BRST Cohomology and Hilbert Spaces of Non-Abelian Models in the Decoupled Path Integral Formulation

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Abstract

The existence of several nilpotent Noether charges in the decoupled formulation of two-dimensional gauge theories does not imply that all of these are required to annihilate the physical states. We elucidate this matter in the context of simple quantum mechanical and field theoretical models, where the structure of the Hilbert space is known. We provide a systematic procedure for deciding which of the BRST conditions is to be imposed on the physical states in order to ensure the equivalence of the decoupled formulation to the original, coupled one.
1 Introduction

Bosonization techniques have proven useful in solving two–dimensional quantum field theories. In particular the Schwinger and Thirring models have been solved in this way [1]. In the path integral framework the solubility of the Schwinger model manifests itself in the factorization of the partition function in terms of a free massive positive metric field, a free negative metric zero mass field and free massless fermions [1, 2]. It has long been realized that this factorization can be understood as a chiral change of variables in the path integral [3]. In this decoupled formulation the physical Hilbert space of gauge invariant observables of the original model is recovered by implementing the BRST conditions associated with the usual gauge fixing procedure and the chiral change of variables [1]. These conditions are identical to those originally obtained by Lowenstein and Swieca [5] on the operator level, stating that the physical Hilbert space be annihilated by the sum of the currents of the negative metric fields and the free fermions.

Similar bosonization techniques have been applied to Quantum Chromodynamics in 1+1–dimensions (QCD$_2$). A review can be found in [1, 6]. Analogous to the case of the Schwinger model, it has recently been shown that for a suitable choice of integration variables the partition function of QCD$_2$ factorizes in terms of free massless fermions, ghosts, negative level Wess Zumino Witten fields and fields describing massive degrees of freedom [7]. In the case of one flavor the sector corresponding to the massless fields was found [4] to be equivalent to that of a conformally invariant, topological $G/G_1$ coset model with $G$ the relevant gauge group. As a result of the gauge fixing and the decoupling procedure there were shown to exist several nilpotent charges [9] associated with BRST-like symmetries. These charges were found to be second class. This raises the question as to whether all or just some of these charges are required to annihilate the physical states.

Assuming the ground state(s) of QCD$_2$ to be given by the state(s) of the conformally invariant sector, (as is the case in the Schwinger model), the solution of the corresponding cohomology problem led to the conclusion [4] that the ground state of QCD$_2$ with gauge group $SU(2)$ and one flavor is 2 times twofold degenerate, and corresponding to the primaries of the $(U(1) \times SU(2)_1)/SU(2)_1$ coset describing the conformal sector (with $U(1)$ playing a spectator role). Since there are however also BRST constraints linking the coset-sector to the sector of massive excitations, the above hypothesis is not necessarily realized.

In ref. [10] the idea of smooth bosonization was introduced whereby two
dimensional path integral bosonization is formulated in terms of a gauge fixing procedure. To accomplish this, a “bosonization” gauge symmetry with an associated “bosonization” BRST symmetry were introduced. It was argued in ref. [11] that the most natural choice of gauge to recover the canonical bosonization dictionary is to “gauge fix” the fermions in a $U(N)/U(N)$ coset model. In fact, non-abelian smooth bosonization has only been achieved by this choice of “gauge” [11, 12]. The main result of this approach to bosonization [11] is that the free fermion partition function factorizes into the partition function of a $U(N)/U(N)$ coset model and a WZW model. An interpretation of this result along the lines mentioned above for the case of $QCD_2$ would lead one to conclude that the spectrum of free $U(2)$ fermions is two-fold degenerate. This conclusion is, however, wrong since the “bosonization” BRST links the $U(N)/U(N)$ coset model to the ”matter” sector described by the WZW model [11] (see section 3). Therefore the BRST constraints play an essential role in identifying the correct spectrum.

The above examples illustrate that extreme care has to be taken with regard to the implementation of the BRST symmetries when identifying the physical states. In particular the identification of the BRST symmetries on the decoupled level can be misleading as not all the charges associated with these symmetries are generally required to annihilate the physical states in order to ensure equivalence with the original, coupled formulation. In section 2 we illustrate this point using a simple quantum mechanical model for which the Hilbert space is known and the BRST symmetries and associated cohomology problem are very transparent and easy to solve. These considerations generalize to the field theoretic case. As examples we discuss the non-abelian bosonization of free fermions in section 3 and some aspects of $QCD_2$, as considered in ref. [13], in section 4.

It is of course well known that the BRST symmetries on the decoupled level originate from the changes of variables made to achieve the decoupling, as has been discussed in the literature before (see e.g. ref. [13, 23, 24]). The aspect we want to emphasize here is the role these symmetries, and the associated cohomology, play in constructing the physical subspace on the decoupled level, i.e., a subspace isomorphic to the Hilbert space of the original coupled formulation. In particular we want to stress that the mere existence of a nilpotent symmetry does not imply that the associated charge must annihilate the physical states. We are therefore seeking a procedure to decide on the latter issue. We formulate such a procedure in section 2 and apply it in sections 3 and 4. It amounts to implementing the changes of variables by
inserting an appropriate identity in terms of auxiliary fields in the functional integral. BRST transformation rules for these auxiliary fields are then introduced such that the identity amounts to the addition of a $Q$-exact form to the action. The dynamics is therefore unaltered on the subspace of states annihilated by the corresponding BRST charge. We must therefore require that this charge annihilate the physical states. The auxiliary fields are then systematically integrated out and the BRST transformation rules "followed" through the use of equations of motion. This simple procedure allows for the identification of the BRST charges on the decoupled level which are required to annihilate the physical subspace isomorphic to the Hilbert space of the coupled formulation.

The paper is organized as follows: In section 2 we illustrate the points raised above in terms of a simple quantum mechanical model. Using this model as example we also formulate a general procedure to identify the BRST charges required to annihilate the physical states such that equivalence with the coupled formulation is ensured on the physical subspace. In sections 3 and 4 we apply this procedure to the non-abelian bosonization of free Dirac fermions in 1+1 dimensions and to QCD$_2$, respectively.

## 2 Quantum mechanics

In this section we discuss a simple quantum mechanical model to illustrate the points raised in the introduction. Since the emphasis is on the structures of the Hibert spaces on the coupled and decoupled levels, and the role which the BRST symmetries and associated cohomology play in this respect, we introduce the model in second quantized form and only then set up a path integral formulation using coherent states. The model is then bosonized (decoupled) and the BRST symmetries and cohomologies are discussed.

Consider the model with Hamiltonian

$$\hat{H} = 2g \hat{J} \cdot \hat{\tilde{J}}$$  \hspace{1cm} (2.1)

where $\hat{J}^a$ are the generators of a $SU(2)$ algebra. We realize this algebra on Fermion Fock space in the following way \cite{14}:
\[ \hat{J}^+ = \sum_{m>0} a_m^\dagger a_{-m}, \]
\[ \hat{J}^- = \sum_{m>0} a_{-m} a_m, \]
\[ \hat{J}^0 = \frac{1}{2} \sum_{m>0} (a_m^\dagger a_m - a_{-m} a_{-m}^\dagger). \]

Here \( a_m^\dagger (a_m) \), \(|m| = 1, 2, \ldots N_f \) are fermion creation (annihilation) operators. The operators (2.2) provide a reducible representation of the usual commutation relations of angular momentum and the representations carried by Fermion Fock space are well known [15]. The index \(|m|\) plays the role of flavor with \( N_f \) the number of flavors. The Hamiltonian (2.1) thus describes a \( SU(2) \) invariant theory with \( N_f \) flavors, as will become clear in the Lagrange formulation discussed below.

The spectrum of the Hamiltonian \( \hat{H} \) is completely known, the eigenvalues of \( \hat{H} \) being given by

\[ E_j = 2 g_j (j + 1) \]

where each eigenvalue is \( g_j (2j + 1) \)-fold degenerate, with \( g_j \) the number of times the corresponding irreducible representation occurs. In particular, for \( N_f = 1 \) the spectrum of \( \hat{H} \) consists of a doublet, as well as two singlets describing a two-fold degenerate state of energy \( E = 0 \). For positive \( g \) this corresponds to a two-fold degenerate ground state.

We express the vacuum–to–vacuum amplitude associated with \( \hat{H} \) as a functional integral over Grassmann variables. For this purpose introduce the fermionic coherent state [16]

\[ | \chi > = \exp \left[ - \frac{1}{2} \sum_{m>0} (\chi_m^\dagger \chi_m + \chi_{-m}^\dagger \chi_{-m} + \chi_m a_{m}^\dagger - \chi_{-m} a_{-m}^\dagger) \right] | 0 > \]

(2.4)

where \( \chi_m^\dagger, \chi_m \) are complex valued Grassmann variables and

\[ a_m | 0 > = a_{-m}^\dagger | 0 > = 0, \quad \forall m > 0. \]

We obtain the path integral representation of the vacuum–to–vacuum transition amplitude by following the usual procedure [17] and using the completeness relation for the coherent states [16]. We find

5
\[ Z = \int [d\eta^\dagger] [d\eta] e^{i \int dt L_F} \]  

(2.6a)

where \( L_F \) is the "fermionic" Lagrangian

\[ L_F = \eta_f^\dagger (i \partial_t + m) \eta_f - g \operatorname{tr} j^2 \]  

(2.6b)

and a summation over the flavor index \( f (f = 1, 2, \ldots, N_f) \) is implied. The mass \( m = -3g \) arises from normal ordering with respect to the Fock vacuum \( |0> \) defined in (2.5). Furthermore, \( \eta_f \) denotes the two-component spinor

\[ \eta_f = \begin{pmatrix} \chi_f \\ \chi_{-f} \end{pmatrix}, \]  

(2.7a)

and \( j, j^a_f \) are the "currents"

\[ \dot{j} = \sum_f j_f, \quad j_f = j^a_f t^a, \]  

(2.7b)

\[ j^a_f = \eta_f^\dagger t^a \eta_f \]

where the \( SU(2) \) generators are normalized as \( \operatorname{tr}(t^a t^b) = \delta^{ab} \).

Introducing the field

\[ B = B^a t^a, \]  

(2.8)

we can write the partition function as

\[ Z = \int [d\eta^\dagger] [d\eta] [dB] e^{i \int dt L} \]  

(2.9a)

\[ L = \eta_f^\dagger (i \partial_t + B + m) \eta_f + \frac{1}{2g} \operatorname{tr} B^2 \]  

(2.9b)

The bosons and fermions can be decoupled by making the change of variables

\[ B = V i \partial_t V^{-1} \]  

(2.10a)

where \( V \) are group valued fields in the fundamental representation of \( SU(2) \). Simultaneously we make the change of variables
\[ \psi_f = V^{-1} \eta_f . \] (2.10b)

The Jacobian associated with the transformation is (there are no anomalous contributions in 0 + 1 dimensions):

\[ J = \det (D_t^{adj}(V)) = \det (\partial_t) \] (2.11)

where \( D_t^{adj}(V) \) is the covariant derivative in the adjoint representation

\[ D_t^{adj}(V) = \partial_t + [V i \partial_t V^{-1}, ] . \] (2.12)

Representing the determinant in terms of ghosts we obtain for the partition function the factorized form

\[ Z = Z_F^{(0)} Z_{gh}^{(0)} Z_V = \int [d\psi^\dagger] [d\psi] [dV] [db] [dc] e^{i \int dt \mathcal{L}^{(0)}} \] (2.13a)

with

\[ \mathcal{L}^{(0)} = \mathcal{L}_F^{(0)} + \mathcal{L}_{gh}^{(0)} + \mathcal{L}_V \] (2.13b)

where

\[ \mathcal{L}_F^{(0)} = \psi_f^\dagger (i \partial_t + m) \psi_f , \]

\[ \mathcal{L}_{gh}^{(0)} = \mathrm{tr} b i \partial_t c , \] (2.13c)

\[ \mathcal{L}_V = \frac{1}{2g} \mathrm{tr} (V i \partial_t V^{-1})^2 . \]

Here \( b \) and \( c \) are Lie algebra valued ghost fields \( b = b^a t^a \) and \( c = c^a t^a \).

In arriving at the above decoupled form of the partition function, we have not mentioned the effect of the change of variables on the boundary of the path integral. It is important to realize that the decoupling of the partition function is not affected by the implied change of the boundary condition since the transformation (2.10) is local.

The Hilbert space associated with the factorized partition function is the direct product of fermion, boson and ghost Fock spaces and is clearly much larger than that of the original interacting model. It is therefore natural to ask what conditions should be imposed on this direct product space to recover a subspace isomorphic to the original Hilbert space. As these conditions are of a group theoretical nature, it is necessary to first clarify the
precise content of these Hilbert spaces from a representation theory point of
view before the above mentioned isomorphism can be established.

As we have seen the states in the Hilbert space (Fermion Fock space)
of the original interacting model can be labeled by \( |\alpha j m > \) where \( j, m \)
labels the \( SU(2) \)-flavor representations and weights, respectively, and \( \alpha \) is a
multiplicity index.

On the decoupled level the Hilbert space is the direct product of fermion,
bo son and ghost Fock spaces. Upon canonical quantization it is again clear
that states in the (free) fermionic sector can be labeled by
\( |\alpha j m >_F \) where the allowed values of \( \alpha \) and \( j \) coincide with those of
the interacting model. The ghosts \( c \) and \( b \) are canonically conjugate fields. We take \( b \)
as the annihilation and \( c \) as the creation operator. The ghost vacuum is then
defined by \( b^a |0> = 0 \). Defining the ghost number operator by
\( N_{gh} = c^a b^a \), we see that \( b^a \) carries ghost number \(-1\) and \( c^a \) ghost number \( 1 \).
Since we are interested in the physical sector which is built on the ghost vacuum
(ghost number zero), we do not need to analyze the representation theory content
of the ghost sector in more detail. It is therefore only the bosonic sector
that requires a detailed analysis.

Turning to the bosonic Lagrangian of (2.13c) we note the presence of a
left and right global symmetry

\[
\begin{align*}
V & \rightarrow LV, \\
V & \rightarrow VR,
\end{align*}
\]
where \( L \) and \( R \) are \( SU(2) \) matrices in the fundamental representation cor-
responding to left- and right-transformations, respectively. Following the
Noether construction the conserved currents generating these symmetries
are identified as

\[
\begin{align*}
L^a &= \frac{i}{g} \text{tr} i V V^{-1} V \partial_t V^a, \\
R^a &= \frac{i}{g} \text{tr} V \partial_t (V V^{-1}) V t^a,
\end{align*}
\]
respectively. In phase space this reads

\[
\begin{align*}
L^a &= \text{tr} i V \tilde{\pi}_V t^a, \\
R^a &= \text{tr} i \pi_V V t^a,
\end{align*}
\]
with \( \pi_V \) the momentum canonically conjugate to \( V \), and "tilde" denoting
"transpose". The following Poisson brackets are easily verified
\{L^a, L^b\}_P = -f^{abc} L^c, \\
\{R^a, R^b\}_P = f^{abc} R^c, \\
\{L^a, P^b\}_P = 0, \\
\{L^a, V\}_P = -i t^a V, \\
\{R^a, V\}_P = -i V t^a,
\hspace{1cm}(2.17)

where \(f^{abc}\) are the \(SU(2)\) structure constants.

Canonical quantization proceeds as usual. The Hilbert space of this system is well known, and corresponds to that of the rigid rotator [18]. Hence the Wigner \(D\)-functions \(D_{MK}^I\) provide a realization in terms of square integrable functions on the group manifold. It is important to note that the Casimirs of the left and right symmetries both equal \(I(I+1)\). Furthermore \(M\) and \(K\) label the weights of the left and right symmetries, respectively. To determine the allowed values of \(I\) one notes from (2.17) that \(V\) transforms as the \(j = \frac{1}{2}\) representation under left and right transformations. Thus \(V\) acts as a tensor operator connecting integer and half–integer spins. It follows that \(I\) can take the values \(I = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots\). We therefore conclude that on the decoupled level the states have the structure \(|\alpha j m >_F |IMK >_B |gh >\) where the subscripts \(F, B\) refer to the fermionic and bosonic sectors, respectively. The allowed values of the quantum numbers in these sectors are as discussed above.

Returning to the question as to which conditions are to be imposed on the direct product space of the decoupled formalism in order to recover the Hilbert space of the original model, we note by inspection of (2.13) the existence of three BRST symmetries (of which only two are independent). One of them acts in all three sectors and is given by

\[
\delta_1 \psi = c \psi, \\
\delta_1 \psi^\dagger = \psi^\dagger c, \\
\delta_1 V = -V c, \\
\delta_1 V^{-1} = c V^{-1}, \\
\delta_1 b = -j^{(0)} - R + \{b, c\}, \\
\delta_1 c = \frac{1}{2} \{c, c\}.
\hspace{1cm}(2.18)
\]

Here \(\delta_1\) is a variational derivative graded with respect to Grassmann number, \(\{ \}\) denotes a matrix anti–commutator and
\[ j^{(0)} = \sum_f (\psi^*_f \bar{t}^a \psi_f) t^a, \]
\[ R = R^a t^a = \frac{1}{g} (i \partial_t V^{-1}) V. \]

The other BRST symmetries act in the fermion–ghost and boson–ghost sectors, respectively, and are given by

\[
\begin{align*}
\delta_2 \psi &= c \psi, \\
\delta_2 \psi^\dagger &= \psi^\dagger c, \\
\delta_2 b &= -j^{(0)} + \{b, c\}, \\
\delta_2 c &= \frac{1}{2} \{c, c\}.
\end{align*}
\]

as well as

\[
\begin{align*}
\delta_3 V &= -V c, \\
\delta_3 V^{-1} &= c V^{-1}, \\
\delta_3 b &= -R + \{b, c\}, \\
\delta_3 c &= \frac{1}{2} \{c, c\}.
\end{align*}
\]

The above transformations are nilpotent. Note, however, that they do not commute.

Performing a canonical quantization, we define the ghost current

\[ J_{gh} = - : \{b, c\} : \]

where \( : \) denotes normal ordering with respect to the ghost vacuum. The nilpotent charges \( Q_i \) generating the transformations (2.18) – (2.21), i.e., \( \delta_i \phi = [Q_i, \phi] \), with \( \phi \) a generic field and [ , ] a graded commutator, have the general form:

\[
\begin{align*}
Q_1 &= -\text{tr} [c (j^{(0)} + R + \frac{1}{2} J_{gh})], \\
Q_2 &= -\text{tr} [c (j^{(0)} + \frac{1}{2} J_{gh})], \\
Q_3 &= -\text{tr} [c (R + \frac{1}{2} J_{gh})].
\end{align*}
\]

We remarked above that the direct product Hilbert space associated with the factorized form of the partition function is much larger than that of the original interacting model. We now inquire as to which of the above BRST
charges are required to vanish on the physical subspace ($H_{ph}$) in order to establish the isomorphism to the Hilbert space of the original model.

We begin by showing that $Q_1$ is required to vanish on $H_{ph}$. In order to illustrate the method, which will be used repeatedly, we briefly sketch the main steps for the case at hand. To implement the change of variables (2.10a) we make use of the identity

$$1 = \int [dV] \delta (B - V i \partial_t V^{-1}) \det i D_t^{adj} (V)$$

where the covariant derivative was defined in (2.12). Inserting this identity into (2.9a), using the Fourier representation of the Dirac delta and lifting the determinant by introducing Lie–algebra valued ghosts $\tilde{b}$ and $\tilde{c}$ we have

$$Z = \int [d\eta^\dagger] [d\eta] [dB] [d\lambda] [dV] [d\tilde{b}] [d\tilde{c}] e^{i \int dt L'}$$

with

$$L' = L + \Delta L$$

where

$$\Delta L = \text{tr} (\lambda (B - V i \partial_t V^{-1}) - \text{tr} (\tilde{b} i D_t^{adj} (V) \tilde{c})).$$

This Lagrangian is invariant under the BRST transformation

$$\begin{align*}
\delta \eta &= \delta \eta^\dagger = \delta B = \delta \lambda = 0, \\
\delta \tilde{b} &= \lambda, \\
\delta V &= \tilde{c} V, \delta V^{-1} = -V^{-1} \tilde{c}, \\
\delta \tilde{c} &= \frac{1}{2} \{ \tilde{c}, \tilde{c} \}.
\end{align*}$$

One readily checks that this symmetry is nilpotent off–shell. Noting that $\Delta L$ can be expressed as a BRST exact form

$$\Delta L = \delta (\tilde{h} (B - V i \partial_t V^{-1})),$$

we conclude that equivalence with the original model is ensured on the subspace of states annihilated by the corresponding BRST charge.

Next we show that the transformation (2.26) is in fact equivalent to the transformation (2.18). Using the equation of motion for $\lambda$ and $B$ we obtain the BRST transformation rules:
\[ \delta \psi = \delta \psi^\dagger = 0, \]
\[ \delta \tilde{b} = -\frac{1}{g} V i \partial_t V^{-1} - j, \]
\[ \delta \tilde{c} = \frac{1}{2} \{ \tilde{c}, \tilde{c} \}, \]
\[ \delta V = \tilde{c} V, \]
\[ \delta V^{-1} = -V^{-1} \tilde{c}. \]

(2.28)

One readily checks that this is a symmetry of the action with Lagrangian

\[ L = \eta_\dagger (i \partial_t + V i \partial_t V^{-1} + m) \eta_f + \frac{1}{2g} \text{tr} (V i \partial_t V^{-1})^2 - \text{tr} (\tilde{b} i D^{adj}_t (V) \tilde{c}). \]

(2.29)

obtained after integrating out \( \lambda \) and \( B \). Finally we return to the decoupled partition function (2.13) by transforming to the free fermions and ghosts

\[ \psi_f = V^{-1} \eta_f, \]
\[ c = -V^{-1} \tilde{c} V, \]
\[ b = V^{-1} \tilde{b} V. \]

(2.30)

In terms of these variables the BRST transformations (2.28) become those of (2.18). This demonstrates our above claim that the BRST charge generating the transformation (2.18) has to vanish on \( \mathcal{H}_{\text{ph}} \) to ensure equivalence with the original model.

An alternative way of proving the above statement is to note that the decoupled Lagrangian \( L^{(0)} \) of (2.13) can be expressed in terms of the original fermionic Lagrangian \( L_F \) of (2.6b) plus a \( \delta_1 \) exact part. In terms of the free fermions \( \psi_f \) and interacting fermions \( \eta_f = V^{-1} \psi_f \), we may rewrite \( L^{(0)} \) as:

\[ L^{(0)} = \eta_\dagger (i \partial_t + m) \eta_f + g \text{tr} (Rj^{(0)}) + \frac{g}{2} \text{tr} R^2 + \text{tr} (b i \partial_t c) \]

(2.31)

which may be put in the form

\[ L^{(0)} = \eta_\dagger (i \partial_t + m) \eta_f - g \frac{2}{2} \text{tr} j^2 + \frac{1}{2} \text{tr} (b i \partial_t c) - g \frac{2}{2} \delta_1 [\text{tr} b (R + j^{(0)})] . \]

(2.32)

Comparing with (2.6) we see that \( L^{(0)} \) and \( L_F \) just differ by a BRST exact term (up to a decoupled free ghost term). Hence we recover the original
fermion dynamics on the sector which is annihilated by the BRST charge $Q_1$. Therefore only the first of the three BRST symmetries (2.18) – (2.21) has to be imposed on the states. To see what this implies, we now solve the cohomology problem associated with $Q_1$.

As usual we solve the cohomology problem in the zero ghost number sector $b^a \Psi_{ph} = 0$. The condition $Q_1 | \Psi_{ph} = 0$ is then equivalent to (see (2.22) and (2.23))

\[(j^{(0)} + R) | \Psi_{ph} = 0. \tag{2.33}\]

The physical states $| \Psi_{ph} \rangle$ are thus singlets under the total current $J = j^{(0)} + R$. We have already established the general structure of states on the decoupled level and it is now simple to write down the solution of the cohomology problem (2.33); it is

\[| \alpha j m >_{ph} = \sum_M < j M j - M | 0 0 > | \alpha j M >_F | j - M m >_B | 0 >_{gh} \tag{2.34}\]

with $< j M j - M | 0 0 >$ the Glebsch–Gordon coefficients. We note that (2.34) restricts the a priori infinite number of $SU(2)$ representations carried by boson Fock space to those carried by fermion Fock space. Equation (2.34) shows that every state in Fermion Fock space gives rise to exactly one physical state. This establishes the isomorphism between the decoupled formulation and the original model on the physical Hilbert space.

We note that the BRST condition can also be interpreted as a bosonization rule which states that on the physical subspace the following replacements may be made: $j^{(0)} \rightarrow -R$. This bosonization dictionary can be completed by constructing physical operators, i.e., the operators that commute with the BRST charge $Q_1$. Once this has been done, a set of rules result according to which every fermion operator can be replaced by an equivalent bosonic operator. It is easy to check that the following operators are BRST invariant

\[
1, \quad \eta_f^\dagger \eta_f, \\
\eta_f = V \psi_f, \quad \eta_f^\dagger = \eta_f^\dagger V^{-1}, \\
j_f^a = \psi_f^\dagger V^{-1} t^a V \psi_f = \eta_f^\dagger t^a \eta_f, \\
L^a = \frac{i}{g} \text{tr} (V i \partial_t V^{-1} t^a). \tag{2.35}
\]
We recognize in \( \eta_f, L^a \) and \( j^a_7 \) the (physical) fermion fields, boson fields \( B^a = g L^a \), and generators of the \( SU(2) \) color symmetry associated with the partition function (2.6) of the original model. Note that the currents \( R^a = \frac{i}{g} \text{tr} (i \partial_t V^{-1}) V t^a \) appearing in the BRST charge are not BRST invariant. Once the physical operators have been identified, the physical Hilbert space can be constructed in terms of them. In this way the isomorphism (2.34) can also be established.

As we have now demonstrated explicitly the only condition that physical states are required to satisfy in order to ensure the above isomorphism is that \( Q_1 | \Psi >_{ph} = 0 \). It is, however, interesting to examine what further restrictions would result by imposing that a state be annihilated by all three nilpotent charges. Since these charges do not commute, it raises the question as to whether this is a consistent requirement. This leads us to consider the algebra of those BRST charges. One finds

\[
[Q_\alpha, Q_\beta] = K_{(\gamma)} (\alpha, \beta, \gamma \text{ cyclic}),
\]

\[
K_{(\gamma)} = -\frac{1}{2} f^{abc} J^a_{(\gamma)} c^b c^c
\]

where \( J^a_{(\gamma)} = j^a_7, R^a \) and \( j^a_7 + R^a \) for \( \gamma = 1, 2 \) and 3, respectively. The \( K_{(\gamma)} \) are nilpotent and further have the properties \( [K_{(\gamma)}, Q_\alpha] = 0 \) and \( [K_{(\gamma)}, K_{(\gamma')}'] = 0 \).

The \( K_{(\gamma)} \) generate the infinitesimal transformation

\[
[K_1, \psi] = -\frac{1}{2} \{c, c\} \psi, \quad [K_1, b] = \{j, c\},
\]

\[
[K_2, \psi] = \frac{1}{2} \{c, c\} V, \quad [K_2, b] = \{R, c\}
\]

with \( K_1 + K_2 = K_3 \). As before \( [\ , \] denotes a graded commutator and \( \{ \ , \} \) a matrix anti-commutator. All other transformations vanish. They are easily checked to represent a symmetry of the action, as is required by consistency.

From eq (2.37) we note that the conditions \( Q_\alpha | \Psi > = 0 (\alpha = 1, 2, 3) \) can only be consistently imposed if we require \( K_{(\gamma)} | \Psi > = 0 (\gamma = 1, 2, 3) \) as well. The implementation of all three conditions would restrict the physical (ghost number zero) states to be singlets with respect to the physical fermionic currents generating the \( SU(2) \) symmetry. From eq (2.1) we note that for \( g > 0 \) the ground–state is a singlet. There is in fact a double degeneracy since there is a double multiplicity in the singlet sector for an arbitrary number of flavors. By restricting to this subspace one is therefore effectively studying the ground–state sector of the model.
3 Non-abelian bosonization by coset factorization

Consider the partition function of free fermions in the fundamental representation of $U(N)$. As mentioned in the introduction, the approach of ref. \cite{10, 12} to the bosonization of such fermions in two dimensions is most naturally implemented by factoring from the corresponding partition function $Z_F^{(0)}$, a topological $U(N)/U(N)$ coset carrying the fermion and chiral selection rules associated with the fermions, but no dynamics \cite{11}. In factoring out this coset, the bosonization BRST symmetry of ref. \cite{10, 12} is also uncovered.

To emphasize the care with which BRST symmetries have to be implemented in the identification of physical states, we note that if we were to ignore the BRST constraint linking the coset sector to the remaining WZW sector, we would conclude that the spectrum of free fermions is not equivalent to a WZW model, but to the direct product of the coset model and the WZW model. In particular we would conclude that this spectrum is N-fold degenerate. A correct interpretation thus requires a careful analysis of the BRST symmetries associated with the introduction of additional degrees of freedom in the path integral, and those associated with changes of variables. As we show in this section the original spectrum of free fermions is identified, if the BRST symmetries of the physical states are correctly identified.

To discuss the BRST cohomology associated with the bosonization BRST, it is useful to decouple the coset again, that is, we work with the fermionic coset in its decoupled form. The reason for doing this is that it is more convenient to analyze the physical spectrum of the coset model in the decoupled formulation \cite{8}.

As in the quantum mechanical models discussed above, one can proceed with the bosonization procedure and only after the final action has been obtained, the BRST symmetries are identified by inspection. The disadvantage of this procedure, as became abundantly clear in our discussion above, is that one does not recognize which of these BRST charges should be imposed as symmetries of the physical states to ensure equivalence with the original free fermion dynamics. Instead we follow here the procedure used above to identify the relevant BRST charges from first principles.

In subsection 3.1 we review briefly the main results of \cite{11}, showing how the bosonization BRST arises by an argument similar to that of section 2. In subsection 3.2 we proceed to rewrite the coset in the decoupled form, keeping track of the bosonization BRST and the new BRST that arises when the decoupling is performed. In the last part of this section we briefly discuss
the structure of the physical Hilbert space.

3.1 Bosonization BRST

As explained in ref. [11] the partition function of free Dirac fermions in the fundamental representation of $U(N)$ can be written as:

$$Z_F^{(0)} = Z_{U(N)} / U(N) \times Z_{WZW}$$  (3.1.1a)

where $Z_{U(N)} / U(N)$ is the partition function of a $U(N) / U(N)$ coset,

$$Z_{U(N)} / U(N) = \int [d\eta] [d\bar{\eta}] [d(ghosts)] [dB_-] e^{i \int d^2x \left( \eta^\dagger_1 i \partial_+ m + \eta^\dagger_2 \left( \partial_+ \bar{B}_- + B_- \eta_2 \right) \right) e^{i \int d^2x \text{tr} b - i \partial_+ c}} \tag{3.1.1b}$$

and $Z_{WZW}$ is the partition function

$$Z_{WZW} = \int [dg] e^{i \Gamma[g]}$$  (3.1.1c)

of a Wess-Zumino-Witten (WZW) field $g$ of level one, with $\Gamma[g]$ the corresponding action [20]

$$\Gamma[g] = \frac{1}{8\pi} \int d^2x \text{tr} (\partial_\mu g^{-1} \partial^\mu g^{-1}) + \frac{1}{12\pi} \int d^3x \epsilon^{\mu\nu\rho} \text{tr} (g \partial_\mu g^{-1} g \partial_\nu g^{-1} g \partial_\rho g^{-1}) \quad . \tag{3.1.2}$$

Since, as we have seen in section 2, the Hilbert space of the bosonic sector (described by the WZW action in the case in question) is in general much larger than that of the original fermionic description, the question arises as to which constraints must be imposed in order to ensure equivalence of the two formulations. We now show that if the BRST symmetry of the physical states is correctly identified, the original spectrum of free fermions is recovered.

We begin by briefly reviewing the steps leading to the factorized form (3.1.1), with the objective of establishing systematically which of the BRST symmetries should be imposed on the physical states.

We start with the partition function of free Dirac fermions in the fundamental representation of $U(N)$,

$$Z_F^{(0)} = \int [d\eta] [d\bar{\eta}] e^{i \int d^2x \left( \eta^\dagger_1 i \partial_+ m + \eta^\dagger_2 \left( \partial_+ \bar{B}_- + B_- \eta_2 \right) \right)} \tag{3.1.3}$$
Following ref. [11] we enlarge the space by introducing bosonic \( U(N) \) Lie algebra valued fields \( B_- = B_a^a \) (tr\( t^a t^b \) = \( \delta^{ab} \)) via the identity

\[
1 = \int [dB_-] e^{i \int d^2 x [\eta^1_1 B_- \eta^2_2] \delta [B_-]}.
\] (3.1.4)

Using a Fourier representation of the Dirac Delta functional by introducing an auxiliary field \( \lambda_+ \), the partition function (3.1.3) then takes the alternative form

\[
Z^{(0)} = \int [d\eta_1][d\eta_2] e^{i \int \{\eta^1_1 i\partial_+ \eta_1 + \eta^1_2 (i\partial_+ + B_-) \eta_2 + \text{tr} \lambda_+ B_- \}}
\] (3.1.5)

where \( \lambda_+ \) are again \( U(N) \) Lie algebra valued fields.

We now make the change of variable \( \lambda_+ \to g \) defined by \( \lambda_+ = \alpha g^{-1} i\partial_+ g \) where \( g \) are \( U(N) \) group-valued fields. The Jacobian associated with this transformation is ambiguous since we do not have gauge invariance as a guiding principle. For reasons to become apparent later, we choose it to be defined with respect to the Haar measure \( g \delta g^{-1} \). Noting that

\[
\delta(g^{-1} i \partial_+ g) = -g^{-1} i \partial_+ (g \delta g^{-1}) g
\] (3.1.6)

we have for the corresponding Jacobian,

\[
J = \int [d\tilde{b}_-][dc_-] e^{i \int \text{tr} (g \delta g^{-1}) \partial_+ c_-}.
\] (3.1.7)

The partition function (3.1.5) then takes the form

\[
Z^{(0)} = \int [d\eta_1][d\eta_2] \int [dg][dB_-] e^{i \int d^2 x [\eta^1_1 i\partial_+ \eta_1 + \eta^1_2 (i\partial_+ + B_-) \eta_2] + \text{tr} \lambda_+ B_-} \times e^{i \int d^2 x \{\alpha \text{tr} (g \delta g^{-1} i \partial_+ g B_-) + \text{tr} g \delta g^{-1} (i \partial_+ c_-) g\}}.
\] (3.1.8)

There is a BRST symmetry associated with the change of variable \( \lambda_+ \to g \). In order to discover it we systematically perform this change of variable by introducing in (3.1.5) the identity

\[
1 = \int [dg] J \delta [\lambda_+ - \alpha g^{-1} i \partial_+ g]
\] (3.1.9)

where \( J \) is the Jacobian defined in (3.1.7). Using the Fourier representation for the delta functional we are thus led to the alternative form for the partition function,

\[
Z^{(0)} = \int [d\eta_1][d\eta_2] \int [d(ghosts)] \int [dg][dB_-] \int [d\lambda_+][d\rho_-] e^{i S_{aux}} \times e^{i \int d^2 x [\eta^1_1 i\partial_+ \eta_1 + \eta^1_2 (i\partial_+ + B_-) \eta_2 + \text{tr} \lambda_+ B_-]}
\] (3.1.10)
where
\[ S_{\text{aux}} = \int d^2x \text{tr} \{ \rho_-(\lambda_+ - \alpha g^{-1}i\partial_+g) + g\tilde{b}_-g^{-1}i\partial_+c_- \} . \] (3.1.11)

The auxiliary action, \( S_{\text{aux}} \), is evidently invariant under the off-shell nilpotent transformations
\[
\delta_1 B_- = \delta_1 \rho_- = \delta_1 \lambda_+ = \delta_1 \eta_1 = \delta_1 \eta_2 = 0 ,
\delta_1 gg^{-1} = c_- ,
\delta_1 \tilde{b}_- = \alpha \rho_- , \quad \delta_1 c_- = \frac{1}{2} \{ c_- , c_- \} .
\] (3.1.12)

We now observe that \( S_{\text{aux}} \) may be written as
\[
S_{\text{aux}} = \frac{1}{\alpha} \delta_1 \text{tr} \tilde{b}_-(\lambda_+ - \alpha g^{-1}i\partial_+g) .
\] (3.1.13)

Hence \( S_{\text{aux}} \) is BRST exact, so that equivalence of the two partition functions is guaranteed on the (physical) states invariant under the transformations (3.1.12).

Integrating over \( \rho_- \) and \( \lambda_+ \) the BRST transformations (3.1.12) are replaced by
\[
\delta_1 B_- = \delta_1 \eta_1 = \delta_1 \eta_2 = 0 ,
\delta_1 gg^{-1} = c_- ,
\delta_1 \tilde{b}_- = -\alpha B_- , \quad \delta_1 c_- = \frac{1}{2} \{ c_- , c_- \} .
\] (3.1.14)

and the partition function (3.1.10) reduces to (3.1.8). We now further make the change of variables
\[ \eta_2 \rightarrow \eta'_2 = g\eta_2 , \quad \tilde{b}_- \rightarrow b_- = g\tilde{b}_-g^{-1} . \] (3.1.15)

The transformation \( \tilde{b}_- \rightarrow b_- \) has Jacobian one. The Jacobian associated with the transformation \( \eta_2 \rightarrow \eta'_2 \) is, on the other hand, given by
\[
J_F = e^{i[dg] - \frac{1}{4\pi} \int d^2x (B_-g^{-1}i\partial_+g)} .
\] (3.1.16)

Notice that \( J_F \) contains the contribution from the non-abelian, as well as abelian \( U(1) \) anomaly. For the choice \( \alpha = \frac{1}{4\pi} \) the second term in \( \ln J_F \) cancels the term proportional to \( \alpha \) in (3.1.8). Noting that \( g(i\partial_- + B_-)g^{-1} = i\partial_- + B'_- \) with \( B'_- = gB_-g^{-1} + gi\partial_-g^{-1} \), using \( [dB] = [dB'] \), and streamlining the notation by dropping "primes" everywhere, the partition function (3.1.8)
As one readily checks, they represent a symmetry of the partition function (3.1.1a). We see that these BRST conditions couple the matter sector \((g)\) to the coset sector. As we have shown, they must be symmetries of the physical states.

### 3.2 BRST analysis of coset sector

It is inconvenient to analyze the cohomology problem with the \(U(N)/U(N)\) coset realized in the present form as a constrained fermion system. Instead it is preferable to decouple \[8\] in (3.1.1b) the \(B_-\) field from the fermions, in order to rewrite the coset partition function in terms of free fermions, negative level WZW fields and ghosts. As we now show, this procedure leads to an additional BRST symmetry.

Concentrating now on the coset sector we introduce in (3.1.1b) the identity

\[
1 = \int [d\rho_+][dh][db_+] [dc_+] e^{i\tilde{S}_{aux}}
\]

with

\[
\tilde{S}_{aux} = \int d^2x tr(\rho_+[B_- - hi\partial_- h^{-1}]) + tr(\tilde{b}_+ D_-(h)\tilde{c}_+)
\]

where \(h\) is a \(U(N)\) group-valued field, \(D_-(h) = \partial_- + [h\partial_- h^{-1}]\) and, like the \(b_-, c_-\)-ghosts, the \(\tilde{b}_+, \tilde{c}_+\)-ghosts transform in the adjoint representation. Note that, unlike in the previous case, the representation of the coset as a gauged fermionic system has led us to define the Jacobian with respect to the Haar measure \(h\delta h^{-1}\):

\[
\delta(h\partial_- h^{-1}) = \det D_-(h) h\delta h^{-1}.
\]

\(\tilde{S}_{aux}\) is invariant under the off-shell nilpotent transformation

\[
\delta_2 \rho_+ = \delta_2 B_- = 0, \quad h\delta_2 h^{-1} = \tilde{c}_+, \quad \delta_2 \tilde{b}_+ = \rho_+, \quad \delta_2 \tilde{c}_+ = -\frac{1}{2}\{\tilde{c}_+, \tilde{c}_+\}
\]
and $\tilde{S}_{\text{aux}}$ is readily seen to be an exact form with respect to this transformation. As before we thus conclude that the original dynamics is recovered on the subspace annihilated by the corresponding BRST charge.

Following the previous steps we find, upon introducing the identity (3.2.1) in (3.1.1b) and integrating over $\rho_+$ and $B_-$,

$$Z_{U(N)/U(N)} = \int[d\eta][d\bar{\eta}] \int[d\hat{h}][d(\text{ghosts})] e^{i\int d^2x \eta_1 \tilde{\rho} \eta_1 + \bar{\psi}_1 \tilde{\rho} \bar{\psi}_1 + \psi_2 \tilde{\rho} \bar{\psi}_2 + \text{tr}(\tilde{\rho} D_-(h) \bar{c}_-)} \times e^{i\int d^2x (\text{tr}(b_- \tilde{\rho} c_-) + \text{tr}(\bar{b}_- \tilde{\rho} (h) \bar{c}_-))}.$$  

This partition function is seen to be invariant under the BRST transformation

$$h \delta_2 \hat{h}^{-1} = \tilde{c}_+, \quad \delta_2 \tilde{b}_+ = \eta_2 \eta_2, \quad \delta_2 \tilde{c}_+ = -\frac{1}{2} \{\tilde{c}_+, \tilde{c}_+\}, \quad \delta_2 \eta_1 = 0, \quad \delta_2 \eta_2 = 0$$

obtained from (3.2.4) after making use of the equations of motion associated with a general variation in $\rho_+$ and $B_-$. We now decouple the fermions by making the change of variable $\eta_2 \to \psi_2$ with $\eta_2 = h\bar{\psi}_2$. Correspondingly the last variation in (3.2.6) is replaced by $\delta_2 \bar{\psi}_2 = -h^{-1} \tilde{c}_+ h$. Taking account of the Jacobian $\exp -i\Gamma[h]$ associated with this change of variable and setting $\eta_1 = \psi_1$ to further streamline the notation, the coset partition function (3.2.5) reduces to

$$Z_{U(N)/U(N)} = \int[d\eta][d\bar{\eta}] \int[d(\text{ghosts})] \int[d\hat{h}] e^{-i\Gamma[h]} \times e^{i\int d^2x (\psi_1 \tilde{\rho} \psi_1 + \bar{\psi}_2 \tilde{\rho} \bar{\psi}_2 + \text{tr}(\tilde{\rho} D_-(h) \bar{c}_-))}.$$  

Finally we also decouple the ghosts $\tilde{b}_+, \tilde{c}_+$ by making the change of variables $\tilde{b}_+ \to b_+, \tilde{c}_+ \to c_+$ defined by

$$h \tilde{b}_+ = hb_+ h^{-1}, \quad \tilde{c}_+ = hc_+ h^{-1}.$$  

Only the $SU(N)$ part of $h$ contributes a non-trivial Jacobian. Setting $h = v\hat{h}$ with $v \in U(N)$ and $\hat{h} \in SU(N)$, we have

$$[db_+][dc_+] = e^{-iC_V \Gamma[h]}[db_+][dc_+]$$  

where $C_V$ is the quadratic Casimir in the adjoint representation. Making further use of the Polyakov-Wiegmann identity one has $\Gamma[h] = \Gamma[\hat{h}] + \Gamma[v]$ and our final result for the coset partition function reads

$$Z_{U(N)/U(N)} = \int[d\eta][d\bar{\eta}] \int[d(\text{ghosts})] \int[dv][d\hat{h}] e^{iS_{U(N)/U(N)}}$$  

(3.2.10)
with
\[
S_{U(N)/U(N)} = -\Gamma[v] - (1 + C_V)\Gamma[\hat{h}] + \int d^2x \left\{ \psi_1^i i\partial_- \psi_1 + \psi_2^i i\partial_+ \psi_2 \right\} + \int d^2x \left\{ \text{tr}(b_- i\partial_+ c_-) + \text{tr}(b_+ i\partial_- c_+) \right\}.
\]

(3.2.11)

Notice that \(v\) and \(\hat{h}\) correspond to level -1 and \(-(1+C_V)\) fields, respectively. Notice also that the ghost term contains the \(SU(N)\) as well as \(U(1)\) contributions.

In terms of the new variables the BRST conditions (3.2.6) read (notice in particular the changes with regard to the first and last variations)

\[
\begin{align*}
  h^{-1}\delta_2 h &= -c_+ , \\
  \delta_2 \psi_1 &= 0 , \quad \delta_2 \psi_2 &= c_+ \psi_2 , \\
  \delta_2 b_+ &= \psi_2 \psi_2^0 - \frac{1}{4\pi} v^{-1} i\partial_+ v - \frac{(1+C_V)}{4\pi} \hat{h}^{-1} i\partial_+ \hat{h} + \{b_+, c_+\} , \\
  \delta_2 c_+ &= \frac{1}{2} \{c_+, c_+\} .
\end{align*}
\]

(3.2.12)

Notice that we have included in the transformation law for \(\delta_2 b_+\) an anomalous piece proportional to \((1+C_V)\), in order to compensate the contribution coming from the variation of the (anomalous) first two terms in (3.2.11) arising from the Jacobians of the transformations.

Finally, returning to the transformation laws (3.1.17), and recalling that, according to (3.2.2) \(B_- = hi\partial_- h^{-1}\), these transformations are to be replaced by

\[
\begin{align*}
  g\delta_1 g^{-1} &= -c_- , \quad h\delta_1 h^{-1} = -c_- , \\
  \delta_1 \psi_1 &= \delta_1 \psi_2 = 0 , \\
  \delta_2 b_- &= \frac{1}{2\pi} g i\partial_- g^{-1} - \frac{(1+C_V)}{4\pi} hi\partial_- h^{-1} + \{b_-, c_-\} , \\
  \delta_1 c_- &= \frac{1}{2} \{c_-, c_-\} .
\end{align*}
\]

(3.2.13)

where an anomalous piece has again been included in the variation for \(b_-\) in order to compensate for the corresponding contribution coming from the Jacobian in (3.2.9).

The corresponding BRST charges are obtained via the usual Noether construction, and are found to be of the general form

\[
\Omega_{\pm} = \text{tr}c_{\pm} \{\Omega_{\pm} - \frac{1}{2} \{c_{\pm}, c_{\pm}\}\}.
\]

(3.2.14)

for the \(SU(N)\) and \(U(1)\) pieces separately. For the \(U(1)\) piece the anticommutator of the ghosts vanishes, of course. Setting \(g = u\hat{g}, u \in U(1)\),
\[
\dot{g} \in SU(N) \text{ and noting that } \Gamma[g] = \Gamma[u] + \Gamma[\dot{g}], \text{ we find }
\]
\[
\Omega_- = \frac{1}{4\pi} u i \partial_- u^{-1} - \frac{1}{4\pi} v i \partial_- v^{-1},
\Omega_+ = \text{tr}(\psi_2 \psi_2^\dagger) - \frac{1}{4\pi} v^{-1} i \partial_+ v,
\Omega_2^a = \text{trt}^a \left[\frac{1}{4\pi} g i \partial_- g^{-1} - \frac{(1+C\nu)}{4\pi} h i \partial_- h^{-1} + \{b_-, c_\} \right],
\Omega_+^a = \text{trt}^a \left[\psi_2 \psi_2^\dagger - \frac{(1+C\nu)}{4\pi} h^{-1} i \partial_+ h + \{b_+, c_+ \} \right]
\]

where \( a = 1, \ldots, N^2 - 1 \). All these operators are required to annihilate the physical states, as we have seen. By going over to canonical variables and using the results of ref. [8], one easily verifies that these constraints are first class with respect to themselves (vanishing central extension). Indeed, define (tilde stands for “transpose”)
\[
\tilde{\Pi}^g = \frac{1}{4\pi} \partial_0 g^{-1}, \quad \tilde{\Pi}^h = -\frac{(1+C\nu)}{4\pi} \partial_0 h^{-1}.
\]

Canonical quantization then implies the Poisson algebra (see ref. [1, 21] for derivation; \( g \) stands for a generic field)
\[
\{g_{ij}(x), \tilde{\Pi}^g_{kl}(y)\}_P = \delta_{ik} \delta_{jl} \delta(x^1 - y^1),
\{\tilde{\Pi}^g_{ij}(x), \tilde{\Pi}^g_{kl}(y)\}_P = -\frac{1}{4\pi} \left( \partial_j g_{jk} g_{li}^{-1} - g_{jk} g_{li}^{-1} \partial_j \right) \delta(x^1 - y^1).
\]

In terms of canonical variables, we have for the constraints (3.2.15)
\[
\Omega_+ = \text{tr}[\psi_2 \psi_2^\dagger - i \tilde{\Pi}^h h - \frac{1}{4\pi} h^{-1} i \partial_1 h],
\Omega_- = \text{tr}[i g \tilde{\Pi} - \frac{1}{4\pi} g i \partial_1 g^{-1} + i h \tilde{\Pi} + \frac{1}{4\pi} h i \partial_1 h^{-1}]
\]
\[
\Omega_+^a = \text{trt}^a [\psi_2 \psi_2^\dagger - i \tilde{\Pi}^h h - \frac{(1+C\nu)}{4\pi} h^{-1} i \partial_1 h + \{b_+, c_+ \}]
\Omega_-^a = \text{trt}^a [i g \tilde{\Pi}^g - \frac{1}{4\pi} g i \partial_1 g^{-1} + i h \tilde{\Pi}^h + \frac{(1+C\nu)}{4\pi} h i \partial_1 h^{-1} + \{b_-, c_- \}]
\]

With the aid of the Poisson brackets (3.2.17) it is straightforward to verify that \( \Omega_\pm \) and \( \Omega_\pm^a \) are first class:
\[
\{\Omega_\pm(x), \Omega_\pm(y)\}_P = 0, \quad \{\Omega_\pm^a(x), \Omega_\pm^b(y)\}_P = -f_{abc} \Omega_\pm^c \delta(x^1 - y^1).
\]

Hence the corresponding BRST charges are nilpotent.

The physical Hilbert space is now obtained by solving the cohomology problem associated with the BRST charges \( \Omega_\pm \) in the ghost-number zero
sector. This can be done in two ways. One can either solve the cohomology problem in the Hilbert space explicitly, i.e., find the states which are annihilated by the BRST charges, but which are not exact, as was done in section 2. Alternatively one can construct the physical operators, which commute with the BRST charges, in terms of which the physical Hilbert space can be constructed. Here we prefer to follow the second approach as it is more transparent. For completeness let us indicate how the analysis would proceed in the first approach.

Although technically more involved, the analysis parallels that of section 2. One first notes that on the decoupled level the Hilbert space is the direct product of four sectors, namely, a free fermion sector, positive - and negative level WZW sectors and a ghost sector. Each of the sectors is again the direct product of left and right moving sectors. For the matter fields the left and right moving sectors are not independent, but have to be combined in a specific way, namely, they must belong to the same representation of the Kac-Moody algebra [22]. This is analogous to the quantum mechanical model discussed in section 2 where the left and right symmetries also belong to the same SU(2) representation. Using this and the results of ref. [8] one finds that the constraints (3.2.18) relate the representations and weights of the various sectors in such a way that a one-to-one correspondence is established between the states of the free fermion model and the physical states annihilated by the BRST charges (3.2.18).

This equivalence is even more transparent in the second approach where one requires that the physical operators commute with the constraints. Making use of the Poisson brackets (3.2.17), we have

\[
\{\Omega^a_-(x), g^{-1}(y)\}_P = i(g^{-1}(x)t^a)\delta(x^1 - y^1),
\]

\[
\{\Omega^a_+(x), h(y)\}_P = -i(t^a h(y))\delta(x^1 - y^1)
\] (3.2.20)

\[
\{\Omega^a_- (x), \psi(y)\}_P = -i(t^a \psi)\delta(x^1 - y^1),
\]

\[
\{\Omega^a_+ (x), h(y)\}_P = i(h(y)t^a)\delta(x^1 - y^1)
\] (3.2.21)

From the Poisson brackets (3.2.20) follows that the fields \( h \) and \( g \) can occur in physical observables only in the combinations \( g^{-1}h \). From the other two Poisson brackets (3.2.21) follows that the fermion field can only occur in the combination \( h\psi \). Putting things together we conclude that physical fermion field corresponds to the local product \( g^{-1}h\psi \). Turning back our set of transformations on the original fermion field, we see that the BRST conditions establish in this way a one-to-one correspondence between the fields of the decoupled formulation and the original free fermion field:
\[ \eta = h^{-1}g\psi. \] This establishes the equivalence of the decoupled partition function (3.1.1a), subject to the BRST conditions, to the fermionic one as given by (3.1.3). The coset factor in (3.1.1a) merely encodes the selection rules of the partition function (3.1.3), but carries no dynamics.

4 \quad QCD_{2} \text{ in the local decoupled formulation}

As a final example we prove deductively that the BRST charges associated with the currents (2.40) and (2.49) of ref. [9] must annihilate the physical states in order to ensure equivalence with the original formulation. To this end we start from the partition function

\[ Z = \int [dA_+][dA_-] \int [d\psi][d\bar{\psi}] e^{iS[A,\psi,\bar{\psi}]} \] (4.1)

with

\[ S[A,\psi,\bar{\psi}] = -\frac{1}{4} \text{tr} F_{\mu\nu} F^{\mu\nu} + \psi_1^\dagger (i\partial_+ + eA_+) \psi_1 + \psi_2^\dagger (i\partial_- + eA_-), \] (4.2)

where \( F_{\mu\nu} \) is the chromoelectric field strength tensor, and \( \partial_\pm = \delta_0 \pm \delta_1 \), \( A_\pm = A_0 \pm A_1 \).

We parametrize \( A_\pm \) as follows:

\[ eA_+ = U^{-1}i\partial_+ U, \quad eA_- = Vi\partial_- V^{-1} \] (4.3)

and change variables from \( A_\pm \) to \( U,V \) by introducing the identities

\[ 1 = \int [dU] \det iD_+(U) \delta(eA_+ - U^{-1}i\partial_+ U), \]
\[ 1 = \int [dV] \det iD_-(V) \delta(eA_- - Vi\partial_- V^{-1}) \] (4.4)

in the partition function (4.1). Here \( D_+(U) \) and \( D_-(V) \) are the covariant derivatives in the adjoint representation:

\[ D_+(U) = \partial_+ + [U^{-1}\partial_+ U, ], \]
\[ D_-(V) = \partial_- + [V\partial_- V^{-1}, ]. \] (4.5)

Exponentiating as usual the corresponding functional determinants in terms of ghost fields and representing the delta functions as a Fourier integral, we obtain for (4.1)

\[ Z = \int [dA_+][dA_-][d\psi][d\bar{\psi}] \int [dU][dV][d\lambda_+][d\lambda_-] \int [d(ghosts)] \\
\times e^{iS[A,\psi,\bar{\psi}]} \times e^{i\int \lambda_-(eA_- - V^{-1}i\partial_- V)} \times e^{i\int \lambda_+(eA_+ - U^{-1}i\partial_+ U)} \times \int b_- iD_+(U)c_- \]
\times \int b_+ iD_-(V)c_+ \] (4.6)
We follow again the procedure of ref. [23]. The partition function (4.6) is seen to be invariant under the transformations

\[ \delta_1 \lambda_+ = 0, \delta_1 A_- = 0, \]
\[ V \delta_1 V^{-1} = c_+, \]
\[ \delta_1 \psi_2 = 0, \]
\[ \delta_1 b_+ = \lambda_+, \]
\[ \delta_1 c_+ = -\frac{1}{2} \{c_+, c_+\} \]

and

\[ \delta_2 \lambda_- = 0, \delta A_+ = 0, \]
\[ U^{-1} \delta_2 U = c_- , \]
\[ \delta_2 \psi_1 = 0, \]
\[ \delta_2 b_- = \lambda_-, \]
\[ \delta_2 c_- = -\frac{1}{2} \{c_-, c_-\} . \]

These transformations are off-shell nilpotent. It is easily seen that in terms of the graded variational derivatives \( \delta_{1,2} \), the effective action in (4.6) can be rewritten as

\[ S_{eff} = S[A, \psi, \bar{\psi}] + \Delta_1 + \Delta_2 \] (4.8)

where

\[ \Delta_1 = \delta_1 [b_- (e A_+ - U^{-1} i \partial_+ U)], \]
\[ \Delta_2 = \delta_2 [b_+ (e A_- - V i \partial_- V^{-1})] \] (4.9)

are \( Q_1 \) and \( Q_2 \) exact with \( Q_1/Q_2 \) the BRST Noether charges associated with the respective transformations. Hence the physical states must belong to \( \text{kern} Q_1/\text{Im} Q_1 \) and \( \text{kern} Q_2/\text{Im} Q_2 \) if \( S_{eff} \) is to be equivalent to the original action \( S[A, \psi, \bar{\psi}] \).

Integrating over \( A_{\pm} \) and \( \lambda_{\pm} \) the partition function and BRST transformations reduce to

\[ Z = \int [dU][dV] \int [d\psi][d\bar{\psi}] \int [d(ghosts)] \]
\[ \times e^{\frac{i}{2} \int \text{tr}(F_{01})^2} e^{\int (U \psi_1)^d_i \partial_+(U \psi_1) + i \int (V^{-1} \psi_2)^d_i \partial_+(V^{-1} \psi_2)} \]
\[ \times e^{i \int b_- i D_+(U) c_-} e^{i \int b_+ i D_-(V) c_-} \] (4.10)

and

\[ V \delta_1 V^{-1} = c_+, \]
\[ \delta_1 \psi_2 = 0, \]
\[ \delta_1 b_+ = -\frac{1}{2} D_+(U) F_{01} + \psi_2 \bar{\psi}_2, \]
\[ \delta_1 c_+ = -\frac{1}{2} \{c_+, c_+\} . \] (4.11a)
\[ U^{-1}\delta_2 U = c_-, \]
\[ \delta_2 \psi_1 = 0, \]
\[ \delta_2 b_- = \frac{1}{2}D_-(V)F_{01} + \psi_1 \psi_1^\dagger, \]
\[ \delta_1 c_- = -\frac{1}{2}\{c_-, c_-\}, \]
\[ (4.11b) \]

respectively. As one readily checks, the partition function (4.10) is invariant under these (nilpotent) transformations which, as we have seen, must also leave \( \mathcal{H}_{\text{phys}} \) invariant.

We now decouple the fermions and ghosts by defining
\[
\begin{align*}
\psi_1^{(0)} &= U\psi_1, & \quad \psi_2^{(0)} &= V^{-1}\psi_2, \\
b_-^{(0)} &= Ub_- U^{-1}, & \quad c_-^{(0)} &= Uc_- U^{-1}, \\
b_+^{(0)} &= V^{-1}c_+ V, & \quad c_+^{(0)} &= V^{-1}c_+ V.
\end{align*}
\]

Making a corresponding transformation in the measure, we have
\[
\begin{align*}
[d\psi_1][d\psi_2] &= e^{-i\Gamma[U,V][d\psi_1^{(0)}][d\psi_2^{(0)}]} \\
[d(\text{ghosts})] &= e^{-iC_V\Gamma[U,V][d(\text{ghosts}^{(0)})]} \tag{4.13}
\end{align*}
\]

where \( \Gamma[g] \) is the Wess-Zumino-Witten (WZW) functional (3.1.2). We thus arrive at the decoupled partition function \[ 6, 7, 9 \]
\[ Z = Z_F^{(0)} Z_{gh}^{(0)} Z_{U,V} \tag{4.14a} \]

where
\[
Z_F^{(0)} = \int [d\psi^{(0)}][d\bar{\psi}^{(0)}]e^{i\int \bar{\psi}\partial\psi}, \tag{4.14b}
\]
\[
Z_{gh}^{(0)} = \int [d(\text{ghosts}^{(0)})]e^{i\int b_-^{(0)} i\partial_- c_-^{(0)} - b_+^{(0)} i\partial_+ c_+^{(0)}} \tag{4.14c}
\]

and
\[
Z_{U,V} = \int [dU][dV]e^{-i(1+C_V)\Gamma[U,V]}e^{\frac{i}{2}\int u(F_{01})^2}. \tag{4.14d}
\]

We rewrite \( F_{01} \) in terms of \( U \) and \( V \) by noting that
\[
F_{01} = -\frac{1}{2}[D_+(U)V i\partial_- V^{-1} - \partial_- (U^{-1} i\partial_+ U)] = \frac{1}{2}[D_-(V)U^{-1} i\partial_+ U - \partial_+(V i\partial_- V^{-1})] \tag{4.15}
\]

and making use of the identities
\[
\begin{align*}
D_-(V)B &= V[\partial_- (V^{-1}BV)]V^{-1} \\
D_+(U)B &= U^{-1}[\partial_+ (UBU^{-1})]U
\end{align*}
\]

\[
(4.16)\]
as well as
\[ U^{-1}[\partial_+(U\partial_-U^{-1})]U = -\partial_-(U^{-1}\partial_+U). \quad (4.17) \]
We thus obtain the alternative expressions
\[ F_{01} = -\frac{1}{2}U^{-1}[\partial_+(\Sigma\partial_-\Sigma^{-1})]U \\
= \frac{1}{2}\Sigma V[\partial_-(\Sigma^{-1}\partial_+\Sigma)]V \quad (4.18) \]
where \( \Sigma \) is the gauge invariant quantity \( \Sigma = UV \). The term \( -(1+C_V)\Gamma[UV] \) in the effective action arising from the change of variables is of quantum origin and must be explicitly taken into account when rewriting the BRST transformations laws (4.11a), (4.11b) in terms of the decoupled variables. Its contribution to these transformations is obtained by noting that \( (\delta = \delta_1 + \delta_2) \)
\[ -(1+C_V)\delta\Gamma[UV] = \frac{1+C_V}{4\pi} \int \text{tr} \left\{ [(UV)^{-1}i\partial_+(UV)] \ i\partial_-c_+^{(0)} \right. \\
\left. + [(UV)i\partial_-(UV)^{-1}] \ i\partial_+c_+^{(0)} \right\}. \quad (4.19) \]
We thus find, making use of (4.18) and the identities (4.16), (4.17),
\[
\delta V^{-1}V = c_+^{(0)}, \\
\delta \psi_1^{(0)} = c_+^{(0)}\psi_2^{(0)}, \\
\delta b_+^{(0)} = -\frac{1}{2}\Sigma^{-1} [\partial_+^2 (\Sigma i\partial_-\Sigma^{-1})] \Sigma - \left( \frac{1+C_V}{4\pi} \right) \Sigma^{-1}i\partial_+\Sigma \\
+ \psi_2^{(0)}\psi_2^{(0)+} + \left\{ b_+^{(0)}, c_+^{(0)} \right\}, \\
\delta c_+^{(0)} = \frac{1}{2} \left\{ c_+^{(0)}, c_+^{(0)} \right\} \quad (4.20a) \\
\delta U^{-1}U = c_-^{(0)}, \\
\delta \psi_1^{(0)} = c_-^{(0)}\psi_1^{(0)}, \\
\delta b_-^{(0)} = -\frac{1}{2}\Sigma [\partial_+^2 (\Sigma^{-1}i\partial_+\Sigma)] \Sigma^{-1} - \left( \frac{1+C_V}{4\pi} \right) \Sigma i\partial_+\Sigma^{-1} \\
+ \psi_1^{(0)}\psi_1^{(0)+} + \left\{ b_-^{(0)}, c_-^{(0)} \right\}, \\
\delta c_-^{(0)} = \frac{1}{2} \left\{ c_-^{(0)}, c_-^{(0)} \right\} \quad (4.20b). \\
\]
Notice the change in the transformation law for \( V \) and \( U \), as well as the change in sign in the transformation of \( c_\pm^{(0)} \).

We now perform a gauge transformation \( U \to UG^{-1}, V \to GV \), taking us to the gauge \( U = 1 \) (\( G = U \)): \( U \to 1, \quad V \to \Sigma \). The decoupled fields evidently remain unaffected by this gauge transformation. The transformation laws for \( V \) and \( U \) above are replaced by a single transformation law
\[ \delta \Sigma^{-1}\Sigma = c_+^{(0)} - \Sigma^{-1}c_-^{(0)}\Sigma. \quad (4.21) \]
Making once more use of the identities (4.16) and (4.17), we finally obtain for the BRST transformations for the decoupled fields in the $U = 1$ gauge ($e_+^{(0)}$ and $e_-^{(0)}$ are to be regarded as independent “parameters”):

\[
\begin{align*}
\delta \Sigma^{-1} \Sigma &= e_+^{(0)}, \\
\delta \psi_1^{(0)} &= e_+^{(0)} \psi_2^{(0)}, \\
\delta b_+^{(0)} &= -\frac{1}{2} \Sigma^{-1} \left[ \partial_+^2 \left( \Sigma i \partial_- \Sigma^{-1} \right) \right] \Sigma + \psi_2^{(0)} \psi_2^{(0)+} + \left\{ b_+^{(0)}, c_+^{(0)} \right\}, \\
\delta c_+^{(0)} &= \frac{1}{2} \left\{ e_+^{(0)}, e_+^{(0)} \right\}.
\end{align*}
\]  

(4.22a)

\[
\begin{align*}
\Sigma \delta \Sigma^{-1} &= -e_-^{(0)}, \\
\delta \psi_2^{(0)} &= e_-^{(0)} \psi_2^{(0)}, \\
\delta b_-^{(0)} &= -\frac{1}{2} \Sigma^{-1} \left[ \partial_-^2 \left( \Sigma^{-1} i \partial_+ \Sigma \right) \right] \Sigma^{-1} + \psi_1^{(0)} \psi_1^{(0)+} + \left\{ b_-^{(0)}, c_-^{(0)} \right\}, \\
\delta c_-^{(0)} &= \frac{1}{2} \left\{ e_-^{(0)}, e_-^{(0)} \right\}.
\end{align*}
\]  

(4.22b)

Using again the identities (4.16) and (4.17), we have

\[
\Sigma [\partial^2 (\Sigma^{-1} \partial_+ \Sigma)] \Sigma^{-1} = D_-(\Sigma)(\partial_+(\Sigma \partial_- \Sigma^{-1})).
\]

(4.23)

Comparing our results with those of ref. [9], we see that we have recovered the BRST conditions of the local formulation (eqs. (2.27) and (2.41) of ref. [9]) after identification of $V$ with $\Sigma$ in the $U = 1$ gauge). This establishes that the transformations (4.22a) and (4.22b) indeed have to be a symmetry of the physical states, as has been taken for granted in ref. [9].

5 Conclusion

Much interest has been devoted recently to gauged WZW theories and $QCD_2$ in a formulation in which various sectors of the theory appear decoupled on the level of the partition function, and are only linked via BRST conditions associated with nilpotent charges. In particular, in the case of $QCD_2$ one is thus led via the Noether construction to several such conserved charges; however not all them are required to vanish on the physical subspace. In order to gain further insight into the question as to which BRST conditions must actually be imposed in order to ensure equivalence of the decoupled formulation to the original coupled one, we have examined this question in the context of simple fermionic models. We have in particular exhibited a general procedure for deciding which of the BRST conditions are valid.
to be imposed, and have thereby shown that this selects in general a sub-
set of nilpotent charges. By solving the corresponding cohomology problem
we have shown that one recovers the Hilbert space structure of the original
models.

We have further demonstrated that the requirement that all of the nilpo-
tent charges should vanish generally implies a restriction to a subspace of
the physical Hilbert space. On this subspace the full set of nilpotent opera-
tors, though non-commuting, could be consistently imposed to vanish. For
$QCD_2$ this means that the vacuum degeneracy obtained in ref. [4] by solv-
ing the cohomology problem in the conformally invariant sector, described
by a $G/G$ topological coset theory presumes, a priori, that the ground state
of $QCD_2$ lies in the conformally invariant, zero-mass sector of the theory

We have emphasized the difference in the "currents" involved in the
BRST conditions and the currents generating the symmetries of the original
coupled formulation: With respect to the former, physical states have to be
singlets, whereas these states belong to the irreducible representations with
respect to the latter.

Finally, we have clarified the BRST symmetries underlying the non-
abelian bosonization of free fermions. The role these symmetries play in
assuring equivalence with the original free fermion dynamics has also been
elucidated.

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