Line Solutions for the Euler and Euler-Poisson Equations with Multiple Gamma Law

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Abstract

In this paper, we study the Euler and Euler-Poisson equations in $\mathbb{R}^N$, with multiple $\gamma$-law for pressure function:

$$P(\rho) = e^s \sum_{j=1}^m \rho^{\gamma_j},$$

(1)

where all $\gamma_{i+1} > \gamma_i \geq 1$, is the constants. The analytical line solutions are constructed for the systems. It is novel to discover the analytical solutions to handle the systems with mixed pressure function. And our solutions can be extended to the systems with the generalized multiple damping and pressure function.

Key Words: Multiple Gamma Law, Euler Equations, Euler-Poisson Equations, Analytical Solutions, Navier-Stokes Equations, Global Solutions, External forces, Free Boundary, Multiple Damping

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1 Introduction

The $N$-dimensional Euler equations can be formulated as the follows:

\[
\begin{aligned}
\rho_t + \nabla \cdot (\rho \vec{u}) &= 0, \\
\rho [\vec{u}_t + (\vec{u} \cdot \nabla) \vec{u}] + \nabla P(\rho) &= -\delta \rho \nabla \Phi - \rho \vec{F}(t), \\
S_t + \vec{u} \cdot \nabla S &= 0, \\
\Delta \Phi(t, \vec{x}) &= \alpha(N) \rho,
\end{aligned}
\]

(2)

with $\vec{x} = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N$,

and $\rho = \rho(t, \vec{x}), \vec{u}(t, \vec{x}) \in \mathbb{R}^N$ and $S = S(t)$ are the density, the velocity and the entropy respectively.

And $\alpha(N)$ is a constant related to the unit ball in $\mathbb{R}^N$: $\alpha(1) = 2; \alpha(2) = 2\pi$ and for $N \geq 3$,

\[
\alpha(N) = N(N - 2)Vol(N) = N(N - 2)\frac{\pi^{N/2}}{\Gamma(N/2 + 1)},
\]

(3)

where $Vol(N)$ is the volume of the unit ball in $\mathbb{R}^N$ and $\Gamma$ is a Gamma function.

When $\delta = 1$, the system can model fluids that are self-gravitating, such as gaseous stars. For $N = 3$, the equations (2) are the classical (non-relativistic) descriptions of a galaxy in astrophysics. See [1] and [2], for details about the systems.

When $\delta = -1$, the system is the compressible Euler-Poisson equations with repulsive forces. The equation (2), is the Poisson equation through which the potential with repulsive forces is determined by the density distribution of the electrons. In this case, the system can be viewed as a semiconductor model. See [3] and [10] for detailed analysis of the system.

When $\delta = 0$, the potential forces are ignored. The system is called the Euler equations. See [4] and [13] for detailed analysis of the system.

Here $P = P(\rho)$ is the pressure, the $\gamma$-law on the pressure for the single gas, i.e.

\[
P(\rho) = e^{\gamma} \rho^\gamma,
\]

(4)

is a universal hypothesis. The constant $\gamma = c_p/c_v \geq 1$, where $c_p$ and $c_v$ are the specific heats per unit mass under constant pressure and constant volume respectively, is the ratio of the specific heats. In particular, the fluid is called isothermal if $\gamma = 1$. More generally, the pressure function
of mixed gases, can be expressed by the multiple $\gamma$-law ([11], [14], and [15]):

$$P(\rho) = P_1 + P_2 + \ldots P_N = e^\gamma \sum_{j=1}^{m} \rho \gamma_j,$$

(5)

where all $\gamma_{j+1} > \gamma_j \geq 1$, is the constant, $m$ is a positive integer.

The system with the multiple $\gamma$-law can reflect the better approximation of the real situations. For example, the fluids in the stars mix with many types of gases in models of gaseous stars in astrophysics [15].

Here the time-dependent external force $\vec{F}(t) = (F_1(t), F_2(t), \ldots, F_N(t)) \in C^0$ are coupled in the systems.

The system (2) can be rewritten in scalar form,

$$\begin{aligned}
\rho \left( \frac{\partial \rho}{\partial t} + \sum_{k=1}^{N} u_k \frac{\partial \rho}{\partial x_k} + \rho \sum_{k=1}^{N} \frac{\partial u_k}{\partial x_k} \right) = 0, \\
\rho \left( \frac{\partial \rho}{\partial t} + \sum_{k=1}^{N} u_k \frac{\partial \rho}{\partial x_k} + \frac{\partial}{\partial x_i} \left( e^S \sum_{j=1}^{m} \rho \gamma_j \right) \right) = -\rho \frac{\partial \Phi}{\partial x_i} + \rho F_i(t), \text{ for } i = 1, 2, \ldots, N, \\
S_t + \sum_{k=1}^{N} u_k \frac{\partial S}{\partial x_k} = 0.
\end{aligned}$$

(6)

For the Euler equations (6), ($\delta = 0$), with $\vec{F}(t) = 0$, in radial symmetry:

$$\begin{aligned}
\rho(t, \vec{x}) = \rho(t, r) \quad \text{and} \quad \vec{u} = \frac{\vec{x}}{r} V(t, r) := \frac{\vec{x}}{r} V,
\end{aligned}$$

(7)

with $r = \left( \sum_{i=1}^{N} x_i^2 \right)^{1/2}$,

there exists a family of solutions for the Euler equations (6), ($\delta = 0$), without the external force ($\vec{F} = 0$), for $\gamma > 1$, [9]

$$\begin{aligned}
\rho(t, r) &= \begin{cases} 
\frac{y(r/a(t))^{1/(\gamma-1)}}{a(t)^N}, \text{ for } y(\frac{r}{a(t)}) \geq 0; \\
0, \text{ for } y(\frac{r}{a(t)}) < 0 
\end{cases}, \\
V(t, r) &= \frac{a(t)}{a(t)} r, \quad S(t, r) = \ln K, \\
\dot{a}(t) &= \frac{-\lambda}{a(t)^{1+N(\gamma-1)}}, \quad a(0) = a_0 > 0, \quad \dot{a}(0) = a_1, \\
y(x) &= (\gamma-1)\lambda 2^{2\gamma-1} x^2 + \alpha^{\gamma-1},
\end{aligned}$$

(8)

where $K$ is a positive number,

for $\gamma = 1$, [18]

$$\begin{aligned}
\rho(t, r) &= \frac{y(r/a(t))}{a(t)^N}, \quad V(t, r) = \frac{\dot{a}(t)}{a(t)} r, \quad S(t, r) = \ln K, \\
\dot{a}(t) &= \frac{-\lambda}{a(t)}, \quad a(0) = a_0 > 0, \quad \dot{a}(0) = a_1, \\
y(x) &= \frac{\lambda}{2K} x^2 + \alpha,
\end{aligned}$$

(9)
where $\lambda$, $\alpha$, $a_0$ and $a_1$ are constants.

The separation method for analytical solutions, were used to handle other similar systems with single $\gamma$ functions:

$$P(\rho) = e^\rho^\gamma,$$  \hspace{1cm} (10)

or without pressure function, in [7], [5], [9], [12], [16], [17], [18], [19] and [20].

It is very natural to extend the results for the system with multiple $\gamma$ function (6):

$$P(\rho) = e^{\sum_{j=1}^{m} \rho^{\gamma_j}},$$  \hspace{1cm} (11)

In this article, we have obtained a class of line solutions to the Euler equations (6) ($\delta = 0$) and Euler-Poisson equations (6) ($\delta = \pm 1$), with multiple Gamma function, in the following theorems:

**Theorem 1** For the Euler equations with multiple Gamma function, ($\delta = 0$) (6), we have the family of the solutions,

$$\begin{align*}
\rho(t, \vec{x}) &= f \left( \sum_{i=1}^{N} C_i (x_i - a_i(t)) \right), \\
\vec{u}(t, \vec{x}) &= (\dot{a}_1(t), \dot{a}_2(t), ..., \dot{a}_N(t)), \\
S(t, \vec{x}) &= \ln \left[ g \left( \sum_{i=1}^{N} C_i (x_i - a_i(t)) \right) \right],
\end{align*}$$  \hspace{1cm} (12)

where

$$\begin{align*}
\dot{a}_i(t) &= F_i(t) + C_i \xi, \quad \text{for} \quad i = 1, 2, ..., N, \\
a(0) &= a_0, \quad \dot{a}(0) = a_1,
\end{align*}$$  \hspace{1cm} (13)

(1) for $\gamma_1 > 1$, with

$$\begin{align*}
\xi + \dot{g}(z) \sum_{j=1}^{m} f(z)^{\gamma_j-1} + g(z) \sum_{j=1}^{m} \gamma_j f(z)^{\gamma_j-2} \dot{f}(z) &= 0, \\
g(z) &> 0, \quad \text{for} \quad z \in (-\infty, \infty),
\end{align*}$$  \hspace{1cm} (14)

where $C_1, C_2, ..., C_N, \xi, g_0$ and $g_1$ are arbitrary constants; and $f \geq 0$ is an arbitrary $C^1$ function;

(2) for $\gamma_1 = 1$, with

$$\begin{align*}
\xi + \rho C_i \dot{g}(z) + g(z) \frac{\dot{f}(z)}{f(z)} C_i + \dot{g}(z) \sum_{j=2}^{m} f(z)^{\gamma_j-1} + g(z) \sum_{j=2}^{m} \gamma_j f(z)^{\gamma_j-2} \dot{f}(z) &= 0, \\
g(z) &> 0, \quad \text{for} \quad z \in (-\infty, \infty),
\end{align*}$$  \hspace{1cm} (15)

$f > 0$ is an arbitrary $C^1$ function.
We notice that the velocity \( \bar{u}(t, \bar{x}) \) of the solutions (12) and (18), are only time-dependent functions:

\[
\bar{u}(t, \bar{x}) = (\bar{a}_1(t), \bar{a}_2(t), \ldots, \bar{a}_N(t)).
\]

It is different from the conventional velocity for the analytical solutions:

\[
\tilde{u}(t, \tilde{x}) = \frac{\dot{a}(t)}{a(t)} \tilde{x}.
\]

Moreover, we need some modification to have the corresponding results for the Euler-Poisson equations.

**Theorem 2** For the Euler-Poisson equations with multiple Gamma function, \( (\delta = \pm 1) \), (10), we have the family of the solutions,

\[
\begin{aligned}
\rho(t, \bar{x}) &= \frac{\sum_{i=1}^{N} C_i^2}{\alpha(N)} \tilde{f} \left( \sum_{i=1}^{N} C_i \left( x_i - a_i(t) \right) \right), \\
\bar{u}(t, \bar{x}) &= (\bar{a}_1(t), \bar{a}_2(t), \ldots, \bar{a}_N(t)), \\
S(t, \bar{x}) &= \ln \left[ \tilde{g} \left( \sum_{i=1}^{N} C_i \left( x_i - a_i(t) \right) \right) \right]
\end{aligned}
\]

where

\[
\begin{aligned}
\bar{a}_i(t) &= F_i(t) + C_i \xi + d_i(t), \text{ for } i = 1, 2, \ldots, N, \\
a(0) &= a_0, \quad \dot{a}(0) = a_1,
\end{aligned}
\]

(1) for \( \gamma_1 > 1 \), with

\[
\begin{aligned}
\xi + \frac{\alpha(N)}{\sum C_i^2} \dot{g}(z) \sum_{j=1}^{m} \bar{f}(z) \gamma_j^{-1} + \frac{\alpha(N)}{\sum C_i^2} g(z) \sum_{j=1}^{m} \gamma_j \bar{f}(z) \gamma_j^{-2} \dot{\bar{f}}(z) + \delta \ddot{\bar{f}}(z) &= 0, \\
g(z) &= 0, \text{ for } z \in (-\infty, \infty),
\end{aligned}
\]

where \( C_1, C_2, \ldots, C_N, \xi, g_0 \) and \( g_1 \) are arbitrary constants with \( \sum_{i=1}^{N} C_i^2 > 0 \); \( \bar{f} \geq 0 \) is an arbitrary \( C^3 \) function; and \( d_1, d_2, \ldots, d_N \) are arbitrary \( C^3 \) functions;

(2) for \( \gamma_1 = 1 \), with

\[
\begin{aligned}
\xi + \frac{\alpha(N)}{\sum C_i^2} \dot{g}(z) + \frac{\alpha(N)}{\sum C_i^2} g(z) \ddot{\bar{f}}(z) + \frac{\alpha(N)}{\sum C_i^2} \dot{g}(z) \frac{\dot{\bar{f}}(z)}{\bar{f}(z)} \sum_{j=2}^{m} \bar{f}(z) \gamma_j^{-1} + \frac{\alpha(N)}{\sum C_i^2} g(z) \sum_{j=2}^{m} \gamma_j \bar{f}(z) \gamma_j^{-2} \dot{\bar{f}}(z) + \delta \ddot{\bar{f}}(z) &= 0, \\
g(z) &= 0, \text{ for } z \in (-\infty, \infty),
\end{aligned}
\]

\( \ddot{\bar{f}} > 0 \) is an arbitrary \( C^1 \) function.
Remark 3 The mass of the solutions (12) for the Euler equations, and (18) for the Euler-Poisson equations, in 1-dimensional case, is finite, if

$$\int_{-\infty}^{\infty} f(z) \, dz < \infty,$$

and

$$\int_{-\infty}^{\infty} \ddot{f}(z) \, dz < \infty,$$

respectively.

2 Line Solutions

With regard to the continuity equation of mass (2), we found the following solution structures of the below lemmas fit it well:

Lemma 4 For the mass equation:

$$\rho_t + \nabla \cdot (\rho \vec{u}) = 0,$$

there exist solutions,

$$\rho(t, \vec{x}) = f \left( \sum_{i=1}^{N} C_i (x_i - a_i(t)) \right), \quad \vec{u}(t, \vec{x}) = \vec{u} = (\dot{a}_1(t), \dot{a}_2(t), ..., \dot{a}_N(t)),$$

with the form arbitrary \( f \geq 0 \in C^1 \) and arbitrary \( a_i(t) \in C^1 \).

Proof. For the mass equation, we have

$$\rho_t + \nabla \rho \cdot \vec{u} + \rho \nabla \cdot \vec{u}$$

$$= \frac{\partial}{\partial t} f \left( \sum_{i=1}^{N} C_i (x_i - a_i(t)) \right) + \nabla f \left( \sum_{i=1}^{N} C_i (x_i - a_i(t)) \right) \cdot (\dot{a}_1(t), \dot{a}_2(t), ..., \dot{a}_N(t))$$

$$= f \left( \sum_{i=1}^{N} C_i (x_i - a_i(t)) \right) \left( - \sum_{i=1}^{N} C_i \dot{a}_i(t) \right)$$

$$+ f \left( \sum_{i=1}^{N} C_i (x_i - a_i(t)) \right) \left( C_1 \dot{a}_1(t) + C_2 \dot{a}_2(t) + ... + C_N \dot{a}_N(t) \right)$$

$$= f \left( \sum_{i=1}^{N} C_i (x_i - a_i(t)) \right) \left( - \sum_{i=1}^{N} C_i \dot{a}_i(t) + \sum_{i=1}^{N} C_i \dot{a}_i(t) \right)$$

$$= 0.$$

The proof is completed. ■
Similarly, we have the corresponding lemma for the entropy equation (2):\n
**Lemma 5** For the entropy equation:

\[ S_t + \vec{u} \cdot \nabla S = 0, \]  

there exist solutions,

\[ S(t, \vec{x}) = G \left( \sum_{i=1}^{N} C_i (x_i - a_i(t)) \right), \quad u(t, \vec{x}) = \vec{\dot{a}} = (\dot{a}_1(t), \dot{a}_2(t), ..., \dot{a}_N(t)), \]  

with the form \( f \geq 0 \in C^1 \) and \( a_i(t) \in C^1 \).

**Proof.** For the mass equation, we can obtain:

\[
S_t + \vec{u} \cdot \nabla S = \frac{\partial}{\partial t} G \left( \sum_{i=1}^{N} C_i (x_i - a_i(t)) \right) + (\dot{a}_1(t), \dot{a}_2(t), ..., \dot{a}_N(t)) \cdot \nabla G \left( \sum_{i=1}^{N} C_i (x_i - a_i(t)) \right)
\]

\[
= \dot{G} \left( \sum_{i=1}^{N} C_i (x_i - a_i(t)) \right) \left( - \sum_{i=1}^{N} C_i \dot{a}_i(t) \right)
\]

\[
+ \dot{G} \left( \sum_{i=1}^{N} C_i (x_i - a_i(t)) \right) (C_1 \dot{a}_1(t) + C_2 \dot{a}_2(t) + ... + C_N \dot{a}_N(t))
\]

\[ = 0. \]  

The proof is completed. \( \blacksquare \)

The technique of constructing solutions is to deduce the partial differential equations into ordinary differential equations only. Based on the above lemmas, it is clear to check our solutions for the system.

**Proof of Theorem 1** Our structure of the solutions (12), fits well for the mass equation (2) and the entropy equation (2), from the above lemmas. For the \( i \)-th momentum equation (6), we...
can define $z := \sum_{i=1}^{N} C_i (x_i - a_i(t))$, to have

$$\rho \left( \frac{\partial u_i}{\partial t} + \sum_{k=1}^{N} u_k \frac{\partial u_i}{\partial x_k} \right) + \frac{\partial}{\partial x_i} \rho \sum_{j=1}^{m} \rho \gamma_j + \rho F_i(t)$$

$$= \rho \ddot{a}_i(t) + \frac{\partial}{\partial x_i} e^{\ln g(z)} \sum_{j=1}^{m} f(z)\gamma_j + \rho F_i(t)$$

$$= \rho \ddot{a}_i(t) + e^{\ln g(z)} \sum_{j=1}^{m} f(z)\gamma_j + \rho g(z) \frac{\partial}{\partial x_i} \left( \sum_{j=1}^{m} f(z)\gamma_j \right) + \rho F_i(t)$$

$$= \rho \ddot{a}_i(t) + \rho C_i \dot{g}(z) \sum_{j=1}^{m} f(z)\gamma_j - 1 + g(z) \sum_{j=1}^{m} \gamma_j f(z)\gamma_j - 1 \dot{f}(z) C_i + \rho F_i(t)$$

$$= \rho C_i \left\{ \xi + \dot{g}(z) \sum_{j=1}^{m} f(z)\gamma_j - 1 + g(z) \sum_{j=1}^{m} \gamma_j f(z)\gamma_j - 2 \dot{f}(z) \right\}$$

$$= 0,$$

where we require the following ordinary differential equations:

$$\begin{cases} 
\ddot{a}_i(t) = F_i(t) + C_i \xi, & \text{for } i = 1, 2, ...N, \\
\dot{a}(0) = a_0, & \ddot{a}(0) = a_1, \\
\xi + \dot{g}(z) \sum_{j=1}^{m} f(z)\gamma_j - 1 + g(z) \sum_{j=1}^{m} \gamma_j f(z)\gamma_j - 2 \dot{f}(z) = 0, \\
g(z) > 0, & \text{for } z \in (-\infty, \infty). 
\end{cases}$$
For $\gamma_1 = 1$, we have,

$$\rho \left( \frac{\partial u_i}{\partial t} + \sum_{k=1}^{N} u_k \frac{\partial u_i}{\partial x_k} \right) + \frac{\partial}{\partial x_i} \left( e^S \rho + e^S \sum_{j=2}^{m} \rho^{\gamma_j} \right) + \rho F_i(t)$$

(46)

and

$$= \rho \ddot{a}(t) + \frac{\partial}{\partial x_i} \left( e^{\ln g(z)} \rho \right) + \frac{\partial}{\partial x_i} \left( e^{\ln g(z)} \sum_{j=2}^{m} f(z)^{\gamma_j} \right) + \rho F_i(t)$$

(47)

and

$$= \rho \ddot{a}(t) + \frac{\partial}{\partial x_i} g(z) f(z) + g(z) \frac{\partial}{\partial x_i} f(z)$$

(48)

and

$$+ \left( \frac{\partial}{\partial x_i} g(z) \right) C_i \sum_{j=2}^{m} f(z)^{\gamma_j} + g(z) \frac{\partial}{\partial x_i} \left( \sum_{j=2}^{m} f(z)^{\gamma_j} \right) + \rho F_i(t)$$

(49)

and

$$= \rho \ddot{a}(t) + \rho C_i g(z) f(z) + g(z) \dot{f}(z) C_i$$

(50)

and

$$+ \rho C_i g(z) \sum_{j=2}^{m} f(z)^{\gamma_j-1} + g(z) \sum_{j=2}^{m} \gamma_j f(z)^{\gamma_j-1} \dot{f}(z) C_i + \rho F_i(t)$$

(51)

and

$$= \rho C_i \left\{ \xi + \rho C_i \dot{g}(z) + g(z) \frac{\dot{f}(z)}{f(z)} C_i + \dot{g}(z) \sum_{j=2}^{m} f(z)^{\gamma_j-1} + g(z) \sum_{j=2}^{m} \gamma_j f(z)^{\gamma_j-2} \dot{f}(z) \right\}$$

(52)

and

$$= 0,$$

(53)

with

$$\left\{ \xi + \rho C_i \dot{g}(z) + g(z) \frac{\dot{f}(z)}{f(z)} C_i + \dot{g}(z) \sum_{j=2}^{m} f(z)^{\gamma_j-1} + g(z) \sum_{j=2}^{m} \gamma_j f(z)^{\gamma_j-2} \dot{f}(z) = 0, \right.$$

(54)

$$g(z) > 0, \text{ for } z \in (-\infty, \infty).$$

The proof is completed. 

With similar analysis, we can derive the corresponding theorem for the Euler-Poisson equations.

**Proof of Theorem 2** We can use the previous lemmas again to handle the mass equation and entropy equation again. Then, we can define the potential function as:

$$\Phi(t, \bar{x}) = f(z) + \sum_{i=1}^{N} d_i(t)x_i,$$

(55)

where $z := \sum_{i=1}^{N} C_i \left( x_i - a_i(t) \right)$.

We differentiate (55) twice to obtain:

$$\Delta \Phi(t, \bar{x}) = \ddot{f}(z) \sum_{i=1}^{N} C_i^2 = \alpha(N) \rho,$$

(56)
where in our solution

\[
\rho = \frac{\sum_{i=1}^{N} C_i^2}{\alpha(N)} \tilde{f}(z). \tag{57}
\]

For the \(i\)-th momentum equation (6), we have:

\[
\rho \left( \frac{\partial u_i}{\partial t} + \sum_{k=1}^{N} u_k \frac{\partial u_i}{\partial x_k} \right) + \frac{\partial}{\partial x_i} \left( e^S \sum_{j=1}^{m} \rho^{\gamma_j} \right) + \delta \rho \frac{\partial}{\partial x_i} \Phi + \rho F_i(t) \tag{58}
\]

\[
= \rho \ddot{a}_i(t) + \frac{\partial}{\partial x_i} \left( e^{\ln g(z)} \sum_{j=1}^{m} \tilde{f}(z)^{\gamma_j} \right) + \delta \rho \frac{\partial}{\partial x_i} \left( f(z) + \sum_{i=1}^{N} d_i(t) x_i \right) + \rho F_i(t) \tag{59}
\]

\[
= \rho \ddot{a}_i(t) + C_i \dot{g}(z) \sum_{j=1}^{m} \tilde{f}(z)^{\gamma_j} + g(z) \frac{\partial}{\partial x_i} \sum_{j=1}^{m} f(z)^{\gamma_j} + \delta \rho \left( \dot{f}(z) C_i + d_i(t) \right) + \rho F_i(t) \tag{60}
\]

\[
= \rho \ddot{a}_i(t) + \rho C_i \dot{g}(z) \sum_{j=1}^{m} \tilde{f}(z)^{\gamma_j} + g(z) \sum_{j=1}^{m} \gamma_j \tilde{f}(z)^{\gamma_j - 1} \dot{f}(z) C_i \tag{61}
\]

\[
+ \delta \rho \left( \dot{f}(z) C_i + d_i(t) \right) + \rho F_i(t) \tag{62}
\]

\[
= \rho C_i \left\{ \xi + \frac{\alpha(N)}{\sum_{i=1}^{N} C_i^2} \dot{g}(z) \sum_{j=1}^{m} \tilde{f}(z)^{\gamma_j - 1} + \frac{\alpha(N)}{\sum_{i=1}^{N} C_i^2} g(z) \sum_{j=1}^{m} \gamma_j \tilde{f}(z)^{\gamma_j - 2} \dot{f}(z) + \delta \ddot{f}(z) \right\} \tag{63}
\]

\[
= 0, \tag{64}
\]

where we require the following ordinary differential equations:

\[
\begin{align*}
\ddot{a}_i(t) &= F_i(t) + C_i \xi + \delta d_i(t), \text{ for } i = 1, 2, \ldots, N, \\
a(0) &= a_0, \quad \dot{a}(0) = a_1, \\
\xi + \frac{\alpha(N)}{\sum_{i=1}^{N} C_i^2} \dot{g}(z) \sum_{j=1}^{m} f(z)^{\gamma_j - 1} + \frac{\alpha(N)}{\sum_{i=1}^{N} C_i^2} g(z) \sum_{j=1}^{m} \gamma_j f(z)^{\gamma_j - 2} \dot{f}(z) + \delta \ddot{f}(z) &= 0, \tag{65}
\end{align*}
\]

\[
g(z) > 0, \text{ for } z \in (-\infty, \infty).
\]
For $\gamma_1 = 1$, we get,

$$\rho \left( \frac{\partial u_i}{\partial t} + \sum_{k=1}^{N} u_k \frac{\partial u_i}{\partial x_k} \right) + \frac{\partial}{\partial x_i} \rho \sum_{j=1}^{m} \gamma_j + \delta \rho \frac{\partial}{\partial x_i} \Phi + \rho F_i(t)$$

(66)

$$= \rho \ddot{u}_i(t) + \frac{\partial}{\partial x_i} \left( \rho \left( e^{\ln g(z)} \dot{f}(z) \right) + \frac{\partial}{\partial x_i} \left( e^{\ln g(z)} \sum_{j=1}^{m} \dot{f}(z)^{\gamma_j} \right) \right)$$

(67)

$$+ \delta \rho \frac{\partial}{\partial x_i} \left( \dot{f}(z) \sum_{i=1}^{N} d_i(t) x_i \right) + \rho F_i(t)$$

(68)

$$= \rho \ddot{u}_i(t) + C_i \dot{g}(z) \ddot{f}(z) + g(z) \frac{\partial}{\partial x_i} f(z)$$

(69)

$$+ C_i \dot{g}(z) \sum_{j=2}^{m} \ddot{f}(z)^{\gamma_j} + \rho \sum_{j=2}^{m} \frac{\partial}{\partial x_i} f(z)^{\gamma_j} + \delta \rho \left( \dot{f}(z)C_i + d_i(t) \right) + \rho F_i(t)$$

(70)

$$= \rho \ddot{u}_i(t) + C_i \dot{g}(z) \ddot{f}(z) + g(z) \frac{\partial}{\partial x_i} f(z)$$

(71)

$$\rho C_i \dot{g}(z) \sum_{j=2}^{m} \ddot{f}(z)^{\gamma_j} + g(z) \sum_{j=2}^{m} \gamma_j \ddot{f}(z)^{\gamma_j} + \frac{\partial}{\partial x_i} f(z)C_i + \delta \rho \left( \dot{f}(z)C_i + d_i(t) \right) + \rho F_i(t)$$

(72)

$$= 0,$$

(73)

where we require the following ordinary differential equations:

$$\begin{align*}
\left\{ \begin{array}{l}
\xi + \frac{\alpha(N)}{\Sigma C_i^2} \dot{g}(z) + \frac{\alpha(N)}{\Sigma C_i^2} g(z) \dot{f}(z) + \frac{\alpha(N)}{\Sigma C_i^2} \dot{g}(z) \sum_{j=2}^{m} \ddot{f}(z)^{\gamma_j} + \frac{\partial}{\partial x_i} f(z)C_i + \delta \rho \left( \dot{f}(z)C_i + d_i(t) \right) + \rho F_i(t) \\
\quad + \frac{\partial}{\partial x_i} f(z)C_i + \delta \rho \left( \dot{f}(z)C_i + d_i(t) \right) + \rho F_i(t) \\
\quad + \frac{\partial}{\partial x_i} g(z) C_i + \delta \rho \left( \dot{f}(z)C_i + d_i(t) \right) + \rho F_i(t) \\
\end{array} \right\}
\end{align*}$$

(74)

$$g(z) > 0, \text{ for } z \in (-\infty, \infty).$$

The proof is completed. \(\blacksquare\)

**Remark 6** The ordinary differential equations (72),

$$\left\{ \begin{array}{l}
\ddot{a}_i(t) = F_i(t) + C_i \xi + d_i(t), \text{ for } i = 1, 2, \ldots, N, \\
a(0) = a_0, \ \dot{a}(0) = a_1,
\end{array} \right.$$

(75)

are solved by

$$a_i(t) = \int_{0}^{t} \int_{0}^{s} \left( F_i(\eta) + d_i(\eta) \right) d\eta ds + \frac{C_i \xi t^2}{2} + a_1 t + a_0.$$

(76)
Remark 7 In fact, our solutions (12) and (13), can be easily extended to the systems, with the generalized multiple and nonlinear, damping and Gamma pressure function:

\[
\begin{aligned}
\frac{\partial \rho}{\partial t} + \sum_{k=1}^{N} u_k \frac{\partial \rho}{\partial x_k} + \rho \sum_{k=1}^{N} \frac{\partial u_k}{\partial x_k} &= 0, \\
\rho \left( \frac{\partial u_i}{\partial t} + \sum_{k=1}^{N} u_k \frac{\partial u_i}{\partial x_k} \right) + \sum_{i=1}^{n} \beta_i \rho \left[ \left( \sum_{i=1}^{N} u_i^2 \right)^{1/2} \right]^{p_i-1} u_i + \frac{\partial}{\partial x_i} \left( e^S \sum_{j=1}^{m} \lambda_j \rho^j \right) &= 0,
\end{aligned}
\]

...(77)

where \( \beta_1, \beta_2, ..., \beta_n; \lambda_1, \lambda_2, ..., \lambda_m; \) and \( p_1, p_2, ..., p_n \geq 0, \) are constants.

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