Comments on:
“Quantum mechanics as a gauge theory of metaplectic spinor fields” by M. Reuter

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Abstract

We point out how some mathematically incorrect passages of [1] can be formulated in a rigorous way. The fibre bundle approach to quantum mechanics of [2–6] is compared with the one contained in loc. cit.
1. Introduction

The purpose of these notes is the comparison of fibre bundle approach to (non-relativistic) quantum mechanics developed in our investigation [2–6] and the one contained in [1] and the correction of some incorrect mathematical statements, definitions and expressions in [1]. We have to emphasize that here we shall comment only on some technical mathematical details of [1]. What concerns the ‘physical’ part of this interesting paper, we agree with the author’s own conclusions and will not concern with it in this text. We hope that the presented here material will help to the improvement of certain not exactly mathematical rigorous places in [1]. So, these remarks may be considered as a mathematical appendix to loc. cit. in which are given more or less complete instructions how this work can be make mathematically rigorous.

Here we freely make use of the notation and terminology of [2–6] to which papers the reader is referred for details and explanations. The references to sections, equations, footnotes etc. from the parts of the series [2–6] are denoted by the corresponding sequential reference numbers in these parts preceded by the Roman number of the part in which it appears and a dot as a separator. For instance, Sect. I.5 and (IV.2.11) mean respectively section 5 of part I, i.e. of [2], and equation (2.11) (equation 11 in Sect. 2) of part IV, i.e. of [5].

At this point we want to say a few words on the possibility to identify the Hilbert bundle’s base\(^1\) \(M\) with the phase space of certain system which case is taken as a base for a bundle approach to quantum mechanics in [1]. Our generic opinion is that the phase space is not a ‘suitable’ candidate for a bundle’s base, the reason being the Heisenberg uncertainty principle by virtue of which the points of the phase space have no physical meaning [7, chapter IV]. This reason does not apply if as a base is taken the phase space of some observer as, by definition, the observers are treated as classical objects (systems). Therefore one can set the base \(M\) of the Hilbert bundle \((F, \pi, M)\) to be the phase space of some observer. Then the reference path \(\gamma: J \to M\) can be interpreted as the observer’s phase-space trajectory which, generally, can have self-intersections. The further treatment of this case is the same as of \(M = \mathbb{E}^3\). Regardless of the above-said, one can always identify \(M\) with the system’s phase space, if it exists, as actually \(M\) is a free parameter in [2–6].

An interesting bundle approach to quantum mechanics is contained in [1]. In it the evolution of a quantum system is described in a Hilbert bundle over the system’s phase space with the ordinary system’s Hilbert space as a (typical) fibre which is, some times, identified with the fibre over an arbi-

\(^1\)Here and below, when talking about a Hilbert bundle we mean the one used for the fibre bundle description of quantum mechanics [2–6].
trary fixed phase-space point. The evolution itself is presented as a parallel transport in the bundle space generated via non-dynamical linear (and symplectic) connection which is closely related to the symplectic structure of the phase space. An important feature of [1] is that in it the bundle structure is derived from the physical content of the paper. In this sense [1] can be considered as a good motivation for the general constructions in [2–6].

Before comparing the mathematical results of [1] with the ones of [2–6] in Sect. 3, we will pay attention in Sect. 2 on some incorrect ‘bundle’ expressions in [1] which, however, happily do not influence most of the conclusions made on their base.

2. Critical remarks

In this section we point to and show possible ways for improving of a number of mathematically non-rigorous or wrong expressions, assertions, and definitions in [1]. Once again we emphasize that all this concerns only the ‘bundle’ part of the mathematical structure of loc. cit. and does not deal with its physical contents.

First of all, expressions like $\partial_a|\psi\rangle_\phi$ and $d|\psi\rangle_\phi(s)/ds$ (in the notation of [2–6] $\phi \in M (= $ system’s phase space) and $\phi: J \to M, \phi: s \mapsto \phi(s) \ s \in J$ respectively and $|\psi\rangle_\phi \in F_\phi$ is an element in the fibre over $\phi$, i.e. $|\psi\rangle$ is a section of the bundle) are not defined as the defining them (conventional) limits contain differences of elements of different fibres which are undefined objects per. ce. Generally this situation is the same as outlined at the beginning of Sect. II.2. For the same reason the difference $|\psi\rangle_{\phi+\delta\phi} - |\psi\rangle_\phi$ in [1, equation (4.10)] is senseless. Almost the same is the situation with $\partial_\phi \mathcal{O}_f$ in [1, equations (4.13) and (5.9)] (see also [1, equation (5.21)]) where $\mathcal{O}_f$ is, in the terminology of [2–6], a bundle morphism corresponding to a dynamical variable whose classical analogue is a classical observable $f: M \to \mathbb{R}$. Since $\mathcal{O}_f: \phi \mapsto \mathcal{O}_f(\phi): F_\phi \to F_\phi$, the derivative $\partial_\phi \mathcal{O}_f|_\phi$ can not be defined as $\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} (\mathcal{O}_f(\phi + \varepsilon a) - \mathcal{O}_f(\phi))$ with $(\varepsilon a)^b = \varepsilon \delta_a^b$ and needs special redefinition. The same arguments are applicable to $\partial_\phi \varepsilon, \partial_\phi \Gamma_a$ and $\partial_\phi A$ appearing in [1, equations (3.8), (3.9), (3.14), (3.21)]. All of these deficiencies can easily be corrected by rewriting the corresponding equations and definitions in component form, i.e., formally, by adding to them the required component indices (however see below). Besides, when defining the components, e.g. of $|\psi\rangle \in \mathcal{F}$ (= $\mathcal{V}$ in [1]), the author improperly transfers the notation from the typical fibre $\mathcal{F}$ to the fibres $F_\phi$ over $M$. For example, if $\{|x\}\}$ (in the notation of [2–6]: $\{f_x\}$ with $x$ in system’s configuration space) is a basis in $\mathcal{F}$ and $\{|x\}$ – in $\mathcal{F}^*$, $\langle x := \langle |x\rangle$ (in the notation of [2–6]: $\{f^x\}$, $f^x := \langle f_x\rangle^*$, then the author writes [1, equations (1.16) and (1.17)]:

\footnote{See, for instance, [1, equations (3.1), (3.9), (3.14), (3.51), (3.52), (4.40), (4.41), (5.46), (5.47)].}
ψx(φ) = ⟨x|ψ⟩φ and χx(φ) = φ⟨χ|x⟩ for the components of |ψ⟩φ ∈ Fφ and φ(χ) ∈ F∗x. This is incorrect by two reasons: (i) {⟨x|} is a basis in F, not in Fφ, so the inner product, e.g., ⟨x|ψ⟩φ is not defined, and (ii) since the inner product (dual pairing) ⟨·|·⟩ is defined as a map ⟨·|·⟩: F∗ × F → ℂ, it can not be used (directly) for the definition of the components of |ψ⟩φ ∈ Fφ. (The same remark is true for [1, equation (1.20)] defining the components of a bundle morphism (family of operators or (1,1)-multispinor field in author’s terminology).) This confusion can be met and further in the text (see, e.g [1, equations (3.3), (3.10), (4.4), (5.37)]). A lucky exception of this rule is [1, first equation (4.42)] (regardless of the fact that its l.h.s. is not defined). In fact, in the framework of [1], this equation is the corner-stone for solving the above problems with undefined scalar products and, at the end, with the components of the vectors in Fφ, φ ∈ M. For this purpose, the only thing one has to do is to define the l.h.s. of [1, first equation (4.42)] through its r.h.s., i.e. this equation has to be converted into definition.3

In this way the following six problems find natural solutions: (i) The (typical) fibre F is identified with the fibre Fφ0 for arbitrarily fixed point φ0 in the phase-space and the homeomorphisms lφ: Fφ → F are given through lφ−1: |ψ⟩φ0 → |ψ⟩φ via [1, equation (4.41)]. (ii) A fibre inner (scalar) product [·|·]: F∗ × F → ℂ is rigorously defined on any fibre Fφ, φ ∈ M by [1, first equation (4.42)]. (iii) Choosing a basis {⟨x⟩φ} in Fφ, we define the components of, e.g, |ψ⟩φ ∈ Fφ by ψx(φ) := |⟨x⟩φ⟩∗, |ψ⟩φ with |x⟩φ∗ being the dual of |x⟩φ. (iv) Applying steps (ii) and (iv), we can rewrite all equations of [1] containing inner products or vectors’ (or multispinors’) components in such a way that they obtain rigorous mathematical meaning. (v) The previous point makes strict the above-pointed solution of the problems with derivatives like ∂a|ψ⟩φ. (vi) The [1, second equation (4.42)], which includes the ‘background-quantum split symmetry’, becomes a consequence of [1, equations (4.40) and (4.41)].

The so-described procedure allows us to take off the above-pointed problems and to give a strict mathematical sense to the (most of the) results of [1].

In [1] nowhere a precise definition is given of what exactly a Hilbert bundle is (the author talks about Hilbert spaces attached to the phase space points etc.) regardless of the fact that this concept appears many times in the paper. So, somewhere at the beginning of this work must be said that a Hilbert bundle is a collection (F, π, M) of a phase space M (in the concrete case), a map π: F → M, and F = ∪φ∈MFφ where Fφ = π−1(φ) are Hilbert spaces homeomorphic to the system’s conventional Hilbert space F. Besides, for the concrete purposes of [1], to this collection should be added

3This is possible on any linearly connected subset of M containing the fixed basic point φ0. The so obtained scalar products in the fibres over this set are path-independent and self-consistent by virtue of the used in [1] ‘Abelian’ connection.
the structure group $G$ of unitary operators acting on $\mathcal{F}$.

And the last serious problem of [1] deserving mentioning. In the paragraph following [1, equation (3.4)] we see a mixing of the meaning of active and passive transformations of the fibres and where they are acting. By definition, passive are the transformations that change only the vectors’ components and are due to changes of the bases, while the active ones are fibres’ diffeomorphisms. The author writes: “In all fibres $\mathcal{V}_\phi (= F_\phi$ in the notation of [2–6] - B.I.) we may perform independent changes of their base by means of gauge transformation $U : M \rightarrow G, \phi \mapsto U(\phi)$”, where he defines $G$ as the group of all unitary operators on the fibre $F$ ($V$ in his notation). Two incorrect things are presented here: First, since $U(\phi) : F \mapsto F$ by definition, the operator $U(\phi)$ can not act on the fibre $F_\phi$ over $\phi$ as this is a different space.\(^4\) And second, the operator $U(\phi) : F \mapsto F$ changes not only the bases in $F$ but also all its vectors, i.e. it is not a simple change of the bases (passive transformation) in $F$, but an active transformation in $F$. The conclusion is that $U(\phi)$ does not act on $F_\phi$ at all and it is not a simple change of the base neither in $F$ nor in $F_\phi$. What the author really wants to do, we hope, is the following. Let $U : \phi \mapsto U(\phi)$ (be a bundle morphism) with $U(\phi) : F_\phi \mapsto F_\phi$ being a unitary operator on $F_\phi$. (The operator $U(\phi)$ not only changes the bases in $F_\phi$, it transforms $|\psi\rangle_\phi \in F_\phi$ into $|\psi\rangle'_\phi := U(\phi)|\psi\rangle_\phi$ (see [1, equation (3.5)]).) Furthermore, the author claims that under $U(\phi)$ from the covariance of the connection [1, equation (3.6)] follows its transformation law [1, equation (3.7)] (containing the undefined term $\partial_a U(\phi)$). Two important remarks are in order here. The defined by [1, equation (3.1)] covariant derivative $\nabla_a \equiv \nabla_a (\Gamma)$ (connection $\Gamma$) is not correct due to the involved in it undefined term $\partial_a |\psi\rangle_\phi$, but this can be repaired as described already. And next, the transformations $U(\phi) : F_\phi \mapsto F_\phi$ do not act on the connection at all, they leave it unchanged! What the author really does is that he defines by [1, equation (3.6)]\(^5\) a new covariant derivative $\nabla_a (\Gamma')$ (connection $\Gamma'$) associated to (the bundle morphism) $U$ and having the natural property $\nabla_a (\Gamma') |\psi\rangle'_\phi = U(\phi) (\nabla_a (\Gamma) |\psi\rangle_\phi)$. Explicitly this new connection is given by [1, equation (3.7)] which is equivalent to the mentioned its property. Another possibility is to consider the connection components (coefficients) with respect to two fields of local bases whose vectors are connected via $U$. In this case $\nabla_a$ transforms as a vector (with respect to the index $a$), this law replaces [1, equation (3.6)], and the connections’ coefficients transform according to [1, equation (3.7)] provided in it all operators are replaced with their matrices in the bases mentioned.

Ending with the critical comments on [1], we conclude: most of the final results and conclusions of this interesting paper are valid provided

\(^4\)The existence of a homeomorphism between $F$ and $F_\phi$ does not influence this conclusion; it can only help to define correctly a representation of $G$ on $F_\phi$.

\(^5\)The index $a$ of $\nabla$ in the r.h.s. of this equation is missing.
the above-pointed (and other minor) corrections are made in it. Below we shall suppose that this is carefully done. On this base we will compare \[1\] with \[2–6\].

3. Conclusions

The main common point between \[1\] and \[2–6\] is the consistent application of the fibre bundle theory to (nonrelativistic) quantum mechanics. But the implementation of this intention is quite different: in \[1\] we see a description of quantum mechanics in a new, but ‘frozen’, geometrical background based on a non-dynamical linear connection deduced from the symplectical structure of the system’s phase space, while in the series \[2–6\] is used a ‘dynamical geometry’ (linear transport along paths, which may turn to be a parallel one generated by a linear connection) whose properties depend on the system’s Hamiltonian, i.e. on the physical system under consideration itself.

The fact that in \[1\] the system’s phase space is taken as a base of the used Hilbert bundle is not essential since nothing can prevent us from making the same choice as, actually, the base is not fixed in \[2–6\]. In \[1\] is partially considered the dynamics of multispinor fields. This is an interesting problem, but, since it is not primary related to conventional quantum mechanics, we think it is out of the scope of our works \[2–6\]. The methods of its solution are outlined in \[1\] and can easily be incorporated within the bundle quantum mechanics of \[2–6\].

The fields of (metaplectic) spinors used in \[1\] are simply sections of the Hilbert bundle, while the “world-line spinors” in of loc. cit. are sections along paths in the terminology of \[2–6\]. The family of operators \(\mathcal{O}\) or \(\mathcal{O}(\phi)\) \[1, \text{equations (4.8) and (4.9)}\] acting on \(F\) are actually bundle morphisms.

A central role in both works plays the ‘principle of invariance of the mean values’: the mean values (mathematical expectations) of the morphisms corresponding to the observables (dynamical variables) are independent of the way they are calculate. We have used this assumption many times in \[2–6\] (see, e.g., Sections II.3, III.2, and IV.2, in particular, equations (II.3.3), (III.2.5), (III.2.11), (III.2.28), and (IV.2.17)) without explicitly formulating it as a ‘principle’. But if one wants to build axiomatically the bundle quantum mechanics, he will be forced to include this principle (or an equivalent to it assertion) into the basic scheme of the theory. In \[1\] ‘the invariance of the mean values’ is mentioned several times and it is used practically in the form of the ‘background-quantum split symmetry’ principle, explained in \[1, \text{sect. 4}\] (see, e.g., \[1, \text{equation (4.18)}\] and the comments after it). Its particular realizations are written as \[1, \text{equation (4.17) and second equation (4.42)}\] which are equivalent to it in the corresponding context. A consequence of the mean-value invariance is the ‘Abelian’ character of the
compatible with it connections, expressed by [1, equation (4.14)], which is a special case of our result [8, equation (4.4)]. In [1] the mean values are independent of the point at which they are determined. In the bundle quantum mechanics of [2–6] this is not generally the case as different points correspond to different time values (see, e.g., (II.3.3)). This difference clearly reflects the dynamical character of the approach of [2–6] and the ‘frozen’ geometrical one of [1]. In any case, the principle we are talking about is so important that without it the equivalence between the bundle and conventional forms of quantum mechanics can not be established.

In both works the quantum evolution is described via appropriate transport along paths: In [1, see, e.g., equations (3.54) and (4.53)] this is an ‘Abelian’ parallel transport along curves, whose holonomy group is $U(1)$ [1, equation (4.38)], while in the investigation [2–6] is employed a transport along paths uniquely determined by the Hamiltonian (see Sect. I.5) which, generally, need not to be a parallel translation.

Now we turn our attention on the bundle equations of motion: in [2–6] we have a single bundle Schrödinger equation (II.2.24) (see also its matrix version (II.2.11)), while in [1, equation (5.54)] there is an infinite number of such equations, one Schrödinger equation in each fibre $F_\phi$ for the system’s state vector $|\psi(t)\rangle_\phi$ at every point $\phi \in M$. Analogous is the situation with the statistical operator (compare our equation (IV.2.17) or (IV.2.15) with [1, equation (4.56)]). This drastic difference is due to the different objects used to describe systems states: for the purpose we have used sections along paths (see Sect. I.4), while in [1] are utilized (global) sections of the bundle defined via (I.4.3) (cf. [1, equation (4.41)]). Hence, what actually is done in [1] is the construction of an isomorphic images of the quantum mechanics from the fibre $F$ in every fibre $F_\phi$, $\phi \in M$ (see the comments after (I.4.3)).

To summarize the comments on part of the mathematical structures in [1]: It contains a fibre bundle description of quantum mechanics. The state vectors are replaced by (global) sections of a Hilbert bundle with the system’s phase space as a base and their (bundle) evolution is governed through Abelian parallel transport arising from the symplectic structure of the phase space. Locally, in any fibre of the bundle, the evolution is presented by a Schrödinger equation, specific for each fibre of the bundle. The work contains a number of incorrect mathematical constructions which, however, can be corrected so that the final conclusions remain valid. Some ideas of the paper are near to the ones of [2–6] but their implementation and development is quite different in these investigations.

The style and mathematical language of [1] are typical for the high energy and particle physics literature in which the mathematical ‘details’ we

\footnote{Note that the appearing in [1, equations (4.54)–(4.56)] operator $\mathcal{O}_H$ is an analogue of our matrix-bundle Hamiltonian (see Sect. II.2).}
are emphasizing in these notes are “understood and hardly ever mentioned explicitly"\(^7\). In this sense, the present work can be considered as an appendix to *loc. cit.* in which is pointed how some its passages can be translated into a manner suitable for mathematicians or mathematical physicists.

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\(^7\)Quotation from an e-mail massage of M. Reuter to the author (September 27, 1998).
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