INJECTIVITY ON ONE LINE

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Abstract. Let $k$ be an algebraically closed field of characteristic zero. Let $H : k^2 \to k^2$ be a polynomial mapping such that the Jacobian $\text{Jac} H$ is a non-zero constant. In this note we prove, that if there is a line $l \subset k^2$ such that $H|_l : l \to k^2$ is an injection, then $H$ is a polynomial automorphism.

1. Main result

Let $k$ be an algebraically closed field of characteristic zero. Put $k^* = k \setminus \{0\}$. By $\text{Aut} k^2$ we denote the group of polynomial automorphisms i.e. all mappings $H = (f, g) : k^2 \to k^2$, $f, g \in k[x, y]$ for which there exists an inverse polynomial mapping. Remind that for $H \in \text{Aut} k^2$ the Jacobian $\text{Jac} H$ is a non-zero constant.

The famous Keller conjecture states that any polynomial mapping which has the non-zero constant Jacobian is a polynomial automorphism.

Theorem 1.1. Let $H : k^2 \to k^2$ be a polynomial mapping such that $\text{Jac} H \in k^*$. If there exists a line $l \subset k^2$ such that $H|_l : l \to k^2$ is injective then $H$ is a polynomial automorphism.

The proof of the above theorem is given in Section 2 of this note. Let us note here that a weaker result (injectivity on three lines) has been proved recently in [4].

2. Proof of the theorem

The proof of our result is based on the Abhyankar–Moh theorem and on some properties of Newton’s polygon which we quote below.

If $f$ is a polynomial in $k[x, y]$ then $S_f$ denotes the support of $f$, that is, $S_f$ is the set of integer points $(i, j)$ such that the monomial $x^i y^j$ appears in $f$ with a non-zero coefficient. We denote by $N_f$ the convex hull (in the real space $\mathbb{R}^2$) of $S_f \cup \{(0, 0)\}$. The set $N_f$ is called (see [2]) Newton’s polygon of $f$.

Theorem 2.1. ([2], [3, theorem 3.4.]) Let $f, g$ be polynomials in $k[x, y]$. Assume that $\text{Jac}(f, g)$ is a non-zero constant and $\deg f > 1$, $\deg g > 1$. Then the polygons $N_f$ and $N_g$ are similar, that is $N_g = \frac{\deg g}{\deg f} N_f$.

The following lemma is easy to check, so we omit the proof.

Lemma 2.2. Let $f, g$ be polynomials in $k[x, y]$ with $\text{Jac}(f, g) \in k^*$. If $\deg f \leq 1$ or $\deg g \leq 1$ then the mapping $(f, g)$ is a polynomial automorphism.
Lemma 2.3. Let $H = (f, g)$ be a polynomial mapping such that $\text{Jac} H$ is a non-zero constant. If $H(x, 0) = (x, 0)$ then $H \in \text{Aut} k^2$.

Proof. First suppose that $\deg f > 1$, $\deg g > 1$. We have

\begin{align*}
(1) & \quad f(x, 0) = x \\
(2) & \quad g(x, 0) = 0.
\end{align*}

From (1) the point $(1, 0)$ belongs to the polygon $N_f$, so by theorem 2.1 $(\deg g, 0) \in N_g$. This means that the polynomial $g$ contains some monomials of the form $x^i$, $i > 0$ with non-zero coefficients but this is a contradiction with (2).

We have $\deg f \leq 1$ or $\deg g \leq 1$ and by lemma 2.2 $H$ is a polynomial automorphism.

Proof of theorem 1.1. Without loss of generality we may assume that the line $l$ has an equation $y = 0$. Otherwise we replace $H$ by $H \circ L$ where $L$ is an affine automorphism such that $L(\{y = 0\}) = l$.

Put $\gamma(x) = H(x, 0)$. By assumptions of the theorem the mapping $\gamma : k \to k^2$ is an injection and $\gamma'(x) \neq 0$ for $x \in k$, hence $\gamma$ is an embedding of the line in the plane. By Abhyankar-Moh theorem [1] there exist an automorphism $H_1 \in \text{Aut} k^2$ such that $\gamma(x) = H_1(x, 0)$. Let $G = H_1 \circ H$. We get $\text{Jac} G = \text{Jac} H_1^{-1} \text{Jac} H$ is a non-zero constant and $G(x, 0) = (x, 0)$, so by lemma 2.3 $G \in \text{Aut} k^2$. Therefore $H = H_1 \circ G$ is a polynomial automorphism.

References

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**Injektywność na jednej prostej**

Streszczenie.

Niech $k$ będzie ciàdem algebraicznie domkniętym charakterystyki zero. Niech $H : k^2 \to k^2$ będzie odwzorowaniem wielomianowym którego Jakobian $\text{Jac} H$ jest stałą różną od zera. W pracy dowodzimy, że jeśli istnieje prosta $l \subset k^2$ na której $H|_l : l \to k^2$ jest injekcją, to $H$ jest automorfizmem wielomianowym.