The Large N Harmonic Oscillator as a String Theory

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We propose a duality between the large-$N$ gauged harmonic oscillator and a novel string theory in two dimensions.

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1. Introduction

Since ’t Hooft’s work thirty years ago [1] it is generally believed that large-\(N\) gauge theories admit a dual string theory description. Given the utility of such dualities for both sides, and the difficulty of finding them, it is of interest to find more examples which are under precise control. The aim of this paper is to explore a new and remarkably simple duality of this kind. We address an old question that was resurrected recently [2]: what is the string theory dual of the large-\(N\) gauged harmonic oscillator?

A clue comes from the fact that the large \(N\) inverted harmonic oscillator is dual to a certain string theory in two dimensions (for reviews see [3, 4, 5]). Therefore, we should find the meaning in string theory of rectifying the inverted matrix potential. This is done in section 4, after discussing some of the properties and symmetries of the large-\(N\) harmonic oscillator in sections 2 and 3. The physics of the harmonic oscillator and the inverted oscillator are very different. The latter has a continuous spectrum, while in the former the spectrum is discrete. This implies that, unlike in the standard 2d-string/matrix model duality, there is no need to take a double scaling limit to find a continuum dual string description. Finite \(N\) corresponds to nonzero string coupling constant. In this sense, the duality we propose works more like the AdS/CFT duality [6] than the standard 2d-string/matrix model duality. Since the matrix model is well-defined at finite \(N\), we will be able to define and study interesting finite-coupling effects in this bosonic string theory.

The observables of interest in the large-\(N\) harmonic oscillator are the overlap amplitudes between resonances. In section 5 we explain how the resonances come about on the string theory side. In section 6 we calculate these overlap amplitudes on both sides of the duality and show that the large-\(N\) limit of the harmonic oscillator exactly agrees with the relevant sphere amplitudes on the string theory side. In this section, we also make contact with a normal matrix model. In section 7 we show that the duality passes non-trivial tests involving the \(1/N\) corrections to the leading large-\(N\) behavior. Section 8 is devoted to a heuristic picture of the duality.

In addition to providing a new example of open/closed string duality, the relation we propose is interesting for a completely different reason. The matrix harmonic oscillator is closely related to the quantum hall effect (QHE) and therefore the dual stringy description might be useful for understanding various open questions in the QHE, like what is the effective description of the quantum phase transition from one plateau to the next. In section 9 we briefly speculate on this and other possible applications and generalizations.
2. The Matrix Model

The model we wish to study is the gauged quantum mechanics of an $N \times N$ matrix harmonic oscillator,

$$S = \frac{1}{2} \int dt \text{ tr } ((D_0 X)^2 - X^2). \quad (2.1)$$

The derivative $D_0 = \partial_0 + [A_0, \cdot]$ is covariant with respect to gauged $U(N)$ conjugations $X \mapsto \Omega X \Omega^\dagger$. The gauge field acts as a Lagrange multiplier that projects onto singlet states, which in turn describe $N$ free fermions in the harmonic oscillator potential.

This model is, of course, solvable. The possible energy levels of a single fermion are

$$E_j = j + \frac{1}{2}. \quad (2.2)$$

Since there are $N$ fermions the vacuum energy is

$$E_0 = \frac{1}{2} + \frac{3}{2} + \ldots + \frac{(2N - 1)}{2} = \frac{N^2}{2}. \quad (2.3)$$

The Hilbert space is spanned by states labelled by $N$ integers $k_n$ such that $0 \leq k_1 < k_2 < \ldots < k_N$, and the eigenvalues of the Hamiltonian are

$$H |k_1, k_2, \ldots, k_N\rangle = \left( \frac{N}{2} + \sum_{n=1}^{N} k_n \right) |k_1, k_2, \ldots, k_N\rangle. \quad (2.4)$$

Despite the fact that this model is free something quite interesting is happening in the large-$N$ limit: excitations above the ground state are most easily described in terms of a chiral boson, as emphasized recently in [2]. A nice way to see this is to consider the partition function

$$Z = \text{tr } q^H = q^{\frac{N}{2}} \sum_{k_1=0}^{\infty} \sum_{k_2=k_1+1}^{\infty} \ldots \sum_{k_N=k_{N-1}+1}^{\infty} q^{\sum_{n=1}^{N} k_n}. \quad (2.5)$$

Performing the sums sequentially we get [8]

$$Z = q^{N^2/2} \prod_{n=1}^{N} \frac{1}{1 - q^n}. \quad (2.6)$$

This is exactly the partition function of a two-dimensional chiral boson with $\alpha_0 = N$ whose excitations are truncated at level $N$. Namely the Hamiltonian is

$$H = \frac{\alpha_0^2}{2} + \sum_{n=1}^{N} \alpha_{-n} \alpha_n, \quad (2.7)$$
and we are using stringy conventions for the commutators

$$[\alpha_m, \alpha_n] = n\delta_{n+m}. \tag{2.8}$$

Later on we shall argue that this chiral boson is (up to normalization) equivalent to the target-space field of the string description. Note that for finite $N$, the momentum modes of the chiral boson are truncated in a clean way. This is another example of the stringy exclusion principle [9,10], about which we will say more in section 5.

To understand this boson description in more detail, we introduce matrix raising and lowering operators

$$a^i_j = \frac{1}{\sqrt{2}} (X^i_j + iP^i_j), \quad a^{i\dagger}_j = \frac{1}{\sqrt{2}} (X^i_j - iP^i_j), \tag{2.9}$$

where $P$ is the momentum conjugate to the matrix $X$. These operators satisfy

$$[a^i_j, a^{i\dagger}_k] = \delta^i_k \delta^j_l, \quad [H, a] = -a, \quad [H, a^{\dagger}] = a^{\dagger}. \tag{2.10}$$

The vacuum is defined by $a^j_j|0\rangle = 0$. The states

$$|\{m_i\}\rangle \equiv c_{\{m_i\}} \prod_{i=1}^r \text{tr} (a^{i\dagger} m_i)|0\rangle, \tag{2.11}$$

with $m_1 \geq m_2 \geq ... \geq m_r$ provide a useful basis for the Hilbert space. From (2.10) we see that indeed the spectrum is evenly-spaced. Note that the stringy exclusion is the statement that

$$m_i \leq N \tag{2.12}$$

in order that these states be linearly independent.

An orthonormal basis of states can be constructed from (2.11) as

$$|R(\{m_i\})\rangle = c_{\{m_i\}} \chi_R(\{m_i\})(a^{\dagger})|0\rangle \tag{2.13}$$

(see e.g. [11] or [12]) where $R(\{m_i\})$ is the representation of $U(N)$ corresponding to the Young tableau with columns of lengths $(m_1, m_2, ..., m_r)$ and $\chi_R(U)$ is the character of $U$.

\footnote{1 We thank D. Berenstein for pointing out a misstatement in an earlier version, and the latter reference.}
in this representation. The bound \( m_i \leq N \) guarantees that this is indeed a tableau for a \( U(N) \) representation.

In terms of the free-fermion description, the wavefunction for the state (2.13) is a Slater determinant constructed as follows. Look at the tableau \( \{ m_i \} \) sideways, define \( k_n \) to be the row-lengths; note that there are at most \( N \) rows by (2.12). The many-fermion wavefunction is then

\[
\langle z_1 | \otimes \langle z_2 | \cdots \langle z_N | \psi_k \rangle_{k_n + N - n + 1} (z_l)
\]

where \( \psi_n(z) = \langle n | z \rangle = H_n(z)e^{-|z|^2/2} \) are the single-particle harmonic-oscillator wavefunctions. For example, for the empty tableau, this fills the lowest \( N \) energy levels with fermions. The state \( \text{tr} \left( a^\dagger k | 0 \right) \) corresponds to exciting \( k \) fermions by one level each – it makes a hole in the fermi sea at level \( N - k \). For \( m = N \), the hole is at the lowest state. For \( m > N \), there is no such single-particle description, in accord with the stringy exclusion principle.

This is the same Hilbert space as that of two-dimensional Yang-Mills theory on a cylinder (see e.g. [13]). The Hamiltonian for the matrix oscillator is, however, the number of boxes in the tableau,

\[
(H - E_0)|\{ m_i \}\rangle = \sum_{i=1}^r m_i|\{ m_i \}\rangle
\]

rather than the second Casimir \( C_2(R) \).

3. Symmetries

As is well-known (see e.g. §12 of [4]) there are infinitely many conserved charges associated with the harmonic oscillator. These charges generate the \( w_\infty \) algebra that controls much of the physics of the system, and was studied quite intensively in the case of the inverted harmonic oscillator [14] in relation with the \( c = 1 \) theory. Although most of our discussion can be obtained from these papers by plugging i’s in the right places, we find it useful to be explicit because the harmonic oscillator physics is quite different from the inverted oscillator physics.

At large \( N \), a semi-classical description exhibits most of the relevant physics in a simple way. Semi-classically, the eigenvalues form a Fermi sea in phase space. It is most convenient to parametrize the phase space by

\[
U = \frac{(X + iP)}{\sqrt{2}}, \quad V = U^\dagger = \frac{(X - iP)}{\sqrt{2}},
\]

(3.1)
which satisfy the Poisson bracket

\[ \{U,V\}_{PB} = i. \tag{3.2} \]

The Hamiltonian is \( H = UV \), and thus

\[ U(t) = e^{-it}U(0), \quad V(t) = e^{it}V(0), \tag{3.3} \]

which implies that following charges are conserved

\[ Q_{n,m} = e^{i(n-m)t}U^nV^m. \tag{3.4} \]

These charges form the \( w_\infty \) algebra

\[ \{Q_{n,m}, Q_{n',m'}\}_{PB} = i(nm' - mn') Q_{n+n'-1,m+m'-1}. \tag{3.5} \]

The ground state is obtained by filling states in the phase space up to the Fermi surface

\[ UV = N, \tag{3.6} \]

so that the area in units of the Poisson brackets (3.2) is equal to the number of fermions.

Excitations above the ground state can be described using area-preserving transformations of the phase space. The area is preserved since it corresponds to the number of fermions. Area-preserving transformations can be described using a single scalar function \( h(U, V) \), a basis for which are

\[ h_{nm} = U^nV^m, \tag{3.7} \]

\( n \) and \( m \) must be integers in order to preserve the connectivity of the fermi surface. \( h_{nm} \) determines a vector field that is associated with an infinitesimal area-preserving transformation of the U-V plane,

\[ \vec{B}_{nm} = \frac{\partial h_{nm}}{\partial U} \partial V - \frac{\partial h_{nm}}{\partial V} \partial U. \tag{3.8} \]

The Lie brackets of the \( \vec{B}_{nm} \)’s form a \( w_\infty \) algebra.

The vector field \( \vec{B}_{nm} \) clearly depends both on \( n \) and \( m \). However, when acting on the ground state (3.6), \( \vec{B}_{nm} \) depends only on \( |s| = |n - m| \). Let us see how this comes about. Eq. (3.8) implies that \( h_{nm} \) generates the following infinitesimal deformation

\[ \delta V = \epsilon \partial_U h = \epsilon n U^{n-1}V^m, \quad \delta U = -\epsilon \partial_V h = -\epsilon m U^n V^{m-1}. \tag{3.9} \]
Therefore, to leading order in $\epsilon$, we find that

$$(V + \delta V)(U + \delta U) = UV + \epsilon(n - m)U^nV^m. \tag{3.10}$$

If $UV$ is to begin with a constant (like in the ground state) then (3.10) implies that the deformation of the Fermi surface is

$$\delta(UV) \sim \epsilon \left( \frac{U}{V} \right)^{s/2}. \tag{3.11}$$

The only dependence on $r = n + m$ is in the numerical constant. Parameterizing the phase space using polar coordinates (see fig. 1)

$$U = re^{i\theta}, \quad V = re^{-i\theta}, \tag{3.12}$$

we can write the variation in the fermi level (3.11) as

$$\delta(\theta) \sim \epsilon \text{Re}(e^{is\theta}) = \epsilon \cos(|s|\theta). \tag{3.13}$$

If we act more than once with $\vec{B}_{nm}$ on the ground state then both $s$ and $r$ matter as is clear from the $w_\infty$ algebra. For example, both 1 and $UV$ act trivially on the ground state but only 1 acts trivially on every state.

**Fig. 1:** For the right-side-up oscillator, the Fermi sea is a compact droplet. Ripples on the Fermi surface (and holes) travel in circles with unit angular velocity.

This can be conveniently phrased at the quantum level. Using the bosonic description one can excite the ground state by acting with single trace operators. For example, up to a normalization constant, the $\alpha_n$ of the chiral boson, which appeared in (2.7), correspond to

$$\text{tr} \left((a^\dagger)^n\right), \tag{3.14}$$
with $a$ as in (2.9). Momentarily we shall claim that these are dual to excitations of the closed string "tachyon". As should be clear from the semi-classical discussion above, there are other single trace operators in the theory that involve both $a^\dagger$ and $a$. These are associated with the $h_{nm}$ with $m \neq 0$. For example, consider the operator

$$\text{tr} (a(a^\dagger)^2).$$ (3.15)

Acting with this operator on the ground state has the same effect as acting with $\text{tr} (a^\dagger)$. Namely, both take the ground state to the first excited state. However clearly these operators are not the same. In fact we will see that it is natural to relate these operators to discrete states in the dual stringy description.

At this point we encounter a small puzzle. At the quantum level there seem to be many more excitations than at the semi-classical level, which clearly makes no sense. To see this we note that there are many different single trace operators that at the semi-classical level are associated with the same $h_{nm}$. The simplest example is $\text{tr} (aa^\dagger aa^\dagger)$ and $\text{tr} (a^2a^\dagger a^\dagger 2)$. On the one hand, since in a non-Abelian theory the order inside the trace matters, these are different operators. On the other hand semi-classically they are both associated with $h_{2,2}$. The fact that the theory is gauged resolves this issue: in one dimension there is no electric or magnetic field and so the current that couples to $A$ must vanish on-shell,  

$$j = [X, D_0X] = i[a, a^\dagger] = 0.$$ (3.16)

This implies that operators which differ by such commutators actually give the same result when acting on physical states. As a result, the inequivalent single-trace operators are in 1-to-1 correspondence with $w_\infty$ generators. In section 5 we discuss further these operators at the quantum level.

Eq. (3.16) is reminiscent of a normal matrix model (for a recent discussion in relation with the $c = 1$ theory see [13]). We will elaborate on this connection in §6.

2 Note that here we are using matrix commutator $[a, a^\dagger]_j^i$ which is not to be confused with the canonical commutator $[a^j_i, a_k^l]_i$ in (2.10).
4. The dual string theory

Generally speaking it is not an easy task to find the stringy dual description of some large-$N$ gauge theory. In our case a natural candidate comes from the well-studied duality between 2d string theory with a Liouville direction and matrix quantum mechanics. The matrix quantum mechanics dual to 2d string theory is closely related to the one that we are considering. The kinetic term is the same and the sign of the potential term is flipped. Namely, the free fermions experience the celebrated inverted quadratic potential, rather then the harmonic oscillator potential of our case. Starting from that well-tested duality, what we have to do is figure out what it means, on the string theory side, to flip the potential, and see if what we get makes sense. In this section we discuss the meaning of flipping the potential on the string theory side. In the rest of the paper we test our conjecture for the dual stringy description.

As was emphasized in [3] the curvature of the potential in the matrix model is related to the tension of the dual string theory by
\[ U(x) = \frac{1}{2\alpha'} x^2. \] (4.1)

So flipping the potential means that we have to take \( \alpha' \rightarrow -\alpha' \). (4.2)

In dimension larger than two this would cause an instability due to the massive modes that are now tachyonic. However, in 2d there are no massive modes (other than some discrete states that will play an important role shortly) so we do not have that problem. A better way to say this is that instead of (4.2) what we could do is to keep \( \alpha' \) positive and Wick rotate all dimensions. In \( D \) dimensions this means that there are \( D - 1 \) time-like directions, which causes problems if \( D > 2 \). But in two dimensions this just means that the previously spatial dimension is now the time direction and vice-versa.

Let us see what happens if we apply that logic to the \( c = 1 \) theory. Before we flip the potential the dual string theory could be viewed as a tensor product of two CFT’s, a free time-like boson \( X \) (with \( c = 1 \)), which is identified with the quantum mechanics time, and a space-like Liouville field, \( \varphi \), with \( c = 25 \). Namely,
\[ T(z) = \frac{1}{\alpha'} (: \partial X \partial X : - : \partial \varphi \partial \varphi :) + \frac{Q}{\sqrt{\alpha'}} \partial^2 \varphi, \quad Q = b + 1/b = 2. \] (4.3)
Adding the Liouville term, we end up with the following world sheet action

\[ S = \frac{1}{4\pi\alpha'} \int d^2 \sigma \sqrt{g} \left( -\partial_\alpha X \partial^\alpha X + \partial_\alpha \varphi \partial^\alpha \varphi + \sqrt{\alpha'} R^{(2)} \varphi + \mu_0 e^{2b \varphi/\sqrt{\alpha'}} \right). \]  

(4.4)

According to the reasoning above, to find the candidate dual to the large-\(N\) harmonic oscillator we could either apply (4.2) or double Wick rotate

\[ X \to iX, \quad \varphi \to i\varphi. \]  

(4.5)

Either way we get a CFT whose stress tensor is

\[ T(z) = \frac{1}{\alpha'} (: \partial \varphi \partial \varphi : - : \partial X \partial X :) + i \frac{Q}{\sqrt{\alpha'}} \partial^2 \varphi. \]  

(4.6)

The energy-momentum tensor is now complex which implies that the world-sheet theory is non-unitary. This, however, does not mean that the target space theory is inconsistent, because of the large redundancy of the worldsheet description. It is worth pointing out in this context that a similar CFT (with spatial \(\varphi\) and no \(X\)) is used to realize the minimal models in the Coulomb gas formalism (for a review see [16], chapter 9). The total central charge of (4.6) is

\[ c = 2 - 6i^2 Q^2 = 26, \]  

(4.7)

where the minus sign comes from the fact that now the Liouville direction is time-like. The worldsheet action is

\[ S = \frac{1}{4\pi\alpha'} \int d^2 \sigma \sqrt{g} \left( \partial_\alpha X \partial^\alpha X - \partial_\alpha \varphi \partial^\alpha \varphi + i \sqrt{\alpha'} R^{(2)} \varphi + \mu_0 e^{i2b \varphi/\sqrt{\alpha'}} \right). \]  

(4.8)

Notice that in the presence of the Liouville interaction this is not an analytic continuation [17]. During the continuation, the contour of integration of the \(\varphi\) field passes over infinitely many troughs of the Liouville potential, and the two theories are therefore not equivalent. As is clear from (4.1) and (2.1), to compare this string theory with the harmonic oscillator it is most convenient to work with \(\alpha' = 1\), which we use in the rest of the paper.

The coupling of \(\varphi\) to the world-sheet curvature implies that the string coupling constant is

\[ g_s = e^{i2\varphi}. \]  

(4.9)

Notice that unlike in the usual linear dilaton case there is no separation into weakly coupled and strongly coupled regions. The absolute value of \(g_s\) is one everywhere. This seems to suggest that there is no good expansion parameter in this theory. That is indeed the case
in the free theory, $\mu_0 = 0$. However, as we shall see in the next section, in the interacting theory $1/\mu_0$ is the parameter that controls the genus expansion.

From (4.9) we also see that now, again unlike the usual case, we can compactify $\varphi$. The allowed radii of compactification seem to be

$$R = m/2,$$

(4.10)

where $m$ is an integer. However, since non-perturbatively there are D-branes and open strings effects in the theory, the open-string coupling constant should be well-defined. Since $g_0^2 = g_s$ we find that $m$ is an even number. So the smallest possible radius is 1

$$\varphi \sim \varphi + 2\pi.$$

(4.11)

The dual string theory we propose to the large-$N$ harmonic oscillator is (4.8) with this periodicity condition (4.11).

A useful way to think about the radius of the $\varphi$ direction is that $\varphi$ is dual to $\theta$ of (3.12), which has the same periodicity. The relation we are proposing between the large-$N$ harmonic oscillator and that string theory is simple: $X$ should be identified with the quantum mechanics time and $\varphi$ should be identified with the angular variable, $\theta$, in phase-space. This seems to make sense since excitations in both descriptions travel at the same speed (1 in our units) regardless of their energy. Note, however, that on the string theory side, at least naively, there are both left-movers and right-movers (in the target space), while on the quantum mechanics side there are only left-movers. In the next section we will see how to resolve this apparent contradiction.

5. Correlation functions

In this section we describe the map between the quantum mechanics and the string theory degrees of freedom and compare their correlation functions. This section is devoted to a more qualitative illustration of the dictionary between the two sides of the duality. More details can be found in the next section.

Let us start with the quantum mechanics side. The observables of interest are the overlaps between different states

$$\mathcal{O}(\text{bra}; \text{ket}) = \langle \text{bra} | \text{ket} \rangle.$$

(5.1)
Since the time evolution in the theory is trivial, in principle, all other gauge invariant observables are determined by these overlap amplitudes. The simplest case corresponds to what we loosely speaking call a two-point function, namely (5.1) with

\[ \langle \text{bra} \rangle = \langle 0 | \text{tr} (a^k_1), \quad | \text{ket} \rangle = \text{tr} ( (a^\dagger)^{k_2} ) |0 \rangle. \]  

By counting index lines, we see that the leading contribution in the large-\(N\) limit goes like

\[ O(k_1; k_2) \sim \delta_{k_1, k_2} N^{k_1}. \]  

As usual there are \(1/N\) corrections. What is special about this large-\(N\) theory is that here the \(1/N\) corrections are truncated after a finite number of terms. Namely,

\[ O(k_1; k_2) = \delta_{k_1, k_2} \sum_{l=0}^{[k_1/2]} C_l(k_1) N^{k_1-2l}, \]  

where the \(C_l\)'s depend on \(k_1\) but not on \(N\). Later on we shall see how this comes about on the string theory side.

The analog of a three-point function is (5.1) with

\[ \langle \text{bra} \rangle = \langle 0 | \text{tr} (a^k), \quad | \text{ket} \rangle = \text{tr} ( (a^\dagger)^{k_1} \text{tr} ( (a^\dagger)^{k_2} ) |0 \rangle. \]  

In that case the leading contribution in the large-\(N\) limit is

\[ O(k; k_1, k_2) \sim \delta_{k, k_1+k_2} N^{k-1}. \]  

We now turn to the string theory side. The ghost-number-two cohomology contains the tachyon vertex operator

\[ T^\pm_k = c\bar{c} \ e^{-ik(X \pm \phi)} e^{i2b\phi}, \]  

where the factor \(e^{i2b\phi}\) is the string coupling that multiplies the tachyon wave function, and \(k\) is an integer due to the periodicity (4.11). There are four kinds of modes, with the following interpretations

\[
\begin{align*}
T^+_k & : \text{ incoming leftmover,} & T^-_k & : \text{ incoming rightmover,} \\
T^+_k & : \text{ outgoing leftmover,} & T^-_k & : \text{ outgoing rightmover.}
\end{align*}
\]
Note that to make the relation with the matrix model simple, the energy in the $X$ direction (rather than in the $\varphi$ direction which is actually the time direction in this background) determines in our terminology if a wave is incoming or outgoing.

S-matrix amplitudes are constructed in the usual fashion, and take the form

$$A(k_1, k_2, \ldots, k_{n^+}; k_{n^+ + 1}, \ldots, k_{n^+ + n^-}) = \int D\sigma D\varphi e^{-S} \prod_{i=1}^{n^+ + n^-} d\sigma \sqrt{g} e^{ik_i \varphi \pm ik_i X} e^{i2b\varphi},$$

where it is understood that the first $n^+$ tachyons have positive chirality, and the remaining $n^-$ have negative chirality. The action $S$ is given by (4.8) (with $\alpha' = 1$).

At first sight it seems hard to do calculations in this theory, since the worldsheet action contains an interaction term. However, we can expand the exponent in powers of $\mu_0$ and get

$$A(k_1, k_2, \ldots, k_{n^+}; k_{n^+ + 1}, \ldots, k_{n^+ + n^-}) = \int D\sigma D\varphi \sum_{n=0}^{\infty} \frac{\mu_0^n}{n!} \left( \int d\sigma \sqrt{g} e^{i2b\varphi} \right)^n e^{-S_0} \prod_{i=1}^{n^+ + n^-} d\sigma \sqrt{g} e^{ik_i \varphi \pm ik_i X} e^{i2b\varphi},$$

where

$$S_0 = \frac{1}{4\pi} \int d\sigma \sqrt{g} \left( \partial_\alpha X \partial^\alpha X - \partial_\alpha \varphi \partial^\alpha \varphi + iR^{(2)} Q\varphi \right).$$

Eq. (5.9) now takes the form of a sum over amplitudes in the free theory with the same tachyon (or any other closed string mode) insertions plus additional insertions of the screening operator, $\int d\sigma \sqrt{g} e^{i2b\varphi}$.

To proceed it is convenient to follow and decompose $\varphi = \varphi^0 + \tilde{\varphi}$ and $X = X^0 + \tilde{X}$ and integrate first the zero modes, $\varphi^0$ and $X^0$. The integral over $X^0$ is a delta function which imposes energy conservation

$$k^+_{tot} + k^-_{tot} = 0, \quad k^+_{tot} = \sum_{i=1}^{n^+} k_i, \quad k^-_{tot} = \sum_{i=n^+ + 1}^{n^+ + n^-} k_i.$$

The integral over $\varphi^0$ imposes the condition

$$2 - 2g - (n + n^+ + n^-) + \frac{1}{2}(k^+_{tot} - k^-_{tot}) = 0,$$

where $g$ is the genus and $n$ is the number of insertions of the screening operator. This relation is the equivalent of momentum conservation in the $\varphi$ direction. The last two terms need no explanation. The rest of the terms are usually called the background charge, as
they are induced by the coupling of $\varphi_0$ to the integral of the world sheet curvature which is

$$\chi = 2 - 2g - h,$$

where $h$ is the number of holes; in our case $h$ corresponds to the total number of closed string insertions. Combining (5.12) and (5.11) we get

$$k_{tot}^- = k_{tot}^+ = 2 - 2g - (n + n_+ + n_-). \tag{5.13}$$

This relation implies that $1/\mu_0$ is indeed the genus expansion parameter: Given a certain amplitude determined by $k_i, n^+$ and $n^-$, we see that as we increase $g$, we decrease $n$, the power of $\mu_0$. The phases arising from the complex dilaton conspire to make $\mu_0^{-1}$ the string coupling constant. In fact, comparing (5.13) to the 't Hooft counting of powers of $N$, we see that we can identify

$$\mu_0 \sim N. \tag{5.14}$$

Now we are ready to compare some stringy amplitudes with the quantum results mentioned at the beginning of this section. Let us start with $1 \rightarrow 1$ amplitudes. There are two possible ways to satisfy the energy conservation (5.11) and momentum conservation (5.12) conditions on the sphere ($g = 0$). The first is to take $n = 0$ and $k_{tot}^- = k_{tot}^+ = 0$. This can be achieved by having two particles with the same chirality and opposite energy (say $n^+ = 2$ and $n^- = 0$). Such amplitudes vanish since they involve only two closed string insertions, which do not suffice to saturate the ghost zero modes on the sphere. Indeed there is no dual quantum mechanical amplitude for these scattering amplitudes.

The second way to saturate (5.11), (5.12) is more interesting. We take one particle with positive chirality and energy $k$ and another particle with the negative chirality and energy $-k$. So $n^+ = 1$ and $n^- = 1$. From (5.13) we see that $n = k$ and so the amplitude scales like

$$A(k; -k) \sim \mu_0^k \sim N^k. \tag{5.15}$$

This amplitude does not vanish since (due to the insertions of the screening operators) it involves more than two closed string insertions on the sphere. It is natural to make the following identification between the closed string modes and the quantum mechanics operators

$$T_{k>0}^+ \Leftrightarrow \text{tr} \left((a^\dagger)^k\right), \quad T_{k<0}^- \Leftrightarrow \text{tr} \left(a^k\right). \tag{5.16}$$
Indeed we see that (5.15) is in agreement with the harmonic oscillator result (5.3). Notice that amplitudes that involve \( T_{k<0}^+ \) and \( T_{k>0}^- \) vanish. This resolves the puzzle arising from the naive expectation that we should have twice as many stringy modes as excitations of the Fermi surface, raised at the end of the previous section. Namely, just like in the harmonic oscillator, we only see tachyon wavefunctions with \( p_\varphi = -i\partial_\varphi \) negative. In this 'imaginary' version of Liouville theory, there is a sense in which the Seiberg bound becomes the fact that the target-space field is chiral.

**Stringy exclusion**

As usual in string theory there are corrections from higher-genus worldsheets. Since \( n \geq 0 \) we find from (5.13) non-vanishing amplitudes only for

\[
g \leq \left\lfloor \frac{k}{2} \right\rfloor,
\]

which exactly agrees with (5.4). So our string theory has the remarkable property that the perturbative corrections to a given amplitude truncate after a finite number of terms. As it is clear from (5.17) there are more and more corrections as we increase \( k \). Although the single-trace states \( \text{tr} \ a^{ik}|0\rangle \) remain orthogonal for any \( k \leq N \) (by energy conservation), this suggests that, as in ten dimensions [24], the best description of these states at \( k \sim N \) may not be in terms of a perturbative closed string. It seems likely that a geometric D-brane mechanism may again explain the interesting finite \( N \) truncation of the spectrum. In fact, models that are closely related to the harmonic oscillator have been considered in [25,11] in relation with giant gravitons in AdS spaces. A similar phenomenon of a UV cutoff on the target space momentum of order \( 1/g_s \) has recently been observed in the context of topological strings [20].

**Three-point functions**

Next we turn to “three-point functions,” namely, the \( 1 \to 2 \) scattering amplitudes. From (5.13) we see that the sphere contribution to such amplitudes scales like

\[
A(k; -k_1, -k_2) \sim \delta_{k,k_1+k_2} \mu_0^{k-1}.
\]

Again we find agreement with the harmonic oscillator scaling (5.6). At the level of the discussion of this section it seems possible to have \( k_1 > 0 \) and \( k_2 < 0 \) such that (5.18) appear not to vanish. This would contradict the proposed duality. A closer look in the next section will show that such amplitudes in fact vanish, in agreement with the discussion below (5.16).

\footnote{In the next section we shall see how this comes about in higher point functions as well.}
5.1. Discrete states

The closed-string ghost-number-two cohomology contains other states, known as discrete states, that are the remnants of the massive modes of the string. In the usual $c = 1$ theory they appear as non-normalizable modes with imaginary energy (or real Euclidean energy) and imaginary Liouville momentum. In our case they become normalizable propagating modes that are as important as the tachyon modes discussed above.

Let us review how the discrete states come about. The simplest way to think about them is in terms of the chiral $SU(2)$ current algebra

$$J^\pm(z) = e^{\pm 2iX(z)}, \quad J^3(z) = i\partial X(z). \quad (5.19)$$

The highest weight fields with respect to this algebra are $\Psi_{j,j}(z) = e^{2ijX}$ where $j = 0, 1/2, 1, ...$. With the help of $J^+_0 = \oint \frac{dz}{2\pi i} e^{-2iX}$ we can now form representations of the $SU(2)$ algebra

$$\Psi_{j,m}(z) \sim (J^+_0)^{j-m}\Psi_{j,j}(z), \quad m = -j, -j + 1, ..., j. \quad (5.20)$$

The closed string vertex operators (with dimension $(1, 1)$ and ghost number two) in the theory are

$$S_{j,m}(z, \bar{z}) = Y_{j,m}Y_{j,m}, \quad Y_{j,m} = e^{2i(1-j)\varphi}. \quad (5.21)$$

The $S_{j,j}$’s are the tachyon vertex operators that we have already discussed. The rest are new fields that are called the discrete states. In our case this name is a bit misleading since the tachyon modes are discrete as well. This analysis differs from the usual $c = 1$ case only by the inclusion of appropriate factors of $i$. Again we emphasize that this is a crucial $i$ since it turns the discrete states into a physical propagating modes.

It is easy to see that the energy in the $X$ direction of $S_{j,m}$ is $2m$ while the momentum in the $\varphi$ direction is $2j$. Therefore, when computing $e.g.$ the $1 \rightarrow 1$ amplitude we get, instead of (5.11) and (5.12),

$$2m^+ + 2m^- = 0, \quad 2 - 2g - (n + 1 + 1) + \frac{1}{2}(2j^+ + 2j^-) = 0, \quad (5.22)$$

and so the sphere amplitude scales like

$$A(m^-, j^-; m^+, j^+) \sim \delta_{m^+ + m^-} \mu_0^{j^+ + j^-}. \quad (5.23)$$

---

4 The theory also has non-trivial closed string cohomology at ghost number 0 that forms the ground ring, and at ghost number 1, which is associated with the conserved charges dual to (3.4) 27.
On the quantum mechanics side we can follow the same steps. The $SU(2)$ generators are

\[ J^+ = \frac{1}{2} \text{tr} \left( (a^\dagger)^2 \right), \quad J^- = \frac{1}{2} \text{tr} \left( a^2 \right), \quad J^3 = \frac{1}{4} \text{tr} \left( aa^\dagger + a^\dagger a \right). \]  

(5.24)

The highest-weight states with respect to this $SU(2)$ algebra are $\gamma_{j,j} = c_j \text{tr} \left( (a^\dagger)^{2j} \right)$, where $j = 0, 1/2, 1, \ldots$. Again we can form a representation of the algebra by commuting $j - m$ times $\gamma_{j,j}$ with $J^-$ to get $\gamma_{j,m}$. Up to ordering issues that are not relevant for the qualitative discussion of this section we find

\[ \gamma_{j,m} \sim \text{tr} \left( a^{j-m}(a^\dagger)^{j+m} \right). \]  

(5.25)

These operators satisfy a commutator algebra which at large $N$ approaches the semi-classical $w_\infty$ algebra (3.5). Even at finite $N$, the algebra closes, using the gauge equivalence (3.16). The precise details of the finite $N$ corrections to (3.5), however, remain to be determined.

The two-point function associated with the discrete states takes the form

\[ \mathcal{O}(m^-, j^-; m^+, j^+) \sim \langle 0 | \text{tr} \left( a^{j^-+m^-}(a^\dagger)^{j^--m^-} \right) \text{tr} \left( a^{j^+-m^+}(a^\dagger)^{j^++m^+} \right) | 0 \rangle \]  

(5.26)

\[ \sim \delta_{m^++m^-} N^{j^++j^-}, \]

again in agreement with the string theory result (5.23).

6. A closer look at the correlation functions

In this section we consider some of the correlation functions discussed in the previous section in more detail. Let us again start with the harmonic oscillator. It is a simple calculation to show that the $1 \to 1$ amplitude (5.3) is

\[ \mathcal{O}(k_1; k_2) = \delta_{k_1,k_2} N^{k_1} (1 + \mathcal{O}(N^{-2})). \]  

(6.1)

That follows from the fact that there are $k_1$ planar ways to commute the $a$’s through the $a^\dagger$. Somewhat more involved combinatorics show that for the $1 \to 2$ overlap amplitudes, (5.6), the planar limit gives

\[ \mathcal{O}(k; k_1, k_2) = \delta_{k,k_1+k_2} k_1 k_2 N^{k-1}. \]  

(6.2)
The simplest way to calculate $1 \rightarrow m$ overlap amplitudes

$$O(k; k_1, k_2, ... k_m) = \langle 0 | \text{tr} (a^k) \text{tr} ((a^\dagger)^{k_1}) \text{tr} ((a^\dagger)^{k_2}) ... \text{tr} ((a^\dagger)^{k_m}) | 0 \rangle,$$  \hspace{1cm} (6.3)

in the large-$N$ limit is to use the $w_\infty$ algebra $m$ times to move $\text{tr} (a^k)$ to the right. From (3.5) we get

$$O(k; k_1, k_2, ... k_m) = \delta_{k,k_1+k_2+...k_m} k(k-1)(k-2)...(k-m+2)k_1k_2...k_mN^{k-m+1}. \hspace{1cm} (6.4)$$

To match this on the string theory side, we have to compute the relevant sphere amplitudes in full detail. As explained in the previous section the relevant calculation is equivalent to scattering amplitudes in the free theory ($\mu_0 = 0$) with extra insertions of the screening operator. In particular, on the sphere we get for the $1 \rightarrow m$ tachyon scattering amplitude

$$A(k; -k_1, -k_2, ..., -k_m)_{\mu_0} = \frac{\mu_0^n}{n!} A(k; 0, 0, ..., 0, -k_1, -k_2, ..., -k_m)_{\text{free}}, \hspace{1cm} (6.5)$$

where the zeros indicate the insertion of the screening operator $n = k - m + 1$ times. In critical string theory one cannot write down a closed formula for such amplitudes. However, in 2D this can be done using various approaches\[23,28,29,30]. The result of the most general $1 \rightarrow r$ amplitude is

$$A(k; -k_1, -k_2, ..., -k_r)_{\text{free}} = \prod_{i=1}^{r} \frac{\Gamma(1 - k_i)}{\Gamma(k_i)} \frac{\pi^{r-2}}{(r-2)!}.$$ \hspace{1cm} (6.6)

Combining this with (6.5) we find

$$A(k; -k_1, -k_2, ..., -k_m)_{\mu_0} = \frac{\pi^{k-1}}{(k-m+1)!(k-1)!} \left( \frac{\mu_0 \Gamma(1)}{\Gamma(0)} \right)^{k-m+1} \prod_{i=1}^{m} \frac{\Gamma(1 - k_i)}{\Gamma(k_i)}.$$ \hspace{1cm} (6.7)

This expression contains some zeros and some infinities that must be dealt with before comparing with the harmonic oscillator results. There are two kinds of zeros. The first comes from the $(1/\Gamma(0))^{k-m+1}$ and is due to the screening operators. Recall that to obtain

5 Since we are interested in the large-$N$ limit we can simply replace the Poisson bracket in (3.5) by a commutator.

6 The motivation for studying scattering amplitudes in this theory was as a tool for finding the scattering amplitudes in the usual $c = 1$ theory.
finite results in the Liouville theory one needs to renormalize the cosmological constant when \( b \to 1 \) (see e.g. \cite{31}). To be precise we need to take \( \mu_0 \to \infty \) and \( b \to 1 \) while keeping
\[
\mu = \pi \mu_0 \frac{\Gamma(b^2)}{\Gamma(1 - b^2)},
\]
(6.8)
fixed. This is exactly the combination that appears in (6.7) so we can replace the \( \mu_0 \frac{\Gamma(1)}{\Gamma(0)} \) in that equation by \( \mu/\pi \). A precise way to think about this procedure is the following.

Take \( b = 1 + \epsilon/2 \) and twist the boundary condition on the tachyon field \( T(\varphi) = e^{2\pi i \epsilon} T(\varphi + 2\pi) \). That twisting has the effect of lifting the tachyon zero mode to \( k = \epsilon \) so that the \( \mu_0 \frac{\Gamma(1)}{\Gamma(0)} \) in (6.7) becomes \( \mu_0 \frac{\Gamma(1 + \epsilon)}{\Gamma(\epsilon)} \) and in the limit \( \epsilon \to 0 \) we simply get \( \mu/\pi \).

The formula (6.7) also vanishes when one of the outgoing momenta, \( -k_i \), is positive. This is exactly as it should be since there are no operators on the harmonic oscillator side that correspond to \( T_{k^2 > 0} \) (see discussion after eq. (5.18)).

Infinites arise from the \( \Gamma \) functions when \( k_i > 0 \). The reason for these infinites is that the amplitudes are sitting on a resonance. Physically there is no region of interaction, the particles are always nearby and they keep interacting forever. In the harmonic oscillator side we avoided this by not propagating in time each of the operators. To realize the same effect on the string theory side we need to introduce leg factors. Heuristically, these can be viewed as putting back the external legs that are missing in the amputated S-matrix elements calculated by string theory amplitudes. The way to find the relevant leg-factors is to match the two-point function on both sides of the duality. Then one can test the duality by comparing the higher point functions. The relevant leg-factors are\(^7\)
\[
f_{\text{ket}}(k) = -\frac{1}{\pi} \frac{\Gamma(k)}{\Gamma(-k)}, \quad (k < 0) \quad \text{and} \quad f_{\text{bra}}(k) = \pi \Gamma(k) \Gamma(k + 1), \quad (k > 0). \quad (6.9)
\]
Putting all of this together one finds that the relation between the string theory scattering amplitudes and the matrix model overlap amplitudes for the \( 1 \to m \) processes is
\[
\mathcal{O}(k; k_1, k_2, ... k_m) = f_{\text{bra}}(k) \prod_{i=1}^{m} f_{\text{ket}}(k_i) A(k; -k_1, -k_2, ... -k_m) \mu,
\]
(6.10)
\(^7\) The leg factors for \( k < 0 \) are familiar from the usual matrix model 2d string duality (in Euclidean signature). We could have taken the same leg-factors for \( k > 0 \) (since \( \Gamma(-n + \epsilon) = \frac{1}{\epsilon \Gamma(n+1)} + \mathcal{O}(\epsilon^0) \)) but that would have made the relation with the matrix model a bit more clumsy.
with \( \mu = N \). This relation holds in the \( 1 \to 1 \) cases by construction. The fact that it holds for the other cases \( (m > 1) \), that have non-trivial dependence on the \( k \)'s (see eq. (6.4)) is an encouraging consistency check of the proposed duality.

So far we did not consider the winding modes. Since \( \varphi \) is compactified, modular invariance implies that there must be winding modes in the theory. But there are no candidate dual operators for the winding modes in the quantum mechanics.\(^8\) This is exactly the same puzzle we have encountered with the momentum modes with opposite chirality. In that case the resolution was that scattering amplitudes with at least one opposite-momentum mode vanish. The same is expected to happen for the winding modes. As in the case of the momentum modes with opposite chirality, we do not know how to prove this in general, but we demonstrate it here for some cases. Consider, for example, scattering amplitudes of \( m \) winding modes. The analog of (5.12) in that case is

\[
2 - 2g - (n + m) = 0, \tag{6.11}
\]

where again \( n \) is the number of insertions of the screening operator. We see that no amplitude with \( m > 2 \) can satisfy this relation. For \( m = 1 \) and \( m = 2 \) the relation can be satisfied on the sphere with \( n = 1 \) and \( n = 0 \) respectively. But both cases vanish since they involve only two insertions of closed strings on the sphere.

### 6.1. Normal matrix models and matrix integral representation

We end this section with a discussion on a relation with normal matrix models. A normal matrix model (NMM) is an integral over complex matrices \( Z \) such that \([Z, Z^\dagger] = 0\). The fact that \( Z \) commutes with its conjugate implies that, much like in the case of a single matrix, in many interesting cases the model can be described in terms of the eigenvalues of \( Z \) and \( Z^\dagger \). The NMM that appears to be related to the quantum mechanics described here takes the form

\[
Z_{\text{NMM}}(J, J^\dagger) = \int_{[Z, Z^\dagger] = 0} d^{N^2} Z d^{N^2} Z^\dagger e^{-\text{tr} \left( Z^\dagger Z + Z J^\dagger + Z^\dagger J \right)}. \tag{6.12}
\]

The normal-matrix constraint in the path integral can be imposed with a Lagrange multiplier matrix \( A \),

\[
\delta^{N^2} \left( i[Z, Z^\dagger] \right) = \int dA \ e^{\text{tr} \left[ Z, Z^\dagger \right] A}, \tag{6.13}
\]

\(^8\) The usual suspects are the Wilson loops. But since \( X \) is not compactified there are no such operators in the gauged quantum mechanics.
so that (6.12) becomes

\[ Z_{\text{NMM}}(J, J^\dagger) = \int d^Nz d^Nz' e^{-\text{tr} (Z^\dagger Z + [Z, Z^\dagger]A + ZJ^\dagger + Z^\dagger J)}. \] (6.14)

We recognize the first two terms in the exponent as the Hamiltonian of the gauged harmonic oscillator (when one does not fix the gauge \( A = 0 \)). This strongly indicates a relation with the quantum mechanics, where the NMM should be viewed as the Hamiltonian reduction of the quantum mechanics. One expects to be able to compute (at least in the large-\( N \) limit) the overlap amplitudes discussed above using the NMM. In particular, by comparing Wick contractions, one should find a relation of the form

\[
\prod_i \text{tr} \left( \partial_{J_i}^k \partial_{J_i}^{p_i} \right) |_{J=J^\dagger=0} \ln Z_{\text{NMM}}(J, J^\dagger) = \langle T \left( \prod_i \text{tr} a_i^{+k_i} a_i^{p_i} \right) \rangle \tag{6.15}
\]

where \( T \) indicates a certain ordering prescription.

Here we mention one interesting way to think about this connection. Recall the Wigner phase-space integral representation of expectation values in one-dimensional quantum mechanics:

\[
\langle \psi | \hat{O} | \psi \rangle = \int dx dp \ W_{\psi}(x, p) W^*_\psi(x, p), \tag{6.16}
\]

where

\[
W_{\psi}(x, p) \equiv \int dy e^{ipy} \langle x + \frac{y}{2} | \hat{O} | x - \frac{y}{2} \rangle, \tag{6.17}
\]

and the Wigner function of the state \( \psi \) is

\[
W_{\psi}(x, p) \equiv W_{|\psi\rangle \langle \psi|} = \int dy e^{ipy} \psi^* \left( x + \frac{y}{2} \right) \psi \left( x - \frac{y}{2} \right). \tag{6.18}
\]

For the case of the harmonic oscillator this representation is particularly interesting. To compute the vacuum expectation values of some operator we simply use the fact that the ground state wave function is \( \psi(x) = \psi_0(x) = e^{-x^2/2} \), and eq. (6.16) becomes

\[
\langle \psi_0 | \hat{O} | \psi_0 \rangle = \int dx dp \ e^{-H(x, p)} W_{\psi_0}(x, p). \tag{6.19}
\]

This quantum expectation value is equal to a classical statistical average at inverse temperature \( \beta = 1 \), the resonant frequency. The generalization of these formulae to \( U(N) \) matrix quantum mechanics look very similar to (6.12). There are, however, various subtleties involving ordering (related to the ordering in (6.15)) and gauge fixing that should be clarified.
7. Beyond tree level

So far on the string theory side we have considered in detail only sphere amplitudes. In this section we first test our proposal at the one-loop level and then discuss some non-perturbative aspects.

7.1. Torus vacuum energy

Let us consider the simplest one-loop amplitude. This is the one-loop contribution to the ground state energy. On the string theory side the ground state energy is the expectation value of the zero momentum graviton which is one of the discrete states discussed in section 5, $S_{1,0}$. The energy conservation condition is automatic (since $m = 0$) and the $\phi$ momentum conservation gives

$$n = 2 - 2g. \quad (7.1)$$

This means that there are two possible perturbative contributions. The first comes from the sphere and since $n = 2$ it scales like $N^2$. The second comes from the torus with $n = 0$, and it therefore scales like $N^0$.

On the quantum mechanics side the ground state energy is $N^2/2$ with no additional constant that scales like $N^0$. This suggests that the one-loop vacuum energy in this string theory should be zero. This seems unlikely since this is a bosonic string theory and it is hard to see what could possibly cancel the positive vacuum energy. On the other hand this is a somewhat unconventional string theory and so it is worthwhile to do the calculation and see what we get.

A simpler way to compute the ground state energy is to calculate the one particle irreducible contribution to the partition function with $X$ compactified, $X \sim X + \beta$. In the limit $\beta \rightarrow \infty$ we have

$$\ln Z = -\beta E_0, \quad (7.2)$$

from which we can read off $E_0$. Thus what we have to do is to compute the torus partition function. That calculation is relatively simple since the background charge on the torus is zero and so it does not involve the Liouville term at all. As far as this calculation is concerned $X$ and $\varphi$ are two free scalar fields. In fact this calculation was already done in the context of the usual $c = 1$ theory [20][21][22]. The motivation for that calculation
was to study the theory at finite temperature. In that case the Liouville direction is not compactified and $X$ is compactified $X \sim X + 2\pi R$. The result of that calculation is \[ Z_{\text{torus}} = \frac{1}{12\sqrt{2}} \left( \frac{R}{\sqrt{\alpha'}} + \frac{\sqrt{\alpha'}}{R} \right), \] \[(7.3)\]

where $V_L$ is the volume in the Liouville direction.

Since in that calculation $X$ and $\varphi$ are two free scalars we can interchange their role to find the partition function in our case. Interchanging their role means that now $\varphi$ is compactified and $V_L \to V_X = \beta$, \[(7.4)\]

which gives

\[
\lim_{\beta \to \infty} \frac{Z_{\text{torus}}}{\beta} = -\frac{i}{12\sqrt{2}} \left( \frac{R}{\sqrt{\alpha'}} - \frac{\sqrt{\alpha'}}{R} \right).
\] \[(7.5)\]

The $i$ and the minus sign come from the fact that $\varphi$ is a timelike direction. Another way to see this is to take $\alpha' \to -\alpha'$ in (7.3). Since in our case $R = 1$ in units where $\alpha' = 1$ we find that

\[
\lim_{\beta \to \infty} \frac{Z_{\text{torus}}}{\beta} = 0,
\] \[(7.6)\]
as predicted by the duality with the quantum mechanics.

This is an interesting result for several reasons. First, it is obtained by summing over contributions of all of the physical states of the string. In addition to the chiral momentum modes with nonzero S-matrix amplitudes, these include winding modes and momentum modes with the opposite chirality that, we argued, do not appear on external legs. This is reminiscent of the ghost structure of loop amplitudes in a gauge theory, and one is tempted to attribute this behavior to the underlying $w_\infty$ symmetry. Second, one should always take notice when finding zero for the quantum correction to the cosmological constant, especially in a bosonic theory.

\[9\] Since $\varphi$ is time-like this is not the self-dual radius. Acting on a time-like direction, T-duality takes $R$ to $-\alpha'/R$, in order that the partition function be invariant.
7.2. Hints for nonperturbative effects

The fact that the perturbative expansion for all scattering amplitudes is truncated seems to imply that there is no room for non-perturbative effects in this theory. Indeed instantons are usually related to the non-convergence of the perturbative expansion, which is not an issue with finitely many terms. On the string theory side this means that there are no D-instantons. Below we provide some evidence, from the quantum mechanics side, that there are, however, D-particles in the theory.

Consider the matrix model at finite temperature. Assume that the temperature is low so that the contribution of the zero modes of the gauge fields can be neglected. The free energy is then

\[ F(q, N) = \ln Z_N(q) = \frac{1}{2} \beta N^2 - \sum_{n=1}^{N} \ln(1 - q^n), \quad q = e^{-\beta}. \]  

(7.7)

Keeping track of the \( N \)-dependence, this can be rewritten (a similar equation appears in QCD2 [10]) as

\[ F(q, N) = \frac{\beta N^2}{2} + f(\beta) + \sum_{n=1}^{\infty} C_n \left( e^{-n\beta N} \right), \quad f(\beta) = \sum_{n=1}^{\infty} \ln(1 - q^n). \]  

(7.8)

The first term we recognize as the ground state energy. The second term is associated with the leading torus contribution to the free energy when \( \beta \) is large but finite. There are no further perturbative corrections in accord with the scaling rule (5.13). There are, however, non-perturbative effects. The fact that these scale like \( e^{-w_N} \) implies that there are D-particles in the spectrum which contribute to the partition sum when their worldlines wrap the thermal circle. The fact that there are no terms of the form \( e^{-N} \) implies that, as was argued above, there are no D-instantons. It would be very interesting to verify this statement using worldsheet techniques.

Another reason to study D-branes in this theory is the following. We motivated our search for the string dual of the harmonic oscillator by noticing that reversing the sign of the string tension in the usual \( c = 1 \) theory rights the upside-down oscillator. This identification of \(-\frac{1}{\alpha'}\) with the mass of the open-string tachyon relies on the D-brane physics of the 2d string theory. The analogous understanding for this 'imaginary Liouville' theory remains to be developed. The branes which describe the matrix eigenvalues in this case are closely related to the instantonic branes of the usual \( c = 1 \) theory [32], which experience trajectories of the form \( z = \tilde{\lambda} \cos t \). We note that the gauging of the \( U(N) \) symmetry should again be motivated by the presence of a null open-string descendant. A more detailed understanding of these issues will allow a clear interpretation of the relation discussed in this paper as open/closed duality along the lines of [33,32,34].

23
8. The worldsheet as a phase-space

So far we have presented quite a bit of evidence that supports the proposed duality. What is still missing is a simple intuitive picture of how the duality works. Below we give a heuristic description of how, we believe, the closed string world-sheet is related to the harmonic oscillator phase-space.

It is reasonable that $\mu$ on the string theory side is actually the conjugate variable to $N$. Namely, it is the chemical potential. At large $\mu$, these agree $\langle \hat{N} \rangle = \mu$. There are two reasons to suspect that this is the case. First, that is the way the usual duality between 2d strings and the matrix model works. Second, at the moment, it is not clear what effect on the string theory side could force $\mu$ to be an integer.\(^\text{10}\) Assuming that this is the case we see that something quite interesting is happening. From the world-sheet point of view $\mu$ is the conjugate variable to the area of the world-sheet. From the target space point of view it is conjugate to $N$ which is the area of the phase-space. This seems to suggests that the worldsheet should be related (via some non-local map) with the phase-space of the harmonic oscillator.

At first sight this seems unlikely since the phase-space has a boundary while the closed string word-sheet does not. However, recall the nature of the calculations we did on the quantum mechanics side that had a simple interpretation on the string theory side. We were computing overlap amplitudes

$$\langle \text{bra}|\text{ket} \rangle.$$  \hspace{1cm} (8.1)

At the semi-classical level this is equivalent to exciting two copies of the Fermi sea, and gluing them together along their Fermi surfaces to form a closed manifold, which could be viewed as the closed-string worldsheet (see fig. 2).

\(^{10}\) This argument is not too convincing since there is a counterexample to the reasoning, namely the duality of [35]. In this topological case, the open-string side of the duality implies that the closed string coupling constant should be quantized. But the closed string dual seems to make sense for all values of $g_s$. 

\[24\]
Fig. 2: Gluing together two copies of the phase space along the Fermi surface produces a closed Fermi sea, which could be viewed as the closed string world sheet.

This closed two-dimensional space might be interpreted as the closed string worldsheet. The analog of the target-space energy conservation condition on the world-sheet is the level matching between the left and right movers

\[ E_{\text{bra}} = E_{\text{ket}} \Leftrightarrow N_L = N_R. \tag{8.2} \]

It should be interesting to see how precise this can be made. In particular, it would be nice to make contact with [36] where it was illustrated how a closed string world-sheet could be realized in the large-\(N\) limit of a free field theory.

9. Discussion

There are various generalizations and applications of the duality proposed here that might be interesting to study. Here we mention some of them.

\(c < 1\)

The simplest generalization of the string theory described here (other than the supersymmetric generalization) is to consider minimal models (with \(c < 1\)) times a time-like Liouville direction with an imaginary linear dilaton so that the total central charge is 26. One way to think about this theory is to start with the usual minimal strings where the Liouville direction is space-like and the linear dilaton is real and Wick rotate the Liouville direction.\(^{11}\) That theory is quite amusing since the minimal models have a Coulomb gas description in terms of a space-like scalar with an imaginary linear dilaton. So we end up

\(^{11}\) This is different than taking \(\alpha' \rightarrow -\alpha'\) since we do not "Wick rotate" the minimal model.
with a string theory in 2D Minkowski space-time with an imaginary dilaton that depends on a certain linear combination of the time and the space directions. The string theory considered here is a special case in which the imaginary linear dilaton depends only on the time-like direction, \( \varphi \). The fact that the \( \varphi \) direction is compact prevents one from relating the theories by a boost. It is likely that these string theories are dual to some matrix models. The results of the present paper seem to indicate that these matrix models should not involve double scaling. It should be interesting to explore the relation between these theories and the usual minimal strings, especially in light of recent progress (see e.g. [37,38]).

**Topological strings**

The fact that the perturbative corrections to the amplitudes in this string theory truncate after a finite number of terms may seem to conflict with one’s intuition about unitarity. This confusion is resolved however, when one realizes that these amplitudes do not have an imaginary part. In fact, as is obvious on the matrix model side, they are integers that are equal to the number of diagrams with given topology, fixed by the external lines and the genus. This makes a connection with topological string seem inevitable. Given that deep connections have been found between noncritical strings and topological strings on noncompact Calabi-Yaus (e.g. [39,40]) and further that 2d Yang-Mills also has a topological string description [41], it seems likely that there is a topological string description of the matrix harmonic oscillator.

**Stringy compactification**

The fact that the string theory considered in this paper seems to makes sense despite its strange properties suggests that generalizations to higher dimension might be interesting to explore. Perhaps these could even lead to a new class of stringy compactification with phenomenological applications. For example consider string theory on \( \mathbb{R}^d \times S^1 \) where the radius of the \( S^1 \) is \( R \). The usual phenomenological problem with such a compactification is that the radius of the \( S^1 \) is not fixed and as a result the \( d \) dimensional physics contains a massless scalar that (under reasonable assumptions) contradicts observation. Fixing such moduli is a non-trivial problem in string theory. Here we see to encounter a simple way to deal with this: Suppose that we turn on an imaginary linear dilaton in the \( S^1 \) direction. As discussed in section 4, periodicity of the string coupling constant fixes \( R \) to...
be an integer $m$. For $m \gg 1$ we find that the contribution of that direction to the central charge is almost 1. And so the difference, which scales like $1/m^2$, could easily be cancelled against some other fluxes in the theory without involving significant $\alpha'$ corrections.

Needless to say, one should be careful here. The fact that in 2d having a complex string coupling constant seems to work does not mean that the same will happen in higher dimensions. In particular, in 2d only the zero mode of the dilaton appears in the cohomology, $S_{1,0}$, which makes the unitarity condition much less restrictive.

The quantum Hall effect

The matrix model studied in this paper is closely related to the quantum hall effect (for a beautiful review of the QHE see [42]). The simplest way to see this is at the level of the classical phase-space. If we replace $P$ by $B_z Y$ (where $B_z$ is the magnetic field and $Y$ is interpreted as a new spatial direction), we end up with the familiar cyclotron motion around the origin $X = Y = 0$. The fact that the center of the cyclotron motion is fixed means that translation invariance in the $X,Y$ plane is broken, which is crucial for realizing the QHE. Namely, this is the analog of the Anderson localization in the presence of an impurity located at the origin of a sample in the shape of a disk. At the quantum level, this is clearest when considering the problem in the symmetric gauge $\vec{A} = -\frac{1}{2} \vec{r} \times \vec{B}$. In that gauge it is convenient to label states by their energy $n + 1/2$ (at the nth Landau level) and their angular momentum $m$. There is no further degeneracy in the problem. The states we have in the harmonic oscillator correspond to $n = m$, which means that in each Landau level we have exactly one state.

A long-standing problem in the QHE is to find a CFT description of the quantum phase transition from one plateau to another. The jump from one plateau to the next is associated with occupying a new Landau level. The analogous transition in our case is of adding a new eigenvalue to the system, which means on the string theory side adding a D-brane. This might be viewed as an indication that the CFT that describes the phase transition in the QHE is the one considered here, but on the disk rather than on the sphere. This fits neatly with the heuristic picture of the previous section. Now the worldsheet has a boundary that is associated with the disk shaped sample. All the interesting physics associated with the edge states should correspond to open strings inserted at the boundary

\footnote{Recall that due to the large magnetic field $m$ cannot be negative. This translates to the chiral nature of the tachyon in our case.}
of the worldsheet. The fact that much progress has been made recently in understanding the open string spectrum of D-branes in the usual Liouville theory \([13,14,15]\) suggests that this speculation could be tested in the near future.

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