Local measurement uncertainties impose a limit on non-local quantum correlations

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Abstract. In quantum mechanics, joint measurements of non-commuting observables are only possible if a minimal unavoidable measurement uncertainty is accepted. On the other hand, correlations between non-commuting observables can exceed classical limits, as demonstrated by the violation of Bell’s inequalities. Here, the relation between the uncertainty limited statistics of joint measurements and the limits on expectation values of possible input states is analyzed. It is shown that the experimentally observable statistics of joint measurements explain the uncertainty limits of local states, but result in less restrictive bounds when applied to identify the limits of non-local correlations between two separate quantum systems. A tight upper bound is obtained for the four correlations that appear in the violation of Bell’s inequalities and the statistics of pure states saturating the bound is characterized. The results indicate that the limitations of quantum non-locality are a necessary consequence of the local features of joint measurements, suggesting the possibility that quantum non-locality could be explained in terms of the local characteristics of quantum statistics.

Keywords: quantum measurement, uncertainty relations, quantum non-locality, entanglement, Bell’s inequality, Cirel’son bound
1. Introduction

Scientific investigations must be rooted in reproducible observations. In quantum theory, this requires a closer look at the mechanisms of measurement and control, since it is not immediately obvious how the theoretical formalism describes these mechanisms. In recent years, there has been a renewed interest in the non-classical aspects of measurement statistics, motivated to a great extent by an increasing variety of experimental realizations of quantum measurements [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19]. At the center of these investigations is the uncertainty principle and the associated possibility of jointly measuring complementary observables that do not have a joint representation in the theoretical description of quantum systems [20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33]. The results of these investigations show that the Hilbert space formalism imposes a set of well defined rules on joint measurements, and the derivation of these rules from the structure of Hilbert space may help to explain the physics behind the strange and unexpected features of quantum theory.

Of particular importance would be a better understanding of the violation of Bell’s inequalities, which is widely recognized as the most compelling proof that the outcomes of quantum measurements cannot be explained in terms of local physical properties assigned to the individual systems [34]. It should be obvious that the violation of Bell’s inequalities is only possible because there is no uncertainty free joint measurement of the observables concerned [35, 36, 37], but the specific implications of measurement uncertainties for non-local correlations are somewhat less clear [38, 39, 40]. However, we can expect that the violation of Bell’s inequalities will be limited by the finite amount of uncertainty in joint measurements, since the experimentally observable non-local correlations must stay within their classical bounds. It has already been shown that the Cirel’son bound [41], which identifies the quantum mechanical limit of Bell’s inequality violations, can be explained in terms of the uncertainty limits of conditional quantum states obtained from measurements of one of the two correlated systems [42]. Motivated by this result, it has also been shown that the Cirel’son bound can also be derived from the uncertainty limits of a joint measurement of two collective observables in Bell’s inequalities [43]. Having established a fundamental connection between uncertainty relations and non-local quantum correlations, it may be interesting to consider the relation between the information obtained in local joint measurements of two quantum systems and the non-local correlations between these two systems. Specifically, the question is whether the quantum mechanical bounds on non-local correlations can be explained in terms of the local measurement uncertainties of two joint measurements applied independently to each of the two quantum correlated systems.

In the following, I will show that the analysis of this problem can be simplified significantly by the identification of a class of joint measurement of two complementary properties of a qubit for which the complete output statistics of the joint measurement can be derived from the statistics observed in separate precise measurements of
the complementary properties. The rule concerning the experimentally observed correlations between the two values of ±1 is simply that the average of their product will always be zero. This experimentally verifiable fact explains the need for uncertainties in quantum mechanics, since it would predict negative experimental probabilities for any combination of uncertainty free inputs and measurements. Importantly, the problem of negative probability predictions is not resolved by the measurement uncertainties alone. Once measurement uncertainties are included, the positivity of joint measurements defines a corresponding uncertainty bound that apply to the possible statistics of the input states. This means that the statistics observed by precise measurements of individual observables are limited by the statistical rules that apply to joint measurements involving that observable. It can thus be shown that the uncertainty bounds for quantum states can be derived from the positivity bounds defined by the state-independent statistical properties of joint measurements.

The main result of the present paper is that the requirement of positivity for experimental probabilities observed in a combination of two local joint measurements performed on two separate quantum systems defines a bound on the non-local correlations between the two systems (Eq. (13) below). This bound is saturated by entangled states that violate Bell’s inequalities, demonstrating that the restrictions imposed by a combination of two measurement uncertainties is weaker than the limitations imposed by local realism. At the same time, the saturation of the bound indicates that it is a tight bound that fully explains the statistical limits of the non-local expectation values involved. The Cirel’son bound, which describes the upper limit for violations of Bell’s inequalities, is obtained from a weaker version of the bound that applies only to the specific linear combinations of correlations that appear in Bell’s inequalities. The positivity of experimental probabilities for a combination of local joint measurements thus provides a more precise description of statistical limits for non-local correlations than criteria based merely on the non-locality described by Bell’s inequalities.

2. Statistics of joint measurements for qubits

Quantum theory sets very precise conditions for the realization of measurements that are simultaneously sensitive to two non-commuting observables. In the most simple case of a two level system, it is possible to formulate a few simple rules that completely determine the experimentally observable statistics of measurement outcomes with respect to the results of separate measurements of the observables concerned. For a pair of observables $\hat{X}$ and $\hat{Y}$ with eigenvalues of ±1 and mutually unbiased eigenstates, it is possible to construct a set of joint measurements characterized by visibilities $V_x$ and $V_y$, where the experimentally obtained averages of the measurement outcomes are related to the expectation values of the input state by

$$\langle x\rangle_{\text{exp.}} = V_x \langle \hat{X} \rangle,$$

$$\langle y\rangle_{\text{exp.}} = V_y \langle \hat{Y} \rangle.$$  \hspace{1cm} (1)
However, these two relations are not sufficient to derive the joint probabilities \( P_{\text{exp}}(x, y) \), since the average value of the product \( \langle xy \rangle_{\text{exp}} \) needs to be determined as well. It should be noted that this third expectation value cannot be obtained in separate measurements of \( \hat{X} \) and \( \hat{Y} \) and therefore represents a piece of experimental evidence about the relation between \( \hat{X} \) and \( \hat{Y} \) that can only be obtained in joint measurements [19, 32]. It is therefore important to understand how quantum theory determines this additional piece of information.

For complementary properties of two level systems, it is possible to construct joint measurements that are only sensitive to the expectation values \( \langle \hat{X} \rangle \) and \( \langle \hat{Y} \rangle \) that span the equatorial plane of the Bloch vector. For this class of joint measurements, the experimental average of the product \( xy \) is exactly equal to zero,

\[
\langle xy \rangle_{\text{exp}} = 0. \tag{2}
\]

This relation is indeed satisfied by a wide range of experimental realizations of joint measurements, such as the sequential measurements used to study Leggett-Garg inequality violations and measurement uncertainties [11, 12, 15, 16, 17, 19, 20, 21]. We should therefore consider Eq.(2) as an experimentally confirmed property of a widely used class of joint measurements. Conditions (1) and (2) completely determine the four outcome probabilities \( P_{\text{exp}}(x, y) \) of the joint measurement. The results read

\[
P_{\text{exp}}(+1, +1) = \frac{1}{4} \left( 1 + V_x \langle \hat{X} \rangle + V_y \langle \hat{Y} \rangle \right)
\]

\[
P_{\text{exp}}(+1, -1) = \frac{1}{4} \left( 1 + V_x \langle \hat{X} \rangle - V_y \langle \hat{Y} \rangle \right)
\]

\[
P_{\text{exp}}(-1, +1) = \frac{1}{4} \left( 1 - V_x \langle \hat{X} \rangle + V_y \langle \hat{Y} \rangle \right)
\]

\[
P_{\text{exp}}(-1, -1) = \frac{1}{4} \left( 1 - V_x \langle \hat{X} \rangle - V_y \langle \hat{Y} \rangle \right). \tag{3}
\]

Because of condition (2), these probabilities cannot describe an uncertainty free measurement. Specifically, the expectation value \( \langle xy \rangle_{\text{exp}} \) would have to be equal to +1 for an uncertainty free measurement of an input with \( x = +1 \) and \( y = +1 \). If condition (2) is imposed, the logical contradiction between the actual values and the condition results in negative probabilities in Eq.(3). It is therefore possible to identify the uncertainty limits required by Eq.(2) quantitatively using the requirement of positivity for the probabilities in Eq.(3). The result is a limit on visibilities and expectation values given by

\[
|V_x \langle \hat{X} \rangle| + |V_y \langle \hat{Y} \rangle| \leq 1. \tag{4}
\]

The use of condition (2) thus establishes a relation between the measurement uncertainties represented by \( V_x \) and \( V_y \) and the quantum state uncertainties represented by \( \langle \hat{X} \rangle \) and \( \langle \hat{Y} \rangle \). If we complete the description of the joint measurement by adding the uncertainty limit of \( V_x \) and \( V_y \), it is possible to derive the uncertainty limit for \( \langle \hat{X} \rangle \) and \( \langle \hat{Y} \rangle \) in quantum state preparation from the requirement that the statistics of the joint measurement must always be positive.
If we consider the class of joint measurements defined by condition (2) and the uncertainty limit of the visibilities \[ V_x^2 + V_y^2 \leq 1, \] (5)
then the requirement that the experimentally observed probabilities in Eq.(3) must always be positive results in the condition
\[ \langle \hat{X} \rangle^2 + \langle \hat{Y} \rangle^2 \leq 1 \] (6)
for all possible input states. Thus the curved surface of the Bloch sphere can be derived from the statistical properties of joint measurements given by condition (2) and the measurement uncertainty in Eq.(5).

It is important to remember that the impossibility of uncertainty free joint measurements is an essential element of quantum theory [37]. In the derivation above, uncertainties emerge as a result of a single non-classical condition, Eq.(2), which defines a necessary statistical relation between the measurement outcomes \( x \) and \( y \). Since this statistical relation does not depend on the input state, it is difficult to reconcile it with any measurement independent assignment of values to \( x \) and \( y \). This difficulty results in statistical limitations of quantum state statistics that are actually more restrictive than hidden variable theories. It is therefore interesting to ask what kind of restrictions we can obtain for quantum correlations that exceed the bounds of such hidden variable theories. Specifically, it should be interesting to analyze the specific bounds imposed by a pair of joint measurements on the quantum correlations that appear in Bell’s inequalities.

3. Observation of two qubit correlations by joint measurements

If two local joint measurements are performed independently on a pair of qubits, the correlations between the two qubits will be observed in the joint measurement statistics \( P_{\text{exp}}(x_A, y_A, x_B, y_B) \). In general, there are fifteen independent statistical moments that characterize this probability distribution over sixteen possible outcomes. To simplify the problem somewhat, the analysis can be limited to situations where all local expectation values are zero. The marginal probabilities for qubit \( A \) and for qubit \( B \) are then characterized by probabilities of \( 1/4 \) for all four possible outcomes of \((x_A, y_A)\) or \((x_B, y_B)\). This assumption eliminates six of the fifteen statistical moments, leaving only nine moments of the distribution for the following analysis. These nine statistical moments can all be expressed as correlations between \((x_A, y_A, x_A y_A)\) and \((x_B, y_B, x_B y_B)\). However, Eq.(2) implies that expectation values that involve either \(x_A y_A\) or \(x_B y_B\) will be zero. If we represent an arbitrary contribution from system \( i \) by \( f_i \), the rule for correlations between the outcomes of joint measurements reads
\[ \langle x_A y_A f_B \rangle_{\text{exp}} = 0, \]
\[ \langle f_A x_B y_B \rangle_{\text{exp}} = 0. \] (7)
In total, these are five more conditions that determine the joint measurement statistics for the two independently performed local measurements of the two qubits. The remaining four statistical moments are determined by the correlations between \((x_A, y_A)\) and \((x_B, y_B)\), which can be related to the input expectation values between \((\hat{X}_A, \hat{Y}_A)\) and \((\hat{X}_B, \hat{Y}_B)\) using the visibilities of local measurements as defined by Eq. (1),

\[
\begin{align*}
\langle x_A x_B \rangle_{\text{exp.}} &= V_x(A) V_x(B) \langle \hat{X}_A \otimes \hat{X}_B \rangle, \\
\langle x_A y_B \rangle_{\text{exp.}} &= V_x(A) V_y(B) \langle \hat{X}_A \otimes \hat{Y}_B \rangle, \\
\langle y_A x_B \rangle_{\text{exp.}} &= V_y(A) V_x(B) \langle \hat{Y}_A \otimes \hat{X}_B \rangle, \\
\langle y_A y_B \rangle_{\text{exp.}} &= V_y(A) V_y(B) \langle \hat{Y}_A \otimes \hat{Y}_B \rangle.
\end{align*}
\] (8)

The probabilities \(P_{\text{exp.}}\) for the sixteen outcomes of the two joint measurements \((x_A, y_A, x_B, y_B)\) thus depend only on the four correlations between local observables that can also be observed in separate measurements of \(\hat{X}_A\) or \(\hat{Y}_A\) and \(\hat{X}_B\) or \(\hat{Y}_B\).

It is in principle a straightforward matter to derive explicit expressions for the experimentally observable probabilities. To keep the expression simple, it may be sufficient to show one explicit example for which the positivity limit is particularly easy to see,

\[
P_{\text{exp.}}(+1, +1, -1, -1) = \frac{1}{16} \left( 1 - V_x(A) V_x(B) \langle \hat{X}_A \otimes \hat{X}_B \rangle - V_x(A) V_y(B) \langle \hat{X}_A \otimes \hat{Y}_B \rangle - V_y(A) V_x(B) \langle \hat{Y}_A \otimes \hat{X}_B \rangle - V_y(A) V_y(B) \langle \hat{Y}_A \otimes \hat{Y}_B \rangle \right) \geq 0.
\] (9)

For other values of \((x_A, y_A, x_B, y_B)\), the signs in front of each correlation are determined by the corresponding products of \((x_A, y_A)\) and \((x_B, y_B)\).

It is highly non-trivial that the complete probability distribution of a joint measurement of \((x_A, y_A, x_B, y_B)\) can be derived from the separately observed expectation values of precise projective measurements on \(\hat{X}_A\) or \(\hat{Y}_A\) and \(\hat{X}_B\) or \(\hat{Y}_B\). As a result of the product conditions (7), there is no additional degree of freedom that would allow an adjustment of the relation given by Eq. (9). It is therefore impossible to obtain correlations between the outcomes of precise projective measurements that would result in the prediction of a negative probability \(P_{\text{exp.}}(x_A, y_A, x_B, y_B)\) for any possible outcome of the two joint measurements. The positivity of joint measurements thus imposes statistical bounds on non-local correlations between the physical properties of two separate qubits.

4. Analysis of the statistical bounds for non-local correlations

Since the probabilities of all possible experimental outcomes must be positive, it is possible to identify statistical bounds that relate local (single qubit) measurement uncertainties to non-local (two qubit) expectation values. For the probability in Eq. (9), this statistical bound is given by

\[
V_x(A) V_x(B) \langle \hat{X}_A \otimes \hat{X}_B \rangle + V_x(A) V_y(B) \langle \hat{X}_A \otimes \hat{Y}_B \rangle
\]
Limit on non-local quantum correlations

\[ + V_y(A)V_x(B)\langle \hat{Y}_A \otimes \hat{X}_B \rangle + V_y(A)V_y(B)\langle \hat{Y}_A \otimes \hat{Y}_B \rangle \leq 1. \] (10)

All other bounds can be obtained by flipping the signs in front of each visibility \( V_m(S) \) to find the bound for an outcome with opposite sign in \( m_S \). Since the visibilities \( V_m(S) \) are bounded by the uncertainty relation given in Eq.(5), the bound for the non-local correlations of possible input states can be found by defining visibilities that satisfy the uncertainty bound of Eq.(5). Since visibilities are always positive, the values of \( (x_A, y_A, x_B, y_B) \) for the probability that defines the positivity bound can be included in the definition of the visibilities, so that all combinations of measurement outcomes and visibilities can be represented by trigonometric functions,

\[
\begin{align*}
x_A V_x(A) &= \cos(\alpha) \\
y_A V_y(A) &= \sin(\alpha) \\
-x_B V_x(B) &= \cos(\beta) \\
-y_B V_y(B) &= \sin(\beta).
\end{align*}
\] (11)

The statistical bound for the four correlations between the two qubits can then be expressed by a single inequality that must be satisfied for all possible values of \( \alpha \) and \( \beta \),

\[
\cos(\alpha + \beta) \left( \langle \hat{X}_A \otimes \hat{X}_B \rangle - \langle \hat{Y}_A \otimes \hat{Y}_B \rangle \right) \\
+ \sin(\alpha + \beta) \left( \langle \hat{X}_A \otimes \hat{Y}_B \rangle + \langle \hat{Y}_A \otimes \hat{X}_B \rangle \right) \\
+ \cos(\alpha - \beta) \left( \langle \hat{X}_A \otimes \hat{X}_B \rangle + \langle \hat{Y}_A \otimes \hat{Y}_B \rangle \right) \\
- \sin(\alpha - \beta) \left( \langle \hat{X}_A \otimes \hat{Y}_B \rangle - \langle \hat{Y}_A \otimes \hat{X}_B \rangle \right) \leq 2. \] (12)

Since the sums and the differences of \( \alpha \) and \( \beta \) can be varied independently, it is easy to identify the bound that applies to the correlations. Specifically, the optimal choice of visibilities results in contributions that correspond to the lengths of two dimensional vectors in the Bloch space of multi-qubit expectation values,

\[
\sqrt{\left( \langle \hat{X}_A \otimes \hat{X}_B \rangle - \langle \hat{Y}_A \otimes \hat{Y}_B \rangle \right)^2 + \left( \langle \hat{X}_A \otimes \hat{Y}_B \rangle + \langle \hat{Y}_A \otimes \hat{X}_B \rangle \right)^2} \\
+ \sqrt{\left( \langle \hat{X}_A \otimes \hat{X}_B \rangle + \langle \hat{Y}_A \otimes \hat{Y}_B \rangle \right)^2 + \left( \langle \hat{X}_A \otimes \hat{Y}_B \rangle - \langle \hat{Y}_A \otimes \hat{X}_B \rangle \right)^2} \leq 2. \] (13)

This bound is the central result of the present paper. It describes a tight statistical limit for the four non-local correlations between a pair of qubits that also appear in Bell’s inequalities. However, the present bound is based on the necessary positivity of experimentally observable probabilities, and not on the hypothetical possibility of assigning hidden variables to unobserved quantities. That is why the bound given by Eq.(13) must always be valid and cannot be violated by any quantum state whatsoever. It is therefore closely related to the Cirel’son bound [11], which describes the maximal violation of Bell’s inequalities permitted by quantum mechanics. In fact, the Cirel’son bound is included in the bound given by Eq.(13), as will be shown in the following.

To obtain a better intuitive understanding of the bound in Eq.(13), it may be useful to derive bounds that are less tight but easier to interpret. The most obvious
simplification is to focus on only one of the two square roots on the right hand side of the inequality, which allows us to remove the square root to obtain
\[
\left(\langle \hat{X}_A \otimes \hat{X}_B \rangle - \langle \hat{Y}_A \otimes \hat{Y}_B \rangle \right)^2 + \left(\langle \hat{X}_A \otimes \hat{Y}_B \rangle + \langle \hat{Y}_A \otimes \hat{X}_B \rangle \right)^2 \leq 4. \tag{14}
\]
This simplification of the bound already brings us closer to the term on the left hand side of a Bell’s inequality. Specifically, the Bell’s inequality defines a bound for the sum of the two terms that are squared on the left hand side of Eq.(14). Using the basic mathematical properties of squares, it is possible to identify the bound that applies only to the sum of the terms, which can be saturated when the difference of the two terms is exactly equal to zero. This bound for the left hand side of the Bell’s inequality reads
\[
\langle \hat{X}_A \otimes \hat{X}_B \rangle + \langle \hat{X}_A \otimes \hat{Y}_B \rangle + \langle \hat{Y}_A \otimes \hat{X}_B \rangle - \langle \hat{Y}_A \otimes \hat{Y}_B \rangle \leq 2\sqrt{2}, \tag{15}
\]
which is equal to the Cirel’son bound that described the maximal violation of Bell’s inequalities allowed by quantum mechanics [41].

The relation between the Cirel’son bound and joint measurability has been identified previously by Banik and coworkers [43]. However, the joint measurement considered in that work was a non-local measurement of the two correlations that are squared on the right hand side of Eq.(14). In the present analysis, it is shown that the same bound can be derived from local joint measurements on the separate qubits. This means that the measurement outcomes assign four individual values to each of the local spin components. It is therefore possible to obtain more detailed information on the non-local correlations between the qubits by considering the results of the local joint measurement in detail. In particular, it is possible to identify a specific joint probability that drops to zero at the bound, and this probability will be associated with a very specific assignment of values to all of the relevant local observables. The present approach can thus provide new insights into the relation between local statistics and non-local correlations that are not available if the bounds are derived from non-local measurements.

5. Saturation of the bound

Before taking a closer look at the statistics observed at the bound, it may be good to relate the bound to the quantum states that saturate it. In particular, it is important to demonstrate that the bound is tight and cannot be improved upon by the addition of more terms. To do so, it is necessary to identify the density matrix elements that correspond to the correlations in Eq.(13). Using the conventional definition of the computational basis \{ | 0 \rangle, | 1 \rangle \} so that \( \hat{X} | 0 \rangle = | 1 \rangle \) and \( \hat{Y} | 0 \rangle = i | 1 \rangle \), the correlations can be identified with specific off-diagonal elements of the density matrix using
\[
\begin{align*}
(\hat{X}_A \otimes \hat{X}_B - \hat{Y}_A \otimes \hat{Y}_B) - i(\hat{X}_A \otimes \hat{Y}_B + \hat{Y}_A \otimes \hat{X}_B) &= 4 \langle 0 | \langle 00 | \langle 00 | \langle 01 | (16)
(\hat{X}_A \otimes \hat{X}_B + \hat{Y}_A \otimes \hat{Y}_B) - i(\hat{X}_A \otimes \hat{Y}_B - \hat{Y}_A \otimes \hat{X}_B) &= 4 \langle 0 | \langle 01 | \langle 10 | .
\end{align*}
\]
The bound given by Eq. (13) is therefore equal to the mathematical limit on two coherences in separate subspaces of the density matrix,

$$|\langle 00 | \hat{\rho} | 11 \rangle| + |\langle 10 | \hat{\rho} | 01 \rangle| \leq \frac{1}{2}.$$  

(17)

This bound is saturated by any mixed state of equal superpositions of $|00\rangle$ and $|11\rangle$ and equal superpositions of $|10\rangle$ and $|01\rangle$.

The simplified bound of Eq. (14) is the bound in the subspace of $|00\rangle$ and $|11\rangle$, saturated by any state of the form

$$|\psi_{\text{sat.}}\rangle = \frac{1}{\sqrt{2}} (|00\rangle + \exp(i\phi)|11\rangle),$$  

(18)

with correlations of

$$\langle \hat{X}_A \otimes \hat{X}_B \rangle = -\langle \hat{Y}_A \otimes \hat{Y}_B \rangle = \cos(\phi),$$

$$\langle \hat{X}_A \otimes \hat{Y}_B \rangle = \langle \hat{Y}_A \otimes \hat{X}_B \rangle = \sin(\phi).$$  

(19)

The bound thus identifies a maximal two qubit coherence. Note that the Cirel’son bound given by Eq. (15) is only saturated for $\phi = \pi/4$, demonstrating that the bounds derived here are significantly tighter than the Cirel’son bound itself.

6. Experimental statistics at the bound

The upper bound on non-local correlations between two separate qubits derived above originates from the requirement that the experimentally observable measurement probabilities of joint measurements must remain positive for all physical input states. This means that the bounds can be tested experimentally by showing that the saturation of the bound corresponds to a probability of zero in a corresponding joint measurement. It is interesting to relate this observation to the previous result that the Cirel’son bound can be explained by the uncertainty limit for conditional statistics observed in correlated qubits [42]. For a quantum state at the bound, there exists a combination of joint measurements so that at least one of the outcome probabilities is zero. If the measurement of system A is treated as part of a conditional quantum state preparation in B, the conditional probabilities in system B describe the joint measurement of a single qubit with one of the four joint probabilities at zero. From the discussion in Sec. 3 it is clear that the joint probability of zero marks the uncertainty limit for $\hat{X}_B$ and $\hat{Y}_B$ in system B. Therefore the present results may also serve as a generalization of the conditional uncertainty bound introduced by Oppenheim and Wehner [42] to a wider variety of possible remote state preparations. The interpretation of the bound as a conditional uncertainty bound may also help to explain why Bell’s inequality violations are possible even though the local uncertainty bounds are more restrictive than hidden variable theories. Since the joint measurement in A used to select the conditional state in B is also uncertainty limited, the conditional reduction of uncertainties in B is lower than the reduction caused by a hypothetical discovery of the hidden variables that determine the exact values of $\hat{X}_A$ and $\hat{Y}_A$ simultaneously. Measurement uncertainties...
thus identify a gap between the tighter bounds of hidden local realism and the more permissive bounds of observable local reality.

Finally, it may also be helpful to look at the experimentally observed correlations in more detail. Banik and coworkers derived the Cirel’son bound by arguing that the experimental probabilities of a collective (non-local) measurement were limited by Bell’s inequalities and demonstrating that the minimal factor by which the initial correlations were reduced was $\sqrt{2}$ [13]. The present argument is quite different, since the positivity bounds of the individual outcome probabilities are more detailed and precise than the collective bounds imposed by Bell’s inequalities. We can therefore expect that the experimental probabilities at the bound do not saturate Bell’s inequalities. For the state in Eq. (18), the experimentally observed correlation is

$$\langle x_A x_B \rangle_{\text{exp}} + \langle y_A x_B \rangle_{\text{exp}} + \langle x_A y_B \rangle_{\text{exp}} - \langle y_A y_B \rangle_{\text{exp}} = \cos(\alpha - \beta) \cos(\phi) + \sin(\alpha + \beta) \sin(\phi),$$

where $0 < \alpha < \pi/2$ and $0 < \beta < \pi/2$ describe the visibilities of the two local measurements according to Eq. (11). The maximal value of this correlation is $\sqrt{2}$, obtained for $\alpha = \beta = \pi/4$, which describes measurements with equal visibilities of $V_x = V_y = 1/\sqrt{2}$ in both systems. This actually seems to be the highest experimental value for this correlation in any joint measurement, indicating that the Bell bound of two cannot be achieved by the experimental statistics of local joint measurements. In fact, the experimental correlation is even lower for measurements that result in an experimental probability of zero. In this case, the measurement visibilities must satisfy $\alpha + \beta = \phi$ and the maximal Bell’s inequality correlation is obtained for $\alpha = \beta = \phi/2$.

In the presence of an experimental outcome probability of zero, the correlation for the state in Eq. (18) is given by

$$\langle x_A x_B \rangle_{\text{exp}} + \langle y_A x_B \rangle_{\text{exp}} + \langle x_A y_B \rangle_{\text{exp}} - \langle y_A y_B \rangle_{\text{exp}} = 1 + \cos(\phi) - \cos^2(\phi).$$

The maximal experimentally observable value of the Bell’s inequality correlation in the presence of a measurement outcome with probability zero is 1.25, achieved at $\cos(\phi) = 1/2$. Interestingly, the corresponding state does not saturate the Cirel’son bound, since the Bell’s inequality violation of the correlations shown in Eq. (19) is only $(1 + \sqrt{3}) \approx 2.73$ for this value of $\phi$. This result highlights the difference between the bound described by Bell’s inequalities and the actual quantum mechanical bound required to obtain only positive experimental probabilities.

The problem with Bell’s inequalities seems to be that they do not really identify any relevant quantum mechanical limit at all. Local uncertainty limits indicating entanglement are well below the Bell’s inequality bound and the Cirel’son bound is well above it. Even the experimentally observable probabilities of joint measurements fail to come close to the bound of local realism. The reason for the latter failure to achieve the Bell’s inequality bound is that the positivity of experimentally observable probabilities must accommodate not just the Bell’s inequality sum of correlations, but
also a number of other conditions, as summarized by the comprehensive bound on non-local correlations given by Eq. 13. It may therefore be justified to conclude that the statistical bounds imposed by local joint measurements are more fundamental than the quantum non-locality evidenced by a violation of Bell’s inequalities.

7. Conclusions

Uncertainty bounds on joint measurability are a necessary condition for the observation of non-classical correlations between non-commuting observables 35, 36, 37. As shown above, it is possible to trace the non-classical relation between complementary physical properties of qubit systems to a simple rule for the probabilities of measurement outcomes observed in uncertainty limited joint measurements of these properties, which is that the average product of the two measurement outcomes of ±1 must always be zero (Eq. 2). When combined with the uncertainty limit of Eq. 5, this simple rule not only determines the uncertainty limit of the qubit Bloch sphere, but also imposes collective statistical limits on pairs of qubits that relate specifically to the maximal values of two qubit correlations. The limit given by Eq. 13 explains the statistical signature of maximal two qubit coherence as a consequence of the uncertainty limits of local joint measurements. This limit naturally includes the Cirel’son bound that describes the quantum mechanical limit of Bell’s inequality violations as only one component of the more general limit on two qubit correlations.

Most significant is the realization that the characteristic features of quantum non-locality can be explained as a consequence of local relations between non-commuting observables that appear directly in the experimental data of joint measurements of the local systems. Joint measurements can therefore reveal aspects of quantum physics that seemed to be hidden by statistical uncertainties. In this context, quantum non-locality merely appears as an illustration of the fact that the relations between non-commuting physical properties seem to be at odds with a joint assignment of realities. It is therefore possible (and perhaps even likely 14, 45) that the violation of Bell’s inequalities is merely a natural consequence of the local structure of quantum statistics.

Acknowledgments

This work was supported by JSPS KAKENHI Grant Number 26220712.

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Limit on non-local quantum correlations

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