Abstract

We use an entropy based method to study two graph maximization problems. We upper bound the number of matchings of fixed size $\ell$ in a $d$-regular graph on $N$ vertices. For $\frac{2\ell}{N}$ bounded away from 0 and 1, the logarithm of the bound we obtain agrees in its leading term with the logarithm of the number of matchings of size $\ell$ in the graph consisting of $\frac{N}{2d}$ disjoint copies of $K_{d,d}$. This provides asymptotic evidence for a conjecture of S. Friedland et al. We also obtain an analogous result for independent sets of a fixed size in regular graphs, giving asymptotic evidence for a conjecture of J. Kahn. Our bounds on the number of matchings and independent sets of a fixed size are derived from bounds on the partition function (or generating polynomial) for matchings and independent sets.

1 Introduction

Given a $d$-regular graph $G$ on $N$ vertices and a particular type of subgraph, a natural class of problems arises: “How many subgraphs of this type can $G$ contain?” In this paper we give upper bounds on the number of partial matchings of a fixed fractional size, and on the number of independent sets
of a fixed size, in a general $d$-regular graph, and we show that our bounds are asymptotically matched at the logarithmic level by the graph consisting of $\frac{N}{2d}$ disjoint copies of $K_{d,d}$. (See [2] and [3] for graph theory basics.)

Let $G$ be a bipartite graph on $N$ vertices with partition classes $A$ and $B$ and with $|A| = |B|$. Suppose that the degree sequence of $A$ is given by $\{r_i\}_{i=1}^{|A|}$. A result of Brégman concerning the permanent of 0-1 matrices [3] (see also [1]) gives a bound on the number of perfect matchings in $G$:

**Theorem 1.1 (Brégman)** Let $M_{\text{perfect}}(G)$ be the set of perfect matchings in $G$. Then

$$|M_{\text{perfect}}(G)| \leq \prod_{i=1}^{|A|} (r_i!)^{\frac{1}{r_i}}.$$ 

When $r_i = d$ for all $i$ and $|A|$ is divisible by $d$, equality in the above theorem is achieved by the graph consisting of $\frac{N}{2d}$ disjoint copies of the complete bipartite graph $K_{d,d}$, so we know that among $d$-regular bipartite graphs on $N$ vertices, with $2d|N$, this graph contains the greatest number of perfect matchings. (Wanless [12] has considered the case when $2d$ is not a multiple of $N$, obtaining lower bounds on $|M_{\text{perfect}}(G)|$ and some structural results on the maximizing graphs in this case.)

Friedland et al. [6] propose an extension of this observation, which they call the Upper Matching Conjecture. Write $m_\ell(G)$ for the number of matchings in $G$ of size $\ell$, and write $DK_{N,d}$ for the graph consisting of $\frac{N}{2d}$ disjoint copies of $K_{d,d}$.

**Conjecture 1.2** For any $N$-vertex, $d$-regular graph $G$ with $2d|N$ and any $0 \leq \ell \leq N/2$,

$$m_\ell(G) \leq m_\ell(DK_{N,d}).$$

In this note we upper bound the logarithm of the number of $\ell$-matchings of a regular graph and show that, at the level of the leading term, this upper bound is achieved by the disjoint union of the appropriate number of copies of $K_{d,d}$. We will use the parameterization $\alpha = \frac{2\ell}{N}$, and refer interchangeably to a matching of size $\ell$ or a matching whose size is an $\alpha$-fraction of the maximum possible matching size. In what follows, $H(x) = -x \log x - (1-x) \log(1-x)$ is the usual binary entropy function. (All logarithms in this note are base 2.)
Theorem 1.3 Let $G$ be a $d$-regular graph on $N$ vertices and $\ell$ an integer satisfying $0 \leq \ell \leq \frac{N}{2}$. Set $\alpha = \frac{2\ell}{N}$. The number of matchings in $G$ of size $\ell$ satisfies
\[
\log(m_\ell(G)) \leq \frac{N}{2} \left[ \alpha \log d + H(\alpha) \right].
\]
This bound is tight up to the first order term: for fixed $\alpha \in (0, 1)$,
\[
\log(m_\ell(DK_{N,d})) \geq \frac{N}{2} \left[ \alpha \log d + 2H(\alpha) + \alpha \log \left( \frac{\alpha}{e} \right) + \Omega \left( \frac{\log d}{d} \right) \right],
\]
with the constant in the $\Omega$ term depending on $\alpha$.

In [7] an asymptotic variant of Conjecture 1.2 is presented. Let $\{G_k\}$ be a sequence of $d$-regular bipartite graphs with $|V_k|$, the number of vertices of $G_k$, growing to infinity, and fix $\alpha \in [0, 1]$. Set
\[
h_{\{G_k\}}(\alpha) = \limsup \frac{\log m_\ell_k(G_k)}{|V_k|}
\]
where the limit is over all sequences $\{\ell_k\}$ with $2\ell_k/|V_k| \to \alpha$. The Asymptotic Upper Matching Conjecture asserts that
\[
h_{\{G_k\}}(\alpha) \leq h_{\{kK_{d,d}\}}(\alpha)
\]
where $kK_{d,d}$ is the graph consisting of $k$ disjoint copies of $K_{d,d}$. Theorem 1.3 shows that for each fixed $\alpha$, there is a constant $c_\alpha$ (independent of $d$) with $h_{\{G_k\}}(\alpha) \leq h_{\{kK_{d,d}\}}(\alpha) + c_\alpha$.

We show similar results for the number of independent sets in $d$-regular graphs. A point of departure for our consideration of independent sets is the following result of Kahn [10]. For any graph $G$ write $I(G)$ for the set of independent sets in $G$ and write $i_t(G)$ for the set of independent sets of size $t$ (i.e., with $t$ vertices).

Theorem 1.4 (Kahn) For any $N$-vertex, $d$-regular bipartite graph $G$,
\[
|I(G)| \leq |I(K_{d,d})|^{N/2d}.
\]
Note that when $2d|N$, we have $|I(K_{d,d})|^{N/2d} = |I(DK_{N,d})|$. Kahn [10] proposes the following natural conjecture.
Conjecture 1.5 For any $N$-vertex, $d$-regular graph $G$ with $2d|N$ and any $0 \leq t \leq N/2$, 

$$i_t(G) \leq i_t(DK_{N,d}).$$

We provide asymptotic evidence for this conjecture.

Theorem 1.6 For $N$-vertex, $d$-regular $G$, and $0 \leq t \leq N/2$,

$$i_t(G) \leq \begin{cases} 
2^N \left(H\left(\frac{\alpha}{2}\right) + \frac{\beta}{2}\right) & \text{in general} \\
2^N \left(H\left(\frac{\alpha}{2}\right) + \frac{\beta}{2} \log_d\left(1 - \frac{d}{N}\right)^d\right) & \text{if } G \text{ is bipartite} \\
2^t \left(\frac{N}{t}\right) & \text{if } G \text{ has a perfect matching.}
\end{cases} \quad (1)$$

On the other hand,

$$i_t(DK_{N,d}) \geq \begin{cases} 
(1 - \frac{c}{2}) \left(\frac{N}{t}\right) 2^N \left(\frac{\alpha}{2} + \frac{\beta}{2} (1 - \frac{\alpha}{N})^d\right) & \text{for any } c > 1 \\
2^t \left(\frac{N}{t}\right) \prod_{k=1}^{t-1} \left(1 - \frac{2kd}{N}\right) & \text{for } t \leq \frac{N}{2d}.
\end{cases} \quad (2)$$

If $N$, $d$ and $t$ are sequences satisfying $t = \alpha \frac{N}{2}$ for some fixed $\alpha \in (0, 1)$ and $G$ is a sequence of $N$-vertex, $d$-regular graphs, then from (1) we obtain

$$\log i_t(G) \leq \begin{cases} 
\frac{N}{2} \left[H\left(\alpha\right) + \frac{\beta}{2}\right] & \text{in general} \\
\frac{N}{2} \left[H\left(\alpha\right) + \frac{1}{d}\right] & \text{if } G \text{ is bipartite,}
\end{cases}$$

whereas if $N = \omega(d \log d)$ and $d = \omega(1)$ then taking $c = 2$ in the first bound of (2) and using Stirling’s formula to analyze the behavior of $\binom{N/2}{t}$, we obtain the near matching lower bound

$$\log i_t(DK_{N,d}) \geq \frac{N}{2} \left[H\left(\alpha\right) + \frac{1}{d}(1 + o(1))\right].$$

If $N = o\left(d\left(1 - \alpha\right)^d\right)$ and $G$ is bipartite, then the gap between our bounds on $i_t(G)$ and $i_t(DK_{N,d})$ is just a multiplicative factor of $O(\sqrt{N})$; indeed, in this case (taking any $c = \omega(1)$) we obtain from the first bound of (2) that

$$i_t(DK_{N,d}) \geq (1 - o(1)) \left(\frac{N}{t}\right) 2^N \left[H(\alpha) + \frac{\beta}{2}\right].$$

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For smaller sets, whose sizes scale with $N/d$ rather than $N$, the final bounds in (1) and (2) come into play. Specifically, for any $N$, $t$ and $d$

$$i_t(DK_{N,d}) \geq \begin{cases} 
(\frac{N}{t})^{2t(1+o(1))} & \text{if } t = o\left(\frac{N}{d}\right) \\
(1+o(1))\left(\frac{N}{t}\right)^{2t} & \text{if } t = o\left(\sqrt{\frac{N}{d}}\right) 
\end{cases}$$

(3)

Note that in the latter case, for $G$ with a perfect matching we have $i_t(G) \leq (1+o(1))i_t(DK_{N,d})$. To obtain (3) from (2) we use

$$\prod_{k=1}^{t-1} \left(1 - \frac{2kd}{N}\right) \geq \exp\left\{-\frac{4d}{N}\sum_{k=1}^{t-1} k\right\} \geq \exp\left\{-\frac{2dt(t-1)}{N}\right\}.$$

2 Counting Matchings

Given a graph $G$ and a nonnegative real number $\lambda$, we can form weighted matchings of $G$ by assigning each matching containing $\ell$ edges weight $\lambda^\ell$. The weighted partition function, $Z^\text{match}_\lambda(G)$, gives the total weight of matchings. Formally,

$$Z^\text{match}_\lambda(G) := \sum_{m \in \mathcal{M}(G)} \lambda^{|m|} = \sum_{k=0}^{\frac{N}{2}} m_k(G) \lambda^k.$$

(This is often referred to as the generating function for matchings or the matching polynomial). We will prove Theorem 1.3 by showing a bound on the partition function, and then using that bound to limit the number of matchings of a particular weight (size).

Lemma 2.1 For all $d$-regular graphs $G$, $Z^\text{match}_\lambda(G) \leq (1 + d\lambda)^{\frac{N}{2}}$

This lemma is easily proven in the bipartite case; the difficulty arises when we want to prove the same bound for general graphs. Indeed, if $G$ is a bipartite graph with bipartition classes $A$ and $B$, we can easily see that the right hand side above counts a superset of weighted matchings. Elements in this superset are sets of edges no two of which are adjacent to the same element of $A$ (but with no restriction on incidences with $B$).

Proof of Lemma 2.1 To prove this lemma, we will use the following result of Friedgut [5], which describes a weighted version of the information theoretic Shearer’s Lemma.
Theorem 2.2 (Friedgut) Let $H = (V, E)$ be a hypergraph, and $F_1, F_2, \ldots, F_r$ subsets of $V$ such that every $v \in V$ belongs to at least $t$ of the sets $F_i$. Let $H_i$ be the projection hypergraphs: $H_i = (V, E_i)$, where $E_i = \{e \cap F_i : e \in E\}$. For each edge $e \in E$, define $e_i = e \cap F_i$, and assign each $e_i$ a nonnegative real weight $w_i(e_i)$. Then

$$\left( \sum_{e \in E} \prod_{i=1}^{r} w_i(e_i) \right)^t \leq \prod_{i} \sum_{e_i \in E_i} w_i(e_i)^t$$

The first step in applying this theorem is to define appropriate variables. Let $G = (V, E)$ be a $d$-regular graph, with its vertex set $\{v_1, v_2, \ldots, v_N\}$. We will use $G$ to form an associated matching hypergraph, $H = (E, \mathcal{M})$, where the vertex set of the hypergraph is the edge set of $G$, and $\mathcal{M}$ is the set of matchings in $G$. Let $F_i$ be the set of edges incident to a vertex $v_i \in V$. Note that each edge in $E$ is covered twice by $\bigcup_{i=1}^{N} F_i$, so we may take $t = 2$. We define the trace sets, $E_i = \{F_i \cap m : m \in \mathcal{M}\}$, as the set of possible intersections of a matching with the set of edges incident with $v_i$. Let $m_i = m \cap F_i$. Then for all $i$, assign

$$w_i(m_i) = \begin{cases} 1 & \text{if } m_i = \emptyset \\ \sqrt{\lambda} & \text{else} \end{cases}$$

With these definitions we have $\sum_{m \in E_i} w_i(m_i)^2 = 1 + d\lambda$, and for a fixed $m$, $\prod_{i} w_i(m_i) = \sqrt{\lambda^{2|m|}}$. Putting these expressions into Theorem 2.2, we have that

$$(Z_{\lambda}^{\text{match}}(G))^{2} = \left( \sum_{m \in \mathcal{M}} \lambda^{|m|} \right)^2 \leq \prod_{i=1}^{N} (1 + d\lambda).$$

Therefore,

$$Z_{\lambda}^{\text{match}}(G) \leq (1 + d\lambda)^{\frac{N}{2}}.$$
are all real and negative, and so we can write \( Z^\text{match}_\lambda(G) = \prod_{i=1}^{\nu(G)} (1 + \alpha_i \lambda) \) for some positive \( \alpha_i \)'s with \( \sum \alpha_i = (Z^\text{match}_\lambda(G))'|_{\lambda=0} = |E(G)| = \frac{Nd}{2}, \) where \( \nu(G) \) is the size of the largest matching of \( G \). Applying the arithmetic mean - geometric mean inequality to this expression we obtain
\[
Z^\text{match}_\lambda(G) \leq \left( 1 + \lambda \frac{\sum \alpha_i}{\nu(G)} \right)^{\nu(G)} = \left( 1 + \lambda \frac{Nd}{2\nu(G)} \right)^{\nu(G)} \leq (1 + d\lambda)^{\frac{N}{2}}.
\]

**Proof of Theorem 1.3** We begin with the upper bound. We may assume \( 0 < \ell < N/2 \), since the extreme cases \( \ell = 0, N/2 \) are obvious. For fixed \( \ell \), a single term of the partition function \( Z^\text{match}_\lambda(G) \) is bounded by the whole sum, and so by Lemma 2.1 we have
\[
m_\ell(G) \lambda^\ell \leq Z^\text{match}_\lambda(G) \leq (1 + d\lambda)^{\frac{N}{2}}
\]
and
\[
m_\ell(G) \leq (1 + d\lambda)^{\frac{N}{2}} \left( \frac{1}{\lambda} \right)^\ell.
\]  
We take
\[
\lambda = \frac{\ell}{d \left( \frac{N}{2} - \ell \right)}
\]
to minimize the right hand side of (4) and obtain the upper bound in Theorem 1.3 (in the case \( \ell = \frac{\alpha N}{2} \)):
\[
\log(m_\ell(G)) \leq \log \left( \frac{N}{d \left( \frac{N}{2} - \ell \right)} \right)^{\frac{N}{2}} \left( \frac{d \left( \frac{N}{2} - \ell \right)}{\ell} \right)^\ell = \frac{N}{2} \left( \frac{2\ell}{N} \log d + H(2\ell/N) \right) = \frac{N}{2} (\alpha \log d + H(\alpha)).
\]

We now turn to the lower bound. We begin by observing
\[
m_\ell(DK_{N,d}) = \sum_{a_1, \ldots, a_{N/2d}, \sum_{i=1}^{\ell} a_i = \ell} \prod_{i=1}^{N/2d} \binom{d}{a_i} a_i!
\]
Here the \( a_i \)'s are the sizes of the intersections of the matching with each of the components of \( DK_{N,d} \), and the term \( \binom{d}{a_i} a_i! \) counts the number of matchings of size \( a_i \) in a single copy of \( K_{d,d} \). (The binomial term represents
the choice of \(a_i\) endvertices for the matching from each partition class, and the factorial term tells us how many ways there are to pair the endvertices from the top and bottom to form a matching.)

From Stirling’s formula we have that there is an absolute constant \(c \geq 1\) such that for any \(d \geq 1\) and \(0 < a < d\),

\[
\log \left( \frac{d}{a} \right)^2 a! \geq a \log d + a \log \frac{a}{d} - a \log e + 2H(a/d)d - \log cd, \tag{6}
\]

and we may verify by hand that (3) holds also for \(a = 0, d\). Combining (5) and (6) we see that \(\log(m_\ell(DK_{N,d}))\) is bounded below by

\[
\frac{N}{2} \left( \frac{2\ell}{N} \log d - \frac{2\ell}{N} \log e - \frac{\log cd}{d} + \frac{2}{N} N^{N/2d} \sum_{i=1}^{N} \left( a_i \log \frac{a_i}{d} + 2H(a_i/d)d \right) \right) \tag{7}
\]

for any valid sequence of \(a_i\)'s. To get our lower bound in the case \(\ell = \alpha N\), we consider (6) for that sequence of \(a_i\)'s in which each \(a_i\) is either \(\lfloor \alpha d \rfloor\) or \(\lceil \alpha d \rceil\). Note that by the mean value theorem, there is a constant \(c_\alpha > 0\) such that both

\[
\log \frac{\lfloor \alpha d \rfloor}{d}, \ \log \frac{\lceil \alpha d \rceil}{d} \geq \log \alpha - \frac{c_\alpha}{d}
\]

and

\[
H \left( \frac{\lfloor \alpha d \rfloor}{d} \right), \ H \left( \frac{\lceil \alpha d \rceil}{d} \right) \geq H(\alpha) - \frac{c_\alpha}{d}.
\]

(Here we use \(\lvert \frac{\lfloor \alpha d \rfloor}{d} - \alpha \rvert, \ \lvert \frac{\lceil \alpha d \rceil}{d} - \alpha \rvert \leq \frac{1}{d}\) and \(\alpha \neq 0, 1\).) Putting these bounds into (7) we obtain

\[
\log(m_\ell(DK_{N,d})) \geq \frac{N}{2} \left( \alpha \log d + 2H(\alpha) + \alpha \log \left( \frac{\alpha}{e} \right) + \Omega \left( \frac{\log d}{d} \right) \right),
\]

with the constant in the \(\Omega\) term depending on \(\alpha\). \(\square\)
3 Counting Independent Sets

In this section we prove the various assertions of Theorem 1.6. We begin with the second bound in (1). We use a result from [8], which states that for any $\lambda > 0$ and any $d$-regular $N$-vertex bipartite graph $G$, the weighted independent set partition function satisfies

$$Z^\text{ind}_\lambda(G) := \sum_{I \in \mathcal{I}(G)} \lambda^{|I|} \leq \left(2(1 + \lambda)^d - 1\right)^{\frac{N}{2d}}.$$ (8)

Choose $\lambda$ so that $\frac{\lambda N}{2(1+\lambda)} = t$. Noting that $i_t(G)\lambda^\frac{tN}{1+t}$ is the contribution to $Z^\text{ind}_\lambda(G)$ from independent sets of size $t$ we have

$$i_t(G) \leq \frac{Z^\text{ind}_\lambda(G)}{\lambda^\frac{tN}{1+t}} \leq \left(2(1 + \lambda)^d - 1\right)^{\frac{N}{2d}} \left(1 - \frac{1}{2(1 + \lambda)^d - 1}\right)^{\frac{N}{2d}} = 2^{\frac{N}{2d} \left(1 + \lambda\right)^{\frac{N}{2d}}} \left(1 - \frac{1}{2(1 + \lambda)^d - 1}\right)^{\frac{N}{2d}} = 2^{H \left(\frac{N}{2d} + \frac{N}{2d} \left(1 - \frac{1}{2(1 + \lambda)^d - 1}\right)^{\frac{N}{2d}}\right)}.$$ (9)

We use (8) to make the critical substitution in (9).

To obtain the first bound in (1) we need the following analog of (8) for $G$ not necessarily bipartite:

$$Z^\text{ind}_\lambda(G) \leq 2^{\frac{N}{2d} \left(1 + \lambda\right)^{\frac{N}{2d}}}.$$ (10)

From (10) we easily obtain the claimed bound, following the steps of the derivation of the second bound in (1) from (8). We prove (10) by using a more general result on graph homomorphisms. For graphs $G = (V_1, E_1)$ and $H = (V_2, E_2)$ set

$$\text{Hom}(G, H) = \{f : V_1 \to V_2 : \{u, v\} \in E_1 \Rightarrow \{f(u), f(v)\} \in E_2\}.$$ 

That is, $\text{Hom}(G, H)$ is the set of graph homomorphisms from $G$ to $H$. Fix a total order $\prec$ on $V(G)$. For each $v \in V(G)$, write $P_{\prec}(v)$ for $\{w \in V(G) : w \prec v\}$.
$\{w, v\} \in E(G), w \prec v$ and $p_\prec(v)$ for $|P_\prec(v)|$. The following natural generalization of a theorem of J. Kahn is due to D. Galvin (see [11] for a proof).

**Theorem 3.1** For any $d$-regular and $N$-vertex graph $G$ (not necessarily bipartite) and any total order $\prec$ on $V(G)$,

$$|\text{Hom}(G, H)| \leq \prod_{v \in V(G)} |\text{Hom}(K_{p_\prec(v), p_\prec(v)}, H)|^{\frac{1}{d}}.$$ 

If $G$ is bipartite with bipartition classes $\mathcal{E}$ and $\mathcal{O}$ and $\prec$ satisfies $u \prec v$ for all $u \in \mathcal{E}, v \in \mathcal{O}$ then Theorem 3.1 reduces to the main result of [8].

To prove (10), we first note that (by continuity) it is enough to prove the result for $\lambda$ rational. Let $C$ be an integer such that $C\lambda$ is also an integer, and let $H_C$ be the graph which consists of an independent set of size $C\lambda$ and a complete looped graph on $C$ vertices, with a complete bipartite graph joining the two. As described in [8] we have, for any graph $G$ on $N$ vertices,

$$|\text{Hom}(G, H_C)| = C^N Z_\lambda^{\text{ind}}(G).$$

For $G$ $d$-regular and $N$-vertex, we apply Theorem 3.1 twice to obtain

$$Z_\lambda^{\text{ind}}(G) = \frac{|\text{Hom}(G, H_C)|}{C^N} \leq \prod_{v \in V(G)} \frac{|\text{Hom}(K_{p_\prec(v), p_\prec(v)}, H_C)|^{\frac{1}{d}}}{C^N} = \prod_{v \in V(G)} \left( C^{2p_\prec(v)} Z_\lambda^{\text{ind}}(K_{p_\prec(v), p_\prec(v)}) \right)^{\frac{1}{d}} \leq \frac{C^{2 \sum_{v \in V(G)} p_\prec(v)}}{C^N} \prod_{v \in V(G)} \left( 2(1 + \lambda)^{p_\prec(v)} \right)^{\frac{1}{d}} = 2^{\frac{N}{d}} \frac{C^{2 \sum_{v \in V(G)} p_\prec(v)}}{d} \left( 1 + \lambda \right)^{\frac{\sum_{v \in V(G)} p_\prec(v)}}.$$ 

Now noting that

$$\sum_{v \in V(G)} p_\prec(v) = |E(G)| = \frac{Nd}{2}$$

we obtain

$$Z_\lambda(G) \leq 2^{\frac{N}{d}} (1 + \lambda)^{\frac{N}{d}}.$$
as claimed.

We now turn to the third bound in (1). Fix a perfect matching of $G$ joining a set of vertices $A \subseteq V(G)$ of size $N/2$ to the set $B := V(G) \setminus A$. Let $f$ be the bijection from subsets of $A$ to subsets of $B$ that moves the set along the chosen matching. Every independent set in $G$ of size $t$ is of the form $I_A \cup I_B$ where $I_A \subseteq A$, $I_B \subseteq B$, $f(A) \cap B = \emptyset$ and $|A| + |B| = t$. We therefore count all the independent sets of size $t$ (and more) by choosing a subset of $A$ of size $t$ ($\binom{N/2}{t}$ choices) and a subset of this set to send to $B$ via $f$ ($2^t$ choices).

To obtain the first bound in (2), we introduce a probabilistic framework and use Markov’s inequality. If we divide a set of size $N/2$ into $N/2d$ blocks of size $d$ and choose a uniform subset of size $t$, then the probability that this set misses a particular block is $\binom{N/2-d}{t}/\binom{N/2}{t}$. Let $X$ be a random variable representing the number of blocks that the $t$-set misses. Let $b_k$ equal the number of $t$-sets which miss exactly $k$ blocks. Then $P(X = k) = b_k/\binom{N/2}{t}$. Let $\chi_A$ be the indicator variable for the event $A$. Then

$$X = \sum_{i=0}^{\frac{N}{2d}} \chi\{\text{block } i \text{ empty}\}$$

and by linearity of expectation the expected number of blocks missed satisfies

$$\mu := \mathbb{E}(X) = \frac{N}{2d} \frac{\binom{N/2-d}{t}}{\binom{N/2}{t}} \leq \frac{N}{2d} \left(1 - \frac{2t}{N}\right)^d. \quad (11)$$

From Markov’s inequality we have

$$\sum_{k=0}^{c\mu} P(X = k) = P(X \leq c\mu) \geq \left(1 - \frac{1}{c}\right).$$

We substitute the previously discussed value for $P(X = k)$, yielding the inequality

$$\sum_{k=0}^{c\mu} b_k \geq \left(1 - \frac{1}{c}\right) \binom{N/2}{t}. \quad (12)$$

How many independent sets of size $t$ does $DK_{N,d}$ have? To choose an independent set from $DK_{N,d}$ of size $t$, we first create a bipartition $E \cup O$ of $DK_{N,d}$ by choosing (arbitrarily) one of the bipartition classes of each of the
$N/2d$ $K_{d,d}$'s of $DK_{N,d}$ to be in $\mathcal{E}$. We then choose a subset of $\mathcal{E}$ of size $t$. The number of subsets of $\mathcal{E}$ which have empty intersection with exactly $k$ of the $K_{d,d}$'s that make up $DK_{N,d}$ is precisely $b_k$. Each of these subsets corresponds to $2^{N/2-k}$ independent sets in $DK_{N,d}$. Combining this observation with (11) and (12) we obtain the first bound in (2):

$$i_t(DK_{N,d}) = 2^{N/2} \sum_{k=0}^{\infty} 2^{-k} b_k$$

$$\geq 2^{N/2 - c \mu} \sum_{k=0}^{\infty} b_k$$

$$\geq \left(1 - \frac{1}{c} \right) \left(\frac{N/2}{t}\right) 2^{N/2 - \frac{c}{2}(1 - \frac{1}{c})^t}. $$

Finally we turn to the second bound in (2). We obtain the claimed bound by considering all of the independent sets whose intersection with each component of $DK_{N,d}$ has size either 0 or 1:

$$i_t(DK_{N,d}) \geq (2d)^t \left(\frac{N/2}{t}\right).$$

After a little algebra, the right hand side above is seen to be exactly the right hand side of the second bound in (2).

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