Distributed optimal control applied to Fluid Structure Interaction problems

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Abstract. In this paper we focus on an optimal control problem based on Lagrangian multipliers and adjoint variables applied to the steady state Fluid-structure interaction problem. FSI systems consist of one or more solid structures that deform by interacting with a surrounding fluid flow. We consider the Fluid-structure interaction problem written in variational monolithic form in order to balance automatically solid and liquid forces at the interface. The objective of the control problem is the minimization of a velocity matching functional and it is obtained with a distributed force control that deforms the shape of the solid domain. This distributed control for Fluid-structure interaction systems can be easily used to control the shape of the solid domain and the flow effects of the fluid-solid interactions. The optimality system is derived from the first order optimality conditions by taking the Fréchet derivatives of the Lagrangian with respect to all the variables involved. The optimal solution is then obtained through a steepest descent algorithm applied to the optimality system. To support this approach we report a few numerical tests where the distributed force deforms the solid structure in order to match the desired velocity profile. The results obtained show the feasibility and robustness of this approach and its possible use in many industrial relevant applications.

1. Introduction

Fluid structure interaction (FSI) systems are commonly studied in many engineering application such as the design of floating structures, wind turbines, parachutes, injection systems, valves and in the study of biological systems [1, 2, 3, 4, 5]. Usually those kind of problems are solved in a forward approach in order to evaluate the tensional state of a mechanical component due to a surrounding fluid flow taking into account the effects of the solid deformations on the motion of the interior flows. However, in recent years optimization has become a key aspect in the field of engineering, in order to improve the performance of devices or reduce the cost of production processes. Nowadays several approaches to optimization are available, such as single and multi-objective, adjoint or sensitivities based method, evolutionary algorithms and many others, see [6, 7, 8, 9, 10, 11]. In this paper we would like to study the inverse FSI problem by using an optimal control approach based on adjoint variables which have been proven to be a good approach for the optimal control of complex systems [12, 8]. We adopt a monolithic approach to solve the FSI system to obtain a stable and well defined solution in a finite element setting.

Inside this framework the objective of our control problem is the minimization of a velocity field tracking functional in a specific region of the fluid domain with a distributed force that acts in the solid domain and changes the location of both the solid and fluid domains. Through Lagrange multipliers formalism we reformulate the physically constrained minimization problem.
into an unconstrained one. The optimality system is then obtained by imposing the first order necessary conditions which consists in setting to zero the Fréchet derivatives of the Lagrangian taken with respect to all the variables involved. In order to solve the optimality system we use a simple steepest descent algorithm. In the next section we first recall some basic notation about the functional spaces of interest. Then we present the FSI system together with the boundary and interface conditions. Finally the optimality system is obtained and the iterative algorithm used is presented. In order to support the proposed approach we perform some numerical tests with different values of the regularization parameter.

2. Mathematical formulation

In this section we discuss the mathematical formulation of the FSI problem and we derive the optimality system. We briefly recall some notations about functional spaces used in this paper, for a detailed description one can consult [13, 14]. We use standard notation $H^s(\Omega)$ for the Sobolev spaces with norm $\| \cdot \|_s (H^0(\Omega) = L^2(\Omega)$ and $\| \cdot \|_0 = \| \cdot \|)$. Let $H^s_0(\Omega)$ be the space of all functions in $H^s(\Omega)$ that vanish on the boundary of the bounded open set $\Omega$ and $H^{-s}(\Omega)$ be the dual space of $H^s_0(\Omega)$. The trace space for the functions in $H^1(\Omega)$ is denoted by $H^{1/2}(\Gamma)$.

Let us consider a domain $\Omega \subset \mathbb{R}^N$ which consists of a structure domain $\Omega_s$ and a fluid domain $\Omega_f$ so that $\Omega = \Omega_s \cup \Omega_f$. The deformed solid domain $\Omega_s(l)$ is expressed through the solid displacement field $l$ as

$$\Omega_s(l) = \{ x \in \mathbb{R}^3 \mid x = x_0 + l \}, \quad (1)$$

where the vector $x_0$ defines the initial solid domain $\Omega_s(0)$, taken as the reference domain. For our steady state FSI problem we consider the interaction of a Newtonian fluid, whose behavior is described by the Navier-Stokes equations, with a St.Venant-Kirchhoff material. The mathematical model in strong form of the FSI problem is then the following

$$\nabla \cdot v = 0 \quad \text{on} \quad \Omega_f, \quad (2)$$

$$\rho_f (v \cdot \nabla) v - \nabla \cdot T = 0 \quad \text{on} \quad \Omega_f, \quad (3)$$

$$\nabla \cdot S(l) = f \quad \text{on} \quad \Omega_s, \quad (4)$$

with $v$ being the fluid velocity field and $\rho_f$ its density. We denoted with $f$ the distributed force that acts in the solid region only. The constitutive relations for the fluid stress tensor $T$ in the Newtonian incompressible case and for the solid Cauchy stress tensor $S$ read

$$T(p_f, v_f) = -p_f I + \mu_f (\nabla v + \nabla v^T), \quad (5)$$

$$S(l) = \lambda_s (\nabla \cdot l) I + \mu_s \nabla l, \quad (6)$$

where $p_f$ is the fluid pressure, $\mu_f$ the dynamic viscosity of the fluid while $\lambda_s$ and $\mu_s$ are the solid Lamé parameters. To complete the system (2-4) it is necessary to define the appropriate boundary and interface conditions, which are

$$v = v_0 \quad \text{on} \quad \Gamma_{fd},$$

$$l = l_0 \quad \text{on} \quad \Gamma_{sd},$$

$$T \cdot n = 0 \quad \text{on} \quad \Gamma_{fn},$$

$$S \cdot n = 0 \quad \text{on} \quad \Gamma_{sn},$$

$$T \cdot n = S \cdot n \quad \text{on} \quad \Gamma_i,$$

$$v = 0 \quad \text{on} \quad \Gamma_i, \quad (7)$$
where $\Gamma_{fd}$ and $\Gamma_{sd}$ are the surfaces where Dirichlet boundary conditions are imposed for the
fluid velocity and solid displacement, $\Gamma_{fn}$ and $\Gamma_{sn}$ are the surfaces where standard homogeneous
outflow boundary conditions are imposed for both the fields. By $\Gamma_i$ we denote the interface
between the solid and fluid domain, $\Gamma_i = \Omega_s \cap \Omega_f$.

In this work we study a fluid velocity matching profile problem in a region of the liquid
domain by using a distributed force that moves and deforms the solid. This goal is expressed in
terms of the following objective functional

$$
\mathcal{J}(v,f) = \frac{1}{2} \int_{\Omega_d} w ||v - v_d||^2 d\Omega + \frac{1}{2} \beta \int_{\Omega_c} ||f||^2 d\Omega 
$$

where $v_d$ is the desired velocity on the controlled domain region $\Omega_d \subset \Omega_f$ and $w$ a weight
function of the coordinates $x$ used to give more importance to some parts of $\Omega_d$. The functional
is completed by a regularization term which is needed to obtain a control function $f$ in the
space of square integrable functions $L^2(\Omega_c)$, with $\Omega_c$ being the solid region where the control
can act. The regularization parameter $\beta$ plays a fundamental role for the numerical solution
of the minimization problem. If a too high value of $\beta$ is chosen the regularization contribution
dominates over the objective one and the objective cannot be achieved well, while a lack of
regularization leads to singular solutions or convergence issues in the numerical solution of the
problem.

To obtain the optimality system we write the full Lagrangian of the problem which
is composed of the functional (8) and state equations (2-4) multiplied by the appropriate
Lagrangian multipliers, the so-called adjoint variables. By doing so we transform a constrained
minimization problem into an unconstrained one.

$$
\mathcal{L}(p,v,\mathbf{l},p_a,\mathbf{v},\mathbf{s},\beta) = \mathcal{J}(v,f) + \int_{\Omega_f} (\nabla \cdot v)p_a d\Omega + \int_{\Omega_s} (\nabla \cdot v)p_a d\Omega + \\
+ \int_{\Omega_f} [\rho f (v \cdot \nabla)v + \nabla p - \nabla \cdot (\mu_f \nabla v)] \cdot v_a d\Omega + \\
+ \int_{\Omega_s} [-\nabla p + \nabla \cdot (\mu_s \nabla \mathbf{l}) + f] \cdot v_a d\Omega + \int_{\Omega_s} \nabla^2 \mathbf{l} \cdot \mathbf{l} a d\Omega + \\
+ \int_{\Gamma_i} \mathbf{s} a \cdot (\mathbf{l} - \frac{v}{h}) d\Gamma + \int_{\Omega_s} \beta a \cdot [v - h(1 - \mathbf{l})] d\Omega .
$$

In (9) we have introduced the auxiliary mesh displacement $\mathbf{l}$ defined as

$$
\nabla^2 \mathbf{l} = 0 \quad \text{on} \ \Omega_s \\
\mathbf{l} = 1 - \frac{v}{h} \quad \text{on} \ \Gamma_i \\
\mathbf{l} = 0 \quad \text{on} \ \Gamma_s - \Gamma_i ,
$$

through which we can extend the velocity field in the whole domain $\Omega$, with $h$ positive and constant,

$$
v = \begin{cases} 
  h(1 - \mathbf{l}) & \text{on} \ \Omega_s , \\
  v & \text{solution of (2)-(3) on} \ \Omega_f .
\end{cases}
$$

It is clear that at steady state $\Omega/l(1) = \Omega f(\mathbf{l})$ and $\Omega^s(1) = \Omega^s(\mathbf{l})$ with $v = 0$ on the interface $\Gamma_i$.

In order to derive the optimality system we impose the minimization necessary condition

$$
\delta \mathcal{L} = 0 .
$$
Since the total variation of the Lagrangian has to be zero and each variation is independent from the others then each variation has to be zero. By taking the Fréchet derivatives of the Lagrangian (9) with respect to the adjoint variables the weak form of the state system (2-4) is obtained, together with the correct boundary and interface conditions. When the derivatives are taken with respect to the state variables, after some term rearrangement, the following adjoint system in weak form is recovered

$$\int_{\Omega_f} (\nabla \cdot v_a) \delta p \, d\Omega = 0 \quad \forall \delta p \in L^2(\Omega_f)$$  \hspace{1cm} (15)

$$\int_{\Omega_f} [(\rho' (\delta v \cdot \nabla)) \cdot v_a + (\rho' (v \cdot \nabla)\delta v) \cdot v_a + \mu' \nabla \delta v : \nabla v_a] \, d\Omega +$$

$$+ \frac{1}{h} \int_{\Omega_a} [\mu_s \nabla v_a : \nabla \delta v + \lambda_s (\nabla \cdot v_a)(\nabla \cdot \delta v)] \, d\Omega +$$

$$+ \int_{\Omega_f} (\nabla \cdot \delta v)p_a \, d\Omega + \int_{\Omega_d} w(v - v_d)\delta v \, d\Omega = 0 \quad \forall \delta v \in H^1_{f, \Gamma_d, \Gamma_{sd}}(\Omega)$$ \hspace{1cm} (16)

$$\int_{\Omega_a} \nabla \phi_a : \nabla \phi_d \, d\Omega + \int_{\Omega_a} \mu \nabla \nabla v_a : \nabla \phi_d \, d\Omega = 0 \quad \forall \phi_d \in H^1_{f, \Gamma_d}(\Omega_a)$$ \hspace{1cm} (17)

The shape derivatives with respect to the fluid and solid domain have been taken into account and simplified. Furthermore the equation for the adjoint displacement \( l_a \) can be neglected since we do not need it to determine the force \( f \). Finally, when considering the Fréchet derivatives of the Lagrangian (9) with respect to the control parameter \( f \) we obtain the control equation

$$f = \frac{v_a}{\beta}.$$  \hspace{1cm} (18)

By performing integrations by parts on the system (15-17) it is possible to determine the strong form of the adjoint system together with the boundary conditions for the adjoint variables. After performing the integration by parts, we recover the adjoint state \((v_a^f, v_a^s, p_a) \in H^1_{0; \Omega_f - \Gamma_i}(\Omega^f) \times H^2(\Omega^f) \times H^1_{0; \partial \Omega - \Gamma_i}(\Omega^s) \times H^2(\Omega^s) \times L^2(\Omega^f) \cap H^1(\Omega^s)\), by solving

$$\nabla \cdot v_a^f = 0,$$  \hspace{1cm} (19)

$$- \rho_l (\nabla v)^T v_a^f + \rho_l [(v \cdot \nabla)v_a^f] + \nabla p_a - \nabla \cdot (\mu \nabla v_a^f) = w(v - v_d),$$  \hspace{1cm} (20)

$$\nabla \cdot S(v_a^s) = 0.$$  \hspace{1cm} (21)

with boundary conditions defined as

$$v_a^s = v_a^f \quad \text{on} \quad \Gamma_i,$$

$$S(v_a^s) \cdot n = T(v_a^f) \cdot n \quad \text{on} \quad \Gamma_i,$$

$$\mu_l (\nabla v_a) \cdot n = -(v \cdot n) v_a \quad \text{on} \quad \Gamma_f^d \cup \Gamma_{sd},$$

$$p_a = 0 \quad \text{on} \quad \Gamma_f^d \cup \Gamma_{sd}.$$  \hspace{1cm} (22)

The optimality system is then composed by the state (2-4) and adjoint (15-17) systems coupled with the control equation (18). Due to the strong non-linearity of the problem a one-shot solution of the optimal system can not be performed. In this work we use a segregate approach for the solution of the optimality system, solving the state system, the adjoint system and the control equation separately and iteratively, see for example [15]. By doing so we can also use the same solver with only minimal modifications to solve the adjoint and state systems. The
Algorithm 1 Description of the Steepest Descent algorithm.

1. Set a state \((v^0, p^0, l^0)\) satisfying (2-4) \(\triangleright \) Setup of the state - Reference case
2. Compute the functional \(\mathcal{J}^0\) in (8)
3. Set \(r^0 = 1\)
   
   for \(i = 1 \rightarrow i_{\text{max}}\) do
     4. Solve the system (15)-(17) to obtain the adjoint state \((v_i^a, p_i^a)\) \(\triangleright \) Line search
     5. Set the control update \(\delta f^i = (f^i - f^{i-1}) - \frac{v_i^a}{\beta}\)
     6. Set \(r^i = r^0\)
        while \(\mathcal{J}^i(f^{i-1} + r^i \delta f^i) > \mathcal{J}^{i-1}(f^{i-1})\) do
          7. Set \(r^i = \rho r^i\)
          8. Solve (2-4) for the state \((v^i, f^i, l^i)\) with \(f^i = f^{i-1} + r^i \delta f^i\)
             if \(r^i < \text{toll}\) then
               Line search not successful \(\triangleright \) End of the algorithm
             end if
        end while
     if \(||\mathcal{J}^i(f^{i-1} + r^i \delta f^i) - \mathcal{J}^{i-1}(f^{i-1})||/\mathcal{J}^{i-1}(f^{i-1}) < \tau\) then
       9. Convergence reached \(\triangleright \) End of the algorithm
     end if
   end for

iterative algorithm adopted to minimize the objective functional is the simple Steepest Descent method described in Algorithm 1.

It starts solving the state system with no control obtaining the reference case \((v^0, p^0, l^0)\). Then it is determined the gradient direction from the solution of the control equations (18), in which \(v_a\) is known once that the adjoint system (15-17) has been solved. The core of the algorithm consists of the backtracking line search process, where the state system is solved iteratively reducing the value of the control parameters until a functional reduction is obtained. The step length \(r\) determines how far from the current state solution we are moving along the gradient direction given by \(\delta f\). This algorithm comes to an end either when two consecutively computed functionals are almost equal and no further improvement can be achieved or when \(r\) becomes lower than a tolerance value \(\text{toll} = 10^{-7}\). In this case the optimal solution is the one obtained at the previous iteration step.

This algorithm requires several solutions of the state and adjoint systems in order to find the optimal control, however it does not need a great amount of memory which is limited to a standard CFD simulation. We implemented this algorithm in an in-house finite element code which is parallelized by using openMPI libraries and uses a multigrid solver with mesh-moving capability. We have used standard quadratic-linear elements for all the variables except the pressure which is assumed linear to satisfy the BBL inf-sup condition. The displacements are approximated with standard quadratic elements.

3. Numerical results

In this section we present the results obtained by applying Algorithm 1 to a two-dimensional test case. In Figure 1 we have reported the domain considered with the geometrical properties. The origin of the reference system is the bottom left corner (point \(A\)). The solid region \(\Omega_s\) is colored with darker gray while the fluid region \(\Omega_f\) with lighter gray. In red is the fluid controlled region \(\Omega_d = [0.465, 0.5]m \times [0.16, 0.2]m\) which is located near the channel exit. We impose pressure boundary conditions with vanishing tangential velocity on the left and right surfaces, \(p_{AD} = 10000Pa\) and \(p_{BC} = 9750Pa\), respectively. On the lower boundary \(AB\) we prescribe a no-slip condition, while the upper surface \(CD\) is a symmetry axis and we set a homogeneous Neumann boundary condition for all the state and adjoint variables apart from
Figure 1. Case study. Domain overview with the solid region ($\Omega_s$), the fluid region ($\Omega_f$) and the controlled region ($\Omega_d$).

Figure 2. Reference geometry with velocity (m/s) profile in the fluid region (on the top). Pressure (Pa) profile and iso-lines in the fluid region (on the bottom).

$u_y$ and $u_{ay}$ that vanish. All the boundaries are fixed with the exception of the interface $\Gamma_i$, that may move with fixed endpoints $H$ and $I$. The physical properties are $\rho_s = \rho_l = 1000\,Kg/m^3$ with $\mu_l/\rho_l = 0.07Pa\,m^3/Kg$ and $\mu_s = 76250Pa$ so that the flow is not turbulent and the solid can bend easily. Young’s modulus $E_s$ and Poisson’s coefficient $\xi$ are equal to 183000$Pa$ and $\xi = 0.2$, respectively.

According to Algorithm 1, we start solving the state system (2-4) assuming $f = 0$ in order to
**Figure 3.** Velocity ($m/s$) profile in the fluid region and force ($N$) vectors (on the top), $\beta = 10^{-9}$. Pressure ($Pa$) field and iso-lines in the fluid region (on the bottom), $\beta = 10^{-9}$.

**Figure 4.** On the left axial velocity $u_x$, profile on a line at $x = 0.49$. On the right $u_x$ on a line at $y = 0.18$. Result (A) obtained with no control, (B) with $\beta = 10^{-7}$ and (C) with $\beta = 10^{-9}$.

obtain the reference case shown in Figure 2. The velocity profile is reported on the top of this Figure, while on the bottom the pressure field with iso-lines is shown. Since the solid obstacle
reduces the fluid domain cross section the fluid has to accelerate to satisfy the mass conservation equation. Furthermore, the solid object is responsible for the majority of the pressure losses as it can be seen on the bottom of Figure 2.

Our optimal control problem consists in finding the optimal solid deformation so that the axial component of the velocity ($v_x$) matches a desired profile over the controlled domain $\Omega_d$. The solid is deformed by a distributed force $f$ which is the control parameter acting in the whole solid domain, see equation (4). In the reference case with no control we obtained a mean value of $v_x$ over $\Omega_d$ equal to $\bar{v}_x = 0.037 \text{ m/s}$. We then choose a constant target value $v_{xtd} = 0.065 \text{ m/s}$ in the whole controlled domain, thus requiring a higher fluid velocity near the channel outlet. The objective functional becomes

$$J(v_x, f) = \frac{1}{2} \int_{\Omega_d} w(v_x - 0.065)^2 d\Omega + \frac{1}{2} \beta \int_{\Omega_s} ||f||^2 d\Omega$$

in which we set

$$w = \begin{cases} 1 & x \in \Omega_d, \\ 0 & \text{otherwise}. \end{cases}$$

Once that the reference case has been set up it is possible to solve the optimality system with the technique described in Algorithm 1.
We now report the results obtained for different values of the regularization parameter $\beta = 10^{-5}, 10^{-7}, 10^{-9}$. On the top of Figure 3 we reported the velocity profile and the force acting in the solid domain, and on the bottom of the same Figure is shown the pressure profile. The force bends the solid to the right so that the channel cross section is larger than that of the reference case. Since the total pressure drop between the channel inlet and outlet is fixed by the boundary conditions ($250 \text{ Pa}$), if the pressure losses induced by the obstacle decrease then the fluid velocity has to increase. In particular it is worth noticing that, due to the regularization term in (23), it is more effective to apply an intense force near the tip of the solid instead of a weaker but more distributed one. In Figure 4 it is shown the axial component of the velocity field ($v_x$) on a vertical line at $x = 0.49$ on the left and on a horizontal one at $y = 0.18$ on the right, for different values of the regularization $\beta$. Both lines cross the controlled region $\Omega_d$. By reducing the value of $\beta$ the control can act more strongly and the velocity is higher than that of the reference case. In Figure 5 is reported the control force for different amount of regularization. The force acting on the solid is very small in the case $\beta = 10^{-5}$, while becomes more relevant when reducing $\beta$, leading to a higher solid deformation. The adjoint velocity profile is reported in Figure 6 with arrows and colors related to the $v_a$ magnitude. The source term of the adjoint velocity, which is the difference between the actual and the desired velocity in the area $\Omega_d$ is the driving force that deforms the solid.

| $\beta$ | $\infty$ | $10^{-5}$ | $10^{-7}$ | $10^{-9}$ |
|--------|--------|--------|--------|--------|
| $J(u) \cdot 10^8$ | 113.69 | 111.68 | 32.75 | 1.10 |
| $v_x$ | 0.0367 | 0.0375 | 0.0584 | 0.0651 |

Table 1. Objective functionals $J$ and average axial velocity over the controlled region $\Omega_d$ computed with no control ($\beta = \infty$) and different $\beta$ values.

Finally, in Table 1 we reported the functionals and the average axial velocity over the controlled region as computed in the reference and in the controlled case with different $\beta$. The functional has been reduced in every case and in particular with $\beta = 10^{-9}$, in which the axial velocity is very close to the desired value $0.065$.

4. Conclusions
In this work we have presented an optimal control method for the fluid structure interaction problem based on adjoint variables. The objective of the optimal control problem is the velocity matching in a specific region of the fluid domain through a distributed control acting in the solid to change the shape of the solid domain. We have adopted a monolithic variational formulation to satisfy automatically the coupling conditions at the fluid-solid interface. Furthermore, we have extended the velocity field to the solid domain to couple adjoint variables and forces on the interface. The optimality system has been derived by imposing the first order necessary conditions to the full Lagrangian and solved iteratively with a steepest descent algorithm in order to use the same solver for state and adjoint system. The results obtained show the feasibility and robustness of this approach. In future works we plan to assess this kind of control to more realistic geometries such as wing shape optimization to show the feasibility of this approach in real complex cases.
References

1. Bazilevs Y, Takizawa K and Tezduyar T 2013 Computational fluid-structure interaction (John Wiley & Sons)
2. Formaggia L, Quarteroni A and Veneziani A 2009 Cardiovascular Mathematics, vol. 1 (Springer, Heidelberg)
3. Cerroni D, Da Via R and Manservisi S 2018 J. Comput. Phys. 354 646–671
4. Cerroni D, Gionni D, Manservisi S and Mengini F 2018 Lecture Notes 84 217–231
5. Aulisa E, Cervone A, Manservisi S and Seshaiyer P 2009 Commun. Comput. Phys. 6 319–341
6. Gunzburger M D 2003 Perspectives in flow control and optimization vol 5 (Siam)
7. Nocedal J and Wright S 2006 Numerical optimization (Springer Science & Business Media)
8. Gunzburger M and Manservisi S 2000 Comput Methods Appl Mech Eng 189(3) 803–823
9. Rowley C W and Williams D R 2006 Annu. Rev. Fluid Mech. 38 251–276
10. Kim J and Bewley T R 2007 Annu. Rev. Fluid Mech. 39 383–417
11. Manservisi S and Menghini F 2016 Comput. and Fluids 125 130–143
12. Yan Y and Keyes D E 2015 Journal of Computational Physics 281 759–786
13. Adams R A 1975 Sobolev spaces (Academic Press, New York)
14. Brenner S and Scott R 2007 The mathematical theory of finite element methods vol 15 (Springer Science)
15. Cerroni D and Manservisi S 2016 J. Computat. Phys. 313 13–30