1. Introduction

Let \((S, L)\) be a smooth, primitively polarized complex \(K3\) surface of genus \(p \geq 2\), with \(L\) a globally generated, indivisible line bundle with \(L^2 = 2p - 2\). We denote by \(K_p\) the moduli space (or stack) of smooth primitively polarized \(K3\) surfaces of genus \(p\), which is smooth, irreducible, of dimension 19. Its elements correspond to the isomorphism classes \([S, L]\) of pairs \((S, L)\) as above. We will often abuse notation and denote \([S, L]\) simply by \(S\).

For \(m \geq 1\), the arithmetic genus of the curves in \(|mL|\) is \(p(m) = m^2(p - 1) + 1\). Let \(\delta\) be an integer such that \(0 \leq \delta \leq p(m)\). We consider the quasi–projective scheme (or stack) \(V_{p,m,\delta}\) (or simply \(V_{m,\delta}\), when \(p\) is understood), called the \((m, \delta)\)–universal Severi variety, parametrizing all pairs \((S, C)\), with \((S, L) \in K_p\) and \(C \in |mL|\) a reduced and irreducible curve, with only \(\delta\) nodes as singularities.

One has the projection

\[ \phi_{m,\delta} : V_{m,\delta} \to K_p \]

whose fiber over \((S, L) \in K_p\) is the variety, denoted by \(V_{m,\delta}(S)\), called the Severi variety of \(\delta\)-nodal irreducible curves in \(|mL|\). The variety \(V_{m,\delta}(S)\) is well-known to be smooth, pure of dimension \(g_{m,\delta} := p(m) - \delta = m^2(p - 1) + 1 - \delta\). We will often write \(g\) for \(g_{m,\delta}\) if no confusion arises: this is the geometric genus of any curve in \(V_{m,\delta}\).

One has the obvious moduli map

\[ \psi_{m,\delta} : V_{m,\delta} \longrightarrow M_g, \]

where \(M_g\) is moduli space of smooth genus-\(g\) curves, sending a curve \(C\) to the class of its normalization. Our objective in this paper is to find conditions on \(p\), \(m\) and \(\delta\) (or equivalently \(g\)) ensuring the existence of a component \(V\) of \(V_{m,\delta}\), such that \(\psi_{m,\delta}|_V\) is either (a) generically finite onto its image, or (b) dominant onto \(M_g\). Note that (a) can happen only for \(g \geq 11\) and (b) only for \(g \leq 11\).

We collect our results in the following statement, which solves the problem for all but finitely many triples \((p, g, m)\). For instance, for \(m \geq 5\) and \(p \geq 7\) the result yields that the moduli map has maximal rank on some component for any \(g\).

**Theorem 1.1.** With the above notation, one has:

(A) For the following values of \(p \geq 3\), \(m\) and \(g\) there is an irreducible component \(V\) of \(V_{m,\delta}\), such that the moduli map \(V \to M_g\) is dominant:

- \(m = 1\) and \(0 \leq g \leq 7\);
- \(m = 2, p \geq g - 1\) and \(0 \leq g \leq 8\);
- \(m = 3, p \geq g - 2\) and \(0 \leq g \leq 9\);
- \(m = 4, p \geq g - 3\) and \(0 \leq g \leq 10\);
For the following values of $p$, $m$ and $g$ there is an irreducible component $V$ of $V_{m,\delta}$, such that the moduli map $V \to \overline{M}_g$ is generically finite onto its image:

- $m \geq 5$, $p \geq g - 4$ and $0 \leq g \leq 11$.

(B) To prove this, it suffices to exhibit some specific curve in the universal Severi variety such that a component of the fiber of the moduli map at that curve has the right dimension, i.e., $\min\{0, 22 - 2g\}$.

To do this, we argue by degeneration, i.e., we consider partial compactifications (cf. [Fr2, Ku, PP]). This is essentially obtained by adding to $\overline{M}_g$ a divisor $\mathcal{S}_p$ parametrizing reducible $K3$ surfaces of genus $p$ that can be realised in $\mathbb{P}^p$ as the union of two rational normal scrolls intersecting along a normal elliptic curve. In §3 we start the analysis of the case $m = 1$, and we introduce the curves in $\mathcal{V}_{1,\delta}$ over the reducible surfaces in $\mathcal{S}_p$ that we use for proving our results. Section 5 is devoted to the study of the fiber of the moduli map for the curves introduced in the previous §3 this is the technical core of this paper. In §6 we prove the $m = 1$ part of Theorem 1.1. In order to work out the cases $m > 1$ of Theorem 1.1 which is done in §§8–9 we need to introduce more limit curves, and this is what we do in §7.

The paper is organized as follows. The short §2 is devoted to preliminary results, and we in particular define a slightly broader notion of Severi variety including the cases of curves with more than $\delta$ nodes and of reducible curves. In §3 we introduce the partial compactification $\mathcal{S}_p$ we use (cf. [Fr2, Ku, PP]). This is essentially obtained by adding to $\overline{M}_g$ a divisor $\mathcal{S}_p$ parametrizing reducible $K3$ surfaces of genus $p$ that can be realised in $\mathbb{P}^p$ as the union of two rational normal scrolls intersecting along a normal elliptic curve. In §3 we start the analysis of the case $m = 1$, and we introduce the curves in $\mathcal{V}_{1,\delta}$ over the reducible surfaces in $\mathcal{S}_p$ that we use for proving our results. Section 5 is devoted to the study of the fiber of the moduli map for the curves introduced in the previous §3 this is the technical core of this paper. In §6 we prove the $m = 1$ part of Theorem 1.1. In order to work out the cases $m > 1$ of Theorem 1.1 which is done in §§8–9 we need to introduce more limit curves, and this is what we do in §7.

The study of Severi varieties is classical and closely related to modular properties. For the case of nodal plane curves the traditional reference is Severi’s wide exposition in [Sev, Anhang F], although already in Enriques–Chisini’s famous book [EC, vol. III, chap. III, §33] families of plane nodal curves with general moduli have been considered. The most important result on this subject is Harris’ proof in [Har] of the so-called Severi conjecture, which asserts that the Severi variety of irreducible plane curves of degree $d$ with $\delta$ nodes is irreducible.

In recent times there has been a growing interest in Severi varieties for $K3$ surfaces and their modular properties. In their seminal works [MM, Mu1, Mu2] Mori and Mukai proved that in the case of smooth curves in the hyperplane section class, i.e., what we denoted here by $V_{p,1,0}$, the modular map: (a) dominates $M_p$, for $p \leq 9$ and $p = 11$, whereas this is false for $p = 10$, and (b) is generically finite between $V_{p,1,0}$ and its image in $M_p$ for $p = 11$ and $p \geq 13$, whereas this is false for $p = 12$. In [CLM] one gives a different proof of these results, proving, in addition, that $V_{p,1,0}$ birationally maps to its image in $M_p$ for $p = 11$ and $p \geq 13$. The case of $V_{p,m,0}$, with $m > 1$, has been studied in [CLM2].

As for $\delta > 0$, in [FKPS] one proves that $V_{p,1,\delta}$ dominates $M_9$ for $2 \leq g < p \leq 11$. Quite recently, Kemeny, inspired by ideas of Mukai’s, and using geometric constructions of appropriate curves on $K3$ surfaces with high rank Picard group, proved in [Ke] that there is an irreducible component of $V_{p,m,\delta}$ for which the moduli map is generically finite onto its image for all but finitely many values of $p$, $m$ and $\delta$. Kemeny’s results partly intersect with part (B) of our Theorem 1.1 his results are slightly stronger than ours for $m \leq 4$, however our results are stronger and in fact optimal for $m \geq 5$ (if $p \geq 7$). Moreover part (A), which is also optimal for $m \geq 5$ (except for some very low values of $p$), is completely new.

To finish, it is the case to mention that probably the most interesting open problems on the subject are the following:

Questions 1.2. For $S \in \mathcal{S}_p$ general, is the Severi variety $V_{p,m,\delta}(S)$ irreducible? Is the universal Severi variety $V_{p,m,\delta}$ irreducible?

So far it is only known that $V_{p,1,\delta}$ is irreducible for $3 \leq g \leq 12$ and $g \neq 11$ (see [CD]).
Terminology and conventions. We work over \( \mathbb{C} \). For \( X \) a Gorenstein variety, we denote by \( O_X \) and \( \omega_X \) the structure sheaf and the canonical line bundle, respectively, and \( K_X \) will denote a canonical divisor of \( X \). If \( x \in X \), then \( T_{X,x} \) denotes the Zariski tangent space to \( X \) at \( x \). For \( Y \subset X \) a subscheme, \( J_{Y/X} \) (or simply \( J_Y \) if there is no danger of confusion) will denote its ideal sheaf whereas \( N_{Y/X} \) its normal sheaf. For line bundles we will sometimes abuse notation and use the additive notation to denote tensor products. Finally, we will denote by \( L^* \) the inverse of a line bundle \( L \) and by \( \equiv \) the linear equivalence of Cartier divisors.

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2. Preliminaries

To prove our main results, we will need to consider nodal, reducible curves. To this end we will work with a slightly broader notion of Severi variety than the one from the introduction.

Let \(|D|\) be a base point free complete linear system on a smooth K3 surface \( S \). Consider the quasi-projective scheme \( V_{D,\delta}(S) \) parametrizing pairs \((C,\nu_C)\) such that
1. \( C \) is a reduced (possibly reducible) nodal curve in \(|D|\);
2. \( \nu_C \) is a subset of \( \delta \) of its nodes, henceforth called the marked nodes (called assigned nodes in \( \text{[Ta]} \)), such that the normalization \( \tilde{C} \to C \) at \( \nu_C \) is 2-connected.

Then, by \( \text{[Ta]} \) Thms. 3.8 and 3.11, one has:
(i) \( V_{D,\delta}(S) \) is smooth of codimension \( \delta \) in \(|D|\);
(ii) in any component of \( V_{D,\delta}(S) \), the general pair \((C,\nu_C)\) is such that \( C \) is irreducible with precisely \( \delta \) nodes.

We leave it to the reader to verify that the conditions in \( \text{[Ta]} \) are in fact equivalent to ours. We call \( V_{D,\delta}(S) \) the Severi variety of nodal curves in \(|D|\) with \( \delta \) marked nodes. This definition is different from others one finds in the literature (and even from the one in the introduction!). Usually, in the Severi variety one considers only irreducible curves \( C \) with exactly \( \delta \) nodes. With our definition we consider the desingularization of a partial compactification of the latter variety. This will be useful for our purposes.

We define the \((m,\delta)\)-universal Severi variety \( V_{p,m,\delta} \) (or simply \( V_{m,\delta} \) when \( p \) is understood) to be the quasi-projective variety (or stack) parametrizing triples \((S,C,\nu_C)\), with \( S = (S,L) \in K_p \) and \((C,\nu_C) \in V_{[m\delta]}\). It is smooth and pure of dimension \( 19 + p(m) - \delta = 19 + g_{m,\delta} \) and the general element in any component is a triple \((S,C,\nu_C)\) with \( C \) irreducible with exactly \( \delta \) nodes. We may simplify notation and identify \((S,C,\nu_C) \in V_{p,m,\delta}\) with the curve \( C \), when the surface \( S \) and the set of nodes \( \nu_C \) are intended.

One has the projection
\[
\phi_{m,\delta} : V_{m,\delta} \longrightarrow K_p
\]
whose fiber over \( S \in K_p \) is the Severi variety \( V_{m,\delta}(S) := V_{[mL,\delta]}(S) \). Similarly, we have a moduli map
\[
\psi_{m,\delta} : V_{m,\delta} \longrightarrow \mathcal{M}_g,
\]
where we recall that \( g = g_{m,\delta} := p(m) - \delta \), which sends a curve \( C \) to the stable model of the partial normalization of \( C \) at the \( \delta \) marked nodes.

For \( S \in K_p \) and integers \( 0 \leq \delta \leq \delta' \leq p(m) \), one has a correspondence
\[
X_{m,\delta,\delta'}(S) := \{ (C,C') \in V_{m,\delta}(S) \times V_{m,\delta'}(S) \mid C = C' \text{ and } \nu_C \subseteq \nu_{C'} \}
\]
with the two projections
\[
V_{m,\delta}(S) \xrightarrow{p_1} X_{m,\delta,\delta'}(S) \xrightarrow{p_2} V_{m,\delta'}(S),
\]
which are both finite onto their images. Precisely:

- $p_2$ is surjective, étale of degree $(\delta')^g$, hence $\dim(X_{m,\delta',\delta}(S)) = g_{m,\delta'}$;
- $p_1$ is birational onto its image, denoted by $V_{m,\delta,\delta'}(S)$, which is pure with
  $$\dim(V_{m,\delta,\delta'}(S)) = g_{m,\delta'} = \dim(V_{m,\delta}(S)) - (\delta' - \delta).$$

Roughly speaking, the variety $V_{m,\delta,\delta'}(S)$ is the proper subvariety of $V_{m,\delta}(S)$ consisting of curves with at least $\delta'$ nodes, $\delta$ of them marked. The general point of any component of $V_{m,\delta,\delta'}(S)$ corresponds to a curve with exactly $\delta'$ nodes. So one has the filtration

$$V_{m,\delta,p(m)}(S) \subset V_{m,\delta,p(m)-1}(S) \subset \ldots \subset V_{m,\delta,\delta+1}(S) \subset V_{m,\delta}(S) = V_{m,\delta}(S)$$

in which each variety has codimension 1 in the subsequent.

**Remark 2.1.** Given a component $V$ of $V_{m,\delta}(S)$ and $\delta' > \delta$, there is no a priori guarantee that $V \cap V_{m,\delta',\delta}(S) \neq \emptyset$. If this is the case, then each component of $V \cap V_{m,\delta',\delta}(S)$ has codimension $\delta' - \delta$ in $V$ and we say that $V$ is $\delta'$–complete. If $V$ is $\delta'$–complete, then it is also $\delta''$–complete for $\delta < \delta'' < \delta'$. If $V$ is $p(m)$–complete we say it is fully complete, i.e., $V$ is fully complete if and only if it contains a point parametrizing a rational nodal curve.

If $V$ is a component of $V_{m,\delta}(S)$ and $W$ a component of $V_{m,\delta'}(S)$, such that $\dim(X_{m,\delta',\delta}(S) \cap (V \times W)) = g_{m,\delta'}$, then $X_{m,\delta,\delta'} \cap (V \times W)$ dominates a component $V'$ of $V_{m,\delta,\delta'}$ contained in $V$, whose general point is a curve in $W$ with $\delta$ marked nodes. In this case we will abuse language and say that $W$ is included in $V$.

Of course one can make a relative version of the previous definitions, and make sense of the subscheme $\mathcal{V}_{m,\delta,\delta'} \subset V_{m,\delta}$, which has dimension $19 + g_{m,\delta'}$, of the filtration

$$\mathcal{V}_{m,\delta,p(m)} \subset \mathcal{V}_{m,\delta,p(m)-1} \subset \ldots \subset \mathcal{V}_{m,\delta,\delta+1} \subset V_{m,\delta},$$

of the definition of a $\delta'$–complete component $\mathcal{V}$ of $V_{m,\delta}$, for $\delta' > \delta$, of fully complete components, etc.

For $\delta' > \delta$, the image of $\mathcal{V}_{m,\delta,\delta'}$ via $\psi_{m,\delta}$ sits in the $(\delta' - \delta)$–codimensional locus $\Delta_{g_{m,\delta},\delta'} - \delta$ of $(\delta' - \delta)$–nodal curves in $\overline{M}_{g_{m,\delta}}$.

### 3. Stable limits of $K3$ surfaces

In this section we consider some reducible surfaces (see [CLM1]), which are limits of smooth, polarized $K3$ surfaces, in the sense of the following:

**Definition 3.1.** Let $R$ be a compact, connected analytic variety. A variety $\mathcal{Y}$ is said to be a deformation of $R$ if there exists a proper, flat morphism

$$\pi: \mathcal{Y} \to \mathbb{D} = \{ t \in \mathbb{C} \mid |t| < 1 \}$$

such that $R = \mathcal{Y}_0 := \pi^*(0)$. Accordingly, $R$ is said to be a flat limit of $\mathcal{Y}_t := \pi^*(t)$, for $t \neq 0$.

If $\mathcal{Y}$ is smooth and if any of the $\pi$–fibers has at most normal crossing singularities, $\mathcal{Y} \xrightarrow{\pi} \mathbb{D}$ is said to be a semi-stable deformation of $R$ (and $R$ is a semi-stable limit of $\mathcal{Y}_t$, $t \neq 0$). If, in addition, $\mathcal{Y}_t$ is a smooth $K3$ surface, for $t \neq 0$, then $R$ is a semi-stable limit of $K3$ surfaces.

If, in the above setting, one has a line bundle $\mathcal{L}$ on $\mathcal{Y}$, with $L_t = \mathcal{L}|\mathcal{Y}_t$ for $t \neq 0$ and $L = \mathcal{L}|R$, then one says that $(R, L)$ is a limit of $(\mathcal{Y}_t, L_t)$ for $t \neq 0$.

Let $p = 2l + \varepsilon \geq 3$ be an integer with $\varepsilon = 0, 1$ and $l \in \mathbb{N}$. If $E' \subset \mathbb{P}^p$ is an elliptic normal curve of degree $p + 1$, we set $L_{E'} := \mathcal{O}_{E'}(1)$. Consider two generic line bundles $L_1, L_2 \in \text{Pic}^2(E')$ with $L_1 \neq L_2$. We denote by $R'_i$ the rational normal scroll of degree $p - 1$ in $\mathbb{P}^p$ described by the secant lines to $E'$ spanned by the divisors in $|L_i|$, for $1 \leq i \leq 2$. We have

$$R'_i \simeq \begin{cases} \mathbb{P}^1 \times \mathbb{P}^1 & \text{if } p = 2l + 1 \text{ is odd and } \mathcal{O}_{E'}(1) \not\sim (l + 1)L_i, \text{ for } i = 1, 2 \\ \mathbb{P}^1 & \text{if } p = 2l \text{ is even.} \end{cases}$$
The surfaces $R'_1$ and $R'_2$ are $\mathbb{P}^1$-bundles on $\mathbb{P}^1$. We denote by $\sigma_i$ and $F_i$ a minimal section and a fiber of the ruling of $R'_i$, respectively, so that $\sigma_i^2 = \varepsilon - 1$ and $F_i^2 = 0$, and

$$L_{R'_i} := \mathcal{O}_{R'_i}(1) \simeq \mathcal{O}_{R'_i}(\sigma_i + lF_i), \quad \text{for} \quad 1 \leq i \leq 2.$$  

By [CLM1] Thm. 1], $R'_1$ and $R'_2$ intersect transversely along $E'$, which is anticanonical on $R'_i$, i.e.

$$E' \equiv -K_{R'_i} \equiv 2\sigma_i + (3-\varepsilon)F_i \quad \text{for} \quad 1 \leq i \leq 2.$$  

Hence $R' = R'_1 \cup R'_2$ has normal crossings and $\omega_{R'}$ is trivial. We set $L_{R'} := \mathcal{O}_{R'}(1)$. The first cotangent sheaf $T_{R'}^1$ (cf. [FT 2, § 1]) is the degree 16 line bundle on $E'$

$$T_{R'}^1 \simeq \mathcal{N}_{E'/R'_1} \otimes \mathcal{N}_{E'/R'_2} \simeq L_{R'}^\otimes 4 \otimes (L_1 \otimes L_2)^\otimes (3-2\varepsilon),$$

the last isomorphism coming from (1) and (2).

The surface $R'$ is a flat limit of smooth $K3$ surfaces in $\mathbb{P}^p$. Namely, if $\mathcal{H}_p$ is the component of the Hilbert scheme of surfaces in $\mathbb{P}^p$ containing $K3$ surfaces $S$ such that $[S, \mathcal{O}_S(1)] \in \mathcal{K}_p$, then $R'$ sits in $\mathcal{H}_p$ and, for general choices of $E', L_1, L_2$, the Hilbert scheme $\mathcal{H}_p$ is smooth at $R'$ (see [CLM1]). However, the fact that $T_{R'}^1$ is non-trivial implies that $R'$ is not a semi-stable limit of $K3$ surfaces: indeed, the total space of every flat deformation of $R'$ to $K3$ surfaces in $\mathcal{H}_p$ is singular along a divisor $T \in [T_{R'}^1]$ (cf. [FT 2, Prop. 1.11 and § 2]). More precisely (see again [CLM1] for details), if

$$T_{\mathcal{H}_p, R'} \simeq H^0(R', \mathcal{N}_{R'}/\mathbb{P}^p) \to H^0(T_{R'}^1).$$

If $T$ is reduced (this is the case if (1) is general enough), then the tangent cone to $R'$ at each of the 16 points of $T$ has rank 4. In this case, by blowing up $R'$ at the points of $T$, the exceptional divisors are rank 4 quadric surfaces and, by contracting each of them along a ruling on one of the two irreducible components of the strict transform of $R'$, one obtains a small resolution of singularities $\Pi : R \to R'$ and a semi-stable degeneration $\pi : R \to \mathbb{D}$ of $K3$ surfaces, with central fiber $R := R_1 \cup R_2$, where $R_i = \Pi^{-1}(R'_i)$, for $i = 1, 2$ and still $\omega_R \simeq \mathcal{O}_R$. Note that we have a line bundle on $R$, defined as $L := \Pi^*(\mathcal{O}_{\mathbb{P}^p}(1)))$. So $\pi : R \to \mathbb{D}$ is a deformation of polarized $K3$ surfaces, and we set $L_R := \mathcal{L}|_{R} = \Pi_{R}^*(L_{R'})$.

We will abuse notation and terminology by identifying curves on $R'$ with their proper transforms on $R$ (so we may talk of lines on $R$, etc.).

We will set $E = R_1 \cap R_2$: then $E \simeq E'$ (and we will often identify them). We have $T_{R}^1 \simeq \mathcal{O}_E$. If the divisor $T$ corresponding to the deformation (1) is not reduced, the situation can be handled in a similar way, but we will not dwell on it, because we will not need it.

The above limits $R$ of $K3$ surfaces are stable, Type II degenerations according to the Kulikov-Persson-Pinkham classification of semi-stable degenerations of $K3$ surfaces (see [Kul, PP]).

By [ET2 Thm. 4.10] there is a normal, separated partial compactification $\overline{\mathcal{K}}_p$ of $\mathcal{K}_p$ obtained by adding to $\mathcal{K}_p$ a smooth divisor consisting of various components corresponding to the various kinds of Type II degenerations of $K3$ surfaces. One of these components, which we henceforward call $\mathcal{G}_p$, corresponds to the degenerations we mentioned above. Specifically, points of $\mathcal{G}_p$ parametrize isomorphism classes of pairs $(R', T)$ with $R' = R'_1 \cup R'_2$ as above and $T \in [T_{R'}^1]$, cf. [ET2 Def. 4.9]. Since all our considerations will be local around general members of $\mathcal{G}_p$, where the associated
$T$ is reduced, we may and will henceforth assume (after substituting $\overline{K}_p$ and $\mathcal{S}_p$ with dense open subvarieties) that $\overline{K}_p$ is smooth,

$\overline{K}_p = K_p \cup \mathcal{S}_p$

and that $T$ is reduced for all $(R', T) \in \mathcal{S}_p$. Thus, again since all our considerations will be local around such pairs, we may identify $(R', T)$ with the surface $R = R_1 \cup R_2$, as above, with $R_1 \simeq R'_1$ and $R_2$ the blow–up of $R'_2$ at the 16 points of $T$ on $E'$. If $T = p_1 + \ldots + p_{16}$, we denote by $\mathcal{C}_i$ the exceptional divisor on $R_2$ over $p_i$, for $1 \leq i \leq 16$, and we set $\mathcal{C} := \mathcal{C}_1 + \cdots + \mathcal{C}_{16}$.

To be explicit, let us denote by $\mathcal{R}'_p \subset \mathcal{H}_p$ the locally closed subscheme whose points correspond to unions of scrolls $R' = R'_1 \cup R'_2$ with $L_1, L_2$ general as above and such that $(R', T) \in \mathcal{S}_p$ for some $T \in |T^1_{R'}|$. One has an obvious dominant morphism $\mathcal{R}'_p \rightarrow M_1$ whose fiber over the class of the elliptic curve $E'$, modulo projectivities $\text{PGL}(p + 1, \mathbb{C})$, is an open subset of $\text{Sym}^2(\text{Pic}^2(E')) \times \text{Pic}^{p+1}(E')$ modulo the action of $\text{Aut}(E')$. Hence

$$\dim(\mathcal{R}'_p) = p^2 + 2p + 3,$$

whereas

$$\dim(\mathcal{H}_p) = p^2 + 2p + 19,$$

so that for $R' \in \mathcal{R}'_p$ general, the normal space $N_{R'/\mathcal{R}'_p} = T_{\mathcal{H}_{p,R'}}/T_{\mathcal{R}'_p,R'}$ has dimension 16. The map $[\mathcal{I}]$ factors through a map

$$N_{R'/\mathcal{R}'_p} \rightarrow H^0(T^1_{R'})$$

which is an isomorphism (see [CLM1]).

We denote by $\mathcal{R}_p$ the $\text{PGL}(p+1, \mathbb{C})$–quotient of $\mathcal{R}'_p$, which, by the above argument, has dimension $\dim(\mathcal{R}_p) = 3$. By definition, there is a surjective morphism

$$\pi_p : \mathcal{S}_p \rightarrow \mathcal{R}_p$$

whose fiber over (the class of) $R'$ is a dense, open subset of $|T^1_{R'}|$, which has dimension 15 (by [6], given $L_1, L_2$ and $L_{E'}$, then $\mathcal{O}_{E'}(T)$ is determined). This, by the way, confirms that $\dim(\mathcal{S}_p) = 18$.

The universal Severi variety $\mathcal{V}_{m,\delta}$ has a partial compactification $\overline{\mathcal{V}}_{m,\delta}$ (see [CK1] Lemma 1.4]), with a morphism

$$\overline{\phi}_{m,\delta} : \overline{\mathcal{V}}_{m,\delta} \rightarrow \overline{K}_p$$

extending $\phi_{m,\delta}$, where the fiber $\overline{\mathcal{V}}_{m,\delta}(R)$ over a general $R = R_1 \cup R_2 \in \mathcal{S}_p$ consists of all nodal curves $C \in [mL|R|]$, with $\delta$ marked, non–disconnecting nodes in the smooth locus of $R$. (There exist more refined partial compactifications of $\mathcal{V}_{m,\delta}$, for instance by adding curves with tacnodes along $E$, as considered in [Ch, Gk], following [Ra]; we will consider such curves in [7] but we do not need them now). The total transform $C$ of a nodal curve $C' \in [mL|R'|]$ with $h$ marked nodes in the smooth locus of $R'$ and $k$ marked nodes at points of $T$, with $h + k = \delta$, lies in $\overline{\mathcal{V}}_{m,\delta}(R)$, since it contains the exceptional divisors on $R_2$ over the $k$ points of $T$, and has a marked node on each of them off $E$. As for the smooth case, $\overline{\mathcal{V}}_{m,\delta}(R)$, is smooth, of dimension $g = p(m) – \delta$. We can also consider $\delta' – \delta$ codimensional subvarieties of $\overline{\mathcal{V}}_{m,\delta}(R)$ of the form $\overline{\mathcal{V}}_{m,\delta,\delta'}(R)$, with $\delta < \delta'$, etc.

Finally, there is an extended moduli map

$$\overline{\psi}_{m,\delta} : \overline{\mathcal{V}}_{m,\delta} \longrightarrow \overline{M}_g.$$

We end this section with a definition, related to the construction of the surfaces $R'$, that we will need later.

**Definition 3.2.** Let $E$ be a smooth elliptic curve with two degree–two line bundles $L_1$ and $L_2$ on it.

For any integer $k \geq 0$, we define the automorphism $\phi_{k,E}$ on $E$ that sends $x \in E$ to the unique point $y \in E$ satisfying

\begin{align*}
(6) & \quad \mathcal{O}_E(x + y) \simeq L_2 \otimes L_1^{k+1} \otimes (L_1^*)^{k+1}, \quad k \text{ odd}; \\
(7) & \quad \mathcal{O}_E(x - y) \simeq (L_2 \otimes L_1^*)^{k+1}, \quad k \text{ even}.
\end{align*}
For any $x \in E$, we define the effective divisor $D_{k,E}(x)$ to be the degree $k + 1$ divisor
\begin{equation}
D_{k,E}(x) = x + \phi_{1,E}(x) + \cdots + \phi_{k,E}(x).
\end{equation}

4. Limits of nodal hyperplane sections on reducible $K3$ surfaces

Given $R \in \Theta_p$, with $R = R_1 \cup R_2$ as explained in the previous section, we will now describe
\begin{equation}
\text{certain curves in } W_{1,\delta} \text{ lying on } R.
\end{equation}

Definition 4.1. Let $d, \ell$ be non–negative integers such that $\ell \leq 16$. Set $\delta = d + \ell$ and assume
$\delta \leq p - \epsilon$, where
\begin{equation}
\epsilon = \begin{cases} 
1 & \text{if } \ell = 0, \\
0 & \text{if } \ell > 0.
\end{cases}
\end{equation}

We define $W_{d,\ell}(R)$ to be the set of reduced curves in $\overline{\mathcal{M}}_{\delta}(R)$:
(a) having exactly $d_1 := \lfloor \frac{d+1}{2} \rfloor$ nodes on $R_1 - E$ and $d_2 := \lfloor \frac{d}{2} \rfloor$ nodes on $R_2 - E - \epsilon$, hence they split off $d_i$ lines on $R_i$, for $i = 1, 2$;
(b) such that the union of these $d = d_1 + d_2$ lines is connected;
(c) containing exactly $\ell$ among the irreducible components of $\epsilon$, so they have $\ell$ further nodes, none
of them lying on $E$.

These curves have exactly $\delta$ nodes on the smooth locus of $R$, which are the marked nodes. We
write $W_{\delta}(R)$ for $W_{\delta,0}(R)$.

For any curve $C$ in $W_{d,\ell}(R)$, we denote by $\mathcal{C}$ the connected union of $d$ lines as in (b), called the line
chain of $C$, and by $\gamma_i$ the irreducible component of the residual curve to $\mathcal{C}$ and to the $\ell$ components
of the exceptional curve $\epsilon$ on $R_i$, for $i = 1, 2$.

Curves in $W_{d,\ell}(R)$ are total transforms of curves on $R' = \pi_p(R)$ with $d$ marked nodes on $R' \setminus E'$
and passing through $\ell$ marked points in $T$. Members of $W_{\delta}(R)$ are shown in the following pictures,
which also show their images via the moduli map $\overline{\psi}_\delta$ (provided $g \geq 3$ if $\delta$ is odd and $g \geq 2$ if $\delta$
\begin{equation}
\text{is even}$. The curves $\gamma_i \subset R_i$, $i = 1, 2$, are mapped each to
\begin{equation}
\text{one of the two rational components of the image curve in } \overline{\mathcal{M}}_g. \text{The situation is similar for curves in}
\end{equation}

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig1.png}
\caption{Members of $W_{\delta}(R)$ when $\delta$ is odd (left) and even (right).}
\end{figure}

\begin{equation}
\text{The condition } \delta \leq p - \epsilon \text{ guarantees that the normalization of the curves in } W_{d,\ell}(R) \text{ at the } \delta \text{ marked}
\end{equation}

\begin{equation}
\text{nodes is connected.}
\end{equation}

\begin{equation}
\text{The points } P \text{ and } Q \text{ in the picture are called the distinguished points of the curve in } W_{d,\ell}(R). \text{They satisfy}
\end{equation}

\begin{equation}
\text{the relation } \phi_{d,E}(P) = Q \text{ (equivalently, } \phi_{d,E}(Q) = P \text{ if } d \text{ is odd}). \text{Note that } P, Q \in \gamma_1
\end{equation}

\begin{equation}
\text{if } \delta \text{ is odd, whereas } P \in \gamma_1 \text{ and } Q \in \gamma_2 \text{ if } \delta \text{ is even.} 
\end{equation}
We will consider the pair of distinguished points as an element \( P + Q \in \text{Sym}^2(E) \) when \( d \) is odd, whereas we will consider the pair as an ordered element \((P, Q) \in E \times E\) when \( d \) is even, since we then can distinguish \( P \) and \( Q \), as \( P \in \gamma_1 \) and \( Q \in \gamma_2 \).

Recalling (8), we note that \( D_{d, E}(P) \) (equivalently, \( D_{d, E}(Q) \) if \( d \) is odd) is the (reduced) divisor of intersection points of the line chain of the curve in \( W_\delta(R) \) with \( E \).

**Proposition 4.2.** Assume \( 0 \leq \delta \leq p - 1 \). Then \( W_\delta(R) \) is a smooth, dense open subset of a component of dimension \( g = p - \delta \) of \( \overline{V}_{1, \delta}(R) \).

*Proof.* If \( \delta \) is even we denote by \( G_2^\delta \subset E \times E \) the graph of the translation by \( \frac{\delta}{2}(L_1 - L_2) \in \text{Pic}^0(E) \). Then we have surjective morphisms

\[
g_{p, \delta} : W_\delta(R) \longrightarrow \begin{cases} P + Q \in |\frac{\delta}{2}L_2 - \frac{\delta-1}{2}L_1| & \text{if} \ \delta \text{ is odd,} \\ (P, Q) \in G_2^\delta & \text{if} \ \delta \text{ is even.} \end{cases}
\]

Take any point \( \eta = P + Q \) or \( \eta = (P, Q) \) in the target of \( g_{p, \delta} \). Set \( D_\eta := D_{\delta, E}(P) \), which is an effective divisor of degree \( \delta + 1 \) on \( E \). Then \( D_{\eta} \) defines a unique line chain \( C_{\eta} \) on \( R \) and the fiber \( \gamma_{p, \delta}(\eta) \) is a dense, open subset of the projective space \(|L \otimes \mathcal{J}_{C_{\eta}}/R| \simeq |L' \otimes \mathcal{J}_{C_{\eta}}/R'| \simeq |L_{E'} \otimes \mathcal{O}_{E'}(-D_{\eta})|\). Since

\[
f := \dim(|L_{E'} \otimes \mathcal{O}_{E'}(-D_{\eta})|) = p - \delta - 1 \geq 0,
\]

then \( W_\delta(R) \) is irreducible of dimension \( f + 1 = p - \delta = g \). The smoothness of \( \overline{V}_{1, \delta}(R) \) at the points of \( W_\delta(R) \) follows, as usual, because the \( \delta \) marked nodes are non-disconnecting (cf. [CK] Rmk. 1.1)).

The same argument proves the following:

**Proposition 4.3.** Let \( d, \ell \) be non--negative integers such that \( 0 < \ell \leq 16 \). Set \( \delta = d + \ell \) and assume \( \delta \leq p \). Then \( W_{d, \ell}(R) \) is a union of smooth, open dense subsets of components of dimension \( g = p - \delta \) of \( \overline{V}_{1, \delta}(R) \).

The number of components of \( W_{d, \ell}(R) \) is \( \binom{16}{\ell} \), depending on the choice of subdivisors of degree \( \ell \) of \( T \).

We can finally consider the universal family \( \mathcal{W}_{d, \ell} \), irreducible by monodromy, parametrizing all pairs \((R, C)\) with \( R \in \mathfrak{S}_p \) and \( C \in W_{d, \ell}(R) \) with the map \( \mathcal{W}_{d, \ell} \rightarrow \mathfrak{S}_p \), whose general fiber is smooth of dimension \( g \), so that \( \dim(W_{d, \ell}) = g + 18 \). Note that \( \overline{V}_{1, \delta} \) is smooth along \( W_{d, \ell} \).

We will only be concerned with the cases \( \ell = 0 \) and \( \ell = 1 \), and mostly with \( \ell = 0 \).

**Definition 4.4.** For \( \delta \leq p \), we denote by \( \mathcal{V}_\delta^* \) the unique irreducible component of \( \overline{V}_{1, \delta} \) containing \( \mathcal{W}_\delta \) if \( \delta < p \) and \( W_{p - 1, 1} \) if \( \delta = p \). We denote by \( \mathcal{V}_\delta^*(S) \) the fiber of \( \mathcal{V}_\delta^* \) over \( S \in \mathcal{K}_p \). We denote by \( \psi_\delta^* : \mathcal{V}_\delta^* \rightarrow \overline{\mathcal{M}_g} \) and \( \phi_\delta^* : \mathcal{V}_\delta^* \rightarrow \overline{\mathcal{K}_p} \) the restrictions of \( \overline{V}_\delta \) and \( \overline{\mathcal{V}_\delta} \), respectively.

**Remark 4.5.** It is useful to notice that, for \( R \in \mathfrak{S}_p \) one has

\[
W_{\delta+1}(R) \subset \overline{W}_\delta(R), \quad \text{for} \ 0 \leq \delta \leq p - 1,
\]

and

\[
W_{\delta, 1}(R) \subset \overline{W}_{\delta, 0}(R) = \overline{W}_\delta(R), \quad \text{for} \ 0 \leq \delta \leq p - 1.
\]

Hence we also have

\[
\mathcal{V}_{\delta+1}^* \subset \mathcal{V}_\delta^*, \quad \text{for} \ 0 \leq \delta \leq p - 1,
\]

i.e., \( \mathcal{V}_\delta^* \) is fully complete (see Remark 2.1). It follows by monodromy that for general \( S \in \mathcal{K}_p \), every component of \( \mathcal{V}_\delta^*(S) \) is fully complete.
moduli of nodal curves on K3 surfaces

5. Special fibers of the moduli map

This section is devoted to the study of the dimension of the fibers of the moduli map $\psi_3^*$ over curves arising from reducible K3 surfaces.

For any $R' \in \mathcal{R}_p$, we define varieties of nodal curves analogously to Definition 4.1: we denote by $W_3(R')$ the set of reduced curves in $|\mathcal{L}_{R'}| = |\mathcal{O}_{R'}(1)|$ having exactly $\left\lfloor \frac{2}{3} \right\rfloor$ nodes on $R'_1 - E'$ and $\left\lceil \frac{2}{3} \right\rceil$ nodes on $R'_2 - E'$, so that they split of a total of $\delta$ lines, and such that the union of these $\delta$ lines is connected. Clearly, if $R' = \pi_p(R)$ for some $R = (R', T) \in \mathcal{S}_p$, then there is a dominant, injective map

$$W_3(R) \longrightarrow W_3(R')$$

mapping a curve in $W_3(R)$ to its isomorphic image on $R'$. The image of this map is precisely the set of curves in $W_3(R')$ not passing through any of the points in $T$.

The concepts of line chain, distinguished pair of points and curves $\gamma_i$ associated to an element of $W_3(R')$ are defined in the same way as for the curves in $W_3(R)$.

As above, we can consider the irreducible variety $\mathcal{W}_3^*$ parametrizing all pairs $(R', C')$ with $R' \in \mathcal{R}_p$ and $C' \in W_3(R')$ and there is an obvious (now surjective!) map $\mathcal{W}_3 \to \mathcal{W}_3^*$.

More generally, we may define a universal Severi variety of nodal curves with $\delta$ marked nodes on the smooth locus over $\mathcal{R}_p$, and we denote by $\mathcal{Y}_3^*_{\mathcal{R}_p}$ the component containing $\mathcal{W}_3^*$. There is a modular map $\psi_3^* : \mathcal{Y}_3^*_{\mathcal{R}_p} \to \overline{\mathcal{M}}_g$ as before. Moreover, defining $\mathcal{Y}_3^*_{\mathcal{S}_p}$ to be the open set of curves in $(\phi_3^*)^{-1}(\mathcal{S}_p) \cap \mathcal{Y}_3^*$ not containing any exceptional curves (note that $\mathcal{W}_3 \subset \mathcal{Y}_3^*_{\mathcal{S}_p}$), we have the surjective map

$$\pi_3 : \mathcal{Y}_3^*_{\mathcal{S}_p} \longrightarrow \mathcal{Y}_3^*_{\mathcal{R}_p}$$

sending $(C, R) \in \mathcal{Y}_3^*_{\mathcal{S}_p}$, where $C \subset R$ and $R = (R', T) \in \mathcal{S}_p$, to $(C', R')$, where $C'$ is the image of $C$ in $R'$ under the natural contraction map $R \to R'$.

Summarizing, we have the commutative diagrams

\begin{equation}
\begin{array}{c}
\mathcal{M}_g \\
\downarrow \\
\mathcal{Y}_3^*_{\mathcal{R}_p} \\
\downarrow \psi_3^* \\
\downarrow \pi_3 \\
\mathcal{Y}_3^*_{\mathcal{S}_p} \\
\downarrow \phi_3^* \\
\mathcal{S}_p \\
\downarrow \pi_p \\
\mathcal{K}_p
\end{array}
\end{equation}

and

\begin{equation}
\begin{array}{c}
\mathcal{W}_3 \\
\downarrow \\
\mathcal{Y}_3^*_{\mathcal{R}_p} \\
\downarrow \pi_3 \\
\mathcal{Y}_3^*_{\mathcal{S}_p} \\
\downarrow \phi_3^* \\
\mathcal{S}_p \\
\downarrow \pi_p \\
\mathcal{K}_p
\end{array}
\end{equation}

The fibers of $\pi_3$ are all 15-dimensional (isomorphic to a dense open subset of $|\mathcal{L}_{R'}^*|$). This shows that the fibers of $\psi_3^*$ over curves that come from reducible K3 surfaces have at least one component that has dimension at least 15. As $15 \geq 22 - 2g$ (the expected dimension of the general fiber) for $g \geq 4$, this makes it difficult to prove Theorem 4.1 directly by semicontinuity around elements of $\mathcal{Y}_3^*_{\mathcal{S}_p}$. We will circumvent this problem in the next section, to prove Theorem 4.1 for $m = 1$.

To this end we start by studying the fibers of the restriction of $\psi_3^*$ to $\mathcal{W}_3^*$. 

Proposition 5.1. Let $p \geq 3$ and $g \geq 3$. Then the general fiber of $(\psi_\delta')|_{\mathcal{W}_\delta'}$ has dimension at most $\max\{0,7-g\}$.

Proof. We will assume $\delta \geq 1$. The case $\delta = 0$ is easier, can be treated in the same way and details can be left to the reader.

Set $\mathbb{Q} = \mathbb{P}^1 \times \mathbb{P}^1$, with $\pi_i : \mathbb{Q} \to \mathbb{P}^1$ for $i = 1,2$ the projections to the factors. Define

$$Q_\delta^2 := \begin{cases} \text{Sym}^2(\mathbb{Q}) & \text{if } \delta \text{ is odd,} \\ \mathbb{Q}^2 & \text{if } \delta \text{ is even.} \end{cases}$$

and denote its elements by

$$(P,Q)_\delta := \begin{cases} P + Q \in \text{Sym}^2(\mathbb{Q}) & \text{if } \delta \text{ is odd,} \\ (P,Q) \in \mathbb{Q}^2 & \text{if } \delta \text{ is even.} \end{cases}$$

Step 1: A basic construction. Consider the locally closed incidence scheme

$$I_\delta \subset | - K_\mathbb{Q} | \times \text{Sym}^g(\mathbb{Q}) \times \mathbb{Q}_\delta^2$$

formed by all triples $(E,Q_1 + \cdots + Q_g,(P,Q)_\delta)$, with $E \in | - K_\mathbb{Q} |$, $Q_1,\ldots,Q_g \in E \subset \mathbb{Q}$, $P,Q \in E$, such that:

- $E$ is smooth;
- $Q = \phi_{\delta,E}(P)$ (equivalently, $P = \phi_{\delta,E}(Q)$ for odd $\delta$);
- $\pi_i(Q_1 + \cdots + Q_g)$ is reduced (in particular $Q_1,\ldots,Q_g$ are distinct);
- $\pi_i(D_{\delta,E}(P))$ and $\pi_i(Q_1 + \cdots + Q_g)$ have no points in common, for $i = 1,2$.

The divisor $D_{\delta,E}(P)$ is the one in Definition 3.2 with $L_i$ defined by the projection $\pi_i|_{E}$, for $i = 1,2$.

Recall that $D_{\delta,E}(P) = D_{\delta,E}(Q)$ when $\delta$ is odd.

The projection of $I_\delta$ to $| - K_\mathbb{Q} |$ is dominant, with general fiber of dimension $g + 1$, hence

$$\dim(I_\delta) = g + 9.$$ 

Next we define a surjective map

$$\vartheta_\delta : I_\delta \to \mathcal{W}_\delta'$$

as follows: let $\xi = (E,Q_1 + \cdots + Q_g,(P,Q)_\delta) \in I_\delta$ and set $D_\xi = D_{\delta,E}(P) + Q_1 + \cdots + Q_g$, which is a degree $p + 1$ divisor on $E$. Then $\Omega_{E}(D_\xi)$ defines, up to projective transformations, an embedding of $E$ as an elliptic normal curve $E' \subset \mathbb{P}^p$. The degree $2$ line bundles $L_i$ defined by the projections $\pi_i|_{E}$ define rational normal scrolls $R'_i \subset \mathbb{P}^p$, for $i = 1,2$, hence the surface $R' = R'_1 \cup R'_2$. The hyperplane section of $R'$ cutting out $D_\xi$ on $E$ defines a curve $C'_\delta \subset W_\delta(R')$. One defines $\vartheta_\delta(\xi) = C'_\delta$.

The map $\vartheta_\delta$ is surjective. Take a curve $C'$ on $R' = R'_1 \cup R'_2$ corresponding to a point of $\mathcal{W}_\delta'$ and consider $E = E' = R'_1 \cap R'_2$ together with the $2 : 1$ projection maps $f_i : E \to \gamma_i$, where $\gamma_i$ is the residual curve to the line chain of $C'$ on $R'_i$, cf. Definition 4.1. Up to choosing an isomorphism $\gamma_i \equiv \mathbb{P}^1$, the map $f_1 \times f_2$ embeds $E$ as an anticanonical curve in $\mathbb{Q}$. Let $Q_1,\ldots,Q_g$ be the unordered images of the intersection points $\gamma_1 \cap \gamma_2 \subset E$ and $(P,Q)_\delta$ be the pair of distinguished points, ordered if $\delta$ is even, and unordered if $\delta$ is odd. This defines a point $\xi = (E,Q_1 + \cdots + Q_g,(P,Q)_\delta) \in I_\delta$ such that $\vartheta_\delta(\xi) = C'$. This argument shows that the fibers of $\vartheta_\delta$ are isomorphic to $(\text{Aut}(|\mathbb{P}^1|))^2$.

We have the commutative diagram

$$\begin{array}{ccc} I_\delta & \xrightarrow{\vartheta_\delta} & \mathcal{W}_\delta' \\
\sigma_\delta \downarrow & & \downarrow (\psi_\delta'|_{\mathcal{W}_\delta'}) \\
\mathcal{M}_g & & \end{array}$$

where $\sigma_\delta$ is the composition map. It can be directly defined as follows. Take $\xi = (E,Q_1 + \cdots + Q_g,(P,Q)_\delta) \in I_\delta$. Call $\Gamma_1$ and $\Gamma_2$ the two $\mathbb{P}^1$s such that $Q = \Gamma_1 \times \Gamma_2$. We have the
g points \( P_{i,j} = \pi_i(Q_j) \in \Gamma_i \), for \( i = 1, 2 \) and \( 1 \leq j \leq g \). If \( \delta \) is even, we get additional points \( P_{1,g+1} = \pi_1(P) \in \Gamma_1 \) and \( P_{2,g+1} = \pi_2(Q) \in \Gamma_2 \). If \( \delta \) is odd we get two more points \( P_{1,g+1} = \pi_1(P), P_{1,g+2} = \pi_1(Q) \) on \( \Gamma_1 \). Then \( \Gamma = \sigma_\delta(\xi) \) is obtained:

- by gluing \( \Gamma_1 \) and \( \Gamma_2 \) with the identification of \( P_{1,j} \) with \( P_{2,j} \), for \( 1 \leq j \leq g + 1 \), if \( \delta \) is even;
- by first identifying \( P_{1,g+1} \) and \( P_{1,g+2} \) on \( \Gamma_1 \) and then gluing with \( \Gamma_2 \) by identifying the points \( P_{1,j} \) with \( P_{2,j} \), for \( 1 \leq j \leq g \), if \( \delta \) is odd.

**Step 2: The fiber.** By Step 1, to understand the general fiber of \( (\psi_\delta)_*|_{W_\delta} \), it suffices to understand the general fiber of \( \sigma_\delta \). This is what we do next.

Consider a general element \( \Gamma \) in \( \psi_\delta(W'_\delta) \). Then \( \Gamma = X_1 \cup X_2 \) is a union of two rational components (recall Figure 13 this is where we use the hypothesis \( g \geq 3 \)), and we denote by \( \Gamma_1 \) and \( \Gamma_2 \) their normalizations, both isomorphic to \( \mathbb{P}^1 \), so that \( \Gamma_1 \times \Gamma_2 \simeq \mathbb{Q} \), up to the action of \( (\text{Aut}(\mathbb{P}^1))^2 \) (which will contribute to the fiber of \( \sigma_\delta \)).

If \( \delta \) is odd, then \( X_2 = \Gamma_2 \simeq \mathbb{P}^1 \), whereas \( X_1 \) has one node. On \( \Gamma_1 \times \Gamma_2 \simeq \mathbb{Q} \) we have the \( g \) points \( Q_j = (x_j, y_j) \), for \( 1 \leq j \leq g + 1 \). Then the \( \sigma_\delta \)-fiber of \( \Gamma \) is the union of \( g + 1 \) components obtained in the following way. Choose an index \( j \in \{1, \ldots, g + 1\} \), e.g., \( j = g + 1 \). Then the corresponding component of the fiber consists of all \( (E, Q_{1} + \cdots + Q_{g}, (P, Q)_{\delta}) \in I_\delta \) such that \( E \) is a smooth curve in \( |− K_Q \otimes J_{Q_1 \cup \cdots \cup Q_g}| \), satisfying the additional condition

\[
Q = \phi_{\delta, E}(P) \quad \text{and} \quad \pi_1(P) + \pi_1(Q) = z_1 + z_2.
\]

If \( \delta \) is even, then \( X_1 \simeq \Gamma_1 \simeq \mathbb{P}^1 \), for \( i = 1, 2 \), and the intersection points between the two components yield \( g + 1 \) points \( Q_j = (x_j, y_j) \in \mathbb{Q} \), for \( 1 \leq j \leq g + 1 \). Then the \( \sigma_\delta \)-fiber of \( \Gamma \) is the union of \( g + 1 \) components obtained in the following way. Choose an index \( j \in \{1, \ldots, g + 1\} \), e.g., \( j = g + 1 \). Then the corresponding component of the fiber consists of all \( (E, Q_{1} + \cdots + Q_{g}, (P, Q)_{\delta}) \in I_\delta \) such that \( E \) is a smooth curve in \( |− K_Q \otimes J_{Q_1 \cup \cdots \cup Q_g}| \), satisfying the additional condition

\[
Q = \phi_{\delta, E}(P) \quad \text{and} \quad \pi_1(P) = x_{g+1}, \pi_2(Q) = y_{g+1}.
\]

Either way, up to the action of \( (\text{Aut}(\mathbb{P}^1))^2 \), the \( \sigma_\delta \)-fiber of \( \Gamma \) is contained in \( |− K_Q \otimes J_{Q_1 \cup \cdots \cup Q_g}| \), and, by assumption, we know that there are smooth curves \( E \) in \( |− K_Q \otimes J_{Q_1 \cup \cdots \cup Q_g}| \). Hence \( \dim(|− K_Q \otimes J_{Q_1 \cup \cdots \cup Q_g}|) = \max\{0, 8 − g\} \), which proves the assertion if \( g \geq 8 \).

To finish the proof, we have to prove that for \( g \leq 7 \) and for a general choice of \( (E, Q_{1} + \cdots + Q_{g}, (P, Q)_{\delta}) \in I_\delta \), condition (11) or (12) defines a proper, closed subscheme of \( |− K_Q \otimes J_{Q_1 \cup \cdots \cup Q_g}| \).

Let \( \delta \) be odd. Let us fix a general point \( P \in \mathbb{Q} \) and let us take a general curve \( E \) in \( |− K_Q \otimes J_P| \), which neither contains \( Q \) nor its conjugate on \( E \) via \( L_1 \). The curves \( E \) and \( E' \) intersect at 7 distinct points off \( P \). Choose \( Q_1, \ldots, Q_g \) among them. Then \( E, E' \in |− K_Q \otimes J_{Q_1 \cup \cdots \cup Q_g}| \), and \( E \) satisfies (11) (with \( z_1 = \pi_1(P), z_2 = \pi_1(Q) \)), whereas \( E' \) does not.

The proof for \( \delta \) even is similar and can be left to the reader.

---

**Corollary 5.2.** Let \( p \geq 3 \), \( g \geq 3 \) and \((R_0, C_0) \in W_\delta \). Set \((R_0', C_0') := \pi_\delta((R_0, C_0)) \in W'_\delta \) and \( \Gamma_0 := (\psi'_\delta)(R_0, C_0) = (\psi'_\delta)(R_0', C_0') \), cf. (9) and (11).

There is at least one component \( V_0 \) of \((\psi'_\delta)^{-1}(\Gamma_0)\) containing the whole fiber \( \pi^{-1}_\delta((R_0, C_0')) \), and any such \( V_0 \) satisfies

(i) \( V_0 \subseteq W'_\delta \);
(ii) \( \dim(\pi_\delta(V)) \leq \max\{0, 7 − g\} \).

**Proof.** (a) As mentioned above, the fiber \( \pi^{-1}_\delta((R_0, C_0')) \) is isomorphic to a dense open subset of \(|T_{R_0'}^1|\), and it is contained in \((\psi'_\delta)^{-1}(\Gamma_0) \cap W_\delta \). Hence there is at least one component \( V_0 \) of \((\psi'_\delta)^{-1}(\Gamma_0)\) containing the whole fiber \( \pi^{-1}_\delta((R_0, C_0')) \) and any such contains some \((R, C) \in W_\delta \) such that \( R = (R', T) \) for a general element \( T \in |T_{R'}^1| \). Due to the generality of \( T \), the surface \( R \) lies off the closure of the Noether–Lefschetz locus in \( K_p \) (see [CLMI]). Thus, there can be no element \((S, C) \in V_0 \) with
S smooth, because \( \text{Pic}(S) \cong \mathbb{Z} \), whereas \( C \) is reducible, being birational to \( \Gamma \). So, if \( (S, C) \in V_0 \), then \( S \in \mathcal{G}_p \). Thus \( V_0 \subseteq \mathcal{V}_p^* \cap (\phi_p^*)^{-1}(\mathcal{G}_p) \).

If \( \dim(V_0) = 0 \), there is nothing more to prove. So assume there is a disc \( \mathcal{D} \subset V_0 \) parametrizing pairs \( (R_t, C_t) \) such that \( \psi_d^*(R_t, C_t) = \Gamma_0 \), with \( R_t \in \mathcal{G}_p \) and \( C_t \in V_0(R_t) \), for all \( t \in \Delta \), with \( R_0 = R \). Consider \( R'_t = \pi_p(R_t) \in \mathcal{R}_p \), and the divisor \( T_t \) on \( E'_t \) along which the modification \( \pi_t : R_t \to R'_t \) takes place. As \( \pi_0(C_0) \cap T_0 = \emptyset \), we may assume that \( \pi_t(C_t) \cap T_t = \emptyset \) for all \( t \in \Delta \). Hence \( C_t \simeq \pi_t(C_t) \subset R'_t \), which has \( \delta \) marked nodes on the smooth locus of \( R'_t \), for all \( t \in \Delta \). These nodes lie on lines contained in \( \pi_t(C_t) \). As \( \pi_0(C_0) \) contains precisely one connected chain of lines (i.e., its line chain, see Definition 4.1), the same holds for \( \pi_t(C_t) \) for all \( t \in \Delta \). Hence \( \pi_t(C_t) \in W_\delta(R'_t) \), so that \( C_t \in W_\delta(R_t) \). This proves (i). Assertion (ii) then follows from Proposition 5.1. \( \square \)

6. The moduli map for \( m = 1 \)

In this section we will prove the part of Theorem 6.1 concerning the case \( m = 1 \). We circumvent the problem of the “superabundant” fibers of \( \psi'_d \) remarked in the previous section by passing to (a component of) the universal Severi variety over the Hilbert scheme \( \mathcal{H}_p \), which is the pullback of the universal Severi variety \( \mathcal{V}_p^* \) over the locus of smooth, irreducible \( K3 \) surfaces.

Proposition 6.1. The map \( \psi'_d \) is generically finite for \( g \geq 15 \).

Proof. There exists a dense, open subset \( \mathcal{H}_p^o \subset \mathcal{H}_p \), disjoint from \( \mathcal{R}_p' \), such that there is a dominant moduli morphism

\[
\mu_p : \mathcal{H}_p^o \longrightarrow \mathcal{K}_p \subset \mathcal{K}_p.
\]

We consider the cartesian diagram

\[
\begin{array}{ccc}
\mathcal{M}_p^o & \xrightarrow{\nu_p} & \mathcal{V}_p^* \\
\Phi_p^o \downarrow & & \downarrow \phi_p^o \\
\mathcal{H}_p^o & \xrightarrow{\mu_p} & \mathcal{K}_p \\
\end{array}
\]

which defines, in the first column, (a component of) a universal Severi variety over \( \mathcal{H}_p^o \). Accordingly, we can then consider the moduli map \( \Psi_p^o \) fitting in the commutative diagram

\[
\begin{array}{ccc}
\mathcal{M}_p^o & \xrightarrow{\nu_p} & \mathcal{V}_p^* \\
\Psi_p^o \downarrow & & \downarrow \psi_p^o \\
\mathcal{M}_p^o & \xrightarrow{\psi_p} & \mathcal{M}_p \\
\end{array}
\]

and it suffices to prove that the general fiber of \( \Psi_p^o \) has dimension \( p^2 + 2p \). We prove this by semicontinuity. To this end we introduce a partial compactification \( \mathcal{M}_p^c \) of \( \mathcal{M}_p^o \) fitting into the commutative diagram

\[
\begin{array}{ccc}
\mathcal{M}_p^c & \xrightarrow{\phi_p} & \mathcal{M}_p \\
\Phi_p^c \downarrow & & \downarrow \phi_p \\
\mathcal{M}_p^o & \xrightarrow{\phi_p} & \mathcal{M}_p \\
\end{array}
\]

The fiber of \( \Phi_p \) over any element \( R' \in \mathcal{R}_p^c \subset \mathcal{H}_p \) contains the varieties \( W_\delta(R') \) by Proposition 4.2 (and Definition 4.3). Possibly after substituting \( \mathcal{M}_p^c \) with a dense, open subset (still containing the varieties \( W_\delta(R') \)), we have a natural moduli map \( \Psi_p^c : \mathcal{M}_p^c \to \mathcal{M}_p^c \) extending \( \Psi_p^c \).

Take general \( R' \in \mathcal{R}_p^c \) and \( C'_0 \in W_\delta(R') \) and set \( \Gamma := \Psi_p^c((R', C'_0)) = \psi_p^c((R', C'_0)), \) cf. (9) and (10). By semicontinuity, it suffices to prove that the fiber \( \psi_p^c^{-1}(\Gamma) \) has dimension \( p^2 + 2p \) in a neighborhood
of \((R', C'_0)\). In other words, letting \(W_0 \subset \Psi_\delta^{-1}(\Gamma)\) be the component containing \((R', C'_0)\), it suffices to prove that \(\dim(W_0) = p^2 + 2p\).

If \(\Phi_\delta(W_0) \subset \mathcal{R}'_p\), then under the \(\operatorname{PGL}(p + 1, \mathbb{C})\)-quotient map

\[
\Phi_\delta^{-1}(\mathcal{R}'_p) \xrightarrow{\psi_\delta} \mathcal{V}_{\delta, \mathcal{R}'_p} \\
\mathcal{R}'_p \longrightarrow \mathcal{R}_p
\]

the variety \(W_0 \subset \Phi_\delta^{-1}(\mathcal{R}'_p)\) is mapped (with fibers of dimension \(p^2 + 2p\)) to a variety \(V'_0 \subseteq (\psi_\delta)^{-1}(\Gamma) \subseteq \mathcal{V}_{\delta, \mathcal{R}'_p}\) intersecting \(W'_0\). The component \(V'_0\) of \((\psi_\delta)^{-1}(\Gamma)\) containing \((\pi_\delta)^{-1}(V'_0)\) (cf. \([9]\) and \([10]\)) lies in \(W_\delta\) and satisfies \(\dim(\pi_\delta(V'_0)) = 0\) by Corollary \([6,2]\). Therefore \(\dim(V'_0) = 0\) by the surjectivity of \(\pi_\delta\), hence \(\dim(W_0) = p^2 + 2p\), as desired.

The remaining case to be treated is when \(\Phi_\delta(W_0) \not\subseteq \mathcal{R}'_p\), that is, \(W_0\) contains a one-parameter family containing \((R', C'_0)\) and such that its general point is a pair \((S, C)\) with \(S \not\in \mathcal{R}'_p\).

If \(S\) is singular along a curve, then, as it degenerates to \(R'\), it must have double points along a smooth elliptic curve of degree \(p + 1\), hence its general hyperplane sections are of arithmetic genus \(p\) with \(p + 1\) double points, so they must be reducible. It follows that \(S \in \mathcal{R}'_p\) a contradiction.

If \(S\) has (at most) isolated singularities, then, as it degenerates to \(R'\), it must have isolated rational double points, so it is a projective model of a smooth \(K3\) surface. Thus we may find a one-dimensional family \((S_t, C_t)_{t \in \mathbb{D}}\) of points in \(\mathcal{V}'_0\) parametrized by the disc \(\mathbb{D}\), such that \(R := S_0 \in \mathfrak{S}_p\) and \(\pi_p(R) = R'\) (that is, \(R\) is a birational modification of \(R'\)), \(C_0\) is the total transform of a \(C'_0\) on \(R'\), \(\psi_\delta^*(S_t, C_t) = \Gamma\) for all \(t \in \mathbb{D}\), and \(S_t\) is smooth for all \(t \in \mathbb{D} - \{0\}\).

Since \(\Gamma\) has \(g + 1\) nodes, the curves \(C_t\) (including \(C_0\)) have at least \(g + 1\) unmarked nodes mapping to the nodes of \(\Gamma\) through the process of partial normalization at the \(\delta\) marked nodes and possibly semi-stable reduction. On \(C_0\) they must lie on the smooth locus of \(R\). (Indeed, it is well-known, and can be verified by a local computation, that a node of a limit curve lying on \(E\) smooths as \(R\) smooths, cf. e.g. [Ch., G §2] or [1, Pf. of Lemma 3.4].) This means that these nodes of \(C_0\) lie on the exceptional curves of the morphism \(R \to R'\). Therefore the \(g\) intersection points \(\gamma_1 \cap \gamma_2\) (all lying in the elliptic curve \(E'\)) plus one (at least) among the intersections of the line cycle of \(C'_0\) with the elliptic curve \(E'\) are contained in the divisor \(T \in |T_{R'}|\). Hence we have \(g + 1 \leq \deg(W) = 16\). This is a contradiction if \(g \geq 16\).

Assume finally that \(g = 15\). The above argument shows that the \(g\) intersection points \(\gamma_1 \cap \gamma_2\) plus a point \(x = \phi_{k, E'}(P)\) are the zeroes of a section of \(T_{R'}\), for some integer \(0 \leq k \leq \delta\) (cf. Definition \([3,2]\).

If \(\delta\) is odd, the \(g\) intersection points \(\gamma_1 \cap \gamma_2\) are in \(|L_{E'} \otimes (L^*_2)^{\delta+1}|\). It follows that

\[
\mathcal{O}_{E'}(x) \simeq T^{1}_{R'} \otimes L^*_2 \otimes L_2^{\delta+1}
\]

which implies that \(x\) is uniquely determined, hence also the line chain of \(C'_0\) is, contradicting the fact that \(C'_0\) is general in \(W_\delta(R')\).

If \(\delta\) is even, the \(g\) intersection points \(\gamma_1 \cap \gamma_2\) are in

\[
|L_{E'} \otimes \mathcal{O}_{E'}(-P) \otimes (L^*_1)^{\delta+1}| = |L_{E'} \otimes \mathcal{O}_{E'}(-Q) \otimes (L^*_2)^{\delta+1}|.
\]

Hence

\[
(13) \quad \mathcal{O}_{E'}(x) \simeq T^{1}_{R'} \otimes L^*_2 \otimes \mathcal{O}_{E'}(P) \otimes L^*_1 \otimes T^{1}_{R'} \otimes L^*_2 \otimes \mathcal{O}_{E'}(Q) \otimes L^*_2^{\delta+1}.
\]

If \(k\) is odd, combining \([10]\) (with \(y\) replaced by \(P\)) with \([10,13]\) yields

\[
\mathcal{O}_{E'}(2P) \simeq L_{E'} \otimes (T^{1}_{R'})^* \otimes L_2^{\delta+1} \otimes (L^*_1)^{\delta+1+1}.
\]
which implies only finitely many choices for \( P \), and we get a contradiction as before. If \( k \) is even, then \( \mathcal{F} \), with \( x \) and \( y \) replaced by \( P \) and \( y \) respectively, combined with \( \mathbf{13} \) yields

\[
L_{E'} \simeq T_{R'}^\ast \otimes L_2^\otimes \frac{b}{2} \otimes L_1^\otimes \frac{b}{2},
\]

which, together with \( \mathbf{13} \) gives a non-trivial relation between \( L_{E'}, L_1 \) and \( L_2 \), a contradiction. \( \square \)

The following lemma is trivial and its proof can be left to the reader:

**Lemma 6.2.** Let \( \mathcal{V} \) be an irreducible, \((\delta + 1)\)-complete, component of \( \mathcal{V}_{m,\delta} \). If \( \psi_{m,\delta|\mathcal{V}} \) is dominant, then there is an irreducible component \( \mathcal{W} \subseteq \mathcal{V}_{m,\delta + 1} \) such that \( \psi_{m,\delta + 1|\mathcal{W}} \) is also dominant.

Using this lemma, and taking into account Remark \( \mathbf{4.5} \) to prove the dominance assertion for \( m = 1 \) in Theorem \( \mathbf{1.1} \) it suffices to prove that \( \psi^\ast_{\delta} \) is dominant for \( g = 7 \). We actually prove more (notation as in Proposition \( \mathbf{5.1} \)).

**Proposition 6.3.** For \( p \geq 3 \) and \( 3 \leq g \leq 7 \), the map \( \psi^\ast_{\delta} \) is dominant.

**Proof.** We use the same notation as in the proof of Proposition \( \mathbf{6.1} \). It suffices to prove that the general fiber of \( \Psi \) has dimension \( p^2 + 2p + 22 - 2g \).

Let \( \Psi_{\delta}^{-1}(\Gamma) \) be the fiber over \( [\Gamma] \in \mathbb{M}_g \), where \( \Gamma := \Psi_{\delta}((R', C'_0)) \) with \( R' \in \mathcal{R}'_p \) and \( C'_0 \in W_\delta(R') \) general points. We are interested in the reduced tangent cone \( \mathfrak{N} \) to \( \Psi_{\delta}^{-1}(\Gamma) \) at \( (R', C'_0) \). In order to provide an upper bound for the dimension of \( \mathfrak{N} \), we consider the intersection \( \Psi_{\delta}^{-1}(\Gamma) \cap \Phi_{\delta}^{-1}(R'_p) \) with its reduced induced scheme structure. By the generality of \( R' \) and \( C'_0 \), the tangent space \( \mathfrak{T} \) to this variety at \( (R', C'_0) \) has dimension \( \leq p^2 + 2p + \max\{0, 7 - g\} \), arguing as in the proof of Proposition \( \mathbf{6.1} \) using Corollary \( \mathbf{5.2} \). By standard deformation theory, we have that \( \mathfrak{T} \) is exactly the intersection of \( \mathfrak{N} \) with the kernel of the linear map

\[
L : T_{\Psi_{\delta}^{-1}(\Gamma),(R', C'_0)} \longrightarrow T_{\mathfrak{N}, R'},
\]

defined by composing the differential \( d_{(R', C'_0)}(\Phi_{\delta}|_{\Psi_{\delta}^{-1}(\Gamma)}) \) of \( \Phi_{\delta} \) at \( (R', C'_0) \) with the map \( \mathbf{3} \). By the proof of Proposition \( \mathbf{6.1} \) we have that \( \text{dim}(\mathfrak{N}) \) has dimension at most \( 16 - g - 1 = 15 - g \). Hence we deduce that

\[
\text{dim}(\mathfrak{N}) \leq p^2 + 2p + \max\{0, 7 - g\} + 15 - g = p^2 + 2p + \max\{15 - g, 22 - 2g\},
\]

which is bounded above by \( p^2 + 2p + 22 - 2g \) precisely when \( g \leq 7 \), proving the proposition. \( \square \)

At this point, the proof of Theorem \( \mathbf{1.1} \) for \( m = 1 \) is complete.

7. More limits of nodal curves on reducible \( K3 \) surfaces

In order to treat the case \( m > 1 \) we need to consider more limits of nodal curves on surfaces in \( \mathfrak{S}_p \). They are similar to the ones constructed by X. Chen in [Ch, §3.2].

**Definition 7.1.** For each \( m \geq 1 \), \( R \in \mathfrak{S}_p \) and the corresponding \( R' := \pi_p(R) \in \mathcal{R}_p \), we define \( U'_m(R') \subset |mL_{R'}| \) to be the locally closed subset of curves of the form

\[
C' = C'_1 \cup C'_2 \cup \cdots \cup C'_{m - 1} \cup C'_m \cup C'_1 \cup C'_2 \cup \cdots \cup C'_{m - 1} \cup C'_m
\]

where:

- \( C'_i \subset R'_i \) for \( i = 1, 2 \) and \( 1 \leq j \leq m - 1 \);
- \( C'_i \subset |\sigma| \) for \( 1 \leq j \leq m - 1 \) and \( C'_m \subset |\sigma_1 + mlF_i| \) if \( p = 2l + 1 \) is odd;
- \( C'_i \subset |\sigma_i + F_i| \) for \( 1 \leq j \leq m - 1 \) and \( C'_m \subset |\sigma_1 + (ml - m + 1)F_i| \) if \( p = 2l \) is even;
- there are points \( P, Q_0, \ldots, Q_{2m}, Q \) on \( E' \) such that

\[
C'_j \cap E' = Q_{2j - 2} + Q_{2j - 1}, \quad C'_j \cap E' = Q_{2j - 1} + Q_{2j}, \quad \text{for} \quad 1 \leq j \leq m - 1
\]

\[
C'_m \cap E' = 2mlQ + P + Q_{2m - 2}, \quad C'_m \cap E' = 2mlQ + P + Q_0,
\]
if $p$ is odd, and

$$C_j^1 \cap E' = Q_{2j-2} + 2Q_{2j-1}, \quad C_j^2 \cap E' = 2Q_{2j-1} + Q_{2j}, \quad 1 \leq j \leq m - 1$$

$$C_m^1 \cap E' = (2ml - 2m + 1)Q + P + Q_{2m-2}, \quad C_m^2 \cap E' = (2ml - 2m + 1)Q + P + Q_0,$$

if $p$ is even.

We denote by $U_m(R) \subset |mL_R|$ the set of total transforms of the curves in $U'_m(R')$ passing through two points of $T$ different from $Q$.

We remark that the curves in $U'_m(R')$ (or $U_m(R)$) are tacnodal at $Q$ and, for even $p$, also at $Q_i$ with $i$ odd. Figure 7 depicts a member of $U'_3(R')$ for odd $p$. The intersections on $E'$ (which are not all shown in the figure) are marked with dots and are all transversal except at $Q$. The case of $p$ even is similar, except that the intersections at $Q_1$ and $Q_3$ are tangential.

**Lemma 7.2.** We have $\dim(U'_m(R')) = 2$ and $\dim(U_m(R)) = 0$.

**Proof.** It is easily seen that any of the three maps $U'_m(R') \to E^2$ mapping $C'$ to $(Q, P), (Q, Q_0)$ and $(Q, Q_{2m-2})$, respectively, is an isomorphism, proving the lemma. \hfill \Box

**Proposition 7.3.** For any $m \geq 1$, the curves in $U_m(R)$ deform to a set $U_m(S)$ of rational nodal curves in $|mL|$ on the general $[S, L] \in K_p$.

More precisely, there is a 19-dimensional subvariety $U_m \subset V_{m,p(m)}$ with a partial compactification $\overline{U}_m \to \overline{K}_p$ whose fiber over $R \in \mathcal{S}_p$ is $U_m(R)$ and whose fiber $U_m(S)$ over a general $S \in K_p$ is a nonempty finite subset of $V_{m,p(m)}(S)$.

**Proof.** This follows the lines of proof in [Ch. §3.2] or [GK. Proof of Thm. 1.1]. For the reader’s convenience, we outline here the main ideas, without dwelling on details.

Take a curve $C \in U_m(R)$ that is the total transform of a general curve $C' \in U'_m(R')$.

If $p = 2l + 1$ is odd, then $C$ has a total of $2lm(m - 1) + 2$ nodes on the smooth locus of $R$:
- two nodes occur on the two exceptional curves of $\mathfrak{e}$ that $C$ contains;

**Figure 2.** Sketch of a member of $U'_3(R')$ when $p$ is odd.
the remaining nodes are the points $C_j^i \cap C_m^i$ for $i = 1, 2, 1 \leq j \leq m - 1$, a total of $2(m - 1)$ times

$$C_j^i \cdot C_m^i = \alpha_i \cdot (\alpha_i + mF_i) = ml.$$ 

Moreover, the point $Q$ on $C$ is a $2ml$–tacnode, which can be (locally) deformed to $2ml - 1$ nodes, for a general deformation of $R$ to a K3 surface (see [Ra], [CH] Sect. 2.4] or [GK, Thm 3.1] for a generalization of this result). Since, by construction, $C$ does not contain subcurves lying in $[hL_R]$ for any $1 \leq h \leq m - 1$ (this is like saying that the aforementioned singularities of $C$ are non–disconnecting), one checks that, for a general deformation of $[R, L_R]$ to a K3 surface $[S, L]$, the curve $C$ deforms to an irreducible, nodal curve in $[mL]$ with a total of

$$\left[2mn(m-1) + 2\right] + \left[2m - 1\right] = 2m^2 + 1 = p(m)$$

nodes, as claimed.

The case $p$ even is similar and can be left to the reader. \hfill \Box

8. Domination of the moduli map for $m \geq 2$

In this section we will use Proposition 6.3 and the curves in Definition 7.1 to prove part $(A)$ of Theorem 1.1 for $m \geq 2$. We assume $p \geq 3$ in the whole section.

The following Proposition fixes a gap in [CK1, Prop. 1.2], whose proof is incomplete, cf. [CK2]:

**Proposition 8.1.** Let $S \in \mathcal{K}_p$ be general and $m \geq 1$ be an integer. Let $V$ be a fully complete component of $V_{m, \delta}(S)$. Then the moduli map

$$\psi_{\delta}|_V : V \to \overline{M}_{g_m, \delta}$$

is generically finite. In particular this applies to any component of $V_{\delta}^*(S)$.

**Proof.** By fully completeness, we have the filtration

$$\emptyset \neq V_{[m, \delta, p]}(S) \cap V \subset V_{[m, \delta, p-1]}(S) \cap V \subset \ldots \subset V_{[m, \delta, 1]}(S) \cap V \subset V$$

Set $V_i = \psi_\delta(V_{[m, \delta, p-i]}(S) \cap V)$, for $0 \leq i \leq g_m - 1$, and $V_{g_m, \delta} = \psi_\delta(V)$. Then we have

$$\emptyset \neq V_0 \subset V_1 \subset \ldots \subset V_{g_m, \delta} \subset V_{g_m, \delta}.$$

Since $\dim(V_0) = 0$, we have $\dim(V_{g_m, \delta}) \geq g_m, \delta$, which yields the first assertion. The final assertion follows by Remark 1.5 \hfill \Box

Next we need a technical construction and a lemma. Consider the commutative diagram

$$
\begin{array}{ccc}
V_{\delta}^* & \xrightarrow{\psi_{\delta}^*} & \overline{M}_{g_1, \delta} \\
\phi_{\delta} \downarrow & & \downarrow \varphi_{\delta} \\
\mathcal{K}_p & \xrightarrow{\psi_{\delta}} & \overline{M}_{g_1, \delta}
\end{array}
$$

where the triangle is the Stein factorization of $\psi_{\delta}^*$. We will abuse notation and we will identify an element $\Gamma \in \overline{M}_{g_1, \delta}$ with its image in $\overline{M}_{g_1, \delta}$.

Assume $g_1, \delta \leq 7$. By Proposition 6.3, given a general $\Gamma \in \text{Im}(\overline{\psi}_{\delta})$, the fiber $\overline{\psi}_{\delta}^{-1}(\Gamma)$ is irreducible of dimension

$$\dim(\overline{\psi}_{\delta}^{-1}(\Gamma)) = \dim(V_{\delta}^*) - \dim(\overline{M}_{g_1, \delta}) = 22 - 2g_1, \delta.$$ 

Set $T(\Gamma) := \phi_{\delta}^{-1}(\overline{\psi}_{\delta}^{-1}(\Gamma))$ which is irreducible and, by Proposition 8.1, one has

$$\dim(T(\Gamma)) = \dim(\overline{\psi}_{\delta}^{-1}(\Gamma)) = 22 - 2g_1, \delta.$$

Consider a general $S \in \mathcal{K}_p$. For any pair $(s, a)$ of positive integers, with $s \leq 4$ (hence $g_1, \delta + s \leq 11$), there are on $S$ finitely many curves of the form

$$Z = B_1 + \cdots + B_{s-1} + B_s, \text{ with } B_i \in V_{\delta}^*(S), \text{ for } 1 \leq i \leq s - 1, \text{ and } B_s \in U_a(S),$$

where $U_a$ is the set of}$\psi_{\delta}$-unobstructed curves in $V_{\delta}^*(S)$, with $a \geq 1$. By fully completeness, we have the filtration

$$\emptyset \neq V_{[g_1, \delta, p]}(S) \cap V \subset V_{[g_1, \delta, p-1]}(S) \cap V \subset \ldots \subset V_{[g_1, \delta, 1]}(S) \cap V \subset V$$

Set $V_i = \psi_\delta(V_{[g_1, \delta, p-i]}(S) \cap V)$, for $0 \leq i \leq g_1 - 1$, and $V_{g_1, \delta} = \psi_\delta(V)$. Then we have

$$\emptyset \neq V_0 \subset V_1 \subset \ldots \subset V_{g_1, \delta} \subset V_{g_1, \delta}.$$
where $U_a(S)$ is as in Proposition 6.3 (cf. Definition 6.1).

Given a general $(S, C) \in \mathcal{V}_g^1$, set $\Gamma := \tilde{\psi}_C(S, C)$, which is the (class of the) normalization $\tilde{C}$ of $C$. For all $S' \in T(\Gamma)$, we have a morphism $\Gamma \to S'$, whose image we denote by $C'$, and each of the curves $B_i$ on $S'$ cuts out on $C'$ a divisor, which can be pulled-back to a divisor $g_i$ on $\Gamma$, for $1 \leq i \leq s$. Consider the following variety, parametrized by the points in $T(G)$

$$\mathcal{J} = \{(d_1, \ldots, d_s) \in (\text{Sym}^2(\Gamma))^s | d_1 \leq g_i, \text{ for } 1 \leq i \leq s \} \subset T(\Gamma)$$

**Lemma 8.2.** In the above setting:

(i) $\mathcal{J} = (\text{Sym}^2(\Gamma))^s$;

(ii) for the general $S' \in T(\Gamma)$ the curves of the form $C' + Z$ (with $Z$ as in (15)) are nodal.

**Proof.** We specialize $(S, C) \in \mathcal{V}_g^1$ to $(R_0, C_0) \in \mathcal{W}_g$, set $\Gamma := \psi_C(S, C) \in \mathcal{M}_{g,1}$, and $\Gamma_0 = \psi_C(R_0, C_0) \in \mathcal{M}_{g,1}$, and prove the assertions (i) and (ii) in the limit situation for $\Gamma_0$ and $T(\Gamma_0)$.

Each $S' \in T(\Gamma_0)$ is such that $S'$ contains a curve $C' \in \mathcal{V}_g^1(S')$ with $\psi_C(S', C') = \Gamma_0$. By the proof of Proposition 6.3 all components of $T(\Gamma_0) = \phi^*(\tilde{\psi}^{-1}(\Gamma_0))$ have dimension at most $22 - 2g_{1,\delta}$. By (14) we conclude that all components of $T(\Gamma_0)$ have dimension precisely $22 - 2g_{1,\delta}$. In particular, this applies to the component $T_0 := \phi^*(V_0)$, cf. (9), where $V_0$ is a component of $(\psi_C^{-1}(\Gamma_0))$ satisfying the conditions in Corollary 5.2. We recall that $V_0$ contains $(R_0, C_0)$ and satisfies $\dim(\pi_0(V_0)) \leq \max\{0, 7 - g_{1,\delta}\}$ by Corollary 5.2(b), and $T_0$ lies in the specialization of $T(G) = \phi^*(\tilde{\psi}^{-1}(\Gamma))$. Therefore, recalling again (9), we have

$$\dim(\pi_0(T_0)) = \dim(\pi_0(\phi^*(V_0))) \leq \dim(\pi_0(V_0)) \leq \max\{0, 7 - g_{1,\delta}\}$$

and the general fiber of $\pi|_{T_0}$ has dimension

$$f \geq \dim(T_0) - \max\{0, 7 - g_{1,\delta}\} = \dim(T_0) - \max\{0, 7 - g_{1,\delta}\} = \min\{22 - 2g_{1,\delta}, 15 - g_{1,\delta}\} \geq 2s$$

(remember the assumptions on $s$ and $g_{1,\delta}$).

Then, inside $T_0$ we can find a subscheme $\mathcal{T}_0$ of dimension at least $2s$ consisting solely of surfaces that are birational modifications of a fixed $R'$ and it is therefore parametrized by a family $\mathcal{X}$ (of dimension at least $2s$) of divisors in $\mathcal{T}_0$. For $R$ corresponding to the general point of $\mathcal{X}$, the curves in $V_0^p(R)$ are limits of rational nodal curves in $V_0^p(S')$ for general $S' \in T(G)$, and similarly the curves in $U_a(R)$ are limits of rational nodal curves in $U_a(S')$ (see Proposition 7.3).

Now $\mathcal{W}_g(R')$ is a $g_{1,\delta}$-dimensional family of curves. Its general element contains a line chain, that varies in a 1-dimensional, base point free system, so the lines of these chains are in general different from lines contained in the general member of $\mathcal{U}_s^1(R')$. Fixing a line chain $\mathfrak{C}$, the family of curves in $\mathcal{W}_g(R')$ containing it is a linear system $\mathcal{L}_\mathfrak{C}$ of dimension $g_{1,\delta} - 1 \geq 1$, whose general element, minus $\mathfrak{C}$, is a smooth rational curve on every $R'_i$, so that $\mathcal{L}_\mathfrak{C}$ is base point free off $\mathfrak{C}$. Given the general element $(C_0, B'_1, \ldots, B'_{s-1}, B'_s) \in \mathcal{W}_g(R') \times (\text{Sym}^2(C_0))^s \times U_a(R')$, set $Z'_0 = B'_1 + \cdots + B'_{s-1} + B'_s$. Then $C_0 + Z'_0$ is nodal off the tacnodes of $B'_s$. Moreover, since the curves $B'_i$ move in an at least two-dimensional algebraic system for $1 \leq i \leq s$, $Z'_0$ cuts out on $C_0$ an algebraic system $\mathcal{S}$ and the associated variety $\mathfrak{X}'$ equals $(\text{Sym}^2(C_0))^s$. Hence, for the general $s$-tuple of pairs of points of $C_0$, we may find a $(B'_1, \ldots, B'_{s-1}, B'_s) \in (\text{Sym}^2(R'))^s \times U_a(R')$ such that $Z'_0$ contains the given points on $C_0$ and is nodal off the tacnodes of $B'_s$. Each $B'_i$ in $Z'_0$ cuts $E'$ in $p + 1$ points for $1 \leq i \leq s - 1$ and in at least 3 points off the tacnodes if $i = s$. Since $\dim(\mathfrak{X}') \geq 2s$, we can find a surface $R$ corresponding to a point of $\mathfrak{X}$ for which the modification $R \to R'$ involves two of the intersection points of each $B'_i$ with $E'$ for $1 \leq i \leq s$ (off the tacnodes if $i = s$). We denote by $Z_0$ the total transform of $Z'_0$ on $R$. Then $R \subset T_0$, and we denote by $B_{0,i}$ the total transform of each $B'_i$ on $R$, for $1 \leq i \leq s$. Then such a curve is the limit of a nodal rational curve in $V_0^p(S')$ for $1 \leq i \leq s - 1$, and $B_{0,s}$ is the limit of a nodal rational curve in $U_a(S')$.

Since (i) and (ii) hold in the limiting situation for $R$, $C_0$ and $Z_0$, they also hold in general. \hfill \square

Let $s \leq 4$ and $a$ be positive integers. Then for $[S, L] \in \mathcal{K}_p$ general we can consider the locally closed subset $V_{(S, a)}(S)$ in $[(s + a)L]$ consisting of all nodal curves of the form $C + Z$, with $(S, C) \in \mathcal{W}_g(S')$. \hfill \square
Note that $V_{(\delta,s,a)}(S)$ is non-empty for $[S,L]$ general by Lemma 8.2 and $\dim(V_{(\delta,s,a)}(S)) = g_{1,\delta}$. In the same way, we have the universal family $V_{(\delta,s,a)} \to \mathcal{K}_p$, with fiber $V_{(\delta,s,a)}(S)$ over $S$.

Marking all the nodes of a curve $C + Z$ but two in the divisor $g_i$ cut out by $B_i$ on $C$, for $1 \leq i \leq s$, one checks that the normalizations are 2-connected, hence these curves lie in $V_{m,\zeta}(S)$, where $m = s + a$ and $\zeta = p(m) - g_{1,\delta} - s$, with stable models in $\overline{M}_{g_{1,\delta} + s}$ as shown in Figure 3 below.

![Figure 3](image)

**Figure 3.** Members of $V_{(\delta,s,a)}(S)$, $s = 2$, with marked nodes the ones without dots, and the stable model.

In this way we determine (at least) an irreducible, fully complete component $V^*_{m,\zeta}$ of $\mathcal{V}_{m,\zeta}$ with its moduli map

$$\psi^*_{m,\zeta} : \mathcal{V}_{m,\zeta} \to \overline{M}_{g_{1,\delta} + s}.$$  

(Note that $V^*_{1,\delta} = V^*_\delta$.)

**Proposition 8.3.** Let $p \geq 3$. Let $g$ and $s$ be positive integers with $3 \leq g \leq 7$, $g \leq p$ and $s \leq 4$. Set $\zeta = p(m) - g - s$. Then, for any integer $m \geq s + 1$, the map $\psi^*_{m,\zeta}$ is dominant.

**Proof.** We keep the notation introduced above. Take a general $\Gamma \in \mathcal{M}_g$ and let $(x_i, y_i)$, for $1 \leq i \leq s$, be general pairs of points on $\Gamma$. By Proposition 6.3 and Lemma 8.2 we may construct nodal curves $\overline{C} = C + B_1 + \cdots + B_s \in V_{(\delta,s,m-s)}(S)$, with $x_i + y_i \leq g_i$, for $1 \leq i \leq s$, where $\delta = p - g$, $(S,C) \in V^*_{1,\delta} = V^*_\delta$ and $\Gamma = \psi^*_\delta(C)$. The stable model $\overline{\Gamma}$ of the curve obtained by normalizing $\overline{C}$ at all the marked nodes, is the curve obtained by pairwise identifying the inverse images of the points $x_i, y_i$ on $\Gamma$, for $1 \leq i \leq s$. This is, by construction, a general member of the $s$-codimensional locus $\Delta_{s,g+s}$ of irreducible $s$-nodal curves in $\overline{M}_{g+s}$, which therefore sits in the closure of the image of $\psi^*_{m,\zeta}$.

We claim that any component of $V_{(\delta,s,m-s)}$ is a whole component of $(\psi^*_{m,\zeta})^{-1}(\Delta_{s,g+s})$. Indeed, if it were not, then the general member of the component containing it would be a curve $C^*$ specializing
Therefore, the general fiber of \( \dim(V_{(\delta,s,m-s)}) \) is
\[
\dim(V_{(\delta,s,m-s)}) - \dim(V_{(\delta,s,g+s)}) = (g + 19) - (3(g + s) - 3 - s) = 22 - 2(g + s),
\]
hence the general fiber of \( \psi^*_{m,\xi} \) has at most this dimension, so
\[
\dim(\operatorname{Im}(\psi^*_{m,\xi})) \geq [g + s + 19] - [22 - 2(g + s)] = 3(g + s) - 3
\]
proving the assertion. \( \square \)

**Corollary 8.4.** With \( g := p(m) - \delta \), the map \( \psi^*_{m,\delta} \) is dominant if:
• \( m = 2 \), \( 0 \leq g \leq 8 \) and \( p \geq g - 1 \);
• \( m = 3 \), \( 0 \leq g \leq 9 \) and \( p \geq g - 2 \);
• \( m = 4 \), \( 0 \leq g \leq 10 \) and \( p \geq g - 3 \);
• \( m \geq 5 \), \( 0 \leq g \leq 11 \) and \( p \geq g - 4 \).

**Proof.** For a fixed \( m \geq 2 \), we apply Proposition 8.3 with \( s = \min\{4, m - 1\} \) and Lemma 6.2 because \( V^*_m \) is fully complete. \( \square \)

This proves Theorem 1.1A for \( m \geq 2 \).

9. GENERIC Finiteness of the Moduli Map for \( m \geq 2 \)

In this section we will use Proposition 6.1 and Corollary 8.4 to prove generic finiteness of \( \psi^*_{m,\delta} \) for \( m > 1 \) as in Theorem 1.1.

The following lemma is trivial and the proof can be left to the reader:

**Lemma 9.1.** If there exists a component \( V \subseteq V_{m,\delta} \) such that \( \psi^*_{m,\delta}|_V \) is generically finite onto its image, then for each component \( W \subseteq V_{m,\delta-1} \) such that \( V \) is included in \( W \) (see Remark 2.7), the map \( \psi^*_{m,\delta-1}|_W \) is generically finite onto its image.

**Lemma 9.2.** Assume that \( \gamma \) and \( \mu \) are integers such that \( \psi^*_{m,p(\mu)-\gamma} \) is generically finite onto its image on a component \( V' \) of \( V_{m,p(\mu)-\gamma} \). Let \( a \geq 1 \) and \( b \geq 0 \) be integers such that there is a component \( V'' \) of \( V_{\alpha,p(\alpha)-b} \) satisfying:
(i) for general \( (S,C) \in V' \) and \( (S,D) \in V'' \), \( D \) intersects \( C \) transversally;
(ii) for general \( \alpha \) such that \( W \subseteq V_{m,\delta} \), the restriction \( \psi|_{V_{m,\delta-1}\alpha|_W} \) is generically finite, where \( V''(S) \) denotes the fiber of \( (\alpha,\beta)|_W : V'' \to V_{\alpha,\beta} \) over \( \alpha \).

Then for \( m = \mu + a \) and \( \delta = p(\mu + a) - (\gamma + b + 1) \), the map \( \psi^*_{m,\delta} \) is generically finite onto its image on some component \( V \) of \( V_{m,\delta} \).

**Proof.** Pick general elements in \( V' \) and \( V'' \) like in (i), with \( \tilde{C} = \psi^*_{m,p(\mu)-\gamma}(C) \in M_{\gamma} \) and \( \tilde{D} = \psi^*_{m,p(\mu)-b}(D) \in M_{b} \). Then \( (S,C + D) \) corresponds to a point of a component \( V \). If we consider as marked nodes of \( C + D \) all of its nodes but two in \( C \cap D \). Let \( \psi = \psi_m \). This map sends \( C + D \) to \( \tilde{C} + \tilde{D} \) at two point (with a further contraction of \( \tilde{D} \) if \( b = 0 \)).

We denote by \( B \subseteq V \) the subset of curves of the form \( C' + D' \) where \( C' \) and \( D' \) are in \( V' \) and \( V'' \), respectively. Then \( \dim(B) = 19 + \gamma + b \).

Because of the hypotheses, the map \( \psi|_B \) is generically finite onto its image. Hence \( \dim(\operatorname{Im}(\psi|_B)) = 19 + \gamma + b \). Since the general element of \( \operatorname{Im}(\psi) \) is smooth, we must have \( \dim(\operatorname{Im}(\psi)) \geq 20 + \gamma + b \). Therefore, the general fiber of \( \psi \) has dimension at most
\[
\dim(V) - (20 + \gamma + b) = [19 + (\gamma + b + 1)] - (20 + \gamma + b) = 0,
\]
and the result follows. \( \square \)
Corollary 9.3. The map \( \psi_{m,\delta} \) is generically finite on some component of \( V_{m,\delta} \) in the following cases, with \( g := p(m) - \delta \):

- \( 2 \leq m \leq 5, \ g \geq 16 \) and \( p \geq 15 \);
- \( m \geq 5, \ p \geq 7 \) and \( g \geq 11 \).

Proof. Let \( m \geq 2 \). If \( p \geq 15 \), we apply Lemma 9.2 with \( \mu = 1, \ \gamma = 15, \ a = m - 1 \geq 1, \ b = 0, \ V' = V^*_{p-15} \), the component for which we proved generic finiteness of \( \psi_{p-15} \) in Proposition 8.1 and \( V'' = U_0 \subseteq V_{a,p(\delta)} \) the component consisting of rational curves that degenerate to a curves of the type \( U_n(R) \) as in Definition 7.1. Condition (ii) of Lemma 8.2 is satisfied and an argument as in the proof of Lemma 8.2 shows that also condition (i) is satisfied. Then Lemma 9.2 implies that \( \psi_{m,\delta} \) is generically finite on a suitable component of the universal Severi variety for \( g = 16 \). Lemma 9.1 yields that \( \psi_{m,\delta} \) is generically finite on some component for all \( g \geq 16 \), as stated.

If \( m \geq 5, \ p \geq 7 \) and \( g = 11 \), the map \( \psi_{m,\delta} \) is generically finite on some component of \( V_{m,\delta} \) by Corollary 8.4; hence Lemma 9.1 yields the same for all \( g \geq 11 \).

This proves Theorem 11.1(B) for \( m \geq 2 \).

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