Conformal Non-Abelian Thirring Models

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Abstract

The Lie-Poisson structure of non-Abelian Thirring models is discussed and the Hamiltonian quantization of these theories is carried out. The consistency of the Hamiltonian quantization with the path integral method is established. It is shown that the space of non-Abelian Thirring models contains the nonperturbative conformal points which are in one-to-one correspondence with general solutions of the Virasoro master equation. A BRST nature of the master equation is clarified.

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1 Introduction

Thirring models appear to be of a great interest in string theory. This interest has mainly come from the idea that the space of all conformal Thirring models is the space of string compactifications. The Abelian Fermionic Thirring models have been considered as the most appropriate candidates to describe all toroidal string compactifications [1]. In their turn the non-Abelian (Lefton-Righton) Thirring models have been proposed for the description of the most general (left-right asymmetric) string compactifications on group manifolds [2, 3]. Therefore, the space of all conformal Thirring models seems to collect all symmetric string vacua, which could form the multitude of conformal backgrounds appropriate to the formulation of background independent closed string field theory [4].

The natural coordinates in the Thirring model-space might be the coupling constants of the current-current interaction. The whole model-space, however, may have additional dimensions parameterized by some extra variables coming from the geometric formulation of the Thirring model [5, 6]. Therefore, it would be illuminating if one could explore the Thirring model at all the possible values of its couplings. However, this seems to be beyond our present analytical abilities. Most of the difficulty resides in the highly non-linear character of the current-current interaction of the Thirring theory. Given our present knowledge, the theory is tractable only when it possesses either affine symmetry or quantum group symmetry (which might turn out to be a sort of deformation of the former.) In this paper we will not discuss the quantum group symmetry of Thirring models but rather affine symmetries. We will show that affine symmetries are intimately related to the conformal invariance of the Thirring model.

We will begin with a description of the two formulations of the non-Abelian Thirring model. Namely, Fermionic and Bosonic formulations. We will show when these two classically distinguished theories become equivalent at the quantum level. In sect. 3 we will discuss the Lie-Poisson structure of the classical Bosonic Thirring model. We will demonstrate the interconnection between this algebraic structure of the Thirring model and self-duality of its two-dimensional fields valued in Lie algebras. Sect. 4 contains the Hamiltonian quantization of the non-Abelian Thirring model on the basis of the Lie-
Poisson structure of the classical theory. It will be shown that the given quantization will be consistent as long as the conformal symmetry is present. We will find that consistency requires particular values of the Thirring coupling constants which are in one-to-one correspondence with solutions of the so-called Virasoro master equation [7]. The Virasoro master equation describes in conformal field theory the most general embedding of the Virasoro algebra into the affine algebra through the affine-Virasoro construction [7]. The affine-Virasoro construction in its turn is the most general bilinear of the affine currents. Thus, the non-Abelian Thirring models provide a natural sigma model interpretation to conformal theories based on general solutions of the master equation. It is interesting that the so-called Sugawara solution of the Virasoro master equation has been derived by Dashen and Frishman from the isoscalar non-Abelian Thirring model two decades ago [12]. Therefore we call all conformal points, which are solutions of the master equation, Dashen-Frishman conformal points [6,10,11]. In sect. 5 we will derive these same Dashen-Frishman conformal points from the Fermionic Thirring model. In the process we will justify the conformal symmetry at the Dashen-Frishman conformal points by using the path integral method. In sect. 6 we will show that the Dashen-Frishman conformal points appear to be a consistence condition for the BRST quantization of the Bosonic-Thirring model coupling to the two dimensional gravity. Finally, sect. 7 contains some concluding remarks.

In the appendix, we will discuss the representation of the affine-Virasoro construction for the affine group $SU(2)$. The $SU(2)$ case is interesting because the minimal conformal series can be described with the $SU(2)$ non-Abelian Fermionic Thirring models [11].

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*Note that our field-theoretic realization of the affine-Virasoro construction [6,10,11] differs from attempts in [8,9].

†In addition to the Dashen-Frishman conformal points non-Abelian Thirring models may have so-called “Higgs conformal points” [13] which are descendants of Dashen-Frishman conformal points. Higgs conformal points come into being due to duality symmetry which Thirring models possess at the quantum level [2, 13].
2 Fermionic and Bosonic non-Abelian Thirring models

Let us start with a description of the Fermionic non-Abelian Thirring model. The action is given by

\[ S_F = \frac{1}{4\pi} \int d^2 z (\bar{\psi}_L \partial \psi_L + \bar{\psi}_R \partial \psi_R - S_{a\bar{a}} J_L^a J_R^{\bar{a}}), \]  

(2.1)

where \( \psi_L \) and \( \psi_R \) are complex Weyl spinors (in general carrying a flavor) transforming as the fundamental representations of given groups \( G_L \) and \( G_R \) respectively. The last term in (2.1) describes the general interaction between fermionic currents

\[ J_L^a = \bar{\psi}_L t^a \psi_L, \quad J_R^{\bar{a}} = \bar{\psi}_R t^\bar{a} \psi_R. \]  

(2.2)

Here \( t^a, t^\bar{a} \) are the generators in the Lie algebras \( G_L, G_R \).

\[ [t^a, t^{b}] = i f_{abc} t^c, \quad a, b, c = 1, 2, \ldots, \text{dim} G_L, \]  

(2.3)

\[ [t^\bar{a}, t^{\bar{b}}] = i f_{\bar{a}\bar{b}\bar{c}} t^\bar{c}, \quad \bar{a}, \bar{b}, \bar{c} = 1, 2, \ldots, \text{dim} G_R. \]

\( S_{a\bar{a}} \) is a coupling constant matrix.

We have used also the following notations

\[ \partial \equiv \partial/\partial z, \quad \bar{\partial} \equiv \partial/\partial \bar{z}, \]

\[ z = (t^E + ix)/\sqrt{2}, \quad \bar{z} = (t^E - ix)/\sqrt{2}, \]

where \( t^E = it \). We will use \( z \) and \( \bar{z} \) to denote Euclidean coordinates, whereas \( x \) and \( t \) will signify Minkowski coordinates. We follow the convention \( d^2 z \equiv idz d\bar{z} = -dxdt^E/2. \)

In addition to the non-Abelian interaction, we always can include the \( U(1) \) current-current interaction in the Fermionic action. The \( U(1) \) currents are defined as

\[ J_L = \bar{\psi}_L \psi_L, \quad J_R = \bar{\psi}_R \psi_R. \]  

(2.4)

where a sum over all the internal indices is assumed.

It is worth mentioning that the classical Fermionic non-Abelian Thirring model obviously possesses the global \( G_L \times G_R \) invariance provided the coupling matrix \( S_{a\bar{a}} \) also
transforms as the adjoint representation of the $G_L \times G_R$ group. Therefore, the physically
distinguished couplings are defined as

\[ \tilde{S}_{a\bar{a}} = \{S_{a\bar{a}}\}/\text{ad}(G_L \times G_R), \]  

(2.5)

where $\{S_{a\bar{a}}\}$ is a set of all the consistent values of the Thirring couplings $S_{a\bar{a}}$. However
at the quantum level the given symmetry can be broken or reduced to a smaller one. We
will show that the exact symmetry of the conformal points is the diagonal subgroup of
the group $G_L \times G_R$. Due to this symmetry, the space of Thirring models ought perhaps
to be a coset.

The action of the Bosonic Thirring model is formulated as follows

\[ S_B = \int \left[ L_L(k_L, g_L) + L_R(k_R, g_R) + L_{\text{int}}(g_L, g_R; S) \right], \]  

(2.6)

where these three terms respectively are given by

\[
\begin{align*}
4\pi L_L(k_L, g_L) &= -k_L \left[ (1/2) \text{tr}_L |g_L^{-1}dg_L|^2 + (i/3)d^{-1}\text{tr}_L(g_L^{-1}dg_L)^3 \right], \\
4\pi L_R(k_R, g_R) &= -k_R \left[ (1/2) \text{tr}_R |g_R^{-1}dg_R|^2 + (i/3)d^{-1}\text{tr}_R(g_R^{-1}dg_R)^3 \right], \\
L_{\text{int}}(g_L, g_R; S) &= -(k_Lk_R/4\pi)\text{tr}_L\text{tr}_Rg_L^{-1}\partial g_L S \bar{\partial} g_R g_R^{-1}dzd\bar{z},
\end{align*}
\]

(2.7)

with the coupling $S$ belonging to the direct product $G_L \otimes G_R$. Here the fields $g_L$ and $g_R$
take their values in the Lie groups $G_L$ and $G_R$, respectively. $k_L$, $k_R$ are central elements
in the affine algebras $\hat{G}_L$, $\hat{G}_R$. The symbols $\text{tr}_L$, $\text{tr}_R$ indicate tracing over the group indices
of $G_L$, $G_R$.

The point to be made is that the non-Abelian Bosonic Thirring model in eq. (2.6)
becomes equivalent to the Lefton-Righton Thirring model [13] when the fields $g_L$, $g_R$ obey
the conditions

\[
\begin{align*}
\partial_+ g_R g_R^{-1} &= -k_L \text{tr}_L S \ g_L^{-1}\partial_+ g_L, \\
g_L^{-1}\partial_- g_L &= -k_R \text{tr}_R S \partial_- g_R g_R^{-1},
\end{align*}
\]

(2.8)

where we have used the light cone coordinates

\[
\begin{align*}
x^+ &= x + t, & x^- &= x - t, \\
\partial_+ &\equiv \partial/\partial x^+, & \partial_- &\equiv \partial/\partial x^-.
\end{align*}
\]
In the limit $S \to 0$, equations (2.8) go to the self-duality conditions for the non-Abelian fields $g_L$, $g_R$. Therefore, we will call equations (2.8) self-duality conditions. At the quantum level we will show that the conformal points of the non-Abelian Bosonic Thirring model with the action given by eq. (2.6) are in one-to-one correspondence to the conformal points of the Lefton-Righton Thirring model [13].

Classically the theories (2.1) and (2.6) are distinguished, whatever conditions we may impose upon them. However, at the quantum level the fermionic and bosonic non-Abelian Thirring models become indistinguishable under the following conditions: 1) the two Weyl spinors $\psi_i^R$ and $\bar{\psi}_i^L$ carry flavor indices $\bar{i} = 1, ..., k_R$ and $i = 1, ..., k_L$ and 2) the coupling constant matrix $S$ is invertible. When these conditions are fulfilled the statistical sums of the two models are identical [14]. Note that the second condition is also necessary for the Lefton-Righton Thirring model to be written in first order form [13]. Since we are going to use the Fermi-Bose equivalence in what following, we recall the main steps of its proof.

It has been shown in [13] that the partition function of the Bosonic Thirring model possesses the property

$$Z_B(k_L, k_R; S) = J Z_B(k'_L, k'_R; S') Z_B(k_L, k_R; 0) Z_{gh}, \quad (2.9)$$

where

$$Z_B(k_L, k_R; S) = \int \mathcal{D}g_L \mathcal{D}g_R \exp^{-S_B}, \quad (2.10)$$

$$Z_{gh} = \int \mathcal{D}b \mathcal{D}\bar{b} \mathcal{D}c \mathcal{D}\bar{c} \exp \left[ - \int d^2 z (b \partial c + \bar{b} \partial \bar{c}) \right],$$

with $(b, c)$ and $(\bar{b}, \bar{c})$ Grassmann odd auxiliary fields from the adjoint representations of $\mathcal{G}_L$ and $\mathcal{G}_R$ respectively. The constant $J$ in eq. (2.9) is a Jacobian factor due to the change in the measure of the auxiliary fields [2]. The relation (2.9) allows us to see how the Bosonic Thirring model partition function transforms under inversions of the Thirring couplings

$$S_{a\bar{a}} \to S'_{a\bar{a}} = - (k'_L k'_R S_{ab} S_{\bar{b}\bar{b}})^{-1} S_{b\bar{a}} \quad (2.11)$$

and simultaneous mirror reflections of the central charges

$$k_L \to k'_L = -k_R - c_2(G_R)/2, \quad k_R \to k'_R = -k_L - c_2(G_L)/2, \quad (2.12)$$
where $c_2(G_L)$ and $c_2(G_R)$ are quadratic Casimir operator eigenvalues referring to the adjoint representations of $G_L$ and $G_R$ respectively.

To link the partition function of the Bosonic Thirring model with the partition function of the Fermionic Thirring model we consider an equivalent dual formulation of the Fermionic Thirring Lagrangian

$$\tilde{L}_F(\psi_R, \psi_L; A_+, A_-; S) = \bar{\psi}_L \partial \psi_L + \bar{\psi}_R \partial \psi_R + A_- J^a_L + A_+ J^a_R + (S^{-1})_{a\bar{a}} A^a A^\bar{a}. \quad (2.13)$$

It is easy to see that this theory gives rise to the Fermionic Thirring model after eliminating the auxiliary fields $A_-$ and $A_+$ by using their algebraic equations of motion. Due to the algebraic character of the auxiliary fields, the dual equivalence must hold also at the quantum level. Then the Fermionic functional integrals in the partition function

$$Z_F = \int \mathcal{D} \psi_L \mathcal{D} \psi_R \exp^{-S_F} \quad (2.14)$$

can be computed by the chiral anomalies resulting in the non-local functional of the auxiliary fields [15]. These non-local expressions transform to the WZNW models after making the change

$$A_- \rightarrow \partial g_R g^{-1}_R, \quad A_+ \rightarrow g^{-1}_L \partial g_L. \quad (2.15)$$

Arising within the process the partition function of the Bosonic Thirring model leads us via property (2.9) to the remarkable identity [14]

$$\frac{Z_B(k_L, k_R; S)}{Z_B(k_L, k_R; 0)} = \frac{Z_F(k_L, k_R; S)}{Z_F(k_L, k_R; 0)}. \quad (2.16)$$

Apparently, in the limit $S = 0$ the identity (16) becomes trivial. This is not surprising because as we demonstrated in [10] in order to fermionize the WZNW models (or $S = 0$ Bosonic Thirring model) with arbitrary levels, we have to use the Fermionic Thirring model at the so-called isoscalar Dashen-Frishman conformal points, not at $S_{a\bar{a}} = 0$. We will discuss this procedure in sect. 6 of the present paper. Meanwhile, when $S \neq 0$, the identity (2.16) is very fruitful since it allows us to establish an equivalence between the conformal points of the Fermionic and Bosonic versions of the Thirring model, and to clarify its geometrical meaning [5,4]. Furthermore, we can easily show that the ratio of the partition functions for the non-Abelian Bosonic Thirring model is equal to the ratio of the
partition functions of the Lefton-Righton Thirring model. Indeed the partition functions of the non-Abelian and Lefton-Righton Thirring models differ only by the ghost partition function. This ghost partition function does not depend on the coupling constants. Therefore, the ghost contributions are the same in the nominator and denominator of eq. (2.16) and, hence, cancel each other.

It follows also from formula (2.16) that only chargeless combinations of fermions contribute to the normalized partition function. Therefore, to preserve the Lorentz symmetry at the quantum level, it would be sufficient to keep it apparent only for the mentioned composite chargeless fields. While fermions themselves might not be of any certain spin.

Later on we will discuss how this phenomena may affect the existence of massive deformations of conformal non-Abelian Thirring models.

3 Lie-Poisson structure of Hamiltonian system

Identity (2.16) will continue to be somewhat formal until we are able to calculate the functional integrals for the non-Abelian Bosonic and Fermionic Thirring models. Apparently, this seems to be very difficult for arbitrary values of the coupling constant matrix $S_{a\bar{a}}$. However, it might be possible at some particular values of $S_{a\bar{a}}$ at which the theory could be quantized nonperturbatively.

All currently known nonperturbative quantum methods are essentially based on some symmetries which can be promoted through Poisson brackets to the quantum level. Therefore, the first thing we have to learn about the non-Abelian Thirring model is to find its phase space symmetries. This will be a subject of the present section. Specifically we will focus on the symmetries of the non-Abelian Bosonic Thirring model.

The non-Abelian Bosonic Thirring model is a highly nonlinear field theory. Therefore, its analysis is very complicated. However, its symmetries can be uncovered from another simpler model which has a very nice geometrical structure [5, 6]. This geometrical theory is described by the following action

$$ S_G = \int \alpha_L + \int \alpha_R + S_H, $$

(3.17)
where $\alpha_L$ and $\alpha_R$ are canonical one-forms associated to the nondegenerate closed symplectic structures defined on the coadjoint orbits of the affine groups $\hat{G}_L$ and $\hat{G}_R$ respectively [17, 18]. The relations between $\alpha_L$, $\alpha_R$ and $\omega_L$, $\omega_R$ are given locally by

$$d\alpha_L = \omega_L, \quad d\alpha_R = \omega_R.$$  \hspace{1cm} (3.18)

The last term in eq. (3.17) is defined by a Hamiltonian in the phase space with the symplectic forms $\omega_L$ and $\omega_R$. The explicit expressions for the symplectic forms are

$$\omega_L = \left(\frac{k_L}{\pi}\right) \int \text{tr}_{L}(Lg_L^{-1}dg_L \wedge g_L^{-1}dg_L - dL \wedge g_L^{-1}dg_L)dx^+dx^-,$$

$$\omega_R = \left(\frac{k_R}{\pi}\right) \int \text{tr}_{R}(Rdg_Rg_R^{-1} \wedge dg_Rg_R^{-1} - dR \wedge dg_Rg_R^{-1})dx^+dx^-,$$  \hspace{1cm} (3.19)

where $L$ and $R$ are fields conjugated to $g_L$ and $g_R$ respectively.

The corresponding Poisson brackets between the canonical variables are found by inverting $\omega_L$ and $\omega_R$ [19]. We find

$$\{g_L^1(x^+), g_L^2(y^+)\} = 0,$$
$$\{L^1(x^+), g_L^2(y^+)\} = -2\gamma_L C_L g_L^2(y^+)\delta(x^+ - y^+), \quad \gamma_L = \pi/k_L,$$
$$\{L^1(x^+), L^2(y^+)\} = \left(\frac{\gamma_L}{2}\right) [C_L, L^1(x^+) - L^2(y^+)]\delta(x^+ - y^+) + \gamma_L C_L \delta'(x^+ - y^+);$$

$$\{g_R^1(x^-), g_R^2(y^-)\} = 0,$$
$$\{R^1(x^-), g_R^2(y^-)\} = -2\gamma_R C_R g_R^2(y^-)\delta(x^- - y^-), \quad \gamma_R = \pi/k_R,$$
$$\{R^1(x^-), R^2(y^-)\} = \left(\frac{\gamma_R}{2}\right) [C_R, R^1(x^-) - R^2(y^-)]\delta(x^- - y^-) + \gamma_R C_R \delta'(x^- - y^-).$$

Here $\{A^1(x), B^2(y)\}$ denotes either the 2 dim $G_L \times 2$ dim $G_L$ or 2 dim $G_R \times 2$ dim $G_R$ matrix of all Poisson brackets $A$ and $B$, arranged in the same fashion, as in the product of matrices

$$A^1 = A \otimes I$$

and

$$B^2 = I \otimes B,$$
with $I$ the unity either in $\mathcal{G}_L$ or $\mathcal{G}_R$. $C_L$ and $C_R$ are constant matrices given by

$$C_L = \sum_a t^a \otimes t^a, \quad C_R = \sum_\bar{a} t^{\bar{a}} \otimes t^{\bar{a}}. \quad (3.21)$$

The dynamics in the phase space with the symplectic structures determined by the first two terms in eq. (3.17), is defined by the last term in the action. Let us consider the following choice for $S_H$

$$S_H = \frac{1}{2} \int dx^+ dx^- \mathcal{H} \quad (3.22)$$

with the Hamiltonian density $\mathcal{H}$ given by

$$\mathcal{H} = -\frac{\pi}{\gamma_L \gamma_R} \langle S, L \otimes R \rangle. \quad (3.23)$$

Here the symbol $\langle , \rangle$ implies the double tracing over group indices of the Lie groups $G_L$ and $G_R$.

Given the Hamiltonian density we find the Hamiltonian

$$H_L = \int dx^- \mathcal{H} \quad (3.24)$$

in the phase space of variables $g_L$, L, and

$$H_R = \int dx^+ \mathcal{H} \quad (3.25)$$

in the phase space of variables $g_R$, R, respectively. These Hamiltonians yield the dynamical equations

$$\partial_- g_L + (\pi/\gamma_R) g_L (tr_R S \ R) = 0, \quad (3.26)$$

$$\partial_+ g_R + (\pi/\gamma_L) (tr_L S \ L) g_R = 0.$$

If the coupling constant matrix $S$ is invertible, then we can solve these equations to express $L$ and $R$ in terms of $\partial_+ g_R$ and $\partial_- g_L$ respectively. Therefore, after substitution of the expressions for $L$ and $R$ in the functional in eq. (3.17), we get an action in terms of the fields $g_L$ and $g_R$ only. It turns out that this action yields the same equations of motion as the action of the non-Abelian Bosonic Thirring model upon using the self-duality conditions given by eqs. (2.8). In other words, the dynamics of the constrained
non-Abelian Thirring model should be similar to the dynamics of the Hamiltonian system with the Hamiltonian density as in eq. (3.23) and the Poisson structure given by eqs. (3.20). Thus, the direct quantization of the Hamiltonian equations should provide the quantization to the starting Lagrangian Thirring model. The conditions when such a quantization can be carried out will be the topic of the next section.

4 Hamiltonian quantization

Based on the results obtained in the previous section we may quantize the non-Abelian Bosonic Thirring model by the Hamiltonian method. The method will work as long as the algebraic Poisson structure given by eqs. (3.20) are consistent with the Hamiltonian equations. In attempts to promote the classical Lie-Poisson structure to the quantum level we are giving ourselves an account of possible quantum deformations coming in the quantum Poisson brackets and quantum Hamiltonians. We do not know a systematical way to control nonperturbative corrections to classical structures. However, if such deformations come into being, they have to occur self-consistently, i.e. the quantum corrections should not destroy the dynamical equations. Therefore, we will assume the classical Hamiltonian equations as exact quantum ones up to a certain normal ordering of composite operators.

We want to quantize the Hamiltonian system by promoting the classical Poisson structure (3.17) to the quantum level. First of all, we postulate the following quantum brackets

\[
\begin{align*}
\left[ L^1(x^+), L^2(y^+) \right] &= \left( \gamma_L / 2 \right) \left[ C_L, L^1(x^+) - L^2(y^+) \right] \delta(x^+ - y^+) + \gamma_L C_L \delta'(x^+ - y^+), \\
\left[ R^1(x^-), R^2(y^-) \right] &= \left( \gamma_R / 2 \right) \left[ C_R, R^1(x^-) - R^2(y^-) \right] \delta(x^- - y^-) + \gamma_R C_R \delta'(x^- - y^-).
\end{align*}
\]

(4.27)

If there are quantum corrections, they should result in a certain renormalization of the operators $L$ and $R$. Let us consider one obvious quantum effect. In the conformal regime we want the operators $L$ and $R$ to be scaling (but not necessarily Virasoro primary) operators. In turn, due to the Poisson structure, these operators are required to have classical canonical scaling weights. Respectively $L$ has wight $(1,0)$ and $R$ has weight...
Then it is not hard to show that the following equations should hold

\[ \partial_- L|0\rangle = \text{null vector}, \]
\[ \partial_+ R|0\rangle = \text{null vector}. \]  

(4.28)

So, at the quantum level the operators \( L \) and \( R \) become to be analytical. (The given deformations of the classical Hamiltonian equations originate from the quantum effects similar to the chiral anomaly which modifies the classical conserved chiral current.) Hence, the quantum Poisson brackets realize affine algebras. Thus, the hidden classical affine symmetry \([5, 13]\) becomes apparent at the quantum level and we can identify the affine generators with the renormalized operators \( L \) and \( R \). However, the affine symmetry is not necessarily a symmetry of physical states. We will see in the next section that a highest weight affine representation, in general, appears to be nondegenerate in energy. Nevertheless, the classical integrability on “the second level” (see eqs. (4.38)) may entail the full conformal invariance as a byproduct of the fact that the Virasoro algebra of the conformal group belongs to the enveloping algebra of the affine algebra.

The important point to be made is that eqs. (4.28) can be thought of as another pair of Hamiltonian equations. It is interesting that at the quantum level in the conformal regime the r.h.s. of the classical Hamiltonian equations reduces to null vectors.

As a consequence of the Lie algebra structure of the brackets given by eqs. (4.27) there are only two different ways of fixing the quantum brackets between \( g_L \), \( g_R \) and \( L \), \( R \) consistently with the corresponding Jacobi identities. Namely, the first one is to keep the classical structures as in eqs. (3.20). The second one is to admit an exchange of representations for \( g_L \) and \( g_R \) as follows

\[ [L_1(x^+), g_R^2(y^+)] = -2\gamma_L C_L g_R^2(y^+) \delta(x^+ - y^+), \]
\[ [R_1(x^-), g_L^2(y^-)] = -2\gamma_R C_R g_L^2(y^-) \delta(x^- - y^-), \]  

(4.29)

The second choice works when \( G_L = G_R = G \). More generally one can consider some embedding of one Lie group into another.
Both options are equally good on the basis of symmetry arguments. So, we have to turn to the dynamical equations to make a choice. Obviously, the Hamiltonian equations (3.26), (4.28) in the conformal regime should admit the following representation

\[
[L_{-1}, L] = 0, \\
[L_{-1}, g_R] = -\left(\pi/\gamma_L\right) tr_L : S L g_R :;
\]

\[
(4.30)
\]

\[
[\bar{L}_{-1}, \bar{R}] = 0, \\
[\bar{L}_{-1}, g_L] = -\left(\pi/\gamma_R\right) tr_R : g_L S R :;
\]

where we defined normal ordering by the rule

\[
:L g_R : (z) = \oint \frac{dw}{2\pi i} \frac{L(w) g_R(z)}{w - z},
\]

\[
(4.31)
\]

\[
:g_L R : (\bar{z}) = \oint \frac{d\bar{w}}{2\pi i} \frac{R(\bar{w}) g_L(\bar{z})}{\bar{w} - \bar{z}},
\]

with products \(L(w)g_R(z)\) and \(g_L(\bar{z})R(\bar{w})\) being understood as T-ordered operator product expansions [20]. The operators \(L_{-1}\) and \(\bar{L}_{-1}\) are the generators of translations. By definition

\[
L_{-1} = \oint \frac{dz}{2\pi i} T(z), \\
\bar{L}_{-1} = \oint \frac{d\bar{z}}{2\pi i} \bar{T}(\bar{z}),
\]

with \(T(z)\) and \(\bar{T}(\bar{z})\) the holomorphic and antiholomorphic components of the energy-momentum tensor of the conformal Thirring model under consideration. It is interesting to point out that eqs. (4.30) appear to be a quantum field theoretic generalization of the isotropic rotator which possesses quantum group symmetries [30]. Therefore, it might be possible to look for integrable deformations of the Bosonic Thirring model following in the manner of ref. [30].

So, what we have to do is to find the operators \(T\) and \(\bar{T}\) which produce the r.h.s. of eqs. (4.30). With the first Lie-Poisson structure as in eqs. (3.20) we fail to get any expressions for \(T\) and \(\bar{T}\) obeying the Hamiltonian equations. While the second structure given by eqs. (4.29) admits solutions for the operators \(T\) and \(\bar{T}\) to exist. It is easy to check that the following operators

\[
T(z) = L_{ab} : \bar{L}^a \bar{L}^b :, \quad \bar{L} = k_L L;
\]
\[ \tilde{T}(\bar{z}) = L_{ab} : \bar{R}^a \bar{R}^b : , \quad \bar{R} = k_R R \]
satisfy the Hamiltonian equations (4.32) provided that
\[ L_{ab} = S_{ab}. \] (4.34)

However, this is not the whole story. The operators in eqs. (4.33) should form two copies of the Virasoro algebra. Otherwise, they will not make sense of the components of the conformal energy-momentum tensor. We can prove that the operators \( T \) and \( \tilde{T} \) give rise to the Virasoro algebras, if and only if the matrix \( L_{ab} \) satisfies the Virasoro master equation \[ L_{ab} = 2L_{ac}G^{cd}L_{db} - L_{cd}L_{ef}f_a^{ce}f_b^{df} - L_{cd}f_a^{ce}f_b^{df}L_{ef} - L_{cd}f_a^{ce}f_b^{df}L_{ef} - L_{be}, \] (4.35)
with \( f_a^{bc} \) and \( G^{ab} \) respectively the structure constants and general Killing metric of the Lie algebra \( \mathcal{G} \) and Lie group \( G \) respectively. The Virasoro master equation may have many solutions [7, 25]. The entire space of solutions possesses the symmetry under transformations from the diagonal of \( G_L \times G_R \). This comes transparently from the affine-Virasoro construction itself and also can be verified with the invariance of the master equation under the following transformations

\[ L_{ab} \rightarrow L_{ab} + x_h (f_b^{\underline{hk}}L_{\underline{kb}} + f_b^{\underline{hk}}L_{\underline{ak}}) + \mathcal{O}^2(x), \]
where \( x_h \) are the infinitesimal parameters. Accordingly, the conformal symmetry of the non-Abelian Thirring model is held on the orbits which are built by acting with the global diagonal group on the “physically not equivalent” solutions of the master equation. In section 6 we will show that components of the energy-momentum tensor in the form given by eqs. (4.33) appear naturally in the course of coupling the Bosonic Thirring model to the 2D gravity.

\(^\dagger\)The Hamiltonian associated with the energy-momentum tensor in (4.33) can be viewed as a Hamiltonian of the Siegel invariant Lefton-Righton Thirring model [3] taken in the Floreanini-Jackiw gauge (R. Floreanini and R. Jackiw, Phys. Rev. Lett. 59 (1987) 1873). As a matter of fact we deal with the same Hamiltonian system.
Thus, there is a one-to-one correspondence between solutions of the Virasoro master equation and the conformal points of the Bosonic non-Abelian Thirring model. Indeed, given a solution of the master equation we can define a value of the Thirring coupling constant at which the theory can be quantized in a fashion consistent with the conformal invariance. The given conformal points we will call Dashen-Frishman conformal points [10]. In the process the conformal symmetry emerges not as a byproduct of the Hamiltonian quantization but as its essential ingredient.

At the same time the conformal points may tell us something about the model beyond the conformal regime. Going off the conformal regime means giving mass to the fundamental fields. However, massive terms can be consistent with the Lorentz symmetry only if the massive fields have a certain Lorentz spin. Obviously, if the fields do not have the correct spin at the conformal points, they cannot get it in the vicinity of the conformal points. In such a case there may not be a smooth way away from the conformal phase to the massive phase. In other words, the theory may turn out to be well defined only at the conformal points. On the other hand, it may happen that some of the fields do have the correct spin. For example, some of the components of an affine group multiplet may have a right spin. Then the theory can be moved out of the conformal points. We will show in next section that in Thirring models at the Dashen-Frishman conformal points there is always at least one fundamental field with the correct Lorentz spin.

5 Conformal non-Abelian Fermionic Thirring model

We now go on to discuss conformal points of the non-Abelian Fermionic Thirring model described by the action in eq. (2.1). In this case the theory is already first order and its quantization is more straightforward compared to the Bosonic version. The classical Lagrangian yields the following equation of motion for $\psi_L$

$$\bar{\partial} \psi_L = S_{aa} t^a J^a_R \psi_L$$

and a similar one for $\psi_R$. At the quantum level this equation makes sense provided the normal ordering of its r.h.s. exists.
The classical equations of motion entail the following relations

\[
\partial \bar{J}_L^a = i f_{ce}^b S_{ba} \bar{J}_L^c \bar{J}_R^a, \\
\partial \bar{J}_R^\bar{a} = i f_{\bar{e}\bar{c}}^\bar{b} S_{\bar{a}b} \bar{J}_L^a \bar{J}_R^\bar{c},
\]

(5.37)

where the currents \( J_L, J_R \) are given by eqs. (2.2). Since the Fermionic current-current interaction does not contain a time derivative, the fields \( J_L, J_R \) form a standard current Poisson algebra [12] similar to (3.20). Eqs. (5.37) are rather reminiscent of the Lax pair representations of integrable systems. Nevertheless, we cannot associate a curl free local current to the given system, with the exception of the isoscalar case \( S_{a\bar{a}} = \lambda \delta_{a\bar{a}} \). In the isoscalar case the system of eqs. (5.37) can be presented as a zero curvature condition for a conserved local current.

It is not difficult, however, to find a completely integrable subsystem of the currents. This is formulated in terms of the fields

\[
X = G_{ab} \bar{J}_L^a J_L^b, \quad \bar{X} = G_{\bar{a}\bar{b}} J_R^{\bar{a}} \bar{J}_R^{\bar{b}}.
\]

(5.38)

which obviously satisfy the analytisity conditions

\[
\partial X = 0, \quad \partial \bar{X} = 0.
\]

(5.39)

Thus, nonanalytic parts of the currents \( J_L, J_R \) are irrelevant in analyzing the system of eqs. (5.38) and (5.39).

From now on we will be interested only in the conformal regime of the model in question. It means that its quantum energy-momentum tensor should consist of holomorphic \( T \) and antiholomorphic \( \bar{T} \) components forming two copies of the Virasoro algebra. In order to elucidate the expressions for \( T \) and \( \bar{T} \), we have to carefully investigate the symmetry and dynamics of the theory under consideration.

The current algebra is consistent with the scale symmetry as long as the Fermionic currents are the scaling fields of the canonical weights. At the quantum level we can prove that the scaling properties result in analyticity conditions for the current operators. These equations are very similar to eqs. (4.28). Thus, in the quantum regime the starting
current algebra transforms to two copies of the affine algebra of the composite operators $J_L$ and $J_R$. Hence, these operators can be treated as generators of local symmetries and we can require the following local commutation relations \[21, 12\]

\[
[J_L(z), \psi(w, \bar{w})] = (a + \bar{a} \gamma_5) \psi(w, \bar{w}) \delta(z - w),
\]

(5.40)

\[
[J^a_L(z), \psi(w, \bar{w})] = -\frac{1}{2}(1 + \delta \gamma_5) t^a \psi(w, \bar{w}) \delta(z - w)
\]

and the similar ones for $J_R$ and $J^a_R$. Here $J_L, J_R$ are the $U(1)$ currents given by eqs. (2.4). These currents can always be defined when we are dealing with the complex fermions.

The parameters $a$ and $\bar{a}$ in eq. (5.40) are not fixed by the symmetry until we demand certain requirements for the spin of $\psi$. The constant $\delta$ in turn must be either +1 or -1 from the Jacoby identity. By a redefinition of the field $\psi \rightarrow \gamma_1 \psi$, we can always choose $\delta$ to be, say, +1. However, this will entail a change of the Lorentz representation of $\psi$ in the equations of motion. Therefore, it is more convenient to consider the commutators with different $\delta$’s. The sign of $\delta$ cannot be fixed from the symmetry and, therefore, should depend on the dynamics.

The commutation relations in eqs. (5.40) reflect a simple fact that the field $\psi$ is an affine primary field. This, however, does not imply the field $\psi$ is a conformal primary one. Note that until now all known conformal field theories have Virasoro primary as their fundamental fields. In the case of the non-Abelian Thirring models, we are dealing with a more general situation.

The commutators in eqs. (5.40) provide us with the operator product expansions

\[
J_L(z)\psi(w, \bar{w}) = \frac{(a + \bar{a}\gamma_5)\psi(w, \bar{w})}{z - w} + \text{reg.},
\]

(5.41)

\[
J^a_L(z)\psi(w, \bar{w}) = \frac{1}{2}(1 + \delta \gamma_5) t^a \psi(w, \bar{w}) \frac{1}{z - w} + \text{reg}.
\]

Here the regular parts of the OPE’s depend on the specific properties of the representation $\psi$ with respect to the conformal transformations. To clarify this point we have to construct the Virasoro generators by using the affine currents. In general, it could be done in many ways [22]. However, in the case under consideration the Virasoro generators have to be consistent with the equations of motion.
To proceed we have to define normal ordering between the currents and the fields. In sect. 4 we made use of normal ordering as in eqs. (4.31). This prescription can be extended to a general case of two operators $A$ and $B$ when one of them is analytic (holomorphic in the case at hand). So, we define normal ordering between $A$ and $B$ in the following fashion

$$ : A B : (z) = \oint \frac{dw}{2\pi i} \frac{A(w)B(z)}{w-z}, \quad (5.42)$$

where as usual $A(w)B(z)$ is understood as a time-ordered OPE in the framework of the radial quantization (after corresponding Euclideanization of the Minkowski space-time).

The point to be made is that the l.h.s. of eq. (5.36) is a certain translation of $\psi$. Hence, the equation of motion has to follow from the commutation of the Fermionic field with the proper energy-momentum tensor. Otherwise, the theory will be inconsistent dynamically. By definition the operators of translations are given by the following generators of the conformal algebras

$$ \partial \rightarrow L_{-1} = \oint \frac{dz}{2\pi i} T(z), \quad (5.43)$$

$$ \bar{\partial} \rightarrow \bar{L}_{-1} = \oint \frac{d\bar{z}}{2\pi i} \bar{T}(\bar{z}),$$

with $T(z)$ and $\bar{T}(\bar{z})$ the holomorphic and antiholomorphic components of the energy-momentum tensor.

All in all, the quantum equation of motion should be as follows

$$ [L_{-1}, \psi_R] = S_{ab} : J^a_L J^b_R \psi_R :. \quad (5.44)$$

We have assumed that $G_L = G_R = G$. Although one can try to consider the case when $G_L \neq G_R$ provided one of the groups can be embedded into another.

The aim is to construct the operator $L_{-1}$ and $\bar{L}_{-1}$ so that they obey all the quantum equations of motion. It turns out that the quantum canonical brackets with $\delta = +1$ are incompatible with the full system of equations of motion. Therefore we have to consider $\delta = -1$. In this case it is not hard to check that eq. (5.44) is fulfilled with $T$ given by

$$ T = L_{ab} : J^a_L J^b_L : + \kappa : J_L J_L :, \quad (5.45)$$
provided that
\[ L_{ab} = S_{ab}. \] (5.46)

The last term in eq. (5.45) originates from the free Fermionic theory and survives in the interacting model because the \( U(1) \) current commutes with the non-Abelian currents. The magnitude of the constant \( \kappa \) does not affect any observables in the theory.

Thus, with \( \delta = -1 \) we are able to define all quantum operators entering the quantum equations of motion. More conditions are required for \( L_{-1} \) and \( \bar{L}_{-1} \) to be the translation operators. The operators \( L_{-1} \), \( \bar{L}_{-1} \) will enjoy this property, if \( T \) and \( \bar{T} \) form Virasoro algebras. With the fact that the currents \( J^a_L \) satisfy the affine algebra, we can show that \( T \) forms the Virasoro algebra, if and only if the matrix \( L_{ab} \) is a solution of the Virasoro master equation (4.35). So, the Bosonic and Fermionic non-Abelian Thirring models share the same Dashen-Frishman conformal points in the full correspondence with eq. (2.16). This result implies that the partition functions in eq. (2.16) is perhaps calculable at the Dashen-Frishman conformal points.

In general, we do not know how to handle the partition function at the non-perturbative Dashen-Frishman conformal points. However, at the particular conformal points corresponding to the isoscalar case we can gain some insight.

Let us consider the simplest case when \( k_L = k_R = k \). By using eqs. (2.9) and (2.16), we can obtain the following formula for the Fermionic partition function
\[ Z_F(k, k; S) = J Z_F(k, k; 0) Z_B \left( -(k + \frac{1}{2} c_2(G)), -(k + \frac{1}{2} c_2(G)); S' \right) Z_{gh}, \] (5.47)
where \( S' \) is given by eq. (2.11). The partition function of the Bosonic Thirring model possesses a useful property
\[ Z_B(k, k; S = 1/k) = \frac{1}{Z_B(k, k; 0)}, \] (5.48)
which is a direct consequence of the Polyakov-Wiegmann formula. Keeping in mind the given property one can easily prove the following identity
\[ Z_F(k, k; S = 1/(k + \frac{1}{2} c_2(G))) = J Z_F(k, k; 0) Z_B^{\frac{1}{2}} \left( -(k + \frac{1}{2} c_2(G)), -(k + \frac{1}{2} c_2(G)); 0 \right) Z_{gh}. \] (5.49)
We have used the fact that the coupling constant $S'$ associated with the coupling $S^* = \frac{1}{(k + \frac{1}{2}c_2(G))}$ is given by

$$S' = -\frac{1}{(k + \frac{1}{2}c_2(G))}.$$  \hfill (5.50)

Therefore, we can use relation (5.48) to obtain eq. (5.49). The latter signifies that the coupling constant $S^*$ corresponds to the conformal point of the Fermionic Thirring model since on the r.h.s of identity (5.49) we have a product of the conformal partition functions. The given conformal point is nothing but the isoscalar Dashen-Frishman conformal point [12] generalized to the case of spinors with $k$ flavors. Note that the presented proof of the conformal symmetry of the Fermionic Thirring model at the isoscalar Dashen-Frishman fixed point is essentially nonperturbative.

At the same time, it may be instructive to check the conformal symmetry by another method. Namely, one can use the $1/N$ method. Indeed, let us consider the $SU(N)$ non-Abelian Fermionic Thirring model with the isoscalar current-current interaction

$$S_{int} = -\lambda \int d^2 z \ J_L^a \ J_R^a,$$  \hfill (5.51)

with $\lambda$ being a coupling constant. When the fermions do not have flavor, we should be able to treat this theory by the $1/N$-expansion method in the limit of large $N$.

An isoscalar solution of the Virasoro master equation (4.35) in the case under consideration reads [12]

$$\lambda = \frac{4\pi}{(N + 1)}.$$  \hfill (5.52)

In the limit when $N$ is large the following ratio

$$4\pi N \ \lambda = \bar{\lambda} = 1$$  \hfill (5.53)

holds. Hence, one can use the $1/N$-expansion method to explore the theory at the given value of the coupling constant. Obviously, if the model had conformal points at nontrivial values of $\lambda$, then the corresponding renormalization group $\beta$-function should vanish at these points for each order in $1/N$. Actually, one can use the results obtained in [23, 24].

It is known that the isoscalar Thirring model is equivalent to the Gross-Neveu theory for a small coupling [14]. A puzzle is that Gross and Neveu have shown that their model does not allow nontrivial conformal points to exist. At the same time, we just proved above
that the Dashen-Frishman model does have a nontrivial conformal point. An explanation of the paradox might be as follows. In the small vicinity of zero coupling constant there may exist a phase transition which prohibits Fierz transformations to be used at the values of $\lambda$ comparable with the value given by eq. (5.52). Therefore, the nontrivial conformal point of the non-Abelian Thirring model can be missed in the Gross-Neveu theory. Our conjecture is that the Gross-Neveu model describes the non-Abelian Thirring model in the phase of very small couplings. Whereas when $\lambda$ approaches the critical value from the right side on a parametrical line the Dashen-Frishman isoscalar model seems to be equivalent to the Wilson’s theory of $N$ scalar fields in $d = 4 - \epsilon$ dimensions in the limit $\epsilon \to 2$. Wilson has calculated the dependence of the coupling $\lambda$ on the cut-off in the given limit and he established that the model has a nontrivial fixed point when $\bar{\lambda} = 1$ (in our normalization). Thus, the conformal point in eq. (5.52) which follows from the Hamiltonian quantization could be in a favorable agreement with the conformal point in eq. (5.53) following from the $1/N$-expansion method.

Of course, the $1/N$-expansion method fails to be appropriate for most of other Dashen-Frishman conformal points. However, one could try to apply it to the case when the number of colors is fixed but the number of flavors goes to infinity. This situation corresponds to the case of the affine algebra with a large level.

From the point of view of the Hamiltonian quantization, the consistency of a solution of the Virasoro master equation with the conformal invariance of the quantum field theory is enough to justify the conformal symmetry of the non-Abelian Thirring models at all other Dashen-Frishman conformal points.

It is noteworthy that the non-Abelian Fermionic Thirring model at the isoscalar conformal points corresponding to the affine-Sugawara construction [7] yields the proper Fermionic Lagrangian description of the representation described by the WZNW model on affine $\hat{G}$. Thus, in order to fermionize the WZNW model with the level permitted by the bifermionic currents, we have to take fermions described not by the free Lagrangian but the isoscalar Thirring Lagrangian.

Now we would like to discuss some features of the fundamental fields at the Dashen-
Frishman conformal points. We begin with the vacuum of affine $G$

\[ J^a_m |0\rangle = L_m |0\rangle = 0, \tag{5.54} \]

where $J^a_m$ and $L_m$ are defined as

\[ J^a_L(z) = \sum_{m=-\infty}^{\infty} J^a_m z^{-m-1}, \quad a = 1, \ldots, \dim G, \tag{5.55} \]

\[ L_{ab} : J^a_L(z) J^b_L(z) : = \sum_{m=-\infty}^{\infty} L_{m} z^{-m-2}. \]

Due to the property in eq. (5.41), the field $\psi^a_R$ has to obey

\[ J^a_{m \geq 0} \psi^a_R(0) |0\rangle = \delta^a_{m,0} (t^a)^{\alpha}_{\beta} \psi^\alpha_R(0) |0\rangle. \tag{5.56} \]

Consider now the action of the $L_{ab} : J^a_L J^b_L$ on the affine primary states. It is easily verified with eq. (5.54) that

\[ L_0 \psi^a_R(0) |0\rangle = \Delta^a_\beta \psi^\beta_R |0\rangle, \tag{5.57} \]

where

\[ \Delta^a_\beta = L_{ab} \left( t^a t^b \right)^{\alpha}_{\beta}, \tag{5.58} \]

is called the conformal weight matrix [25]. There exists an eigenbasis of affine primary fields in which the conformal weight matrix is diagonal [26]. When the field $\psi$ is in the fundamental representation, we may think of the fundamental fields $\psi^a$ as the eigenbasis of the affine primary fields.

The information about the Lorentz spin of $\psi_R$ resides in the matrix $\Delta^a_\beta$. Namely, the Lorentz spin operator is given by

\[ s = L_0 - \bar{L}_0. \tag{5.59} \]

Since in the $\{\psi^a\}$ eigenbasis the conformal weight matrix has a diagonal form, the eigen-matrix of the spin operator also can be arranged to be diagonal. Thus, in general, different components of an affine multiplet should be of a different spin. Due to this fact, the components of an affine multiplet describe different conformal highest weight representations with different background energies. Therefore, the underlying affine symmetries are not
generally symmetries of the physical states, since the affine generators may not commute with the Hamiltonian.

Since the matrix $L_{ab}$ is fixed by the conformal symmetry, we cannot change the eigenvalues of $s$ by tuning $L_{ab}$. However, there is one free parameter - the parameter $a$ ($\bar{a} = a$ in eq. (5.40) when the $U(1)$ current-current interaction is omitted in the starting Lagrangian). Via the operator $\bar{L}_0$ this parameter enters the eigenmatrix of $s$. This enables us to set the spin of one of the components of an affine multiplet to any value we want. This is a consequence of the fact that the diagonal of the $U(1) \times U_{\gamma_5}(1)$ symmetry is conserved at the quantum level at arbitrary values of the Thirring coupling constants. Therefore, in the case of the Fermionic theory, both at the classical and quantum levels one of the fermions may have Lorentz spin equal to $1/2$. This is very important, if one wants to consider the Thirring model beyond conformal points. Indeed, the existence of the fundamental fields of undeformed Lorentz spin allows massive terms to appear. This may also mean that the space of all Thirring models is a connected multitude.

6 A BRST nature of the master equation

In this section we are going to show that the Bosonic Thirring model can be viewed as a gauge invariant theory such that the action in eq. (2.6) corresponds to a particular gauge choice in the gauge model. Interestingly, one gauge symmetry comes into being due to the chirality conditions given by eqs. (2.18). This is the Siegel gauge symmetry [31] arising in the process of including the chirality constraints in the Lagrangian. This symmetry is anomalous at the quantum level. However, there is a remedy to cure this problem [32, 33]. Due to Siegel’s symmetry all auxiliary lagrange multipliers become pure gauge degrees of freedom and can be set to zero values both at the classical and quantum levels [32, 33]. Presently we will be convinced of an importance of the given symmetry for understanding the equivalence between the Bosonic Thirring model and the affine-Virasoro construction within the BRST approach. In order to see more local symmetries, we should look at the

§In the appendix we will discuss the situation when all components of an affine multiplet have the same spin such that the affine field becomes simultaneously a Virasoro primary.
global symmetries of the Bosonic Thirring model.

Let us forget for a while about the interaction term in eq. (2.6). Then each of the two WZNW models should possess global symmetries generated by the following conserved currents

\[ J_n^{a_1a_2...a_n} = P_n^{a_1a_2...a_n}(J, \partial), \]

where \( P_n \) is a polynom of order \( n \) in \( J^a \) and \( \partial \) with \( J^a \) being the affine current obeying the equation of motion

\[ \bar{\partial} J^a = 0. \]

(6.61)

The currents presented in eq. (6.60) may form a very rich algebra. In this paper we want to concentrate on a particular subalgebra of this big algebra formed by the currents \( J_1^a = J^a \) and

\[ T = L_{ab}J_2^{ab} = L_{ab}J^aJ^b \]

(6.62)

without \( \partial \)-dependent terms. The current \( T \) carries spin two and is a natural candidate on a role of the energy-momentum tensor. There is also an antiholomorphic component

\[ \bar{T} = L_{ab}J^aJ^b. \]

(6.63)

Note that in general the current \( \bar{T} \) may go with a different matrix \( \bar{L}_{ab} \). However in what follows we will be restricted to the case when \( \bar{L}_{ab} = L_{ab} \).

The very important point to be made is that the classical currents \( T \) and \( \bar{T} \) form a closed algebra provided the matrix \( L_{ab} \) obeys the following algebraic equation

\[ L_{ab} = 2L_{ac}G^{cd}L_{db}, \]

(6.64)

which is easily identified with the classical limit of the Virasoro master equation [8].

Correspondingly, transformation properties of fields follow from the formulas

\[ \delta_T \psi = \epsilon \oint \frac{dw}{2\pi i} T(w)\psi, \]

\[ \delta_{\bar{T}} \psi = \bar{\epsilon} \oint \frac{d\bar{w}}{2\pi i} T(\bar{w})\psi, \]

(6.65)
where $\psi$ is a field and the product on the r.h.s is understood as an OPE; $\epsilon$, $\bar{\epsilon}$ are constant parameters. The given definition is suitable for WZNW models since group elements are affine primary fields whose OPE’s with the affine currents are known.

In order to gauge the symmetry associated with the conserved currents $T$ and $\bar{T}$ within the WZNW model, one has to introduce a set of new fields $h$, $\bar{h}$ coupling to the group element trough the currents $T$ and $\bar{T}$ respectively. The procedure is rather straightforward and to a great extent is reminiscent of the method [34] used to construct the gauge theory of the $W$-gravity. We are not going into all details of this method. For us it is important to point out that the gauge fields $h$, $\bar{h}$ can be identified with a metric of the 2D gravity [34]. Thus, we can conclude that the 2D gravity can couple to the classical WZNW model in as many ways as a number of solutions of the equation (6.64) can be found. At the quantum level one can expect to get more restrictions on classically admitted solutions.

Let us turn to the Bosonic Thirring model. Now the theory has to describe the interaction between two chiral WZNW models coupling to the 2D gravity. Such an interaction can be constructed consistently with the group of two dimensional diffeomorphisms with a method developed in ref. [3].

The interaction term of the Bosonic Thirring model spoils the analiticy properties of the affine currents. Therefore, the energy-momentum tensor in general will be different from the affine-Virasoro form. It is quite amusing that when the coupling constant matrix $S_{ab}$ coincides with the momentum matrix $L_{ab}$, i.e.

$$S_{ab} = L_{ab},$$

the components of the energy-momentum tensor of the Bosonic Thirring theory acquire the affine-Virasoro construction form given by

$$T_L \equiv 4\pi \left( \frac{\delta S}{\delta h} \right)_{h,\bar{h}=0} = L_{ab}J^a_LJ^b_L, \quad T_R \equiv 4\pi \left( \frac{\delta S}{\delta \bar{h}} \right)_{h,\bar{h}=0} = L_{ab}J^a_RJ^b_R,$$

where $J_L = kg_L^{-1}\partial_+g_L + k\partial_+g_Rg_R^{-1}$, $J_R = k\partial_-g_Rg_R^{-1} + kg_L^{-1}\partial_-g_L$. To get these expressions one has to use the classical master equation (6.64). It is quite natural that eqs. (6.67) appear to be similar to the usual spin-spin Hamiltonian interaction of a system of two identical rotators [30]. Obviously, $T_L$ and $T_R$ satisfy the analiticity conditions

$$\bar{\partial}T_L = 0, \quad \partial T_R = 0.$$
So that nonanalytical parts of the currents $J_L, J_R$ become irrelevant in eqs. (6.68).

At the quantum level instead of the classical $T_L$ and $T_R$ we consider the quantum energy-momentum tensors with corresponding ghost contributions. Given the quantum $T_L$ and $T_R$ one can construct the BRST operator $Q$ following the standard scheme [35]. It is well known that the nilpotence of $Q$ guarantees the conformal symmetry of the system. In its turn the conformal symmetry entails the analiticity of the quantum currents $J_L$ and $J_R$. After that we come to the affine-Virasoro construction and the Virasoro master equation. However, now we have one more restriction coming from the nilpotence of the BRST operator. Namely,

$$c = 26,$$  \hspace{0.5cm}  (6.69)

where $c$ is the Virasoro central charge of the affine-Virasoro construction

$$c = 2G^{ab}L_{ab}.$$  \hspace{0.5cm}  (6.70)

The last restriction is not very severe, since the Bosonic Thirring model is to be considered as a conformal model describing a compactification of a certain string theory. Therefore, in a whole theory the nilpotence of $Q$ will result in the condition

$$c_0 + c = 26,$$  \hspace{0.5cm}  (6.71)

where $c_0$ is a total Virasoro central charge of a noncompact part of a given string.

Thus, we have proved that equations (4.34), (4.35), (6.66) appear to be necessary and sufficient conditions of the conformal invariance of the Bosonic Thirring model. Due to the identity (2.16), the Fermionic Thirring model should share the same conformal conditions.

Note that in ref. [8] authors discussed a gauge invariant action for the affine-Virasoro construction by utilizing one WZNW model. We found that it is more convenient to consider two interacting chiral WZNW theories since the chirality conditions taken together with the gravity constraints fix completely the so-called K-conjugate invariance of the affine-Virasoro construction [7, 8]. Moreover, this way we were able to discover new class of conformal quantum field models which could be useful for description of new string compactifications.

* A possibility to derive the Virasoro master equation as a condition of conformal invariance of a certain
7 Conclusion

Following observations of the Lie-Poisson structure and the existence of nontrivial conformal points in non-Abelian Thirring models [5, 6, 10], we have derived a theory of conformal non-Abelian Thirring models both for Bosonic and Fermionic versions. We have shown that these models can be quantized in a conformally invariant fashion at the values of the Thirring coupling constants which are solutions to the Virasoro master equation. Due to this fact, the conformal non-Abelian Thirring models seem to provide the algebraic affine-Virasoro construction with a natural Lagrangian description. Moreover, since Thirring models have a nice interpretation in string theory [2, 3], we can expect to get explicit conformal sigma models corresponding to the affine-Virasoro construction.

In this paper, we discussed non-Abelian Thirring models only in the conformal regime. However, for the realization of the background independent string field theory formulation program it is very important to investigate Thirring models beyond conformal points. We argued that massive deformations are not prohibited by the Lorentz symmetry even though the quantum fermions, in general, are no longer of Lorentz spin 1/2. Therefore, we hope that the whole multitude of Thirring models can be realized as a connected space of theories.

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Appendix

The peculiar algebraic properties of the affine-Virasoro construction entail unwanted conformal representations of the Virasoro algebra. Some general features of the representations of the affine-Virasoro constructions on a general affine $\hat{G}$ can be found in [25-28]. A sigma model was discussed in [36]. However, obtained results do not seem to go beyond a classical limit of the master equation. To such an extent of accuracy, there are no contradictions between our approach and the method of beta functions in [36].
Generally since the affine-Virasoro construction is made with the affine currents it would seem to be true that Virasoro representations carried by the affine-Virasoro construction should be described in terms of the representations of the underlying affine algebra. Such a situation occurs, for example, in the WZNW model [29]. However, a consideration of the generic affine-Virasoro construction makes this intuition not so obvious [26]. The complications are partly caused by the fact that since we are dealing with the affine-Virasoro construction, affine primary fields are no longer Virasoro primary fields in general [27].

Surprisingly, the affine-Virasoro construction on the affine $SU(2)$ does appear an exception to the rule. This comes about due to the following fortunate quaternionic identity

$$2t^a t^b = \eta^{ab} + i f_{c}^{ab} t^c,$$

which holds when $t^a$ are the $SU(2)$ generators in the fundamental representation of the Lie algebra $SU(2)$. In this case, the conformal weight matrix $\Delta^\alpha_\beta$ takes the form

$$\Delta^\alpha_\beta = \left(\frac{c}{4k}\right) \delta^\alpha_\beta,$$

where the constant $c$ is the Virasoro central charge in the $SU(2)$ affine-Virasoro construction

$$c = 2G^{ab} L_{ab}.$$

In this case the Lorentz spin of $\psi$ is given by

$$s = c - 4\kappa a^2.$$

Therefore, we are able to treat the field $\psi$ as a spinor when

$$4a^2 \kappa = c - 1/2.$$

Eqs. (2) and (4) remain true for all the possible solutions of the Virasoro master equation of the generic $SU(2)$ affine-Virasoro construction when the latter acts on the space of states in the fundamental representation of the Lie algebra $su(2)$. This means that the $SU(2)$ affine primary fields, say $\psi^\alpha$, can be also considered as the Virasoro primary fields provided the matrix $L_{ab}$ obeys the Virasoro master equation.
Keeping in mind the identity (2), we obtain the OPE between the $SU(2)$ affine Virasoro construction $T = L_{ab} : J^a J^b :$ and the affine primary field $\psi$

\[
T(z)\psi^\alpha(w, \bar{w}) = \Delta \left( \frac{1}{(z-w)^2} + \frac{\partial}{\Delta(z-w)} \right) \psi^\alpha(w, \bar{w}) + \text{reg.} \tag{A.6}
\]

where

\[
\Delta = c/4k,
\tag{A.7}
\]

\[
J^a(z)\psi^\alpha(w, \bar{w}) = (t^a)^\alpha_\beta \left( \frac{1}{(z-w)} \psi^\beta(w, \bar{w}) + \frac{1}{2\Delta} \partial \psi^\beta(w, \bar{w}) \right) + O(z-w).
\]

Let us consider the following composite field

\[
S_{ab}(t^a)^\beta_\alpha \psi^\alpha_R = \partial \psi^\beta_R - S_{ab}(t^a)^\beta_\alpha : J^a \psi^\alpha_R :.
\tag{A.8}
\]

This field might appear in the process of quantization of the classical equation of motion of the non-Abelian Fermionic Thirring model

\[
\partial \psi^\alpha_R = S_{ab}(t^b)^\alpha_\beta J^a \psi^\beta_R.
\tag{A.9}
\]

The quantization will be consistent provided the l.h.s. of eq. (8) is a null vector. By using the OPE’s in eqs. (6), (7), one can derive the correlator of the given composite fields [26]

\[
\langle \psi^\alpha_a(T, z) \psi^\beta_b(\bar{T}, w) \rangle = \left( \frac{G_{ab} + \frac{2\Delta-1}{2\Delta} t_b t_a - t_a t_b}{(z-w)^{2\Delta+2}} \right)^{\alpha\beta}, \tag{A.10}
\]

where $T$ refers to the fundamental representation of $G = SU(2)$. Then, by straightforward calculation, one can obtain the explicit expression for the correlator

\[
K^{\alpha\beta} = \langle L_{ab}\psi^\gamma(t, z)(t^b)^\alpha_\gamma L_{cd}\psi^\sigma(\bar{t} w)(t^d)^\beta_\sigma \rangle
\tag{A.11}
\]

\[
= \frac{1/2}{(z-w)^{2\Delta+2}} \left( L_{ab}(t^a t^b)^{\beta\alpha} - (1/\Delta)L_{ab}L_{cd}(t^d t^a t^b)^{\beta\alpha} \right).
\]

Here $\bar{t}$ is the complex conjugate representation defined as

\[
(t^a)^\beta_\alpha = -\eta_\alpha\gamma \eta^{\beta\sigma}(t^a)^\gamma_\sigma.
\tag{A.12}
\]
where $\eta_{\alpha\beta}$ is the metric which is used to rise and lower indices of the field $\psi$. Now it is clear that this correlator vanishes when $t^a$ are in the fundamental representation of $SU(2)$.

Thus, the l.h.s of eq. (8) is nothing but a null vector. Note that in order to derive eq. (11) we have to use the Virasoro master equation. Therefore, the vanishing of the correlator in eq. (11) can be considered as another way for the master equation to be arrived.

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