A SPECTRAL MULTIPLIER THEOREM ASSOCIATED WITH A SCHRÖDINGER OPERATOR

YOUNG HUN HONG

Abstract. We establish a spectral multiplier theorem associated with a Schrödinger operator \( H = -\Delta + V(x) \) in \( \mathbb{R}^3 \). We present a new approach employing the Born series expansion for the resolvent. This approach provides an explicit integral representation for the difference between a spectral multiplier and a Fourier multiplier, and it allows us to treat a large class of Schrödinger operators without Gaussian heat kernel estimates. As an application to nonlinear PDEs, we show the local-in-time well-posedness of a 3d quintic nonlinear Schrödinger equation with a potential.

1. Introduction

1.1. Statement of the main theorem. We establish a spectral multiplier theorem associated with a Schrödinger operator \( H = -\Delta + V(x) \) in \( \mathbb{R}^3 \) for a large class of short-range potentials \( V(x) \). Precisely, we assume that \( V \in \mathcal{K}_0 \cap L^{3/2,\infty} \), where \( \mathcal{K}_0 \) is the norm closure of bounded, compactly supported functions with respect to the global Kato norm

\[
\|V\|_{\mathcal{K}} := \sup_{x \in \mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|V(y)|}{|x - y|} dy,
\]

and \( L^{3/2,\infty} \) denotes the standard weak \( L^{3/2} \)-space. We also assume that \( H \) has no eigenvalue or resonance on the positive real-line \( [0, +\infty) \). By a resonance, we mean a complex number \( \lambda \) such that the equation \( \psi + (-\Delta - \lambda \pm i0)^{-1} V\psi = 0 \) has a slowly decaying solution \( \psi \in L^{2-s} \) for any \( s > \frac{1}{2} \), where \( L^{2,s} = \{ \langle x \rangle^{s} f \in L^2 \} \).

Under the above assumptions, it is known that \( H \) is self-adjoint on \( L^2 \) and that its spectrum \( \sigma(H) \) is purely absolutely continuous on the positive real-line \( [0, +\infty) \) and has at most finitely many negative eigenvalues \([3]\). Moreover, for a bounded Borel function \( m : \sigma(H) \to \mathbb{C} \), one can define an \( L^2 \)-bounded operator \( m(H) \) via the functional calculus.

The main theorem of this paper says that the operator \( m(H) \) extends to an \( L^p \)-bounded operator for all \( 1 < p < \infty \) under a suitable regularity assumption on a symbol \( m \). Let \( \chi \in C^\infty_c(\mathbb{R}) \) be a standard dyadic partition of unity function such that \( \chi \) is supported in \([\frac{1}{2}, 2]\) and \( \sum_{N \in 2^\mathbb{Z}} \chi(\frac{y}{2^N}) = 1 \) on \( (0, +\infty) \). For \( s > 0 \) and a symbol \( m : \sigma(H) \to \mathbb{C} \), we define

\[
\|m\|_{H(s)} := \sup_{t > 0} \|\chi(t\lambda)m((t\lambda)^2)\|_{W^{s,2}((0, +\infty))} < \infty
\]

where \( W^{s,2} \) is the \( L^2 \)-Sobolev space of order \( s \).

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Theorem 1.1 (Spectral multiplier theorem). Let $V \in \mathcal{K}_0 \cap L^{3/2,\infty}$. If $H = -\Delta + V$ has no eigenvalue or resonance on $[0, +\infty)$ and a symbol $m : \sigma(H) \to \mathbb{C}$ satisfies $\|m\|_{\mathcal{H}(0)} < \infty$, then

$$\|m(H)\|_{L^p \to L^p} \lesssim \|m\|_{\mathcal{H}(0)}, \quad 1 < p < \infty.$$  

Remark 1.2. When $V = 0$, the spectral multiplier theorem is simply the classical Hörmander-Mikhlin multiplier theorem [5].

Remark 1.3 (Spectral multiplier theorem with the heat kernel estimate). The spectral multiplier theorem has been studied extensively for general positive-definite self-adjoint operators obeying the Gaussian heat kernel estimate (see [6] and references therein). For Schrödinger operators $H = -\Delta + V$ in $\mathbb{R}^3$, it can be read as follows. Let $V = V_+ - V_-$ with $V_+, V_- \geq 0$. If $V_+$ is in local Kato class, that is,

$$\lim_{r \to 0^+} \sup_{x \in \mathbb{R}^3} \int_{|x-y| \leq r} \frac{|V(y)|}{|x-y|} dy = 0,$$

$V_- \in \mathcal{K}_0$ and $\|V_-\|_\mathcal{K} < 4\pi$, then the kernel of the semigroup $e^{-tH}$, denoted by $e^{-tH}(x,y)$, obeys the Gaussian heat kernel estimates, precisely, there exists $c > 0$ such that

$$e^{-tH}(x,y) \leq t^{-3/2} e^{-\frac{|x-y|^2}{ct}}, \quad t > 0.$$  

[22, 3]. The spectral multiplier theorem for $H$ then follows from [6] Theorem 3.1]. Note that the heat kernel estimate always fails unless $H$ is positive-definite. Theorem 1.1 improves this result in that it allows $H$ to have negative eigenvalues.

Remark 1.4 (Spectral multiplier theorem with the wave operator). In [25], Yajima proved that the (forward-in-time) wave operator, defined by

$$W := s-lim_{t \to +\infty} e^{-itH} e^{-it(-\Delta)},$$

is bounded on $L^p$ for all $1 \leq p \leq \infty$, provided that $|V(x)| \lesssim \langle x \rangle^{-5-}$ and zero is not an eigenvalue or a resonance of $H$. Later, in [2], Beceanu extended it to a larger class

$$B := \left\{ V : \sum_{k=-\infty}^{\infty} 2^{k/2} \|V(x)\|_{L^2(2^k \leq |x| < 2^{k+1})} < \infty \right\}.$$  

The spectral multiplier theorem then follows from the boundedness and intertwining property of the wave operator. Theorem 1.1 improves this consequence, because the potential class $\mathcal{K}_0 \cap L^{3/2,\infty}$ is larger than $B$.

In this paper, we present a new approach employing the Born series expansion for the resolvent, which allows us to treat a large class of Schrödinger operators without Gaussian heat kernel estimates. Let $P_c$ be the spectral projection to the continuous spectrum. Considering the spectral multiplier $m(H)P_c$ as a perturbation of the Fourier multiplier $m(-\Delta)$,
we generate formal series expansions for the low, medium and high frequencies of the difference \((m(H)P_c - m(-\Delta))\) whose terms have explicit integral representations via the free resolvent formula

\[(\Delta - z)^{-1} f(x) = \int_{\mathbb{R}^3} \frac{e^{i\sqrt{z}|x-y|}}{4\pi |x-y|} f(y) dy.
\]

We estimate each term, and summing them up, we prove the spectral multiplier theorem. Surprisingly, in spite of the singular integral nature of both \(m(H)P_c\) and \(m(-\Delta)\), their difference is not a singular integral operator. This observation is essential, since it allows us to avoid using the classical Calderon-Zygmund theory for the complicated operator \(m(H)\) (see Remark 4.4).

1.2. **Application to NLS.** The choice of the potential class \(K_0 \cap L^{3/2, \infty}\) in the main theorem is motivated by the following nonlinear application. Let’s recall the following Strichartz estimates:

**Proposition 1.5** (Strichartz estimates). If \(V \in K_0\) and \(H\) has no eigenvalue or resonance on \([0, +\infty)\), then

\[
\|e^{-itH}P_c f\|_{L_t^q L_x^r} \lesssim \|f\|_{L^2},
\]

\[
\| \int_0^t e^{-i(t-s)H} P_c F(s) ds \|_{L_t^q L_x^r} \lesssim \| F \|_{L_t^6 L_x^3},
\]

where \(2/q + 3/r = 3/2\) and \(2 \leq q, r \leq \infty\).

**Proof.** Beceanu-Goldberg [3] proved the dispersive estimate

\[(1.3) \quad \|e^{-itH}P_c\|_{L^1 \rightarrow L^\infty} \lesssim |t|^{-3/2},
\]

Strichartz estimates then follow by the argument of Keel-Tao [18]. \(\Box\)

**Remark 1.6.** The dispersive estimate of the form \((1.3)\) was first proved by Journé-Soffer-Sogge under suitable assumptions on potentials [17]. The assumptions has been relaxed by Rodnianski-Schlag [19], Goldberg-Schlag [11] and Goldberg [9, 10]. Recently, Beceanu-Goldberg established \((1.3)\) for a scaling-critical potential class \(K_0\) [3, 13].

A natural question is then whether one can use the above Strichartz estimates to show the local-in-time well-posedness (LWP) for a 3d nonlinear Schrödinger equation

\[(NLS_{p}^0) \quad iu_t + \Delta u - Vu \pm |u|^{p-1}u = 0; \ u(0) = u_0,
\]

where \(1 < p \leq 5\), for the potential class \(K_0\). However, if one tries to show LWP by a contraction mapping argument [5, 23], one will realize there is a subtle problem, mainly because the linear propagator \(e^{-itH}\) does not commute with the differential operators from the Sobolev norms.

In the energy-subcritical case \((1 < p < 5)\), this problem can be solved by the norm equivalence between two inhomogeneous Sobolev norms [15, Lemma 3.2]:
Lemma 1.7 (Norm equivalence: inhomogeneous case \cite{15}). If $V \in K_0 \cap L^{3/2,\infty}$, then there exists $\alpha \gg 1$ such that for $0 \leq s \leq 2$ and $1 < r < \frac{3}{\alpha}$,

$$\| (a + H)^{\frac{s}{2}} f \|_{L^r} \sim \| f \|_{W^{s,r}}.$$ 

**Sketch of Proof.** Choosing $\alpha \gg 1$, one can make $(a + H)$ satisfy the Gaussian heat kernel estimate, and the spectral multiplier theorem \cite{6} thus implies boundedness of imaginary power operators. The norm equivalence then follows from the argument of \cite{7}. □

By the norm equivalence, one can switch from one norm to another during in a contraction mapping argument, and we thus establish LWP:

**Theorem 1.8** (LWP: energy-subcritical case \cite{15}). Let $1 < p < 5$. Suppose that $V \in K_0 \cap L^{3/2,\infty}$ and $H$ has no eigenvalue or resonance on the positive real-line $[0, +\infty)$. Then $\text{NLS}_V^p$ is locally well-posed in $H^1$.

Consider the energy-critical case $p = 5$. Recall that if $V = 0$, the equation is locally well-posed in the homogeneous space $\dot{H}^1$. One may expect that the same is true in the presence of a potential. Now, we have to make use of Theorem 1.1 since $H$ does not satisfy the Gaussian heat kernel estimate by itself. We then establish the norm equivalence and LWP of an energy-critical equation:

**Lemma 1.9** (Norm equivalence). If $V \in K_0 \cap L^{3/2,\infty}$ and $H$ has no eigenvalue or resonance on the positive real-line $[0, +\infty)$, then for $0 \leq s \leq 2$ and $1 < r < \frac{3}{s}$,

$$\| H^{\frac{s}{2}} P_c (\Delta)^{-\frac{s}{2}} f \|_{L^r} \lesssim \| f \|_{L^r}, \| (\Delta)^{\frac{s}{2}} H^{-\frac{s}{2}} P_c f \|_{L^r} \lesssim \| f \|_{L^r}.$$ 

**Theorem 1.10** (LWP: energy-critical case). If $V \in K_0 \cap L^{3/2,\infty}$ and $H$ has no eigenvalue or resonance on the positive real-line $[0, +\infty)$, then $\text{NLS}_V^5$ is locally well-posed in $\dot{H}^1$.

**Remark 1.11.** (i) The range of $r$ in the norm equivalence (Lemma 1.7 and 1.9) is sharp. See the counterexample in \cite{21}.

(ii) Throughout the paper, we assume that $V$ is contained in $L^{3/2,\infty}$. This extra assumption is not necessary for Strichartz estimates (Proposition 1.5), but is necessary in the interpolation step in the proof of the norm equivalence.

1.3. **Organization of the paper.** The outline of the proof of Theorem 1.1 is given in §2: we decompose the spectral representation of the difference $(m(H)P_c - m(-\Delta))$ into the low, medium and high frequencies, and then analyze them separately in §4-6. In §7, we establish LWP of a 3d energy quintic nonlinear Schrödinger equation with a potential.

1.4. **Notations.** For an integral operator $T$, its integral kernel is denoted by $T(x, y)$. We denote by $A^\ast = ^*B$ the formal identity which will be proved later.

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2. Reduction to the Key Lemma

Suppose that $V \in \mathcal{K}_0$ and $H$ has no eigenvalue or resonance on $[0, +\infty)$. For $z \notin \sigma(H)$, we define the resolvent by $R_V(z) := (H - z)^{-1}$, and denote

$$R_V^+(\lambda) := \lim_{\epsilon \to 0^+} R_V(\lambda \pm i\epsilon).$$

By the Stone’s formula, the spectral multiplier operator $m(H)P_c$ is represented by

$$m(H)P_c = \frac{1}{2\pi i} \int_0^\infty m(\lambda)[R_V^+(\lambda) - R_V^-(\lambda)]d\lambda = \frac{1}{\pi} \int_0^\infty m(\lambda) \Im R_V^+(\lambda)d\lambda.$$

Then, by the identity

$$R_V^+(\lambda) = R_0^+(\lambda)(I + VR_0^+(\lambda))^{-1} = R_0^+(\lambda) - R_0^+(\lambda)(I + VR_0^+(\lambda))^{-1}VR_0^+(\lambda),$$

we split $m(H)P_c$ into the pure and the perturbed parts:

$$m(H)P_c = \frac{1}{\pi} \int_0^\infty m(\lambda) \Im R_0^+(\lambda)d\lambda - \frac{1}{\pi} \int_0^\infty m(\lambda) \Im[R_0^+(\lambda)(I + VR_0^+(\lambda))^{-1}VR_0^+(\lambda)]d\lambda,$$

$$= m(-\Delta) + Pb,$$

where

$$Pb := -\frac{1}{\pi} \int_0^\infty m(\lambda) \Im[R_0^+(\lambda)(I + VR_0^+(\lambda))^{-1}VR_0^+(\lambda)]d\lambda.$$

For the pure part $m(-\Delta)$, we apply the classical Hörmander multiplier theorem [16]:

$$\|m(-\Delta)\|_{L^p \to L^p} \lesssim \|m\|_{H^{3/2+}}, 1 < p < \infty.$$

To analyze the perturbed part, we further decompose it into dyadic pieces. Let $\chi$ be the smooth dyadic partition of unity function chosen in (1.1), and decompose

$$Pb = \sum_{N \in 2^\mathbb{Z}} Pb_N$$

where

$$(2.1) \quad Pb_N := -\frac{1}{\pi} \int_0^\infty m(\lambda)\chi_N(\sqrt{\lambda}) \Im[R_0^+(\lambda)(I + VR_0^+(\lambda))^{-1}VR_0^+(\lambda)]d\lambda.$$

For a small dyadic number $N_0$ and a large dyadic number $N_1$ to be chosen later, we denote the low (high, resp) frequency part by

$$Pb_{\leq N_0} := \sum_{N \leq N_0} Pb_N, \quad (Pb_{\geq N_1} := \sum_{N \geq N_1} Pb_N, \text{ resp}).$$

In the next four sections, we will show the following lemma:

Lemma 2.1 (Key lemma). Suppose that $V \in \mathcal{K}_0 \cap L^{3/2, \infty}$ and $H$ has no eigenvalue or resonance on $[0, +\infty)$.

(i) (High frequency) There exists $N_1 = N_1(V) \gg 1$ such that

$$\|Pb_{\geq N_1}\|_{L^p \to L^p, \infty} \lesssim \|m\|_{H(6)}, 1 < p \leq 2.$$

(ii) (Low frequency) For large $m$, there is $\sigma(m)$ such that

$$\|Pb_{\leq m}\|_{L^p \to L^p, \infty} \lesssim \|m\|_{H(6)}, 1 < p \leq 2.$$
(ii) (Low frequency) There exists \( N_0 = N_0(V) \ll 1 \) such that
\[
\|P_{b \leq N_0}\|_{L^{p,1} \rightarrow L^{p,\infty}} \lesssim \|m\|_{\mathcal{H}(s)}, \quad 1 < p \leq 2.
\]

(iii) (Medium frequency) For \( N_0 < N < N_1 \),
\[
\|P_{b\geq N}\|_{L^{p,1} \rightarrow L^{p,\infty}} \lesssim_{N_0,N_1} \|m\|_{\mathcal{H}(s)}, \quad 1 < p \leq 2.
\]

Here, \( L^{p,1} \) and \( L^{p,\infty} \) are the Lorentz spaces (see Appendix A).

Proof of Theorem 1.1, assuming Lemma 2.1. By Lemma 3.6 (below), it suffices to show the boundedness of \( m(H)P_c \). Indeed, summing the estimates in Lemma 2.1, we obtain that \( P_b \) is bounded from \( L^{p,1} \) to \( L^{p,\infty} \) for \( 1 < p \leq 2 \). Hence, by the interpolation theorem (Corollary A.5), \( P_b \) is bounded on \( L^p \) for \( 1 < p \leq 2 \), and so is \( m(H)P_c = m(-\Delta) + P_b \). Observe that if \( m \in \mathcal{H}(s) \), then \( m \in \mathcal{H}(s) \). Hence, by duality, \( m(H)P_c = \bar{m}(H)^*P_c \) is bounded on \( L^p \) for \( 2 \leq p < \infty \).

3. Preliminaries

3.1. Resolvent estimates. We prove kernel estimates for \( VR_0^+ (\lambda) \), \( V(R_0^+ (\lambda) - R_0^-(\lambda_0)) \), \( (VR_0^+(\lambda))^4 \) and \( (I + VR_0^+ (\lambda))^{-1} \), all of which will play as building blocks for \( P_{bN} \).

Lemma 3.1. Suppose that \( V \in \mathcal{K}_0 \).

(i) \( \|VR_0^+(\lambda)\|_{L^1 \rightarrow L^1} \leq \frac{|V|_\mathcal{K}}{4\pi} \) for \( \lambda \gg 0 \).

(ii) Define the difference operator by \( B_{\lambda,\lambda_0} := V(R_0^+ (\lambda) - R_0^+(\lambda_0)) \). For \( \epsilon > 0 \), there exist \( \delta > 0 \) and an integral operator \( B = B_\epsilon \in \mathcal{L}(L^1) \) such that \( |B_{\lambda,\lambda_0}(x,y)| \leq B(x,y) \) for \( |\lambda - \lambda_0| \leq \delta \) and \( \lambda, \lambda_0 \gg 0 \), and \( \|B(x,y)\|_{L^\infty_{\xi}L^1_x} \leq \epsilon \).

(iii) For \( \epsilon > 0 \), there exist \( N_1 \gg 1 \) and an integral operator \( D = D_\epsilon \in \mathcal{L}(L^1) \) such that \( \|VR_0^+(\lambda))^4(x,y)\| \leq D(x,y) \) for \( \lambda \geq N_1 \) and \( \|D(x,y)\|_{L^\infty_{\xi}L^1_x} \leq \epsilon \).

Proof. (i) By the free resolvent formula and the Minkowski inequality, we have
\[
\|VR_0^+(\lambda)f\|_{L^1} \leq \int_{\mathbb{R}^3} \left\| \frac{V(x)e^{i\sqrt{\lambda}|x-y|}}{4\pi|x-y|} \right\|_{L^1_x} |f(y)|dy \leq \frac{|V|_\mathcal{K}}{4\pi} \|f\|_{L^1}.
\]

(ii) For \( \epsilon > 0 \), decompose \( V = V_1 + V_2 \) such that \( V_1 \) is bounded and compactly supported and \( \|V_2\|_{\mathcal{K}} \leq \epsilon \). We choose \( \delta > 0 \) such that \( |\sqrt{\lambda} - \sqrt{\lambda_0}| \leq \epsilon \|V_1\|_{L^1} \) for all \( \lambda, \lambda_0 \gg 0 \) with \( |\lambda - \lambda_0| \leq \delta \). By the mean-value theorem,
\[
|B_{\lambda,\lambda_0}(x,y)| \leq \frac{|V_1(x)(e^{i\sqrt{\lambda}|x-y|} - e^{i\sqrt{\lambda_0}|x-y|})|}{4\pi|x-y|} + \frac{|V_2(x)(e^{i\sqrt{\lambda}|x-y|} - e^{i\sqrt{\lambda_0}|x-y|})|}{4\pi|x-y|}
\leq \frac{|V_1(x)||\sqrt{\lambda} - \sqrt{\lambda_0}|}{4\pi} + \frac{|V_2(x)|}{2\pi|x-y|} \leq \frac{\epsilon |V_1(x)|}{4\pi \|V_1\|_{L^1}} + \frac{|V_2(x)|}{2\pi|x-y|} =: B_\epsilon(x,y).
\]

Then,
\[
\|B_\epsilon(x,y)\|_{L^\infty_{\xi}L^1_x} \leq \frac{\epsilon}{4\pi} + \frac{|V_2|_{\mathcal{K}}}{2\pi} \leq \epsilon.
\]
(iii) Similarly, for \( \epsilon > 0 \), decompose \( V = V_1 + V_2 \) such that \( V_1 \) is bounded and compactly supported and \( \|V_2\|_\mathcal{K} \leq \epsilon \|V\|_\mathcal{K}^{-3} \). We then write

\[
\|(VR_0^+)(\lambda))^4(x, y)\| \leq \|(VR_0^+(\lambda))^4(x, y)\| + \|(VR_0^+(\lambda))^4(x, y) - (V_1R_0^+(\lambda))^4(x, y)\|
\]

Observe that, by the fractional integration inequalities, the Hölder inequalities in the Lorentz spaces (Lemma A.2) and the free resolvent estimate \( \|R_0^+(\lambda)\|_{L^{3/2,1}} \leq \langle \lambda \rangle^{-1/4} \) [12, Lemma 2.1], we have

\[
\left\|R_0^+(\lambda)(V_1R_0^+(\lambda))^3f\right\|_{L^\infty} \leq \left\|R_0^+(\lambda)(V_1R_0^+(\lambda))^2f\right\|_{L^{3/2,1}} \leq \left\|V_1\right\|_{L^{3/2,1}} \left\|R_0^+(\lambda)(V_1R_0^+(\lambda))^2f\right\|_{L^{3/2,1}} \leq \langle \lambda \rangle^{-1/4} \left\|V_1\right\|_{L^{3/2,1}} \left\|f\right\|_{L^4} \lesssim \langle \lambda \rangle^{-1/4} \|f\|_{L^4}.
\]

Taking \( f \to \delta(-y) \), we obtain that \( |R_0^+(\lambda)(V_1R_0^+(\lambda))^3(x, y)| \to 0 \) as \( \lambda \to +\infty \). Thus, there exists \( N_1 = N_1(\epsilon, \ell) \) such that if \( \lambda \geq N_1 \), then

\[
\|(V_1R_0^+(\lambda))^4(x, y)\| \leq \frac{\epsilon \|V_1(x)\|}{2\|V\|_{L^1}} =: D_1(x, y).
\]

For the second term, we split

\[
(VR_0^+(\lambda))^4(x, y) - (V_1R_0^+(\lambda))^4(x, y) = (V_2R_0^+(\lambda))(V_1R_0^+(\lambda))^3(x, y) + (V_1R_0^+(\lambda))V_2R_0^+(\lambda)(VR_0^+(\lambda))^2(x, y)
\]

But, since the kernel of \( R_0^+(\lambda) \) is bounded by the kernel of \( R_0^+(0) = (-\Delta)^{-1} \), we have

\[
\left\|(VR_0^+(\lambda))^4(x, y) - (V_1R_0^+(\lambda))^4(x, y)\right\| \leq \|(V_2R_0^+(0))(V_1R_0^+(0))^3\|_{L^\infty} + \|(V_1R_0^+(0))V_2R_0^+(0)(VR_0^+(0))^2\|_{L^\infty} + \|(V_1R_0^+(0))^2V_2R_0^+(0)(VR_0^+(0))^2\|_{L^\infty} + \|(V_1R_0^+(0))^3V_2R_0^+(0)(VR_0^+(0))^2\|_{L^\infty} =: D_2(x, y).
\]

Therefore, we get the upper bound

\[
\|(VR_0^+(\lambda))^4(x, y)\| \leq D(x, y) := D_1(x, y) + D_2(x, y),
\]

and as in (ii), one can check that \( \|D(x, y)\|_{L^\infty L^1} \leq \epsilon \).

By algebra, the resolvent \( R_V^+(\lambda) \) can be written as

\[
R_V^+(\lambda)^{-1} = \left( I + VR_0^+(\lambda) \right)^{-1}.
\]

The following lemmas say that \( (I + VR_0^+(\lambda)) \) is invertible in \( \mathcal{L}(L^1) \) for \( \lambda > 0 \), its inverse \( (I + VR_0^+(\lambda))^{-1} \) is uniformly bounded in \( \mathcal{L}(L^1) \), and is the sum of the identity map and an integral operators:

**Lemma 3.2** (Invertibility of \( (I + VR_0^+(\lambda)) \)). If \( V \in \mathcal{K}_0 \) and \( H \) has no eigenvalue or resonance on \( [0, +\infty) \), then \( (I + VR_0^+(\lambda)) \) is invertible in \( \mathcal{L}(L^1) \) for \( \lambda \geq 0 \).
Proof. If it is not invertible, there exists \( \varphi \in L^1 \) such that \((I + VR_0^+(\lambda))\varphi = 0\). Then, \( \psi := R_0^+(\lambda)\varphi \) solves \( \psi + R_0^+(\lambda)V\psi = 0 \), and
\[
\| \langle x \rangle^{-s} \psi \|_{L^2} = \| \langle x \rangle^{-s} R_0^+(\lambda) \varphi \|_{L^2} \leq \int_{\mathbb{R}^3} \frac{1}{\langle x \rangle^s |x - y|} \| \varphi(y) \| dy \leq \| \varphi \|_{L^1}
\]
for any \( s > \frac{1}{2} \). Hence, \( \lambda \) is an eigenvalue or a resonance (contradiction!).

**Lemma 3.3** (Uniform bound for \((I + VR_0^+(\lambda))^{-1}\)). If \( V \in \mathcal{K}_0 \) and \( H \) has no eigenvalue or resonance on \([0, +\infty)\), then \( S_\lambda := (I + VR_0^+(\lambda))^{-1} : [0, +\infty) \to \mathcal{L}(L^1) \) is uniformly bounded.

**Proof.** Iterating the resolvent identity, we get the formal identity:

\[
(I + VR_0^+(\lambda))^{-1} = (I - VR_0^+(\lambda))^{-1} = \sum_{n=0}^{\infty} (VR_0^+(\lambda))^n.
\]

Indeed, by Lemma 3.1 (iii), \( \| (VR_0^+(\lambda))^4 \|_{L^1 \to L^1} < \frac{1}{2} \) for all sufficiently large \( \lambda \). Hence, the formal identity (3.1) makes sense, and \((I + VR_0^+(\lambda))^{-1}\) is uniformly bounded for all sufficiently large \( \lambda \). Thus, it suffices to show that \((I + VR_0^+(\lambda))^{-1}\) is continuous. To see this, we fix \( \lambda_0 \geq 0 \) and write

\[
(I + VR_0^+(\lambda_0))^{-1} = (I + VR_0^+(\lambda_0))^{-1} - (I + VR_0^+(\lambda_0))^{-1} = \sum_{n=0}^{\infty} (-S_\lambda B_{\lambda, \lambda_0})^n.
\]

Then, by Lemma 3.1 (ii), we have

\[
\| (I + VR_0^+(\lambda))^{-1} - (I + VR_0^+(\lambda_0))^{-1} \|_{L^1 \to L^1} \leq \sum_{n=1}^{\infty} \| S_\lambda \|_{L^1 \to L^1} \| B_{\lambda, \lambda_0} \|_{L^1 \to L^1} \to 0 \text{ as } \lambda \to \lambda_0.
\]

Therefore, the formal identity (3.2) makes sense, and \((I + VR_0^+(\lambda))^{-1}\) is continuous.

**Lemma 3.4.** If \( V \in \mathcal{K}_0 \) and \( H \) has no eigenvalue or resonance on \([0, +\infty)\), then \( \tilde{S}_\lambda := (S_\lambda - I) = (I + VR_0^+(\lambda))^{-1} - I : [0, +\infty) \to \mathcal{L}(L^1) \) is not only uniformly bounded but also an integral operator with kernel \( \tilde{S}_\lambda(x, y) \):

\[
\tilde{S}_\lambda := \sup_{\lambda \geq 0} \| \tilde{S}_\lambda \|_{L^1 \to L^1} = \sup_{\lambda \geq 0} \| \tilde{S}_\lambda(x, y) \|_{L^\infty_y L^1_x} < \infty.
\]

**Proof.** By algebra, we have

\[
\tilde{S}_\lambda = (I + VR_0^+(\lambda))^{-1} - I = -(I + VR_0^+(\lambda))^{-1}VR_0^+(\lambda) = -S_\lambda VR_0^+(\lambda).
\]

Consider \( F_V(x; y, \lambda) := V(x) \frac{e^{\sqrt{\lambda} |x - y|}}{4\pi |x - y|} \) as a function of \( x \) with parameters \( y \in \mathbb{R}^3 \) and \( \lambda \in \mathbb{R} \), which is bounded in \( L^1_y \) uniformly in \( y \) and \( \lambda \). Hence, by Lemma 3.3, \( s_0(x; y, \lambda) :=
\]
\[ -[S\lambda F_V(x; y, \lambda)](x) \text{ is also a uniformly bounded } L^1 - \text{"function," in other word,} \]
\[ \hat{S} := \sup_{\lambda \geq 0} \sup_{y \in \mathbb{R}^3} \| s_0(x; y, \lambda) \|_{L^1_x} < \infty. \]

Then, by the Fubini theorem and the duality, we write
\[
\int_{\mathbb{R}^3} \left( \int_{\mathbb{R}^3} s_0(x; y, \lambda) f(y) dy \right) g(x) dx = -\int_{\mathbb{R}^3} \langle [S\lambda F_V(x; y, \lambda)], g(x) \rangle_{L^2_x} f(y) dy
\]
\[
= -\int_{\mathbb{R}^3} \langle F_V(x; y, \lambda), (S\lambda^k g)(x) \rangle_{L^2_x} f(y) dy = -\int_{\mathbb{R}^3} \left( \int_{\mathbb{R}^3} F_V(x; y, \lambda) f(y) dy \right) (S\lambda^k g)(x) dx
\]
\[
= -\langle VR_0^\lambda(\lambda)f, S\lambda g \rangle_{L^2} = -\langle S\lambda VR_0^\lambda(\lambda)f, g \rangle_{L^2} = \langle \hat{S}\lambda f, g \rangle_{L^2}.
\]

We thus conclude that \( \hat{S}\lambda(x, y) \) is an integral operator satisfying (3.3). \( \square \)

### 3.2. Spectral projections and eigenfunctions.

Let \( \chi \) be the dyadic partition of unity function chosen in (1.1), and let \( \check{c}\lambda N(\lambda) \in C^\infty_c(\mathbb{R}) \) such that \( \check{c}\lambda N(\lambda) = \chi(\frac{\lambda}{2N}) \) if \( \lambda \geq 0 \); \( \check{c}\lambda N(\lambda) = 0 \) if \( \lambda < 0 \). By functional calculus, we define the Littlewood-Paley projections by \( P_N = \check{c}\lambda N(H), P_{\leq N} = \sum_{N < N_0} P_N, P_{N_0 < N_1} = \sum_{N_0 < N < N_1} P_N \) and \( P_{N_1} = \sum_{N \geq N_1} P_N \).

**Lemma 3.5.** Suppose that \( V \in K_0 \cap L^{3/2,\infty} \) and \( H \) has no eigenvalue or resonance on \( [0, +\infty) \). Let \( \mathcal{G} := \{ f \in L^1 \cap L^{\infty} : P_r f = P_{N_0 < N_1} f \text{ for some } N_0, N_1 > 0 \} \). For \( 1 < r < \infty \), \( \mathcal{G} \) is dense in \( L^r \).

**Proof.** \( L^1 \cap L^{\infty} \) is dense in \( L^r \). Fix \( f \in L^1 \cap L^{\infty} \). We claim that \( \lim_{N_0 \to 0} \| P_{N_0} f \|_{L^r} = 0 \). By the spectral theory, \( \lim_{N_0 \to 0} \| P_{N_0} f \|_{L^2} = 0 \). On the other hand, replacing \( \check{c}\lambda N \) by \( \sum_{N < N_0} \check{c}\lambda N \) in the proof of [14] Corollary 1.6], one can show that \( \| P_{N_0} f \|_{L^1} \) and \( \| P_{N_0} f \|_{L^\infty} \) are bounded uniformly in \( N_0 \). Hence the claim follows from the interpolation. By the same argument, one can show that \( \lim_{N_1 \to \infty} \| P_{N_1} f \|_{L^r} = 0 \). Thus, \( \mathcal{G} \) is dense in \( L^r \). \( \square \)

**Lemma 3.6** (Boundedness of eigenfunctions). Suppose that \( V \in K_0 \cap L^{3/2,\infty} \) and \( H \) has no eigenvalue or resonance on \( [0, +\infty) \). Let \( \psi_j \) be an eigenfunction corresponding to the negative eigenvalue \( \lambda_j \).

(i) For all \( 1 < p < \infty \), \( \psi_j \in L^p \) and \( P_{\lambda_j} \) is bounded on \( L^p \), where \( P_{\lambda_j} \) is the spectral projection onto the point \( \{ \lambda_j \} \).

(ii) \( \nabla \psi_j \in L^r \) for \( 1 \leq r < 3 \).

**Proof.** (i) We prove the lemma following the argument of [2]. We decompose \( V = V_1 + V_2 \) such that \( V_1 \) is compactly supported and bounded and \( \| V_2 \|_K \leq 1 \). Then,
\[
\psi_j + R_0(\lambda_j)V\psi_j = \psi_j + R_0(\lambda_j)(V_1 + V_2)\psi_j = 0
\]
\[
\Rightarrow \psi_j = -(I + R_0(\lambda_j)V_2)^{-1}R_0(\lambda_j)V_1\psi_j = -\sum_{n=0}^{\infty} (-R_0(\lambda_j)V_2)^n R_0(\lambda_j)V_1\psi_j.
\]
Observe that, since $V_1 \in C_c^\infty$ and $\lambda_j < 0$, $R_0(\lambda_j)V_1 \psi_j$ is exponentially decreasing. Indeed, for sufficiently small $\epsilon > 0$, by the fractional integration inequality and the Hölder inequality in the Lorentz spaces (Lemma 4.2), we have

$$|e^{\epsilon|x|}(R_0(\lambda_j)V_1 f)(x)| \leq e^{\epsilon|x|} \int_{\mathbb{R}^3} \frac{e^{-\sqrt{-\lambda_j}|x-y|}}{4\pi|x-y|} |V_2(y)||\psi_j(y)|dy$$

$$\leq \int_{\mathbb{R}^3} \frac{e^{-(\sqrt{-\lambda_j}-\epsilon)|x-y|}}{4\pi|x-y|} e^{\epsilon|y|} |V_2(y)||\psi_j(y)|dy \leq \|e^{\epsilon|x|}V_2\psi_j\|_{L^{3/2,1}} \lesssim \|e^{\epsilon|y|}V_2\|_{L^{6,2}} \|\psi_j\|_{L^2}.$$ 

Similarly, one can check that $e^{\epsilon|y|}R_0(\lambda_j)V_2 e^{-\epsilon|y|}$ is bounded on $L^\infty$ and its operator norm is less then 1. Thus, we prove that

$$\|e^{\epsilon|x|}\psi_j\|_{L^2} \leq \left( \sum_{n=0}^{\infty} \|e^{\epsilon|x|}R_0(\lambda_j)V_2 e^{-\epsilon|y|}\|_{L^\infty \rightarrow L^\infty} \right) \|e^{\epsilon|y|}R_0(\lambda_j)V_1 \psi_j\|_{L^2} < \infty.$$ 

Therefore, $\psi_j \in L^p$ and $P_\lambda f = \langle \psi_j, f \rangle_{L^2} \psi_j$ is bounded on $L^p$ for all $1 \leq p \leq \infty$.

(ii) Since $\lambda_j < 0$, by the inhomogeneous Sobolev inequality, we get

$$\|\nabla \psi_j\|_{L^1} = \|\nabla R_0^+(\lambda)\psi_j\|_{L^1} \lesssim \|V\psi_j\|_{L^1} \lesssim \|V\|_{L^{3/2,\infty}} \|\psi_j\|_{L^{3,1}} < \infty,$$

$$\|\nabla \psi_j\|_{L^3} = \|\nabla R_0^+(\lambda)\psi_j\|_{L^3} \lesssim \|V\psi_j\|_{W^{-1,3}} \lesssim \|V\psi_j\|_{\dot{W}^{-1,3}} \lesssim \|V\|_{L^{3/2,\infty}} \|\psi_j\|_{L^{6,-3}} < \infty.$$ 

Thus, interpolation gives (ii).

4. High Frequency Estimate: Proof of Lemma 2.1 (i)

4.1. Construction of the formal series expansion. Let $N_1 > 1$ to be chosen later in Lemma 4.3. For $N \geq N_1$, we construct a formal series expansion of the kernel of $Pb_N$ as follows. Iterating the resolvent identity, we generate a formal series expansion

$$(I + VR_0^+(\lambda))^{-1} = \sum_{n=0}^{\infty} (-VR_0^+(\lambda))^n.$$ 

Plugging this formal series into (2.1), we write

$$Pb_N = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{\pi} \int_0^{\infty} m(\lambda)\chi_N(\sqrt{\lambda}) \text{Im}[R_0^+(\lambda)(VR_0^+(\lambda))^n VR_0^+(\lambda)]d\lambda.$$ 

By the free resolvent formula (1.2) (for the first and the last free resolvents) and Fubini, the kernel of $Pb_N$ is written as

$$Pb_N(x, y) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{\pi} \int_0^{\infty} m(\lambda)\chi_N(\sqrt{\lambda})$$

$$\times \text{Im} \left[ \iint_{\mathbb{R}^6} \frac{e^{i\sqrt{\lambda}|x-\tilde{x}|}}{4\pi|x-\tilde{x}|} (VR_0^+(\lambda))^n(x, \tilde{y})V(\tilde{y}) e^{i\sqrt{\lambda}|\tilde{y}-y|} \frac{d\tilde{x}d\tilde{y}}{4\pi|x-\tilde{x}|} d\lambda \right]$$

$$= \iint_{\mathbb{R}^6} \frac{V(\tilde{y})}{16\pi^3|x-\tilde{x}|} \left\{ \sum_{n=0}^{\infty} (-1)^{n+1} Pb_N^n(x, x, \tilde{y}, y) \right\} d\tilde{x}d\tilde{y},$$

as
Lemma 4.1 (Decay estimate for $P^N_b(x, \tilde{x}, \tilde{y}, y)$)

\[
P^N_b(x, \tilde{x}, \tilde{y}, y) = \int_0^\infty m(\lambda) \chi_N(\sqrt{\lambda}) \Im \{ e^{i\sqrt{\lambda}|x-\tilde{x}|+|\tilde{y}-y|} (VR^+_0(\lambda))^n(\tilde{x}, \tilde{y}, y) \} d\lambda.
\]

The series expansion (4.1) makes sense only formally at this moment, but it is expected to be absolutely convergent for large $N$ by Lemma 3.1 (iii).

4.2. Kernel estimates for $P^N_b$. First, we prove that the intermediate kernel $P^N_b(x, \tilde{x}, \tilde{y}, y)$ decays away from $x = \tilde{x}$ and $\tilde{y} = y$:

**Lemma 4.1** (Decay estimate for $P^N_b$). There exists $K^N_{\text{dec}}(\tilde{x}, \tilde{y})$ such that for $s_1, s_2 \geq 0$,

\[
|P^N_b(x, \tilde{x}, \tilde{y}, y)| \leq \frac{N^2 \|m\|_{\mathcal{H}(s_1+s_2)} K^N_{\text{dec}}(\tilde{x}, \tilde{y})}{\langle N(x-x) \rangle^{s_1} \langle N(y-y) \rangle^{s_2}}.
\]

**Proof.** By abuse of notation, we denote by $\chi$ the even extension of itself. By making change of variables $\lambda \mapsto N^2 \lambda^2$, we write the above oscillatory integral as

\[
N^2 \int_0^\infty 2\lambda m(N^2 \lambda^2) \chi(\lambda) \sin(N\lambda\sigma) d\lambda = N^2 \int_{\mathbb{R}} \lambda m(N^2 \lambda^2) \chi(\lambda) e^{i\lambda N\sigma} d\lambda
\]

\[
= N^2 (m(N^2 \lambda^2) \chi(\lambda))^{\vee}(N\sigma) = \frac{N^2}{\langle N\sigma \rangle ^s} (\langle \nabla \rangle ^s (m(N^2 \lambda^2) \chi(\lambda)))^{\vee}(N\sigma).
\]

But, since

\[
\| (\langle \nabla \rangle ^s (m(N^2 \lambda^2) \chi(\lambda)))^{\vee} \|^L^x \leq \| m(N^2 \lambda^2) \chi(\lambda) \|^W^{s,1} \leq \| m(N^2 \lambda^2) \chi(\lambda) \|^W^{s,2} \leq \| m \|_{\mathcal{H}(s)},
\]

we obtain the lemma. \qed

**Proof of Lemma 4.1**. First, using the free resolvent formula, we write

\[
P^N_b(x, \tilde{x}, \tilde{y}, y)
\]

\[
= \int_0^\infty m(\lambda) \chi_N(\sqrt{\lambda}) \Im \left\{ \int_{\mathbb{R}^{3(n-1)}} \prod_{k=1}^{n} V(x_k) \prod_{k=0}^{n+1} e^{i\sqrt{\lambda}|x_k-x_{k+1}|} d\lambda \right\} d\lambda
\]

\[
= \int_{\mathbb{R}^{3(n-1)}} \prod_{k=1}^{n} V(x_k) \prod_{k=1}^{n+1} \frac{e^{i\sqrt{\lambda}|x_k-x_{k+1}|}}{4\pi|x_k-x_{k+1}|} \left\{ \int_0^\infty m(\lambda) \chi_N(\sqrt{\lambda}) \Im \{ e^{i\sqrt{\lambda}\sigma_n} \} d\lambda \right\} d\lambda,
\]

where $x_0 := x$, $x_1 := \tilde{x}$, $x_{n+1} := \tilde{y}$, $x_{n+2} := y$, $dx_{(2,n)} := dx_2 \cdots dx_n$ and $\sigma_n := \sum_{j=0}^{n} |x_j - x_{j+1}|$. Then, by Lemma 4.2 with $s = s_1 + s_2$ and the trivial inequality

\[
|x_0 - x_1|, |x_{n+1} - x_{n+2}| \leq \sigma_{n+1} = \sum_{j=0}^{n+1} |x_j - x_{j+1}|,
\]

where
we obtain that
\[ |\text{Pb}_N(x, \bar{x}, \bar{y}, y)| \leq \frac{N^2 \|m\|_{H(s_1+s_2)}K_{dec}^n(\bar{x}, \bar{y})}{\langle N(x - \bar{x}) \rangle^{s_1} \langle N(y - y) \rangle^{s_2}}, \]
where
\[ K_{dec}^n(\bar{x}, \bar{y}) := \int_{\mathbb{R}^{3(n-1)}} \frac{\prod_{k=1}^n |V(x_k)|}{\prod_{k=1}^n 4\pi |x_k - x_{k+1}|} d\mathbf{x}(2n). \]
By Lemma 3.1 (i), we conclude that \( K_{dec}^n(\bar{x}, \bar{y}) \|_{L^2_y L^1_x} \leq (\|K\|_4/4\pi)^n. \)

Next, we prove that the formal series (4.1) is convergent for large \( N \):

**Lemma 4.3** (Summability for \( \text{Pb}_N \)). There exist a large number \( N_1 = N_1(V) \gg 1 \) and \( K_{sum}^n(\bar{x}, \bar{y}) \) such that for \( N \geq N_1 \),
\begin{align}
(4.6) & |\text{Pb}_N(x, \bar{x}, \bar{y}, y)| \leq N^2 \|m\|_{H(0)}K_{sum}^n(\bar{x}, \bar{y}), \\
(4.7) & \|K_{sum}^n(\bar{x}, \bar{y})\|_{L^\infty_y L^1_x} \leq \|V\|_{K}^{-n}.
\end{align}

**Proof.** For \( \epsilon := \|V\|_{K}^{-4} \), choose \( N_1 \gg 1 \) and an operator \( D \) from Lemma 3.1 (iii). Set
\[ K_{sum}^n(\bar{x}, \bar{y}) := |(D\frac{1}{2}1(|V|(-\Delta)^{-1})n-4\frac{1}{21}1)(\bar{x}, \bar{y})| \]
where \( |a| \) is the largest integer less than or equal to \( a \). Then, by definition (see (4.2)), it is easy to check (4.6). Moreover, (4.7) follows from Lemma 3.1 (i) and (iii).

### 4.3. Proof of Lemma 2.1 (i)

Choose \( N_1 \) from Lemma 4.3. We want to show that
\[ \|\text{Pb}_{\geq N_1}\|_{L^{\frac{3}{2\gamma - 1}} \rightarrow L^{\frac{3}{2\gamma}}_{\tilde{x}, \tilde{y}}} \lesssim \|m\|_{H(0)}, \ 0 < \epsilon < \frac{3}{7}. \]

**Step 1. Kernel estimate for \( \text{Pb}_{\geq N_1} \)** We claim that there exists \( K(\bar{x}, \bar{y}) \in L^\infty_y L^1_x \) such that
\[ |\text{Pb}_{\geq N_1}(x, y)| \leq \|m\|_{H(0)} \int_{\mathbb{R}^6} \frac{K(\bar{x}, \bar{y})|V(\bar{y})|}{|x - \bar{x}|^{3-\epsilon}|\bar{y} - y|^{1+\epsilon}} d\bar{x} d\bar{y}. \]
Interpolating (4.3) (with \( s_1 = 4 \) and \( s_2 = 2 \)) and (4.6), we get
\[ |\text{Pb}_N^\epsilon(x, \bar{x}, \bar{y}, y)| = |\text{Pb}_N^\epsilon(x, \bar{x}, \bar{y}, y)|^{1/2}|\text{Pb}_N(x, \bar{x}, \bar{y}, y)|^{1/2} \lesssim \frac{N^2 \|m\|_{H(0)}}{\langle N(x - \bar{x}) \rangle^{2} \langle N(y - y) \rangle} K_{dec}^n(\bar{x}, \bar{y})^{1/2}K_{sum}^n(\bar{x}, \bar{y})^{1/2}. \]
Define
\[ K(\bar{x}, \bar{y}) = \sum_{n=0}^{\infty} K_{dec}^n(\bar{x}, \bar{y})^{1/2}K_{sum}^n(\bar{x}, \bar{y})^{1/2}. \]
Going back to the definition of \( \text{Pb}_N(x, y) \) in (4.2), we see that
\[ |\text{Pb}_N(x, y)| \lesssim \int_{\mathbb{R}^6} \frac{N^2 \|m\|_{H(0)} K(\bar{x}, \bar{y})|V(\bar{y})|}{|x - \bar{x}|\langle N(x - \bar{x}) \rangle^{2}|\bar{y} - y|\langle N(\bar{y} - y) \rangle} d\bar{x} d\bar{y}. \]
Summing in $N$, we get
\[ |\mathbf{P}_{b \geq N_1}(x, y)| \leq \int_{\mathbb{R}^6} \frac{\|m\|_{\mathcal{H}(6)}}{|x - \tilde{x}|} \frac{|\tilde{y} - y|}{|\tilde{y} - y|} \left\{ \sum_{N \geq N_1} \frac{N^2}{\langle N(x - \tilde{x})^2 \rangle} \right\} d\tilde{x}d\tilde{y}. \]

Observe that, by the definition (4.9), the Hölder inequality, (4.4) and (4.7), we have
\[ \|K(\tilde{x}, \tilde{y})\|_{L^1_x} \leq \sum_{n=0}^{\infty} \left( \frac{\|V\|_K}{4\pi} \right)^{n/2} \|V\|_{K}^{-n/2} = \sum_{n=0}^{\infty} \frac{1}{(4\pi)^{n/2}} < \infty. \]

For (4.8), it suffices to show that
\[ \sum_{N \in 2\mathbb{Z}} \frac{N^2}{\langle N^2 \rangle} \leq \frac{1}{|x|^{2-\epsilon}|y|^{\epsilon}}. \]

Fix $x, y \in \mathbb{R}^3$, and consider the following four cases:

(Case 1: $N < \min(|x|^{-1}, |y|^{-1})$)
\[ \sum_{\text{Case 1}} \frac{N^2}{\langle N^2 \rangle} \leq \sum_{\text{Case 1}} N^2 \leq \min \left( \frac{1}{|x|}, \frac{1}{|y|} \right)^2 \leq \frac{1}{|x|^{2-\epsilon}|y|^{\epsilon}}. \]

(Case 2: $|x|^{-1} \leq N < |y|^{-1}$)
\[ \sum_{\text{Case 2}} \frac{N^2}{\langle N^2 \rangle} \leq \sum_{\text{Case 2}} \frac{N^2}{\langle N^2 \rangle x^2} \leq \sum_{\text{Case 2}} \frac{1}{|x|^{2-\epsilon}} \sum_{\text{Case 2}} N^2 \leq \frac{1}{|x|^{2-\epsilon}|y|^{\epsilon}}. \]

(Case 3: $|y|^{-1} \leq N < |x|^{-1}$)
\[ \sum_{\text{Case 3}} \frac{N^2}{\langle N^2 \rangle} \leq \sum_{\text{Case 3}} \frac{N^2}{\langle N^2 \rangle y^2} = \sum_{\text{Case 3}} \frac{N^2}{|y|^2} \leq \frac{1}{|y|^{2-\epsilon}} \sum_{\text{Case 3}} N^{2-\epsilon} \leq \frac{1}{|x|^{2-\epsilon}|y|^{\epsilon}}. \]

(Case 4: $N \geq \max(|x|^{-1}, |y|^{-1})$)
\[ \sum_{\text{Case 4}} \frac{N^2}{\langle N^2 \rangle} \leq \frac{1}{|x|^{2}|y|^2} \sum_{\text{Case 4}} \frac{1}{\langle N \rangle} \leq \frac{1}{|x|^{2}|y|^2} \sum_{\text{Case 4}} \frac{1}{|x|^{2}|y|^2} \max \left( \frac{1}{|x|}, \frac{1}{|y|} \right)^{-1} \leq \frac{1}{|x|^{2-\epsilon}|y|^{\epsilon}}. \]

Summing them up, we prove (4.10).

(Step 2. Proof of Lemma 2.1 (ii)) Let $T_K$ be an integral operator with kernel $K(x, y)$ (so, $T_K$ is bounded on $L^1$). By (4.8), we have
\[ \|\mathbf{P}_{b \geq N_1} f\|_{L^1_{\mathbb{R}^6 \times \mathbb{R}^6}} \leq \|m\|_{\mathcal{H}(6)} \|\nabla|^{-\epsilon} T_K(|V|\|\nabla|^{-2-\epsilon})(|f|)\|_{L^1_{\mathbb{R}^6 \times \mathbb{R}^6}}. \]

Thus, by the fractional integration inequality and Hölder inequality in the Lorentz spaces (see Appendix A), we prove
\[ \|\nabla|^{-\epsilon} T_K(|V|\|\nabla|^{-2-\epsilon})(|f|)\|_{L^1_{\mathbb{R}^6 \times \mathbb{R}^6}} \leq \|T_K(|V|\|\nabla|^{-2-\epsilon})(|f|)\|_{L^1_{\mathbb{R}^6 \times \mathbb{R}^6}} \]
\[ \leq \|V\|\|\nabla|^{-2-\epsilon})(|f|)\|_{L^1_{\mathbb{R}^6 \times \mathbb{R}^6}} \leq \|V\|_{L^{3/2, \infty}}\|\nabla|^{-2-\epsilon})(|f|\|_{L^{3, 1}} \leq \|f\|_{L^{3, 1}}. \]

(4.11)
Remark 4.4. In (4.11), we only used the fractional integration inequality and the Hölder inequality. Note that after applying the fractional integration inequality, we always have the $L^{p,q}$-norm with smaller $p$ on the right hand side, although we want to show the $L^{\frac{3}{2},\infty} - L^{\frac{3}{2},\infty}$ boundedness. Hence, one must have at least one chance to raise the number $p$ to compensate the decease of $p$ caused by the fractional integration inequalities. In (4.11), the potential $V$ plays such a role with the Hölder inequality. This is the main reason we keep one extra potential term $V$ in the spectral representation by considering the perturbation $m(H)P_\epsilon - m(-\Delta)$ instead of $m(H)P_\epsilon$, and introducing intermediated kernels $Pb_N^\epsilon(x,\tilde{x},\tilde{y},y)$, even though they look rather artificial.

5. Low Frequency Estimate: Proof of Lemma 2.1 (ii)

5.1. Construction of the formal series expansion. We prove Lemma 2.1 (ii) by modifying the argument in Section 4. Note that for small $N$, the formal series expansion (4.1) is not convergent, since $(VR_0^+(\lambda))^4$ in (4.1) is not small anymore. For convergence, we introduce a new series expansion for $(I + VR_0^+(\lambda))^{-1}$:

$$
(I + VR_0^+(\lambda))^{-1} = (I + VR_0^+(\lambda_0) + B_{\lambda,\lambda_0})^{-1} = [(I + B_{\lambda,\lambda_0}S_{\lambda_0})(I + VR_0^+(\lambda_0))^{-1}$$

$$ii = "(I + VR_0^+(\lambda_0))^{-1}(I + B_{\lambda,\lambda_0}S_{\lambda_0})^{-1}" = S_{\lambda_0} \sum_{n=0}^{\infty} (-B_{\lambda,\lambda_0}S_{\lambda_0})^n,$$

where $B_{\lambda,\lambda_0} = V(R_0^+(\lambda) - R_0^+(\lambda_0))$ and $S_{\lambda_0} = (I + VR_0^+(\lambda_0))^{-1}$.  

Plugging the formal series (5.1) with $\lambda_0 = 0$ into (2.1), we write

$$
Pb_N^{\epsilon} = " \sum_{n=0}^{\infty} \frac{(-1)^n}{\pi} \int_0^\infty m(\lambda)\chi_N(\sqrt{\lambda}) \text{Im}[R_0^+(\lambda)S_0(B_{\lambda,0}S_0)^nVR_0^+(\lambda)]d\lambda.$$

By the free resolvent formula (1.2) (for the first and the last free resolvents) and Fubini, the kernel of $Pb_N$ is written as

$$
Pb_N(x,y)^{\epsilon} = " \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{\pi} \int_0^\infty m(\lambda)\chi_N(\sqrt{\lambda})$$

$$\times \text{Im}\left[\int_{\mathbb{R}^6} e^{i\sqrt{\lambda}|x - \bar{x}|} [S_0(B_{\lambda,0}S_0)^n](\bar{x},\bar{y})V(\bar{y}) \frac{e^{i\sqrt{\lambda}|\bar{y} - y|}}{4\pi|\bar{y} - y|} d\bar{x}d\bar{y}\right]d\lambda$$

$$= \int_{\mathbb{R}^6} \int_{\mathbb{R}^6} \frac{V(\bar{y})}{16\pi^3|x - \bar{x}|}\left[\sum_{n=0}^{\infty} (-1)^{n+1}Pb_N^\epsilon(x,\bar{x},\bar{y},y)\right]d\bar{x}d\bar{y},$$

where

$$
Pb_N^\epsilon(x,\bar{x},\bar{y},y) = \int_0^\infty m(\lambda)\chi_N(\sqrt{\lambda}) \text{Im}[e^{i\sqrt{\lambda}(|x - \bar{x}| + |\bar{y} - y|)} [S_0(B_{\lambda,0}S_0)^n](\bar{x},\bar{y})]d\lambda.$$  

By Lemma 3.1 (ii), $B_{\lambda,0}$ in (5.2) is small for sufficiently small $N$. This fact will guarantee the convergence of the formal series.
5.2. Reduction to the kernel estimates for $\text{Pb}_N^\alpha$. We will show the analogues of Lemma 4.1 and 4.3. The proof of Lemma 2.2 will follow from exactly the same argument to show Lemma 2.1. Let $\tilde{S}$ be the positive number given by (3.3).

Lemma 5.1 (Decay estimate for $\text{Pb}_N^\alpha$). There exists $\tilde{K}_{\text{dec}}^n(x, y)$ such that for $s_1, s_2 \geq 0$,

\begin{equation}
|\text{Pb}_N^\alpha(x, \tilde{x}, \tilde{y}, y)| \leq \frac{N^2\|m\|_{H(s_1+s_2)}\tilde{K}_{\text{dec}}^n(x, \tilde{y})}{\langle N(x - \tilde{x}) \rangle^{s_1}\langle N(y - y) \rangle^{s_2}}.
\end{equation}

\begin{equation}
\|\tilde{K}_{\text{dec}}^n(x, \tilde{y})\|_{L_2^\tilde{S} L_2^\tilde{S}} \leq (\tilde{S} + 1)^{n+1}\left(\frac{V}{2\pi}\right)^n.
\end{equation}

Proof. First, splitting $B_{\lambda,0} = VR_0^+(\lambda) - VR_0^+(0)$ in

\begin{equation}
\text{Pb}_N^\alpha(x, \tilde{x}, \tilde{y}, y) = \int_0^\infty m(\lambda)\chi_N(\sqrt{\lambda}) \text{Im}[e^{i\sqrt{\lambda}|x - \tilde{x}| + |\tilde{y} - y|}][S_0(B_{\lambda,0}S_0)^n](\tilde{x}, \tilde{y})d\lambda,
\end{equation}

we write $\text{Pb}_N^\alpha(x, \tilde{x}, \tilde{y}, y)$ as the sum of $2^n$ copies of

\begin{equation}
\int_0^\infty m(\lambda)\chi_N(\sqrt{\lambda}) \text{Im}[e^{i\sqrt{\lambda}|x - \tilde{x}| + |\tilde{y} - y|}][S_0 VR_0^+(\alpha_1\lambda)S_0 \cdots VR_0^+(\alpha_n\lambda)S_0](\tilde{x}, \tilde{y})d\lambda
\end{equation}

up to $\pm$, where $\alpha_k = 0$ or $1$ for each $k = 1, \ldots, n$. Next, splitting all $S_0$ into $I$ and $\tilde{S}_0$ in (5.6), we further decompose (5.6) into the sum of $2^{n+1}$ kernels.

Among them, let us consider the two representative terms:

\begin{equation}
\text{Im} \int_0^\infty m(\lambda)\chi_N(\sqrt{\lambda}) e^{i\sqrt{\lambda}|x - \tilde{x}| + |\tilde{y} - y|} [S_0 VR_0^+(\alpha_1\lambda)S_0 \cdots VR_0^+(\alpha_n\lambda)S_0](\tilde{x}, \tilde{y})d\lambda,
\end{equation}

\begin{equation}
\text{Im} \int_0^\infty m(\lambda)\chi_N(\sqrt{\lambda}) e^{i\sqrt{\lambda}|x - \tilde{x}| + |\tilde{y} - y|} [VR_0^+(\alpha_1\lambda) \cdots VR_0^+(\alpha_n\lambda)](\tilde{x}, \tilde{y})d\lambda.
\end{equation}

For the first term, by the free resolvent formula (1.2), we write (5.7) in the integral form:

\begin{equation}
\text{Im} \int_0^\infty \int_{\mathbb{R}^6n} m(\lambda)\chi_N(\sqrt{\lambda}) \prod_{k=1}^{n+1} \tilde{S}_0(x_{2k-1}, x_{2k}) \prod_{k=1}^{n+1} V(x_{2k}) \prod_{k=1}^{n+1} 4\pi|x_{2k} - x_{2k+1}|d\lambda d\lambda d\lambda d\lambda
\end{equation}

where $x_0 := x$, $x_1 := \tilde{x}$, $x_{2n+2} := \tilde{y}$, $x_{2n+3} := y$, $d\lambda(2n) := dx_2 \cdots dx_n$, $\bar{\sigma}_n := \sum_{k=0}^{n+1} \alpha_k|2\lambda - x_{2k+1}|$ and $\alpha_0 = \alpha_{n+1} = 1$. Then, by Lemma 1.2 with $s = s_1 + s_2$ and $|x_0 - x_1, |x_{2n+2} - x_{2n+3}| \leq \bar{\sigma}_n$, we obtain that

\begin{equation}
\int_0^\infty m(\lambda)\chi_N(\sqrt{\lambda}) \text{Im}(e^{i\sqrt{\lambda}\bar{\sigma}_{n+1}})d\lambda \leq \frac{N^2\|m\|_{H(s_1+s_2)}}{\langle N(x_0 - x_1) \rangle^{s_1}\langle N(x_{2n+2} - x_{2n+3}) \rangle^{s_2}}.
\end{equation}

Applying (5.9) to (5.7), we get the arbitrary polynomial decay away from $x_0 = x_1$:

\begin{equation}
|\text{(5.7)}| \equiv \frac{N^2\|m\|_{H(s_1+s_2)}\tilde{K}_{\text{dec}}^n(x, \tilde{y})}{\langle N(x_0 - x_1) \rangle^{s_1}\langle N(x_{2n+2} - x_{2n+3}) \rangle^{s_2}} = \frac{N^2\|m\|_{H(s_1+s_2)}\tilde{K}_{\text{dec}}^n(x, \tilde{y})}{\langle N(x - \tilde{x}) \rangle^{s_1}\langle N(y - y) \rangle^{s_2}}.
\end{equation}
where

\[
K_{(5.7)}^n(\tilde{x}, \tilde{y}) := \int_{\mathbb{R}^{2n}} \prod_{k=1}^{n+1} |\tilde{S}_0(x_{2k-1}, x_{2k})| \prod_{k=1}^{n} |V(x_{2k})| \, dx_{(2n+1)}
\]

\[
= (4\pi)^{-n} [ |\tilde{S}_0|||V|(-\Delta)^{-1}|\tilde{S}_0||]^n(\tilde{x}, \tilde{y})
\]

and $|\tilde{S}_0|$ is the integral operator with kernel $|\tilde{S}_0(x, y)|$. We claim that

\[
\|K_{(5.7)}^n(\tilde{x}, \tilde{y})\|_{L^\infty\rightarrow L^1} \leq \tilde{S}^{n+1}(\|V\|/4\pi)^n.
\]

Indeed, since $\|\tilde{S}_0||(V|(-\Delta)^{-1}|\tilde{S}_0)||^n f\|_{L^1} \leq \tilde{S}^{n+1}(\|V|/4\pi)^n\|f\|_{L^1}$ and $|\tilde{S}_0||(V|(-\Delta)^{-1}|\tilde{S}_0)||^n$ is an integral operator, sending $f \to \delta(\cdot - y)$, we prove the claim.

Similarly, we write (5.8) as

\[
\text{Im} \int_{0}^{\infty} \int_{\mathbb{R}^{2n-3}} m(\lambda) \chi_N(\sqrt{\lambda}) \prod_{k=1}^{n} V(x_k) \left\{ \int_{0}^{\infty} m(\lambda) \chi_N(\sqrt{\lambda}) \text{Im}(e^{i\sqrt{\lambda} x_{k+1}}) \, d\lambda \right\} \, dx_{(2n)}
\]

where $x_0 := x, x_1 := \tilde{x}, x_{n+1} := \tilde{y}$, $x_{n+2} := y$, $\alpha_0 = \alpha_{n+2} = 1$ and $\tilde{\alpha}_n := \sum_{k=0}^{n} \alpha_k |x_k - x_{k+1}|$.

Then, by Lemma 4.2 with $s = s_1 + s_2$ and $|x_0 - x_1|, |x_{n+1} - x_{n+2}| \leq \tilde{\alpha}_n$, we obtain that

\[
|\left\langle (5.8) \right\rangle| \leq \frac{N^2\|m\|_{\mathcal{H}(s_1 + s_2)} K_{(5.8)}^n(\tilde{x}, \tilde{y})}{\left\langle N(x_0 - x_1)^{s_1} N(x_{n+1} - x_{n+2})^{s_2} \right\rangle} = \frac{N^2\|m\|_{\mathcal{H}(s_1 + s_2)} K_{(5.8)}^n(\tilde{x}, \tilde{y})}{\left\langle N(x - \tilde{x})^{s_1} N(\tilde{y} - y)^{s_2} \right\rangle}
\]

where

\[
K_{(5.8)}^n(\tilde{x}, \tilde{y}) := \int_{\mathbb{R}^{2n-3}} \prod_{k=1}^{n} |V(x_k)| \prod_{k=1}^{n} 4\pi |x_k - x_{k+1}| \, dx_{(2n)} = (4\pi)^{-n} [ |V|(-\Delta)^{-1}|\tilde{S}_0||]^n(\tilde{x}, \tilde{y}).
\]

Then by the definition of the global Kato norm, we prove that

\[
\|K_{(5.8)}^n(\tilde{x}, \tilde{y})\|_{L^\infty\rightarrow L^1} \leq (\|V\|/4\pi)^n.
\]

Similarly, we estimate other kernels, and define $K_{\text{sum}}^n(\tilde{x}, \tilde{y})$ as the sum of all $2^{2n+1}$ many upper bounds including $K_{(5,7)}(\tilde{x}, \tilde{y})$ or $K_{(5,8)}(\tilde{x}, \tilde{y})$. Then, $K_{\text{sum}}^n(\tilde{x}, \tilde{y})$ satisfies (5.4) and (5.5).

\[\square\]

**Lemma 5.2** (Summability for $\text{Pb}_N$). There exist a small number $N_0 = N_0(V) \ll 1$ and $K_{\text{sum}}^n(\tilde{x}, \tilde{y})$ such that for $N \leq N_0$,

\[
|\text{Pb}_N^n(x, \tilde{x}, \tilde{y}, y)| \leq N^2\|m\|_{\mathcal{H}(6)} K_{\text{sum}}^n(\tilde{x}, \tilde{y}),
\]

\[
\|K_{\text{sum}}^n(\tilde{x}, \tilde{y})\|_{L^\infty\rightarrow L^1} \leq (\tilde{S} + 1)^{-n+1} \|V\|^{-n}.
\]

**Proof.** Let $\epsilon := ((\tilde{S} + 1)^2 \|\tilde{V}\|)^{-1}$ (see (3.30)). Then from Lemma 3.1 (ii), we get $N_0 := \delta = \delta(\epsilon) > 0$ an integral operator $B$ such that $|B_{\lambda,0}(x, y)| \leq B(x, y)$ for $0 \leq \lambda \leq N_0$, and

\[
\|B\|_{L^1\rightarrow L^1} \leq ((\tilde{S} + 1)^2 \|\tilde{V}\|)^{-1}.
\]

We define

\[
K_{\text{sum}}^n(\tilde{x}, \tilde{y}) := [(I + \tilde{S}_0)(B(I + \tilde{S}_0))]^{n}(\tilde{x}, \tilde{y}),
\]
where \(|\hat{S}_0|\) is the integral operator with \(|\hat{S}_0(x, y)|\). Then, by definitions (see (5.2)), \(\tilde{K}_{\text{sum}}^n(\tilde{x}, \tilde{y})\) satisfies (5.10). For (5.11), splitting \((I + |\hat{S}_0|)\) into \(I\) and \(|\hat{S}_0|\) in \(\tilde{K}_{\text{sum}}^n(\tilde{x}, \tilde{y})\), we get \(2^{n+1}\) terms:

\[
\tilde{K}_{\text{sum}}^n(\tilde{x}, \tilde{y}) = [|\hat{S}_0|(B|\hat{S}_0|)^n](\tilde{x}, \tilde{y}) + \cdots + B^n(\tilde{x}, \tilde{y}).
\]

For example, we consider \(|\hat{S}_0|(B|\hat{S}_0|)^n\) and \(B^n\). Since both \(|\hat{S}_0|\) and \(B\) are integral operators, by Lemma 3.4 and (5.12), we obtain

\[
\|(|\hat{S}_0|(B|\hat{S}_0|)^n)(\tilde{x}, \tilde{y})\|_{L_0^1 L_y^1} = \|(|\hat{S}_0|(B|\hat{S}_0|)^n\|_{L_1^1} \leq \hat{S}^{n+1}(\hat{S} + 1)^2 \|V\|_{\mathcal{K}}^{-n},
\]

\[
\|B^n(\tilde{x}, \tilde{y})\|_{L_0^1 L_y^1} = \|B^n\|_{L_1^1} \leq (\hat{S} + 1)^2 \|V\|_{\mathcal{K}}^{-n}.
\]

Similarly, we estimate other \(2^{n+1} - 2\) terms. Summing them up, we prove (5.11). \(\square\)

6. MEDIUM FREQUENCY ESTIMATE: PROOF OF LEMMA 2.1 (iii)

The proof closely follows from that of Lemma 2.1 (ii), so we only sketch the proof. Let \(\epsilon := ((\hat{S} + 1)^2 \|V\|_{\mathcal{K}})^{-1}\) and take \(\delta = \delta(\epsilon) > 0\) from Lemma 3.1 (ii). We choose a partition of unity function \(\psi \in C^\infty_c\) such that \(\text{supp } \psi \subset [-\delta, \delta]\), \(\psi(\lambda) = 1\) if \(|\lambda| \leq \frac{\delta}{3}\) and \(\sum_{j=1}^{\infty} \psi(-\lambda_j) \equiv 1\) on \((0, +\infty)\), where \(\lambda_j = j\delta\).

Let \(N_0\) and \(N_1\) be dyadic numbers chosen in the previous sections. For \(N_0 \leq N \leq N_1\), we first decompose \(\chi_N(\sqrt{\lambda})\) in \(Pb_N\) (see (2.1)) into \(\chi_N(\sqrt{\lambda}) = \sum_{j=N/2^8}^{2N/\delta} \lambda_j^N(\lambda)\) where \(\lambda_j(\lambda) = \chi_N(\sqrt{\lambda})\psi(\lambda - \lambda_j)\). Plugging the formal series (5.11) with \(\lambda_0 = \lambda_j\) into each integral, we write the kernel of \(Pb_N\) as

\[
(6.1)\quad Pb_N(x, y) = \int_{\mathbb{R}^6} \frac{V(\tilde{y})}{16\pi^3|x - \tilde{x}|} \left|\sum_{n=0}^{\infty} (-1)^{n+1} pb_N(x, \tilde{x}, \tilde{y}, y)\right| d\tilde{x}d\tilde{y},
\]

where

\[
(6.2)\quad pb_N(x, \tilde{x}, \tilde{y}, y) = \sum_{j=N/2^8}^{2N/\delta} \int_0^{\infty} m(\lambda) \lambda_j^N(\sqrt{\lambda}) \text{Im} [e^{i\sqrt{\lambda}(|x - \tilde{x}| + |y - y|)}] S_{\lambda_j}(B_{\lambda_j} S_{\lambda_j})^n(\tilde{x}, \tilde{y}) d\lambda.
\]

By the arguments in the previous sections, for Lemma 2.1 (iii), it suffices to show the following two lemmas:

**Lemma 6.1** (Decay estimate for \(pb_N^n\)). For \(N_0 < N < N_1\), there exists \(\tilde{K}_{\text{sum}, \text{dec}}^n(\tilde{x}, \tilde{y})\) such that for \(s_1, s_2 \geq 0\),

\[
(6.3)\quad |pb_N^n(x, \tilde{x}, \tilde{y}, y)| \lesssim \frac{N^2 \|m\|_{\mathcal{H}(s_1 + s_2)} \tilde{K}_{\text{sum}, \text{dec}}^n(\tilde{x}, \tilde{y})}{\langle N(x - \tilde{x}) \rangle^{s_1} \langle N(y - y) \rangle^{s_2}}
\]

\[
(6.4)\quad \|\tilde{K}_{\text{sum}, \text{dec}}^n(\tilde{x}, \tilde{y})\|_{L_0^1 L_y^1} \leq (\hat{S} + 1)^{n+1} \left(\frac{\|V\|_{\mathcal{K}}}{2\pi}\right)^n.
\]
Proof. Consider
\begin{equation}
\int_0^\infty m(\lambda)\chi_N^2(\lambda) \text{Im}[e^{i\sqrt{N}|x-\tilde{x}|+|\tilde{y}-y|} \{S_{\lambda_i}(B_{\lambda,\lambda_i}S_{\lambda_j})^n \}(\tilde{x}, \tilde{y})]d\lambda
\end{equation}
among $O(N)$-many integrals in (6.2). As we did in Lemma 1.2, we show that
\begin{equation}
\left| \int_0^\infty m(\lambda)\chi_N^2(\lambda) \text{Im}(e^{i\sqrt{N}\sigma})d\lambda \right| \leq N_0N_1 \frac{N\|m\|_{H(\sigma)}}{\langle N\sigma \rangle^s}.
\end{equation}
Repeating the proof of Lemma 5.1 (but replacing $S_0$ and $B_{\lambda,0}$ by $S_{\lambda_j}$ and $B_{\lambda,\lambda_j}$ and applying (6.6) instead of Lemma 4.2), one can find $\tilde{K}_{N,j,dec}(\tilde{x}, \tilde{y})$ such that for $s_1, s_2 \geq 0$,
\begin{equation*}
|\langle x-\tilde{x} \rangle^{s_1} \langle y-\tilde{y} \rangle^{s_2} \tilde{K}_{N,j,dec}(\tilde{x}, \tilde{y}) \rangle \lesssim (\tilde{S} + 1)^{n+1} \left( \frac{\|V\|_K}{2\pi} \right)^n.
\end{equation*}
Define
\begin{equation*}
\tilde{K}_{N,dec}(\tilde{x}, \tilde{y}) := \delta \sum_{j=N/\delta}^{2N/\delta} \tilde{K}_{N,j,dec}(\tilde{x}, \tilde{y}),
\end{equation*}
then it satisfies (6.3) and (6.4). \qed

Lemma 6.2 (Summability for $P_{B_n}$). For $N_0 < N < N_1$, there exists $\tilde{K}_{N,\text{sum}}(\tilde{x}, \tilde{y})$ such that
\begin{align}
\text{6.7} \quad |P_{B_{\lambda}}(x, \tilde{x}, \tilde{y}, y)| & \leq N^2\|m\|_{H(\sigma)} \tilde{K}_{N,\text{sum}}(\tilde{x}, \tilde{y}), \\
\text{6.8} \quad \|\tilde{K}_{N,\text{sum}}(\tilde{x}, \tilde{y})\|_{L_2^\infty L_2^1} & \leq (\tilde{S} + 1)^{-n+1}\|V\|_K^{-n}.
\end{align}

Proof. Consider (6.5). By the choice of $\epsilon$ and $\delta$ and Lemma 3.1 (ii), there exists an integral operator $B$ such that $|B_{\lambda,\lambda_j}(x, y)| \leq B(x, y)$ for $|\lambda - \lambda_j| < \delta$, $\lambda, \lambda_j \geq 0$, and $\|B\|_{L^1 \to L^1} \leq ((\tilde{S} + 1)^2\|V\|_K)^{-1}$. Let $\tilde{S}_{\lambda_j}$ is the integral operator with integral kernel $|\tilde{S}_{\lambda_j}(x, y)|$. Then, we have
\begin{equation*}
|\langle x-\tilde{x} \rangle^{s_1} \langle y-\tilde{y} \rangle^{s_2} \tilde{K}_{N,\text{sum}}(\tilde{x}, \tilde{y}) \rangle \lesssim (\tilde{S} + 1)^{n+1} \left( \frac{\|V\|_K}{2\pi} \right)^n.
\end{equation*}
Define
\begin{equation*}
\tilde{K}_{\text{sum}}(\tilde{x}, \tilde{y}) := \delta \sum_{j=N/\delta}^{2N/\delta} [(I + |\tilde{S}_{\lambda_j}|)(B(I + |\tilde{S}_{\lambda_j}|))^n](\tilde{x}, \tilde{y}),
\end{equation*}
then it satisfies (6.7) and (6.8). \qed

7. Application to the Nonlinear Schrödinger Equation

7.1. Norm equivalence. Following the argument of [7], we begin with the boundedness of the imaginary power operators. For $\alpha \in \mathbb{R}$, the imaginary power operator $H^{i\alpha}P_n$ is defined as a spectral multiplier of symbol $\lambda^{i\alpha}1_{[0, +\infty)}$. We consider $H^{i\alpha}P_n$ instead of $H^{i\alpha}$ just for convenience’s sake. By Lemma 3.6, the boundedness of $H^{i\alpha}P_n$ is equivalent to that of $H^{i\alpha}$.
Lemma 7.1 (Imaginary power operator). If $V \in \mathcal{K}_0 \cap L^{3/2,\infty}$ and $H$ has no eigenvalue or resonance on $[0, +\infty)$, then for $\alpha \in \mathbb{R}$,

$$\|H^{\alpha} P_c\|_{L^r \to L^r} \lesssim \langle \alpha \rangle^6, \; 1 < r < \infty.$$ 

Proof. Since $\|\lambda^{\alpha}1_{[0, +\infty)}\|_{H(\alpha)} \lesssim \langle \alpha \rangle^6$, the lemma follows from Theorem 1.11 \hfill \Box

Proposition 7.2 (Norm equivalence). If $V \in \mathcal{K}_0 \cap L^{3/2,\infty}$ and $H$ has no eigenvalue or resonance on $[0, +\infty)$, then for $0 \leq s \leq 2$ and $1 < r < \frac{3}{s}$,

$$\|H^s P_c(-\Delta)^{-\frac{s}{2}} f\|_{L^r} \lesssim \|f\|_{L^r}, \quad (7.1)$$

$$\|(-\Delta)^{\frac{s}{2}} H^{-s} P_c f\|_{L^r} \lesssim \|f\|_{L^r}. \quad (7.2)$$

Proof. (7.1): Pick $f, g \in L^1 \cap L^\infty$ such that supp $\hat{f} \subset B(0, R) \setminus B(0, r)$, $P_{n \leq s} N g = P_c g$ for some $R, r, N, n > 0$. Note that by Lemma 3.5, the collection of such $f$ (resp) is dense in $L^r$ ($L^r$, resp). We define

$$F(z) := \langle H^s P_c(-\Delta)^{-\frac{s}{2}} f, g \rangle_{L^2} = \langle (-\Delta)^{-\Re z - i\Im z} f, H^{-\Re z} g \rangle_{L^2}.$$ 

Indeed, $F(z)$ is well-defined, since $(-\Delta)^{-\Re z - i\Im z} f, H^{-\Re z} g \in L^2$. Moreover, $F(z)$ is continuous on $S = \{z : 0 \leq \Re z \leq 1\} \subset \mathbb{C}$, and it is analytic in the interior of $S$. We claim that $H P_c(-\Delta)^{-1}$ is bounded on $L^r$ for $1 < r < \frac{3}{2}$. Indeed, by Lemma 3.6 (i),

$$\|H P_c(-\Delta)^{-1} f\|_{L^r} \lesssim \|(-\Delta + V)(-\Delta)^{-1} f\|_{L^r} \lesssim \|f\|_{L^r} + \|V(-\Delta)^{-1} f\|_{L^r}.$$ 

By the Hölder inequality (Lemma A.2) and the Sobolev inequality in the Lorentz norms (Corollary A.6), we have

$$\|V(-\Delta)^{-1} f\|_{L^r} \lesssim \|V\|_{L^{3/2,\infty}} \|(\cdot)^{-1} f\|_{L^{\frac{3r}{3-2r}} L^{\frac{3r}{2r}}} \lesssim \|f\|_{L^r}.$$ 

Hence, by the claim and Proposition 7.1 we get

$$F(1 + i\alpha) \lesssim \|H^{1+i\alpha} P_c(-\Delta)^{-1-i\alpha} f\|_{L^r} \|g\|_{L^{r'}} \lesssim \langle \alpha \rangle^6 \|f\|_{L^r} \|g\|_{L^{r'}}, \quad (1 < r < \frac{3}{2}),$$

$$F(i\alpha) \lesssim \|H^{i\alpha} P_c(-\Delta)^{-i\alpha} f\|_{L^r} \|g\|_{L^{r'}} \lesssim \langle \alpha \rangle^6 \|f\|_{L^r} \|g\|_{L^{r'}}, \quad (1 < r < \infty).$$

Therefore (7.1) follows from the Stein’s complex interpolation theorem.

(7.2): Pick $f$ and $g$ as above, and consider

$$G(z) := \langle (-\Delta)^{\frac{s}{2}} H^{-\frac{s}{2}} P_c, g \rangle_{L^2}.$$ 

We claim that $(-\Delta) H^{-1} P_c g$ is bounded on $L^r$ for $1 < r < \frac{3}{2}$. By the triangle inequality,

$$\|(-\Delta) H^{-1} P_c g\|_{L^r} = \|(H - V) H^{-1} P_c g\|_{L^r} \lesssim \|P_c g\|_{L^r} + \|V H^{-1} P_c g\|_{L^r}.$$ 

By Lemma 3.6 (i), $\|P_c g\|_{L^r} \lesssim \|g\|_{L^r}$. By the Hölder inequality in the Lorentz norms (Lemma A.2) and the Sobolev inequality associated with $H$ \hfill \Box

\[\text{Theorem 1.9}], we get

$$\|V H^{-1} P_c g\|_{L^r} \lesssim \|V\|_{L^{3/2,\infty}} \|H^{-1} P_c g\|_{L^{\frac{3r}{3-2r}}} \lesssim \|V\|_{L^{3/2,\infty}} \|g\|_{L^r},$$

Repeating the above argument with the complex interpolation, we complete the proof. \hfill \Box
7.2. **Local well-posedness.** Now we are ready to show the local-in-time well-posedness (LWP) of a 3d quintic nonlinear Schrödinger equation

\[(\text{NLS}_V^5) \quad iu_t + \Delta u - Vu \pm |u|^4u = 0; \quad u(0) = u_0.\]

**Theorem 7.3** (LWP). If \(V \in K_0 \cap L^{3/2, \infty}\) and \(H\) has no eigenvalue or resonance on \([0, +\infty)\), then \(\text{NLS}_V^5\) is locally well-posed in \(H^1\): for \(A > 0\), there exists \(\delta = \delta(A) > 0\) such that for an initial data \(u_0 \in H^1\) obeying

\[\|\nabla u_0\|_{L^2} \leq A \text{ and } \|e^{-itH}u_0\|_{L_{t\in[0,T_0]}^{10}L_{x}^{20}} < \delta,\]

\(\text{NLS}_V\) has a unique solution \(u \in C_t(I; \dot{H}^1_x)\), with \(I = [0, T) \subset [0, T_0]\), such that

\[\|\nabla u\|_{L_{t\in I}^{10}L_{x}^{30/13}} < \infty \text{ and } \|u\|_{L_{t\in I}^{10}L_{x}^{12}} < 2\delta.\]

**Proof.** (Step 1. Contraction mapping argument) Let \(\psi_j\) be the eigenfunction corresponding to the negative eigenvalue \(\lambda_j\) normalized so that \(\|\psi_j\|_{L^2} = 1\). Choose small \(T \in (0, T_0)\) such that \(\|\psi_j\|_{L_{t\in I}^{10}L_{x}^{10}}, \|\psi_j\|_{L_{t\in I}^{2}L_{x}^{2}} \ll 1\) for all \(j\), where \(I = [0, T]\). For notational convenience, we omit the time interval \(I\) in the norm \(\cdot \|_{L_{t\in I}^{p}L_{x}^{q}}\) if there is no confusion. Following a standard contraction mapping argument \([4, 23]\), we aim to show that

\[\Phi_{u_0}(v)(t) := e^{-itH}u_0 \pm i \int_0^t e^{-i(t-s)H}(|v|^4v(s))ds\]

is a contraction map on

\[B_{a,b} := \{v : \|v\|_{L_{t,x}^{10}} \leq a, \; \|\nabla v\|_{L_{t,x}^{10}L_{x}^{30/13}} \leq b,\}\]

where \(a, b\) and \(\delta\) will be chosen later.

We claim that \(\Phi_{u_0}\) maps from \(B_{a,b}\) to itself. We write

\[
\|\Phi_{u_0}(v)\|_{L_{t,x}^{10}} \leq \|e^{-itH}u_0\|_{L_{t,x}^{10}} + \left\| \int_0^t e^{-i(t-s)H}P(|v|^4v(s))ds \right\|_{L_{t,x}^{10}}
\]

\[+ \sum_{j=1}^J \left\| \int_0^t e^{-i(t-s)H}(\langle |v|^4v(s), \psi_j \rangle_{L^2} \psi_j)ds \right\|_{L_{t,x}^{10}} = I + II + \sum_{j=1}^J III_j.\]

By assumption, \(I \leq \delta\). For \(II\), by the Sobolev inequality associated with \(H\) \([14, \text{Theorem } 1.9]\), Strichartz estimates (Proposition 1.5) and the norm equivalence, we get

\[II \leq \left\| \int_0^t e^{-i(t-s)H}P(|v|^4v(s))ds \right\|_{L_{t,x}^{10}L_{x}^{30/13}} \leq \|H^{1/2}P(|v|^4v)\|_{L_{t,x}^{6/5}} \lesssim \|\nabla(|v|^4v)\|_{L_{t,x}^{2}L_{x}^{2/5}} \lesssim \|v\|_{L_{t,x}^{10}}^{4} \|\nabla v\|_{L_{t,x}^{10}L_{x}^{20/13}} \leq a^4b.\]

For the last term, by the Hölder inequality, the choice of \(T\) and (7.3), we obtain

\[III_j = \left\| \int_0^t e^{-i(t-s)\lambda_j}(\langle |v|^4v(s), \psi_j \rangle_{L^2} \psi_j)ds \right\|_{L_{t,x}^{10}} \leq \left( \int_0^T |\langle |v|^4v(s), \psi_j \rangle_{L^2} \psi_j|ds \right) \|\psi_j\|_{L_{t,x}^{10}}
\]

\[\leq \|\nabla(|v|^4v)\|_{L_{t,x}^{2}L_{x}^{2/5}} \|\nabla \psi_j\|_{L_{t,x}^{2}L_{x}^{2}} \leq \|\nabla(|v|^4v)\|_{L_{t,x}^{2}L_{x}^{2/5}} \|\psi_j\|_{L_{t,x}^{2}} \leq a^4b.\]
Therefore, we prove that
\( \| \Phi_{u_0}(v) \|_{L_{t,x}^{10/7}} \leq \delta + Ca^4b. \)

Next, we write
\[
\| \nabla \Phi_{u_0}(v) \|_{L_{t,x}^{10/7}} \leq \| \nabla P_c \Phi_{u_0}(v) \|_{L_{t,x}^{10/7}} + \sum_{j=1}^{J} \| \nabla P_{\lambda_j} \Phi_{u_0}(v) \|_{L_{t,x}^{10/7}} = \tilde{I} + \sum_{j=1}^{J} \tilde{I}_j.
\]

For \( \tilde{I} \), by the norm equivalence, Strichartz estimates and (7.3), we obtain
\[
\tilde{I} \leq \| H^{1/2} P_c \Phi_{u_0}(v) \|_{L_{t,x}^{10/7}} \leq \| H^{1/2} P_c u_0 \|_{L^2} + \| H^{1/2} P_c (|v|^4 v) \|_{L_{t,x}^{2/5}} \\
\leq \| \nabla u_0 \|_{L^2} + \| H^{1/2} P_c (|v|^4 v) \|_{L_{t,x}^{2/5}} \leq A + a^4b.
\]

For \( \tilde{I}_j \), by the Hölder inequality, (7.4) and Lemma 3.6 we get
\[
\tilde{I}_j \leq \langle \Phi_{u_0}(v), \psi_j \rangle_{L^2} \| \psi_j \|_{L^{30/13}} \leq \| \Phi_{u_0}(v) \|_{L_{t,x}^{10/7}} \| \psi_j \|_{L_{t,x}^{10/7}} \leq \delta + a^4b.
\]

Collecting all, we prove that
\( \| \nabla \Phi_{u_0}(v) \|_{L_{t,x}^{10/7}} \leq CA + Ca^4b. \)

Let \( b = 2AC, \ a = \min((2C)^{-\frac{1}{4}}, (2C^3)^{-\frac{1}{4}}) \) and \( \delta = \frac{\theta}{2} (\Rightarrow Ca^4b \leq AC \text{ and } Ca^3b \leq \frac{1}{2}). \)

Then, by (7.4) and (7.5), \( \Phi_{u_0} \) maps from \( B_{a,b} \) to itself. Similarly, one can show that \( \Phi_{u_0} \) is contractive in \( B_{a,b} \). Thus, we conclude that there exists unique \( u \in B_{a,b} \) such that
\[
\Phi_{u_0}(u) = e^{-itH}u_0 + i \int_0^t e^{-i(t-s)H} (|u|^4 u)(s) ds.
\]

(Step 2. Continuity) In order to show that \( u(t) \in C_t(I; \dot{H}^1) \), we write
\[
u(t) = e^{-itH}(P_c u_0 + \sum_{j=1}^{J} P_{\lambda_j} u_0) \pm i \int_0^t e^{-i(t-s)H} (P_c (|u|^4 u)(s) + \sum_{j=1}^{J} P_{\lambda_j} (|u|^4 u)(s)) ds
\]
\[
= e^{-itH} P_c u_0 + \sum_{j=1}^{J} e^{-it\lambda_j} P_{\lambda_j} u_0 \pm i \int_0^t e^{-i(t-s)H} P_c (|u|^4 u)(s) ds
\]
\[
\pm i \sum_{j=1}^{J} \int_0^t e^{-i(t-s)\lambda_j} P_{\lambda_j} (|u|^4 u)(s) ds
\]
\[
=: I(t) + \sum_{j=1}^{J} II_j(t) + III(t) + \sum_{j=1}^{J} IV_j(t).
\]

For \( I(t) \), by the norm equivalence and \( L^2 \)-continuity of \( e^{-itH} \), we have
\[
\| I(t) - I(t_0) \|_{\dot{H}^1} \leq \| (e^{-itH} - e^{-it_0H}) H^{1/2} P_c u_0 \|_{L^2} \to 0 \text{ as } t \to t_0,
\]

since \( \| H^{1/2} P_c u_0 \|_{L^2} \leq \| u \|_{\dot{H}^{1}} < \infty. \) \( II_j(t) \) is continuous in \( \dot{H}^1 \), since
\[
\| P_{\lambda_j} u_0 \|_{\dot{H}^1} = \| \langle u_0, \psi_j \rangle_{L^2} \| \psi_j \|_{\dot{H}^1} \leq \| u_0 \|_{\dot{H}^1} \| \psi_j \|_{\dot{H}^{-1}} \leq \| u_0 \|_{\dot{H}^1} \| \psi_j \|_{L^6} < \infty.
\]
For \(III(t)\), by the norm equivalence, Strichartz estimates and (7.3), we have
\[
\|III(t) - III(t_0)\|_{H^1} \lesssim \|H^{1/2}(III(t) - III(t_0))\|_{L^2} \lesssim \|H^{1/2}p_c(\{|u|^4u\})\|_{L^2_{s[e(t_0,t)]}L^{6/5}} \to 0
\]
as \(t \to t_0\). For \(IV_j(t)\), by the Hölder inequality and (7.3), we write
\[
\|IV_j(t) - IV_j(t_0)\|_{H^1} \lesssim \|\psi_j\|_{H^1} \|\nabla (|u|^4u)(s)\|_{L^2_{s[e(t_0,t)]}L^{6/5}} \|\nabla^{-1}\psi_j\|_{L^2_{s[e(t_0,t)]}L^{6}} \\
\lesssim \|\nabla (|u|^4u)(s)\|_{L^2_{s[e(t_0,t)]}L^{6/5}} \|\psi_j\|_{L^2_{s[e(t_0,t)]}L^{6}} \to 0 \text{ as } t \to t_0.
\]
Collecting all, we conclude that \(u(t)\) is continuous in \(\dot{H}^1\).

\[\square\]

**Appendix A. Lorentz Spaces and Interpolation Theorem**

Following [23], we summarize some properties of the Lorentz spaces. Let \((X, \mu)\) be a measure space. The Lorentz (quasi) norm is defined by
\[
\|f\|_p,q := \begin{cases} p^{1/q} \lambda\mu(\{|f| \geq \lambda\})^{1/p} L^q((0, +\infty), \frac{d\lambda}{\lambda}) & \text{when } 1 \leq p < \infty \text{ and } 1 \leq q \leq \infty; \\ \|f\|_{L^\infty} & \text{when } p = q = \infty. \end{cases}
\]

**Lemma A.1** (Properties of the Lorentz spaces). Let \(1 \leq p \leq \infty\) and \(1 \leq q, q_1, q_2 \leq \infty\).

(i) \(L^{p,q} = L^p\), and \(L^{p,\infty}\) is the standard weak \(L^p\)-space.

(ii) If \(q_1 \leq q_2\), \(L^{p,q_1} \subset L^{p,q_2}\).

**Lemma A.2** (Hölder). If \(1 \leq p, p_1, p_2, q, q_1, q_2 \leq \infty\), \(\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}\) and \(\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}\), then
\[
\|fg\|_{L^{p,q}} \lesssim \|f\|_{L^{p_1, q_1}} \|g\|_{L^{p_2, q_2}}.
\]

**Lemma A.3** (Dual characterization of \(L^{p,q}\)). If \(1 < p < \infty\) and \(1 \leq q \leq \infty\), then
\[
\|f\|_{L^{p,q}} \sim \sup_{\|g\|_{L^{p', q'}} \leq 1} \left| \int_X fg d\mu \right|.
\]

A measurable function \(f\) is called a *sub-step function* of height \(H\) and width \(W\) if \(f\) is supported on a set \(E\) with measure \(\mu(E) = W\) and \(|f(x)| \lesssim H\) almost everywhere. Let \(T\) be a linear operator that maps the functions on a measure space \((X, \mu_X)\) to functions on another measure space \((Y, \mu_Y)\). We say that \(T\) is *restricted weak-type* \((p, \tilde{p})\) if
\[
\|Tf\|_{L^{p,\infty}} \lesssim HW^{1/p}
\]
for all sub-step functions \(f\) of height \(H\) and width \(W\).

**Theorem A.4** (Marcinkiewicz interpolation theorem). Let \(T\) be a linear operator such that
\[
\langle Tf, g \rangle_{L^2} = \int_Y Tf \tilde{g} d\mu_Y
\]
is well-defined for all simple functions \(f\) and \(g\). Let \(1 \leq p_0, p_1, \tilde{p}_0, \tilde{p}_1 \leq \infty\). Suppose that \(T\) is restricted weak-type \((p_i, \tilde{p}_i)\) with constant \(A_i > 0\) for \(i = 0, 1\). Then,
\[
\|Tf\|_{L^{p_0, q}} \lesssim A_0^{-\theta} A_1^\theta \|f\|_{L^{p_0, q}}.
\]
where \(0 < \theta < 1, \frac{1}{p_0} = \frac{1 - \theta}{p_1} + \frac{\theta}{p_1}, \frac{1}{p_0} = \frac{1 - \theta}{p_1} + \frac{\theta}{p_1}, \tilde{p}_0 > 1\) and \(1 \leq q \leq \infty\).

In this paper, we use the interpolation theorem of the following form:

**Corollary A.5** (Marcinkiewicz interpolation theorem). Let \(T\) be a linear operator. Let \(1 \leq p_1 < p_2 \leq \infty\). Suppose that for \(i = 0, 1\), \(T\) is bounded from \(L^{p_1,1}\) to \(L^{p_1,\infty}\). Then \(T\) is bounded on \(L^p\) for \(p_1 < p < p_2\).

**Proof.** The corollary follows from Theorem A.4, since \(T\) is restricted weak-type \(p_1, p\):

\[
\|T\|_{L^p} = p_i \int_0^\infty \mu(|f| \geq \lambda)^{1/p_i} d\lambda \leq p_i \int_0^H W^{1/p_i} d\lambda = p_i H W^{1/p_i},
\]

for a sub-step function \(f\) of height \(H\) and width \(W\). \(\square\)

**Corollary A.6** (Fractional integration inequality in the Lorentz spaces).

\[
(A.1) \quad \left\| \int_{\mathbb{R}^d} \frac{f(y)}{|x - y|^{d-s}} dy \right\|_{L^{q,r}(\mathbb{R}^d)} \leq \|f\|_{L^{p,r}},
\]

where \(1 < p < q < \infty, 1 \leq r \leq \infty\) and \(\frac{1}{q} = \frac{1}{r} - \frac{s}{d}\). At the endpoints, we have

\[
(A.2) \quad \left\| \int_{\mathbb{R}^d} \frac{f(y)}{|x - y|^{d-s}} dy \right\|_{L^{q,\infty}(\mathbb{R}^d)} \leq \|f\|_{L^1}, \quad \left\| \int_{\mathbb{R}^d} \frac{f(y)}{|x - y|^{d-s}} dy \right\|_{L^{\infty}(\mathbb{R}^d)} \leq \|f\|_{L^{d/s,1}}.
\]

**Proof.** (A.2) follows from [20] Theorem 1, p.119] and duality. Corollary A.5 then gives (A.1). \(\square\)

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Brown University

E-mail address: yhhong@math.brown.edu