Abstract. This is a survey on trace constructions on various operator algebras with an emphasis on regularized traces on algebras of pseudodifferential operators. For motivation our point of departure is the classical Hilbert space trace which is the unique semifinite normal trace on the algebra of bounded operators on a separable Hilbert space. Dropping the normality assumption leads to the celebrated Dixmier traces.

Then we give a leisurely introduction to pseudodifferential operators. The parameter dependent calculus is emphasized and it is shown how this calculus leads naturally to the asymptotic expansion of the resolvent trace of an elliptic differential operator.

The Hadamard partie finie regularization of an integral is explained and used to extend the Hilbert space trace to the Kontsevich-Vishik canonical trace on pseudodifferential operators of non-integral order.

Then the stage is well prepared for the residue trace of Wodzicki-Guillemin and its purely functional analytic interpretation as a Dixmier trace by Alain Connes.

We also discuss existence and uniqueness of traces for the algebra of parameter dependent pseudodifferential operators; the results are surprisingly different.

Finally, we will discuss the analogue of the regularized traces on the symbolic level and study the de Rham cohomology of $\mathbb{R}^n$ with coefficients being symbol functions. This generalizes a recent result of S. Paycha concerning the characterization of the Hadamard partie finie integral and the residue integral in light of the Stokes property.

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2000 Mathematics Subject Classification. Primary 58J42; Secondary 58J40, 58J35, 58B34.

Key words and phrases. pseudodifferential operator, noncommutative residue, Dixmier trace.

The author gratefully acknowledges the hospitality of the Department of Mathematics at The University of Colorado at Boulder where this paper was written.

The author was partially supported by the Hausdorff Center for Mathematics (Bonn).
Traces on an algebra are important linear functionals which come up in various incarnations in various branches of mathematics, e.g. group characters, norm and trace in field extensions, many trace formulas, to mention just a few.

On a separable Hilbert space $\mathcal{H}$ there is a canonical trace (tracial weight, see Section 2) $\text{Tr}$ defined on non–negative operators by

\begin{equation}
\text{Tr}(T) := \sum_{j=0}^{\infty} \langle Te_j, e_j \rangle,
\end{equation}

where $(e_j)_{j \geq 0}$ is an orthonormal basis. This is the unique semifinite normal trace on the algebra $\mathcal{B}(\mathcal{H})$ of bounded operators on $\mathcal{H}$. In the 1930’s MURRAY and VON NEUMANN [MuvN36, MuvN37, vN40, MuvN43] studied traces on weakly closed $\ast$–subalgebras (now known as von Neumann algebras) of $\mathcal{B}(\mathcal{H})$. They showed that on a von Neumann factor there is up to a normalization a unique semifinite normal trace.

GUillemin [Gui85] and Wodzicki [Wod84, Wod87] discovered independently that a similar uniqueness statement holds for the algebra of pseudodifferential operators on a compact manifold. The residue trace, however, has nothing to do with the Hilbert space trace: it vanishes on trace class operators.

In the 60s DIXMIER [Dix66] had already proved that the uniqueness statement for the Hilbert space trace fails if one gives up the assumption that the trace is normal.

In the late 80’s and early 90’s then the Dixmier trace had a celebrated comeback when ALAIN CONNES [Con88] proved that in important cases the residue trace coincides with a Dixmier trace.

The aim of this note is to survey some of these results. We will not touch von Neumann algebras, however, any further.

The paper is organized as follows:

In Section 2 our point of departure is the classical Hilbert space trace. We give a short proof that it is up to a factor the unique normal tracial weight on the algebra $\mathcal{B}(\mathcal{H})$ of bounded operators on a separable Hilbert space $\mathcal{H}$.

Then we reproduce Dixmier’s very elegant construction which shows that non–normal tracial weights are abundant. We do confine ourselves however
to those Dixmier traces which will later turn out to be related to the residue trace.

Section 2 presents the basic calculus of pseudodifferential operators with parameter on a closed manifold.

In Section 4 we pause the discussion of pseudodifferential operators and look at the problem of extending the Hilbert space trace to pseudodifferential operators of higher order. A pseudodifferential operator $A$ of order $< - \dim M$ on a closed manifold $M$ is of trace class and its trace is given by integrating its Schwartz kernel $k_A(x, y)$ over the diagonal

$$\text{Tr}(A) = \int_M k_A(x, x) dx.$$  

We will show that the classical Hadamard partie finie regularization of integrals allows to extend Eq. (1.2) to all pseudodifferential operators of non–integral order. This is the celebrated Kontsevich-Vishik canonical trace.

Section 5 on asymptotic analysis then shows how the parameter dependent pseudodifferential calculus leads naturally to the asymptotic expansion of the resolvent trace of an elliptic differential operator. For the resolvent of elliptic pseudodifferential operators a refinement, due to Grubb and Seeley, of the parametric calculus is necessary. Without going into the details of this refined calculus we will explain why additional log $\lambda$ terms appear in the asymptotic expansion of $\text{Tr}(B(P - \lambda)^{-N})$ if $B$ or $P$ are pseudodifferential rather than differential operators. These log $\lambda$ terms are at the heart of the noncommutative residue trace. The straightforward relations between the resolvent expansion, the heat trace expansion and the meromorphic continuation of the $\zeta$–function, which are based on the Mellin transform respectively a contour integral method, are also briefly discussed.

In Section 6 we state the main result about the existence and uniqueness of the residue trace. We present it in a slightly generalized form due to the author for log–polyhomogeneous pseudodifferential operators. A formula for the relation between the residue trace of a power of the Laplacian and the Einstein–Hilbert action due to Kalau–Walze \cite{KaWa95} and Kastler \cite{Kas95} is proved in an example.

Then we give a proof of Connes’ Trace Theorem which states that on pseudodifferential operators of order minus $\dim M$ on a closed manifold $M$ the residue trace is proportional to the Dixmier trace.

Having seen the significance of the parameter dependent calculus it is natural to ask whether the algebras of parameter dependent pseudodifferential operators have an analogue of the residue trace. Somewhat surprisingly the results for these algebras are quite different: there are many traces on this algebra, however, there is a unique symbol–valued trace from which many other traces can be derived. This result resembles very much the center valued trace in von Neumann algebra theory. Furthermore, in contrast to the non–parametric case the $L^2$–Hilbert space trace extends to a trace on the whole algebra. This part of the paper surveys results from a joint paper with Markus J. Pflaum \cite{LePf00}.

Finally, in the short Section 7 we will discuss the analogue of the regularized traces on the symbolic level and announce a generalization of a recent result of S. Paycha concerning the characterization of the Hadamard partie
finie integral and the residue integral in light of the Stokes property. The
result presented here allows one to calculate de Rham cohomology groups of
forms on $\mathbb{R}^n$ whose coefficients lie in a certain symbol space. We will show
that both the Hadamard partie finie integral and the residue integral provide
an integration along the fiber on the cone $\mathbb{R}_+^* \times M$ and as a consequence
there is an analogue of the Thom isomorphism.

ACKNOWLEDGMENTS. I would like to thank the organizers of the confer-
ence on Motives, Quantum Field Theory and Pseudodifferential Operators
for inviting me to contribute these notes. Also I would like to thank the
anonymous referee for taking his job very seriously and for making very de-
tailed remarks on how to improve the paper. I think the paper has benefited
considerably from those remarks.

2. The Hilbert space trace (tracial weight)

2.1. Basic definitions. Let $\mathcal{H}$ be a separable Hilbert space. Denote
by $\mathcal{B}(\mathcal{H})$ the algebra of bounded operators on $\mathcal{H}$. Let $\mathcal{A}$ be a $C^*$–
subalgebra, that is, a norm closed self-adjoint ($a \in \mathcal{A} \Rightarrow a^* \in \mathcal{A}$) sub-
algebra. It follows that $\mathcal{A}$ is invariant under continuous functional calculus,
e.g. if $a \in \mathcal{A}$ is non–negative then $\sqrt{a} \in \mathcal{A}$.

Denote by $\mathcal{A}_+ \subset \mathcal{A}$ the set of non–negative elements. $\mathcal{A}_+$ is a cone in
the following sense:

(1) $T \in \mathcal{A}_+, \lambda \in \mathbb{R}_+ \Rightarrow \lambda T \in \mathcal{A}_+$,
(2) $S, T \in \mathcal{A}_+, \lambda, \mu \in \mathbb{R}_+ \Rightarrow \lambda S + \mu T \in \mathcal{A}_+$.

A weight on $\mathcal{A}$ is a map

\begin{equation}
\tau : \mathcal{A}_+ \longrightarrow \mathbb{R}_+ \cup \{\infty\}, \quad \mathbb{R}_+ := [0, \infty),
\end{equation}

such that

\begin{equation}
\tau(\lambda S + \mu T) = \lambda \tau(S) + \mu \tau(T), \quad \lambda, \mu \geq 0, \ S, T \in \mathcal{A}_+.
\end{equation}

A weight is called tracial if

\begin{equation}
\tau(T^*T) = \tau(T^*T), \quad T \in \mathcal{A}_+.
\end{equation}

It follows from (2.3) that for a unitary $U \in \mathcal{A}$ and $T \in \mathcal{A}_+$

\begin{equation}
\tau(UTU^*) = \tau((UT^{1/2})(UT^{1/2})^*) = \tau((UT^{1/2})^*(UT^{1/2})) = \tau(T).
\end{equation}

implies that $\tau$ is monotone in the sense that if $0 \leq S \leq T$ then

\begin{equation}
\tau(T) = \tau(S) + \tau(T - S) \geq \tau(S).
\end{equation}

Remark 2.1. In the literature tracial weights are often just called traces.
We adopt here the convention of KADISON and RINGROSE [KaRi97] Chap.
8.

We reserve the word trace for a linear functional $\tau : \mathcal{B} \longrightarrow \mathbb{C}$ on a $\mathbb{C}$–
algebra $\mathcal{B}$ which satisfies $\tau(AB) = \tau(BA)$ for $A, B \in \mathcal{B}$. A priori a tracial
weight $\tau$ is only defined on the positive cone of $\mathcal{A}$ and it may take the value
$\infty$. Below we will see that there is a natural ideal in $\mathcal{A}$ on which $\tau$ is a
trace.
2.1.1. The canonical tracial weight on bounded operators on a Hilbert space. Let \((e_j)_{j \in \mathbb{Z}_+}\) be an orthonormal basis of the Hilbert space \(\mathcal{H}\); \(\mathbb{Z}_+ := \{0, 1, 2, \ldots\}\). For \(T \in \mathcal{B}(\mathcal{H})\) put
\[
\text{Tr}(T) := \sum_{j=0}^{\infty} \langle Te_j, e_j \rangle.
\]

\(\text{Tr}(T)\) is indeed independent of the choice of the orthonormal basis and it is a tracial weight on \(\mathcal{B}(\mathcal{H})\) (Pedersen [Ped89, Sec. 3.4]).

2.1.2. Trace ideals. We return to the general set-up of a tracial weight on a \(C^*\)-subalgebra \(\mathcal{A} \subset \mathcal{B}(\mathcal{H})\). Put
\[
\mathcal{L}^1(\mathcal{A}, \tau) := \{ T \in \mathcal{A}_+ \mid \tau(T) < \infty \}
\]
and denote by \(\mathcal{L}^1(\mathcal{A}, \tau)\) the linear span of \(\mathcal{L}^1(\mathcal{A}, \tau)\). Furthermore, let
\[
\mathcal{L}^2(\mathcal{A}, \tau) := \{ T \in \mathcal{A} \mid \tau(T^*T) < \infty \}.
\]

Using the inequality
\[
(S + T)^*(S + T) \leq (S + T)^*(S + T) + (S - T)^*(S - T) = 2(S^*S + T^*T)
\]
and the polarization identity
\[
4T^*S = \sum_{k=0}^{3} i^k (S + i^k T)^*(S + i^k T)
\]
one proves exactly as for the tracial weight \(\text{Tr}\) in [Ped89, Sec. 3.4]:

**Proposition 2.2.** \(\mathcal{L}^1(\mathcal{A}, \tau)\) and \(\mathcal{L}^2(\mathcal{A}, \tau)\) are two-sided self-adjoint ideals in \(\mathcal{A}\).

Moreover for \(T, S \in \mathcal{L}^2(\mathcal{A}, \tau)\) one has \(TS, ST \in \mathcal{L}^1(\mathcal{A}, \tau)\) and
\[
\tau(ST) = \tau(TS).
\]
The same formula holds for \(T \in \mathcal{L}^1(\mathcal{A}, \tau)\) and \(S \in \mathcal{B}(\mathcal{H})\).

In particular \(\tau \mid \mathcal{L}^p(\mathcal{A}, \tau), p = 1, 2,\) is a trace.

2.2. Uniqueness of \(\text{Tr}\) on \(\mathcal{B}(\mathcal{H})\). As for finite-dimensional matrix algebras one now shows that up to a normalization there is a unique trace on the ideal of finite rank operators.

**Lemma 2.3.** Let \(\mathcal{FR}(\mathcal{H})\) be the ideal of finite rank operators on \(\mathcal{H}\). Any trace \(\tau : \mathcal{FR}(\mathcal{H}) \to \mathbb{C}\) is proportional to \(\text{Tr} \mid \mathcal{FR}(\mathcal{H})\).

**Proof.** Let \(P, Q \in \mathcal{B}(\mathcal{H})\) be rank one orthogonal projections. Choose \(v \in \text{im} P, w \in \text{im} Q\) with \(\|v\| = \|w\| = 1\) and put
\[
T := \langle v, \cdot \rangle w.
\]
Then \(T \in \mathcal{FR}(\mathcal{H})\) and \(T^*T = P, TT^* = Q\). Consequently \(\tau\) takes the same value \(\lambda_\tau\geq 0\) on all orthogonal projections of rank one.

If \(T \in \mathcal{FR}(\mathcal{H})\) is self-adjoint then \(T = \sum_{j=1}^{N} \mu_j P_j\) with rank one orthogonal projections \(P_j\). Thus
\[
\tau(T) = \lambda_\tau \sum_{j=1}^{N} \mu_j = \lambda_\tau \text{Tr}(T).
\]
Since each $T \in \mathcal{FR}(\mathcal{H})$ is a linear combination of self-adjoint elements of $\mathcal{FR}(\mathcal{H})$ we reach the conclusion. \hfill \Box

The properties of $\text{Tr}$ we have mentioned so far are not sufficient to show that a tracial weight on $\mathcal{B}(\mathcal{H})$ is proportional to $\text{Tr}$. The property which implies this is normality:

**Proposition 2.4.** 1. $\text{Tr}$ is normal, that is, if $(T_n)_{n \in \mathbb{Z}_+} \subset \mathcal{B}_+(\mathcal{H})$ is an increasing sequence with $T_n \rightarrow T \in \mathcal{B}_+(\mathcal{H})$ strongly then $\text{Tr}(T) = \sup_{n \in \mathbb{Z}_+} \text{Tr}(T_n)$.

2. Let $\tau$ be a normal tracial weight on $\mathcal{B}(\mathcal{H})$. Then there is a constant $\lambda_\tau \in \mathbb{R}_+ \cup \{\infty\}$ such that for $T \in \mathcal{B}_+(\mathcal{H})$ we have $\tau(T) = \lambda_\tau \text{Tr}(T)$.

**Remark 2.5.** In the somewhat pathological case $\lambda = \infty$ the tracial weight $\tau_\infty$ is given by

$$
\tau_\infty(T) = \begin{cases} 
\infty, & T \in \mathcal{B}_+(\mathcal{H}) \setminus \{0\}, \\
0, & T = 0.
\end{cases}
$$

In all other cases $\tau$ is semifinite, that means for $T \in \mathcal{B}_+(\mathcal{H})$ there is an increasing sequence $(T_n)_{n \in \mathbb{Z}_+}$ with $\tau(T_n) < \infty$ and $T_n \nrightarrow T$ strongly. Here, $T_n$ may be chosen of finite rank.

**Proof.** 1. Let $(e_k)_{k \in \mathbb{Z}_+}$ be an orthonormal basis of $\mathcal{H}$. Since $T_n \rightarrow T$ strongly we have $(T_ne_k, e_k) \nrightarrow (Te_k, e_k)$. The Monotone Convergence Theorem for the counting measure on $\mathbb{Z}_+$ then implies

$$
\text{Tr}(T) = \sum_{k=0}^\infty (Te_k, e_k) = \sup_{n \in \mathbb{Z}_+} \sum_{k=0}^\infty (T_ne_k, e_k) = \sup_{n \in \mathbb{Z}_+} \text{Tr}(T_n).
$$

2. Let $\tau : \mathcal{B}_+(\mathcal{H}) \rightarrow \mathbb{R}_+ \cup \{\infty\}$ be a normal tracial weight. As in the proof of Lemma 2.3 one shows that $\tau \upharpoonright \mathcal{FR}(\mathcal{H}) = \lambda_\tau \text{Tr} \upharpoonright \mathcal{FR}(\mathcal{H})$ for some $\lambda_\tau \in \mathbb{R}_+ \cup \{\infty\}$.

Choose an increasing sequence of orthogonal projections $(P_n)_{n \in \mathbb{Z}_+}$, rank $P_n = n$. Given $T \in \mathcal{B}_+(\mathcal{H})$ the sequence of finite rank operators $(T^{1/2}P_n T^{1/2})_{n \in \mathbb{Z}_+}$ is increasing and it converges strongly to $T$. Since $\tau$ is assumed to be normal we thus find

$$
\tau(T) = \sup_{n \in \mathbb{Z}_+} \tau(T^{1/2}P_n T^{1/2}) = \sup_{n \in \mathbb{Z}_+} \lambda_\tau \text{Tr}(T^{1/2}P_n T^{1/2}) = \lambda_\tau \text{Tr}(T). \quad \Box
$$

**Remark 2.6.** The uniqueness of the trace $\text{Tr}$ we presented here is in fact a special case of a rich theory of traces for weakly closed self-adjoint subalgebras of $\mathcal{B}(\mathcal{H})$ (von Neumann algebras) due to Murray and von Neumann [MuvN36, MuvN37, vN40, MuvN43].

**2.3. The Dixmier Trace.** In view of Proposition 2.3 it is natural to ask whether there exist non-normal tracial weights on $\mathcal{B}(\mathcal{H})$. A cheap answer to this question would be to define for $T \in \mathcal{B}_+(\mathcal{H})$

$$
\tau(T) := \begin{cases} 
\text{Tr}(T), & T \in \mathcal{FR}(\mathcal{H}), \\
\infty, & T \notin \mathcal{FR}(\mathcal{H}).
\end{cases}
$$
Then \( \tau \) is certainly a non–trivial non–normal tracial weight on \( \mathcal{B}(\mathcal{H}) \).

To make the problem non–trivial, one should ask whether there exists a non–trivial non–normal tracial weight on \( \mathcal{B}(\mathcal{H}) \) which vanishes on trace class operators. This was answered affirmatively by J. Dixmier in the short note \cite{Dix66}. We briefly describe Dixmier’s very elegant argument.

Denote by \( \mathcal{K}(\mathcal{H}) \) the ideal of compact operators. We abbreviate
\[
L^p(\mathcal{H}) := L^p(\mathcal{B}(\mathcal{H}), \text{Tr}),
\]
see Section 2.1.2. A compact operator \( T \) is in \( L^1(\mathcal{H}) \) if and only if
\[
\sum_{j=1}^{\infty} \mu_j(T) < \infty.
\]
Here \( \mu_j(T), j \geq 1 \), denotes the sequence of eigenvalues of \( |T| \) counted with multiplicity.

By \( L^{(1,\infty)}(\mathcal{H}) \supset L^1(\mathcal{H}) \) one denotes the space of \( T \in \mathcal{K}(\mathcal{H}) \) for which
\[
\sum_{j=1}^{N} \mu_j(T) = O(\log N), \quad N \to \infty.
\]

For an operator \( T \in L^{(1,\infty)}(\mathcal{H}) \) the sequence
\[
\alpha_N(T) := \frac{1}{\log(N+1)} \sum_{j=1}^{N} \mu_j(T), \quad N \geq 1,
\]
is thus bounded.

**Proposition 2.7** (J. Dixmier \cite{Dix66}). Let \( \omega \in l^\infty(\mathbb{Z}_+ \setminus \{0\})^* \) be a linear functional satisfying
\[\begin{align*}
(1) & \quad \omega \text{ is a state, that is, a positive linear functional with } \\
& \quad \omega(1,1,\ldots) = 1. \\
(2) & \quad \omega((\alpha_N)_{N\geq1}) = 0 \text{ if } \lim_{N \to \infty} \alpha_N = 0. \\
(3) & \quad \omega(\alpha_1,\alpha_2,\alpha_3,\ldots) = \omega(\alpha_1,\alpha_2,\alpha_2,\alpha_2,\ldots).
\end{align*}\]

Put for non–negative \( T \in L^{(1,\infty)}(\mathcal{H}) \)
\[
\text{Tr}_\omega(T) := \omega\left(\frac{1}{\log(N+1)} \sum_{j=1}^{N} \mu_j(T)\right)_{N\geq1}
\]
\[
=: \lim_{\omega} \frac{1}{\log(N+1)} \sum_{j=1}^{N} \mu_j(T).
\]

Then \( \text{Tr}_\omega \) extends by linearity to a trace on \( L^{(1,\infty)}(\mathcal{H}) \). If \( T \in L^1(\mathcal{H}) \) is of trace class then \( \text{Tr}_\omega(T) = 0 \). Furthermore,
\[
\text{Tr}_\omega(T) = \lim_{N \to \infty} \frac{1}{\log(N+1)} \sum_{j=1}^{N} \mu_j(T),
\]
if the limit on the right hand side exists.

Finally, by putting \( \text{Tr}_\omega(T) = \infty \) if \( T \in \mathcal{B}_+(\mathcal{H}) \setminus L^{(1,\infty)}(\mathcal{H}) \) one extends \( \text{Tr}_\omega \) to \( \mathcal{B}_+(\mathcal{H}) \) and hence one obtains a non–normal tracial weight on \( \mathcal{B}(\mathcal{H}) \).
Proof. Let us make a few comments on how this result is proved: First the existence of a state $\omega$ with the properties (1), (2), and (3) can be shown by a fixed point argument; in this simple case even Schauder’s Fixed Point Theorem would suffice. Alternatively, the theory of Cesàro means leads to a more constructive proof of the existence of $\omega$, Connes [Con94, Sec. 4.2.\gamma].

Next we note that (1) and (2) imply that if $(\alpha_N)_{N \geq 1}$ is convergent then $\omega((\alpha_N)_{N \geq 1}) = \lim_{N \to \infty} \alpha_N$. Thus changing finitely many terms of $(\alpha_N)_{N \geq 1}$ (i.e. adding a sequence of limit 0) does not change its $\omega$–limit. Together with the positivity of $\omega$ this implies

\[(2.20) \text{ if } \alpha_N \leq \beta_N \text{ for } N \geq N_0 \text{ then } \omega((\alpha_N)_{N \geq 1}) \leq \omega((\beta_N)_{N \geq 1}).\]

The previously mentioned facts imply furthermore

\[(2.21) \liminf_{N \to \infty} \alpha_N \leq \omega((\alpha_N)_{N \geq 1}) \leq \limsup_{N \to \infty} \alpha_N.\]

Now let $T_1, T_2 \in \mathcal{L}^{(1, \infty)}$ be non–negative operators and put

\[
\alpha_N := \frac{1}{\log(N+1)} \sum_{j=1}^{N} \mu_j(T_1), \quad \beta_N := \frac{1}{\log(N+1)} \sum_{j=1}^{N} \mu_j(T_2),
\]

\[
(2.22) \quad \gamma_N := \frac{1}{\log(N+1)} \sum_{j=1}^{N} \mu_j(T_1 + T_2).
\]

Using the min-max principle one shows the inequalities

\[(2.23) \quad \sum_{j=1}^{N} \mu_j(T_1 + T_2) \leq \sum_{j=1}^{N} \mu_j(T_1) + \mu_j(T_2) \leq \sum_{j=1}^{2N} \mu_j(T_1 + T_2),\]

cf. Hersch [Her61a, Her61b], thus

\[(2.24) \quad \gamma_N \leq \alpha_N + \beta_N,\]

\[(2.25) \quad \alpha_N + \beta_N \leq \frac{\log(2N+1)}{\log(N+1)} \gamma_{2N}.\]

\[(2.24) \text{ gives } \omega((\gamma_N)_{N \geq 1}) \leq \omega((\alpha_N)_{N \geq 1}) + \omega((\beta_N)_{N \geq 1}).\]

The proof of the converse inequality makes essential use of the crucial assumption (2.17). Together with (2.25) and (2.20) we find

\[(2.26) \quad \omega((\alpha_N)_{N \geq 1}) + \omega((\beta_N)_{N \geq 1}) \leq \omega(\gamma_2, \gamma_4, \gamma_6, \ldots) = \omega(\gamma_2, \gamma_2, \gamma_4, \gamma_4, \ldots),\]

so, in view of (2.27) (2), it only remains to remark that

\[\lim_{N \to \infty} (\gamma_{2N} - \gamma_{2N-1}) = 0.\]

Thus $Tr_\omega$ is additive on the cone of positive operators. Since $Tr_\omega(T)$ depends only on the spectrum, it is certainly invariant under conjugation by unitary operators. Now it is easy to see that $Tr_\omega$ extends by linearity to a trace on $\mathcal{L}^{(1, \infty)}(\mathcal{H})$. The other properties follow easily. ☐
3. Pseudodifferential operators with parameter

3.1. From differential operators to pseudodifferential operators. Historically, pseudodifferential operators were invented to understand differential operators. Suppose given a differential operator

\[ P = \sum_{|\alpha| \leq d} p_\alpha(x) i^{-|\alpha|} \frac{\partial^{|\alpha|}}{\partial x^\alpha} \]

in an open set \( U \subset \mathbb{R}^n \). Representing a function \( u \in C_0^\infty(U) \) in terms of its Fourier transform

\[ u(x) = \int_{\mathbb{R}^n} e^{i(x,\xi)} \hat{u}(\xi) d\xi, \quad d\xi = (2\pi)^{-n} d\xi, \]

where \( \hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-i(x,\xi)} u(x) dx \), we find

\[ Pu(x) = \int_{\mathbb{R}^n} e^{-i(x,\xi)} p(x,\xi) \hat{u}(\xi) d\xi \]

\[ = \int_{\mathbb{R}^n} \left( \int_U e^{i(x-y,\xi)} p(x,\xi) u(y) dy \right) d\xi \]

\[ =: \text{Op}(p)u(x). \]

Here

\[ p(x,\xi) = \sum_{|\alpha| \leq d} p_\alpha(x) \xi^\alpha \]

denotes the complete symbol of \( P \). The right hand side of (3.3) shows that \( P \) is a pseudodifferential operator with complete symbol function \( p(x,\xi) \).

Note that \( p(x,\xi) \) is a polynomial in \( \xi \). One now considers pseudodifferential operators with more general symbol functions such that inverses of differential operators are included into the calculus. E.g., a first approximation to the resolvent \( (P-\lambda^d)^{-1} \) is given by \( \text{Op}((p(\cdot,\cdot)-\lambda^d)^{-1}) \). For constant coefficient differential operators this is indeed the exact resolvent.

Let us now describe the most commonly used symbol spaces. In view of the resolvent example above we are going to consider symbols with an auxiliary parameter.

3.2. Basic calculus with parameter. We first recall the notion of conic manifolds and conic sets from Duistermaat [Du96, Sec. 2]. A conic manifold is a smooth principal fiber bundle \( \Gamma \rightarrow B \) with structure group \( \mathbb{R}^* := (0, \infty) \). It is always trivializable. A subset \( \Gamma \subset \mathbb{R}^N \setminus \{0\} \) which is a conic manifold by the natural \( \mathbb{R}^* \)-action on \( \mathbb{R}^N \setminus \{0\} \) is called a conic set. The base manifold of a conic set \( \Gamma \subset \mathbb{R}^N \setminus \{0\} \) is diffeomorphic to \( ST := \Gamma \cap S^{N-1} \). By a cone \( \Gamma \subset \mathbb{R}^N \) we will always mean a conic set or the closure of a conic set in \( \mathbb{R}^N \) such that \( \Gamma \) has nonempty interior. Thus \( \mathbb{R}^N \) and \( \mathbb{R}^N \setminus \{0\} \) are cones, but only the latter is a conic set. \( \{0\} \) is a zero–dimensional cone.

3.2.1. Symbols. Let \( U \subset \mathbb{R}^n \) be an open subset and \( \Gamma \subset \mathbb{R}^N \) a cone. A typical example we have in mind is \( \Gamma = \mathbb{R}^n \times \Lambda \), where \( \Lambda \subset \mathbb{C} \) is an open cone.

We denote by \( S_m(U;\Gamma) \), \( m \in \mathbb{R} \), the space of symbols of Hörmander type \((1,0)\) (Hörmander [Hör71], Grigis–Sjöstrand [GrSj94]). More
precisely, $S^m(U; \Gamma)$ consists of those $a \in C^\infty (U \times \Gamma)$ such that for multi-
indices $\alpha \in \mathbb{Z}^\times_+, \gamma \in \mathbb{Z}^N_+$ and compact subsets $K \subset U, L \subset \Gamma$ we have an estimate
\[(3.5) \quad |\partial^\alpha_x \partial^\gamma_{\xi} a(x, \xi)| \leq C_{\alpha, \gamma, K, L} (1 + |\xi|)^{m - |\gamma|}, \quad x \in K, \xi \in L^c.\]
Here $L^c = \{ \xi \mid \xi \in L, t \geq 1 \}$. The best constants in (3.5) provide a set of semi-norms which endow $S^\infty (U; \Gamma) := \bigcup_{m \in \mathbb{C}} S^m(U; \Gamma)$ with the structure of a Fréchet algebra. We mention the following variants of the space $S^*$:

3.2.2. Classical symbols $CS^m(U; \Gamma)$. A symbol $a \in S^m(U; \Gamma)$ is called classical if there are $a_{m-j} \in C^\infty (U \times \Gamma)$ with
\[(3.6) \quad a_{m-j}(x, r\xi) = r^{-m-j} a_{m-j}(x, \xi), \quad r \geq 1, |\xi| \geq 1,\]
such that for $N \in \mathbb{Z}_+$
\[(3.7) \quad a - \sum_{j=0}^{N-1} a_{m-j} \in S^{m-N}(U; \Gamma).\]
The latter property is usually abbreviated $a \sim \sum_{j=0}^\infty a_{m-j}$.

Many authors require the functions in (3.6) to be homogeneous everywhere on $\Gamma \setminus \{ 0 \}$. Note however that if $\Gamma = \mathbb{R}^p$ and $f : \Gamma \to \mathbb{C}$ is a function which is homogeneous of degree $\alpha$ then $f$ cannot be smooth at 0 unless $\alpha \in \mathbb{Z}_+$. So such a function is not a symbol in the strict sense. We prefer the functions in the expansion (3.7) to be smooth everywhere and homogeneous only for $r \geq 1$ and $|\xi| \geq 1$.

The space of classical symbols of order $m$ is denoted by $CS^m(U; \Gamma)$. In view of the asymptotic expansion (3.7) we have $CS^{m'}(U; \Gamma) \subset CS^m(U; \Gamma)$ only if $m - m' \in \mathbb{Z}_+$ is a non–negative integer.

3.2.3. log–polyhomogeneous symbols $CS^{m,k}(U; \Gamma)$. $a \in S^m(U; \Gamma)$ is called log–polyhomogeneous (cf. Lesch [Les99]) of order $(m, k)$ if it has an asymptotic expansion in $S^\infty (U; \Gamma)$ of the form
\[(3.8) \quad a \sim \sum_{j=0}^\infty a_{m-j} \quad \text{with} \quad a_{m-j} = \sum_{l=0}^{k} b_{m-j,l},\]
where $a_{m-j} \in C^\infty (U \times \Gamma)$ and $b_{m-j,l}(x, \xi) = b_{m-j,l}(x, \xi/|\xi|)|\xi|^{m-j} \log^l |\xi|$ for $|\xi| \geq 1$.

By $CS^{m,k}(U; \Gamma)$ we denote the space of log–polyhomogeneous symbols of order $(m, k)$. Classical symbols are those of log degree $0$, i.e. $CS^0(U; \Gamma) = CS^{m,k}(U; \Gamma)$.

3.2.4. Symbols which are holomorphic in the parameter. If $\Gamma = \mathbb{R}^n \times \Lambda$, where $\Lambda \subset \mathbb{C}$ is a cone one may additionally require symbols to be holomorphic in the $\Lambda$ variable. This aspect is important if one deals with the resolvent of an elliptic differential operator since the latter depends analytically on the resolvent parameter. This class of symbols is not emphasized in this paper.
3.2.5. Pseudodifferential operators with parameter. Fix \( a \in S^m(U; \mathbb{R}^n \times \Gamma) \) (respectively \( \in CS^m(U; \mathbb{R}^n \times \Gamma) \)). For each fixed \( \mu_0 \in \Gamma \) we have \( a(\cdot, \cdot, \mu_0) \in S^m(U; \mathbb{R}^n) \) (respectively \( \in CS^m(U; \mathbb{R}^n) \)) and hence we obtain a family of pseudodifferential operators parametrized over \( \Gamma \) by putting

\[
\left[ \text{Op}(a(\mu_0)) u \right](x) := \left[ A(\mu_0) u \right](x)
\]

\[
= \int_{\mathbb{R}^n} e^{i(x,\xi)} a(x,\xi,\mu_0) \hat{u}(\xi) \, d\xi
\]

\[
= \int_{\mathbb{R}^n} \int_U e^{i(x-y,\xi)} a(x,\xi,\mu_0) u(y) \, dy \, d\xi.
\]

(3.9)

Note that the Schwartz kernel \( K_{A(\mu_0)} \) of \( A(\mu_0) = \text{Op}(a(\mu_0)) \) is given by

\[
K_{A(\mu_0)}(x, y, \mu_0) = \int_{\mathbb{R}^n} e^{i(x-y,\xi)} a(x,\xi,\mu_0) \, d\xi.
\]

(3.10)

In general the integral is to be understood as an oscillatory integral, for which we refer the reader to [Shu01], [GrSj94]. The integral exists in the usual sense if \( m + n < 0 \).

The extension to manifolds and vector bundles is now straightforward. Although historically it took quite a while until the theory of singular integrals had evolved into a theory of pseudodifferential operators on vector bundles over smooth manifolds (CALDERÓN-ZYGMUND [CaZy57], SEELEY [Sec59, Sec65], KOHN-NIRENBERG [KoNi65]). For a smooth manifold \( M \) and a vector bundle \( E \) over \( M \) we define the space \( CL^m(M, E; \Gamma) \) of classical parameter dependent pseudodifferential operators between sections of \( E \) in the usual way by patching together local data:

**Definition 3.1.** Let \( E \) be a complex vector bundle of finite fiber dimension \( N \) over a smooth closed manifold \( M \) and let \( \Gamma \subset \mathbb{R}^p \) be a cone. A **classical pseudodifferential operator of order** \( m \) **with parameter** \( \mu \in \Gamma \) is a family of operators \( B(\mu) : \Gamma^\infty(M; E) \rightarrow \Gamma^\infty(M; E), \mu \in \Gamma, \) such that locally \( B(\mu) \) is given by

\[
\left[ B(\mu) u \right](x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_U e^{i(x-y,\xi)} b(x,\xi,\mu) u(y) \, dy \, d\xi
\]

with \( b \) an \( N \times N \) matrix of functions belonging to \( CS^m(U, \mathbb{R}^n \times \Gamma) \).

\( CL^{m,k}(M, E; \Gamma) \) is defined similarly, although we will discuss \( CL^{m,k} \) only in the non–parametric case. Of course, operators may act between sections of different vector bundles \( E, F \). In that case we write \( CL^{m,k}(M, E, F; \Gamma) \).

**Remark 3.2.** 1. In case \( \Gamma = \{0\} \) we obtain the usual (classical) pseudodifferential operators of order \( m \) on \( U \). Here we write \( CL^m(M, E) \) instead of \( CL^m(M, E; \{0\}) \) respectively \( CL^m(M, E, F; \{0\}) \).

2. Parameter dependent pseudodifferential operators play a crucial role, e.g., in the construction of the resolvent expansion of an elliptic operator (GILKEY [Gil95]).

A **pseudodifferential operator with parameter** is more than just a map from \( \Gamma \) to the space of pseudodifferential operators, cf. Corollary 3.8 and Remark 3.9.

To illustrate this let us consider a single elliptic operator \( A \in CL^m(U) \).

For simplicity let the symbol \( a(x, \xi) \) of \( A \) be positive definite. Then we can
consider the “parametric symbol” $b(x, \xi, \lambda) = a(x, \xi) - \lambda^m$ for $\lambda \in \Lambda := \mathbb{C} \setminus \mathbb{R}_+$. However, in general $b$ lies in $\mathrm{CS}^m(U; \Lambda)$ only if $A$ is a differential operator. The reason is that $b$ will satisfy the estimates (3.3) only if $a(x, \xi)$ is polynomial in $\xi$, because then $\partial_\beta^2 a(x, \xi) = 0$ if $|\beta| > m$. If $a(x, \xi)$ is not polynomial in $\xi$, however, (3.5) will in general not hold if $\beta > m$.

This problem led Grubb and Seeley [GrSe95] to invent their calculus of \textit{weakly parametric} pseudodifferential operators. $b(x, \xi, \lambda) = a(x, \xi) - \lambda^m$ is weakly parametric for any elliptic $A$ with positive definite leading symbol (or more generally if $A$ satisfies Agmon’s angle condition). The class of weakly parametric operators is beyond the scope of this survey, however.

3. The definition of the parameter dependent calculus is not uniform in the literature. It will be crucial in the sequel that differentiating by the parameter reduces the order of the operator. This is the convention, e.g. of Gilkey [Gi95] but differs from the one in Shubin [Shu01]. In Lesch–Pflaum [LePf00] Sec. 3] it is shown that parameter dependent pseudodifferential operators can be viewed as translation invariant pseudodifferential operators on $U \times \Gamma$ and therefore our convention of the parameter dependent calculus contains Melrose’s suspended algebra from [Mel95].

\textbf{Proposition 3.3.} CL\textsuperscript{**}(M, E; \Gamma) is a bi–filtered algebra, that is,

$$AB \in \mathrm{CL}^{m+k+k'}(M, E; \Gamma)$$

for $A \in \mathrm{CL}^{m-k}(M, E; \Gamma)$ and $B \in \mathrm{CL}^{m', k'}(M, E; \Gamma)$.

The following result about the $L^2$–continuity of a parameter dependent pseudodifferential operator is crucial. We denote by $L^2_s(M, E)$ the Hilbert space of sections of $E$ of Sobolev class $s$.

\textbf{Theorem 3.4.} Let $A \in \mathrm{CL}^m(M, E; \Gamma)$. Then for fixed $\mu \in \Gamma$ the operator $A(\mu)$ extends by continuity to a bounded linear operator $L^2_s(M, E) \longrightarrow L^2_{s-m}(M, E)$, $s \in \mathbb{R}$.

Furthermore, for $m \leq 0$ one has the following uniform estimate in $\mu$: for $0 \leq \vartheta \leq 1$, $\mu_0(1)$, there is a constant $C(s, \vartheta)$ such that

$$||A(\mu)||_{s, s+\vartheta|\mu|} \leq C(s, \vartheta, \mu_0)(1 + |\mu|)^{-(1-\vartheta)|\mu|}, \quad |\mu| \geq |\mu_0|, \mu \in \Gamma.$$ 

Here $||A(\mu)||_{s, s+\vartheta|\mu|}$ denotes the norm of the operator $A(\mu)$ as a map from the Sobolev space $L^2_s(M, E)$ into $L^2_{s-m}(M, E)$.

If $\Gamma = \mathbb{R}^n$ then we can omit the $\mu_0$ in the formulation of the Theorem (i.e. $\mu_0 = 0$). For a proof of Theorem 3.4 see e.g. Shubin [Shu01] Theorem 9.3].

3.2.6. \textbf{The parametric leading symbol}. The leading symbol of a classical pseudodifferential operator $A$ of order $m$ with parameter is now defined as follows: if $A$ has complete symbol $a(x, \xi, \mu)$ with expansion $a \sim \sum_{j=0}^{\infty} a_m \xi^j$

then

$$\sigma_A^m(x, \xi, \mu) := \lim_{r \to \infty} r^{-m} a(x, r\xi, r\mu)$$

(3.11)

$$= (|\xi|^2 + |\mu|^2)^{m/2} a_m(x, \frac{(\xi, \mu)}{\sqrt{|\xi|^2 + |\mu|^2}}).$$
Lemma 3.6. Let \( \sigma^m_\lambda \) has an invariant meaning as a smooth function on \( T^*M \times \Gamma \setminus \{(x,0,0) \mid x \in M\} \) which is homogeneous in the following sense:
\[
\sigma^m_\lambda(x, r\xi, r\mu) = r^m \sigma^m_\lambda(x, \xi, \mu) \text{ for } (\xi, \mu) \neq (0,0), \ r > 0.
\]
This symbol is determined by its restriction to the sphere in \( S(T^*M \times \Gamma) = \{(\xi, \mu) \in T^*M \times \Gamma \mid |\xi|^2 + |\mu|^2 = 1\} \) and there is an exact sequence
\[
0 \rightarrow \text{CL}^{m-1}(M; \Gamma) \hookrightarrow \text{CL}^m(M; \Gamma) \xrightarrow{\rho} C^\infty(S(T^*M \times \Gamma)) \rightarrow 0;
\]
the vector bundle \( E \) being omitted from the notation just to save horizontal space.

Example 3.5. Let us look at an example to illustrate the difference between the parametric leading symbol and the leading symbol for a single pseudodifferential operator. Let
\[
a(x, \xi) = \sum_{|\alpha| \leq m} a_\alpha(x) \xi^\alpha
\]
be the complete symbol of an elliptic differential operator. Then (cf. Remark (3.2) 2.)
\[
b(x, \xi, \lambda) = a(x, \xi) - \lambda^m
\]
is a symbol of a parameter dependent (pseudo)differential operator \( B(\lambda) \) with parameter \( \lambda \) in a suitable cone \( \Lambda \subset \mathbb{C} \). The parameter dependent leading symbol of \( B \) is \( \sigma^m_B(x, \xi, \lambda) = a_m(x, \xi) - \lambda^m \) while for fixed \( \lambda \) the leading symbol of the single operator \( B(\lambda) \) is \( \sigma^m_{B(\lambda)}(x, \xi) = a_m(x, \xi) = \sigma^m_B(x, \xi, \lambda = 0) \).

In fact we have in general:

Lemma 3.6. Let \( A \in \text{CL}^m(M, E; \Gamma) \) with parameter dependent leading symbol \( \sigma^m_A(x, \xi, \mu) \). For fixed \( \mu_0 \in \Gamma \) the operator \( A(\mu_0) \in \text{CL}^m(M, E) \) has leading symbol \( \sigma^m_{A(\mu_0)}(x, \xi) = \sigma^m_A(x, \xi, 0) \).

Proof. It suffices to prove this locally in a chart \( U \) for a scalar operator \( A \). Since the leading symbols are homogeneous it suffices to consider \( \xi \) with \( |\xi| = 1 \).

So suppose that \( A \) has complete symbol \( a(x, \xi, \mu) \) in \( U \). Write \( a(x, \xi, \mu) = a_m(x, \xi, \mu) + \tilde{a}(x, \xi, \mu) \) with \( \tilde{a} \in \text{CS}^{m-1}(U; \mathbb{R}^n \times \Gamma) \) and \( a_m(x, r\xi, r\mu) = r^m a_m(x, \xi, \mu) \) for \( r \geq 1, |\xi|^2 + |\mu|^2 \geq 1 \). Then for fixed \( \mu_0 \in \Gamma \) we have \( \tilde{a}(\cdot, \cdot, \mu_0) \in \text{CS}^{m-1}(U; \mathbb{R}^n) \) and hence \( \lim_{r \to \infty} r^{-m} \tilde{a}(x, r\xi, \mu_0) = 0 \). Consequently
\[
\sigma^m_{A(\mu_0)}(x, \xi) = \lim_{r \to \infty} r^{-m} a_m(x, r\xi, \mu_0) = \lim_{r \to \infty} a_m(x, \xi, \mu_0/r) = a_m(x, \xi, 0). \quad \square
\]
3.2.7. Parameter dependent ellipticity. This is now defined as the invertibility of the parametric leading symbol. The basic example of a pseudodifferential operator with parameter is the resolvent of an elliptic differential operator (cf. Remark 3.2 and Example 3.5). The following two results can also be found in [Shu01, Section II.9].

**Theorem 3.7.** Let $M$ be a closed manifold and $E,F$ complex vector bundles over $M$. Let $A \in \text{CL}^m(M,E,F;\Gamma)$ be elliptic. Then there exists a $B \in \text{CL}^{-m}(M,F,E;\Gamma)$ such that $AB - I \in \text{CL}^{-\infty}(M,F;\Gamma)$, $BA - I \in \text{CL}^{-\infty}(M,E;\Gamma)$.

Note that in view of Theorem 3.4 this implies the estimates

$$
\|B(\mu)A(\mu) - I\|_{s,t} + \|A(\mu)B(\mu) - I\|_{s,t} \leq C(s,t,N)(1 + |\mu|)^{-N}
$$

for all $s,t \in \mathbb{R}$, $N > 0$. This result has an important implication:

**Corollary 3.8.** Under the assumptions of Theorem 3.7, for each $s \in \mathbb{R}$ there is a $\mu_0 \in \Gamma$ such that for $|\mu| \geq |\mu_0|$ the operator $A(\mu) : L^2_s(M,E) \to L^2_{s-m}(M,F)$ is invertible.

**Proof.** In view of (3.15) there is a $\mu_0 = \mu_0(s)$ such that

$$
\|(BA - I)(\mu)\|_s < 1 \text{ and } \|(AB - I)(\mu)\|_{s-m} < 1,
$$

for $|\mu| \geq |\mu_0|$ and hence $AB : L^2_s \to L^2_s$ and $BA : L^2_{s-m} \to L^2_{s-m}$ are invertible. \qed

**Remark 3.9.** This result causes an interesting constraint on those pseudodifferential operators which may appear as special values of an elliptic parametric family. Namely, if $A \in \text{CL}^m(M,E,F;\Gamma)$ is parametric elliptic then for each $\mu$ the operator $A(\mu) \in \text{CL}^m(M,E,F)$ is elliptic. Furthermore, by the previous Corollary and the stability of the Fredholm index we have $\text{ind} A(\mu) = 0$ for all $\mu$.

4. Extending the Hilbert space trace to pseudodifferential operators

We pause the discussion of pseudodifferential operators and look at the Hilbert space trace $\text{Tr}$ on pseudodifferential operators.

4.1. $\text{Tr}$ on operators of order $< - \dim M$. Consider the local situation, i.e. a compactly supported operator $A = \text{Op}(a) \in \text{CL}^{m,k}(U,E)$ in a local chart.

If $m < - \dim M$ then $A$ is trace class and the trace is given by integrating the kernel of $A$ over the diagonal:

$$
\text{Tr}(A) = \int_U \text{tr}_{E_y}(k_A(x,x))\,dx
$$

(4.1)

$$
= \int_U \int_{\mathbb{R}^n} \text{tr}_{E_y}(a(x,\xi))\,d\xi\,dx,
$$

where we have used (3.10).

The right hand side is indeed coordinate invariant. To explain this consider a coordinate transformation $\kappa : U \to V$. Denote variables in $U$ by $x,y$
and variables in $V$ by $\tilde{x}, \tilde{y}$. It is not so easy to write down the symbol of $\kappa_{\ast}A$. However, an amplitude function (these are “symbols” which depend on $x$ and $y$, otherwise the basic formula still holds) for $\kappa_{\ast}A$ is given by

$$
(\tilde{x}, \tilde{y}, \xi) \mapsto a(\kappa^{-1}\tilde{x}, \phi(\tilde{x}, \tilde{y})^{-1}\xi)^{\det D\kappa^{-1}(\tilde{x}, \tilde{y})}|\det \phi(\tilde{x}, \tilde{y})|
$$

cf. [Shu01] Sec. 4.1, 4.2, where $\phi(\tilde{x}, \tilde{y})$ is smooth with $\phi(\tilde{x}, \tilde{y}) = D\kappa^{-1}(\tilde{x})^t$.

Comparing the trace densities in the two coordinate systems requires a linear coordinate change in the $\xi$-variable. Indeed,

$$
\text{Tr}(\kappa_{\ast}A) = \int_V \int_{\mathbb{R}^n} \text{tr}_{E_x}(a(\kappa^{-1}\tilde{x}, \phi(\tilde{x}, \tilde{y})^{-1}\xi))d\xi d\tilde{x}
$$

$$
= \int_V \int_{\mathbb{R}^n} \text{tr}_{E_{\tilde{x}}}(a(\kappa^{-1}\tilde{x}, \xi))d\xi |\det D\kappa^{-1}(\tilde{x})|d\tilde{x},
$$

$$
= \int_U \int_{\mathbb{R}^n} \text{tr}_{E_{\tilde{x}}}(a(x, x, \xi))d\xi dx = \text{Tr}(A).
$$

Therefore, the trace of a pseudodifferential operator $A \in \text{CL}^{m,k}(M, E)$ of order $m < -\dim M =: -n$ on the closed manifold $M$ may be calculated from the complete symbol of $A$ in coordinates as follows. Choose a finite open cover by coordinate neighborhoods $U_j$, $j = 1, \ldots, r$, and a subordinated partition of unity $\varphi_j$, $j = 1, \ldots, r$. Furthermore, let $\psi_j \in \text{C}_0^\infty(U_j)$ with $\psi_j \varphi_j = \varphi_j$. Denoting by $a_j(x, \xi)$ the complete symbol in the coordinate system on $U_j$ we obtain

$$
\text{Tr}(A) = \sum_{j=1}^r \text{Tr}(\varphi_j A \psi_j) = \sum_{j=1}^r \int_{U_j} \varphi_j(x) \text{tr}_{E_x}(a_j(x, x, \xi))d\xi dx.
$$

A priori the previous argument is valid only for operators of order $m < -n$. However, the symbol function $a_j(x, \xi)$ is rather well–behaved in $\xi$. If for a class of pseudodifferential operators we can regularize $\int_{\mathbb{R}^n} a_j(x, \xi)d\xi$ in such a way that the change of variables works then indeed extends the trace to this class of operators. Such a regularization is provided by:

### 4.2. The Hadamard partie finie regularized integral

The problem of regularizing divergent integrals is in fact quite old. The method we are going to present here goes back to HADAMARD who used his method to regularize integrals which arose when solving the wave equation [Had32].

Given a function $f \in \text{CS}^{m,k}(\mathbb{R}^p)$, e.g. $a(x, \cdot)$ above for fixed $x$. Then $f$ has an asymptotic expansion

$$
f(x) \sim_{|x| \to \infty} \sum_{j=0}^k \sum_{l=0}^{k-j} f_{jl}(x/|x|)|x|^{m-j} \log^l |x|.
$$

Integrating over balls of radius $R$ gives the asymptotic expansion

$$
\int_{|x| \leq R} f(x)dx \sim_{R \to \infty} \sum_{j=0}^{k+1} \sum_{l=0}^{k-j} f_{jl} R^{m+n-j} \log^l R.
$$
The regularized integral $\int_{\mathbb{R}^p} f(x)dx$ is, by definition, the constant term in this asymptotic expansion. Some authors call the regularized integral \textit{partie finie integral} or \textit{cut–off integral}.

It has a couple of peculiar properties, cf. [Mel95], which were further investigated in [Les99 Sec. 5] and [LePf00]. The most notable features are a modified change of variables rule for linear coordinate changes and, as a consequence, the fact that Stokes’ theorem does not hold in general:

**Proposition 4.1.** [Les99 Prop. 5.2] Let $A \in \text{GL}(p, \mathbb{R})$ be a regular matrix. Furthermore, let $f \in \text{CS}^{m,k}(\mathbb{R}^p)$ with expansion (4.5). Then we have the change of variables formula

\[
\int_{\mathbb{R}^p} f(A\xi)d\xi = |\det A|^{-1} \left( \int_{\mathbb{R}^p} f(\xi)d\xi + \sum_{l=0}^{k} \frac{(-1)^{l+1}}{l+1} \int_{S^{p-1}} f_{-p,l}(\xi) \log^{l+1} |A^{-1}\xi|d\xi \right).
\]

The following proposition, which substantiates the mentioned fact that Stokes’ Theorem does not hold for $\int_{\mathbb{R}^p}$, was stated as a Lemma in [LePf00]. A couple of years later it was rediscovered by MANCHON, MAEDA, and PAYCHA [MMP05], [Pay05].

**Proposition 4.2.** [LePR00 Lemma 5.5] Let $f \in \text{CS}^{m,k}(\mathbb{R}^p)$ with asymptotic expansion (4.5). Then

\[
\int_{\mathbb{R}^p} \frac{\partial f}{\partial \xi_j} d\xi = \int_{S^{p-1}} f_{1-p,k}(\xi)\xi_j d\text{vol}_S(\xi).
\]

We will come back to this below when we discuss the residue trace.

### 4.3. The Kontsevich–Vishik canonical trace

Using the Hadamard partie finie integral we can now follow the scheme outlined in Subsection 4.1. Let $A \in \text{CL}^{a,k}(M, E)$ be a log–polyhomogeneous pseudodifferential operator on a closed manifold $M$. If $a \notin \mathbb{Z}$ we put, using the notation of (4.4) and (4.3),

\[
\text{TR}(A) := \sum_{j=1}^{\infty} \int_{U_j} \int_{\mathbb{R}^n} \varphi_j(x) \text{tr}_{E_x}(a_j(x, \xi)) d\xi dx.
\]

By Proposition 4.1 one shows exactly as in (4.3) that TR($A$) is well–defined.

In fact we have (essentially) proved the following:

**Theorem 4.3** (KONTSEVICH–VISHIK [KoVi95], [KoVi94], LESCH [Les99 Sec. 5]). There is a linear functional TR on

\[
\bigcup_{a \in \mathbb{C} \setminus \{-n, -n+1, -n+2, \ldots\}, k \geq 0} \text{CL}^{a,k}(M, E)
\]

such that

(i) In a local chart TR is given by (4.1), with $\int_{\mathbb{R}^n}$ to be replaced by the cut–off integral $\int_{\mathbb{R}^n}$.

(ii) TR $\upharpoonright \text{CL}^{a,k}(M, E) = \text{Tr} \upharpoonright \text{CL}^{a,k}(M, E)$ if $a < - \dim M$.

(iii) TR([$A, B$]) = 0 if $A \in \text{CL}^{a,k}(M, E), B \in \text{CL}^{b,l}(M, E)$, $a+b \notin \mathbb{Z}$.
We mention a stunning application of this result [KoVi95, Cor. 4.1]. Let G be a domain in the complex plane and let \( A(z), B(z) \) be holomorphic families of operators in \( \text{CL}^{k,0}(M, E) \) with \( \text{ord} A(z) = \text{ord} B(z) = z \). We do not formalize the notion of a holomorphic family here. What we have in mind are e.g. families of complex powers \( A(z) = A^z \). Assume that \( G \) contains points \( z \) with \( \text{Re} z < - \dim M \). Then \( \text{TR}(A(z)) \) is the analytic continuation of \( \text{Tr}(A(.)) \mid G \cap \{ z \in \mathbb{C} \mid \text{Re} z < - \dim M \} \); a similar statement holds for \( B(z) \).

If for a point \( z_0 \in G \setminus \{-n, -n+1, \ldots \} \) we have \( A(z_0) = B(z_0) \) we can conclude that the value of the analytic continuation of \( \text{Tr}(A(.)) \mid G \cap \{ z \in \mathbb{C} \mid \text{Re} z < - \dim M \} \) to \( z_0 \) coincides with the value of the corresponding analytic continuation of \( \text{Tr}(B(.)) \mid G \cap \{ z \in \mathbb{C} \mid \text{Re} z < - \dim M \} \). Namely, we obviously have \( \text{TR}(A(z_0)) = \text{TR}(B(z_0)) \). The author does not know of a direct proof of this fact.

Proposition 4.4 shows that if \( A \) is of integral order additional terms show up when making the linear change of coordinates \( (4.3) \), indicating that \( \text{TR} \) cannot be extended to a trace on the algebra of pseudodifferential operators. The following no go result shows that the order constraints in Theorem 4.3 are indeed sharp:

**Proposition 4.4.** There is no trace \( \tau \) on the algebra \( \text{CL}^0(M) \) of classical pseudodifferential operators of order 0 such that \( \tau(A) = \text{Tr}(A) \) if \( A \in \text{CL}^{-\infty}(M) \).

**Proof.** We reproduce here the very easy proof: from Index Theory we use the fact that on \( M \) there exists an elliptic system \( T \in \text{CL}^0(M, \mathbb{C}^r) \) of non–vanishing Fredholm index; in general we cannot find a scalar elliptic operator with non–trivial index. Let \( S \in \text{CL}^0(M, \mathbb{C}^r) \) be a pseudodifferential parametrix (cf. Theorem 3.7) such that \( I - ST, I - TS \in \text{CL}^{-\infty}(M, \mathbb{C}^r) \). \( \tau \) and \( \text{Tr} \) extend to traces on \( \text{CL}^0(M, \mathbb{C}^r) = \text{CL}^0(M) \otimes M(r, \mathbb{C}) \) via \( \tau(A \otimes X) = \tau(A) \text{Tr}(X), A \in \text{CL}^0(M), X \in M(r, \mathbb{C}) \) and \( \text{Tr}(X) \) is the usual trace on matrices. Since smoothing operators are of trace class one has

\[
(4.9) \quad \text{ind} T = \text{Tr}(I - ST) - \text{Tr}(I - TS)
\]

and we arrive at the contradiction

\[
0 \neq \text{ind} T = \text{Tr}(I - ST) - \text{Tr}(I - TS) = \tau(I - ST) - \tau(I - TS) = \tau([T,S]) = 0. \tag*{\square}
\]

5. Pseudodifferential operators with parameter: Asymptotic expansions

We take up Section 3 and continue the discussion of pseudodifferential operators with parameter.

5.1. The Resolvent Expansion. The following result is the main technical result needed for the residue trace. It goes back to MINAKSHISUNDARAM and PLEIJEL [MiPl49] who follow carefully HADAMARD’s method of the construction of a fundamental solution for the wave equation [Had32]. It is at the heart of the Local Index Theorem and therefore has received much attention. In the form stated below it is essentially due to SEELEY
The (straightforward) generalization to log–polyhomogeneous symbols was done by the author \cite{Les99}. Of the latter the published version contains annoying typos, the arxiv version is correct.

**Theorem 5.1.** 1. Let $U \subset \mathbb{R}^n$ open, $\Gamma \subset \mathbb{R}^p$ a cone, and $a \in \text{CS}^{m,k}(U; \Gamma)$, $m + n < 0$, $A = \text{Op}(a)$. Let $k_A(x; \mu) := \int_{\mathbb{R}^n} a(x, \xi, \mu) d\xi$ be the Schwartz kernel (cf. Eq. (3.10)) of $A$ on the diagonal. Then $k_A \in \text{CS}^{m+n,k}(U; \Gamma)$. In particular there is an asymptotic expansion

\begin{equation}
    k_A(x, x; \mu) \sim_{|\mu| \to \infty} \sum_{j=0}^{\infty} \sum_{l=0}^{k} c_{m-j,l}(x, \mu/|\mu|) |\mu|^{m+n-j} \log^k |\mu|.
\end{equation}

2. Let $M$ be a compact manifold, $\dim M =: n$, and $A \in \text{CL}^{m,k}(M, E; \Gamma)$. If $m + n < 0$ then $A(\mu)$ is trace class for all $\mu \in \Gamma$ and $\text{Tr} A(\cdot) \in \text{CS}^{m+n,k}(\Gamma)$. In particular,

$$
\text{Tr} A(\mu) \sim_{|\mu| \to \infty} \sum_{j=0}^{\infty} \sum_{l=0}^{k} c_{m-j,l}(\mu/|\mu|) |\mu|^{m+n-j} \log^k |\mu|.
$$

3. Let $P \in \text{CL}^{m}(M, E)$ be an elliptic classical pseudodifferential operator and assume for simplicity that with respect to some Riemannian structure on $M$ and some Hermitian structure on $E$ the operator $P$ is self–adjoint and non–negative. Furthermore, let $B \in \text{CL}^{b,k}(M, E)$ be a pseudodifferential operator. Let $\Lambda = \{ \lambda \in \mathbb{C} \mid |\arg \lambda| \geq \varepsilon \}$ be a sector in $\mathbb{C} \setminus \mathbb{R}_+$. Then for $N > (b + n)/m, n := \dim M$, the operator $B(P - \lambda)^{-N}$ is of trace class and there is an asymptotic expansion

\begin{equation}
\text{Tr}(B(P - \lambda)^{-N}) \sim_{\lambda \to \infty} \sum_{j=0}^{\infty} \sum_{l=0}^{k+1} c_{j,l} \lambda^{\frac{n+b-j}{m} - N} \log^l \lambda + \sum_{j=0}^{\infty} d_j \lambda^{-j-N}, \quad \lambda \in \Lambda.
\end{equation}

Furthermore, $c_{j,k+1} = 0$ if $(j - b - n)/m \notin \mathbb{Z}_+$. 

**Proof.** We present a proof of 1. and 2. and sketch the proof of 3. in a special case.

Since $a \in \text{CS}^{m,k}(U; \Gamma)$ we have Eq. (3.3). Thus we write

\begin{equation}
a = \sum_{j=0}^{N} \alpha_{m-j} + R_N,
\end{equation}

with $R_N \in \text{S}^{m-N}(U; \Gamma)$. In fact, $R_N \in \text{S}^{m-N-1+\varepsilon}(U; \Gamma)$ for every $\varepsilon > 0$, but we don’t need this below. Now pick $L \subset \Gamma, K \subset U$, compact and a
multi–index $\alpha$. Then for $x \in K$ the kernel $k_{A,N}$ of $R_N$ satisfies

$$
|\partial_\mu^\alpha k_{A,N}(x, x; \mu)| = \left|\int_{\mathbb{R}^n} \partial_\mu^\alpha R_N(x, \xi, \mu) d\xi\right|
\leq C_{\alpha,K,L} \int_{\mathbb{R}^n} (1 + (|\xi|^2 + |\mu|^2)^{1/2})^{m-|\alpha|-N} d\xi
\leq C_{\alpha,K,L} (1 + |\mu|)^{m+n-|\alpha|-N}.
$$

(5.4)

Now consider one of the summands of (3.8). We write it in the form

$$
b_{m-j,l}(x, \xi, \mu) = \tilde{b}_{m-j,l}(x, \xi, \mu) \log^j(|\xi|^2 + |\mu|^2),
$$

with

$$
\tilde{b}_{m-j,l}(x, r\xi, r\mu) = r^{m-j} \tilde{b}_{m-j,l}(x, \xi, \mu), \quad \text{for } r \geq 1, |\xi|^2 + |\mu|^2 \geq 1.
$$

(5.6)

Then the contribution $k_{m-j,l}$ of $b_{m-j,l}$ to the kernel of $A$ satisfies

$$
k_{m-j,l}(x, x; r\mu) = \int_{\mathbb{R}^n} \tilde{b}_{m-j,l}(x, \xi, r\mu) \log^j(|\xi|^2 + r^2|\mu|^2) d\xi
= r^{m-j} \int_{\mathbb{R}^n} \tilde{b}_{m-j,l}(x, r^{-1}\xi, \mu) (\log r^2 + \log(|r^{-1}\xi|^2 + |\mu|^2))^j d\xi
= r^{m-n-j} \int_{\mathbb{R}^n} \tilde{b}_{m-j,l}(x, \xi, \mu) (\log r^2 + \log(|\xi|^2 + |\mu|^2))^j d\xi,
$$

(5.7)

proving the expansion (5.1).

2. follows simply by integrating (5.1). In view of (5.4) the expansion (5.1) is uniform on compact subsets of $U$ and hence may be integrated over compact subsets. Covering the compact manifold $M$ by finitely many charts then gives the claim.

3. We cannot give a full proof of 3. here; but we at least want to explain where the additional log terms in (5.2) come from. Note that even if $B \in C^{1,\alpha}(M, E)$ is classical there are log terms in (5.2). In general the highest log power occurring on the rhs of (5.2) is one higher than the log degree of $B$.

For simplicity let us assume that $P$ is a differential operator. This ensures that $(P - \lambda^m)^{-N}$ (note the $\lambda^m$ instead of $\lambda$) is in the parametric calculus (cf. Remarks 3.2 2., 3.5). We first describe the local expansion of the symbol of $B(P - \lambda^m)^{-N}$. To obtain the claim as stated one then has to replace $\lambda^m$ by $\lambda$ and integrate over $M$: choose a chart and denote the complete symbol of $B$ by $b(x, \xi)$ and the complete parametric symbol of $(P - \lambda^m)^{-N}$ by $q(x, \xi, \lambda)$. Then the symbol of the product is given by

$$
(b \ast q)(x, \xi, \lambda) \sim \sum_{\alpha \in \mathbb{Z}_+^n} \frac{i^{-\alpha}}{\alpha!} \left(\frac{\partial_\xi^\alpha b(x, \xi)}{\partial_\xi^\alpha q(x, \xi, \lambda)}\right).
$$

(5.8)
Expanding the rhs into its homogeneous components gives
\[
(b \ast q)(x, \xi, \lambda) \sim \sum_{j=0}^{\infty} \sum_{|\alpha|+\ell\cdot j = j} i^{-\alpha} \frac{\alpha!}{\alpha!} \left( \partial^\alpha_x b_{\ell-j}(x, \xi) \right) \left( \partial^\alpha_x q_{-mN-\ell}(x, \xi, \lambda) \right).
\]

The contribution to the Schwartz kernel of \( B(P - \lambda^m)^{-N} \) of a summand is given by

\[
\frac{i^{-\alpha}}{\alpha!} \int_{\mathbb{R}^n} \left( \partial^\alpha_x b_{\ell-j}(x, \xi) \right) \left( \partial^\alpha_x q_{-mN-\ell}(x, \xi, \lambda) \right) d\xi.
\]

The asymptotic expansion of (5.10) will be singled out as Lemma 5.2 below. The proof of it will in particular explain why the highest possible log-power in (5.2) is one higher than the log-degree of \( B \).

The following expansion Lemma is maybe of interest in its own right. Its proof will explain the occurrence of higher log powers in the resolvent respectively heat expansions. The homogeneous version of the Lemma can again be found in [GrSc95]. We generalize it here slightly to the log–polyhomogeneous setting (cf. [Les99]).

**Lemma 5.2.** Let \( B \in C^\infty(\mathbb{R}^n) \) and \( Q \in C^\infty(\mathbb{R}^n \times [1, \infty)) \) and assume that \( B, Q \) have the following properties

\[
B(\xi) = \tilde{B}(\xi/|\xi|)|\xi|^b \log^k |\xi|, \quad |\xi| \geq 1,
\]

\[
Q(r\xi, r\lambda) = r^q Q(\xi, \lambda), \quad r \geq 1, \lambda \geq 1,
\]

\[
|Q(\xi, 1)| \leq C(|\xi| + 1)^{-q},
\]

where \( b, q \in \mathbb{R} \) and \( b + q + n < 0 \). Then the following asymptotic expansion holds:

\[
F(\lambda) = \int_{\mathbb{R}^n} B(\xi)Q(\xi, \lambda) d\xi \sim_{\lambda \to \infty} \sum_{j=0}^{k+1} c_j \lambda^{n+b+n} \log^j \lambda + \sum_{j=0}^{\infty} d_j \lambda^{g-j}.
\]

\( c_{k+1} = 0 \) if \( b \) is not an integer \( \leq -n \).

The coefficients \( c_j, d_j \) will be explained in the proof.

**Proof.** The integral on the lhs of (5.12) exists since \( b + q + n < 0 \).

We split the domain of integration into the three regions:

1. \( 1 \leq \lambda \leq |\xi|, |\xi| \geq 1 \), and \( 1 \leq |\xi| \leq \lambda \).
Here we are in the domain of homogeneity and a change of variables yields

\[
\int_{\lambda \leq |\xi|} B(\xi)Q(\xi, \lambda) d\xi = \lambda^q \int_{1 \leq |\xi|} \tilde{B}(\xi/|\xi|)|\xi|^b (\log^k |\xi|) Q(\xi/\lambda, 1) d\xi
\]

\[
= \lambda^{q+b+n} \int_{1 \leq |\xi|} \tilde{B}(\xi/|\xi|)|\xi|^b (\log \lambda + \log |\xi|)^k Q(\xi, 1) d\xi,
\]

\[
= \sum_{j=0}^k \alpha_j \lambda^{q+b+n} \log^j \lambda,
\]

giving a contribution to the coefficient \(c_j\) for \(0 \leq j \leq k\).

\([\xi] \leq 1\): For the remaining two cases we employ the Taylor expansion of the smooth function \(\eta \mapsto Q(\eta, 1)\) about \(\eta = 0\):

\[
Q(\eta, 1) = \sum_{j=0}^N Q_j(\eta) + R_N(\eta),
\]

where \(Q_j(\eta) \in \mathbb{C}[\eta_1, \ldots, \eta_n]\) are homogeneous polynomials of degree \(j\) and \(R_N\) is a smooth function satisfying \(R_N(\eta) = O(|\eta|^{N+1})\), \(\eta \to 0\). Respectively, for \(\xi \in \mathbb{R}^n, \lambda \geq 1\),

\[
Q(\xi, \lambda) = Q(\xi/\lambda, 1) \lambda^q = \sum_{j=0}^N Q_j(\xi) \lambda^{q-j} + R_N(\xi/\lambda) \lambda^q.
\]

Plugging (5.15) into the integral for \(|\xi| \leq 1\) we find

\[
\int_{|\xi| \leq 1} B(\xi)Q(\xi, \lambda) d\xi = \lambda^q \sum_{j=0}^N \int_{1 \leq |\xi|} B(\xi)Q_j(\xi) d\xi \lambda^{q-j} + O(\lambda^{q-N-1}), \quad \lambda \to \infty,
\]

giving a contribution to the coefficient \(d_j\).

\(1 \leq |\xi| \leq \lambda\): We again use the Taylor expansion (5.15) with \(N\) large enough such that \(b + N + 1 > -n\) to ensure \(\int_{|\xi| \leq 1} |\xi|^b \log^j |\xi| |R_N(\xi)| d\xi < \infty\) for all \(j\). Let \(B^h(\xi) := B(\xi/|\xi|)|\xi|^b \log^k |\xi|\) be the homogeneous extension of \(B(\xi)\) to all \(\xi \neq 0\). Then

\[
\int_{|\xi| \leq 1} (|B(\xi)| + |B^h(\xi)|) \lambda^q |R_N(\xi/\lambda)| d\xi = O(\lambda^{q-N-1}), \quad \lambda \to \infty,
\]
and thus
\[
\int_{1 \leq |\xi| \leq \lambda} B(\xi) \lambda^q R_N(\xi/\lambda) d\xi = \int_{0 \leq |\xi| \leq \lambda} B^h(\xi) \lambda^q R_N(\xi/\lambda) d\xi + O(\lambda^{q-N-1})
\]
\[
= \int_{|\xi| \leq 1} \tilde{B}(\xi/|\xi|)|\xi|^h \left( \log \lambda + \log |\xi| \right)^k R_N(\xi) d\xi \lambda^{q+b+n} + O(\lambda^{q-N-1}), \quad \lambda \to \infty.
\]
(5.18)

So the contribution of the “remainder” $R_N$ to the expansion is not small, rather it contributes to the coefficient $c_j$ of the $\lambda^{q+b+n} \log^j \lambda$ term for $0 \leq j \leq k$. Note that so far we have not obtained any contribution to the coefficient $c_{k+1}$.

Such a contribution will show up only now when we finally deal with the summands in the Taylor expansion. Using polar coordinates we find
\[
\int_{1 \leq |\xi| \leq \lambda} B(\xi) Q_j(\xi) d\xi \lambda^{q-j} = \lambda^{q-j} \int_1^\lambda \int_{S^{n-1}} \tilde{B}(\omega) r^b (\log^k r) Q_j(r\omega) r^{n-1} \mathrm{vol}_{S^{n-1}}(\omega) dr d\omega
\]
\[
= C_j \lambda^{q-j} \int_1^\lambda r^{b+n-1+j} \log^k r dr
\]
(5.19)
\[
= C_j \lambda^{q-j} \begin{cases}
\sum_{\sigma=0}^k a_{\sigma} \lambda^{b+n+j} \log^\sigma \lambda + \beta_j, & b + n + j \neq 0,
\frac{1}{k+1} \log^k \lambda, & b + n + j = 0.
\end{cases}
\]

As a side remark note the explicit formula
\[
\int_1^\lambda r^a \log^k r dr = \sum_{j=0}^k \frac{(-1)^j k!}{(k-j)\alpha^{k+1}} \lambda^{a+1} \log^{k-j} \lambda + \frac{(-1)^{k+1} k!}{(\alpha+1)^{k+1}}, \quad \alpha \neq -1,
\]
\[
\frac{1}{k+1} \log^k \lambda, \quad \alpha = -1.
\]
(5.20)

The constant term in (5.20) respectively $\beta_j$ on the rhs of (5.19) was omitted in \cite[Eq. 3.16]{Les99}. Fortunately the error was inconsequential for the formulation of the expansion result because $\beta_j$ is just another contribution to the coefficient $d_j$.

\[\square\]

5.2. Resolvent expansion vs. heat expansion. From the resolvent expansion one can easily derive the heat expansion and the meromorphic continuation of the $\zeta$–function. In fact under a mild additional assumption the resolvent expansion can be derived from the heat expansion of the meromorphic continuation of the $\zeta$–function (cf. e.g. \cite[Theorem 5.1.4 and 5.1.5]{Les97}, \cite[Lemma 2.1 and 2.2]{BrLe99}).

Let $B, P$ be as above. Next let $\gamma$ be a contour in the complex plane as sketched in Figure \ref{fig1}. Then $B e^{-tP}$ has the following contour integral representation:
Figure 1. Contour of integration for calculating $Be^{-tP}$
from the resolvent.

$$Be^{-tP} = -\frac{1}{2\pi i} \int_{\gamma} e^{-t\lambda} B(P - \lambda)^{-1} d\lambda$$

(5.21)

$$= -(-t)^{-N+1}(N-1)! \frac{1}{2\pi i} \int_{\gamma} e^{-t\lambda} B(P - \lambda)^{-N} d\lambda.$$  

Taking the trace on both sides and plugging in the asymptotic expansion
of $\text{Tr}(B(P - \lambda)^{-N})$ one easily finds

(5.22) $\text{Tr}(Be^{-tP}) \sim_{t\to0^+} \sum_{j=0}^{\infty} \sum_{l=0}^{k+1} a_{j,l}(B,P)t^{\frac{j+b-n}{m}} \log^l t + \sum_{j=0}^{\infty} \tilde{d}_j(B,P)t^j.$

$a_{j,k+1} = 0$ if $(j-b-n)/m \notin \mathbb{Z}.$

5.3. Heat expansion vs. $\zeta$–function. Finally we briefly explain how
the meromorphic continuation of the $\zeta$–function can be obtained from the
heat expansion. As before let $B \in CL^{k,k}(M,E)$ and let $P \in CL^{m}(M,E)$ be
an elliptic operator which is self–adjoint with respect to some Riemannian
structure on $M$ and some Hermitian structure on $E.$ Furthermore, assume
that $P \geq 0$ is non–negative. Let $\Pi_{\ker P}$ be the orthogonal projection onto
$\ker P$ and put for $\text{Re } s > 0$

(5.23) $P^{-s} := (I - \Pi_{\ker P})(P + \Pi_{\ker P})^{-s}.$

I.e. $P^{-s} \upharpoonright \ker P = 0$ and for $\xi \in \text{im } P$ we let $P^{-s}\xi$ be the unique $\eta \in \ker P^\perp$ with $P^s\eta = \xi.$ The $\zeta$–function of $(B, P)$ is defined (up to a $\Gamma$–factor) as the Mellin transform of the heat trace $\text{Tr}(B(I - \Pi_{\ker P})e^{-tP}):$

(5.24) $\zeta(B, P; s) = \text{Tr}(BP^{-s})$

$$= \frac{1}{\Gamma(s)} \int_{0}^{\infty} t^{s-1} \text{Tr}(B(I - \Pi_{\ker P})e^{-tP}) dt, \quad \text{Re } s \gg 0.$$
Tr\((B(I - \Pi_{\ker P})e^{-tP})\) decays exponentially as \(t \to \infty\). The meromorphic continuation is thus obtained by plugging the short time asymptotic expansion (5.22) into the rhs of (5.24) (cf. e.g. [Les97], Sec. II.1):

\[
\Gamma(s)\zeta(B, P; s) = \int_0^1 t^{s-1} \text{Tr}(Be^{-tP})dt - \frac{1}{s} \text{Tr}(B\Pi_{\ker P}) + \text{Entire function}(s),
\]

(5.25)

where the formal sum on the right is meant to display the principal parts of the Laurent series at the poles of \(\Gamma(s)\zeta(B, P; s)\).

The \(\Gamma\)-function has simple poles in \(\mathbb{Z}_{-}=\{0, -1, -2, \ldots \}\), hence the \(d'_j\) do not contribute to the poles of \(\zeta(B, P; s)\). The \(a_{jl}\) depend linearly on the \(a_{jl}\) and consequently \(a'_{l,k+1} = 0\) if \((n + b - j)/m\) is not a pole of the \(\Gamma\)-function. Let us summarize:

**Theorem 5.3.** Let \(M\) be a compact closed manifold of dimension \(n\). Let \(B \in \text{CL}^{b,k}(M, E)\) and let \(P \in \text{CL}^m(M, E)\) be an elliptic operator which is self-adjoint with respect to some Riemannian structure on \(M\) and some Hermitian structure on \(E\). Then the \(\zeta\)-function \(\zeta(B, P; s)\) is meromorphic for \(s \in \mathbb{C}\) with poles of order at most \(k+1\) in \((n + b - j)/m\).

6. Regularized traces

### 6.1. The Residue Trace (Noncommutative Residue)

We have seen in Proposition 4.4 that the Hilbert space trace \(\text{Tr}\) cannot be extended to all classical pseudodifferential operators.

However, in his seminal papers [Wod84], [Wod87] M. Wodzicki was able to show that, up to a constant, the algebra \(\text{CL}^*(M)\) has a unique trace which he called the noncommutative residue; we prefer to call it residue trace. The residue trace was independently discovered by V. Guillemin [Gui85] as a byproduct of his axiomatic approach to the Weyl asymptotics. In [Les99] the author generalized the residue trace to the algebra \(\text{CL}^*(M, E)\). Strictly speaking there is no residue trace on the full algebra \(\text{CL}^*(M, E)\). Rather one has to restrict to operators with a given bound on the log degree.

In detail: let \(A \in \text{CL}^{a,k}(M, E)\) and let \(P \in \text{CL}^m(M, E)\) elliptic, non-negative and invertible, cf. Subsection 5.3. Put

\[
\text{Res}_k(A, P) := m^{k+1} \text{Res}_{k+1} \text{Tr}(AP^{-s})|_{s=0} = m^{k+1}(-1)^{k+1}(k + 1)! \times \text{coefficient of } \log^{k+1}t \text{ in the asymptotic expansion of } \text{Tr}(Ae^{-tP}) \text{ as } t \to 0.
\]

(6.1)

In [Les99] it was assumed in addition that the leading symbol of \(P\) is scalar. This assumption allows one to use Duhamel’s principle and to systematically exploit the fact that the order of a commutator \([A, P]\) is at
most ord $A + \text{ord } P - 1$. Using the resolvent approach it was shown in Grubb [Gru05] that for defining $\operatorname{Res}_k$ and to derive its properties one does not need to assume that $P$ has scalar leading symbol.

The main properties of $\operatorname{Res}_k$ can now be summarized as follows:

**Theorem 6.1** (Wodzicki–Guillemin; log–polyhomogeneous case [Les99]).

Let $A \in \text{CL}^{a,k}(M, E)$ and let $P \in \text{CL}^{m}(M, E)$ be elliptic, non–negative and invertible.

1. $\operatorname{Res}_k(A, P) =: \operatorname{Res}_k(A)$ is independent of $P$, i.e.

$$\operatorname{Res}_k : \text{CL}^{\ast, k}(M, E) \rightarrow \mathbb{C}$$

is a linear functional.

2. If $A \in \text{CL}^{a,k}(M, E), B \in \text{CL}^{b,l}(M, E)$ then $\operatorname{Res}_k([A, B]) = 0$. In particular, $\operatorname{Res} := \operatorname{Res}_0$ is a trace on $\text{CL}^{\ast}(M, E)$.

3. For $A \in \text{CL}^{a,k}(M, E)$ the $k$-th residue $\operatorname{Res}_k(A)$ vanishes if $a \notin -\dim M + \mathbb{Z}_+$. 

4. In a local chart one puts

$$\omega_k(A)(x) = \frac{(k + 1)!}{(2\pi)^n} \left( \int_{|\xi| = 1} \text{tr}_{E_x}(a_{-n,k}(x, \xi)) |d\xi| \right) |dx|.$$ 

Then $\omega_k(A) \in \Gamma^{\infty}(M, \Omega)$ is a density (in particular independent of the choice of coordinates), which depends functorially on $A$. Moreover

$$\operatorname{Res}_k(A) = \int_M \omega_k(A).$$

5. If $M$ is connected and $n = \dim M > 1$ then $\operatorname{Res}_k$ induces an isomorphism $\text{CL}^{a,k}(M)/[\text{CL}^{a,k}(M), \text{CL}^{1,0}(M)] \rightarrow \mathbb{C}$. In particular, $\operatorname{Res}$ is up to scalar multiples the only trace on $\text{CL}^{\ast}(M)$.

**Example 6.2.** 1. Let $A$ be a classical pseudodifferential operator of order $-n = - \dim M$ which is assumed to be elliptic, non–negative and invertible. To calculate the residue trace of $A$ we may use $P := A^{-1}$. Thus

$$\operatorname{Res}(A) = n \operatorname{Res} \text{Tr}(A^{1+s})|_{s=0} = n \operatorname{Res} \zeta(A^{-1}; s)|_{s=1} > 0,$$

where $\zeta(A^{-1}; s) = \zeta(I, A^{-1}; s)$ is the $\zeta$–function of the elliptic operator $A^{-1}$. The positivity follows from Eq. (6.2).

2. Let $\Delta$ be the Laplacian on a closed Riemannian manifold $(M, g)$. Then the heat expansion [Gil95] (with $B = I$ and $P = \Delta$) simplifies: since $\Delta$ is a differential operator there are no log terms and by a parity argument every other heat coefficient vanishes [Gil95]. Thus we have an asymptotic expansion

$$\text{Tr}(e^{-t\Delta}) \sim_{t \rightarrow 0} \sum_{j=0}^{\infty} a_j(\Delta)t^{|j-n|/2}, \quad a_{2j+1}(\Delta) = 0.$$ 

The $a_j(\Delta)$ are enumerated such that (6.5) is consistent with (6.22). The first few $a_j(\Delta)$ have been calculated although the computational complexity
increases drastically with \( j \) (cf. e.g. [Gil95]). One has
\[
a_0(\Delta) = c_n \text{vol}(M)
\]
(6.6)
\[
a_2(\Delta) = c'_n \int_M \text{scal}(M, g) d\text{vol}.
\]
The latter is known as the Einstein-Hilbert action in the physics literature. Therefore the following relation between the heat coefficients (and in particular the EH action) and the residue trace has received some attention from the physics community, e.g. Kalau–Walze [KaWa95], Kastler [Kas95]. We find for real \( \alpha \)
\[
\text{Res}(\Delta^\alpha) = 2 \lim_{s \to 0} s \text{Tr}(\Delta^{\alpha-s})
\]
(6.7)
\[
= 2 \lim_{s \to 0} s \zeta(I, \Delta; s - \alpha)
\]
\[
= 2 \lim_{s \to 0} \frac{s}{\Gamma(s - \alpha)} \int_0^1 t^{s-\alpha-1}(\text{Tr}(e^{-t\Delta}) - \dim \ker \Delta) dt
\]
(6.8)
\[
= 2 \sum_{j=0}^\infty \lim_{s \to 0} \frac{a_j(\Delta)s}{\Gamma(s - \alpha)(s - \alpha + \frac{j-n}{2})}
\]
(6.9)
\[
= \begin{cases} 
2a_0(\Delta) \frac{1}{\Gamma(\frac{n}{2})}, & \alpha = \frac{i-n}{2} < 0, \\
0, & \text{otherwise}.
\end{cases}
\]
Here we have used that the \( \zeta \)-function of \( \Delta \) has only simple poles (cf. Theorem 5.3). Furthermore, in (6.7) we use that due to the exponential decay of \( (\text{Tr}(e^{-t\Delta}) - \dim \ker \Delta) \) the function \( s \mapsto \int_0^1 t^{s-\alpha-1}(\text{Tr}(e^{-t\Delta}) - \dim \ker \Delta) dt \) is entire and hence does not contribute to the residue at \( s = 0 \). Furthermore, note that the sum in (6.8) is finite.

In view of (6.6) we have the following special cases of (6.9):
\[
\text{Res}(\Delta^{-n/2}) = 2a_0(\Delta) \frac{1}{\Gamma(\frac{n}{2})} = c_n \text{vol}(M),
\]
(6.10)
\[
\text{Res}(\Delta^{1-n/2}) = c'_n \text{EH}(M, g),
\]
(6.11)
where EH denotes the above mentioned Einstein-Hilbert action. It is formula (6.11) which caused physicists to become enthusiastic about this business. Needless to say, the calculation we present here goes through for any Dirac Laplacian. One only has to replace the scalar curvature in (6.6) by the second local heat coefficient, which can be calculated for any Dirac Laplacian. We wanted to show that the relation between the heat asymptotic and the poles of the \( \zeta \)-function, which is an easy consequence of the Mellin transform, leads to a straightforward proof of (6.11). There also exist “hard” proofs of this fact which check that the local Einstein-Hilbert action coincides with the residue density of the operator \( \Delta^{1-n/2} \) [KaWa95], [Kas95].

6.2. Connes’ Trace Theorem. The famous trace Theorem of Connes gives a relation between the Dixmier trace and the Wodzicki–Guillemin residue trace for pseudodifferential operators of order minus dim \( M \). It was extended by Carey et. al. [CPS03], [CRSS07] to the von Neumann algebra setting.
Theorem 6.3 (Connes’ Trace Theorem [Con88]). Let $M$ be a closed manifold of dimension $n$ and let $E$ be a smooth vector bundle over $M$. Furthermore let $P \in \mathrm{CL}^{-n}(M,E)$ be a pseudodifferential operator of order $-n$. Then $P \in \mathcal{L}^{(1,\infty)}(L^2(M,E))$ and for any $\omega$ satisfying the assumptions of the previous Proposition one has

(6.12) $\text{Tr}_\omega(P) = \frac{1}{n} \text{Res } P.$

We give a sketch of the proof of Connes’ Theorem using a Tauberian argument. This was mentioned without proof in [Con94, Prop. 4.2.4] and has been elaborated in various ways by many authors. The argument we present here is an adaption of an argument in [CPS03] to the type I case.

Let us mention the following simple version of Ikehara’s Tauberian Theorem:

Theorem 6.4 ([Shu01, Sec. II.14]). Let $F : [1,\infty) \to \mathbb{R}$ be an increasing function such that

1. $\zeta_F(s) = \int_1^\infty \lambda^{-s}dF(\lambda)$ is analytic for $\text{Re } s > 1$,
2. $\lim_{s \to 1^+} (s-1)^{-1} \zeta_F(s) = L$.

Then

(6.13) $\lim_{\lambda \to \infty} \frac{F(\lambda)}{\lambda} = L.$

Corollary 6.5. Let $F : [1,\infty) \to \mathbb{R}$ be an increasing function such that

$\int_1^\infty e^{-t\lambda}dF(\lambda) = \frac{L}{t} + O(t^{\varepsilon-1}), t \to 0^+, \text{ for some } \varepsilon > 0.$

Then Ikehara’s Theorem applies to $F$ and (6.13) holds.

Proof. The $\zeta$–function of $F$ satisfies

$$
\zeta_F(s) = \int_1^\infty \lambda^{-s}dF(\lambda)
= \int_1^\infty \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1}e^{-t\lambda}dF(\lambda)
= \int_0^1 \frac{t^{s-1}}{\Gamma(s)} \int_1^\infty e^{-t\lambda}dF(\lambda)dt + \text{holomorphic near } s = 1
\sim \frac{1}{\Gamma(s)} \frac{L}{s-1} \text{ near } s = 1.
$$

Proof of Connes’ Trace Theorem. Each $P \in \mathrm{CL}^{-n}(M,E)$ is a linear combination of at most 4 non–negative operators: to see this we first write $P = \frac{1}{2}(P+P^*) + \frac{1}{2}(P-P^*)$ as a linear combination of two self–adjoint operators. So consider a self–adjoint $P = P^*$. We choose an elliptic operator $Q \in \mathrm{CL}^{-n}(M,E)$ with $Q > 0$ and positive definite leading symbol. Since we are on a compact manifold it then follows that $c \cdot Q - P \geq 0$ for $c$ large enough. Hence $P = c \cdot Q - (c \cdot Q - P)$ is the desired decomposition of $P$ as a difference of non–negative operators.

So it suffices to prove the claim for a non–negative operator $P$. Then $P + \varepsilon Q$ is elliptic and invertible for each $\varepsilon > 0$. By an approximation
argument we are ultimately left with the problem of proving the claim for an elliptic positive operator \( P \in \text{CL}^{-n}(M,E) \).

Let \( \mu_1 \geq \mu_2 \geq \mu_3 \geq \cdots > 0 \) be the eigenvalues of \( P \) counted with multiplicity. We consider the counting function

\[
F(\lambda) = \# \{ j \in \mathbb{N} \mid \mu_j^{-1} \leq \lambda \}.
\]

The associated \( \zeta \)-function

\[
\zeta_F(s) = \int_1^{\infty} \lambda^{-s} dF(\lambda) = \text{Tr}(P^s) - \sum_{\mu_j > 1} \mu_j^s
\]
is, up to the entire function \( \sum_{\mu_j > 1} \mu_j^s \), the \( \zeta \)-function of the elliptic operator \( P^{-1} \). Thus by Theorem 5.3 the function \( \zeta_F \) is holomorphic for \( \text{Re} \, s > 1 \) and it has a meromorphic extension to the complex plane, and 1 is a simple pole with

\[
\lim_{s \to 1} (s - 1) \zeta_F(s) = \frac{1}{n} \text{Res}(P) \neq 0,
\]
cf. Example 6.2 1. Thus Ikehara’s Theorem 6.4 applies to \( F \) and hence

\[
\lim_{\lambda \to \infty} \frac{F(\lambda)}{\lambda} = \frac{1}{n} \text{Res}(P).
\]

**Claim:**

\[
\lim_{j \to \infty} j \mu_j = \frac{1}{n} \text{Res}(P) =: L.
\]

To see this let \( \varepsilon > 0 \) be given. Then there exists a \( \lambda_0 \) such that for \( \lambda \geq \lambda_0 \)

\[
1 - \varepsilon \leq \frac{F(\lambda)}{\lambda L} \leq 1 + \varepsilon.
\]

Thus

\[
\exists \lambda_0 \forall \lambda \geq \lambda_0 \quad (1 - \varepsilon)\lambda L \leq \# \{ j \in \mathbb{N} \mid \mu_j^{-1} \leq \lambda \} \leq (1 + \varepsilon)\lambda L.
\]

Hence for \( j \geq (1 + \varepsilon)\lambda L \) we have \( \mu_j^{-1} \geq \lambda \) and for \( j \leq (1 - \varepsilon)\lambda L \) we have \( \mu_j^{-1} \leq \lambda \). For a given fixed \( j_0 \) large enough we therefore infer

\[
(1 - \varepsilon)L \leq j \mu_j \leq (1 + \varepsilon)L, \quad j \geq j_0,
\]

proving the Claim.

Now consider

\[
\beta(u) = \int_1^{e^u} \lambda^{-1} dF(\lambda) = \sum_{\mu_j \geq e^{-u}} \mu_j.
\]

We check that Ikehara’s Tauberian Theorem applies to \( \beta \):

\[
\int_1^{\infty} e^{-s \lambda} d\beta(\lambda) = \int_1^{\infty} e^{-(s+1)\lambda} dF(e^\lambda)
\]

\[
= \int_1^{\infty} x^{-(s-1)} dF(x) = \zeta_F(1 + s)
\]

\[
= \frac{\text{Res}(P)}{ns} + O(1), \quad s \to 0.
\]
Thus Corollary 6.5 implies

\[
(6.24) \quad \frac{1}{u} \sum_{\mu_j \geq e^{-u}} \mu_j = \frac{\beta(u)}{u} \xrightarrow{u \to \infty} \frac{1}{n} \text{Res}(P).
\]

To infer Connes’ Trace Theorem from (6.24) we choose \( j_0 \) such that (6.21) holds for \( \varepsilon = 1/2 \) and \( j \geq j_0 \). Then put for \( N \) large enough \( u_N := \log \frac{N}{1+\varepsilon} \). Hence we have \( \mu_j \geq \mu_N \geq e^{-u_N} \) for \( 1 \leq j \leq N \) and thus

\[
(6.25) \quad \frac{1}{\log(N+1)} \sum_{j=1}^{N} \mu_j \leq \frac{1}{\log(N+1)} \sum_{\mu_j \geq \text{exp}(-u_N)} \mu_j
\]

\[
= \frac{u_N}{\log N + 1} \sum_{\mu_j \geq \text{exp}(-u_N)} \mu_j \to L, \quad \text{for } N \to \infty,
\]

by (6.24) and since \( u_N/\log(N+1) \to 1 \). This proves

\[
(6.26) \quad \limsup_{N \to \infty} \frac{1}{\log(N+1)} \sum_{j=1}^{N} \mu_j \leq L = \frac{1}{n} \text{Res}(P).
\]

Arguing with \( u_N = \log \frac{N}{(1+\varepsilon)L} \) instead of \( u_N = \log \frac{N}{(1-\varepsilon)L} \) one shows

\[
(6.27) \quad \liminf_{N \to \infty} \frac{1}{\log(N+1)} \sum_{j=1}^{N} \mu_j \geq L = \frac{1}{n} \text{Res}(P),
\]

and Connes’ Trace Theorem is proved. \( \square \)

The attentive reader might have noticed that we did not use the full strength of the Claim (6.18). We only used that there exist positive constants \( c_1, c_2 \) such that \( c_1 \leq j \mu_j \leq c_2 \) for \( j \geq j_0 \).

### 6.3. Parametric case: The symbol valued trace.

In contrast to Proposition 4.4 the situation is entirely different for the algebra of parametric pseudodifferential operators.

Fix a compact smooth manifold \( M \) without boundary of dimension \( n \). Denote the coordinates in \( \mathbb{R}^p \) by \( \mu_1, \ldots, \mu_p \) and let \( \mathbb{C}[\mu_1, \ldots, \mu_p] \) be the algebra of polynomials in \( \mu_1, \ldots, \mu_p \). By a slight abuse of notation we denote by \( \mu_j \) also the operator of multiplication by the \( j \)-th coordinate function. Then we have maps

\[
(6.28) \quad \partial_j : \text{CL}^{m}(M, E; \mathbb{R}^p) \to \text{CL}^{m+1}(M, E; \mathbb{R}^p),
\]

\[
\mu_j : \text{CL}^{m}(M, E; \mathbb{R}^p) \to \text{CL}^{m-1}(M, E; \mathbb{R}^p).
\]

Also \( \partial_j \) and \( \mu_j \) act naturally on the parametric symbols over the one–point space \( \text{CS}^\ast\ast(\mathbb{R}^p) := \text{CS}^\ast\ast(\{\text{pt}\}; \mathbb{R}^p) \) and on polynomials \( \mathbb{C}[\mu_1, \ldots, \mu_p] \). Thus they act on the quotient \( \text{CS}^\ast\ast(\mathbb{R}^p)/\mathbb{C}[\mu_1, \ldots, \mu_p] \). After these preparations we can summarize one of the main results of [LeP00].

Let \( E \) be a smooth vector bundle on \( M \) and consider \( A \in \text{CL}^{m}(M, E; \mathbb{R}^p) \) with \( m+n < 0 \). Then for \( \mu \in \mathbb{R}^p \) the operator \( A(\mu) \) is trace class; hence we may define the function \( \text{TR}(A) : \mu \mapsto \text{Tr}(A(\mu)) \). The map \( \text{TR} \) is obviously...
tracial, i.e. $\text{TR}(AB) = \text{TR}(BA)$, and commutes with $\partial_j$ and $\mu_j$. In fact, the following theorem holds.

**Theorem 6.6.** [LeP00] Theorems 2.2, 4.6 and Lemma 5.1] *There is a unique linear extension*

$$\text{TR} : \text{CL}^\bullet(M, E; \mathbb{R}^p) \to \text{CS}^\bullet\bullet(\mathbb{R}^p)/\mathbb{C}[\mu_1, \ldots, \mu_p]$$

of TR to operators of all orders such that

1. $\text{TR}(AB) = \text{TR}(BA)$, i.e. TR is tracial.
2. $\text{TR}(\partial_j A) = \partial_j \text{TR}(A)$ for $j = 1, \ldots, p$.

This unique extension TR satisfies furthermore:

3. $\text{TR}(\mu_j A) = \mu_j \text{TR}(A)$ for $j = 1, \ldots, p$.
4. $\text{TR}(\text{CL}^m(M, E; \mathbb{R}^p)) \subset \text{CS}^{m+p,1}(\mathbb{R}^p)/\mathbb{C}[\mu_1, \ldots, \mu_p]$.

This Theorem is an example where functions with log–polyhomogeneous expansions occur naturally. Note that although an operator $A \in \text{CL}^m(M, E; \mathbb{R}^p)$ has a homogeneous symbol expansion without log terms the trace function $\text{TR}(A)$ is log–polyhomogeneous.

**Sketch of Proof.** The main observation for the proof is that differentiating by the parameter (6.28) lowers the degree and hence differentiating often enough we obtain a parametric family of trace class operators:

Given $A \in \text{CL}^m(M, E; \mathbb{R}^p)$, then $\partial^\alpha A \in \text{CL}^{m-|\alpha|}(M, E, \mathbb{R}^p)$ is of trace class if $m - |\alpha| + \dim M < 0$. Now integrate the function $\text{TR}(\partial^\alpha A)(\mu)$ back. Since we mod out polynomials this procedure is independent of $\alpha$ and the choice of anti–derivatives. This integration procedure also explains the possible occurrence of log terms in the asymptotic expansion and hence why TR ultimately takes values in $\text{CS}^\bullet\bullet(\mathbb{R}^p)$. For details, see [LeP00, Sec. 4]. □

TR is not a trace in the usual sense since it maps into a quotient space of the space of parametric symbols over a point. However, composing any linear functional on $\text{CS}^\bullet\bullet(\mathbb{R}^p)/\mathbb{C}[\mu_1, \ldots, \mu_p]$ with TR yields a trace on $\text{CL}^\bullet(M, E; \mathbb{R}^p)$. A very natural choice for such a trace is the Hadamard partie finie integral introduced in Subsection 4.2. Let us first note that for a polynomial $P(\mu) \in \mathbb{C}[\mu_1, \ldots, \mu_p]$ of degree $r$ the function

\[
\int_{|\mu| \leq R} P(\mu) d\mu = \sum_{j=p}^{p+r} a_j R^j
\]

is a polynomial of degree $p + r$ without constant term. In particular

\[
\int_{\mathbb{R}^p} P(\mu) d\mu = 0
\]

and hence $\int_{\mathbb{R}^p}$ induces a linear functional on the quotient space $\text{CS}^\bullet\bullet(\mathbb{R}^p)/\mathbb{C}[\mu_1, \ldots, \mu_p]$.

Thus putting for $A \in \text{CL}^\bullet(M, E; \mathbb{R}^p)$

\[
\text{TR}(A) := \int_{\mathbb{R}^p} \text{TR}(A)(\mu) d\mu
\]
we obtain a trace $\tilde{\text{Tr}}$ on $\text{CL}^\bullet(M,E;\mathbb{R}^p)$ which extends the natural trace on operators of order $< - \dim M - p$

\[(6.32) \quad (\int \text{Tr})(A) := \int_{\mathbb{R}^p} \text{Tr}(A(\mu))d\mu.\]

However, since $\int$ is not closed on $\text{CS}^\bullet\bullet(\mathbb{R}^p)$ (Prop. 4.2), $\tilde{\text{Tr}}$ is not closed on $\text{CL}^\bullet(M,E;\mathbb{R})$. Therefore we obtain derived traces

\[(6.33) \quad \partial_j \tilde{\text{Tr}}(A) := \tilde{\text{TR}}_j(A) := \int_{\mathbb{R}^p} \text{TR}(\partial_j A)(\mu)d\mu.\]

The relation between $\text{TR}$ and $\tilde{\text{TR}}_j$ can be explained more elegantly in terms of differential forms on $\mathbb{R}^p$ with coefficients in $\text{CL}^\infty(M,E;\mathbb{R}^p)$. Let $\Lambda^\bullet := \Lambda^\bullet(\mathbb{R}^p)^* = \mathbb{C}[d\mu_1, \ldots, d\mu_p]$ be the exterior algebra of the vector space $(\mathbb{R}^p)^*$ and put

\[(6.34) \quad \Omega_p := \text{CL}^\infty(M,E;\mathbb{R}^p) \otimes \Lambda^\bullet.\]

Then, $\Omega_p$ consists of pseudodifferential operator-valued differential forms, the coefficients of $d\mu_i$ being elements of $\text{CL}^\infty(M,E;\mathbb{R}^p)$.

For a $p$-form $A(\mu)d\mu_1 \wedge \ldots \wedge d\mu_p$ we define the regularized trace by

\[(6.35) \quad \tilde{\text{TR}}(A(\mu)d\mu_1 \wedge \ldots \wedge d\mu_p) := \int_{\mathbb{R}^p} \text{TR}(A)(\mu)d\mu_1 \wedge \ldots \wedge d\mu_p.\]

On forms of degree less than $p$ the regularized trace is defined to be 0. $\tilde{\text{TR}}$ is a graded trace on the differential algebra $(\Omega_p, d)$. In general, $\tilde{\text{TR}}$ is not closed. However, its boundary,

\[\text{TR} := d\tilde{\text{TR}} := \tilde{\text{TR}} \circ d,\]

called the formal trace, is a closed graded trace of degree $p - 1$. It is shown in [LePo00, Prop. 5.8], [Mel95, Prop. 6] that $\text{TR}$ is symbolic, i.e. it descends to a well-defined closed graded trace of degree $p - 1$ on

\[(6.36) \quad \partial \Omega_p := \text{CL}^\infty(M,E;\mathbb{R}^p)/\text{CL}^{-\infty}(M,E;\mathbb{R}^p) \otimes \Lambda^\bullet.\]

The properties of the formal trace $\tilde{\text{TR}}$ resemble those of the residue trace.

Denoting by $r$ the quotient map $\Omega_p \to \partial \Omega_p$ we see that Stokes’ formula with ‘boundary’

\[(6.37) \quad \text{TR}(d\omega) = \tilde{\text{TR}}(r\omega)\]

now holds by construction for any $\omega \in \Omega$.

Finally we mention an interesting linear form on $\text{CS}^\bullet\bullet(\mathbb{R}^p)/\mathbb{C}[\mu_1, \ldots, \mu_p]$ in the spirit of the residue trace. Let

\[(6.38) \quad \Omega^r \text{CS}^\bullet\bullet(\mathbb{R}^p) = \text{CS}^\bullet\bullet(\mathbb{R}^p) \otimes \Lambda^\bullet\]

be the $r$–forms on $\mathbb{R}^p$ with coefficients in $\text{CS}^\bullet\bullet(\mathbb{R}^p)$. We extend the notion of homogeneous functions to differential forms in the obvious way. If $\omega = f d\mu_{i_1} \wedge \cdots \wedge d\mu_{i_r}$ is a form of degree $r$ and $f \in \text{CS}^{a,b}(\mathbb{R}^p)$ then we define the total degree of $\omega$ to be $r + a$. The exterior derivative preserves the
total degree and each \( \omega \in \Omega^a \mathbb{C}^\bullet \mathbb{C}^\bullet (\mathbb{R}^p) \) of total degree \( a \) has an asymptotic expansion

\[
\omega \sim \sum_{j=0}^{\infty} \omega_{a-j}
\]

where \( \omega_{a-j} \) are forms of total degree \( a-j \) which are log–polyhomogeneous in the sense of (3.8), see (3.6). More concretely, if \( f \in \mathbb{C}^\bullet \mathbb{C}^\bullet (\mathbb{R}^p) \), then for \( \omega = f d\mu_1 \wedge \ldots \wedge d\mu_p \) we have

\[
\omega_{a+r-j} = f_{a-j,l}.
\]

Accordingly we define \( \omega_{a+r-j,l} := f_{a-j,l} \).

Finally let \( X = \sum_{j=1}^{p} \mu_j \frac{\partial}{\partial \mu_j} \) be the Liouville vector field on \( \mathbb{R}^p \).

After these preparations we put for \( \omega = f d\mu_1 \wedge \ldots \wedge d\mu_p \in \Omega \mathbb{C}^\bullet \mathbb{C}^\bullet (\mathbb{R}^p) \)

\[
\text{res}(\omega) := \frac{1}{(2\pi)^p} \int_{S^{p-1}} i_X (\omega_0) = \frac{1}{(2\pi)^p} \int_{S^{p-1}} f_{-p,0} d\text{vol}_S.
\]

On forms of degree \( < p \) we put \( \text{res}(\omega) = 0 \).

**Proposition 6.7.** If \( f \in \mathbb{C}[\mu_1, \ldots, \mu_p] \) is a polynomial then

\[
\text{res}(f d\mu_1 \wedge \ldots \wedge d\mu_p) = 0.
\]

If \( \omega \in \Omega A \mathbb{C}^\bullet,0 (\mathbb{R}^p) \) then \( \text{res}(d\omega) = 0 \).

The second statement is due to Manchon, Maeda and Paycha [MMP05].

**Proof.** For \( f \in \mathbb{C}[\mu_1, \ldots, \mu_p] \) the component of homogeneity degree 0 of \( f d\mu_1 \wedge \ldots \wedge d\mu_p \) is obviously 0.

Using Cartan’s identity we have

\[
\text{res}(d\omega) = \int_{S^{p-1}} i_X(d\omega_0) = \int_{S^{p-1}} (i_X d + d i_X)(\omega_0)
\]

(6.42)

\[
= \int_{S^{p-1}} L_X \omega_0 = 0,
\]

since the Lie derivative of a form of homogeneity degree 0 with respect to the Liouville vector field \( X \) is 0.

Composing the res functional with TR we obtain another trace on the algebra \( \mathbb{C}^\bullet \mathbb{C}^\bullet (M, E; \mathbb{R}^p) \) which despite of the previous Proposition is not closed. The point here is that the range of TR is not contained in \( \mathbb{C}^\bullet (\mathbb{R}^p) \) but rather in \( \mathbb{C}^\bullet,1 (\mathbb{R}^p) \).

The significance of this functional and its relation to the noncommutative residue is still to be clarified.

**7. Differential forms whose coefficients are symbol functions**

Proposition 6.7 says that the res functional on \( \Omega \mathbb{C}^\bullet (\mathbb{R}^n) \) descends to a linear functional on the \( n \)-th de Rham cohomology of differential forms with coefficients in \( \mathbb{C}^\bullet (\mathbb{R}^n) \). In Paycha [Pay05] it is shown that the space of linear functionals on \( \mathbb{C}^\bullet (\mathbb{R}^n) \) having the Stokes property is one–dimensional. From this statement in fact the uniqueness of the residue trace can be derived. Translated into our terminology this means that the dual of the \( n \)-th de Rham cohomology group of \( \mathbb{R}^n \) with coefficients in \( \mathbb{C}^\bullet (\mathbb{R}^n) \) is spanned
Definition 7.1. Let \( \mathcal{A} \subset C^\infty([0, \infty)) \) be a Fréchet space with the following properties.

1. \( C^\infty_c([0, \infty)) \subset \mathcal{A} \subset C^\infty([0, \infty)) \) are continuous embeddings. \( C^\infty([0, \infty)) \) carries the usual Fréchet topology of uniform convergence of all derivatives on compact sets and \( C^\infty_c(\mathbb{R}) \) has the standard LF-space topology as inductive limit of the Fréchet spaces \( \{ f \in C^\infty([0, \infty)) \mid \text{supp } f \subset [0, N] \} \) \( N \in \mathbb{N} \).

We denote by \( \mathcal{A}_0 = \{ f \in \mathcal{A} \mid \text{supp } f \subset (0, \infty) \} \).

2. The derivative \( \partial := \frac{d}{dx} \) maps \( \mathcal{A} \) into \( \mathcal{A} \).

3. There is a non–trivial linear functional \( \hat{f} : \mathcal{A} \to \mathbb{C} \) with the following properties:
   
   (a) The restriction of \( \hat{f} \) to \( C^\infty([0, \infty)) \) is a multiple of the integral \( \int_0^\infty \). That is, there is a \( \lambda \in \mathbb{C} \) such that for \( f \in C^\infty([0, \infty)) \)
   we have \( \hat{f} f = \lambda \int_0^\infty f(x)dx \).

   (b) \( \hat{f} \) is closed on \( \mathcal{A}_0 \). That is, for \( f \in \mathcal{A}_0 \) we have \( \hat{f} f = 0 \).

   (c) If \( f \in \mathcal{A}_0 \) and \( \hat{f} f = 0 \) then the function \( F := \int f \in \mathcal{A}_0 \).

Remark 7.2. It follows from (1) that if \( \chi \in C^\infty([0, \infty)) \) with \( \chi(x) = 1, x \geq x_0 \) and \( f \in \mathcal{A} \) then \( \chi f \in \mathcal{A} \) because \( (1-\chi)f \in C^\infty_c([0, \infty)) \subset \mathcal{A} \).

2. Since \( \mathcal{A} \) is Fréchet it follows from (1) and (2) and the Closed Graph Theorem that \( \frac{d}{dx} : \mathcal{A} \to \mathcal{A} \) is continuous.

3. If \( \lambda \) in (3a) is nonzero we can renormalize \( \hat{f} \) such that \( \lambda = 1 \). Thus we are left with two major cases: \( \lambda = 1 \) and \( \lambda = 0 \). In the first case \( \hat{f} \) is a regularization of the ordinary integral while in the second case \( \hat{f} \) is an analogue of the residue trace. This will be explained below in the examples.

Example 7.3. 1. The Schwartz space \( \mathcal{S}(\mathbb{R}) \), \( \hat{f} f = f \).

2. Let \( \text{CS}^a([0, \infty)) \), \( a \in [0, \infty) \) be the classical symbols of order \( a \). This space carries a natural Fréchet topology. If \( a \notin \{-1, 0, 1, \ldots\} \) then let \( f \) be the regularized integral in the partie finie sense described in Subsection 4.2. This integral is continuous with respect to the Fréchet topology on \( \text{CS}^a([0, \infty)) \).

   If \( a \in \{-1, 0, 1, \ldots\} \) then let \( f \) be the residue integral (cf. (5.2)), i.e. if

   \[
   f(x) \sim_{x \to \infty} \sum_{j=0}^{\infty} f_{a-j} x^{a-j}
   \]

   then

   \[
   \hat{f} \int f := f_{-1}.
   \]
One can vary this example. With some care one can also deal with log-polyhomogeneous symbols. Moreover, there are classes of symbols of integral order where the regularized integral has the Stokes property \cite{Pay05}. These “odd class symbols” also fit into the present framework.

From now on \mathcal{A} will always denote a Fréchet space as in Def. 7.1.

Starting from \mathcal{A} we can construct associated spaces of functions on \mathbb{R}^n respectively on cones over a manifold.

Let \( M \) be an oriented compact manifold. By \( \mathcal{A}_0(0, \infty) \times M \) we denote the space of functions \( f \in C^\infty([0, \infty) \times M) \) such that

- There is an \( \varepsilon > 0 \) such that \( f(r, p) = 0 \) for \( r < \varepsilon, p \in M \).
- For fixed \( p \in M \) we have \( f(\cdot, p) \in \mathcal{A} \).

Note that for \( f \in \mathcal{A}_0([0, \infty) \times M) \) the map \( M \to \mathcal{A}, p \mapsto f(\cdot, p) \) is smooth. This follows from the Closed Graph Theorem.

As a consequence we have a continuous integration along the fiber

\[
(7.3) \quad \int_{([0, \infty) \times M)/M} : \mathcal{A}_0([0, \infty) \times M) \longrightarrow C^\infty(M), \quad f \mapsto \int f(\cdot, p).
\]

We put

\[
(7.4) \quad \mathcal{A}_0(\mathbb{R}^n) = \{ \pi^* f \mid f \in \mathcal{A}_0([0, \infty) \times S^{n-1}) \},
\]

where \( \pi : \mathbb{R}^n \setminus \{0\} \longrightarrow [0, \infty) \times S^{n-1}, x \mapsto (\|x\|, x/\|x\|) \) is the polar coordinate diffeomorphism.

Furthermore we put \( \mathcal{A}(\mathbb{R}^n) := C^\infty_0(\mathbb{R}^n) + \mathcal{A}_0(\mathbb{R}^n) \). \( \mathcal{A}_0(\mathbb{R}^n) \) carries a natural LF-topology while \( \mathcal{A}(\mathbb{R}^n) \) carries a natural Fréchet topology.

**Remark 7.4.** Composing the integral (7.3) with an integral over \( M \) yields a natural integral on \( \mathcal{A}_0([0, \infty)) \times M \). In the case of \( M = S^{n-1} \) and the standard integral on \( S^{n-1} \) this integral even extends to an integral on \( \mathcal{A}(\mathbb{R}^n) \) which has the Stokes property. If \( \mathcal{A} = CS^\infty([0, \infty)) \) the so constructed integral on \( \mathcal{A}(\mathbb{R}^n) \) is the Hadamard regularized integral if \( a \not\in \{-1, 0, 1, \ldots\} \) and the residue integral if \( a \in \{-1, 0, 1, \ldots\} \). Thus our approach allows us to discuss these two, a priori rather different, regularized integrals within one common framework.

Finally we denote by \( \Omega^k \mathcal{A}_0([0, \infty) \times M) \) the space of differential forms whose coefficients are locally in \( \mathcal{A}_0([0, \infty) \times U) \) for any chart \( U \subset M \). A more global description in terms of projective tensor products is also possible:

\[
(7.5) \quad \mathcal{A}_0([0, \infty) \times M) = \mathcal{A}_0 \otimes_{\pi} C^\infty(M),
\]

respectively

\[
(7.6) \quad \Omega^k \mathcal{A}_0([0, \infty) \times M) = (\mathcal{A}_0 \oplus \mathcal{A}_0 dr) \otimes_{\pi} \Omega^k(M).
\]

By Def. 7.1 (2) the exterior derivative maps \( \Omega^k \mathcal{A}_0(X) \) to \( \Omega^{k+1} \mathcal{A}_0(X) \) for \( X = [0, \infty) \times M \), respectively \( X = \mathbb{R}^n \). The corresponding cohomology groups are denoted by \( H^k \Omega^\bullet \mathcal{A}_0(X) \). Our goal is to calculate these cohomology groups.

**Definition 7.5.** We call the \( \mathcal{A} \) of type I if \( \lambda \) in Def. 7.1 (3a) is 1 and of type II if \( \lambda \) is 0.
Lemma 7.6. \( \mathcal{A} \) is of type II if and only if the constant function 1 is in \( \mathcal{A} \). Moreover we have for \( k = 0, 1 \)

\[
H^k \mathcal{A}([0, \infty)) \simeq \begin{cases} 
0 & \text{if } \mathcal{A} \text{ is of type I}, \\
\mathbb{C} & \text{if } \mathcal{A} \text{ is of type II}.
\end{cases}
\]

\( H^k \mathcal{A}([0, \infty)) \) (obviously) vanishes for \( k \geq 2 \). Furthermore \( f \) induces an isomorphism \( H^1 \mathcal{A}_0([0, \infty)) \simeq \mathbb{C} \).

7.2. Integration along the fiber and statement of the main result.

7.2.1. Integration along the fiber. The integration \( (7.3) \) extends to an integration along the fiber of differential forms as follows (cf. \[BoTu82\]):

A \( k \)-form \( \omega \in \Omega^k \mathcal{A}_0([0, \infty) \times M) \) is, locally on \( M \), a sum of differential forms of the form

\[
\omega = f_1(r, p)\pi^* \eta_1 + f_2(r, p)\pi^* \eta_2 \wedge dr
\]

with \( f_j \in \mathcal{A}_0([0, \infty) \times M), \eta_1 \in \Omega^k(M), \eta_2 \in \Omega^{k-1}(M) \). For such forms we put

\[
\pi_* \omega := \left( \int_{(0, \infty) \times M} f_2 \right) \pi^* \eta_2.
\]

Lemma 7.7. \( \pi_* \) extends to a well-defined homomorphism

\[
\Omega^k \mathcal{A}_0([0, \infty) \times M) \rightarrow \Omega^{k-1} \mathcal{A}_0([0, \infty) \times M).
\]

Furthermore, \( \pi_* \) commutes with exterior differentiation, i.e.

\[
d_M \circ \pi_* = \pi_* \circ d_{\mathbb{R}^+ \times M}.
\]

For the proof of this Lemma the closedness of \( f \) is crucial.

7.2.2. Statement of the main result. We are now able to state our main result:

Theorem 7.8. Type I: If \( \mathcal{A} \) is of type I then the natural inclusion \( \Omega^n(\mathbb{R}^n) \hookrightarrow \Omega^* \mathcal{A}(\mathbb{R}^n) \) of compactly supported forms induces an isomorphism in cohomology.

Type II: If \( \mathcal{A} \) is of type II then

\[
H^k \mathcal{A}(\mathbb{R}^n) \simeq \begin{cases} 
\mathbb{C} & k = 0, 1, n, \\
0 & \text{otherwise}.
\end{cases}
\]

In both cases \( f \) induces an isomorphism \( H^n \mathcal{A}(\mathbb{R}^n) \rightarrow \mathbb{C} \).

Remark 7.9. 1. The groups \( H^k \mathcal{A}(\mathbb{R}^n) \) can be described more explicitly. Namely, the natural inclusion \( \Omega^* \mathcal{A}_0(\mathbb{R}^n) \hookrightarrow \Omega^* \mathcal{A}(\mathbb{R}^n) \) induces isomorphisms

\[
H^k \mathcal{A}_0(\mathbb{R}^n) \rightarrow H^k \mathcal{A}(\mathbb{R}^n)
\]

for \( k \geq 1 \). Furthermore, integration along the fiber induces isomorphisms

\[
\pi_* : H^k \mathcal{A}_0(\mathbb{R}^n) \rightarrow H^{k-1}(S^{n-1}), \text{ for } k \geq 1.
\]

Thus there is a natural extension of integration along the fiber to closed forms \( \pi_* : \Omega^k \mathcal{A}(\mathbb{R}^n) \rightarrow \Omega^{k-1}(S^{n-1}) \). The isomorphisms \( H^k \mathcal{A}_0(\mathbb{R}^n) \rightarrow \mathbb{C}, \ k = 1, n \) are given by integration along the fiber.
2. This Theorem generalizes the results of [Pay05, Sec. 1] on the characterization of the residue integral and the regularized integral in terms of the Stokes property.

3. The proof of the Theorem is based on the Thom isomorphism below.

7.2.3. The Thom isomorphism. We consider again a Fréchet space $\mathcal{A}$ as in Def. 7.1. Having established integration along the fiber the Thom isomorphism is proved along the lines of the classical case of smooth compactly supported forms. The result is as follows:

**Theorem 7.10.** Let $\mathcal{A}$ be a Fréchet algebra as in Def. 7.1. Let $M$ be a compact oriented manifold of dimension $n$. Furthermore let

$$\pi_* : \Omega^k \mathcal{A}_0([0, \infty) \times M) \to \Omega^{k-1}([0, \infty) \times M)$$

be integration along the fiber as defined in Section 7.2.1. Then $\pi_*$ induces an isomorphism

$$H^k \mathcal{A}_0([0, \infty) \times M) \to H^{k-1}_{dR}(M)$$

for all $k \geq 0$ (meaning $H^0 \mathcal{A}_0([0, \infty) \times M) \simeq \{0\}$.)

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