Abstract. In this article, we aim to recover locally conservative and $H(\text{div})$ conforming fluxes for the linear Cut Finite Element Solution with Nitsche’s method for Poisson problems with Dirichlet boundary condition. The computation of the conservative flux in the Raviart-Thomas space is completely local and does not require to solve any mixed problem. The $L^2$-norm of the difference between the numerical flux and the recovered flux can then be used as a posteriori error estimator in the adaptive mesh refinement procedure. Theoretically we are able to prove the global reliability and local efficiency. The theoretical results are verified in the numerical results. Moreover, in the numerical results we also observe optimal convergence rate for the flux error.

Key words. CutFEM, a Posteriori Error Estimation, Flux Recovery, Adaptive Mesh Refinement

AMS subject classifications. 68Q25, 68R10, 68U05

1. Introduction. Cut finite element method (CutFEM) may be regarded as a fictitious domain method. The finite element fictitious domain method was introduced in [28] as an approach to simplify the meshing problem and then further improved in [8, 31, 14, 5, 30]. The challenges for such methods are typically that if the mesh is cut in an unfavorable way the system can be ill-conditioned and accuracy can be lost. Several approaches have been proposed to handle this problem, all based on the idea of extending the stability of the solution in the bulk up to the boundary. In [31], boundary fluxes are evaluated using the gradient extended from internal elements. In [33, 32, 13, 4], agglomeration of adjacent elements is used. Finally, in [12, 15], the weakly consistent ghost penalty term was proposed that serves purpose.

The purpose of this paper is to design and analyze a locally conservative flux in the Raviart-Thomas space of order 0 and 1 for the linear CutFEM. We will base our discussion on the approximation of Poisson’s problem [15] in two dimensions. In addition, the CutFEM uses Nitsche’s method [37] to impose Dirichlet boundary conditions and a ghost penalty term [12] to enhance stability in the boundary zone.

One important application for the flux recovery is that the $L^2$-norm of the difference between the numerical flux and the recovered flux can be used in the a posteriori error estimation. The main motivation for studying the a posteriori error estimation is the application of adaptive mesh refinement (AMR) procedure. It is well known that AMR is extremely useful for problems with singularities, discontinuities, sharp derivatives etc.. And it has been extensively studied in the last several decades, see e.g., [40, 3]. In [17], a residual based a posterior error estimator for the CutFEM method was studied. One drawback of residual based error estimation is that its reliability constants are unknown and usually not polynomial-robust and problem dependent. On the other hand, it is well known that the difference between the numerical flux and a locally conservative (equilibrate) flux automatically yields an upper bound for the true energy error with a reliability constant being exactly 1.

Thanks to the sharp reliability, equilibrate flux recovery has been extensively studied for various finite element methods on fitted meshes in the last decade. It is

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well known that for discontinuous Galerkin methods, a locally element-wise equilibrate flux can be easily obtained thanks to the fact that the test functions are completely local [6, 24, 1, 10, 7]. For nonconforming finite element methods of odd order, a local element-wise construction can be also easily obtained by taking advantage of the nonconforming local basis functions [35, 19]. For second order nonconforming finite element method, an explicit construction is designed in [34]. However, for nonconforming methods of general even orders and conforming finite element methods, local element-wise (explicit) flux recovery is not straightforward and usually local problems on star patches need to be solved [2, 25, 7]. In [38], a conservative flux is obtained by adding a piecewise constant correction through minimizing a weighted global $L^2$-norm.

We note that the method introduced in [7] is applicable to various finite element methods, and furthermore, designed in a framework that fully takes advantages of the local basis functions for each method. For conforming finite elements on fitted meshes, this method only requires solving local problems that do not involve any hybrid mixed problem which are required e.g., in [11, 20, 25]. In this work, we use a similar approach for CutFEM. For the interior elements not cut by the boundary, the recovered flux is similar to that in [7] although it needs extra treatment for the ghost penalty term. Unlike the fitted methods whose mesh is an exact partition of the computational domain, one major challenge for cut finite elements is that the domain cuts the background mesh in an arbitrary fashion. For cut elements, the recovery of the flux becomes complex due to non-standard terms in the variational formulation and partial intersections between the mesh and the domain, and, therefore, needs to be carefully designed to avoid artificial error. We divide boundary elements into two types, i.e., the set of boundary elements that are fitted and not fitted to the mesh, and design the flux differently.

Contrary to the classical approach, we consider the domain with non-polygonal boundary. In order to achieve the same accuracy as the classical fitted method, boundary geometry and boundary data for CutFEM need to be approximated to similar accuracy. It is therefore important in this context to derive error estimators that are able to integrate both the discretization error of the method and the discretization error of the geometry. In this work, we approximate the physical geometry by a piece-wise affine polygonal domain. In [17], a boundary correction error was separated that particularly estimates the geometry approximation error. The computation of this term, however, is not trivial and requires the construction of a sub mesh. Nevertheless, numerical results have shown that such error is not necessary to compute since the boundary approximation error can already be captured by the residual based error estimator in [17]. In this paper, we also discard such term and numerical results also confirm that our error estimator is able to catch both the boundary approximation errors and the discretization error due to the numerical method.

Although we herein restrict the discussion to the case of piecewise affine approximation spaces from [15], we believe that the ideas introduced can be extended for instance to the high order case discussed in [16] as regards the CutFEM method, and in [7] as regards the flux reconstruction in two dimensions. For other works treating a posteriori error estimation and cut cell techniques we refer to [26], where a finite volume method was considered, and [39] and [22] where cut cell methods were applied.

This paper is organized as follows. In section 2, the model problem and the CutFEM are introduced. In section 3, we design the local conservative flux by introducing an auxiliary mixed method and establish its well-posedness. In section 4, we apply the conservative flux in the a posteriori error estimation and establish its reliability and
efficiency. Finally, we show the results of several numerical experiments in section 5.

2. Model problem and the Cut Finite Element Method.

2.1. The continuous problem. Let $\Omega$ be a domain in $\mathbb{R}^d \ (d = 2)$ with Lipschitz continuous, piecewise smooth boundary $\partial \Omega$ with exterior unit normal $n$. We consider the problem: find $u : \Omega \to \mathbb{R}$ such that

\[ -\Delta u = f \quad \text{in } \Omega, \]

\[ u = g \quad \text{on } \partial \Omega, \]

where $f \in L^2(\Omega)$ and $g \in H^{1/2}(\partial \Omega)$. For the sake of simplicity, we only consider the model problem of Laplacian operator with Dirichlet boundary conditions. However, the technique could be generalized to other boundary conditions and more complex elliptic operators.

Define the spaces

\[ H^1_g(\Omega) = \{ v \in H^1(\Omega) : v = g \text{ on } \partial \Omega \} \] and \[ H^1_0(\Omega) = \{ v \in H^1(\Omega) : v = 0 \text{ on } \partial \Omega \}. \]

Then the weak formulation for this problem renders to find $u \in H^1_g(\Omega)$ such that

\[ a(u, v) = (f, v)_\Omega, \quad \forall v \in H^1_0(\Omega), \]

where $a(u, v) = (\nabla u, \nabla v)_\Omega$. It follows from the Lax-Milgram lemma that there exists a unique solution $u \in H^1_g(\Omega)$ to this problem.

2.2. The mesh, discrete domain, and finite element spaces. Assume that $\partial \Omega$ is composed of a finite number of smooth surfaces $\Gamma_i$, such that $\partial \Omega = \bigcup_i \bar{\Gamma}_i$. We let $\rho$ be the signed distance function such that

\[ \rho(x) = \begin{cases} < 0 & \text{if } x \in \Omega, \\ = 0 & \text{if } x \in \partial \Omega, \\ > 0 & \text{if } x \in \bar{\Omega}^c, \end{cases} \]

where $\bar{\Omega}^c$ is the complement of the closure of $\Omega$. We define $U_\delta(\partial \Omega) = \{ x \in \mathbb{R}^d : |\rho(x)| < \delta \}$ be the tubular neighborhood $\{ x \in \mathbb{R}^d : |\rho(x)| < \delta \}$ of $\partial \Omega$. Choose $\Omega_0 \subset \mathbb{R}^d$ be the background domain (see e.g., the square outline of the entire mesh in Figure 1) such that it is polygonal, $\Omega \subset \Omega_0$ and $U_{\delta_0}(\partial \Omega) \subset \Omega_0$ where $\delta_0$ is chosen such that $\rho$ is well defined in $U_{\delta_0}(\partial \Omega)$. Let $T_{0,h}$ be a partition of $\Omega_0$ into shape regular triangles or tetrahedra (see e.g., the mesh in Figure 1). Note that this setting allows meshes with locally dense refinement.

Given a subset $\omega$ of $\Omega_0$, let $T_h(\omega)$ be the sub-mesh defined by

\[ T_h(\omega) = \{ K \in T_{0,h} : K \cap \omega \neq \emptyset \}, \]

i.e., the sub-mesh consisting of elements that have non-zero intersection with $\omega$, and let

\[ \triangle_h(\omega) = \bigcup_{K \in T_h(\omega)} K, \]

which is the union of all elements in $T_h(\omega)$.
For each $T_{0,h}$, let $\Omega_h$ (see Figure 1 for an example) be a polygonal domain approximating $\Omega$. We assume that $\partial\Omega_h \subset U_{\delta_0}(\partial\Omega)$, i.e., $\partial\Omega_h$ is within the distance of $\delta_0$ to $\partial\Omega$. Moreover, we also require that the maximum distance between the two domains is small enough so that $\Omega_h$ is a sufficiently good approximation to $\Omega$. More details will be given later.

Let the active mesh be defined by

\begin{equation}
T_h := T_h(\Omega_h)
\end{equation}

i.e., the sub-mesh consisting of elements that intersect $\Omega_h$, and let

\begin{equation}
\triangle_h := \triangle_h(\Omega_h).
\end{equation}

Since $\partial\Omega_h$ cut the active mesh $T_h$ in an arbitrary fashion, we denote by $T^b_h$ the set of elements that are “cut” by $\partial\Omega_h$, i.e.,

$$T^b_h = \{K \in T_{0,h} : K \cap \partial\Omega_h \neq \emptyset\} \subset T_h.$$

We further assume that $\Omega_h$ is constructed in such a way that for each $K \in T^b_h$, the intersection $K \cap \partial\Omega_h$ is a subset of a $d-1$ dimensional hyperplane, i.e., a line segment in two dimensions. Under the assumptions on $\partial\Omega$, and if we also assume that $h$ is sufficiently small, then for any element $K \in T^b_h$ there exists an element $K' \in T_h \setminus T^b_h$ such that $\text{dist}(K, K') = O(h_K)$ where $O(\cdot)$ denotes the Landau big-O.

Also we denote by $E$ the set of all facets of $T_h$, and by $E_I$ and $E_\partial$ the set of all interior and boundary facets with respect to $T_h$, respectively. It is obvious that $E_I \cup E_\partial = E$. For each $F \in E$ denote by $n_F$ an unit vector normal to $F$ and by $h_F$ (or $|F|$) the diameter of $F$. If $F \in E_\partial$, then $n_F$ is the outward unit normal vector. For each $K \in T_h$ denote by $h_K$ the diameter of $K$ and by $E_K$ the set of all facets of $K$.

On the boundary $\partial\Omega_h$, let $n_h$ be the outer normal to $\partial\Omega_h$. For each $\Omega_h$, we assume that, for $\delta_0$ small enough, there exist a vector function $\nu_h : \partial\Omega_h \to \mathbb{R}^d$, $|\nu_h| = 1$, and $g_h : \partial\Omega_h \to \mathbb{R}$, such that the function, $p_h(x, \varsigma) := x + \varsigma \nu_h(x)$, is well defined and satisfies $p_h(x, g_h(x)) \in \partial\Omega$ for all $x \in \partial\Omega_h$. Here $g_h$ is the distance from the approximate to the physical boundary in the direction $\nu_h$. The mapping $p_h$ allows us to impose the boundary data defined on the physical boundary on the
approximate boundary. The existence of the vector-valued function \( \nu_h \) is known to hold on Lipschitz domains, see Grisvard [29]. We further assume that

\[
p_h(x, \varsigma) \in U_{h_0}(\partial \Omega), \quad \forall x \in \partial \Omega_h, \ 0 \leq \varsigma \leq \varrho_h(x).
\]

For convenience, we will drop the second argument, \( \varsigma \), of \( p_h \) below whenever it takes the value \( \varrho_h(x) \) and then \( p_h \) denotes the map \( p_h : \partial \Omega_h \to \partial \Omega \). Moreover, we assume that the following assumption is satisfied

\[
\| \varrho_h \|_{L^\infty(\partial \Omega_h \cap K)} \leq O(h_K) \quad \forall K \in T_h^b.
\]

The above assumption immediately implies that \( \partial \Omega_h \cap K \) is within the distance of \( O(h_K) \) of \( \partial \Omega \), i.e.,

\[
\| \varrho \|_{L^\infty(\partial \Omega_h \cap K)} \leq O(h_K) \quad \forall K \in T_h^b.
\]

This assumption is necessary for the constant in the a posteriori error estimates to be independent of the geometry/mesh configuration see [17]. It is however not enough to guarantee the optimal a priori error estimates, which requires \( \| \varrho \|_{L^\infty(\partial \Omega_h \cap K)} \leq O(h_K^2) \), and \( \| n \circ p_h - n_h \|_{L^\infty(\partial \Omega_h \cap K)} \leq O(h_K) \), see [16]. It is noted that in theory we can assume neither \( \Omega_h \subset \Omega \) nor \( \Omega \subset \Omega_h \). However, for simplicity, we assume \( \Omega \subset \Omega_h \). This will help skip the analysis for the a posteriori error estimation on the part of data approximation error of \( f \). Such error can be eventually ignored in the computation since it is of higher order.

2.3. The Cut Finite Element Method. In this subsection we recall the Cut-FEM introduced in [15]. We begin with some necessary notation. For any \( F \in \mathcal{E}_I \), let \( K_F^+ \) and \( K_F^- \) be those two elements sharing \( F \) as a common facet such that the outer normal of \( K_F^+ \) coincides with \( n_F \). For any discontinuous function \( v \), define the jump of \( v \) across the facet \( F \) by

\[
[v]_F := v_F^+ - v_F^- \quad \text{and} \quad v_F^\pm(x) = \lim_{s \to 0^\pm} v(x \mp sn_F).
\]

The set of facets associated with cut elements is defined by

\[
\mathcal{E}_g := \{ F \in \mathcal{E}_I : (K_F^+ \cup K_F^-) \cap \partial \Omega_h \neq \emptyset \}.
\]

The index \( g \) above refers to the ghost penalization, defined on every \( F \in \mathcal{E}_g \).

For each \( K \in T_h \), we define a sign function \( s_K \) defined on \( \mathcal{E}_K \) such that

\[
s_K(F) = \begin{cases} 
1 & \text{if } n_F = n_K|_F, \\
-1 & \text{if } n_F = -n_K|_F.
\end{cases}
\]

The conforming linear finite element space is then defined as

\[
CG_h := \{ v \in H^1(\Delta_h) : v|_K \in P_1(K) \quad \forall K \in T_h \}.
\]

We also define the following forms:
\[ a_0(w, v) := (\nabla w, \nabla v)_{\Omega_h} - \langle \partial_{n_h} w, v \rangle_{\partial \Omega_h} - \langle w, \partial_{n_h} v \rangle_{\partial \Omega_h} + \sum_{K \in T_h^b} \frac{\beta}{h_K} \langle w, v \rangle_{\Gamma_K}, \]
\[ j_h(w, v) := \gamma \sum_{F \in E_u} h_F \left[ [\partial_{n_F} w], [\partial_{n_F} v] \right]_F, \]
\[ a_h(w, v) := a_0(w, v) + j_h(w, v), \]
\[ l_h(v) := (f, v)_{\Omega_h} - \langle g_h, \partial_{n_h} v \rangle_{\partial \Omega_h} + \sum_{K \in T_h^b} \frac{\beta}{h_K} \langle g_h, v \rangle_{\Gamma_K}, \]

where \( \Gamma_K = K \cap \partial \Omega_h, \partial_{n_h} := n_h \cdot \nabla, \gamma \) and \( \beta \) are positive constants, \( g_h \) is an approximation of \( g \) defined on \( \partial \Omega_h \). A natural choice is that \( g_h(x) = g \circ p_h \). Note that we only assumed that \( \Omega \subset \Omega_h \). In \( \Omega_h \setminus \Omega \) where \( f \) is not originally defined, \( f|_{\Omega_h \setminus \Omega} \) is defined to be an appropriate extension.

**(Remark 2.1)** The stabilizing term \( j_h(w, v) \), which is the so-called ghost penalty term, is introduced to extend the coercivity of \( a_h(\cdot, \cdot) \) to all of \( \Delta_h \), see [12, 36]. Thanks to this property, one may prove that the condition number of the linear system is uniformly bounded regardless of the arbitrary boundary-mesh intersection.

**(Remark 2.2)** In order to guarantee the coercivity of the bilinear form \( a_h(\cdot, \cdot) \) in (2.9), \( \beta \) has to be chosen large enough.

The finite element method is then to find \( u_h \in CG_h \) such that
\[ a_h(u_h, v) = l_h(v) \quad \forall v \in CG_h \]
where \( a_h \) and \( l_h \) are defined in (2.9).

For \( v \in H^1(\Delta_h) \), define the continuous and discrete energy norms respectively by
\[ |||v|||^2 := \|\nabla v\|^2_{\Omega_h} + \|\partial_{n_h} v\|^2_{H^{-1/2}(\partial \Omega)} + \|h^{-1/2} v\|^2_{\partial \Omega} \]
and
\[ |||v|||^2 := \|\nabla v\|^2_{\Omega_h} + \sum_{K \in T_h^b} h_K \|\partial_{n_h} v\|^2_{\Gamma_K} + \sum_{K \in T_h^b} h_K^{-1} \|v\|^2_{\Gamma_K} + j_h(v, v). \]

In (2.11), \( h \) denotes the piecewise constant mesh size function. From [12], we have for \( \beta \) sufficiently large the following coercivity result,
\[ a_h(v, v) \geq C|||v|||^2_h \quad \forall v \in CG_h, \]
which, together with the uniform \( ||| \cdot |||_h \)-continuity of \( a_h(\cdot, \cdot) \) on \( CG_h \), implies that (2.10) has a unique solution.

**2.4. Some important inequalities.** Below we list the well known trace and inverse inequalities [21, Section 1.4.3],

\[ ||v||_{\partial K} \leq h_K^{-1/2} \|v\|_K + h^{1/2} \|\nabla v\|_K \quad \forall v \in H^1(K), \quad \forall K \in T_h, \]
\[ h_K^{-1} ||v_h||_{\partial K} + ||\nabla v_h||_K \leq h_K^{-1} ||v_h||_K \quad \forall v_h \in P_1(K), \quad \forall K \in T_h. \]
The following irregular trace inequality can be found in [30]

\begin{equation}
\|v_h\|_{\Gamma_k} \lesssim h_K^{-1/2}\|v_h\|_K + h_K^{1/2}\|\nabla v_h\|_K \quad \forall v_h \in \mathbb{P}_1(K), \quad \forall K \in \mathcal{T}_h,
\end{equation}

where the hidden constant is independent of the boundary-mesh intersection. Here and below we use the notation \(\lesssim\) to denote less or equal up to a generic constant that is independent of the mesh-geometry configuration.

In the following Lemma, we also provide a Poincaré-type inequality for the boundary elements.

**Lemma 2.3.** Let \(v \in H_0^1(\Omega)\). Then for any \(K\) such that \(K \cap \partial \Omega \neq \emptyset\) or \(K \cap \partial \Omega_h \neq \emptyset\), there exists a local convex neighborhood \(S_K\) of \(K\) such that \(v\) vanishes on a non-empty subset of \(\partial S_K\) and

\begin{equation}
\|v\|_K \lesssim h_K\|\nabla v\|_{S_K},
\end{equation}

where we defined \(v\) outside \(\Omega\) using the trivial extension \(v_{|\Omega \setminus \Omega} = 0\).

The proof of the lemma can be found in [17].

**3. Mixed formulation.** In this section, we introduce an auxiliary mixed formulation for the cut finite element method. The aim is not to solve the global mixed problem. Instead, our goal is to establish the connections between the mixed and CutFEM formulation, and, with the help of those connections, to locally recover a conservative flux in the \(H(\text{div})\) space. The idea generates from [7] in which classical finite element methods with fitted meshes were studied.

Define the discontinuous finite element space on \(\Delta_h\) by

\begin{equation}
DG_h := \{v \in L^2(\Delta_h) : v|_K \in \mathbb{P}_1(K) \forall K \in \mathcal{T}_h\}.
\end{equation}

Also define the average operator:

\[
\{w\} = \begin{cases} 
0.5(w_F^+ + w_F^-), & F \in \mathcal{E}_I, \\
w, & F \in \mathcal{E}_\partial.
\end{cases}
\]

We also denote by \(N\) the set of all vertices in \(\mathcal{T}_h\), and by \(N_I\) and \(N_\partial\) the sets of all interior and boundary vertices on \(\mathcal{T}_h\). Obviously, we have \(N_I \cap N_\partial = \emptyset\). For each \(N \in \mathcal{N}\), define \(\mathcal{T}_N\) and \(\mathcal{E}_N\) be the sets of all elements and all facets sharing \(N\) as a vertex, respectively. For each \(N \in \mathcal{N}\), we define a signed function \(s_N\) on \(\mathcal{E}_N\) such that \(s_N(F) = 1\) if \(n_F\) is oriented counter-clockwise in \(\mathcal{T}_N\), otherwise \(s_N(F) = -1\). We further define the following space:

\[
M_h = \left\{ \mu \in L^2(\mathcal{E}_I) : \mu|_F \in \mathbb{P}_1(F) \forall F \in \mathcal{E}_I, \sum_{F \in \mathcal{E}_N} s_N(F) h_F \mu|_F(N) = 0 \forall N \in \mathcal{N}_I \right\}.
\]

Here \(\mathbb{P}_1(F)\) is the space of linear functions defined on \(F \in \mathcal{E}\). For the sake of brevity, we use the following notations for piecewise integration,

\[
\int_{\mathcal{T}_h} = \sum_{K \in \mathcal{T}_h} \int_K, \quad \int_{\mathcal{E}} = \sum_{F \in \mathcal{E}} \int_F.
\]
As regards the second one, we first note that

\[ (3.2) \]

so we clearly have that

\[ (3.3) \]

where

\[ (3.4) \]

The auxiliary mixed formulation is defined as follows: find \((u_h, \theta_h)\) \(\in DG_h \times M_h\) such that

\[ (3.5) \]

where

\[ (3.6) \]

Indeed, any function \(w_h\) in \(CG_h\) satisfies \([w_h]_F = 0\) for any \(F \in \mathcal{E}_I\), and hence belongs to \(\text{ker}(b)\). Reciprocally, for any \(w_h\) in \(\text{ker}(b)\), we can choose \(\mu_h \in M_h\) defined by \((\mu_h)_F = h^{-1}_F ||v||_F\) for any \(F \in \mathcal{E}_I\), which yields \([w_h]_F = 0\) and hence, \(w_h \in CG_h\). It is then obvious to see that the solution \(u_h\) for (2.10) coincides with the solution for (3.3) if it is well defined.

### 3.1. Well-posedness of the mixed finite element approximation.

**Lemma 3.1 (continuity).** We have the following continuity results for the bilinear forms:

\[
b(\mu, v) \lesssim ||\mu||_{M_h} ||v||_{h,*}, \quad \forall \mu \in M_h, \forall v \in DG_h,
\]

\[
ad_h(v, w) \lesssim ||v||_{h,*} ||w||_{h,*}, \quad \forall v, w \in DG_h.
\]

**Proof.** The proof of the first assertion is trivial by Cauchy-Schwartz and the definitions of the norms:

\[
b(\mu, v) \lesssim \sum_{F \in \mathcal{E}_I} \|h_{1/2}^{-1} \mu \|_F \|h_{-1/2}^{-1} v \|_F \| \leq \|\mu\|_{M_h} ||v||_{h,*}.\]

As regards the second one, we first note that \(j_h(v, v) \lesssim \int_{\partial h} |\nabla v|^2\) for any \(v \in DG_h\), so we clearly have that \(||\cdot||_h \lesssim ||\cdot||_{h,*}\) on \(DG_h\), hence the continuity of \(a_h(\cdot, \cdot)\). The
remaining terms of $\tilde{a}_h(\cdot,\cdot)$ are bounded as follows:

$$\quad |\langle \{\partial_n f\}v, [w]\rangle_{E_I h} | \leq \left( \sum_{F \in E_I} h_F \|\{\partial_n f\}v\|^2_F \right)^{1/2} \left( \sum_{F \in E_I} h_F^{-1} [w]^2_F \right)^{1/2}$$

$$\quad \lesssim \left( \int_{\Omega} |\nabla v|^2 \right)^{1/2} \left( \int_{E_I} h_F^{-1} [w]^2 \right)^{1/2} \quad \forall v, w \in DG_h,$$

which ensures the continuity of $\tilde{a}_h(\cdot,\cdot)$.

**Lemma 3.2 (inf-sup condition).** We also have the inf-sup result for the bilinear form:

$$\inf_{\mu \in M_h} \sup_{v \in DG_h} \frac{b(\mu, v)}{|||v|||_{h,s} \|\mu\|_{M_h}} \geq C,$$

where the constant is independent of the mesh size and domain-mesh intersection.

**Proof.** We prove by construction. To define a linear function $v \in DG_h$, it suffices to define $v_K(N) := v|_K(N)$ for all $K \in T_h$ and for all $N \in N_K$. For each $N \in N_I$, we let $\{K_i\}_{i=1}^{n_N}$ be the clockwise oriented elements in $T_N$ where $n_N \geq 1$ is the number of elements in $T_N$. We also let $F_i = K_{i-1} \cap K_i$, $i = 1, \cdots, n_N(K_0 = K_{n_N})$. We let $v_{K_i}(N) = 0$. The rest are defined such that

$$v_{K_i}(N) - v_{K_i-1}(N) = h_{F_i} s_{N}(F_i) \mu_{F_i}(N), \quad i = 2, \cdots, n_N.$$

It is easy to check that (3.6) is compatible. Moreover, it is easy to check that

$$[v]|_{F_i}(N) = h_{F_i} \mu_{F_i}(N), \quad i = 1, \cdots, n_N.$$

For $N \in N_{\partial}$ such that $E_N \cap E_I \neq \emptyset$, we define $v_{K_i}, i = 1, \cdots, n_N$ in the same way (where we assume that $K_1$ has a boundary facet). Note that there are only $n_N - 1$ interior facets in this case. And we also have

$$[v]|_{F_i}(N) = h_{F_i} \mu_{F_i}(N), \quad i = 2, \cdots, n_N.$$

Finally, for $N \in N_{\partial}$ such that $E_N \cap E_I = \emptyset$, we simply put $v_K(N) = 0$. Combining (3.7) and (3.8) gives that

$$[v]|_F = h_F \mu_F \quad \forall F \in E_I.$$

We then have that

$$b(\mu, v) = \sum_{F \in E_I} \sum_{N \in N \cap F} \frac{h_F^2}{2} \mu_F(N)^2 \gtrsim \|\mu\|_{M_h}^2.$$

The last inequality follows from the equivalence of norms in a finite dimensional space. Immediately from (3.8), we also have the following bound:

$$\sum_{i=1}^{n_N} v_{K_i}(N)^2 \lesssim \sum_{F \in E_N \cap E_I} h_F^2 \mu_F(N)^2.$$
By a direct computation, (3.11), and norm equivalence in a finite dimensional space, we have

\[
\sum_{K \in T_h} \frac{h}{h} \sum_{N \in N_h} v_N \approx \sum_{N \in N_h} \frac{h}{h} \mu(N)^2 \lesssim \|\mu\|^2_{M_h}.
\]

(3.12)

Immediately, we also have, by using (2.16) for the last estimate,

\[
\int_{E_i} \frac{h}{h}^{-1} [v]^2 ds = \int_{E_i} h F \mu F ds = \|\mu\|^2_{M_h},
\]

(3.13)

We also have, thanks to (2.16),

\[
h_K \|\partial_n h v\|^2_{T_K} ds \lesssim \|\nabla v\|^2_{K}, \quad \forall K \in T_h^b.
\]

(3.14)

(3.5) is then the direct consequence of (3.10) and (3.15). This completes the proof of the lemma.

3.2. Local construction of \( \theta_h \). In this subsection, we aim to compute the solution \( \theta_h \) to (3.3) based on the solution \( u_h \) of (2.10) through solving local problems, following [7]. Firstly note that

\[
b(\theta_h, w) = r(w) := l_h(w) - \tilde{a}_h(u_h, w) \quad \forall w \in D_{H_h}.
\]

(3.16)

Note that \( r(w) = 0 \) for any \( w \in C_{H_h} \).
Let $N \in \mathcal{N}$ such that $\mathcal{E}_N \cap \mathcal{E}_I \neq \emptyset$. We define $\theta_N \in M_h$ on $\mathcal{E}_N \cap \mathcal{E}_I$ such that, for each $K \in \mathcal{T}_N$,

\begin{equation}
\begin{aligned}
b(\theta_N, \lambda_N \chi_K) &= r(\lambda_N \chi_K) \\
b(\theta_N, \lambda_M \chi_K) &= 0 \quad \text{if } M \in \mathcal{N}_K \text{ and } M \neq N,
\end{aligned}
\end{equation}

where $\lambda_M$ denotes the barycentric basis function corresponding to the vertex $M$. On $\mathcal{E}_I \setminus \mathcal{E}_N$, we impose that $\theta_N$ is null.

Recall from the definition of $M_h$ that the condition $\theta_N \in M_h$ means that $\theta_N$ has to satisfy a constraint at any interior node. The last equation of (3.17) gives that $\theta_N|_F(M) = 0$ for all $F \in \mathcal{E}_N$ and $M \in F, M \neq N$, so the constraint is obviously satisfied at the node $M$, for $M \in \mathcal{N}_I$. Hence, $\theta_N$ only has to satisfy the constraint equation at the node $N$, for $N \in \mathcal{N}_I$.

Also note that $\sum_{K \subset \mathcal{T}_N} r(\lambda_N \chi_K) = r(\lambda_N) = 0$. It is then easy to check that the system introduced by (3.17) is compatible.

**Lemma 3.4.** For any $N \in \mathcal{N}$, the system (3.17) has a unique solution $\theta_N$ in $M_h$.

**Proof.** We first assume that $N \in \mathcal{N}_I$. Because of the compatibility condition, the system (3.17) is a one-dimensional kernel. Indeed, let $\Psi_N$ defined on any $F \in \mathcal{E}_N$ by

\begin{equation}
\begin{aligned}
\Psi_N|_F(N) &= h_F^{-1} s_N(F), \\
\Psi_N|_F(M) &= 0, \quad \forall M \in F, M \neq N,
\end{aligned}
\end{equation}

whereas on $F \in \mathcal{E}_I \setminus \mathcal{E}_N$, we set $\Psi_N|_F \equiv 0$. It is then easy to check that for all $K \in \mathcal{T}_N$,

\begin{equation}
\begin{aligned}
b(\Psi_N, \lambda_N \chi_K) &= 0, \\
b(\Psi_N, \lambda_M \chi_K) &= 0 \quad \text{if } M \in \mathcal{N}_K \text{ and } M \neq N.
\end{aligned}
\end{equation}

Thus $\text{span}\{\Psi_N\}$ is the kernel of the system (3.17). However, it is obvious that $\Psi_N$ does not satisfy the constraint equation $\sum_{F \in \mathcal{E}_N} s_N(F) h_F \mu|_F(N) = 0$ for $N \in \mathcal{N}_I$. Thus (3.17) has a unique solution $\theta_N \in M_h$ under the constraint for $N \in \mathcal{N}_I$. For $N \in \mathcal{N}_I$, note that there are $n_N - 1$ interior facets. The solution is then unique since there are $n_N - 1$ unknowns with $n_N - 1$ independent equations. This completes the proof of the lemma.

**Lemma 3.5.** Let $\theta_N$ be defined in (3.17) and $\theta_h$ be the solution of (3.3). Then we have that

\begin{equation}
\theta_h = \sum_{N \in \mathcal{N}} \theta_N.
\end{equation}

**Proof.** From (3.16), it suffices to prove that

\begin{equation}
\sum_{N \in \mathcal{N}} b(\theta_N, \lambda_M \chi_K) = r(\lambda_M \chi_K), \quad \forall M \in \mathcal{N}, \forall K \in \mathcal{T}_M.
\end{equation}

By (3.17), we have that

\begin{equation}
\sum_{N \in \mathcal{N}} b(\theta_N, \lambda_M \chi_K) = \sum_{N \in \mathcal{N}_K} b(\theta_N, \lambda_M \chi_K) = b(\theta_M, \lambda_M \chi_K) = r(\lambda_M \chi_K).
\end{equation}

This completes the proof of the lemma.
3.3. Computation of $\theta_N$ for $N \in \mathcal{N}$. For each $N \in \mathcal{N}$, recall that we let $\{K_{1,N}\}_{i=1}^{n_N}$ be the clockwise oriented elements in $\mathcal{T}_N$ and $n_N \geq 1$ is the number of elements in $\mathcal{T}_N$. When $N \in \mathcal{N}_I$, we let $F_i = K_{i-1,N} \cap K_{i,N}$, $i = 1, \cdots, n_N (K_0 = K_{n_N})$. When $N \in \mathcal{N}_B$, note that there are $n_N - 1$ interior facets in $\mathcal{E}_N$. Also let $M_i$ denotes the other vertices of $F_i$.

We firstly deal with the case $N \in \mathcal{N}_I$. From the second equation in (3.17), it is easy to see that $\theta_N(M_i) = 0$ for $i = 1, \cdots, n_N$. We then let

$$\theta_{i,N} := \theta_N|_{F_i}(N), \quad r_{i,N} := r(\lambda_N \chi_{K_i}), \quad a_{i,N} := s_N(F_i)h_{F_i}. $$

From the first equation in (3.17), a straight computation gives that

$$a_{i,N} \theta_{i,N} - a_{i+1} \theta_{i+1,N} = 2r_{i,N}, \quad i = 1, \cdots, n_N - 1. \tag{3.22}$$

Note that we only used $n_N - 1$ equations since the last one is linearly dependent. The constraint provides the last equation if $N \in \mathcal{N}_I$:

$$\sum_{i=1}^{n_N} a_{i,N} \theta_{i,N} = 0. \tag{3.23}$$

Combining (3.22) and (3.23) yields a $n_N \times n_N$ non-singular local system. It is helpful to denote the unknowns by $\bar{\theta}_{i,N} := a_{i,N} \theta_{i,N}$, such that the matrix of the previous local system has constant coefficients and depends only on $n_N$.

If $N \in \mathcal{N}_B$, note that $\theta_N|_{F}(N) = 0$ for $F \in \mathcal{E}_B$, i.e., $\theta_{1,N} = \theta_{n_N,N} = 0$. Hence we can explicitly compute $\theta_{i,N}$ for $i = 1, \cdots, n_N - 1$. Thus no local $n_N \times n_N$ problem is required to solve for boundary elements.

By a simple calculation, we also have the following estimate:

$$\sum_{i=1}^{n_N} \bar{\theta}_{i,N}^2 \lesssim \sum_{i=1}^{n_N} r_{i,N}^2 \iff \{\bar{\theta}_N\} \lesssim \{r_N\}, \tag{3.24}$$

where $\{r_N\}$ and $\{\bar{\theta}_N\}$ are the vectors in $\mathbb{R}^{n_N}$ formed by $(r_{i,N})_i$ and $(\bar{\theta}_{i,N})_i$, respectively. Here $\|\cdot\|$ denotes the Euclidean 2-norm.

We also obtain the following estimate for $\{r_N\}$ which will be used later. From the definition of $r_{i,N}$, integration by parts, (2.16), $\{u_{N}\} |_{\mathcal{E}_I} = 0$ and direct computations on norms of $\lambda_N \chi_{K_i}$, we have

$$r_{i,N} = r(\lambda_N \chi_{K_i}) = (f, \lambda_N \chi_{K_i})_{\Omega_h} - \langle \nabla u_h, \nabla (\lambda_N \chi_{K_i}) \rangle_{\Omega_h} - (g_h - u_h, \partial_{n_h}(\lambda_N \chi_{K_i}))_{\partial \Omega_h} + (\lambda_N \chi_{K_i}, \partial_{n_h} u_h)_{\partial \Omega_h} + \langle \partial_{n_{F}}(u_h), [\lambda_N \chi_{K_i}] \rangle_{\mathcal{E}_{F} \cap \partial \Omega_h} + \sum_{K \in \mathcal{T}_h} \beta h_K^{-1} \langle g_h - u_h, \lambda_N \chi_{K_i} \rangle_{\Gamma_K} - \gamma \sum_{F \in \mathcal{E}_{G}} h_F \langle [\partial_{n_{F}}(\lambda_N \chi_{K_i})], [\partial_{n_{F}} u_h] \rangle_{F}$$

$$= (f, \lambda_N \chi_{K_i})_{\Omega_h} - \langle g_h - u_h, \partial_{n_h}(\lambda_N \chi_{K_i}) \rangle_{\Gamma_{K_i}} + \beta h_K^{-1} \langle g_h - u_h, \lambda_N \chi_{K_i} \rangle_{\Gamma_{K_i}} - \sum_{F \in \mathcal{E}_{F} \cap \partial \Gamma_{K_i}} h_F \langle [\partial_{n_{F}}(\lambda_N \chi_{K_i})], [\partial_{n_{F}} u_h] \rangle_{F} \lesssim h_{K_i} \|f\|_{\mathcal{E}_{f} \cap \partial \Omega_h} + h_{K_i}^{-1/2} \|g_h - u_h\|_{\Gamma_{K_i}} + \sum_{F \in \mathcal{E}_{F} \cap \partial \Gamma_{K_i}} h_F^{1/2} \|\partial_{n_{F}} u_h\|_{F}.$$
We then define the global recovered flux by Proposition 2.3.4 in [9]:

\[ \| \tilde{\theta}_N \| \lesssim \| r_N \| \lesssim \sum_{F \in \mathcal{E}_N \cap \mathcal{E}_I} h_F^{1/2} \| \partial_{n_F} u_h \|_F + \sum_{K \in \mathcal{T}_h} \left( h_K \| f \|_{K \cap \Omega_h} + h_K^{-1/2} \| g_h - u_h \|_{V_K} \right). \]

3.4. Flux reconstruction. In this subsection, we locally recover a flux for each element \( K \in \mathcal{T}_h \). The element-wise construction is completely explicit and based on the computation of \( u_h \) and \( \theta_h \). Denote the \( H(\text{div}; \Delta_h) \) conforming Raviart-Thomas (RT) space of index 1 with respect to \( \mathcal{T}_h \) by

\[ \text{RT}_h = \{ \tau \in H(\text{div}; \Delta_h) : \tau|_K \in \text{RT}_1(K), \forall K \in \mathcal{T}_h \}, \]

where \( \text{RT}_1(K) = \mathbb{P}_1(K)^d + x \mathbb{P}_1(K) \). On a triangular element \( K \in \mathcal{T}_h \), a vector-valued function \( \tau \) in \( \text{RT}_1(K) \) is characterized by the following degrees of freedom (see Proposition 2.3.4 in [9]):

\[ \int_K \tau \cdot \zeta \, dx, \quad \forall \zeta \in \mathbb{P}_0(K)^d, \]

and

\[ \int_F (\tau \cdot n_F) \, w \, ds, \quad \forall w \in \mathbb{P}_1(F) \text{ and } \forall F \in \mathcal{E}_K. \]

For each element \( K \in \mathcal{T}_h \), we define \( \sigma_K \in \text{RT}_1(K) \) such that for all \( \zeta \in \mathbb{P}_0(K)^d \) and for all \( F \in \mathcal{E}_K \) and \( w \in \mathbb{P}_1(F) \) it satisfies:

\[ \langle \sigma_K, \zeta \rangle_K = \langle \nabla u_h, \zeta \rangle_K + \gamma \sum_{F \in \mathcal{E}_I \cap \mathcal{E}_K} h_F \langle [\partial_{n_F} u_h], [\zeta \cdot n_F] \rangle_F + \langle g_h - u_h, \zeta \cdot n_h \rangle_{\Gamma_K}, \]

\[ \langle \sigma_K \cdot n_F, w \rangle_F = \langle [\partial_{n_F} u_h], w \rangle_F - b_F(\theta_h, w) \quad \text{if } F \in \mathcal{E}_I, \]

\[ \langle \sigma_K \cdot n_F, w \rangle_F = \langle [\partial_{n_F} u_h], w \rangle_F \quad \text{if } F \in \mathcal{E}_0 \setminus \Gamma_K, \]

\[ \langle \sigma_K \cdot n_F, w \rangle_F = \langle \partial_{n_F} u_h, w \rangle_F + \frac{\beta}{h_K} \langle g_h - u_h, w \rangle_{\Gamma_K} \quad \text{if } F = \Gamma_K, \]

where

\[ b_F(\theta_h, w) = \frac{h_F}{2} \sum_{N \in \mathcal{N}_F} \theta_{h|F}(N) w(N). \]

Remark 3.6. One can also reconstruct the flux in the Raviart-Thomas space of index 0 similarly to (3.27). Note that there are now no interior degrees of freedom thus the first equation in (3.27) is not needed, whereas the edge degrees of freedom are tested with \( w \in \mathbb{P}_0(F) \).

Remark 3.7. Note that if \( F = \Gamma_K \in \mathcal{E}_K \), we then have that

\[ \sigma_K \cdot n_F = \partial_{n_F} u_h + \frac{\beta}{h_K} (g_h - u_h). \]

We then define the global recovered flux by

\[ \sigma_h = \sum_{K \in \mathcal{T}_h} \sigma_K. \]
Recall that on a cut element \( K \in \mathcal{T}_h^b \), \( f \) is only defined on \( K \cap \Omega_h \). We next introduce an extension of \( f \) to the whole cut element \( K \). For each \( K \in \mathcal{T}_h^b \) and \( K \not\subset \Omega_h \), we extend \( f \) to \( K \cap \Omega_h^\circ \) such that \( f|_{K \cap \Omega_h^\circ} \) is linear and also satisfies that for all \( w \in P_1(K) \),

\[
(3.30) \quad (f,w)_{K \cap \Omega_h^\circ} = \beta h_K^{-1} (g_h - u_h, w)_{\Gamma_K} + \sum_{F \in \mathcal{E}_K \cap \mathcal{E}_I} \frac{1}{2} \langle g_K(F) [\partial_{n_F} u_h], [w] \rangle_{F \cap \Omega_h^\circ} .
\]

Note that if the right-hand side is 0, this represents a zero extension.

Let \( \Pi \) be the \( L^2 \) projection operator onto the space \( DG_h \) defined on \( \mathcal{T}_h \).

**Lemma 3.8.** Let \( \sigma_h \) be defined in (3.28). Then we have that \( \sigma_h \in RT_h \) and

\[
(3.30) \quad -\nabla \cdot \sigma_h = \Pi_i(f) \quad \forall K \in \mathcal{T}_h .
\]

**Proof.** By its definition, it is easy to see that \( \sigma_h \in RT_h \). Firstly, we note that for all \( w \in DG_h \) we have

\[
(3.31) \quad b(\theta_h, w) = l_h(w) - \tilde{a}_h(u_h, w) \\
= (f,w)_{\Omega_h} - (\nabla u_h, \nabla w)_{\Omega_h} - (g_h - u_h, \partial_n w)_{\partial \Omega_h} + \langle w, \partial_n u_h \rangle_{\partial \Omega_h} \\
+ \sum_{K \in \mathcal{T}_h^b} \frac{\beta}{h_K} (g_h - u_h, w)_{\Gamma_K} - \gamma \sum_{F \in \mathcal{E}_p \cap \mathcal{E}_K} h_F \langle [\partial_{n_F} u_h], [\partial_{n_F} w] \rangle_F \\
+ \langle \{ \partial_{n_F} u_h \}, [w] \rangle_{\mathcal{E}_p \cap \Omega_h} .
\]

since \([u_h] = 0\). To prove (3.30), we first consider the case of \( K \in \mathcal{T}_h^{int} \), i.e., \( K \subset \Omega_h \), \( K \cap \partial \Omega_h = \emptyset \). Let \( w \in DG_h \) such that \( w|_K \in P_1(K) \) and \( w \) vanishes elsewhere. From (3.31) we have

\[
(3.32) \quad b(\theta_h, w) = (f,w)_K - (\nabla u_h, \nabla w)_K - \gamma \sum_{F \in \mathcal{E}_p \cap \mathcal{E}_K} h_F \langle [\partial_{n_F} u_h], [\partial_{n_F} w] \rangle_F \\
+ \langle \{ \partial_{n_F} u_h \}, [w] \rangle_{\mathcal{E}_p \cap \Omega_h} .
\]

By integration by parts, (3.27), the fact that \( b(\theta_h, w) = \sum_{F \in \mathcal{E}_K} b_F(\theta_h, [w]) \), and (3.32), we have

\[
\begin{align*}
(\nabla \cdot \sigma_h, w)_K &= - (\sigma_h, \nabla w)_K + (\sigma_h, n_K, w)_{E_K} \\
&= - (\sigma_h, \nabla w)_K + (\sigma_h, n_K, [w])_{E_K} \\
&= -(\nabla u_h, \nabla w)_K - \gamma \sum_{F \in \mathcal{E}_p \cap \mathcal{E}_K} h_F \langle [\partial_{n_F} u_h], [\partial_{n_F} w] \rangle_F \\
&\quad + \langle \{ \partial_{n_F} u_h \}, [w] \rangle_{E_K} - b(\theta_h, w) = -(f,w)_K ,
\end{align*}
\]

which yields (3.30) for all interior elements.

Now consider the second case when \( K \in \mathcal{T}_h^b \). Again, let \( w \in DG_h \) such that \( w|_K \in P_1(K) \) and \( w \) vanishes elsewhere. From (3.31) we now have

\[
(3.34) \quad b(\theta_h, w) = (f,w)_{K \cap \Omega_h} - (\nabla u_h, \nabla w)_{K \cap \Omega_h} - (g_h - u_h, \partial_n w)_{\Gamma_K} + \langle w, \partial_n u_h \rangle_{\Gamma_K} \\
+ \beta h_K^{-1} (g_h - u_h, w)_{\Gamma_K} - \gamma \sum_{F \in \mathcal{E}_p \cap \mathcal{E}_K} h_F \langle [\partial_{n_F} u_h], [\partial_{n_F} w] \rangle_F \\
+ \langle \{ \partial_{n_F} u_h \}, [w] \rangle_{\mathcal{E}_p \cap \Omega_h} .
\]
Applying integration by parts on $K \cap \Omega_h^c$ gives
\begin{equation}
\langle w, \partial_{n_h} u_h \rangle_{\Gamma_K} = - (\nabla u_h, \nabla w)_{K \cap \Omega_h^c} + (\partial_{n_h} u_h, w)_{\partial(K \cap \Omega_h^c) \setminus \Gamma_K}
\end{equation}
\begin{equation}
= - (\nabla u_h, \nabla w)_{K \cap \Omega_h^c} + (\partial_{n_h} u_h, \{w\})_{\mathcal{E}_I \cap \mathcal{E}_K \cap \Omega_h^c} + (\partial_{n_h} u_h, w)_{\mathcal{E}_K \cap \mathcal{E}_B}
\end{equation}
which, combining with the following equation,
\begin{equation}
\langle \partial_{n_F} u_h, \{w\} \rangle_{\mathcal{E}_I \cap \mathcal{E}_K \cap \Omega_h^c} = \langle \{\partial_{n_F} u_h\}, \{w\} \rangle_{\mathcal{E}_I \cap \mathcal{E}_K \cap \Omega_h^c} + \frac{1}{2} \langle \mathcal{S}(F) \{\partial_{n_F} u_h\}, \{w\} \rangle_{\mathcal{E}_I \cap \mathcal{E}_K \cap \Omega_h^c},
\end{equation}
implies
\begin{equation}
\langle w, \partial_{n_h} u_h \rangle_{\Gamma_K} = - (\nabla u_h, \nabla w)_{K \cap \Omega_h^c} + \{\partial_{n_F} u_h\}, \{w\} \rangle_{\mathcal{E}_I \cap \mathcal{E}_K \cap \Omega_h^c} + \frac{1}{2} \langle \mathcal{S}(F) \{\partial_{n_F} u_h\}, \{w\} \rangle_{\mathcal{E}_I \cap \mathcal{E}_K \cap \Omega_h^c} + (\partial_{n_K} u_h, \mathcal{E}_K \cap \mathcal{E}_B).
\end{equation}
Note that the previous relation also holds in the case $\Gamma_K = \mathcal{E}_K \cap \mathcal{E}_B$.

Combining all above with the definitions in (3.29) and (3.27) gives
\begin{equation}
b(\theta_h, w) = (f, w)_K - (\nabla u_h, \nabla w)_K - (g_h - u_h, \partial_{n_h} w)_{\Gamma_K}
\end{equation}
\begin{equation}
-\gamma \sum_{F \in \mathcal{E}_I \cap \mathcal{E}_K} h_F \langle \{\partial_{n_F} u_h\}, \{\partial_{n_F} w\} \rangle_F + \{\partial_{n_F} u_h\}, \{w\} \rangle_{\mathcal{E}_K}
\end{equation}
\begin{equation}
= (f, w)_K - (\mathbf{\sigma}_h, \nabla w)_K + \{\partial_{n_F} u_h\}, \{w\} \rangle_{\mathcal{E}_K}.
\end{equation}
Again, we have $b(\theta_h, w) = \sum_{F \in \mathcal{E}_K} b_F(\theta_h, \{w\})$. By using (3.27), the previous equality gives
\begin{equation}
0 = (f, w)_K - (\mathbf{\sigma}_h, \nabla w)_K + \{\mathbf{\sigma}_h \cdot n_F\}, \{w\} \rangle_{\mathcal{E}_K}
\end{equation}
so we obtain (3.30) thanks to the integration by parts formula. This completes the proof of the lemma.

Remark 3.9. For the cut elements, it is not obvious to construct a flux that is both locally conservative in the cut part $K \cap \Omega_h$ and, at the same time, maintains continuous normal flux. The technique of applying integration by parts in (3.36) renders the problem to be a more regular problem and completes partial elements to full elements.

4. Application in the a posteriori error estimation. For the sake of simplicity, we assume in this section that $f|_{K} \in \mathbb{P}_1(K)$ on any $K \in \mathcal{T}_h \setminus \mathcal{T}_h^c$. In the adaptive procedure, we define the following local error indicators
\begin{equation}
\eta_{K,1} = \|\mathbf{\sigma}_h - \nabla u_h\|_K, \quad \eta_{K,2} = \|\mathbf{\sigma}_h - \nabla u_h\|_{K \cap \Omega_h} \quad \forall K \in \mathcal{T}_h
\end{equation}
and the corresponding estimators:
\begin{equation}
\eta_1 = \sqrt{\sum_{K \in \mathcal{T}_h} \eta_{K,1}^2}, \quad \eta_2 = \sqrt{\sum_{K \in \mathcal{T}_h} \eta_{K,2}^2}.
\end{equation}
4.1. Reliability. Let $\hat{e} \in H^1(\Omega)$ be the lifting such that $\hat{e} = e := u - u_h$ on $\partial \Omega$ and
\[
\|\hat{e}\|_{H^1(\Omega)} = \|e\|_{H^{1/2}(\partial \Omega)}.
\]

**Theorem 4.1 (Reliability).** Let $\sigma_h$ be given by (3.28) and $u_h$ be the CutFEM solution in (2.10). We have the following reliability result:
\[
\|\nabla(u - u_h)\|_\Omega \leq \|\sigma_h - \nabla u_h\|_\Omega + 2\|\nabla \hat{e}\|_\Omega + C\epsilon,
\]
where the constant $C$ is independent of the mesh size and mesh-domain intersection, and
\[
\epsilon = \sqrt{\sum_{K \in \mathcal{T}_h^b} h_K^2 \|f - \Pi_1(f)\|_{K \cap \Omega}^2}.
\]

**Remark 4.2.** Thanks to the assumption that $\Omega \subset \Omega_h$, we have that
\[
\|\sigma_h - \nabla u_h\|_\Omega \leq \|\sigma_h - \nabla u_h\|_{\Omega_h} \leq \|\sigma_h - \nabla u_h\|_{\Delta_h}.
\]
Therefore both $\eta_1$ and $\eta_2$ could serve as the error estimator in the AMR procedure.

**Proof.** By triangle inequality, we firstly have the following bound:
\[
\|\nabla e\|_\Omega \leq \|\nabla(e - \hat{e})\|_\Omega + \|\nabla \hat{e}\|_\Omega
\]
\[
= \sup_{v \in H^1_0(\Omega)} \frac{(\nabla(e - \hat{e}), v)}{\|\nabla v\|_\Omega} + \|\nabla \hat{e}\|_\Omega
\]
\[
\leq \sup_{v \in H^1_0(\Omega)} \frac{(\nabla e, v)}{\|\nabla v\|_\Omega} + 2\|\nabla \hat{e}\|_\Omega.
\]

To bound $(\nabla e, \nabla v)_\Omega$ we have
\[
|(\nabla e, \nabla v)_\Omega| = |(\nabla u - \sigma_h, \nabla v)_\Omega + (\sigma_h - \nabla u_h, \nabla v)_\Omega|
\]
\[
\leq \sum_{K \in \mathcal{T}_h^b} |(f - \Pi_1(f), v)_K| + |(\sigma_h - \nabla u_h, \nabla v)_\Omega|
\]
\[
\lesssim \sum_{K \in \mathcal{T}_h^b} \|f - \Pi_1(f)\|_{K \cap \Omega} \|v\|_K + \|\sigma_h - \nabla u_h\|_\Omega \|\nabla v\|_\Omega
\]
\[
\leq C\epsilon \|\nabla v\|_\Omega + \|\sigma_h - \nabla u_h\|_\Omega \|\nabla v\|_\Omega
\]
where we used (2.17) for the last inequality. This completes the proof of the theorem. \(\Box\)

**Remark 4.3.** It is useful to note that for any $K \in \mathcal{T}_h^b$, we have that
\[
\|\Pi_1(f)\|_K \lesssim |K|^{-1/2} \int_K |f|.
\]
Indeed, denoting by $(a_i)_{1 \leq i \leq d+1}$ the values taken by $\Pi_1(f)$ at the vertices of $K$ and by $\lambda_i$ the corresponding nodal basis functions on $K$, we have using that $0 \leq \lambda_i \leq 1,$
\[
|K| \sum_{i=1}^{d+1} a_i^2 \simeq \|\Pi_1(f)\|_K^2 = \int_K f \left( \sum_{i=1}^{d+1} a_i \lambda_i \right)
\]
\[
\lesssim \int_K |f| \left( \sum_{i=1}^{d+1} |a_i| \right) \lesssim |K|^{-1/2} \|\Pi_1(f)\|_K \int_K |f|,
\]
which yields the desired estimate.
Remark 4.4. By the definition of $f$ in (3.29), $f|_{\Omega_h^e}$ can be bounded as follows using Cauchy-Schwarz inequality:

$$|f, w|_{K \cap \Omega_h^e} \lesssim \beta h_K^{-1} \|g_h - u_h\|_{\Gamma_K} \|w\|_{\Gamma_K} + \sum_{F \in E_{I} \cap \mathcal{E}_K} \|\partial_{n_F} u_h\|_{F \cap \Omega_h^e} \|w\|_{F \cap \Omega_h^e},$$

for any $w \in \mathbb{P}_1(K \cap \Omega_h^e)$. Therefore, we have

$$|K \cap \Omega_h^e|^{1/2} \|f|_{K \cap \Omega_h^e} \lesssim \beta h_K^{-1} |\Gamma_K|^{1/2} \|g_h - u_h\|_{\Gamma_K} + \sum_{F \in E_{I} \cap \mathcal{E}_K} |F \cap \Omega_h^e|^{1/2} \|\partial_{n_F} u_h\|_{F \cap \Omega_h^e}.$$

Hence, using that

$$\|f - \Pi_1(f)\|_{K \cap \Omega} \leq \|f - \Pi_1(f)\|_{K \cap \Omega_h} \leq \|f\|_{K \cap \Omega_h} + \|\Pi_1(f)\|_K$$

as well as (4.6) and

$$\|\Pi_1(f)\|_K \lesssim \frac{1}{|K|^{1/2}} \int_K |f| \lesssim \frac{|K \cap \Omega_h^e|^{1/2}}{|K|^{1/2}} \|f\|_{K \cap \Omega_h} + \frac{|K \cap \Omega_h^e|^{1/2}}{|K|^{1/2}} \|f\|_{K \cap \Omega_h^e},$$

we obtain the next bound for $\varepsilon$:

$$\varepsilon^2 \lesssim \sum_{K \in \mathcal{T}_h} \left( h_K^2 \|f\|_{K \cap \Omega_h}^2 + h_K^{-2} |\Gamma_K| \|g_h - u_h\|_{\Gamma_K}^2 + \sum_{F \in E_{I} \cap \mathcal{E}_K} |F \cap \Omega_h^e| \|\partial_{n_F} u_h\|_{F \cap \Omega_h^e}^2 \right).$$

Remark 4.5. From the above estimate, we observe that $\varepsilon$ can be bounded by the classical residual based error estimator. Moreover, the constants are uniformly bounded and independent of the domain-mesh intersection. We further note that $\varepsilon$ can also be bounded by $\eta$ with an additional higher order oscillation term,

$$\text{osc}_1 = \sqrt{h_K^2 \|f - \Pi_{0,K \cap \Omega_h} f\|_{K \cap \Omega_h}^2 + h_K^{-2} |\Gamma_K| \|g_h - u_h - \Pi_{0,K \cap \Omega_h} (g_h - u_h)\|_{\Gamma_K}^2}.$$
Lemma 4.6. Let $K$ be a given element in $\mathcal{T}_h$. Then the following local efficiency result holds:

$$ \|\sigma_h - \nabla u_h\|_{K \cap \Omega_h} \leq \|\sigma_h - \nabla u_h\|_{K} \lesssim \|\nabla (u - u_h)\|_{\hat{K}} $$

$$ + \sum_{N \in \mathcal{T}_h} \left( \sum_{F \in \mathcal{E}_N \cap \mathcal{E}_g} \frac{h_F^{1/2}}{\|\partial_{\nu_F} u_h\|_F} + \sum_{K' \in \mathcal{T}_N \cap \mathcal{T}_h} h_{K'} \|f\|_{K' \cap \Omega_h} + h_{K'}^{-1/2} \|g_h - u_h\|_{\Gamma_{K'}} \right), $$

where $\hat{K}$ is a local neighborhood of $K$ that does not contain elements in $\mathcal{T}_h^b$ and the efficiency constant does not depend on the mesh size nor the domain-mesh intersection.

Proof. By the degrees of freedom for $RT^1(K)$ space, we have the following bound:

$$ \|\sigma_h - \nabla u_h\|_{K} \lesssim \sum_{F \in \mathcal{E}_K} h_F^{1/2} \|\langle \sigma_h - \nabla u_h \rangle \cdot \eta_F \|_{F} + \|\Pi_0 (\sigma_h - \nabla u_h)\|_{K}, $$

where $\Pi_0$ is the $L^2$ projection onto the piecewise constant space on $\mathcal{T}_h$. Let $F \in \mathcal{E}_K$ and $p \in \mathbb{P}_1(F)$ arbitrary, then from (3.27) we have:

$$ \langle (\sigma_h - \nabla u_h) \cdot \eta_F, p \rangle_F = \langle \{\partial_{n_F} u_h\} - \nabla u_h \cdot \eta_F, p \rangle_F - b_F(\theta, p) $$

$$ \lesssim \|\{\partial_{n_F} u_h\}\|_{F} \|p\|_F, $$

where $\theta \in E$. Let

$$ \theta = \frac{\partial_{n_F} u_h}{h_{\Gamma} \|\nabla u_h\|_{\Gamma}^2} $$

(4.10)

$$ \langle (\sigma_h - \nabla u_h) \cdot \eta_F, p \rangle_F = \begin{cases} 0 & \text{if } F \in \mathcal{E}_T \setminus \Gamma_K, \\ \frac{\beta}{h_{\Gamma}} \langle g_h - u_h, p \rangle_{\Gamma_K} & \text{if } F = \Gamma_K. \end{cases} $$

Then applying Cauchy-Schwartz inequality and (3.26) gives, for $F \in \mathcal{E}_T$,

$$ b_F(\theta, p) = \sum_{N \in \mathcal{N}_F} b_F(\theta, p) \lesssim h_F^{-1/2} \left( \sum_{N \in \mathcal{N}_F} \|\{\partial_{n_F} u_h\}\|_{F} \right) \|p\|_F $$

$$ \lesssim \left( \sum_{F' \in \bigcup_{N \in \mathcal{N}_F} \mathcal{E}_N \cap \mathcal{E}_T} \|\partial_{n_{F'}} u_h\|_{F'} + \sum_{K \in \bigcup_{N \in \mathcal{N}_F} \mathcal{T}_N} \left( h_K^{1/2} \|f\|_{K \cap \Omega_h} + h_K^{-1/2} \|g_h - u_h\|_K \right) \right) \|p\|_F. $$

Combining (4.10) and (4.11) gives, for any $F \in \mathcal{E}$,

$$ \|\sigma_h - \nabla u_h\|_{F} \leq \sup_{p \in \mathbb{P}_1(F)} \frac{\langle (\sigma_h - \nabla u_h) \cdot \eta_F, p \rangle_F}{\|p\|_F} $$

$$ \lesssim \left( \sum_{F' \in \bigcup_{N \in \mathcal{N}_F} \mathcal{E}_N \cap \mathcal{E}_T} \|\partial_{n_{F'}} u_h\|_{F'} + \sum_{K \in \bigcup_{N \in \mathcal{N}_F} \mathcal{T}_N} \left( h_K^{1/2} \|f\|_{K \cap \Omega_h} + h_K^{-1/2} \|g_h - u_h\|_K \right) \right) \|p\|_F. $$

By the definition of $\sigma_h$ in (3.27) and Cauchy-Schwartz inequality, we also have

$$ \|\Pi_0 (\sigma_h - \nabla u_h)\|_{K} = \sup_{p \in \mathbb{P}_1(K)} \frac{\langle \sigma_h - \nabla u_h, \nabla p \rangle_K}{\|\nabla p\|_K} $$

$$ \lesssim \sum_{F \in \mathcal{E}_T \cap \mathcal{E}_K} h_F^{1/2} \|\partial_{n_F} u_h\|_{F} + h_K^{-1/2} \|g_h - u_h\|_K. $$

(4.13)
Combining (4.9), (4.12) and (4.13), we have

\[ \|\sigma_h - \nabla u_h\|_K \]

(4.14)

\[ \lesssim \sum_{N \in \mathcal{N}_K} \left( \sum_{F \in \mathcal{F}_N \cap \mathcal{E}_I} h_F^{1/2}\|\partial_n F u_h\|_F + \sum_{K' \in \mathcal{T}_N} h_{K'} \|f\|_{K' \cap \Omega_h} + h_{K'}^{-1/2} \|g_h - u_h\|_{\Gamma_{K'}} \right). \]

For regular facets and elements, there holds the following classical local efficiency results (see [40, 18]):

(4.15)

\[ h_F^{1/2}\|\partial_n F u_h\|_F \lesssim \|\nabla (u - u_h)\|_{K_F^+ \cup K_F^-} + h_K \|f - \Pi_1(f)\|_{K_F^+ \cup K_F^-} \quad \forall F \in \mathcal{E}_I \setminus \mathcal{E}_g, \]

\[ h_K \|f\|_K \lesssim \|\nabla (u - u_h)\|_K + h_K \|f - \Pi_1(f)\|_K \quad \forall K \in \mathcal{T}_h \setminus \mathcal{T}_h^b. \]

(4.8) is then a direct consequence of (4.14), (4.15) and of the hypothesis \( f = \Pi_1(f) \) on any \( K \in \mathcal{T}_h \setminus \mathcal{T}_h^b \). This completes the proof of the lemma.

The following lemma, which follows from theorem 4.5 in [17], gives the efficiency result for the irregular error terms. Define

\[ \text{osc}(f) = \left( \sum_{K \in \mathcal{T}_h^b} h_K^2 \left( \|f - f_K\|_{\omega_K \cap \Omega_h}^2 + \|f\|_{(\Omega \setminus \Omega) \cap K}^2 \right) \right)^{1/2}, \]

where \( \omega_K \) is the union of all elements sharing a common vertex with \( K \) and \( f_K = \arg\min_{c \in \mathcal{R}} h_K \|f - c\|_{\omega_K \cap \Omega_h} \).

**Lemma 4.7.** We have the best approximation result for the irregular terms

\[ j_h(u_h, v_h) + \sum_{K \in \mathcal{T}_h^b} h_K^2 \|f\|_{K \cap \Omega_h}^2 + \sum_{K \in \mathcal{T}_h^b} h_K^{-1} \|u_h - g_h\|_{\Gamma_K}^2 \]

(4.16)

\[ \leq C_e \inf_{v_h \in \mathcal{CG}_h} \left( \|u - u_h\|^2 + j(v_h, v_h) + \sum_{K \in \mathcal{T}_h^b} h_K^{-1} \|v_h - g_h\|_{\Gamma_K}^2 + \text{osc}(f)^2 \right), \]

where the constant \( C_e \) does not depend on the mesh size nor the domain-mesh intersection and \( \text{osc}(f) \) can be regarded as a higher order oscillation term.

Thanks to the previous result, we can easily deduce an error bound for the flux error \( \|\sigma - \sigma_h\|_\Omega \), where \( \sigma = \nabla u \).

**Lemma 4.8.** Assume \( u \in H^2(\Omega) \). Then one has:

\[ \|\sigma - \sigma_h\|_\Omega \lesssim |u - u_h|_{1,\Omega} + h|u|_{2,\Omega} \]

(4.17)

\[ + C_e \inf_{v_h \in \mathcal{CG}_h} \left( \|u - v_h\|^2 + j(v_h, v_h) + \sum_{K \in \mathcal{T}_h^b} h_K^{-1} \|v_h - g_h\|_{\Gamma_K}^2 + \text{osc}(f)^2 \right) \]

**Proof.** We have, using that \( \Omega \subset \Omega_h \), that

\[ \|\sigma - \sigma_h\|_\Omega \leq \|\nabla u - \nabla u_h\|_\Omega + \|\sigma_h - \nabla u_h\|_{\Omega_h}. \]
It is therefore sufficient to bound \(\|\sigma_h - \nabla u_h\|_K\) for any \(K \in T_h\). For this purpose, we use (4.14). The triangle inequality together with norm equivalence in a discrete space and standard interpolation results give, for any \(F \in E_I \setminus E_g\), that:

\[
(4.18) \quad h_F^{1/2} \left\| \partial_{n_F} (u - u_h) \right\|_F \lesssim h_F^{1/2} \left\| \partial_{n_F} (u - R_h u) \right\|_F + |R_h u - u_h|_{1,K_F^+ \cup K_F^-} + |u - R_h u|_{1,K_F^+ \cup K_F^-} + h |u|_{2,K_F^+ \cup K_F^-},
\]

where \(R_h\) is the continuous, piecewise linear Lagrange interpolation operator. Together with (4.14), the second estimate of (4.15) and Lemma 4.7, this gives (4.17) which completes the proof of the lemma. Note that in the proof, we do not need the requirement that \(f\) is piecewise linear.

\[\square\]

**Remark 4.9.** When \(\Omega = \Omega_h\), we refer to [15] for the a priori error estimate of \(\|\nabla (u - u_h)\|_\Omega \lesssim h |u|_{2,\Omega}\). In the case when \(\Omega \neq \Omega_h\), the same order can be achieved but with some additional inconsistency error of higher order regarding the geometry approximation, which can be bounded using similar techniques to [17].

5. **Numerical results.** In this section, we present several numerical examples to validate the performance of the a posteriori error estimator in the adaptive mesh refinement procedure. The adaptive mesh refinement procedure is set as follows:

Solve \(\rightarrow\) Estimate \(\rightarrow\) Mark \(\rightarrow\) Refine \(\rightarrow\) Solve.

For the penalty parameters in the finite element method, we set \(\beta = 10\) and \(\gamma = 0.1\). For the refinement strategy, we use the Dörfler marking strategy [23] and the refinement rate is set to be ten percent. Regarding the domain approximation, let \(\rho\) be the level set function that satisfies \(\rho = 0\) on \(\partial \Omega\) and negative (positive) inside (outside) the domain \(\Omega\). Let \(\rho_h\) be the nodal interpolation of \(\rho\) with respect to \(T_{0,h}\). Then we define

\[
(5.1) \quad \partial \Omega_h = \{x : \rho_h(x) = 0\}.
\]

We can easily check that (2.6) holds.

In the adaptive procedure, we compare the error estimators \(\eta_1\) and \(\eta_2\) defined in (4.2) with the residual based error estimator (see [17]) defined as follows,

\[
(5.2) \quad \eta_{K,\text{res}} = \sqrt{h_K^2 \left\| f \right\|_{K \cap \Omega_h}^2 + h_K^{-1} \beta^2 \left\| g_h - u_h \right\|_{1,K}^2 + \sum_{F \in E_K \cap E_I} \frac{h_F}{2} \left\| \partial_{n_F} u_h \right\|_F^2}.
\]

The global residual based error estimator is then defined by

\[
(5.3) \quad \eta_{\text{res}} = \left( \sum_{K \in T_h} \eta_{K,\text{res}}^2 \right)^{1/2}.
\]

**Example 5.1.** In this example, we test a problem with a strong interior peak. The exact solution has the following representation:

\[
u(x, y) = \exp(-100((x - 0.5)^2 + (y - 0.5)^2)).
\]

This function has a strong peak at the point \((0.5, 0.5)\).
Note that the boundary of the domain is regular, thus we have that $\eta_{K,1} = \eta_{K,2}$. Moreover, the function value is very smooth and almost vanishes on the boundary. The purpose of this example is to test the efficacy of our adaptive algorithm for Nitsche’s method on a regular domain.

In the numerical scheme, $g$ and $f$ are approximated by their interpolations into the continuous piecewise linear space. We firstly test the convergence of the method on uniform meshes. The results are plotted in Figure 2(e) which show optimal convergence rates (order 1) for both the true error $\|\nabla(u - u_h)\|$ and the flux error $\|\nabla u - \sigma_h\|$.

In the adaptive mesh refinement (AMR) procedure, we start with a $5 \times 5$ initial mesh. The marking percent is set to be 25%, i.e., the ordered elements (from the one with largest error indicator) that accounts for the first 25% of the total error estimator get refined. With the stopping criteria that the total number of degree of freedoms (DOFs) be not greater than 5000, the final meshes generated by $\eta_{1,K}$ and $\eta_{res,K}$ are provided in Figure 2(a) and Figure 2(b). From Figure 2(c) and Figure 2(d), we observe optimal convergence rates for both estimators. However, the efficiency index, which is defined by $\eta = \|\nabla(u - u_h)\|$ of $\eta_1$ is more accurate with mean values 1.42 and 1.68 for Figure 2(c) and Figure 2(d), respectively, comparing to that of $\eta_{res}$ with mean values 5.75 and 5.10 for Figure 2(c) and Figure 2(d), respectively.

Example 5.2. In this example, we test the Franke function [27] on the unit square domain,

$$u(x, y) = \frac{3}{4} \exp\left(-\frac{(9x - 2)^2}{4} - \frac{(9y - 2)^2}{4}\right) + \frac{3}{4} \exp\left(-\frac{(9x + 1)^2}{49} - \frac{(9y + 1)^{10}}{10}\right) + \frac{1}{2} \exp\left(-\frac{(9x - 7)^2}{4} - \frac{(9y - 3)^2}{4}\right) - \frac{1}{5} \exp\left(-\frac{(9x - 4)^2}{4} - (9y - 7)^2\right).$$

This function has two peaks at $(2/9, 2/9)$ and $(7/9, 1/3)$ and one sink at $(4/9, 7/9)$. Since the boundary of the domain is regular, the purpose of this example is again to
test the efficacy of our algorithm for Nitsche’s method on regular domain. However, the solution on the boundary is more volatile than in Example 5.1 and our numerical results show that this boundary volatility potentially causes extra challenges for the efficiency of Nitsche’s method that imposes the Dirichlet boundary condition weakly. The optimal convergence results on uniform meshes for the errors $\|\nabla(u - u_h)\|$ and $\|\sigma - \sigma_h\|$ are verified in Figure 3(e). With the same initial mesh and marking strategy as in Example 5.1, and with the stopping criteria that the total number of DOFs be not greater than 7500, the final meshes generated using $\eta_{1,K}$ and $\eta_{res,K}$ are provided respectively in Figure 3(a) and (b). Both meshes are similar with DOFs centered around the peaks and sinks. Since the solution is more volatile on some parts of the boundary, we also observe dense refinements on some right and upper parts of the boundary. However, the mesh in Figure 3(a) puts relatively more DOFs on the boundary comparing to Figure 3(b) on the boundary.

From Figure 3(c) and Figure 3(d), we observe optimal convergence for both the true error and the estimators in the overall pattern, however, with occasional oscillations, for both cases. Such oscillation is uniquely caused by the Nitsche method since it imposes the Dirichlet boundary condition weakly. Again, we observe that $\eta_1$ is more accurate than $\eta_{res}$ for most regular (non-oscillating) iterations. Nevertheless, it seems that $\eta_1$ has a stronger magnifying effect for the oscillation.

Example 5.3. In this example, we test our algorithm on an irregular domain. The level set of the problem has a flower shape (see e.g., Figure 4(a)) that has the following representation:

$$\rho = \min(\rho_0, \rho_1, \cdots, \rho_8)$$

with

$$\begin{align*}
\rho_0(x, y) &= x^2 + y^2 - r^2, \\
\rho_i(x, y) &= (x - x_i)^2 + (y - y_i)^2 - r_i^2, \\
\rho_1(x, y) &= (x - x_1)^2 + (y - y_1)^2 - 2, \\
\rho_2(x, y) &= (x - x_2)^2 + (y - y_2)^2 - 2, \\
\rho_3(x, y) &= (x - x_3)^2 + (y - y_3)^2 - 2, \\
\rho_4(x, y) &= (x - x_4)^2 + (y - y_4)^2 - 2, \\
\rho_5(x, y) &= (x - x_5)^2 + (y - y_5)^2 - 2, \\
\rho_6(x, y) &= (x - x_6)^2 + (y - y_6)^2 - 2, \\
\rho_7(x, y) &= (x - x_7)^2 + (y - y_7)^2 - 2, \\
\rho_8(x, y) &= (x - x_8)^2 + (y - y_8)^2 - 2
\end{align*}$$
for \( i = 1, \cdots, 8 \), and
\[
x_i = r(\cos(\pi/8) + \sin(\pi/8)) \cos(i\pi/4), \quad y_i = r(\cos(\pi/8) + \sin(\pi/8)) \cos(i\pi/4).
\]
The domain boundary is defined to be the zero level set, i.e., \( \Omega = \{ x \in \mathbb{R}^2 : \rho(x) \leq 0 \} \).
The data are given such that \( g = 0 \) on \( \partial \Omega \) and
\[
f(x, y) = \begin{cases} 
10 & \text{if } (x - x_1)^2 + (y - y_1)^2 \leq r_1^2/2, \\
0 & \text{otherwise.}
\end{cases}
\]

In the numerical scheme, we take \( g_h = 0 \) and \( f \) is approximated by its \( L^2 \) projection into the discontinuous piecewise constant space. We start with a \( 8 \times 8 \) crossed mesh on the rectangular domain \((-4, 4) \times (-4, 4)\). With the stopping criteria that the total number of DOFs be not greater than 7000 and marking percent set to be 15\%, the final meshes obtained by \( \eta_1, \eta_2 \) and \( \eta_{\text{res}} \) are given in Figure 4(a), (b) and (c), respectively. We observe similar meshes for the three cases and DOFs are centered around the heat source. From Figure 4(c), (d) and (e), we observe optimal convergence rate for all error estimators. In this example, \( \eta_1 \) is very close to \( \eta_2 \) since there are no dense refinement on the boundary, and \( \eta_{\text{res}} \) is relatively bigger.

![Fig. 4. Example 5.3. Final meshes and convergence of error estimators](image)

**Example 5.4.** In this example, we consider the reentrant problem whose solution has the following polar representation:
\[
u(r, \theta) = r^\alpha \sin(\alpha \theta),
\]
with \( \alpha = \pi/\omega \) and \( \omega \) being the angle of the reentrant corner. In this example, we take \( \omega = 3/2\pi \). The domain is set to be \( \Omega = ([-1, 1]^2 \setminus [0, 1] \times [-1, 0]) \cap B(0.95) \), where \( B(0.95) \) is the ball with center \((0, 0)\) and radius 0.95. It is easy to check that \( f = 0 \) in \( \Omega \).
In the numerical scheme, we extend \( f \) outside of \( \Omega \) by 0 and take \( g_h \) to be the conforming linear interpolation of \( u \) with respect to \( T_h \). The optimal convergence results on uniform meshes for the errors \( \| \nabla (u - u_h) \| \) and \( \| \sigma - \sigma_h \| \) are verified in Figure 5(e).

In the AMR procedure, we choose to use the initial mesh \( 10 \times 10 \) on the regular domain \((-1, 1) \times (-1, 1)\). With the marking percent set to be 10\% and the stopping criteria set such that the maximal number of degrees of freedom does not exceed 5000, the final meshes generated by \( \eta_{2,K} \) and \( \eta_{\text{res}} \) are given in Figure 5(a) and Figure 5(b). The corresponding convergence rate of the estimators are presented in Figure 5(c) and Figure 5(d). We again observe optimal convergence for both AMR procedures. This again indicates that the estimators work equivalently effective for problems with reentrant singularity on the boundary. However, \( \eta_2 \) is more accurate than \( \eta_{\text{res}} \). For Figure 5(c), the mean ratio for \( \eta_{\text{res}}/\| \nabla (u - u_h) \|_\Omega \) is 4.1, whereas the mean ratio for \( \eta_1/\| \nabla (u - u_h) \|_\Omega \) and \( \eta_2/\| \nabla (u - u_h) \|_\Omega \) is 2.4 and 1.5, respectively. The corresponding ratios for Figure 5(d) are similar. We note that for this example, using \( \eta_{1,K} \) generates almost the same mesh as \( \eta_{2,K} \). However, \( \eta_2 \) is more accurate than \( \eta_1 \) in both cases.

![Mesh by \( \eta_{2,K} \)](image)

![Mesh by \( \eta_{\text{res},K} \)](image)

![Errors by \( \eta_{2,K} \)](image)

![Errors by \( \eta_{\text{res},K} \)](image)

![Errors on uniform meshes](image)

**Fig. 5.** Example 5.4. Final meshes and convergence of error estimators

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