Understanding zero-point energy in the context of classical electromagnetism

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Abstract

Today’s textbooks of electromagnetism give the particular solution to Maxwell’s equations involving the integral over the charge and current sources at retarded times. However, the texts fail to emphasise that the choice of the incoming-wave boundary conditions corresponding to solutions of the homogeneous Maxwell equations must be made based upon experiment. Here we discuss the role of these incoming-wave boundary conditions for an experimenter with a hypothetical charged harmonic oscillator as his equipment. We describe the observations of the experimenter when located near a radio station or immersed in thermal radiation at temperature $T$. The classical physicists at the end of the 19th century chose the incoming-wave boundary conditions for the homogeneous Maxwell equations based upon the experimental observations of Lummer and Pringsheim which measured only the thermal radiation which exceeded the random radiation surrounding their measuring equipment; the physicists concluded that they could take the homogeneous solutions to vanish at zero temperature. Today at the beginning of the 21st century, classical physicists must choose the incoming-wave boundary conditions for the homogeneous Maxell equations to correspond to the full radiation spectrum revealed by the recent Casimir force measurements which detect all the radiation surrounding conducting parallel plates, including the radiation absorbed and emitted by the plates themselves. The random classical radiation spectrum revealed by the Casimir force measurements includes electromagnetic zero-point radiation, which is missing from the spectrum measured by Lummer and Pringsheim, and which cannot be eliminated by going to zero temperature. This zero-point radiation will lead to zero-point energy for all systems which have electromagnetic interactions. Thus the choice of the incoming-wave boundary conditions on the homogeneous Maxwell equations is intimately related to the ideas of zero-point energy and non-radiating ground states which are introduced in classes of modern physics.
Keywords: classical zero-point radiation, thermal radiation, zero-point energy, incoming-wave boundary conditions on the homogeneous Maxwell equations

1. Introduction—‘the entire theoretical content of classical electrodynamics’

In physics classes today, the subjects of classical electromagnetism and of modern physics are taught as though there were virtually no connection between these two areas. Thus electromagnetism calculates the electric and magnetic fields due to charges and currents, while modern physics introduces the unfamiliar ideas of zero-point energy and non-radiating ground states. Here we wish to point out that this continuation of the historical separation is not something which arises in nature. Some of the ideas of modern physics can be understood easily and usefully in the context of classical electrodynamics.

When the full Maxwell equations are finally displayed in textbooks of classical electromagnetism, there is sometimes a generalising comment on electrodynamics. For example, in section 7.3.3 of his *Introduction to Classical Electrodynamics*, Griffiths [1] states that Maxwell’s equations together with the Lorentz force law \( \mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \) ‘summarise the entire theoretical content of classical electrodynamics’. Griffiths goes on to note that some further information is required regarding matter, and, in a footnote, he mentions the need for boundary conditions regarding the large-distance fall-off of electromagnetic fields of bounded charge and current distributions [1]. However, there is no attention paid to the incoming-wave boundary conditions on the homogeneous Maxwell equations where the charge density \( \rho \) and the current density \( \mathbf{J} \) are taken to vanish. The failure to consider the incoming-wave boundary conditions on the homogeneous Maxwell equations is a serious lapse in electromagnetism textbooks and represents a continuation of a failure of classical physicists which goes back to the turn of the 20th century. The content of classical electrodynamics actually includes a crucial choice regarding the incoming-wave boundary conditions on the solutions of the homogeneous Maxwell equations, and this choice must be made based upon experiment. In the discussion to follow, we will try to make the important nature of this boundary condition clear to students of classical electrodynamics, and will show its connection to ideas of zero-point energy.

2. Plane waves and charged harmonic oscillators

2.1. Plane waves as solutions of the homogeneous Maxwell equations

In order to connect classical electrodynamics with the ideas of zero-point energy and non-radiating ground states, it is useful to consider plane waves and charged harmonic oscillator systems within classical electrodynamics. Plane waves are solutions of the homogeneous Maxwell equation (1) where \( \rho \) and \( \mathbf{J} \) are taken to vanish. All textbooks of electromagnetism introduce plane waves (see [1], p 397)

\[
\begin{align*}
\nabla \cdot \mathbf{E} &= \rho / \varepsilon_0, \quad \nabla \cdot \mathbf{B} = 0, \\
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t},
\end{align*}
\]

1 We are using SI units at the request of the editor.
\[ E(r, t) = \vec{\varepsilon}E_0 \cos(\vec{k} \cdot r - \omega t + \theta), \]
\[ B(r, t) = (\vec{k} \times \vec{\varepsilon}E_0/c) \cos(\vec{k} \cdot r - \omega t + \theta), \]  
(2)

where \( \vec{\varepsilon} \) is a unit vector orthogonal to the wave vector \( \vec{k} \), \( E_0 \) sets the scale for the wave, \( \omega = c\kappa \) is the angular frequency, and \( \theta \) provides a determination of phase. The introduction of a plane wave means precisely making a choice regarding the incoming-wave boundary conditions on the homogeneous Maxwell equations.

### 2.2. Use of the scalar and vector potentials

Now when discussing the sources of radiation in classical electrodynamics, it is convenient to calculate the scalar and vector potentials \( \Phi \) and \( \mathbf{A} \), which are connected to the fields by \( \mathbf{E} = -\nabla \Phi - \partial \mathbf{A} / \partial t \) and \( \mathbf{B} = \nabla \times \mathbf{A} \). Maxwell’s equations in terms of the potentials, written in the Lorentz (or Lorenz) gauge, take the form (see [1], p 442)

\[
\begin{align*}
\nabla^2 \Phi - \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} &= -\rho / \varepsilon_0, \\
\nabla^2 \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} &= -\mu_0 \mathbf{J}.
\end{align*}
\]

The scalar and vector potentials for the plane wave in equation (2) can be written as

\[
\Phi(r, t) = 0, \\
\mathbf{A}(r, t) = \frac{1}{\omega} \vec{\varepsilon}E_0 \sin(\vec{k} \cdot r - \omega t + \theta). \]

These plane-wave potentials in equation (4) provide a solution of the homogeneous Maxwell equations for the potentials in equation (3).

### 2.3. General solution of Maxwell’s equations

In terms of the scalar and vector potentials, the general solution of Maxwell’s equations takes the form

\[
\begin{align*}
\Phi(r, t) &= \Phi^m(r, t) + \int d^3r' \int d^3t' \frac{\delta(t - t' - |r - r'|/c)}{4\pi|r - r'|} \frac{\rho(r', t')}{\varepsilon_0} \\
&= \Phi^m(r, t) + \int d^3r' \frac{\rho(r, t - |r - r'|/c)}{4\pi \varepsilon_0|r - r'|}, \\
\mathbf{A}(r, t) &= \mathbf{A}^m(r, t) + \int d^3r' \int d^3t' \frac{\delta(t - t' - |r - r'|/c)}{4\pi|r - r'|} \mu_0 \mathbf{J}(r', t') \\
&= \mathbf{A}^m(r, t) + \int d^3r' \frac{\mu_0 \mathbf{J}(r, t - |r - r'|/c)}{4\pi|r - r'|},
\end{align*}
\]

where \( \Phi^m(r, t) \) and \( \mathbf{A}^m(r, t) \) are solutions of the homogeneous Maxwell’s equations, and the integrals provide the fields due to the sources at the retarded time. If the electrodynamic system included plane waves as in equation (4), then these would appear as \( \Phi^m(r, t) \) and \( \mathbf{A}^m(r, t) \).

The general solution of a system of linear differential equations is always written in the form of equations (5) and (6) which includes both the general solution of the homogeneous differential equations and a particular solution of the inhomogeneous differential equations.
However, for some reason, the textbooks of classical electromagnetism never write the general solution of Maxwell’s equations in this correct form\textsuperscript{2} [2, 3]. For some reason, the homogeneous solution part involving $\Phi^m(\mathbf{r}, t)$ and $\mathbf{A}^m(\mathbf{r}, t)$ is always omitted in textbooks, leaving only the integral over the sources multiplying the retarded Green function for the scalar wave equation in all spacetime\textsuperscript{3} [4–7]. One suspects that $\Phi^m(\mathbf{r}, t)$ and $\mathbf{A}^m(\mathbf{r}, t)$ are omitted because of an unspoken theoretical prejudice that all radiation arises from accelerating charges at a finite time. Many physicists cannot imagine any situation other than this prejudiced view. In their thinking, the Universe at the time of its formation contained only matter, and all radiation was emitted later. However, mathematically, it is possible to have homogeneous solutions as in equations (5) and (6). In the view which allows homogeneous solutions, the Universe at the time of its formation contained both radiation and matter. Clearly, $\Phi^m(\mathbf{r}, t)$ and $\mathbf{A}^m(\mathbf{r}, t)$ should be determined by experimental observation, not by prejudiced assumption.

\textbf{2.4. Charged harmonic oscillator system}

One of the simplest systems used in classical electrodynamics is the charged harmonic oscillator system, which can be pictured as composed of two charges $\pm e$ with one charge $e$ having mass $m$ attached at the end of a spring of spring-constant $k$ (with the other end attached to a charge $-e$ fixed in a wall), so that (neglecting the effects of damping) the oscillator has natural frequency of oscillation $\omega_0 = (k/m)^{1/2}$. Such charged harmonic oscillator systems are often used in discussions of dispersion and absorption in electromagnetism texts (see [1], section 9.4.3). The equation of motion for a small charged harmonic oscillator system located on the x-axis with equilibrium position at the origin of coordinates is (see [1], p 420, equation (9.154))

$$m \frac{d^2x}{dt^2} = -m\omega_0^2 x - m\gamma \frac{dx}{dt} + eE_0(0, t),$$

where $-m\gamma dx/dt$ is the damping force which is proportional to the velocity, and $eE_0(0, t)$ represents the force applied to the oscillator by the electric field which exists in the region. If the electric field driving the oscillator corresponds to the plane wave given in equations (2) and (4), then the steady-state solution for the displacement of the oscillator is (see [1], p 420, equation (9.157))

$$x(t) = \Re x(t) = \Re \frac{e\epsilon_0 E_0 \exp[-i(\omega t + \theta)]}{m(-\omega^2 + \omega_0^2 + i\omega\gamma)}.$$  

We notice that the oscillator has an oscillating position whose time-average is zero $\langle x \rangle = 0$, but whose mean square is given by

$$\langle x^2 \rangle = \frac{e^2\epsilon_0^2 E_0^2}{2m^2[(-\omega^2 + \omega_0^2)^2 + (\omega\gamma)^2]}.$$  

If we take the time derivative in equation (8) to obtain $\dot{x}(t)$, then the velocity is oscillating with an average which vanishes $\langle \dot{x} \rangle = 0$ and a mean square given by

\textsuperscript{2} One of the few accounts which gives the full correct form is Coleman’s report [2]. The report was reprinted twenty years later in [3].

\textsuperscript{3} I have never encountered a textbook of classical electromagnetism which gives the correct general solution. See, for example, [4–7]. The texts by both Jackson and Zangwill indeed note that the general solution of the scalar wave equation includes a solution of the homogeneous equation; but the homogeneous solutions have been dropped when the electromagnetic potentials are presented.
The energy of the oscillator is accordingly

\[ \langle \mathcal{E} \rangle = \left\{ \frac{1}{2} mx^2 + \frac{1}{2} m\omega_0^2 x^2 \right\} = \frac{(\omega^2 + \omega_0^2) e^2 e^2 E_0^2}{4m[(-\omega^2 + \omega_0^2)^2 + (\omega\gamma)^2]} . \] (11)

The oscillator acts as a small electric dipole oscillator with electric dipole moment 
\[ \mathbf{p}(t) = i e x(t) , \] whose charge density and current density can be approximated as (see [7], p 727)

\[ \rho(r, t) = -\nabla \cdot [\mathbf{p}(t)\delta^3(r)] , \] (12)

\[ \mathbf{J}(r, t) = \frac{d\mathbf{p}}{dt} \delta^3(r) . \] (13)

The charge and current distributions are sources for electromagnetic fields as given by equations (5) and (6). In the context of Maxwell’s equations for the electromagnetic fields of our system, the plane wave appearing in equations (2) and (4) and also appearing as the driving force in equation (7) is a homogeneous solution of Maxwell’s equations in equations (5) and (6) while the radiation emitted by the oscillator depends on the charge and current distributions in equations (12) and (13), giving

\[ \Phi(r, t) = 0 + \int d^3r' \int dt' \frac{\delta(t - t' - |r - r'|/c)}{4\pi\varepsilon_0| r - r'| \cdot [-\nabla' \cdot [i e x(t')\delta^3(r')]]} \]

\[ = -\nabla \left[ \frac{i e x(t - r/c)}{4\pi\varepsilon_0} \right] , \] (14)

\[ \mathbf{A}(r, t) = \frac{1}{\omega} \hat{E}_0 \sin(k \cdot r - \omega t + \theta) + \int d^3r' \int dt' \frac{\delta(t - t' - |r - r'|/c)}{4\pi\varepsilon_0 c^2 | r - r'|} \left\{ \frac{d\mathbf{p}}{dt} \right\} \delta^3(r) \]

\[ = \frac{1}{\omega} \hat{E}_0 \sin(k \cdot r - \omega t + \theta) + \frac{\mathbf{p}(t - r/c)}{4\pi\varepsilon_0 c^2} . \] (15)

Here the potentials \( \Phi(r, t) \) and \( \mathbf{A}(r, t) \) allow us to obtain the full electromagnetic fields associated with the system consisting of the plane wave and the oscillator.

It must be emphasised that the charged harmonic oscillator located in the plane wave is both absorbing energy from the incident plane wave and is also losing energy to emitted radiation. The energy loss and energy gain are in balance when the oscillator has the energy given in equation (11). The oscillator does not radiate away all its energy and collapse to zero energy because of the presence of the driving force of the plane wave corresponding to a solution of the homogeneous Maxwell’s equations.

3. Understanding incoming-wave boundary conditions on the homogeneous Maxwell equations in nature

3.1. An experimenter with a charged harmonic oscillator

Having introduced plane waves as examples of solutions of the homogeneous Maxwell equations and charged harmonic oscillators as simple classical electrodynamic systems, we
wish to explore the role of the incoming-wave boundary conditions on Maxwell’s equations in nature. Let us imagine an experimenter with a charged harmonic oscillator in his laboratory. As far as the experimenter is concerned, the homogeneous solutions of Maxwell’s equations $\Phi^{\text{in}}$ and $A^{\text{in}}$ are the fields which exist before he turns on his equipment and which are not at his control. If an experimenter is located in a place where the only electromagnetic fields are those which are produced by his own equipment, then he takes the incoming-wave homogeneous solutions of Maxwell’s equations to vanish, and he can arrange to start his experiments with his harmonic oscillator at rest at his origin of coordinates $x(t) = 0, \dot{x}(t) = 0$. However, if the experimenter is in a location where a nearby radio station is emitting radiation, then the experimenter will find that his charged oscillator has a non-zero amplitude due to forcing by the electromagnetic fields of the radio station. If the experimenter is confined to a small laboratory compared to the distance to the radio station, the experimenter would describe the electromagnetic fields of the radio station as plane waves satisfying the homogeneous Maxwell equations while the radiation emitted by his oscillator would give additional radiation. His oscillator is responding to the homogeneous solution which is not under his direct control. Indeed, the physical situation corresponds exactly to the electromagnetic fields following from the potentials given in equations (14) and (15). If the radio station turned off for the night, then the homogeneous fields $\Phi^{\text{in}}$ and $A^{\text{in}}$ for the experimenter would drop to zero, and the oscillator would emit all its energy as radiation and would collapse back to $x(t) = 0, \dot{x}(t) = 0$.

Of course, the need for the experimenter to include the solutions of the homogeneous Maxwell equations $\Phi^{\text{in}}$ and $A^{\text{in}}$ will depend upon the relative magnitudes of the energy delivered to his oscillator by his own equipment compared to the energy delivered to his oscillator by the homogeneous solution of Maxwell’s equations. If the energy delivered to the oscillator by his own equipment is overwhelmingly larger than the energy delivered by the homogeneous solution, then he can ignore the contribution of the homogeneous solution. This latter situation, appropriate for strong sources, seems to be that envisioned by the textbooks of classical electromagnetism which omit entirely the terms $\Phi^{\text{in}}$ and $A^{\text{in}}$ involving the homogeneous solution.

Now the resonant circuits measured by students in our elementary physics courses and indeed our modern radio receivers act like little harmonic oscillators with natural resonant frequencies $\omega_0$, analogous to the harmonic oscillator of our imagined experimenter. The energy delivered by a student-lab signal generator is overwhelmingly larger than that present in the ambient radiation in the laboratory, and so the effects of the homogeneous solution are not discussed. However, in our elementary labs, one sometimes asks students to touch the input lead to an oscilloscope to show that the laboratory classroom is actually filled with electromagnetic waves (usually generated by the classroom wiring or the fluorescent lamps) which could be described approximately as solutions $\Phi^{\text{in}}$ and $A^{\text{in}}$ of the homogeneous Maxwell equations which are not under the control of the equipment on the laboratory table.

### 3.2. An experimenter in thermal radiation

Suppose now that the experimenter with his harmonic oscillator is interested in making delicate measurements of very low-energy phenomena. We imagine that he is located not near a radio station but rather in a room at non-zero temperature $T$, so that the solution of the homogeneous Maxwell equations corresponds to thermal electromagnetic radiation. Within classical physics, thermal radiation is described as random electromagnetic radiation with a characteristic ‘thermal’ spectrum corresponding to the spectrum of blackbody radiation. Taking periodic boundary conditions for a very large box of volume $V$, we can treat thermal
radiation as a sum over plane waves of all wave vectors. Following the form in equation (4), we can choose the scalar potential \( \Phi_\lambda^a(r, t) = 0 \) and the vector potential as

\[
A_\lambda^a(r, t) = \sum_{\mathbf{k}, l=1}^2 \frac{1}{\omega} \mathbf{e}(\mathbf{k}, \lambda) \left( \frac{2U(\omega, T)}{\epsilon_0 V} \right)^{1/2} [\sin(\mathbf{k} \cdot \mathbf{r} - \omega t + \theta(\mathbf{k}, \lambda))].
\]

where the wave vectors \( \mathbf{k} \) correspond to \( \mathbf{k} = \hat{i}2\pi/a + \hat{j}m2\pi/a + \hat{k}n2\pi/a \) with \( l, m, n \) running over all positive and negative integers, \( a \) is a length such that \( a^3 = V \), and the two mutually-orthogonal polarisation vectors \( \mathbf{e}(\mathbf{k}, \lambda) \) are orthogonal to the wave vectors \( \mathbf{k} \). Since thermal radiation is isotropic in the inertial frame of its container, the amplitude \( |U(\omega, T)|^{1/2} \) depends only on the frequency \( \omega = c|\mathbf{k}| = ck \), and the constants are chosen so that \( U(\omega, T) \) is the energy per normal mode appropriate for the thermal radiation spectrum in classical physics. In order to describe the randomness of the radiation, the phases \( \theta(\mathbf{k}, \lambda) \) are chosen as random variables uniformly distributed on \( (0, 2\pi) \), independently distributed for each \( \mathbf{k} \) and \( \lambda \).

The electric and magnetic fields of the thermal radiation in the experimenter’s laboratory would set his dipole oscillator into oscillation, just as the electric and magnetic fields of the radio station set the oscillator into oscillation according to equation (7). For thermal radiation as given in equation (16) and the oscillator located at the coordinate origin, the steady-state solution corresponding to equation (8) becomes

\[
x(t) = \text{Re} \sum_{\mathbf{k}, \lambda=1}^2 \epsilon_\lambda(\mathbf{k}, \lambda) \left( \frac{2U(\omega, T)}{\epsilon_0 V} \right)^{1/2} \frac{e \exp \{i[-\omega t + \theta(\mathbf{k}, \lambda)]\}}{m(-\omega^2 + \omega_0^2 + i\omega\gamma)}.
\]

Again, the experimenter would describe the forcing electromagnetic fields acting on his oscillator as solutions of the homogeneous Maxwell’s equations while the electromagnetic fields radiated by his oscillator would be given by the retarded fields associated with the charges densities and current densities of the oscillator motion, just as indicated in equations (5) and (6).

In thermal radiation, the mean displacement of the oscillator and the mean velocity are both zero \( \langle x(t) \rangle = 0, \langle \dot{x}(t) \rangle = 0 \), but the mean squares are non-zero. We can find the mean-square displacement by averaging over time or averaging over the random phases at a fixed time. Since the random phases \( \theta(\mathbf{k}, \lambda) \) are distributed randomly and independently for each mode, we have the averages

\[
\langle \exp \{i[-\omega t + \theta(\mathbf{k}, \lambda)]\} \exp \{i[-\omega't + \theta(\mathbf{k}', \lambda')\] \rangle = 0
\]

and

\[
\langle \exp \{i[-\omega t + \theta(\mathbf{k}, \lambda)]\} \exp \{-i[-\omega't + \theta(\mathbf{k}', \lambda')\] \rangle = \delta_{\mathbf{k}\mathbf{k}'}\delta_{\lambda\lambda'}
\]

which give

\[
\langle x^2 \rangle = \sum_{\mathbf{k}, \lambda=1}^2 \epsilon_\lambda^2(\mathbf{k}, \lambda) \left( \frac{2U(\omega, T)}{\epsilon_0 V} \right)^{2/2} \frac{e^2}{2m^2[(-\omega^2 + \omega_0^2)^2 + (\omega\gamma)^2]},
\]

\[
\langle \dot{x}^2 \rangle = \sum_{\mathbf{k}, \lambda=1}^2 \epsilon_\lambda^2(\mathbf{k}, \lambda) \left( \frac{2U(\omega, T)}{\epsilon_0 V} \right)^{2/2} \frac{e^2\omega^2}{2m^2[(-\omega^2 + \omega_0^2)^2 + (\omega\gamma)^2]}.
\]
and the average energy of the oscillator

$$\langle \mathcal{E}(\omega_0, T) \rangle = \sum_{\mathbf{k}, \lambda}^2 \epsilon^2_{\lambda}(\mathbf{k}, \lambda) \left( \frac{2U(\omega, T)}{\epsilon_0 V} \right) \frac{e^2(\omega^2 + \omega_0^2)}{4m[(-\omega^2 + \omega_0^2)^2 + (\omega \gamma)^2]}.$$  \hspace{1cm} (22)

Now we are imagining that the box (or laboratory) which holds the thermal radiation is sufficiently large so that the normal modes are very closely spaced, and therefore the sum over normal modes can be replaced by an integral

$$\langle \mathcal{E}(\omega_0, T) \rangle = \frac{a^3}{2\pi} \int \frac{d^2k}{2\pi} \sum_{\lambda=1}^{2N} \epsilon^2_\lambda(\mathbf{k}, \lambda) \left( \frac{2U(\omega, T)}{\epsilon_0 V} \right) \frac{e^2(\omega^2 + \omega_0^2)}{4m[(-\omega^2 + \omega_0^2)^2 + (\omega \gamma)^2]}$$ \hspace{1cm} (23)

which is sharply peaked at $\omega = \omega_0$.

Although the textbooks of classical electrodynamics are often interested in a damping force corresponding to the absorption of energy by the material forming the oscillator, we are interested in the simplest possible case where the damping term represents an approximation to the radiation damping force $F_r = [e^2/(6\pi\epsilon_0 c^3)]d^3x/dt^3 \approx -\omega_0^2[e^2/(6\pi\epsilon_0 c^3)]dx/dt$, so that $\gamma = \omega_0^2[e^2/(6\pi\epsilon_0 c^3)]$. In this case, we can integrate over all angles so that $\epsilon^2_\lambda(\mathbf{k}, \lambda)$ contributes a factor of $1/3$ for each polarisation, and then approximate the integral over $k = \omega/c$ by extending the lower limit to minus infinity, setting $\omega = \omega_0$ in every term except for $(-\omega^2 + \omega_0^2) \approx 2\omega_0(\omega_0 - \omega)$, and using the definite integral

$$\int_{-\infty}^{\infty} \frac{dx}{a^2x^2 + b^2} = \frac{\pi}{ab}$$ \hspace{1cm} (24)

to obtain for the average oscillator energy $^4$ [8]

$$\langle \mathcal{E}(\omega_0, T) \rangle = U(\omega_0, T).$$ \hspace{1cm} (25)

Thus the average energy $\langle \mathcal{E}(\omega_0, T) \rangle$ of the oscillator with resonant frequency $\omega_0$ is the same as the average energy $U(\omega_0, T)$ of the radiation normal mode at the same frequency.

4. Random radiation and experiment

4.1. Random classical radiation known at the end of the 19th century

The description we have given for the classical electrodynamics involving a charged harmonic oscillator corresponds exactly to that involved in the thinking of the physicist at the end of the 19th century. It parallels the classical development given by Planck in his theory of thermal radiation [9]. Now the experiments of Lummer and Pringsheim [10] in 1899 were fitted by Planck [11] in 1900 to a spectrum for thermal radiation

$$U_p(\omega, T) = \frac{\hbar\omega}{\exp[\hbar\omega/k_BT] - 1} = \frac{1}{2} \frac{\hbar\omega}{2k_BT} \coth \left( \frac{\hbar\omega}{2k_BT} \right) - \frac{1}{2} \frac{\hbar\omega}{2}$$ \hspace{1cm} (26)

which is now termed the Planck spectrum. For a non-zero value of temperature $T$, the Planck radiation spectrum given in equation (26) goes to zero for large values of frequency $\omega$ because of the exponential function appearing in the denominator. At low frequencies $\omega$, the spectrum in equation (26) goes over to

$^4$ The calculation for the average energy of the oscillator given here is the traditional calculation dating back to the early twentieth century. See, for example, [8].
The low-frequency value $U_p(\omega, T) \approx k_B T - \frac{1}{2} \hbar \omega + O(\hbar \omega/k_B T)$ for $k_B T \gg \hbar \omega$. (27)

The low-frequency value $U_p(\omega, T) \approx k_B T$ fits with the equipartition result of non-relativistic classical statistical mechanics for a harmonic oscillator at temperature $T$.

Now an experimenter in a laboratory at room temperature would have a difficult time measuring the thermal radiation of his harmonic oscillator. At room temperature, the energy per normal mode gives $k_B T = 1/40 \text{ eV}$. This energy is vastly smaller than the energies of the macroscopic harmonic oscillators of our elementary mechanics courses or of the resonant circuits in our elementary electromagnetism classes. Rather, this value corresponds to energy at the atomic scale. We have no macroscopic evidence regarding a single oscillator at this scale. On the other hand, if we go to high enough temperatures for a single oscillator to have a macroscopic energy, the thermal energy would destroy the harmonic oscillator system. Lummer and Pringsheim’s experiments escaped the limitations of a single oscillator by measuring the thermal energy received by a macroscopic surface, and so involved an enormous number of oscillators.

4.2. Random classical radiation known at the beginning of the 21st century

When continuing our story of the experimenter with his harmonic oscillator in his laboratory, we will want to use not only the experimental information available at the end of the 19th century but also the experimental information available at the beginning of the 21st century. It turns out that the thermal experiments of 1899 contained a crucial limitation which confused the classical electromagnetic theorists at the beginning of the 20th century. The Lummer–Pringsheim experiments of 1899 measured only the random radiation of their sources which was above the random radiation surrounding their measuring devices. If their sources were at the same temperature as their measuring devices, the measuring devices would have registered no signal at all. Today, in contrast to the end of the 19th century, random classical radiation measurements are available which are of an entirely different character from those of Lummer and Pringsheim.

Textbooks of classical electromagnetism point out that radiation falling on a conducting surface will place a force on the surface (see [1], p 400). Today it is possible to make measurements of the forces, termed Casimir forces\textsuperscript{5} [12, 13] between parallel conducting plates due to any radiation which is present surrounding the plates. These force measurements allow the measurement of all the radiation falling on the plates. In contrast to the limitation present in the thermal measurements at the end of the 19th century, any random radiation emitted, absorbed or reflected by the plates themselves contributes to the forces between the plates and will be registered by force measurements on the plates. The forces may be small, but since the area of the plates can be made large, the forces are measurable as macroscopic forces. For plates of area 1 cm\textsuperscript{2}, and separations of half a micron, the force is measured as 1/5 dyne.

In the measurements of Casimir forces, it is the wavelengths roughly comparable to the plate separation which give the dominant force contribution. Thus at large parallel-plate separations, the long wavelength radiation corresponding to the Rayleigh–Jeans spectrum contributes. At small separations, the short-wavelength high-frequency waves provide the dominant force, and, according to the Planck expression (26) corresponding to Lummer and Pringsheim’s thermal measurements, the radiation goes to zero at low temperatures. However, when the Casimir-force measurements between parallel conducting plates are actually carried

\textsuperscript{5} The literature on Casimir forces is vast. A systematic organisation of the references is given by Lamoreau [13].
out at low temperature, it is found that the forces between the plates do not correspond to the spectrum of Lummer and Pringsheim’s thermal measurements. Rather the forces between the plates do not go to zero as the temperature is decreased [14]. Thus instead of the spectrum given in equation (26), the spectrum of random electromagnetic radiation found from modern Casimir force experiments contains an additional part at low temperatures. The spectrum of random electromagnetic radiation which fits both the thermal measurements of Lummer and Pringsheim and also the Casimir force measurements corresponds to an energy per normal mode

\[ U_C(\omega, T) = \frac{\hbar \omega}{2k_B T} \coth \left( \frac{\hbar \omega}{2k_B T} \right) = \frac{1}{\exp[\hbar \omega/(k_B T)] - 1} + \frac{1}{2} \hbar \omega. \]  

(28)

At high temperatures, this spectrum again goes over to the equipartition result

\[ U_C(\omega, T) \approx k_B T + O(\hbar \omega/k_B T) \text{ for } k_B T \gg \hbar \omega. \]  

(29)

However, at low temperatures, the spectrum has an energy per normal mode

\[ U_C(\omega, T) \approx (1/2)\hbar \omega + O(\hbar \omega/k_B T) \text{ for } k_B T \ll \hbar \omega \]  

(30)

corresponding to a zero-point spectral energy per normal mode \( U_C(\omega, 0) = U_{zp}(\omega) \) with

\[ U_{zp}(\omega) = (1/2)\hbar \omega. \]  

(31)

Let us now return to our hypothetical experimenter with his charged harmonic oscillator. Since we have no direct macroscopic measurements of a single harmonic oscillator in thermal radiation, we must infer the behaviour of the oscillator of our imagined experimenter. Clearly, we wish to use the best possible experimental information in order to infer this behaviour. The latest experimental information indicates that the random radiation which would influence the oscillator of our imagined experimenter in his laboratory does not correspond to the 1899 thermal spectrum in equation (26) but rather to the full spectrum of equation (28).

### 4.3. Classical electromagnetic zero-point radiation

The radiation energy per normal mode given in equation (28) corresponds closely to the spectrum found in the 1899 measurements of Lummer and Pringsheim, but it also includes additional random radiation which exists even as the temperature \( T \) goes to zero. This random radiation which exists at zero-temperature is termed zero-point radiation. Since here we are describing nature with a purely classical electromagnetic theory, we will term this radiation ‘classical electromagnetic zero-point radiation’.

Based on the 1899 thermal spectrum in equation (26), the classical physicists at the beginning of the 20th century believed that they could take the solutions of the homogeneous Maxwell equations to vanish by screening their laboratories and going to the absolute zero of temperature. Lorentz makes this explicit in his monograph *The Theory of Electrons*\(^6\) [15]. Today at the beginning of the 21st century, a classical physicist cannot make this assumption. Our imagined experimenter in his laboratory may cool his laboratory close to absolute zero, yet he can not escape the random zero-point radiation revealed by experimental measurements of Casimir forces. Any classical description of electromagnetism in accord with nature must include zero-point radiation which corresponds to the solution of the homogeneous Maxwell equations relevant at \( T = 0 \).

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\(^6\) Traditional classical electron theory is described by Lorentz [15]. This volume is a republication of the second edition of 1915 based on Lorentz’s Columbia University lectures of 1909. On page 20 and on page 240, note 6, Lorentz gives his explicit assumption on the boundary conditions for Maxwell’s equations; the assumption excludes the possibility of classical electromagnetic zero-point radiation.
4.4. Classical zero-point energy and classical ground states

It must be emphasised that the classical electromagnetic zero-point radiation appearing in the experiments on Casimir forces will also influence the charged harmonic oscillator of our experimenter in his laboratory. If the experimenter is far from any radio station and he also cools his laboratory to near absolute zero, he will still find a zero-point motion for his harmonic oscillator given by

\[ \langle x^2 \rangle = \frac{\hbar}{2m\omega}, \quad \langle \dot{x}^2 \rangle = \frac{\hbar\omega}{2m}, \]

and

\[ \langle E(\omega_0, 0) \rangle = \frac{\hbar \omega}{2}. \]  \hspace{1cm} (32)

The classical harmonic oscillator shows a random zero-point motion derived from the random electromagnetic radiation which exists even at the zero of temperature. From the point of view of classical electromagnetism, the zero-point energy of an oscillator is not something intrinsic to the oscillator and the state of the charged oscillator in zero-point radiation does not involve a non-radiating situation. Rather in the natural classical electromagnetic point of view, the zero-point electromagnetic radiation (corresponding to a solution of the homogeneous Maxwell equations) produces an oscillation of the oscillator which involves a balance between the absorption and emission of radiation.

4.5. Zero-point radiation versus thermal radiation in classical electromagnetism

It should be noted that although zero-point radiation in equation (31) and thermal radiation at \( T \geq 0 \) appearing in equation (28) are treated on the same footing in classical electromagnetism, they have different effects because of the differences in the electromagnetic spectra at zero temperature and at finite temperature. The spectrum of zero-point radiation (31) holding at zero temperature is Lorentz invariant, scale invariant, and invariant under adiabatic compression [16]. The zero-point radiation spectrum is isotropic in any inertial frame. In free space, the zero-point spectrum can not give rise to velocity-dependent forces on particles because of its Lorentz-invariant character. The correlation functions involving zero-point radiation depend upon only the geodesic separation between the spacetime points at which it is evaluated [17]. In contrast, the spectrum of thermal radiation (28) (including the zero-point radiation) at a non-zero temperature \( T > 0 \) has a preferred reference frame, that unique frame in which the spectrum is isotropic. At finite non-zero temperature in free space, the thermal spectrum gives velocity-dependent forces on any system having electromagnetic interactions which is moving through the thermal radiation. The differences in the spectra provide a qualitative classical way for thinking about many aspects of thermal versus zero-point energy.

5. Discussion

5.1. Connections between classical electromagnetism and modern physics

The description of classical electromagnetic physics which we have given here is fully in accord with classical electromagnetic theory and with experiment. However, it also shows a direct connection with certain basic aspects of a course in modern physics. Harmonic oscillators are treated not only in electromagnetism texts but also in texts of modern physics and of quantum mechanics, where a quantum oscillator has a zero-point energy in its ground state. If the quantum harmonic oscillator is charged, then the oscillator ground state is said to have zero-point energy but involves a non-radiating state. Although quantum physics and classical electromagnetism describe harmonic oscillators differently, they both arrive at the same results for the mean-square displacement, mean-square velocity, and average energy.
Indeed, one can prove a general connection between the results of quantum theory and the results of classical electrodynamics with classical electromagnetic zero-point radiation for free fields and harmonic oscillator systems [18]. The classical electromagnetic theory including classical zero-point radiation can give descriptions of Casimir forces, Van der Waals forces, harmonic oscillator systems, specific heats of solids, diamagnetism, blackbody radiation, and the absence of atomic collapse in hydrogen

\[19, 20\]. The theory gives a satisfying classical perspective on a number of phenomena. However, it should be emphasised that the classical electromagnetic theory here is not equivalent to quantum theory and is not a substitute for quantum theory. We still do not know the areas of agreement and disagreement between nature and classical electrodynamics including classical zero-point radiation. Our ignorance regarding the results of the classical theory compared to quantum theory is perhaps not surprising. It should noted that quantum theory has been developed for over a century by a vast number of physicists. The inclusion of classical zero-point radiation within classical electrodynamics has been explored systematically only within the last half-century by a tiny group of physicists.

5.2. Treatment of thermal radiation in classical and quantum physics

It is noteworthy that quantum physics makes a clear distinction between zero-point energy and the thermal energy of excited states, whereas classical electrodynamics regards both these energies in equation (28) as having the same character. Quantum ideas first arose in an attempt to interpret the experimental data on blackbody radiation at the turn of the 20th century, and, on this account, quantum theory makes a distinction between the thermal ‘photons’ which give the Planck spectrum in equation (26) and the zero-point energy which involves no photons. On the other hand, classical electrodynamics with classical electromagnetic zero-point radiation arose largely from attempts to understand Casimir forces and has no basis for any such distinction between zero-point energy and thermal energy. The experimental data for Casimir forces requires that the full spectrum of random classical radiation corresponds to equation (28), including both zero-point radiation and the thermal contributions. Thus within classical electrodynamics, zero-point radiation is a continuous part of the random radiation spectrum which includes thermal radiation. Indeed, the treatment of thermal fluctuations and the calculations of Casimir forces and of van der Waals forces illustrate the distinctions between the classical and quantum points of view [19, 20]. Within quantum theory, thermal fluctuations involve the wave-particle duality of nature, whereas within classical electrodynamics, thermal fluctuations involve simply the fluctuations of the full random electromagnetic field above the zero-point background [21]. Within quantum physics, Casimir forces and van der Waals forces are calculated using both the ground state zero-point energy and then separate contributions from the thermal photons responsible for the spectrum in equation (26). Within classical electrodynamics, Casimir forces and van der Waals forces are calculated once using the full spectrum of random classical electromagnetic given in equation (28). For these force calculations, the classical electromagnetic calculations are distinctly easier yet arrive at exactly the same results as the quantum calculations [19, 20].

\[7\] Classical electron theory with classical electromagnetic zero-point radiation is often termed ‘stochastic electrodynamics’. See the monograph by de la Pena and Cetto [19]. A more recent brief survey regarding classical zero-point radiation is given by Boyer [20].
5.3. Does classical electromagnetic zero-point radiation exist?

Some physicists are strongly antagonistic to the idea of ‘classical’ zero-point energy; they insist the classical zero-point energy does not exist and that Planck’s constant is a ‘quantum constant.’ Does classical zero-point energy actually exist? The answer is analogous to that as to whether or not gravitational forces exist. In Newtonian physics, the planets follow orbits around the Sun due to gravitational forces. Thus in the context of Newtonian physics, we would say that gravitational forces certainly exist. However, within general relativity, planets follow their paths around the Sun due to the curvature of spacetime, and we do not speak of the existence of gravitational forces because their role has been replace by that of curved space. Both quantum physics and classical physics allow the idea of zero-point energy. If one wishes to describe nature in terms of classical physics, then classical zero-point energy certainly exists, arising from the classical electromagnetic zero-point radiation which is the incoming-wave boundary condition on the homogeneous Maxwell equations, as described in the present article. In quantum physics, the corresponding role is played by ‘quantum zero-point energy.’

The objection that Planck’s constant $h$ is a ‘quantum constant’ and hence is not allowed in classical theory represents a misunderstanding of the nature of fundamental physical constants. Planck’s constant $h = 2\pi\hbar$ was introduced into physics in 1899 as a parameter in the fit to the experimental data on blackbody radiation [22]. Thus the constant $\hbar$ (or $h$) is a fundamental constant of nature which needs not have a connection to any particular theory. Within quantum physics, $\hbar$ sets the scale for the quantum of action. Within classical electrodynamics, $h$ sets the scale of random classical radiation needed to describe Casimir forces, and so sets the scale of classical electromagnetic zero-point radiation; the zero-point energy of the radiation reappears as zero-point energy for all systems which have electromagnetic interactions.

References

[1] Griffiths D J 2013 Introduction to Electrodynamics 4th edn (New York: Pearson) p 338, section 7.3.3
[2] Coleman S 1961 Classical Electron Theory from a Modern Standpoint (Santa Monica, CA: RAND Corp.) p 13, equation (2)
[3] Teplitz D (ed) 1982 Electromagnetism: Paths to Research (New York: Springer) pp 183–210, ch 6
[4] See, for example, Griffiths D J 2013 Introduction to Electrodynamics 4th edn (New York: Pearson) section 10.2.1, equation (10.26)
[5] Eyges L 1972 The Classical Electromagnetic Field (Reading, MA: Addison-Wesley) p 186, equations (11.45), (11.46)
[6] Jackson J D 1999 Classical Electrodynamics 3rd edn (New York: Wiley) p 246, equation (6.48)
[7] Zangwill A 2013 Modern Electrodynamics (New York: Cambridge University Press) p 724–5, equations (20.57) and (20.58)
[8] See, for example, Lavenda B H 1991 Statistical Physics: A Probabilistic Approach (New York: Wiley-Interscience) pp 73–4, section 2.2.1
[9] Planck M 1959 The Theory of Heat Radiation (New York: Dover)
[10] Lummer O and Pringsheim E 1899 Die vertheilung der energie im spectrum des schwarzen köpers und des blanken plattins Verh. Dtsch. Phys. Ges. 1 215–35
[11] Planck M 1900 Über eine Verbesserung der Wien’schen Spektargleichung Verh. Dtsch. Phys. Ges. 2 202–4
[12] Casimir H B G 1948 On the attraction between two perfectly conducting plates Proc. Kon. Ned. Akad. Wetensch. 51 793–5
[13] Lamoreau S K 1999 Resource letter CF-1: Casimir force Am. J. Phys. 67 850–61
[14] Sparnaay M J 1958 Measurement of the attractive forces between flat plates *Physica* **24** 751–64
Lamoreaux S K 1997 Demonstration of the Casimir force in the 0.6 to 6 μm range *Phys. Rev. Lett.* **78** 5–8
Lamoreaux S K 1998 Demonstration of the Casimir force in the 0.6 to 6 μm range *Phys. Rev. Lett.* **81** 5475–6
Mohideen U 1998 Precision measurement of the Casimir force from 0.1 to 0.9 μm *Phys. Rev. Lett.* **81** 4549–52
Chan H B, Aksyuk V A, Kleiman R N, Bishop D J and Capasso F 2001 Quantum mechanical actuation of microelectromechanical systems by the Casimir force *Science* **291** 1941–4
Bressi G, Carugno G, Onofrio R and Ruoso G 2002 Measurement of the Casimir force between parallel metallic surfaces *Phys. Rev. Lett.* **88** 041804

[15] Lorentz H A 1952 *The Theory of Electrons* (New York: Dover)

[16] Boyer T H 1975 Random electrodynamics: the theory of classical electrodynamics with classical electromagnetic zero-point radiation *Phys. Rev. D* **11** 790–808

[17] Boyer T H 2013 Contrasting classical and quantum vacuum states in non-inertial frames *Found. Phys.* **43** 923–47

[18] Boyer T H 1975 General connection between random electrodynamics and quantum electrodynamics for free electromagnetic fields and for dipole oscillator systems *Phys. Rev. D* **11** 809–30

[19] de la Pena L and Cetto A M 1996 *The Quantum Dice—An Introduction to Stochastic Electrodynamics* (Dordrecht: Kluwer)

[20] Boyer T H 2011 Any classical description of nature requires classical electromagnetic zero-point radiation *Am. J. Phys.* **79** 1163–7

[21] Boyer T H 1969 Classical statistical thermodynamics and electromagnetic zero-point radiation *Phys. Rev.* **186** 1304–18

[22] Gillispie C C (ed) 1975 *Dictionary of Scientific Biography* vol 11 (New York: Scribners) p 11