APPLICATION OF LIE TRANSFORMATION GROUP METHODS TO CLASSICAL THEORIES OF PLATES AND RODS

V. VASSILEV and P. DJONDJOROV
Institute of Mechanics, Bulgarian Academy of Sciences
Acad. G. Bontchev St., Block 4, 1113 Sofia, Bulgaria

Abstract—In the present paper, a class of partial differential equations related to various plate and rod problems is studied by Lie transformation group methods. A system of equations determining the generators of the admitted point Lie groups (symmetries) is derived. A general statement of the associated group-classification problem is given. A simple intrinsic relation is deduced allowing to recognize easily the variational symmetries among the "ordinary" symmetries of a self-adjoint equation of the class examined. Explicit formulæ for the conserved currents of the corresponding (via Noether’s theorem) conservation laws are suggested. Solutions of group-classification problems are presented for subclasses of equations of the foregoing type governing stability and vibration of plates, rods and fluid conveying pipes resting on variable elastic foundations and compressed by axial forces. The obtained group-classification results are used to derive conservation laws and group-invariant solutions readily applicable in plate statics or rod dynamics.

1. INTRODUCTION

A wide variety of classical theories of plates and rods rest on linear fourth-order partial differential equations in one dependent and two independent variables. Some of them, such as the Poisson-Kirchhoff type theories for small bending of plates, are developed within the framework of the linear elastostatics to determine the state of equilibrium of thin elastic plates in terms of the transversal displacement of the plate middle plane, the derived governing equations providing the background for solving problems concerning stability and vibration of such structural elements. Others (among them – the dynamic theory of Bernoulli-Euler beams, for instance) are deduced on the ground of the linear elastodynamics to describe the dynamic behaviour of rods in terms of the transversal displacement of the rod axis.

The aim of the present paper is to study the invariance properties (symmetries) of the equations of the foregoing type with respect to local Lie groups of point transformations of the involved independent and dependent variables. The work is motivated both by the aforesaid wide applicability of the equations in question in structural mechanics, and by the remarkable efficiency demonstrated by the symmetry methods, especially when applied to differential equations arising in physics and engineering.

Actually, once the invariance properties of a given differential equation are established, several important applications of its symmetries arise. First, it is possible to distinguish classes of solutions to this equation invariant under the transformations of symmetry groups admitted. The determination of such a group-invariant solution assumes solving a reduced equation involving less independent variables than the original one. Typical examples of group-invariant solutions are axisymmetric solutions, self-similar solutions, travelling waves, etc., which have proved to be quite useful in many branches of physics and engineering.

For a self-adjoint differential equation another substantial application of its symmetries is

\footnote{E-mail: vassil@bgcict.acad.bg}
\footnote{E-mail: padjon@bgcict.acad.bg}
\footnote{In this work, following Antman (1984) we use "rod" as a generic name for "arch", "bar", "beam", "ring", "column", "tube", "pipe", etc. We employ "rod" in the intuitive sense of a slender solid body.}
available. As is well known, the self-adjoint equations are the Euler-Lagrange equations of a certain action functional. If a one-parameter symmetry group of such an equation turned out to be its variational symmetry as well, that is a symmetry of the associated action functional, then Noether’s theorem guarantees the existence of a conservation law for the smooth solutions of this equation. Needless to recall or discuss here the fundamental role that the conserved quantities and conservation laws (or the corresponding integral relations, i.e. the balance laws) have played in natural sciences, but it is worthy to point out that the available conservation laws (balance laws) should not be overlooked (as it is often done) in the examination of discontinuous solutions (acceleration waves, shock waves, etc.) or in the numerical analysis (when constructing finite difference schemes or verifying numerical results, for instance) of any system of differential equations of physical interest. It should be remarked also that the path-independent integrals (such as the well known $J$, $L$, and $M$-integrals) related to the conservation laws are basic tools in fracture analysis of solids and structures.

The aforementioned and many other applications of the symmetries of differential equations and variational problems as well as the foundations of the Lie transformation group methods, including the basic notions, statements and techniques, can be found in the books by Ovsiannikov (1982), Ibragimov (1985), Bluman and Kumei (1989) and Olver (1993) (see also the references given in these books). In the present paper, however, as fare as the application of the symmetry groups of the equations studied is concerned, our attention is restricted to the constructing of group-invariant solutions and conservation laws. Of course, the first task is to find these symmetry groups, and as here we do not deal with a single differential equation but with a class of differential equations, this means to solve a group-classification problem.

The layout of the paper is as follows. A detailed description of the differential equations to be studied as well as the variational statement for the self-adjoint equations among them are given in Section 2. Several examples of mechanical systems governed by such equations complete this Section. In Section 3, a system of equations determining the generators of the symmetry groups admitted by the equations of the class considered is derived, the invariance properties inherent to the whole class because of its linearity and homogeneity being taken into account, and then the general statement of the associated group-classification problem is given. After that, the variational symmetries of the self-adjoint equations of the examined class are investigated. A simple intrinsic relation allowing to recognize easily the variational symmetries among the "ordinary" point Lie symmetries of such an equation is deduced, and explicit formulae for the conserved currents of the conservation laws corresponding to the variational symmetries via Noether’s theorem are suggested. Group-classification results, conservation laws and group-invariant solutions are presented in Section 4 for differential equations governing stability and vibration of plates of Poisson-Kirchhoff type. Similar results are displayed in Section 5 for equations governing vibration of rods on a variable elastic foundation and dynamic stability of fluid conveying pipes or rods compressed by axial forces. In the reminder of this Section, conservation laws for rods derived in the present contribution are compared to other ones reported in the literature.

2. BASIC EQUATIONS

Consider a fourth-order homogeneous linear partial differential equation

$$A^{\alpha\beta\gamma\delta}(x)w_{\alpha\beta\gamma\delta} + A^{\alpha\beta\gamma}(x)w_{\alpha\beta\gamma} + A^{\alpha\beta}(x)w_{\alpha\beta} + A^{\alpha}(x)w_{\alpha} + A(x)w = 0,$$

(1)

in two independent variables $x = (x^1, x^2)$ and one dependent variable $w(x)$. Here and throughout: Greek indices have the range 1, 2, and the usual summation convention over a repeated index (one subscript and one superscript) is employed; $w_{\alpha_1\alpha_2...\alpha_k}$ ($k = 1, 2, ...$) denote the $k$-th order partial derivatives of the dependent variable, i.e.

$$w_{\alpha_1\alpha_2...\alpha_k} = \frac{\partial^k w}{\partial x^{\alpha_1}\partial x^{\alpha_2}...\partial x^{\alpha_k}} (k = 1, 2, ...).$$
Further, a similar notation will be used for the partial derivatives of any other function of the
variables \(x^1, x^2\) but, in this case, the indices indicating the differentiation will be preceded
by a coma, e.g.

\[
A^{\alpha\beta\gamma\delta}_{\alpha_1\alpha_2...\alpha_k} = \frac{\partial k A^{\alpha\beta\gamma\delta}_{\alpha_1\alpha_2...\alpha_k}}{\partial x^{\alpha_1} \partial x^{\alpha_2}...\partial x^{\alpha_k}} \quad (k = 1, 2, ...).
\]

The coefficients of equation (1) are supposed to be smooth functions possessing as many
derivatives as may be required on a certain domain of interest, and to be symmetric under
any permutation of their indices, i.e.

\[
A^{\alpha\beta\gamma\delta} = A^{\beta\gamma\delta\alpha} = A^{\gamma\delta\alpha\beta} = A^{\delta\alpha\beta\gamma}, \quad A^{\alpha\beta\gamma} = A^{\beta\gamma\alpha} = A^{\gamma\alpha\beta}, \quad A^{\alpha\beta} = A^{\beta\alpha}.
\]

Using the total derivative operators

\[
D\alpha = \frac{\partial}{\partial x^\alpha} + w_\alpha \frac{\partial}{\partial w} + w_{\alpha\mu} \frac{\partial}{\partial w_\mu} + w_{\alpha\mu\nu} \frac{\partial}{\partial w_{\mu\nu}} + w_{\alpha\mu\nu\sigma} \frac{\partial}{\partial w_{\mu\nu\sigma}} + \cdots,
\]

the equation (1) may be written in the form

\[
D[w] = 0,
\]

where \(D\) is the linear differential operator given by the expression

\[
D = A^{\alpha\beta\gamma\delta} D_{\alpha} D_{\beta} D_{\gamma} D_{\delta} + A^{\alpha\beta\gamma} D_{\alpha} D_{\beta} D_{\gamma} + A^{\alpha\beta} D_{\alpha} D_{\beta} + A^\alpha D_{\alpha} + A.
\]

An equation of form (2) is the Euler-Lagrange equation associated with a certain variational
problem involving only one dependent variable if and only if the differential operator \(D\) is self-adjoint, that is

\[
D = D^*,
\]

where \(D^*\) is the (formal) adjoint operator of \(D\) (cf. Olver, 1993). The explicit form of \(D^*\) is

\[
D^* = D_{\alpha} D_{\beta} D_{\gamma} D_{\delta} A^{\alpha\beta\gamma\delta} - D_{\alpha} D_{\beta} D_{\gamma} A^{\alpha\beta\gamma} - D_{\alpha} D_{\beta} A^{\alpha\beta} - D_{\alpha} A^\alpha + A.
\]

In such a case, (3) can be associated with the variational problem for the functional

\[
A[w] = \int \frac{1}{2} w D[w] dx^1 dx^2,
\]

since the application of the Euler operator

\[
E = \frac{\partial}{\partial w} - D_{\mu} \frac{\partial}{\partial w_\mu} + D_{\mu} D_{\nu} D_{\lambda} \frac{\partial}{\partial w_{\mu\nu\lambda}} + D_{\mu} D_{\nu} D_{\sigma} D_{\tau} \frac{\partial}{\partial w_{\mu\nu\sigma\tau}} - \cdots
\]
on the Lagrangian density

\[
L = \frac{1}{2} w D[w],
\]
yields

\[
D[w] = E(L),
\]
due to (3) and (4).

Let us give several examples of plate and rod structures whose governing equations are
self-adjoint and belong to the class specified above. Henceforward, when regarding plates
the variables \(x^1, x^2\) will represent the coordinates of the plate middle plane. As for the rod
problems, \(x^1\) will be associated with the spatial variable along the rod axis, and \(x^2\) – with
the time. In both cases, the dependent variable \(w\) will represent the transversal displacement field.

**Example 1. Small bending of plates resting on elastic foundations.** Consider a thin
elastic plate of variable bending rigidity \(D(x)\) resting on an elastic foundation of Winkler type with variable modulus \(k(x)\) and subjected to an edge loading leading to the appearance
of nonuniform membrane stresses $N^{\alpha\beta}(x)$. In this physical situation, the equation governing the small bending of the plate assumes the following form

$$\Delta [Dw] - [(1 - \nu)\varepsilon^{\alpha\mu}\varepsilon^{\beta\nu}D_{\alpha\beta} - N^{\mu\nu}]w_{\mu\nu} + kw = 0,$$

(8)

the membrane stress tensor $N^{\alpha\beta}$ being symmetric, $N^{\alpha\beta} = N^{\beta\alpha}$, and divergence free, i.e. $N^{\mu\nu} = 0$. Here: $\nu$ is Poisson’s ratio; $\Delta$ is the Laplace operator, that is $\Delta \equiv \delta^{\alpha\beta}\partial^2/\partial x^\alpha\partial x^\beta$, where $\delta^{\alpha\beta}$ is the Kronecker delta symbol ($\delta^{11} = \delta^{22} = 1$, $\delta^{12} = \delta^{21} = 0$) and $\varepsilon^{\alpha\beta}$ is the alternating symbol ($\varepsilon^{11} = \varepsilon^{22} = 0$, $\varepsilon^{12} = -\varepsilon^{21} = 1$).

Example 2. Elastodynamics of Bernoulli-Euler beams. Consider a nonhomogeneous Bernoulli-Euler beam with bending rigidity $B(x)$ and inertia term $H(x)$. The differential equation governing the small vibration of such a beam is (Chien et al., 1993):

$$Bw_{1111} + 2B_1 w_{11} + B_{11} w_{11} + Hw_{22} = 0.$$

Example 3. Elastic beams resting on elastic foundations. Consider an elastic beam of constant bending rigidity $K$ and constant mass density $m$ (mass of the beam per unit length), resting on an elastic foundation with variable modulus $k(x)$. Suppose it is subjected to a constant follower force $p$. Then, according to the Bernoulli-Euler theory, the differential equation for small transverse vibration of the beam is (see Smith and Herrmann, 1972):

$$K w_{1111} + pw_{11} + k(x) w + mw_{22} = 0.$$

(9)

Example 4. Pipes conveying fluid. Consider an elastic circular-cylindrical pipe of uniform outer radius of the pipe cross section, which is supposed to be small in comparison with certain characteristic pipe length (for a simply supported pipe say the length of the span). Let the pipe conveys inviscid incompressible fluid with a flow velocity $U = const$. Then, the equation of motion is (see Gregory and Paidoussis, 1966):

$$EJ w_{1111} + MU^2 w_{11} + 2MU w_{12} + (m + M) w_{22} = 0,$$

(10)

where $E$ is Young’s modulus of the pipe material, $J$ is the (axial) moment of inertia of the pipe cross section, $m$ is the mass (constant) of the pipe per unit length, $M$ is the mass (also constant) of the fluid per unit length.

Combining and generalizing equations (1) and (10) presented in Examples 3 and 4, in Section 5 we will pay particular attention to the differential equations of the form

$$\gamma w_{1111} + \chi^{\alpha\beta} w_{\alpha\beta} + \kappa(x) w = 0,$$

(11)

where $\gamma \neq 0$ and $\chi^{\alpha\beta}$ are real constants, while $\kappa(x)$ is a smooth function.

3. SYMMETRIES AND CONSERVATION LAWS

Consider a local one-parameter Lie group of point transformations acting on some open subset $\Omega$ of the space $\mathbf{R}^3$ representing the independent and dependent variables $x^1, x^2, w$ involved in our basic equation (2). The infinitesimal generator of such a group is a vector field $X$ on $\mathbf{R}^3$,

$$X = \xi^\mu (x, w) \frac{\partial}{\partial x^\mu} + \eta (x, w) \frac{\partial}{\partial w},$$

(12)

whose components $\xi^\mu(x, w)$ and $\eta(x, w)$ are supposed to be functions of class $C^\infty$ on $\Omega$. By virtue of Theorem 2.31 (Olver, 1993), a vector field $X$ of form (12) generates a point Lie symmetry group of equation (2) if and only if there exists a function $\lambda$ depending on $x$, $w$ and derivatives of $w$ (that is a differential function) such that the following infinitesimal criterion of invariance,

$$\chi_k (D[w]) - \lambda D[w] = 0,$$

(13)

holds; here $\chi_k$ denote the $k$-th prolongation of $X$ (Ovsiannikov, 1982).
The invariance criterion (13) leads, through the standard computational procedure (see, e.g. Ovsiannikov, 1982 or Olver, 1993), to the following results:

(i) each equation of form (9) being linear and homogeneous is invariant under the point Lie groups generated by the vector fields

$$X_0 = u \frac{\partial}{\partial w}, \quad X_u = u(x) \frac{\partial}{\partial w},$$

where $u(x)$ is an arbitrary solution of the equation considered, the invariance criterion (13) being fulfilled with $\lambda = 1$ for $X_0$, and $\lambda = 0$ for the generators $X_u;

(ii) an equation of form (2) admits other vector fields (12), in addition to the aforementioned (14), if and only if they have the special form

$$X = \xi^\mu (x) \frac{\partial}{\partial x^\mu} + \sigma (x) w \frac{\partial}{\partial w},$$

the functions $\xi^\mu (x)$ and $\sigma (x)$ being nontrivial solutions of the following system of determining equations (called further the DE system for easy reference):

$$\xi^\mu \alpha \beta \gamma \delta + (\sigma - \lambda) A^{\alpha \beta \gamma \delta} - A^{\alpha \beta \gamma \delta} \xi^\mu - A^{\alpha \beta \gamma \delta} s^\mu - A^{\mu \beta \gamma \delta} \xi^\alpha = 0,$$

$$4A^{\alpha \beta \gamma \delta} s^\mu - 2A^{\alpha \beta \mu \nu} \xi^\gamma_{,\mu} - 2A^{\alpha \gamma \mu \nu} \xi^\beta_{,\nu} - 2A^{\beta \gamma \mu \nu} \xi^\alpha_{,\mu} + \xi^\mu A^{\alpha \beta \gamma} + (\sigma - \lambda) A^{\alpha \beta \gamma} - A^{\alpha \beta \mu} \xi^\gamma_{,\mu} - A^{\alpha \mu \gamma} \xi^\beta_{,\mu} - A^{\beta \mu \gamma} \xi^\alpha_{,\mu} = 0,$$

$$6A^{\alpha \beta \mu \nu} s^\sigma - 2A^{\alpha \mu \sigma \nu} \xi^\beta_{,\mu \sigma} - 2A^{\beta \mu \sigma \nu} \xi^\alpha_{,\mu \sigma} + 3A^{\alpha \beta \mu} (3/2) A^{\alpha \mu \nu} \xi^\gamma_{,\mu \nu} - (3/2) A^{\beta \mu \nu} \xi^\alpha_{,\mu \nu} + \xi^\mu A^{\alpha \beta} + (\sigma - \lambda) A^{\alpha \beta} - A^{\alpha \mu \beta} - A^{\beta \mu \alpha} - A^{\mu \beta \alpha} = 0,$$

$$4A^{\alpha \mu \sigma \nu} s^\mu - A^{\mu \sigma \nu} \xi^\alpha + 3A^{\alpha \mu \sigma} s^\nu - A^{\mu \sigma \nu} \xi^\alpha + \xi^\mu A^{\alpha \mu} + (\sigma - \lambda) A^{\alpha} - A^{\mu \alpha} = 0,$$

for a certain function $\lambda$ depending on $x^1$ and $x^2$ only. (Here, by a trivial solution we mean not only $\xi^\mu = 0$, $\sigma = 0$, but also $\xi^\mu = 0$, $\sigma = c = const \neq 0$, since the latter leads to the vector field $cX_0$ generating the same group as $X_0$ which is already identified to be admitted by each equation of the type considered.)

Thus, given an equation of form (9), the question is whether there exist vector fields $X \neq cX_0$ of form (13) which leave it invariant, and the answer depends on whether the respective DE system has at least one nontrivial solution. In this context the coefficients of the equation are supposed to be known functions, and thereby (16) – (20) constitute an over-determined system of linear homogeneous partial differential equations with respect to the unknowns $\xi^\mu$ and $\sigma$. Therefore, as a rule, it turns out possible to find in an explicit form some (or even all) nontrivial solutions of the DE system, and thus to determine several (all) additional point Lie symmetry groups inherent to the equation in question.

It should be remarked that various equations of form (2) admit only the point Lie groups generated by the vector fields (14) with $u(x)$ being any solution of the respective equation. For instance, it is easy to check that all equations of the form (14) such that $\chi^{\alpha \beta} = \delta^{\alpha \beta}$ and $k(x) = p(x)$, where $p(x)$ is an arbitrary polynomial of $x^1$ and $x^2$, belong to this variety. Without too much difficulties one can ascertain that the same holds true for the equations of the form (8) with $D = const$, $N^{\alpha \beta} = \delta^{\alpha \beta}$ and $k(x) = p(x)$.

On the other hand, there are equations of the foregoing type which are invariant under a larger group; an immediate example is the biharmonic equation, $\Delta^2 w = 0$, which admits
the seven-parameter group generated by the linear combinations of \( X_0 \) and the following six additional basic vector fields (cf. Ovsiannikov, 1972):

\[
X_1 = \frac{\partial}{\partial x^1}, \quad X_3 = x^2 \frac{\partial}{\partial x^1} - x^1 \frac{\partial}{\partial x^2}, \quad X_5 = 2x^1 x^2 \frac{\partial}{\partial x^1} - \left[ (x^1)^2 - (x^2)^2 \right] \frac{\partial}{\partial x^2} + 2x^2 w \frac{\partial}{\partial w},
\]

\[
X_2 = \frac{\partial}{\partial x^2}, \quad X_4 = x^1 \frac{\partial}{\partial x^1} + x^2 \frac{\partial}{\partial x^2}, \quad X_6 = \left[ (x^1)^2 - (x^2)^2 \right] \frac{\partial}{\partial x^2} + 2x^1 x^2 \frac{\partial}{\partial x^2} + 2x^1 w \frac{\partial}{\partial w}.
\]

An important problem naturally arises in the light of the above note. It may be placed in the category of the so-called group-classification problems (see Ovsiannikov, 1982) and consist in determination of all those equations of the type considered that admit a larger group together with this group itself. Its most general statement assumes all functions \( A^{\alpha\beta\gamma\delta}(x) \), \( A^{\alpha\beta}(x) \), \( A^{\alpha}(x) \), \( A(x) \), \( \xi^\alpha(x) \) and \( \sigma(x) \) involved in the determining equations (16) – (20) to be regarded as unknown variables and to find all solutions of this system. Here we are not going to study this rather complicated nonlinear problem in general. However, in Sections 4 and 5, restricting our attention to the equations of form (10), \( D = \text{const} \), and (11), respectively, we will examine the corresponding group-classification problems.

Let us now specialize to the case of self-adjoint equations of form (10). Suppose that

\[
D[w] = 0, \quad D = D^*,
\]

is such an equation. Then, of particular interest are its variational symmetries – the Lie groups generated by the so-called infinitesimal divergence symmetries (see Definition 4.33 in Olver, 1993) of any variational functional with (21) as the associated Euler-Lagrange equation. (Note that if two functionals lead to the same Euler-Lagrange equation, then they have the same collection of infinitesimal divergence symmetries.) This interest is motivated by the fact that, in virtue of Noether’s theorem, each variational symmetry of a given self-adjoint equation corresponds to a conservation law admitted by the smooth solutions of the equation. Thus, if a vector field \( X \) of form (12) is found to generate a variational symmetry of equation (21), then Noether’s theorem implies the existence of a conserved current, which, in the present case, is a couple of differential functions \( P^\alpha \) such that

\[
D_{\alpha} P^\alpha = Q D[w],
\]

where \( Q \) is the characteristic of \( X \); by definition

\[
Q = \eta - w_\mu \xi^\mu.
\]

The total divergence of the conserved current \( P^\alpha \) vanishes on the smooth solutions of (21) and so we have the conservation law

\[
D_{\alpha} P^\alpha = 0,
\]

being its expression in characteristic form, and \( Q \) – its characteristics. Therefore, to derive the conservation laws of the foregoing type, one can proceed by first determining the variational symmetries of equation (21), and then using their characteristics (23) to find, from (22), explicit expressions for the corresponding conserved currents.

Having analyzed earlier the invariance properties of the whole class of equations (10), it is convenient to base the determination of the variational symmetries of equation (21) on the following observation. A vector field \( X \) of form (12) generates a variational symmetry of equation (21) if and only if \( X \) is an infinitesimal symmetry of this equation, that is (13) holds, and

\[
\frac{\partial}{\partial x^4} \left( D[w] \right) + \left( \frac{\partial \eta}{\partial w} + D_\mu \xi^\mu \right) D[w] = 0.
\]

This is a consequence of Lemma 4.34 and Proposition 5.55 (Olver, 1993), see also Lemma 7.46 (Olver, 1995). Subtracting (13) from (22) we can replace the latter with

\[
\left( \frac{\partial \eta}{\partial w} + D_\mu \xi^\mu + \lambda \right) D[w] = 0,
\]
and as $\mathcal{D}[w]$ is not supposed to vanish identically we arrive at the conclusion that
\[
\frac{\partial \eta}{\partial w} + D_\mu \xi^\mu + \lambda = 0, \tag{26}
\]
is a necessary and sufficient condition for an infinitesimal symmetry admitted by a self-adjoint equation of form (2) to be its infinitesimal variational symmetry as well. It should be remarked that the same holds true for any self-adjoint partial differential equation in one dependent variable $w$ and $n$ independent variables $x = (x^1, \ldots, x^n)$; of course, in the general case the summation index $\mu$ will take the values $1, \ldots, n$ in both formulae (12) and (26). For a vector field of form (15) the relation (26) simplifies, and reads
\[
\sigma + \xi^\mu + \lambda = 0. \tag{27}
\]

Thus to find the variational symmetries of an equation of form (21), it suffices to check which of its "ordinary" symmetries satisfy the additional requirement (26). For instance, the result (i) implies that $X_0 = w \partial / \partial w$ does not generate a variational symmetry of any equation of form (21), while a vector field $X_u = u(x) \partial / \partial w$ generates a variational symmetry of an equation of form (21) whenever $u(x)$ is its solution (this is a common property of all systems of linear homogeneous partial differential equations, see Section 5.3 in Olver, 1993).

Thus to find the variational symmetries of an equation of form (21), it suffices to check which of its "ordinary" symmetries satisfy the additional requirement (26). For instance, the result (i) implies that $X_0 = w \partial / \partial w$ does not generate a variational symmetry of any equation of form (21), while a vector field $X_u = u(x) \partial / \partial w$ generates a variational symmetry of an equation of form (21) whenever $u(x)$ is its solution (this is a common property of all systems of linear homogeneous partial differential equations, see Section 5.3 in Olver, 1993).

Suppose one has established that a vector field $X$ with characteristic $Q$ generates a variational symmetry of a given equation of form (21), and now wishes to find the conserved current $P^\alpha$ of the corresponding conservation law (2). For this purpose one can use formulae (5.150) and (5.151) given by Olver (1993) which express (in an explicit form) a null Lagrangian as a divergence. Indeed, in this case the right hand side of (22) is a total divergence or, in other words, a null Lagrangian. However, bearing in mind the recommendation of Olver (1993) to use these formulae only as a last resort since "the homotopy formula (5.151) can rapidly become unmanageable", in the present paper we suggest another way for determination of the sought conserved currents.

Our starting point is the so-called Noether identity (cf. Ibragimov, 1985):
\[
X_{\infty}(\mathcal{L}) + (D_\alpha \xi^\alpha) \mathcal{L} = Q \mathcal{E}(\mathcal{L}) + D_\alpha N^\alpha(\mathcal{L}), \tag{28}
\]
which holds for any differential function $\mathcal{L}$ and vector field $X$ of the types considered here. In (28), $N^\alpha$ are the differential operators given by the expressions
\[
N^\alpha = \xi^\alpha + Q \left\{ \frac{\partial}{\partial w_\alpha} + \sum_{s \geq 1} (-1)^s D_{\nu_1} \cdots D_{\nu_s} \frac{\partial}{\partial w_{\alpha\nu_1\cdots\nu_s}} \right\} \tag{29}
\]
\[
+ \sum_{r \geq 1} (D_{\mu_1} \cdots D_{\mu_r} Q) \left\{ \frac{\partial}{\partial w_{\alpha\mu_1\cdots\mu_r}} + \sum_{s \geq 1} (-1)^s D_{\nu_1} \cdots D_{\nu_s} \frac{\partial}{\partial w_{\alpha\mu_1\cdots\mu_r\nu_1\cdots\nu_s}} \right\},
\]
where $Q = \eta - \xi^\alpha w_\alpha$ being the characteristic of the vector field $X$. Setting $\mathcal{L} = L$ in (28), and taking into account (1) and (4), after a little manipulation we obtain the identity
\[
D_\mu N^\mu(-w \mathcal{D}[w]) = -w X_{\frac{1}{4}}(\mathcal{D}[w]) - \{\eta + (D_\mu \xi^\mu) w - 2Q\} \mathcal{D}[w], \tag{30}
\]
valid for any self-adjoint differential operator $\mathcal{D}$ of form (3) and vector field of form (4).

In particular, for $X_v = v(x) \partial / \partial w$, where $v(x)$ is an arbitrary smooth function, we have
\[
\xi^\alpha = 0, \quad Q = \eta = v, \tag{31}
\]
and hence
\[
X_v(\mathcal{D}[w]) = \mathcal{D}[v], \tag{32}
\]
since $D$ is a linear differential operator. Substituting (31) and (32) into (30) we obtain
\[ D_\mu N^\mu (-w D[w]) = v D[w] - w D[v], \]
which is nothing but the reciprocity relation associated with the equation $D[w] = 0$. Under the additional assumption $v = u(x)$, where $u(x)$ is an arbitrary smooth solution of the latter equation, the reciprocity relation (33) becomes
\[ D_\mu N^\mu (-w D[w]) = u D[w]. \]

Taking into account (34) we can give now the following general formula for the conserved currents $P^\alpha$ of the conservation laws with characteristics $Q = u$ corresponding to the infinitesimal variational symmetries $X_u = u \partial/\partial w$ of equation (21):
\[ P^\alpha = P^\alpha_{(u)} + G^\alpha, \]
where
\[ P^\alpha_{(u)} = N^\alpha (-w D[w]), \]
and $G^\alpha$ is any null divergence. Of course,
\[ D_\mu P^\mu_{(u)} = u D[w], \]
and
\[ D_\mu P^\mu_{(u)} = 0, \]
on the smooth solutions of the equation (21).

Next, let $X$ be an infinitesimal variational symmetry of equation (21) with characteristic $Q = w \sigma - w_\mu \xi^\mu$. Then, on account of (25), (30) takes the form
\[ D_\mu N^\mu \left( -\frac{1}{2} w D[w] \right) = Q D[w], \]
and hence we can write down the following explicit formula for the conserved currents $P^\alpha$ of the conservation laws with characteristics $Q = w \sigma - w_\mu \xi^\mu$ corresponding to the aforementioned variational symmetries of equation (21), namely
\[ P^\alpha = B^\alpha + G^\alpha, \]
\[ B^\alpha = N^\alpha (-\frac{1}{2} w D[w]) + \frac{1}{2} D_\mu (w \xi^\alpha A^{\mu\beta\gamma\delta} D_\beta D_\gamma D_\delta w - w \xi^\mu A^{\alpha\beta\gamma\delta} D_\beta D_\gamma D_\delta w), \]
where, as before, $G^\alpha$ is any null divergence. Of course,
\[ D_\mu B^\mu = Q D[w], \]
and on the smooth solutions of respective equation (21) we have
\[ D_\mu B^\mu = 0, \]
Let us remark that the special null divergence
\[ \frac{1}{2} D_\mu \left( w \xi^\alpha A^{\mu\beta\gamma\delta} D_\beta D_\gamma D_\delta w - w \xi^\mu A^{\alpha\beta\gamma\delta} D_\beta D_\gamma D_\delta w \right), \]
is used in the expression (37) for the conserved current $B^\alpha$ to cut the fourth-order derivatives of the dependent variable $w$ away since in practice we are usually interested in conserved currents which involve derivatives of order not higher than $k - 1$, where $k$ is the order of the equation considered. Making use of (29) it is easy to check that the right-hand side of (37) incorporates derivatives of $w$ of order less than fourth. In the subsequent Sections just (35) and (37) will be referred to as the expressions for the conserved currents of the conservation laws with characteristics $Q = u$ and $Q = w \sigma - w_\mu \xi^\mu$, respectively, derived for equations of the form (21).
To summarize, given an equation of form (21), the crucial point on the way of deriving conservation laws admitted by its smooth solutions is to find vector fields of form (15) generating "ordinary" point Lie symmetries of the given equation. For that purpose, we should look for solutions of the respective DE system (16) – (20). Once such vector fields are found, it is easy to check which of their linear combinations satisfy the requirement (26) and hence generate variational symmetries of the equation considered. Now, using the characteristics of these symmetries we first construct the operators $N^{\alpha}$ from formulae (29) and then calculate from (37) the conserved currents of the corresponding conservation laws.

4. SYMMETRIES, CONSERVATION LAWS AND GROUP-INARIANT SOLUTIONS OF PLATE EQUATIONS

In Section 2 (Example 1), we have quoted the self-adjoint equation (8) describing the small bending of a plate resting on an elastic foundation. Many problems concerning stability and vibration of isotropic thin elastic plates are studied on the ground of this type of equations. Here, we analyze the invariance properties of a generic equation of this form under the assumption that the bending rigidity of the plates considered is uniform, that is $D = \text{const}$. In this case (8) may be written as follows

$$A^{\alpha\beta\gamma\delta}w_{\alpha\beta\gamma\delta} + A^{\alpha\beta}(x)w_{\alpha\beta} + A(x)w = 0,$$

with

$$A^{\alpha\beta\gamma\delta} = \frac{1}{3}(\delta^{\alpha\beta}\delta^{\gamma\delta} + \delta^{\alpha\gamma}\delta^{\beta\delta} + \delta^{\alpha\delta}\delta^{\beta\gamma}), \quad A^{\alpha\mu} = 0,$$

assuming that

$$A^{\alpha\beta} = \frac{1}{D}N^{\alpha\beta}, \quad A = \frac{1}{D}k.$$

In view of the general results of Section 3, it is clear that $X_u = u(x)\partial/\partial w$ generates a variational symmetry of any equation of form (38) whenever $u(x)$ is its solution, while $X_0 = w\partial/\partial w$ alone could never generate a variational symmetry of an equation of form (38), though it always is its infinitesimal point Lie symmetry. Substituting (39) into the determining equations (16) – (20) and taking into account that $A^{\alpha\beta} = 0, A^{\alpha} = 0$, we obtain, after a straightforward computation, that an equation of form (38) is invariant under a point Lie group generated by a vector field $X$ of form (15), $X \neq cX_0$, if and only if

$$\sigma = \frac{1}{2}S_{,\mu}$$

$$\delta^{\alpha\mu}S_{,\mu} + \delta^{\mu\beta}S_{,\mu} - \delta^{\alpha\beta}S_{,\mu} = 0,$$

$$\xi^{\mu}A_{\mu} - A^{\alpha\mu}S_{,\mu} - 2\xi^{\mu}A^{\alpha\beta}S_{,\mu} = 0,$$

$$A^{\alpha\nu}S_{,\mu\nu} - A^{\mu\nu}S_{,\mu\nu} = 0,$$

$$2\xi^{\mu}A_{,\mu} + 4\xi^{\mu}A + A^{\mu\nu}\xi_{,\mu\nu} = 0.$$

At that,

$$\lambda = -\frac{3}{2}S_{,\mu}.$$

Substituting expression (40) into (15), and expressions (40) and (45) into condition (27), we immediately arrive at the conclusion that the generator of such a group is a vector field of form

$$X = \xi^{\mu}\frac{\partial}{\partial \xi^{\mu}} + \frac{1}{2}S_{,\mu}w\frac{\partial}{\partial w},$$

each such symmetry of (38) being variational symmetry of the latter equation as well, and hence there exist a conservation law with characteristic $Q = (1/2)\xi^{\mu}w - w_{,\mu}\xi^{\mu}$ and conserved current $B^{\alpha}$ given by (37) admitted by the smooth solutions of the equation considered.
Thus to derive the conservation laws, which correspond to the variational symmetries of an equation of form (38), it suffices to know the results of the group classification of the class of equations in question; of course, the same holds true for the derivation of group-invariant solutions to (38). This group-classification problem is studied in Vassilev (1988), (1991) and (1997). The classification results presented below are obtained in these works.

It is shown that the scalar fields

\[ s_1 = A_{\mu\nu}\delta_{\mu\nu}, \quad s_2 = (8A - \delta_{\alpha\mu}\delta_{\beta\nu}A^{\alpha\beta}A^{\mu\nu})^{1/2}, \quad s_3 = (\delta_{\mu\nu}s_{(1, \mu}s_{(1, \nu})^{1/3}, \quad (47) \]

are of key importance for the group classification of the considered class of equations. These scalar fields are called the invariants of equation (38) since here they play a role similar to that of Laplace's and Cotton's invariants in the group classification of the second-order linear partial differential equations (see Ovsiannikov, 1982 and Ibragimov, 1985). The following two properties of the scalar fields (47) give us both an additional reason to call them invariants of (38) and explicit expressions for the invariants of groups admitted by (38). First, if an equation of form (38) admits a vector field of form (46), then

\[ \xi^\mu s_{(j)} + \xi^\nu s_{(j, \mu)} = 0 \quad (j = 1, 2, 3), \]

and hence \( U_{(j)} = w\sqrt{s_{(j)}} \) are invariants of the corresponding Lie group whenever \( s_{(j)} \neq 0 \).

Second, if an equation of form (38) admits a vector field of form (46) and is such that at least one of its invariants (47) is an invariant of the corresponding symmetry group. Note, that the invariants \( s_{(k)} \) and \( s_{(l)} \) of such an equation of form (38) provide two couples of functionally independent invariants, namely \( U_{(k)} = w\sqrt{s_{(k)}} \) and \( s_{(k)}/s_{(l)} \) as well as \( U_{(l)} = w\sqrt{s_{(l)}} \) and \( s_{(k)}/s_{(l)} \), of the admitted symmetry group, both couples being readily applicable for constructing group-invariant solutions to the respective equation. However, if even one of the invariants (47) of an equation of form (38) is not identically equal to zero, then this equation admits at most a 3-parameter group with generators of form (46). On the other hand, if all invariants (47) of an equation of form (38) are identically equal to zero, then this equation admits a 6-parameter group with generators of form (46). Below, the latter case is set out in detail.

Let \( \omega(z) \neq \text{const} \) be an analytic function of the complex variable \( z = x^1 + ix^2 \), and let \( E_\omega \) be the equation of the form (38) with coefficients

\[ A^{11} = -A^{22} = 4 \text{ Re} \{ \phi \}, \quad A^{12} = A^{21} = -4 \text{ Im} \{ \phi \}, \quad A = 4\phi\bar{\phi}, \quad (48) \]

where \( \phi \) is the Schwarzian derivative of the function \( \omega \), that is

\[ \phi = \left( \frac{\omega''}{\omega'} \right)' - \frac{1}{2} \left( \frac{\omega''}{\omega'} \right)^2, \quad (49) \]

\( \bar{\phi} \) is the complex conjugated of \( \phi \), and the prime is used to denote differentiation with respect to the variable \( z \). Substituting (38) into (47) one can see that all invariants of \( E_\omega \) are identically equal to zero. Then, taking into account the DE system (11)–(14), (48) and (19), one can verify by direct computing that \( E_\omega \) admits the 6-parameter group generated by the vector fields

\[ Z_{(j)} = \xi^\mu_{(j)} \frac{\partial}{\partial x^\mu} + \frac{1}{2} \xi^\mu_{(j, \nu)} w \frac{\partial}{\partial w} \quad (j = 1, \ldots, 6), \]

the functions \( \xi^\mu_{(j)} \) being given by the expressions

\[
\begin{align*}
\xi^1_{(1)} &= \text{ Re } \{ \omega_1 \}, & \xi^2_{(1)} &= \text{ Im } \{ \omega_1 \}, \\
\xi^1_{(2)} &= \text{ Re } \{ i\omega_1 \}, & \xi^2_{(2)} &= \text{ Im } \{ i\omega_1 \}, \\
\xi^1_{(3)} &= \text{ Re } \{ \omega_2 \}, & \xi^2_{(3)} &= \text{ Im } \{ \omega_2 \}, \\
\xi^1_{(4)} &= \text{ Re } \{ i\omega_2 \}, & \xi^2_{(4)} &= \text{ Im } \{ i\omega_2 \}, \\
\xi^1_{(5)} &= \text{ Re } \{ \omega_3 \}, & \xi^2_{(5)} &= \text{ Im } \{ \omega_3 \}, \\
\xi^1_{(6)} &= \text{ Re } \{ i\omega_3 \}, & \xi^2_{(6)} &= \text{ Im } \{ i\omega_3 \}.
\end{align*}
\]
where
\[
\omega_1 = \frac{1}{\omega'}, \quad \omega_2 = \omega, \quad \omega_3 = \frac{\omega^2}{\omega'}.
\] (50)

It should be remarked that each equation of form (38) which admits a 6-parameter group with generators of form (43), is of type \(E_\omega\), meaning that it can be generated in the above manner using a suitable analytic function \(\omega\). The coefficients of each equation of this type are of the form \(A^\alpha{}^\beta = \delta^\alpha{}^\mu \delta^\beta{}^\nu \varphi_{,\mu\nu}, A = (1/8) \delta^\alpha{}^\mu \delta^\beta{}^\nu \varphi_{,\alpha\beta} \varphi_{,\mu\nu}\), where \(\varphi\) is a harmonic function, that is \(\delta^\alpha{}^\beta \varphi_{,\alpha\beta} = 0\), and vice versa. It is noteworthy that each equation with variable coefficients of type \(E_\omega\) can be mapped to an equation with constant coefficients belonging to the same family. It is easy to verify by direct computing that the equation \(E_\omega\) corresponding to an analytic function \(\omega\) whose Schwarzian derivative is not constant transforms to a constant coefficients one under the following change of the independent and dependent variables:

\[
y^\alpha = f^\alpha (x^1, x^2), \quad W = wU (x^1, x^2),
\] (51)

\[
f^1 (x^1, x^2) = \text{Re} \left\{ \int f^{-1} dz \right\}, \quad f^2 (x^1, x^2) = \text{Im} \left\{ \int f^{-1} dz \right\}, \quad U (x^1, x^2) = \left( f^{-1} \right)^{-1/2},
\]

where \(f\) is any linear combination of the functions (50) such that \(f \neq 0\), i.e.

\[
f = k_1 \omega_1 + k_2 \omega_2 + k_3 \omega_3,
\] (52)

where \(k_1, k_2,\) and \(k_3\) are complex constants such that \(k_1^2 + k_2^2 + k_3^2 \neq 0\).

Consider, as a simple example, the equation \(E_\omega\) corresponding to \(\omega = z \sqrt{x/2}\), where \(x\) is a positive real constant. In this case (43) gives \(\phi = 4 (2 - x) z^2\) and hence, according to (38), the coefficients of \(E_\omega\) read

\[
A^{11} = -A^{22} = (2 - x) \frac{(x^1)^2 - (x^2)^2}{[(x^1)^2 + (x^2)^2]^2}, \quad A^{12} = A^{21} = (2 - x) \frac{2x^1 x^2}{[(x^1)^2 + (x^2)^2]^2}.
\]

\[
A = (2 - x)^2 \frac{1}{4 [(x^1)^2 + (x^2)^2]^2}.
\] (53)

Using the function \(f = z\) obtained from (52) for \(\omega = z \sqrt{x/2}\), \(k_1 = k_3 = 0, k_2 = 1 + \sqrt{x/2}\), we introduce, according to (51), the new independent and dependent variables

\[
y^1 = \frac{1}{2} \ln \left[ (x^1)^2 + (x^2)^2 \right], \quad y^2 = \arctan \left( \frac{x^2}{x^1} \right), \quad W = w \left[ (x^1)^2 + (x^2)^2 \right]^{-1/2}.
\] (54)

Note that the inverse transformations are given by the expressions

\[
x^1 = e^{y^1} \cos y^2, \quad x^2 = e^{y^1} \sin y^2, \quad w = W \left[ (x^1)^2 + (x^2)^2 \right]^{1/2}.
\] (55)

Under the change of the variables according to (54), the considered equation \(E_\omega\) transforms to the following one,

\[
\delta^\alpha{}^\beta \delta^{\mu\nu} \frac{\partial^4 W}{\partial y^\rho \partial y^\sigma \partial y^\beta \partial y^\gamma} - x^2 \left( \frac{\partial^2 W}{\partial y^1 \partial y^1} + \frac{\partial^2 W}{\partial y^2 \partial y^2} + \frac{1}{4} x^2 W \right) = 0,
\] (56)

which belongs to the same class (since it corresponds to the analytic function \(\omega = e^z \sqrt{x/2}\)) but whose coefficients are constant.
Equation (56) admits the 6-parameter group of variational symmetries generated by the basic vector fields

\[ V_a = \frac{\partial}{\partial y^a}, \]
\[ V_3 = e^{\theta y^1} \cos (\theta y^2) \frac{\partial}{\partial y^1} + e^{\theta y^1} \sin (\theta y^2) \frac{\partial}{\partial y^2} + \theta e^{\theta y^1} w \cos (\theta y^2) \frac{\partial}{\partial w}, \]
\[ V_4 = -e^{\theta y^1} \cos (\theta y^2) \frac{\partial}{\partial y^1} + e^{\theta y^1} \cos (\theta y^2) \frac{\partial}{\partial y^2} - \theta e^{\theta y^1} w \sin (\theta y^2) \frac{\partial}{\partial w}, \]
\[ V_5 = e^{-\theta y^1} \cos (\theta y^2) \frac{\partial}{\partial y^1} - e^{-\theta y^1} \sin (\theta y^2) \frac{\partial}{\partial y^2} - \theta e^{-\theta y^1} w \cos (\theta y^2) \frac{\partial}{\partial w}, \]
\[ V_6 = e^{-\theta y^1} \cos (\theta y^2) \frac{\partial}{\partial y^1} + e^{-\theta y^1} \sin (\theta y^2) \frac{\partial}{\partial y^2} - \theta e^{-\theta y^1} w \cos (\theta y^2) \frac{\partial}{\partial w}, \]

where \( \theta = \sqrt{\kappa/2} \). These vector fields give rise to six linearly independent conservation laws for equation (56). The characteristics of these conservation laws are

\[ Q(j) = \frac{1}{2} W \frac{\partial}{\partial y^\mu} V_j (y^\mu) - W \mu V_j (y^\mu) \quad (j = 1, \ldots, 6). \]

Here, \( V_j \) are regarded as operators acting on the functions \( \zeta : \mathbb{R}^2 \to \mathbb{R} \). The corresponding conserved currents can be easily calculated from (37).

Finally, let us remark that each one-parameter group generated by a linear combination of the basic vector fields \( V_j \) can be used for constructing group-invariant solutions of equation (56). Consider, for instance, the group \( H (V_3 + V_5) \) generated by the vector field \( V_3 + V_5 \). The functions \( s = \sin (\theta y^2) / \cosh (\theta y^1) \) and \( u = W / \cosh (\theta y^1) \) constitute a complete set of invariants for this group and hence, following the well known algorithm (Ovsiannikov, 1982; Olver, 1993), we seek the \( H (V_3 + V_5) \)-invariant solutions of equation (56) in the form

\[ W = u(s) \cosh (\theta y^1), \quad s = \frac{\sin (\theta y^2)}{\cosh (\theta y^1)}. \]  

(57)

Substituting (57) into (56), we get the reduced equation

\[ (s^2 - 1)^2 \frac{d^4 u}{ds^4} + 8 s (s^2 - 1) \frac{d^3 u}{ds^3} + 4 (3s^2 - 1) \frac{d^2 u}{ds^2} = 0. \]

The general solution to this ordinary differential equation is

\[ u(s) = C_1 + C_2 \ln \left( \frac{s+1}{s-1} \right) + C_3 s + C_4 s \ln \left( \frac{s+1}{s-1} \right), \]

where \( C_1, C_2, C_3 \) and \( C_4 \) are real constants. Hence the \( H (V_3 + V_5) \)-invariant solutions of equation (56) are given by the expression

\[ W (y^1, y^2) = C_1 \cosh (\theta y^1) + C_2 \cosh (\theta y^1) \ln \left[ \frac{\sin (\theta y^2) + \cosh (\theta y^1)}{\sin (\theta y^2) - \cosh (\theta y^1)} \right] \]

\[ + C_3 \sin (\theta y^2) + C_4 \sin (\theta y^2) \cosh (\theta y^1) \ln \left[ \frac{\sin (\theta y^2) + \cosh (\theta y^1)}{\sin (\theta y^2) - \cosh (\theta y^1)} \right]. \]

Using the inverse transformations (53), one can convert the above solutions of equation (56) into solutions of the equation \( E_{\omega}, \omega = z \sqrt{\kappa/2} \), with variable coefficients (53).
4. SYMMETRIES, CONSERVATION LAWS AND GROUP-INVARIANT SOLUTIONS OF ROD EQUATIONS

In Section 2, combining and generalizing Examples 3 and 4 we have introduced the class of self-adjoint partial differential equations

\[ A^{1111}w_{1111} + A^{\alpha\beta}w_{\alpha\beta} + Aw = 0, \]  

with coefficients

\[ A^{1111} = \gamma, \quad A^{\alpha\beta} = \chi^{\alpha\beta}, \quad A = \kappa(x), \]

where \( \gamma = \text{const} \neq 0, \chi^{\alpha\beta} \) are arbitrary constants (but \((\chi^{12})^2 + (\chi^{22})^2 \neq 0\), otherwise (58) degenerates and becomes an ordinary differential equation), and \( \kappa(x) \) is an arbitrary function. Equations of this special type are used by many authors to study applied engineering problems concerning dynamics and stability of both elastic beams resting on elastic foundations (see e.g. Smith and Herrmann, 1972) and pipes conveying fluid (see e.g. Gregory and Paidoussis, 1966). In the present Section we first examine the point Lie symmetries of (58) and solve the corresponding group-classification problem. Then we derive conservation laws and group-invariant solutions of various rod equations of form (58).

Consider the group-classification problem. In view of the results (i) and (ii) of Section 3, it is clear that each equation of form (58) is invariant under the point Lie groups generated by the vector fields \( X_0 = w\partial/\partial w \) and \( X_\alpha = u(x)\partial/\partial w \) where \( u(x) \) is any smooth solution of the respective equation and the objective is to find those equations of the type considered which admit vector fields \( X \) of form (15), \( X \neq cX_0, c = \text{const} \neq 0. \)

Substituting (59) into (60) – (64), and taking into account that \( A^\alpha, A^{\alpha\beta\gamma} \) and all \( A^{\alpha\beta\gamma\delta} \) except \( A^{1111} \) are equal to zero, we obtain after a little manipulation the following system of determining equations for the functions \( \xi^\mu(x) \) and \( \sigma(x) \) associated with the sought vector fields \( X \) of form (15):

\[
\begin{align*}
\xi^1_{,1} &= 0, \quad 2\sigma_{,1} - 3\xi^1_{,11} = 0, \quad (60) \\
5\gamma\xi^1_{,111} + 2\alpha^{11}\xi^1_{,1} - 2\chi^{12}\xi^1_{,2} &= 0, \quad (61) \\
\chi^{22}(2\xi^1_{,1} - \xi^2_{,2}) &= 0, \quad \chi^{12}(3\xi^1_{,1} - \xi^2_{,2}) - \chi^{22}\xi^1_{,2} = 0, \quad (62) \\
2\chi^{12}\sigma_{,2} - \chi^{22}\xi_{,22} &= 0, \quad 2\chi^{12}\sigma_{,1} + \chi^{22}(2\sigma_{,2} - \xi^2_{,22}) = 0, \quad (63) \\
\sigma_{,1111} + \chi^{\alpha\beta}\sigma_{,\alpha\beta} + \xi^\mu\kappa_{,\mu} + 4\xi^1_{,1}\kappa &= 0, \quad (64)
\end{align*}
\]

the auxiliary function \( \lambda \) being expressed by

\[ \lambda = \sigma - 4\xi^1_{,1}. \]  

We look for the equations of form (58) whose coefficients \( \gamma, \chi^{\alpha\beta} \) and \( \kappa(x) \) are such that system (60) – (64) possesses solutions different from the trivial one \( \xi^\mu = 0, \sigma = \text{const} \neq 0. \)

Observing the system of determining equations (60) – (64) we see that for the purposes of the group-classification it is convenient to divide the equations of form (58) into three subclasses depending on whether the coefficients \( \chi^{\alpha\beta} \) of the given equation are such that:

(A) \( \chi^{22} \neq 0, \det(\chi^{\alpha\beta}) \neq 0; \) (B) \( \chi^{22} \neq 0, \det(\chi^{\alpha\beta}) = 0 \) or (C) \( \chi^{22} = 0, \det(\chi^{\alpha\beta}) \neq 0. \) This covers all possibilities except for \( \chi^{22} = 0, \det(\chi^{\alpha\beta}) = 0 \) when (58) becomes an ordinary differential equation that falls outside our interest in the present paper.

For convenience we introduce the notation

\[ Y_\alpha = \frac{\partial}{\partial x^\alpha}, \quad Y_3 = \left(x^1 + \frac{x^{12}}{\chi^{22}}x^2\right)\frac{\partial}{\partial x^1} + 2x^2\frac{\partial}{\partial x^2}, \quad Y_4 = \left(x^1 + \frac{x^{11}}{\chi^{12}}x^2\right)\frac{\partial}{\partial x^1} + 3x^2\frac{\partial}{\partial x^2}. \]

Let \( \chi^{22} \neq 0. \) Then, the first equation (15) is equivalent to

\[ 2\xi^1_{,1} - \xi^2_{,2} = 0. \]
Differentiating (60) with respect to \( x^1 \) and taking into account the first equation (60) we obtain \( \chi_{11}^1 = 0 \). Hence, (61) and the second equation (62) reduce to
\[
\chi_{11}^{11} \xi_1^1 - \chi_{12}^{12} \xi_2^1 = 0, \quad \chi_{11}^{12} \xi_1^1 - \chi_{22}^{22} \xi_2^1 = 0. \tag{67}
\]

Consider the subclass (A): \( \chi_{22} \neq 0, \det(\chi^{\alpha\beta}) \neq 0 \). Then, the first equation (60) together with (66) and (67) lead to \( \xi_{11}^1 = \xi_{12}^2 = \xi_{21}^2 = \xi_{22}^2 = 0 \), i.e., \( \xi^\alpha = c^\alpha = \text{const} \). Consequently, the second equations of (60) and (63) imply \( \sigma = c = \text{const} \). At this, the first equation (53) is satisfied and (54) becomes
\[
c^\alpha \kappa_{,\alpha} = 0. \tag{68}
\]
All this means that when \( \chi_{22} \neq 0 \) and \( \det(\chi^{\alpha\beta}) \neq 0 \) the DE system has only the trivial solution unless
\[
\kappa(x) = f(\beta_1 x^1 - \beta_2 x^2), \tag{69}
\]
for a certain smooth function \( f \neq \text{const} \) and certain constants \( \beta^\alpha \) such that \( (\beta_1)^2 + (\beta_2)^2 \neq 0 \). In this latter case, the DE system has the nontrivial solution
\[
\xi^1 = \beta_1, \quad \xi^2 = \beta_2, \quad \sigma = 0, \tag{70}
\]
and so the differential equations of that kind admit additionally the one-parameter symmetry group associated with the vector field \( \beta_1 Y_1 + \beta_2 Y_2 \). In the case \( \kappa = \text{const} \), (53) is satisfied for any couple of constants \( c^\alpha \), and hence such equations admit the 2-parameter symmetry group with generators \( Y_1 \) and \( Y_2 \). This completes the analysis of subclass (A).

Consider now the subclass (B): \( \chi_{22} \neq 0, \det(\chi^{\alpha\beta}) = 0 \). Differentiating the second equation (67) successively with respect to \( x^1 \) and \( x^2 \) we obtain
\[
\chi_{12}^{11} \xi_{11}^1 - \chi_{22}^{22} \xi_{22}^1 = 0, \quad \chi_{12}^{12} \xi_{12}^2 - \chi_{22}^{22} \xi_{22}^2 = 0,
\]
which, on account of \( \xi_{11}^1 = 0 \), leads to \( \xi_{12}^2 = \xi_{22}^2 = 0 \). This result, together with (66) imply \( \xi_{22}^2 = 0 \), and the nontrivial solution of (60) – (64) is now obvious:
\[
\xi^1 = c^1 + c^3 \left( x^1 + \frac{\chi_{12}^{12}}{\chi_{22}^{22}} x^2 \right), \quad \xi^2 = c^2 + 2c^3 x^2, \quad \sigma = 0, \tag{71}
\]
with \( c^1, c^2 \) and \( c^3 \) – arbitrary constants. Equation (64) reduces to
\[
\xi^\alpha \kappa_{,\alpha} + 4\xi^1 \kappa = 0. \tag{72}
\]
For an arbitrary \( \kappa(x) \) it leads to \( \xi^\alpha = 0 \); it is easily verified that for \( \kappa(x) = x^1 + x^2 x^2, \xi^\alpha = 0 \) is the only solution of (72).

If \( \kappa(x) \) is a function of form (69), then (72) implies (71) as a nontrivial solution to the system (60) – (64). Therefore, a generic equation of this kind admits only the one-parameter group generated by \( \beta_1 Y_1 + \beta_2 Y_2 \), unless \( \kappa(x) \) is of one of the following two special forms. The first one is
\[
\kappa(x) = \kappa_0 \left( \beta + x^2 \right)^{-2}, \quad \kappa_0 = \text{const} \neq 0, \quad \beta = \text{const}, \tag{73}
\]
when the nontrivial solution of (60) – (64) is (71) with \( c^2 = 2\beta, \ c^3 = 1, \ c^1 \) – arbitrary, and hence the equations of subclass (B) with \( \kappa(x) \) of form (73) admit the 2-parameter symmetry group generated by the vector fields \( Y_1 \) and \( 2\beta Y_2 + Y_3 \). The second special form of the function \( \kappa(x) \) is
\[
\kappa(x) = \kappa_0 \left( \beta + x^1 - \frac{\chi_{12}^{12}}{\chi_{22}^{22}} x^2 \right)^{-4}, \quad \kappa_0 = \text{const} \neq 0, \quad \beta = \text{const}, \tag{74}
\]
when the respective differential equations admit the 2-parameter group spanned over the vector fields \( \beta Y_1 + Y_3 \) and \( (\chi_{12}^{12}/\chi_{22}^{22}) Y_1 + Y_2 \).
Another extension of the symmetry group is possible if there exist two constants \( \beta^1 \) and \( \beta^2 \), as well as a smooth function \( f \neq 0 \) such that

\[
\kappa(x) = (\beta^2 + x^2)^{-2} f(y), \quad y = (\beta^2 + x^2)^{-1/2} \left( \beta^1 + x^1 - \frac{\chi_{12}}{\chi_{22}} x^2 \right).
\]

If \( \kappa(x) \) is of form (73), then the nontrivial solution of the determining equations (76) – (79) is of form (78), with \( c^1 = \beta^1 + 2 (\chi^{12}/\chi^{22}) \beta^2, c^2 = 2 \beta^2, c^3 = 1 \) and the differential equations of that sort admit the one-parameter group, generated by

\[
\left( \beta^1 + 2 \frac{\chi_{12}}{\chi_{22}} \beta^2 \right) Y_1 + 2 \beta^2 Y_2 + Y_3
\]

only, except for the cases \( f(y) = \kappa_0 y^{-4} (\kappa_0 = \text{const}) \), when \( \kappa(x) \) takes the form (74), and \( f(y) = \text{const} \neq 0 \) when \( \kappa(x) \) becomes (75).

The differential equations of form (75) with \( \chi^{22} \neq 0 \), \( \det(\chi^{\alpha\beta}) = 0 \) admit the 2-parameter group generated by \( Y_1 \) and \( Y_2 \) when \( \kappa(x) = \text{const} \neq 0 \) or the 3-parameter group with generators \( Y_1, Y_2 \) and \( Y_3 \) when \( \kappa(x) = 0 \).

Finally, consider the subclass (C): \( \chi^{22} = 0, \det(\chi^{\alpha\beta}) \neq 0 \). Substituting \( \chi^{22} = 0 \) in the determining equations (76) – (79), the latter simplify to

\[
\xi^2 = 0, \quad \xi^1_{,1} = 0, \quad \chi^{11} \xi^1_{,1} - \chi^{12} \xi^2_{,2} = 0, \quad 3 \xi^1_{,1} - 2 \xi^2_{,2} = 0, \quad \sigma_{,1} = 0, \quad \sigma_{,2} = 0,
\]

and their nontrivial solution is easily obtained:

\[
\xi^1 = c^1 + c^3 \left( x^1 + \frac{\chi_{11}}{\chi_{12}} x^2 \right), \quad \xi^2 = c^2 + 3c^3 x^2, \quad \sigma = 0,
\]

where \( c^i \) are arbitrary constants (note that if \( \chi^{22} = 0 \), then \( \det(\chi^{\alpha\beta}) \neq 0 \) assumes \( \chi^{12} \neq 0 \)). Equation (74) takes the form (72) and for an arbitrary \( \kappa(x) \) it leads to \( \xi^2 = 0 \). If \( \kappa(x) \) is of the form (78), such equations admit the one-parameter group generated by \( \beta^1 Y_1 + \beta^2 Y_2 \) only, except for two special forms of \( \kappa(x) \), namely

\[
\kappa(x) = \kappa_0 (\beta + x^2)^{-4/3},
\]

and

\[
\kappa(x) = \kappa_0 \left( \beta + 2 x^1 - \frac{\chi_{11}}{\chi_{12}} x^2 \right)^{-4}.
\]

where \( \kappa_0 \) and \( \beta \) are constants. In the case (78), the nontrivial solution of the determining equation (74) – (75) is of form (77) with \( c^1 = 3\beta, c^3 = 1, c^1 \) arbitrary, and the differential equation considered admits the 2-parameter group spanned over the vector fields \( Y_1 \) and \( 3\beta_2 Y_2 + Y_4 \). In the case (78) the group admitted is also a 2-parameter one, but generated by \( \beta^1 Y_1 + Y_4 \) and \( (\chi^{11}/\chi^{12}) Y_1 + 2 Y_2 \).

Another extension of the symmetry group is possible if

\[
\kappa(x) = (\beta^2 + x^2)^{-4/3} f(y), \quad y = (\beta^2 + x^2)^{-1/3} \left( \beta^1 + 2 x^1 - \frac{\chi_{11}}{\chi_{12}} x^2 \right),
\]

where \( \beta^\alpha \) are constants and \( f \neq 0 \) is a smooth function. In this case, the nontrivial solution of (76) – (79) is of form (77) with \( c^1 = \beta^1 + 3(\chi^{11}/\chi^{12}) \beta^2, c^2 = 6 \beta^2, c^3 = 1 \) and we conclude that the equations of subclass (C) with \( \kappa(x) \) of form (74) admit only the one-parameter group generated by \( \beta^1 + 3(\chi^{11}/\chi^{12}) \beta^2 Y_1 + 6 \beta^2 Y_2 + Y_3 \), except for \( f(y) = \kappa_0 y^{-4} (\kappa_0 = \text{const}) \), when (77) coincides with (78) or \( f(y) = \text{const} \neq 0 \) when \( \kappa(x) \) has the form (77).

Evidently, the differential equations of form (78) with \( \chi^{22} = 0, \det(\chi^{\alpha\beta}) \neq 0 \) admitt: the 2-parameter group with generators \( Y_1 \) and \( Y_2 \) when \( \kappa(x) = \text{const} \neq 0 \), and the 3-parameter group associated with \( Y_1, Y_2 \) and \( Y_3 \) when \( \kappa(x) = 0 \).

The results of the above group-classification analysis are summarized in Table 1, where the equations invariant under larger groups are given through their coefficients together with the generators of the associated symmetry groups.
Having completely solved the group-classification problem, our next step is to identify the variational symmetries of those equations of form (58) which have been found to admit larger symmetry groups. For this purpose, we are to apply the condition (26) to the linear combinations of (14) and the vector fields presented in Table 1, the respective functions \( \lambda \) being given by (65). Thus, we found that all vector fields quoted under numbers 1, 3, 5, 7, 8, 9 and 11 generate variational symmetries as well. In case \# 2 the variational symmetries are associated with \( \beta^4 + 2(\chi^{12}/\chi^{22})\beta^2 Y_1 + 2\beta^2 Y_2 + Y_3 + (1/2)X_0 \). Similarly, in case \# 4 the variational symmetries are generated by \( Y_1 \) and \( 2\beta Y_2 + Y_3 + (1/2)X_0 \), in case \# 6 – by \( (\chi^{12}/\chi^{22})Y_1 + Y_2 \) and \( \beta Y_1 + Y_3 + (1/2)X_0 \), and in case \# 10 – by \( Y_1, Y_2 \) and \( Y_3 + (1/2)X_0 \).

Once the variational symmetries of the equations (58) are identified, we can derive the corresponding conservation laws. The conserved currents of the conservation laws for the equations given in Table 1 are computed using (67), the above notes concerning the corresponding variational symmetries being taken into account. The obtained conservation laws are listed in Table 2 (in the same order as in Table 1) in terms of the differential functions:

| #  | Coefficients | Generators |
|----|--------------|------------|
| 1  | \( \kappa(x) = (\beta^2 x^2 - \beta^3 x^3) \) | \( \beta^4 Y_1 + \beta^2 Y_2 \) |
| 2  | \( \chi^{22} \neq 0, \det(\chi^{\alpha\beta}) = 0, \kappa(x) = (\beta^2 + x^3)^{-2} f(y), \) \( y = (\beta^2 + x^3)^{-1/2} \left[ \beta^1 + x^1 - (\chi^{12}/\chi^{22}) x^2 \right] \) | \( [\beta^1 + 2(\chi^{12}/\chi^{22})\beta^2] Y_1 + 2\beta^2 Y_2 + Y_3 \) |
| 3  | \( \chi^{22} = 0, \det(\chi^{\alpha\beta}) \neq 0, \kappa(x) = (\beta^2 + x^3)^{-4/3} f(y), \) \( y = (\beta^2 + x^3)^{-1/3} \left[ \beta^1 + 2x^1 - (\chi^{11}/\chi^{12}) x^2 \right] \) | \( [\beta^1 + 3(\chi^{11}/\chi^{12})\beta^2] Y_1 + 2Y_2 + 2Y_4 \) |
| 4  | \( \chi^{22} \neq 0, \det(\chi^{\alpha\beta}) = 0, \kappa(x) = \kappa_0 (\beta + x^2)^{-2}, \) Y_1, 2\beta Y_2 + Y_3 |
| 5  | \( \chi^{22} = 0, \det(\chi^{\alpha\beta}) \neq 0, \kappa(x) = \kappa_0 (\beta + x^2)^{-4/3} \) Y_1, 3\beta Y_2 + Y_4 |
| 6  | \( \chi^{22} \neq 0, \det(\chi^{\alpha\beta}) = 0, \kappa(x) = \kappa_0 (\beta + x^2)^{-4} \) \( \beta^1 Y_1 + Y_3, \) \( (\chi^{12}/\chi^{22})Y_1 + Y_2 \) |
| 7  | \( \chi^{22} = 0, \det(\chi^{\alpha\beta}) \neq 0, \kappa(x) = \kappa_0 (\beta + 2x^1 - (\chi^{11}/\chi^{12}) x^2)^{-4} \) \( \beta^1 Y_1 + 2Y_4, \) \( (\chi^{11}/\chi^{12})Y_1 + 2Y_2 \) |
| 8  | \( \chi^{22} \det(\chi^{\alpha\beta}) \neq 0, \kappa(x) = \text{const} \) Y_1, Y_2 |
| 9  | \( \chi^{22} \det(\chi^{\alpha\beta}) = 0, \kappa(x) = \text{const} \neq 0 \) Y_1, Y_2 |
| 10 | \( \chi^{22} \neq 0, \det(\chi^{\alpha\beta}) = 0, \kappa(x) = 0 \) Y_1, Y_2, Y_3 |
| 11 | \( \chi^{22} = 0, \det(\chi^{\alpha\beta}) \neq 0, \kappa(x) = 0 \) Y_1, Y_2, Y_4 |

Table 1. Equations of form (58) invariant under larger symmetry groups.
Table 2. Conservation laws for equations of form (58)

| # | Conservation laws |
|---|-------------------|
| 1 | $D_\alpha[\beta^1 B_{(1)}^\alpha + \beta^2 B_{(2)}^\alpha] = 0$ |
| 2 | $D_\alpha[(\beta^1 + 2(\chi^{12}/\chi^{22})\beta^2) B_{(1)}^\alpha + 2\beta^2 B_{(2)}^\alpha + B_{(3)}^\alpha] = 0$ |
| 3 | $D_\alpha[(\beta^1 + 3(\chi^{11}/\chi^{12})\beta^2) B_{(1)}^\alpha + 6\beta^2 B_{(2)}^\alpha + 2B_{(4)}^\alpha] = 0$ |
| 4 | $D_\alpha B_{(1)}^\alpha = 0, \ D_\alpha[2\beta B_{(2)}^\alpha + B_{(3)}] = 0$ |
| 5 | $D_\alpha B_{(1)}^\alpha = 0, \ D_\alpha[3\beta B_{(2)}^\alpha + B_{(4)}] = 0$ |
| 6 | $D_\alpha[\beta B_{(1)}^\alpha + B_{(3)}^\alpha] = 0, \ D_\alpha[(\chi^{12}/\chi^{22}) B_{(1)}^\alpha + B_{(2)}^\alpha] = 0$ |
| 7 | $D_\alpha[\beta B_{(1)}^\alpha + 2B_{(4)}^\alpha] = 0, \ D_\alpha[(\chi^{11}/\chi^{12}) B_{(1)}^\alpha + 2B_{(2)}^\alpha] = 0$ |
| 8 | $D_\alpha B_{(1)}^\alpha = 0, \ D_\alpha B_{(2)}^\alpha = 0$ |
| 9 | $D_\alpha B_{(1)}^\alpha = 0, \ D_\alpha B_{(2)}^\alpha = 0$ |
| 10 | $D_\alpha B_{(1)}^\alpha = 0, \ D_\alpha B_{(2)}^\alpha = 0, \ D_\alpha B_{(3)}^\alpha = 0$ |
| 11 | $D_\alpha B_{(1)}^\alpha = 0, \ D_\alpha B_{(2)}^\alpha = 0, \ D_\alpha B_{(4)}^\alpha = 0$ |

According to the general results of Section 3, each equation (58) admits conservation laws with characteristics $Q = u(x)$, where $u(x)$ is any smooth solution of the equation considered. These conservation laws are of the form (35), that is

$$D_\alpha P_{(u)}^\alpha = 0,$$

the corresponding conserved currents $P_{(u)}^\alpha$ being given by the expression (35). Here, on account of (59), (35) simplifies and reads

$$P_{(u)}^\alpha = \chi^{\alpha}u(ww_{\mu} - u_{\mu}w) + \delta^{1\alpha}\gamma(ww_{111} + u_{11}w_1 - u_{111}w - u_{1}w_{11}).$$ (80)

Let us now specialize to the differential equation

$$EJw_{1111} + mw_{22} = 0,$$ (81)

governing the dynamics of a classic homogeneous Bernoulli-Euler beam. Here $EJ$ is the bending rigidity of the beam and $m$ is the mass of the beam per unit length. According to the above analysis, (81) admits the following six linearly independent infinitesimal variational symmetries:

$$Y_1, \ Y_2, \ Y_3 + \frac{1}{2}X_6, \ Y_5 = \frac{\partial}{\partial w}, \ Y_6 = x^1 \frac{\partial}{\partial w}, \ Y_7 = x^2 \frac{\partial}{\partial w},$$ (82)

where $Y_5, Y_6$ and $Y_7$ are vector fields of the type $X_u = u(x) \partial/\partial w$ corresponding to the solutions $u = 1, u = x^1$ and $u = x^2$ of (81), respectively. Here, the independent variables $x^1$ and $x^2$ are the spatial variable along the rod axis and the time, respectively, so that the conservation laws admitted by the smooth solutions of equation (81) may be written in the more familiar form

$$\frac{\partial \Psi}{\partial x^2} + \frac{\partial P}{\partial x^1} = 0,$$

where $\Psi$ and $P$ denote the density and flux of the conservation law, respectively. The densities and fluxes of the conservation laws for (81) associated with the vector fields (82) together with their physical interpretation are presented in Table 3.
namely those associated with the solutions to \((81)\) of the form

\[
\Psi(1) = mw_1w_2 \\
P(1) = (1/2)[EJ(2w_1w_{111} - w_1^2) - mw_2^2]
\]

Only a part of these conservation laws are identified and presented in (Chien et al., 1993), the conserved currents of the conservation laws for \((81)\) with characteristics considered in Chien et al. (1993) when

\[
B
\]

into account the aforementioned restrictions we observe that:

\[
\Psi(2) = (1/2) (EJw_{11}^2 + mw_2^2) \\
P(2) = EJ(w_{11}w_{12} - w_2w_{111})
\]

\[
\Psi(3) = x^1\Psi(1) + 2x^2\Psi(2) + mw_2 \]

\[
P(3) = x^1P(1) + 2x^2P(2) + (1/2)EJ(w_{1111} - w_1w_{11})
\]

\[
\Psi(5) = mw_2, \quad P(5) = EJw_{111}
\]

\[
\Psi(6) = x^1mw_2, \quad P(6) = EJ(x^1w_{111} - w_{11})
\]

\[
\Psi(7) = m(x^2w_2 - w), \quad P(7) = EJx^2w_{111}
\]

Conservation laws in the dynamics of rods are considered in many papers (see e.g. Antman, 1984; Kienzler, 1986; Chien et al., 1993; Maddocks and Dichmann, 1994; Tabarrok et al., 1994; Djondjorov, 1995). However, the particular form of the differential equations examined in the present study allows comparison with the results reported by Chien et al. (1993) and by Maddocks and Dichmann (1994) only.

Chien et al. (1993) derive conservation laws for the statics and dynamics of rods employing a technique called by the authors Neutral Action (NA) method. The conservation laws for rod equations established in this Section could be compared to their ones only for the differential equation \((81)\) which coincides with the equation

\[
Bu_{1111} + 2B_1w_{111} + B_{11}w_{11} + Hw_{22} = 0,
\]

considered in Chien et al. (1993) when \(B = EJ\) and \(H = m\). The comparison shows that the conserved currents of the conservation laws for \((81)\) with characteristics other than \(Q = u(x)\) obtained by Chien et al. (1993) coincide with ours presented in Table 3. As for the conserved currents of the conservation laws for \((81)\) with characteristics \(Q = u(x)\), where \(u(x)\) is any solution of \((81)\), our general formula \((80)\) implies

\[
P^\alpha(u) = \delta^{2\alpha} m(uw_2 - u_2w) + \delta^{1\alpha} EJ(uw_{111} + u_{11}w_{11} - u_{111}w_1 - w_1w_{11}).
\]

Only a part of these conservation laws are identified and presented in (Chien et al., 1993), namely those associated with the solutions to \((81)\) of the form

\[
u(x) = C_1(x^1)^3 + C_2(x^1)^2 + C_3x^1 + C_4 + C_5x^2, \quad C_i = \text{const} \quad (i = 1, \ldots, 5),
\]

while, in fact, equation \((81)\) has an infinite-dimensional space of solutions.

Five conservation laws in the dynamics of rods are reported in (Maddocks and Dichmann, 1994) within a general nonlinear direct theory. The restricted version of this theory describing small planar bending of an uniform inextensible unshearable isotropic elastic rod with a linear constitutive law, the rotatory inertia of the rod cross section being neglected, is exactly the classic Bernoulli-Euler theory for homogeneous beams whose governing equation is \((81)\). Rewriting the conservation laws in (Maddocks and Dichmann, 1994) taking into account the aforementioned restrictions we observe that: \((1)\) the conservation law for the total angular momentum (formula 2.14 in Maddocks and Dichmann, 1994) degenerates to the well known basic relation of Bernoulli-Euler theory \(Q = \partial M/\partial x^1\) (here \(Q\) and \(M\) denote shear force and bending moment, respectively, see Washizu, 1982); \((2)\) the density
and flux of the conservation law associated with material isotropy (formula 4.5 in Maddocks and Dichmann, 1994) vanish identically; (3) the conservation law corresponding to material homogeneity (formula 3.2 in Maddocks and Dichmann, 1994) reduces to conservation of the wave momentum (see Table 3); (4) the expressions for the densities and fluxes of energy (formula 2.19 in Maddocks and Dichmann, 1994) and linear momentum (formula 2.12 in Maddocks and Dichmann, 1994) conservation laws coincide with the respective ones presented in Table 3. The set of conservation laws with characteristics $Q = u(x)$, where $u(x)$ is any solution of \( \dot{u}(x) \), as well as the conservation law associated with the variational scaling symmetry $Y_3 + (1/2)X_0$ (see Table 3) have no analogues in (Maddocks and Dichmann, 1994).

Three interesting kinds of group-invariant solutions to certain equations of the class \( \mathcal{E}_j \) are identified below. First of them corresponds to vector fields $cY_1 \mp Y_2$, where $c = \text{const}$. These group-invariant solutions are travelling waves

$$w = U(s), \quad s = x^1 \pm cx^2,$$

admissible only for equations \( \mathcal{E}_j \) with $\kappa(x^1, x^2) = f(s)$. The reduced equations determining such group-invariant solutions are

$$\gamma \frac{d^4U}{ds^4} + (\chi^{11} \pm 2\chi^{12} c + \chi^{22} c^2) \frac{d^2U}{ds^2} + f(s)U = 0.$$  

The second one corresponds to the vector field $Y_3$ and is of the form

$$w = U(s), \quad s = x^1(x^2)^{-1/2} - \frac{\chi^{12}}{\chi^{22}} (x^2)^{1/2}.$$  

The vector field $Y_3$ is admitted only if $\kappa(x^1, x^2) = (x^2)^{-2} f(s)$ (see cases # 2, 4, 6 and 10 in Table 1). The reduced equations for these invariant solutions are

$$4\gamma \frac{d^4U}{ds^4} + \chi^{22} s^2 \frac{d^2U}{ds^2} + 3\chi^{22} \frac{dU}{ds} + 4f(s)U = 0.$$  

The third kind of group-invariant solutions corresponds to the vector field $Y_4$:

$$w = U(s), \quad s = 2x^1(x^2)^{-1/3} - \frac{\chi^{11}}{\chi^{12}} (x^2)^{2/3}.$$  

The vector field $Y_4$ is admitted only if $\kappa(x^1, x^2) = (x^2)^{-4/3} f(s)$ (see cases # 3, 5, 7 and 11 in Table 1). The reduced equations for the invariant solution under consideration are

$$48\gamma \frac{d^4U}{ds^4} - 4\chi^{12} s^2 \frac{d^2U}{ds^2} - 4\chi^{12} \frac{dU}{ds} + 3f(s)U = 0.$$  

Obviously, the latter two kinds of group-invariant solutions could be reduced to self-similar solutions if $\chi^{12} = 0$ or $\chi^{11} = 0$, respectively.

6. CONCLUDING REMARKS

In this paper, Lie transformation group methods have been applied to the class of partial differential equations \( \mathcal{E}_j \). This class is of interest for structural mechanics since the governing equations of various classical plate and rod theories belong to it; the examples given in Section 2 illustrate this fact. In the context of structural mechanics, the results of the group analysis of equations \( \mathcal{E}_j \) give a number of attractive possibilities. Here, the established point Lie symmetries of \( \mathcal{E}_j \) are used to construct group-invariant solution to the governing equations of several plate and rod models, to derive conservation laws revealing important features of such models and to find transformations simplifying the differential structure of equations associated with particular plate problems.
First of all, the well known computational procedure for finding the most general point Lie symmetry group has been applied to the foregoing class of equations. As a result, the system of equations (16) – (20) is derived determining the equations of the type considered that admit a larger group together with the generators of this group; naturally, all equations of this class being linear and homogeneous admit the point Lie groups generated by the vector fields (14). The system (16) – (20) allows the associated group-classification problem to be stated and examined.

In Section 4, this problem is solved for the plate equations (38) in terms of their invariants \( s(1), s(2) \) and \( s(3) \) defined by (47). The equations of form (38) with \( s(1) \equiv s(2) \equiv s(3) \equiv 0 \) are found to admit the largest symmetry groups. It is noteworthy that each such equation with variable coefficients can be transformed, using a suitable change of variables, to an equation with constant coefficients belonging to the same class. An example of such a transformation is given at the end of Section 4 where, in addition, a class of group-invariant solutions to the equation considered is presented.

The group-classification problem for the rod equations (58) is entirely solved in Section 5. All equations of that kind admitting point Lie symmetry groups, in addition to the ones generated by (14), are determined and presented in Table 1 together with the generators of the respective groups. The largest symmetry groups are admitted by the equations of form (58) whose coefficients are such that \( \chi^{22} \det(\chi^{\alpha\beta}) = 0, \kappa(x) \equiv 0 \). The most interesting group-invariant solutions for equations (58) are identified and the corresponding reduced equations are presented at the end of Section 5.

Once the "ordinary" point Lie symmetries of an equation of form (1) are determined, one can easily find, using the general criterion (26), which of them are variational symmetries of this equation. Then, (28) and (30) provide explicit expressions for the conserved currents of the conservation laws associated through Noether's theorem with the established variational symmetries. These expressions will involve derivatives of the dependent variable of lowest possible order, which is important in view of their application in structural mechanics. The reciprocity relation valid for each equation of form (1) is given explicitly by formula (33).

In Section 4, it is shown, using the consequence (27) of the general criterion (26), that each point Lie symmetry of a plate equation of form (38) generated by a vector field of form (46) is variational symmetry of this equation. Therefore, each such symmetry gives rise to a conservation law with characteristic \( Q = (1/2)\xi_{,\mu}w - w_{,\mu}\xi^\mu \) and conserved current given by (37) admitted by the smooth solutions of the respective equation.

The conservation laws for the rod equations listed in Table 1 are given in Table 2. Inspecting these results one can see that the equations for unsupported rods and rods on Winkler foundations admit two independent conservation laws associated with the wave momentum \( D_\alpha B_{(1)}^\alpha = 0 \) and energy \( D_\alpha B_{(2)}^\alpha = 0 \). Equations (9) and (10) governing the stability of unsupported axially compressed beams and fluid conveying pipes belong to this class. Rod equations with \( \kappa(x) = 0 \) and \( \det(\chi^{\alpha\beta}) = 0 \) admit a supplementary conservation law \( D_\alpha B_{(3)}^\alpha = 0 \) associated with the infinitesimal scaling symmetry \( Y_3 \). Such an equation is (81) governing the vibration of the classic Bernoulli-Euler beam. Table 3 contains several physically important conservation laws for this equation. A comparison between the conservation laws derived here for equation (81) and the relevant results in (Chien et al., 1993) and (Maddocks and Dichmann, 1994) is presented in Section 5.

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