1 Introduction

The Polaron is the quantum of a "Polarization" field \([N] [L,S][M,V]\). It is generated by the "contact" interaction \([D1][D2]\) of a massive charged particle with the electromagnetic field. The contact interaction is a process that takes place in a point. It is only defined in the "particles version" (photons and electrons) of Quantum Electrodynamics and takes the place of "delta" interaction between Quantized Fields.

In second quantization one may describe polarization as follows. A polarizable massive charged particle (e.g. a spin \(\frac{1}{2}\) particle) emits a photon in a point. The interaction reverses the polarization of the photon and the spin of the particle. The photon is "immediately" reabsorbed. If the initial state of the field is not polarized the final result is the polarization of the massive charged particle. The process can be described as a contact interaction of the charged particles with two photons, the one which is created and the one that is absorbed. The interaction is invariant under the Lorentz group.

We prove that in this (particle) representation of the electromagnetic field (by photons) the interaction produces, both in the relativistic and the non relativistic case, polarized bound states of the massive particle. Each bound state is described by an element of \(L(R^3)\) times the vacuum in the c.c.r. (canonical commutation relation) representation of the photon field which is inequivalent to the Fock representation.

In the relativistic case representations associated to different bound states are inequivalent. Pictorially on can describe the bound states as a massive particle surrounded by a cloud made of an uncountable numbers of photons.

The bound states are critical points of the energy functional (Pekar functional) \([M,V]\). Notice that in \([M,V]\), as in our analysis, variational methods are employed and perturbation theory is not used.

The process takes place in a point and can be repeated arbitrary many times in different points. The result is the production of a Polarization Field. The bound states are the Polars \([N] [L,S][M,V]\).

For the role played by the representations of the canonical commutation relations (c.c.r.) we call these bound states Nelson Polars.

We study both the relativistic and the non relativistic case for the massive particle. Since the number of massive particles is conserved, sectors with a different number of massive particles can be treated separately. We concentrate first on the one particle sector.

In the case of a single massive particle we will prove that in the non relativistic case the Polarization Field is a field of single bound states with binding energy proportional the strength \(c\) of the interaction. In the relativistic case there are infinitely many bound states with eigenvalues \(-c\frac{1}{\sqrt{n}}\) (so that the sum of the bound state energies diverges.
To each bound state corresponds a different representation of the canonical commutation relations (c.c.r.) of the photons; all are inequivalent to the Fock representation.

To describe mathematically the "contact" interaction we make use as in [D1][D2] of Gamma convergence [Dal] a variational tool introduced by E. de Giorgi in the sixties in the study of finely fragmented materials. This method is only outlined here; it is presented in more details in [D1][D2].

It consists in choosing as hamiltonian a suitable self-adjoint extension of the free hamiltonian restricted to function that vanish in a neighbourhood of the "contact manyfold".

The method is non-perturbative and takes the place of renormalization in the Fields presentation of the electromagnetic field.

2 Contact interaction and the existence of bound states

If the attractive interaction is sufficiently strong, bound states may be present even if the "region of interaction" is infinitely small (a point).

We shall consider zero range interactions, i.e contact interactions (we shall presently give precise definitions).

These interactions produce bound states; they represent the polarizability of the system and therefore it is natural to call them Polarons.

We shall describe them explicitly as non-Fock representation of the c.c.r. of the E-M field. For this explicit representation we call them Nelson Polarons.

The spin structure of the massive particle is essential for the interaction to produce polarization; in fact this structure allows the interaction Hamiltonian to be a scalar.

The process is the following: a massive charged quantum particles of spin $\frac{1}{2}$ emits a photon at a point (a contact interaction); the spin of the particle and the polarization of the photon are reversed (so that the interaction can be taken invariant under rotation) and then the photon is absorbed.

This process can occur everywhere. The result is a field of polarized particles, i.e. a field of Polarons.

Since the interaction is by contact, we are in the setting of contact interactions described in [D1][D2].

We will use the contact interaction instead of the delta function that describes interaction between field in the Field Theory description of Electrodynamics.

And we will make use of Gamma convergence (a variational tool) instead of renormalization (a formal procedure)

Let $x_0, x_1, x_2 \in \mathbb{R}^3$ be the coordinates of the massive particle and of the two mass zero particles (photons) with opposite orientations.

Define the symmetric operator $H_0$ as the free hamiltonian restricted to functions that vanish in a neighbourhood of the contact manyfold $\Gamma \equiv \{x_0 = x_1\} \cap \{x_0 = x_2\}$.

Both particles have an "internal degree of freedom " (spin for the massive particle and polarizability for the photon) but the hamiltonian is a scalar. One has the vacuum as invariant state.

For simplicity in the following we neglect the "internal structures" and we take all particles to be scalars.
Since the interaction takes place in a point, we are neglecting only a two-by-two matrix $g\sigma_1$ where $g$ is strength of the interaction and the real Pauli matrix $\sigma_1$ inverts the spin.

In the non-relativistic the free Hamiltonian of the zero mass particle is a second order differential operator

In the relativistic case it is a differential operator of order one; in the "scalar version" that we are using it is a positive pseudo-differential operator of order one.

The weak contact hamiltonian is defined as the self-adjoint extension extension that has in its domain functions that take a fixed value $c$ at $\Gamma$.

Integration by parts shows the the interaction potential is formally a (non tempered) distribution supported by a point (in polar coordinates it is a primitive of the delta function). It is not a bone-fide potential.

In the relativistic case the interaction is of strong contact; in the domain of the extension there are function with a $\frac{1}{|x_i-x_j|}$ behaviour at the boundary.

The potential is formally a delta function.

In both cases we must give an interpretation to the formal potential.

Still as quadratic form in both cases the potential is defined on continuous function and it is bounded below.

On the other hand the free Hamiltonian in both cases is defines on absolutely continuous functions a quadratic form which is positive (the difficulties came from integration by parts).

Therefore in both cases the quadratic form of the total Hamiltonian is bounded below and therefore if it defines a self-adjoint operator this self-adjoint operator is bounded below.

It remains to be proved that the Hamiltonian exists and it is a self-adjoint extension of the free Hamiltonian. In both cases (weak and strong contact) a constant $c$ (the strength of the interaction) defines the extension.

The interaction can take place in every point and therefore these extensions are natural candidates to produce a field.

This field is a polarizability field; we will see that it is a field of bound states.

We shall prove that in the non relativistic case (weak contact) there is only one bound state with eigenvalue that is proportional to the coupling constant of the process.

This bound state belongs to a non-Fock representation (which we give explicitly) of the canonical commutation relations (c.c.r ) of the free e.m.field.

in the relativistic case (strong contact) there are infinitely many bound states with eigenvalues $-\frac{c}{\sqrt{n}}$ where $c$ is a function of the coupling.

The bound states can be expressed as product of a function $\Phi_n(x) \in L^2(R^3)$ times the vacuum of a representation of the zero mass field that depends on $n$ and on the strength of the interaction.

The field can be quantized; the quanta are the Polarons. The name Polaron is chosen since they represent the "polarization field" due to creation and instantaneous absorption of a photon at a point.

We call them "Nelson Polarons" because E.Nelson [N] was the first to describe the process in approximately the same way as is presented here.

3 The field of Polarons

Each massive particle interacts separately with the mass zero field and therefore we can restrict attention to the sector in which there is only one massive particle.
The entire system is translation invariant and we choose the reference frame in which the interaction of the massive particle takes place at the origin.

Later we comment briefly on the general case of $N$ massive particles.

The expression we have given above for the interaction potential were formal, since the "step function at Gamma and the "delta function" are not bona fide potentials.

We shall give a precise meaning to the interaction and discuss in detail the self adjoint extensions.

We shall also prove that the resulting Hamiltonian is the limit, in strong resolvent sense of Hamiltonians with potentials $V^\varepsilon(x_i - x_j), x \in \mathbb{R}^3$ that scale as $V^\varepsilon(|y|) = c_1^\varepsilon V(|y|^\varepsilon)$ in the non relativistic case and as $V^\varepsilon(|y|) = c_3^\varepsilon V(|y|)$ in the relativistic case.

The Polaron is the "quantum" of the "polarization field" due to the contact interaction.

All bound states can be represented as the massive particles surrounded by a "cloud" of uncountably many photons.

Since photons are identical particles this cloud represents a Photon condensate; the ground state is rotation invariant and could be seen as a ball of light.

We will prove that each bound state of the particle is associated to the vacuum of a different (inequivalent) representation of the canonical commutation relations (c.c.r) for the quantised zero mass field (recall that for zero mass field there are uncountably many inequivalent representations.)

Different values of the coupling constant correspond to inequivalent representation. In the relativistic case the representations associated to different bound states are inequivalent and inequivalent to that of the non-relativistic case.

We treat separately the relativistic and non-relativistic cases.

In absence of interactions there is no ground state (it would be the tensor product of the vacuum of the field with a zero momentum state of the particle).

If there is interaction the bound states are the product of a bound state of the massive particle times the vacuum of a suitable representation of the zero mass. The representation depends on the bound state of the particle.

We remark that in the non relativistic case only one bound state is produced. In the relativistic case there infinitely many bound states with energies that scale as $-c_1\sqrt{n}$. In both cases the the wave function of the bound states are not in $L^1(\mathbb{R}^3)$.

It is known that representations of the c.c.r. associated with different states $\Phi_1$ and $\Phi_2$ are equivalent only if $|\Phi_1 - \Phi_2|_1 < \infty$.

In the present case representations associated to different bound states are inequivalent.

For each bound state $\Phi(x), x \in \mathbb{R}^3$ the representation is the direct integral of representations of the c.c.r., each associated to the value taken at $x$ by the wave function $\Phi(x)$ of the bound state of the particle.

For the proof we make use of the theory of self-adjoint extensions of symmetric operators and of Gamma convergence, a variational tool [Dal] [D2] introduced by E. de Giorgi in the study of finely fragmented composites.

As intermediate step in the proof, in order to clarify the nature of the "contact potentials", we use a map (we call it "Krein map") that is "fractioning" and "mixing"; this is the reason why a natural tool is Gamma convergence.

**REMARK**

If there is no interaction there is no bound state. The energy of the bound state (or of the lowest bound state in the relativistic case) can therefore be taken as (small) parameter $\varepsilon$. 

that measures the strength of the interaction. It would is natural to take this strength as "perturbation parameter".

But the proof we give here is variational (through Gamma convergence of the resolvent of the Hamiltonian) and not through expansion of the Hamiltonian in series of powers of this small parameter.

We prove that the hamiltonian of our system is the limit in strong resolvent sense when $\epsilon \to 0$ of hamiltonians in which the interaction is not a "contact" (zero range) interaction but is described by potentials $V^\epsilon(|x-x_k|) = \frac{1}{\epsilon^{m_p}} V\left(\frac{|x-x_k|}{\epsilon^{m_p}}\right)$, $k = 1, 2$, $x \in \mathbb{R}^3$ where $m = 2$ in the non relativistic case and $m = 3$ in the relativistic case.

There is no convergence of quadratic forms and no convergence of a perturbation expansion in $\epsilon$.

Variational methods (Gamma convergence) are used to prove convergence of the hamiltonians. Notice that the photons have "mass zero" and states with a non-denumerably large number of photons can still have finite energy and momentum.

In the proofs we shall make use of compactness in the (Frechet topology given by the Sobolev norms and the number of massive particles.

This allows the use Gamma convergence (a variational instrument originally introduced in Quantum Mechanics [D1][D2]) in place of renormalization (a formal scheme).

For massive fields as yet no topology has been found that allows the use Gamma convergence as a substitute of renormalization.

Remark that in our system one of the particles is massive and the analysis is done in each sector of $n$ massive particles. There is no natural field associated to this massive particle. For massive particle there is no way to construct a representation the c.c.r. that is uniformly bounded energy and momentum.

It is plausible that the renormalization group (a scaling group of the energies) could be combined with Gamma converge to provide a non perturbative tool that can be used also in the massive case. It would perhaps correspond to the choice of a suitable representation of the c.c.r. for each $N$ particle sector.

For completeness we mention that a Polaron obtained by "abstract boundary conditions" is presented in [La].

A precursor of the theory presented here may be found in the work of Dimock and Rajeev [D,Ra]. These Authors introduction the "heat kernel renormalization", an operation that for the potential coincides with the Krein map we introduce here. But nothing is done to the kinetic part and (therefore) Gamma convergence is not introduced.

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4 The non relativistic case

We prove now that in the non relativistic case the weak contact (Nelson) Polaron is a bound state and we give its wave function $\Phi(x)$. We prove also that $\Phi(x) \notin L^1(\mathbb{R}^3)$.

We begin describing weak contact of the massive particle with any two of the zero mass non-relativistic particles.

The free Hamiltonian is written in Fourier transform $H_0 = \frac{p^2}{2m} + p_1^2 + p_2^2$.

Let $H_0^\epsilon$ be the symmetric operator defined by restricting $H_0$ to functions that have support away from the "boundary" $x = x_1 \cup x = x_2$.
By definition a hamiltonian of weak contact interaction is a self-adjoint extension of the symmetric operator \( H_0 \) (the operator \( H_0 \) restricted to functions that vanish in some neighbourhood of the boundary).

Weak contact extensions are parametrised by the value \( c \) taken at the boundary by functions in the domain.

Formally they can be described by a step potential \( W_c \) that acts only at the boundary. The one-particle hamiltonian is the "point interaction hamiltonian" introduced in [A]. It is the limit as \( \epsilon \to 0 \) of potentials \( V^\epsilon(x-x_1) \) that scale as \( V^\epsilon(y) = \frac{1}{\epsilon^2} V(y/\epsilon^2) \), \( y \in \mathbb{R}^3 \) with \( |V|_1 = C \).

In \( \mathbb{R}^3 \) the behaviour of the wave function at the boundary and the fact that the "potential" acts only at the boundary implies that there is a zero energy resonance.

Recall that we are studying a system of a particle in interaction of two identical Bose particles. This system has no zero energy resonances since the symmetric product of the wave functions of two zero energy resonances is a bound state in the relative coordinates. Indeed when considering the inverse of the resolvent a symmetric two-by-two matrix with zero on the diagonal has one negative (and one positive) eigenvalue.

Therefore the resolvent is regular at the origin and the extension has a bound state (and only one in the non relativistic case).

We study the system by using first, as intermediate step, a map (called in [D1][D2] "Krein map" \( K \)) to a space of more singular function (called there "Minlos space" \( \mathcal{M} \)). The map acts differently on the kinetic part (a self-adjoint operator) and on the potential term (a quadratic form) and it is "mixing" (with respect to the two particle channels).

For the kinetic part one has \( H_0 \to \sqrt{H_0} \). On the singular "potential" \( W_C \) at the boundary the map it acts as \( W_C \to W_C H_0^{-1} \).

One verifies that in \( \mathcal{M} \) the kinetic and potential terms have the same "singular" action on functions that have support in a very small neighbourhood of the "contact manyfold \( \{x-x_1\} \cap \{x-x_2\} = 0 \)."

It follows [D,R] that in \( \mathcal{M} \) there is a one-parameter family of self-adjoint operators each with a single bound state; the bound states of the family cover the interval \([-C,0] \). Notice that in \( \mathcal{M} \) both the kinetic and the potential are represented by a family of self-adjoint operators.

Inverting now the Krein map one has in physical a well ordered family of quadratic form bounded below.

Due to the change in metric topology the forms are only weakly closed; they are strictly convex since the interaction affects only the s-wave.

We have noticed that the "Krein map" is "fractioning" (the space \( \mathcal{M} \) is a space of more singular functions) and "mixing" (it is not diagonal in the particle "channels").

This suggest to make use of Gamma convergence [Dal], a variational method introduced by E. de Giorgi in the study of homogenisation i.e. the study of finely fragmented materials (for a variational analysis of the Polaron problem see [M,V]).

In a topological space \( Y \) the Gamma limit of a sequence of strictly decreasing convex quadratic forms \( F_n \) is the quadratic form \( F(y) \) characterised by the following relations

\[
\forall y \in Y, y_n \to y, F(y) = \liminf F_n(y), \forall x \in Y_n \forall \{x_n\} : F(x) \leq \limsup_n F_n(x_n) \tag{1}
\]

The first condition implies that \( F \) is a common lower bound for the \( F_n \), the second implies that the bound is optimal.
In our case the Gamma limit exists because there are no zero energy resonances and therefore the quadratic forms belong to a compact set for the Frechet topology given by the Sobolev semi-norms.

Gamma convergence selects the most negative form (to which by definition the others Gamma converge); being a minimum this limit form can be closed strongly [K] and defines a self-adjoint operator $\hat{H}_c$ (the Gamma limit).

Recall that $C$ is the value taken at the boundary by functions in the domain of the chosen extension.

A strictly decreasing sequence in a compact region of a Frechet space has a unique limit. Gamma convergence implies resolvent convergence [Dal] but not convergence of the quadratic forms; the sequence of quadratic forms converges only if it is uniformly bounded.

It is easy to verify that $\hat{H}_c$ is also the Gamma limit of the approximating Hamiltonians

$$H^\epsilon = H_0 + \int V^\epsilon(x - y_1)\Psi(y_1)dy_1 + \int V^\epsilon(x - y_2)\Psi(y_2)dy_2$$

with potentials that scale as $V^\epsilon(x - y) = \frac{1}{\epsilon}V(\frac{x - y}{\epsilon})$ of constant $L^1$ norm $|V(x)|_1 = C$.

Indeed the sequence of $\epsilon$-dependent quadratic forms is a decreasing sequence that has as lower bound the quadratic form $H_C$. Therefore $H_c$ is the limit, in strong resolvent sense, of the sequence of hamiltonians $H^\epsilon$.

Denote by $\Psi_\epsilon(x), x \in R^3 \times R^3$ the bound state for the approximation hamiltonian. One has $|\Psi_\epsilon - \Psi_2| \to 0$ in $L^2(R^3 \times R^3)$ but the limit vector is the symmetric part of the product of two zero energy resonances and therefore it is not in $L^1(R^3 \times R^3)$.

Therefore for any value of the parameter $c$ (the value at the boundary) the representation we have obtained for the field of zero mass particles is not equivalent to the Fock representation and to different values of $c$ correspond inequivalent representations of the c.c.r. (recall that it is the $L^1$ norm and not the $L^2$ norm that provides equivalence of representations).

5 The relativistic case

The difference with the previous case is that here also the hamiltonian of the relativistic particles is a first order differential operator.

The free hamiltonian is now $\sqrt{|p|^2 + m + |p_1| + |p_2|}$.

The massive particles is in weak contact separately with the other two.

The weak contact is still defined by the requirement that the function of the domain of the extension take a finite value at the boundary.

This is the same condition as in the non-relativistic case but now the kinetic energy is a first order (pseudo-differential) operator.

Also here we can take as reference the frame in which the particle of mass $m$ is at rest.

We perform next a separate change of coordinates for the relativistic particles. In the description in polar coordinates $\{p, \theta, \phi\}, p \in R^+$ we take as new coordinates $P = \sqrt{p}$ so that $p = P^2$.

This turns the relativistic two-particle hamiltonian into the non-relativistic form; it turns weak contact into strong contact. Indeed changing the coordinates changes also the form of the boundary condition.

Formally this interaction is described by a delta function.
Once again we introduce a Krein map \[ D_1, D_2 \], a map which is mixing and fractioning and acts differently of the kinetic and potential terms and does not break rotational invariance. We recall briefly its structure.

The action on the kinetic is \( H_0 \rightarrow \sqrt{H_0} \). The action on the potential term is \( V \rightarrow V\frac{1}{\sqrt{H_0}} \), where \( V = \delta(x - x_1) + \delta(x - x_2) \).

The “Minlos space” \( \mathcal{M} \) is a space of more singular functions. In \( \mathcal{M} \) the kinetic term is \( \sqrt{H_0} \) and the potential term has in each channel a pole singularity in position variables.

Therefore \( [D, R] \) if the potential is strong enough there is a one parameter family of self-adjoint operators unbounded below each with an infinite number of bound states with eigenvalues that scale as \( -\frac{1}{n} \).

"Inverting" the Krein map one has now, due to the change in metric topology, a one parameter family of weakly closed well ordered strictly convex quadratic forms bounded below; the forms are strictly convex because the interaction modifies only the s-wave.

Again the condition for existence of the Gamma limit is that the sequence be contained in a compact set for the topology of \( \mathcal{Y} \) (so that a Palais-Smale converging sequence exists).

Once again the topology is the Frechet topology defined by the Sobolev semi-norms and compactness is assured also in this case by the absence of zero energy resonances. Therefore the Gamma limit exists.

We use Gamma convergence to select the infimum.

Since it corresponds to a minimum by a theorem of Kato \([K]\) the limit form admits strong closure and defines a self-adjoint hamiltonian. It represents separate separate strong contact of a particle of mass \( m \) with two non relativistic zero mass particles.

It is a self-adjoint operator with an infinite number of bound states \( \Phi_n(x) \), \( n = 0, 1, \ldots \) where \( x \) are the coordinate of the massive particle.

The energies of the bound states scale as \( -c\frac{1}{\sqrt{n}} \). Only the ground state has a wave function that can be chosen real.

Notice that the choice of different coordinates for the relativistic particle does not change the metric of the massive particle.

Each bound state \( \Phi_n \) defines a representation of the field of the identical zero mass particles; the representations are inequivalent since \( \Phi_n - \Phi_m \notin L^1(\mathbb{R}^3) \) for \( n \neq m \).

The negative part of the spectrum of the system is now given by an infinite collection of states, a state of the massive particle with wave function \( \Phi_n \) decorated with a cloud of zero mass particles. The representations associated to different bound state are inequivalent.

Each representation is the direct integral of irreducible representations of the zero mass field parametrised by \( x \in \mathbb{R}^3 \) and associated to the value \( \Phi_n \) at the point \( x \) of the wave function of the particle of mass \( m \).

6 Representations of the zero mass field associated to a bound state of the massive particles.

Here we give the representation of the zero mass field (photons) associated to the bound states of the massive particles.

In the reference frame we have chosen the wave function of each bound state has its barycentre at the origin.
Let $\Phi(x)$ be the wave function of the bound state normalised to $\|\Phi\|_2 = 1$
One can associate to the wave function $\Phi(x)$ a representation of the field which is the direct integral over $x$ of the representation of the c.c.r. in which the field of annihilation operators $A^\Phi_x$ is defined by

$$A^\Phi_x(y) = a(y) - \Phi(x)$$

where the field $a(y)$ is in the Fock representation.

In these notation the field is $\Phi(x) = A(x) + A^*(x)$, $A(x) = A^\Phi_x$

For each value of $x$ the operators $A^\Phi_x(y)$ and $(A^\Phi_x(y))^*$ satisfy the same c.c.r. as the operators $a(y), a^*(y)$ but it is known that the representations associated to two different wave functions $\Phi_1, \Phi_2$ are inequivalent if $\Phi_1 - \Phi_2 \notin L^1$.

Since the coupling with the field in linear if one writes the Hamiltonian as a function of the field $A^\Phi_x(y)$ one obtains

$$H = H_0 + \int \omega(p)(A^\Phi_x)^*(p)A^\Phi_x(p)dp$$

(in the Theoretical Physics Literature this operation takes the name of ”completing the square”).

In order to minimise the energy one must choose for every $x$ the vacuum of the Fock representation for $A^\Phi_x$

The ground state $\Omega(x)$ satisfies then for every value of $x$ the equation $A^\Phi_x \Omega = 0$

In the reference frame we have chosen the self-adjoint operator $\hat{H}$ has as ground state the massive particle with wave function $\Phi(x)$ times the direct integral over $x$ of representations that depends on the coordinate of the particle of mass one. [N] [G,W], [L,S], [S].

### 7 The case of dimension two

We have treated the case of dimension 3 ; a similar analysis of the Polaron can be made for dimension two; here the delta function has the same scaling properties as the laplacian and therefore in this sense the contact is weak. There are no zero energy resonances.

Separate contact interaction with two particles can be analysed as in three dimensions.

The Krein map is again constructed with $\sqrt{H_0}$ and is again mixing and fractioning.

Gamma convergence is used as in the case of strong contact in three dimensions.

In two dimensions the Polarization field is a field of single bound states.

### 8 On the particle formulation (photons and electrons) of Quantum Electrodynamics

In the previous analysis we have introduced photons (zero mass particles) i.e. we have adopted the particles formulation of Q.F.T.

As a consequence we have introduced interactions that in same sense ”substitute” interactions between fields, i.e ”contact interactions” between particles.

Contact interactions are ”local interactions” that take the place of a ”delta potential” (that is ill defined for particles).
In a description in which particles are represented by functions (probability waves) local interactions are self-adjoint extensions of the symmetric operator defined as the free hamiltonian restricted to functions that vanish in some neighbourhood of a "contact manyfold ".

The analysis we have made here extends trivially to the separate contact interaction of the field of zero mass particles with an arbitrary (but finite) number N particles of mass \( m > 0 \) i.e to Quantum Electrodynamics in its particles version. Since the interaction takes place in a point and the particles are represented by functions in \( L^2(\mathbb{R}^3) \), we may safely assume that different massive particles interact separately and independently with two distinct pairs of zero mass particles, the photons.

The massive particles are can be relativistic or non relativistic. Correspondingly the contact interaction with the photon may be weak or strong.

In the particle interpretation of the zero mass field there may be states of bounded energy and momentum that contain uncountably many particles.

Traditionally one calls Fock representation (or Fock sector) the one in which there is a countable number of zero mass particles. The states of this representation are obtained by acting countably many times on the vacuum with an operator that adds a (zero mass) particle.

We have proved that there are states in which the massive particle is surrounded by a cloud of zero mass particles. In the non relativistic case there is only one such state, in the relativistic case there is a infinite number. These clouds are not in the Fock sector and are therefore they cannot be found in a Fock space analysis of the field.

A trace of them remains in the "polarizability field" and in its quanta, the "Polarons".

We stress that these results are obtained in the "particle" formulation of Quantum Electrodynamics in which the interaction is not described by a delta function but rather a contact interaction, weak or strong.

We have considered only the one-particle sector. The analysis extends to sectors of an arbitrary number of particles because sectors corresponding to different number of massive particles are not coupled by the contact interaction.

The Hamiltonian is the limit of hamiltonians with potentials with support that shrinks to a point, but no rate of convergence can be given in the parameter that controls the size of the support.

One can add to the contact interaction any regular interaction \([K,K]\) without altering the results obtained with contact interactions.

We stress that the method of analysis we follow here is variational and not perturbative. Gamma convergence is a substitute for renormalization theory, a formal procedure that is used in the Field Theory version of electrodynamics.

We recall that it is argued in [D1][D2] that the semiclassical limit of weak contact interaction is Coulomb interaction and that this is also the non relativistic limit.

Subtraction of a zero point energy is the only renormalization needed in the Field Theory formulation of Quantum Electrodynamics in the non relativistic case.

9 References

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