An Approximation to the Gröbner Basis of Ideals of Perturbed Points: Part I.

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Abstract

We develop a method for approximating the Gröbner basis of the ideal of polynomials which vanish at a finite set of points, when the coordinates of the points are known with only limited precision. The method consists of a preprocessing phase of the input points to mitigate the effects of the input data uncertainty, and of a new “numerical” version of the Buchberger-Möller algorithm to compute an approximation $\overline{GB}$ to the exact Gröbner basis. This second part is based on a threshold-dependent procedure for analyzing from a numerical point of view the membership of a perturbed vector to a perturbed subspace. With a suitable choice of the threshold, the set $\overline{GB}$ turns out to be a good approximation to a “possible” exact Gröbner basis or to a basis which is an “attractor” of the exact one. In addition, the polynomials of $\overline{GB}$ are “sufficiently near” to the polynomials of the extended basis, introduced by Stetter, but they present the advantage that $LT(\overline{GB})$ coincides with the leading terms of a “possible” exact case. The set of the preprocessed points, approximation to the unknown exact points, is a pseudozero set for the polynomials of $\overline{GB}$.

Keywords: Ideal of points, Gröbner basis, perturbed data, numerical algorithm.

1 Introduction

Let $P = K[x_1, \ldots, x_s]$ be the polynomial ring in $s$ indeterminates over a field $K$. In this paper we develop a method for approximating the Gröbner basis of the ideal $I \subset P$ of polynomials which vanish at a finite set of points, when the coordinates of the points are known with only limited precision. In particular we analyze the case $K = \mathbb{R}$.

In the exact case, that is when we deal with a set of unperturbed points $P$, the problem has been deeply analyzed by several authors (see for example [2], [9], [10], [11]) and the Gröbner basis of the ideal $I(P)$ of the points $P$ can be computed, for example, by the Buchberger-Möller algorithm ([11], [10]). It is

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well known that, given a term ordering, each set of \( m \) points \( \mathcal{P} \) corresponds to a closed set \( \mathcal{N} = \{ t_1, \ldots, t_m \} \) such that each polynomial \( g \) of the Gröbner basis \( \text{GB} \) of \( \mathcal{I}(\mathcal{P}) \) has the form

\[
g = t - \sum_{t_i < t} c_i t_i,
\]

with leading term \( t \) and suitable coefficients \( c_i \). The set \( \mathcal{N} \), called normal set, is the basis of the quotient ring \( \mathcal{P}/\mathcal{I}(\mathcal{P}) \) and the leading term \( t \) belongs to the corner set \( \mathcal{C}[\mathcal{N}] \).

Nevertheless, it is also well known that the problem of computing the Gröbner basis of an ideal of points is “discontinuous” ([10], [17]), i.e. small perturbations of the points can cause structural changes in the basis, as illustrated in Example 1.1. For this reason exact methods applied blindly to perturbed “real-world” data can produce meaningless results.

**Example 1.1.** Given the term ordering DegLex and the exact points \( p_1 = (1, 1), p_2 = (3, 2) \) and \( p_3 = (5, 3) \), the Gröbner basis of the ideal \( \mathcal{I} \) of polynomials in \( \mathbb{R}[x, y] \) vanishing at these points is

\[
\text{GB} = \{ x - 2y + 1, y^3 - 6y^2 + 11y - 6 \}, \quad \text{with normal set } \mathcal{N} = \{ 1, y, y^2 \}.
\]

If data errors perturb the point \( p_3 \) so that the set of input points is \( \hat{\mathcal{P}} = \{ p_1, p_2, (5.1, 3) \} \), then the Gröbner basis \( \hat{\text{GB}} \) of the ideal \( \mathcal{I}(\hat{\mathcal{P}}) \) is:

\[
\hat{\text{GB}} = \{ y^2 - 20x + 37y - 18, xy - 43x + 81y - 39, x^2 - 90.1x + 172.2y - 83.1 \}, \quad \text{with normal set } \hat{\mathcal{N}} = \{ 1, y, x \}.
\]

This is an obvious result since the points \( \hat{\mathcal{P}} \) are not aligned and so \( \hat{\text{GB}} \) structurally differs from \( \text{GB} \). Nevertheless, since the points \( \hat{\mathcal{P}} \) are not aligned because of data errors, from the computational point of view it is more interesting to approximate the Gröbner basis \( \text{GB} \) than to compute \( \hat{\text{GB}} \), given as input the set \( \hat{\mathcal{P}} \).

In the general case, if the input points arise from real-world data, their coordinates are approximations to the unknown exact values, and only an error estimation is known. For this reason, each point \( \hat{\mathcal{P}} \) of \( \hat{\mathcal{P}} \) represents a “cloud” of points. More precisely, given an input point \( \hat{\mathcal{P}} \) known with uncertainty \( s_0 \), each point \( \hat{\mathcal{P}} \) which differs from \( \hat{\mathcal{P}} \) componentwise by less than \( s_0 \) can be chosen as an input point computationally equivalent to \( \hat{\mathcal{P}} \), and so it will be called “admissible input point”. Analogously, given a set of perturbed input points \( \hat{\mathcal{P}} \), an “admissible input set” is a set consisting of admissible input points computationally equivalent to \( \hat{\mathcal{P}} \). Obviously, given an error estimation \( s_0 \), there exist several admissible input sets. These remarks combined with the structural discontinuity show that the computation of the Gröbner basis of ideals of perturbed points is a very tricky problem. In fact, since the exact points
are unknown and the Gröbner basis can change choosing different admissible input sets, it can be difficult to identify the case to be approximated. For this reason, we define some criteria to choose the “reference” Gröbner basis to approximate, pointing out its structure, that is its leading terms and the supports of its polynomials.

We define “reference basis” a Gröbner basis $GB$ generating an ideal of polynomials vanishing at some admissible input set, whose structure represents all the admissible input sets. Since the structure of a Gröbner basis and the elements of its normal set are closely connected, we require that the normal set of a reference basis can be associated to all the admissible input sets, that is, analogously to the exact case, we require that its terms provide independent vectors if evaluated on the points of any admissible input set. A normal set with this property will be called “reference normal set”. Note that a reference basis could coincide with the exact basis, since the set of exact points belongs to the cloud of the admissible input sets.

In order to approximate a reference basis, our method computes a set of polynomials $\overline{GB}$, whose supports belong to $\overline{N} \cup C(\overline{N})$, where $\overline{N}$ is a reference normal set. In some particular case, $\overline{GB}$ is a reference Gröbner basis and, in general, it is a good approximation to a reference basis. In fact, we show that, in the general case, $\overline{GB} = \{\overline{g}_1, \ldots, \overline{g}_n\}$ structurally corresponds to a reference basis $GB = \{g_1, \ldots, g_n\}$, that is there is an one-to-one correspondence between $\overline{GB}$ and $GB$ which preserves the leading terms and the supports of each polynomial. In addition, for each $i = 1 \ldots n$, $\overline{g}_i \in \overline{GB}$ and $g_i \in GB$ have “similar” coefficients and we estimate the difference between $\overline{g}_i$ and $g_i$. Moreover, we show that the elements of $\overline{GB}$ are “near” to the polynomials of the extended basis $GB_E$ introduced by Stetter in [12].

Unfortunately, some “degenerate” cases can occur, when there are no reference bases, that is when all the Gröbner bases, with reference normal set, are associated to points which are not admissible input points. Nevertheless, in these degenerate cases the numerical tests point out that also the normal set of the exact points cannot represent, from a numerical point of view, all the admissible input sets. In addition, the tests suggest that the computed normal set $\overline{N}$ seems to be the numerically stable normal set “nearest” to the exact one. The analyzed degenerate examples are not presented in this paper since the degenerate cases will be studied in a future work, whereas in this article we shall limit ourselves to “non-degenerate” cases.

Our method consists of two parts. First of all, the input points are preprocessed, in order to mitigate some negative effects of the data errors. The possible “splittings” of the coordinates due to the errors are removed and the preprocessing algorithm computes a new admissible input set $\overline{F}$, by replacing with a unique suitable value the perturbed coordinates which differ by less than the error estimation. The preprocessed points turn out to be a set of pseudozeros [13] for the polynomials of $\overline{GB}$.

The second part of our algorithm constitutes the main innovative part of this paper. It is a numerical version of the Buchberger-Möller algorithm which computes a set of polynomials $\overline{GB}$, approximation to a reference basis. The
numerical Buchberger-Möller algorithm is based on the following notion of “approximate” linear dependence of vectors. From the numerical point of view, a vector \( v \) depends on the vectors \( \{v_1, \ldots, v_k\} \) if it is “sufficiently” near to an element of the subspace \( \text{Span}\{v_1, \ldots, v_k\} \). Since this “numerical” dependence of vectors is defined choosing a threshold \( \epsilon \), the numerical version of the Buchberger-Möller algorithm is threshold-dependent. We have to choose a suitable value of the threshold for computing a good approximation to a reference basis. The threshold choice is one of the trickier aspects of the implementation of our method.

This paper is organized as follows. In Section 2 we describe the threshold-dependent approximation algorithm and in Section 3 we present an heuristic approach for the choice of this threshold, whereas a more detailed analysis is left to future works. The approximation properties of the computed set \( \overline{GB} \) are analyzed in Section 4, and a comparison with some results presented by Stetter is described in Section 5. In Section 6 we report two examples in order to illustrate and clarify explicitly the behaviour of our method. Appendix A contains a description of the strategy for preprocessing the input points used in our examples and Appendix B presents two error upper bounds useful in the estimation of the goodness of the basis approximation.

**Notation.** Later on “G-basis” means “Gröbner basis”. We denote the points with the letter \( p \) and the tuple of points with \( \mathcal{P} = \{p_1, p_2, \ldots\} \). If \( g(x_1, \ldots, x_s) \) is a polynomial in \( \mathbb{R}[x_1, \ldots, x_s] \), \( g(p) \) is the evaluation of \( g \) at \( p \) and \( g(\mathcal{P}) \) is the vector whose components are the values of \( g \) on the elements of \( \mathcal{P} \). In addition we denote with \( \|v\|_2 \) the 2-norm of a vector \( v \).

## 2 The approximation method

In this section we describe our method for approximating a reference basis with a set of polynomials \( \overline{GB} \), when we deal with perturbed data in non-degenerate cases, that is when a reference basis exists.

Our method computes the set of polynomials \( \overline{GB} \), working on a set of points obtained after preprocessing the perturbed input data. The goal of the preprocessing phase is to minimize the damage caused by possible “splittings” of the data due to the input perturbations, replacing by a unique value the coordinates which differ by less than the error estimation, since they are coincident from the computational point of view. The preprocessed set \( \overline{\mathcal{P}} \) depends on the preprocessing technique; the strategy used in our tests is described in Appendix A.

Even if the exact Buchberger-Möller (BM) algorithm \([4]\) applied to the preprocessed data sometimes computes a good approximation to a reference basis, this is not true in general. The simple preprocessing of the input data is not sufficient to obtain a numerically significant set of polynomials, and so it is necessary to develop a new approximation algorithm.
It is well known that, given a term ordering $\sigma$ and a set of $m$ points $P = \{p_1, \ldots, p_m\}$, not perturbed by errors, the BM algorithm computes the $\sigma$-Gröbner basis $GB$ of the ideal of the polynomials which vanish at the set $P$. At each step, after choosing the term $t$, the BM algorithm checks if the vector $t(P)$ is linearly independent of the vectors $\{t_1(P), \ldots, t_k(P)\}$, where $\{t_1, \ldots, t_k\}$ is the subset of the normal set computed at the previous steps, and $t >_{\sigma} t_i, i = 1, \ldots, k$. If this is the case, $t$ is a new term of the normal set. Otherwise, the BM algorithm builds a new polynomial $g$ of the G-basis $GB$ such that

$$g = t - \sum_{i=1}^{k} c_i t_i,$$

where $c_i, i = 1, \ldots, k$, are the components of the vector $c$ that satisfies

$$M_k(P)c = t(P),$$

where $M_k(P)$ is the matrix $[t_1(P), \ldots, t_k(P)]$, whose columns are linearly independent, by construction.

The above test of linear dependence is crucially affected by even very small variations of the input data. Therefore, when we deal with approximate input points, different choices of admissible data may lead to different choices in the BM algorithm. Nevertheless, if the data are affected by errors, it is possible to check the linear dependence of perturbed vectors from a numerical point of view. Intuitively, a perturbed vector $v$ belongs, in some approximate sense, to a subspace $W$, spanned by perturbed vectors, if $v$ is “sufficiently near” to an element of $W$. This idea of “numerical” membership of a vector to a subspace, formalized and analyzed by the author in [7], is based on the following definition.

**Definition 2.1** Given a subspace $W$ and a threshold $\epsilon > 0$, a vector $v$ “numerically” lies in $W$ if

$$\frac{\|r\|_2}{\|v\|_2} \leq \epsilon,$$

where $r$ is the component of $v$ orthogonal to $W$. In this case we write $v \in_\epsilon W$.

By exploiting this concept of “numerical” membership of a subspace, we develop the numerical **Approximate Buchberger Möller** algorithm (denoted by ABM) for computing the approximation $\overline{GB}$ to a reference basis. Working on the preprocessed input points $P$, the ABM algorithm checks, at each step, if a perturbed vector lies numerically in a subspace spanned by a perturbed basis. If the vector “numerically” lies in the subspace, then the algorithm builds a new polynomial of $\overline{GB}$, even if the vector does not exactly belong to the subspace. Otherwise the ABM algorithm inserts a new element into the normal set $N$.

The ABM algorithm can be described as follows. We check the numerical membership of a vector $v$ to a subspace $W = \text{Span}\{w_1, \ldots, w_k\}$ computing the residual $r \in W^\perp$ of the least square problem $[w_1 \ldots w_k]x = v$, since, in this way, we can more easily estimate the difference between $\overline{GB}$ and a reference basis, as shown in Section 4.
The ABM Algorithm.

**Input.** The term ordering \(\sigma\), the preprocessed input points \(\overline{P} = \{\overline{p}_1, \ldots, \overline{p}_m\}\) and the threshold \(\epsilon\).

**Output.** The set of polynomials \(\overline{GB}\) and the set of power products \(\overline{N}\).

Let \(\overline{t}_1, \ldots, \overline{t}_k\) be the terms of \(\overline{N}\) computed at the previous steps and let \(\overline{t}\) be the current power product, \(\overline{t} >_\sigma \overline{t}_1, \ldots, \overline{t}_k\), chosen analogously to the BM algorithm.

1. Given the matrix \(\overline{M}_k(\overline{P}) = [\overline{t}_1(\overline{P}), \ldots, \overline{t}_k(\overline{P})]\), solve the least square problem
   \[
   \overline{M}_k(\overline{P})\overline{c} = \overline{t}(\overline{P}).
   \] (2)

2. Let \(\overline{r} = \overline{t}(\overline{P}) - \overline{M}_k\overline{c}\) be the residual of the problem (2), then
   - if \(\|\overline{r}\|_2 / \|\overline{t}(\overline{P})\|_2 > \epsilon\), put the term \(\overline{t}\) into the set \(\overline{N}\);  
   - otherwise, put the polynomial \(\overline{g} = \overline{t} - \sum_{i=1}^k c_i \overline{t}_i\) into \(\overline{GB}\).  

Note that, analogously to the exact case, the supports of the polynomials of \(\overline{GB}\) consist of terms of \(\overline{N} \cup \mathcal{C}N\) and the leading terms belong to \(\mathcal{C}\overline{N}\).

Obviously, the output sets \(\overline{GB}\) and \(\overline{N}\) of the ABM algorithm depend on the value of \(\epsilon\) and we have to choose a suitable value of the threshold so that \(\overline{GB}\) is a good approximation to a reference basis. A detailed analysis of the choice of suitable thresholds will be presented in a future work. In the next section we illustrate an heuristic approach based on the analysis of some properties of \(\overline{N}\) and \(\overline{GB}\).

### 3 The threshold choice

Intuitively, if we choose too small a value of \(\epsilon\), the ABM algorithm might not recognize numerically dependent vectors and so the set \(\overline{GB}\) could be very sensitive to data perturbations. Vice versa, if the value of the threshold is too large a vector may be considered as numerically belonging to a subspace even if it is too “far” from it. In this section we present an heuristic strategy for choosing a suitable value of the threshold analyzing the computed output sets. If the value of \(\epsilon\) does not produce satisfactory output sets, we repeat the procedure with a new threshold.

As mentioned earlier, \(\overline{GB}\) is a good approximation to a reference basis \(GB\) if each polynomial of \(\overline{GB}\) corresponds to a polynomial of \(GB\) with the same support and similar coefficients and vice versa. In order to obtain the same structure of a reference basis, since we have excluded degenerate cases, we require that \(\overline{N}\) is the normal set of a reference basis, and so we have to choose a value of \(\epsilon\) such that the following properties hold.
P1. We require that \( \overline{N} \) has \( m \) elements. In this case, since the ABM algorithm analyzes the power products using the same strategy of the exact BM algorithm, \( \overline{N} = \{ \overline{t}_1, \ldots, \overline{t}_m \} \) turns out to be, by construction, a closed set. For this reason, as is well known \[17\], \( \overline{N} \) is the basis of the quotient ring \( P/I(\overline{P}) \) for each polynomial ideal \( I(\overline{P}) \) with \( m \) simple zeros \( \overline{P} \) such that \( \{ \overline{t}_1(\overline{P}), \ldots, \overline{t}_m(\overline{P}) \} \) are linearly independent vectors, that is \( \overline{N} \) is a normal set associated to such set \( \overline{P} \).

P2. We require that for each term \( \overline{t} \) the relative residual \( \| \overline{r} \|_2 / \| \overline{t}(\overline{P}) \|_2 \) of the problem (2) with right side \( \overline{t}(\overline{P}) \) must be sufficiently greater or less than the threshold, that is well separated from it. This condition guarantees that small data perturbations do not affect each residual sufficiently to change its position with respect to \( \epsilon \) and so \( \overline{N} \) turns out to be invariant for admissible input sets. It follows that the elements of \( \overline{N} \) provide independent vectors if evaluated on admissible input sets, and so, if P1 holds, \( \overline{N} \) turns out to be invariant for admissible input sets, i.e. it is a reference normal set.

P3. We require that, given the term ordering \( \sigma \), \( \overline{N} \) is the normal set of a \( \sigma \)-Gröbner basis \( GB \) of an ideal of admissible input points, that is \( \overline{N} \) is the normal set of the reference G-basis \( GB \).

The properties P1 and P2 can be directly checked on the set \( \overline{N} \) and on the computed relative residuals, whereas it can be more difficult to investigate if \( \overline{N} \) satisfies P3, since several cases can occur.

If the following system, consisting of the polynomials of \( \overline{GB} \),
\[
\begin{align*}
\overline{g}_1 &= 0 \\
\vdots \\
\overline{g}_n &= 0,
\end{align*}
\]
vanishes at \( \overline{P} \) then \( \overline{GB} \) is the \( \sigma \)-Gröbner basis of the ideal \( I(\overline{P}) \) with reference normal set \( \overline{N} \), since in this case the ABM and BM algorithms have the same behaviour. Since \( \overline{P} \) is an admissible input set, then \( \overline{GB} \) is a reference basis and the property P3 is obviously satisfied.

Otherwise we can check the property P3 using the following intuitive idea. The value of the threshold is suitably chosen if \( \overline{GB} \) approximates the reference basis \( GB = \{g_1, \ldots, g_n\} \), with normal set \( \overline{N} \), corresponding to a set of points \( \mathcal{P} \) near to \( \overline{P} \), that is if the function:
\[
\overline{F} : p \mapsto (\overline{g}_1(p), \ldots, \overline{g}_n(p)),
\]
is a good approximation to the function:
\[
F : p \mapsto (g_1(p), \ldots, g_n(p)),
\]
which vanishes at \( \mathcal{P} \). For this reason, if the threshold is suitably chosen, we can suppose that Newton method \[6\] works analogously if applied to the polynomial system \( \overline{F}(p) = 0 \) or to \( F(p) = 0 \). Since Newton iteration applied to
the equation $F(p) = 0$ computes descent directions for $\|F(p)\|_2$ we can assume that it computes descent directions for $\|F(p)\|_2$ too. For this reason the set $\tilde{P}^+$, computed by Newton iteration working on $\tilde{F}$ with initial points $\tilde{P}$, can be considered a good approximation to $P$. It follows that the property P3 is “numerically” verified if the set $\tilde{P}^+$ is sufficiently “near” to $\tilde{P}$.

Since the properties P1–P3 must be checked on the computed sets $\tilde{N}$ and $\tilde{GB}$, we need a value of the threshold to begin the computation. The first value of the threshold can be chosen as follows. We apply the exact BM algorithm to the preprocessed points $\tilde{P}$ and we compute for each term $t$ of the exact normal set $\tilde{N}_0$ the relative residual of the least square problem with right side $t(\tilde{F})$. We choose the first value of $\epsilon$ in order to eliminate from $\tilde{N}_0$, by means the ABM algorithm, the power products with too small relative residuals.

The following example illustrates the effects of different thresholds.

**Example 3.1**

Let $\tilde{P} = \{(1, 1), (3, 2), (5, 1, 3)\}$ be the perturbed points of Example 1.1. Since the preprocessing phase does not change the perturbed points, we have $\tilde{P} = P$, corresponding to the exact normal set $\tilde{N}_0 = \{1, y, x\}$. Since the relative residuals of the terms $y$ and $x$ computed on $\tilde{P}$ are equal to 0.37 and 0.0068, respectively, we choose $\epsilon > 0.0068$ to obtain a set $\tilde{GB}$ different from the exact G-basis of $I(\tilde{P})$.

Choosing $0.0068 < \epsilon < 0.37$, the polynomial $x - 2.05y + 1.06$ is inserted into $\tilde{GB}$ and the term $y^2$ is analyzed. Since the relative residual of $y^2$ is equal to 0.0824 we have two possibilities.

If $0.0824 < \epsilon < 0.37$, the nonzero relative residual of $y^2$ is treated as if it were equal to zero and the following set of polynomials is computed:

$$\tilde{GB}_1 = \{ x - 2.05y + 1.06, y^2 - 4y + 3.3 \}.$$  

Since $\tilde{GB}_1$ vanishes at $\tilde{P}_1^+ = \{(4.7072, 2.8165), (1.3595, 1.1835)\}$ which differs in cardinality from $\tilde{P}$, we conclude that $\tilde{GB}_1$ is not a good approximation to $GB$ and that the threshold has not been suitably chosen.

If $0.0068 < \epsilon < 0.0824$, then the polynomial $y^2 - 4y + 3.3$ does not belong to $\tilde{GB}_2$ and the term $y^2$ is a new element of the normal set. So in this case the ABM algorithm computes the set $\tilde{GB}_2$:

$$\tilde{GB}_2 = \{ x - 2.05y + 1.06, y^3 - 6y^2 + 11y - 6 \}.$$  

G-basis of the ideal of points $\tilde{P}_2^+ = \{(0.98\overline{3}, 1), (3.0\overline{3}, 2), (5.08\overline{3}, 3)\}$, very near to the set of points $\tilde{P}$, and so we consider the threshold suitably chosen.

Note that $\tilde{GB}_2$ is very similar to the G-basis of the ideal of the exact points of Example 1.1.

**Remark.** If the ABM algorithm cannot compute normal sets which satisfy properties P1–P3, then we are dealing with “degenerate” cases. The numerical
tests suggest that, in the degenerate examples, the relative least square residuals for the exact normal sets associated to all the admissible input points are small. Unfortunately, since it is rather difficult, if not impossible, to distinguish the case where the small residuals computed on the preprocessed points are caused by data perturbations from the case where they are related to degenerate problems, we cannot check, \emph{a priori}, if we are dealing with degenerate cases.

4 \hspace{1em} Comparison with a reference basis

In this section we estimate the difference between a reference basis and the set GB computed, given the term ordering $\sigma$, by the ABM algorithm working on the preprocessed points $\overline{P}$. We assume that the value of the threshold $\epsilon$ has been suitably chosen and that “non-degenerate” problems are analyzed, that is the computed set $\overline{N}$ is a reference normal set and there exists a reference $\sigma$-Gröbner basis $GB$ with normal set $\overline{N}$. We show that $\overline{GB}$ is a good approximation to a reference basis $GB$.

First of all, note that if at each step of the ABM algorithm the analyzed relative residual is equal to zero or greater than the threshold, then the ABM algorithm works on the preprocessed points $\overline{P}$ like the BM algorithm. In this case the set $\overline{GB}$ is the exact G-basis of the ideal $I(\overline{P})$ and so, since $\overline{P}$ is an admissible input set, $\overline{GB}$ coincides with a reference basis.

In the general case, it can happen that, at some step of the ABM algorithm, the relative residual associated to $t$ is greater than zero and less than or equal to $\epsilon$. In this case the polynomial $g = t - \sum_{i=1}^{k} c_i t_i$ is the new element of the set $\overline{GB}$, even though it does not vanish at the points $\overline{P}$, since $g(\overline{P}) = t(\overline{P}) - r$, where $r$ is the nonzero residual of (2), and so $\overline{GB}$ is not the G-basis of the ideal $I(\overline{P})$. Nevertheless, since $\overline{N}$ is also the normal set of a reference basis $GB$, $\overline{GB}$ structurally corresponds to $GB$. In addition, in the following we prove that the polynomials of $\overline{GB}$ and $GB$ have similar coefficients. For these reasons, as mentioned earlier, the computed set $\overline{GB}$ can be considered a good approximation to a reference basis.

Let $P$ be the admissible input points which are zeros of the polynomials of a reference basis $GB$. By construction, if $g \in \overline{GB}$ corresponds to $g \in GB$, the vectors $\overline{c}$ and $c$ of the coefficients of $\overline{g}$ and $g$ satisfy respectively

$$M_k(\overline{P})\overline{c} = \overline{t}(\overline{P}) - \overline{\tau} \quad \text{and} \quad M_k(P)c = t(P),$$

where $\overline{\tau}$ is the nonzero residual of (2), and so $\overline{GB}$ is not the G-basis of the ideal $I(\overline{P})$. Nevertheless, since $\overline{N}$ is also the normal set of a reference basis $GB$, $\overline{GB}$ structurally corresponds to $GB$. In addition, in the following we prove that the polynomials of $\overline{GB}$ and $GB$ have similar coefficients. For these reasons, as mentioned earlier, the computed set $\overline{GB}$ can be considered a good approximation to a reference basis.

From the analysis of the sensitivity of a least square problem (2) using the preprocessed points. From the analysis of the sensitivity of a least square problem (6), (8), since the residual of the system (1) is zero, we have

$$\frac{||\overline{\tau} - c||_2}{||c||_2} \leq \frac{K_2(M_k(P))}{1 - \epsilon M K_2(M_k(P))} [\epsilon + \epsilon M], \quad (5)$$
where

\[ \epsilon_t = \frac{\| \tilde{t}(\mathcal{P}) - \tilde{t}(\mathcal{P}) \|_2}{\| \tilde{t}(\mathcal{P}) \|_2}, \quad \epsilon_M = \frac{\| \overline{M_k(\mathcal{P})} - \overline{M_k(\mathcal{P})} \|_2}{\| \overline{M_k(\mathcal{P})} \|_2}, \]

and \( K_2(\overline{M_k(\mathcal{P})}) \) is the 2-condition number of the matrix \( \overline{M_k(\mathcal{P})} \) [3]. It is well known [3] that, in general, the upper bound [3] overestimates the sensitivity of the linear systems. Nevertheless, the formula [3] suggests that \( \epsilon_t \) and \( \epsilon_M \) can be interpreted as a measure of goodness of approximation. If \( \delta_R \) is the maximum relative error which perturbs the coordinates of the points \( \mathcal{P} \) with respect to \( \mathcal{P} \), we show, in Appendix B, that

\[ \epsilon_t \leq \text{Deg}(\tilde{t})\delta_R \quad \text{and} \quad \epsilon_M \leq \sqrt{k}d_M\delta_R, \]

where \( \text{Deg}(\tilde{t}) \) is the degree of the term \( \tilde{t} \) and \( d_M \) is the maximum degree of the terms \( \tilde{t}_1, \ldots, \tilde{t}_k \). Since both sets \( \overline{\mathcal{P}} \) and \( \mathcal{P} \) are admissible data perturbations, the differences between their coordinates is not large and so the value \( \delta_R \) is small. We can conclude that, with a suitable choice of the threshold \( \epsilon \), the algorithm ABM, working on the preprocessed points \( \overline{\mathcal{P}} \), computes, for non-degenerate problems, a good approximation to a reference basis.

**Remark.** Note that, under small perturbations of \( \overline{\mathcal{P}} \), the matrix \( \overline{M_k(\mathcal{P})} \) of the system (2) preserves its full rank, since \( \overline{\mathcal{N}} \) is a reference normal set, but its elements slightly change. For this reason, choosing different admissible input sets, we obtain sets of polynomials which differ only in their coefficients, in a continuous way, and which do not present structural differences.

## 5 Extended basis and pseudozeros

In his book “Numerical Polynomial Algebra” [17] and in several papers ([12, 13, 14, 15, 16]) Stetter analyzes some aspects of Polynomial Algebra from the numerical point of view, when perturbed data and floating point arithmetic are used. In this section we compare the set \( \overline{GB} \) with the *extended basis* developed by Stetter. Moreover, we show that, in the terminology of Stetter, \( \overline{\mathcal{P}} \) is a pseudozero set for the polynomials of \( \overline{GB} \).

### 5.1 The extended basis

In [12], Stetter analyzes the problem of computing the G-basis of an ideal determined by a system of polynomials with perturbed coefficients, pointing out that the G-basis can be structurally altered by the data uncertainty. The numerical instabilities occur, analogously to the case of ideals of points, when the zeros of the perturbed polynomials are “near” to a set of points associated to a normal set \( \mathcal{N}^\rho \) which differs from the perturbed normal set \( \tilde{\mathcal{N}}^\rho \). In order to solve the numerical instabilities, Stetter introduces the notion of extended basis \( \overline{GB_E} \), that is an approximation to the G-basis, computed using the normal set \( \mathcal{N}^\rho \)
instead of \( \hat{N}^\rho \). The computed set \( GB_E \) is not a G-basis in general, but, if there exists a G-basis \( GB \) structurally corresponding to \( GB_E \), then \( GB_E \) is a “numerically stable” basis of the ideal associated to the input polynomials (see theorem 4.1 in [12]).

Modifying the technique proposed by Stetter, it is possible to compute the extended basis for ideals of perturbed points too. Given the set of \( m \) points \( \overline{P} \), the normal set \( N^\rho \) coincides with the normal set \( \overline{N} = \{ \overline{t}_1, \ldots, \overline{t}_m \} \) computed by the ABM algorithm, because \( N^\rho \) characterizes all the admissible input sets. Following Stetter’s method, each polynomial \( g_E \in GB_E \) corresponds to a power product \( \overline{t} \in C[\overline{N}] \) and can be written as

\[
g_E = \overline{t} - \sum_{j=1}^{m} c_j^E \overline{t}_j, \tag{6}
\]

using all the terms of \( \overline{N} \). The vector \( c_E \) of the coefficients of \( g_E \) is the solution of the system

\[
\overline{M}_m(\overline{P}) c_E = \overline{t}(\overline{P}), \quad \text{with} \quad \overline{M}_m(\overline{P}) = \begin{bmatrix} \overline{t}_1(\overline{P}), & \ldots, & \overline{t}_m(\overline{P}) \end{bmatrix}, \tag{7}
\]

and so \( g_E \) vanishes at \( \overline{P} \).

Since for each \( \overline{t} \in C[\overline{N}] \) there exist both a polynomial \( \overline{g} \in GB \) and a polynomial \( g_E \in GB_E \), there exists an one-to-one correspondence between \( GB \) and \( GB_E \). In addition, since the vector \( \overline{c} \) of the coefficients of \( \overline{g} \) is the least square solution of the problem (2), we have that

\[
\overline{g}(\overline{P}) = \overline{t}(\overline{P}) - \overline{M}_k(\overline{P}) \overline{c}, \quad \text{and from (7),} \quad \overline{g}(\overline{P}) = \overline{M}_m(\overline{P}) c_E - \overline{M}_k(\overline{P}) \overline{c}.
\]

Since the matrix \( \overline{M}_k(\overline{P}) \) consists of the first \( k \) columns of \( \overline{M}_m(\overline{P}) \), we have

\[
\overline{g}(\overline{P}) = \overline{M}_m(\overline{P}) c_E - \overline{M}_m(\overline{P}) \begin{bmatrix} \overline{c} \\ 0 \end{bmatrix} = \overline{M}_m(\overline{P}) \Delta c, \quad \text{where} \quad \Delta c = c_E - \begin{bmatrix} \overline{c} \\ 0 \end{bmatrix}.
\]

Since \( \Delta c = (\overline{M}_m(\overline{P}))^{-1} \overline{g}(\overline{P}) \) and \( c_E = (\overline{M}_m(\overline{P}))^{-1} \overline{g}(\overline{P}) \), we obtain [8]

\[
\frac{\|\Delta c\|_2}{\sigma_{\max}} \leq \|\Delta c\|_2 \quad \text{and} \quad \frac{\|\overline{g}(\overline{P})\|_2}{\sigma_{\min}} \leq \|\Delta c\|_2 \leq \frac{\|\overline{t}(\overline{P})\|_2}{\sigma_{\max}} \leq \frac{\|\Delta c\|_2}{\sigma_{\min}},
\]

where \( \sigma_{\max} \) and \( \sigma_{\min} \) are respectively the minimum and the maximum singular values of \( \overline{M}_m(\overline{P}) \). Since the ABM algorithm computes an element of \( GB \) if the relative residual of the least square problem (2) is less than the threshold \( \epsilon \), we can derive the following upper bound for the differences of the coefficients of the elements of \( GB \) and of \( GB_E \):

\[
\frac{\|\Delta c\|_2}{\|c_E\|_2} \leq \frac{\|\overline{g}(\overline{P})\|_2}{\|\overline{t}(\overline{P})\|_2} \frac{\sigma_{\max}}{\sigma_{\min}} \leq \epsilon K_2(\overline{M}_m(\overline{P})),
\]

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where $\kappa_2(M_m(\mathcal{P})) = \sigma_{\text{max}}/\sigma_{\text{min}}$ is the 2-condition number of $M_m(\mathcal{P})$. Since in general the previous upper bound overestimates the relative error on the coefficients, we can conclude that $GB$ is “near” to the extended basis $GB_E$.

**Remark.** The power product $\tilde{t} \in \mathbb{C}[\mathbb{N}]$ is probably not the leading term of the polynomial $g_E$ defined by (6). In fact, if $\tilde{t}_k < \tilde{t} < \tilde{t}_{k+1}$, since in general $\tilde{t}(\mathcal{P})$ is not in the span of the first $k$ columns of $M_m(\mathcal{P})$ in the classical sense, then the last $m - k$ components of $c_E$ are different from zero. In contrast, by construction, $\tilde{t}$ is the leading term of $\tilde{g} \in GB$ and so, with respect to the extended basis, $GB$ has the advantage that $\text{LT}(GB) = \mathbb{C}[\mathbb{N}]$, that is $\text{LT}(GB)$ coincides with the leading term of a reference basis.

### 5.2 Pseudozeros

Even if sometimes $\mathcal{P}$ is a zero set for the polynomials of $GB$, in general the elements of $GB$ do not vanish at $\mathcal{P}$, but we show that the preprocessed points are pseudozeros for the polynomials of $GB$.

Intuitively, given a tolerance $\delta$, a point $p$ is a pseudozero of a polynomial $g$ if it is the exact zero of a polynomial $\tilde{g}$ whose coefficients differ from those of $g$ by less than $\delta$. This idea, formalized by Stetter in [13], can be generalized for a set of points.

**Definition 5.1** Let $S$ be a set of power products.

A point $p$ is a pseudozero of $g = \sum_{x^j \in S} a_j x^j$ with respect to $S$ and tolerance $\delta$ if $p$ is an exact zero of some $\tilde{g} \in N_S(g, \delta)$, where

$$N_S(g, \delta) = \{ \tilde{g} : \tilde{g}(x) = \sum_{x^j \in S} \tilde{a}_j x^j, |\tilde{a}_j - a_j| < \delta \}.$$

A set of points $\mathcal{P}$ is a set of pseudozeros of $g$ with respect to $S$ and tolerance $\delta$ if it is a set of exact zeros of some $\tilde{g} \in N_S(g, \delta)$.

A set of points $\mathcal{P}$ is a set of pseudozeros for a system of $k$ polynomials $g_1, \ldots, g_k$, whose supports belong to $S$, with respect to $S$ and tolerances $\mathcal{D} = \{ \delta_1, \ldots, \delta_k \}$ if it is a set of pseudozeros with tolerance $\delta_i$ for each polynomial $g_i$, $i = 1 \ldots k$, of the system.

The set $\mathcal{P}$ of the preprocessed points is a pseudozero set for $GB$ with respect to $S = \mathbb{N} \cup \mathbb{C}[\mathbb{N}]$, as shown in the following theorem.

**Theorem 5.2** The points of $\mathcal{P}$ are pseudozeros of the system $\mathcal{G}$ consisting of the polynomials $GB = \{ \tilde{g}_1, \ldots, \tilde{g}_m \}$ with respect to $S$ and with tolerances $\mathcal{D} = \{ \delta_1, \ldots, \delta_m \}$, where

$$S = \mathbb{N} \cup \mathbb{C}[\mathbb{N}] \quad \text{and} \quad \delta_i = \frac{\|\tilde{g}_i(\mathcal{P})\|_2}{\sigma_{\text{min}}}.$$
Proof. Given a polynomial \( g_i = \overline{t} - \sum_{j=1}^{k} \overline{c}_j \overline{t}_j \in GB \), there exists \( g_E = \overline{t} - \sum_{j=1}^{m} \overline{c}_E_j \overline{t}_j \in GB_E \), vanishing at \( \overline{P} \), such that

\[
|c_E^j - c_j| \leq \frac{\|I(\overline{P})\|_2}{\sigma_{\text{min}}} = \delta_i.
\]

Since the supports of \( \overline{g} \) and \( g_E \) are contained in \( S = \overline{N} \cup C[\overline{N}] \) and \( g_E \) belongs to \( N_S(\overline{g}; \delta_i) \) we have that \( \overline{P} \) is a pseudozero set of \( GB \).

Each value \( \delta_i \) is small with respect to the norm of the vector \( \overline{c} \) of the coefficients of \( \overline{g}_i \). In fact, if \( \overline{g}_i \in GB \) and \( g_E \in GB_E \) have similar coefficients, from the results of the previous section we have

\[
\delta_i \leq \epsilon K(M_m(\overline{P}))\|c_E\| \quad \text{and so} \quad \delta_i \leq \epsilon K(M_m(\overline{P}))\|\overline{c}\| + O(\epsilon^2).
\]

The conclusion follows, since the previous upper bound is, in general, an overestimation and \( \epsilon \) is a small value of the threshold.

6 Numerical examples

In this section we present two numerical examples which illustrate how the ABM algorithm approximates a reference basis. From the perturbed points, after the preprocessing phase, the ABM algorithm computes in Example 6.1 a reference basis and in Example 6.2 an approximation to it. In these examples we use the term ordering DegLex; in addition, the coordinates of the points and the coefficients of the polynomials are displayed with a finite number of digits, but all computations are performed in exact arithmetic using CoCoA 4.2 [5]. Moreover, in the following examples the threshold in the test of the ABM algorithm has been suitably chosen. A more exhaustive survey of numerical experiments will be presented in a forthcoming paper.

Example 6.1

Given the perturbed points \( \overline{P} \) known with uncertainty \( s_0 = 0.1 \),

\[
\overline{P} = \{ (-2.45,-3.6), (-0.53,-1.45), (1.5,0.45), (3.5,2.5) \},
\]

the exact, DegLex-Gröbner basis of the ideal \( I(\overline{P}) \) is

\[
\widehat{GB} = \begin{cases} 
xy - 0.97550y^2 + 1.45450x - 2.48031y - 1.54307 \\
x^2 - 0.95161y^2 + 0.89208x - 2.94110y - 2.07192 \\
y^3 + 0.64549y^2 + 91.31103x - 98.56564y - 92.83385
\end{cases}
\]

The exact G-basis \( \widehat{GB} \) cannot be a reference basis, since there is a small relative residual associated to a power product of its normal set \( \widehat{N} \), as pointed out in the following table.

The small relative residual associated to the power product \( x \) means that the perturbed points are “almost aligned” and so a normal set corresponding
Let $\hat{ \mathcal{GB}}$ be the set of perturbed points with uncertainty $s_0 = 0.1$, 
\[
\hat{\mathcal{P}} = \{(0, 4.1), (0.05, -4.03), (3.1, 0.1), (-3, 0.03), (2.37, 2.5), (-2.4, -2.33), (2.31, -2.486), (-2.4, 2.4)\}.
\]

The exact DegLex-Gröbner basis of the ideal $\mathcal{I}(\hat{\mathcal{P}})$ is
\[
\hat{\mathcal{GB}} = \begin{cases} 
    x^2y - 0.00164xy^2 + 0.52911y^3 + 0.15329x^2 - 0.01032xy \\
    + 0.05231y^2 - 0.02767x - 8.75015y - 1.47114, \\
    x^3 + 0.00405xy^2 + 0.00120y^3 - 1.38130x^2 - 0.04581xy \\
    - 0.72231y^2 - 9.17153x + 0.03536y + 11.91429, \\
    y^4 - 2.50589xy^2 - 0.34459y^3 - 178.88212x^2 + 1.01537xy \\
    - 117.12255y^2 + 17.87987x + 11.35992y + 1663.42803, \\
    xy^3 - 0.09269xy^2 + 0.04595y^3 + 1.13505x^2 - 5.90939xy \\
    + 0.65885y^2 + 0.28526x - 1.06888y - 9.86020 
\end{cases}
\]

The exact normal set $\hat{\mathcal{N}}$ of $\hat{\mathcal{GB}}$ cannot be a reference normal set, since the relative residual associated to its power product $x^2$ is very small, as reported in the following table.

Since $\hat{\mathcal{N}}$ can change structurally for small data perturbations, $\hat{\mathcal{GB}}$ is not a reference basis. The ABM algorithm, using the threshold $\epsilon = 0.1$ and working
on the preprocessed points $\hat{P}$

$$\hat{P} = \{(0, 4.065), (0, -4.065), (3.05, 0), (-3.05, 0), (2.405, 2.405), (-2.405, -2.405), (2.405, -2.405), (-2.405, 2.405)\}$$

computes a set of power products $N = \{1, y, x, y^2, xy, y^3, xy^2\}$ which is a normal set associated to relative residuals greater than 0.22. Since all the relative residuals are sufficiently greater or less than the threshold, $N$ turns out to be a reference normal set. Moreover, the ABM algorithm computes the set of polynomials

$$\overline{GB} = \begin{cases} 
    x^2 + 0.55840y^2 - 9.13935, \\
    xy^3 - 5.78402xy, \\
    y^3 - 22.30825y^3 + 95.57653y, 
\end{cases}$$

whose zero set $\{(3.02313, 0), (-3.02313, 0)\}$ is not an admissible input set. Nevertheless, Newton method applied to the function $\overline{F}$, defined by (4), follows descent directions for $\|\overline{F}\|$. For this reason the Newton iteration computes the admissible input set

$$\overline{P}^+ = \{(0, 4.065), (0, -4.065), (3.02, 0), (-3.02, 0), (2.41, 2.41), (-2.41, -2.41), (2.41, -2.41), (-2.41, 2.41)\}.$$ 

This fact suggests that the threshold has been suitably chosen and that $N$ and $\overline{GB}$ are a good approximation to the exact case. In fact, the set $\overline{GB}$ approximates the set $GB$

$$GB = \begin{cases} 
    x^2 + 0.5625y^2 - 9, \\
    xy^3 - 5.76xy, \\
    y^3 - 21.76y^3 + 92.16y 
\end{cases}$$

which is a reference basis, since it is a DegLex-Gröbner basis, with normal set $N$, of the ideal of the admissible input points

$$\{(0, 4), (0, -4), (3, 0), (-3, 0), (2.4, 2.4), (-2.4, -2.4), (2.4, -2.4), (-2.4, 2.4)\}. \quad \Diamond$$

**Appendix A: the preprocessing phase**

In this appendix we summarize the preprocessing phase of the perturbed points $\hat{P}$. This procedure finds the coordinates which differ by less than the error estimation and replaces them by a single representative value. For this reason
the set $\mathcal{P}$ of the preprocessed points is equivalent to $\hat{\mathcal{P}}$ from the computational point of view, even if it can happen that the cardinality of $\mathcal{P}$ is less than the cardinality of $\hat{\mathcal{P}}$. In order to preserve or to get back the "geometrical symmetries" of the exact points, the preprocessing algorithm works on the absolute values of the coordinates of all the points, assembled together in a set $Y$, and it computes a preprocessed set $\hat{Y}$. The preprocessed points $\mathcal{P}$ are built from the set $\hat{Y}$ in an obvious way.

Let $s_0$ be an estimate of the errors on the data and let $Y = \{y_1, \ldots, y_n\}$ be the set of the absolute values of the coordinates of the input points, sorted in non-decreasing order. If $(k + 1)$ elements $y_j, \ldots, y_{j+k}$ of $Y$ are such that the intersection

$$X_j^k = \bigcap_{i=j}^{j+k} [y_i - s_0, y_i + s_0] = [y_{j+k} - s_0, y_j + s_0],$$

is not empty, then any element $\overline{y}_j \in X_j^k$ can represent, from the computational point of view, the values $y_j, \ldots, y_{j+k}$, since it differs from them by less than $s_0$. For this reason we replace $y_j, \ldots, y_{j+k}$ with a value $\overline{y}_j \in X_j^k$ in the set $\hat{Y}$ of the preprocessed coordinates.

The preprocessing algorithm for building the set $\hat{Y}$ can be described as follows. First of all, each element of $Y$ less than $s_0$ is removed from $Y$ and the value $0$ is inserted into $\hat{Y}$. After this step, the algorithm computes for each element $y_j \in Y$ the largest non empty intersection $X_j^k$, and then it processes the set $X_j^k$ which contains the maximum number of elements of $Y$, that is the set with the maximum index $k$. If there are several intersections with this property, the set with the minimum index $j$ is chosen. If $X_j^k$ is the intersection to process, all the elements $y_j, \ldots, y_{j+k} \in X_j^k$ are removed from $Y$ and the middle point $\overline{y}_j$ of $X_j^k$ is inserted in $\hat{Y}$.

The preprocessing algorithm ends when $Y$ is empty or when it contains elements which differ by more than $s_0$. In this case such values do not require any preprocessing treatment and so they are removed from $Y$ and inserted directly into $\hat{Y}$.

Even though it can happen that $[\overline{y}_i - s_0, \overline{y}_i + s_0] \cap [\overline{y}_{i+1} - s_0, \overline{y}_{i+1} + s_0] \neq \emptyset$, we decided not to repeat the preprocessing phase on the set $\hat{Y}$, since all the new coordinates differ by more than $s_0$. In fact, if the set $X_j^k$ is processed,

$$[y_{j-1} - s_0, y_{j-1} + s_0] \cap X_j^k = \emptyset, \quad [y_{j+k+1} - s_0, y_{j+k+1} + s_0] \cap X_j^k = \emptyset,$$

and so the middle point $\overline{y}_j$ of $X_j^k$ differs from $y_{j-1}$ and $y_{j+k+1}$ by more than $s_0$. Since $y_{j+k+1} - y_{j-1} > 2s_0$, the preprocessing phase works separately on the sets $Y_1 = \{y_1, \ldots, y_{j-1}\}$ and $Y_2 = \{y_{j+k+1}, \ldots, y_n\}$. The values obtained by preprocessing $Y_1$ and $Y_2$ are, respectively, less than the maximum of $Y_1$ and greater than the minimum of $Y_2$, and so they differ from $\overline{y}_j$ by more than $s_0$. Analogously, analyzing the sets $Y_1$ and $Y_2$ separately, we can show that all the preprocessed values differ to each other by more than $s_0$, that is they are “well
and 

\[ \left\| \delta_{1} \right\| \leq \left\| \delta_{2} \right\| , \] 

which points out that it is computationally better to use the set \( \mathcal{P} \) than the set \( \mathcal{P}^{*} \) as input points.

**Appendix B: an upper bound for \( \epsilon_{t} \) and \( \epsilon_{M} \)**

In this section we present an upper bound for the values \( \epsilon_{t} \) and \( \epsilon_{M} \), useful for estimating how well the set \( \mathcal{G} \mathcal{B} \) approximates a reference basis.

First of all, we analyze the sensitivity of a term evaluated at a perturbed point. Let \( t = x_{1}^{j_{1}} \ldots x_{k}^{j_{n}} \) be a power product belonging to \( \mathbb{R}[x_{1} \ldots x_{n}] \) and let \( \text{Deg}(t) = j_{1} + \ldots + j_{n} \) be its degree. If \( \mathcal{P} = \{ p_{1}, \ldots, p_{s} \} \) is a perturbation of the point \( p = (p_{1}, \ldots, p_{s}) \) such that \( p_{i} = p_{i}(1 + \delta_{i}), \ |\delta_{i}| \leq \delta_{p}, \ i = 1 \ldots s \), then

\[ t(\mathcal{P}) = t_{i}^{j_{1}} \ldots t_{s}^{j_{n}} = p_{i}^{j_{1}}(1 + \delta_{1})^{j_{1}} \ldots p_{s}^{j_{n}}(1 + \delta_{s})^{j_{n}} = t(p)(1 + \delta_{1})^{j_{1}} \ldots (1 + \delta_{s})^{j_{n}}. \]

By a first-order error analysis, ignoring the errors of higher order, we obtain

\[ |t(\mathcal{P}) - t(p)| \leq |t(p)| (|j_{1}| + \ldots + |j_{n}|) \leq |t(p)| \text{Deg}(t) \delta_{p}. \]

Now, we can easily upper bound \( \epsilon_{t} \) as follows. Let \( \mathcal{P} = \{ p_{1}, \ldots, p_{m} \} \) be a set of points perturbation of \( \mathcal{P} = \{ p_{1}, \ldots, p_{m} \} \), such that \( p_{i} = (p_{1}^{(i)}, \ldots, p_{s}^{(i)}) \) and \( \mathcal{P}_{i} = (p_{1}^{(i)}(1 + \delta_{1}^{(i)}), \ldots, p_{s}^{(i)}(1 + \delta_{s}^{(i)})) \), where \( |\delta_{j}^{(i)}| \leq \delta_{R}, \ i = 1 \ldots m \) and \( j = 1 \ldots s \). Then we have

\[ \left\| t(\mathcal{P}) - t(\mathcal{P}_{i}) \right\| F_{2} \leq \sum_{i=1}^{m} \left\| t(p_{i}) \right\| ^{2} \text{Deg}(t)^{2} \delta_{R}^{2} \leq \text{Deg}(t)^{2} \delta_{R}^{2} \left\| t(\mathcal{P}) \right\| F_{2} \]

that is \( \epsilon_{t} \leq \text{Deg}(t) \delta_{R} \).

Analogously, we can also upper bound \( \epsilon_{M} \). Let \( M(\mathcal{P}) \) and \( M(\mathcal{P}_{i}) \) be the matrices whose columns are generated by the terms \( t_{1} \ldots t_{k} \) evaluated, respectively, at \( \mathcal{P} \) and \( \mathcal{P}_{i} \). If \( d_{M} = \max\{ \text{Deg}(t_{1}), \ldots, \text{Deg}(t_{k}) \} \) is the maximum degree of the terms \( t_{1} \ldots t_{k} \), we obtain

\[ \left\| M(\mathcal{P}) - M(\mathcal{P}_{i}) \right\| F_{2} \leq \delta_{M} \sum_{j=1}^{k} \left\| t_{j}(\mathcal{P}) - t_{j}(\mathcal{P}_{i}) \right\| F_{2} \leq d_{M} \delta_{R} \left\| M(\mathcal{P}) \right\| F_{2}, \]

where \( \left\| \cdot \right\| _{F} \) is the Frobenius matrix norm. Since, given an \( m \times k \) matrix \( A \), we have \( \left\| A \right\| _{2} \leq \left\| A \right\| _{F} \leq \sqrt{k} \left\| A \right\| _{2} \), then

\[ \epsilon_{M} = \frac{\left\| M(\mathcal{P}) - M(\mathcal{P}_{i}) \right\| F_{2}}{\left\| M(\mathcal{P}) \right\| F_{2}} \leq \sqrt{k} \frac{\left\| M(\mathcal{P}) - M(\mathcal{P}_{i}) \right\| F_{2}}{\left\| M(\mathcal{P}) \right\| F_{2}} \leq \sqrt{k} d_{M} \delta_{R}. \]

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