We investigate exact and analytic solutions in $f(T)$ gravity within the context of a Friedmann–Lemaître–Robertson–Walker background space with nonzero spatial curvature. For the power law theory $f(T) = T^n$, we find that the field equations admit an exact solution with a linear scalar factor for negative and positive spatial curvature. That Milne-like solution is asymptotic behaviour for the scale factor near the initial singularity for the model $f(T) = T + f_0 T^n - 2\Lambda$. The analytic solution for that specific theory is presented in terms of Painlevé Series for $n > 1$. Moreover, from the value of the resonances of the Painlevé Series we conclude that the Milne-like solution is always unstable while for large values of the independent parameter, the field equations provide an expanding universe with a de Sitter expansion of a positive cosmological constant. Finally, the presence of the cosmological term $\Lambda$ in the studied $f(T)$ model plays no role in the general behavior of the cosmological solution and the universe immerge in a de Sitter expansion either when the cosmological constant term $\Lambda$ in the $f(T)$ model vanishes.

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Keywords: Teleparallel cosmology; exact solutions; open universe

1. INTRODUCTION

Teleparallel theory of gravity [1, 2] and its modifications [3–10] have attracted the attention of cosmologists over the last year because it can provide a geometric explanation for
the explanation of the recent observations \[11, 12\], for a recent review we refer the reader to \[13\]. The modified teleparallel theories of gravity belong to the family of theories in which Lorentz symmetry is violated \[14, 15\]. In teleparallelism, the fundamental connection is the curvature-less Weitzenböck connection \[16\] while the torsion scalar $T$ is used for the definition of the gravitational Action Integral \[17\]. In contrary to General Relativity in which the Levi-Civita connection and the Ricci scalar $R$ are the fundamental geometric objects of the theory.

When the gravitational Action Integral is linear on the torsion scalar $T$, then the theory is equivalent to General Relativity and it is know as TEGR. However, there are various originally proposed gravitational Lagrangians such is the Teleparallel dark energy theory, the $f(T)$ theory and its extensions, see for instance \[7–10, 18–23\].

In this study we consider the $f(T)$ gravity in a Friedmann–Lemaître–Robertson–Walker background space. The theory has been proposed originally as a geometric dark energy candidate in a spatially flat background space \[3\]. However, during the last years it has been found that it can explain various eras of the cosmological history. In $f(T)$ cosmology, the field equations are of second-order and they have only one dependent variable, the scale factor of the underlying geometry. Recently, in \[24\] the case of nonzero spatial curvature for the background space has been considered. The authors investigated the existence of bounces and static solutions, while the conditions for the existence of these solutions were investigated.

We focus on the existence of exact and analytic solutions for the cosmological field equations in $f(T)$ cosmology where the spatial curvature is nonzero. A background space with nonzero spatial curvature is not excluded by the inflationary scenario \[25, 26\]. Indeed, inflation is immune to negative curvature while the energy density which corresponds to the curvature can be nonzero in the pre-inflationary era \[27, 28\]. Moreover, we shall focus on the existence of the de Sitter expansion as described by a positive cosmological constant. Such an expansion is necessary because it provides a rapid expansion for the size of the universe such that the latter effectively loses its memory on the initial conditions, which means that the de Sitter expansion solves the “flatness”, “horizon” and monopole problem \[29, 30\]. The cosmic “no-hair” conjecture states that all expanding universes with a positive cosmological constant admit as an asymptotic solution the de Sitter universe \[31, 32\]. The plan of the paper is as follows.
In Section 2, we present the field equations for the cosmological model of our consideration. In Section 3, we consider the power-law \( f(T) = T^n \) theory where we find a generalized Milne solution that exists for the field equations for negative and positive spatial curvature. A more general function \( f(T) \) is considered in Section 4. Specifically, we select the \( f(T) = T + f_0 T^n - 2\Lambda \) model, which has been proposed as a dark energy candidate. We write the field equations where we find that they admit a movable singularity. Near the singularity and for \( n > 1 \) the field equations are dominated by the \( T^n \) term which means that the generalized Milne solution describes the scale factor at the singularity. Furthermore, we study if the field equations admit the Painlevé property. We find that for \( n > 1 \) the field equations pass the Painlevé test and the resonances give that the analytic solution can be expressed by a Right Painlevé Series. The latter indicates an expanding universe which leads to a de Sitter expansion. Finally, in Section 5, we summarize our results and we draw our conclusions.

2. \( f(T) \) COSMOLOGY WITH NONZERO CURVATURE

The modified teleparallel modified theory of gravity known as \( f(T) \) theory is a second-order theory which violates the Lorentz symmetry for any nonlinear function \( f \). The fundamental geometric objects of the theory are the vierbein fields \( e^i(x^k) \) which define the unholonomic frame of the theory. For the vierbein fields it follows \( g(e_i, e_j) = \eta_{ij} \), while in a coordinate system \( e^i(x^k) = h_i^\mu(x^k)dx^i \) such that \( g_{\mu\nu}(x^k) = \eta_{ij} h_i^\mu(x^k)h_j^\nu(x^k) \) where \( h_i^\mu \) is the dual basis of the theory. The invariant which is used for the definition of the Lagrangian in teleparallelism is the scalar \( T \) for the curvatureless Weitzenböck connection. Because of the existence of the unholonomic tensor the non-null torsion is defined as

\[
T_{\mu\nu}^\beta = \hat{\Gamma}_{\nu\mu}^\beta - \hat{\Gamma}_{\mu\nu}^\beta = h_i^\beta (\partial_{\mu} h_i^\nu - \partial_{\nu} h_i^\mu).
\]

where the scalar \( T \) is given by the expression

\[
T = \frac{1}{2} (K_{\mu\nu}^{\beta} + \delta_{\beta}^\mu T_{\theta}^{\beta\nu} - \delta_{\beta}^\nu T_{\theta}^{\beta\mu}) T_{\mu\nu}^{\beta}.
\]

where \( K_{\beta}^{\mu\nu} \) equals the difference of the Levi Civita connection in the holonomic and the unholonomic frame and it is defined as \( K_{\beta}^{\mu\nu} = -\frac{1}{2} (T_{\beta}^{\mu\nu} - T_{\beta}^{\nu\mu} - T_{\beta}^{\mu\nu}) \).
The Action Integral in \( f(T) \) theory is defined as

\[
S_f(T) = \frac{1}{16\pi G} \int d^4x e^f(T) + S_m, \tag{3}
\]

in which \( S_m \) is the Action Integral of the matter source and \( G \) is Newton’s constant. In the case where \( f(T) \) is a linear function, the teleparallel equivalence of General relativity, with or without the cosmological constant, is recovered.

Variation with respect to the vierbein fields of the gravitational Action Integral (3) provides the field equations

\[
e^{-1} \partial_\mu(eS_i^{\mu\nu}f'(T) - h_i^\lambda T^\beta_{\mu\lambda}S^{\nu\mu}_\beta f'(T)) + S_i^{\mu\nu} \partial_\mu(T)f''(T) + \frac{1}{4} h_i^\nu f(T) = 4\pi G h_i^\beta T^\nu_{\beta} \tag{4}
\]

in which \( f'(T) = \frac{df(T)}{dT} \), \( f''(T) = \frac{d^2f(T)}{dT^2} \), \( T_{\mu\nu} \) includes the contribution of the matter source in the field equation and \( S_i^{\mu\nu} = h_i^\lambda S^{\mu\nu}_{\lambda} \) where \( S^{\mu\nu}_{\lambda} = \frac{1}{2}(K^{\mu\nu}_{\lambda} + \delta^{\mu\nu}_\theta T^{\theta}_{\lambda} - \delta^{\mu\nu}_\theta T^{\theta}_{\lambda}) \).

In the case of a spatially flat Friedmann–Lemaître–Robertson–Walker (FLRW) universe

\[
ds^2 = dt^2 - a^2(t) (dx^2 + dy^2 + dz^2), \tag{5}
\]

then when we select the diagonal unholonomic frame \( e^A = (dt, a(t) dx, a(t) dy, a(t) dz) \), the field equations (4) are in agreement with the Action integral (3).

However, when we consider a nonzero spatial curvature, \( K \neq 0 \), in the FLRW universe,

\[
ds^2 = dt^2 - a^2(t) (dr^2 + \sin^2(\phi) (d\theta^2 + \sin^2 \theta d\phi^2)) , \quad K = 1, \tag{6}
\]

\[
ds^2 = dt^2 - a^2(t) (dr^2 + \sinh^2(\phi) (d\theta^2 + \sin^2 \theta d\phi^2)) , \quad K = -1, \tag{7}
\]

a nondiagonal frame should be considered. Indeed, the selected unholonomic frame is \( e_i^A = (dt, a(t) E^r (K), a(t) E^\theta (K), a(t) E^\phi (K)) \), in which for \( K = 1 \),

\[
E^r (K = 1) = - \cos \theta dr + \sin r \sin \theta (\cos r d\theta - \sin r \sin \theta d\phi), \tag{8}
\]

\[
E^\theta (K = 1) = \sin \theta \cos \phi dr - \sin r (\sin r \sin \phi - \cos r \cos \theta \cos \phi) d\theta + \\
- \sin r \sin \theta (\cos r \sin \phi + \sin r \cos \theta \cos \phi) d\phi, \tag{9}
\]

\[
E^\phi (K = 1) = - \sin \theta \sin \phi dr - \sin r (\cos r \cos \phi + \cos r \cos \theta \sin \phi) d\theta + \\
- \sin r \sin \theta (\cos r \cos \phi - \sin r \cos \theta \sin \phi) d\phi. \tag{10}
\]
On the other hand, for \( K = -1 \), \( E^r(K) \), \( E^\theta(K) \) and \( E^\phi(K) \) are defined as

\[
E^r(K = -1) = \cos \theta dr + \sinh r \sin \theta (-\cosh r d\theta + i \sinh r \sin \theta d\phi),
\]

\[
E^\theta(K = -1) = -\sin \theta \cos \phi dr + \sinh r (i \sinh r \sin \phi - \cos r \cos \theta \cos \phi) d\theta + 
+ \sinh r \sin \phi (\cosh r \sin \phi + i \sinh r \cos \theta \cos \phi) d\phi,
\]

\[
E^\phi(K = -1) = \sin \theta \sin \phi dr + \sinh r (i \sinh r \cos \phi + \cosh r \cos \theta \sin \phi) d\theta + 
+ \sinh r \sin \theta (\cosh r \cos \phi - \sinh r \cos \theta \sin \phi) d\phi.
\]

For this frame, the scalar \( T \) is derived

\[
T = 6 \left( \frac{K}{a^2} - H^2 \right), \quad H = \frac{\dot{a}}{a}, \quad \dot{a} = \frac{da}{dt}
\]

while the field equations (14) for a isotropic fluid source with energy density \( \rho \) and pressure \( p \) are

\[
6 \left( H^2 + \frac{K}{a^2} \right) f' + (f(T) - Tf') = 2\rho,
\]

\[
-4f' \left( 2\dot{H} + 3H^2 \right) + 4 \left( \dot{H} + Ka^{-2} \right) \left( 12H^2 f'' + f' \right) - f(T) = 2p,
\]

where we observe that of linear function \( f(T) \), the usual Friedmann’s equations are recovered.

We continue with the investigation of analytic and exact solutions for the field equations (15), (16) for functional forms of \( f(T) \) of specific interests.

### 3. GENERALIZED MILNE UNIVERSE

In the following we assume that the spacetime is vacuum, that is, \( \rho = 0 \) and \( p = 0 \). Therefore, gravitational field equations become

\[
6 \left( H^2 + \frac{K}{a^2} \right) f' + (f(T) - Tf') = 0,
\]

\[
-4f' \left( 2\dot{H} + 3H^2 \right) + 4 \left( \dot{H} + Ka^{-2} \right) \left( 12H^2 f'' + f' \right) - f(T) = 0.
\]

From the latter, we are able to define the effective energy density for the geometric fluid to be \( \rho_{f(T)} = -\frac{(f(T)-Tf')}{2f'} \).
In the following we consider the power-law function $f(T) = T^n$, and we investigate the existence of Milne-like solutions.

Indeed for the power-law $f(T)$ function, the field equation (17), (18) are written as follows

$$a^{-2n} (K - \dot{a}^2)^{n-1} (K + (2n - 1) \dot{a}^2) = 0 ,$$

$$a^{-2n} (K - \dot{a}^2)^{n-2} ((2n - 3) (K - \dot{a}^2) (K + (2n - 1) \dot{a}^2) - 2na (K - (2n - 1) \dot{a}^2) \ddot{a}) = 0 .$$

Hence, for $n > 1$, we have the (positive scale factor) solutions $a_A(t) = \sqrt{K} t$ and $a_B(t) = \sqrt{\frac{K}{1-2n}} t$. The first solution, namely $a_A(t)$, is a real function when $K > 0$, while $a_B(t)$ is a real solution and physically accepted when $K < 0$. The second solution $a_B(t)$ describes the generalized Milne universe, since $K < 0$, where the geometric terms which follow by $f(T)$ in the field equations mimic the curvature filed such that to modify the effective curvature from $K \rightarrow \frac{K}{1-2n}$. Moreover, $a_A(t)$ is a new solution for positive curvature where the geometric fluid mimics the curvature term with a negative effective energy density.

Conversely, $n < 1$, there exists only the generalized Milne solution $a_B(t)$ but in this case the solution exists for $K > 0$. These solutions are the general solutions for the field equations.

It is important to mention that in contrary to the spatially flat FLRW universe in which the vacuum solution in $f(T)$ gravity is the Minkowski universe [34], that is, the vacuum solution of General Relativity; in the presence of the curvature term the vacuum solutions are different from that of General Relativity.

4. ANALYTIC SOLUTIONS

We proceed by considering the more general functions $f(T)$ with special interests in cosmology and astrophysics. Specifically, for the function $f(T)$ we consider the dark energy model $f(T) = T + T^n$ known as dark torsion [36], and the case of the dark torsion model with the cosmological constant $f(T) = T + f_0 T^n - 2\Lambda$ which has been constraint by local gravitational systems [37].

For these two models, we observe that Milne or Milne-like exact solutions do not exist as for the power-law theory. Thus, other mathematical techniques should be applied for the derivation of solutions. We follow the approach applied in [34] on the application of the
singularity analysis for the construction of analytic cosmological solutions.

The necessary property in order for the singularity analysis to work is the existence of a movable pole for the differential equation. Hence, we consider \( a(t) = \phi_0(t)\phi(t)^p \), where \( \phi(t)_{t\to t_0} = 0 \), and \( p \) should be a negative rational number. Parameter \( p \) and function \( \phi_0(t) \) are derived by choosing the master equation of our study, i.e. equation (18) to be dominated by the singular terms. This is known as the first step in the Ablowitz-Ramani-Segur (ARS) algorithm [38–40]. In singularity analysis the solution is expressed in terms of Painlevé Series, thus it is necessary to find the integration constants and the step of the Painlevé Series as we move far from the singularity.

The second step of the algorithm is based on the determination of the resonances which provide information about the position of the integration constants of the solution [41]. Hence, for the leading order behaviour \( a_l(t) \) that we have found, we replace \( a(t) = a_l(t)\left(1 + \varepsilon \phi(t)^S\right) \) and we linearize around \( \varepsilon^2 \to 0 \). From the leading-order terms of the latter equation we extract a polynomial where by assuming that it is zero, we find the values of the resonances \( S \). The number of independent values of \( S \) should be equal to the order of the differential equation, since as we mentioned before the resonances are related with the integration of constants of the solution. Furthermore, the resonance \( S = -1 \) should exist in order for the singularity to be a simple pole. The third and final step of the ARS algorithm, known as consistency test, describes how to write the Painlevé Series and replace it in the differential equation in order to test that it is a solution. For more details on the ARS algorithm and for an extended discussion we refer the reader to [42].

### 4.1. Dark torsion \( f(T) = T + f_0 T^n \)

For the dark torsion model with \( f(T) = T + f_0 T^n \), the modified Friedmann equations become

\[
0 = 12H^2 \left(1 + nf_0 T^{n-1}\right) + T + f_0 T^n ,
\]

\[
0 = -4 \left(1 + nf_0 T^{n-1}\right) \left(2\dot{H} + 3H^2\right) - T - f_0 T^n
\]

\[
+ 4 \left(1 + nf_0 T^{n-1}\right) \left(12(n-1)H^2 + T\right) \left(H + Ka^{-2}\right) .
\]

We replace scalar \( T \) from (14), and we assume that the leading-order behaviour is described by the \( a(t) = \phi_0(t)\phi(t)^p \). Usually, \( \phi(t) \) is assumed to be the linear function
\[ \phi(t) = (t - t_0), \text{ however that it is not necessary}. \]

For \( n > 1 \), from equation (22) we find that the leading-order terms is for \( p = 1 \). However, \( p = 1 \), it is not acceptable because such value for \( p \) does not provide a singular behaviour. In order to overpass that we follow the approach described in [42] and we consider the new variable \( b(t) = (a(t))^{-1} \).

Hence, in the new variable we find that the leading-order behaviour of equation (22) is \( b(t) = \Phi_0(t) \Phi(t)^p \), where \( p = -1 \) and \( K = \left( \frac{\Phi(t)}{\Phi_0(t)} \right)^2 \) or \( K = - (1 - 2n) \left( \frac{\Phi(t)}{\Phi_0(t)} \right)^2 \). In the case where \( \Phi(t) \) is a linear function, then the leading-order behaviour is that of the generalized Milne solution which is described by the power-law term \( T^m \) of the model. From the second step of the ARS algorithm we find the resonances \( S = -1 \) and \( S = 1 \), that is, the analytic solution is expressed in terms of the Painlevé Series

\[
\Phi_1(t) = \frac{1}{2\sqrt{K}} \frac{\ddot{\Phi}}{\Phi}, \tag{25}
\]

For positive value \( K \), i.e. for the leading-order behaviour with \( K = \left( \frac{\dot{\Phi}(t)}{\Phi_0(t)} \right)^2 \) we find that \( \Phi_1(t) \) is an arbitrary function, and

\[
\Phi_2(t) = \frac{\left( 216 f_0 K (\Phi_1)^2 - 72 f_0 \sqrt{K} \Phi_1 - 1 \right)}{72 f_0 \sqrt{K} \Phi_3} \frac{\ddot{\Phi}^2 + 72 f_0 \dddot{\Phi} + 24 f_0 \dot{\Phi} \left( 9 \sqrt{K} \Phi_1 \ddot{\Phi} - \Phi^{(3)} \right)}{72 f_0 \sqrt{K} \Phi_3}, \tag{24}
\]

Moreover, from the constraint equation (21) it follows that

\[
\Phi_1(t) = - \frac{1}{2\sqrt{K}} \frac{\ddot{\Phi}}{\Phi}. \tag{26}
\]

For negative curvature and for \( K = -3 \left( \frac{\dot{\Phi}(t)}{\Phi_0(t)} \right)^2 \) we apply the same procedure and we find the constraint equation for the integration function

\[
\Phi_1(t) = - \frac{1}{2} \sqrt{-3} \frac{\ddot{\Phi}}{K \Phi} \tag{26}
\]

and

\[
\Phi_2(t) = \left( 1 - 24 f_0 \left( 5 K \Phi_1^2 + 3 \sqrt{-3 K} \Phi_1 \right) \right) \frac{\ddot{\Phi}^2 + 144 f_0 \dddot{\Phi} + 24 f_0 \dot{\Phi} \left( 5 \sqrt{-3 K} \Phi_1 \ddot{\Phi} - 3 \Phi^{(3)} \right)}{72 f_0 \sqrt{-3 K} \Phi^3}, \tag{27}
\]
On the other hand, for \( n < 1 \) we replace \( b(t) = (a(t))^{-1} \) and we find the leading-order behaviour \( b(t) = \sqrt{\frac{1-2n}{K}} \Phi(t)^{-1} \), which is valid for \( K > 0 \), when \( \Phi(t) \) is a real function. We apply the same procedure as before and we find the resonances to be \( S = -1 \) and \( S = 2n - 3 \), that is, the second resonance is negative which indicates that the analytic solution is expressed in terms of the Left Painlevé Series

\[
b(t) = \Phi_0(t) (\Phi(t))^{-1} + \Phi_1(t) (\Phi(t))^{-2} + \Phi_2(t) (\Phi(t))^{-3} + \Phi_3(t) (\Phi(t))^{-4} + \ldots. \tag{28}
\]

However, by replacing in the field equations the latter solution we find that the latter Painlevé Series does not solve the differential equations, thus, the singularity analysis fails.

Let us assume now the case with linear function \( \Phi(t) = (t - t_0) \) where the leading-order behaviour is that of the generalized Milne universe, and let us write the analytic solutions. For \( n > 1 \), the analytic solution is expressed in terms of the Puiseux Series

\[
b(t) = b_0 (t - t_0)^{-1} + b_1 + b_2 (t - t_0) + b_3 (t - t_0)^2 + \ldots. \tag{29}
\]

Hence, for \( n = 2 \), and \( K = 1 \) we find the coefficients

\[
b_0 = 1, \quad b_1 = 0, \quad b_2 = -\frac{1}{72f_0}, \quad b_3 = 0, \quad b_4 = \frac{1}{3240f_0^2}, \quad b_5 = 0, \quad \text{etc.} \tag{30}
\]

On the other hand for \( K = -1 \) it follows

\[
b_0 = \sqrt{3}, \quad b_1 = 0, \quad b_2 = \frac{1}{27\sqrt{3}f_0}, \quad b_3 = 0, \quad b_4 = \frac{17}{77760\sqrt{3}f_0^2}, \quad b_5 = 0, \quad \text{etc.} \tag{31}
\]

Someone can use another function \( \Phi(t) \) to write the analytic solution. However, for every smooth function \( \Phi(t) \) near the singularity \( \Phi(t) \simeq t - t_0 \) such that \( \Phi(t_0) \simeq 0 \).

For \( n > 1 \), from (29) we observe that as far as we move from the singularity, then the right parts of the Series dominate, which indicate that when \( t_0 > 0 \), the behaviour \( (t - t_0)^{-1} \) is unstable. As far as the scale factor \( a(t) \) is concerned, it can be easily constructed by using function \( b(t) \).

Indeed for \( n = 2 \) and \( K = 1 \) it follows

\[
a(t) = (t - t_0) \left( 1 + \frac{1}{72f_0} (t - t_0)^2 - \frac{1}{8640f_0^2} (t - t_0)^4 + \ldots \right), \tag{32}
\]

while for \( K = -1 \) we calculate the scale factor

\[
a(t) = \frac{(t - t_0)}{\sqrt{3}} \left( 1 - \frac{1}{216f_0} (t - t_0)^2 - \frac{1}{19440f_0^2} (t - t_0)^4 + \ldots \right). \tag{33}
\]
4.2. Dark torsion with cosmological constant $f(T) = T + f_0 T^n - 2\Lambda$

In the presence of the cosmological constant in the dark torsion model, the singularity analysis provides the same results as before, with $\Lambda = 0$. The only difference is in the values of the coefficients of the analytic solution where the cosmological constant is introduced. In order to make it clear we present the solution for the scale factor $a(t)$ for $n = 2$ and $K = 1$, $K = -1$ as before.

For $K = 1$, the scale factor is

$$a(t) = (t - t_0) \left( 1 + \frac{1}{72f_0} (t - t_0)^2 - \frac{1 + 12f_0\Lambda}{8640f_0^2} (t - t_0)^4 + ... \right), \quad (34)$$

$$a(t) = \sqrt{\frac{3}{2}} \left( 1 - \frac{1}{216f_0} (t - t_0)^2 - \frac{1 + 9f_0\Lambda}{19440f_0^2} (t - t_0)^4 + ... \right). \quad (35)$$

from which it is obvious the the contribution of the cosmological constant is significant far from the movable singularity.

5. CONCLUSIONS

In this study we investigated the existence of exact and analytic solutions for the modified teleparallel $f(T)$ gravitational theory in a FLRW background space with nonzero spatial curvature. For the $f(T)$ function we assumed the proposed power-law models. We found that there exists a movable singularity for the field equations which describes a singular scaling solution for the field equations. The asymptotic behaviour for the scale factor is that of a Milne-like scale factor, for negative and positive spatial curvature for the background space. These generalized Milne solutions are the analytic solutions for the power-law $f(T) = T^n$ theory.

However, for the $f(T) = T + f_0 T^n - 2\Lambda$ model, for the derivation of the analytic solution, because of the existence of the movable singularity, we applied the singularity analysis and specifically the ARS algorithm. We proved that the modified Friedmann equations possess the Painlevé property and the analytic solution for the scalar factor are expressed in Right Puiseux Series, where the first term is the generalized Milne behaviour. Because the Puiseux solutions are Right, that is, as far as we move from the initial singularity the scale factor increases, consequently, we can easily derive the attractor for the field equations and the
solution for large values of the independent parameter. In addition, the presence of the cosmological constant term plays no role in the general evolution for the dynamical system.

For large values of a nonconstant scale factor, from \( (14) \) it follows \( T \simeq -6H^2 \), that is, equation \( (21) \) becomes

\[
0 \simeq H^2 \left( -2 \left( 1 + nf_0 (-6H^2)^{n-1} \right) - 1 + f_0 (-6H^2)^{n-1} \right).
\]

(36)

Hence, for it follows \( H(a) = -6 \left( \frac{1}{f_0 (1-2n)} \right)^\frac{1}{n-1} \), for \( n \neq \frac{1}{2} \). Thus, this model leads to an expanding universe, as we found from the singularity analysis without necessary to consider a cosmological constant term.

It is important to mention here that the limit of General Relativity is not recovered for a nonlinear function \( f(T) \). Indeed, General Relativity is recovered in \( f(T) \) gravity, when it holds \( f(T) \big|_{T \to 0} = 0 \), and \( T f'(T) \big|_{T \to 0} = 0 \). Thus, from \( (14) \) it is clear that the General Relativity vacuum solution does not hold when \( T = 0 \).

We found that in the presence of the spatial curvature in \( f(T) \) gravity the universe becomes inflationary, while a generalized Milne-like exact solution was found in the power-law model. In a future study we plan to consider the asymptotic scale factor near to the movable singularity as a toy model for the cosmological observations with a background space with spatial curvature.

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