THE MODULARITY OF K3 SURFACES WITH NON-SYMPLECTIC GROUP ACTIONS

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Abstract. We consider complex K3 surfaces with a non-symplectic group acting trivially on the algebraic cycles. Vorontsov and Kondō classified those K3 surfaces with transcendental lattice of minimal rank. The purpose of this note is to study the Galois representations associated to these K3 surfaces. The rank of the transcendental lattices is even and varies from 2 to 20, excluding 8 and 14. We show that these K3 surfaces are dominated by Fermat surfaces, and hence they are all of CM type. We will establish the modularity of the Galois representations associated to them. Also we discuss mirror symmetry for these K3 surfaces in the sense of Dolgachev, and show that a mirror K3 surface exists with one exception.

1. Introduction

Let $X$ be an algebraic K3 surface, and let $S_X$ and $T_X$ be the lattices of algebraic and transcendental cycles on $X$, respectively. The automorphism group $\text{Aut}(X)$ acts on these lattices. Let $O(S_X)$ denote the group of isometries of $S_X$. Then the kernel $H_X := \text{Ker}(\text{Aut}(X) \to O(S_X))$ is a finite cyclic group. If $g$ is a generator of $H_X$, and if $\omega$ is a nowhere vanishing holomorphic 2-form on $X$, then $g$ acts on $\omega$ by multiplication by a primitive $k$-th root of unity for some $k$. Let $\phi(k)$ denote the Euler $\phi$-function. Then $\phi(k)$ divides $\text{rank}(T_X)$. We are interested in the special situation where $\text{rank}(T_X) = \phi(k)$. There are two cases (cf. Thm. 1, 2):

(I) $T_X$ is unimodular (i.e. $\det(T_X) = \pm 1$). Then there are exactly six values for $k$, namely, $k \in \{66, 44, 42, 36, 28, 12\}$. Conversely, for each $k$, there exists a unique K3 surface $X$ up to isomorphism with these properties. These K3 surfaces can be defined over $\mathbb{Q}$ by Weierstrass equations.

(II) $T_X$ is non-unimodular. Then there are exactly ten values for $k$, namely, $k \in \{3, 9, 27, 5, 25, 7, 11, 13, 17, 19\}$. Conversely, for each $k$, there exists a unique K3 surface $X$ up to isomorphism with these properties. With one exception, these K3 surfaces are defined by Weierstrass equations over $\mathbb{Q}$. For $k = 25$, the K3 surface is defined as a double sextic over $\mathbb{Q}$, i.e. a double cover of $\mathbb{P}^2$ branched along a sextic curve.
Our goal of this paper is to establish the modularity of the Galois representations associated to the transcendental lattices of these K3 surfaces. We note that the rank of \( T_X \) takes even values from 2 to 20, excluding the values 8 and 14. We show in Corollary\(^1\) that the K3 surfaces are all of CM type. Indeed they are Delsarte surfaces. As such, they are realized as quotients of Fermat surfaces of appropriate degrees (Lemma\(^1\)). We verify the modularity explicitly in Lemma\(^2\). We study the Galois representations associated to the transcendental lattices of our K3 surfaces (Lemma\(^4\)). The paper concludes with a discussion of mirror symmetry for these K3 surfaces in the sense of Dolgachev \(^3\), and its arithmetic aspects. All K3 surfaces with one exception \((k = 3)\) admit mirror partners (Lemma\(^5\)). When \( k \neq 3, 7, 11, 17, 19 \), there are mirrors that are again Delsarte surfaces (Lemma\(^6\)).

K3 surfaces with non-symplectic automorphisms of 2-power order are classified in a recent article by the second author \(^{14}\). The paper also considers analogous arithmetic questions.

2. Classification of K3 surfaces with non-symplectic group actions

Let \( X \) be an algebraic K3 surface over \( \mathbb{C} \). Let \( H^0(X, \Omega^2_X) \) be the space of global holomorphic 2-forms on \( X \), which is of complex dimension 1, i.e., \( H^0(X, \Omega^2_X) \cong \mathbb{C}\omega \). Let \( S_X \) and \( T_X \) be the lattices of algebraic and transcendental cycles of \( X \), respectively. Let \( \text{Aut}(X) \) be the automorphism group of \( X \). Since any automorphism \( g \in \text{Aut}(X) \) preserves the one-dimensional vector space \( H^0(X, \Omega^2_X) = \mathbb{C}\omega \), there is a non-zero complex number \( \alpha(g) \in \mathbb{C}^* \) such that \( g^*(\omega) = \alpha(g)\omega \). Nikulin proved that \( \alpha(\text{Aut}(X)) \) is a finite cyclic group \(^{11, \text{Thm. 3.1}}\). Moreover, if \( k = |\alpha(\text{Aut}(X))| \), then \( \phi(k) \mid \text{rank}(T_X) \).

Let \( O(L) \) denote the group of isometries of a lattice \( L \). It is a consequence of the Torelli theorem that the natural representation

\[ \rho : \text{Aut}(X) \rightarrow O(H^2(X, \mathbb{Z})) \]

is faithful, i.e. the induced map \( \text{Aut}(X) \rightarrow O(S_X) \times O(T_X) \) is injective. We are interested in automorphisms which act trivially on \( S_X \). We define the subgroup \( H_X \subseteq \text{Aut}(X) \) by the following short exact sequence

\[ 1 \rightarrow H_X \rightarrow \text{Aut}(X) \rightarrow O(S_X) \rightarrow 1. \]

Since \( \rho \) is faithful, \( H_X \) can be identified with its image under \( \alpha \), a finite cyclic group. Hence if \( k = |H_X| \), then \( \phi(k) \) divides the rank of \( T_X \). In the following, we let always \( g_k \) denote a generator for the cyclic group \( H_X \).

There are only finitely many values for \( k \) arising this way. The classification of all possible choices for such \( k \) was announced by Vorontsov \(^{15}\). It was proved later by Vorontsov using the theory of cyclotomic fields over \( \mathbb{Q} \), and by Kondō \(^5\) using the theory of elliptic surfaces and results of Nikulin \(^{11}\) on finite automorphism groups of K3 surfaces.

**Theorem 1** (Unimodular cases). Let \( X \) be an algebraic K3 surface over \( \mathbb{C} \).

Assume that \( T_X \) is unimodular (i.e., \( \det(T_X) = \pm 1 \)). Denote \( k = |H_X| \). Let \( \Sigma = \{66, 44, 42, 36, 28, 12\} \). Then the following assertions hold.

(a) \( k \) is a divisor of an element in \( \Sigma \). In particular \( k \leq 66 \).

(b) Furthermore, assume that \( \phi(k) = \text{rank}(T_X) \). Then \( k \in \Sigma \).
(c) Conversely, for each \( k \in \Sigma \), there exists a unique K3 surface \( X \) (up to isomorphism) satisfying the properties that \( |H_X| = k \) and \( \text{rank}(T_X) = \phi(k) \). For these values of \( k \), \( S_X \) and \( T_X \) are given as follows.

\[
\begin{array}{c|c|c|c}
 k & S_X & \text{rank}(S_X) & T_X & \text{rank}(T_X) \\
\hline
66 & U_2 & 2 & U_2 \oplus (-E_8)^2 & 20 \\
44 & U_2 & 2 & U_2 \oplus (-E_8)^2 & 20 \\
42 & U_2 \oplus (-E_8) & 10 & U_2 \oplus (-E_8) & 12 \\
36 & U_2 \oplus (-E_8) & 10 & U_2 \oplus (-E_8) & 12 \\
28 & U_2 \oplus (-E_8)^2 & 18 & U_2^8 & 4 \\
\hline
\end{array}
\]

Table 1. Unimodular non-symplectic K3 surfaces

Here \( U_2 \) denotes the hyperbolic plane and \( E_8 \) is the positive-definite unimodular root lattice of rank 8. In this notation, the K3 lattice is \( H^2(X, \mathbb{Z}) = U_2^2 \oplus (-E_8)^2 \).

The K3 surfaces in Theorem 1 can all be defined over \( \mathbb{Q} \). Kondō \cite{5} exhibits defining equations in terms of Weierstrass models. We reproduce them in the following table up to some signs. We also list the action of a generator \( g_k \) of \( H_X \) on \( X \). In terms of the local coordinates of the Weierstrass model, the action of \( g_k \) is given by

\[
(x, y, t) \mapsto (\zeta_k^\alpha x, \zeta_k^\beta y, \zeta_k^\gamma t)
\]

where \( \zeta_k \) is a primitive \( k \)-th root of unity, and \( \alpha, \beta, \gamma \in \mathbb{Z}/k\mathbb{Z} \).

\[
\begin{array}{c|c|c}
 k & X & g_k \\
\hline
66 & y^2 = x^3 - t(t^{11} + 1) & (x, y, t) \mapsto (\zeta_{66}^\alpha x, \zeta_{66}^\beta y, \zeta_{66}^\gamma t) \\
44 & y^2 = x^3 + x + t^{14} & (x, y, t) \mapsto (-x, \zeta_{44}^\alpha y, \zeta_{44}^\beta t) \\
42 & y^2 = x^3 - t^3(t^{1} + 1) & (x, y, t) \mapsto (\zeta_{42}^\alpha x, \zeta_{42}^\beta y, \zeta_{42}^\gamma t) \\
36 & y^2 = x^3 - t^3(t^{13} + 1) & (x, y, t) \mapsto (\zeta_{36}^\alpha x, \zeta_{36}^\beta y, \zeta_{36}^\gamma t) \\
28 & y^2 = x^3 + x + t^4 & (x, y, t) \mapsto (-x, \zeta_{28}^\alpha y, \zeta_{28}^\beta t) \\
12 & y^2 = x^3 + t^4(t^{2} + 1) & (x, y, t) \mapsto (\zeta_{12}^\alpha x, \zeta_{12}^\beta y, -t) \\
\hline
\end{array}
\]

Table 2. Elliptic fibrations of unimodular K3 surfaces

The corresponding theorem in the non-unimodular case reads as follows:

**Theorem 2** (Non-unimodular cases). Let \( X \) be an algebraic K3 surface over \( \mathbb{C} \). Assume that \( T_X \) is non-unimodular. Let \( \Omega = \{3, 9, 27, 5, 25, 7, 11, 13, 17, 19\} \). Denote \( k = |H_X| \). Then the following assertions hold.

(a) Suppose that \( \text{rank}(T_X) = \phi(k) \). Then \( k \in \Omega \).

(b) Conversely, for each \( k \in \Omega \), there exists a unique algebraic K3 surface \( X \) with \( |H_X| = k \) and \( \text{rank}(T_X) = \phi(k) \). The non-unimodular lattices \( S_X, T_X \) are given in the following table.
Remark 1. In the general non-unimodular case, there is a divisibility result similar to (a) in Theorem 1: Let $k = |H_X|$ without the assumption $\text{rank}(T_X) = \phi(k)$. Then $k \in \Omega \cup \{1, 2, 4, 8, 16\}$ by [5] Cor. 6.2]. The cases where $k = 2^j$ are analyzed by the second author in [14].

In a weaker form, this classification was announced by Vorontsov [18]. Kondō gave a proof of part (a) in [5]. The uniqueness of part (b) was established by Machida and Oguiso [9] for $k = 25$ and by Oguiso and Zhang [12] for the remaining cases. The lattices will be calculated in the next section.

Explicit defining equations for these K3 surfaces were given by Kondō [5]. All K3 surfaces but one are elliptic with section and thus defined by Weierstrass equations. The exception is the K3 surface corresponding to the case $k = 25$. This is only defined as a double sextic. The following table reproduces the defining affine equations from [5] up to sign changes. For the $H_X$-action we employ the same convention as before.

Remark 2. Though the K3 surfaces in Theorem 1 and Theorem 2 are unique up to isomorphism, there are several ways of defining these K3 surfaces. For instance, for $k = 66$, we may take a weighted K3 surface:

$$x_0^2 + x_1^3 + x_2^{11} x_3 + x_3^{12} = 0 \subset \mathbb{P}(6, 4, 1, 1)$$

of degree 12. Letting $x_2 = 1$, we obtain the affine piece $x_0^2 + x_1^3 + x_3 + x_3^{12} = 0$. This is birationally equivalent to the elliptic surface defined by Kondō, $y^2 = x^3 + t(t^{11} - 1)$. This was also pointed out to us by Y. Goto. We will elaborate on two examples in more detail in Remarks 5, 6.

Some of the K3 surfaces in Theorem 1 and 2 offer geometric interpretations of the symplectic group actions. We elaborate on three cases in the next two examples.
By [6], they naturally inherit an automorphism of order three from Example 2. Weil rank is always positive, this automorphism acts non-trivially on the Néron-Severi lattice.

The minimal resolution of the quotient $X/\iota$ is isomorphic to the Kummer surface of the Jacobian of the following genus 2 curve:

$$C : \quad u^2 = v^6 + 4v.$$ 

Here $C$ is equipped with an automorphism of order five, $v \mapsto \zeta_5 v, u \mapsto \zeta_5^3 u$.

### Table 4. Equations of non-unimodular K3 surfaces

| $k$ | $X$ | $g_k$ |
|-----|-----|-------|
| 19  | $y^2 = x^3 + t^3 x - t$ | $(x, y, t) \mapsto (\zeta_{19} x, \zeta_{19} y, \zeta_{19}^2 t)$ |
| 17  | $y^2 = x^3 + t^3 x - t^2$ | $(x, y, t) \mapsto (\zeta_{17} x, \zeta_{17} y, \zeta_{17} t)$ |
| 13  | $y^2 = x^3 + t^3 x - t$ | $(x, y, t) \mapsto (\zeta_{13} x, \zeta_{13} y, \zeta_{13}^2 t)$ |
| 11  | $y^2 = x^3 + t^3 x - t^2$ | $(x, y, t) \mapsto (\zeta_{11} x, \zeta_{11} y, \zeta_{11} t)$ |
| 7   | $y^2 = x^3 + t^3 x - t^3$ | $(x, y, t) \mapsto (\zeta_7^3 x, \zeta_7 y, \zeta_7 t)$ |
| 5   | $y^2 = x^3 + t^3 x - t^4$ | $(x, y, t) \mapsto (\zeta_5^2 x, \zeta_5 y, \zeta_5 t)$ |
| 21  | $y^2 = x^3 - t(t^3 + 1)$ | $(x, y, t) \mapsto (\zeta_{27}^3 x, \zeta_{27} y, \zeta_{27}^6 t)$ |
| 9   | $y^2 = x^3 - t^3(t^3 + 1)$ | $(x, y, t) \mapsto (\zeta_9^3 x, \zeta_9^2 y, \zeta_9^5 t)$ |
| 3   | $y^2 = x^3 + t^3(t - 1)^2$ | $(x, y, t) \mapsto (\zeta_3 x, y, t)$ |
| 25  | $y^2 = u^3 + uv^3 - 1$ | $(u, v, y) \mapsto (\zeta_{25}^5 u, \zeta_{25} v, y)$ |

### Example 1.
The surfaces for $k = 3$ and $k = 12$ naturally occur in a one-dimensional family

$$X_\lambda : \quad y^2 = x^3 + t^5(t^2 + 2\lambda t + 1).$$

The K3 surfaces $X_\lambda$ are double coverings of the Kummer surfaces for $E_0 \times E$ where $E_0$ is the elliptic curve with $j(E_0) = 0$ and $E$ is equipped with an automorphism of order five, unless $E \cong E_0$. The double covering exhibits the Shioda-Inose structure on $X_\lambda$.

On the one hand, $X_\lambda$ admits an automorphism of order $k = 12$ if and only if the elliptic curve $E$ admits an automorphism of order four, i.e. $j(E) = 1728$. This is the case $\lambda = 0$. On the other hand, every $X_\lambda$ admits an automorphism of order $k = 3$, since the elliptic fibration is trivial:

$$g_3 : x \mapsto \zeta_3 x.$$ 

Here $\text{rank}(T_{X_\lambda}) = \phi(k) = 2$ if and only if $E_0$ and $E$ are isogenous. However, unless $E \cong E_0$, the elliptic fibration $\iota$ has Mordell-Weil rank two with non-trivial $g_k$-action. Hence the only case with $\rho(X_\lambda) = 20$ and trivial $g_k$-action on $S_{X_\lambda}$ is $E \cong E_0$. This corresponds to $\lambda = \pm 1$ as in the table.

By similar arguments, we can rule out the Kummer surfaces themselves, although they naturally inherit an automorphism of order three from $E_0$: Since the Mordell-Weil rank is always positive, this automorphism acts non-trivially on the Néron-Severi lattice.

### Example 2.
A similar picture involving Shioda-Inose structures arises for $k = 5$: By [6], $X$ admits a Nikulin involution $\iota$

$$x \mapsto x^3/t^8, \quad y \mapsto x^3 y/t^{12}, \quad t \mapsto -x/t^3.$$ 

The minimal resolution of the quotient $X/\iota$ is $\mathbb{Q}$-isomorphic to the Kummer surface of the Jacobian of the following genus 2 curve:

$$C : \quad u^2 = v^6 + 4v.$$ 

Here $C$ is equipped with an automorphism of order five, $v \mapsto \zeta_5 v, u \mapsto \zeta_5^3 u$. 
3. Algebraic and transcendental cycles

In this section, we compute the lattices $S_X, T_X$ of algebraic and transcendental cycles on the K3 surfaces $X$ from Theorem 2. We use the theory of the discriminant form as developed by Nikulin in [10] and Mordell-Weil lattices after Shioda [16].

We start by computing the Néron-Severi lattices for the K3 surfaces in Theorem 2. For all but $k = 25$, we will use the elliptic fibration given by Kondo [5]. For $k = 3, 9, 27$, there is no section. By the formula of Shioda and Tate, $S_X$ agrees with the trivial lattice which is generated by the zero section $O$, a general fiber $F$ and fiber components not meeting the zero section. After identifying the singular fibers with the corresponding Dynkin diagrams, we deduce the claimed shape.

In the other cases, there is a non-torsion section $P$ that we give in the next table. Then the lattice $S_X$ is encoded in the intersection number $(P.O)$ and in the fiber components that $P$ meets. After Shioda [16], this information can be expressed through the height of $P$:

$$h(P) = 4 + 2(P.O) - \sum_v \text{corr}_v(P).$$

Here the sum runs over all reducible fibers and the correction terms behave as follows: On the one hand,

$$\text{corr}_v(P) = 0 \iff P \text{ meets the identity component of the fiber } F_v.$$

Otherwise we identify the singular fibers with the corresponding Dynkin diagrams. For $A_{n-1}$ (i.e. $III, IV, I_n$ in Kodaira’s notation), we number the components $\Theta_i$ cyclically such that $O$ meets $\Theta_0$. The entry in the following table for $A_{n-1}$ lists the correction term if $P$ meets $\Theta_j$ ($j > 0)$:

| fiber type | $E_6$ | $E_7$ | $A_{n-1}$ |
|------------|-------|-------|-----------|
| $\text{corr}_v(P)$ | $4/3$ | $3/2$ | $(n-j)/n$ |

In [16], the height was introduced to endow the Mordell-Weil group modulo torsion with the structure of a positive definite lattice (though not integral in general). For the K3 surfaces in consideration, the Mordell-Weil rank can only be one, since the ranks of $S_X$ and $T_X$ add up to 22. Moreover there cannot be torsion in the Mordell-Weil group. Hence the discriminant of $S_X$ is given by the following formula:

$$\text{disc}(S_X) = h(P) \prod_v \text{disc}(F_v).$$

The following table lists the reducible singular fibers, the MW-generator and its height plus the resulting discriminant for the elliptic K3 surfaces in Theorem 2. Some sections involve the imaginary unit $i = \sqrt{-1}$.

For $k = 17, 19$, the Néron-Severi lattice $S_X$ given in Theorem 2 is exactly the hyperbolic plane $U_2$ generated by $O$ and $F$, plus its orthogonal complement. The orthogonal projection $\pi$ takes $P$ to the divisor class of
Table 5. Reducible singular fibers and sections of non-unimodular elliptic K3 surfaces

| $k$ | reducible fibers | $P$ | $h(P)$ | $\text{disc}(S_X)$ |
|-----|-----------------|-----|--------|-------------------|
| 19  | $III$           | $(1/P^k, 1/P^k)$ | 19/2 | -19               |
| 17  | $III, IV$       | $(0, i t^4)$     | 17/6 | -17               |
| 13  | $III^*$         | $(1/t^4, 1/t^6)$ | 13/2 | -13               |
| 11  | $IV, III^*$     | $(0, i t^4)$     | 11/6 | -11               |
| 7   | $IV^*, III^*$   | $(0, i t^4)$     | 7/6  | -7                |
| 5   | $III^*, IV^*$   | $(t^4, t^6)$     | 5/2  | -5                |
| 3   | $IV, III^*, II^*$ | -            | -    | -3                |

Theorem 3 (Nikulin [10, Cor. 1.9.4]). The genus of an even integral non-degenerate lattice is determined by its signature and discriminant form.

For the other surfaces, we shall use the discriminant form to determine the abstract shape of the lattices. This approach will also suffice to find the lattices $T_X$ of transcendental cycles for all surfaces from Theorem 2.

Given an even integral non-degenerate lattice $L$, we denote its dual by $L^\vee$. In [10], Nikulin introduced a quadratic form on the quotient $L^\vee/L$ which he called discriminant form:

$$ q_L : L^\vee/L \to \mathbb{Q} \mod 2\mathbb{Z} $$

$$ x \mapsto x^2 $$

Consider the above cases of Mordell-Weil rank one ($k = 5, 7, 11, 13, 17, 19$). Then $L^\vee/L$ is cyclic of order $k$. The discriminant form maps the canonical generator to $-1/h(P)$. This is abbreviated by the notation

$$ q_{S_X} = \mathbb{Z}/k\mathbb{Z} \left( \frac{1}{h(P)} \right) $$

In each case, it is immediate that the claimed lattice in Theorem 2 has exactly the same discriminant form. By [2, §15, Cor. 22], the discriminant has small enough absolute value so that there is only one class per genus. Hence the lattices are isomorphic by Theorem 3.

In the non-elliptic case $k = 25$, we proceed as follows: Since $T_X$ has rank 20, we know that $\rho(X) = 2$. We find generators of $S_X$ on the double sextic model from Theorem 2. Consider the line

$$ \ell = \{ u = 0 \} \subset \mathbb{P}^2. $$
This line meets the branch curve of the double cover with multiplicity six at one point. The pull-back $\pi^* \ell$ to the double cover $X$ splits into two rational curves

$$\ell_{\pm} = \{u = 0, y = \pm \sqrt{-1}\} \subset X.$$ 

These lines intersect with multiplicity three at the preimage of the above point. Since $(\pi^* \ell)^2 = 2$, we deduce $\ell_{\pm}^2 = -2$. This gives the intersection matrix from Theorem 2. As its determinant $-5$ is squarefree and $\rho(X) = 2$, we have $S_X = \langle \ell_{\pm} \rangle$.

**Example 3.** For $k = 7$, there is a geometric way to see the abstract shape of the Néron-Severi lattice $S_X$: We find an elliptic fibration on $X$ with trivial lattice $S_X$. In terms of the Weierstrass equation above, we blow up three times at $(0, 0, 0)$. One chart then is given by

$$x = t^3 x', \quad y = t^3 y'.$$

After dividing by the common multiple, the resulting equation is

$$y'^2 = x'^3 + t^2 + x'.$$

We want to use $x' = x/t^3$ as the coordinate of the base curve $\mathbb{P}^1$. A change of variables gives rise to the Weierstrass form

$$y'^2 = t^3 + t^2 + x'^7.$$

This has the claimed trivial lattice $U_2 \oplus (-A_6) \oplus (-E_8)$ with trivial action of the non-symplectic automorphism $x' \mapsto \zeta_7 x'$.

**Example 4.** A similar approach works for $k = 11$. By [5, Lem. 2.1, 2.2], the surface $X$ from Theorem 2 admits an elliptic fibration with section and a singular fiber of type $I_{11}$. We claim that this fibration is given as follows

$$Z : \quad y^2 = x^3 + x^2 + t^{11}.$$

By definition, $Z$ has the claimed trivial lattice $U_2 \oplus (-A_{10})$ with trivial operation of the non-symplectic automorphism

$$\varphi : \quad t \mapsto \zeta_{11} t.$$ 

However, we did not find an explicit transformation between the above Weierstrass form and the one from Theorem 2. Instead we can use the uniqueness of Theorem 2 to prove that the K3 surfaces are isomorphic.

To see this, we only need that $\phi(11) = 10 | \text{rank}(T_Z)$. Since $T_Z$ has rank $22 - \rho(Z) \leq 10$, we deduce equality and $\rho(Z) = 12$. In particular, $\varphi$ operates trivially on $S_Z$. By uniqueness, $Z$ is isomorphic over $\mathbb{C}$ to the K3 surface from Theorem 2.

It remains to prove the lattices $T_X$ of transcendental cycles in Theorem 2. This only requires the following result:

**Theorem 4** (Nikulin [10, Prop. 1.6.1]). Let $N$ be an even integral unimodular lattice. Let $L$ be a primitive non-degenerate sublattice and $M = L^\perp$. Then

$$q_L = -q_M.$$ 

On a K3 surface $X$, the Néron-Severi lattice $S_X$ always embeds primitively into $H^2(X, \mathbb{Z})$. Hence the proof of the transcendental lattices in Theorem 2 is an easy application of Theorem 4. We use that there is one class per genus in each case. For $k \neq 3$, $T_X$ is indefinite, so we deduce the claim from [2 §15, Cor. 22]. For $k = 3$, $T_X$ is positive definite, so the claim follows from class group theory.
4. Delsarte surfaces

The K3 surfaces listed in Theorem 1 and Theorem 2 are all Delsarte surfaces except for \( k = 3 \). Indeed, each surface is defined by a sum of four monomials in the affine 3-space. Hence such a surface is covered by a Fermat surface of a suitable degree. The main result of this section is to determine the corresponding Fermat surface and a covering map for each of these K3 surfaces.

From now on, we let
\[
F_m : x_0^m + x_1^m + x_2^m + x_3^m = 0 \subset \mathbb{P}^3
\]
denote the Fermat surface of degree \( m \geq 4 \).

**Theorem 5.** Let \( X \) be one of the K3 surfaces listed in Theorem 1 and Theorem 2. Let \( k = |H_X| \). If \( k \neq 3 \), then \( X \) is covered by the Fermat surface of the following degree \( m \):
\[
\begin{cases}
m = k, & \text{if } X \text{ is unimodular}, \\
m = 2k, & \text{if } X \text{ is non-unimodular}.
\end{cases}
\]

For each K3 surface, we will prove Theorem 5 by explicitly giving the covering map on the affine model \( \{x_0 \neq 0\} \) of \( F_m \) with coordinates \( U = \frac{x_1}{x_0}, V = \frac{x_2}{x_0}, W = \frac{x_3}{x_0} \):
\[
F_m : U^m + V^m + W^m + 1 = 0.
\]
A method to compute these covering maps goes back to Shioda in [15].

(I) Unimodular cases

**\( k = 66 \)** The elliptic surface is given by the Weierstrass equation
\[
y^2 = x^3 - t(t^{11} + 1) = x^3 - t^{12} - t.
\]
It is more convenient to work in the chart at \( \infty \) where \( t \neq 0 \). The local parameter is \( s = 1/t \). Then the equation is transformed to
\[
y^2 = x^3 - 1/s^{12} - s^{11}/s^{12}.
\]
Introduce new coordinates \( \xi := s^4 x, \eta := s^6 y \). The resulting equation is
\[
\eta^2 = \xi^3 - 1 - s^{11}.
\]
The covering map from the affine Fermat surface of degree 66 is as follows:
\[
\eta \mapsto U^{33}, \quad \xi \mapsto -V^{22}, \quad s \mapsto W^6.
\]

**\( k = 44 \)** The covering map from the affine Fermat surface of degree 44 is as follows:
\[
\{U^{44} + V^{44} + W^{44} + 1 = 0\} \rightarrow \{y^2 = x^3 + x + t^{11}\}
\]
\[
y \mapsto U^{22}V^{11}, \quad x \mapsto -V^{22}, \quad t \mapsto -W^{4}V^{2}.
\]

**\( k = 42 \)** The argument is very similar to the case \( k = 66 \). The same transformations lead to
\[
\eta^2 = \xi^3 - 1 - s^{7}.
\]
The covering map from the affine Fermat surface of degree 42 is as follows:
\[
\eta \mapsto U^{21}, \quad \xi \mapsto -V^{14}, \quad s \mapsto W^6.
\]
The covering map from the affine Fermat surface of degree 36 is as follows:
\[ y \mapsto U^{18} W^{15}, \quad x \mapsto -V^{12} W^{10}, \quad t \mapsto W^{6}. \]

The covering map from the affine Fermat surface of degree 28 is as follows:
\[ y \mapsto U^{14} V^{7}, \quad x \mapsto -V^{14}, \quad t \mapsto -W^{4} V^{2}. \]

The covering map from the affine Fermat surface of degree 12 is as follows:
\[ y \mapsto U^{6} W^{15}, \quad x \mapsto -V^{4} W^{10}, \quad t \mapsto -W^{6}. \]

(II) Non-unimodular cases

The covering map from the affine Fermat surface of degree 38 is as follows:
\[ y \mapsto U^{19} W^{3}/V, \quad x \mapsto -V^{12} W^{2}, \quad t \mapsto W^{6}/V^{2}. \]

The covering map from the affine Fermat surface of degree 34 is as follows:
\[ y \mapsto U^{17} W^{6}/V^{2}, \quad x \mapsto -V^{10} W^{4}, \quad t \mapsto W^{6}/V^{2}. \]

The covering map from the affine Fermat surface of degree 26 is as follows:
\[ y \mapsto U^{13} W^{3}/V, \quad x \mapsto -V^{8} W^{2}, \quad t \mapsto W^{6}/V^{2}. \]

The covering map from the affine Fermat surface of degree 22 is as follows:
\[ y \mapsto U^{11} W^{6}/V^{2}, \quad x \mapsto -V^{6} W^{4}, \quad t \mapsto W^{6}/V^{2}. \]

The covering map from the affine Fermat surface of degree 14 is as follows:
\[ y \mapsto U^{7} V^{8}/W^{24}, \quad x \mapsto -V^{10}/W^{16}, \quad t \mapsto V^{2}/W^{6}. \]

The covering map from the affine Fermat surface of degree 10 is as follows:
\[ y \mapsto U^{5} V^{7}/W^{21}, \quad x \mapsto -V^{8}/W^{14}, \quad t \mapsto V^{2}/W^{6}. \]

The covering map from the affine Fermat surface of degree 54 is as follows:
\[ y \mapsto U^{27} W^{3}, \quad x \mapsto -V^{18} W^{2}, \quad t \mapsto W^{6}. \]

The covering map from the affine Fermat surface of degree 18 is as follows:
\[ y \mapsto U^{9} W^{15}, \quad x \mapsto -V^{6} W^{10}, \quad t \mapsto W^{6}. \]

Affinely, the covering map from the Fermat surface of degree 50 is as follows:
\[ y \mapsto U^{25}, \quad u \mapsto -V^{10}, \quad v \mapsto W^{10}/V^{2}. \]
5. Motivic decomposition

Shioda studied Delsarte surfaces in [15]. He computed their Picard numbers through the covering Fermat surfaces. Here we recall his argument and apply it to the K3 surfaces in Theorem 1 and 2. Note that we will actually be most interested in the transcendental lattices.

Let $\mu_m$ be the group of $m$-th roots of unity and $\Delta$ the image of the diagonal inclusion $\mu_m \hookrightarrow \mu_4^m$. Then the quotient group $M = \mu_4^m / \Delta$ operates by multiplication of coordinates on the Fermat surface $F_m$.

This group operation induces a decomposition of the second cohomology $H^2(F_m)$ into one-dimensional eigenspaces with character. More generally true for Fermat varieties of any dimension, this result is due to Weil [20]. The decomposition is best described in terms of the character group of $M$:

$$A_m := \{ \alpha = (a_0, a_1, a_2, a_3) \in (\mathbb{Z}/m\mathbb{Z})^4 \mid a_i \neq 0 \pmod{m}, \sum_{i=0}^{3} a_i \equiv 0 \pmod{m} \}.$$ 

It is well-known that $H^2(F_m)$ decomposes into the subspace $V_0$ of the hyperplane class $H$ and one-dimensional subspaces $V(\alpha)$ for each $\alpha \in A_m$:

$$H^2(F_m) = V_0 \oplus \bigoplus_{\alpha \in A_m} V(\alpha).$$

We now describe a criterion whether $V(\alpha)$ is algebraic: If we choose representatives $0 < a_i < m$, then this gives a well-defined map

$$|\alpha| = \frac{1}{m} \sum_{i=0}^{3} a_i.$$ 

Let $u \in (\mathbb{Z}/m\mathbb{Z})^*$ operate on $\alpha \in A_m$ coordinatewise by multiplication. Then we define a subset $B_m$ of $A_m$ as follows:

$$B_m = \{ \alpha \in A_m; |u \cdot \alpha| = 2 \forall u \in (\mathbb{Z}/m\mathbb{Z})^* \}.$$ 

**Criterion:** The eigenspace $V(\alpha)$ is algebraic if and only if $\alpha \in B_m$.

**Remark 3.** The criterion is based on the following two facts:

1. The action of $u \in \text{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q}) \simeq (\mathbb{Z}/m\mathbb{Z})^*$ sends $V(\alpha)$ to $V(u \cdot \alpha)$.
2. The eigenspace $V(\alpha)$ with character $\alpha \in A_m$ contributes to the Hodge group $H^{2-|\alpha|}(F_m)$.

Using the above criterion, one can easily determine the transcendental and algebraic part of $H^2(F_m)$ and in particular the Picard number. We shall now see how this carries over to the K3 surfaces in Theorem 1 and 2.

**Lemma 1.** Each K3 surface from Theorem 1 and 2 except for $k = 3$ is birationally equivalent to a quotient of the covering Fermat surface in Theorem 5.

**Proof:** In the proof of Theorem 3, we have exhibited the explicit covering map $\pi$ between the K3 surface $X$ and the Fermat surface $F_m$. We define the following subgroup $G$ of the automorphism group $M = \mu_4^m / \Delta$ (viewed as $\mu_3^m$ operating on the affine coordinates of $F_m$):

$$G = \{ g \in M; \pi = \pi \circ g \}. $$
It is immediate that \( X \) is birationally equivalent to the quotient \( F_m/G \).

**Example 5.** In some cases, the precise shape of \( G \) is visible directly. For instance, if \( k = 66 \), then
\[
G = \mu_{33} \times \mu_{22} \times \mu_6.
\]
In other cases such as \( k = 44 \), some more work is required to determine \( G \) abstractly, but we will not go into the details here.

Let \( X \) be one of the K3 surfaces from Theorem 1 or 2 with \( k \neq 3 \) and \( F_m \) the covering Fermat surface. The key property for the analysis of \( H^2(X) \) lies in the Lefschetz number \( \lambda(X) \) which gives the dimension of the transcendental subspace of \( H^2(X) \):
\[
\lambda(X) = b_2(X) - \rho(X).
\]
Namely, \( \lambda(X) \) is a birational invariant of algebraic surfaces. Hence
\[
\lambda(X) = \lambda(F_m/G).
\]
Thus we can identify the transcendental part of \( H^2(X) \) with the transcendental part of \( H^2(F_m) \) that is invariant under \( G \). Here we shall further use the decomposition \((3)\) of \( H^2(F_m) \):

**Criterion:** Let \( \alpha = (a_0, a_1, a_2, a_3) \in A_m \). Then \( V(\alpha) \) is \( G \)-invariant if and only if
\[
\prod_{i=1}^{3} \zeta_i^{a_i} = 1 \quad \forall g = (\zeta_1, \zeta_2, \zeta_3) \in G.
\]
Let \( \mathcal{S}_G = \{ \alpha \in A_m; V(\alpha) \text{ is } G \text{-invariant} \} \). Then
\[
H^2(F_m)^G = V_0 \oplus \bigoplus_{\alpha \in \mathcal{S}_G} V(\alpha).
\]
In consequence, the transcendental part \( T(X) \) of \( H^2(X) \) can be identified with
\[
T(X) = \bigoplus_{\alpha \in \mathcal{S}_G \setminus \mathcal{B}_m} V(\alpha).
\]

In the following tables, we list the character sets \( \mathcal{S}_G \setminus \mathcal{B}_m \) for all K3 surfaces from Theorem 1 and 2 except for \( k = 3 \). We use the shorthand \( \alpha = [a_1, a_2, a_3] \) for \( \alpha = (a_0, a_1, a_2, a_3) \), corresponding to the affine chart \( \{ x_0 \neq 1 \} \), since this determines \( a_0 \) uniquely.

**Remark 4.** 1. For each \( k \), it is easily checked that the set \( \mathcal{S}_G \setminus \mathcal{B}_m \) constitutes a single \((\mathbb{Z}/m\mathbb{Z})^\times\)-orbit.
2. We also see by Remark 3.2 that the first character in each entry is the unique one of Hodge type \((0,2)\).

### 6. Modularity

In this section, we will prove the modularity of all K3 surfaces from Theorem 1 and 2. We compute their \( \zeta \)-functions over finite fields explicitly in terms of Jacobi sums.

For \( k = 3 \), the K3 surface is singular \((\rho = 20)\). Hence its modularity follows from a result by the first author \([7]\) (cf. Rem. 5). For all other K3 surfaces from Theorem 1 and 2, we will use the covering Fermat surface from section 3 and the motivic decomposition from section 4 to prove modularity in Lemma 2.
### Table 6. Motivic decomposition for unimodular K3 surfaces

| $k$ | $\mathcal{O}_\tau \setminus \mathcal{B}_m$ |
|-----|---------------------------------|
| 66  | $[6, 33, 22], [6, 33, 44], [12, 33, 22], [12, 33, 44], [18, 33, 22], [18, 33, 44], [24, 33, 22], [24, 33, 44], [30, 33, 22], [30, 33, 44], [36, 33, 22], [36, 33, 44], [42, 33, 22], [42, 33, 44], [48, 33, 22], [48, 33, 44], [54, 33, 22], [54, 33, 44], [60, 33, 22], [60, 33, 44]$ |
| 44  | $[1, 22, 24], [3, 22, 28], [5, 22, 32], [7, 22, 36], [9, 22, 40], [13, 22, 4], [15, 22, 8], [17, 22, 12], [19, 22, 16], [21, 22, 20], [23, 22, 24], [25, 22, 28], [27, 22, 32], [29, 22, 36], [31, 22, 40], [35, 22, 4], [37, 22, 8], [39, 22, 12], [41, 22, 16], [43, 22, 20]$ |
| 42  | $[6, 21, 14], [6, 21, 28], [12, 21, 14], [12, 21, 28], [18, 21, 14], [18, 21, 28], [24, 21, 14], [24, 21, 28], [30, 21, 14], [30, 21, 28], [36, 21, 14], [36, 21, 28]$ |
| 36  | $[1, 18, 12], [5, 18, 24], [7, 18, 12], [11, 18, 24], [13, 18, 12], [17, 18, 24], [19, 18, 12], [23, 18, 24], [25, 18, 12], [29, 18, 24], [31, 18, 12], [35, 18, 24]$ |
| 28  | $[1, 14, 16], [3, 14, 20], [5, 14, 24], [9, 14, 4], [11, 14, 8], [13, 14, 12], [15, 14, 16], [17, 14, 20], [19, 14, 24], [23, 14, 4], [25, 14, 8], [27, 14, 12]$ |
| 12  | $[1, 6, 4], [5, 6, 8], [7, 6, 4], [11, 6, 8]$ |

### Table 7. Motivic decomposition for non-unimodular K3 surfaces

| $k$ | $\mathcal{O}_\tau \setminus \mathcal{B}_m$ |
|-----|---------------------------------|
| 19  | $[19, 1, 35], [19, 3, 29], [19, 5, 23], [19, 7, 17], [19, 9, 11], [19, 11, 5], [19, 13, 37], [19, 15, 31], [19, 17, 25], [19, 21, 13], [19, 23, 7], [19, 25, 1], [19, 27, 33], [19, 29, 27], [19, 31, 21], [19, 33, 15], [19, 35, 9], [19, 37, 3]$ |
| 17  | $[17, 2, 28], [17, 4, 22], [17, 6, 16], [17, 8, 10], [17, 10, 4], [17, 12, 32], [17, 14, 26], [17, 16, 20], [17, 18, 14], [17, 20, 8], [17, 22, 2], [17, 24, 30], [17, 26, 24], [17, 28, 18], [17, 30, 12], [17, 32, 6]$ |
| 13  | $[13, 1, 23], [13, 3, 17], [13, 5, 11], [13, 7, 5], [13, 9, 25], [13, 11, 19], [13, 15, 7], [13, 17, 1], [13, 19, 21], [13, 21, 15], [13, 23, 9], [13, 25, 3]$ |
| 11  | $[11, 2, 16], [11, 4, 10], [11, 6, 4], [11, 8, 20], [11, 10, 14], [11, 12, 8], [11, 14, 2], [11, 16, 18], [11, 18, 12], [11, 20, 6]$ |
| 7   | $[7, 2, 8], [7, 4, 2], [7, 6, 10], [7, 8, 4], [7, 10, 12], [7, 12, 6]$ |
| 25  | $[25, 2, 40], [25, 4, 30], [25, 6, 20], [25, 8, 10], [25, 12, 40], [25, 14, 30], [25, 16, 20], [25, 18, 10], [25, 22, 40], [25, 24, 30], [25, 26, 20], [25, 28, 10], [25, 32, 40], [25, 34, 30], [25, 36, 20], [25, 38, 10], [25, 42, 40], [25, 44, 30], [25, 46, 20], [25, 48, 10]$ |
| 5   | $[5, 1, 7], [5, 3, 1], [5, 7, 9], [5, 9, 3]$ |
| 27  | $[1, 36, 27], [5, 18, 27], [9, 36, 27], [11, 18, 27], [13, 36, 27], [15, 18, 27], [19, 36, 27], [23, 18, 27], [25, 36, 27], [29, 18, 27], [31, 36, 27], [35, 18, 27], [37, 36, 27], [41, 18, 27], [43, 36, 27], [47, 18, 27], [49, 36, 27], [53, 18, 27]$ |
| 9   | $[1, 6, 9], [5, 12, 9], [7, 6, 9], [11, 12, 9], [13, 6, 9], [17, 12, 9]$ |

It goes back to Weil [19] that the Fermat surface $\mathcal{F}_m$ is of CM type in the following sense: Over the cyclotomic field $\mathbb{Q}(\zeta_m)$, the Galois representation of $H^2(\mathcal{F}_m)$ splits into one-dimensional subrepresentations corresponding to the eigenspaces $V_0, V(\alpha) (\alpha \in \mathfrak{A}_m)$ from section 4. These subrepresentations are associated to Hecke Grössencharacters and can be described in terms of Jacobi sums. Here the Jacobi sums are determined by the characters $\alpha$. In particular, the $V(\alpha)$ are pairwise non-isomorphic as $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_m))$ representations.
By Lemma 1 (and by [7] for \(k = 3\)), we immediately obtain the

**Corollary 1.** The K3 surfaces in Theorem 1 and 2 are also of CM type.

In fact, we can even determine the \(\zeta\)-function of all K3 surfaces in Theorem 1 and 2. For this, we recall Weil’s theorem [19] for the covering Fermat surface. To state the result, we recall the basic setup involving Jacobi sums.

We fix the degree \(m\) and a prime \(p\). Let \(q = p^r\) such that \(q \equiv 1 \mod m.\)

On the field \(\mathbb{F}_q\) of \(q\) elements, we fix a character \(\chi : \mathbb{F}_q^* \to \mathbb{C}^*\) of order exactly \(m\). For any \(\alpha = (a_0, a_1, a_2, a_3) \in \mathfrak{A}_m\), we then define the Jacobi sum

\[
j(\alpha) = \sum_{v_1, v_2, v_3 \in \mathbb{F}_q^*} \chi(v_1)^{a_1} \chi(v_2)^{a_2} \chi(v_3)^{a_3},
\]

\(v_1 + v_2 + v_3 = -1\).

**Theorem 6** (Weil). In the above notation, the Fermat surface \(F_m\) over \(\mathbb{F}_q\) has the following \(\zeta\)-function:

\[
\zeta(F_m/\mathbb{F}_q, T) = \frac{1}{(1 - T) P(T) (1 - q^2 T)}
\]

where

\[
P(T) = (1 - q T) \prod_{\alpha \in \mathfrak{A}_m} (1 - j(\alpha) T).
\]

We will now use Theorem 6 to determine the \(\zeta\)-functions of all K3 surfaces from Theorem 1 and 2. Since every K3 surface \(X\) has \(b_1(X) = 0,\)

\[
\zeta(X/\mathbb{F}_q, T) = \frac{1}{(1 - T) R(T) (1 - q^2 T)}
\]

where \(R(T)\) is the reciprocal characteristic polynomial of \(\text{Frob}_q^*\) on \(H^2(X)\). (Here we use without specifying an appropriate Weil cohomology, say \(\ell\)-adic étale cohomology for some prime \(\ell \neq p\) on the base change \(\bar{X}\) over the algebraic closure \(\overline{\mathbb{F}_q}\).)

Over \(\mathbb{Q}\), the polynomial \(R(T)\) factors according to algebraic and transcendental part of \(H^2(X)\):

\[
R(T) = R_a(T) R_t(T).
\]

Thanks to Theorem 6, the factor \(R_t(T)\) corresponding to the transcendental part is determined by Lemma 1 and the motivic decomposition [14].

**Lemma 2.** Let \(X\) be one of the K3 surfaces from Theorem 1 or 2 with \(k \neq 3\). Fix the above setup over \(\mathbb{F}_q\) with \(q \equiv 1 \mod m.\) Then

\[
R_t(T) = \prod_{\alpha \in \mathfrak{A} \setminus \mathfrak{B}_m} (1 - j(\alpha) T).
\]
Remark 5. The singular K3 surface from Theorem 2 with $k = 3$ is modular by [7]. The affine model in section 2 is associated to the modular form of weight 3 and level 27 given in [13, Table 1]. Over $\mathbb{F}_p$ ($p \neq 2, 3$), the reciprocal characteristic polynomial $R_3(T)$ is given as follows:

$$R_3(T) = \begin{cases} 
1 - (\pi^2 + \bar{\pi}^2)T + p^2T^2, & \text{if } p = \pi \bar{\pi} \text{ in } \mathbb{Z}[3\zeta_3], \\
1 - p^2T^2, & \text{if } p \text{ is inert in } \mathbb{Q}(\sqrt{-3}).
\end{cases}$$

Because of this explicit description, we did not check how to express $R_3(T)$ in terms of Jacobi sums.

For the $\zeta$-functions of the K3 surfaces $X$ in Theorem 1 and 2, it remains to determine the reciprocal characteristic polynomial $R_a(T)$ of $\text{Frob}_q^*$ on the algebraic part of $H^2(X)$, i.e. on $S_X$.

Lemma 3. Let $X$ a K3 surface in Theorem 1 or 2. Consider the affine model given in section 2 over some field $K$ of characteristic coprime to $2k$. Then $S_X$ is generated by algebraic cycles over $K(\sqrt{-1})$.

Proof: We prove the claim for $K = \mathbb{Q}$. The lemma then follows by smooth base change.

If $k \neq 25$, then we have already studied an elliptic fibration on $X$. It is easy to see that all fiber components are defined over $\mathbb{Q}(\sqrt{-1})$. Since the same holds for the sections by Table 3 the claim follows.

For $k = 25$, we saw that $S_X$ is generated by two lines which are conjugate in $\mathbb{Q}(\sqrt{-1})$. This proves the Lemma for $K = \mathbb{Q}$ and consequently for any field $K$ such that $X$ defines a (smooth) K3 surface over $K$. □

Some non-unirational surfaces really require the extension by $\sqrt{-1}$. Explicitly, the local Euler factors take the following shape:

Corollary 2. Let $X$ a K3 surface in Theorem 1 or 2. Consider the affine model from section 2 over some finite field $\mathbb{F}_q$. Let

$$n_+ = \begin{cases} 
0, & q \equiv 1 \pmod{4} \text{ or } k \neq 7, 9, 11, 17, 25, 27, \\
1, & q \equiv 3 \pmod{4} \text{ and } k = 25, 27, \\
2, & q \equiv 3 \pmod{4} \text{ and } k = 9, 11, 17, \\
3, & q \equiv 3 \pmod{4} \text{ and } k = 7.
\end{cases}$$

Define $n_- = 22 - \phi(k) - n_+$. Then

$$R_a(T) = (1 - qT)^{n_+} (1 + qT)^{n_-}.$$

Proof: In the first cases, $\sqrt{-1} \in \mathbb{F}_q$ or all generators of $S_X$ can be defined over $\mathbb{Q}$. Hence $\text{Frob}_q^*$ acts as multiplication by $q$ on $S_X$.

In all other cases, $\text{Gal}(\mathbb{F}_q(\sqrt{-1})/\mathbb{F}_q)$ acts nontrivially on the section, components of the fibres of type $IV$ or $IV^*$ resp. on the lines $\ell_\pm$. The dimension $n_-$ of the ($-1$)-space of the Galois action is easily verified. □

For $q \equiv 1 \pmod{m}$, we combine Lemma 2 (or Remark 5) and Corollary 2 to obtain the $\zeta$-function of all K3 surfaces in Theorem 1 and 2. In particular, we deduce their modularity.
7. Galois representations

In this section we will study the Galois representations of dimension \( \phi(k) \) associated to the transcendental parts of these K3 surfaces.

First we recall the result of Nikulin [11]. Most of the following statements hold true in full generality, but we only state the special case of minimal rank of the transcendental lattice which is relevant to our issues.

**Theorem 7** (Nikulin). Let \( X \) be an algebraic K3 surface. Then \( H_X \) is a finite cyclic group of order \( k \). Assume that \( \phi(k) = \text{rank}(T_X) \). Thus \( k \) takes the values listed in Theorem 1 and Theorem 2.

(a) For those values of \( k \), the ring \( \mathbb{Z}[\zeta_k] \) is a PID, i.e. is of class number one. (See [8], and also [9].)

(b) The representation of \( H_X = \langle g_k \rangle \) in \( T_X \otimes \mathbb{Q} \) is isomorphic to a direct sum of irreducible representations of \( H_X \) of dimension one.

(c) Let \( \Phi_k(x) \) denote the \( k \)-th cyclotomic polynomial. Regard \( T_X \otimes \mathbb{Q} \) as a \( \mathbb{Z}[\langle g_k \rangle] \)-module via the natural action of \( g_k \) on \( T_X \). Then \( T_X \otimes \mathbb{Q} \) is a torsion free \( \mathbb{Z}[\langle g_k \rangle]/\langle \Phi_k(g_k) \rangle \)-module. Identifying \( \mathbb{Z}[\langle g_k \rangle]/\langle \Phi_k(g_k) \rangle \) with the ring of integers \( \mathbb{Z}[\zeta_k] \), we derive an isomorphism

\[
T_X \otimes \mathbb{Q} \simeq \mathbb{Q}[\zeta_k] \quad \text{as } \mathbb{Z}[\zeta_k]-\text{modules.}
\]

We also recall the result of Zarhin on the Hodge groups of complex K3 surfaces. Recall that \( T_X \) is the orthogonal complement of \( S_X \) with respect to the cup product which we denote by \( \langle \cdot, \cdot \rangle \).

**Theorem 8** (Zarhin). Let \( X \) be a complex K3 surface, and \( \text{Hdg} \subset \text{Aut}(H^2(X, \mathbb{Q})) \) be the Hodge group. Then

(a) \( T_X \) is an irreducible \( \text{Hdg} \)-module,

(b) \( E := \text{End}_{\text{Hdg}}(T_X) \) is a commutative field. \( E \) has an involution induced by \( \langle \cdot, \cdot \rangle \) with totally real fixed field \( E_0 \), and either \( E = E_0 \) or \( E \) is a totally complex quadratic extension of \( E_0 \), and \( \langle \cdot, \cdot \rangle \) induces a symmetric, respectively, Hermitian form \( \Phi : T_X \times T_X \to E \),

(c) \( \text{Hdg} = \text{SO}(T_X, \Phi) \), respectively, \( U(T_X, \Phi) \),

(d) Let \( U(X) \) be the image of \( \text{Aut}(X) \) in \( \text{Aut}(T_X) \). Then \( U(X) \) is contained in the roots of unity of \( E \), and hence is cyclic of order \( n \) for some \( n \), and \( \phi(n)\left| \left| E : \mathbb{Q} \right| = \dim_{\mathbb{Q}}(T_X) \right. \)

We will elaborate this theorem for our examples of K3 surfaces.

**Corollary 3.** Let \( X \) be a K3 surface in Theorem 1 or Theorem 2. Then \( [E : \mathbb{Q}] = \phi(k) = \dim_{\mathbb{Q}}(T_X) \), so that \( E \) is a cyclotomic field, and hence a CM field, over \( \mathbb{Q} \) of degree \( \phi(k) \).

This gives a Hodge theoretic proof that our K3 surfaces in Theorem 1 and Theorem 2 are all of CM type. (Confer Corollary 1.)

Now we will study the Galois representations associated to our K3 surfaces. The main point is the comparison of the Hodge structure and some piece of the Galois representation.

**Lemma 4.** Let \( X \) be one of the K3 surfaces in Theorem 1 and Theorem 2. Then for each \( k \), the Galois representation associated to \( T_X \) has dimension \( \phi(k) \), and is
irreducible over \( \mathbb{Q} \). In fact this \( \text{Gal} (\overline{\mathbb{Q}}/\mathbb{Q}) \) representation is induced from the Jacobi sum Grossencharacters corresponding to the first character associated to \( X \) in the table of Section 5.

**Proof:** Since \( H^2(X) \) is a submotive of \( H^2(F_m) \), the results of Section 5, in particular Remarks 3 and 4 show that the \( \text{Gal} (\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_m)) \) representation defined by \( T_X \) is a sum of one dimensional representations which are simply transitively permuted by \( \text{Gal} (\mathbb{Q}(\zeta_m)/\mathbb{Q}) \). The claim follows \( \square \)

**Remark 6.** Let \( K \) be a finite field extension of \( \mathbb{Q} \). Since the \( \text{Gal} (\overline{\mathbb{Q}}/\mathbb{Q}) \) representation given by \( T_X \) is induced from a character (of \( \text{Gal} (\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_m)) \)), it follows from Mackey’s formula that the restriction of this representation to \( \text{Gal} (\mathbb{Q}(\zeta_m)/K) \) is a direct sum of cyclic representations. In particular the \( L \)-function of the motive \( T_X \otimes K \) and with it the \( L \)-function of \( H^2(X \otimes K) \) is modular for any \( K \).

**Remark 7.** A related modularity result is obtained by invoking the automorphic induction of Arthur and Clozel [1]. The first result we proved is the \( GL(1) \)-modularity for a CM motive over \( \mathbb{Q}(\zeta_k) \). The automorphic induction of Arthur and Clozel then takes this \( GL(1) \) automorphic representation to a \( GL(\varphi(k)) \) automorphic cuspidal representation over \( \mathbb{Q} \) having the same \( L \)-function (for the standard representation).

8. **Arithmetic mirror symmetry**

In this section, we will show that for all values except \( k = 3 \), a mirror K3 surfaces exist. For some of them, we interpret mirror symmetry arithmetically.

In the literature, there are several variants of mirror symmetry for K3 surfaces. Here we employ the notion of mirror symmetry for lattice polarized K3 surfaces introduced by Dolgachev [3], based on the Arnold strange duality.

**Definition 1.** Let \( X \) be an algebraic K3 surface. Then a K3 surface \( \tilde{X} \) is called a mirror of \( X \) if

\[
T_X = U_2 \oplus S_{\tilde{X}}.
\]

(6)

Usually mirror symmetry is exhibited on the level of families of K3 surfaces. Here we are only interested in the existence of a mirror K3 surface. Then we will investigate arithmetic properties.

**Lemma 5.** For each of the K3 surfaces listed in Theorem 1 and 2 except for \( k = 3 \), a mirror K3 surface exists.

**Proof:** We give an abstract proof, based on the following fact: If \( T_X \) admits an orthogonal splitting

\[
T_X = U_2 + M
\]

as in (7), then \( M \) embeds primitively into \( H^2(X, \mathbb{Z}) \). Since \( M \) has signature \((1, 19 - \rho(X))\), there is a K3 surface \( \tilde{X} \) with \( S_{\tilde{X}} = M \) by the work of Nikulin [11]. By definition, \( \tilde{X} \) is a mirror of \( X \). Hence we have to show that for each K3 surface in Theorem 1 and 2 except for \( k = 3 \), \( T_X \) admits an orthogonal splitting (7).
For each K3 surface in Theorem 1 and 2 except for $k = 3, 11, 19$, an orthogonal splitting (7) is given in the tables in section 2. For $k = 3$, no such splitting exists since $T_X$ is positive definite. For $k = 11, 19$, we use a primitive embedding

$$A_2 \hookrightarrow E_8.$$ 

In the present cases, this induces a primitive embedding

$$(2) \oplus (-A_2) \hookrightarrow T_X.$$ 

Hence the claim follows from the fact that

$$U_2 \hookrightarrow (2) \oplus (-A_2).$$

To see this, choose generators $b$ of the left summand and $a_1, a_2$ for the right summand with intersection form $\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$. Then $U_2$ can be identified with the sublattice $\langle b + a_1, b + a_1 + a_2 \rangle$. □

**Remark 8.** For each surface $X$ in Theorem 1 and 2 except for $k = 3$, the mirror construction is in fact symmetric, i.e. $X$ is a mirror of $\check{X}$. This follows from lattice theory as used in section 3, since there is only one class per genus.

For many surfaces, we can write down a mirror explicitly. For instance, for the K3 surfaces in Theorem 1, we have the following mirror pairs:

$$\{k = 12\} \leftrightarrow \{k = 44, 66\}$$

$$\{k = 28, 36, 42\} \leftrightarrow \{k = 28, 36, 42\}$$

The same applies to some of the K3 surface in Theorem 2:

$$k = 9 \leftrightarrow k = 27, \quad k = 5 \leftrightarrow k = 25, \quad k = 13 \leftrightarrow k = 13.$$ 

These relations give rise to an arithmetical mirror symmetry:

**Lemma 6.** For each K3 surface in Theorem 1 and 2 except for $k = 3, 7, 11, 17, 19$, there is a mirror which is a Delsarte surface. In particular, this mirror K3 surface is again of CM-type and modular.

No arithmetical mirror symmetry seems to be known for the remaining K3 surfaces. Nonetheless we can look for mirrors explicitly (except for $k = 3$). Here we briefly comment on this for two cases:

For $k = 7, 17$, mirror symmetry is particularly easy, since any mirror $\check{X}$ admits an elliptic fibration with section due to the embedding

$$U_2 \hookrightarrow S_{\check{X}}.$$ 

For $k = 7$, e.g., we require exactly one reducible fibre with root lattice $A_1$ (i.e. type $I_2$ or $III$) and a section $P$ meeting this fibre in the non-zero component, but not intersecting the zero section. These conditions give rise to a 16-dimensional family of K3 surfaces as follows.

The $A_1$ fibre (located at $\infty$) is encoded in the general Weierstrass equation with polynomial coefficients $b(t), c(t) \in K[t]$

$$y^2 = x^3 + a t^4 x^2 + b(t) x + c(t), \quad a \in K, \deg(b) \leq 7, \deg(c) \leq 10.$$ (8)
We require a section meeting the non-zero component of the $A_1$ fiber at $\infty$, but not intersecting the zero section. Any such section $P$ can be given polynomially as

$$P = (X(t), Y(t)), \quad \deg(X(t)) \leq 3, \quad \deg(Y(t)) \leq 5.$$  

By inspection, the polynomials $X(t), Y(t), a, b(t)$ determine the remaining coefficient $c(t)$ of the Weierstrass form uniquely. Taking the three normalisations due to Möbius transformations and rescaling $x, y$ into account, we derive a 16-dimensional family of elliptic K3 surfaces as claimed. Any general member of this family serves as a mirror of $X$.

However, at this time we have no knowledge about the $\zeta$-functions of K3 surfaces of Picard number $\rho(X) = 4$ unless they are Delsarte surfaces. In the above family, there is only one member with exactly one reducible fiber (of type $A_1$) which is a Delsarte surface:

$$W : y^2 = x^3 + t^7 x + 1.$$  

Here the section $P$ is given by $(0, 1)$. Nonetheless, $W$ is not a mirror of $X$ since it has Picard number $\rho(W) = 16$. This can be proved along the lines of section 5 by interpreting $W$ as a quotient of the Fermat surface $F_m$ of degree $m = 42$. Hence no arithmetical mirror of $X$ is known.

Similarly for $k = 17$, we can write down a mirror family of elliptic K3 surfaces with trivial lattice

$$U_2 \oplus (-A_1) \oplus (-A_2) \oplus (-E_8)$$  

and a section $P$ of height $h(P) = 17/6$, but then there is no non-degenerate Delsarte surface in this family at all.

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