Magnetic Moment Density from Lack of Smoothness of the Ernst Potential

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In this talk it is shown a way for constructing magnetic surface sources for stationary axisymmetric electrovac spacetimes possessing a non-smooth electromagnetic Ernst potential. The magnetic moment density is related to this lack of smoothness and its calculation involves solving a linear elliptic differential equation. As an application the results are used for constructing a magnetic source for the Kerr-Newman field.

1 Introduction

The Darmois’ junction conditions provide a way for constructing surface sources for non-smooth spacetimes. Whenever the extrinsic curvature of a hypersurface contained in the spacetime is discontinuous, a distributional stress tensor can be assigned to it. Furthermore, if the electromagnetic curvature is discontinuous across a hypersurface, a distributional electromagnetic source can be located on it. The existence of a timelike (axial) Killing field allows the calculation of the mass (angular momentum) surface density in terms of the discontinuity of the extrinsic curvature of the hypersurface. An expression for the electric charge density is also obtained for stationary fields, but so far, no expression has been provided for the magnetic moment density. In the next section we shall follow a different approach. Instead of considering the discontinuities of the electromagnetic curvature we shall study the discontinuities of the electromagnetic Ernst potential, restricting ourselves to stationary axisymmetric electrovac spacetimes. This approach follows closely the classical theory of potential, where a discontinuous (non-smooth) scalar potential gives rise to a dipole (monopole) source for the field that derives from the potential. As an example, we shall apply the formalism to the Kerr-Newman spacetime.

2 Magnetic moment surface density

As it has already been stated, we shall just study stationary axisymmetric electrovac spacetimes. Let us consider the metric,

\[ ds^2 = -e^{2U}(dt - A d\phi)^2 + e^{-2U} \{ e^{2k}(d\rho^2 + dz^2) + \rho^2 d\phi^2 \}, \]

(1)
written in Weyl coordinates. \( U, A, k \) are functions of \( \rho \) and \( z \).

The electric, \( E \), and magnetic field, \( B \), as viewed from an orthonormal coframe \(^6\) whose timelike one-form is given by \( \theta^0 = e^U (dt - A d\phi) \), can be written, according to Maxwell equations, in terms of a complex function, \( \Phi \),

\[
f = E + i B = -e^{-U} d\Phi,
\]

the Ernst electromagnetic potential\(^5\). This scalar potential \( \Phi \) satisfies one of the Ernst equations\(^5\), which in this notation can be written as,

\[
L \Phi \equiv \frac{1}{\sqrt{g}} \partial_\mu \left\{ N \sqrt{g} \left( e^{-2U} g^\mu\nu - i \frac{i}{\rho} A e^\mu\nu \right) \partial_\nu \Phi \right\} = 0,
\]

where \( g \) is the metric induced by (1) on the hypersurfaces \( t = \text{const.} \), \( N = (-4 g^{tt})^{-\frac{1}{2}} \) is the lapse function and \( \epsilon \) is the Levi-Civita tensor on the surfaces of constant time, \( t \), and azimuthal angle, \( \phi \). For simplicity the whole equation has been written as the action of a differential operator, \( L \), on the potential.

This is a consequence only of the Maxwell equations in the curved spacetime whose metric is given by Eq. 1, regardless of whether the electromagnetic field is the source of the gravitational field.

Since we will be interested in compact sources, we shall only consider metrics which are asymptotically flat in some coordinates \((t, r, \theta, \phi)\),

\[
ds^2 = - \left( 1 - \frac{2m}{r} \right) \left( dt + \frac{2J \sin^2 \theta}{r} d\phi \right)^2 + \left( 1 + \frac{2m}{r} \right) \left\{ dr^2 + (r^2 + c_1 r) \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \right\} + O \left( \frac{1}{r^2} \right),
\]

where \( m \) is the total mass of the source and \( J \) is the total angular momentum and \( c_1 \) is just a constant (its value is \(-2m\) for Kerr-Newman metrics). The electromagnetic potential of the compact source will have the following asymptotic expansion,

\[
\Phi = \frac{e}{r} + \frac{M \cos \theta}{r^2} + \frac{c_2}{r^2} + O(r^{-3}),
\]

where \( e \) is the total charge, \( M \) is a complex constant whose real and imaginary parts are, respectively, the electric and magnetic dipole moment and \( c_2 \) is a constant that may arise in some choices of coordinates.

As it was done in\(^6, 7, 8, 9\), we shall introduce a function \( Z \) that satisfies the following elliptic differential equation out of the surface source and behaves at infinity like the cartesian coordinate \( z \),

\[
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where $L^+ Z = 0$ and $Z = (r + c_3) \cos \theta + O(r^{-1})$, \( (6) \)

The region where the electromagnetic surface source is located will be taken as the surface, $S$, where the electromagnetic potential (or its derivative), $\Phi$, is discontinuous, just as it is done in the classical theory of potential. Without loss of generality we shall assume that this surface is closed. The three-space, $\Omega$, defined by any of the hypersurfaces $t = const.,$ excluding $S$, will be divided into two regions, $\Omega^+$ and $\Omega^-$, respectively the outer and inner parts of $\Omega$ with respect to $S$.

Taking into account Eq. 3 and Eq. 6 the following Green identity can be written,

$$0 = \int_{\Omega} \sqrt{g} (Z L \Phi - \Phi L^+ Z) \, dx^1 \, dx^2 \, dx^3 = \int_{\partial \Omega} dS N \left\{ e^{-2U} \left( Z \frac{d\Phi}{dn} - \Phi \frac{dZ}{dn} \right) + i \frac{A}{\rho} (Z * d\Phi(n) + \Phi * dZ(n)) \right\}, \quad (7)$$

using the Stokes theorem. The two-dimensional Hodge dual on the surfaces of constant time and azimuthal angle is denoted by $*$ and $n$ is the unitary outer normal to $S$.

The boundary $\partial \Omega^+$ is formed by $S$ and the sphere at infinity whereas $\partial \Omega^-$ is the surface $S$. The integral at infinity can be performed with the information we get from the asymptotic behaviours. The discontinuity of the integrand on $S$,

$$\sigma_M = \frac{1}{4\pi} \left\{ N \left\{ e^{-2U} \left( \Phi \frac{dZ}{dn} - Z \frac{d\Phi}{dn} \right) - i \frac{A}{\rho} (Z * d\Phi(n) + \Phi * dZ(n)) \right\} \right\}, \quad (8)$$

denoted by squared brackets, can be interpreted as the electromagnetic dipole moment surface density of the source for this field. Its real part is the electric dipole density and its imaginary part is the magnetic moment density.

3 \hspace{1em} The Kerr-Newman spacetime

As an example for this formalism, the magnetic dipole density for a surface in the Kerr-Newman spacetime will be calculated. We shall follow $^3$ and restrict the range of the Boyer-Lindquist coordinate $r$ to positive values. Points on the hypersurface $r = 0$ with coordinates $(t, \phi, 0, \theta)$ and $(t, \phi, 0, \pi - \theta)$ are identified. Hence the Ernst potential,
\[ \Phi = \frac{e}{r - ia \cos \theta}, \]  
(9)

will be discontinuous on \( r = 0 \).

The Kerr-Newman metric in Boyer-Lindquist coordinates induces the line element,

\[ ds^2_2 = a^2 \cos^2 \theta \, d\theta^2 + \sin^2 \theta \left( a^2 - e^2 \tan^2 \theta \right) \, d\phi^2, \]  
(10)
on the surface \( r = 0 \). We just need a solution of Eq. 6,

\[ Z = (r - 2m) \cos \theta + \frac{e^2 \cos \theta + ia \, m \, \cos^2 \theta}{r + ia \, \cos \theta}, \]  
(11)
in order to calculate the magnetic moment surface density,

\[ \sigma_M = \frac{\left( e^2 \cos^2 \theta + e^2 + a^2 \cos^2 \theta \right) i \, e}{2 \pi a^2 \cos^3 \theta \sqrt{a^2 - e^2 \tan^2 \theta}}, \]  
(12)

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