ON THE CAUCHY PROBLEM FOR INTEGRO-DIFFERENTIAL EQUATIONS WITH SPACE-DEPENDENT OPERATORS IN GENERALIZED HÖLDER CLASSES

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ABSTRACT. Parabolic integro-differential Kolmogorov equations with different space-dependent operators are considered in Hölder-type spaces defined by a scalable Lévy measure. Probabilistic representations are used to prove continuity of the operator. Existence and uniqueness of the solution are established and some regularity estimates are obtained.

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1. Introduction

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and $\nu$ be a Lévy measure on $\mathbb{R}_0^d = \mathbb{R}^d \setminus \{0\}$ that is of order $\alpha$, i.e.

$$\alpha := \inf \{ \sigma \in (0, 2) : \int_{|y| \leq 1} |y|^\sigma \nu(dy) < \infty \}.$$  

We denote by $J(ds, dy)$ a Poisson random measure on $(\Omega, \mathcal{F}, P)$ such that $E[J(ds, dy)] = \nu(dy) ds$, and denote by $Z^\nu_t$ the Lévy process

$$Z^\nu_t = \int_0^t \int_{\mathbb{R}_0^d} \chi_\alpha(y) y \tilde{J}(ds, dy) + \int_0^t \int_{\mathbb{R}_0^d} (1 - \chi_\alpha(y)) y J(ds, dy).$$  

Here $\chi_\alpha(y) := 1_{\alpha \in (1, 2)} + 1_{\alpha = 1} 1_{|y| \leq 1}$, and

$$\tilde{J}(ds, dy) := J(ds, dy) - \nu(dy) ds$$

is the compensated Poisson measure.

This work is a continuation of [15], in which we studied the Cauchy problem for the following parabolic-type Kolmogorov equations in generalized Hölder spaces $\tilde{C}^\beta(\mathbb{R}^d)$ endowed with norms $\| \cdot \|_\beta$ (see Section 2.2):

$$\partial_t u(t, x) = L^\nu u(t, x) - \lambda u(t, x) + f(t, x), \lambda \geq 0,  
 u(0, x) = 0, \quad (t, x) \in [0, T] \times \mathbb{R}^d,$$

where $L^\nu$ is the infinitesimal generator of $Z^\nu_t$. Namely, for any $\varphi \in C_0^\infty(\mathbb{R}^d)$,

$$L^\nu \varphi(x) := \int |\varphi(x+y) - \varphi(x) - \chi_\alpha(y) y \cdot \nabla \varphi(x)| \nu(dy).$$

A notion of scaling functions was utilized in [15] to include some recent popular models of $\nu$ (cf. [7, 8, 18]).

**Definition 1.** A continuous function $w : (0, \infty) \to (0, \infty)$ is called a scaling function if

$$\lim_{r \to 0} w(r) = 0, \quad \lim_{R \to \infty} w(R) = \infty$$

and if there is a nondecreasing continuous function $l(\varepsilon), \varepsilon > 0$ such that

$$\lim_{\varepsilon \to 0} l(\varepsilon) = 0$$

and

$$w(\varepsilon r) \leq l(\varepsilon) w(r), \quad \forall r, \varepsilon > 0.$$

$l$ is called the scaling factor of $w$.

For any Lévy measure $\nu$, any $R > 0$ and $\forall B \in \mathcal{B}(\mathbb{R}_0^d)$,

$$\nu_R(B) := \int_B (y/R) \nu(dy),$$

$$\tilde{\nu}_R(dy) := w(R) \nu_R(dy).$$

We can always normalize $w$ by a constant so that $w(1) = 1$ and $\tilde{\nu}_1(dy) = \nu(dy)$. It was imposed in [15] for $\nu$:  

$$\nu(\mathbb{R}_0^d) = \int_0^\infty \nu(dy) = 1.$$
A(w,l) (i) (Non-degeneracy) Suppose $\tilde{\nu}_R(dy) \geq \mu^0(dy), R > 0$ for some Lévy measure $\mu^0$ that is supported on the unit ball $B(0)$, with $\mu^0$ satisfying
\begin{equation}
\int |y|^2 \mu^0(dy) + \int |\xi|^d [1 + \nu(\xi)]^{d+3} \exp\{-\zeta^0(\xi)\} d\xi < \infty,
\end{equation}
where
\begin{align*}
\nu(\xi) & = \int \chi_\alpha(y) |y| [(|\xi| |y|) \wedge 1] \mu^0(dy), \\
\zeta^0(\xi) & = \int [1 - \cos (2\pi \xi \cdot y)] \mu^0(dy).
\end{align*}
In addition, for all $\xi \in S_{d-1} = \{\xi \in \mathbb{R}^d : |\xi| = 1\}$, there is a constant $c_1 > 0$, such that
\begin{equation}
\int_{|y| \leq 1} |\xi \cdot y|^2 \mu^0(dy) \geq c_0.
\end{equation}
(ii) (Symmetry) If $\alpha = 1$, then
\begin{equation}
\int_{r < |y| < R} y \nu(dy) = 0 \quad \text{for all } 0 < r < R < \infty.
\end{equation}
(iii) (Scalability) There exist constants $\alpha_1 \geq \alpha_2$ such that $\alpha_1, \alpha_2 \in (0, 1)$ if $\alpha \in (0, 1)$, $\alpha_1, \alpha_2 \in (1, 2]$ if $\alpha \in (1, 2)$, $\alpha_1 \in (1, 2]$ and $\alpha_2 \in [0, 1)$ if $\alpha = 1$, and
\begin{equation}
\int_{|y| \leq 1} |y|^{\alpha_1} \tilde{\nu}_R(dy) + \int_{|y| > 1} |y|^{\alpha_2} \tilde{\nu}_R(dy) \leq N_0.
\end{equation}
The $N_0 > 0$ above is uniform with respect to $R$.
(iv) (Scalability) Suppose $\varsigma(r) := \nu(|y| > r)$, $r > 0$ is continuous in $r$ and
\begin{equation}
\int_0^1 s\varsigma(rs) \varsigma(r)^{-1} ds \leq C_0
\end{equation}
for some positive $C_0$ independent of $r$.

Under $A(w,l)$, $Z_t^{\nu}$ possesses a smooth density function whose regularity estimates were derived in [9]. Moreover, $Z_t^{\nu}$ is approximately distributed as $\frac{1}{R} Z_{w(R)}, R > 0$. This property gives a uniform description of Lévy measures that were considered in [18], [7] and [8]. In [18], $\nu$ is assumed to be confined by two $\alpha$-stable measures of the same order, namely,
\begin{equation}
\int_{S_{d-1}} \int_0^\infty 1_B(rw) \frac{dr}{r^{1+\alpha}} \Sigma_1(dw)
\end{equation}
for any Borel measurable set $B$. They also assumed $\Sigma_1$ and $\Sigma_2$ are two finite measures defined on the unit sphere and $\Sigma_1$ is nondegenerate. In
this situation, \( \nu \) satisfies \( A(w, l) \) with \( w(r) = l(r) = r^{\alpha}, \ r > 0 \). Another interesting class of Lévy measures was investigated in [7] and [8], where

\[
\nu(B) = \int_0^\infty \int_{|w|=1} 1_B(rw) a(r, w) j(r) r^{d-1} \delta dw \ dr, \quad \forall B \in \mathcal{B}(\mathbb{R}_0^d),
\]

\( \delta \) is a finite measure on the unit sphere, and

\[
j(r) = \int_0^\infty (4\pi t)^{-d/2} \exp \left( -\frac{r^2}{4t} \right) \Lambda(dt), \ r > 0,
\]

with \( \Lambda(dt) \) being a measure on \((0, \infty)\) such that \( \int_0^\infty (1 + t) \Lambda(dt) < \infty \). Let \( \phi(r) = \int_0^\infty (1 - e^{-r^2}) \Lambda(dt), \ r \geq 0 \) be the associated Bernstein function. They imposed

**H.** there is a function \( \rho_0(w) \) defined on the unit sphere such that \( \rho_0(w) \leq a(r, w) \leq 1, \forall r > 0 \), and for all \( |\xi| = 1 \),

\[
\int_{S_d-1} |\xi \cdot w|^2 \rho_0(w) \geq c > 0.
\]

**G.**

(i) There is \( C > 1 \) such that

\[
\frac{1}{C} \phi(r^{-2}) r^{-d} \leq j(r) \leq C \phi(r^{-2}) r^{-d}.
\]

(ii) There are \( 0 < \sigma_1 \leq \sigma_2 < 1 \) and \( C > 0 \) such that for all \( 0 < r \leq R \)

\[
C^{-1} \left( \frac{R}{r} \right)^{\sigma_1} \leq \frac{\phi(R)}{\phi(r)} \leq C \left( \frac{R}{r} \right)^{\sigma_2}.
\]

It can be verified that \( H \) and \( G \) produce Lévy measures of \( A(w, l) \)-type with \( w(r) = j(r)^{-1} r^{-d}, r > 0 \), and

\[
l(r) = \begin{cases} 
C r^{2\sigma_1} & \text{if } r \leq 1, \\
C r^{2\sigma_2} & \text{if } r > 1
\end{cases}
\]

for some \( C > 0 \). (See [7] [8] [9] [10] for details and examples.)

Write \( H_T = [0, T] \times \mathbb{R}_d \). In this note, we consider the following parabolic integro-differential equation:

\[
(1.11) \quad \partial_t u(t, x) = \mathcal{L} u(t, x) - \lambda u(t, x) + f(t, x), \ \lambda \geq 0, \\
u(0, x) = 0, \ (t, x) \in H_T,
\]

where \( \mathcal{L} = A + Q \) or \( \mathcal{L} = G + Q \), and for any function \( \varphi \in C_0^\infty(\mathbb{R}_d) \),

\[
A\varphi(x) := \int [\varphi(x + y) - \varphi(x) - \chi_\alpha(y) y \cdot \nabla \varphi(x)] \rho(t, x, y) \nu(dy),
\]

\[
G\varphi(x) := \int [\varphi(x + G(x)y) - \varphi(x) - \chi_\alpha(y) G(x)y \cdot \nabla \varphi(x)] \nu(dy),
\]

\[
Q\varphi(x) := 1_{\alpha \in (1, 2]} b(t, x) \cdot \nabla \varphi(x) + p(t, x) \varphi(x) + \int_{\mathbb{R}_d^2} [\varphi(x + q(t, x, y)) - \varphi(x)] \nu_2(dy),
\]

(1.12) \quad -\varphi(x) - \nabla \varphi(x) \cdot q(t, x, y) 1_{\alpha \in (1, 2]} 1_{|y| \leq 1} \varphi(t, x, y) \nu_2(dy).
We assume for the underlying Lévy measure $\nu$: $\mathcal{A}(w, l, \gamma)$. (i) $\nu$ satisfies $\mathcal{A}(w, l)$.
(ii) There is $\varepsilon \in (0, 1)$ such that for any $\beta' \in (0, \beta + \varepsilon)$,
\[
\int_0^1 l(t)^{\beta'} \frac{dt}{t} + \int_1^\infty l(t)^{\beta'} \frac{dt}{t^2} + 1_{\alpha \in \{1, 2\}} \int_0^1 l(t)^{1+\beta'} \frac{dt}{t^2} < \infty.
\]
(iii) Set $\gamma(t) = \inf\{s > 0 : l(s) \geq t\}$ for $t > 0$. There exist $0 < \delta < \min\left(\frac{1}{2}, \beta\right)$ and $0 < \delta' < \min\left(\frac{1}{2}, \varepsilon\right)$ for the $\varepsilon$ in (ii) such that
\[
1_{\alpha \in \{0, 1\}} \int_1^\infty t^{\delta} \gamma(t)^{-1} \frac{dt}{t} < \infty,
\]
\[
1_{\alpha = 1} \left( \int_0^1 t^{\delta} \gamma(t)^{1-1} dt + \int_1^\infty t^{-\delta} \gamma(t)^{-1} + t^{\delta} \gamma(t)^{-2} dt \right) < \infty,
\]
\[
1_{\alpha \in \{1, 2\}} \left( \int_0^1 t^{-\delta} \gamma(t)^{-1} dt + \int_1^\infty t\delta \gamma(t)^{-2} + t^{-\frac{1}{2} + \delta} \gamma(t)^{-1} dt \right) < \infty.
\]

Suppose the kernel function $\rho$ satisfies $\mathcal{H}(K, \beta)$. (i) There is $K > 0$ so that for all $t \in [0, T]$,
\[
|\rho(t, x, y)| \leq K, \forall x, y \in \mathbb{R}^d,
\]
\[
|\rho(t, x_1, y) - \rho(t, x_2, y)| \leq Kw(|x_1 - x_2|)\beta, \forall y \in \mathbb{R}^d.
\]
(ii) If $\alpha = 1$, then for all $x \in \mathbb{R}^d$, $\forall r \in (0, 1), \forall t \in [0, T]$,
\[
\int_{r < |y| < 1} y\rho(t, x, y) \nu(dy) = 0.
\]

We assume for the main part $\mathcal{G}$:
\[
\mathcal{G}(c_0, K, \beta). \quad \text{(i) $G(z), z \in \mathbb{R}^d$ is an invertible and uniform continuous $d\times d$-matrix, and $G(z) \neq G(z')$ if $z \neq z'$.}
\]
\[
\text{(ii) $|\det G(z)| \geq c_0, ||G(z)|| \leq K, } \forall z \in \mathbb{R}^d \text{ for some } c_0, K > 0.
\]
\[
\text{(iii) For the same } K, g(z, z') \leq Kw(|z - z'|)\beta, \forall z, z' \in \mathbb{R}^d, \text{ where } G_{z, z'} := ||G(z) - G(z')||
\]

For the same $w, l, K, \beta$, we assume the lower order part $\mathcal{Q}$ satisfies:
\[
\mathcal{B}(K, \beta). \quad \text{(i) $|b(t, \cdot)|_p + |p(t, \cdot)|_p + |g(t, \cdot, y)|_p \leq K, } \forall y \in \mathbb{R}^d, \forall t \in [0, T].
\]
\[
\text{(ii) For all } \alpha \in (0, 2), z' \in \mathbb{R}^d, \forall t \in [0, T], \text{ $q(t, \cdot, y) \neq 0$ if } y \neq 0. \text{ Besides,}
\]
\[
\lim_{\varepsilon \to 0} \sup_{t, z'} \int_{|q(t, z', y)| \leq \varepsilon} \left( w(|q(t, z', y)|) + 1_{\alpha = 1} |q(t, z', y)| \right) \nu_2(dy) = 0.
\]
\[
\text{(iii) For all } \alpha \in (0, 1), z' \in \mathbb{R}^d, \forall t \in [0, T],
\]
\[
\int_{\mathbb{R}^d} 1_{\alpha < 1} (w(|q(t, z', y)|) \wedge 1) + 1_{\alpha = 1} (|q(t, z', y)| \wedge 1) \nu_2(dy) \leq K.
\]
(iv) For all \( \alpha \in (1, 2), z' \in \mathbb{R}^d, \forall t \in [0, T], \)
\[
\int_{|y| \leq 1} w(|q(t, z', y)|) \nu_2(\,dy\,) + \int_{|y| > 1} (|q(t, z', y)| \wedge 1) \nu_2(\,dy\,) \leq K.
\]

(v) For all \( z', h \in \mathbb{R}^d, \forall t \in [0, T], \alpha \in (1, 2), \)
\[
\int_{|y| \leq 1} w(|q(t, x + h, y)|^\beta \,|q(t, x + h, y) - q(t, x, y)|) \nu_2(\,dy\,) \leq K w(|h|)^\beta,
\]
\[
\int_{|y| \leq 1} w(|q(t, x + h, y) - q(t, x, y)|^\beta) \nu_2(\,dy\,) \leq K w(|h|)^\beta,
\]
\[
\int_{|y| > 1} (|q(t, z' + h, y) - q(t, z', y)| \wedge 1) \nu_2(\,dy\,) \leq K w(|h|)^\beta.
\]

If \( \alpha \in (0, 1), \)
\[
\int_{\mathbb{R}^d} w(|q(t, z' + h, y) - q(t, z', y)|) \wedge 1 \nu_2(\,dy\,) \leq K w(|h|)^\beta.
\]

And if \( \alpha = 1, \)
\[
\int_{\mathbb{R}^d} (|q(t, z' + h, y) - q(t, z', y)| \wedge 1) \nu_2(\,dy\,) \leq K w(|h|)^\beta.
\]

The main conclusion of this paper is

**Theorem 1.1.** Let \( \tilde{A}(w, l, \gamma), B(K, \beta) \) and \( H(K, \beta) \) (resp. \( G(c_0, K, \beta) \)) hold. If \( f(t, x) \in \dot{C}^\beta(H_T), \beta \in (0, \frac{1}{\alpha}) \), then there is a unique solution \( u(t, x) \in \dot{C}^{1+\beta}(H_T) \) to (1.11) with \( \mathcal{L} = \mathcal{A} + \mathcal{Q} \) (resp. \( \mathcal{L} = \mathcal{G} + \mathcal{Q} \)). Moreover, there exists a constant \( C \) depending on \( c_0, c_1, N_1, K, \beta, d, T, \mu, \nu \) such that
\[
|u|_\beta \leq C (\lambda^{-1} \wedge T) |f|_\beta,
\]
\[
|u|_{1+\beta} \leq C |f|_\beta.
\]

And there is a constant \( C \) depending on \( c_0, c_1, N_1, K, \beta, d, T, \mu, \nu \) such that for all \( 0 \leq s < t \leq T, \alpha \in [0, 1] \) and \( \kappa + \beta > 1, \)
\[
|u(t, \cdot) - u(s, \cdot)|_{\kappa+\beta} \leq C |t - s|^{1-\kappa} |f|_\beta.
\]

Due to generality of the measure \( \nu \) we are considering, the Lévy symbol \( \psi^\nu(\xi), \xi \in \mathbb{R}^d \) of the process \( Z^\nu_t \) is generally not smooth in \( \xi \). This was already an obstacle for applying the standard Fourier multiplier theorem to solutions of equations with space-independent coefficients, and it continues to be a difficulty in this work. Thus, probabilistic representations are used instead and continuity of the operators are proved in that approach. Then we apply continuation of parameters, which was also used in [13] and [14], to show well-posedness of the Cauchy problem. In [14], a parabolic-type Kolmogorov equation with an operator \( \mathcal{L} = \mathcal{A} + \mathcal{Q} \) was considered in the
standard Hölder-Zygmund space, where $Q$ is the lower order part and the principal part
\[
Au (t, x) := \int [u (t, x + y) - u (t, x) - \chi_{|y| \leq 1} y \cdot \nabla u (t, x)] \rho (t, x, y) \frac{dy}{|y|^{d+\alpha}}.
\]

With more flavor of probability, in [13] a stochastic parabolic integro-differential equation with operators
\[
Lu (t, x) := \int [u (t, x + y) - u (t, x) - 1_{|y| \leq 1} y \cdot \nabla u (t, x)] \nu (t, x, dy)
+ 1_{\alpha = 2a^{ij} (t, x) \partial^2_{ij} u (t, x)} + 1_{\alpha \geq 1} \tilde{b}^i (t, x) \partial_i u (t, x) + l (t, x) u (t, x)
\]
was studied in Hölder spaces. A deterministic model with a similar operator was addressed in the little Hölder-Zygmund spaces in [12]. Besides, the Cauchy problem for a second order linear SPDE was considered in [11] and [16] in standard Hölder classes.

The outline of this note is as follows.

In section 2, notation is introduced. Definitions of function spaces and results on norm equivalence from [15] are briefly mentioned at the convenience of readers. In section 3, we show continuity of the operators by using probability representations. In section 4, we derive some a priori estimates and prove the main theorem by applying continuation of parameters. Other auxiliary results are collected in the Appendix section.

2. Notation and Function Spaces

2.1. Basic Notation. We use $\mathbb{N}$ for the set of nonnegative integers, $\mathbb{N}_+$ for $\mathbb{N} \setminus \{0\}$, and $\mathbb{R}$ for the real part of a complex-valued quantity.

For a function $u = u (t, x)$ on $H_T = [0, T] \times \mathbb{R}^d$, $\partial_t u := \partial u / \partial t$, $\partial_i u := \partial u / \partial x_i$, $\partial^2_{ij} u := \partial^2 u / \partial x_i \partial x_j$. The gradient of $u$ with respect to $x$ is denoted by $\nabla u$, and $D |\gamma| u := \partial |\gamma| u / \partial x_1^{\gamma_1} \ldots \partial x_d^{\gamma_d}$, where $\gamma = (\gamma_1, \ldots, \gamma_d) \in \mathbb{N}^d$ is a multi-index.

As usual, $C_0^\infty (\mathbb{R}^d)$ denotes the set of infinitely differentiable functions on $\mathbb{R}^d$ whose derivative of arbitrary order is finite, $S (\mathbb{R}^d)$ is the space of rapidly decreasing functions on $\mathbb{R}^d$ and $S' (\mathbb{R}^d)$ denotes the space of continuous functionals on $S (\mathbb{R}^d)$. It is well-known that Fourier transform is a bijection on $S' (\mathbb{R}^d)$. We adopt the normalized definition for Fourier and its inverse transforms in this note, i.e.,

\[
F \varphi (\xi) = \hat{\varphi} (\xi) := \int e^{-i2\pi x \cdot \xi} \varphi (x) \, dx,
\]
\[
F^{-1} \varphi (x) = \check{\varphi} (x) := \int e^{i2\pi x \cdot \xi} \varphi (\xi) \, d\xi, \quad \varphi \in S (\mathbb{R}^d).
\]

For any Lévy measure $\nu$ we may symmetrize it as below.

\[
\tilde{\nu} (dy) := \frac{1}{2} (\nu (dy) + \nu (-dy)).
\]
As a convention, $C$ is a positive constant that represents different values in various contexts. Explicit dependence on certain quantities may be indicated when necessary.

### 2.2. Function Spaces of Generalized Smoothness

Our primary function spaces of generalized smoothness in this note are $\tilde{C}^{\beta} (\mathbb{R}^d)$, $\beta \in (0, 1/\alpha)$ endowed with the norm
\[
|u|_\beta = \sup_{t,x} |u(t, x)| + \sup_{t,x,h \neq 0} \frac{|u(t, x + h) - u(t, x)|}{w(|h|)^{\beta}} := |u|_0 + |u|_\beta < \infty
\]
and $\tilde{C}^{1+\beta} (\mathbb{R}^d)$, $\beta \in (0, 1/\alpha)$ with the norm
\[
|u|_{1+\beta} := |u|_0 + |L^\mu u|_0 + [L^\mu u]_\beta < \infty,
\]
where $\mu$ is a reference measure satisfying $A(w, l)$ for the same $w$ and $l$ as $\nu$, and $L^\mu$ is the associated operator defined as (1.3).

By [15] Proposition 1], these generalized Hölder norms are equivalent to the norm of generalized Besov spaces $\tilde{C}^{\beta, \infty} (\mathbb{R}^d)$:
\[
|u|_{\beta, \infty} = \sup_{j \in \mathbb{N}} w (N^{-j})^{-\beta} |u * \varphi_j|_0 < \infty, \quad \beta \in (0, \infty).
\]

Given the choice of [15], in above definition $\varphi_j \in S (\mathbb{R}^d)$ for any $j \in \mathbb{N}$ and $\sum_{j=0}^\infty \mathcal{F} \varphi_j = 1$. Moreover, when $j \geq 1$, $\mathcal{F} \varphi_j = \phi (N^{-j} \cdot)$ for some $N$ such that $l(N^{-1}) < 1 < l(N)$ and for some $\phi \in C_0^\infty (\mathbb{R}^d)$ so that $\text{supp} (\phi) = \{ \xi : N^{-1} \leq |\xi| \leq N \}$.

Set $\kappa \in [0, 1]$ and $\beta > 0$. Denote the Lévy symbol associated with $L^\mu$ by
\[
\psi^\mu (\xi) = \int \left[ e^{i2\pi \xi y} - 1 - i2\pi \chi_\alpha (y) \xi \cdot y \right] \mu (dy), \xi \in \mathbb{R}^d,
\]
and denote
\[
\psi^{\mu, \kappa} = \begin{cases} 
\psi^\mu & \text{if } \kappa = 1, \\
-(-\Re \psi^\mu)^\kappa & \text{if } \kappa \in (0, 1), \\
1 & \text{if } \kappa = 0.
\end{cases}
\]

Then the auxiliary space $C^{\mu, \kappa, \beta} (\mathbb{R}^d)$ is a class of functions whose norm
\[
|u|_{\mu, \kappa, \beta} := |u|_0 + |L^{\mu, \kappa} u|_{\beta, \infty} < \infty,
\]
where
\[
L^{\mu, \kappa} u := \mathcal{F}^{-1} [\psi^{\mu, \kappa} \mathcal{F} u], \quad u \in S' (\mathbb{R}^d).
\]

Set
\[
(I - L^\mu)^\kappa u = \begin{cases} 
(I - L^\mu) u & \text{if } \kappa = 1, \\
\mathcal{F}^{-1} \left[ (1 - \Re \psi^\mu)^\kappa \mathcal{F} u \right] & \text{if } \kappa \in [0, 1).
\end{cases}
\]
Another auxiliary space $\tilde{C}^{\mu, \kappa, \beta} (\mathbb{R}^d)$ is introduced as the collection of functions whose norm
\[
||u||_{\mu, \kappa, \beta} := |(I - L^\mu)^\kappa u|_{\beta, \infty} < \infty.
\]
Lemma 3. [15, Lemma 10] Let \( C \) be a Lévy measure satisfying (iii) in \( A(w,l) \) and \( L^{\nu,R,\kappa} \) be defined as (2.2). Then for any \( \varphi(x) \in C^\infty_b(\mathbb{R}^d) \),
\[
L^{\nu,R,\kappa}_\varphi(x) = C \int_0^\infty t^{-\kappa-1} \mathbf{E} \left[ \varphi \left( x + Z^\nu_\kappa t \right) - \varphi(x) \right] dt, \quad \kappa \in (0,1),
\]
where \( C^{-1} = \int_0^\infty t^{-\kappa-1} \left( 1 - e^{-t} \right) dt \) and
\[
\nu_R(dy) = \frac{1}{2} (\nu_R(dy) + \nu_R(-dy)), \quad R > 0.
\]
Besides, \( L^{\nu,R,\kappa}_\varphi \in C^\infty_b(\mathbb{R}^d) \). If furthermore \( \varphi(x) \in S(\mathbb{R}^d) \), then \( |L^{\nu,R,\kappa}_\varphi|_{L_1} < C' \) for some \( C' > 0 \) uniform w.r.t. \( R \).

Lemma 2. [15, Lemma 9] Let \( a > 0 \) and \( \nu \) be a Lévy measure satisfying (iii) in \( A(w,l) \). Then \( aI - L^\nu \) defines a bijection on \( C^\infty_b(\mathbb{R}^d) \). Moreover, for all \( C^\infty_b(\mathbb{R}^d) \) functions \( \varphi \), the following representations hold.
\[
\varphi(x) = \int_0^\infty e^{-at} \mathbf{E} (aI - L^\nu) \varphi(x + Z^\nu_t) dt,
\]
\[
(aI - L^\nu)^{-1} \varphi(x) = \int_0^\infty e^{-at} \mathbf{E} \varphi(x + Z^\nu_t) dt, \quad x \in \mathbb{R}^d.
\]

Lemma 3. [15, Lemma 10] Let \( a > 0 \) and \( \kappa \in (0,1) \). Suppose \( \nu \) is a Lévy measure satisfying (iii) in \( A(w,l) \). Then \( (aI - L^\nu)^\kappa \) is a bijection on \( C^\infty_b(\mathbb{R}^d) \). Moreover, for all \( C^\infty_b(\mathbb{R}^d) \) functions \( \varphi \),
\[
(aI - L^\nu)^\kappa \varphi(x) = C \int_0^\infty t^{-\kappa-1} \left[ \varphi(x) - e^{-at} \mathbf{E} \varphi(x + Z^\nu_t) \right] dt,
\]
\[
(aI - L^\nu)^{-\kappa} \varphi(x) = C' \int_0^\infty t^{\kappa-1} e^{-at} \mathbf{E} \varphi(x + Z^\nu_t) dt,
\]
where \( C^{-1} = \int_0^\infty t^{-\kappa-1} \left( 1 - e^{-t} \right) dt, \quad C'^{-1} = \int_0^\infty t^{\kappa-1} e^{-t} dt \) and \( Z^\nu_t \) is the Lévy process associated with \( \nu \).

Remark: Lemmas 1 and 3 imply that \( L^{\mu,\kappa}, (aI - L^\nu)^\kappa, (aI - L^\nu)^{-\kappa} \), \( \kappa \in (0,1) \) are closed operations in \( C^\infty_b(\mathbb{R}^d) \). Therefore, they may be all extended to \( \kappa \in (1,2) \) through composition of operators. It was shown in [15, Corollary 2] that (2.4) also holds for \( \kappa \in (1,2) \).

Lemma 4. [15, Lemma 6 and Proposition 6] Let \( \beta > 0 \) and \( \kappa \in [0,2) \). Suppose \( \nu \) is a Lévy measure satisfying \( A(w,l) \). Then (2.2) is well-defined for all \( \nu \) and all \( u \in C^\kappa_{\kappa,\infty}(\mathbb{R}^d) \),
\[
L^\nu,\kappa u(x) = \lim_{n \to \infty} L^\nu,\kappa u_n(x), \quad x \in \mathbb{R}^d,
\]
and this convergence is uniform with respect to \( x \). Moreover,
\[
|L^\nu,\kappa u|_0 \leq |L^\nu,\kappa u|_{\kappa,\infty} \leq C |u|_{\kappa+\beta,\infty}
\]
for some \( C > 0 \) independent of \( u \).
Based on Lemmas \[ \text{[1,4]} \] norm equivalence were established.

**Proposition 1.** \[ \text{[15, Theorems 3.2 and 3.3]} \] Let \( \nu \) be a Lévy measure satisfying \( A(w,l) \), \( \beta > 0, \kappa \in (0,1] \). Then norms \( |u|_{\nu,\kappa,\beta}, \|u\|_{\nu,\kappa,\beta} \) and \( |u|_{\kappa+\beta,\infty} \) are mutually equivalent.

3. CONTINUITY OF THE OPERATOR

In this section, we study respectively operators that have a kernel depending on the spatial variable \( x \) and operators that have space-dependent coefficients. The first lemma explains the relation between generalized regularity and the ordinary smoothness.

**Lemma 5.** Let \( \beta, \delta \in (0, \infty), \sigma \in [0,1) \) and \( k \) be a positive integer so that
\[
\int_0^1 l(t)^\beta t^{-k-1} dt < \infty.
\]

a) Any function \( u \in \tilde{C}^{\beta, \infty}_{\kappa,\infty}(\mathbb{R}^d) \) is \( k \)-times continuously differentiable and there is \( C \) depending only on \( N, \beta \) so that for any multi-index \( |\gamma| \leq k \) and any \( \sigma \in [0,1) \) with \( |\gamma| + \sigma \leq k \),
\[
|\partial^\sigma D^\gamma u|_0 \leq C |u|_{\beta,\infty} \int_0^1 l(t)^\beta t^{-|\gamma| - \sigma - 1} dt.
\]
Moreover,
\[
\partial^\sigma D^\gamma u = D^\gamma \partial^\sigma u = \sum_{j=0}^{\infty} (\partial^\sigma D^\gamma u) * \varphi_j
\]
converges uniformly.

b) Any function \( u \in \tilde{C}^{\beta+\delta, \infty}_{\kappa,\infty}(\mathbb{R}^d) \) is \( k \)-times continuously differentiable and there is \( C \) depending only on \( N, \beta \) so that for any multi-index \( |\gamma| \leq k \) and any \( \sigma \in [0,1) \) with \( |\gamma| + \sigma \leq k \),
\[
|\partial^\sigma D^\gamma u|_{\delta,\infty} \leq C |u|_{\beta+\delta,\infty} \int_0^1 l(t)^\beta t^{-|\gamma| - \sigma - 1} dt.
\]

**Proof.** Recall properties of the convolution functions \( \varphi_j, j \in \mathbb{N} \). If we write
\[
\hat{\varphi}_j = \varphi_{j-1} + \varphi_j + \varphi_{j+1}, j \geq 2,
\]
\[
\hat{\varphi}_1 = \hat{\varphi}_0 + \varphi_1 + \varphi_2,
\]
then,
\[
\mathcal{F}\hat{\varphi}_j(\xi) = \hat{\varphi}_j(\xi) = \mathcal{F}\hat{\varphi}(N^{-j}\xi), \quad \xi \in \mathbb{R}^d, j \geq 1,
\]
where
\[
\mathcal{F}\hat{\varphi}(\xi) = \phi(N\xi) + \phi(\xi) + \phi(N^{-1}\xi).
\]
Note that \( \phi \) is necessarily 0 on the boundary of its support. Then,
\[
\varphi_j = \varphi_j * \hat{\varphi}_j, j \geq 0,
\]
\[
\hat{\varphi}_j(x) = N^{jd}\hat{\varphi}(N^j x), j \geq 1.
\]
And then,

\[ u = \sum_{j=0}^{\infty} u * \varphi_j = \sum_{j=0}^{\infty} \tilde{\varphi}_j * u * \varphi_j. \]

a) We only show cases in which \(|\gamma| = 1\). The proof for other higher orders is an application of induction on \(\gamma\). Denote

\[ (\partial^\alpha D^\gamma \tilde{\varphi})_j (x) = N^j d (\partial^\alpha D^\gamma \tilde{\varphi})(N^j x), \ x \in \mathbb{R}^d, j \geq 1. \]

Then

\[ \sum_{j=1}^{\infty} \partial^\alpha D^\gamma (\tilde{\varphi}_j * u * \varphi_j) = \sum_{j=1}^{\infty} N(|\gamma|+\sigma)j (\partial^\alpha D^\gamma \tilde{\varphi})_j * u * \varphi_j. \]

Since

\[ \sum_{j=0}^{\infty} w \left( \frac{(N-j)^\beta}{(N-j)^{|\gamma|+\sigma}} \right) \leq w(N)^\beta \int_0^{\infty} \frac{l(N-x)^\beta}{(N-x)^{|\gamma|+\sigma}} dx \leq C \int_0^{1} l(t)^\beta \frac{dt}{t^{\gamma+\sigma}} < \infty, \]

we have

\[ \sum_{j=0}^{\infty} |\partial^\alpha D^\gamma (\tilde{\varphi}_j * u * \varphi_j)|_0 \leq C \sum_{j=0}^{\infty} w \left( \frac{(N-j)^\beta}{(N-j)^{|\gamma|+\sigma}} \right) w(N^{-j})^{-\beta} |u * \varphi_j|_0 \]

\[ \leq C |u|_{\beta,\infty} \int_0^{1} l(t)^\beta t^{-|\gamma|-\sigma-1} dt < \infty. \]

Therefore, \(\sum_{j=0}^{\infty} \partial^\alpha D^\gamma (\tilde{\varphi}_j * u * \varphi_j) \in C(\mathbb{R}^d)\) converges uniformly and therefore it converges in the weak topology of \(S'(\mathbb{R})\). By continuity of the Fourier transform,

\[ D^\gamma \partial^\alpha u = \partial^\alpha D^\gamma u = \sum_{j=0}^{\infty} \partial^\alpha D^\gamma (\tilde{\varphi}_j * u * \varphi_j) = \sum_{j=0}^{\infty} (\partial^\alpha D^\gamma u) * \varphi_j. \]

Moreover,

\[ |\partial^\alpha D^\gamma u|_0 \leq \sum_{j=0}^{\infty} |\partial^\alpha D^\gamma (\tilde{\varphi}_j * u * \varphi_j)|_0 \leq C |u|_{\beta,\infty} \int_0^{1} l(t)^\beta t^{-|\gamma|-\sigma-1} dt. \]

b) From a),

\[ w \left( \frac{(N-j)^{-\delta}}{(N-j)^{|\gamma|+\sigma}} \right) |(\partial^\alpha D^\gamma u) * \varphi_j|_0 \]

\[ \leq C \sum_{j=0}^{\infty} w \left( \frac{(N-j)^{\beta}}{N^{|\gamma|+\sigma}} \right) w \left( \frac{(N-j)^{-\delta}}{(N-j)^{-|\gamma|-\sigma}} \right) |u * \varphi_j|_0 \]

\[ \leq C |u|_{\beta+\delta,\infty} \int_0^{1} l(t)^\beta t^{-|\gamma|-\sigma-1} dt, \ \forall j \in \mathbb{N}. \]

And the conclusion follows. \(\square\)

**Remark:** As a conclusion of Lemma 5 and \(\tilde{A}(w, l, \gamma)\) (iii), if \(u \in \tilde{C}_{\infty,\infty}^{1+\beta}(\mathbb{R}^d), \beta > 0\) and \(\alpha \in [1, 2]\), then \(u\) has classical first-order derivatives.
Lemma 6. Let $\kappa \in (0, 2)$ and $\mu$ be the reference measure. Then for any function $\varphi \in C_b^\infty (\mathbb{R}^d)$,

\[
(aI - L^\mu)\kappa \varphi \rightarrow -L^\mu,\kappa \varphi, \kappa \in (0, 1),
\]
\[
(aI - L^\mu)\kappa \varphi \rightarrow L^\mu,\kappa \varphi, \kappa \in (1, 2).
\]

uniformly as $a \rightarrow 0_+$.

Proof. Apparently, $(aI - L^\mu) \varphi (x) \rightarrow -L^\mu \varphi (x)$ uniformly as $a \rightarrow 0$. Use the representation (2.3) for $\kappa \in (0, 1)$:

\[
(aI - L^\mu)\kappa \varphi (x) = C (\kappa) \int_0^\infty t^{-\kappa-1} e^{-at} \left[ \varphi (x) - E \varphi (x + Z^\mu_t) \right] dt + a^\kappa \varphi (x),
\]

where $C (\kappa)^{-1} = \int_0^\infty t^{-\kappa-1} (1 - e^{-t}) dt$. Note (2.3), then

\[
|(aI - L^\mu)\kappa \varphi (x) + L^\mu,\kappa \varphi (x)|
\]

\[
\leq C (\kappa) \int_0^\infty t^{-\kappa-1} |e^{-at} - 1| |\varphi (x) - E \varphi (x + Z^\mu_t)| dt + a^\kappa |\varphi (x)|
\]

\[
\leq 2C (\kappa) |\varphi|_0 \left[ a^\kappa \int_0^a t^{-\kappa-1} (1 - e^{-t}) dt + \int_1^\infty t^{-\kappa-1} (1 - e^{-at}) dt + a^\kappa \right]
\]

\[
\rightarrow 0 \text{ uniformly as } a \rightarrow 0_+, \forall x \in \mathbb{R}^d.
\]

To be precise, for any $\varepsilon > 0$, there is $\delta > 0$ such that

\[
|(aI - L^\mu)\kappa \varphi + L^\mu,\kappa \varphi|_0 < \varepsilon |\varphi|_0
\]

whenever $0 < a < \delta$. Besides,

\[
(aI - L^\mu)2\kappa \varphi - L^\mu,2\kappa \varphi
\]

\[
= [[(aI - L^\mu)\kappa + L^\mu,\kappa] \circ [(aI - L^\mu)\kappa + L^\mu,\kappa] \circ \varphi
\]

\[- 2 (aI - L^\mu)\kappa \circ L^\mu,\kappa \varphi - 2L^\mu,2\kappa \varphi.
\]

By arguments above, when $0 < a < \delta$,

\[
||[(aI - L^\mu)\kappa + L^\mu,\kappa] \circ [(aI - L^\mu)\kappa + L^\mu,\kappa] \circ \varphi|_0 \leq \varepsilon^2 |\varphi|_0,
\]

and

\[
-2 (aI - L^\mu)\kappa \circ L^\mu,\kappa \varphi - 2L^\mu,2\kappa \varphi \rightarrow 0 \text{ uniformly as } a \rightarrow 0_+.
\]

Therefore,

\[
(aI - L^\mu)2\kappa \varphi \rightarrow L^\mu,2\kappa \varphi \text{ uniformly as } a \rightarrow 0_+.
\]

\[
\Box
\]

The following derivation is needed in next two lemmas. Given (1.8),

\[
\psi^\mu (\xi) = w (R)^{-1} \psi^\mu_R (R\xi), \xi \in \mathbb{R}^d, \forall R \in \mathbb{R}_+.
\]

Using the Lévy-Khintchine formula, we obtain

\[
p (t, z) = R^{-d} p^R \left( w (R)^{-1} t, R^{-1} z \right), z \in \mathbb{R}^d, \forall R \in \mathbb{R}_+,
\]
where \( p(t, z), z \in \mathbb{R}^d \) denotes the density function of \( Z_t^R \) if \( \kappa = 1 \) and that of \( Z_t^\beta \) if \( \kappa \in (0, 1) \cup (1, 2) \), \( p^R(t, z), z \in \mathbb{R}^d \) denotes the density of \( Z_t^R := Z_t^\beta \) if \( \kappa = 1 \) and \( Z_t^\beta \) otherwise. Existence of \( p^R(t, z) \) is guaranteed by Lemma 1 in Appendix.

**Lemma 7.** Let \( \kappa \in (0, 2), \beta \in (0, \infty) \) and \( \mu \) be the reference measure. Assume

\[
(3.1) \quad \int_1^\infty t^{\kappa-1} \gamma(t)^{-1} \, dt < \infty.
\]

Then for any function \( \varphi \in \tilde{C}^\kappa_{\infty, \infty}(\mathbb{R}^d) \) and any \( R > 0 \),

\[
\varphi(x + y) - \varphi(x) = C(\kappa) w(R) \int_0^\infty t^{\kappa-1} \int |L^{\mu, \kappa} \varphi(x + Rz)| \, dz \, dt,
\]

where \( p^R(t, x), x \in \mathbb{R}^d \) follows the definition above. In particular,

\[
(3.3) \quad |\varphi(x + y) - \varphi(x)| \leq C w(|y|^{\kappa}) |L^{\mu, \kappa} \varphi|_0, \forall x, y \in \mathbb{R}^d.
\]

**Proof.** We first assume \( \varphi \in C^\infty_b(\mathbb{R}^d) \cap \tilde{C}^\kappa_{\infty, \infty}(\mathbb{R}^d) \). By (2.4),

\[
\varphi(x + y) - \varphi(x) = C \int_0^\infty t^{\kappa-1} e^{-at} [p^R(x)] \int (aI - L^{\mu})^{\kappa} \varphi(x + y + Z_t) - (aI - L^{\mu})^{\kappa} \varphi(x + Z_t) \, dt
\]

where \( Z_t = Z_t^\mu \) if \( \kappa = 1 \) and \( Z_t = Z_t^\beta \) otherwise, and \( p(t, x) \) denotes the probability density function of \( Z_t \). Recall that Lemma 18 claims

\[
\int |\nabla p(t, z)| \, dz < C' \gamma(t)^{-1}.
\]

Let \( a \to 0 \) under (3.1). By Lemma 6 for all \( \kappa \in (0, 2) \),

\[
\varphi(x + y) - \varphi(x) = C(\kappa) \int_0^\infty t^{\kappa-1} \int |L^{\mu, \kappa} \varphi(x + z)| \, dz \, dt
\]

Now consider \( \varphi \in \tilde{C}^\kappa_{\infty, \infty}(\mathbb{R}^d) \). By [15] Proposition 5 and Lemma 4, there is a sequence of functions \( v_n \in C^\infty_b(\mathbb{R}^d) \cap \tilde{C}^\kappa_{\infty, \infty}(\mathbb{R}^d) \) such that

\[
\lim_{n \to \infty} |L^{\mu, \kappa} v_n - L^{\mu, \kappa} \varphi|_0 = 0, \forall \kappa \in [0, 2).
\]
Moreover,
\[ v_n (x + y) - v_n (x) = C (\kappa) w (R)^\kappa \int_0^\infty t^{\kappa - 1} \int L^{\mu,\kappa} v_n (x + Rz) \cdot [p^R (t, z - R^{-1}y) - p^R (t, z)] \, dz \, dt. \]
Pass the limit on both sides. Then (3.2) holds for \( C \nabla \).

Denote
\[ \nabla^\alpha u (x; y) = u (x + y) - u (x) - \chi_\alpha (y) y \cdot \nabla u (x). \]

Lemma 8. Let \( \kappa \in (0, 2), \beta \in (0, \infty) \) and \( \mu \) be the reference measure. Assume
\[ \int_1^{\infty} 1_{\alpha \in [1, 2]} \int_0^1 t^{\kappa - 1} \gamma (t)^{-1} \, dt < \infty, \]

Then for all \( \alpha \in (0, 1) \cup (1, 2) \) and any function \( \varphi \in \tilde{C}^{+\kappa,\beta}_{\infty,\infty} (\mathbb{R}^d) \),
\[ \nabla^\alpha \varphi (x; y) = C (\kappa) w (R)^\kappa \int_0^\infty t^{\kappa - 1} \int L^{\mu,\kappa} \varphi (x + z) \nabla^\alpha p (t, z; -y) \, dz \, dt \]
\[ = C (\kappa) w (R)^\kappa \int_0^\infty t^{\kappa - 1} \int L^{\mu,\kappa} \varphi (x + Rz) \]
\[ \nabla^\alpha p^R (t, z; -R^{-1}y) \, dz \, dt, \forall R > 0. \]

Moreover,
\[ |\nabla^\alpha \varphi (x; y)| \leq C w (|y|)^\kappa |L^{\mu,\kappa} \varphi|_0, \forall x, y \in \mathbb{R}^d. \]

If \( \alpha = 1 \), then (3.7) hold for \( |y| \leq 1 \) and all \( \varphi \in \tilde{C}^{+\kappa,\beta}_{\infty,\infty} (\mathbb{R}^d) \).

Proof. Let \( a > 0 \). Similarly as in Lemma 7 we first consider \( \varphi \in C^\infty_b (\mathbb{R}^d) \cap \tilde{C}^{+\kappa,\beta}_{\infty,\infty} (\mathbb{R}^d) \). By (2.3),
\[ \nabla^\alpha \varphi (x; y) = C \int_0^\infty t^{\kappa - 1} e^{-at} \int (aI - L^\mu)^\kappa \varphi (x + z) \nabla^\alpha p (t, z; -y) \, dz \, dt, \]
where \( C = (\int_0^\infty t^{\kappa - 1} e^{-at} \, dt)^{-1} \). Let \( a \to 0 \) under assumptions (3.4), (3.5). Then for all \( \kappa \in (0, 2) \),
\[ \nabla^\alpha \varphi (x; y) = C w (R)^\kappa \int_0^\infty t^{\kappa - 1} \int L^{\mu,\kappa} \varphi (x + z) \nabla^\alpha p (t, z; -y) \, dz \, dt \]
\[ = C w (R)^\kappa \int_0^\infty t^{\kappa - 1} \int L^{\mu,\kappa} \varphi (x + Rz) \nabla^\alpha p^R (t, z; -R^{-1}y) \, dz \, dt. \]

For general \( \varphi \in \tilde{C}^{+\kappa,\beta}_{\infty,\infty} (\mathbb{R}^d) \). By [15] Proposition 5 and Lemma 4 there is a sequence of functions \( v_n \in C^\infty_b (\mathbb{R}^d) \cap \tilde{C}^{+\kappa,\beta}_{\infty,\infty} (\mathbb{R}^d) \) such that
\[ \lim_{n \to \infty} |L^{\mu,\kappa} v_n - L^{\mu,\kappa} \varphi|_0 = 0, \forall \kappa \in (0, 2), \]
and
\[ \nabla^\alpha v_n (x; y) = C w (R)^\kappa \int_0^\infty t^{\kappa - 1} \int L^{\mu, \kappa} v_n (x + R z) \nabla^\alpha p^R (t, z; -R^{-1} y) \, dz dt. \]

Passing the limit on both sides, we obtain (3.6) for all functions in \( C^{\kappa + \beta} (\mathbb{R}^d) \).

Setting \( R = |y|, y \neq 0, \) we have
\[ \nabla^\alpha \varphi (x; y) = C (\kappa) w (|y|)^\kappa \int_0^\infty t^{\kappa - 1} \int L^{\mu, \kappa} \varphi (x + |y| z) \nabla^\alpha p^{|y|} (t, z; -|y|^{-1} y) \, dz dt. \]

(3.8) then follows from (3.8), (3.4) and (3.5). \( \square \)

Now we are ready to prove the stronger continuity of the operator. Choose \( \eta (x) \in C_0^\infty (\mathbb{R}^d) \) such that \( 0 \leq \eta (x) \leq 1, \forall x \in \mathbb{R}^d, \text{supp} (\eta) \subseteq \{ x : |x| \leq 2 \}, \) and \( \eta (x) \equiv 1 \) on \( B_1 (0) \). \( \eta_{m, z} (x) := \eta (m (x - z)), m \geq 1, \)

3.1. Operators with Space-Dependent Kernels. Let \( \nu \) be a Lévy measure satisfying \( \tilde{A} (w, l, \gamma) \). We now consider \( \rho (t, x, y) \nu (dy) \), where \( \rho (t, x, y) \) satisfies \( H(K, \beta) \). Obviously, \( \rho (t, x, y) \nu (dy) \) is a Lévy measure for each fixed \( x \in \mathbb{R}^d \) and \( t \in [0, T] \). Denote
\[ L_{t, z} u (x) \]
\[ = \int \left[ u (x + y) - u (t, x) - \chi_\alpha (y) y \cdot \nabla u (x) \right] \rho (t, z, y) \nu (dy), \]
and
\[ \langle u, \eta_{m, z} \rangle_{t, z} \]
\[ = \int \left[ u (x + y) - u (x) \right] \left[ \eta_{m, z} (x + y) - \eta_{m, z} (x) \right] \rho (t, z, y) \nu (dy). \]

Lemma 9. Let \( \nu \) be a Lévy measure satisfying \( \tilde{A} (w, l, \gamma) \) and \( \rho \) be a bounded measurable function. \( \beta \in (0, 1/\alpha) \). \( u \in C^{1 + \beta} (\mathbb{R}^d) \). Then there is \( \beta' \in (0, \beta) \) such that
\[ |L_{t, z} u|_0 \leq C \sup_{t, z, y} |\rho (t, z, y)| |u|_{1 + \beta', \infty}, \]
\[ |L_{t, z} u|_{\beta} \leq C \sup_{t, z, y} |\rho (t, z, y)| |u|_{1 + \beta, \infty}. \]

where \( C \) does not depend on \( t, z \) or \( u \).

Proof. Recall parameters introduced in \( A(w, l) \). Clearly,
\[ |L_{t, z} u|_0 \leq \sup_{t, z, y} |\rho (t, z, y)| \left( \int_{|y| \leq 1} |\nabla^\alpha u (x; y)| \nu (dy) + \int_{|y| > 1} |\nabla^\alpha u (x; y)| \nu (dy) \right). \]

Choose \( \kappa \in (0, 1) \) sufficiently small so that \( \lim_{r \to \infty} w (r)^\kappa / r^{\alpha_2} = 0 \). According to [15, Lemma 1], such a \( \kappa \) must exist. Then by (3.3) and
\( \tilde{A}(w, l, \gamma)(\text{iii}), \) for all \( \alpha \in (0, 1], \)
\[
\int_{|y| > 1} |\nabla^\alpha u(x; y)| \nu(dy) \leq C |L^{\mu, \kappa}u|_0 \int_{|y| > 1} w(|y|)^\kappa \nu(dy) \\
\leq C (d, \kappa, \alpha) |L^{\mu, \kappa}u|_0, \forall x \in \mathbb{R}^d.
\]

If \( \alpha \in (1, 2), \) then we use Lemma 5 and \( A(w, l). \)
\[
\int_{|y| > 1} |\nabla^\alpha u(x; y)| \nu(dy) \leq C (d, \alpha) \left(|u|_0 + |u|_{1, \infty}\right), \forall x \in \mathbb{R}^d.
\]

It follows from [15 Proposition 4] and Lemma 4 that for all \( \alpha \in (0, 2) \) and any \( \beta' \in (0, \delta), \)
\[
\int_{|y| > 1} |\nabla^\alpha u(x; y)| \nu(dy) \leq C (d, \kappa, \alpha, \beta') |u|_{1+\beta', \infty}, \forall x \in \mathbb{R}^d.
\]

On the other hand, suggested by \( \tilde{A}(w, l, \gamma)(\text{iii}), \) we apply (3.7) by setting \( \beta' \in (0, \delta) \) if \( \alpha \neq 1 \) and \( \beta' = \delta \) if \( \alpha = 1. \) Then
\[
\int_{|y| > 1} |\nabla^\alpha u(x; y)| \nu(dy) \\
\leq C (d, \beta', \alpha) |L^{\mu, 1+\beta'}u|_0 \int_{|y| \leq 1} w(|y|)^{1+\beta'} \nu(dy), \forall x \in \mathbb{R}^d.
\]

By Lemma 17 (c) and Lemma 4
\[
\int_{|y| \leq 1} |\nabla^\alpha u(x; y)| \nu(dy) \leq C (d, \beta', \alpha) |u|_{1+(\beta+\beta')/2, \infty}, \forall x \in \mathbb{R}^d.
\]

Now consider \( |L_z u(x_1) - L_z u(x_2)|. \) Set \( a = |x_1 - x_2|. \) Then,
\[
|L_t z u(x_1) - L_t z u(x_2)| \\
\leq \sup_{t, z, y} |\rho(t, z, y)| \int_{|y| \leq a} |\nabla^\alpha (u(x_1; y) - u(x_2; y))| \nu(dy) \\
+ \sup_{t, z, y} |\rho(t, z, y)| \int_{|y| > a} |\nabla^\alpha (u(x_1; y) - u(x_2; y))| \nu(dy).
\]

Denote \( \zeta(r) = \nu(|y| > r) \) and take \( \beta' \in (0, \delta) \) if \( \alpha \neq 1 \) and \( \beta' = \delta \) if \( \alpha = 1. \) Apply (3.8), [15 Proposition 1], Lemmas 4 and 17 (a).
\[
\int_{|y| \leq a} |\nabla^\alpha (u(x_1; y) - u(x_2; y))| \nu(dy) \\
\leq C \sup_{z} \left|L^{\mu, 1+\beta'}u(x_1 + z) - L^{\mu, 1+\beta'}u(x_2 + z)\right| \int_{|y| \leq a} w(|y|)^{1+\beta'} \nu(dy) \\
\leq -C \left[L^{\mu, 1+\beta'}u\right]_{\beta-\beta'} w(a)^{\beta-\beta'} \int_{0}^{a} \zeta(r)^{-1-\beta'} d\zeta(r) \\
\leq C |u|_{1+\beta, \infty} w(a)^{\beta-\beta'} \zeta(r)^{-\beta'} \int_{0}^{a} \leq C |u|_{1+\beta, \infty} w(a)^{\beta}.
\]
Recall $\~A(w,l,\gamma)$ and set $\kappa = \beta + \min(\delta,\varepsilon)/2$ if $\alpha \neq 1$ and $\kappa = \beta + (\delta' + \varepsilon)/2$ if $\alpha = 1$. Apply (3.8), [15, Proposition 1] and Proposition 1

$$
\int_{|y|>a} |\nabla^\alpha (u(x_1;y) - u(x_2;y))| \nu(dy)
\leq C \sup_z |L^{\mu,1+\beta-\kappa} u(x_1 + z) - L^{\mu,1+\beta-\kappa} u(x_2 + z)| \int_{|y|>a} w(|y|)^{1+\beta-\kappa} \nu(dy)
\leq C |u|_{1+\beta,\infty} w(a)^\kappa \int_{|y|>a} w(|y|)^{1+\beta-\kappa} \nu(dy).
$$

Similarly as above,

$$
\int_{|y|>a} |\nabla^\alpha (u(x_1;y) - u(x_2;y))| \nu(dy)
\leq -C |u|_{1+\beta,\infty} w(a)^\kappa \int_a^\infty \zeta(r)^{-1-\beta+\kappa} \nu(r)
\leq -C |u|_{1+\beta,\infty} w(a)^\kappa \zeta(r)^{-\beta}|_a^\infty \leq C |u|_{1+\beta,\infty} w(a)^\beta.
$$

As a conclusion, $|L_{t,z}u|_{\beta} \leq C \sup_{t,z,y} |\rho(t,z,y)||u|_{1+\beta,\infty}$. \qed

**Corollary 1.** Let $\nu$ be a Lévy measure satisfying $\~A(w,l,\gamma)$ and $\rho$ satisfy $H(K,\beta)$, $\beta \in (0,1/\alpha)$. Then for any $u \in \tilde{C}^{1+\beta}_{\infty,\infty}(\mathbb{R}^d)$,

$$
|Au|_{\beta} \leq C \left( \sup_{t,z,y} |\rho(t,z,y)||u|_{1+\beta,\infty} + \sup_{t,y} |\rho(t,\cdot,y)||u|_{1+\beta',\infty} \right),
$$

where $\beta' \in (0,\beta)$ and $C$ does not depend on $u$.

**Proof.** Obviously, $|Au|_0 \leq \sup_{t,z} |L_{t,z}u|_0 \leq C \sup_{t,z,y} |\rho(t,z,y)||u|_{1+\beta',\infty}$ for some $\beta' \in (0,\beta)$. Meanwhile,

$$
|L_{t,x+y} u(x + y) - L_{t,x} u(x)|
\leq |L_{t,x+y} u(x + y) - L_{t,z} u(x + y)| + |L_{t,z} u(x + y) - L_{t,x} u(x)|
\leq C \sup_{t,y} |\rho(t,\cdot,y)||u|_{1+\beta',\infty} w(|y|)^\beta + C \sup_{t,z,y} |\rho(t,z,y)||u|_{1+\beta,\infty} w(|y|)^\beta.
$$

Namely, $|Au|_{\beta} \leq C \left( \sup_{t,y} |\rho(t,\cdot,y)||u|_{1+\beta',\infty} + \sup_{t,z,y} |\rho(t,z,y)||u|_{1+\beta,\infty} \right)$. \qed

**Lemma 10.** Let $\nu$ be a Lévy measure satisfying $\~A(w,l,\gamma)$ and $\rho$ satisfy $H(K,\beta)$. $\beta \in (0,1/\alpha)$. Then for any $u \in \tilde{C}^{1+\beta}_{\infty,\infty}(\mathbb{R}^d)$ and any $\varepsilon \in (0,1)$,

$$
(3.11) \quad \sup_{t,z} |\langle u, \eta_{m,z} \rangle_{t,z}|_{\beta,\infty} \leq C \varepsilon |m|^{1+\beta} \left( |u|_{1+\beta,\infty} + C\varepsilon |u|_0 \right),
$$

where $C\varepsilon$ depends on $\varepsilon$ but is independent of $u$.

**Proof.** Direct computation shows that for $\kappa \in (0,1)$,

$$
L^{\mu,\kappa} \eta_{m,z}(x) = w(m^{-1})^{-\kappa} L^{\mu-1-\kappa} \eta(m(x-z)).
$$
By $\tilde{A}(w, l, \gamma)$, there is $\kappa \in (1/2, 1)$ such that $\int_1^\infty t^{\kappa - 1} \gamma(t)^{-1} dt < \infty$. Apply (3.3) with such a $\kappa$.

Similarly, we use (3.3) with such a $\kappa$.

\[
|\langle u, \eta_{m,z} \rangle_{t, z}|_0 \leq C \int |u(x + y) - u(x)| |\eta_{m,z}(x + y) - \eta_{m,z}(x)| \nu(dy) \\
\leq C w(m^{-1})^{-\kappa} |L^{t, \kappa}u|_0 |L^{\tilde{m}^{-1}, \kappa} \eta|_0 \int_{|y| \leq 1} w(|y|)^{2\kappa} \nu(dy) + C |u|_0.
\]

According to Lemmas [17, 4] and [15, Proposition 4],

\[
|\langle u, \eta_{m,z} \rangle_{t, z}|_0 \leq C w(m^{-1})^{-\kappa} (|L^{t, \kappa}u|_0 + |u|_0) \\
\leq C l(m)^\kappa \left( \varepsilon |u|_{1+\beta, \infty} + C \varepsilon |u|_0 \right).
\]

For the difference estimate, let us set $a = |x_1 - x_2|$ and denote

\[
|\langle u, \eta_{m,z} \rangle_{t, z}(x_1) - \langle u, \eta_{m,z} \rangle_{t, z}(x_2)| \\
\leq |\int_{|y| \leq 1} [u(x_1 + y) - u(x_1) - u(x_2 + y) + u(x_2)] \\
\cdot [\eta_{m,z}(x_1 + y) - \eta_{m,z}(x_1)] \rho(t, z, y) \nu(dy) | \\
+ |\int_{|y| \leq 1} [u(x_2 + y) - u(x_2)] [\eta_{m,z}(x_1 + y) - \eta_{m,z}(x_1)] \\
- [u(x_2 + y) - u(x_2)] [\eta_{m,z}(x_2 + y) - \eta_{m,z}(x_2)] \rho(t, z, y) \nu(dy) | \\
+ |\int_{|y| > 1} \{ [u(x_1 + y) - u(x_1)] [\eta_{m,z}(x_1 + y) - \eta_{m,z}(x_1)] \\
- [u(x_2 + y) - u(x_2)] [\eta_{m,z}(x_2 + y) - \eta_{m,z}(x_2)] \} \rho(t, z, y) \nu(dy) | \\
:= I_1 + I_2 + I_3.
\]

Similarly, we use (3.3). Then for some $\kappa \in (1/2, 1)$,

\[
I_1 \leq C w(m^{-1})^{-\kappa} \sup_z |L^{t, \kappa}u(x_1 + z) - L^{t, \kappa}u(x_2 + z)||L^{\tilde{m}^{-1}, \kappa} \eta|_0 \\
\leq C w(m^{-1})^{-\kappa} w(a)^\beta |L^{t, \kappa}u|_{\beta, \infty} \\
\leq C l(m)^\kappa w(a)^\beta \left( \varepsilon |u|_{1+\beta, \infty} + C \varepsilon |u|_0 \right).
\]

For the same $\kappa$,

\[
I_2 \leq C w(m^{-1})^{-\kappa} \sup_z |L^{\tilde{m}^{-1}, \kappa} \eta(m(x_1 - z)) - L^{\tilde{m}^{-1}, \kappa} \eta(m(x_2 - z))||L^{t, \kappa}u|_0 \\
\leq C w(m^{-1})^{-\kappa} l(m)^\beta w(a)^\beta |u|_{n+\beta, \infty} \\
\leq C l(m)^{1+\beta} w(a)^\beta \left( \varepsilon |u|_{1+\beta, \infty} + C \varepsilon |u|_0 \right).
\]
Besides,

\[ I_3 \leq C \int_{|y|>1} |[u(x_1 + y) - u(x_1)] [\eta_{m,z}(x_1 + y) - \eta_{m,z}(x_1) - \eta_{m,z}(x_2) + \eta_{m,z}(x_2)]| \nu(dy) \]

\[ + C \int_{|y|>1} |[u(x_1 + y) - u(x_2 + y) - u(x_1) + u(x_2)] [\eta_{m,z}(x_2 + y) - \eta_{m,z}(x_2)]| \nu(dy) \]

\[ := I_{31} + I_{32}, \]

where

\[ I_{31} \leq C |u|_0 \int_{|y|>1} |\eta_{m,z}(x_1 + y) - \eta_{m,z}(x_2 + y) - \eta_{m,z}(x_1) + \eta_{m,z}(x_2)| \nu(dy) \leq C |u|_0 l(m)^{1+\beta} w(a)^\beta, \]

and obviously,

\[ I_{32} \leq C w(a)^\beta |u|_{1+\beta,\infty} \leq C w(a)^\beta \left( \varepsilon |u|_{1+\beta,\infty} + C\varepsilon |u|_0 \right). \]

Summarizing, for any \( z \in \mathbb{R}^d \),

\[ [(u, \eta_{m,z})_{t,z}]_\beta \leq Cl(m)^{1+\beta} \left( \varepsilon |u|_{1+\beta,\infty} + C\varepsilon |u|_0 \right). \]

It follows immediately from \( [15, \text{Proposition 1}] \) that

\[ \sup_{t,z} |(u, \eta_{m,z})_{t,z}|_\beta \leq C \sup_{t,z} |(u, \eta_{m,z})_{t,z}|_\beta \leq Cl(m)^{1+\beta} \left( \varepsilon |u|_{1+\beta,\infty} + C\varepsilon |u|_0 \right). \]

3.2. Operators with Space-Dependent Coefficients. In this section, we study the operator

\[ \mathcal{G}\varphi(x) := \int [\varphi(x + G(x)y) - \varphi(x) - \chi_\alpha(y) G(x)y \cdot \nabla \varphi(x)] \nu(dy). \]

We define the norm of an \( d \times d \)-invertible matrix function \( G(x), x \in \mathbb{R}^d \) to be its operator norm, i.e.,

\[ |G(x)| := \sup_{y \in \mathbb{R}^d, |y|=1} |G(x)y|, \]

and

\[ \|G\| := \sup_{x \in \mathbb{R}^d} |G(x)\|. \]

If all entries of \( G \) are constants, then \( G \) is viewed as a constant function and definitions above apply. Note \( \|G\| \) being finite implies finiteness of each entry. If furthermore \( |\det G(z)| \geq c_0 \) for some \( c_0 > 0 \), then \( \|G^{-1}\| \) is also finite.
Lemma 11. Let \( G \) be an invertible \( d \times d \)-matrix and \( \beta > 0 \). \( f \in \mathcal{C}^\beta_{\infty,\infty}(\mathbb{R}^d) \). Define \( g(x) := f(Gx), x \in \mathbb{R}^d \). Then,

\[
(3.12) \quad |g|_{\beta,\infty} \leq C |f|_{\beta,\infty}
\]

for some \( C \) only depending on \( \|G^{-1}\| \) and \( \|G\| \).

Proof. Consider the mapping \( T : \mathbb{R}^d \to \mathbb{R}^d \) such that \( T(x) = Gx \). Then \( T^{-1}(x) = G^{-1}x \). Clearly, both \( T \) and \( T^{-1} \) are continuous and \( \|T\|^{-1} = \|G\|^{-1} \leq \|T^{-1}\| = \|G^{-1}\| \). For any \( j \neq 0 \),

\[
|g * \varphi_j|_0 = \sup_x \left| \int f(Gy) \varphi_j(x-y) \, dy \right| = \sup_x \frac{1}{|\det G|} \left| \int f(y) \varphi_j(G^{-1}x-G^{-1}y) \, dy \right| = \| \mathcal{F}^{-1}[\phi_j(G\xi) \cdot \mathcal{F}f] \|_0.
\]

Note \( \phi_j(G\cdot) \) is supported on \( \{ \xi : N^{j-1} \leq |G\xi| \leq N^{j+1} \} \subset \{ \xi : N^{j-1} ||G||^{-1} \leq \xi | \leq N^{j+1} ||G||^{-1} \} \). Denote

\[
n(j) = \min\{i : N^{j+1} \|G^{-1}\| \leq N^i \} \vee 1,
n(m(j)) = \max\{i : N^i \leq N^{j-1} \|G^{-1}\| \} \vee 0.
\]

Then \( n(j) = n(1) + j - 1 \), \( m(j) = m(1) + j - 1 \), and that \( n(j) - m(j) \leq n(1) - m(1) + 1 \) which is independent of \( j \). Moreover,

\[
\phi_j(G\xi) = \phi_j(G\xi) \sum_{i=m(j)}^{n(j)} \phi_i(\xi).
\]

Therefore,

\[
|g * \varphi_j|_0 = \| \mathcal{F}^{-1}\left[ \phi_j(G\xi) \sum_{i=m(j)}^{n(j)} \phi_i(\xi) \cdot \mathcal{F}f \right] \|_0 \\
\leq (n(1) - m(1) + 1) \sup_j |f * \varphi_j|_0 \| \mathcal{F}^{-1}[\phi_j(G\xi)] \|_{L^1(\mathbb{R}^d)} \\
\leq Cw(N^{-j})^\beta |f|_{\beta,\infty}.
\]

Similarly, if \( j = 0 \),

\[
|g * \varphi_0|_0 = \| \mathcal{F}^{-1}[\widehat{\varphi}_0(G\xi) \cdot \mathcal{F}f] \|_0,
\]

and \( \text{supp}(\widehat{\varphi}_0(G\xi)) = \{ \xi : |G\xi| \leq N \} \subset \{ \xi : |\xi| \leq N \|G^{-1}\| \} \). Denote \( k = \min\{i : N \|G^{-1}\| \leq N^i \} \vee 1 \). Then,

\[
\widehat{\varphi}_0(G\xi) = \varphi_0(G\xi) \left( \varphi_0(\xi) + \sum_{i=1}^k \phi_i(\xi) \right).
\]
Therefore,

\[ |g \ast \varphi_j|_0 = \left| F^{-1} \left[ \widehat{\varphi_0} (G\xi) \left( \widehat{\varphi_0} (\xi) + \sum_{i=1}^{k} \phi_i (\xi) \right) F_j \right] \right|_0 \]

\[ \leq C \sup_j |f \ast \varphi_j|_0 \left| F^{-1} \left[ \widehat{\varphi_0} (G\xi) \right] \right|_{L^1 (\mathbb{R}^d)} \]

\[ \leq C w (N^{-j})^{\beta} |f|_{\beta, \infty} . \]

Summarizing, \( |g|_{\beta, \infty} \leq C |f|_{\beta, \infty} \).

\[ \square \]

**Proposition 2.** Let \( \nu \) be a Lévy measure satisfying \( A(w,l) \) and \( G \) be an invertible \( d \times d \)-matrix. For any function \( f \in C^2_b (\mathbb{R}^d) \),

\[ Lf (x) := \int [f (x + Gy) - f (x) - \chi_\alpha (y) G y \cdot \nabla f (x)] \nu (dy) . \]

Then for \( \beta \in (0,1/\alpha) \), there exists \( C \) depending on \( \|G^{-1}\| \) and \( \|G\| \) such that

\[ |Lf|_{\beta, \infty} \leq C |f|_{1+\beta, \infty} . \]

**Proof.** If \( g (x) := f (Gx) \). Then \( Lf (Gx) = L^\nu g (x) \). By previous continuity and equivalence results and Lemma 11,

\[ |Lf|_{\beta, \infty} \leq C |Lf (G \cdot)|_{\beta, \infty} = |L^\nu g|_{\beta, \infty} \leq C |g|_{1+\beta, \infty} \leq C |f|_{1+\beta, \infty} . \]

\[ \square \]

Let us denote

\[ \nabla^{\alpha; z} u (x; y) = u (x + G (z) y) - u (x) - \chi_\alpha (y) G (z) y \cdot \nabla u (x) , \]

(3.13) \[ L_z u (x) = \int \nabla^{\alpha; z} u (x; y) \nu (dy) , \]

\[ \bar{G}_{z,z'} := \|G (z) - G (z')\| , \]

and

\[ g (z, z') = \begin{cases} 
\frac{w \left( \bar{G}_{z,z'}^{-1} \right)}{w \left( \bar{G}_{z,z'}^{-1} \right)} & \text{if } \alpha \in (0,1) , \\
\frac{w \left( \bar{G}_{z,z'}^{-1} \right)}{w \left( \bar{G}_{z,z'}^{-1} \right)} & \text{if } \alpha = 1 , \\
\bar{G}_{z,z'} & \text{if } \alpha \in (1,2) . 
\end{cases} \]

**Lemma 12.** Let \( \beta \in (0,1/\alpha) \), \( \nu \) be a Lévy measure satisfying \( \tilde{A}(w,l,\gamma) \), and \( G(z), \forall z \in \mathbb{R}^d \) satisfy \( G(c_0, K, \beta) \). If \( u \in C^{1+\beta} (\mathbb{R}^d) \), then,

\[ |L_z u - L_{z'} u|_0 \leq C g (z, z') |u|_{1+\beta', \infty} , \]

\[ |L_z u - L_{z'} u|_{\beta} \leq C \left( \bar{G}_{z,z'} \right)^{\sigma} |u|_{1+\beta, \infty} \]

for some \( \beta' \in (0, \beta) , \sigma \in (0,1) \). \( C \) is independent \( z, z' \) and \( u \).
Proof. Write for simplicity $G = G(z)$, $G' = G(z')$ and $\bar{G} = \bar{G}_{z,z'}$. Use (3.6).

\[
\nabla^{\alpha,z} u(x; y) - \nabla^{\alpha,z'} u(x; y) = cw(R)^\kappa \int_0^\infty t^{\kappa-1} \int L^{\mu,\kappa} u(x + R\theta) \cdot \left[ \nabla^{\alpha,z} p^R(t, \theta; -R^{-1}Gy) - \nabla^{\alpha,z'} p^R(t, \theta; -R^{-1}G'y) \right] d\theta dt
\]

\[
:= D(\kappa, R)
\]

with $\kappa$ and $R$ to be determined according to our needs. If $\alpha \in (0, 1)$, by $\tilde{A}(w, l, \gamma)$, we may split the integral as follows.

\[
|L_z u(x) - L_{z'} u(x)| \\
\leq \int_{|y| \leq 1} |D(\kappa, \bar{G}|y|)| \nu(dy) + \int_{|y| > 1} |u(x + Gy) - u(x + G'y)| \nu(dy)
\]

\[
:= I_1 + I_2
\]

for some $\kappa \in (1, 1 + \delta)$. By Lemmas 17 and 18 in Appendix,

\[
I_1 \leq C |L^{\mu,\kappa} u|_0 \int_{|y| \leq 1} \int_0^\infty t^{\kappa-1} \left( 1 \land \gamma(t) \right)^{-1} \bar{G}^{-1} |y|^{-1} |Gy - G'y| dt
\]

\[
w(G|y|)^\kappa \nu(dy)
\]

\[
\leq C w(\bar{G}^{-1})^{-1} |L^{\mu,\kappa} u|_0 \int_{|y| \leq 1} w(|y|)^\kappa \tilde{\nu}_{\bar{G}^{-1}}(dy)
\]

\[
\leq C w(\bar{G}^{-1})^{-1} |L^{\mu,\kappa} u|_0.
\]

Besides,

\[
I_2 \leq C w(\bar{G}^{-1})^{-1} \int_{|y| > 1} |u|_0 \tilde{\nu}_{\bar{G}^{-1}}(dy) \leq C w(\bar{G}^{-1})^{-1} |u|_0.
\]

Therefore when $\alpha \in (0, 1)$, there exists $\beta' \in (0, \beta)$ such that

\[
|L_z u - L_{z'} u|_0 \leq C w(\bar{G}^{-1})^{-1} |u|_{1+\beta', \infty}.
\]

If $\alpha = 1$, we write instead

\[
|L_z u(x) - L_{z'} u(x)|
\]

\[
\leq \int_{|y| \leq 1} |D(1 + \delta, |y|)| \nu(dy) + \int_{|y| > 1} \int_{|y| \leq 1, |y| > 1} |u(x + Gy) - u(x + G'y)| \nu(dy)
\]

\[
+ \int_{|y| > 1, |y| > 1} |u(x + Gy) - u(x + G'y)| \nu(dy) := I_3 + I_4 + I_5,
\]

where

\[
I_3 \leq C \left| L^{\mu,1+\delta} u \right|_0 \int_{|y| \leq 1} w(|y|)^{1+\delta} \int_0^\infty t^{\kappa-1} G \left( \gamma(t)^{-1} \land \gamma(t)^{-2} \right) dt \nu(dy)
\]

\[
\leq CG \left| L^{\mu,1+\delta} u \right|_0.
\]
Meanwhile, similarly as \(I_1, I_2\), we have

\[
 w (G^{-1}) (I_4 + I_5)
 \leq C \left( \left| \mu^{1-\delta'} \right|_0 \int_{|y| \leq 1} w \left( |y|^{1-\delta'} \tilde{\nu}_{G^{-1}} (dy) \right) + \int_{|y| > 1} |u_0| \tilde{\nu}_{G^{-1}} (dy) \right)
\]

\[
 \leq C \left( \frac{1}{\delta'} \right) \left| \mu^{1-\delta'} \right|_0 + \frac{1}{\delta'} \left( \left| \mu^{1-\delta'} \right|_0 w (\tilde{G})^{-\delta'} + |u_0| \right)
\].

Thus for \(\alpha = 1\), there is \(\beta' \in (0, \beta)\) so that

\[
(3.15) \quad |L_2 u - L_2 u_0| \leq \frac{C}{\delta'} \left( \tilde{G} \vee w (\tilde{G}^{-1})^{-1} w (\tilde{G})^{-\delta'} \right) |u|_{1+\beta', \infty}.
\]

Next, we discuss the case \(\alpha \in (1, 2)\). Split the integral as

\[
|L_2 u (x) - L_2 u_0 (x)|
\leq \int_{|y| \leq 1} |D (\kappa, |y|)| \nu (dy) + \int_{|y| > 1} \left| \nabla^{\alpha, z} u (x; y) - \nabla^{\alpha, z'} u (x; y) \right| \nu (dy)
\]

\[
:= I_6 + I_7.
\]

Then as how we estimated \(I_3\), we have \(I_6 \leq \tilde{C} |L^{\mu, \kappa} u|_0\) for some \(\kappa \in (1, 1 + \delta)\). Clearly, \(I_7 \leq \tilde{C} \tilde{G} |\nabla u|_0\). Thus for \(\alpha \in (1, 2)\), there is \(\beta' \in (0, \beta)\) so that

\[
(3.16) \quad |L_2 u - L_2 u_0| \leq \tilde{C} \tilde{G} |u|_{1+\beta', \infty}.
\]

We now estimate the difference. Without loss of generality, we set \(|x_1 - x_2| = a \in (0, 1)\). By Lemma 8,

\[
\nabla^{\alpha, z} u (x_1; y) - \nabla^{\alpha, z'} u (x_1; y) - \nabla^{\alpha, z} u (x_2; y) + \nabla^{\alpha, z'} u (x_2; y)
\]

\[
= C w (R)^{\kappa} \int_0^\infty \kappa^{-1} \left[ \mu^{1-\delta} (x_1 + R \partial) - \mu^{1-\delta} (x_2 + R \partial) \right]
\cdot \left[ \nabla^{\alpha, z} p^R (t, \partial; -R^{-1} y) - \nabla^{\alpha, z'} p^R (t, \partial; -R^{-1} y) \right] \, d\partial dt
\]

\[
:= \tilde{D} (\kappa, R)
\]

with \(\kappa\) and \(R\) to be determined. Then,

\[
|L_2 u (x_1) - L_2 u (x_1) - L_2 u (x_2) + L_2 u (x_2)|
\leq \int_{|y| \leq a} |\tilde{D} (\kappa, |y|)| \nu (dy) + \int_{|y| > a} |\tilde{D} (\kappa', |y|)| \nu (dy)
\]

\[
:= I_8 + I_9.
\]

Denote \(\zeta (r) = \nu (|y| > r)\). By Lemma 18 and [15 Lemma 1], for all \(\alpha \in (0, 2)\), there is \(\beta' \in (0, \beta)\) and \(\sigma \in (0, 1)\) such that

\[
I_8 \leq -C \left[ \mu^{1+\beta'} u \right]_{\beta-\beta'} \tilde{G}^{\sigma} w (a)^{\beta-\beta'} \int_0^\alpha \zeta (r)^{-1-\beta'} \, d\zeta (r)
\]

\[
\leq C |u|_{1+\beta', \infty} \tilde{G}^{\sigma} w (a)^{\beta-\beta'} \zeta (r)^{-\beta'} |\alpha| \leq C |u|_{1+\beta', \infty} \tilde{G}^{\sigma} w (a)^{\beta}.
\]
Recall $\tilde{A}(w,l,\gamma)$. Using the symmetry assumption for $\alpha = 1$ and non-degeneracy of $G$, we can set $\kappa = \beta + \min(\delta, \varepsilon)/2$ if $\alpha \neq 1$ and $\kappa = \beta + (\delta' + \varepsilon)/2$ if $\alpha = 1$, there is $\beta' \in (0, \beta)$ and $\sigma \in (0, 1)$ such that

$$I_9 \leq C \left[ L^{\mu, 1+\beta - \kappa} u \right]_\kappa \bar{G}^\sigma w(a)^\kappa \int_{|y| > a} w(|y|)^{1+\beta - \kappa} \nu(dy)$$

$$\leq -C |u|_{1+\beta, \infty} \bar{G}^\sigma w(a)^\kappa \int_a^\infty \varsigma(r)^{-1-\beta + \kappa} d\varsigma(r)$$

$$\leq C |u|_{1+\beta, \infty} \bar{G}^\sigma w(a)^\kappa \varsigma(a)^{\kappa - \beta} \leq C |u|_{1+\beta, \infty} \bar{G}^\sigma w(a)^\beta .$$

This ends the proof. 

**Corollary 2.** Let $\beta \in (0, 1/\alpha)$, $\nu$ be a Lévy measure satisfying $\tilde{A}(w, l, \gamma)$, and $G(z), \forall z \in \mathbb{R}^d$ satisfy $G(c_0, K, \beta)$. If $u \in \bar{C}^{1+\beta}(\mathbb{R}^d)$, then,

$$|Gu - L_z u|_{\beta} \leq C \left( (\bar{G}_{x,z})^\sigma |u|_{1+\beta, \infty} + |u|_{1+\beta', \infty} \right)$$

for some $\beta' \in (0, \beta)$. $C$ is independent of $x, z$ and $u$.

**Proof.** First by Lemma 12, $|Gu - L_z u|_0 \leq \sup_{x, x'} |L_z u - L_z u|_0 \leq C |u|_{1+\beta', \infty}$ for some $\beta' \in (0, \beta)$. In the meantime,

$$|L_{x+y} u(x+y) - L_z u(x+y) - L_x u(x) + L_z u(x)|$$

$$\leq |L_x u(x+y) - L_z u(x+y) - L_x u(x) + L_z u(x)|$$

$$+ |L_{x+y} u(x+y) - L_x u(x+y)|$$

$$\leq C (\bar{G}_{x,z})^\sigma |u|_{1+\beta, \infty} w(|y|)^\beta + C |u|_{1+\beta', \infty} w(|y|)^\beta .$$

Namely, $|Gu - L_z u|_{\beta} \leq C \left( (\bar{G}_{x,z})^\sigma |u|_{1+\beta, \infty} + |u|_{1+\beta', \infty} \right)$. 

For $\eta_{m, z}$ introduced in previous section, we denote

$$\langle u, \eta_{m, z} \rangle_z$$

$$= \int [u(x + G(z)y) - u(x)] [\eta_{m, z}(x + G(z)y) - \eta_{m, z}(x)] \nu(dy).$$

**Lemma 13.** Let $\nu$ be a Lévy measure satisfying $\tilde{A}(w, l, \gamma)$ and $\|G(z)\| \leq K, \forall z \in \mathbb{R}^d$ for some $K > 0$. $\beta \in (0, 1/\alpha)$. Then for any $u \in \bar{C}^{1+\beta}_{1, \infty}(\mathbb{R}^d)$ and any $\varepsilon \in (0, 1)$,

$$\sup_z |\langle u, \eta_{m, z} \rangle_z|_{\beta, \infty} \leq Cl(m)^{1+\beta} \left( \varepsilon |u|_{1+\beta, \infty} + C_\varepsilon |u|_0 \right),$$

where $C_\varepsilon$ depends on $\varepsilon$ but is independent of $u$. 

\[ (3.17) \] 
\[ (\langle u, \eta_{m, z} \rangle_z) \] 
\[ = \int [u(x + G(z)y) - u(x)] [\eta_{m, z}(x + G(z)y) - \eta_{m, z}(x)] \nu(dy). \]
Proof. We proceed in the same manner as in Lemma [10] First, since \( \|G(z)\| \)
is uniformly bounded, there is \( \kappa \in (1/2, 1) \) such that
\[
\langle u, \eta_{m,z} \rangle_{z} \leq \frac{1}{\kappa} \frac{M}{M^*} \|L_{\mu}^{\kappa}u\|_0 \|L_{\bar{\mu}}^{m-1,\kappa} \eta\|_0 \int \langle u, \eta \rangle_{\nu} dy + C \|u\|_0 \\
\leq C l(m)^{\kappa} \left( \varepsilon \|u\|_{1+\beta,\infty} + C \varepsilon \|u\|_0 \right).
\]
For the difference, again, let us set \( a = |x_1 - x_2| \in (0, 1) \) and estimate
\[
\langle u, \eta_{m,z} \rangle(x_1) - \langle u, \eta_{m,z} \rangle(x_2)
\leq \left| \int_{|y| \leq 1} [u(x_1 + G(z)y) - u(x_1) - u(x_2 + G(z)y) + u(x_2)] \cdot [\eta_{m,z}(x_1 + G(z)y) - \eta_{m,z}(x_1)] \nu(dy) \right|
+ \left| \int_{|y| > 1} [u(x_2 + G(z)y) - u(x_2)] [\eta_{m,z}(x_1 + G(z)y) - \eta_{m,z}(x_1)] - [\eta_{m,z}(x_2 + G(z)y) + \eta_{m,z}(x_2)] \nu(dy) \right|
+ \left| \int_{|y| > 1} [u(x_1 + G(z)y) - u(x_1) - u(x_2 + G(z)y) + u(x_2)] \cdot [\eta_{m,z}(x_1 + G(z)y) - \eta_{m,z}(x_1)] \nu(dy) \right|
:= I_1 + I_2 + I_3 + I_4.
\]
Then \( (8.3) \) implies that
\[
I_1, I_2 \leq C l(m)^{1+\beta} \|w(a)\| \left( \varepsilon \|u\|_{1+\beta,\infty} + C \varepsilon \|u\|_0 \right),
\]
where \( C \) depends on \( K \). Meanwhile,
\[
I_3 \leq C \|u\|_0 \int_{|y| > 1} |\eta_{m,z}(x_1 + G(z)y) - \eta_{m,z}(x_2 + G(z)y) - \eta_{m,z}(x_1) + \eta_{m,z}(x_2)| \nu(dy) \leq C \|u\|_0 l(m)^{\beta} \|w(a)\|^{\beta},
\]
and obviously,
\[
I_4 \leq C w(a)\|u\|_{\beta,\infty} \leq C w(a)\| (\varepsilon \|u\|_{1+\beta,\infty} + C \varepsilon \|u\|_0) \!
\]
Summarizing,
\[
\sup_z \langle u, \eta_{m,z} \rangle \leq C l(m)^{1+\beta} \left( \varepsilon \|u\|_{1+\beta,\infty} + C \varepsilon \|u\|_0 \right),
\]
and thus,
\[
\sup_z \langle u, \eta_{m,z} \rangle z|_{\beta, \infty} \leq C l(m)^{1+\beta} \left( \varepsilon |u|_{1+\beta, \infty} + C_\varepsilon |u|_0 \right).
\]

\[\square\]

3.3. Lower Order Operators. For any function \( u \in C^2_0 (H_T) \), denote
\[
Q(t,z,z')u(t,x)
\]
\[
:= 1_{\alpha \in (1,2)} b(t,z) \cdot \nabla u(t,x) + p(t,z) u(t,x) + \int_{R^d_0} [u(t,x + q(t,z',y))] - u(t,x) - \nabla u(t,x) \cdot q(t,z',y) 1_{\alpha \in (1,2) \leq 1} q(t,z,y) \nu_2(dy)
\]
\[
:= 1_{\alpha \in (1,2)} b(t,z) \cdot \nabla u(t,x) + p(t,z) u(t,x) + \tilde{Q}(t,z,z')u(t,x).
\]

Lemma 14. Let \( B(K, \beta) \) hold. \( \beta \in (0, 1/\alpha) \). Then for any \( u \in C^{1+\beta}_{\infty, \infty} (H_T) \) and any \( \varepsilon \in (0, 1) \), there exists \( \beta' \in (0, \beta) \),

\[
(3.19) \sup_{t,z,z'} \left| \tilde{Q}_{t,z,z'}u(t,\cdot) \right|_0 \leq C \sup_{t,z,y} |\rho (t,z,y)| |u|_{1+\beta', \infty},
\]

\[
(3.20) \sup_{t,z,z'} \left[ \tilde{Q}_{t,z,z'}u(t,\cdot) \right]_{\beta'} \leq \sup_{t,z,y} |\rho (t,z,y)| \left( \varepsilon |u|_{1+\beta, \infty} + C_\varepsilon |u|_0 \right),
\]

where \( C, C_\varepsilon \) are independent of \( u \).

Proof. We split the integral.
\[
\left| \tilde{Q}_{t,z,z'}u(t,x) \right|_0 \leq C \sup_{t,z,y} |\rho (t,z,y)| \int_{|y| \leq 1} 1_{\alpha \in (1,2)} |\nabla^\alpha u(t,x; q(t,z',y))| \nu_2(dy)
\]
\[
+ C \sup_{t,z,y} |\rho (t,z,y)| \int_{R^d_0} 1_{\alpha \in (0,1]} |u(t,x + q(t,z',y)) - u(t,x)| \nu_2(dy)
\]
\[
+ C \sup_{t,z,y} |\rho (t,z,y)| \int_{|y| > 1} 1_{\alpha \in (1,2)} |u(t,x + q(t,z',y)) - u(t,x)| \nu_2(dy)
\]
\[
:= C \sup_{t,z,y} |\rho (t,z,y)|(I_1 + I_2 + I_3).
\]

Use assumptions \( \tilde{A}(w,l,\gamma) \) and \( B(K, \beta) \). By Lemma \[8\]
\[
I_1 \leq C |L^\mu u|_0 \int_{|y| \leq 1} w(\|q(t,z',y)\|) \nu_2(dy) \leq C |L^\mu u|_0.
\]

Take \( \beta' \in (0, \delta) \). Then by Lemma \[7\]
\[
I_2 \leq C 1_{\alpha \in (0,1)} \int_{R^d_0} \left( |L^{\mu,1+\beta'} u|_0 w(\|q(t,z',y)\|^{1+\beta'} \wedge |u|_0) \right) \nu_2(dy)
\]
\[
+ C 1_{\alpha = 1} \int_{R^d_0} (|\nabla u|_0 \|q(t,z',y)\| \wedge |u|_0) \nu_2(dy)
\]
\[
\leq C \left( 1_{\alpha \in (0,1)} |L^{\mu,1+\beta'} u|_0 + 1_{\alpha = 1} |\nabla u|_0 + |u|_0 \right).
\]
Clearly, $I_3 \leq C (|\nabla u|_0 + |u|_0)$. Summarizing, there exists $\beta' \in (0, \beta)$ so that
\[
\sup_{t,z,z'} |\tilde{Q}_{t,z,z'} u (t, x) - \tilde{Q}_{t,z,z'} u (t, x_1)|_0 \leq C \sup_{t,z,y} |\rho (t, z, y)| |u|_{1+\beta',\infty}.
\]

Meanwhile,
\[
\sup_{t,z,y} |\rho (t, z, y)|^{-1} \left| \tilde{Q}_{t,z,z'} u (t, x_1) - \tilde{Q}_{t,z,z'} u (t, x_2) \right|
\leq C \int_{|y| \leq 1} 1_{\alpha \in (1,2)} \left| \nabla^\alpha u (t, x_1; q (t, z', y)) - \nabla^\alpha u (t, x_2; q (t, z', y)) \right| \nu_2 (dy)
+ C \int_{|y| > 1} 1_{\alpha \in (1,2)} |u (t, x_1 + q (t, z', y)) - u (t, x_1)| + u (t, x_2 + q (t, z', y))
+ u (t, x_2) \nu_2 (dy)
:= C (I_4 + I_5 + I_6).
\]

Set $|x_1 - x_2| = a$. Then for any $\epsilon \in (0,1)$,
\[
I_4 = \int_{|q(t,z',y)| \leq |\epsilon|, |y| \leq 1} \left| \nabla^\alpha u (t, x_1; q (t, z', y)) - \nabla^\alpha u (t, x_2; q (t, z', y)) \right| \nu_2 (dy)
+ \int_{|q(t,z',y)| > |\epsilon|, |y| \leq 1} \left| \nabla^\alpha u (t, x_1; q (t, z', y)) - \nabla^\alpha u (t, x_2; q (t, z', y)) \right| \nu_2 (dy)
:= I_{41} + I_{42}.
\]

By $B(K, \beta)$, for any $\epsilon \in (0,1)$, there exists $\epsilon \in (0,1)$ such that
\[
I_{41} = w (a)^\beta |L^\mu u (t, \cdot)|_{\beta} \int_{|q(t,z',y)| \leq |\epsilon|, |y| \leq 1} w (|q (t, z', y)|) \nu_2 (dy)
\leq C w (a)^\beta |u (t, \cdot)|_{1+\beta,\infty} \int_{|q(t,z',y)| \leq |\epsilon|} w (|q (t, z', y)|) \nu_2 (dy)
\leq \frac{\epsilon}{5} w (a)^\beta |u (t, \cdot)|_{1+\beta,\infty}.
\]

There also exists $\kappa \in (0,1)$ so that
\[
I_{42} = w (a)^\beta |L^\mu u (t, \cdot)|_{\beta} \int_{|q(t,z',y)| > |\epsilon|, |y| \leq 1} w (|q (t, z', y)|) \nu_2 (dy)
\leq Cl (\epsilon^{-1})^{1-\kappa} w (a)^\beta |u (t, \cdot)|_{\kappa+\beta,\infty} \int_{|y| \leq 1} w (|q (t, z', y)|) \nu_2 (dy)
\leq Cl (\epsilon^{-1})^{1-\kappa} w (a)^\beta |u (t, \cdot)|_{\kappa+\beta,\infty}.
\]

Applying [15] Proposition 4, we can always attain
\[
I_{42} \leq w (a)^\beta \left( \frac{\epsilon}{5} |u|_{1+\beta,\infty} + C_{\epsilon} |u|_0 \right),
\]
which concludes $I_4 \leq w (a)^\beta \left( \frac{2\alpha}{5} |u|_{1+\beta,\infty} + C_\epsilon |u|_0 \right)$.

In the mean time, by $B(K, \beta)$, Lemma 5 and Proposition 4,

$$I_5 \leq 1_{\alpha \in (1,2)}w (a)^\beta \int_{\|q\|>1} [\nabla u (t, \cdot)]_{\beta} |q (t, z', y)| \wedge |u (t, \cdot)|_{\beta} \nu (dy)$$

$$\leq C 1_{\alpha \in (1,2)}w (a)^\beta \left( \frac{\epsilon}{5} |u|_{1+\beta,\infty} + C_\epsilon |u|_0 \right).$$

Besides, for any $\epsilon \in (0,1)$,

$$I_6 \leq \int_{\|q(t,z',y)\|\leq \epsilon} 1_{\alpha \in (0,1)} |u (t, x_1 + q (t, z', y)) - u (t, x_1)$$

$$- u (t, x_2 + q (t, z', y)) + u (t, x_2) |\nu_2 (dy)$$

$$+ \int_{\|q(t,z',y)\|> \epsilon} 1_{\alpha \in (0,1)} |u (t, x_1 + q (t, z', y)) - u (t, x_1)$$

$$- u (t, x_2 + q (t, z', y)) + u (t, x_2) |\nu_2 (dy)$$

$$:= I_{61} + I_{62}.$$  

We first discuss the case $\alpha \in (0,1)$. Applying Lemma 7 we have

$$I_{61} \leq C [L^\mu u (t, \cdot)]_{\beta} w (a)^\beta \int_{\|q(t,z',y)\|\leq \epsilon} 1_{\alpha \in (0,1)} |w \left( |q (t, z', y)| \right) |\nu_2 (dy)$$

$$\leq C w (a)^\beta |u (t, \cdot)|_{1+\beta,\infty} \int_{\|q(t,z',y)\|\leq \epsilon} 1_{\alpha \in (0,1)} |w \left( |q (t, z', y)| \right) |\nu_2 (dy).$$

On the other hand,

$$I_{62} \leq C w (a)^\beta \int_{\|q(t,z',y)\|> \epsilon} \left[ L^\mu \frac{1}{2} u (t, \cdot) \right]_{\beta} w \left( |q (t, z', y)| \right) \frac{2}{\epsilon} \wedge |u|_{\beta} \nu_2 (dy)$$

$$\leq C w (a)^\beta \frac{1}{\epsilon} \left( \frac{2}{\epsilon} \right) |u (t, \cdot)|_{1+\beta,\infty} \int_{\|q(t,z',y)\|> \epsilon} \left( w \left( |q (t, z', y)| \right) \wedge |u|_{\beta} \right) \nu_2 (dy).$$

As what we did for $I_4$, by choosing an appropriate $\epsilon$, we have

$$I_6 \leq w (a)^\beta \left( \frac{\epsilon}{5} |u|_{1+\beta,\infty} + C_\epsilon |u|_0 \right),$$

If $\alpha = 1$, then

$$I_{61} \leq C [\nabla u (t, \cdot)]_{\beta'} w (a)^{\beta'} \int_{\|q(t,z',y)\|\leq \epsilon} \left| q (t, z', y) \right| \nu_2 (dy)$$

$$\leq C w (a)^{\beta'} |u (t, \cdot)|_{1+\beta,\infty} \int_{\|q(t,z',y)\|\leq \epsilon} \left| q (t, z', y) \right| \nu_2 (dy)$$

for all $\beta' \in (0, \beta)$. Examining the proof of Lemma 5 we find that this constant $C$ is uniformly bounded under $\hat{A} (w, l, \gamma)(ii)$ for all $\beta' \in (0, \beta)$. Thus,

$$I_{61} \leq C w (a)^\beta |u (t, \cdot)|_{1+\beta,\infty} \int_{\|q(t,z',y)\|\leq \epsilon} \left| q (t, z', y) \right| \nu_2 (dy).$$
Estimate $I_{62}$ in the same way as above. By choosing an appropriate $\epsilon$, we arrive at

$$I_6 \leq w(a)^\beta \left( \frac{\epsilon}{5} |u|_{1+\beta,\infty} + C_\epsilon |u|_0 \right).$$

As a summary, for all $\alpha \in (0, 2)$ and any $\epsilon \in (0, 1)$, there exists $C_\epsilon$ that is independent of $u$ so that

$$\sup_{t,z,z'} \left[ \tilde{Q}_{t,z,z'}^{\cdot} u(t, \cdot) \right]_{\beta} \leq \sup_{t,z,y} \left[ \tilde{Q}_{t,z,\cdot}^{\cdot} u(t, \cdot) \left( \epsilon |u|_{1+\beta,\infty} + C_\epsilon |u|_0 \right) \right].$$

\[ \text{Lemma 15.} \] Let $B(K, \beta)$ hold. $\beta \in (0, 1/\alpha)$. Then for any $u \in \tilde{C}^{1+\beta}_{\infty, \infty}(H_T)$ and any $\epsilon \in (0, 1)$, there exists $C_\epsilon$ independent of $u$ such that

\[ |QU(t, \cdot)|_{\beta, \infty} \leq \epsilon |u(t, \cdot)|_{1+\beta, \infty} + C_\epsilon |u(t, \cdot)|_0. \]

\textbf{Proof.} Note that $QU(t, x) = Q_{t,x,x}^{\cdot} u(t, x)$. By Lemmas 14 and 5

$$|QU(t, \cdot)|_0 \leq C |u(t, \cdot)|_{1+\beta', \infty}$$

for some $\beta' \in (0, \beta)$. Meanwhile, for any $x, h \in \mathbb{R}^d$,

$$|Q_{t,x,x+h,x}^{\cdot} u(t, x + h) - Q_{t,x,x}^{\cdot} u(t, x)|$$

$$\leq 1_{\alpha \in (1, 2)} |b(t, x + h) \nabla u(t, x + h) - b(t, x) \nabla u(t, x)|$$

$$+ |p(t, x + h) u(t, x + h) - p(t, x) u(t, x)|$$

$$+ \left| \int_{t,x}^{t,x} u(t, x + h) - \tilde{Q}_{t,x,x}^{\cdot} u(t, x) \right|.$$
for some $\beta' \in (0, \beta)$, and
\[
\left| \bar{Q}_{t,x} u(t, x + h) - \hat{Q}_{t,x} u(t, x) \right| \\
\leq C \sup_{t, z, y} |q(t, z, y)| w(|h|)^\beta \left( \varepsilon |u|_{1+\beta, \infty} + C \varepsilon |u|_0 \right).
\]
Besides,
\[
\left| \bar{Q}_{t,x} u(t, x + h) - \hat{Q}_{t,x} u(t, x + h) \right| \\
\leq C \int_{|y| \leq 1} \left| \nabla \alpha u(t, x + h; q(t, x, h, y)) - \nabla \alpha u(t, x + h; q(t, x, y)) \right| \nu_2(dy) \\
+ C \int_{|y| > 1} \left| \nabla \alpha u(t, x + h; q(t, x, y)) - u(t, x + h; q(t, x, y)) \right| \nu_2(dy) \\
+ C \int_{\mathbb{R}^d} \left| u(t, x + h; q(t, x, y)) - u(t, x + h; q(t, x, y)) \right| \nu_2(dy) \\
:= C \left( I_1 + I_2 + I_3 \right).
\]
Similarly as what we did in Lemma [14]
\[
I_1 \leq C \int_{|y| \leq 1} \int_0^1 |\nabla u(t, x + h + \theta q(t, x, h, y)) - \nabla u(t, x + h)| d\theta \\
|q(t, x + h, y) - q(t, x, y)| \nu_2(dy) \\
+ C \int_{|y| \leq 1} \int_0^1 |\nabla u(t, x + h + \theta q(t, x, y)) - \nabla u(t, x + h + \theta q(t, x, y))| d\theta |q(t, x, y)| \nu_2(dy) \\
\leq C |\nabla u(t, \cdot)|_\beta \int_{|y| \leq 1} w(|q(t, x, h, y)|)^\beta |q(t, x, h, y) - q(t, x, y)| \nu_2(dy) \\
+ C |\nabla u(t, \cdot)|_\beta \int_{|y| \leq 1} w(|q(t, x, h, y) - q(t, x, y)|)^\beta |q(t, x, y)| \nu_2(dy) \\
\leq C |u(t, \cdot)|_{\kappa + \beta', \infty} w(|h|)^\beta
\]
for some $\kappa \in (0, 1)$.
\[
I_2 \leq C \int_{|y| > 1} |\nabla u(t, \cdot)|_0 \left| q(t, z' + h, y) - q(t, z', y) \right| \wedge |u|_0 \nu_2(dy) \\
\leq C |u(t, \cdot)|_{1+\beta', \infty} w(|h|)^\beta
\]
for some $\beta' \in (0, \beta)$. And
\[
I_3 \leq C \int_{\mathbb{R}^d} 1_{\alpha \in (0, 1)} \left( |L^\mu u(t, \cdot)|_0 w \left( |q(t, z' + h, y) - q(t, z', y)| \right) \wedge |u|_0 \right) \nu_2(dy) \\
+ C \int_{\mathbb{R}^d} 1_{\alpha = 1} \left( |\nabla u(t, \cdot)|_0 \left| q(t, z' + h, y) - q(t, z', y) \right| \wedge |u|_0 \right) \nu_2(dy) \\
\leq C 1_{\alpha \in (0, 1)} |u(t, \cdot)|_{1+\beta', \infty} w(|h|)^\beta
\]
for some \( \beta' \in (0, \beta) \).

Summarizing, we obtain (3.21). \( \square \)

4. Proof of the Main Result

4.1. Auxiliary Results. In this section, we state or prove well-posedness for integro-differential equations with space-independent operators.

Theorem 4.1. [15, Theorem 1.1] Let \( \beta \in (0, \infty), \lambda \geq 0 \) and \( \nu \) be a Lévy measure satisfying \( A(w, l) \). If \( f(t, x) \in \tilde{C}^{\beta}_{\infty, \infty}(H_T) \). Then there is a unique solution \( u \in (t, x) \in \tilde{C}^{1+\beta}_{\infty, \infty}(H_T) \) to

\[
\partial_t u(t, x) = Lu(t, x) - \lambda u(t, x) + f(t, x), \lambda \geq 0,
\]

\[
u(0, x) = 0, (t, x) \in H_T,
\]

where for any function \( \varphi \in C^2_b(\mathbb{R}^d) \),

\[
L\varphi(x) := \int [\varphi(x + y) - \varphi(x) - \chi_\alpha(y) y \cdot \nabla \varphi(x)] \nu(dy).
\]

Moreover, there exists a constant \( C \) depending on \( \kappa, \beta, d, T, \mu, \nu \) such that

\[
|u|_{\beta, \infty} \leq C(\lambda^{-1} \wedge T) |f|_{\beta, \infty},
\]

\[
|u|_{1+\beta, \infty} \leq C |f|_{\beta, \infty}
\]

And there is a constant \( C \) depending on \( \kappa, \beta, d, T, \mu, \nu \) such that for all \( 0 \leq s < t \leq T, \kappa \in [0, 1] \),

\[
|u(t, \cdot) - u(s, \cdot)|_{\kappa+\beta, \infty} \leq C |t-s|^{-\kappa} |f|_{\beta, \infty}.
\]

Theorem 4.2. Let \( \nu \) be a Lévy measure, \( \alpha \in (0, 2), \beta \in (0, 1), \lambda \geq 0 \). \( G \) is an invertible \( d \times d \)-matrix. Assume that \( f(t, x) \in \tilde{C}^{\beta}_{\infty, \infty}(H_T) \). Then there is a unique solution \( u \in (t, x) \in \tilde{C}^{1+\beta}_{\infty, \infty}(H_T) \) to

\[
\partial_t u(t, x) = Lu(t, x) - \lambda u(t, x) + f(t, x), \lambda \geq 0,
\]

\[
u(0, x) = 0, (t, x) \in H_T,
\]

where for any function \( \varphi \in C^2_b(\mathbb{R}^d) \),

\[
L\varphi(x) := \int [\varphi(x + Gy) - \varphi(x) - \chi_\alpha(y) Gy \cdot \nabla \varphi(x)] \nu(dy).
\]

Moreover, there exists a constant \( C \) depending on \( \kappa, \beta, d, T, \mu, \nu \), \( \|G^{-1}\|, \|G\| \) such that

\[
|u|_{\beta, \infty} \leq C(\lambda^{-1} \wedge T) |f|_{\beta, \infty},
\]

\[
|u|_{1+\beta, \infty} \leq C |f|_{\beta, \infty}
\]

And there is a constant \( C \) depending on \( \kappa, \beta, d, T, \mu, \nu \), \( \|G^{-1}\|, \|G\| \) such that for all \( 0 \leq s < t \leq T, \kappa \in [0, 1] \),

\[
|u(t, \cdot) - u(s, \cdot)|_{\kappa+\beta, \infty} \leq C |t-s|^{-\kappa} |f|_{\beta, \infty}.
\]
Proof. We first assume \( f(t, x) \in C^\infty_b(H_T) \cap \tilde{C}_b^{\beta, \infty}(H_T) \).

Existence. Denote \( F(r, Z^\nu_r) = e^{-\lambda(r-s)} f(s, x + GZ^\nu_r - GZ^\nu_s), s \leq r \leq t \), and apply the Itô formula to \( F(r, Z^\nu_r) \) on \([s, t]\).

\[
e^{-\lambda(t-s)} f(s, x + GZ^\nu_t - GZ^\nu_s) - f(s, x) = -\lambda \int_s^t F(r, Z^\nu_r) \, dr + \int_s^t \int \chi_\alpha(y) y \cdot \nabla F(r, Z^\nu_r) \, J(dr, dy)
+ \int_s^t \left[ F(r, Z^\nu_r + y) - F(r, Z^\nu_r) - \chi_\alpha(y) y \cdot \nabla F(r, Z^\nu_r) \right] J(dr, dy).
\]

Take expectation for both sides.

\[
e^{-\lambda(t-s)} \mathbb{E} f(s, x + GZ^\nu_t - GZ^\nu_s) - f(s, x) = -\lambda \int_s^t e^{-\lambda(r-s)} \mathbb{E} f(s, x + GZ^\nu_r - GZ^\nu_s) \, dr
+ \int_s^t Le^{-\lambda(r-s)} \mathbb{E} f(s, x + GZ^\nu_r - GZ^\nu_s) \, dr.
\]

Integrate both sides over \([0, t]\) with respect to \( s \) and obtain

\[
\int_0^t e^{-\lambda(t-s)} \mathbb{E} f(s, x + GZ^\nu_t - GZ^\nu_s) \, ds - \int_0^t f(s, x) \, ds
= -\lambda \int_0^t \int_0^r e^{-\lambda(r-s)} \mathbb{E} f(s, x + GZ^\nu_r - GZ^\nu_s) \, dsdr
+ \int_0^t L \int_0^r e^{-\lambda(r-s)} \mathbb{E} f(s, x + GZ^\nu_r - GZ^\nu_s) \, dsdr,
\]

which shows \( u(t, x) = \int_0^t e^{-\lambda(t-s)} \mathbb{E} f(s, x + GZ^\nu_t - GZ^\nu_s) \, ds \) solves (4.6) in the integral sense. As a result of the dominated convergence theorem and Fubini's theorem, \( u \in C^\infty_b(H_T) \). And by the equation, \( u \) is continuously differentiable in \( t \).

Uniqueness. Suppose there are two solutions \( u_1, u_2 \) solving the equation, then \( u := u_1 - u_2 \) solves

\[
(4.10) \quad \partial_t u(t, x) = Lu(t, x) - \lambda u(t, x),
\]

\( u(0, x) = 0 \).

Apply the Itô formula to \( v(t-s, Z^\nu_s) := e^{-\lambda s} u(t-s, x + GZ^\nu_s), 0 \leq s \leq t \), over \([0, t]\) and take expectation for both sides of the resulting identity, then

\[
u(t, x) = -\mathbb{E} \int_0^t e^{-\lambda s} \left[ (\partial_t u - \lambda u + Lu) \circ (t-s, x + GZ^\nu_{s-}) \right] \, ds = 0.
\]
Estimates for Smooth Inputs. Denote \( g(t, x) = f(t, Gx), x \in \mathbb{R}^d \).
Then by Lemma 11 for any \( \beta \in (0, 1) \),
\[
|u(t, \cdot)|_{\beta, \infty} \leq \left| \int_0^t e^{-\lambda(t-s)} E f(s, G \cdot + GZ^\nu_{t-s}) \, ds \right|_{\beta, \infty} \\
= \left| \int_0^t e^{-\lambda(t-s)} E g(s, \cdot + Z^\nu_{t-s}) \, ds \right|_{\beta, \infty} \\
\leq C (\lambda^{-1} \wedge T) |g(t, \cdot)|_{\beta, \infty},
\]
\[
\leq C (\lambda^{-1} \wedge T) |f(t, \cdot)|_{\beta, \infty}.
\]

Similarly, we can prove \((4.18), (4.19)\).

Estimates for Hölder Inputs. This part of proof is identical to section 5 of [13].
\( \square \)

4.2. Proof of Theorem 1.1. We aim at providing a unifying proof for both \( \mathcal{L} = \mathcal{A} + \mathcal{Q} \) and \( \mathcal{L} = \mathcal{G} + \mathcal{Q} \). Before that, we claim

Lemma 16. Let \( \beta \in (0, 1/\alpha) \) and \( f, g \in \mathring{C}^\beta (\mathbb{R}^d) \). Then
\[
|fg|_{\beta} \leq |f|_0|g|_0 + |f|_0|g|_0|f|_\beta, \\
|f|_0 = \sup_{z} |\eta_{m,z}f|_0, \quad \forall k \in \mathbb{N}_+,
\]
and for some positive constant \( C \) that does not depend on \( m \),
\[
|f|_{\beta} \leq Cl(m)^{\beta} |f|_0 + \sup_{z} |\eta_{m,z}f|_\beta, \\
\sup_{z} |\eta_{m,z}f|_{\beta} \leq Cl(m)^{\beta} |f|_0 + |f|_{\beta}.
\]

Proof. Proof for the first two is identical to that for the standard Hölder norm. By monotonicity of the scaling factor,
\[
|f|_{\beta} \leq C \sup_{|x-y| \geq \frac{1}{m}} \left( \frac{1}{|x-y|} \right)^{\beta} |f|_0 + \sup_{|x-y| \leq \frac{1}{m}} \frac{|f(x) - f(y)|}{w(|x-y|)^{\beta}}
\leq Cl(m)^{\beta} |f|_0 + \sup_{z} \frac{|\eta_{m,z}(x) f(x) - \eta_{m,z}(y) f(y)|}{w(|x-y|)^{\beta}}
\leq Cl(m)^{\beta} |f|_0 + \sup_{z} |\eta_{m,z}f|_{\beta}.
\]

Meanwhile,
\[
\sup_{z} |\eta_{m,z}f|_{\beta} \leq |f|_0 |\eta|_0 + |f|_0 \sup_{z} |\eta_{m,z}|_{\beta} + |\eta|_0 |f|_{\beta}
\leq Cl(m)^{\beta} |f|_0 + |f|_{\beta}.
\]
\( \square \)

Without introducing much confusion, we will use \( L_z \) to represent \((3.9)\) and \((3.13)\) at the same time, and \( \langle u, \eta_{m,z} \rangle_z \) for both \((3.10)\) and \((3.17)\). We will also use \( |\cdot|_{\beta} \) and \( |\cdot|_{\beta, \infty} \) interchangeably, which is justified by \( \tilde{A}(w, l, \gamma)(ii) \).
Estimates and Uniqueness. Let \( u \in C^{1+\beta}_{\infty, \infty} (H_T) \) be a solution to (4.11), either \( \mathcal{L} = A + Q \) or \( \mathcal{L} = \mathcal{G} + Q \). Obviously,
\[
\partial_t (\eta_{m,z} u) = \eta_{m,z} (L_z u) - \lambda(\eta_{m,z} u) + \eta_{m,z} f + \eta_{m,z} \left( (\mathcal{L} - L_z) u \right),
\]
where by elementary derivation,
\[
\eta_{m,z} (L_z u) = L_z (\eta_{m,z} u) - u (L_z \eta_{m,z}) - \langle u, \eta_{m,z} \rangle_z,
\]
therefore, \( \eta_{m,z} u \) solves
\[
\partial_t (\eta_{m,z} u) = L_z (\eta_{m,z} u) - \lambda(\eta_{m,z} u) - u (L_z \eta_{m,z}) + \eta_{m,z} f + \eta_{m,z} \left( (\mathcal{L} - L_z) u \right) - \langle u, \eta_{m,z} \rangle_z \tag{4.12}
\]
(1.11), either \( \lambda \) is either \( L \) where \( \mathcal{L} = A + Q \) or \( \mathcal{L} = \mathcal{G} + Q \). Clearly,
\[
\| \eta_{m,z} u \|_{1+\beta, \infty} \leq C \left( \| u (L_z \eta_{m,z}) \|_{\beta, \infty} + \| \eta_{m,z} f \|_{\beta, \infty} + \| \eta_{m,z} \left( (\mathcal{L} - L_z) u \right) \|_{\beta, \infty} + \| \langle u, \eta_{m,z} \rangle_z \|_{\beta, \infty} \right),
\]
for some \( C \) independent of \( \lambda \). Consequently, \( \| L_z \eta_{m,z} \|_{\beta, \infty} \leq C \| \eta_{m,z} \|_{1+\beta, \infty} \leq Cl(m)^{1+\beta} \). Then by Lemma 16 and Proposition 4,
\[
\| \eta_{m,z} [L_z \eta_{m,z}] \|_{\beta, \infty} \leq C \| u (L_z \eta_{m,z}) \|_{\beta},
\]
\[
\leq \| u \|_{\beta} \| L_z \eta_{m,z} \|_0 + \| u \|_0 \| L_z \eta_{m,z} \|_{\beta},
\]
\[
\leq Cl(m)^{1+\beta} \| u \|_{\beta, \infty} \leq Cl(m)^{1+\beta} \left( \varepsilon \| u \|_{1+\beta} + C \varepsilon \| u \|_0 \right).
\]
Apply Lemma 16 again,
\[
| \eta_{m,z} [(\mathcal{L} - L_z) u] \|_{\beta, \infty} \leq C \left( l(m)^\beta \left( (\tilde{\mathcal{L}} - L_z) u \right)_0 + l(m)^\beta \| Q u \|_0 + \left| (\tilde{\mathcal{L}} - L_z) u \right|_{\beta, \infty} + \| Q u \|_{\beta, \infty} \right),
\]
where \( \tilde{\mathcal{L}} \) is either \( A \) or \( \mathcal{G} \). Then by Lemmas 9, 12 and Corollaries 12
\[
| \eta_{m,z} [(\mathcal{L} - L_z) u] \|_{\beta, \infty} \leq Cl(m)^\beta \left( \varepsilon \| u \|_{1+\beta, \infty} + C \varepsilon \| u \|_0 \right) + CF(m, x, z) \| u \|_{1+\beta, \infty},
\]
where
\[
F(m, x, z) := \sup_{t, y, |x-z| \leq 2/m} \| \rho(t, x, y) - \rho(t, z, y) \|_0
\]
if \( \mathcal{L} = A + Q \), and \( F(m, x, z) := \sup_{|x-z| \leq 2/m} \| G(x) - G(z) \|_\sigma \) if \( \mathcal{L} = \mathcal{G} + Q \).
Combining Lemmas 10 and 13 we obtain

\[
|u(L_z \eta_{m,z})|_{\beta,\infty} + |\eta_{m,z} f|_{\beta,\infty}
+ |\eta_{m,z} [(L - L_z) u]|_{\beta,\infty} + |\langle u, \eta_{m,z} \rangle z|_{\beta,\infty}
\leq \ell(m)^{1+\beta} \left( \varepsilon |u|_{1+\beta,\infty} + C \varepsilon |u|_0 + |f|_{\beta,\infty} \right)
+ CF(m, x, z) |u|_{1+\beta,\infty}.
\]

(4.15)

An immediate conclusion of this estimate is

\[
|\eta_{m,z} u|_{\beta,\infty} \leq C \left( \lambda^{-1} \wedge T \right) \left( F(m, x, z) |u|_{1+\beta,\infty} + \ell(m)^{1+\beta} \left( \varepsilon |u|_{1+\beta,\infty} + C \varepsilon |u|_0 \right) \right),
\]

(4.16)

where \( C \) does not depend on \( \lambda, m \). Thus,

\[
|u|_0 \leq C \sup_z |\eta_{m,z} u|_{\beta,\infty}
\leq C \left( \lambda^{-1} \wedge T \right) \ell(m)^{1+\beta} \left( |u|_{1+\beta,\infty} + |f|_{\beta,\infty} \right),
\]

(4.17)

Combining (4.14), (4.15), (4.17), we then have

\[
|\eta_{m,z} u|_{1+\beta,\infty}
\leq \varepsilon \ell(m)^{1+\beta} |u|_{1+\beta,\infty} + C \varepsilon \left( \lambda^{-1} \wedge T \right) \ell(m)^{2+2\beta} \left( |u|_{1+\beta,\infty} + |f|_{\beta,\infty} \right)
+ CF(m, x, z) |u|_{1+\beta,\infty}.
\]

(4.18)

On the other hand, by Lemma 16

\[
|L^\mu u|_{\beta,\infty} \leq C |L^\mu u|_{\beta,\infty} \leq Cl(m)^{\beta} \left( |L^\mu u|_0 + C \sup_z |\eta_{m,z} L^\mu u|_{\beta,\infty} \right)
\leq Cl(m)^{\beta} |L^\mu u|_0 + C \sup_z |\eta_{m,z} L^\mu u|_{\beta,\infty}.
\]

Let \( \rho(z, y) = 1, \nu(dy) = \mu(dy) \) in Lemma 9 and utilize (4.12).

\[
\sup_z |\eta_{m,z} L^\mu u|_{\beta,\infty}
\leq \sup_z |\eta_{m,z} u|_{1+\beta,\infty} + \sup_z |u(L^\mu \eta_{m,z})|_{\beta,\infty} + \sup_z |\langle u, \eta_{m,z} \rangle z|_{\beta,\infty}
\leq \sup_z |\eta_{m,z} u|_{1+\beta,\infty} + Cl(m)^{1+\beta} \left( \varepsilon |u|_{1+\beta,\infty} + C \varepsilon |u|_0 \right),
\]

(4.19)
where $C$ does not depend on $\lambda, m$. Combining (4.17), (4.18), we obtain

$$|u|_{1+\beta,\infty} \leq C \left( |u_0| + |L^\mu u|_{\beta,\infty} \right)$$

$$\leq C \left( \lambda^{-1} \wedge T \right) l \left( m \right)^{1+\beta} \left( |u|_{1+\beta,\infty} + |f|_{\beta,\infty} \right) + C \sup_z |\eta_{m,z}u|_{1+\beta,\infty}$$

$$+Cl \left( m \right)^{1+\beta} \left( \varepsilon |u|_{1+\beta,\infty} + C\varepsilon |u_0| \right)$$

$$\leq \varepsilon l \left( m \right)^{1+\beta} |u|_{1+\beta,\infty} + C \left( \lambda^{-1} \wedge T \right) l \left( m \right)^{1+\beta} \left( |u|_{1+\beta,\infty} + |f|_{\beta,\infty} \right)$$

$$+C\varepsilon l \left( m \right)^{1+\beta} |f|_{\beta,\infty} + CF \left( m, x, z \right) |u|_{1+\beta,\infty}.$$

In the inequality above, we first set $m$ sufficiently large so that

$$CF \left( m, x, z \right) |u|_{1+\beta,\infty} \leq \frac{1}{4} |u|_{1+\beta,\infty}.$$

For such an $m$, we then select $\varepsilon$ such that $\varepsilon l \left( m \right)^{1+\beta} < 1/4$. At last, we choose $\lambda$ large enough so that for such $m, \varepsilon$, $C \left( \lambda^{-1} \wedge T \right) l \left( m \right)^{2+2\beta} < 1/4$. As a summary, with appropriate choice of $m, \varepsilon, \lambda$, $|u|_{1+\beta,\infty} \leq C \left( \lambda \right) |f|_{\beta,\infty}$.

We need $\lambda$ to be sufficiently large though, say $\lambda \geq \lambda_0$. To completely relax this constraint, let us consider $v \left( t, x \right) := e^{\left( \lambda-\lambda_0 \right)t} u \left( t, x \right), \lambda \geq 0$, where $u$ solves (1.11). Then $v$ is a solution to

$$\partial_t v \left( t, x \right) = \mathcal{L} v \left( t, x \right) - \lambda_0 v \left( t, x \right) + e^{\left( \lambda-\lambda_0 \right)t} f \left( t, x \right), \lambda \geq 0,$$

$$v \left( 0, x \right) = 0, \quad (t, x) \in H_T,$$

and

$$|v|_{1+\beta,\infty} = \left| e^{\left( \lambda-\lambda_0 \right)t} u \right|_{1+\beta,\infty} \leq C_{\lambda_0} \left| e^{\left( \lambda-\lambda_0 \right)t} f \right|_{\beta,\infty}.$$

Namely, $|v|_{1+\beta,\infty} \leq C_{\lambda_0} |f|_{\beta,\infty}$. Note $C_{\lambda_0}$ is uniform with respect to $\lambda$.

Now we can conclude from (4.16), (4.17) and Lemma 16 that

$$|u|_{\beta,\infty} \leq Cl \left( m \right)^{\beta} |u_0| + C \sup_z |\eta_{m,z}u|_{\beta,\infty} \leq C \left( \lambda^{-1} \wedge T \right) |f|_{\beta,\infty},$$

where $C$ does not depend on $\lambda, u, f$.

Again, according to Theorems 4.1, 4.2 and (4.15), (4.13), there is a constant $C$ depending on $\kappa, \beta, d, T, \mu, \nu$ such that for all $0 \leq s < t \leq T$, $\kappa \in [0,1],$

$$|\eta_{m,z}u \left( t, \cdot \right) - \eta_{m,z}u \left( s, \cdot \right)|_{1+\beta,\infty} \leq C \left( t-s \right)^{-\kappa} \left( |u \left( Lz\eta_{m,z} \right)|_{\beta,\infty} + |\eta_{m,z}f|_{\beta,\infty} + |\eta_{m,z} \left( \mathcal{L} - Lz \right) u|_{\beta,\infty} + \left| \langle u, \eta_{m,z} \rangle z \right|_{\beta,\infty} \right) \leq Cl \left( m \right)^{1+\beta} \left( t-s \right)^{-\kappa} |f|_{\beta,\infty}.$$
Apply Lemma 16 and repeat derivation (4.19) for the difference function,

\[
|u(t,\cdot) - u(s,\cdot)|_{1+\beta,\infty} \\
\leq C \left( |u(t,\cdot) - u(s,\cdot)|_0 + |L^\mu u(t,\cdot) - L^\mu u(s,\cdot)|_{\beta,\infty} \right) \\
\leq C \sup_z |\eta_{m,z} u(t,\cdot) - \eta_{m,z} u(s,\cdot)|_0 \\
+ C l(\varepsilon) \sup_z |\eta_{m,z} |L^\mu u(t,\cdot) - \eta_{m,z} L^\mu u(s,\cdot)|_{\beta,\infty} \\
\leq \left( Cl(\varepsilon) + C l(\beta,1+2\beta) \right) |\eta_{m,z} u(t,\cdot) - \eta_{m,z} u(s,\cdot)|_{1+\beta,\infty} \\
+ C l(\varepsilon) |\mu|_{\beta,\infty}.
\]

Choose \(\varepsilon\) such that \(C l(\varepsilon)(1+2\beta) < 1/2\). Then we arrive at 

\[
|u(t,\cdot) - u(s,\cdot)|_{1+\beta,\infty} \\
\leq \left( Cl(\beta) + C l(\beta,1+2\beta) \right) |\eta_{m,z} u(t,\cdot) - \eta_{m,z} u(s,\cdot)|_{1+\beta,\infty} \\
\leq C(t-s)^{1-\beta} |f|_{\beta,\infty}.
\]

Uniqueness of the solution is a direct consequence of these estimates.

**Existence.** Let \(V(H_T)\) be the linear space that for any \(v \in V(H_T)\), there exists a unique \(f \in C_{\infty,\infty}^1(H_T)\) such that \(v(t,x) = \int_0^t f(s,x) \, ds\). Equip \(V(H_T)\) with norm \(|v|_V := |f|_{\beta,\infty}\). Let \(U(H_T)\) be the linear space that for any \(u \in U(H_T)\), there is \(g \in C_{\infty,\infty}^1(H_T)\) such that \(u(t,x) = \int_0^t g(s,x) \, ds\). Endow \(U(H_T)\) with norm \(|u|_U := |u|_{1+\beta,\infty}\). Then \(V(H_T)\) is a normed linear space and \(U(H_T)\) is a Banach space. Define for \(\theta \in [0,1]\),

\[
\mathcal{T}_\theta u(t,x) = \theta \left( u(t,x) - \int_0^t (Lu(s,x) - \lambda u(s,x)) \, ds \right) \\
+ (1-\theta) \left( u(t,x) - \int_0^t (\lambda u(s,x) - \lambda u(s,x)) \, ds \right) \\
\]  

\[
:= u(t,x) - \int_0^t (Lu(s,x) - \lambda u(s,x)) \, ds,
\]

where \(\mathcal{L}_\theta = \theta \mathcal{L} + (1-\theta) \lambda \). Take \(u \in U(H_T)\). Then \(u(t,x) := \int_0^t g(s,x) \, ds\) for some \(g \in C_{\infty,\infty}^1(H_T)\). Clearly, for any \(\theta \in [0,1]\), \(u\) solves

\[
u(t,x) = \int_0^t [\mathcal{L}_\theta u(s,x) - \lambda u(s,x) + (g(s,x) - \mathcal{L}_\theta u(s,x) + \lambda u(s,x))] \, ds.
\]

Therefore,

\[
\mathcal{T}_\theta u(t,x) = \int_0^t [g(s,x) - \mathcal{L}_\theta u(s,x) + \lambda u(s,x)] \, ds,
\]

where by Lemma 16, Proposition 2 and Corollary 1

\[
|\mathcal{T}_\theta u|_V = |g - \mathcal{L}_\theta u + \lambda u|_{\beta,\infty} \leq C |u|_{1+\beta,\infty} < \infty.
\]
Then, \( T_0[H^T] \subset V[H^T] \). Meanwhile, by estimates we derived above, there is \( C \) independent of \( u, \theta \) such that

\[
|u|_U = |u|_{1+\beta, \infty} \leq C |g - \mathcal{L}_0u + \lambda u|_{\beta, \infty} \leq C |T_0u|_V .
\]

Theorem 1.1 says \( T_0 \) maps \( U \) onto \( V \). By Theorem 5.2 in [4], so does \( T_1 \).

5. Appendix

Lemma 17. [15] Lemma 2] Let \( \nu \) be a Lévy measure and \( w \) be the scaling function which \( \nu \) satisfies \( A(w, l) \) for. Then,

a) there are constants \( C_1, C_2 > 0 \) such that

\[
C_1 \varsigma(r) \leq w(r) \leq C_2 \varsigma(r) , \quad \forall r > 0 .
\]

b) \( \int_{|y| \leq 1} w(|y|) \nu(dy) = +\infty \).

c) For any \( \varepsilon > 0 \), \( \int_{|y| \leq 1} w(|y|)^{1+\varepsilon} \nu(dy) < \infty \).

d) For any \( \varepsilon > 0 \), \( \int_{|y| \leq 1} |y|^\varepsilon w(|y|) \nu(dy) < \infty \).

Lemma 18. [9] Lemma 5] Let \( \nu \) be a Lévy measure satisfying \( A(w, l) \). \( Z_t^{\nu_R} \) is the Lévy process associated to \( \tilde{\nu}_R, R > 0 \). For each \( t, R, Z_t^{\nu_R} \) has a bounded and continuous density function \( p^R(t, x), t \in (0, \infty), x \in \mathbb{R}^d \). And \( p^R(t, x) \) has bounded and continuous derivatives up to order 4. Meanwhile, for any multi-index \( |\vartheta| \leq 4 \),

\[
\int \left| \partial^\vartheta p^R(t, x) \right| dx \leq C \gamma(t)^{-|\vartheta|},
\]

\[
\sup_{x \in \mathbb{R}^d} \left| \partial^\vartheta p^R(t, x) \right| \leq C \gamma(t)^{-d-|\vartheta|},
\]

where \( C > 0 \) is independent of \( t, R \). For any \( \beta \in (0, 1) \) such that \(|\vartheta| + \beta < 4\),

\[
\int \left| \partial^\vartheta \partial^{\vartheta} p^R(t, x) \right| dx \leq C \gamma(t)^{-|\vartheta| - \beta} .
\]

For any \( a > 0 \), there is a constant \( C > 0 \) independent of \( t, R \), so that

\[
\int_{|x| > a} \left| \partial^\vartheta p^R(t, x) \right| dx \leq C \left( \gamma(t)^{2-|\vartheta|} + t \gamma(t)^{-|\vartheta|} \right) .
\]

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