On the $\tau$-functions of the reduced Ostrovsky equation and the $A_2^{(2)}$ two-dimensional Toda system

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Abstract

The reciprocal link between the reduced Ostrovsky equation and the $A_2^{(2)}$ two-dimensional Toda (2D-Toda) system is used to construct the $N$-soliton solution of the reduced Ostrovsky equation. The $N$-soliton solution of the reduced Ostrovsky equation is presented in the form of pfaffian through a hodograph (reciprocal) transformation. The bilinear equations and the $\tau$-function of the reduced Ostrovsky equation are obtained from the period 3-reduction of the $B_\infty$ or $C_\infty$ 2D-Toda system, i.e. the $A_2^{(2)}$ 2D-Toda system. One of the $\tau$-functions of the $A_2^{(2)}$ 2D-Toda system becomes the square of a pfaffian which also becomes a solution of the reduced Ostrovsky equation. There is another bilinear equation which is a member of the 3-reduced extended BKP hierarchy. Using this bilinear equation, we can also construct the same pfaffian solution.

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1. Introduction

In this paper, we study the $N$-soliton solutions of the reduced Ostrovsky equation [1, 2]

$$\partial_t (\partial_t + u \partial_x) u - 3u = 0, \quad (1.1)$$

which is a special case ($\beta = 0$) of the Ostrovsky equation

$$\partial_t (\partial_t + u \partial_x + \beta \partial_x^3) u - \gamma u = 0. \quad (1.2)$$

The Ostrovsky equation was originally derived as a model for weakly nonlinear surface and internal waves in a rotating ocean [1, 2]. Later, the same equation was derived from different physical situations by several authors [2–4]. Note that the reduced Ostrovsky equation (1.1) is sometimes called the Vakhnenko equation or the Ostrovsky–Hunter equation [4–8]. Vakhnenko et al constructed the $N$ (loop) soliton solution of the reduced Ostrovsky equation by using
Differentiating the reduced Ostrovsky equation (1.1) with respect to \( x \), we obtain
\[
\frac{u_{xx}}{u_{ttx}} + 3u_{ux}u_{uxx} + u_{uxxx} - 3u_{xx} = 0, 
\]
which is known as the short wave limit of the Degasperis–Procesi (DP) equation [9, 10]. This equation is derived from the DP equation [11] by taking a short wave limit \( \epsilon \to 0 \) with
\[
\begin{align*}
U &= \epsilon^2 (u + \epsilon u_1 + \cdots), \\
T &= \epsilon^{-1} t, \\
X &= \epsilon x.
\end{align*}
\]
Using this connection, Matsuno constructed the \( N \)-soliton solution of the short wave model of the DP equation, i.e. the reduced Ostrovsky equation, from the \( N \)-soliton solution of the DP equation [10, 12, 13]. This \( N \)-soliton formula is equivalent to the one obtained by Vakhnenko. Hone and Wang pointed out that there is a reciprocal link between the reduced Ostrovsky equation and the first negative flow in the Sawada–Kotera hierarchy [9].

In this paper, we show the reciprocal link between the reduced Ostrovsky equation and the \( A_2^{(2)} \) two-dimensional Toda (2D-Toda) system and investigate their \( \tau \)-functions. Using this reciprocal link, we construct the \( N \)-soliton solution of the reduced Ostrovsky equation in the form of pfaffian. The bilinear equations and the \( \tau \)-functions of the reduced Ostrovsky equation are systematically obtained from the period 3-reduction of the \( B_\infty \) or \( C_\infty \) 2D-Toda system.

One of the \( \tau \)-functions of the \( A_2^{(2)} \) 2D-Toda system becomes the square of a pfaffian, by which the \( N \)-soliton solution of the reduced Ostrovsky equation is expressed. We also show that another bilinear equation which is a member of the 3-reduced extended BKP hierarchy, or the so-called negative Sawada–Kotera hierarchy, can give rise to the reduced Ostrovsky equation by a hodograph transformation. Using this bilinear equation, we can also construct the same pfaffian solution.

2. The reduced Ostrovsky equation, the period 3-reduction of the \( B_\infty \) 2D-Toda system and the 3-reduced extended BKP hierarchy

2.1. The \( 2D \)-Toda system of \( B_\infty \)-type and its period 3-reduction

The \( 2D \)-Toda system of \( A_\infty \)-type, which is also called the Toda field equation or the 2D-Toda lattice, is given as follows [14–17]:
\[
\frac{\partial^2 \theta_n}{\partial x_1 \partial x_{-1}} = -\sum_{m \in \mathbb{Z}} a_{n,m} e^{-\theta_n}, \quad n \in \mathbb{Z},
\]
where the matrix \( A = (a_{n,m}) \) is the transpose of the Cartan matrix for the infinite-dimensional Lie algebra \( A_\infty \) [18].

The \( A_\infty \) 2D-Toda system (2.1) may be written as
\[
\frac{\partial^2 \theta_n}{\partial x_1 \partial x_{-1}} = e^{-\theta_{n-1}} - 2e^{-\theta_n} + e^{-\theta_{n+1}}.
\]
The \( A_\infty \) 2D-Toda system (2.1) is transformed into the bilinear form
\[
-(\frac{1}{2} D_n D_{n-1} - 1) \tau_n \cdot \tau_n = \tau_{n-1} \tau_{n+1},
\]
through the dependent variable transformation
\[
\theta_n = -\ln \frac{\tau_{n+1} \tau_{n-1}}{\tau_n^2}.
\]
Here \( D_n \) is the Hirota \( D \)-operator which is defined as
\[
D_n a(x) \cdot b(x) = (\partial_x - \partial_{x'})^n a(x)b(x')|_{x=x'}.
\]
Lemma 2.1 (Ueno–Takasaki [19], Babich–Matveev–Sall [20], Hirota [21], Nimmo–Willox [18]). The bilinear equations of the 2D-Toda lattice hierarchy including (2.3) have the following Gram-type determinant solution:

\[ \tau_n = \det \left( \psi_{i,j}^{(n)} \right)_{1 \leq i,j \leq M}, \]  

(2.6)

where

\[ \psi_{i,j}^{(n)} = c_{i,j} + (-1)^n \int_{-\infty}^{x_1} \psi_{i}^{(n)} \psi_{j}^{(n)} \, dx_1. \]  

(2.7)

Here \( c_{i,j} \) are the constants, and \( \psi_{i,j}^{(n)} \) and \( \hat{\psi}_{i,j}^{(n)} \) satisfy

\[ \frac{\partial \psi_{i,j}^{(n)}}{\partial x_k} = \psi_{i,j}^{(n+k)}, \quad \frac{\partial \hat{\psi}_{i,j}^{(n)}}{\partial x_k} = (-1)^{k-1} \hat{\psi}_{i,j}^{(n+k)}, \]  

(2.8)

for \( k = \pm 1, \pm 2, \pm 3, \ldots \).

For example, the linear independent set of functions \( \{ \psi_{1}^{(n)}, \hat{\psi}_{1}^{(n)} \} \) such that

\[ \psi_{1}^{(n)} = p^n e^{\xi_1}, \quad \hat{\psi}_{1}^{(n)} = q^n e^{\eta_1}, \]

\[ \xi_1 = p_0 x_1 + \frac{1}{2} x_2 + \frac{1}{3} x_3 + \cdots + \xi_0 \] and \( \eta_1 = q_0 x_1 + \frac{1}{2} x_2 + \frac{1}{3} x_3 + \cdots + \eta_0 \) for \( i, j = 1, 2, \ldots, M \), gives the \( M \)-soliton solution of the \( A_\infty \) 2D-Toda system.

Proof. See [21]. \( \square \)

We impose the \( B_\infty \)-reduction \( \theta_0 = \theta_{1-n} \) on the \( A_\infty \) 2D-Toda system (2.1), i.e. fold the infinite sequence \( \{ \ldots, \theta_{-2}, \theta_{-1}, \theta_0, \theta_1, \ldots \} \) in the midpoint between \( \theta_0 \) and \( \theta_1 \) [22, 19, 18, 23]. Under this constraint, we have \( \theta_0 = \theta_1, \theta_{-1} = \theta_2, \theta_{-2} = \theta_3, \ldots \).

For \( n = 1 \),

\[ \frac{\partial^2 \theta_1}{\partial x_1 \partial x_{-1}} = e^{-\theta_1} - 2e^{-\theta_0} + e^{-\theta_2} \]

\[ = e^{-\theta_0} - e^{-\theta_1} \]

\[ = e^{-\theta_0} - 2e^{-(\theta_0 + \ln 2)}. \]

For \( n = 2 \),

\[ \frac{\partial^2 \theta_2}{\partial x_1 \partial x_{-1}} = e^{-\theta_1} - 2e^{-\theta_0} + e^{-\theta_3} \]

\[ = 2e^{-(\theta_0 + \ln 2)} - 2e^{-\theta_0} + e^{-\theta_1}. \]

After redefining \( \theta_1 \) by \( \theta_1 + \ln 2 \rightarrow \theta_1 \), we obtain the \( B_\infty \) 2D-Toda system [14, 17]:

\[ \frac{\partial^2 \theta_n}{\partial x_1 \partial x_{-1}} = - \sum_{m \in \mathbb{Z} \geq 1} a_{n,m} e^{-\theta_m}, \quad \text{for} \ n \in \mathbb{Z} \geq 1, \]  

(2.9)

where the matrix \( A = (a_{n,m}) \) is the transpose of the Cartan matrix for the infinite-dimensional Lie algebra \( B_\infty \) [18].

The \( B_\infty \) 2D-Toda system (2.9) is transformed into the bilinear equations

\[ -\left( \frac{1}{2} D_n D_{x_{-1}} - 1 \right) \tau_1 \cdot \tau_1 = \tau_2 \cdot \tau_2, \]  

(2.10)

\[ -\left( \frac{1}{2} D_n D_{x_{-1}} - 1 \right) \tau_n \cdot \tau_n = \tau_{n-1} \tau_{n+1}, \quad \text{for} \ n \geq 2, \]  

(2.11)

through the dependent variable transformation

\[ \theta_1 = - \ln \frac{\tau_2}{\tau_1}, \quad \text{and} \quad \theta_n = - \ln \frac{\tau_{n+1} \tau_{n-1}}{\tau_n^2}, \]  

(2.12)

for \( n \geq 2 \).
Lemma 2.2. The bilinear equations of the $B_{\infty}$ 2D-Toda system (2.10) and (2.11) have the N-soliton solution which is expressed as

$$\tau_n = \det(\psi_{i,j}^{(n)})_{1 \leq i,j \leq 2N},$$

where

$$\psi_{i,j}^{(n)} = c_{i,j} - 2(-1)^n \int_{-\infty}^{x_1} \phi_i^{(n)} \phi_j^{(-n+1)} \, dx_1,$$

$$\phi_i^{(n)} = p_i^n e^{\xi_i}, \quad \xi_i = p_i x_1 + \frac{1}{p_i^3} x_3 + \frac{1}{p_i^3} x_3 + \cdots + \xi_0,$$

and $c_{i,j} = -c_{j,i}, c_{i,i} = 0$.

Proof. Imposing the $B_{\infty}$ reduction $\tau_n = \tau_{1-n}$, i.e. folding the sequence of the $\tau$-functions $\{\ldots, \tau_{-2}, \tau_{-1}, 0, \tau_1, \tau_2, \tau_3, \ldots\}$ in the midpoint between $\tau_0 = 0$ and $\tau_1$, we have $\tau_0 = \tau_1, \tau_{-1} = \tau_2, \tau_{-2} = \tau_3, \ldots$ [22, 19, 18, 23]. Thus, we obtain the bilinear equations (2.10) and (2.11) from the $A_{\infty}$ 2D-Toda bilinear equation (2.3).

To impose the $B_{\infty}$ reduction on the Gram-type determinant solution of the $A_{\infty}$ 2D-Toda system, we impose the constraint $\phi_j^{(n)} = -2\frac{d}{dx} \psi_j^{(n)} = -2\psi_j^{(n+1)}, c_{i,j} = -c_{j,i}, c_{i,i} = 0, M = 2N$ and $x_{2k+1} = 0$ for every integer $k$. With this constraint, each element of the Gram-type determinant has the following property:

$$\psi_{i,j}^{(n)} = c_{i,j} - 2(-1)^n \int_{-\infty}^{x_1} \phi_i^{(n)} \phi_j^{(-n+1)} \, dx_1 = -c_{i,j} + 2(-1)^{1-n} \int_{-\infty}^{x_1} \phi_j^{(-n+1)} \phi_i^{(n)} \, dx_1 = -\psi_{j,i}^{(-1-n)}.$$

Then the $\tau$-function satisfies $\tau_n = \tau_{1-n}$. Therefore, the N-soliton solution of the $B_{\infty}$ 2D-Toda lattice is expressed by the above Gram-type determinant.

Definition 2.3. Let $A = (a_{i,j})_{1 \leq i,j \leq 2N}$ be a $2N \times 2N$ skew-symmetric matrix. The pfaffian of $A$ is defined by

$$\text{pf}(A) = \text{pf}(a_{i,j})_{1 \leq i,j \leq 2N} = \text{pf}(1, 2, \ldots, 2N) = \sum_{i_1 < i_2 < \cdots < i_N} \text{sgn} (\pi) a_{i_1,j_1} a_{i_2,j_2} \cdots a_{i_N,j_N},$$

where $\pi = (1_1 \ 2 \ 3 \ \cdots \ \frac{2N-1}{2} \ \frac{2N}{2} \ \cdots \ \frac{2N-1}{2} \ \frac{2N}{2})$ is a permutation of $\{1, 2, \ldots, 2N + 1, 2N\}$.

The pfaffian can be computed recursively by

$$\text{pf}(1, 2, \ldots, 2N, 2N + 1) = \sum_{i=2}^{2N} (-1)^i \text{pf}(1, 2, \ldots, i, i+1, \ldots, 2N - 1, 2N).$$

For a skew-symmetric matrix $A$, we have $[\text{pf}(A)]^2 = \det(A)$.

Lemma 2.4. The $\tau$-function of the bilinear equations (2.10) and (2.11) is written in the form of pfaffian,

$$\tau_1 = \tau^2, \quad \tau = \text{pf}(1, 2, \ldots, 2N - 1, 2N),$$

where $\text{pf}(i, j) = c_{i,j} + \int_{-\infty}^{x_1} D_{x_i} \phi_i^{(0)} \phi_j^{(0)} \, dx_1, \phi_i^{(n)} = p_i^n \phi_i^{(1)}, \phi_j^{(0)} = p_j^1 \phi_j, \xi_i = p_i^{-1} x_{i-1} + p_i x_1 + p_i^1 x_3 + \frac{1}{p_i^1} x_3 + \cdots + \xi_j^0, c_{i,j} = -c_{j,i}, c_{i,i} = 0.$
Lemma 2.5. The bilinear equations

\[-(\frac{1}{2}D_{\alpha}D_{\alpha,-1} - 1)\tau_1 \cdot \tau_1 = \tau_1 \tau_2,\]

\[-(\frac{1}{2}D_{\alpha}D_{\alpha,-1} - 1)\tau_2 \cdot \tau_2 = \tau_1^2\]

have the N-soliton solution which is expressed as

\[\tau_n = \det(\psi_{i,j}^{(n)})_{1 \leq i,j \leq 2N},\]

where

\[\psi_{i,j}^{(n)} = c_{i,j} + \frac{2p_i}{p_i + p_j} \left( \frac{p_i}{p_j} \right)^{n-1} e^{\xi_i + \xi_j}, \quad \xi_i = p_i x_i + \frac{1}{p_i} x_{i-1} + \xi_0,\]

and \(c_{i,j} = \delta_{j,2N+1-i} c_i, \ c_i = -c_{2N+1-i}\) \(p_i^2 - p_i p_{2N+1-i} + p_{2N+1-i} = 0\).

\[\Box\]

Proof. See [21, 24].

Corollary 2.6. The \(\tau\)-function \(f = \tau_1\) of the bilinear equations (2.20) and (2.21) is written in the form of pfaffian,

\[f = \tau_1 = \tau^2, \quad \tau = pf(1, 2, \ldots, 2N - 1, 2N),\]

where

\[pf(i, j) = c_{i,j} + \frac{p_i - p_j}{p_i + p_j} e^{\xi_i + \xi_j}, \quad \xi_i = p_i x_i + \frac{1}{p_i} x_{i-1} + \xi_0,\]

and \(c_{i,j} = \delta_{j,2N+1-i} c_i, \ c_i = -c_{2N+1-i}\) \(p_i^2 - p_i p_{2N+1-i} + p_{2N+1-i} = 0\).
Proof. Using lemmas 2.4 and 2.5, we obtain the above formula.

Let $k_i = p_i + p_{2N+i-1}$. From $p_i^2 - p_ip_{2N+i-1} + p_{2N+i-1}^2 = 0$, we obtain $p_i = \frac{1}{6}(3k_i + iv\sqrt{3k_i})$, $p_{2N+i-1} = \frac{1}{6}(3k_i - iv\sqrt{3k_i})$, $p_ip_{2N+i-1} = \frac{k_i^2}{6}$ and $\frac{1}{p_i} + \frac{1}{p_{2N+i-1}} = \frac{3}{k_i}$. Thus,

$$\xi_i + \xi_{2N+i-1} = k_i x_1 + \frac{3}{k_i} x_{-1} + \xi_0 + \xi_{2N+i-1}.$$

In the pfaffian solution, all phase functions can be expressed by the summation of $\xi_i + \xi_{2N+i-1}$. So the phase functions can be expressed by the parameters $\{k_i\}$ ($i = 1, 2, \ldots, N$). Each coefficient of exponential functions can be normalized to 1 after absorption into phase constants or can be rewritten by the parameters $\{k_i\}$. Thus, it is possible to rewrite the above $\tau$-function by using the parameters $\{k_i\}$ instead of $\{p_i\}$.

Examples: For $N = 1$,

$$\tau = \text{pf}(1, 2) = c_1 + \frac{p_1 - p_2}{p_1 + p_2} e^{\xi_1 + \xi_2}. \quad (2.23)$$

Letting $c_1 = 1$ and $e^{\gamma_1} = \frac{p_1 - p_2}{p_1 + p_2}$, the $\tau$-function can be rewritten as

$$\tau = \text{pf}(1, 2) = 1 + e^{\xi_1 + \xi_2 + \gamma_1}, \quad (2.24)$$

where

$$\xi_1 + \xi_2 = k_1 x_1 + \frac{3}{k_1} x_{-1} + \xi_10 + \xi_{20}.$$

For $N = 2$,

$$\tau = \text{pf}(1, 2, 3, 4) = \text{pf}(1, 2)\text{pf}(3, 4) - \text{pf}(1, 3)\text{pf}(2, 4) + \text{pf}(1, 4)\text{pf}(2, 3)$$

$$= \frac{p_1 - p_2}{p_1 + p_2} e^{\xi_1 + \xi_2} + \frac{p_3 - p_4}{p_3 + p_4} e^{\xi_3 + \xi_4} - \frac{p_1 - p_3}{p_1 + p_3} e^{\xi_1 + \xi_3} - \frac{p_2 - p_4}{p_2 + p_4} e^{\xi_2 + \xi_4} + \left( \frac{p_1 - p_2}{p_1 + p_2} e^{\xi_1 + \xi_2} \right) \left( \frac{p_3 - p_4}{p_3 + p_4} e^{\xi_3 + \xi_4} \right)$$

$$= c_1 c_2 + c_2 \left( \frac{p_1 - p_2}{p_1 + p_2} e^{\xi_1 + \xi_2} \right) + \left( \frac{p_3 - p_4}{p_3 + p_4} e^{\xi_3 + \xi_4} \right)$$

$$= \left( \frac{p_1 - p_2}{p_1 + p_2} e^{\xi_1 + \xi_2} \right) \left( \frac{p_3 - p_4}{p_3 + p_4} e^{\xi_3 + \xi_4} \right)$$

Letting $c_1 = c_2 = 1$ and $e^{\gamma_1} = \frac{p_1 - p_2}{p_1 + p_2}$ and $e^{\gamma_2} = \frac{p_3 - p_4}{p_3 + p_4}$, the above $\tau$-function becomes

$$\tau = 1 + e^{\xi_1 + \xi_2 + \gamma_1} + e^{\xi_3 + \xi_4 + \gamma_2} + b_{12} e^{\xi_1 + \xi_2 + \xi_3 + \xi_4 + \gamma_1 + \gamma_2},$$

where

$$b_{12} = \frac{p_1 - p_2}{p_1 + p_2} + \frac{p_3 - p_4}{p_3 + p_4} = \frac{p_1 - p_3}{p_1 + p_3} + \frac{p_2 - p_4}{p_2 + p_4} + \frac{p_1 - p_4}{p_1 + p_4} + \frac{p_2 - p_3}{p_2 + p_3}$$

$$= \frac{(k_1 - k_2)^2(k_3^2 - k_1 k_2 + k_1^2)}{(k_1 + k_2)^2(k_3^2 + k_1 k_2 + k_2^2)}.$$

Then,

$$\xi_1 + \xi_4 = k_1 x_1 + \frac{3}{k_1} x_{-1} + \xi_10 + \xi_{40}, \quad \xi_2 + \xi_3 = k_2 x_1 + \frac{3}{k_2} x_{-1} + \xi_20 + \xi_{30}.$$
The $N$-soliton solution is written in the following form:

$$
\tau = \sum_{\mu=0,1} \exp \left[ \sum_{i=1}^{N} \mu_i \eta_i + \sum_{i<j}^{(N)} \mu_i \mu_j \ln b_{ij} \right],
$$

where

$$
b_{ij} = \frac{(k_i - k_j)^2 (k_i^2 - k_i k_j + k_j^2)}{(k_i + k_j)^2 (k_i^2 + k_i k_j + k_j^2)} \text{ for } i < j, \quad \eta_i = \xi_i + \xi_{2N+1-i},
$$

(2.25)

and

$$
\sum_{\mu=0,1} \text{ means the summation over all possible combinations of } \mu_i = 0 \text{ or } 1 \text{ for } i = 1, 2, \ldots, N, \text{ and } \sum_{i<j}^{(N)} \text{ means the summation over all possible combinations of } N \text{ elements under the condition } i < j. \text{ This is consistent with the results in [5, 6].}
$$

**Lemma 2.7.** The $\tau$-function $f$ of (2.20) and (2.21) gives the solution of the reduced Ostrovsky equation through the dependent variable transformation

$$
u = -\left(\ln f\right)_{x_1x_1} = -2\left(\ln \tau\right)_{x_1x_1},
$$

and the hodograph (reciprocal) transformation

$$
\begin{align*}
x &= x_{-1} + \int_{-\infty}^{x_{1}} u(x', x_{-1}) \, dx' \\
t &= x_{1} - \left(\ln f\right)_{x_1}.
\end{align*}
$$

(2.26)

**Proof.** The bilinear equation (2.20) is rewritten as

$$
-\left(\ln f\right)_{x_1x_1} = \frac{g}{f} - 1.
$$

(2.27)

Let

$$
\rho = \frac{f}{g} = \frac{1}{1 - \left(\ln f\right)_{x_1x_1}} \quad \text{and} \quad u = -\left(\ln f\right)_{x_1x_1}.
$$

(2.28)

Then (2.27) becomes

$$
\frac{u_{x_{-1}}}{\rho} = \left(\frac{1}{\rho}\right)_{x_1}.
$$

(2.29)

This is rewritten as

$$
\left(\ln \rho\right)_{x_1} = -\rho u_{x_{-1}}.
$$

(2.30)

The bilinear equations (2.20) and (2.21) are written as

$$
-\left(\ln f\right)_{x_1x_1} + 1 = \frac{1}{\rho},
$$

(2.31)

$$
-\left(\ln g\right)_{x_1x_1} + 1 = \rho^2.
$$

(2.32)

Subtracting (2.32) from (2.31), we obtain

$$
-\left(\ln \rho\right)_{x_1x_1} = \frac{1}{\rho} - \rho^2,
$$

(2.33)

which leads to

$$
\rho^3 = 1 + \rho \left(\ln \rho\right)_{x_1x_1}.
$$

(2.34)
Using (2.30), it becomes
\[ \rho^3 = 1 - \rho (\rho u_{x_{-1}})_{x_{-1}}. \]  

(2.35)

Let us consider the hodograph (reciprocal) transformation
\[ \begin{cases} 
  x = x_{-1} + \int_{-\infty}^{x_1} u(x', x_{-1}) \, dx', \\
  t = x_1 - (\ln f)_{x_1},
\end{cases} \]

(2.36)

This yields
\[ \begin{cases} 
  \frac{\partial x}{\partial x_{-1}} = 1 - (\ln f)_{x_{-1} x_{-1}}, \\
  \frac{\partial x}{\partial x_1} = -(\ln f)_{x_1 x_1},
\end{cases} \]

(2.37)

and
\[ \begin{cases} 
  \partial_{x_{-1}} = \frac{1}{\rho} \partial_x, \\
  \partial_{x_1} = \partial_t + u \partial_x.
\end{cases} \]

(2.38)

Applying the hodograph (reciprocal) transformation to (2.30) and (2.35), we obtain
\[ \begin{cases} 
  (\partial_t + u \partial_x) \ln \rho = -u_x, \\
  \rho^3 = 1 - u_{xx}.
\end{cases} \]

(2.39)

This is equivalent to
\[ (\partial_t + u \partial_x) \ln (1 - u_{xx}) = -3u_x, \]

(2.40)

which can be written as
\[ (\partial_t + u \partial_x)(1 - u_{xx}) = -3u_x(1 - u_{xx}). \]

(2.41)

This is nothing but the short wave limit of the DP equation which is equivalent to the reduced Ostrovsky equation.

\[ \square \]

Remark 2.8. Setting \( \rho = \frac{1}{H} \) in (2.33), we obtain
\[ (\ln \mathcal{H})_{x_{-1} x_{-1}} = \mathcal{H} = \frac{1}{H^2}. \]

(2.42)

This is the Tzitzeica equation which is one of prime integrable systems that appears in classical differential geometry [23–26]. This equation is sometimes called the Dodd–Bullough–Mikhailov equation since this was found independently by Dodd, Bullough and Mikhailov [27, 15, 16].

Theorem 2.9. The N-soliton solution of the reduced Ostrovsky equation is given by the following formula:

\[ u = -(\ln f)_{x_{1} x_{1}} = -2(\ln \tau)_{x_{1} x_{1}}, \]

(2.43)

\[ f = \tau^2, \quad \tau = pf(1, 2, \ldots, 2N - 1, 2N), \]

where

\[ pf(i, j) = c_{i,j} + \frac{p_i - p_j}{p_i + p_j} e^{\xi_i + \xi_j}, \quad \xi_i = p_i x_1 + \frac{1}{p_i} x_{-1} + \xi_{0i}, \]

and

\[ \xi_{0i} = c_{i,0} + \frac{p_i}{p_i + p_j} e^{\xi_j}. \]
and $c_{i,j} = \delta_{j,2N+1-i}c_i, c_i = -c_{2N+1-i}, p_i^2 - p_i p_{2N+1-i} + p_{2N+1-i}^2 = 0,$
\[
\begin{align*}
x &= x_{-1} + \int_{-\infty}^{x_1} u(x_1', x_{-1}) \, dx_1' \\
&= x_{-1} - \ln f(x_1), \\
t &= x_1,
\end{align*}
\] (2.44)

**Proof.** From corollary 2.6 and lemma 2.7, we obtain this theorem. \qed

**Corollary 2.10.** The $N$-soliton solution of the reduced Ostrovsky equation is given by the following formula:
\[
u = -2(\ln \tau)_{x_1, x},
\]
\[
\tau = \sum_{\mu=0,1} \exp \left[ \sum_{i=1}^{N} \mu_i \eta_i + \sum_{i<j}^{(N)} \mu_i \mu_j \ln b_{ij} \right],
\] (2.45)

\[
b_{ij} = \frac{(k_i - k_j)^2 (k_i^2 + k_i k_j + k_j^2)}{(k_i + k_j)^2 (k_i^2 - k_i k_j + k_j^2)} \quad \text{for} \quad i < j, \quad \eta_i = k_i x_1 + \frac{3}{k_i} x_{-1} + \eta_{0i},
\]

and
\[
\begin{align*}
x &= x_{-1} + \int_{-\infty}^{x_1} u(x_1', x_{-1}) \, dx_1' \\
&= x_{-1} - \ln f(x_1), \\
t &= x_1,
\end{align*}
\] (2.46)

Here $\sum_{\mu=0,1}$ means the summation over all possible combinations of $\mu_i = 0$ or $1$ for $i = 1, 2, \ldots, N,$ and $\sum_{i<j}^{(N)}$ means the summation over all possible combinations of $N$ elements under the condition $i < j.$

Note that the form of the $N$-soliton solution in corollary 2.10 is equivalent to the one obtained by Morrison, Parkes and Vakhnenko in [6].

**Remark 2.11.** It is well known that the 3-reduced B-Toda system is exactly the same as the 3-reduced C-Toda system. Thus, the above $\tau$-function can be obtained by the period 3-reduction of $C_{\infty}$ 2D-Toda system. For more details, see the appendix.

2.2. The reduced Ostrovsky equation and the 3-reduced extended BKP hierarchy

Consider a member of the extended BKP hierarchy (the BKP hierarchy with negative time variables) [21, 22, 28–32],
\[
\left[ (D_{x_1} - D_{x_1}^0) D_{x_{-1}} + 3 D_{x_1}^2 \right] \tau = 0,
\] (2.47)

which is the bilinear equation of a nonlinear partial differential equation
\[
w_{x_{-1}, x_{1}} - w_{x_1, x_1} - 3(w_{x_1} w_{x_{-1}}) x_1 + 3w_{x_1, x_1} = 0,
\] (2.48)

through the dependent variable transformation $u = 2(\ln \tau)_{x_1}.$ [21].

The $N$-soliton solution of (2.47) is expressed in the form of pfaffian.
Lemma 2.13 (Date–Jimbo–Kashiwara–Miwa [28], Hirota [21, 31, 32]). The bilinear equation (2.47) has the pfaffian solution
\[
\tau = pf(1, 2, \ldots, 2N - 1, 2N),
\]
where \( pf(i, j) = c_{i,j} + \int_{-\infty}^{\xi_i} D_{x_i} h_i(x) \cdot h_j(x) \, dx_i \), \( c_{i,j} = -c_{j,i}, \ c_{i,i} = 0, \ x = (\ldots, x_{-3}, x_{-1}, x_1, x_3, x_5, \ldots) \) and \( h_i(x) \) satisfies the linear differential equation
\[
\frac{\partial}{\partial x_i} h_i(x) = \frac{\partial^2}{\partial x_i^2} h_i(x). \tag{2.49}
\]

Here we define
\[
\frac{\partial}{\partial x_{-1}} h_i(x) = \int_{-\infty}^{\xi_i} h_i(x) \, dx_i, \tag{2.50}
\]
for \( n = -1 \). Particularly, we could choose
\[
h_i(x) = \exp\left( p_i^{-1} x_{-1} + p_i x_1 + p_i^3 x_3 + \xi_i^0 \right). \tag{2.51}
\]

Alternatively, the \( N \)-soliton solution of (2.47) can be expressed as
\[
\tau = \sum_{\mu_i=0,1} \exp \left[ \sum_{i=1}^{N} \mu_i \eta_i + \sum_{1 \leq i < j \leq N} B_{i,j} \mu_i \mu_j \right], \tag{2.52}
\]
where \( \sum_{\mu_i=0,1} \) denotes the summation over all possible combinations of \( \mu_i = 0, 1 \) for \( i = 1, 2, \ldots, N \), and \( \sum_{1 \leq i < j \leq N} \) is the sum over all pairs \( i, j \) (\( i < j \)) chosen from \( \{1, 2, \ldots, 2N\} \). Here \( \exp(\eta_i) \) is defined as
\[
\exp(\eta_i) = \exp(\xi_i + \hat{\xi}_i),
\]
where \( \xi_i = p_i^{-1} x_{-1} + p_i x_1 + p_i^3 x_3 + \xi_i^0 \) and \( \hat{\xi}_i = q_i^{-1} x_{-1} + q_i x_1 + q_i^3 x_3 + \hat{\xi}_i^0 \), and \( B_{i,j} \) is given by
\[
e^{B_{i,j}} = h_{i,j} = \frac{(p_i - p_j)(p_i - q_j)(q_i - p_j)(q_i - q_j)}{(p_i + p_j)(p_i + q_j)(q_i + p_j)(q_i + q_j)},
\]

Proof. See [21]. \( \square \)

Lemma 2.12. The bilinear equation
\[
(D_{x_i} D_{x_i}^3 - 3D_{x_i}^2) \tau \cdot \tau = 0 \tag{2.53}
\]
has the following \( N \)-soliton solution:
\[
\tau = pf(1, 2, \ldots, 2N - 1, 2N),
\]
where
\[
 pf(i, j) = c_{i,j} + \frac{p_i - p_j}{p_i + p_j} e^{\xi_i + \hat{\xi}_i}, \quad \xi_i = p_i x_1 + p_i^{-1} x_{-1} + \xi_i^0,
\]
and \( c_{i,j} = \delta_{j,2N+1-i} c_i, \ c_i = -c_{2N+1-i}, \ p_i^2 - p_i p_{2N+1-i} + p_{2N+1-i}^2 = 0. \)

Proof. Since all phase functions are expressed by the summation of \( \xi_i + \hat{\xi}_{2N+1-i} \), \( x_3 \) is eliminated from the \( \tau \)-function by imposing the constraints \( p_i^2 - p_i p_{2N+1-i} + p_{2N+1-i}^2 = 0 \) for \( i = 1, 2, \ldots, N \) and \( c_{i,j} = \delta_{j,2N+1-i} c_i, \ c_i = -c_{2N+1-i} \), on the \( N \)-soliton solution of the BKP \( \tau \)-function in lemma 2.12 [22, 28–30]. Thus, \( \partial_3 \tau = 0 \) is satisfied. \( \square \)
Lemma 2.14. The reduced Ostrovsky equation (1.1) is transformed into the bilinear equation
\[(D_{x_1} D_{x_1}^3 - 3 D_{x_1}^2) \tau \cdot \tau = 0,\] (2.54)
through the dependent transformation
\[u = -w_{x_1} = -2(\ln \tau)_{x_1, x_1},\] (2.55)
and the hodograph (reciprocal) transformation (2.26).

Proof. Applying (2.38) to the reduced Ostrovsky equation (1.1), we obtain
\[\rho u_{x_1 x_{-1}} = 3u,\] (2.56)
which yields
\[u_{x_1, x_{-1}} = \frac{3}{\rho} u.\] (2.57)
Let us introduce \(w\) such that \(u = -w_{x_1} = -(\ln f)_{x_1, x_1}\). From (2.31), we obtain
\[\frac{1}{\rho} = 1 - w_{x_{-1}}.\] (2.58)
Using this relation, (2.57) can be rewritten as
\[w_{x_1, x_{-1}} + 3w_{x_{-1}} w_{x_1} - 3w_{x_1} = 0.\] (2.59)
This equation is transformed into a bilinear equation
\[(D_{x_1} D_{x_1}^3 - 3 D_{x_1}^2) \tau \cdot \tau = 0,\] (2.60)
through the dependent transformation
\[w = 2(\ln \tau)_{x_1}.\] (2.61)
Note that the hodograph (reciprocal) transformation (2.26) can be expressed as
\[
\begin{cases}
    x = x_{-1} - 2(\ln \tau)_{x_1}, \\
    t = x_1.
\end{cases}
\] (2.62)

From lemmas 2.13 and 2.14, we can also prove theorem 2.9.

Remark 2.15. Vakhnenko and Parkes found the hodograph (reciprocal) transformation (2.62) and the bilinear equation (2.60). It is noted that we can switch \(x_{-1}\) and \(x_1\), i.e. the alternative hodograph (reciprocal) transformation \(x = x_1 - (\ln f)_{x_{-1}} = x_1 - 2(\ln \tau)_{x_{-1}}, t = x_{-1}\) can be used instead of the above hodograph (reciprocal) transformation.

Remark 2.16. Morrison, Parkes and Vakhnenko pointed out that the bilinear equation (2.60) is a special case of the Ito equation [6, 33]. Hone and Wang pointed out that the reduced Ostrovsky equation is linked to the first negative flow of the Sawada–Kotera hierarchy. The bilinear equation (2.60) is nothing but the bilinear equation of the first negative flow of the Sawada–Kotera hierarchy which can be obtained from the BKP hierarchy with the 3-reduction, i.e. this is related to the \(A_2^{(1)}\) affine Lie algebra [22].

Remark 2.17. The relationship between the Tzitzeica equation and the 3-reduced BKP hierarchy was pointed out by Lambert et al [34].
3. Conclusions

In this paper, we have shown the reciprocal link between the reduced Ostrovsky equation and the $A_{(2)}^{(2)}$ 2D-Toda system. Using this reciprocal link, we have constructed the $N$-soliton solution of the reduced Ostrovsky equation in the form of pfaffian. The bilinear equations and the $\tau$-function of the reduced Ostrovsky equation have been obtained from the period 3-reduction of the $B_\infty$ or $C_\infty$ 2D-Toda system. One of the $\tau$-functions of the $A_{(2)}^{(2)}$ 2D-Toda system has been shown to be the square of a pfaffian, by which the $N$-soliton solution of the reduced Ostrovsky equation is represented. We have shown that the bilinear equation for a member of the 3-reduced extended BKP hierarchy (the first negative flow of the Sawada–Kotera hierarchy) can also give rise to the reduced Ostrovsky equation, thus, the same pfaffian solution.

It should be noted that the approach in this paper is similar to the one in our previous paper about the short wave model of the Camassa–Holm equation [35]. We proposed integrable semi-discrete and integrable fully discrete analogues of the short wave model of the Camassa–Holm equation based on bilinear equations and a determinant solution. Thus, we can construct integrable discrete analogues of the reduced Ostrovsky equation (or the short wave model of the DP equation) by using the result in this paper. We would also like to comment on the $N$-soliton solution of the DP equation which can be obtained by using the reduction of the 2D-Toda system developed in this paper. Our approach gives the $N$-soliton solution of the DP equation in a much simpler way compared to Matsuno’s method [12, 13]. We will report the detail in a forthcoming paper.

Appendix. The period 3-reduction of the $C_\infty$ 2D-Toda system

In this appendix, we show that the $\tau$-function obtained in section 2 can be obtained from the period-reduction of the $C_\infty$ 2D-Toda system.

We impose the $C_\infty$-reduction $\theta_n = \theta_{-n}$ ($n \geq 0$) on the $A_\infty$ 2D-Toda system, i.e. fold the infinite sequence $\{\ldots, \theta_{-2}, \theta_{-1}, \theta_0, \theta_1, \theta_2, \ldots\}$ in $\theta_0$ [22, 19, 18, 23]. Then we obtain the $C_\infty$ 2D-Toda system [14, 17]

$$\frac{\partial^2 \theta_n}{\partial x_1 \partial x_{-1}} = -\sum_{m \in \mathbb{Z}_{\geq 0}} a_{n+1, m+1} e^{-\theta_m}, \quad n \in \mathbb{Z}_{\geq 0}, \tag{A.1}$$

where the matrix $A = (a_{n,m})$ is the transpose of the Cartan matrix for the infinite-dimensional Lie algebra $C_\infty$ [18].

The $C_\infty$ 2D-Toda system (A.1) is transformed into the bilinear equations

$$-(\frac{1}{2} D_{\theta_0} D_{\theta_{-1}} - 1) \tau_0 \cdot \tau_0 = \tau_1^2, \tag{A.2}$$
$$-(\frac{1}{2} D_{\theta_n} D_{\theta_{n-1}} - 1) \tau_n \cdot \tau_n = \tau_{n-1} \tau_{n+1}, \quad \text{for} \quad n \geq 1, \tag{A.3}$$

through the dependent variable transformation

$$\theta_0 = -\ln \frac{\tau_1^2}{\tau_0^2}, \quad \text{and} \quad \theta_n = -\ln \frac{\tau_{n+1} \tau_{n-1}}{\tau_n^2} \quad \text{for} \quad n \geq 1. \tag{A.4}$$

Lemma A.1. The bilinear equations (A.2) and (A.3) have the $N$-soliton solution which is expressed as

$$\tau_n = \det(\psi_{i,j}^{(n)})_{1 \leq i,j \leq 2N}, \tag{A.5}$$

for $n \geq 1$. 

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where
\[
\psi_{i,j}^{(n)} = c_{i,j} + (-1)^n \int_{-\infty}^{t_i} \varphi_j^{(n)}(\varphi_j^{(-n)}) \, dx_1,
\]
\[
\varphi_j^{(n)} = p_i^e e^{\xi_j}, \quad \xi_j = \frac{x_j}{p_i} + \frac{1}{p_i} x_{-1} + p_i^3 x_3 + \frac{1}{p_i} x_{-3} + \cdots + \xi_0,
\]
and \(c_{i,j} = c_{j,i}\).

**Proof.** Imposing the \(C_\infty\) reduction \(\tau_n = \tau_{-n}\), i.e. folding the sequence of the \(\tau\)-functions \([\ldots, \tau_{-2}, \tau_{-1}, \tau_0, \tau_1, \tau_2, \ldots]\) in \(\tau_0\), we have \(\tau_{-1} = \tau_1, \tau_{-2} = \tau_2, \tau_{-3} = \tau_3, \ldots\) \([22, 19, 18, 23]\). Thus, we obtain the bilinear equations (A.2) and (A.3) from the 2D-Toda bilinear equation (2.3).

To impose the \(C_\infty\) reduction on the Gram-type determinant solution of the \(A_\infty\) 2D-Toda system, we impose the constraint \(\varphi_j^{(n)}(\varphi_j^{(-n)}) = \psi_j^{(i)}, c_{i,j} = c_{j,i}, M = 2N\) and \(x_2k \equiv 0\) for every integer \(k\). With this constraint, each element of the Gram-type determinant has the following property:
\[
\psi_{i,j}^{(n)} = c_{i,j} + (-1)^n \int_{-\infty}^{t_i} \varphi_j^{(n)}(\varphi_j^{(-n)}) \, dx_1 = c_{j,i} + (-1)^n \int_{-\infty}^{t_i} \varphi_j^{(n)}(\varphi_j^{(-n)}) \, dx_1
\]
\[
= \psi_j^{(i)}. \tag{A.7}
\]

Then the \(\tau\)-function satisfies \(\tau_n = \tau_{-n}\). Therefore, the \(N\)-soliton solution of the \(C_\infty\) 2D-Toda system is expressed by the above Gram-type determinant.

**Lemma A.2.** The bilinear equations
\[
-(\frac{1}{2} D_\alpha D_{\alpha \tau_1} - 1) \tau_0 \cdot \tau_0 = \tau_1^2, \tag{A.8}
\]
\[
-(\frac{1}{2} D_\alpha D_{\alpha \tau_1} - 1) \tau_1 \cdot \tau_1 = \tau_1 \tau_0 \tag{A.9}
\]
have the \(N\)-soliton solution which is expressed as
\[
\tau_n = \det(\psi_{i,j}^{(n)})_{1 \leq i, j \leq 2N}, \tag{A.10}
\]
where
\[
\psi_{i,j}^{(n)} = c_{i,j} + (-1)^n \int_{-\infty}^{t_i} \varphi_j^{(n)}(\varphi_j^{(-n)}) \, dx_1, \quad \varphi_j^{(n)} = p_i^e e^{\xi_j}, \quad \xi_j = \frac{x_j}{p_i} + \frac{1}{p_i} x_{-1} + \xi_0.
\]

**Proof.** Imposing a period 3-reduction \(\tau_0 = \tau_{n+3}\) on the sequence of the \(\tau\)-functions \([\ldots, \tau_3, \tau_2, \tau_1, \tau_0, \tau_1, \tau_2, \tau_3, \ldots]\), it becomes \([\ldots, \tau_0, \tau_1, \tau_1, \tau_0, \tau_1, \tau_0, \tau_0, \ldots]\). Thus, we obtain two bilinear equations (A.8) and (A.9).

In Lie algebraic terms, this corresponds to the reduction to the affine Lie algebra \(A_2^{(2)}\) from the infinite-dimensional Lie algebra \(C_\infty\).

To impose a period 3-reduction on the \(N\)-soliton solution, we add a constraint \(p_1^3 + p_{2N+1}^3 = 0\) (\(p_i \neq -p_{2N+1-i}\)) for \(i = 1, 2, \ldots, N\) and \(c_{i,j} = \delta_{j,2N+1-i}\alpha_i\). To show \(\tau_{n+3} = \tau_n\), we manipulate \(\tau_n\) as follows:
\[
\tau_n = \det(\delta_{j,2N+1-i}\alpha_i + \frac{1}{p_i + p_j} \left( \frac{p_i}{p_j} \right)^n e^{(-\xi + \xi_j)} )_{1 \leq i, j \leq 2N}
\]
\[
eq e^{3 \sum_{i=1}^{2N} \delta_{j,2N+1-i}\alpha_i} \left( \frac{p_i}{p_j} \right)^n e^{-\xi + \xi_j} + \frac{1}{p_i + p_j} )_{1 \leq i, j \leq 2N}
\]
\[
eq e^{\sum_{i=1}^{2N} \delta_{j,2N+1-i}\alpha_i} \left( \frac{p_{2N+1-i}}{p_i} \right)^n e^{-\xi + \xi_j} + \frac{1}{p_i + p_j} )_{1 \leq i, j \leq 2N}
\]
Since \((-\frac{p_n+1}{p_i})^3 = 1\), \(\tau_{n+3} = \tau_n\) is satisfied.

\[\square\]
Lemma A.3. The $\tau$-function $\tau_1$ of the bilinear equations (A.8) and (A.9) is written in the form of pfaffian,

$$\tau_1 = \frac{1}{2^{2N} \prod_{i=1}^{2N} p_k} \tau^2,$$

where $\tau^2 = \text{pf}(1, 2, \ldots, 2N - 1, 2N)$, \hspace{1cm} (A.11)

where $\text{pf}(i, j) = c_{i,j} + \frac{p_i - p_j}{p_i + p_j} e^{x_i + x_j}, c_{i,j} = p_{j-1} x_{i+1} + p_i x_j + \xi_i^0, c_{i,j} = \delta_{j,2N+1-i} c_i, c_i = -c_{2N+1-i}$. Let $p_i^2 = p_i p_{2N+1-i} + p_{2N+1-i}^2 = 0$.

Proof. Let $\alpha_i = \alpha_{2N+1-i}$. For $n = 1$, \hspace{1cm} (A.11)

$$\tau_1 = \text{det}(\Psi_{i,j})_{1 \leq i,j \leq 2N} = \text{det}\left(\delta_{j,2N+1-i} \alpha_i - \frac{1}{p_i + p_j} p_i e^{x_i + x_j}\right)_{1 \leq i,j \leq 2N}$$

$$= \frac{1}{2^{2N} \prod_{i=1}^{2N} p_k} \text{det}\left(2\delta_{j,2N+1-i} \alpha_i \frac{p_j^2}{p_i} - \frac{2p_j}{p_i + p_j} e^{x_i + x_j}\right)_{1 \leq i,j \leq 2N}$$

$$= \frac{1}{2^{2N} \prod_{i=1}^{2N} p_k} \text{det}\left(2\delta_{j,2N+1-i} \alpha_i \frac{p_j^2}{p_i} - \frac{2p_j}{p_i + p_j} e^{x_i + x_j}\right)_{1 \leq i,j \leq 2N}$$

$$= \frac{1}{2^{2N} \prod_{i=1}^{2N} p_k} \text{det}\left(c_{i,j} + \frac{p_i - p_j}{p_i + p_j} e^{x_i + x_j} - e^{x_i + x_j}\right)_{1 \leq i,j \leq 2N},$$

where $c_{i,j} = 2\delta_{j,2N+1-i} \alpha_i \frac{p_j^2}{p_i}$. Then we note

$$c_{i,j} = 2\delta_{j,2N+1-i} \alpha_i \frac{p_j^2}{p_i} - \frac{2p_j}{p_i + p_j} e^{x_i + x_j}.$$\hspace{1cm} (A.11)

Introducing $c_i = 2\delta_i \frac{p_i^2}{p_i}$, we can write $c_{i,j} = \delta_{j,2N+1-i} c_i$ and $c_i = -c_{2N+1-i}$. \hspace{1cm} (A.11)

Since the $2N \times 2N$ matrix $(c_{i,j} + \frac{p_i - p_j}{p_i + p_j} e^{x_i + x_j})_{1 \leq i,j \leq 2N}$ is skew-symmetric, we obtain

$$\tau_1 = \frac{1}{2^{2N} \prod_{i=1}^{2N} p_k} [\text{pf}(1, 2, \ldots, 2N - 1, 2N)]^2,$$

where $\text{pf}(i, j) = c_{i,j} + \frac{p_i - p_j}{p_i + p_j} e^{x_i + x_j}$. Here we used the formula \hspace{1cm} (A.11)

$$\text{det}(a_{i,j} - y_{i,j})_{1 \leq i,j \leq 2N} = \text{det}(a_{i,j})_{1 \leq i,j \leq 2N} = [\text{pf}(1, 2, \ldots, 2N)]^2,$$

where $a_{i,j} = -a_{j,i}, a_{i,i} = 0, \text{pf}(i, j) = a_{i,j}$. \hspace{1cm} (A.11)

Letting $f = \tau_1$ and $g = \tau_0$, we obtain the bilinear equations (2.20) and (2.21) from the bilinear equations (A.9) and (A.8). As shown in \hspace{1cm} (A.11) the period 3-reduction of the $C_\infty$ 2D-Toda system gives the same result as the period 3-reduction of the $B_\infty$ 2D-Toda system after the relabelling indices.

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