Interpreting type theory in a quasicategory: a Yoneda approach

El Mehdi Cherradi

IRIF - CNRS - Université Paris Cité
MINES ParisTech - Université PSL

Abstract

We make use of a higher version of the Yoneda embedding to construct, from a given quasicategory, a simplicially enriched category, as a subcategory of a well-behaved simplicial model category, whose simplicial nerve is equivalent to the former quasicategory. We then show that, when the quasicategory is locally cartesian closed, it is possible to further endow such a simplicial category with enough structure for it to provide a model of Martin-Löf type theory. This correspondence restricts so that elementary higher topoi are seen to model homotopy type theory.

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Introduction

Categories have been known to provide, under suitable assumptions, a logical framework for reasoning. By this we mean that some classes of categories provide models for logics (in the sense of syntactic systems for deduction): for instance, propositional logic is modeled by boolean algebras and Martin-Löf type theory by locally cartesian closed categories.

The connection between logics and categories is important and has been studied, notably using a fibrational approach (see [Jac99]), which has resulted in categorical notions such as that of comprehension categories to connect the syntax of (dependent) type theory and the world of categories. From a semantical point-of-view, there is a well-known correspondence between (the models of) Martin-Löf type theory and locally cartesian closed categories ([Hof94]).

Another remarkable direction for the study of categories has consisted in weakening some axioms given as equations by shifting from an equality to an equivalence (for instance as for bicategories), which has led to make use of the general theory of abstract homotopy in order to develop homotopy coherent version of many categorical constructions. \((\infty, 1)\)-categories ([Joy08]) consist, for instance, in a homotopy-enabled version of usual categories (referred to as 1-categories). A similar trend has affected type theory, leading to homotopy type theory, a version of dependent type theory where proof of equality between terms is thought as a path in a space, hence providing types with the structure of spaces (or \(\infty\)-groupoids) from the point of view of homotopy theory.

Motivated by this 1-categorical result, the natural question that arose about the link between \((\infty, 1)\)-categories and homotopy type theory has recently found some answers extending the previous logical correspondence to quasicategories, through the simplicial model for homotopy type theory ([KL12]) and more generally the so-called internal language of \((\infty, 1)\)-topoi (the higher version of Grothendieck topoi) ([Shu19]).

Models of \((\infty, 1)\)-categories such as quasicategories come with an inbuilt notion of homotopy, which is compatible with all the structure corresponding to some usual structure of 1-categories (for instance limits). However, while this is very convenient to talk about homotopy coherent properties of quasicategories, there happens to be a mismatch with the usual approach to type theory which involves “strict” features, even when it comes to describing the syntax of homotopy type theory.

Unsurprisingly, interpreting type theory in a higher topos involves a rigidification step to make sense of the expected strict rules of type theory. This can account for the fact that the current results only apply to \((\infty, 1)\)-categories which can be presented by a model category: such a model category works as rigidified substitute of, let say, a quasicategory, and presents the very same data in a 1-categorical way (which is more directly suitable to
interpret type theory).

One way we may think about the problem is that of coherence. Morphisms (which are central to a categorical interpretation of type theory) in an \((\infty, 1)\)-category exist, but there is no internal way to reason about them strictly as everything works up to homotopy: for instance, composition of morphisms is “only” defined up to homotopy. Therefore, if we want to deal with morphisms and their composites, we need to be able to compose them (so that associativity and unitality are strict) in a coherent fashion. This coherence problem is reminiscent of the splitting procedure for a Grothendieck fibration introduced by Bénabou ([Bén80], but also [Gir71]), whose importance for solving the usual coherence problem arising from pullbacks so as to model substitution in type theory has been noticed, see [Hof94].

Following Bénabou’s idea, we may want to rely on a categorical device with inbuilt composition to replace morphisms (and objects). But this is precisely the role played by the slice categories, or, equivalently, the representable presheaves! Indeed, the Yoneda embedding allows us to see any small category \(C\) as a full subcategory of \(\text{Set}^{C^{\text{op}}}\) (or of \(\text{Cat}_{/C}\) through the Grothendieck construction).

This paper puts this idea at work in the setting of higher categories. This gives rise to a general construction to rigidify a quasicategory (that is to turn it into an equivalent, as an \((\infty, 1)\)-category, simplicial category). This rigidification can be exploited to interpret in full generality a large fragment MLTT in any quasicategory. This gives access to a type-theoretic account of the quasicategory and its logical properties. Although the latter is the true aim of the paper, the rigidification process is interesting on its own as it gives a more concrete alternative to the rigidification functor \(S\) defined by Lurie ([Lur09]) as the left adjoint to the simplicial nerve \(N_\Delta\) (see also [DS11]).

Our main result consists in providing models of Martin-Löf type theory for a more general class of quasicategories (the so-called internal logic of a quasicategory) than in the current literature. Having shown that structure such as limits, dependent products and object classifiers in a quasicategory can be carried over to a corresponding simplicial category to model the associated type-theoretic notions, we most notably observe that elementary higher topoi (that is, locally cartesian closed quasicategories with finite limits and colimits, a subobject classifier and enough universes, see [Ras18a]) provide models for homotopy type theory. We are also able to characterize the internal language of the class of locally cartesian closed quasicategory as an intensional dependent type theory, as conjectured in [KL18].
Overview of the paper

In the first section, we construct the rigidification of a given quasicategory and observe that it is indeed equivalent, as an \((\infty, 1)\)-category, to this quasicategory (precisely the simplicial nerve of the rigidification is related to the quasicategory which we start with by a zig-zag of equivalences of quasicategories).

In the second section, we study the case where the quasicategory we start from is locally cartesian closed: we show that we can add to the rigidification some strict structure (mainly pullbacks and dependent products along fibration as well as path objects) while preserving the equivalence of \((\infty, 1)\)-categories. At this point, we are able to see that the internal logic of a locally cartesian closed category is a type theory with dependent sums and products, and intensional identity types.

The third section investigates the case of univalent universes, showing how they can be carried over to the rigidification of the quasicategory.

The fourth section briefly discusses a strategy to use initial structure in the quasicategory to construct (higher) inductive types.

In the fifth section, we are able to conclude that elementary higher topoi further model homotopy type theory.

Finally, in the sixth section, we make the observation that the relative category of models of Martin-Löf’s type theory with dependent sums, identity types, and extensional dependent products is DK-equivalent to the relative category of locally cartesian closed quasicategories.

We also briefly recall, in an appendix, some standard constructions regarding the categorical model of type theory in the context of a (full subcategory of a) model category.

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1 A Yoneda-style rigidification

Quasicategories and (fibrant) simplicial categories are known to provide equivalent models for \((\infty, 1)\)-categories. This is witnessed by a Quillen equivalence where the right Quillen functor is the usual simplicial nerve (also called homotopy coherent nerve) of a simplicial category that we write

\[ N_\Delta : \text{Cat}_\Delta \to \text{SSet} \]
that goes from simplicially enriched categories to simplicial sets.

We will write $\mathcal{C}$ for its left adjoint, following Lurie’s notation in \cite{Lur09}.

From now on in this paper, we fix a quasicategory $\mathcal{C}$. The unit

$$\mathcal{C} \to \mathbf{N}_\Delta(\mathcal{C}(\mathcal{C}))$$

is an equivalence of quasicategories, that is a weak equivalence between fibrant object in the Joyal model structure for simplicial sets, or, more concretely, a functor that is fully faithful (i.e. induces weak equivalence of simplicial sets on the hom-spaces) and that is essentially surjective on objects (every object in the codomain is equivalent to an object in the image).

We think of $\mathcal{C}(\mathcal{C})$ as a rigidified version of $\mathcal{C}$ following Dugger’s terminology, see \cite{DS11}. Lurie showed that, in particular, there is a canonical equivalence of simplicial set (with respect to the Quillen model structure)

$$\text{Hom}_{\mathbf{N}_\Delta(\mathcal{C}(\mathcal{C}))}(x, y) \to \text{Hom}_\mathcal{C}(x, y)$$

However, the precise description of the hom-spaces $\text{Hom}_{\mathcal{C}(\mathcal{C})}(x, y)$ is not very easy to work with, although a nice description relying on the notion of necklace is spelled out in \cite{DS11}.

As mentioned in the introduction, the following result, a simple consequence of the fibrational Yoneda lemma (that is the Yoneda lemma as seen through the Grothendieck construction), suggests an alternative way.

**Proposition 1.1.** Let $\mathcal{C}$ be a small category. Then the functor $\mathcal{C} \to \text{Cat}/\mathcal{C}$ defined on objects by $x \mapsto \mathcal{C}/x$ is fully faithful and thus exhibits $\mathcal{C}$ as a full subcategory of $\text{Cat}/\mathcal{C}$.

We therefore propose, as a first step, the following construction:

**Definition 1.1.** We define $\mathcal{C}_{fb}$ as the full simplicial subcategory of $\mathbf{SSet}/\mathcal{C}$ with objects the slice categories $\mathcal{C}/x \to \mathcal{C}$ equipped with the projection for every object $x$ of $\mathcal{C}$. Note that this is a subcategory of fibrant-cofibrant objects with respect to the contravariant model structure on $\mathbf{SSet}/\mathcal{C}$.

We now expect the following result:

**Proposition 1.2.** The simplicial nerve $\mathbf{N}_\Delta(\mathcal{C}_{fb})$ is equivalent to $\mathcal{C}$.

The assignment $x \mapsto \mathcal{C}/x$ does not seem to extend to a functor of quasicategories (i.e. a morphism of simplicial sets) in a canonical way.

However, the proof of Proposition 1.1 suggests that we should rely on both the $(\infty, 1)$–Yoneda lemma as well as the higher version of the usual Grothendieck construction (the (un)straightening construction). Both of these constructions are studied in details in \cite{Lur09}.

Actually, we will deduce the expected result stated in Proposition 1.2 directly from the following one, which is due to Lurie (HTT 5.1.1.1):
Proposition 1.3. There is a span of equivalences

\[ \text{Fun}(C^{\text{op}}, S) \leftrightarrow N_\Delta((\text{SSet}^{\Delta(\text{cop})})^\circ) \to N_\Delta(\text{RFib}(\mathcal{C})) \]

where \( S \) is the quasicategory of spaces (obtained by applying the simplicial nerve to the simplicial category of Kan complexes), \( \text{RFib}(\mathcal{C}) \) denotes the full simplicial subcategory of \( \text{SSet}_{/C} \) spanned by the right fibrations and \( (\text{SSet}^{\Delta(\text{cop})})^\circ \) the full subcategory of \( \text{SSet}^{\Delta(\text{cop})} \) spanned by the objects that are fibrant-cofibrant for the global projective model structure.

We will see that it is enough to study the restriction of these two maps to the full subcategory of \( N_\Delta(\text{SSet}^{\Delta(\text{cop})}) \) spanned by (fibrant-cofibrant replacements of) the representable simplicial functors \( \text{Hom}_{\Delta}(\mathcal{C}, x) \) (for \( x \) an object of \( \mathcal{C} \)).

First, we recall the statement of two important results, which can be also be found in [Lur09].

Theorem 1.4. There is a Quillen adjunction

\[ \text{SSet}_{/\mathcal{C}} \xrightarrow{\text{St}} \text{SSet}^{\Delta(\text{cop})} \]

between the contravariant model structure on \( \text{SSet}_{/\mathcal{C}} \) and the global projective model structure on \( \text{SSet}^{\Delta(\text{cop})} \). The right (resp. left) adjoint is called the unstraightening (resp. straightening).

Proposition 1.5. Consider a fibrant replacement functor \( R : \text{SSet} \to \text{SSet} \) for the Quillen model structure. The assignment

\[ (X, Y) \to R(\text{Hom}_{\Delta}(\mathcal{C}, X)) \]

defines a simplicial functor from \( \mathcal{C}(\text{cop}) \times \mathcal{C}(\mathcal{C}) \) to the simplicial category \( \text{Kan} \) of Kan complexes. Composing on the left with the natural map

\[ \mathcal{C}(\text{cop}) \times \mathcal{C} \to \mathcal{C}(\text{cop}) \times \mathcal{C}(\mathcal{C}) \]

and transposing along the adjunction \( \mathcal{C} \dashv N_\Delta \), we get a map of simplicial sets

\[ \mathcal{C}(\text{cop}) \times \mathcal{C} \to S \]

The corresponding map

\[ j : \mathcal{C} \to \text{Fun}(\mathcal{C}(\text{cop}), S) \]

is called the Yoneda embedding. It is fully faithful and preserves (small) limits.
Proof of Proposition 1.2. Firstly, the morphism

$$(\text{SSet}\mathcal{C}^{(\mathcal{C}^{\text{op}})})^\circ \to N_\Delta(\mathcal{RFib}(\mathcal{C}))$$

is obtained by applying the nerve to a functor of simplicial categories

$$g : (\text{SSet}\mathcal{C}^{(\mathcal{C}^{\text{op}})})^\circ \to \mathcal{RFib}(\mathcal{C})$$

induced by the un straightening functor (see [Lur09] or [HHR21] for details on this functor).

The right fibration $\mathcal{Un}(F')$ associated by this unstraightening functor to (a fibrant-cofibrant replacement $F'$ of) a representable functor

$$F = \text{Hom}_{\mathcal{C}}(\mathcal{C}, x)$$

is fiberwise equivalent to the canonical right fibration $\mathcal{C}/x \to \mathcal{C}$. This is implied by the fact that there is a canonical weak homotopy equivalence

$$\text{St}(\mathcal{C}/x)(y) \to \text{Hom}_{\mathcal{C}}(\mathcal{C}, y, x)$$

as argued in Proposition 2.2.4.1 of [Lur09] (St being the straightening functor). It follows that $\mathcal{C}/x \to \mathcal{C}$ is fiberwise equivalent (hence contravariantly equivalent) to the fibrant-cofibrant replacement of $F$.

Therefore, the essential image by $g$ of the subcategory of fibrant-cofibrant replacements of the representable functors is a simplicial category equivalent to $\mathcal{C}_{fb}$.

Secondly, the morphism $f : N_\Delta((\text{SSet}\mathcal{C}^{(\mathcal{C}^{\text{op}})})^\circ) \to \text{Fun}(\mathcal{C}^{\text{op}}, \mathcal{S})$ is obtained by Yoneda from the map (natural in $\mathcal{D}$)

$$\text{Hom}_{\text{Cat}}(\mathcal{D}, (\text{SSet}\mathcal{C}^{(\mathcal{C}^{\text{op}})})^\circ) \to \text{Hom}_{\text{Cat}}(\mathcal{D} \times \mathcal{C}^{\text{op}}, (\text{SSet})^\circ)$$

resulting from the evaluation map $(\text{SSet}\mathcal{C}^{(\mathcal{C}^{\text{op}})})^\circ \times \mathcal{C}^{(\mathcal{C}^{\text{op}})} \to (\text{SSet})^\circ$.

When restricted to the (fibrant-cofibrant replacements of) representable simplicial functors, the image of $f$ is constituted of functors $\mathcal{C}^{\text{op}} \to \mathcal{S}$ which are essentially representable in the sense that they are in the essential image of Yoneda embedding (by the very definition of the latter).

Therefore, when restricting to the simplicial functors which are representable up to equivalence (this is indicated by the $\text{rep}$ underscript below), the zig-zag

$$\mathcal{C} \to \text{Fun}(\mathcal{C}^{\text{op}}, \mathcal{S})\text{rep} \leftarrow N_\Delta((\text{SSet}\mathcal{C}^{(\mathcal{C}^{\text{op}})})^\circ)\text{rep} \to N_\Delta(\mathcal{RFib}(\mathcal{C}))\text{rep} \leftarrow N_\Delta(\mathcal{C}_{fb})$$

constitutes a zig-zag of equivalences. This concludes the proof.

The construction of $\mathcal{C}_{fb}$ provides a rather concrete rigidified version of $\mathcal{C}$, however the contravariant model structure (which corresponds in a sense
to the projective global model structure on simplicial presheaves) lacks some important properties that we will need in the rest of the paper.

For this purpose, our preferred alternative will be the following variation which is one step closer to $\mathcal{C}$ in the previous zig-zag. This allows us to trade the “concreteness” of $\mathcal{C}_{\text{fb}}$ for a better homotopy behavior (given by the global injective model structure on simplicial presheaves, which is an excellent model structure in the sense of [LS20]).

**Definition 1.2.** For every object $x$ of $\mathcal{C}$, which we also think of as an object of $\mathcal{C}(\mathcal{C})$, we write $\overline{y}(x)$ for an injectively fibrant replacement (hence a fibrant-cofibrant object) in $\text{SSet}^{\mathcal{C}(\mathcal{C})}$ of the representable simplicial functor $\text{Hom}(-, x)$. We write $\overline{\mathcal{C}}$ for the full subcategory of $\text{SSet}^{\mathcal{C}(\mathcal{C})}$ spanned by the $\overline{y}(x)$.

As before, the simplicial nerve of $\overline{\mathcal{C}}$ is equivalent to $\mathcal{C}$.

## 2 Adding strict structure

In this section, we additionally suppose that $\mathcal{C}$ is locally cartesian closed. From now on, the only model structure we consider on $\text{SSet}^{\mathcal{C}(\mathcal{C})}$ is the global injective one.

**Definition 2.1.** We define $\overline{\mathcal{C}}^\ast$ as the smallest full subcategory of $\text{SSet}^{\mathcal{C}(\mathcal{C})}$ containing $\overline{\mathcal{C}}$ and stable under the following operations:

- Taking pullbacks along fibrations
- Forming the objects of the form $\Pi_f(x)$ where $f$ is a fibration and $\Pi_f$ the right adjoint to the pullback functor along $f$
- Forming the (cofibration,trivial fibration) and (trivial cofibration, fibration) factorizations of a morphism.

We claim the following:

**Proposition 2.1.** $\mathcal{C}$ is (equivalent to) the simplicial nerve of $\overline{\mathcal{C}}^\ast$.

The idea of the proof is that the strict structure added by Definition 2.1 is already present up to equivalence in the sense that, for instance, the pullback of a cospan with objects of the forms $\text{Hom}(-, \overline{y}(x_i))$ should be equivalent to $\text{Hom}(-, \overline{y}(p))$, where $p$ is the pullback of a corresponding diagram in $\mathcal{C}$, since homotopy pullbacks in $\text{SSet}^{\mathcal{C}(\mathcal{C})}$ yield pullbacks when applying the simplicial nerve.

**Proof of Proposition 2.1.** Observe that $\overline{\mathcal{C}}^\ast$ is the colimit of the tower

$$
\overline{\mathcal{C}}^{0} \to ... \to \overline{\mathcal{C}}^{i} \to ...
$$
where $\mathcal{C}_i = \mathcal{C}$ and $\mathcal{C}_i + 1$ is the full subcategory of $\mathbf{SSet}^{\mathfrak{C}(\text{op})}$ obtained by adjoining to $\mathcal{C}$ the objects of the form $f^*(a)$, $\Pi f(a)$ for $f$ a fibration and $a$ an object in $\mathcal{C}$, as well as adjoining the object appearing in the (cofibration,trivial fibration) and (trivial cofibration, fibration) factorizations of a morphism $f$ in $\mathcal{C}$. This is because, since the three operations we consider only involve finitely many objects and arrows, we only need $\omega$ steps of the construction to stabilize. $\mathcal{C}$ is therefore the full subcategory of $\mathbf{SSet}^{\mathfrak{C}(\text{op})}$ such that $x$ is an object of $\mathcal{C}$ if and only if it is an object of any of the $\mathcal{C}_i$.

It will be enough to prove that the canonical inclusion $\iota : \mathcal{C} \to \mathcal{C}$ is an equivalence of simplicially enriched categories. Since this functor is by definition full and faithful, it will be enough to check that all the objects of $\mathcal{C}$ are equivalent to an object of $\mathcal{C}$.

If $a$ is of the form $\mathfrak{y}(x)$ for $x$ an object of $\mathcal{C}$, it is already in the image of $\iota$.

Suppose that for a natural number $i$ we have proved that every object of $\mathcal{C}_i$ is equivalent to an object of $\mathcal{C}$. Then any cospan $u$ with limit $p$ in $\mathcal{C}$ of the form

$$
\begin{array}{ccc}
C & \xrightarrow{G} & B \\
\downarrow & & \downarrow_{F} \\
B & \xrightarrow{F} & A
\end{array}
$$

where $F$ is a fibration, is equivalent to a cospan $u'$

$$
\begin{array}{ccc}
\mathfrak{y}(c) & \xrightarrow{g} & \mathfrak{y}(a) \\
\downarrow \downarrow & & \downarrow \downarrow \\
\mathfrak{y}(b) & \xrightarrow{f} & \mathfrak{y}(a)
\end{array}
$$

Now taking $f'$ and $g'$ to be morphisms in $\mathfrak{C}$ obtained by pulling/pushing the morphisms $f$ and $g$ (seen as edges of the simplicial nerve of $\mathfrak{C}$) along the zig-zag establishing the equivalence between $\mathfrak{C}$ and $\mathbf{N}_\Delta(\mathfrak{C})$ we get a cospan $u''$ in $\mathfrak{C}$.

Note $d$ for its limit in $\mathfrak{C}$. Since the Yoneda embedding and the equivalences of $\infty$–categories preserve limits, $\mathfrak{y}(d)$ is a limit of the cospan $u'$ in $\mathbf{N}_\Delta(\mathbf{SSet}^{\mathfrak{C}(\text{op})})$. The limit $p$ of $u$ computed in $\mathbf{SSet}^{\mathfrak{C}(\text{op})}$ is a homotopy limit, so it yields a limit in the $\mathbf{N}_\Delta(\mathbf{SSet}^{\mathfrak{C}(\text{op})})$. $p$ and $\mathfrak{y}(d)$ are therefore seen to be equivalent as limits of equivalent cospans.

A similar reasoning applies to the dependent product along a fibration $\Pi f(a)$ since dependent product are also preserves by the Yoneda embedding and since the defining adjunction yields a derived adjunction (and even
an adjunction of $\infty$-categories) since the functor $\Pi_f$ is right Quillen. The
(cofibration, trivial fibration) and (trivial cofibration, fibration) factorizations
case is evident.

This proves the result for $i + 1$ and concludes the proof.

Remark 2.1. Note that $\overline{C}^*$ is still a small full subcategory $\text{SSet}^{\text{C}(\text{op})}$ whose
objects are fibrant-cofibrant objects of the latter. However, without further
assumption, there need not exist a cardinal $\kappa$ such that all objects of $\overline{C}^*$
are $\kappa$-presentable because applying a (trivial cofibration, fibration) or (cofi-
bration, trivial fibration) factorization to a morphism between $\kappa$-presentable
objects need not yield a $\kappa$-presentable third object. In the next section, we
will implicitly make this hypothesis, this can be justified in two different
ways:

- We consider algebraic weak factorization systems for the model struc-
ture on $\text{SSet}^{\text{C}(\text{op})}$ as introduced in [Gar09]. This ensures that we can
have a bound of compactness for the objects added in the construction
of $\overline{C}^*$.

- We replace $\text{SSet}$ by a larger category $\text{SSET}$ if needed. This can be
achieved by considering a set-theoretical universe with one inaccessible

We choose to retain the first approach which is more scalable.

Note that, in any case, the focus of the paper is not to provide a detailed
account of the axiomatic strength required in our set-theoretic foundations
for the constructions to make sense in term of size, so we will not further
discuss this matter here.

Remark 2.2. In this section and the following ones, we will not be explicitly
concerned with the coherence problems arising from the fact pullbacks (that
is substitutions) only preserves the structure in weak sense. It will therefore
be implicit that we apply a splitting operation to the comprehension category
modeling the type theory we are working with (precisely, the left-splitting
introduced in [LW13]).

Theorem 2.2. $\overline{C}^*$ yields an interpretation of MLTT.

Proof. The simplicial category $\overline{C}^*$ actually consists in simplicial $\pi$-tribe, as
defined in [Joy17], which can be used as the basis of a comprehension cate-
gory $(\overline{C}^*)_{\text{fib}} \to \overline{C}^*$, where $(\overline{C}^*)_{\text{fib}}$ is the full subcategory of $(\overline{C}^*)^{\rightarrow}$ spanned by
the fibrations. □
3 Universes and univalence

In this section, we further strengthen our hypothesis by supposing that \( C \) admits a univalent map \( \pi : \tilde{u} \to u \) classifying a pullback-stable class of morphism \( S \).

The goal is to construct an object that consists in a universe for \( S \) in an adequate sense. This is more involved than adding, for instance, pullbacks along fibrations because the construction of universes in a model category will be done relying on some sophisticated technology which has been introduced by Shulman in [Shu19].

We start by recalling what it means for a universe to be univalent, first in a quasicategory, then in a model category. The first definition stems from Theorem 6.28 in [Ras18c] (in a situation where it is formulated in terms of complete segal spaces rather than quasicategories):

**Definition 3.1.** Let \( p : e \to b \) be a map in \( C \). Note \( \mathcal{O}_C := \text{Fun}(\Delta_1, C) \) and \( \mathcal{O}_C^{(p)} \) for the subcategory whose objects are the map that are pullbacks of \( p \) and whose morphisms are pullbacks squares.

There is a canonical map of right fibrations \( \alpha : C_{/b} \to \mathcal{O}_C^{(p)} \) induced by Yoneda (this map essentially takes a morphism \( x \to b \) to the morphism \( p^*e \to x \) obtained by pullback from \( p \)). The map \( p : e \to b \) will be said to be univalent if \( \alpha \) is an equivalence.

Note that, equivalently, the map \( p \) is univalent when \( p \) is a final object of \( \mathcal{O}_C^{(p)} \).

**Remark 3.1.** In particular, \( p \) being univalent implies that if a morphism \( q \) is displayed in two ways as a pullback of \( p \) as in the diagrams below

\[
\begin{array}{ccc}
  y & \xrightarrow{p} & e \\
  \downarrow q & & \downarrow p \\
  x & \xrightarrow{\phi} & b \\
\end{array}
\quad
\begin{array}{ccc}
  y & \xrightarrow{p} & e \\
  \downarrow q & & \downarrow p \\
  x & \xrightarrow{\psi} & b \\
\end{array}
\]

then \( \phi \) and \( \psi \) are homotopic.

The second definition (or rather characterization), due to Voevodsky (see [Gam15] and [Joy18]), is the following:

**Definition 3.2.** In a model category \( \mathcal{M} \), a fibration \( P : E \to B \) is said to be univalent if for every fibration \( Q : Y \to X \), the pair of maps \( (\phi, \phi') \) in the following homotopy pullback diagram is unique up to homotopy when such a pair exists:
where the dotted map is a weak equivalence.

Our goal is now the following:

**Proposition 3.1.** There exists a univalent universe \( \mathcal{U} \) in \( \text{SSet}^{(\mathcal{E}(\text{cop}))} \) classifying the fibrations equivalent to a morphism corresponding to an edge in \( S \).

Moreover, \( \mathcal{U} \) is essentially representable in the sense that it is equivalent to an object of \( \mathcal{C} \).

Here the notion of equivalence between morphisms in \( \text{SSet}^{(\mathcal{E}(\text{cop}))} \) refers to isomorphisms in the homotopy category \( \text{Ho}((\text{SSet}^{(\mathcal{E}(\text{cop}))})\to) \)

We will deduce this result directly from theorem 5.22 in [Shu 19] by constructing an appropriate notion of fibred structure (as introduced in section 3 of that paper, the definition being recalled below) in \( \mathcal{E} = \text{SSet}^{(\mathcal{E}(\text{cop}))} \). Our starting point for a universal fibration is the following naive candidate:

**Definition 3.3.** Define \( \pi : \tilde{U} \to U \) in \( \mathcal{E} \) to be a fibration between fibrant objects corresponding to the universal map \( \tilde{u} \to u \) in \( \mathcal{E} \). This means that \( \pi \) is equivalent to the morphism \( \text{Hom}(\cdot, \tilde{u}) \to \text{Hom}(\cdot, u) \) in the previous sense.

In a locally cartesian closed category, it is possible to construct an object representing the isomorphism between two objects (in the sense that, for instance, the object \( \text{Iso}(A, B) \) has global elements which are naturally bijective with the set of isomorphisms between \( A \) and \( B \)).

In a suitable model category, a similar construction, though more sophisticated, can be carried out to represent not just isomorphisms but weak equivalences in general. The construction of such an object is detailed in Section 4 of [Shu13]. We record the corresponding result in the following lemma.

**Lemma 3.2.** Consider two fibration between fibrant objects \( E_1 \to B \) and \( E_2 \to B \) in \( \mathcal{E} \). There is an object \( \text{Equiv}_B(E_1, E_2) \) over \( B \) such that for any fibrant object \( Y \) and maps \( f : Y \to B \), morphisms from \( f \) to \( \text{Equiv}_B(E_1, E_2) \) over \( B \) are naturally bijective with weak equivalences \( f^* E_1 \to f^* E_2 \) over \( Y \).
In the preceding lemma, although the fibrancy conditions are required to get the correspondence between morphisms \( f \to \text{Equiv}_B(E_1, E_2) \) and weak equivalences \( f^*E_1 \to f^*E_2 \), the construction of the object \( \text{Equiv}_B(E_1, E_2) \) itself is still possible in the general case. We will make use of the following definition:

**Definition 3.4.** In the general case, let us call virtual equivalence any map \( E_1 \to E_2 \) corresponding to a lift of \( B \) to \( \text{Equiv}_B(E_1, E_2) \).

Such virtual equivalences are stable under pullback along a map \( C \to B \) because the definition of \( \text{Equiv}_B(E_1, E_2) \) is compatible with change of base.

**Definition 3.5.** We define a (large) groupoid-valued pseudofunctor \( \text{Rh}_\pi : \mathcal{E}^{op} \to \text{GPD} \) as follows:

- For \( X \) an object, \( \text{Rh}_\pi(X) \) has as its underlying (large) set the set of diagrams of this form:

  \[
  \begin{array}{c}
  Z \\
  \downarrow h \\
  Y \downarrow \downarrow \downarrow \\
  X \to U \\
  \end{array}
  \]

  \[
  \begin{array}{c}
  \tilde{U} \\
  \downarrow \pi \\
  U \\
  \end{array}
  \]

  where \( h \) is a virtual equivalence. The morphisms are the (natural) isomorphisms between two such diagrams above the map \( \pi : \tilde{U} \to U \).

  We will refer to such diagrams as homotopy classification diagrams.

- For \( X' \to X \) a morphism, and a homotopy classification diagram for \( X \), we construct one for \( X' \) as follows:

  \[
  \begin{array}{c}
  Z' \\
  \downarrow h' \\
  Y' \downarrow \downarrow \downarrow \\
  X' \to U \\
  \end{array}
  \]

  \[
  \begin{array}{c}
  Z \\
  \downarrow h \\
  Y \downarrow \downarrow \downarrow \\
  X \to U \\
  \end{array}
  \]

  Clearly \( h' \) is a virtual equivalence if \( h \) is so.

This construction is indeed pseudofunctorial (similarly to the situation where one considers ordinary classification diagrams).

We briefly recall the following definitions from [Shu19]:
Definition 3.6. Write \( \text{PSH}(\mathcal{E}) \) for the 2-category of contravariant pseudo-functors from \( \mathcal{E} \) to \( \text{GPD} \), with pseudonatural transformations between them.

A strict discrete fibration in \( \text{PSH}(\mathcal{E}) \) is a strictly natural transformation \( X \to Y \) such that each component \( X(A) \to Y(A) \) is a discrete fibration.

From now on, we define \( \mathcal{E}^{\text{op}} \to \text{GPD} \), the core self-indexing of \( \mathcal{E} \), by mapping \( Y \) to the core of \( \mathcal{E}/Y \).

Definition 3.7. A notion of fibred structure is a strict discrete fibration \( \phi : F \to E \) with smalls fiber (where \( F \) is an object of \( \text{PSH}(\mathcal{E}) \) and \( E \) is the core self-indexing defined previously).

Given such a fibration, an \( F \)-structure on a morphism \( f : X \to Y \) in \( \mathcal{E} \) is an element of the fiber of \( \phi(Y) \) above \( f \). When \( f \) is equipped with an \( F \)-structure, it is called an \( F \)-algebra. An \( F \)-morphism from \( f \) to \( f' \) is a pullback square (exhibiting \( f' \) as a pullback of \( f \)) such that \( f' \) is equipped with the \( F \)-algebra structure induced from that of \( f \) from the function between fibers given by \( \phi \).

Assigning to a homotopy classification diagram (1) for \( X \) the morphism \( Y \to X \) allows us to define (up to the strictification process for a fibration discussed in Section 2 of [Shu19]) a notion of fibred structure \( \phi : \text{Rh}_\pi \times_E \text{Fib} \to E \).

There are three important properties for a notion of fibred structure to verify in order to construct a universe for it from Theorem 5.22 in [Shu19]. We recall the definition then give the corresponding proof for \( \text{Rh}_\pi \times_E \text{Fib} \), the notion of fibred structure obtained from \( \text{Rh}_\pi \) by restricting to those morphisms that are fibration.

Definition 3.8. A notion of fibred structure \( \phi : F \to E \) is locally representable if the pullback of representable pseudofunctors along \( \phi \) is representable. Concretely, this means that given any map \( X \to Z \) in \( \mathcal{E} \) (seen by Yoneda as morphism of \( \text{Hom}(\cdot, X) \to E(Z) \)), there is a map \( \phi_X^F : F_X \to Z \) such that, for any \( g : Y \to Z \), there is a natural bijection between \( F \)-structure on \( g^*X \) and lifts of \( g \) to \( F_X \).

Lemma 3.3. \( \phi : \text{Rh}_\pi \times_E \text{Fib} \to E \) is a locally representable notion of fibred structure.

Proof. Since \( \text{Fib} \) is known to be locally representable (Example 3.17 in [Shu19]), it is enough to show that so is \( \text{Rh}_\pi \) (by Example 3.11 in [Shu19]).

The proof follows directly from Example 3.15 of [Shu19], replacing the object of isomorphisms with an object of weak equivalences.

Actually, the very notion of virtual equivalence has been defined so that \( \text{Equiv}_{Z \times U}(X \times U, Z \times \tilde{U}) \) represents \( \text{Rh}_{\pi,X} \).

Definition 3.9. A notion of fibred structure \( \phi : F \to E \) is relatively acyclic if for any pullback as in the diagram:
where $i$ is a cofibration and $f$ and $f'$ are $F$-algebras, there exist a new $F$-structure on $f'$ making the diagram an $F$-morphism.

**Lemma 3.4.** $\phi : \text{Rh}_\pi \times_\mathcal{E} \text{Fib} \to \mathcal{E}$ is a relatively acyclic notion of fibred structure.

**Proof.** Consider a pullback diagram

\[
\begin{array}{ccc}
X' & \xrightarrow{g} & X \\
\downarrow f' & & \downarrow f \\
Y' & \xrightarrow{i} & Y
\end{array}
\]

where $F$ and $F'$ are $\text{Rh}_\pi$–algebras and $i$ is a cofibration.

We consider a (small enough) universal fibration $\tilde{V} \to V$ that classifies $F'$ and $F$ and write $\text{Eq}(U, V) := \text{Equiv}_{U \times V}(\tilde{U} \times V, U \times \tilde{V})$.

Let $Y \to U \times V$ be map so that the first component is that obtained from the $\text{Rh}_\pi$–algebra structure and the second exhibits $F$ as a pullback of the previous universal fibration. We also have a map $Y' \to U \times V$ where the first component is obtained by precomposing $Y \to U$ with $Y' \to Y$ and the second component being the classifying map for $F'$. Form the two following (trivial cofibration,fibration) factorization: $Y \to Y_f \to U \times V$ and $Y' \to Y'_f \to U \times V$.

We have a lift in the following diagram

\[
\begin{array}{ccc}
Y' & \xrightarrow{\sim} & Y \\
\downarrow & & \downarrow \\
Y'_f & \xrightarrow{\sim} & *
\end{array}
\]

that we can further factor as cofibration followed by a trivial fibration:

\[
Y'_f \xrightarrow{\sim} Y_{f,2} \xrightarrow{\sim} Y_f
\]

This induces a diagram:
where $X_f \times Z_f$ is obtained by pullback of $\tilde{U} \times \tilde{V} \rightarrow U \times V$, $Z'_f$ by pullback of $\tilde{U} \rightarrow U$, and $X'_f$ by pullback of $X_f \rightarrow Y_{f,2}$.

In particular, we have a commutative diagram where the three face are pullbacks:

Furthermore, the map $Y \rightarrow \text{Eq}(U,V)$ (above $U \times V$) can be extended along the trivial cofibration $Y \rightarrow Y_{f,2}$ and similarly for $Y'$.

It is clear that providing a new $\text{Rh}_\pi$-algebra structure on $X_f \rightarrow Y_{f,2}$ inducing that of $X'_f \rightarrow Y'_f$ will also give us a new $\text{Rh}_\pi$-algebra structure on
$F$ inducing that on $F'$. Hence we are back in the original setting with the additional assumption that the objects we consider are all fibrant.

The first pullback diagram we considered is hence a homotopy pullback corresponding in $\text{Fun}(\mathcal{C}^{\text{op}}, S)$ to a pullback

\[
\begin{array}{ccc}
  x' & \xrightarrow{g} & x \\
  \downarrow{f'} & \quad & \downarrow{f} \\
  y' & \xrightarrow{i} & y
\end{array}
\]

Likewise, the homotopy classification diagrams for $F$ and $F'$ correspond to pullbacks in $\text{Fun}(\mathcal{C}^{\text{op}}, S)$:

\[
\begin{array}{ccc}
  x' & \xrightarrow{\sim} & 1 \\
  \downarrow{f'} & \quad & \downarrow{f} \\
  y' & \xrightarrow{\sim} & y
\end{array}
\]

Univalence of $u$ in $\mathcal{E}$ implies the existence of a homotopy (in the sense given by the global injective model structure) from $H \circ I$ and $H'$ (where $H$ and $H'$ are morphisms in $\mathcal{E}$ corresponding to $h$ and $h'$).

Since $I$ is a cofibration, the homotopy extension property (that is the right lifting property of a trivial fibration projection $\text{Path}(u) \to u$ against $I$) implies the existence of $H''$ such that $H' = H'' \circ I$.

This $H''$ gives a new $\text{Rh}_\pi$—structure on $F$ which induces the given structure on $F'$. This proves that $\phi$ is relatively acyclic.

\[\square\]

**Definition 3.10.** A notion of fibred structure $\phi : F \to E$ is homotopy invariant if every $F$-algebra is a fibration and for any diagram

\[
\begin{array}{ccc}
  X' & \xrightarrow{\sim} & X \\
  \downarrow{f'} & \quad & \downarrow{f} \\
  Y' & \xrightarrow{\sim} & Y
\end{array}
\]
where $f$ and $f'$ are fibrations and the horizontal maps are weak equivalence, $f$ can be equipped with an $F$-structure if and only so can $f'$ be.

**Lemma 3.5.** $\phi : \text{Rh}_\pi \times_E \text{Fib} \to E$ is a homotopy invariant notion of fibred structure.

**Proof.** Consider the following diagram where $f$ and $f'$ are fibrations and the horizontal maps weak equivalences

$$
\begin{array}{ccc}
X' & \sim & X \\
\downarrow f' & \downarrow & \downarrow f \\
Y' & \sim & Y
\end{array}
$$

Suppose that $f$ has $\text{Rh}_\pi$-structure. Form the (trivial cofibration, fibration) factorizations of the classifying maps $Y' \to V$ and $Y \to U \times V$. Keeping the notation from the proof of the previous lemma, the situation is then the following:

$$
\begin{array}{ccc}
X' & \sim & X' \\
\downarrow f' & \downarrow & \downarrow f \\
Y' & \sim & Y
\end{array}
\begin{array}{ccc}
\tilde{V} & \sim & \tilde{V} \\
\downarrow & \downarrow & \downarrow \\
\tilde{U} \times \tilde{V} & \sim & \tilde{V}
\end{array}
\begin{array}{ccc}
X_f & \sim & X_f \\
\downarrow g & \downarrow & \downarrow g \\
Y_f & \sim & Y_f
\end{array}
\begin{array}{ccc}
\text{Eq}(U, V) & \sim & \text{Eq}(U, V) \\
\downarrow & \downarrow & \downarrow \\
U \times V & \sim & U \times V
\end{array}
$$

where $X'_f$ and $X_f$ are obtained by pullback of $\tilde{V} \to V$.

The map $Y' \to \text{Eq}(U, V)$ now extends to $Y$ to give an $\text{Rh}_\pi$-structure on $X_f \to Y_f$. Now form the following two pullbacks square where $Z_f$ is part of the $\text{Rh}_\pi$-structure on $X_f \to Y_f$. 

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$X'_f \sim Z'_f$ is an equivalence, so we get an $\text{Rh}_{\pi}$-structure on $X'_f \to Y'_f$. Transferring this structure to $X' \to Y'$ is now just a matter of precomposing the map $Y'_f \to \text{Equiv}_{U \times V}(\tilde{U} \times V, U \times V)$ by $Y' \to Y'_f$.

The reverse direction is completely analogous since weak equivalences between fibrant-cofibrant objects admit an inverse up to homotopy.

We are now in a position to deduce, by theorem 5.22 in [Shul19], that there is a univalent universe $\pi' : \mathcal{U} \rightarrow \mathcal{U}$ classifying the fibration that can be equipped with an $\text{Rh}_{\pi}$-structure (and that are small enough, i.e. $\kappa$-presentable as briefly discussed in Remark 2.1). Write $\mathcal{C}^{**}$ for the smallest full subcategory of $\text{SSet}_{/\mathcal{C}}$ containing the objects of $\mathcal{C}$, $\mathcal{U}$ and $\mathcal{V}$ and stable under the appropriate constructions (as in Definition 2.1). Those morphism classified by $\pi'$ hence include the fibrations, between objects in $\mathcal{C}^{**}$, that are equivalent to morphisms induced by an edge in $\mathbf{S}$.

**Proposition 3.6.** The object $\mathcal{U}$ is equivalent to $U$ (hence the object $\tilde{\mathcal{U}}$ is equivalent to $\tilde{U}$).

**Proof.** By definition of $\pi'$, there is a pullback square:

\[
\begin{array}{ccc}
\tilde{U} & \sim & \tilde{U} \\
\pi' \downarrow & & \downarrow \pi' \\
U & \sim & \mathcal{U}
\end{array}
\]

Because $\pi'$ can be equipped with an $\text{Rh}_{\pi}$-structure, we also have a homotopy classification diagram:

\[
\begin{array}{ccc}
\mathcal{U} & \sim & Z_{\mathcal{U}} \\
\mathcal{U} \downarrow & & \mathcal{U} \downarrow \\
\mathcal{U} & \sim & \mathcal{U}
\end{array}
\]
Univalence will imply that \( \phi \) and \( \psi \) are homotopy inverse of each other. Precisely, the characterization of a fibration being univalent imply that the two bottom arrow in the following two diagrams (\( \phi \circ \psi \) and \( id_{\widetilde{U}} \)) belong to the same contractible space (hence they are homotopic):

\[
\begin{array}{ccc}
\widetilde{U} & \xrightarrow{\sim} & Z_{\widetilde{U}} \\
\downarrow^{\pi'} & & \downarrow^{\pi} \\
U & \xrightarrow{\psi} & U \\
\phi & & \phi \\
\end{array}
\quad
\begin{array}{ccc}
\widetilde{U} & \xrightarrow{\sim} & \widetilde{U} \\
\downarrow^{\pi'} & & \downarrow^{\pi'} \\
U & \xrightarrow{id_{\widetilde{U}}} & U \\
\end{array}
\]

Likewise, the two following diagrams correspond to two pullback squares in \( \text{Fun}(\text{C}^{\text{op}}, S) \) (hence in \( \mathcal{C} \)), so that \( \psi \circ \phi \) and \( id_{U} \) are also homotopic by univalence of the morphism \( \tilde{u} \rightarrow u \).

\[
\begin{array}{ccc}
\tilde{U} & \xrightarrow{\sim} & Z_{\widetilde{U}} \\
\downarrow^{\pi} & & \downarrow^{\pi} \\
U & \xrightarrow{\psi} & U \\
\phi & & \phi \\
\end{array}
\quad
\begin{array}{ccc}
\tilde{U} & \xrightarrow{\sim} & \tilde{U} \\
\downarrow^{\pi'} & & \downarrow^{\pi'} \\
U & \xrightarrow{id_{U}} & U \\
\end{array}
\]

Since \( \mathcal{U} \) and \( \mathcal{Y} \) are equivalent to objects already in \( \mathcal{C} \), we deduce the following:

**Theorem 3.7.** \( \mathcal{C}^{**} \) yields an interpretation of \( \text{HoTT} \) with one univalent universe.

Also note that the inclusion \( \mathcal{C}^{*} \hookrightarrow \mathcal{C}^{**} \) is an equivalence.

## 4 Adding disjoint sum, pushout and \( W \)-types

Substitution, as well as \( \Sigma \)-types, identity types and \( \Pi \)-types, are arguably at the core of Martin-Löf type theory, and, as such, required a specific treatment involving, for instance, strict pullbacks and dependent products. On the other hand, some other (higher) inductive type can be interpreted following a more general pattern relying mainly on the existence of some structure in the quasicategory \( \mathcal{C} \). For the definition of the types we are looking to add up, we follow [LS20] and [LW15].

We start with disjoint sum (\( + \)-types).
Proposition 4.1. Suppose that C has finite coproducts.

Consider two types A and B in context Γ corresponding to two fibrations
A → Γ and B → Γ.

It is possible to construct in SSet_{fin} an associated +-type A + B whose
underlying object in essentially representable.

Proof. Let a, b and γ be objects of C such that γ(a), γ(b) and γ(γ) are
equivalent to A, B and Γ (these exist since all the objects of C** are equivalent
to objects of the form γ(x) for x an object of C). Let γ(a) → γ(γ) and
γ(b) → γ(γ) be maps equivalent to the fibrations A → Γ and B → Γ,
explicitly obtained by composing the latter fibrations with the equivalences
A ≃ γ(a), B ≃ γ(b) and Γ ≃ γ(γ).

We will use both the universal properties of the coproduct in C and in
SSet_{fin} and we will rely on the correspondence (unique up to homotopy)
between morphisms x → z in C and maps γ(x) → γ(z) in SSet_{fin}.

Form the (cofibration, trivial fibration) factorization of the morphism
A ∐ B → γ(a) ∐ γ(b) → γ(a ∐ b) obtained from the universal property of the
coproduct in SSet_{fin}:

\[ A ∐ B \quad \xrightarrow{(A + B)_0} \quad (A + B) \quad \xrightarrow{\sim} \quad γ(a ∐ b) \]

After that, form the (trivial cofibration, fibration) factorization of the morphism
(A + B)_0 → γ(a ∐ b) → γ(γ) → Γ, where the second morphism in this
composite results from the universal property of the coproduct in C applied
to the pair of maps a → γ and b → γ:

\[ (A + B)_0 \quad \xrightarrow{\sim} \quad A + B \quad \xrightarrow{\sim} \quad Γ \]

Also consider a section A + B → (A + B)_0 of the trivial cofibration. We now
have two injections (inl, inr) : A ∐ B → (A + B)_0 → A + B.

Suppose a type C (with an equivalence C ≃ γ(c)) in context Γ, A + B
(so there is an edge c → a ∐ b) and a morphism d : A ∐ B → C such that
d(c) = (inl, inr). These corresponds to morphisms a → c and b → c in C.
There is an induced morphism (defined up to homotopy) a ∐ b → c which is
moreover a section of c → a ∐ b. We deduce the existence of a morphism
case_0 : A + B → (A + B)_0 → γ(a ∐ b) → γ(c) → C which is a section of
p_C ∘ d = (inl, inr) is homotopic to d.

Firstly, consider a homotopy H_0 : A + B ⊗ Δ_1 → A + B from p_C ∘ case_0
to id_{A + B}. The second component of the lift K_0 in the following diagram
yields the expected section case_0′.
Secondly, consider a homotopy

\[ H : A \amalg B \to \text{Path}(C) \]

from \( \text{case}_0 \circ (\text{inl}, \text{inr}) \) to \( d \). We take this homotopy in \( \text{SSet}^{E^{(\text{op})}_{/A+B}} \), which is possible because the composite \( a \amalg b \to c \to a \amalg b \) being the identity morphism \( a \amalg b \to a \amalg b \) really corresponds to a homotopy in \( E_{a \amalg b} \) (or equivalently in \( E_{/a \amalg b} \)). We consider a solution \( K \) to the following lifting problem:

\[
\begin{array}{c}
A \amalg B & \xrightarrow{H} & \text{Path}(C) \\
\downarrow & \searrow & \downarrow \\
A + B & \xrightarrow{K} & C
\end{array}
\]

The second component of the homotopy \( K \) is a morphism \( \text{case} : A + B \to C \) such that \( \text{case} \circ (\text{inl}, \text{inr}) = d \).

We record a similar result for pushout types:

**Proposition 4.2.** Suppose that \( E \) has pushouts.

Consider three types \( A, B_1 \) and \( B_2 \) in context \( \Gamma \) corresponding to fibrations \( A \to \Gamma \) and \( B_i \to \Gamma \). Let \( f_1 : A \to B_1 \) and \( f_2 : A \to B_2 \) be morphisms above \( \Gamma \).

It is possible to construct in \( \text{SSet}^{E^{(\text{op})}} \) an associated pushout type \( D \) whose underlying object in essentially representable.

**Proof.** The proof is almost identical to that of Proposition 4.1 using the homotopy pushout \( B_1 \amalg A \otimes \Delta_1 \amalg B_2 \) instead of the coproduct in \( \text{SSet}^{E^{(\text{op})}} \). It is easy to see that we can construct a \( \Delta_1 \)-typal pushout as in Definition 4.1 of [LS20].

The previous proof pattern actual seems very general. For instance, we can extend it to \( W \)-types:

**Proposition 4.3.** Consider two types \( A \) and \( B \), in context \( \Gamma \) and \( \Gamma', A \) respectively, corresponding to two fibrations \( H : A \to \Gamma \) and \( F : B \to A \). These corresponds (up to equivalence) to maps \( h : a \to \gamma \) and \( f : b \to a \) in \( E \).
Suppose that the polynomial endofunctor \( p \) associated with \( h, f \) and \( g := h \circ f \) (as in the diagram below) admits an initial algebra.

Then we can construct in \( \mathbf{SSet}^{\mathcal{C}(\operatorname{cop})} \) the corresponding \( W \)-type, and the underlying object is essentially representable.

Proof. Consider the initial algebra \( px \to x \) in \( \mathcal{C} \). Since \( H^* \) corresponds to \( h^* \) up to equivalence (in the sense that \( H^* \gamma(x) \simeq \gamma(h^*x) \)) and similarly for \( F^* \), it is clear that \( \gamma(x) \) can be made into a \( P \)-algebra \( l : P\gamma(x) \to \gamma(x) \), where \( P \) is the following polynomial functor corresponding to \( p \):

\[
\begin{array}{cccccc}
\mathbf{SSet}^{\mathcal{C}(\operatorname{cop})}_{/\Gamma} & \xrightarrow{H^*} & \mathbf{SSet}^{\mathcal{C}(\operatorname{cop})}_{/A} & \xrightarrow{F^*} & \mathbf{SSet}^{\mathcal{C}(\operatorname{cop})}_{/B} & \xrightarrow{G^*} & \mathbf{SSet}^{\mathcal{C}(\operatorname{cop})}_{/\Gamma}
\end{array}
\]

We form the (cofibration, trivial fibration) factorization of this algebra:

\[
\begin{array}{ccc}
P\gamma(x) & \xrightarrow{} & W_0 \\
\sim & & \sim \\
& & \gamma(x)
\end{array}
\]

Then the (trivial cofibration, fibration) factorization of \( W_0 \to \Gamma \):

\[
\begin{array}{ccc}
W_0 & \xrightarrow{\sim} & W \\
\sim & & \sim \\
& & \Gamma
\end{array}
\]

\( W \) can then be shown to be the expected \( W \)-type. To check this, let us first build the map \( \text{fold} : \Gamma, A, \Pi_B W[p_A](= PW) \to \Gamma, W \) (over \( \Gamma \)).

But we do have a map \( \text{fold}_0 : \gamma(px) \to \gamma(x) \) (given by the algebra structure stemming from the one internal to \( \gamma \) on \( x \)). This map induces \( \text{fold} \) as the composite \( PW \to \gamma(px) \to W \) (where the first map is an equivalence deduced from the equivalence \( W \simeq \gamma(x) \)).

For the elimination rule, consider a type \( C \) in context \( \Gamma, W \) (and an equivalence \( C \simeq \gamma(c) \)) and a square of the following form:

\[
\begin{array}{ccc}
\Gamma, A, \Pi_B W, \Pi_B C(= P\Sigma_W C) & \xrightarrow{} & \Gamma, A, \Pi_B W \\
\downarrow D & & \downarrow \text{fold} \\
\Gamma, W, C & \xrightarrow{} & \Gamma, W
\end{array}
\]

There is a corresponding commutative square in \( \mathcal{C} \):

\[
\begin{array}{ccc}
pc & \xrightarrow{} & px \\
\downarrow d & & \downarrow d \\
c & \xrightarrow{} & x
\end{array}
\]
By initiality of $px \to x$, we get a square (over $x$):

\[
\begin{array}{c}
px \downarrow \downarrow d \\
x \rightarrow_{wrec} c
\end{array}
\]

where $wrec$ is section of $c \to x$. The corresponding map $wrec_0$ is only a section of the projection $pC : \Gamma, W, C \to \Gamma, W$ up to homotopy, but we can replace it with a homotopic map that is a real section:

We consider a homotopy $H_0$ from $pC \circ wrec_0$ to $id_W$. The second component of the lift in the following diagram gives the section $wrec'_0$.

\[
\begin{array}{c}
W \xrightarrow{wrec_0} C \\
\sim \downarrow \downarrow \downarrow H_0 \\
W \otimes \Delta_1 \xrightarrow{wrec'_0} W
\end{array}
\]

However, the section only satisfies the computation rule up to homotopy (the corresponding square in $SSet_{\mathcal{C}(\text{op})}$ does not commute strictly). Thus we may consider a corresponding homotopy $H : p\overline{\gamma}(x) \to \text{Path}(C)$ fitting in a commutative square (in $SSet_{\mathcal{C}(\text{op})}$):

\[
\begin{array}{c}
\overline{\gamma}(px) \xrightarrow{H} \text{Path}(C) \\
\sim \downarrow \downarrow \downarrow K_0 \\
W \xrightarrow{wrec'_0} C
\end{array}
\]

The second component of $K$ yields the expected map $wrec$ satisfying the computation rule:

\[
\begin{array}{c}
PW \xrightarrow{\text{fold}} P\Sigma WC \\
\downarrow \downarrow \downarrow D \\
W \xrightarrow{wrec} C
\end{array}
\]

Remark 4.1. Given a concrete description of the associated syntax, we believe that any cell monad with parameters (in the sense of [LS20]) inside of $\mathcal{C}$ admitting an initial algebra yields a corresponding higher inductive type in
\textbf{SSet}^{\mathcal{C}(\omega^p)} using the previous proof pattern. However, we do not further investigates this question in the present paper.

5 Elementary higher topoi

The following definition for the notion of elementary higher topos was originally proposed by Shulman ([Shu17]) and further studied by Rasekh (see [Ras18a]):

\textbf{Definition 5.1.} An elementary higher topos is a quasicategory \( \mathcal{E} \) such that

1. \( \mathcal{E} \) is finitely complete et cocomplete.
2. \( \mathcal{E} \) is locally cartesian closed.
3. \( \mathcal{E} \) admits a subobject classifier.
4. For every morphism \( f : x \to y \) in \( \mathcal{E} \), there exists an object classifier \( p : V \to U \) classifying \( f \) such that the class of morphisms it classifies is closed under composition, fiberwise finite limits and dependent product.

At this point, we have studied how the properties 1, 2, and 4 in Definition 5.1 can be used to provide features of the type theory we construct from the quasicategory \( \mathcal{C} \) (here \( \mathcal{E} \)). Namely, we have that:

- Pullbacks provide substitutions (Section 2), the terminal object plays the role of the empty context and pushout give the so-called “pushout types” (Section 4).
- Local cartesian closure yields \( \Pi \)-types (Section 2).
- Objects classifier, or rather equivalently univalent map (see Theorem 3.32 of [Ras18a]), provide univalent universes (Section 3).

We now consider property 3, that is the existence of a subobject classifier. The link between such an object and the propositional resizing axiom is known (this is discussed in Section 8 of [Ras18b] for instance).

We will use the following definition

\textbf{Definition 5.2.} A model of type theory (e.g. a tribe) satisfies propositional resizing if there exists a fibration \( s : \tilde{V} \to V \) classifying the fibrations that are \((-1)\)-truncated.

The following proposition follows from Section 3 of the present papers, in the special case where the class \( \mathcal{S} \) of morphism classified by the universe is the class of fibrations that are homotopy monomorphisms ((\(-1\))-truncated maps).
Proposition 5.1. Suppose $\mathcal{C}$ has a subobject classifier. Then there is an essentially representable object of $\text{SSet}^{\mathcal{C}^{\text{op}}}$ which witnesses propositional resizing in the sense of Definition 5.2. 

Observing that a family of univalent universes inside $\mathcal{C}$ yields a family of univalent universes in $\mathcal{C}^{\ast\ast}$ (as in Section 3), Theorem 3.7 has now the following corollary:

Theorem 5.2. Every elementary higher topos provides a model for homotopy type theory.

Example 5.1. In his paper [Ane21], Anel makes the observation that the quasicategory $\mathcal{S}_{\text{coh}}^{<\infty}$ of truncated coherent spaces has properties that ought to make it an example of elementary higher topos although it does not admit all finite colimits and its universes are not closed under type constructors (dependent sum and product). In particular, it is not satisfy the axioms of Definition 5.1. Our work shows that, nonetheless, the quasicategory $\mathcal{S}_{\text{coh}}^{<\infty}$ provides an interpretation for a homotopy type theory which lacks some higher inductive types and whose universes are not closed under type constructors.

6 The internal language of LCC quasicategories

This section is based on results obtained in [KS19] (more precisely Theorem 9.10) and [Kap15] (Theorem 5.8). The latter result is required to see that the functor $\text{Ho}_{\infty}: \text{weCat} \to \text{SSet}$, which can be implemented in several ways (for instance as the hammock localization followed by the simplicial nerve), restricts suitably (so that the statement of the following proposition makes sense).

Proposition 6.1. The functor

$$\text{Ho}_{\infty}: \text{CompCat}_{\Sigma, \Pi, \text{ext}, \text{Id}} \to \text{QCat}_{\text{lcc}}$$

is a DK-equivalence.

Proof. This functor is known to induce weak homotopy equivalences on homspaces (this is proved in [KS19]), so it is enough to prove that it is essentially surjective on objects.

Starting from a locally cartesian closed quasicategory $\mathcal{C}$, the first two sections have described how to construct a corresponding simplicial category which consist in a simplicial $\pi$-tribe, whose simplicial nerve is equivalent to $\mathcal{C}$, and where Martin-Löf type theory with dependent sum and product (which are extensional) as well as intensional identity types can be interpreted. The only point that is worth mentioning is that the weak equivalences in the sense of a categorical model (see Definition 9.4 in [KS19]) coincide with the
weak equivalences in the sense of the tribe structure (hence also in the sense of the simplicially enriched category structure), that is because the identity types are defined from (relative) path objects.

This proves that the functor is indeed essentially surjective on objects.

The previous proposition can be rephrased by saying that the internal language of locally cartesian closed quasicategories is an dependent type theory with dependent sums, dependent products that are extensional and intensional identity types. Hence, this proves the conjecture formulated in [KL18].
Appendix - Some details on the interpretation of MLTT

In this appendix, we quickly recall the usual interpretation of the core of MLTT in a (full subcategory of a) suitable model category, or, more generally, in simplicial $\pi$-tribe $\mathcal{T}$.

**Definition 6.1** (Interpretation). We interpret the core of MLTT as follows:

- Each context $\Gamma$ is interpreted by an object of $\mathcal{T}$ noted $[[\Gamma]]$.
- The empty context is interpreted by the terminal object $id : C \to C$.
- Each substitution $\Gamma \vdash s : \Delta$ is interpreted by a morphism $[[\Gamma]] \to [[\Delta]]$.
- Each dependent type $\Gamma \vdash A$ is interpreted by a fibration $p_A : [[\Gamma, A]] \to [[\Gamma]]$.
- Each term $\Gamma \vdash a : A$ is interpreted by a section of $p_A$.

**Proposition 6.2** (Substitution). Substitution can be modeled by pullback: if $\Gamma \vdash s : \Delta$ is a substitution and $\Delta \vdash A$ is a type in context $\Delta$, there is a substituted type $\Gamma \vdash s^* A$.

**Proof.** Since fibrations are stable under pullback, pulling back $s$ along $p_A : [[\Delta, A]] \to [[\Delta]]$ yields a fibration $p_A' : [[\Gamma, s^* A]] \to [[\Gamma]]$. Given a term $\Delta \vdash a : A$, the universal property of the pullback provides a term $a'$ of type $s^* A$. □

**Proposition 6.3** (Dependent sum). Given types $\Gamma \vdash A$ (associated to a projection $p_A$) and $\Gamma, A \vdash B$ (associated to a projection $p_B$), there is a sum-type $\Gamma \vdash \Sigma A B$ provided by the composite $p_{\Sigma AB} = p_A \circ p_B$.

**Proof.** The composite of two fibrations indeed yields a fibration. By construction $\Gamma, A, B$ is the same as the context $\Gamma, \Sigma A B$. The identity morphism associated to the object $[[\Gamma, A, B]]$ yields the substitution $\Gamma, A, B \vdash \text{pair}_{A,B} : \Sigma A B$.

Given a term $\Gamma \vdash a : A$ (given by a section $s_A$ of $p_A$) and a term $\Gamma \vdash B[a]$ (as section $s_B$ of $p_B$); $s_B \circ s_A$ gives a section of $p_{\Sigma AB}$ which is the term $(a, b)$. For elimination, suppose given a type $\Gamma, \Sigma A B \vdash C$ (corresponding to the projection $p_C$) and a term $\Gamma, x : A, y : B \vdash d(x, y) : C$ (as a section of the pullback of $p_C$ along $\text{pair}_{A,B}$). This gives immediately a term $\Gamma, z : \Sigma A B \vdash d(z) : C$ which is the exact same section (the pullback is trivial).

The compatibility with substitution stems from the Beck-Chevalley condition satisfied by the adjunctions of the form $f_! \dashv f^*$.

**Proposition 6.4** (Dependent product). Given types $\Gamma \vdash A$ (associated to a projection $p_A$) and $\Gamma, A \vdash B$ (associated to a projection $p_B$), there is a product-type $\Gamma \vdash \Pi A B$ provided by the dependent product $\Pi p_A p_B$. 

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Proof. The dependent product \( p_{\Pi A B} = \Pi_{p_A} p_B \) is still a fibration. By transposition, a term \( \Gamma, a : A \vdash b : B \) given as section \( s_B \) of \( p_B \) is equivalent to the data of a section \( s_{\Pi} \) of \( p_{\Pi A B} \).

Explicitly, given a term \( \Gamma \vdash f : \Pi A B \) given as a section \( s_{\Pi} \) of \( p_{\Pi} \), and a term \( \Gamma \vdash a : A \) (a section \( s_A \)). The applied term \( \Gamma, a : A \vdash f(a) : B \) is interpreted by the composite of \( s_A \) and of the transpose \( s'_{\Pi} \) of \( s_{\Pi} \). In particular \( p_B \circ s'_{\Pi} = s_A \).

Compatibility with substitution is as before.

Proposition 6.5 (Identity types). Given types \( \Gamma \vdash A \) (associated to a projection \( p_A \)), there is a identity-type \( \Gamma, A, p^* A \vdash Id_A \) provided by the path object \( Path(p_A) \) (obtained by factoring the dependent diagonal \( A \rightarrow \Delta_{p_A} A \) as \( (i_0, i_1) \circ \text{refl}_A \)).

Proof. The so defined identity type comes by definition with a fibration \( p_{Id_A} = Path(p_A) \rightarrow p^*_A A \) a section \( \text{refl}_A \) of \( p_{p^*_A A} \circ p_{Id_A} \) (the first part of the factorization).

Given a type \( \Gamma, A, p^*_A A, Id_A \vdash C \) with terms given as a section \( d : \Gamma, A \rightarrow \Gamma, A, p^*_A A, Id_A, C \) such that \( p_C \circ d = \text{refl}_A \), there is a section \( J_{C,d} : \Gamma, A, p^*_A A, Id_A \rightarrow \Gamma, A, p^*_A A, Id_A, C \) given as a solution of the following lifting problem (hence \( J_{C,d} \circ \text{refl}_A = d \)).

\[
\begin{array}{ccc}
\Gamma, A & \xrightarrow{d} & \Gamma, A, p^*_A A, Id_A, C \\
\downarrow \text{refl}_A & & \downarrow p_C \\
\Gamma, A, p^*_A A, Id_A & \xrightarrow{id} & \Gamma, A, p^*_A A, Id_A \\
\end{array}
\]
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