REAL HYPERSURFACES IN COMPLEX TWO-PLANE
GRASSMANNIANS WITH RECURRENT RICCI TENSOR

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ABSTRACT. In this paper, we have introduced a new notion of generalized
Tanaka-Webster Reeb recurrent Ricci tensor in complex two-plane Grassman-
nians $G_2(\mathbb{C}^{m+2})$. Next, we give a non-existence property for real hypersurfaces
$M$ in $G_2(\mathbb{C}^{m+2})$ with such a condition.

INTRODUCTION

The complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ is a kind of Hermitian sym-
metry spaces of compact irreducible type with rank 2. It consists of all complex
two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. Remarkably, it is equipped with both a
Kähler structure $J$ and a quaternionic Kähler structure $\mathfrak{J}$ (not containing $J$) sat-
sifying $JJ_\nu = J_\nu J$ ($\nu = 1, 2, 3$), where $\{J_\nu\}_{\nu = 1, 2, 3}$ is an orthonormal basis of $\mathfrak{J}$.
When $m = 1$, $G_2(\mathbb{C}^3)$ is isometric to the two-dimensional complex projective space
$\mathbb{C}P^2$ with constant holomorphic sectional curvature eight. When $m = 2$, we note
that the isomorphism Spin(6) $\simeq$ SU(4) yields an isometry between $G_2(\mathbb{C}^4)$ and the
real Grassmann manifold $G_2^+(\mathbb{R}^6)$ of oriented two-dimensional linear subspaces in
$\mathbb{R}^6$. In this paper, we assume $m \geq 3$ (see Berndt and Suh [2] and [3]).

Let $M$ be a real hypersurface of $G_2(\mathbb{C}^{m+2})$, that is, a submanifold of $G_2(\mathbb{C}^{m+2})$
with real codimension one and $T_p M$ stands for the tangent space of $M$ at $p \in M$.
The induced Riemannian metric on $M$ will also be denoted by $g$, and $\nabla$ denotes
the Riemannian connection of $(M, g)$. Let $N$ be a local unit normal vector field
of $M$ and $A$ the shape operator of $M$ with respect to $N$.

By using the result of Alekseevskii [1], Berndt and Suh [2] have classified all real
hypersurfaces with these invariant properties in $G_2(\mathbb{C}^{m+2})$ as follows:

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Theorem A. Let $M$ be a real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then both $[\xi]$ and $Q^\perp$ are invariant under the shape operator of $M$ if and only if

(A) $M$ is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, or

(B) $m$ is even, say $m = 2n$, and $M$ is an open part of a tube around a totally geodesic $H^{P^n}$ in $G_2(\mathbb{C}^{m+2})$.

In the case of (A) (resp., (B)), we say that $M$ is of Type (A) (resp., Type (B)).

Furthermore, the real hypersurface $M$ is said to be Hopf if $A[\xi] \subset [\xi]$, or equivalently, the Reeb vector field $\xi$ is principal with principal curvature $\alpha = g(A\xi, \xi)$. In this case, the principal curvature $\alpha = g(A\xi, \xi)$ is said to be a Reeb curvature of $M$.

By using Theorem A, many geometers have given some characterizations for Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with geometrical quantities; shape operator, normal (or structure) Jacobi operator, Ricci tensor, and so on. The Ricci tensor $S$ of $M$ in $G_2(\mathbb{C}^{m+2})$ is given by

$$g(SX, Y) = \sum_{i=1}^{4m-1} g(R(e_i, X)Y, e_i),$$

where $\{e_1, \ldots, e_{4m-1}\}$ denotes a basis of the tangent space $T_pM$ of $M$, $p \in M$, in $G_2(\mathbb{C}^{m+2})$ (see [23]).

Now we define the notion of recurrent, which is weaker than the usual parallelism. The notion of recurrent for a $(1,1)$ type tensor field $T$ has a close relation to holonomy group. For a 1-form $\omega$ on $M$ is defined by $\nabla T = T \otimes \omega$, (see [10]).

Let us consider a notion of recurrent (resp., Reeb recurrent) Ricci tensor $S$ for a real hypersurface $M$ in $G_2(\mathbb{C}^{m+2})$ defined by

(C-1) $\nabla_X S = \omega(X) S$

for any $X$ in $TM$.

Motivated by such a notion, we want to introduce another new notion of Reeb recurrent Ricci tensor. It is weaker than usual parallel Ricci tensor and is defined by

(C-2) $\nabla_\xi S = \omega(\xi) S$.

Now we say that if $S$ satisfies the condition (C-2), it is a proper Reeb recurrent if $\omega(\xi)$ is non-vanishing, i.e., $\omega(\xi) \neq 0$. Then (C-1) (resp., (C-2)) means $[\nabla_X S, S] = \omega(X)[S, S] = 0$ (resp., $[\nabla_\xi S, S] = 0$) for any tangent vector field $X$ defined on $M$ (see [21]). Its geometrical meaning is that the eigenspaces of the Ricci operator $S$ of $M$ are parallel along any curve $\gamma$ (resp., Reeb flow). Here, the eigenspaces are said to be parallel if they are invariant with respect to any parallel translations along $\gamma$ (resp., Reeb flow) (for detailed examples, see [26], [27], [11]). There are many examples of Recurrent Ricci tensor in pseudo-Riemannian manifolds [27, Example 4, p. 13].

In this paper, we give a complete classification of real hypersurfaces $M$ in $G_2(\mathbb{C}^{m+2})$ with recurrent (resp., Reeb recurrent) Ricci tensor as follows:

Theorem 1. There do not exist any Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with proper Reeb recurrent Ricci tensor if the Reeb curvature is non-vanishing.
Remark 1. When \( \omega(\xi) = 0 \), the Reeb recurrent Ricci tensor is equivalent to Reeb parallel Ricci tensor, so by using the result of [23], \( M \) is locally congruent to one of the following:

(i) a tube over a totally geodesic \( G_2(\mathbb{C}^{m+1}) \) in \( G_2(\mathbb{C}^{m+2}) \) with radius \( r \neq \frac{\pi}{4\sqrt{2}} \), or

(ii) a tube over a totally geodesic quaternionic projective space \( \mathbb{H}P^n \), \( m = 2n \), in \( G_2(\mathbb{C}^{m+2}) \) with radius \( r \) such that \( \cot^2(2r) = \frac{1}{2m-1} \) and \( \xi \)-parallel eigenspaces \( T_{\cot r} \) and \( T_{\tan r} \).

On the other hand, if we use the result in [24], we can assert another non-existence property for real hypersurfaces in \( G_2(\mathbb{C}^{m+2}) \) with Recurrent Ricci tensor as follows:

Corollary 1. There do not exist any Hopf hypersurfaces in \( G_2(\mathbb{C}^{m+2}) \), \( m \geq 3 \), with recurrent Ricci tensor.

Next, we consider a new connection which is different from the usual Levi-Civita connection, so called, the generalized Tanaka-Webster (in short, GTW) connection. Even though this connection does not satisfy torsion free condition, it is deeply related to the contact structure (see [6], [7]).

Let us consider a notion of the GTW recurrent Ricci tensor \( S \) for a real hypersurface \( M \) in \( G_2(\mathbb{C}^{m+2}) \) defined by

\[
\hat{\nabla}^{(k)}_X S = \omega(X)S
\]

for any \( X \) in \( TM \), where \( \omega \) denotes a 1-form defined on \( M \).

Similarly, we may also define GTW Reeb parallel Ricci tensor as follows

\[
\hat{\nabla}^{(k)}_{\xi} S = \omega(\xi)S.
\]

We say that the condition (C-4) is said to be a proper GTW Reeb recurrent if the 1-form \( \omega(\xi) \) is non-vanishing, i.e., \( \omega(\xi) \neq 0 \). We can classify real hypersurfaces \( M \) in \( G_2(\mathbb{C}^{m+2}) \) with GTW Reeb recurrent Ricci tensor as follows:

Theorem 2. There do not exist any Hopf hypersurfaces in \( G_2(\mathbb{C}^{m+2}) \), \( m \geq 3 \), \( (\alpha \neq 2k) \) with proper GTW Reeb recurrent Ricci tensor.

Remark 2. When \( \omega(\xi) \) identically vanishes, that is, \( \omega(\xi) = 0 \), then the GTW Reeb recurrent Ricci tensor is equivalent to GTW Reeb parallel Ricci tensor; therefore, by using the result of [15], \( M \) is locally congruent to one of the following:

(i) a tube over a totally geodesic \( G_2(\mathbb{C}^{m+1}) \) in \( G_2(\mathbb{C}^{m+2}) \) with radius \( r \) such that \( r \neq \frac{1}{2\sqrt{2}} \cot^{-1}(\frac{k}{\sqrt{2}}) \), or

(ii) a tube over a totally geodesic \( \mathbb{H}P^n \), \( m = 2n \), in \( G_2(\mathbb{C}^{m+2}) \) with radius \( r \) such that \( r = \frac{1}{2} \cot^{-1}(\frac{1}{\sqrt{2m-1}}) \).

Using the result in [20], we can assert another non-existence property for real hypersurfaces \( M \) in \( G_2(\mathbb{C}^{m+2}) \) as follows:

Corollary 2. There do not exist any Hopf hypersurfaces in \( G_2(\mathbb{C}^{m+2}) \), \( m \geq 3 \), \( (\alpha \neq 2k) \) with GTW recurrent Ricci tensor.
In Sections 1, 2 complete proofs of Theorem 1 and Corollary 1 will be given respectively. In Sections 3 and 4, the proofs of Theorem 2 and Corollary 2 will be given. Main references for Riemannian geometric structures of $G_2(C^m+2), m \geq 3$ will be explained in detail (see [1], [2], [3], and [14]).

1. The proper Reeb recurrent Ricci tensor

From now on, let $M$ represent a real hypersurface in $G_2(C^m+2), m \geq 3$, and $S$ denote the Ricci tensor of $M$. Hereafter, unless otherwise stated, we consider that $X$ and $Y$ are any tangent vector fields on $M$ and $N$ denotes the normal vector field of $M$. $\omega$ stands for any 1-form on $M$. For the Kähler structure $J$ and the quaternionic Kähler structure $J = \text{span}\{J_\nu\}_{\nu=1,2,3}$, we may put

$$JX = \phi X + \eta(X)N, \quad J_\nu X = \phi_\nu X + \eta_\nu(X)N$$

where $\phi X$ (resp., $\phi_\nu X$) is the tangential part of $JX$ (resp., $J_\nu X$) and $\eta(X) = g(X, \xi)$ (resp., $\eta_\nu(X) = g(X, \xi_\nu)$) is the coefficient of normal part of $JX$ (resp., $J_\nu X$). In this case, we call $\phi$ the structure tensor field of $M$. In [19], the Ricci tensor $S$ of a real hypersurface $M$ in $G_2(C^m+2), m \geq 3$, is given by

$$SX = \sum_{i=1}^{4m-1} R(X, e_i)e_i$$

$$= (4m+7)X - 3\eta(X)\xi + hAX - A^2X$$

$$+ \sum_{\nu=1}^{3} \left\{ -3\eta_\nu(X)\xi_\nu + \eta_\nu(\xi)\phi_\nu X - \eta_\nu(\phi X)\phi_\nu \xi - \eta(X)\eta_\nu(\xi)\xi_\nu \right\},$$

(1.1)

where $h$ denotes the trace of the shape operator $A$, that is, $h = \text{Tr} A$.

In [15], the covariant derivative of $S$ is given by

$$(\nabla_X S) Y = -3g(\phi AX, Y)\xi - 3\eta(Y)\phi AX$$

$$- 3 \sum_{\nu=1}^{3} \left\{ g(\phi_\nu AX, Y)\xi_\nu + \eta_\nu(Y)\phi_\nu AX \right\}$$

$$+ \sum_{\nu=1}^{3} \left\{ 2g(\phi AX, \xi_\nu)\phi_\nu Y + g(AX, \phi_\nu Y)\phi_\nu \xi \right.$$

$$- \eta(Y)g(AX, \xi_\nu)\phi_\nu \xi - \eta_\nu(\phi Y)g(AX, \xi)\xi_\nu - \eta_\nu(\phi X)\phi_\nu AX$$

$$- \eta(Y)g(\phi AX, \xi_\nu)\xi_\nu - \eta_\nu(\phi AX, \xi)\xi_\nu \right\}$$

$$+ (Xh)AY + h(\nabla_X A)Y - (\nabla_X A)AY - A(\nabla_X A)Y.$$  

(1.2)

Thus, $(\nabla_X S)Y = \omega(X)SY$ is embodied as follows:
Proof. To show this fact, we consider that the Reeb vector field ξ or the distribution Q or the distribution R, then the Reeb vector field ξ belongs to either the distribution Q or the distribution Q⊥.

Lemma 1.1. Let M be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. If M has Reeb recurrent Ricci tensor, then the Reeb vector field ξ belongs to either the distribution Q or the distribution Q⊥.

Proof. To show this fact, we consider that the Reeb vector field ξ satisfies

(*) $\xi = \eta(X_0)X_0 + \eta(\xi_1)\xi_1$

As a special case, we may consider Reeb directional derivative of the Ricci tensor. If the Ricci tensor of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ is Reeb recurrent, then it is defined by

(C-2) $(\nabla_\xi S)Y = \omega(\xi)SY$.

Under the condition of being Hopf, (C-2) is specified:

\[-3\alpha \sum_{\nu=1}^{3} \left\{ g(\phi_\nu, Y)\xi_\nu + \eta_\nu(Y)\phi_\nu \xi \right\} + \alpha \sum_{\nu=1}^{3} \left\{ g(\xi, \phi_\nu, \phi Y)\phi_\nu \xi - \eta(Y)\eta_\nu(\xi)\phi_\nu \xi \nu \right\} \]

\[+ (Xh)AY + h(\nabla X\xi)Y - (\nabla X\xi)AY - A(\nabla X\xi)Y = \omega(\xi) \left[ (4m + 7)Y - 3\eta(Y)\xi + hAY - A^2Y \right. \]

\[+ \sum_{\nu=1}^{3} \left\{ \right. -3\eta_\nu(Y)\xi_\nu + \eta_\nu(\xi)\phi_\nu \phi Y - \eta_\nu(\phi Y)\phi_\nu \phi Y - \eta_\nu(\phi Y)\phi_\nu \xi - \eta(Y)\eta_\nu(\xi)\xi_\nu \left. \right\}. \]
for some unit vectors $X_0 \in \mathcal{Q}$ and $\xi_1 \in \mathcal{Q}^\perp$. Putting $Y = \xi$ in (1.3), by (1) and using basic formulas in [12, Section 2], it follows that

\begin{equation}
-4\alpha\eta_1(\xi)\phi_1 \xi = \alpha(h\xi + h(\xi_0)\xi + 2\alpha(\xi_0)\xi) = \omega(\xi)\{4m + 4 + \alpha + \alpha^2\xi - 4\eta_1(\xi)\xi_0\},
\end{equation}

where we have used $(\nabla_\xi A)\xi = (\xi_0)\xi$ and $(\nabla_\xi A)A\xi = \alpha(\xi_0)\xi$.

Taking the inner product of (1.5) with $\phi_2 \xi$, we have

\begin{equation}
-4\alpha\eta_1(\xi)\eta_2^2(X_0) = 0.
\end{equation}

From this, we have the following three cases.

**Case 1:** $\alpha = 0$.

By the equation $Y\alpha = (\xi_0)\eta(Y) - 4\sum_{\nu=1}^3 \eta_\nu(\xi)\eta_\nu(\phi Y)$ in [2, Lemma 1], we obtain easily that $\xi$ belongs to either $\mathcal{Q}$ or $\mathcal{Q}^\perp$ (see [19]).

**Case 2:** $\eta(\xi_1) = 0$.

By the notation (1) related to the Reeb vector field, we see that $\xi$ belongs to the distribution $\mathcal{Q}$.

**Case 3:** $\eta(X_0) = 0$.

This case implies that $\xi$ belongs to the distribution $\mathcal{Q}^\perp$.

Accordingly, summing up these cases, the proof is completed.

\[\square\]

**Lemma 1.2.** Let $M$ be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$. If the Reeb vector field $\xi$ belongs to $\mathcal{Q}^\perp$, then the Ricci tensor $S$ and the shape operator $A$ commute with each other, that is, $SA = AS$.

(see [17, Lemma 1.2].)

**Lemma 1.3.** Let $M$ be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with nonvanishing Reeb curvature (i.e., $\alpha \neq 0$). When the Reeb vector field $\xi$ belongs to $\mathcal{Q}^\perp$, if $M$ has Reeb recurrent Ricci tensor, that is, $(\nabla_\xi S)X = \omega(\xi)SX$, then $M$ must have commuting Ricci tensor $S\phi = \phi S$.

**Proof.** From the Codazzi equation in [2] and by differentiating $A\xi = \alpha\xi$, we obtain

\begin{equation}
(\nabla_\xi A)X = (X_0)\xi + \alpha(\phi AX - A\phi X + \phi \xi + 2\eta_3(\xi)\xi_2 - 2\eta_2(\xi)\xi_3).
\end{equation}

[12, Lemma A, (3.3)] is essential equation for proving this lemma:

\begin{equation}
\alpha A\phi X + \alpha\phi AX - 2A\phi AX + 2\phi X = 2 \sum_{\nu=1}^3 \left\{ - \eta_\nu(X)\phi_\nu - \eta_\nu(\phi X)\xi_\nu \right\} - \eta_\nu(\xi)\phi_\nu X + 2\eta(X)\eta_\nu(\xi)\phi_\nu + 2\eta_\nu(\phi X)\eta_\nu(\xi).
\end{equation}

Using (1.7) and (1.8), we get $(\nabla_\xi A)X = \frac{\alpha}{2}\phi AX - \frac{\alpha}{2}A\phi X + (\xi_0)\eta(X)\xi$, which changes (1.2) into

\begin{equation}
(\xi h)AX + \frac{\alpha}{2}(\phi AX - A\phi X) - \frac{\alpha}{2}(\phi A^2 X - A^2 \phi X) + (h - 2\alpha)(\xi_0)\eta(X)\xi = \omega(\xi)SX.
\end{equation}

Here replacing $X$ by $\phi X$ in (1.1) (resp., applying $\phi$ to (1.1)), we have
By the equation of Codazzi [2] and [2, Proposition 3] we obtain
\begin{equation}
\phi SX = (4m + 7)\phi X - \phi_1 X + 2\eta_2 (X)\xi_3 - 2\eta_3 (X)\xi_2 + hA\phi X - A^2 \phi X,
\end{equation}
\begin{equation}
\phi SX = (4m + 7)\phi X - \phi_1 X + 2\eta_2 (X)\xi_3 - 2\eta_3 (X)\xi_2 + hA\phi X - A^2 \phi X.
\end{equation}

Combining equations in (1.10), we obtain
\begin{equation}
S\phi X - \phi SX = hA\phi X - A^2 \phi X - h\phi AX + \phi A^2 X.
\end{equation}

Using (1.11), (1.9) becomes
\begin{equation}
(\xi h)AX + \frac{\alpha}{2}(\phi SX - S\phi X) + (h - 2\alpha) (\xi\alpha)\eta (X) \xi = \omega (\xi) SX.
\end{equation}

Substituting $X$ to $AX$ into (1.12) and applying $A$ to (1.12), we have
\begin{equation}
\begin{cases}
(\xi h)A^2 X + \frac{\alpha}{2}(\phi S - S\phi) AX + \alpha (h - 2\alpha)(\xi\alpha)\eta (X) \xi = \omega (\xi) SAX, \\
(\xi h)A^2 X + \frac{\alpha}{2} A(\phi S - S\phi) X + \alpha (h - 2\alpha)(\xi\alpha)\eta (X) \xi = \omega (\xi) ASX.
\end{cases}
\end{equation}

By combining equations in (1.13) and using Lemma 1.2 we get
\begin{equation}
(\phi S - S\phi) A = A(\phi S - S\phi).
\end{equation}

If the Reeb vector field $\xi$ belongs to $Q^\perp$ and $A\xi = \alpha\xi$ on $M$, $A(\phi S - S\phi) = (\phi S - S\phi) A$ is equivalent to $S\phi = \phi S$ on $M$ (see [17, Lemma 1.5]).

Summing up above lemmas [1.2] [1.3] [23, Theorem 1.1] [3, Theorem] and [2, Theorem], we conclude that if $M$ is a Hopf hypersurface in $G_2(C^{m+2})$ on which holds (C-2), then $M$ satisfies the condition of being a model space of Type (A) (shortly, $M_A$).

From this together with Theorem A in the introduction we know that any real hypersurface in $G_2(C^{m+2})$ with Reeb recurrent Ricci tensor and $\xi \in Q^\perp$ is congruent to a tube over a totally geodesic $G_2(C^{m+1})$ in $G_2(C^{m+2})$. Now let us check if real hypersurfaces $M_A$ satisfy the condition of Reeb recurrent Ricci tensor.

By virtue of [25], we have

**Remark 1.4.** If $\omega (\xi) = 0$, the Ricci tensor $S$ of real hypersurfaces $M_A$ in $G_2(C^{m+2})$ satisfies the Reeb parallel condition.

So we may consider only $\omega (\xi) \neq 0$. We assume that $M_A$ satisfies (C-2).

By the equation of Codazzi [2] and [2, Proposition 3] we obtain $X \in T_xM_A = T_\alpha \oplus T_\beta \oplus T_\lambda \oplus T_\mu$,
\begin{equation}
(\nabla_\xi S)X = -h(\nabla_\xi A)X + (\nabla_\xi A)AX + A(\nabla_\xi A)X
\end{equation}
and
\begin{equation}
(\nabla_\xi A)X = \frac{\alpha}{2} \phi AX - \frac{\alpha}{2} A\phi X + (\xi\alpha)\eta (X) \xi
\end{equation}

\begin{equation}
\begin{cases}
0 & \text{if } X \in T_\alpha, \\
0 & \text{if } X \in T_\beta = \text{span}\{\xi_\ell | \ell = 2, 3\}, \\
0 & \text{if } X \in T_\lambda, \\
0 & \text{if } X \in T_\mu.
\end{cases}
\end{equation}
From these two equations, it follows that

\[
(\nabla S)X = \begin{cases} 
0 & \text{if } X = \xi \in T_{\alpha}, \\
0 & \text{if } X = \xi_{\ell} \in T_{\beta} = \text{span}\{\xi_{\ell}| \ell = 1, 2, 3\}, \\
0 & \text{if } X \in T_{\gamma}, \\
0 & \text{if } X \in T_{\mu}.
\end{cases}
\]

Consider

\[
SX = \begin{cases} 
(4m + h\alpha - \alpha^{2})\xi & \text{if } X = \xi \in T_{\alpha}, \\
(4m + 6 + h\beta - \beta^{2})\xi_{\ell} & \text{if } X = \xi_{\ell} \in T_{\beta} = \text{Span}\{\xi_{\ell}| \ell = 1, 2, 3\}, \\
(4m + 6 + h\lambda - \lambda^{2})X & \text{if } X \in T_{\gamma}, \\
(4m + 8)X & \text{if } X \in T_{\mu}.
\end{cases}
\]

If we consider a non-zero tangent vector field \(X \in T_{\mu}\), then we get \(\omega(\xi)(4m + 8)X = 0\), which means \(\omega(\xi) = 0\). This is a contradiction.

**Remark 1.5.** If \(\omega(\xi) \neq 0\), the Ricci tensor \(S\) of real hypersurfaces \(M_{A}\) in \(G_{2}(\mathbb{C}^{m+2})\) does not satisfy the Reeb recurrent condition.

Summing up all cases mentioned above, we can assert that if \(\omega(\xi) = 0\), then \(S\) of real hypersurfaces \(M_{A}\) in \(G_{2}(\mathbb{C}^{m+2})\) satisfies the Reeb recurrent condition.

For \(\xi \in Q\), by [14 Main Theorem], we know \(g(AQ, Q) = 0\).

We know that a Hopf hypersurface \(M\) in \(G_{2}(\mathbb{C}^{m+2})\) with Reeb recurrent Ricci tensor and \(\xi \in Q\) is a real hypersurface of type \((B)\) (denoted by \(M_{B}\)) in \(G_{2}(\mathbb{C}^{m+2})\), that is, a tube over a totally geodesic \(\mathbb{HP}^{n}\). We will check if such a tube satisfies the notion of Reeb recurrent Ricci tensor. We assume that \(M_{B}\) satisfies \((C.2)\).

In order to do this, let us calculate the fundamental equation related to the covariant derivative of \(S\) of \(M_{B}\) along the direction of \(\xi\). On \(T_{x}M_{B}, x \in M_{B}\), since \(\xi \in Q\) and \(h = \text{tr}(A) = \alpha + (4m - 1)\beta\) is a constant, equation \((C.2)\) is reduced to

\[
(\nabla S)X = -4\alpha \sum_{\nu=1}^{3} \left\{ \eta_{\nu}(\phi X)\xi_{\nu} - \eta_{\nu}(X)\phi_{\nu}\xi \right\} \\
- h(\nabla A)X + (\nabla A)X + A(\nabla A)X.
\]

Moreover, by the equation of Codazzi [2] and [2 Proposition 2], we obtain that for any \(X \in T_{2}M_{B} = T_{\alpha} \oplus T_{\beta} \oplus T_{\gamma} \oplus T_{\lambda} \oplus T_{\mu}\)

\[
(\nabla A)X = \alpha\phi AX - A\phi AX + \phi X - \sum_{\nu=1}^{3} \left\{ \eta_{\nu}(X)\phi_{\nu}\xi + 3g(\phi_{\nu}\xi, X)\xi_{\nu} \right\}
\]

Consider

\[
(1.20) \quad \nabla_{\xi}X = \begin{cases} 
0 & \text{if } X \in T_{\alpha}, \\
0 & \text{if } X \in T_{\beta} = \text{Span}\{\xi_{\ell}| \ell = 1, 2, 3\}, \\
-4\xi_{\ell} & \text{if } X \in T_{\gamma} = \text{Span}\{\phi_{\xi_{\ell}}| \ell = 1, 2, 3\}, \\
(\alpha \lambda + 2)\phi X & \text{if } X \in T_{\lambda}, \\
(\alpha \mu + 2)\phi X & \text{if } X \in T_{\mu}.
\end{cases}
\]
Combining (1.19) and (1.20), it follows that

\begin{equation}
(\nabla \xi S) X = \begin{cases}
0 & \text{if } X = \xi \in T_\alpha \\
\alpha(4 - h\beta + \beta^2)\phi\xi & \text{if } X = \xi \in T_\beta \\
4(\alpha + h - \beta)\xi & \text{if } X = \phi\xi \in T_\gamma \\
(h - \beta)(-\alpha\lambda - 2)\phi X & \text{if } X \in T_\lambda \\
(h - \beta)(-\alpha\mu - 2)\phi X & \text{if } X \in T_\mu.
\end{cases}
\end{equation}

From (1.1) and \[2, Proposition 2\], we obtain the following

\begin{equation}
SX = \begin{cases}
(4m + 4 + h\alpha - \alpha^2)\xi & \text{if } X = \xi \in T_\alpha \\
(4m + 4 + h\beta - \beta^2)\xi & \text{if } X = \xi \in T_\beta \\
(4m + 8)\phi\xi & \text{if } X = \phi\xi \in T_\gamma \\
(4m + 7 + h\lambda - \lambda^2)X & \text{if } X \in T_\lambda \\
(4m + 7 + h\mu - \mu^2)X & \text{if } X \in T_\mu.
\end{cases}
\end{equation}

For the case \(X = \xi\) in \((C, 2)\), we have 0 = \(\omega(\xi)(-8n + 8)\xi\) which means \(\omega(\xi) = 0\). For \(X \in T_\gamma\) and \(X \in T_\mu\), we have \(h = \beta - \alpha\) and \(h = \beta\) must be hold. However, this derives \(\alpha = 0\) which gives a contradiction.

Remark 1.6. The Ricci tensor \(S\) of real hypersurfaces of Type \((B)\) in \(G_2(\mathbb{C}^{m+2})\) does not satisfy the recurrent condition \((C, 2)\).

Hence summing up these considerations, we give a complete proof of our Theorem 1 in the introduction.

2. The recurrent Ricci tensor

Let us assume that the Ricci tensor of a Hopf hypersurface \(M\) in \(G_2(\mathbb{C}^{m+2})\) is recurrent. It is given by

\begin{equation}
(\nabla_X S)Y = \omega(X)SY
\end{equation}

In this section, we prove Corollary 2, given in the introduction. By virtue of lemma 1.1, we know that if \(M\) has recurrent Ricci tensor, then the Reeb vector field \(\xi\) belongs to either \(Q\) or \(Q^\perp\).

Next let us consider the case, \(\xi \in Q^\perp\). Accordingly, we may put \(\xi = \xi_1\).

Lemma 2.1. Let \(M\) be a Hopf hypersurface in \(G_2(\mathbb{C}^{m+2})\), \(m \geq 3\) with vanishing Reeb curvature, that is, \(\alpha = 0\). If the Reeb vector field \(\xi\) belongs to \(Q^\perp\) and \(M\) has recurrent Ricci tensor, then the shape operator \(A\) and the structure tensor field \(\phi\) commutes with each other i.e., \(A\phi = \phi A\).

Proof. Putting \(Y = \xi\) into equation (1.8) and using (1.6), we have

\begin{equation}
-6\phi AX + hA\phi AX + A^2\phi AX = 4m\omega(X)\xi.
\end{equation}

Taking the inner product of (2.2) with \(\xi\), we have \(\omega(X) = 0\). Thus, (2.2) becomes

\begin{equation}
-6\phi AX + hA\phi AX + A^2\phi AX = 0.
\end{equation}

Given that \(\xi = \xi_1\), (1.8) becomes
Applying $A$ to (2.4), and using (1.11), we have

$$A^2 \phi AX = 2 A \phi X.$$ (2.5)

Thus, we have

$$-6 \phi AX + h A \phi AX + 2 A \phi X = 0.$$ (2.6)

Taking the symmetric part of (2.6), we have

$$6 A \phi X - h A \phi AX - 2 \phi AX = 0.$$ (2.7)

Combining (2.6) and (2.7), we have

$$A \phi = \phi A.$$ □

Summing up lemmas [1.2, 1.3, 2.1, [3, Theorem] and [2, Theorem 2], we know that any connected Hopf hypersurface in $G_2(C^{m+2})$ with recurrent Ricci tensor is locally congruent to a real hypersurface $M_A$ if the Reeb vector field $\xi$ belongs to the distribution $Q^\perp$. Now we check the converse problem: whether a Hopf hypersurface $M_A$ satisfies the given condition (2.1) or not. So we assume that $M_A$ satisfies (2.1).

Putting $Y = \xi$ into (1.3), we obtain

$$-6 \phi AX + (h - \alpha) \phi AX + h A \phi AX + A^2 \phi AX = \omega(X)(4m + h \alpha - \alpha^2) \xi.$$ (2.8)

Taking $X \in T_\lambda$, we have

$$\lambda \{ -6 + (h - \alpha) \alpha + h \lambda + \lambda^2 \} \phi X = \omega(X)(4m + h \alpha - \alpha^2) \xi,$$ (2.9)

where we have used $\phi T_\lambda \subset T_\lambda$ in Type A.

Thus $\lambda \{ -6 + (h - \alpha) \alpha + h \lambda + \lambda^2 \} \phi X$ and $\omega(X)(4m + h \alpha - \alpha^2) \xi$ should vanish respectively. Using $\lambda \neq 0$ from [2, Proposition 3], as $\phi X$ cannot be vanishing, we have

$$-6 + \alpha(h - \alpha) + h \lambda + \lambda^2 = 0.$$ (2.10)

Taking $X \in T_\beta$, (2.8) becomes

$$\beta \{ -6 + (h - \alpha) \alpha + h \beta + \beta^2 \} \phi X = \omega(X)(4m + h \alpha - \alpha^2) \xi$$ (2.11)

where we have used $\phi T_\beta \subset T_\beta$ in Type A.

Thus $\beta \{ -6 + (h - \alpha) \alpha + h \beta + \beta^2 \} \phi X$ and $\omega(X)(4m + h \alpha - \alpha^2) \xi$ should be vanishing respectively. Using $\beta \neq 0$ from [2, Proposition 3], as $\phi X$ cannot be vanishing, we also have

$$-6 + \alpha(h - \alpha) + h \beta + \beta^2 = 0.$$ (2.12)

Using $\beta - \lambda \neq 0$ and combining (2.10) and (2.12), we have

$$h + \lambda + \beta = 0.$$ (2.13)
Combining (2.10) and (2.13), and applying
\[
\alpha = \sqrt{8} \cot(\sqrt{8}r), \quad \beta = \sqrt{2} \cot(\sqrt{2}r), \quad \lambda = -\sqrt{2} \tan(\sqrt{2}r), \quad \mu = 0
\]
with some \( r \in (0, \pi/\sqrt{8}) \) (see [2, Proposition 2]).

We have
\[
(2.14) \quad 0 = -6 + \alpha(h - \alpha) + h\lambda + \lambda^2 = 4 + 2 \left\{ \tan(\sqrt{2}r) - \cot(\sqrt{2}r) \right\}^2 + \alpha^2 > 0.
\]
This gives a contradiction.

**Remark 2.2.** The Ricci tensor \( S \) of real hypersurfaces \( M_A \) in \( G_2(\mathbb{C}^{m+2}) \) does not satisfy the recurrent condition.

For \( \xi \in \mathcal{Q} \), by [14, Main Theorem], we know \( g(AQ, Q^\perp) = 0 \). By virtue of Remark 1.6, Hopf hypersurface \( M_B \) does not satisfy the given condition.

3. The GTW Reeb recurrent Ricci tensor

In this section, we prove our Theorem 2, given in the introduction. Related to Levi-Civita connection \( \nabla \), the generalized Tanaka-Webster connection (from now on, GTW connection) for contact metric manifolds was introduced by Tanno [29] as a generalization of the connection defined by Tanaka in [28] and, independently, by Webster in [31]. The Tanaka-Webster connection is defined as a canonical affine connection on a non-degenerate, pseudo-Hermitian CR-manifold. A real hypersurface \( M \) in a Kähler manifold has an (integrable) CR-structure associated with the almost contact structure \((\phi, \xi, \eta, g)\) induced on \( M \) by the Kähler structure; however, in general, this CR-structure is not guaranteed to be pseudo-Hermitian. Cho defined GTW connection for a real hypersurface of a Kähler manifold by
\[
\widehat{\nabla}^{(k)}_X Y = \nabla_X Y + F_X^{(k)} Y,
\]
where constant \( k \in \mathbb{R} \setminus \{0\} \) and \( F_X^{(k)} Y = g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y \). \( F_X^{(k)} \) is a skew-symmetric \((1,1)\) type tensor, that is, \( g(F_X^{(k)} Y, Z) = -g(Y, F_X^{(k)} Z) \) for any tangent vector fields \( X, Y, \) and \( Z \) on \( M \), and is said to be Tanaka-Webster (or k-th-Cho) operator with respect to \( X \). In particular, if the real hypersurface satisfies \( A\phi + \phi A = 2k\phi \), then the GTW connection \( \widehat{\nabla}^{(k)} \) coincides with the Tanaka-Webster connection (see [6, 7, 8]).

The Ricci tensor \( S \) is said to be generalized Tanaka-Webster parallel (in short, GTW parallel) if the covariant derivative in GTW connection \( \widehat{\nabla}^{(k)} \) of \( S \) along any \( X \) vanishes, that is, if \( (\widehat{\nabla}^{(k)}_X S) Y = 0 \).

From the definition of \( \widehat{\nabla}^{(k)} \) and \( (\widehat{\nabla}^{(k)}_X S) Y \), we have
\[
(\widehat{\nabla}^{(k)}_X S) Y = (\nabla_X S) Y + F_X^{(k)}(SY) - SF_X^{(k)} Y = \omega(X) SY.
\]

The condition (3.1) is specified as follow:
\[
(\hat{\nabla}^{(k)}_X S)Y = (\nabla_X S)Y \\
+ g(\phi AX, SY)\xi - \eta(SY)\phi AX - k\eta(X)\phi SY \\
- g(\phi AX, Y)S\xi + \eta(Y)S\phi AX + k\eta(X)S\phi Y \\
= \omega(X)SY.
\] (3.2)

The Ricci tensor \(S\) is said to be \textit{GTW Reeb parallel} if the covariant derivative in GTW connection \(\hat{\nabla}^{(k)}\) of \(S\) along the Reeb direction vanishes, that is, if \((\hat{\nabla}^{(k)}_\xi S)Y = 0\). Furthermore, GTW Reeb recurrent Ricci tensor is given by
\[
(\hat{\nabla}^{(k)}_\xi S = \omega(\xi)S. \tag{C-4}
\]

**Lemma 3.1.** Let \(M\) be a Hopf hypersurface in \(G_2(\mathbb{C}^{m+2}), m \geq 3\). If \(M\) has GTW Reeb recurrent Ricci tensor, then the Reeb vector field \(\xi\) belongs to either the distribution \(Q\) or the distribution \(Q^\perp\).

**Proof.** We write
\[
(*) \quad \xi = \eta(X_0)X_0 + \eta(\xi_1)\xi_1
\]
for some unit vectors \(X_0 \in Q\) and \(\xi_1 \in Q^\perp\).

Putting \(Y = \xi\) into (C-4) and applying \(\phi\) to (C-4), we have
\[
-4(\alpha + k)\eta(\xi)\{\xi_1 + \eta(\xi)\xi\} = -4\omega(\xi)\eta(\xi)\phi\xi_1. \tag{3.3}
\]

Taking an inner product with \(X_0\), we have
\[
-4(\alpha + k)\eta^2(\xi)\eta(X_0) = 0. \tag{3.4}
\]
From this, we have the following three cases.

**Case 1:** \(\alpha = -k\).

By the equation \(Y = (\xi_0)\eta(Y) - 4\sum \eta(\xi)\eta(\phi Y)\) in [2, Lemma 1], we obtain easily that \(\xi\) belongs to either \(Q\) or \(Q^\perp\) (see [19]).

**Case 2:** \(\eta(\xi_1) = 0\).

By the notation (\ref{case1}) related to the Reeb vector field, we see that \(\xi\) belongs to the distribution \(Q\).

**Case 3:** \(\eta(X_0) = 0\).

This case implies that \(\xi\) belongs to the distribution \(Q^\perp\).

Accordingly, summing up these cases, it completes the proof of our Lemma. \(\square\)

As we know,
\[
(\hat{\nabla}^{(k)}_\xi S)Y = (\nabla_\xi S)Y + k(S\phi - \phi S)Y. \tag{3.5}
\]

Next let us consider the case, \(\xi \in Q^\perp\). Accordingly, we may put \(\xi = \xi_1\).

**Lemma 3.2.** Let \(M\) be a Hopf hypersurface in \(G_2(\mathbb{C}^{m+2}), m \geq 3\). When the Reeb vector field \(\xi\) belongs to the distribution \(Q^\perp\), if \(M\) has the GTW Reeb recurrent Ricci tensor, that is, \((\hat{\nabla}^{(k)}_\xi S)X = \omega(\xi)SX\) \((\alpha \neq 2k)\), then \(S\phi = \phi S\).
Proof. Using (1.10) and (1.12), then (3.1) becomes

\[
(\xi h)AX + \left(\frac{\alpha}{2} - k\right)(\phi SX - S\phi X) + (h - 2\alpha)\eta(\xi)\eta(X)\xi = \omega(\xi)SX.
\]

Substituting \(X\) to \(AX\) into (3.6) and applying \(A\) to (3.6) and combining them, we have \((\phi S - S\phi)A = A(\phi S - S\phi)\). By [12, Lemma 1.5], we have \(\phi S = \phi S\).

\[\Box\]

Summing up these discussions, we conclude that if a Hopf hypersurface \(M\) in complex two-plane Grassmannians \(G_2(\mathbb{C}^{m+2})\), \(m \geq 3\), satisfying \((\hat{\nabla}_\xi^{(k)}SY) = \omega(X)SY\) then \(M\) is of Type (A). Hereafter, let us check whether \(S\) of a model space of \(M_A\) satisfies the Reeb parallelism with respect to \(\hat{\nabla}^{(k)}\) by [2, Proposition 3] (see [12]). From these two equations, it follows that

\[
(\hat{\nabla}_\xi^{(k)}S)X = \begin{cases}
0 & \text{if } X = \xi \in T_\alpha \\
0 & \text{if } X = \xi \in T_\beta = \text{Span}\{\xi_\ell | \ell = 1, 2, 3\} \\
0 & \text{if } X \in T_\lambda \\
0 & \text{if } X \in T_\mu.
\end{cases}
\]

Consider (1.19) and \(X = \xi \in T_\alpha\); thus, \(S\xi = (4m + h\alpha - \alpha^2)\xi\). Thus, \(\omega(\xi) = 0\). Summing up all cases mentioned above, we can assert that if \(\omega(\xi) = 0\), then \(S\) of \(M_A\) in \(G_2(\mathbb{C}^{m+2})\) is GTW Reeb parallel.

Remark 3.3. The Ricci tensor \(S\) of real hypersurfaces \(M_A\) in \(G_2(\mathbb{C}^{m+2})\) satisfies the GTW Reeb parallel condition if \(\omega(\xi) = 0\).

For \(\xi \in Q\), by [14, Main Theorem], we know \(g(AQ, Q^\perp) = 0\).

Now let us consider our problem for a model space \(M_B\). In order to do this, let us calculate the fundamental equation related to the covariant derivative of \(S\) of \(M_B\) along the direction of \(\xi\) in GTW connection. On \(T_xM_B\), \(x \in M_B\), since \(\xi \in Q\) and \(h = \text{Tr}(A) = \alpha + (4n - 1)\beta\) is a constant, (C-4) is reduced to

\[
(\hat{\nabla}_\xi^{(k)}S)X = \left(4k - \alpha\right)\sum_{\nu=1}^{3} \left\{\eta_\nu(\phi X)\xi_\nu - \eta_\nu(X)\phi_\nu\xi\right\} - h(\nabla_\xi A)X + (\nabla_\xi A)AX + A(\nabla_\xi A)X \\
+ kh\phi AX - k\phi A^2 X - khA\phi X + kA^2 \phi X.
\]

Moreover, by the equation of Codazzi [2 and [2, Proposition 2] we obtain that for any \(X \in T_xM_B\)

\[
(\nabla_\xi A)X = \alpha \phi AX - A\phi AX + \phi X - \sum_{\nu=1}^{3} \left\{\eta_\nu(X)\phi_\nu\xi + 3g(\phi_\nu\xi, X)\xi_\nu\right\}
\]

(3.8)

\[
= \begin{cases}
0 & \text{if } X \in T_\alpha \\
\alpha\beta\phi\xi_\ell & \text{if } X \in T_\beta = \text{Span}\{\xi_\ell | \ell = 1, 2, 3\} \\
-4\xi_\ell & \text{if } X \in T_\gamma = \text{Span}\{\phi\xi_\ell | \ell = 1, 2, 3\} \\
(\alpha\lambda + 2)\phi X & \text{if } X \in T_\lambda \\
(\alpha\mu + 2)\phi X & \text{if } X \in T_\mu.
\end{cases}
\]
From these two equations, it follows that

\[
(\nabla_{\xi} S) X = \begin{cases} 
0 & \text{if } X = \xi \in T_{\alpha} \\
(\alpha - k)(4 - h\beta + 2\beta^2)\phi \xi & \text{if } X = \xi \in T_{2\beta} \\
4(\alpha - k) + (h - 3)(4 + k\beta)\phi \xi & \text{if } X = \xi \in T_{7} \\
(h - \beta)(k\lambda - k\mu - \alpha\lambda - 2)\phi X & \text{if } X \in T_{\lambda} \\
(h - \beta)(k\mu - k\lambda - 2)\phi X & \text{if } X \in T_{\mu}.
\end{cases}
\]  

(3.9)  

Therefore, we see that \( M \) has Reeb parallel GTW-Ricci tensor, when \( \alpha \) and \( h \) satisfies the conditions \( \alpha = k \) and \( h - \beta = 0 \), which means \( r = \frac{1}{2} \cot^{-1} \left( \frac{k}{4(2n - 1)} \right) \). Moreover, this radius \( r \) satisfies our condition \( \alpha \neq 2k \). Secondly, we check whether a model space \( M \) satisfies the condition of GTW Reeb recurrent Ricci tensor. In this case, (3.5) becomes

\[
-3\phi AX - \sum_{\nu = 1}^{3} g(AX, \xi_{\nu})\phi_{\nu} \xi + (h - \alpha)\alpha \phi AX + hA\phi AX + A^2\phi AX = \omega(X)(4m + h\alpha - \alpha^2)\xi.
\]  

(3.10)  

Taking the inner product with \( \xi \), we get \( (4m + h\alpha - \alpha^2)\omega(X) = 0 \) which means

\[
-3\phi AX - \sum_{\nu = 1}^{3} g(AX, \xi_{\nu})\phi_{\nu} \xi + (h - \alpha)\alpha \phi AX + hA\phi AX + A^2\phi AX = 0.
\]  

(3.11)  

**Remark 3.4.** The Ricci tensor \( S \) of any real hypersurface \( M \) in \( G_2(C^{m+2}) \) satisfies the GTW Reeb parallel condition.

Consider \( X = \xi_{1} \in T_{\beta} \), we get

\[
(-2 + h\alpha - \alpha^2 - h\beta)\phi \xi_{1} = 0.
\]  

(3.12)  

The coefficient of left term is less than 0, i.e., \(-2 + h\alpha - \alpha^2 - h\beta = -2 - 4(4n - 2) - (4n - 1)\beta^2 < 0 \). This means \( \alpha \xi_{1} = 0 \) which makes a contradiction.

**Remark 3.5.** The Ricci tensor \( S \) of a real hypersurface \( M \) in \( G_2(C^{m+2}) \) does not satisfy the Proper GTW Reeb recurrent condition.

4. GTW recurrent Ricci tensor

By virtue of 3.1, if \( M \) has the GTW recurrent Ricci tensor 3.2, then the Reeb vector field \( \xi \) belongs to either \( Q \) or \( Q^\perp \). In addition, by virtue of lemma 3.2, if \( \xi \) belongs to \( Q^\perp \), we have \( S\phi = \phi S \). Now we check the converse problem whether a real hypersurface \( M \) satisfies the given condition 3.2 or not.

Putting \( Y = \xi \) into 3.2, we get

\[
(\nabla_{X} S)\xi + F_{X}^{(k)}(S\xi) - SF_{X}^{(k)}\xi = \omega(X)S\xi.
\]  

(4.1)  

Taking the inner product of (4.1) with \( \xi \), consider \( (\nabla_{\xi} S)\xi = 0 \), \( F_{X}^{(k)} \) is skew symmetric and \( S\xi = (4m + h\alpha - \alpha^2)\xi \), we have \( (4m + h\alpha - \alpha^2)\omega(X) = 0 \), where \( h = \alpha + 2\beta + (2m - 2)(\lambda + \mu) \).
\[4m + h\alpha - \alpha^2 = 4m + 2\alpha\beta + (2m - 2)\alpha\lambda\]
\[= 4\{\cot^2(\theta) + (m - 1)\tan^2(\theta)\}\]
\[\geq 8\sqrt{(m - 1)} > 0.\] 

This gives
\[(4.3) \quad \omega(X) = 0.\]

Putting \(Y = \xi\) into (4.1), we have
\[(4.4) \quad -6\phi AX + (h - \alpha)\alpha\phi AX + hA\phi AX + A^2\phi AX - \sigma\phi AX + S\phi AX = 0,\]
where \(\sigma = 4m + h\alpha - \alpha^2.\)

Putting \(X \in T_\lambda\) into (4.1), we have
\[2h\lambda = 0\] which means
\[(4.5) \quad h = 0.\]

Consider \(Y = \xi_3 \in T_\mu\) into (4.3), by (4.4) and (4.5), we have
\[\begin{align*}
-4\alpha\eta(X)\xi - 3\phi_1 AX + \phi_3\phi AX - \beta(\nabla_X A)\xi_3 \\
-\beta A(\nabla_X A)\xi_3 + (6 - \beta^2 + \alpha^2)\eta_\beta(AX)\xi
\end{align*}\]
\[(4.6) \quad = 0.\]

Taking the inner product with \(\xi_2\) of (4.6), we have \(3\beta\eta_\beta(X) = 0.\) This means \(3\beta\xi_3 = 0,\) and gives a contradiction. Putting \(Y \in T_\mu\) into (4.1), we have

**Remark 4.1.** The Ricci tensor \(S\) of a real hypersurface \(M_A\) in \(G_2(\mathbb{C}^{m+2})\) does not satisfy the GTW recurrent condition.

Now we check the converse problem, that is, a real hypersurface \(M_B\) satisfies the given condition (3.2) or not. Hereafter, let us check whether \(M_B\) satisfies the condition of GTW recurrent Ricci tensor.

(1.3) becomes
\[\begin{align*}
-3\phi AX - \sum_{\nu=1}^3 g(AX, \xi_\nu)\phi_\nu\xi + (h - \alpha)\alpha\phi AX + hA\phi AX + A^2\phi AX \\
= \omega(X)(4m + h\alpha - \alpha^2)\xi.
\end{align*}\]

Taking the inner product of (4.7) with \(\xi,\) we get \(\omega(X) = 0,\) which means
\[\begin{align*}
-3\phi AX - \sum_{\nu=1}^3 g(AX, \xi_\nu)\phi_\nu\xi + (h - \alpha)\alpha\phi AX + hA\phi AX + A^2\phi AX = 0.
\end{align*}\]

Consider \(X = \xi_1 \in T_\beta\) into above equation, we get
\[(4.8) \quad (-2 + h\alpha - \alpha^2 - h\beta)\phi_1 = 0.\]

Since \(-2 + h\alpha - \alpha^2 - h\beta = -2 - 4(4m - 2) - (4m - 1)\beta^2 < 0,\) (4.8) means \(\phi_1 = 0.\) This is a contradiction.
Remark 4.2. The Ricci tensor $S$ of real hypersurfaces $M_B$ in $G_2(\mathbb{C}^{m+2})$ does not satisfy the GTW recurrent condition.

Summing up these assertions, we give a complete proof of Corollary 2 in the introduction.

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