Invariants of Forth Order Linear Differential Operators

V. V. Lychagin¹* and V. A. Yumaguzhin²,³**

(Submitted by J. S. Krasil’shchik)

¹V. A. Trapeznikov Institute of Control Sciences, Russian Academy of Sciences, Moscow, 117997 Russia
²Program Systems Institute, Russian Academy of Sciences, Pereslavl-Zalessky, Yaroslavl oblast, 152021 Russia
³Institute of Control Sciences, Russian Academy of Sciences, Moscow, 117997 Russia

Received April 29, 2020; revised May 13, 2020; accepted May 24, 2020

Abstract—In this paper, we study scalar the forth order linear differential operators over an oriented 2-dimensional manifold. We investigate differential invariants of these operators and show their application to the equivalence problem.

DOI: 10.1134/S1995080220120288

Keywords and phrases: 4th order linear partial differential operator, jet bundle, differential invariant, equivalence problem.

1. INTRODUCTION

In the papers [3–5], we analyzed the equivalence of k-order linear differential operators acting on a line bundle over smooth n-dimensional manifold. In [3], we investigated the case of arbitrary n and k = 2, in [4], we investigated the case of n = 2 and k = 3, and in [5], we investigated the case n ≥ 2 and k ≥ 3 for constant type operators.

It is shown in [5] that a stationary Lie algebra of symbol of regular operator is trivial at every point when n ≥ 2 and k ≥ 3 and therefore codimension of regular orbit of this symbol is

\[ c(n, k) = \left( \frac{n + k - 1}{k} \right) - n^2. \]  

(1)

It is easy to check that \( c(n, k) \geq n \) for all \( n \geq 2 \) and \( k \geq 3 \), with the exception for the following three cases: \( n = 2, k = 3 \); \( n = 2, k = 4 \); \( n = 3, k = 3 \). It follows, see [5], that the field of natural invariants of regular operator is generated by zero order invariants in non exceptional cases.

The case \( n = 2, k = 3 \) was investigated in [4].

In this paper, we consider the case, \( n = 2, k = 4 \). In this case \( c(2, 4) = 1 \). This means that there is a unique independent differential invariant of zero order for the regular symbols.

Essentially, this invariant has long been known. Indeed, in the Hilbert lectures [2], p. 57, two relative invariants for a forth degree homogeneous polynomial of two variables \( a_0 x^4 + 4a_1 x^3 y + 6a_2 x^2 y^2 + 4a_3 x y^3 + a_4 y^4 \) are found:

\[ I_2 = a_0 a_4 - 4a_1 a_3 + 3a_2^2, \quad I_3 = a_0 a_2 a_4 - a_0 a_3^2 - a_2^2 a_4 + 2a_1 a_2 a_3 - a_2^3. \]

These relative invariants have weights 2 and 3 respectively. Hence,

\[ I_0 = I_2^2 / I_2^3 \]  

(2)

*E-mail: Valentin.Lychagin@uit.no
**E-mail: yuma@diffiety.botik.ru
is an invariant for forth degree homogeneous polynomials in two variables. Thus \( I_0 \) is a rational invariant of a principal symbol of a forth order linear differential operator on two dimensional manifolds and hence \( I_0 \) is a zero order rational differential invariant of this operator. We use this invariant in the following way.

If \( I_0 \) is not constant, then it generates the second differential invariant of 1-st order. Using both of these invariants as natural coordinates, we get the full classification of 4th order operators with the non constant invariant \( I_0 \) w.r.t. the group of diffeomorphism of the manifold.

If \( I_0 \) is constant, then we have the case of the constant type 4th order operators which is considered in [5]. For such operators, there exist the Wagner connections in the manifold. Using this connection, we reduce the classification problem for the constant type 4th order operators to the classification problem set of four symmetric tensors of orders 4, 3, 2, 1, 0. This allows us to find the differential invariants for such kind of operators and solve the equivalence problem for them.

1.1. Notations

In this paper, we use the same notations as in [3–5].

Let \( M \) be \( n \)-dimensional manifold.

By \( \tau : TM \to M \) and \( \tau^* : T^*M \to M \) we denote respectively tangent and cotangent bundles over \( M \). Let \( \Sigma_k(M) = C^\infty(S^k(\tau)) \) be the module of symmetric \( k \)-vectors, \( \Sigma^k(M) = C^\infty(S^k(\tau^*)) \) the module of symmetric \( k \)-forms, \( \Omega_k(M) = C^\infty(\Lambda^k(\tau)) \) the module of skew-symmetric \( k \)-vectors, and \( \Omega^k(M) = C^\infty(\Lambda^k(\tau^*)) \) the module of exterior \( k \)-forms.

We denote by \( \text{Diff}_k(1) \) a left \( C^\infty(M) \)-module of linear differential operators of order \( \leq k \), acting on the trivial line bundle \( 1 : M \times \mathbb{R} \to M \), and by \( \chi_k : \text{Diff}_k(1) \to M \), we denote the bundle of these differential operators, thus \( C^\infty(\chi_k) = \text{Diff}_k(1) \).

By \( \mathcal{F}(M) \) we denote a multiplicative group of smooth functions on \( M \) without zeros on \( M \), by \( \mathcal{G}(M) \) will be denoted a group of diffeomorphisms of \( M \).

We denote by \( \text{GL}(V) \) a group of all linear transformations of vector space \( V \).

By principal symbol \( \sigma_k(A) \) of operator \( A \in \text{Diff}_k(1) \) we mean the equivalence class

\[ \sigma_k(A) \equiv A \mod \text{Diff}_{k-1}(1) \in \Sigma_k(M). \]

2. SYMBOLS OF DIFFERENTIAL OPERATORS ON 2-DIMENSIONAL MANIFOLDS

Let \( M \) be an oriented 2-dimensional manifold and \( x, y \) be local coordinates in \( M \).

An operator \( A \in \text{Diff}_4(1) \) has the following form in the coordinates \( x, y \),

\[
A = a_0 \partial_x^4 + 4a_1 \partial_x^3 \partial_y + 6a_2 \partial_x^2 \partial_y^2 + 4a_3 \partial_x \partial_y^3 + a_4 \partial_y^4
+ b_0 \partial_x^3 + 3b_1 \partial_x^2 \partial_y^2 + 3b_2 \partial_x \partial_y^3 + b_3 \partial_y^4 + c_0 \partial_x^2 + 2c_1 \partial_x \partial_y + 2c_2 \partial_y^2 + d_0 \partial_x + d_1 \partial_y + e_0.
\]

Its principal symbol \( \sigma_4(A) \in \Sigma_4(M) \) is the following symmetric 4-vector in the coordinates \( x, y \),

\[
\sigma_4(A) = a_0 \partial_x^4 + 4a_1 \partial_x^3 \cdot \partial_y + 6a_2 \partial_x^2 \cdot \partial_y^2 + 4a_3 \partial_x \cdot \partial_y^3 + a_4 \partial_y^4,
\]

where we denoted by \( \cdot \) the symmetric product and by \( \partial_a^k \) the symmetric product of \( k \) copies of \( \partial_a \).

In canonical coordinates \( x, y, p_x, p_y \) on \( T^*M \) this tensor is a forth degree homogeneous polynomial (Hamiltonian)

\[
\sigma_4(A) = a_0 p_x^4 + 4a_1 p_x^3 p_y + 6a_2 p_x^2 p_y^2 + 4a_3 p_x p_y^3 + a_4 p_y^4
\]

on \( T^*M \).

There are three possibilities for roots of this polynomial: all roots are real, two roots are real and other are complex, and all roots are complex.

Let \( \mathcal{D}(A) \) be a discriminant of \( \sigma_4(A) \). Then \( \mathcal{D}(A) = 0 \) if and only if the symbol \( \sigma_4(A) \) has multiple roots, or characteristics We say that the symbol \( \sigma_4(A) \) is regular if \( \mathcal{D}(A) \neq 0 \).

In this paper, we consider only regular operators, i.e., operators with regular symbols. One can check that: \( \mathcal{D}(A) > 0 \) if and only if all roots of \( \sigma_4(A) \) are distinct and real or all roots are distinct and complex, \( \mathcal{D}(A) < 0 \) if and only if two roots of \( \sigma_4(A) \) are distinct and real and other two are complex.
Lemma 1.

1. If symbol $\sigma_4(A)$ has two distinct real roots then there are local coordinates $x, y$ such that
   \[\sigma_4(A) = \partial_x \cdot \partial_y \cdot (\alpha_0 \partial_x^2 + 2\alpha_1 \partial_x \cdot \partial_y + \alpha_2 \partial_y^2).\]  

2. If symbol $\sigma_4(A)$ has complex root then there are local coordinates $x, y$, which we call isothermal coordinates, such that
   \[\sigma_4(A) = (\partial_x^2 + \partial_y^2) \cdot (\alpha_0 \partial_x^2 + 2\alpha_1 \partial_x \cdot \partial_y + \alpha_2 \partial_y^2).\]  

Remark that if $z = x + iy$, $\bar{z} = x - iy$ are complex coordinates in a domain of the isothermal coordinates, then symbol (5) has form
   \[\sigma_4(A) = \partial_z \cdot \partial_{\bar{z}} \cdot (\alpha_0 \partial_z^2 + 2\alpha_1 \partial_z \cdot \partial_{\bar{z}} + \alpha_2 \partial_{\bar{z}}^2).\]  

similar to (4).

Corollary 1.

1. If symbol $\sigma_4(A)$ is defined by (4) or (6), then its discriminant is defined up to a positive numerical factor by the formula $D(A) = \alpha_0^2 \alpha_2^2 (9\alpha_1^2 - 16\alpha_0 \alpha_2)$.

2. If symbol $\sigma_4(A)$ is defined by (5), then its discriminant is defined up to a positive numerical factor by the formula $D(A) = (4\alpha_1^2 + (\alpha_0 - \alpha_2)^2)^2 (\alpha_0 \alpha_2 - \alpha_1^2)$.

3. INVARIANTS OF NON CONSTANT TYPE DIFFERENTIAL OPERATORS

3.1. Constant Type Operators

Let $V$ be a 2-dimensional vector space and let $\varpi \subset S^k(V)$ be a regular $GL(V)$-orbit (i.e., $\varpi$ is defined by equations $I_1 = c_1, \ldots, I_m = c_m$, where $I_i$ are independent $GL(V)$-invariants in a neighborhood of $\varpi$, $c_i$ are constant, and $m$ is codimension of $\varpi$).

Recall, see [5], that:

1. a symbol $\sigma \in \Sigma_4(M)$ has a constant type $\varpi$ if for any point $q \in M$ and any isomorphism $\varphi : T_q(M) \to V$ the image $\varphi_*(\sigma) \in S^k(V)$ belongs to $\varpi$;

2. an operator $A \in \text{Diff}_4(1)$ has the constant type $\varpi$ if its symbol $\sigma_4(A)$ has the constant type $\varpi$.

Remark that a symbol $\sigma \in \Sigma_4(M)$ has a constant type if and only if its zero order rational differential invariant $I_0(\sigma)$ is a constant.

3.2. The Bundle of Differential Operators

In the bundle $\chi_4 : \text{Diff}_4(1) \to M$ we will use the following canonical local coordinates $(x, y, u^\alpha)$, where $(x, y)$ are local coordinates in $M$ and $u^\alpha$ are fiber wise coordinates in bundle $\chi_4$. Here $\alpha = (\alpha_1, \alpha_2)$ is the multi index of length $0 \leq |\alpha| = \alpha_1 + \alpha_2 \leq 4$.

In these coordinates the section $S_A : M \to \text{Diff}_4(1)$ that corresponds to operator $A = \sum_{|\alpha| \leq 4} a^\alpha(x, y) \partial_x^{\alpha_1} \partial_y^{\alpha_2} \in \text{Diff}_4(1)$, has the form $u^\alpha = a^\alpha(x, y)$.

Denote by $\pi_1 : J^1(\chi_4) \to M$ the vector bundles of $l$-jets of sections of bundles $\chi_4$, or, in other words, bundles of $l$-jets of the 4-order scalar differential operators.

We will denote by $[A]_l^p$ $l$-jets of operators at a point $p \in M$.}

---

**LOBACHEVSKII JOURNAL OF MATHEMATICS** Vol. 41 No. 12 2020
Bundles $\chi_4$, as well as bundles $\pi_l$, are natural in the sense that the action of the diffeomorphism group $G(M)$ is lifted to automorphisms of these bundles in the natural way:
\[
\varphi^{(0)} : [A]^t_p \mapsto [\varphi_*(A)]^t_{\varphi(p)}
\]
for any diffeomorphism $\varphi \in G(M)$.

The total differential operator of order 4, see [5],
\[
\square: C^\infty(J^l(\chi_4)) \to C^\infty(J^{l+4}(\chi_4)), \quad l = 0, 1, \ldots,
\]
is defined by the formula
\[
j_{4+l}(S_A)^*(\square(f)) = A(j_l(S_A)^*(f)),
\]
for all functions $f \in C^\infty(J^l(\chi_4))$ and operators $A \in \text{Diff}_4(1)$. It is easy to check that in the standard jet-coordinates in the bundles $\pi_l$ this operator has the following form
\[
\square = \sum_{|\alpha| \leq 4} a^\alpha \frac{d^{\alpha_1}}{dx^1} \frac{d^{\alpha_2}}{dy^2}.
\]

The main property of this operator is its naturality: $\varphi^{(4+l)*} \circ \square = \square \circ \varphi^{(l)*}$ for all diffeomorphisms $\varphi \in G(M)$.

Let $J = (J_1, J_2)$ be a pair of natural differential invariants. We say that they are in general position if $dJ_1 \wedge dJ_2 \neq 0$. Let $T$ be an invariant, then $d\hat{T} = \hat{I}_1 dJ_1 + \hat{I}_2 dJ_2$, for some rational functions $\hat{I}_i$, which are called Tresse derivatives. We will denote them by $d\hat{T}/dJ_i$. They are invariants by the construction, having, as a rule, higher order then the invariant $T$.

3.3.

Let $A \in \text{Diff}_4(1)$ be a non constant type regular operator, $\sigma_4(A) \in \Sigma_4(M)$ be its symbol, and $I_0(A)$ be its zero order rational differential invariant defined in local coordinates $x, y$ of $M$ by (2),
\[
I_0(A) = \frac{I_3^2}{I_2^3},
\]
where $I_3 = a_0a_1 - 4a_1a_3 + 3a_2^2, I_2 = a_0a_2a_4 - a_0a_3^2 - a_1^2a_4 + 2a_1a_2a_3 - a_2^3$, and $a_0, \ldots, a_4$ are coefficients of $\sigma_4(A)$, see (3).

Assume that $dI_0(A) \neq 0$ in some open domain of $M$. Then the convolution $\langle dI_0(A)^{4}, \sigma(A) \rangle$ of symmetric differential 4-form $dI_0(A)^{4}$ and symbol $\sigma_4(A)$ is a first order rational differential invariant of $A$. We denote it by $I_1(A)$,
\[
I_1(A) = \langle dI_0(A)^{4}, \sigma(A) \rangle.
\]

3.4. The Field of all Natural Rational Differential Invariants of Non Constant Type Operators

We say that 2-jet $\theta_2 \in J^2\chi_4$ is regular if $d\hat{T_0} \wedge \hat{d\hat{T}}_1 \neq 0$ at the point $\theta_2$. Denote by $S_2 \subset J^2\chi_4$ the set of all singular points of $J^2\chi_4$, i.e.,
\[
S_2 = \{\theta_2 \in J^2\chi_4 | (d\hat{T_0} \wedge \hat{d\hat{T}}_1)_{\theta_2} = 0\}.
\]

In the regular domain $J^2\chi_4 \setminus S_2$ the invariants $I_0$ and $I_1$ are in general position. From [5], Theorem 11, we get the following statement.

**Theorem 1.** The field of all natural rational invariants of non constant operators $A \in \text{Diff}_4(1)$ is generated by the invariants $I_0, I_1$, and the Tresse derivatives
\[
\frac{d^{\alpha_1}}{dI_0^{\alpha_1}} \frac{d^{\alpha_2}}{dI_1^{\alpha_2}}
\]
of invariants $J_\alpha = \square(I_0^{\alpha_1} \cdot I_1^{\alpha_2})$ with $0 \leq |\alpha| \leq 4$.

The field of rational natural invariants separates, see [8, 10], regular orbits in the jet spaces of differential operators of non constant type.
4. INVARIANTS OF CONSTANT TYPE DIFFERENTIAL OPERATORS

4.1. The Wagner Connection

Theorem 2. A regular symbol \( \sigma \in \Sigma_4 \) has the constant type if and only if it has a linear connection \( \nabla^\sigma \) in the tangent bundle to \( M \) such that

\[
\nabla_X^\sigma(\sigma) = 0
\]

for all vector fields \( X \) on \( M \).

Proof. Suppose that \( \sigma \in \Sigma_4(M) \) is regular and has a constant type \( \varpi \subset \Sigma_4(M) \). Then, (see [5], Corollary 9), for any point \( p \in M \) there are a neighborhood \( \mathcal{O}_p \) and a unique linear isomorphisms \( A_{p,p'} : T_pM \to T_{p'}M \), for all \( p' \in \mathcal{O}_p \), such that \( (A_{p,p'})_* (\sigma_{p'}) = \sigma_p \). It follows that there are a unique linear isomorphisms \( A_{p,p'} : T_pM \to T_{p'}M \), for all \( p' \in M \), such that \( (A_{p,p'})_* (\sigma_{p'}) = \sigma_p \). Therefore, there is a unique linear connection \( \nabla^\sigma \) on manifold \( M \) such that \( \nabla_X^\sigma(\sigma) = 0 \) for all vector fields \( X \) on \( M \). We call it Wagner connection, see [4].

Inversely, let \( \sigma \in \Sigma_4(M) \) and let \( \nabla^\sigma \) be its Wagner connection, and \( p \in M \). By \( \{(e_1)_p, (e_2)_p\} \) we denote a base in the tangent space \( T_p(M) \), then transferring each vector \( (e_i)_p \) in parallel along the Wagner connection at every point in \( M \), we get the frame \( \{e_1, e_2\} \) on \( M \). In the terms of this frame, \( \sigma = a_0 e_1^4 + 4a_1 e_1^3 \cdot e_2 + 6a_2 e_1^2 \cdot e_2^2 + 4a_3 e_1 \cdot e_2^3 + a_4 e_2^4 \), where \( a_i \in \mathbb{R} \). This means that \( \sigma \) has a constant type.

From construction of Wagner connection, we get

Corollary 2. The curvature tensor of the Wagner connection \( \nabla^\sigma \) is equal to zero.

4.1.1. Coordinates. Suppose that the invariant \( I_0 \) of \( \sigma \) is a constant. Here we express Christoffel symbols \( \Gamma^i_{jk} \) of the Wagner connection \( \nabla^\sigma \) in terms of coefficients of \( \sigma \).

Let \( \partial_1 = \partial_x \), \( \partial_2 = \partial_y \). Then the symbol \( \sigma \) can be rewritten in the form \( \sigma = a_1^{i_1} \partial_1 \cdot \partial_{i_1} \), where is the summation over repeated indices, \( i_1, \ldots, i_4 = 1, 2 \), and coefficients \( a_1^{i_1} \) are symmetric in superscripts. Now condition (8) is the following

\[
\nabla_{\partial_1}^\sigma a_1^{i_1} = \partial_1 a_1^{i_1} + \Gamma^{i_1}_{ml} a_m^{i_2} a_{i_3}^{i_4} + \ldots + \Gamma^{i_4}_{ml} a_m^{i_3} a_{i_2}^{i_1} + \Gamma^{i_2}_{ml} a_m^{i_1} a_{i_3}^{i_4} = 0, \quad l = 1, 2.
\]

System (9) consist of 10 linear algebraic equations on 8 unknown functions \( \Gamma^i_{jk} \).

Let us fix some component \( a_{k_1 k_2 k_3 k_4} \) of \( \sigma \), for example \( a^{1122} \). Excluding two equations \( \partial_1 a_{k_1 k_2 k_3 k_4} + \ldots = 0 \) and \( \partial_2 a_{k_1 k_2 k_3 k_4} + \ldots = 0 \) from system (9), we get the system of 8 equations.

One can check that the determinant of this system is proportional to the discriminant \( D(\sigma) \) of the symbol \( \sigma \) and therefore this system has a unique solution. Moreover, one can check that this solution is independent of choice of the component \( a_{k_1 k_2 k_3 k_4} \).

The solution satisfies also to the excluded equations \( \partial_1 a_{k_1 k_2 k_3 k_4} + \ldots = 0 \) and \( \partial_2 a_{k_1 k_2 k_3 k_4} + \ldots = 0 \). Indeed, one can check that substituting the solution in the excluded equations, we get two expressions which are \( \partial_1 I_0 \) and \( \partial_2 I_0 \) respectively.

Example 3. Let regular symbol \( \sigma \in \Sigma_4(M) \) be defined in local coordinate by formula (4),

\[
\sigma = \partial_x \cdot \partial_y \cdot (\alpha_0 \partial_2^2 + 2\alpha_1 \partial_x \cdot \partial_y + \alpha_2 \partial_y^2).
\]

Then non zero components \( \Gamma^i_{jk} \) of its Wagner connection \( \nabla^\sigma \) are defined in these coordinates by the formulas:

\[
\Gamma^1_{11} = (-3\alpha_2 \partial_x \partial_x \alpha_0 + \alpha_0 \partial_x \alpha_2)/(8\alpha_0 \alpha_2), \quad \Gamma^1_{21} = (-3\alpha_2 \partial_2 \partial_y \alpha_0 + \alpha_0 \partial_y \alpha_2)/(8\alpha_0 \alpha_2),
\]
\[
\Gamma^2_{12} = (\alpha_2 \partial_x \partial_x \alpha_0 - 3\alpha_0 \partial_x \partial_x \alpha_0)/(8\alpha_0 \alpha_2), \quad \Gamma^2_{22} = (\alpha_2 \partial_2 \partial_y \alpha_0 - 3\alpha_0 \partial_y \partial_y \alpha_2)/(8\alpha_0 \alpha_2).
\]
4.2. Group-Type Symbols

Let $M$ be a connected and simply connected manifold, $\sigma$ be a regular symmetric 4-vector, and $\nabla^\sigma$ the Wagner connection. Assume that this connection is complete. We assume also that the torsion tensor $T^\sigma$ of Wagner connection $\nabla^\sigma$ is parallel, i.e.,

$$d_{\nabla^\sigma}(T^\sigma) = 0.$$  

Then, it is easy to check that the 2-dimensional vector space $g^\sigma$ of all parallel vector fields on $M$, is a Lie algebra with respect to the bracket

$$X, Y \in g^\sigma \rightarrow T^\sigma(X, Y) \in g^\sigma.$$  

For this 2-dimensional Lie algebra $g^\sigma$, we have: $T^\sigma(X, Y) = [X, Y]$ for all $X, Y \in g^\sigma$, here $[X, Y]$ is the usual bracket of vector fields,

1. $g^\sigma$ is either commutative or solvable,
2. $g^\sigma$ is commutative if and only if $T^\sigma = 0$, and
3. $g^\sigma$ is solvable if and only if $T^\sigma \neq 0$.

If $g^\sigma$ is solvable, then there is basis $X, Y \in g^\sigma$ such that $[X, Y] = X$.

**Theorem 4.** Let $\sigma \in \Sigma_4(M)$ be a regular symbol and let $\nabla^\sigma$ be the corresponding Wagner connection with parallel torsion tensor $T^\sigma$. Then:

1. Symbol $\sigma$ is locally equivalent to the symbol with constant coefficients
   $$\sigma = c_0 e^{4y} + 4c_1 e^{3y} \partial_x \cdot \partial_y + 6c_2 e^{2y} \partial_y^2 + 4c_3 e^y \partial_y^3 + c_4 \partial_y^4, \quad c_i \in \mathbb{R},$$
   if and only if $T^\sigma = 0$.

2. Symbol $\sigma$ is locally equivalent to the symbol
   $$\sigma = c_0 e^{4y} \partial_x^4 + 4c_1 e^{3y} \partial_x^3 \partial_y + 6c_2 e^{2y} \partial_x^2 \partial_y^2 + 4c_3 e^y \partial_x \partial_y^3 + c_4 \partial_y^4, \quad c_i \in \mathbb{R},$$
   if and only if $T^\sigma \neq 0$.

**Proof.** (1) The condition $T^\sigma = 0$ means that $[X, Y]=0$ for any parallel vector fields $X$ and $Y$ on $M$.

Let vector fields $X, Y$ be parallel and linearly independent at every point. Then there exist local coordinates $x, y$ in $M$ such that $X = \partial_x$ and $Y = \partial_y$. It follows that all components $\Gamma_{jk}^i$ of the Wagner connection $\nabla^\sigma$ are equal to zero in domain of these coordinates $x, y$. From equations (9), we get that coefficients of $\sigma$ in the coordinates $x, y$ are constants.

(2) Let $T^\sigma \neq 0$ and $d_{\nabla^\sigma}(T^\sigma) = 0$. Let vector fields $X, Y$ be a basis in the algebra $g^\sigma$ and $[X, Y] = X$. Then there are local coordinates $x, y$ in $M$ such that $X = \partial_x$ and $Y = \alpha(x, y) \partial_x + \beta(x, y) \partial_y$. From the condition $[X, Y] = X$, we get that $\alpha(x, y) = x + c, \ c \in \mathbb{R}$, and $\beta$ is independent of $x$. Linear independence of $X$ and $Y$ means that $\beta \neq 0$ everywhere. Thus, we can take $Y = x \partial_x + \partial_y$.

The parallelism of each vector field $X = \partial_x$ and $Y = x \partial_x + \partial_y$ gives us that $\Gamma_{21}^1 = -1$ and all other components $\Gamma_{jk}^i$ are equal to zero.

Now, system (9) is the following,

$$\partial_x a_i = 0, \quad \partial_y a_i - (4 - i)a_i = 0, \quad i = 0, 1, 2, 3, 4.$$  

Whence it follows that

$$a_0 = c_0 e^{4y}, \quad a_1 = c_1 e^{3y}, \quad a_2 = c_2 e^{2y}, \quad a_3 = c_3 e^y, \quad a_4 = c_4, \quad c_i \in \mathbb{R}.$$  

□
4.3. Symbols and Quantization

Let $\Sigma = \oplus_{k \geq 0} \Sigma^k(M)$ be the graded algebra of symmetric differential forms and let $\nabla$ be a Wagner connection of a regular symbol from $\Sigma_4(M)$. Then the covariant differential

$$d_{\nabla} : \Omega^1(M) \rightarrow \Omega^1(M) \otimes \Omega^1(M)$$

define derivation $d_{\nabla}^a : \Sigma \rightarrow \Sigma^{a+1}$ of degree one in graded algebra $\Sigma$. Namely, these derivations are defined by their actions on generators:

$$d_{\nabla}^a = d : C^\infty(M) \rightarrow \Omega^1(M) = \Sigma^1,$$
$$d_{\nabla}^a : \Omega^1(M) = \Sigma^1 \rightarrow \Omega^1(M) \otimes \Omega^1(M) \xrightarrow{\text{Sym}} \Sigma^2.$$

Let now $\sigma_k \in \Sigma_k(M)$ be a symbol. We define a differential operator $Q(\sigma_k) \in \text{Diff}_k(1)$ as follows:

$$Q(\sigma_k)(h) \overset{\text{def}}{=} \frac{1}{k!} \left\langle \sigma_k, \left( d_{\nabla}^a \right)^k(h) \right\rangle,$$

where $h \in C^\infty(M), \left( d_{\nabla}^a \right)^k(h) \in \Sigma^k(M)$, and $\langle \cdot, \cdot \rangle$ is the natural convolution

$$\Sigma_k(M) \otimes \Sigma^k(M) \rightarrow C^\infty(M).$$

Remark that the value of the symbol of the derivation $d_{\nabla}^a$ at a covector $\theta$ equals the symmetric product by $\theta$ into the module $\Sigma$. We get that the symbol of operator $Q(\sigma_k)$ equals $\sigma_k$ because the symbol of a composition of operators equals the composition of symbols.

We call this operator $Q(\sigma_k)$ a quantization of symbol $\sigma_k$. Morphism $Q : \Sigma_k \rightarrow \text{Diff}_k(1)$ splits exact sequence

$$0 \rightarrow \text{Diff}_{k-1}(1) \rightarrow \text{Diff}_k(1) \xrightarrow{\sigma_k} \Sigma^k(M) \rightarrow 0$$

by the construction.

Let now $A \in \text{Diff}_4(1)$ and $\sigma_4(A)$ be its symbol. Then operator $A - Q(\sigma_4(A))$ has order 3, and let $\sigma_3(A)$ be its symbol. Then operator $A - Q(\sigma_4(A)) - Q(\sigma_3(A))$ has order 2. Repeating this process we get subsymbols $\sigma_i(A) \in \Sigma_i(M), 0 \leq i \leq 3$, such that $A = Q(\sigma_4(A))$, where

$$\sigma_4(A) = \sigma_4(A) + \sigma_3(A) + \ldots + \sigma_0(A)$$

is a total symbol of the operator, and

$$Q(\sigma(A)) = Q(\sigma_4(A)) + Q(\sigma_3(A)) + \ldots + Q(\sigma_0(A)).$$

4.3.1. Coordinates. Let $x_1, x_2$ be local coordinates in a neighborhood $O \subset M$. Denote by $x_1, x_2, w_1, w_2$ induced standard coordinates in the tangent bundle over $O$. Then $d_{\nabla}^a(dx_k) = -\sum \Gamma_i^k d x_i \otimes d x_j$, where $\Gamma_i^k$ are the Christoffel symbols of the Wagner connection $\nabla$. Thus, in coordinates $x, w$ we have $d_{\nabla}^a(w_k) = -\sum \Gamma_i^k w_i w_j$ and the derivation $d_{\nabla}^a$ has the form

$$d_{\nabla}^a = \sum w_i \partial x_i - \sum \Gamma_i^k w_i w_j \partial w_k.$$

4.4. Differential Invariants of Constant Type Scalar Differential Operators

Here we give (see [5]), a description of the field of rational differential invariants for a fourth-order linear scalar differential operators of constant type, as well as for its symbol.

4.4.1. Differential invariants of a constant type symbol. Let $\pi : S^4T(M) \rightarrow M$ be the bundle of symmetric 4-vectors (symbols) and let $\nu_4 \in \Sigma_4(\pi)$ be the universal symbol (of order 0). We denote by $\mathcal{O}_0 \subset J^0(\pi)$ the domain of regular symbols. The symbols having the constant type $\nu$ constitute a subbundle $\pi_\nu : E^\nu \rightarrow M$ of the bundle $\pi_{|\mathcal{O}_0} : \mathcal{O}_0 \rightarrow M$ of regular symbols. Then the Wagner connection defines a total covariant differential

$$\tilde{d}_{\nabla}^a : \Sigma^1(\pi_\nu) \rightarrow \Sigma^1(\pi_\nu) \otimes \Omega^1(\pi_\nu),$$

LOBACHEVSKII JOURNAL OF MATHEMATICS Vol. 41 No. 12 2020
over the domain of regular symbols, and, by the construction \( \hat{\vartheta}_\alpha(\nu_4) = 0 \). Let \( T^\omega \in \Omega^2(\pi^\omega) \otimes \Sigma_1(\pi^\omega) \) be the total torsion of the connection and \( \theta^\omega \in \Omega^1(\pi^\omega) \) be the torsion form. Then, applying the total differential of the dual (to Wagner) connection
\[
\hat{\vartheta}_\alpha : \Omega^1(\pi^\omega) \longrightarrow \Omega^1(\pi^\omega) \otimes \Omega^1(\pi^\omega),
\]
we get tensor
\[
\hat{\vartheta}_\alpha(\theta^\omega) \in \Omega^1(\pi^\omega) \otimes \Omega^1(\pi^\omega).
\]
Taking the symmetric \( g^\omega \) and antisymmetric \( a^\omega \) parts of this tensor, we get tensors
\[
g^\omega \in \Sigma^2(\pi^\omega), \quad a^\omega \in \Omega^2(\pi^\omega).
\]
Assuming that tensor \( g^\omega \) is non degenerated, we get a total operator
\[
A^\omega \in \Sigma^1(\pi^\omega) \otimes \Omega^1(\pi^\omega),
\]
instead of \( a^\omega \), and horizontal 1-forms
\[
\theta^\omega_1 = \theta^\omega, \quad \theta^\omega_2 = A^\omega(\theta^\omega_1).
\]
Remark that the torsion \( T^\omega \) and torsion form \( \theta^\omega \) have order 1 and therefore, tensors: \( g^\omega, \ a^\omega, \ A^\omega \) and \( \theta^\omega_2 \) have order 2.

We say that a domain \( \mathcal{O}^\omega_2 \subset J^2(\pi^\omega) \) consists of regular 2-jet of symbols if the tensor \( g^\omega \) is non degenerated and \( \theta^\omega_1 \wedge \theta^\omega_2 \neq 0 \).

Let \( (e^\omega_1, e^\omega_2) \) be the frame of horizontal vector fields \( e^\omega_i \in \Sigma^1(\pi^\omega) \) dual to coframe \( (\theta^\omega_1, \theta^\omega_2) \). Then coefficients \( J^\omega_{\alpha} \) in the decomposition of universal symbol \( \nu_4 \) in this frame
\[
\nu_4 = \sum_{|\alpha|=4} J^\omega_{\alpha}(e^\omega_1)^{\alpha_1} \cdot (e^\omega_2)^{\alpha_2},
\]
are rational functions over regular domain \( \mathcal{O}^\omega_2 \) and invariants of the diffeomorphism group \( \mathcal{G}(M) \).

**Theorem 5.** The field of rational natural invariants of symbols having degree 4 and constant type \( \pi^\omega \) is generated by invariants \( J^\omega_{\alpha} \), \( |\alpha| = 4 \), and invariant derivations \( e^\omega_i \), \( i = 1, 2 \).

### 4.4.2. Differential invariants of a constant type scalar operator.

Let \( \chi^\omega_4 : \text{Diff}_4^\omega(1) \rightarrow \mathcal{M} \) be the bundle of scalar differential operator of order 4 having constant type \( \pi^\omega \) and let \( \text{Diff}^\omega(1) \) be its module of smooth sections. By \( \hat{\mathcal{O}}^\omega_2 \subset J^2(\chi^\omega_4) \) we denote the domain, where 2-jets of symbols are regular in the above sense, i.e. 2-jets of symbols belong to regular domain \( \mathcal{O}^\omega_2 \).

Denote by \( \tau^\omega : S^l T \rightarrow M \) a bundle of symbols having degree 4 and constant type \( \pi^\omega \), and let \( \tau : S^l T \rightarrow M \) be bundles of symbols of degree \( l = 0, 1, 2, 3 \) and let \( \tau^{(4)} = \tau^\omega \oplus \tau_3 \oplus \tau_2 \oplus \tau_1 \oplus \tau_0 \) be the bundle of total symbols with principle symbol having of constant type \( \pi^\omega \). Consider differential operator
\[
\mu_\omega : J^{l+1}(\chi^\omega_4) \rightarrow \tau^{(4)},
\]
which sends differential operators \( A \in \text{Diff}(1) \) having regular 2-jet \( [A]^2_{\mu} \in \hat{\mathcal{O}}^\omega_2 \) to the total symbol
\[
\sigma^{(4)}(A) = (\sigma_4(A), \sigma_3(A), ..., \sigma_0(A))
\]
with respect to the Wagner connection that corresponds to the regular principal symbol \( \sigma_4(A) \).

It follows from the construction of the Wagner connection that this operator has order 5 and is natural, i.e. commutes with the action of the diffeomorphism group \( \mathcal{G}(M) \).

Regularity conditions allow us to construct invariant coframe (10) and then by decomposing (11) the total symbol in this coframe to find natural rational invariants \( J^\omega_{\alpha} \), where \( |\alpha| \leq 4 \), on the 5-jet bundle \( J^5(\chi^\omega_4) \).

It follows from (11) that invariants \( J^\omega_{\alpha} \) and invariant derivations \( e^\omega_i \) generate the field of natural invariants of total symbols. Therefore, applying the prolongations of \( \mu_\omega \)
\[
\mu_l^{(4)} : J^{5+l}(\chi^\omega_4) \rightarrow J^l(\tau^{(4)}),
\]
we will get natural invariants of differential operators of the constant type.
Theorem 6. The field of natural differential invariants of linear scalar differential operators of order 4 having constant type $\varpi$ is generated by the basic invariants $\mu_4^\omega(J_\alpha^\omega)$, $|\alpha| \leq 4$, and invariant derivatives $e_i^\omega$, $i = 1, 2$.

In this paper, we did not discuss in detail the equivalence problem for operators from $\text{Diff}_4(1)$ and corresponding linear differential equations w.r.t. of the group $G(M)$.

Moreover, we omitted a description of field of differential $\text{Aut}(\xi)$-invariants for constant type differential operators acting on the line bundle $\xi$. In addition, we omitted a discussion of equivalence problem for these operators and corresponding differential equations w.r.t. of the automorphism group $\text{Aut}(\xi)$.

All of this can be done similar to [5].

FUNDING

This work is supported by the Russian Foundation for Basic Research under grant 18-29-10013 mk.

REFERENCES

1. D. Alekseevskij, V. Lychagin, and A. Vinogradov, “Basic ideas and concepts of differential geometry,” in Encyclopedia of Mathematical Sciences, Geometry 1 (Springer, Berlin, 1991), Vol. 28.
2. D. Hilbert, Theory of Algebraic Invariants (Cambridge Univ. Press, Cambridge, 1993).
3. V. V. Lychagin and V. A. Yumaguzhin, “Classification of second order linear differential operators and differential equations,” J. Geom. Phys. 130, 213–228 (2018).
4. V. V. Lychagin and V. A. Yumaguzhin, “On equivalence of third order linear differential operators on two-dimensional manifolds,” J. Geom. Phys. 146, 103507 (2019).
5. V. V. Lychagin and V. A. Yumaguzhin, “On structure of linear differential operators, acting in line bundles,” J. Geom. Phys. 148, 103549 (2020).
6. I. S. Krasilschik, V. V. Lychagin, and A. M. Vinogradov, Geometry of Jet Spaces and Differential Equations (Gordon and Breach, New York, 1986).
7. B. Kruglikov and V. Lychagin, "Geometry of Differential Equations," in Handbook of Global Analysis, Ed. by D. Krupka and D. Saunders (Elsevier, Amsterdam, 2008), pp. 725–772.
8. B. Kruglikov and V. Lychagin, “Global Lie-Tresse theorem,” Selecta Math. (N.S.) 22, 1357–1411 (2016).
9. Valentin Lychagin, “Quantum mechanics on manifolds,” Acta Appl. Math. 56, 231–251 (1999).
10. M. Rosenlicht, “A remark on quotient spaces,” An. Acad. Brasil. Ci. 35, 487–489 (1963).
11. V. V. Wagner, “Two dimensional space with cubic metric,” Uch. Zap. SGU 1 (XIV) (1) (1938).