Super Schrödinger algebra in AdS/CFT

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Abstract

We discuss (extended) super Schrödinger algebras obtained as subalgebras of the superconformal algebra psu(2,2|4). The Schrödinger algebra with two spatial dimensions can be embedded into so(4,2). In the superconformal case the embedded algebra may be enhanced to the so-called super Schrödinger algebra. In fact, we find an extended super Schrödinger subalgebra of psu(2,2|4). It contains 24 supercharges (i.e., 3/4 of the original supersymmetries) and the generators of so(6), as well as the generators of the original Schrödinger algebra. In particular, the 24 supercharges come from 16 rigid supersymmetries and half of 16 superconformal ones. Moreover, this superalgebra contains a smaller super Schrödinger subalgebra, which is a supersymmetric extension of the original Schrödinger algebra and so(6) by eight supercharges (half of 16 rigid supersymmetries). It is still a subalgebra even if there are no so(6) generators. We also discuss super Schrödinger subalgebras of the superconformal algebras, osp(8|4) and osp(8\textsuperscript{*}|4).
1 Introduction

AdS/CFT correspondence [1–3] is increasingly important from fundamental aspects in string theory and its applications to some realistic systems such as QCD, quark-gluon plasma and condensed matter physics. The rigorous proof of the correspondence has not been given yet, and it is still a conjecture. But it is firmly supported by enormous evidence without any obvious failures. In recent years the integrability of the AdS superstring [4] has played an important role in testing the AdS/CFT beyond supergravity approximation. With this integrability and symmetry argument, the S-matrix of the AdS superstring has now been proposed [5].

It is, however, technically difficult to treat the full AdS superstring [6], and it is still a nice direction to look for a solvable limit of the AdS superstring. The Penrose limit [7] is a well-known example and the resulting theory is a pp-wave string [8] which is exactly solvable [9] with the light-cone gauge fixing. The solvability was exploited to construct the BMN operator correspondence [10]. Another example is a non-relativistic limit in the target spacetime [11]. In this limit, the world-sheet theory of the AdS superstring becomes a set of free theories on AdS$_2$ geometry. It is also exactly solvable [12, 13] and it has well been studied [14–16].

As a first step towards a new solvable limit of the AdS superstring, we focus upon a Schrödinger algebra [17, 18] obtained as a subalgebra of the conformal algebra so(4,2). A Schrödinger algebra is a non-relativistic analog of the conformal algebra. When the Schrödinger algebra has $d$ spatial dimensions, it is embedded into a conformal algebra so($d+2,2$) as a subalgebra. In this paper we discuss (extended) super Schrödinger subalgebra of the superconformal algebra psu(2,2)$|4)$. In the superconformal case the embedded Schrödinger algebra may be enhanced to a supersymmetric version of the Schrödinger algebra. It should be called (extended) super Schrödinger algebra, where the word “extended” implies the additional bosonic generators coming from so(6) other than the original Schrödinger.

As a new result, we find an extended super Schrödinger subalgebra of psu(2,2)$|4). It

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1 Although some supersymmetric extensions of Schrödinger algebra are discussed in [19–22], there would be no overlap with our result. After this paper, less supersymmetric Schrödinger algebras have been found in [23]. One of them coincides with the symmetry of [24].
contains 24 supercharges (i.e., 3/4 of the original supercharges) and those of so(6), as well as the generators of the original Schrödinger algebra. In particular, the 24 supercharges come from 16 rigid supersymmetries and half of 16 superconformal ones. This superalgebra further contains a smaller super Schrödinger subalgebra, which is composed of generators of the original Schrödinger algebra, eight supercharges (half of 16 rigid supersymmetries) and those of so(6). It is still a subalgebra even if there are no so(6) generators. We also discuss super Schrödinger algebras in the superconformal algebras related to the AdS$_{4/7} \times S^{7/4}$, namely osp(8|4) and osp(8$^*$|4). The results are similar to the case of psu(2,2$|$4).

Recently the non-relativistic CFT [25–28] is discussed in relation to fermions at unitarity. The gravity background preserving the Schrödinger symmetry has been found in [29, 30] and it is a candidate of the gravity dual of cold atoms. It is given by the geometry which is conformally equivalent to an asymptotically plane-wave background. It would be an interesting issue to embed this background into string theory as a proper background, and discuss the spectrum of the string theory. Hoping that our results may be useful in this direction, we leave it as a future problem.

This paper is organized as follows. In section 2 we introduce the superconformal algebra psu(2,2$|$4) and identify (extended) super Schrödinger subalgebras. In section 3 we discuss other superconformal algebras, osp(8|4) and osp(8$^*$|4). Section 4 is devoted to a conclusion and discussions.

2 Super Schrödinger algebras in psu(2,2$|$4)

We begin with the superconformal algebra psu(2,2$|$4) and then find (extended) super Schrödinger subalgebras of it.

2.1 Decomposition of psu(2,2$|$4)

The (anti-)commutation relation of the super-AdS$_5 \times S^5$ algebra, psu(2,2$|$4), is composed as follows. The bosonic part contains the so(4,2)

\[
[P_a, P_b] = J_{ab} , \quad [J_{ab}, P_c] = \eta_{bc} P_a - \eta_{ac} P_b ,
\]

An application to the aging phenomena is also discussed in [31].
\[ [J_{ab}, J_{cd}] = \eta_{bc} J_{ad} + 3\text{-terms} \quad (a = 0, 1, 2, 3, 4) , \]

and the so(6)
\[ [P_{a'}, P_{b'}] = -J_{a'b'} , \quad [J_{a'b'}, P_{c'}] = \delta_{b'c'} P_{a'} - \delta_{a'c'} P_{b'} , \]
\[ [J_{a'b'}, J_{c'd'}] = \delta_{b'c'} J_{a'd'} + 3\text{-terms} \quad (a' = 5, 6, 7, 8, 9) , \]

where \( \eta_{ab} = \text{diag}(-1, +1, +1, +1, +1) \) and \( \delta_{a'b'} = \text{diag}(+1, +1, +1, +1, +1) \). Then the fermionic generator \( Q \) satisfies
\[ [P_a, Q] = -\frac{1}{2} Q \Gamma_a I \sigma_2 , \quad [P_{a'}, Q] = \frac{1}{2} Q \Gamma_{a'} J \sigma_2 , \quad [J_{AB}, Q] = \frac{1}{2} Q \Gamma_{AB} , \]
\[ \{ Q^T, Q \} = 2i CT A h_+ P_A + i C T A \sigma_2 h_+ P_B - i C T A' J \sigma_2 h_+ P_{A'} , \]

where \( A = (a, a') \) and \( I \equiv \Gamma^{01234} , \quad J \equiv \Gamma^{56789} . \)

Here \( \Gamma^A \)'s are \((9+1)\)-dimensional gamma-matrices and \( C \) is the charge conjugation matrix satisfying
\[ \Gamma^T_A = -C \Gamma_A C^{-1} . \]

The fermionic generator \( Q \) is a pair of Majorana-Weyl spinors in \((9+1)\)-dimensions with the same chirality. \( h_+ \) is the chirality projector defined by \( h_+ = \frac{1}{2}(1 + \Gamma_{01...9}) . \)

Let us define
\[ \tilde{P}_\mu = \frac{1}{2}(P_\mu - J_{4\mu}) , \quad \tilde{K}_\mu = \frac{1}{2}(P_\mu + J_{4\mu}) , \quad \tilde{D} = P_4 , \]
\[ \tilde{J}_{\mu\nu} = J_{\mu\nu} , \quad \tilde{Q} = Q p_- , \quad \tilde{S} = Q p_+ , \]

where \( a = (\mu, 4) \) and \( \mu = 0, 1, 2, 3 \). Here the projectors \( p_\pm \) are
\[ p_\pm \equiv \frac{1}{2}(1 \pm \Gamma^A I \sigma_2) = \frac{1}{2}(1 \pm \Gamma^{0123} i \sigma_2) . \]

Note that
\[ p_\pm^T C = C p_\mp , \quad \Gamma^{0123} i \sigma_2 p_\pm = \pm p_\pm , \quad [p_\pm, h_+] = 0 . \]

Then the (anti-) commutation relations are\(^3\)
\[ [\tilde{P}_\mu, \tilde{D}] = -\tilde{P}_\mu , \quad [\tilde{K}_\mu, \tilde{D}] = \tilde{K}_\mu , \quad [\tilde{P}_\mu, \tilde{K}_\nu] = \frac{1}{2} \tilde{J}_{\mu\nu} + \frac{1}{2} \eta_{\mu\nu} \tilde{D} , \]

\(^3\) We suppress trivial commutators.
\[ [\tilde{J}_{\mu\nu}, \tilde{P}_\rho] = \eta_{\nu\rho} \tilde{P}_\mu - \eta_{\mu\rho} \tilde{P}_\nu, \quad [\tilde{J}_{\mu\nu}, \tilde{K}_\rho] = \eta_{\nu\rho} \tilde{K}_\mu - \eta_{\mu\rho} \tilde{K}_\nu, \]
\[ [\tilde{J}_{\mu\nu}, \tilde{J}_{\rho\sigma}] = \eta_{\nu\rho} \tilde{J}_{\mu\sigma} + 3\text{-terms}, \]
\[ \{ \tilde{Q}^T, \tilde{Q} \} = 4iC\Gamma^\mu p_- h_+ \tilde{P}_\mu, \quad \{ \tilde{S}, \tilde{S} \} = 4iC\Gamma^\mu p_- h_+ \tilde{K}_\mu, \]
\[ \{ \tilde{Q}^T, \tilde{S} \} = iC\Gamma^{\mu\nu} \tilde{\sigma}_2 p_+ h_+ \tilde{J}_{\mu\nu} + 2iC\Gamma^d p_+ h_+ \tilde{D} = \]
\[ + 2iC\Gamma^{\mu d'} p_+ h_+ p_{d'} - iC\Gamma^{d'\mu'} \tilde{J} i\sigma_2 p_+ h_+ J_{d'\mu'}, \]
\[ [\tilde{P}_\mu, \tilde{S}] = -\frac{1}{2} \tilde{Q} \Gamma_\mu 4, \quad [\tilde{K}_\mu, \tilde{Q}] = \frac{1}{2} \tilde{S} \Gamma_\mu 4, \quad [\tilde{D}, \tilde{Q}] = \frac{1}{2} \tilde{Q}, \quad [\tilde{D}, \tilde{S}] = -\frac{1}{2} \tilde{S}, \]
\[ [\tilde{J}_{\mu\nu}, \tilde{Q}] = \frac{1}{2} \tilde{Q} \Gamma_{\mu\nu}, \quad [\tilde{J}_{\mu\nu}, \tilde{S}] = \frac{1}{2} \tilde{S} \Gamma_{\mu\nu}, \quad [\tilde{J}_{\mu\nu}, \tilde{J}_{\rho\sigma}] = \frac{1}{2} \tilde{J}_{\mu\rho} \tilde{J}_{\nu\sigma}, \quad [\tilde{J}_{\mu\nu}, \tilde{K}_{\rho\sigma}] = \frac{1}{2} \tilde{K}_{\mu\rho} \tilde{K}_{\nu\sigma}, \quad (2.6) \]
\[ [\tilde{P}_{d'}, \tilde{Q}] = \frac{1}{2} \tilde{Q} \Gamma_{d'} i\sigma_2, \quad [\tilde{J}_{d'\mu}, \tilde{Q}] = \frac{1}{2} \tilde{Q} \Gamma_{d'\mu}, \]
\[ [\tilde{P}_{d'}, \tilde{S}] = \frac{1}{2} \tilde{S} \Gamma_{d'} i\sigma_2, \quad [\tilde{J}_{d'\mu}, \tilde{S}] = \frac{1}{2} \tilde{S} \Gamma_{d'\mu}, \quad (2.7) \]

and so (6) in (22). This is \( \mathcal{N} = 4 \) superconformal algebra in four dimensions. \( \tilde{Q} \) are 16 supercharges while \( \tilde{S} \) are 16 superconformal charges.

In order to clarify the embedding of the Schrödinger algebra into the conformal algebra, let us further decompose the generators as follows:

\[ P_\pm = \frac{1}{\sqrt{2}} (\tilde{P}_0 \pm \tilde{P}_3), \quad K_\pm = \frac{1}{\sqrt{2}} (\tilde{K}_0 \pm \tilde{K}_3), \quad J_{i\pm} = \frac{1}{\sqrt{2}} (\tilde{J}_{i0} \pm \tilde{J}_{i3}), \]
\[ D = \frac{1}{2} (\tilde{D} - J_{03}), \quad D' = \frac{1}{2} (\tilde{D} + J_{03}), \quad P_i = \tilde{P}_i, \quad K_i = \tilde{K}_i, \quad J_{ij} = \tilde{J}_{ij}, \quad (2.8) \]

where \( \mu = (0, i, 3) \) with \( i = 1, 2 \). The (anti-)commutation relations can straightforwardly be written down. The bosonic part is

\[ [J_{ij}, J_{k\pm}] = \eta_{jk} J_{i\pm} - \eta_{ik} J_{j\pm}, \quad [J_{ij}, P_k] = \eta_{jk} P_i - \eta_{ik} P_j, \quad [J_{ij}, K_k] = \eta_{jk} K_i - \eta_{ik} K_j, \]
\[ [J_{i\pm}, J_{j\pm}] = J_{ij} \pm \eta_{ij} (D' - D), \]
\[ [P_i, K_j] = \frac{1}{2} J_{ij} + \frac{1}{2} \eta_{ij} (D' + D), \quad [P_i, K_{i\pm}] = \frac{1}{2} J_{i\pm}, \quad [P_{i\pm}, K_i] = -\frac{1}{2} J_{i\pm}, \]
\[ [D, J_{i\pm}] = \pm \frac{1}{2} J_{i\pm}, \quad [D', J_{i\pm}] = \pm \frac{1}{2} J_{i\pm}, \]
\[ [P_i, J_{j\pm}] = \eta_{ij} P_{i\pm}, \quad [K_i, J_{j\pm}] = \eta_{ij} K_{i\pm}, \quad [J_{i\pm}, P_{i\pm}] = -P_i, \quad [J_{i\pm}, K_{i\pm}] = -K_i, \]
\[ [P_+, K_-] = -D', \quad [P_-, K_+] = -D, \]
\[ [D, P_-] = P_-, \quad [D, P_+] = \frac{1}{2} P_i, \quad [D, K_+] = -K_-, \quad [D, K_-] = -\frac{1}{2} K_i, \]
\[ [D', P_-] = P_+, \quad [D', P_+] = \frac{1}{2} P_i, \quad [D', K_+] = -K_-, \quad [D', K_-] = -\frac{1}{2} K_i, \quad (2.9) \]

and so (6) in (22). The (anti-)commutation relations including the fermionic generators are

\[ \{ \tilde{Q}^T, \tilde{Q} \} = 4iC\Gamma^+ p_- h_+ P_+ + 4iC\Gamma^- p_- h_+ P_- + 4iC\Gamma^d p_- h_+ P_i, \]
\[ \{ \bar{S}^T, \bar{S} \} = 4i CT^+ p_+ h_+ K_+ + 4i CT^- p_+ h_+ K_- + 4i CT^i p_+ h_+ K_i , \]
\[ \{ \bar{Q}^T, \bar{S} \} = i CT^{ij} T i \sigma_3 p_+ h_+ J_{ij} + 2i CT^{ij} T i \sigma_2 p_+ h_+ J_{+i} + 2i CT^{ij} T i \sigma_2 p_+ h_+ J_{ij} - 2i CT^4 \Gamma^- p_+ h_+ D' - 2i CT^4 \Gamma^+ p_+ h_+ D \]
\[ + 2i CT^{a'b'} p_+ h_+ P_{a'V} - i CT^{a'b'} J i \sigma_2 p_+ h_+ J_{a'V} , \]
\[ [K_\pm, \bar{Q}] = -\frac{1}{2} \bar{S} T^+ \Gamma_4 , \quad [K_i, \bar{Q}] = \frac{1}{2} \bar{S} T_i \Gamma_4 , \quad [P_\pm, \bar{S}] = \frac{1}{2} \bar{Q} \Gamma^+ \Gamma_4 , \quad [P_i, \bar{S}] = -\frac{1}{2} \bar{Q} \Gamma_i \Gamma_4 , \]
\[ [J_{ij}, \bar{Q}] = \frac{1}{2} \bar{Q} \Gamma_{ij} , \quad [J_{ij}, \bar{S}] = \frac{1}{2} \bar{S} \Gamma_{ij} , \quad [J_{ij}, \bar{Q}] = \frac{1}{2} \bar{Q} \Gamma_i \Gamma_4 , \quad [J_{ij}, \bar{S}] = -\frac{1}{2} \bar{S} \Gamma_i \Gamma_4 , \]
\[ [D, \bar{Q}] = -\frac{1}{4} \bar{Q} \Gamma^+ \Gamma^- , \quad [D, \bar{S}] = \frac{1}{4} \bar{S} \Gamma^+ \Gamma^- . \]

2.2 Super Schrödinger subalgebras

From the superconformal algebra obtained in the previous subsection, it is easy to see that the following subset of the generators

\[ \{ J_{ij}, J_{i+}, D, P_\pm, P_i, K_+ \} \quad (2.10) \]

forms the Schrödinger algebra. Its commutation relations are

\[ [J_{ij}, J_{k+}] = \eta_{jk} J_{i+} - \eta_{ik} J_{j+} , \quad [J_{ij}, P_k] = \eta_{jk} P_i - \eta_{ik} P_j , \quad [P_i, K_+] = \frac{1}{2} J_{i+} , \]
\[ [P_i, J_{j+}] = \eta_{ij} P_+ , \quad [J_{i+}, P_-] = -P_i , \quad [P_-, K_+] = -D , \]
\[ [D, J_{i+}] = -\frac{1}{2} J_{i+} , \quad [D, P_-] = P_- , \quad [D, P_i] = \frac{1}{2} P_i , \quad [D, K_+] = -K_+ . \quad (2.11) \]

This subalgebra may be enhanced to a supersymmetric subalgebra of psu(2,2|4), including the so(6) part. It should be called (extended) super Schrödinger subalgebra.

We will look for super Schrödinger subalgebras of psu(2,2|4). We begin with the anticommutation relation \{ \bar{S}^T, \bar{S} \}, which contains \( K_- \) and \( K_i \) in the right-hand side. \( K_- \) and \( K_i \) are not in (2.10). For some part of \( \bar{S} \) to be supercharges of a super Schrödinger

\[ \begin{align*}
[J_{ij}, K_i] &= -i \eta_{jk} K_i + i \eta_{ik} K_j , \quad [J_{ij}, P_k] = -i \eta_{jk} P_i + i \eta_{ik} P_j , \\
[P_i, C] &= -i K_i , \quad [P_i, K_i] = i \eta_{ij} N , \quad [K_i, H] = i P_i , \quad [H, C] = -i D , \\
[D, K_i] &= -i K_i , \quad [D, H] = 2i H , \quad [D, P_i] = i P_i , \quad [D, C] = -2i C .
\end{align*} \]

\[ \text{Denoting } (P_-, P_+, J_{i+}, D, K_+) \text{ as } (iH, iN, K_i, -\frac{i}{2} \bar{D}, \frac{1}{2} C), \text{ we arrive at the expression used in the literature} \]
algebra, it is necessary to introduce some projectors which project out the terms including $K_-$ and $K_i$ in \{$\tilde{S}^T, \tilde{S}$\}. An example is the light-cone projectors

$$\ell_\pm = \frac{1}{2} (1 \pm \Gamma^{03}) = -\frac{1}{2} \Gamma^\pm \Gamma^\mp, \tag{2.12}$$

which commute with $h_+$ and $p_\pm$. Then $\tilde{S}$ can be decomposed as

$$\tilde{S} = S + S', \quad S = \tilde{S} \ell_-, \quad S' = \tilde{S} \ell_+, \tag{2.13}$$

and then \{$$S^T, S$$\} contains only $K_+$, namely \{$$S^T, S$$\} $\sim K_+$. One may expect that $S$ is a supercharge of a super Schrödinger algebra. We will see below that this is the case.

The anti-commutation relations between $\tilde{Q}$ and $S$ are

$$\{\tilde{Q}^T, \tilde{Q}\} = 4i CT^+ p_- h_+ P_+ + 4i CT^- p_+ h_- P_- + 4i CT^i p_- h_+ P_i, \quad \{S^T, S\} = 4i CT^+ \ell_- p_+ h_+ K_+,$$

$$\{\tilde{Q}^T, S\} = i CT^{ij} \xi_i \sigma_2 \ell_- p_+ h_+ J_{ij} + 2i CT^{ij} \xi_i \sigma_2 \ell_- p_+ h_+ J_{ij} - 2i CT^4 \Gamma^l \ell_- p_+ h_+ D + 2i CT^{a'} \ell_- p_+ h_+ P_{a'} - i CT^{a'b'} \xi_2 \ell_- p_+ h_+ J_{a'b'}, \tag{2.14}$$

where $\Gamma^\pm \ell_\pm = 0$ has been used. The so(6) generators appear in the right-hand side of \{$$\tilde{Q}^T, S$$\} and hence those should be added to (2.10).

Next the commutation relations between (2.10) including so(6) generators, and (\tilde{Q}, S) are

$$[K_+, \tilde{Q}] = -\frac{1}{2} SG^- \Gamma_4; \quad [P_-, S] = \frac{1}{2} \tilde{Q} \ell_+ \Gamma^4 \Gamma_4; \quad [P_i, S] = -\frac{1}{2} \tilde{Q} \ell_- \Gamma_{i4},$$

$$[J_{ij}, \tilde{Q}] = \frac{1}{2} \tilde{Q} \Gamma_{ij}; \quad [J_{ij}, S] = \frac{1}{2} SG_{ij}; \quad [J_{ij}, \tilde{Q}] = -\frac{1}{2} \tilde{Q} \Gamma_{ij} \Gamma^-, \quad [D, \tilde{Q}] = -\frac{1}{4} \tilde{Q} \Gamma_+ \Gamma^-; \quad [D, S] = \frac{1}{4} SG^- \Gamma^+,$$

$$[P_{a'}, \tilde{Q}] = \frac{1}{2} \tilde{Q} \Gamma_{a'} \xi i \sigma_2; \quad [J_{a'b'}, \tilde{Q}] = \frac{1}{2} \tilde{Q} \Gamma_{a'b'}; \quad [P_{a'}, S] = \frac{1}{2} SG_{a'} \xi i \sigma_2; \quad [J_{a'b'}, S] = \frac{1}{2} SG_{a'b'} \tag{2.15}.$$

Now $\tilde{Q}$ and $S$ only are contained in the right-hand sides of the commutation relations, and hence the set of generators

$$\{J_{ij}, J_{i+}, D, P_\pm, P_i, K_+, P_{a'}, J_{a'b'}, \tilde{Q}, S\}$$

forms an extended super Schrödinger algebra. The bosonic subalgebra is a direct sum of the Schrödinger algebra and so(6). The word “extended” is attached due to the presence
of the extra so(6) in addition to the original Schrödinger algebra. The number of the supercharge is 24 since 1/4 supercharges have been projected out.

Finally we shall comment on super subalgebras of the above extended super Schrödinger algebra. $\tilde{Q}$ is also decomposed as

$$\tilde{Q} = Q + Q', \quad Q = \tilde{Q}\ell_-, \quad Q' = \tilde{Q}\ell_+. \quad (2.16)$$

Substituting it into the commutation relations, it is easy to find that the set of generators, so(6) generators and $Q$, forms a subalgebra of the extended super Schrödinger algebra. The bosonic generators form the Schrödinger algebra and so(6). The (anti-)commutation relations including $Q$ are

$$\{Q^T, Q\} = 4iC\Gamma^+\ell_-p_-h_+P_+, \quad [J_{ij}, Q] = \frac{1}{2}Q\Gamma_{ij}, \quad (2.17)$$

$$[P_{a'}, Q] = \frac{1}{2}Q\Gamma_{a'}J_i\sigma_2, \quad [J_{a',j}, Q] = \frac{1}{2}Q\Gamma_{a'j}. \quad (2.18)$$

It is possible to further reduce the algebra keeping only (2.10) and $Q$. The (anti-)commutation relations are (2.11) and (2.17). This is a supersymmetric extension of the Schrödinger algebra with eight supersymmetries, of which bosonic subalgebra is the Schrödinger algebra only.

2.3 Interpretation of so(6) in (2+1) dimensions

The so(6) in (2.2) corresponds to the isometry of $S^5$. The supercharge $Q$ is a pair of 16 component Majorana-Weyl spinors in $(9 + 1)$-dimensions. Under the reduction (2.4), $Q$ is decomposed into a pair of four four-component spinors, supercharges $\tilde{Q}$ and superconformal charges $\tilde{S}$, in $(3 + 1)$-dimensions. As seen in (2.6), so(6) $\cong$ su(4) rotates four of $\tilde{Q}$ and four of $\tilde{S}$, separately. Thus so(6) is related to the su(4) R-symmetry of the $\mathcal{N} = 4$ superconformal algebra in $(3 + 1)$-dimensions. By the reduction, (2.7), (2.13) and (2.16), four of $\tilde{Q}$ reduce to a pair of four two-component spinors, $Q$ and $Q'$, in $(2 + 1)$-dimensions. Similarly $\tilde{S}$ reduces to a pair of four two-component spinors, $S$ and $S'$. As can be seen from (2.14), so(6) rotates $Q$, $Q'$, $S$ and $S'$ separately. Thus even in the extended super Schrödinger algebra, so(6) acts as su(4) R-symmetry. The commutation relations in (2.17) mean that so(6) $\cong$ su(4) acts on four of two-component supercharges in $(2+1)$-dimensions.
3 Other cases - osp(8|4) and osp(8*|4)

We consider the super-AdS$_{q+2} \times S^{9-q}$ algebras with $q = 2$ and 5, that is, osp(8|4) and osp(8*|4). These are the super isometries of the near-horizon geometries of M2-brane and M5-brane, respectively. For osp(8|4) and osp(8*|4), super Schrödinger subalgebras are found in the same way as psu(2,2|4).

3.1 osp(8|4) and osp(8*|4)

Let us first introduce the superconformal algebras, osp(8|4) and osp(8*|4).

The (anti-)commutation relations of the super-AdS$_4 \times S^7$ algebra, i.e., osp(8|4), are

\[
[P_a, P_b] = 4J_{ab} , \quad [J_{ab}, P_c] = \eta_{bc}P_a - \eta_{ac}P_b , \quad [J_{ab}, J_{cd}] = \eta_{bc}J_{ad} + 3\text{-terms} ,
\]

\[
[P_{a'}, P_{b'}] = -J_{a'b'} , \quad [J_{a'b'}, P_{c'}] = \delta_{b'c'}P_{a'} - \delta_{a'c'}P_{b'} ,
\]

\[
[J_{a'b'}, J_{c'd'}] = \delta_{b'c'}J_{a'd'} + 3\text{-terms} ,
\]

\[
[P_a, Q] = -Q IT_a , \quad [P_{a'}, Q] = -\frac{1}{2} QT_{a'} , \quad [J_{AB}, Q] = \frac{1}{2} Q \Gamma_{AB} ,
\]

\[
\{Q^T, Q\} = -2C \Gamma^A P_A + 2C \Gamma^{a'b} J_{ab} - C \Gamma^{a'b'} J_{a'b'} , \quad I \equiv \Gamma^{0123} ,
\]

where $a = 0, 1, 2, 3$, $a' = 4, \ldots, 9, \hat{a}$ and $A = (a, a')$. The commutation relations in (3.1) and (3.2) are those of so(3,2) and so(8), respectively. The gamma matrices $\Gamma^A$'s are (10 + 1)-dimensional ones and the charge conjugation matrix $C$ satisfies the relation $\Gamma^T_A = -C \Gamma_A C^{-1}$. The fermionic generator $Q$ is a 32 component Majorana spinor in (10 + 1)-dimensions.

On the other hand, the (anti-)commutation relations of the super-AdS$_7 \times S^4$ algebra, i.e., osp(8*|4), are given by

\[
[P_a, P_b] = J_{ab} , \quad [J_{ab}, P_c] = \eta_{bc}P_a - \eta_{ac}P_b , \quad [J_{ab}, J_{cd}] = \eta_{bc}J_{ad} + 3\text{-terms} ,
\]

\[
[P_{a'}, P_{b'}] = -4J_{a'b'} , \quad [J_{a'b'}, P_{c'}] = \delta_{b'c'}P_{a'} - \delta_{a'c'}P_{b'} ,
\]

\[
[J_{a'b'}, J_{c'd'}] = \delta_{b'c'}J_{a'd'} + 3\text{-terms} ,
\]

\[
[P_a, Q] = -\frac{1}{2} QT_a , \quad [P_{a'}, Q] = -Q IT_{a'} , \quad [J_{AB}, Q] = \frac{1}{2} Q \Gamma_{AB} ,
\]

\[
\{Q^T, Q\} = -2C \Gamma^A P_A - C \Gamma^{a'b} J_{ab} + 2C \Gamma^{a'b'} J_{a'b'} , \quad I \equiv \Gamma^{789} ,
\]

where $a = 0, \ldots, 6$, $a' = 7, 8, 9, \hat{a}$ and $A = (a, a')$. The commutation relations in (3.4) and (3.5) are those of so(6,2) and so(5), respectively.
3.2 Bosonic part

Let us scale generators as follows

$$P_a \rightarrow 2P_a \text{ for } q = 2 \quad \text{and} \quad P_{a'} \rightarrow 2P_{a'} \text{ for } q = 5 \ . \quad (3.7)$$

Then the bosonic subalgebra takes the standard form

$$[P_a, P_b] = J_{ab} \ , \quad [J_{ab}, P_c] = \eta_{bc}P_a - \eta_{ac}P_b \ , \quad [J_{ab}, J_{cd}] = \eta_{bc}J_{ad} + 3\text{-terms} \ , \quad (3.8)$$

$$[P_{a'}, P_{b'}] = -J_{a'b'} \ , \quad [J_{a'b'}, P_{c'}] = \delta_{b'c'}P_{a'} - \delta_{a'c'}P_{b'} \ ,$$

$$[J_{a'b'}, J_{c'd'}] = \delta_{b'c'}J_{a'd'} + 3\text{-terms} \ , \quad (3.9)$$

where $a = 0, \cdots , q + 1$ and $a' = q + 2, \cdots, 9, \sharp$. The commutation relation in (3.8) is $\text{so}(q + 1,2)$, while that of (3.9) is $\text{so}(10 - q)$. By decomposing the generators as

$$\tilde{P}_\mu = \frac{1}{2}(P_\mu - J_{\mu q+1}) \ , \quad \tilde{K}_\mu = \frac{1}{2}(P_\mu + J_{\mu q+1}) \ , \quad \tilde{D} = P_{q+1} \ , \quad \tilde{J}_{\mu \nu} = J_{\mu \nu} \ , \quad (3.10)$$

$$a = (\mu, q + 1) \text{ with } \mu = 0, \cdots, q \ ,$$

the commutation relations in (3.8) become

$$[\tilde{P}_\mu, \tilde{D}] = -\tilde{P}_\mu \ , \quad [\tilde{K}_\mu, \tilde{D}] = \tilde{K}_\mu \ , \quad [\tilde{P}_\mu, \tilde{K}_\nu] = \frac{1}{2} \tilde{J}_{\mu \nu} + \frac{1}{2} \eta_{\mu \nu} \tilde{D} \ ,$$

$$[\tilde{J}_{\mu \nu}, \tilde{P}_\rho] = \eta_{\nu \rho} \tilde{P}_\mu - \eta_{\mu \rho} \tilde{P}_\nu \ , \quad [\tilde{J}_{\mu \nu}, \tilde{K}_\rho] = \eta_{\nu \rho} \tilde{K}_\mu - \eta_{\mu \rho} \tilde{K}_\nu \ ,$$

$$[\tilde{J}_{\mu \nu}, \tilde{J}_{\rho \sigma}] = \eta_{\nu \rho} \tilde{J}_{\mu \sigma} + 3\text{-terms} \ . \quad (3.11)$$

This is the conformal algebra in $d = q + 1$ dimensions. Further decomposition \footnote{Note that $J_{ij}$ vanishes for $q = 2$ since $i = 1$.}

$$P_\pm = \frac{1}{\sqrt{2}}(\tilde{P}_0 \pm \tilde{P}_q) \ , \quad K_\pm = \frac{1}{\sqrt{2}}(\tilde{K}_0 \pm \tilde{K}_q) \ , \quad J_{i \pm} = \frac{1}{\sqrt{2}}(\tilde{J}_{i0} \pm \tilde{J}_{iq}) \ ,$$

$$D = \frac{1}{2}(\tilde{D} - J_{0q}) \ , \quad D' = \frac{1}{2}(\tilde{D} + J_{0q}) \ , \quad P_i = \tilde{P}_i \ , \quad K_i = \tilde{K}_i \ , \quad J_{ij} = \tilde{J}_{ij} \ , \quad (3.12)$$

$$\mu = (0, i, q) \text{ with } i = 1, \cdots, q - 1 \ ,$$

leads to

$$[J_{ij}, J_{kl}] = \eta_{jk} J_{il} + 3\text{-terms} \ , \quad [J_{ij}, J_{k \pm}] = \eta_{jk} J_{i \pm} - \eta_{ik} J_{j \pm} \ ,$$

$$[J_{ij}, P_k] = \eta_{jk} P_i - \eta_{ik} P_j \ , \quad [J_{ij}, K_k] = \eta_{jk} K_i - \eta_{ik} K_j \ , \quad [J_{i \pm}, J_{j \mp}] = J_{ij} \pm \eta_{ij}(D' - D) \ ,$$
\[ [P_i, K_j] = \frac{1}{2} J_{ij} + \frac{1}{2} \eta_{ij} (D' + D), \quad [P_i, K_\pm] = \frac{1}{2} J_{i \pm}, \quad [P_\pm, K_i] = -\frac{1}{2} J_{i \pm}, \]
\[ [D, J_{\pm}] = \mp \frac{1}{2} J_{i \pm}, \quad [D', J_{\pm}] = \pm \frac{1}{2} J_{i \pm}, \]
\[ [P_i, J_{\pm}] = \eta_{ij} P_\pm, \quad [K_i, J_{\pm}] = \eta_{ij} K_\pm, \quad [J_{i \pm}, P_+] = -P_i, \quad [J_{i \pm}, K_\pm] = -K_i, \]
\[ [P_+, K_-] = -D', \quad [P_-, K_+] = -D, \]
\[ [D, P_-] = P_-, \quad [D, P_i] = \frac{1}{2} P_i, \quad [D, K_+] = -K_+, \quad [D, K_i] = -\frac{1}{2} K_i, \]
\[ [D', P_+] = P_+, \quad [D', P_i] = \frac{1}{2} P_i, \quad [D', K_-] = -K_-, \quad [D', K_i] = -\frac{1}{2} K_i. \quad (3.13) \]

Here the following set of the generators

\[ \{J_{ij}, J_{++}, D, P_\pm, P_i, K_+\} \]

forms a subalgebra of so(q+1, 2), whose commutation relations are

\[ [J_{ij}, J_{kl}] = \eta_{jk} J_{il} + 3\text{-terms}, \quad [J_{ij}, J_{k+}] = \eta_{jk} J_{i+} - \eta_{ik} J_{j+}, \quad [J_{ij}, P_k] = \eta_{jk} P_i - \eta_{ik} P_j, \]
\[ [P_i, K_+] = \frac{1}{2} J_{i+}, \quad [P_i, J_{++}] = \eta_{ij} P_+, \quad [J_{++}, P_-] = -P_i, \quad [P_-, K_+] = -D, \]
\[ [D, J_{++}] = -\frac{1}{2} J_{i+}, \quad [D, P_-] = P_-, \quad [D, P_i] = \frac{1}{2} P_i, \quad [D, K_+] = -K_+ . \quad (3.14) \]

This is nothing but the Schrödinger algebra with \( q - 1 \) spatial directions (see footnote 3).

### 3.3 Fermionic part

The remaining task is to consider the fermionic part. Here the cases with \( q = 2 \) and \( q = 5 \) are discussed at once. For simplicity, let us rescale \( Q \) as \( Q \to \sqrt{2}Q \) for \( q = 2 \) and take the rescaling (3.7) for the bosonic generators.

Then the (anti-)commutation relations including \( Q \), (3.3) and (3.6), are rewritten as

\[ \{Q^T, Q\} = -2C \Gamma^a P_a + \epsilon C \Gamma^{ab} J_{ab} - \beta C \Gamma^{a' b'} P_{a'} - \epsilon \beta C \Gamma^{a' b'} J_{a' b'}, \]
\[ [P_a, Q] = -\frac{1}{2} Q \Gamma_a, \quad [J_{ab}, Q] = \frac{1}{2} Q \Gamma_{ab}, \]
\[ [P_{a'}, Q] = -\frac{1}{2} Q \Gamma_{a'}, \quad [J_{a' b'}, Q] = \frac{1}{2} Q \Gamma_{a' b'}, \quad (3.15) \]

where \( \epsilon, \beta \) and \( I \) are defined as, respectively,

\[ \epsilon = \begin{cases} 1 & \text{for } q = 2, \\ -1 & \text{for } q = 5, \end{cases} \quad \beta = \begin{cases} 1 & \text{for } q = 2, \\ 4 & \text{for } q = 5, \end{cases} \quad I = \begin{cases} \Gamma^{0123} & \text{for } q = 2, \\ \Gamma^{789} & \text{for } q = 5. \end{cases} \quad (3.16) \]
Under the decomposition (3.10) and
\[
\mathcal{Q} = \tilde{Q} + \tilde{\mathcal{S}}, \quad \tilde{Q} = \mathcal{Q}p_-, \quad \tilde{\mathcal{S}} = \mathcal{Q}p_+, \quad p_\pm = \begin{cases} \frac{1}{2}(1 \pm \Gamma^{012}) & \text{for } q = 2, \\ \frac{1}{2}(1 \pm \Gamma^{6789}) & \text{for } q = 5, \end{cases}
\]
the (anti-)commutation relations in (3.15) are rewritten as
\[
\begin{align*}
\{\tilde{Q}^T, \tilde{Q}\} &= -4CT^\mu p_- \tilde{P}_\mu, \quad \{\tilde{\mathcal{S}}^T, \tilde{\mathcal{S}}\} = -4CT^\mu p_+ \tilde{K}_\mu, \\
\{\tilde{Q}^T, \tilde{\mathcal{S}}\} &= \epsilon CTT^{\mu\nu} p_+ J_{\mu\nu} - 2CT^{q+1} p_+ \tilde{D} - \beta C\Gamma^{a'a'} p_+ P_{a'} - \epsilon\frac{\beta}{2} CTT^{a'\nu} p_+ J_{a'\nu}, \\
[\tilde{P}_\mu, \tilde{\mathcal{S}}] &= \frac{1}{2} \tilde{Q} \Gamma_{\mu q+1}, \quad [\tilde{K}_\mu, \tilde{Q}] = -\frac{1}{2} \tilde{\mathcal{S}} \Gamma_{\mu q+1}, \quad [\tilde{D}, \tilde{Q}] = \frac{1}{2} \tilde{Q}, \quad [\tilde{D}, \tilde{\mathcal{S}}] = -\frac{1}{2} \tilde{\mathcal{S}}, \\
[\tilde{J}_{\mu\nu}, \tilde{Q}] &= \frac{1}{2} \tilde{Q} \Gamma_{\mu\nu}, \quad [\tilde{J}_{\mu\nu}, \tilde{\mathcal{S}}] = \frac{1}{2} \tilde{\mathcal{S}} \Gamma_{\mu\nu}, \quad [J_{a'\nu}, \tilde{Q}] = \frac{1}{2} \tilde{Q} \Gamma_{a'\nu}, \quad [J_{a'\nu}, \tilde{\mathcal{S}}] = \frac{1}{2} \tilde{\mathcal{S}} \Gamma_{a'\nu}, \\
[P_{a'}, \tilde{Q}] &= -\frac{1}{2} \tilde{Q} TT_{a'}, \quad [P_{a'}, \tilde{\mathcal{S}}] = -\frac{1}{2} \tilde{\mathcal{S}} TT_{a'},
\end{align*}
\]
where the following relations have been used
\[
p^T_{\pm} C = \begin{cases} C_{p_\pm} & \text{for } q = 2, \\ C_{p_+} & \text{for } q = 5. \end{cases}
\]
The (anti-)commutation relations in (3.11), (3.9) and (3.18) form the \((q+1)\)-dimensional superconformal algebra. \(\tilde{Q}\) are 16 supercharges and \(\tilde{\mathcal{S}}\) are 16 superconformal charges.

Further decomposition of the generators as (3.12) reduces the (anti-)commutation relations in (3.18) to
\[
\begin{align*}
\{\tilde{Q}^T, \tilde{Q}\} &= -4CT^+ p_- P_+ - 4CT^- p_- P_- - 4CT^i p_- P_i, \\
\{\tilde{\mathcal{S}}^T, \tilde{\mathcal{S}}\} &= -4CT^+ p_+ K_+ - 4CT^- p_+ K_- - 4CT^i p_+ K_i, \\
\{\tilde{Q}^T, \tilde{\mathcal{S}}\} &= \epsilon CTT^{ij} p_+ J_{ij} + 2\epsilon CTT^{ij} p_+ J_{i+} + 2\epsilon CTT^{ij} p_+ J_{i-} \\
&\quad + 2CT^{q+1} \Gamma^{+}\Gamma_- p_+ D' + 2CT^{q+1} \Gamma^- \Gamma_+ p_+ D - \beta C\Gamma^{a'a'} p_+ P_{a'} - \epsilon\frac{\beta}{2} CTT^{a'\nu} p_+ J_{a'\nu}, \\
[K_{\pm}, \tilde{Q}] &= \frac{1}{2} \tilde{\mathcal{S}} \Gamma^\pm \Gamma_{q+1}, \quad [K_{i}, \tilde{Q}] = -\frac{1}{2} \tilde{\mathcal{S}} \Gamma_{iq+1}, \\
[P_{\pm}, \tilde{\mathcal{S}}] &= -\frac{1}{2} \tilde{Q} \Gamma^\pm \Gamma_{q+1}, \quad [P_{i}, \tilde{\mathcal{S}}] = \frac{1}{2} \tilde{Q} \Gamma_{iq+1}, \\
[J_{ij}, \tilde{Q}] &= \frac{1}{2} \tilde{Q} \Gamma_{ij}, \quad [J_{ij}, \tilde{\mathcal{S}}] = \frac{1}{2} \tilde{\mathcal{S}} \Gamma_{ij}, \quad [J_{i\pm}, \tilde{Q}] = -\frac{1}{2} \tilde{Q} \Gamma_{i\pm}, \quad [J_{i\pm}, \tilde{\mathcal{S}}] = -\frac{1}{2} \tilde{\mathcal{S}} \Gamma_{i\pm}, \\
[D, \tilde{Q}] &= -\frac{1}{4} \tilde{Q} \Gamma^+ \Gamma^- , \quad [D, \tilde{\mathcal{S}}] = \frac{1}{4} \tilde{\mathcal{S}} \Gamma^- \Gamma^+, \\
[D', \tilde{Q}] &= -\frac{1}{4} \tilde{Q} \Gamma^- \Gamma^+ , \quad [D', \tilde{\mathcal{S}}] = \frac{1}{4} \tilde{\mathcal{S}} \Gamma^+ \Gamma^-, \\
[J_{a'\nu}, \tilde{Q}] &= \frac{1}{2} \tilde{Q} \Gamma_{a'\nu}, \quad [J_{a'\nu}, \tilde{\mathcal{S}}] = \frac{1}{2} \tilde{\mathcal{S}} \Gamma_{a'\nu},
\end{align*}
\]
\[ [P_{a'}, \tilde{Q}] = -\frac{1}{2} \tilde{Q} \Gamma_{a'}, \quad [P_{a'}, \tilde{S}] = -\frac{1}{2} \tilde{S} \Gamma_{a'}, \quad (3.20) \]

where \( \Gamma^\pm = \frac{1}{\sqrt{2}}(\Gamma^0 \pm \Gamma^q) \) have been introduced.

As seen in the previous subsection, \( \{P_\pm, P_i, K_+, J_{ij}, J_{i+}, D\} \) forms the Schrödinger algebra (3.14). In the following, we shall show that

\[ \{P_\pm, P_i, K_+, J_{ij}, J_{i+}, D, \, P_{a'}, J_{a'\nu}, \, \tilde{Q}, \, S\} \quad (3.21) \]

is a subalgebra of \( \text{osp}(8|4) \) for \( q = 2 \) or \( \text{osp}(8^*|4) \) for \( q = 5 \). Here \( S \) can be read off from the following decomposition of \( \tilde{S} \)

\[ \tilde{S} = S + S', \quad S = \tilde{S} \ell_-, \quad S' = \tilde{S} \ell_+, \quad \ell_\pm = \frac{1}{2}(1 \pm \Gamma^0_q) = -\frac{1}{2} \Gamma^\pm \Gamma^\mp. \quad (3.22) \]

Note that \( \ell_\pm \) commute with \( p_\pm \). The superalgebra (3.21) contains the Schrödinger algebra and \( \text{so}(10 - q) \) as its bosonic subalgebra. Thus this superalgebra should also be referred to as an extended super Schrödinger algebra.

First derive the anti-commutation relations containing \( \tilde{Q} \) and \( S \) only,

\[ \{\tilde{Q}^T, \tilde{Q}\} = -4C \Gamma^p_\pm P_\pm - 4C \Gamma^- P_- - 4C \Gamma^i P_i, \quad (3.23) \]

\[ \{\tilde{S}^T, S\} = -4C \Gamma^\pm \ell_\pm P_+, \quad \{\tilde{Q}^T, S\} = \epsilon C \Gamma^{ij} \ell_\pm p_\pm J_{ij} + 2\epsilon C \Gamma^{ij} \ell_\pm p_\pm J_{i+} + 2C \Gamma^q \Gamma^\mp \ell_\pm p_\pm D \]

\[ -\beta C \Gamma^{a'\nu} \ell_\pm p_\pm J_{a'\nu} + \epsilon \frac{1}{2} C \Gamma^{a''} \ell_\pm p_\pm J_{a''}. \quad (3.24) \]

The generators appearing in the right-hand sides are contained in (3.21). Next we derive commutation relations between bosonic generators and fermionic generators in (3.21).

\[ [K_+, \tilde{Q}] = \frac{1}{2} \tilde{Q} \Gamma^- \Gamma_{q+1}, \quad [P_-, S] = -\epsilon \frac{1}{2} \tilde{Q} \ell_+ \Gamma^+ \Gamma_{q+1}, \quad [P_i, S] = \epsilon \frac{1}{2} \tilde{Q} \ell_- \Gamma_{iq+1}, \]

\[ [J_{ij}, \tilde{Q}] = \frac{1}{2} \tilde{Q} \Gamma_{ij} \Gamma^- \Gamma_{q+1}, \quad [J_{ij}, S] = \frac{1}{2} \tilde{Q} \Gamma_i \Gamma^+ \Gamma_{q+1}, \quad [J_{i+}, \tilde{Q}] = \frac{1}{2} \tilde{Q} \Gamma_i \Gamma^+ \Gamma_{q+1}, \]

\[ [D, \tilde{Q}] = -\frac{1}{4} \tilde{Q} \Gamma^+ \Gamma^- \Gamma_{q+1}, \quad [D, S] = \frac{1}{4} \tilde{Q} \Gamma_i \Gamma^+ \Gamma_{q+1}, \]

\[ [J_{a'\nu}, \tilde{Q}] = \frac{1}{2} \tilde{Q} \Gamma_{a'\nu} \Gamma^- \Gamma_{q+1}, \quad [J_{a'\nu}, S] = \frac{1}{2} \tilde{Q} \Gamma_{a'\nu} \Gamma^+ \Gamma_{q+1}, \]

\[ [P_{a'}, \tilde{Q}] = -\frac{1}{2} \tilde{Q} \Gamma^a \Gamma_{a'}, \quad [P_{a'}, S] = -\frac{1}{2} \tilde{Q} \Gamma^a \Gamma_{a'}. \quad (3.24) \]

Here \( \tilde{Q} \) and \( S \) only appear in the right-hand sides of the commutators in (3.24). Thus we have found an extended super Schrödinger subalgebra of \( \text{osp}(8|4) \) and \( \text{osp}(8^*|4) \). The
number of the remaining supercharges is 24 since 1/4 supercharges have been projected out.

Note that the extended super Schrödinger subalgebra further contains a smaller super subalgebra formed by the set of the generators,

\[ \{P_\pm, P_i, K_+, J_{ij}, J_{i+}, D, P_{a'}, J_{a'b'}, Q\} , \tag{3.25} \]

where the generator \( Q \) is obtained via a projection,

\[ \tilde{Q} = Q + Q' , \quad Q = \tilde{Q}_- \quad Q' = \tilde{Q}_+ . \]

The commutation relations including \( Q \) are

\[ \{Q^T , Q\} = -4CT^+\ell_- p_- P_+ , \quad [J_{ij} , Q] = \frac{1}{2}Q\Gamma_{ij} , \]
\[ [J_{a'b'} , Q] = \frac{1}{2}Q\Gamma_{a'b'} , \quad [P_{a'} , Q] = -\frac{1}{2}Q\Gamma_{a'} . \tag{3.26} \]

The superalgebra \( (3.25) \) can further be reduced to a super subalgebra

\[ \{P_\pm, P_i, K_+, J_{ij}, J_{i+}, D, Q\} . \tag{3.27} \]

This is a super Schrödinger algebra with eight supercharges. Its bosonic subalgebra contains the original Schrödinger algebra only.

The interpretation of \( \text{so}(10 - q) \) is similar to the case of \( \text{so}(6) \) explained in subsection 2.3. The \( \text{so}(8) \) for \( q = 2 \) acts on eight two-component spinors \( \tilde{Q} \) and eight two-component spinors \( \tilde{S} \) in \( (2 + 1) \)-dimensions, separately. In \( (1 + 1) \)-dimensions, as \( p_\pm \) is the chirality projector, eight two-component spinors reduce to a pair of eight one-component spinors. The \( \text{so}(8) \) acts on four of eight one-component spinors separately. On the other hand, the \( \text{so}(5) \cong \text{sp}(4) \) for \( q = 5 \) acts on a pair of four four-component Weyl spinors in \( (5 + 1) \)-dimensions where we note \( p_\pm \) is the chirality projector in \( (5 + 1) \)-dimensions. The \( \text{sp}(4) \) rotation imposes a symplectic Majorana condition on the spinors, and we are left with a pair of four four-component symplectic Majorana-Weyl spinors. The reduction to \( (4 + 1) \)-dimensions reduces them to four of four four-component spinors subject to the \( \text{sp}(4) \) rotation. Thus we are left with four of two four-component symplectic Majorana spinors in \( (4 + 1) \)-dimensions. The \( \text{so}(5) \cong \text{sp}(4) \) acts on the four sets of spinors, separately. Thus \( \text{so}(10 - q) \) acts as R-symmetry even in the extended super Schrödinger algebra.
4 Conclusion and Discussion

We have found (extended) super Schrödinger algebras contained in the $\text{psu}(2,2|4)$. An extended super Schrödinger subalgebra contains, as well as generators of the original Schrödinger algebra, 24 supercharges (16 rigid supersymmetries and half of 16 superconformal) and generators of $\text{so}(6)$. It also contains a smaller subalgebra. It is composed of generators of the original Schrödinger algebra, eight supercharges (half of 16 rigid supersymmetries) and the generators of $\text{so}(6)$. It is still a subalgebra even if there are no $\text{so}(6)$ generators. We have also discussed super Schrödinger subalgebras of the superconformal algebras, $\text{osp}(8|4)$ and $\text{osp}(8^*|4)$. The results are similar to the case of $\text{psu}(2,2|4)$.

One may find another super Schrödinger subalgebra other than the ones found here. It is a nice subject to completely classify (extended) super Schrödinger subalgebras of the superconformal algebras. It would also be interesting to find super Schrödinger subalgebras of the other superconformal algebras besides $\text{psu}(2,2|4)$, $\text{osp}(8|4)$ and $\text{osp}(8^*|4)$.

The next issue is to consider the coset construction by using the supergroup of the super Schrödinger algebra obtained here. In the usual coset construction the isometry group is divided by the local Lorentz symmetry, as is well known for the $\text{AdS}_5 \times \text{S}^5$. The local Lorentz symmetry should be replaced by something different in the case of the Schrödinger group. By finding an appropriate coset, it might be possible to reproduce the asymptotically plane-wave background found in [29, 30]. Our super Schrödinger algebra directly appeared from the $\text{psu}(2,2|4)$ and hence it may lead to a supersymmetric background constructed from the $\text{AdS}_5 \times \text{S}^5$ somehow.

It would also be important to construct the CFT action, which has the super Schrödinger symmetry obtained here as the maximal symmetry. It would be a clue to shed light on the non-relativistic holographic relation.

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