Piezoelectricity and Piezomagnetism: Duality in Two-Dimensional Checkerboards

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Abstract

The duality approach in 2-dim two-component regular checkerboards was extended onto piezoelectricity and piezomagnetism problems. There are found a relation for effective piezoelectric and piezomagnetic modules for the checkerboard with $p6'mm'$-plane symmetry group (dichromatic triangle).

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1. Introduction.

Duality transformation in a 2-dim heterogeneous composites discovered by Keller $^1$ and Dykhne $^2$ has restricted number of the physical contexts where it could be applied. The dual symmetry is based upon the simple observation that any 2-dim divergence-free field when rotated locally at each point by 90° becomes curl-free and vice versa. This leads to the fact that effective physical properties of 2-dim two-component composites like an electric conductivity $\hat{\sigma}_{\text{ef}}$, thermal conductivity $\hat{\kappa}_{\text{ef}}$ and other 2-nd rank symmetric tensors $\hat{v}_{\text{ef}}$ don’t depend in some sense of composite’s micro-structure, being universal $^3$, and can be described making use of similarity of the dual problems

\[
\mathbf{Y} = \mathbf{\hat{v}} \cdot \mathbf{X} , \quad \text{div}\mathbf{Y} = \text{rot}\mathbf{X} = 0 , \quad \hat{v}_{ij} = \hat{v}_{ji} \tag{1}
\]

with duality relation for non-isotropic structures $^4$, $^5$

\[
\det\hat{v}_{\text{ef}} = \sqrt{\det\hat{v}_a \cdot \det\hat{v}_b} , \tag{2}
\]

where subscripts "a" and "b" correspond to the a and b components of composite respectively, while "ef" denotes an effective medium.

The further attempts $^6$ of extension the dual symmetry onto 2-dim elasticity

\[
\mathbf{u}_{ij} = \hat{\kappa}_{ij}^{\text{KL}} \cdot \tau_{kl} , \sum_j \partial_j \tau_{ij} = 0 , \tag{3}
\]

\[
\partial_{yy}^2 u_{xx} + \partial_{xx}^2 u_{yy} = 2 \partial_{xy}^2 u_{xy} \tag{4}
\]

had shown an absence of duality in the sense mentioned above. The main problem on this way makes unreducibility of an equation $^7$ for the strain tensor $\mathbf{u}_{ij}$ to the curl-free form. Therefore one can not write a duality relation for the 4-th rank compliance tensor $\hat{\kappa}_{ij}^{\text{KL}}$.

In this paper we consider the physical phenomena dealt with 3-rd rank tensors, when a high symmetry of 2-dim two-component checkerboard makes it possible to exploited the duality transformation. The most known are piezoelectricity and piezomagnetism.

2. Symmetry considerations.

Let us consider a homogeneous medium submitted under mechanical stress $\mathbf{\tau}$ which produces a dielectric displacement $\mathbf{D}$ and magnetic induction $\mathbf{B}$

\[
D^i = \hat{\gamma}_{jk}^i \cdot \tau_{jk} , \quad B^i = \hat{\beta}_{jk}^i \cdot \tau_{jk} ,
\]

\[
\text{div } \mathbf{D} = \text{div } \mathbf{B} = 0 , \sum_j \partial_j \tau_{ij} = 0 , \tag{5}
\]

where the piezoelectric $\hat{\gamma}_{jk}^i$ and piezomagnetic $\hat{\beta}_{jk}^i$ coefficients are a polar and an axial tensors of the rank 3 respectively with the inner symmetry $[V^2] \times V$ in Jahn notations $^8$.

\[
\hat{\gamma}_{jk}^i = \hat{\gamma}_{kj}^i , \quad \hat{\beta}_{jk}^i = \hat{\beta}_{kj}^i .
\]

The piezoelectric properties in 3-dim media are possessed by the crystals belonging to 20 point symmetry groups, the restrictions came only from the crystallographic standpoint, the piezoelectricity vanishes for all medium with point groups contained an inversion and for cubic group O. The piezomagnetism is certainly possible only in the crystals with the magnetic symmetry belonging to 90 (31 ferromagnetic and 59 anti-ferromagnetic of the 1-st type) magnetic groups $^9$. The coexistence of piezoelectric and piezomagnetic properties in anisotropic media was discussed in $^8$. A wide class of materials where the piezoelectric and piezomagnetic properties co-exist is listed in $^8$. Practically all such materials are synthetic compounds.

In 2-dim media we have a drastic decrease in a number of symmetries which enabled to find both piezoelectric and piezomagnetic properties. These are 5 ferromagnetic point groups: $C_1$, $C_s$, $C_3$, $C_s(C_1)$, $C_{3v}(C_3)$ and 1 anti-ferromagnetic of the 1-st type group $C_{3c}$. In general case of group $C_1$ there are only 6+6=12 independent piezomodules due to the inner symmetry $[V^2] \times V$. In 2-dim medium it is convenient to represent each of tensors $\hat{\gamma}_{jk}^i$, $\hat{\beta}_{jk}^i$ by two non-symmetric tensors of the 2-nd rank...
\[ \hat{g}_{11} = \begin{pmatrix} \gamma_1 & \gamma_3 \\ \gamma_2 & \gamma_4 \end{pmatrix}, \quad \hat{g}_{12} = \begin{pmatrix} \gamma_3 & \gamma_5 \\ \gamma_4 & \gamma_6 \end{pmatrix}, \]
\[ \hat{g}_{21} = \begin{pmatrix} \beta_1 & \beta_3 \\ \beta_2 & \beta_4 \end{pmatrix}, \quad \hat{g}_{22} = \begin{pmatrix} \beta_3 & \beta_5 \\ \beta_4 & \beta_6 \end{pmatrix}, \]

where \( \gamma_1 = \gamma_x^x, \gamma_2 = \gamma_y^x, \gamma_3 = \gamma_x^y, \gamma_4 = \gamma_y^y, \gamma_5 = \gamma_y^x, \gamma_6 = \gamma_y^y \). The notations for \( \beta_{jk} \) are chosen respectively, i.e., \( \beta_1 = \beta_x^x, \) etc.

An increase of symmetry reduces a number of piezomodules in the following way:

\[ C_s \cdot C_s(C_1) - 6 \]
\[ C_3 = 4 \]
\[ \hat{g}_{11}(C_3) = \begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_2 & -\gamma_1 \end{pmatrix}, \quad \hat{g}_{12}(C_3) = \begin{pmatrix} -\gamma_1 & -\gamma_2 \\ \gamma_2 & \gamma_1 \end{pmatrix}, \]
\[ \hat{g}_{21}(C_3) = \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_2 & -\beta_1 \end{pmatrix}, \quad \hat{g}_{22}(C_3) = \begin{pmatrix} -\beta_1 & -\beta_2 \\ \beta_2 & \beta_1 \end{pmatrix}. \]

where \( \gamma, \gamma_k, \beta, \beta_k \) are real numbers.

Up to this moment we have not specified a crystallographic type of two-component composite since according to Mendelson \( \mathbb{F} \) a duality relation is universal there. Nevertheless an arbitrariness of the micro-structure can make algebraic equations arisen from the duality relations completely unsolvable, e.g., in general case two-component composite has 12 effective piezoelectric (6) and piezomagnetic (6) modules although the number of the corresponding equations can be much less. Therefore it is worth to specify most simple cases.

According to the Curie principle \( \mathbb{G} \) the point symmetry group \( G_{df} \) of the physical phenomenon in composed medium is a maximal common subgroup of the micro-structure group \( G_{st} \) and the inner symmetry groups \( G_a, G_b \) of this phenomenon in both components "a", "b".

\[ G_{df} = \max \{ G_{st} \bigcap G_a \bigcap G_b \} . \]

We will look for 2-dim checkerboards with those regular dichromatic tessellation by polygons which are compatible with point group of the symmetry \( C_3 \). From 46 dichromatic plane mosaics \( \mathbb{H} \) there is compatible only dichromatic triangle which possesses a \( p6'mm' \)-plane group (see Fig. \( \mathbb{I} \)). The choice of the \( C_3 \)-symmetry is not accidental but concerned with the special properties of the transport tensors \( \hat{g}_{kl} \) which we would discuss in the next section.

**3. Duality relations.**

Let us consider a regular 2-dim two-component checkerboard with equal concentrations of the components submitted under mechanical stress \( \hat{\tau} \) which produces a dielectric displacement \( \hat{D} \) and magnetic induction \( \hat{B} \) defined in \( \mathbb{J} \). In order to make a similarity between \( \mathbb{J} \) and \( \mathbb{I} \) more clear we will define the following vectors

\[ \hat{t}_1 = (\tau_{xx}, \tau_{xy}), \quad \hat{t}_2 = (\tau_{yx}, \tau_{yy}), \quad \text{div} \hat{t}_k = 0 \]

that makes \( \mathbb{I} \) more simple

\[ \hat{D} = \hat{g}_{11} \cdot \hat{t}_1 + \hat{g}_{12} \cdot \hat{t}_2, \quad \hat{B} = \hat{g}_{21} \cdot \hat{t}_1 + \hat{g}_{22} \cdot \hat{t}_2. \]

The further approach is based on the validity of the matrix identities for the transport tensors \( \hat{g}_{kl}(C_3) \) from \( \mathbb{J} \):

\[ \hat{g}_{11}(C_3) = \hat{M} \cdot \hat{g}_{21}(C_3), \quad \hat{g}_{12}(C_3) = \hat{M} \cdot \hat{g}_{22}(C_3), \]

where

\[ \hat{M} = \frac{1}{\beta_1^2 + \beta_2^2} \begin{pmatrix} \beta_1 \gamma_1 + \beta_2 \gamma_2 & \beta_2 \gamma_1 - \beta_1 \gamma_2 \\ \beta_2 \gamma_1 - \beta_1 \gamma_2 & \beta_1 \gamma_1 + \beta_2 \gamma_2 \end{pmatrix}. \]

This leads to unexpected relation

\[ \hat{D} = \hat{M} \cdot \hat{g}_{21}(C_3) \cdot \hat{t}_1 + \hat{M} \cdot \hat{g}_{22}(C_3) \cdot \hat{t}_2 = \hat{M} \cdot \hat{B}. \]

The next transformation will bring a latter equation \( \mathbb{K} \) in the completely similar to \( \mathbb{J} \) relation between divergence-free field \( \hat{D} \) and curl-free field \( \hat{\mathbb{K}} \)

\[ \hat{D} = \hat{M} \cdot \hat{\mathbb{K}}^{-1} \cdot \hat{\mathbb{K}} \]

\[ \text{div} \hat{D} = \text{rot} \hat{\mathbb{K}}, \quad \hat{\mathbb{K}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \]

where \( \hat{\mathbb{K}} \) is a rotation by 90° operator and \( \hat{\mathbb{K}} = \hat{M} \cdot \hat{\mathbb{K}}^{-1} \) is an anti-symmetric positively defined matrix

\[ \hat{\mathbb{K}} = \frac{1}{\beta_1^2 + \beta_2^2} \begin{pmatrix} \beta_1 \gamma_2 - \beta_2 \gamma_1 & \beta_1 \gamma_1 + \beta_2 \gamma_2 \\ -((\beta_1 \gamma_1 + \beta_2 \gamma_2) \beta_2 \gamma_1 - \beta_1 \gamma_2) \end{pmatrix}. \]
Applying now the duality transformation to (16) we will make use of duality relation (2)

\[ \det \hat{L}_{\text{ef}} = \sqrt{\det \hat{L}_a \cdot \det \hat{L}_b}. \]  

(17)

After simple algebra we obtain finally

\[ \frac{\gamma_{1\text{ef}}^2 + \gamma_{2\text{ef}}^2}{\beta_{1\text{ef}}^2 + \beta_{2\text{ef}}^2} = \sqrt{\frac{\gamma_{1a}^2 + \gamma_{2a}^2}{\beta_{1a}^2 + \beta_{2a}^2} \cdot \frac{\gamma_{1b}^2 + \gamma_{2b}^2}{\beta_{1b}^2 + \beta_{2b}^2}}. \]

(18)

If the inner symmetry \( G_a, G_b \) increases up to the \( C_{3v} \) the duality relation (17) looks more simple

\[ \frac{\gamma_{\text{ef}}^2}{\beta_{\text{ef}}^2} = \frac{|\gamma_a \gamma_b|}{|\beta_a \beta_b|}. \]

(19)

Note that the equations (12) do not permit to write the duality relations for the piezoelectric \( \gamma_{jk}^i \) and piezomagnetic \( \beta_{jk}^i \) tensors separately.

4. Conclusion.

In the present paper we have considered the coexisting simultaneously piezoelectric and piezomagnetic phenom-
ena in 2-dim two-component composites with triangular tessellation of the plane (\( p6'mm' \)-plane symmetry group). The duality approach in 2-dim two-component regular checkerboards was extended onto physical problem dealt with 3-rd rank tensors.

In the conclusion we will name some compounds which have a trigonal symmetry and where the piezoelectricity and the piezomagnetism coexist \( [9] \). These are the rare-earth maganites having the overall formula \( RMnO_3 \), where \( R = Y, Ho, Er, Tm, Yb, Lu, \) or \( Sc \). The Mn atoms lie inside the bipyramidal bonds, while the rare-earth atoms lie inside the bipyramids. The trigonal structure in these compounds arises from the smallness of the ionic radii of the rare-earth ions and the presence of covalent \( Mn-O \) bonds.

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