1. Introduction. T. E. Harris wrote the first definitive book \cite{5} on branching processes, published in 1963. It covered much of the work on the subject up to that time, a sizeable part due to Harris himself. It identified the subject of branching processes and resulted in a great deal of interest in the subject, among both mathematicians and statisticians. Between 1963 and 1970, a vast number of papers on branching processes appeared in many good journals specializing in probability theory and mathematical statistics, and by 1971 more books on the subject appeared both in the U.S. and elsewhere \cite{15,18}. Harris himself moved on to work on other beautiful topics such as percolation and interacting particle systems. As with branching processes, his work in these other areas was profound. T. E. Harris was pioneer par excellence, creating many areas of research in which he laid the foundations that others built on. In what follows, we present a brief account of Harris’s contribution to branching processes.

Harris’s 1947 PhD dissertation at the mathematics department of Princeton University was on branching processes, titled “Some theorems on Bernoulli multiplicative processes.” This was followed in 1948 by his basic paper \cite{6} in the Annals of Mathematical Statistics. In \cite{6}, he used the term branching processes, a term which had also been used by Russian mathematicians; he treated the single type discrete time branching process. He also coined the term Galton–Watson branching process for this process. His main focus in \cite{6} was on the supercritical case; we now give a description of this work.
2. Single type, discrete time case. Let \( \{p_j\}_{j \geq 0} \) be a probability distribution. Let \( \{\xi_{n,k}\}_{n \geq 0, k \geq 1} \) be an array of nonnegative integer valued random variables that are i.i.d. (independent and identically distributed) with distribution \( \{p_j\}_{j \geq 0} \). Let \( Z_0 \) be a positive integer. Now set

\[
Z_1 = \sum_{k=1}^{Z_0} \xi_{0,k}
\]

and for \( n \geq 1 \),

\[
Z_{n+1} = \sum_{k=1}^{Z_n} \xi_{n,k} \text{ if } Z_n > 0 \text{ and } 0 \text{ if } Z_n = 0.
\]

Then the sequence \( \{Z_n\}_{n \geq 0} \) is called a Galton–Watson branching process with initial population \( Z_0 \) and offspring distribution \( \{p_j\}_{j \geq 0} \). Clearly, \( \{Z_n\}_{n \geq 0} \) is a Markov chain with time homogeneous transition probabilities and the nonnegative integers as the state space. The transition probabilities are given by

\[
p_{ij} = P(\sum_{r=1}^{i} \xi_r = j) \quad \text{for } i \geq 1 \quad \text{and } \quad p_{00} = 1,
\]

where \( \{\xi_r\}_{r \geq 1} \) are i.i.d. with distribution \( \{p_k\}_{k \geq 0} \).

One can interpret the sequence \( \{Z_n\}_{n \geq 0} \) as follows. If \( Z_n \) is thought of as the number of individuals in the \( n \)th generation, then each one of them produces a random number of children with distribution \( \{p_j\}_{j \geq 0} \) independently of others in the \( n \)th generation as well as any past ancestors. The total number \( Z_{n+1} \) of all these individuals is the size of the \( (n+1) \)st generation.

An important parameter in determining how the sequence \( \{Z_n\}_{n \geq 0} \) behaves for \( n \) large is the offspring mean \( m \equiv \sum_j j p_j \). Here are some basic results.

**Theorem 2.1.** Let \( 0 < m \equiv \sum_{j=1}^{\infty} j p_j < \infty \) and let \( P(0 < Z_0 < \infty) = 1 \). Then:

(i) \( m < 1 \Rightarrow P(Z_n \to 0 \text{ as } n \to \infty) = 1 \),

(ii) \( m = 1, p_1 < 1 \Rightarrow P(Z_n \to 0 \text{ as } n \to \infty) = 1 \),

(iii) \( m > 1 \Rightarrow P(Z_n \to 0 \text{ as } n \to \infty | Z_0 = 1) \equiv q < 1 \),

where \( q \) is the unique root of the equation

\[
s = f(s) \equiv \sum_{j=0}^{\infty} p_j s^j, \quad 0 \leq s < 1.
\]

Further, \( P(Z_n \to \infty \text{ as } n \to \infty | Z_0 = 1) = 1 - q \), and for any \( k \geq 1 \), \( P(Z_n \to 0 \text{ as } n \to \infty | Z_0 = k) = q^k \).

Harris in his book [5] notes in that in 1874 in [19], Galton and Watson did notice that the extinction probability \( q \) satisfied \( q = f(q) \), but failed to
notice that if \( m > 1 \) the relevant root is less than one. Galton and Watson’s work was motivated by the problem of the survival of British peerage names, posed by Galton in the *London Times* in the 1870s.

In his paper \([6]\), which is based on his doctoral thesis, Harris focused mainly on the *supercritical case*, that is, \( m > 1 \). The case \( m = 1 \) is called *critical* and \( m < 1 \) is the *subcritical case*. Let \( \{p_j\} \), \( m \), \( \{Z_n\} \) be as in Theorem 1.

**Theorem 2.2** (Supercritical case, \([6, 7]\)). Assume \( p_0 = 0, p_1 < 1, m > 1, \sum_{j=1}^{\infty} j^2 p_j < \infty \) and \( 0 < Z_0 < \infty \). Let \( W_n \equiv Z_n/m^n, n \geq 0 \). Then there exists a nonnegative random variable \( W \) such that:

(i) \( E((W_n - W)^2 | Z_0) \to 0 \) as \( n \to \infty \),

(ii) \( P(W = 0) = 0 \),

(iii) \( W \) has an absolutely continuous distribution on \((0, \infty)\) with a continuous density,

(iv) \( E(W | Z_0 = 1) = 1 \).

Harris \([5]\) observes that J. L. Doob seems to have been the first to note that \( \{W_n\}_{n \geq 0} \) is a martingale and, being nonnegative, converges a.s. as \( n \to \infty \). Kesten and Stigum \([13]\) improved on this, as follows.

**Theorem 2.3** \([13]\). Let \( p_0 = 0, p_1 < 1, Z_0 < \infty, 1 < m \), and \( W_n = \frac{Z_n}{m^n} \). Then:

(i) \( \sum_{j=1}^{\infty} j (\log j)p_j < \infty \Rightarrow W_n \to W \) a.s. and in mean, where \( P(W = 0) = 0, E(W | Z_0 = 1) = 1 \) and \( W \) has an absolutely continuous distribution on \((0, \infty)\).

(ii) \( \sum_{j=1}^{\infty} j (\log j)p_j = \infty \Rightarrow W_n \to 0 \), a.s.

The work of A. N. Kolmogorov \([14]\) in 1938 and A. M. Yaglom \([20]\) in 1947 (see \([5]\)) led to the following.

**Theorem 2.4** (Critical case). Suppose \( m = 1, p_1 < 1 \) and \( \sum_{j=1}^{\infty} j^2 p_j < \infty \). Then, as \( n \to \infty \),

(i) \( nP(Z_n > 0 | Z_0 = 1) \to \frac{\sigma^2}{2} \), where \( \sigma^2 \equiv \sum_{j=1}^{\infty} j^2 p_j - 1 \),

(ii) \( P(\frac{Z_n}{n} > x | Z_0 = 1, Z_n > 0) \to e^{-2/\sigma^2 x}, \) for all \( 0 < x < \infty \).

**Theorem 2.5** (Subcritical case). Let \( m < 1 \). Then for all \( j \geq 1, \lim_{n} P(Z_n = j | Z_n > 0) \equiv b_j \) exists, \( 0 < b_j < \infty \) and \( \sum_{j=1}^{\infty} b_j = 1 \).

In his book \([5]\), Harris presents extensions of Theorems 2.1, 2.2, 2.4 and 2.5 to the multitype (finite type) case. In \([13]\), Kesten and Stigum established the analog of Theorem 2.3 above for the multitype Galton–Watson process. See Athreya and Ney \([1]\) for details; see also Sevastyanov \([18]\) and Mode \([15]\).
3. **Single type, age dependent case.** In 1948 Harris, with Richard Bellman [3, 10], formulated the theory of age dependent branching processes, where each individual lives a random length of time and on death creates a random number of individuals, and all individuals live and reproduce independently of each other. Assuming all moments on the offspring distribution and an absolutely continuous life time distribution, they established an integral equation for the probability generating function of \( Z(t) \), the population size at time \( t \). They showed that in the supercritical case, \( Z(t) e^{-\alpha t} \) converges in probability to a limit random variable \( W \), where \( \alpha \) is the Malthusian parameter defined by \( m \int_{0}^{\infty} e^{-\alpha u} dG(u) = 1 \), with \( G(\cdot) \) being the distribution function of the lifetime of an individual. They further showed that \( W \) is nontrivial and has an absolutely continuous distribution on \((0, \infty)\). There are analogs of Theorems 2.3 and 2.4 for this case, as well.

Conditions for the supercritical case were relaxed by later authors; see Athreya and Ney [1].

4. **General type case.** Harris also considered branching processes with arbitrary type space by using the point process approach. Here in any generation, one has a finite point process an some type space \( X \). The basic branching property of independent production is retained. An individual located at \( x \in X \) produces children according to a point process over \( X \) whose distribution depends on \( x \). All individuals act independently of each other. For this, Harris used the method of moment generating functions. In [9], he established the analog of Theorem 2.2 in this context, and applied this to nuclear cascades and related processes, as well as a one-dimensional neutron model. For details on this, see Harris's book [5]. Harris mentions that J. E. Moyal worked on similar ideas. In the 1970s, Jagers and his collaborators in Sweden developed this topic further in great detail (see [11]). See also Ney [16, 17].

5. **Cosmic-ray cascades.** Harris studied the theory of cosmic-rays cascades and supplemented the work of nuclear physicists; Chapter 7 of his 1963 book [5] deals with this topic. We present a brief summary of Harris’s work on cosmic-ray cascades as discussed in his paper [8]. Here are the model assumptions:

   (1) A photon of positive energy \( \varepsilon \), moving through homogeneous material, has probability \( \lambda dt + o(dt) \) of being transformed in the thickness interval \((t, t + dt)\) into two electrons, positive or negative, which receive energies \( \varepsilon U \) and \( \varepsilon(1 - U) \), respectively, where \( U \) is a random variable with an absolutely continuous distribution in \((0, 1)\). Note that the role of time parameter is played by the thickness of the material.

   (2) An electron loses (by “collision” or “ionization”) a deterministic amount of energy \( \beta t \) in an interval of length \( t \).
Harris shows that the following integro-differential equation holds:

\[ \partial f_2(s, \varepsilon, t) = \int_0^1 \left[ f_1 \left( s, \frac{\varepsilon}{u}, t \right) f_2 \left( s, \frac{\varepsilon}{1 - u}, t \right) - f_2(s, \varepsilon, t) \right] k(u) \, du, \]

with \( f_1(s, \varepsilon, 0) = f_1(s, 1, t) = f_2(s, 1, t) = 1 \) for \( t > 0 \) and \( f_2(s, \varepsilon, 0) = s \) for \( \varepsilon < 1 \). Harris shows that the earlier results of Bartlett and Kendall [2] and of Janossy [12] could be deduced from the above.

Harris introduces a vector valued Markov process \((I(t), \zeta(t)), t \geq 0\), where \( I(t) \) is the condition of a single particle at time \( t \) which can be a photon (\( I = 1 \)) or an electron (\( I = 2 \)) and has energy \( \zeta(t) \). He then derives the limiting distribution of the process \((I(t), \zeta(t))\) as \( t \to \infty \) (assuming \( \beta = 0 \)) and is able to deduce the earlier results of other authors as special cases.

Harris also obtained results for cascades with \( \beta > 0 \). In particular, he shows that when \( \beta > 0 \), the energy \( \varepsilon_1(t) \) of an electron at time \( t \) can be represented by

\[ \varepsilon_1(t) = \max \left\{ 0, \varepsilon_0(t) \left( 1 - \beta \int_0^t \frac{ds}{\varepsilon_0(s)} \right) \right\}. \]

6. Concluding remarks. T. E. Harris was deeply involved in the development of all aspects of contemporary branching process theory. He laid a rigorous foundation to areas where it had been lacking. His 1963 book [5]
is a beautiful and major work of scholarship. One can substantially credit its publication the explosion of work on branching processes in the 1960s and 1970s and up to the present. It set the impetus and direction of research on the subject for many years. The present authors owe T. E. Harris a deep debt of gratitude for this.

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