Convergence of the Ricci flow toward a unique soliton

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Abstract

We will consider a \( \tau \)-flow, given by the equation \( \frac{d}{dt}g_{ij} = -2R_{ij} + \frac{1}{\tau}g_{ij} \) on a closed manifold \( M \), for all times \( t \in [0, \infty) \). We will prove that if the curvature operator and the diameter of \( (M, g(t)) \) are uniformly bounded along the flow and if one of the limit solitons is integrable, then we have a convergence of the flow toward a unique soliton, up to a diffeomorphism.

1 Introduction

The Ricci flow equation

\[
\frac{d}{dt}g_{ij} = -2R_{ij},
\]

has been introduced by R. Hamilton in his seminal paper [6]. We will refer to this equation as to an unnormalized Ricci flow. A normalized Ricci flow is given by the equation

\[
\frac{d}{dt}\tilde{g}_{ij} = -2R(\tilde{g})_{ij} + \frac{2}{n}r\tilde{g}_{ij},
\]

where \( r = \frac{1}{\text{Vol}(M)} \int_M R(\tilde{g})dV_{\tilde{g}} \). This equation is sometimes more convenient to consider, since a volume of a manifold is being fixed along the normalized Ricci flow and a volume collapsing case can not happen in a limit, if the limit exists.
A natural question that arises in studying the evolution equations, in particular the Ricci flow equation, is under which conditions a solution will exist for all times, that is under which conditions it will avoid the singularities at finite times. The other question one can ask is if there exists a limit of the solutions when we approach infinity and how we can describe the metrics obtained in the limit. In the case of dimension three with positive Ricci curvature and dimension four with positive curvature operator we know (due to R. Hamilton) that the solutions of the Ricci flow equation, in both cases exist for all times, converging to Einstein metrics. In general, we can not expect to get an Einstein metric in the limit. We can expect to get in the limit a solution to the Ricci flow equation which moves under one-parameter subgroup of the symmetry group of the equation. These kinds of solutions are called solitons. Since the Ricci flow equation is a gradient flow of Perelman’s functional $W$, it is natural to expect that a soliton in the limit is unique up to diffeomorphisms.

Our goal in this paper is to prove the following theorem.

**Theorem 1.** Let $(g_{ij})_t = -2R_{ij} + \frac{1}{\tau}g_{ij}$ be a Ricci flow on a closed manifold $M$ with uniformly bounded curvature operators and diameters for all $t \in [0, \infty)$. Assume also that some limit soliton is integrable. Then there is an 1-parameter family of diffeomorphisms $\phi(t)$, a unique soliton $h(t)$ and constants $C$, $\delta$, $t_0$ such that $|\phi(t)^*g(t) - h(0)|_{k,\alpha} < Ce^{-\delta t}$, for all $t \in [t_0, \infty)$. Moreover, if $\psi(t)$ is a diffeomorphism such that $h(t) = \psi^*h(0)$, then $|(\phi\psi)^*g(t) - h(t)|_{C^0} < Ce^{-\delta t}$.

The ideas for the proof of Theorem 1 have been inspired by those of Cheeger and Tian in [3].

**Outline of the proof of Theorem 1**

In order to deal with this problem, we will first construct a gauge on time intervals of an arbitrary length, so that in the chosen gauge the $\tau$-flow equa-
tion becomes strongly parabolic. We will look at the solutions of a strictly parabolic equation. It will turn out that our metrics (in the right gauge) will satisfy a strictly parabolic equation that is almost linear and therefore their behavior is modeled on the behavior of the solutions of the linear equation. There are 3 types of the solutions of our strictly parabolic equation,

- the solutions that have an exponential growth,
- the solutions that have an exponential decay,
- the solutions that change very slowly.

Roughly speaking, the integrability condition means that the solutions of a linearized deformation equation for solitons arise from a curve of metrics satisfying the same soliton equation. To deal with those slowly changing solutions we will use the integrability condition to change the reference soliton metric so that at the end we deal only with the cases of either a growth or a decay. We will rule out the possibility of the exponential growth, by using the fact that our flow sequentially converges toward solitons and by using the similar arguments established by L.Simon in [16] and also later used by Cheeger and Tian in [3]. We will be left with the exponential decay which will allow us to continue our gauge up to infinity.

The organization of the paper is as follows. In section 2 we will give a necessary background and notation. In section 4, using the sequential convergence of the $\tau$-flow (that has been proved in [13]), we will construct a gauge on time intervals of an arbitrary length, so that in the chosen gauge the $\tau$-flow equation becomes strongly parabolic. In section 5 we will use the integrability assumption to prove that a soliton that we get in the limit is unique, up to a diffeomorphism.

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2 Background

Perelman’s functional $\mathcal{W}$ and its properties will play an important role in the paper. $M$ will always denote a closed manifold. $\mathcal{W}$ has been introduced in [11].

$$\mathcal{W}(g, f, \tau) = (4\pi \tau)^{-\frac{n}{2}} \int_M e^{-f}[\tau(|\nabla f|^2 + R) + f - n]dV_g.$$  

We will consider this functional restricted to $f$ satisfying

$$\int_M (4\pi \tau)^{-\frac{n}{2}} e^{-f}dV = 1.$$  (1)

$\mathcal{W}$ is invariant under simultaneous scalings of $\tau$ and $g$ and under a diffeomorphism change, i.e. $\mathcal{W}(g, f, \tau) = \mathcal{W}(c\phi^*g, \phi^*f, c\tau)$ for a constant $c > 0$ and a diffeomorphism $\phi$. Perelman showed that the Ricci flow can be viewed as a gradient flow of a functional $\mathcal{W}$, which is one of the reasons why this functional plays an important role throughout [11]. Let $\mu(g, \tau) = \inf \mathcal{W}(g, f, \tau)$ over smooth $f$ satisfying (1). It has been showed by Perelman that $\mu(g, \tau)$ is achieved by some smooth function $f$ on a closed manifold $M$, that $\mu(g, \tau)$ is negative for small $\tau > 0$ and that it tends to zero as $\tau \to 0$.

We will explain the motivation why we have decided to study this flow instead of a normalized one in which a volume of a manifold has been fixed along the flow. First of all, there is a simple reparametrization that allows us to go from a $\tau$-flow to an unnormalized flow and many smoothing regularity properties that have been proved for the unnormalized flow continue to hold for a $\tau$-flow as well. For example, Hamilton’s compactness theorem also holds for the $\tau$-flow. This is because Shi’s estimates hold for $\tau$-flow as well, and therefore, since we have a uniform curvature bound on the solutions to a $\tau$-flow, we may assume uniform bounds on all covariant derivatives of the curvature, $|D^p Rm| \leq C(p)$. The reparametrization that we use to go from a $\tau$-flow to an unnormalized flow is as follows. Let $c(s) = 1 - \frac{x}{\tau}$ and $t(s) = -\tau \ln(1 - \frac{x}{\tau})$. Let $\tilde{g}(s) = c(s)g(t(s))$. $\tilde{g}(s)$ is a solution to an unnormalized Ricci flow. On the other hand we have that
\( W(g(t(s)), f(t(s)), \tau) = W(\bar{g}(s), \bar{f}(s), \tau - s) \). By the monotonicity formula for \( W \) we have that the later quantity is increasing along an unnormalized Ricci flow and therefore the former quantity is increasing along the \( \tau \) flow as well. The monotonicity formula for a \( \tau \)-flow gets the simpler form; \( W(g(t), f(t), \tau) \) is increasing along the \( \tau \)-flow, while \( f(t) \) changes by the evolution equation \( \frac{df}{dt} = -\Delta f + |\nabla f|^2 - R + \frac{n}{2\tau} \), and \( \tau \) is just a constant. The fact that \( \tau \) is now a constant will be very useful in taking the limits of the minimizers for \( W \).

One of the most important properties of \( W \) is the monotonicity formula.

**Theorem 2 (Perelman).** \( \frac{d}{dt} W = \int_M 2\tau |R_{ij} + \nabla_i \nabla_j f - \frac{1}{2\tau} g_{ij}|^2 (4\pi \tau)^{-\frac{n}{2}} e^{-f} dV \geq 0 \) and therefore \( W \) is increasing along the flow described by the following equations

\[
\begin{align*}
\frac{d}{dt} g_{ij} &= -2R_{ij}, \\
\frac{d}{dt} f &= -\Delta f + |\nabla f|^2 - R + \frac{n}{2\tau}, \\
\dot{\tau} &= -1.
\end{align*}
\]

One of the very important applications of the monotonicity formula is the noncollapsing theorem for the Ricci flow that has been proved by Perelman in [11].

**Definition 3.** Let \( g_{ij}(t) \) be a smooth solution to the Ricci flow \( (g_{ij})_t = -2R_{ij}(t) \) on \([0, T)\). We say that \( g_{ij}(t) \) is locally collapsing at \( T \), if there is a sequence of times \( t_k \to T \) and a sequence of metric balls \( B_k = B(p_k, r_k) \) at times \( t_k \), such that \( \frac{r_k^n}{t_k} \) is bounded, \( |\text{Rm}|(g_{ij}(t_k)) \leq r_k^{-2} \) in \( B_k \) and \( r_k^{-n}\text{Vol}(B_k) \to 0 \).

**Theorem 4 (Perelman).** If \( M \) is closed and \( T < \infty \), then \( g_{ij}(t) \) is not locally collapsing at \( T \).

The corollary of Theorem [4] is
Corollary 5. Let $g_{ij}(t), t \in [0, T)$ be a solution to the Ricci flow on a closed manifold $M$, where $T < \infty$. Assume that for some sequences $t_k \to T$, $p_k \in M$ and some constant $C$ we have $Q_k = |Rm|(x,t) \leq CQ_k$, whenever $t < t_k$. Then a subsequence of scalings of $g_{ij}(t_k)$ at $p_k$ with factors $Q_k$ converges to a complete ancient solution to the Ricci flow, which is $\kappa$-noncollapsed on all scales for some $\kappa > 0$.

We would like to recall a definition of a soliton that will appear in later sections.

Definition 6. A Ricci soliton $g(t)$ is a solution to a Ricci flow equation that moves by 1-parameter group of diffeomorphisms $\phi(t)$, i.e. $g(t) = \phi(t)^*g(0)$.

The equation for a metric to move by a diffeomorphism in the direction of a vector field $V$ is $2\text{Ric}(g) = \mathcal{L}_V(g)$, or $R_{ij} = g_{ik}D_jV^k + g_{jk}D_iV^k$. If the vector field $V$ is the gradient of a function $f$, we say that the soliton is the gradient Ricci soliton. Moreover, we can consider the solutions to the Ricci flow that move by diffeomorphisms and also shrink or expand by a factor at the same time. The stationary solutions of the unnormalized Ricci flow are the Ricci flat metrics. The Ricci solitons are the generalizations of those, namely they are the stationary solutions to the Ricci flow equations, up to diffeomorphisms.

3 Uniqueness of a limit soliton

In [13] we have proved the sequential convergence of a $\tau$-flow with uniformly bounded curvatures and diameters toward the solitons. In this section we will assume that one of the limit solitons is integrable, in order to prove the uniqueness of a soliton in the limit, up to a diffeomorphism. We will first construct a gauge in which a $\tau$-flow becomes a strictly parabolic flow. Similar ideas to those in [3] will help us finish the proof of Theorem 1.
3.1 The construction of a gauge

To construct the right gauge, assume for simplicity that we are in a situation when \( g(t) \to h \) as \( t \to \infty \), where \( h \) is an Einstein metric, with the Einstein constant \( \frac{1}{2\tau} \). We will see how we construct a gauge so that our modified Ricci flow equation becomes strictly parabolic on time intervals of an arbitrary length, if we go sufficiently far in time direction. This construction applies to our more general case, just with minor modifications and only for simplicity reasons we have decided to consider a case of an Einstein metric in a limit.

The main purpose of this section is to prove the following Proposition that will be reformulated in the next section for our more general setting.

**Proposition 7.** Let \( A > 0 \) be an arbitrary real number, \( k \) an integer and \( 0 < \alpha < 1 \). There exists \( \epsilon_0(A,k) \) such that for every \( \epsilon < \epsilon_0 \) there exists \( s_0 = s_0(A,k,h,\epsilon) \), such that for all \( t_0 \geq s_0 \) the equation

\[
\frac{d}{dt}\phi = \Delta_{g(t),h}\phi,
\]

has a solution \( \phi(t) \), so that it is a diffeomorphism, \( |\phi(t) - \text{Id}|_{k,\alpha,h} < \epsilon \) and \( |\phi^*g(t) - h_0|_{k,\alpha} < \epsilon \), for every \( t \in [t_0, t_0 + A] \). \( \phi_{t_0} \) is chosen to be a diffeomorphism so that \( \delta_{\phi^*(t_0)h}(g(t_0)) = 0 \).

**Definition 8.** Let \( \phi : M \to M \) be a smooth function. Define \( e(\phi - \text{Id}) = g^{ij}h_{kl}(\phi^k_i - \text{Id}^k_i)(\phi^l_j - \text{Id}^l_j) \). Define \( E(\phi - \text{Id}) = \int_M e(\phi - \text{Id}) \) and \( F_l = \phi^l - \text{Id}^l \).

Throughout the proof of Proposition 7 we will have a tendency to use the same symbol for different uniform constants.

**Proof of Proposition 7.** Fix \( A > 0 \). Let \( \epsilon > 0 \) be very small (we will see later how small we want to take it). We know that for \( s_0 \) sufficiently big we can make \( |g(t) - h| \) as small as we want, and therefore we have that \( \delta_{\phi(t_0)*h}g(t_0) = 0 \) implies that \( |\phi(t_0) - \text{Id}|_{k+2,\alpha,h} < \epsilon/1000 \) on \( M \) (see for more details). Choose some \( t_0 \geq s_0 \). We can make \( |F(t_0)|_{N,\alpha,h} \), for say
$N \gg k$ as small as we want by choosing $s_0$ sufficiently big. Since $g(t) \to h$ as $t \to \infty$, the coefficients and the initial data of harmonic map flow (2) are uniformly bounded and uniformly close to each other for $t_0$ big enough. This implies that there exists a uniform constant $\delta_1 > 0$ so that a solution to (2) exists on $t \in [t_0, t_0 + \delta_1)$, for all $t_0 \geq s_0$. For the same reasons there exists some $\delta > 0$ such that $|F(t)|_{W^2, N, g(t)} < \epsilon$, for $t \in [t_0, t_0 + \delta)$. We can assume that we have chosen $N$ big enough so that as a consequence of Sobolev embedding theorems we have that $|F|_{k, \alpha, g(t)} < \tilde{\epsilon}$ (where $\tilde{\epsilon}$ differs from $\epsilon$ by a Sobolev embedding constant) for all $t \in [t_0, t_0 + \delta)$ and all $t_0 \geq s_0$. We want to show that the estimate $|F|_{k, \alpha} < \tilde{\epsilon}$ continues to hold past time $t_0 + \delta$, until $\delta < A$. Then $|F|_{k, \alpha} < \tilde{\epsilon}$ continues to hold past time $t_0 + \delta$, until $\delta < A$. This actually gives a uniform upper bound on the energy densities on whole manifold $M$. To see this, notice that a bound $|F|_{k, \alpha} < \tilde{\epsilon}$ implies that $e(\phi - \text{Id}) \leq C\tilde{\epsilon}$. Since

$$e(\phi - \text{Id}) = e(\phi) + e(\text{Id}) - 2g^{ij}h_{kl}\text{Id}^i_k\phi^j_l,$$

by the Schwartz inequality for quadratic forms and the interpolation inequality we get that

$$e(\phi) \leq C\tilde{\epsilon} + C + 2(g^{ij}h^{kl}\phi^k_i\phi^l_j)^{1/2}(g^{ij}h^{kl}\text{Id}^i_k\text{Id}^l_j)^{1/2} \leq C\tilde{\epsilon} + C + \eta e(\phi),$$

for some $\eta < 1$, which implies that $e(\phi) \leq \tilde{C}$. By the results proved by Eells and Sampson in [5] there exists $\delta$, depending on $(M, h)$ and the uniform bound on the energy densities $\tilde{C}$, so that for every $s \in [t_0, t_0 + \delta)$ a solution to a harmonic map flow (2) can be extended to $[s, s + \delta]$. If $t_0 + \delta + \tilde{\delta} < t_0 + A$, we can repeat the procedure above for a solution $\phi(t)$, on time interval $[t_0, t_0 + \delta + \tilde{\delta})$ to get that the energy density estimates with the same constant $\tilde{C}$ hold past time $t_0 + \delta + \tilde{\delta}$. Since all our estimates depend only on $A$ and the uniform bounds on geometries $g(t)$, we can iterate the argument till we reach time $t_0 + A$, for every $t_0 \geq s_0$. As a result, we will get $\phi(t)$, a solution to (2), such that $|\phi(t) - \text{Id}|_{k, \alpha} < \tilde{\epsilon}$ for all $t \in [t_0, t_0 + A]$. 

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We know that \((\Delta g(t), h) \gamma = g^{\alpha \beta} (\Gamma(h)_{\alpha \beta} - \Gamma(g)_{\alpha \beta})\) and that \(\frac{d}{dt} \text{Id} = 0\). Therefore, we have

\[
\frac{d}{dt}(\phi^k - \text{Id}^k) = \Delta g(t), h(\phi^k - \text{Id}^k) + g^{ij}(\Gamma^k_{ij}(h) - \Gamma^k_{ij}(g)),
\]

(3)

where we can choose \(s_0\) so big, that the last term is arbitrarily small (since \(g(t) \to h\)). We will see later how small we want to make it, for now we can say it is less than some \(\epsilon_1 > 0\).

Before we start establishing the estimates on \(F = \phi - \text{Id}\), we will occupy ourselves with the problem of replacing equation (3) which in terms of local coordinates on \(M\) is a local system of equations, by some much more global system. Passing to a global system of equations will make establishing the estimates on \(F\) much easier. We will follow a discussion in [5].

Since \(M\) is compact, there exists an embedding \(\omega : M \to R^q\) and due to Eells and Sampson ([5]) it is always possible to construct a smooth Riemannian metric \(g'' = (g''_{ab})_{1 \leq a, b \leq q}\) on a tubular neighborhood \(N\) of \(M\) in \(R^q\), such that \(N\) is Riemannian fibered. They actually meant that if \(\pi : N \to M\) is a projection map, it suffices to construct an appropriate smooth inner product in each space \(R^q(p)\) for all \(p \in M\), for which we can translate that tangent space to any point \(m \in N\) along the straight line segment (that is contained in \(N\)) from \(p = \pi(m)\) to \(m\). Following the arguments of section 7 in [5] we find that the evolution equation (3), given in local coordinates is satisfied by \(\phi - \text{Id}\) if and only if \(W - \tilde{W}\), where \(W = \omega \circ \phi\) and \(\tilde{W} = \omega \circ \text{Id}\) satisfies

\[
\frac{d}{dt}(W^c - \tilde{W}^c) = \Delta(W^c - \tilde{W}^c) + \pi^c_{ab}(W_i^a - \tilde{W}_i^a)(W_j^b - \tilde{W}_j^b)g^{ij} + \frac{\partial \omega^c}{\partial y_k} g^{ij}(\Gamma_{ij}^k(h) - \Gamma_{ij}^k(g)),
\]

(4)

where \((y_1, \ldots, y_n)\) are the local coordinates on \(M\). Moreover, since \(M\) is compact, the projection \(\pi\) satisfies (see [5])

\[
|\pi^c_{ab}|_{k+1, \alpha} \leq C,
\]
on $M$ and there are constants $A_1$ and $A_2$ so that
\[ A_1 ds_0^2 \leq ds^2 \leq A_2 ds_0^2, \]
where $ds_0^2$ denotes the line element induced on $M$ by the usual metric on $R^q$. These estimates immediately imply that
\[ \left| \frac{\partial^k \pi^c_{ab}}{\partial y^k} W^a_i W^b_j g^{ij} \right| \leq C(k)e(\phi), \]
where also $e(\phi) = g''_{ab} W^a_i W^b_j g^{ij}$, $e(\phi - \text{Id}) = g''_{ab}(W - W)g^{ij}$. Moreover, if $\tilde{F}^c = W^c - \tilde{W}^c$ then $\left| \frac{\partial^k}{\partial y^k} \tilde{F}^a_i \tilde{F}^b_j g^{ij} \right| \leq Ce(\phi - \text{Id})$.

The evolution equation for $e(\phi - \text{Id})$ (see for details [5] and [9]) is
\[ \frac{d}{dt} e(\phi - \text{Id}) = \Delta e(\phi - \text{Id}) - 2|D^2(\phi - \text{Id})|^2 + 2\text{Rm}(D(\phi - \text{Id}), D(\phi - \text{Id}), D(\phi - \text{Id}), D(\phi - \text{Id})) - \frac{1}{\tau} e(\phi - \text{Id}) + g^{ij} h_{kl}(\phi^k_j - \text{Id}^k_j)[g^{pq} (\Gamma^l_{pq}(h) - \Gamma^l_{pq}(g))] \]
where $\text{Rm}(D(\phi - \text{Id}), D(\phi - \text{Id}), D(\phi - \text{Id}), D(\phi - \text{Id})) = g^{ik} g^{jl} R_{pqmn} D_i(\phi^p - \text{Id}^p) D_j(\phi^q - \text{Id}^q) D_k(\phi^m - \text{Id}^m) D_l(\phi^n - \text{Id}^n)$ and $|D^2(\phi - \text{Id})|^2 = g^{ik} g^{jl} h_{pq} D^2_{ij}(\phi^p - \text{Id}^p) D^2_{kl}(\phi^q - \text{Id}^q)$. Applying the Schwarz inequality for quadratic forms and using the fact that $2\sqrt{\tau} (g^{pq} (\Gamma^l_{pq}(h) - \Gamma^l_{pq}(g)))_i$ can be made arbitrarily small by choosing $s_0$ sufficiently big (e.g. smaller than $\frac{2\epsilon}{1000}$), the last term in inequality (5) can be estimated as
\[ g^{ij} h_{kl}(\phi^k_j - \text{Id}^k_j)[g^{pq} (\Gamma^l_{pq}(h) - \Gamma^l_{pq}(g))]_i \leq \frac{e(\phi - \text{Id})^\frac{1}{2}}{2\sqrt{\tau}}(2\epsilon)/1000. \]
Factor of 1000 (that we can increase if necessary) is chosen so that after multiplying $\frac{2\epsilon}{1000}$ by at most a polynomial expression in $A$ (which will become more apparent later in the proof of Proposition [7]) can be made again much smaller than $\epsilon$. Therefore, for $t \in [t_0, t_0 + \delta)$ we have that

**Claim 9.** There exists $C$, small $\epsilon$ and sufficiently big $s_0$ such that for all $t_0 \geq s_0$

1. $e(\phi - \text{Id}) < \epsilon_1$,
2. \( E(\phi - \text{Id})(s) < \epsilon_1, \)

for all \( s \) belonging to a time interval starting at \( t_0 \) at which \( \phi \) exists, where \( \epsilon_1 \) is a constant that can be made much smaller than \( \epsilon \).

Proof. By using the interpolation inequality in (5), we get

\[
\frac{d}{dt} e(\phi - \text{Id}) \leq \Delta e(\phi - \text{Id}) + C \epsilon^4 - \frac{1}{\tau} e(\phi - \text{Id}) + \frac{1}{2\tau} e(\phi - \text{Id}) + C \frac{\epsilon^2}{1000^2}
\]

since we can start with \( \epsilon \) as small as we want, in particular we may choose \( \epsilon \) so that

\[
C \epsilon^4 + C \frac{\epsilon^2}{1000^2} < \frac{\epsilon}{1000}
\]

Let \( f(t) = \max_M e(\phi - \text{Id})(t) \). Then

\[
\frac{d}{dt} f \leq -\frac{1}{2\tau} f + \frac{\epsilon}{1000},
\]

\[
\frac{d}{dt} f \leq -\frac{1}{2\tau} (f - \frac{\tau \epsilon}{500}).
\]

If we choose \( s_0 \) big enough, we may assume that \( f(t_0) < \frac{\tau \epsilon}{500} \text{Vol}_h(M) \). If \( f(t) \geq \frac{\tau \epsilon}{500} \text{Vol}_h(M) \) for some \( t > t_0 \), then \( f(t) \) is nonincreasing (because \( \frac{d}{dt} f(t) \leq 0 \) and since it starts as \( f \leq \frac{\tau \epsilon}{500} \text{Vol}_h(M) \)), it will remain so forever while \( \phi \) exists. Denote by \( \epsilon_1 = \frac{\tau \epsilon}{500} \max_t \text{Vol}_{g(t)}(M) \).

\[
E(\phi - \text{Id})(s) = \int_U e(\phi - \text{Id})(s) dV_g(s) < \epsilon_1.
\]  

(6)

By Claim \( \# \) \( e(W - \bar{W}) \) can be made much smaller than \( \epsilon \) whenever \( \phi \) is defined (if \( t_0 \) is big enough and \( \epsilon \) is small enough). The conditions \( |F|_{W^{2,N}} < \epsilon \) and \( |F|_{k,\alpha} < \tilde{\epsilon} \) actually mean that for \( \bar{F} \) we make an assumption that \( |\bar{F}|_{W^{2,N}} < \epsilon \) and \( |\bar{F}|_{k,\alpha} < \tilde{\epsilon} \), for \( t \in [t_0, t_0 + \delta] \) (these \( \epsilon \) and \( \tilde{\epsilon} \) can be slightly different from those for \( F \)). In order to finish the proof of Proposition \( \# \) it is enough to show that \( |\bar{F}|_{k,\alpha} < \tilde{\epsilon} \) continues to hold past time \( t_0 + \delta \), for
$t_0$ big enough. From now on we will consider a globally defined evolution equation

$$\frac{d}{dt}(\tilde{F}^c) = \Delta \tilde{F}^c + \pi^c_{ab} \tilde{F}^a \tilde{F}^b g_{ij} + \frac{\partial \omega^c}{\partial y^k} g^{ij} (\Gamma^k_{ij}(h) - \Gamma^k_{ij}(g)), \quad (7)$$

**Step 9.1.** $\int_M |\tilde{F}^c|^2 dV_g(t)$ and $\int_{t_0}^{t_0+\delta} \int_M |\nabla \tilde{F}^c|^2 dV_g(t)$ can be made much smaller than $\epsilon$, for all $t \in [t_0, t_0 + \delta)$ and for all $t_0$ big enough.

Multiply the equation (7) by $\tilde{F}^c$ and integrate it over $M$ against the metric $g(t)$.

$$\frac{1}{2} \frac{d}{dt} \int (\tilde{F}^c)^2 dV_g(t) < \int_M (\tilde{F}^c)^2 \left( \frac{n}{2} - R \right) dV_g(t) - \int_M |\nabla \tilde{F}^c|^2 dV_g(t) + \epsilon_1 \int_M |\tilde{F}^c| dV_g(t) +$$

$$+ C \left( \int_M e(\tilde{F})^2 dV_g(t) \right)^{1/2} \left( \int_M (\tilde{F}^c)^2 dV_g(t) \right)^{1/2}$$

$$\leq \epsilon_1 \epsilon - \int_M |\nabla \tilde{F}^c|^2 + \epsilon_1 \left[ \int_M |\tilde{F}^c| dV_g(t) \right] + C \epsilon \epsilon_1 \int_M (\tilde{F}^c)^2 dV_g(t)$$

(8)

since

$$\int_M \tilde{F}^c \frac{\partial \omega^c}{\partial y_l} g^{ij} (\Gamma_l^k_{ij} - \Gamma_k^{ij}(g)) dV_g(t) \leq \epsilon_1 \int_M |\tilde{F}^c| dV_g(t),$$

$$\int_M \tilde{F}^c g^{ij} \pi^c_{ab} \tilde{F}^a \tilde{F}^b \leq C \left( \int_M e(\tilde{F})^2 \right)^{1/2} \left( \int_M (\tilde{F}^c)^2 \right)^{1/2}$$

$$< C \epsilon \epsilon_1 \left[ \int (\tilde{F}^c)^2 \right]^{1/2}.$$

In the above estimates we have used the energy estimates (6), the fact that $g(t) \to h$ as $t \to \infty$ uniformly on $M$ and that $|\tilde{F}|_{W^{2, N}} < \epsilon$ for $t \in [t_0, t_0 + \delta)$ (which implies $|\tilde{F}|_{C^{k, \alpha}} < \tilde{\epsilon}$ for sufficiently big $N$). For those reasons, $\epsilon_1 << \epsilon$ is a constant that can be made much smaller than $\epsilon$, by taking $\epsilon$ small and $s_0$ big. Integrate (8) in $t$.

$$\frac{1}{2} \sup_{t \in [t_0, t_0 + \delta]} \int (\tilde{F}^c)^2(t) dV_g(t) + \sup_{t \in [t_0, t_0 + \delta]} \int_{t_0}^{t} \int_M |\nabla \tilde{F}^c|^2 dV_g(t) \leq \int \frac{1}{2} (\tilde{F}^c)^2(t_0) dV_h + C A \epsilon \epsilon_1.$$
Since for big \( t_0 \) the first integral on the right hand side of the previous inequality can be made much smaller than \( \epsilon \), it follows that for big \( t_0 \) and small \( \epsilon \),

\[
\sup_{t \in [t_0, t_0 + \delta]} \int_M (\tilde{F}^c)^2(t) dV_g(t) < \tilde{\epsilon},
\]

\[
\sup_{t \in [t_0, t_0 + \delta]} \int_M |\nabla \tilde{F}^c|^2 dV_g(t) < \tilde{\epsilon},
\]

for some constant \( \tilde{\epsilon} << \epsilon \) and these estimates depend on \( A \).

**Step 9.2.** \( \sup_{t \in [t_0, t_0 + \delta]} \int_0^t \int_M \left| \frac{d}{dt} \tilde{F}^c \right|^2 dV_g(t) \) and \( \sup_{t \in [t_0, t_0 + \delta]} \int_0^t \int_M |\nabla^2 \tilde{F}^c|^2 dV_g(t) \) can be made much smaller than \( \epsilon \) for big enough \( s_0 \) which depends on \( A \) and on the rate of convergence of \( g(t) \) to \( h \), for small enough \( \epsilon \).

\[
\frac{d}{dt} \tilde{F}^c = \Delta \tilde{F}^c + H^c, \quad \text{where} \quad H^c = \frac{\partial \omega^c}{\partial y^l} g^{ij} (\Gamma(h)_{ij}^l - \Gamma(g)_{ij}^l) + g^{ij} \pi_{ab}^{\omega} \tilde{F}^a_i \tilde{F}^b_j.
\]

Then

\[
(H^c)^2 = (\Delta \tilde{F}^c)^2 + \left( \frac{d}{dt} \tilde{F}^c \right)^2 - 2 \Delta \tilde{F}^c \frac{d}{dt} \tilde{F}^c.
\]

\[
- \int_M \frac{d}{dt} \tilde{F}^c \Delta \tilde{F}^c = \int_M g^{ij} \nabla_i (\frac{d}{dt} \tilde{F}^c) \nabla_j \tilde{F}^c
\]

\[
= \frac{1}{2} \frac{d}{dt} \int_M |\nabla \tilde{F}^c|^2 + \frac{1}{2} \int_M g^{ip} g^{jq} (-2R_{pq} + \frac{1}{\tau} g_{pq}) |\nabla \tilde{F}^c|^2
\]

\[
- \frac{1}{2} \int_M |\nabla \tilde{F}^c|^2 (\frac{n}{2\tau} - R).
\]

\[
\int_M (\Delta \tilde{F}^c)^2 = \int_M |\nabla^2 \tilde{F}^c|^2 + \int_M g^{ij} g^{ks} \nabla_j \tilde{F}^c R^k_{kp} \nabla_p \tilde{F}^c.
\]

Combining (9), (10) and (11) we get

\[
\int_M \left| \frac{d}{dt} \tilde{F}^c \right|^2 + \int_M |\nabla^2 \tilde{F}^c|^2 + \frac{d}{dt} \int_M |\nabla \tilde{F}^c|^2 \leq \int_M (H^c)^2 + C \int_M |\nabla \tilde{F}^c|^2 dV_g(t),
\]

(12)
we start with $\epsilon$ that can be made much smaller than $\epsilon$ (6). We will sometimes use the same constant $\epsilon$

Notice also that since $e$ because of Step 9.1, the fact that for big $t$

Multiply this equation by $\hat{\epsilon}$

Step 9.3.

Let $\hat{\epsilon} = \frac{d}{dt}\tilde{F}^c$. Then

\[
\frac{d}{dt}\hat{\epsilon} = \Delta \tilde{F}^c + g^{ij}g^{jk}(2R_{pq} - \frac{1}{r}g_{pq})\nabla \tilde{F}^c + \frac{d}{dt}H^c - g^{ij}\frac{d}{dt}\Gamma_{ij}^k \nabla \tilde{F}^c.
\]

Multiply this equation by $\hat{\epsilon}$ and integrate it over $M$.

\[
\frac{1}{2} \frac{d}{dt} \int_M (\hat{\epsilon})^2 - \frac{1}{2} \int_M (\hat{\epsilon})^2(\frac{M}{2\pi} - R) = -\int_M |\nabla \hat{\epsilon}|^2 + \int_M g^{ij}g^{jk}(2R_{pq} - \frac{1}{r}g_{pq})\nabla \nabla \hat{\epsilon}^c + \int_M \hat{\epsilon} \frac{d}{dt}H^c - \int_M g^{ij}\frac{d}{dt}\Gamma_{ij}^k \nabla \hat{\epsilon}^c.
\]
Integrate it in $t$ to get

$$\frac{1}{2} \int_M (\hat{F}^c(t))^2 + \int_{t_0}^t \int_M |\nabla \hat{F}^c|^2 \leq \frac{1}{2} \int_M (\hat{F}^c(t_0))^2 + \frac{1}{2} \int_{t_0}^t \int_M (\hat{F}^c)^2 \left( \frac{n}{2\tau} - R \right) +$$

$$+ \epsilon_1 \left( \int_{t_0}^t \int_M |\nabla \hat{F}^c|^2 \right)^{1/2} \left( \int_{t_0}^t \int_M |\hat{F}^c|^2 \right)^{1/2}$$

$$+ \left( \int_{t_0}^t \int_M (\hat{F}^c)^2 \right)^{1/2} \left( \int_{t_0}^t \int_M \frac{d}{dt} H^c \right)^{1/2} +$$

$$+ C \left( \int_{t_0}^t \int_M |\nabla \hat{F}^c|^2 \right)^{1/2} \left( \int_{t_0}^t \int_M (\hat{F}^c)^2 \right)^{1/2}. \quad (15)$$

Notice that

$$\int_{t_0}^t \int_M \left( \frac{d}{dt} H^c \right)^2 \leq C \left( \int_{t_0}^t \int_M \left( \frac{\partial \omega^c}{\partial y^i} g^{ij} (\Gamma(h)_{ij} - \Gamma(g)_{ij}) \right)^2 \right)$$

$$+ \int_{t_0}^t \int_M \left( (\frac{d}{dt} g^{ij}) \pi_{ab} (\hat{F}^a_i \hat{F}^b_j) \right)^2 \quad (16)$$

$$+ \int_{t_0}^t \int_M \left( g^{ij} \pi_{ab} (\frac{d}{dt} \hat{F}^a_i \hat{F}^b_j) \right)^2, \quad (17)$$

where

$$\int_{t_0}^t \int_M \left( \frac{d}{dt} (g^{ij} \partial \omega^c) (\Gamma(h)_{ij} - \Gamma(g)_{ij}) \right)^2 < \epsilon_1,$$

$$\int_{t_0}^t \int_M \left( (\frac{d}{dt} g^{ij}) \pi_{ab} (\hat{F}^a_i \hat{F}^b_j) \right)^2 \leq \epsilon_1,$$

if $t_0$ is big enough, since $g(t) \to h$ uniformly on $M$ and $\frac{d}{dt} g^{ij} = g^{pi} g^{qj} (2R_{pq} - \frac{1}{r} g_{pq})$, and

$$\int_{t_0}^t \int_M \left( g^{ij} \pi_{ab} (\frac{d}{dt} \hat{F}^a_i \hat{F}^b_j) \right)^2 \leq C \epsilon \int_{t_0}^t \int_M |\nabla \hat{F}^c|^2 < \frac{1}{2} \int_{t_0}^t \int_M |\nabla \hat{F}^c|^2,$$

if we choose $\epsilon$ small enough, such that $C \epsilon < \frac{1}{2}$, since $|\nabla_j \hat{F}| < \epsilon$ for $t \in [t_0, t_0 + \delta]$.

$$\int_M |\hat{F}^c|^2 dV_{g(t)} \leq C \left( \int_{t_0}^t \int_M |\nabla^2 \hat{F}^c|^2 + C \int_{t_0}^{t_0 + \delta} \int_M |\nabla \hat{F}^c|^2 + \epsilon_1 \right) \leq C(\epsilon_1 + \epsilon_1),$$

by Step 9.2. The assertion of Step 9.3 follows now immediately from (15).
From the estimate (16) we can now get (using the estimates of Steps 9.1, 9.2 and 9.3) that \( \int_{t_0}^{t} \int_U (\frac{d}{dt} H^c)^2 \) can be much smaller than \( \epsilon \). Consider the equation
\[
\frac{d}{dt} \hat{F}^c = \Delta \hat{F}^c + \hat{H}^c,
\]
(18)
where \( \hat{F}^c = \frac{d}{dt} \tilde{F}^c \) and \( \hat{H}^c = g^{ij} g^{jq}(2R_{pq} - \frac{1}{7} q_{pq}) \nabla_i \nabla_j \hat{F}^c + \frac{d}{dt} H^c - g^{ij} \nabla_k \hat{F}^c \frac{d}{dt} (\Gamma^k_{ij}) \).

Since \( g(t) \to h \), where \( \text{Ric}(h) = \frac{1}{4\pi} h \), by using the previous estimates, we can easily see that \( \int_{t_0}^{t} \int_U (\frac{d}{dt} H^c)^2 dV_g(s) ds \) can be made much smaller than \( \epsilon \). In the same manner as we have obtained the estimates in step 9.2 for \( \tilde{F}^c \), we can get the following estimates for \( \hat{F}^c = \frac{d}{dt} \tilde{F}^c \) by considering the evolution equation (18).

\[
\sup_{t \in [t_0, t_0 + \delta]} \int_U |\nabla \frac{d}{dt} \hat{F}^c|^2,
\]
\[
\int_{t_0}^{t_0 + \delta} \int_U |\nabla^2 \frac{d}{dt} \hat{F}^c|^2,
\]
\[
\int_{t_0}^{t_0 + \delta} \int_U (\frac{d^2}{dt^2} \hat{F}^c)^2,
\]
can be made much smaller than \( \epsilon \) for big \( t_0 \).

We have that \( \Delta \hat{F}^c = \frac{d}{dt} \hat{F}^c - H^c \) where \( W^{1,2} \) norm of \( \frac{d}{dt} \hat{F}^c \) and \( L^2 \) norm of \( H^c \) can be made much smaller than \( \epsilon \). By elliptic regularity theory we can get that \( W^{2,2} \) norm of \( \hat{F}^c \) can be made much smaller than \( \epsilon \) (since it can be estimated in terms of \( W^{1,2} \) norm of \( \frac{d}{dt} \hat{F}^c \) and \( L^2 \) norm of \( H^c \)). Using that and the fact that \( |\hat{F}|_{1,\alpha,g} < \epsilon \) notice that
\[
\int_M |\nabla H^c|^2 \leq C(\epsilon_1 + \int_M g^{rs} (\nabla_r (g^{ij} \pi^c_{ab}) \hat{F}_i^a \hat{F}_j^b)) (\nabla_s g^{ij} \pi^c_{a'b'} \hat{F}_i^{a'} \hat{F}_j^{b'}) + \int_M g^{rs} (g^{ij} \pi^c_{ab} \nabla_r \hat{F}_i^a \hat{F}_j^b) (g^{ij} \pi^c_{a'b'} \nabla_s \hat{F}_i^{a'} \hat{F}_j^{b'}) \leq C(\epsilon_1 + \epsilon \sum_a \int_M |\nabla \hat{F}_i^a|^2 + \epsilon \sum_b \int_M |\nabla^2 \hat{F}_i^b|^2) \leq \tilde{\epsilon},
\]
for some small constant \( \tilde{\epsilon} \), that can be assumed to be much smaller than \( \epsilon \), since \( W^{2,2} \) norm of \( \hat{F}^c \) can be made much smaller than \( \epsilon \). By elliptic
regularity theory this implies that $W^{3,2}$ norm of $\tilde{F}^c$ can be made much smaller than $\epsilon$ for $t_0$ very big.

We can continue our proof by studying the equation $\frac{d}{dt} \tilde{F}^c = \Delta \tilde{F}^c + \tilde{H}^c$. $|\tilde{F}^c|_{2,\alpha} < \epsilon$ for $t \in [t_0, t_0 + \delta)$. By a standard parabolic regularity we can get the higher order estimates of $\tilde{F}^c$, by constants that are comparable to $\epsilon$. Therefore, by the similar analysis as above we can get that $W^{3,2}$ norms of $\tilde{F}^c$ can be made much smaller than $\epsilon$, since from the estimates that we have got till this point we can again easily get that $W^{1,2}$ norm of $\tilde{H}^c$ can be made much smaller than $\epsilon$. Consider again the equation

$$\Delta \tilde{F}^c = \frac{d}{dt} \tilde{F}^c - \tilde{H}^c. \quad (19)$$

We know that $W^{3,2}$ norm of $\frac{d}{dt} \tilde{F}^c$ and $W^{3,2}$ norms of $\tilde{F}^c$ can be made much smaller than $\epsilon$. Let’s check that $W^{3,2}$ norm of $\tilde{H}^c$ can be made much smaller than $\epsilon$ as well. In order for it to be true it is enough to check that $\int_M |\nabla^3 (g^{ij} \pi_a^c G_i^a F_j^b)|^2$ can be made much smaller than $\epsilon$.

$$\int |\nabla^3 (g^{ij} \pi_a^c G_i^a F_j^b)|^2 \leq C(\epsilon) \int_M |\nabla \tilde{F}|^2 + C \sum_{a,b} \int |\nabla^4 \tilde{F}^a|^2 |\nabla \tilde{F}^b|^2 + C \sum_{a,b} \int_M |\nabla^3 \tilde{F}^a|^2 |\nabla^2 \tilde{F}^b|^2,$$

since $|\tilde{F}|_{W^{2,N}, g(t)} < \epsilon$ for all $t \in [t_0, t_0 + \delta)$ and all $t_0 \geq s_0$. From here, again by elliptic regularity theory applied to equation $(19)$, it follows that $W^{5,2}$ norm of $\tilde{F}^c$ can be made much smaller than $\epsilon$.

We can continue the proof in a similar manner as above, by taking the higher order derivatives of our original equation $\Delta \tilde{F}^c = \frac{d}{dt} \tilde{F}^c - \tilde{H}^c$ in $t$, using the estimates that we get on the way and then go backward to our original equation to improve a regularity of $\tilde{F}^c$. As a result, we can get (performing the previously described procedure sufficiently many times) that $|\tilde{F}|_{W^{N,2}, g(t)} < \epsilon$ continues to hold past time $t_0 + \delta$.

So far we have proved that for every $A > 0$ and an integer $k$ there exists $\epsilon_0 = \epsilon_0(A, k)$ such that for every $\epsilon < \epsilon_0$ we can find $s_0 = s_0(A, \epsilon, k)$, so that
∀ \ t_0 \geq s_0 \ there \ exists \ a \ solution \ of
\[ \frac{d}{dt} \phi(t) = \Delta_{g(t), h(t)} \phi(t) \]  \hspace{1cm} (20)
\[ \phi(t_0) = \phi_{t_0}, \]
for all \( t \in [t_0, t_0 + A] \) and \( |\phi - \text{Id}|_{k, \alpha} < \epsilon \).

We want to show that these maps \( \phi(t) : M \to M \) are actually diffeomor-
phisms which will imply that we have constructed an 1- parameter family
of gauges such that for \( \bar{g}(t) = (\phi(t)^*)^{-1}g(t) \) the linearization of the Ricci-
DeTurck flow
\[ \frac{d}{dt} \bar{g} = -2\bar{R}_{ij} + \frac{1}{2} \bar{g}_{ij} + \nabla_i W_j + \nabla_j W_i, \]
with \( \bar{g}(t_0) = (\phi_{t_0}^{-1})^*g(t_0) \) is strictly parabolic (\( W_j = \bar{g}_{jk}\bar{g}_{pq}(\Gamma^k_{pq}(\bar{g}) - \Gamma(h)^k_{pq}) \)).

**Corollary 10.** Adopt the notation from Proposition 7. \( \phi(t) \) are diffeomor-
phisms for all \( t \in [t_0, t_0 + A] \) and all \( t_0 \geq s_0 \).

**Proof.** Fix any \( t_0 \geq s_0 \). Consider the equation
\[ \frac{d}{dt} \tilde{g}_{ij} = -2\tilde{R}_{ij} + \frac{1}{2} \tilde{g}_{ij} + \nabla_i V^j + \nabla_j V^i, \]  \hspace{1cm} (21)
\[ \tilde{g}(t_0) = (\phi_{t_0}^{-1})^*g(t_0), \]
where \( V^k = \tilde{g}^{pq}(\Gamma^k_{pq}(\tilde{g}) - \Gamma(h)^k_{pq}) \). This is a strictly parabolic system of equations and therefore there exists some \( \delta > 0 \) so that a solution \( \tilde{g} \) exists for all times \( t \in [t_0, t_0 + \delta] \). On the other hand, look at the system
\[ \frac{d}{dt} \psi(t) = -V \circ \psi(t), \]  \hspace{1cm} (22)
\[ \psi(t_0) = \phi_{t_0}. \]
Vector fields \( V(t) \) are defined for \( t \in [t_0, t_0 + \delta] \) and therefore the system \((22)\) has a solution \( \psi(t) \) for all those times. It is easy to show (a classical result) that all \( \psi(t) \) are diffeomorphisms for \( t \in [t_0, t_0 + \delta] \). The simple computation (due to the fact that \( g(t) \) is a solution of the Ricci flow equation) shows that
\[ \frac{d}{dt} \psi(t) = \Delta_{g(t), h(t)} \psi(t), \] 18
with \( \psi(t_0) = \phi_{t_0} \). Because of the uniqueness of a harmonic map flow with the same initial data (we know that our solutions are smooth and uniformly bounded, so the uniqueness follows by the arguments of Eells and Sampson in [5]), we have that \( \psi(t) = \phi(t) \) for all \( t \in [t_0, t_0 + \delta] \). This means \( \phi(t) \) is a diffeomorphism for \( t \in [t_0, t_0 + \delta] \) and \( \tilde{g}(t) = (\phi(t)^{-1})^* g(t) \). We know that for all \( t \in [t_0, t_0 + A] \), for \( t_0 \) sufficiently big, we have that \( |\phi(t) - \text{Id}|_{k,\alpha,h} < \epsilon \). Therefore, \( |\phi^{-1} - \text{Id}|_{k,\alpha} \) can be made small which implies that \( |\tilde{g}(t) - g(t)| \) can be made very small, comparable to \( \epsilon \), for all \( t \in [t_0, t_0 + \delta] \). We want to extend a solution \( \tilde{g}(t) \) of (21) all the way up to \( t_0 + A \). Since \( |\tilde{g}(t) - h| < \bar{\epsilon} \) and since our flow (21) is strictly parabolic, there exists \( t_1 = t_1(h, \bar{\epsilon}) \) so that for every \( t \in [t_0, t_0 + \delta] \), a solution to (21) exists for all times \( s \in [t, t + t_1] \). That means we can extend our solution past time \( t_0 + \delta \). Since our estimates on \( |\tilde{g}(t) - h| \) for those times for which a solution \( \tilde{g}(t) \) exists are independent of \( \delta \leq A \), we can easily extend our solution all the way up to \( t_0 + A \), with \( |\tilde{g}(t) - h| \) staying very small (comparable to \( \epsilon \)) for all \( t \in [t_0, t_0 + A] \). Existence of \( \tilde{g}(t) \) for \( t \in [t_0, t_0 + A] \) gives that \( \phi(t) \) stays a diffeomorphism for all times up to \( t_0 + A \), because it solves the equation (22).

3.2 The integrable case

The proofs in this subsection are motivated by those in [3], where Cheeger and Tian have considered the uniqueness problem of tangent cones under the assumption of integrability of one of the tangent cones and under some curvature and volume bounds.

Remark 11. So far we have proved that for every \( A > 0 \) and an integer \( k \) there exists \( \epsilon_0 = \epsilon_0(A,k) \) such that for every \( \epsilon \leq \epsilon_0 \) there exists \( s_0 = s_0(\epsilon,A,k) \) with the property that for every \( t_0 \geq s_0 \) there is an 1-parameter family of diffeomorphisms \( \phi(t) \) so that
1. $\phi^{-1}$ solves a harmonic map flow equation

$$\frac{d}{dt} \phi^{-1} = \Delta_{g,h} \phi^{-1},$$

$$\phi^{-1}(t_0) = \phi_{t_0},$$

where $\delta_{\phi_{t_0}^* h}(g(t_0)) = 0$, for $t \in [t_0, t_0 + A]$,

2. $\tilde{g} = \phi^* g$ solves strictly parabolic equation on $[t_0, t_0 + A]$

$$\frac{d}{dt} \tilde{g} = -2\text{Ric}(\tilde{g}) + \frac{1}{\tau} \tilde{g} + \nabla_i V_j + \nabla_j V_i,$$

where $V^i = \tilde{g}^{pq}(\Gamma^i_{pq}(\tilde{g}) - \Gamma^i_{pq}(h))$. We will say that $\tilde{g}$ is in a standard form around $h$. We will denote by $P_{h_0}(\tilde{g}) = \nabla_i V_j + \nabla_j V_i$.

3. $|\phi - \text{Id}|_{k,\alpha} < \epsilon$.

4. $|\tilde{g} - h|_{k,\alpha} < \epsilon$.

From now on, we will simply write $\phi g$ instead of $\phi^* g$. By the assumptions of Theorem 1 there exists a limit soliton, say $h(t)$ which is integrable. There is a sequence $t_i$ such that $g(t_i + t) \to h(t)$ as $i \to \infty$ and

$$R_{ij} + \nabla_i \nabla_j f - \frac{1}{2\tau} h_{ij}(t) = 0,$$

for some function $f$. From before we know that $f(t)$ is a minimizer for $\mathcal{W}$ with respect to a metric $h(t)$, for every $t$. Let $\psi(t)$ be 1-parameter family of diffeomorphisms induced by a vector field $-\nabla f$. Then $h(t) = \psi^*(t) h_0$, where $h_0 = h(0)$. Since $h_0 = (\psi^{-1})^* h(t)$, it satisfies the equation

$$0 = \frac{d}{dt} h_0 = -2\text{Ric}(h_0) + \frac{1}{\tau} h_0 - \mathcal{L}_{\psi^* \frac{d}{dt} \psi^{-1}} h_0, \quad (23)$$

From $\psi \circ \psi^{-1} = \text{Id}$, by taking a time derivative, we see that $\psi^* \frac{d}{dt} \psi^{-1} + \psi^* \mathcal{L}_{\frac{d}{dt} \psi} \psi^{-1} = 0$ and since $\psi$ is a diffeomorphism, we get that

$$\frac{d}{dt} \psi^{-1} = -\mathcal{L}_{\frac{d}{dt} \psi} \psi^{-1} = \mathcal{L}_{\nabla f(\psi)} \psi^{-1}. \quad (24)$$

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Since \( \{f(t)\}_{0 \leq t < \infty} \) are the minimizers for \( W \), there are uniform \( C^{k+2,\alpha} \) estimates on \( f(t) \). Since \( \frac{d}{dt} \psi = -\nabla f(\psi) \), there are uniform \( C^{k+1,\alpha} \) bounds on \( \psi \), for \( t \in [0, B] \). This together with (24) yields \( |\psi^{-1}|_{k,\alpha} \leq C(B) \), for \( t \in [0, B] \). Let \( \tilde{g}(t) = \psi^{-1}g(t) \). Then \( \tilde{g}(t) \) satisfies the equation
\[
\frac{d}{dt} \tilde{g} = -2\text{Ric}(\tilde{g}) + \frac{1}{\tau} \tilde{g} - \mathcal{L}_{\psi} \# \psi^{-1} \tilde{g},
\]
and
\[
|\tilde{g}(t_i + t) - h_0|_{k,\alpha} \leq |\psi^{-1}| |g(t_i + t) - h(t)| \leq C(B) |g(t_i + t) - h(t)| \to 0,
\]
when \( i \to \infty \), uniformly on \( M \times [0, B] \) (that implies \( \tilde{g}(t_i + t) \to h_0 \) uniformly on compact subsets of \( M \times [0, \infty) \)). The proof of Proposition 11 after minor modifications can be used to get the following result that tell us how to find an appropriate gauge in the case of convergence toward the solitons instead of Einstein metrics.

**Theorem 12.** For every \( L > 0 \) and an integer \( k \), there exists \( \epsilon_0 = \epsilon_0(L, k) \) such that for every \( \epsilon < \epsilon_0 \) we can find \( i_0 = i_0(L, \epsilon, k) \), so that whenever \( i \geq i_0 \) there is a gauge \( \phi(t) \) on \( M \times [t_i, t_i + L] \) such that \( \phi g \) is in a standard form around \( h_0 \) (see Remark 14 below), \( |\phi \tilde{g} - h_0|_{k,\alpha} < \epsilon \) and \( |\phi - \text{Id}|_{k,\alpha} < \epsilon \).

**Definition 13.** A limit soliton \( h(0) \) is said to be **integrable** if for every solution \( a \) of a linearized deformation equation
\[
\frac{d}{du} [\text{Ric}_{g_u} + \mathcal{L}_{\psi} \# \psi^{-1} g_u - \frac{1}{\tau} (g_u)_{ij}] |_{u=0} = 0,
\]
with \( g_0 = h_0 \) there exists a path of solitons \( h_u \), satisfying the soliton equation
\[
\text{Ric}_{h_u} + \mathcal{L}_{\psi} \# \psi^{-1} h_u - \frac{1}{\tau} (h_u)_{ij} = 0, \tag{25}
\]
with \( u \in (-\epsilon, \epsilon) \) and \( h_0 = h(0) \) such that
\[
\frac{d}{du} |_{u=0} h_u = a.
\]
Remark 14. In the context of Theorem 1, to say that $\bar{g}(t)$ is in a standard form around $h_0$ means that $\bar{g}$ satisfies the following equation

$$\frac{d}{dt} \bar{g} = -2\text{Ric}(\bar{g}) + \frac{1}{\tau} \bar{g} + P_{h_0}(\bar{g}) - \mathcal{L}_{\psi} \psi^{-1} \bar{g},$$

(26)

where $P_{h_0}(\bar{g}) = \nabla_i V_j + \nabla_j V_i$ and $V^k = \bar{g}^{pq} (\Gamma^k_{pq}(\bar{g}) - \Gamma^k_{pq}(h_0))$. We will write $h_0$ for $h(0)$ in a further discussion.

Choose $i_0, \phi$ as in Theorem 12 with $3L$ instead of $L$. Denote by $\| \cdot \|_{a,b} = \int_a^b | \cdot |$, where $| \cdot |$ is just the $L^2$ norm. Let $\pi$ denote an orthogonal projection on the subspace $\ker(-\frac{d}{dt} + \Delta + \frac{1}{\tau} + U)_{M \times [t_{i_0}, t_{i_0} + L]}$, with respect to norm $\| \cdot \|_{t_{i_0}, t_{i_0} + L}$, where $U$ is a linear first-order expression that comes out after linearizing the equation (26). Let $g_1$ be a suitable chosen soliton. Denote by $k = \phi g - g_1$ and put $\pi k = (\pi k)_{\uparrow} + (\pi k)_{\downarrow} + (\pi k)_0$. The integrability assumption on $h_0$ enters when we choose $g_1$ so that $(\pi k)_0 = 0$. Look at the explanation for $(\cdot)_{\uparrow}$, $(\cdot)_{\downarrow}$ and $(\cdot)_0$, just after the equation (28) below.

Lemma 15. Let $h_0$ be an integrable limit soliton. Then if $\tau < \tau(n, L)$, for any cylinder $M \times [t_{i_0}, t_{i_0} + L]$ there is a soliton $g_1$ satisfying $P_{h_0}(g_1) = 0$ and equation (25), and such that $(\pi k)_0 = 0$. Moreover, if

$$\sup_{[t_{i_0}, t_{i_0} + L]} | \phi g(t) - h_0 | < \tau,$$

then

$$\| g_1 - h_0 \|_{t_{i_0}, t_{i_0} + L} \leq 2\| \pi (\phi g(t) - h_0) \|_{t_{i_0}, t_{i_0} + L}.\quad (27)$$

Proof. The proof of this lemma follows the proof of Lemma 5.56 in [3]. The integrability assumption implies that the set of metrics $\bar{g}$ satisfying

$$\text{Ric}(\bar{g}) - \frac{1}{\tau} \bar{g} + \mathcal{L}_{\psi} \psi^{-1} \bar{g} = 0,$$

$$P_{h_0}(\bar{g}) = 0,$$

has a natural smooth manifold structure near $h_0$. Let $\mathcal{V}$ be a sufficiently small Euclidean neighborhood of $h_0$. The tangent space to $\mathcal{V}$ at $h_0$ is naturally identified with

$$\mathcal{K} = \{ a \in \ker(-\frac{d}{dt} + \Delta + \frac{1}{\tau} + U) | P_{h_0} a = 0 \}.$$
Define $\psi : \mathcal{V} \to \mathcal{K}$ by

$$
\psi(\tilde{g}) = \sum_i \langle \tilde{g}, B_i \rangle B_i,
$$

where $B_i$ is an orthonormal basis for $\mathcal{K}$ with respect to a natural inner product. $\psi$ is a smooth map and the differential of $\psi$ is the identity map. We can use now the implicit function theorem and Lemma 16 to finish the proof of the Lemma 15. \qed

The inequality (27) implies that $|g_1 - h_0| \leq 2 \sup_{[t_0, t_0 + L]} |\pi(\phi g(t)) - h_0|$, where $\cdot$ is just the usual $L^2$ norm. The linearization of the right hand side of the equation $\frac{d}{dt}\phi g = Q(\phi g)$, satisfied by $\phi g$, where $\phi$ is a gauge chosen as in Theorem 12 is $DQ(k) = \Delta k + \frac{1}{r} k + U$, where $U$ is a linear first-order expression in $k$ and a Laplacian and $U$ are with respect to metric $\phi g$. Let $F$ be a solution of

$$
\frac{d}{dt} F = \mathcal{L} F,
$$

where $\mathcal{L} = \Delta + \frac{1}{r} + U$ and the Laplacian and $U$ are this time given with a respect to a fixed metric (in our case we will take metric $h_0$). Let $\{\lambda_k\}$ be the set of eigenvalues of $\mathcal{L}$. We can write $F = F_\uparrow + F_\downarrow + F_0$, where $F_\uparrow(t) = \sum_{\lambda_k < 0} a_k e^{-\lambda_k t}$, $F_\downarrow(t) = \sum_{\lambda_k > 0} a_k e^{-\lambda_k t}$, and $F_0$ is a projection of $F$ to a kernel of $\mathcal{L}$.

The basic parabolic estimates (for example similarly as in [16] and [3]) yield the following.

**Lemma 16.** There exists $\tau > 0$ such that for any solution $\eta$ of (28) with $|g_1 - h_0|_{k+2,\alpha} \leq \tau$, we have that

$$
\sup_{(t_0, t_0 + L)} |\eta|_{k, \alpha} \leq C \sup_{(t_0, t_0 + L)} |\eta|,
$$

where the first norm is $C^{k,\alpha}$ norm and the last norm is $L^2$ norm.

**Lemma 17.** There exists $\alpha > 1$ such that

$$
\sup_{[L,2L]} |F_\uparrow| \geq \alpha \sup_{[0,L]} |F_\uparrow|,
$$

(29)

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\[ \sup_{[L,2L]} |F_\uparrow| \leq \alpha^{-1} \sup_{[0,L]} |F_\downarrow|. \]  

The norms considered above are standard \( L^2 \) norms.

**Proof.** We will prove only (30), since the proof of (30) is similar. Let \( \delta = \min\{\lambda_k \neq 0\} > 0 \).

\[ \sup_{[L,2L]} |F_\downarrow| - \alpha \sup_{[0,L]} |F_\downarrow| = \sup_{[0,L]} \sum_{\lambda_k < 0} a_k^2 e^{-2\lambda_k t} e^{-2\lambda_k L} - \alpha \sup_{[0,L]} \sum_{\lambda_k < 0} a_k^2 e^{-2\lambda_k t} \]

\[ \geq \sup_{[0,L]} \sum_{\lambda_k < 0} a_k^2 e^{-2\lambda_k t} (e^{2\delta L} - \alpha), \]

which is positive, if \( e^{2\delta > \alpha} \). We can choose \( \alpha = e^{\delta L} > 1 \). \( \square \)

**Lemma 18.** There exists \( \beta < \alpha \) such that if

\[ \sup_{[L,2L]} |F| \geq \beta \sup_{[0,L]} |F|, \]  

then

\[ \sup_{[2L,3L]} |F| \geq \beta \sup_{[L,2L]} |F|, \]  

and if

\[ \sup_{[2L,3L]} |F| \leq \beta^{-1} \sup_{[L,2L]} |F|, \]  

then

\[ \sup_{[L,2L]} |F| \leq \beta^{-1} \sup_{[0,L]} |F|. \]  

Moreover, if \( F_0 = 0 \) at least one of \( (32), (34) \) holds.

The proof of Lemma 18 is almost the same to the proof of analogous lemma (5.31) in [3]. We can choose \( \beta \) to be of order \( e^{\frac{L}{4}} \).

Let \( \eta = \phi g - g_1 \), where \( \phi \) is chosen as in Theorem 12 and \( g_1 \) is a soliton as in Lemma 15 which does not depend on \( t \) for a considered time interval of length \( L \).
Lemma 19.

\[
\frac{d}{dt}(\phi g - g_1) = \Delta_{h_0}(\phi g - g_1) + \frac{1}{\tau}(\phi g - g_1) + F(\phi g, h_0, g_1) + U(\phi g - g_1)
\]

where \( |F(\phi g, h_0, g_1)|_{k,\alpha} \leq C(|g_1 - h_0| + |\eta|_{k,\alpha})|\nabla^2 \eta|_{k-2,\alpha} + C(|\nabla (g_1 - h_0)|_{k-1,\alpha} + |\nabla \eta|_{k-1,\alpha})|\nabla \eta|_{k-1,\alpha} \) and \( U \) is a first order linear expression in \( \phi - g_1 \).

Proof. Since both \( \phi g \) and \( g_1 \) are in a standard form around \( h_0 \) (recall that \( P_{h_0}(g_1) = 0 \)), by using a formula for linearization of a second order operator \(-2\text{Ric}(\phi g) + P_{h_0}(\phi g)\), we get

\[
\frac{d}{dt}(\phi g - g_1) = (-2\text{Ric}(\phi g) + P_{h_0}(\phi g) - L_{\phi g}^* \psi^* \phi g) -
\]

\[
- (-2\text{Ric}(g_1) + P_{h_0}(g_1) - L_{\phi g}^* \psi^* g_1) + \frac{1}{\tau}(\phi g - g_1)
\]

\[
\Delta_{\phi g}(\phi g - g_1) + \frac{1}{\tau}(\phi g - g_1) + U(\phi g - g_1) + \tilde{F}(\phi g, g_1),
\]

where \( |\tilde{F}(\phi g, g_1)|_{k,\alpha} \leq C(|\eta|_{k,\alpha}|\nabla^2 \eta|_{k-2,\alpha} + |\nabla \eta|_{k,\alpha}^2) \), by a similar computation to a computation in [3]. Furthermore, \( \Delta_{\phi g} \eta = \Delta_{h_0} \eta + (\Delta_{\phi g} - \Delta_{h_0}) \eta \) and since \( |\phi g - h_0|_{k,\alpha} \leq C(|\eta|_{k,\alpha} + |g_1 - h_0|_{k,\alpha}) \), we have that \( |(\Delta_{\phi g} - \Delta_{h_0}) \eta| \leq C(|\eta|_{k,\alpha} + |g_1 - h_0|_{k,\alpha})|\nabla^2 \eta|_{k,\alpha} \). The Lemma 19 now follows.

We assume that \( |g_1 - h_0|_{k,\alpha} < \epsilon \). Let \( k \) be a solution to (35). Then we have the following Proposition.

Proposition 20. There exists \( \epsilon_0 > 0 \), depending on the uniform bounds on the geometries \( g(t) \), such that if \( \epsilon < \epsilon_0 \), then if

\[
\sup_{[L,2L]} |k| \geq \beta \sup_{[0,L]} |k|,
\]

then

\[
\sup_{[2L,3L]} |k| \geq \beta \sup_{[L,2L]} |k|,
\]

and if

\[
\sup_{[2L,3L]} |k| \leq \beta^{-1} \sup_{[L,2L]} |k|.
\]
then
\begin{equation}
\sup_{[L, 2L]} |k| \leq \beta^{-1} \sup_{[0, L]} |k|,
\end{equation}

Moreover, if \((\pi k)_0 = 0\), at least one of (38), (40) holds.

Proof. Assume there exist a sequence of gauges \(\phi_i\) and constants \(\tau_i \to 0\), such that \(|\eta_i|_{k, \alpha} = |\phi_i g - h|_{k, \alpha} \leq \tau_i \to 0\), but for which none of the assertions in Proposition 20 holds. Let \(\psi_i = \frac{m}{\sup_{[L, 2L]} |\eta_i|} \). Then in view of Lemma 16 from standard compactness results (as in [3]) we get that for a subsequence \(\psi_i \to \psi\) and
\[ \frac{d}{dt} \psi = \Delta h \psi + U(\psi) + \frac{1}{\tau} \psi, \]
where \(\psi\) has a property that contradicts Lemma 18. Recall that \(\beta\) is of order \(e^{\frac{1}{4}}\).

Proof of Theorem 7. We will adopt the notation from above. Take some \(L > 0\) big enough (we will see later how big we want to make it) and choose \(\epsilon_0 > 0\) as in Theorem 12 so that the Theorem holds for \(\epsilon_0\), and \(3L\). For every \(\epsilon < \epsilon_0\) there exists \(i_0\) such that for every \(i \geq i_0\) there exists a gauge \(\phi\) so that \(\phi\) satisfies all the conditions in Theorem 12 that is \(\phi g\) is in a standard form around \(h_0\), \(|\phi g - h_0|_{k, \alpha} < \epsilon\) and \(\frac{d}{dt}\phi g| < \epsilon\) on \(M \times [t_i, t_i + 3L]\), where \(\epsilon\) is comparable to \(\epsilon\). For each \(t_i\) pick up the largest possible \(L'\) (we will omit emphasizing a dependence of \(L'\) on \(i\) and we will call it just \(L'\), since it is irrelevant for further discussion) such that \((**\) \(\phi\) is defined on \(M \times [t_i, t_i + L']\), \(\phi g\) is in a standard form around \(h\) and \(|\phi g - h|_{k, \alpha} < \epsilon\) and \(\min_{[t_i, t_i + 3L]} |\phi g - h|_{k, \alpha} < \frac{\epsilon}{1000}\). Divide \([t_i, t_i + L']\) into the subintervals of length \(L\) and assume that \(N\) is the largest number such that \([t_i + (N - 1)L, t_i + NL] \subset [t_i, t_i + L']\).

Notice that for \(L\) chosen above, from the proof of Theorem 12 all the estimates that we have got on \(|\phi - \text{Id}|_{k, \alpha}\) in the previous subsection depend on a polynomial in \(L\) (call it \(q(L)\)), whose coefficients depend only on a dimension, an integer \(k\) and the uniform bounds on geometries \(g(t)\). By the
estimates established in Proposition 7, we can increase $i_0$ if necessary, so that

1. For every $i \geq i_0$ we can find a gauge on $M \times [t_i, t_i + 3L]$, such that
\[ \sup_{[t_i, t_i + 3L]} |\phi^g(t) - h_0| < \frac{\epsilon}{1000 e^{L/\delta}}. \]

2. If the initial data $\phi(s)$ is such that $|\phi(s) - \text{Id}| < \frac{\epsilon}{e^{L/\delta}}$ and $|\phi(s)^* g(s) - h_0| < \frac{\epsilon}{e^{L/\delta}}$, where $s \in [t_i, t_i + L']$, for $i \geq i_0$, then $\phi$ can be extended to interval $[s, s + 3L]$ such that $\sup_{[s, s + 3L]} |\phi^g - h_0| < \frac{\epsilon}{100 e^{L/\delta}}$ (we might need increase $i_0$ for this to hold). Polynomial $p(L)$ can be any polynomial with leading coefficient 1 and with a degree that is e.g. one more than a degree of $q(L)$.

3. If the initial data is such that $|\phi(s) - \text{Id}| < \frac{\epsilon}{p(L)}$ and $|\phi(s)^* g(s) - h_0| < \frac{\epsilon}{p(L)}$, where $s \in [t_i, t_i + L']$, for $i \geq i_0$, then $\phi$ can be extended on interval $[s, s + 3L]$ such that $\sup_{[s, s + 3L]} |\phi^g - h_0| < \epsilon$.

We want to show that there exists $i$ (for sufficiently big $L$, so that above holds) such that a corresponding $L' = \infty$. Assume that for all $i \geq i_0$ and all $\epsilon > 0$, $L' < \infty$. Denote by $I_j = [t_i + jL, t_i + jL + L]$. Assume that $\epsilon$ is small enough so that we can apply Lemma 15 that is for every $j$ there exists a soliton $g_j$ such that $(\pi (\phi^g - g_j))_0 = 0$ on $I_j$ and therefore by Proposition 20, $\phi^g - g_j$ either satisfies a growth condition $(37) \Rightarrow (38)$ or a decay condition $(39) \Rightarrow (40)$. Moreover, $|g_j - h_0| \leq 2 \sup_{I_j} |\pi (\phi^g - h_0)| \leq C \sup_{I_j} |\phi^g - h_0|$. We need to consider two cases.

**Case 1.** Assume that for all $i_0$ and all $i \geq i_0$, where $i_0 = i_0(L)$ is chosen as in Theorem 12 for $L$ big enough (so that (1), (2) and (3) hold), and for all the intervals $I_j$ (that are defined with respect to $t_j$; we want to omit double indices) for which we have $\sup_{I_j} |\phi^g - h_0| \leq \frac{\epsilon}{1000 p(L)}$, $\phi^g - g_j$ satisfies a decay condition on $I_j$ (recall that $L^2$ norms are considered in a growth and a decay condition).
By using Proposition 20 inductively, we get that
\[
\sup_{I_l} |\phi g - g_j| \leq \frac{1}{\beta^l} \sup_{I_1} |\phi g - g_j|,
\]
for all \( l \leq j \). Moreover, \( \sup_{I_l} |\phi g - g_j| \leq \sup_{I_1} |\phi g - h_0| + |g_j - h_0| \leq \sup_{I_1} |\phi g - h_0| + 2 \sup_{I_j} |\phi g - h_0| < \frac{3\epsilon}{100p(L)}, \) which yields
\[
\sup_{I_l} |\phi g - g_j| \leq \frac{1}{\beta^l} \frac{3\epsilon}{100p(L)}.
\]

By Lemma 10 we may assume that \( \sup_{I_l} |\phi g - g_j|_{k+2,\alpha} \leq \frac{\epsilon}{\beta^i} \). Whenever we increase \( L \) (the necessity for \( L \) being increased will depend only on the uniform estimates), we can choose an appropriate \( \epsilon_0 \) as in Theorem 12 and take any \( \epsilon < \epsilon_0 \). Each time we do that we might have to increase \( i_0 \) (depending on \( \epsilon < \epsilon_0 \)). Therefore, on \( M \times I_l \), for \( l \leq j \) we have

\[
|\frac{d}{dt}\phi g|_{k,\alpha} = |\frac{d}{dt}(\phi g - g_j)|_{k,\alpha}
\]
\[= (-2\text{Ric}(\phi g) + 2\text{Ric}(g_j)) + \frac{1}{\tau}(\phi g - g_j) + (P_{h_0}(\phi g) - P_{h_0}(g_j)) + \mathcal{L}_{\psi} \frac{d}{dt} \psi^{-1}(g_j - \phi g)
\]
\[\leq C \sup_{I_l} |\phi g - g_j|_{k+2,\alpha} < C \frac{\epsilon}{\beta^l}.
\]

For every \( l \leq j \), since \( \frac{d}{dt}\phi g = \frac{d}{dt}(\phi g - h_0) \), we have that
\[
\sup_{I_l} |\phi g - h_0|_{k,\alpha} \leq 2L \sup_{I_1 \cup I_{l-1}} \sup_{I_{l-1}} |\frac{d}{dt}\phi g|_{k,\alpha} + \sup_{I_{l-1}} |\phi g - h_0|_{k,\alpha}
\]
\[\leq 2LC \frac{\epsilon}{\beta^{l-1}} + 2LC \frac{\epsilon}{\beta^{l-2}} + \cdots + 2LC \frac{\epsilon}{\beta} + \sup_{I_2} |\phi g - h_0|_{k,\alpha}
\]
\[\leq \sup_{I_2} |\phi g - h_0|_{k,\alpha} + \frac{2LC\epsilon}{\beta - 1},
\]
which can be made smaller than \( \frac{\epsilon}{\beta^{j+1}} \) for \( L \) chosen big enough at the beginning. By condition 2 for big values of \( i \) we can extend \( \phi \) on \( I_{j+1} \) so that \( \sup_{I_{j+1}} |\phi g - h_0| < \frac{\epsilon}{100p(L)} \) and it has to coincide with our previously constructed \( \phi \) on \( I_{j+1} \). We can continue a described procedure by looking now
at intervals $I_j$ and $I_{j+1}$ replaced by intervals $I_{j+1}$ and $I_{j+2}$ respectively. If we repeat this sufficiently many times, we will reach the interval $I_{N-1}$ with

$$\sup_{I_{N-1}} |\phi g - h_0|_{k,\alpha} < \frac{\epsilon}{100p(L)}.$$ 

By condition 3, we will now be able to extend $\phi$ (for sufficiently big values of $i$) to interval $[t_i + (N-1)L, t_i + (N+1)L]$, with $\sup_{[t_i+(N-1), t_i+(N+1)L]} |\phi g - h_0| < \epsilon$ holding. Since $(N+1)L > L'$, this estimate contradicts a maximality of $L'$ with properties (**) Therefore, either there exists $i$ such that a corresponding $L' = \infty$, or we have a following case holding.

**Case 2.** There are some $L, i$ and $j$ for which $\sup_{L_j} |\phi g - h_0|_{k,\alpha} < \frac{\epsilon}{100p(L)}$, and $\phi g - h_0$ satisfies a growth condition on $I_j$ ($I_j$ is defined with respect to $t_i$).

By using Proposition 20 inductively, we would have that

$$\sup_{I_{N-1}} |\phi g - g_j| < \frac{1}{\beta} \sup_{I_N} |\phi g - g_j|$$

$$\leq \frac{1}{\beta} (\sup_{I_N} |\phi g - h_0| + |g_j - h_0|)$$

$$\leq \frac{1}{\beta} (\sup_{I_N} |\phi g - h_0| + 2 \sup_{I_j} |\phi g - h_0|)$$

$$< \frac{3\epsilon}{\beta}.$$ 

Moreover, if we use Lemma 16 together with the estimate

$$\sup_{I_{N-1}} |\phi g - h_0| \leq \sup_{I_{N-1}} |\phi g - g_j| + |g_j - h_0|$$

$$\leq \sup_{I_{N-1}} |\phi g - g_j| + 2 \sup_{I_j} |\phi g - h_0|$$

$$< \frac{3\epsilon}{\beta} + \frac{\epsilon}{100p(L)},$$

which can be made smaller than $\frac{\epsilon}{p(L)}$, by condition 3, we can extend $\phi$ to an interval $[t_i + (N-1)L, t_i + (N+1)L]$ (if $i$ is big enough), with $|\phi g - h_0|_{k,\alpha} < \epsilon$
holding. We again get a contradiction as in the previous case if we assume
$L' < \infty$ for all $i$.

Therefore, there exists $i_0$ such that a gauge $\phi$ can be constructed on
$M \times [t_{i_0}, t_{i_0} + L')$, satisfying properties (***) and such that a corresponding
$L' = \infty$. Consider again $I_j = [t_{i_0} + jL, t_{i_0} + jL + L]$ and the corresponding
gauge $g_j$ that are found by Lemma 15 such that for $k_j = \phi g - g_j$ we have that
$(\pi k_j)_{i_0} = 0$ on $M \times I_j$. Notice that a decay condition (39 $\Rightarrow$ 40) holds for
all $j$. If there existed some $j$ for which it were not true, by using Proposition
20 inductively and standard parabolic estimates (Lemma 16), we would find
that

$$
\epsilon > \sup_{[t_{i_0} + (N-1)L, t_{i_0} + NL]} |\phi g - g_j| \geq \beta^{N-j} \sup_{I_j} |\phi g - g_j|,
$$

for all $N$ and we would get a contradiction by letting $N$ tend to infinity (if
$\sup_{I_j} |\phi g - g_j| = 0$, our metric $g(t_{i_0} + jL)$ would be a soliton satisfying (23)
and it would stay so for all later times which is not an interesting case). This
means we have a decay for all times if we do not start with a soliton.

After passing to a subsequence, we may assume that for some metric
g$\infty$ that satisfies a soliton type equation

$$
\lim_{p \to \infty} \sup_{I_{jp}} |k_{jp}|_{k, \alpha'} = 0.
$$

Claim 21. $\lim_{p \to \infty} \sup_{I_{jp}} |k_{jp}|_{k, \alpha} = 0$.

Proof. If it were not the case, there would exist a subsequence of $j_p$ (denote
it by the same symbol) such that $\phi g - g_{j_p}$ would satisfy a growth condition,
that is

$$
\sup_{[t_{i_0} + (N-1)L, t_{i_0} + NL]} |\phi g - g_{j_p}| \geq \beta^{(N-j_p)} \sup_{I_{jp}} |\phi g - g_{j_p}|,
$$

for all $N$, where $\beta$ can be taken to be $e^{\frac{L^2}{4}}$ and by taking $N \to \infty$ we
immediately get a contradiction, since $\sup_{[t_{i_0} + (N-1)L, t_{i_0} + NL]} |\phi g - g_{j_p}| < C\epsilon$.  

30
As in the proof of the claim above, we get that \( \phi g - g_{jp} \) has to satisfy a decay condition for all \( p \). By Claim 21, by using Proposition 20 inductively and by standard parabolic estimates (Lemma 16) we find that for some \( c > 0 \),

\[
|\phi g - g_\infty|_{k,\alpha} \leq ce^{-\frac{5(t-t_{i0})}{4}},
\]

for \( t \in [t_{i0} + (N - 1)L, t_{i0} + NL] \) and for all \( N > 0 \), that is

\[
|\phi g - g_\infty|_{k,\alpha} \leq Ce^{-ct}, \tag{41}
\]

for all \( t \geq t_{i0} \). (41) implies that \( |g(t) - \phi^{-1}g_\infty|_{C^0} < Ce^{-ct} \). \( \phi^{-1}g_\infty \) is a soliton that moves by diffeomorphisms \( \phi(t)^{-1} \) and therefore is determined by metric \( \phi^{-1}(t_{i0})g_\infty \). Since \( h_0 \) is a limit soliton of metrics \( g(t_i) \), \( h_0 \) and \( \phi^{-1}(t_{i0})g_\infty \) differ only by a diffeomorphism, that is \( \eta \phi^{-1}(t_{i0})\phi(t) \) for some diffeomorphism \( \eta \). Let finally \( \phi' = \eta \phi^{-1}(t_{i0})\phi(t) \). Then,

\[
|\phi' g(t) - h_0|_{k,\alpha} < Ce^{-ct},
\]

that is \( \phi' g(t) \) converges to a soliton \( h_0 \) exponentially as \( t \to \infty \). We know that \( h(t) = \psi(t)h_0 \) and therefore,

\[
|\psi \phi' g(t) - h(t)|_{C^0} \leq Ce^{-ct}.
\]

This finishes the proof of Theorem 1. □

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