^- f(x) \geq 0,
\]

for \( s_0 < s \) and \( 1 \leq |x| \leq R \), where \( p = p(n, \lambda, \Lambda, b_{\text{min}}, b_{\text{max}}, R) \), \( s_0 = s_0(n, \lambda, \Lambda, b_{\text{min}}, b_{\text{max}}, R) \).

Proof Consider the following elementary inequalities:

\[
(a_2 + a_1)^{-l} + (a_2 - a_1)^{-l} \geq 2a_2^{-l} + l(l + 1)a_1^2a_2^{-l-2}
\]

and

\[
(a_2 + a_1)^{-l} \geq a_2^{-l}\left(1 - \frac{a_1}{a_2}\right).
\]

where \( 0 < a_1 < a_2 \) and \( l > 0 \). Suppose without loss of generality that \( b_1 = b_{\text{max}} \). Taking into account the inequalities (4.12) and (4.13), we estimate, for \( |y| < \frac{1}{2} \),

\[
\delta(f, e_1, y) := |e_1 + y|^{-p} + |e_1 - y|^{-p} - 2
\]

\[
= \left(1 + |y|^2 + 2y_1\right)^{-\frac{p}{2}} + \left(1 + |y|^2 - 2y_1\right)^{-\frac{p}{2}} - 2
\]

\[
\geq 2\left(1 + |y|^2\right)^{-\frac{p}{2}} + p(p + 2)y_1^2\left(1 + |y|^2\right)^{-\frac{p+4}{2}} - 2
\]

\[
\geq 2\left(1 - \frac{p}{2}|y|^2\right) + p(p + 2)y_1^2 - p(p + 4)\left(\frac{p + 2}{2}\right)y_1^2|y|^2 - 2
\]

\[
= p\left[-|y|^2 + (p + 2)y_1^2 - (p + 4)\left(\frac{p + 2}{2}\right)y_1^2|y|^2\right].
\]

If \( 1 \leq |x| \leq R \), there is a rotation \( T_x : \mathbb{R}^n \to \mathbb{R}^n \) such that \( x = |x|Te_1 \). Then, changing variables, we obtain

\[
\mathcal{M}^- f(x) = q_{\text{max}}, |x|^{-p} |\det T_x| \left[ \int_{\mathbb{R}^n} \frac{\lambda \delta^+(f, e_1, y) - \lambda \delta^{-}(f, e_1, y)}{\sum_{i=1}^n (|x|T_x y)_i^{b_i} y_i} dy \right].
\]
Thus, we can estimate

\[
|x|^{p-n}M^{-f(x)} = q_{\max,s} \int_{B_{1/4}(0)} \frac{\Lambda \delta^+(f, e_1, y) - \lambda \delta^-(f, e_1, y)}{\left(\sum_{i=1}^{n} ||x|(T_x y)_i|^b\right)^{\frac{1}{2}}} dy
\]

\[
+ q_{\max,s} \int_{B_{1/4}(0)} \frac{\Lambda \delta^+(f, e_1, y) - \lambda \delta^-(f, e_1, y)}{\left(\sum_{i=1}^{n} ||x|(T_x y)_i|^b\right)^{\frac{1}{2}}} dy
\]

\[
\geq q_{\max,s} \int_{B_{1/4}(0)} \frac{2p\lambda(p + 2)y_1^2}{\left(\sum_{i=1}^{n} ||x|(T_x y)_i|^b\right)^{\frac{1}{2}}} dy
\]

\[
- q_{\max,s} \int_{B_{1/4}(0)} \frac{2p\Lambda y^2}{\left(\sum_{i=1}^{n} ||x|(T_x y)_i|^b\right)^{\frac{1}{2}}} dy
\]

\[
- q_{\max,s} \int_{B_{1/4}(0)} \frac{\Lambda_1 p(p + 4)(p + 2)y^4}{\left(\sum_{i=1}^{n} ||x|(T_x y)_i|^b\right)^{\frac{1}{2}}} dy
\]

\[
+ q_{\max,s} \int_{B_{1/4}(0)} \frac{-\lambda 2^{p+1}}{\left(\sum_{i=1}^{n} ||x|(T_x y)_i|^b\right)^{\frac{1}{2}}} dy
\]

\[
:= I_1 + I_2 + I_3 + I_4,
\]

where $I_1$, $I_2$, $I_3$ and $I_4$ represent the three terms on the right-hand side of the above inequality.

Changing variables again, we get

\[
\int_{B_{1/4}(0)} \frac{y_1^2}{\left(\sum_{i=1}^{n} ||x|(T_x y)_i|^b\right)^{\frac{1}{2}}} dy = \int_{B_{1/4}(0)} \frac{y_1^2}{\left(\sum_{i=1}^{n} ||x|(T_x y)_i|^b\right)^{\frac{1}{2}}} dy
\]

\[
= \int_{B_{1/4}(0)} \frac{||x|-1}{||y||^{c+x}} \frac{1}{|x|^{-n}} dy
\]

\[
= |x|^{-n} \int_{B_{1/4}(0)} \frac{(T_x^1 y, x_1)^2}{||y||^{c+x}} dy
\]

\[
= |x|^{-n} \int_{B_{1/4}(0)} \frac{(y, x)^2}{||y||^{c+x}} dy.
\]

Moreover, without loss of generality, we can assume that

\[
x \in \{y \in \mathbb{R}^n : x_1 \geq 0\} \quad \text{and} \quad x_1 \geq \frac{1}{n}.
\]

From Lemma 2.1 there exists $r_0 = r_0(n, b_{\min}, b_{\max}) \in (0, 1)$ such that $E_{r_1} \subset B_{1/4}$. Then, from (4.15) we estimate
\[ p^{-1}I_1 \geq q_{\max,s}n^{-1} \lambda (p + 2) |x|^{-(n + 2)} \int_{E_{n+1}} \frac{y_i^2}{\|y\|^{c+s}} dy, \]
\[ \geq c(n, b_{\min}, b_{\max}) R^{-(n+2)} n^{-1} \lambda (p + 2) \int_{\partial B_1} y_i^2 d\nu(y) \left[ \frac{\int_{0}^{\theta_0} q_{\max,s}}{1 - 2^{-q_{\max,s}}} \right] \]
\[ \geq C_3 \left[ (p + 2) \int_{\partial B_1} y_i^2 d\nu(y) \right], \]
where \( C_3 = C_3(n, \lambda, \Lambda, b_{\min}, b_{\max}, R) > 0. \) Let \( C = C(n, b_{\max}, b_{\min}) \) be a positive constant such that \( B_{1/4}(0) \subset E_{C,1}. \) Then, for \( |x| \geq 1 \) we get
\[ p^{-1}I_2 \geq - C_4 q_{\max,s} \int_{B_{1/4}(0)} \frac{|y|^2}{\left( \sum_{i=1}^{n} \| (T_{x}y) \|_{h_i} \right)^{2p}} dy \]
\[ = - C_4 q_{\max,s} |\det T_x^{-1}| \int_{B_{1/4}(0)} \frac{|y|^2}{\|y\|^{c+s}} dy \]
\[ = - C_4 R^{-n} q_{\max,s} \int_{B_{1/4}(0)} \frac{|y|^2}{\|y\|^{c+s}} dy \]
\[ \geq - C_5 q_{\max,s} \int_{E_{C,1}} \frac{|y|^2}{\|y\|^{c+s}} dy, \]
where \( C_4 = C_4(n, \lambda, \Lambda, b_{\min}, b_{\max}, R). \) We have also
\[ q_{\max,s} \int_{E_{C,1}} \frac{|y|^2}{\|y\|^{c+s}} dy = q_{\max,s} \sum_{k=1}^{\infty} \int_{E_{k+1} \setminus E_{k+1,1}} \frac{|y|^2}{\|y\|^{c+s}} dy \leq C_5, \]
where \( r_k := C2^{-k} \) and \( C_5 = C_5(n, \lambda, \Lambda, b_{\max}, b_{\min}). \) Moreover, we have
\[ I_3 \geq - C_6 q_{\max,s} \int_{E_{C,1}} \frac{|y|^4}{\|y\|^{c+s}} dy \]
\[ \geq - C_7 \left[ \underbrace{\frac{\int_{E_{C,1}} \frac{|y|^4}{\|y\|^{c+s}} dy}_{q_{\max,s}}}_{1 - 2^{-\left( \frac{16}{b_{\max}} - s \right)}}, \right] \]
and, if \( r_1 = r_1(r_0) > 0 \) is such that \( \Theta_{r_1} \subset E_{r_1,1}, \) we obtain
\[ I_4 \geq - C_8 q_{\max,s} \int_{\mathbb{R}^n \setminus E_{r_1}} \frac{|y|^4}{\|y\|^{c+s}} dy \]
\[ \geq - C_8 q_{\max,s} \int_{\mathbb{R}^n \setminus \Theta_{r_1}} \frac{|y|^4}{\|y\|^{c+s}} dy \]
\[ \geq - C_9 \left( \frac{1}{b_{\max}^{s}} \right) \]
for positive constants \( C_7 = C_7(n, \lambda, \Lambda, b_{\min}, b_{\max}, p) \) and \( C_8 = C_8(n, \lambda, \Lambda, b_{\min}, b_{\max}, p). \) Choosing \( p = p(n, \lambda, \Lambda, b_{\min}, b_{\max}, R) > 0 \) such that
\[ C_3(p + 2) \int_{\partial B_1} y_1^2 d\nu(y) - C_4 C_5 > 0 \]

and combining (4.14), (4.18) and (4.19), there is a positive constant \( s_0 = s_0(n, \lambda, \Lambda, b_{\text{min}}, b_{\text{max}}, R) < \frac{4}{b_{\text{max}}} \) such that

\[ |x|^{p-n} M f(x) \geq C_9 > 0, \]

for a positive constant \( C_9 = C_9(n, \lambda, \Lambda, b_{\text{min}}, b_{\text{max}}, R) \).

As in [5], from Lemma 4.8 we get the following results:

**Corollary 4.9** Given \( s_0 \in \left( 0, \frac{4}{b_{\text{max}}} \right) \), and \( R > 1 \), there exist \( \kappa > 0 \) and \( p > 0 \) such that the function

\[ f(x) = \min(\kappa^{-p}, |x|^{-p}) \]

satisfies

\[ \mathcal{M}^\gamma f(x) \geq 0, \]

for \( 1 \leq |x| \leq R \) and \( s_0 < s < \frac{4}{b_{\text{max}}} \), where \( p = p(n, \lambda, \Lambda, b_{\text{min}}, b_{\text{max}}, R) \) and \( \kappa = \kappa(n, \lambda, \Lambda, b_{\text{min}}, b_{\text{max}}, s_0, R) \).

**Corollary 4.10** Given \( r > 0 \), \( R > 1 \) and \( s_0 \in \left( 0, \frac{4}{b_{\text{max}}} \right) \), there exist \( \kappa > 0 \) and \( p > 0 \) such that the function

\[ g(x) = \min(\kappa^{-p}, |T_{\beta,r}^{-1} x|^{-p}) \]

satisfies

\[ \mathcal{M}^\gamma g(x) \geq 0 \]

for \( s_0 < s < \frac{4}{b_{\text{max}}} \) and \( x \in E_{r,R} \setminus E_{r,1} \), where \( p = p(n, \lambda, \Lambda, b_{\text{min}}, b_{\text{max}}, R) \) and \( \kappa = \kappa(n, \lambda, \Lambda, b_{\text{min}}, b_{\text{max}}, s_0, R) \).

**Lemma 4.11** Given \( s_0 \in \left( 0, \frac{4}{b_{\text{max}}} \right) \), there is a function \( \Psi : \mathbb{R}^n \to \mathbb{R} \) satisfying

1. \( \Psi \) is continuous in \( \mathbb{R}^n \);
2. \( \Psi = 0 \) for \( x \in \mathbb{R}^n \setminus E_{\frac{1}{2},3\sqrt{n}} \);
3. \( \Psi > 3 \) for \( x \in \mathcal{R}_{\frac{4}{3}} \);
4. \( \mathcal{M}^\gamma \Psi(x) > -\varphi(x) \) for some positive function \( \varphi \in C_0\left( E_{\frac{1}{4},1} \right) \) for \( s_0 < s < \frac{4}{b_{\text{max}}} \).

### 4.3 Harnack inequality and regularity

The next lemma is the fundamental tool towards the proof of the Harnack inequality. It bridges the gap between a pointwise estimate and an estimate in measure, see [5].
Lemma 4.12 Let $0 < s_0 < \frac{4}{b_{\text{max}}}$ If $s \in \left(s_0, \frac{4}{b_{\text{max}}} \right)$, then there exist constants $\varepsilon_0 > 0$, $0 < \varsigma < 1$, and $M > 1$, depending only on $s_0, s, \lambda, b_{\text{min}}, b_{\text{max}}$, and $n$, such that if

1. $u \geq 0$ in $\mathbb{R}^n$;
2. $u(0) \leq 1$;
3. $\mathcal{M}^- u \leq \varepsilon_0$ in $E_{\frac{b_{\text{max}}}{2k}^n}$.

then

$$|\{u \leq M\} \cap Q_1| > \varsigma.$$ 

The next lemma is fundamental to iterate Lemma 4.12 and to get the $L_\infty$ decay of the distribution function $\lambda_u := |\{u > t\} \cap B_1|$. Since our scaling is anisotropic, the following Calderón–Zygmund decomposition is performed with $n$-dimensional rectangles that satisfy the covering lemma of Caffarelli–Calderón (Lemma 4.4). We can then apply Lebesgue’s differentiation theorem having these $n$-dimensional rectangles as a differentiation basis, see Lemma 5.2 in [5].

Lemma 4.13 Let $u$ be as in Lemma 4.12. Then

$$|\{u > M^k\} \cap Q_1| \leq C(1 - \varsigma)^k, \quad k = 1, \ldots,$$

where $M$ and $\varsigma$ are as in Lemma 4.12. Thus, there exist positive constants $d$ and $\varepsilon$, depending only on $s_0, s, \lambda, \Lambda, b_{\text{min}}, b_{\text{max}}$, and $n$ such that

$$|\{u \geq t\} \cap Q_1| \leq dt^{-\varepsilon}, \quad \forall t > 0.$$ 

Using standard covering arguments we get the following theorem.

Theorem 4.14 Let $u \geq 0$ in $\mathbb{R}^n$, $u(0) \leq 1$ and $\mathcal{M}^- u \leq \varepsilon_0$ in $B_2$. Suppose that $s_0 < s < \frac{4}{b_{\text{max}}}$ for some $s_0 > 0$. Then

$$|\{u \geq t\} \cap B_1| \leq Ct^{-\varepsilon}, \quad \forall t > 0,$$

where $C = C(n, \lambda, \Lambda, b_{\text{max}}, b_{\text{min}}, s_0, s) > 0$ and $\varepsilon = \varepsilon(n, \lambda, \Lambda, b_{\text{max}}, b_{\text{min}}, s_0, s) > 0$.

Remark 4.15 For each $l > 0$, we will denote $E_{l,2}^l := E_{l,2}$. Let $u \geq 0$ in $\mathbb{R}^n$ and $\mathcal{M}^- u \leq C_0$ in $E_{l,2}^l$, with $0 < r \leq 1$. We consider the anisotropic scaling

$$v(x) = \frac{u(T_{j,\beta, r}x)}{u(0) + C_0r^{\frac{\delta}{2}}}, \quad x \in \mathbb{R}^n,$$

where $T_{j,\beta, r} : \mathbb{R}^n \to \mathbb{R}^n$ is defined by
We find \( v \geq 0 \) in \( \mathbb{R}^n \), \( v(0) \leq 1 \) and \( T_{j,\beta,r}(B_2) = E_{r,2}^j \). Moreover, changing variables, we estimate

\[
M^{-} v(x) = \frac{r_j^{\beta}}{u(0) + C_0 r_j^{\beta}} M^{-}(T_{j,\beta,r} x) \leq 1, 
\]

for all \( x \in B_2 \).

Then, using the anisotropic scaling \( T_{j,\beta,r} \) and Theorem 4.14 we have the following scaled version.

**Theorem 4.16** (Pointwise Estimate) Let \( u \geq 0 \) in \( \mathbb{R}^n \) and \( M^{-} u \leq C_0 \) in \( E_{r,2}^j \). Suppose that \( s_0 < s < \frac{4}{b_{\text{max}}} \) for some \( s_0 > 0 \). Then

\[
|\{ u \geq t \} \cap E_{r,1}^j | \leq C |E_{r,1}^j| \left( u(0) + C_0 r_j^{\beta} \right)^{\epsilon} t^{-\epsilon} \quad \forall t > 0, 
\]

where \( C = C(n, \lambda, \Lambda, b_{\text{min}}, b_{\text{max}}, s_0, s) > 0 \) and \( \epsilon = \epsilon(n, \lambda, \Lambda, b_{\text{min}}, b_{\text{max}}, s_0, s) > 0 \).

We are now ready to prove the Harnack inequality.

**Theorem 4.17** (Harnack Inequality) Let \( u \geq 0 \) in \( \mathbb{R}^n \), \( M^{-} u \leq C_0 \), and \( M^{+} u \geq -C_0 \) in \( B_2 \). Suppose that \( s_0 < s < \frac{4}{b_{\text{max}}} \) for some \( s_0 > 0 \). Then

\[
u \leq C(u(0) + C_0) \quad \text{in} \quad B_{\frac{1}{2}}, 
\]

**Proof** Without loss of generality, we can suppose that \( u(0) \leq 1 \) and \( C_0 = 1 \). Let

\[
\tau = \frac{c b_{\text{max}}}{2 \epsilon},
\]

where \( \epsilon > 0 \) is as in Theorem 4.14. For each \( \vartheta > 0 \), we define the function

\[
f_{\vartheta}(x) := \vartheta (1 - |x|)^{-\tau}, \quad x \in B_1.
\]

Let \( t > 0 \) be such that \( u \leq f_{\vartheta} \) in \( B_1 \). There is an \( x_0 \in B_1 \) such that \( u(x_0) = f_t(x_0) \). Let \( d := (1 - |x_0|) \) be the distance from \( x_0 \) to \( \partial B_1 \).

We will estimate the portion of the ellipsoid \( E_{r,1}^{\text{max}}(x_0) \) covered by \( \{ u > \frac{u(x_0)}{2} \} \) and by \( \{ u < \frac{u(x_0)}{2} \} \). As in [6], we will prove that \( t > 0 \) cannot be too large. Thus, since

\[
\tau \leq \frac{C(n, b_{\text{min}}, b_{\text{max}})}{\epsilon},
\]

we conclude the proof of the theorem. By Theorem 4.14, we have
\[
\left\{ u > \frac{u(x_0)}{2} \right\} \cap B_1 \leq C \left| \frac{2}{u(x_0)} \right|^\varepsilon = C t^{-\varepsilon} d^n \leq C_1 t^{-\varepsilon} r_{\text{max}}^{-\varepsilon},
\]
where \( r = \frac{d}{2} \). Thus, we get
\[
\left\{ u > \frac{u(x_0)}{2} \right\} \cap E_{\text{max}}^{\max}(x_0) \leq C_1 t^{-\varepsilon} |E_{\text{max}}^{\max}|. \tag{4.20}
\]

Now we will estimate \( \left\{ u > \frac{u(x_0)}{2} \right\} \cap E_{\text{max}}^{\max}(x_0) \), where \( 0 < \theta < 1 \). Since
\[
|x| \leq |x - x_0| + |x_0|, \quad \forall x \in \mathbb{R}^n,
\]
we have
\[
(1 - |x|) \geq \left[ d - \frac{d\theta}{2} \right],
\]
for \( x \in B_{r \theta}(x_0) \). Hence, if \( x \in B_{r \theta}(x_0) \), we get
\[
u(x) \leq f_t(x) \leq t(1 - |x|)^{-\varepsilon} \leq u(x_0) \left( 1 - \theta \frac{d}{2} \right)^{-\varepsilon}.
\]

Then, since \( M^+ u \geq -1 \), the function
\[
v(x) = \left( 1 - \theta \frac{d}{2} \right)^{-\varepsilon} u(x_0) - u(x)
\]
satisfies
\[
v \geq 0 \quad \text{in } B_{r \theta}(x_0) \quad \text{and} \quad M^+ v \leq 1.
\]

We will consider the function \( w := v^+ \). For \( x \in \mathbb{R}^n \) we have
\[
M^+ w(x) = M^+ v(x) + (M^+ w(x) - M^+ v(x))
\]
and
\[
\frac{M^+ w(x) - M^+ v(x)}{q_{\text{max}, s}} = \lambda \int_{\mathbb{R}^n} \frac{\delta^+(w, x, y) - \delta^+(v, x, y)}{\|y\|^{c+s}} dy + \Lambda \int_{\mathbb{R}^n} \frac{\delta^-(v, x, y) - \delta^-(w, x, y)}{\|y\|^{c+s}} dy
\]
\[
= I_1 + I_2,
\]
where \( I_1 \) and \( I_2 \) represent the two terms in the right-hand side above. Using the elementary equality
\[
v^+(x + y) = v(x + y) + v^-(x + y),
\]
and denoting \( \delta_w := \delta(w, x, y) \) and \( \delta_v := \delta(v, x, y) \), we obtain
\[
\delta^+_w = \delta_v + v^-(x - y) + v^+(x + y).
\]
Thus, taking in account that
\[ \delta_w^+ \geq \delta_v^+ \text{ and } \delta_v = \delta_v^+ - \delta_v^- \]

we estimate

\[
I_1 = -\lambda \int_{\{\delta_v^+ \geq \delta_v^+, \text{ and } \delta_v \}} \frac{\delta_v^-}{||y||^{c+s}} dy
+ \lambda \int_{\{\delta_v^+ > \delta_v^+, \text{ and } \delta_v \}} \frac{v^-(x+y) + v^-(x-y)}{||y||^{c+s}} dy
\leq \Lambda \int_{\{\delta_v^+ > 0\}} \frac{v^-(x+y) + v^-(x-y)}{||y||^{c+s}} dy.
\]

(4.21)

Analogously, we get

\[
I_2 = \Lambda \int_{\{\delta_v^+ > 0\} \cap \{\delta_v^+ \neq \delta_v^-\}} \frac{\delta_v^- - \delta_w^-}{||y||^{c+s}} dy
+ \Lambda \int_{\{\delta_v^+ = 0\} \cap \{\delta_v^+ \neq \delta_v^-\}} \frac{v^-(x+y) + v^-(x-y)}{||y||^{c+s}} dy
\leq \Lambda \int_{\{\delta_v^+ > 0\} \cap \{\delta_v^+ \neq \delta_v^-\}} \frac{-\delta_v^- - \delta_v^-}{||y||^{c+s}} dy.
\]

(4.22)

We also have

\[
-\delta_v^- - \delta_w^- = 2v(x) - (v(x+y) + v(x-y)) - \delta_w^-
= 2v(x) - [(v^+(x+y) + v^+(x-y))
-(v^-(x+y) + v^-(x-y))]
= (-\delta_w^- - \delta_w^-) + v^+(x+y) + v^+(x-y)
= -\delta_w^+ + v^+(x+y) + v^+(x-y).
\]

(4.23)

Then, from (4.23) and (4.22), we obtain

\[
I_2 \leq -\Lambda \int_{\{\delta_v^+ > 0\} \cap \{\delta_v^+ \neq \delta_v^-\}} \frac{\delta_w^+}{||y||^{c+s}} dy
+ \Lambda \int_{\{\delta_v^+ > 0\} \cap \{\delta_v^+ \neq \delta_v^-\}} \frac{v^-(x+y) + v^-(x-y)}{||y||^{c+s}} dy
\leq \Lambda \int_{\{\delta_v^+ > 0\}} \frac{v^-(x+y) + v^-(x-y)}{||y||^{c+s}} dy.
\]

(4.24)

Hence, using (4.21), (4.24), and changing variables, we find

\[
\frac{\mathcal{M}^- w(x) - \mathcal{M}^- v(x)}{q_{\text{max}, s}} \leq \Lambda \int_{B^n} \frac{v^-(x+y) + v^- (x-y)}{||y||^{c+s}} dy
= -2\Lambda \int_{\{v(x+y) < 0\}} \frac{v(x+y)}{||y||^{c+s}} dy.
\]

Moreover, if \( x \in B^\tau(x_0) \), we have
\[\frac{M^-w(x) - M^-v(x)}{q_{\text{max},s}} \leq 2\Lambda \int_{\mathbb{R}^n \setminus B_{d}(x_0 - x)} \frac{-v(x + y)}{||y||^{c+s}} dy \]
\[\leq 2\Lambda \int_{\mathbb{R}^n \setminus B_{d}(x_0 - x)} \frac{(u(x + y) - (1 - \frac{\theta}{2})^{-\tau}u(x_0))^+}{||y||^{c+s}} dy.\]

If \( t > 0 \) is the largest value such that \( u(x) \geq t \left(1 - |4x|^2\right) \), then there is a point \( x_1 \in B_{\frac{1}{2}} \) such that \( u(x_1) = \left(1 - |4x_1|^2\right) \). Moreover, since \( u(0) \leq 1 \), we get \( t \leq 1 \). Then, we have

\[q_{\text{max},s} \int_{\mathbb{R}^n} \frac{\delta^-(u, x_1, y)}{||y||^{c+s}} dy \leq q_{\text{max},s} \int_{\mathbb{R}^n} \frac{\delta^-(\left(1 - |4x|^2\right), x_1, y)}{||y||^{c+s}} dy \leq C,\]

where the constant \( C > 0 \) is independent of \( s \). Moreover, since \( M^- u(x_1) \leq 1 \), we find

\[q_{\text{max},s} \int_{\mathbb{R}^n} \frac{\delta^+(u, x_1, y)}{||y||^{c+s}} dy \leq C.\]

Recall that \( u(x_1 - y) \geq 0 \) and \( u(x_1) \leq 1 \). Thus, we obtain

\[q_{\text{max},s} \int_{\mathbb{R}^n} \frac{(u(x_1 + y) - 2)^+}{||y||^{c+s}} dy \leq C.\]

Since \( t > 0 \) is large enough, we can suppose that \( u(x_0) > 2 \). Let

\[x \in E_{\text{max}}^{\max}(x_0) \subset B_{d}(x_0) \subset B_{\frac{2}{3}}(x_0)\]

and

\[y \in \mathbb{R}^n \setminus B_{\rho}(x_0 - x) \subset \mathbb{R}^n \setminus E_{\rho,1}^{\max}(x_0 - x).\]

Then, we have the inequalities

\[||y + x + x_1|| \leq C(||y|| + ||x|| + ||x_1||) \leq C||y|| + 2C\]

and

\[|y_1| \geq |(y - (x_0 - x_1))_1| - |(x_0 - x)_1| \geq \frac{(r\theta)^{b_{\text{max}}}}{n^{1/2}} - \frac{(r\theta)^{b_{\text{max}}}}{2n} \geq \frac{(r\theta)^{b_{\text{max}}}}{n} - \frac{(r\theta)^{b_{\text{max}}}}{2n} \geq \frac{(r\theta)^{b_{\text{max}}}}{2n}.\]

Then, taking into account the obvious equalities

\[u(x + y) - \left(1 - \frac{\theta}{2}\right)^{-\tau} u(x_0) = u(x + x_1 + y - x_1) - \left(1 - \frac{\theta}{2}\right)^{-\tau} u(x_0),\]
and
\[ \frac{1}{\|y\|^{c+s}} = \frac{1}{\|y + x + x_1\|^{c+s}} \frac{\|y + x + x_1\|^{c+s}}{\|y\|^{c+s}}, \]
we estimate
\[ 2\Lambda \int_{\mathbb{R}^n \setminus E_{\max}^{x_1}(x_0 - x)} \frac{(u(x + y) - (1 - \frac{\theta}{2})^{-\tau} u(x_0))}{\|y\|^{c+s}} dy \leq C_1(\theta r)^{-\frac{\max(c+s)}{2}}. \]
Thus, we have
\[ \mathcal{M}^{-} w \leq C_1(\theta r)^{-\frac{\max(c+s)}{2}} \text{ in } E_{\max}^{x_1}(x_0). \]
Applying Theorem 4.16 to \( w \) in \( E_{\max}^{x_1}(x_0) \subset B_{\frac{r}{2}}(x_0 - x) \) and using that
\[ w(x_0) = \left( \left( 1 - \frac{\theta}{2} \right)^{-\tau} - 1 \right) u(x_0), \]
we get
\[ \left\{ u > \frac{u(x_0)}{2} \right\} \setminus E_{\max}^{x_1/2} \]
\[ = \left\{ w > \left( \left( 1 - \frac{\theta}{2} \right)^{-\tau} - \frac{1}{2} \right) u(x_0) \right\} \setminus E_{\max}^{x_1/2} \]
\[ \leq C \left| E_{\max}^{x_1/2} \right| \left[ \left( \left( 1 - \frac{\theta}{2} \right)^{-\tau} - \frac{1}{2} \right) u(x_0) + C_1(\theta r)^{-\frac{\max(c+s) + \epsilon \max}{2}} \right]^\epsilon \]
\[ \cdot \left[ \left( \left( 1 - \frac{\theta}{2} \right)^{-\tau} - \frac{1}{2} \right) u(x_0) \right]_{-\epsilon} \]
\[ = C \left| E_{\max}^{x_1/2} \right| \left[ \left( \left( 1 - \frac{\theta}{2} \right)^{-\tau} - \frac{1}{2} \right) u(x_0) + C_1(\theta r)^{-\frac{\max}{2}} \right]^\epsilon \]
\[ \cdot \left[ \left( \left( 1 - \frac{\theta}{2} \right)^{-\tau} - \frac{1}{2} \right) u(x_0) \right]_{-\epsilon}. \]
(4.25)
Thus, using (4.25) and the elementary inequalities
\[ \left[ \left( \left( 1 - \frac{\theta}{2} \right)^{-\tau} - \frac{1}{2} \right) u(x_0) + C_1(\theta r)^{-\frac{\max}{2}} \right]^\epsilon \]
\[ \leq \left( \left( 1 - \frac{\theta}{2} \right)^{-\tau} - \frac{1}{2} \right)^\epsilon u(x_0) + C_1(\theta r)^{-\frac{\max + \epsilon}{2}} \]
and
\[ \left( 1 - \frac{\theta}{2} \right)^{-\tau} - \frac{1}{2} = \left( 1 - \frac{\theta}{2} \right)^{-\frac{\epsilon \max}{2}} - \frac{1}{2} \geq \frac{1}{2}, \]
for \( \theta > 0 \) sufficiently small, and yet
\[ C_3 \theta^{-\frac{c_{\text{max}}}{2}} r^{-\frac{c_{\text{max}}}{2}} u(x_0)^{-\varepsilon} \left( \left( 1 - \frac{\theta}{2} \right)^{-\varepsilon} - 1 \right) \leq C_4 \theta^{-\frac{c_{\text{max}}}{2}} r^{-\frac{c_{\text{max}}}{2}} u(x_0)^{-\varepsilon} \leq C_5 \theta^{-\frac{c_{\text{max}}}{2}} r^{-\varepsilon} d(1-\varepsilon)^{1/2} \leq C_6 \theta^{-\frac{c_{\text{max}}}{2}} r^{-\varepsilon}, \]

we obtain

\[ \left\{ u > \frac{u(x_0)}{2} \right\} \cap E_{\text{max}}^{\frac{n}{n+2}} \leq C E_{\text{max}}^{\frac{n}{n+2}} \left[ \left( \left( 1 - \frac{\theta}{2} \right)^{-\varepsilon} - 1 \right) + \theta^{-\frac{c_{\text{max}}}{2}} r^{-\varepsilon} \right]. \]

Now we choose \( \theta > 0 \) sufficiently small such that

\[ C E_{\text{max}}^{\frac{n}{n+2}} \left[ \left( 1 - \frac{\theta}{2} \right)^{-\varepsilon} - 1 \right] \leq \frac{1}{4} E_{\text{max}}^{\frac{n}{n+2}}. \]

Having fixed \( \theta > 0 \) (independently of \( t \)), we take \( t > 0 \) sufficiently large such that

\[ C E_{\text{max}}^{\frac{n}{n+2}} \theta^{-\frac{c_{\text{max}}}{2}} r^{-\varepsilon} \leq \frac{1}{4} E_{\text{max}}^{\frac{n}{n+2}}. \]

Then, using (4.25), we find

\[ \left\{ u > \frac{u(x_0)}{2} \right\} \cap E_{\text{max}}^{\frac{n}{n+2}} \leq \frac{1}{4} E_{\text{max}}^{\frac{n}{n+2}}. \]

Hence, we have, for \( t > 0 \) large,

\[ \left\{ u \leq \frac{u(x_0)}{2} \right\} \cap E_{\text{max}}^{\frac{n}{n+2}} \geq c \theta^{-\frac{c_{\text{max}}}{2}} E_{r,1} \geq c_2 E_{r,1}^{\frac{n}{n+2}}, \]

which is a contradiction to (4.20). \( \square \)

As a consequence of the Harnack inequality we obtain the \( C^\gamma \) regularity.

**Theorem 4.18** (\( C^\gamma \) estimates) Let \( u \) be a bounded function such that

\[ \mathcal{M}^- u \leq C_0 \quad \text{and} \quad \mathcal{M}^+ u \geq -C_0 \quad \text{in} \quad B_1. \]

If \( 0 < s_0 < s < \frac{4}{c_{\text{max}}} \), then there is a positive constant \( 0 < \gamma < 1 \), that depends only \( n, \lambda, A, b_{\text{min}}, b_{\text{max}}, s_0, \) and \( s \), such that \( u \in C^\gamma(B_{1/2}) \) and

\[ |u|_{C^\gamma(B_{1/2})} \leq C \left( \sup_{B_1} |u| + C_0 \right), \]

for some constant \( C > 0 \).

The next results are consequences of the arguments used in [5, 6] and Theorem 4.18. As in [5, 6], if we suppose a modulus of continuity of \( K_{a\beta} \) in measure, then as to make sure that
faraway oscillations tend to cancel out, we obtain the interior $C^{1,\gamma}$ regularity for solutions of equation $Iu = 0$.

**Theorem 4.19** ($C^{1,\gamma}$ estimates) Suppose that $0 < s_0 < s < \frac{1}{b_{\text{max}}}$, there exists a constant $\tau_0 > 0$, that depends only on $\lambda$, $\Lambda$, $n$, $b_{\text{min}}$, $b_{\text{max}}$, $s_0$ and $s$, such that

$$\int_{\mathbb{R}^n \setminus B_{\tau_0}} \frac{|K(y) - K(y - h)|}{|h|} \, dy \leq C_0, \quad \text{whenever } |h| \leq \frac{\tau_0}{2}.$$

If $u$ is a bounded function satisfying $Iu = 0$ in $B_1$, then there is a constant $0 < \gamma < 1$, that depends only on $n$, $\lambda$, $\Lambda$, $b_{\text{min}}$, $b_{\text{max}}$, $s_0$ and $s$, such that $u \in C^{1,\gamma}(B_{1/2})$ and

$$|u|_{C^{1,\gamma}(B_{1/2})} \leq C \sup_{\mathbb{R}^n} |u|,$$

for some constant $C = C(n, \lambda, \Lambda, b_{\text{min}}, b_{\text{max}}, s_0, s, C_0) > 0$.

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**References**

1. Bucur, C., Valdinoci, E.: Nonlocal diffusion and applications. arxiv:1504.08292v6
2. Caffarelli, L.A., Calderón, C.P.: Weak type estimates for the Hardy–Littlewood maximal functions. Stud. Math. 49, 217–223 (1974)
3. Caffarelli, L.A., Calderón, C.P.: On Abel summability of multiple Jacobi series. Colloq. Math. 30, 277–288 (1974)
4. Chaker, J., Kassmann, M.: Nonlocal operators with singular anisotropic kernels. Commun. Partial Differ. Equ. 45(1), 1–31 (2020)
5. Caffarelli, L., Leitão, R., Urbano, J.M.: Regularity for anisotropic fully nonlinear integro-differential equations. Math. Ann. 360(3–4), 681–714 (2014)
6. Caffarelli, L.A., Silvestre: Regularity theory for fully nonlinear integro-differential equations. Comm. Pure Appl. Math. 62, 597–638 (2009)
7. Caffarelli, L., Silvestre, L.: An extension problem related to the fractional Laplacian. Commun. Partial Differ. Equ. 45(32), 1245–1260 (2007)
8. Caffarelli, L., Salsa, S., Silvestre, L.: Regularity estimates for the solution and the free boundary to the obstacle problem for the fractional Laplacian. Invent. Math. 171(2), 425–461 (2008)
9. Caffarelli, L., Teimurzyan, R., Urbano, J.M.: Fully nonlinear integro-differential equations with deforming kernels. Commun. Partial Differ. Equ. 45(8), 847–871 (2020)
10. De Giorgi, E.: Sulla differenziabilita e l’analiticità delle estremali degli integrali multipli regolari. Mem. Accad. Sci. Torino. Cl. Sci. Fis.Mat. Nat. (3) 3, 25–43 (1957)
11. Hanyga, A., Magin, R.L.: A new anisotropic fractional model of diffusion suitable for applications of diffusion tensor imaging in biological tissues. Proc. R. Soc. A 470, 20140319 (2014)
12. Hanyga, A., Seredynska, M.: Anisotropy in high-resolution diffusion-weighted MRI and anomalous diffusion. J. Magn. Reson. 220, 85–93 (2012). https://doi.org/10.1016/j.jmr.2012.05.001
13. Landis, E.M.: Second Order Equations of Elliptic and Parabolic Type, Translations of Mathematical Monographs, vol. 171. American Mathematical Society, Providence (1998)
14. Leitão, R.: Almgren’s frequency formula for an extension problem related to the anisotropic fractional Laplacian. Rev. Mat. Iberoam. 36(3), 641–660 (2020)
15. Meerschaert, M., Magin, R., Ye, A.: Anisotropic fractional diffusion tensor imaging. J. Vib. Control 22(9), 2211–2221 (2016). https://doi.org/10.1177/1077546314568696
16. Orovio, A., Teh, I., Schneider, J., Burrage, K., Grau, V.: Anomalous diffusion in cardiac tissue as an index of myocardial microstructure. IEEE Trans. Med. Imaging 35(9), 2200–2207 (2016). https://doi.org/10.1109/TMI.2016.2548503
17. Reich, N.: Anisotropic operator symbols arising from multivariate jump processes. Integral Equ. Oper. Theory 63, 127–150 (2009)
18. Silvestre, L.: Hölder estimates for solutions of integro-differential equations like the fractional Laplace. Indiana Univ. Math. J. 55, 1155–1174 (2006)
19. Silvestre, L.: Regularity of the obstacle problem for a fractional power of the Laplace operator. Commun. Pure Appl. Math. 60(1), 67–112 (2007)