PARTICLES WITH SPIN IN STATIONARY FLAT SPACETIMES

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Abstract. We construct stationary flat three-dimensional Lorentzian manifolds with singularities that are obtained from Euclidean surfaces with cone singularities and closed one-forms on these surfaces. In the application to (2+1)-gravity, these spacetimes correspond to models containing massive particles with spin. We analyse their geometrical properties, introduce a generalised notion of global hyperbolicity and classify all stationary flat spacetimes with singularities that are globally hyperbolic in that sense. We then apply our results to (2+1)-gravity and analyse the causality structure of these spacetimes in terms of measurements by observers. In particular, we derive a condition on observers that excludes causality violating light signals despite the presence of closed timelike curves in these spacetimes.

1. Introduction

Flat three-dimensional Lorentzian manifolds with conical singularities were first introduced in the physics literature on (2+1)-dimensional gravity, where they model (2+1)-dimensional spacetimes that contain massive point particles with spin.

The first models of (2+1)-gravity with particles were derived in [Sta63] and [SD84]. Their physical properties and their quantisation were studied in the subsequent publications [SD88, Car89, dSG90, tH93b, tH93a, tH96], which led to a large body of work on the classical aspects and quantisation of the models, for an overview see [Car03].

As they are models that include matter and still are amenable to quantisation, these models play an important role in the research subject of quantum gravity. Since they allow one to investigate the quantisation of gravity coupled to matter, they have been studied extensively in the physics literature. Another reason why these models are of interest in quantum gravity is their causality structure. It was shown in [SD84] that the presence of massive point particles with spin leads to the presence of closed timelike curves in these models which, however, can be removed by excising a small cylinder around each particle. Additionally, closed timelike curves can be generated dynamically when two spinless massive particles approach each other with sufficiently high speed ("Gott pairs"). A detailed investigation of this phenomenon has been given in [Got91, SD92b], with the conclusion that these dynamically generated closed, timelike curves are not physically meaningful since they are present only for very short times, and, in particular, "time machines" are excluded [SD92c, Des93].

Despite their relevance, many geometrical properties of the models including massive point particles with spin are not fully understood even on the classical level. Although their properties have been investigated in the physics literature, they are very few results concerning the underlying mathematical structures. The closest treatment in the mathematics literature is the study of geometric Riemannian manifolds, mostly in the case of euclidean, spherical or hyperbolic surfaces with conical singularities ([Tro07], [Mas06, MT02]) sometimes in relation to...
the work on billiards, and in the 3-dimensional case the work devoted to the Orbifold Theorem (CHK00, BLP05).

A similar treatment in the Lorentzian case is still at the beginning. This includes in particular the causality issues arising in these models as well as the lack of systematic definitions and classification. A systematic investigation of the mathematical features of three-dimensional Lorentzian spacetimes with particles has been initiated only recently and is mainly concerned with the case of constant negative curvature (BB09, BS09, LS09, KS07, BBS09).

In this article, we investigate flat stationary Lorentzian spacetimes with a general number of massive particles with spin. We construct examples of stationary flat Lorentzian spacetimes with particles that are based on Euclidean surfaces with cone singularities and closed one-forms on these surfaces. We introduce a generalised notion of global hyperbolicity that can be applied to these models despite the fact that they contain closed timelike curves. Based on this notion of global hyperbolicity, we classify the flat stationary globally hyperbolic Lorentzian spacetimes with particles and give a detailed analysis of their geometrical properties.

The last section of the article is dedicated to a problem that is of high relevance to physics, namely the question, how the presence of particles manifests itself in measurements by observers that probe the geometry of the spacetime by exchanging “test light rays.” This idea is very natural from the viewpoint of general relativity, whose physical interpretation was formulated in terms of light rays exchanged by observers from the beginning. It is also of special relevance to quantum gravity in four dimensions, as it is hoped that quantum gravity effects might manifest itself in cosmic microwave background radiation and thus be determined by means of light rays. The (2+1)-dimensional models considered in this work share many properties with the cosmological models investigated in (3+1) dimensions.

We show that light signals exchanged by observers correspond naturally to piecewise geodesic curves on the underlying Euclidean surfaces with cone singularities. We demonstrate how an observer can construct the relevant parameters that describe the spacetime from such measurements: the positions, masses and spins of such particles as well as their velocity with the respect to the observer.

Building on these results, we investigate the causality issues associated with closed, timelike curves in spacetimes containing particles with spin. In particular, this allows one to establish a condition on the observers that excludes paradoxical signals, i.e. signals that are received before they are emitted. In physical terms, this implies that observers that stay away a sufficient distance from each particle, will not experience paradoxical light signals, even if the light signals themselves enter a region around the particle which contains closed, timelike curves.

Note that these models do not involve the dynamically generated closed, timelike curves that are investigated in Got91, SD92b, SD92c, Des93, since the spacetimes under consideration are stationary. Instead, the spacetimes investigated in this paper contain closed timelike curves that are due to the presence of massive point particles with spin. Rather than investigating the impact of dynamically generated closed timelike curves and determining the spacetime regions in which they occur, we focus on the impact of these closed timelike curves on observers at a sufficient distance from the particles and on light signals exchanged by such observers.

2. The model: a single particle with spin

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The name “test light rays” is motivated by the fact that they play a role similar to the test masses used in general relativity. They are lightlike geodesics in a given spacetime rather than actual lightlike point sources, and we neglect their impact on the stress energy tensor. This is different from the treatment in SD92a, which considers solutions of the Einstein equations in (2+1) dimensions with lightlike point sources.
2.1. Definition. In the following we denote by $\mathbb{R}^{1,2}$ the three-dimensional Minkowski space and by $\Delta$ a timelike geodesic in $\mathbb{R}^{1,2}$. We choose a suitable coordinate system $(x, y, t)$, in which the Minkowski norm takes the form $ds^2 = dx^2 + dy^2 - dt^2$ and $\Delta$ is given by the equation $x = y = 0$. We also introduce spatial polar coordinates $(r, \theta)$, $r \in \mathbb{R}^+$, $\theta \in [0, 2\pi)$ which are given in terms of the spatial coordinates $x, y$ by $x = r \cos \theta$, $y = r \sin \theta$.

The model for a single particle introduced in [SD84] depends on two real parameters, an angle $0 < \theta_0 < 2\pi$, in the following referred to as deficit or apex angle, and a parameter $\sigma \in \mathbb{R}$, in the following referred to as spin. The names for these parameters are motivated by their physical interpretation. It is shown in [SD84], see also [Car03], that the metric for a single point particle in $\mathbb{R}^{2,1}$ is that of a cone with a deficit angle $\theta_0 = 2\pi - \mu$, where $\mu \in [0, 2\pi)$ is the mass of the particle in units of the Planck mass. Moreover, it is shown there that the resulting spacetime has a non-trivial asymptotic angular momentum that is given by $\sigma \in \mathbb{R}$. The parameter $\sigma$ which therefore is viewed as an internal angular momentum or spin of the particle in units of $\hbar$ [SD84].

The associated flat Lorentzian manifold is constructed as follows. The equations $\theta = 0, \theta = \theta_0$ define two timelike half-planes, which both have the timelike geodesic $\Delta$ as their boundary and which we denote, respectively, $P_0, P_1$. These half-planes bound the region

$$W_{\theta_0} := \{r > 0, 0 \leq \theta \leq \theta_0\},$$

that we call a wedge of angle $\theta_0$. We glue $P_0$ to $P_1$ by the map $(r, \theta, t) \mapsto (r, \theta + \theta_0, t + \sigma)$, which is a restriction of the elliptic isometry $g : (r, \theta, t) \mapsto (r, \theta + \theta_0, t + \sigma)$. The gluing of the wedge is pictured in Figure 1. The result is a manifold $M'_{\theta_0, \sigma}$, that is naturally equipped with a flat Lorentz metric and homeomorphic to $\mathbb{R}^3$ minus a line. Note that the time orientation of Minkowski space induces a time orientation on $M'_{\theta_0, \sigma}$, namely the one for which the coordinate $t$ increases along future oriented causal curves.

We now demonstrate how a (singular) line can be added to $M'_{\theta_0, \sigma}$ to obtain a manifold homeomorphic to $\mathbb{R}^3$. For this, one is tempted to extend the gluing defined above to the closed
wedge $\bar{W}_{\theta_0} := \{ r \geq 0, 0 \leq \theta \leq \theta_0 \}$ in such a way that points of the form $(0,0,t)$ are identified with $(0,0,t+\sigma)$. This is possible if $\sigma = 0$ and in that case yields a singular flat spacetime $M_{\theta_0,0}$ that contains a singular line characterised by the condition $r = 0$. It corresponds to a $(2+1)$-dimensional spacetime with a single particle of mass $\mu = 2\pi - \theta_0$ and vanishing spin $\sigma = 0$.

However, this procedure does not work in the case $\sigma \neq 0$. For non-vanishing spin $\sigma$, the quotient of $\Delta$ by this isometry is a circle and not a line. When equipped with the quotient topology, it is no longer a manifold. Indeed, an open disc in the $xy$-plane that is centred at the point $(0,0,0)$ corresponds to a union $\bigcup_{n=0}^{\infty} D_n$ of infinitely many circular sectors $D_n = \{(r,\theta,t)|r < R_0, 0 \leq \theta \leq \theta_0, t = n\sigma, n \in \mathbb{Z}\}$ that are identified along the line segments given by $\theta = 0$ and $\theta = \theta_0$.

A more transparent description of spacetimes containing particles with non-vanishing spin is obtained by introducing a new set of coordinates $(r,\alpha,\tau)$ that includes the radial coordinate $r$ as well as

$$\tau = \theta_0 t - \sigma \theta, \quad \alpha = \frac{2\pi}{\theta_0} \tau.$$

As the coordinate $\alpha$ has the range $0 \leq \alpha < 2\pi$, it induces a map $\varphi : \mathbb{R}^3 \setminus \{ r = 0 \} \to M'_{\theta_0,\sigma}$.

The pull-back by $\varphi$ of the flat metric on $M'_{\theta_0,\sigma}$ to $\mathbb{R}^3 \setminus \{ r = 0 \}$ is given by

$$ds^2 = -\frac{1}{\theta_0^2} d\tau^2 - \frac{\sigma}{\pi \theta_0} d\alpha d\tau + \frac{r^2 \theta_0^2 - \sigma^2}{4\pi^2} d\alpha^2 + dr^2.$$

In the following we denote by $M_{\theta_0,\sigma}$ the manifold $\mathbb{R}^3$ equipped with this metric outside the singular line given by $r = 0$. It contains (an isometric copy of) $M'_{\theta_0,\sigma}$. Note that this formula can be extended to the case $\theta_0 \geq 2\pi$. In geometrical terms, this amounts to the following construction. We consider the wedge $W_{\theta_0}$ not as embedded in Minkowski space, but as embedded in the universal cover of $\mathbb{R}^{1,2} \setminus \Delta$. In other words, we introduce a coordinate system $(r,\theta,t)$, where $\theta$ is no longer defined modulo $2\pi$ but now parameterises the entire real line. The resulting flat singular spacetime $M_{\theta_0,\sigma}$ is then given as a $n$-branched cover along the singular line over $M_{\theta_0/n,\sigma}$, where $n$ is chosen so that $\theta_0/n$ is less than $2\pi$. In this description, the mass parameter $\mu = 2\pi - \theta_0$ can become negative or vanish. In particular, the limit case $\theta_0 = 2\pi$ yields a massless particle with non-vanishing spin $\sigma$.

2.2. Closed timelike curves (CTCs) and the CTC surface. In contrast to the coordinate $t$, the coordinate $\tau$ on $M_{\theta_0,\sigma}$ is not a time function. Introducing the variable $r_0 = \sigma/\theta_0$, we can rewrite the metric (2) as

$$ds^2 = -\frac{1}{\theta_0^2} d\tau^2 - \frac{r_0}{\pi} d\alpha d\tau + \left(\frac{\theta_0}{2\pi}\right)^2 (r^2 - r_0^2) d\alpha^2 + dr^2.$$

For a given value of $\tau$, the circle $C_{\tau,r} = \{(r,\alpha,\tau)|0 \leq \alpha \leq 2\pi\}$ of constant radius $r$ is spacelike if $r > |r_0|$, timelike if $r < |r_0|$, and null for $r = |r_0|$. This implies in particular that it defines a closed timelike curve (CTC) for $r < |r_0|$ and a closed lightlike curve for $r = |r_0|$. In the following we will therefore refer to $|r_0|$ as the CTC radius, to the surface $H = \{|r_0|,\alpha,\tau)|0 \leq \alpha \leq 2\pi, \tau \in \mathbb{R}\}$ of constant radius $|r_0|$ as the CTC surface. We call the domain $U = \{(r,\alpha,\tau)|0 < r < |r_0|, 0 \leq \alpha \leq 2\pi, \tau \in \mathbb{R}\}$ the CTC region and the region $M^*_{\theta_0,\sigma} = \{(r,\alpha,\tau)|r > |r_0|, 0 \leq \alpha \leq 2\pi, \tau \in \mathbb{R}\}$ the interior region of the spacetime. The latter is a manifold with boundary, whose boundary is the CTC surface $H$. It is the complement of the CTC region, which is diffeomorphic to $\mathbb{D}^2 \times \mathbb{R}$, where $\mathbb{D}^2$ denotes the open disc in $\mathbb{R}^2$. On
the CTC surface the metric \[ ds^2|_H = -\frac{1}{\theta_0^2} \, d\tau^2 - \frac{r_0}{\pi} \, d\alpha \, d\tau = -\left( \frac{1}{\theta_0^2} \, d\tau + \frac{r_0}{\pi} \, d\alpha \right) \, d\tau. \]

Note that \( d\tau \) does not vanish along spacelike curves in \( H \). It follows that a non-timelike curve in \( H \) cannot close up unless it is contained in a circle in \( H \) characterised by the condition \( \tau = \) constant. Such circles are lightlike but they are not geodesics. In the following we will call them null circles on the CTC surface. Note that the future of a point \( x = (|r_0|, \alpha, \tau_0) \) in the CTC surface \( H \), i.e. the points in \( H \) that can be connected to \( x \) via a future directed timelike curves in \( H \), is the region \( x_+ = \{(r_0, \alpha, \tau) | \tau > \tau_0 \} \subset H \) above the null circle containing \( x \). The future in \( H \) of a point on a given null circle therefore coincides with the future in \( H \) of this null circle.

The CTC region \( U \) contains many closed timelike curves (CTCs). Note, however, that it does not contain closed timelike geodesics. It follows from the expression for the metric, that in order to close up, timelike curves must have an acceleration, which is related to the spin parameter \( \sigma \). The smaller the value of the spin parameter, the bigger the acceleration associated with CTCs must be, and it tends to infinity in the limit of vanishing spin. Due to the presence of CTCs, the CTC region \( U \) exhibits quite pathological causality relations. The future (or the past) inside \( U \) of any point in \( U \) is the entire CTC region. Its future (or past) in \( M'_{\theta_0, \sigma} \) is the entire manifold \( M'_{\theta_0, \sigma} \).

In contrast to the CTC region, the causality structure of the interior region \( M'^{*}_{\theta_0, \sigma} \) is well-behaved. As the coordinate \( \tau \) defines a time function on \( M'^{*}_{\theta_0, \sigma} \), \( M'^{*}_{\theta_0, \sigma} \) contains no CTCs. Of course, this does not exclude that a timelike curve starts in the interior region, enters the CTC region and then returns to its starting point in the interior region. However, the absence of CTCs in the interior region implies that any closed timelike curve with a starting point in the interior region \( M'_{\theta_0, \sigma} \) must enter the CTC region.

2.3. Killing vector fields. The group of time orientation and orientation preserving isometries of \( M'_{\theta_0, \sigma} \) is an abelian group of dimension two. It is generated by rotations \((r, \alpha, \tau) \mapsto (r, \alpha + \alpha_0, \tau)\) and by translations \((r, \alpha, \tau) \mapsto (r, \alpha, \tau + \tau_0)\). In particular, \( M'_{\theta_0, \sigma} \) is stationary: the translation along \( \tau \) induces an isometry between the level sets of \( \tau \). However, if the spin \( \sigma \) is nonzero, \( \theta_0 \) is not static because the lapse term \(-\frac{\pi}{\theta_0} d\alpha d\tau \) in (3) does not vanish.

The CTC region, CTC surface and the interior region are distinguished by the Killing vector \( \partial_\alpha \) associated with the rotations. The CTC region is characterised by the condition that \( \partial_\alpha \) is timelike, the CTC surface is the locus where \( \partial_\alpha \) is lightlike, and the interior region is the region where \( \partial_\alpha \) is spacelike.

2.4. Cauchy surfaces. As the CTC region around the particles contains closed timelike curves, \( M'_{\theta_0, \sigma} \) is far from being globally hyperbolic. However, the level surfaces of the coordinate \( \tau \) are Cauchy surfaces for the interior region in the sense that any inextendible causal curve in \( M'_{\theta_0, \sigma} \) that is contained in \( M'^{*}_{\theta_0, \sigma} \) must intersect every level set of \( \tau \). We express this property by saying that \( M'^{*}_{\theta_0, \sigma} \) is globally hyperbolic relatively to its boundary. As observed in Section 2.2, the only non timelike loops in the CTC surface \( H = \partial M'^{*}_{\theta_0, \sigma} \) are the null circles. The boundary of any Cauchy surface in the interior region \( M'^{*}_{\theta_0, \sigma} \) therefore must coincide with one of these null circles.

2.5. The developing map. The universal covering \( \tilde{M}'_{\theta_0, \sigma} \) of \( M'_{\theta_0, \sigma} \) is homeomorphic to the manifold obtained by by taking an infinite number of copies \( W_i^{\theta_0} \), \( i \in \mathbb{Z} \), of the wedge \( W_{\theta_0} \) introduced in Section 2.1 and gluing them along the associated planes \( P_i^{0}, \ P_i^{1} \) via the the elliptic isometry \( g : (r, \theta, t) \mapsto (r, \theta + \theta_0, t + \sigma) \): \( P_i^{1} \sim P_i^{0+1} \) for all \( i \in \mathbb{Z} \). The covering map
Figure 2. The defining wedge of the Euclidean plane with a cone singularity for a) \( \theta_0 > \pi \) and b) \( \pi/2 < \theta_0 < \pi \). The solid line in b) corresponds to a geodesic loop in \( \mathbb{R}^2 \) with winding number 1.

\( p : \tilde{M}'_{\theta_0,\sigma} \rightarrow M'_{\theta_0,\sigma} \) is the map induced by the isometry \( W^i_{\theta_0} \cong W_{\theta_0} \). Denote by \( g^i \) the elliptic isometry obtained by applying the elliptic isometry \( g \) \( i \) times: \( g^i : (r, \theta, t) \mapsto (r, \theta + i\theta_0, t + i\sigma) \).

Then the maps \( W^i_{\theta_0} \rightarrow g^i(W_{\theta_0}) \) together define a (local) isometry \( D : \tilde{M}'_{\theta_0,\sigma} \rightarrow \mathbb{R}^{1,2} \setminus \Delta \). This map is the developing map of the Minkowski structure on \( M'_{\theta_0,\sigma} \). It is equivariant with respect to the natural actions of \( \pi_1(M'_{\theta_0,\sigma}) \cong \mathbb{Z} \) on \( \tilde{M}'_{\theta_0,\sigma} \) and on \( \mathbb{R}^{1,2} \setminus \Delta \). The first action is the one that maps every \( W^i_{\theta_0} \) onto \( W^i_{\theta_0}+1 \), and the second action is the one induced by \( g \).

Note that the map \( D \) is never a homeomorphism. When \(|i|\) increases, the wedges \( p(W^i_{\theta_0}) = g^i(W_{\theta_0}) \) wrap around the line \( \Delta \), and for \(|i| > 2\pi/\theta_0 \) overlap with the initial wedge \( W_{\theta_0} \). This overlapping is a perfect matching if and only if \( 2\pi/\theta_0 \) is rational, in which case \( M'_{\theta_0,\sigma} \) might be seen as a finite quotient of \( \mathbb{R}^{1,2} \setminus \Delta \). This reflects a general pattern that is also present in the case of Minkowski spacetimes with multiple particles. The developing maps for these spacetimes are not one-to-one. Moreover, as we will see in the following, the developing maps of spacetimes with at least two particles are surjective. The developing maps are thus quite pathological, which reflects the fact that the regular part of these manifolds cannot be obtained as a quotient of a region of the Minkowski space.

2.6. Geodesics. To investigate the properties of the geodesics in \( M'_{\theta_0,\sigma} \), it is useful to introduce the Euclidean plane with a cone singularity of cone angle \( \theta_0 \), which in the following will be denoted by \( \mathbb{R}^2 \). The definition is analogous to the one of the manifold \( M_{\theta_0,\sigma} \). Consider the wedge of angle \( \theta_0 \) in the Euclidean plane \( \mathbb{R}^2 \setminus \{0\} \): \( \hat{W}_{\theta_0} = \{(r,\theta) \mid r > 0, 0 \leq \theta \leq \theta_0\} \) and glue the two sides of this wedge via the identification \((r,\theta) \sim (r,\theta_0)\). Alternatively, the Euclidean plane with a cone singularity is obtained as the completion of the following metric on \( \mathbb{R}^2 \setminus \{0\} \) given in polar coordinates

\[
(4) \quad ds^2_E = \left(\frac{\theta_0}{2\pi}\right)^2 r^2 d\theta^2 + dr^2.
\]
The vertical projection \( p_0 : \mathbb{R}^{1,2} \setminus \Delta \rightarrow \mathbb{R}^2 \setminus \{0\} \) then induces a map \( M_{\theta_0, \sigma} \rightarrow \mathbb{R}^2_{\theta_0} \). Denote by \( \pi_0 : \tilde{M}_{\theta_0, \sigma} \rightarrow \mathbb{R}^2_{\theta_0}, \pi_0 = p_0 \circ D \) the composition of this projection with the developing map. Let now \( c : (a, b) \rightarrow M_{\theta_0, \sigma} \) a geodesic path (timelike, lightlike or spacelike). Then \( c \) lifts to a geodesic path \( \tilde{c} : (a, b) \rightarrow \tilde{M}_{\theta_0, \sigma} \). As the developing map \( D \) is a local isometry, the image \( \tilde{c} := D \circ \tilde{c} \) is a geodesic path in \( \mathbb{R}^{1,2} \setminus \Delta \) and its projection \( p_0 \circ \tilde{c} = \pi_0 \circ \tilde{c} \) is a geodesic path in \( \mathbb{R}^2_{\theta_0} \). Note that this path is constant if and only if the geodesic \( \tilde{c} \) is parallel to \( \Delta \).

The path \( \pi_0 \circ \tilde{c} \) is a geodesic loop in \( \mathbb{R}^2_{\theta_0} \) if and only if there exists a timelike geodesic \( \Delta \) parallel to \( \Delta \) in \( \mathbb{R}^{1,2} \setminus \Delta \) such that both its starting and endpoint of \( \tilde{c} \) lie on \( \Delta' \). As we will see in the following, a lightlike geodesic \( \tilde{c} \) with this property corresponds to a returning lightray, i.e. a lightray sent out by an observer with worldline \( \Delta' \) in the following, a lightlike geodesic \( \bar{\Delta} \) in \( \mathbb{R}^2_{\theta_0} \) with the parameter \( \theta_0 \). As the developing map \( D \) is a local isometry, the image \( \tilde{\bar{\Delta}} := D \circ \bar{\Delta} \) is a geodesic path in \( \mathbb{R}^{1,2} \setminus \Delta \) and its projection \( p_0 \circ \tilde{\bar{\Delta}} = \pi_0 \circ \tilde{\bar{\Delta}} \) is a geodesic path in \( \mathbb{R}^2_{\theta_0} \). Note that this path is constant if and only if the geodesic \( \tilde{\bar{\Delta}} \) is parallel to \( \Delta \).

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2.7. CTC cylinders. In the following section, we will extend our model obtain a more general notion of flat Lorentzian spacetimes with a particles. For this we will need to consider the interior region as a manifold with boundary that is given by the CTC surface. We introduce the following definition.

**Definition 2.1.** Let \( h \) be a positive real number and \( a \in \mathbb{R} \). A CTC cylinder of height \( h \) based at \( a \) is the region in \( U \) between the two level sets \( \tau^{-1}(a), \tau^{-1}(a + h) \) of \( \tau \). The past (future) complete CTC cylinder based at \( a \) is the past (future) in \( U \) of the level set \( \tau^{-1}(a) \).

Note that all CTC cylinders for a given value of \( h \) are isometric. In contrast to the quantity \( h \), the parameter \( a \) therefore has no intrinsic geometrical meaning. Similarly, all past and future complete cylinders are isometric to the entire CTC region, which implies in particular that they are complete.

3. **Global hyperbolicity**

3.1. **Definition.** We are now ready to give a general definition of flat Lorentzian manifolds with particles and to define a modified notion of global hyperbolicity, which will allow us to restrict the class of Lorentzian manifolds with particles under consideration.

**Definition 3.1.** A flat Lorentzian manifold with particles is a three-dimensional manifold \( M \) with an embedded closed 1-submanifold \( \Delta \) (not necessarily connected), such that \( M \setminus \Delta \) is endowed with a flat Lorentzian metric and for every \( x \in \Delta \) there exists a neighbourhood \( U \) of \( x \) in \( M \) such that \( U \setminus \Delta \) is isometric to the neighbourhood of a point on the singular line (the particle) in \( M_{\theta_x, \sigma_x} \) with the singular line itself removed\(^2\).

This definition provides us with a very general notion of a flat Lorentzian spacetime with particles and thus potentially with a large class of examples. However, there is no hope of obtaining a global understanding of flat spacetimes with particles without suitable additional hypotheses. In Riemannian geometry, it is customary to impose as such an additional hypothesis the compactness of the ambient manifold. However, this condition is not suited to the Lorentzian context, since it implies issues with the causal structure such which are undesirable.

\(^2\)Observe that the map \( x \mapsto (\theta_x, \sigma_x) \) is then necessarily locally constant on \( \Delta \).
from both the mathematics and the physics point of view. Such issues arise even in the much simpler situation of flat Lorentzian manifolds without particles (cf. [Gal84, Sán06]).

Instead, the standard condition imposed in Lorentzian geometry is the requirement of global hyperbolicity. This implies the existence of a Cauchy surface, and an especially favourable situation is the case where the Cauchy surfaces are compact. This is the point of view we will adopt in the following. However, the fact that the manifolds under consideration exhibit closed timelike curves in the CTC region requires that we modify our concept of global hyperbolicity in a suitable way. The central idea is to consider the flat Lorentzian manifold as a surface with boundaries that are given by the CTC surfaces associated to particles. The appropriate notion of a Cauchy surface is that of a spacelike surface with lightlike boundaries, the latter corresponding to its intersection with the CTC surfaces.

**Definition 3.2.** A **globally hyperbolic** flat Lorentzian spacetime $M$ with $n$ particles is a flat Lorentzian manifold with particles $(M, \Delta)$ such that $\Delta = \{d_1, \ldots, d_n\}$ is the disjoint union of $n$ lines $d_1, \ldots, d_n$ and there exist disjoint neighbourhoods $V_1, \ldots, V_n$ of the singular lines $d_1, \ldots, d_n$ such that:

1. each neighbourhood $V_i$ is isometric to a CTC cylinder of height $h_i$ in $M_{\theta_i, \sigma_i}$.
2. the complement $M^*$ of the disjoint union $\bigcup_{i=1}^n V_i$ is a flat Lorentzian manifold with boundary that admits a Cauchy surface, i.e. an embedded surface with boundary $S$ with spacelike interior, such that the boundary components of $S$ are null circles in the CTC surfaces $\partial V_i$, and such that every inextendible causal curve in $M \setminus \Delta$ that is contained in $M^*$ intersects $S$.

If moreover the Cauchy surface $S$ can be selected to be compact, then $M$ is called **spatially compact**.

Note that this definition is quite restrictive regarding the CTC region around the particles. This is due to the following reasons. Firstly, we want the particles to be hidden behind a “CTC surface” $\partial V_i$, and the CTC regions $V_i$ around each particle therefore must be sufficiently big so that they reach the CTC surfaces in the associated one-particle models. Secondly, we need the
interior region to be globally hyperbolic and hence foliated by Cauchy surfaces. This induces a foliation of the CTC surfaces \( \partial V_i \subset M \) around each particle by non-timelike closed curves, and hence by null circles. In order to obtain a notion of globally hyperbolic flat Lorentzian manifold with particles that fulfils each of these requirements, we then have to assume that each surface \( \partial V_i \) is the boundary of a CTC cylinder.

Given a flat Lorentzian spacetime with particles that is globally hyperbolic in the sense of Definition\textsuperscript{3.2}, one can add to each CTC cylinder the entire CTC region in the corresponding one-particle model, and this completion has no impact on the geometry of its interior part \( M^* \). However, in the following, we take the viewpoint that the specific geometry of the CTC region is irrelevant itself and only of interest through its effect on geodesics that enter a connected component \( V_i \) of the CTC region from the interior region \( M^* \) and then return to \( M^* \). Such a geodesic has to be contained in the CTC cylinder bounded by \( \partial V_i \). What happens outside the CTC cylinder inside the CTC region is therefore not relevant to our situation except through its effects on geodesics outside the CTC region.

In the following we will focus on the situation in which the spins \( \sigma_i \) are small compared to the cone angles \( \theta_i \) so that the scale of the CTC radii \( |r_i| = \frac{|\sigma_i|}{\theta_i} \) is small compared with the global geometry of the more classical globally hyperbolic interior region \( M^* \). In the limit case, where one or more spins \( \sigma_i \) tend to zero, the associated CTC regions \( V_i \) become empty. In that situation, one can extend the notion of causal curves by including curves that contain components of the singular lines. In this setting, our notion of global hyperbolicity requires that there is a closed Cauchy surface intersected by all inextendible curves that are causal in that sense.

### 3.2. Doubling the spacetime along the CTC surface.

Classical results involving global hyperbolicity are not available for spacetimes with boundary such as the interior region \( M^* \) in Definition\textsuperscript{3.2}. However, we can nevertheless relate these spacetimes to the classical framework by employing the following “doubling the spacetime” trick.

Let \( M \) be a singular flat spacetime satisfying the first condition in Definition\textsuperscript{3.2} and denote by \( M^* = M \setminus \bigcup^n_{i=1} V_i \) its interior region. For each singular line \( d_i \), let \( \theta_i \) be the cone angle around \( d_i \), \( \sigma_i \) the spin and let \( r_i = \sigma_i/\theta_i \) denote the associated CTC radius. The isometry between the boundary components \( V_i \) of \( M \) and the CTC cylinders defines a local coordinate system \((r, \alpha, \tau)\) in a neighbourhood \( U_i \subset M^* \) of each boundary component, in which the metric takes the form\textsuperscript{2} (with \( \theta_0 \) replaced by \( \theta_i \) and \( \sigma \) by \( \sigma_i \)). Define a new coordinate \( \xi \) on \( U_i \) through the condition \( r = |r_i| \cosh \xi, \xi \geq 0 \) in the interior region near each surface \( \partial V_i \). In terms of these coordinates the metric\textsuperscript{2} for each particle takes the form

\[
- \frac{1}{\theta_i^2} \, d\tau^2 - \frac{r_i}{\pi} \, d\sigma d\tau + r_i^2 (\sinh \xi)^2 \left[ \left( \frac{\theta_i}{2\pi} \right)^2 \, d\alpha^2 + d\xi^2 \right]
\]

When the CTC cylinder is a finite \( h_i \)-cylinder one can prescribe the coordinate \( \tau \) to vary in \([-h_i/2, h_i/2]\). As the number of particles is finite, there exists an \( \epsilon_0 > 0 \) such that the subsets of the neighbourhoods \( U_i \) characterised by the condition \( \xi \leq \epsilon_0 \) define \( n \) solid pairwisely disjoint cylinders that contain all surfaces \( \partial V_i \).

Consider two copies \( M^*_1, M^*_2 \) of \( M^* \) and glue them along their boundaries in the obvious way. More precisely, let \( f_1 : M^* \to M^*_1 \) and \( f_2 : M^* \to M^*_2 \) be two identifications and consider the union of \( M^*_1 \) and \( M^*_2 \) with \( f_1(p) \) and \( f_2(p) \) identified for every \( p \) in \( \partial M^* \). We get a manifold \( M^d \), containing a surface \( T \) (the locus where the glueing has been performed) and two embeddings \( \tilde{f}_1 : M^* \to M^d, \tilde{f}_2 : M^* \to M^d \). In the following, \( M^* \) is referred to as the doubling of \( M^* \) along \( T \). A neighbourhood of every connected component of \( T \) in \( M^d \) can be parametrized by coordinates \((\alpha, \tau, \xi)\) but where now \( \xi \) is allowed to vary in \([-\epsilon_0, \epsilon_0]\), hence to have negative
values. Positive values of \( \xi \) correspond to points in the first copy \( \hat{M}_1^* := \hat{f}_1(M_1^*) \) whereas negative values represent points in \( \hat{M}_2^* := \hat{f}_1(M_2^*) \). The surface \( \hat{T} = \hat{f}_{1,2}(T) \) is characterised by \( \xi = 0 \).

The manifold \( M^* \) is equipped with a metric \( g_0 \) which, however, becomes degenerate on \( \hat{T} \). Nevertheless, it is still reasonable to study the causality properties of such a degenerate cone field. A convenient way to do so is to consider \( g_0 \) as the limit of non-degenerate Lorentzian metrics. For this we introduce a bump function \( \eta : [0, +\infty) \rightarrow [0, 1] \), which is a non-increasing smooth function that vanishes on the interval \((1, \infty)\) and takes constant value 1 on \([0, 1/2]\). For every \( 0 < \epsilon < \epsilon_0 \), we define

\[
(5) \quad g_\epsilon := -\frac{1}{\theta_i^2} d\tau^2 - \frac{r_i^2}{\pi} d\alpha d\tau + \left( \frac{\theta_i}{2\pi} \right)^2 d\alpha^2 + d\xi^2
\]

Then \( g_\epsilon \) is a Lorentzian metric on \( M^d \), equal to the flat metric on the region \( \xi \geq \epsilon \), and converges with respect to the \( C^0 \)-norm to the degenerate flat metric \( g_0 \) for \( \epsilon \to 0 \).

One can allow the coordinate \( \xi \) in (5) to take values on the entire real line. In this case, it defines a Lorentzian metric that becomes degenerate for \( \epsilon = 0 \) and approximates the doubling \( M_{\theta,\sigma_1}^d \) of the interior region \( M_{\theta,\sigma_1}^* \).

**Lemma 3.3.** Every spacetime \((M_{\theta,\sigma_1}^d, g_\epsilon)\) for \( \epsilon > 0 \) is globally hyperbolic, and every level set of \( \tau \) is a Cauchy surface.

**Proof.** The level sets of \( \tau \) are spacelike for every \( g_\epsilon \), hence \( \tau \) is a time function. Let \( c : I \rightarrow M_{\theta,\sigma_1}^d \) be an inextendible \( g_\epsilon \)-causal curve. Then it is also \( g_0 \)-causal. The map \( \xi \mapsto |\xi| \) induces an isometric branched covering \( s^d : M_{\theta,\sigma_1}^d \rightarrow M_{\theta,\sigma_1}^* \) that preserves the coordinate \( \tau \). As \( M_{\theta,\sigma_1}^* \) is globally hyperbolic relatively to its boundary (cf. section 2.4) the image of \( c \) by \( s^d \) must intersect every level set of \( \tau \). The lemma follows. \( \square \)

### 3.3. A criterion for global hyperbolicity.

**Proposition 3.4.** Let \( M \) be a singular flat spacetime satisfying the first condition in Definition 3.2. Assume that the closure of the interior region \( M^* \) contains no closed causal curves except the null circles in the surface \( \partial V_i \), and that for all \( p, q \in M^* \), the intersection \( J^+(p) \cap J^-(q) \) between the causal future of \( p \) and the causal past of \( q \) is either compact or empty. Then \( M^* \) admits a Cauchy surface.

**Proof.**

1. This lemma is well-known in the non-degenerate case and is at the foundation of the notion of global hyperbolicity. To prove it for the degenerate case, we first observe that the metric \( g_0 \) has no closed causal curves (CCC) except the null circles. Indeed, consider the map \( s : M^d \rightarrow M^* \) (a branched cover) that sends the points \( \hat{f}_1(p) \) and \( \hat{f}_2(p) \) to \( p \). It is an isometry with respect to the metric \( g_0 \). If \( c \) is a CCC in \( M^d \) for the metric \( g_0 \), its image \( \hat{s}(c) \) under \( s \) is a CCC for the flat metric in \( M^* \) and hence, by hypothesis, a null circle. Now observe that \( g_0 \) weakly dominates all the metrics \( g_\epsilon \), in the sense that every causal curve for \( g_\epsilon \) is also causal for the degenerate metric \( g_0 \). A direct calculation shows that the null circles in the CTC surfaces are spacelike for \( g_\epsilon \). It follows that the metrics \( g_\epsilon \) have no closed causal curves.

For every point \( p \) in \( M^d \), denote by \( J^+_\epsilon(p) \) the causal past (-) and future (+) of \( p \) in \( M^d \) with respect to the metric \( g_\epsilon \) and by \( J^\pm(p) \) its causal past (-) and future (+) with respect to \( g_0 \). As every causal curve for \( g_\epsilon \) is a causal curve for \( g_0 \), the intersection \( J^+_\epsilon(p) \cap J^-_\epsilon(q) \) is contained in \( J^+(p) \cap J^-(q) \) for all \( p, q \in M^d \). Assume that \( J^+_\epsilon(p) \cap J^-_\epsilon(q) \) is not empty. Then \( s : M^d \rightarrow M^* \) maps \( J^+_\epsilon(p) \cap J^-_\epsilon(q) \) into a closed subset of \( J^+(s(p)) \cap J^-(s(q)) \). On the other hand, \( s \) is a proper map. As \( J^+(s(p)) \cap J^-(s(q)) \) is compact by hypothesis, the same holds for \( J^+_\epsilon(p) \cap J^-_\epsilon(q) \). As
\[ J^+_\epsilon(p) \cap J^-_\epsilon(q) \] is a closed subset of \( J^+(p) \cap J^-(q) \), it is therefore also compact. This proves that there exists an \( \epsilon_0 > 0 \) such that the metric \( g_\epsilon \) is globally hyperbolic for all \( \epsilon < \epsilon_0 \).

2. For every \( \epsilon < \epsilon_0 \) let \( S_\epsilon \) be a Cauchy surface for \( g_\epsilon \). Denote by \( \widehat{S}_\epsilon \) the intersection of the Cauchy surface \( S_\epsilon \) with the interior region \( M^* \), considered as a subset of \( M^4 \). Denote by \( K_\epsilon \) the region in \( M^* \) that is characterised by the condition \( \xi \leq \epsilon \) and by \( K_\epsilon^1, \ldots, K_\epsilon^n \) its connected components. Recall that \( g_\epsilon \) is equal to \( g_0 \) outside \( K_\epsilon \). The intersection of \( \widehat{S}_\epsilon \) with every connected component \( K_\epsilon^i \) is the graph of a map \((\alpha, \xi) \mapsto f_{i,\epsilon}(\alpha, \xi)\) which takes values in \((-h_i/2, h_i/2)\).

**Claim:** There is a compact \( g_0 \)-spacelike hypersurface \( S \) and a positive real number \( \epsilon < \epsilon_0 \) such that \( S \) coincides with a Cauchy surface \( S_\epsilon \) of \((M^4, g_\epsilon)\) in the region \( M^4 \setminus K_\epsilon \).

To prove the claim, we first assume that the spacetime admits only one particle \((n = 1)\). We fix a point \( x \) in the CTC cylinder characterised by the condition \( \xi = 0 \), which is the boundary of the interior region \( M^* \). Without loss of generality, we select \( x \) such that its \( \tau \)-coordinate vanishes. Then, we can assume without loss of generality that the Cauchy surfaces \( S_\epsilon \) have been chosen in such a way that they all contain \( x \).

By applying the Ascoli-Arzelà Theorem to \( f_{i,\epsilon} \), one then obtains directly that there is a subsequence of the sequence of surfaces \( S_{1/k}, k \in \mathbb{N} \), which converges to a \( g_0 \)-spacelike hypersurface \( S_\infty \) in the region \( K_{\epsilon_0} = \{ \xi \leq \epsilon_0 \} \). Note, however, that outside the region \( K_{\epsilon_0} \), these surfaces may escape to infinity when \( k \to \infty \). This issue can be addressed as follows: for \( \epsilon \) sufficiently small, one can extend the part of \( S_\epsilon \) outside \( K_{\epsilon_0} \) by a surface approximating \( S_\infty \), which is \( g_0 \)-spacelike (details are left to the reader). We then obtain a compact surface \( S \) which, as required, is \( g_0 \)-spacelike and coincides with \( S_\epsilon \) outside of \( K_{\epsilon_0} \) (recall that \( g_\epsilon \) and \( g_0 \) coincide there). This proves the claim for \( n = 1 \).

Consider now the case \( n \geq 2 \). Fix a point \( x_1 \) in the CTC cylinder in the first component \( K_{1,0}^0 \), and assume that every \( S_\epsilon \) contains \( x_1 \). Reasoning as above, we construct a surface \( \Sigma_1 \) which coincides with \( S_{\epsilon_1} \) (for some \( \epsilon_1 \)) outside \( K_{1,0}^0 \) and is \( g_0 \)-spacelike in \( K_{1,0}^0 \). Denote by \( \Sigma_1^+, \Sigma_1^- \) the two compact surfaces obtained by pushing in the future (respectively in the past) the surface \( \Sigma_1 \) in such a way that the resulting surfaces are \( g_0 \)-spacelike in \( K_{1,0}^0 \) and \( g_{\epsilon_1} \)-spacelike outside \( K_{1,0}^0 \). We consider the region \( W_1 \) between \( \Sigma_1^+ \) and \( \Sigma_1^- \). As the surfaces \( \Sigma_1^+ \) are \( g_\epsilon \)-spacelike for every \( \epsilon, W_1 \setminus (K_{1,0}^0 \cup \ldots \cup K_{n,0}^n) \) is globally hyperbolic for every \( g_\epsilon \). It follows that the surfaces \( S_\epsilon \) can be selected in such a way that they all lie in \( W_1 \), with the possible exception of the region \( K_{2,0}^0 \cup \ldots \cup K_{n,0}^n \).

We now drop the condition \( x_1 \in S_\epsilon \) and replace it by an analogous condition for the second connected component: we impose that all surfaces \( S_\epsilon \) contain a given point \( x_2 \) in the CTC cylinder in \( K_{2,0}^0 \). Repeating the argument above, we obtain two disjoint surfaces \( \Sigma_2^+, \Sigma_2^- \) which

- are chosen in such a way that \( \Sigma_2^+ \) lies in the future of \( \Sigma_2^- \)
- are \( g_0 \)-spacelike in the region \( K_{1,0}^0 \cup K_{2,0}^0 \),
- are \( g_{\epsilon_2} \)-spacelike in \( K_{3,0}^0 \cup \ldots \cup K_{n,0}^n \),
- lie between the surfaces \( \Sigma_1^+ \) and \( \Sigma_1^- \) outside of \( K_{2,0}^0 \cup \ldots \cup K_{n,0}^n \).

We now impose as a condition that the Cauchy surfaces \( g_\epsilon \) lies between \( \Sigma_2^- \) and \( \Sigma_2^+ \), with the possible exception of region \( K_{3,0}^0 \cup \ldots \cup K_{n,0}^n \). Iterating this process, we obtain two compact surfaces \( \Sigma_n^+, \Sigma_n^- \) which are \( g_0 \)-spacelike everywhere. We conclude the proof of the claim as in the case \( n = 1 \).

3. After proving the claim, we resume our proof of Proposition 3.3. To conclude this proof, we show that the surface \( \widehat{S} = S \cap M^* \) is a Cauchy surface for \( g_0 \). Let \( \gamma : I \to M^*, t \mapsto \gamma(t) \) be an inextendible future oriented causal curve in \( M^* \) for the metric \( g_0 \). Assume without loss
of generality that \( \gamma(0) \) lies in the past of \( \hat{S}_\epsilon \) for the metric \( g_\epsilon \). By way of contradiction, assume that \( \gamma \) never intersects \( \hat{S} \).

Define \( t_0 = \sup\{t \in I \mid \gamma(s) \notin K_0 \ \forall s \in I, s \leq t \} \). By definition of \( K_0 \), this implies that for all \( t \leq t_0 \), \( \gamma(t) \) lies in the region where \( g_0 \) and \( g_\epsilon \) are equal. Hence the restriction of \( \gamma \) to the interval \( I \cap (-\infty, t_0] \) is a causal curve with respect to \( g_\epsilon \). As the surface \( \hat{S}_\epsilon \) is a Cauchy surface for \( g_\epsilon \) and coincides with \( \hat{S} \) outside \( K_0 \), it follows from the assumption that \( t_0 \) must be finite. Moreover, \( \gamma(t_0) \) lies in the past of the Cauchy surface \( \hat{S}_\epsilon \) and therefore under the graph of \( \hat{f} \).

However, by hypothesis, \( K_0 \cap \gamma(I) \), cannot intersect the graph of \( \hat{f} \). It follows that \( \gamma \) must exit the region \( K_0 \) and that its exit point is still in the past of \( \hat{S}_\epsilon \) with respect to \( g_\epsilon \). Let \( T = \sup\{t \in I : \gamma(t) \in K_0 \cap J^-_{\gamma}(\hat{S}_\epsilon) \} \) and let \( (t_n)_{n \in \mathbb{N}} \) be an increasing sequence such that \( t_n \rightarrow T \). Observe that \( T \) might be infinite. Every point \( \gamma(t_n) \) lies in the future of \( \gamma(0) \). As \( g_\epsilon \) is globally hyperbolic, \( J^+(\gamma(0)) \cap J^-_{\gamma}(\hat{S}_\epsilon) \) is compact and the sequence \( \gamma(t_n) \) converges. As \( \gamma \) is inextendible, it follows that \( T \) is finite, and the limit must be \( \lim_{n \to \infty} \gamma(t_n) = \gamma(T) \). In particular, this implies that \( T \) is finite and \( \gamma(T) \) lies in \( K_0 \cap J^-_{\gamma}(\hat{S}_\epsilon) \). By hypothesis, \( \gamma(T) \) is not in \( \hat{S}_\epsilon \) because \( \hat{S}_\epsilon \cap K_0 \subset S \). This implies that for some \( t > T \), \( \gamma(t) \) is still in the past of \( \hat{S}_\epsilon \). But the argument above, proving that \( t_0 \) is finite, implies that \( \gamma \) should meet \( K_0 \) once more, which is a contradiction.

This implies that every inextendible causal curve with respect to \( g_0 \) must intersect the surface \( \hat{S} \). Hence \( \hat{S} \) is a Cauchy surface, and \( M^* \) is globally hyperbolic. \( \square \)

4. Construction of stationary flat spacetimes

4.1. Euclidean surfaces with cone singularities. After discussing the one-particle model and introducing a notion of global hyperbolicity, we will now construct examples of flat Lorentzian spacetimes with particles. The resulting spacetimes are stationary and the construction is based on Euclidean surfaces with cone singularities. In the following, we denote by \( \mathbb{R}^2_\theta \) for \( \theta > 0 \) the Euclidean plane with one singular point of cone angle \( \theta \), that is \( \mathbb{R}^2 \setminus \{0\} \) with the metric given in \( \{1\} \).

**Definition 4.1.** A Euclidean metric with cone singularities on a closed orientable surface \( \Sigma \) consists of a finite number of points \( p_1, \ldots, p_n \) (the cone singularities) together with an assignment of positive real numbers \( \theta_i > 0 \) (the cone angles) to \( p_i \) for \( i = 1, \ldots, n \), and a flat Riemannian metric on \( \Sigma^* = \Sigma \setminus \{p_1, \ldots, p_n\} \), such that every point \( p_i \) admits a neighbourhood \( U_i \) in \( \Sigma \) so that \( U_i \setminus \{p_i\} \) is isometric to a ball in \( \mathbb{R}^2_{\theta_i} \) centred at the singular point.

Note that the quantities \( \mu_i := 2\pi - \theta_i \), which in \((2+1)\)-gravity are interpreted as masses of the particles, are usually referred to as apex curvatures in the mathematics literature (see for example [Thu98]). They are subject to the relation

\[
2\pi \chi(\Sigma) = \sum_{i=1}^{n} \mu_i
\]

where \( \chi(\Sigma) \) denotes the Euler characteristic of \( \Sigma \). In particular, if all the cone angles satisfy \( \theta_i \leq 2\pi \), then the surface \( \Sigma \) is either the sphere of the torus, and the torus arises only if there is no singularity.

Observe that the flat Euclidean structure naturally defines a conformal structure on \( \Sigma^* \) and the punctures \( p_i \) correspond to the cusps of this conformal structure. Consequently, the flat Euclidean structure equips \( \Sigma \) with the structure of a Riemann surface. That the converse is also true follows from a theorem by Troyanov.
Theorem 4.2 ([Tro86]). Let \( p_1, \ldots, p_n \) be a collection of \( n \) points in \( \Sigma \), and \( \theta_1, \ldots, \theta_n \) positive real numbers such that
\[
2\pi \chi(\Sigma) = \sum_{i=1}^{n} (2\pi - \theta_i)
\]
Then, for any conformal structure on \( \Sigma \), there is an Euclidean metric on \( S \) with cone singularities of cone angles \( \theta_i \) at each \( p_i \) that induces the given conformal structure. This singular Euclidean metric is unique up to a rescaling factor - in particular, it is unique if we require the total volume to be equal to 1.

The study of singular Euclidean surfaces is a very traditional topic in mathematics. For instance, it is related to billiards. A way of investigating a billiard in a polygon in the Euclidean plane is to consider it as the geodesic flow of the singular flat Euclidean metric on the sphere, which is obtained by taking the double of the polygon along its sides (see [MT02]).

An important case is the one in which all cone angles are rational. For instance, if all of these angles are multiples of \( \pi \), the associated singular Euclidean metric is directly related to holomorphic quadratic differentials.

On the other hand, the “positive curvature case”, in which all cone angles are less than \( 2\pi \) and the Euclidean surface is a sphere sphere with conical singularities, there is always a geodesic triangulation of the singular Euclidean surface \( \Sigma \). This implies that \( \Sigma \) can be obtained by gluing triangles in the Euclidean plane along their sides (see [Thu98, Proposition 3.1]). In particular, when all the cone angles are rational, the associated flat surface is an orbifold. It is obtained as a quotient of a closed Euclidean surface without cone singularities by the action of a finite group of isometries.

4.2. Stationary flat spacetimes with particles. We now construct globally hyperbolic flat spacetimes with particles based on Euclidean surfaces with cone singularities. The simplest and most obvious example are static spacetimes, which are obtained as a direct product of the Euclidean surface with cone singularities with \( \mathbb{R} \).

Definition 4.3 (Static spacetimes with particles). Let \( \Sigma \) be a closed Euclidean surface with conical singularities \( p_1, \ldots, p_n \) of angles \( \theta_1, \ldots, \theta_n \). We denote by \( ds_0^2 \) the flat metric on the regular part \( \Sigma^* \). Then the product \( M = \Sigma \times \mathbb{R} \) contains the open subset \( M^* = \Sigma^* \times \mathbb{R} \) and can be equipped with the Lorentzian metric \( ds_0^2 - dt^2 \), where the coordinate \( t \) parametrises \( \mathbb{R} \). This metric is locally flat, and can be considered as the regular part of a flat singular metric on \( M \) where the lines \( \{p_i\} \times \mathbb{R} \) are spinless particles of cone angle \( \theta_i \).

Observe that these spacetimes are static: the vertical vector field \( \partial_t \) is a Killing vector field, orthogonal to the level sets of \( t \). As the spacetime is static, \( t \) is a Cauchy time function and the levels of \( t \) are compact and hence complete. This implies directly that the static singular flat spacetime \( M \) is globally hyperbolic.

To obtain a more interesting class of examples, we consider a closed 1-form \( \omega \) on \( \Sigma^* \), where \( \Sigma^* \) is the regular part of a singular flat Euclidean surface as in Definition 4.3. We consider again the direct product \( \Sigma^* \times \mathbb{R} \) but now equipped with the metric
\[
 ds_\omega = ds_0^2 - (\omega + dt)^2 = ds_0^2 - \omega^2 - dt^2 - 2\omega dt
\]
instead of \( ds_0^2 - dt^2 \). Note that this defines a flat Lorentzian metric on \( M^* = \Sigma^* \times \mathbb{R} \). On any subset \( U \times \mathbb{R} \) where \( U \subset \Sigma^* \) is contractible, the form \( \omega \) is exact, i. e. given as the differential \( \omega = df \) of a map \( f : U \to \mathbb{R} \). This implies that the metric \( ds_\omega \) on \( V \times \mathbb{R} \) is simply the pull-back of \( ds_0^2 \) under the diffeomorphism \( (x, t) \mapsto (x, t + f(x)) \) and hence a flat Lorentzian metric on \( V \times \mathbb{R} \). Moreover, this argument shows that \( ds_\omega^2 \) only depends, up to isometry, on the cohomology class of \( \omega \).
We now fix open pairwisely disjoint neighbourhoods $U_i \subset \Sigma$ around every singular point $p_i$ such that $U^*_i := U_i \setminus \{p_i\}$ is isometric to a ball in $\mathbb{R}^2_{p_i}$ centred at the singular point. In a suitable polar coordinate system, the metric on $U_i$ then takes the form

$$ds_i^2 = dr^2 + \left(\frac{\theta_i}{2\pi}\right)^2 r^2 d\alpha^2.$$ 

Denote by $\omega_i$ the closed 1-form $d\omega/2\pi$ in $U^*_i$. As it generates the first cohomology group, the restriction of $\omega$ to $U^*_i$ is cohomologous to $\sigma_i \omega_i$, where $\sigma_i = \int_{\gamma_i} \omega$ and $\gamma_i$ is a loop in $U^*_i$ that makes one positive turn around $p_i$. Therefore, there exists a function $f_i: U^*_i \to \mathbb{R}$ such that $\omega = \sigma_i \omega_i + df_i$. Let $f: \Sigma^* \to \mathbb{R}$ be a function whose restriction to $U^*_i$ coincides with $f_i$ for all $i \in \{1, \ldots, n\}$. Such a function can be constructed by means of bump functions $b_i: \sigma^* \to \mathbb{R}$, which satisfy $b_i|_{U^*_i} = 1$ and $b_i|_{U^*_i'} = 0$ for all $i \neq j$. The function is then given by $f = \sum_{i=1}^n b_if_i$. On every neighbourhood $U^*_i$, the one form $\omega' = \omega - df$ then coincides with $\sigma_i\omega_i$. This implies that the associated metric $ds^2_i$ defined as in (5) and restricted to $V_i^* := U^*_i \times \mathbb{R}$ takes the form

$$ds^2_{V_i} = dr^2 + \left(\frac{\theta_i}{2\pi}\right)^2 r^2 d\alpha^2 - \left(\frac{\sigma_i}{2\pi}\right)^2 d\alpha^2 - \frac{1}{\pi} d\alpha dt - dt^2$$

We recognise, up to a rescaling factor $1/\theta_i$ for the coordinate $t$, the metric (3) for a particle with spin $\sigma_i$. Hence $(M, ds^2_i)$ is the regular part of a singular flat Lorentzian metric with particles on $\Sigma \times \mathbb{R}$ according to Definition 3.1. We are therefore free to make the following definition.

**Definition 4.4** (Stationary spacetimes with particles). Let $\Sigma$ be a closed Euclidean surface with conical singularities $p_1, \ldots, p_n$ of angles $\theta_1, \ldots, \theta_n$ and $ds^2_0$ the flat metric on its regular part $\Sigma^*$. Let $\omega$ be a closed 1-form on $\Sigma^*$. Then the flat stationary singular spacetime $(M, \omega)$ associated with $M = \Sigma \times \mathbb{R}$ and $\omega$ is the product $M^* = \Sigma^* \times \mathbb{R}$ equipped with the flat Lorentzian metric (6).

Note that we did not state that the stationary spacetimes defined in this way are globally hyperbolic in the sense of Definition 3.2, as this is not the case in general. We will give a necessary and sufficient set of conditions for global hyperbolicity in Section 4.3.

### 4.3. Geometrical properties of stationary flat singular spacetimes.

#### 4.3.1. Geometrical interpretation of the closed 1-form. Let $(M, ds^2_\omega)$ a singular flat spacetime as in Definition 4.4 defined by a Euclidean closed surface $\Sigma$ with cone singularities and a closed 1-form $\omega$ on the regular part $\Sigma^*$. Then it is immediate from (6) that the translations along the $t$-coordinate are isometries and the vertical vector field $\partial_t$ is a Killing vector field, which is timelike everywhere. The space of trajectories of this vector field is the space of vertical lines; it is naturally identified with $\Sigma$. Actually, the projection on the first factor is a (trivial) $\mathbb{R}$-principal fibration $\nu: M \to \Sigma$. The orthogonal complements of $\partial_t$ for $ds^2_\omega$ define a plane field transverse to this fibration restricted over $\Sigma^*$, i.e. a connection on this restricted $\mathbb{R}$-bundle. More precisely, $\omega$ is the 1-form relating this connection to the trivial product connection on $\Sigma^* \times \mathbb{R}$ which arises from the static metric $ds_0^2 - dt^2$. As the curvature $d\omega + \omega \wedge \omega$ vanishes, the connection associated with $ds^2_\omega$ is flat. This can also be deduced in a more elementary way from the fact that horizontal planes characterised by the condition $t = \text{constant}$ are orthogonal to the Killing vector field $\partial_t$.

That the metric $ds^2_\omega$ only depends on the cohomology class of $\omega$ reflects the fact that the trivialisation of the $\mathbb{R}$-principal fibration $\nu$ is unique only up to gauge transformations. The latter are translations in the fibers and hence determined by a function $f: \Sigma' \to \mathbb{R}$. They result in a change of $\omega$ by a coboundary $\omega \mapsto \omega + df$. 

As a direct application of Mayer-Vietoris sequence, one finds that a tuple of real numbers \((\sigma_1, \ldots, \sigma_n)\) can be realised as the residues of a closed 1-form around the cone singularities \(p_1, \ldots, p_n\) on the surface \(\Sigma\) if and only if the sum \(\sigma_1 + \ldots + \sigma_n\) vanishes. Moreover, once the residues \(\sigma_1, \ldots, \sigma_n\) are prescribed, the 1-form \(\omega\) is unique up to a closed 1-form on the closed surface \(\Sigma\). In particular, when \(\Sigma\) is a sphere, then the residues \(\sigma_1, \ldots, \sigma_n\) determine the cohomology class \([\omega]\).

In the application to \((2+1)\)-gravity, the residues \(\sigma_1, \ldots, \sigma_n\) correspond to the spins of \(n\) massive particles associated with the singularities. If each particle has positive mass, then it follows from the discussion in Section 4.1 that the resulting Euclidean surface \(\Sigma\) is a sphere. In that case, the associated stationary spacetime is then determined uniquely by the spins of the particles which must satisfy the condition \(\sigma_1 + \ldots + \sigma_n = 0\).

4.3.2. Geodesics and Completeness. If the closed 1-form is exact, \(\omega = df\), which is always true locally, the associated metric \(ds_\omega^2\) is given as the pull-back of \(ds_0^2\) under the diffeomorphism \((x, t) \mapsto (x, t + f(x))\). This allows one to give the following simple description of the geodesics in \((M, ds_\omega^2)\). A geodesic \(g\) in \((M, ds_\omega^2)\) is a path

\[
g : t \mapsto \left( c(t), \quad t_0 + \lambda \int_0^t \omega(c(s)) ds \right),
\]

where \(\lambda \in \mathbb{R}\) and \(t \mapsto c(t)\) is a geodesic in \(\Sigma\) parametrised by arc length. The geodesic \(g\) is timelike if \(|\lambda| > 1\), lightlike if \(|\lambda| = 1\), and spacelike if \(|\lambda| < 1\). Note that our notion of geodesic is not the one of a curve that minimises a length functional. In particular, we do not exclude that the geodesic \(c\) in \(\Sigma\) contains the singular points, which implies that the geodesic \(g\) can go through the singular lines. The singular lines themselves can be considered as geodesics according to this definition.

Using this notion of geodesics, we obtain a natural definition of geodesic completeness that allows us to directly deduce that the stationary spacetimes in Definition 4.4 are complete. We call a stationary spacetime \((M, ds_\omega^2)\) geodesically complete if inextensible geodesics are defined on the entire real line \(\mathbb{R}\). This leads to the following proposition.

**Proposition 4.5.** The spacetime \((M, ds_\omega^2)\) is geodesically complete.

**Proof.** As our notion of geodesics allows them to traverse the singularities, the underlying Euclidean surface \((\Sigma, ds_0^2)\) is geodesically complete, since it is compact. As the geodesics of \((M, ds_\omega^2)\) take the form \(g : t \mapsto (c(t), t_0 + \lambda \int_0^t \omega(c(s)) ds)\) where \(\lambda \in \mathbb{R}\) and \(t \mapsto c(t)\) is a geodesic in \(\Sigma\) parametrised by arc length, this also holds for the geodesics in \((M, ds_\omega^2)\). \(\square\)

We are now ready to investigate under which conditions the stationary flat spacetimes with particles are globally hyperbolic. The answer to this question is provided by the following proposition.

**Proposition 4.6.** The flat stationary singular spacetime \((M, ds_\omega^2)\) is globally hyperbolic if and only if the following three conditions are satisfied:

1. For every singular point \(p_i\), the Euclidean ball centred at \(p_i\) and of CTC radius \(|r_i| = |\sigma_i|/\theta_i\) is embedded, i.e., the injectivity radius at \(p_i\) for the singular Euclidean metric is greater than \(r_i\).
2. For every pair \((p_i, p_j)\) of singular points the sum of their CTC radii is greater than their Euclidean distance \(|r_i| + |r_j| < d(p_i, p_j)\).
3. Let \(S_0\) be the set of points \(p \in \Sigma\) for which the Euclidean distance \(d(p, p_i)\) from each singular point \(p_i\) is strictly greater than the CTC radius \(|r_i|\). Then the absolute value
of the integral of $\omega$ along any closed loop $\gamma$ in $S_0$ is strictly smaller than the Euclidean length $\ell_0(\gamma)$ of $\gamma$

$$|\int_\gamma \omega| < \ell_0(\gamma).$$

The first two conditions are immediately recognised as equivalent to the first condition in Definition 3.2 which states that the CTC regions associated to particles are disjoint CTC cylinders. The third condition implies that if the residues of $\omega$ are sufficiently small, then $(M, ds_0^2)$ is globally hyperbolic in the sense of Definition 3.2. Note that if $\gamma_i$ is the circle of radius $|r_i|$ and centre $p_i$, we have the equality

$$|\int_{\gamma_i} \omega| = |\sigma_i| = \theta_i|r_i| = \ell_0(\gamma_i).$$

The inequality in the third condition must therefore become an equality at the boundary of $S_0$.

Proof. The conditions are necessary: Assume that $(M, ds_0^2)$ is globally hyperbolic in the sense of Definition 3.2. Then the CTC regions around the particles must be pairwise disjoint, and every particle must be contained in a CTC cylinder. It is clear that these properties imply items (1) and (2). Moreover, by definition of global hyperbolicity, the interior region of $M$ must contain a Cauchy surface $S$, which intersects every vertical line in exactly one point. This implies that the Cauchy surface $S$ is the graph of a function $f: \mathcal{S}_0 \to \mathbb{R}$, where $\mathcal{S}_0$ is the closure of $S_0$. As this graph is spacelike, we have for every non-vanishing tangent vector $v$ of $S_0$

$$0 < ds_0^2(v) - (\omega(v) + df(v))^2$$

and therefore

$$-ds_0(v) < \omega(v) + df(v) < ds_0(v).$$

The inequality in the condition (3) is then obtained directly by integration along any closed loop $\gamma$ in $S_0$, since the integral of $df$ along such a loop vanishes.

The conditions are sufficient: As already observed, the first two conditions equivalent to the statement that every particle is surrounded by a CTC cylinder, and that these cylinders are disjoint. The inequality in condition (3) means precisely that there are no closed causal curves (CCCs) in $S_0 \times \mathbb{R}$, hence the only CCC in $M^*$ are the null circles in the CTC surface. It remains to construct a Cauchy surface for $M^*$.

Let now $q = (x, t_0)$ be a point in the closure $\overline{M^*}$ of $M^*$. For every $y$ in $\overline{S}_0$, select a path $c: [a, b] \to \mathcal{S}_0$ such that $c(a) = x$ and $c(b) = y$. Then the curve $t \mapsto (c(t), t_0 + \int_{[a, t]} ds_0(\dot{c}(s)) - \omega(\dot{c}(s))ds)$ is a future oriented lightlike curve in $\overline{M^*}$ joining $q$ to a point $q(y)$ in the vertical line above $y$. Hence, there exists a $t \in \mathbb{R}$ such that $(y, t)$ lies in the causal future $J^+(q)$ of $q$. Similarly, there exists a $t' \in \mathbb{R}$ such that $(y, t')$ lies in $J^-(q)$.

As there is no CCC above $S_0$, we must have $t' < t$, except if $x, y$ lies in the same boundary component of $S_0$, in which case we must have $t = t'$. (Otherwise we could construct a CTC in $M^*$). In particular, there is an upper bound for $t'$, and a lower bound for $t$. In other words, for any $q \in \overline{M^*}$ there are two maps $f_q^+ : \mathcal{S}_0 \to \mathbb{R}$ such that a point $p = (y, t)$ in $S_0 \times \mathbb{R}$ lies in $J^+(q)$ (respectively $J^-(q)$) if and only if $t \geq f_q^+(y)$ (respectively $t \leq f_q^-(y)$).

Observe that the future $I^+(q)$ of $q$ in $\overline{M^*}$ is the set of points $(y, t)$ with $t > f_q^+(y)$: by adding small $t$-components to the lightlike curves considered above one can obtain lightlike curves joining $q$ to $(y, t)$ for any $t > f_q^+(p)$. As $I^+(p)$ is open, it follows that $f_q^+$ is upper semi-continuous. Similarly, $f_q^-$ is lower semi-continuous. As $\mathcal{S}_0$ is compact, the set of points $(x, t)$ such that $f_p^-(x) \leq t \leq f_q^+(x)$ is compact. But this set is precisely the intersection
Proposition \(3.4\) then implies the existence of a Cauchy surface and hence the global hyperbolicity of \(M\).

In addition to establishing criteria for the global hyperbolicity of the stationary singular spacetime \((M, ds_0^2)\) this proposition provides a characterisation of the 1-forms associated with globally hyperbolic spacetimes. We obtain the following corollary.

**Corollary 4.7.** Let \((M, ds_0^2)\) be the singular flat spacetime associated to a closed singular Euclidean surface \(\Sigma\) and a closed 1-form \(\omega\) on \(\Sigma^*\). If \((M, ds_0^2)\) is globally hyperbolic, then \(\omega\) is cohomologous to a 1-form \(\omega_0\) such that at every point \(x\) of \(S_0\), the operator norm of \(\omega_0\), as a linear form on \(T_xS_0\), and with respect to the norm \(ds_0\) is less than one.

**Proof.** Consider a smooth map \(f_0: S_0 \to \mathbb{R}\) whose graph is a Cauchy surface in \(M^* = S_0 \times \mathbb{R}\), and take \(\omega_0 = \omega + df_0\). The fact that the graph of \(f_0\) is spacelike is equivalent to the condition that for any path \(\gamma: [a, b] \to S_0\) (not necessarily a loop!) the following inequality holds

\[
\int_{\gamma} \omega_0 < \ell_0(\gamma)
\]

As this inequality holds for any path \(\gamma: [a, b] \to S_0\), we obtain an infinitesimal version by differentiation: the evaluation of \(\omega\) on any tangent vector \(v\) is bounded by the \(ds_0\)-norm of \(v\).

The corollary follows.

4.4. Classification of stationary flat singular spacetimes. We already observed that the spacetimes \((M, ds_0^2)\) are stationary. We will now show that there is a converse statement that allows one to relate any stationary flat spacetime with a compact spatial surface and particles to a spacetime as in Definition 4.4.

**Proposition 4.8.** Let \(M\) be a flat globally hyperbolic spatially compact spacetime with particles. Assume that \(M\) is stationary, i.e. that the regular region \(M^*\) admits a Killing vector field \(X\) which is timelike everywhere. Then \(M\) embeds isometrically in the flat singular spacetime \((M, ds_0^2)\) associated to a singular Euclidean surface \(\Sigma\) and a closed 1-form \(\omega\) on the regular part \(\Sigma^*\) of \(\Sigma\).

**Proof.** Denote by \(M^*\) the regular part of \(M\) given as the complement of the singular lines, and by \(\hat{M}^*\) the interior region that is the complement of the CTC regions. Let \(\hat{S}\) be a compact Cauchy surface in \(M^*_0\). We denote by \(\hat{M}, \hat{M}^*, \hat{M}_0^*, \hat{S}\), respectively, the universal coverings of these manifolds. Then there are natural inclusions \(\hat{S} \subset \hat{M}_0^* \subset \hat{M}^*\).

By definition, the regular part \(M^*\) is locally modelled on Minkowski space \(\mathbb{R}^{1,2}\). Let \(\mathcal{D}: \hat{M}^* \to \mathbb{R}^{1,2}\) be the associated developing map, and let \(\rho: \Gamma \to SO_0(1, 2) \times \mathbb{R}^{1,2}\) be the holonomy representation, where \(\Gamma\) is the fundamental group of \(M^*\). The Killing vector field \(X\) induces a local flow \(\phi^t\) on \(M^*\). This flow is isometric and maps closed timelike curves to closed timelike curves. It follows that \(\phi^t\) preserves every CTC region \(U_i\) around the singular lines \(d_i\). Let \(\hat{\phi}^t\) be the lift of of \(\phi^t\) to \(\hat{M}^*\), which is generated by the lift \(\hat{X}\) of \(X\). Then the orbit space of \(\hat{\phi}^t\) is naturally identified with \(\hat{S}\).

It is well-known that local isometries on open subsets of \(\mathbb{R}^{1,2}\) extend to isometries of \(\mathbb{R}^{1,2}\). This implies that the lift \(\hat{X}\) is the pull-back \(\mathcal{D}_*X_0\) of a Killing vector field \(X_0\) on \(\mathbb{R}^{1,2}\). Let \(\tilde{U}_1\) be a lift of the region \(U_1\) around \(d_1\) to \(M^*\). Then there exists an isometry of \(\mathbb{R}^{2,1}\) whose composition with the developing map \(\mathcal{D}\) maps \(\tilde{U}_1\) into cylinder \(\{(r, \theta, t) : 0 \leq \theta \leq 2\pi, r^2 \leq r_0^2, a < t < b\}\) in Minkowski space. Note that we admit also cylinders with \(a\) and/or \(b\) are infinite. It follows that the flow associated with \(X_0\) must preserve this cylinder and, consequently, \(X_0\) is a linear combination \(a\partial_\theta + b\partial_t\).
Suppose $\alpha \neq 0$. Then, the only timelike line invariant under the flow of $X_0$ is the vertical line $\Delta_0$ characterised by the condition $r = 0$. But for every $\gamma$ in $\Gamma$, $\rho(\gamma)(\Delta_0)$ is also a timelike $X_0$-invariant line: $\Delta_0$ is therefore preserved by the entire holonomy group $\rho(\Gamma)$, which therefore is contained in the subgroup of rotations with axis $\Delta_0$. In particular, the holonomy preserves the vertical vector field $\partial_t$. The pull-back $\mathcal{D}^*(\partial_t)$ is a Killing vector field, which is timelike everywhere. Hence we can replace $X_0$ by this Killing vector field, since it satisfies the same hypothesis.

We can therefore assume $\alpha = 0$ and identify $X_0$ with the pull-back $\mathcal{D}^*(\partial_t)$ up to an homothety. This implies that every element of $\rho(\Gamma)$ preserves the vertical lines in $\mathbb{R}^{1,2}$. Elements of $\rho(\Gamma)$ must therefore be given as the composition of a rotation around a vertical axis and a translation. Let $\mathbb{R}^2$ be the horizontal plane characterised by the condition $t = 0$ and let $\pi_0: \mathbb{R}^{1,2} \to \mathbb{R}^2$ be the orthogonal projection. This projection is equivariant under the action of $\Gamma$ on $\mathbb{R}^{1,2}$ through the holonomy representation $\rho$ and its isometric action on $\mathbb{R}^2$. More precisely, for every $\gamma \in \Gamma$, we have $\pi \circ \rho(\gamma) = \tilde{\rho}(\gamma) \circ \pi$ where $\tilde{\rho}(\gamma)$ is the isometry of $\mathbb{R}^{1,2}$ which has the same linear part as $\rho(\gamma)$ (some rotation) and whose translation part is the projection of the translation part of $\rho(\gamma)$.

Then, the restriction of $\overline{\mathcal{D}} = \pi_0 \circ \mathcal{D}$ to $\tilde{S}$ and $\tilde{\rho}: \Gamma \to \text{Iso}(\mathbb{R}^2)$ are the developing map and holonomy representation of a Euclidean structure on $S$. By investigating the behaviour of of this maps in a neighbourhood of each singular line $d_i$, one finds that this Euclidean structure extends to a singular Euclidean metric on a closed surface $\Sigma$ that is obtained from $S$ by adding to the boundary of $S$ round disks with one cone-angle singularity.

The maximal trajectories of $\phi^i$ are timelike inextendible curves in $M^*$. As it is a fiber of a fibration $\nu: M^*_0 \to S$, every trajectory of $\phi^i$ in the interior region $M^*_0$ intersects $S$ in exactly one point. By considering this fibration in every CTC region, one finds that it extends naturally on the regular region $M^*$ to a fibration $\nu: M^* \to \Sigma^*$, whose fibers are trajectories of $\phi^i$. In particular, the fibers are homeomorphic to $\mathbb{R}$: there is a global section $\sigma: \Sigma^* \to M^*$. For every $p$ in $M^*_0$, let $\Psi(x)$ be the pair $(\nu(p), t(p))$, where $t(p)$ is the unique real number $t$ such that $p = \phi^i(\sigma(\nu(x)))$: it defines an embedding $\Psi: M^*_0 \to \Sigma^* \times \mathbb{R}$ such that $\nu = p_1 \circ \Psi$, where $p_1$ is the projection on the first factor.

The orthogonal complement to $X_0$ with respect to the flat metric defines a plane field on $M^*$ which is transverse to $\nu$ and invariant under vertical translations. It therefore extends to a connection on the trivial $\mathbb{R}$-principal bundle $p_2: \Sigma^* \times \mathbb{R} \to \Sigma^*$, which is invariant under $\mathbb{R}$-translations on the fiber. This connection is flat, since in $\mathbb{R}^{1,2}$ the plane field orthogonal to $\partial_t/\partial r$ is integrable.

Let $\omega$ be the 1-form relating this connection to the trivial connection on $\Sigma^* \times \mathbb{R}$. As both connections are flat, it is a closed 1-form. This implies that the bundle embedding $\Psi: M^* \to \Sigma^* \times \mathbb{R}$ maps the plane field orthogonal to $X_0$ on the plane field in $\Sigma^* \times \mathbb{R}$ which is orthogonal to the fibers for the metric $ds^2_\Sigma$ (cf. section [4.3.1]). It follows quite easily that $\Psi$ is an isometric embedding.

□

Remark 4.9. As an immediate corollary of this is proposition, we obtain that the stationary flat singular spacetimes $(M, ds^2_\Sigma)$ are maximal; i. e. they admit no proper isometric extensions.

5. Observers, particles and null geodesics

In this section, we illustrate how our description of stationary flat globally hyperbolic spacetimes with particles allows one to clarify their causality properties and to derive general results about the outcome of measurements by observers. These issues play an important role in the physical interpretation of the theory. The question if spacetimes containing particles with spin
are admissible models in (2+1)-dimensional general relativity or if the presence of closed causal
curves makes them unsuitable has been subject to much debate in the physics community.

Generally, the presence of closed timelike curves (CTCs) in a spacetime is problematic from
the physics point of view because it leads to “grandfather paradoxes”. Each point in the
spacetime corresponds to a physical event, and a timelike curve defines the set of the events
experienced by an observer. Given two points \( p, q \) on a closed timelike curve, it becomes
impossible to determine which of the associated events happened to the observer before the
other one, since - by definition - each of them lies in the future and in the past of the other one.

In the models under consideration, the CTCs are confined to a small region around each
particle. Although this region is not inaccessible or hidden behind a horizon, one is tempted to
argue that the presence of CTCs in a small region around the particles is unproblematic if one
intends to model the large scale behaviour of the spacetime and is interested only in observers
that are located at a sufficient distance from the particles.

However, this reasoning is too naive because it clashes with one of the fundamental notions
of general relativity, namely the principle that observers can communicate with each other
by exchanging light signals. Such light signals are modelled by future directed null geodesics,
which can enter the regions in which closed timelike curves occur and remerge from them, even
if the associated observers are located at a large distance from such regions. This can lead
to light signals which are received by observers before they are emitted and again give rise to
causality paradoxes.

In order to establish if flat globally hyperbolic spacetimes with particles are admissible physics
models, it is therefore necessary to analyse carefully and in detail how the presence of particles
with spin affects light signals passed between different observers. However, to our knowledge
there is no systematic investigation of this issue in the literature. In the following, we will show
how this question can be resolved for the stationary flat globally hyperbolic spacetimes with
particles.

5.1. Null geodesics in the one-particle model. To illustrate how the presence of particles
with spin manifests itself in the measurements of observers, we start with an informal discussion
based on the one-particle model introduced in Section 2. As in Section 2, we denote by \( \theta_0 = 2\pi - \mu \)
the associated apex angle, by \( \sigma \) the spin of the particle, and assume that the singular
line associated with the particle is given in radial coordinates by the equation \( r = 0 \).

We consider observers who probe the geometry of the spacetime by emitting and receiving
lightrays. For simplicity, we will restrict attention to observers which do not undergo accel-
eration, and we will require that they do not collide with the particle. Such observers are
characterised uniquely by a worldline, a future directed timelike geodesic in the complement of
the singular line.

A lightray is modelled by a future directed null geodesic in \( M \). Note that we will not require
that this null geodesic avoids the CTC region or the singular line, which in physical terms
means that light can pass through the particle. Moreover, the lightrays under consideration are
“test lightrays” in analogy to the test masses used in thought experiments on general relativity.
This means we consider them as hypothetical lightrays in a given spacetime and neglect their
contribution to the stress-energy tensor.

In the presence of particles, it is possible that a lightray that is emitted by an observer at a
given time returns to the observer. Such a returning lightray corresponds to a future directed
null geodesic \( \gamma : [0, 1] \to M \) which intersects the observer’s worldline twice. As shown in Section
2.6, such geodesics exist only for large masses \( \mu = 2\pi - \theta_0 > \pi \) which correspond to cones with
apex angles \( \theta_0 < \pi \). We therefore restrict attention to this case.
To obtain an explicit description of returning light rays, we need a concrete parametrisation for the future directed timelike geodesic which characterises the observer. Making use of the symmetry of the one-particle system under rotations around the $t$-axis, we assume that in rectangular coordinates $(x, y, t)$ on $\mathbb{R}^{2,1}$ this worldline takes the form
\[ g(t) = (0, d, 0) + t(\sinh \vartheta, 0, \cosh \vartheta) \quad \vartheta, d \in \mathbb{R}. \]

Note that the parameter $d$ gives the minimal distance between the particle and observer as it appears in the reference frame of the particle. The parameter $\vartheta$ defines the relative velocity $v$ of observer relative to the particle: $v = \tanh \vartheta$. The parameter $t$ coincides with the eigentime of the observer, i.e. the time shown on a clock carried by the observer. Note that the origin of this time variable is chosen in such a way that the distance between the observer and the particle is minimal at $t = 0$.

In the universal cover, returning light rays are characterised by future directed null geodesic segments whose endpoints lie on two different lifts of the observer’s worldline as indicated. The lifts of the observer’s worldline to the universal cover can be parametrised as
\[ g_m(t) = R_m^* g(t) + m\sigma(0, 0, 1), \]
where $m \in \mathbb{Z}$ and $R$ is the rotation around the $t$-axis by an angle $\theta_0$. It follows from the discussion in Section 2.6 that a null geodesic connecting two different lifts $g_m$ and $g_n$ exists if and only if $0 < |m - n| < \frac{\pi}{\theta_0}$. A light signal emitted by the observer at eigentime $t - \Delta t$ and received at eigentime $t$ corresponds thus to a future directed null geodesic from $g_m(t - \Delta t)$ to $g_0(t)$ with $0 < |m| < \frac{\pi}{\theta_0}$. Such a returning light ray is depicted in Figure 4.

The requirement that $g(t) - g_m(t - \Delta t)$ is lightlike defines a quadratic equation in $\Delta t$. A short calculation shows that the solution of this equation that corresponds to a future-directed lightray is given by
\[ \Delta t_m = m\sigma \cosh \vartheta + \sinh \vartheta T_m + \sqrt{(m\sigma \sinh \vartheta + \cosh \vartheta T_m)^2 + S_m^2} \quad (7) \]
where
\[ T_m = 2 \sin \frac{m\theta_0}{2} \left( t \sinh \vartheta \sin \frac{m\theta_0}{2} + d \cos \frac{m\theta_0}{2} \right) \]
\[ S_m = 2 \sin \frac{m\theta_0}{2} \left( t \sinh \vartheta \cos \frac{m\theta_0}{2} - d \sin \frac{m\theta_0}{2} \right). \]
This defines the return time of the signal, the interval \( \Delta t_m \) of eigentime elapsed between the emission and the reception if the lightray. Note, however, that depending on the sign of the spin and on the direction into which the lightray travels around the particle, the first and second term in this formula can become negative. A short calculation shows that the return time \( \Delta t_m > 0 \) if and only if
\[ \sqrt{t^2 \sinh^2 \vartheta + d^2} > \frac{m\sigma}{2\sin \frac{m\theta_0}{2}}. \]
From the concavity of the sine function it follows that this condition is satisfied for all admissible values of \( m \) if
\[ \sqrt{t^2 \sinh^2 \vartheta + d^2} > \frac{\pi|r_0|}{2}. \]
Conditions (8), (9) have a direct physical interpretation. The term \( \sqrt{t^2 \sinh^2 \vartheta + d^2} \) on the left-hand-side gives the distance of the observer from the particle at the moment at which the observer receives the returning lightray and with respect to the momentum rest frame of the particle. If this distance is smaller than the term on the right-hand side, the effect of the spin can become dominant and results in a negative return time: \( \Delta t_m < 0 \). This corresponds to a light signal that is received before it is emitted thus violating causality. Note that causality violating signals can arise even if the observer does not enter the CTC zone associated with the particle. Similarly, it is irrelevant if the observer had entered or will enter the CTC zone at an earlier or later time. What determines if causality violating light signals can be received at a given time is the distance of the observer from the particle at that time.

In order to exclude causality violating light signals for all times \( t \) and values of \( m \), one therefore needs to impose that the minimum distance \( |d| \) of the observer from the particle viewed from the reference frame of the particle satisfies the condition
\[ |d| > \frac{\pi|r_0|}{2} =: r_c. \]
Note that this condition represents a considerable weakening with respect to the condition used in the definition of global hyperbolicity. In Definition 3.2 only causal curves which \( do \ not \ enter \) the CTC zones around the particles are admissible. In this example, we consider a closed, piecewise geodesic causal curve (obtained by composing the light ray with the segment of the timelike geodesic which characterises the observer). This curve is such that the the closest point of the timelike segment lies outside of a circle with radius \( r_c \) around the particle but whose lightlike segment may enter the CTC zone.

It is instructive to consider how measurements with returning light signals allow the observer to obtain information about the spacetime, i.e. to determine the position, mass and spin of the particle. For simplicity we restrict attention to observers which are at rest with respect to the particle. Such observers are characterised by the condition \( \vartheta = 0 \). For such an observer, determining the return time reduces to a two dimensional problem that can be solved by elementary geometry. The return time (7) takes the form
\[ \Delta t_m = m\sigma + 2d\left| \sin \frac{m\theta_0}{2} \right| = 2m + l, \]
where \( l \) is the length of the straight line in the \( t = 0 \)-plane that connects its intersection point with the observer’s worldline and and its image under the rotation \( R \) as shown in Figure 5. The directions from which the observer receives the returning lightrays are given by the vertical
Figure 5. Different lifts of the observer and the projections of returning null geodesics. The points $g_0, g_1, g_{-1}$ correspond to the intersection points of the lifted worldlines with the plane characterised by $t = 0$. The solid lines correspond to the projections of returning lightrays.

projection of the lightlike vector $g(t) - g_m(t - \Delta t_m)$ on the plane $t = 0$-plane. The angle $\alpha_m$ between the direction of the particle as viewed by the observer and the direction of the returning lightray is therefore given by

$$\alpha_m = \frac{\text{sgn}(m)(\pi - |m|\theta_0)}{2}$$

This allows the observer to draw conclusions about the location of the particle and to determine its mass $\mu = 2\pi - \theta_0$ and spin $\sigma$. For this it is sufficient that the observer determines the direction and the return time for the first two lightrays which are given by $m = \pm 1$. The observer then obtains the direction of the particle by constructing the bisector of the angle between the directions of these two returning lightrays. A measurement of this angle allows him to determine the particle’s mass via (12). The time elapsed between the return of the two lightrays yields the particle’s spin $\sigma$ via (11), and by taking the sum $\Delta t_1 + \Delta t_{-1} = 4d\cos\alpha_1$, the observer obtains his distance $d$ from the particle. By emitting and receiving returning lightrays, the observer can thus determine all parameters associated with the model: the position of the particle, its mass $\mu = 2\pi - \theta_0$ and its spin $\sigma$.

5.2. Null geodesics in stationary flat spacetimes with particles. We will now generalise our discussion from the previous subsection to general stationary flat globally hyperbolic spacetimes with particles and to more general light signals emitted and received by several observers. Let $(M, ds^2_\omega)$ be a stationary flat spacetime with particles as in Definition 4.4, which is globally hyperbolic, i. e. satisfies the conditions of Proposition 4.6.

The discussion from Section 4 shows that there are no closed causal curves contained in the interior region. However, by definition, there is a region around each particle which the metric takes the same form as the one-particle model. If the apex angles associated with the particles are sufficiently small, this allows for the existence of light signals that enter the CTC region and...
that return to the observer before they are emitted. In analogy to the one-particle model, one can show that such signals are not present if the observer is sufficiently far from each particle.

However, this does not address the issue of causality violating signals in full generality. In general relativity, it is traditional to consider light signals that are passed back and forth between several different observers. Such light signals play a fundamental role in the physical interpretation of the theory since they allow observers to synchronise clocks and to measure distances and relative velocities.

To demonstrate that these spacetimes have acceptable causality properties, we therefore need to establish physically reasonable conditions that ensure that measurements conducted by a “team of observers” located in the interior region of the spacetime do not result in light signals that return to an observer before they are emitted. As we will show in the following, this is guaranteed if one imposes that all of the observers are located sufficiently far away from each particle.

To derive this result, we give a precise definition of the relevant physical concepts of observers, light rays and light signals. For simplicity, we will restrict attention to observers whose worldlines are parallel to the singular lines that define the particles.

**Definition 5.1 (Observers and light signals).**

Let \((M, ds^2)\) be a stationary globally hyperbolic flat spacetime with particles, and denote by \(M^*\) its regular part, i.e. the complement of its singular lines.

1. A **stationary observer** in \(M\) is a future directed vertical geodesic in \(M^*\).
2. A light ray sent from a stationary observer \(g_1\) to a stationary observer \(g_2\) in \(M\) is a future directed null geodesic \(\gamma : [0, 1] \to M\) with \(\gamma(0) \in g_1\) and \(\gamma(1) \in g_2\).
3. A **light signal** sent from a stationary observer \(g_1\) to a stationary observer \(g_2\) in \(M\) is a piecewise geodesic curve \(\gamma : [0, 1] \to M\) with \(\gamma(0) \in g_1\) and \(\gamma(1) \in g_2\) for which there exists a subdivision \([0, 1] = \bigcup_{j=0}^{N} [s_j, s_{j+1}]\), \(s_0 = 0\), \(s_{N+1} = 1\) such that \(\gamma|_{[s_j, s_{j+1}]}\) is a future directed null geodesic.

By identifying \(M\) with \(\Sigma \times \mathbb{R}\), one can express every light signal \(\gamma : [0, 1] \to M\) from \(g_1\) to \(g_2\) as a function \(\gamma(s) = (c(s), t(s))\) where \(c : [0, 1] \to \Sigma\) is a piecewise geodesic curve on \(\Sigma\) with \(c(0) = g_1 \cap \Sigma\) and \(c(1) = g_2 \cap \Sigma\). Note that the function \(t : [0, 1] \to \mathbb{R}\) is characterised uniquely up to a global constant by the requirement that \(\gamma\) can be subdivided into future directed null geodesics.

From the definition, it is clear that the concept of a light signal encompasses precisely the situation discussed above. Each point \(c(s_j)\) on \(\Sigma\) for \(j \in \{0, ..., N + 1\}\) corresponds to an observer, which is given by the unique vertical line through \(c(s_j)\). The first observer at \(c(0)\) emits a light ray at \(t(0) = 0\), that is described by the null geodesic \(\gamma|_{[s_0, s_1]}\). The second observer at \(c(s_1) \in \Sigma\) receives this light ray at \(t(s_1)\) and immediately emits another light ray, \(\gamma|_{[s_1, s_2]}\), which is received by the third observer at \(t(s_2)\) and so on, until the last light ray is received by the observer at \(t(1)\). This situation is depicted in Figure 6.

Piecewise geodesic curves on the surface \(\Sigma\) thus have a natural general relativistic interpretation: they define groups of observers that transmit a signal by sending and receiving lightrays.

The question is now if by passing such light signals between different observers, it is possible to construct a light signal that returns to the first observer and which is such that \(t(1) < t(0)\). As we can identify \(t(1) - t(0)\) with the time elapsed between the emission of the light signal and its reception as shown on a clock carried by this observer, the condition \(t(1) < t(0)\) states that the light signal is received before it is emitted, which is an obvious violation of causality.

**Definition 5.2.** Let \((M, ds^2)\) be a stationary globally hyperbolic flat spacetime with particles. A **returning light signal** for a stationary observer \(g\) is a light signal from \(g\) to \(g\). It can be
Figure 6. Returning light signal in a stationary flat spacetime \((M, ds^2_ω)\). The black solid lines depict the worldlines of stationary observers in \(M\), the red solid line corresponds to a returning light signal. The shaded areas around the points on \(Σ\) correspond to the CTC regions around each particle, and the grey dashed line to the associated piecewise geodesic curve on \(Σ\).

expressed as a curve \(γ : [0, 1] → M, γ(s) = (c(s), t(s))\), where \(c : [0, 1] → Σ\) is a closed piecewise geodesic curve with \(c(0) = c(1)\). The light signal is called a paradoxical if \(t(1) < t(0)\).

We will now show that there is direct and intuitive condition that rules out paradoxical light signals and which depends only on the location of the observers. In other words, as long as the observers do not come too close to the particles, there is no possibility of paradoxical light signals. This implies that although causality is violated near the particles, observers which are located at a certain distance from them do not encounter signals that are received before they are emitted.

**Proposition 5.3.** Let \((M, ds^2_ω)\) be a stationary globally hyperbolic spatially compact flat spacetime with particles defined in terms of a Euclidean metric \(ds^2_0\) with cone singularities and a closed 1-form \(ω\) on a closed surface \(Σ\). For every singular line \(d_i\), denote by \(r^i_ω := \frac{π}{2}|r_i| = \frac{π|σ_i|}{2θ_i}\) the CTC radius rescaled by \(π/2\). Let \(γ : [0, 1] → M, γ(s) = (c(s), t(s))\) be a returning light signal given in terms of a closed piecewise geodesic curve \(c : [0, 1] → Σ\) with a subdivision as in Definition 5.1. Assume that \(c\) is not constant and

\[
d_{ds^2_0}(c(s_j), p_i) > r^i_ω \quad ∀ j ∈ \{0, ..., N + 1\}, i ∈ \{0, ..., n\}.
\]

Then \(γ\) does not give rise to a paradox, i.e. it satisfies \(t(1) - t(0) > 0\).

To prove the proposition, we note that its content can easily be reformulated as a statement on closed, piecewise geodesic curves on the Euclidean surface with cone singularities. We have:

**Remark 5.4.** Proposition (5.3) is equivalent to the following statement. Let \(c : [0, 1] → Σ\) be a closed curve for which there exists a subdivision \([0, 1] = \bigcup_{j=0}^{N} [s_j, s_{j+1}], s_0 = 0, s_{N+1} = 1\) such...
that $c|_{[s_j, s_{j+1}]}$ is a geodesic, $c(s_0) = c(s_n)$ and which satisfies condition (13). Then:

$$l(c) > \int_c \omega$$

Indeed, combining this inequality with the corresponding inequality involving the 1-form $-\omega$ we obtain:

$$l(c) > \left| \int_c \omega \right|$$

By means of this remark, we can now give a direct proof of Proposition 5.3

**Proof of proposition 5.3.** For every cone singularity $p_i$, let $D_i$ be the disk of radius $r^i_c$ centred at $p_i$. We can assume that up to a coboundary, $\omega$ coincides in each disc $D_i$ with $\frac{\omega}{r^i_c}d\theta = r^i_c d\theta$, where $\theta$ is the angular coordinate in the wedge of angle $\theta_i$.

According to Corollary 4.7, one can adjust $\omega$, without perturbing the previous property, so that the integration of $\omega$ along a curve contained in the interior region - a fortiori, outside the disks $D_i$ - cannot exceed the length of the curve.

Let $c : [0, 1] \rightarrow \Sigma$ be a closed piecewise geodesic curve satisfying the assumptions of Remark 5.4. Then there is a subdivision of $[0, 1]$ such that every interval $[a, b]$ of this subdivision is either contained in the complement of the disks $D_i$, or in such a disk.

We have just seen that in the first case, we have the inequality:

$$\int_{c|_{[a,b]}} \omega < l(c|_{[a,b]})$$

Consider now the second case. In this case, the endpoints $c(a)$ and $c(b)$ lie on the boundary $\partial D_i$. As the corner points $c(s_i)$ lie outside $D_i$, the restriction of $c$ to $[a, b]$ is a geodesic arc. Consequently, there is an integer $m$ such that the lift $\tilde{c}$ of $c|_{[a,b]}$ to the $m$-branched cover $D^m_i$ of $D_i$ is a minimising geodesic arc. Choose a polar coordinate system $(r, \theta)$ in $D_i$ such that $\theta(\tilde{c}(a)) = 0$ and let $\alpha$ be the angular coordinate of $\tilde{c}(b)$. According to Section 2.6, we have $0 < \alpha < \pi$. According to our choice of $\omega$, the lift $\tilde{\omega}$ of $\omega$ in $D^m_i$ is the variation of the angular coordinate multiplied by the factor $r^i_c/2\pi$. This implies

$$\int_{c|_{[a,b]}} \omega = \int_{\tilde{c}|_{[a,b]}} \tilde{\omega} = r^i_c \alpha$$

As $\tilde{c}|_{[a,b]}$ is a chord of angle $\alpha$ of a circle of radius $r^i_c$, its length is given by

$$l(c|_{[a,b]}) = l(\tilde{c}|_{[a,b]}) = 2r^i_c \sin \frac{\alpha}{2} = |r_i| \pi \sin \frac{\alpha}{2}$$

As $\frac{\alpha}{2} < \frac{\pi}{2}$, we have $\alpha < \pi \sin \frac{\alpha}{2}$. It follows that the inequality (15) also holds in this case. This implies that on each subinterval $[a, b]$ of the subdivision of $[0, 1]$, the inequality (15) is satisfied. By summing over all subintervals, we obtain the inequality (14).

$$\square$$

6. Outlook and conclusions

In this article, we give a systematic investigation of the causality structure of flat, stationary (2+1)-dimensional Lorentzian manifolds with particle singularities and clarify its implications in physics. As these manifolds contain closed timelike curves, the usual methods established for globally hyperbolic manifolds cannot be applied directly but have to be replaced by suitable generalisations.

By introducing a generalised notion of global hyperbolicity adapted to manifolds with particle singularities, we are able to classify all stationary flat Lorentzian (2+1)-dimensional manifolds with particles which are globally hyperbolic in that sense. This classification result characterises
flat, stationary globally hyperbolic (2+1)-dimensional Lorentzian manifolds in terms of two-dimensional Euclidean surfaces with cone singularities and closed one-forms on these surfaces and thus provides an explicit and simple description.

It turns out that this description is particularly well-suited for the investigation of the causality structure of these manifolds from a physics point of view. It allows one to systematically address the question how the presence of massive point particles with spin manifests itself in measurements performed by observers in the spacetime.

We show how an observer in the spacetime can use the results of measurements with returning lightrays to determine the mass, spin, position and relative velocity of the particles and investigate more general light signals exchanged between several observers. It turns out that the latter have a natural interpretation in terms of piecewise geodesic loops on the underlying surface with conical singularities.

This allows us to derive a general condition on the observer that excludes paradoxical light signals which return to an observer before they are omitted. In physics terms, our result implies that if all observers stay sufficiently far away from the particles, no causality violating light signals will occur, no matter how often the light signals exchanged between them enter spacetime regions which contain closed timelike curves.

It would be interesting to extend these results to more complete classification of flat, globally hyperbolic 3d Lorentzian manifolds with particle singularities, which also take into account the non-stationary case. As currently very little is known about these manifolds, a first step would be the construction and study of relevant examples which generalise the examples currently known in the physics literature. One could then attempt to classify these manifolds under suitable additional assumptions using the generalised notion of global hyperbolicity introduced in this paper.

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