ELEMENTARY PROOF OF CONGRUENCES MODULO 25 FOR BROKEN $k$-DIAMOND PARTITIONS

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Abstract. Let $\Delta_k(n)$ denote the number of $k$-broken diamond partitions of $n$. Quite recently, the second author proved an infinite family of congruences modulo 25 for $\Delta_k(n)$ with the help of modular forms. In this paper, we aim to provide an elementary proof of this result.

1. Introduction

The notion of broken $k$-diamond partitions was introduced by Andrews and Paule [1] in 2007. They showed that the generating function of $\Delta_k(n)$, the number of broken $k$-diamond partitions of $n$, is given by

$$\sum_{n=0}^{\infty} \Delta_k(n) q^n = \frac{(q^2; q^2)_\infty (q^{2k+1}; q^{2k+1})_\infty}{(q; q)_\infty^3 (q^{2(2k+1)}; q^{2(2k+1)})_\infty}.$$

Throughout this paper, we assume that $|q| < 1$ and adopt the customary $q$-series notation:

$$(a; q)_\infty := \prod_{n=0}^{\infty} (1 - aq^n).$$

The following two congruences modulo 5 were obtained by S.H. Chan [5] and later re-discovered by Radu [14]:

$$\Delta_2(25n + 14) \equiv 0 \pmod{5},$$
$$\Delta_2(25n + 24) \equiv 0 \pmod{5}.$$

In fact, Chan extended these congruences to

$$\Delta_2 \left( 5^{\alpha + 1}n + \frac{11 \times 5^\alpha + 1}{4} \right) \equiv 0 \pmod{5}, \quad (1.1)$$
$$\Delta_2 \left( 5^{\alpha + 1}n + \frac{19 \times 5^\alpha + 1}{4} \right) \equiv 0 \pmod{5}. \quad (1.2)$$

Hirschhorn [8] subsequently gave simple proofs of (1.1) and (1.2).

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Furthermore, other infinite families of congruences modulo 5 satisfied by $\Delta_2(n)$ have been discovered by many authors. The interested readers may refer to Radu [14] and Xia [16].

Quite recently, with the help of modular forms, the second author [15, Theorem 2] proved the following infinite family of congruences modulo 25 for $\Delta_k(n)$.

**Theorem 1.1.** For all $n \geq 0$,

$$\Delta_k(125n + 99) \equiv 0 \pmod{25}, \quad \text{if} \quad k \equiv 62 \pmod{125}. \quad (1.3)$$

Our main purpose of this paper is to provide an elementary proof of Theorem 1.1.

2. Preliminaries

For notational convenience, we denote

$$E_j = (q^j; q^j)^\infty.$$

We also write

$$R(q) := \frac{(q; q_5^\infty)(q^4; q^5_5)^\infty}{(q^2_5; q^5_5)(q^3; q^5_5)^\infty}.$$

From [7, Eq. (8.4.4)], one has the following 5-dissection identity.

**Lemma 2.1.**

$$\frac{1}{E_1} = \frac{E_5}{E_5^{25}} \left( \frac{1}{R(q^5)^4} + \frac{q}{R(q^5)^3} + \frac{2q^2}{R(q^5)^2} + \frac{3q^3}{R(q^5)} + 5q^4 
- 3q^5 R(q^5) + 2q^6 R(q^5)^2 - q^7 R(q^5)^3 + q^8 R(q^5)^4 \right). \quad (2.1)$$

We now absorb the ideas of [2] with some refinements. The first ingredient from [2] is the following three relations.

**Lemma 2.2 (Lemma 1.3, [2]).** Let $x = \frac{1}{R(q)}$ and $y = \frac{1}{R(q^2)}$, then

$$\frac{x^2}{y} - \frac{y}{x^2} = \frac{4q}{K}, \quad (2.2)$$

$$xy^2 - \frac{q^2}{xy^2} = K, \quad (2.3)$$

$$\frac{y^3}{x} + \frac{q^2 x}{y^3} = K - 2q + \frac{4q^2}{K}, \quad (2.4)$$

where $K = \frac{E_2 E_5^5}{E_1 E_5^{10}}$. 


Now, for $\alpha \in \mathbb{Z}_{\geq 0}$ and $\beta \in \mathbb{Z}$, we define

$$P(\alpha, \beta) := x^{\alpha+2\beta} y^{2\alpha-\beta} + \frac{(-1)^{\alpha+\beta} q^{2\alpha}}{x^{\alpha+2\beta} y^{2\alpha-\beta}}.$$  \hspace{2cm} (2.5)

It is not hard to observe that

$$P(0, 0) = 2, \hspace{2cm} (2.6)$$

$$P(0, 1) = \frac{x^2}{y} - \frac{y}{x^2} = \frac{4q}{K}, \hspace{2cm} (2.7)$$

$$P(1, 0) = xy^2 - \frac{q^2}{xy^2} = K, \hspace{2cm} (2.8)$$

$$P(1, -1) = \frac{y^3}{x} + \frac{q^2 x}{y^3} = K - 2q + \frac{4q^2}{K}. \hspace{2cm} (2.9)$$

With the help of the following recurrence relations along with the initial conditions (2.6)–(2.9), one may easily express $P(\alpha, \beta)$ in terms of $K$ and $q$ for arbitrary $\alpha \in \mathbb{Z}_{\geq 0}$ and $\beta \in \mathbb{Z}$.

**Lemma 2.3.** For $\alpha \in \mathbb{Z}_{\geq 0}$ and $\beta \in \mathbb{Z}$,

$$P(\alpha, \beta + 1) = \frac{4q}{K} P(\alpha, \beta) + P(\alpha, \beta - 1), \hspace{2cm} (2.10)$$

$$P(\alpha + 2, 0) = KP(\alpha + 1, 0) + q^2 P(\alpha, 0), \hspace{2cm} (2.11)$$

$$P(\alpha + 2, -1) = \left( K - 2q + \frac{4q^2}{K} \right) P(\alpha + 1, 0) - q^2 P(\alpha, 1). \hspace{2cm} (2.12)$$

**Proof.** We first notice that

$$P(\alpha, \beta) P(0, 1) = \left( x^{\alpha+2\beta} y^{2\alpha-\beta} + \frac{(-1)^{\alpha+\beta} q^{2\alpha}}{x^{\alpha+2\beta} y^{2\alpha-\beta}} \right) \left( \frac{x^2}{y} - \frac{y}{x^2} \right)$$

$$= \left( x^{\alpha+2(\beta+1)} y^{2\alpha-(\beta+1)} + \frac{(-1)^{\alpha+(\beta+1)} q^{2\alpha}}{x^{\alpha+2(\beta+1)} y^{2\alpha-(\beta+1)}} \right)$$

$$- \left( x^{\alpha+2(\beta-1)} y^{2\alpha-(\beta-1)} + \frac{(-1)^{\alpha+(\beta-1)} q^{2\alpha}}{x^{\alpha+2(\beta-1)} y^{2\alpha-(\beta-1)}} \right)$$

$$= P(\alpha, \beta + 1) - P(\alpha, \beta - 1).$$

This gives (2.10).

Next, it follows from (2.5) and (2.8) that

$$P(\alpha + 1, 0) P(1, 0) = \left( x^{\alpha+1} y^{2(\alpha+1)} + \frac{(-1)^{\alpha+1} q^{2(\alpha+1)}}{x^{\alpha+1} y^{2(\alpha+1)}} \right) \left( xy^2 - \frac{q^2}{xy^2} \right)$$

$$= \left( x^{\alpha+2} y^{2(\alpha+2)} + \frac{(-1)^{\alpha+2} q^{2(\alpha+2)}}{x^{\alpha+2} y^{2(\alpha+2)}} \right) - q^2 \left( x^\alpha y^{2\alpha} + \frac{(-1)^\alpha q^{2\alpha}}{x^\alpha y^{2\alpha}} \right)$$

$$= P(\alpha + 2, 0) - q^2 P(\alpha, 0),$$
which is equivalent to (2.11).

At last, we have
\[
P(\alpha + 1, 0)P(1, -1) = \left( x^{\alpha+1}y^{2(\alpha+1)} + \frac{(-1)^{\alpha+1}q^{2(\alpha+1)}}{x^{\alpha+1}y^{2(\alpha+1)}} \right) \left( \frac{y^3}{x} + \frac{q^2x}{y^3} \right)
\]
\[
= \left( x^{\alpha}y^{2\alpha+5} + \frac{(-1)^{\alpha+1}q^{2\alpha+4}}{x^{\alpha}y^{2\alpha+5}} \right) + q^2 \left( x^{\alpha+2}y^{2\alpha-1} + \frac{(-1)^{\alpha+1}q^{2\alpha}}{x^{\alpha+2}y^{2\alpha-1}} \right)
\]
\[
= P(\alpha + 2, -1) + q^2P(\alpha, 1).
\]

This yields (2.12). \qed

The other ingredient we request from [2] states as follows:

Lemma 2.4. If \( K = \frac{E_2E_5^2}{E_1E_{10}^5} \), then
\[
K - 3q - \frac{4q^2}{K} = \frac{E_1^2E_2^2}{E_5^2E_{10}^2}.
\]

Proof. Eq. (2.13) is an immediate consequence of [2, Eqs. (1.7) and (2.2)]. \qed

3. Proof of Theorem 1.1

Let
\[
\sum_{n=0}^{\infty} c(n)q^n = \frac{E_2}{E_1^3}.
\]

We shall show

Theorem 3.1. For any integer \( n \geq 0 \),
\[
c(125n + 99) \equiv 0 \pmod{25}.
\]

One readily sees that Theorem 1.1 is a direct consequence of (3.1) since if \( k \equiv 62 \pmod{125} \), then \( 2k + 1 \) is a multiple of 125.

3.1. The first 5-dissection: Identities of Baruan and Begum. According to [2, Eqs. (2.8) and (2.9)], we have
\[
\sum_{n=0}^{\infty} c(5n + 4)q^n = 41 \frac{E_3^{10}}{E_1^4E_2^2E_5^2} + 860q \frac{E_6^{10}}{E_1^5E_2E_5^2} + 6800q^2 \frac{E_9^{10}}{E_1^8E_3^2}
\]
\[
+ 24000q^3 \frac{E_2E_4^{12}}{E_1^{11}E_5^3} + 32000q^4 \frac{E_2^2E_5^{15}}{E_1^{14}E_6^5}.
\]

Notice that the coefficients 6800, 24000 and 32000 are multiples of 25. Moreover,
\[
860q \frac{E_6^{10}}{E_1^5E_2E_5^2} \equiv 10q \frac{E_6^{10}}{E_2E_5^2} \pmod{25},
\]
which contains no terms of the form \( q^{5n+4} \).
Let
\[ \sum_{n=0}^{\infty} \tilde{c}(n)q^n = \frac{E_3^3}{E_1^3 E_2^2 E_4}. \]  
(3.3)

To show (3.1), it suffices to prove that
\[ \tilde{c}(25n + 19) \equiv 0 \pmod{25}. \]  
(3.4)

### 3.2. The second 5-dissection: Cubic partition pairs.

We now observe that
\[ \sum_{n=0}^{\infty} b(n)q^n \]  
(3.5)
can be treated as the generating function of cubic partition pairs (cf. [18]). In particular, Zhao and Zhong [18] proved that
\[ b(5n + 4) \equiv 0 \pmod{5}. \]  
(3.6)

For other interesting arithmetic properties of \( b(n) \), we refer to [6, 9, 11–13, 19].

The following dissection identity will play an important role.

**Lemma 3.2.** We have
\[ \sum_{n=0}^{\infty} b(5n + 4)q^n = \frac{E_5^{10} E_6^{10}}{E_5^{12} E_6^{12}} \cdot \left( \frac{1}{R(q^5)} + \frac{q^2}{R(q^5)^3} + \frac{2q^4}{R(q^5)^2} + \frac{3q^6}{R(q^5)} + 5q^4 \right)^2 \]
\[ \times \left( \frac{1}{R(q^{10})^4} + \frac{q^2}{R(q^{10})^3} + \frac{2q^4}{R(q^{10})^2} + \frac{3q^6}{R(q^{10})} + 5q^8 \right)^2 \]
\[ \times \left( \frac{1}{R(q^{10})^4} + \frac{q^2}{R(q^{10})^3} + \frac{2q^4}{R(q^{10})^2} + \frac{3q^6}{R(q^{10})} + 5q^8 \right)^2. \]

Extracting terms of the form \( q^{5n+4} \), dividing by \( q^4 \) and replacing \( q^5 \) by \( q \) yields
\[ \sum_{n=0}^{\infty} b(5n + 4)q^n = \frac{E_5^{10} E_6^{10}}{E_5^{12} E_6^{12}} \left( 5 \left( x^8 y^6 + \frac{q^5}{x^8 y^6} \right) + 10 \left( x^6 y^7 - \frac{q^8}{x^6 y^7} \right) \right) \]
where $x$ and $y$ are as defined in Lemma 2.
In view of (2.5), we may rewrite the above identity as

$$
\frac{E_{12}^{12} E_{21}^{12}}{E_{10}^{0} E_{10}^{10}} \sum_{n=0}^{\infty} b(5n+4)q^n = \left(5P(4, 2) + 10P(4, 1) + 20P(4, 0)\right)
+ q\left(40P(3, 2) + 100P(3, 1) + 80P(3, 0) + 40P(3, -1) - 20P(3, -2)\right)
+ q^2\left(20P(2, 3) + 135P(2, 2) + 320P(2, 1) + 540P(2, 0) + 150P(2, -1)
+ 135P(2, -2) + 40P(2, -3) + 5P(2, -4)\right)
+ q^3\left(-40P(1, 3) + 150P(1, 2) + 320P(1, 1) + 540P(1, 0) - 320P(1, -1)
- 320P(1, -2) - 100P(1, -3) - 10P(1, -4)\right)
+ q^4\left(20P(0, 4) - 80P(0, 3) + 540P(0, 2) - 540P(0, 1) + 225\right).
$$
Using Lemma 2.3 and the initial conditions (2.6)–(2.9) to express each summand $P(\cdot, \cdot)$ in terms of $K$ and $q$, we may further simplify the above identity as

$$
\frac{E_{12}^{12}E_{10}^{12}}{E_{5}^{10}E_{10}^{10}} \sum_{n=0}^{\infty} b(5n + 4)q^n = 35K^4 + 280K^3q + 1905K^2q^2 + 1760Kq^3 + 13825q^4
$$

$$
- \frac{7040q^5}{K} + \frac{30480q^6}{K^2} - \frac{17920q^7}{K^3} + \frac{8960q^8}{K^4}
$$

$$
= 35 \left(K - 3q - \frac{4q^2}{K}\right)^4 + 700q \left(K - 3q - \frac{4q^2}{K}\right)^3
$$

$$
+ 6875q^2 \left(K - 3q - \frac{4q^2}{K}\right)^2 + 31250q^3 \left(K - 3q - \frac{4q^2}{K}\right) + 78125q^4
$$

$$
= 35 \frac{E_8^8E_2^8}{E_5^5E_{10}^{10}} + 700q \frac{E_8^6E_6^6}{E_5^5E_{10}^{10}} + 6875q^2 \frac{E_5^4E_4^4}{E_5^5E_{10}^{10}} + 31250q^3 \frac{E_2^2E_2^2}{E_5^5E_{10}^{10}} + 78125q^4,
$$

where we use (2.13) in the last identity. Lemma 3.2 follows readily. \qed

**Remark 3.3.** It is easy to see that (3.6) is a direct consequence of Lemma 3.2.

**Remark 3.4.** Let $a(n)$ count the number of cubic partitions of $n$, which were introduced by H.C. Chan [3]. Its generating function is

$$
\sum_{n=0}^{\infty} a(n)q^n = \frac{1}{E_1E_2}.
$$

Using modular forms, H.H. Chan and Toh [4] and independently Xiong [17] found an infinite family of congruences modulo powers of 5 for $a(n)$. We notice that, by similar arguments to the proof of Lemma 3.2, it is not hard to prove that

$$
\sum_{n=0}^{\infty} a(5n + 2)q^n = 3 \frac{E_5E_{10}^{10}}{E_1^3E_2^2} + 25q \frac{E_5^3E_{10}^{10}}{E_1^4E_2^4} + 125q^2 \frac{E_5^6E_{10}^{10}}{E_1^7E_2^6}.
$$

This gives an elementary proof of an identity due to Xiong [17]. We also learnt from Michael Hirschhorn that he was able to give an completely elementary proof [10] of the infinite family of congruences modulo powers of 5 for $a(n)$ obtained by H.H. Chan and Toh as well as Xiong.

In view of (3.3) and (3.7), we have

$$
\sum_{n=0}^{\infty} \tilde{c}(5n + 4)q^n = 35 \frac{E_8^2E_{10}^{10}}{E_1^3E_2^2} + 700q \frac{E_8^4E_{10}^{10}}{E_1^4E_2^4} + 6875q^2 \frac{E_8^6E_{10}^{10}}{E_1^7E_2^6}
$$

$$
+ 31250q^3 \frac{E_8^8E_{10}^{10}}{E_1^{11}E_2^2} + 78125q^4 \frac{E_8^{10}E_8^{10}}{E_1^{13}E_2^2}.
$$

(3.9)
3.3. **The final punch.** Now (3.4) is almost trivial from (3.9). We have

\[
\sum_{n=0}^{\infty} \tilde{c}(5n + 4)q^n \equiv 10 \frac{E_3^2 E_2^2}{E_1^2 E_2} \equiv 10 \frac{E_3 E_2}{E_2} \pmod{25},
\]

which contains no terms of the form \(q^{5n+3}\). We therefore arrive at (3.4). Consequently, we have

\[
c(125n + 99) \equiv 0 \pmod{25},
\]

and hence complete the proof of Theorem 3.1.

4. **Closing remarks**

In light of (3.2), one has

\[
\sum_{n=0}^{\infty} c(5n + 4)q^n \equiv \frac{E_3^3}{E_1^2 E_2 E_5^2} \equiv \frac{E_3 E_2}{E_5^2} \pmod{5}. \quad (4.1)
\]

It follows from [15, Eq. (9)] that

\[
c(25n + 24) \equiv 0 \pmod{5},
\]

which is equivalent to

\[
\Delta_k(25n + 24) \equiv 0 \pmod{5}, \quad \text{if} \quad k \equiv 12 \pmod{25}.
\]

This is discovered by the second author [15, Theorem 1].

Moreover, we have

\[
\sum_{n=0}^{\infty} \Delta_2(5n + 4)q^n = 41 \frac{E_{10}^3}{E_1^2 E_2^3 E_5} + 860q^2 \frac{E_{10}^6}{E_1^4 E_2^2 E_5^2} + 6800q^3 \frac{E_{10}^9}{E_1^7 E_2 E_5^3}
\]

\[
+ 24000q^4 \frac{E_{10}^{12}}{E_1^{12} E_5^5} + 32000q^4 \frac{E_2 E_{10}^{15}}{E_1^{15} E_5^5},
\]

from which it follows that

\[
\sum_{n=0}^{\infty} \Delta_2(5n + 4)q^n \equiv 41 \frac{E_{10}^3}{E_1^2 E_2^3 E_5} = 41 \frac{E_2^2 E_{10}^2}{E_1 E_2^2 E_5} \equiv \psi(q) \psi(q^5) \pmod{5},
\]

where \(\psi(q) := \sum_{n=0}^{\infty} q^{n(n+1)/2}\) is one of Ramanujan’s classical theta functions. This is first proved by Hirschhorn [8, Eq. (2.9)].

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