On the generalized spinor field classification: Beyond the Lounesto Classification

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Abstract. In this paper we advance into a generalized spinor field classification, based on the so-called Lounesto classification. The program developed here is based on an existing freedom on the spinorial dual structures definition, which, in certain simple physical and mathematical limit, allows us to recover the usual Lounesto classification. The protocol to be accomplished give full consideration in the understanding of the underlying mathematical structure, in order to satisfy the quadratic algebraic relations known as Fierz-Pauli-Kofink identities, and also to provide physical observables. As we will see, such identities impose a given restriction on the number of possible spinor field classes in the classification. We also expose a mathematical device known as Clifford algebra deformation, which ensures real spinorial densities and holds the Fierz-Pauli-Kofink quadratic relations.

I. INTRODUCTION

The well known Lounesto’s spinor field classification is a comprehensive and exhaustive categorization based on the bilinear covariants that discloses the possibility of a large variety of spinors, comprising regular and singular spinors which includes the cases of Dirac, Weyl, and Majorana as very particular spinor fields [1]. Although the Lounesto’s standard spinor field classification is restricted to the $U(1)$ gauge symmetry, one can find other categorizations by splitting off classes of charged and neutral spinors, and then implementing more general gauge symmetries. For instance, in [2] the authors studied and classified the spinor doublets by considering the gauge symmetry in electroweak theory. An extended non-Abelian spinor field classification that encompasses the $SU(2) \times U(1)$ gauge symmetry can be found in [3].

The Dirac equation, one of the most remarkable achievements in the history of physics conceived from the purely theoretical reasoning, is a covariant first-order derivative field equation which suitably describes spin one-half particles, as well as the matter/antimatter duality [4]. Hundreds of textbooks usually show the dual structure for the fermionic spin one-half Dirac field (or also known as eigenspinors of parity operator), only by elucidating a given structure without at least show explicitly how it actually emerges, or even mentioning the fact that it may not be unique. Now, if other physical fields describing relevant particles exist, a natural question is if they must obey the same dual structure commonly used for Dirac fields, or not? In other words, is the Dirac structure so fundamental? Both questions are rarely asked in the physics literature. Undertaking a deep analysis of spinorial duals could help us to get closer to answer the above-mentioned questions. The algebraic theory of spinor duals makes use of the rich and well known structure of Clifford algebras to specify all possible duals for arbitrary algebras of any dimension and space(time) signature [5]. However, when the theory of the mass-dimension-one (Elko) fields was proposed, it was necessary to revisit some fundamental aspects of the Quantum Field Theory, such as spinorial dual theory and (deformation of) the Clifford algebras, always aiming to retrieve physical information [6, 7].

Elko spinor, proposed in [8], is a spin-1/2 fermionic field endowed with mass dimension one, built upon a complete set of eigenspinors of the charge conjugation operator, which has the property of being neutral with respect to

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gauge interactions. On its earlier formulation, these fields were quantum objects which carried a representation of subgroups of the Lorentz group $HOM(2)$ and $SIM(2)$ [9], and corresponding semi-direct extension encompassing translation. Recently, a redefinition in the spinor adjoint has lead to a theory endowed with full Lorentz (Poincaré) symmetry. The main features of this formulation, along with the theory of duals may be found in Ref.[10]. As Elko spinor fields have mass dimension one, there is nothing that precludes the appearance of mass dimension one spinor fields further in classes (4) and (6) in Lourenço’s spinor field classification [11].

When we deal with the theory of spinors regardless of their nature, for instance eigenspinors of parity or charge conjugation operator, the dual structure is of paramount importance, firstly because much of the physics associated to a spinor field is unveiled from its covariants bilinears by the simple reason that single fermions are not directly experienced [7], and also to study the field dynamics, spin sums, couplings, Fermi-Dirac statistic among other relevant quantities. Then, it is necessary to have a well-defined dual structure.

In the context of the bilinear structures, it is indeed important to pay attention to the subtleties of Clifford algebra when associating real numbers to the bilinear covariants [7, 12]. All the protocol developed in this paper is based on two fundamental works written by Crawford [13, 14], where he worked out several important formalizations concerning the bispinor algebra, developing a rigorous mathematical mechanism to obtain real bilinear covariants. For a more complete understanding of this subject the reader is also referred to [15–17].

In this communication we look for the underlying mathematical approach that allows us to define a spinorial dual structure, by making correct use of the Crawford mechanism to evaluate the bilinear covariants, using the correct composition law of the basis vector of the Clifford algebra (also taking into account the Dirac normalization), and verifying if given forms satisfy the algebraic Fierz-Pauli-Kofink identities (henceforth, we will call FPK identities). In the case they do not satisfy FPK identities, we present a similar procedure based on the deformation of the Clifford algebra, which allows us to ensure the FPK identities. To the best of our hope, the program to be accomplished here may shed some light on the methods to define a correct spinorial dual structure.

The paper is organized as follows: in section II we define the basic concepts concerning the construction of the spinorial duals, and then we define a general dual structure. For a book-keeping purpose, section III is reserved for a complete and deep overview on the well-known Lourenço classification, by showing the main aspects of the classification and the underlying algebraic structure behind it. In section IV we start the program proposed and, from a general dual structure, we define a general spinor field classification holding the same algebraic structure as Lourenço classification does. In section V we present a mathematical mechanism which ensures real quadratic forms and also guarantees the FPK identities. Finally, in section VI we conclude. Also some appendices are reserved.

II. PROEM: A BRIEF OVERVIEW ON SPINORIAL DUALS

Spinors may be defined in several different ways. In the context of Clifford algebra, the spinors are defined to be elements of a left minimal ideal, whereas in the context of group theory we say that the spinors are carriers of the fundamental representation of the group. Spinors are used extensively in physics [18], and it is widely accepted that they are more fundamental than tensors (when the spacetime itself is represented by a manifold endowed with a Riemannian - or Lorenzian - metric structure).

The idea that the usual Dirac dual cannot be applied to every spinor is sharp enough to force the development of an accurate criteria in the formalization of spinor duals [5]. As asserted in [19], regarding to the physical observables, authors in [20] classify Elko spinors as type-5. However, the Elko norm is defined taking into account the Elko dual structure, and then all the physical quantities should carry the same dual structure rather than the Dirac one. This fact suggests the necessity of constructing bilinear forms using the correct dual structure. Until the present moment, there are three well-established dual structures in the theory of spinor. They are: the Dirac dual given by

$$\bar{\psi}_h = [\Xi_D(p)\psi_h]^\gamma_0, \tag{1}$$

where the lower index $h$ concerns the helicity, and the operator $\Xi_D(p)$ can be given as, $\Xi_D(p) = 1$ or $\Xi_D(p) \neq 1$

1 Authors hasten to advise the reader that the most general prescription to write the dual structure is the following: $\bar{\psi}_h = [\Xi(p)\psi_h]^\eta$. Moreover, the norm under the requirement of Lorentz invariance, e.g., $\bar{\psi}_h\psi_h = \bar{\psi}_h\psi_h$, where the primed amounts stand for the Lorentz transformed ones, provide $\{\kappa, \eta\} = 0$ and $\{\zeta, \eta\} = 0$, in which $\kappa$ and $\zeta$ stand for the boost and rotation generators. Then, the only way to satisfy simultaneously both conditions is setting $\eta = \gamma_0$. As we will see later, another important fact is that the $\gamma_0$ matrix, present in the dual structure, is responsible for assuring the reality character of bilinear forms [19].
and the Elko dual given by [5, 6]

\[ -\frac{S/A}{D} \lambda_h = [\Xi_E(p)\lambda_h^{S/A}]^\dagger \gamma_0 \mathcal{O}, \]  

(2)

where the operator \( \Xi_E(p) \) is responsible to change the spinor helicity of the Elko spinors [10]. In general, the \( \Xi(p) \) operators, presented in (1) and (2), must satisfy certain conditions. Denoting the spinor space by \( S \), then the \( \Xi(p) \) operator is such that [25]

\[ \Xi(p) : S \rightarrow S \]

\[ \psi_h \mapsto \psi_{h'}, \]  

(3)

Furthermore, \( \Xi(p) \) has to be idempotent, \( \Xi^2(p) = 1 \), ensuring an invertible mapping [25]. From (3), we have the following possibilities: \( h = h' \), for which \( \Xi(p) = 1 \) and stands for the Dirac usual case, \( \Xi(p) \neq 1 \) stands for the non-standard Dirac adjoint [24], and finally \( h \neq h' \) leading to a more involved operator present in the mass-one theory [6]. Thus the purpose of the present paper is to invoke a mathematical procedure for determining the spinorial dual structure based on the general form \( \sim = [\Xi_{E}(p)\psi]^\dagger \gamma_0 \), by analysing the bilinear forms and the related FPK identities. First of all, if the dual structure provides bilinear forms which ensure the FKP quadratic relations, then it will be sufficient to have a well-defined theory. If not, we will show a method of deformation of the Clifford algebra so that FPK identities are respected by the bilinear forms obtained from the new dual structure.

### III. BASIC CONCEPTIONS ON THE LOUNESTO CLASSIFICATION AND SPINORIAL DENSITIES

Spinors in the Minkowski spacetime, \( \mathcal{M}(4, \mathbb{C}) \), are elements belonging to the spinor bundle on \( \mathcal{M}(4, \mathbb{C}) \) carrying the so-called \((1/2, 0) \oplus (0, 1/2)\) representations of the Lorentz group [2]. Suppose \( \psi \) to be a given spinor field which belongs to a section of the vector bundle \( \mathbf{P}_{\text{Spin}^\mathbb{C}_3}(\mathcal{M}) \times \rho \mathbb{C}^4 \) where \( \rho \) stands for the entire representation space \( D^{(1/2,0)} \oplus D^{(0,1/2)} \), or a given sector of such [13, 14]. The bilinear quantities associated to \( \psi \) read

\[
\begin{align*}
\sigma &= \bar{\psi} \psi, \quad \text{(scalar)} \\
\omega &= i\bar{\psi} \gamma_5 \psi, \quad \text{(pseudo-scalar)} \\
J &= J_{\mu} \theta^\mu = \bar{\psi} \gamma_\mu \psi \theta^\mu, \quad \text{(vector)} \\
K &= K_{\mu} \theta^\mu = -\bar{\psi} \gamma_5 \gamma_\mu \psi \theta^\mu, \quad \text{(axial-vector)} \\
S &= S_{\mu\nu} \theta^{\mu\nu} = \bar{\psi} i \varepsilon_{\mu\nu} \psi \theta^\mu \wedge \theta^\nu, \quad \text{(bi-vector)}
\end{align*}
\]

(4)

where \( \gamma_5 := -i\gamma_{0123} \) and \( \gamma_{\mu\nu} := \gamma_\mu \gamma_\nu \). Denoting by \( \eta_{\mu\nu} \) the Minkowski metric, the set \( \{ 1, \gamma_I \} \) (where \( I \in \{ \mu, \mu, \mu\mu, 5 \} \)) is a composed index) is a basis for the Minkowski spacetime \( \mathcal{M}(4, \mathbb{C}) \) satisfying \( \gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\eta_{\mu\nu} 1 \), and \( \bar{\psi} = \psi^\dagger \gamma_0 \) stands for the adjoint spinor with respect to the Dirac dual. Here, we are considering the space-time metric given by diagonal \((1, -1, -1, -1)\). The elements \( \{ \theta^\mu \} \) are the dual basis of a given inertial frame \( \{ e_\mu \} = \{ \frac{\partial}{\partial x^\mu} \} \), with \( \{ x^\mu \} \) being the global spacetime coordinates. Through the analysis of the above forms it is possible to construct a specific classification, namely Lounesto classification, which provides six disjoint classes for the fermionic spin one-half fields [12]. Commonly, it is always expected to relate (4) to physical observables. For space-time of dimension \( N = 2n \), the corresponding spinor, \( \psi \), has \( D = 2^n \) complex components which necessarily satisfy a system of \((D - 1)^2\) quadratic relations (FPK identities). From a mathematical point of view, the bilinear identities, also called the Pauli identities, or in the most general form the FPK identities, are the direct consequences of the completeness of the Dirac matrices.

Bilinear identities are usually considered as a useful tool for transforming different mathematical expressions involving products of the electron wave functions [26, 27]. We shall emphasize that the transformation properties (under \( \mathcal{C}, \mathcal{P}, \mathcal{T} \) or any other Lorentz transformation) of the bilinears are important in constructing relativistic theories [28, 29].

The electromagnetic and kinematic properties of the electron are well-described in the Dirac theory in terms of quantities constructed as bilinear expressions involving the \( \psi \) spinor [29–34]. So, we shall emphasize that, exclusively in the Dirac theory, the above bilinear covariants have particular interpretations. In this context, the mass of the particle is related to \( \sigma \), and the pseudo-scalar \( \omega \) is relevant for parity-coupling (the pseudo-scalar quantity interacts with a pseudo-scalar meson \( \pi^0 \) preserving parity [28]). In addition, \( \sigma \) appears as mass and self-interaction terms.

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2 The presence of the abstract \( \mathcal{O} \) operator stands for the new operators \( \mathcal{A} \) and \( \mathcal{B} \), built in Ref [6]. Such operators were recently proposed and they are responsible to ensure the Lorentz invariance and locality of the Elko fields.
in the Lagrangian, whereas $\omega$, being $CP$-odd, might probe $CP$ features [2]. In addition, the current four-vector $J$ gives the current of probability$^3$, $K$ is an axial vector current$^4$, and $S$ is associated with the distribution of intrinsic angular momentum. The bilinear covariants are physically interpreted in the Dirac theory. In fact, $eJ_0$ is the charge density, whereas $eCJ_k(i,j,k = 1,2,3)$ is identified to the (electric) current density [26]. The quantity $\frac{\partial}{\partial m^2}S_{ij}$ is the magnetic moment density, while $\frac{\partial}{\partial m^2}S_{0ij}$ is the electric moment density [21, 39–41]. The $\frac{2}{3}K_\mu$ is interpreted as chiral current (spin density), conserved when $m = 0$ [1] or when the spinor belongs to Dirac type-2 according to Lounesto classification.

For the Dirac theory, the two currents $J_\mu$ and $K_\mu$ are the Noether currents corresponding to the two transformations [22]

$$\psi(x) \rightarrow \psi'(x) = e^{i\alpha}\psi(x),$$

$$\psi(x) \rightarrow \psi'(x) = e^{i\alpha\gamma_0}\psi(x).$$

The first of these is a symmetry of the Dirac lagrangian and explicitly provides the Noether conserved current $J_\mu$. Such symmetry also provides the $N$ conserved quantity,

$$N = \int d^3x\psi^\dagger\psi,$$ (7)

which corresponds to the particle-number operator for the fermion case [42] (usually interpreted as the conserved charge in the Dirac theory [21]). The second, called as chiral transformation, is a symmetry of the derivative term in the lagrangian but not the mass term. Thus, the Noether theorem confirms that the axial vector current is conserved only if $m = 0$ [22].

We stress that such labels are given accordingly to the way that each one of these quantities (the time-component and space component) behaves under Lorentz transformations (for more details see chapter 17 in reference [43]. Besides, all the above structures are only valid for spinors which have the same dual structure as Dirac spinor does. These mathematical structures mandatorily obey the so-called FPK identities, given by [44]

$$J^2 = \sigma^2 + \omega^2,$$

$$J_\mu K_\nu - K_\mu J_\nu = -\omega S_{\mu\nu} - \frac{\sigma}{2}\epsilon_{\mu\nu\alpha\beta}S^{\alpha\beta},$$

$$J_\mu K^\mu = 0,$$

$$J^2 = -K^2.$$ (8)

The above identities are fundamental not only for classification, but also to guarantee the inversion theorem [13]. Within the Lounesto classification scheme, a non vanishing $J$ is crucial, since it enables to define the so called boomerang [45], which has an ample geometrical meaning in order to assert that there are precisely six different classes of spinors. This is a prominent consequence of the definition of a boomerang. As far as the boomerang is concerned, it is not possible to exhibit more than six types of spinors, according to the bilinear covariants. Indeed, Lounesto spinor classification splits them into regular and singular spinors. The regular spinors are those which have at least one of the bilinear covariants $\sigma$ and $\omega$ non-null. Singular spinors, on the other hand, have $\sigma = 0 = \omega$, consequently the Fierz identities are normally replaced by the more general conditions [13, 14]

$$Z^2 = 4\sigma Z,$$

$$Z\gamma_\mu Z = 4J_\mu Z,$$

$$Zi\gamma_\mu Z = 4\omega Z,$$

$$Zi\gamma_\mu\gamma_\nu Z = 4S_{\mu\nu} Z,$$

$$Z\gamma_\mu\gamma_\nu Z = 4K_\mu Z.$$ (9)

$^3$ The Dirac current $J_D$ is a future-oriented vector ($J_D^\mu > 0$) whereas the vector $K_D$ is space-like ($K_D^2 < 0$), such that $K_D^2 = -J_D^2$ [12]. Elko spinors provide the following relations $J_E^\mu < 0$ and $K_E^2 > 0$ which stand for a space-like and time-like vector respectively. Remarkably, in [35], authors stated that both $J_M$ and $K_M$, for Majorana spinors, are light-like, and the reason lies in the following observation: from the point of view of a first quantized theory, the Majorana field can not carry any electric or magnetic charge, i.e., the physical currents $\epsilon_MJ_M$ and $q_MK_M$ are null because $\epsilon_M = q_M = 0$.

$^4$ If $\psi$ satisfies the Dirac equation we have $\partial_\mu J^\mu = 0$, thus $J^\mu$ is always conserved, and similarly we have $\partial_\mu K^\mu = 2m\bar{\psi}\gamma_\mu\psi = 2m\omega$. If $m = 0$ or $\omega = 0$ this current is also conserved [22] (see also [36] for further comments). The Dirac massless case is also discussed in [21, 28, 37]. In [38] it is proposed that the massless Dirac equation describes (massless) neutrinos which carry pair of opposite magnetic charges.
When an arbitrary spinor $\xi$ satisfies $\xi^*\psi \neq 0$ and belongs to $\mathbb{C} \otimes \mathcal{C}_{1,3}$ — or equivalently when $\xi^\dagger \gamma_0 \psi \neq 0 \in \mathcal{M}(4, \mathbb{C})$ — it is possible to recover the original spinor $\psi$ from its aggregate $Z$. Such relation is given by $\psi = Z \xi$, and the aggregate reads

$$Z = \sigma + J + iS + K\gamma_5 - iw\gamma_5.$$  \hspace{1cm} (10)

Hence, using (10) and taking into account that we are dealing with singular spinors, it is straightforward to see that the aggregate can be recast as

$$Z = J(1 + is + h\gamma_{0123}),$$  \hspace{1cm} (11)

where $s$ is a space-like vector orthogonal to $J$, and $h$ is a real number [12]. The multivector as expressed in (11) is a boomerang. By inspecting the condition $Z^2 = 4hZ$, we see that $Z^2 = 0$ for singular spinors. However, in order to the FPK identities hold, it is also necessary that both conditions $J^2 = 0$ and $(s + h\gamma_{0123})^2 = -1$ must be satisfied [46].

In total there are nine FPK identities, but these four above are the main ones since the others can be constructed from them. As can be seen the physical requirement of reality can always be satisfied for the Dirac spinors bilinear covariants [13], by a suitable deformation of the Clifford basis leading to physical appealing quantities. However, the same assertion cannot be stated for the mass-dimension-one spinors, for which the FPK relations are violated [7]. We suspect that this fact is due to the new dual structure associated to these spinors. It is worth to mention that the main difference between the Crawford deformation [13, 14] and the one to be accomplished here is that in the former case, the spinors are understood as Dirac spinors, i.e., spinorial objects endowed with single helicity [7], while the protocol that will be developed here is a general procedure that can fits in any case.

IV. ON THE SET-UP OF A GENERALIZED SPINOR CLASSIFICATION

A natural path to classify fermions resides on the Lounesto classification. Such classification is built up taking into account the 16 bilinear forms, encompassing fermions such as Dirac, Type-4 spinors [47], Majorana$^5$ (neutrino), and Weyl$^6$ (massless neutrino) [12], and it labels the spinors accordingly to the bilinear forms value. This specific classification is based on geometric identities known as FPK identities, given in (8) and (9), and displays exclusively six disjoint classes of fields. This fact is due to the restriction imposed by the FPK identities. Then, it covers all the possibilities of fields restricted by this geometrical constraint.

The quantum field theory literature usually takes the Dirac dual as the standard one, with no suspicion or need of alternative dual structures as being potentially interesting [5]. However, the development of the theory of dark spinors presented the need for a review of dual structures [6]. A very important and peculiar feature that concerns to the Lounesto classification lies in the fact that such classification takes into account (exclusively) the Dirac dual structure. However, if fields with a more involving dual structure exists, how it may be classified? Is it possible to fit them within Lounesto classification? Maybe there is a naive freedom in Lounesto classification which allows one to develop a more general classification, taking into account the appropriated dual structure. In this section we look towards develop a generalized spinor classification which allows us, in a certain and appropriate mathematical limit, to recover the usual Lounesto classification. For the sake of completeness, we will consider the same rigorous mathematical procedure and conditions as Lounesto did, however we will impose a general spinorial dual structure.

Suppose $\psi$ to be a given (algebraic) spinor field which belongs to a section of the vector bundle $\mathcal{P}_{Spin_{1,3}}(\mathcal{M}) \times \rho \mathbb{C}^4$ where $\rho$ stands for the entire representation space $D^{(1/2,0)} \oplus D^{(0,1/2)}$. Such spinor can be described as follows

$$\psi = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix},$$  \hspace{1cm} (12)

where the components $a, b, c, d \in \mathbb{C}$. Differing from what it was developed by Lounesto, here we extend the dual structure to cover all possibilities of spinorial duals. Now, we define the new dual structure as

$$\psi^\dagger \equiv [\Xi_G(p)\psi]^\dagger \gamma_0,$$  \hspace{1cm} (13)

$^\dagger$ As shown in [35] Majorana spinors can not satisfy the Dirac equation.

$^6$ The Weyl spinors are eigenspinors of the chirality operator $\gamma_{0123}\psi_{\pm} = \pm i\psi_{\pm}$ [12].
where the general $\Xi_G(p)$ operator is defined as

$$
\Xi_G(p) = \begin{pmatrix}
m_{11} & m_{12} & m_{13} & m_{14} \\
m_{21} & m_{22} & m_{23} & m_{24} \\
m_{31} & m_{32} & m_{33} & m_{34} \\
m_{41} & m_{42} & m_{43} & m_{44} 
\end{pmatrix},
$$

with the components $m_{ij} \in \mathbb{C}$. This operator, \textit{a priori}, must obey the following two main constraints: $\Xi_G^2(p) = 1$ and also $\Xi_G^{-1}(p)$ has to exist, ensuring an invertible map. Notice that if one imposes $\Xi_G(p) \equiv 1$ we retrieve the usual Lounesto classification. Finally, the general dual structure is defined as follow

$$
\psi = \begin{pmatrix}
am_{3,1} & bm_{3,2} & cm_{3,3} & dm_{3,4} \\
am_{4,1} & bm_{4,2} & cm_{4,3} & dm_{4,4} \\
am_{1,1} & bm_{1,2} & cm_{1,3} & dm_{1,4} \\
am_{2,1} & bm_{2,2} & cm_{2,3} & dm_{2,4} 
\end{pmatrix}^T,
$$

where the over line stands for the complex conjugation. Now, by using this structure one can evaluate the 16 bilinear amounts

$$
\sigma = \psi \psi, \quad \tilde{\varepsilon} = -i \psi \gamma_5 \psi, \quad \tilde{J} = \psi \gamma_\mu \psi \theta^\mu, \quad \tilde{K} = -i \psi \gamma_5 \gamma_\mu \psi \theta^\mu, \quad \tilde{S} = i \psi \gamma_\mu \gamma_5 \psi \theta^\mu \wedge \theta^\nu,
$$

which are explicitly given in Appendix A. We highlight that the operator $\Xi_G(p)$ is dimensionless, in such a way, it does not affect the units of the bilinear forms. So far, such spinorial densities allow us to define a generalized spinor classification, providing the first six disjoint classes with the vector $\tilde{J} \neq 0$ and the remaining three with $\tilde{J} = 0$ (as developed in [48]). Then, it reads

1. $\tilde{\sigma} \neq 0, \quad \tilde{\omega} \neq 0, \quad \tilde{K} \neq 0, \quad \tilde{S} \neq 0,$
2. $\tilde{\sigma} \neq 0, \quad \tilde{\omega} = 0, \quad \tilde{K} \neq 0, \quad \tilde{S} \neq 0,$
3. $\tilde{\sigma} = 0, \quad \tilde{\omega} \neq 0, \quad \tilde{K} \neq 0, \quad \tilde{S} \neq 0,$
4. $\tilde{\sigma} = 0 = \tilde{\omega}, \quad \tilde{K} \neq 0, \quad \tilde{S} \neq 0,$
5. $\tilde{\sigma} = 0 = \tilde{\omega}, \quad \tilde{K} = 0, \quad \tilde{S} \neq 0,$
6. $\tilde{\sigma} = 0 = \tilde{\omega}, \quad \tilde{K} = 0, \quad \tilde{S} = 0,$
7. $\tilde{\sigma} = 0 = \tilde{\omega}, \quad \tilde{K} \neq 0, \quad \tilde{S} \neq 0, \quad \tilde{Z} = i(\tilde{S} + \tilde{K} \gamma_{0123}),$
8. $\tilde{\sigma} = 0 = \tilde{\omega}, \quad \tilde{K} = 0, \quad \tilde{S} \neq 0, \quad \tilde{Z} = i \tilde{S},$
9. $\tilde{\sigma} = 0 = \tilde{\omega}, \quad \tilde{K} \neq 0, \quad \tilde{S} = 0, \quad \tilde{Z} = i \tilde{K} \gamma_{0123},$

where we have three regular classes and six singular classes. For spinors respecting the Dirac dynamics $\tilde{J}$ is the conserved current, and the last three classes describe spinors obeying only the Klein-Gordon equation [48]. We stress that, up to date, classes with $\tilde{J} = 0$ have only mathematical significance, since no physical entities were observed.

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7 Here we replaced the notation of the bilinear forms built upon a new dual structure by tilde ones.
Through an exhaustive analysis of the FPK geometric identities, we can guarantee that these are the only nine possible classes to be constructed.

In view of this fact, with the bilinear forms computed in [7], Elko spinor fields belong to class-2 of the above spinor field classification; belonging to a regular class. It is important to highlight why we are looking for such brand new classification. Notice that all the physics built upon the spinors takes into account the spinorial dual structure, e.g., the orthonormal relations, the interactions presented in the interaction lagrangian (or Hamiltonian) density among other relevant physical processes. Therefore, the need for a real interpretation of such bilinear densities arises. Then, the statement presented in section 11.1 of [28], shall shed some light for the interpretation that we are searching for. If we know the interaction Lagrangian or Hamiltonian density for a given process, we can determine its behaviour under a specific Lorentz transformation which would then tell us whether such transformation will be conserved in processes mediated through such interactions. Moreover, interactions must be hermitian and manifestly Lorentz invariant.

Thus, for the general classification developed above, $\sigma$ still stands for the invariant length (even known as mass term). Moreover, the four-vector $\tilde{J}$ represents the electric current density for charged particles, whereas for neutral particles it can be interpreted as the effective electromagnetic current four-vector [52] (see Appendix C). The bilinear $\tilde{K}$ shall be related with the spin alignment due to a coupling with matter or electromagnetic field. Finally, $\tilde{S}$ is related to the electromagnetic moment density for charged particles, although for neutral particles we can infer that such physical quantity may correspond to the momentum spin-density, or even it may represent spin precession (spin oscillation) in the presence of matter or electromagnetic fields [53]. Such effect is caused by the neutral particle interaction with matter polarized by external magnetic field or, equivalently, by the interaction of the induced magnetic moment of a neutral particle with the magnetic field [54, 55]. The meaning of the $\tilde{\omega}$ bilinear is, in this very moment, not clear enough for us to get a physical interpretation. The observations above are a hint towards the physical meaning of the bilinears in this general classification.

Based on the discussion developed in [56], which states the following: "By evading the Weinberg no go theorem, and by understanding the underlying general structure of duals and adjoints, we find a whole new range of possibilities to extend the standard model of high energy physics. Many of these new constructs shall carry a natural darkness, and would be endowed with new properties with respect to the parity, charge conjugation, and time reversal. Mass dimension one fermions are only a first concrete example of this new programme. As is evident from the vast literature on the subject various classes of spinors can be defined. Some of these, for instance, depend on the bilinear covariants and the relevant duals...", we stress that given above classification aims to fit all the possibilities of existing particles and also new particles that may raise in the Standard Model of high energy physics.

A. Defining the antisymmetrization for the bilinear forms

The fermion fields, as quantum mechanical operators, satisfy anticommutation relations to reflect the Fermi-Dirac statistics. In this case, the appropriate operator ordering for the current is the antisymmetrization (also equivalent to normal ordering in this case) [28]. Let us next calculate how the bilinears transform under charge conjugation, to accomplish such task. First, we have to define the following properties

\begin{align}
C^{-1}\gamma_\mu C &= -\gamma_\mu^T, \\
C^{-1}\gamma_5 C &= -\gamma_5^T, \\
C^{-1}\gamma_5\gamma_\mu C &= (\gamma_5\gamma_\mu)^T, \\
C^{-1}\gamma_\mu\gamma_5 C &= -(\gamma_\mu\gamma_5)^T,
\end{align}

which hold the same also for the Elko deformed set elements $\{\tilde{\Gamma}_i\}$ presented in [7]. Now, we are able to define the antisymmetrized bilinear forms, as follows

$$
(\bar{\psi}\tilde{\Gamma}_i\psi)^\text{anti} = \frac{1}{2}(\bar{\psi}\tilde{\Gamma}_i\psi - \psi^T\tilde{\Gamma}_i^T\bar{\psi}^T).
$$

8 For a complete review on the Elko interactions, please, check the following references [49–51]
9 The no go theorem states the impossibility of constructing another spin-1/2 quantum field without violating Lorentz symmetries, and locality, besides Dirac field [57, 58].
The most relevant analysis to be accomplished here is related with the following bilinear form

\[ J_{\mu}^{anti} = \frac{1}{2}(\bar{\psi} \gamma_\mu \psi - \bar{\psi} \gamma_\mu \psi^c), \tag{22} \]

where \( \psi^c \) stands for the charged conjugated spinor.

As usual, for the Dirac spinors we have that \( C \psi_D = -\psi_D \), for Majorana \( C \psi_M = \psi_M \) and for Elko \( C \lambda^{S/A} = \pm \lambda^{S/A} \). Therefore, in the Dirac case, \( J_{\mu}^{anti} \) match \( J_\mu \) given in (4), whereas for neutral particles like Majorana and Elko we have the following

\[ J_{\mu}^{anti} = 0, \tag{23} \]

and

\[ K_{\mu}^{anti} = 0. \tag{24} \]

Then, Majorana [28] and Elko [6] fermions are charge neutral, and hence they cannot have any electromagnetic interaction, which is in agreement with what was discussed about the electric and magnetic charges, \( e_M \) and \( q_M \), in [35].

V. A DETOUR ON THE CLIFFORD ALGEBRA BASIS DEFORMATION: DIRAC NORMALIZATION AND REAL SPINORIAL DENSITIES ISSUE

This section is reserved for a more careful and formal analysis regarding the general bilinear forms. Here we specify a protocol that should be followed if all the previous prescription do not appreciate the FPK identities, or if the calculated bilinear forms are not real quantities. So, we will employ the Dirac normalization procedure [13] in order to deform the basis of Clifford algebra, and consequently ensuring real bilinear forms which satisfy the FPK identities. Consider the well-known constitutive relation of the Clifford algebra

\[ \{ \gamma_\mu, \gamma_\nu \} = 2\eta_{\mu\nu}\mathbb{1}, \quad \mu, \nu = 0, 1, 2, \ldots, N - 1, \tag{25} \]

where \( \eta_{\mu\nu} \) is a \( N = 2n \) even-dimensional space-time metric, diagonal \( \{1, -1, \ldots, -1\} \). The generators of the Clifford algebra are the identity \( \mathbb{1} \) and the vectors \( \gamma_\mu \) (commonly represented by square matrices), and product of the vector basis, given by [13],

\[ \tilde{\gamma}_{\mu_1 \mu_2 \ldots \mu_{N-1}} \equiv \frac{1}{M!}\gamma_{\mu_1 \mu_2 \ldots \mu_N}\gamma^{\mu_{N-1} \mu_{N-2} + 1 \mu_{N-3} + 2 \ldots \mu_N}. \tag{26} \]

As one can easily check, the lowest value of \( M \) is two, which stands for the smallest possible combination, and the highest is \( M = N \). Moreover, the elements that form the real Clifford algebra basis are given by

\[ \{ \Gamma_i \} \equiv \{ 1, \gamma_\mu, \tilde{\gamma}_{\mu_1 \mu_2 \ldots \mu_{N-2}}, \ldots, \tilde{\gamma}_\mu, \tilde{\gamma} \}, \tag{27} \]

where \( \tilde{\gamma} \equiv \tilde{\gamma}_{\mu_1 \mu_2 \ldots \mu_{N-1}} \).

The existence of the \( \Xi_G(p) \)-operator, appearing in the general definition of the spinorial dual structure, makes necessary to adequate the Clifford algebra basis, in order to get the right appreciation of the FPK relations. We highlight that, for the Dirac spinorial case, the set (27) is suitable deformed in order to provide real bilinear covariants. The first two main structures arising from the Clifford algebra basis are defined as

\[ \tilde{\sigma} \equiv \tilde{\psi} \mathbb{1} \psi, \tag{28} \]

\[ \tilde{J}_\mu \equiv \bar{\psi} \gamma_\mu \psi, \tag{29} \]

where \( \tilde{\psi} \) is defined in (13). The requirement of reality \( \tilde{\sigma} = \tilde{\sigma} \) automatically leads to \( \gamma_0 = \Xi_E(p)\gamma_0 \Xi_E(p) \), since \( \Xi_E^2(p) = 1 \). This constraint is satisfied, in such a way that (28) is real. Requiring the same condition on (29), leads to the following constraint \( \gamma_0 \gamma_\mu = \Xi_E(p)\gamma_0 \gamma_\mu \Xi_E(p) \), which cannot be fulfilled by the Clifford vector basis. We highlight that if one performs any change in \( \gamma_\mu \), it may lead to a change in the constitutive relation of Clifford algebra (25) which in general leads to inconsistencies, and the we would be forced to abandoned the present approach. Nevertheless, it is important to emphasize that this modification must be excluded. As a matter of fact, if one imposes \( \Xi_E(p) = 1 \) associating the given structure with Dirac spinors, one simply obtains \( \gamma_0^{-1} \gamma_\mu \gamma_0 = \gamma_\mu \).
Hereupon, we may deform the usual basis in order to redefine the bilinear covariants, and then satisfying the FPK relations. From equation (26), and by considering that the norm for the spinors must assume real values, we have
\[
\left[ \tilde{\psi} \gamma_{\mu_1 \mu_2 \ldots \mu_{N-M}} \psi \right]^2 = (-1)^{M(M-1)/2} \tilde{\psi} \Xi_G(p) \gamma_{\mu_1 \mu_2 \ldots \mu_{N-M}} \Xi_G(p) \psi,
\]
so that
\[
\tilde{\gamma}_{\mu_1 \mu_2 \ldots \mu_{N-M}} = (i^{M(M-1)/2}/M!) \Xi_G(p) \gamma_{\mu_1 \ldots \mu_{N-M}} \gamma^{\mu_{N-M+1} \ldots \mu_N} \Xi_G(p).
\]
From the last equation, we can define the bispinor Clifford algebra basis as in (27). For instance, let us consider the four-dimensional space-time, i.e., \( N = 4 \). In this specific case the basis is given by
\[
M = 4 \Rightarrow \tilde{\gamma} = -i \Xi_G(p) \gamma_5 \Xi_G(p),
\]
\[
M = 3 \Rightarrow \tilde{\gamma}_\mu = -i \Xi_G(p) \gamma_\mu \gamma_5 \Xi_G(p),
\]
\[
M = 2 \Rightarrow \tilde{\gamma}_{\mu\nu} = i \Xi_G(p) \gamma_\mu \gamma_\nu \Xi_G(p).
\]
Thus, the new elements that compose the Clifford algebra basis read
\[
\{ \tilde{\Gamma}_i \} \equiv \{ 1, \gamma_\mu, \Xi_G(p) \gamma_\mu \gamma_5 \Xi_G(p), \Xi_G(p) \gamma_5 \Xi_G(p), \Xi_G(p) \gamma_5 \Xi_G(p) \}.
\]
Now, by using now the well-defined real Clifford algebra basis, we can construct all the bilinear quantities as follows,
\[
\tilde{\sigma} = \tilde{\psi} \gamma_5 \psi, \quad \text{for} \quad \Gamma_i = 1,
\]
\[
\tilde{\omega} = -i \tilde{\psi} \Xi_G(p) \gamma_5 \Xi_G(p) \psi, \quad \text{for} \quad \Gamma_i = \tilde{\gamma},
\]
\[
\tilde{J} = \tilde{\psi} \gamma_\mu \theta_\mu, \quad \text{for} \quad \Gamma_i = \gamma_\mu,
\]
\[
\tilde{K} = -\tilde{\psi} \Xi_G(p) \gamma_\mu \gamma_5 \Xi_G(p) \psi \theta_\mu, \quad \text{for} \quad \Gamma_i = \tilde{\gamma}_\mu,
\]
\[
\tilde{S} = i \tilde{\psi} \Xi_G(p) \gamma_\mu \gamma_5 \Xi_G(p) \psi \theta_\mu \wedge \theta_\nu, \quad \text{for} \quad \Gamma_i = \tilde{\gamma}_{\mu\nu}.
\]
After some algebra, it can be shown that the above modifications are sufficient to assure that the FPK identities (8) are fully satisfied. Notice that if one imposes \( \Xi_G(p) = 1 \) we recover all the usual Crawford deformation and the bilinear forms in (36) recast into (4). It is worth pointing out that the above deformation procedure does not necessarily provide a set of real bilinear quantities. All the forms in (4) were built upon spin one-half fermions under the Dirac dual structure. However, as it was shown, this may not be unique, and if more dual spinorial structures exist, then new classifications and bilinears forms must be constructed and analyzed.

VI. FINAL REMARKS

In the present communication we have developed a general spinor field classification based on a general dual structure consistent with the FPK quadratic relations. The bilinear forms play a very important role, not only in classifying spinors but also to provide a physical interpretation. Thus, we have used the general procedure to interpret the physics related to the eigenspinors of parity operator and eigenspinors of charge conjugation operator. In fact, given a spinor related to some of the mentioned symmetries, such a relation will provide a physical interpretation of the bilinear forms, as the case of the Dirac, Majorana and Weyl spinors. Note that only a small group of spinors are physically relevant and then, the spinors which are not related to above mentioned symmetries should be analyzed in other scenarios, to ascertain if they can possibly carry physical information, or if only exist by algebraic construction.

As one can easily see, the Lounesto classification is a very particular case of the one proposed in here. Such general classification encompasses exclusively nine classes. This restriction is due to the FPK algebraic relations, which can never be evaded. In this construction we look for accommodating all types of particles, both charged and neutral. Although we have a found restricted set of fermionic spin-1/2 particles that can be classified, we still seek to develop a classification where any other particle can be fitted. As highlighted in [56], a new range of possibilities still open windows to compose the standard model of high energy physics. This line of reasoning began when a new dual structure was presented in the literature, the Elko’s dual structure. Thus, we wonder whether or not these spinors should be classified within Lounesto classification, since they evade the dual spinorial structure imposed in such classification. Under these circumstances, we suppose that each spinor that has a different dual structure may present different interactions, different mass dimensionality, and then it needs an specific spinorial classification.
Regarding to the interpretation of the bilinear forms, as we have already mentioned in the course of the paper, for an exact understanding we need to know the mass dimension of the spinor field, and evidently the possible couplings. In such a way, we would be able to infer about the physical interpretation of the bilinear by following the same path that Lounesto used for the classification of Dirac spinors. As previously was thought, neutral particles do not carry any electric current or can not electromagnetically interact, as shown in section IV A, so in this sense the current and the axial-current vanish in the antisymmetrization programme, as already expected. Albeit still several studies related to neutral particles are in development (Appendix C) and the results obtained are just simple observations, we expect that this may help us by bringing a coherent interpretation to the bilinear forms for such particles, and somehow making an association with the effective current density, momentum and spin precession in the presence of matter fields and even electromagnetic fields.

To conclude the program presented here, we also proposed a subtle deformation on the Clifford algebra. Such a procedure is intended to use the general spinorial dual on the construction of the 16 basis elements. This mathematical procedure undergoes on a deformation of the basis of Clifford algebra in order to provide new bilinear forms (via Dirac normalization method), leading to real spinorial densities, and also quantities that hold the FPK identities. Thus, such a mechanism is mathematically necessary if everything we have developed so far does not provide real spinorial densities or does not satisfy FPK identities.

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Appendix A: Generalized spinorial densities

This section was reserved for the explicit forms of the bilinear densities built upon the spinors defined in Sect IV, which can be written in the following way (the over bar stands for the complex conjugate):

\[ \tilde{\sigma} = \frac{am_{3,1} + bm_{3,2} + cm_{3,3} + dm_{3,4}}{am_{4,1} + bm_{4,2} + cm_{4,3} + dm_{4,4}} a + \frac{am_{4,1} + bm_{4,2} + cm_{4,3} + dm_{4,4}}{am_{4,1} + bm_{4,2} + cm_{4,3} + dm_{4,4}} b + \frac{am_{1,1} + bm_{1,2} + cm_{1,3} + dm_{1,4}}{am_{2,1} + bm_{2,2} + cm_{2,3} + dm_{2,4}} c + \frac{am_{2,1} + bm_{2,2} + cm_{2,3} + dm_{2,4}}{am_{2,1} + bm_{2,2} + cm_{2,3} + dm_{2,4}} d, \] (A1)

\[ \tilde{\omega} = -i(\frac{am_{3,1} + bm_{3,2} + cm_{3,3} + dm_{3,4}}{am_{4,1} + bm_{4,2} + cm_{4,3} + dm_{4,4}} a - i(\frac{am_{1,1} + bm_{1,2} + cm_{1,3} + dm_{1,4}}{am_{2,1} + bm_{2,2} + cm_{2,3} + dm_{2,4}} c + i(\frac{am_{2,1} + bm_{2,2} + cm_{2,3} + dm_{2,4}}{am_{2,1} + bm_{2,2} + cm_{2,3} + dm_{2,4}} d, \] (A2)

\[ \tilde{J}_0 = \frac{(am_{1,1} + bm_{1,2} + cm_{1,3} + dm_{1,4}) a + (am_{2,1} + bm_{2,2} + cm_{2,3} + dm_{2,4}) b}{(am_{3,1} + bm_{3,2} + cm_{3,3} + dm_{3,4}) c + (am_{4,1} + bm_{4,2} + cm_{4,3} + dm_{4,4}) d}, \] (A3)

\[ \tilde{J}_1 = -\frac{(am_{1,1} + bm_{1,2} + cm_{1,3} + dm_{1,4}) a - (am_{2,1} + bm_{2,2} + cm_{2,3} + dm_{2,4}) b}{(am_{3,1} + bm_{3,2} + cm_{3,3} + dm_{3,4}) c + (am_{4,1} + bm_{4,2} + cm_{4,3} + dm_{4,4}) d}, \] (A4)

\[ \tilde{J}_2 = -i(\frac{am_{1,1} + bm_{1,2} + cm_{1,3} + dm_{1,4}}{am_{4,1} + bm_{4,2} + cm_{4,3} + dm_{4,4}} a + i(\frac{am_{1,1} + bm_{1,2} + cm_{1,3} + dm_{1,4}}{am_{4,1} + bm_{4,2} + cm_{4,3} + dm_{4,4}} b + i(\frac{am_{4,1} + bm_{4,2} + cm_{4,3} + dm_{4,4}}{am_{3,1} + bm_{3,2} + cm_{3,3} + dm_{3,4}} c - i(\frac{am_{3,1} + bm_{3,2} + cm_{3,3} + dm_{3,4}}{am_{2,1} + bm_{2,2} + cm_{2,3} + dm_{2,4}} d, \] (A5)
\[ \tilde{J}_3 = \frac{(am_{1,1} + bm_{1,2} + cm_{1,3} + dm_{1,4})a + (am_{2,1} + bm_{2,2} + cm_{2,3} + dm_{2,4})b}{(am_{3,1} + bm_{3,2} + cm_{3,3} + dm_{3,4})c - (am_{4,1} + bm_{4,2} + cm_{4,3} + dm_{4,4})d}, \tag{A6} \]

\[ \tilde{K}_0 = \frac{(am_{1,1} + bm_{1,2} + cm_{1,3} + dm_{1,4})a + (am_{2,1} + bm_{2,2} + cm_{2,3} + dm_{2,4})b}{(am_{3,1} + bm_{3,2} + cm_{3,3} + dm_{3,4})c - (am_{4,1} + bm_{4,2} + cm_{4,3} + dm_{4,4})d}, \tag{A7} \]

\[ \tilde{K}_1 = \frac{-i(am_{2,1} + bm_{2,2} + cm_{2,3} + dm_{2,4})a - (am_{1,1} + bm_{1,2} + cm_{1,3} + dm_{1,4})b}{(am_{3,1} + bm_{3,2} + cm_{3,3} + dm_{3,4})c - (am_{4,1} + bm_{4,2} + cm_{4,3} + dm_{4,4})d}, \tag{A8} \]

\[ \tilde{K}_2 = \frac{-i(am_{2,1} + bm_{2,2} + cm_{2,3} + dm_{2,4})a + i(am_{1,1} + bm_{1,2} + cm_{1,3} + dm_{1,4})b}{(am_{3,1} + bm_{3,2} + cm_{3,3} + dm_{3,4})c + i(am_{3,1} + bm_{3,2} + cm_{3,3} + dm_{3,4})d}, \tag{A9} \]

\[ \tilde{K}_3 = \frac{-i(am_{1,1} + bm_{1,2} + cm_{1,3} + dm_{1,4})a + (am_{2,1} + bm_{2,2} + cm_{2,3} + dm_{2,4})b}{(am_{3,1} + bm_{3,2} + cm_{3,3} + dm_{3,4})c + (am_{4,1} + bm_{4,2} + cm_{4,3} + dm_{4,4})d}, \tag{A10} \]

\[ \tilde{S}_{01} = \frac{-i(am_{4,1} + bm_{4,2} + cm_{4,3} + dm_{4,4})a - i(am_{3,1} + bm_{3,2} + cm_{3,3} + dm_{3,4})b}{(am_{2,1} + bm_{2,2} + cm_{2,3} + dm_{2,4})c + i(am_{1,1} + bm_{1,2} + cm_{1,3} + dm_{1,4})d}, \tag{A11} \]

\[ \tilde{S}_{02} = \frac{(am_{4,1} + bm_{4,2} + cm_{4,3} + dm_{4,4})a - (am_{3,1} + bm_{3,2} + cm_{3,3} + dm_{3,4})b}{(am_{2,1} + bm_{2,2} + cm_{2,3} + dm_{2,4})c + (am_{1,1} + bm_{1,2} + cm_{1,3} + dm_{1,4})d}, \tag{A12} \]

\[ \tilde{S}_{03} = \frac{-i(am_{3,1} + bm_{3,2} + cm_{3,3} + dm_{3,4})a + i(am_{4,1} + bm_{4,2} + cm_{4,3} + dm_{4,4})b}{(am_{1,1} + bm_{1,2} + cm_{1,3} + dm_{1,4})c - i(am_{2,1} + bm_{2,2} + cm_{2,3} + dm_{2,4})d}, \tag{A13} \]

\[ \tilde{S}_{12} = \frac{(am_{3,1} + bm_{3,2} + cm_{3,3} + dm_{3,4})a - (am_{4,1} + bm_{4,2} + cm_{4,3} + dm_{4,4})b}{(am_{1,1} + bm_{1,2} + cm_{1,3} + dm_{1,4})c - (am_{2,1} + bm_{2,2} + cm_{2,3} + dm_{2,4})d}, \tag{A14} \]

\[ \tilde{S}_{13} = \frac{-i(am_{4,1} + bm_{4,2} + cm_{4,3} + dm_{4,4})a + i(am_{3,1} + bm_{3,2} + cm_{3,3} + dm_{3,4})b}{(am_{2,1} + bm_{2,2} + cm_{2,3} + dm_{2,4})c + i(am_{1,1} + bm_{1,2} + cm_{1,3} + dm_{1,4})d}, \tag{A15} \]

\[ \tilde{S}_{23} = \frac{(am_{4,1} + bm_{4,2} + cm_{4,3} + dm_{4,4})a + (am_{3,1} + bm_{3,2} + cm_{3,3} + dm_{3,4})b}{(am_{2,1} + bm_{2,2} + cm_{2,3} + dm_{2,4})c + (am_{1,1} + bm_{1,2} + cm_{1,3} + dm_{1,4})d}. \tag{A16} \]
All the above spinorial densities satisfy the FPK identities presented in (8) and also in (9). Some relations between $\Xi_G(p)$ and the $\gamma_5$ matrices are given below:

$$\{\Xi_G(p), \gamma_5\} = \begin{pmatrix}
2m_{1,1} & 2m_{1,2} & 0 & 0 \\
2m_{2,1} & 2m_{2,2} & 0 & 0 \\
0 & 0 & -2m_{3,3} & -2m_{3,4} \\
0 & 0 & -2m_{4,3} & -2m_{4,4}
\end{pmatrix}, \quad (A17)$$

and

$$[\Xi_G(p), \gamma_5] = \begin{pmatrix}
0 & 0 & -2m_{1,3} & -2m_{1,4} \\
0 & 0 & -2m_{2,3} & -2m_{2,4} \\
2m_{3,1} & 2m_{3,2} & 0 & 0 \\
2m_{4,1} & 2m_{4,2} & 0 & 0
\end{pmatrix}. \quad (A18)$$

Appendix B: On the Noether’s conserved quantities regarding Elko spinors

This appendix is reserved for a detailed approach on the Noether’s conserved quantities related to the Elko spinors. As highlighted on the Ref [6] (similarly for the neutrino [59]), the Elko global transformation is given by

$$\lambda \rightarrow \lambda' = e^{i\gamma_5\beta \lambda}, \quad (B1)$$

where $\beta \in \mathbb{R}$. Such global transformation leads the lagrangian invariant under certain conditions and also preserves the charge conjugacy. The Elko lagrangian

$$\mathcal{L}' = \partial_\mu \bar{\lambda}' \gamma^\mu \lambda' - m^2 \bar{\lambda}' \lambda', \quad (B2)$$

remains invariant under the global transformation (B1), if the following conditions are satisfied

$$\tilde{\lambda} \gamma_5 \lambda = 0, \quad (B3)$$

$$\partial_\mu \tilde{\lambda} \gamma_5 \partial^\mu \lambda = 0. \quad (B4)$$

The relation in (B4) comes from a total derivative like

$$\partial_\mu (\tilde{\lambda} \gamma_5 \partial^\mu \lambda) = \partial_\mu \tilde{\lambda} \gamma_5 \partial^\mu \lambda - m^2 \bar{\lambda} \gamma_5 \lambda, \quad (B5)$$

in which the second term on the right hand side is null. Recognizing the remaining terms as a 4-divergence we obtain the desired results.

Performing the above global transformation on the Elko spinors, Noether’s theorem provides a conserved quantity, namely $J^E_\mu$ (with no association with the bilinear forms in (4)), which reads

$$J^E_\mu = i[(\partial_\mu \bar{\lambda}) \gamma_5 \lambda + \bar{\lambda} \gamma_5 (\partial_\mu \lambda)]. \quad (B6)$$

Recognizing the term between the brackets as $\partial_\mu \omega$, one is able to write

$$J^E_\mu = \partial_\mu \omega. \quad (B7)$$

Using (36), we conclude that $\omega = 0$ for Elko, and then $J^E_\mu = 0$, or equivalently $\partial^\mu J^E_\mu = 0$. For the Dirac case, the conserved quantity reads

$$J_\mu = \bar{\psi} \gamma_\mu \psi, \quad (B8)$$

which also is conserved,

$$\partial^\mu J_\mu = 0. \quad (B9)$$
Moreover, the corresponding $N$ conserved quantity, given by

$$N = \int J_0 dV,$$

(B10)

vanishes because $J^E_0 = 0$.

On the other hand, the Noether current in the Dirac case does coincide with the bilinear form $J_\mu$, contrasting with the Elko case. As mentioned above, all the physical interpretation for the Dirac bilinear covariants, as well as for the parameters and coupling constants (like the $e$ charge), comes from the evaluation and observation of Dirac conserved quantities, which are directly related to the Dirac spinor dynamics. Due to this reason, the Elko spinors (type-4, type-5 and type-6 spinors fields) can not carry the same interpretation for the bilinear forms. Of course, each spinor has its proper kind of conserved quantity and couplings parameters, depending on its mass-dimension and the dynamics.

Now, gauging the symmetry, the equation (B1) reads

$$\lambda' = e^{i \gamma_5 \beta(x)} \lambda,$$

(B11)

and for the dual spinor\(^{10}\)

$$\lambda' = \lambda e^{i \gamma_5 \beta(x)},$$

(B12)

where the parameter $\beta(x)$ is an arbitrary space-time dependent function. Considering the infinitesimal transformation, we can write

$$\delta \lambda = i \beta(x) \gamma_5 \lambda,$$

(B13)

and

$$\delta \lambda' = i \beta(x) \lambda' \gamma_5.$$

(B14)

Similarly

$$\delta (\partial_\mu \lambda) = i \partial_\mu \beta(x) \gamma_5 \lambda + i \beta(x) \gamma_5 \partial_\mu \lambda,$$

(B15)

and then,

$$\delta (\partial^\mu \lambda') = i \partial^\mu \beta(x) \lambda' \gamma_5 + i \beta(x) \partial^\mu \lambda' \gamma_5.$$

(B16)

From these relations and the conditions (B3) and (B4), we obtain that the variation of the Elko lagrangian vanishes, namely

$$\delta \mathcal{L} = 0.$$

(B17)

Since these fields have suppressed coupling with the Standard Model particles, then, it is not necessary to introduce auxiliary fields, as for the scalar field case. The absence of a $U(1)$ symmetry means that there is no conserved quantity which we can assign to the electric charge [60].

Now we are restricted for some details on the Elko $J$ and $K$ currents, looking towards some features on the Elko conserved quantities. From the Elko Dirac-like equation

$$(i \Xi_E(p) \gamma^\mu \partial_\mu \pm m) \lambda_\alpha = 0,$$

(B18)

we get for the dual structure

$$\lambda_\alpha (i \gamma^\mu \partial_\mu \Xi_E(p) \mp m) = 0.$$

(B19)

Now, from these equations, and taking into account, the specific relations satisfied by the Elko operator, namely $[\Xi_E(p), \gamma_5] = 0$ and $[\Xi_E(p), \gamma^\mu p_\mu] = 0$, we can prove that\(^{11}\)

$$\partial_\mu K^\mu = 2im \lambda_\alpha \gamma_5 \Xi_E(p) \lambda_\alpha = 0.$$

(B20)

Then, The $K^\mu$ quantity in (B20) is always conserved (independently of the mass term), unlike the Dirac currents presented in footnote 3.

---

\(^{10}\) The transformation given in (B12) holds also for the new dual introduced in [6].

\(^{11}\) In the equation (B20) we have omitted the global factor “$\pm$”. The signal should be defined as soon as the spinor type $S$ or $A$ and also the helicities $\alpha = \{ \pm, \mp \}$ are well established.
Appendix C: Some Remarks on neutral particles charge and electromagnetic properties

For neutrinos it is well-known that the electric charge is zero, and then there are no electromagnetic interactions at tree level. However, such interactions can arise at the quantum level from loop diagrams at higher order of the perturbative expansion of the interaction. In the one-photon approximation, the electromagnetic interactions of a neutrino field can be described by the effective interaction lagrangian

\[ \mathcal{L}_{\text{em}} = J_{\mu} A^\mu, \]  

(C1)

where \( J_{\mu} \) is the neutrino effective electromagnetic current four-vector [52, 61]. It is worth pointing out that neutrino electromagnetic properties exist even if neutrinos are elementary particle without an internal structure. Such properties are generated via quantum loop effects. Then, the neutrino’s charge and magnetic form factors have a different origin from the neutron charge and magnetic form factors (also known as Dirac and Pauli form factors), which are mainly due to its internal quark structure [52]. As it was believed, neutrino electric charge is exactly zero. This is true in the standard model, but in extensions of the standard model neutrinos may be millicharged particles [62]. However, to the best of our knowledge, neutrino electromagnetic properties are known to be small with rather stringent upper bounds obtained in laboratory experiments or from astrophysical observations. Thus, we stress that experimental and theoretical studies of electromagnetic properties related to neutrino still open windows which could shed some light to new physics beyond the standard model [52]. A neutrino millicharge might have specific phenomenological consequences in astrophysics because new electromagnetic processes are opened due to a nonzero charge. For a complete understanding concerning neutrino charge and other related subjects, the reader can check references [63–68] and references therein.
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