Theory of adiabatic fluctuations : third-order noise

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Abstract

We consider the response of a dynamical system driven by external adiabatic fluctuations. Based on the ‘adiabatic following approximation’ we have made a systematic separation of time-scales to carry out an expansion in $\alpha|\mu|^{-1}$, where $\alpha$ is the strength of fluctuations and $|\mu|$ is the damping rate. We show that probability distribution functions obey the differential equations of motion which contain third order terms (beyond the usual Fokker-Planck terms) leading to non-Gaussian noise. The problem of adiabatic fluctuations in velocity space which is the counterpart of Brownian motion for fast fluctuations, has been solved exactly. The characteristic function and the associated probability distribution function are shown to be of stable form. The linear dissipation leads to a steady state which is stable and the variances and higher moments are shown to be finite.

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I. Introduction

In this paper we have discussed the stochastic dynamics of a system driven by external, adiabatic fluctuations. The opposite counterpart of these processes correspond to stochastic processes with fast fluctuations which are more frequently encountered in physical and chemical sciences. The classic and celebrated problem of the latter kind is the century-old paradigm of Brownian motion first correctly formulated by Einstein \[1,2\]. In dealing with fast stochastic processes one essentially examines the average motion of the system subjected to fast fluctuations (which may be of external or of internal type) with the following separation of time-scales in mind. If \(\tau_c\) is the correlation time of fluctuations which is the shortest timescale in the dynamics, compared to coarse-grained timescale \(\Delta t\) over which one follows the average evolution, then

\[
\tau_c \ll \Delta t \ll \frac{1}{|\mu|},
\]

where \(|\mu|^{-1}\) refers to the inverse of the damping rate (or inverse of the largest eigenvalue of the “unperturbed” system). Herein we analyze the average dynamics of a general multivariate nonlinear system subjected to external, adiabatically slow fluctuations. We have derived the equation of motion for evolution of the probability distribution function in phase space on a coarse-grained timescale \(\Delta t\) assuming that \(\Delta t\) satisfies the following inequality

\[
\frac{1}{|\mu|} \ll \Delta t \ll \tau_c.
\]

(II)
The slow fluctuations characterized by very long correlation time have received a lot of attention of various workers over the years [2,3]. While the treatment of stochastic differential equations with fast fluctuation is based on the assumption that there is a very short correlation time such that one is allowed to make an appropriate expansion in $\alpha \tau_c$, where $\alpha$ is the strength of fluctuation, simplified assumption for dealing with long correlation time is rather relatively scarce. In general, the problem of long correlation time is handled at the expense of severe restriction on the type of stochastic behavior. For instance, several authors [2,3] have tried the linear and nonlinear models within the framework of simple Markov processes of the type, dichotomic process, two-state Markov process, random telegraphic process, etc. Our strategy here is to follow a perturbative approach, pertaining to the separation of the timescale (II) without keeping any above-mentioned restriction on the type of stochastic behavior. Based on the ‘adiabatic following approximation’ [4] we have recently [5] carried out an expansion in $\alpha |\mu|^{-1}$ to obtain a linear differential equation for the average solution. In this paper we extend this analysis to treat nonlinear stochastic differential equations for construction of appropriate master equations. The perturbative expansion is essentially a counterpart of expansion in $\alpha \tau_c$ as dealt in the case of fast fluctuations [2]. The difference between the two expansion schemes lies in identification of two distinct shortest time-scales in the dynamics of the two cases. In the case of fast fluctuations it is $\tau_c$, whereas the corresponding role is played by $|\mu|^{-1}$ in adiabatic fluctuations.

We have shown that the equation of motion in phase space for probability distribu-
tion function contains beyond the ordinary Fokker-Planck terms, third order derivative terms. As shown by Pawula [6] for one dimensional case, an equation with third order derivative terms is in contradiction to the positivity for transition probability for short time. However it is well known that finite derivative terms of order larger than two may be quite useful in different occasions [7-8], e.g., in the treatment of optical bistability described in terms of quasi-distribution function of Wigner in quantum optics [7] or explaining trimolecular reactions using Poisson representation of Fokker-Planck equation, also in one dimensional random walk with boundary within a scheme of expansion of master equation. Although at this stage of development a clear general probabilistic interpretation in terms of any real stochastic process is lacking [7] one can identify the noise terms with distinct characteristics for such processes. In a similar spirit we are led to the conclusion in the present context that adiabatic fluctuations give rise to third order non-Gaussian noise terms beyond the usual Fokker-Planck terms.

The central result of this paper is the solution of the problem of adiabatic fluctuations in velocity space, which is the counterpart of Brownian dynamics for rapid fluctuations. We have shown that the characteristic function obeys a simple third order differential equation. This can be solved exactly to obtain a probability distribution of stable form which for small arguments displays a power law behavior. It is also important to note that the linear dissipation leads to a stable steady state distribution. However, the fluctuation being external the energy supplied by this cannot be balanced by dissipation and as such there is no fluctuation-dissipation relation in this
case. Furthermore, the non-Gaussian statistical characteristics can be obtained from
the calculation of variances and higher moments which are shown to be finite. We thus
conclude that although in many cases third order noise makes the probabilistic consid-
eration truly difficult, the systems driven by adiabatic fluctuations display a distinct
non-Gaussian stochastic behavior is amenable to understanding in simple probabilistic
terms. Occasionally wherever possible we allow ourselves a fair comparison with Levy
processes [9,11] and point out the essential differences.

The outlay of the paper is as follows: In the next section we review the basic
aspects of adiabatic fluctuations in linear processes as dealt in our earlier paper [5].
The two basic assumptions, e.g., the adiabatic following approximation and the de-
coupling approximation as well as validity and convergence of perturbative expansion
were discussed in detail in the earlier paper [5]. To make this paper self-contained and
readable we review its salient features. In Sec.III we extend the treatment to nonlinear
equations. The equations in phase space have been derived in Sec.IV. The counter-
part of Brownian motion in velocity space for slow fluctuations have been treated in
Sec.V. Explicit solution for the probability distribution function and the approach to
equilibrium have been discussed. The paper is concluded in Sec.VI.
II. Linear processes with adiabatic fluctuations

To begin with we have considered the following linear equation,

\[ \dot{u} = \{A_0 + \alpha A_1(t)\}u, \]

(1)

where \( u \) is a vector with \( n \) components, \( A_0 \) is a constant matrix of dimension \( n \times n \) with negative real eigenvalues and \( A_1(t) \) is a random matrix, \( \alpha \) is a parameter which measures the strength of fluctuation.

It is convenient to assume that \( A_1(t) \) is a stationary process with \( \langle A_1(t) \rangle = 0 \). Eq.(1) sets the two time scales of the system, measured by the inverse of the largest eigenvalue of the matrix \( A_0 \) and the time scale of fluctuations of \( A_1(t) \) (more precisely correlation time of fluctuation). In the problem of Brownian motion where one deals with very fast fluctuations such that correlation time \( \tau_c \) is essentially the shortest time scale in the dynamics, one thus follows the evolution of the average \( \langle u \rangle \) on a coarse-grained timescale.

Before proceeding further we now make two remarks at this stage: First, since in the present context we have considered a stochastic process in which the fluctuations are weak and adiabatically slow, \( A_1(t) \) is an adiabatic stochastic process. Therefore the usual procedure of systematic expansion in \( \alpha \tau_c \) which relies on smallness of \( \tau_c \), is not valid. We thus take resort to a different approach based on an expansion in \( \alpha |\mu|^{-1} \), where \( |\mu| \) refers to the largest eigenvalue of \( A_0 \) matrix. Second, we do not make any
a priori assumption about the nature of the stochastic process, such as, Gaussian or dichotomic etc. The only assumption that have been made about the stochastic process is that the inverse of the damping rate is much shorter compared to the correlation time of fluctuations $A_1(t)$.

As a first step we introduce an interaction representation as given by,

$$u(t) = \exp(A_0 t)v(t), \tag{2}$$

and applying it to Eq.(1) we obtain,

$$\dot{v} = \alpha V(t)v, \tag{3}$$

where,

$$V(t) = \exp(-A_0 t)A_1(t)\exp(A_0 t). \tag{4}$$

On integration Eq.(3) yields,

$$v(t) = v(0) + \alpha \int_0^t V(t')v(t')dt'. \tag{5}$$

On iterating Eq.(5) once, we are led to an ensemble average equation of the form,

$$\langle v(t) \rangle = v(0) + \alpha^2 \int_0^t dt' \int_0^{t'} dt'' \langle V(t')V(t'')v(t'') \rangle. \tag{6}$$

The equation is still exact since no second order approximation has been used.

Now taking the time derivative of $v(t)$ we arrive at the following integrodifferential equation in which the initial value $v(0)$ no longer appears,

$$\frac{d}{dt}\langle v(t) \rangle = \alpha^2 \int_0^t \langle V(t)V(t')v(t') \rangle dt'. \tag{7}$$
Making use of a change of integration variable $t' = t - \tau$ and reverting back to the original representation we obtain

$$\frac{d}{dt} \langle u(t) \rangle = A_0 \langle u \rangle + \alpha^2 \int_0^t \langle A_1(t) \exp(A_0 \tau) A_1(t - \tau) u(t - \tau) \rangle d\tau. \quad (8)$$

The adiabatic following assumption (see the discussion at the end of this section), that $A_1(t)$ and the components of $u(t)$ vary slowly on the scale of inverse of $A_0$, can now be utilized. Following Crisp [4] we note that a Taylor series expansion of $A_1(t - \tau) u(t - \tau)$ in the average $\langle \ldots \rangle$ of the $\alpha^2$-term in Eq.(8) allows us to reduce the above equation to the following form,

$$\frac{d}{dt} \langle u(t) \rangle = A_0 \langle u \rangle + \alpha^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \langle A_1(t) I_n \frac{d^n}{dt^n}[A_1(t) u(t)] \rangle. \quad (9)$$

$I_n$ can also be written as

$$I_n = n! D E_{n+1} D^{-1}.$$  

Eq.(9) can then be rewritten in the form

$$\frac{d}{dt} \langle u(t) \rangle = A_0 \langle u \rangle + \alpha^2 \sum_{n=0}^{\infty} (-1)^n \langle A_1(t) D E_{n+1} D^{-1} \frac{d^n}{dt^n}[A_1(t) u(t)] \rangle,$$  

where we use

$$I_n = n! D E_{n+1} D^{-1}. \quad (11)$$
Here $D$ is a matrix which diagonalises $A_0$ and

$$E_{n+1} = \begin{pmatrix}
\frac{1}{\mu_{11}^{n+1}} & 0 \\
\cdot & \cdots \\
0 & \frac{1}{\mu_{jj}^{n+1}}
\end{pmatrix}$$

and $\mu_{jj}$ are the eigenvalues of $A_0$.

Although the Eq.(10) involves an infinite series it is expected to yield useful result in the adiabatic following approximation. Under this approximation the quantity $[A_1(t)u(t)]$ varies very little (such that $\frac{d}{dt} (A_1 u)$ in Eq.(10) is small) and also since $|\mu_{jj}|$ in $E_{n+1}$ is large the series in Eq.(10) (which is thus an expansion in $\alpha |\mu|^{-1}$) converges rapidly. Keeping only the two lowest order terms we arrive at,

$$\frac{d}{dt} \langle u(t) \rangle = A_0 \langle u \rangle + \alpha^2 \langle A_1(t)X_1A_1(t)u(t) \rangle - \alpha^2 \langle A_1(t)X_2A_1(t)\dot{u}(t) \rangle - \alpha^2 \langle A_1(t)X_2A_1(t)\ddot{u}(t) \rangle$$

(12)

where,

$$X_{n+1} = D E_{n+1} D^{-1}.$$
\[
\frac{d}{dt}\langle u(t) \rangle = \left\{ A_0 + \alpha^2 \left[ \langle A_1(t)X_1A_1(t) \rangle - \langle A_1(t)X_2A_1(t) \rangle \right] - \langle A_1(t)X_2A_1(t) \rangle \right\} \langle u(t) \rangle .
\]

Thus the average of \( u(t) \) obeys a nonstochastic differential equation in which the effect of weak adiabatic fluctuations is accounted for by renormalizing \( A_0 \) through the addition of constant terms of the order of \( \alpha^2 \).

The implementation of Bourret’s decoupling approximation [10] is a major step for almost any treatment of multiplicative noise up to date [2,3,12]. This is because of the fact that the average of a product of stochastic quantities does not factorize into the product of averages, although it has been argued that good approximations can be derived by assuming such factorization. In the case of fast fluctuations it has been justified if the driving stochastic noise has a short correlation time such that Kubo number \( \alpha^2 \tau_c \) is very small in the cummulant expansion scheme (an expansion in \( \alpha \tau_c \)). The factorization has been shown to be exact in the limit of zero correlation time and in some cases of specific noise processes [3,12] and the solution for the average can be found exactly.

In contrast to cummulant expansion (valid in the case of fast fluctuation which relies on an expansion in \( \alpha \tau_c \)) the present scheme of adiabatic following approximation results in a perturbation series, where the \( n \)-th term is of order \( \alpha \frac{\sigma_n}{\pi \tau_c} [A(t)u(t)]/\mu_{jj}^{n+1} \) and the convergence of the series is assured since the numerator varies little in the scale of
1/|μ_{j,j}^{n+1}|. Eq.(13) is a result of decoupling approximation employed in this expansion scheme. If one neglects the free motion due to $A_0$ term then Eq.(13), which gives the lowest order evolution, asserts that

$$\frac{d}{dt} \langle u \rangle \sim \frac{\alpha^2}{|\mu|} \langle u \rangle .$$

The contribution of $|\mu|^{-1}$ is derived from $X_1$ of the first term in Eq.(13), (i.e., due to $E_{n+1}$ matrix). Note that because of full integration over $\tau$ in moving from Eq.(8) to Eq.(9) correlation time $\tau_c$ does not appear in Eq.(13) and the time-scale set by the dynamics is $|\mu|^{-1}$ only. For a fast process on the other hand the counterpart of the last relation is [12]

$$\frac{d}{dt} \langle u \rangle \sim \alpha^2 \tau_c \langle u \rangle .$$

It is also easy to calculate the relative error made in the decoupling approximation. We first note that Eq.(13) is obtained from Eq.(8). To the second order it means omitting terms of the order $(\alpha \Delta t)^3$ and higher (where $\Delta t$ is the coarse-grained time-scale over which $\langle u \rangle$ evolves). As the lower bound of $\Delta t$ is determined by $|\mu|^{-1}$, it implies that we neglect terms of the order $(\alpha |\mu|^{-1})^3$ in the evolution equation. The relative error made in the decoupling approximation is thus of the order $(\alpha |\mu|^{-1})^3$ which is well within the order of lowest order evolution. We thus see that the adiabatic expansion is an expansion in $\alpha |\mu|^{-1}$ and the decoupling approximation in the slow fluctuation is valid where $\alpha^2 |\mu|^{-1}$ is very small. Thus $u(t)$ in the average (in the right
hand side of Eq.(12) is realized as an average \( \langle u(t) \rangle \) (which varies in the coarse-grained timescale \( \Delta t \)) in Eq.(13) pertaining to the separation of the time-scales in the inequality (II) in Sec.I.

Before closing this section a few pertinent points regarding the notion of “adiabatic following approximation” and its genesis may be noted. The notion has acquired special relevance in the quantum optical context where one is concerned with a two-level atom interacting with single mode electromagnetic field. The model is described by the standard Bloch equations, where the field strength varies slowly on the time-scale of inverse of the damping constant or the frequency detuning between the atom and the field. If the field is varying adiabatically enough, then the population inversion of the Bloch vector components would follow the field adiabatically in going from ground to upper state, i.e., the ground state population is adiabatically inverted. The term “adiabatic following” is thus used to describe collectively the associated experimental phenomena [19].
III. Probabilistic considerations : Extension to nonlinear equations

We now generalize Eq.(1) to a stochastic nonlinear differential equation written in the following form

\[ \dot{u}_\nu = F_\nu(\{u_\nu\}, t; \xi(t)), \quad \nu = 1, 2, \ldots, N. \quad (14) \]

The above equation determines a stochastic process with some initial given condition \( \{u_\nu(0)\} \). \( \xi(t) \) is the adiabatic stochastic process. It may be pointed out that the treatment given in the last section cannot be extended directly to this equation to obtain an equation for average \( \langle u \rangle \) since nonlinearity in Eq.(14) results in higher moments. However, it is possible to transform the nonlinear problem to a linear one if one considers the motion of a representative point-\( u \) in \( n \)-dimensional space \( (u_1 \ldots u_n) \) as governed by Eq.(14). The equation of continuity, which expresses the conservation of points determines the variation of density in time,

\[ \frac{\partial \rho(u, t)}{\partial t} = -\sum_\nu \frac{\partial}{\partial u_\nu} F_\nu(\{u_\nu\}, t; \xi(t)) \rho(u, t) \quad (15) \]

or more compactly

\[ \frac{\partial \rho}{\partial t} = -\nabla \cdot F \rho. \quad (16) \]

Eq.(15) is a linear stochastic differential equation and is an ideal candidate for the method discussed in the last section for the linear case. We emphasize here that the
basis of the present analysis is essentially the two approximations as introduced earlier and no further approximation is needed to extend the analysis to nonlinear domain.

\[ F = G(u; t; \xi(t)) = G_0(u) + \alpha G_1(u; t; \xi(t)) , \] (17)

where \( G_0(u) \) is the constant part and \( G_1(u; t; \xi(t)) \) is the random part with \( \langle G_1(t) \rangle = 0; \alpha \) is the parameter defined earlier which measures the strength of fluctuation. Eq.(16) therefore takes the following form,

\[ \dot{\rho}(u, t) = (A_0 + \alpha A_1)\rho(u, t) , \] (18)

where \( A_0 = -\nabla \cdot F_0 \) and \( A_1 = -\nabla \cdot F_1 \). The symbol \( \nabla \) is used for the operator that differentiates everything that comes after it with respect to \( u \).

With the above identification of \( A_0 \) and \( A_1 \) we are now in a position to apply the fundamental Eq.(9) derived in the earlier section, to Eq.(18). We have

\[ \frac{\partial}{\partial t} P(u, t) = \left[ -\nabla \cdot F_0 P + \alpha^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (\nabla \cdot F_1 I_n \frac{d^n}{dt^n} (\nabla \cdot F_1 P)) \right] , \] (19)

where \( \langle \rho(u, t) \rangle = P(u, t) \) and also

\[ I_n = \int_0^\infty d\tau e^{-\tau \nabla \cdot F_0 \tau^n} . \] (20)

Adiabatic following approximation may now be invoked again in the spirit of earlier treatment in Sec.II to obtain
\[
\frac{\partial}{\partial t} P(u, t) = - \nabla \cdot \left[ F_0 + \alpha^2 \langle F_1 I_0 \nabla \cdot F_1 \rangle - \alpha^2 \langle F_1 I_1 \nabla \cdot \dot{F}_1 \rangle \right] P(u, t) ,
\]

where we keep terms of the order of \( \alpha^2 \) for \( n = 0 \) and 1 of the series in Eq.(19).

Our next task is to simplify further the expressions for the averages in Eq.(21). To this end we first note that the operator \( \exp(-\tau \nabla \cdot F_0) \) provides the solution of the equation

\[
\frac{\partial f(u, t)}{\partial t} = -\nabla \cdot F_0 f(u, t) ,
\]

\((f \) signifies the unperturbed part of \( P) \) which can be found explicitly in terms of characteristic curves. The equation

\[
\dot{u} = F_0(u)
\]

for fixed \( t \) determines a mapping from \( u(\tau = 0) \) to \( u(\tau) \), i.e., \( u \to u^\tau \) with inverse \((u^\tau)^{-\tau} = u\). The solution of Eq.(22) is

\[
f(u, t) = f(u^{-\tau}, 0) \left| \frac{d(u^{-\tau})}{d(u)} \right| = e^{-t \nabla \cdot F_0} f(u, 0) ,
\]

\(|d(u^{-\tau})/d(u)|\) being a Jacobian determinant. The effect of \( \exp(-t \nabla \cdot F_0) \) or \( f(u) \) is as follows

\[
\exp(-t \nabla \cdot F_0)f(u, 0) = f(u^{-\tau}, 0) \left| \frac{d(u^{-\tau})}{d(u)} \right| .
\]

The relation (24) may be used to simplify the average in Eq.(21). We thus have

\[
\langle \nabla \cdot F_1 I_0 \nabla \cdot F_1 \rangle = \nabla \cdot \int_0^\infty \langle F_1 \nabla_{-\tau} \cdot F_1 (u^{-\tau}) \rangle \left| \frac{du^{-\tau}}{du} \right| d\tau ,
\]

\(15\)
\[
\langle \nabla \cdot \mathbf{F}_1 \nabla \cdot \mathbf{F}_1 \rangle = \nabla \cdot \int_0^\infty \tau \langle \mathbf{F}_1 \nabla_{-\tau} \cdot \mathbf{F}_1(u^{-\tau}) \rangle \left| \frac{du^{-\tau}}{du} \right| d\tau, \tag{26}
\]

\[
\langle \nabla \cdot \mathbf{F}_1 \nabla \cdot \mathbf{F}_1 \nabla \cdot \mathbf{F}_0 \rangle = \nabla \cdot \int_0^\infty \tau \langle \mathbf{F}_1 \nabla_{-\tau} \cdot \mathbf{F}_1(u^{-\tau}) \nabla_{-\tau} \cdot \mathbf{F}_0(u^{-\tau}) \rangle \left| \frac{du^{-\tau}}{du} \right| d\tau. \tag{27}
\]

The use of Eq.(25)-(27) reduces Eq.(21) to a more tractable form,

\[
\frac{\partial}{\partial t} P(u, t) = - \nabla \cdot \left\{ F_0 - \alpha^2 \int_0^\infty \langle \mathbf{F}_1 \nabla_{-\tau} \cdot \mathbf{F}_1(u^{-\tau}) \rangle \left| \frac{du^{-\tau}}{du} \right| d\tau \right. \\
+ \alpha^2 \int_0^\infty \tau \langle \mathbf{F}_1 \nabla_{-\tau} \cdot \mathbf{F}_1(u^{-\tau}) \rangle \left| \frac{du^{-\tau}}{du} \right| d\tau \\
- \alpha^2 \int_0^\infty \tau \langle \mathbf{F}_1 \nabla_{-\tau} \cdot \mathbf{F}_1(u^{-\tau}) \nabla_{-\tau} \cdot \mathbf{F}_0(u^{-\tau}) \rangle \left| \frac{du^{-\tau}}{du} \right| d\tau \left\} \right\} P(u, t), \tag{28}
\]

where \( \nabla_{-\tau} \) denotes the differential with respect to \( u_{-\tau} \). Eq.(28) is our basic result in this section. The equation is second order in \( \alpha \), i.e., of the order of \( \alpha^2 |\mu|^{-1} \), where \( |\mu| \) refers to the eigenvalue of \( A_0 \). In our earlier communication [5] we have shown the convergence of the series in \( \alpha |\mu|^{-1} \), pertaining to the separation of the time-scales implied in II in Sec.I. We also remark that it is possible to extend the treatment to higher order, in general. It is also to be noted that the equation involves three differentiation of \( P(u, t) \) with respect to the components of \( u \) and is a third order equation. The appearance of third-order noise beyond the usual Fokker-Planck terms is a characteristic of the process we consider here. We discuss this aspect in more detail.
in the following two sections.

IV. Adiabatic stochasticity in phase space

We now consider the motion of a particle in one dimension subjected to a force $K(x)$ depending on the position $x$, a frictional force $-\beta \dot{x}$ and a stochastic force $\alpha \xi(t)$. Here $\beta$ is a measure of damping of the system and $\alpha$ is the strength of adiabatically slow fluctuations $\xi(t)$. We thus write

$$m \ddot{x} + \beta \dot{x} = K(x) + \alpha \xi(t).$$

The corresponding problem of fast fluctuation $\alpha \xi(t)$ was studied by Kramers [13] as a model of simple chemical reactions and by Bixon and Zwanzig [14] as a model for fluctuating nonlinear systems.

For simplicity we set $m = 1$ and $\dot{x} = v$. Then the two components of $u$ in this example are $x$ and $v$. Taking Eq.(17) into account we have

$$\begin{align*}
F_{0x} &= v \\
F_{1x} &= 0 \\
F_{0v} &= -\beta v + K(x) \\
F_{1v} &= \alpha \xi(t)
\end{align*}$$

By considering a small variation of $v$ in time $\tau$, one obtain (from the unperturbed version of Eq.(29)) the Jacobian determinant for the “unperturbed” mapping $u \to u^\tau$

$$\left| \frac{du^{-\tau}}{du} \right| = \left| \frac{d(x^{-\tau}, v^{-\tau})}{d(x, v)} \right| = 1 + \beta \tau + \mathcal{O}(\tau^2)$$

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and

\[
\begin{align*}
\frac{\partial}{\partial v_{\tau}} &= (1 - \beta \tau) \frac{\partial}{\partial v} + \tau \frac{\partial}{\partial x} + \mathcal{O}(\tau^2) \\
\frac{\partial}{\partial x_{\tau}} &= \frac{\partial}{\partial x} + \tau \frac{\partial K(x)}{\partial x} \frac{\partial}{\partial v} + \mathcal{O}(\tau^2)
\end{align*}
\]  

(32)

The Eq.(28) now reduces to the following form

\[
\begin{align*}
\frac{\partial}{\partial t} P(x, v, t) &= -\nabla \cdot \left\{ F_0 - \alpha^2 \int_0^\infty \langle F_1 \nabla_{-\tau} \cdot F_1(x-\tau, v-\tau) \rangle \left| \frac{d(x-\tau, v-\tau)}{d(x, v)} \right| d\tau \\
&+ \alpha^2 \int_0^\infty \tau \langle F_1 \nabla_{-\tau} \cdot \dot{F}_1(x-\tau, v-\tau) \rangle \left| \frac{d(x-\tau, v-\tau)}{d(x, v)} \right| d\tau \\
&- \alpha^2 \int_0^\infty \tau \langle F_1 \nabla_{-\tau} \cdot F_1(x-\tau, v-\tau) \rangle \left| \frac{d(x-\tau, v-\tau)}{d(x, v)} \right| d\tau \right\} P(x, v, t). (33)
\end{align*}
\]

Making use of relations (30-32) one may reduce the terms on the right hand side of Eq.(33) to more simplified forms. Thus

\[
\begin{align*}
-\nabla \cdot F_0 P(x, v, t) &= -v \frac{\partial P}{\partial x} + \beta \frac{\partial}{\partial v}(vP) - K(x) \frac{\partial P}{\partial v}, \\
\frac{\alpha^2}{\nabla} \int_0^\infty \langle F_1 \nabla_{-\tau} \cdot F_1(x-\tau, v-\tau) \rangle \left| \frac{d(x-\tau, v-\tau)}{d(x, v)} \right| d\tau P(x, v, t) &= \alpha^2 \tilde{c}_0 \frac{\partial^2 P}{\partial v^2} + \alpha^2 \tilde{c}_1 \frac{\partial^2 P}{\partial v \partial x}, \\
\frac{\alpha^2}{\nabla} \int_0^\infty \tau \langle F_1 \nabla_{-\tau} \cdot \dot{F}_1(x-\tau, v-\tau) \rangle \left| \frac{d(x-\tau, v-\tau)}{d(x, v)} \right| d\tau P(x, v, t) &= -\alpha^2 \tilde{c}_2 \frac{\partial^2 P}{\partial v^2},
\end{align*}
\]
\[ \alpha^2 \nabla \cdot \int_0^\infty \tau (F_1 \nabla_{-\tau} \cdot F_1 (x^{-\tau}, v^{-\tau}) \nabla_{-\tau} \cdot F_0 (x^{-\tau}, v^{-\tau})) \left[ \frac{d(x^{-\tau}, v^{-\tau})}{d(x, v)} \right] d\tau P(x, v, t) \]

\[ = \alpha^2 \bar{c}_1 \left[ 2 \frac{\partial^2 P}{\partial v \partial x} + v \frac{\partial^3 P}{\partial v^2 \partial x} + K(x) \frac{\partial^3 P}{\partial v^3} - \beta \frac{\partial^3}{\partial v^3} (vP) \right], \quad (37) \]

where

\[ \begin{align*}
\bar{c}_0 &= \int_0^\infty \langle \xi(t) \xi(t-\tau) \rangle d\tau \\
\bar{c}_1 &= \int_0^\infty \tau \langle \xi(t) \xi(t-\tau) \rangle d\tau \\
\bar{c}_2 &= \int_0^\infty \tau \langle \xi(t) \dot{\xi}(t-\tau) \rangle d\tau 
\end{align*} \quad (38) \]

The final equation for the average motion corresponding to an adiabatic stochastic evolution in phase space is,

\[ \frac{\partial}{\partial t} P(x, v, t) = -v \frac{\partial P}{\partial x} + \beta \frac{\partial}{\partial v} (vP) - K(x) \frac{\partial P}{\partial v} + \alpha^2 (\bar{c}_0 - \bar{c}_2) \frac{\partial^2 P}{\partial v^2} + 3 \alpha^2 \bar{c}_1 \frac{\partial^2 P}{\partial v \partial x} \]

\[ + \alpha^2 \bar{c}_1 \left[ v \frac{\partial^3 P}{\partial v^2 \partial x} + K(x) \frac{\partial^3 P}{\partial v^3} - \beta \frac{\partial^3}{\partial v^3} (vP) \right]. \quad (39) \]

The remarkable departure from the standard form of Fokker-Planck equation is thus evident in Eq.(39) since it contains third derivative terms. The magnitude of their contribution is dependent on how much ‘unperturbed’ \( x \) and \( v \) vary during \( \tau \) which is of the order of \(|\mu|^{-1}\). We also point out that in the above derivation Bourret’s decoupling approximation [10] has been used as in the treatment of linear equation in Sec.II.
V. Adiabatic fluctuations in velocity space

We now consider the motion of a particle with velocity $v$ in presence of fluctuations $\alpha \xi(t)$ which is adiabatically slow. The equation of motion is given by

$$\dot{v} = -\beta v + \alpha \xi(t).$$

(40)

The corresponding problem of a Brownian particle with fast fluctuations is a century-old problem in physical science, in general. Following the procedure described in the earlier section we first identify the perturbed and the unperturbed part of $F$, i.e.,

$$F_0 = -\beta v, \quad F_1 = \alpha \xi(t)$$

(41)

and calculate the Jacobian $\frac{|dv-\tau|}{dv}$ for the mapping $v \rightarrow v^\tau$ for the ‘unperturbed’ motion

$$\left| \frac{dv-\tau}{dv} \right| = e^{\beta \tau}.$$  

(42)

Also note that

$$\nabla \tau = e^{-\beta \tau} \frac{\partial}{\partial v}.$$  

(43)

The evolution of the probability distribution function $P(v, t)$ is then given by (terms of the order $\alpha^2$)

$$\frac{\partial}{\partial t} P(v, t) = \beta \frac{\partial}{\partial v}(vP) + \alpha^2 c_1 \frac{\partial^2 P}{\partial v^2} - \alpha^2 c_3 \frac{\partial^3}{\partial v^3}(vP),$$

(44)
where
\[
\begin{align*}
  c_{12} &= c_1 - c_2 \\
  c_1 &= \int_0^\infty \langle \xi(t)\xi(t-\tau) \rangle d\tau \\
  c_2 &= \int_0^\infty \tau \langle \xi(t)\dot{\xi}(t-\tau) \rangle d\tau \\
  c_3 &= \int_0^\infty \tau \langle \xi(t)\dot{\xi}(t-\tau) \rangle d\tau
\end{align*}
\]  

(45)

While in the absence of the third term, the first two terms on the right hand side of Eq.(44) correspond to drift and diffusion terms in the Fokker-Planck description of an Ornstein-Uhlenbeck process, the third derivative term precludes the possibility of any straight-forward interpretation of the equation. Similar equations with third order noise, although not very common, however may be encountered [7] in the treatment of trimolecular reactions and also in quantum optics describing optical bistability in terms of associated Wigner distribution function for the reduced density operator in symmetrical ordering for the radiation field.

We now return to the Eq.(44) which after some modification becomes
\[
\frac{\partial}{\partial t} P(v, t) = \beta \frac{\partial}{\partial v}[v P(v, t)] + D_1 \frac{\partial^2 P(v, t)}{\partial v^2} - \beta D_2 v \frac{\partial^3 P(v, t)}{\partial v^3},
\]

(46)

where
\[
\begin{align*}
  D_1 &= \alpha^2(c_{12} - 3c_3\beta) \\
  D_2 &= \alpha^2 c_3
\end{align*}
\]  

(47)

We now transform the Eq.(46) to Fourier space by defining the conditional probability \(P(v|v_0, 0)\) and its Fourier transform as
\[
P(v, t|v_0, 0) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk e^{ikv} \hat{P}(k, t|v_0, 0),
\]

(48)
to obtain
\[ \frac{\partial}{\partial t} \tilde{P}(k, t|v_0, 0) = -\beta(k + D_2 k^3) \frac{\partial \tilde{P}}{\partial k} - (D_1 + 3\beta D_2) k^2 \tilde{P} \, . \] (49)

The linear partial differential equation (49) can be solved by the method of characteristics. For the initial condition (at time \( t = 0 \))
\[ P(v, 0|v_0, 0) = \delta(v - v_0) \, , \] (50)
the solution is
\[ \tilde{P}(k, t|v_0, 0) = \frac{1}{(1 + B k^2) A} \exp \left[ -i k v_0 \sqrt{\frac{f(t)}{1 + B k^2}} \right] \, , \] (51)
where
\[ f(t) = e^{-2\beta t} \]
\[ A = \frac{c_{12}}{2 \beta c_3} \]
\[ \text{and} \quad B = \alpha^2 c_3 \{ 1 - f(t) \} \] (52)

It is easy to check that Eq.(51) satisfies
\[ \tilde{P}^*(k, t|v_0, 0) = \tilde{P}(-k, t|v_0, 0) \]
and the characteristic function (51) is of stable form.

The conditional probability density \( P(v, t|v_0, 0) \) is obtained by inverse Fourier transformation of Eq.(51) and is given by,
\[ P(v, t|v_0, 0) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk \frac{1}{(1 + B k^2)^A} \exp \left[ i k v - i k v_0 \sqrt{\frac{f(t)}{1 + B k^2}} \right] \, . \] (53)
It is evident that $P(v, t|v_0, 0)$ is a Fourier transform of a stable characteristic function. Hence the solution (53) forms a stable distribution in the variable $v$. The Eq.(53) is one of the important results of this paper.

Although an explicit expression for $P(v, t|v_0, 0)$ is difficult to obtain, closed-form solutions for $P(v, t|v_0, 0)$ for stationary state can be easily obtained. In the long time limit the characteristic function (51) reduces to its asymptotic form

$$\tilde{P}(k, \infty) = \frac{1}{(1 + D_2 k^2)^A},$$

which results a steady state distribution of stable form. Explicitly for small $A$, i.e., large $\beta$ this is given by

$$P_{st}(v) = \frac{|v|^{A+1}}{2^A D_2^A \Gamma(A) v^2} e^{-v^\beta/2}.$$  

It is interesting to note that the dominant behavior of $P_{st}(v)$ for small $v$. This is given by a power law of the form

$$P_{st}(v) \sim \frac{|v|^{A+1}}{v^2} \left\{ \begin{array}{c} \sim |v|^{-1+A} \end{array} \right\}.$$  

Such power law behavior is also apparent for Levy processes [9,11] but for the large $v$ regime.

Although explicit solution for probability distribution $P(v, t|v_0, 0)$ is difficult to obtain for arbitrary time, however, a few statistical properties of the process can be obtained from the calculation of variances and higher moments. For convenience, we
define such moments by subtracting the mean motion of the variables, i.e., we calculate the moments of \( \Delta v = v - v_0 e^{-\beta t} \). Thus we write
\[
\langle |\Delta v|^m \rangle = \int_{-\infty}^{+\infty} (v - v_0 e^{-\beta t})^m P(v, t|v_0, 0), \tag{57}
\]
or more explicitly
\[
\langle |\Delta v|^m \rangle = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk \frac{1}{(1 + Bk^2)^{\frac{3}{2}}} \exp \left[ ikv_0 e^{-\beta t} \left\{ 1 - \left( 1 + Bk^2 \right)^{-\frac{1}{2}} \right\} \right] 
\int_{-\infty}^{+\infty} d\Delta v |\Delta v|^m e^{ik\Delta v}. \tag{58}
\]
After some algebra we get
\[
\langle |\Delta v|^m \rangle = (-i)^m \int_{-\infty}^{+\infty} dk \frac{1}{(1 + Bk^2)^{\frac{3}{2}}} \exp \left[ ikv_0 e^{-\beta t} \left\{ 1 - \left( 1 + Bk^2 \right)^{-\frac{1}{2}} \right\} \right] \frac{\partial^m \delta(k)}{\partial k^m}. \tag{59}
\]
In principle, using the property of Dirac \( \delta \)-function and appropriate integrations any moment can be calculated from the above relation. We quote the results explicitly for the first three moments.
\[
\begin{align*}
\text{For } m = 1 & \quad \langle |\Delta v| \rangle = 0 \\
\text{For } m = 2 & \quad \langle |\Delta v|^2 \rangle = \frac{\alpha^2 c_2}{\beta} (1 - e^{-2\beta t}) \\
\text{For } m = 3 & \quad \langle |\Delta v|^3 \rangle = 3\alpha^2 c_3 v_0 e^{-\beta t} (1 - e^{-2\beta t})
\end{align*}
\tag{60}
\]
It is thus evident that unlike Levy processes [9,11] the moments are finite.

We thus observe that because of the linear dissipation \( \beta \), a system driven by adiabatic noise reaches a steady state which is stable. However, since the noise is of external origin, the outward flow of energy due to linear dissipation cannot balance the inward
flow of energy supplied by the adiabatic fluctuations and hence a fluctuation-dissipation relation cannot be conceived in this case.

VI. Conclusions

In conclusion, we consider herein a dynamical system driven by external adiabatic fluctuations. Based on the ‘adiabatic following approximation’ we have made a systematic separation of time-scales to carry out an expansion in $\alpha |\mu|^{-1}$ to obtain a linear differential equation for the average solution, where $\alpha$ is the strength of fluctuation and $|\mu|$ is the largest eigenvalue of the unperturbed system. The main results of this study can be summarized as follows:

(i) The probability distribution functions obey the differential equations of motion which contain third-order terms beyond the usual Fokker-Planck terms. The adiabatic fluctuations thus may give rise to non-Gaussian noise.

(ii) We have examined in detail the corresponding equation in velocity space and the characteristic function is shown to obey a simple third-order differential equation which can be solved exactly in closed form. The characteristic function is found to be of stable form.

(iii) Although third-order noise, in general, leads to serious interpretative difficulties in terms of truly probabilistic considerations in several cases, we show that in the present problem of adiabatic stochasticity in velocity space, statistical properties of the
processes are more transparent. It is of special interest to note that in contrast to Levy process all the variances and higher moments are finite and probability distribution is of stable form.

(iv) Because of linear dissipation, the system driven by adiabatic fluctuations reaches a stable steady state.

(v) For small arguments the probability distribution function obeys a power law behavior which is reminiscent of Levy processes.

The stochastification by adding rapid fluctuating terms had been applied earlier to a wide variety of physical problems described by linear relaxation equations [15], hydrodynamic equations [16], Maxwell equations in a medium [17], Boltzmann equation [18] etc. Our present analysis shows that the present method might reveal interesting consequences in such cases where the added fluctuating terms, in question, are adiabatically slow. We hope to address such issues in future communications.

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References

1. Selected Papers on Noise and Stochastic Processes, edited by N. Wax (Dover, New York, 1954).

2. Stochastic Processes in Physics and Chemistry, N. G. van Kampen (North-Holland, Amsterdam, 1992).

3. R. C. Bourret, U. Frisch and A. Pouquet, Physica 65, 303 (1973); O. J. Heilman and N. G. van Kampen, Physica 77, 279 (1974); M. R. Cruty and K. C. So, Phys. Fluids. 16, 1765 (1973); A. Brissaud and U. Frisch, J. Math. Phys. 15, 524 (1974); H. F. Arnoldus and G. Nienhus, J. Phys. B 16, 2325 (1983); J. Phys. A 19, 1629 (1986); M. Rahman, Phys. Rev. E52, 2486 (1995); R. Walser, H. Ritsch, P. Zoller and J. Cooper, Phys. Rev. A45, 468 (1992); V. Berdichevsky and M. Gitterman, Europhysics Letts. 36, 161 (1996)

4. M. D. Crisp, Phys. Rev. A8, 2128 (1973).

5. S. K. Banik and D. S. Ray, J. Phys. A: Math. and General, 31, 3937 (1998).

6. R. F. Pawula, Phys. Rev. 162, 186 (1967).

7. See, for example, C. W. Gardiner, Handbook of Stochastic Methods, p.299, (Springer-Verlag, Berlin, 1983).
8. G. Ryskin, Phys. Rev. E56, 5123 (1997); N. G. van Kampen and I. Oppenheim, J. Math. Phys. 13, 842 (1972).

9. B. J. West and V. Seshadri, Physica 113A, 203 (1982).

10. R. C. Bourret, Can. J. Phys. 40, 782 (1962); Nuovo. Cim. 26, 1 (1962).

11. Levy Flights and Related Topics in Physics, edited by M. F. Shlesinger, G. M. Zaslavsky and U. Frisch (Springer, New York, 1995); B. J. West and W. Deering, Phys. Rep. 246, 1 (1994).

12. N. G. van Kampen, Phys. Rep. 24, 171 (1976).

13. H. A. Kramers, Physica 7, 284 (1940).

14. M. Bixon and R. Zwanzig, J. Stat. Phys. 3, 245 (1971).

15. L. Onsager and S. Machlup, Phys. Rev. 91, 1505 (1953); 91, 1512 (1953).

16. Fluid Mechanics, L. D. Landau and E. M. Lifshitz, (Pergamon, Oxford, 1959) Ch. 17.

17. Electrodynamics, L. D. Landau and E. M. Lifshitz, (Pergamon, Oxford, 1960) Ch. 13.

18. R. F. Fox and G. E. Uhlenback, Phys. Fluids. 13, 2881 (1970).

19. D. Grischkowsky, Phys. Rev. Letts. 24, 866 (1970); D. Grischkowsky, E. Courtens and J. Armstrong, Phys. Rev. Letts. 31, 422 (1973).