The Kerr medium as an SU(2) system

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Abstract

The Kerr medium in the presence of damping and associated with SU(1,1) symmetry, is solved using the techniques of Thermo field Dynamics (TFD). These TFD techniques, well studied earlier \[^3\] , help us to exactly solve the Kerr medium as a spin damped system associated with SU(2) symmetry. Using TFD, the association with SU(2) is exploited to express the dynamics of the system as a Schrodinger-like equation, whose solution is obtained using the appropriate disentanglement theorem.

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These considerations are extended to a system with multi-mode coupled nonlinear oscillators.

**Keywords:** Kerr Medium, Thermo field Dynamics, Master equation, SU(2) symmetry and Disentanglement theorem.

## 1 Introduction

In nonlinear optics, one of the most important processes is propagation in the Kerr medium. This medium is used extensively for the generation of nonclassical light, for example, squeezed light \[1\] which is very commonly used in quantum optics. Another type of nonclassical light generated by this medium are the cat states known as the Yurke-Stoler states \[2\].

While considering various aspects of the Kerr medium, a natural thing that arises is the presence and effect of damping. The dynamics for this system has been studied and solved in \[3\]. In this paper, the authors have also solved the problem of the Kerr medium with \( N \) coupled oscillators in the presence of damping \[4\]. In both these cases the method of thermo-field dynamics is used for solving the master equation.

The method of thermo field dynamics \[5, 6, 7, 8, 9\] is developed in finite temperature field theory. This formalism has two salient features. First, the solving of the master equation is reduced to solving a Schröedinger-type equation. Thus all the techniques available for solving the Schroedinger equation are applicable here. Second, the thermal coherent state under the master equation evolution goes over to a thermal coherent state. Thus, the application of this method in solving the master equation gives a very simple and elegant approach.

A brief description of thermofield dynamics (TFD) is given below. In TFD the state \(|\rho^{\alpha}\rangle, 1/2 \leq \alpha \leq 1\), is considered a state vector in the extended Hilbert space \( \mathcal{H} \otimes \mathcal{H}^{*} \). Then the average of any operator \( \hat{A} \) with respect to \( \rho \) reduces to a scalar product,

\[
\langle A \rangle = \text{Tr}(A\rho) = \langle \rho^{1-\alpha} | A | \rho^{\alpha} \rangle,
\]

\[1\]
where the state $|\rho^\alpha\rangle$ is given by

$$|\rho^\alpha\rangle = \hat{\rho}^\alpha |I\rangle,$$

(2)

where

$$\hat{\rho}^\alpha = \rho^\alpha \otimes I,$$

(3)

and $|I\rangle$ is the resolution of the identity

$$|I\rangle = \sum |N\rangle\langle N| = \sum |N\rangle \otimes |\bar{N}\rangle \equiv \sum |N, \bar{N}\rangle,$$

(4)

in terms of a complete orthonormal set $\{|N\rangle\}$ in $\mathcal{H}$. The state vector $|I\rangle$ takes a normalized vector to another normalized vector in the extended Hilbert space $\mathcal{H} \otimes \mathcal{H}^*$. From now on, we work in, $\alpha = 1$ representation[3, 4, 10]. In this representation, for any hermitian operator $A$, one has

$$\langle A \rangle = Tr(A\rho) = \langle A | \rho \rangle.$$  

(5)

By choosing the number state $|n\rangle$ for $|N\rangle$ we introduce the creation and annihilation operators $a^\dagger, \tilde{a}^\dagger, a, \tilde{a}$ and their actions as follows

$$a|n, m\rangle = \sqrt{n}|n-1, m\rangle, \quad a^\dagger|n, m\rangle = \sqrt{n+1}|n+1, m\rangle, \quad (6)$$

$$\tilde{a}|n, m\rangle = \sqrt{m}|n, m-1\rangle, \quad \tilde{a}^\dagger|n, m\rangle = \sqrt{m+1}|n, m+1\rangle. \quad (7)$$

Further, the operators $a$ and $a^\dagger$ commute with $\tilde{a}$ and $\tilde{a}^\dagger$. It is clear from the above that $a$ acts on the vector space $\mathcal{H}$ and $\tilde{a}$ acts on vector space $\mathcal{H}^*$. From the expression for $|I\rangle$ in terms of the number states

$$|I\rangle = \sum_{n, \bar{n}} |n, \bar{n}\rangle,$$

(8)

it follows that

$$a|I\rangle = \tilde{a}^\dagger |I\rangle, \quad a^\dagger |I\rangle = \tilde{a} |I\rangle,$$

(9)

and hence for any operator one has

$$A|I\rangle = \tilde{A}^\dagger |I\rangle,$$

(10)
where $\tilde{A}$ is obtained from $A$ by making the replacements (called tilde conjugation rules) $a \rightarrow \tilde{a}, a^\dagger \rightarrow \tilde{a}^\dagger$, along with complex numbers going to their conjugates. For a more detailed discussion on TFD see ref[3, 6, 8, 9].

1.1 The Master Equation

Given any master equation of the form

$$\frac{\partial}{\partial t} \rho(t) = -\frac{i}{\hbar}(H \rho - \rho H) + L \rho, \quad (11)$$

one converts this into a problem in TFD by applying $|I\rangle$ from the right to get

$$\frac{\partial}{\partial t}|\rho(t)\rangle = -i\tilde{H}|\rho\rangle \quad (12)$$

where

$$-i\tilde{H} = i(H - \tilde{H}) + L. \quad (13)$$

In TFD $-i\tilde{H}$ is tildian i.e. it reflects the hermiticity property of the density operator. Thus the problem of solving the master equation is reduced to solving a Schröedinger equation.

As illustration, consider the master equation for a damped linear harmonic oscillator. In TFD [3, 4, 10, 11] this is given by

$$\frac{\partial}{\partial t}|\rho(t)\rangle = \left[ \left( \frac{\kappa\bar{n} + 1}{2} \left( 2a\tilde{a} - a^\dagger \tilde{a}^\dagger \right) + \frac{\kappa\bar{n}}{2} \left( 2a^\dagger \tilde{a}^\dagger - aa^\dagger - \tilde{a}^\dagger \tilde{a} \right) \right) - i\omega(a^\dagger a - \tilde{a}^\dagger \tilde{a}) \right]|\rho(t)\rangle, \quad (14)$$

where $\kappa$, $\bar{n}$ are as defined in [3, 4, 10, 11]. This model of the master equation is considered for many dissipative systems.

We define the following operators

$$\mathcal{K}_3 = \frac{1}{2} \left( a^\dagger a + \tilde{a}^\dagger \tilde{a} + 1 \right), \quad \mathcal{K}_z = a^\dagger \tilde{a}^\dagger, \quad \mathcal{K}_- = a\tilde{a} \quad (15)$$
and note that these satisfy the $SU(1, 1)$ algebra

$$[K_3, K_+] = K_+, \quad [K_3, K_-] = K_-, \quad [K_+, K_-] = 2K_3,$$  \hfill (16)

The operator $K_- = (a^\dagger a - \tilde{a}^\dagger \tilde{a})$ is here the Casimir operator.

In terms of these $SU(1, 1)$ generators the master equation (14) becomes

$$\frac{\partial}{\partial t} |\rho(t)\rangle = -i\hat{H}|\rho\rangle = \left(-i\omega K_0 + \kappa(\bar{n} + 1)K_- + \kappa\bar{n}K_+ - \kappa(2\bar{n} + 1)K_3 + \frac{\kappa\bar{n}}{2}\right)|\rho(t)\rangle.$$  \hfill (17)

It is clear that the master equation above is like a Schrödinger equation associated with $SU(1, 1)$ symmetry. Its solution is given by the evolution operator $e^{-i\hat{H}t}$, with the $\hat{H}$ as given above. One can then apply the disentanglement theorem (for $SU(1,1)$) and solve for arbitrary initial conditions.

In this paper, we look at systems for which $SU(2)$ operators (generators) can be introduced; thus these systems will be associated with $SU(2)$ symmetry. The TFD method is used, and the Schrödinger type of equation thus obtained is considered using the disentanglement theorem for $SU(2)$.

## 2 The Kerr Medium

In this section, we consider the Kerr medium in presence of damping as a spin system. The Hamiltonian for the Kerr medium is

$$H = \omega a^\dagger a + \chi(a^\dagger a)^2$$  \hfill (18)

where $\omega$ is the frequency of the oscillator, and $\chi$ is the coupling factor which depends on the Kerr medium.

The evolution of this Hamiltonian is given by the master equation

$$\frac{\partial}{\partial t} \rho = -i[H, \rho]$$  \hfill (19)
Applying $|I\rangle$ on (19) from the right and using (9) this master equation for $\rho$ goes over to a Schrödinger-like equation for the state $|\rho\rangle$

$$\frac{\partial}{\partial t}|\rho\rangle = -i\hat{H}|\rho\rangle,$$  \hspace{1cm} (20)

where

$$-i\hat{H} = -i\omega (a^\dagger a - \tilde{a}^\dagger \tilde{a}) - i\chi [(a^\dagger a)^2 - (\tilde{a}^\dagger \tilde{a})^2]$$  \hspace{1cm} (21)

This Hamiltonian can be rewritten as

$$-i\hat{H} = -i\omega (a^\dagger a - \tilde{a}^\dagger \tilde{a}) - i\chi [(a^\dagger a + \tilde{a}^\dagger \tilde{a})(a^\dagger a - \tilde{a}^\dagger \tilde{a})]$$  \hspace{1cm} (22)

Then one can identify this Hamiltonian (22) with an $SU(1,1)$ system

$$-i\hat{H} = -i\omega K_0 - 2i\chi K_0 \left( K_3 - \frac{1}{2} \right).$$  \hspace{1cm} (23)

with $K_0$ the Casimir operator. The formalism of thermofield dynamics thus confirms the $SU(1,1)$ nature of the non-linear oscillator, with the above $K_0$ as the Casimir operator, and the $K_3$ generator as given above. This $SU(1,1)$ system has been studied in detailed in [3, 4].

It is interesting that when the roles of $K_0$ and $K_3$ are (somewhat) reversed, we have an identification with $SU(2)$, implying that the Kerr medium can be viewed as a $SU(2)$ system.

To see this we define the following operators

$$S_0 = a^\dagger a + \tilde{a}^\dagger \tilde{a}, \hspace{1cm} S_3 = \frac{(a^\dagger a - \tilde{a}^\dagger \tilde{a})}{2}, \hspace{1cm} S_+ = a^\dagger \tilde{a}, \hspace{1cm} S_- = a\tilde{a}^\dagger.$$  \hspace{1cm} (24)

These satisfy the $SU(2)$ algebra

$$[S_3, S_+] = S_+, \hspace{1cm} [S_3, S_-] = -S_-, \hspace{1cm} [S_+, S_-] = 2S_3,$$  \hspace{1cm} (25)

with $S_0 = (a^\dagger a + \tilde{a}^\dagger \tilde{a})$ as the Casimir operator. The Hamiltonian (22) for the Kerr medium is thus identified with having $SU(2)$ symmetry. In terms of these generators, we have

$$\hat{H} = \omega S_3 + \chi S_0 S_3.$$  \hspace{1cm} (26)
Hence, by adding a spin damping to this Hamiltonian (26) one has
\[ \hat{H}_D = \omega S_3 + \chi S_0 S_3 + i\gamma S_+ + i\gamma S_- - i\gamma S_3 \] (27)
where \( \gamma \) is the decay parameter of the dissipative cavity. This, when substituted in the Schrödinger equation in (20) results in its solution, which is written as
\[ |\rho(t)\rangle = \exp(\gamma_+ S_+ + \gamma_3 S_3 + \gamma_- S_-)|\rho(0)\rangle, \] (28)
where
\[ \gamma_+ = \gamma t; \quad \gamma_- = \gamma t; \quad \gamma_3 = -(i\omega + \gamma + i\chi S_0)t. \] (29)
Using the disentangling theorem \[12, 13\], one has
\[ |\rho(t)\rangle = \exp(\Gamma_+ S_+) \exp(\Gamma_3 S_3) \exp(\Gamma_- S_-) |\rho(0)\rangle, \] (30)
where
\[ \Gamma_\pm = \left( \frac{\sinh \lambda}{\lambda \cosh \lambda - \gamma_3 \sinh \lambda} \right) \gamma_\pm; \quad \Gamma_3 = \ln \left( \frac{\lambda}{\lambda \cosh \lambda - \gamma_3 \sinh \lambda} \right) \] (31)
with
\[ \lambda^2 = \gamma_3^2 + \gamma_+ \gamma_- \] (32)
To go further, we expand the state \( |\rho(0)\rangle \) in the basis of number states,
\[ |\rho(0)\rangle = \sum_{m,n=0}^{\infty} \rho_{m,n}(0)|m,n\rangle \] (33)
We substitute this expansion in (30) and use the explicit expressions for \( S_\pm, S_3 \). Taking care that the repeated actions of \( a, \tilde{a} \) on the number states \( |m,n\rangle \) is limited by the values of \( m, n \), we get the result
\[ |\rho(t)\rangle = \sum_{m,n=0}^{\infty} \rho_{m,n}(0) \sum_{r=0}^{m} \frac{(\Gamma_-)^r}{r!} \sum_{s=0}^{n} \frac{(\Gamma_+)^s}{s!} \exp \left[ \frac{\Gamma_3 (m - n - 2r)}{2} \right] \] (34)
\[ \times \sqrt{(m)_r (n + 1)^{(r)}(n + r)_s (m - r + 1)^{(s)}} \, |(m - r + s), (n + r - s)\rangle \]
where we have used the Pochhammer notation \[14\], defined by \((J)_n = J(J - 1)(J - 2) \cdots (J - n + 1)\) and \((J)^{(n)} = J(J + 1)(J + 2) \cdots (J + n - 1)\). This completes the solution.
3 The Master Equation for Coupled Non-Linear Oscillators

We generalize the above work for \(N\) coupled oscillators in the presence of a Kerr medium. The dynamics is given by

\[
\frac{\partial}{\partial t} \rho = -i \sum_{i=1}^{N} \omega_i [a_i^\dagger a_i, \rho] - i \sum_{i,j=1}^{N} \chi_{ij} \left[ (a_i^\dagger a_j)(a_j^\dagger a_i), \rho \right],
\]

where the \(\chi_{ij}\) indicates the coupling among the oscillators and may depend on the Kerr medium. We take \(\chi_{ij} = \chi_{ji}\). Applying \(|I\rangle\) on (35) from the right and using (9) this master equation for \(\rho\) goes over to a Schrödinger-like equation for the state \(|\rho\rangle\)

\[
\frac{\partial}{\partial t} |\rho\rangle = -i \hat{H} |\rho\rangle,
\]

where

\[
- i \hat{H} = -i \sum_{i=1}^{N} \omega_i \left( a_i^\dagger a_i - \tilde{a}_i^\dagger \tilde{a}_i \right) - i \sum_{i,j=1}^{N} \chi_{ij} \left[ a_i^\dagger a_j a_j^\dagger a_i - \tilde{a}_i^\dagger \tilde{a}_j \tilde{a}_j^\dagger \tilde{a}_i \right].
\]

This Hamiltonian can be rewritten as

\[
- i \hat{H} = -i \sum_{i=1}^{N} \omega_i \left( a_i^\dagger a_i - \tilde{a}_i^\dagger \tilde{a}_i \right) - i \sum_{i,j=1}^{N} \chi_{ij} \left( a_i^\dagger a_j a_j^\dagger a_i + \tilde{a}_i^\dagger \tilde{a}_j \tilde{a}_j^\dagger \tilde{a}_i \right) - i \sum_{i,j=1}^{N} \chi_{ij} \left( a_i^\dagger a_j a_j^\dagger a_i - \tilde{a}_i^\dagger \tilde{a}_j \tilde{a}_j^\dagger \tilde{a}_i \right)
\]

It has been shown in [4] that this Hamiltonian is related to \(SU(1,1)\), with appropriate definitions of the group generators. The case of damped coupled oscillators has also been considered, and the solution to the evolution equation obtained using the disentanglement theorem for \(SU(1,1)\).

Here we study the relation between this system and \(SU(2)\). By including damping to this Hamiltonian one has

\[
- i \hat{H}_D = \sum_{i=1}^{N} \left[ -i \omega_i S_i^3 - i \sum_{j=1}^{N} \chi_{ij} S_j^0 S_i^3 + \gamma_i S_i^+ S_i^- + \gamma_i S_i^+ S_i^- \right],
\]

where \(\gamma\) is the decay parameter for the dissipative cavity. The operators

\[
S_i^3 = \frac{(a_i^\dagger a_i - \tilde{a}_i^\dagger \tilde{a}_i)}{2}, \quad S_i^+ = a_i^\dagger \tilde{a}_i, \quad S_i^- = a_i \tilde{a}_i^\dagger
\]
satisfy the SU(2) algebra

\[
\begin{align*}
\left[ S_i^3, S_j^3 \right] &= S_i^3, \\
\left[ S_i^+ S_j^- \right] &= -S_i^-, \\
\left[ S_i^+, S_j^- \right] &= 2S_i^3,
\end{align*}
\]  

(41)

with $S_0^i = (a_i^\dagger a_i + \tilde{a}_i^\dagger \tilde{a}_i)$ the Casimir operator for SU(2).

The substitution of the Hamiltonian (39) in (36) leads to the solution of (36),

\[
|\rho(t)\rangle = \prod_{i=1}^N \exp\left(\gamma_i^+ S_i^+ + \gamma_i^3 S_i^3 + \gamma_i^- S_i^-\right) |\rho(0)\rangle,
\]  

(42)

where

\[
\begin{align*}
\gamma_i^+ &= \gamma_i t; \\
\gamma_i^- &= \gamma_i t; \\
\gamma_i^3 &= -\left(\omega_i + \gamma_i + i \sum_{j=1}^N \chi_{ij} S_0^j\right) t.
\end{align*}
\]  

(43)

Using the disentangling theorem, one has

\[
|\rho(t)\rangle = \prod_{i=1}^N \exp\left(\Gamma_i^+ S_i^+\right) \exp\left(\Gamma_i^3 S_i^3\right) \exp\left(\Gamma_i^- S_i^-\right) |\rho(0)\rangle,
\]  

(44)

where

\[
\begin{align*}
\Gamma_i^\pm &= \left(\frac{\sinh \phi_i}{\phi_i \cosh \phi_i - \gamma_{i3} \sinh \phi_i}\right) \gamma_i^\pm, \\
\Gamma_i^3 &= \ln\left(\frac{\phi_i}{\phi_i \cosh \phi_i - \gamma_{i3} \sinh \phi_i}\right),
\end{align*}
\]  

(45)

with

\[
\phi_i^2 = \gamma_{i3}^2 + \gamma_i^+ \gamma_i^-.
\]  

(46)

Thus the evolution of the state for the coupled nonlinear oscillator system is given in terms of the generators of SU(2).

As earlier, we make an expansion in the basis of number states. We first introduce

\[
\mathbf{m} = (m_1, m_2, m_3, \ldots, m_N) \quad \mathbf{n} = (n_1, n_2, n_3, \ldots, n_N).
\]

Then we make the expansion for $|\rho(0)\rangle$ as

\[
|\rho(0)\rangle = \sum_{\mathbf{m,n}=0}^\infty \rho_{\mathbf{m,n}}(0) |\mathbf{m}, \mathbf{n}\rangle.
\]  

(47)

It is to be understood that the sum above is over all the $m_i$ and the $n_i$, $(i = 1, 2, 3 \ldots, N)$. 


Using this in (44) one gets

$$|\rho(t)\rangle = \prod_{i=0}^{N} \sum_{m_i,n_i=0}^{\infty} \rho_{m_i,n_i}(0) \sum_{r_i=0}^{m_i} (\Gamma_{-r_i})^{r_i} \frac{(n_i+r_i)!}{r_i!} \sum_{s_i=0}^{(r_i+s_i)!} \frac{(\Gamma_{+s_i})^{s_i}}{s_i!} \exp \left[ \Gamma_{id} \frac{(m_i-n_i-2r_i)}{2} \right]$$

$$\times \sqrt{(m_i)_r (n_i+1)_r (n_i+r_i)_s (m_i-r_i+1)^{s_i}} \langle (m_i-r_i+s_i), n_i+r_i-s_i) \rangle$$

(48)

here $(J)_n = J(J-1)(J-2)\cdots(J-n+1)$ and $(J)^n = J(J+1)(J+2)\cdots(J+n-1)$ is the Pochhammer’s notation. This completes the solution.

4 Conclusion

In this paper, using techniques from thermofield dynamics we have considered the Kerr medium as an SU(2) system with damping. This system has been solved exactly for arbitrary initial conditions using these techniques. The case of $N$ coupled oscillators in the presence of damping is also considered. It would be gratifying if these results can be experimentally verified.

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