Electromagnetic radiative corrections to pionic beta decay $\pi^+ \rightarrow \pi^0 e^+ \nu_e$

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Pionic beta decay $\pi^+ \rightarrow \pi^0 e^+ \nu_e$ is analyzed in chiral perturbation theory with virtual photons and leptons. All electromagnetic corrections up to $O(\alpha^2)$ are taken into account. Theoretical results are confronted with preliminary data from a PSI measurement and a value for the CKM matrix element $|V_{ud}|$ is given. Although the precision is presently still below the one of existing determinations of $|V_{ud}|$, an analysis of pionic beta decay, based on a systematic treatment within a low-energy effective field theory, may become a useful alternative.

1. Introduction

One of the cornerstones of the Standard Model is the CKM matrix $V$. Its matrix elements $V_{ij}$ determine the strengths of transitions between quark flavours $i$ and $j$. The $V_{ij}$ are therefore of eminent importance, and their precise determination is a major task of today’s particle physics.

Unitarity implies a series of relations among the matrix elements. In particular, one expects the first-row CKM matrix elements to satisfy

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1, \quad (1)$$

where mixing among the first-generation quarks $u$ and $d$ is strongest, and mixing between the first and third generation is very small and can be neglected at the present precision [1].

The strength of an $s$ to $u$ transition is given by $V_{us}$ which is measured in $K^{0}\bar{K}^{0}$ decays. Taking into account a recent analysis of radiative corrections in $K^{0}\bar{K}^{0}$ decays in chiral perturbation theory [2], the Particle Data Group (PDG) leaves the central value of $|V_{us}|$ unchanged but increases the error and gives finally $|V_{us}| = 0.2196 \pm 0.0026$ [3].

The most precise up-to-date knowledge of $|V_{ud}|$ comes either from analyses of nuclear beta decays, in particular super-allowed Fermi decays, or (not as precise) from neutron decay.

From beta decays of nuclei one finds $|V_{ud}| = 0.9740 \pm 0.0005$, where the uncertainty depends on structure-dependent radiative corrections [4].

Interactions with the nuclear medium could give the quarks different effective masses, which might lead to an enhanced value of $|V_{ud}|$. The PDG compensates for this possibility by doubling the aforementioned error [1].

An extraction of $|V_{ud}|$ from neutron decay benefits on one hand from fewer theoretical problems, but suffers at the same time from the fact that not only the neutron lifetime but also the ratio of axial-vector and vector couplings $g_A/g_V$ comes into play. Averaging over neutron decay experiments (see [1] and references therein), the PDG gives $|V_{ud}| = 0.9725 \pm 0.0013$, and a final combined value of $|V_{ud}| = 0.9734 \pm 0.0008$ with a remarkably small error is found [1].

With these values one finds a $2.2\sigma$ deviation from unitarity in [1]: $|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 - 1 = -0.0042 \pm 0.0019$. Adopting a conservative line of reasoning, one may speculate if the given error of $|V_{ud}|$ might be too optimistic, and look for a cleaner theoretical alternative to determine $|V_{ud}|$.

Pionic beta ($\pi\beta$) decay $\pi^+ \rightarrow \pi^0 e^+ \nu_e$ combines the advantages of the traditional approaches, i.e., it is a pure vector transition without any complications due to nuclear structure. However, practical utilization of $\pi\beta$ decay is severely complicated by its branching ratio of $\text{BR}_{\pi\beta} \sim 1 \times 10^{-8}$ [1].

In spite of the rareness of the decay, a measurement of $\text{BR}_{\pi\beta}$ is being performed at PSI, aiming to achieve a precision of 0.5% in a first stage [1]. Of course, this first accuracy will not be enough to compete with nuclear or neutron decays, but a third independent determination of $|V_{ud}|$ may certainly be useful.

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At this level of accuracy radiative corrections have to be included in the theoretical analysis, and we employ chiral perturbation theory as the ideal framework for low-energy hadron physics. This article is based on [3] which in turn is based on recent work on $K_{l3}$ decays [2].

2. Kinematics and decay amplitude

Neglecting radiative corrections for the moment, the decay amplitude of $\pi\beta$ decay

$$\pi^+(p_+) \rightarrow \pi^0(p_0) \ e^+(p_e) \ \nu_e(p_\nu)$$

is written in terms of two form factors $f_\pm^{(0)}$:

$$\mathcal{M} = G_F V_{ud}^* \rho_y \left[ (p_\mu^+ + p_\mu_0^0) f_+^{(0)} + (p_\mu^- - p_\mu_0^0) f_-^{(0)} \right] .$$

where $l_\mu = \bar{u}(p_\nu)\gamma_\mu(1 - \gamma_5)v(p_e)$ is the lepton current. $G_F$ is Fermi’s coupling constant.

At leading order, $f_+^{(0)} = 1$ and $f_-^{(0)} = 0$. At next-to-leading order, the form factors develop a dependence on the variable $t = (p_+ - p_0)^2$.

$f_-^{(0)}$ turns out to be proportional to the difference of squared pion masses $(m_{\pi^+}^2 - m_{\pi^0}^2)$. Additionally, $f_-^{(0)}$ terms in the squared decay amplitude come always along with a factor of $r_e = m_e^2/m_{\pi^+}^2 \simeq 1.35 \times 10^{-5}$. These observations justify to neglect $f_-^{(0)}$ in the amplitude.

The decay rate $\Gamma_{\pi\beta}$ for $\pi\beta$ decay,

$$\Gamma_{\pi\beta} = \int_\mathcal{D} dy \ dz \ \rho^{(0)}(y, z) ,$$

is expressed in terms of a phase space density $\rho^{(0)}$ which depends on two variables $y$ and $z$ defined in the c.m.s. by

$$y = \frac{2E_e}{m_{\pi^+}} \text{ and } z = \frac{2E_{\pi^0}}{m_{\pi^+}} .$$

$\mathcal{D}$ denotes the kinematically allowed area of $(y, z)$. The phase space density itself is given by

$$\rho^{(0)}(y, z) = N \times |f_+^{(0)}(t)|^2 A^{(0)}(y, z) ,$$

where $A^{(0)}(y, z)$ is a kinematical density

$$A^{(0)}(y, z) = \left[ 4(-1 + z + 2y - yz - y^2) - 4r_0 
+ r_e(4y + 3z - 3 + r_0 - r_e) \right] ,$$

and $N$ in (5) and $r_0$ in (8) are defined by

$$N = \frac{|V_{ud}|^2 G_F^2 m_{\pi^+}^5}{64\pi^3} , \quad r_0 = m_{\pi^0}^2/m_{\pi^+}^2 .$$

At $\mathcal{O}(p^4)$, meson loops and counterterms from the chiral Lagrangian $\mathcal{L}_1$ generate the aforementioned $t$-dependence $f_+^{(0)}(t) = 1 + \delta f_+^{(0)}(t)$, $\delta f_+^{(0)}(t) = \frac{1}{F^2} \left[ 2tL_6^{(\mu)}(\mu) + 2h_2^r \right.
+ \left. h_4^r(t, \mu) \right]$.

The functions $h_2^r(t, \mu)$ comprise one-loop corrections from the particles $P$ and $Q$ [4]. Both the renormalized low-energy coupling (LEC) $L_6^{(\mu)}$ and $h_4^{r}(t, \mu)$ depend individually on the renormalization scale, but $\delta f_+^{(0)}(t)$ is independent of $\mu$.

3. Radiative corrections

Radiative corrections manifest themselves in virtual photon exchange, new electromagnetic counterterms, and in the emission of real photons.

Photon exchange generates one-loop diagrams as a result of which a new dynamical variable $u = (p_+ - p_e)^2$ and infrared divergences appear in the amplitude $\mathcal{M}$. Counterterms from Lagrangians $\mathcal{L}_c^{\pi^\pm}$ and $\mathcal{L}_{\text{opt}}$ contribute three new LECs $X_1, X_6^{(\mu)}(\mu)$, and $K_{12}^{(\mu)}$.

Virtual radiative corrections induce a change of the form factor from $f_+^{(0)}(t)$ to $f_+^{(t, u, \lambda)} = f_+^{(0)}(t) + \delta f_+^{\text{em}}(u, \lambda)$, where $\lambda$ plays the role of a fictitious photon mass to regularize infrared divergences.

Only showing local contributions explicitly and omitting remaining one-loop contributions, the radiative corrections of $\mathcal{O}(\alpha)$ read

$$\delta f_+^{\text{em}}(u, \lambda) = \delta f_+^{\text{em}}(u, \lambda) + \delta f_+^{\text{em}}(u, \lambda) ,$$

$$\delta f_+^{\text{em}} = 4\pi\alpha \left\{ \frac{2}{3} X_1 - \frac{1}{2} X_6^{(\mu)}(\mu) + 2K_{12}^{(\mu)} 
- \frac{1}{32\pi^2} \left[ 3 + \ln \frac{m_e^2}{m_{\pi^+}^2} + 3 + \ln \frac{m_{\pi^+}^2}{m_e^2} \right] \right\} .$$

As before, $\delta f_+^{\text{em}}(u, \lambda)$ is independent of the renormalization scale.

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1Jaus also [2] addressed the problem of radiative corrections to $\pi\beta$ decay in a constituent quark model.

2$\mathcal{O}(c^2p^0)$ corrections are absorbed in the meson masses.

3The omitted one-loop part $\delta f_+^{(t, u, \lambda)}$ containing in particular the infrared divergences can be found in [4].
As it was shown in [2], it is useful to rearrange the complete form factor in the following way:

$$f_+(t, u, \lambda) = F_+(t) \times \left[ 1 + \delta f_{\text{em}}^{\text{ren}}(u, \lambda) \right] ,$$

(11)

by factoring-out the $t$-dependence in $F_+(t) \equiv f_+^{(0)}(t) + \delta f_{\text{em}}^{\text{ren, local}}$ and isolating both infrared divergences and $u$-dependence. This allows us to write the rate in the presence of virtual radiative corrections in combination with real photon emission formally in exactly the same way as in [1], (6), i.e., in terms of a generalized form factor $F_+(t)$ and a generalized infrared finite kinematical density.

The treatment of real photon emission and cancellation of infrared divergences was discussed in detail in [2]. Working at order $\alpha$, it is sufficient to consider the emission of only one real photon. We adopt a scheme proposed by Ginsberg [11] and integrate over the entire phase space of undetected particles, i.e., the photon and the neutrino, but restrict the $(y, z)$ space to the non-radiative three-particle decay case. Furthermore, a new variable $x = (p_\nu + p_\gamma)^2$ is defined and integrated over.

The phase space density for the radiative decay $\pi^+ \to \pi^0 e^+ \nu_e \gamma$ is given by

$$\rho^\gamma(y, z) = \frac{m_{\pi^+}}{2\pi^3 \rho_\pi} \int_x^{x_{\text{max}}} dx \int \frac{d^3 p_\mu}{E_\mu} \frac{d^3 p_\gamma}{E_\gamma} \times \delta(4)(p_\mu - p_0 - p_\nu - p_\gamma) \sum_{\text{pol}} |\mathcal{M}^\gamma|^2 ,$$

(12)

where $\mathcal{M}^\gamma$ denotes the leading-order amplitude of the radiative decay, respectively. Upon combining the phase space densities of the mother process and the associated radiative decay, one arrives at an infrared finite phase space density

$$\rho(y, z) = \rho^{(0)}(y, z) + \rho^\gamma(y, z)$$

(13)

which we may write as already anticipated, as

$$\rho(y, z) = \mathcal{N} \times |F_+(t)|^2 \times A(y, z) ,$$

(14)

in terms of a generalized kinematical density $A(y, z)$ and the form factor $F_+(t)$. Apart from electromagnetic local contributions, radiative corrections from virtual photon exchange [2] and real photon emission [2] are entirely included in $A(y, z) [2,3].$

4. Results and conclusions

Before presenting our results, a few comments seem advisable. In general, low-energy couplings of chiral Lagrangians parametrize short-distance physics that is not explicitly dealt with in the effective theory.

According to [12], it turns out that all semileptonic charged current amplitudes are affected by universal short-distance physics corrections when expressed in terms of the muon decay constant. Relating therefore $G_F$ with the muon lifetime, it is possible to trace back some of the high-energy origin of the LEC $X_6$ to these universal corrections [2]. Hence, we may write $X_6^\prime(\mu) = X_6^{\text{SD}} + X_6^\prime(\mu)$, and define $e^2 X_6^{\text{SD}} = 1 - S_{\text{ew}}(m_\mu, m_Z)$ to contain the short-distance enhancement factor $S_{\text{ew}}(m_\mu, m_Z) = 1.0232$ from [2]. The remaining part $X_6^\prime(\mu)$ is now expected to be of the order of $\sim 10^{-3}$. Extracting the short-distance physics from $\rho(y, z)$ and expanding until $\mathcal{O}(t)$, we may write the infrared safe inclusive decay rate as

$$\Gamma_{\pi^+B(\gamma)} = \mathcal{N} S_{\text{ew}}(m_\mu, m_Z) \times |F_+(0)|^2 I(\lambda_+) .$$

(15)

The linear approximation works extremely well, since $m_e^2 \leq t \leq (m_{\pi^+} - m_{\mu})^2 \simeq 21.1 \text{ MeV}^2$.

$$I(\lambda_+) = \int_D dy dz A(y, z) \left[ 1 + \frac{t}{m_{\pi^+}} \chi_+ \right]$$

(16)

contains the slope parameter $\chi_+$ from the expansion in $t$.

To estimate $F_+(0)$, we use the ‘classical’ value of $L_5^\prime(m_\rho) = (6.9 \pm 0.7) \times 10^{-3}$, quoted in [2], and take Moussallam’s estimate [13] of $K_{12}^\prime(m_\rho) = (-4.0 \pm 0.5) \times 10^{-3}$.

Since essentially nothing is known about the $X_1$, we resort to naive dimensional analysis and use $|X_1|, |\tilde{X}_6^\prime(m_\rho)| \leq 6.3 \times 10^{-3}$.

Finally, $F_+(0)$ is found to be $1.0046 \pm 0.0005$. The error is completely due to uncertainties in the electromagnetic LECs and is extremely small. The slope parameter’s uncertainty comes from the error of $L_5^\prime(m_\rho)$, and we obtain $\chi_+ = (0.037 \pm 0.003)$.

4In a combined analysis of $K_{12}^\prime$ and $K_{13}^\prime$ decays it is possible to extract a value for $X_1$ [14].
The phase space factor $I(\lambda_\pi)$ is calculated to be $I(\lambda_\pi) \simeq 7.3832 \times 10^{-8}$ with a tiny error.

We arrive at the following results. Radiative corrections turn out to be at the $\leq 1\%$ level. The corrected phase space $I(\lambda_\pi)$ enhances the rate by $\sim 0.1\%$, the form factor $F_{+}(0)$ by $\sim 0.9\%$.

Using the PDG’s value of $|V_{ud}| = 0.9734 \pm 0.0008$, we arrive at a $\pi\beta$ branching ratio of

$$BR_{\pi\beta} = (1.0376 \pm 0.0003) \times 10^{-8}. \quad (17)$$

Turning the procedure upside down, we use the $\pi\beta$ branching ratio together with our calculation of radiative corrections to extract

$$|V_{ud}| = 9600.8 \times \sqrt{\frac{BR_{\pi\beta}(\gamma)}{F_{+}(0)}}. \quad (18)$$

The error assigned to our value of $|V_{ud}|$ is

$$\Delta|V_{ud}| = |V_{ud}| \times \left| \frac{\Delta F_{+}(0)}{F_{+}(0)} + \frac{1}{2} \Delta BR_{\pi\beta}(\gamma) \right|. \quad (19)$$

For the PDG’s value of the branching ratio $BR_{PDG} = (1.025 \pm 0.034) \times 10^{-8}$, we get

$$|V_{ud}|_{\text{PDG}} = 0.9675 \pm 0.0005\,(\text{th}) \pm 0.017\,(\text{exp}). \quad (20)$$

Obviously, the theory is very well under control and it is the experiment that limits the precision.

The situation is much improved if one uses new PSI data on $\pi\beta$ decay [4]. With the recent, however still preliminary, branching ratio of $BR_{PSI} = (1.044 \pm 0.016) \times 10^{-8}$, we find

$$|V_{ud}|_{\text{PSI}} = 0.9765 \pm 0.0005\,(\text{th}) \pm 0.0075\,(\text{exp}). \quad (21)$$

From the theorist’s point of view, we wish to emphasize that the inclusion of radiative corrections is extremely straightforward and particularly simple. The remarkably small theoretical error can practically entirely be attributed to the low-energy couplings $X_1$ and $X_7'(m_{\rho})$.

Moreover, the approach is completely equivalent to the one employed in $K_{\ell 3}$ decays [2], which puts both processes on an equally sound footing. Note finally that the central value of $|V_{ud}|$ in (21) matches the unitarity constraint [1] very well, but the error is, of course, much too large to allow for more stringent final conclusions.

It is fair to say that the present experimental precision of $\pi\beta$ decay cannot yet compete with nuclear beta decays, and even if the final precision will drastically increase it remains rather unlikely to reach a competitive accuracy. Nevertheless, $\pi\beta$ decay could become a valuable alternative source for the determination of $|V_{ud}|$, in particular if previous error estimates turned out to have been too optimistic.

REFERENCES

1. The Particle Data Group, K. Hagiwara et al., Phys. Rev. D66, 010001 (2002).
2. V. Cirigliano, M. Knecht, H. Neufeld, H. Rupertsberger and P. Talavera, Eur. Phys. J. C23, 121 (2002).
3. W.K. McFarlane et al., Phys. Rev. D32, 547 (1985).
4. PIBETA Collaboration, M. Bychkov et al., PSI Scientific Report 2001, Vol.1, p. 8, eds. J. Gobrecht et al., Villigen PSI (2002).
5. V. Cirigliano, M. Knecht, H. Neufeld and H. Pichl, hep-ph/0209228.
6. W. Jaus, Phys. Rev. D63, 053009 (2001).
7. J. Gasser and H. Leutwyler, Nucl. Phys. B250, 465 (1985); ibid. B250, 517 (1985).
8. R. Urech, Nucl. Phys. B433, 234 (1995).
9. M. Knecht, H. Neufeld, H. Rupertsberger and P. Talavera, Eur. Phys. J. C12, 469 (2000).
10. E.S. Ginsberg, Phys. Rev. 162, 1570 (1967); ibid. 187, 2280(E) (1969).
11. A. Sirlin, Rev. Mod. Phys. 50, 573 (1978).
12. W.J. Marciano and A. Sirlin, Phys. Rev. Lett. 71, 3629 (1993).
13. B. Moussallam, Nucl. Phys. B504, 381 (1997).
14. V. Cirigliano, $|V_{us}|$ and $|V_{ud}|$ from Semileptonic Decays of K and $\pi$, talk presented at DPF 2002, May 24-28 2002, Williamsburg, Virginia.