Effective Theory For Quantum Spin System In Low Dimension - Beyond Long-Wavelength Limit

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Abstract

An effective theory for quantum spin system in low dimension is constructed in the finite-q regime. It is shown that there are field configurations for which Wess-Zumino term contributes to the partition functions as topological term for ferromagnet as well as antiferromagnet in both one and two dimensional lattice, in contrast to the long wave length regime.
PACS: 75.10.Jm

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Introduction

Quantum spin system in low dimension has drawn a lot of attention from the theorists after the discoveries of high temperature superconductivity and quantum Hall effect. In a recent paper\textsuperscript{1}, we presented a unified scheme for analysing the topological terms in the effective action corresponding to the long wave length limit for the XY-like anisotropic quantum Heisenberg ferromagnet and antiferromagnet in one and two spatial dimensions for any value of the spin. The present work originated from the apparent discrepancy between the neutron scattering experimental results\textsuperscript{2} and theoretical predictions based on analysis\textsuperscript{3} in the long wave length regime. Here we generalize our previous formalism and analysis to medium wave length regime and characterize the excitations. The importance of the extension of the theoretical analysis to the short wave length regime was also highlighted recently by Tai Kai Ng\textsuperscript{4}. Our results provide possible explanation of some of the qualitative features observed in neutron scattering experiments.

Mathematical Formulation

We analyse the quantum actions for XXZ ferromagnet and anti-ferromagnet in both 1D and 2D by 'spin coherent state' formalism\textsuperscript{1,5} in the 'medium wave-length' limit. This is the lowest order correction to the results obtained in the long wave length limit. In particular we would like to investigate the existence of topological term in the effective action corresponding to finite q-regime. As in the previous paper\textsuperscript{1} we choose the anisotropy of the spin models to be XY-like.

We perform all the calculations on the lattice with a finite lattice parameter \(a\) in the so called 'medium wavelength' limit implying that the Wess-Zumino term is kept up to order \(a^2\) (in the long wave-length limit this was up to order \(a\) ) and the Hamiltonian is kept up to order \(a^3\) or \(a^4\) according to whether the spin system is ferromagnetic or antiferromagnetic (in the long-wavelength limit this was up to order \(a^2\); for antiferromagnets the terms of the order \(a^3\) turn out to be surface terms and hence vanish due to periodic boundary conditions ).

Calculations

The quantum Euclidean action \(S_E[n]\), for spin coherent fields \(n\), is given by\textsuperscript{1,5}:

\[
S_E[n] = -is\sum_r S_{WZ}[n(r)] + \frac{s\delta t}{4} \sum_r \int_0^\beta dt (\partial_t n(r))^2 + \int_0^\beta dt H(n) 
\]  

(1)

where \(s\) is the magnitude of the spin and

\[
H(n) = \langle n|H(s)|n\rangle; |n\rangle = \prod_{\sigma \tau} |n(r)\rangle
\]  

(2)

\(H(s)\) being the spin Hamiltonian in the representation \(s\). The Wess-Zumino term \(S_{WZ}\) is given by\textsuperscript{5}

\[
S_{WZ}[n(r)] = \int_0^\beta dt \int_0^1 d\tau n(r,t,\tau) \cdot \partial_t n(r,t,\tau) \land \partial_\tau n(r,t,\tau) = \mathcal{A}
\]  

(3)

with \(n(t,0) \equiv n(t); n(t,1) \equiv n_0; n(0,\tau) \equiv n(\beta,\tau); t \in [0,\beta], \tau \in [0,1]\).

In (3) \(\mathcal{A}\) is the area of the cap bounded by the trajectory \(\Gamma\) parametrized by \(n(t)\) on the sphere

\[
n(r) \cdot n(r) = 1
\]  

(4)
Here \( |n \rangle \) is the spin coherent state as defined in Refs.1,5. The spin Hamiltonian for XXZ Heisenberg ferromagnet (antiferromagnet) is given by

\[
H(S) = -g \sum_{\langle r, r' \rangle} \langle r \rangle \cdot \langle r' \rangle - g\lambda z \sum_{\langle r, r' \rangle} S_3(r) S_3(r')
\]  

with \( g \geq 0 \), (or \( g \leq 0 \)) \( 0 \leq \lambda z \leq 1 \), where \( r \) and \( r' \) run over the lattice and \( \langle r, r' \rangle \) signifies nearest neighbour interaction and \( S = (\langle \hat{S}, S_3 \rangle) \).

**Linear Chain**

Using eqn.(1) the quantum Euclidean action for ferromagnet in the medium-wavelength limit can be written as:

\[
S_E[n] = -i8\pi f[n] - i a^2 N \sum_{i=1}^{\beta} dt[m \cdot \partial_t m \wedge \partial_x m + l \cdot \partial_t m \wedge \partial_x m]
\]

\[
+ (-\frac{1}{2} \partial_x^2 m + \partial_x l) \cdot m \wedge \partial_t m + a \partial_x^2 m \partial_x l [2ia]
\]

\[
+ gs^2 a^2 \sum_{i=1}^{\beta} dt[(\partial_x m)^2 + 2a\partial_x m \partial_x l + \rho C_3^2]
\]

\[
+ gs^2 (1 - \lambda z)a^2 \sum_{i=1}^{\beta} dt[2m_3^2 + 4al_3 m_3 + 2a^2 l_3^2] [2ia]
\]

where we have used the Hamiltonian in the spin coherent field\( |n \rangle \) representation:

\[
H(n) = -gs^2 \sum_{i=1}^{2N} \langle n(ia) \cdot \langle n((i+1)a) - gs^2 \lambda z \sum_{i=1}^{2N} n_3(ia)n_3((i+1)a)
\]

with \( n(ia) = m(ia) + al(ia) \), \( m \) being the order parameter field and \( l \) the fluctuation around \( m \). The first term in (6) viz.,\( F[n] \) is a fraction and depends on the \( n \)-field configuration. However this being always a fraction it has no bearing on the winding number of a topological sector. The second term on the right hand side of (6) [denoted by \( WZ \) in eqn.(8)] arises only from the Wess-Zumino part of the action given by (1). This being a term in the order parameter field \( m \), the fractional term \( f[n] \) does not contribute to this term while integrating out the fluctuation field \( l \) from the partition function. We analytically continue action (6) to Minkowski one by replacing \( t \) by \( it \) and \( \beta \) by \( iT \). We take the truncated action without the fractional term and integrate out the field \( l \). The effective action in the order parameter field \( m \) takes the following form:

\[
S_M = [saWZ + sa^2 \sum_{i=1}^{N} \int_0^T dt \left( B_1^2 + B_2^2 + \rho C_3^2 \right)]
\]
the Hamiltonian (10) as well as in the Wess-Zumino term

\[ -\frac{sa^2}{2} \sum_{i=1}^{N} \int_{0}^{T} dt \mathbf{m} \cdot \partial_t \mathbf{m} \wedge \partial_x^2 \mathbf{m} \]

\[ -gs^2a^2\sum_{i=1}^{N} \int_{0}^{T} dt ((\partial_x \mathbf{m})^2 + \lambda_z(\partial_x m_3)^2) \]

\[ -2gs^2(1-\lambda_z)\sum_{i=1}^{N} \int_{0}^{T} dt m_3^2 \] (2ia)

(8)

In eqn.(8) we have defined \( WZ = \sum_{i=1}^{N} \int_{0}^{T} dt(\mathbf{m} \cdot \partial_t \mathbf{m} \wedge \partial_x \mathbf{m}) \);

\( B_k = -\frac{1}{2} \mathbf{m} + \frac{3}{2} \partial_2 \mathbf{m} \wedge \partial_x \mathbf{m} + gs_0 \partial_x^2 \mathbf{m} \)

\( k = 1, 2 \); \( C_k = -\frac{1}{2} \mathbf{m} + \frac{3}{2} \partial_2 \mathbf{m} \wedge \partial_x \mathbf{m} + \frac{gs_0 \lambda_z}{\rho} \partial_x^2 \mathbf{m} \) and \( \rho = 1 + 2gs(1 - \lambda_z) \).

Now we analytically continue the expression on right hand side of eqn.(8) to Euclidean space and demand finiteness of the action. As a consequence the boundary conditions on the \( \mathbf{m} \)-fields are quite general. However the class of field configurations satisfying the boundary conditions \( \mathbf{m}(x, t) \rightarrow \mathbf{m}_0 \) at \( \infty \) [Ref.1] together with \( m_3 \rightarrow 0 \) at \( \infty \) keeps the action finite and makes the WZ term in eqn.(8) a topological one.\(^5\) The quantum action for antiferromagnet in the medium wavelength limit can be written as :

\[ S_E[\mathbf{n}] = -isa\sum_{i=1}^{N} \int_{0}^{\beta} dt \]

\[ [\mathbf{m} \cdot \partial_t \mathbf{m} \wedge \partial_x \mathbf{m} + 2l \cdot \mathbf{m} \wedge \partial_x \mathbf{m}] \] (2ia)

\[ -isa^2\sum_{i=1}^{N} \int_{0}^{\beta} dt \left[ -\frac{1}{2} \mathbf{m} \cdot \partial_t \mathbf{m} \wedge \partial_x^2 \mathbf{m} \right] \]

\[ -m \cdot \partial_t \mathbf{m} \wedge \partial_x l - m \cdot \partial_x \mathbf{m} \wedge \partial_t l \]

\[ +2l \cdot \mathbf{m} \wedge \partial_t l + l \cdot \partial_t \mathbf{m} \wedge \partial_x \mathbf{m} \] (2ia)

\[ +gs^2a^2\sum_{i=1}^{N} \int_{0}^{\beta} dt [4l^2 + (\partial_x \mathbf{m})^2] \]

\[ -\frac{a^2}{12} (\partial_x^2 \mathbf{m})^2 + a^2 l \cdot \partial_x^2 l \] (2ia)

\[ -gs^2(1-\lambda_z)a^2\sum_{i=1}^{N} \int_{0}^{\beta} dt (m_3^2 + a^2 l_3^2) \]

\[ +2gs^2(1-\lambda_z)\sum_{i=1}^{N} \int_{0}^{\beta} dt (m_3^2 + a^2 l_3^2) \] (2ia)

(9)

where we have used in eqn.(1)

\[ H(\mathbf{n}) = gs^2\sum_{i=1}^{2N} \mathbf{n}(ia) \cdot \mathbf{n}((i+1)a) + gs^2\lambda_z\sum_{i=1}^{2N} n_3(ia)n_3((i+1)a) \] (10)

The expression of the above action (9) is different from (6) due to the fact that in the Hamiltonian (10) as well as in the Wess-Zumino term \( S_{WZ}[\mathbf{n}(ia)] \) we have done the staggering operation: \( \mathbf{n}(ia) \rightarrow (-1)^ia\mathbf{n}(ia) \) and the staggered field \( \mathbf{n}(ia) \) is then set \( \mathbf{n}(ia) = \mathbf{m}(ia) + (-1)^ia\mathbf{l}(ia) \). We neglect terms like \( \mathbf{l} \cdot \partial_x^2 \mathbf{l} \) in the above action since \( \mathbf{m} \) is slowly varying and \( \mathbf{l} \) behaves as a derivative of \( \mathbf{m} \) and as a result these terms are smaller with respect to terms involving derivatives of \( \mathbf{m} \) within the terms of the order \( a^4 \).

Then we continue the action (9) to Minkowski one in a similar fashion as in the case of
ferromagnet and integrate out the field \( l \) from the partition function. The effective action in the order parameter field takes the following form:

\[
S_M = [saWZ + \frac{1}{16g}\sum_{i=1}^{N}\int_0^T dt (D_1^2 + D_2^2 + \frac{2}{1+\lambda}D_3^2)]
\]

\[
- gs^2a^2\sum_{i=1}^{N}\int_0^T dt (E_1^2 + E_2^2 + \lambda_3E_3^2)
\]

\[
-sa^2\sum_{i=1}^{N}\int_0^T dt \frac{1}{2} \mathbf{m} \cdot \partial_t \mathbf{m} \wedge \partial_x^2 \mathbf{m}
\]

\[
-2gs^2(1-\lambda_3)\sum_{i=1}^{N}\int_0^T dt m_3^2(2ia)
\]

where \( D = 2\mathbf{m} \wedge \partial_t \mathbf{m} + a\mathbf{m} \wedge \partial_t \partial_x \mathbf{m}, k = 1, 2 \)

\[
E_l = (\partial_x m_l)(\partial_x m_l) - \frac{1}{12}(\partial_x^2 m_l)(\partial_x^2 m_l), l = 1, 2, 3.
\]

Again the argument similar to the one in the case of ferromagnet, the field configurations satisfying \( \mathbf{m} \rightarrow \mathbf{m}_0 \) at \( \infty \) with \( m_3 \rightarrow 0 \) at \( \infty \) makes the WZ term in eqn.(11) a topological one.

At this stage we point out that in the long wavelength limit i.e., when we keep terms up to order \( a \) in the WZ-part and up to order \( a^2 \) in the Hamiltonian part and integrate out the fluctuation \( l \) in the partition function we get usual effective action in \( \mathbf{m} \). This leads to non-linear sigma model with a topological term in the case of antiferromagnet and a different model with a topological term in the case of ferromagnet.

**Two dimensional square lattice**

The Minkowskian action for ferromagnet in the two dimensional square lattice can be written in the form:

\[
S_M = 8\pi s\chi[\mathbf{n}] + sa\sum_{i,j=1}^{N}\int_0^T dt m_3 \cdot \partial_t \mathbf{m} \wedge \partial_x \mathbf{m}
\]

\[
m_3 \cdot \partial_t \mathbf{m} \wedge \partial_x \mathbf{m}(2ia,2ja) - sa^2\sum_{i,j=1}^{N}\int_0^T dt[2] + 2l \cdot
\]

\[
\frac{m_3}{a} - \frac{3}{2}M + \frac{1}{2}\mathbf{m} \cdot \partial_t \mathbf{m} \wedge \nabla^2 \mathbf{m}(2ia,2ja)
\]

\[
-2gs^2a^2\sum_{i,j=1}^{N}\int_0^T dt [(\partial_x m_1)^2 + (\partial_x m_2)^2 + (\partial_y m_1)^2]
\]

\[
(\partial_y m_2)^2 - 2al_3 \cdot \nabla^2 \mathbf{m}(2ia,2ja) - 2gs^2\lambda_3a^2\sum_{i,j=1}^{N}\int_0^T dt[(\partial_x m_3)^2 + (\partial_y m_3)^2 - 2al_3\nabla^2 m_3](2ia,2ja) - 2gs^2(1-\lambda_3)
\]

\[
\sum_{i,j=1}^{N}\int_0^T dt[4m_3^2 + 8am_3l_3 + 4a^2l_3^2](2ia,2ja)
\]

(12)

Here \( \chi[\mathbf{n}] \) is a proper fraction depending on the field configuration. \( \mathbf{M} = \partial_t \mathbf{m} \wedge (\partial_x \mathbf{m} + \partial_y \mathbf{m}) \) and \( m_i, l_i, M_i \) stand for the \( i \)th components of \( \mathbf{m}, l, \mathbf{M} \) in the spin space. \( \tilde{\mathbf{m}}, \tilde{l}, \tilde{\mathbf{M}} \) stand
Using the same argument as used in the case of ferromagnetic chain we neglect the fraction number $\chi[n]$ and work with the reduced action. After integrating out the field $l$, the effective action in the order parameter field $m$ takes the form:

\[
S_M = saWZ + sa^2 \sum_{i,j=1}^{N} \int_0^T dt[T_1^2 + T_2^2 + \alpha R_3^2](2ia, 2ja)
- \frac{sa^2}{2} \sum_{i,j=1}^{N} \int_0^T dt[m \cdot \partial_t m \wedge \nabla^2 m](2ia, 2ja)
-2gs^2a^2 \sum_{i,j=1}^{N} \int_0^T dt[(\partial_x \tilde{m})^2 + (\partial_y \tilde{m})^2]
+ \lambda_x(\partial_x m_3)^2 + \lambda_y(\partial_y m_3)^2](2ia, 2ja)
- \frac{1}{2}sa^2 \sum_{i,j=1}^{N} \int_0^T dt(m \cdot \partial_t m \wedge \nabla^2 m)(2ia, 2ja)
-8gs^2(1 - \lambda_z) \sum_{i,j=1}^{N} \int_0^T dt m_{3x}^2(2ia, 2ja)
\] (13)

where $WZ = [m \cdot \partial_t m \wedge \partial_x m + m \cdot \partial_t m \wedge \partial_y m](2ia, 2ja)$,
$T_k = \frac{1}{m}m_k - \frac{3}{2}M_k - 2gsa\nabla^2 m_k$, $k = 1, 2$
and $R_3 = \frac{1}{m}m_3 - \frac{3}{2}M_3 - \frac{2gsa\lambda_z}{\alpha}\nabla^2 m_3$ Here $\nabla^2$ stands for $\partial_x^2 + \partial_y^2$ in two dimensional space and $\alpha = 1 + 8gs(1 - \lambda_z)$.

We write down the action for the antiferromagnet neglecting the terms $\partial_x l^2, \partial_y l^2$ as compared to terms involving $m$ or derivatives of $m$ within $a^4$ order in a manner similar to the case of antiferromagnetic chain as:

\[
S_M = sa^2 \sum_{i,j=1}^{N} \int_0^T dt[a \cdot m \wedge \partial_t m](2ia, 2ja) + sa^2 \sum_{i,j=1}^{N} \int_0^T dt[-2l \cdot \partial_t m \wedge m + \partial_x m \vee \partial_y m + \partial_t m \wedge m + 1 \cdot \partial_t m \wedge \partial_y m]
+ \partial_y m](2ia, 2ja) + sa^2 \sum_{i,j=1}^{N} \int_0^T dt[\partial_x m \cdot \partial_t m \wedge \partial_y m \cdot \partial_t m \wedge \partial_x m](2ia, 2ja)
-2gs^2a^2 \sum_{i,j=1}^{N} \int_0^T dt[8l^2 + (\partial_x m)^2 + (\partial_y m)^2](2ia, 2ja) - 2gs^2a^2(1 - \lambda_z) \sum_{i,j=1}^{N} \int_0^T dt[8l_3^2 + (\partial_x m_3)^2 + (\partial_y m_3)^2](2ia, 2ja)
- \frac{1}{6}(\partial_x m_3)^2 + \frac{1}{6}(\partial_y m_3)^2](2ia, 2ja)
+ \frac{1}{6}(\partial_x m_3)^2 + \frac{1}{6}(\partial_y m_3)^2](2ia, 2ja)
\] (14)

After integrating out the fluctuation field $l$ we have the following form of the effective
action:

\[
S_M = \left(\frac{so^2}{2}\right)WZ + \frac{1}{4g} \sum_{i,j=1}^{N} \int_0^{T} dt \left( K_1^2 + K_2^2 + \frac{4}{1 + 3\lambda_z}K_3^2 \right)
- 2gs^2a^2 \sum_{i,j=1}^{N} \int_0^{T} dt \left( J_1^2 + J_2^2 + \lambda_z J_3^2 \right)
+ \frac{gs^2a^4}{6} \sum_{i,j=1}^{N} \int_0^{T} dt \left( P_1^2 + P_2^2 + \lambda_z P_3^2 \right)
- \frac{gs^2a^4(1 - \lambda_z)}{12} \sum_{i,j=1}^{N} \int_0^{T} dt \left( P_3^2 + +12(\partial_x \partial_y m_3)^2 \right)
- 4gs^2(1 - \lambda_z) \sum_{i,j=1}^{N} \int_0^T dt \left( m_3 \right)^2(2ia, 2ja)
\]

(15)

Where \( WZ = \sum_{i,j=1}^{N} \int_0^{T} dt \left[ \partial_x (m \cdot \partial_t m \wedge \partial_y m) + \partial_y (m \cdot \partial_t m \wedge \partial_x m) \right] \),
\( K = \left[ m \wedge \partial_t m + \frac{a}{2} (m \wedge \partial_x m + m \wedge \partial_y m) + \frac{a^2}{4} \mathbf{M} \right] \),
\( M \) has been defined before in the action for ferromagnet.

\( J_k^2 = (\partial_x m_k)^2 + (\partial_y m_k)^2 \), \( P_k^2 = (\partial_x^2 m_k)^2 + (\partial_y^2 m_k)^2 \)
with \( k = 1, 2, 3 \)

To see the role of WZ-term in ferromagnet as well as antiferromagnet we go back to Euclidean forms of the actions (13) and (15) and demand finiteness of the actions. As before the class of \( m \)-field configurations satisfying boundary conditions \( m \rightarrow m_0 \) at \( \infty \) and \( m_3 \rightarrow 0 \) at \( \infty \) makes the WZ-term topological in both the cases. The periodic boundary condition on the space lattice and time makes \( m \) periodic but not necessarily their derivatives. Thus the WZ-term on the right hand side of eqn.(15) is a topological term as it is derivative of the winding number which has been discussed in Ref.1.

Conclusions and comments

The lattice structure has to be explicitly dealt with as we are working with finite-q (in the medium and short wave length regime). The definition of topological term is however exact only when we go to continuum limit(\( q \rightarrow 0 \)).

There should be a cross-over from non-topological to topological behaviour in physical manifestation for 2D quantum Heisenberg antiferromagnet as we increase the magnitude of the wave vector (i.e., q-value) to go from long wave length regime to short wave length regime. The neutron scattering experiments performed on the Copper Oxides in the antiferromagnetic phase, seem to probe the medium and the short wave-length regime and exhibit some of the features of topological excitations in accordance with our inference.

The quantum Kosterlitz-Thouless-like scenario may hold in the finite-q regime too, as the topological excitations continue to exist in this regime for a 2D quantum Heisenberg ferromagnet.

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