Group Structure of an Extended Poincare Group

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Abstract

In previous papers we extended the Lorentz and Poincare groups to include a set of Dirac boosts that give a direct correspondence with a set of generators which for spin 1/2 systems are proportional to the Dirac matrices. The groups are particularly useful for developing general linear wave equations beyond spin 1/2 systems. In this paper we develop explicit group properties of the extended Poincare group to obtain group parameters that will be useful for physical calculations in systems which manifest the group properties. The inclusion of space-time translations will allow future explorations of the gauge properties inherent in the group structure.

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1 Introduction

One of the most useful properties of Dirac’s equation\[1\] for spinor fields is that the equations satisfy a linear energy-momentum relationship in the field equations. In previous papers\[2\][3][4] we have developed a set of group generators and field equations that generalize Dirac’s formulation. The equations have appropriate correspondence with the Dirac equation for spin 1/2 systems.

In this paper, we will develop the group structure elements for the extended Poincare group developed in reference\[3\]. We will build on the extended Lorentz group structure elements developed in reference\[4\]. We will explicitly calculate those group structure elements that will be relevant for calculations involving systems which have the extended Poincare group as a local gauge pre-symmetry. In future submissions we will examine the quantum geometrodynamics of systems with extended Poincare pre-symmetry.

2 Group Theoretic Conventions

We will continue to utilize the group conventions and parameterization developed for the extended Lorentz subgroup of this extended Poincare group developed in reference \[4\]. In particular, the generators of infinitesimal transformations for operator representations are given by

\[
i X_r \equiv \frac{\partial}{\partial M^r} S(M) \bigg|_{M=I}.
\] (2.1)

and Lie structure matrices are defined by

\[
\Theta^s_r(M) \equiv \frac{\partial \Phi^s(M'; M)}{\partial M'^r} \bigg|_{M'=I},
\] (2.2)

where \(\Phi^s(M'; M)\) is the group composition (closure) element resulting from the operation \(M'\) on element \(M\). The generators transform under the representations of the group as given by the relation

\[
S(M^{-1}) X_r S(M) = \Theta^s_r(M) X_s
\] (2.3)

where the matrices \(\Theta\) given by

\[
\Theta^s_r(M) \equiv \frac{\partial}{\partial M'^r} \Phi^s(M^{-1} ; \Phi(M'; M)) \bigg|_{M'=I}
\] (2.4)

form a fundamental representation of the group. The Lie structure matrices are useful to determine transformation properties of gauge fields\[5\].

We will therefore calculate the group matrix elements \(\Theta\) and \(\Theta^s\) that determine the transformation properties of generators and gauge fields for this group. In subsequent papers, these elements will be used in developing dynamical models for physical systems.
3 Extended Group Commutation Relations

As developed previously, the commutation relationships between the generators of this extended Lorentz group are given by

\[
\begin{align*}
[J_j, J_k] &= i \epsilon_{jkm} J_m \\
[J_j, K_k] &= i \epsilon_{jkm} K_m \\
[K_j, K_k] &= -i \epsilon_{jkm} J_m \\
[\Gamma^0, \Gamma^k] &= i K_k \\
[\Gamma^0, J_k] &= 0 \\
[\Gamma^0, K_k] &= -i \Gamma^k \\
[\Gamma^j, \Gamma^k] &= -i \epsilon_{jkm} J_m \\
[\Gamma^j, J_k] &= i \epsilon_{jkm} \Gamma^m \\
[\Gamma^j, K_k] &= -i \delta_{jk} \Gamma^0 \\
[\Gamma^j, \Gamma^k] &= -i \epsilon_{jkm} \Gamma^m \\
[\Gamma^j, J_k] &= i \epsilon_{jkm} \Gamma^m \\
[\Gamma^j, K_k] &= -i \delta_{jk} \Gamma^0
\end{align*}
\]  

An extended Lorentz group Casimir operator can be constructed in the form

\[
C_{\Lambda} = J \cdot J - K \cdot K + \Gamma^0 \Gamma^0 - \Gamma \cdot \Gamma.
\]  

For the extended Poincare group, the non-vanishing extended translation commutators involving the operators \( \hat{P}_\mu \) and \( \hat{\mathcal{G}} \) are given by

\[
\begin{align*}
[J_j, P_k] &= i \epsilon_{jkm} P_m \\
[K_j, P_0] &= -i P_j \\
[K_j, P_k] &= -i \delta_{jk} P_0 \\
[\Gamma^\mu, P_\nu] &= \pm i \delta^\mu_\nu \mathcal{G} \\
[\Gamma^\mu, \mathcal{G}] &= \pm i \eta^{\mu\nu} P_\nu
\end{align*}
\]  

where the upper signs were used in the previous results\[^3\]. We can construct an extended Poincare group Casimir operator given by

\[
C_{(\mu)} = g^2 - \eta^{\beta\nu} P_\beta P_\nu.
\]  

To be consistent with the group element \( \mathcal{G} \) being the generator for representations as defined in Equation 2.1, we will choose the lower signs in Equation 3.17

\[
\begin{align*}
[\Gamma^\mu, P_\nu] &= -i \delta^\mu_\nu \mathcal{G} \\
[\Gamma^\mu, \mathcal{G}] &= -i \eta^{\mu\nu} P_\nu
\end{align*}
\]
4 Extended Poincare Group Structure

We will construct finite element transformations in the extended Poincare group in what follows. This group will have rotations, general Lorentz transformations, the extended Lorentz transformations, and extended translations as subgroups. A general group element is characterized by the 15 parameters given by \( \{ \alpha, \vec{a}, \vec{\omega}, u, \theta \} \), where \( \alpha \) is the element conjugate to the extended scalar translation generator \( \mathcal{G} \), \( a^\mu \) is the element conjugate to the momentum \( P_\mu \), \( \omega^\mu \) is the element conjugate to the Dirac boost generator \( \Gamma^\mu \), \( \beta^m \) (which is directly related to the four-velocity parameter \( u^m \)) is the element conjugate to the Lorentz boost generator \( K_m \), and \( \theta^m \) is the element conjugate to the angular momentum \( J_m \).

4.1 General Lorentz Transformations

The Lorentz group is a subgroup of the extended Poincare group being developed. The four-velocity \( \vec{u} \) is as usual defined using \( u^0 = \sqrt{1 + |u|^2} \). The three-components of the four-velocity are related to the parameters conjugate to the (Lorentz) boost generators \( K_m \) by \( \vec{u} \equiv \vec{\beta} \tanh \beta \). We will establish our convention for the Lorentz transformation matrices on four-vectors. The general Lorentz transformation matrix is defined by

\[
\Lambda^\mu_{\ \nu}(u, \theta) \equiv L_{\Gamma^\nu}^{\Gamma^\beta}(u) R_{\Gamma^\nu}^{\Gamma^\beta}(\theta),
\]

where \( L_{\Gamma^\nu}^{\Gamma^\beta}(u) \) and \( R_{\Gamma^\nu}^{\Gamma^\beta}(\theta) \) are directly related to the 4-Lorentz boost and rotation matrices defined in reference [4]. Since we now have 4-generators with both covariant and contravariant properties, we will carefully exhibit all transformation properties. This matrix has the usual properties of a Lorentz transformation, leaving invariant the group metric \( \eta_{\nu\rho} \) obtained from the structure constants of the extended Lorentz group, which is proportional to the usual Minkowski metric \( \eta_{\mu\nu} \). The inverse of this matrix can directly be shown to result from lowering and raising its indeces using the Minkowski metric. The inverse element of this representation of the Lorentz group satisfies

\[
\{ u, \theta \}^{-1} = \{-R(-\theta)u, -\theta\} = \{-uR^{-1}(\theta^{-1}), -\theta\}
\]

The form of the infinitesimal 4-Lorentz generators

\[
(J_m)^\mu_{\ \nu} = \left. \frac{\partial}{\partial \theta^m} \Lambda^\mu_{\ \nu}(0, \theta) \right|_{\theta=0}
\]

\[
(K_m)^\mu_{\ \nu} = \left. \frac{\partial}{\partial u^m} \Lambda^\mu_{\ \nu}(u, 0) \right|_{u=0}
\]

has non-vanishing elements given by

\[
(J_m)^j_{\ k} = \epsilon_{mjk}
\]

\[
(K_m)^0_{\ k} = -\delta_{m,k} = (K_m)^k_{\ 0}
\]
4.2 Extended Translations

Group transformations parameterized by elements $a^\mu$ conjugate to the generators $P_\mu$ and $\alpha$ conjugate to the generator $\mathcal{G}$ will be referred to as extended translations. The extended translations all mutually commute, so that this subset is an abelian subgroup.

4.2.1 Extended Lorentz Transformations on Extended Translations

We examine the effect of extended Lorentz transformations on the extended translations. Utilizing Equation (2.3) for this representation, we can read off various elements of the fundamental representation. For pure Lorentz transformations, we obtain new matrix elements in addition to those obtained previously for the extended Lorentz group:

$$\Theta^{(XL)}_{P_\mu}(0, \bar{0}, \bar{0}, \bar{u}, \bar{\theta}) = \Theta^{(XL)}(0, \bar{u}^{-1}, \bar{\theta}^{-1}) = \Lambda^\beta_\mu(\bar{u}^{-1}, \bar{\theta}^{-1}) = \Lambda^\beta_\mu(\bar{u}, \bar{\theta}) \quad (4.5)$$

where $\Theta^{(XL)}_{P_\mu}(\bar{\omega}, \bar{u}, \bar{\theta})$ are the fundamental representation matrices for the extended Lorentz group as given in reference 4. These relations demonstrate that $T^\mu P_\mu$ and $\mathcal{G}$ are Lorentz invariants.

For pure Dirac boosts, the additional fundamental representation elements are given by

$$\Theta^{P_\mu}_{\mu}(0, \bar{0}, \bar{\omega}, \bar{u}, \bar{\theta}) = \delta^\beta_\mu + (\cos(\omega) - 1)\frac{\omega^\alpha}{\omega^\alpha} \equiv S^\beta_\mu(-\bar{\omega})$$

$$\Theta^{P_\mu}_{\mu}(0, \bar{0}, \bar{\omega}, \bar{u}, \bar{\theta}) = -\sin(\omega)\frac{\omega^\alpha}{\omega^\alpha} \equiv S^\beta_\mu(-\bar{\omega})$$

$$\Theta^{P_\mu}_{\mu}(0, \bar{0}, \bar{\omega}, \bar{u}, \bar{\theta}) = -\sin(\omega)\frac{\omega^\alpha}{\omega^\alpha} \equiv S^\beta_\mu(-\bar{\omega})$$

where $\omega^\alpha \equiv \eta^{\alpha\beta}\omega_\beta$ and $\omega \equiv \sqrt{\bar{\omega} \cdot \bar{\omega}} = \sqrt{-\omega_\alpha \omega^\alpha}$.

4.2.2 Extended Translations on Extended Lorentz Transformations

There are additional fundamental representation matrix elements due to the non-vanishing commutation of the extended translation generators with the Dirac boost generators. Most of these elements are unity on the diagonal. The only other non-vanishing elements are given by

$$\Theta^{P_\mu}_{\mu}(\alpha, \bar{0}, \bar{0}, \bar{0}, \bar{0}) = \alpha\eta^{\mu\beta} \quad (4.7)$$

$$\Theta^{\mathcal{G}}_{\mu}(0, \bar{a}, \bar{0}, \bar{0}, \bar{0}) = a^\mu$$

The general fundamental transformation matrix for the extended Poincare group is then given by

$$\Theta_{(XL)}(\alpha, \bar{a}, \bar{\omega}, \bar{u}, \bar{\theta}) = \Theta_p^{(XL)}(\alpha, \bar{0}, \bar{0}, \bar{0}, \bar{0}) \Theta^{(XL)}_{P_\mu}(0, \bar{a}, \bar{0}, \bar{0}, \bar{0}) \Theta^{(XL)}_{\mathcal{G}}(0, \bar{0}, \bar{0}, \bar{0}, \bar{0})$$

$$\Theta_{(XL)}^{(XL)}(\alpha, \bar{0}, \bar{0}, \bar{0}, \bar{0}) \Theta^{(XL)}_{\mathcal{G}}(0, \bar{a}, \bar{0}, \bar{0}, \bar{0}) \Theta^{(XL)}_{P_\mu}(0, \bar{0}, \bar{0}, \bar{0}, \bar{0})$$

$$\Theta^{(XL)}_{(XL)}(\bar{\omega}, \bar{u}, \bar{\theta}) \quad (4.8)$$

where $\Theta^{(XL)}_{(XL)}$ are extended Lorentz group representation matrices from reference 4 expanded to include Equations (4.5) and (4.7).
4.3 Extended Poincare Group Transformations

To develop a description of the group structure of this extended Poincare group, we will explicitly demonstrate the group composition rule and Lie structure elements of the complete group. The representation we have developed has been constructed by sequential pure rotation, Lorentz boost, Dirac boost, space-time translation, and extended scalar translation:

$$M(\alpha, \bar{a}, \bar{\omega}, \bar{u}, \bar{\theta}) \equiv C(\alpha)V(\bar{a})W(\bar{\omega})L(\bar{u})R(\bar{\theta})$$

(4.9)

The group composition rule then defines elements in this same manner

$$M(\alpha_2, \bar{a}_2, \bar{\omega}_2, \bar{u}_2, \bar{\theta}_2) M(\alpha_1, \bar{a}_1, \bar{\omega}_1, \bar{u}_1, \bar{\theta}_1) \equiv C(\alpha_2, \bar{a}_2, \bar{\omega}_2, \bar{u}_2, \bar{\theta}_2) C(\alpha_1, \bar{a}_1, \bar{\omega}_1, \bar{u}_1, \bar{\theta}_1)$$

$$W(\bar{\omega}_2(\alpha_2, \bar{a}_2, \bar{\omega}_2, \bar{u}_2, \bar{\theta}_2 ; \alpha_1, \bar{a}_1, \bar{\omega}_1, \bar{u}_1, \bar{\theta}_1)) L(\bar{u}_2(\alpha_2, \bar{a}_2, \bar{\omega}_2, \bar{u}_2, \bar{\theta}_2 ; \alpha_1, \bar{a}_1, \bar{\omega}_1, \bar{u}_1, \bar{\theta}_1))$$

$$R(\bar{\theta}_2(\alpha_2, \bar{a}_2, \bar{\omega}_2, \bar{u}_2, \bar{\theta}_2 ; \alpha_1, \bar{a}_1, \bar{\omega}_1, \bar{u}_1, \bar{\theta}_1))$$

(4.10)

The group composition elements can be expressed using previously constructed functions in terms of pure extended Lorentz transformations and extended translations:

$$\alpha(\alpha_2, \bar{a}_2, \bar{\omega}_2, \bar{u}_2, \bar{\theta}_2 ; \alpha_1, \bar{a}_1, \bar{\omega}_1, \bar{u}_1, \bar{\theta}_1) = \alpha_2 + \alpha_1 (\Theta(-\bar{\omega}_2))_{\mu}^G + a_2^\nu (\Theta(-\bar{\omega}_2)\Theta(-\bar{u}_2)\Theta(-\bar{\theta}_2))_{\mu}^G$$

$$a^\mu(\alpha_2, \bar{a}_2, \bar{\omega}_2, \bar{u}_2, \bar{\theta}_2 ; \alpha_1, \bar{a}_1, \bar{\omega}_1, \bar{u}_1, \bar{\theta}_1) = a_2^\mu + \alpha_1 (\Theta(-\bar{\omega}_2))_{\mu}^\nu + a_1^\nu (\Theta(-\bar{\omega}_2)\Theta(-\bar{u}_2)\Theta(-\bar{\theta}_2))_{\mu}^\nu$$

$$\bar{\omega}(\alpha_2, \bar{a}_2, \bar{\omega}_2, \bar{u}_2, \bar{\theta}_2 ; \alpha_1, \bar{a}_1, \bar{\omega}_1, \bar{u}_1, \bar{\theta}_1) = \tilde{\omega}(XL)(\tilde{\omega}_2(\alpha_2, \bar{a}_2, \bar{\omega}_2, \bar{u}_2, \bar{\theta}_2 ; \tilde{\omega}_1, \bar{u}_1, \bar{\theta}_1))$$

$$\tilde{\omega}_2(\alpha_2, \bar{a}_2, \bar{\omega}_2, \bar{u}_2, \bar{\theta}_2 ; \alpha_1, \bar{a}_1, \bar{\omega}_1, \bar{u}_1, \bar{\theta}_1)$$

$$\tilde{\theta}(\alpha_2, \bar{a}_2, \bar{\omega}_2, \bar{u}_2, \bar{\theta}_2 ; \alpha_1, \bar{a}_1, \bar{\omega}_1, \bar{u}_1, \bar{\theta}_1)$$

(4.11)

where fundamental representation matrices with only one argument presume the vanishing of all other elements, and \(\tilde{\omega}(XL), \tilde{u}(XL), \tilde{\theta}(XL)\) are the extended Lorentz group composition rules obtained in reference\textsuperscript{4}. The inverse element can then be shown to be given by

$$\left\{ \alpha, \bar{a}, \omega, \bar{u}, \bar{\theta} \right\}^{-1} = \left\{ -\alpha S_{\bar{\theta}}(\omega^{-1}) - \bar{a} \cdot \Lambda^{-1}(\bar{u}^{-1}, \bar{\theta}^{-1}) \cdot S_{\bar{\theta}}(\omega^{-1}) , -\alpha S_{\bar{\theta}}(\omega^{-1}) - \bar{a} \cdot \Lambda^{-1}(\bar{u}^{-1}, \bar{\theta}^{-1}) S(\omega^{-1}) - \omega \Lambda(\bar{u}^{-1}, \bar{\theta}^{-1}), -\omega R^{-1}(\bar{\theta}^{-1}), \bar{\theta}^{-1} = -\bar{\theta} \right\}$$

(4.12)

where \((\bar{a} \Lambda^{-1})^\mu = a^\nu \Lambda^\nu \mu \) and \((\bar{\omega} \Lambda)_\mu = \omega^\nu \Lambda^\nu \mu \). More explicitly, the inverse extended translation elements can be expressed

$$\alpha^{-1} = -\alpha (\Theta(-\bar{\omega}^{-1}))_{\mu}^G - a^\nu (\Theta(-\bar{\theta}^{-1}))_{\mu}^\nu$$

$$a^{-1})^\mu = -\alpha (\Theta(-\bar{\omega}^{-1}))_{\mu}^\nu - a^\nu (\Theta(-\bar{\theta}^{-1}))_{\mu}^\nu$$

(4.13)

where representation matrices of single arguments assume all other arguments vanish.
4.4 Lie Structure Elements

Finally, we can use the composition rules given in Equation 4.11 along with the definition given in Equation 2.2 to explicitly develop the Lie structure matrices

\[
\begin{align*}
\Theta^\alpha_\nu(\alpha, \vec{a}, \vec{\omega}, u, \theta) &= 1 \\
\Theta^\omega_{\mu}(\alpha, \vec{a}, \vec{\omega}, u, \theta) &= a^\mu \\
\Theta^a_{\beta}(\alpha, \vec{a}, \vec{\omega}, u, \theta) &= \delta^a_\beta \\
\Theta^\omega_{a}(\alpha, \vec{a}, \vec{\omega}, u, \theta) &= \alpha \eta^{\mu\beta} \\
\Theta^a_{\nu}(\alpha, \vec{a}, \vec{\omega}, u, \theta) &= \delta^a_\mu a^j + \delta^a_\nu a^0 \\
\Theta^\omega_{\nu}(\alpha, \vec{a}, \vec{\omega}, u, \theta) &= \Theta^{(XL)}_{\nu}(\vec{\omega}, u, \theta) \\
\Theta^a_{\nu}(\alpha, \vec{a}, \vec{\omega}, u, \theta) &= \Theta^{(XL)}_{a}(\vec{\omega}, u, \theta) \\
\Theta^\omega_{\nu}(\alpha, \vec{a}, \vec{\omega}, u, \theta) &= \Theta^{(XL)}_{\nu}(\vec{\omega}, u, \theta),
\end{align*}
\]

where \(\Theta^{(XL)}\) are the extended Lorentz group Lie structure matrices given in reference [4]. We will be able to utilize these matrices in general gauge transformations for systems which have a local gauge symmetry in this group.

5 Conclusions

We have demonstrated a group representation of the extended Poincare group developed in reference [3]. The fundamental representation matrices and Lie structure matrices have been explicitly calculated. The group structure elements allow direct construction of finite dimensional unitary vector representations under which quantum states transform as developed in reference [3]. The set of extended translations are invariant under rotations (for massive systems), which is the little group for appropriately chosen standard states. Several Lorentz invariants can be constructed from the set of generators of this group, which provides a rich unified structure for the dynamics of physical systems. The implications of this group structure on the quantum geometrodynamics of systems exhibiting this symmetry or pre-symmetry will be further explored in future explorations.

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