Stability Analysis and Stabilization of Switched Systems With Average Dwell Time: A Matrix Polynomial Approach

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ABSTRACT This paper focuses on stability analysis and stabilization of a switched system under average dwell time criteria in the continuous-time domain. The matrix polynomial approach is applied to the switched system to construct a continuous Lyapunov function and design a more effective controller, which can not only improve the system performance but also lessen the system conservativeness. The stability problem is studied with square matricial representation for the first time. The new method is more applicable than previous work under average dwell time switching with the time-varying controller gains. In addition, new sufficient conditions of stability and stabilization are derived to guarantee the global uniform exponential stability of the switched system. A numerical example is provided to show the advantages of the new method.

INDEX TERMS Switched systems, average dwell time, matrix polynomial, stability, stabilization.

I. INTRODUCTION

As the basic framework of physical or man-made systems, switched systems have been widely studied in recent decades [1], [2]. A switched system is a hybrid dynamical system consisting of a finite number of subsystems and a switching signal that orchestrates the switching between subsystems [3]–[10]. The switching signal plays an important role in stability analysis and stabilization of switched systems. Most research results focus on exploiting switching signals to guarantee the global uniform exponential stability of the switched system consisting of various subsystems [11]–[15]. In a certain sense, the switching events in control systems could be classified into uncontrolled or controlled [1]. The stability analysis of the switched system with various subsystems under uncontrolled switching is available. However, the state of subsystems will be changed; for example, the stable subsystem may transform into an unstable subsystem because of noise or equipment failure in some cases [16]. Then, the system state and stability will become uncertain. To address this problem, controllers are proposed. However, for the switched system under controller switching signals, finding suitable switching signals is a challenge which represents the hottest topic in the field [2], [17]–[19].

The typical categories of switching signals include dwell time (DT), average dwell time (ADT) and mode-dependent average dwell time (MDADT) [20]–[26]. DT is the time interval between two successive switching instants [13]. ADT switching refers to the case in which the number of switches in a finite interval is bounded and the average time between consecutive switchings is not less than a constant [27]. In switched systems, each subsystem is also called a mode. MDADT switching means that the ADT of each subsystem is not less than a constant. Multiple Lyapunov functions (MLFs) are always used in the stability analysis and stabilization of the switched system [5], [22], [25], [28]. With MLFs, the switched linear system with stability of all subsystems is globally uniformly exponentially stable if the DT is sufficiently large. Some researchers aim to determine the lower bounds of the DT to guarantee stability of the system. However, dwell time switching is still restrictive. To address this problem, ADT switching and MDADT switching are proposed. It is shown that ADT and MDADT switching are more general and flexible than DT switching in related stability analyses and control syntheses [24], [29]–[33].

With MLFs, ADT switching and MDADT switching are widely used in switched systems [29], [30]. However,
the existing works face difficulty in further lessening conservativeness for switched systems under ADT switching or MDADT switching. In a previous paper [34], the matrix polynomial approach is used to reconstruct the Lyapunov function. Inspired by this, this paper seeks to exploit the matrix polynomial approach to the switched system with respect to improving system performance and lessening conservativeness. The matrix polynomial is widely used in robustness analysis [35], filter design [36], periodic piecewise systems [34] and T-S fuzzy systems [37]. The matrix polynomial method can be used to construct Lyapunov functions for uncertain systems to improve system performance and lessen conservativeness by introducing more freedom variables [35]. However, there are few applications of applying the matrix polynomial approach to construct Lyapunov functions in stability analysis and stabilization of switched systems. Thus, exploiting the matrix polynomial approach in switched systems motivates the present study. This paper attempts to apply the matrix polynomial approach to stability analysis and stabilization of switched systems under ADT switching.

In this paper, the main contribution is the application of the matrix polynomial approach to stability analysis and stabilization of the switched system under ADT criteria, which further lessens the conservativeness of the system. In addition, the continuous Lyapunov function and time-varying controller gain are constructed by a matrix polynomial approach, which improves the robustness of the system. Finally, the stability and stabilization conditions of the continuous-time switched linear systems with ADT switching are derived. In conclusion, the matrix polynomial approach can improve the system performance in comparison with general ADT switching [29].

The structure of this paper is as follows. Section II provides preliminaries, including the description of the system and basic definitions. Section III is the main body of this paper. With the matrix polynomial approach, the stability and stabilization conditions are derived. The time-varying controller of the switched system is proposed. Section IV presents a numerical example to verify the results obtained from section III. Section V presents the conclusion.

Notations: The notation used in this paper is fairly standard. The superscript “T” stands for matrix transposition. \( \mathbb{R} \) denotes the set of real numbers. \( \mathbb{R}^n \) denotes the \( n \)-dimensional Euclidean vector space. \( \mathbb{N} \) and \( \mathbb{N}^+ \) represent the set of non-negative and positive integers, respectively. \( \mathbb{S}^n \) denotes an \( n \)-dimensional symmetric matrix. The notation \( || \cdot || \) refers to the Euclidean norm. \( P > 0 (P \geq 0) \) means that the matrix \( P \) is real symmetric and positive definite (semi-positive definite), \( I_n \in \mathbb{S}^n \) stands for the identity matrix. The notation \( x^q, x \in \mathbb{R}^n, q \in \mathbb{N}^+ \) denotes \( x_1^{q_1}x_2^{q_2} \cdots x_n^{q_n} \).

II. PRELIMINARIES
Consider the continuous-time switched linear system:

\[
\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t),
\]

where \( A_i(t), \forall \sigma(t) = i \in \mathcal{I} \) denotes the system state matrix of the \( i \)-th subsystem. Define index set \( \mathcal{I} := \{1, 2, \ldots, N\} \). \( N \in \mathbb{N}^+ \) is the number of subsystems. \( \sigma(t) \) is the switching signal, which is defined as \( \sigma(t) : [0, \infty) \rightarrow \mathcal{I} \). \( x(t) \in \mathbb{R}' \) is the state vector and \( x(0) \) is the initial state. \( u(t) \) is the control input. The two-matrix pair \( (A_i, B_i), \forall i \in \mathcal{I} \), represents the \( i \)-th subsystem.

Definition 1: [4] Consider the continuous-time switched system (1) with \( u(t) = 0 \). For any initial state \( x(0) \) and a given switching signal \( \sigma(t) \), if there exist two positive constants \( c > 0, \gamma > 0 \) and \( \forall t \geq t_0 \) satisfies \( ||x(t)|| \leq ce^{-\gamma(t-t_0)}||x(t_0)|| \). The system (1) is then globally uniformly exponentially stable (GUES).

The control input \( u(t) \) in (1) is used to achieve system stability or certain performances for certain switching signals. In this paper, the state feedback is considered with \( u(t) = K_{\sigma(t)}x(t) \), where \( K_{\sigma(t)} \), \( \forall \sigma(t) = i \in \mathcal{I} \) is the time-varying controller gain to be determined. In addition, the resulting closed-loop system can be expressed as

\[
\dot{x}(t) = (A_i + B_iK_{\sigma(t)})x(t), \forall i \in \mathcal{I}.
\]
Even degree matrix polynomials can be represented via matrices. The matrix polynomial with degree 2m can be rewritten as:

\[ M(x) = (x^{[m]} \otimes I_r)^T (H + L(\alpha))(x^{[m]} \otimes I_r) \]  

(7)

where \( H \in \mathbb{S}^{r \times (n \times m)} \) is a square matricial representation (SMR) matrix of \( H \) with respect to \( x^{[m]} \) and (7) is an SMR of \( H \).

\[ \sigma(n, m) = \frac{(n + m - 1)!}{(n - 1)!m!} \]  

(8)

The matrix \( L \in \mathbb{S}^{r \times (n \times m)} \) is a linear parameterization of the set:

\[ \mathcal{L} = \{ L_i \in \mathbb{S}^{r \times (2d)} : (e_i^{[d]}(t) \otimes I_r) L_i(e_i^{[d]}(t) \otimes I_r) = 0 \}, \]  

(9)

whose dimension is given by

\[ \omega(2, d, r) = \frac{1}{2} r(\sigma(2, d)\sigma(2, d) + 1) - (r + 1)\sigma(2, 2d), \]  

(10)

where \( \alpha \in \mathbb{R}^{d \times (2d \times r)} \). The algorithm for constructing the matrices \( H \) and \( L(\alpha) \) is provided in paper [35]. With the SMR, the positivity of \( M(x) \) can be transformed into the testing positivity of matrix \( H \).

Lemma 1: [30] Consider the continuous-time switched system (1). There are two given constants, \( \lambda > 0 \) and \( \mu > 1 \). Suppose there exists a function \( V(\sigma(t)) : \mathbb{R}^n \rightarrow \mathbb{R} \) and two positive scalars \( t_1, t_2 \) such that, \( \forall t \in \mathbb{I} \)

\[ t_1(\|x(t)\|) \leq V(t(x(t)) \leq t_2(\|x(t)\|), \]  

(11)

\[ V(t) \leq -\lambda V(t), \quad \lambda > 0 \]  

(12)

and \( \sigma(t_1) = i, \quad \sigma(t_2) = j, \quad i \neq j \)

\[ V(t_1) \leq \mu V(t_2), \quad \mu > 1, \]  

(13)

then the switched system is GUES for any switching signal with ADT

\[ \tau_a > \tau_a^* = \frac{\ln \mu}{\lambda}, \]  

(14)

where \( \tau_a^* \) represents the minimal value of ADT.

III. MAIN RESULTS

In this section, the matrix polynomial is used in the Lyapunov function for the stability analysis and stabilization of the switched system. The details of the matrix polynomial are shown as follows.

The matrix polynomial is the special form of the homogeneous polynomial. Choose the Lyapunov function:

\[ V(t) = x^T(t)P(t)x(t), \quad t \in [t_n, t_{n-1}^-], \]  

(15)

where \( P_i(t) > 0 \) is a time-varying Lyapunov matrix function and \( t_n, n \in \mathbb{N} \) denotes the switching instant.

Inspired by a previous paper [34], the Lyapunov matrix function \( P_i(t) \), \( t \in [t_n, t_{n-1}^-] \) can be written as a homogeneous matrix polynomial

\[ P_i(t) = P_{i,0}2^d + P_{i,1}(t-t_n)2^{d-1} + \cdots + P_{i,2d}(t-t_n)^{2d}. \]  

(16)

It is noticed that the Lyapunov function degenerates into multiple Lyapunov functions when \( t - t_n = 0 \). The introduction of \( t - t_n \) makes the Lyapunov function more general and applicable. In addition, it can be represented via SMR [35], which makes it easier to find feasible solutions.

Defining

\[ \epsilon_i^{[d]}(t) = (1, t - t_n, \cdots, (t-t_n)^d), \]  

(17)

where \( t-t_n \) is the basic element of the vector \( \epsilon_i^{[d]}(t) \in \mathbb{R}^{d+1} \), then homogeneous matrix polynomial \( P_i(t) \) of degree 2d can be represented as

\[ P_i(t) = (\epsilon_i^{[d]}(t) \otimes I_r) (H_i + L_i(\alpha))(\epsilon_i^{[d]}(t) \otimes I_r), \]  

(18)

where \( H_i \in \mathbb{S}^{r \times (2d \times 2)} \) and \( L_i(\alpha) \) meets the condition (9).

\[
H_i = \begin{bmatrix}
P_{i,0} & \frac{1}{2}P_{i,1} & 0 & \cdots & 0 \\
\ast & P_{i,2} & \frac{1}{2}P_{i,3} & \cdots & 0 \\
\ast & \ast & P_{i,4} & \cdots & 0 \\
\ast & \ast & \ast & \ddots & \ddots \\
\ast & \ast & \ast & \ast & \ddots & \ast \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast & P_{i,2d-2} & \frac{1}{2}P_{i,2d-1} \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & P_{i,2d}
\end{bmatrix},
\]  

(19)

Remark 1: It is obvious that \( t - t_n \geq 0, \quad t \in [t_n, t_{n-1}^-] \). The given matrix \( H_i \) (19) is the simplest form in this paper. The matrix \( H_i \) is associated with the basic element of vector \( \epsilon_i^{[d]}(t) \). We can also choose \( t \) as the basic element of vector \( \epsilon_i^{[d]}(t) \). However, this will make the form of matrix \( H_i \) and the result of the stability condition more complex. According to equality (9), the matrix \( L_i(\alpha) \) can be a zero matrix, but it will be difficult to determine the flexible solutions. The matrix \( L_i \) cannot affect the feasible solution \( P_{i,k}, \quad i \in \mathbb{I}, \quad k = 0, 1, \cdots, 2d \); it can only calculate the feasible solution more easily.

With the matrix polynomial approach, the sufficient condition is given as follows, which guarantees that the switched system is globally uniformly exponentially stable.

Theorem 1: Consider the switched system (1) with \( u_i(t) = 0 \). If there exist parameters \( \lambda > 0, \mu > 1 \), matrices \( P_{i,k}, \quad i \in \mathbb{I}, \quad k = 0, 1, \cdots, 2d \) and vectors \( \alpha, \beta, \gamma \), for \( \forall t \in [t_n, t_{n-1}^-] \), such that

\[
H_i(P_{i,k}) + L_i(\alpha) > 0,
\]  

(20)

\[
\Gamma_i(P_{i,k}) + L_i(\beta) < 0,
\]  

(21)

\[
\Psi_i(P_{i,k}) + L_i(\gamma) > 0,
\]  

(22)
where $L_i(\alpha)$, $L_i(\beta)$, $L_i(\gamma)$ meet the condition (9), the detail of matrices $H_i$, $\Gamma_i$ and $\Psi_i$ is given as follows. Double superscript $(m,n)$ represents $m$th row, $n$th column block,

$$m = n = 1$$

$$H_{i}^{(m,n)} = P_{i0},$$

$$\psi_{i}^{(m,n)} = P_{i0} - \frac{1}{\mu} P_{j0}, \quad i \neq j,$$

$$\Gamma_{i}^{(m,n)} = A_{i}^T P_{i0} + P_{i0} A_{i} + \lambda P_{i0} + P_{i1}, \quad 2 \leq m = n \leq d$$

$$H_{i}^{(m,n)} = P_{i2(m-1)},$$

$$\psi_{i}^{(m,n)} = P_{i2(m-1)},$$

$$\Gamma_{i}^{(m,n)} = (2m-1)P_{i2m-1} + A_{i}^T P_{i2(m-1)} + P_{i2(m-1)} A_{i} + \lambda P_{i2(m-1)},$$

$$m = n = d + 1$$

$$H_{i}^{(m,n)} = P_{i2d},$$

$$\psi_{i}^{(m,n)} = P_{i2d},$$

$$\Gamma_{i}^{(m,n)} = A_{i}^T P_{i2d} + P_{i2d} A_{i} + \lambda P_{i2d},$$

$$n = m + 1, \quad 1 \leq m \leq d$$

$$H_{i}^{(m,n)} = \frac{1}{2} P_{i2m-1},$$

$$\psi_{i}^{(m,n)} = \frac{1}{2} P_{i2m-1},$$

$$\Gamma_{i}^{(m,n)} = \frac{1}{2} (2mP_{i2m} + A_{i}^T P_{i2m-1} + P_{i2m-1} A_{i} + \lambda P_{i2m-1}),$$

$$m = n = 1, \quad 1 \leq n \leq d$$

$$H_{i}^{(m,n)} = \frac{1}{2} P_{i2n-1},$$

$$\psi_{i}^{(m,n)} = \frac{1}{2} P_{i2n-1},$$

$$\Gamma_{i}^{(m,n)} = \frac{1}{2} (2nP_{i2n} + A_{i}^T P_{i2n-1} + P_{i2n-1} A_{i} + \lambda P_{i2n-1}),$$

(23)

then the system (1) is GUES.

Proof: Choosing $\epsilon_i^{[d]}(t)$ as (17) shows, with the considered Lyapunov function (15) with $P_i(t)$ given as (16), for $\forall t \in [t_n, t_n^+]$ and $\forall i \in J$, one has

$$V_i(t) = x(t)^T P_i(t) x(t)$$

$$= x(t)^T (P_{i0} + P_{i1}(t_{n+1} - t) + \cdots + P_{i2d}(t_{n+1} - t)^{2d}) x(t).$$

(24)

The $P_i(t)$ can be expressed as matrix form via SMR. Based on equality (18), the Lyapunov function $V_i(t)$ can be written as

$$V_i(t) = x(t)^T P_i(t) x(t)$$

$$= x(t)^T (\epsilon_i^{[d]}(t) \otimes I_r) (H_i + L_i(\alpha)) (\epsilon_i^{[d]}(t) \otimes I_r) x(t).$$

(25)

With the given condition (20), the inequality $V_i(t) > 0$ is satisfied, which is the basic condition of the theorem 1.

Similarly, for $\forall t \in [t_n, t_n^+]$, the inequality (12) can be rewritten as

$$\dot{V}(t) + \lambda \nu(t)$$

$$= x(t)^T (A_{i}^T P_{i(t)}(t) + P_{i(t)} A_{i} + \dot{P}_{i(t)}(t) + \lambda P_{i(t)}(t)) x(t)$$

$$= x(t)^T (A_{i}^T P_{i0} + P_{i0} A_{i} + \cdots + P_{i2d}(t_{n+1} - t)^{2d}) x(t)$$

$$+ \lambda (P_{i1}(t) + P_{i2d}(t_{n+1} - t)^{2d}) x(t)$$

$$= \frac{d^2}{2} \sum_{i=0}^{2d} \lambda_i (t_{n+1} - t)^{2i} x(t)$$

(26)

The inequality (12) holds through the condition (21).

The inequality (13) is represented as

$$\mu V_i(t_{n}) - V_i(t_{n}^{+}) > 0$$

$$\Rightarrow x(t)(\mu P_{i}(t_{n}) - P_{i}(t_{n}^{+}))(x(t)^T) > 0.$$  (27)

At the switching instant $t \to t_{n}^{+}$, $t - t_{n+1} = 0, P_{i}(t_{n}^{+}) = P_{i0}.$ When $t \to t_{n}^{+}, P_{i}(t_{n}) = P_{i0} + P_{i1}(t - t_{n}) + \cdots + P_{i2d}(t - t_{n})^{2d}$. The equality (27) is thus equal to

$$\mu P_{i0} - P_{i}(t_{n}) > 0$$

$$\Rightarrow \mu (P_{i0} + P_{i1}(t - t_{n}) + \cdots + P_{i2d}(t - t_{n})^{2d}) - P_{i0}$$

$$\Rightarrow \mu (P_{i0} - \frac{1}{\mu} P_{i0} + P_{i1}(t - t_{n}) + \cdots + P_{i2d}(t - t_{n})^{2d}) > 0$$

$$\Rightarrow (\epsilon_i^{[d]}(t) \otimes I_r) (\Psi_i + L_i(\gamma)) (\epsilon_i^{[d]}(t) \otimes I_r) > 0$$  (28)

By condition (22), the inequality (13) is obtained. According to the lemma 1, the switched system is GUES in this paper. This proof is completed.

Remark 2: The matrix $L_i(\alpha), L_i(\beta), L_i(\gamma)$ is associated with the vector $\epsilon_i^{[d]}(t)$. According to the algorithm of constructing matrix $L_i$ [35], the vectors $\alpha$, $\beta$, $\gamma$ determine the matrix $L_i$. They can be the same, but this does not benefit the flexible solution $P_{i,k}$. When $t = t_{n} = 0$, the Lyapunov function (15) transforms into MLFs. The Lyapunov function (15) thus introduces more variables than MLFs, which can identify the feasible solution easily and effectively. However, it is a challenge to obtain stability and the stabilization condition.

Next, controllers with time-varying gains are designed to stabilize the continuous-time switched linear system.

Theorem 2: Consider the continuous-time switched linear system (2). Let $\lambda > 0, \mu > 1$ be given constants. If there exist constant matrices $W_{i,k}, \forall i \in J, k = 0, 1, \cdots, 2d$ and vector $\alpha$, $\beta$, $\gamma$, for $\forall t \in [t_n, t_{n+1}^+]$, such that

$$H_i(Q_{i,k}) + L_i(\alpha) > 0,$$  (29)

$$\Psi_i(Q_{i,k}) + L_i(\beta) > 0,$$  (30)

$$\Phi_i(Q_{i,k}, Z_{i,k}, R) > 0,$$  (31)

where $L_i(\alpha)$, and $L_i(\beta)$ meet the condition (9), and the matrices $H_i(Q_{i,k})$ and $\Psi_i(Q_{i,k})$ are given as follows. It is noted that the vectors $\alpha$, $\beta$ in Theorem 2 have different form than
the one in Theorem 1. Double superscript \((m,n)\) represents the \(m\)th row, \(n\)th column block.

\[
m = n = 1
\]

\[
H_{i}^{(m,n)} = Q_{i,0}
\]

\[
\Psi_{i}^{(m,n)} = Q_{i,0} - \frac{1}{\mu}Q_{i,0}, \quad i \neq j
\]

\[
2 \leq m = n \leq d
\]

\[
H_{i}^{(m,n)} = Q_{i,2(m-1)}
\]

\[
\Psi_{i}^{(m,n)} = Q_{i,2(m-1)}
\]

\[
m = n = d + 1
\]

\[
H_{i}^{(m,n)} = Q_{i,2d}
\]

\[
\Psi_{i}^{(m,n)} = Q_{i,2d}
\]

\[
n = m + 1, \quad 1 \leq m \leq d
\]

\[
H_{i}^{(m,n)} = Q_{i,2m-1}
\]

\[
\Psi_{i}^{(m,n)} = Q_{i,2m-1}
\]

\[
m = n + 1, \quad 1 \leq n \leq d
\]

\[
H_{i}^{(m,n)} = Q_{i,2n-1}
\]

\[
\Psi_{i}^{(m,n)} = Q_{i,2n-1}
\]

(32)

then there exists a set of stabilizing controllers such that system (2) is \(GUES\) for any switching signal with ADT satisfying (14). Moreover, if (29), (30) and (31) have flexible solutions, the time-varying controller gain can be given by

\[
K_{i}(t) = K_{i,0} + K_{i,1}(t - t_{n}) + \cdots + K_{i,2d}(t - t_{n})^{2d}
\]

\[
K_{i,k} = Z_{i,k}R^{-1}, \quad \forall i \in I, \quad k = 0, 1, \cdots, 2d
\]

(33)

**Proof:** Without loss of generality, letting controller

\[
u_{i}(t) = K_{i}(t)x(t), \quad \text{the system (1) can be obtained as}
\]

\[
\dot{x}(t) = A_{ci}(t)x(t)
\]

(34)

where \(A_{ci} = A_{i} + B_{i}K_{i}(t)\).

For \(\forall i \in [n_{1}, n_{2}+1]\), choose the Lyapunov function \(V_{i}(t) = x(t)^{T}P_{i}(t)x(t), \quad P_{i}(t) > 0\).

Considering the condition (12), it is equal to:

\[
\dot{V}_{i}(t) + \lambda V_{i}(t) < 0
\]

\[
\Rightarrow x(t)(A_{ci}^{T}(t)P_{i}(t) + P_{i}(t)A_{ci}(t))
\]

\[
+ P_{i}(t) + \lambda P_{i}(t)x^{T}(t) < 0
\]

\[
\Rightarrow A_{ci}^{T}(t)P_{i}(t) + P_{i}(t)A_{ci}(t) + \dot{P}_{i}(t) + \lambda P_{i}(t) < 0
\]

\[
\Rightarrow A_{ci}^{T}(t)P_{i}(t) + P_{i}(t)A_{ci}(t) + \dot{P}_{i}(t) + \lambda P_{i}(t) < 0
\]

\[
\Rightarrow A_{ci}^{T}(t)P_{i}(t) + P_{i}(t)A_{ci}(t) + \dot{P}_{i}(t) + \lambda P_{i}(t) < 0
\]

\[
\Rightarrow (A_{ci}^{T}(t) + \lambda P_{i}(t))< 0
\]

\[
\Rightarrow \Phi_{i}(Q_{i,k}, Z_{i,k}, R) < 0
\]

(37)

The matrix \(\Phi_{i} \in \mathbb{S}^{4(2d+1)}\) is a diagonal matrix, and each principal diagonal element is given as (38), as shown at the bottom of the next page, where \(R = J^{-1}, \quad Q_{i,k} = (J^{T})^{-1}P_{i,k}J^{-1}, \quad Z_{i,k} = K_{i,k}J^{-1}, \quad k = 0, 1, \cdots, 2d\). With the condition (31), it is obtained that the requirement (12) holds.

The Lyapunov matrix function \(P_{i}(t) > 0\) is proven as follows:

\[
P_{i}(t) > 0
\]

\[
\Rightarrow P_{i,0} + P_{i,1}(t - t_{n}) + \cdots + P_{i,2d}(t - t_{n})^{2d} > 0
\]

\[
\Rightarrow (J^{T})^{-1}(P_{i,0} + P_{i,1}(t - t_{n}) + \cdots
\]

(36)
\[ + P_{1,2d}(t - t_n)^{2d})J_l^{-1} > 0 \]
\[ \Rightarrow Q_{i,0} + Q_{i,1}(t - t_n) + \cdots + Q_{i,2d}(t - t_n)^{2d} > 0 \]
\[ \Rightarrow H_l(Q_{i,k}) + L_l(\alpha) > 0 \quad (39) \]

The proof of condition (12) is similar with Theorem 1:
\[ \mu P_j(t_n^-) - P_j(t_n^+) > 0 \]
\[ \Rightarrow (J^T)^{-1}(P_{j,0} - \frac{1}{\mu}P_{j,1}(t - t_n) \]
\[ + \cdots + P_{j,2d}(t - t_n)^{2d})J_l^{-1} > 0 \]
\[ \Rightarrow (Q_{j,0} - \frac{1}{\mu}Q_{j,1}(t - t_n) \]
\[ + \cdots + Q_{j,2d}(t - t_n)^{2d}) > 0 \]
\[ \Rightarrow \Psi_l(Q_{i,k}) + L_l(\gamma) > 0 \quad (40) \]

Based on condition (30), the inequality (13) holds. Therefore, the GUES conditions are obtained for any switching signal with ADT satisfying (14). Additionally, if the inequalities (29), (30) and (31) have feasible solutions, the admissible controller gains can be given by (33). This proof is completed.

**Corollary 1:** (ADT with MLFs [29]) Consider the continuous-time switched linear system (1). Choose the multiple Lyapunov function \( V_i(t) = x^T(t)P_ix(t) \). Let \( \lambda_i > 0 \), \( \mu_i > 1 \), \( i \in I \) be given constants. If there exist matrices \( U_i > 0 \), and \( T_i \), \( i \in I \), such that
\[
A_iU_i + B_iT_i + U_iA_i^T + T_iB_i^T + \lambda U_i < 0,
\]
\[
U_j < \mu U_i, \quad \forall i, j \in I, \quad i \neq j.
\]
then there exists a set of stabilizing controllers such that system (1) is GUES for any switching signal with ADT satisfying
\[ \tau_a \geq \tau_a^* = \frac{\ln \mu}{\lambda}. \]

Moreover, if (41) and (42) have a solution, the controller gains can be given by
\[ K_i = T_iU_i^{-1}, \]
where \( U_i = P_i^{-1} \).

**Remark 3:** It is difficult to identify a feasible solution to establish a closed-loop feedback system with \( u_i(t) = K_i(t)x(t) \). To address this problem, most existing works introduced inverse matrix \( U_i^{-1} = P_i \) to cope with \( P_i(t) \) and \( K_i(t) \) with MLFs. However, this is not suitable for the time-varying controller in this paper. This paper thus uses a slack nonsingular matrix \( J \) to address this problem, which can effectively identify a flexible solution for the continuous Lyapunov function.

**Remark 4:** MDADT switching is the special case of ADT switching where the average dwell time of each subsystem is identical. From the proof process of Theorem 1 and Theorem 2, the matrix polynomial can also be applied to switched systems under MDADT switching. Compared with existing works [29], [38], the proposed approach constructs the discretized Lyapunov function to introduce more free freedom variables, which can find feasible solutions easily and further lessen conservativeness.

Next, two switching signals are presented to verify the results based on Theorem 2. Switching signals 1: constants controller based on ADT, as in corollary 1 [29]. Switching signals 2: time-varying controller based on Theorem 2.

**IV. NUMERICAL EXAMPLE**

In this section, a numerical example in the continuous-time domain will be presented to verify the effectiveness of the proposed approach.

Consider the continuous-time switched linear system (1) with three subsystems. These subsystems are given
\[
A_1 = \begin{bmatrix} 3.9 & 1.5 \\ 2.5 & 2.3 \end{bmatrix}, \quad B_1 = \begin{bmatrix} -0.2 \\ 0.1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1.4 & 0.3 \\ 2.1 & -2.7 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 2.2 & 0.1 \\ -2 & -0.4 \end{bmatrix}, \quad B_3 = \begin{bmatrix} -0.1 \\ 0.1 \end{bmatrix}.
\]

The purpose of this paper is to design a set of time-varying stabilizing controllers and find the admissible switching signals with ADT such that the resulting closed-loop system is stable. To illustrate the advantages of the proposed ADT switching, the results of ADT switching proposed in this paper and general ADT switching with MLFs are presented. By different switching signals and setting the same parameters, the computation results for the system with two switching schemes are listed in Table 1.

Let the initial system state \( x_0 = [1 -1] \). Next, the resulting closed-loop systems under two different ADT switchings are presented. Fig. 1, 2 and 3 are the state response comparisons of three closed-loop subsystems with \( \Gamma_1 \) and \( \Gamma_2 \) controllers.

This shows the state responses of three subsystems with controllers. It is obvious that the subsystem state response
TABLE 1. Computation results for the system under two switching schemes.

| Switching schemes | ADT switching with MLFs | ADT switching with matrix polynomial approach |
|-------------------|------------------------|---------------------------------------------|
| Criteria for controller design | Corollary 1 in [29] | Theorem 2 in the paper |
| Controller gains | $\Gamma_1$ : $K_1 = [-13.4329 - 26.7600]$ |
| | $K_2 = [-30.3287 - 7.6060]$ |
| | $K_3 = [57.1878 19.2350]$ |
| Switching signals | $\tau_0 = 0.1140$ |
| | $\tau_0^* = 0.1140$ |

$\lambda = 1.6, \mu = 1.2$ $\lambda = 1.6, \mu = 1.2$

converges to zero quickly under ADT switching proposed in Theorem 2, which is faster than under general ADT switching in Corollary 1. It is evident that there is a better transient behavior in the state response of closed-loop subsystem 1 controlled by $\Gamma_1$, which potentially improves the state response. However, there exists better transient behavior in the state responses of three closed-loop subsystems controlled by $\Gamma_2$. This means that the state response under controller $\Gamma_2$ is better than under $\Gamma_1$, so the matrix polynomial approach is effective.

The system state responses of general ADT switching in Corollary 1 and ADT switching in Theorem 2 are shown in Fig. 4 and 5, respectively, for the same initial state condition and switching sequences.

It can be seen from the curves that the state response of the closed-loop system fluctuated under the ADT switching scheme proposed in Corollary 1, and is smooth under the ADT switching scheme in Theorem 2. However, the state response under controller $\Gamma_2$ is faster than under controller $\Gamma_1$. In short, the application of the matrix polynomial approach in switched systems is successful and practical.

Remark 5: From Table 1, the given parameters are identical. However, the degree $d$ is equal to 1. The value of average dwell time is related with the parameter $d$, which means that we can obtain smaller ADT when $d$ increases [34], [39]. In addition, the matrix polynomial approach need not consider the features of subsystems, which is more applicable than general ADT switching in Corollary 1.

Remark 6: The time-varying controller gain is applied to the switched system with the matrix polynomial approach. Compared with the constant controller gain, the gain value can be quickly adjusted. In addition, the applications of time-varying controller gain in switched systems are few.
V. CONCLUSION

This paper studies the stability and stabilization problems for the continuous-time switched linear system under ADT criteria. The Lyapunov function is constructed with a matrix polynomial, which can further lessen conservativeness and improve system performance. Stabilizing controllers are then designed with time-varying polynomial controller gains, which is more applicable in practice. It turns out that the proposed new approach is effective and successful. Next, the sufficient conditions of stability and stabilization are obtained. Finally, a numerical example is provided to verify the effectiveness of the proposed method. In the future, it will be considered that the matrix polynomial approach be applied to the switched system under MDADT switching and extended to $H_{\infty}$ and $l_2$ control.

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The linear interpolation method can build a time-varying controller, but it cannot introduce more variables than the proposed method [38]. Therefore, the proposed design is more general and effective in practice.
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