A DIAGRAMMATICAL APPROACH TO HOPF MONADS

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ABSTRACT. Given a Hopf algebra in a symmetric monoidal category with duals, the category of modules inherits the structure of a monoidal category with duals. If the notion of algebra is replaced with that of monad on a monoidal category with duals then Bruguières and Virelizier showed when the category of modules inherits this structure of being monoidal with duals, and this gave rise to what they called a Hopf monad. In this paper it is shown that there are good diagrammatic descriptions of dinatural transformations which allows the three-dimensional, object-free nature of their constructions to become apparent.

INTRODUCTION

Overview. An algebra, i.e., a monoid in the category of vector spaces, has an associated category of modules (or representations, if you prefer). A Hopf algebra is an algebra equipped with extra structure which ensures that its category of modules inherits the monoidal structure and duals from the category of vector spaces. These notions work similarly in braided monoidal categories other than that of vector spaces. A monad on a monoidal category can be thought of as a generalization of an algebra (or monoid) in that category and has an associated category of modules (also known, confusingly, as its category of algebras). Brugières and Virelizier [1] defined, following Moerdijk [7], a Hopf monad structure which ensures that a monad’s category of modules is monoidal with duals.

The goal of this paper is to put some of the work of Brugières and Virelizier into a diagrammatic context, which also means to put it into appropriate framework of monoidal two-categories. One of the purposes of this was to make their constructions essentially object-free. To do this it was necessary to do various things including using an object-free formulation of categories with duals, which here means describing evaluation and coevaluation as dinatural transformations and then extending string diagrammatics to include dinatural transformations, something which appears to work rather well. Such dinatural transformations exist in the landscape of the monoidal two-category of categories, and so are three-dimensional in nature, thus are better manipulated, I would argue, using the three-dimensional algebra presented here.

In terms of results on Hopf monads, many of the results are just slight simplifications of those of Brugières and Virelizier. The example of a strong monoidal functor with a left adjoint is given a more explicit treatment, this example being of primary importance to me. My motivation lies in the specific case of a Hopf monad on the derived category of coherent sheaves on a complex manifold; the monad coming from a strong monoidal functor with a left adjoint.

Three-dimensional string diagrams. The utility of string diagram notation in describing adjunctions and monads is well-known (see for example [4] though it undoubtedly has its roots in the Australian school). Adjunctions and monads live in the two-category of categories and their counterparts in the monoidal category
A monad \( T : \mathcal{C} \to \mathcal{C} \) on a category is an endofunctor together with a \textit{multiplication} natural transformation \( T^2 \Rightarrow T \) and a \textit{unit} natural transformation \( \text{Id}_\mathcal{C} \Rightarrow T \) satisfying some appropriate associativity and unital conditions. One then has the category of \( T \)-modules, consisting of pairs \( (m, r) \) where \( m \) is an object in \( \mathcal{C} \) and \( r : T(m) \to m \) is an action map in a suitable sense. If \( \mathcal{C} \) is a monoidal category then it is natural to ask if the tensor product of any two \( T \)-modules can be given a natural \( T \)-module structure. For instance, if \( T \) is of the form \( A \otimes - \) for some algebra object \( A \) of \( \mathcal{C} \) then this occurs when \( A \) has the structure of a bialgebra. The question was considered by Moerdijk \cite{Moerdijk} and he showed that lifting the monoidal structure on \( \mathcal{C} \) to a monoidal structure on the module category corresponds precisely to giving \( T \) some extra structure to form what he called, following operad terminology, a Hopf monad. However in this arena this is not the best nomenclature and a better term would be either \textit{bimonad}, by analogy with bialgebras, or, more accurately, but less succinctly, \textit{opmonoidal monad}, as the extra structure is the same as making \( T \) into a monad in the category of opmonoidal functors, as was observed by McCrudden \cite{McCrudden}.

A natural progression from this is to ask if duals from \( \mathcal{C} \) lift to duals on the category of modules, so if \( \mathcal{C} \) is a monoidal category with duals and \( M \) is object of \( \mathcal{C} \) with an action of the bimonad \( T \) on it, does the dual \( M^\vee \) also naturally have an action of \( T \) on it? In the case where \( T = A \otimes - \) is a monad on, say, the category of vector spaces for a bialgebra \( A \), then \( A \) being a \textit{Hopf} algebra, i.e., having an antipode, suffices to make the module category have a lift of the duality on vector spaces. Bruguière and Virelizier defined the notion of antipode for a bimonad on a category with duals, such that it corresponds to the duals lifting to the module category.

An example. As explained in Section 4, a strong monoidal functor with a left adjoint gives rise to a Hopf monad by composing it with its left adjoint. Suppose \( G \) is a finite group, then a good example of such a functor is the functor \( \Delta^* \) from the category of representations of \( G \times G \) to the category of representations of \( G \) defined by restricting to the diagonal copy of \( G \). This is a strong monoidal functor and has a left adjoint \( \Delta_* \), which is induction from \( G \) to \( G \times G \) along the diagonal embedding. Then \( \Delta^* \Delta_* \) is a Hopf monad on the category of representations of \( G \), in fact it is basically the tensor-with-the-group-algebra functor, the group algebra (with the conjugation action) is a Hopf algebra in this category. Similar examples arise in other areas such as when a complex manifold takes the place of the finite group and the derived category of coherent sheaves takes the place of the representation category. These examples naturally arise in topological quantum field theory. Details will appear elsewhere.

Synopsis. In the first section we recall the notion of string diagrams and how they are useful for denoting adjoints. We then enhance the notation in a third dimension to denote products of categories. We then see how this is used with monoidal and
opmonoidal functors and natural transformations. Finally opposite categories are introduced to the notation.

In Section 2 we recall the string diagrammatic approach to monads and see how Moerdijk’s notion of bimonad fits in. In Section 3 we recall the notion of dinatural transformations and how they are used in the definition of a monoidal category with duals.

Section 4 is where we get to the definition of Hopf monad and Bruguières and Virilizier’s theorem about their categories of modules being monoidal with duals. Finally, following [1], we see how a strong monoidal functor with a left adjoint gives rise to a Hopf monad.

**Terminology.** There is a split in terminology between category theorists and, say, quantum algebraists, in that an object in a monoidal category with a unital, associative multiplication is called a monoid by the former and an algebra by the latter, primarily because the typical categories for the two groups of mathematicians are respectively the category of sets and the category of vector spaces. We will stick to the latter terminology as we are close to areas in which Lie algebra objects and universal enveloping algebra objects are considered. Similarly it makes sense from this perspective to talk of *modules* over a monad, rather than *algebras* over a monad as the category theorists prefer. My apologies go to any reader to whom this seems ridiculous, but there will be a good proportion of the audience to whom it will seem very sensible.

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1. **The diagrammatics of natural transformations**

In this section we introduce the basic string diagram notation for natural transformations. In particular we will be interested in monoidal categories and opmonoidal functors, so will need to extend the notation in an extra dimension. We will also need to consider contravariant functors so we end this section with a way of denoting opposite categories.

1.1. **String diagrams.** Firstly, here is a very quick reminder on the use of string diagrams to represent natural transformations. These diagrammatics have a history going back to Feynmann and Penrose, but were formalized in the context of monoidal categories by Joyal and Street [3], and this formalism extends to arbitrary two-categories; though here we will just be interested in the two-category of categories, functors and natural transformations. The idea of string diagrams for natural transformations is that the usual globular pictures of functors and natural transformations are replaced by their Poincaré duals, so that categories are represented by two-cells, functors by one-cells, and natural transformations by zero-cells. The basic example is as follows:

\[
\text{becomes}
\]

A more complicated example is the following:

\[
\text{becomes}
\]
Note that the identity functor is omitted from the string diagram notation.

The standard example of the utility of string diagrams is with regard to adjoint functors and this will turn out to be useful to us later. So suppose that $F: C \to D$ and $U: D \to C$ form an adjunction $F \dashv U$, then we have the unit and counit natural transformations, $\varepsilon: F \circ U \Rightarrow \text{Id}_D$ and $\eta: \text{Id}_C \Rightarrow U \circ F$, which are drawn as follows:

$$
\varepsilon \equiv \begin{array}{c} \text{D} \\
\text{C} \\
\text{U} \\
\text{F} \\
\text{D}
\end{array} 
\quad \eta \equiv \begin{array}{c} \text{U} \\
\text{F} \\
\text{C}
\end{array}
$$

Then the required conditions on the unit and counit are drawn as

$$
\begin{array}{c}
\text{D} \\
\text{F} \\
\text{C} \\
\text{U} \\
\text{C}
\end{array} = \begin{array}{c}
\text{D} \\
\text{F} \\
\text{C}
\end{array} \quad \text{and} \quad
\begin{array}{c}
\text{D} \\
\text{U} \\
\text{F} \\
\text{C} \\
\text{U}
\end{array} = \begin{array}{c}
\text{C} \\
\text{U} \\
\text{D}
\end{array}
$$

where the vertical line marked with $F$ means the identity natural transformation on the functor $F$.

1.2. Monoidal categories and monoidal functors. The notation above can be enhanced to also encode cartesian products of categories, so can denote, for instance, functors of the form $\otimes: C \times C \to C$. In this context we can think of the two-category $\mathcal{C}at$ with its cartesian product as being a one-object three-category, and so should expect to have to use three-dimensions in our notation.

We will use the direction out of the page for the ‘product direction’. So, for example, given functors $G, H: C \times C' \to C''$ and $K: C'' \to C'''$, we denote $G$ and $K \circ H$ as follows.

A natural transformation $\theta: G \Rightarrow K \circ H$ will then be denoted

$$
\begin{array}{c}
\text{C''} \\
\text{G} \\
\text{C'} \\
\text{C}
\end{array} \quad \text{or} \quad
\begin{array}{c}
\text{C''} \\
\text{K} \\
\text{C''} \\
\text{C'} \\
\text{C}
\end{array}
$$

the latter being used if the labels are clear from the context.

Note that diagrams are read front to back, right to left and bottom to top.

1.2.1. Monoidal categories. A monoidal category $(\mathcal{C}, \otimes, 1)$ consists, as is well known, of a category $\mathcal{C}$, a functor $\otimes: \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, and a unit object $1$ — here considered as a functor $1: * \to \mathcal{C}$ from the one object, one morphism category $*$ — together with an associativity natural transformation $\alpha: \otimes \circ (1 \times \otimes) \Rightarrow \otimes \circ (\otimes \times 1)$, and unit natural isomorphisms $\nu_1: \otimes \circ (1 \times \text{Id}_\mathcal{C}) \Rightarrow \text{Id}_\mathcal{C}$ and $\nu_\ast: \otimes \circ (\text{Id}_\mathcal{C} \times 1) \Rightarrow \text{Id}_\mathcal{C}$. The natural transformations have to satisfy the pentagon and triangle relations.
The associativity will be drawn as

\[ \alpha \equiv \]

\[ C \times C \times C \to C \]

but coherence means that notationally we can draw three-fold tensor products with the understanding that to make sense they must be resolved into the composition of two-fold tensor products, but that, up to canonical identification, it is independent of the choice of resolution, so we could instead draw the following triple tensor product \( - \otimes - \otimes - : C \times C \times C \to C \).

The category \( \star \) is the unit object for the product \( \times \) on \( \mathcal{C} \mathcal{a}t \); we will make the canonical identifications \( C \times \star \cong C \cong \star \times C \) and thus allow ourselves to denote \( \star \) by the empty surface. So for instance the unit \( 1 : \star \to C \) will be denoted by the picture on the left, from which the picture on the right will be understood.

\[ C \equiv \star \]

The unit natural isomorphisms \( \nu_l : \otimes \circ (1 \times \text{Id}_C) \Rightarrow \text{Id}_C \) and \( \nu_r : \otimes \circ (\text{Id}_C \times 1) \Rightarrow \text{Id}_C \) are drawn as

\( \nu_l \equiv \)

\( \nu_r \equiv \)

The inverses of these are drawn the same but the other way up.

1.2.2. Monoidal and opmonoidal functors. If \( (\mathcal{C}, \otimes, 1) \) and \( (\mathcal{D}, \otimes, 1) \) are monoidal categories then a monoidal functor \( M : \mathcal{D} \to \mathcal{C} \) — sometimes called a weak- or lax-monoidal functor — means a functor equipped with a natural family of morphisms \( M(d_1) \otimes M(d_2) \to M(d_1 \otimes d_2) \), parameterized by pairs of objects, and a morphism \( 1 \to M(1) \) satisfying coherence conditions. In other words, there are natural transformations \( \sigma^2_M : \otimes \circ (M \times M) \Rightarrow M \circ \otimes \) and \( \sigma^0_M : 1 \Rightarrow M \circ 1 \), satisfying some constraints. These natural transformations will be drawn as follows:

\[ \sigma^2_M \equiv \]

\[ \sigma^0_M \equiv \]
The conditions that they are required to satisfy are the diagrammatically appealing:

(Mondl1)

(Mondl2)

(Mondl3)

The first condition together with associativity of the tensor product means that again we can unambiguously draw triple tensor products, with the understanding that such a triple tensor product should be resolved into one of the two compositions of ordinary tensor products. Thus we think of the two natural transformations pictured as a single natural transformation $(\cdot \otimes \cdot \otimes \cdot) \circ (M \times M \times M) \Rightarrow M \circ (\cdot \otimes \cdot \otimes \cdot)$, drawn as

A functor $M$ as above is said to be strong monoidal if both the natural transformations drawn above are natural isomorphisms.

Similarly an opmonoidal functor $Q: C \to D$ — sometimes called a comonoidal functor — is a functor equipped with natural transformations $\sigma^Q_2: Q \circ \otimes \Rightarrow \otimes \circ (Q \times Q)$ and $\sigma^Q_0: Q \circ 1 \Rightarrow 1$, drawn as

which satisfy the above relations inverted, which will be called (Opmond1–3).

Note the important situation in which $U$ and $F$ form an adjoint pair $F \dashv U$ of functors between monoidal categories. Then there is a bijection between pairs of natural transformations making $U$ monoidal and pairs of natural transformations
making $F$ opmonoidal; or more informally, $U$ is monoidal if and only if $F$ is opmonoidal. To see this, suppose that $U$ is monoidal, then an opmonoidal structure is defined on $F$ using the following two natural transformations:

The proof that they satisfy the requisite relations is easily derived diagrammatically.

1.2.3. Opmonoidal natural transformations. We will need the notion of an opmonoidal natural transformation. So suppose that $P, Q : \mathcal{C} \rightarrow \mathcal{D}$ are two opmonoidal functors between monoidal categories, then an opmonoidal natural transformation $\theta : P \Rightarrow Q$ is a natural transformation which commutes with the opmonoidal structure transformations, in other words, the following relations hold:

$\theta_{Q} \circ \theta_{P} = \theta_{Q \circ P}$ (OpmonNT1)

$\theta_{Q} \circ \theta_{P} = \theta_{Q \circ P}$ (OpmonNT2)

1.3. Opposite categories and contravariance. Later on we will be interested in looking at duals in categories. There is a slight problem from the point of view of diagrams in that a duality $\nabla$ on a category $\mathcal{C}$ is actually a contravariant functor, so is not a functor $\mathcal{C} \rightarrow \mathcal{C}$ but rather can be considered a covariant functor $\mathcal{C}^{op} \rightarrow \mathcal{C}$. This means it will be extremely convenient to be able to denote opposites in the diagrammatic language. First it is imperative to think about the operation of taking opposites inside the two-category $\mathcal{Cat}$. Given a category $\mathcal{C}$ the category $\mathcal{C}^{op}$ is the category whose objects are in canonical bijection with the objects of $\mathcal{C}$ but whose arrows are reversed. This means given a functor $G : \mathcal{C} \rightarrow \mathcal{D}$ one obtains a functor $G^{op} : \mathcal{C}^{op} \rightarrow \mathcal{D}^{op}$. However, given a natural transformation $\theta : G_{1} \Rightarrow G_{3} \circ G_{2}$ one obtains a natural transformation $\theta^{op} : G_{3}^{op} \circ G_{2}^{op} \Rightarrow G_{1}^{op}$. This is an easy and informative exercise for the reader. Put another way, this gives a two-functor $^{op} : \mathcal{Cat} \rightarrow \mathcal{Cat}^{co}$ where $\mathcal{Cat}^{co}$ denotes the two-category of categories with the two-morphisms reversed.

In traditional notation we get a correspondence as follows:

```
\begin{equation*}
\begin{array}{c}
G_{3} \\
\Downarrow \theta \\
G_{2} \\
\Downarrow \\
G_{1}
\end{array}
\end{equation*}
```

gives rise to

```
\begin{equation*}
\begin{array}{c}
G_{3}^{op} \\
\Downarrow \theta^{op} \\
G_{2}^{op} \\
\Downarrow \\
G_{1}^{op}
\end{array}
\end{equation*}
```

To get this into the string pictures we will adopt the useful convention that a shaded region means that it is the opposite category, and the functors and natural
transformations will be the opposites of their labels. Thus

\[
\begin{array}{c}
G_3 \xrightarrow{\theta} G_2 \\
D \xrightarrow{\varnothing} G_1
\end{array}
\quad \text{gives rise to} \quad
\begin{array}{c}
D^{\text{op}} \xrightarrow{\varnothing^{\text{op}}} C^{\text{op}}_1 \\
G_3^{\text{op}} \xrightarrow{\varnothing^{\text{op}}} C^{\text{op}}_2
\end{array}
\quad \text{denoted}
\begin{array}{c}
\begin{array}{c}
D \xrightarrow{\varnothing} \quad G_1 \\
G_3
\end{array} \quad \text{and} \quad
\begin{array}{c}
C^{\text{op}}_1 \\
C^{\text{op}}_2
\end{array}
\end{array}
\]

Note the essential difference that “shaded” natural transformations are “turned upside-down”. When we get to dinatural transformations, the shading can be given the interpretation of the ‘other-side’ of the surface.

This notational convention means that a contravariant functor \( \vee \) on a category \( C \) can be denoted in the following way:

\[
\begin{array}{c}
C \\
\vee
\end{array}
\]

Note that the conditions needed to be satisfied by a duality on a monoidal category will be stated in terms of dinatural transformations, so we will not do that properly until later.

2. Monads and bimonads

In this section we will look at monads and bimonads using string diagrams. In particular we consider the category of modules over a monad from this perspective; this can be seen as a ‘formal’ or two-categorical point of view in that we discuss the category of modules over a monad on a category without talking about the internal structure of the category. We go on to look at bimonads and how the category of modules in this case is monoidal, analogous to the category of modules for a bialgebra. Finally we will see how a bimonad arises from a pair of adjoint functors.

2.1. Monads. This will be a quick diagrammatic recap on monads. A monad on a category \( C \) is an endofunctor \( T: C \to C \) together with natural transformations \( \mu: T^2 \Rightarrow T \) and \( \iota: \text{Id}_C \Rightarrow T \), known as the multiplication and unit and drawn as

\[
\mu \equiv \begin{array}{c}
T \\
T \\
T
\end{array} \quad \text{and} \quad \iota \equiv \begin{array}{c}
T
\end{array}
\]

These have to satisfy the associativity and unit laws, namely

\[
\begin{array}{c}
\begin{array}{c}
T
\end{array} \\
\begin{array}{c}
T
\end{array} \quad \Rightarrow \quad \begin{array}{c}
T
\end{array}
\end{array}
\quad \text{(Monad1)}
\]

\[
\begin{array}{c}
\begin{array}{c}
T \\
T
\end{array} \\
\begin{array}{c}
T
\end{array} \quad = \quad \begin{array}{c}
T
\end{array}
\end{array}
\quad \text{(Monad2)}
\]

There is an associated category \( TC \) of \( T \)-modules, this is sometimes written \( C^T \). The objects of this category are pairs \( (m, (r: T(m) \to m)) \) where \( m \) is an object
of \( \mathcal{C} \), such that the diagrams

\[
\begin{array}{c}
\xymatrix{T \circ T(m) \ar[r]^<<<<<<<<<{T \circ m} & T(m) \\
m \ar[r]^<<<<<<<<<{\mu_m} & T(m) \\
T(m) \ar[r]^<<<<<<<<<{r} & m}
\end{array}
\quad \text{and} \quad
\begin{array}{c}
m \ar[r]^<<<<<<<<<{\iota_m} & T(m) \\
\ar[r]^<<<<<<<<<{\text{id}_m} & \text{id}_m \\
m \ar[r]^<<<<<<<<<{\iota_m} & m
\end{array}
\]

both commute, as one would expect from anything befitting the name ‘module’.

The morphisms in the category of \( T \)-modules are morphisms between the underlying objects of \( \mathcal{C} \) that commute with the \( T \)-action. The example that I have in mind here is where \( \mathcal{C} \) is a monoidal category, \( A \) is a unital algebra object in \( \mathcal{C} \) and \( T \) is the monad \( A \otimes - \); in this case the category of \( T \)-modules is precisely the category of \( A \)-modules in the usual sense.

We fit the category of modules into the graphical calculus by considering associated functors and natural transformations. An object of the category of modules consists of an object of \( \mathcal{C} \) and an action morphisms, an alternative view of such a pair is that we have a forgetful functor \( U_T: \mathcal{C} \to \mathcal{C} \) given by \( U_T(m, r) := m \), which just forgets the action, and we have a natural transformation \( \rho: T \circ U_T \Rightarrow U_T \) defined by \( \rho(m, r) := r \) which encodes the action. We will denote the forgetful functor \( U_T \) by a dashed-dotted line and draw the natural transformation \( \rho \) as follows.

\[
\begin{array}{c}
\xymatrix{\rho & \mathcal{C} \ar[r]^<<<<<<<<<{T C} & \mathcal{T C} \\
& \mathcal{T C} \ar[l]^<<<<<<<<<{U_T} \\
& \mathcal{C} \ar[l]^<<<<<<<<<{T U_T}
\end{array}
\]

The module conditions above become the following:

\[
\begin{array}{c}
\xymatrix{T \circ U_T \ar[r]^<<<<<<<<<{T U_T} & U_T \circ T \\
U_T \circ T \ar[r]^<<<<<<<<<{T U_T} & T \circ U_T}
\end{array}
\quad \text{(Module1)}
\quad \text{and}
\quad \begin{array}{c}
\xymatrix{T \circ U_T \ar[r]^<<<<<<<<<{T U_T} & U_T \circ T \\
U_T \circ T \ar[r]^<<<<<<<<<{T U_T} & T \circ U_T}
\end{array}
\quad \text{(Module2)}
\]

We also have a free module functor \( F_T: \mathcal{C} \to \mathcal{T C} \). This is given on objects by

\[
F_T(x) := (T(x), (\mu_{T(x)}: T(T(x)) \to T(x))).
\]

Note that \( T \) is precisely the composite \( U_T \circ F_T \), so we have identity natural transformations \( \text{Id}: U_T \circ F_T \Rightarrow T \) and \( \text{Id}: T \Rightarrow U_T \circ F_T \) which we draw as

\[
\begin{array}{c}
\xymatrix{T \ar[r]^<<<<<<<<<{U_T} & \mathcal{C} \ar[r]^<<<<<<<<<{T} & \mathcal{T C} \\
\mathcal{C} \ar[r]^<<<<<<<<<{F_T} & \mathcal{T C} \ar[l]^<<<<<<<<<{U_T}
\end{array}
\quad \text{and}
\quad \begin{array}{c}
\xymatrix{T \ar[r]^<<<<<<<<<{U_T} & \mathcal{C} \ar[r]^<<<<<<<<<{T} & \mathcal{T C} \\
\mathcal{T C} \ar[r]^<<<<<<<<<{F_T} & \mathcal{T C} \ar[l]^<<<<<<<<<{U_T}
\end{array}
\]

Here it should be observed that the graphical language fails to distinguish between identity natural transformations and natural isomorphisms.

Note that the multiplication \( \mu \) on \( T \) is recovered from these identifications together with the action natural transformation \( \rho \) in the following way:

\[
\begin{array}{c}
\xymatrix{T \ar[r]^<<<<<<<<<{U_T} & \mathcal{C} \ar[r]^<<<<<<<<<{T} & \mathcal{T C} \\
T \ar[r]^<<<<<<<<<{T U_T} & \mathcal{T C} \ar[l]^<<<<<<<<<{U_T}
\end{array}
\quad \text{and}
\quad \begin{array}{c}
\xymatrix{T \ar[r]^<<<<<<<<<{U_T} & \mathcal{T C} \ar[r]^<<<<<<<<<{T} & \mathcal{T C} \\
T \ar[r]^<<<<<<<<<{U_T} & \mathcal{T C} \ar[l]^<<<<<<<<<{U_T}
\end{array}
\]

Here it should be observed that the graphical language fails to distinguish between identity natural transformations and natural isomorphisms.
2.2. Monads from adjoint functors. A standard way of obtaining monads is via pairs of adjoint functors. Suppose that $F$ and $U$ form such a pair, $F \dashv U$, then $U \circ F : \mathcal{C} \to \mathcal{C}$ forms a monad. The multiplication and unit of the monad are obtained from the unit and counit of the adjunction in the following easily drawn fashion:

$$
\mu \equiv \begin{array}{c}
\quad U \\
\quad F & \quad U \\
\quad F & \quad U \\
\quad F \\
\end{array}
$$

$$
\iota \equiv \begin{array}{c}
\quad U \\
\quad F \\
\end{array}
$$

If the reader has not seen this before then they should immediately verify diagrammatically that the axioms of a monad are satisfied.

It should be noted that every monad $T$ actually arises in this way, as the composite of a left and a right adjoint; for example, there is an adjunction $F_T \dashv U_T$ between the free and forgetful functors described above. In general there will be several different adjoint decompositions of a monad.

2.3. Bimonads. We now bring monads and monoidal categories together. Suppose that $T : \mathcal{C} \to \mathcal{C}$ is a monad on a monoidal category $(\mathcal{C}, \otimes, 1)$. We can then ask the question “Under what circumstances does the monoidal structure on $\mathcal{C}$ lift to a monoidal structure on the category of $T$-modules $\mathcal{C}^T$?” Or, we could ask the weaker and less precise question “Given two $T$-modules $(m, (r : T(m) \to m))$ and $(m', (r' : T(m') \to m'))$ how do we obtain a natural $T$-module structure on $m \otimes m'$?”

The answer to the first question (and hence the second) was found by Moerdijk, but before stating the answer we should introduce the following piece of terminology.

**Definition.** A monad $T : \mathcal{C} \to \mathcal{C}$ on a monoidal category has a bimonad (or op-monoidal monad) structure if the functor $T$ has an op-monoidal structure with respect to which both the multiplication $\mu$ and the unit $\iota$ are op-monoidal natural transformations (in the sense of Section 1.2.3).

Bimonads are so named because of the analogy with bialgebras given by Theorem 1 below, though they were called Hopf monads in his original paper. Before stating the Theorem it is worth unpacking this rather concise definition a little. A monad has a bimonad structure if firstly there is an op-monoidal structure for $T$, that is there are specified natural transformation

$$
\sigma^T_2 : \otimes \circ (T \times T) \Rightarrow T \circ \otimes \quad \text{and} \quad \sigma^T_0 : T \circ 1 \Rightarrow 1,
$$
drawn as follows, in which hopefully it is clear where the hidden lines go.

These natural transformations must satisfy the axioms (Opmondl1–3) and the multiplication and unit must be op-monoidal with respect to this which means that the following must hold:

$$
\text{(BM1–2)}
$$
The result of Moerdijk can now be stated.

**Theorem 1** (Moerdijk [7]). Let $T$ be a monad on a monoidal category $C$ and let $TC$ be its category of modules. Then specifying a lift of the monoidal structure on $C$ to a monoidal structure on $TC$ is precisely the same as specifying a bimonad structure on $T$.

We will spend the rest of this section seeing why this is true. To do this, the following is an extremely useful observation.

**Proposition 2.** For $T : C \to C$ a monad, $TC$ its category of modules and $D$ any category, specifying a functor $H : D \to TC$ is precisely the same as specifying a functor $U_H : D \to C$ and a natural transformation $\rho_H : T \circ U_H \Rightarrow U_H$, drawn as

\[
\begin{array}{c}
T \circ U_H \\
\downarrow \\
U_H
\end{array}
\]

such that the following two relations are satisfied:

\[ (R1-2) \quad T \circ U_H = U_H \quad \text{and} \quad T \circ U_H = U_H. \]

**Proof (sketch).** This is just the fact that $TC$ consists of pairs $(m, r)$, where $m \in C$ and $r : T(m) \to m$ is an action. So given such a functor $H$ define $U_T H$ to be $U_T \circ H$ and $\rho_H : T \circ U_T \circ H \Rightarrow U_T \circ H$ to be $\rho \circ \text{id}_H$. Conversely, given such a pair, define $H(d) := (U_T H(d), (\rho_H^d : T(U_T H(d)) \to U_T H(d))).$ □

Now we will see how to prove Moerdijk’s Theorem. To lift the monoidal structure of $C$ to a monoidal structure on $TC$ we need to specify the tensor product and unit on $TC$, together with the associativity and unital natural transformations. We will concentrate on the tensor product.

For a functor $\otimes^T : C \times TC \to TC$ to be a lift of a functor $\otimes : C \times C \to C$ (with respect to the forgetful functor $U_T : TC \to C$) the following diagram must commute.

\[
\begin{array}{ccc}
TC \times TC & \xrightarrow{\otimes^T} & TC \\
U_T \times U_T \downarrow & & \downarrow U_T \\
C \times C & \xrightarrow{\otimes} & C
\end{array}
\]

Algebraically this means that $U_T \circ \otimes^T = \otimes \circ (U_T \times U_T)$, so in the notation of Proposition 2 we have $U \otimes^T = \otimes \circ (U_T \times U_T)$. By the proposition then, lifting the functor $\otimes$ is precisely the same as specifying a natural transformation

\[
\rho_{\otimes} : T \circ \otimes \circ (U_T \times U_T) \Rightarrow \otimes \circ (U_T \times U_T)
\]

which satisfies the two conditions of the lemma. (Intuitively this says that to specify a lifted tensor product of two modules we need to specify an action of $T$ on the tensor product of the two objects underlying the modules.) We draw the natural
transformation as

\[ \rho \otimes \equiv \]

In general for an adjunction \( F \dashv U \), and with \( G \) and \( H \) functors with suitable source and target, there is a bijection between sets of natural transformations:

\[ \text{Nat}(G, H \circ F) \cong \text{Nat}(G \circ U, H). \]

Hence, because there is the adjunction \( F_T \dashv U_T \) and the monad factorizes as \( T = U_T \circ F_T \), for any functors \( G, H : \mathcal{C} \to \mathcal{D} \) there are the following identifications of sets of natural transformations

\[ \text{Nat}(G, H \circ T) = \text{Nat}(G, H \circ U_T \circ F_T) \cong \text{Nat}(G \circ U_T, H \circ U_T). \]

In the diagrammatic notation, the isomorphism between the two outside sets is given by

\[ \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
G
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
H
\end{array}
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\begin{array}{c}
\begin{array}{c}
T
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\frac{}{\Rightarrow} \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
G
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\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
H
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\begin{array}{c}
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\begin{array}{c}
U_T
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\end{array}
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\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
T
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\frac{}{\Rightarrow} \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
G
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H
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U_T
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F_T
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And suppose that $U$ has a left adjoint $F : C \to D$ then, as mentioned in Section 1.2.2, $F$ is opmonoidal; and as $U$ is also opmonoidal, so the composite $U \circ F$ is opmonoidal, with the opmonoidal structure given explicitly as follows:

$$
\sigma_{U \circ F}^2 \equiv \begin{array}{c}
\begin{array}{c}
U \\
F
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
U \\
F
\end{array}
\end{array}
\begin{array}{c}
F \\
F
\end{array} \\
\begin{array}{c}
F \\
F
\end{array} \\
U
\end{array}
\quad \quad \quad \quad \quad \sigma_{U \circ F}^0 \equiv \begin{array}{c}
\begin{array}{c}
FU \\
F
\end{array}
\end{array}
$$

Of course, $U \circ F$ is also a monad, and it is very easy to see in this pictorial language that it is an opmonoidal monad, ie. that the product and unit of the monad are opmonoidal transformations.

For instance, the following proves that (BM1) holds:

Thus in the case that $U$ is strongly monoidal, $U \circ F$ is a bimonad.

Note that if $T$ is a bimonad, then the category $T C$ of $T$-modules is monoidal and in the decomposition $T = U_T \circ F_T$ the forgetful functor $U_T$ is strongly monoidal.

3. The diagrammatics of dinatural transformations

In this section we introduce dinatural transformations so that we can give a diagrammatic description of duality on a monoidal category, the point being that evaluation and coevaluation are dinatural rather than natural transformations.

3.1. Motivating example: a monoidal category with duals. The first question to address is “What is an appropriate notion of a monoidal category with duals?” To simplify the situation, we will just consider left duals: right duals can be handled similarly. So suppose that $(C, \otimes, 1)$ is a monoidal category, a left duality on $C$ will be a functor $\vee : C^{op} \to C$ together with evaluation and coevaluation maps for every object $a$ in the category,

$$
\text{ev}_a : \vee a \otimes a \to 1 \quad \text{and} \quad \text{coev}_a : 1 \to a \otimes \vee a,
$$

such that for every morphism $f : a \to a'$ in the category the following naturality conditions hold:

$$
\text{ev}_a \circ (\vee f \otimes \text{id}) = \text{ev}_{a'} \circ (\text{id} \otimes f) \quad \text{and} \quad (f \otimes \text{id}) \circ \text{coev}_a = (\text{id} \otimes \vee f) \circ \text{coev}_{a'},
$$
and such that the following “snake” relations hold,

\[(\text{id}_a \otimes \text{ev}_a) \circ (\text{coev}_a \otimes \text{id}_a) = \text{id}_a\]
\[(\text{ev}_a \otimes \text{id}_a) \circ (\text{id}_a \otimes \text{coev}_a) = \text{id}_{va}.$

One would like to interpret \(\text{ev}\) and \(\text{coev}\) as some sort of natural transformations, so that, for instance, \(\text{ev}\) would be a natural transformation from the “functor” \(\mathcal{C} \to \mathcal{C}\) given by \(a \mapsto \gamma a \otimes a\) to the functor \(\mathcal{C} \to \mathcal{C}\) given by \(a \mapsto 1\): however the former is not a functor. So we consider the functor \(\mathcal{C}^{\text{op}} \times \mathcal{C} \to \mathcal{C}\); \((b, a) \mapsto \gamma b \otimes a\) and the functor \(1: * \to \mathcal{C}\). Evaluation is then a family of morphisms \(\text{ev}_a: \gamma a \otimes a \to 1\), indexed by the objects of \(\mathcal{C}\), such that the following diagrams commute:

3.2. Dinatural transformations. Motivated by the above example, we are led to Eilenberg and Kelly’s notion of dinatural transformation. Suppose \(P: \mathcal{C} \times \mathcal{C}^{\text{op}} \times A \to B\) and \(Q: A \times D^{\text{op}} \times D \to B\) are two functors, then a dinatural transformation \(\beta: P \Rightarrow Q\) is a family of morphisms \(\beta_{c,a,d}: P(c, c, a) \to Q(a, d, d)\) which satisfies the following naturality condition. If \(f: a \to a', g: c \to c'\) and \(h: d \to d'\) are morphisms in \(A, C\) and \(D\) respectively then the diagram below commutes.

Note that each of the categories \(A, C, D\) and \(D\) can be products of other categories or can indeed be the terminal category \(*\) and can all be permuted in the definition. Thus the case of evaluation described above occurs when \(\mathcal{C} = \mathcal{B}\) and \(A = D = *\), and a usual natural transformation is the case where \(\mathcal{C} = \mathcal{D} = *\).

We can then denote such a dinatural transformation as follows. For functors \(P: \mathcal{C} \times \mathcal{C}^{\text{op}} \times A \to B\) and \(Q: A \times D^{\text{op}} \times D \to B\)
a dinatural transformation $\beta: P \Rightarrow Q$ is denoted

\[
\begin{array}{c}
A \\
B \\
P \\
\beta \\
Q \\
C \\
D
\end{array}
\]

where as usual the diagram is read upwards.

The first thing to note is that in the case of a natural transformation, when $C = D = \star$, we recover the usual string diagram notation.

The next thing to note is the right-hand profile of the surface. This is the so-called Eilenberg-Kelly graph of the dinatural transformation, consisting of arcs with the end-points of an arc labelled by the same category. These graphs are important in the composition of dinatural transformations as we will see below.

It can also be pointed out that a dinatural transformation can actually be written as a natural transformation. For example a dinatural transformation $\beta: P \Rightarrow Q$ as above is equivalent to a natural transformation

$$\text{Hom}_C(-, -) \times \text{Hom}_A(-, -) \times \text{Hom}_D(-, -) \Rightarrow \text{Hom}_B(P(-, -, -), Q(-, -, -))$$

between functors from $C^{op} \times C \times A^{op} \times A \times D^{op} \times D$ to Set, so this also allows a diagrammatic description, but it is rather messier and the composition is not as straightforward.

### 3.3. Vertical composition.

In order to make sense of the snake condition on $\text{ev}$ and $\text{coev}$ we will need to define the vertical composition of dinatural transformations. If $\beta': P \Rightarrow Q$ and $\beta: Q \Rightarrow R$ are two dinatural transformations then the composite $\beta \circ \beta'$ can not always be defined; however the composite can be defined, resulting in a dinatural transformation, in the case that the composite of the Eilenberg-Kelly graphs contains no loops, as I will now explain.

This is best illustrated by our motivating example. If we have a monoidal category $C$ with a functor $\vee: C^{op} \rightarrow C$ and dinatural transformations $\text{ev}: \vee \otimes \text{Id}_C \Rightarrow 1$ and $\text{coev}: 1 \Rightarrow \text{Id}_C \otimes \vee$, where, for instance $\vee \otimes \text{Id}_C$ means $\circ \circ (\text{Id}_C \times \vee)$. Then we can draw these dinatural transformations as

\[
\begin{array}{c}
\text{ev} = \\
\text{coev} =
\end{array}
\]

We then have a dinatural transformation $\text{id} \otimes \text{ev}: \text{Id}_C \otimes (\vee \otimes \text{Id}_C) \Rightarrow \text{Id}_C$ whose components are given by $(\text{id} \otimes \text{ev})_{a'', a} := \text{id}_{a''} \otimes \text{ev}_a: a'' \otimes (\vee \otimes a) \rightarrow a''$. This should be drawn with binary tensor products, but, by identifying $\circ \circ (\otimes \times \text{Id}_C)$ with a triple tensor product $- \otimes - \otimes -$, it will be drawn as

\[
\begin{array}{c}
\text{id} \otimes \text{ev} =
\end{array}
\]
Similarly we have $\text{coev} \otimes \text{id}: \text{Id}_C \Rightarrow (\text{Id}_C \otimes \gamma) \otimes \text{Id}_C$ with its components given by $[\text{coev} \otimes \text{id}]_{a',a} := \text{coev}_{a'} \otimes \text{id}_a: a \rightarrow (a' \otimes \gamma(a')) \otimes a$ which is drawn as

$$\text{coev} \otimes \text{id} \equiv \includegraphics[width=0.2\textwidth]{diagram1.png}.$$ 

The two dinatural transformations pictured above can be vertically composed to give a dinatural transformation, which is actually a natural transformation, $\text{id} \otimes \text{ev} \circ (\text{coev} \otimes \text{id}): \text{Id}_C \Rightarrow \text{Id}_C$ given by

$$[(\text{id} \otimes \text{ev}) \circ (\text{coev} \otimes \text{id})]_a := (\text{id} \otimes \text{ev})_{a,a} \circ (\text{coev} \otimes \text{id})]_{a,a}: a \rightarrow a$$

and drawn as

$$\includegraphics[width=0.2\textwidth]{diagram2.png}.$$ 

One of the conditions for $(\gamma, \text{ev}, \text{coev})$ to form a duality on the monoidal category $C$ is that the above should be the identity natural transformation. The two conditions are drawn in the next subsection, below.

More generally, we can define the composite of two dinatural transformations provided the composite Eilenberg-Kelly graph has no loops. For example we can form a dinatural transformation from two dinatural transformations of the following form. Suppose we have functors

$$P: \text{pt} \rightarrow B, \quad R: C \times C^{\text{op}} \rightarrow B, \quad Q: C^{\text{op}} \times C \times C^{\text{op}} \rightarrow B$$

together with dinatural transformations $\beta': P \Rightarrow Q$ and $\beta: Q \Rightarrow R$ which pair up the categories as pictured below, then there is a composite $\beta \circ \beta': P \Rightarrow R$.
However, we can not form a dinatural transformation from the composite $\text{ev} \circ \text{coev}$ as we would get a loop in the Eilenberg-Kelly graph as can be seen here:

See [2] for more details.

3.4. Definition of a monoidal category with left duals. We can now state the definition of a monoidal category with left duals in this language. Suppose that $C$ is a monoidal category, $\vee: C^{\text{op}} \to C$ is a functor and $\text{ev}: \vee \otimes \text{Id}_C \Rightarrow \text{Id}_C$ and $\text{coev}: \text{Id}_C \Rightarrow \text{Id}_C \otimes \vee$ are dinatural transformations drawn as

Then $(\vee, \text{ev}, \text{coev})$ forms a left duality of $C$ if the following snake relations hold:

3.5. Composition with natural transformations. Suppose that $F: \tilde{A} \to A$, $G: \tilde{C} \to C$, $H: \tilde{D} \to D$, and $K: B \to \tilde{B}$ are functors and that $\beta: P \Rightarrow Q$ is a dinatural transformation of the above form then we get a dinatural transformation

$$\tilde{\beta}: K \circ P \circ (G \times G^{\text{op}} \times F) \Rightarrow K \circ Q \circ (F \times H^{\text{op}} \times H),$$
given by $\tilde{\beta}_{a,c,d} := K(\beta_{F(a),G(c),H(d)})$. This is denoted graphically as

Note that in the case that $\beta$ is an ordinary natural transformation this recovers the ordinary horizontal composition of natural transformations.

Suppose now that we have natural transformations $\phi$, $\gamma$, $\theta$, and $\kappa$, together with a dinatural transformation $\beta: P \Rightarrow Q$ of the above form then the following dinatural transformations are equal:

This just follows from the definitions. In traditional notation it is quite a mess to write down, thus this does show how nicely the diagrammatics capture the essence of composition of dinatural transformations.

4. Hopf monads

In this section we use the diagrammatic language to first give Bruguières and Virilizier’s definition of a Hopf monad and a minor simplification of their proof that such thing is equivalent to a lift of duals to the module category. We then use the diagrammatics to be more explicit than them in their example of a monad coming from a strongly monoidal functor with a left adjoint.

4.1. Hopf monads. The difference between a bialgebra and a Hopf algebra is that the latter has an antipode. In the current context, the principal consequence of this is that the vector space dual of a module over a Hopf algebra carries a canonical action. More precisely, the duality on the base category of vector spaces lifts to a duality on the category of modules. It is this property that we wish to examine for monads, and we can do this by asking the question “What structure is required of a bimonad on a monoidal category with duals so that the category of modules has a lift of the duals?” This was answered by Bruguières and Virilizier and here we will describe their solution in the diagrammatic language, something that they themselves would have liked to have done.

Bruguières and Virilizier gave the definition of ‘left antipode’. We will restate this definition using the diagrammatic notation developed above.
Definition. If \( T \) is a bimonad on a monoidal category with left duals then a left antipode for \( T \) is a natural transformation \( S : T \circ \circ \circ T \Rightarrow \circ \), denoted as follows,

\[
S \equiv \begin{array}{c}
\text{Diagram 1}
\end{array}
\]

which satisfies the following two relations:

\[
\text{(HM1)} = \begin{array}{c}
\text{Diagram 2}
\end{array}
\]

\[
\text{(HM2)} = \begin{array}{c}
\text{Diagram 3}
\end{array}
\]

Here the dinatural transformation parts of the diagrams are the evaluation and coevaluation of the duality on the category.

A bimonad equipped with a left antipode is called a (left) Hopf monad.

The question asked above is then fully answered by the following theorem which tells us that, in a certain specific sense, Hopf monads are analogous to Hopf algebras.

**Theorem 3** (Bruguières and Virelizier [1]). Suppose that \( T \) is a bimonad on a monoidal category \( C \) with a left duality and that \( TC \) is the monoidal category of \( T \)-modules. Then specifying a lift of the left duality on \( C \) to a left duality on \( TC \) is the same as specifying a left antipode for \( T \).

The rest of this section will consist of a diagrammatic proof of the above theorem. We essentially translate Bruguières and Virelizier’s proof into our diagrammatic language, with some minor simplification making the proof more transparent.

We begin with a lemma similar to results in Section 2.3 on bimonads.

**Lemma 4.** If \( T \) is a monad on a category \( C \) and \( \circ : C^{op} \rightarrow C \) is a functor then lifts of this to a functor \( \circ : TC^{op} \rightarrow TC \) on the category of modules correspond to natural transformations \( T \circ \circ \circ T \Rightarrow \circ \), drawn as

\[
\begin{array}{c}
\text{Diagram 4}
\end{array}
\]

which satisfy

\[
\text{(HM0)} = \begin{array}{c}
\text{Diagram 5}
\end{array} \quad \text{and} \quad \begin{array}{c}
\text{Diagram 6}
\end{array}
\]
Proof. The functor $\wedge$ being a lift of the functor $\vee$ means that the following diagram commutes.

\[
\begin{array}{ccc}
TC^{\text{op}} & \xrightarrow{\wedge} & TC \\
U_T^{\text{op}} & \downarrow & U_T \\
C^{\text{op}} & \xrightarrow{\vee} & C
\end{array}
\]

By Lemma 2, specifying a functor $\wedge: TC^{\text{op}} \to TC$ is the same as specifying a functor $U_T \circ \wedge: TC^{\text{op}} \to TC$ and a natural transformation $T \circ U_T \circ \wedge \Rightarrow U_T \circ \wedge$ satisfying the two module conditions. But as the above diagram commutes, we know that $U_T \circ \wedge = \vee \circ U_T^{\text{op}}$, so we just need to specify a natural transformation $T \circ \vee \circ U_T^{\text{op}} \Rightarrow \vee \circ U_T^{\text{op}}$, drawn as

and which satisfies the two module conditions. Analogously to the bimonad case we can use the identity $T = U_T \circ F_T$ and the reversed adjunction $U_T^{\text{op}} \dashv F_T^{\text{op}}$ to obtain a bijection

\[\text{Nat}(G \circ U_T^{\text{op}}, H \circ U_T^{\text{op}}) \cong \text{Nat}(G \circ T^{\text{op}}, H).\]

So the above natural transformation corresponds to a natural transformation $S: T \circ \vee \circ T^{\text{op}} \Rightarrow \vee$:

\[S \equiv \begin{array}{c}
T \\
\xrightarrow{U_T} \\
\xrightarrow{\vee} \\
T
\end{array}\]

The original natural transformation is recovered in the following way:

\[T \equiv \begin{array}{c}
T \\
\xrightarrow{U_T} \\
\xrightarrow{\vee} \\
T
\end{array}\]

Concretely, we can recover the lift $\wedge$ from $S$ via

\[\wedge(m, r) = (\vee m, S_m \circ T(\vee r)).\]

The two module conditions translate to (HMO) as required. \(\square\)

In order to show that such a lift of a left duality is itself a left duality we need to show that $\text{ev}$ and $\text{coev}$ define $T$-module maps. This is precisely where (HM1) and (HM2) come into play.

Theorem 5. Suppose that $T$ is a bimonad on a monoidal category $C$ with duals, and that $S: T \circ \vee \circ T^{\text{op}} \Rightarrow \vee$ is a natural transformation satisfying the conditions (HMO), so it gives rise to a functor $\wedge: TC^{\text{op}} \to TC$.

1. The evaluation dinatural transformation on $C$ lifts to a dinatural transformation on the module category $TC$ if and only if (HM1) is satisfied.
2. The coevaluation dinatural transformation on $C$ lifts to a dinatural transformation on the module category $TC$ if and only if (HM2) is satisfied.
Proof. Consider the evaluation case. For \( \text{ev} \) to lift to an evaluation on \( TC \) its components must be maps of \( T \)-modules, that is they must commute with the \( T \)-action, so for each \( T \)-module \((m, r) \in TC \) the following diagram must commute.

\[
\begin{array}{c}
T(\lhd m \otimes m)^{\text{ev}_m} \rightarrow T(1) \\
\downarrow \quad \downarrow \\
\lhd m \otimes m \rightarrow 1
\end{array}
\]

Here, of course, the action on \( \lhd m \otimes m \) is using \( S \) and the bimonad structure.

In terms of the diagrammatic calculus this means that the following must hold:

Now we can use the same machinery as before to remove \( U_T \) from the statement: namely using we have a bijection

\[
\text{Dinat} \left( G \circ (U_T \times U_T), H \right) \cong \text{Dinat} \left( G \circ (T^{\text{op}} \times \text{Id}), H \right).
\]

Via this correspondence, the above equality becomes

Moving things ‘over the top’ this condition is seen to be equivalent to

which is, by the properties of \( U_T \) and \( F_T \) from Section 2 is just

and this \((\text{HM1})\) as required.

The coevaluation case is similar. \( \square \)

We have now seen that if \( T \) is a bimonad on a monoidal category with duals then specifying a lift of the duality on \( C \) to a duality on \( TC \) the category of \( T \)-modules is equivalent to specifying a natural transformation \( T \circ \lhd \circ T^{\text{op}} \Rightarrow \lhd \) such that conditions \((\text{HM0}), (\text{HM1}) \) and \((\text{HM2}) \) are satisfied. To prove Theorem 3 we just need to see that \((\text{HM0})\) is actually a redundant condition.
Theorem 6. Suppose that $T$ is a bimonad on a monoidal category with duals, then any natural transformation $T \circ \vee \circ T^{op} \Rightarrow \vee$ which satisfies $(HM_1)$ and $(HM_2)$ also satisfies $(HM_0)$.

Proof. We first prove that the following equation holds:

\[
\begin{array}{cc}
\text{LHS} := & \text{RHS}.
\end{array}
\]

This is true for the following reason:

\[
\begin{array}{cc}
\text{LHS} := & \text{RHS}.
\end{array}
\]

Also, by $(HM_1)$ we have

\[
\begin{array}{cc}
= & .
\end{array}
\]
Thus

\[
\text{Duality}_2 = \text{††} = \text{†} = \text{††} = \text{Duality}_2 = 0.
\]

\[\square\]

Now Theorem 5 and Theorem 6 immediately imply Theorem 3 as required.

4.2. **Hopf monads from adjoint functors.** Suppose that there is an adjunction \(F \dashv U\) where \(U : D \to C\) is a strong monoidal functor between monoidal categories with (left) duals. We know from Section 2.4 that \(U \circ F\) is a bimonad; we will see that it is also naturally a Hopf monad, i.e., that it naturally comes equipped with an antipode. This is due to Brugières and Virelizier but we make the structure more explicit than in their paper.

4.2.1. **Overview.** The key point for the definition of the antipode is that if \(U\) is strong monoidal then it commutes with taking duals. More precisely, we will see below that there is a natural isomorphism \(\text{†} \circ U^\text{op} \cong U \circ \text{†}\) and we will draw the mutually inverse transformations as

\[
\text{Duality}_2 \quad \text{ and } \quad \text{Duality}_2
\]

We will also see below how to define these from the monoidal structure of \(U\), but first we can use these together with the unit and counit of the adjunction to define \(S : U \circ F \circ \text{†} \circ U^\text{op} \circ F^\text{op} \Rightarrow \text{†}\), the antipode for the bimonad \(U \circ F\), in the following
way:

\[
S := \begin{array}{c}
\text{Diagram}
\end{array}
\]

This will be shown to indeed be an antipode in Section 4.2.3, but the reader is invited to check diagrammatically that this satisfies \((\text{HM0})\).

4.2.2. **Strong monoidal functors commute with taking duals.** We will now show that \(U\) commutes with taking duals, i.e., that there exists a natural isomorphism \(\vee \circ U^{\text{op}} \cong U \circ \vee\). On the level of objects the idea is that for \(d \in \mathcal{D}\) the objects \(\vee U^{\text{op}}(d)\) and \(U(\vee d)\) are both left duals of \(U(d)\), and so they are canonically isomorphic via the standard yoga: namely, being somewhat fastidious, we have

\[
\begin{align*}
\vee U^{\text{op}}(d) &\to \vee U^{\text{op}}(d) \otimes 1 \to \vee U^{\text{op}}(d) \otimes U(1) \to \vee U^{\text{op}}(d) \otimes U(d \otimes \vee d) \\
&\to \vee U^{\text{op}}(d) \otimes (U(d) \otimes U(\vee d)) \to (\vee U^{\text{op}}(d) \otimes U(d)) \otimes U(\vee d) \\
&\to 1 \otimes U(\vee d) \to U(\vee d).
\end{align*}
\]

This gives rise to a natural isomorphism \(\vee \circ U^{\text{op}} \Rightarrow U \circ \vee\) constructed from the dinatural transformations \(\text{ev}\) and \(\text{coev}\) as follows:

\[
\begin{array}{c}
\text{Diagram}
\end{array}
\]

The inverse transformation is defined similarly.

Using these, we can now show that \(U\) also commutes with evaluation and coevaluation.

**Theorem 7.** If \(U\) is a strong monoidal functor as above then \(U\) commutes with evaluation and coevaluation in the following sense:

\[
\begin{array}{c}
\text{Diagram}
\end{array}
\]

In more traditional notation this is expressing the commutativity of the following diagrams:

\[
\begin{array}{c}
\text{Diagram}
\end{array}
\]

**Proof.** We will consider the evaluation case as the coevaluation case is similar. The left hand diagram is seen to commute as soon as its left hand arrow is unpacked as in the following diagram.
4.2.3. The antipode and Hopf monad. We can now prove that the natural transformation $S$ defined above does give a left antipode for the bimonad $U \circ F$. This is essentially the proof of Theorem 3.14 in [1].

**Theorem 8.** If $F \dashv U$ is an adjunction where $U$ is a strong monoidal functor between monoidal categories with left duals, then the natural transformation $S : T \circ \lor \Rightarrow \lor$ (defined in Section 4.2.1) is a left antipode for the bimonad $U \circ F$.

**Proof.** By Theorem 6, it suffices to show that $(HM1)$ and $(HM2)$ are satisfied. I will just give the proof of $(HM1)$; the proof of $(HM2)$ is analogous.

Thus we get that in this case $U \circ F$ is indeed a Hopf monad.

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