PRIMES IN HIGHER-ORDER PROGRESSIONS ON AVERAGE

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Abstract. In this paper, we establish some theorems on the distribution of primes in higher-order progressions on average.

1. Introduction

The Bateman-Horn conjecture [2] suggests that if \( x^\ell + u \in \mathbb{Z}[x] \) be irreducible polynomial with \( u \) be an even number and the degree \( \ell \geq 1 \), then

\[
\sum_{m \leq X} \Lambda(m) \Lambda(m^\ell + u) \sim \prod_p \left( 1 - \frac{1}{p} \right)^{-2} \left( 1 - \frac{n_\ell(p, u)}{p} \right) X,
\]

where \( \Lambda \) denotes the von Mangoldt function, \( p \) stands for primes and \( n_\ell(p, u) \) being the number of solutions of the congruence \( x(x^\ell + u) \equiv 0 \pmod{p} \).

If \( \ell = 1 \), the asymptotic formula in (1.1) is the twin prime conjecture. However, even the simple case seems beyond the current approach. In 1970, Lavrik [12] proved that if \( \ell = 1 \), then given any \( A > 0 \), (1.1) holds for all even integer \( u \geq 1 \) not exceeding \( X \) with at most \( O(X(\log X)^{-A}) \) exceptions.

In [1], S. Baier and L. Zhao established certain theorems for the Bateman-Horn conjecture for quadratic polynomials on average. Their main result states the following. Given \( A, B > 0 \), we have, for \( x^2(\log x)^{-A} \leq y \leq x^2 \),

\[
\sum_{k \leq y} \left| \sum_{\mu^2(k) = 1} \Lambda(n^2 + k) - \mathcal{G}(k)x \right|^2 \ll yx^2(\log x)^{-B},
\]

where

\[
\mathcal{G}(k) = \prod_{p > 2} \left( 1 - \frac{1}{p - 1} \left( \frac{-k}{p} \right) \right)
\]

with \( \left( \frac{-k}{p} \right) \) being the Legendre symbol. In [5], F. Too and L. Zhao established similar results for the cubic cases.

In this paper, we shall study the asymptotic formula in (1.1) on average. Our main results are as follows.

Theorem 1.1. Let integer \( \ell \geq 2 \). For any \( A > 0 \), there exists a \( B = B_\ell(A) \) such that

\[
\sum_{1 \leq u \leq y} \left| \sum_{m \leq \sqrt{X}} \Lambda(m^\ell + u) \Lambda(m) - \mathcal{G}_\ell(u)\sqrt{X} \right|^2 \ll \frac{yX^{3/2}}{\log A X}
\]
holds for any \( y \in (X^{1-\frac{1}{2\ell}}(\log X)^B, X) \), where
\[
\mathfrak{S}_\ell(u) = \prod_{p | u} \frac{p - \varrho_\ell(p, u)}{p - 1 - \varrho_\ell(p, u)} \prod_p \left(1 - \frac{\varrho_\ell(p, u) - 1}{p - 1 - (p - 1)^2}\right),
\]
p stands for primes and \( \varrho_\ell(p, u) \) being the number of solutions of the congruence \( x^\ell + u \equiv 0 \pmod{p} \).

By similar arguments, we have the following theorem which improves the results in [1] and [5].

**Theorem 1.2.** Let integer \( \ell \geq 2 \). For any \( A > 0 \) and \( \varrho_\ell(p, u) \) as defined in the Theorem 1.1. Then there exists a \( B' = B'_\ell(A) \) such that
\[
\sum_{1 \leq u \leq y} \left| \sum_{m \leq \sqrt{X}} \Lambda(m^\ell + u) - \mathfrak{S}_\ell(u)\sqrt{X} \right|^2 \ll \frac{yX^{\frac{2}{3}}}{\log^4 X}
\]
holds for any \( y \in (X^{1-\frac{1}{2\ell}}(\log X)^B', X) \) with
\[
\mathfrak{S}'_\ell(u) = \prod_p \left(1 - \frac{\varrho_\ell(p, u) - 1}{p - 1}\right)
\]
and the product being taken over all primes.

The primary technique used in the proof of Theorem 1.1 is the circle method and the using of a variant of Weyl’s inequality. The main difficulty in this application of the circle method is with the singular series. As for the asymptotic conjecture (1.1), the coefficient \( \mathfrak{S}_\ell(u) \) involves the using of Dedekind zeta functions associated to suitable algebraic number fields of the form \( \mathbb{Q} \left( \sqrt{u} \right) \). On the other hand, let \( p, q \) denote primes and we observe that left of (1.1) means that one can give an estimate for
\[
\# \{ q \in \mathbb{N} : q^\ell + u = p, q \leq X \} = \# \{ q \in \mathbb{N} : p - q^\ell = u, q \leq X \}.
\]
Which similar with the Hardy-Littlewood conjecture [7], say every sufficiently large number is either an \( \ell \)-th power or a sum of a prime number and an \( \ell \)-th power, for \( \ell = 2, 3 \). When the circle method be used, in fact there is no big difference between them. Therefore when \( \ell \geq 2 \), the singular series similar to the singular series of Zaccagnini [14], which first give a crude estimates for the kinds of singular series. In [10], Kawada announced that he could obtain an asymptotic formula for the number of representations of numbers as the sum of a prime and an \( \ell \)-th power on average, and give a detailed proof in [11] by use of the analytic properties of the Dedekind zeta function. Based on this result and under Generalized Riemann Hypothesis, Brüdern [4] give an asymptotic formula for the number of representations of numbers as the sum of a prime and an \( \ell \)-th power of a prime on average.

Furthermore, combined with the work of Perelli, Zaccagnini [13] and Bauer [3], we can have a good treatment for the minor arcs. Hence we get the proof of our main theorem.

**Notation.** Notation is standard or otherwise introduced when appropriate. The symbols \( \mathbb{Z} \) and \( \mathbb{Q} \) denote the set of integers and rational numbers, respectively. \( e(z) = e^{2 \pi i z} \), the letter \( p \) always denotes a prime. The symbol \( \mathbb{Z}_q \) represents shorthand for the groups \( \mathbb{Z}/q\mathbb{Z} \). Also, the shorthand for the multiplicative group composed by reduced residue classes \( (\mathbb{Z}/q\mathbb{Z})^\ast \) is \( \mathbb{Z}_q^\ast \). Denote by \( \varphi \) and \( \Lambda \) the Euler and von Mangoldt functions, respectively. For a large number \( X \), denote \( L = \log X \). For the sake of simplicity, we set
\[
I_\ell(\alpha, z) = \sum_{z < m^\ell \leq 2^\ell z} e\left(m^\ell \alpha\right), \quad J_\ell(\alpha, z) = \sum_{z < m^\ell \leq 2^\ell z} \Lambda(m) e\left(m^\ell \alpha\right),
\]
Proof. This is due to Theorem 1, Corollary 1 and Corollary 2 of \[ \lambda \] integer when Lemma 2.2. Let 

$$ I(\alpha, z) = \sum_{m \leq 2^l z} e(-m\alpha) \quad \text{and} \quad J(\alpha, z) = \sum_{m \leq 2^l z} \Lambda(m) e(-m\alpha). $$

Further, we set 

$$ \lambda(q, u) = \frac{1}{q} \sum_{a \in \mathbb{Z}_q} \sum_{h \in \mathbb{Z}_q} e\left( \frac{a(h^\ell + u)}{q} \right) \quad \text{and} \quad A(q, u) = \frac{1}{\varphi(q)} \sum_{a \in \mathbb{Z}_q^*} \sum_{h \in \mathbb{Z}_q^*} e\left( \frac{a(h^\ell + u)}{q} \right). $$

It is easily seen that both \( \lambda(q, u) \) and \( A(q, u) \) are multiplicative function with respect to positive integer \( q \). It is obvious that 

$$ \lambda(q, u) = \prod_{p|q} (g_e(p, u) - 1) $$

and 

$$ A(q, u) = \frac{q}{\varphi(q)} \prod_{p|(q, u) - 1} \left( g_e(p, u) - 1 + \frac{1}{p} \right) \prod_{p|(q, u)} \left( g_e(p, u) - 2 + \frac{1}{p} \right) $$

when \( q \) is square-free. Also, for any \( z \geq 1 \), we always set 

$$ \mathcal{G}_e(u, z) = \sum_{q \leq z} \frac{\mu(q)}{\varphi(q)} \lambda(q, u), \quad \mathcal{G}_e(u, z) = \sum_{q \leq z} \frac{\mu(q)}{\varphi(q)} A(q, u), $$

$$ \mathcal{P}_e(u, z) = \prod_{p|u, p \leq z} \left( \frac{p - g_e(p, u)}{p - 1 - g_e(p, u)} \right) \prod_{p \leq z} \left( 1 - \frac{g_e(p, u) - 1}{p} - \frac{g_e(p, u)}{p - 1} \right) $$

and 

$$ \mathcal{P}'_e(u, z) = \prod_{p \leq z} \left( 1 - \frac{g_e(p, u) - 1}{p} \right). $$

2. Preliminary lemma

We shall need the following well-known results in analytic number theory.

**Lemma 2.1.** Let \( 1 \leq z \leq y, v \in \mathbb{Z} \setminus \{0\} \) and integer \( \ell \geq 2 \). Then we have 

$$ \left| \mathcal{G}_e^j(nv, z) - \mathcal{G}_e^j(nv) \right|^2 \ll_{v, \ell} yz^{-1/(2000^2)}. $$

**Proof.** This is due to Theorem 1, Corollary 1 and Corollary 2 of [11].

**Lemma 2.2.** Let \( |u| \geq 1, \ell \in \mathbb{Z}_{\geq 1} \) and \( x^\ell + u \) is irreducible over \( \mathbb{Q}[x] \). Then we have 

$$ \left| \sum_p \frac{\varrho(p, u) - 1}{p} \right| \leq O_\ell(1) + 4\ell \log \log(2|u|). $$

**Proof.** Let \( D_u \) be the discriminant of \( \mathbb{Q}[\sqrt{u}] \). It is easily seen that \( |D_u| \leq \ell^\ell |u|^{\ell - 1} \). Hence by Landau prime ideal theorem (see [9, Theorem 5.33]) and partial summation we have 

$$ \left| \sum_p \frac{\varrho(p, u) - 1}{p} \right| \leq (\ell - 1) \sum_{p \leq y} \frac{1}{p} + \left| \sum_{p > y} \frac{\varrho(p, u) - 1}{p} \right| $$

$$ \leq (\ell - 1) \log \log y + O_\ell(1) + O\left( \sqrt{D_u} \exp\left( -c_\ell \sqrt{\log y} \log y \right) \right), $$

where \( c_\ell \) is an absolute constant depending only on \( \ell \). Setting \( y = \exp((\log(2|u|))^{1/4}) \) we obtain that 

$$ \left| \sum_p \frac{\varrho(p, u) - 1}{p} \right| \leq 4\ell \log \log(2|u|) + O_\ell \left( 1 + \frac{D_u^{1/2}}{|u|^{-(1/4) \log(2|u|)}} \log \log(2|u|) \right). $$
Thus if \(|u| \geq \exp(\ell/c_\ell)\), then we obtain that
\[
\left| \sum_p \vartheta(p, u) - 1 \right| \leq 4\ell \log \log(2|u|) + O_\ell(1).
\]
Thus we get the proof of the lemma. \(\square\)

**Lemma 2.3.** Let \(\alpha = \frac{a}{q} + \beta, |\beta| \leq X^{-1}L^B, q \leq L^B\) and \(B \geq 1\). Also let \(z \in (XL^{-B}, X]\). We have
\[
R_\ell(\alpha, z) := J_\ell(\alpha, z) - I_\ell(\beta, z)B_\ell(q, a)/\varphi(q) \ll z^{1/\ell}L^{-10B^2},
\]
\[
R'_\ell(\alpha, z) := I_\ell(\alpha, z) - I_\ell(\beta, z)B'_\ell(q, a)/q \ll z^{1/\ell}L^{-10B^2},
\]
where
\[
B_\ell(q, a) = \sum_{h \in \mathbb{Z}_q^*} e\left(\frac{ah^\ell}{q}\right) \quad \text{and} \quad B'_\ell(q, a) = \sum_{h \in \mathbb{Z}_q} e\left(\frac{ah^\ell}{q}\right).
\]

**Proof.** It is easily seen that
\[
J_\ell(\alpha, z) = \sum_{h \in \mathbb{Z}_q^*} e\left(\frac{ah^\ell}{q}\right) \sum_{z^{1/\ell} < m \leq 2z^{1/\ell}} \Lambda(m)e(m^\ell\beta) + O(\log q \log z)
\]
\[
= \frac{1}{\varphi(q)} \sum_{h \in \mathbb{Z}_q^*} e\left(\frac{ah^\ell}{q}\right) \sum_{z^{1/\ell} < m \leq 2z^{1/\ell}} e\left(\frac{m^\ell\beta}{q}\right) + R_\ell(\alpha, z),
\]
by partial summation and where
\[
R_\ell(\alpha, z) \ll (\log z)^2 + \varphi(q)(1 + |\beta|z) \max_{\ell \leq x \leq 2z^{1/\ell}} \max_{m \equiv h \mod q} \sum_{m \leq x} \Lambda(m) - x/\varphi(q).
\]
Then by \([9, \text{Corollary 5.29}]\), we get the estimate of \(R_\ell(\alpha, z)\). The estimate of \(R'_\ell(\alpha, z)\) is similar and we omit its detail. \(\square\)

**Lemma 2.4.** Let \(\alpha = \frac{a}{q} + \lambda\) with \((a, q) = 1\) and \(|\lambda| \leq q^{-2}\). Then for each integer \(\ell \geq 2\) and any \(A > 0\) there exists a \(B_{m, \ell}(A) > 0\) such that for \(B \geq B_{m, \ell}(A)\) the estimate
\[
\int_y^{2y} dt \left| \sum_{t < m^\ell \leq t + H} \Lambda(m)e(m^\ell a/q) \right|^2 \ll_{\ell, A} H^2 y^{3/2 - 1} L^{-A - 2}
\]
holds for \(L^B < q \leq HL^{-B}y^{1-1/(2\ell)}L^B < H \leq y\) and \(y \geq \sqrt{X}\).

**Proof.** This is quoted from \([3, \text{Lemma 3.3}]\). \(\square\)

**Lemma 2.5.** Let \(a, q\) be positive integers with \((a, q) = 1\). Then for each integer \(\ell \geq 2\), there exists a \(B'_{m, \ell}(A) > 0\) such that for \(B \geq B'_{m, \ell}(A)\) the estimate
\[
W_\ell(y, H) := \int_y^{2y} dt \left| \sum_{t < m^\ell \leq t + H} e(m^\ell a/q) \right|^2 \ll_{\ell, A} H^2 y^{3/2 - 1} L^{-A - 2}
\]
holds for \(L^B < q \leq HL^{-B}y^{1-1/\ell}L^B < H \leq y\) and \(y \geq \sqrt{X}\).
Proof. It is easily seen that
\[
W_\ell(y, H) = \sum_{y < m_1^\ell, m_2^\ell \leq 2y + H} e \left( (m_1^\ell - m_2^\ell) a/q \right) \int_{\max(m_1^\ell - H, m_2^\ell - H)}^{\min(m_1^\ell, m_2^\ell)} dx
\]
\[
= \sum_{y < m_1^\ell, m_2^\ell \leq 2y + H} e \left( (m_1^\ell - m_2^\ell) a/q \right) \left( H - |m_1^\ell - m_2^\ell| \right)
\]
\[
= \sum_{k} \sum_{n} (H - |P_\ell(n, k)|) 1_{y < (n+k)^\ell \leq 2y + H} e(P_\ell(n, k)a/q)
\]
\[
\ll \sum_{k \leq H^{\ell/t - 1}} H y^{1/\ell} \max_{y^{1/\ell} < x \leq 2y^{1/\ell}} \left| \sum_{y^{1/\ell} \leq n \leq x} e(P_\ell(n, k)a/q) \right|
\]
\[
\ll \sum_{k \leq H^{\ell/t - 1}} H y^{1/\ell} \max_{2y^{1/\ell} / x \leq 2y^{1/\ell}} \left| \sum_{n \leq x} e(P_\ell(n, k)a/q) \right|
\]
where \( P_\ell(n, k) = (n + k)^\ell - n^\ell \), then by [13, Lemma], we have
\[
W_\ell(y, H) \ll_{\ell, D} H^2 y^{2 \ell - 1} \left( \left( \frac{LD}{q} \right)^{2 \ell - t} + \left( \frac{LD}{y^{1/\ell}} \right)^{2 \ell - t} + \left( \frac{qLD}{H} \right)^{2 \ell - t} + \left( \frac{LD}{L^\ell} \right)^{2 \ell - t} \right)
\]
holds for any \( D > 1 \). By setting \( D = B/2 \) and \( B_{m, \ell}(A) = 2^{\ell - 1}(A + 2) + 2\ell^2 \), we obtain the proof of the lemma. \( \square \)

3. The proof of the main results

We first denote
\[
S_\ell(y, X) = \sum_{1 \leq u \leq y} \sum_{m^\ell \leq X} \Lambda(m^\ell + u) \Lambda(m) - \mathcal{S}_\ell(u) \sum_{m^\ell \leq X} 1
\]
and
\[
S_\ell'(y, X) = \sum_{1 \leq u \leq y} \sum_{m^\ell \leq X} \Lambda(m^\ell + u) - \mathcal{S}_\ell'(u) \sum_{m^\ell \leq X} 1
\]
Then, by sum over dyadic intervals process one has
\[
S_\ell(y, X) \ll \sum_{1 \leq u \leq y} \sum_{m^\ell \leq XL^{-B}} \Lambda(m^\ell + u) \Lambda(m) - \mathcal{S}_\ell(u) \sum_{m^\ell \leq XL^{-B}} 1
\]
\[
+ \sum_{1 \leq u \leq y} \sum_{XL^{-B} < m^\ell \leq X} \Lambda(m^\ell + u) \Lambda(m) - \mathcal{S}_\ell(u) \sum_{XL^{-B} < m^\ell \leq X} 1
\]
\[
\ll yX^{2/\ell} \left( L^{-B} + BL \right) \sup_{X/L^B \leq z \leq X/2^\ell} F_\ell(y, z),
\]
where $B \geq 2$ and

$$F_\ell(y,z) = \sum_{u \leq y} \left| \sum_{z < m^\ell \leq 2^\ell z} \left( \Lambda(m^\ell + u)\Lambda(m) - \mathcal{G}_\ell(u) \right) \right|^2.$$  

Similarly, we have

$$(3.2) \quad S'_\ell(y,X) \ll yX^{2/\ell}L^{-B} + BL \sup_{X/L^B \leq z \leq X/2^\ell} F_\ell(y,z)$$

for any $B \geq 2$, where

$$F_\ell(y,z) = \sum_{u \leq y} \left| \sum_{z < m^\ell \leq 2^\ell z} \left( \Lambda(m^\ell + u) - \mathcal{G}_\ell'(u) \right) \right|^2.$$  

We define the major arcs as

$$(3.3) \quad J_{q,a} = \left( a/q - L^{B^2}/X, a/q + L^{B^2}/X \right],$$

where $1 \leq a \leq q$. It is obvious that the interval $J_{q,a}$ are pairwise disjoint. Setting

$$(3.4) \quad \mathcal{M} = \bigcup_{q \leq L^{B^2} \leq a \leq q} J_{q,a} \quad \text{and} \quad m = \left( L^{B^2}/X, 1 + L^{B^2}/X \right] \setminus \mathcal{M},$$

where $*$ means that $(a,q) = 1$. Application of the circle method gives

$$\sum_{z < m^\ell \leq 2^\ell z} \Lambda(m^\ell + u)\Lambda(m) = \left\{ \int_{\mathcal{M}} + \int_{m} \right\} \int J(\alpha, z)J_\ell(\alpha, z)e(u\alpha) \, d\alpha.$$  

Therefore,

$$F_\ell(y,z) = \sum_{u \leq y} \left| \sum_{z < m^\ell \leq 2^\ell z} \left( \Lambda(m^\ell + u)\Lambda(m) - \mathcal{G}_\ell(u) \right) \right|^2$$

$$\ll \sum_{u \leq y} \left| \int_{\mathcal{M}} J(\alpha, z)J_\ell(\alpha, z)e(u\alpha) \, d\alpha - \sum_{z < m^\ell \leq 2^\ell z} \mathcal{G}_\ell(u) \right|^2$$

$$+ \sum_{u \leq y} \left| \int_{m} J(\alpha, z)J_\ell(\alpha, z)e(u\alpha) \, d\alpha \right|^2 := S_{\mathcal{M}}(y,z) + S_m(y,z).$$

Similarly, we have

$$(3.6) \quad F_\ell'(y,z) \ll S'_{\mathcal{M}}(y,z) + S'_m(y,z),$$

where

$$S'_{\mathcal{M}}(y,z) = \sum_{u \leq y} \left| \int_{\mathcal{M}} J(\alpha, z)J_\ell(\alpha, z)e(u\alpha) \, d\alpha - \sum_{z < m^\ell \leq 2^\ell z} \mathcal{G}_\ell'(u) \right|^2$$

and

$$S'_m(y,z) = \sum_{u \leq y} \left| \int_{m} J(\alpha, z)J_\ell(\alpha, z)e(u\alpha) \, d\alpha \right|^2.$$  

We shall prove the following lemmas, from which, (3.1), (3.2), (3.5) and (3.6) the results of our two theorems follow.
Lemma 3.1. For any $A > 0$, there exists a $B_{\ell,1}(A) > 0$ such that for $B \geq B_{\ell,1}(A)$, then
\[ S_m(y, z) \ll \gamma z^{2/\ell} L^{-A} \]
holds for all $z^{1-1/(2\ell)} L^B \leq y \leq z$ with $XL^{-B} \leq z \leq X/2^\ell$.

For any $A > 0$, there exists a $B'_{\ell,1}(A) > 0$ such that for $B \geq B'_{\ell,1}(A)$, then
\[ S'_m(y, z) \ll \gamma z^{2/\ell} L^{-A} \]
holds for all $z^{1-1/(2\ell)} L^B \leq y \leq z$ with $X/L^B \leq z \leq X/2^\ell$.

Lemma 3.2. Let $z \in [X/L^B, X/2^\ell]$ and $y \in (z^\delta, z]$ with $\delta \in (0, 1)$ be fixed. We have
\[ S_m(y, z) \ll \delta, A \gamma z^{2/\ell} L^{-A} \]
and
\[ S'_m(y, z) \ll \delta, A \gamma z^{2/\ell} L^{-A} \]
for any $A > 0$, $B = \max(2000\ell^2 (12\ell + A), 2^\ell (10\ell + A) + c(\ell))$ with $c(\ell)$ an absolute constant depending only on $\ell$.

4. THE MINOR ARCS

In this section, we shall prove Lemma 3.1. Firstly, we have
\[ S_m(y, z) \ll \int_m |J(\alpha, z)J_\ell(\alpha, z)|^2 d\alpha \ll \sup_{\alpha \in m} |J_\ell(\alpha, z)|^2 \]
by Bessel’s inequity. Then the classical result
\[ J_\ell(\alpha, z) \ll z^{1/\ell} (\log z)^{-B} \]
holds for all $\alpha \in m$ and any $B \geq 0$. This implies that if $y \in (zL^{-B}, z]$ then
\[ S_m(y, z) \ll z^{1+2/\ell} L^{-2B} \ll \gamma z^{2/\ell} L^{-B}. \]

If $y \in (z^{1-1/(2\ell)} L^B, zL^{-B}]$, then
\[ S_m(y, z) = \sum_{u \leq y} \left| \int_m J(\alpha, z)J_\ell(\alpha, z)e(u\alpha) d\alpha \right|^2 \]
\[ = \int_m d\beta \int_m J(-\alpha, z)J_\ell(-\alpha, z) \sum_{u \leq y} e(u(\alpha - \beta)) d\alpha \]
\[ \ll \int_m d\beta |J(\beta, z)J_\ell(\beta, z)| \int_m |J(\alpha, z)J_\ell(\alpha, z)| \min \left( \frac{1}{\| \alpha - \beta \|} \right) d\alpha. \]

Splitting the unit interval in $H = [y] + 1$ adjacent, disjoint intervals $H_i$ of length $H^{-1}$, we obtain that
\[ S_m(y, z) \ll \sum_{1 \leq i, j \leq H} \sum_{1 \leq i, j \leq H} \int_m \int_m \beta |J(\beta, z)J_\ell(\beta, z)| \int_m \int_m \beta |J_\ell(\beta, z)|. \]

By Cauchy’s inequity, we have
\[ S_m(y, z) \ll y \sum_{1 \leq i \leq H} \left( \int_{m \cap H_i} d\beta |J(\beta, z)J_\ell(\beta, z)| \right)^2 \sum_{1 \leq j \leq H} \frac{1}{1 + |i - j|} \]
\[ \ll y \log y \sum_{1 \leq i \leq H} \left( \int_{m \cap H_i} d\beta |J(\beta, z)|^2 \int_{m \cap H_i} d\beta |J_\ell(\beta, z)|^2 \right) \]
\[ \ll yL \int_m |J(\alpha, z)|^2 d\alpha \max_{1 \leq i \leq H} \int_{m \cap H_i} d\beta |J_\ell(\beta, z)|^2. \]
For $\beta = a/q + \lambda \in m \cap H_i (1 \leq i \leq H)$, there exist $q$, $a$ and $\lambda$ satisfying $\beta = a/q + \lambda$, $L^B \leq q \leq HL^{-B}$, $|\lambda| \leq L^{-B}$ and $(q, a) = 1$. Applying Gallagher’s lemma (see [6, Lemma 1]) we have

$$\int_{m \cap H_i} d\beta |J(\beta, z)|^2 \ll \int_{|\lambda| \leq \frac{1}{H}} d\lambda \left| \sum_{z < m^\ell \leq 2^\ell z} \Lambda(m) e \left( \frac{a}{q} m^\ell \right) \left( m^\ell \lambda \right) \right|^2$$

$$\ll \frac{1}{H^2} \int_{\mathbb{R}} dx \left| \sum_{x \leq m^\ell \leq x + H/2} 1_{z < m^\ell \leq 2^\ell z} \Lambda(m) e \left( \frac{a}{q} m^\ell \right) \right|^2$$

$$= \frac{1}{H^2} \int_{z - H/2}^{2^\ell z} dx \left| \sum_{x \leq m^\ell \leq x + H/2} 1_{z < m^\ell \leq 2^\ell z} \Lambda(m) e \left( \frac{a}{q} m^\ell \right) \right|^2.$$

Namely,

$$\int_{m \cap H_i} d\beta |J(\beta, z)|^2 \ll \frac{1}{H^2} \left( \int_{z}^{2^\ell z - H/2} dx \left| \sum_{x \leq m^\ell \leq x + H/2} \Lambda(m) e \left( \frac{a}{q} m^\ell \right) \right|^2 + H(Hz^{1/2}) \right)$$

$$\ll \frac{1}{H^2} \sum_{j=0}^{\ell - 1} \int_{2^j z}^{2^{j+1} z} dx \left| \sum_{x \leq m^\ell \leq x + H/2} \Lambda(m) e \left( \frac{a}{q} m^\ell \right) \right|^2 + Hz^{2/2}. $$

Then by Lemma 2.4 and notice that $y \leq zL^{-B}$, one has

$$S_m(y, z) \ll yL(z^{2/2}L^{-A - 2} + z^{2/2}L^{-B}) \int_{0}^{1} |J(\alpha, z)|^2 d\alpha \ll yz^2L^{-A}$$

holds for any $A > 0$ if $B \geq \max(B_m, \ell(A), A) + 2$. Combining (4.1) and above, we get

$$S_m(y, z) \ll yz^{2}L^{-A}$$

holds for any $B \geq \max(B_m, \ell(A), A) + 2$. Finally, using Lemma 2.5 in place of Lemma 2.4, it is not difficult to obtain the proof of the estimate of $S'_m(y, z)$.

### 5. The major arcs

In this section we consider the estimates for $S_{2m}(y, z)$ and $S'_{2m}(y, z)$. For $S_{2m}(y, z)$, notice that the definition of $B_\ell(q, a)$ (see Lemma 2.3), the fact

$$B_1(q, -a) = \sum_{h \in \mathbb{Z}_q^*} e \left( -\frac{ah}{q} \right) = \mu(q) \text{ if } (q, a) = 1,$$
(3.3) and (3.4) implies that

\[
\int_{\mathfrak{M}} J(\alpha, z)J_\ell(\alpha, z)e(ua) \, d\alpha - \mathcal{G}_\ell(u) \sum_{z < m^\ell \leq 2^\ell z} 1
\]

\[
= \sum_{q \leq L^{\beta/2}} \sum_{a \in \mathbb{Z}_q} e\left(\frac{au}{q}\right) \int_{-L^{\beta/2}/X}^{L^{\beta}/X} d\lambda J(\alpha, z) \left( J_\ell(\alpha, z) - \frac{B_\ell(q, a)}{\varphi(q)} I_\ell(\lambda, z) \right) e(u\lambda)
\]

\[
+ \sum_{q \leq L^{\beta/2}} \sum_{a \in \mathbb{Z}_q} e\left(\frac{au}{q}\right) \frac{B_\ell(q, a)}{\varphi(q)} \int_{-L^{\beta/2}/X}^{L^{\beta}/X} d\lambda I_\ell(\lambda, z) \left( J(\alpha, z) - \frac{\mu(q)}{\varphi(q)} I(\lambda, z) \right) e(u\lambda)
\]

\[
+ \sum_{q \leq L^{\beta/2}} \sum_{a \in \mathbb{Z}_q} e\left(\frac{au}{q}\right) \frac{\mu(q)B_\ell(q, a)}{\varphi(q)^2} \int_{-L^{\beta/2}/X}^{L^{\beta}/X} d\lambda I(\lambda, z)I_\ell(\lambda, z) e(u\lambda) - \mathcal{G}_\ell(u) \sum_{z < m^\ell \leq 2^\ell z} 1.
\]

Namely,

\[
\int_{\mathfrak{M}} J(\alpha, z)J_\ell(\alpha, z)e(ua) \, d\alpha - \mathcal{G}_\ell(u) \sum_{z < m^\ell \leq 2^\ell z} 1
\]

\[
= \int_{\mathfrak{M}} d\alpha J(\alpha, z)R_\ell(\alpha, z)e(ua) + \int_{\mathfrak{M}} d\alpha R(\alpha)J_\ell(\alpha, z)e(ua)
\]

\[
+ \left( \mathcal{G}_\ell(u, L^{\beta/2}) \int_{-L^{\beta}/X}^{L^{\beta}/X} d\lambda I(\lambda, z)I_\ell(\lambda, z) e(u\lambda) - \mathcal{G}_\ell(u) \sum_{z < m^\ell \leq 2^\ell z} 1 \right)
\]

\[= I_1(u, z) + I_2(u, z) + I_3(u, z),\]

where

\[
R(\alpha) = \sum_{m \leq 2^\ell z} \Lambda(m)e(-m\alpha) - \frac{\mu(q)}{\varphi(q)} \sum_{m \leq 2^\ell z} e(-m\lambda)
\]

with \(\alpha = a/q + \lambda\). Therefore, we have

\[
S_{\mathfrak{M}}(y, z) = \sum_{u \leq y} |I_1(u, z) + I_2(u, z) + I_3(u, z)|^2
\]

\[\ll \sum_{u \leq y} \left( |I_1(u, z)|^2 + |I_2(u, z)|^2 + |I_3(u, z)|^2 \right)\]

by Cauchy’s inequality. Similarly, we obtain that

\[
S_{\mathfrak{M}}'(y, z) \ll \sum_{u \leq y} \left( |I_1'(u, z)|^2 + |I_2'(u, z)|^2 + |I_3'(u, z)|^2 \right),
\]

where

\[
I_1'(u, z) = \int_{\mathfrak{M}} d\alpha J(\alpha, z)R_\ell'(\alpha, z)e(ua) \quad I_2'(u, z) = \int_{\mathfrak{M}} d\alpha R(\alpha)I_\ell(\alpha, z)e(ua)
\]
and
\[ \mathcal{I}_3'(u, z) = \mathcal{G}'(u, L^B) \int_{-\frac{L^B}{X}}^{\frac{L^B}{X}} d\lambda I(\lambda, z)I_\ell(\lambda, z)e(u\lambda) - \mathcal{G}'(u) \sum_{z < \ell^\ell < 2^\ell z} 1. \]

We have firstly
\[ \sum_{u \leq y} |\mathcal{I}_1(u, z)|^2 = \sum_{u \leq y} \left| \sum_{q \leq L^B} \sum_{a \in \mathbb{Z}^q} e\left(\frac{au}{q}\right) \int_{-\frac{L^B}{X}}^{\frac{L^B}{X}} d\lambda J(\lambda, z)R(\alpha, z)e(u\lambda) \right|^2 \leq \sum_{u \leq y} \left| \sum_{q \leq L^B} \varphi(q)X \sup_{\alpha \in \mathbb{R}} |R(\alpha, z)| X^{-1}L^B \right|^2 \ll_B yz^2 L^{-B^2}
\]
by Lemma 2.3. Also, from lemma 2.3 we obtain that
\[ \sum_{u \leq y} |\mathcal{I}_2(u, z)|^2 = \sum_{u \leq y} \left| \sum_{q \leq L^B} \sum_{a \in \mathbb{Z}^q} e\left(\frac{au}{q}\right) \int_{-\frac{L^B}{X}}^{\frac{L^B}{X}} d\lambda J(\lambda, z)R(\alpha, z)e(u\lambda) \right|^2 \ll yz^2 \sum_{q \leq L^B} \frac{L^B}{X} \sup_{\alpha \in \mathbb{R}} \max_{z \leq 2X} |R_1(-\alpha, z)| X^{-1} \ll_B yz^2 L^{-B^2}. \]

Note that \( I(\lambda, z) \ll |\lambda|^{-1} \), Hua’s inequity (see [8, Theorem 4])
\[ \int_0^1 |I_\ell(\lambda, z)|^{2\ell} d\lambda \ll \ell z^{-\frac{2\ell}{\ell - 1}} L^{c(\ell)}, \]
where \( c(\ell) \) is an absolute constant depending only on \( \ell \). Then the using of Hölder’s inequality gives
\[ \int_{-\frac{L^B}{X} < |\lambda| \leq \frac{1}{2}} d\lambda |I(\lambda, z)I_\ell(\lambda, z)| \ll \left( \int_{-\frac{L^B}{X} < |\lambda| \leq \frac{1}{2}} d\lambda |I(\lambda, z)|^{2\ell} \right)^{\frac{2\ell-1}{2\ell}} \left( \int_0^1 d\lambda |I_\ell(\lambda, z)|^{2\ell} \right)^{\frac{1}{2\ell}} \ll_{\ell} \left( L^B / X \right)^{(1-\frac{2\ell}{2\ell - 1})\frac{2\ell-1}{2\ell}} \left( \frac{z^\ell}{z^\ell - 1} L^{c(t)} \right)^{\frac{1}{2\ell}} \ll_{\ell} z^{-\frac{1}{2\ell}} L^{-\frac{B-c(t)}{2\ell}}. \]

On the other hand,
\[ \int_{-\frac{1}{2}}^{\frac{1}{2}} d\lambda I(\lambda, z)I_\ell(\lambda, z)e(u\lambda) = z\frac{1}{\ell} + O(1). \]
Setting $B \geq \max(2, c(\ell))$, we obtain that
\[
\sum_{u \leq y} |T_3(u, z)|^2 = \sum_{u \leq y} \left| \mathcal{G}_\ell(u, L^B) \int_{|\lambda| \leq z_y} \dalpha I(\alpha, z)I_\ell(\alpha, z) e(u\alpha) - \mathcal{G}_\ell(u) \sum_{z < m \leq 2z_y} 1 \right|^2 
\ll z^{\frac{7}{2}} \sum_{u \leq y} \left| \mathcal{G}_\ell(u, L^B) - \mathcal{G}_\ell(u) \right|^2 + z^{\frac{7}{2}} L^{-\frac{B-\delta(l)}{2\ell}} \sum_{u \leq y} |\mathcal{G}_\ell(u)|^2.
\]
We can conclude from the above estimates that
\[
(5.1) \quad S_{2\Re}(y, z) \ll yz^2 L^{-B} + z^2 \sum_{u \leq y} \left| \mathcal{G}_\ell(u, L^B) - \mathcal{G}_\ell(u) \right|^2 + z^{\frac{7}{2}} L^{-\frac{B-\delta(l)}{2\ell}} \sum_{u \leq y} |\mathcal{G}_\ell(u)|^2
\]
for $B \geq \max(2, c(\ell))$.

We now prove the following crude estimates for $\mathcal{G}_\ell(u)$ and $\mathcal{G}_\ell'(u)$.

**Lemma 5.1.** For all integer $|u| \in (0, X]$, we have
\[
\mathcal{G}_\ell(u) \ll \ell L^{5\ell} \quad \text{and} \quad \mathcal{G}_\ell'(u) \ll \ell L^{5\ell}.
\]

**Proof.** We just prove the estimate for $\mathcal{G}_\ell(u)$, the proof for $\mathcal{G}_\ell'(u)$ is similar. Note that $0 \leq \varrho_\ell(p, u) \leq \ell$, we have
\[
\mathcal{G}_\ell(u) = \prod_{p | u} \frac{1}{p - 1} \prod_{p | u, p \geq \ell} \left( 1 + \frac{1}{p - 1 - \ell} \right) \prod_{p > 2\ell} \left( 1 - \frac{\varrho_\ell(p, u) - 1}{p - 1} \right)
\ll \ell \exp \left( \sum_{p | u, p > 2\ell} \frac{1}{p} \right) \exp \left( \sum_{p > 2\ell} \frac{1 - \varrho_\ell(p, u)}{p} \right).
\]
Then by Lemma 2.2, we trivially have
\[
\mathcal{G}_\ell(u) \ll \ell \exp \left( \log \log |u| + 4\ell \log \log (2|u|) + O_\ell(1) \right) \ll (\log (2|u|))^{5\ell} \ll L^{5\ell}
\]
holds for all $|u| \in [1, X] \cap \mathbb{Z}$. Which complete the proof of the lemma.

**Lemma 2.1.** Lemma 5.1, (5.1) and the crude estimates $\mathcal{G}_\ell(u, x) \ll x$ for $x > 0$ implies that
\[
S_{2\Re}(y, z) \ll yz^2 L^{-B} + z^2 L^{4B+10\ell} + z^2 \sum_{L^{2B} < u \leq y} \left| \mathcal{G}_\ell(u, L^B) - \mathcal{G}_\ell(u) \right|^2 + yz^2 L^{-\frac{B-\delta(l)}{2\ell} + 10\ell}.
\]
Notice that $y \geq z^\delta \geq (X/L^B)^\delta \gg B, X^{\delta/2}$, we have
\[
(5.2) \quad S_{2\Re}(y, z) \ll \ell, B \ yz^2 L^{-\frac{B-\delta(l)}{2\ell} + 10\ell} + z^2 \sum_{L^{2B} < u \leq y} \left| \mathcal{G}_\ell(u, L^B) - \mathcal{G}_\ell(u) \right|^2.
\]
Similarly, we have
\[
S'_{2\Re}(y, z) \ll \ell, B \ yz^2 L^{-\frac{B-\delta(l)}{2\ell} + 10\ell} + z^2 \sum_{L^{2B} < u \leq y} \left| \mathcal{G}_\ell'(u, L^B) - \mathcal{G}_\ell'(u) \right|^2.
\]
From Lemma 2.1 we obtain the estimate for $S'_{2\Re}(y, z)$ immediately, say
\[
S'_{2\Re}(y, z) \ll \ell, B \ yz^2 L^{-\frac{B-\delta(l)}{2\ell} + 10\ell} + z^2 L y (L^B)^{-\frac{1}{2000\ell^2}} \ll \ell, A \ yz^2 L^{-A}
\]
by setting $B = \max(2000\ell^2(1 + A), 2^\ell(10\ell + A) + c(\ell))$. For get the estimate for $S_{2M}(y, z)$, we need the following lemma.

Lemma 5.2. Let positive real numbers $x$ and $y$ be sufficiently large. We have

$$\sum_{u \leq y} |\mathcal{P}_\ell(u, x) - \mathcal{S}_\ell(u, x)|^2 \ll \ell yx^{-1} \log^2 x + x^{4\log x}$$

and

$$\sum_{u \leq y} |\mathcal{P}_\ell'(u, x) - \mathcal{S}_\ell'(u, x)|^2 \ll \ell yx^{-1} \log^2 x + x^{4\log x}.$$  

Proof. We denote by $P(x) = \prod_{p \leq x} p$, $S(u, x) = \mathcal{P}_\ell(u, x) - \mathcal{S}_\ell(u, x)$ and let $V > x$. Clearly,

$$S(u, x) = \sum_{x < q \leq V} \frac{\mu(q)}{\varphi(q)} A(q, u) + \sum_{q > V \atop q | P(x)} \frac{\mu(q)}{\varphi(q)} A(q, u).$$

Let $\lambda_x = \log^{-1} x > 0$. We have the following estimate

$$\sum_{q > V \atop q | P(x)} \frac{\mu(q)}{\varphi(q)} A(q, u) \ll \sum_{q > V \atop q | P(x)} \frac{\mu(q)^2}{\varphi(q)} |A(q, u)| \leq \frac{1}{V^{\lambda_x}} \sum_{q | P(x)} \frac{q^{\lambda_x} \mu(q)^2}{\varphi(q)} |A(q, u)|$$

$$= \frac{1}{V^{\lambda_x}} \prod_{p \leq x} \left(1 + \frac{p^{\lambda_x} |A(p, u)|}{p - 1}\right) \leq \frac{1}{V^{\lambda_x}} \prod_{p \leq x} \left(1 + \frac{e |A(p, u)|}{p - 1}\right).$$

Setting $V = \exp(\log^2 x)$ and notice that

$$A(p, u) = \begin{cases} \frac{p - 1}{p - 1}(\varphi(p) - 1) & u \equiv 0 \pmod{p} \\ \frac{p}{p - 1}(\varphi(p) - 1) + \frac{1}{p - 1} & u \not\equiv 0 \pmod{p} \end{cases}$$

we obtain

$$|A(p, u)| \leq \frac{fp}{p - 1}.$$  

Therefore we get

$$\sum_{u \leq y} \left|\sum_{q > V \atop q | P(x)} \frac{\mu(q)}{\varphi(q)} A(q, u)\right|^2 \ll \frac{1}{x^2} \prod_{p \leq x} \left(1 + \frac{e}{p}\right)^2 \ll yx^{-2} \log^{2\ell} x.$$  

One the other hand,

$$\sum_{u \leq y} \left|\sum_{x < q \leq V \atop q | P(x)} \frac{\mu(q)}{\varphi(q)} A(q, u)\right|^2 = \sum_{u \leq y} \sum_{x < q \leq V \atop q \mid P(x)} \frac{\mu(q_1)\mu(q_2)}{\varphi(q_1)\varphi(q_2)} A(q, u)\overline{A(q, u)}$$

$$= \sum_{u \leq y} \sum_{x < q \leq V \atop q \mid P(x)} \frac{\mu(q_1)^2}{\varphi(q_1)^2} |A(q, u)|^2 + T_R(x)$$

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with
\[ T_R(x) = \sum_{x < q_1, q_2 \leq x} \sum_{q_1, q_2 \mid P(x)} \frac{\mu(q_1)\mu(q_2)}{\varphi(q_1)^2\varphi(q_2)^2} \sum_{\substack{a_1 \in \mathbb{Z}_{q_1}^+ \\ a_2 \in \mathbb{Z}_{q_2}^+}} e \left( \frac{a_1h_1}{q_1} - \frac{a_2h_2}{q_2} \right) \sum_{u \leq y} \left( \frac{a_1q_2 - a_2q_1}{q_1q_2} \right) \]

\[ \leq \sum_{x < q_1, q_2 \leq V} \sum_{q_1, q_2 \mid P(x)} \frac{\mu(q_1)^2\mu(q_2)^2}{\varphi(q_1)^2\varphi(q_2)^2} \sum_{\substack{a_1 \in \mathbb{Z}_{q_1}^+ \\ a_2 \in \mathbb{Z}_{q_2}^+}} \left( \sum_{j=1,2} h_j \right)^2 \left( \frac{|q_1q_2|}{(q_1q_2)^2} \right) \]

\[ \leq \sum_{x < q_1, q_2 \leq V} q_1q_2 \left( \sum_{x < q \leq V, q \mid P(x)} q \right)^2 \ll V^4 \leq x^{4\log x}, \]

where the obvious fact \( q_1q_2 \mid (a_1q_2 - a_2q_1) \) has been used. Moreover,
\[ \sum_{x < q \leq V, q \mid P(x)} \frac{\mu(q)^2}{\varphi(q)^2} |A(q, u)|^2 \ll x^{-1} \sum_{x < q < V} \frac{\mu(q)^2}{\varphi(q)^2} |A(q, u)|^2 \ll x^{-1} \sum_{q \mid P(x)} \frac{\mu(q)^2}{\varphi(q)^2} |A(q, u)|^2 \]

\[ \ll x^{-1} \prod_{p \leq x} \left( 1 + \frac{p|A(p, u)|^2}{(p-1)^2} \right) \ll x^{-1}\log^2 x. \]

Hence we obtain that
\[ \sum_{u \leq y} |S(u, x)|^2 \ll \epsilon xy^{-1}\log^2 x + x^{4\log x} + yx^{-2}\log^2\epsilon x \ll \epsilon xy^{-1}\log^2 x + x^{4\log x}. \]

Similarly,
\[ \sum_{u \leq y} |\mathcal{P}_\epsilon(u, x) - \mathcal{S}_\epsilon(u, x)|^2 \ll \epsilon xy^{-1}\log^2 x + x^{4\log x}. \]

Which completes the proof of the lemma.

Under Lemma 5.2, we have the following estimate for \( \mathcal{S}_\epsilon(u, L^B) \).

**Lemma 5.3.** Let \( y \leq X \) be sufficiently large. We have
\[ \sum_{L^{2B} \leq u \leq y} |\mathcal{S}_\epsilon(u, L^B) - \mathcal{S}_\epsilon(u)|^2 \ll \epsilon yL^{-B/(2000\epsilon^2) + 10}\epsilon + 3. \]

**Proof.** First of all, by Lemma 5.2 it is clear that
\[ \sum_{L^{2B} \leq u \leq y} |\mathcal{S}_\epsilon(u, L^B) - \mathcal{S}_\epsilon(u)|^2 \ll \epsilon yL^{-B + 1} + \sum_{L^{2B} \leq u \leq y} |\mathcal{S}_\epsilon(u) - \mathcal{P}_\epsilon(u, L^B)|^2. \]

Note that
\[ \mathcal{P}_\epsilon(u, x) = \mathcal{P}_\epsilon(u, x)f_\epsilon(u, x), \]
where
\[ f_\epsilon(u, x) = \prod_{p|u, p \leq x} \left( 1 - \frac{1}{p - \epsilon(p, u)} \right)^{-1} \prod_{p \leq x} \left( 1 - \frac{q_\epsilon(p, u)}{(p-1)(p - \epsilon(p, u))} \right). \]

Let \( x \to \infty \), then
\[ \mathcal{S}_\epsilon(u) = \mathcal{S}_\epsilon(u)f_\epsilon(u), \]
where \( f_\epsilon(u) = \lim_{x \to \infty} f_\epsilon(u, x) \). It is easily seen that
\[ f_\epsilon(u, x) \ll \epsilon \log(|u| + 2) \]

\[ \therefore \sum_{L^{2B} \leq u \leq y} |\mathcal{S}_\epsilon(u, L^B) - \mathcal{S}_\epsilon(u)|^2 \ll \epsilon yL^{-B + 1} + \sum_{L^{2B} \leq u \leq y} |\mathcal{S}_\epsilon(u) - \mathcal{P}_\epsilon(u, L^B)|^2. \]
for all \( x > 0 \) and integer \( u \neq 0 \). Hence by Lemma 5.1, we obtain
\[
\sum_{L^{2B} < u \leq y} |\mathfrak{P}_\ell(u, L^B) - \mathfrak{S}_\ell(u)|^2 = \sum_{L^{2B} < u \leq y} |\mathfrak{P}'_\ell(u, L^B)f_\ell(u, L^B) - \mathfrak{S}'_\ell(u)f_\ell(u)|^2 \\
\ll L^2 \sum_{L^{2B} < u \leq y} |\mathfrak{P}'_\ell(u, L^B) - \mathfrak{S}'_\ell(u)|^2 + L^{10\ell}R_f.
\]

where it is not difficult prove that
\[
R_f = \sum_{u \leq y} \left| f_\ell(u, L^B) - f_\ell(u) \right|^2 \ll_\ell yL^{-B+2}.
\]

By Lemma 5.2, we obtain that
\[
\sum_{L^{2B} < u \leq y} \left| \mathfrak{P}'_\ell(u, L^B) - \mathfrak{S}'_\ell(u) \right|^2 \ll_\delta,\ell, B yL^{-B+1} + \sum_{L^{2B} < u \leq y} \left| \mathfrak{S}'_\ell(u) - \mathfrak{S}'_\ell(u, L^B) \right|^2.
\]

Then, the following is obvious by Lemma 2.1. \( \square \)

Finally, using Lemma 5.3 and setting \( B = \max(2000\ell^2(12\ell + A), 2^\ell(10\ell + A) + c(\ell)) \) in (5.2) completes the proof of Lemma 3.2.

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