The Discrete Sell or Hold Problem with Constraints on Asset Values

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Abstract

The discrete sell or hold problem (DSHP), which is introduced in [2], is studied under the constraint that each asset can only take a constant number of different values. We show that if each asset can take only two values, the problem becomes polynomial-time solvable. However, even if each asset can take three different values, DSHP is still NP-hard. An approximation algorithm is also given under this setting.

1 Introduction

There are three key factors in asset management, which are asset allocation, security selection and timing [4]. A common scenario faced by an asset management team is to meet various kinds of capital requirements, such as customer withdraw needs or regulation requirements, at the end of a fiscal year. In order to achieve those capital goals, an asset manager may need to sell part of its portfolio holdings to generate as much capital as possible. Dealing with this scenario, an asset manager should take security selection (which assets to sell), and timing (when to sell an asset) into consideration.

Since the future price of a financial asset is a stochastic process, the timing issue poses a lot of challenges to asset managers. Instead of studying when to sell an asset in a continuous time setting, a simplified model is to study whether we should sell an asset now or at a specified future date. The problem, which is called the discrete sell or hold problem (DSHP), is introduced by He et. al. [2]. It can be modeled as a two-stage stochastic combinatorial optimization problem. It is shown that DSHP is NP-hard to solve.

However, the model in [2] does not impose any constraints on the possible values that an asset can take. A natural conjecture is that the number of different values an asset can take would have great impact on the complexity of DSHP. Following this idea, we can further simplify the model by assuming that each asset can take only a constant number of different values. In particular, we study the case when an asset can take only three different values. This case coincides with the idea of binomial tree model, which is a standard finance textbook model [3] for asset pricing. In the binomial tree model, the current value of asset is $v$; in the future stage, it can take two possible values: a high value scenario $v_u$ and a low value scenario $v_d$. Thus, it is very interesting to study the discrete sell or hold problem under this simplified model. Furthermore, the case that an asset can take only two possible values is also studied.

The organization of the paper is as follows: the mathematical formulation of the discrete sell or hold problem is given in section 2, the complexity of the problem with constraints on
asset values is studied in section 3; an approximation algorithm is presented in section 4; the conclusion is in section 5.

2 The Problem Formulation

The discrete sell or hold problem (DSHP) deals with a decision problem of two stages. There are \( n \) assets \( A = \{1, 2, ..., n\} \) to manage. At the first stage, the value of each asset \( c_i \) is known. However, at the second stage, there are \( m \) scenarios. Under each scenario \( j \), every asset \( i \) has value \( f_{ij} \). We plan to sell \( k \) assets to generate as much revenue as possible. Each asset can be hold or sold. If it is sold, we have to decide whether we should sell it at the first stage or the second stage. The problem can be formulated as an integer programming problem as follows.

\[
\begin{align*}
\max & \sum_{i=1}^{n} c_i x_i + \sum_{j=1}^{m} p_j \sum_{i=1}^{n} f_{ij} y_{ij} \\
\text{s.t.} & \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} y_{ij} = k \quad j = 1, ..., m \\
& x_i + y_{ij} \leq 1 \quad i = 1, ..., n, j = 1, ..., m \\
& x_i \in \{0, 1\}, y_{ij} \geq 0
\end{align*}
\]

Since the discrete sell or hold problem deals with uncertainty in the future state, it belongs to a category of problems called \textit{combinatorial stochastic programming problem}, which is extensively studied in computer science and operation research literatures [5, 6, 7, 8, 9].

3 DSHP with Constraints on Asset Values

3.1 The Case when Assets can take only two values

**Proposition 1.** If \( c_i \in \{v_{\max}, v_{\min}\}, i = 1, ..., n \) and \( f_{ij} \in \{v_{\max}, v_{\min}\}, i = 1, ..., n, j = 1, ..., m \), DSHP is solvable in \( O(nm) \) time.

**Proof.** Suppose \( C = \{i | c_i = v_{\max}\} \). We will construct an optimal solution to the DSHP as follows: if \( |C| < k \), all the assets in \( C \) are sold at the first stage; for each scenario at the second stage, the most valuable \( k - |C| \) assets are sold; Otherwise, if \( |C| \geq k \), arbitrary \( k \) assets in \( C \) are sold at the first stage while nothing is sold at the second stage.

We will show that the above solution is an optimal one. For \( i \in \{1, .., n\} \), if \( c_i = v_{\min} \), the revenue of selling asset \( i \) at the second stage is always at least as good as selling it at the first stage. Thus, in order to maximize the total revenue, we don’t have to sell asset \( i \) at the first stage. If \( c_i = v_{\max} \), suppose that in an optimal solution \( O \), asset \( i \) is not sold at the first stage, and some assets are sold for each scenario \( j \in \{1, ..., m\} \) at the second stage. Then, we can construct another solution \( O' \) that asset \( i \) is sold at the first stage while one asset \( r_j \) is removed from the sold asset list of the optimal solution \( O \) for each scenario \( j \). For any choice of \( r_j \), the solution \( O' \) is at least as good as \( O \). Thus, if \( c_i = v_{\max} \), asset \( i \) should be sold at the first stage to achieve the optimal revenue. Combining both cases, it is easy to see that the solution we construct above is an optimal solution. \( \square \)

**Corollary 2.** For an asset \( i \) that \( c_i < \sum_{j=1}^{m} p_j f_{ij} \), it should not be sold at the first stage in an optimal solution of DSHP.
### 3.2 The Case when Assets can take three values

**Definition 3.** Given a graph $G = (V, E)$, the dominating set of $G$ is a set of vertices $D$ such that $\forall v \in V$, either $v \in D$ or $\exists u \in D$ such that $(v, u) \in E$.

The problem of finding the minimum dominating set for a regular planar graph with degree 4, which is called the MDS-RPG4 problem, is NP-hard [1]. We will reduce the MDS-RPG4 problem to DSHP and show that solving DSHP is NP-hard as well.

The construction of the reduction is: There are $n$ assets and $n$ scenarios, each of which has probability $\frac{1}{n}$ to realize. For each asset $i$, its value at the first stage is 1, while its value at the second stage under each scenario is defined according to the graph $G = (V, E)$ as following:

- $\forall i, f_{ii} = 1 - B$;
- $\forall i, j$, if $(i, j) \in E$, $f_{ij} = 1 - B$;
- $\forall i, j$, if $(i, j) \notin E$, $f_{ij} = 1 + S$.

Here we make $B, M$ and $S$ satisfies the following condition:

$$\frac{d}{n-d} < \frac{S}{B} < \frac{d+1}{n-d-1}.$$  

Where $d$ is the degree of the regular graph $G$. It is easy to check that $\frac{d}{n-d} < \frac{d+1}{n-d-1}$ holds for any $d \in [0, n-1)$. In particular, $d = 4$ here. Thus, we have a matrix $M$ like below:

|   | 1 | 2 | ... | n |
|---|---|---|-----|---|
| 1 | 1 | . | ... | . |
| 2 | 1 | . | ... | . |
| ... | 1 | ... | ... | ... |
| n | 1 | . | ... | . |

Note that in the instance of DSHP, each asset can take only three values $\{1, 1 - B, 1 + S\}$.

**Theorem 4.** DSHP is NP-hard, even if all the assets can take only three different values.

**Proof.** We will reduce the MDS-RPG4 problem to DSHP with $k = n - 1$, where $n = |V|$. W.L.O.G, we assume that graph $G$ is connected.

Suppose in an optimal solution the set of $V \setminus D$ assets are sold at the first stage, we will show that $D$ is a minimum dominating set. First, we show that $D$ must be a dominating set. Otherwise, there is a node $i$ in $V \setminus D$ that is not dominated by any node in $D$. Since the graph is connected, $i$ should be connected to some node $h$ in $V \setminus D$. We will construct another solution for DSHP: sell assets in $V \setminus (D \cup \{h\})$ at the first stage. Since asset $h$ is not sold at the first stage, we define a baseline perturbed solution as selling asset $h$ under each scenario at the second stage. Note that in the $h$th column of the matrix (which represents the $h$th scenario of the second stage), in the original solution, some asset $g$ in $D$ is not sold under this scenario (remember $k = n - 1$ and there must be one asset left unsold). Thus, we can make further improvement to the baseline perturbed solution by selling asset $g$ instead of asset $h$ under this scenario. Note that since $h$ is not dominated by any node in $D$, $M_{gh} = 1 + S$ while $M_{hh} = 1 - B$. This scenario is showed briefly in the following matrix.

|   | ... | $h$ | ... |
|---|-----|-----|-----|
| ... | 1 | ... | ... |
| $h$ | 1 | $1-B$ | ... |
| ... | 1 | ... | ... |
| $g$ | 1 | $1+S$ | ... |
| ... | 1 | ... | ... |
Let \( R \) be the revenue generated by the original optimal solution and \( R' \) be the revenue generated by the new solution. We get,

\[
R' = R - 1 + \frac{1}{n} \left[ \sum_j f_{ij} - (1 - B) + (1 + S) \right]
\]

\[
= R - 1 + \frac{1}{n} \left[ n - dB + (n - d - 1)S - B - (1 - B) + (1 + S) \right]
\]

\[
= R + \frac{1}{n} \left[ -dB + (n - d)S \right]
\]

This contradicts to the assumption that \( D \) is not a dominating set and selling assets in \( V \setminus D \) induces an optimal solution. Thus, \( D \) must be a dominating set.

Next, we will show that if selling assets in \( V \setminus D \) at the first stage induces an optimal solution for DSHP, \( D \) must be a minimum dominating set. Otherwise, suppose there is another dominating set \( D' \) with \( |D'| < |D| \). Since both \( D \) and \( D' \) are dominating sets, at the second stage, for each scenario \( s \), there exists an asset(node) \( g \) in \( D \) or \( D' \) such that \( M_{gs} = 1 - B \). Let the revenues generated by both sets be \( R \) and \( R' \). Thus, We get

\[
R = n - |D| + \frac{1}{n} \left[ |D|(n - dB + (n - d - 1)S - B) - n(1 - B) \right]
\]

and

\[
R' = n - |D'| + \frac{1}{n} \left[ |D'| (n - dB + (n - d - 1)S - B) - n(1 - B) \right]
\]

Thus,

\[
R' - R = \left( |D| - |D'| + \frac{1}{n} \left[ (|D'| - |D|)(n - dB + (n - d - 1)S - B) \right] \right)
\]

\[
= \frac{1}{n} \left( (|D'| - |D|)(- (d + 1)B + (n - d - 1)S) \right)
\]

\[
> 0
\]

This contradicts to the assumption that \( D \) is a minimum dominating set and selling assets in \( V \setminus D \) induces an optimal solution. Thus, \( D \) must be a minimum dominating set. Combining the above two steps, we can get that in an optimal solution of DSHP, if the set of \( V \setminus D \) assets are sold at the first stage, \( D \) must be a minimum dominating set.

Now we show the reverse direction. Suppose \( D \) is a minimum dominating set of \( G \), we will show that selling assets in \( V \setminus D \) at the first stage will induce an optimal solution \( OPT \) for DSHP. Let the revenue generated by \( OPT \) is \( R' \). Otherwise, suppose there is another optimal solution \( OPT' \) with revenue \( R' > R \). According to the argument above, \( OPT' \) will induce a smaller dominating set than \( D \) (Note that dominating sets with the same size will generate the same revenue). This contradicts to the assumption that \( D \) is a minimum dominating set. Therefore, if \( D \) is a minimum dominating set of \( G \), selling assets in \( V \setminus D \) at the first stage will induce an optimal solution for DSHP.

In all, we have reduced the MDS-RPG4 problem to DSHP. Thus, DSHP is NP-hard, even if all the assets can take only three different values.
4 The Approximation Algorithm

He et. al. \cite{2} gives a max$\{\frac{1}{2}, \frac{k}{n}\}$ algorithm for DSHP. Here we assume that each asset can take only three different values $\{V_S, V_M, V_L\}$, which satisfies the following relationship $V_S < V_M < V_L$. We present another approximation algorithm below.

**Algorithm 1 A Heuristic for DSHP when assets can take only three different values**

1. Suppose at the first stage, there are $t_L$ assets with value $V_L$ and $t_M$ assets with value $V_M$. If $t_L + t_M \leq k$, sell all of them; otherwise, sell $k$ of them.
2. If $(t_L + t_M) < k$, under each scenario for the second stage, sell the top $k - (t_L + t_M)$ most expensive assets that have not been sold at the first stage.

**Proposition 5.** Algorithm 1 is a $\frac{V_L}{d}$-approximation algorithm for DSHP when assets can take only three different values $\{V_S, V_M, V_L\}$. Here $V_S < V_M < V_L$.

**Proof.** Let $ALG_1(I)$ be the optimal value that Algorithm 1 gets on an instance $I$ while $OPT(I)$ is the optimal value of the DSHP on instance $I$. It is easy to see that every asset sold at the first stage in Algorithm 1 has value of either $V_L$ or $V_M$.

If $(t_L + t_M) \geq k$, $ALG_1(I) \geq kV_M$ and $OPT(I) \leq kV_L$. Therefore, $OPT(I) \leq \frac{V_L}{V_M} ALG_1(I)$. If $(t_L + t_M) < k$, let $F_{OPT(I)}$ and $F_{ALG_1(I)}$ (the size of it is $t_L + t_M$) be the sets of assets that are sold in the optimal solution and in Algorithm 1 respectively, while $V_{F_{OPT(I)}}$ and $V_{F_{ALG_1(I)}}$ be the values generated in the first stage respectively. It is easy to see that $F_{OPT(I)} \subseteq F_{ALG_1(I)}$.\footnote{If some asset $i$ with value $V_S$ is sold at the first stage in an optimal solution, we can make asset $i$ sold at the second stage instead of the first stage without changing the optimal value.}

At the second stage, under each scenario, the top $k - |F_{OPT(I)}|$ most expensive assets in $S_{OPT(I)} = A/F_{OPT(I)}$ should be sold in the optimal solution. Let $V_{S_{OPT(I)}}$ be the value generated at the second stage in optimal solution. Moreover, the top $k - (t_L + t_M)$ most expensive assets in $S_{ALG_1(I)} = A/F_{ALG_1(I)}$ should be sold in Algorithm 1. Note $S_{ALG_1(I)} \subseteq S_{OPT(I)}$. Let $V_{S_{ALG_1(I)}}$ be the value generated at the second stage by Algorithm 1. Moreover, we define $d = (k - |F_{OPT(I)}|) - (k - |F_{ALG_1(I)}|)$. Then we can get the following relationship:

$$V_{S_{OPT(I)}} \leq d \cdot V_L + V_{S_{ALG_1(I)}}$$

Note that if the above relationship holds for each scenario of the second stage, the relationship will hold for each stage when we consider the expected value of revenue generated across all scenarios at the second stage. Thus, we abuse the notation a little bit such that $V_{S_{OPT(I)}}$ and $V_{S_{ALG_1(I)}}$ refer to both revenues for each scenario and the expected revenue across all scenarios. Therefore,

$$OPT(I) = V_{F_{OPT(I)}} + V_{S_{OPT(I)}}$$

$$\leq \frac{V_L}{V_M} V_{F_{ALG_1(I)}} \cdot \frac{|F_{OPT(I)}|}{|F_{ALG_1(I)}|} + d \cdot V_L + V_{S_{ALG_1(I)}}$$

$$\leq \frac{V_L}{V_M} [V_{F_{ALG_1(I)}} \cdot \frac{|F_{OPT(I)}|}{|F_{ALG_1(I)}|} + d \cdot V_M + V_{S_{ALG_1(I)}}]$$

$$\leq \frac{V_L}{V_M} [V_{F_{ALG_1(I)}} (|F_{OPT(I)}| + d) + V_{S_{ALG_1(I)}}]$$

$$= \frac{V_L}{V_M} [V_{F_{ALG_1(I)}} + V_{S_{ALG_1(I)}}]$$

$$= \frac{V_L}{V_M} ALG_1(I)$$
Therefore, Algorithm 1 is a $\frac{V_M}{V_L}$-approximation algorithm.

In the next, we will show the approximation ratio $\frac{V_M}{V_L}$ is also tight. Consider the following example. There are four assets to sell and three scenarios at the second stage. Let $k = 1$. Algorithm 1 will sell the second asset at the first stage, which generates revenue $V_M$. However, in an optimal solution, we should not sell any asset at the first stage. Nevertheless, we should sell the most expensive one under each scenario at the second stage, which generates revenue $V_L$.

|   | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 1 | $V_S$ | $V_L$ | $V_S$ | $V_S$ |
| 2 | $V_M$ | $V_S$ | $V_S$ | $V_S$ |
| 3 | $V_S$ | $V_S$ | $V_L$ | $V_S$ |
| 4 | $V_S$ | $V_S$ | $V_S$ | $V_L$ |

Thus, the approximation ratio $\frac{V_M}{V_L}$ is tight.

Note that the $\frac{V_M}{V_L}$ approximation ratio could be better than the $\max\{\frac{1}{2}, \frac{k}{n}\}$ approximation ratio achieved by algorithms in [2] under many scenarios. For instance, if $V_M > 0.5V_L$ and $k < 0.5n$, our algorithm will perform better. In a real world, the annual return of an asset will usually be much less than 100% (i.e. $\frac{V_M}{V_L} \geq 0.5$) while an asset manager seldom closes his positions on half of its portfolio holdings (i.e. $\frac{k}{n} < 0.5$). Thus, our algorithm will have a good chance to perform better in practice.

5 Conclusion

We studied DSHP under the constraint that each asset can take only a constant number of different values in this paper. We show that if each asset can take only two values, the problem becomes polynomial-time solvable. However, even if each asset can take three different values, DSHP is still NP-hard. A $\frac{V_M}{V_L}$ approximation algorithm is also given under this setting.

There are quite a few interesting open problems raised in this line of work. First of all, we can extend our model to the multistage case instead of two stages. It would be interesting to design an approximation algorithm for that case. Another interesting direction is to impose some restrictions on the relationships among prices of all the assets, such that the prices of assets correlate with each other in some way, and study the complexity in that case.

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