Temporal decay of strong solutions for generalized Newtonian fluids with variable power-law index

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Abstract

We consider the motion of a power-law-like generalized Newtonian fluid in $\mathbb{R}^3$, where the power-law index is a variable function. This system of nonlinear partial differential equations arises in mathematical models of electrorheological fluids. The aim of this paper is to investigate the decay properties of strong solutions for the model, based on the Fourier splitting method. We first prove that the $L^2$-norm of the solution has the decay rate $(1 + t)^{-\frac{3}{4}}$. If the $H^1$-norm of the initial data is sufficiently small, we further show that the derivative of the solution decays in $L^2$-norm at the rate $(1 + t)^{-\frac{5}{4}}$.

Keywords: Non-Newtonian fluid, variable exponent, electrorheological fluid, temporal decay

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1 Introduction

We are interested in studying the decay properties for solutions of a system of nonlinear partial differential equations (PDEs) modelling the rheological response of electrorheological fluids. The electrorheological fluid is a viscous fluid with a special property: When it is disposed to an electro-magnetic field, the viscosity exhibits a significant change. For instance, there exist some electrorheological fluids whose viscosity changes by a factor of 1000 as a response to the application of an electric field within 1ms. These days, some useful electrorheological fluids were found with the quality and potential for a wide range of scientific and industrial applications, including for example, clutches, shock absorbers and actuators. In this paper, we consider the reduced model for the incompressible electrorheological fluids, which consists of the following system of PDEs:

$$\partial_t u + (u \cdot \nabla)u - \text{div} S(Du) + \nabla \pi = 0 \quad \text{in } Q_T = (0, T) \times \mathbb{R}^d,$$

$$\text{div} u = 0 \quad \text{in } Q_T = (0, T) \times \mathbb{R}^d,$$

where the extra stress tensor is of the form

$$S(Du) = (1 + |Du|^2)^{\frac{p(t,x)-2}{2}} Du.$$  \hfill (1.3)

Indeed, the exponent $p(\cdot)$ depends on the magnitude of the electric field $|E|$, which is the solution of quasi-static Maxwell’s equations. However, since the equations for $|E|$ decouple from (1.1)-(1.2), we can consider $p(\cdot)$ as a given function and restrict ourselves to the study of the equations (1.1)-(1.3). In the above system of equations, $u : Q_T \to \mathbb{R}^d$, $p : Q_T \to \mathbb{R}$, denote the velocity field and pressure respectively, and $Du$ is the symmetric velocity gradient, i.e. $Du = \frac{1}{2}(\nabla u + (\nabla u)^T)$. Here we prescribe the initial condition

$$u(0, x) = u_0(x) \quad \text{in } \mathbb{R}^d.$$  

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Such electrorheological models were studied in [17, 18], where the detailed description of the model is presented and existence theory and numerical approximation are developed.

In the present paper, we shall investigate the decay properties of the model (1.1)-(1.3). Regarding this matter, for the Navier–Stokes equations, there are a large number of contributions; see, for example, [19, 20] where the Fourier splitting method was developed, and [9] with the references therein for related results such as upper and lower bounds on the decay rate for various norms and classes of initial data. For the magnetohydrodynamics equations where the Navier–Stokes equations are coupled with Maxwell’s equations, see [12, 23, 1, 6]. On the other hand, for the non-Newtonian fluid flow model, the algebraic $L^2$ decay of solutions was examined in [3, 4], and the similar results for the non-Newtonian fluids combined with Maxwell’s equations were studied in [10, 11].

To the best of our knowledge, there is no result for the decay properties of solutions to generalized Newtonian fluids with a variable power-law index. In this paper, we shall study the $L^2$ decay rates for the $L^2$-norm and $H^1$-seminorm of strong solution to the electrorheological fluids.

2 Preliminaries and main theorem

In this section, we first introduce some notations and preliminaries which will be used throughout the paper. For two vectors $a$ and $b$, $a \cdot b$ denotes the scalar product, and $C$ denotes a generic positive constant, which may differ at each appearance. For $1 < p < \infty$, we have $|a|^p = \left(\sum |a_i|^p \right)^{1/p}$, and $(\cdot)^{1/p}$ is a continuous function.

Next we introduce the minimum regularity of the exponent $p(\cdot)$, which guarantees the validity of various results from the theory of classical Lebesgue space: log-\textit{Hölder} continuity.

\textbf{Definition 2.2.} We call a function $p : \Omega \to \mathbb{R}$ is locally log-\textit{Hölder continuous} on $\Omega$ if there exists $C_1$ satisfying for all $x, y \in \Omega$,

$$|p(x) - p(y)| \leq \frac{C_1}{\log(e + 1/|x - y|)}.$$  \hfill (2.1)

We also say that $p$ satisfies the log-\textit{Hölder decay condition} if there exist constants $p_{\infty} \in \mathbb{R}$ and $C_2 > 0$ such that for all $x \in \Omega$,

$$|p(x) - p_{\infty}| \leq \frac{C_2}{\log(e + |x|)}.$$  \hfill (2.2)

Lemma 2.1. Assume that $1 < q < \infty$. Then there exists a positive constant $C > 0$ depending on $q$ such that for any $u \in W^{1,q}(\mathbb{R}^d)$, we have

$$\|\nabla u\|_q \leq C\|Du\|_q.$$
We say that $p$ is \textit{globally log-Hölder continuous} in $\Omega$ if it satisfies both (2.1) and (2.2). We call $C_{\log}(p) := \max\{C_1, C_2\}$ a \textit{log-Hölder constant} of $p$.

\textbf{Definition 2.3.} We define the class of log-Hölder continuous variable exponents:

$$\mathcal{P}^{\log}(\Omega) := \left\{ p \in \mathcal{P}(\Omega) : \frac{1}{p} \text{ is globally log-Hölder continuous} \right\}.$$  

If $\Omega$ is unbounded, we define $p_\infty$ by $\frac{1}{p_\infty} := \lim_{|x| \to \infty} \frac{1}{p(x)}$.

Note that, since $p \mapsto \frac{1}{p}$ is bilipschitz form $[p^-, p^+]$ to $\left[ \frac{1}{p^+}, \frac{1}{p^-}\right]$, if $p \in \mathcal{P}(\Omega)$ and $p^+ < \infty$, we can show that $p \in \mathcal{P}^{\log}(\Omega)$ if and only if $p$ is globally Hölder continuous. For further information, see [2] as an extensive source of information for the variable-exponent spaces.

Next, we introduce the notation

$$\mathcal{D}u := (1 + |Du|^2)^{\frac{1}{2}},$$

and define the following energies:

$$I_p(u)(t) = \int_{\mathbb{R}^d} (\mathcal{D}u)^{p(\cdot)-2} |Du|^2 \, dx,$$

$$J_p(u)(t) = \int_{\mathbb{R}^d} (\mathcal{D}u)^{p(\cdot)-2} \|\nabla u\|^2 \, dx.$$  

In this paper, for the sake of simplicity, we consider the case of three space dimensions. Note that the proof for the general case follows in a similar manner. The existence of strong solutions of the model (1.1)-(1.3) can be found in some literatures. For example, in [3], it was shown that the local strong solutions exist under the condition (1.1)-(1.3) can be found in some literatures. For example, in [3], it was shown that the local strong solutions exist under the condition $p > \frac{11}{5}$. For the variable-exponent case, we can show the existence of strong solutions in $\mathbb{R}^3$ to the equations (1.1), (1.2) with $p_{\infty}$ where the exponent $p(\cdot)$ only depends on the spatial variable $x \in \mathbb{R}^3$. Indeed, if we assume $p \in W^{1,\infty}(\mathbb{R}^3) \cap \mathcal{P}^{\log}(\mathbb{R}^3)$ with $p^- \geq \frac{11}{5}$, by following the argument presented in [16] we can show in a straightforward manner that the global strong solutions of (1.1), (1.2) and (2.3) exist. Furthermore, based on the method presented in [13], we can also show the existence of global strong solutions to (1.1), (1.2) and (2.3) provided that $p \in W^{1,\infty}(\mathbb{R}^3) \cap \mathcal{P}^{\log}(\mathbb{R}^3)$ and $p^- > \frac{5}{3}$, under the smallness assumption of the initial data. The existence of global strong solutions of the model (1.1)-(1.3) in $\mathbb{R}^3$ where $p = p(t, x)$ depends on both time and space variables is still open.

In this paper, we consider the aforementioned global-in-time strong solutions of (1.1), (1.2) and (2.3) with the condition $p^- \geq \frac{11}{5}$. Note however, that our proofs of Theorem 2.4 and Theorem 2.5 also work for the space-time-dependent power-law index, and hence we can extend the current results to the case of the equations (1.1)-(1.3) if the existence of corresponding strong solutions for $p = p(t, x)$ is guaranteed. Here strong solutions of the model (1.1), (1.2) and (2.3) mean that $u \in L^\infty((0, T); H^1(\mathbb{R}^3) \cap L^2((0, T); H^2(\mathbb{R}^3)))$, $|\nabla u| \in L^{p(\cdot)}(Q_T) \cap L^\infty((0, T); L^{p(\cdot)}(\mathbb{R}^3))$ and $\partial_t u \in L^2(Q_T)$ with the following energy inequalities:

$$\sup_{0 \leq t \leq T} \|u(t)\|^2 + \int_0^T I_p(u)(t) \, dt \leq \|u_0\|^2,$$  

(2.4)

$$\sup_{0 \leq t \leq T} \|\nabla u(t)\|^2 + \int_0^T J_p(u)(t) \, dt \leq C(\|u_0\|_{H^1}).$$  

(2.5)
Now we are ready to state our main theorems. Note here that the condition $p \in P^{\log}(\mathbb{R}^3) \cap W^{1,\infty}(\mathbb{R}^3)$ is required for the existence of strong solutions, and not for the proof of Theorem 2.4 and hence will be omitted in the statement of the theorem. On the other hand, the condition $p \in W^{1,\infty}(\mathbb{R}^3)$ is needed for the proof of Theorem 2.5.

**Theorem 2.4.** Suppose that $u_0 \in L^1(\mathbb{R}^3) \cap H^1(\mathbb{R}^3)$, and assume that $p^+ \geq \frac{11}{5}$. Then for the strong solutions $u$ of (1.1), (1.2) and (2.3) we have

$$
\|u(t)\|_2 \leq C(1 + t)^{-\frac{3}{2}} \quad \forall t > 0,
$$

where the constant $C > 0$ depends on the $L^1$ and $H^1$-norms of $u_0$.

**Theorem 2.5.** Suppose that $u_0 \in L^1(\mathbb{R}^3) \cap H^1(\mathbb{R}^3)$, and assume that $p \in W^{1,\infty}(\mathbb{R}^3)$ with $\frac{11}{5} \leq p^- \leq p^+ < \frac{5}{3}$. Then there exists a small number $\varepsilon > 0$ such that if $\|u_0\|_{H^1} < \varepsilon$, then for the strong solutions $u$ of (1.1), (1.2) and (2.3) we have

$$
\|\nabla u(t)\|_2 \leq C(1 + t)^{-\frac{3}{2}} \quad \forall t > 0,
$$

where the constant $C > 0$ depends on the $L^1$ and $H^1$-norms of $u_0$.

3 Proof of Theorem 2.4

In this section, we aim to prove Theorem 2.4. Note that the following lemma also holds for the case $p = p(t,x)$ and the proof for this general case is exactly same as the proof below by adding $t$ to the exponent. We first define $G(Du) := ((Du)^p(x) - 1) Du$ and rewrite the equations (1.1) as

$$
\partial_t u + (u \cdot \nabla) u - \text{div} G(Du) - \Delta u + \nabla \pi = 0.
$$

**Lemma 3.1.** Suppose that $p^- \geq \frac{11}{5}$ and $u$ is sufficiently smooth. Then there exist positive constants $C_1$, $C_2$ and $C_3$ such that for almost all time $t \in (0,T)$, the following inequality holds:

- **(Case 1)** $p^- \geq 3$:

$$
\int_0^t \int_{\mathbb{R}^3} |G(Du)| \, dx \, ds \leq C_1,
$$

- **(Case 2)** $\frac{11}{5} \leq p^- < 3$:

$$
\int_0^t \int_{\mathbb{R}^3} |G(Du)| \, dx \, ds \leq C_2 + C_3 \left( \int_0^t \|u(s)\|_{L^{2\beta/\alpha}} \, ds \right)^{\frac{2-\beta}{\alpha}}.
$$

where $\alpha = \frac{7-p^-}{4}$ and $\beta = \frac{5p^- - 11}{4}$.

**Proof.** We first note that the inequality $(1 + s)^{\alpha} - 1 \leq s + s^\alpha$ for $s \geq 0$ and $\alpha > 0$. Then we have

$$
\int_{\mathbb{R}^3} |G(Du)| \, dx \leq \int_{\mathbb{R}^3} \left( 1 + |Du|^p(x) - 1 \right) |Du| \, dx
$$

$$
\leq \int_{\mathbb{R}^3} (|Du| + |Du|^p(x) - 1) |Du| \, dx
$$

$$
= \int_{\mathbb{R}^3} |Du|^2 \, dx + \int_{\mathbb{R}^3} |Du|^{p(x) - 1} \, dx
$$

$$
\leq I_p(u) + \int_{\{|Du| \geq 1\}} |Du|^{p(x) - 1} \, dx + \int_{\{|Du| < 1\}} |Du|^{p(x) - 1} \, dx
$$

$$
\leq I_p(u) + \int_{\mathbb{R}^3} |Du|^{p(x)} \, dx + \int_{\mathbb{R}^3} |Du|^{p^- - 1} \, dx
$$

$$
\leq I_p(u) + \int_{\mathbb{R}^3} |Du|^{p^- - 1} \, dx.
$$
It remains to estimate the second term on the right-hand side. For $p^- \geq 3$, by (2.4) and the interpolation inequality,

$$
\int_0^t \| \nabla u(s) \|_{p^--1}^{p^-} \, ds \lesssim \int_0^t \| \nabla u(s) \|_2^2 \| \nabla u(s) \|_{p^-}^{p^-} \, ds \lesssim \| \nabla u \|_{L^2((0,T);L^2)}^{p^-} \| \nabla u \|_{L^{p^-}((0,T);L^{p^-})} < \infty.
$$

For $\frac{11}{5} \leq p^- < 3$, as shown in [4], by (2.5) and Gagliardo–Nirenberg interpolation inequality (see, for example, [7])

$$
\int_0^t \| \nabla u(s) \|_{p^--1}^{p^-} \, ds \lesssim \int_0^t \| u(s) \|_2^3 \| \nabla^2 u(s) \|_2^3 \, ds
\lesssim \left( \int_0^t \| u(s) \|_2^{\frac{2-\beta}{2-\alpha}} \, ds \right)^{\frac{2-\beta}{2}} \left( \int_0^t \| \nabla^2 u(s) \|_2^\frac{\beta}{2} \, ds \right)^{\frac{\beta}{2}}
\lesssim \left( \int_0^t \| u(s) \|_2^{\frac{2-\beta}{2-\alpha}} \, ds \right)^{\frac{2-\beta}{2}},
$$

where $\alpha = \frac{7-p^-}{4}$ and $\beta = \frac{5p^- - 11}{4}$. \hfill \Box

Furthermore, we need the following estimate, where $\hat{f}$ denotes the Fourier transformation of $f$.

**Lemma 3.2.** Suppose that $u_0 \in H^1(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$ and $p^- \geq \frac{11}{5}$. Then for a strong solution $u$ to the equations (1.1), (1.2) and (1.3), we have the following:

(Case 1) $p^- \geq 3$:

$$
|\tilde{u}(t, \xi)| \leq C|\tilde{u}_0(\xi)| + C|\xi| \left( 1 + \int_0^t \| u(s) \|_2^3 \, ds \right).
$$

(Case 2) $\frac{11}{3} \leq p^- < 3$:

$$
|\tilde{u}(t, \xi)| \leq C|\tilde{u}_0(\xi)| + C|\xi| \left( 1 + \left( \int_0^t \| u(s) \|_2^{\frac{2-\beta}{2-\alpha}} \, ds \right)^{\frac{2-\beta}{2}} + \int_0^t \| u(s) \|_2^3 \, ds \right).
$$

**Proof.** If we take the Fourier transformation on (3.1), we have

$$
\hat{u}_t + |\xi|^2 \hat{u} = F(t, \xi) \quad \text{and} \quad \hat{u}_0(\xi) := \hat{u}(0, \xi) = \hat{u}_0,
$$

where

$$
F(t, \xi) := \nabla \cdot G(t, \xi) - (u \cdot \nabla)u(t, \xi) - \nabla \pi(t, \xi).
$$

Note that

$$
|\hat{u}_0(\xi)| \leq \left| \int_{\mathbb{R}^3} e^{-ix \cdot \xi} u_0(x) \, dx \right| \leq \int_{\mathbb{R}^3} |u_0(x)| \, dx \leq C,
$$

For the stress tensor term,

$$
|\nabla \cdot G(t, \xi)| = \left| \int_{\mathbb{R}^3} e^{-ix \cdot \xi} \nabla \cdot G(Du) \, dx \right| \leq |\xi| \int_{\mathbb{R}^3} |G(Du)| \, dx.
$$

Next, by Hölder’s inequality with (1.2), we have

$$
|(u \cdot \nabla)u(t, \xi)| = \left| \int_{\mathbb{R}^3} e^{-ix \cdot \xi} (u \otimes u) \, dx \right| \leq |\xi||u(t) \otimes u(t)|_1 \leq |\xi||u(t)|_2^2.
$$

\hfill \Box
Finally, taking divergence operator on (1.1) gives
\[ \Delta \pi = \sum_{i,j} \frac{\partial^2}{\partial x_i \partial x_j}(-u_i u_j + G(Du)_{ij}), \]
and therefore, by Hölder’s inequality, we have
\[ |\nabla \pi(t, \xi)| \leq |\xi||G(Du)||_1 + |\xi||u(t) \otimes u(t)||_1 \leq |\xi||G(Du)||_1 + |\xi||u(t)||_2^2. \quad (3.9) \]
Now, it follows from (3.4) that
\[ \text{Proof of Theorem 2.4.} \]
Next, let us assume that \( f(0) = 1, f(t) > 0 \text{ and } f'(t) > 0. \) Then for some constant \( C_0 > 0, \) we have
\[ \frac{d}{dt} \left( f(t) \int_{\mathbb{R}^3} |\hat{u}(t, \xi)|^2d\xi \right) + C_0 f(t) \int_{\mathbb{R}^3} |\xi|^2 |\hat{u}(t, \xi)|^2 d\xi \leq f'(t) \int_{\mathbb{R}^3} |\hat{u}(t, \xi)|^2 d\xi. \quad (3.10) \]
If we define the set \( L(t) = \{ \xi \in \mathbb{R}^3 : C_0 |\xi|^2 f(t) \leq f'(t) \} \) where \( C_0 > 0 \) is the constant appearing in (3.10), we obtain
\[ C_0 f(t) \int_{L(t)^c} |\xi|^2 |\hat{u}(t, \xi)|^2 d\xi \geq C_0 f(t) \int_{L(t)^c} |\xi|^2 |\hat{u}(t, \xi)|^2 d\xi \geq f'(t) \int_{L(t)} |\hat{u}(t, \xi)|^2 d\xi - f'(t) \int_{L(t)} |\hat{u}(t, \xi)|^2 d\xi, \]
and therefore, from (3.10), we deduce
\[ \frac{d}{dt} \left( f(t) \int_{\mathbb{R}^3} |\hat{u}(t, \xi)|^2 d\xi \right) \leq f'(t) \int_{L(t)} |\hat{u}(t, \xi)|^2 d\xi. \]
Integrating the above inequality over \((0, t)\) yields
\[ f(t) \int_{\mathbb{R}^3} |\hat{u}(t, \xi)|^2 d\xi \leq \int_{\mathbb{R}^3} |\hat{u}(0, \xi)|^2 d\xi + \int_0^t f'(s) \int_{L(s)} |\hat{u}(s, \xi)|^2 d\xi ds. \quad (3.11) \]
(Case 1) \( p^- \geq 3: \) Now we set \( f(t) = (1 + t)^3. \) Then by Lemma 3.2 with (3.6), (3.11) and the energy inequality (2.4), we obtain that
\[ (1 + t)^3 \int_{\mathbb{R}^3} |\hat{u}(t, \xi)|^2 d\xi \leq \int_{\mathbb{R}^3} |\hat{u}_0(\xi)|^2 d\xi + C \int_0^t (1 + s)^2 \int_{L(s)} |\hat{u}_0(\xi)|^2 d\xi ds \]
\[ + C \int_0^t (1 + s)^2 \int_{L(s)} |\xi|^2 \left( 1 + \int_0^s ||u(\tau)||_2^2 d\tau \right)^2 d\xi ds \]
\[ \leq C + C \int_0^t (1 + s)^{\frac{3}{2}} ds + C \int_0^t (1 + s)^{\frac{3}{2}} ds + C \int_0^t (1 + s)^{\frac{5}{2}} ds \]
\[ \leq C + C(1 + t)^{\frac{3}{2}} + C(1 + t)^{\frac{5}{2}}. \]
From Plancherel’s theorem, we have

\[ \|u(t)\|_2^2 \leq C(1 + t)^{-\frac{1}{2}}. \]  

(3.12)

Now if we substitute (3.12) into (3.2), and repeat the same process, we finally obtain the desired estimate.

(Case 2) \( \frac{1}{2} \leq p^- < 3 \): In this case, we first note that \( \frac{1}{2} < \frac{2-\beta}{2} < 1 \) and \( \frac{4\alpha}{2-\beta} > 2 \). Then by Hölder’s inequality and (2.4), we obtain

\[
\int_0^t (1 + s)^2 \int_{L(s)} |\xi|^2 \left( 1 + \left( \int_0^s \|u(\tau)\|_{L^2}^{2\alpha/\beta} \,d\tau \right)^{\frac{2-\beta}{\beta}} + \int_0^s \|u(\tau)\|_{L^2}^2 \,d\tau \right)^2 \,d\xi \,ds \\
\leq C \int_0^t (1 + s)^2 \int_{L(s)} |\xi|^2 \,d\xi \,ds + C \int_0^t (1 + s)^2 \int_{L(s)} |\xi|^2 \left( \int_0^s \|u(\tau)\|_{L^2}^{4\alpha/\beta} \,d\tau \right)^{\frac{2-\beta}{\beta}} \,d\xi \,ds \\
+ C \int_0^t (1 + s)^2 \int_{L(s)} |\xi|^2 s^{\frac{2-\beta}{2}} \left( \int_0^s \|u(\tau)\|_{L^2}^\beta \,d\tau \right)^{\frac{2-\beta}{\beta}} \,d\xi \,ds \\
\leq C \int_0^t (1 + s)^2 \int_{L(s)} |\xi|^2 \,d\xi \,ds + C \int_0^t (1 + s)^2 \int_{L(s)} |\xi|^2 s^{\frac{2-\beta}{2}} \left( \int_0^s \|u(\tau)\|_{L^2}^\beta \,d\tau + C \right) \,d\xi \,ds \\
+ C \int_0^t (1 + s)^2 \int_{L(s)} |\xi|^2 s^{\frac{2-\beta}{2}} \left( \int_0^s \|u(\tau)\|_{L^2}^\beta \,d\tau \right) \,d\xi \,ds \\
\leq C(1 + t)^{\frac{3}{2}} + C(1 + t)^{\frac{3}{2}} \left( \int_0^t \|u(\tau)\|_{L^2}^\beta \,d\tau \right).
\]

Now, by again from Lemma 3.2, (3.6) and (3.11) with \( f(t) = (1 + t)^3 \), we have that

\[ (1 + t)^3 \|u(t)\|_2^2 = (1 + t)^3 \int_{\mathbb{R}^3} |\widetilde{u}(t, \xi)|^2 \,d\xi \leq C(1 + t)^{\frac{3}{2}} + C(1 + t)^{\frac{3}{2}} \left( \int_0^t \|u(\tau)\|_{L^2}^\beta \,d\tau \right). \]

This yields

\[ (1 + t)^{\frac{3}{2}} \|u(t)\|_2^2 \leq C + C \int_0^t (1 + \tau)^{\frac{3}{2}} \|u(\tau)\|_{L^2}^\beta (1 + \tau)^{-\frac{3}{2}} \,d\tau, \]

and therefore, by Gronwall’s inequality, we obtain the desired decay estimate.

\[ \square \]

4 Proof of Theorem 2.5

We begin with the following a priori estimate.

**Lemma 4.1.** Assume that \( p \in W^{1,\infty}(\mathbb{R}^3) \) with \( p^+ < \frac{8}{3} \). Then there exists a small number \( \delta > 0 \) such that if

\[ \sup_{0 \leq t \leq T} \|u(t)\|_{H^1} < 2\delta, \]

(4.1)

we have the following differential inequality: for almost all time \( t \in (0, T) \),

\[ \frac{d}{dt} \|\nabla u(t)\|_2^2 + \|\nabla^2 u(t)\|_2^2 \leq 0. \]

(4.2)
Proof. We shall derive some formal inequalities which are essential for the correct arguments. For the detailed arguments, see for example, [3, 13]. We first differentiate formally with respect to the spatial variable $x_j$ and take scalar product with $\frac{\partial u}{\partial x_j}$. Summing over $j = 1, 2, 3$ yields the following a priori estimate (see [3]):

$$\frac{1}{2} \frac{d}{dt} \| \nabla u \|^2_2 + \| \nabla^2 u \|^2_2 + J_p(u) \leq \int_{\mathbb{R}^3} (u \cdot \nabla) u \cdot \Delta u \, dx + \int_{\mathbb{R}^3} |\nabla p| |\overline{D}u|^{p(x)-2} \log(\overline{D}u) |Du| \| \nabla^2 u \| \, dx$$

$$\leq \|u\|_3 \|\nabla u\|_6 \|\nabla^2 u\|_2 + \int_{\mathbb{R}^3} \log(\overline{D}u) |Du| \| \nabla^2 u \| \, dx$$

$$+ \int_{\mathbb{R}^3} |Du|^{p^+-2} \log(\overline{D}u) |Du| \| \nabla^2 u \| \, dx$$

$$: = I_1 + I_2 + I_3.$$  

Note that the logarithmic term appears above when we differentiate the stress tensor with the variable exponent $p(x)$. By the interpolation inequality and the Sobolev embedding,

$$I_1 \lesssim \|u\|^\frac{1}{2} \|u\|^\frac{1}{6} \|\nabla u\| \|\nabla^2 u\|_2 \lesssim \|u\|^\frac{1}{2} \|\nabla u\|^\frac{1}{2} \|\nabla^2 u\|^\frac{1}{2}.$$  

Next, by the inequality $\log(\overline{D}u) \leq C_\alpha |Du|^\alpha$ for any $0 < \alpha \leq 1$, we have

$$I_2 \lesssim \int_{\mathbb{R}^3} |Du|^\frac{2}{3} |Du| \| \nabla^2 u \| \, dx \lesssim \|Du\|^\frac{2}{3} \|Du\|^\frac{2}{3} \|\nabla^2 u\|_2 \lesssim \|\nabla u\|^\frac{2}{3} \|\nabla^2 u\|^\frac{2}{3}.$$  

Finally, due to the condition $p^+ < \frac{8}{3},$

$$I_3 \lesssim \int_{\mathbb{R}^3} |Du|^\frac{2}{3} |Du| \| \nabla^2 u \| \, dx \lesssim \|\nabla u\|^\frac{2}{3} \|\nabla^2 u\|^\frac{2}{3}.$$  

Altogether, we conclude that there exist positive constants $C_1$ and $C_2$ such that

$$\frac{d}{dt} \| \nabla u \|^2_2 + \| \nabla^2 u \|^2_2 \leq C_1 \|u\|^\frac{4}{3} \|\nabla u\|^\frac{4}{3} \|\nabla^2 u\|^2_2 + C_2 \|\nabla u\|^\frac{4}{3} \|\nabla^2 u\|^\frac{4}{3} \|\nabla^2 u\|^\frac{2}{3},$$  

and hence

$$\frac{d}{dt} \| \nabla u \|^2_2 + (1 - C_1 \|u\|^\frac{4}{3} \|\nabla u\|^\frac{4}{3} \|\nabla^2 u\|^\frac{2}{3} - C_2 \|\nabla u\|^\frac{4}{3} \|\nabla^2 u\|^\frac{4}{3} \|\nabla^2 u\|^\frac{2}{3}) \| \nabla^2 u \| \leq 0.$$  

Therefore, we obtain the desired inequality if $\sup_{0 \leq t \leq T} \|u(t)\|_{H^1} < 2\delta$ for sufficiently small $\delta > 0$. □

Lemma 4.2. Assume that $p \in W^{1,\infty}(\mathbb{R}^3)$ with $\frac{1}{3} \leq p^- \leq p^+ < \frac{8}{3}$. Then there exists a small number $\varepsilon > 0$ such that if $\|u_0\|_{H^1} < \varepsilon$, we have

$$\sup_{0 \leq t \leq T} \|u(t)\|_{H^1} < 2\delta,$$

where $\delta > 0$ is the constant defined in Lemma 4.1.

Proof. With the same argument as above and by Hölder’s inequality and Young’s inequality, we have

$$\frac{1}{2} \frac{d}{dt} \| \nabla u \|^2_2 + \| \nabla^2 u \|^2_2 + J_p(u) \leq \|\nabla u\|^3_3 + C \int_{\mathbb{R}^3} (\overline{D}u)^{p(x)-2} \log(\overline{D}u) |Du| \| \nabla^2 u \| \, dx$$

$$\leq \|\nabla u\|^3_3 + \varepsilon J_p(u) + C \int_{\mathbb{R}^3} (\overline{D}u)^{p^+} \log^2(\overline{D}u)$$

$$\leq \|\nabla u\|^3_3 + \varepsilon J_p(u) + C \int_{\mathbb{R}^3} \log^2(\overline{D}u) \, dx + C \int_{\mathbb{R}^3} |Du|^\frac{8}{3} \log^2(\overline{D}u) \, dx$$

$$\leq C \|\nabla u\|^3_3 + C \|\nabla u\|^3_3 + \varepsilon J_p(u).$$
Next, for $\frac{1}{3} \leq p^-$, we have that (see [13])

$$\|\nabla u\|^3_2 \leq C \|\nabla u\|_{p^-}^p \|\nabla u\|^2_2 + \varepsilon J^-_p(u) \leq C \|\nabla u\|_{p^-}^p \|\nabla u\|^2_2 + \varepsilon J_p(u).$$

Therefore, we finally have

$$\frac{d}{dt}\|\nabla u\|^2_2 \leq C(1 + \|\nabla u\|_{p^-}^p)\|\nabla u\|^2_2,$$

which yields by Gronwall’s inequality

$$\|\nabla u\|^2_2 \leq \|\nabla u_0\|^2_2 \exp\left(\int_0^t (1 + \|\nabla u(s)\|_{p^-}^p) \, ds\right).$$

Thanks to (2.4), $\|\nabla u(t)\|_{p^-}^p$ is integrable with respect to time, and hence, there exists small $T^* > 0$ such that

$$\sup_{0 \leq t \leq T^*} \|\nabla u\|^2_2 \leq 2\|\nabla u_0\|^2_2. \tag{4.4}$$

Now, suppose that $\|u_0\|_{H^1} < \frac{\varepsilon}{\sqrt{2}}$. Then by (4.4) and (2.4),

$$\sup_{0 \leq t \leq T^*} \|u\|_{H^1} < 2\delta. \tag{4.5}$$

Then by Lemma 4.1

$$\frac{d}{dt}\|\nabla u\|^2_2 \leq 0,$$

which implies that

$$\|\nabla u(T^*)\|^2_2 \leq \sup_{0 \leq t \leq T^*} \|\nabla u\|^2_2 \leq \|\nabla u_0\|^2_2 < \frac{\delta^2}{2}. \tag{4.6}$$

Next, we consider the original problem (1.1)–(1.2) for $t \geq T^*$ with the initial data $u(T^*)$. With the same argument as above,

$$\sup_{T^* \leq t \leq 2T^*} \|\nabla u\|^2_2 \leq 2\|\nabla u(T^*)\|^2_2 < \delta^2,$$

and hence

$$\sup_{T^* \leq t \leq 2T^*} \|u\|_{H^1} < 2\delta.$$

By Lemma 4.1 again, we have

$$\|\nabla u(2T^*)\|^2_2 \leq \sup_{T^* \leq t \leq 2T^*} \|\nabla u\|^2_2 \leq \|\nabla u(T^*)\|^2_2 < \frac{\delta^2}{2}.$$

If we repeat the same process for $(n - 1)T^* < t < nT^*$ with $n \in \mathbb{N}$, we finally obtain that

$$\sup_{0 \leq t \leq T^*} \|u\|_{H^1} < 2\delta.$$

\[\square\]

**Proof of Theorem 2.3** Let $L(t) = \{\xi \in \mathbb{R}^3 : |\xi| \leq f(t)\}$ where $f(t) = \left(\frac{1}{1+t}\right)^{\frac{1}{2}}$. By Plancherel’s theorem, we have

$$\|\nabla^2 u(t)\|^2_2 = \int_{\mathbb{R}^3} |\xi|^4 |\hat{u}(\xi, t)|^2 \, d\xi \geq |f(t)|^2 \int_{L(t)^c} |\xi|^2 |\hat{u}(\xi, t)|^2 \, d\xi$$

$$= |f(t)|^2 \int_{\mathbb{R}^3} |\xi|^2 |\hat{u}(\xi, t)|^2 \, d\xi - |f(t)|^2 \int_{L(t)^c} |\xi|^2 |\hat{u}(\xi, t)|^2 \, d\xi$$

$$\geq |f(t)|^2 \|\nabla u(t)\|^2_2 - |f(t)|^4 \int_{L(t)^c} |\hat{u}(\xi, t)|^2 \, d\xi$$

$$\geq |f(t)|^2 \|\nabla u(t)\|^2_2 - |f(t)|^4 \|u(t)\|^2_2.$$
From Lemma (4.1) and (4.2), we can deduce that for almost all $t \in (0, T)$,

$$\frac{d}{dt} \|\nabla u(t)\|_2^2 + \|\nabla^2 u(t)\|_2^2 \leq 0.$$

Therefore, we obtain

$$\frac{d}{dt} \|\nabla u(t)\|_2^2 + \frac{1}{1+t} \|\nabla^2 u(t)\|_2^2 \leq \left( \frac{1}{1+t} \right)^2 \|u(t)\|_2^2.$$

Then for $\ell > \frac{5}{2}$, by Theorem 2.4

$$\frac{d}{dt} \left( (1+t)^\ell \|\nabla u(t)\|_2^2 \right) \leq (1+t)^{\ell-2} \|u(t)\|_2^2 \leq C(1+t)^{\ell-2-\frac{3}{2}}.$$

By integrating the above inequality over time, we have the desired decay estimate. \qed

References

[1] R. Agapito and M. Schonbek. Non-uniform decay of MHD equations with and without magnetic diffusion. *Comm. Partial Differential Equations*, 32(10-12):1791–1812, 2007.

[2] L. Diening, P. Harjulehto, P. Hästö, and M. Růžička. *Lebesgue and Sobolev spaces with variable exponents*, volume 2017 of *Lecture Notes in Mathematics*. Springer, Heidelberg, 2011.

[3] L. Diening and M. Růžička. Strong solutions for generalized Newtonian fluids. *J. Math. Fluid Mech.*, 7(3):413–450, 2005.

[4] B.-Q. Dong. Decay of solutions to equations modelling incompressible bipolar non-Newtonian fluids. *Electron. J. Differential Equations*, pages No. 125, 13, 2005.

[5] B. Guo and P. Zhu. Algebraic $L^2$ decay for the solution to a class system of non-Newtonian fluid in $\mathbb{R}^n$. *J. Math. Phys.*, 41(1):349–356, 2000.

[6] P. Han and C. He. Decay properties of solutions to the incompressible magnetohydrodynamics equations in a half space. *Math. Methods Appl. Sci.*, 35(12):1472–1488, 2012.

[7] D. Henry. *Geometric theory of semilinear parabolic equations*, volume 840 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York, 1981.

[8] R. Kajikiya and T. Miyakawa. On $L^2$ decay of weak solutions of the Navier-Stokes equations in $\mathbb{R}^n$. *Math. Z.*, 192(1):135–148, 1986.

[9] T. Kato. Strong $L^p$-solutions of the Navier-Stokes equation in $\mathbb{R}^m$, with applications to weak solutions. *Math. Z.*, 187(4):471–480, 1984.

[10] H. K. Kim, K. Kang, and J.-M. Kim. Existence and temporal decay of regular solutions to non-Newtonian fluids combined with Maxwell equations. *Nonlinear Anal.*, 180:284–307, 2019.

[11] J.-M. Kim. Temporal decay of strong solutions to the magnetohydrodynamics with power-law type nonlinear viscous fluid. *J. Math. Phys.*, 61(1):011504, 6, 2020.

[12] H. Kozono. On the energy decay of a weak solution of the MHD equations in a three-dimensional exterior domain. *Hokkaido Math. J.*, 16(2):151–166, 1987.

[13] J. Málek, J. Nečas, M. Rokyta, and M. Růžička. *Weak and measure-valued solutions to evolutionary PDEs*, volume 13 of *Applied Mathematics and Mathematical Computation*. Chapman & Hall, London, 1996.
[14] Š. Nečasová and P. Penel. $L^2$ decay for weak solution to equations of non-Newtonian incompressible fluids in the whole space. In *Proceedings of the Third World Congress of Nonlinear Analysts, Part 6 (Catania, 2000)*, volume 47, pages 4181–4192, 2001.

[15] M. Oliver and E. S. Titi. Remark on the rate of decay of higher order derivatives for solutions to the Navier-Stokes equations in $\mathbb{R}^n$. *J. Funct. Anal.*, 172(1):1–18, 2000.

[16] M. Pokorný. Cauchy problem for the non-Newtonian viscous incompressible fluid. *Appl. Math.*, 41(3):169–201, 1996.

[17] M. Růžička. *Electrorheological fluids: modeling and mathematical theory*, volume 1748 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2000.

[18] M. Růžička. Modeling, mathematical and numerical analysis of electrorheological fluids. *Appl. Math.*, 49(6):565–609, 2004.

[19] M. E. Schonbek. $L^2$ decay for weak solutions of the Navier-Stokes equations. *Arch. Rational Mech. Anal.*, 88(3):209–222, 1985.

[20] M. E. Schonbek. Large time behaviour of solutions to the Navier-Stokes equations. *Comm. Partial Differential Equations*, 11(7):733–763, 1986.

[21] M. E. Schonbek. Lower bounds of rates of decay for solutions to the Navier-Stokes equations. *J. Amer. Math. Soc.*, 4(3):423–449, 1991.

[22] M. E. Schonbek. Asymptotic behavior of solutions to the three-dimensional Navier-Stokes equations. *Indiana Univ. Math. J.*, 41(3):809–823, 1992.

[23] M. E. Schonbek, T. P. Schonbek, and E. Suli. Large-time behaviour of solutions to the magnetohydrodynamics equations. *Math. Ann.*, 304(4):717–756, 1996.

[24] M. Wiegner. Decay results for weak solutions of the Navier-Stokes equations on $\mathbb{R}^n$. *J. London Math. Soc. (2)*, 35(2):303–313, 1987.