Categorical localization for the coherent-constructible correspondence

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Abstract

We prove a microlocal counterpart of categorical localization for Fukaya categories in the setting of the coherent-constructible correspondence.

1 Introduction

Kontsevich’s homological mirror symmetry (HMS) conjecture [Kon95] states that two categories associated to a mirror pair are equivalent. For a Calabi-Yau (CY) variety, a mirror is also CY and the conjecture is a quasi-equivalence between the dg category of coherent sheaves over one and the (derived) Fukaya category of the other one.

For non-CYs, mirrors need not to be varieties. For a Fano toric variety which we will focus on, its mirror is a Landau-Ginzburg (LG) model, which is a holomorphic function over $(\mathbb{C}^\times)^n$ which can be read from the defining fan of the toric variety [HV00]. Fukaya-type (A-brane) category for an LG model is known to be the Fukaya-Seidel category [Sei01] when the LG-model is a Lefschetz fibration. Hence, for a smooth Fano fan, HMS predicts a quasi-equivalence between the dg category of coherent sheaves \( \text{coh} X_{\Sigma} \) over the toric variety \( X_{\Sigma} \) and the Fukaya-Seidel category \( \text{Fuk}(W_{\Sigma}) \) of the Laurent function \( W_{\Sigma} \):

\[
\text{coh} X_{\Sigma} \cong \text{Fuk}(W_{\Sigma}),
\]

which are now proved in many cases (e.g. [AKO06, AKO08, Ued06, UY13]).

When a variety is not complete, its derived category of coherent sheaves has infinite-dimensional nature (non-proper dg category). Accordingly, its mirror Fukaya-type category also should have infinite-dimensional nature. Such a construction is known to be (partially) wrapped Fukaya categories, whose hom-spaces could have infinitely many generators (intersection points) formed by quadratically increasing Hamiltonian isotopy [AS10, Syl16].

There is a relation between Fukaya-Seidel and wrapped Fukaya, that is, the latter is obtained by categorical localization of the former. This point of view is due to Seidel [Sei08]. Let \( X \) be a variety and \( D \) be a divisor of \( X \). Then, the dg category of coherent sheaves over \( X\setminus D \) is obtained by the quotient of the dg category of \( X \) by the objects supported on \( D \):

\[
\text{coh}(X\setminus D) \cong \text{coh} X / \text{coh}_D X.
\]

In [Sei08], he expected that the mirror operation of this is the identification of \( \text{id} \) and Serre functor when \( D \) is a canonical divisor. Hence, wrapped Fukaya mirror to \( X\setminus D \) is obtained by this identification for Fukaya-Seidel mirror to \( X \) [AS, AC]. Our point of view is more closely related to Sylvan’s [Syl16]. He associated the partially wrapped Fukaya category \( W_s(M) \) for a symplectic manifold \( M \) with symplectic stops \( s \) which designate directions...
not to be wrapped. Removing some of symplectic stops $r$ allow Lagrangians to wrap to the corresponding directions. The resulting category can be described as

$$W_{s|s}(M) \simeq W_s(M)/B_r$$  \hspace{1cm} (1.3)$$

where $B_r$ is the full subcategory spanned by Lagrangians near $r$.

Our aim of this paper is relating (1.2) and an analogue of (1.3) in the microlocal world. Microlocal study of A-brane categories (Fukaya-type categories) started from the work of Nadler and Zaslow [Nad09, NZ09]. Their theorem shows that the infinitesimally wrapped Fukaya category of a cotangent bundle is quasi-equivalent to the dg category of constructible sheaves over its base space. This theorem motivates us to use some types of the dg category of constructible sheaves for A-branes in homological mirror symmetry. For mirrors of toric varieties, Fang-Liu-Treumann-Zaslow [FLTZ11, FLTZ12] provide the following candidates of such categories as the full subcategories of the dg categories of the constructible sheaves over real tori.

Let $M$ be a free abelian group of rank $n$ and $N$ be its dual. Let further $\Sigma$ be a smooth fan defined in $N_R := N \otimes_{\mathbb{Z}} \mathbb{R}$. Set $T^n := M_R/M$. We identify $T^n \times N_R$ with $T^*T^n$, and set

$$\Lambda_{\Sigma} := \bigcup_{\sigma \in \Sigma} p(\sigma^\perp) \times (-\sigma) \subset T^*T^n, \hspace{1cm} (1.4)$$

where $p: M_R \to T^n$ is the quotient map and $\sigma^\perp \subset M_R$ is the orthogonal subspace of $\sigma \subset N_R$. It is known that there is a fully faithful morphism $\kappa_\Sigma: \text{coh}(X_\Sigma) \to \text{Sh}_{\Lambda_{\Sigma}}(T^n)$ where the RHS is the full sub dg category of the quasi-constructible sheaves over $T^n$ spanned by objects whose microsupports are contained in $\Lambda_{\Sigma}$.

**Conjecture 1.1** (The coherent-constructible correspondence [FLTZ11, FLTZ12]). Suppose that $\Sigma$ is complete. Then $\kappa_\Sigma$ induces a quasi-equivalence

$$\text{coh}(X_\Sigma) \simeq \text{Sh}_{\Lambda_{\Sigma}}(T^n) \hspace{1cm} (1.5)$$

where the RHS is the full sub dg category of the constructible sheaves over $T^n$ spanned by objects whose microsupports are contained in $\Lambda_{\Sigma}$.

This is considered as a microlocal counterpart of (1.1). In the course of this paper, we will assume a slightly stronger version for a (not necessarily complete) $\Sigma$.

**Conjecture 1.2.** The morphism $\kappa_\Sigma$ extends to a quasi-equivalence

$$\text{Qcoh } X_\Sigma \simeq \text{Sh}_{\Lambda_{\Sigma}}(T^n) \hspace{1cm} (1.6)$$

where LHS is the dg category of quasi-coherent sheaves over $X_\Sigma$.

Conjecture 1.1 is proved in some special cases [Tre10, SS16, Kuw15] and in the equivariant version [FLTZ11]. Conjecture 1.2 is sketched in [Vai] and also proved in [Kuw] independently. The former follows from the latter (Corollary 2.12 below).

Conjecture 1.2 induces the equivalence between the subcategories of compact objects. The compact objects of LHS are coherent sheaves $\text{coh } X_\Sigma$ and that of RHS are wrapped constructible sheaves $\text{Sh}_{\Lambda_{\Sigma}}^w(T^n)$ of Nadler [Nad16], which are introduced as a microlocal counterpart of (partially) wrapped Fukaya categories.

To state an analogue of (1.3), we also need the notion of microlocal skyscraper sheaves which is also due to Nadler. Roughly speaking, for a point of a Lagrangian of a cotangent
bundle, a microlocal skyscraper sheaf represents the microlocal stalk functor at the point. For a 1-dimensional cone \( \rho \in \Sigma \), let \( \Sigma^\rho_c \) be the complement of the star neighborhood of \( \rho \) in \( \Sigma \). Set \( B_\rho \) be the full subcategory of \( \text{Sh}_{w}^{\Lambda \Sigma}(T^n) \) split-generated by microlocal skyscraper sheaves over regular points of \( \Lambda \Sigma \setminus \Lambda \Sigma^\rho_c \).

Then, our main theorem is the following.

**Theorem 1.3.** Assuming Conjecture 1.2 for \( \Sigma \). There is a quasi-equivalence

\[
\text{Sh}_{\Lambda \Sigma}^w(M_{\mathbb{R}}/M) \simeq \text{Sh}_{\Lambda \Sigma}^w(M_{\mathbb{R}}/M)/B_\rho.
\]

such that the induced morphism \( \text{Sh}_{\Lambda \Sigma}^w(M_{\mathbb{R}}/M) \to \text{Sh}_{\Lambda \Sigma}^w(M_{\mathbb{R}}/M) \) fits into the diagram

\[
\begin{array}{ccc}
\text{coh} X_{\Sigma} & \xrightarrow{\kappa_{\Sigma}} & \text{Sh}_{\Lambda \Sigma}^w(M_{\mathbb{R}}/M) \\
\downarrow & & \downarrow \\
\text{coh} X_{\Sigma^\rho_c} & \xrightarrow{\kappa_{\Sigma^\rho_c}} & \text{Sh}_{\Lambda \Sigma^\rho_c}^w(M_{\mathbb{R}}/M).
\end{array}
\]

In appendix, we also prove the following.

**Theorem 1.4.** If \( \dim \Sigma = 2 \), Conjecture 1.2 is true.

The proof is independent of that of [Vai, Kuw] and based on the fact Conjecture 1.1 is true for \( \dim \Sigma = 2 \) [Kuw15] and an explicit proof of “constructible = wrapped” for 2-dimensional complete \( \Sigma \).

The structure of the paper is as follows. In Section 2, we prepare microlocal sheaf theory and give a review of the coherent-constructible correspondence for the later sections. In Section 3, we will prove our main theorem. In Appendix, we give a proof of Conjecture 1.2 for \( \dim \Sigma = 2 \).

**Notation**

We denote the space of morphisms of dg-categories by \( \text{hom}^*(-, -) \). In this paper, any (co)limits in \( \text{dg} \) categories are homotopical sense. Cocompleteness always means filtered cocompleteness. All functors are derived. All morphisms between \( \text{dg} \) categories are meant to be morphisms in the homotopy category of the \( \text{dg} \) category of \( \text{dg} \) categories localized at quasi-equivalences (Dwyer-Kan equivalences).

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2 Preliminaries

2.1 Microlocal theory of sheaves

Let $Z$ be a real analytic manifold. We denote by $\text{Sh}^\wedge(Z)$ the dg category of complexes of sheaves of $\mathbb{C}$-vector spaces on $Z$ whose cohomology sheaves are quasi-$\mathbb{R}$-constructible in the sense of [FLTZ11] (weakly constructible in the sense of [KS90], large constructible in the sense of [Nad16]). Let $\text{Sh}^c(Z) \subset \text{Sh}^\wedge(Z)$ be the full dg subcategory consisting of $\mathbb{R}$-constructible complexes. In what follows, we consider only (quasi-) $\mathbb{R}$-constructible sheaves and simply write (quasi-) constructible.

First we recall the definition of microsupports of sheaves (see Kashiwara-Schapira [KS90, Section 5.1]). Although the treatment of [KS90] is limited to bounded complexes, we can obtain a similar result for unbounded complexes by using [Spa88]. See, for example, [Sch03].

Definition 2.1. Take a point $z \in Z$. Let $f$ be a real analytic function defined in a neighborhood of $z$ satisfying $f(z) = 0$. Recall the local cohomology was defined as

$$\Gamma_{\{f \geq 0\}}(F) \simeq \Gamma(B; F) \to \Gamma(B \cap \{f < 0\})[-1],\tag{2.1}$$

where $B \subset Z$ is a sufficiently small open ball centered at $z$.

For any $F \in \text{Sh}^\wedge(Z)$, we define its microsupport (or singular support) $\text{SS}(F) \subset T^*Z$ as follows:

$$(z_0; \zeta_0) \notin \text{SS}(F) \iff \text{there is an open neighborhood } U \text{ of } (z_0; \zeta_0) \text{ such that}$$

for any $z \in Z$ and any $C^\infty$-function $f$ defined in a neighborhood of $z$ with $(z; df(z)) \in U$, one has $\Gamma_{\{f \geq 0\}}(F)_z \simeq 0$.

Note that $\text{SS}(F) \cap T^*_Z Z = \text{Supp}(F)$. For $F \in \text{Sh}^\wedge(Z)$, the microsupport $\text{SS}(F)$ is a closed conic Lagrangian subvariety of $T^*Z$.

The following proposition is a very important tool in the microlocal theory of sheaves (for the original version, see [KS90, Proposition 2.7.2]).

Proposition 2.2 (The non-characteristic deformation lemma). Let $F \in \text{Sh}^\wedge(Z)$ and $\{U_t\}_{t \in \mathbb{R}}$ be a family of subanalytic open subsets of $Z$. Assume the following conditions:

(i) $U_t = \bigcup_{s < t} U_s$ ( $t \in \mathbb{R}$).

(ii) For any $(s, t) \in \mathbb{R}^2$ with $s \leq t$, $U_t \setminus U_s \cap \text{Supp}(F)$ is compact.

(iii) Setting $Z_s := \bigcap_{t > s} (U_t \setminus U_s)$, for any $(s, t) \in \mathbb{R}^2$ with $s \leq t$ and $z \in Z_s \setminus U_t$ we have

$$\Gamma_{X \setminus U_t}(F)_z = 0.\tag{2.2}$$

Then for any $t \in \mathbb{R}$, one has a quasi-isomorphism

$$\Gamma\left(\bigcup_{t \in \mathbb{R}} U_t; F\right) \xrightarrow{\sim} \Gamma(U_t; F).\tag{2.3}$$

By the definition of microsupports and the non-characteristic deformation lemma, we obtain the following proposition called the microlocal Morse lemma.
**Proposition 2.3** (Microlocal Morse lemma). Let $F \in \mathbf{SH}^\wedge(Z)$ and $\psi: Z \to \mathbb{R}$ be a real analytic function. Let $a < b$ in $\mathbb{R}$ and assume that

(i) $\psi$ is proper on $\text{Supp}(F)$.
(ii) For any $z \in \psi^{-1}([a, b))$, $d\psi(z) \notin \text{SS}(F)$.

Then the natural restriction morphism

$$\Gamma(\psi^{-1}((-\infty, a)); F) \longrightarrow \Gamma(\psi^{-1}((-\infty, b)); F)$$

(2.4)

is a quasi-isomorphism.

We introduce the notion of microlocal stalk (microlocal type in the sense of [KS90]). Let $\Lambda \subset T^*Z$ be a conic Lagrangian subvariety and $p = (z_0; \zeta_0) \in \Lambda$ a point in $\Lambda$. Assume that $\Lambda$ is smooth near $p$. Take a real analytic function $f$ in a neighborhood of $z_0$ satisfying the following three conditions:

(i) $f(z_0) = 0$,
(ii) $df(z_0) = p$,
(iii) the graph $\Gamma_d$ and $\Lambda$ intersect transversely at $p$.

**Definition 2.4.** Let $\Lambda, p, f$ as above and $F \in \mathbf{SH}^\wedge(Z)$. Assume $\text{SS}(F) \subset \Lambda$. We define the microlocal stalk at $p$ associated with $f$ by

$$\phi_{p,f}(F) := \Gamma_{\{f \geq 0\}}(F)_{z_0} \simeq \Gamma_{\{f \geq 0\}}(B; F),$$

where $B \subset Z$ is a sufficiently small open ball centered at $z_0$.

Note that $B$ depends only on $\Lambda, p$, and $f$. That is, we can take the same $B$ for any object $F$ with $\text{SS}(F) \subset \Lambda$ in (2.5).

### 2.2 Nadler’s wrapped constructible sheaves

For a closed conic Lagrangian subvariety $\Lambda \subset T^*Z$, we denote by $\mathbf{SH}^\wedge_{\Lambda}(Z)$ (resp. $\mathbf{SH}^\wedge_{\mathrm{reg}}(Z)$) the full subcategory of objects $F \in \mathbf{SH}^\wedge(Z)$ (resp. $\mathbf{SH}^\wedge(Z)$) satisfying $\text{SS}(F) \subset \Lambda$. The homotopy category of $\mathbf{SH}^\wedge_{\Lambda}(Z)$ (resp. $\mathbf{SH}^\wedge_{\mathrm{reg}}(Z)$) is equivalent to the derived category of quasi-constructible (resp. constructible) complex of sheaves microsupported in $\Lambda$.

Let $\mathbf{SH}^\wedge_{\Lambda}(Z)$ be the full dg subcategory of compact objects in $\mathbf{SH}^\wedge_{\Lambda}(Z)$ whose objects are called wrapped constructible sheaves, which is defined by Nadler [Nad16].

Let $p \in \Lambda_{\mathrm{reg}}$ be a smooth point of $\Lambda$ and $f$ be a real analytic test function satisfying the conditions (i)–(iii) stated just before Definition 2.3. The microlocal stalk functor $\phi_{p,f}: \mathbf{SH}^\wedge_{\Lambda}(Z) \to \mathbf{Mod}(k)$ preserves limits since it is a composite of functors $\Gamma_{\{f \geq 0\}}(*)$ and $\Gamma(B; *)$, which are both right adjoint functors. Moreover $\phi_{p,f}$ preserves colimits since

$$\Gamma_{\{f \geq 0\}}(B; F) \simeq \text{hom}(\mathbb{C}_{\{f \geq 0\} \cap B}, F)$$

(2.6)

and $\mathbb{C}_{\{f \geq 0\} \cap B}$ is constructible (see Proposition 2.11). Hence $\phi_{p,f}$ admits a right adjoint functor $\phi^\vee_{p,f}: \mathbf{Mod}(k) \to \mathbf{SH}^\wedge_{\Lambda}(Z)$ and $\phi^\vee_{p,f}$ preserves compact objects.

**Definition 2.5.** Let $\Lambda, p, f$ as above. Define the microlocal skyscraper $F_{p,f} := \phi^\vee_{p,f}(k) \in \mathbf{SH}^\wedge_{\Lambda}(Z)$ as the object representing the microlocal stalk functor:

$$\phi_{p,f}(F) = \text{hom}(F_{p,f}, F) \quad (F \in \mathbf{SH}^\wedge_{\Lambda}(Z)).$$

(2.7)
Nadler [Nad16] gives a more geometric characterization of $\text{Sh}_{\Lambda}^w(Z)$ by using microlocal skyscrapers.

**Lemma 2.6** ([Nad16, Lemma 3.15]). The dg category $\text{Sh}_{\Lambda}^w(Z)$ is split-generated by the microlocal skyscrapers $F_{p,f}$ ($p \in \Lambda_{\text{reg}}$).

### 2.3 Preliminaries on CCC

In this section, we review the coherent-constructible correspondence.

#### 2.3.1 Formulation

Let $M$ be a free abelian group of rank $n$ and $N$ the dual abelian group of $M$. Let $\Sigma$ be a smooth fan in $N_R$. We denote by $X_\Sigma$ the smooth toric variety associated with $\Sigma$. We also set

\[
\Lambda_{\Sigma} := \bigcup_{\sigma \in \Sigma} \sigma^\perp \times (-\sigma) \subset T^*M_R,
\]

where $p: M_R \cong \mathbb{R}^n \rightarrow M_R/M =: T^n$ is the projection. Note that $\Lambda_{\Sigma}$ is a conic Lagrangian subvariety in $T^*T^n$.

Denote the dg category of coherent sheaves on $X_\Sigma$ by $\text{coh}_{X_\Sigma}$ which is localized at quasi-isomorphisms. We also denote by $\text{Sh}_{\Lambda_{\Sigma}}^c(T^n)$ the dg category of $\mathbb{R}$-constructible objects on $T^n$ whose microsupports are contained in $\Lambda_{\Sigma}$ (which is also localized at quasi-isomorphisms). When $\Sigma$ is complete, there is a well-defined fully faithful morphism (cf. [FLTZ11, Tre10])

\[
\kappa_{\Sigma}: \text{coh}_{X_\Sigma} \rightarrow \text{Sh}_{\Lambda_{\Sigma}}^c(T^n).
\]

We will recall the construction of $\kappa_{\Sigma}$ in the next section.

Fang-Liu-Treumann-Zaslow formulated the following conjecture called the coherent-constructible correspondence and its equivariant version and proved the latter [FLTZ11, Tre10].

**Conjecture 2.7** (The coherent-constructible correspondence, CCC). For a smooth complete fan $\Sigma$, the morphism (2.9) induces a quasi-equivalence of dg categories.

In some cases, proofs are known.

**Theorem 2.8** ([Tre10, SS16, Kuw15]). If $\Sigma$ is cragged or $\dim \Sigma = 2$, the morphism $\kappa_{\Sigma}$ is a quasi-equivalence.

We conjecture a variant of Conjecture 2.7 to include non-complete $\Sigma$'s. Note that there exists a fully faithful morphism $\kappa_{\Sigma}: \text{coh}_{X_\Sigma} \rightarrow \text{Sh}_{\Lambda_{\Sigma}}^w(T^n)$ by the construction of $\kappa_{\Sigma}$ even when $\Sigma$ is not complete.

**Conjecture 2.9.** The morphism $\kappa_{\Sigma}$ induces a quasi-equivalence

\[
\kappa_{\Sigma}: \text{coh}_{X_\Sigma} \overset{\sim}{\rightarrow} \text{Sh}_{\Lambda_{\Sigma}}^w(T^n).
\]

There is also a “quasi-version” (or ind-version) of the above conjecture.
Conjecture 2.10. The morphism $\kappa_\Sigma$ induces a quasi-equivalence

$$\kappa_\Sigma: \text{Qcoh } X_\Sigma \xrightarrow{\sim} \text{Sh}^\wedge_{\Lambda_\Sigma}(T^n)$$

(2.11)

A proof of Conjecture 2.10 is sketched in [Val] and independently proved in [Kuw]. By taking compact objects of (2.11), Conjecture 2.10 is equivalent to Conjecture 2.10.

Proposition 2.11. The objects of $\text{Sh}^\wedge_{\Lambda_\Sigma}(T^n)$ are compact in $\text{Sh}^\wedge_{\Lambda_\Sigma}(T^n)$.

Proof. Let $F \in \text{Sh}^\wedge_{\Lambda_\Sigma}(T^n)$ be a constructible object. Take a stratification $S$ consisting connected strata such that $\Lambda_\Sigma \subset \bigsqcup_{S \in S} T^*_S T^n =: \Lambda_S$. Then, $\text{Sh}^\wedge_{\Lambda_S}(T^n)$ is equivalent to a dg category of dg modules over a quiver path algebra (e.g. [STZ14]). Since $F$ is locally constant and has finite-dimensional stalk on each stratum $S$, it is a perfect object in $\text{Sh}^\wedge_{\Lambda_S}(T^n)$. Since $\Lambda_\Sigma \subset \Lambda_S$, the result follows. $\square$

Corollary 2.12. Conjecture 2.7 for $\Sigma$ follows from Conjecture 2.9 for $\Sigma$.

Proof. This follows from the fact that the functor $\kappa_\Sigma|_{\text{coh}}$ is a fully faithful functor into $\text{Sh}^\wedge \subset \text{Sh}^\wedge$ ([Tre10]). $\square$

In particular, wrapped constructible sheaves along $\Lambda_\Sigma$ coincide with constructible sheaves along $\Lambda_\Sigma$ for complete $\Sigma$ under Conjecture 2.10. In the appendix of this paper, we give a proof of this coincidence for $\dim \Sigma = 2$ without assuming Conjecture 2.10.

2.3.2 Construction of $\kappa_\Sigma$

First we recall the construction of $\kappa_\Sigma$ due to Fang-Liu-Treumann-Zaslow [FLTZ11; FLTZ12].

For $\sigma \in \Sigma$, we set

$$\Theta(\sigma) := p^! C_{\text{Int}(\sigma^\vee)}[n] \in \text{Qcoh } X_\Sigma,$$

$$\Theta'(\sigma) := i_{U_\sigma}^! \mathcal{O}_\sigma \in \text{Sh}^\wedge_{\Lambda_\Sigma}(T^n),$$

where $p: M_\mathbb{R} \rightarrow T^n$ is the projection, $C_{\text{Int}(\sigma^\vee)}$ is the constant sheaf on $\text{Int}(\sigma^\vee) \subset M_\mathbb{R}$, the interior of the polar dual of $\sigma$, $i_{U_\sigma}: U_\sigma \hookrightarrow X_\Sigma$ is the affine toric coordinate of $X_\Sigma$ corresponding to $\sigma$, and $\mathcal{O}_\sigma$ is the structure sheaf of $U_\sigma$.

For $\sigma, \tau \in \Sigma$, there exists an isomorphism

$$H^i \text{hom}^\bullet_{\text{Sh}^\wedge_{\Lambda_\Sigma}(T^n)}(\Theta(\sigma), \Theta(\tau)) \simeq \begin{cases} \mathbb{C}[^\sigma \cap M] & \text{when } \sigma \supset \tau \text{ and } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

(2.13)

For $m \in \sigma^\vee \cap M$, the corresponding morphism in LHS of the above is given by

$$\Theta(\sigma) := p^! C_{\text{Int}(\sigma^\vee)}[n] \simeq p^! C_{\text{Int}(\sigma^\vee + m)}[n] \xrightarrow{p^!(i_{\sigma^\vee + m, \sigma^\vee})}[n] \xrightarrow{p^! C_{\text{Int}(\tau^\vee)}[n]} =: \Theta(\tau),$$

(2.14)

where $i_{\sigma^\vee + m, \sigma^\vee}$ is the canonical morphism $C_{\text{Int}(\sigma^\vee + m)} \rightarrow C_{\text{Int}(\tau^\vee + m)}$ induced by the inclusion $\sigma^\vee + m \hookrightarrow \tau^\vee$.

Similarly, for $\sigma, \tau \in \Sigma$, there exists an isomorphism

$$H^i \text{hom}^\bullet_{\text{Qcoh } X_\Sigma}(\Theta'(\sigma), \Theta'(\tau)) \simeq \begin{cases} \mathbb{C}[^\sigma \cap M] & \text{when } \sigma \supset \tau \text{ and } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

(2.15)

For $m \in \sigma^\vee \cap M$, the corresponding morphism in LHS of the above is given by the restriction morphism $\theta'_m: i_{U_\sigma}^! \mathcal{O}_\sigma \rightarrow i_{U_\tau}^! \mathcal{O}_\tau$. 7
We denote by $\Gamma(\Sigma)$ the dg category whose set of objects is $\Sigma$ and hom-spaces are defined by

$$\text{hom}_\Gamma(\Theta(\sigma), \Theta(\tau)) := \begin{cases} \mathbb{C}[\sigma^\vee \cap M] & \text{when } \sigma \supset \tau \text{ and } i = 0, \\ 0 & \text{otherwise}, \end{cases}$$

with trivial differentials. Then there exist quasi-equivalent dg functors $\Gamma(\Sigma) \to \Theta'_\Sigma$ and $\Gamma(\Sigma) \to \Theta'_\Sigma$, where $\Theta_{\Sigma}$ (resp. $\Theta'_{\Sigma}$) is the full subcategory of $\text{Qcoh} X_{\Sigma}$ (resp. $\text{Sh}^\wedge_{\Lambda_{\Sigma}}(T^n)$) spanned by $\{\Theta(\sigma) \mid \sigma \in \Sigma\}$ (resp. $\{\Theta'(\sigma) \mid \sigma \in \Sigma\}$). Hence we have a quasi-equivalent morphism $\kappa_{\Sigma} : \text{Perf} \Theta_{\Sigma} \to \text{Perf} \Theta'_{\Sigma}$, where $\text{Perf} A$ denotes the full subcategory of the category of dg modules over a dg category $A$ spanned by perfect modules.

By using Čech resolution, it follows that $\text{coh} X_{\Sigma} \subset \text{Perf} \Theta_{\Sigma}$. Hence $\kappa_{\Sigma}$ is restricted to $\kappa_{\Sigma} : \text{coh} X_{\Sigma} \to \text{Sh}^\wedge_{\Lambda_{\Sigma}}(T^n)$. We denote this morphism by $\kappa_{\Sigma}$. Since $\text{Sh}^\wedge_{\Lambda_{\Sigma}}(T^n)$ is cocomplete and $\kappa_{\Sigma}$ is finite-colimit preserving (since it is induced from an equivalence), there also exists $\kappa_{\Sigma} : \text{Qcoh} X_{\Sigma} \to \text{Sh}^\wedge_{\Lambda_{\Sigma}}(T^n)$ by abuse of notation.

### 2.3.3 Properties of the morphism $\kappa_{\Sigma}$

In this section, we list three properties of $\kappa_{\Sigma}$, which we will later use in this paper.

First one is the naturality. Let $\Sigma'$ be a subfan of $\Sigma$. Then we have $\Lambda_{\Sigma'} \subset \Lambda_{\Sigma}$, accordingly $\text{Sh}^\wedge_{\Lambda_{\Sigma'}}(T^n) \subset \text{Sh}^\wedge_{\Lambda_{\Sigma}}(T^n)$. We also have $\Theta_{\Sigma} \subset \Theta_{\Sigma'}$, $\Theta'_{\Sigma} \subset \Theta'_{\Sigma'}$ and $\Gamma(\Sigma') \subset \Gamma(\Sigma)$. Hence, by the definition of $\kappa_{\Sigma}$, $\kappa_{\Sigma'}$ can be obtained as the restriction of $\kappa_{\Sigma}$. In other words, we have

**Proposition 2.13** (FLTZ11).

$$\kappa_{\Sigma'}(\text{coh} X_{\Sigma'}) \subset \text{Sh}^\wedge_{\Lambda_{\Sigma'}}(T^n).$$

The second property is the monoidality. Let $m : T^n \times T^n \to T^n$ be the multiplication. We define the convolution product $\text{Sh}^\wedge(T^n) \times \text{Sh}^\wedge(T^n) \to \text{Sh}^\wedge(T^n)$ by $E \ast F := m_!(E \boxtimes F)$. This operation defines a monoidal structure on $\text{Sh}^\wedge(T^n)$.

**Proposition 2.14** (FLTZ11). For any $\mathcal{E}, \mathcal{F} \in \text{coh} X_{\Sigma}$, then we have

$$\kappa_{\Sigma}(\mathcal{E} \otimes \mathcal{F}) = \kappa_{\Sigma}(\mathcal{E}) \ast \kappa_{\Sigma}(\mathcal{F}).$$

The third property is naturality for direct products. Let $\Sigma$ and $\Sigma'$ be smooth fans. The product of two fans is defined as

$$\Sigma \times \Sigma' := \{\sigma \times \sigma' \mid \sigma \in \Sigma, \sigma' \in \Sigma'\},$$

which associates the product of toric varieties $X_{\Sigma} \times X_{\Sigma'}$ (e.g. CLSI11). It is clear from the definitions that $\Gamma(\Sigma \times \Sigma') = \Gamma(\Sigma) \times \Sigma(\Sigma')$, $\Theta(\sigma) \boxtimes \Theta'(\sigma') = \Theta(\sigma \times \sigma')$, and $\Theta'(\sigma) \boxtimes \Theta'(\sigma') = \Theta'(\sigma \times \sigma')$. As a consequence, we have the following.

**Proposition 2.15.** In the above notation, we have

$$\kappa_{\Sigma \times \Sigma'} \simeq \kappa_{\Sigma} \times \kappa_{\Sigma'}. $$
2.3.4 Mirrors of points in toric varieties

In this section, we describe the image of the structure sheaves of points in toric varieties under the functor $\kappa_C$. This subject should be strongly related to T-duality picture of mirror symmetry \cite{FLTZ12, SYZ01}.

For a point $x \in X^\Sigma$, take the smallest open toric subvariety $U_\sigma$ which contains $x$. Let $\hat{\rho}_1, \ldots, \hat{\rho}_r$ be the 1-dimensional faces (rays) of $\sigma^\vee$ the polar dual of $\sigma$. Then, the primitive generators of $\hat{\rho}_i$’s can be extended to a basis $m_1, \ldots, m_n$ of $M$, where $m_1, \ldots, m_r$ correspond to $\hat{\rho}_1, \ldots, \hat{\rho}_r$, respectively. Then the open toric subvariety $U_\sigma$ can be identified as

$$U_\sigma = \text{Spec } \mathbb{C}[X_1, \ldots, X_r, X_{r+1}^\pm, \ldots, X_n^\pm] \simeq \mathbb{C}^r \times (\mathbb{C}^*)^{n-r}. \quad (2.21)$$

In the following, we will view the RHS of the above isomorphism as the coordinate of $U_\sigma$. Since $U_\sigma$ is the smallest open toric varieties containing $x$, from the first coordinate to the $r$th coordinate of $x$ are zero: $x = (0, \ldots, 0, x_{r+1}, \ldots, x_n)$ and $x_i \neq 0$ for $i \geq r + 1$.

Since $m_1, \ldots, m_n$ is a basis of $M$, we can factorize $M_\mathbb{R}$ and $T^n$ as $M_\mathbb{R} \simeq \mathbb{R}^r \times \mathbb{R}^{n-r}$ and $T^n \simeq T^r \times T^{n-r}$.

Let $\mathcal{C}_{(X_1, \ldots, X_r)}$ be the proper push-forward to $T^r$ of the constant sheaf on $\mathbb{R}^r$ supported on $[0,1]^r$. Let further $\mathcal{C}_{(X_{r+1}, \ldots, X_n)}$ be the locally constant sheaf on $T^{n-r}$ whose monodromy along $m_i$ corresponds to the multiplication by $x_i$. Here we take a set of generators of the fundamental group $\pi_1(T^{n-r})$ as $m_{r+1}, \ldots, m_n$. We view the exterior product $P_x: = \mathcal{C}_{(X_1, \ldots, X_r)} \boxtimes \mathcal{C}_{(X_{r+1}, \ldots, X_n)}$ as a sheaf on $T^n = M_\mathbb{R}/M$.

**Proposition 2.16.** Under the above notations, we have

$$\kappa_C(\mathcal{O}_x) \simeq P_x. \quad (2.22)$$

Under T-duality picture, the big algebraic torus $(\mathbb{C}^*)^n$ in $X^\Sigma$ should be viewed as the non-degenerate torus fibration structure \cite{FLTZ12} and the structure sheaves on these points are corresponding to the local systems on $T^n$ which are expected to be equivalent to the torus fibers with $U(1)$-local system in the Fukaya category of $T^*T^n$. On the other hand, points in toric divisors are corresponding to degenerate local systems which are “local systems with 0 or $\infty$ monodromies” and such sheaves have non-compact Lagrangians as these microsupports. This is expected to have some relations with other descriptions of SYZ mirror to non-flat torus fibrations (e.g. \cite{AAK16}).

**Proof.** On $U_\sigma$, we have the Koszul resolution of the skyscraper sheaf $\mathcal{O}_x$ on $x$:

$$0 \leftarrow \mathcal{O}_x \leftarrow \mathcal{O}_{U_\sigma} \leftarrow \mathcal{O}_{U_\sigma}^{\oplus n} \leftarrow \bigwedge^2(\mathcal{O}_{U_\sigma}^{\oplus n}) \leftarrow \bigwedge^3(\mathcal{O}_{U_\sigma}^{\oplus n}) \leftarrow \cdots \mathcal{O}_{U_\sigma} \leftarrow 0. \quad (2.23)$$

As $R$-modules ($R := \mathbb{C}[X_1, \ldots, X_r, X_{r+1}^\pm, \ldots, X_n^\pm]$), this resolution is

$$0 \leftarrow R/(X_1, \ldots, X_r, X_{r+1} - x_{r+1}, \ldots, X_n - x_n) \leftarrow R \leftarrow R^{\oplus n} \leftarrow \cdots \leftarrow 0 \quad (2.24)$$

and the morphism is the internal differential by $i_{\Sigma X_i - x_i}$. This resolution is the exterior tensor product of the resolutions

$$0 \leftarrow \mathbb{C}[X_i]/(X_i - x_i) \leftarrow \mathbb{C}[X_i] \leftarrow \mathbb{C}[X_i] \leftarrow 0 \quad (2.25)$$

where the morphism $\mathbb{C}[X_i] \rightarrow \mathbb{C}[X_i]$ is given by the multiplication of $X_i - x_i$. Let $\Sigma_\mathbb{C}$ be the fan of $\mathbb{C}$. Then, we have

$$\kappa_{\Sigma_\mathbb{C}}(\mathbb{C}[X_i]/(X_i - x_i)) = \mathbb{C}(X_i - x_i). \quad (2.26)$$

The Proposition 2.20 implies $\kappa_{\Sigma_\mathbb{C}} = \kappa_{\Sigma_\mathbb{C}}^{\oplus n}$, which gives the conclusion. \hspace{1cm} $\square$
2.4 Seidel’s categorical localization of \( \text{coh} X \)

**Proposition 2.17** ([Sei08, p.87 Complements of divisors]). Let \( X \) be a smooth toric variety over \( \mathbb{C} \), \( D \subset X \) be a toric divisor, and \( U := X \setminus D \) be its complement.

(i) Let \( s \) be the canonical section of the line bundle \( \mathcal{O}(D) \). Then for any \( F \in \text{coh} X \), one has

\[
j_* F|_U \simeq \colim_i F \otimes \mathcal{O}(D)^{\otimes i},
\]

where \( j : U \hookrightarrow X \) is the inclusion and the colimit is formed with respect to multiplication with \( s \).

(ii) Set

\[
\text{coh}_D X := \{ F \in \text{coh} X \mid \text{Supp} F \subset D \},
\]

then the restriction functor

\[
\text{coh} X / \text{coh}_D X \rightarrow \text{coh} U
\]

is an quasi-equivalence of dg categories.

**Proof.** First, we assume that \( X \) is quasi-projective.

(i) By [Sei08], we have

\[
\colim_i \Gamma(V; F \otimes \mathcal{O}(D)^{\otimes i}) \simeq \Gamma(U \cap V; F|_U)
\]

for any open set \( V \) of \( X \) and obtain the desired isomorphism.

(ii) This is given in [Sei08] for quasi-projective cases.

Finally, we let \( X \) be a (not necessarily quasi-projective) toric variety. In our case, \( D \) is not arbitrary but a toric divisor. Hence the star neighborhood of the corresponding ray to \( D \) in \( \Sigma \) gives a open quasi-projective toric subvariety \( X' \) and the complement of the star neighborhood never intersects \( D \). Hence, we can apply the above argument to \( X' \) and glue it to the complement of \( X' \).

\[ \square \]

3 Main theorem

Let \( \Sigma \) be a smooth fan in \( N_\mathbb{R} \) and \( \rho \in \Sigma \) be a 1-dimensional cone in \( \Sigma \). Denote the toric divisor defined as the closure of the \( T \)-orbit \( T_\rho \) associated with \( \rho \) by \( D_\rho \). Set \( \Sigma_\rho^c := \Sigma \setminus \{ \tau \in \Sigma \mid \rho \preceq \tau \} \) the complement of star neighborhood of \( \rho \) and denote the open inclusion by \( i : X_{\Sigma_\rho^c} = X_\Sigma \setminus D_\rho \hookrightarrow X_\Sigma \).

Let further \( B_\rho \) be the subcategory of \( \text{Sh}_w^{\Lambda_\Sigma}(T^n) \) split-generated by microlocal skyscraper sheaves \( \{ F_{p,f} \mid p \in (\Lambda_\Sigma \setminus \Lambda_{\Sigma_\rho^c})_{\text{reg}} \} \) and \( \text{coh}_{D_\rho} X_\Sigma \) be the full dg subcategory of \( \text{coh} X_\Sigma \) consisting of objects supported in \( D_\rho \).

**Theorem 3.1.** Assuming Conjecture [2.10] for \( \Sigma \). There is a quasi-equivalence

\[
\text{Sh}_w^{\Lambda_{\Sigma_\rho^c}}(T^n) \simeq \text{Sh}_w^{\Lambda_{\Sigma_\rho^c}}(T^n)/B_\rho
\]

such that the induced morphism \( \text{Sh}_w^{\Lambda_{\Sigma_\rho^c}}(T^n) \rightarrow \text{Sh}_w^{\Lambda_{\Sigma_\rho^c}}(T^n) \) fits into the diagram

\[
\begin{array}{ccc}
\text{coh} X_\Sigma & \xrightarrow{i_*} & \text{Sh}_w^{\Lambda_{\Sigma_\rho^c}}(T^n) \\
i^* \downarrow & & \downarrow \\
\text{coh} X_{\Sigma_\rho^c} & \xrightarrow{\kappa_{\Sigma_\rho^c}} & \text{Sh}_w^{\Lambda_{\Sigma_\rho^c}}(T^n).
\end{array}
\]
A proof of this theorem is given in the end of this section. Throughout this section, we assume Conjecture 2.10 for \( \Sigma \).

Let \( \mathcal{N}_\rho \) be a dg subcategory \( \kappa_\Sigma(\text{coh}_{\mathcal{D}_\rho}X_\Sigma) \subset \text{Sh}_{\Sigma_c}(T^\mu) \). Set \( D := D_\rho \) for simplicity. Define \( F_D := \kappa_\Sigma(\mathcal{O}(D)) \) and

\[
\beta(F) := \text{colim}_l F \ast (F_D)^{l1} \in \text{Sh}_{\Sigma_c}(T^\mu)
\]

for \( F \in \text{Sh}_{\Sigma_c}(T^\mu) \). Assuming Conjecture 2.10 by Proposition 2.17 \( \beta \circ \kappa_\Sigma = \kappa_\Sigma \circ j_*j^* \)

where \( j: X_{\Sigma_c} \hookrightarrow X_\Sigma \) is the inclusion. In other words,

\[
\beta(F) \simeq F \ast \kappa_\Sigma(\mathcal{O}_{X_{\Sigma_c}}).
\]

**Lemma 3.2.** (0) \( F \ast F' \in \mathcal{N}_\rho \) if \( F' \in \mathcal{N}_\rho \).

1. \( \text{Cone}((F_D)^{l1} \to (F_D)^{l2}) \in \mathcal{N}_\rho \) (\( l_2 > l_1 \)).
2. \( \beta(F) \simeq 0 \) if \( F \in \mathcal{N}_\rho \).

**Proof.** These are clear from the definition of \( \mathcal{N}_\rho \), Conjecture 2.10 the monoidality (Proposition 2.17) and the cocontinuity of \( \kappa_\Sigma \).

Consider the following commutative diagram of dg functors

\[
\begin{array}{ccc}
\text{Sh}_{\Sigma}(T^\mu) & \xrightarrow{\beta} & \text{Ind}(\text{Sh}_{\Sigma}(T^\mu)) = \text{Sh}_{\Sigma_c}(T^\mu) \\
\downarrow & & \downarrow \iota \\
\text{Sh}_{\Sigma}(T^\mu)/\mathcal{N}_\rho & \xrightarrow{\beta_{\mathcal{N}_\rho}} & \text{Sh}_{\Sigma_c}(T^\mu),
\end{array}
\]

where \( \text{Ind} \) is the ind-completion (cf. [KS01], [KS06]). The bottom arrow is defined by using Lemma 3.2 (2). By the cocompleteness of \( \text{Sh}_{\Sigma_c}(T^\mu) \), \( \beta_{\mathcal{N}_\rho} \) extends to \( \text{Ind}(\text{Sh}_{\Sigma}(T^\mu))/\mathcal{N}_\rho) \):

\[
\begin{array}{ccc}
\text{Sh}_{\Sigma}(T^\mu)/\mathcal{N}_\rho & \xrightarrow{\beta_{\mathcal{N}_\rho}} & \text{Sh}_{\Sigma_c}(T^\mu) \\
\downarrow & & \downarrow \Phi \\
\text{Ind}(\text{Sh}_{\Sigma}(T^\mu))/\mathcal{N}_\rho.
\end{array}
\]

There exists a canonical candidate for a quasi-inverse of \( \Phi \). Let \( \tilde{q}: \text{Ind}(\text{Sh}_{\Sigma_c}(T^\mu)) \to \text{Ind}(\text{Sh}_{\Sigma_c}(T^\mu)/\mathcal{N}_\rho) \) be the morphism induced by \( \text{Sh}_{\Sigma_c}(T^\mu) \to \text{Sh}_{\Sigma}(T^\mu)/\mathcal{N}_\rho \). Define \( \Psi \) to be the composite of \( i: \text{Sh}_{\Sigma_c}(T^\mu) \hookrightarrow \text{Sh}_{\Sigma}(T^\mu) \simeq \text{Ind}(\text{Sh}_{\Sigma_c}(T^\mu)) \) and \( \tilde{q} \):

\[
\Psi := \tilde{q} \circ i: \text{Sh}_{\Sigma_c}(T^\mu) \hookrightarrow \text{Sh}_{\Sigma}(T^\mu) \simeq \text{Ind}(\text{Sh}_{\Sigma_c}(T^\mu)) \to \text{Ind}(\text{Sh}_{\Sigma_c}(T^\mu)/\mathcal{N}_\rho).
\]

**Lemma 3.3.** The functor \( \Psi \) is a quasi-inverse to \( \Phi \). Hence, \( \text{Ind}(\text{Sh}_{\Sigma}(T^\mu)/\mathcal{N}_\rho) \simeq \text{Sh}_{\Sigma_c}(T^\mu) \). In particular, there is a quasi-equivalence

\[
\beta_{\mathcal{N}_\rho}: \text{Sh}_{\Sigma}(T^\mu)/\mathcal{N}_\rho \xrightarrow{\simeq} \text{Sh}_{\Sigma_c}(T^\mu) \subset \text{Sh}_{\Sigma_c}(T^\mu).
\]

by taking compact objects.
**Proof.** Denote by $[F]$ the image of $F \in \text{Sh}_{\Lambda^w}(T^n)$ in $\text{Sh}_{\Lambda^w}(T^n)/N_\rho$. Any object of \text{Ind}(\text{Sh}_{\Lambda^w}(T^n)/N_\rho) is of the form of $\lim\limits_{\to} [F]$ $(F_i \in \text{Sh}_{\Lambda^w}(T^n))$, where $\lim\limits_{\to}$ is formal limit (cf. [KS06]). By construction, the functors $\Phi$ and $\tilde{q}$ are described as

$$\Phi: \lim\limits_{\to} [F_i] \mapsto \colim_i \lim\limits_{\to} F_i \ast (F_D)^{\ast l}, \quad (3.9)$$

$$\tilde{q}: \lim\limits_{\to} [F_i] \mapsto \lim\limits_{\to} [F_i]. \quad (3.10)$$

Moreover, the equivalence $\text{Ind}(\text{Sh}_{\Lambda^w}(T^n)) \xrightarrow{\sim} \text{Sh}_{\Lambda^w}$ can be described as

$$\lim\limits_{\to} F_i \mapsto \colim_i F_i, \quad (3.11)$$

Hence, the functor $\Psi \circ \Phi: \text{Ind}(\text{Sh}_{\Lambda^w}(T^n)/N_\rho) \to \text{Ind}(\text{Sh}_{\Lambda^w}(T^n)/N_\rho)$ is

$$\Psi \circ \Phi: \lim\limits_{\to} [F_i] \to \lim\limits_{\to} \lim\limits_{\to} [F_i \ast (F_D)^{\ast l}]. \quad (3.12)$$

By Lemma 3.2 (0) and (1), $F_i \to F_i \ast (F_D)^{\ast l}$ is an isomorphism in $\text{Sh}_{\Lambda^w}(T^n)/N_\rho$. Therefore the image of $\lim\limits_{\to} [F_i]$ under $\Psi \circ \Phi$ is isomorphic to itself and $\Psi \circ \Phi \simeq \text{id}$. Next we consider $\Phi \circ \Psi$. Let $G \in \text{Sh}_{\Lambda^w}$ and write $G \simeq \colim_i F_i$ $(F_i \in \text{Sh}_{\Lambda^w}(T^n))$. By construction,

$$\Phi \Psi(G) = \colim_i \beta(F_i) = \colim_i \lim\limits_{\to} F_i \ast (F_D)^{\ast l}. \quad (3.13)$$

Consider a distinguished triangle

$$F_i \to F_i \ast (F_D)^{\ast l} \to \text{Cone}(F_i \to F_i \ast (F_D)^{\ast l}) \xrightarrow{+1}. \quad (3.14)$$

By Lemma 3.2 (0) and (1), $\text{Cone}(F_i \to F_i \ast (F_D)^{\ast l}) \in N_\rho$. Taking colimits, we obtain a distinguished triangle

$$\colim_i F_i \to \colim_i \beta(F_i) \to \colim_i \text{Cone}(F_i \to F_i \ast (F_D)^{\ast l}) \xrightarrow{+1}. \quad (3.15)$$

Here $\colim_i F_i \simeq G \in \text{Sh}_{\Lambda^w}(T^n)$ and $\beta(F_i) \in \text{Sh}_{\Lambda^w}(T^n)$ by Proposition 2.7.3. By the distinguished triangle and the triangular inequality for microsupports, we get

$$\colim_i \text{Cone}(F_i \to F_i \ast (F_D)^{\ast l}) \simeq \text{Sh}_{\Lambda^w}(T^n). \quad (3.16)$$

**Lemma 3.4.** We have

$$\bot \text{Sh}_{\Lambda^w}(T^n) \cap \text{Sh}_{\Lambda^w}(T^n) = N_\rho. \quad (3.17)$$

As a corollary,

$$\text{hom}_{\text{Sh}_{\Lambda^w}(N_\rho), \text{Sh}_{\Lambda^w}(\text{Sh}_{\Lambda^w}(T^n))} = 0 \quad (3.18)$$

where $\overline{\text{N}_\rho}$ is the filtered cocomplete closure of $N_\rho$.  

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\textbf{Proof.} First, we show that \( \mathcal{Sh}_{X_{\Sigma}}(T^n) \cap \mathcal{Sh}_{\Lambda_{\Sigma}}(T^n) \supset \mathcal{N}_{\rho} \). It is enough to show that

\[
\text{hom}_{\text{coh}(X_{\Sigma})}(\mathcal{F}, \kappa_{\Sigma}^{-1}H) = 0
\]  

(3.19)

for any \( \mathcal{F} \in \text{coh}_{D_{\rho}} X_{\Sigma} \) and \( H \in \mathcal{Sh}_{X_{\Sigma}}(T^n) \). Set \( \mathcal{G} := \kappa_{\Sigma}^{-1}H \). Since \( \mathcal{F} \) is coherent, we have

\[
\text{Hom}(\mathcal{F}, \mathcal{G}) \simeq \text{hom}_{\mathcal{O}_x}(\mathcal{F}_x, \mathcal{G}_x).
\]  

(3.20)

For \( x \notin D_{\rho} \), \( \mathcal{F}_x \simeq 0 \) and RHS vanishes. It is enough to show that \( \text{hom}_{\mathcal{O}_x}(\mathcal{G}_x, \mathcal{O}_x) \simeq 0 \) for \( x \in D_{\rho} \) since \( \text{coh}(\{x\}) \simeq (\mathcal{O}_x) \simeq \text{Vect}_\mathbb{C} \). Denote by \( i_x : \{x\} \hookrightarrow X \) the inclusion. Then we have the isomorphisms

\[
\text{Hom}_{\mathcal{O}_x}(\mathcal{G}_x, \mathcal{O}_x) \simeq \text{hom}_{\mathcal{O}_x}(i_x^* \mathcal{G}, \mathcal{O}_x)
\]

\[
\simeq \text{hom}_{\mathcal{O}_X}(\mathcal{G}, \mathcal{O}_x)
\]

\[
\simeq \text{hom}_{\mathcal{Sh}(H, \kappa_{\Sigma}(\mathcal{O}_x))}.
\]

Fix an integral basis of \( N \) including the primitive vector of \( \rho \). Then, we have the dual integral basis of \( M \) which has \( v_\rho \) the dual to \( \rho \). As we described in Proposition 2.16, \( \kappa(\mathcal{O}_x) \) is of the form

\[
\tilde{p}(\mathcal{L} \boxtimes \mathbb{C}_{\{0,1\}}).
\]  

(3.21)

where the second component of the exterior tensor is the direction \( v_\rho \). the first part are spanned by the others , \( \mathcal{L} \) is a local system on \( T^{n-r} \) and \( \tilde{p} : T^{n-r} \times \mathbb{R}^r \to T^n \) is the quotient map. Then we obtain an isomorphism

\[
\text{hom}(H, \kappa(\mathcal{O}_x)) \simeq \text{hom}(\tilde{p}^{-1}H, \mathcal{L} \boxtimes \mathbb{C}_{\{0,1\}}).
\]  

(3.22)

Here by Corollary 6.5 of [KS90], \( \text{SS} (\text{Hom}(\tilde{p}^{-1}H, \mathcal{L} \boxtimes \mathbb{C}_{\{0,1\}})) \) does not contain \( \rho \). Thus we can apply the microlocal Morse lemma ( Proposition 2.3 ) to the function \( \psi(t, x) := (\nu'_r, x) \ ( (t, x) \in T^{n-r} \times \mathbb{R}^r , ) \) where \( \nu'_r \) is the vector in \( \mathbb{R}^r \) corresponding to \( v_\rho \). Noticing that \( \text{Hom}(\tilde{p}^{-1}H, \mathcal{L} \boxtimes \mathbb{C}_{\{0,1\}}) \) has a compact support, we obtain \( \text{hom}(H, \kappa(\mathcal{O}_x)) \simeq \text{hom}(\tilde{p}^{-1}H, \mathcal{L} \boxtimes \mathbb{C}_{\{0,1\}}) = 0 \). As a corollary, the second line in the statement follows.

Conversely, for \( E \in \mathcal{Sh}_{\Lambda_{\Sigma}}(T^n) \cap \mathcal{Sh}_{\Lambda_{\Sigma}}(T^n) \), we want to see that \( \kappa_{\Sigma}^{-1}(E)_x \simeq 0 \) for \( x \in X_{\Sigma} \). This is implied by

\[
0 \simeq \text{hom}_{\mathcal{Sh}(\kappa_{\Sigma}(\mathcal{O}_x), E)} \simeq \text{hom}_{\mathcal{X}_\Sigma}(\mathcal{O}_x, \kappa_{\Sigma}^{-1}(E))
\]

\[
\simeq \text{hom}_{\mathcal{X}_\Sigma}(\kappa_{\Sigma}^{-1}(E), \mathcal{O}_x)^{\vee}
\]

\[
\simeq \text{hom}_{\mathcal{O}_x}(\kappa_{\Sigma}^{-1}(E)_x, \mathcal{O}_x)^{\vee}.
\]  

(3.23)

\[
\blacksquare
\]

By Lemma 3.4

\[
\text{hom}(\text{colim}_i \text{colim}_l \text{Cone}(F_i \to F_i \ast (F_D)^{*l}), H) \simeq 0
\]

for any \( H \in \mathcal{Sh}_{\Lambda_{\Sigma}}(T^n) \). Therefore we have \( \text{colim}_i \text{colim}_l \text{Cone}(F_i \to F_i \ast (F_D)^{*l}) \simeq 0 \) and an isomorphism

\[
G \xrightarrow{\sim} \text{colim}_i \beta(F_i).
\]  

(3.24)

This shows \( \Phi \circ \Psi \simeq \text{id.} \)

\[
\blacksquare
\]
Lemma 3.5. There is a quasi-equivalence

\[ \text{coh}_{D_p}(X_{\Sigma}) \xrightarrow{\sim} N_\rho \simeq B_\rho \] (3.25)

induced by the restriction of \( \kappa_{\Sigma} \).

Proof. For a microlocal skyscraper sheaf \( F_{p,f} \in \text{Sh}_{\text{w}}(T^n) \) at \( p \in (\Lambda_{\Sigma} \backslash \Lambda_{\Sigma_{\rho}})^{\text{reg}} \), this is contained in \( N_\rho \) by Lemma 3.4. Suppose that \( F \in N_{\rho} \) satisfies \( \text{hom}(F_{p,f}, F) = 0 \) for any \( p \in (\Lambda_{\Sigma} \backslash \Lambda_{\Sigma_{\rho}})^{\text{reg}} \) and \( f \). Then, \( \text{SS}(F) \subset \Lambda_{\Sigma_{\rho}} \). By Lemma 3.4, we conclude that \( F = 0 \).

Proof of Theorem 3.1. The desired equivalence and the diagram simultaneously follow from Proposition 2.17 and Lemma 3.3 and Lemma 3.5.

A Wrapped constructible sheaves are constructible sheaves for mirrors to complete toric surfaces

In this appendix, we prove that wrapped constructible sheaves coincide with constructible sheaves for dim \( \Sigma = 2 \) without assuming Conjecture 2.10. We consider such a proof is useful for further developments of theory of wrapped constructible sheaves and decide to place it here.

Theorem A.1. Assume \( \Sigma \) is a smooth complete fan and dim \( \Sigma = 2 \). Then, the wrapped constructible sheaves along \( \Lambda_{\Sigma} \) are constructible.

This theorem is concluded after the propositions in this section and Proposition 2.11. We also place a proof of Conjecture 2.9 for dim \( \Sigma = 2 \) as a corollary in the end of this section.

Let \( \mathcal{S} \) be the coarsest stratification of \( T^2 \) such that \( \Lambda_{\Sigma} \subset \Lambda_{\mathcal{S}} := \bigcup_{S \in \mathcal{S}} T^n_S T^2 \). Take \( F \in \text{Sh}_{\Lambda_{\Sigma}}(T^2) \). Let \( x \in M_\mathbb{R} \) be a point over \([x] \in T^2 \) and define \( C_x \subset M_\mathbb{R} \) by

\[ C_x := \bigcap_{\rho \in \Sigma(1), c \in \mathbb{Z}} H_{\rho \geq c}, \] (A.1)

where

\[ H_{\rho \geq c} := \{ m \in M_\mathbb{R} \mid \langle m, \rho \rangle \geq c \}. \] (A.2)

We say \( C_x \) is the closure of \( x \) with respect to \( \Lambda_{\Sigma} \).

Proposition A.2. If \( \text{hom}(F, p(C_x)) \) is perfect, \( F_x \) is also perfect.

Proof. We denote the inclusion by \( t_{C_x} : C_x \rightarrow M_\mathbb{R} \). Note that the finite-dimensionality of \( \text{hom}(F, p_{tC_x} C_x) \) is equivalent to that of \( \text{hom}(C_{\text{Int}(C_x)}, \mathbb{D}_{tC_x}^{-1} p^{-1} F) = \Gamma(\text{Int}(C_x), \mathbb{D}_{tC_x}^{-1} p^{-1} F) \).

We define a continuous family of polyhedral open sets \( \{ \mathcal{C}_t \}_{t \in (0,1)} \) in \( \text{Int}(C_x) \) by

\[ \mathcal{C}_t := x + t(\text{Int}(C_x) - x) \] (A.3)

where the multiplication and the sum are taken with respect to the structure of the vector space of \( M_\mathbb{R} \). This family satisfies
(i) $\mathcal{C}_t$ is congruent to $\text{Int}(C_x)$ for any $t \in (0, 1)$,
(ii) $\cup_{t \in (0, 1)} \mathcal{C}_t = \text{Int}(C_x)$,
(iii) $\{x\} = \cap_{t \in (0,1)} \mathcal{C}_t$, and
(iv) the conditions (1) and (2) of Proposition 2.2.

If the conormal cone of a face of $\mathcal{C}_t$ intersects $-\Lambda_{\Sigma}$ on $y \in M_\mathbb{R}$, then $y \in \rho^\perp$ for some $\rho \in \Sigma(1)$. This contradicts to the minimality of $C_x$. Hence, (3) of Proposition 2.2 is also satisfied and we have

$$\Gamma(\text{Int}(C), D_{\mathcal{C}_t}^{-1}F) \simeq \Gamma(\mathcal{C}_t, D_{\mathcal{C}_t}^{-1}p^{-1}F)$$  \hspace{1cm} (A.4)

for any $t \in (0, 1)$. By the constructibility of $\mathbb{D}(F)$, for a sufficiently small $t$, we have

$$\Gamma(\mathcal{C}_t, D_{\mathcal{C}_t}^{-1}p^{-1}F) \simeq (D_{\mathcal{C}_t}^{-1}p^{-1}F)_x \simeq (\mathbb{D}F)_x.$$  \hspace{1cm} (A.5)

Hence, we can conclude that hom$(\mathcal{C}_{\text{Int}(C_x)}, D_{\mathcal{C}_t}^{-1}p^{-1}F)$ is isomorphic to $(\mathbb{D}F)_x$, hence $F_x$ is finite-dimensional.

Hence, it suffices to show the following to prove Theorem A.1.

**Proposition A.3.** Assume that $F$ is a compact object of $\text{Sh}_{\Lambda_{\Sigma}}(T^2)$. Let $x \in M_\mathbb{R}$ be a point and $C_x$ be the closure of $x$ with respect to $\Lambda_{\Sigma}$. If $\Sigma$ is complete, hom$(F, p_!C_x)$ is perfect.

First, we will prove the following lemma.

**Lemma A.4.** Assume that $F$ is a compact object of $\text{Sh}_{\Lambda_{\Sigma}}(T^2)$. Let $C$ be a closed subset of $M_\mathbb{R}$ which satisfies $\text{SS}(p_!C) \subset \Lambda_{\Sigma}$. Then, hom$(F, p_!C)$ is perfect.

**Proof.** For an open set $U \subset T^2$, the morphism

$$\text{hom}(F, p_!C) \longrightarrow \text{hom}(F(U), (p_!C)(U))$$  \hspace{1cm} (A.6)

induces the dual morphism

$$\text{hom}(F(U), (p_!C)(U))^\vee \longrightarrow \text{hom}(F, p_!C)^\vee.$$  \hspace{1cm} (A.7)

Here by the boundedness of $C$, $(p_!C)(U)$ is finite-dimensional. Hence we obtain an isomorphism

$$\text{hom}(F(U), (p_!C)(U))^\vee \otimes (p_!C)(U) \simeq \text{hom}(\text{hom}(F(U), (p_!C)(U)), (p_!C)(U)).$$  \hspace{1cm} (A.8)

Combining this with the evaluation map

$$F(U) \longrightarrow \text{hom}(\text{hom}(F(U), (p_!C)(U)), (p_!C)(U)),$$  \hspace{1cm} (A.9)

we get a morphism

$$e_C(U) : F(U) \longrightarrow \text{hom}(F, p_!C)^\vee \otimes (p_!C)(U)$$  \hspace{1cm} (A.10)

and $e_C \in \text{hom}(F, \text{hom}(F, p_!C)^\vee \otimes p_!C)$. Here $\text{hom}(F, p_!C)^\vee \simeq \text{colim}_V V$, where $V$ ranges through the family of finite-dimensional subspace of $\text{hom}(F, p_!C)^\vee$. Since tensor products commutes with colimits, one has an isomorphism

$$(\text{colim}_V V) \otimes p_!C \simeq \text{colim}_V (V \otimes p_!C).$$  \hspace{1cm} (A.11)
Moreover by the compactness of \( F \), we have
\[
\text{hom}(F, \text{hom}(F, p_！C_C) \otimes p_！C_C) \simeq \text{hom}(F, \text{colim}(V \otimes p_！C_C)) \\
\simeq \underset{V}{\text{colim}} \text{ hom}(F, V \otimes p_！C_C).
\]

(A.12)

Therefore there is a finite-dimensional subspace \( V_0 \) satisfying \( e_S \in \text{hom}(F, V_0 \otimes p_！C_C) \). By construction, \( \langle e_C, \varphi \rangle = \varphi \) for any \( \varphi \in \text{hom}(F, p_！C_C) \), which implies that \( \text{hom}(F, p_！C_C) \) is finite-dimensional. \( \square \)

To prove Proposition A.3, we give some explanations of constant sheaves on locally closed polyhedral sets.

**Definition A.5.**

(i) A locally closed subset \( Z \) of \( M_\mathbb{R} \) is a \textit{locally closed rational polyhedral set} if there exists \( \rho_1, \ldots, \rho_r \in \mathbb{N} \) and \( c_1, \ldots, c_r \in \mathbb{Z} \) for some \( r \) such that
\[
Z = \bigcap_{i=1}^{i'} H_{\rho_i \geq c_i} \cap \bigcap_{i=i'+1}^{r} H_{-\rho_i < c_i},
\]
for some \( i' \). We sometimes omit “rational” for short.

(ii) A locally closed polyhedral set \( Z \) is a \( \Lambda_\Sigma \)-polyhedral if \( \rho_i \in \Sigma(1) \) for any \( i \leq i' \) and \( -\rho_i \in \Sigma(1) \) for any \( i > i' \) in the above expression.

(iii) A face of locally closed polyhedral set is a face of the closure, which is a polyhedron.

(iv) For a face \( f \) of locally closed polyhedral set \( Z \), there exists the subset \( R \subset \{1, \ldots, r\} \) such that \( f \subset p_{\rho_i}^+ \) for \( i \in R \). Then, we refer \( \{\rho_i\}_{i \in R} \) as \textit{defining rays of} \( f \).

For \( \text{dim} \Sigma = 2 \), we define further notions. Let \( Z \subset \mathbb{R}^2 \) be a locally closed polyhedral set and \( v \) be a vertex of \( Z \). Let \( f_0 \) and \( f_1 \) be facets of \( Z \) containing \( v \) and \( \rho_0 \) and \( \rho_1 \) be defining rays of \( f_0 \) and \( f_1 \), respectively.

We define the length of \( v \) as
\[
l(v) := \# \{ \rho \in \Sigma(1) \mid \rho \subset \text{Int}(\text{Cone}(f_1, f_2)) \}.
\]

(A.14)

Note that if \( l(v) = 0 \) for any vertices of \( Z \), \( Z \) is \( \Lambda_\Sigma \)-polyhedral.

**Definition A.6.** For a \( \Lambda_\Sigma \)-polyhedral set \( Z \), a face \( f \) of \( Z \) is a \( \Lambda_\Sigma \)-face (or belongs to \( \Lambda_\Sigma \)) if there exists a point \( x \) in \( \text{Int}(f) \) where the microsupport at \( x \) of \( C_Z \) is in \( \Lambda_\Sigma \). Otherwise, we say \( f \) is \textit{non-}\( \Lambda_\Sigma \).

**Proof of Proposition A.3.** We shall show that \( \Gamma(\text{Int}(C_x), \mathbb{D}!C_x^{-1}p^{-1}F) \) is perfect for any compact object \( F \in \text{Sh}^{\text{Sh}}_{\Lambda_\Sigma}(T^2) \). We will construct an object \( C_x \in \text{Sh}^{\text{Sh}}_{\Lambda_\Sigma}(T^2) \) such that
\[
\text{hom}(C_x, G) \simeq \text{hom}(\text{C}_{\text{Int}(C_x)}, G) \simeq \Gamma(\text{Int}(C_x), G),
\]

(A.15)

where \( G := \mathbb{D}!C_x^{-1}p^{-1}F \). By Lemma A.4, this completes our proof. Note that \( \text{SS}(G) \subset \widetilde{\Lambda}_\Sigma \).

If \( \text{dim} C_x = 1 \), after restricting to the affine span of \( C_x \), we can take \( C_x \) as the constant sheaf on the minimal open segment connecting points in \( M \) which contains \( C_x \).

Hence, we assume \( \text{dim} C_x = 2 \). Then non-\( \Lambda_\Sigma \) faces of \( C_x \) are vertices. Suppose that a vertex \( v \) of \( C_x \) is non-\( \Lambda_\Sigma \). Let \( \rho_0, \rho_1 \in \Sigma(1) \) be the defining rays of \( v \). If \( \text{Int}(\text{Cone}(\rho_0, \rho_1)) \) contains no elements of \( \Sigma(1) \) (i.e. \( l(v) = 0 \)), the smoothness assumption implies that \( v \) belongs to \( \Lambda_\Sigma \). This contradicts to the assumption. Hence, we can take \( \rho_2 \in \Sigma(1) \) such
that \( \text{Int}(\text{Cone}(\rho_0, \rho_2)) \) contains no \( \Sigma(1) \)'s. Note that a sufficiently small neighborhood of \( v \) is translation of a neighborhood of 0 of \( H_{\rho_0 > 0} \cap H_{\rho_1 > 0} \). We set
\[
D_1 := H_{\rho_0 > 0} \cap H_{\rho_1 > 0} \cap H_{\rho_2 < c},
\]
where \( c \) is the smallest integer which makes \( D_1 \) non-empty. Then, \( D_1 \) is a 2-dimensional locally closed simplex such that those 3 vertices are

(i) \( v \),

(ii) a \( \Lambda_\Sigma \)-vertex formed by \( \rho_0 \) and \( \rho_2 \), and

(iii) a vertex \( v' \) with \( l(v') < l(v) \).

Thus, there exists an extension map \( \mathbb{C}_{D_1} \to \mathbb{C}_{\text{Int}(C_x)} \) (cf. [STZ14]). We can repeat this process for other non-\( \Lambda_\Sigma \) vertices of \( C_x \) and \( D_1 \). Moreover the sum of lengths of vertices strictly decreases in this inductive process. Hence, this process will stop in finitely many steps. \( \square \)

Now we prove Conjecture 2.9 (or equivalently Conjecture 2.10) for a smooth not necessarily complete 2-dimensional fan \( \Sigma \).

Theorem A.7. If \( \dim \Sigma = 2 \), Conjecture 2.9 is true. In other words, \( \kappa_\Sigma \) induces a quasi-equivalence
\[
\kappa_\Sigma : \text{coh } X_\Sigma \xrightarrow{\sim} \text{Sh}^w_{\Lambda_\Sigma}(T^2)
\]
for any smooth (not necessarily complete) fan \( \Sigma \).

Proof. By completion and induction, it suffices to show the following: for a smooth complete fan \( \tilde{\Sigma} \) with \( \dim \tilde{\Sigma} = 2 \) and a 1-dimensional cone \( \rho \) in \( \tilde{\Sigma} \), one has the quasi-equivalence
\[
\kappa_{\tilde{\Sigma}_\rho} : \text{coh } (X_{\tilde{\Sigma}_\rho}) \xrightarrow{\sim} \text{Sh}^w_{\Lambda_{\tilde{\Sigma}_\rho}}(T^2).
\]
Here we used the same notation as in Section 3. By taking ind-categories in the equivalence of Theorem 2.8 by Theorem A.1, we see that Conjecture 2.10 for \( \Sigma \) holds. Hence the theorem follows from Theorem 3.1. \( \square \)

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