Absence of the non-percolating phase for percolation on the non-planar Hanoi network

Takehisa Hasegawa
Graduate School of Information Sciences, Tohoku University, 6-3-09, Aramaki-Aza-Aoba, Sendai, 980-8579, Japan

Tomoki Nogawa
Department of Mathematics, Tohoku University, 6-3-09, Aramaki-Aza-Aoba, Sendai, Miyagi 980-8579, Japan

(Dated: May 6, 2014)

We investigate bond percolation on the non-planar Hanoi network (HN-NP), which was studied in [Boettcher et al. Phys. Rev. E 80 (2009) 041115]. We calculate the fractal exponent of a subgraph of the HN-NP, which gives a lower bound for the fractal exponent of the original graph. This lower bound leads to the conclusion that the original system does not have a non-percolating phase, where only finite size clusters exist, and the percolating phase, where a unique giant component almost surely exists, at a unique critical point. Monte Carlo simulations support our conjecture.

PACS numbers: 89.75.Hc 64.60.aq 89.65.-s

I. INTRODUCTION

Percolation is the simplest model exhibiting a phase transition [1]. Many results for percolation on Euclidean lattices have been reported. It is well known that bond percolation with open bond probability \( p \) on the \( d(\geq 2) \)-dimensional Euclidean lattice shows a second order transition between the non-percolating phase, where only finite size clusters exist, and the percolating phase, where a unique giant component almost surely exists, at a unique critical point \( p_c \). However, this may not be the case for non-Euclidean lattices.

Complex networks have been actively studied in recent years [2–4]. Among extensive researches carried out on complex networks, percolation on various networks has played an important role in clarifying the interplay between network topology and critical phenomena [5]. Percolation on uncorrelated networks (represented by the configuration model [6]) is well described by the local tree approximation; there is a phase transition between the non-percolating phase and the percolating phase just as in Euclidean lattice systems, but its critical exponents depend crucially on the heterogeneity of the degree distribution of the network [6]. On the other hand, several authors [8–11] have reported that percolation on networks constructed with certain growth rules exhibits quite a different phase transition from that of uncorrelated networks and Euclidean lattices, referred to as an infinite order transition. Percolation on NAGs is defined to be transitive graphs with a positive Cheeger constant. Percolation on NAGs (with one end) exhibits the following three phases depending on the value of \( p \): the non-percolating phase \((0 \leq p < p_c)\), the critical phase \((p_c < p < p_c')\), where infinitely many infinite clusters exist, and the percolating phase \((p > p_c)\). Here, an infinite cluster is defined to be a cluster whose size is of order \( O(N^\alpha) \) \((0 < \alpha \leq 1)\). It is called a giant component when \( \alpha = 1 \). In the critical phase, \( 0 < \alpha < 1 \), the system is always in a critical state where \( n_s \) satisfies a power law \([12]\).

All previous studies [8–11, 14, 15, 19] of percolation on growing networks and hierarchical small-world networks indicate \( 0 = p_{c1} < p_{c2} < 1 \), except in the following case. Boettcher et al. investigated bond percolation on the non-planar Hanoi network (HN-NP) using the renormalization group technique [20]. They concluded that there are two critical probabilities \( p_{c1} \) and \( p_{c2} \) between zero and one: \( 0 < p_{c1} < p_{c2} < 1 \).

In this paper, we reconsider this model. We show analytically that the fractal exponent of a subgraph, which is a lower bound for that of the HN-NP, takes a non-zero value at all \( p(\neq 0) \), indicating that \( p_{c1} = 0 \). This means that the system is either in the critical phase or the percolating phase, not in the non-percolating phase, in contrast to the result of [20]. The Monte Carlo simulations support our analytical prediction.
The HN-NP consists of a one-dimensional chain and long-range edges. The HN-NP with generation $L(\geq 2)$ is constructed as follows [20]. (i) Consider a chain of $N_L = 2^L + 1$ nodes. Here, each node $i (= 0, 1, 2, \ldots, N_L - 1)$ connects to node $i + 1$. We call these edges the backbones. (ii) For each combination of $i$ and $j (= 0, 1, 2, \ldots, L - 2)$ and $k (= 0, 1, 2, \ldots, 2^{L-1}-1)$, nodes $(4i+2^k)$ and $(4j+1+2^k)$ are connected to $(4i+3)^2$ and $(4j+4)^2$, respectively. We call these edges the shortcuts. The schematic of the HN-NP with $L = 4$ generations is shown in Fig. 1. At generation $L$, the number of backbones is $2^L$ and the number of shortcuts is $2^L - 2$ (the total number of edges $E_L = 2^{L+1} - 2$). The geometrical properties of the HN-NP are as follows [20]: (i) the degree distribution $p_k$ decays exponentially as $p_{2m+3} \propto 2^{-m}$, (ii) the average degree $\langle k \rangle = 2E_L/N_L = (2^{L+2} - 4)/(2^{L+1}) \approx 4$ (for $L \gg 1$), (iii) the mean shortest path length $l$ increases logarithmically with $N_L$ as $l \propto \log N_L$, and (iv) the clustering coefficient is zero.

Boettcher et al. studied bond percolation on the HN-NP with open bond probability $p$ [20]. In the HN-NP with $L$ generations, they considered the renormalization of four parameters: $R_L$ (the probability that three consecutive points $a, b, c$ of a chain are connected), $S_L$ (the probability of $bc$ being connected, but not $a$), $U_L$ (the probability of $ac$ being connected, but not $b$), and $N'_L$ (the probability that there are no connections among $a$, $b$, and $c$). From the renormalization group flow for $R_L$ they determined the two critical probabilities as $p_{c1}^{BCZ} \approx 0.319445$ and $p_{c2}^{BCZ} \approx 0.381966$.

II. MODEL

The HN-NP consists of a one-dimensional chain and long-range edges. The HN-NP with generation $L(\geq 2)$ is constructed as follows [20]. (i) Consider a chain of $N_L = 2^L + 1$ nodes. Here, each node $i (= 0, 1, 2, \ldots, N_L - 1)$ connects to node $i + 1$. We call these edges the backbones. (ii) For each combination of $i$ and $j (= 0, 1, 2, \ldots, L - 2)$ and $k (= 0, 1, 2, \ldots, 2^{L-1}-1)$, nodes $(4i+2^k)$ and $(4j+1+2^k)$ are connected to $(4i+3)^2$ and $(4j+4)^2$, respectively. We call these edges the shortcuts. The schematic of the HN-NP with $L = 4$ generations is shown in Fig. 1. At generation $L$, the number of backbones is $2^L$ and the number of shortcuts is $2^L - 2$ (the total number of edges $E_L = 2^{L+1} - 2$). The geometrical properties of the HN-NP are as follows [20]: (i) the degree distribution $p_k$ decays exponentially as $p_{2m+3} \propto 2^{-m}$, (ii) the average degree $\langle k \rangle = 2E_L/N_L = (2^{L+2} - 4)/(2^{L+1}) \approx 4$ (for $L \gg 1$), (iii) the mean shortest path length $l$ increases logarithmically with $N_L$ as $l \propto \log N_L$, and (iv) the clustering coefficient is zero.

Boettcher et al. studied bond percolation on the HN-NP with open bond probability $p$ [20]. In the HN-NP with $L$ generations, they considered the renormalization of four parameters: $R_L$ (the probability that three consecutive points $a, b, c$ of a chain are connected), $S_L$ (the probability of $bc$ being connected, but not $a$), $U_L$ (the probability of $ac$ being connected, but not $b$), and $N'_L$ (the probability that there are no connections among $a$, $b$, and $c$). From the renormalization group flow for $R_L$ they determined the two critical probabilities as $p_{c1}^{BCZ} \approx 0.319445$ and $p_{c2}^{BCZ} \approx 0.381966$.

III. ANALYTICAL CALCULATION FOR THE SKELETON OF THE HN-NP

The fractal exponent $\psi_{\text{max}}(p)$ is useful to determine phase behavior. It is defined to be $\psi_{\text{max}}(p) = \lim_{N_L \to \infty} \log N_L \cdot s_{\text{max}}(N_L; p)$. A non-percolating phase, a critical phase, and a percolating phase are characterized by $\psi_{\text{max}}(p) = 0$, $0 < \psi_{\text{max}}(p) < 1$, and $\psi_{\text{max}}(p) = 1$, respectively. Unfortunately, it seems difficult to evaluate $s_{\text{max}}(N_L; p)$ directly for the HN-NP. Instead, we focus on a subgraph of the HN-NP and evaluate its fractal exponent.

We extract a subgraph from the HN-NP with $L$ generations by removing the backbones. Because the resulting subgraph has no cycles and the number of shortcuts is $2^L - 2 = N_L - 3$, this subgraph is composed of three disconnected trees. Indeed, nodes $i = 0, 2^{L-1}$, and $2^L$ belong to the three different trees. We call these the root nodes. Here the graphs isomorphic to these trees and having root nodes $i = 0, 2^{L-1}$, and $2^L$ will be called the skeletons $T_a(L), T_b(L),$ and $T_c(L)$, respectively. Clearly, $T_a(L)$ and $T_c(L)$ are also isomorphic to each other, $T_a(L) \approx T_c(L)$. At $L = 2$, $T_a(2)$ is composed of nodes 0 and 3 and the edge between them, $T_c(2)$ is one isolated node $i = 2$, and $T_c(2)$ is composed of nodes 1 and 4 and the edge between them. The skeletons $T_a(L), T_b(L),$ and $T_c(L)$ for arbitrary $L$ are given recursively as follows. First, we consider two subgraphs of the sets of nodes $\{0, 1, \ldots, 2^{L-1}\}$ and $\{2^{L-1}, 2^{L-1}+1, \ldots, 2^L\}$ after removing the backbones from the HN-NP with $L$ generations. By symmetry, both subgraphs consist of the skeletons $T_a(L-1), T_b(L-1),$ and $T_c(L-1)$ (Fig. 2a). Here the root nodes of the latter subgraph are $i = 2^{L-1}$ (for $T_a(L-1)$), $3 \times 2^{L-2}$ (for $T_b(L-1)$), and $2^L$ (for $T_c(L-1)$). Note that the skeletons $T_a(L), T_b(L),$ and $T_c(L)$ are given by adding the two long-range edges $\{0, 3 \times 2^{L-2}\}$ and $\{2^{L-2}, 2^L\}$, and taking into account the connection of node $2^{L-1}$ (Fig. 2b), we have

\begin{align*}
T_a(L) & \approx T_a(L-1) \approx R_1(T_a(L-1), T_b(L-1)), \\
T_b(L) & \approx R_2(T_c(L-1), T_c(L-1)),
\end{align*}

where the operation $R_1(x, y)$ adds the edge between root nodes of the skeletons $x$ and $y$, and $R_2(x, y)$ merges two root nodes of the skeletons $x$ and $y$ into one.
We now calculate the mean size of a cluster including the root node (the root cluster size) for each skeleton. We denote the root cluster sizes of \( T_p(L) \) by \( s_a(L) \) and \( s_b(L) \), respectively. Because of the recursive structure of the skeletons, the root cluster sizes \( s_a(L) \) and \( s_b(L) \) also satisfy recursive relations:

\[
s_a(L+1) = s_a(L) + ps_b(L), \quad s_b(L+1) = 2s_a(L) - 1,
\]

where the initial conditions are \( s_a(2) = 1 + p \) and \( s_b(2) = 1 \). Then, we find that

\[
s_a(L) = \frac{1}{2} \left( 1 + \sqrt{1 + 8p} \right)^{L+1} - \frac{1 - \sqrt{1 + 8p}}{2} \left( \frac{1 + \sqrt{1 + 8p}}{2} \right)^{L+1},
\]

\[
s_b(L) = \frac{1}{2} \left( 1 + \sqrt{1 + 8p} \right)^{L} - \frac{1 - \sqrt{1 + 8p}}{2} \left( \frac{1 + \sqrt{1 + 8p}}{2} \right)^{L}.
\]

For \( L \gg 1 \), we obtain \( s_a(L) \propto N_L^{\psi_{\text{skeleton}}(p)} \), where

\[
\psi_{\text{skeleton}}(p) = \log_2(1 + \sqrt{1 + 8p}) - 1.
\]

We expect \( \psi_{\text{skeleton}}(p) = \psi_{\text{max}}^{\text{HN-NP}}(p) \) because the roots are hubs. In fact, we performed Monte Carlo simulations for the bond percolation on the skeletons. Our numerical result of \( \psi_{\text{skeleton}}(p) \) shows a good correspondence with Eq. (7) except near \( p = 0 \) (not shown). According to Eq. (7), \( \psi_{\text{skeleton}}(p) \) increases continuously from \( \psi_{\text{skeleton}}(0) = 0 \) to \( \psi_{\text{skeleton}}(1) = 1 \). This means that the subsystem consisting only of the shortcuts is in the critical phase for all \( p \neq 0, 1 \), like the growing random tree [15]. Because the HN-NP is obtained by adding the backbones to the skeletons, the clusters in the skeletons become larger. Therefore, the entire system permits a critical phase even for infinitesimal \( p \), i.e., the non-percolating phase does not exist except at \( p = 0 \).

IV. NUMERICAL CHECK

In the previous section, we evaluated the root cluster size of the skeleton to show that its fractal exponent \( \psi_{\text{root}}^{\text{skeleton}}(p) \) takes a non-zero value for all \( p > 0 \). Because the skeleton is just a subgraph of the HN-NP, \( \psi_{\text{root}}^{\text{skeleton}}(p) \) is a lower bound for the fractal exponent of the largest cluster of the HN-NP \( \psi_{\text{max}}^{\text{HN-NP}}(p) \), i.e., \( \psi_{\text{root}}^{\text{skeleton}}(p) \geq \psi_{\text{root}}^{\text{max}} \). For bond percolation on the HN-NP, \( \psi_{\text{max}}^{\text{HN-NP}}(p) > 0 \) when \( p > 0 \), implying that \( p_c = 0 \). To check our prediction, we performed Monte Carlo simulations of bond percolation on the HN-NP. The number of generations is \( L = 13, 14, \cdots, 20 \), and the number of percolation trials is 100000 for each \( p \).

Figures (a) and (b) show the results for the order parameter \( m(N_L;p) \) and the fractal exponent of the largest cluster \( \psi_{\text{max}}^{\text{HN-NP}}(N_L;p) \), respectively. Here the fractal exponent \( \psi_{\text{max}}^{\text{HN-NP}}(N_L;p) \) at a finite generation \( L \) is evaluated as

\[
\psi_{\text{max}}^{\text{HN-NP}}(N_L;p) \approx \frac{\log s_{\text{max}}(N_{L+1};p) - \log s_{\text{max}}(N_{L-1};p)}{\log N_{L+1} - \log N_{L-1}}.
\]

We also plot the fractal exponent \( \psi_{\text{root}}^{\text{skeleton}}(p) \) of the skeleton (Eq. (4), shown as the thick-dashed line) and \( p_{c, \text{BCZ}} \) (shown as vertical lines) in Fig. (3).

From Fig. (b), we see that \( \psi_{\text{root}}^{\text{skeleton}}(p) \) is actually the lower bound of \( \psi_{\text{max}}^{\text{HN-NP}}(p) \), implying that \( p_c = 0 \). In particular, \( \psi_{\text{max}}^{\text{HN-NP}}(N_L;p) \) coincides with \( \psi_{\text{root}}^{\text{skeleton}}(p) \) for \( p \lesssim 0.26 \) (except near \( p = 0 \), where finite size effects are not negligible). For \( p \geq 0.26 \), \( \psi_{\text{max}}^{\text{HN-NP}}(N_L;p) \) is considerably greater than \( \psi_{\text{root}}^{\text{skeleton}}(p) \), and reaches unity at \( p = p_c \). At a glance, in the large size limit, \( \psi_{\text{max}}^{\text{HN-NP}}(N_L;p) \) seems to change continuously with \( p < p_c \). However, we speculate that in the large size limit \( \psi_{\text{max}}^{\text{HN-NP}}(N_L;p) \) (i) coincides with \( \psi_{\text{root}}^{\text{skeleton}}(p) \) in the entire region below \( p_{c, \text{BCZ}} \), (ii) jumps to a higher value at \( p = p_{c, \text{BCZ}} \), and (iii) increases monotonically up to unity for \( p_c < p \leq p_{c, \text{BCZ}} \).
$p < p^{BCZ}_{c1}$ means that the partial ordering (in the sense that the largest cluster is $O(N^\alpha)$ with $\alpha < 1$) in this region is essentially governed by the shortcuts. Because Boettcher et al. \cite{20} considered renormalization of the connecting probability of consecutive points of the backbones, we would expect their first critical probability $p^{BCZ}_{c1}$ to be the probability above which the backbones become relevant. Thus, we expect that there is a transition between critical phases, in the sense that the fractal exponent jumps, implying a qualitative change in the criticality, while it is very difficult to judge whether such a transition exists or not by finite size simulations. Such a jump in the fractal exponent has already been observed in site-bond percolation on the decorated (2,2)-flower \cite{21}. In addition, our numerical result shows that $\psi^{HN-NP}(p)$ reaches unity smoothly at $p^{BCZ}_{c2}$. This indicates that the phase transition to the percolating phase is discontinuous, similarly as in \cite{13}.

Finally, we discuss the cluster size distribution function, $n_s(p)$, below $p^{BCZ}_{c2}$. Figure 4(a) shows $n_s(p)$ for several values of $p$ with $0 < p < p^{BCZ}_{c2}$. In the critical phase, we expect a power law for $n_s(p)$:

$$n_s(p) \propto s^{-\tau(p)}, \quad (9)$$

where

$$\tau(p) = 1 + \psi_{\text{max}}(p)^{-1}, \quad (10)$$

and a corresponding scaling form:

$$n_s(N; p) = N^{-\psi_{\text{max}}(p)\tau(p)} f(sN^{-\psi_{\text{max}}(p)}), \quad (11)$$

where the scaling function $f(\cdot)$ behaves as

$$f(x) \sim \begin{cases} \text{rapidly decaying func.} & \text{for } x \gg 1, \\ x^{-\tau(p)} & \text{for } x \ll 1. \end{cases} \quad (12)$$

We tested this scaling for $0 < p \lesssim 0.26$ and $p^{BCZ}_{c1} < p < p^{BCZ}_{c2}$ and obtained excellent collapses (Fig 4(b)). We would also expect that $n_s$ to be fat-tailed for $0.26 \lesssim p < p^{BCZ}_{c2}$ because $n_s(p)$ for the skeletons perfectly obeys Eqs. (9) and (10) via Eq. (11) for $0 < p < 1$ (not shown), and $n_s$ is broader when we add the backbones to the skeletons, i.e., for the original HN-NP.

V. SUMMARY

In this paper, we have studied bond percolation on the HN-NP. Our results give the two critical probabilities as $p_{c1} = 0(< p^{BCZ}_{c1})$ and $p_{c2} = p^{BCZ}_{c2}$, implying that the system has only a critical phase and a percolating phase, and does not have a non-percolating phase for $p > 0$. As far as we know, all complex network models with a critical phase have only the critical phase and the percolating phase (\cite{8,13,15,19} for percolation and \cite{12,13,19,22,27} for spin systems). It will be challenging to clarify the origin of such universal behavior.

Acknowledgments

TH acknowledges the support through Grant-in-Aid for Young Scientists (B) (No. 24740054) from MEXT, Japan.
[1] D. Stauffer and A. Aharony, *Introduction to Percolation Theory* (Taylor and Francis, London, 1994).
[2] R. Albert and A.-L. Barabási, Rev. Mod. Phys. 74, 47 (2002).
[3] M. E. J. Newman, SIAM review 45, 167 (2003).
[4] A. Barrat, M. Barthélemy, and A. Vespignani, *Dynamical processes on complex networks* (Cambridge University Press, Cambridge, 2008).
[5] S. N. Dorogovtsev, A. V. Goltsev, and J. F. F. Mendes, Rev. Mod. Phys. 80, 1275 (2008).
[6] M. Molloy and B. Reed, Random Struct. Algor. 6, 161 (1995).
[7] R. Cohen, D. Ben Avraham, and S. Havlin, Phys. Rev. E 66, 036113 (2002).
[8] D. S. Callaway, J. E. Hopcroft, J. M. Kleinberg, M. E. J. Newman, and S. H. Strogatz, Phys. Rev. E 64, 041902 (2001).
[9] S. N. Dorogovtsev, J. F. F. Mendes, and A. N. Samukhin, Phys. Rev. E 64, 066110 (2001).
[10] L. Zalanyi, G. Csárdi, T. Kiss, M. Lengyel, R. Warner, J. Tobochnik, and P. Érdi, Phys. Rev. E 68, 066104 (2003).
[11] T. Hasegawa, M. Sato, and K. Nemoto, Phys. Rev. E 82, 046101 (2010).
[12] M. Hinczewski, and A. N. Berker, Phys. Rev. E 73, 066126 (2006).
[13] S. Boettcher, V. Singh, and R. M. Ziff, Nature Communications 3, 787 (2012).
[14] T. Hasegawa, T. Nogawa, and K. Nemoto, arXiv: 1009.6009 (2010).
[15] T. Hasegawa and K. Nemoto, Phys. Rev. E 81, 051105 (2010).
[16] I. Benjamini and O. Schramm, Electron. Comm. Probab. 1, 71 (1996).
[17] R. Lyons, J. Math. Phys. 41, 1099 (2000).
[18] T. Nogawa and T. Hasegawa, J. Phys. A: Math. Theor. 42, 145001 (2009).
[19] A. N. Berker, M. Hinczewski, and R. R. Netz, Phys. Rev. E 80, 041118 (2009).
[20] S. Boettcher, J. L. Cook, and R. M. Ziff, Phys. Rev. E 80, 041115 (2009).
[21] T. Hasegawa, M. Sato, and K. Nemoto, Phys. Rev. E 85, 017101 (2012).
[22] M. Bauer, S. Coulomb, and S. N. Dorogovtsev, Phys. Rev. Lett. 94, 200602 (2005).
[23] E. Khajeh, S. N. Dorogovtsev, and J. F. F. Mendes, Phys. Rev. E 75, 041112 (2007).
[24] S. Boettcher and C. T. Brunson, Phys. Rev. E 83, 021103 (2011).
[25] S. Boettcher and C. Brunson, Front. Physiol. 2, 102 (2011).
[26] T. Nogawa, T. Hasegawa, and K. Nemoto, Phys. Rev. Lett. 108, 255703 (2012).
[27] T. Nogawa, T. Hasegawa, and K. Nemoto, Phys. Rev. E 86, 030102 (2012).