Topological invariant variables in QCD

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Abstract

We show that the class of functions of topologically nontrivial gauge transformations in QCD includes a zero-mode of the Gauss law constraint. The equivalent unconstrained system compatible with Feynman’s integral is derived in terms of topological invariant variables, where the zero-mode is identified with the winding number collective variable and leads to the dominance of the Wu-Yang monopole. Physical consequences of Feynman’s path integral in terms of the topological invariant variables are studied.

1. Introduction

The anomalous decay processes in both electrodynamics and chromodynamics are described by the spatial integral of the product of the magnetic and electric field strength tensors. In non-Abelian field theory, this integral is known as the time derivative of the winding number functional.

One of the main differences between QCD and QED is the fact of the non-invariance of the QCD winding number functional with respect to gauge transformations with stationary matrices (considered as maps of the coordinate space into the color group one [1]). This is a result of the nontrivial topological structure of the group of stationary gauge transformations. The condition that the functional for the degree of the map is a finite number (normalization) determines the ‘winding number’ class of functions on which the gauge transformations act. These functions decrease at spatial infinity as $O(1/r)$. This means that the dynamic gluon fields behave similar to the fields of point charges (monopoles).

This difference between the classes of functions for dynamic fields in QED and QCD is the principal peculiarity of the gauge theory of strong interactions. The present paper is devoted to a discussion of the dynamical consequences of the ‘winding number’ class of functions in QCD.

The present approach is based on the fact that the winding number class of functions contains the zero-mode of the Gauss law constraint [2, 3]. This zero-mode is identified with the winding number as a collective variable [4, 5].

We derive an equivalent unconstrained system compatible with the simplest canonical quantization scheme in the form of Feynman’s path integral. From this path integral representation of the generating functional a unitary perturbation theory for non-Abelian gauge theories can be obtained (see [6, 7, 8]) which is similar intuitive Faddeev-Popov (FP) scheme [9].

When comparing the scheme which will be developed in this paper with the instanton approach [10] the following basic differences are observed: (i) the approach is formulated in the Minkowski space instead of the Euclidean one, and (ii) the perturbation theory is formulated with physical states instead of the classical vacuum ones.

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The paper is organized as follows. In Section 2 we discuss the statement of the problem in detail. Section 3 is devoted to the derivation of the equivalent unconstrained system with the zero-mode in the Yang-Mills theory. In Section 4, the generating functional for Green functions in the form of the Feynman integral is constructed for QCD as the basis for a solution of the problems of the hadronization and confinement.

2. The winding number class of functions

We consider the winding number functional

$$X[A] = -\frac{1}{8\pi^2} \int_V d^3x \epsilon^{ijk} Tr \left[ \hat{A}_i \partial_j \hat{A}_k - \frac{2}{3} \hat{A}_i \hat{A}_j \hat{A}_k \right]$$  \hspace{1cm} (1)

for the gluon gauge fields $\hat{A}_\mu = g \frac{\lambda^a}{2} A^a_\mu$, in the non-Abelian $SU_c(3)$ theory with the action functional

$$W = \int d^4x \left\{ \frac{1}{2} (G_{0i}^a)^2 - B_i^a \right\} + \bar{\psi} [i\gamma^\mu (\partial_\mu + \hat{A}_\mu) - m \psi] ,$$  \hspace{1cm} (2)

where $\psi$ and $\bar{\psi}$ are the fermionic quark fields. We use the conventional notations for the non-Abelian electric and magnetic fields

$$G_{0i}^a = \partial_0 A_i^a - D_i^{ab}(A) A_0^b , \quad B_i^a = \epsilon_{ijk} \left( \partial_j A_k^a + \frac{g}{2} f^{abc} A_j^b A_k^c \right) ,$$  \hspace{1cm} (3)

as well as the covariant derivative $D_i^{ab}(A) := \delta^{ab} \partial_i + g f^{abc} A_i^c$.

The action (2) is invariant with respect to gauge transformations $u(t, \vec{x})$

$$\hat{A}_i^a := u(t, \vec{x}) (\hat{A}_i + \partial_i) u^{-1}(t, \vec{x}) , \quad \psi^u := u(t, \vec{x}) \psi .$$  \hspace{1cm} (4)

It is well-known [8] that the fixation of the gauge in the classical equations of motion leaves the ambiguity in the choice of initial data for the gauge fields due to the invariance of the action with respect to the gauge transformations (4), with stationary matrices $u(t, \vec{x}) = v(\vec{x})$.

The group of the stationary gauge transformations $v(\vec{x})$ in the coordinate space is topologically nontrivial and represents the group of three-dimensional paths lying on the three-dimensional space of the $SU_c(3)$-manifold with the homotopy group $\pi_3(SU_c(3)) = Z$. The whole group of the stationary gauge transformations is split into topological classes marked by the integer number $n$ (the degree of the map) which counts how many times a three-dimensional path turns around the $SU(3)$-manifold when the coordinate $x_i$ runs over the space where it is defined. The stationary transformations $v^n(\vec{x})$ with $n = 0$ are called the small ones; and those with $n \neq 0$

$$\hat{A}_i^{(n)} := v^n(\vec{x}) \hat{A}_i(\vec{x}) v^n(\vec{x})^{-1} + L^n_i , \quad L^n_i = v^n(\vec{x}) \partial_i v^n(\vec{x})^{-1}$$  \hspace{1cm} (5)

the large ones.

In QCD, the winding number functional (1) is not invariant with respect to large gauge transformations (5)

$$X[A^{(n)}] = X[A] + N_1[A, n] + N_2[n] ,$$  \hspace{1cm} (6)

where

$$N_1[A, n] = \frac{1}{8\pi^2} \int d^3x \epsilon^{ijk} Tr [\partial_i (\hat{A}_j L^n_k)] ,$$  \hspace{1cm} (7)

$$N_2[n] = -\frac{1}{24\pi^2} \int d^3x \epsilon^{ijk} Tr [L^n_i L^n_j L^n_k] = n .$$  \hspace{1cm} (8)
It determines the degree of a map $N_2[n]$ (see Eqs. (3.33), (3.36) in [11]). The degree of a map $N_2[n] = n$ as the condition of the normalization means that the large transformations are given in the class of functions with the spatial asymptotics $O(1/r)$. Such a function $L_i^n$ is given by

$$v^{(n)}(\vec{x}) = \exp(n\Phi(\vec{x})), \quad \Phi = -i\pi \frac{\lambda_3^a x^a}{r} f_0(r),$$

where the antisymmetric SU(3) matrices are denoted as

$$\lambda_A^1 := \lambda^2, \quad \lambda_A^2 := \lambda^5, \quad \lambda_A^3 := \lambda^7,$$

and $r = |\vec{x}|$. The function $f_0(r)$ satisfies the boundary conditions

$$f_0(0) = 0, \quad f_0(\infty) = 1,$$

so that the functions $L_i^n$ disappear at spatial infinity $\sim O(1/r)$ but can have nonvanishing surface integrals in Eq. (6). We call the class of functions (9) the ‘winding number’ class of functions.

The present paper is based on the evident fact that the dynamical field $A_i$ and its transformations $L_i^n$ both belong to the winding number class of functions. The second fact is that the winding number class of functions includes the topological excitations of the gluon system as a whole in the form of the zero-mode ($Z^a$) of the non-Abelian Gauss law constraint

$$\frac{\delta W}{\delta A_0} = 0 \quad \Rightarrow \quad D^{ab}_i(A)c^{b}_0 = -j_a^0,$$

where $j_a^\mu = g\bar{\psi} \gamma_\mu \lambda_a^A \psi$ is the quark current. The Gauss law takes the form of an inhomogeneous equation for the time-like component $A_0$ of the gauge field

$$(D^2(A))^{bc}_i A_0^c = D^{ac}_i(A)\partial_0 A_c^i + j_a^0.$$

A general solution of this inhomogeneous equation is a sum of the solution $Z^a$ of the homogeneous equation

$$(D^2(A))^{ab} Z^b = 0,$$

i.e., a zero mode of the Gauss law constraint, and a particular solution $\tilde{A}_0^a$ of the inhomogeneous one

$$A_0^a = Z^a + \tilde{A}_0^a.$$

It is the central point of our paper, that the zero-mode $Z^a$ at the spatial infinity can be represented in the form of the product of a new topological variable $\hat{N}(t)$ and a phase $\Phi_0(\vec{x})$

$$\hat{Z}(t, \vec{x}) |\text{asymptotics} = \hat{N}(t) \hat{\Phi}_0(\vec{x}) ,$$

where the phase $\Phi_0(\vec{x})$ belongs to the winding number class of functions (9)

$$\hat{\Phi}_0 = -i\pi \frac{\lambda_3^a x^a}{r} f_0(r), \quad f_0(0) = 0, \quad f_0(\infty) = 1.$$
with the boundary conditions
\[ f_1(0) = 0, \quad f_1(\infty) = 1. \] (19)

In this case, the single one-parametric variable \( N(t) \) reproduces the topological degeneracy of all field variables, provided the separation of the zero-mode phase factors of the topological degeneracy from the topological invariant variables \( A^I, \psi^I \) of perturbation theory (i.e., the variables without topological degeneracy). This separation is fulfilled by the gauge transformations
\[ 0 = U_Z (\hat{Z} + \partial_0) U_Z^{-1} , \]
\[ \hat{A}_i = U_Z (\hat{A}^I + \partial_i) U_Z^{-1} , \quad \psi = U_Z \psi^I , \]
where the spatial asymptotics of \( U_Z \) is
\[ U_Z = T \exp\left[ \int dt' \hat{\Phi}(t', \vec{x}) \right]_{\text{asymptotics}} = \exp[ N(t) \hat{\Phi}_0(\vec{x})] . \] (21)

A known example of fields \((\Phi_i, \Phi_0)\) which satisfy Eq. (17) is the Bogomol’ny-Prasad-Sommerfield (BPS) monopole \((\Phi^{BPS}_i)\) with the Higgs field \((\Phi^{BPS}_0)\) \([12]\)
\[ \Phi^{BPS}_i = -i \lambda_a A^{\hat{A}^a_k} x^k f^{BPS}_1 (r) , \quad f^{BPS}_1 (r) = 1 - \frac{r}{\epsilon \sinh(r/\epsilon)} , \]
\[ \hat{\Phi}^{BPS}_0 = -i \pi \lambda_a A^{\hat{A}^a_x} x^a f^{BPS}_0 (r) , \quad f^{BPS}_0 (r) = \frac{1}{\tanh(r/\epsilon)} - \frac{\epsilon}{r} , \]
in the limit \( \epsilon \to 0 \), where \( \epsilon \) is the size of the monopole. For \( \epsilon \to 0 \) the BPS monopole goes over into the Wu-Yang monopole \([3]\) without singularity at the origin (as the solution of classical equations).

In this case, the winding number functional \([1]\) takes the form of the sum of the invariant term and the new topological variable \( N(t) \)
\[ X[A] = X[A^I] + N_1[\Phi_i, N(t)] + N_2[N(t)] = X[A^I] + N(t) . \] (24)
This can be seen \([11]\) from the fact that for noninteger \( n = N(t) \) the responses of the winding number \([1]\) are equal to
\[ N_1[\Phi_i, N(t)] = \sin[2\pi N(t)]/2\pi , \]
and
\[ N_2[N(t)] = \{N(t) - \sin[2\pi N(t)]/2\pi\} , \]
so that the sinus terms cancel each other in the sum \([24]\). The function \( N(t) \) represents a one-dimensional parametrization of the transition between different maps of the homotopy group.

The topological degeneration of initial data means that the points \( N(t) \) and \( N(t) - n \) are physically equivalent. Thus, the configuration space of the physical variables of QCD has the topology of a cylinder where the role of the degree of freedom which rotates around the cylinder is played by the topological variable \( N(t) \) as the zero mode of the first class constraint. This zero mode can be revealed by only the explicit resolving the Gauss law constraint \([3]\). In the contrast to the zero mode of the second class constraint (i.e., gauge) treated as the Gribov ambiguity \([14]\), the zero mode of the first class constraint is the inexorable consequence of internal dynamics, which can be implicitly lost by the standard gauge-fixing scheme.

The present paper is devoted to the derivation of an equivalent unconstrained system with the topological variable \( N(t) \) and to the construction of the generating functional for a unitary perturbation theory.
3. Yang-Mills theory

3.1. Constraining

Recall that for a long time the problem of quantization of non-Abelian constrained systems was considered as the greatest challenge of relativistic quantum field theory [15]. The success in the description of the quantum dynamics of such systems was the unitary perturbation theory in the form of the Faddeev-Popov functional integral [9] used for the proof of the renormalizability of the Standard Model (Nobel Prize of 1999 for 't Hooft and Veltman). However, the foundation of the intuitive Faddeev-Popov integral for the Yang-Mills (YM) theory was achieved by Faddeev [6] on the way of the construction of an 'equivalent unconstrained system' compatible with the simplest canonical scheme of quantization in the form of the Feynman integral over independent physical variables

\[ Z_{YM}^* = \int \prod_{t,x} \left\{ \frac{3}{2\pi} \left[ d^2 A^T_a d^2 E^T_a \right] \right\} \exp \left\{ i W_{YM}^* + i \int d^4 x J^a_i A^T_i \right\} . \] (25)

The action of this equivalent unconstrained system

\[ W_{YM}^* = \int d^4 x \left\{ E^a_k T \cdot A^T_k - \frac{1}{2} \left\{ (E^2_k T)^2 + B^2_k (A^T) + (\partial_k \sigma^a)^2 \right\} \right\} , \] (26)

where \( \sigma^a \) satisfies the equation

\[ D(A)^{bc}_i \partial_i \sigma^c = -g e^{abc} A^a_i E^c_i , \] is derived by the constraining

\[ W_{YM}^* = W_{YM}(\text{constraint}) \] (27)

the initial YM action in the first order formalism

\[ W_{YM} = \int d^4 x \left\{ F^a_0 G^a_0 - \frac{1}{2} \left[ F^2_0 + B^2_i \right] \right\} \] (28)

The constraining (27) means the substitution of the explicit solution of the Gauss law constraint (29)

\[ D_i^{ab}(A) F^b_i = 0 \] (29)

obtained by decomposing the electrical components of the field strength tensor \( F_{0i} \) into transverse \( F_{0i}^T = E_i^T \) and longitudinal \( F_{0i}^L = -\partial_i \sigma^a \) parts

\[ F_{0i}^a = E^a_i T - \partial_i \sigma^a; \quad \partial_k E^b_k = 0 . \] (30)

This decomposition is compatible with the perturbative gauge

\[ \partial_i A^a_i := \partial_i A^T_i = 0 . \] (31)

3.2. Constraining with the zero mode

To formulate an equivalent unconstrained system for the YM theory in the winding number class of functions in the presence the zero mode \( Z^b \) of the Gauss law constraint (29)

\[ A^a_0 = Z^a + \tilde{A}^a_0, \quad F_{0k}^a = -D_k^{ab}(A) Z^b + \tilde{F}_{0k}^a \quad (D^2(A))^{ab} Z^b = 0 ) \] (32)

we shall follow two principles of the Faddeev foundation [6]: i) the constraint-shell action (27)

\[ W_{YM}(\text{constraint}) = W_{YM}[Z] + \tilde{W}_{YM}[	ilde{F}] , \] (33)
and ii) the choice of the gauge
\[
D_{i}^{ab}(\Phi_{i})\tilde{A}_{i}^{b} := 0
\]
compatible with the perturbation theory around the SU(2) monopole \(\Phi_{i}^{a}\)
\[
\tilde{F}_{0k} = U_{z}\tilde{F}_{0k}U_{z}^{-1}, \quad \tilde{A}_{i} = U_{z}(\tilde{A}_{i}^{a} + \partial_{i})U_{z}^{-1}, \quad \tilde{A}_{i}^{a}(t, \vec{x}) = \Phi_{i}(\vec{x}) + \tilde{A}_{i}(t, \vec{x}).
\]
To present the action in the form (33), we can use the evident decomposition
\[
F^{2} = (-DZ + \tilde{F})^{2} = (DZ)^{2} - 2F_{0}DZ + (\tilde{F})^{2} = \partial(\sqrt{\partial(ZDZ)}) - 2\partial(\tilde{F}Z) + (\tilde{F})^{2}
\]
and the Gauss Eqs. \(D\tilde{F} = 0\) and \(D^{2}Z = 0\) which show that the zero mode part \(W_{YM}\) of the constraint-shell action (33) is the sum of two surface integrals
\[
W_{YM}[Z] = \int dt \int d^{3}x [\frac{1}{2}\partial_{i}(Z^{a}D_{i}^{ab}(A)Z^{b}) - \partial_{i}(\tilde{F}_{0i}Z^{a})] = W^{0} + W',
\]
where the first one \(W^{0}\) is the kinetic term and the second one \(W'\) describes the coupling of the zero-mode to the local excitations. These surface terms are determined by the asymptotics of the fields \((Z^{a}, \tilde{A}^{a}_{i})\) at spatial infinity [13], [8] which we denoted, in the SU(3) case, by \((\tilde{N}(t)\Phi_{0}^{a}(\vec{x}), \Phi_{i}^{a}(\vec{x}))\). The fluctuations \(\tilde{F}_{0i}\) belongs to the class of multipoles and since the surface integral over monopole-multipole couplings vanishes, the fluctuation part of the second term obviously drops out. The substitution of the solution with the asymptotics (13) into the first surface term Eq. (37) leads to the zero-mode action
\[
W^{0} = \frac{I}{2} \int dt \hat{N}(t)^{2},
\]
where
\[
I = \int d^{3}x \partial_{i} [\Phi_{0}^{a}(\vec{x})\partial_{i}\Phi_{0}^{a}(\vec{x})] = (\frac{2\pi}{g})^{2}4\pi\epsilon,
\]
and
\[
\epsilon = \lim_{r_{v}\to\infty} r_{v}^{2}f_{0}(r_{v}).
\]
\(r_{v}\) is a value of \(r\) at a boundary of a finite spatial volume.

The action for the equivalent unconstrained system of the local excitations,
\[
\tilde{W}_{YM}[\tilde{F}] = \int d^{4}x \left\{ E_{k}^{a} \cdot \tilde{A}^{ab}_{k} - \frac{1}{2} \left\{ E_{k}^{a} + B_{k}^{a}(A^{b}) + [D_{k}^{ab}(\tilde{\Phi})\tilde{\sigma}^{b}]^{2} \right\} \right\},
\]
is obtained by decomposing the electrical components of the field strength tensor \(F_{0i}^{a}\) into transverse \(F_{0i}^{aT} = E_{i}^{a}\) and longitudinal \(F_{0i}^{aL} = -D_{i}^{ab}(\tilde{\Phi})\tilde{\sigma}^{b}\) parts, so that
\[
F_{0i}^{aT} = E_{i} - D_{i}^{ab}(\tilde{\Phi})\tilde{\sigma}^{b}; \quad D_{k}^{ab}(\tilde{\Phi})E_{k}^{b} = 0.
\]
Here the function \(\tilde{\sigma}^{b}\) is determined from the Gauss equation (24)
\[
((D^{2}(\tilde{\Phi}))^{ab} + g\epsilon^{adc}A_{i}^{d}D_{i}^{cb}(\tilde{\Phi}))\tilde{\sigma}^{b} = -g\epsilon^{abc}\tilde{A}_{i}^{a}E_{i}^{c}.
\]

### 3.3. Feynman path integral

The Feynman path integral over the independent variables includes the integration over the topological variable \(N(t)\)
\[
Z_{YM}^{\text{Feynman}}[J] = \int dN(t)\tilde{Z}_{YM}[J^{U}],
\]
where
where
\[
\tilde{Z}[J^U] = \prod_{t,x} \frac{3}{2\pi} \int \prod_{a=1}^8 [d^2 A^a_d d^2 E_a] \exp \left\{ i\mathcal{W}_{YM} + i\mathcal{W}_Y + iS[J^U] \right\} .
\] (45)

As we have seen above, the functionals \( \tilde{W}, S \) are given in terms of the variables which contain the nonperturbative phase factors \( U = U_Z \) (21) of the topological degeneration of initial data. These factors disappear in the action \( \tilde{W} \), but not in the source
\[
S[J^U] = \int d^4x J^a I^a, \quad \hat{A}_i = U(\hat{A}_i^a + \partial_i U)^{-1}
\] (46)
what reflects the fact of the topological degeneration of the physical fields.

Thus, instead of the instanton averaging over 'interpolations between different vacua', in order to remove the topological degeneracy, all Green functions should be averaged over values of the topological variable and all possible angles of orientation of the monopole unit vector \( \vec{n} = \vec{x}/r \) (18) in the group space. This averaging leads to complete destructive interference of colored Green functions and corresponding color amplitudes. The complete destructive interference of color amplitudes can be interpreted as confinement in the spirit of the Feynman quark-parton duality 2, 16.

4. Quantum chromodynamics

4.1. Feynman path integral

QCD differs from the YM theory by the group SU(3) and quark fields.

The decomposition of the spatial components of the gluon fields \( A^a_I = \Phi^a_I + \tilde{A}^a_I \) into the static monopole \( \Phi^a_I \) and the fluctuations can be used to define a perturbation theory with respect to the dynamical gluon fields \( \tilde{A}^a_I \). The adequate gauge for this perturbation theory is the covariant Coulomb gauge defined by
\[
D^a_{ic}(\Phi^b_J) \tilde{A}^c_i = 0 .
\] (47)

After the separation of the phase factors of the topological degeneracy, we obtain the similar path integral in terms of perturbative fields \( (E, A^I, \bar{\psi}^I, \psi^I) \)
\[
Z_{QCD}^{Feynman}[J, \eta, \bar{\eta}] = \int \prod_t N(t) e^{i\mathcal{W}_0} \tilde{Z}[J^U, \eta^U, \bar{\eta}^U],
\] (48)

where
\[
\tilde{Z}[J^U, \eta^U, \bar{\eta}^U] = \int \prod_{t,x} \left\{ [d\bar{\psi}^I d\psi^I] \prod_{a=1}^8 [d^2 A^a_d d^2 E_a] \right\} \exp \left\{ i\mathcal{W}' + i\tilde{W} + iS[J^U, \eta^U, \bar{\eta}^U] \right\} ,
\] (49)

the zero-mode actions \( \mathcal{W}_0 \) and \( \mathcal{W}' \) are defined by the expressions similar to (37) and (38), in the first order formalism obtained by the Lagrange transformation
\[
\frac{1}{2} G^2_{0i} = G_{0i} F_{0i} - \frac{1}{2} F^2_{0i}.
\]

The quark part of \( \mathcal{W}' \) will be discussed later. The action for the equivalent unconstrained system of the local excitations
\[
\tilde{W} = \int d^4x \left\{ E_k^a \cdot \hat{A}^a_k - \frac{1}{2} \left\{ E^2_k + B^2_k(A^I) + [D^a_{kb}(\Phi) \sigma^b]^2 \right\} + \bar{\psi}^I [i\gamma_\mu \partial^\mu - m] \psi^I + \bar{\psi}^I \right\},
\] (51)
is obtained by decomposing the electrical components of the field strength tensor $F^a_0$ (similarly (35)) into transverse $F^a_0 = E_i$ and longitudinal $F^a_0 = -D^a_{0i}(\Phi)\tilde{\sigma}^b$ parts, so that

$$F^a_0 = E_i - D^a_i(\Phi)\tilde{\sigma}^b; \quad D^a_{0i}(\Phi)E^b_k = 0.$$  \hfill (52)

The function $\tilde{\sigma}^b$ is determined from the Gauss equation (11)

$$\left((D^2(\Phi))^{ab} + g_j^{abc}A^c_k(\Phi)\right)\tilde{\sigma}^b = -j^b_{\text{tot},0} \quad \text{with} \quad j^b_{\text{tot},0} = g_j^{abc}A^c_k(\Phi)E^b_k + j^b_0.$$  \hfill (53)

Equation (53) can be formally solved by introducing a Green function $G^{ab}(\vec{x}, \vec{y})$ defined as the solution to

$$\left((D^2(\Phi))^{ab} + g_j^{abc}A^c_k(\Phi)\right)G^{bc}(\vec{x}, \vec{y}) = \delta^{ac}\delta^3(\vec{x} - \vec{y}).$$  \hfill (54)

Then $\tilde{\sigma}^a$ can be expanded in a power series in $g$

$$\tilde{\sigma}^b(t, \vec{x}) = -\int d^3y G^{bc}(\vec{x}, \vec{y})j^c_{\text{tot},0}(t, \vec{y}) - \int d^3y d^3z G^{bc}(\vec{x}, \vec{y})g_j^{cde}A^d_k(t, \vec{y})G^{ef}(\vec{y}, \vec{z})j^f_{\text{tot},0}(t, \vec{z}) - ...$$  \hfill (55)

in analogy to the standard perturbation theory \cite{7}. The calculation of the Green function $G^{ab}(\vec{x}, \vec{y})$ is given in the Appendix.

**4.2. Dominance of the Wu-Yang monopole**

The action $W^0$ in the path integral (49) (given by (38)) describes a free ‘rotator’ with the momentum spectrum \cite{2}

$$P_0 = \hat{N}M_0 = (2\pi k + \theta), \quad k = 0, 1, 2, ..., \quad (56)$$

which follows from the constraint on the wave function for physically equivalent points $N$ and $N + 1$

$$\Psi(N + 1) = \exp(i\theta)\Psi(N).$$

The action for the ‘rotator’ (38) compensates the action for the monopole background

$$W[\Phi_t] = -\frac{1}{2} \int dt \int d^3x [B^a_t(\Phi_t)]^2$$

in the Minkowski space for finite values of the momentum of the rotator, so that the perturbation theory begins from the zero value of the Minkowski action, in the contrast to the instanton background. This can be interpreted as the dominance of the monopole background.

For the self-dual BPS ansatz (22) the dominant value of the momentum of the rotator is

$$\frac{P_0}{2\pi} = \frac{4\pi}{g^2}. \quad (57)$$

**4.3. Hadronization**

In the lowest order of this perturbation theory, we can rewrite the instantaneous interaction term in (51) in the form of a current-current interaction

$$W_{\text{int}} = -\frac{1}{2} \int d^4x [D^a_k(\Phi)\tilde{\sigma}^b]^2 = \frac{1}{2} \int d^4x \int d^3y \tilde{\sigma}^a_{\text{tot},0}(t, \vec{x})G^{ab}(\vec{x}, \vec{y})j^b_{\text{tot},0}(t, \vec{y})). \quad (58)$$

Following the QED perturbation theory of ‘radiative corrections’ one can formulate a similar perturbation theory of ‘dynamical corrections’ in QCD. This ‘dynamical perturbation theory’ is based on the decomposition of the action $\tilde{W}$ for the dynamical variables (17) into the ‘free’ part $\tilde{W}_0$, the instantaneous interaction $W_{\text{int}}$ (58) and the residual dynamical interaction $\tilde{W}_{\text{int}}$

$$\tilde{W} = \tilde{W}_0 + W_{\text{int}} + \tilde{W}_{\text{int}}. \quad (59)$$
The lowest order of the ‘dynamical perturbation theory’ $\tilde{W}_{\text{int}} = 0$ describes gluons and quarks in the monopole field and their instantaneous bound states. Next orders of this perturbation theory contain ‘dynamical corrections’ which describe matrix elements of transitions between these states.

Bound states are obtained in a similar way as in QED (i.e., by the Schwinger-Dyson and Bethe-Salpeter equations with the instantaneous interaction $C^{bc}(\vec{x}, \vec{y})$) given in the Appendix. In the field of the monopole the instantaneous quark-quark potential is the sum of a Coulomb potential and a rising one. It is well-known that the latter one leads to spontaneous chiral symmetry breaking and to mesonic bound states.

4.4. Frozen gluon approximation and $U_A(1)$- problem

The ‘frozen gluon’ approximation (FGA) follows from the Feynman path integral if we neglect the dynamical gluon fields: $\tilde{A}_a^i = 0$ (67).

The FGA generating functional (45) takes the form

$$\tilde{Z}_{\text{FGA}}[\eta, \bar{\eta}] = \int \prod_{t, x} \left\{ [d\bar{\psi}d\psi] \right\} \exp \left\{ iW + i\tilde{W}_{\text{FGA}} + iS[\eta, \bar{\eta}] \right\},$$

where

$$\tilde{W}_{\text{FGA}} = \tilde{W}_0 + \mathcal{W}_{\text{int}}$$

(61) depends only on the quark fields.

The bilocal linearization of the four quark interaction $W_{\text{int}}$ using the Hubbard-Stratonovich transformation leads to an effective bilocal meson action [17]. This meson action includes Abelian anomalies in the pseudoscalar isosinglet ($\eta_0$-meson) channel [4, 5, 18].

In our case, these anomalies include the time derivative of topological variable $N(t)$ due to interaction of the quark fields with the monopole and the zero-mode $\tilde{W}'$ (see eq. (37)).

Neglecting all mesonic channels except the $\eta_0$-meson one, we get in FGA nothing but the well-known gauge-invariant expression

$$\tilde{W}^{\eta_0}_{\text{anomaly}}[\eta, \bar{\eta}] = C_\eta \int dt \bar{\eta}(t, 0)I_c\gamma_5 \eta(t, 0)\frac{g^2}{16\pi^2} \int d^3x G_{\mu\nu}^a G_{\mu\nu}^a,$$

(62)

where $\eta, \bar{\eta}$ are fermion sources, $C_\eta$ is a constant. In the BPS monopole selfdual field \([9], [8], (22)\) with $G_{\mu_0}^a = \tilde{N} D_i^{ab}(\Phi)\Phi_0^b$ and $2\pi\epsilon D_i^{ab}(\Phi)\Phi_0^b = B^a_i(\Phi)$ we obtain the normalizable zero-mode

$$\frac{g^2}{16\pi^2} \int d^3x G_{\mu\nu}^a G_{\mu\nu}^a = \tilde{N}(t),$$

(63)
as

$$\frac{g^2}{8\pi^2} \int d^3D_i^{ab}(\Phi)\Phi_0^b B^a_i(\Phi) = 1 .$$

The physical anomaly term $\tilde{W}_{\text{anomaly}}$ in the $\eta_0$-meson channel takes the form \([19]\)

$$\tilde{W}^{\eta_0}_{\text{anomaly}}[\eta, \bar{\eta}] = C_\eta \int dt \bar{\eta}_{0}(t)\tilde{N}, \quad \eta_{0}(t) = \bar{\eta}(t)I_c\gamma_5 \eta(t).$$

(64)

Following Veneziano \([19]\), we can identify $\eta_0$ with the field of the $\eta_0$-meson at rest. The effective action including the anomaly term (62) and the zero-mode one (38) is then

$$W_{\text{eff}} = \int dt \left[ \frac{\tilde{N}^2 T}{2} + \eta_{0}C_\eta\tilde{N} + \frac{\epsilon^2 m_0^2 V}{2} \right],$$

(65)
where $m_0$ is the standard current quark mass contribution to the $\eta_0$ meson mass. The diagonalization of this effective action leads to an additional mass of $\eta_0$-meson

$$\Delta m^2 V = \frac{C_\eta^2}{T} \Rightarrow T = \frac{C_\eta^2}{\Delta m^2 V} = \frac{(2\pi)^2 \epsilon}{g^2} 4\pi.$$

(66)
The finite contribution of the zero-mode to the mass of the $\eta_0$-meson entails the disappearance of the mass of the collective topological variable and leads to a stable perturbation theory in the infinite volume limit, as in this limit the singularity-free BPS monopole converts into the Wu-Yang one without singularities at the origin.

### 4.5. Relativistic covariance

Recall that Dirac has also obtained the unconstrained form of QED in terms of gauge invariant variables for QED as functionals of the initial gauge fields by explicitly resolving the Gauss law constraint. The resulting unconstrained formulation of QED coincides with the one obtained in the Coulomb gauge with the physical phenomena of electrostatics, ‘dressed’ electrons, and two transverse photon degrees of freedom. In QED, in terms of the Dirac variables, the Poincaré symmetry is realized which is mixing with the gauge symmetry. This mixing was interpreted in 1930 by Heisenberg and Pauli (with reference to an unpublished note by von Neumann) as the transition from the Coulomb gauge with respect to the time axis in the rest frame \( (l_\mu^0 = (1, 0, 0, 0)) \) to the Coulomb gauge with respect to the time axis in the moving frame \( l_\mu = l_\mu^0 + \delta L l_\mu^0 = (Ll_\mu^0) \). The Coulomb interaction has the covariant form

\[
W_C = \int dxdy \frac{1}{2} j_l^T(x) V_C(z^\perp) j_l^T(y) \delta(l \cdot z),
\]

where \( j_l^T = e \bar{\psi}^T \gamma_l \psi^T \), \( z_\mu^\perp = z_\mu - l_\mu(z \cdot l) \), \( z_\mu = (x - y)_\mu \). This transformation law and the relativistic covariance of this formulation of QED has been proven by Zumino on the level of the algebra of generators of the Poincaré group.

In QED we can change variables to construct the generating functional of the Green functions in any gauges including the Lorentz invariant ones. The invariance of the corresponding Green functions under a change of variables (which generates the Ward-Taylor-Slavnov identities) is guaranteed by the Dirac factors in source terms, which restore the Coulomb gauge Feynman rules in any Lorentz invariant gauge. So, the Coulomb interaction and electrostatics are consequences of the identification of the physical degrees of freedom which correspond to an explicit solution of the Gauss law, but not primarily to the choice of the gauge. For example, if one would omit the Dirac factors in the source terms in relativistically invariant Lorentz gauge formulations of QED, one would get the Wick-Cutkosky bound states formed by gauge propagators with light-cone singularities with a spectrum different from the observed one which corresponds to the instantaneous Coulomb interaction.

In QCD, the moving Lorentz frame corresponds to the moving Wu-Yang monopole. We should only manage to realize this transformation law on the level of operators.

### 5. Conclusion

In this paper, we have shown that there is a collective topological excitation in the gluon spectrum as the zero-mode of Gauss’ constraint from the winding number class of functions of the large gauge transformations.

This topological excitation leads to the dominance of the Wu-Yang monopole in the Feynman path integral compatible with the equivalent unconstrained system obtained in this work.

In the field of the Wu-Yang monopole the instantaneous quark-quark potential is the sum of a Coulomb-type potential and a rising one. The latter one leads to spontaneous chiral symmetry breaking and to mesonic bound states.

The $\eta_0$-meson mixes with the zero-mode so that after diagonalization of this low-energy action a mass shift of the $\eta_0$-meson is obtained which resolves the $U_A(1)$ problem.

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Colored amplitudes contain additional phase factors which depend on the zero-mode. Averaging the colored amplitudes over the zero-mode parameters can lead to the phenomenon of complete destructive interference \[2, 16\], so that the color amplitudes disappear. The colorless ones of the type of the expectation values of the electroweak currents do not vanish since the zero-mode phase factors are absent.

According to Heisenberg, Pauli \[21\] and Zumino \[22\], the relativistic covariance is established by a rotation of the timelike axis so that the non-Abelian Coulomb-gauge field moves together with the relativistic bound states. Recently, Faddeev and Niemi \[24\] constructed a similar relativistically covariant form of the Wu-Yang monopole.

In summary, the present scheme for the introduction of topological global variables is a promising tool for the investigation of the challenging properties of QCD such as the meson spectrum, chiral symmetry breaking, quark and gluon confinement, and the \( U_A(1) \) anomaly.

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A The Green function

We can calculate the instantaneous the Green function \([68]\)
\[
(D^2(h))^{ab}(\vec{x})G^{bc}(\vec{x},\vec{y}) = \delta^{ac}\delta^3(x-y) .
\]

In the presence of the Wu-Yang monopole we have
\[
(D^2)^{ab}(\vec{x}) = \delta^{ab}\Delta - \frac{n^a n^b + \delta^{ab}}{r^2} + 2\left(\frac{n_a}{r}\partial_b - \frac{n_b}{r}\partial_a\right),
\]
and \(n_a(x) = x_a/r; \ r = |\vec{x}|\). Let us decompose \(G^{ab}\) into a complete set of orthogonal vectors in color space
\[
G^{ab}(\vec{x},\vec{y}) = [n^a(x)n^b(y)V_0(z) + \sum_{\alpha=1,2} e_{\alpha}^a(x)e_{\alpha}^b(y)V_1(z)]; \ (z = |\vec{x} - \vec{y}|).
\]

Substituting the latter into the first equation, we get
\[
\frac{d^2}{dz^2}V_n + \frac{2}{z}\frac{d}{dz}V_n - \frac{n}{z^2}V_n = 0 \ \ \ n = 0, 1 .
\]

The general solution for the last equation is
\[
V_n(z) = d_n z^{l_1^n} + c_n z^{l_2^n}, \tag{69}
\]
where \(d_n, c_n\) are constants, and \(l_1^n, l_2^n\) can be found as roots of the equation \((l^n)^2 + l^n = n\), i.e.
\[
l_1^n = -1 + \sqrt{1 + 4n} / 2; \ \ l_2^n = -1 + \sqrt{1 + 4n} / 2 . \tag{70}
\]

It is easy to see that for \(n = 0\) we get the Coulomb-type potential \(d_0 = -1/4\pi\), and for \(n = 1\) the ‘golden section’ potential with
\[
l_1^1 = -1 + \sqrt{5} / 2 \approx -1.618; \ \ l_2^1 = -1 + \sqrt{5} / 2 \approx 0.618 . \tag{71}
\]

The last potential (in the contrast with the Coulomb-type one) leads to spontaneous chiral symmetry breaking and can be used as the potential for the ‘hadronization’ of quarks and gluons in QCD \[16, 17\].
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