SHORT COMMUNICATION

Comments on the Bellman functional for linear time-delay systems

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Abstract
In this note, we present some complementary results on the infinite horizon optimal control for linear time-delay systems. We formally establish some properties of the matrices arising in the Bellman functional, and we prove that any admissible linear control law generates only a class of Bellman functionals for addressed systems. Additionally, experimental results are presented, which illustrate a satisfactory control performance when is compared with an optimal proportional integral industrial control.

KEYWORDS
Bellman functional, optimal control, time-delay systems

1 INTRODUCTION

The optimal control problem for linear time-delay systems have been widely studied. The infinite horizon case for systems with state delay was studied by Krasovskii and Ross. Among important results on the topic, one can mention the relationship between the Pontryaguin maximum principle and dynamic programming, properties of the optimal control in the frequency domain, the proposal of a suboptimal control law for state delay systems, the finite horizon control problem for time delay systems, as well as more recent contributions. In the work of Krasovskii, sufficient optimality conditions were given and the general form of the Bellman functional was suggested. This functional was the starting point for the explicit characterization of the optimal control which was introduced, a few years later, by Ross. However, neither this form for the Bellman functional, nor some of its properties were formally justified. Moreover, Ross and Flügge-Lotz mentioned that, given a class of candidates for the optimal control for this problem, there is a corresponding class of Bellman functionals. In this paper, as our main contribution, we prove that for any candidate for the optimal admissible linear control, there exists only one class of Bellman functionals, which involves only three terms. As a consequence, the sufficient optimality conditions lead to the well known optimal control structure which involves two terms, the first one depends on the instantaneous state and the second is a distributed time delay. Clearly, if a candidate with more terms is proposed for the optimal control, the optimal values for the gains of these additional terms will reduce to zero. Another contribution of our work is the proof of the properties of the matrices of the Bellman functional, which were not given by Ross and Flügge-Lotz. These properties are useful when using the optimality sufficient conditions, when constructing numerically the above mentioned tree matrices and when proving the existence and uniqueness of the solution of the Riccati equation analogue for time delay linear systems in the scalar and matrix cases.

The note is organized as follows: We briefly recapitulate the known results on optimal control of time-delay systems in Section 2. In Section 3, we give the full proof of some properties of the matrices appearing in the Bellman functional. In Section 4, it is demonstrated that any admissible linear control law with general structure, candidate to be optimal control, leads to a single class of Bellman functionals. We validate our results in Section 5 with experimental results for...
the optimal control of a temperature process, which is compared to an optimally tuned industrial proportional integral (PI) control. We end the note with some short concluding remarks.

We denote the space of \( \mathbb{R}^n \)-valued piecewise-continuous functions on \([-h, 0]\) by \( \text{PC}([-h, 0], \mathbb{R}^n) \). For a given initial function \( \varphi(\theta), x_\theta(\varphi) \) denotes the state of the delay system \( x(t + \theta, \varphi), \theta \in [-h, 0] \), with delay \( h > 0 \); when the initial condition is not crucial, the argument \( \varphi \) is omitted. The Euclidian norm for vectors is represented by \( \| \cdot \| \). The set of piecewise continuous functions is equipped with the norm \( \| \varphi \|_h = \sup_{\theta \in [-h, 0]} \| \varphi(\theta) \| \). The notations \( Q > 0 \) indicates that matrix \( Q \) is positive definite. By \( V(x_\theta) \|_{\varphi} \), we denote the time derivative of the functional \( V(x_\theta) \) along the trajectories of system (*), when the control law is \( u^* \).

## 2 Preliminaries and Problem Statement

Consider time-delay systems of the form

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bx(t-h) + Du(t), \\
\varphi &\in \text{PC}([-h, 0], \mathbb{R}^n)
\end{align*}
\]

where the matrices \( A, B \in \mathbb{R}^{nxr}, D \in \mathbb{R}^{nxr} \) are constant, the state \( x(t) \) is in \( \mathbb{D} \), the space of solutions which contains the trivial one, and the control vector \( u(t) \) belongs to \( \mathbb{R}^r, r \leq n \).

Let the following quadratic performance index be given:

\[
J = \int_{0}^{\infty} g(x_t, u(t))dt = \int_{0}^{\infty} (x^T(t)Qx(t) + u^T(t)Ru(t))dt,
\]

with \( Q \in \mathbb{R}^{nxn}, R \in \mathbb{R}^{rxr} \), \( Q = Q^T > 0, R = R^T > 0 \).

The optimal control problem consists in the synthesis of the optimal control \( u^*(t) \) that minimizes the quadratic performance index (2) subject to (1).

Admissible controls for this problem satisfy:

1. \( u = u(x_t) \), in other words, the control is a function of the system state, \( x_t \).
2. The functional \( u(x_t) \) is such that solutions to (1) exist and are unique for \( t \geq 0 \) and for all initial conditions \( \varphi \).
3. The trivial solution of (1) in closed-loop with control law \( u = u(x_t) \) is asymptotically stable.
4. For \( u = u(x_t) \) and all initial conditions \( \varphi \) the performance index has a finite value.

In accordance with the above definition, the general linear admissible control

\[
u_{L}(x_t) = \int_{-h}^{0} d\eta(\theta)x(t + \theta)
\]

in closed-loop with system (1) is an exponentially stable system given by

\[
\dot{x}(t) = Ax(t) + Bx(t-h) + D \int_{-h}^{0} d\eta(\theta)x(t + \theta).
\]

Here \( \eta(\theta) \) is a matrix function of bounded variation on \([-h, 0]\) of appropriate dimensions.\(^{13}\) Notice that the notation given in (3), implies that the control \( u_{L}(\cdot) \) is linear respect to \( x_t \). The Cauchy formula of (4) is:

\[
x(t, \varphi) = K(t)\varphi(0) + \int_{-h}^{0} K(t-s-h)B\varphi(s)d\sigma + \int_{-h}^{0} \int_{-h}^{\sigma} K(t-s-\theta)Dd\eta(\theta)\varphi(\sigma)d\sigma, \quad \forall t \geq 0,
\]

where \( K(t) \) is the fundamental matrix of (4), such that \( K(\theta) = 0 \) for all \( \theta < 0 \), \( K(0) = I_n \), and

\[
\dot{K}(t) = AK(t) + BK(t-h) + D\int_{-h}^{0} d\eta(\theta)K(t + \theta).
\]
Now, the matrix function $\hat{K}(t, \sigma)$ is defined as:

$$\hat{K}(t, \sigma) = K(t - \sigma - h)B + \int_{-h}^{\sigma} K(t - \sigma + \zeta) Dd\eta(\zeta),$$

then, the solution (5) can express as follows:

$$x(t, \varphi) = K(t)\varphi(0) + \int_{-h}^{0} \hat{K}(t, \sigma)\varphi(\sigma)d\sigma, \forall t \geq 0 \quad (7)$$

The sufficient conditions showcasing the Bellman equation for an optimal control for time-delay system are given next:

**Theorem 1** (Ross\(^3\)). *If there exists an admissible control \(u^* = u^*(x_t)\) and a scalar continuous non negative function \(V(x_t)\), \(V = 0\) for all \(x_t = 0\) such that

$$\dot{V}(x_t) \bigg|_{u = u^*} + g(x_t, u^*(x_t)) = 0, \forall t \geq 0, \quad (8)$$

$$\dot{V}(x_t) \bigg|_{u = u^*} + g(x_t, u^*(x_t)) \leq \dot{V}(x_t) \bigg|_{u = u(t)} + g(x_t, u(t)), \forall t \geq 0, \quad (9)$$

for all admissible \(u(t)\), then \(u^*(t)\) is an optimal control. Furthermore \(V(\varphi) = J(\varphi, u^*)\) is the optimal value of the performance index \(J\).*

**Proof of Theorem 1.** For an initial state \(\varphi\), integrating (8) from 0 to \(\infty\) (which is valid due to the asymptotic stability) gives

$$V[\varphi] = \int_{0}^{\infty} g(x_t, u^*(x_t))dt = J[\varphi, u^*]. \quad (10)$$

From (9), if \(u(x_t)\) is any other admissible control, then

$$\dot{V}(x_t) \bigg|_{u = u^*} \geq -g(x_t, u(x_t)). \quad (11)$$

Integrating both sides of this inequality (again using the asymptotic stability property of admissible controls) yields

$$V[\varphi] \leq \int_{0}^{\infty} g(x_t, u(x_t))dt = J[\varphi, u]. \quad (12)$$

Consequently, \(J[\varphi, u^*] \leq J[\varphi, u]\), and we conclude that \(u^*\) is optimal among admissible controls. \(\blacksquare\)

**Remark 1.** The functional \(V(x_t)\), called Bellman functional, is used to establish necessary and sufficient conditions of optimality for time-delay systems. In fact, as reported by Ross,\(^3\) the necessary conditions for optimality can be stated as follows: Given an optimal structure \(u^*(x_t)\), the optimal value of the performance index given by \(V(\varphi)\), is the representation of the performance index with respect to the control \(u^*\), then \(V(x_t)\) satisfies the Bellman equation given by (8), see Ross,\(^3\) and Knowles for delay free nonlinear systems.\(^{14}\) Notice that for time delay linear systems, the necessary conditions given in Ross\(^3\) involve the explicit construction of \(V(\varphi)\). The sufficient conditions of optimality of Theorem 1, can be formulated as: If there exists a control \(u^*\) and a positive definite functional \(V(.)\) which satisfies expressions (8) and (9) for all admissible control \(u\), then \(u^*\) is optimal.

### 3 Construction of the Bellman Functional

The construction of the Bellman functional for time-delay systems is presented in the following proposition. The proof of this proposition was not given in the Ross article.\(^3\)
**Proposition 1** (Ross\(^1\)). If \( u_t = u_L(x_t) \), \( \forall t \geq 0 \), is an admissible linear control, \( \varphi \) is an initial condition function on \([- h, 0]\), then the function

\[
V_L(\varphi) = J(\varphi, u_L) = \int_0^\infty (x^T(t)Qx(t) + u_L(t)^TRu_L(t))dt,
\]

(13)

**Proof of Proposition 1.** Replacing \( x(t, \varphi) \) under the integral by the Cauchy formula (7) into the admissible control structure (3) gives

\[
u_L(x_t) = \int_{-h}^{0} d\eta(\theta) \left( K(t + \theta)\varphi(0) + \int_{-h}^{0} \dot{K}(t + \theta, \sigma)\varphi(\sigma)d\sigma \right).
\]

(15)

Substituting the expressions (7) and (15) into the performance index (2) yields

\[
J = \int_0^\infty \left( \begin{pmatrix} K(t)\varphi(0) + \int_{-h}^{0} \dot{K}(t, \sigma)\varphi(\sigma)d\sigma \end{pmatrix}^T Q \begin{pmatrix} K(t)\varphi(0) + \int_{-h}^{0} \dot{K}(t, \sigma)\varphi(\sigma)d\sigma \end{pmatrix} + \left( \int_{-h}^{0} d\eta(\theta) \left( K(t + \theta)\varphi(0) + \int_{-h}^{0} \dot{K}(t + \theta, \sigma)\varphi(\sigma)d\sigma \right) \right)^T R \left( \int_{-h}^{0} d\eta(\theta) \left( K(t + \theta)\varphi(0) + \int_{-h}^{0} \dot{K}(t + \theta, \sigma)\varphi(\sigma)d\sigma \right) \right) \right) dt.
\]

Algebraic calculations lead to

\[
J(\varphi, u_L) = \varphi^T(0)\Pi_0\varphi(0) + 2\varphi^T(0)\int_{-h}^{0} \Pi_1(\theta)\varphi(\theta)d\theta + \int_{-h}^{0} \int_{-h}^{0} \varphi^T(\xi)\Pi_2(\xi, \theta)\varphi(\theta)d\xi d\theta.
\]

(14)

where

(i) \( \Pi_0 > 0 \) is a symmetric positive matrix.

(ii) \( \Pi_1(\theta) \) is defined on \([- h, 0]\).

(iii) \( \Pi_2(\xi, \theta) \) is defined on \( \xi, \theta \in [- h, 0] \),

\[
\Pi_2(\xi, \theta) = \Pi_2(\theta, \xi).
\]

hence the form (14) is established.

The proof of the properties of the matrices \( \Pi_0, \Pi_1(\theta), \) and \( \Pi_2(\xi, \theta) \) are given next.

To prove property (i), notice that the expression of matrix \( \Pi_0 \) given by (16) rewrites as

\[
\Pi_0 = \int_0^\infty \begin{bmatrix} K(t) \\ \int_{-h}^{0} d\eta(\delta_1)K(t + \delta_1) \end{bmatrix}^T \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} K(t) \\ \int_{-h}^{0} d\eta(\delta_2)K(t + \delta_2) \end{bmatrix} dt.
\]

(16)

(17)

(18)
As $Q, R > 0$, $\Pi_0$ is a symmetric positive matrix.

To prove property (ii), consider the expression (17) for $\Pi_1(\theta)$. As $K(t)$ is the fundamental matrix of the exponentially stable closed-loop system (5), it is also exponentially stable, hence all the integral summands of (17) are well defined on $[-h, 0]$.

Finally, for (iii), consider the expression (18) for $\Pi_2(\theta, \xi)$. Using the arguments for ii) (18) is also well defined on $[-h, 0]$.

The expressions $\Pi_2^T(\xi, \theta)$ and $\Pi_2(\theta, \xi)$ obtained from (18),

$$
\Pi_2^T(\xi, \theta) = \int_0^\infty \left( \dot{K}^T(t, \theta)Q\dot{K}(t, \xi) + \int_{-h}^0 \int_{-h}^0 \dot{K}^T(t + \delta_1, \theta)\eta^T(\delta_1)R\eta(\delta_2)\dot{K}(t + \delta_2, \xi) \right) dt,
$$

and

$$
\Pi_2(\theta, \xi) = \int_0^\infty \left( \dot{K}^T(t, \theta)Q\dot{K}(t, \xi) + \int_{-h}^0 \int_{-h}^0 \dot{K}^T(t + \delta_1, \theta)\eta^T(\delta_1)R\eta(\delta_2)\dot{K}(t + \delta_2, \xi) \right) dt,
$$

are equal, which concludes the proof. 

**Remark 2.** Notice that for any linear optimal control structure candidate, the resulting class of functionals $V[\varphi]$ always involves exactly three terms, characterized by three matrices satisfying properties (i)–(iii). The formal proof of this result, which is proposed here, can be useful in the study of the optimal control problem for more general classes of delay systems, such as those of neutral type.

### 4 | ON THE FORM OF THE OPTIMAL CONTROL FOR TIME-DELAY SYSTEMS

At this point, we conclude that the form of the functional for the general control (3) is also a tree term functional of the form (14). Its expression in terms of the state $x_i$ is

$$
V(x_i) = x^T(t)\Pi_0 x(t) + 2x^T(t)\int_{-h}^0 \Pi_1(\theta)x(t + \theta)d\theta + \int_{-h}^0 \int_{-h}^0 x^T(t + \xi)\Pi_2(\xi, \theta)x(t + \theta)d\theta d\xi, \quad \forall t \geq 0. \tag{21}
$$

According to (13) $V(x_i) \geq 0$. Assuming that $x_i$ is a trajectory of system (1), and defining

$$
H(x_i, u) = \dot{V}(x_i)|_{u\text{-admissible}} + x^T(t)Qx(t) + u^T(t)Ru(t), \tag{22}
$$

where the time derivative of (21) along the trajectories of the system (1) is

$$
\dot{V}(x_i)|_{u\text{-admissible}} = (Ax(t) + Bx(t - h) + Du(t))^T\Pi_0 x(t) + x^T(t)\Pi_0(Ax(t) + Bx(t - h) + Du(t))
$$

$$
+ (Ax(t) + Bx(t - h) + Du(t))^T \int_{-h}^0 \Pi_1(\theta)x(t + \theta)d\theta + x^T(t)\int_{-h}^0 \Pi_1(\theta)\frac{\partial}{\partial \theta}x(t + \theta)d\theta
$$

$$
+ \int_{-h}^0 \int_{-h}^0 \frac{\partial}{\partial \xi}(x^T(t + \xi)\Pi_2(\xi, \theta)x(t + \theta)d\theta d\xi + \int_{-h}^0 \int_{-h}^0 x^T(t + \xi)\Pi_2(\xi, \theta)\frac{\partial}{\partial \theta}(x(t + \theta))d\theta d\xi. \tag{23}
$$

Substituting (23) into (22), implies that

$$
H(x_i, u) = x^T(t)(A^T\Pi_0 + \Pi_0 A + Q)x(t) + 2x^T(t)\Pi_0 Bx(t - h) + 2x^T(t)\Pi_0 Du(t) + 2x^T(t)A^T \int_{-h}^0 \Pi_1(\theta)x(t + \theta)d\theta
$$

$$
+ 2x^T(t - h)B^T \int_{-h}^0 \Pi_1(\theta)x(t + \theta)d\theta + 2u^T(t)D^T \int_{-h}^0 \Pi_1(\theta)x(t + \theta)d\theta + x^T(t)\int_{-h}^0 \Pi_1(\theta)\frac{\partial}{\partial \theta}x(t + \theta)d\theta
$$
By the fundamental theorem of calculus of variations\textsuperscript{15} 

\[
\min_{u\text{--admissible}} H(x_t, u) = H(x^*_t, u^*),
\]

\[
\frac{\partial}{\partial u} H(x_t, u) = 2D^T \Pi_0 x(t) + 2D^T \int_{-h}^{0} \Pi_1(\theta)x(t + \theta)d\theta + 2Ru(t) = 0,
\]

then, we have that 

\[
u^*(t) = -R^{-1}D^T \Pi_0 x(t) - R^{-1}D^T \int_{-h}^{0} \Pi_1(\theta)x(t + \theta)d\theta. \quad (25)
\]

Moreover, as 

\[
\frac{\partial^2}{\partial u^2} H(x_t, u) = 2R > 0,
\]

we conclude that \(u^*(t)\) is a local minimum of (22).

The necessary and sufficient conditions for and optimal control for time-delay systems are given in the seminal result reminded bellow:

**Theorem 2** (Ross\textsuperscript{3}). A linear control law

\[
u^*(t) = -R^{-1}D^T \Pi_0 x(t) - R^{-1}D^T \int_{-h}^{0} \Pi_1(\theta)x(t + \theta)d\theta. \quad (26)
\]

provides the global minimum of the performance index (2) for the dynamical system (1) if:

(a) \(u^*(x_t)\) is a stabilizing control law (since \(u^*\) is linear, stability and admissibility are equivalent)

(b) \(\Pi_0\) is a symmetric positive definite matrix which, together with the \(n \times n\) array \(\Pi_1(\theta)\) of functions defined on \([-h, 0]\), and the \(n \times n\) array, \(\Pi_2(\xi, \theta)\) of functions in two variables having domain \(\xi, \theta \in [-h, 0]\), satisfies the relations:

1. \(A^T \Pi_0 + \Pi_0 A - \Pi_0 DR^{-1}D^T \Pi_0 + \Pi_1(0) + \Pi_1(0)Q = 0,\)
2. \(\frac{d\Pi_1(\theta)}{d\theta} = (A^T - \Pi_0 DR^{-1}D^T) \Pi_1(\theta) + \Pi_2(\theta, \theta), -h \leq \theta \leq 0,\)
3. \(\frac{d\Pi_2(\xi, \theta)}{d\xi} + \frac{d\Pi_2(\xi, \theta)}{d\theta} = -\Pi_1(\xi)DR^{-1}D^T \Pi_1(\theta), -h \leq \xi \leq 0, -h \leq \theta \leq 0,\)
4. \(\Pi_1(-h) = \Pi_0 B,\)
5. \(\Pi_2(-h, \theta) = B^T \Pi_1(\theta), -h \leq \theta \leq 0.\) \quad (27)

Furthermore, under these conditions, the representation of (2) in terms of the initial function is

\[
J(\varphi, u^*) = \varphi^T(0) \Pi_0 \varphi(0) + 2\varphi^T(0) \int_{-h}^{0} \Pi_1(\theta) \varphi(\theta)d\theta + \int_{-h}^{0} \int_{-h}^{0} \varphi^T(\xi) \Pi_2(\xi, \theta) \varphi(\theta) d\xi d\theta. \quad (28)
\]

**Proof of Theorem 2.** For \(u^*(t)\) in Equation (25) to stabilize system (1) and to be a global minimum, it has to satisfy the Hamilton–Jacobi–Bellman equation (8). If \(t = 0, x_0 = \varphi,\) then

\[
\dot{V}(\varphi) \bigg|_{u = u^*} + \varphi^T(0) Q \varphi(0) + u^T(0) Ru^*(0) = 0. \quad (29)
\]
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where

\[
\dot{V}(\varphi) \bigg|_{u_*} = \varphi^T(0) (A^T \Pi_0 + \Pi_0 A) \varphi(0) + 2 \varphi^T(0) \Pi_0 B \varphi(-h) + 2 u^*_T(0) D^T \Pi_0 \varphi(0) + 2 \varphi^T(0) A^T \int_{-h}^{0} \Pi_1(\theta) \varphi(\theta) \, d\theta \\
+ 2 \varphi^T(-h) B^T \int_{-h}^{0} \Pi_1(\theta) \varphi(\theta) \, d\theta + 2 \varphi^T(0) A^T \int_{-h}^{0} \Pi_1(\theta) \varphi(\theta) \, d\theta \\
+ 2 \varphi^T(-h) B^T \int_{-h}^{0} \Pi_1(\theta) \varphi(\theta) \, d\theta + 2 u^*_T(0) D^T \int_{-h}^{0} \Pi_1(\theta) \varphi(\theta) \, d\theta \\
+ 2 \varphi^T(0) \int_{-h}^{0} \Pi_1(\theta) \frac{d\varphi(\theta)}{d\theta} \, d\theta + \int_{-h}^{0} \int_{-h}^{0} \frac{d\varphi^T(\xi)}{d\xi} \Pi_2(\xi, \theta) \varphi(\theta) \, d\theta \, d\xi + \int_{-h}^{0} \int_{-h}^{0} \varphi^T(\xi) \Pi_2(\xi, \theta) \frac{d\varphi(\theta)}{d\theta} \, d\theta \, d\xi,
\]

and

\[
u^*(\varphi) = -R^{-1} D^T \Pi_0 \varphi(0) - R^{-1} D^T \int_{-h}^{0} \Pi_1(\theta) \varphi(\theta) \, d\theta.
\]

Substituting (30) and (31) into left-hand of (29) gives

\[
\dot{V}(\varphi) \bigg|_{u_*} + \varphi^T(0) Q \varphi(0) + u^*_T(0) R u^*(0) \\
= \varphi^T(0) (A^T \Pi_0 + \Pi_0 A - \Pi_0 DR^{-1} D^T \Pi_0 + \Pi_1^T(0) + \Pi_1(0) + Q) \varphi(0) \\
+ 2 \varphi^T(0) \int_{-h}^{0} \left( (A^T - \Pi_0 DR^{-1} D^T) \Pi_1(\theta) + \Pi_2(0, \theta) - \frac{d\Pi_1(\theta)}{d\theta} \right) \varphi(\theta) \, d\theta \\
- \int_{-h}^{0} \int_{-h}^{0} \varphi^T(\xi) \left( \frac{\partial \Pi_2(\xi, \theta)}{\partial \xi} + \frac{\partial \Pi_2(\xi, \theta)}{\partial \theta} + \Pi_1^T(\xi) DR^{-1} D^T \Pi_1(\theta) \right) \varphi(\theta) \, d\theta \, d\xi \\
+ 2 \varphi^T(0) (\Pi_0 B - \Pi_1(-h)) \varphi(-h) + 2 \varphi^T(-h) \int_{-h}^{0} (B^T \Pi_1(\theta) - \Pi_2(-h, \theta)) \varphi(\theta) \, d\theta.
\]

Therefore, the Equation (29) is satisfied if only if the expressions 1–5 are true.

Remark 3. If the Bellman functional has only three terms, when the minimization procedure is used (by using the sufficient conditions for optimality), it always gives the optimal control structure given by (31). Moreover, if a candidate with more terms is proposed for the optimal control \(u^*(\cdot)\), the optimal values for the gains of these terms are zero.

5 EXPERIMENTAL RESULTS

Our experimental prototype (40cm × 20cm × 20cm) emulates a real atmospheric dehydrator. The platform has a drying section with a wind tunnel as output and a pipe that recycles the hot air into the system and induces a state delay in the mathematical model. The main parts of the dehydrator are: a temperature sensor LM35 with a measurement rate of 10 mV/°C; a fan producing a constant air flow with velocity of 2.1 m/s; an electrical grid (actuator) as heat source; a control voltage in the range 0–120 Vrms of AC power, which regulates the temperature inside the chamber. The optimal control for time delay systems is programmed on a MyRIO-National Instruments target which uses LabVIEW software and sampling time of 500 milliseconds. For performance evaluation, the optimal control for delayed systems is compared to an optimally tuned PI control implemented with an industrial PID Honeywell DC1040 with following characteristics: maximum precision of ±1°C of cold junction compensation, ±5% of maximum deviation in the linear output 4–20 mA, automatic compensation of dead zone, and a thermocouple J type of extended rate in the input. A sketch of the experimental platform is shown in Figure 1.

For the optimal control strategy, the process is modeled as

\[
\dot{x}(t) = a_0 x(t) + a_1 x(t-h) + bu(t),
\]

where \(h\) is the delay induced by the hot air recycling loop, the state variable \(x(t)\) is the temperature value, the control input \(u(t)\) is the energy applied to the actuator and the initial condition is \(x=17\)°C. It is reported in the literature
that the poor performance observed\(^\text{16}\) when using a linear model of the form (32) can be improved\(^\text{17,18}\) by using a nonlinear model with nominal linear part given by (32). Here, we introduce an adaptive scheme based on the online Recursive Least Square method\(^\text{19}\). the estimated model parameters converge to \(a_0 = -0.079931\), \(a_1 = 0.053467\), and \(b = 0.002374\). The delay \(h = 4\) seconds is heuristically estimated by comparison of different measurements in the recirculating tube. The estimation error converges to 0.000149. In each cycle, the plant parameters are identified, but the optimal control given by (25) is recalculated only if the temperature has a deviation greater than \(\pm 0.3\%\) of the set point (SP) fixed to 25°C. It is worthy of mention that a similar criterion is used in some commercial products, such as the PID Honeywell UDC 6000\(^\text{20}\). We chose \(Q = 15\) and \(R = 1\) in the performance index (2), and for the identified parameters, we solve the set of Equation (27) numerically as in Ross\(^\text{3}\). The digital implementation of the optimal control is explained in López-Labra et al.\(^\text{16}\) Figure 2 displays the time behavior of the plant parameters and their convergence.

For the optimal PI control implemented with an industrial Honeywell DC1040 controller, the mathematical model is assumed to be of the form

\[
G(s) = \frac{Y(s)}{U(s)} = \frac{ke^{-\tau s}}{Ts + 1},
\]

with parameters \(k = 0.01455\), \(T = 150\) seconds, and \(\tau = 3\) seconds obtained from the step response (120 VCA rms, applied to the grid, representing 100% of the control signal). The measured initial condition is 17°C. These parameters are used to tune the optimal PI controller via the linear quadratic regulator with state extension approach\(^\text{21}\). Here, instead of a predictor, as proposed by He\(^\text{21}\), we use the dead time compensation function of the DC1040 controller. For a quadratic performance index with \(Q=\text{diag}[15, 15]\) and \(R = 1\), the resulting optimal PI, has gains \(K_P = 79.51\) and \(K_I = 3.873\) (the derivative action is set to zero).

The temperature, control signals, and error signals of both controllers, shown in Figure 3, indicate that the optimal control for time delay systems presents a better performance: no overshoot and smaller convergence error.

Table 1 shows the closed-loop system performance measured by integral absolute error (IAE) criterion and the energy consumption of both controllers.

Clearly, this table gives experimental evidence to conclude that the optimal control for time delay systems has a better performance than the optimal PI control. In contrast with previous results,\(^\text{16}\) it is possible to use a linear model only for the prototype, if it is improved with an adaptive scheme.
Temperature response, control, and error signals with set point (SP) adjusted to 25°C using both controllers

TABLE 1 Comparative table of numerical values for the IAE and energy consumption for the optimal control and optimal PI control

| Control strategy  | IAE-optimal control | Energy consumption (Wh) |
|-------------------|---------------------|-------------------------|
| Optimal control   | 718.68              | 57.26                   |
| Optimal PI control| 905.85              | 55.55                   |

6  | CONCLUDING REMARKS

In this note, we have proved some interesting complementary results concerning some properties of the Bellman functional, and on the form of the optimal control. It appears that establishing formally that the functional has a given structure is a crucial step in the solution of each of these problems. The experimental comparison made between the optimal control for time delay systems (with adaptive criterion) and an Optimal PI controller implemented in an industrial PID, shows the interest of our proposal.

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DATA AVAILABILITY STATEMENT

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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