Exact string solutions and duality

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Abstract
We review known exact classical solutions in (bosonic) string theory. The main classes of the solutions are ‘cosets’ (gauged WZW models), ‘plane wave’-type backgrounds (admitting a covariantly constant null Killing vector) and ‘F-models’ (backgrounds with two null Killing vectors generalising the ‘fundamental string’ solution). The recently constructed $D = 4$ solutions with Minkowski signature are given explicitly. We consider various relations between these solutions and, in particular, discuss some aspects of the duality symmetry.

To appear in: Proceedings of the 2nd Journée Cosmologie,
Observatoire de Paris, June 2-4, 1994 (World Scientific, Singapore)

July 1994

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1. Introduction

One of the central problems in string theory is to study the space of exact classical solutions. This may clarify the formal structure of string theory and also may be relevant for understanding the implications of string theory (assuming there exist regions where string coupling is small so that perturbative and non-perturbative corrections to string equations of motion can be ignored).

To be able to discuss issues of singularities and short distance structure one should be interested not just in solutions of the leading-order low-energy string effective equations (derived under the assumption of small curvature $|\alpha' R| \ll 1$) but in the ones that are exact in $\alpha'$. In general it is not enough to know just a set of ‘massless’ background fields ($G_{\mu\nu}, B_{\mu\nu}, \phi$). Since strings are extended objects test strings ‘feel’ geometry in a way which is different from point particles. Different (dual) point-particle geometries may appear to be equivalent from the string point of view. In order to give an adequate description of a first-quantised string propagation, i.e. a ‘string-geometric’ interpretation of a given solution one needs to know the corresponding 2d conformal field theory (CFT). The knowledge of CFT includes information about equations for all string modes in a given background and thus goes beyond a specification of one particular conformal $\sigma$-model.

The tree-level effective action for the ‘massless’ fields is given by an infinite powers series in $\alpha'$. While the solutions of the leading-order equations are numerous and straightforward to find, only a small subset of them is known to be exact to all orders in $\alpha'$. Moreover, only some of these exact solutions have known CFT interpretation.

Given that the general structure of the effective action is unknown, to find the exact solutions one needs to use some indirect methods. The key property is that the string solutions, i.e. the stationary points of the effective action, must correspond to conformally invariant sigma models.[1,2]

There are three possible approaches that were used to construct exact string solutions:

(i) Start with a particular leading-order solution and show that $\alpha'$-corrections are absent or take some explicitly known form due to some special properties of this background. Such are some ‘plane wave’-type backgrounds, or, more generally, backgrounds with a covariantly constant Killing vector, see e.g. [3,4,5,6,7].

(ii) Start with a known Lagrangian CFT and represent it as a conformal $\sigma$-model. Essentially the only known example of such construction is based on gauged WZW models, see e.g. [8,9,10,11].
(iii) Start directly with a leading-order sigma model path integral and prove that there exists such a definition of the sigma model couplings (i.e. a ‘renormalisation scheme’) for which this sigma model is conformal to all loop orders. This strategy works [7] for so-called ‘F-models’ introduced in [12,7] and their generalisations [13]. A particular example of F-model is the ‘fundamental string’ (FS) solution [14].

These three classes of exact solutions are not completely independent. Namely, a subclass of plane waves can be identified with gauged WZW models based on non-semisimple groups [15,16]. Also, a different subclass of backgrounds with a covariantly constant null Killing vector (called ‘K-models’ in [7]) is dual to F-models. Finally, a subclass of F-models can be interpreted [17] as special gauged WZW models (based on maximally non-compact group and nilpotent subgroup as a gauged one); similar interpretation is true for the generic $D = 3$ F-model [7].

There exists also a more general class of exact solutions (with just one null Killing vector) which generalises both the K-models and F-models [13].

We shall start in section 2 with a discussion of the structure of the string effective action stressing the importance of the field redefinition ambiguity (‘scheme dependence’).

In section 3 we shall review some aspects of the duality symmetry and consider several $D = 3$ examples where the duality is exact to all orders in $\alpha'$, illustrating in which sense different string modes ‘feel’ different geometries.

Exact solutions corresponding to gauged WZW models will be considered in section 4. We shall argue that there exists a ‘leading-order’ scheme in which the background fields do not depend on $\alpha'$. We shall mention some known solutions with $D = 4$.

Solutions with a covariantly constant null Killing vector will be the topic of section 5. We shall start with the simplest plane wave backgrounds (some of which will be related to gauged WZW models based on non-semisimple groups) and discuss several generalisations, in particular, K-model dual to the fundamental string solution and $D = 4$ hybrid K-model with the transverse part represented by the euclidean $D = 2$ background.

Section 6 will be devoted to a new class of exact solutions described by F-models. We shall start with the F-models with flat transverse part and consider a subclass of them (in particular, with $D = 4$) which corresponds to specific nilpotently gauged WZW models and thus admits a direct CFT interpretation. We shall also present the $D = 4$ F-model with the $D = 2$ euclidean black hole as the transverse part which generalises the $D = 4$ fundamental string solution.

A more general class of exact string solutions which contains both the K-models and F-models will be briefly discussed in section 7.

In section 8 we shall make some concluding remarks, in particular, on the problem of finding the CFT interpretation of the exact solutions.
2. Structure of the string effective action

String effective action contains terms of all orders in $\alpha'$ and thus depends on an infinite number of parameters. Some of these parameters are unambiguous, i.e. are determined by the string $S$-matrix while others can be changed by local field redefinitions. The latter represent the freedom of changing the ‘scheme’. This ‘scheme dependence’ is an important novel property of the string equations of motion which needs to be taken into account in the discussion of exact string solutions: equivalent exact solutions may look very different in different schemes. The aim of this section is to discuss the general structure of the tree-level string theory effective action (EA) emphasizing a possibility to use the field redefinition freedom to put higher order $\alpha'^n$-corrections in the simplest form. In particular, the EA can be chosen in such a form (a ‘scheme’) that all $\alpha'$-corrections vanish once we specialise to the case of a $D = 2$ background. In such a scheme the $D = 2$ metric-dilaton EA is thus known explicitly, i.e. is given by the leading-order terms. There also exists a scheme in which the $D = 3$ limit of the EA has all $\alpha'$-corrections depending only on the derivatives of the dilaton but not on the curvature or antisymmetric tensor.

2.1. ‘Scheme dependence’ of the effective action

Let us first recall a few basic facts about the string effective action [18,19]. Given a tree-level string $S$-matrix (in $D = 26$) we can try to reproduce its massless sector by a local covariant field-theory action $S(G, B, \phi)$ for the metric, antisymmetric tensor and dilaton. Subtracting the massless exchanges from the string scattering amplitudes and expanding the massive ones in powers of $\alpha'$ gives an infinite series of terms in $S$ of all orders in $\alpha'$.

The form of such action is not unique: a class of actions related by field redefinitions which are local, covariant, background-independent, power series in $\alpha'$ (depending on dilaton only through its derivatives not to mix different orders of string loop perturbation theory) will correspond to the same string $S$-matrix [20,21]. Given some representative in a class of equivalent EA’s we refer to other equivalent actions as corresponding to different ‘schemes’. The reason for this terminology is that the extremality conditions for the effective action are equivalent to the conditions of conformal invariance of the sigma model representing string action in a background [1,2] and the related ambiguity in the sigma model Weyl anomaly coefficients or ‘$\beta$-functions’ can be interpreted as being a consequence of different choices of a renormalisation scheme [20]. This implies that some coefficients of the $\alpha'^n$-terms in the EA will be unambiguous (being fixed by the string $S$-matrix) while many others will be ‘scheme-dependent’.

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Though one possible way of determining the EA is to start with perturbative massless string scattering amplitudes on a flat $D = 26$ background, $S$ must actually be background-independent. In particular, its unambiguous coefficients are universal (e.g. they do not depend on the dimension $D$). This is implied by the equivalence between the effective equations of motion and the string sigma model Weyl invariance conditions (which are background-independent). To make this equivalence precise in any dimension $D$ we need only to add to the EA one $D$-dependent (‘central charge’) term $\sim \int d^D x \sqrt{G} \exp(-2\phi) (D - 26)$. Given such a background-independent EA [18,19,2]

$$S = \int d^D x \sqrt{G} \ e^{-2\phi} \left\{ \frac{2(D - 26)}{3\alpha'} - \left[ R + 4(\partial_\mu \phi)^2 - \frac{1}{12} (H_{\mu\nu\lambda})^2 \right] + O(\alpha') \right\}, \quad (2.1)$$

it would be useful to choose a scheme (i.e. the values of ambiguous coefficients) in which $S$ has the simplest possible form. For example, the correspondence with the Weyl anomaly coefficients of a string sigma model implies that there exists a scheme in which (2.1) does not contain other higher-order dilatonic terms. This follows also from the general argument [22] based on the path integral representation for the EA [19] and was checked directly at the $\alpha'$-order [23,24] by comparing with string $S$-matrix (as we shall note below show in three dimensions one can do just the opposite, i.e., have only dilaton terms as higher order corrections).

For general $D \geq 4$ the massless sector of string $S$-matrix is non-trivial so no simple scheme is expected to exist. To order $\alpha'$ one can choose a ‘standard’ scheme in which the effective action has the form [23]

$$S = \int d^D x \sqrt{G} \ e^{-2\phi} \left\{ \frac{2(D - 26)}{3\alpha'} - \left[ R + 4\nabla^2 \phi - 4(\nabla \phi)^2 - \frac{1}{12} (H_{\mu\nu\lambda})^2 \right] \right\} - \frac{1}{4} \alpha' \left[ R_{\mu\nu\lambda\kappa} - \frac{1}{2} R^{\mu\nu\kappa\lambda} H_{\mu\nu} H_{\kappa\lambda} \right]
+ \frac{1}{24} H_{\mu\nu\lambda} H_{\rho\sigma} H_{\mu\rho\sigma} - \frac{1}{8} (H_{\mu\alpha\beta} H_{\nu}^{\alpha\beta})^2 \right\} + O(\alpha'^3) \right\}. \quad (2.2)$$

### 2.2. Effective action in $D \leq 3$

It is possible to arrive at a more definitive conclusion about a simplest possible scheme by specialising to the low dimensional cases of $D = 2$ and $D = 3$ [4]. More precisely, we would like to find an EA (defined for general $D$) such that its $\alpha'^n$-terms take a simple form in the limit $D \to 2, 3$. 


Given that (2.1) is background-independent (in particular, its higher-order coefficients do not depend on $D$) we are free to take $(G, B)$ in (2.1) to correspond to a generic $D = 2$ or $D = 3$ background. Since the basic fields $(G, B)$ are second rank tensors, higher order terms which involve ‘irreducible’ contractions of tensors of rank greater than two cannot be altered by field redefinitions. But in $D \leq 3$ the Riemann tensor can be expressed in terms of the Ricci tensor, and $H_{\mu\nu\lambda} = \epsilon_{\mu\nu\lambda} H$. Thus all possible covariant structures in the EA will have the ‘reducible’ form of products of scalars, vectors, or at most, second-rank tensors.

This is a necessary condition for a higher order term to be removed by a field redefinition, but it is not sufficient. It has been shown [25] that some combinations of a priori ambiguous coefficients in the EA are actually redefinition-invariant (unambiguous) and thus are uniquely determined by the string $S$-matrix. In fact, it is easy to show that one cannot find a scheme in which there is no $\alpha'$-term in the $D = 3$ EA [4]. However in $D = 2$ (where $H_{\mu\nu\lambda} = 0$) one can write $(R_{\mu\nu\kappa\lambda})^2 = 2(R_{\mu\nu})^2$ so the $\alpha'$ term now vanishes when the leading order equations are satisfied. Using the results of [23] one can show that this term can indeed be removed by a field redefinition.

More generally, suppose we compute the scattering amplitudes for the dilaton and graviton (in general $D$) directly in the string frame where the dilaton and graviton mix in the propagator. Since there are no transverse degrees of freedom for the string in $D = 2$, there are no dynamical degrees of freedom in the $(G, \phi)$ system, and the limit $D \to 2$ of the scattering amplitudes is trivial. That means that the on-shell limits of unambiguous terms in the EA must vanish identically. Hence there exists a choice of the EA (in generic $D$) such that higher order terms in it vanish in the $D \to 2$ limit.

A similar statement is not true in $D = 3$ since there is one transverse degree of freedom for the string which could yield higher order corrections to the scattering amplitudes and hence to the EA. However one can express these corrections solely in terms of the dilaton.

Since in $D = 2$ since there is a scheme in which all $\alpha'$ corrections to the EA vanish, all backgrounds which solve the leading-order equations are in fact exact solutions. This conclusion is not so surprising: the $D = 2$ ‘black hole’ background [26,8] represents the generic solution of the leading-order equations, and given that the corresponding CFT is

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1 We are not including massive modes (tachyon, etc) in the effective action since we are mostly interested in general $D$. It is true, however, that the tachyon becomes massless in $D = 2$ and thus one can define an effective action which will involve it as well.
known \((SL(2,R)/U(1)\) coset \[8\]) one can find explicitly \[22\] a local covariant background-independent redefinition from the ‘CFT scheme’ \[11\] (where the background fields are \(\alpha'\)-dependent) to the ‘leading-order’ scheme. It also follows that in this scheme the \(D = 2\) sigma model Weyl anomaly coefficients just have their leading-order form.

As for \(D = 3\), in the scheme where \(\alpha'\)-corrections are proportional to the derivatives of \(\phi\) the solutions of the leading-order equations which have \(\phi = \text{const}\) remain exact to all orders. It is easy to see that the only leading-order solution with \(\phi = \text{const}\) in \(D = 3\) is the constant curvature anti-DeSitter space with the parallelising \(H_{\mu\nu}\)-torsion corresponding to the \(SL(2,R)\) WZW model or its possible cosets over discrete subgroups (in particular, the \(D = 3\) black hole of \[27\]). For solutions with \(\phi \neq \text{const}\) the best what we can hope for is to find a scheme in which a particular leading-order solution does not receive \(\alpha'\)-corrections. This was shown \[28\] to be the case (to \(\alpha'^2\) order) for the charged black string background \[29\], i.e. \(SL(2,R) \times R/R\) coset model. It was also found \[7\] that there exists a scheme where the leading order solution for the \(D = 3\) F-model is also conformal to the next order in \(\alpha'\).

Another implication of the above discussion of the structure of the EA concerns an exact form of the abelian duality transformations which map string solutions into solutions (see next section). The leading-order duality \[30\] is a symmetry of the leading-order terms in the EA \[31,32\] and thus is the exact symmetry in the simplest scheme in \(D = 2\). In fact, one can check directly \[33\] that while for a general \(D\) there does not exist a scheme in which the leading order duality remains a symmetry at \(\alpha'\)-order without been modified by the derivative \(O(\alpha')\)-term \[32\], such scheme does exist in \(D = 2\).\[2\]

3. Duality symmetry

String theory is a remarkable extension of General Relativity which is consistent at the quantum level. Being a theory of extended objects string theory is radically different from the Einstein theory in its description of space-time geometry. Both the diffeomorphism invariance and the Einstein action appear only as large-distance approximations to the full string symmetry group and string field theory action. The space-time metric \(G_{\mu\nu}\) provides

\[2\] The locality of the \(\alpha'\)-correction \[32\] to the duality (and the existence of several examples where leading-order duality relates exact solutions to exact solutions) strongly suggests that it can also be interpreted as some generalised change of the scheme provided one somehow relaxes the condition of covariance of the redefinition.
only a crude description of the true ‘string geometry’ when probed by point-like string states.

The metric and its curvature are not adequate characterisations of the string geometry (i.e. they are not invariants of the string symmetry group): spaces with different curvature may represent equivalent string backgrounds. This can be demonstrated on the example of particular string symmetry – space-time duality symmetry (see, e.g., [34,35,30,36,32,37,38,39]).

In this section we shall present some examples of string backgrounds related by duality emphasizing that even in the simplest point-like approximation an invariant content of the string geometry is described not just by the metric but by a combination of the metric, antisymmetric tensor and dilaton with all the three fields playing complementary roles. Particular properties of the metric (e.g. its curvature and causal structure) are not, in general, invariant under duality. Different string modes may ‘see’ different ‘sides’ of the same string geometry. For example, a flat space (with some compact dimensions) viewed as a solution of the string theory can be equivalent to a curved space with a non-trivial dilaton and/or antisymmetric tensor. While the momentum string modes move in the original flat geometry, the winding modes present in the ‘flat’ string spectrum propagate in a dual curved background. This is certainly very different from what we are used to in the field theory of gravity – Einstein theory.

The examples of backgrounds we shall discuss have null Killing vectors and are exact string solutions (to all orders in $\alpha'$).

3.1. General remarks on duality

The space-time duality is one of the most fundamental properties of string theory, a direct consequence of the one-dimensional extended nature of strings. In two dimensions a scalar field is dual in two-dimensional sense to another scalar field ($\partial_m x = \epsilon_{mn} \partial^n \tilde{x}$). This formal relation implies a relation between correlators of vertex operators on a flat background and thus a space–time symmetry of the string $S$-matrix. The off-shell extension of the latter – the low-energy effective action is then invariant under the simultaneous duality transformation of the ‘massless’ string background fields and interchanging of the momentum and winding modes. In particular, the duality maps isometric string backgrounds which solve the effective string equations into other isometric solutions. In the case when the direction of the isometry is compact the two dual solutions correspond to the same conformal field theory, i.e. are equivalent from the string theory point of view.
From a more general point of view, duality provides a relation between processes at small and large scales and should be closely connected with the existence of the fundamental string scale $L_{\text{Planck}} \sim \sqrt{\alpha'}$, a “stringy” uncertainty principle ($\Delta x \sim E^{-1} + \alpha' E$) and existence of critical values of field strengths and temperature. The duality is likely to play an important role in understanding string theory predictions for black hole physics and cosmology (see, e.g., [40,41,11]). In the simplest case of the torus compactification the duality manifests itself as a symmetry of the spectrum of states under the interchanging of the momentum and winding modes and inverting the radius of the torus $M_{nm}^2 (r) = n^2 r^{-2} + m^2 r^2 \alpha'^{-2} + \ldots$, $\tilde{r} = \alpha'/r$. If the one-dimensional torus of radius $r$ is parametrized by the periodic coordinate $x(\tau, \sigma) = r \theta$, then introducing the dual string coordinate, $\tilde{x}(\tau, \sigma) = \tilde{r} \tilde{\theta}$, $\partial_a x = \epsilon_{ab} \partial^b \tilde{x}$, $x = x_0 + p \tau + \tilde{p} \sigma + \ldots$, $\tilde{x} = \tilde{x}_0 + \tilde{p} \tau + p \sigma + \ldots$, $p = \sqrt{\alpha'n}/r$, $\tilde{p} = mr/\sqrt{\alpha'}$, one can understand duality as a symmetry under which a process in the $x$–space (torus of radius $r$) corresponds to a dual process in the $\tilde{x}$–space (torus of radius $\tilde{r} = \alpha'/r$) with momentum and winding modes interchanged. The picture of momentum and winding modes which correspond to the same string vacuum but move in two different dual geometries generalizes to the case of non-trivial curved geometries.

A fundamental feature of string theory is that even a large-scale approximation to the geometry is described not only by the metric but also by the antisymmetric tensor and the scalar dilaton field. The action of a string in such a background is

$$I = \frac{1}{4\pi \alpha'} \int d^2 z \sqrt{\gamma} \left[ \partial_m x^\mu \partial^m x^\nu G_{\mu\nu}(x) + i \epsilon^{mn} \partial_m x^\mu \partial_n x^\nu B_{\mu\nu}(x) + \alpha' R^{(2)} \phi(x) \right].$$

$\gamma$ is a world sheet metric which decouples once the background fields satisfy the string field equations which can be derived from the effective action (2.1). The classical string probe follows the ‘geodesic’ equations corresponding to the action of a string in such a background (3.1). One of the consequences of the extended nature of a string is that though a point-like (or zero mode) string state follows the geodesics of the metric, other string modes do feel the antisymmetric tensor background. First-quantised test string also feels the dilaton background [3].

An important lesson of the discussion of duality is that the antisymmetric tensor and dilaton play the roles very similar to that of the metric. In particular, some metric properties of a background can be exchanged for some properties of the antisymmetric tensor and/or dilaton background. If (3.1) is invariant under an abelian isometry, $x^1 \rightarrow$
then choosing the coordinates \( \{ x^\mu \} = \{ x^1 \equiv \theta, x^i \} \) one finds the dual action \( \tilde{I} \) (1) for \( \{ \tilde{x}^\mu \} = \{ \tilde{x}^1 \equiv \tilde{\theta}, x^a \} \) with \( M \equiv G_{11} \)

\[
\tilde{G}_{11} = M^{-1}, \quad \tilde{G}_{1a} = M^{-1} B_{1a}, \quad \tilde{G}_{ab} = G_{ab} - M^{-1} (G_{1a} G_{1b} - B_{1a} B_{1b}) ,
\]

(3.2)

\[
\tilde{B}_{1a} = M^{-1} G_{1a}, \quad \tilde{B}_{ab} = B_{ab} - M^{-1} (B_{1a} G_{1b} - G_{1a} B_{1b}) , \quad \tilde{\phi} = \phi - \frac{1}{2} \ln M .
\]

These transformation rules generalise to the case of \( N \) abelian isometries. If \( (G, B, \phi) \) is a solution of the string equations then \( (\tilde{G}, \tilde{B}, \tilde{\phi}) \) is also a solution (in general, corrected by \( \alpha' \) terms [32]) and (assuming \( \theta \) is periodic) the two dual string actions correspond to the same CFT which is behind a given string background [38].

3.2. Duality between curved and flat space-times: \( D = 3 \) examples

To illustrate the above remarks let us now consider the backgrounds with a covariantly constant null Killing vector [3,4,6] which provide a possibility to study duality exactly to all orders in \( \alpha' \). There exists a simple class of such ‘plane-wave’ backgrounds \( (i, j = 1, ..., N) \)

\[
ds^2 = G_{\mu\nu} dx^\mu dx^\nu + G_{ij}(u) d\theta^i d\theta^j + dx^a dx_a ,
\]

(3.3)

\[
B_{\mu\nu} = (0, B_{ij}(u)) , \quad \phi = \phi(u) ,
\]

where \( \theta^i \) are periodic and \( x^a \) are ‘spectator’ flat coordinates (which we shall ignore in what follows). The fields in (3.3) are subject to just one string field equation or the conformal invariance constraint and represent exact classical solutions of string theory which transform into each other under the full duality group \( O(N, N|Z) \) [42].

In the case of the flat space \( (G_{ij} = \delta_{ij}) \) the conformal invariance of (1) is maintained by the balance of contributions of the antisymmetric tensor and dilaton fields. Some duality transformations may rotate a flat metric into curved backgrounds. For example, a curved \( (R_{iuju} \neq 0) \) four-dimensional solution \( G_{ij} = \left( \frac{1}{f}, \frac{f}{1 + f^2} \right) , \quad B_{ij} = 0 , \quad f = f(u) , \quad \phi = \phi(u) \), is related by the duality rotation in the \( \theta^1 \)-direction to the flat metric one \( \tilde{G}_{ij} = \delta_{ij} , \quad \tilde{B}_{ij} = -f \epsilon_{ij} , \quad \tilde{\phi} = \phi \). Thus a curved space is equivalent (from the string theory point of view) to the one with a flat metric but non-trivial antisymmetric tensor and dilaton, i.e. both backgrounds give different space-time representations of the same exact string solution.

To clarify further the meaning of the relation of flat spaces to curved ones under duality let us now consider an even simpler example with \( B_{ij} = 0 \): a flat \( D = 3 \) background

\[
ds^2 = -dt^2 + dx^2 + (x - t)^2 d\theta^2 = du dv + u^2 d\theta^2 , \quad \phi = \phi_0 = \text{const} .
\]

(3.4)
The dual background

\[ ds^2 = du dv + u^{-2} d\tilde{\theta}^2, \quad \tilde{\phi} = \phi_0 - \ln u, \quad (3.5) \]

has a non-zero curvature, \( R_{u\theta u\theta} = -2u^{-4} \) and is also an exact solution. Though flat, the background (3.4) (called ‘null orbifold’ in [43]) is quite non-trivial (it does not have a conical singularity at \( u = 0 \)). As in the case of the flat torus it is possible to demonstrate explicitly that (3.4) and (3.5) represent the same string solution, i.e. the same conformal theory. Representing the metric (3.4) in flat coordinates

\[ ds^2 = -2 dU dV + dX^2 \]

with the identification \((U, V, X) = (U, V + \frac{w}{2} X + \frac{1}{2} w^2 X, X + w U)\), \( w = m \pi, m = 0, 1, 2, \ldots \), it is easy to solve the corresponding string equations and to show that in addition to the standard momentum modes there are winding modes [17]. Quantising the string canonically and replacing the winding number \( m \) by \(-\frac{i}{2} \frac{\partial}{\partial \tilde{\theta}}\) one finds for the zero mode parts of the standard constraints on the string states

\[ (L_0^+ - L_0^-) T = \frac{1}{4} \frac{\partial^2}{\partial \tilde{\theta}^2} T = 0, \quad T(u, v, \theta, \tilde{\theta}) = T(u, v, \theta) + \tilde{T}(u, v, \tilde{\theta}), \quad (3.6) \]

\[ \frac{2}{\alpha'} (L_0^+ + L_0^- - 2) T = \left( 2 \frac{\partial^2}{\partial u \partial v} - \frac{1}{u^2} \frac{\partial^2}{\partial \theta^2} - \frac{u^2}{16 \alpha'^2} \frac{\partial^2}{\partial \tilde{\theta}^2} + \frac{1}{u} \frac{\partial}{\partial v} - \frac{4}{\alpha'} \right) T = 0. \quad (3.7) \]

Eq.(11) restricted to a solution of the first constraint takes the form of a Klein-Gordon-type equation in a metric-dilaton background

\[ (-D^2 + 2 D^\mu \phi_\mu - 4/\alpha') T = 0. \]

We conclude [17] that while the momentum point-like state \( T(u, v, \theta) \) propagates in the original flat background (3.4) the winding state \( \tilde{T}(u, v, \tilde{\theta}) \) ‘feels’ the dual curved background (3.5). The backgrounds (3.4) and (3.5) thus represent two complementary images of the same string geometry as ‘seen’ by different point-like string states.

It is useful compare (3.4) with the standard flat \( D = 3 \) background

\[ ds^2 = -dt^2 + dx^2 + dy^2 = -dt^2 + dr^2 + r^2 d\theta^2, \quad \phi = \phi_0, \quad (3.8) \]

which is related by the leading-order duality to (see also [37,38])

\[ ds^2 = -dt^2 + dr^2 + r^{-2} d\tilde{\theta}^2, \quad \tilde{\phi} = \phi_0 - \ln r. \quad (3.9) \]

The dual metric is curved and singular; it solves the leading-order conformal invariance equations but is corrected by \( \alpha' \)-terms so that in contrast to the above plane wave example
(where both the original and the duality transformed backgrounds are known explicitly) here one is unable to make definitive conclusions, e.g., about the fate of the \( r = 0 \) singularity. Moreover, the string spectrum on this flat background does not contain genuine winding modes so that one cannot reproduce the dual background as a geometry felt by some string modes on the original background (as was the case for the torus and the null orbifold (3.4)).

Another instructive example of a flat \( D = 3 \) space with a curved dual counterpart is provided by a plane-wave type one (but now outside the class (3.3))

\[
ds^2 = d\theta dt + x d\theta^2 + dx^2, \quad \phi = \phi_0,
\]

where \( \theta \) is assumed to be periodic. This is the simplest example of a ‘\( K \)-model’. Performing the duality transformation with respect to \( \theta \) we get a background with curved metric and non-constant antisymmetric tensor and dilaton described by the following sigma model (which is the simplest example of a \( F \)-model)

\[
\bar{I} = \frac{1}{\pi \alpha'} \int d^2 z (x^{-1} \partial u \partial v + \partial x \partial x), \quad \bar{\phi} = \phi_0 - \frac{1}{2} \ln x, \quad u = \bar{\theta} - t, \quad v = \bar{\theta} + t.
\]

The space (3.10) is flat and regular (‘null manifold’) but has rather bizarre causal properties. It appears that the acausality of the metric (3.10) is lost in the process of the duality transformation since the metric in (3.11)

\[
ds^2 = x^{-1} du dv + dx^2 = x^{-1} (d\bar{\theta}^2 - dt^2) + dx^2
\]

does not have closed time-like geodesics. However, a peculiar causal structure of (3.10) is now reflected in the presence of the time-like component of the antisymmetric tensor in (3.11). The causal structure of the string geometry is thus represented in different ways in its dual images.

As in the case of the flat torus and the null orbifold (3.4) it is possible to identify the dual background (3.11) as the geometry ‘probed’ by the winding modes of the string spectrum corresponding to the flat space (3.10). Solving the string equations using flat coordinates for (3.10) and quantising the theory one finds the expressions for the constraints which are similar to (3.6),(3.7) and concludes that while the momentum zero-mode string state \( T(\theta, t, z) \) satisfies the Klein-Gordon equation (3.11) corresponding to the original flat background (3.10) the winding state \( T(\bar{\theta}, t, z) \) propagates in the curved metric-dilaton background (3.11).
4. Exact solutions corresponding to gauged WZW models

The general bosonic sigma model describing string propagation in a ‘massless’ background is given by (3.1), i.e. in conformal gauge

\[ I = \frac{1}{\pi \alpha'} \int d^2z L , \quad L = (G_{\mu\nu} + B_{\mu\nu})(x) \partial x^\mu \partial x^\nu + \alpha' \mathcal{R}\phi(x) , \]  

(4.1)

where \( G_{\mu\nu} \) is the metric, \( B_{\mu\nu} \) is the antisymmetric tensor and \( \phi \) is the dilaton (\( \mathcal{R} \) is related to the worldsheet metric \( \gamma \) and its scalar curvature by \( \mathcal{R} \equiv \frac{1}{4} \sqrt{\gamma} R^{(2)} \)). The action of the ungauged WZW model for a group \( G \) is

\[ I = kI_0 , \quad I_0 \equiv \frac{1}{2\pi} \int d^2z \text{ Tr} (\partial g^{-1} \bar{\partial} g) + \frac{i}{12\pi} \int d^3z \text{ Tr} (g^{-1} dg)^3 , \]  

(4.2)

can be put into the general sigma model form (4.1) by introducing the coordinates on the group manifold. Then \( G_{\mu\nu} \) is the group space metric, \( H_{\mu\nu\lambda} = 3 \partial_{[\mu} B_{\nu\lambda]} \) is the parallelising torsion and dilaton \( \phi = \phi_0 = \text{const} \) (and \( k = 1/\alpha' \)). In a special scheme (where the \( \beta \)-function of the general model (4.1) is proportional to the generalised curvature) the resulting sigma model is finite at each order of \( \alpha' \)-perturbation theory [45,46] and corresponds to the well-known current algebra CFT [44].

The action of the gauged WZW model [47]

\[ I(g, A) = kI_0(g) + \frac{k}{\pi} \int d^2z \text{ Tr} (-A \bar{\partial} g g^{-1} + \bar{A} g^{-1} \partial g + g^{-1} Ag \bar{A} - A\bar{A}) , \]  

(4.3)

is invariant under the vector gauge transformations with parameters taking values in a subgroup \( H \) of \( G \). Parametrising \( A \) and \( \bar{A} \) in terms of \( h \) and \( \bar{h} \) from \( H \), \( A = h \partial h^{-1} \), \( \bar{A} = \bar{h} \bar{\partial} \bar{h}^{-1} \) one can represent (4.3) as the difference of the two manifestly gauge-invariant terms: the ungauged WZW actions corresponding to the group \( G \) and the subgroup \( H \),

\[ I(g, A) = kI_0(h^{-1} g \bar{h}) - kI_0(h^{-1} \bar{h}) . \]  

(4.4)

This representation implies that the gauged WZW model corresponds to a conformal theory (coset CFT [48,49]). Fixing a gauge on \( g \) and changing the variables to \( g' = h^{-1} g \bar{h} , \ h' = h^{-1} \bar{h} \) we get a sigma model on the group space \( G \times H \) which is conformal to all orders in a particular ‘leading-order’ scheme. That means that the 1-loop group space solution remains exact solution in that scheme. Replacing (4.4) with the ‘quantum’ action with renormalised levels \( k \rightarrow k + \frac{1}{2} c_G \) and \( k \rightarrow k + \frac{1}{2} c_H \) does not change this conclusion. This replacement corresponds to starting with the theory formulated in the ‘CFT’ scheme in which, e.g., the exact central charge of the WZW model is reproduced by the first non-trivial correction [50,28] and the metric \((k + \frac{1}{2} c_G) G_{\mu\nu} \) is the one that appears in the CFT Hamiltonian \( L_0 \) considered as a Klein-Gordon operator.
4.1. ‘CFT’ and ‘leading-order’ schemes

To obtain the corresponding sigma model in the ‘reduced’ $G/H$ configuration space (with coordinates being parameters of gauge-fixed $g$) one needs to integrate out $A, \tilde{A}$ (or, more precisely, the WZW fields $h, \tilde{h}$). This is a non-trivial step and the form of the result depends on a choice of a scheme in which the original ‘extended’ $(g, h, \tilde{h})$ WZW theory is formulated.

Suppose first the latter is taken in the leading-order scheme with the action (4.4). Then the result of integrating out $A, \tilde{A}$ and fixing a gauge takes the form of the sigma model (4.1) where the sigma model metric and dilaton are then given by (see [7] and refs. there)

$$G'_{\mu\nu} = G_{\mu\nu} - 2\alpha' \partial_\mu \phi \partial_\nu \phi, \quad \phi = \phi_0 - \frac{1}{2} \ln \det F.$$  (4.5)

$G_{\mu\nu}$ is the metric obtained by solving for $A, \tilde{A}$ at the classical level and $\phi_0$ is the original constant dilaton. Since the $\alpha'$-term in the metric can be eliminated by a field redefinition we conclude that there exists a leading-order scheme in which the leading-order gauged WZW sigma model background $(G, B, \phi)$ remains an exact solution. The leading-order scheme for the ungauged WZW sigma model is thus related to the leading-order scheme for the gauged WZW $\sigma$-model by an extra $2\alpha' \partial_\mu \phi \partial_\nu \phi$ redefinition of the metric. This provides a general explanation for the observations in [50,28] about the existence of a leading-order scheme for particular $D = 2, 3$ gauged WZW models.

If instead we start with the $(g, h, \tilde{h})$ WZW theory in the CFT scheme, i.e with the action

$$I(g, A) = (k + \frac{1}{2} c_G) \left[ I_0(h^{-1} g \tilde{h}) - \frac{k + \frac{1}{2} c_H}{k + \frac{1}{2} c_G} I_0(h^{-1} \tilde{h}) \right],$$  (4.6)

then the resulting sigma model couplings will explicitly depend on $1/k \sim \alpha'$ (and will agree with the coset CFT operator approach results [11,51,52]). While in the WZW model the transformation from the CFT to the leading order scheme is just a simple rescaling of couplings, this transformation becomes non-trivial at the level of gauged WZW $\sigma$-model. It is the ‘reduction’ of the configuration space resulting from integration over the gauge fields $A, \tilde{A}$ that is responsible for a complicated form of the transformation law between the ‘CFT’ and ‘leading-order’ schemes in the gauged WZW $\sigma$-models (in particular, this transformation involves dilaton terms of all orders in $1/k$, see below).³

³ An exception is provided by some $\sigma$-models obtained by nilpotent gauging: here the second term in (4.6) is absent by construction [12]. The background fields do not receive non-trivial $1/k$ corrections even in the CFT scheme, i.e. the relation between the leading-order and CFT schemes is equivalent to the one for the ungauged WZW model. The same is true for the $D = 3$ $F$-model or the extremal limit of the $SL(2, R) \times R/R$ coset.
The basic example is the $SL(2, R)/R$ gauged WZW model (or $D = 2$ black hole) \[8, 53, 54\]. The (euclidean) background in the leading-order scheme

\[ ds^2 = dx^2 + \tanh^2 bx \ d\theta^2 , \quad \phi = \phi_0 - \ln \cosh bx , \]  

(4.7)
is related to the background in the CFT scheme \[11\]

\[ ds^2 = dx^2 + \frac{\tanh^2 bx}{1 - p \tanh^2 bx} \ d\theta^2 , \]

(4.8)
\[ \phi = \phi_0 - \ln \cosh bx - \frac{1}{4} \ln (1 - p \tanh^2 bx) , \]

(4.9)
\[ p \equiv \frac{2}{k} , \quad \alpha'b^2 = \frac{1}{k-2} , \quad D - 26 + 6\alpha'b^2 = \frac{3k}{k-2} - 1 - 26 = 0 , \quad D = 2 , \]

(4.10)
by the following local covariant redefinition \[50\]

\[ G^{(lead)}_{\mu\nu} = G^{(cft)}_{\mu\nu} - \frac{2\alpha'\partial_\mu \phi \partial_\nu \phi}{1 + \frac{1}{2}\alpha'R} + \frac{2\alpha' (\partial \phi)^2 G_{\mu\nu}}{1 + \frac{1}{2}\alpha'R} , \]

(4.11)
\[ \phi^{(lead)} = \phi^{(cft)} - \frac{1}{4} \ln (1 + \frac{1}{2}\alpha'R) . \]

It should be emphasized that the two backgrounds related by (4.11) describe the same string geometry since the probe of the geometry is the tachyon field and the tachyon equation remains the same differential equation implied by coset CFT (even though it looks different being expressed in terms of different $G$ and $\phi$).

Similar remarks apply to the a particular $D = 3$ solution – the ‘charged black string’ corresponding to $[SL(2, R) \times R]/R$ gauged WZW model \[29\]. Its form in the leading-order scheme can be obtained by making a duality rotation of the neutral black string, i.e. the direct product of the $D = 2$ black hole and a free scalar theory.

4.2. $D = 4$ solutions

In general, the backgrounds obtained from the gauged WZW models have very few (at most, abelian) symmetries and thus cannot directly describe non-trivial $SO(3)$-symmetric backgrounds which are of interest in connection with $D = 4$ cosmology and black holes. Let us mention the explicitly known $D = 4$ solutions with $(-, +, +, +)$ signature. A class of axisymmetric black-hole-type and anisotropic cosmological backgrounds is found by starting with $[G_1 \times G_2]/[H_1 \times H_2]$ gauged WZW model with various combinations of $G_i = SL(2, R)$ or $SU(2)$ and $H_i = R$ or $U(1)$ \[10, 55, 56, 57\]. The leading-order form of these
backgrounds can be obtained by applying the $O(2, 2)$ duality transformation to the direct product of the Euclidean and Minkowski $D = 2$ black holes or their analytic continuations. The stationary black-hole-type background is \[ \text{(10)} \]

$$
\begin{align*}
  ds^2 &= -\frac{g_1(y)}{g_1(y)g_2(z) - q^2} \, dt^2 + \frac{g_2(z)}{g_1(y)g_2(z) - q^2} \, dx^2 + dy^2 + dz^2, \\
  B_{tx} &= \frac{q}{g_1(y)g_2(z) - q^2}, \quad \phi = \phi_0 - \frac{1}{2} \ln \left( \sinh^2 y \sinh^2 z \left[ g_1(y)g_2(z) - q^2 \right] \right), \\
  B_{yz} &= \frac{q}{g_1(y)g_2(z) - q^2}, \quad \phi = \phi_0 - \frac{1}{2} \ln \left( \cos^2 t \cosh^2 x \left[ g_1(t)g_2(x) + q^2 \right] \right),
\end{align*}
$$

where $g_1 = \coth^2 y$, $g_2 = \coth^2 z$. The cosmological background \[ \text{(56)} \] can be obtained by the analytic continuation and renaming the coordinates

$$
\begin{align*}
  ds^2 &= -dt^2 + dx^2 + \frac{g_1(t)}{g_1(t)g_2(x) + q^2} \, dy^2 + \frac{g_2(x)}{g_1(t)g_2(x) + q^2} \, dz^2, \\
  B_{tx} &= \frac{q}{g_1(x)g_2(t) + q^2}, \quad \phi = \phi_0 - \frac{1}{2} \ln \left( \cos^2 t \cosh^2 x \left[ g_1(t)g_2(x) + q^2 \right] \right),
\end{align*}
$$

where $g_1 = \tan^2 t$, $g_2 = \tanh^2 x$. A non-trivial (‘non-direct-product’) $D = 4$ background without isometries is obtained from $SO(2, 3)/SO(1, 3)$ gauged WZW model \[ \text{(38)} \].

Two other classes of $D = 4$ Minkowski signature ‘coset’ solutions will be discussed in more detail in the following sections 5.1 and 6.2. The first includes plane-wave-type backgrounds which are obtained from non-semisimple versions of $R \times SU(2)$ and the above $[SL(2, R) \times SL(2, R)]/\{R \times R\}$, etc. ‘product’ cosets \[ \text{(15,16)} \]. The second contains $F$-models (backgrounds with 2 null Killing vectors) which follow from nilpotent gauging of rank 2 maximally non-compact groups \[ \text{(12)} \].

5. Solutions with covariantly constant null Killing vector

The string action in a background with a covariantly constant null Killing vector can be represented in the following general form ($i, j = 1, ..., N$)

$$
L = \partial u \bar{\partial} v + K(u, x) \partial u \bar{\partial} u + A_i(u, x) \partial u \bar{\partial} x^i + \bar{A}_i(u, x) \bar{\partial} u \partial x^i + (G_{ij} + B_{ij})(u, x) \partial x^i \bar{\partial} x^j + \alpha' \mathcal{R} \phi(u, x),
$$

$$
I = \frac{1}{\pi \alpha'} \int d^2 z \, L, \quad \mathcal{R} \equiv \frac{1}{4} \sqrt{\mathcal{g}} R^{(2)}.
$$
$K, A_i, A_\bar{i}$ can be eliminated (locally) by a coordinate and $B_\mu{}^\nu$-gauge transformation \cite{59} so that the general form of the Lagrangian is

$$L = \partial u \bar{\partial} v + (G_{ij} + B_{ij})(u, x) \partial x^i \bar{\partial} x^j + \alpha' R \phi(u, x),$$

(5.2)

with $K, A_i, A_\bar{i}$ been now ‘hidden’ in a possible general coordinate and gauge transformation. If the ‘transverse’ space is trivial (for fixed $u$) one may use (5.2) with ‘flat’ $G_{ij}, B_{ij}$ taken in the most general frame, or may choose a special frame where $G_{ij} = \delta_{ij}, B_{ij} = 0$ and then go back to (5.1) (which may be preferable for a global coordinate choice).

There are two possibilities to satisfy the conditions of conformal invariance. The first is realised in the case when the ‘transverse’ theory is conformally invariant. The second option is to include the term linear in $v$ to the dilaton field \cite{6}. Then the conformal invariance conditions are satisfied provided the ‘transverse’ couplings $G, B$ depend on $u$ according to the standard RG equations with the $\beta$-functions of the ‘transverse’ theory. Since these $\beta$-functions are not known in a closed form (except for some supersymmetric sigma model cases \cite{60}) one is unable to determine the explicit all-order form of the solution. For this reason here we shall consider only the first possibility.

5.1. Plane waves

The simplest special case is that of the ‘plane-wave’ backgrounds for which $G, B, \phi$ in (5.2) do not depend on $x$ (i.e., in particular, the transverse metric is flat). The conformal invariance condition then reduces to one equation

$$-\frac{1}{2} G^{ij} \dddot{G}_{ij} + \frac{1}{4} G^{ij} G^{mn}(\dddot{G}_{im} \dddot{G}_{jn} - \dddot{B}_{im} \dddot{B}_{jn}) + 2 \dddot{\phi} = 0. \tag{5.3}$$

$G_{ij}(u), B_{ij}(u)$ and $\phi(u)$ satisfying (5.3) represent exact classical solutions of string theory which transform into each other under the full duality symmetry group $O(N, N|Z)$ \cite{42}.

A subclass of such models admits a coset CFT interpretation in terms of gauged WZW models for non-semisimple groups \cite{15,16}. The latter can be obtained from standard semisimple gauged WZW models by taking special singular limits. The singular procedure involves a coordinate transformation and a rescaling of $\alpha'$ and is carried out directly at the level of the string action \cite{16}. As a result, one is able not only to obtain the plane-wave

\footnote{It is similar to the transformation in \cite{61} which maps any space-time into some plane wave.}
background fields but also to relate the corresponding CFT’s (i.e. to construct a ‘nonsemisimple’ coset CFT from a ‘semisimple’ coset CFT) and thus to give a CFT description to this subclass of plane-wave solutions.

In particular, a set of $D = 4$ plane wave solutions that can be obtained in this way from gauged $[SU(2) \times SL(2, R)]/[U(1) \times R]$ WZW models is given by (cf. (4.13))

$$ds^2 = du dv + \frac{g_1(u')}{g_1(u')g_2(u) + q^2} dx_1^2 + \frac{g_2(u)}{g_1(u')g_2(u) + q^2} dx_2^2,$$

$$B_{12} = \frac{q}{g_1(u')g_2(u) + q^2}, \phi = \phi_0 - \frac{1}{2} \ln \left( f_1^2(u')f_2^2(u)[g_1(u')g_2(u) + q^2] \right),$$

where $u' = au + d, (a, d = \text{const})$ and the functions $g_i, f_i$ can take any pairs of the following values

$$g(u) = 1, u, \tanh^2 u, \tan^2 u, u^{-2}, \coth^2 u, \cot^2 u,$$

$$f(u) = 1, 1, \cosh u, \cos u, u, \sinh u, \sin u.$$  

A particular case is $g_1 = 1, g_2 = u^2$ (this background is dual to flat space). Another special case is the $E_2$ WZW model of Nappi and Witten

$$L = \partial v \bar{\partial} u + \partial x_1 \bar{\partial} x_1 + \partial x_2 \bar{\partial} x_2 + 2 \cos u \partial x_1 \bar{\partial} x_2,$$

which can be obtained by a singular limit from the WZW action for $SU(2) \times R$.

For $B_{ij} = 0$ (5.2) can be transformed into the familiar form (i.e. (5.1) with $G_{ij} = \delta_{ij}, A_i = \bar{A}_i = B_{ij} = 0$)

$$L = \partial u \bar{\partial} v + K(u, x) \bar{\partial} u \bar{\partial} u + \partial x^i \bar{\partial} x_i + \alpha' R \phi(u).$$

The fact that this model has a covariantly constant null vector can be used to give a simple geometrical argument that leading order solutions are exact when $G_{ij} = \delta_{ij}, B_{ij} = 0$, and $\phi$ depends only on $u$ [4]. This is because the curvature contains two powers of the constant null vector $l$, and derivatives of $\phi$ are also proportional to $l$. One can thus show that all higher order terms in the equations of motion vanish identically. The one-loop conformal invariance condition

$$-\frac{1}{2} \partial^i \partial_i K + 2 \partial_u^2 \phi = 0,$$

is then the exact one and is solved by the standard plane-wave ansatz [3,4] $K = w_{ij}(u) x^i x^j$.

A different rotationally symmetric solution exists for $\phi = \text{const}$

$$K = 1 + \frac{M}{r^{D-4}} , \ D > 4; \ K = 1 - M \ln r , \ D = 4; \ r^2 \equiv x_i x^i > 0 , \ \phi = \text{const}.$$  

This background is dual to the fundamental string solution [62] and describes a string boosted to the speed of light.

---

$E_2^c$ is a central extension of the Euclidean algebra in two dimensions admitting a non-degenerate invariant bilinear form.
5.2. Generalised plane waves

If one sets \( G_{ij} = \delta_{ij}, \ B_{ij} = 0 \) in (5.1) but keeps \( A_i, \bar{A}_i \) non-vanishing the conditions of conformal invariance take the form [6]

\[
\partial^j F_{ij} = 0, \quad \partial^j \bar{F}_{ij} = 0, \quad (5.10)
\]

\[
-\frac{1}{2} \partial^2 K + F^{ij} \bar{F}_{ij} + \partial^i \partial_u (A_i + \bar{A}_i) + 2 \partial_u^2 \phi + O(\alpha'^s (\partial^s F)^2, \alpha'^s (\partial^s \bar{F})^2) = 0.
\]

(5.11)

It is clear that one-loop expression simplifies if \( A_i = 0 \) or \( \bar{A}_i = 0 \). Similar simplification happens at the two-loop level [63]. Plane waves with \( \bar{A}_i = 0 \) were shown to preserve (half of) supersymmetry and to have certain \( \alpha' \)-corrections to vanish in the context of superstring effective action [64]. In fact, it is possible to show [13] that such ‘chiral’ plane wave backgrounds with \( \bar{A}_i = 0 \) (or \( A_i = 0 \))

\[
L = \partial u \bar{\partial} v + K(u, x) \partial u \bar{\partial} u + A_i(u, x) \partial u \bar{\partial} x^i + \partial x^i \bar{\partial} x^i + \alpha' R \phi(u, x),
\]

(5.12)

are exact solutions of bosonic string theory provided \( K, A_i, \phi \) satisfy the one-loop conformal invariance conditions,

\[
-\frac{1}{2} \partial^2 K + \partial^i \partial_u A_i + 2 \partial_u^2 \phi = 0, \quad \partial^j F_{ij} = 0, \quad (5.13)
\]

i.e. all higher order terms in (5.11) are actually of the \( FF \)-structure \( O(\alpha'^s \partial^s F \partial^k \bar{F}) \), i.e. vanish for \( \bar{A}_i = 0 \).

5.3. K-model

Another special case of (5.1) is found when \( K, G_{ij}, B_{ij}, \phi \) do not depend on \( u \). This is a generalisation of the plane wave solutions called the ‘K-model’ in [7]

\[
L_K = \partial u \bar{\partial} v + K(x) \partial u \bar{\partial} u + (G_{ij} + B_{ij})(x) \partial x^i \bar{\partial} x^j + \alpha' R \phi(x).
\]

(5.14)

In general, suppose that the transverse space is known to be an exact string solution in some scheme. Then the model (5.14) is be conformal for \( K = 0 \). But the curvature of the metric with \( K \neq 0 \) is equal to the curvature of the metric with \( K = 0 \) plus a term of the form \( (\nabla \nabla K) l l \). Unfortunately, this can result in nontrivial corrections to the equations of motion at each order of \( \alpha' \). These corrections will be linear in \( K \), so one learns that the
exact equation for $K$ will also be linear. The exact form of the equation for $K$ turns out to be the following \[^7\]

$$-\omega K + \partial^i \phi \partial_i K + 2 \partial^2_\phi \phi = 0 , \quad \omega = \frac{1}{2} \nabla^2 + O(\alpha') , \quad (5.15)$$

where $\omega$ is the scalar anomalous dimension operator which in general contains $(G_{ij}, B_{ij})$-dependent corrections to all orders in $\alpha'$ (only a few leading $\alpha'^m$-terms in it are known explicitly, see e.g. \[^50\] and refs. there).

Given an exact string solution $(G_{ij}, B_{ij}, \phi)$, in general, we would still be unable to determine the exact expression for $K$ because of the unknown higher order terms in (5.13). There are, however, special cases when this is possible. The simplest one is that of the flat transverse space with the dilaton being linear in the coordinate $x$, $\phi = \phi_0(u) + b_i(u)x^i$ which is an obvious generalisation of the plane wave case (5.8). It is possible to obtain more interesting new exact solutions when the CFT behind the ‘transverse’ space solution $(G_{ij}, B_{ij}, \phi)$ is nontrivial but still known explicitly \[^7\]. In fact, in that case the structure of the ‘tachyonic’ operator $\omega$ is determined by the zero mode part of the CFT Hamiltonian, or $L_0$-operator. Fixing a particular scheme (e.g., the ‘CFT’ one where $L_0$ has the standard Klein-Gordon form with the dilaton term) one is able, in principle, to establish the form of the background fields $(G_{ij}, B_{ij}, \phi)$ and $K$. This produces ‘hybrids’ of gauged WZW and plane wave solutions.

To obtain a *four dimensional* hybrid solution \[^7\] one must start with a two dimensional conformal $\sigma$-model. Essentially the only non-trivial possibility is the $SL(2, R)/U(1)$ gauged WZW model which describes the two dimensional euclidean black hole, i.e. the resulting solution will be the *generic* $D=4$ $K$-model. To construct this new solution we must use all the zero-mode information provided by the $SL(2, R)/U(1)$ coset: the exact metric and dilaton and the form of the tachyon equation. This will give us the all-order form of all the functions in the $D=4$ model.

The exact background fields of the $SL(2, R)/U(1)$ model in the CFT scheme were given in (4.8),(4.9). In the CFT scheme the tachyonic equation has the standard uncorrected form, so that the function $K(x)$ must satisfy

$$-\frac{1}{2} \nabla^2 K + \partial^i \phi \partial_i K = -\frac{1}{2\sqrt{G}e^{-2\phi}}\partial_i(\sqrt{G}e^{-2\phi}G^{ij}\partial_j)K = 0 . \quad (5.16)$$

A particular solution of (5.16) with $K$ depending only on $x$ and not on $\theta$ is

$$K = a + m \ln \tanh bx . \quad (5.17)$$
The constants $a, m$ can be absorbed into a redefinition of $u$ and $v$, so that the full exact $D = 4$ metric is

$$ds^2 = dudv + \ln \tanh bx \ du^2 + dx^2 + \frac{\tanh^2 bx}{1 - p \tanh^2 bx} \ d\theta^2,$$  \hspace{1cm} (5.18)

while the dilaton is unchanged. This metric is asymptotically flat, being a product of $D = 2$ Minkowski space with a cylinder at infinity.

The solution for $K$ (5.17) is the same in the ‘leading-order’ scheme where the metric and dilaton do not receive $\alpha'$ corrections. The point is that the tachyon operator remains the same differential operator, it is only its expression in terms of the new $G, \phi$ that changes. Thus, in the ‘leading-order’ scheme we get the following exact $D = 4$ solution

$$ds^2 = dudv + \ln \tanh bx \ du^2 + dx^2 + \tanh^2 bx \ d\theta^2, \quad \phi = \phi_0 - \ln \cosh bx.$$  \hspace{1cm} (5.19)

In addition to the covariantly constant null vector $\partial/\partial v$, this solution has two isometries corresponding to shifts of $u$ and $\theta$. Hence we can consider two different types of duals. Dualizing with respect to $\theta$ gives

$$ds^2 = dudv + \ln \tanh bx \ du^2 + dx^2 + \coth^2 bx \ d\theta^2, \quad \phi = \phi_0 - \ln \sinh bx.$$  \hspace{1cm} (5.20)

Starting with the general solution for $K$ (5.17) one finds the $u$-dual background [7] which will be discussed in section 6.3.

6. **$F$-model solutions**

The simplest $F$-model [12,7] describes a family of backgrounds with metric and antisymmetric tensor characterized by a single function $F(x)$ and dilaton $\phi(x)$:

$$ds^2 = F(x)dudv + dx_i dx^i, \quad B_{uv} = \frac{1}{2}F(x).$$  \hspace{1cm} (6.1)

The two functions $F$ and $\phi$ depend only on the transverse coordinates $x^i$. The leading order string equations then reduce to [12]

$$\partial^2 F^{-1} = 2b^i \partial_i F^{-1}, \quad \phi = \phi_0 + b_i x^i + \frac{1}{2} \ln F(x),$$  \hspace{1cm} (6.2)

where $b_i$ is a constant vector. Some of the solutions to (6.2) were shown to correspond to gauged WZW models where the subgroup being gauged is nilpotent [12] (see section 6.2
It was argued that like ungauged WZW models they should not receive non-trivial higher order corrections even in the CFT scheme. Though it is unlikely that all solutions to (6.2) can be obtained from a gauged WZW model one can show [7] that there exists a scheme in which all of these solutions are exact and receive no $\alpha'$ corrections. Since the equation for $F^{-1}$ is linear, linear combinations of these solutions yield new exact solutions.

One of the most interesting solutions in this class is the ‘fundamental string’ [14] which has $b_i = 0$ and

$$F^{-1} = 1 + \frac{M}{r^{D-4}} , \quad D > 4 \; ; \; \quad F^{-1} = 1 - M \ln r , \quad D = 4 , \quad r^2 = x_i x^i ,$$

(6.3)

where $D$ is the number of spacetime dimensions. This solution describes the field outside of a straight fundamental string located at $r = 0$.

6.1. General $F$-model

The general $F$-model has a curved transverse space, i.e. is defined as [7]

$$L_F = F(x)\partial u \partial \bar{v} + (G_{ij} + B_{ij})(x) \partial^i x^j + \alpha' R \phi(x) .$$

(6.4)

This model is related by leading order duality to $K$-model (5.14): the dual of it with respect to $u$ is (6.4) with $F = K^{-1}$. The $F$-model has a large symmetry group. It is invariant under the three Poincare transformations in the $u,v$ plane. Moreover, it is invariant under the infinite-dimensional symmetry $u \to u + f(\tau + \sigma)$, $v \to v + h(\tau - \sigma)$, i.e. it has two chiral currents.

The all-order conformal invariance conditions for $F$-model are satisfied [7] provided one is given a conformal ‘transverse’ theory $(G', B', \phi')$ and

$$G_{ij} = G'_{ij} + \frac{1}{2} \alpha' \partial_i \ln F \partial_j \ln F , \quad \phi = \phi' + \frac{1}{2} \ln F , \quad B_{ij} = B'_{ij} ,$$

(6.5)

with $F$ satisfying

$$-\omega' F^{-1} + \partial^i \phi' \partial_i F^{-1} = -\frac{1}{2} \nabla^2 F^{-1} + O(\alpha') + \partial^i \phi' \partial_i F^{-1} = 0 .$$

(6.6)

Here $\omega'$ is the anomalous dimension operator depending on $G'$. When $(G', B, \phi')$ correspond to a known CFT this equation can be written down explicitly to all orders in $\alpha'$. Since the relation between $G$ and $G'$ is local and the transverse theory is, in general,
defined modulo local coupling redefinitions, one can argue [1] that there exists a (‘leading-order’) scheme in which the exact solution is represented by the conformal transverse theory $(G, B, \phi')$ and $F$ satisfying (6.6).

The simplest example of the conformal transverse model is the flat space with a linear dilaton, i.e. the corresponding $F$-model is represented (in the leading-order scheme) by

$$G_{ij} = \delta_{ij}, \quad \phi = \phi_0 + b_i x^i + \frac{1}{2} \ln F.$$  

(6.7)

In this scheme, the exact form of the equation for the function $F$ is simply

$$-\frac{1}{2} \partial^2 F^{-1} + b^i \partial_i F^{-1} = 0,$$

(6.8)

i.e. the model

$$L_F = F(x) \partial u \bar{\partial} v + \partial x^i \bar{\partial} x_i + \alpha' \mathcal{R}(\phi_0 + b_i x^i + \frac{1}{2} \ln F),$$

(6.9)

with $F$ satisfying (6.8) is conformally invariant to all orders, i.e. gives an exact string solution.

In this scheme the leading-order duality is exact since the leading-order dual to (6.9) is the $K$-model

$$L_K = \partial u \bar{\partial} v + F^{-1}(x) \partial x^i \bar{\partial} x_i + \alpha' \mathcal{R}(\phi_0 + b_i x^i),$$

(6.10)

which represents an exact solution if $F$ solves (6.8) (cf.(5.8) with $\phi = 0$). In particular, one concludes that there exists a scheme in which the FS solution (6.3) is a classical string solution to all orders in $\alpha'$.

6.2. $F$-models obtained by nilpotent gauging of WZW models

The conclusion about the existence of a scheme where the $F$-model (6.9) represents an exact string solution is consistent with the result of [12] that the particular $F$-model with

$$F^{-1} = \sum_{i=1}^{N} \epsilon_i e^{\alpha_i \cdot x}, \quad \phi = \phi_0 + \rho \cdot x + \frac{1}{2} \ln F,$$

(6.11)

can be obtained from a $G/H$ gauged WZW model. Here the constants $\epsilon_i$ take values 0 or $\pm 1$, $\alpha_i$ are simple roots of the algebra of a maximally non-compact Lie group $G$ of rank $N = D - 2$ and $\rho = \frac{1}{2} \sum_{s=1}^{m} \alpha_s$ is half of the sum of all positive roots. $H$ is a nilpotent subgroup of $G$ generated by $N-1$ simple roots (this condition on $H$ is needed to get
models with one time-like direction). The flat transverse coordinates $x^i$ are the Cartan subalgebra directions.

The corresponding ‘null’ gauging is based on the Gauss decomposition and directly applies only to the groups with the algebras that are the ‘maximally non-compact’ real forms of the classical Lie algebras (real linear spans of the Cartan-Weyl basis), i.e. $sl(N+1, R)$, $so(N, N+1)$, $sp(2N, R)$, $so(N, N)$. These WZW models can be considered as natural generalisations of the $SL(2, R)$ WZW model. For such groups there exists a real group-valued Gauss decomposition

$$g = \exp\left(\sum_{\Phi_+} u^\alpha E_\alpha\right) \exp\left(\sum_{i=1}^N x^i H_i\right) \exp\left(\sum_{\Phi_+} v^\alpha E_-\alpha\right).$$

$\Phi_+$ and $\Delta$ are the sets of the positive and simple roots of a complex algebra with the Cartan-Weyl basis consisting of the step operators $E_\alpha$, $E_-\alpha$, $\alpha \in \Phi_+$ and $N (= \text{rank} G)$ Cartan subalgebra generators $H_\alpha = \alpha^i H_i$, $\alpha \in \Delta$.

The four dimensional ($D = 2 + N = 4$) models are obtained for each of the rank 2 maximally non-compact groups: $SL(3, R)$, $SO(2, 3) = Sp(4, R)$, $SO(2, 2) = SL(2, R) \oplus SL(2, R)$ and $G_2$. In the rank 2 case the corresponding background is parametrised by a $2 \times 2$ Cartan matrix or by two simple roots with components $\alpha_{1i}$ and $\alpha_{2i}$ and one parameter $\epsilon = \epsilon_2/\epsilon_1$ with values $\pm 1$,

$$L = \frac{1}{e^{\alpha_{1} \cdot x} + \epsilon e^{\alpha_{2} \cdot x}} \partial u \partial \bar{v} + \partial x_1 \bar{\partial} x_1 + \partial x_2 \bar{\partial} x_2$$

$$+ \alpha' \mathcal{R}[\phi_0 + \frac{1}{2}(\alpha_1 \cdot x + \alpha_2 \cdot x) - \frac{1}{2} \ln(e^{\alpha_1 \cdot x} + \epsilon e^{\alpha_2 \cdot x})].$$

In addition to the Poincare symmetry in the $u, v$ plane this model is also invariant under a correlated constant shift of $x^i$ and $u$ (or $v$). For example, in the case of $SL(3, R)$ $\alpha_1 = (\sqrt{2}, 0)$, $\alpha_2 = (-\frac{1}{\sqrt{2}}, \frac{3}{\sqrt{2}})$. For $\epsilon = 1$ the curvature is non-singular in the coordinate patch used in (6.1), but there is also a horizon so one should first consider the geodesic completion. It is likely that there is still no curvature singularity, as in the case of the $SL(2, R)$ group space.

The presence of the two chiral currents implies that the classical equations of these $\sigma$-models (i.e. of the string propagation in these backgrounds) reduce to the Toda equation for $x^i$

$$\partial \bar{\partial} x_i + \frac{1}{2} \chi \partial_i F^{-1} = 0, \quad F \bar{\partial} v = \nu(\bar{z}), \quad F \partial u = \mu(z), \quad \chi \equiv \nu(\bar{z})\mu(z).$$

(6.13)
χ can be made constant by a coordinate transformation. Then the solutions (including
the solutions of the constraints) can be expressed in terms of the Toda model solutions.

Let us mention also that there exists another example of F-model which corresponds
to a gauged WZW model: the general F-model in \(D = 3\). Solving (6.8) in \(D = 3\) one
finds that \(F^{-1} = a + me^{kx}\). As shown in [7], this model can be obtained from a special
\(SL(2, R) \times R/R\) gauged WZW model and, at the same time, is the extremal limit of the
charged black string [29] solution. The case of \(a = 0\) corresponds to the ungauged \(SL(2, R)\)
WZW model (with the action written in the Gauss decomposition parametrisation).

6.3. \(D = 4\) F-model with 2-dimensional euclidean black hole as the transverse part

It is possible to find another exact \(D = 4\) F-model solution by taking the transverse
two dimensional theory to be non-trivial, i.e. represented by the \(D = 2\) euclidean black
hole model. This solution is related to the \(K\)-model with \(K\) given by (5.17) by the \(u\)-duality
transformation [6]

\[
ds^2 = (a + m \ln \tanh bx)^{-1}dudv + dx^2 + \tanh^2 bx \, d\theta^2 ,
\]

\[
B_{uv} = \frac{1}{2}(a + m \ln \tanh bx)^{-1} , \quad \phi = \phi_0 - \ln \cosh bx + \frac{1}{2} \ln F .
\]

This background can be viewed as a generalisation of the fundamental string (6.3) in four
dimensions, \(F^{-1} = 1 - M \ln r\). In addition to the usual singularity at \(r = 0\) there is another
singularity outside the string at nonzero \(r\). The solution (6.14) has the same singularity
at the origin (and hence can be viewed as the field outside a fundamental string) but is
regular elsewhere and even asymptotically flat. The original fundamental string solution
can be recovered by taking the limit \(b \to 0\) which is consistent since the central charge
condition is now imposed only at the level of the full \(D = 4\) solution.

7. Further generalisations

Let us consider the following generalisation of both the \(K\)-model and F-model with
flat transverse space (cf. (5.12),(6.4))

\[
L = F(x)\partial u \bar{\partial}v + K(x)\partial u \bar{\partial}u + \tilde{A}_i(x)\partial u \bar{\partial}x^i + \partial x^i \bar{\partial}x^i + \alpha' R \phi(x) ,
\]

or \((\tilde{A}_i = FA_i)\)

\[
L = F(x)\partial u[\bar{\partial}v + A_i(x)\bar{\partial}x^i] + K(x)\partial u \bar{\partial}u + \partial x^i \bar{\partial}x^i + \alpha' R \phi(x) .
\]
This model has one null Killing vector, is invariant under the gauge transformations of \( A_i \) combined with a shift of \( v \) and is ‘self-dual’. In fact, it transforms into itself under the duality transformation along a direction in the \((u,v)\)-plane

\[
F' = \frac{F}{K + qF}, \quad K' = \frac{1}{K + qF}, \quad A'_i = A_i, \quad q = \text{const}.
\] (7.3)

Just as for the \( F \)-model it is possible to prove \cite{13} that there exists a scheme in which this model is conformally invariant to all orders provided the one-loop conformal invariance conditions are satisfied. The latter are given by

\[
\partial^2 F^{-1} = 0, \quad \partial^2 (KF^{-1}) = 0, \quad \phi = \phi_0 + \frac{1}{2} \ln F(x), \quad \partial_i F^{ij} = 0.
\] (7.4)

There exists also a straightforward generalisation to the case when \( K \) depends on \( u \). As a result, one is able to find many new interesting exact solutions \cite{13}, in particular, various generalisations of the fundamental string solution (some of them were already obtained as the leading-order solutions in \cite{65,66,67}). In the case of solutions with the spherical symmetry in the transverse space the solution for \( K \) is \( K = k + cF \) so that one can redefine \( v \) to make \( K = k = \text{const} \).

8. Concluding remarks

We have discussed several new classes of exact classical solutions in bosonic string theory. They can be embedded into the closed superstring theory and thus also in the heterotic string theory (introducing appropriate gauge field background equal to the spin connection). All of the \( K \)-model and \( F \)-model solutions and their generalisations have at least one null Killing vector. This indicates that backgrounds with null Killing vectors play special role in string theory. Though we have given several examples when such backgrounds can be derived from a gauged WZW model and thus correspond to a coset CFT, the question about CFT interpretation of general \( K \)-model and \( F \)-model-type solutions remains open. The fact that the exact expressions for these backgrounds are known explicitly certainly suggests that the underlying CFT’s should exist and have well-defined properties.

This may not, however, apply to the fundamental string solution. A peculiarity of the FS solution is that it admits two possible interpretations. It can be considered as a vacuum solution (an extremal limit of a general class of charged string solutions \cite{68}) of
the effective equations which is valid only outside the core \( r = 0 \). Alternatively, it can be derived (as, in fact, was done in the original approach of \( \cite{14} \)) as a solution corresponding to the combined action (we separate the constant part of the dilaton field \( g_0 = e^{\phi_0} \))

\[
\hat{S} = \frac{1}{g_0^2} S_{\text{eff}}(\varphi) + I_{\text{str}}(x; \varphi), \quad I_{\text{str}} = I_0(x) + V_i \varphi^i,
\]

(8.1)

containing both the effective action for the background fields \( \varphi^i = (G, B, \phi) \) (condensates of massless string modes) and the action \( I_{\text{str}} \) of a source string interacting with the background. The source action leads to the \( \delta(r) \)-term in the Laplace equation for \( F^{-1} \). Though such mixture of actions looks strange from the point of view of perturbative string theory, \( \hat{S} \) can in fact be considered as describing a non-perturbative ‘thin handle’ (or ‘wormhole’) approximation to quantum string partition function \( \cite{69} \):

\[
Z = \sum_{n=0}^{\infty} g_0^{2n-2} \int [dm]_n \int_{M_n} [dx] \exp(-I_0[x])
\]

(8.2)

\[
\approx \int [d\varphi] \int_{M_0} [dx] \exp[-\frac{1}{g_0^2} S_{\text{eff}}(\varphi) - I_{\text{str}}(x; \varphi)] + ...
\]

Here \( m \) denote moduli, \( M_n \) are 2-surfaces of genus \( n \) and dots indicate contributions of other parts of moduli spaces. Extrema of \( \hat{S} \) can thus be viewed as some non-perturbative solitonic solutions in string theory. This raises the question if they can actually be described by a conformal field theory since we only know that solutions of the tree-level effective equations \( \delta S_{\text{eff}}/\delta \varphi = 0 \) correspond to CFT’s.

Given that the effective string field equations contain terms of all orders in \( \alpha' \) their general solution can be represented as (\( \varphi \) stands for a set of ‘massless’ fields): \( \varphi = \varphi_0 + \alpha' \varphi_1 + \alpha'^2 \varphi_2 + ... \). Here \( \varphi_0 \) is the leading-order solution while \( \varphi_n \) are, in general, non-local (involving inverse Laplacians) functionals of \( \varphi_0 \). It may happen that for some special \( \varphi_0 \) the higher order corrections are local, covariant functionals of \( \varphi_0 \). In that case one is able to change the scheme (i.e. to make a local covariant field redefinition) so that in the new scheme \( \varphi_0 \) is actually an exact solution. It may be a bit disappointing to learn that all of the presently explicitly known exact string solutions are of that type, i.e. we do not yet know an example of a string solution with truly non-trivial \( \alpha' \)-dependence (the one which cannot be absorbed into a local field redefinition). It may happen that for some leading-order solutions (e.g. Schwarzschild) which can of course be deformed order by order in
\( \alpha' \) to make them satisfy the full string equations there is no regular (satisfying standard axioms) CFT which corresponds to the resulting \( \alpha' \)-series.

9. Acknowledgements

I would like to thank G. Horowitz, R. Kallosh, C. Klimčík and K. Sfetsos for useful discussions and collaborations. I acknowledge also the support of PPARC.

\footnote{Superstring solutions corresponding to \( \sigma \)-models with \( n = 2 \) world sheet supersymmetry are examples of the situation where this does not happen. The corresponding \( \beta \)-function equations contain non-trivial \( \alpha'^3 \) - and higher- order corrections \cite{70} which cannot be redefined away by a local redefinition of the metric since otherwise one would eliminate the \( R^4 \) - term in the effective action \cite{21} and thus change the string S-matrix. The fact that there exists a local redefinition of the Kahler potential \cite{71} seems to indicate that the \( n = 2 \) case is special compared to the general \( n = 1 \). In fact, here there are good reasons to expect that the resulting \( \alpha' \)-deformed solution corresponds to a regular \( n = 2 \) superconformal theory \cite{72}.}
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