The cat’s cradle, stirring, and topological complexity

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There are several physical situations in which the ‘tangling’ of a loop is relevant: the game of cat’s cradle is a simple example, but a more important application involves the stirring of a fluid by rods. Here we discuss how elementary topology constrains the types of mappings that can occur on a surface, for example when the surface is the domain of a two-dimensional fluid.

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In the children’s game of cat’s cradle, a loop of string is first anchored around both hands. The fingers of each hand are then inserted into the opposite hand’s loop and pulled apart, resulting in the situation shown in Figure 1. The motion of the fingers allows us to predict in advance just how long a piece of string we will need: the more intricate the motion, the more string is required. In fact the length of string is not a bad first definition of ‘topological complexity,’ and we will see how this simple concept can be applied to the study of dynamical systems, particularly in two dimensions. The game of cat’s cradle also suggests a connection between topology and combinatorics: after all, the number of distinct cat’s cradle shapes that can be made is essentially a difficult combinatorial problem. Throughout this short note, we will explore the interplay between topology and combinatorics, and see what it reveals about dynamical systems on surfaces.

On the practical side of things, this sort of approach to topological complexity is useful when studying stirring devices. Figure 2 shows a top-down view of such a device: the black disks are stirring rods immersed in a two-dimensional white fluid, assumed very viscous. The rods are moved following the prescription shown in the inset. The result is that an initial loop gets wrapped around the rods in a topologically complex manner, and is stretched exponentially (blue line). This is very good for the ultimate goal, which is to enhance the chemical reaction rate or blending of solutes: the blue line represents the contact area between the two reagents. The reaction rate is limited by the growth of this contact area, so having it grow exponentially is desirable.

Two handles are better than one

Figure 3 shows the double-torus $S_2$, beloved of low-dimensional topologists. The subscript 2 on $S_2$ stands for the number of handles, or genus. Though it isn’t encountered very much in the real world, it is a surface that’s complex enough to be interesting, unlike the humble
ordinary torus $S_1$ which is a bit too trivial. We are interested in transformations of a surface $S$, i.e., continuous mappings of $S$ to itself. Such a mapping, if also invertible, is called a homeomorphism of $S$. (We assume that the surface is orientable, and that the mapping preserves this orientation.) Now in topology one only cares about gross features, so we assume that two mappings are equivalent if they can be continuously deformed into each other: they are then called isotopic. An equivalence class of mappings under isotopy is called an isotopy class. The set of all such classes forms the mapping class group $\text{MCG}(S)$ of $S$, with the group operation obtained from composition of mappings.

The mapping class group of the torus $S_1$ is particularly simple: after we allow for isotopy, all that matters is how many times a mapping $\phi : S_1 \to S_1$ has caused us to wind around each of the two periodic directions of the torus. This winding is best illustrated by looking at the action of a mapping on two nontrivial disjoint closed curves drawn on the surface, as in Figure 4. The red loop around the thin direction of the torus is labeled $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, the blue loop $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. The mapping in Figure 4 can be written as $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$: its linear action transforms the red loop to $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and the blue loop to $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, which indicates the number of twists of each image loop around each periodic direction of the torus. In fact, in general we have $\text{MCG}(S_1) \simeq \text{PSL}_2(\mathbb{Z})$ (the “projective” version of the group of invertible two-by-two matrices with determinant 1), since we can wind in positive as well as negative directions, and the action of $\phi$ on these loops is commutative in the case of the torus. Note that since the loops initially only intersect once, then their images only intersects once as well.

According to the celebrated Thurston–Nielsen (TN) classification theorem there are only three types of isotopy classes: finite-order, reducible, and pseudo-Anosov. A mapping $\phi : S \to S$ is finite-order if $\phi^k$ is isotopic to the identity map, for some integer $k > 0$. The mapping $\phi$ is reducible if it leaves a set of disjoint closed curves invariant: the surface is carved up such that the pieces map to each other. Finite-order mappings are not interesting to us because they are too simple; reducible ones are not interesting because one can then reapply the TN theorem to the individual pieces (with boundaries) until the mapping is no longer reducible.

It is the third type — pseudo-Anosov — that is most dynamically interesting. For such a mapping, in addition to leaving no nontrivial closed curve “untouched” (otherwise it would be reducible), the action on loops causes their length to grow exponentially. The asymptotic factor by which the winding numbers of loops is multiplied (under repeated iteration) is called the dilatation, $\lambda > 1$. Its logarithm is the topological entropy. The topological entropy is
closely related to the growth rate of loops, as discussed for the cat’s cradle and stirring device examples in the introduction. The positive entropy of a pseudo-Anosov mapping tells us that loops will grow exponentially under repeated iteration, with growth rate \( \log \lambda \).

For the torus \( S_1 \), the TN classification is achieved merely by looking at the trace of an element of \( \text{MCG}(S_1) \cong \text{PSL}_2(\mathbb{Z}) \). If the absolute value of the trace is greater than 2, then the mapping is pseudo-Anosov; if less than 2, finite-order; if equal to 2, reducible. One can check this easily by using the Cayley–Hamilton theorem for two-by-two matrices of unit determinant. The mapping of the torus in Figure 4, for instance, is pseudo-Anosov with \( \lambda \approx 2.618 \).

Of polynomials and indices

Now let us return to the double-torus \( S_2 \). The mapping class group in this case is more complicated than for the torus, but we can get a handle on part of it by considering \( \text{MCG}^+(S_2) \) — the subset of the mapping class group that also leaves invariant a vector field (as opposed to a line field in the general case). In that case the mapping class group can be deduced from the action on the four loops drawn in Figure 3, so that \( \text{MCG}^+(S_2) \subset \text{SL}_4(\mathbb{Z}) \). But this time we do not have equality: there are many matrices that do not correspond to an element of the mapping class group. But which ones?

The matrices we want are in the symplectic group \( \text{Sp}_4(\mathbb{Z}) \). One important feature of these matrices is that their characteristic polynomial

\[
P(x) = x^4 + c_1 x^3 + c_2 x^2 + c_3 x + c_4
\]

satisfies \( c_3 = c_1, \ c_4 = 1 \), so that the polynomial is reciprocal — the coefficients read the same from left to right as from right to left. The largest root (in magnitude) of \( P(x) \) is the dilatation \( \lambda \); for a pseudo-Anosov mapping it is always real. We appeal to the Lefschetz fixed point theorem, which in our case says

\[
L(\phi) = 2 - \text{Tr}(\phi^*_s) = \sum_{p \in \text{Fix}(\phi)} \text{Ind}(\phi, p).
\]

Here, \( L(\phi) \) is an integer called the Lefschetz number of \( \phi \); \( \phi_s \in \text{Sp}_4(\mathbb{Z}) \) is a matrix representing the isotopy class of \( \phi \) by looking at its action on the loops in Figure 3; \( \text{Tr}(\phi^*_s) \) is the trace; \( \text{Fix}(\phi) \) is the set of isolated fixed points of \( \phi \); and \( \text{Ind}(\phi, p) \) is the topological index of \( \phi \) at the isolated fixed point \( p \). The topological index is given by the signed number of turns of a vector joining a point \( x \) and its image \( \phi(x) \) as \( x \) travels counterclockwise around a small circle enclosing the singularity (see Figure 5). The index can only assume two simple values: it is \( 1 - N \) if \( \phi \) fixes the \( 2N \) ‘hyperbolic sectors’ around a fixed point, and it is +1 (independent of \( N \)) if \( \phi \) permutes the sectors. This is due to the severe restrictions imposed by continuity and orientation-preservation.
Now comes the first crucial ingredient: the Cayley–Hamilton theorem and Newton’s formulas tell us how to find $\text{Tr}(\phi_x^k)$ recursively by looking at the polynomial $P(x)$:

$$\text{Tr}(\phi_x^k) = -\sum_{m=1}^{k-1} c_m \text{Tr}(\phi_x^{k-m}) - kc_k,$$

where $c_k = 0$ for $k > n$. Hence, we can calculate the Lefschetz numbers of all powers of $\phi$ directly from the polynomial. For example, for the polynomial $P(x) = x^4 - x^3 - x^2 - x + 1$ with dilatation $\lambda \simeq 1.72208$, we find

$$L(\phi) = 1, \quad L(\phi^2) = -1, \quad L(\phi^3) = -5, \quad L(\phi^4) = -5, \quad L(\phi^5) = -14, \quad L(\phi^6) = -25 \ldots .$$

The second crucial ingredient is that the index $\text{Ind}(\phi, p)$ at a fixed point $p$ is heavily constrained. Around a fixed point, the vector field that remains invariant under $\phi$ can have a singularity, near which the mapping’s action can be divided into $2N$ hyperbolic sectors as in Figure 5 with $N$ even. Assuming the polynomial $P(x)$ has a positive dominant root, then as mentioned above there are two cases: $\text{Ind}(\phi, p) = 1 - N < 0$ if $\phi$ fixes the sectors at $p$ (left in Figure 5); $\text{Ind}(\phi, p) = 1$ if it permutes the sectors (right in Figure 5). For technical reasons, a permutation involves only $N/2$ of the sectors.

Let’s see what we can glean from this for the case of the polynomial above: on a surface of genus two, the Euler–Poincaré formula dictates

$$\sum_{p \in \text{Fix}(\phi)} (2 - N_p) = 2\chi_2 = -4$$

where $\chi_2 = -2$ is the Euler characteristic of the surface $S_2$, and $2N_p$ is the number of hyperbolic sectors at a fixed point $p$.

Note first that if $N = 2$ the fixed point contributes nothing to the left side (from now on we drop the $p$ subscript from $N_p$), so according to the formula we can have any number of fixed points of this type. These are called regular fixed points, and can only have a index of $-1$ ($N/2 = 1$, so the only permutation of sectors is the identity). Then, ignoring regular fixed points, the only possibilities for singularities are either one with $N = 6$, or two with $N = 4$ (recall $N$ must be even).

First consider the case of two singularities with $N = 4$. Since $L(\phi) = 1$, at least one of the singularities must be fixed with its sectors permuted, since this is the only way to get a positive index. When $N = 4$ the permutation involves only two sectors, so $\phi^2$ will fix the sectors and the singularity will have an index of $1 - N = -3$. However, the other singularity can contribute at most $+1$ to $L(\phi^2)$, so this is incompatible with the Lefschetz number $L(\phi^2) = -1$. We conclude that this polynomial cannot correspond to a pseudo-Anosov mapping involving two singularities with $N = 4$. 
Now consider the second case: one singularity with \( N = 6 \). Since \( L(\phi) = 1 \), the sectors must be permuted to get a positive index. When \( N = 6 \) the permutation involves \( N/2 = 3 \) sectors (Figure 5). Therefore, \( \phi^2 \) also permutes the sectors (so that the singularity contributes \(+1\) to the Lefschetz number) and \( \phi^3 \) fixes the sectors (so the singularity contributes \( 1 - N = -5 \)). Then, in order to match the Lefschetz number \( L(\phi^2) = -1 \) there must be two regular fixed points, which together make a regular period-2 orbit. Regular fixed points must appear at a multiple of their period (that is, a period-\( k \) orbit becomes \( k \) fixed points of \( \phi^k, \phi^{2k}, \text{etc.} \)). Hence, we conclude that a period-2 regular orbit exists, so that \( L(\phi^2) = 1 - 2 \cdot (-1) = -1 \), as required.

Next, for the third iterate \( \phi^3 \), the Lefschetz number is \( L(\phi^3) = -5 \): this is fine since the 3 sectors of our \( N = 6 \) singularity are now unpermuted, and the index is thus \( 1 - N = 1 - 6 = -5 \). This means that there are no regular orbits of period 3. For the fourth iterate, we have \( L(\phi^4) = -5 \) again, but this time our main singularity has its sectors permuted, so its index is \(+1\). Our earlier period-2 orbits appear at this iterate and contribute \(-2\), and that leaves a deficit of \(-4\), which is accounted for by invoking a single period-4 orbit.

Let’s look at one final iterate: for the fifth one \( L(\phi^5) = -14 \), so something big is happening. Our main singularity with \( N = 6 \) still has its sectors permuted, so its contribution is \(+1\). No other periodic orbits that we’ve encountered before can contribute, since 5 is a prime iterate. Hence, the deficit of \(-14 - (+1) = -15 \) must be entirely made up by three period-5 orbits.

So far the tally is (i) a single fixed point with \( N = 6 \); (ii) a regular \((N = 2)\) period-2 orbit; (iii) a regular period-4 orbit; and (iv) three regular period-5 orbits. Of course, we can keep going, but we have made our point: the list of Lefschetz numbers allows us to deduce which periodic orbits must appear at each iterate. Moreover, it is clear that the list is not arbitrary: there are many sequences of Lefschetz numbers that would be inconsistent, in the sense that no compatible periodic-orbit structure exists. What is remarkable is that all this information can be obtained from a simple polynomial.

The littlest pseudo-Anosov

The tools discussed so far allow us to answer an interesting question: for a given surface of genus \( g \), what is the ‘simplest’ pseudo-Anosov mapping? By simplest, we mean the one with the least dilatation \( \lambda \), which we call \( \delta^+_g \) (the ‘+’ superscript corresponds to \( \text{MCG}^+ \)). This is a question that has occupied some low-dimensional topologists for 20 years, ever since Penner first posed it. Originally it wasn’t clear how this ‘minimizer’ should behave with the genus, but better and better examples were constructed, and currently the best general bound stands at \( \delta^+_g \lesssim (2+\sqrt{3})^{1/g} \), implying that the minimizer converges to unity for large genus.\[^{11}\] In fact, this bound is sharp for \( g = 1 \)\[^{15}\] and \( 2 \)\[^{11,13,16,17,18}\] but explicit pseudo-Anosov mappings have been found with dilatation below the bound for \( 3 \leq g \leq 5 \).\[^{11,19}\] In genus 5, the pseudo-Anosov mapping associated with \( \delta^+_5 \) has the mysterious Lehmer’s number \( \simeq 1.17628 \) as a dilatation.\[^{19,20}\] Moreover, the minimum dilatations for genus 2 to 5 are all Salem numbers: these are roots of polynomials with only one eigenvalue outside the unit circle in the complex plane.\[^{21}\] Nobody knows what happens in general for higher genus.

In Ref.\[^{11}\] we used a very direct method to prove, amongst others things, the minimality of the Salem numbers \( \delta^+_2 \simeq 1.40127, \delta^+_4 \simeq 1.28064, \) and \( \delta^+_5 \simeq 1.17628 \). Here, we illustrate the method for genus 3: Start with the polynomial associated with the candidate pseudo-Anosov
mapping, whose dilatation is believed to give \( \delta_3^+ \), in this case

\[
P(x) = x^6 - x^4 - x^3 - x^2 + 1
\]

with dilatation \( \simeq 1.40127 \). Now how many degree 6, reciprocal, integer-coefficient polynomials have a largest real root that is less than this dilatation? A simple computer search tells us that the list is very small: there is only one, which can be factored as \( (x^3 - x - 1)(x^3 + x^2 - 1) \), with Lefschetz numbers

\[
L(\phi) = 3, \quad L(\phi^2) = -1, \quad L(\phi^3) = -3, \quad L(\phi^4) = 3, \quad L(\phi^5) = -7, \quad L(\phi^6) = -1 \ldots
\]

where \( \phi \) is the hypothetical pseudo-Anosov mapping. Since there are a finite number of possible singularities (because of the Euler–Poincaré formula with \( \chi_3 = -4 \)), we can check if this sequence of Lefschetz numbers is possible. Because \( L(\phi) = 3 \), \( \phi \) must fix at least three singularities. For instance, assume that \( \phi \) fixes one singularity with \( N = 6 \), permuting its sectors as in Figure 5 and two with \( N = 4 \) with their \( N/2 \) sectors permuted. Then \( \phi^2 \) will fix the sectors of the two singularities with \( N = 4 \), and each singularity will contribute \( 1 - 4 = -3 \) to the Lefschetz number, for a total of \(-6\). But from above, \( L(\phi^2) = -1 \), which is not possible: there is no way to make the Lefschetz number more positive by adding regular orbits, since they contribute negatively. Hence, \( \phi \) cannot fix these three types of singularities. There is only one other possible case — four singularities with \( N = 4 \) — which we can rule out as well. We conclude that the polynomial \( (x^3 - x - 1)(x^3 + x^2 - 1) \) is not associated with any pseudo-Anosov mapping. Thus, we have the minimizer \( \delta_3^+ \simeq 1.40127 \). In order to truly complete the proof, one needs to explicitly construct the action of this pseudo-Anosov mapping on the surface (see Ref. 11).

In the end, is this numerological exercise worth it? Well, it certainly simplifies the study of pseudo-Anosov mappings on surfaces. In fact, even these abstract surfaces of genus \( g \) have practical significance: pseudo-Anosov mappings of disks, which can be associated to a stirring device such as that in Figure 2, can be lifted to surfaces of genus \( g \) by a ‘branched cover.’ For such applications, the biggest challenge is to develop practical tools (such as train track\(^2\)) that can handle singularities with an odd number of sectors, which are mathematically more challenging.

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For the experts, we are avoiding here the rigorous definition of pseudo-Anosov mappings in terms of invariant measured foliations, hopefully without too much harm.

Actually, for the torus the ‘pseudo’ is dropped because of the absence of singularities.

The reducible case is rather degenerate for the torus, since there is an infinity of reducing curves.

Such a collection of loops is a basis of the first homology group $H_1(S_2, \mathbb{Z})$. Note that we are leaving out a surjective group homomorphism from $\text{MCG}(S_2)$ to $\text{Sp}_4(\mathbb{Z})$.

The symplectic group consists of matrices that preserve a skew-symmetric quadratic form.

We are assuming $P(x)$ has a positive largest root; for negative root a parallel argument can be made.

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