Improved estimators for the rate parameter of gamma model using asymptotic properties

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ABSTRACT

In this paper we proposed three estimators namely linear shrinkage, preliminary test and shrinkage preliminary test for the rate parameter of univariate gamma. The salient feature of the proposed estimators is the admissibility property that is defined on belief of the uncertain prior information. Expressions for bias and relative efficiency under method of moment have been derived using asymptotic theory. A Monte Carlo simulation study shows that the proposed estimators are more efficient and minimally biased when prior information is close to the neighbourhood of the rate parameter.

1. Introduction

In the theory of estimation, parameters describe underlying physical setting in such a way that their estimated value affects the distribution of the measured data. In many areas of statistics the main goal or objective is to estimate parameters in a model given observed data in order to explain a real physical phenomenon.

The gamma distribution plays a crucial role in the foundation of distributions within the context of mathematical statistics and many applied areas. The gamma distribution has been used in the areas such as engineering, computer science, pharmacy, forensic science etc. Over the last few decades, the gamma distribution has become one of the most important techniques for modeling life-testing situations, Chou and Huang (2003). In spite of many uses of the gamma distribution, there have been very few distributional assessment procedures developed, Chou and Huang (2004).

The two-parameter gamma distribution has been used widely in many studies including reliability studies, survival analysis, and in many applications particularly when data are complete or incomplete. Some authors (Bhunya et al. (2007)) have considered the problem of estimating parameters of the gamma distribution. One of the inferential problems is to find an estimator which is able to reveal the true parameter in a population at an economic cost in other words much efficient. In parameter estimation, efficient estimators gives us the best possible or optimal estimator of a parameter of interest. The estimator with relatively small variance is known to be efficient.

The method of moment estimation (MMEs) a known classical or standard method of estimation have been widely used in practice. The MMEs was first suggested by Karl Pearson (1894). Theorists like Ibragimov et al. (1981) have shown that the method of moment estimator (MME) is consistent, not necessarily unique, can be bias and not as efficient in most cases. The MME have been used to estimate the two-parameter gamma model. A new and modern approach known as testimators proposed by Ahmed (1989)Ahmed (1991)Ahmed (1992) Ahmed (1997)Ahmed and Khan (1993)Ahmed and Khan (1997) Ahmed and Saleh (1988)Ahmed et al. (2007)Alam et al. (2012) considers experience or prior knowledge of parameters in statistical models. These testimators have been proven to be more efficient compared to the classical estimators in many conditional and univariate models.

There have been fewer studies on proposing new estimators for the parameters of the gamma distribution in recent times. Salman et al. (2014), in their paper, proposed a preliminary test single stage shrinkage estimator for the scale parameter of the gamma distribution when the prior information about the scale parameter is available. From their numerical analysis, they showed that the suggested estimator is more efficient than the classical estimators when the prior information is close to the neighbourhood of the scale parameter. However, they considered a small sample and the implication of the efficiency is because of the use of the Maximum Likelihood estimation method. However, the small sample issue can be addressed by using estimators under the method of moment assumption.

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The main objective of this paper is to propose a linear shrinkage estimator (LSE), pretest (PE) and linear shrinkage pretest (SPE) under MMEs for rate parameter of the gamma distribution and analyze the performance for efficiency in comparison with MME using a Monte Carlo simulation technique. Specific objective is to do a comparison among the three (3) proposed estimators. The RStudio statistical software package version 1.0.136 was used for the simulation exercise.

This paper is organised as follows: In section 2, we proposed the three (3) estimators. The third section deals with the derivation of bias and Relative Efficiency (RE) of the proposed estimators under asymptotic properties. In the fourth section, we have compared the bias, RE of the estimators and MME of the rate parameter in a simulation study. We state our discussion and conclusions in section 5.

2. Methods

2.1. Definitions

Let $T_n = t(X_1, … ,X_n)$ be an estimator of a parameter $\theta$, $\theta \in \Theta \subset \mathbb{R}$ where $\Theta$ the parameter space and $\mathbb{R}$ is the real line. The bias of an estimator $T_n$ is defined as:

$$\text{Bias} = E(T_n) - \theta$$

(1)

An estimator with minimum bias is often chosen over other estimators. In general, the risk function $R(T_n, \theta)$ of an estimator $T_n$ is defined as the expected loss of the estimator $T_n$ in estimating the parameter $\theta$.

$$R(T_n, \theta) = E(L(T_n, \theta))$$

(2)

where $L(T_n, \theta) = T_n - \theta$ is the loss function which defines the amount of loss for using $T_n$ in estimating the parameter $\theta$. A good estimation strategy is the strategy that minimizes the risk function in Eq. (2). The Mean Squared Error (MSE) is a risk function, corresponding to the expected value of the squared error loss. In practice the MSE is used in place of the risk function shown in Eq. (3), the reason that the parameters are fixed. The MSE of an estimator $T_n$ is defined as:

$$\text{MSE}(T_n) = E(T_n - \theta)^2 = \text{var}(T_n) + (\text{Bias}(T_n))^2$$

(3)

The MSE is a measure of the quality of an estimator it is always non-negative, and values close to zero are better.

Relative Efficiency was used to compare different estimators. The Relative Efficiency (RE) of the new estimator $T'_n$ is defined as:

$$\text{RE}(T'_n/T_n) = \frac{\text{MSE}(T_n)}{\text{MSE}(T'_n)}$$

(4)

The condition under which the new estimator is relatively more efficient as compared to the classical estimator is RE $< 1$ in Eq. (4).

2.2. Estimation strategies under MMEs

Given that $X \sim \text{Gamma}(\alpha, \beta)$, the gamma density is presented as:

$$f(x, \alpha, \beta) = \beta^\alpha \Gamma(\alpha) x^{\alpha-1} e^{-\beta x} I_{(0,\infty)}(x)$$

(5)

where $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} \, dx$, $x > 0$, $\alpha > 0$, $\beta > 0$ and $\Gamma(\alpha) = (\alpha-1)!$ for any positive integer $\alpha$. The mean, variance and $r^{th}$ moment of gamma by Hazewinkel and Michiel (2001) are respectively given as:

$$E(X) = \frac{\alpha}{\beta} \quad \text{Var}(X) = \frac{\alpha}{\beta^2} \quad E[X^r] = \frac{(\alpha + r - 1) \cdot \alpha}{\beta} \quad \text{for } r > 1$$

Given a random sample $X_1, X_2, … ,X_n$ from the gamma population, under MMEs the $r^{th}$ population moment is equated to the $r^{th}$ sample moment:

$$E(X'|X) = \frac{1}{n} \sum_{i=1}^n X_i'$$

(6)

Over here the rate parameter $\beta$ is unknown with an assumed shape parameter $\alpha_0$. We equate the first sample moment to the first population moment of the gamma distribution under unrestricted MME to obtain Eq. (6).

$$\hat{\beta}_{\text{MME}} = \frac{\alpha_0}{X}$$

(7)

where $\sum_{i=1}^n = \frac{1}{n} \sum_{i=1}^n X_i$. Applying the theorem of approximations for mean ratio of random variables, it is shown Wiens and Beaulieu (2003), that the unrestricted estimator $\hat{\beta}_{\text{MME}}$ is unbiased:

$$E(\hat{\beta}_{\text{MME}}) = \beta$$

2.2.1. Asymptotic distribution of MME

Applying the Central Limit Theorem (CLT) to determine the convergence in distribution of the estimator $T_n$:

$$\sqrt{n}(T_n - \theta) \xrightarrow{d} N(0, \sigma^2_{T_n})$$

(7)

Lemma 1. The delta method proposed by Dorfman (1938) was used to obtain an approximate distribution of the estimator $T_n$. From the CLT in Eq. (7), given function $g$ which is continuous and differentiable, suppose that $g(\mu)$ exist and different from zero, then;

$$\sqrt{n}(g(X_n)) - g(\mu) \xrightarrow{d} N(0, \sigma^2((g'(\mu))^2))$$

where $\mu$ and $\sigma^2$ are the mean and variance respectively of the statistics $X_n$.

Theorem 1. The asymptotic distribution of $\hat{\beta}_{\text{MME}}$ by the CLT and delta method is given as

$$\sqrt{n}(\hat{\beta}_{\text{MME}} - \beta) \xrightarrow{d} N\left(0, \frac{\beta^2}{\alpha_0}\right)$$

Proof:

From Eq. (6) and Lemma 1, $g(\mu)$ is given as: $g(\mu) = \frac{\alpha_0}{\beta}$

The first derivative of $g(\mu)$ is $g'(\mu) = -\frac{\alpha_0}{\beta^2}$

The variance of $\hat{\beta}_{\text{MME}}$ by the delta method is obtained:

$$\left(\frac{\alpha_0}{\beta}\right)^2 \left(\frac{\alpha_0}{\beta^2}\right)^2 = \frac{\alpha_0^2}{\beta^2} \frac{\alpha_0}{\beta}$$

(8)

we substitute $\mu$ by $\frac{\alpha_0}{\beta}$ in Eq. (8) to obtain:

$$\left(\frac{\alpha_0}{\beta}\right)^2 \left(\frac{\alpha_0}{\beta^2}\right)^2 = \frac{\beta^2}{\alpha_0}$$

hence by delta method:

$$\sqrt{n}(\hat{\beta}_{\text{MME}} - \beta) \xrightarrow{d} N\left(0, \frac{\beta^2}{\alpha_0}\right)$$

A test statistic is constructed from normal theory as:

$$z = \frac{\hat{\beta}_{\text{MME}} - \beta}{\sqrt{\frac{\beta^2}{\alpha_0}}} \quad \text{with a chi-square approximation given as:}$$
2.2.1. Preliminary test estimation with MME. According to Mosteller (1948), pretest procedure is an alternative scheme to the hypothesis test to determine whether the prior information is appropriate to reveal the true existence of the unknown parameter. This estimator was proposed to improve the performance in terms of MSE and RE. The preliminary test estimator (PE) for the simple hypothesis is defined as:

\[ H_0 : \beta = \beta_0 \]
\[ H_1 : \beta \neq \beta_0 \]

The general form of PE in terms of MMEs of the gamma rate parameter is expressed as:

\[
\hat{\beta}_{\text{MME}} = \begin{cases} 
\hat{\beta}_0 & \text{for } \chi^2 > d_n \\
\hat{\beta}_{\text{MME}} & \text{for } \chi^2 < d_n
\end{cases}
\]  

(9)

where \( \chi^2 \) represents the test statistic in the pretest estimator, \( d_n \) is the threshold of the test statistic and \( \beta_0 \) is the uncertain prior information (UPI) of the rate parameter. We write the PE explicitly as

\[
\hat{\beta}_{\text{PE}} = \beta_0 I_{[0,\chi^2]}(\chi^2) + \hat{\beta}_{\text{MME}} I_{[\chi^2,\infty]}(\chi^2)
\]  

(10)

2.2.1.2. Linear shrinkage estimation with MME. Ahmed (1989) and Ahmed (1991) proposed both the Linear Shrinkage and Shrinkage Preliminary test estimators. By our definition, the general form of linear shrinkage estimator (LSE) in terms of MMEs for the rate parameter of the gamma distribution is given as:

\[
\hat{\beta}_{\text{LSE}} = \pi \hat{\beta}_0 + (1 - \pi) \hat{\beta}_{\text{MME}}
\]  

(11)

where \( \pi \) represents the shrinkage weight factor or shrinkage parameter which specifies the degree of belief in the UPI, \( \pi \in (0,1) \). Once past experience or knowledge of the rate parameter is known and added to the unrestricted estimator suddenly becomes biased and renders the null hypothesis more conservative.

2.2.1.3. Shrinkage preliminary test estimation with MME. The shrinkage pretest estimator (SPE) is defined by adding two parts of the information from the unrestricted estimator into the preliminary test form. From Eq. (9), the form of the linear shrinkage pretest estimator (SPE) is defined as:

\[
\hat{\beta}_{\text{SPE}} = \begin{cases} 
\hat{\beta}_{\text{LSE}} & \text{for } \chi^2 \leq d_n \\
\hat{\beta}_{\text{MME}} & \text{for } \chi^2 > d_n
\end{cases}
\]

We write the SPE explicitly as:

\[
\hat{\beta}_{\text{SPE}} = \hat{\beta}_{\text{LSE}} I_{[0,\chi^2]}(\chi^2) + \hat{\beta}_{\text{MME}} I_{[\chi^2,\infty]}(\chi^2)
\]  

(12)

2.2.2. Derivation of asymptotic bias and relative efficiency of proposed estimators

We introduce here the Lemmas by Judge and Bock (1978) to help in the proof of Theorem 2.

**Lemma 2.** If \( X \) is a normally distributed random variable with mean \( \mu \) and variance 1. Then

\[
E(\hat{\beta}_{PE}(x^2|x^2)) = \mu^2 P\left(\chi^2(3,\sigma^2) < d_n\right) + \mu^2 P\left(\chi^2(5,\sigma^2) < d_n\right)
\]

**Lemma 3.** If \( X \) is a normally distributed random variable with mean \( \mu \) and variance 1. Then

\[
E(\hat{\beta}_{PE}(x^2|x^2)) = \mu^2 P\left(\chi^2(3,\sigma^2) < d_n\right)
\]

We give the general definitions for Asymptotic Distributed Bias (ADB), Asymptotic MSE (AMSE) and Asymptotic Relative Efficiency (ARE) of estimators \( T_n, T_n^* \) as follows:

\[
ADB(T_n) = \lim_{n \to \infty} E(\sqrt{n}(T_n - \theta))
\]
\[
AMSE(T_n) = \lim_{n \to \infty} E(n(T_n - \theta)^2)
\]
\[
ARE(T_n) = \frac{AMSE(T_n)}{AMSE(T_n^*)}
\]

**Theorem 2.** The ADB of \( \hat{\beta}_{PE}, \hat{\beta}_{LSE} \) and \( \hat{\beta}_{SPE} \) are as follows:

\[
ADB(\hat{\beta}_{PE}) = -\delta^2 P\left(\chi^2(3,\sigma^2) \leq d_n\right)
\]
\[
ADB(\hat{\beta}_{LSE}) = -\pi \delta
\]
\[
ADB(\hat{\beta}_{SPE}) = -\pi \delta^2 P\left(\chi^2(3,\sigma^2) \leq d_n\right)
\]

where \( \delta = \sqrt{n}(\beta - \beta_0) \) and \( d_n \) represents the critical value threshold of the chi-square distribution.

**Proof.**

We implore theorems from Ahmed, S.E. and Khan, S.M. (1993). including regularity conditions, under local alternative,

\[
K_n : \beta_n = \beta_0 + \frac{\delta^n}{\sqrt{n}} \delta \text{ is fixed}
\]

Under the above condition, **Theorem 1** is redefined as:

\[
\sqrt{n}(\hat{\beta}_{MME} - \beta_0) \xrightarrow{d} N\left(\delta, \frac{\delta^2}{n(\beta_0)}\right)
\]

Given the test statistic from **Theorem 1**, \( D_n = \frac{n\delta^2(\hat{\beta}_{MME} - \beta_0)}{\beta^2} \)

AD of PE:
$$A DB(\hat{\beta}^PE) = \lim_{n \to \infty} \sqrt{n}(\hat{\beta}^PE - \beta)$$

$$= \lim_{n \to \infty} \sqrt{n} \left( \mathbb{E}\left(\hat{\beta}^PE \left| \chi^2_{\alpha} \leq d_n \right.\right) + \hat{\beta}^P(1 - \mathbb{I}(\chi^2_{\alpha} \leq d_n)) - \beta \right)$$

$$= \lim_{n \to \infty} \sqrt{n} \left( \mathbb{E}\left(\hat{\beta}^PE \left| \chi^2_{\alpha} \leq d_n \right.\right) + \mathbb{E}(\hat{\beta}^P - \beta \mid \chi^2_{\alpha} \leq d_n) - \beta \right)$$

$$= - \lim_{n \to \infty} \sqrt{n} \mathbb{E}\left(\hat{\beta} - \beta \mid \chi^2_{\alpha} \leq d_n \right)$$

From Lemma 2

$$AD B(\hat{\beta}^{LSE}) = - \delta \mathbb{P} \left( \chi^2_{\alpha} \leq d_n \right)$$

$$A D B(\hat{\beta}^{SPE}) = - \delta \mathbb{P} \left( \chi^2_{\alpha} \leq d_n \right)$$

### Table 1. Bias of estimators under MMEs for $\alpha = 0.005$ at a scale of $\times 10^{-3}$

| $\delta$ | $\pi$ | 0.9 | 0.5 | 0.1 |
|---------|-------|-----|-----|-----|
| $n$     | $\hat{\beta}^{LSE\,\text{MME}}$ | $\hat{\beta}^{PE\,\text{MME}}$ | $\hat{\beta}^{SPE\,\text{MME}}$ | $\hat{\beta}^{LSE\,\text{MME}}$ | $\hat{\beta}^{PE\,\text{MME}}$ | $\hat{\beta}^{SPE\,\text{MME}}$ | $\hat{\beta}^{LSE\,\text{MME}}$ | $\hat{\beta}^{PE\,\text{MME}}$ | $\hat{\beta}^{SPE\,\text{MME}}$ |
| 0.00    | 200   | 0.055 | 0.193 | 0.351 | 0.154 | 0.509 |
| 0.01    | 200   | 8.947 | 7.195 | 8.053 | 7.363 | 0.303 |
| 0.02    | 200   | 17.926 | 8.621 | 9.660 | 4.463 | 1.338 |
| 0.03    | 200   | 29.957 | 5.515 | 4.186 | 1.557 | 0.439 |
| 0.04    | 200   | 35.933 | 1.560 | 1.560 | 0.467 | 0.341 |
| 0.05    | 200   | 44.937 | 0.220 | 0.220 | 0.143 | 0.174 |
| 0.06    | 200   | 53.923 | 0.732 | 0.732 | 0.305 | 0.732 |
| 0.07    | 200   | 62.925 | 0.746 | 0.746 | 0.305 | 0.732 |
| 0.08    | 200   | 71.903 | 0.972 | 0.972 | 0.712 | 0.972 |
| 0.09    | 200   | 80.926 | 0.739 | 0.739 | 0.833 | 0.739 |

### Table 2. Bias of estimators under MMEs for $\alpha = 0.05$ at a scale of $\times 10^{-4}$

| $\delta$ | $\pi$ | 0.9 | 0.5 | 0.1 |
|---------|-------|-----|-----|-----|
| $n$     | $\hat{\beta}^{LSE\,\text{MME}}$ | $\hat{\beta}^{PE\,\text{MME}}$ | $\hat{\beta}^{SPE\,\text{MME}}$ | $\hat{\beta}^{LSE\,\text{MME}}$ | $\hat{\beta}^{PE\,\text{MME}}$ | $\hat{\beta}^{SPE\,\text{MME}}$ | $\hat{\beta}^{LSE\,\text{MME}}$ | $\hat{\beta}^{PE\,\text{MME}}$ | $\hat{\beta}^{SPE\,\text{MME}}$ |
| 0.00    | 200   | 0.55 | 4.30 | 4.83 | 4.93 | 4.17 | 5.35 |
| 0.01    | 200   | 89.47 | 41.21 | 20.55 | 5.25 | 46.37 | 0.11 |
| 0.02    | 200   | 179.27 | 31.75 | 12.20 | 13.38 | 31.75 | 3.44 |
| 0.03    | 200   | 269.57 | 6.70 | 1.82 | 26.14 | 7.93 | 3.06 |
| 0.04    | 200   | 359.34 | 5.09 | 5.78 | 34.01 | 4.91 | 6.48 |
| 0.05    | 200   | 449.37 | 6.33 | 6.33 | 44.30 | 6.33 | 6.33 |
| 0.06    | 200   | 539.23 | 7.72 | 7.72 | 53.05 | 7.72 | 7.72 |
| 0.07    | 200   | 629.25 | 7.52 | 7.52 | 63.23 | 7.52 | 7.52 |
| 0.08    | 200   | 719.03 | 9.72 | 9.72 | 71.25 | 9.72 | 9.72 |
| 0.09    | 200   | 809.26 | 7.39 | 7.39 | 83.35 | 7.39 | 7.39 |

### Table 3. Bias of estimators under MMEs for $\alpha = 0.10$ at a scale of $\times 10^{-4}$

| $\delta$ | $\pi$ | 0.9 | 0.5 | 0.1 |
|---------|-------|-----|-----|-----|
| $n$     | $\hat{\beta}^{LSE\,\text{MME}}$ | $\hat{\beta}^{PE\,\text{MME}}$ | $\hat{\beta}^{SPE\,\text{MME}}$ | $\hat{\beta}^{LSE\,\text{MME}}$ | $\hat{\beta}^{PE\,\text{MME}}$ | $\hat{\beta}^{SPE\,\text{MME}}$ | $\hat{\beta}^{LSE\,\text{MME}}$ | $\hat{\beta}^{PE\,\text{MME}}$ | $\hat{\beta}^{SPE\,\text{MME}}$ |
| 0.00    | 200   | 0.55 | 5.10 | 5.27 | 4.93 | 5.06 | 5.44 |
| 0.01    | 200   | 89.47 | 27.72 | 31.38 | 5.25 | 31.38 | 1.61 |
| 0.02    | 200   | 179.27 | 11.84 | 13.97 | 13.38 | 13.97 | 5.22 |
| 0.03    | 200   | 269.57 | 0.18 | 2.01 | 26.14 | 0.27 | 3.83 |
| 0.04    | 200   | 359.34 | 6.19 | 6.40 | 34.01 | 6.14 | 6.60 |
| 0.05    | 200   | 449.37 | 6.33 | 6.33 | 44.30 | 6.33 | 6.33 |
| 0.06    | 200   | 539.23 | 7.72 | 7.72 | 53.05 | 7.72 | 7.72 |
| 0.07    | 200   | 629.25 | 7.52 | 7.52 | 63.23 | 7.52 | 7.52 |
| 0.08    | 200   | 719.03 | 9.72 | 9.72 | 71.25 | 9.72 | 9.72 |
| 0.09    | 200   | 809.26 | 7.39 | 7.39 | 83.35 | 7.39 | 7.39 |
Figure 1. Simulation study: the left panel shows the bias and the right panel shows the relative efficiency of the proposed estimators for $\pi = 0.9, 0.5, 0.1; \alpha = 0.005$
Figure 2. Simulation study: the left panel shows the bias and the right panel shows the relative efficiency of the proposed estimators for $\pi = 0.9, 0.5, 0.1; \alpha = 0.05$. 
Figure 3. Simulation study: the left panel shows the bias and the right panel shows the relative efficiency of the proposed estimators for $\pi = 0.9, 0.5, 0.1; \sigma = 0.10$.
2.2.2.1. Relative Efficiency. One of the key objective of this paper is to compare the MSE of the proposed estimators to the MSE of the MME. The proofs of AMSE of the proposed estimators are shown in the appendices. We derive the RE as follows:

\[
\text{ARE}(\tilde{\beta} : \beta) = \frac{\sigma^2}{\beta^2 / n_{00} + \delta^2} - \frac{(2 - \pi)\sigma^2 + \sigma^2}{\beta^2 / n_{00}} \]

\[
= \frac{\sigma^2}{\beta^2 / n_{00} + \delta^2} - \frac{(2 - \pi)\sigma^2 + \sigma^2}{\beta^2 / n_{00} + \delta^2} \]

\[
= \frac{\sigma^2}{\beta^2 / n_{00} + \delta^2} - \frac{(2 - \pi)\sigma^2 + \sigma^2}{\beta^2 / n_{00} + \delta^2} \]

\[
\text{ARE}(\tilde{\beta} : \beta) = \frac{\sigma^2}{\beta^2 / n_{00} + \delta^2} - \frac{(2 - \pi)\sigma^2 + \sigma^2}{\beta^2 / n_{00} + \delta^2} \]

\[
= \frac{\sigma^2}{\beta^2 / n_{00} + \delta^2} - \frac{(2 - \pi)\sigma^2 + \sigma^2}{\beta^2 / n_{00} + \delta^2} \]

Where \( \sigma^2 \) is the asymptotic variance of the classical estimator \( \hat{\beta}_{MME} \) defined under Theorem 1. The proposed estimators performs better than the classical MME if for each ARE is greater than one (1).

3. Results of simulation study

In this results, random samples where generated from a gamma distribution with a known shape parameter value of one (1), with varying rate parameter from 0.1 to 5. The number of simulations for each were initially varied and it was observed that fifty (50) of each set of observations were adequate, since a further increase in that number did not change the result significantly. Five thousand (5000) Monte Carlo simulation samples of size two hundred (200) were drawn from this gamma distribution. The initial guess or the conjecture for the rate parameter was set \( \hat{\beta}_0 = 0.1 \), increased in steps of 0.01 for a total of 50 changing values. Following Choi and Wette (1969) who showed that the bias of MME are always positive, the sample counterpart of the bias is formulated as:

\[
\text{bias}(\hat{\beta}) = \frac{1}{s} \sum_{i=1}^{50} (\hat{\beta}_i - \beta) \]

where \( \hat{\beta}_i \) is the average value of the estimated rate parameter in each \( i \)th simulation and \( s \) represents the number of simulations. In the simulation, delta(\( \delta \)) was defined as:

\[
\delta = |\beta - \beta_0| \]

Here 50 steps are used to calculate the different delta values and eventually truncate the last few steps to keep only the first 10 steps (i.e. from \( \delta = 0 \)). The size of the test statistic \( \alpha \) was varied for each simulation that is, \( \alpha = 0.005, 0.05, 0.10 \). The shrinkage parameter, \( \alpha \) was varied for each simulation at \( \alpha = 0.9, 0.5, 0.1 \). These changes are made to observe different scenarios for the quality of statistical inference. The simulation results are shown in Tables 1, 2, and 3 for varying shrinkage weights and test sizes.

From Tables 1, 2, and 3, we observe generally how the changes in values of \( \alpha \) and \( \alpha \) affects the bias of the proposed estimators. Theoreti-
cally, these estimators are bias compared to the unrestricted estimator. For each $\alpha$ value the bias of $\hat{\beta}_{MME}^{PE}$ increases as $\delta$ increases. This is expected because theoretically the bias of the LSE grows linearly. Bias of $\hat{\beta}_{MME}^{PE}$ and $\hat{\beta}_{MME}^{SPE}$ increases to a certain $\delta$ and decrease to a bound as $\delta$ increases. It is observed that from Tables 1, 2, and 3 the biases further reduces for $\hat{\beta}_{MME}^{PE}$ and $\hat{\beta}_{MME}^{SPE}$ over increase hypothesis error. There is a similarity of bias for both $\hat{\beta}_{MME}^{PE}$ and $\hat{\beta}_{MME}^{SPE}$ for $\delta > 0.05$ overall shrinkage weights.

Figure 1(a) shows the bias function of $\hat{\beta}_{MME}^{LSE}$ increases monotonically as $\delta$ increases over all shrinkage weights. As the shrinkage weight reduces the bias of $\hat{\beta}_{MME}^{LSE}$ and $\hat{\beta}_{MME}^{SPE}$ decreases in general. The bias functions of $\hat{\beta}_{MME}^{PE}$ and $\hat{\beta}_{MME}^{SPE}$ increases over small delta values and reach a peak where the null hypothesis is rejected, while the bias estimates approaches the unbiasedness of the unrestricted estimator $\hat{\beta}_{MME}$ as delta increases. From Figure 1(b), the RE of the proposed estimators shows efficient results about the neighbourhood of $\delta = 0$. The $\hat{\beta}_{MME}^{LSE}$ performs worse than $\hat{\beta}_{MME}$ for larger values of delta. At certain delta values the RE of $\hat{\beta}_{MME}^{PE}$ and $\hat{\beta}_{MME}^{SPE}$ are below one (1) and eventually converges to one (1) for larger delta values.

Figure 2(a) shows the bias of $\hat{\beta}_{MME}^{SPE}$ gets closer to zero over increase in delta($\delta$) as $\alpha$ decreases. The bias of $\hat{\beta}_{MME}$ which does not depend on the $\alpha$ shows similar trends in Figure 1(a). From Figure 2(b) the RE of $\hat{\beta}_{MME}^{LSE}$ are smaller for shrinkage weights 0.1 compared to weights 0.5, 0.9 about the neighbourhood of delta equal to 0.

Figure 3(a) shows that the bias of $\hat{\beta}_{MME}^{PE}$ and $\hat{\beta}_{MME}^{SPE}$ converges to zero as delta($\delta$) increases. We observe generally that the bias of the proposed estimators improves significantly over increase in hypothesis error (alpha). Results of RE of the proposed estimators are presented for $\alpha = 0.005, 0.05, 0.10$ in Tables 4-6 (appendix). Simulation study have shown that maximum efficiency of all the proposed estimators relative to the unrestricted estimator $\hat{\beta}_{MME}$ occurred at $\delta = 0$ for moderate samples.

3.1. Application

In recent years, roundabouts have gradually gained great popularity worldwide as they represent a type of intersection control without traffic signals which, by making use of a circular geometric layout, establishes a self-regulated intersection control system. According to the Highway Capacity manual, sixth edition, the UPI for minimum critical gap acceptance is 4.1s. A data on critical gap of four exits of a roundabout (Ashanti region, Ghana) was used as an application.

In Figure 4, the gamma model was fitted to the critical gap acceptance data with a nonsignificant Kolmogorov-Smirnov test (1.021, $p = 0.714$) indicating a good fit. A 5% hypothesis error with a shrinkage weight of 0.9 was used to estimate the critical gap using the three proposed estimators and the unrestricted MME.

From Table 7, generally the estimates are consistent however, the standard errors (s.e) of the proposed estimators are relatively small compared to the MME. Thus, we have demonstrated the efficiency gained using the proposed estimators under the MME assumption.

### Table 7. Critical gap estimation.

| Exit Route          | MME | LSE | PE | SPE |
|---------------------|-----|-----|----|-----|
| Santasi-Ahodwo      | 3.59(1.232) | 4.05(0.0123) | 3.59(0.1010) | 3.59(0.0999) |
| Santasi-Patasi      | 4.41(3.386) | 4.13(0.0339) | 4.41(0.0541) | 4.41(0.0455) |
| Santasi-Santasi     | 4.91(2.496) | 4.18(0.0250) | 4.91(0.0372) | 4.91(0.0297) |
| Santasi-Bekwai      | 4.09(1.277) | 4.10(0.0128) | 4.10(0.4571) | 4.09(0.5965) |

4. Discussion and conclusion

The three proposed estimators were developed with the classical estimation MME. Under the MME strategy, it was assumed that the shape parameter of the two-parameter gamma distribution is known. Formulation of the estimators depended on the uncertain prior information (UPI). The UPI may be derived from past experience, or in a typical conjecture. The asymptotic behaviour of the proposed estimators with MME very well rest on the asymptotic normality assumptions of MME. Simulation study was carried out to investigate the performance of the proposed estimators’ based on Bias and RE. The results of bias with MME shown in Tables 1, 2, and 3. We observe that bias of LSE increases over larger deviations away from the true parameter. Bias function of both PE and SPE start from delta of zero(0) increases to a point, then decreases gradually to zero(0). The decrease is due to the rejection of the null hypothesis. In light of the above discussion, none of the three estimators LSE, PE and SPE is uniformly superior with respect to the other. Generally, the propose estimators are efficient compared to the classical MME around the neighborhood of delta value of zero(0).

In conclusion, we found that the proposed estimators PE and SPE are minimally bias when the uncertain prior information is rejected. However, there is efficiency gain for all proposed estimators when the difference between the true rate parameter and the uncertain prior information are so close. Therefore, we conclude that even though the unrestricted method of moment estimator is unbiased in the estimation of the rate parameter for a known shape, there is maximum efficiency gain for the proposed estimators at $\delta = 0$ with minimal bias.

Declarations

**Author contribution statement**

N. K. Frempong: Conceived and designed the experiments; Performed the experiments; Analyzed and interpreted the data; Contributed reagents, materials, analysis tools or data; Wrote the paper.

F. K. Bukari: Analyzed and interpreted the data; Contributed reagents, materials, analysis tools or data; Wrote the paper.

I. K. Dontwi: Analyzed and interpreted the data; Wrote the paper.

R. K. Avuglah: Conceived and designed the experiments; Performed the experiments; Wrote the paper.

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**Data availability statement**

The authors do not have permission to share data.
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The authors declare no conflict of interest.

Additional information

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References

Ahmed, S.E., 1989. “Estimation theory under uncertain prior information” Pakistan. J. Stat. 5, 211–228.
Ahmed, S.E., 1991. To Pool or Not Pool: the Discrete Data. North-Holland, Statistics and Probability Letters II, pp. 233–237.
Ahmed, S.E., 1992. Shrinkage preliminary test estimation in multivariate normal distributions. J. Stat. Comput. Simulat. 43, 177–195.
Ahmed, S.E., 1997. Asymptotic shrinkage estimation. The regression case. Appl. Stat. Sci. II, 113–139.
Ahmed, S.E., Khan, S.M., 1993. Improved estimation of the Poisson parameter. Statistica, anno L11 n. 2, 265–277.
Ahmed, S.E., Khan, S.M., 1997. Shrinkage estimation in randomized response model. InterStat (September issue), 1–20.
Ahmed, S.E., Saleh, E., 1988. Estimation strategy using a preliminary test in some univariate normal models. Soochow J. Mathem. 114, 135–165.
Ahmed, S.E., Doksum, K.A., Hossain, S., Yon, J., 2007. Shrinkage, pretest and absolute penalty estimators in partially linear models. Aust. N. Z. J. Stat. 49, 435–454.

Alam, A.T.M.J., Rahman, M.S., Saadat, A.H.M., Hug, M.M., 2012. Gamma distribution and its application of spatially monitoring meteorological drought in barind, Bangladesh. J. Environ. Sci. and Natural Resources 5 (2), 287–293.
Bhunya, P.K., Berndsson, R., Ojha, C.S.P., Mishra, S.K., 2007. Suitability of gamma, chi-square, Weibull, and beta distributions as synthetic unit hydrographs. J. Hydrol. 334, 28–36.
Choi, S., Wette, R., 1969. Maximum likelihood estimation of the parameters of the gamma distribution and their bias. Technometrics 11 (4), 683–690.
Chou, C.W., Huang, W.J., 2003. Characterizations of the gamma distribution via conditional moments. Sankhya 65, 271–283.
Chou, Chao-Wei, Huang, Wen-Jang, 2004. On characterizations of the gamma and generalized inverse Gaussian distributions. Stat. Probabil. Letter. 69 (4), 381–388.
Dorfman, R., 1938. A Note on the Method for finding variance formulae. Biometric Bulletin 1, 129–137.
Gamma-distribution. In: Hazewinkel and Michiel (Ed.), 2001. Encyclopedia of Mathematics. Springer.
Ibragimov, I.A., Has’minskii, R.Z., Kotz, S., 1981. Statistical Estimation: Asymptotic Theory. Springer-Verlag, New York, 1981, vol. 2.
Judge, G.G., Bock, M.E., 1978. The Statistical Implications of Pre-test and Stein-Rule Estimators in Econometrics. North-Holland, Amsterdam.
Mosteller, F., 1948. On pooling data. Journal of American Statistics Association (43), 231–242.
Pearson, K., 1894. Science and Monte Carlo. Fortn. Rev. New Series (55), 183–193.
Salman, A.N., Ali, A.H., Salman, M.D., 2014. Preliminary test single stage shrinkage estimator for the scale parameter of gamma distribution. Am. J. Mathem. Stat. 4 (3), 131–136.
Wien, D.P., Cheng, Beaulieu, N.C., 2003. A class of methods of moments estimators for the two-parameter gamma family. Pak.J.Statistics 19, 129–141.