Quantum Fluctuations and Curvature Singularities in Jackiw-Teitelboim gravity

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Abstract

The Jackiw-Teitelboim gravity with the matter degrees of freedom is considered. The classical model is exactly solvable and its solutions describe non-trivial gravitational scattering of matter wave-packets. For huge amount of the solutions the scattering space-times are free of curvature singularities. However, the quantum corrections to the field equations inevitably cause the formation of (thunderbolt) curvature singularities, vanishing only in the limit $\hbar \to 0$. The singularities cut the space-time and disallow propagation to the future. The model is inspired by the dimensional reduction of 4-d pure Einstein gravity, restricted to the space-times with two commuting space-like Killing vectors. The matter degrees of freedom also stem from the 4-d ansatz. The measures for the continual integrations are judiciously chosen and one loop contributions (including the graviton and the dilaton ones) are evaluated. For the number of the matter fields $N = 24$ we obtain even the exact effective action, applying the DDK-procedure. The effective action is nonlocal, but the semiclassical equations can be solved by using some theory of the Hankel transformations.
1 Introduction

The Einstein gravity is undoubtedly beautiful and physically relevant theoretical construction which, at the same time, brought many novel mathematical structures into theoretical physics. The notions like the black-hole, the horizon, the curvature singularities or the gravitational collapse enriched our conceptual world, but they also posed new challenging problems, yet to be understood. For instance, what is the fate of matter in the last part of the gravitational collapse? Following the overwhelming success of the quantum theory in describing the world of subatomic distances, physicists feel that an appropriate theory of quantum gravity has to provide the correct solutions to the problems and improve our understanding of the phenomena, occurring in very strong gravitational fields near the curvature singularities. Unfortunately, such generally accepted and technically applicable quantum theory of gravity does not exist yet. The considerable breakthrough was reached due to string theory, where the consistent perturbative $S$ matrix for scattering of gravitons and excitations of other fields can be obtained [1]. However, the dynamics of a gravitational collapse or the status of quantum black holes remain nonunderstood. Though there is a common belief that quantum effects should smear the singular behaviour of the classical theory, there is no sufficiently established quantitative evidence for such a conjecture.

The string boom had also an indirect, but very important influence on the subject of quantum gravity. The models of $1 + 1$ dimensional theories of gravity coupled to a dilaton field $\phi$ have arisen in string theory [2, 3]. These models possess black hole solutions and they motivated Callan, Giddings, Harvey and Strominger [4] to investigate the “CGHS” model of 2-d dilaton gravity coupled to conformal matter. The model is of interest as a “toy” model of quantum gravity in two dimensions which contains gravitational collapse, black holes, cosmic censorship and Hawking radiation. Moreover, the model is very similar to that obtained by the dimensional reduction of the spherically symmetric gravitational system in $3 + 1$ dimensions, hence, one may expect the relevance of the $1 + 1$ results to the $3 + 1$ physics. Recently, many authors have been investigating the quantum dynamics of the black holes by using the $CGHS$ model [5]-[13]. The issues of particular interest are the backreaction of the Hawking radiation on the metric and the endpoint of the black hole evaporation. The problem is far simpler than the original $3 + 1$ dimensional one and powerful methods of conformal field theory in two
dimensions can be used.

The *CGHS* model and its variants \[15,16\] and also other 2-d dilaton gravities have been studied in the literature \[17\], particularly in the context of the noncritical string theory \[18\]. The models can be typically obtained by the dimensional reduction of the higher-dimensional pure metric gravities. This fact suggests the following *CGHS*-like scenario for “addressing” the four-dimensional quantum problems: one finds the corresponding dilaton gravity model in 1+1 dimensions and attempts to quantize it. Though the 1+1 quantum theory may still be complicated enough to prevent exact solvability (like *CGHS* is), usually it is far simpler than its 4-d counterpart. In this contribution, we adopt the scenario and address the quantum dynamics of the colliding gravitational waves. The fact, that the nonlinear character of the Einstein equations results in the formation of curvature singularities after collisions of gravitational waves, is known only since seventies \[19,20\] and, perhaps, it is less familiar to nonspecialists than the fact that a black hole is formed as the consequence of gravitational collapse. However, the colliding-waves problem keeps attracting many relativists \[21\]-\[24\] without an interruption, since the discoveries of the first colliding-waves space-times by Szekeres \[19\] and Khan and Penrose \[20\]. The problem of main interest for us will constitute in the following: What is the quantum status of the those scattering space-times? As we have mentioned above, one usually expects that the curvature singularities should be smeared by the effects of quantum fluctuations. Our quantitative analysis will show, however, the surprising result: the quantum curvature singularities are even worse than the classical ones and even classically nonsingular spacetimes are destabilized by quantum curvature singularities.

Apart from the physical questions which our analysis will try to answer, the model to be considered in this paper is of interest also for some more theoretical reasons. Indeed, while in *CGHS* and related theories \[17,18\] the matter degrees of freedom are added by hand after the dimensional reduction, in our model the matter degrees of freedom also stem from the four-dimensional theory. This fact should even increase the relevance of our results for the 4-d case. There is another pleasant thing, namely, not only the matter loops, but also one loop dilaton and graviton contributions can be evaluated, yielding the one loop effective action. We need not to perform \(1/N\) expansion, but we can take \(\hbar\) as the natural parameter of expansion. Moreover, for the critical number of matter fields (\(N=24\)) our result will be
nonperturbative and exact. But the good news is not exhausted by that, it turns out, moreover, that the semiclassical equations can be solved and the behaviour of curvature singularities is under control.

The plan of the paper is as follows. In section 2 we introduce the 2-d matter-dilaton model motivated by the dimensional reduction of the 3 + 1 dimensional system with two commuting space-like Killing vectors. Then we find the classical equations of motion in the conformal gauge. The dilaton field turns out to satisfy the standard d’Alambert wave equation, hence, we introduce a sort of the “light cone” gauge. In this gauge the matter fields obey the Gowdy cylindrical wave equation \[25\], the general solution of which can be given by the decomposition into the Fourier-Bessel and Fourier-Neumann modes. The corresponding metric we find explicitly by integrating the remaining equations. We show that the Neumann modes cause the formation of the (classical) curvature singularities which close the space-time to the future, while the appropriate superpositions of the Bessel modes describe the collisions of the wave packets travelling against each other with the velocity of light. The corresponding spacetimes are everywhere regular with the out region in which the scattered wave packets travel to the opposite space infinities. In section 3 we discuss the quantization of the model. We choose the standard Polyakov measure for the functional integration over the metrics and the reparametrization invariant measures for the dilaton and the matter fields integration. Then we compute the one loop effective action. The effective action is nonlocal even in the conformal gauge, due to the presence of the direct matter-dilaton coupling in the action. The one loop effective field equations are localized by going to the dilaton “light-cone” gauge. The renormalization requires a purely dilatonic counterterm, the contribution of which makes finite one infinite constant in the semiclassical field equations. The actual computation requires the knowledge of the functional derivatives of the determinant of the (Gowdy) wave operator with respect to the dilaton and metric. They are evaluated by using the heat kernel regularization and some theory of the Hankel transformations in the Appendix. In section 4 we solve the semiclassical field equations. We perform the detailed analysis of the scalar curvature of the space-times, obtained by solving the semiclassical equations. We show that the contribution of the quantum fluctuations to the effective action inevitably generates the curvature singularities in the semiclassical spacetimes. These singularities may disappear only in the limit $\hbar \to 0$, thus indicating, that the classical regular scattering space-times are
in fact unstable from the quantum point of view. We end up with short conclusions and outlook.

2 The model and its classical dynamics

2.1 The dimensional reduction

The form of the four-dimensional metric describing the collisions of collinear gravitational waves is given by [24]

\[ ds^2 = -2\phi^{-\frac{1}{2}}e^\mu dudv + \phi(e^{\sqrt{2}\kappa Q}dx^2 + e^{-\sqrt{2}\kappa Q}dy^2), \]

(1)

where the metric functions \( \mu, \phi \) and \( Q \) are invariant on the \((x, y)\) plane of symmetry. The 4-d vacuum Einstein equations for the metric (1) consist of the constraints

\[ -\phi_{uu} + \mu_u \phi_u = \kappa \phi Q_u^2 \]

(2)

\[ -\phi_{vv} + \mu_v \phi_v = \kappa \phi Q_v^2 \]

(3)

and the evolution equations

\[ \phi_{uv} = 0 \]

(4)

\[ (\phi Q_u)_v + (\phi Q_v)_u = 0 \]

(5)

\[ -\mu_{uv} = \kappa Q_u Q_v. \]

(6)

It is not difficult to demonstrate, that the same set of the constraints and the evolution equations follows from the following 2-d action

\[ S = \frac{1}{2\kappa} \int d^2\xi \sqrt{-g}\phi (R - \kappa g^{\alpha\beta} \partial_\alpha Q \partial_\beta Q), \]

(7)

after fixing the conformal gauge

\[ ds^2 = -2e^\mu dudv. \]

(8)

The action (7) can be interpreted as the Jackiw-Teitelboim gravity [26] where the cosmological constant is replaced with the kinetic term of the matter. The matter is coupled to the dilaton and possesses all dynamical degrees of freedom of the theory.
2.2 The solutions of the field equations

In what follows, we shall consider the model (7) with an arbitrary number of matter fields. The classical dynamics does not change dramatically, but the properties of the quantum theory will depend on that number. The action reads

\[ S = \frac{1}{2\kappa} \int d^2 \xi \sqrt{-g} \phi (R - \kappa g^{\alpha \beta} \partial_\alpha Q^j \partial_\beta Q^j) \]  

(9)

and the constraints and the evolution equations in the conformal gauge get obviously modified

\[ - \phi_{uu} + \mu_u \phi_u = \kappa \phi Q^j_u Q^j_u \]  

(10)

\[ - \phi_{vv} + \mu_v \phi_v = \kappa \phi Q^j_v Q^j_v \]  

(11)

\[ \phi_{uv} = 0 \]  

(12)

\[ (\phi Q^j_u)_v + (\phi Q^j_v)_u = 0 \]  

(13)

\[ - \mu_{uv} = \kappa Q^j_u Q^j_v. \]  

(14)

The general solution of (12) reads

\[ \phi = f(u) + g(v) \]  

(15)

If \( \phi_v = 0 \) (or \( \phi_u = 0 \)), then from (11) (or (10)) it follows that \( Q^j_v = 0 \) (or \( Q^j_u = 0 \)) and from (13) \( \mu_{uv} = 0 \). Since the scalar curvature \( R \) is given by

\[ R = 2e^{-\mu} \mu_{uv} \]  

(16)

and in two dimension the curvature tensor reads

\[ R_{\alpha \beta \gamma \delta} = \frac{1}{2} (g_{\alpha \gamma} g_{\beta \delta} - g_{\alpha \delta} g_{\beta \gamma}) R \]  

(17)

we may conclude that arbitrary functions \( \phi(u) \) and \( Q^j(u) \) (or \( \phi(v) \) and \( Q^j(v) \)) are solutions of the field equations and the corresponding space-time is flat. Such solutions obviously describe the matter excitations propagating in one direction with the velocity of light.

If neither \( \phi_v = 0 \) nor \( \phi_u = 0 \) we can (at least locally) perform the conformal transformation
\[ U = \sqrt{2} f(u), \quad V = \sqrt{2} g(v) \]  \hspace{1cm} (18)

Such a transformation is obviously the symmetry transformation of the set of the field equations (10-14), hence we may fix this residual symmetry by the claim

\[ \phi = \frac{U + V}{\sqrt{2}} \equiv t \]  \hspace{1cm} (19)

We may call this gauge fixing the “dilaton” gauge. In the gauge the field equations become

\[ \mu_U = \kappa(U + V)Q^j_U Q^j_U \]  \hspace{1cm} (20)
\[ \mu_V = \kappa(U + V)Q^j_V Q^j_V \]  \hspace{1cm} (21)

\[ Q^j_{tt} + \frac{1}{t} Q^j_t - Q^j_{\sigma\sigma} = 0 \]  \hspace{1cm} (22)
\[ -\mu_{UV} = \kappa Q^j_U Q^j_V \]  \hspace{1cm} (23)

where

\[ \sigma \equiv \frac{V - U}{\sqrt{2}} \]  \hspace{1cm} (24)

We observe that the linear equation for the matter fields \( Q^j \) does not contain the metric function \( \mu \). We may call this equation by the name of Gowdy, who studied cosmological models with plane symmetry [25], governed locally by (22). The general solution of the Gowdy equation (which tend to zero at the spatial infinities) is given by the following mode expansion [24, 28].

\[ Q^j(t, \sigma) = \int_0^\infty d\omega \text{Re}[A^j(\omega)J_0(\omega t)e^{-i\omega \sigma} + B^j(\omega)N_0(\omega t)e^{-i\omega \sigma}], \]  \hspace{1cm} (25)

where \( A^j \) and \( B^j \) are (complex) distributions ensuring the proper behaviour at the space infinity and \( J_0 \) and \( N_0 \) are the Bessel and Neumann functions of zero order, respectively. This mode expansion can be easily found by using the Fourier transformation in the variable \( \sigma \) in (22). The resulting
ordinary differential equation in the variable \( t \) is then the Bessel equation. We should note, at this place, that in higher dimensions some additional (so called “solitonic”) terms are considered at the r.h.s. of (25). They do not vanish at the space infinity and in the limit of the weak gravitational coupling \( \kappa \to 0 \) those solutions diverge and do not approach the non-interacting matter solutions [30]. We shall not consider this “solitonic” sector in this paper and prescribe the boundary conditions, mentioned above.

It remains to solve the equations (20), (21) and (23), which determine the metric function \( \mu \). Combining (20) with (21), we obtain

\[
\mu_\sigma = 2\kappa t Q^i_\sigma Q^j_\sigma
\]

(26)

\[
\mu_t = \kappa t (Q^i_\sigma Q^j_\sigma + Q^j_\sigma Q^i_\sigma)
\]

(27)

We use the fact [27] that for \( F \) and \( G \), fulfilling the Bessel equations

\[
x^2 \frac{d^2F}{dx^2} + x \frac{dF}{dx} + (\lambda^2 x^2 - n^2) F = 0
\]

\[
x^2 \frac{d^2G}{dx^2} + x \frac{dG}{dx} + (\nu^2 x^2 - n^2) G = 0
\]

it holds

\[
\int_a^b dx (\lambda^2 - \nu^2) x F G = \left[ x(F \frac{dG}{dx} - G \frac{dF}{dx} ) \right]_a^b
\]

(28)

This formula enables us to integrate the products of the Bessel functions. The result of the integration gives the explicit form of the metric function\(^\dagger\)

\[
\mu = \kappa \int_0^\infty d\omega_1 d\omega_2 \omega_1 \omega_2 \ t \text{ Re} \left[ \frac{-1}{\omega_1 + \omega_2} G^i_1(\omega t) G^j_0(\omega t) e^{-i(\omega_1 + \omega_2)\sigma} \right.
\]

\[
+ \left. \frac{1}{2} \frac{1}{\omega_1 - \omega_2} (G^i_1(\omega t) G^j_0(\omega t) e^{i(\omega_1 - \omega_2)\sigma} - G^j_1(\omega t) G^i_0(\omega t) e^{-i(\omega_1 - \omega_2)\sigma}) \right],
\]

(29)

\(^\dagger\)It appears, that this result is new even from the point of view of 4-d theory of colliding waves.
where
\[ G_{0(1)}^{j}(\omega t) = A^{j}(\omega)J_{0(1)}^{j}(\omega t) + B^{j}(\omega)N_{0(1)}^{j}(\omega t) \]  
(30)

We note, that the classical equations (12-14) turn out to be “iteratively” linear. Indeed, solving the linear Eq.(12) and inserting its solution \( \phi \) into Eq.(13), we get again the linear equation. After solving it, we insert \( Q^{j} \) into Eq.(14) and get the linear equation for \( \mu \). Such a structure of the equations gives the classical integrability and will be also important later for the quantization.

### 2.3 Curvature singularities and the global structure

We start our analysis of the curvature singularities with the formula for the scalar curvature. Following from Eqs. (16) and (23), we have
\[ R = \kappa e^{-\mu} (-Q^{j}_{t}Q^{j}_{t} + Q^{j}_{\sigma}Q^{j}_{\sigma}). \]  
(31)

Near \( t \to 0^{+} \) we have
\[ J_{0}(t) \sim 1 - \frac{t^{2}}{4} + \cdots \]  
(32)
\[ N_{0}(t) \sim (1 - \frac{t^{2}}{4}) \ln t + \cdots \]  
(33)

hence,
\[ Q^{j}_{t} \sim \frac{1}{t} \left( \int_{0}^{\infty} d\omega \omega \Re[B^{j}(\omega)e^{-i\omega t}] \right) + \text{bounded} \equiv \frac{E^{j}}{t} + \text{bounded}, \]  
(34)
\[ Q^{j}_{\sigma} \sim \ln t \left( \int_{0}^{\infty} d\omega \omega \Re[-iB^{j}(\omega)e^{-i\omega t}] \right) + \text{bounded} \equiv H^{j} \ln t + \text{bounded}, \]  
(35)
\[ \mu \sim \kappa \ln t \ E^{j} E^{j} + \text{bounded}. \]  
(36)

Inserting Eqs. (34), (35) and (36) into Eq.(31), we have
\[ R \sim t^{-\kappa E^{j} E^{j}} \left[ -\frac{E^{j} E^{j}}{t^{2}} + H^{j} H^{j} (\ln t)^{2} + \cdots \right]. \]  
(37)

We conclude, that the regularity of the (classical) space-times requires both \( E^{j} \) and \( H^{j} \) to be equal zero, or, equivalently,
\[ B^j(\omega) = 0. \]  
(38)

Consider now (regular) space-times, given by

\[ A^j(\omega, \omega_0, \rho) = |a_j| e^{i\phi_j} \sqrt{\frac{\omega}{4\rho}} e^{-\frac{(\omega - \omega_0)^2}{2\rho}}, \quad B^j(\omega) = O, \]  
(39)

where \(\phi_j\) is real and \(\rho\) and \(\omega_0\) are real positive parameters. Note that for \(\rho \to 0\)

\[ A^j(\omega, \omega_0, \rho) \to |a_j| e^{i\phi_j} \frac{\omega_0 \pi}{\sqrt{2}} \delta(\omega - \omega_0). \]  
(40)

From Eq. (25) for \(B^j = 0\) we obtain

\[ Q^j = \int_0^\infty d\omega \text{Re} \left[ |a_j| e^{i\phi_j} \sqrt{\frac{\omega}{4\rho}} e^{-\frac{(\omega - \omega_0)^2}{2\rho}} J_0(\omega t) e^{-i\omega \sigma} \right]. \]  
(41)

From the well-known formula for the asymptotic behaviour of the Bessel functions for \(t \to \pm \infty\) \[27\]

\[ J_0(\omega t) = \sqrt{\frac{2}{\pi \omega |t|}} \cos \left( \omega t \mp \frac{\pi}{4} \right) + \cdots, \]  
(42)

we have for \(t \to \pm \infty\)

\[ Q^j = \frac{|a_j|}{\sqrt{|t|}} \left\{ e^{-\frac{\pi}{8}(t-\sigma)^2} \cos [\omega_0 (t - \sigma) + \phi_j - \frac{\pi}{4}] 
+ e^{-\frac{\pi}{8}(t+\sigma)^2} \cos [\omega_0 (t + \sigma) - \phi_j - \frac{\pi}{4}] \right\}. \]  
(43)

Now the physical interpretation of this solution is obvious. At \(t \to -\infty\) we have two wave packets propagating against each other by the velocity of light; at \(t \to \infty\) the two scattered packets propagate apart from each other with the gained phase shift, indicated in Eq. (43). Because \(J_0(\omega t)\) and its derivatives are bounded functions \[27\], we may use the Riemann-Lebesgue lemma and from Eq. (11) conclude that for \(t = \text{const}\) and \(\sigma \to \pm \infty\), \(Q^j\) and all its derivatives with respect to \(t\) and \(\sigma\) vanish. Hence, by using the constraints (26) and (27) and the formula (31) for the scalar curvature, we observe that the space-time is flat in this limit. For the cases \(\sigma = \text{const}, \)
\( R \neq 0, \quad R \text{ bounded} \)

Figure 1: The scalar curvature of the regular classical space-times.

\[ t \to \pm \infty; \quad t + \sigma = \text{const}, \quad t - \sigma \to \pm \infty \quad \text{and} \quad t - \sigma = \text{const}, \quad t + \sigma \to \pm \infty, \]

we use the asymptotic expression (43), the constraints (26), (27) and the formula (31) to arrive at the same conclusion. Therefore, for the “wave-packet” choice (39) the corresponding space-time is asymptotically flat (see Figure 1), it has the same topology as the two-dimensional Minkowski space-time and is free from curvature singularities. We shall not need the explicit form of the metric, which, nevertheless, can be obtained by performing the integral (29) with the choice (39). We should end up the classical analysis with some important comments.

First of all, it does not seem unexpected that for the collisions of the localized wave packets travelling with the velocity of light, the space-time is asymptotically flat in the space-like and time-like directions. What looks more surprising is the fact that the same is true for the null infinities. The reason is simple: in two dimensions a single propagating wave does not curve the space-time (cf. the analysis after (Eq.17)). In higher dimensions this is not true [29, 30], but in that case the curvature is given by the shape of the wave front in the transverse directions. Since there are no transverse directions in two dimensions, our result could be anticipated.
The second comment is closely related to the first one. It concerns the regularity of the initial data. In the higher dimensional case the following problem was studied [23, 30]. If initially regular gravitational waves interact, will a curvature singularity be formed? The criterion for the regularity of the incoming data can be naturally formulated: one requires the boundedness of the amplitude of the wave, which itself is defined by means of the components of the Riemann tensor of the corresponding metric in the so called parallelly propagated orthonormal frame [23, 30]. However, in the two-dimensional case, incoming waves do not curve the spacetime and this criterion of regularity fails. But certainly we should not consider all incoming waves as regular (with bounded amplitude). One might require that the scalar fields $Q^i$ itself should be bounded, but, on the other hand, we could rescale it by an arbitrary function of another scalar field, the dilaton $\phi$, and we would get the classically equivalent dynamical theory with the different condition of incoming regularity. Fortunately, it appears to be a natural candidate for the amplitude of the wave. The matter part of the action (9) suggests the following inner product on the space of fields $Q$ (see also Eq.(50))

$$(Q_1, Q_2) = \int d^2 \xi \sqrt{-g} \phi Q_1 Q_2.$$  

Hence, our condition of the incoming regularity reads

$$\lim_{U \to \infty} (V \to \infty) \phi Q^i Q^j = \text{finite}$$  

In the case of the wave-packets (39) we get

$$\lim_{U \to \pm \infty} \phi Q^i Q^j = \sum_j |a_j|^2 e^{-2\rho V^2} \cos^2[\sqrt{2}\omega_2 V - \phi_j \mp \pi/4] = \text{finite}$$  

and similarly for $V \to \pm \infty$.

3 The quantization

3.1 The functional measures and the effective action

Define the generating functional of the model $W[J_g, J_\phi, J_j]$ by

$$W[J_g, J_\phi, J_j] \equiv \int \frac{D_g g_{\alpha\beta} D_g \phi D_g \phi Q^i}{\text{Vol}(\text{Diff})}$$  

11
where \( J_g \) is a scalar source, \( J_\phi \) and \( J_j \) are scalar densities and \( \text{Vol}(\text{Diff}) \) is the volume of the group of diffeomorphisms. We define the functional measures by the following norms

\[
\|\delta g\|^2 = \int d^2\xi \sqrt{-g} g^{\alpha\gamma} g^{\beta\delta} \delta g_{\alpha\beta} \delta g_{\gamma\delta}, \quad (48)
\]

\[
\|\delta \phi\|^2 = \int d^2\xi \sqrt{-g} \delta \phi^2, \quad (49)
\]

\[
\|\delta Q_j\|^2 = \int d^2\xi \sqrt{-g} \phi \delta Q_j^i Q^j. \quad (50)
\]

Eq. (48) defines the usual de Witt-Polyakov norm \[31\] and Eq. (49) gives the standard reparametrization invariant measure for a scalar field. The norm (50) is given by the form of the matter part of the classical action, much in the same way as the norm for the quantization of the standard non-linear \( \sigma \)-model with the coordinates \( X^A \) of the target and a metric \( H_{AB}(X) \), i.e. \[32\]

\[
\|\delta X^j\|^2 = \int d^2\xi \sqrt{-g} H_{AB}(X(\xi)) \delta X^A(\xi) \delta X^B(\xi). \quad (51)
\]

We return to Eq. (47) and we fix the conformal gauge

\[
ds^2 = -2e^{\mu} du dv. \quad (52)
\]

By using the standard Faddeev-Popov procedure, we obtain

\[
W[J] = \int D\mu D\mu \phi D\mu \phi D\mu Q^j \exp \left\{ \frac{i}{\hbar} \int d^2\xi \frac{1}{2\kappa} \mu \delta^2 \mu \right\} \exp \left\{ \frac{i}{\hbar} \int d^2\xi \left[ \frac{\phi}{2\kappa} (\delta^2 \mu - \kappa \delta Q^j \delta Q^j) - J_g \delta^2 \mu + J_\phi \delta \phi + J_j \delta Q_j \right] \right\}, \quad (53)
\]

\[
\times \exp \left\{ \frac{i}{\hbar} \int d^2\xi \left[ \frac{\phi}{2\kappa} (\delta^2 \mu - \kappa \delta Q^j \delta Q^j) - J_g \delta^2 \mu + J_\phi \delta \phi + J_j \delta Q_j \right] \right\}; \quad (54)
\]

where \( \delta^2 \) is the Minkowski d’Alambertian and \( \delta Q^j \delta Q^j \) means the Minkowski metric scalar product. The Weyl anomaly term comes from the Faddeev-Popov determinant and the measure \( D\mu \mu \) is given by the following norm

\[
\|\delta \mu\|^2 = \int d^2\xi e^{\mu} \delta \mu^2. \quad (55)
\]
We standardly suppose that the exponential term in the Weyl anomaly is eventually (after including all contributions) cancelled by tuning of the 2-d cosmological constant counter-term.

Define now the effective action $\Gamma$ by the following prescription

\[
\frac{\hbar}{i} \ln W[J] \equiv Z[J] \equiv \Gamma[\mu_c, \phi_c, Q^j_c] - J_g \partial^2 \mu_c + J_\phi \phi_c + J_j Q^j_c,
\]

(56)

where

\[
\partial^2 \mu_c \equiv -\frac{\delta Z}{\delta J_g}, \quad \phi_c \equiv \frac{\delta Z}{\delta J_\phi}, \quad Q^j_c \equiv \frac{\delta Z}{\delta J_j}.
\]

(57)

We wish to compute the one loop effective action $\Gamma_1$. In order to do that, we have first to determine $Z_1$ (the generating functional for the connected Green functions) from (54) and then to perform the Legendre transformation (56) and (57). We note, that the dependences of the measures on the fields $\mu$ and $\phi$ are of the order $O(\bar{\hbar})$ with respect to the classical action in the exponent. Hence, the loop diagrams with the vertices coming from the measures will be of the order $O(\bar{\hbar}^2)$ and may be neglected in the one loop approximation. Therefore, we may write

\[
W_{\text{semicl.}}[J] = \int D\mu D\phi D\mu, \phi, Q^j \exp \left\{ \frac{26}{96\pi} \int d^2 \xi \mu \partial^2 \mu_J \right\}
\]

\[
\exp \left\{ i \int d^2 \xi \left[ \frac{\phi}{2\kappa} \left( -\partial^2 \mu_J - \kappa \partial Q^j \partial Q^j \right) - J_g \partial^2 \mu + J_\phi \phi + J_j Q^j \right] \right\},
\]

(58)

where $\mu_J$ and $\phi_J$ are the saddle point values of the exponent, given by the equations

\[
- \partial^2 \mu_J - \kappa \partial Q^j \partial Q^j + 2\kappa J_\phi = 0,
\]

(59)

\[
\phi_J + 2\kappa J_g = 0,
\]

(60)

\[
\partial(\phi_J \partial Q^j_J) + J_j = 0.
\]

(61)

We observe that in the one loop approximation, we can consider the measures to be independent on the field integration variables (but, of course, dependent on the Schwinger currents). Now we evaluate the integral (58). In the (second) exponent, there stands the quadratic form in the variables $\phi$ and $\mu$. Moreover, the norms defining the measures has the same $\mu_J$-dependence., i.e.
\[ \|\delta \mu\|^2 = \int d^2 \xi e^{\mu_j} \delta \mu^2; \|\delta \phi\|^2 = \int d^2 \xi e^{\mu_j} \delta \phi^2. \] (62)

Therefore, we can easily perform the Gaussian integration over \( \mu \) and \( \phi \) with the result

\[
W_{\text{semicl.}}[J] = \int D_{\mu, \phi} Q^j \exp \left\{ i \frac{24}{96\pi} \int d^2 \xi \mu_j \partial^2 \mu_j \right\}
\exp \left\{ \frac{1}{\hbar} \int d^2 \xi \left( -2\kappa J_g J_\phi + \kappa J_g \partial Q^j \partial Q^j + J_j Q^j \right) \right\}. \] (63)

The integration over \( Q^j \) is again Gaussian, hence we obtain the closed expression for the semiclassical generating functional

\[
W_{\text{semicl.}}[J_g, J_\phi, J_j] = \det^{-N/2} \left[ -i \frac{\kappa}{\hbar} e^{-\mu_c} \left[ \frac{1}{2\kappa J_g} \partial (J_g \partial) \right] \right]
\exp \left\{ i \frac{24}{96\pi} \int d^2 \xi \mu_c \partial^2 \mu_c + \frac{1}{\hbar} \int d^2 \xi \left( -2\kappa J_g J_\phi + J_j \frac{1}{4\kappa \partial (J_g \partial) J_j} \right) \right\}, \] (64)

where we used the definition (50) of the measure \( D_{\mu, \phi} Q^j \) and Eq.(60). The Legendre transformation (56) and (57) can be easily performed and we obtain the following expression for the one loop effective action

\[
\Gamma_1(\mu_c, \phi_c, Q^j_c) = \int d^2 \xi \frac{\phi_c}{2\kappa} (-\partial^2 \mu_c - \kappa \partial Q^j_c \partial Q^j_c) + \frac{\hbar}{96\pi} \int d^2 \xi \mu_c \partial^2 \mu_c
+i \hbar \frac{N}{2} \ln \det \left[ - i \frac{\kappa}{2\hbar} e^{-\mu_c} \frac{1}{\phi_c} \partial (\phi_c \partial) \right] + O(\hbar^2). \] (65)

We recognize the classical action and two quantum corrections. The first one is the Weyl anomaly, while the second one is the nonlocal term depending on \( \mu_c \) and \( \phi_c \). The expression (63) can also be written in the manifestly covariant way, i.e.

\[
\Gamma_1(g_{c,\alpha\beta}, \phi_c, Q^j_c) = \int d^2 \xi \sqrt{-g_c} \frac{\phi_c}{2\kappa} \left( R_c - \kappa g_c^{\alpha\beta} \partial_\alpha Q^j_c \partial_\beta Q^j_c \right)
+i \hbar \frac{24}{96\pi} \int d^2 \xi \sqrt{-g_c} R_c \left( \frac{1}{\sqrt{-g_c}} \partial_\alpha \sqrt{-g_c} g_c^{\alpha\beta} \partial_\beta \right)^{-1} R_c
+i \hbar \frac{N}{2} \ln \det \left[ - i \frac{1}{2\hbar} \sqrt{-g_c} \phi_c \partial_\alpha \left( \sqrt{-g_c} \phi_c g_c^{\alpha\beta} \partial_\beta \right) \right] + O(\hbar^2). \] (66)
3.2 The case \( N = 24 \)

For \( N = 24 \), the quadratic term of the Weyl anomaly vanishes. The dilaton gravities usually get simplified and more precise results can be obtained in that case [33]. This happens also in our model. We show that the semiclassical effective action \( \Gamma_1 \) (65) is the exact quantum effective action of the theory for \( N = 24 \). We use the DDK-approach [34, 35] to establish this result. The dependences of the functional measures on the field \( \mu \) read

\[
D_{\mu} = (D_0\mu) \exp \left[ -\frac{i}{48\pi} \int d^2\xi \frac{1}{2} \mu \partial^2 \mu \right],
\]

(67)

\[
D_\mu \phi = (D_0\phi) \exp \left[ -\frac{i}{48\pi} \int d^2\xi \frac{1}{2} \mu \partial^2 \mu \right],
\]

(68)

\[
D_{\mu,\phi} Q^i = (D_{0,\phi} Q^i) \times \exp \left[ -\frac{i}{48\pi} \int d^2\xi \left\{ \frac{1}{2} \mu \partial^2 \mu + \frac{3}{2} \mu [2 \partial^2 \ln |\phi| + (\partial \ln |\phi|)^2] \right\} \right].
\]

(69)

The formulas (67) and (68) are fairly standard [36, 37], however, the relation (69) deserves some comment. Indeed, it can be explicitly derived by computing the Jacobian, which relates both measures, with some regularization procedure. We shall use the heat kernel regularization and use the defining formula (50) to write

\[
D_{\mu,\phi} Q = D_{0,\phi} Q \sqrt{\det L},
\]

(70)

where \( L \) is the diagonal operator

\[
L(\xi_1, \xi_2) = e^{\mu(\xi_1)} \delta(\xi_1, \xi_2).
\]

(71)

Note that the \( \delta \)-function \( \delta(\xi_1, \xi_2) \) is to be understood in the sense of the scalar product (50) with \( \mu = 0 \), i.e.

\[
\delta(\xi_1, \xi_2) = \frac{1}{\phi(\xi_1)} \delta(\xi_1 - \xi_2).
\]

(72)

Clearly

\[
\delta \ln \det L = \delta \text{Tr} \ln L = \int d^2 \xi \phi(\xi) \delta(\xi, \xi) \delta \mu(\xi),
\]

(73)
where $\delta(\xi, \xi)$ is the meaningless quantity. As in [36, 37], we replace it by the heat kernel of the covariant Laplacian, but in our case with respect to the scalar product (50), i.e.

$$
\delta \varepsilon(\xi, \xi) = \frac{1}{\phi(\xi)} \langle \xi | \exp \left\{ -\varepsilon \left[ -\frac{i}{\hbar} \frac{1}{\sqrt{-g\phi}} \partial_\alpha (\sqrt{-g\phi} g^{\alpha\beta} \partial_\beta) \right] \right\} | \xi \rangle = \frac{1}{\phi(\xi)} (-i) \left\{ \frac{1}{24\pi} \partial^2 \mu + \frac{1}{16} [2\partial^2 \ln |\phi| + (\partial \ln |\phi|)^2] \right\}. 
$$

(74)

We obtained the last equality in the conformal gauge, by combining the formulas (115) and (121) of the Appendix. Now we insert (74) into (73) and in a straightforward way we arrive at the formula (69).

We may also check the validity of the formula (69) for a particular integrand. Indeed, let us compute the integral

$$
\int D_{\mu,\phi} Q \exp \left\{ \frac{i}{2\hbar} \int d^2\xi Q \partial(\phi \partial) Q \right\} = \det^{-1/2} \left[ -\frac{i}{2\hbar} e^{-\mu/\phi} \partial(\phi \partial) \right]. 
$$

(75)

We have (see Appendix Eqs.(122,123))

$$
\det^{-1/2} \left[ -\frac{i}{2\hbar} e^{-\mu/\phi} \partial(\phi \partial) \right] = \det^{-1/2} \left[ -\frac{i}{2\hbar} \frac{1}{\phi} \partial(\phi \partial) \right] 
$$

$$
\times \exp \left\{ -\frac{i}{48\pi} \int d^2\xi \left\{ \frac{1}{2} \mu \partial^2 \mu + \frac{3}{2} \mu [2\partial^2 \ln |\phi| + (\partial \ln |\phi|)^2] \right\} \right\}. 
$$

(76)

Because

$$
\det^{-1/2} \left[ -\frac{i}{2\hbar} \frac{1}{\phi} \partial(\phi \partial) \right] = \int D_{0,\phi} Q \exp \left\{ \frac{i}{2\hbar} \int d^2\xi Q \partial(\phi \partial) Q \right\}, 
$$

(77)

Eqs.(75), (76) and (77) obviously match with the formula (69).

After this digression, we now compute the effective action for the case $N = 24$. We use the defining formula (54) for the generating functional in the conformal gauge and insert the field dependences of the measures (57), (58) and (59) in it. We obtain

$$
W[J] = \int D_{0,\mu} D_{0,\phi} D_{0,\phi} Q^j \exp \left\{ -\frac{24i}{32\pi} \int d^2\xi \mu [2\partial^2 \ln |\phi_c| + (\partial \ln |\phi_c|)^2] \right\} 
$$

$$
\exp \left\{ \frac{i}{\hbar} \int d^2\xi \left[ \frac{\phi}{2\kappa} (-\partial^2 \mu - \kappa \partial Q^j \partial Q^j) - J_g \partial^2 \mu + J_{\phi} \phi + J_Q Q_j \right] \right\} 
$$

16
The integration over $Q_j$ is Gaussian and over $\mu$ gives the $\delta$-function, therefore

$$W[J] = \int D\phi \delta(2\partial^2\phi + 2\kappa\partial^2J_g + \frac{3\hbar\kappa}{2\pi}[2\partial^2 \ln |\phi| + (\partial \ln |\phi|)^2])$$

$$\times \det^{-12}[ - \frac{i}{2\hbar} \phi(\partial(\phi)] \exp \left[ \frac{i}{\hbar} \int d^2 \xi (J_\phi \phi - J_j \frac{1}{2\partial(\phi(\partial) J_j])} \right] =$$

$$= \det^{-12}[ - \frac{i}{2\hbar} \phi(J_g) \partial(\phi(J_g)\partial)]$$

$$\times \exp \left[ \frac{i}{\hbar} \int d^2 \xi (J_\phi \phi(J_g) - J_j \frac{1}{2\partial(\phi(J_g)\partial) J_j]}) \right],$$

where the dependence of $\phi(J_g)$ on $J_g$ is dictated by the $\delta$-function in (79). We stress that the formula (79) gives the exact generating functional. Performing the Legendre transformation (56) and (57) we obtain the exact effective action

$$\Gamma(\mu_c, \phi_c, Q^j_c) = \int d^2 \xi \phi_c (-\partial^2\mu_c - \kappa\partial Q^j_c \partial Q^j_c)$$

$$- \frac{3}{4\pi} \int d^2 \xi \mu_c[2\partial^2 \ln |\phi_c| + (\partial \ln |\phi_c|)^2] + 12i\hbar \ln \det[- \frac{i}{2\hbar} \phi_c \partial(\phi_c(\partial)])].$$

Comparing the result with Eq.(65), we conclude that for $N = 24$ the semiclassical approximation is, in fact, exact.

4 Quantum curvature singularities

4.1 The semiclassical field equations

We obtain the semiclassical field equations by varying the one loop effective action $\Gamma_1$ with respect to the classical fields $\mu_c, \phi_c$ and $Q^j_c$. We have

$$- \frac{1}{2\kappa} \partial^2 \phi_c + \hbar \frac{24}{48\pi} \partial^2 \mu_c + \hbar \frac{iN}{2\mu_c} \delta \frac{1}{\mu_c} \partial(\phi_c(\partial)] = 0,$$

$$\partial(\phi_c(\partial Q^j_c) = 0,$$

$$- \partial^2 \mu_c - \kappa \partial Q^j_c \partial Q^j_c + \hbar \kappa N \delta \frac{1}{\phi_c} \partial(\phi_c(\partial)] = 0.$$
The solutions of the semiclassical equations have expansions

\[ \mu_c = \mu_{c,0} + \hbar \mu_{c,1} + O(\hbar^2), \tag{84} \]

\[ \phi_c = \phi_{c,0} + \hbar \phi_{c,1} + O(\hbar^2), \tag{85} \]

\[ Q^i_c = Q^i_{c,0} + \hbar Q^i_{c,1} + O(\hbar^2). \tag{86} \]

Because we know just the first loop corrections to the effective action, the \(O(\hbar^2)\)-terms in the field expansions (84), (85) and (86) are irrelevant in the one loop approximation. Our next task will consist of the determination of \(\mu_{c,1}, \phi_{c,1}\) and \(Q^i_{c,1}\) from the semiclassical equations (81), (82) and (83), when \(\mu_{c,0}, \phi_{c,0}\) and \(Q^i_{c,0}\) is a given classical solution. Since the “lndet” terms in the one loop field equations are already of the order \(O(\hbar)\), it is enough to compute the functional derivatives of lndet at the classical solution \(\mu_{c,0}\) and \(\phi_{c,0}(=t)\).

The actual calculation requires the knowledge of the heat kernels of elliptic operators, some theory of the Hankel transformations and some integration of the Bessel functions. The details are presented in the Appendix here we list only the final result

\[ \frac{\delta}{\delta \mu_c} \text{lndet} \left[ -\frac{i}{2\hbar} e^{-\mu_c} \frac{1}{\phi_c} \partial(\phi_c \partial) \right] = \frac{i}{24\pi} \partial^2 \mu_c + \frac{i}{16\pi} \left[ 2 \partial^2 \ln |\phi_c| + (\partial \ln |\phi_c|)^2 \right]. \tag{87} \]

\[ \frac{\delta}{\delta \phi_c(\xi)} \text{lndet} \left[ -\frac{i}{2\hbar} e^{-\mu_c} \frac{1}{\phi_c} \partial(\phi_c \partial) \right]|_{\phi_c=t} = \frac{i}{8\pi t} \left[ \partial^2 \mu_c - \partial(\mu_c \partial \ln |t|) \right] - \frac{i}{8\pi t^3} \ln \left( \frac{\hbar t^2}{\epsilon \Omega} \right), \tag{88} \]

where \(\Omega\) is a finite constant and \(\epsilon \rightarrow 0\) is the ultraviolet cut-off. The following counter-term is needed for the cancelation of the UV divergence

\[ \mathcal{L}_{\text{count.}} \sim \hbar \sqrt{-g} g^{\alpha\beta} \phi^{-2} \partial_\alpha \phi \partial_\beta \phi. \tag{89} \]

This is the vertex of the new type, as it is usual in the renormalization of 2-d field theories. Its appropriate tuning replaces the product \(\epsilon \Omega\) in the semiclassical equations by some finite constant, but it cannot remove the logarithmical dependence on \(t^2\).
Inserting the evaluated functional derivatives (87) and (88) and the $\hbar$-expansion (84), (85) and (86) of the fields into the semiclassical equations (81), (82) and (83) we obtain the equations for $\mu_{c,1}, \phi_{c,1}$ and $Q^i_{c,1}$

$$-\frac{1}{2\kappa} \partial^2 \phi_{c,1} + \frac{24 - N}{48\pi} \partial^2 \mu_{c,0} - \frac{N}{32\pi t^2} = 0,$$

(90)

$$\partial(\phi_{c,1} \partial Q^i_{c,0}) + \partial(t \partial Q^i_{c,1}) = 0,$$

(91)

$$(-\partial^2 \mu_{c,1} - 2\kappa \partial Q^i_{c,1} \partial Q^i_{c,0}) = \frac{N\kappa}{8\pi} \left( -\frac{1}{t} \partial^2 \mu_{c,0} + \frac{1}{t^2} \partial_t \mu_{c,0} - \frac{1}{t^3} \mu_{c,0} - \frac{1}{t^3} \ln \frac{\hbar t^2}{\text{const}} \right).$$

(92)

This system of equations can be solved in a similar way as the classical system (12), (13) and (14) was solved. Indeed, because we know the Green function of the Minkowski d’Alambertian, we find from Eq. (90) the general form of $\phi_{c,1}$, by adding arbitrary solution of the homogeneous equation to one particular solution of the full equation. Inserting $\phi_{c,1}$ into (91), we obtain the linear (inhomogeneous) Gowdy equation for $Q^i_{c,1}$. Since we know the eigenvalues and the eigenfunctions of the Gowdy operator, we know also its Green function and, eventaully, the general form of $Q^i_{c,1}$. Finally, putting $Q^i_{c,1}$ into (92), we obtain the linear inhomogeneous d’Alambert equation for $\mu_{c,1}$, the general solution of which can be easily found. We conclude that our semiclassical equations (90), (91) and (92) are exactly solvable. For our purposes, there is no need to write down the explicite (and somewhat cumbersome) formulas. Instead of that we shall concentrate on the behaviour of the general solution near $t \sim 0$. We shall show, somewhat surprisingly, that even when we consider a regular classical solution $\mu_{c,0}, \phi_{c,0}$ and $Q^i_{c,0}$, the corresponding solution $\mu_{c,1}, \phi_{c,1}$ and $Q^i_{c,1}$ possesses necessarily the curvature singularity at $t = 0$. Such space-times are therefore classically regular but the quantum fluctuations induce the scalar curvature singularity, proportional to $\hbar$. Hence, the quantum effects not only do not smear the classical curvature singularities, they even destabilize the regular space-times! We present the corresponding quantitative analysis in the next subsection.

### 4.2 The scalar curvature of the semiclassical space-times

Let us study the behaviour of the scalar curvature $R$ near $t \sim 0$ for the space-times which solve the semiclassical field equations. In this subsection
we omit the index “c” of the fields $\mu_c, \phi_c$ and $Q^j_c$. We choose such classical metric field $\mu_0$, that the classical space-time is nonsingular. From Eqs.(25) and (29) for $B^j = 0$, it follows for $t \sim 0$

$$\mu_0 \sim t^2 f(\sigma) + \cdots,$$  

$$Q_0 = g(\sigma) + t^2 h(\sigma) + \cdots,$$  

where $f(\sigma), g(\sigma)$ and $h(\sigma)$ are functions, the concrete form of which is not relevant for our purposes and dots denote the subleading terms, also irrelevant for our analysis of the curvature singularities. Hence, from the Eq.(90) we find the behaviour of $\phi_1$ near $t \sim 0$

$$\phi_1 \sim -\frac{N\kappa}{16\pi} \ln|t| + \text{const } f(\sigma) t^2 + F(U) + G(V) + \cdots.$$  

The functions $F(U)$ and $G(V)$ cannot be specified from this equation, however, we may change our “dilaton” gauge condition (19) by the prescription

$$t + \bar{h}[F(U) + G(V)] \to t.$$  

The one loop effective action (66) is invariant under this transformation, hence, the semiclassical equations (81),(82) and (83) remain unchanged. Moreover, the classical solution at which the functional derivatives of the determinants are to be evaluated change just by the terms of order $\bar{h}$. This effect, of course, remain unseen in the one loop approximation, because the “Indet” terms are already of the first order in $\bar{h}$-expansion. Thus, all subsequent analysis goes through and we can omit the $F(U) + G(V)$ term and write without a loss of generality

$$\phi_1 \sim -\frac{N\kappa}{16\pi} \ln|t| + \text{const } t^2 + \cdots$$  

Now we insert $\phi_1$ into Eq.(91). We obtain

$$(-\frac{\partial^2}{\partial t^2} - \frac{1}{t} \frac{\partial}{\partial t} + \frac{\partial^2}{\partial \sigma^2}) Q_1 = \frac{H(\sigma)}{t} \ln|t| - \frac{N\kappa h(\sigma)}{8\pi t} + \text{bounded},$$  

where $H(\sigma)$ is some function of irrelevant shape. It is easy to determine the behaviour of a particular solution of (98) near $t \sim 0$. It is given by

$$Q_{1,\text{par}} = -H(\sigma) t \ln|t| + \left[ 2H(\sigma) + \frac{N\kappa h(\sigma)}{8\pi} \right] t + \cdots$$  

20
A general solution of the $Q_1$ equation near $t \sim 0$ is then given by Eq.(99) plus an arbitrary solution (25) of the homogeneous Gowdy equation. From (99), (25) and (94) then easily follows that

$$\partial Q^j_1 \partial Q^j_0 \sim C(\sigma) \ln |t| + \text{bounded.}$$

(100)

The function $C(\sigma)$ vanishes, if the Neumann modes are absent in the "homogeneous" part of $Q_1$. Inserting (100) into the remaining semiclassical evolution equation (92), we obtain

$$-\partial^2\mu_1 = 2\kappa C(\sigma) \ln |t| - \frac{N\kappa}{8\pi} \left[ \frac{f(\sigma)}{t} + \frac{1}{t^3 \ln \text{const}} \right] + \text{bounded},$$

(101)

hence

$$\mu_1 = -\frac{N\kappa}{16\pi t} \ln \frac{\hbar t^2}{\text{const}} + \rho(U) + \nu(V) + \text{bounded.}$$

(102)

In the classical case, the arbitrary integration functions $\rho(U)$ and $\nu(V)$ are determined from the constraints (20) and (21). In the semiclassical case they have to be determined from the given boundary conditions. The situation is fully analogous to that occurring in the CGHS model [4], where the semiclassical contribution to the constraints come from the Polyakov nonlocal action

$$S_P = \bar{\hbar} \frac{24 - N}{96\pi} \int d^2\xi \sqrt{-g} R \left( \frac{1}{\sqrt{-g}} \partial_{\alpha} \sqrt{-g} g^{\alpha\beta} \partial_{\beta} \right)^{-1} R,$$

(103)

We can make the generaly covariant action (103) local at the cost of introducing a new auxiliary field $Z$ (in a similar but not identical way as in [14]). It reads

$$S_P = \hbar \frac{24 - N}{48\pi} \int d^2\xi \sqrt{-g} \left[ \frac{1}{2} g^{\alpha\beta} \partial_{\alpha} Z \partial_{\beta} Z + R Z \right].$$

(104)

The contribution to the constraints are then obtained from $\delta g^{uu}$ and $\delta g^{vv}$ variations of the action, in the conformal gauge. We have

$$\frac{1}{\sqrt{-g}} \delta S_P = \frac{1}{\hbar} \frac{24 - N}{48\pi} \left( \frac{1}{2} Z_u Z_u - Z_{uu} + \mu_u Z_u \right),$$

(105)

$$\frac{1}{\sqrt{-g}} \delta S_P = \frac{1}{\hbar} \frac{24 - N}{48\pi} \left( \frac{1}{2} Z_v Z_v - Z_{vv} + \mu_v Z_v \right),$$

(106)

We get rid of the auxiliary field $Z$ using the equation of motion

$$\partial^2(\mu + Z) = 0,$$

(107)
\[ \mu = -Z + \mu^+(u) + \mu^-(v) \]  

and
\[ \frac{1}{\sqrt{-g}} \frac{\delta S_P}{\delta g^{uu}} = \hbar \frac{24 - N}{48\pi} (\mu_{uu} - \frac{1}{2} \mu_u \mu_u) + T^+(u), \]  
\[ \frac{1}{\sqrt{-g}} \frac{\delta S_P}{\delta g^{vv}} = \hbar \frac{24 - N}{48\pi} (\mu_{vv} - \frac{1}{2} \mu_v \mu_v) + T^-(v), \]  
The functions \( T^+(u) \) and \( T^-(v) \) are undetermined, because \( \mu^+(u) \) and \( \mu^-(v) \) are not known.

In our present model the Polyakov nonlocal action is the part of the one loop semiclassical action (c.f.(66)). Therefore, the unknown functions \( \rho(U) \) and \( \nu(V) \) can be specified only by fixing the boundary conditions. The \( \delta g^{uu} \) and \( \delta g^{vv} \) variations of the remaining undet part of the effective action of our model cannot influence this conclusion and we shall not consider them.

Finally we are ready to write down the scalar curvature of the semiclassical space-times. It reads (see Eq.(16))
\[ R(\bar{h}) = -e^{-\mu(\bar{h})} \partial^2 \mu(\bar{h}) = R(0) + \hbar (-R(0)\mu_1 - e^{-\mu_0} \partial^2 \mu_1) + O(\hbar^2) = \]  
\[ = R(0) + \hbar \left\{ R(0) \frac{N \kappa}{16\pi t} \ln \frac{\hbar t^2}{\text{const}} - R(0)(\rho(U) + \nu(V)) \right\} \]  
\[ + \hbar \left\{ - e^{-\mu_0} 2 \kappa C(\sigma) \ln |t| + e^{-\mu_0} \frac{N \kappa}{8\pi t} \left( f(\sigma) + \frac{1}{t^2} \ln \frac{\hbar t^2}{\Omega} \right) \right\} \]  
+ bounded + O(\hbar^2). \]  

Clearly, whatever the functions \( \rho(U) \) and \( \nu(V) \) may be, the semiclassical space-time is obviously singular. The singularity occurs at \( t = 0 \) and all timelike observers will run into it - which is referred to as a “thunderbolt” in Ref.[11]. We arrived at the remarkable conclusion, that while at the classical level there existed the nonsingular spacetimes, at the semiclassical level all space-times are necessarily singular. From Eq.(37) and (111) we also learn that the singular behaviour of classical and quantum curvature is different, hence no cancelation of a classical curvature singularity due to quantum effects may occur\(^2\). If the classical spacetime is regular, then the formula

\(^2\)I am grateful to R.Jackiw for a comment on this point.
(111) says, that the corresponding semiclassical space-time is plagued by the curvature singularity proportional to $\hbar$. Schematically

$$R = \text{regular} + \hbar \text{ singular} \cdots$$

(112)

We conclude, that the quantum effects destabilize the classical space-times and lead to even more severe curvature singularities than the classical dynamics does. There remain only one possibility to avoid this conclusion in the framework of the present model, which may seem quite unnatural, however. It consists in introducing by hand into the effective action several finite counter-terms of new type, which would be fine-tuned so that to cancel the divergent terms in (101). But also keeping this possibility in mind we may conclude that the quantum instabilities in our model are generic.

5 Conclusions and outlook

We attempted to give a detailed description of the classical and quantum dynamics of the Jackiw-Teitelboim gravity with the cosmological constant replaced by the kinetic term of matter fields. We showed that the classical solutions of the model have natural physical interpretation, namely they describe the collisions of the wave packets of matter. For huge class of such solutions the corresponding classical space-times are topologically trivial, asymptotically flat and free of curvature singularities. Then we computed the semiclassical effective action of the model; for the case $N = 24$ we, in fact, got the exact expression. The effective field equations turned out to be manageable from the technical point of view. We have solved them and provided a simple analysis of the semiclassical solutions near $t \sim 0$. The surprising result followed: the scalar curvature acquires the quantum correction which is necessarily singular. Hence, the quantum fluctuations do not smear classical curvature singularities, in fact they do just the opposite thing, they plague the regular classical space-times with quantum curvature singularities. Because for $N = 24$ we obtained the result starting from the exact effective action, our conclusion does not seem to be an artefact of the semiclassical approximation.

We believe that the model which we investigated is also interesting from the field theoretical point of view. At the classical level it is completely integrable and iteratively linear in the sense of subsection (2.2). This kind of
“linearity” played the decisive role in the evaluation of the continual integral, in a similar way as it was reported recently in the context of 2+1 Chern-Simmons theory \[38\]. The fact that for \( N = 24 \) that computation gives the exact result suggests an existence of a deeper algebraic structure in the model\[4\]. Moreover, it turns out that the model possesses an unexpected and interesting geometric structure. Indeed, the action \( (7) \) with the included matter field can be interpreted as the Jackiw-Teitelboim action (without a cosmological constant) in the non-commutative geometry of the “two sheet” manifolds \( Y \times Z_2 \), where \( Y \) is the 2-d space-time and \( Z_2 \) is the internal space containing just two points \[39, 40\]. The matter field plays the geometrical role of the distance between the two points in the internal space.

Our present model has also connections to the string theory on the curved backgrounds and to the exact 2-d conformal field theories. Indeed, in the conformal gauge, the classical action reads

\[
S = -\frac{1}{2\kappa} \int d^2 \xi (\partial \mu \partial \phi + \kappa \phi \partial Q^j \partial Q^j). \tag{113}
\]

This is obviously an action of the non-linear \( \sigma \)-model where \( \mu, \phi \) and \( Q^j \) are the coordinates of the target manifold with the metric

\[
ds^2 = -d\mu d\phi + \kappa \phi dQ^j dQ^j. \tag{114}
\]

It is not difficult to see that the metric \( (114) \) (it is written in the so called Rosen coordinates) describes a single gravitational plane wave propagating on \( N + 2 \)-dimensional target! \[29, 30\]. In other words, a single gravitational wave in \( N + 2 \)-dimensional target yields the \( \sigma \)-model action describing collisions of the two gravitational waves in two dimensions. Generalizing the work \[41\] Brooks has shown that adding the target dilaton background in the critical target dimension to such a \( \sigma \)-model, one obtains an exact conformal field theory \[42\]. It would be interesting to study our present model from this point of view. All the mentioned features of the model look quite promising for further investigations and we shall certainly return to those problems elsewhere.

I thank J. Fröhlich, K. Gawędzki and R. Jackiw for enlightening comments.

\[\text{3} \text{This comment is due to K.Gawędzki.} \]
6 Appendix

In this appendix, we evaluate the functional derivatives of the determinant in (65), which were needed for obtaining the explicit form of the semiclassical field equations. In evaluating the Traces we carefully keep in mind the definition of the scalar product (50).

Start with the derivative with respect to $\mu_c$.

$$
\frac{\delta}{\delta \mu_c}(\xi) \text{ln det} \left[ -\frac{i}{2\hbar} e^{-\mu_c} \frac{1}{\phi_c} \partial (\phi_c \partial) \right] = \\
= \frac{\delta}{\delta \mu_c}(\xi) (-1) \int_\varepsilon^\infty \frac{d\tau}{\tau} \text{Tr} \exp \left\{ -\tau \left[ -\frac{i}{2\hbar} e^{-\mu_c} \frac{1}{\phi_c} \partial (\phi_c \partial) \right] \right\} = \\
= \int_\varepsilon^\infty d\tau <\xi| e^{-\frac{i}{2\hbar} e^{-\mu_c} \frac{1}{\phi_c} \partial (\phi_c \partial)} \exp \left\{ -\tau \left[ -\frac{i}{2\hbar} e^{-\mu_c} \frac{1}{\phi_c} \partial (\phi_c \partial) \right] \right\} |\xi> \\
= -<\xi| \exp \left\{ -\varepsilon \left[ -\frac{i}{2\hbar} e^{-\mu_c} \frac{1}{\phi_c} \partial (\phi_c \partial) \right] \right\} |\xi>. \quad (115)
$$

We have used the heat-kernel regularization [43, 44] based on the following representation

$$
\text{ln } x = -\int_\varepsilon^\infty \frac{d\tau}{\tau} e^{-\tau x} + (\text{an } x \text{ independent constant}) + O(\varepsilon x). \quad (116)
$$

The “bras” and “kets” in Eq.(115) has to be understood in the standard sense. The asymptotic expression for the heat kernel for small $\varepsilon$ was obtained for an arbitrary elliptic operator in two dimensions [44]. If $M$ has the form

$$
M = -\frac{1}{\sqrt{-g}} (\nabla_\alpha + B_\alpha) \sqrt{-g} g^{\alpha \beta} (\nabla_\beta + B_\beta) - B_0, \quad (117)
$$

then\footnote{We put the sign + in front of $R$ (see also Alvarez [45] Eq. (4.38)), because $R$ is given by Eq.(10).}

$$
<\xi| e^{-M \varepsilon} |\xi> = \frac{1}{4\pi \varepsilon} \sqrt{-g} + \frac{1}{24\pi} R \sqrt{-g} + \frac{1}{4\pi} B_0 \sqrt{-g} + O(\varepsilon). \quad (118)
$$
Performing the Wick rotation to the Minkowski time, we can use Eq. (118) for evaluating the heat kernel (115). In our case

\[ B_\alpha = \frac{1}{2} \partial_\alpha \ln |\phi_c|, \]  

\[ B_0 = e^{-\mu_c} \left[ -\frac{1}{2} \partial^2 \ln |\phi_c| - \frac{1}{4} (\partial \ln |\phi_c|)^2 \right], \]  

hence

\[ \frac{\delta}{\delta \mu_c} \text{Indet} \left[ -\frac{i}{2\hbar} e^{-\mu_c} \frac{1}{\phi_c} \partial (\phi_c \partial) \right] = \frac{1}{24\pi} \partial^2 \mu_c + \frac{i}{16\pi} [2\partial^2 \ln |\phi_c| + (\partial \ln |\phi_c|)^2]. \]

Note that the functional derivative with respect to \( \mu_c \) is the local expression. We also did not consider the first term in the r.h.s. of Eq. (118), which is eventually to be cancelled by the two-dimensional cosmological constant counter-term.

Next we compute the variation with respect to \( \phi_c \). First of all we note that Eq. (121) implies

\[ \delta \frac{\partial}{\partial \phi_c} \left\{ \text{Indet} \left[ -\frac{i}{2\hbar} e^{-\mu_c} \frac{1}{\phi_c} \partial (\phi_c \partial) \right] \right\} = \frac{i}{8\pi \phi_c} [\partial^2 \mu_c - \partial (\mu_c \partial \ln |\phi_c|)] \\
+ \frac{\delta}{\delta \phi_c} \left\{ \text{Indet} \left[ -\frac{i}{2\hbar} [\partial^2 + (\partial \ln |\phi_c|) \partial] \right] \right\}. \]

Now we have

\[ \delta \text{Indet} \left\{ -\frac{i}{2\hbar} [\partial^2 + (\partial \ln |\phi_c|) \partial] \right\} |_{\phi_c=t} = \]

26
\[-\delta \int_{\epsilon}^{\infty} \frac{d\tau}{\tau} \text{Tr} \exp \left\{ -\tau \left[ -\frac{i}{2\hbar} \left( \partial^2 + (\partial \ln |\phi_c|)\partial \right) \right] \right\}_{\phi_c = t} =
\]

\[\int_{\epsilon}^{\infty} d\tau \text{Tr} \left\{ -\frac{i}{2\hbar} (\partial \phi_c / t) \partial \exp \left\{ \tau \left[ \frac{1}{2\hbar} (\alpha - i) (\partial_t^2 + \frac{1}{t} \partial_t) + \frac{1}{2\hbar} (\alpha + i) \partial_{\sigma}^2 \right] \right\} \right\}
\]

where \( \alpha > 0 \) is a (small) “Euclidean” cut-off, damping the oscillatory behaviour of the exponent. Now we wish to evaluate the last Trace in Eq. (125).

Form the basis of the space of fields \( Q \) as follows

\[\Psi_{\pm, k, p}(t, \sigma) = \theta(\pm t) J_0(kt) \frac{1}{\sqrt{2\pi}} e^{-ip\sigma}, k > 0, \quad (126)\]

where \( \theta(t) \) is the usual step function. Using the theory of the Hankel transformations [27], it is easy to establish the relations of orthogonality

\[\int_{R^2} dt d\sigma \ t \ \Psi_{\pm, k, p}^*(t, \sigma) \Psi_{\pm, k', p'}(t, \sigma) = (-1)^{\pm 1 - \frac{1}{2}} \frac{1}{k} \delta(k - k') \delta(p - p') \quad (127)\]

and the relation of completeness

\[\int_{0}^{\infty} kd\kappa \int_{-\infty}^{\infty} dp (\Psi_{+, kp}^*(\kappa) \Psi_{+, kp}(\kappa') - \Psi_{-, kp}^*(\kappa) \Psi_{-, kp}(\kappa')) = \frac{1}{t} \delta(t - t') \delta(\sigma - \sigma'). \quad (129)\]

(Note that in the r.h.s. of Eq. (129) stands the \( \delta \)-function \( \delta(\xi, \xi') \) with the respect to the inner product (50) for \( \mu = 0 \) and \( \phi = t \).) Therefore, the Trace of an operator \( O \) is given by

\[\text{Tr} O = \int_{R^2} dt d\sigma \int_{0}^{\infty} kd\kappa \int_{-\infty}^{\infty} dp (\Psi_{+, kp}^*(\kappa) O \Psi_{+, kp}(\kappa) - \Psi_{-, kp}^*(\kappa) O \Psi_{-, kp}(\kappa)) \quad (130)\]

Using Eq. (130), we can easily evaluate the Trace in Eq. (125). We have

\[\delta \text{ndet} \left\{ -\frac{i}{2\hbar} \left( \partial^2 + (\partial \ln |\phi_c|)\partial \right) \right\}_{\phi_c = t} =
\]

\[\int_{\epsilon}^{\infty} d\tau \int_{R^2} dt d\sigma \int_{0}^{\infty} kd\kappa \int_{-\infty}^{\infty} dp \exp \left\{ -\tau \left[ \frac{1}{2\hbar} (\alpha - i) k^2 + \frac{1}{2\hbar} (\alpha + i) p^2 \right] \right\}
\]
\[(\Psi_{+kp}(\xi)(-\frac{i}{2h}(\frac{\partial}{\partial t}\delta\phi_c)\partial)\Psi_{+kp}(\xi) - (\Psi_{-kp}(\xi)(-\frac{i}{2h}(\frac{\partial}{\partial t}\delta\phi_c)\partial)\Psi_{-kp}(\xi)) = \]
\[= \int_{\epsilon}^{\infty} d\tau \int_{\mathbb{R}^2} dt d\sigma t \int_{0}^{\infty} dk \int_{-\infty}^{\infty} dp \exp \left\{ -\frac{\tau}{2h}[(\alpha - i)k^2 + (\alpha + i)p^2] \right\} \]
\[\frac{i}{2h}(\partial_t \delta\phi_c) \frac{1}{2} \partial_t (\Psi_{+kp}(\xi)\Psi_{+kp}(\xi) - \Psi_{-kp}(\xi)\Psi_{-kp}(\xi)). \quad (131)\]

Now the formula (see [27])
\[\int_{0}^{\infty} r dr e^{-\rho^2 r^2} J_0(\lambda r) J_0(\mu r) = \frac{1}{2\rho^2} e^{-\frac{(\lambda^2 + \mu^2)}{4\rho^2}} J_0(\frac{i\lambda\mu}{2\rho^2}) \quad (132)\]
and the standard Gaussian integration explicitly give the integrals over \(k\) and \(p\). We obtain (for \(t \neq 0\))
\[\delta \text{indet} \left\{ -\frac{i}{2h} [\partial^2 + (\partial \ln |\phi_c|)\partial] \right\}_{\phi_c = t} = \int_{\epsilon}^{\infty} d\tau \int_{\mathbb{R}^2} dt d\sigma t \frac{i}{4h} \partial_t (\delta \phi_c) \]
\[\sqrt{\frac{h}{(\alpha + i)2\pi}} \partial_t \left[ \frac{h}{(\alpha - i)\tau} \right] e^{-\frac{\rho^2}{2\tau^2}} J_0(\frac{i\rho t^2}{\tau(\alpha - i)}) (\theta^2(t) - \theta^2(-t)). \quad (133)\]

We can rewrite Eq. (133) as follows
\[\frac{\delta}{\delta \phi_c(t, \sigma)} \text{indet} \left\{ -\frac{i}{2h} [\partial^2 + (\partial \ln |\phi_c|)\partial] \right\}_{\phi_c = t} = \]
\[= -[\theta(t) - \theta(-t)] \int_{0}^{h} dp \rho^{-1/2} \frac{i}{4\sqrt{2\pi}(\alpha - i)} t \partial_t [t \partial_t (e^{-\frac{\rho^2}{4\tau^2}} J_0(\frac{i\rho t^2}{\alpha - i}))] \]
\[= -[\theta(t) - \theta(-t)] \frac{i}{4\sqrt{2\pi}(\alpha - i)} \frac{1}{t} \partial_t \partial_t (\frac{1}{t} \int_{0}^{h\tau^2} dp \rho^{-1/2} e^{-\frac{\rho^2}{4\pi}} J_0(\frac{i\rho}{\alpha - i})) \]
\[= -\frac{i}{4\sqrt{2\pi}(\alpha - i)} \frac{1}{t} \partial_t \partial_t (\frac{1}{t} \int_{0}^{h\tau^2} dp \rho^{-1/2} e^{-\frac{\rho^2}{4\pi}} J_0(\frac{i\rho}{\alpha - i})). \quad (134)\]

Now we can decompose the integral over \(\rho\)
\[\int_{0}^{h\tau^2} dp \rho^{-1/2} \frac{e^{-\frac{\rho^2}{4\pi}}}{\sqrt{\rho}} J_0(\frac{i\rho}{\alpha - i}) = \quad (135)\]
\[\text{const} + \int_{1}^{h\tau^2} dp \rho e^{-\frac{\rho^2}{4\pi}} J_0(\frac{i\rho}{\alpha - i}) - \frac{\alpha - i}{\sqrt{2\pi} \alpha - i} e^{-\frac{\rho^2}{4\pi}} + \int_{1}^{h\tau^2} dp \rho \sqrt{\frac{\alpha - i}{2\pi}}.\]
From the asymptotic behaviour of the Bessel functions \[27\], we conclude that the first integral in the r.h.s. of Eq.(135) is convergent for \(\varepsilon \to 0\). Hence

\[
\frac{\delta}{\delta \phi_c(t,\sigma)} \ln \det \left\{ -\frac{i}{2\hbar} \left[ \partial^2 + (\partial \ln |\phi_c|) \partial \right] \right\}_{|\phi_c=t} = \frac{i}{8\pi t} \ln \left( \frac{\hbar t^2}{\varepsilon \text{ const}} \right) = -\frac{i}{8\pi t^3} \ln \left( \frac{\hbar t^2}{\varepsilon \Omega} \right),
\]

where \(\Omega\) is a finite constant.

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