Identification of moving sinusoidal wave loads for sensor structural configuration by finite element inverse method

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Abstract: In this paper, a beam structure of composite materials with elastic foundation supports is established as the sensor model, which propagates moving sinusoidal wave loads. The inverse Finite Element Method (iFEM) is applied for reconstructing moving wave loads which are compared with true wave loads. The conclusion shows that iFEM is accurate and robust in the determination of wave propagation. This helps to seek a suitable new wave sensor method.

1. Introduction

By detecting and localizing sound sources, acoustic sensor surveillance systems are commonly utilized in many fields as a countermeasure for accidents or crimes. These systems collect data from a ranging mechanism using acoustic sensors, by measuring the times-of-flight (TOF) or difference-in-times-of-flight (DTOF) in order to compute the positions. Nevertheless, data collected by the ranging mechanism are often influenced or corrupted by different parameters (e.g., environment noise, delays associated with transducers, responses, and latency of signal processing algorithms), which lead to errors in the estimation of positions [1,2,3]. The goal of this research is to develop a single sensor capable of locating an acoustic source. Unlike acoustic sensors network, this new sensor can give continuous measurement in nature, and it utilizes new tools to reconstruct the wave impinging on the sensor’s surface. Zhang et al. [4] first applied the Tikhonov force reconstruction method to identify moving sinusoidal wave loads, and attempted to develop it as a new sensor method. However, due to complexity of moving arbitrary waves, it is difficult to identify them well just by using the conventional Tikhonov combined with the L-curve method. Then Zhang et al. [5] applied the Arnoldi-Tikhonov regularization method coupled with the Generalized Cross-Validation (GCV) for seeking the regularization parameter to identify arbitrary waves of different amplitudes and waveforms. In general, inverse results in their work are in agreement with true values. However, there are still some significant deviations between them. Besides, inverse results obtained from inverse methods above have narrower range of wave frequencies for identification and are easier to be affected by sensor structure geometry. To improve the accuracy of inverse results, we try to apply the inverse finite element method for sensor structural configuration study. Many inverse finite element method (iFEM) applications are available in the literature. For example, Gunawan [6] applied iFEM to identify the parameters of a simple viscoelastic model in form of the Prony series expansion; Gherlone et al [7] used iFEM for shape sensing: reconstruction of structural displacements by using surface-measured strains. However, until now iFEM has not been applied to reconstruct acoustic wave loads as the
sensor method. In this paper, iFEM is presented to reconstruct the distribution of moving acoustic wave, then inverse results are compared with true loads. We will find that iFEM is practical in a micro dimension wave sensor model.

In the following sections, a description of iFEM needed for identifying a moving sinusoidal wave load from sensing responses is provided. Its results are evaluated by true wave loads.

2. Moving wave loads identification by using iFEM

To illustrate the practicability and precision of the presented approach in estimating unknown input moving wave loads, a composite beam structure with elastic foundation supports is investigated. As known in Figure 1, a wave load \( f(x,t) \) travels at a constant speed \( c \) along an Euler-Bernoulli beam with a finite length. The lateral response of the composite beam is governed by the following partial differential equation \[5\].

\[
\mu_{eq} \frac{\partial^2 W(x,t)}{\partial t^2} + 2 \mu_{eq} \omega \frac{\partial W(x,t)}{\partial t} + (EI) \frac{\partial W^f(x,t)}{\partial x^4} + C_f W(x,t) = f(x,t),
\]

where \( L, \mu, \omega, (EI), C \) are the beam length, equivalent linear mass density, damping circular frequency, equivalent flexural rigidity and foundation elasticity constant respectively. \( w(x,t) \) denotes beam lateral displacement. Here we let the mass and bending stiffness of the elastic foundation neglected for simplicity.

For a composite sandwich structure shown in Figure 1, the equivalent flexural rigidity and the equivalent mass respectively can be expressed as [8]

\[
\mu_{eq} = 2B \left[ \rho h + \rho f H - \frac{h f}{h + f} \right],
\]

\[
(EI)_{eq} = \frac{2B}{3} \left[ E_f h^3 + E_f \left( H^3 - h^3 \right) \right],
\]

where \( B \) is the sandwich beam width, \( H \) and \( h \) are terms related with thickness shown in Figure 1, \( \rho \) is the material density and \( E \) is Young’s modulus. Subscripts “c” and “f” represents quantities of the core material and the face material respectively, \( E_f \) can be expressed as [8]

\[
\frac{1}{E_f} = \frac{1}{E_{f1}} \cos^4 \theta + \frac{1}{E_{f2}} \sin^2 \theta \cos^2 \theta + \frac{1}{E_{f3}} \sin^4 \theta ,
\]

where \( E_{f1}, E_{f2}, G_{f12} \) and \( \nu_{12} \) are mechanical properties of face lamina along their principal directions; \( \theta \) is the fiber orientation angle measured from the \( x \)-axis in counter-clock wise direction. To solve equation (1), we discretize the entire beam into \( \hat{m} \) beam finite elements of length \( L_e = L/\hat{m} \). Within a finite element, the beam lateral displacement is

\[
W_e(x,t) = \sum_{i=1}^{4} N_{d_i}(x) q_i(x) + N_{d_4}(x) q_4(x),
\]

where \( N_{d_i}(x) \) \( (i=1,2,3,4) \) are interpolation functions described in [9]. Suppose that the lateral displacement vector \( \{ q(t) \} \) is a collection of nodal variables \( q_i \), where \( i=1,2,3,4 \). We may obtain the following equations of motion using the Lagrange equations
\[ [M]\{\ddot{q}\}+[C]\{\dot{q}\}+[K+K_f]\{q\} = \{F(t)\}, \quad (6) \]

where \{\ddot{q}\}, \{\dot{q}\} and \{q\} are the beam element acceleration, velocity and displacement vectors respectively, and \{F(t)\} represents the external force vector which depends on moving wave loads. \([C], [M]\) and \([K+K_f]\) are the beam element damping matrix, mass matrix, beam structural stiffness and foundation base stiffness matrix respectively. Details can be referred to [9]. For the external force vector \{F(t)\}, its vector element can be expressed as follows
\[
F_i(t) = \int_{t}^{t+\Delta t} f(x,t)Nd_i(x)dx \quad (i = 1,2,3,4), \quad (7)
\]

the structural system described in equation (6) can be solved by the Newmark \(\beta\) method of direct integration [9]. For a typical time step from \(t\) to \(t+\Delta t\), the acceleration and velocity at \(t+\Delta t\) are related to the displacement at \(t+\Delta t\) as follows
\[
\{\ddot{q}\}_{t+\Delta t} = a_0 \{\{q\}_{t+\Delta t} - \{q\}_t\} - a_2 \{\dot{q}\}_t - a_3 \{\ddot{q}\}_t, \quad (8)
\]

and
\[
\{\dot{q}\}_{t+\Delta t} = \{\dot{q}\}_t + a_0 \{\ddot{q}\}_t + a_2 \{\dot{q}\}_t + a_3 \{q\}_t. \quad (9)
\]

where the integration parameters are given by
\[
a_0 = \frac{1}{\beta \Delta t}, \quad a_1 = \frac{\gamma}{\beta \Delta t}, \quad a_2 = \frac{1}{\beta \Delta t} \cdot \frac{1}{2}, \quad a_3 = \frac{\Delta t (1-\gamma)}{2 \beta}, \quad \text{and} \quad a_4 = \gamma \Delta t. \quad \text{In this study,} \quad \beta = 0.25 \quad \text{and} \quad \gamma = 0.5 \quad \text{are selected, which implies a constant average acceleration with unconditional numerical stability.} \quad \{\ddot{q}_{t+\Delta t}\} \quad \text{in equation (8) can be substituted into equation (6), and equations for} \quad \{\ddot{q}_{t+\Delta t}\} \quad \text{and} \quad \{\ddot{q}_{t+\Delta t}\} \quad \text{can be expressed in terms of the unknown vector} \quad \{q_{t+\Delta t}\}. \quad \text{If the response at time} \quad t \quad \text{is determined, then the displacement at} \quad t+\Delta t \quad \text{may be obtained by solving the following algebraic equations:}
\[
(a_0[M] + a_0a_7[C] + [K+K_f])\{\ddot{q}\}_{t+\Delta t} = \{F\} + [M]\{a_0\{q\}_t + a_2 \{\dot{q}\}_t + a_3 \{\ddot{q}\}_t\} + [C]\{(-1+a_7a_0)\{q\}_t + a_2a_3 \{\dot{q}\}_t + a_3a_4 \{\ddot{q}\}_t\}. \quad (10)
\]

once the displacement vector \{q_{t+\Delta t}\} is found, the velocity and acceleration vectors may be obtained using equations (8) and (9). 

The procedure of applying the inverse finite element method to solve our problem is as follows. Assume that displacement velocity and acceleration at a beam element nodes are known, we can use equation (6) and equation (7) to identify moving wave loads from the inverse approach. Using the composite Simpson’s rule for numerical integration, the right term in equation (7) can be expressed as
\[
\Pi(j) \approx \frac{h}{3} \left\{ f(0,j)Nd_i(0) + 2 \sum_{m=1}^{(N/2)-1} f(x_{2m},j)Nd_i(x_{2m}) + 4 \sum_{m=1}^{(N/2)} f(x_{2m-1},j)Nd_i(x_{2m}) + f(L,j)Nd_i(x_N) \right\}. \quad (11)
\]

For a beam FEM model of 15 elements, if each element is divided into 10 segments, we have in total 151 integration points. Note that \(h = \frac{L}{L_1} \quad \text{and} \quad x_k = o + k \frac{L}{N} (k=0,1,2,...,N), \quad \text{where} \quad N = 15 \times 10. \quad \text{The residual error associated with approximation in equation (11) is}
\[ \text{Error} = \frac{Lh^4}{180} \left| f(x,t)Nd_j(x) \right|^4 \leq 1.05 \times 10^{-5}. \]  \hfill (12)

Note that
\[ \begin{bmatrix} a(0,0), a(x_1,0), \ldots, a(x_N,0), a(0,1), a(x_1,1), \ldots, a(x_N,1), \ldots, a(0,N), a(x_1,N), \ldots, a(x_N,N) \end{bmatrix}^T = \sum_{i=1}^{4} \begin{bmatrix} s_0 & 0 & \ldots & 0 & B_\varepsilon & 0 & \ldots & 0 \end{bmatrix} \times \begin{bmatrix} s_0 & s_1 & \ldots & s_N \end{bmatrix} \] \hfill (13)

\[ \begin{bmatrix} f(0,0), f(x_1,0), \ldots, f(x_N,0), f(0,1), f(x_1,1), \ldots, f(x_N,1), \ldots, f(0,N), f(x_1,N), \ldots, f(x_N,N) \end{bmatrix}, \text{where} \]

\[ S_j = Nd_j(x_j) \text{ and } B_\varepsilon = \begin{bmatrix} \frac{h_e}{3} & \frac{4h_e}{3} & \frac{2h_e}{3} & \frac{4h_e}{3} & \frac{2h_e}{3} & \frac{2h_e}{3} & \ldots & \frac{h_e}{3} \end{bmatrix}. \]

Equation (12) can be simplified as
\[ V_{[(N+1)\times(N+1)]} = G_{[(N+1)\times(N+1)]} \times F_{[(N+1)\times(N+1)]} \] \hfill (14)

where
\[ V_{[(N+1)\times(N+1)]} = \begin{bmatrix} a(0,0), a(x_1,0), \ldots, a(x_N,0), a(0,1), a(x_1,1), \ldots, a(x_N,1), \ldots, a(0,N), a(x_1,N), \ldots, a(x_N,N) \end{bmatrix}^T. \]

Here \( a(x,j) \) represents the left hand side term of equation (6). The other two terms in equation (14) are given by
\[ G_{[(N+1)\times(N+1)]} = \sum_{i=1}^{4} \begin{bmatrix} s_0 & 0 & \ldots & 0 & B_\varepsilon & 0 & \ldots & 0 \end{bmatrix} \times \begin{bmatrix} s_0 & s_1 & \ldots & s_N \end{bmatrix} \]

and
\[ F_{[(N+1)\times(N+1)]} = \begin{bmatrix} f(0,0), f(x_1,0), \ldots, f(x_N,0), f(0,1), f(x_1,1), \ldots, f(x_N,1), \ldots, f(0,N), f(x_1,N), \ldots, f(x_N,N) \end{bmatrix}^T. \]

It should be noted that in this study, we assume that time dependent response such as displacement, velocity and acceleration at distributing \( N+1 \) fixed points on the beam are known and can be determined beforehand. Through iFEM inverse process, moving wave loads will be reconstructed correctly regardless of arrival time of the wave. Equation (14) is expressed as an inverse problem and \( F = G^{-1}V \) is about ill-posed matrix problem. To solve this problem and find a useful and stable solution, the Moore-Penrose inverse method is applied. Since time dependent responses are known and often polluted with random noise disturbance. We develop the random noise filter algorithm to deal with responses before they are used for the inverse process.

### 3. Moving wave load identification

In this section, we use moving wave loads propagating our sensor model for tests. We suppose that the wave load begins to progressively step on the beam sensor until it is entirely on the beam. It is expressed by
\[ f(x,t) = A(x) \sin \left( \frac{2\pi}{cT}(x-ct) \right) \times \left\{ 1 - \hat{H}(x-ct) \right\}, \quad (0 \leq t \leq NT) \] \hfill (15a)

In this equation, \( \hat{N} \) is the number of half-cycles, \( T \) is single half-cycle time period and \( \hat{H} \) is the Heaviside step function. The speed of wave is denoted by \( c \), which for the cases considered here will be the speed of sound in air. Amplitude is \( A(x) \), here \( x \) represents the location along beam sensor longitudinal direction. This equation represents a discrete number of half-cycles in a traveling sinusoidal wave. Once the load is completely on the sensor, the force is expressed as
\[ f(x,t) = -A(x) \sin \left( \frac{2\pi}{cT} (x-ct) \right) \times \left[ H(x-c(t-\hat{N}T)) - H(x-c(t+\hat{N}T)) \right], (\hat{N}T \leq t \leq L/c) \] (15b)

until it reaches the other end of the beam. Finally, the load begins to leave the sensor, as

\[ f(x,t) = -A(x) \sin \left( \frac{2\pi}{cT} (x-ct) \right) \times \left[ H(x-c(t+\hat{N}T)) - H(x-c(t-\hat{N}T)) \right], (L/c \leq t \leq \hat{N}T + L/c) \] (15c)

Note that equations (15a–c) represent a sinusoidal wave traveling along the beam sensor in parallel to the beam axial direction. Because the intent here is to demonstrate the effectiveness of the proposed method, waves incident at oblique angles are not considered. Nevertheless, there is no restriction in the approach to parallel traveling waves. The force reconstruction method is effective, no matter what angle the acoustic waves move across the beam sensor.

Suppose that the beam sensor is with sandwich-structured composite as shown in Figure 1, its relative material properties are listed in Table 1.

**Table 1. Composite materials**

| Material          | \( \rho \) (kg/m\(^3\)) | E11 (GPa) | E22 (GPa) | G12 (GPa) | \( \nu_{12} \) |
|-------------------|-----------------|-----------|-----------|-----------|---------------|
| Carbon/Epoxy      | 1600            | 177       | 10.8      | 76        | 0.270         |
| Foam              | 11.2            | 85.53E-05 | 85.53E-05 | 31.40E-05 | 0.362         |

Initial values of other parameters are listed in Table 2.

**Table 2. Initial parameters**

| Item       | Description                          | Units | Value   |
|------------|--------------------------------------|-------|---------|
| \( L \)    | Beam length                          | M     | 0.5     |
| \( B \)    | Beam width                           | M     | 0.03    |
| \( H \)    | Half sandwich Beam thickness          | m     | 0.0005  |
| \( C \)    | Wave speed                           | ms\(^{-1}\) | 343     |
| \( T \)    | Half-cycle duration                   | S     | 0.0005  |
| \( TH \)   | Foundation thickness                  | M     | 3       |
| \( C_f \)  | Foundation Young’s modulus           | MPa   | 2.3     |
| \( K_E \)  | The number of half-cycles             |       | 1       |
| \( H \)    | Thickness                             | M     | 0.5\( H \) |

We assume that the wave load amplitude \( A(x) \) is constant and equal to 1, and beam structural damping is neglected for simplicity. To simulate noisy measured data, we construct a vector \( V = V_{\text{true}} + \epsilon \) where \( \epsilon \) is a random noise vector containing pseudorandom entries chosen from a normal distribution, with zero mean and standard deviation equal to one. Figure 2 shows moving wave loads obtained by using the iFEM method. Figure 2(a) and 2(b) represent the comparison of 2-d wave load inverse results with corresponding true values at beam location \( x=0.2317\)m and certain time \( t=1.0\times10^{-03}\)s. We notice that the waveform has good agreement with the true load, but the inverse wave amplitude is slightly larger. Figure 2(c) and 2(d) shows 3-D inverse results and influence of different noise levels on inverse results at the beam mid-span. From Figure 2(d), we notice that inverse results are much less influenced by background noise, even for a noise-to-signal ratio as high as 20%.
In addition, iFEM can always work well even no matter how we vary beam structural dimensions such as thickness.

Figure 2. Moving wave loads inverse results by using iFEM method

Figure 3. 3-D moving wave inverse results by the iFEM method through varying beam thickness from h=0.001m to h=0.03m.

From Figure 3, we find that even though we reduced the beam thickness from h=0.03m to h=0.001m, inverse results are not affected if the iFEM method is used. Figure 4 shows the minimum beam sensor length for the inverse methods to identify moving acoustic loads. From this figure, we can find that the minimum beam sensor length for iFEM is 0.5mm. From the comparison, we can draw the conclusion that the iFEM method is very practical to be applied for the new micro dimension sensor method for wave prediction.

Figure 4. Minimum beam sensor length for the inverse methods to identify moving acoustic loads. The iFEM can be used to identify wide range frequencies of moving wave loads which propagate through the sensor. Figure 5 shows different frequencies of wave loads identified by the iFEM.
Figure 5. Minimum and maximum frequencies of sinusoidal wave loads identified by iFEM method

From Figure 6, we know that iFEM method can well identify sinusoidal wave load frequencies between 1 Hz and 1MHz. This demonstrates that iFEM method is capable of predicting the wide range of wave frequencies, and it is very practical to be a new wave sensor method.

4. Conclusions
In this paper, we also apply the iFEM to identify moving loads, iFEM inverse results are compared with corresponding true wave loads. Furthermore, iFEM inverse results are compared with those of the analytical inverse method by changing some sensor structural parameters and noise levels. It is proved that iFEM is suitable in identifying wave loads, especially in the micro dimension sensor wave prediction and therefore it is very practical as a new sensor method.

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