Point-like source solutions in modified gravity with a critical acceleration

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Abstract We consider equations of modified gravity involving critical accelerations and find its solutions for the point-like source by suggesting the appropriate symmetry of metrics in the empty space-time.

Keywords Modified Newtonian dynamics · critical acceleration · metrics

1 Introduction

The modified Newtonian dynamics [1] is a successful phenomenological setting for the description of ”dark matter effects” at galactic scales [2]. Its concept is based on the introduction of a universal critical acceleration $g_0$ as the manifestation of empirical regularities in dark matter halos, so that the gravitation law is crucially modified at accelerations less than $g_0$ with the ordinary visible or baryonic sources in the following way

$$g \zeta \left( \frac{g}{g_0} \right) = -\nabla \phi_M,$$

where the interpolating function $\zeta$ is originally set to be equal to

$$\zeta(y) = \left( \frac{y^2}{1+y^2} \right)^{\frac{1}{3}},$$

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while $\phi_M$ is the gravitational potential of visible matter with density $\rho_M$: 
\[ \Delta \phi_M = 4\pi G \rho_M. \]

In the Newtonian limit $y \gg 1$ of super-critical accelerations one finds corrections negligible for the Solar system, for instance. In the deep MONDian limit $y \ll 1$ of sub-critical accelerations the point-like source of mass $M$ produces the enforced acceleration in the same direction as the Newtonian one,
\[ g = \frac{1}{r} \sqrt{g_0 GM}. \]

This law reproduces flat rotation curves in dark galactic halos of disc galaxies without introduction of any dark matter.

However, actually we cannot a priori fix a precise form of the function $\zeta(y)$, since the main features of the galactic dynamics are predicted with an impressive accuracy using only the asymptotic behavior of $\zeta(y)$: $\zeta(y) \to 1$ for $y \gg 1$ and $\zeta(y) \to y$ for $y \ll 1$. So, there are some families of interpolating functions suitable for the same predictions, say,
\[ \zeta(y) \mapsto \zeta_n(y) = \frac{y}{(1 + y^n)^{\frac{1}{n}}} \]

The experimental data of galaxy disks [3] prefer for $n$ posed between 1 and 2. For a complicated $\gamma$-family [4]
\[ \zeta(y) \mapsto \xi_\gamma(y) = \left(1 - \exp\left(-\sqrt{y^{\gamma} \zeta_\gamma(y)}\right)\right)^{-\frac{1}{\gamma}}, \]
combined data on galaxies and the Solar system can be well described at $\gamma = 1$, too. In this respect, we accept the form of (2) in order to reach the consistency with the data at the galaxy scale of distances at the rather simple functional expression.

In addition to the asymptotic flatness of the rotation curves in spiral galaxies, the most impressive successes of modification (1) are the baryonic Tully-Fisher relation (BFTR), Faber-Jackson relation and Freeman limit observed as the dynamical regularities in galaxies. Let us shortly mention these regularities.

With a remarkably little intrinsic scatter, BTFR reads of
\[ \log M_b = s \log V_f - \log b \]
for $M_b = M_* + M_g$ being the sum of masses of visible stars and gas in the disk galaxy, and $V_f$ being the flat limit of the rotation curve, while $b = Gg_0$ as it directly follows from (3), and the slope $s = 4$. Moreover, the BTFR is a remarkably persistent relation, as it holds for both low and high surface brightness galaxies.

The Faber-Jackson relation holds with quite a good accuracy and suggests that for quasi-isothermal elliptical galaxies with dispersion of radial velocity $\sigma$ and mass $M$
\[ \sigma^4 \sim GM g_0. \]
This follows from the Jeans equation
\[ \frac{d\sigma^2}{dr} + \sigma^2 \frac{2\beta + \alpha}{r} = -g(r) \] (8)
at constant \( \alpha = \frac{d\ln \rho}{d\ln r} \) and \( \beta = 1 - \frac{(\sigma^2 + \sigma^2 \kappa)}{2\sigma^2} \), so that
\[ \frac{d(\sigma^2 \rho)}{dr} = -\rho \sqrt{GMg_0} \Rightarrow \sigma^4 = \frac{GMg_0}{\alpha^4}. \] (9)

Moreover, the Milgrom’s law provides an additional stability for stellar systems regardless of the form of interpolating function. This is known as the Freeman limit: the disks with the surface density \( \Sigma \leq \frac{g_0}{G} \) have enhanced stability [5] and the number of disks with \( \Sigma \geq \frac{g_0}{G} \) decreases exponentially. So, since the bulk of the disk is in the weak-accelerated regime, the acceleration \( g \sim \sqrt{M} \), instead of \( g \sim M \) in the Newtonian regime. Thus, \( \frac{\delta g}{g} = \frac{\delta M}{2M} \) instead of \( \frac{\delta g}{g} = \frac{\delta M}{M} \), leading to the weaker response to the small perturbations. Numerical simulations show, that MOND has an effect, similiar to the dark matter halo in stabilizing the galaxy disk [6].

So, the critical acceleration \( g_0 \) is observed phenomenological quantity, and it is measured with high accuracy in the gas-reach galaxies [7]

\[ g_0 = (1.21 \pm 0.14) \cdot 10^{-10} \text{ m s}^{-2} \] (10)

The generalization of nonrelativistic relation (1) to the field theory is given, for instance, in the Tensor-Vector-Scalar theory (TeVeS) by J. Bekenstein [11], wherein he introduces additional gravitational vector and scalar fields replacing the dark components in the general relativity (GR). Other models are discussed in [2,12]. However, those models involve ad hoc new degrees of freedom. Moreover, they have some deep conceptual problems like instabilities [2]. Recently, a nonlocal origin of modified gravity has been also discussed [13].

The origin of critical acceleration has been considered in numerous articles [13,15,16,17,18,19,20].

So, in order to investigate generic features of modified gravity with the critical acceleration, we prefer for a more phenomenological approach by investigating the approximate equations and their generic consequences if applicable. In this way, we can rewrite the law of modified gravity in (1) with interpolating function (2) in terms of Ricci curvature tensor as
\[ \int d^3r R_{\mu}^0 \cdot \zeta \left( \frac{\int d^3r R_{\mu}^0}{\int d^3r K_{\mu}^0} \right) = \int d^3r R_{\mu}^0, \] (11)
where the bar tensor is defined in terms of energy-momentum tensor \( T_{\mu}^\nu \) for the visible matter,
\[ R_{\mu}^\nu = 8\pi G \left( T_{\mu}^\nu - \frac{1}{2} \delta_{\mu}^\nu T \right), \quad T = T_{\mu}^\mu, \] (12)
and the external curvature is given by
\[ K_0^0 = g_0 \frac{2}{r}, \tag{13} \]
while the integration in (11) operates in the sphere of radius \( r \). Since in the non-relativistic limit \( R_0^0 \approx \Delta \phi \) at \( g_0 \approx 1 + 2\phi \), we can easily reproduce the deep MONDian limit with the “dark matter mass” \( M_{DM} = \int d^3 r \rho_{DM} \) entering \( \Delta \phi = 4\pi G(\rho_M + \rho_{DM}) \), so that \( GM_{DM} \approx r\sqrt{GMg_0} \). The Newtonian limit is also restored. The main feature of (11) is the presence of external Ricci tensor \( K_\mu^\nu \) as the critical divider of two regimes.

Note that the equation (11) takes into account the symmetry of space-time: the rotations and stationarity. It does not represent the full system of equations for a modified gravity with the critical acceleration, but it gives the relativistic generalization of MONDian dynamics in the symmetric case. In addition to (11) we have to require the conservation law for the energy-momentum tensor of matter, of course. Nevertheless, for instance, we can get the relativistic static solution in the modified gravity with the critical acceleration in the case of point-like source, that conserve both the spherical symmetry and stationarity. This symmetry in general relativity provides the metrics of the form
\[ ds^2 = f(r) dt^2 - \frac{1}{f(r)} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \tag{14} \]
due to the symmetry of source energy-momentum tensor: \( T_0^0 = T_r^r \) \[21]. It is natural to extend this symmetry of metrics to the case of modified gravity with the point-like source to find the function \( f(r) \).

It is spectacular that (11) can be extrapolated in cosmology with a little modification in order to conserve the spatial homogeneity at extra-galactic scales in large scale structure of Universe as it was done in \[22\]: one can simply replace the external Ricci curvature by de Sitter one, \( K_0^0 \mapsto \bar{K}_0^0 = 3g_0' \), while the integration inside the sphere is canceled, that gives
\[ R_0^0 \cdot \zeta \left( \frac{R_0^0}{K_0^0} \right) = \bar{R}_0^0. \tag{15} \]

In this paper we find the point-like source solutions of (11) in the symmetric case of (14) for both external Ricci tensors of \( K_0^0 \) and \( \bar{K}_0^0 \).

2 The modified solutions

In the symmetric case of (14), when \( R_0^0 = \Delta f/2 \), the integrating over the volume of finite sphere with radius \( r \) yields
\[ \int d^3 r R_0^0 = \frac{1}{2} \int dS \cdot \nabla f(r) = \frac{1}{2} \int r^2 d\Omega \frac{df}{dr} \frac{dr}{r} = 2\pi r^2 \frac{df}{dr}. \tag{16} \]
Therefore
\[ \left( \int d^3 r R_0^0 \right)^2 = \frac{\left( \frac{2 \pi r^2 df}{dr} \right)^4}{\left( \frac{2 \pi r^2 df}{dr} \right)^2 + \left( \int d^3 r K_0^0 \right)^2}, \] (17)
hence, the general solution of (17) is
\[ \frac{df}{dr} = \frac{\int d^3 r R_0^0}{2 \pi r^2 \sqrt{2}} \left\{ 1 + \sqrt{1 + \left( \frac{2 \int d^3 r K_0^0}{\int d^3 r R_0^0} \right)^2} \right\}^{\frac{1}{2}}. \] (18)

At \( K_0^0 \equiv 0 \), we reproduce the limit of general relativity, of course. However, complete equations of GR allow us to get the more definite expression with the same condition of symmetry,
\[ \frac{df}{dr} = -\frac{d}{dr} \frac{2GM_{\text{tot}}(r)}{r}, \] (19)
at \( M_{\text{tot}}(r) = 4\pi \int dr r^2 \rho(r) \) being the total mass enclosed in the sphere of radius \( r \).

For MOND we get
\[ \int d^3 r K_0^0 = 4\pi g_0 r^2, \]
while for the case of de Sitter external Ricci tensor we find
\[ \int d^3 r K_0^0 = 4\pi g_0^\prime r^3. \]

So, the solutions in MOND and its modification derived at cosmic scales should differ essentially.

2.1 De Sitter vacuum state

Considering a solution of modified gravity, we can analyze basic physical features of changing. So, the vacuum with the positive density of energy \( \rho_A \) corresponds to de Sitter space. Then,
\[ R_0^0 = -8\pi G \rho_A, \]
yielding
\[ \int d^3 r R_0^0 = -\frac{32\pi^2}{3} G \rho_A r^3. \]

Therefore, MOND breaks down the vacuum homogeneity, since
\[ \frac{df^{\text{MOND}}}{dr} = -\frac{16\pi}{3} \rho_A G r \frac{1}{\sqrt{2}} \left\{ 1 + \sqrt{1 + \left( \frac{3g_0}{4\pi G \rho_A r} \right)^2} \right\}^{\frac{1}{2}}, \] (20)
and the vacuum metrics,

\[ f^{\text{dS}} = 1 - \frac{8\pi}{3} G \rho_A r^2 \]

transformed into the solution, that can be treated in the framework of general relativity at \( r \to 0 \) as the metric given by a singular negative density of matter \( \rho_m \sim -1/\sqrt{r} \). This behavior artificially introduces a point of world center.

In contrast, when the external Ricci tensor is defined by de Sitter space, too, i.e. \( \bar{K}^0_0 = 3\bar{g}^0_0 = -8\pi G \bar{\rho}_A \), the modified metric scales like the metric of de Sitter space at the modified density,

\[ \frac{df^{\text{dS}}}{dr} = -\frac{16\pi}{3\sqrt{2}} \rho_A G r \left\{ 1 + \sqrt{1 + \left( \frac{\bar{\rho}_A}{\rho_A} \right)^2} \right\}^{\frac{1}{2}}, \tag{21} \]

i.e.

\[ \rho_A \mapsto \rho'_A = \frac{\rho_A}{\sqrt{2}} \left\{ 1 + \sqrt{1 + \left( \frac{\bar{\rho}_A}{\rho_A} \right)^2} \right\}^{\frac{1}{2}}. \]

It is interesting to note, that there is the solution with \( \rho'_A = \bar{\rho}_A \), if \( \bar{\rho}_A = x \rho_A \), while \( x \) satisfies the following relation:

\[ \sqrt{1 + \sqrt{1 + x}} = \sqrt{2}x, \tag{22} \]

that gives

\[ x = \frac{\sqrt{5}}{2} \approx 1.12. \tag{23} \]

It is important to stress that the modified gravity with the scaled critical acceleration conserves the vacuum, i.e. de Sitter space remains de Sitter space. This fact is important for the cosmological extrapolation of MOND, since the cosmic evolution starts with the description of homogeneous Universe.

2.2 Point-like source

The black hole with mass \( M \) corresponds to

\[ \int d^3r \bar{g}^0_0 = 4\pi GM. \tag{24} \]

Therefore, the MONDian solution is given by

\[ \frac{df^{\text{MOND}}}{dr} = \frac{2GM}{r^2} \frac{1}{\sqrt{2}} \left\{ 1 + \sqrt{1 + \left( \frac{2\bar{g}_0 r^2}{GM} \right)^2} \right\}^{\frac{1}{2}}, \tag{25} \]
while the modification with the scaled critical acceleration results in

\[
\frac{df}{dr}^{\text{mdS}} = \frac{2GM}{r^2} \frac{1}{\sqrt{2}} \left\{ 1 + \sqrt{1 + \left( \frac{2g_0'r^3}{GM} \right)^2} \right\}. \tag{26}
\]

So, at \( g_0'r_S = g_0 \) solutions (25) and (26) can be matched at \( r = r_S \).

We expect that effects of modified gravity are essential at galactic scales. In order to test this expectation in the Solar system, we estimate the corrections to the magnitude of the acceleration of Neptune \( g = |\frac{df}{dr}| \) in both cases of modification, because the Neptune is the most distant planet from the Sun that has an almost circular orbit, since

\[
\frac{\text{Aphelion-Perihelion}}{\text{Aphelion}} \approx 0.02. \tag{27}
\]

Expanding at \( \frac{g'^2}{GM} \ll 1 \) in the MOND case, we find

\[
\frac{df}{dr}^{\text{MOND}} \approx \frac{2GM_\odot}{r^2} \left\{ 1 + \frac{1}{2} \left( \frac{g_0'r^3}{GM_\odot} \right)^2 \right\} = \frac{r_g}{r^2} \left\{ 1 + 2 \left( \frac{g_0'r^3}{r_g} \right)^2 \right\}, \tag{28}
\]

where \( r_g = 2GM_\odot \) is the gravitational radius of the Sun. Substituting the semi-major axis of Neptune \( r \mapsto a \approx 4.5 \times 10^9 \) km, we get

\[
\frac{\Delta g}{g} \approx 2 \left( \frac{g_0'r^3}{r_g} \right)^2 \approx 1.6 \times 10^{-10}. \tag{29}
\]

The limit of deep MOND (\( \frac{g''^2}{GM} \gg 1 \)) of flat rotation curves with \( g = V_f^2/r \) is also recovered

\[
\frac{df}{dr}^{\text{MOND}} \approx 2g \approx \frac{2}{r} \sqrt{GMg_0}. \tag{30}
\]

In the case of scaled critical acceleration the limit \( \frac{2g_0'r^3}{GM} \ll 1 \) yields

\[
\frac{df}{dr}^{\text{mdS}} \approx \frac{2GM_\odot}{r^2} \left\{ 1 + \frac{1}{2} \left( \frac{r^3g_0'}{GM_\odot} \right)^2 \right\}, \tag{31}
\]

and the correction to the gravitational acceleration is equal to

\[
\frac{\Delta g}{g} \approx \frac{1}{2} \left( \frac{r^3g_0'}{GM_\odot} \right)^2 = \frac{32\pi^2}{9} \left( \frac{r^3\rho_\Lambda}{M_\odot} \right)^2 \approx 3 \times 10^{-36}, \tag{32}
\]

where we estimate the slope of critical acceleration by \( g_0' = 8\pi G\rho_\Lambda/3 \) at the observed value of cosmological constant \( \rho_\Lambda \).

At \( \frac{2g_0'r^3}{GM} \gg 1 \) we obtain

\[
\frac{df}{dr}^{\text{mdS}} \approx 2g \approx 2\sqrt{\frac{GMg_0'}{r}}, \tag{33}
\]

that covers the limit of \( g_0 = g_0'r \) again in agreement with the consideration in cosmology [22].
2.3 Black hole surrounded by quintessence

The point-like source of black hole can be generalized to the case of symmetric metric (14) by adding some sources with $T^0_0 = T^r_r$ and constant equation of state parameter $w_q [9]$, when in general relativity

$$ f = 1 - \frac{2GM}{r} + \sum_q \left(\frac{r_q}{r}\right)^{3w_q+1}, $$

so that

$$ \int d^3r R^0_0 = 4\pi GM - 2\pi \sum_q r_q(3w_q + 1) \left(\frac{r_q}{r}\right)^{3w_q}, \quad (34) $$

where $r_q$ is the quintessence parameter of length dimension.

For instance, the charged black hole corresponds to the quintessence of static electromagnetic field with $w_q \rightarrow \frac{1}{3}$ and $r^2_q = GQ^2$, so that

$$ \int d^3r R^0_0 = 4\pi GM - \frac{4\pi GQ^2}{r} = 4\pi GMA(r), \quad A(r) = \left(1 - \frac{Q^2}{Mr}\right). \quad (35) $$

Then, we find the modifications of Reissner–Nordstrøm solutions

$$ \frac{df^{\text{MOND}}}{dr} = \frac{2GM A(r)}{r^2} \left\{ 1 + \sqrt{1 + \left(\frac{2g^0r^2}{GMA(r)}\right)^2} \right\}^{\frac{1}{2}}, \quad (36) $$

and

$$ \frac{df^{\text{mdS}}}{dr} = \frac{2GM A(r)}{r^2} \left\{ 1 + \sqrt{1 + \left(\frac{2g^0r^3}{GMA(r)}\right)^2} \right\}^{\frac{1}{2}}. \quad (37) $$

At large distances

$$ \frac{df^{\text{MOND}}}{dr} \approx 2\frac{\sqrt{GMg^0}}{r} \left(1 - \frac{Q^2}{2Mr}\right), \quad \frac{df^{\text{mdS}}}{dr} \approx 2\sqrt{GMg^0} \left(1 - \frac{Q^2}{2Mr}\right). $$

At $w_q = -\frac{1}{3}$ and $M = 0$ we recover the case of global monopole [10], when $f = 1 - \kappa$ and it is not changed by the modification with the critical acceleration.

3 Conclusion

We have considered solutions of modified gravity with critical acceleration for the case of symmetry in the metric as it is relevant to the symmetry in the general relativity for the gravitational sources.

First, we note, that homogeneous distribution of matter in the de Sitter space is not consistent with the original MOND, while the critical acceleration scaled with distance conserves the de Sitter structure of space. This fact
does not mean that the MOND is incorrect, it only points to that MOND has got a restricted area of applicability, i.e. it well works for inhomogeneous distributions of matter of island kind. Moreover, it could mean that there is a regulator of regimes in the modified gravity with the critical acceleration, that would be an extra field inherently related to the gravity.

Second, we have found the metrics for point-like sources of gravity in the modified models of gravity, i.e. the modified Schwarzschild black holes and Reissner–Nordstrøm black holes with the electric charge. The global monopole solution is not changed by the gravity with the critical acceleration. We have compared corrections to the gravitational acceleration in far regions of Solar system, for example, at the distance of Neptune, so that gravity with the scaled critical acceleration has got a much smaller correction.

In order to study more general solutions of modified gravity with the critical acceleration, we need a complete set of relevant gravitational equations, that suggests the involvement of models to the moment.

Next, conditions of transition from the fixed critical acceleration to the scaled one require further investigations. Moreover, one could expect that Newtonian dynamics might be restored at extremely small accelerations \[23,24\].

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