A Free Energy Foundation of Semantic Similarity in Automata and Languages

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Abstract. This paper develops a free energy theory from physics including the variational principles for automata and languages and also provides algorithms to compute the energy as well as efficient algorithms for estimating the nondeterminism in a nondeterministic finite automaton. This theory is then used as a foundation to define a semantic similarity metric for automata and languages. Since automata are a fundamental model for all modern programs while languages are a fundamental model for the programs’ behaviors, we believe that the theory and the metric developed in this paper can be further used for real-world programs as well.

1 Introduction

Semantic similarity between two software systems plays a central role in studying software evolution, software plagiarism and in a more general context of semantic mining of an executable object. Clearly, a syntactic metric of such similarity defined on the source codes of two programs (i.e., the source codes look similar) is far from being enough to catch the semantic similarity. The reasons are not hard to see: plagiarized software can have a completely different “look” even though its semantics is not much modified from the original true copy. We shall also notice that the semantic metric that we are looking for shall be an invariant on the dynamic behaviors (instead of the source codes) of the software systems.

Automata are a fundamental model for all modern software systems while languages (sets of words) are a model for the requirements (i.e., behaviors specifications as sequences or words of events) of the systems. Hence, it is meaningful to study the semantic similarity metric from the fundamental; e.g., finite automata and regular languages. Such studies may provide a hint of inspiration for studying more general software systems and more importantly, finite automata themselves are useful as well in software design (i.e., statecharts[15]).

We now take a new and physical view on a run of a finite-state program (i.e., a finite automaton $M$). When $M$ runs, it receives input symbols while each input symbols “drives” $M$ to transit from the current state to the next. We now imagine the automaton as a gas molecule while each state that the run passes resembles an observation of the molecule’s physical position, speed, etc. Hence, a
run of $M$ corresponds to an observation sequence, called a microstate in physics, of the molecule. Clearly, the semantic similarity metric that we look for must rely on an invariant on the runs of an automaton $M$. This invariant can be found in the thermodynamic formalism that provides a mathematical structure to rigorously deduce, in a many-particle system (e.g. gas), from a microscopic behavior to a macroscopic characteristic \cite{23,24,29,14}. Even though a microstate is highly random, the formalism resolves the challenge of how the highly random microstates in the system demonstrate almost stable macroscopic characteristic such as free energy. Returning back to the automaton, the free energy computed can be treated as an invariant on the runs (i.e., dynamic behaviors) of the automaton $M$. Notice that the free energy is measured on an equilibrium; being interpreted in a software system, a program’s long-term behavior is still highly dynamic and even random, however the dynamics itself is invariant! This thermodynamic view of software systems provides the physical foundation of our semantic similarity metric. However, in order to build up the foundation, we need first develop a free energy theory for automata and languages, and the semantic similarity metric is then a direct and natural by-product of the theory. Notice that the purpose of developing the theory is not limited to semantic similarity; it shall be considered part of a new development in the traditional automata theory that is of more than 60 years of history. We shall expect a wider range of applications of the theory itself, in addition to the similarity metric.

**Related work.** Delvenne \cite{11} and Koslicki and Thompson \cite{18,17} are among the first works to directly use the thermodynamic formalism in Computer Science. In particular, Delvenne \cite{11} computes page rank, using free energy on a random walk on a graph, for web search while a “missing link” from a page to another is present. This work inspires us to study the free energy of finite automata. Koslicki and Thompson \cite{18,17} study topological pressure of a DNA sequence using a thermodynamic formalism, where an interested pattern on a DNA sequence is given a weight. This work inspires us to study free energy for formal languages (sets of words). Shannon’s entropy has widely been considered to have a root in thermodynamics, e.g., Boltzmann equation. In fact, the notion of entropy rate or information rate, originally by Shannon \cite{26} and later by Chomsky and Miller \cite{3}, that computes the information quantity in a string (word), can be treated as a special case of free energy or topological pressure in thermodynamic formalism, as pioneered in Ruelle \cite{23}, Walters \cite{29}, Gurevich \cite{14}, Sarig \cite{24}, etc. Recently, the classical formalism of Shannon’s information rate has been used by the authors in software analysis, testing, and program security analysis \cite{8,20,0,10,16}. In particular, our previous paper \cite{5} on the information rate of a random walk of a graph has been recently used by Naval, Laxmi, Rajarajan, Gaur and Conti in malware detection \cite{21}. We are confident that thermodynamic formalism can find its own applications in various areas of computer science, such as similarity in programs and graphs \cite{6,21,10,12,27,30}. In particular, our previous work on program similarity \cite{6} is based on information rate and the Jaccard index.
The rest of the paper is organized as follows. We first briefly introduce an existing result in thermodynamic formalism, the variational principle [23], in its simplest form in physics. Then, we develop a free energy theory for automata and languages, and, in various cases, provide algorithms to compute the quantity. Finally, we provide a semantic similarity metric based on the free energy theory.

2 A free energy theory of automata and languages

In Computer Science, a program’s semantics can be interpreted as the set of all of its behaviors. In various settings, such a behavior can be simply a run (such as a sequence of state transitions), or a sequence of I/O events. Bearing thermodynamic formalism in mind, we understand the sequence as a microstate of the software system and ask a question:

Given the fact that there are so many (even infinite) microstates in a program, and these microstates are highly “probabilistic” or random, can we use thermodynamic formalism to help us understand some of the program’s macroscopic properties that are stable and free of randomness?

In many ways, some grand challenges in Computer Science have already partially addresses the question. For instance, it is well-known that unit testing (roughly speaking, trying to understand part of an individual microstate) is nowadays relatively easy and can be done automatically in many cases [13]. However, system testing (roughly speaking, trying to understand a global or macroscopic property of the system under test) is very difficult [1]. In the research of P systems [22], an evolution trace of a P system can be considered as a microstate. However, a global property of such a highly dynamic system (which is Turing-complete) can be very difficult to deduce and even be undecidable. Model-checking [4,28] provides algorithms to verify a global property of a finite state transition system through state exploration (or, roughly speaking, microstate exhaustive search), but model-checking theories themselves also show how difficult verification can be because of, for instance, the state-explosion problem [4] in a system of concurrent components.

The answer to the question above is yes but it needs some work. The set of behaviors is a language and we now consider a language $L \subseteq \Sigma^*$ on a finite and nonempty alphabet $\Sigma$ and a function $\psi : \Sigma^* \rightarrow \mathbb{R}$. In this paper, we consider a simplest form of $\psi$: for each $x_0x_1x_2\cdots \in \Sigma^*$, $\psi(x_0x_1x_2\cdots) = U(x_0, x_1)$. Herein, $U : \Sigma \times \Sigma \rightarrow \mathbb{R}$ is a given cost function over words of length 2. The intended purpose of the cost function is to assign a cost to a pattern in a word while the cost can be interpreted as, in a practical setting, an amount of a resource, a priority level, etc.

The cost is abstracted from “potential” in the thermodynamic formalism, which we briefly explain now. An infinite word $\underline{x} = x_0x_1x_2\cdots \in \Sigma^\omega$ is a microstate in thermodynamics. The microstate evolves into $\sigma(\underline{x}) = x_1x_2\cdots$, $\sigma^2(\underline{x}) = x_2\cdots$, $\cdots$, $\sigma^n(\underline{x}) = x_nx_{n+1}\cdots$, etc., as discrete time $n$ evolves. Herein, $\sigma$ is the shift-to-left operator defined in an obvious way. The evolution from $\underline{x}$ to
\(\sigma(x)\) is to break off the first symbol \(x_0\) in \(x\) from the rest. The break-off needs energy which is given by a pre-defined potential \(\psi(x)\). This explanation comes from Sarig’s lecture notes \[25\] and we think it is a best way to illustrate the physics behind the mathematics. Hence, for an \(n\)-step evolution, the total energy needed or the total potential possessed is \((S^n\psi)(x) = \text{def} \sum_{0 \leq i \leq n-1} \psi(\sigma^i(x))\), where \(\sigma^0(x) = x\).

In automata theory, this can also be analogously understood as follows. When an infinite word \(x = x_0x_1x_2 \cdots\) is read (symbol by symbol, from left to the right), the reader reads the first symbol \(x_0\), then the second symbol \(x_1\), etc. Each such symbol-read consumes energy since essentially, it performs a shift-to-left operation in the view of thermodynamics.

In thermodynamics, a particle has high energy if it tends to evolve into one of many choices of different microstates. Similarly, a microstate is of high energy if it is the result of being chosen from many microstates. This is characterized by the Boltzmann equation \(E = k_B \ln W\), where \(W\) is the number of choices, \(k_B\) is the Boltzmann constant, and \(E\) is the Boltzmann entropy. Hence, \(e^{(S^n\psi)(\bar{x})}\) now corresponds to the number of choices for the \(n\)-step evolution of the microstate \(\bar{x}\) (while ignoring the Boltzmann constant). Putting a summation over all \(\bar{x}\), we have the total number of “choices” for all microstates evolved for \(n\)-steps, \(\sum_{\bar{x}} e^{(S^n\psi)(\bar{x})}\). But this formula, in general, has no reason to give a finite value since there are simply an uncountable number of infinite sequences \(\bar{x}\). There are many ways to fix this. One such way is due to Gurevich (see page 63 of \[25\] and \[14\]) where the summation is only over the periodic orbits \(\bar{x}\) that satisfies \(\sigma^n(\bar{x}) = \bar{x}\) considering the fact that the \(n\)-step evolution really concerns the length \(n\) prefix of \(\bar{x}\). As a result, the number of choices is now \(W_n = \sum_{\bar{x} : \sigma^n(\bar{x}) = \bar{x}} e^{(S^n\psi)(\bar{x})}\).

Again, using the Boltzmann equation, the total energy needed per step for all microstates \(\bar{x}\) in \(n\)-step evolution is \(\frac{1}{n} \ln W_n\). Sending \(n \to \infty\), we have \(\lim_{n \to \infty} \frac{1}{n} \ln W_n\) which is the (Gurevich) free energy

\[
\lim_{n \to \infty} \frac{1}{n} \ln \sum_{\bar{x} : \sigma^n(\bar{x}) = \bar{x}} e^{(S^n\psi)(\bar{x})}, \tag{1}
\]

(The limit exists with a weak side condition (see Proposition 3.2 in \[25\]). For the mathematics, see page 63 of \[25\] and \[14\]).

Coming back to Computer Science, we modify the formula in (1) so that the summation is over the (finite) words \(w\) of length \(n\) in a language \(L\) instead of \(\omega\)-words \(\bar{x}\). To do this, we first consider \(w = x_0 \cdots x_{n-1}\) which is the prefix (with length \(n\) of \(\bar{x} = x_0 \cdots x_{n-1} \cdots\). For the term \((S^n\psi)(\bar{x})\) in (1), we modify it slightly into

\[
(U)(w) = \text{def} \sum_{0 \leq i < n-1} U(x_i, x_{i+1}) \tag{2}
\]

while, as we have mentioned earlier, the \(\psi\) now takes the special form \(\psi(x_0x_1 \cdots) = U(x_0, x_1)\). Then the periodic orbits in (1) can be safely replaced with words of length \(n\). We thereby obtain the definition of (Gurevich) free energy of language.
\( L: \)
\[
G_U(L) = \lim_{n \to \infty} \frac{1}{n} \ln \sum_{w \in L, |w| = n} e^{(U)(w)}. \tag{3}
\]
(By convention, \( \ln 0 = 0 \). The limit superior can be replaced by limit in many cases, e.g., when \( L \) is prefix-closed and regular, as we show later.)

Intuitively, the free energy \( G_U(L) \) characterizes the average “cost” per symbol of words in \( L \) with respect to the cost function \( U \). A particularly interesting example is to interpret the cost as “uncertainty”, as shown in a later example. A special case is when \( U = 0 \). In this case, the free energy is simply (modulo a constant) the information rate of \( L \) that was originally proposed by Shannon \[26\] and Chomsky and Miller \[3\], and more recently studied in \[7\]. We shall notice that, once \( U \) is given, the free energy of \( L \) is a constant and its definition does not involve any probabilistic arguments.

We explore how to compute the free energy defined in (3) for various classes of languages. We shall first point out that the free energy is not computable in general, even for simple classes of languages.

**Theorem 1.** The free energy for language \( L' = (\Sigma^* - L)\Sigma^* \), where \( L \) is a context-free language, is not computable.

Later, we will show that the free energy is computable for regular languages. The proof needs the variational principle in thermodynamics, which is briefly introduced as follows.

Consider the compact metric space whose topology is generated from the cylinder sets in the form of \([x_0 \cdots x_{n-1}] = \{\underline{x} : \underline{x} = x_0 \cdots x_{n-1} \cdots \}\), and uses metric \( d(\underline{x}, \underline{y}) = 2^{-\min\{i : x_i \neq y_i\}} \), where \( \underline{x} = x_0 \cdots x_{n-1} \cdots \) and \( \underline{y} = y_0 \cdots y_{n-1} \cdots \). Let \( \mu \) be a probability measure over the space that is invariant under the Markov shift \( \sigma \). In plain English, \( \mu \) defines a Markov chain over the finite state space \( \Sigma \), noticing that, herein, a symbol in \( \Sigma \) is a state of the Markov chain. For the given \( \mu \), one can define the free energy with respect to \( \mu \) as the quantity \( H_\mu + \int \psi d\mu \), where \( H_\mu \) is the Kolmogorov-Sinai entropy of the Markov chain \( \mu \) (intuitively, it quantifies the average randomness, called entropy, on one step of the Markov chain), and \( \int \psi d\mu \) is the average potential on one step of the Markov chain.

In a thermodynamic system, nature tends to make particles move in a way that maximizes the free energy \[25\] as
\[
\sup_{\mu} \{H_\mu + \int \psi d\mu\}, \tag{4}
\]
which is called the free energy (or called Gurevich pressure) of the aforementioned Markov shift \( \sigma \) with potential \( \psi \). We shall point out that the Markov shift itself does not contain any probabilities. For simplicity, we omit much of the mathematics behind the definition which can be found in \[23\].

One of the most important achievements in thermodynamic formalism establishes the variational principle \[23\]. When interpreted on periodic orbits, it says that the free energy, defined in (1), is indeed the free energy on the Markov...
written

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all infinite sequences in the form of
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and a number of designated accepting states
accept.
A run is a sequence of transitions

\[ \tau = (p_0, a_0, p_1)(p_1, a_1, p_2) \cdots (p_{n-1}, a_{n-1}, p_n) \]  

for some \( n \), satisfying \( p_0 \xrightarrow{a_0} p_1 \xrightarrow{a_1} \cdots \xrightarrow{a_{n-1}} p_n \). In \( M \), there is a designated initial
state \( q_{\text{init}} \) and a number of designated accepting states \( q_{\text{accept}} \in F \subseteq Q \). The
run \( \tau \) is initialized if \( p_0 \) in (6) is the initial state. It is an accepting run if it is
initialized and the last state \( p_n \) in (6) is an accepting state. In this case, we say
that the word \( a_0 \cdots a_{n-1} \) is accepted by \( M \). As usual, we use \( L(M) \) to denote
the language accepted by \( M \). \( M \) is deterministic (i.e. a DFA) if, for each \( p \) and
\( a \), there is at most one \( q \) such that \( p \xrightarrow{a} q \). It is well-known that an NFA can be
converted into a DFA such that both automata accept the same language.

Throughout the paper, we assume that \( M \) is cleaned up. That is, all the
states are dropped from \( M \) whenever it cannot be reached from the initial state
or it cannot reach an accepting state.

We now associate a cost function \( V : T \rightarrow \mathbb{R} \) which assigns a cost value to
every transition in \( M \). We write \( M_V \) for the \( M \) associated with cost function \( V \),
called an NFA with cost.

We first assume that \( M \) is strongly connected. That is, every state can reach
every state in \( M \). More precisely, for each \( p \) and \( q \), there is a run in the form of (6) with \( p_0 = p \) and \( p_n = q \). We now define the free energy of \( M_V \). We note
that the results in [14] are defined on strongly connected graphs only. However,
a finite automaton \( M \) is not, strictly speaking, a graph since there could be
multiple transitions from one state to another (while in a graph there is at most
one edge from a node to another). Therefore, we first need to carefully translate
the automaton \( M \) into a Markov shift (a graph) as follows. Let \( \mathcal{G} \) be the set of
all infinite sequences in the form of

\[ \mathcal{L} = p_0(p_0, a_0, p_1)p_1(p_1, a_1, p_2)p_2 \cdots p_{n-1}(p_{n-1}, a_{n-1}, p_n)p_n \cdots \]  

or

\[ \mathcal{T} = (p_0, a_0, p_1)p_1(p_1, a_1, p_2)p_2 \cdots p_{n-1}(p_{n-1}, a_{n-1}, p_n)p_n \cdots \]  

\[ \lim_{n \to \infty} \frac{1}{n} \ln \sum_{\mathcal{L}\sigma^n(\mathcal{L})=\mathcal{X}} e^{(S_n\psi)(\mathcal{X})} = \sup_{\mu} \{ H_\mu + \int \psi d\mu \} \]  

(5)
The supremum on the RHS of (5) is achieved by an equilibrium probability
measure \( \mu^* \) (called Parry measure), which can be computed from a nonnegative
matrix, called Gurevich Matrix [14], constructed from the definition of \( \psi \) when \( \psi 
\text{ is defined as } U \), mentioned earlier. In particular, the LHS of (5) can be computed
from the Perron-Frobenius eigenvalue of the matrix [14].

We now generalize the free energy from Markov shift to a finite automaton.
Let \( M \) be a nondeterministic finite automaton (NFA) with finitely many states
specified by \( Q \) and with alphabet \( \Sigma \). Transitions in \( M \) are specified by a set
\( T \subseteq Q \times \Sigma \times Q \), where each transition \( t \in T \) in the form of \( (p, a, q) \), or simply
written \( p \xrightarrow{a} q \), means that \( M \) moves from state \( p \) to state \( q \) on reading input
symbol \( a \). A run is a sequence of transitions

\[ \tau = (p_0, a_0, p_1)(p_1, a_1, p_2) \cdots (p_{n-1}, a_{n-1}, p_n) \]  

We now associate a cost function \( V : T \rightarrow \mathbb{R} \) which assigns a cost value to
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satisfying $p_0 \xrightarrow{a_0} p_1 \xrightarrow{a_1} \cdots \xrightarrow{a_{n-1}} p_n \cdots$ (we shall note that $\tau$ may not start from the initial state of $M$). In terms of the thermodynamics formalism, we define a potential function $\psi$ such that for each $\tau$ in $\Theta$, its potential $\psi(\tau) = V(p_0, a_0, p_1)$ if $\tau$ is in the form of $(7)$; $\psi(\tau) = 0$ if $\tau$ is in the form of $(8)$. We can similarly define a compact metric space over $\Theta$ whose topology is generated by cylinders and the metric $d$ defined earlier. Let $\mu$ be a $\sigma$-invariant probability measure. Now the Markov shift $\sigma$ on $\Theta$ defines a graph $\hat{M}$ as follows. For all $p, a, q$, $\hat{M}$ has node $p$, node $q$, node $(p, a, q)$, and edges from node $p$ to node $(p, a, q)$ and from node $(p, a, q)$ to node $q$, iff $(p, a, b)$ is a transition in $\hat{M}$. Clearly, $\hat{M}$ is a strongly connected graph.

The free energy $\mathcal{E}(\hat{M})$ is defined as the quantity in $(1)$. It is known that $(1)$ the free energy can be computed as follows. Suppose that the graph $\hat{M}$ has $k$ nodes indexed with $1, \cdots, k$. For a node $q$ (resp. node $(p, a, q)$) in $\hat{M}$, we use $[q]$ (resp. $[(p, a, q)]$) for its index. We now construct a $k \times k$ matrix $M$, called the Gurevich matrix, as follows. For each $i$ and $j,$

- $M_{ij} = 0$ if there is no edge from node $i$ to node $j$ in $\hat{M};$
- $M_{ij} = e^{V(p, a, q)}$ if there is an edge from node $i$ to node $j$ in $\hat{M}$ and $i = [p]$ and $j = [(p, a, q)]$ for some $p, a, q;$
- $M_{ij} = e^0$ if there is an edge from node $i$ to node $j$ in $\hat{M}$ and $i = [(p, a, q)]$ and $j = [q]$ for some $p, a, q.$

Clearly, $M$ is a non-negative and irreducible (since $\hat{M}$ is strongly connected) matrix. Let $\lambda$ denote the spectral radius of $M$, which is obtained as the largest positive real eigenvalue of $M$, which is called the Perron-Frobenius eigenvalue of $M$. Finally, according to $(13)$, the free energy $\mathcal{E}(\hat{M})$ can be efficiently computed as $\ln \lambda.$

**Lemma 1.** (Gurevich theorem) $\mathcal{E}(\hat{M}) = \ln \lambda,$ where $\lambda$ is the Perron-Frobenius eigenvalue of the Gurevich matrix $M$.

Finally, the free energy $\mathcal{E}(M_V)$ for finite automaton $M$ with cost function $V$ is defined as $2 \cdot \mathcal{E}(\hat{M}).$\footnote{The factor 2, intuitively, comes from the fact that we “stretch” by a factor of 2, a run in finite automaton $M$ to correspond it to a walk in graph $\hat{M}$. A somewhat more efficient way to compute $\mathcal{E}(M_V)$ is to construct an $m \times m$ Gurevich matrix $M'$ where $m$ is the number of states in $M$ such that $M'_{ij} = 0$ if there is no transition from $p_i$ to $p_j$ in $M,$ else $M'_{ij} = \sum_{a_i, a_j} e^{V(p_i, a_i, p_j)}.$ Herein, $p_1, \cdots, p_m$ are all states in $M$. One can show that $\mathcal{E}(M_V) = \ln \lambda'$ where $\lambda'$ is the Perron-Frobenius eigenvalue of $M'.$ We omit the details.}
In particular, we use $R$ to denote the set of all runs of $M$ and $A$ to denote the set of all accepting runs of $M$; herein, both sets are languages on transitions in $M$.

**Theorem 2.** *(Variational Principle for Strongly Connected NFA)* Let $M$ be an NFA that is strongly connected and with cost function $V$ on transitions. Then the following equations hold:

$$G_V(R) = \lim_{n \to \infty} \frac{1}{n} \ln \sum_{\tau \in R, |\tau| = n} e^{(V)(\tau)} = \mathcal{E}(M_V). \quad (10)$$

and

$$G_V(A) = \limsup_{n \to \infty} \frac{1}{n} \ln \sum_{\tau \in A, |\tau| = n} e^{(V)(\tau)} = \mathcal{E}(M_V). \quad (11)$$

In particular when every state in $M$ is an accepting state (hence $L(M)$ is prefix closed),

$$G_V(A) = \lim_{n \to \infty} \frac{1}{n} \ln \sum_{\tau \in A, |\tau| = n} e^{(V)(\tau)} = \mathcal{E}(M_V). \quad (12)$$

In general, $M$ is not necessarily strongly connected. However, it is well-known (using the linear-time Tarjan algorithm) that $M$ can be uniquely partitioned into a number of components $M_1, \ldots, M_k$, for some $k \geq 1$, such that

- each $M_i$ is strongly connected, or it is singleton (i.e. it contains only one state that does not have a self-loop transition), and
- each $M_i$ is maximal (i.e. the above condition is no longer true if it is enlarged).

From Lemma[11] the free energy $\mathcal{E}(M_{vi})$ for each component $M_i$ can be computed from its graph representation $\hat{M}_i$ (when $M_i$ is singleton, its free energy is defined to be 0). We now define the free energy of $M_V$ to be $\mathcal{E}(M_V) = \max_i \mathcal{E}(M_{vi})$. Clearly, using the definition of $\mathcal{E}(M_{vi})$ and Lemma[11] we immediately have:

**Theorem 3.** When NFA $M$ is cleaned-up, $\mathcal{E}(M_V)$ is computable.

This definition of $\mathcal{E}(M_V)$ is valid, since, from (11), we can easily show:

**Theorem 4.** *(Variational Principle for NFA)* When NFA $M$ is cleaned-up, we have $G_V(A) = \mathcal{E}(M_V)$, where $A$ is the set of all accepting runs of $M$, and $V$ is a cost function that assigns a cost in $\mathbb{R}$ to a transition in $M$.

We now consider a regular language $L$ associated with a cost function defined earlier as $U : \Sigma \times \Sigma \to \mathbb{R}$. Let $M$ be an NFA with a cost function $V$. We say that $(M, V)$ *implements* $(L, U)$ if $M$ accepts $L$, and, for each $w \in L$ with length at least 2 and each accepting run $\tau$ for $w$ in $M$, we have $(U)(w) = (V)(\tau)$ (i.e. the total cost on $w$ defined by $U$ is kept exactly the same as the total cost of each accepting run $\tau$ defined by $V$). We may ask a number of questions:

**Q1.** Let $M$ be an NFA that accepts $L$. Can we always find a $V$ such that $(M, V)$ implements $(L, U)$?
**Q2.** Can we find an NFA $M$ that accepts $L$ and find a $V$ such that $(M, V)$ implements $(L, U)$?

Q1 has a negative answer, unfortunately, shown in the following example.

**Example 1.** Consider $L = (ab)^*$ with $U(a, b) = 2, U(b, a) = 5$. Clearly, $M$ shown in Figure 1 accepts $L$. However, there are no $V_a$ and $V_b$ (costs assigned on the $a$-transition and the $b$-transition in $M$) such that the resulting $(M, V)$ implements $(L, U)$.

Because of the negative answer to Q1, the positive answer to Q2 is meaningful.

**Example 2.** Following the previous example, we can draw an $M'$ shown in Figure 2 (the costs $V$ are assigned to the transitions in the figure). One can verify that $(M, V)$ implements $(L, U)$.

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**Theorem 5.** For each regular language $L$ associated with a cost function $U$:

1. For each NFA $M$ and cost function $U$ for transitions in $M$ such that $(M, V)$ implements $(L, U)$, we have $G_U(L) \leq E(M_V)$.
2. There is a DFA $M$ and cost function $U$ for transitions in $M$ such that $(M, V)$ implements $(L, U)$. Furthermore, we have $G_U(L) = E(M_V)$.
3. When $L$ is prefix closed, the limit (instead of limsup) in the definition of $G_U(L)$ in formula (3) exists.

It is difficult to show whether a nonregular language $L$ associated with a cost function $U$ has a computable free energy (see also Theorem 1). Below, we show a class of languages where the energy is computable.

We first need a definition. Let $Y$ be a finite set of integer variables. An atomic Presburger formula on $Y$ is either a linear constraint $\sum_{y \in Y} a_uy < b$, or a mod constraint $x \equiv c$ $d$, where $a, b, c$ and $d$ are integers with $0 \leq c \leq d$. A Presburger formula can always be constructed from atomic Presburger formulas using $\neg$ and $\land$. Presburger formulas are closed under quantification. It is well-known that it is decidable whether a Presburger formula is satisfiable.

Let $P(\theta_1, \cdots, \theta_k)$ be a Presburger formula over nonnegative integer variables $\theta_1, \cdots, \theta_k$, for some $k$. For each $1 \leq i \leq k$, we associate $P$ with a regular language...
We use \( r \) to denote \( \langle r_1, \ldots, r_k \rangle \). We then define a language \( L_{P,r} \) as the set of all words in the form of \( w_1 \cdots w_k \) such that each \( w_i \in r_i \), and the lengths \(|w_i|\) of \( w_i \)'s satisfy \( P(|w_1|, \ldots, |w_k|) \). A linear-length language \( L \) is specified by a regular language \( L' \) along with \( P \) and \( r \) such that \( L = L' \cap L_{P,r} \). Intuitively, \( L \) is a subset of given regular language such that each word in the subset is the concatenation of \( k \) subwords, each of which is drawn from a regular language and the lengths of the subwords are constrained by a Presburger formula. For instance, \( L = \{a^nb^2a^n : n \geq 0 \} \) is a linear-length language, which is not context-free. We can show:

**Theorem 6.** For linear-length language \( L \) and cost function \( U \), the free energy \( G_U(L) \) is computable (from \( L \)'s specification and \( U \)). The computability remains even when \( L \) is a finite union of linear-length languages.

Suppose that symbols in \( \Sigma \) are \( a_1, \ldots, a_k \), for some \( k \). Let \( P(\theta_1, \ldots, \theta_k) \) be a Presburger formula over nonnegative integer variables \( \theta_1, \ldots, \theta_k \). Recall that we use \( \#_{a_i}(\alpha) \) to denote the number of appearances of symbol \( a_i \) in word \( \alpha \). For a given regular language \( L \), we use \( L_P \) to denote the set of all words \( \alpha \) in \( L \) such that \( P(\#_{a_1}(\alpha), \ldots, \#_{a_k}(\alpha)) \) holds. Such a language is called a counting language in [7]. Notice that the aforementioned linear-length language can be converted into a counting language through proper symbol renaming. In [7], it was shown that the information rate of counting languages is computable using a complicated convex-optimization approach. Currently, we do not know if a similar technique can be used to show that the free energy of a counting language is computable.

In a linear-length language, we put a Presburger constraint on the lengths of certain subwords. Now, we study the case where we put a similar constraint on the total costs of the subwords. Let \( P(\theta_1, \ldots, \theta_k) \) be a formula in additive theory of rationales over rational variables \( \theta_1, \ldots, \theta_k \), for some \( k \). Recall that \( r \) is an array of \( k \) regular languages \( r_1, \ldots, r_k \). Let \( U \) be a cost function. We define a language \( L_{P,r,U} \) as the set of all words in the form of \( w_1 \cdots w_k \) such that each \( w_i \in r_i \), and the total costs \( c_i = (U)(w_i) \) on \( w_i \)'s satisfy \( P(c_1, \ldots, c_k) \). A linear-cost language \( L \) is specified by a regular language \( L' \) along with \( P \), \( r \) and \( U \) such that \( L = L' \cap L_{P,r,U} \).

**Example 3.** Consider a word \( w \) in the form of \((a+b)^*(c+d)^*(a+b)^*\), with cost function \( U \) given. Conceptually, \( w \) is rewritten as \( w_1w_2w_3 \) with \( w_1 \in r_1 = (a+b)^* \), \( w_2 \in r_2 = (c+d)^* \) and \( w_3 \in r_3 = (a+b)^* \). We use \( c_1, c_2, c_3 \) to denote the total costs on \( w_1, w_2 \) and \( w_3 \), respectively, and add all such \( w \) satisfying \( c_1 > c_2 > c_3 > 0 \) into language \( L \). Then \( L \) is a linear-cost language. Clearly, \( L \) is not context-free in general, even when the cost function is nonnegative.

Currently, we do not know whether the free energy of a linear-cost language is computable or not. Even for the following simple \( L \), the problem is open: the set \( L \) consists of all \( w \) in a given regular language \( L' \) such that the total cost on \( w \) (with respect to a given cost function) is zero. We believe that for such a simple \( L \), the energy is computable by generalizing the seminal proof by Kuich in [19].

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Even when the cost function is nonnegative, the problem is open. However, for nonnegative cost function, we have a computable special case. A restricted linear-cost language is a linear-cost language where the formula \( P(\theta_1, \ldots, \theta_k) \) in defining the language takes the following special form: it is a disjunction over conjunctions of

\[
k_i \cdot \theta_i \sim k_j \cdot \theta_j + l
\]

(13)

where \( \sim \in \{ >, <, =, \geq, \leq \} \), and for each \( i \), a nonnegative integer \( k_i \) is unique throughout a conjunction, while \( l \) is only tied with the formula in (13). For instance, \( (2\theta_1 > 3\theta_2 - 5 \land 2\theta_1 = 4\theta_3 - 8) \lor (7\theta_1 < 3\theta_2 - 12 \land 7\theta_1 > 3\theta_2 + 17) \) is a valid formula, but \( (2\theta_1 > 3\theta_2 - 5 \land 5\theta_1 = 4\theta_3 - 8) \lor (7\theta_1 < 3\theta_2 - 12 \land 7\theta_1 > 3\theta_2 + 17) \) is not. A restricted linear-cost language is not necessarily context-free in general as shown in Example 3. (The following result still holds when we replace each term \( k_i x_i \) with a nonnegative integer linear combination of a subset \( V_i \) of variables \( \theta_1, \ldots, \theta_k \), as long as the combination is unique throughout a conjunction and the sets \( V_i \) are disjoint. We omit the details.)

**Theorem 7.** For restricted linear-cost language \( L \) and nonnegative cost function \( U \), the free energy \( G_U(L) \) is computable (from \( L \)'s specification and \( U \)). The computability remains even when \( L \) is a finite union of restricted linear-cost languages.

We turn back to NFA \( M \) (which is cleaned-up) with a cost function \( V \) on transitions. We show an application of the free energy of \((M, V)\) in estimating nondeterminism in an NFA. There are applications for such an estimation. In software engineering, NFAs can be used to specify a highly nondeterministic system such as a concurrent system where the input symbols are the observable events. Being nondeterministic, the same sequence of input symbols can have many different execution sequences. Hence, an ideal measure for the nondeterminism would be the asymptotic growth rate of the ratio \( f(n)/g(n) \) where \( f(n) \) is the total number of executions of input sequences of length \( n \) while \( g(n) \) is the total number of input sequences of length \( n \). More precisely, we define (slightly different from the above) \( g(n) \) to be the number of words \( \alpha \) of length \( \leq n \) in \( L(M) \), and \( f(n) \) to be the number of initialized runs of all \( \alpha \)'s. Then the nondeterminism in \( M \) is defined by

\[
\lambda_M = \lim_{n \to \infty} \log \frac{f(n) - \log g(n)}{n}.
\]

Clearly, the limit in \( \lambda_M \geq 0 \) exists and is finite (since runs are prefix closed, and \( M \) has no \( \epsilon \)-transitions). In particular, when \( M \) is deterministic, \( \lambda_M = 0 \).

In reality, such a metric is relevant. For instance, it can be used to estimate a form of coverage of extensive testing of a nondeterministic system (e.g. how many paths have already been exercised for an average input sequence). The estimation is important since it is well-known that nondeterministic software systems are notoriously difficult and costly to test. However, in computing \( \lambda \), the difficult part is the asymptotic growth rate of \( \log g(n) \) since currently available algorithms must convert NFA \( M \) into a DFA and use the Chomsky-Miller algorithm [3].
The conversion may cause an exponential blow-up in the number of states in the DFA, which is not tractable. We need a practically efficient algorithm to give an estimation of $\lambda_M$.

We propose an efficient estimation approach based on free energy. For the given NFA $M$, we define a cost function $V$ on transitions of $M$ as follows. Let $k(p,a)$ be the total number of $p''$ such that $p \xrightarrow{a} p''$ is a transition in $M$. For each transition $p \xrightarrow{a} p'$ in $M$, we define $V(p,a,p') = \ln k(p,a)$. Notice that if $k(p,a) = 1$ (in this case, $M$ is deterministic at state $p$ on input $a$), $V(p,a,p') = 0$. Otherwise (i.e., $k(p,a) > 1$. In this case, $M$ is nondeterministic at state $p$ on input $a$), $V(p,a,p') = \ln k(p,a) > 0$.

**Example 4.** Figure 3 shows an NFA with the $V$ assigned on transitions.

![Figure 3](image)

**Fig. 3.** The finite automaton for Example 4 with costs labeled.

We now use the free energy difference $\lambda_M^+ = \mathcal{E}(M_V) - \mathcal{E}(M_0)$ to estimate $\lambda_M$. Herein, $M_0$ is the $M$ where each transition is assigned with cost 0. Notice that $\lambda_M^+ \geq 0$ (roughly, from the Perron-Frobenius theorem applied on the Gurevich matrices for $M_V$ and for $M_0$) and, when $M$ is deterministic, $\lambda_M^+ = \lambda_M = 0$.

We first intuitively explain the meaning behind $\lambda_M^+$. In $M_V$, each transition $p \xrightarrow{a} p'$ is assigned a cost which is the number of nats (information units in natural logarithm) needed to code the transition when $p$ and $a$ are given. Hence, the total cost of an average (initialized) run on a word $\alpha$ will be the total number of nats to code the run (which starts from the known initial state) when $\alpha$ is known. This total cost divided by the length $n$ of the word $\alpha$ will result in nat rate $\delta$ of the code. Notice that, in the definition of free energy of $M$, there are two parts: the average cost per step (which roughly corresponds to the nat rate $\delta$) and the metric entropy (which roughly corresponds to the “average” natural logarithm of the branching factor at a state in $M$). Now, in $M_0$, the free energy is the “average” natural logarithm of the branching factor at a state in $M$. Hence, the $\lambda_M^+ = \mathcal{E}(M_V) - \mathcal{E}(M_0)$ is roughly equal to the nat rate $\delta$ (to encode the transition per step for a given input), which is the intended meaning in $\lambda_M$. When $M$ is
We are now ready to use the theory developed so far to define the similarity $q,p$. That is, $(q,a,p)$ is a transition in $M$ iff $(q,a,q')$ is a transition in $M^1$ and $(p,a,p')$ is a transition in $M^2$, for all states $p,q,,p',q'$. The initial state in $M$ is the pair of initial states in $M^1$ and in $M^2$ and the accepting states in $M$ are all the pairs of an accepting state in $M^1$ and an accepting state in $M^2$. Again, we assume that $M$ is cleaned up. Clearly, $M$ is an NFA accepting $L(M^1) \cap L(M^2)$.

We now define the cost functions $V$ on the $M$ as $V((q,a),(q',p')) = V_1(q,a,q') + V_2(p,a,p')$. The semantic similarity metric $\Delta(M^1_V, M^2_V)$ is defined as the free energy of $M$; i.e., $E(M_V)$. Intuitively, this definition catches the average “shared free energy” per step on the shared accepting runs between $M_1$ and $M_2$. We have

$$0 \leq \Delta(M^1_V, M^2_V) \leq E(M^1_V) + E(M^2_V).$$

### Theorem 8.

For a cleaned-up NFA $M$, $\lambda^+_M \geq \lambda_M$.

We note that in Theorem 8 the upper estimation $\lambda^+_M$ can be efficiently computed from $M$. Finally, we point out that the estimation is asymptotically tight; the proof will be included in the full version of the paper. We shall also point out all the results can be generalized to cost function $U$ over $k > 2$ ($k$ is constant) symbols instead of two symbols.

## 3 A free-energy based similarity metric for automata and languages

We are now ready to use the theory developed so far to define the similarity metric. Let $M^i (i = 1, 2)$ be an NFA with a cost function $V_i$ assigned on edges. Notice that the two automata $M^1$ and $M^2$ work on the same input alphabet $\Sigma$. Consider a word $w = a_0 \cdots a_{n-1}$ accepted by both automata. Suppose that the following transition sequences $(q_0, a_0, q_1)(q_1, a_1, q_2) \cdots (q_{n-1}, a_{n-1}, q_n)$ in $M^1$ and $(p_0, a_0, p_1)(p_1, a_1, p_2) \cdots (p_{n-1}, a_{n-1}, p_n)$ in $M^2$ witness the acceptance. However, the total cost (i.e., the total energy or potential) on the first accepting sequence is defined by $V_1(q_0, a_0, q_1) + \cdots + V_1(q_{n-1}, a_{n-1}, q_n)$ while the total cost on the second is $V_2(p_0, a_0, p_1) + \cdots + V_2(p_{n-1}, a_{n-1}, p_n)$. The sum of the two costs shall tell the deviation of the free energy on the two sequences on the input word $w$. The sum can be expressed on each individual transition as $V_1(q_0, a_0, q_1) + V_2(p_0, a_0, p_1) + \cdots + V_1(q_{n-1}, a_{n-1}, q_n) + V_2(p_{n-1}, a_{n-1}, p_n)$, which again is the free energy on a properly defined (in below) “shared sequence” between the two automata.

We define $M$ to be the Cartesian product of $M^1$ and $M^2$ in a standard way. That is, $((q,p), a, (q',p'))$ is a transition in $M$ iff $(q,a,q')$ is a transition in $M^1$ and $(p,a,p')$ is a transition in $M^2$, for all states $p,q,,p',q'$. The initial state in $M$ is the pair of initial states in $M^1$ and in $M^2$ and the accepting states in $M$ are all the pairs of an accepting state in $M^1$ and an accepting state in $M^2$. Again, we assume that $M$ is cleaned up. Clearly, $M$ is an NFA accepting $L(M^1) \cap L(M^2)$.
To see (14), \( \Delta(M_{V_1}^1, M_{V_2}^2) = \mathcal{E}(M_V) \geq 0 \) is obvious, since the LHS of (14) in Theorem 2 is nonnegative and so is the LHS of the equation in Theorem 4. To show \( \Delta(M_{V_1}^1, M_{V_2}^2) \leq \mathcal{E}(M_{V_1}^1) + \mathcal{E}(M_{V_2}^2) \) in (14), we need some effort. First we assume that \( M \) is strongly connected and hence we can use Theorem 2. Observe that the term \( \sum_{\tau \in A_1, |\tau| = n} e^{(V)(\tau)} \) in (14) in Theorem 2 satisfies, using the definition \( V \), the following inequality

\[
\sum_{\tau \in A_1, |\tau| = n} e^{(V)(\tau)} \leq \sum_{\tau_1 \in A_1, |\tau_1| = n} e^{(V_1)(\tau_1)} \cdot \sum_{\tau_2 \in A_2, |\tau_2| = n} e^{(V_2)(\tau_2)}
\]

where \( A_1 \) and \( A_2 \) are accepting transition sequences of \( M_1 \) and \( M_2 \), respectively. Now we plug-in the RHS of the inequality into (14) and obtain

\[
\limsup_{n \to \infty} \frac{1}{n} \ln \sum_{\tau \in A_1, |\tau| = n} e^{(V)(\tau)} \leq \limsup_{n \to \infty} \frac{1}{n} \ln \sum_{\tau_1 \in A_1, |\tau_1| = n} e^{(V_1)(\tau_1)}
\]

\[
+ \limsup_{n \to \infty} \frac{1}{n} \ln \sum_{\tau_2 \in A_2, |\tau_2| = n} e^{(V_2)(\tau_2)}.
\]

Then, we use Theorem 4 on \( M_{V_1}^1 \) and \( M_{V_2}^2 \) and hence the RHS of the above inequality becomes

\[
\limsup_{n \to \infty} \frac{1}{n} \ln \sum_{\tau \in A_1, |\tau| = n} e^{(V)(\tau)} \leq \mathcal{E}(M_{V_1}^1) + \mathcal{E}(M_{V_2}^2).
\]

Again, using (14) in Theorem 2 on the LHS of the inequality, we have

\[
\mathcal{E}(M_V) \leq \mathcal{E}(M_{V_1}^1) + \mathcal{E}(M_{V_2}^2).
\]

Using Theorem 3 again on the LHS, we finally obtain for a general \( M \) (which may not be strongly connected), the above inequality still holds, which is essentially the result in (14).

In particular, the reader can easily check that the inequality in (14) is tight: when \( M^1 \) and \( M^2 \) are completely independent (i.e., \( L(M^1) \cap L(M^2) = \emptyset \)), the similarity metric \( \Delta(M_{V_1}^1, M_{V_2}^2) = 0 \). However, when \( M^1 \) and \( M^2 \) are the same and the \( V_1 \) and \( V_2 \) are also the same, we have the similarity metric \( \Delta(M_{V_1}^1, M_{V_2}^2) \) reaches the maximum \( \mathcal{E}(M_{V_1}^1) + \mathcal{E}(M_{V_2}^2) \). Intuitively, the metric \( \Delta(M_{V_1}^1, M_{V_2}^2) \) characterizes the “shared free energy” between the two finite state programs \( M_{V_1}^1 \) and \( M_{V_2}^2 \) as follows. Imagine that the two programs (automata) are two moving gas molecules. At each step of observation, the molecules can be highly random and hence it makes little sense to compare every step of the observations to figure out the similarity between the two molecules. The approach we take in defining the metric \( \Delta(M_{V_1}^1, M_{V_2}^2) \) is to “create” a third molecule \( M_V \) that, at each step, possesses the potential as the sum of the first two molecules (this is a reward) whenever the first two molecule share the same orbit (i.e., the same input symbol) – otherwise when the orbit are different, the potential of the third molecule is \(-\infty\) (this is a penalty). Clearly, the dynamics of the third molecule...
Fig. 4. The first figure is an NFA $M^1_{V_1}$ with cost function $V_1$; the second figure is an NFA $M^2_{V_2}$ with cost function $V_2$; the third figure is the Cartesian product $M_V$.

would be very different from the first two. However, as we have shown above, the long term characteristic of free energy of the third molecule reflects the shared free energy between the first two molecules when they follow the same orbit.

We now look at an example of two NFAs $M^1_{V_1}$ and $M^2_{V_2}$ where the cost functions $V_1$ and $V_2$ are labelled in the Figures 4. One can think that the two automata try to specify a genome pattern on the four nucleotides ($G, A, C, T$) in DNA while the costs could be interpreted as probabilities, weights of choices, etc. Notice that the semantics of the automata are the nucleic acid sequences that the automata accept, and those sequences are associated with a cost on each nucleotide. For instance, the following sequence ($A, 0.14)(T, 0.8)(C, 0.65)$ is accepted by $M^1_{V_1}$. Hence, the semantic similarity between the two automata shall concern the similarity between the sequences (with costs) accepted by the two automata instead of the naive similarity on the appearances of the two graphs themselves. Again, we imagine each such sequence accepted as a molecule moving along the orbit specified on the sequence and use the shared free energy between the molecule specified by $M^1_{V_1}$ and the molecule specified by $M^2_{V_2}$ to measure the semantic similarity $\Delta(M^1_{V_1}, M^2_{V_2})$. We compute, using the algorithms in Theorem 4 and the definition of the Cartesian product $M_V$ shown earlier, the results $\mathcal{E}(M^1_{V_1}) = 0.3500, \mathcal{E}(M^2_{V_2}) = 1.4087$, and the similarity metric $\Delta(M^1_{V_1}, M^2_{V_2}) = 1.025$, which indeed satisfies (14).

Notice that $\Delta(M^1_{V_1}, M^2_{V_2})$ can be computed efficiently (which involves only Cartesian product of the two automata, and largest eigenvalues of the Gurevich matrices in Theorems 4 and 2).

Let $L_1$ and $L_2$ be two regular languages associated with two cost functions $U_1$ and $U_2$, respectively. According to Theorem 5 (2), we can construct DFAs $M^1_{V_1}$ and $M^2_{V_2}$ and use $\Delta(M^1_{V_1}, M^2_{V_2})$ to serve as the semantic similarity metric for the two regular languages. However, it does not seem that $\Delta(M^1_{V_1}, M^2_{V_2})$ can be efficiently computed from the regular languages since the known construction from regular languages to deterministic finite automata involves exponential blowup on the state space. There might be other alternative definitions on the metric over regular languages (such as using the estimation of nondeterminism in an NFA shown earlier in the paper) such that the metric can be efficiently computed. We leave this for future work.

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A  Appendix – proofs

Proof of Theorem 1

Proof. Let $L$ be a context-free language on alphabet $\Sigma = \{a, b\}$. Consider the language $L' = (\Sigma^* - L)\Sigma^*$ and $U$ to be the zero function. It is left to the reader to check that, if $L$ is total then $G_U(L') = 0$ otherwise $G_U(L') = \ln 2$. The result follows since totalness of context-free languages is well-known undecidable.

Proof of Theorem 2

Proof. Let $M$ be the Gurevich matrix of $M$. Since $M$ is strongly connected, $M$ is irreducible. From the definition, $M$ is nonnegative with $\lambda > 0$ being its Perron-Frobenius eigenvalue. From the results in [14] (and also the Perron-Frobenius theorem), let $\eta$ (a row vector) and $\xi$ (a column vector) be the unique positive left eigenvector and the unique positive right eigenvector of $M$, respectively, of the eigenvalue $\lambda$, satisfying the normalization condition: $\eta\xi = 1$. (A vector is positive if all its elements are positive reals.)

We now show (10), where we use $\xi$, $\eta$, $\lambda$, $\eta\xi$, and now we use $W(n)$ to denote $\sum_{\tau \in R, |\tau| = n} e^{(V)(\tau)}$. Observe that, for each $n$, by definition,

$$W(n) + W(n + 1) = \|M^{2n}\|$$

that is the sum of all elements in matrix $M^{2n}$. (The term of $2n$ comes from the fact that the Gurevich matrix is constructed from $M$. In $\hat{M}$, by definition, each transition in $M$ is translated into two transitions in $\hat{M}$.) Let $g_{2n}(j)$ be the $j$-th column sum of $M^{2n}$. Recall that $M$ is a $k \times k$ matrix and therefore

$$\|M^{2n}\| = g_{2n}(1) + \cdots + g_{2n}(k).$$

Using the fact that $\xi = (\xi_1, \cdots, \xi_k)$ is a right eigenvector, we have $M^{2n}\xi = \lambda^{2n}\xi$. This gives

$$\xi_1g_{2n}(1) + \cdots + \xi_kg_{2n}(k) = \lambda^{2n}(\xi_1 + \cdots + \xi_k).$$

Since $\xi$ is a positive vector, both $\xi_{\text{min}} = \min(\xi_1, \cdots, \xi_k)$ and $\xi_{\text{max}} = \max(\xi_1, \cdots, \xi_k)$ are positive. Therefore, from (17) and (16), we have $\xi_{\text{min}}\|M^{2n}\| \leq \lambda^{2n}(\xi_1 + \cdots + \xi_k) \leq \xi_{\text{max}}\|M^{2n}\|$. Finally, combining (15), we have, for each $n$,

$$c_0\lambda^{2n} \leq W(n) + W(n + 1) \leq c_1\lambda^{2n},$$

where the positive constants $c_0$ and $c_1$ only depends on $\xi$ (not on $n$). Hence, the LHS of (10), $G_V(R)$, by definition, is $\limsup_{n \to \infty} \frac{1}{n} \ln W(n) \leq \limsup_{n \to \infty} \frac{1}{n} \ln(W(n) + W(n + 1)) \leq 2\ln \lambda$, using (18). On the other hand, from (15), one of $W(n)$ and $W(n + 1)$ must be at least $\frac{1}{2}c_0\lambda^{2n}$. This gives $\liminf_{n \to \infty} \frac{1}{n} \ln W(n) \geq 2\ln \lambda$. Hence, the limit in (10) exists and equals $E(M_V) = 2\ln \lambda$.

To show (11), let

$$Z(n) = \sum_{\tau \in A, |\tau| = n} e^{(V)(\tau)}.$$
By definition, $G_V(A) = \limsup_{n \to \infty} \frac{1}{n} \ln Z(n)$. However, to prove that the limsup is indeed equal to $2 \ln \lambda$, we take two steps. First notice that $G_V(A) \leq G_V(R)$ since $A \subseteq R$. This gives, from (10), $G_V(A) \leq 2 \ln \lambda$. It is left to us to show the limsup is at least $2 \ln \lambda$. Observe that for any fixed constant $N_0$, we have

$$\limsup_{n \to \infty} \frac{1}{n} \ln Z(n) = \limsup_{n \to \infty} \frac{1}{n} \ln (Z(n) + \cdots + Z(n + N_0)),$$

using the converging subsequence definition of limsup. Now, take $n_0$ to be the largest of the lengths of shortest paths (in terms of the number of transitions on a path) from the initial state to each individual state in $M$ and from each individual state in $M$ to each accepting state in $M$. For each of the shortest paths, we obtain the total cost on the path (which is the sum of costs $V(t)$ on the transitions $t$ in $M$ on the path). Let $\epsilon$ be the minimal of all such total costs. Let $N_0 = 2 \cdot n_0$. Notice that both $\epsilon$ and $N_0$ are constants (not depending on $n$). We can show $Z(n) + \cdots + Z(n + N_0) \geq e^\epsilon \cdot W(n) \cdot e^\epsilon$ where $W(n)$ is defined in the proof above. This is because every run of length $n$ in $R$ is a substring of an accepting run with length at most $n_0 + n + n_0 = N_0 + n$. Hence, using (19) and (10) and noticing that $\mathcal{E}(M_V) = 2 \ln \lambda$, we have $\limsup_{n \to \infty} \frac{1}{n} \ln Z(n) \geq 2 \ln \lambda$.

We now show (12), under the assumption that every state in $M$ is an accepting state. That is, under the assumption, the limit of $\lim_{n \to \infty} \frac{1}{n} \ln Z(n)$ exists. From (11), we only need show

$$\liminf_{n \to \infty} \frac{1}{n} \ln Z(n) \geq 2 \ln \lambda. \tag{20}$$

Now, take $n_0$ to be the largest of the lengths of shortest paths from the initial state to each individual state in $M$. For each run of length not more than $n_0$ (which not necessarily start from the initial state), we obtain the total cost on the path. Let $\epsilon$ be the minimal of all such total costs. Observe that

$$Z(n + n_0) \geq \min(e^\epsilon \cdot W(n), e^\epsilon \cdot W(n) \cdot e^\epsilon). \tag{21}$$

This is because every run $\tau$ of length $n$ in $R$ is a substring of an accepting run $\alpha$ with length $n + n_0$. To see this, suppose that the $\tau$ starts with $p$ ends with $q$. If the shortest run $\beta$ from the initial state to the $p$ is with length $l = n_0$ already, then simply take $\alpha$ to be the shortest run $\beta$ concatenated with $\tau$. In this case, the total cost on the $\alpha$ is at least $\epsilon + (V)(\tau)$. On the other hand, if the length of $\beta$ is $l < n_0$, then arbitrarily choose a run $\gamma$ from $q$ with length $n_0 - l$. In this case, we take $\alpha$ to be the $\beta$ concatenated with $\tau$ and then $\gamma$. Clearly, the total cost on the $\alpha$ is at least $\epsilon + (V)(\tau) + \epsilon$. This gives the inequality in (21), using the definitions of $Z(\cdot)$ and $W(\cdot)$. From (21), the result in (20) is immediate, using (11) and noticing that $\mathcal{E}(M_V) = 2 \ln \lambda$.

**Proof of Theorem 5**

*Proof.* (1). The result directly follows from the definition of $G_V(L)$ in (3), the variational principle of NFA in Theorem 4 and the fact that the mapping from a word in $L$ to an accepting run of $w$ in $M$ is many-to-one.

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Let \( M' \) be a DFA accepting \( L \). We now construct the desired DFA \( M \) as follows. \((p, a_1, q, a_2, s)\) is a transition in \( M \) (with cost \( V \) on this transition being \( U(a_1, a_2) \)) iff \( p^a_1q \) and \( q^a_2s \) are transitions in \( M' \). In addition to this, \( p_0 \) is also the initial state in \( M \) with transitions \( p_0 \overset{\alpha}{\to} (p_0, a, p) \) (with cost \( V \) on this transition being \( 0 \)) where it is the initial state in \( M' \) and \( p_0 \overset{\alpha}{\to} p \) is a transition in \( M' \). In particular, Each \( (q, a, s) \) is an accepting state in \( M \) if \( s \) is an accepting state in \( M' \). Clearly, \((L, V)\) implements \((L, U)\). Notice that the mapping from a word in \( L \) to an accepting run of \( w \) in \( M \) is one-to-one. Hence, we have the equation in Theorem 4 (2), from the definition of \( G_U(L) \) in (3), the variational principle of NFA in Theorem 4.

(3). Notice that when \( L \) is prefix closed, the DFA constructed above has this property; every state is an accepting state. The result follows from the above proof and the equation (12) in Theorem 2.

Proof of Theorem 6

Proof. We first need some well-known facts. Let \( N \) be the set of nonnegative integers, \( D \subseteq \mathbb{N}^k \) is a linear set if \( D = \{ d : d = d_0 + s_1d_1 + \cdots + s_md_m, s_1, \ldots, s_m \geq 0 \} \) where \( d_0, \ldots, d_m \) are constant \( k \)-arity vectors in \( \mathbb{N}^k \), for some \( m \geq 0 \). \( D \) is a semilinear set if it is the union of finitely many linear sets. Every finite subset of \( \mathbb{N}^k \) is semilinear – it is a finite union of linear sets whose generators are constant vectors. Clearly, semilinear sets are closed under union and projection. It is also known that semilinear sets are closed under intersection and complementation.

Let \( S \) be a set of \( k \)-tuples in \( \mathbb{N}^k \). \( S \) is Presburger definable if there is a Presburger formula \( P(y_1, \ldots, y_k) \) such that the set of non-negative integer solutions is exactly \( S \). It is well-known that \( S \) is a semi-linear set iff \( S \) is Presburger definable.

If \( \Sigma \) is a semilinear set if \( \Sigma \) is Presburger definable. It can be shown that \( D \subseteq \mathbb{N}^k \) is a semilinear set if \( D \) is Presburger definable.

Let \( \Sigma = \{ a_1, \ldots, a_k \} \) be an alphabet. For each word \( \alpha \in \Sigma^* \), define the \( \alpha \)-orbit of \( \alpha \) to be the vector \( \#(\alpha) = (#_{a_1}(\alpha), \ldots, #_{a_k}(\alpha)) \), where each symbol count \( #_{a_i}(\alpha) \) denotes the number of symbol \( a_i \)'s in \( \alpha \). For a language \( L \subseteq \Sigma^* \), the \( \alpha \)-orbit of \( \alpha \) is the set \( \#(L) = \{ \#(\alpha) : \alpha \in L \} \). The language \( L \) is semilinear if \( \#(L) \) is a semilinear set.

We only proof the case when \( L \) is a linear-length language. The case when \( L \) is a finite union of linear-length languages can be concluded directly from the footnote below.

We now continue with the proof. By definition, \( L = L' \cap L_P \), where \( L' \) is a regular language, \( P(\theta_1, \ldots, \theta_k) \) is a Presburger formula and \( \langle r_1, \ldots, r_k \rangle \) are regular languages in \( r \). Consider the set \( Q = \{ (\theta_1, \ldots, \theta_k) \in \mathbb{N}^k : \theta_i = \alpha_i \in r_i, 1 \leq i \leq k \} \). Since every regular language is a semilinear language, it is not hard to show that \( Q \) is a semilinear set (we omit the details). The semilinear set definable by the Presburger formula \( P(\theta_1, \ldots, \theta_k) \) is also denoted by \( P \). Now, \( P \cap Q \) is a semilinear set, and WLOG, we assume that it is a linear

\[ U(a_1, a_2) \]

\[ a \]

\[ D \]

\[ P \]

\[ U \]

\[ L \]

\[ M \]

\[ Q \]

\[ r \]

\[ \alpha \]

\[ \theta \]

\[ \Sigma \]

\[ \Sigma^* \]

\[ \theta_i \]

\[ \alpha_i \]

\[ r_i \]

\[ k \]

\[ q \]

\[ a \]

\[ b \]

\[ c \]

\[ d \]

\[ e \]

\[ f \]

\[ g \]

\[ h \]

\[ i \]

\[ j \]

\[ k \]

\[ l \]

\[ m \]

\[ n \]

\[ o \]

\[ p \]

\[ q \]

\[ r \]

\[ s \]

\[ t \]

\[ u \]

\[ v \]

\[ w \]

\[ x \]

\[ y \]

\[ z \]
set (instead of the union of several linear sets) \( D \) in the form of \( D = \{ \mathbf{d} : \mathbf{d} = \mathbf{d}_0 + s_1 \mathbf{d}_1 + \cdots + s_m \mathbf{d}_m, s_1, \ldots, s_m \geq 0 \} \) where \( \mathbf{d}_0, \ldots, \mathbf{d}_m \) are constant \( k \)-arity vectors in \( \mathbb{N}^k \), for some \( m \geq 0 \). In particular, WLOG, the vectors are non-zero (i.e., there is at least one positive element in each such vector). Furthermore, the vectors \( \mathbf{d}_1, \ldots, \mathbf{d}_m \) are distinct. To make our presentation more readable and also WLOG, we assume that the first vector \( \mathbf{d}_0 \) is positive (all elements in the vector are positive).

For each constant vector \( \mathbf{d}_i \), we create a finite set \( [\mathbf{d}_i] \) of word tuples \( \mathbf{b} = (\beta_1, \ldots, \beta_k) \) such that \( |\beta_j| = \mathbf{d}_i[j] \). We use \( \langle \mathbf{b} \rangle \) to denote the word \( \beta_1 \cdots \beta_k \). For instance, if \( \mathbf{d}_i = (2, 0, 4) \) and \( \Sigma = \{a, b\} \), then the set \( [\mathbf{d}_i] \) is \( \Sigma^2 \times \Sigma^0 \times \Sigma^4 \). We now consider a sequence \( \gamma \) of word tuples \( \mathbf{b}_0 \mathbf{b}_1 \cdots \mathbf{b}_l \), for some \( l \), drawn from \( \mathbf{d}_0 [\mathbf{d}_1]^* [\mathbf{d}_m]^* \). The sequence \( \gamma \) corresponds to the word \( \langle \gamma \rangle = \langle \mathbf{b}_0 \rangle \langle \mathbf{b}_1 \rangle \cdots \langle \mathbf{b}_l \rangle \). In particular, we use \( \gamma[j] \), projection of \( \gamma \) on the \( j \)-th coordinate, to denote the word \( \mathbf{b}_0[j] \mathbf{b}_1[j] \cdots \mathbf{b}_l[j] \).

Now, we define the set \( \Gamma \) of all the sequences \( \gamma \in [\mathbf{d}_0][\mathbf{d}_1]^*[\mathbf{d}_m]^* \) satisfying \( \gamma[j] \in r_j \), for each \( 1 \leq j \leq k \), and \( \gamma[1] \cdots \gamma[k] \in L \). We write \( \langle \Gamma \rangle = \{ \langle \gamma \rangle : \gamma \in \Gamma \} \). We can claim: (a) \( \Gamma \) is a regular language (on alphabet consisting of tuples in \( \cup_i [\mathbf{d}_i] \)), and (b) \( \langle \Gamma \rangle \) is also a regular language. The claims are not hard to show (whose proof is omitted).

However, the correspondence between \( \gamma \) and \( \langle \gamma \rangle \) is many-to-one (for the given and constant \( \mathbf{d}_0, \ldots, \mathbf{d}_m \) defined in the \( D \)). More specifically, each \( \langle \gamma \rangle \) may correspond to \( O(n^m) \) (where \( m \) is the constant defined in \( D \), and \( n \) is the length of word \( \langle \gamma \rangle \)) many \( \gamma \)’s with \( \langle \gamma \rangle = \langle \gamma' \rangle \). However, we shall also notice that all the \( \langle \gamma \rangle \)'s form exactly \( L \). In other words, each word \( w \in L \) with length \( n \) correspond to at most \( O(n^m) \) many distinct \( \gamma \)'s with \( \langle \gamma \rangle = w \).

We now make a further translation on each word \( \gamma \), in the form of, \( \mathbf{b}_0 \mathbf{b}_1 \cdots \mathbf{b}_l \), in \( \Gamma \), into a new word \( \hat{\gamma} \) in the form of \( \langle \mathbf{b}_0 \rangle \langle \mathbf{b}_1 \rangle \cdots \langle \mathbf{b}_l \rangle \langle \mathbf{b}' \rangle \), where each newly added \( \mathbf{b}' \) is a \( k \)-arity tuple of symbols in \( \Sigma^k \). We further require that the translated \( \hat{\gamma} \) is consistent in the following sense:

- Suppose that \( \mathbf{b}' = (b_1, \ldots, b_k) \). Then, each \( b_j \) is the last symbol of the non-null word \( \mathbf{b}_0[j] \). (Recall that we assume that the vector \( \mathbf{d}_0 \) is positive.)
- For each \( i > 0 \), the following holds. Consider \( \langle \mathbf{b}_{i-1} \rangle \langle \mathbf{b}_i \rangle \langle \mathbf{b}' \rangle \) and \( \langle \mathbf{b}_0, \mathbf{b}_i \rangle \langle \mathbf{b}_i \rangle \langle \mathbf{b}' \rangle \). For each \( j \),
  - if \( \mathbf{b}_j[j] \) is non-null, then \( \mathbf{b}'[j] \) is the last symbol of the word \( \mathbf{b}_j[j] \);
  - if \( \mathbf{b}_j[j] \) is the null word, then \( \mathbf{b}'[j] = \mathbf{b}^{i-1}[j] \).

So, the purpose of these newly introduced symbols in \( \mathbf{b}' \) are to “memorize” the symbols that are most recently “read” on the \( k \) “tracks”. After each word \( \gamma \) in \( \Gamma \) is translated into \( \hat{\gamma} \), we obtain \( \hat{\Gamma} \). Clearly, \( \hat{\Gamma} \) is also regular (on alphabet \( (\cup_i [\mathbf{d}_i]) \times \Sigma^k \)) and the translation between \( \gamma \in \Gamma \) and \( \hat{\gamma} \in \hat{\Gamma} \) is one-to-one and length-preserving.

We now do a final translation from \( \hat{\Gamma} \) into a new language \( \hat{\hat{\Gamma}} \) such that there is a one-to-one and length-preserving correspondence between \( \hat{\Gamma} \) and \( \langle \hat{\Gamma} \rangle \). Consider a word \( \hat{\gamma} \in \hat{\Gamma} \) in the form of \( \langle \mathbf{b}_0 \rangle \langle \mathbf{b}_1 \rangle \cdots \langle \mathbf{b}_l \rangle \langle \mathbf{b}' \rangle \). We shall emphasize again that each “symbol” in the word is drawn from the product set \( (\cup_i [\mathbf{d}_i]) \times \Sigma^k \) and hence \( \hat{\gamma} \) has length \( l \). We now stretch every such symbol longer
by padding it with some new “stutter” symbol “♥” (actually we introduce many such symbols, shown below). For each symbol \((b_j, b^j)\) in \(\hat{\gamma}\), we translate into a word \((b_j, b^j)\) with length \(|b_j|\), where \(|b_j|\) is the “size” of \(b_j\), which is the sum of the lengths of all elements (each element is a word by definition) in the vector \(b_j\). Notice that the stutter symbol \(\forall_{(b_j, b^j)}\) “memorizes” the vector \(b_j\), as well as the most recently “read” symbols in \(b^j\). (The definition is valid since \(|b_j| - 1 \geq 0\) as the vectors \(d_i\) are all non-zero as we have mentioned earlier.) After each symbol in \(\hat{\gamma}\) is translated, we obtain the \(\hat{\gamma}\). After each word \(\hat{\gamma} \in \hat{\Gamma}\) is translated into \(\hat{\gamma}\), we obtain the language \(\hat{\Gamma}\). It is straightforward to check that \(\hat{\Gamma}\) is regular (since \(\hat{\Gamma}\) is regular) and the translation from words \(\langle \gamma \rangle\) all the way to \(\hat{\gamma}\) is length-preserving and one-to-many:

\((\ast)\) each \(w \in L\) with length \(n\) correspond to at most \(O(n^m)\) many distinct \(\gamma\)'s with \(\langle \gamma \rangle = w\).

We now assign a cost function \(U\) to \(\hat{\Gamma}\):

\[- \quad U(\forall_{b_1, b^1}, \forall_{b_1, b^1}) = 0\text{ for each }b_1 \in \cup_i [d_i]\text{ and }b^1 \in \Sigma^k;\]

\[- \quad U((b_1, b^1)\forall_{b_1, b^j}) = K_j = 0\text{ if }b_1[j]\text{ is the null word. Otherwise, }K_j = U(\langle b_1[j] \rangle)\text{, where }U\text{ is the cost function given on }L\text{ in the theorem; (Note that }U(a) = 0\text{ by definition when }a\text{ is a single symbol in }\Sigma.)\]

\[- \quad U(\forall_{b_2, b^2}, (b_1, b^1)) = K_1 + \cdots + K_k\text{, for each }b_1 \in \cup_i [d_i]\text{ and }b^1 \in \Sigma^k,\]

where each \(K_j\) is defined as follows. \(K_j = 0\) if \(b_1[j]\) is the null word. Otherwise, \(K_j = U(b^1[j], b_1[j][1]),\) where \(b_1[j][1]\) is the first symbol in the word \(b_1[j];\)

\[- \quad U((b_1, b^1), (b_2, b^2)) = K_1 + \cdots + K_k\text{, for each }b_1, b_2 \in \cup_i [d_i]\text{ and }b^1, b^2 \in \Sigma^k,\]

where each \(K_j\) is defined as follows.

- if \(b_2[j]\) is the null word, then \(K_j = 0;\)
- if otherwise, then \(K_j = U(b^1[j], b_2[j][1]);\)

The construction of the cost function \(U\) is carefully designed so that the following property is guaranteed: for every \(w \in L\) that is long enough (more precisely, longer than the sum, which is a constant, of all elements in the positive vector \(d_i\) in the definition of \(D\)) and every \(\hat{\gamma} \in \hat{\Gamma}\) with \(\langle \gamma \rangle = w\), the total cost \(U(\langle w \rangle)\) on \(w\) equals the total cost \(U(\langle \hat{\gamma} \rangle)\) on \(\hat{\gamma}\). The “long enough” condition guarantees that the length of \(\hat{\gamma}\) is at least 2. By default, \(U(\langle \hat{\gamma} \rangle)\) is defined to be 0 when the \(\hat{\gamma}\) is a single symbol. Recall that, even though the lengths of \(w\) and \(\hat{\gamma}\) are the same, the correspondence from \(w\) to \(\hat{\gamma}\) is one-to-many (see the statement \((\ast)\) mentioned earlier). To sum up, we finally have,

\[
\limsup_n \frac{1}{n} \ln \sum_{w \in L, |w| = n} O(n^m) \cdot e^{U(w)} \geq \limsup_n \frac{1}{n} \ln \sum_{\hat{\gamma} \in \hat{\Gamma}, |\hat{\gamma}| = n} e^{U(\langle \hat{\gamma} \rangle)}, \quad (22)
\]

and (since \(\hat{\gamma}\)'s in \(\hat{\Gamma}\) with length \(n\) are more than \(w\)'s in \(L\) with length \(n\))

\[
\limsup_n \frac{1}{n} \ln \sum_{w \in L, |w| = n} e^{U(w)} \leq \limsup_n \frac{1}{n} \ln \sum_{\hat{\gamma} \in \hat{\Gamma}, |\hat{\gamma}| = n} e^{U(\langle \hat{\gamma} \rangle)}, \quad (23)
\]
Hence, we immediately have \( G_U(L) = G_U(\hat{\Gamma}) \), noticing that \( \lim_{n \to \infty} \frac{1}{n} \ln O(n^m) = 0 \) for the constant \( m \). The result then follows since, as we have mentioned earlier, \( \hat{\Gamma} \) is a regular language and its free energy \( G_U(\hat{\Gamma}) \) is then computable (Theorem 3) and then Theorem 4.

**Proof of Theorem 7**

Proof. We only show the result when \( L = L' \cap L_{P,R,U} \) and \( P \) takes the form \( k_1 \theta_1 = k_2 \theta_2 = \cdots = k_k \theta_k \) where each \( k_i \) is a positive integer constant; all other forms of the \( P \) can be shown by generalizing the ideas in the following proof.

We first translate a tuple \( \gamma = (w_1, \cdots, w_k) \) such that each \( w_i \in r_i \),

\[
k_1(U)(w_1) = \cdots = k_k(U)(w_k),
\]

(24)

and \( w_1 \cdots w_k \in L' \) (we use \( \Gamma \) to denote the set of all such \( \gamma \)'s) into a word \( \hat{\gamma} = (\hat{w}_1, \cdots, \hat{w}_k) \) as follows. WLOG, we assume that each \( w_i \) has length at least 2. Each \( \hat{w}_i \) is “stretched” from \( w_i \) in the following sense. Suppose that \( w_i = b_0 \cdots b_m \) for some \( m \geq 2 \). Then, for each \( b_j \), \( 1 \leq j < m \), we replace \( b_j \) with \( b_j \cup \{\cdot, \cdot, \cdot\} \) (which has length \( k_j \cdot U(b_j, b_{j+1}) \)). Hence, after the translation, all the \( \hat{w}_i \)'s share the same length \( k_i \cdot U(w_i) + 1 \) (see (24)). That is, we use the length of each \( \hat{w}_i \) to “memorize” the total cost, multiplied by \( k_i \), on \( w_i \).

Then we translate each \( \hat{\gamma} \) into a word \( \hat{\gamma} \) where the \( j \)-th symbol \( \hat{\gamma}[j] \) of the word is the tuple \( (w_1[j], \cdots, w_k[j]) \) of the \( j \)-th symbols for all \( w_i \)'s. \( \hat{\gamma} \) will be perfectly aligned; i.e., each symbol is in \( \Sigma \cup \{\cdot, \cdot, \cdot\} \). Here comes a key step of this proof. We drop all symbols in the form of \( \{\cdot, \cdot, \cdot\} \) from \( \hat{\gamma} \). The result is denoted by \( \tilde{\gamma} \). One shall observe that there are no two distinct \( \gamma_1 \) and \( \gamma_2 \) in \( \Gamma \) such that \( \gamma_1 = \gamma_2 \). Hence, the translation from \( \gamma \) to \( \tilde{\gamma} \) is one-to-one.

We now make a final translation. For each symbol \( (b_1, \cdots, b_k) \) in \( \tilde{\gamma} \), by definition, no all \( b_i \)'s are \( \cdot \). For a symbol \( b \in \Sigma \cup \{\cdot\} \), if it is in \( \Sigma \), we define \([b]_i \) to be the symbol \( (\cdot, \cdot, \cdot, b, \cdot, \cdot, \cdot) \in \Sigma \cup \{\cdot\} \) where the \( b \) appears at the \( i \)-th position. If, however, the \( b \) is \( \cdot \), we simply define \([b]_i \) to be the null word. Now, each symbol \( (b_1, \cdots, b_k) \) in \( \tilde{\gamma} \) is translated into a non-null word \([b_1]_1 \cdots [b_k]_k \). The resulting \( \hat{\gamma} \) is denoted by \( \hat{\gamma} \). Observe that the length of \( \hat{\gamma} \) is exactly the “length” \( |w_1| + \cdots + |w_k| \) of \( \gamma \). We use \( \hat{\Gamma} \) to denote all the translated \( \hat{\gamma} \) from all the \( \gamma \in \Gamma \). It is not hard to show that: (a) \( \hat{\Gamma} \) is a regular language, and (b) the translation from \( \gamma \in \Gamma \) all the way to \( \hat{\gamma} \in \hat{\Gamma} \) is one-to-one and length-preserving.

We can reuse the idea in the proof of Theorem 6 of renaming \( \cdot \) such that the most recently “read” \( \cdot \) symbol in the same row is memorized in the renamed symbol. More precisely, suppose that \( \tilde{\gamma} = b_1 \cdots b_m \) for some \( m \) and each \( b_j \in \Sigma \cup \{\cdot\} \). The \( i \)-th row of \( \tilde{\gamma} \) is then the word \( b_1[i] \cdots b_m[i] \) where each \( b_j[i] \) is the \( i \)-th element in the tuple \( b_j \). By definition, the \( i \)-th row contains at least two non-\( \cdot \) symbols. We now rename all the \( \cdot \)'s in the row as follows. For each \( \cdot \) that is after a symbol \( b \in \Sigma \) (but there is no other symbols in \( \Sigma \) in between) we rename it into \( \cdot \). We leave all the \( \cdot \)'s at the beginning of the row that is not after any symbol in \( \Sigma \) unchanged. After we rename each row in \( \tilde{\gamma} \), we use \( \hat{\gamma} \) to denote the result and use \( \hat{\Gamma} \) to denote all \( \hat{\gamma} \)'s translated from all \( \gamma \in \Gamma \). Notice that the translation from \( \tilde{\gamma} \) to \( \hat{\gamma} \) is one-to-one and length-preserving.
Now, we are ready to show the result by first defining a cost function $u$ that generalizes the given $U$:

- $u(\bigodot_{b_1}, b_2) = U(b_1, b_2)$ where $b_1, b_2 \in \Sigma$;
- $u(\bigodot_{b_1}, \bigodot_{b_1}) = 0$ where $b_1 \in \Sigma$;
- $u(b_1, \bigodot_{b_2}) = 0$ where $b_1, b_2 \in \Sigma$;
- $u(\bigodot, b_2) = 0$ where $b_2 \in \Sigma$;
- $u(\bigodot, \bigodot) = 0$.

(To make $u$ total, for all other cases it takes value 0.) Now for any two $b_1, b_2 \in (\Sigma \cup \{\bigodot\})^k$, we define $U(b_1, b_2) = u(b_1[1], b_2[1]) + \cdots + u(b_1[k], b_2[k])$. Clearly, for each $\gamma = (w_1, \cdots, w_k) \in \Gamma$, $(U)(w_1) + \cdots + (U)(w_k) = (U)(\gamma)$. Hence, the total cost is also preserved by $U$. Notice that each $w = w_1 \cdots w_k$ can only correspond to at most $O(n^k)$ distinct $\gamma$'s, where $n = \lvert w \rvert$. Using the similar argument earlier in the proof of Theorem 8, we have (notice that the absolute difference between $(U)(w)$ and $(U)(w_1) + \cdots + (U)(w_k)$ is bounded by a constant):

$$\limsup_{n \to \infty} \frac{1}{n} \ln \sum_{w \in L, \lvert w \rvert = n} O(n^k) \cdot e^{U}(w) \geq \limsup_{n \to \infty} \frac{1}{n} \ln \sum_{\gamma \in \hat{\Gamma}, \lvert \gamma \rvert = n} e^{U}(\gamma), \quad (25)$$

and (since $\gamma$'s in $\hat{\Gamma}$ with length $n$ are more than $w$'s in $L$ with length $n$)

$$\limsup_{n \to \infty} \frac{1}{n} \ln \sum_{w \in L, \lvert w \rvert = n} e^{U}(w) \leq \limsup_{n \to \infty} \frac{1}{n} \ln \sum_{\gamma \in \hat{\Gamma}, \lvert \gamma \rvert = n} e^{U}(\gamma), \quad (26)$$

Hence, we immediately have $G_U(L) = G_U(\hat{\Gamma})$, noticing that $\lim_{n \to \infty} \frac{1}{n} \ln O(n^k) = 0$ for the constant $k$. The result then follows since $\hat{\Gamma}$ is clearly a regular language and its pre-energy $G_U(\hat{\Gamma})$ is then computable (Theorem 8.2) and then Theorem 8.

**Proof of Theorem 8**

Proof. WLOG, we assume that $M$ is strongly connected. In particular, we are only interested in measuring the growth rate of the numbers of initialized runs (as well as the input words on the runs), we can simply make every state in $M$ an accepting state while doing this without affecting the numbers. This is because the runs and the input words aforementioned are prefix closed.

For the given NFA $M$, let $V$ be the aforementioned cost function assigned on transitions $t$ in $M$. For ease of presentation, we use $V(t)$ to denote $e^{V(t)}$. We first define notations. Let $\alpha$ be any word of length $n$. We write $\alpha \leftarrow \tau$ if $\tau$ is a run of $M$ on $\alpha$ (note that $\tau$ does not necessarily start from the initial state). We write $p \prec \tau$ if $\tau$ starts from state $p$. As usual, $|\tau|$ is the number of transitions in $\tau$. We simply write $(p, a, q) \in M$ if $(p, a, q)$ is a transition in $M$. In particular, by $V(\tau)$, we mean $V(t_1) \cdot V(t_2) \cdots V(t_m)$, where $\tau = t_1, \cdots, t_m$ for some $m$.

We first claim, for each state $p$, and each $\alpha$ with length $n$,

$$\sum_{\tau: |\tau| = n, \alpha \leftarrow \tau, p \prec \tau} (V)(\tau) \geq (\sum_{\tau: |\tau| = n, \alpha \leftarrow \tau, p \prec \tau} 1)^2 \quad (27)$$

\[ 24 \]
We prove the claim by induction on $n$.

Base case ($n = 1$). In this case, (27) becomes, for each $p$ and $\alpha = a \in \Sigma$,

$$\sum_{p' \in (p, a, p') \in M} k(p, a) \geq (\sum_{p' \in (p, a, p') \in M} 1)^2,$$

which holds since both LHS and RHS are equal to $k^2(p, a)$, by the definition of $k(p, a)$.

Induction. We assume that (27) holds for $n$. Then, consider a word $aa\alpha$, for any fixed $a \in \Sigma$, where $\alpha$ is of length $n$. Now, the LHS of (27) becomes

$$\sum_{\tau: |\tau| = n+1, a\alpha \prec \tau \in \tau_p} (V)(\tau),$$

which is, by definition,

$$\sum_{p': (p, a, p') \in M} k(p, a) \cdot \sum_{\tau: |\tau| = n, \alpha \prec \tau, p' \prec \tau} (V)(\tau).$$

Using induction hypothesis, we therefore have,

$$\sum_{\tau: |\tau| = n+1, a\alpha \prec \tau \in \tau_p} (V)(\tau) \geq \sum_{p': (p, a, p') \in M} k(p, a) \cdot (\sum_{\tau: |\tau| = n, \alpha \prec \tau, p' \prec \tau} 1)^2$$

$$\geq k(p, a) \cdot \sum_{p': (p, a, p') \in M} \left(\sum_{\tau: |\tau| = n, \alpha \prec \tau, p' \prec \tau} 1\right)^2$$

(using Jensen’s inequality $\sum_N z_i^2 \geq \frac{1}{N} \cdot (\sum_N z_i)^2$)

$$\geq k(p, a) \cdot \frac{1}{k(p, a)} \cdot \left(\sum_{p': (p, a, p') \in M} \sum_{\tau: |\tau| = n, \alpha \prec \tau, p' \prec \tau} 1\right)^2$$

(noticing that, in above, $k(p, a)$ is the number of $p'$ satisfying $(p, a, p') \in M)$

$$= (\sum_{\tau: |\tau| = n+1, a\alpha \prec \tau \in \tau_p} 1)^2.$$

Hence, the claim follows.

Let $p_0$ be the initial state of $M$. Now, we establish for each $n$,

$$\left(\sum_{\tau: |\tau| = n, p_0 \prec \tau} V(\tau)\right) \cdot \left(\sum_{\alpha: \exists \tau, |\tau| = n, \alpha \prec \tau, p_0 \prec \tau} 1\right) \geq \left(\sum_{\tau: |\tau| = n, p_0 \prec \tau} 1\right)^2.$$  

(29)
Let $K$ be the set of distinct $\alpha$ satisfying $\exists \tau. |\tau| = n, \alpha \leftarrow \tau, p_0 \prec \tau$. The size of the set is denoted by $|K|$. Now, the LHS of (29) is written

$$\left( \sum_{\alpha \in K} \sum_{\tau : |\tau| = n, \alpha \leftarrow \tau, p_0 \prec \tau} V(\tau) \right) \cdot |K|,$$

which is, from the previous claim in (27),

$$\geq \sum_{\alpha \in K} \left( \sum_{\tau : |\tau| = n, \alpha \leftarrow \tau, p_0 \prec \tau} 1 \right)^2 \cdot |K|$$

(using the Jensen’s inequality again)

$$\frac{1}{|K|} \left( \sum_{\alpha \in K} \sum_{\tau : |\tau| = n, \alpha \leftarrow \tau, p_0 \prec \tau} 1 \right)^2 \cdot |K|$$

$$= \left( \sum_{\tau : |\tau| = n, p_0 \prec \tau} 1 \right)^2,$$

which is (28).

Now, from (29), it is direct to have,

$$\frac{1}{n} \ln \sum_{\tau : |\tau| = n, p_0 \prec \tau} V(\tau) - \frac{1}{n} \ln \sum_{\tau : |\tau| = n, p_0 \prec \tau} 1$$

$$\geq \frac{1}{n} \ln \sum_{\tau : |\tau| = n, p_0 \prec \tau} 1 - \frac{1}{n} \ln \sum_{\alpha : \exists \tau. |\tau| = n, \alpha \leftarrow \tau, p_0 \prec \tau} 1.$$

Taking limit on $n$ (all the four limits exist – using the assumption mentioned at the beginning of the proof – and using (12)), we immediately have, $\lambda^+_M = \mathcal{E}(M_V) - \mathcal{E}(M_0) \geq \lambda_M$. The result follows.