Unified Bessel, Modified Bessel, Spherical Bessel and Bessel-Clifford Functions

Banu Yılmaz YASAR and Mehmet Ali ÖZARSLAN
Eastern Mediterranean University, Department of Mathematics Gazimagusa, TRNC, Mersin 10, Turkey
E-mail: banu.yilmaz@emu.edu.tr; mehmetali.ozarslan@emu.edu.tr

Abstract
In the present paper, unification of Bessel, modified Bessel, spherical Bessel and Bessel-Clifford functions via the generalized Pochhammer symbol [Srivastava HM, Çetinkaya A, Kıymaz O. A certain generalized Pochhammer symbol and its applications to hypergeometric functions. Applied Mathematics and Computation, 2014, 226 : 484-491] is defined. Several potentially useful properties of the unified family such as generating function, integral representation, Laplace transform and Mellin transform are obtained. Besides, the unified Bessel, modified Bessel, spherical Bessel and Bessel-Clifford functions are given as a series of Bessel functions. Furthermore, the derivatives, recurrence relations and partial differential equation of the so-called unified family are found. Moreover, the Mellin transform of the products of the unified Bessel functions are obtained. Besides, a three-fold integral representation is given for unified Bessel function. Some of the results which are obtained in this paper are new and some of them coincide with the known results in special cases.

Key words Generalized Pochhammer symbol; generalized Bessel function; generalized spherical Bessel function; generalized Bessel-Clifford function

2010 MR Subject Classification 33C10; 33C05; 44A10

1 Introduction

Bessel function first arises in the investigation of a physical problem in Daniel Bernoulli’s analysis of the small oscillations of a uniform heavy flexible chain [26, 37]. It appears when finding separable solutions to Laplace’s equation and the Helmholtz equation in cylindrical or spherical coordinates. Whereas Laplace’s equation governs problems in heat conduction, in the distribution of potential in an electrostatic field and in hydrodynamics in the irrotational motion of an incompressible fluid [26, 29, 31, 37, 40], Helmholtz equation governs problems in acoustic and electromagnetic wave propagation [31, 32, 51]. Besides, Bessel function and modified Bessel function play an important role in the analysis of microwave and optical transmission in waveguides, including coaxial and fiber [33, 39, 53]. Also, Bessel function appears in the inverse problem in wave propagation with applications in medicine, astronomy and acoustic imaging [17]. On the other hand, spherical Bessel function arises in all problems in three dimensions with spherical symmetry involving the scattering of electromagnetic radiation [9, 31, 36]. In quantum mechanics, it comes out in the solution of the Schrödinger wave equation for a particle in a central potential [14]. Therefore, Bessel function is crucially important for many problems of wave propagation and static potentials. When in solving problems in cylindrical coordinate system, one obtains Bessel function of an integer order, in spherical problems one obtains half-integer orders such as electromagnetic waves in a cylindrical waveguide, pressure amplitudes of inviscid rotational flows, heat conduction in a cylindrical object, modes of vibration of a thin circular (or annular) artificial membrane, diffusion problems on a lattice, solutions to the radial Schrödinger equation (in a spherical and cylindrical coordinates) for a free particle, solving for patterns of an acoustical radiation, frequency-dependent friction in circular pipelines, dynamics of floating bodies, angular resolution. Also, it appears in other problems such as signal processing, frequency modulation synthesis, Kaiser window or Bessel filter. While Bessel function has wide range of applications in mathematical physics such as acoustics, radio physics, hydrodynamics, atomic and nuclear physics, Bessel-Clifford function comes out asymptotic expressions for the Dirac delta function [30]. Regular and irregular Coulomb wave functions are expressed in terms of Bessel-Clifford function [14, 12]. Also, it has an applications in quantum mechanics. Because of these facts, several researchers have studied on some extensions and generalizations of Bessel and Bessel-Clifford functions [34, 16, 19, 21, 22, 23, 24, 35, 13, 68]. In that point, special functions play a remarkably important role. Pochhammer symbol or shifted factorial function is one of the most important and useful function in the theory of special functions. Familiar
definition is given by

\[(\alpha)_n = \alpha(\alpha + 1)(\alpha + 2)...(\alpha + n - 1) \] \hspace{1cm} (1.1)

\[(\alpha)_0 = 1, \quad \alpha \neq 0.\]

Also, factorial function \((\alpha)_n\) can be expressed in terms of a ratio of gamma functions as

\[(\alpha)_n := \frac{\Gamma(\alpha + n)}{\Gamma(\alpha)} \] \hspace{1cm} (1.2)

and using the above definition, one can get

\[(\alpha)_{2n} = 2^{2n}(\frac{\alpha}{2})_n(\frac{\alpha + 1}{2})_n \] \hspace{1cm} (1.3)

where

\[\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}dt, \quad \text{Re}(\alpha) > 0 \] \hspace{1cm} (1.4)

is the usual Euler’s Gamma function. Another important function is the generalized hypergeometric function which is defined by means of Pochhammer symbols as

\[_{p}F_{q}(a_1,a_2,...,a_p; b_1,b_2,..., b_q; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k(a_2)_k...(a_p)_k}{(b_1)_k(b_2)_k...(b_q)_k} \frac{z^k}{k!}, \quad (p \leq q). \] \hspace{1cm} (1.5)

Note that, hypergeometric series given by (1.5) converges absolutely for \(p \leq q\). Substituting \(p = 0\) and \(q = 1\), \(a_2 = 1 + \nu\) and replacing \(z\) with \(-\frac{z^2}{4}\), then it is reduced to

\[0F_1(-; 1 + \nu; -\frac{z^2}{4}) = \sum_{k=0}^{\infty} \frac{1}{(1 + \nu)_k} \frac{(-\frac{z^2}{4})^k}{k!}.\]

The case \(p = 1\) and \(q = 1\), it is reduced to the confluent hypergeometric function which is

\[1F_1(a; b; z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k} \frac{z^k}{k!}.\]

Chaudhry et al. [12] extended the confluent hypergeometric function as

\[F_p(a, b; c; z) := \sum_{n=0}^{\infty} \frac{B_p(b + n, c - b)}{B(b, c - b)} (a)_n \frac{z^n}{n!} \]

\(|z| < 1, \quad p \geq 0; \quad \text{Re}(c) > \text{Re}(b) > 0\)

where \((a)_n\) denotes the Pochhammer symbol given by (1.1) and

\[B_p(x, y) := \int_0^1 t^{x-1}(1 - t)^{y-1}e^{-pt} dt, \]

\[(\text{Re}(p) > 0, \quad \text{Re}(x) > 0, \quad \text{Re}(y) > 0)\]

is the extended Euler’s Beta function. In the case \(p = 0\), extended Beta function is reduced to usual Beta function which is defined by

\[B(x, y) = \int_0^1 t^{x-1}(1 - t)^{y-1} dt, \]

\[(\text{Re}(x) > 0, \quad \text{Re}(y) > 0).\]
An extension of the generalized hypergeometric function \(_{r}F_{s}\) of \(r\) numerator parameters \(a_1, \ldots, a_r\) and \(s\) denominator parameters \(b_1, \ldots, b_s\) was defined by Srivastava et al. in [56] as

\[
_{r}F_{s}\left[\begin{array}{c}
(a_1, \rho), \ a_2, \ldots, a_r; \\
(b_1, \ldots, b_s;)
\end{array}\right] := \sum_{n=0}^{\infty} \frac{(a_1, \rho)_n (a_2)_n \cdots (a_r)_n}{(b_1)_n (b_2)_n \cdots (b_s)_n} \frac{z^n}{n!}
\]

where

\(a_j \in \mathbb{C} \ (j = 1, \ldots, r)\) and \(b_j \in \mathbb{C}/\mathbb{Z}_0^+ \ (j = 1, \ldots, s)\), \(\mathbb{Z}_0^- := \{0, -1, -2, \ldots\}\).

In particular, the corresponding extension of the confluent hypergeometric function \(\,_{1}F_{1}\) is given by

\[
\,_{1}F_{1}\left[(a, \rho); c; z\right] := \sum_{n=0}^{\infty} \frac{(a, \rho)_n}{(c)_n} \frac{z^n}{n!}
\]

Note that, in our main Theorems the above extension of the hypergeometric function is used. Another kind of generalized and extended hypergeometric function was introduced in [61] as

\[
u_{F}\left[\begin{array}{c}
(a_0; p, \{K_l\}_{l \in \mathbb{N}_0}), \ a_2, \ldots, a_u; \\
b_1, \ldots, b_v;
\end{array}\right] := \sum_{n=0}^{\infty} \frac{(a_0; p, \{K_l\}_{l \in \mathbb{N}_0})_n (a_2)_n \cdots (a_u)_n}{(b_1)_n (b_2)_n \cdots (b_v)_n} \frac{z^n}{n!}
\]

where

\[
(\lambda; p, \{K_l\}_{l \in \mathbb{N}_0})_\nu := \frac{\Gamma^{(\{K_l\}_{l \in \mathbb{N}_0})}(\lambda + \nu)}{\Gamma^{(\{K_l\}_{l \in \mathbb{N}_0})}(\lambda)}, \lambda, \nu \in \mathbb{C}
\]

is the generalized and the extended Pochhammer symbol of \((\lambda)_\nu\). Here, \(\Gamma^{(\{K_l\}_{l \in \mathbb{N}_0})}(z)\) is the extended Gamma function which is given by

\[
\Gamma^{(\{K_l\}_{l \in \mathbb{N}_0})}(z) := \int_{0}^{\infty} t^{z-1} \odot (\{K_l\}_{l \in \mathbb{N}_0}; -t - \frac{P}{\tau}) \, dt
\]

\(\odot(\{K_l\}_{l \in \mathbb{N}_0}; z)\) is given by [60]

\[
\odot(\{K_l\}_{l \in \mathbb{N}_0}; z) := \left\{ \begin{array}{ll}
\sum_{l=0}^{\infty} K_{l}^{2} \frac{|z|}{M_{0}} \text{exp}(z) [1 + O(\frac{1}{|z|})] (|z| \to \infty; M_{0} > 0; w \in \mathbb{C}).
\end{array} \right.
\]

Besides, multiple Gaussian hypergeometric function was studied in [55]. Moreover, using the extended Pochhammer symbols, some properties of the generalized and extended hypergeometric polynomials were obtained in [62] [63]. On the other hand, generalization of Gamma, Beta and hypergeometric functions were introduced and studied in [47]. Considering the generalized Beta function, generating functions for the Gauss hypergeometric functions were introduced in [55]. Extended incomplete gamma function was obtained in [13]. Finally, using the generalized Pochhammer function, which was introduced by Srivastava et al. [56], generalized Mittag-Leffler function was introduced and some properties were presented in [70].

The main idea of the present paper is to define unification of Bessel, modified Bessel, spherical Bessel and Bessel-Clifford functions by means of the generalized Pochhammer symbol [56]. Before proceeding, some facts related with the Bessel, modified Bessel, spherical Bessel and Bessel-Clifford functions are presented. Bessel and modified Bessel functions can be expressed in terms of the \(\,_{0}F_{1}(-; 1 + \nu; \frac{-z^2}{4})\) and \(\,_{1}F_{1}(a; b; z)\) functions as

\[
J_{\nu}(z) = \frac{(\frac{z}{2})^{\nu}}{\Gamma(\nu + 1)} \sum_{k=0}^{\infty} \frac{(-1)^k (\frac{z^2}{4})^k}{(\nu + 1)_k} k! = \frac{(\frac{z}{2})^{\nu}}{\Gamma(\nu + 1)} \,_{0}F_{1}(-; 1 + \nu; \frac{-z^2}{4}),
\]

\[
I_{\nu}(z) = \frac{(\frac{z}{2})^{\nu}}{\Gamma(\nu + 1)} \sum_{k=0}^{\infty} \frac{1}{(\nu + 1)_k} \frac{(\frac{z^2}{4})^k}{k!} = \frac{(\frac{z}{2})^{\nu}}{\Gamma(\nu + 1)} \,_{0}F_{1}(-; 1 + \nu; \frac{z^2}{4}),
\]

3
and

\[ J_\nu(z) = \frac{(\frac{\pi}{2})^\nu}{\Gamma(\nu + 1)} e^{-iz} \, _1F_1(\nu + \frac{1}{2}; 2\nu + 1; 2iz), \]

\[ I_\nu(z) = \frac{(\frac{\pi}{2})^\nu}{\Gamma(\nu + 1)} e^{-z} \, _1F_1(\nu + \frac{1}{2}; 2\nu + 1; 2z). \]

Recurrence relations satisfied by usual Bessel and modified Bessel functions are given by \[50\]

\[ \frac{2\nu}{z} J_\nu(z) = J_{\nu-1}(z) + J_{\nu+1}(z), \]

\[ 2 \frac{d}{dz} J_\nu(z) = J_{\nu-1}(z) - J_{\nu+1}(z), \]

\[ \frac{2\nu}{z} I_\nu(z) = I_{\nu-1}(z) - I_{\nu+1}(z), \]

\[ 2 \frac{d}{dz} I_\nu(z) = I_{\nu-1}(z) + I_{\nu+1}(z). \]

Besides, spherical Bessel function of the first kind is defined by means of the Bessel function as follows

\[ j_\nu(z) := \sqrt{\frac{\pi}{2z}} J_{\nu + \frac{1}{2}}(z), \quad \text{Re}(\nu) > -\frac{3}{2}. \]

The following recurrence relations are satisfied by the spherical Bessel function \[6\]

\[ j_{\nu-1}(z) + j_{\nu+1}(z) = \frac{2\nu + 1}{z} j_\nu(z), \]

\[ \nu j_{\nu-1}(z) - (\nu + 1) j_{\nu+1}(z) = (2\nu + 1) \frac{d}{dz} j_\nu(z), \]

\[ \frac{d}{dz} [z^{\nu+1} j_\nu(z)] = z^{\nu+1} j_{\nu-1}(z), \]

\[ \frac{d}{dz} [z^{-\nu} j_\nu(z)] = -z^{-\nu} j_{\nu+1}(z), \]

\[ (\nu - 1) j_{\nu-1}(z) - (\nu + 2) j_{\nu+1}(z) = z \left( \frac{d}{dz} j_{\nu-1}(z) + \frac{d}{dz} j_{\nu+1}(z) \right). \]

Moreover, Bessel-Clifford function of the first kind is defined by means of \(_0F_1(-; n + 1; z)\) as

\[ C_\nu(z) := \frac{1}{\Gamma(n + 1)} \, _0F_1(-; n + 1; z) \]

which is a particular case of Wright function

\[ \Phi(\rho, \beta; z) = \sum_{k=0}^{\infty} \frac{z^k}{k! \Gamma(\rho k + \beta)}, \quad \rho > -1 \text{ and } \beta \in \mathbb{C}. \]

The connection between Bessel-Clifford function and modified Bessel function is given by \[7\]

\[ C_\nu(z) = z^{-\frac{\nu}{2}} I_\nu(2\sqrt{z}), \quad \text{Re}(\nu) > 0. \]

Recurrence relations satisfied by Bessel-Clifford function are given by \[27\]

\[ \frac{d}{dz} C_\nu(z) = C_{\nu+1}(z), \]

\[ z C_{\nu+2}(z) + (\nu + 1) C_{\nu+1}(z) = C_\nu(z), \]

\[ z(2\nu + 4) \frac{d}{dz} C_{\nu+1}(z) + 2z^2 \frac{d^2}{dz^2} C_{\nu+1}(z) = 2z \frac{d^2}{dz^2} C_{\nu-1}(z), \]

\[ (\nu + 1) z \frac{d}{dz} C_{\nu+1}(z) + (\nu + 1)^2 C_{\nu+1}(z) = (\nu + 1) \frac{d}{dz} C_{\nu-1}(z). \]
Recently, Srivastava et al. \cite{56} generalized the Pochhammer function as

\[
(\lambda; \rho)_\nu := \begin{cases} 
\frac{\Gamma_\rho(\lambda + \nu)}{\Gamma(\lambda)} \quad (\text{Re}(\rho) > 0; \lambda, \nu \in \mathbb{C}) \\
\frac{\Gamma(\lambda)}{\Gamma(\lambda)} \quad (\rho = 0; \lambda, \nu \in \mathbb{C})
\end{cases}
\]

where \( \Gamma_\rho \) is the extended Gamma function, which was introduced by Chaudhry and Zubair \cite{13} as

\[
\Gamma_\rho(x) = \int_0^\infty t^{x-1}e^{-t}dt, \quad \text{Re}(\rho) > 0
\]

and hence \((\lambda; \rho)_\nu\) is defined by

\[
(\lambda; \rho)_\nu = \frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda+\nu-1}e^{-\frac{\rho}{t}}dt \tag{1.6}
\]

\((\text{Re}(\nu) > 0; \text{Re}(\lambda + \nu) > 0 \text{ when } \rho = 0)\).

It was proved that \cite{56}

\[
(\lambda; \rho)_{\nu+\mu} = (\lambda)_{\nu}(\lambda + \nu; \rho)_\mu, \lambda, \mu, \nu \in \mathbb{C}. \tag{1.7}
\]

In the light of these definitions and generalizations, unification of four parameter Bessel function is defined by

\[
G_{\nu}^{(b,c)}(z; \rho) := \sum_{k=0}^{\infty} \frac{(-b)^k(c; \rho)_{2k+\nu}}{\Gamma(\nu+k+1)\Gamma(\nu+2k+1)} \left(\frac{z}{2}\right)^{2k+\nu}k!, \tag{1.8}
\]

\((z, c, \nu \in \mathbb{C}, \text{Re}(\nu) > -1, \text{Re}(\rho) > 0).\)

The case \(b = 1\) and \(b = -1\), generalized three parameter Bessel function of the first kind \(J_{\nu}^{(c)}(z; \rho)\) and generalized modified three parameter Bessel function of the first kind \(I_{\nu}^{(c)}(z; \rho)\) are introduced by

\[
J_{\nu}^{(c)}(z; \rho) := \sum_{k=0}^{\infty} \frac{(-1)^k(c; \rho)_{2k+\nu}}{\Gamma(\nu+k+1)\Gamma(\nu+2k+1)} \left(\frac{z}{2}\right)^{2k+\nu}k!,
\]

\[
I_{\nu}^{(c)}(z; \rho) := \sum_{k=0}^{\infty} \frac{(c; \rho)_{2k+\nu}}{\Gamma(\nu+k+1)\Gamma(\nu+2k+1)} \left(\frac{z}{2}\right)^{2k+\nu}k!,
\]

\((z, c, \nu \in \mathbb{C}, \text{Re}(\nu) > -1, \text{Re}(\rho) > 0)\)

respectively. Letting \(c = 1\) and \(\rho = 0\), \(J_{\nu}^{(c)}(z; \rho)\) is reduced to usual Bessel function of the first kind \(J_{\nu}(z)\) and \(I_{\nu}^{(c)}(z; \rho)\) is reduced to usual modified Bessel function of the first kind \(I_{\nu}(z)\). Furthermore, generalized four parameter spherical Bessel function is introduced by

\[
g_{\nu}^{(b,c)}(z; \rho) := \sqrt{\frac{\pi}{2z}} G_{\nu+\frac{1}{2}}^{(b,c-\frac{1}{2})}(z; \rho),
\]

\((\text{Re}(\nu) > -\frac{3}{2}, \text{Re}(c) > \frac{1}{2}, \text{Re}(\rho) > 0).\)

The case \(b = 1, c = \frac{3}{2}\) and \(\rho = 0\), it is reduced to usual spherical Bessel function of the first kind \(j_{\nu}(z)\). Moreover, generalized four parameter Bessel-Clifford function of the first kind is defined by

\[
C_{\nu}^{(b,\lambda)}(z; \rho) := z^{\frac{-\lambda}{2}} \tilde{C}^{(b,\lambda)}(2\sqrt{z}; \rho),
\]

\((\text{Re}(\nu) > -1, \text{Re}(\lambda) > 0, \text{Re}(\rho) > 0).\)
The case \( b = -1, \lambda = 1 \) and \( \rho = 0 \), \( C^{(b,\lambda)}_\nu(z; \rho) \) is reduced to usual Bessel-Clifford function of the first kind \( C_\nu(z) \).

The organization of the paper is as follows: In section 2, generating function, integral representation, Laplace transform and Mellin transform involving the unified four parameter Bessel function are obtained. Moreover, the expansion of the unified four parameter Bessel function in terms of a series of usual Bessel functions is presented. In section 3, derivative properties, recurrence relation and partial differential equation of the unified four parameter Bessel function are found. In section 4, the Mellin transforms involving the products of the unified four parameter Bessel function are obtained. In Section 5, a three-fold integral representation formula for the unified four parameter Bessel function is given. In Section 6, generalized four parameter spherical Bessel and Bessel-Clifford functions are introduced by means of the unified four parameter Bessel function and some properties are obtained such as generating function, integral representation, Laplace transform, Mellin transform, series in terms of usual Bessel functions, recurrence relation and partial differential equation. Some of the corresponding results of the mentioned Theorems are new and some of them coincide with the usual terms of usual Bessel functions.

## 2 Unified Four Parameter Bessel Function

In the following Lemma, the relation between \( G^{(b,c)}_{-\nu}(z; \rho) \) and \( G^{(b,c)}_{\nu}(z; \rho) \) is obtained:

**Lemma 2.1** Let \( \nu \in \mathbb{Z} \). Then the following relation is satisfied by the unified four parameter Bessel function

\[
G^{(b,c)}_{-\nu}(z; \rho) = (-b)^\nu G^{(b,c)}_{\nu}(z; \rho).
\]

**Proof.** It is clear that by (1.8), we have

\[
G^{(b,c)}_{\nu}(z; \rho) = \sum_{k=0}^{\infty} \frac{(-b)^k (c; \rho)_{2k+\nu}}{\Gamma(\nu+k+1)\Gamma(\nu+2k+1)} \frac{(\frac{z}{\rho})^{2k+\nu}}{k!}.
\]

Substituting \(-\nu\) instead of \(\nu\) yields

\[
G^{(b,c)}_{-\nu}(z; \rho) = \sum_{k=\nu}^{\infty} \frac{(-b)^k (c; \rho)_{2k-\nu}}{\Gamma(-\nu+k+1)\Gamma(-\nu+2k+1)} \frac{(\frac{z}{\rho})^{2k-\nu}}{k!}.
\]

Taking \( \nu + k \) for \( k \), we get

\[
G^{(b,c)}_{-\nu}(z; \rho) = (-b)^\nu \sum_{k=0}^{\infty} \frac{(-b)^k (c; \rho)_{2k+\nu}}{\Gamma(\nu+2k+1)\Gamma(\nu+k+1)} \frac{(\frac{z}{\rho})^{2k+\nu}}{k!},
\]

which completes the proof. \( \blacksquare \)

Taking \( b = 1 \) and \( b = -1 \) in Lemma 2.1, the relations of the generalized three parameter Bessel and modified Bessel functions of the first kind are obtained, respectively:

**Corollary 2.1.1** Let \( \nu \in \mathbb{Z} \). The following relations are satisfied by the generalized three parameter Bessel and modified Bessel functions of the first kind

\[
J^{(c)}_{-\nu}(z; \rho) = (-1)^\nu J^{(c)}_{\nu}(z; \rho),
\]

\[
I^{(c)}_{-\nu}(z; \rho) = I^{(c)}_{\nu}(z; \rho).
\]

Letting \( c = 1 \) and \( \rho = 0 \) in Corollary 2.1.1, the relations of the usual Bessel functions are given as follows:
Corollary 2.1.2 Let $\nu \in \mathbb{Z}$. The following relations are satisfied by the usual Bessel and modified Bessel functions

$$
J_{-\nu}(z) = (-1)^\nu J_\nu(z),
I_{-\nu}(z) = I_\nu(z).
$$

In the following theorem, the generating function of the unified four parameter Bessel function is given in terms of the generalized confluent hypergeometric function:

**Theorem 2.2** For $t \neq 0$ and for all finite $z$, $n \in \mathbb{Z}$, we have

$$
{}_1F_1((c; \rho), 1; (t - \frac{b}{t}) \frac{z}{2}) = \sum_{n=-\infty}^{\infty} G_n^{(b,c)}(z; \rho)t^n.
$$

**Proof.** It is clear that we have

$$
\sum_{n=-\infty}^{\infty} G_n^{(b,c)}(z; \rho)t^n = \sum_{n=-\infty}^{-1} G_n^{(b,c)}(z; \rho)t^n + \sum_{n=0}^{\infty} G_n^{(b,c)}(z; \rho)t^n.
$$

Taking $-n - 1$ instead of $n$ in the first summation of the right hand side, we have

$$
\sum_{n=-\infty}^{\infty} G_n^{(b,c)}(z; \rho)t^n = \sum_{n=0}^{\infty} G_{n-1}^{(b,c)}(z; \rho)t^{-n-1} + \sum_{n=0}^{\infty} G_n^{(b,c)}(z; \rho)t^n.
$$

Now, using Lemma 2.1, we get

$$
\sum_{n=-\infty}^{\infty} G_n^{(b,c)}(z; \rho)t^n = \sum_{n=0}^{\infty} (-b)^{n+1} G_{n+1}^{(b,c)}(z; \rho)t^{-n-1} + \sum_{n=0}^{\infty} G_n^{(b,c)}(z; \rho)t^n.
$$

Plugging the series definitions of the unified Bessel function into right hand side, we have

$$
\sum_{n=-\infty}^{\infty} G_n^{(b,c)}(z; \rho)t^n = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-b)^n (c; \rho)^k}{(n + k + 1)!k!(n + k + 1)!} \left(\frac{z}{2}\right)^{2k+n+1} t^{-n-1}
$$

$$
+ \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-b)^{k+n}}{(n + k)!k!(n + 2k)!} \left(\frac{z}{2}\right)^{2k+n} t^n.
$$

Letting $n - 2k$ instead of $n$, we get

$$
\sum_{n=-\infty}^{\infty} G_n^{(b,c)}(z; \rho)t^n = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-b)^{n-k+1} (c; \rho)^{n+1}}{(n + k + 1)!k!(n + 1)!} \left(\frac{z}{2}\right)^{n+1} t^{-n+2k-1}
$$

$$
+ \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-b)^{k+n}}{(n + k)!k!n!} \left(\frac{z}{2}\right)^{n} t^n - 2k.
$$

Taking $n - 1$ instead of $n$ in the first summation of the right side, we have

$$
\sum_{n=-\infty}^{\infty} G_n^{(b,c)}(z; \rho)t^n = \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{(-b)^{n-k} (c; \rho)^n}{(n - k)!k!n!} \left(\frac{z}{2}\right)^{n} t^{-n+2k}
$$

$$
+ (c; \rho) + \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{(-b)^{k+n}}{(n - k)!k!n!} \left(\frac{z}{2}\right)^{n} t^{n-2k}.
$$
Taking into consideration of the following fact, which was proved in [50], (Lemma 12, page 112-113)

\[
\sum_{n=1}^{\infty} \sum_{k=0}^{n-1} A(n-k, n) + \sum_{n=1}^{\infty} \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} A(k, n) = \sum_{n=1}^{\infty} \sum_{k=0}^{n} A(k, n)
\]

we have

\[
\sum_{n=-\infty}^{\infty} G_n^{(b,c)}(z; \rho)t^n = (c; \rho)_0 + \sum_{n=1}^{\infty} \sum_{k=0}^{n} \frac{(-b)^k(c; \rho)_n}{(n-k)!k!n!} \left(\frac{z}{2}\right)^{n-2k}.
\]

Substituting the expansion of \((t - \frac{b}{t})^n\), we get

\[
\sum_{n=-\infty}^{\infty} G_n^{(b,c)}(z; \rho)t^n = \sum_{n=0}^{\infty} \frac{(c; \rho)_n (t - \frac{b}{t})^n (\frac{z}{2})^n}{n!},
\]

\[
= _1 F_1((c; \rho)_1; (t - \frac{b}{t}) \frac{z}{2}).
\]

Substituting \(b = 1\) and \(b = -1\) in Theorem 2.2, the generating functions of the generalized three parameter Bessel and modified Bessel functions of the first kind are obtained, respectively:

**Corollary 2.2.1** For \(t \neq 0\) and for all finite \(z\) and \(n \in \mathbb{Z}\), the generating functions of the generalized three parameter Bessel and modified Bessel functions of the first kind are given by

\[
_1 F_1((c; \rho); 1; (t - \frac{b}{t}) \frac{z}{2}) = \sum_{n=-\infty}^{\infty} J_n^{(c)}(z; \rho)t^n,
\]

\[
_1 F_1((c; \rho); 1; (t + \frac{b}{t}) \frac{z}{2}) = \sum_{n=-\infty}^{\infty} I_n^{(c)}(z; \rho)t^n.
\]

Taking \(c = 1\) and \(\rho = 0\) in Corollary 2.2.1 and considering the facts \(J_n^{(1)}(z; 0) = J_n(z)\), \(I_n^{(1)}(z; 0) = I_n(z)\), the following Corollary is obtained:

**Corollary 2.2.2** For \(t \neq 0\) and for all finite \(z\) and \(n \in \mathbb{Z}\), generating functions of the usual Bessel and modified Bessel functions are given by

\[
e^{\frac{z}{2}(t - \frac{1}{t})} = \sum_{n=-\infty}^{\infty} J_n(z)t^n,
\]

\[
e^{\frac{z}{2}(t + \frac{1}{t})} = \sum_{n=-\infty}^{\infty} I_n(z)t^n.
\]

In the following theorem, the integral representation of the unified four parameter Bessel function is presented:

**Theorem 2.3** The integral formula satisfied by the unified four parameter Bessel function is given by

\[
G_\nu^{(b,c)}(z; \rho) = \frac{\left(\frac{z}{2}\right)^\nu}{[\Gamma(\nu + 1)]^2 \Gamma(c)} \int_{0}^{\infty} t^{\nu+1} e^{-t-\frac{z}{t}} _0 F_3(-; \nu + 1, \nu + \frac{1}{2}, \nu + \frac{2}{2}; -\frac{b^2t^2}{16})dt \quad (2.2)
\]

where \(\text{Re}(c) > 0\) and \(\text{Re}(\nu) > -1\).
Proof. Applying the generalized Pochhammer expansion into (1.8) and using the usual Pochhammer function expansion (1.2), we have

\[
G^{(b,c)}_{\nu}(z;\rho) = \frac{1}{\Gamma(\nu + 1)^2} \sum_{k=0}^{\infty} \frac{(-b)^k}{(\nu + 1)_k(\nu + 1)_{2k} k!} \left( \frac{z}{2} \right)^{2k+\nu} \frac{1}{\Gamma(c)} \int_0^\infty t^{e^{2k+\nu-1}e^{-\frac{t}{\rho}} dt}.
\]

Interchanging the order of summation and integral under the conditions \(\text{Re}(c) > 0\) and \(\text{Re}(\nu) > -1\), we have

\[
G^{(b,c)}_{\nu}(z;\rho) = \frac{(\frac{z}{2})^\nu}{\Gamma(\nu + 1)^2\Gamma(c)} \int_0^\infty \frac{1}{\sum_{k=0}^{\infty} (\nu + 1)_k(\nu + 1)_{2k} k!} \left( -\frac{bz^2t^2}{4} \right)^k \int_0^\infty t^{e^{2k+\nu-1}e^{-\frac{t}{\rho}} dt}.
\]

Letting \(\nu + 1\) in place of \(\alpha\) and taking \(k\) for \(n\) in the duplication formula (1.3), we have

\[
G^{(b,c)}_{\nu}(z;\rho) = \frac{(\frac{z}{2})^\nu}{\Gamma(\nu + 1)^2\Gamma(c)} \int_0^\infty \frac{1}{\sum_{k=0}^{\infty} (\nu + 1)_k(\nu + 1)_{2k} k!} \left( -\frac{bz^2t^2}{16} \right)^k \int_0^\infty t^{e^{2k+\nu-1}e^{-\frac{t}{\rho}} dt}.
\]

Taking into consideration of (1.5), we have

\[
G^{(b,c)}_{\nu}(z;\rho) = \frac{(\frac{z}{2})^\nu}{\Gamma(\nu + 1)^2\Gamma(c)} \int_0^\infty t^{e^{2k+\nu-1}e^{-\frac{t}{\rho}} 0 F_3(-;\nu + 1, \frac{\nu + 1}{2}, \frac{\nu + 2}{2}; -\frac{bz^2t^2}{16}) dt}.
\]

\[\blacksquare\]

Letting \(b = 1\) and \(b = -1\) in Theorem 2.3, the integral representations of the generalized three parameter Bessel and modified Bessel functions of the first kind are found, respectively:

**Corollary 2.3.1** Integral representations satisfied by the generalized three parameter Bessel and modified Bessel functions of the first kind are given by

\[
J^{(c)}_{\nu}(z;\rho) = \frac{(\frac{z}{2})^\nu}{\Gamma(\nu + 1)^2\Gamma(c)} \int_0^\infty t^{e^{2k+\nu-1}e^{-\frac{t}{\rho}} 0 F_3(-;\nu + 1, \frac{\nu + 1}{2}, \frac{\nu + 2}{2}; -\frac{z^2t^2}{16}) dt},
\]

\[
I^{(c)}_{\nu}(z;\rho) = \frac{(\frac{z}{2})^\nu}{\Gamma(\nu + 1)^2\Gamma(c)} \int_0^\infty t^{e^{2k+\nu-1}e^{-\frac{t}{\rho}} 0 F_3(-;\nu + 1, \frac{\nu + 1}{2}, \frac{\nu + 2}{2}; \frac{z^2t^2}{16}) dt},
\]

where \(\text{Re}(c) > 0\) and \(\text{Re}(\nu) > -1\).

Substituting \(c = 1\) and \(\rho = 0\) in Corollary 2.3.1, the following Corollary is obtained:

**Corollary 2.3.2** Integral formulas satisfied by the usual Bessel and modified Bessel functions are given by

\[
J_{\nu}(z) = \frac{(\frac{z}{2})^\nu}{\Gamma(\nu + 1)^2} \int_0^\infty t^{e^{2k+\nu-1}e^{-t}} 0 F_3(-;\nu + 1, \frac{\nu + 1}{2}, \frac{\nu + 2}{2}; -\frac{z^2t^2}{16}) dt,
\]

\[
I_{\nu}(z) = \frac{(\frac{z}{2})^\nu}{\Gamma(\nu + 1)^2} \int_0^\infty t^{e^{2k+\nu-1}e^{-t}} 0 F_3(-;\nu + 1, \frac{\nu + 1}{2}, \frac{\nu + 2}{2}; \frac{z^2t^2}{16}) dt,
\]

where \(\text{Re}(\nu) > -1\).

Note that, some integrals involving usual Bessel functions were obtained in [4, 11, 15, 25, 49]. Taking \(t = \frac{u}{1-a}\) in Theorem 2.3, the following integral representation is found:
Corollary 2.3.3  Integral representation satisfied by the unified four parameter Bessel function is given by

\[ G_{\nu}^{(b,c)}(z; \rho) = \frac{(\frac{z}{2})^\nu}{\Gamma(\nu + 1)^2 \Gamma(c)} \int_0^1 u^{c+\nu - 1} (1 - u)^{-c-\nu-1} e^{-u^2 \rho (1-u)^2} u(1-u) \]

\[ \times _0F_3(-; \nu + 1, \frac{\nu + 1}{2}, \frac{\nu + 2}{2}, \frac{-b z^2 u^2}{(1-u)^2}; 16) du \]

where \( \operatorname{Re}(\nu) > -1 \) and \( \operatorname{Re}(c) > 0 \).

Theorem 2.4  The Laplace transform of the unified four parameter Bessel function is given by

\[ \mathcal{L}\{G_{\nu}^{(b,c)}(t; \rho)\}(s) = \frac{1}{s} \sum_{k=0}^\infty \frac{(-b)^k (c; \rho)_{2k+\nu}}{k! \Gamma(\nu + k + 1)} \left( \frac{1}{2s} \right)^{2k+\nu} \]

where \( \operatorname{Re}(\nu) > -1 \) and \( \operatorname{Re}(s) > 0 \).

Proof.  Laplace transform is given by

\[ \mathcal{L}\{G_{\nu}^{(b,c)}(t; \rho)\}(s) = \int_0^\infty G_{\nu}^{(b,c)}(t; \rho) e^{-st} dt. \]

Substituting the series form of the unified four parameter Bessel function, we have

\[ \mathcal{L}\{G_{\nu}^{(b,c)}(t; \rho)\}(s) = \int_0^\infty \left\{ \sum_{k=0}^\infty \frac{(-b)^k (c; \rho)_{2k+\nu}}{k! \Gamma(\nu + k + 1)) \left( \frac{t}{2} \right)^{2k+\nu} e^{-st} dt. \right\} \]

Replacing the order of summation and integral by the conditions \( \operatorname{Re}(\nu) > 0 \), \( \operatorname{Re}(s) > 0 \) and \( \operatorname{Re}(\nu) > -1 \), we get

\[ \mathcal{L}\{G_{\nu}^{(b,c)}(t; \rho)\}(s) = \sum_{k=0}^\infty \frac{(-b)^k (c; \rho)_{2k+\nu}}{k! \Gamma(\nu + k + 1)) \Gamma(\nu + 2k + 1)) \left( \frac{1}{2} \right)^{2k+\nu} \int_0^\infty t^{2k+\nu} e^{-st} dt. \]

Making the substitution \( st = u \) and using the Gamma functions for the above integral, we get

\[ \mathcal{L}\{G_{\nu}^{(b,c)}(t; \rho)\}(s) = \sum_{k=0}^\infty \frac{(-b)^k (c; \rho)_{2k+\nu}}{k! \Gamma(\nu + k + 1)) \Gamma(\nu + 2k + 1)) 2^{2k+\nu} \frac{\Gamma(\nu + 2k + 1))}{s^{2k+\nu+1}}. \]

Finally, we have

\[ \mathcal{L}\{G_{\nu}^{(b,c)}(t; \rho)\}(s) = \frac{1}{s} \sum_{k=0}^\infty \frac{(-b)^k (c; \rho)_{2k+\nu}}{k! \Gamma(\nu + k + 1)) \left( \frac{1}{2s} \right)^{2k+\nu}. \]

The case \( b = 1 \), \( G_{\nu}^{(b,c)}(t; \rho) \) is reduced to \( J_{\nu}^{(c)}(t; \rho) \) and when \( b = -1 \), \( G_{\nu}^{(b,c)}(t; \rho) \) is reduced to \( I_{\nu}^{(c)}(t; \rho) \). Therefore, the following Corollary is obtained:

Corollary 2.4.1  The Laplace transforms of the generalized three parameter Bessel and modified Bessel functions of the first kind are given by

\[ \mathcal{L}\{J_{\nu}^{(c)}(t; \rho)\}(s) = \frac{1}{s} \sum_{k=0}^\infty \frac{(-1)^k (c; \rho)_{2k+\nu}}{k! \Gamma(\nu + k + 1)) \left( \frac{1}{2s} \right)^{2k+\nu}, \]

\[ \mathcal{L}\{I_{\nu}^{(c)}(t; \rho)\}(s) = \frac{1}{s} \sum_{k=0}^\infty \frac{(c; \rho)_{2k+\nu}}{k! \Gamma(\nu + k + 1)) \left( \frac{1}{2s} \right)^{2k+\nu}, \]

where \( \operatorname{Re}(\nu) > -1 \) and \( \operatorname{Re}(s) > 0 \).
Since $J_{\nu}^{(c)}(z;\rho)$ is reduced to $J_{\nu}(z)$ and $I_{\nu}^{(c)}(z;\rho)$ is reduced to $I_{\nu}(z)$ when $c = 1$ and $\rho = 0$, the following Corollary is presented for usual Bessel and modified Bessel functions:

**Corollary 2.4.2** The Laplace transforms of the usual Bessel and modified Bessel functions are given by

\[
\mathcal{L}\{J_{\nu}(t)\}(s) = \frac{1}{s} \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(2k + \nu + 1)}{k! \Gamma(k + 1)} \left(\frac{1}{2s}\right)^{2k+\nu},
\]

\[
\mathcal{L}\{I_{\nu}(t)\}(s) = \frac{1}{s} \sum_{k=0}^{\infty} \frac{\Gamma(2k + \nu + 1)}{k! \Gamma(k + 1)} \left(\frac{1}{2s}\right)^{2k+\nu},
\]

where $\text{Re}(s) > 0$ and $\text{Re}(\nu) > -1$.

Taking $\nu = 0$ in Corollary 2.4.2 and using the fact that

\[
(1 - z)^{-\alpha} = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} z^n
\]

and applying

\[
(2k)! = 2^{2k} \left(\frac{1}{2}\right)_k k!,
\]

Laplace transform of the $J_0(t)$ and $I_0(t)$ are found.

**Corollary 2.4.3** The Laplace transforms of the $J_0(t)$ and $I_0(t)$ are given by

\[
\mathcal{L}\{J_0(t)\}(s) = \frac{1}{\sqrt{s^2 + 1}}, \quad \text{Re}(s) > 0,
\]

\[
\mathcal{L}\{I_0(t)\}(s) = \frac{1}{\sqrt{s^2 - 1}}, \quad \text{Re}(s) > 0.
\]

Note that the Laplace transform of the modified Bessel functions of the second kind was obtained in [54].

**Theorem 2.5** Mellin transform involving the unified four parameter Bessel function is given by

\[
\mathcal{M}\{e^{-z}G_{\nu}^{(b,c)}(z;\rho); s\}
\]

\[
= \frac{1}{2^\nu \Gamma(\nu + 1)^2} \sum_{k=0}^{\infty} \frac{(-b)_k (c;\rho)_{\nu+2k}}{(\nu + 1)_k (\frac{\nu+1}{2})_k (\frac{\nu+2}{2})_k} \frac{\Gamma(\nu + 2k)}{16^k k!},
\]

where $\text{Re}(\nu) < 1$ and $s + \nu \neq \{0, -1, -2, \ldots\}$.

**Proof.** The Mellin transform is given by

\[
\mathcal{M}\{e^{-z}G_{\nu}^{(b,c)}(z;\rho); s\} = \int_0^\infty e^{-z} z^{s-1} G_{\nu}^{(b,c)}(z;\rho) dz.
\]

Substituting equation (2.2) instead of $G_{\nu}^{(b,c)}(z;\rho)$, we have

\[
\mathcal{M}\{e^{-z}G_{\nu}^{(b,c)}(z;\rho); s\} = \int_0^{\infty} e^{-z} z^{s-1} t^{\nu} \frac{\sqrt{t}}{2^\nu \Gamma(\nu + 1)^2 \Gamma(c)} \int_0^\infty e^{c+\nu-\frac{\sqrt{t}}{2} - \frac{\sqrt{t}}{2}} 0F_3\left(-; \nu + 1, \frac{\nu + 1}{2}, \frac{\nu + 2}{2}; -\frac{b^2 t^2}{16}\right) dt dz.
\]
Inserting the series forms of hypergeometric function, we have

\[ M\{e^{-z}G^{(b,c)}_{\nu}(z; \rho); s\} = \int_0^\infty \int_0^\infty \frac{z^{\nu+1-\nu-1}e^{-z}}{2^{\nu}(\Gamma(\nu+1))^2(\Gamma(c))} \left( \sum_{k=0}^\infty \frac{(-1)^kb^kz^{2k}}{(\nu+1)k(k(\nu+1)k)} \right) \left( \int_0^\infty \frac{e^{s+2\nu-1}e^{-t-\frac{z}{s}}}{\Gamma(c)} \right) dt dz. \]

Interchanging the order of integral and summation which is satisfied under the conditions of the Theorem, we get

\[ M\{e^{-z}G^{(b,c)}_{\nu}(z; \rho); s\} = \frac{1}{2^{\nu}(\Gamma(\nu+1))^2(\Gamma(c))} \sum_{k=0}^\infty \frac{(-1)^kb^kz^{2k}}{(\nu+1)k(k(\nu+1)k)} \Gamma(s+2\nu-1)\left( \int_0^\infty \frac{e^{s+2\nu-1}e^{-t-\frac{z}{s}}}{\Gamma(c)} dt dz \right). \]

Taking into consideration of the generalized Pochhammer function expansion \((c; \rho)_{\nu+2k}\), we get

\[ M\{e^{-z}G^{(b,c)}_{\nu}(z; \rho); s\} = \frac{1}{2^{\nu}(\Gamma(\nu+1))^2(\Gamma(c))} \sum_{k=0}^\infty \frac{(-1)^kb^kz^{2k+s+\nu-1}e^{-z}}{(\nu+1)k(k(\nu+1)k)} \Gamma(s+2\nu-1)\left( \int_0^\infty \frac{e^{s+2\nu-1}e^{-t-\frac{z}{s}}}{\Gamma(c)} dt dz \right). \]

Changing the order of integral and summation under the conditions \(\text{Re}(\nu) > -1\), \(s+\nu \neq \{0, -1, -2, ...\}\) and \(\text{Re}(\rho) > 0\), we get

\[ M\{e^{-z}G^{(b,c)}_{\nu}(z; \rho); s\} = \frac{1}{2^{\nu}(\Gamma(\nu+1))^2(\Gamma(c))} \sum_{k=0}^\infty \frac{(-1)^kb^kz^{2k+s+\nu-1}e^{-z}}{(\nu+1)k(k(\nu+1)k)} \Gamma(s+2\nu-1)\left( \int_0^\infty \frac{e^{s+2\nu-1}e^{-t-\frac{z}{s}}}{\Gamma(c)} dt dz \right). \]

Using the definition of the Gamma function and taking into consideration of (1.2), we have

\[ M\{e^{-z}G^{(b,c)}_{\nu}(z; \rho); s\} = \frac{1}{2^{\nu}(\Gamma(\nu+1))^2(\Gamma(c))} \sum_{k=0}^\infty \frac{(-1)^kb^kz^{2k+s+\nu-1}e^{-z}}{(\nu+1)k(k(\nu+1)k)} \Gamma(s+2\nu-1)\left( \int_0^\infty \frac{e^{s+2\nu-1}e^{-t-\frac{z}{s}}}{\Gamma(c)} dt dz \right). \]

The Mellin transforms of the generalized three parameter Bessel and modified Bessel functions of the first kind can be obtained by substituting \(b = 1\) and \(b = -1\) in Theorem 2.5, respectively. Thus, the following result is clear:

**Corollary 2.5.1** The Mellin transforms involving the generalized three parameter Bessel and modified Bessel functions of the first kind are given by

\[ M\{e^{-z}J^{(c)}_{\nu}(z; \rho); s\} = \frac{1}{2^{\nu}(\Gamma(\nu+1))^2(\Gamma(c))} \sum_{k=0}^\infty \frac{(-1)^kb^kz^{2k}}{(\nu+1)k(k(\nu+1)k)} \Gamma(s+2\nu-1)\left( \int_0^\infty \frac{e^{s+2\nu-1}e^{-t-\frac{z}{s}}}{\Gamma(c)} dt dz \right). \]

\[ M\{e^{-z}I^{(c)}_{\nu}(z; \rho); s\} = \frac{1}{2^{\nu}(\Gamma(\nu+1))^2(\Gamma(c))} \sum_{k=0}^\infty \frac{(-1)^kb^kz^{2k}}{(\nu+1)k(k(\nu+1)k)} \Gamma(s+2\nu-1)\left( \int_0^\infty \frac{e^{s+2\nu-1}e^{-t-\frac{z}{s}}}{\Gamma(c)} dt dz \right). \]

where \(\text{Re}(\nu) > -1\), \(\text{Re}(\rho) > 0\) and \(s+\nu \neq \{0, -1, -2, ...\}\).

Substituting \(c = 1\) and \(\rho = 0\) in Corollary 2.5.1, the Mellin transforms involving the usual Bessel and modified Bessel functions are obtained. Thus, the following Corollary is given:
Corollary 2.5.2  
Mellin transforms involving the usual Bessel and modified Bessel functions are given by

\[ \mathcal{M}\{e^{-z}J_\nu(z); s\} = \frac{1}{2^\nu} \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(s + \nu + 2k)}{\Gamma(\nu + k + 1)2^{2k}k!}, \]
\[ \mathcal{M}\{e^{-z}I_\nu(z); s\} = \frac{1}{2^\nu} \sum_{k=0}^{\infty} \frac{\Gamma(s + \nu + 2k)}{\Gamma(\nu + k + 1)2^{2k}k!}, \]

where \( \text{Re}(\nu) > -1 \) and \( s + \nu \neq \{0, -1, -2, \ldots\} \).

Letting \( \nu = 0 \) in Corollary 2.5.2 and using the fact that

\[ (s)_{2k} = 2^{2k} \left( \frac{s}{2} \right)_k \left( \frac{s + 1}{2} \right)_k, \]

the following result is obtained:

Corollary 2.5.3  
The Mellin transforms involving the usual Bessel function \( e^{-z}J_0(z) \) and modified Bessel function \( e^{-z}I_0(z) \) are given by

\[ \mathcal{M}\{e^{-z}J_0(z); s\} = \Gamma(s) \ {}_2F_1\left( \frac{s}{2}, -\frac{s}{2}; 1; -1 \right), \]
\[ \mathcal{M}\{e^{-z}I_0(z); s\} = \Gamma(s) \ {}_2F_1\left( \frac{s}{2}, -\frac{s}{2}; 1; 1 \right) \]

where \( \text{Re}(s) > 0 \).

Note that the Mellin transforms of products of Bessel and generalized hypergeometric functions were obtained in [46].

Corollary 2.5.4  
Taking \( s = 1 \) in Theorem 2.5, the following integral formula is obtained

\[ \int_0^\infty e^{-z}G^{(b,c)}_\nu(z; \rho)dz = \frac{1}{2^\nu |\Gamma(\nu + 1)|^2} \sum_{k=0}^{\infty} \frac{(-b)^k (c; \rho)_{\nu+2k} \Gamma(\nu + 2k + 1)}{(\nu + 1)_k (\nu + 1)_k (\nu + 1)_k 16^k k!} \]

where \( \text{Re}(\nu) > -1 \) and \( \text{Re}(c) > 0 \).

Taking \( s = 1 \) in Theorem 2.5 and substituting \( b = 1 \) and \( b = -1 \) respectively, the following Corollaries are presented:

Corollary 2.5.5  
Infinite integrals involving the generalized three parameter Bessel and modified Bessel functions of the first kind are given by

\[ \int_0^\infty e^{-z}J^{(c)}_\nu(z; \rho)dz = \frac{1}{2^\nu |\Gamma(\nu + 1)|^2} \sum_{k=0}^{\infty} \frac{(-1)^k (c; \rho)_{\nu+2k} \Gamma(\nu + 2k + 1)}{(\nu + 1)_k (\nu + 1)_k (\nu + 1)_k 16^k k!}, \]
\[ \int_0^\infty e^{-z}I^{(c)}_\nu(z; \rho)dz = \frac{1}{2^\nu |\Gamma(\nu + 1)|^2} \sum_{k=0}^{\infty} \frac{(c; \rho)_{\nu+2k} \Gamma(\nu + 2k + 1)}{(\nu + 1)_k (\nu + 1)_k (\nu + 1)_k 16^k k!}, \]

where \( \text{Re}(\nu) > -1 \) and \( \text{Re}(c) > 0 \).

Integrals involving usual Bessel and modified Bessel functions are given in the following Corollary:
Corollary 2.5.6  Infinite integrals involving the usual Bessel and modified Bessel functions are given by

\[
\begin{align*}
\int_0^\infty e^{-z}J_\nu(z)dz &= \frac{1}{2^{\nu}[\Gamma(\nu + 1)]^2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(\nu + 1)_k (\nu + 2)_k} 16^k k!, \\
\int_0^\infty e^{-z}I_\nu(z)dz &= \frac{1}{2^{\nu}[\Gamma(\nu + 1)]^2} \sum_{k=0}^{\infty} \frac{[\Gamma(\nu + 2k + 1)]^2}{(\nu + 1)_k (\nu + 2)_k} 16^k k!,
\end{align*}
\]

where \( \text{Re}(\nu) > -1. \)

Besides, the Mellin transforms satisfied by unified four parameter Bessel, generalized three parameter Bessel and modified Bessel functions of the first kind, usual and modified Bessel functions are obtained in the following Theorem and corresponding Corollaries, as well.

Theorem 2.6  For the unified four parameter Bessel function, the Mellin transform is given by

\[
\mathcal{M}\{G^{(b,c)}_\nu(z;\rho); s\} = \frac{\Gamma(s)[\Gamma(c + \nu + s)]}{[\Gamma(\nu + 1)]^2[\Gamma(c)]} \sum_{m=0}^{\infty} \frac{(\nu + 2m)}{m!} J_{\nu+2m}(z) \\
\times \binom{c + \nu + s}{2} c + \nu + s + 1; \nu + 1, \nu + 2, -b^2/4)
\]

where \( \text{Re}(\nu) > -1, \text{ Re}(c) > 0, c + \nu + s, \nu + m \notin \{0, -1, -2,\ldots\}. \)

Proof.  The Mellin transform is given by

\[
\mathcal{M}\{G^{(b,c)}_\nu(z;\rho); s\} = \int_0^\infty \rho^{s-1}G^{(b,c)}_\nu(z;\rho)d\rho.
\]

Substituting equation (2.2) for \( G^{(b,c)}_\nu(z;\rho), \) we have

\[
\mathcal{M}\{G^{(b,c)}_\nu(z;\rho); s\} = \int_0^\infty \rho^{s-1} \frac{z^\nu}{2^{\nu}[\Gamma(\nu + 1)]^2[\Gamma(c)]} \int_0^\infty t^{\nu-1}e^{-t} \sum_{m=0}^{\infty} \frac{(\nu + 2m)}{m!} J_{\nu+2m}(z) \\
\times \binom{c + \nu + s}{2} c + \nu + s + 1; \nu + 1, \nu + 2, -b^2/4) dt d\rho.
\]

Interchanging the order of integrals which is valid due to the conditions of the Theorem, we get

\[
\mathcal{M}\{G^{(b,c)}_\nu(z;\rho); s\} = \frac{z^\nu}{2^{\nu}[\Gamma(\nu + 1)]^2[\Gamma(c)]} \int_0^\infty t^{\nu-1}e^{-t} \sum_{m=0}^{\infty} \frac{(\nu + 2m)}{m!} J_{\nu+2m}(z) \\
\times \binom{c + \nu + s}{2} c + \nu + s + 1; \nu + 1, \nu + 2, -b^2/4) dt.
\]

Letting \( \rho = ut \) and using the Gamma function, which is presented in (1.4), in above integral we have

\[
\mathcal{M}\{G^{(b,c)}_\nu(z;\rho); s\} = \frac{\Gamma(s)z^\nu}{2^{\nu}[\Gamma(\nu + 1)]^2[\Gamma(c)]} \int_0^\infty t^{\nu+s-1}e^{-t} \sum_{m=0}^{\infty} \frac{(\nu + 2m)}{m!} J_{\nu+2m}(z) \\
\times \binom{c + \nu + s}{2} c + \nu + s + 1; \nu + 1, \nu + 2, -b^2/4) dt.
\]

Now, writing the expansion of the hypergeometric series, we have

\[
\mathcal{M}\{G^{(b,c)}_\nu(z;\rho); s\} = \frac{\Gamma(s)z^\nu}{2^{\nu}[\Gamma(\nu + 1)]^2[\Gamma(c)]} \int_0^\infty t^{\nu+s-1}e^{-t} \sum_{k=0}^{\infty} \frac{(-b)^k}{(\nu + 1)_k (\nu + 2)_k (\nu + 2k + 1)_k} 16^k k! dt.
\]
Interchanging the order of integral and summation which is true for \( \Re(c) > 0, \Re(\nu) > -1, \Re(s) > 0 \), we get

\[
\mathcal{M}\{G_{\nu}^{(b,c)}(z;\rho); s\} = \frac{\Gamma(s) z^{\nu}}{2^s \Gamma(\nu + 1)^2 \Gamma(c)} \sum_{k=0}^{\infty} \int_0^\infty t^{c+s+2k-1} e^{-t} \frac{(-b)^k z^{2k}}{(\nu + 1)_k (\nu^2 + 2)_k} 2^{4k} k!.
\]

Using the Gamma function, we have

\[
\mathcal{M}\{G_{\nu}^{(b,c)}(z;\rho); s\} = \frac{\Gamma(s) z^{\nu}}{2^s \Gamma(\nu + 1)^2 \Gamma(c)} \sum_{k=0}^{\infty} \frac{\Gamma(c + \nu + s + 2k) (-b)^k z^{2k}}{(\nu + 1)_k (\nu^2 + 2)_k} 2^{4k} k!.
\]

Applying (1.2) for the Gamma function \( \Gamma(c + \nu + s + 2k) \), we have

\[
\mathcal{M}\{G_{\nu}^{(b,c)}(z;\rho); s\} = \frac{\Gamma(s) z^{\nu}}{2^s \Gamma(\nu + 1)^2 \Gamma(c)} \sum_{k=0}^{\infty} \frac{(c + \nu + s + 2k) (-b)^k z^{2k}}{(\nu + 1)_k (\nu^2 + 2)_k} 2^{4k} k!.
\]

By (1.3), instead of \((c + \nu + s)_{2k}\), we can write \(2^{2k} (c+\nu+s)_{2k} = 2^{2k} (\binom{c+\nu+s}{2})_{k} \). Thus, we have

\[
\mathcal{M}\{G_{\nu}^{(b,c)}(z;\rho); s\} = \frac{\Gamma(s) \Gamma(c + \nu + s) z^{\nu}}{2^s \Gamma(\nu + 1)^2 \Gamma(c)} \sum_{k=0}^{\infty} \frac{2^{2k} (\binom{c+\nu+s}{2})_{k} (\nu + 1)_k (\nu^2 + 2)_k} 2^{4k} k!.
\]

In the last step, using the hypergeometric expansion (1.5) and

\[
\left( \frac{z}{2} \right)^\nu = \sum_{m=0}^{\infty} \frac{(\nu + 2m)(\nu + m - 1)!}{m!} J_{\nu+2m}(z)
\]

in [69], we have

\[
\mathcal{M}\{G_{\nu}^{(b,c)}(z;\rho); s\} = \frac{\Gamma(s) \Gamma(c + \nu + s) z^{\nu}}{\Gamma(\nu + 1)^2 \Gamma(c)} \sum_{m=0}^{\infty} \frac{(\nu + 2m) \Gamma(\nu + m) (\nu + m)}{m!} \left( \frac{c+\nu+s}{2}, \frac{c+\nu+s+1}{2}; \nu + 1, \frac{\nu + 1}{2}, \frac{\nu + 2}{2}, -\frac{z^2}{4} \right).
\]

The Mellin transform of the generalized three parameter Bessel and modified Bessel functions of the first kind can be obtained by substituting \(b = 1\) and \(b = -1\) in Theorem 2.6, respectively. Thus, the following result is clear:

**Corollary 2.6.1** For the generalized three parameter Bessel and modified Bessel functions of the first kind, the Mellin transforms are given by

\[
\mathcal{M}\{J_{\nu}^{(c)}(z;\rho); s\} = \frac{\Gamma(s) \Gamma(c + \nu + s) z^{\nu}}{\Gamma(\nu + 1)^2 \Gamma(c)} \sum_{m=0}^{\infty} \frac{(\nu + 2m) \Gamma(\nu + m) (\nu + m)}{m!} \left( \frac{c+\nu+s}{2}, \frac{c+\nu+s+1}{2}; \nu + 1, \frac{\nu + 1}{2}, \frac{\nu + 2}{2}, -\frac{z^2}{4} \right),
\]

\[
\mathcal{M}\{I_{\nu}^{(c)}(z;\rho); s\} = \frac{\Gamma(s) \Gamma(c + \nu + s) z^{\nu}}{\Gamma(\nu + 1)^2 \Gamma(c)} \sum_{m=0}^{\infty} \frac{(\nu + 2m) \Gamma(\nu + m) (\nu + m)}{m!} \left( \frac{c+\nu+s}{2}, \frac{c+\nu+s+1}{2}; \nu + 1, \frac{\nu + 1}{2}, \frac{\nu + 2}{2}, z^2 \right),
\]

where \(\Re(\nu) > -1, \Re(c) > 0, \Re(s) > 0, c + \nu + s, \nu + m \notin \{0, -1, -2, \ldots\}\).
Taking \(c = 1\) in Corollary 2.6.1, it is reduced the following Corollary:

**Corollary 2.6.2** For the functions \(J_{\nu}^{(1,1)}(z; \rho)\) and \(I_{\nu}^{(-1,1)}(z; \rho)\), the Mellin transform is given by

\[
\mathcal{M}\{J_{\nu}^{(1,1)}(z; \rho); s\} = \frac{\Gamma(s)\Gamma(1 + \nu + s)}{\Gamma(\nu + 1)^2} \sum_{m=0}^{\infty} \frac{(\nu + 2m) \Gamma(\nu + m)}{m!} J_{\nu + 2m}(z) \\
\times 2F_3 \left( \frac{\nu + s + 1}{2}, \frac{\nu + s + 2}{2}; \nu + 1, \frac{\nu + 1 + \nu + 2}{2}; -\frac{z^2}{4} \right),
\]

\[
\mathcal{M}\{I_{\nu}^{(-1,1)}(z; \rho); s\} = \frac{\Gamma(s)\Gamma(1 + \nu + s)}{\Gamma(\nu + 1)^2} \sum_{m=0}^{\infty} \frac{(\nu + 2m) \Gamma(\nu + m)}{m!} J_{\nu + 2m}(z) \\
\times 2F_3 \left( \frac{\nu + s + 1}{2}, \frac{\nu + s + 2}{2}; \nu + 1, \frac{\nu + 1 + \nu + 2}{2}; -\frac{z^2}{4} \right),
\]

where \(\text{Re}(\nu) > -1, \ \text{Re}(s) > 0, \ \nu + m \neq \{0, -1, -2, \ldots\}, \ \nu + s \neq \{-1, -2, \ldots\}\).

Letting \(s = 1\) in Theorem 2.6, the following integral representation of the unified four parameter Bessel function is valid:

**Corollary 2.6.3** Infinite integral of the unified four parameter Bessel function is given by

\[
\int_0^\infty G_{\nu}^{(b,c)}(z; \rho) d\rho = \frac{\Gamma(c + \nu + 1)}{\Gamma(\nu + 1)^2 \Gamma(c)} \sum_{m=0}^{\infty} \frac{(\nu + 2m) \Gamma(\nu + m)}{m!} J_{\nu + 2m}(z) \\
\times 2F_3 \left( \frac{c + \nu + 1}{2}, \frac{c + \nu + 2}{2}; \nu + 1, \frac{\nu + 1 + \nu + 2}{2}; -\frac{bz^2}{4} \right),
\]

where \(\text{Re}(\nu) > -1, \ \text{Re}(c) > 0, \ \nu + m \neq \{0, -1, -2, \ldots\}, \ c + \nu \neq \{-1, -2, \ldots\}\).

Substituting \(b = 1\) and \(b = -1\) in Corollary 2.6.3, the following Corollary is found:

**Corollary 2.6.4** Infinite integrals satisfied by the generalized three parameter Bessel and modified Bessel functions of the first kind are given by

\[
\int_0^\infty J_{\nu}^{(c)}(z; \rho) d\rho = \frac{\Gamma(c + \nu + 1)}{\Gamma(\nu + 1)^2 \Gamma(c)} \sum_{m=0}^{\infty} \frac{(\nu + 2m) \Gamma(\nu + m)}{m!} J_{\nu + 2m}(z) \\
\times 2F_3 \left( \frac{c + \nu + 1}{2}, \frac{c + \nu + 2}{2}; \nu + 1, \frac{\nu + 1 + \nu + 2}{2}; -\frac{z^2}{4} \right),
\]

\[
\int_0^\infty I_{\nu}^{(c)}(z; \rho) d\rho = \frac{\Gamma(c + \nu + 1)}{\Gamma(\nu + 1)^2 \Gamma(c)} \sum_{m=0}^{\infty} \frac{(\nu + 2m) \Gamma(\nu + m)}{m!} J_{\nu + 2m}(z) \\
\times 2F_3 \left( \frac{c + \nu + 1}{2}, \frac{c + \nu + 2}{2}; \nu + 1, \frac{\nu + 1 + \nu + 2}{2}; -\frac{z^2}{4} \right),
\]

where \(\text{Re}(\nu) > -1, \ \text{Re}(c) > 0, \ \nu + m \neq \{0, -1, -2, \ldots\}, \ c + \nu \neq \{-1, -2, \ldots\}\).

Letting \(c = 1\) in Corollary 2.6.4, the integral representations of the functions \(J_{\nu}^{(1,1)}(z; \rho)\) and \(I_{\nu}^{(1,1)}(z; \rho)\) are presented, respectively:
Corollary 2.6.5  For the functions $J^{(1,1)}_\nu(z; \rho)$ and $I^{(-1,1)}_\nu(z; \rho)$, the following integrals are valid

$$
\int_0^\infty J^{(1,1)}_\nu(z; \rho) d\rho = \frac{\Gamma(\nu + 2)}{[\Gamma(\nu + 1)]^2} \sum_{m=0}^{\infty} \frac{(\nu + 2m) \Gamma(\nu + m)}{m!} J_{\nu+2m}(z)
$$

$$
\times 2F_3\left(\frac{\nu + 2}{2}, \frac{\nu + 3}{2}; \nu + 1, \frac{\nu + 1}{2}, \frac{\nu + 2}{2}; -\rho^2 \right),
$$

$$
\int_0^\infty I^{(-1,1)}_\nu(z; \rho) d\rho = \frac{\Gamma(\nu + 2)}{[\Gamma(\nu + 1)]^2} \sum_{m=0}^{\infty} \frac{(\nu + 2m) \Gamma(\nu + m)}{m!} J_{\nu+2m}(z)
$$

$$
\times 2F_3\left(\frac{\nu + 2}{2}, \frac{\nu + 3}{2}; \nu + 1, \frac{\nu + 1}{2}, \frac{\nu + 2}{2}; -\rho^2 \right),
$$

where $\text{Re}(\nu) > -1$ and $\nu + m \neq \{0, -1, -2, \ldots\}$.

The expansion of the unified four parameter Bessel function is introduced as a series of Bessel functions in the following Theorem:

Theorem 2.7  Unified four parameter Bessel function is given by a double series of Bessel functions

$$
G^{(b,c)}_{\nu}(z; \rho) = \frac{\Gamma(\mu + 1)}{\Gamma(\nu + 1)^2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^{\nu-\mu+m+n} (-1)^n J_{\mu+m+n}(z)(-m-n)_{b+c}(c; \rho)_{\nu+2n}(\mu + 1)_n
$$

$$
(m + n)! \Gamma(\mu + 1) \Gamma(\nu + 1)(\nu + 1)_n
$$

where $\mu \neq \nu$ and $\text{Re}(\nu) > -1$, $\text{Re}(\mu) > -1$.

Proof.  Taking into consideration of the definition of the unified four parameter Bessel function, we have

$$
G^{(b,c)}_{\nu}(z; \rho) = \sum_{n=0}^{\infty} \frac{(-b)_n(c; \rho)_{\nu+2n}}{n! \Gamma(\nu + 1) \Gamma(2n + \nu + 1)} \left(\frac{z}{2}\right)^{\nu-\mu+n} \Gamma(\mu + 1) \Gamma(\nu + 1)(\nu + 1)_n
$$

Using Gegenbauer expansion [69] (page 143)

$$
\left(\frac{z}{2}\right)^{\mu+n} = \Gamma(\mu + n + 1) \sum_{p=0}^{\infty} \left(\frac{\rho}{\mu+n}\right)^p J_{\mu+n+p}(z),
$$

we have

$$
G^{(b,c)}_{\nu}(z; \rho) = \sum_{n=0}^{\infty} \frac{(-b)_n(c; \rho)_{\nu+2n}}{n! \Gamma(\nu + 1) \Gamma(2n + \nu + 1)} \left(\frac{z}{2}\right)^{\nu-\mu+n} \Gamma(\mu + 1) \Gamma(\nu + 1)(\nu + 1)_n \sum_{p=0}^{\infty} \left(\frac{\rho}{\mu+n}\right)^p J_{\mu+n+p}(z).
$$

Taking $m - n$ for $p$, we have

$$
G^{(b,c)}_{\nu}(z; \rho) = \sum_{m=0}^{\infty} \left(\frac{z}{2}\right)^{\nu-\mu+m} J_{\mu+m}(z) \left(\sum_{n=0}^{\infty} \frac{(-b)_n(c; \rho)_{\nu+2n}}{n!(m - n)! \Gamma(\nu + 1) \Gamma(2n + \nu + 1)} \Gamma(\mu + n + 1) \Gamma(\nu + 1) \Gamma(2n + \nu + 1)\right)
$$

$$
= \sum_{m=0}^{\infty} \left(\frac{z}{2}\right)^{\nu-\mu+m} J_{\mu+m}(z) \left(\sum_{n=0}^{\infty} \frac{(-1)^n m b^n(c; \rho)_{\nu+2n}}{(m - n)! \Gamma(\nu + 1) \Gamma(2n + \nu + 1)m!} \Gamma(\mu + n + 1) \Gamma(\nu + 1) \Gamma(2n + \nu + 1)\right).
$$

Now, using the expansion of

$$
(-m)_n = \frac{(-1)^m}{(m - n)!},
$$

and $(\nu + 1)_n$ and $(\nu + 1)_{2n}$, we get

$$
G^{(b,c)}_{\nu}(z; \rho) = \sum_{m=0}^{\infty} \left(\frac{z}{2}\right)^{\nu-\mu+m} J_{\mu+m}(z) \left(\frac{\Gamma(\mu + 1)}{\Gamma(\nu + 1)^2 m!} \sum_{n=0}^{\infty} \frac{(-m)_n b^n(c; \rho)_{\nu+2n}(\mu + 1)_n}{n!(\nu + 1)_n(\nu + 1)_{2n}}\right).
$$

17
Applying the Cauchy product for the series \([50]\), we get
\[ G^{(b,c)}_{\nu}(z; \rho) = \frac{\Gamma(\mu + 1)}{[\Gamma(\nu + 1)]^2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left( \frac{z}{2} \right)^{\nu-m+n} (-1)^n J_{\mu+m+n}(z) (-m-n)_n (-b)_n (\nu + 2n(\mu + 1))_n (m+n)! n!(\nu + 1)_n(\nu + 1)_{2n}. \]

The expansions of the generalized three parameter Bessel and modified Bessel functions of the first kind are given by follows:

**Corollary 2.7.1** The series expansions of the generalized three parameter Bessel and modified Bessel functions of the first kind are given by
\[
J^{(c)}_{\nu}(z; \rho) = \frac{\Gamma(\mu + 1)}{[\Gamma(\nu + 1)]^2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left( \frac{z}{2} \right)^{\nu-m+n} \frac{J_{\mu+m+n}(z) (-m-n)_n (\nu + 2n(\mu + 1))_n}{(m+n)! n!(\nu + 1)_n(\nu + 1)_{2n}},
\]
\[
I^{(c)}_{\nu}(z; \rho) = \frac{\Gamma(\mu + 1)}{[\Gamma(\nu + 1)]^2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left( \frac{z}{2} \right)^{\nu-m+n} \frac{(-1)^n J_{\mu+m+n}(z) (-m-n)_n (\nu + 2n(\mu + 1))_n}{(m+n)! n!(\nu + 1)_n(\nu + 1)_{2n}},
\]
where \(\mu \neq \nu\) and \(\text{Re}(\nu) > -1\), \(\text{Re}(\mu) > -1\).

Substituting \(c = 1\) and \(\rho = 0\) in Corollary 2.7.1 and then using Chu-Vandermode Theorem \([5]\)
\[
\sum_{n=0}^{m} \frac{(-m)_n (\mu + 1)_n}{(\nu + 1)_n n!} = \frac{(\nu - \mu)_m}{(\nu + 1)_m},
\]
the expansions of the usual Bessel and modified Bessel functions can be obtained.

**Corollary 2.7.2** Series expansions satisfied by the usual and modified Bessel functions are given by
\[
J_{\nu}(z) = \frac{\Gamma(\mu + 1)}{\Gamma(\nu - \mu)} \sum_{m=0}^{\infty} \left( \frac{z}{2} \right)^{\nu-m} \frac{\Gamma(\nu - \mu + m)}{\Gamma(\nu + m)_m} J_{\mu+m}(z), \ \mu \neq \nu, \ \text{Re}(\mu) > -1, \ \nu - \mu \neq \{0, -1, -2, \ldots\},
\]
\[
I_{\nu}(z) = \frac{\Gamma(\mu + 1)}{\Gamma(\nu + 1)} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left( \frac{z}{2} \right)^{\nu-m+n} \frac{(-1)^n J_{\mu+m+n}(z) (-m-n)_n (\mu + 1)_n}{(m+n)! n!(\nu + 1)_n}, \ \mu \neq \nu, \ \text{Re}(\nu) > -1, \ \text{Re}(\mu) > -1.
\]

### 3 Derivative Properties, Recurrence Relation and Partial Differential Equation of the Unified Four Parameter Bessel Function

Derivative properties and recurrence relations of the unified four parameter Bessel function are obtained in the following Theorems.

**Theorem 3.1** A differential recurrence formula satisfied by \(G^{(b,c)}_{\nu}(z; \rho)\) is given by
\[
\frac{\partial}{\partial z} [z^{\nu+1} \frac{\partial}{\partial z} (z^\nu G^{(b,c-1)}_{\nu}(z; \rho))] = (c-1)G^{(b,c)}_{\nu}(z; \rho).
\]

**Proof.** Substituting \(c = 1\) in (1.8) and multiplying with \(z^\nu\) yields
\[
\sum_{k=0}^{\infty} \frac{(-b)_k (c-1; \rho)_{2k+\nu}}{\Gamma(\nu + k + 1) \Gamma(\nu + 2k + 1)} \frac{z^{2k+2\nu}}{2^{2k+\nu} k!}.
\]
Taking derivative with respect to \(z\) of \(z^\nu G^{(b,c-1)}_{\nu}(z; \rho)\), we get
\[
\frac{\partial}{\partial z} [z^\nu G^{(b,c-1)}_{\nu}(z; \rho)] = \sum_{k=0}^{\infty} \frac{(-b)_k (c-1; \rho)_{2k+\nu}}{\Gamma(\nu + k) \Gamma(\nu + 2k + 1)} \frac{z^{2k+2\nu-1}}{2^{2k+\nu-1} k!}.
\]
Then, multiplying the last series with \( z^{-\nu+1} \) and differentiating with respect to \( z \) gives

\[
\frac{\partial}{\partial z}[z^{-\nu+1} \cdot \frac{\partial}{\partial z} \left( z^\nu G_{\nu}^{(b,c-1)}(z; \rho) \right)] = \sum_{k=0}^{\infty} \frac{(-b)^k(c-1; \rho)_{2k+\nu} z^{2k+\nu-1}}{\Gamma(\nu+k) \Gamma(\nu+2k) 2^{2k+\nu-1} k!} z^{2k+\nu-1}.
\]

Using (1.7), we get

\[
\frac{\partial}{\partial z}[z^{-\nu+1} \cdot \frac{\partial}{\partial z} \left( z^\nu G_{\nu}^{(b,c-1)}(z; \rho) \right)] = \sum_{k=0}^{\infty} \frac{(-b)^k(c-1; \nu)_{2k} z^{2k+\nu-1}}{\Gamma(\nu+k) \Gamma(\nu+2k) 2^{2k+\nu-1} k!}.
\]

Multiplying and dividing with the term \((c)_{\nu-1}\) of the right hand side, we have

\[
\frac{\partial}{\partial z}[z^{-\nu+1} \cdot \frac{\partial}{\partial z} \left( z^\nu G_{\nu}^{(b,c-1)}(z; \rho) \right)] = \frac{(c-1)_\nu}{(c)_{\nu-1}} \sum_{k=0}^{\infty} \frac{(-b)^k(c)_\nu (c+\nu-1; \rho)_{2k} z^{2k+\nu-1}}{\Gamma(\nu+k) \Gamma(\nu+2k) 2^{2k+\nu-1} k!}.
\]

Taking into consideration of the unified Bessel function and equation (1.7), yields

\[
\frac{\partial}{\partial z}[z^{-\nu+1} \cdot \frac{\partial}{\partial z} \left( z^\nu G_{\nu}^{(b,c-1)}(z; \rho) \right)] = \frac{(c-1)_\nu}{(c)_{\nu-1}} G_{\nu}^{(b,c-1)}(z; \rho).
\]

By (1.2), we get the result

\[
\frac{\partial}{\partial z}[z^{-\nu+1} \cdot \frac{\partial}{\partial z} \left( z^\nu G_{\nu}^{(b,c-1)}(z; \rho) \right)] = (c-1) G_{\nu}^{(b,c)}(z; \rho).
\]

The following Corollary is presented for the functions \( J_{\nu}^{(c)}(z; \rho) \) and \( I_{\nu}^{(c)}(z; \rho) \).

**Corollary 3.1.1** Recurrence relations satisfied by the generalized three parameter Bessel and modified Bessel functions of the first kind are given by

\[
\frac{\partial}{\partial z}[z^{-\nu+1} \cdot \frac{\partial}{\partial z} \left( z^\nu J_{\nu}^{(c-1)}(z; \rho) \right)] = (c-1) J_{\nu-1}^{(c)}(z; \rho),
\]

\[
\frac{\partial}{\partial z}[z^{-\nu+1} \cdot \frac{\partial}{\partial z} \left( z^\nu I_{\nu}^{(c-1)}(z; \rho) \right)] = (c-1) I_{\nu-1}^{(c)}(z; \rho).
\]

**Theorem 3.2** Recurrence relation satisfied by the unified four parameter Bessel function is given by

\[
\frac{\partial}{\partial z}[z^{\nu+1} \cdot \frac{\partial}{\partial z} \left( z^{-\nu} G_{\nu}^{(b,c-1)}(z; \rho) \right)] = -b(c-1) G_{\nu+1}^{(b,c)}(z; \rho).
\]

**Proof.** Applying the similar process that is used in the previous theorem, one can get the result. ■

Substituting \( b = 1 \) and \( b = -1 \) respectively in Theorem 3.2, the following recurrences are obtained:

**Corollary 3.2.1** Recurrence formulas satisfied by the generalized three parameter Bessel and modified Bessel functions of the first kind are given by

\[
\frac{\partial}{\partial z}[z^{\nu+1} \cdot \frac{\partial}{\partial z} \left( z^{-\nu} J_{\nu}^{(c-1)}(z; \rho) \right)] = -(c-1) J_{\nu+1}^{(c)}(z; \rho),
\]

\[
\frac{\partial}{\partial z}[z^{\nu+1} \cdot \frac{\partial}{\partial z} \left( z^{-\nu} I_{\nu}^{(c-1)}(z; \rho) \right)] = (c-1) I_{\nu+1}^{(c)}(z; \rho).
\]

**Theorem 3.3** For \( c \neq 1 \), we have

\[
\frac{\partial}{\partial \rho} \left[ G_{\nu}^{(b,c)}(z; \rho) \right] = -\frac{1}{c-1} \left( G_{\nu}^{(b,c-1)}(z; \rho) \right).
\]

**Proof.** Taking derivative with respect to \( \rho \) in (2.2), the desired result is obtained. ■

Taking \( b = 1 \) and \( b = -1 \) in Theorem 3.3, the derivatives of the generalized three parameter Bessel and modified Bessel functions of the first kind are calculated, respectively:
Corollary 3.3.1 For $c \neq 1$, we have
\[
\frac{\partial}{\partial \rho} [J^{(c)}_\nu (z; \rho)] = - \frac{1}{c-1} J^{(c-1)}_\nu (z; \rho),
\]
\[
\frac{\partial}{\partial \rho} [I^{(c)}_\nu (z; \rho)] = - \frac{1}{c-1} I^{(c-1)}_\nu (z; \rho).
\]

Theorem 3.4 The recurrence relation satisfied by the unified four parameter Bessel function is
\[
(2 - \nu) \frac{\partial}{\partial z} G^{(b;c-1)}_{\nu-1} (z; \rho) + z \frac{\partial^2}{\partial z^2} G^{(b;c-1)}_{\nu-1} (z; \rho) + b \frac{\partial^2}{\partial z^2} G^{(b;c-1)}_{\nu+1} (z; \rho)
\]
\[
= -(\nu + 2) \frac{\partial}{\partial z} G^{(b;c-1)}_{\nu+1} (z; \rho).
\]

Proof. Taking $\nu + 1$ in place of $\nu$ in (3.1) and letting $\nu - 1$ instead of $\nu$ in (3.2), then comparing these relations, one can get (3.4). \( \square \)

Corollary 3.4.1 Recurrence relations satisfied by the generalized three parameter Bessel and modified Bessel functions of the first kind are given by
\[
(2 - \nu) \frac{\partial}{\partial z} J^{(c-1)}_{\nu-1} (z; \rho) + z \frac{\partial^2}{\partial z^2} J^{(c-1)}_{\nu-1} (z; \rho) + c \frac{\partial^2}{\partial z^2} J^{(c-1)}_{\nu+1} (z; \rho)
\]
\[
= -(\nu + 2) \frac{\partial}{\partial z} J^{(c-1)}_{\nu+1} (z; \rho).
\]

To obtain the recurrence relations satisfied by the usual Bessel and modified Bessel functions, one can substitute $c = 2$ and $\rho = 0$ in Corollary 3.4.1. In that case, $J^{(c-1)}_{\nu-1} (z; \rho)$ is reduced to $J_{\nu-1} (z)$ and $J^{(c-1)}_{\nu+1} (z; \rho)$ is reduced to $J_{\nu+1} (z)$. Under the similar substitutions, $I^{(c-1)}_{\nu-1} (z; \rho)$ is reduced to $I_{\nu-1} (z)$ and $I^{(c-1)}_{\nu+1} (z; \rho)$ is reduced to $I_{\nu+1} (z)$. Besides, considering the equations (1.1) and (1.8) in [50] (page 111), the recurrence relation of the usual Bessel function is obtained. The similar process can be applied to find out the recurrence relation of the modified Bessel function.

Corollary 3.4.2 Recurrence relations satisfied by the usual Bessel and modified Bessel functions are given by
\[
(2 - \nu) \frac{d}{dz} J_{\nu-1} (z) + z \left( \frac{d^2}{dz^2} J_{\nu-1} (z) + \frac{d^2}{dz^2} J_{\nu+1} (z) \right)
\]
\[
= -(\nu + 2) \frac{d}{dz} J_{\nu+1} (z),
\]
\[
(2 - \nu) \frac{d}{dz} I_{\nu-1} (z) + z \left( \frac{d^2}{dz^2} I_{\nu-1} (z) - \frac{d^2}{dz^2} I_{\nu+1} (z) \right)
\]
\[
= (\nu + 2) \frac{d}{dz} I_{\nu+1} (z).
\]

Proof. Taking derivative with respect to $z$ in the following recurrence formula [50]
\[
2\nu J_{\nu} (z) = z [J_{\nu-1} (z) + J_{\nu+1} (z)]
\]

it is directly obtained that
\[
2\nu \frac{d}{dz} J_{\nu} (z) = J_{\nu-1} (z) + J_{\nu+1} (z) + z \frac{d}{dz} J_{\nu-1} (z) + z \frac{d}{dz} J_{\nu+1} (z).
\]
Now, inserting the derivative of \( J_\nu(z) \),
\[
\frac{d}{dz} J_\nu(z) = \frac{1}{2} [J_{\nu-1}(z) - J_{\nu+1}(z)]
\]
in the above recurrence formula and differentiating with respect to \( z \), result is obtained as
\[
(2 - \nu) \frac{d}{dz} J_{\nu-1}(z) + z \frac{d^2}{dz^2} J_{\nu-1}(z) + \frac{d^2}{dz^2} J_{\nu+1}(z) = -(\nu + 2) \frac{d}{dz} J_{\nu+1}(z).
\]
Proof of the recurrence formula satisfied by modified Bessel function can be obtained as a similar way by means of the recurrence formulas given in Section 1. ■

In the following Theorem, the partial differential equation satisfied by the unified four parameter Bessel function is obtained.

**Theorem 3.5** Partial differential equation satisfied by the unified four parameter Bessel function is given by
\[
\frac{\partial^6}{\partial z^4 \partial \rho^2} G^{(b,c)}_\nu(z;\rho) + \frac{5}{z} \frac{\partial^5}{\partial z^3 \partial \rho^2} G^{(b,c)}_\nu(z;\rho) + \frac{(\nu - 2)^2}{z^2} \frac{\partial^4}{\partial z^2 \partial \rho^2} G^{(b,c)}_\nu(z;\rho) = -\frac{b}{z^2} G^{(b,c)}_\nu(z;\rho).
\] (3.5)

**Proof.** Taking \( \nu - 1 \) in place of \( \nu \) in (3.2) yields
\[
\frac{\partial}{\partial z} \left[ z^\nu \frac{\partial}{\partial z} (z^{-\nu+1} G^{(b,c-1)}_{\nu-1}(z;\rho)) \right] = -b(c - 1) G^{(b,c)}_{\nu-1}(z;\rho)
\] (3.6)
and letting \( c - 1 \) in place of \( c \) in (3.1), we get
\[
\frac{1}{(c - 2)} \frac{\partial}{\partial z} \left[ z^{-\nu+1} \frac{\partial}{\partial z} (z^\nu G^{(b,c-2)}_{\nu-1}(z;\rho)) \right] = G^{(b,c-1)}_{\nu-1}(z;\rho).
\] (3.7)
In equation (3.3), substituting \( c - 1 \) in place of \( c \), we get
\[
G^{(b,c-2)}_{\nu}(z;\rho) = -(c - 2) \frac{\partial}{\partial \rho} G^{(b,c-1)}_{\nu}(z;\rho).
\] (3.8)

Using (3.3) for \( G^{(b,c-1)}_{\nu}(z;\rho) \), we get
\[
G^{(b,c-2)}_{\nu}(z;\rho) = -(c - 2) \frac{\partial}{\partial \rho} \left[ -(c - 1) \frac{\partial}{\partial \rho} G^{(b,c)}_{\nu}(z;\rho) \right] = (c - 1)(c - 2) \frac{\partial^2}{\partial \rho^2} G^{(b,c)}_{\nu}(z;\rho).
\] (3.9)
Obtaining \( G^{(b,c-1)}_{\nu-1}(z;\rho) \) from (3.1) and inserting that into (3.6), we get
\[
\frac{\partial}{\partial z} \left[ z^\nu \frac{\partial}{\partial z} \left( \frac{1}{c - 2} \frac{\partial}{\partial z} (z^\nu G^{(b,c-2)}_{\nu}(z;\rho)) \right) \right] = -b(c - 1) G^{(b,c)}_{\nu}(z;\rho).
\] (3.10)
Plugging (3.9) into (3.10), we have
\[
\frac{\partial}{\partial z} \left[ z^\nu \frac{\partial}{\partial z} \left( \frac{1}{c - 2} \frac{\partial}{\partial z} (z^\nu G^{(b,c-2)}_{\nu}(z;\rho)) \right) \right] = -b(c - 1) G^{(b,c)}_{\nu}(z;\rho).
\] (3.11)
Applying the derivatives, we get
\[
\frac{\partial^6}{\partial z^4 \partial \rho^2} G^{(b,c)}_{\nu}(z;\rho) + \frac{5}{z} \frac{\partial^5}{\partial z^3 \partial \rho^2} G^{(b,c)}_{\nu}(z;\rho) + \frac{(\nu - 2)^2}{z^2} \frac{\partial^4}{\partial z^2 \partial \rho^2} G^{(b,c)}_{\nu}(z;\rho) + \frac{b}{z^2} G^{(b,c)}_{\nu}(z;\rho) = 0.
\]

Taking \( b = 1 \), \( b = -1 \) and \( b = 1 \), \( c = 1 \) and \( b = -1 \) and \( c = 1 \) in Theorem 3.5, the following Corollary can be presented:
Corollary 3.5.1  Partial differential equations satisfied by the generalized three parameter Bessel function and modified Bessel function of the first kind and the functions \( J_{\nu}^{(b,c)}(z;\rho) \) and \( I_{\nu}^{(b,c)}(z;\rho) \) are given by

\[
\begin{align*}
\frac{\partial^6}{\partial z^4 \partial \rho^2} J_{\nu}^{(c)}(z;\rho) &+ \frac{5}{z} \frac{\partial^5}{\partial z^3 \partial \rho^2} J_{\nu}^{(c)}(z;\rho) + \frac{(\nu - 2)^2}{z^2} \frac{\partial^4}{\partial z^2 \partial \rho^2} J_{\nu}^{(c)}(z;\rho) + \frac{1}{z^2} \frac{\partial^3}{\partial z \partial \rho^2} J_{\nu}^{(c)}(z;\rho) = 0, \\
\frac{\partial^6}{\partial z^4 \partial \rho^2} I_{\nu}^{(c)}(z;\rho) &+ \frac{5}{z} \frac{\partial^5}{\partial z^3 \partial \rho^2} I_{\nu}^{(c)}(z;\rho) + \frac{(\nu - 2)^2}{z^2} \frac{\partial^4}{\partial z^2 \partial \rho^2} I_{\nu}^{(c)}(z;\rho) - \frac{1}{z^2} \frac{\partial^3}{\partial z \partial \rho^2} I_{\nu}^{(c)}(z;\rho) = 0, \\
\frac{\partial^6}{\partial z^4 \partial \rho^2} J_{\nu}^{(1,1)}(z;\rho) &+ \frac{5}{z} \frac{\partial^5}{\partial z^3 \partial \rho^2} J_{\nu}^{(1,1)}(z;\rho) + \frac{(\nu - 2)^2}{z^2} \frac{\partial^4}{\partial z^2 \partial \rho^2} J_{\nu}^{(1,1)}(z;\rho) + \frac{1}{z^2} \frac{\partial^3}{\partial z \partial \rho^2} J_{\nu}^{(1,1)}(z;\rho) = 0, \\
\frac{\partial^6}{\partial z^4 \partial \rho^2} I_{\nu}^{(1,1)}(z;\rho) &+ \frac{5}{z} \frac{\partial^5}{\partial z^3 \partial \rho^2} I_{\nu}^{(1,1)}(z;\rho) + \frac{(\nu - 2)^2}{z^2} \frac{\partial^4}{\partial z^2 \partial \rho^2} I_{\nu}^{(1,1)}(z;\rho) - \frac{1}{z^2} \frac{\partial^3}{\partial z \partial \rho^2} I_{\nu}^{(1,1)}(z;\rho) = 0,
\end{align*}
\]
respectively.

Theorem 3.6  Let \( \alpha \in \mathbb{R} \) and \( u = G_{\nu}^{(b,c)}(\alpha z;\rho) \). Then the partial differential equation satisfied by \( u \) is given by

\[
\begin{align*}
24 \frac{\partial^6}{\partial z^4 \partial \rho^2} & u + 5z^3 \frac{\partial^5}{\partial z^3 \partial \rho^2} u + (\nu - 2)^2 z^2 \frac{\partial^4}{\partial z^2 \partial \rho^2} u + b\alpha^2 z^2 u = 0. \tag{3.12}
\end{align*}
\]

Proof. Taking into consideration of the following partial derivatives

\[
\begin{align*}
\frac{\partial u}{\partial z} &= \alpha \frac{\partial G_{\nu}^{(b,c)}(\alpha z;\rho)}{\partial z}, \quad \frac{\partial^2 u}{\partial z^2} = \alpha^2 \frac{\partial^2 G_{\nu}^{(b,c)}(\alpha z;\rho)}{\partial z^2}, \quad \frac{\partial^3 u}{\partial z^3} = \alpha^3 \frac{\partial^3 G_{\nu}^{(b,c)}(\alpha z;\rho)}{\partial z^3}, \quad \frac{\partial^4 u}{\partial z^4} = \alpha^4 \frac{\partial^4 G_{\nu}^{(b,c)}(\alpha z;\rho)}{\partial z^4},
\end{align*}
\]
in the partial differential equation (3.5), we get

\[
(\alpha z)^4 \frac{\partial^6}{\partial z^4 \partial \rho^2} G_{\nu}^{(b,c)}(\alpha z;\rho) + 5(\alpha z)^3 \frac{\partial^5}{\partial z^3 \partial \rho^2} G_{\nu}^{(b,c)}(\alpha z;\rho) + (\nu - 2)^2 (\alpha z)^2 \frac{\partial^3}{\partial z^2 \partial \rho^2} G_{\nu}^{(b,c)}(\alpha z;\rho) = -b(\alpha z)^2 G_{\nu}^{(b,c)}(\alpha z;\rho).
\]

Therefore, \( u = G_{\nu}^{(b,c)}(\alpha z;\rho) \) is a solution of (3.12). ■

4  Mellin Transform of Products of Two Unified Four Parameter Bessel Functions

In this Section, the Mellin transform involving products of two unified four parameter Bessel functions are obtained. To obtain this, integral representation of the product of the unified four parameter Bessel functions should be calculated.

Theorem 4.1  Let \( \alpha, \beta \in \mathbb{R} \). Double integral representation satisfied by the multiplication of the unified four parameter Bessel functions is given by

\[
G_{\nu}^{(b,c)}(\alpha z;\rho) G_{\nu}^{(b,c)}(\beta z;\rho) = \frac{(\alpha z)^{\nu}(\beta z)^{\nu}}{2^{\nu + w} \Gamma(c)^2 \Gamma(\nu + 1)^2 \Gamma(w + 1)^2} \int_0^{\infty} \int_0^{\infty} e^{-\nu - 1 u - w - t} \frac{\partial}{\partial t} F_3\left(\begin{array}{c}
-w, -1, \nu + 1, \nu + 2, b\alpha^2 z^2 t^2 \\
-2, \nu + 2, -b\beta^2 z^2 u^2
\end{array}; \frac{16}{16}\right) du dt,
\]
where \( \text{Re}(c) > 0, \text{Re}(\nu) > -1 \) and \( \text{Re}(w) > -1 \).
Proof. Considering series definition of the unified four parameter Bessel function, it can be written as

\[ G_{\alpha}^{(b,c)}(\alpha z; \rho) G_{\alpha}^{(b,c)}(\beta z; \rho) = \sum_{k,l=0}^{\infty} \frac{(-b)^{k+l}(\rho)_{2k+\nu}(\rho)_{2l+\nu}(\alpha z)^{2k+\nu}(\beta z)^{2l+\nu}}{k!l! \Gamma(k+\nu+1) \Gamma(l+\nu+1) \Gamma(2k+\nu+1) \Gamma(2l+\nu+1) 2^{2k+\nu} 2^{2l+\nu}}. \]

Using the generalized Pochhammer function for \((\rho)_{2k+\nu}\) and \((\rho)_{2l+\nu}\), it is clear that

\[ G_{\alpha}^{(b,c)}(\alpha z; \rho) G_{\alpha}^{(b,c)}(\beta z; \rho) = \sum_{k,l=0}^{\infty} \frac{(-b)^{k+l}(\alpha z)^{2k+\nu}(\beta z)^{2l+\nu}}{k!l! \Gamma(k+\nu+1) \Gamma(l+\nu+1) \Gamma(2k+\nu+1) \Gamma(2l+\nu+1) 2^{2k+\nu} 2^{2l+\nu}} \times \frac{1}{[\Gamma(c)]^2} \int_0^{\infty} \int_0^{\infty} e^{u}e^{w}\left(1-\frac{u}{c}\right) \frac{\rho^{2+k+\nu-1}u^{2l+w-1}e^{-t-u-\frac{a}{1}}}{\frac{2-4u}{\rho}} dt du. \]

Now, using (1.2) yields

\[ G_{\alpha}^{(b,c)}(\alpha z; \rho) G_{\alpha}^{(b,c)}(\beta z; \rho) = \frac{\Gamma(\nu+1)}{\Gamma(\nu+1)} \frac{\Gamma(\frac{\nu+2}{2}) \Gamma(\frac{w+1}{2}) \Gamma(\frac{w+2}{2})}{\Gamma(\nu+1) \Gamma(w+1)} \times \sum_{k,l=0}^{\infty} \frac{(-b)^{k+l}(\alpha z)^{2k+\nu}(\beta z)^{2l+\nu}}{k!l! \Gamma(k+\nu+1) \Gamma(l+\nu+1) \Gamma(2k+\nu+1) \Gamma(2l+\nu+1) 2^{2k+\nu} 2^{2l+\nu}} \times \frac{1}{[\Gamma(c)]^2} \int_0^{\infty} \int_0^{\infty} e^{u}e^{w}\left(1-\frac{u}{c}\right) \frac{\rho^{2+k+\nu-1}u^{2l+w-1}e^{-t-u-\frac{a}{1}}}{\frac{2-4u}{\rho}} dt du. \]

Recalling the fact that given double power series

\[ \sum_{m,n=0}^{\infty} \prod_{j=1}^{A} \Gamma(a_j + m\vartheta_j + n\varphi_j) \prod_{j=1}^{B} \Gamma(b_j + m\psi_j) \prod_{j=1}^{B'} \Gamma(b'_j + n\psi'_j) \prod_{j=1}^{D} \Gamma(d_j + m\eta_j) \prod_{j=1}^{D'} \Gamma(d'_j + n\eta'_j) x^m y^n \]

converges absolutely for all complex \(x\) and \(y\) when

\[ 1 + \sum_{j=1}^{C} \delta_j + \sum_{j=1}^{B} \vartheta_j - \sum_{j=1}^{A} \varphi_j - \sum_{j=1}^{D} \psi_j > 0, \]

\[ 1 + \sum_{j=1}^{C} \varepsilon_j + \sum_{j=1}^{B'} \eta'_j - \sum_{j=1}^{A} \varphi'_j - \sum_{j=1}^{D'} \psi'_j > 0, \]

(see [57] in (3.10)). Comparing the given power series with the double series of the right hand side of \(G_{\alpha}^{(b,c)}(\alpha z; \rho) G_{\alpha}^{(b,c)}(\beta z; \rho)\) and replacing the order of the integrals and summations where \(\delta_j = 0, \eta_j = 3, \vartheta_j = 0, \varphi_j = 0, \psi_j = 0\) and \(x = \frac{-b^2s^2}{16}, y = \frac{-b^2\beta^2}{16}\), the following is obtained

\[ G_{\alpha}^{(b,c)}(\alpha z; \rho) G_{\alpha}^{(b,c)}(\beta z; \rho) = \frac{\Gamma(\nu+1)}{\Gamma(\nu+1)} \frac{\Gamma(\frac{\nu+2}{2}) \Gamma(\frac{w+1}{2}) \Gamma(\frac{w+2}{2})}{\Gamma(\nu+1) \Gamma(w+1)} \times \int_0^{\infty} \int_0^{\infty} \frac{(-b)^{k+l}(\alpha z)^{2k+\nu}(\beta z)^{2l+\nu}}{k!l! \Gamma(k+\nu+1) \Gamma(l+\nu+1) \Gamma(2k+\nu+1) \Gamma(2l+\nu+1) 2^{2k+\nu} 2^{2l+\nu}} \times \frac{1}{[\Gamma(c)]^2} \int_0^{\infty} \int_0^{\infty} e^{u}e^{w}\left(1-\frac{u}{c}\right) \frac{\rho^{2+k+\nu-1}u^{2l+w-1}e^{-t-u-\frac{a}{1}}}{\frac{2-4u}{\rho}} dt du. \]
After some calculations in series, summations can be written as

\[
\frac{\Gamma\left(\frac{\nu+1}{2}\right)\Gamma\left(\frac{\nu+2}{2}\right)\Gamma\left(\frac{w+1}{2}\right)\Gamma\left(\frac{w+2}{2}\right)}{\Gamma(\nu+1)\Gamma(w+1)} \times \sum_{k,l=0}^{\infty} \frac{(-bz^2)^k}{16} \left(-\frac{bz^2}{16}\right)^l 2k_u^{2l}
\]

Substituting the last relation inside of the integrals and using usual Pochhammer function expansions, after some calculations in series, summations can be written as

\[
Re\left(\frac{(az)^\nu(\beta z)^w}{2^{\nu+w}[\Gamma(\nu+1)]^2[\Gamma(w+1)]^2} \int_0^\infty \int_0^\infty t^{\nu+1}u^{w-1}e^{-t-u-\frac{\beta z}{t} - \frac{a z}{u}}
\]

Substituting the last relation inside of the integrals and using usual Pochhammer function expansions, the desired result is obtained as

\[
G_{\nu}^{(b,c)}(az;\rho)G_{w}^{(b,c)}(\beta z;\rho) = \frac{(az)^\nu(\beta z)^w}{2^{\nu+w}[\Gamma(\nu+1)]^2[\Gamma(w+1)]^2} \int_0^\infty \int_0^\infty t^{\nu+1}u^{w-1}e^{-t-u-\frac{\beta z}{t} - \frac{a z}{u}}
\]



Corollary 4.1.1 The integral representations satisfied by the functions \(J_{\nu}^{(c)}(az;\rho), J_{w}^{(c)}(\beta z;\rho)\) are given by

\[
J_{\nu}^{(c)}(az;\rho)J_{w}^{(c)}(\beta z;\rho) = \frac{(az)^\nu(\beta z)^w}{2^{\nu+w}[\Gamma(\nu+1)]^2[\Gamma(w+1)]^2} \int_0^\infty \int_0^\infty t^{\nu+1}u^{w-1}e^{-t-u-\frac{\beta z}{t} - \frac{a z}{u}}
\]

\[
I_{\nu}^{(c)}(az;\rho)I_{w}^{(c)}(\beta z;\rho) = \frac{(az)^\nu(\beta z)^w}{2^{\nu+w}[\Gamma(\nu+1)]^2[\Gamma(w+1)]^2} \int_0^\infty \int_0^\infty t^{\nu+1}u^{w-1}e^{-t-u-\frac{\beta z}{t} - \frac{a z}{u}}
\]

where \(\text{Re}(c) > 0, \text{Re}(\nu) > -1\) and \(\text{Re}(w) > -1\).

Corollary 4.1.2 Integral representations satisfied by the usual Bessel and modified Bessel functions...
are given by

\[
J_\nu(\alpha z)J_\nu(\beta z) = \frac{(\alpha z)\nu(\beta z)^w}{2^{\nu+w}[\Gamma(\nu+1)]^2[\Gamma(w+1)]^2} \int_0^\infty \int_0^\infty t^{\nu+w}e^{-t-u} dt du \\
\times 0F_3(-; \nu + 1, \nu + 2; -\frac{\alpha^2 z^2 l^2}{16}) \\
\times 0F_3(-; w + 1, w + 2; -\frac{\beta^2 z^2 u^2}{16}) dt du,
\]

\[
I_\nu(\alpha z)I_\nu(\beta z) = \frac{(\alpha z)\nu(\beta z)^w}{2^{\nu+w}[\Gamma(\nu+1)]^2[\Gamma(w+1)]^2} \int_0^\infty \int_0^\infty t^{\nu+w}e^{-t-u} dt du \\
\times 0F_3(-; \nu + 1, \nu + 2; -\frac{\alpha^2 z^2 l^2}{16}) \\
\times 0F_3(-; w + 1, w + 2; -\frac{\beta^2 z^2 u^2}{16}) dt du,
\]

where \( \Re(\nu) > -1 \) and \( \Re(w) > -1 \).

In the following Theorem, the Mellin transform of the \( zG_{\nu}^{(b,c)}(\alpha z; \rho)G_{w}^{(b,c)}(\beta z; \rho) \) is calculated.

**Theorem 4.2** The Mellin transform of \( zG_{\nu}^{(b,c)}(\alpha z; \rho)G_{w}^{(b,c)}(\beta z; \rho) \) is given by

\[
\mathcal{M}\{ze^{-z}G_{\nu}^{(b,c)}(\alpha z; \rho)G_{w}^{(b,c)}(\beta z; \rho); s\} = \int_0^\infty z^{s-1}ze^{-z}G_{\nu}^{(b,c)}(\alpha z; \rho)G_{w}^{(b,c)}(\beta z; \rho)dz.
\]

Inserting the integral representation of the \( G_{\nu}^{(b,c)}(\alpha z; \rho)G_{w}^{(b,c)}(\beta z; \rho) \) and expanding the hypergeometric series and using the expansion of the generalized Pochhammer functions, we have

\[
\mathcal{M}\{ze^{-z}G_{\nu}^{(b,c)}(\alpha z; \rho)G_{w}^{(b,c)}(\beta z; \rho); s\} = \frac{\alpha^\nu \beta^w}{2^{\nu+w}[\Gamma(\nu+1)]^2[\Gamma(w+1)]^2[\Gamma(c)]^2} \int_0^\infty \int_0^\infty e^{-z s + \nu + w} e^{\nu t + c + w - u e^{-t - u - \frac{c}{c}} - \frac{\nu}{\nu}} \int k!! 16^{k+l} dt dz.
\]

Replacing the order of integrals and summations, we get

\[
\mathcal{M}\{ze^{-z}G_{\nu}^{(b,c)}(\alpha z; \rho)G_{w}^{(b,c)}(\beta z; \rho); s\} = \frac{\alpha^\nu \beta^w}{2^{\nu+w}[\Gamma(\nu+1)]^2[\Gamma(w+1)]^2[\Gamma(c)]^2} \sum_{k,l=0}^{\infty} \int_0^\infty \int_0^\infty e^{\nu + w} e^{\nu + c + w - u e^{-t - u - \frac{\nu}{\nu}} - \frac{\nu}{\nu}} \int k!! 16^{k+l} dt dz.
\]

\[
\times \frac{(-b)^{k+l} \alpha^2 \beta^2 l^{2l} k!!}{(\nu + 1)_k (w + 1)_l (\nu + 2)_k (w + 2)_l} 16^{k+l}.
\]
Now, computing the integrals in terms of the generalized Pochhammer functions and Gamma function, we have

\[ M\{ze^{-z}G^{(b,c)}_\nu(\alpha z; \rho)G^{(b,c)}_w(\beta z; \rho); s\} \]

\[ = \frac{\alpha^\nu \beta^w}{2^{\nu+w}[\Gamma(\nu+1)]^2[\Gamma(w+1)]^2} \sum_{k,l=0}^{\infty} \frac{(-b)^{k+l} \alpha^{2k} \beta^{2l} (c; \rho)_{\nu+2k} (c; \rho)_{w+2l}}{(\nu+1)_k (w+1)_l (\frac{\nu+1}{2})_k (\frac{\nu+1}{2})_l (\frac{w+2}{2})_k (\frac{w+2}{2})_l} k!l!2^{k+l}4^{k+l}. \]

Taking \( s = 1 \) in Theorem 4.2, the following integral representation formula is obtained.

**Corollary 4.2.1** Infinite integral involving of the unified four parameter Bessel functions is given by

\[ \int_0^\infty ze^{-z}G^{(b,c)}_\nu(\alpha z; \rho)G^{(b,c)}_w(\beta z; \rho)dz \]

\[ = \frac{\alpha^\nu \beta^w}{2^{\nu+w}[\Gamma(\nu+1)]^2[\Gamma(w+1)]^2} \sum_{k,l=0}^{\infty} \frac{(-b)^{k+l} \alpha^{2k} \beta^{2l} (c; \rho)_{\nu+2k} (c; \rho)_{w+2l}}{(\nu+1)_k (w+1)_l (\frac{\nu+1}{2})_k (\frac{\nu+1}{2})_l (\frac{w+2}{2})_k (\frac{w+2}{2})_l} k!l!2^{k+l}4^{k+l}, \]

where \( \text{Re}(\nu) > -1 \) and \( \text{Re}(w) > -1 \).

Taking \( b = 1, b = -1, c = 1, \rho = 0 \) and \( b = -1, c = 1, \rho = 0 \) in Corollary 4.2.1, the following integral representations are obtained:

**Corollary 4.2.2** The integral representations involving the multiplication of the generalized three parameter Bessel and modified Bessel functions of the first kind are given by

\[ \int_0^\infty ze^{-z}J^{(c)}_\nu(\alpha z; \rho)J^{(c)}_w(\beta z; \rho)dz \]

\[ = \frac{\alpha^\nu \beta^w}{2^{\nu+w}[\Gamma(\nu+1)]^2[\Gamma(w+1)]^2} \sum_{k,l=0}^{\infty} \frac{(-1)^k \alpha^{2k} \beta^{2l} (c; \rho)_{\nu+2k} (c; \rho)_{w+2l}}{2^k (\nu+1)_k (w+1)_l (\frac{\nu+1}{2})_k (\frac{\nu+1}{2})_l (\frac{w+2}{2})_k (\frac{w+2}{2})_l} k!l!2^{k+l}4^{k+l}, \]

where \( \text{Re}(\nu) > -1 \) and \( \text{Re}(w) > -1 \).

**Corollary 4.2.3** Integrals involving \( J^{(1,1)}_\nu(\alpha z; \rho), J^{(1,1)}_w(\beta z; \rho), I^{(-1,1)}_\nu(\alpha z; \rho) \) and \( I^{(-1,1)}_w(\beta z; \rho) \) are given by

\[ \int_0^\infty ze^{-z}J^{(1,1)}_\nu(\alpha z; \rho)J^{(1,1)}_w(\beta z; \rho)dz \]

\[ = \frac{\alpha^\nu \beta^w}{2^{\nu+w}[\Gamma(\nu+1)]^2[\Gamma(w+1)]^2} \sum_{k,l=0}^{\infty} \frac{(-1)^k \alpha^{2k} \beta^{2l} (1; \rho)_{\nu+2k} (1; \rho)_{w+2l}}{(\nu+1)_k (w+1)_l (\frac{\nu+1}{2})_k (\frac{\nu+1}{2})_l (\frac{w+2}{2})_k (\frac{w+2}{2})_l} k!l!2^{k+l}4^{k+l}, \]

\[ \int_0^\infty ze^{-z}I^{(-1,1)}_\nu(\alpha z; \rho)I^{(-1,1)}_w(\beta z; \rho)dz \]

\[ = \frac{\alpha^\nu \beta^w}{2^{\nu+w}[\Gamma(\nu+1)]^2[\Gamma(w+1)]^2} \sum_{k,l=0}^{\infty} \frac{\alpha^{2k} \beta^{2l} (1; \rho)_{\nu+2k} (1; \rho)_{w+2l}}{(\nu+1)_k (w+1)_l (\frac{\nu+1}{2})_k (\frac{\nu+1}{2})_l (\frac{w+2}{2})_k (\frac{w+2}{2})_l} k!l!2^{k+l}4^{k+l}, \]

where \( \text{Re}(\nu) > -1 \) and \( \text{Re}(w) > -1 \).
For the products of usual Bessel and modified Bessel functions, the following Corollary can be written.

**Corollary 4.2.4** Integral representations satisfied by the usual Bessel and modified Bessel functions $J_{\nu}(\alpha z)$, $J_{w}(\beta z)$ and $I_{\nu}(\alpha z)$, $I_{w}(\beta z)$ are given by

\[
\int_{0}^{\infty} ze^{-z} J_{\nu}(\alpha z) J_{w}(\beta z) dz = \frac{\alpha^{\nu} \beta^{w}}{2^{\nu+w}[\Gamma(\nu+1)]^2} \sum_{k,l=0}^{\infty} (-1)^{k+l} \alpha^{2k} \beta^{2l} \Gamma(\nu+2k+1) \Gamma(\nu+2l+1) \Gamma(\nu+w+2k+2l+2) \frac{(\nu+1)_{k} (w+1)_{l} (\frac{\nu+1}{2})_{k} (\frac{w+1}{2})_{l} k! l! 2^{k+l}}{k! l! 2^{k+l}} ,
\]

where $Re(\nu) > -1$ and $Re(w) > -1$.

Note that, the Mellin transform of the products of Bessel functions was obtained in [45]. Recalling that the integrals

\[
\int_{0}^{\infty} f(x) J_{\nu}(\alpha x) J_{w}(\beta x) dx
\]

arises in problems involving particle motion in an unbounded rotating fluid [20] in magnetohydrodynamic flow, crack problems in elasticity [66] and distortions of nearly circular lipid domains where $\nu, w \in \mathbb{Z}^{+}$ and $\alpha, \beta \in \mathbb{R}^{+}$ [64]. Taking $f(x) = xe^{-x}$ the result of the corresponding Theorem coincides with this integral.

5 A Three-Fold Integral Formula for the Unified Four Parameter Bessel Function

In the following Theorem, a three-fold integral formula for the unified four parameter Bessel function is given:

**Theorem 5.1** A three-fold integral formula satisfied by the unified four parameter Bessel function is given by

\[
G_{\nu}^{(b,c)}(z; \rho) = \frac{(\frac{\pi}{2})^{\nu}}{\Gamma(c)\pi^{\nu} \Gamma(\nu+\frac{1}{2})} \int_{0}^{1} \int_{0}^{c} t^{-\frac{c}{2}} \left( 1-t \right)^{-\frac{c}{2}} u^{-\frac{c}{2}} s^{\nu-1} (1-t)^{\nu-\frac{1}{2}} (1-u)^{\nu-\frac{1}{2}} (1-s)^{-c-\nu-1} \times e^{-z^{2}s(1-s)^{2}} \sum_{0}^{\infty} \frac{(bz^{2} s^{2} u^{2})^{l}}{16(1-s)^{c}} \left( -\frac{1}{4} :\frac{3}{4}:\frac{1}{2} : \frac{b^{2} z^{2} s^{2} u^{2} t^{2}}{16(1-s)^{c}} \right) ds dt du,
\]

where $Re(c) > 0$, $Re(\nu) > -\frac{1}{2}$ and $Re(s) > 0$.

**Proof.** Considering series expansion of the unified four parameter Bessel function as

\[
G_{\nu}^{(b,c)}(z; \rho) = \sum_{k=0}^{\infty} \frac{2^{2k}(-b)^{k}(c; \rho)_{2k}(\frac{\pi}{2})^{\nu} z^{2k} \Gamma(k+\frac{1}{2}) \Gamma(\nu+\frac{1}{2}) \Gamma(2k+\frac{1}{2}) \Gamma(\nu+\frac{1}{2})}{2^{2k} 2^{2k} \Gamma(k+\frac{1}{2}) \Gamma(k+\nu+1) \Gamma(\nu+\frac{1}{2}) \Gamma(2k+\nu+1) \Gamma(2k+\nu+\frac{1}{2}) \Gamma(\nu+\frac{1}{2})},
\]

then using Beta function

\[
B(x; y) = \frac{\Gamma(x) \Gamma(y)}{\Gamma(x+y)}
\]
and applying duplication formula for the Gamma function \([23, 58] \) we have

\[
\sqrt{\pi} \Gamma(2k + 1) = 2^{2k} k! \Gamma(k + \frac{1}{2}),
\]

we have

\[
G^{(b,c)}(z; \rho) = \sum_{k=0}^{\infty} \frac{2^{4k}(-b)^k(c; \rho)_{2k+\nu}(\frac{z}{2})^{\nu} z^{2k} B(k + \frac{1}{2}; \nu + \frac{1}{2}) B(2k + \frac{1}{2}; \nu + \frac{1}{2})}{2^{4k} \sqrt{\pi} \Gamma(2k + 1) \Gamma(\nu + \frac{1}{2})}.
\]

Again, using duplication formula, we get

\[
G^{(b,c)}(z; \rho) = \sum_{k=0}^{\infty} \frac{2^{4k}(-b)^k(c; \rho)_{2k+\nu}(\frac{z}{2})^{\nu} z^{2k} B(k + \frac{1}{2}; \nu + \frac{1}{2}) B(2k + \frac{1}{2}; \nu + \frac{1}{2})}{\sqrt{\pi} \Gamma(4k + 1) \sqrt{\pi} \Gamma(\nu + \frac{1}{2})^2}.
\]

Now, inserting the Beta functions,

\[
B(k + \frac{1}{2}; \nu + \frac{1}{2}) = \int_0^1 t^{k-\frac{1}{2}} (1 - t)^{\nu-\frac{1}{2}} dt,
\]

\[
B(2k + \frac{1}{2}; \nu + \frac{1}{2}) = \int_0^1 u^{2k-\frac{1}{2}} (1 - u)^{\nu-\frac{1}{2}} du
\]

we have

\[
G^{(b,c)}(z; \rho) = \sum_{k=0}^{\infty} \frac{2^{4k}(-b)^k(c; \rho)_{2k+\nu}(\frac{z}{2})^{\nu} z^{2k}}{\pi \Gamma(4k + 1) [\Gamma(\nu + \frac{1}{2})]^2} \int_0^1 t^{k-\frac{1}{2}} (1 - t)^{\nu-\frac{1}{2}} dt \int_0^1 u^{2k-\frac{1}{2}} (1 - u)^{\nu-\frac{1}{2}} du,
\]

\[
= \sum_{k=0}^{\infty} \frac{(-b)^k(c; \rho)_{2k+\nu} z^{2k} 2^{4k} \Gamma(\nu)}{\pi \Gamma(4k + 1) [\Gamma(\nu + \frac{1}{2})]^2} \int_0^1 t^{k-\frac{1}{2}} u^{2k-\frac{1}{2}} (1 - t)^{\nu-\frac{1}{2}} (1 - u)^{\nu-\frac{1}{2}} dtdu,
\]

\[
= \frac{(\frac{z}{2})^\nu}{\pi [\Gamma(\nu + \frac{1}{2})]^2} \sum_{k=0}^{\infty} \frac{(-b)^k(c; \rho)_{2k+\nu} z^{2k} 2^{4k}}{\Gamma(4k + 1)} \int_0^1 \int_0^1 t^{k-\frac{1}{2}} u^{2k-\frac{1}{2}} (1 - t)^{\nu-\frac{1}{2}} (1 - u)^{\nu-\frac{1}{2}} dtdu.
\]

Inserting the generalized Pochhammer function

\[
(c; \rho)_{2k+\nu} = \frac{1}{\Gamma(c)} \int_0^\infty y^{c+2k+\nu-1} e^{-y} \frac{dy}{y},
\]

we have

\[
G^{(b,c)}(z; \rho) = \frac{(\frac{z}{2})^\nu}{\pi [\Gamma(\nu + \frac{1}{2})]^2} \sum_{k=0}^{\infty} \frac{(-b)^k z^{2k} 2^{4k}}{\Gamma(4k + 1)} \frac{1}{\Gamma(c)} \int_0^\infty \int_0^1 t^{k-\frac{1}{2}} u^{2k-\frac{1}{2}} (1 - t)^{\nu-\frac{1}{2}} (1 - u)^{\nu-\frac{1}{2}} dtdu.
\]

Making substitution \( y = \frac{s}{1-s} \) in the first integral on the right side, we have

\[
G^{(b,c)}(z; \rho) = \frac{(\frac{z}{2})^\nu}{\pi [\Gamma(\nu + \frac{1}{2})]^2} \sum_{k=0}^{\infty} \frac{(-b)^k z^{2k} 2^{4k}}{\Gamma(4k + 1)} \int_0^1 \int_0^1 t^{k-\frac{1}{2}} u^{2k-\frac{1}{2}} s^{c+2k+\nu-1} (1 - t)^{\nu-\frac{1}{2}} (1 - u)^{\nu-\frac{1}{2}} (1 - s)^{-c-2k-\nu-1} ds dt du \times e^{-s^2 - \rho(1-s)^2} s(1-s) ds dt du.
\]
Taking into consideration of the Legendre’s duplication formula for the function $\Gamma(4k+1)$

$$\Gamma(4k+1) = \frac{1}{\sqrt{\pi}} \Gamma(2k+\frac{1}{2})^2 \Gamma(2k+1)$$

and applying (1.2) and (1.3), we have

$$\Gamma(4k+1) = 2^{4k} 2^{2k} \left(\frac{3}{4}\right)_k \left(\frac{1}{2}\right)_k k^k \frac{1}{\Gamma(2k+1)}$$

Now, replacing the order of summation and integrals, we get

$$G^{(b,c)}_{\nu}(z;\rho) = \frac{(\frac{z}{2})^{\nu}}{\pi \Gamma(\nu)} \left[ \frac{1}{\Gamma(\nu+\frac{1}{2})^2} \right]^2 \int_0^1 \int_0^1 \Gamma \left( \sum_{k=0}^{\infty} \left( \frac{1}{4} \right)_k \left( \frac{1}{2} \right)_k \frac{1}{k!} \left( -bz^2 s^2 u^2 t^k \right) \right)$$

$$\times t^{-\frac{1}{2}} u^{-\frac{1}{2}} s^{c+\nu-1} (1-t)^{c+\nu-1} (1-u)^{c+\nu-1} (1-s)^{-c-\nu-1}$$

Using hypergeometric series expansion, the result is obtained

$$G^{(b,c)}_{\nu}(z;\rho) = \frac{(\frac{z}{2})^{\nu}}{\pi \Gamma(\nu)} \left[ \frac{1}{\Gamma(\nu+\frac{1}{2})^2} \right]^2 \int_0^1 \int_0^1 \Gamma \left( \sum_{k=0}^{\infty} \left( \frac{1}{4} \right)_k \left( \frac{1}{2} \right)_k \frac{1}{k!} \left( -bz^2 s^2 u^2 t^k \right) \right)$$

$$\times e^{-s^2 s^2 u^2 t} \frac{1}{16(1-s)^2} ds dt du.$$

Letting $b = 1$ and $b = -1$ in Theorem 5.1, the following triple integral representations for the generalized three parameter Bessel and modified Bessel functions of the first kind are obtained:

**Corollary 5.1.1** For the generalized three parameter Bessel and modified Bessel functions of the first kind, we have

$$J^{(c)}_{\nu}(z;\rho) = \frac{(\frac{z}{2})^{\nu}}{\Gamma(\nu)} \left[ \frac{1}{\Gamma(\nu+\frac{1}{2})^2} \right]^2 \int_0^1 \int_0^1 \Gamma \left( \sum_{k=0}^{\infty} \left( \frac{1}{4} \right)_k \left( \frac{1}{2} \right)_k \frac{1}{k!} \left( -z^2 s^2 u^2 t \right) \right)$$

$$\times \left( \frac{1}{16(1-s)^2} \right) ds dt du,$$

$$I^{(c)}_{\nu}(z;\rho) = \frac{(\frac{z}{2})^{\nu}}{\Gamma(\nu)} \left[ \frac{1}{\Gamma(\nu+\frac{1}{2})^2} \right]^2 \int_0^1 \int_0^1 \Gamma \left( \sum_{k=0}^{\infty} \left( \frac{1}{4} \right)_k \left( \frac{1}{2} \right)_k \frac{1}{k!} \left( z^2 s^2 u^2 t \right) \right)$$

$$\times \left( \frac{1}{16(1-s)^2} \right) ds dt du,$$

where $\text{Re}(c) > 0$ and $\text{Re}(\nu) > -\frac{1}{2}$.

Taking $b = 1$, $c = 1$ and $\rho = 0$ and $b = -1$, $c = 1$ and $\rho = 0$ in Corollary 5.1.1, the integral representations of the usual Bessel and modified Bessel functions are given in the following Corollary.

**Corollary 5.1.2** Triple integral formulas satisfied by the usual Bessel and modified Bessel functions
are given by
\[
J_\nu(z) = \frac{(-\frac{z}{2})^\nu}{\pi [\Gamma(\nu + \frac{1}{2})]^2} \int_0^1 \int_0^1 t^{-\frac{1}{2}} u^{-\frac{1}{2}} s^{\nu-\frac{1}{2}} (1-t)^{\nu-\frac{1}{2}} (1-u)^{\nu-\frac{1}{2}} (1-s)^{-\nu-2} \\
\times e^{(\nu-z)} \, _0F_3\left(-\frac{3}{4}, \frac{1}{4}, \frac{1}{2}; \frac{z^2 s^2 u^2 t}{16(1-s)^2}\right) ds dt du,
\]
\[
I_\nu(z) = \frac{(-\frac{z}{2})^\nu}{\pi [\Gamma(\nu + \frac{1}{2})]^2} \int_0^1 \int_0^1 t^{-\frac{1}{2}} u^{-\frac{1}{2}} s^{\nu-\frac{1}{2}} (1-t)^{\nu-\frac{1}{2}} (1-u)^{\nu-\frac{1}{2}} (1-s)^{-\nu-2} \\
\times e^{(\nu-z)} \, _0F_3\left(-\frac{3}{4}, \frac{1}{4}, \frac{1}{2}; \frac{z^2 s^2 u^2 t}{16(1-s)^2}\right) ds dt du,
\]
where Re(\nu) > -\frac{1}{2}.

6 Generalized Four Parameter Spherical Bessel and Bessel-Clifford Functions

In this Section, generalized four parameter spherical Bessel and Bessel-Clifford functions are defined by
\[
g^{(b,c)}_\nu(z;\rho) := \sqrt{\pi} \frac{2}{z} G^{(b,c-\frac{1}{2})}_\nu(z;\rho) \\
(Re(\nu) > -\frac{3}{2}, \, Re(c) > \frac{1}{2}, \, Re(\rho) > 0),
\]
\[
C^{(b,\lambda)}_\nu(z;\rho) := z^{-\frac{\lambda}{2}} C^{(b,\lambda)}_\nu(2\sqrt{z};\rho) \\
(Re(\nu) > -1, \, Re(\lambda) > 0, \, Re(\rho) > 0),
\]
respectively. Taking into consideration of the definition of the unified four parameter Bessel function, the mentioned generalized functions can be written as
\[
g^{(b,c)}_\nu(z;\rho) = \frac{\sqrt{\pi}}{2} \sum_{k=0}^{\infty} \frac{(-b)^k (c-\frac{1}{2};\rho)_{2k+\nu+\frac{1}{2}}}{k! \Gamma(\nu+k+\frac{1}{2}) \Gamma(\nu+2k+\frac{3}{2}) (\sqrt{2})^{2k+\nu}} z^{2k+\nu} \\
(Re(\nu) > -\frac{3}{2}, \, Re(c) > \frac{1}{2}, \, Re(\rho) > 0),
\]
\[
C^{(b,\lambda)}_\nu(z;\rho) = \sum_{k=0}^{\infty} \frac{(-b)^k (\lambda;\rho)_{2k+\nu}}{k! \Gamma(\nu+k+1) \Gamma(\nu+2k+1) (\sqrt{2})^{2k}} z^{2k} \\
(Re(\nu) > -1, \, Re(\lambda) > 0, \, Re(\rho) > 0).
\]

Taking b = 1, c = \frac{3}{2} and \rho = 0, \, g^{(b,c)}_\nu(z;\rho) is reduced to usual spherical Bessel function of the first kind j_\nu(z). Letting b = -1, \, \lambda = 1 and \rho = 0, \, C^{(b,\lambda)}_\nu(z;\rho) is reduced to usual Bessel-Clifford function of the first kind C_\nu(z). Since the proofs of the corresponding Theorems for the generalized four parameter spherical Bessel and Bessel-Clifford functions are similar with the unified four parameter Bessel function, details are omitted. Note that using the monomiality principle, multiplicative and derivative operators of the usual Bessel-Clifford function were introduced and studied in [10] [13] [59].

Lemma 6.1 Let \nu \in \mathbb{Z}. The relationship between C^{(b,\lambda)}_{-\nu}(z;\rho) and C^{(b,\lambda)}_\nu(z;\rho) is given by
\[
C^{(b,\lambda)}_{-\nu}(z;\rho) = (-b)^\nu z^\nu C^{(b,\lambda)}_\nu(z;\rho).
\]

Proof. It can be directly seen from Lemma 2.1 in Section 2.
\textbf{Corollary 6.1.1} Let $\nu \in \mathbb{Z}$. The relation between $C_{-\nu}(z)$ and $C_{\nu}(z)$ is given by

$$C_{-\nu}(z) = z^\nu C_{\nu}(z).$$

\textbf{Theorem 6.2} Let $n \in \mathbb{Z}$. For $t \neq 0$ and for all finite $z$, generating functions of the generalized four parameter spherical Bessel and Bessel-Clifford functions are given by

$$\sum_{n=-\infty}^{\infty} g^{(b, c + \frac{1}{2})}_{n-\frac{1}{2}}(z; \rho) t^n = \sqrt{\frac{\pi}{2 z}} \ _1F_1((c; \rho), 1; (t - \frac{b}{t}) \frac{z}{2}),$$

$$\sum_{n=-\infty}^{\infty} C^{(b, \lambda)}_{n}(z; \rho) t^n = \ _1F_1((\lambda, \rho), 1; t - \frac{b z}{t}).$$

\textbf{Proof.} To prove the generating function for the generalized four parameter spherical Bessel function, we take into consideration of the following relation

$$g^{(b, c)}_{n}(z; \rho) = \sqrt{\frac{\pi}{2 z}} G^{(b, c - \frac{1}{2})}_{n+\frac{1}{2}}(z; \rho).$$

Substituting $n - \frac{1}{2}$ for $n$ and $c + \frac{1}{2}$ for $c$, we have

$$g^{(b, c + \frac{1}{2})}_{n-\frac{1}{2}}(z; \rho) = \sqrt{\frac{\pi}{2 z}} G^{(b, c)}_{n}(z; \rho).$$

Taking summations on both sides of the equality yields

$$\sum_{n=-\infty}^{\infty} g^{(b, c + \frac{1}{2})}_{n-\frac{1}{2}}(z; \rho) t^n = \sqrt{\frac{\pi}{2 z}} \sum_{n=-\infty}^{\infty} G^{(b, c)}_{n}(z; \rho) t^n.$$  

By Theorem 2.2

$$\sum_{n=-\infty}^{\infty} g^{(b, c + \frac{1}{2})}_{n-\frac{1}{2}}(z; \rho) t^n = \sqrt{\frac{\pi}{2 z}} \ _1F_1((c; \rho), 1; (t - \frac{b}{t}) \frac{z}{2}),$$

whence the result. Generating function for the generalized four parameter Bessel-Clifford function can be shown as a similar process that is applied in Theorem 2.2 in Section 2. \hfill \blacksquare

\textbf{Corollary 6.2.1} [\textit{Ib} [27]] Let $n \in \mathbb{Z}$. For $t \neq 0$ and for all finite $z$, generating functions of the usual spherical Bessel and Bessel-Clifford functions are given by

$$\sum_{n=-\infty}^{\infty} j^{(b, c)}_{n}(z) t^n = \sqrt{\frac{\pi}{2 z}} e^{t-\frac{1}{2}},$$

$$\sum_{n=-\infty}^{\infty} C^{(b, \lambda)}_{n}(z) t^n = e^{t+\frac{1}{2}}.$$  

\textbf{Theorem 6.3} Integral representations satisfied by the generalized four parameter spherical Bessel and Bessel-Clifford functions are given by

$$g^{(b, c)}_{\nu}(z; \rho) = \frac{\sqrt{\pi(z^2 + \frac{\nu}{2})}}{2[G(\nu + \frac{1}{2})]^2 G(c - \frac{1}{2})} \int_{0}^{\infty} e^{\nu - t} e^{-\nu t} 0 F_{3}(-; \frac{2 \nu + 3}{2}, \frac{2 \nu + 3}{4}, \frac{2 \nu + 5}{4}; \frac{bz^2 t^2}{16}) dt,$$

(Re($\nu$) $> - \frac{3}{2}$, Re($c$) $> \frac{1}{2}$)

$$C^{(b, \lambda)}_{\nu}(z; \rho) = \frac{1}{[G(\nu + 1)]^2 G(\lambda)} \int_{0}^{\infty} t^{\lambda + \nu - 1} e^{t - \nu} 0 F_{3}(-; \nu + 1, \nu + 2, \nu + 2; \frac{-bz^2 t^2}{4}) dt,$$

(Re($\nu$) $> -1$, Re($\lambda$) $> 0$)

respectively.
Proof. Proofs are clear from Theorem 2.3 in Section 2. 

**Corollary 6.3.1** Integral representations satisfied by spherical Bessel and Bessel-Clifford functions are given by

\[ j_\nu(z) = \frac{\sqrt{\pi} \left(\frac{z}{2}\right)}{2\Gamma\left(\nu + \frac{3}{2}\right)} \int_0^\infty t^{\nu+\frac{1}{2}} e^{-t} 0F_3(-; \frac{2\nu + 3}{2}, \frac{2\nu + 3}{4}, \frac{2\nu + 5}{4}; -\frac{z^2 t^2}{16}) dt, \quad \text{Re}(\nu) > -\frac{3}{2} \]

\[ C_\nu(z) = \frac{1}{\Gamma(\nu + 1)^2} \int_0^\infty t^\nu e^{-t} 0F_3(-; \nu + 1, \frac{\nu + 1}{2}, \frac{\nu + 2}{2}; \frac{zt^2}{4}) dt, \quad \text{Re}(\nu) > -1. \]

**Theorem 6.4** The Laplace transforms of the generalized four parameter spherical Bessel and Bessel-Clifford functions are given by

\[ \mathcal{L}\{g_\nu^{(b,c)}(t; \rho)\}(s) = \frac{\sqrt{\pi}}{2s} \sum_{k=0}^\infty \frac{(-b)^k (c - \frac{1}{2} + \rho)_{2k+\nu+1}}{k! \Gamma(k + \nu + \frac{3}{2}) \Gamma(2k + \nu + \frac{3}{2})} \left(\frac{1}{2s}\right)^{2k+\nu}, \quad \text{Re}(\nu) > -\frac{3}{2}, \quad \text{Re}(c) > \frac{1}{2} \]

\[ \mathcal{L}\{C_\nu^{(b,\lambda)}(t; \rho)\}(s) = \sum_{k=0}^\infty \frac{(-b)^k (\lambda, \rho)_{2k+\nu}}{\Gamma(\nu + k + 1) \Gamma(\nu + 2k + 1) s^{k+1}}, \quad \text{Re}(\nu) > -1, \quad \text{Re}(\lambda) > 0. \]

**Proof.** Laplace transforms of the generalized four parameter spherical Bessel and Bessel-Clifford functions can be calculated in a similar way as in Theorem 2.4 in Section 2. 

**Corollary 6.4.1** The Laplace transforms of the spherical Bessel and Bessel-Clifford functions are given by

\[ \mathcal{L}\{j_\nu(t)\}(s) = \frac{\sqrt{\pi}}{2s} \sum_{k=0}^\infty \frac{(-1)^k \Gamma(\nu + 2k + 1)}{k! \Gamma(k + \nu + \frac{3}{2})} \left(\frac{1}{2s}\right)^{2k+\nu}, \quad \text{Re}(\nu) > -\frac{3}{2} \]

\[ \mathcal{L}\{C_\nu(t)\}(s) = \sum_{k=0}^\infty \frac{1}{\Gamma(\nu + k + 1) s^{k+1}}, \quad \text{Re}(\nu) > -1. \]

**Theorem 6.5** The Mellin transforms of the generalized four parameter spherical Bessel and Bessel-Clifford functions are given by

\[ \mathcal{M}\{e^{-z} g_\nu^{(b,c)}(z; \rho); s\} = \frac{\sqrt{\pi} \Gamma(c)}{2^{\nu+1} \Gamma(\nu + \frac{3}{2})^2 \Gamma(c - \frac{1}{2})} \sum_{k=0}^\infty \frac{(-1)^k b^k (c; \rho)_{\nu+2k} \Gamma(s + \nu + 2k)}{(\nu+2k)_{\nu+2k} k! 16^k k!}, \quad \text{Re}(\nu) > -\frac{3}{2}, \quad \text{Re}(c) > \frac{1}{2}, \quad s + \nu \neq \{0, -1, -2, ...\} \]

\[ \mathcal{M}\{e^{-z} C_\nu^{(b,\lambda)}(z; \rho); s\} = \frac{\Gamma(s)}{\Gamma(\nu + 1)^2} \sum_{k=0}^\infty \frac{(-b)^k (\lambda, \rho)_{\nu+2k} s_k}{(\nu + 1)_k (\nu + 2)_k 16^k k!}, \quad \text{Re}(\nu) > -1, \quad \text{Re}(s) > 0, \quad \text{Re}(\lambda) > 0. \]

**Proof.** Proofs of the corresponding Mellin transforms can be done by Theorem 2.5 in Section 2. 

**Corollary 6.5.1** The Mellin transforms of the spherical Bessel and Bessel-Clifford functions are given
Bessel-Clifford functions are given by

\[ \mathcal{M}\{e^{-z}j_\nu(z); s\} = \frac{\sqrt{\pi}\Gamma(\nu + s)}{2^{\nu+1}\Gamma(\nu + \frac{3}{2})} \frac{\Gamma(\nu + s + 1)}{2}; \frac{2
u + 3}{2}; -1 \]

\[ \text{Re}(\nu) > -\frac{3}{2}, \ s + \nu \neq 0, -1, -2, \ldots \]

\[ \mathcal{M}\{C_\nu(z); s\} = \frac{\Gamma(s)}{\Gamma(\nu + 1)} \frac{\Gamma(s)}{\Gamma(s + 1 - \nu)} ; \frac{2}{s + 1 - \nu}; \frac{2
nu + 3}{2}; -1 \]

\[ \text{Re}(\nu) > -1, \text{ Re}(s) > 0. \]

The following Mellin transforms are also satisfied by the generalized four parameter spherical Bessel and Bessel-Clifford functions, respectively:

**Theorem 6.6** The Mellin transforms of the generalized four parameter spherical Bessel and Bessel-Clifford functions are given by

\[ \mathcal{M}\{g_{\nu}^{(b,c)}(z; \rho); s\} = \frac{\sqrt{\pi}\Gamma(s)}{2\Gamma(\nu + \frac{3}{2})^2} \frac{\Gamma(c + \nu + s)}{2}\sum_{m=0}^\infty \frac{(\nu + 2m)!\Gamma(\nu + m)}{m!} J_{\nu+2m}(z) \]

\[ \times 2F_3\left(\frac{c + \nu + s}{2}, \frac{c + \nu + s + 1}{2}; \frac{2\nu + 3}{2}, \frac{2\nu + 5}{4}; \frac{4}{4}; \frac{-bz^2}{4} \right), \]

\[ (\text{Re}(\nu) > -\frac{3}{2}, \text{ Re}(c) > \frac{1}{2}, \text{ Re}(s) > 0) \]

\[ \mathcal{M}\{C_{\nu}^{(b,\lambda)}(z; \rho); s\} = \frac{\Gamma(s)}{\Gamma(\nu + 1)^2} \frac{\Gamma(\nu + s)}{\Gamma(\nu + s + 1 - \nu)} \frac{2\Gamma(\nu + s)}{2}\sum_{n=0}^\infty \frac{(-m - n)^n(\nu + \frac{3}{2})}{n!(\nu + \frac{3}{2})_n} C_{\nu+2n}(z), \]

\[ (\text{Re}(\nu) > -1, \text{ Re}(\lambda) > 0, \text{ Re}(s) > 0). \]

**Proof.** Proofs can be done in a similar way as Theorem 2.6 in Section 2. \(\blacksquare\)

**Theorem 6.7** Series expansions satisfied by the generalized four parameter spherical Bessel and Bessel-Clifford functions are given by

\[ g_{\nu}^{(b,c)}(z; \rho) = \frac{\sqrt{\pi}}{2\Gamma(\nu + \frac{3}{2})^2} \sum_{m=0}^\infty \sum_{n=0}^\infty (\frac{z}{2})^{\nu-m+n} J_{\nu+m+n}(z) \frac{(-m - n)^n(b^n(c - \frac{1}{2}); \rho)_2n+\nu+1}{n!(\nu + \frac{3}{2})_n} \]

\[ (\text{Re}(\mu) > -1, \text{ Re}(\nu) > -\frac{3}{2}, \text{ Re}(c) > \frac{1}{2}) \]

\[ C_{\nu}^{(b,\lambda)}(z; \rho) = \frac{1}{\Gamma(\nu + 1)^2} \sum_{m=0}^\infty \sum_{n=0}^\infty \frac{(-m - n)^n\lambda^\nu}{\Gamma(\nu + 1)_n} \frac{(-b)^n(\nu + \frac{3}{2})_n}{(m + n)!(-1)^n(\nu + 1)_n}, \]

\[ (\text{Re}(\nu) > -1, \text{ Re}(\lambda) > 0). \]

**Proof.** Formulas can be calculated from Theorem 2.7 in Section 2. \(\blacksquare\)

**Corollary 6.7.1** For spherical Bessel and Bessel-Clifford functions, it can be seen that

\[ j_\nu(z) = \frac{\sqrt{\pi}}{2\Gamma(\nu + \frac{3}{2})^2} \sum_{m=0}^\infty \sum_{n=0}^\infty (\frac{z}{2})^{\nu-m+n} J_{\nu+m+n}(z) \frac{(-m - n)^n(\mu + 1)_n}{n!(\nu + \frac{3}{2})_n}, \text{ Re}(\nu) > -\frac{3}{2} \]

\[ C_\nu(z) = \frac{1}{\Gamma(\nu + 1)} \sum_{m=0}^\infty \sum_{n=0}^\infty \frac{(-m - n)^n\lambda^\nu J_{\nu+m+n}(z)}{(m + n)!(-1)^n(\nu + 1)_n}, \text{ Re}(\nu) > -1. \]

**Theorem 6.8** Recurrence relations satisfied by the generalized four parameter spherical Bessel and Bessel-Clifford functions are given by

\[ \frac{\partial}{\partial z}[z^{\nu+\frac{1}{2}} \frac{\partial}{\partial z}[z^{-\nu} g_{\nu}^{(b,c-1)}(z; \rho)]] = -b\frac{3}{2}(c - \frac{3}{2}) g_{\nu+1}^{(b,c)}(z; \rho), \]

\[ \frac{\partial}{\partial z}[(\sqrt{z})^{\nu+\frac{1}{2}} \frac{\partial}{\partial z}[C_{\nu}^{(b,\lambda-1)}(z; \rho)]] = -b\frac{\lambda - 1}{2} (\sqrt{z})^\nu C_{\nu+1}^{(b,\lambda)}(z; \rho). \]
Proof. Recurrence formulas can be found as in Theorem 3.2 in Section 3. □

**Theorem 6.9** Recurrence relations satisfied by the generalized four parameter spherical Bessel and Bessel-Clifford functions are given by

\[ \frac{\partial}{\partial z} \left[ z^{-\nu + \frac{1}{2}} \frac{\partial}{\partial z} [z^{\nu + 1} g_{\nu}^{(\alpha, \epsilon - 1)}(z; \rho)] \right] = z^{\frac{1}{2}} (c - \frac{3}{2}) g_{\nu - 1}^{(\alpha, \epsilon)}(z; \rho), \]

\[ \frac{\partial}{\partial z} \left[ (\sqrt{z})^{-\nu + \frac{1}{2}} \frac{\partial}{\partial z} \right] \left[ (\sqrt{z})^{2\nu} C_{\nu}^{(b, \lambda - 1)}(z; \rho) \right] = (\sqrt{z})^{\nu - 2} (\frac{\lambda - 1}{2}) C_{\nu - 1}^{(b, \lambda)}(z; \rho). \]

Proof. Recurrence relations can be seen from Theorem 3.1 in Section 3. □

**Theorem 6.10** Derivative formulas satisfied by the generalized four parameter spherical Bessel and Bessel-Clifford functions are given by

\[ \frac{\partial}{\partial \rho} \left[ g_{\nu}^{(\alpha, \epsilon)}(z; \rho) \right] = -\frac{1}{c - \frac{3}{2}} g_{\nu}^{(\alpha, \epsilon - 1)}(z; \rho), \]

\[ \frac{\partial}{\partial \rho} \left[ C_{\nu}^{(b, \lambda)}(z; \rho) \right] = -\frac{1}{\lambda - 1} C_{\nu}^{(b, \lambda - 1)}(z; \rho). \]

Proof. Derivatives can directly seen from Theorem 3.3 in Section 3. □

**Theorem 6.11** Recurrence relations satisfied by the generalized four parameter spherical Bessel and Bessel-Clifford functions are given by

\[ z^2 \left[ \frac{\partial^2}{\partial z^2} g_{\nu - 1}^{(\alpha, \epsilon - 1)}(z; \rho) + b \frac{\partial^2}{\partial z^2} g_{\nu + 1}^{(\alpha, \epsilon - 1)}(z; \rho) \right] + z \left[ \frac{5}{2} - \nu \right] \frac{\partial}{\partial z} g_{\nu - 1}^{(\alpha, \epsilon - 1)}(z; \rho) + b(\nu + 1) \frac{\partial}{\partial z} g_{\nu + 1}^{(\alpha, \epsilon - 1)}(z; \rho) \]

\[ = -\frac{(1 - \nu)}{2} g_{\nu - 1}^{(\alpha, \epsilon - 1)}(z; \rho) - b \frac{(2 + \nu)}{2} g_{\nu + 1}^{(\alpha, \epsilon - 1)}(z; \rho), \]

\[ -b(3\nu + 5) z \frac{\partial}{\partial z} C_{\nu + 1}^{(b, \lambda - 1)}(z; \rho) - b(\nu + 1)^2 C_{\nu + 1}^{(b, \lambda - 1)}(z; \rho) - 2bz^2 \frac{\partial^2}{\partial z^2} C_{\nu + 1}^{(b, \lambda - 1)}(z; \rho) \]

\[ = (\nu + 1) \frac{\partial}{\partial z} C_{\nu - 1}^{(b, \lambda - 1)}(z; \rho) + 2z \frac{\partial^2}{\partial z^2} C_{\nu - 1}^{(b, \lambda - 1)}(z; \rho). \]

Proof. Recurrence formulas satisfied by generalized four parameter spherical Bessel and Bessel-Clifford functions can be proved by Theorem 3.4 in Section 3. □

In Theorem 6.11, taking \( b = -1, \lambda = 2 \) and \( \rho = 0 \) \( C_{\nu - 1}^{(b, \lambda - 1)}(z; \rho) \) is reduced to \( C_{\nu - 1}^{(b, \lambda - 1)}(z; \rho) \) and under the same substitutions \( C_{\nu + 1}^{(b, \lambda - 1)}(z; \rho) \) is reduced to \( C_{\nu + 1}(z; \rho) \). For \( b = 1, \alpha = 5 \) and \( \rho = 0 \), \( g_{\nu - 1}^{(\alpha, \epsilon - 1)}(z; \rho) \) is reduced to \( j_{\nu - 1}(z) \) and under the same substitutions \( g_{\nu + 1}^{(\alpha, \epsilon - 1)}(z; \rho) \) is reduced to \( j_{\nu + 1}(z) \). Hence, the following Corollary is obtained:

**Corollary 6.11.1** Recurrence relations satisfied by the usual spherical Bessel and Bessel-Clifford functions are given by

\[ z^2 \left[ \frac{d^2}{dz^2} j_{\nu - 1}(z) + \frac{d^2}{dz^2} j_{\nu + 1}(z) \right] + z \left[ \frac{5}{2} - \nu \right] \frac{d}{dz} j_{\nu - 1}(z) + (\nu + 1) \frac{d}{dz} j_{\nu + 1}(z) \]

\[ = -\frac{(1 - \nu)}{2} j_{\nu - 1}(z) - \frac{(2 + \nu)}{2} j_{\nu + 1}(z), \]

\[ (3\nu + 5) z \frac{d}{dz} C_{\nu + 1}(z) + (\nu + 1)^2 C_{\nu + 1}(z) + 2z^2 \frac{d^2}{dz^2} C_{\nu + 1}(z) \]

\[ = (\nu + 1) \frac{d}{dz} C_{\nu - 1}(z) + 2z \frac{d^2}{dz^2} C_{\nu - 1}(z). \]

Proof. To obtain the recurrence relation satisfied by spherical Bessel function, the following recurrence relation should be written

\[ \frac{z}{2\nu + 1} \left[ j_{\nu - 1}(z) + j_{\nu + 1}(z) \right] = j_{\nu}(z). \]
Differentiating the above recurrence formula with respect to \( z \) yields
\[
\frac{d}{dz} j_\nu(z) = \frac{1}{2\nu + 1} \left[ j_{\nu-1}(z) + j_{\nu+1}(z) + z\left( \frac{d}{dz} j_{\nu-1}(z) + \frac{d}{dz} j_{\nu+1}(z) \right) \right].
\]
The second recurrence formula satisfied by spherical Bessel function is given by
\[
\frac{d}{dz} j_\nu(z) = \frac{1}{2\nu + 1} \left[ \nu j_{\nu-1}(z) - (\nu + 1) j_{\nu+1}(z) \right].
\]
Comparing the last two equations, one can get
\[
(\nu - 1) j_{\nu-1}(z) - (\nu + 2) j_{\nu+1}(z) = z \left( \frac{d}{dz} j_{\nu-1}(z) + \frac{d}{dz} j_{\nu+1}(z) \right).
\]
Inserting \( j_{\nu-1}(z) \) and \( j_{\nu+1}(z) \)
\[
j_{\nu-1}(z) = z^{-\nu-1} \frac{d}{dz} (z^{\nu+1} j_\nu(z)),
\]
\[
j_{\nu+1}(z) = z^\nu \frac{d}{dz} (z^{-\nu} j_\nu(z)),
\]
and then taking corresponding derivatives gives
\[
(\nu - 1) z j_{\nu-1}(z) - z(\nu + 2) j_{\nu+1}(z) = z^2 \left( \frac{d}{dz} j_{\nu-1}(z) + \frac{d}{dz} j_{\nu+1}(z) \right).
\]
Now, substituting \( j_\nu(z) \) and \( \frac{d}{dz} j_\nu(z) \)
\[
\frac{d}{dz} j_\nu(z) = \frac{z}{2\nu + 1} \left[ j_{\nu-1}(z) + j_{\nu+1}(z) \right],
\]
\[
\frac{d}{dz} j_\nu(z) = \frac{1}{2\nu + 1} \left[ \nu j_{\nu-1}(z) - (\nu + 1) j_{\nu+1}(z) \right]
\]
in above recurrence formula, one can get
\[
(\nu - 1) z j_{\nu-1}(z) - z(\nu + 2) j_{\nu+1}(z) = z^2 \left( \frac{d}{dz} j_{\nu-1}(z) + \frac{d}{dz} j_{\nu+1}(z) \right).
\]
Differentiating with respect to \( z \) in the last recurrence formula and then multiplying with 2 on both sides gives
\[
2z^2 \left( \frac{d^2}{dz^2} j_{\nu-1}(z) + \frac{d^2}{dz^2} j_{\nu+1}(z) \right)
\]
\[
= z \left[ (2\nu - 6) \frac{d}{dz} j_{\nu-1}(z) - (2\nu + 8) \frac{d}{dz} j_{\nu+1}(z) \right] + (2\nu - 2) j_{\nu-1}(z) - (2\nu + 4) j_{\nu+1}(z).
\]
Finally, using the fact that
\[
(\nu - 1) j_{\nu-1}(z) - (\nu + 2) j_{\nu+1}(z) = z \left( \frac{d}{dz} j_{\nu-1}(z) + \frac{d}{dz} j_{\nu+1}(z) \right)
\]
result is obtained.

To get the recurrence formula satisfied by the usual Bessel-Clifford function, the following recurrence formula can be written
\[
z C_{\nu+2}(z) + (\nu + 1) C_{\nu+1}(z) = C_\nu(z).
\]
Taking derivative with respect to \( z \) in above recurrence yields
\[
C_{\nu+2}(z) + z \frac{d}{dz} C_{\nu+2}(z) + (\nu + 1) \frac{d}{dz} C_{\nu+1}(z) = \frac{d}{dz} C_\nu(z).
\]

35
Using the derivative formula
\[ \frac{d}{dz} C_{\nu}(z) = C_{\nu+1}(z) \]
last recurrence formula can be written as
\[ (\nu + 2) \frac{d^2}{dz^2} C_{\nu}(z) + z \frac{d^2}{dz^2} C_{\nu+1}(z) = \frac{d^2}{dz^2} C_{\nu-1}(z). \]

Multiplying with \( z \) on both sides of the above equation gives
\[ (\nu + 2) z \frac{d^2}{dz^2} C_{\nu}(z) + z^2 \frac{d^2}{dz^2} C_{\nu+1}(z) = z \frac{d^2}{dz^2} C_{\nu-1}(z). \]

Substituting
\[ \frac{d^2}{dz^2} C_{\nu}(z) = \frac{d}{dz} C_{\nu+1}(z) \]
gives
\[ (\nu + 2) z \frac{d}{dz} C_{\nu+1}(z) + z^2 \frac{d}{dz} C_{\nu+1}(z) = z \frac{d}{dz} C_{\nu-1}(z). \]

Multiplying by 2 on both sides and then adding the term \((\nu + 1) \frac{d}{dz} C_{\nu-1}(z)\) gives
\[ z(2\nu + 4) \frac{d}{dz} C_{\nu+1}(z) + 2z^2 \frac{d}{dz} C_{\nu+1}(z) + (\nu + 1) \frac{d}{dz} C_{\nu-1}(z) \]
\[ = (\nu + 1) \frac{d}{dz} C_{\nu+1}(z) + 2z \frac{d}{dz} C_{\nu+1}(z). \]

Using the fact that
\[ (\nu + 1) \frac{d}{dz} C_{\nu+1}(z) + (\nu + 1)^2 C_{\nu+1}(z) = (\nu + 1) \frac{d}{dz} C_{\nu-1}(z) \]
which is directly seen by first two recurrence formulas satisfied by Bessel-Clifford function in Section 1, whence the result.

**Theorem 6.12** Partial differential equations satisfied by the generalized four parameter spherical Bessel and Bessel-Clifford functions are given by
\[ (-4\nu^2 - 4\nu + 6) z \frac{\partial^3}{\partial z \partial \rho^2} g_{\nu}^{(b,c)}(z; \rho) + (-4\nu^2 - 4\nu + 39) z^2 \frac{\partial^4}{\partial z^2 \partial \rho^2} g_{\nu}^{(b,c)}(z; \rho) + 28z^3 \frac{\partial^5}{\partial z^3 \partial \rho^2} g_{\nu}^{(b,c)}(z; \rho) \]
\[ + 4z^4 \frac{\partial^6}{\partial z^4 \partial \rho^2} g_{\nu}^{(b,c)}(z; \rho) + (\nu^2 + \nu) \frac{\partial^2}{\partial \rho^2} g_{\nu}^{(b,c)}(z; \rho) \]
\[ = -4b z^2 g_{\nu}^{(b,c)}(z; \rho), \]
\[ (\nu^3 + 4\nu^2 + 5\nu + 2) \frac{\partial^3}{\partial z \partial \rho^2} C_{\nu}^{(b,\lambda)}(z; \rho) + (5\nu^2 + 21\nu + 22) z \frac{\partial^4}{\partial z^2 \partial \rho^2} C_{\nu}^{(b,\lambda)}(z; \rho) \]
\[ + (8\nu + 22) z^2 \frac{\partial^5}{\partial z^3 \partial \rho^2} C_{\nu}^{(b,\lambda)}(z; \rho) + 4z^3 \frac{\partial^6}{\partial z^4 \partial \rho^2} C_{\nu}^{(b,\lambda)}(z; \rho) \]
\[ = -b C_{\nu}^{(b,\lambda)}(z; \rho). \]

**Proof.** Partial differential equations satisfied by the generalized four parameter spherical Bessel and Bessel-Clifford functions can be proved by Theorem 3.5 in Section 3.

**7 Concluding Remarks**

The generalizations of Bessel functions were introduced by some authors in 3, 16, 19, 22, 23, 24, 38, 43, 68. In the present article, defining the unification of the Bessel functions via the generalized Pochhammer function, some potentially useful formulas are obtained. Some of these important formulas are new. In the below table, some special cases of the unified four parameter Bessel function are presented:
### 7.1 Special Cases of the Unified Four Parameter Bessel Function $G_{\nu, c}^{(b,c)}(z; \rho)$

| No | Values of the parameters | Relation Between $G_{\nu, c}^{(b,c)}(z; \rho)$ and its special case |
|----|--------------------------|---------------------------------------------------------------------|
| I  | $b = -1, c = \frac{1}{2}, \nu \rightarrow \nu + \frac{1}{2}, \rho = 0$ | $G_{\nu + \frac{1}{2}}^{(-1,1)}(z) = \frac{\Gamma(\nu+1)}{\sqrt{\pi}\Gamma(\nu+\frac{3}{2})} \frac{1}{2} F_3 \left(\nu+\frac{1}{2}, \nu+2, \nu; \frac{z}{4}, \frac{z^2}{4} \right)$ |
| II | $b = 1, c = \frac{1}{2}, \nu \rightarrow \nu + \frac{1}{2}, \rho = 0$ | $G_{\nu + \frac{1}{2}}^{(1,1)}(z) = \frac{\Gamma(\nu+1)}{\sqrt{\pi}\Gamma(\nu+\frac{3}{2})} \frac{1}{2} F_3 \left(\nu+\frac{1}{2}, \nu+2, \nu; \frac{z}{4}, \frac{z^2}{4} \right)$ |
| III | $b = -1, c = 1, \nu \rightarrow \nu + \frac{1}{2}, \rho = 0$ | $G_{\nu + \frac{1}{2}}^{(-1,1)}(z) = \frac{z^\nu}{\sqrt{\pi}\Gamma(\nu+\frac{3}{2})} \frac{1}{2} F_3 \left(\nu+\frac{1}{2}, \nu+2, \nu; \frac{z}{4}, \frac{z^2}{4} \right)$ |
| IV | $b = 1, c = 1, \nu \rightarrow \nu + \frac{1}{2}, \rho = 0$ | $G_{\nu + \frac{1}{2}}^{(1,1)}(z) = \frac{z^\nu}{\sqrt{\pi}\Gamma(\nu+\frac{3}{2})} \frac{1}{2} F_3 \left(\nu+\frac{1}{2}, \nu+2, \nu; \frac{z}{4}, \frac{z^2}{4} \right)$ |
| V  | $b = -1, c = \frac{3}{2}, \nu \rightarrow \nu - \frac{1}{2}, \rho = 0$ | $G_{\nu - \frac{1}{2}}^{(-1,1)}(z) = \frac{\Gamma(\nu+1)}{\sqrt{\pi}\Gamma(\nu+\frac{3}{2})} \frac{1}{2} F_3 \left(\nu+\frac{1}{2}, \nu+2, \nu; \frac{z}{4}, \frac{z^2}{4} \right)$ |
| VI | $b = 1, c = \frac{3}{2}, \nu \rightarrow \nu - \frac{1}{2}, \rho = 0$ | $G_{\nu - \frac{1}{2}}^{(1,1)}(z) = \frac{\Gamma(\nu+1)}{\sqrt{\pi}\Gamma(\nu+\frac{3}{2})} \frac{1}{2} F_3 \left(\nu+\frac{1}{2}, \nu+2, \nu; \frac{z}{4}, \frac{z^2}{4} \right)$ |
| VII| $b = -1, c = 1, \nu \rightarrow \nu - \frac{1}{2}, \rho = 0$ | $G_{\nu - \frac{1}{2}}^{(-1,1)}(z) = \frac{\Gamma(\nu+1)}{\sqrt{\pi}\Gamma(\nu+\frac{3}{2})} \frac{1}{2} F_3 \left(\nu+\frac{1}{2}, \nu+2, \nu; \frac{z}{4}, \frac{z^2}{4} \right)$ |
| VIII| $b = 1, c = 1, \nu \rightarrow \nu - \frac{1}{2}, \rho = 0$ | $G_{\nu - \frac{1}{2}}^{(1,1)}(z) = \frac{\Gamma(\nu+1)}{\sqrt{\pi}\Gamma(\nu+\frac{3}{2})} \frac{1}{2} F_3 \left(\nu+\frac{1}{2}, \nu+2, \nu; \frac{z}{4}, \frac{z^2}{4} \right)$ |
| IX | $b = -1, c = 1, \nu = 0, \rho = 0$ | $G_{0}^{(-1,1)}(z) = F_1(-1; \frac{z}{2})$ |
| X  | $b = 1, c = 1, \nu = 0, \rho = 0$ | $G_{0}^{(1,1)}(z) = F_1(-1; \frac{z}{2})$ |
| XI | $b = 1, c = 1, \nu = \frac{1}{2}, \rho = 0$ | $G_{0}^{(1,1)}(z) = \sqrt{\frac{2}{\pi}} \sin z = \frac{1}{J_1(z)}$ |
| XII| $b = 1, c = 1, \nu = -\frac{1}{2}, \rho = 0$ | $G_{0}^{(-1,1)}(z) = \sqrt{\frac{2}{\pi}} \cos z = J_{\frac{1}{2}}(z)$ |
| XIII| $b = 1, c = 1, \rho = 0$ | $G_{0}^{(1,1)}(z) = J_{\nu}(z)$ |
| XIV| $b = -1, c = 1, \rho = 0$ | $G_{0}^{(-1,1)}(z) = I_{\nu}(z)$ |
7.2 Graphics

In below, the graphics of the generalized two parameter Bessel functions of the first kind $J_{\nu}^{(c)}(x)$ are drawn for the special cases $\nu = 0$ and $c = 1, 2, 3$, $\nu = \frac{1}{2}$ and $c = 1, 2, 3$, $\nu = \frac{3}{2}$ and $c = 1, 2, 3$ and the graphics of the generalized two parameter spherical Bessel function $g_{\nu}^{(c)}(x)$ are drawn for the special cases $\nu = 0$ and $c = \frac{3}{2}, \frac{5}{2}, \frac{7}{2}$, $\nu = \frac{1}{2}$ and $c = \frac{3}{2}, \frac{5}{2}, \frac{7}{2}$ and $\nu = \frac{3}{2}$ and $c = \frac{3}{2}, \frac{5}{2}, \frac{7}{2}$ respectively.

7.3 Comparison of Graphs of the Generalized Two Parameter Bessel Function $J_{\nu}^{(c)}(x)$ with $J_0(x), J_{\frac{1}{2}}(x)$ and $J_{\frac{3}{2}}(x)$

![Graphs of Bessel Functions](image1.png)

![Graphs of Bessel Functions](image2.png)

![Graphs of Bessel Functions](image3.png)

![Graphs of Bessel Functions](image4.png)

![Graphs of Bessel Functions](image5.png)

![Graphs of Bessel Functions](image6.png)
7.4 Comparison of Graphs of the Generalized Two Parameter Spherical Bessel Function $g^{(c)}_\nu(x)$ with $j_0(x)$, $j_{\frac{1}{2}}(x)$ and $j_{\frac{3}{2}}(x)$
7.5 Application Fields of the Generalized Two Parameter Modified Bessel Function to Digital Signal Processing

Digital signal processing affords greater flexibility, higher performance (in terms of attenuation and selectivity), better time and environment stability and lower equipment production costs than traditional analog techniques. A digital filter is simply a discrete-time, discrete-amplitude convolver. Digital filters are commonly used for audio frequencies for two reasons. First, digital filters for audio are superior in price and performance to the analog alternative. Second, audio analog to digital converters and digital to analog converters can be manufactured with high accuracy and are available at low cost. To design a digital filter, there are some known methods such as Fourier series method, frequency sampling method, window method. Besides, there are other various design methods to design a filter. For instance, Neural Network [8, 35], Genetic Algorithm [48], Particle Swarm Optimization [41, 71], Discrete Cosine Transform [67], Chebychev Criterion [34], etc. Among of them,
Kaiser Window [52] has an advantage of trading-off the transition width against the ripple that is defined by

\[ w_n = \begin{cases} \frac{I_0(\pi \alpha \sqrt{1-(\frac{2n}{N}-1)^2})}{I_0(\pi \alpha)}, & 0 \leq n \leq N - 1 \\ 0, & \text{otherwise} \end{cases} \]

where \( N \) is the length of the sequence, \( I_0 \) is the zeroth order modified Bessel function of the first kind and \( \alpha \) is an arbitrary non-negative real number that determines the shape of the window.

The generalized Kaiser-window function is defined by

\[ w_n^{(c)} = \begin{cases} \frac{I_0^{(c)}(\pi \alpha \sqrt{1-(\frac{2n}{N}-1)^2})}{I_0^{(c)}(\pi \alpha)}, & 0 \leq n \leq N - 1 \\ 0, & \text{otherwise} \end{cases} \]

where \( \text{Re}(c) > 0 \). The case \( c = 1 \), it is reduced to usual Kaiser-window function \( w_n \). Some properties of Kaiser-window function can be applied to generalized Kaiser-window function, as well.

References

[1] Abramowitz M. Regular and irregular Coulomb wave functions expressed in terms of Bessel-Clifford functions. J Math Physics, 1954, 33 : 111-116.

[2] Abramowitz M. Coulomb wave functions expressed in terms of Bessel-Clifford and Bessel functions. J Math Physics, 1951, 29 : 303-308.

[3] Agarwal AK. A generalization of di-Bessel function of Exton. Indian J Pure Appl Math, 1984, 15 : 139-148.

[4] Agarwal P, Jain S, Agarwal S, Nagpal M. On a new class of integrals involving Bessel functions of the first kind. Communications in Numerical Analysis, 2014, 1-7.

[5] Andrews GE, Askey R, Roy R. Special Functions, Cambridge University Press, 1999.

[6] Arfken GB, Weber HJ. Mathematical Methods for Physicists. Sixth Edition, 2005.

[7] Babusci D, Dattoli G, Del Franco M. Lectures on Mathematical Methods for Physics. ENEA Technical Report, 2010.

[8] Bhattacharya D, Antoniou A. Real Time Design of FIR Filters of Feedback Neural Network, 1996, 3 : 1070-1078.

[9] Boyvel LP, Jones AR. Electromagnetic Scattering and Its Applications. Applied Science Publishers, London, 1981.

[10] Caçao I, Ricci PE. Monomiality principle and eigenfunctions of Differential Operators, International Journal of Mathematics and Mathematical Sciences. 2011.

[11] Chaudhry MA, Ahmad M. On improper integrals of products of logarithmic power and Bessel functions. Bulletin Australian Mathematical Society, 1992, 45 : 395-398.

[12] Chaudhry MA, Qadir A, Srivastava HM, Paris RB. Extended hypergeometric and confluent hypergeometric functions. Appl Math Comp 2004, 159 : 589-602.

[13] Chaudhry MA, Zubair SM. Generalized incomplete gamma functions with applications. J Comp Appl Math, 1994, 55 : 99-124.

[14] Chaudhry MA, Zubair SM. Extended incomplete gamma functions with applications. Journal of Mathematical Analysis and Applications, 2002, 274 : 725-745.
[15] Choi J, Agarwal P. Certain unified integrals associated with Bessel functions. Boundary Value Problems, 2013, 1, 95.

[16] Choi J, Agarwal P, Mathur S, Purohit SD. Certain new integral formulas involving the generalized Bessel functions. Bull Korean Math Soc, 2014, 51 : 995-1003.

[17] Colton D, Kress R. Inverse Acoustic and Electromagnetic Scattering Theory. Second ed., Applied Mathematical Sciences, Springer-Verlag, Berlin, 1998.

[18] Dattoli G. Hermite-Bessel and Laguerre-Bessel functions: a by-product of the monomiality principle. In Proceedings of the Melfi School on Advanced Topics in Mathematics and Physics, Advanced Special Functions and Its Applications, Aracne Editrice, Melfi, Italy, 2000, 147-164.

[19] Dattoli G, Maino G. A unified point of view on the theory of generalized Bessel functions. Computers Math Applic, 1995, 30 : 113-125.

[20] Davis AMJ. Drag modifications for a sphere in a rotational motion at small non-zero Reynolds and Taylor numbers: wake interference and possible Coriolis effects. J Fluid Mech, 1992, 237 : 13-22.

[21] Exton H. On a generalization of the Bessel-Clifford equation and an application in quantum mechanics. Riv Mat Univ Parma 1989, 4 : 41-46.

[22] François Swarttouw R. The Hahn-Exton q-Bessel Function, 1992.

[23] Galué L. A generalized Bessel function. Integral Transforms and Special functions, 2010.

[24] Galué L. Evaluation of Some Integrals Involving Generalized Bessel Functions. Integral Transforms and Special Functions, 2001, 12 : 251-256.

[25] Glasser ML, Montaldi E. Some integrals involving Bessel functions. Journal of Mathematical Analysis and Applications, 1994, 183 : 577-590.

[26] Gray A, Mathews GB, MacRobert TM. A Treatise on Bessel functions and their applications to physics. Second ed., Macmillan, London, 1922.

[27] Greenhill AG. The Bessel-Clifford function and its applications. Philosophical Magazine Sixth Series, 1919, 501-528.

[28] Hai NT, Marichev OI, Srivastava HM. A Note on the Convergence of Certain Families of Multiple Hypergeometric Series. Journal of Mathematical Analysis and Applications, 1992, 164 : 104-115.

[29] Happell J, Brenner H. Low Reynolds Number Hydrodynamics with Special Applications to Particulate Media. 2nd edition, Noordhoff International Publishing, Leyden, 1973.

[30] Hayek N, Perez-Acosta F. Asymptotic expressions for the Dirac delta function in terms of the Bessel-Clifford functions. Pure Appl Math Sci, 1993, 37 : 53-56.

[31] Jackson JD. Classical Electrodynamics. Third ed., John Wiley and Sons Inc., New York, 1999.

[32] Jones DS. Acoustic and Electromagnetic Waves. Oxford Science Publications, The Clarendon Press Oxford University Press, New York, 1986.

[33] Kapany NS, Burke JJ. Optical Waveguides. Quantum Electronics-Principles and Applications, Academic Press, New York, 1972.

[34] Karakostantis G, Roy K. An Optimal Algorithm for Low Power Multiplierless FIR Filter Design Using Chebychev Criterion. IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP), Honolulu, Hawaii, USA, 2007, 858-861.
[35] Kaur H, Dhaliwal BS. Design of low pass FIR filter using Artificial Neural Network. 3rd International Conference on Computer and Automation Engineering, Chongqing, China, 2011, 463-466.

[36] Konopinski EJ. Electromagnetic Fields and Relativistic Particles. International Series in Pure and Applied Physics, McGraw-Hill Book Co., New York, 1981.

[37] Koronev BG. Bessel functions and their applications. Analytical Methods for Special Functions, Taylor and Francis Ltd., London-New York, 2002, 8.

[38] Korsch HJ, Klumpp A, Witthaut D. On two-dimensional Bessel functions. Journal of Physics, 2006, 39.

[39] Krivdashlykov SG. Quantum-Theoretical Formalism for Inhomogeneous Graded-Index Waveguides. Akademie Verlag, Berlin-New York, 1994.

[40] Lamb H. Hydrodynamics. Sixth ed., Cambridge University Press, Cambridge, 1932.

[41] Luitel B, Venayagamoorthy GK. Differential Evolution Particle Swarm Optimization for Digital Filter Design. IEEE Congress on Evolutionary Computation (CEC), Hong Kong, China, 2008, 3954-3961.

[42] Magnus W, Oberhettinger F, Tricomi FG. Tables of Integral Transforms. California Institute of Technology Bateman Manuscript Project, 1954.

[43] Mahmoud M. Generalized q-Bessel function and its properties. Advances in Difference Equations, 2013, 121.

[44] Messiah A. Quantum mechanics. North-Holland Publishing Co., Amsterdam, 1961.

[45] Miller AR. On the Mellin transform of products of Bessel and generalized hypergeometric functions. Journal of Computational and Applied Mathematics, 1997, 85: 271-286.

[46] Miller R. On the Mellin transform of products of Bessel and generalized hypergeometric functions. Journal of Computational and Applied Mathematics, 1997, 85: 271-286.

[47] Özergin E, Özarslan MA, Altin A. Extension of gamma, beta and hypergeometric functions. Journal of Computational and Applied Mathematics, 2011, 235: 4601-4610.

[48] Panwar BS, Chand A. Block Linkage Learning Genetic Algorithm: An Efficient Evolutionary Computational Technique for the Design of Ternary Weighted FIR Filters. IEEE World Congress on Computer Science and Information Engineering, Los Angeles, USA, 2009, 810-814.

[49] Rahman M. An integral representation and some transformation properties of q-Bessel functions. Journal of Mathematical Analysis and Applications, 1987, 125: 58-71.

[50] Rainville ED. Special Functions, 1960.

[51] Rayleigh L. The Theory of Sound. Second ed., Dover Publications, New York, 1945.

[52] Singh J, Singh C. Design of Low Pass FIR Filter Using General Regression Neural Network (GRNN). International Journal of Advanced Research in Computer Science and Software Engineering, 2013, 3.

[53] Slater JC. Microwave Transmission. McGraw-Hill Book Co, New York, 1942.

[54] Srivastava HM. The Laplace transform of the modified Bessel function of the second kind. Publications De L’institut mathematique, 1979, 26: 273-282.

[55] Srivastava HM, Agarwal P, Jain S. Generating functions for the generalized Gauss hypergeometric functions. Applied Mathematics and Computation, 2014, 247: 348-352.
[56] Srivastava HM, Çetinkaya A, Kıymaz O. A certain generalized Pochhammer symbol and its applications to hypergeometric functions. Applied Mathematics and Computation, 2014, 226 : 484-491.

[57] Srivastava HM, Doust MC. A Note on the Convergence of Kampe de Feriet’s Double Hypergeometric Series. Mathematische Nachrichten, 1972, 53 : 151-153.

[58] Srivastava HM, Karlsson PW. Multiple Gaussian Hypergeometric Series. John Wiley & Sons, New York, 1985.

[59] Srivastava HM, Manocha HL. A Treatise on Generating Functions. Ellis Horwood Series: Mathematics and Its Applications, UK, 1984.

[60] Srivastava HM, Parmar RK, Chapra P. A class of extended fractional derivative operators and associated generating relations involving hypergeometric functions. Axioms, 2012, 1 : 238-258.

[61] Srivastava R. Some classes of generating functions associated with a certain family of extended and generalized hypergeometric functions. Applied Mathematics and Computation, 2014, 243 : 132-137.

[62] Srivastava R. Some generalizations of Pochhammer’s symbol and their associated families of hypergeometric functions and hypergeometric polynomials. Applied Mathematics and Information Sciences, 2013, 7 : 2195-2206.

[63] Srivastava R, Cho NE. Some extended Pochhammer symbols and their applications involving generalized hypergeometric polynomials. Applied Mathematics and Computation, 2014, 234 : 277-285.

[64] Stone HA, McConnell HM. Hydrodynamics of quantized shape transitions of lipid domains. Proc Roy Soc London, 1994.

[65] Tanzosh J, Stone HA. Motion of a rigid particle in a rotating viscous flow: An integral equation approach. J Fluid Mech, 1994, 275 : 225-256.

[66] Tezer M. On the numerical evaluation of an oscillating infinite series. J Comput Appl Math, 1989, 28 : 383-390.

[67] Tseng CC, Lee SL. Design of Fractional Delay FIR Filter Using Discrete Cosine Transform. IEEE Asia Pacific Conference on Circuits and Systems (APCCAS), Macao, China, 2008, 858-886.

[68] Virchenko NO, Haidey VO. On Generalized $m$–Bessel Functions. Integral Transforms and Special Functions, 1999, 8 : 275-286.

[69] Watson GN. A treatise on the theory of Bessel’s functions, 1922.

[70] Yaşar BY. Generalized Mittag-Leffler Function and Its Properties. New Trends in Mathematical Sciences, 2015, 3 : 12-18.

[71] Zhao L, Zhou L, Huang W. Satisfactory Optimization Design of FIR Digital Filter Based On Adaptive Particle Swarm Optimization. IEEE International Conference on Control and Automation, Guangzhou, CHINA, 2007, 1662-1666.