On approximation of a Dirichlet problem for divergence form operator by Robin problems*

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Abstract

We show that under natural assumptions solutions of Dirichlet problems for uniformly elliptic divergence form operator can be approximated pointwise by solutions of some versions of Robin problems. The proof is based on stochastic representation of solutions and properties of reflected diffusions corresponding to divergence form operators.

Keywords: Robin problem, Dirichlet problem, divergence form operator, stochastic representation, reflected diffusion.

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1 Introduction

Let $D$ be a bounded Lipschitz domain in $\mathbb{R}^d$, $d \geq 3$, and $L$ be the divergence form operator

$$L = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_1} (a_{ij}(x) \frac{\partial}{\partial x_j})$$

with measurable coefficients $a_{ij}: D \to \mathbb{R}$ such that

$$a_{ij} = a_{ji}, \quad \Lambda^{-1} |\xi|^2 \leq \sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \leq \Lambda |\xi|^2, \quad x \in D, \quad \xi \in \mathbb{R}^d, \quad (1.1)$$

for some $\Lambda \geq 1$. For $f: D \to \mathbb{R}$, $g: \partial D \to \mathbb{R}$ and $n \geq 1$ we consider the following boundary-value problem

$$-Lu_n + \lambda u_n = f \quad \text{in } D, \quad -(a\nabla u_n) \cdot n + nu_n = ng \quad \text{on } \partial D, \quad (1.2)$$

where $a = \{a_{ij}\}_{1 \leq i,j \leq d}$ and $n(x)$ is the inward unit normal at $x \in \partial D$. Note that (1.2) is a particular version of Robin problem (also known as Fourier problem, Newton problem or the third boundary-value problem). It is known (see, e.g., [3, Appendix 1, Section 4.4]) that if $f \in L^2(D)$, $g \in H^1(D)$, then for each $n \geq 1$ there exists a unique weak solution of (1.2) and $u_n \to u$ in $H^1(D)$ as $n \to \infty$, where $u$ is the unique weak solution of the Dirichlet problem

$$-Lu + \lambda u = f \quad \text{in } D, \quad u = g \quad \text{on } \partial D. \quad (1.3)$$

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As noted in [8 p. 360], this approximation result is of practical interest, because many numerical schemes for solving boundary value problems (Dirichlet, Neumann, Neumann–Dirichlet) for $L$ consist in computing the solution of a discrete version of (1.2) or its modification with large $n$.

If $f \in L^p(D)$ with $p > d$ and $g \in H^1(D) \cap C(\partial D)$, then $u_n, u$ have continuous versions and one may ask whether $u_n \to u$ for every $x \in \bar{D}$. In this note, we give positive answer to this questions. Our proof is quite simple and is based on stochastic representation of solutions of (1.2), (1.3). But let us stress that in the proof of our positive answer to this question, our proof is quite simple and is based on stochastic representation of solutions of (1.2), (1.3). But let us stress that in the proof of our convergence results we use deep results from [5, 6] (see also [2] for the case $L = (1/2)\Delta$) saying that one can construct a process $M$ on $\bar{D}$ (reflected diffusion) associated with $L$ with a strong Feller resolvent. In fact, in these papers the strong Feller property is proved for less regular and possibly unbounded domains.

2 Preliminaries

In the paper, $D \subset \mathbb{R}^d$, $d \geq 3$, is a bounded Lipschitz domain (for a definition see, e.g., [4 Exercise 5.2.2]), $\bar{D} = D \cup \partial D$. We denote by $m$ or simply by $dx$ the $d$–dimensional Lebesgue measure. $\mathcal{B}(D)$ is the set of Borel subsets of $\bar{D}$. $\mathcal{B}_b(\bar{D})$ (resp. $C(\bar{D})$) is the set of bounded Borel (resp. continuous) functions on $\bar{D}$. To shorten notation, we write $L^2(D)$ instead of $L^2(D; m)$ and $L^2(\partial D)$ instead of $L^2(\partial D; \sigma)$.

We assume that the matrix $a$ satisfies (1.1) and consider the Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(D)$ defined by

$$\mathcal{E}(u, v) = \sum_{i,j=1}^d \int_D a_{ij}(x) \frac{\partial u}{\partial x_i}(x) \frac{\partial v}{\partial x_j}(x) \, dx, \quad u, v \in D(\mathcal{E}) := H^1(D),$$

(2.1)

where $H^1(D)$ is the usual Sobolev space of order 1, and for $\lambda > 0$ set $\mathcal{E}_\lambda(u, v) = \mathcal{E}(u, v) + \lambda(u, v)$, where $(\cdot, \cdot)$ is the usual inner product in $L^2(D; m)$. We denote by $(T_t)_{t>0}$ the strongly continuous semigroup of Markovian symmetric operators on $L^2(\bar{D})$ associated with $\mathcal{E}$ (see [4 Section 1.3]).

In the paper, we define quasi-notions (exceptional sets, quasi-continuity) with respect to $(\mathcal{E}, H^1(D))$. We will say that a property of points in $\bar{D}$ holds quasi everywhere (q.e. in abbreviation) if it holds outside some exceptional set. It is known (see [4 Lemm 2.1.4, Theorem 2.1.3]) that each element of $H^1(\bar{D})$ admits a quasi-continuous $m$-version, which we denote by $\tilde{u}$, and $\tilde{u}$ i.q.e. unique for every $u \in H^1(D)$.

In [6] Theorems 2.1 and 2.2 (see also [5]) it is proved that there exists a conservative diffusion process $M = \{(X, P_x), x \in \bar{D}\}$ on $\bar{D}$ associated with the Dirichlet form (2.1) in the sense that the transition density of $M$ defined as

$$p_t(x, B) = P_x(X_t \in B), \quad t > 0, \ x \in \bar{D}, \ B \in \mathcal{B}(\bar{D}),$$

has the property that

$$P_t f \quad \text{is an } m\text{-version of } T_t f \quad \text{for every } f \in B_b(\bar{D}),$$

where $P_tf(x) = \int_D f(y)p_t(x, dy) = E_x f(X_t)$. Furthermore, $(P_t)_{t>0}$ is strongly Feller in the sense that $P_t(B_b(\bar{D})) \subset C(\bar{D})$ and $\lim_{t \downarrow 0} P_t f(x) = f(x)$ for $x \in \bar{D}$, $f \in C(\bar{D})$. In particular (see [4 Exercise 4.2.1]), the transition density satisfies the following absolute continuity condition: $p_t(x, \cdot) \ll m$ for any $t > 0, \ x \in \bar{D}$. 

2
We denote by \((R_\alpha)_{\alpha>0}\) the resolvent associated with \(\mathbb{M}\) (or with \((P_t)_{t>0}\)), that is
\[
R_\alpha f(x) = E_x \int_0^\infty e^{-\alpha t} f(X_t) \, dt, \quad f \in \mathcal{B}_b(\bar{D}).
\]

Of course
\[
R_\alpha f(x) = \int_{\bar{D}} r_\alpha(x,y) f(y) \, dy, \quad \text{where} \quad r_\alpha(x,y) = \int_0^\infty e^{-\alpha t} p_t(x,y) \, dt.
\]

For a Borel measure \(\mu\) on \(\bar{D}\) we also set
\[
R_\alpha \mu(x) = \int_{\bar{D}} r_\alpha(x,y) \mu(dy), \quad x \in \bar{D}, \quad \alpha > 0,
\]
whenever the integral makes sense.

Let \(\sigma\) denote the surface measure on \(\partial D\). By [6, Lemma 5.1, Theorem 5.1], \(\sigma\) belongs to the space of smooth measures in the strict sense, and hence, by [4, Theorem 5.1.7], there is a unique positive continuous additive functional of \(\mathbb{M}\) in the strict sense with Revuz measure \(\sigma\). In what follows we denote it by \(A\). Note that for any \(g \in \mathcal{B}_b(\bar{D})\) we have
\[
R_\alpha (g \cdot \sigma)(x) = E_x \int_0^\infty e^{-\alpha t} g(X_s) \, dA_s, \quad x \in \bar{D}.
\]
indeed, by [4, Theorem 5.1.3] the above equality holds for \(\mu\)-a.e. \(x \in \bar{D}\), and hence for every \(x \in \bar{D}\), because \(p_t\) satisfies the absolute continuity condition and for any nonnegative \(g \in \mathcal{B}_b(\bar{D})\) both sides of the above equality are \(\alpha\)-excessive functions. Also note that the support of \(A\) is contained in \(\partial D\). Hence
\[
\int_0^t g(X_s) \, dA_s = \int_0^t 1_{\partial D}(X_s) g(X_s) \, dA_s, \quad P_x\text{-a.s., } \quad x \in \bar{D}
\]
(2.2)
(for more details see the beginning of the proof of Lemma 4.1). It follows that in fact the right-hand side of (2.2) is well defined for \(g \in \mathcal{B}_b(\partial D)\).

**Remark 2.1.** If, in addition, \(\frac{\partial a_{ij}}{\partial x_i} \in L^\infty(\bar{D})\), \(i,j = 1,\ldots,d\), then \(X = (X^1,\ldots,X^d)\) has the following Skorohod representation: for \(i = 1,\ldots,d\) and every \(x \in \bar{D}\)
\[
X^i_t - X^i_0 + M^i_t + N^i_t, \quad t \geq 0, \quad P_x\text{-a.s.,}
\]
(2.3)
where \(M^i\) are martingale additive functionals in the strict sense with covariations
\[
\langle M^i, M^j \rangle_t = 2 \int_0^t a_{ij}(X_s) \, ds, \quad t \geq 0, \quad P_x\text{-a.s.,}
\]
and
\[
N^i_t = \sum_{j=1}^d \int_0^t \frac{\partial a_{ij}}{\partial x_j}(X_s) \, ds + \sum_{j=1}^d \int_0^t a_{ij}(X_s) n_j(X_s) \, dA_s, \quad t \geq 0, \quad P_x\text{-a.s.}
\]
In case of the classical Dirichlet form defined by
\[
\mathbb{D}(u,v) = \frac{1}{2} \sum_{i=1}^d \int_D \frac{\partial u}{\partial x_i}(x) \frac{\partial v}{\partial x_i}(x) \, dx, \quad u,v \in H^1(D),
\]
i.e. if $a = \frac{1}{2}I$, the process $M$ is called a reflecting Brownian motion. By Lévy’s characterization of Brownian motion, representation (2.3) reads

$$X_t - X_0 = B^1_t + \frac{1}{2} \int_0^t n(X_s) \, dA_s, \quad t \geq 0, \quad P_\mu\text{-a.s.,}$$

(2.4)

where $B = (B^1, \ldots, B^d)$ is a standard Brownian motion. For the proof of (2.4) see [4, Example 5.2.2] and for the general case (2.3) see [3, Theorem 2.3]. In case $a$ is a general function satisfying (1.2) some representation of $X$ (Lyon’s–Zheng–Skorohod decomposition) is given in [13] (for bounded $C^2$ domain $D$ and $x \in D$).

Let

$$\tau_D = \inf\{t > 0 : X_t \notin D\}, \quad X^D_t = \begin{cases} X_t, & t < \tau_D \\ \partial, & t \geq \tau_D, \end{cases}$$

where $\partial$ is a point adjoined to $\bar{D}$ as an isolated point (cemetery state). We adopt the convention that every function $f$ on $\bar{D}$ is extended to $\bar{D} \cup \partial$ by setting $f(\partial) = 0$.

We denote by $M^\lambda$ the canonical subprocess of $M$ with respect to the multiplicative functional $e^{-\lambda t}$. For its detailed construction we refer to [4, Section A.2]. Here let us only note that we may assume that $M^\lambda = (X^\lambda, P_\mu)$ is defined on the same probability space on which $M$ is defined and

$$X^\lambda_t = \begin{cases} X_t, & t < Z/\lambda \\ \partial, & t \geq Z/\lambda, \end{cases}$$

where $Z$ is a nonnegative random variable independent of $(X_t)_{t \geq 0}$ having exponential distribution with mean 1.

## 3 Weak and probabilistic solutions

For the convenience of the reader, below we recall variational formulation of problems (1.2), (1.3). For more details and comments we refer to [3, Appendix I].

**Definition 3.1.** (i) Let $f \in L^2(D)$, $g \in L^2(\partial D)$. A function $u_n \in H^1(D)$ is called a weak solution of (1.2) if for every $v \in H^1(D)$,

$$\mathcal{E}_\lambda(u_n, v) = \int_D fv \, dx + n \int_{\partial D} (g - u_n)v \, d\sigma.$$  

(3.1)

(ii) Let $f \in L^2(D)$, $g \in H^1(D)$. A function $u \in H^1(D)$ is called a weak solution of (1.3) if $u - g \in H^1_0(D)$ and for every $v \in H^1_0(D)$,

$$\mathcal{E}_\lambda(u, v) = \int_D fv \, dx.$$  

The existence and uniqueness of weak solutions of (1.2), (1.3) is well known. For proofs by classical variational methods we refer for instance to [3, Appendix I]. In Proposition 3.2 below we give proofs by using the probabilistic potential theory. The advantage of using these less classical methods lies in the fact that they provide probabilistic representations of quasi-continuous versions of weak solutions. We would like to stress that the proof of Proposition 3.2 is simply a compilation of known facts. We provide it for completeness and later use.
Proposition 3.2.  (i) Let \( f \in L^2(D), g \in L^2(\partial D) \). Then there exists a unique weak solution \( u_n \) of (1.2) and \( \tilde{u}_n \) defined q.e. on \( D \) by

\[
\tilde{u}_n(x) = E_x \int_0^\infty e^{-\lambda t - nA_t} (f(X_t) + ng(X_t) \, dA_t) \tag{3.2}
\]

is a quasi-continuous m-version of \( u_n \).

(ii) Let \( f \in L^2(D), g \in H^1(D) \). Then there exists a unique weak solution \( u \) of (1.3) and \( \tilde{u} \) defined q.e. on \( D \) by

\[
\tilde{u}(x) = E_x \bigg( e^{-\lambda \tau_D} g(X_{\tau_D}) + \int_0^{\tau_D} e^{-\lambda t} f(X_t) \, dt \bigg) \tag{3.3}
\]

is a quasi-continuous m-version of \( u \).

Proof. (i) Let \((\mathcal{E}^{n\sigma}, D(\mathcal{E}^{n\sigma}))\) denote the form \( \mathcal{E} \) perturbed by the measure \( n\sigma \), that is

\[
\mathcal{E}^{n\sigma}_\lambda(u,v) = \mathcal{E}_\lambda(u,v) + n \int_{\partial D} uv \, d\sigma, \quad u,v \in D(\mathcal{E}^{n\sigma}) := H^1(D) \cap L^2(\tilde{D};\sigma).
\]

By the classical trace theorem, \( D(\mathcal{E}^{n\sigma}) = H^1(D) \), so \( u_n \) is a weak solution of (3.1) if and only if \( u_n \in D(\mathcal{E}^{n\sigma}) \) and

\[
\mathcal{E}^{n\sigma}_\lambda(u_n, v) = \int_D fv \, dx + n \int_{\partial D} gv \, d\sigma, \quad v \in D(\mathcal{E}^{n\sigma}). \tag{3.4}
\]

Therefore we have to show that there is a unique \( u_n \in H^1(D) \) satisfying (3.4). Suppose that \( u_1^n, u_2^n \in H^1(D) \) satisfy (3.4) and let \( u = u_1^n - u_2^n \). Then from (3.4) with test function \( v = u \) we get \( \mathcal{E}^{n\sigma}_\lambda(u,u) = 0 \), hence that \( \mathcal{E}_\lambda(u,u) = 0 \). Clearly, this implies that \( u = 0 \) m-a.e. To prove the existence and its representation, it suffices to note that \( \tilde{u}_n \) can be written in the form

\[
\tilde{u}_n = R^{nA}_\lambda f + nU^\lambda_{n,A} g,
\]

where

\[
R^{nA}_\lambda f(x) = E_x \int_0^\infty e^{-\lambda t - nA_t} f(X_t) \, dt, \quad U^\lambda_{n,A} g(x) = E_x \int_0^\infty e^{-\lambda t - nA_t} g(X_t) \, dA_t,
\]

and then use [4] (6.1.5), (6.1.12). Furthermore, \( \tilde{u}_n \) is quasi-continuous because \( R^{nA}_\lambda f \) is quasi-continuous by [4] Lemma 5.1.5 and \( U^\lambda_{n,A} g \) is quasi-continuous by [4] Lemma 6.1.3.

(ii) With our convention, \( \tilde{u} \) can be equivalently written in the form

\[
\tilde{u} = H^{\lambda}_{\partial D} \tilde{g} + R^D f,
\]

where

\[
H^{\lambda}_{\partial D} \tilde{g}(x) = E_x e^{-\lambda \tau_D} \tilde{g}(X_{\tau_D}), \quad R^D f(x) = \int_0^\infty e^{-\lambda t} f(X^D_t) \, dt.
\]

Let \( H^1_D = \{ u \in H^1(D) : \tilde{u} = 0 \text{ q.e. on } \partial D \} \). It is known (see [4] Exercise 2.3.1) that \( H^1_D = H^1_\partial(D) \). Furthermore, by [4] Theorem 4.3.1], \( H^1_{\partial D} \tilde{g} \) is an m-version of the orthogonal projection of \( g \) on the orthogonal complement of the space \( H^1_D \) in the Hilbert
Proof. Define and only if it satisfies the equation

Let Proposition 3.4. Using this, similarly to the proof of [1, Proposition II.1.1], one can show that \( R^D f \in H^1(D) \) since \( g - H_{\partial D}^\lambda \hat{g} \in H^1_0(D) \) and \( R^D f \in H^1_0(D) \) by [4, Theorem 4.4.1] again. Therefore \( \tilde{u} \) is a weak solution of \((1.3)\). Note that \( \tilde{u} \) is quasi-continuous because \( H_{\partial D}^\lambda \hat{g} \) is quasi-continuous by [4, Theorem 4.3.1] and \( R^D f \) is quasi-continuous by [4, Theorem 4.4.1].

If \( f \in L^p(D) \) with \( p > d \), then \( R^D f \in C(\bar{D}) \) by [6, Theorem 2.1], and if \( g \in B_b(\partial D) \), then \( \lambda \int_0^\infty e^{-\lambda t} g(X_t) \, dA_t \leq \|g\| \int_0^\infty e^{-\lambda t} \, dt, \) \( x \in \bar{D} \). Therefore, under these assumptions on \( f \) and \( g \), the integrals on the right-hand side of (3.2) are well defined for every \( x \in \bar{D} \). Similarly, the right-hand side of (3.3) is well defined for every \( x \in \bar{D} \).

The above remarks and Proposition 3.2 justify the following definition of probabilistic solutions of (1.2), (1.3).

**Definition 3.3.** Let \( f \in L^p(D) \) with \( p > d \) and \( g \in B_b(\partial D) \). The function \( v_n : \bar{D} \to \mathbb{R} \) defined by the right-hand side of (3.2) is called the probabilistic solution of (1.2). The function \( v : \bar{D} \to \mathbb{R} \) defined by the right-hand side of (3.3) is called the probabilistic solution of (1.3).

An equivalent definition of a probabilistic solution of (1.2), resembling (3.1), will be given in Proposition 3.4 below.

For a deep study of connections between probabilistic solutions, weak solutions as well of other kind of solutions to the Dirichlet problem with possibly irregular domain we refer the reader to [8]. Here let us only note that if \( D \) is bounded and Lipschitz (as in the present paper), then it satisfies Poincare’s cone condition. Therefore modifying slightly the proof of [11, Proposition II.1.13] (we use Aronson’s estimates for the transition densities of \( \mathcal{M} \)) one can show that each point \( x \in \partial D \) is regular for \( D^c \), i.e.

\[
P_x(\tau_D = 0) = 1, \quad x \in \partial D.
\]

Using this, similarly to the proof of [11, Proposition II.1.11], one can show that \( H_{\partial D}^\lambda g \in C(\bar{D}) \) if \( g \in C(\partial D) \). For an analytical proof of this well known fact see, e.g., [12]. Furthermore, it is known (see [13, Section 9] or [11]) that if \( f \in L^p(D) \) with \( p > d \), then \( R^D f \in C(\bar{D}) \). Thus \( v \in C(\bar{D}) \) when \( f \in L^p(D) \) with \( p > d \) and \( g \in C(\partial D) \).

**Proposition 3.4.** Let \( f \in L^p(D) \) with \( p > d \) and \( g \in B_b(\partial D) \). Then the probabilistic solution \( v_n \) of (1.2) is continuous. Moreover, \( v_n \in C(\bar{D}) \) is the probabilistic solution if and only if it satisfies the equation

\[
v_n(x) = R_\lambda(f \cdot m + n(g - v_n) \cdot \sigma)(x)
\]

\[
= E_x \int_0^\infty e^{-\lambda t}(f(X_t) \, dt + n(g - v_n)(X_t)(\partial D), \quad x \in \bar{D}.
\]

**Proof.** Define \( u_n, \tilde{u}_n \) as in Proposition 3.2 and set

\[
w_n(x) = R_\lambda(f \cdot m + n(g - \tilde{u}_n) \cdot \sigma)(x)
\]

\[
= \int_D r_\lambda(x, y)f(y) \, dy + n \int_{\partial D} r_\lambda(x, y)(g - \tilde{u}_n)(y) \, \sigma(dy), \quad x \in \bar{D}.
\]
By the remarks following the proof of Proposition 3.2, \( w_n(x) \) is well defined and finite for each \( x \in \bar{D} \). Moreover, there is \( C > 0 \) such that \( |\tilde{u}_n| \leq C \) q.e. Since \( \sigma \) is smooth, \( |\tilde{u}_n| \leq C \) \( \sigma \)-a.e. on \( \partial D \). From this and [10, Theorem 2.1] it follows that in fact \( w_n \in C(\bar{D}) \). For every \( v \in H^1(D) \) we have

\[
E\lambda(w_n, v) = (f, v) + n \int_{\partial D} (g - \tilde{u}_n)v \, d\sigma = (f, v) + n \int_{\partial D} (g - u_n)v \, d\sigma.
\]

By this and (3.1), \( E\lambda(w_n, v) = E\lambda(u_n, v), \ v \in H^1(D) \), which implies that \( w_n = u_n \) \( m \)-a.e., and hence \( w_n = \tilde{u}_n \) q.e. on \( \bar{D} \). From this and (3.7) it follows that \( w_n \) is a continuous solution of (3.6). It is the probabilistic solution of (1.2). To see this, we first note that (3.6), with \( v_n \) replaced by \( w_n \), can be equivalently written as

\[
w_n(x) = E_x \int_0^\infty (f(X_t^\lambda) \, dt + n(g - w_n)(X_t^\lambda) \, dA_t), \quad x \in \bar{D}.
\]

Since the integrals \( E_x \int_0^\infty |f(X_t^\lambda)| \, dt, E_x \int_0^\infty |g - w_n|(X_t^\lambda) \, dA_t \) exist and are finite for each \( x \in \bar{D} \), in much the same way as in [9, Remark 3.3(ii)] we show that there is a martingale additive functional \( M \) such that for each \( x \in \bar{D} \) the pair \((Y^n, M)\), where \( Y_t^n = w_n(X_t^\lambda), \ t \geq 0 \), is a solution of the backward stochastic differential equation

\[
Y_t^n = \int_t^\infty f(X_s^\lambda) \, ds + n \int_t^\infty (g(X_s^\lambda) - Y_s^n) \, dA_s - \int_t^\infty dM_s, \quad t \geq 0, \ P_x\text{-a.s.} \quad (3.9)
\]

Integrating by parts we get

\[
e^{-nA_T} Y_T^n - Y_0^n = -n \int_0^T e^{-nA_t} Y_t^n \, dA_t + \int_0^T e^{-nA_t} dY_t^n, \quad T > 0.
\]

Hence

\[
E_x Y_0^n = E_x e^{-nA_T} Y_T^n + \int_0^T e^{-nA_t} (f(X_t^\lambda) \, dt + g(X_t^\lambda) \, dA_t).
\]

Letting \( T \to \infty \) gives

\[
w_n(x) = E_x Y_0^n = E_x \int_0^\infty e^{-nA_t} (f(X_t^\lambda) \, dt + g(X_t^\lambda) \, dA_t)
\]

for every \( x \in \bar{D} \). This shows that \( v_n \) is continuous and satisfies (3.6), and moreover, any continuous solution of (3.8) coincides with \( v_n \). \( \square \)

Note that (3.6) is a very special case of equation with smooth measure data and (3.9) is the corresponding backward stochastic differential equation (BSDE). More general, semilinear equations of the form (3.6), (3.9) are considered in [10]. Note also that one can prove the existence of a quasi-continuous \( v_n \) satisfying (3.6) for q.e. \( x \in \bar{D} \) by solving the corresponding BSDE, i.e. by probabilistic methods (we do not need know in advance that there is a weak solution \( u_n \)). For a general result of this kind see [10, Theorem 4.3].
4 A convergence result

Recall that $A$ is an additive functional (AF in abbreviation) of $\mathbb{M}$ in the strict sense with Revuz measure $\sigma$. We denote by $F_A$ the support of $A$, i.e.

$$F_A = \{x \in \bar{D} : P_x(A_t > 0 \text{ for all } t > 0) = 1\}.$$

**Lemma 4.1.** $P_x(A_{t\land\tau_D} = 0, t \geq 0) = 1$ and $P_x(A_{t+\tau_D} > 0, t \geq 0) = 1$ for every $x \in \bar{D}$.

**Proof.** In view of (3.5) the first part of the lemma is trivial for $x \in \partial D$. To show it for $x \in D$, we denote by $F$ the quasi-support of $\sigma$. We may and will assume that $F \subset \partial D$ (see [4, p. 190]). Since $A$ is an AF in the strict sense, by [4, Lemma 5.1.11] we have

$$P_x(A_t = (1_{F_A} \cdot A)_t, t > 0) = 1 \text{ for every } x \in \bar{D},$$

where $(1_{F_A} \cdot A)_t = \int_0^t 1_{F_A}(X_s) dA_s$, $t \geq 0$. By [4, Theorem 5.1.5], $F_A = F$, so $P_x(A_t = (1_F \cdot A)_t, t > 0) = 1$ for every $x \in \bar{D}$. Since $F \subset \partial D$, it follows that for $x \in D$, $A_t = 0$ $P_x$-a.s. on $[0, \tau_D)$. Since $A$ is continuous, in fact $A_t = 0$ $P_x$-a.s. on $[0, \tau_D]$ for $x \in D$, which proves the first part of the lemma. Let $B$ be a standard Brownian motion appearing in (2.4). We have $P_y(\tau_D = 0) = 1$ for $y \in \partial D$, where $\tau_D = \inf\{t > 0 : B_t \notin D\}$. From this, (2.4) and the fact that the reflecting Brownian motion is a diffusion with sample paths in $\bar{D}$ it follows that the support of the additive functional appearing in (2.4), which we denote for the moment by $\hat{A}$, equals $\partial D$. Let $\text{Cap}_L$ denote the capacity associated with $\mathcal{E}$ and $\text{Cap}$ the capacity associated with $\mathbb{D}$ (see [4, Section 2.1] for the definitions). Assumption (1.1) implies that $2\lambda^{-1} \text{Cap} \leq \text{Cap}_L \leq 2\lambda \text{Cap}$. Therefore $F$ is a quasi-support of $\sigma$ considered as a smooth measure with respect to $\text{Cap}_L$ if and only if it is a quasi-support of $\sigma$ considered as a smooth measure with respect to $\text{Cap}$. By what has already been proved and [4, Theorem 5.1.5], $F = F_A = \partial D$, so by [4, Theorem 5.1.5] again, $F_A = \partial D$. From this and the definition of $F_A$ we get the second part of the lemma.

**Theorem 4.2.** Assume that $f \in L^p(D)$ with $p > d$ and $g \in C(\partial D)$. Then $v_n(x) \to v(x)$ for every $x \in \bar{D}$.

**Proof.** Recall that $v_n$ is defined by the right-hand side of (3.2). First assume that $x \in D$. By Lemma 4.1 and the dominated convergence theorem, for $x \in D$ we have

$$E_x \int_0^\infty e^{-\lambda t} f(X_t) dt = E_x \int_0^{\tau_D} e^{-\lambda t} f(X_t) dt + E_x \int_{\tau_D}^\infty e^{-\lambda t} f(X_t) dt$$

$$\to E_x \int_0^{\tau_D} e^{-\lambda t} f(X_t) dt = R^D_\lambda f(x) \quad (4.1)$$

as $n \to \infty$. We are going to show that for every $x \in D$,

$$nE_x \int_0^\infty e^{-\lambda t} g(X_t) dA_t = nE_x \int_0^{\tau_D} e^{-\lambda t} g(X_t) dA_t$$

$$\to E_x e^{-\lambda \tau_D} g(X_{\tau_D}) = H^D_{\partial D} g(x) \quad (4.2)$$

as $n \to \infty$. We know that $(P_t)_{t>0}$ is a strongly Feller semigroup on $C(\bar{D})$. Let $(\bar{L}, D(\bar{L}))$ denote its generator. Since $D(\bar{L})$ is dense in $C(\bar{D})$, one can choose a sequence $\{g_k\} \subset D(\bar{L})$ such that $\sup_{x \in \bar{D}} |g_k - g| \leq k^{-1}$. By [7, Theorem 3.6.5], $g_k(X)$ is a semimartingale under $P_x$ for $x \in \bar{D}$. In fact,
is a martingale under $P_x$ for $x \in \bar{D}$. Integrating by parts, for all $k \geq 1$ and $t \geq 0$ we obtain
\[
e^{-\lambda(t+\tau_D) - nA_t + \tau_D} g_k(X_t) - e^{-\lambda \tau_D - nA_{\tau_D}} g_k(X_{\tau_D})
\]
\[= - \int_{\tau_D}^{t+\tau_D} e^{-\lambda s - nA_s} g_k(X_s) d(\lambda s + nA_s) + \int_{\tau_D}^{t+\tau_D} e^{-\lambda s - nA_s} dg_k(X_s)
\]
\[+ \int_{\tau_D}^{t+\tau_D} e^{-\lambda s - nA_s} dM^g_s.
\]
Since $e^{-\lambda t - nA_t} \to 0$ as $t \to \infty$ and $A_{\tau_D} = 0$ $P_x$-a.s., we get
\[nE_x \int_{\tau_D}^{\infty} e^{-\lambda s - nA_s} g_k(X_s) dA_s = E_x e^{-\lambda \tau_D} g_k(X_{\tau_D}) - \lambda E_x \int_{\tau_D}^{\infty} e^{-\lambda s - nA_s} g_k(X_s) ds
\]
\[+ E_x \int_{\tau_D}^{\infty} e^{-\lambda s - nA_s} (\hat{L} g_k)(X_s) ds.
\]
Since $g_k, \hat{L} g_k \in C(\bar{D})$, applying Lemma 4.1 and the dominated convergence theorem shows that the second and third term on the right-hand side of the above equality converge to zero as $n \to \infty$. This proves that
\[nE_x \int_0^{\infty} e^{-\lambda s - nA_s} g_k(X_s) dA_s \to E_x e^{-\lambda \tau_D} g_k(X_{\tau_D}). \quad (4.3)
\]
Furthermore,
\[nE_x \int_{\tau_D}^{\infty} e^{-\lambda s - nA_s} dA_s \leq nE_x e^{-\lambda \tau_D} \int_0^{\infty} e^{-nA_s} dA_s \leq e^{-\lambda \tau_D} (1 - e^{-A_{\infty}}),
\]
so
\[nE_x \int_{\tau_D}^{\infty} e^{-\lambda s - nA_s} g_k - g(X_s) dA_s \leq k^{-1} E_x e^{-\lambda \tau_D}. \quad (4.4)
\]
Clearly, we also have
\[E_x e^{-\lambda \tau_D} |g_k - g|(X_{\tau_D}) \leq k^{-1}. \quad (4.5)
\]
From (4.3)–(4.5) we get (4.2), which together with (4.1) shows the desired convergence for $x \in D$. Since $P_x(\tau_D = 0) = 1$ for $x \in \partial D$, the above arguments also show that $v_n(x) \to E_x g(X_0) = g(x) = v(x)$ for $x \in \partial D$, which completes the proof. \hfill \Box

**Remark 4.3.** (i) Let $f \in L^2(D)$, $g \in C(\partial D)$ and $\tilde{u}_n, \bar{u}$ be defined as in Proposition 3.2. Then $\tilde{u}_n \to u$ q.e. because the proof of Theorem 1.2 shows that then (4.1) holds for q.e. $x \in D$ and (4.2) holds for every $x \in D$. In particular, if $f \in L^2(D)$ and $g \in H^1(D) \cap C(\partial D)$, then $\{u_n\}$ converges q.e. to the weak solution $u$ of (1.3). If $f \in L^2(D), g \in H^1(D)$, then the convergence holds in $H^1(D)$ and hence a.e. For an analytical proof of this fact we refer the reader to [3, Appendix 1, Section 4.2].

(ii) In [3] some results on the rate of convergence of $\{u_n\}$ to $u$ in the norm of $H^1(D)$ are given. Estimating the rate of pointwise convergence of $\{v_n\}$ to $v$ presents a more delicate open problem.
References

[1] R.F. Bass, *Probabilistic Techniques in Analysis*. Springer-Verlag, New York, 1995.

[2] R.F. Bass and P. Hsu, Some potential theory for reflecting Brownian motion in Hölder and Lipschitz domains, *Ann. Probab.* 19 (1991) 486–508.

[3] R. Glowinski, *Numerical methods for nonlinear variational problems*. Springer-Verlag, New York, 1984.

[4] M. Fukushima, Y. Oshima and M. Takeda, *Dirichlet Forms and Symmetric Markov Processes*. Walter de Gruyter, Berlin, New York, 1994.

[5] M. Fukushima and M. Tomisaki, Reflecting Diffusions on Lipschitz Domains with Cusps - Analytic Construction and Skorohod Representation, *Potential Anal.* 4 (1995) 377–408.

[6] M. Fukushima and M. Tomisaki, Construction and decomposition of reflecting diffusions on Lipschitz domains with Hölder cusps, *Probab. Theory Related Fields* 106 (1996) 521–557.

[7] N. Jacob, *Pseudo-Differential Operators and Markov Processes. Vol. III: Markov Processes and Applications*. Imperial College Press, London, 2005.

[8] T. Klimsiak, Trace operator and the Dirichlet problem for elliptic equations on arbitrary bounded open sets, *J. Funct. Anal.* 277 (2019) 1499–1530.

[9] T. Klimsiak and A. Rozkosz, Renormalized solutions of semilinear equations involving measure data and operator corresponding to Dirichlet form, *NoDEA Nonlinear Differential Equations Appl.* 22 (2015) 1911–1934.

[10] T. Klimsiak and A. Rozkosz, Large time behavior of solutions to parabolic equations with Dirichlet operators and nonlinear dependence on measure data, *Potential Anal.* 51 (2019) 255–289.

[11] H. Kunita, General boundary conditions for multi-dimensional diffusion processes, *J. Math. Kyoto Univ.* 10 (1970) 273–335.

[12] W. Littman, G. Stampacchia and H.F. Weinberger, Regular points for elliptic equations with discontinuous coefficients, *Ann. Scuola Norm. Sup. Pisa* 17 (1963) 43–77.

[13] A. Rozkosz and L. Slomiński, Stochastic representation of reflecting diffusions corresponding to divergence form operators, *Studia Math.* 139 (2000) 141–174.

[14] G. Stampacchia, Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus, *Ann. Inst. Fourier* 15 (1965) 189–258.