On the stability of a mixed type functional equation in generalized functions

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Abstract

We reformulate the following mixed type quadratic and additive functional equation with \( n \)-independent variables

\[
2f \left( \sum_{i=1}^{n} x_i \right) + \sum_{1 \leq i, j \leq n \atop i \neq j} f(x_i - x_j) = (n + 1) \sum_{i=1}^{n} f(x_i) + (n - 1) \sum_{i=1}^{n} f(-x_i)
\]

as the equation for the spaces of generalized functions. Using the fundamental solution of the heat equation, we solve the general solution and prove the Hyers-Ulam stability of this equation in the spaces of tempered distributions and Fourier hyperfunctions.

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1. Introduction

In 1940, Ulam [1] raised a question concerning the stability of group homomorphisms as follows:

Let \( G_1 \) be a group and let \( G_2 \) be a metric group with the metric \( d(\cdot, \cdot) \). Given \( \varepsilon > 0 \), does there exist a \( \delta > 0 \) such that if a function \( h : G_1 \to G_2 \) satisfies the inequality

\[
d(h(xy), h(x)h(y)) < \delta \text{ for all } x, y \in G_1,
\]

then there exists a homomorphism \( H : G_1 \to G_2 \) with \( d(h(x), H(x)) < \varepsilon \) for all \( x \in G_1 \)?

In 1941, Hyers [2] firstly presented the stability result of functional equations under the assumption that \( G_1 \) and \( G_2 \) are Banach spaces. In 1978, Rassias [3] generalized Hyers’ result to the unbounded Cauchy difference. After that stability problems of various functional equations have been extensively studied and generalized by a number of authors (see [4-7]). Among them, Towanlong and Nakmahachalasint [8] introduced the following functional equation with \( n \)-independent variables.
\[ 2f \left( \sum_{i=1}^{n} x_i \right) + \sum_{1 \leq i, j \leq n}^{i \neq j} f(x_i - x_j) = (n + 1) \sum_{i=1}^{n} f(x_i) + (n - 1) \sum_{i=1}^{n} f(-x_i), \]

(1.1)

where \( n \) is a positive integer with \( n \geq 2 \). For real vector spaces \( X \) and \( Y \), they proved that a function \( f : X \to Y \) satisfies (1.1) if and only if there exist a quadratic function \( q : X \to Y \) satisfying

\[ q(x + y) + q(x - y) = 2q(x) + 2q(y) \]

and an additive function \( a : X \to Y \) satisfying

\[ a(x + y) = a(x) + a(y) \]

such that

\[ f(x) = q(x) + a(x) \]

for all \( x \in X \). For this reason, equation (1.1) is called the mixed type quadratic and additive functional equation. We refer to [9-14] for the stability results of other mixed type functional equations.

In this article, we consider equation (1.1) in the spaces of generalized functions such as the space \( S'(\mathbb{R}) \) of tempered distributions and the space \( \mathcal{F}'(\mathbb{R}) \) of Fourier hyperfunctions. Making use of similar approaches in [15-20], we reformulate equation (1.1) and the related inequality for the spaces of generalized functions as follows:

\[ 2u \circ A + \sum_{1 \leq i, j \leq n}^{i \neq j} u \circ B_{ij} = (n + 1) \sum_{i=1}^{n} u \circ P_i + (n - 1) \sum_{i=1}^{n} u \circ Q_i, \]

(1.2)

\[ \| 2u \circ A + \sum_{1 \leq i, j \leq n}^{i \neq j} u \circ B_{ij} \| - (n + 1) \sum_{i=1}^{n} u \circ P_i - (n - 1) \sum_{i=1}^{n} u \circ Q_i \| \leq \varepsilon, \]

(1.3)

where \( A, B_{ij}, P_i \) and \( Q_i \) are the functions defined by

\[
\begin{align*}
A(x_1, ..., x_n) &= x_1 + \cdots + x_n, \\
B_{ij}(x_1, ..., x_n) &= x_i - x_j, & 1 \leq i, j \leq n, i \neq j, \\
P_i(x_1, ..., x_n) &= x_i, & 1 \leq i \leq n, \\
Q_i(x_1, ..., x_n) &= -x_i, & 1 \leq i \leq n.
\end{align*}
\]

Here \( \circ \) denotes the pullback of generalized functions and the inequality \( ||v|| \leq \varepsilon \) in (1.3) means that \( |\langle v, \phi \rangle| \leq \varepsilon ||\phi||_{L^1} \) for all test functions \( \phi \).

In order to solve the general solution of (1.2) and prove the Hyers-Ulam stability of (1.3), we employ the heat kernel method stated in section 2. In section 3, we prove that every solution \( u \) in \( \mathcal{F}'(\mathbb{R}) \) (or \( S'(\mathbb{R}) \), resp.) of equation (1.2) is of the form

\[ u = ax^2 + bx \]
for some $a, b \in \mathbb{C}$. Subsequently, in section 4, we prove that every solution $u$ in $\mathcal{F}'(\mathbb{R})$ (or $\mathcal{S}'(\mathbb{R})$, resp.) of the inequality (1.3) can be written uniquely in the form

$$u = ax^2 + bx + \mu(x),$$

where $\mu$ is a bounded measurable function such that $\|\mu\|_{L^\infty} \leq \frac{a^4 + b^2 - 3}{a^4 + b^2 - 2} \, \varepsilon$.

2. Preliminaries

In this section, we introduce the spaces of tempered distributions and Fourier hyperfunctions. We first consider the space of rapidly decreasing functions which is a test function space of tempered distributions.

**Definition 2.1.** [21] The space $\mathcal{S}(\mathbb{R})$ denotes the set of all infinitely differentiable functions $\phi : \mathbb{R} \to \mathbb{C}$ such that

$$\|\phi\|_{\alpha, \beta} = \sup_x |x^\alpha D^\beta \phi(x)| < \infty$$

for all nonnegative integers $\alpha, \beta$.

In other words, $\phi(x)$ as well as its derivatives of all orders vanish at infinity faster than the reciprocal of any polynomial. For that reason, we call the element of $\mathcal{S}(\mathbb{R})$ as the rapidly decreasing function. It can be easily shown that the function $\phi(x) = \exp(-ax^2), a > 0$, belongs to $\mathcal{S}(\mathbb{R})$, but $\psi(x) = (1 + x^2)^{-1}$ is not a member of $\mathcal{S}(\mathbb{R})$. Next we consider the space of tempered distributions which is a dual space of $\mathcal{S}(\mathbb{R})$.

**Definition 2.2.** [21] A linear functional $u$ on $\mathcal{S}(\mathbb{R})$ is said to be a tempered distribution if there exists constant $C \geq 0$ and nonnegative integer $N$ such that

$$|\langle u, \phi \rangle| \leq C \sum_{\alpha, \beta \leq N} \sup_x |x^\alpha D^\beta \phi(x)|$$

(2.1)

for all $\phi \in \mathcal{S}(\mathbb{R})$. The set of all tempered distributions is denoted by $\mathcal{S}'(\mathbb{R})$.

For example, every $f \in L^p(\mathbb{R}), 1 \leq p < \infty$, defines a tempered distribution by virtue of the relation

$$\langle f, \phi \rangle = \int f(x) \phi(x) \, dx, \quad \phi \in \mathcal{S}(\mathbb{R}).$$

Note that tempered distributions are generalizations of $L^p$-functions. These are very useful for the study of Fourier transforms in generality, since all tempered distributions have a Fourier transform, but not all distributions have one. Imposing the growth condition on $\| \cdot \|_{\alpha, \beta}$ in (2.1) a new space of test functions has emerged as follows.

**Definition 2.3.** [22] We denote by $\mathcal{F}(\mathbb{R})$ the set of all infinitely differentiable functions $\phi$ in $\mathbb{R}$ such that

$$\|\phi\|_{A, B} = \sup_{x, \alpha, \beta} \frac{|x^\alpha D^\beta \phi(x)|}{A^\alpha B^\beta \alpha!} < \infty$$

(2.2)

for some positive constants $A, B$ depending only on $\phi$.

It can be verified that the seminorm (2.2) is equivalent to

$$\|\phi\|_{h, k} = \sup_{x, \alpha} \frac{|D^\alpha \phi(x)| \exp |\alpha|}{h^\alpha |\alpha|} < \infty$$

for some constants $h, k > 0$. 

Lee Advances in Difference Equations 2012, 2012:16
http://www.advancesindifferenceequations.com/content/2012/1/16
Page 3 of 11
**Definition 2.4.** [22] The strong dual space of $\mathcal{F}(\mathbb{R})$ is called the Fourier hyperfunctions. We denote the Fourier hyperfunctions by $\mathcal{F}'(\mathbb{R})$.

It is easy to see the following topological inclusions:

\[
\mathcal{F}(\mathbb{R}) \hookrightarrow \mathcal{S}(\mathbb{R}), \quad \mathcal{S}'(\mathbb{R}) \hookrightarrow \mathcal{F}'(\mathbb{R}).
\]

(2.3)

Taking the relations (2.3) into account, it suffices to consider the space $\mathcal{F}'(\mathbb{R})$. In order to solve the general solution and the stability problem of (1.2) in the space $\mathcal{F}'(\mathbb{R})$, we employ the fundamental solution of the heat equation called the heat kernel,

\[
E_t(x) = E(x, t) = \begin{cases} (4\pi t)^{-1/2} \exp(-x^2/4t), & x \in \mathbb{R}, t > 0, \\ 0, & x \in \mathbb{R}, t \leq 0. \end{cases}
\]

Since for each $t > 0$, $E_t(\cdot)$ belongs to the space $\mathcal{F}(\mathbb{R})$, the convolution

\[
\tilde{u}(x, t) = (u * E_t)(x) = \langle u, E_t(x - y) \rangle, \quad x \in \mathbb{R}, \quad t > 0
\]

is well defined for all $u \in \mathcal{F}(\mathbb{R})$. We call $\tilde{u}$ as the Gauss transform of $u$. Semigroup property of the heat kernel

\[
(E_t * E_s)(x) = E_{t+s}(x)
\]

holds for convolution. It is useful to convert equation (1.2) into the classical functional equation defined on upper-half plane. We also use the following famous result called heat kernel method, which states as follows.

**Theorem 2.5.** [23] Let $u \in \mathcal{S}'(\mathbb{R})$. Then its Gauss transform $\tilde{u}$ is a $C^\omega$-solution of the heat equation

\[
(\partial / \partial t - \Delta)\tilde{u}(x, t) = 0
\]

satisfying

(i) There exist positive constants $C, M$ and $N$ such that

\[
|\tilde{u}(x, t)| \leq Ct^{-M}(1 + |x|)^{N}\ln R \times (0, \delta).
\]

(ii) $\tilde{u}(x, t) \to u \text{ as } t \to 0^+$ in the sense that for every $\varphi \in \mathcal{S}(\mathbb{R})$,

\[
\langle u, \varphi \rangle = \lim_{t \to 0^+} \int \tilde{u}(x, t)\varphi(x)dx.
\]

Conversely, every $C^\omega$-solution $U(x, t)$ of the heat equation satisfying the growth condition (2.4) can be uniquely expressed as $U(x, t) = \tilde{u}(x, t)$ for some $u \in \mathcal{S}'(\mathbb{R})$.

Similarly, we can represent Fourier hyperfunctions as initial values of solutions of the heat equation as a special case of the results as in [24]. In this case, the condition (i) in the above theorem is replaced by the following:

For every $\varepsilon > 0$ there exists a positive constant $C_\varepsilon$ such that

\[
|\tilde{u}(x, t)| \leq C_\varepsilon \exp(\varepsilon(|x| + 1/t)) \text{ in } \mathbb{R} \times (0, \delta).
\]

3. General solution in $\mathcal{F}'(\mathbb{R})$

We are now going to solve the general solution of (1.2) in the space of $\mathcal{F}'(\mathbb{R})$ (or $\mathcal{S}'(\mathbb{R})$, resp.). In order to do so, we employ the heat kernel mentioned in the previous section. Convolving the tensor product $E_{\varepsilon_1}(x_1) \ldots E_{\varepsilon_n}(x_n)$ of the heat kernels on both
sides of (1.2) we have
\[
\left( u \circ A \right) \ast \left( E_t \left( x_1 \ldots E_t \left( x_n \right) \right) \right) \left( \xi_1, \ldots, \xi_n \right) \\
= \left( u \circ A, E_t \left( \xi_1 - x_1 \right) \ldots E_t \left( \xi_n - x_n \right) \right) \\
= \left\{ u, \int \cdots \int E_t \left( \xi_1 - x_1 + x_2 + \cdots + x_n \right) E_t \left( \xi_2 - x_2 \right) \ldots E_t \left( \xi_n - x_n \right) \, dx_2 \ldots dx_n \right\} \\
= \left\{ u, \int \cdots \int E_t \left( \xi_1 + \cdots + \xi_n - x_1 - \cdots - x_n \right) E_t \left( x_2 \right) \ldots E_t \left( x_n \right) \, dx_2 \ldots dx_n \right\} \\
= \left\{ u, E_t \ast \ldots \ast E_t \left( \xi_1 + \cdots + \xi_n - x_1 \right) \right\} \\
= \left\{ u, E_t \ast \ldots \ast E_t \left( \xi_1 + \cdots + \xi_n \right) \right\} \\
= \bar{u} \left( \xi_1 + \cdots + \xi_n, t_1 + \cdots + t_n \right),
\]
\[
\left[ \left( u \circ B_t \right) \ast \left( E_t \left( x_1 \right) \ldots E_t \left( x_n \right) \right) \right] \left( \xi_1, \ldots, \xi_n \right) = \bar{u} \left( \xi_1 - \xi_i, t_i + t_j \right),
\]
\[
\left[ \left( u \circ P_t \right) \ast \left( E_t \left( x_1 \right) \ldots E_t \left( x_n \right) \right) \right] \left( \xi_1, \ldots, \xi_n \right) = \bar{u} \left( \xi_i, t_i \right),
\]
\[
\left[ \left( u \circ Q_t \right) \ast \left( E_t \left( x_1 \right) \ldots E_t \left( x_n \right) \right) \right] \left( \xi_1, \ldots, \xi_n \right) = \bar{u} \left( -\xi_i, t_i \right).
\]

where \(\bar{u}\) is the Gauss transform of \(u\). Thus, (1.2) is converted into the following classical functional equation
\[
2\bar{u} \left( \sum_{i=1}^{n} x_i \sum_{i=1}^{n} t_i \right) + \sum_{1 \leq i, j \leq n, i \neq j} \bar{u} \left( x_i - x_j, t_i + t_j \right) \\
= \left( n + 1 \right) \sum_{i=1}^{n} \bar{u} \left( x_i, t_i \right) + \left( n - 1 \right) \sum_{i=1}^{n} \bar{u} \left( -x_i, t_i \right)
\]

for all \(x_1, \ldots, x_n \in \mathbb{R}, t_1, \ldots, t_n > 0\). We here need the following lemma which will be crucial role in the proof of main theorem.

**Lemma 3.1.** A continuous function \(f : \mathbb{R} \times (0, \infty) \to \mathbb{C}\) satisfies the functional equation
\[
2f \left( \sum_{i=1}^{n} x_i \sum_{i=1}^{n} t_i \right) + \sum_{1 \leq i, j \leq n, i \neq j} f \left( x_i - x_j, t_i + t_j \right) \\
= \left( n + 1 \right) \sum_{i=1}^{n} f \left( x_i, t_i \right) + \left( n - 1 \right) \sum_{i=1}^{n} f \left( -x_i, t_i \right)
\]

for all \(x_1, \ldots, x_n \in \mathbb{R}, t_1, \ldots, t_n > 0\) if and only if there exist constants \(a, b, c \in \mathbb{C}\) such that
\[
f(x, t) = ax^2 + bx + ct
\]

for all \(x \in \mathbb{R}, t > 0\).

**Proof.** Putting \((x_1, \ldots, x_n) = (0, \ldots, 0)\) in (3.1) yields
\[
f \left( 0, \sum_{i=1}^{n} t_i \right) + \sum_{1 \leq i < j \leq n} f \left( 0, t_i + t_j \right) = n \sum_{i=1}^{n} f \left( 0, t_i \right)
\]

for all \(t_1, \ldots, t_n > 0\). In view of (3.2) we see that
\[
c = \lim_{t \to 0^+} f(0, t)
\]
exists. Letting \( t_1 = \cdots = t_n \to 0^+ \) in (3.2) gives \( c = 0 \). Setting \( (x_1, x_2, x_3, \ldots, x_n) = (x, y, 0, \ldots, 0) \) and letting \( t_1 = t, t_2 = s, t_3 = \cdots = t_n \to 0^+ \) in (3.1) we have
\[
2f(x + y, t + s) + f(x - y, t + s) + f(-x + y, t + s) = 3f(x, t) + 3f(y, t) + f(-y, s)
\]
(3.3)
for all \( x, y \in \mathbb{R}, t, s > 0 \). Replacing \( x \) and \( y \) with \(-x\) and \(-y\) in (3.3) yields
\[
2f(-x - y, t + s) + f(-x + y, t + s) + f(x - y, t + s) = 3f(-x, t) + 3f(-y, s) + f(x, t) + f(y, s)
\]
(3.4)
for all \( x, y \in \mathbb{R}, t, s > 0 \). We now define the even part and the odd part of the function \( f \) by
\[
f_e(x, t) = \frac{f(x, t) + f(-x, t)}{2}, \quad f_o(x, t) = \frac{f(x, t) - f(-x, t)}{2}
\]
for all \( x \in \mathbb{R}, t > 0 \). Adding (3.3) to (3.4) we verify that \( f_e \) satisfies
\[
f_e(x + y, t + s) + f_e(x - y, t + s) = 2f_e(x, t) + 2f_e(y, s)
\]
(3.5)
for all \( x, y \in \mathbb{R}, t, s > 0 \). Similarly, taking the difference of (3.3) and (3.4) we see that \( f_o \) satisfies
\[
f_o(x + y, t + s) = f_o(x, t) + f_o(y, s)
\]
(3.6)
for all \( x, y \in \mathbb{R}, t, s > 0 \). It follows from (3.5), (3.6) and given the continuity that \( f_e \) and \( f_o \) are of the forms
\[
f_e(x, t) = ax^2 + c_1t, \quad f_o(x, t) = bx + c_2t
\]
for some constants \( a, b, c_1, c_2 \in \mathbb{C} \). Finally we have
\[
f(x, t) = f_e(x, t) + f_o(x, t) = ax^2 + bx + ct,
\]
where \( c = c_1 + c_2 \).

Conversely, if \( f(x, t) = ax^2 + bx + c \) for some \( a, b, c \in \mathbb{C} \), then it is obvious that \( f \) satisfies equation (3.1). \( \square \)

According to the above lemma, we solve the general solution of (1.2) in the space of \( \mathcal{F}'(\mathbb{R}) \) (or \( S'(\mathbb{R}) \), resp.) as follows.

**Theorem 3.2.** Every solution \( u \) in \( \mathcal{F}'(\mathbb{R}))\) or \( S'(\mathbb{R}) \), resp.) of equation (1.2) has the form
\[
u = ax^2 + bx,
\]
for some \( a, b \in \mathbb{C} \).

**Proof.** Convolving the tensor product \( E_t(x_1) \ldots E_t(x_n) \) of the heat kernels on both sides of (1.2) we have
\[
2\tilde{u}\left(\sum_{i=1}^{n} x_i, \sum_{i=1}^{n} t_j\right) + \sum_{1 \leq i, j \leq n, \ i \neq j} \tilde{u}(x_i - x_j, t_i + t_j)
\]
(3.7)
\[
= (n + 1) \sum_{i=1}^{n} \tilde{u}(x_i, t_i) + (n - 1) \sum_{i=1}^{n} \tilde{u}(-x_i, t_i)
\]
for all $x_1, \ldots, x_n \in \mathbb{R}$, $t_1, \ldots, t_n > 0$. It follows from Lemma 3.1 that the solution $\tilde{u}$ of equation (3.7) has the form

$$\tilde{u}(x, t) = ax^2 + bx + ct$$  \hspace{1cm} (3.8)

for some $a, b, c \in \mathbb{C}$. Letting $t \to 0^+$ in (3.8), we finally obtain the general solution of (1.2). □

4. Stability in $F'(\mathbb{R})$

In this section, we are going to state and prove the Hyers-Ulam stability of (1.3) in the space of $F'(\mathbb{R})$ (or $S'(\mathbb{R})$, resp.).

Lemma 4.1. Suppose that $f : \mathbb{R} \times (0, \infty) \to \mathbb{C}$ is a continuous function satisfying

$$|2f \left( \sum_{i=1}^n x_i t_i \right) + \sum_{1 \leq i,j \leq n, i \neq j} f(x_i - x_j, t_i + t_j) - (n + 1) \sum_{i=1}^n f(x_i, t_i) - (n - 1) \sum_{i=1}^n f(-x_i, t_i) | \leq \varepsilon$$

for all $x_1, \ldots, x_n \in \mathbb{R}$, $t_1, \ldots, t_n > 0$, then there exists the unique function $g : \mathbb{R} \times (0, \infty) \to \mathbb{C}$ satisfying equation (3.1) such that

$$|f(x, t) - g(x, t)| \leq \frac{n^2 + n - 3}{n^2 + n - 2} \varepsilon$$

for all $x \in \mathbb{R}$, $t > 0$.

Proof. Putting $(x_1, \ldots, x_n) = (0, \ldots, 0)$ in (4.1) yields

$$\left| f(0, \sum_{i=1}^n t_i) + \sum_{1 \leq i,j \leq n} f(0, t_i + t_j) - n \sum_{i=1}^n f(0, t_i) \right| \leq \frac{\varepsilon}{2}$$

for all $t_1, \ldots, t_n > 0$. In view of (4.2) we see that

$$c : = \lim_{t \to 0^+} f(0, t)$$

exists. Letting $t_1 = \cdots = t_n \to 0^+$ in (4.2) gives

$$|c| \leq \frac{\varepsilon}{n^2 + n - 2}. \hspace{1cm} (4.3)$$

Setting $(x_1, x_2, x_3, \ldots, x_n) = (x, 0, \ldots, 0)$ and letting $t_1 = t_2 = t_3 = \cdots = t_n \to 0^+$ in (4.1) we have

$$\left| f(2x, 2t) + f(0, 2t) - 3f(x, t) - f(-x, t) - c \frac{n^2 + n - 6}{2} \right| \leq \frac{\varepsilon}{2} \hspace{1cm} (4.4)$$

for all $x \in \mathbb{R}$, $t > 0$. Replacing $x$ by $-x$ in (4.4) yields

$$\left| f(-2x, 2t) + f(0, 2t) - 3f(-x, t) - f(x, t) - c \frac{n^2 + n - 6}{2} \right| \leq \frac{\varepsilon}{2} \hspace{1cm} (4.5)$$
for all \( x \in \mathbb{R}, t > 0 \). Let \( f_e \) and \( f_o \) be even and odd part of \( f \) defined in Lemma 3.1, respectively. Using the triangle inequality in (4.4) and (4.5) we get the inequalities

\[
\left| \frac{g_e(2x, 2t)}{4} - g_e(x, t) + \frac{g_o(0, 2t)}{4} \right| \leq \frac{\varepsilon}{8},
\]

(4.6)

\[
\left| \frac{f_e(2x, 2t)}{2} - f_e(x, t) \right| \leq \frac{\varepsilon}{4}
\]

(4.7)

for all \( x \in \mathbb{R}, t > 0 \), where \( g_e(x, t) := f_e(x, t) + \frac{f(n^2 + m - q)}{4} \).

We first consider the even case. Using the iterative method in (4.6) we obtain

\[
\left| \frac{g_e(2^k x, 2^kt)}{4^k} - g_e(x, t) + \sum_{j=1}^{k} \frac{g_e(0, 2^j t)}{4^j} \right| \leq \frac{\varepsilon}{6}
\]

(4.8)

for all \( k \in \mathbb{N}, x \in \mathbb{R}, t > 0 \). Letting \( t_1 = t, t_2 = s, t_3 = \cdots = t_n \to 0^+ \) in (4.2) we have

\[
\left| g_e(0, t + s) - g_e(0, t) - g_e(0, s) \right| \leq \frac{\varepsilon}{4}
\]

(4.9)

for all \( t, s > 0 \). We verify from (4.9) that

\[
h(t) := \lim_{k \to \infty} \frac{g_e(0, 2^k t)}{2^k}
\]

converges and is the unique function satisfying

\[
h(t + s) = h(t) + h(s),
\]

(4.10)

\[
|h(t) - g_e(0, t)| \leq \frac{\varepsilon}{4}
\]

(4.11)

for all \( t, s > 0 \). Combining (4.10) and (4.11) we get

\[
\left| (1 - 2^{-k})h(t) - \sum_{i=1}^{k} \frac{g_e(0, 2^i t)}{4^i} \right| \leq \frac{\varepsilon}{12}
\]

(4.12)

for all \( k \in \mathbb{N}, t > 0 \). Adding (4.8) to (4.12) we have

\[
\left| \tilde{g}_e(x, t) - \frac{\tilde{g}_e(2^k x, 2^k t)}{4^k} \right| \leq \frac{\varepsilon}{4}
\]

(4.13)

for all \( k \in \mathbb{N}, x \in \mathbb{R}, t > 0 \), where \( \tilde{g}_e(x, t) := g_e(x, t) - h(t) \). From (4.1) and (4.13) we verify that

\[
\tilde{G}_e(x, t) := \lim_{k \to \infty} \frac{\tilde{g}_e(2^k x, 2^k t)}{4^k}
\]

is the unique function satisfying equation (3.1) and the inequality

\[
\left| \tilde{g}_e(x, t) - \tilde{G}_e(x, t) \right| \leq \frac{\varepsilon}{4}
\]

(4.14)

for all \( x \in \mathbb{R}, t > 0 \). If we define a function \( q(x, t) := \tilde{G}_e(x, t) + h(t) \), then \( q \) also satisfies (3.1). By Lemma 3.1 and evenness of \( q \) we have
\( q(x, t) = ax^2 + c_1 t \)

for some \( a, c_1 \in \mathbb{C} \). It follows from (4.3) and (4.14) that

\[
|f(x, t) - ax^2 - c_1 t| \leq \frac{n^2 + n - 4}{2(n^2 + n - 2)} \varepsilon \tag{4.15}
\]

for all \( x \in \mathbb{R}, t > 0 \).

Next, we consider the odd case. From (4.7), in the similar manner, we verify that

\[
F_o(x, t) := \lim_{k \to \infty} f_o(2kx, 2kt)
\]

is the unique function satisfying equation (3.1) and the inequality

\[
|F_o(x, t) - f_o(x, t)| \leq \frac{\varepsilon}{2} \tag{4.16}
\]

for all \( x \in \mathbb{R}, t > 0 \). By Lemma 3.1 and oddness of \( F_o \) we have

\[
F_o(x, t) = bx + c_2 t
\]

for some \( b, c_2 \in \mathbb{C} \).

Therefore, from (4.15) and (4.16), we obtain

\[
|f(x, t) - (ax^2 + bx + ct)|
\leq |f(x, t) - (ax^2 + c_1 t)| + |f_o(x, t) - (bx + c_2 t)|
\leq \frac{n^2 + n - 3}{n^2 + n - 2} \varepsilon
\]

for all \( x \in \mathbb{R}, t > 0 \), where \( c = c_1 + c_2 \).

From the above lemma we immediately prove the Hyers-Ulam stability of (1.3) in the space of \( F'(\mathbb{R}) \) (or \( S'(\mathbb{R}) \), resp.) as follows.

**Theorem 4.2.** Suppose that \( u \) in \( F'(\mathbb{R}) \) (or \( S'(\mathbb{R}) \), resp.) satisfies the inequality (1.3), then there exists the unique quadratic additive function \( q(x) = ax^2 + bx \) such that

\[
\|u - q(x)\| \leq \frac{n^2 + n - 3}{n^2 + n - 2} \varepsilon. \tag{4.17}
\]

**Proof.** Convolving the tensor product \( E_t(x_1) \ldots E_t(x_n) \) of the heat kernels on both sides of (1.3) we verify that the inequality (1.3) is converted into

\[
2\bar{u} \left( \sum_{i=1}^{n} x_i \sum_{i=1}^{n} t_i \right) + \sum_{1 \leq i, j \leq n \atop i \neq j} \bar{u}(x_i - x_j, t_i + t_j) - (n + 1) \sum_{i=1}^{n} \bar{u}(x_i, t_i) - (n - 1) \sum_{i=1}^{n} \bar{u}(-x_i, t_i) \leq \varepsilon
\]

for all \( x_1, \ldots, x_n \in \mathbb{R}, t_1, \ldots, t_n > 0 \). According to Lemma 4.1, there exists the unique function \( g(x, t) = ax^2 + bx + ct \) such that
\[ |\dot{u}(x,t) - g(x,t)| \leq \frac{n^2 + n - 3}{n^2 + n - 2} \varepsilon \]  
(4.18)

for all \( x \in \mathbb{R}, t > 0 \). Letting \( t \to 0^+ \) in (4.18) finally we have the stability result (4.17). \( \Box \)

**Remark 4.3.** The above norm inequality \( \|u - q(x)\| \leq \frac{n^2 + n - 3}{n^2 + n - 2} \varepsilon \) implies that \( u - q(x) \) belongs to \( (L^1)' = L^\infty \). Thus, every solution \( u \) of the inequality (4.17) in \( \mathcal{F}'(\mathbb{R}) \) (or \( \mathcal{S}'(\mathbb{R}) \), resp.) can be rewritten uniquely in the form

\[ u = q(x) + \mu(x), \]

where \( \mu \) is a bounded measurable function such that \( \|\mu\|_{L^\infty} \leq \frac{n^2 + n - 3}{n^2 + n - 2} \varepsilon. \)

**Competing interests**

The author declares that they have no competing interests.

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