Analytical Study of Certain Magnetohydrodynamic-\(\alpha\) Models

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Abstract

In this paper we present an analytical study of a subgrid scale turbulence model of the three-dimensional magnetohydrodynamic (MHD) equations, inspired by the Navier-Stokes-\(\alpha\) (also known as the viscous Camassa-Holm equations or the Lagrangian-averaged Navier-Stokes-\(\alpha\) model. Specifically, we show the global well-posedness and regularity of solutions of a certain MHD-\(\alpha\) model (which is a particular case of the Lagrangian averaged magnetohydrodynamic-\(\alpha\) model without enhancing the dissipation for the magnetic field). We also introduce other subgrid scale turbulence models, inspired by the Leray-\(\alpha\) and the modified Leray-\(\alpha\) models of turbulence. Finally, we discuss the relation of the MHD-\(\alpha\) model to the MHD equations by proving a convergence theorem, that is, as the length scale \(\alpha\) tends to zero, a subsequence of solutions of the MHD-\(\alpha\) equations converges to a certain solution (a Leray-Hopf solution) of the three-dimensional MHD equations.

Keywords: subgrid scale models; turbulence models; magnetohydrodynamics; regularizing MHD; magnetohydrodynamic-\(\alpha\) model; Lagrangian-averaged magnetohydrodynamic-\(\alpha\) model; Leray-\(\alpha\) model.

Mathematics Subject Classification: 76D03, 76F20, 76F55, 76F65, 76W05.

1 Introduction

We consider the three-dimensional magnetohydrodynamic (MHD) equations for a homogeneous incompressible resistive viscous fluid subjected to a Lorentz force due to the presence of a magnetic field. The MHD involves coupling Maxwell’s equations governing the magnetic field and the Navier-Stokes equations (NSE) governing the fluid motion. The system has the form

\[
\begin{align*}
\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} - \nu \Delta \mathbf{v} + \nabla \pi + \frac{1}{2} \nabla |B|^2 &= (\mathbf{B} \cdot \nabla) \mathbf{B}, \\
\frac{\partial \mathbf{B}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{B} - (\mathbf{B} \cdot \nabla) \mathbf{v} - \eta \Delta \mathbf{B} &= 0, \\
\nabla \cdot \mathbf{v} &= \nabla \cdot \mathbf{B} = 0,
\end{align*}
\]

(1.1)

where \(\mathbf{v}(x, t)\), the fluid velocity field, \(\mathbf{B}(x, t)\), the magnetic field and \(\pi\), the pressure, are the unknowns; \(\nu > 0\) is the constant kinematic viscosity and \(\eta > 0\) is the constant magnetic diffusivity.

Current scientific methods and tools are unable to compute the turbulent behavior of three-dimensional (3D) fluids and magnetofluids analytically or via direct numerical simulation due to the large range of scales of motion that need to be resolved when the Reynolds number is high. For many purposes, it might be adequate to compute only certain statistical features of the physical phenomenon of turbulence and much effort is being made to produce reliable turbulence models that parameterize the average effects of the fluctuations on the averages, without calculating the former explicitly. Motivated by the remarkable performance of the Navier-Stokes-\(\alpha\) (NS-\(\alpha\)) (also known as the viscous Camassa-Holm equations (VCHE)}
or the Lagrangian-averaged Navier-Stokes-\(\alpha\) (LANS-\(\alpha\)) as a closure model of turbulence in infinite channels and pipes, whose solutions give excellent agreement with empirical data for a wide range of large Reynolds numbers [3–5], the alpha subgrid scale models of turbulence have been extensively studied in recent years (see, e.g., [3–7, 14, 15, 24, 25, 30–32, 39, 48]).

A justification of the inviscid NS-\(\alpha\) model can be found, for example, in [5, 8, 21, 22, 35].

An extension of the NS-\(\alpha\) model to the nondissipative MHD is given, e.g., in [20]. The model was obtained from variational principles by modifying the Hamiltonian associated with the ideal MHD equations subject to the incompressibility constraint. Then the dissipation is introduced in an \textit{ad hoc} fashion in analogy to the NS-\(\alpha\), following [3–5, 15]. Specifically, the flow Lagrangian of the ideal MHD is given by

\[
\mathcal{L}[u, D, B] = \int \left( \frac{1}{2} D |u|^2 - \pi(D - 1) - \frac{1}{2} |B|^2 \right) \, dx
\]

with volume preservation for the pressure. Here the volume element \(D(x, t) = (\det (\partial X/\partial a)(a, t))^{-1}\) at \(x = X(a, t)\), where \(X(a, t)\) is the Lagrangian fluid trajectory, \(\partial X/\partial t(a, t) = u(x, t)\) (see [18]). First, the Lagrangian is averaged and approximated using a form of Taylor’s hypothesis (see, e.g., [19]) to obtain

\[
\bar{\mathcal{L}} = \int \left( \frac{1}{2} D \left( |u|^2 + \alpha^2 |\nabla u|^2 \right) - \pi(D - 1) - \frac{1}{2} |B|^2 + \alpha_M^2 |\nabla B|^2 \right) \, dx,
\]

then the Hamiltonian principle is applied (see, e.g., [22]) to produce an ideal MHD-\(\alpha\) model (eq. (1.2) with \(\nu = \eta = 0\)). Adding viscosity and diffusivity provides the MHD-\(\alpha\) (or the Lagrangian-averaged magnetohydrodynamic-\(\alpha\) (LAMHD-\(\alpha\)) model)

\[
\begin{align*}
\frac{\partial v}{\partial t} + (u \cdot \nabla) v + \sum_{j=1}^{3} v_j \nabla u_j - \nu \Delta v + \nabla p + \sum_{j=1}^{3} (B_s)_j \nabla B_j &= (B_s \cdot \nabla) B, \\
\frac{\partial B_s}{\partial t} + (u \cdot \nabla) B_s - (B_s \cdot \nabla) u - \eta \Delta B &= 0, \\
v &= (1 - \alpha^2 \Delta) u, \\
\nabla \cdot u &= \nabla \cdot v = \nabla \cdot B_s = \nabla \cdot B = 0,
\end{align*}
\]

where \(u\) and \(B_s\) represent the unknown ‘filtered’ fluid velocity and magnetic fields, respectively, \(p\) is the unknown ‘filtered’ pressure, and \(\alpha > 0, \alpha_M > 0\) are lengthscale parameters that represent the width of the filters. At the limit \(\alpha = 0, \alpha_M = 0\), we formally obtain the three-dimensional MHD equations. The LAMHD-\(\alpha\) model was investigated numerically in periodic boundary conditions in two [38, 41] and three [37] space dimensions against direct numerical simulations. In [41] the Kármán-Howarth theorem was extended to LAMHD-\(\alpha\) equations. The LAMHD-\(\alpha\) model was also studied in [26] in the context of convection-driven plane layer geodynamo models.

We tend to think about the \(\alpha\) models as a numerical regularization of the underlying equation, which makes the nonlinearity milder, and hence the solutions of the modified equation are smoother. This is contrary to the hyperviscosity regularization [33] and nonlinear viscosity [28, 29, 43], which lead to unnecessary extra dissipation of the energy of the system. To emphasize this numerical analysis point of view, we observe that recently a Leray-\(\alpha\) model of the inviscid Burgers equation

\[
\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} = 0,
\]

which is

\[
\begin{cases}
\frac{\partial v^\alpha}{\partial \tau} + u^\alpha \frac{\partial v^\alpha}{\partial x} = 0, \\
v^\alpha = u^\alpha - \alpha^2 u^\alpha_{xx},
\end{cases}
\]

has been introduced in [1] and [45]. Regular unique solutions of (1.4) exist globally and it was shown computationally in [1] that the solutions of (1.4) converge to the unique entropy weak solution (see, e.g., [40, 44, 45]) of (1.3). Notice that there is no dissipation in (1.4), and the \(L^\infty\) norm of \(v^\alpha\) is preserved.

On the other hand, the viscous regularizing approach, which is usually taken for the Burgers equation, is achieved by introducing an artificial viscosity term in (1.3) and obtaining the viscous Burgers equation

\[
\frac{\partial v^\varepsilon}{\partial t} - \varepsilon^2 \frac{\partial^2 v^\varepsilon}{\partial x^2} + v^\varepsilon \frac{\partial v^\varepsilon}{\partial x} = 0.
\]
This model gives a smooth solution $\mathbf{v}^\varepsilon$, which converges in the appropriate norms to the unique entropy weak solution (see, e.g., [40]). However, the energy of $\mathbf{v}^\varepsilon$ is decaying in time at a much higher rate than the decay expected for the entropy weak solution. Hence, the advantage of introducing the Leray-$\alpha$ model (1.4) for Burgers equation. This simple example clarifies our numerical approach of why we insist on making the nonlinearity milder instead of adding additional viscous or hyperviscous terms. This approach has been discussed further in [2] in the context of Euler and Navier-Stokes equations.

Filtering the magnetic field, as it is done in [26, 37, 38], is equivalent to introducing hyperdiffusivity for the filtered magnetic field $\mathbf{B}_s$, due to the term $-\eta \alpha^2 \Delta^2 \mathbf{B}_s$ in (1.2), which we think is unnecessary. Taking the numerical analysis point of view discussed above we prove the well-posedness of a certain weak solutions, as in the case for the original MHD equations (1.1). In this case, the term $(\mathbf{u} \cdot \nabla) \mathbf{B}$ leads to the term $\frac{1}{2} \nabla |\mathbf{B}|^2 = (\mathbf{B} \cdot \nabla) \mathbf{B}$.

\[
\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{v} + \sum_{j=1}^{3} \mathbf{v}_j \nabla \mathbf{u}_j - \nu \Delta \mathbf{v} + \nabla p + \frac{1}{2} \nabla |\mathbf{B}|^2 = (\mathbf{B} \cdot \nabla) \mathbf{B},
\]

\[
\frac{\partial \mathbf{B}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{B} - (\mathbf{B} \cdot \nabla) \mathbf{u} - \eta \Delta \mathbf{B} = 0,
\]

\[
v = (1 - \alpha^2 \Delta) \mathbf{u}, \quad \alpha > 0,
\]

\[
\nabla \cdot \mathbf{u} = \nabla \cdot v = \nabla \cdot \mathbf{B} = 0,
\]

instead of the system (1.2).

As $\alpha$ models are some sort of regularizing numerical schemes, we would like to make sure that they inherit some of the original properties of the 3D MHD equations. Formally, three ideal, i.e. $\nu = \eta = 0$, quadratic invariants of the system (1.6) could be identified with the invariants of the original ideal 3D MHD equations under suitable boundary conditions, for instance, in rectangular periodic boundary conditions or in the whole space $\mathbb{R}^3$. Namely, the energy $E^\alpha = \frac{1}{2} \int_\Omega (|\mathbf{v}(x) - |\mathbf{B}(x)|^2) \, dx$, the cross helicity $H^\alpha = \frac{1}{2} \int_\Omega \mathbf{v}(x) \cdot \nabla \mathbf{B}(x) \, dx$, and the magnetic helicity $H^\alpha = \frac{1}{2} \int_\Omega A(x) \cdot \mathbf{B}(x) \, dx$, where $A$ is the vector potential, so that $\mathbf{B} = \nabla \times A$; and they reduce, as $\alpha \to 0$, to the ideal invariants of the MHD equations.

There are other possible alpha subgrid scale models that can be shown to have global existence and uniqueness. For instance, inspired by the Leray-$\alpha$ [6, 7, 17, 23, 48] and modified Leray-$\alpha$ (ML-$\alpha$) [25] models of turbulence, we formulate similar MHD alpha models, we refer to them as Leray-$\alpha$-MHD and ML-$\alpha$-MHD models, respectively. The Leray-$\alpha$ and ML-$\alpha$ models of turbulence reduce to the same closure model for the Reynolds averaged Navier-Stokes equations in turbulent channels and pipes as the NS-$\alpha$ model under the corresponding symmetries [6, 7, 25], which, as we mentioned above, compares successfully with experimental data for a wide range of Reynolds numbers. This comparison means that the Leray-$\alpha$ and the ML-$\alpha$ models as well as NS-$\alpha$ equations could be equally used as subgrid scale models of turbulence.

Specifically, we consider the following version of the three-dimensional Leray-$\alpha$-MHD model

\[
\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{v} - \nu \Delta \mathbf{v} + \nabla p + \frac{1}{2} \nabla |\mathbf{B}|^2 = (\mathbf{B} \cdot \nabla) \mathbf{B},
\]

\[
\frac{\partial \mathbf{B}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{B} - (\mathbf{B} \cdot \nabla) \mathbf{u} - \eta \Delta \mathbf{B} = 0,
\]

\[
v = (1 - \alpha^2 \Delta) \mathbf{u},
\]

\[
\nabla \cdot \mathbf{u} = \nabla \cdot \mathbf{v} = \nabla \cdot \mathbf{B} = 0.
\]

Formally, the term $(\mathbf{B} \cdot \nabla) \mathbf{v}$ comes from requiring in the ideal ($\nu = 0 = \eta$) case the conservation of energy $E^\alpha = \frac{1}{2} \int_\Omega (|\mathbf{v}(x)|^2 + |\mathbf{B}(x)|^2) \, dx$ (under suitable boundary conditions). While the requirement for the system to have an ideal invariant corresponding to the cross helicity $H^\alpha = \frac{1}{2} \int_\Omega \mathbf{v}(x) \cdot \mathbf{B}(x) \, dx$ leads to the term $(\mathbf{u} \cdot \nabla) \mathbf{B}$. Contrary to the MHD-$\alpha$ model (1.6), where we establish the existence and uniqueness, for the 3D Leray-$\alpha$-MHD model (1.7) we are able to establish only the existence of weak solutions, as in the case for the original MHD equations (1.1). In this case, the term $(\mathbf{B} \cdot \nabla) \mathbf{v}$ is problematic as in the usual 3D NSE and MHD. However, in the two dimensional case the existence and uniqueness of weak solutions can be shown (similarly to the proof given for the model (1.6) in section 3).
For the following 2D-Leray-\( \alpha \)-MHD model

\[
\begin{align*}
\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{v} - \nu \Delta \mathbf{v} + \nabla p + \frac{1}{2} \nabla |\mathbf{B}|^2 &= (\mathbf{B} \cdot \nabla) \mathbf{B}, \\
\frac{\partial \mathbf{B}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{B} - (\mathbf{B} \cdot \nabla) \mathbf{u} - \eta \Delta \mathbf{B} &= 0,
\end{align*}
\]

(1.8)

\[
\mathbf{v} = (1 - \alpha^2 \Delta) \mathbf{u}, \\
\nabla \cdot \mathbf{u} = \Delta \cdot \mathbf{v} = \nabla \cdot \mathbf{B} = 0.
\]

For this system, due to the identity \( \int_{\Omega} (\mathbf{u} \cdot \nabla \mathbf{u}) \cdot \Delta \mathbf{u} = 0 \) (for the periodic 2D case and divergence free \( \mathbf{u} \)), the ideal invariant corresponding to the energy is \( E^\alpha = \frac{1}{2} \int_{\Omega} (\mathbf{v}(x) \cdot \mathbf{u}(x) + |\mathbf{B}(x)|^2) \, dx \). At the moment we are unable to find a conserved quantity in the ideal version of (1.8) that can be identified with a cross helicity. The mean-square magnetic potential, given by \( A = \frac{1}{2} \int_{\Omega} |\psi(x)|^2 \, dx \), where \( \mathbf{B} = \nabla^\perp \psi \), is conserved in the ideal case. We note that it appears that there is no conserve quantity that could be identified with energy for the 3D version of (1.8).

The three-dimensional Modified-Leray-\( \alpha \)-MHD model, for which the well-posedness can be proved in a similar way as for the model (1.6), is given by

\[
\begin{align*}
\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{u} - \nu \Delta \mathbf{v} + \nabla p + \frac{1}{2} \nabla |\mathbf{B}|^2 &= (\mathbf{B} \cdot \nabla) \mathbf{B}, \\
\frac{\partial \mathbf{B}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{B} - (\mathbf{B} \cdot \nabla) \mathbf{u} - \eta \Delta \mathbf{B} &= 0,
\end{align*}
\]

(1.9)

\[
\mathbf{v} = (1 - \alpha^2 \Delta) \mathbf{u}, \\
\nabla \cdot \mathbf{u} = \Delta \cdot \mathbf{v} = \nabla \cdot \mathbf{B} = 0,
\]

where the term \( (\mathbf{B} \cdot \nabla) \mathbf{u} \) comes from requiring the conservation of energy (in the ideal case, with periodic boundary conditions or in \( \mathbb{R}^3 \)) \( E^\alpha = \frac{1}{2} \int_{\Omega} (\mathbf{v}(x) \cdot \mathbf{u}(x) + |\mathbf{B}(x)|^2) \, dx \). Also, the system conserves the magnetic helicity \( H^\alpha_M = \frac{1}{2} \int_{\Omega} \mathbf{A}(x) \cdot \mathbf{B}(x) \, dx \). At the moment we are unable to find a conserved quantity in the ideal version of (1.9) which can be identified with a cross helicity.

The main goal of this paper is to establish the global existence, uniqueness and regularity of solutions of the three-dimensional MHD-\( \alpha \) equations (1.6) subject to periodic boundary conditions (similar results also hold in \( \mathbb{R}^3 \)). We emphasize again that we consider a version of the MHD alpha models, where only the velocity field is filtered, while the magnetic field remains unfiltered. We note that in the case of filtering the magnetic field, as in (1.2), one has hyperdiffusivity for the filtered magnetic field \( \mathbf{B}_s \) and the proof of the existence and uniqueness of regular solutions of (1.2) is deduced in a similar way.

We start by introducing some preliminary background and the functional setting in section 2. In section 3 we show the global well-posedness of the MHD-\( \alpha \) subgrid scale model of turbulence (1.6). We remark that using the Gevrey regularity techniques developed in [16] (see also [13]) one can show that the solution of the MHD-\( \alpha \) model becomes instantaneously analytic in space and time. As a result of this Gevrey regularity, one deduces the existence of a dissipation range in the energy spectrum in which the energy decays exponentially fast as a function of the wavenumber, for \( k \) larger than the dissipation length scale (see [11]). One can also establish, in the forced case, the existence of a finite dimensional global attractor, a subject of future work. In section 4 we relate the solutions of the MHD-\( \alpha \) equations to those of the 3D MHD as the length scale \( \alpha \) tends to zero. Specifically, we prove that one can extract subsequences of weak solutions of the MHD-\( \alpha \) equations which converge as \( \alpha \to 0^+ \) (in the appropriate sense) to a Leray-Hopf weak solution of the three-dimensional MHD equations (1.1) on any time interval \([0, T]\), which satisfies the energy inequality

\[
|\mathbf{v}(t)|^2 + |\mathbf{B}(t)|^2 + 2 \int_{t_0}^t (\nu ||\mathbf{v}(s)||^2 + \eta ||\mathbf{B}(s)||^2) \, ds \leq |\mathbf{v}(t_0)|^2 + |\mathbf{B}(t_0)|^2
\]

for almost every \( t_0 \in [0, T] \) and all \( t \in [t_0, T] \). Also, if the initial data is smooth a subsequence of solutions converges for a short interval of time, that depends on the initial data, \( \nu, \eta \) and the domain, to the unique strong solution of the MHD equations on this interval. Thus the \( \alpha \) models can be viewed as a regularizing numerical method. Section 5 contains a discussion summarizing our results.
2 Functional Setting and Preliminaries

Let $\Omega$ be the $L$-periodic three-dimensional box $\Omega = [0, L]^3$. We consider the following MHD-\(\alpha\) subgrid scale turbulence model, which we introduced in (1.6), subject to periodic boundary condition with a basic domain $\Omega$,

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{v} + \sum_{j=1}^{3} \mathbf{v}_j \nabla u_j - \nu \Delta \mathbf{v} + \nabla p + \frac{1}{2} \nabla |\mathbf{B}|^2 = (\mathbf{B} \cdot \nabla) \mathbf{B},$$

(2.1a)

$$\frac{\partial \mathbf{B}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{B} - (\mathbf{B} \cdot \nabla) \mathbf{u} - \eta \Delta \mathbf{B} = 0,$$

(2.1b)

$$\mathbf{v} = (1 - \alpha^2 \Delta) \mathbf{u},$$

(2.1c)

$$\nabla \cdot \mathbf{u} = \nabla \cdot \mathbf{v} = \nabla \cdot \mathbf{B} = 0,$$

(2.1d)

$$\mathbf{u}(x, 0) = \mathbf{u}^{in}(x),$$

(2.1e)

$$\mathbf{B}(x, 0) = \mathbf{B}^{in}(x),$$

(2.1f)

where $\mathbf{u}$ represents the unknown ‘filtered’ fluid velocity vector, $p$ is the unknown ‘filtered’ pressure, and $\alpha > 0$ is a lengthscale parameter which represents the width of the filter. At the limit $\alpha = 0$ we formally obtain the three-dimensional MHD equations (1.1), where $\mathbf{u}$ is the Eulerian velocity field and $p - \frac{1}{2} |\mathbf{u}|^2$ is the pressure. Notice that we chose to smooth only the velocity field and not the magnetic field, thus we do not introduce hyperdiffusivity for the magnetic field, as it is for the filtered magnetic field in (1.2).

We consider initial values with zero spatial means, i.e., we assume that

$$\int_\Omega \mathbf{u}^{in} dx = \int_\Omega \mathbf{B}^{in} dx = 0,$$

(2.2)

then from (2.1a) and (2.1b), after integration by parts, using the spatial periodicity of the solution and the divergence free condition (2.1d) we have $(d/dt) \int_\Omega \mathbf{u} dx = 0, (d/dt) \int_\Omega \mathbf{B} dx = 0$ and $(d/dt) \int_\Omega \mathbf{w} dx = 0.$

Namely, the spatial mean of the solution is invariant under time. Hence, by (2.2), $\int_\Omega \mathbf{v} dx = \int_\Omega \mathbf{u} dx = \int_\Omega \mathbf{B} dx = 0.$

Next, we introduce some notation and background following the mathematical theory of NSEs, see, for instance, [9, 29, 46, 47]. Let $L^p(\Omega)$ and $H^m(\Omega)$ denote the $L^p$ Lebesgue spaces and Sobolev spaces respectively. We denote by $|.|$ the $L^2$-norm, and by $(\cdot, \cdot)$ the $L^2$-inner product. Let $X$ be a linear subspace of integrable functions defined on the domain $\Omega$, we define $\mathcal{X} := \{\varphi \in X : \int_\Omega \varphi dx = 0\}$ and $\mathcal{V} = \{\varphi : \varphi\text{ is a vector valued trigonometric polynomial defined on } \Omega, \text{such that } \nabla \cdot \varphi = 0 \text{ and } \int_\Omega \varphi(x) dx = 0\}$. The spaces $H$ and $V$ are the closures of $\mathcal{V}$ in $L^2(\Omega)$ and in $H^1(\Omega)$ respectively; observe that $H^2$, the orthogonal complement of $H$ in $L^2(\Omega)$ is $\{\nabla p : p \in H^1(\Omega)\}$. Let $P_\sigma : L^2(\Omega) \to H$ be the Helmholtz-Leray projection, and $A = -P_\sigma \Delta$ be the Stokes operator with domain $D(A) = (H^2(\Omega) \cap V)$. In the periodic boundary conditions $A = -\Delta|_{D(A)}$ is a self-adjoint positive operator with compact inverse. Hence the space $H$ has an orthonormal basis $\{w_j\}_{j=1}^{\infty}$ of eigenfunctions of $A, Aw_j = \lambda_j w_j$, with $0 < \lambda_1 \leq \lambda_2 \leq \ldots$, $\lambda_j \sim j^{2/4}L^{-2}$, see, e.g., [9,36]. One can show that $V = D(A^{1/2})$. We denote $(\cdot, \cdot) = (A^{1/2}, A^{1/2})$ and $|.| = |A^{1/2} \cdot|$ the inner product and the norm on $V$, respectively.

Following the notation of the Navier-Stokes equations and those of [15], we denote

$$B(\mathbf{u}, \mathbf{v}) = P_\sigma [(\mathbf{u} \cdot \nabla) \mathbf{v}], \quad \mathbf{u}, \mathbf{v} \in V,$$

$$\tilde{B}(\mathbf{u}, \mathbf{v}) = P_\sigma [(\nabla \times \mathbf{v}) \times \mathbf{u}], \quad \mathbf{u}, \mathbf{v} \in V.$$

Notice that

$$(B(\mathbf{u}, \mathbf{v}), \mathbf{w}) = -(B(\mathbf{w}, \mathbf{u}), \mathbf{v}), \quad \mathbf{u}, \mathbf{v}, \mathbf{w} \in V,$$

and due to the identity

$$b \cdot \nabla a + \sum_{j=1}^{3} a_j \nabla b_j = -b \times (\nabla \times a) + \nabla (a \cdot b),$$

(2.3)

$$\left(\tilde{B}(\mathbf{u}, \mathbf{v}), \mathbf{w}\right) = (B(\mathbf{u}, \mathbf{v}), \mathbf{w}) - (B(\mathbf{w}, \mathbf{v}), \mathbf{u}).$$
The definitions of $B(u, v)$ and $\tilde{B}(u, v)$ and the above algebraic identities may be extended to larger spaces by the density of $V$ in the appropriate space each time the corresponding trilinear forms are continuous. The extensions of the bilinear forms $B$ and $\tilde{B}$ (which we also denote $B$ and $\tilde{B}$) have the following properties

**Lemma 2.1.**

(i) Let $X$ be either $B$ or $\tilde{B}$. The operator $X$ can be extended continuously from $V \times V$ with values in $V'$ (the dual space of $V$). In particular, for every $u, v, w \in V$,

$$|\langle X(u, v), w \rangle| \leq c|u|^{1/2}\|u\|^{1/2}\|v\|\|w\|.$$  \hspace{1cm} (2.4)

Moreover,

$$(B(u, v), w) = - (B(u, w), v), \quad u, v, w \in V,$$  \hspace{1cm} (2.5)

which in turn implies that

$$(B(u, v), v) = 0, \quad u, v \in V.$$  \hspace{1cm} (2.6)

Also

$$\left(\tilde{B}(u, v), w\right) = (B(u, v), w) - (B(w, v), u), \quad u, v, w \in V,$$  \hspace{1cm} (2.7)

and hence

$$\left(\tilde{B}(u, v), u\right) = 0, \quad u, v \in V.$$  \hspace{1cm} (2.8)

(ii) Furthermore, let $u \in D(A), v \in V, w \in H$ and let $X$ be either $B$ or $\tilde{B}$ then

$$|\langle X(u, v), w \rangle| \leq c\|u\|^{1/2}\|Au\|^{1/2}\|v\|\|w\|.$$  \hspace{1cm} (2.9)

(iii) Let $u \in V, v \in D(A), w \in H$ then

$$|\langle B(u, v), w \rangle| \leq c\|u\|\|v\|^{1/2}\|Aw\|^{1/2}\|w\|.$$  \hspace{1cm} (2.10)

(iv) Let $u \in D(A), v \in H, w \in V$, then

$$|\langle B(u, v), w \rangle_V| \leq c\|u\|^{1/2}\|Au\|^{1/2}\|v\|\|w\|.$$  \hspace{1cm} (2.11)

(v) Let $u, v, w \in V$, then

$$|\frac{1}{\langle \tilde{B}(u, v), w \rangle_V}| \leq c\|u\|\|v\|\|w\|^{1/2}\|w\|^{1/2}.$$  \hspace{1cm} (2.12)

(vi) Let $u \in H, v \in V, w \in D(A)$ and let $X$ be either $B$ or $\tilde{B}$ then

$$|\langle X(u, v), w \rangle_{D(A)}| \leq c\|u\|\|v\|\|w\|^{1/2}\|Aw\|^{1/2}.$$  \hspace{1cm} (2.13)

(vii) Let $u \in V, v \in H, w \in D(A)$ then

$$\left|\langle \tilde{B}(u, v), w \rangle_{D(A)}\right| \leq c\left(|u|^{1/2}\|u\|^{1/2}|v||Aw| + |v||u|\|w\|^{1/2}\|Aw\|^{1/2}\right),$$  \hspace{1cm} (2.14)

and hence by Poincaré inequality,

$$\left|\langle \tilde{B}(u, v), w \rangle_{D(A)}\right| \leq c(\lambda_1)^{-1/4}\|u\|\|v\|\|Aw\|.$$  \hspace{1cm} (2.15)

(viii) Let $u \in D(A), v \in H, w \in V$ then

$$\left|\langle \tilde{B}(u, v), w \rangle_V\right| \leq c\left(\|u\|^{1/2}|Au|^{1/2}\|v\|\|w\| + |Au||v||w|^{1/2}\|w\|^{1/2}\right).$$  \hspace{1cm} (2.16)

In this lemma and throughout the paper $c$ denotes a generic scale invariant constant.
Proof. The proof of (i) can be found, for example, in [9, 46, 47] for $B$ and in [15, Lemma 1(iii)] for $\tilde{B}$.

To prove (ii) we first consider the case where $u, v, w \in V$.

\[ |(B(u, v), w)| = \left| \int_{\Omega} (u \cdot \nabla)v \cdot w \, dx \right|, \]
\[ |(\tilde{B}(u, v), w)| = \left| \int_{\Omega} u \times (\nabla \times v) \cdot w \, dx \right|, \]

hence

\[ |(X(u, v), w)| \leq c \|u\|_{L^\infty} \|\nabla v\|_{L^2} \|w\|_{L^2}. \]

By Agmon’s inequality in three-dimensional space, see, e.g., [9],

\[ \|\phi\|_{L^\infty} \leq \|\phi\|_{H^1}^{1/2} \|\phi\|_{H^2}^{1/2}, \]

we obtain

\[ |(X(u, v), w)| \leq c\|u\|^{1/2} |A^1/2|v|w|. \]

Since $V$ is dense in $D(A)$, $V$ and $H$ we conclude the proof of (ii).

To prove (iii) we recall the following Sobolev-Ladyzhenskaya inequalities (see, e.g., [9, 29]) in 3D

\[ \|\phi\|_{L^6} \leq c\|\phi\|, \]
\[ \|\phi\|_{L^3} \leq c\|\phi\|^{1/2} \|\phi\|^{1/2}, \]

for $\phi \in V$. Then we have

\[ |(B(u, v), w)| = \left| \int_{\Omega} (u \cdot \nabla)v \cdot w \, dx \right| \]
\[ \leq c \|u\|_{L^6} \|\nabla v\|_{L^3} \|w\|_{L^2} \]
\[ \leq c\|u\| \|\nabla v\|^{1/2} \|\nabla v\|^{1/2} \|w\| \]
\[ \leq c\|u\| \|v\|^{1/2} |A^1/2|v|w|. \]

The proof of (iv) is a direct result of the (ii) due to the symmetry (2.5). The proof of (v), (vi), (vii), (viii) can be found in [15, Lemma 1 (iii, iv, v, vi)].

Using the above notations and the identity (2.3) we apply $P_\sigma$ to (2.1) to obtain, as for the case of the NSE, the equivalent system of equations (see, e.g., [46] and [12])

\[ \frac{d}{dt} \dot{B}(u, v) + \tilde{B}(u, v) + \nu A v = B(B, B), \]
\[ \frac{d}{dt} \dot{B} + B(u, B) - B(B, u) + \eta A B = 0, \]
\[ u(0) = u^n, \]
\[ B(0) = B^n. \]

Definition 2.2. Let $T > 0$. A weak solution of (2.17) in the interval $[0, T]$, given $u(0) = u^n \in V$ (or equivalently $v^n \in V'$) and $B(0) = B^n \in H$, is a pair of functions $u$, $B$, such that

\[ u \in C([0, T]; V) \cap L^2([0, T]; D(A)) \text{ with } \frac{du}{dt} \in L^2([0, T]; H), \]

(or equivalently $v \in C([0, T]; V') \cap L^2([0, T]; H)$ with $\frac{dv}{dt} \in L^2([0, T]; D(A'))$) and

\[ B \in C([0, T]; H) \cap L^2([0, T]; V) \text{ with } \frac{dB}{dt} \in L^2([0, T]; V'), \]
satisfying
\[
\left\langle \frac{d}{dt}v, w \right\rangle_{D(A)'} + \left\langle \hat{B}(u, v), w \right\rangle_{D(A)'} + \nu(v, Aw) = \left\langle B(B, B), w \right\rangle_{V'},
\]
(3.18a)
\[
\left\langle \frac{d}{dt}B, \xi \right\rangle_{V'} + \left\langle B(u, B), \xi \right\rangle - \left\langle B(B, u), \xi \right\rangle + \eta(B, \xi) = 0
\]
(3.18b)
for every \( w \in D(A), \xi \in V \) and for almost every \( t \in [0, T] \).

The equation (3.18) is understood in the following sense: for almost every \( t_0, t \in [0, T] \)
\[
(v(t), w) - (v(t_0), w) + \int_{t_0}^{t} \left\langle \hat{B}(u(s), v(s)), w \right\rangle_{D(A)'} ds + \nu \int_{t_0}^{t} (v(s), Aw) ds
\]
(3.19a)
\[
= \int_{t_0}^{t} (B(B(s), B(s)), w)_{V'}, ds,
\]
\[
(B(t), \xi) - (B(t_0), \xi) + \int_{t_0}^{t} (B(u(s), B(s)), \xi) ds - \int_{t_0}^{t} (B(B(s), u(s)), \xi) ds
\]
(3.19b)
\[
+ \eta \int_{t_0}^{t} (B(s), \xi) ds = 0.
\]

When \( u^{in} \in D(A) \) (or equivalently \( v^{in} \in H \)) and \( B^{in} \in V \) we call a strong solution of (2.17) in the interval \([0, T]\) the solution that satisfies
\[
B \in C([0, T]; V) \cap L^2([0, T]; D(A)), \quad u \in C([0, T]; D(A)) \cap L^2([0, T]; D(A^{3/2}))
\]
(or equivalently \( v \in C([0, T]; H) \cap L^2([0, T]; D(V)) \)).

3 Global existence and uniqueness

In this section we show the global well-posedness of the MHD-\( \alpha \) model (2.1) or equivalently (2.17).

**Theorem 3.1.** Let \( u^{in} \in V, B^{in} \in H \). Then for any \( T > 0 \) there exists a unique weak solution \( u, B \) of (2.17) on \([0, T]\). Moreover, this solution satisfies
\[
|u(t)|^2 + \alpha^2 \|u(t)\|^2 + |B(t)|^2 + 2 \int_{t_0}^{t} (\nu(\|u(s)\|^2 + \alpha^2|Au(s)|^2) + \eta\|B(s)\|^2) ds
\]
(3.1)
\[
= |u(t_0)|^2 + \alpha^2 \|u(t_0)\|^2 + |B(t_0)|^2, \quad 0 \leq t_0 \leq t \leq T.
\]

We use the Galerkin approximation scheme to prove the global existence and to establish the necessary a priori estimates. Let \( \{w_j\}_{j=1}^{\infty} \) be an orthonormal basis of \( H \) consisting of eigenfunctions of the operator \( A \). Denote \( H_m = \text{span}\{w_1, \ldots, w_m\} \) and let \( P_m \) be the \( L^2 \)-orthogonal projection from \( H \) onto \( H_m \). The Galerkin approximation of (2.17) is the ordinary differential system
\[
\frac{d\nu}{dt} + P_mB(u_m, \nu) + \nu Av_m = P_mB(B_m, B_m)
\]
(3.2a)
\[
\frac{dB_m}{dt} + P_mB(u_m, B_m) - P_mB(B_m, u_m) + \eta AB_m = 0
\]
(3.2b)
\[
u_m = u_m + \alpha^2 Au_m
\]
(3.2c)
\[
u_m(0) = P_mu^{in}
\]
(3.2d)
\[
B_m(0) = P_mB^{in}.
\]
(3.2e)

Since the nonlinear terms are quadratic, hence locally Lipschitz, then by the classical theory of ordinary differential equations, system (3.2) has a unique solution for a short interval of time \((\tau_m, T_m)\). Our goal is to show that the solutions of (3.2) remains finite for all positive times, which implies that \( T_m = \infty \).
3.1 $H^1$-Estimate of $u_m$, $L^2$-Estimate of $B_m$

We take the inner product of (3.2a) with $u_m$ and the inner product of (3.2b) with $B_m$ and use (2.6),(2.8),(2.5) to obtain

\[
\frac{1}{2} \frac{d}{dt} (|u_m|^2 + \alpha^2 |u_m|^2) + \nu (|u_m|^2 + \alpha^2 |Au_m|^2) = (B(B_m, B_m), u_m),
\]

(3.3a)

\[
\frac{1}{2} \frac{d}{dt} |B_m|^2 + \eta |B_m|^2 = -(B(B_m, B_m), u_m).
\]

(3.3b)

Now, by summing up (3.3a) and (3.3b), we have

\[
\frac{1}{2} \frac{d}{dt} (|u_m|^2 + \alpha^2 |u_m|^2 + |B_m|^2) + \nu (|u_m|^2 + \alpha^2 |Au_m|^2 + \eta |B_m|^2) = 0.
\]

(3.4)

We denote $\mu = \min \{\nu, \eta\}$ and obtain

\[
\frac{1}{2} \frac{d}{dt} (|u_m|^2 + \alpha^2 |u_m|^2 + |B_m|^2) + \mu (|u_m|^2 + \alpha^2 |Au_m|^2 + |B_m|^2) \leq 0.
\]

(3.5)

Using Poincaré’s inequality we get

\[
d \frac{dt}{|u_m|^2 + \alpha^2 |u_m|^2 + |B_m|^2} \leq 2\mu \lambda_1 (|u_m|^2 + \alpha^2 |Au_m|^2 + |B_m|^2) \leq 0.
\]

and then by Gronwall’s inequality we obtain

\[
|u_m(t)|^2 + \alpha^2 |u_m(t)|^2 + |B_m(t)|^2 \leq e^{-2\mu \lambda_1 t} (|u_m(0)|^2 + \alpha^2 |u_m(0)|^2 + |B_m(0)|^2).
\]

Hence

\[
|u_m(t)|^2 + \alpha^2 |u_m(t)|^2 + |B_m(t)|^2 \leq k_1 := |u^{in}|^2 + \alpha^2 |u^{in}|^2 + |B^{in}|^2,
\]

(3.6)

for all $t \geq 0$.

This implies that $T_m = \infty$. Indeed, consider $[0, T_m^{max})$, the maximal interval of existence. Either $T_m^{max} = \infty$ and we are done, or $T_m^{max} < \infty$ and we have $\lim_{t \to (T_m^{max})^{-}} (|u_m(t)|^2 + |B_m(t)|^2) = \infty$, a contradiction to (3.6). Hence we have global existence of $u_m$, $B_m$, and hereafter we take an arbitrary interval $[0, T]$.

Integrating (3.4) over the interval $(s, t)$ and using the estimate (3.6) we obtain that, for all $0 \leq s \leq t$,

\[
2 \int_s^t (\nu (|u_m(\tau)|^2 + \alpha^2 |Au_m(\tau)|^2) + \eta |B_m(\tau)|^2) d\tau \leq k_1.
\]

(3.7)

3.2 $H^2$-Estimate of $u_m$, $H^1$-Estimate of $B_m$

By taking the inner product of (3.2a) with $Au_m$ and the inner product of (3.2b) with $AB_m$ we have

\[
\frac{1}{2} \frac{d}{dt} (|u_m|^2 + \alpha^2 |Au_m|^2) + \nu (|Au_m|^2 + \alpha^2 |A^{3/2}u_m|^2) = (B(B_m, B_m), Au_m) - (\tilde{B}(u_m, v_m), Au_m),
\]

(3.8a)

\[
\frac{1}{2} \frac{d}{dt} |B_m|^2 + \eta |AB_m|^2 = (B(B_m, u_m), AB_m) - (B(u_m, B_m), AB_m).
\]

(3.8b)

First, we estimate the nonlinear terms. By (2.12) we have

\[
\left| (\tilde{B}(u_m, v_m), Au_m) \right| \leq c (\lambda_1^{-1} + \alpha^2) |u_m||Au_m|^{1/2} |A^{3/2}u_m|^{3/2}.
\]

(3.9)

To bound the term $|B(u_m, B_m), Au_m|$ we use (2.11)

\[
|B(u_m, B_m), Au_m| \leq c |B_m|^{1/2} |AB_m|^{1/2} |B_m| |A^{3/2}u_m|.
\]

(3.10)

By (2.9) we have

\[
|B(u_m, B_m)| \leq c |B_m|^{1/2} |AB_m|^{3/2}.
\]

(3.11)
and by (2.10)
\[ |B(u_m, B_m)| \leq c\|B_m\|^{1/2}\|u_m\||AB_m|^{3/2}. \] (3.12)

Now, summing up (3.8a) and (3.8b), we obtain
\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \left( \|u_m\|^2 + \alpha^2 |Au_m|^2 + \|B_m\|^2 \right) + \nu \left( |Au_m|^2 + \alpha^2 |A^{3/2}u_m|^2 \right) + \eta |AB_m|^2 \\
= (B(B_m, B_m), Au_m) - \left( \hat{B}(u_m, u_m), Au_m \right) + (B(B_m, u_m), AB_m) - (B(u_m, B_m), AB_m).
\end{align*}
\] (3.13)

By (3.9), (3.10), (3.11) and (3.12) and several applications of Young’s inequality we reach
\[
\frac{d}{dt} \left( \|u_m\|^2 + \alpha^2 |Au_m|^2 + \|B_m\|^2 \right) + \nu \left( |Au_m|^2 + \alpha^2 |A^{3/2}u_m|^2 \right) + \eta |AB_m|^2 \\
\leq c(\alpha^2 \nu)^{-3} \left( \lambda_1^{-1} + \alpha^2 \right)^4 \|u_m\|^4 |Au_m|^2 + c(\alpha^2 \nu)^{-2} \eta^{-1} \|B_m\|^2 |B_m|^4 + c(2) \eta^{-3} \|B_m\|^2 \|u_m\|^4.
\] (3.14)

Integrating over \((s, t)\) and using (3.6), (3.7) we obtain
\[
\|u_m(t)\|^2 + \alpha^2 |Au_m(t)|^2 + \|B_m(t)\|^2 + \int_s^t \left( \nu \left( |Au_m(\tau)|^2 + \alpha^2 |A^{3/2}u_m(\tau)|^2 \right) + \eta |AB_m(\tau)|^2 \right) d\tau \\
\leq \|u_m(s)\|^2 + \alpha^2 |Au_m(s)|^2 + \|B_m(s)\|^2 + K_1.
\] (3.15)

where we denote
\[ K_1 := c \left( \left( \lambda_1^{-1} + \alpha^2 \right)^4 \nu^{-4} \alpha^{-12} + \eta^{-2} \alpha^{-4} (\nu^{-2} + \eta^{-2}) \right) k_1^3. \]

(i) Now, if \( u^in \in D(A), B^in \in V \), we have
\[
\|u_m(t)\|^2 + \alpha^2 |Au_m(t)|^2 + \|B_m(t)\|^2 \\
+ \int_0^t \left( \nu \left( |Au_m(\tau)|^2 + \alpha^2 |A^{3/2}u_m(\tau)|^2 \right) + \eta |AB_m(\tau)|^2 \right) d\tau \\
\leq \|u^in\|^2 + \alpha^2 |Au^in|^2 + \|B^in\|^2 + K_1 := k_2. \] (3.16)

(ii) Otherwise, if \( u^in \notin D(A), B^in \notin V \), we integrate
\[
\|u_m(t)\|^2 + \alpha^2 |Au_m(t)|^2 + \|B_m(t)\|^2 \leq \|u_m(s)\|^2 + \alpha^2 |Au_m(s)|^2 + \|B_m(s)\|^2 + K_1
\]
with respect to \( s \) over \((0, t)\) and use (3.7) to obtain
\[
t \left( \|u_m(t)\|^2 + \alpha^2 |Au_m(t)|^2 + \|B_m(t)\|^2 \right) \leq \frac{1}{2\mu} k_1 + K_1 t,
\]
hence for \( t > 0 \)
\[
\|u_m(t)\|^2 + \alpha^2 |Au_m(t)|^2 + \|B_m(t)\|^2 \leq K_1 + \frac{1}{2t} \mu^{-1} k_1 := k_2 (t), \] (3.17)
and thus
\[
\int_s^t \left( \nu \left( |Au_m(\tau)|^2 + \alpha^2 |A^{3/2}u_m(\tau)|^2 \right) + \eta |AB_m(\tau)|^2 \right) d\tau \leq 2K_1 + \frac{1}{2t} k_1 \mu^{-1} = K_1 + k_2 (s). \] (3.18)
3.3 $H^3$-Estimate of $u_m$

We establish a uniform upper bound for the $H^3$-norm of $u_m$ by providing the estimate for the vorticity $q_m = \nabla \times v_m$. The Galerkin approximation (3.2a) is equivalent to

$$\frac{d}{dt} v_m + \nu A v_m - P_m (u_m \times q_m) = P_m B(B_m, B_m).$$

Taking the curl of the above equation we obtain

$$\frac{d}{dt} q_m + \nu A q_m - \nabla \times P_m (u_m \times q_m) = \nabla \times P_m B(B_m, B_m). \quad (3.19)$$

We use that in periodic boundary conditions

$$\int_\Omega (\nabla \times \phi) \cdot \psi \, dx = \int_{\Omega} \phi \cdot (\nabla \times \psi) \, dx \quad (3.20)$$

and for divergence free vectors

$$\nabla \times (\phi \times \psi) = - (\phi \cdot \nabla) \psi + (\psi \cdot \nabla) \phi. \quad (3.21)$$

Taking the inner product of (3.19) with $q_m$, using that $\nabla \cdot q_m = 0$ and the identities (3.20), (3.21) and (2.6), we reach

$$\frac{1}{2} \frac{d}{dt} |q_m|^2 + \nu |q_m|^2 = (B(q_m, u_m), q_m) + (B(B_m, B_m), \nabla \times q_m).$$

We bound the right hand side using (2.4), Young’s inequality and (3.6)

$$|(B(q_m, u_m), q_m)| \leq c |q_m|^{1/2} |u_m| |q_m|^{3/2}$$

$$\leq c\nu^{-3} \alpha^{-4} k_1^2 |q_m|^2 + \frac{\nu}{4} |q_m|^2$$

and by (2.9)

$$|(B(B_m, B_m), \nabla \times q_m)| \leq c |B_m|^{3/2} |AB_m|^{1/2} |q_m|$$

$$\leq c\nu^{-1} |AB_m|^2 + |B_m|^6 + \frac{\nu}{4} |q_m|^2. \quad (3.22)$$

Note that since $\nabla \cdot v_m = 0$ and due to the periodic boundary conditions we have

$$|q_m| = |\nabla \times v_m| = |\nabla v_m| = |v_m|$$

hence

$$|q_m|^2 \leq |u_m + \alpha^2 A u_m|^2 \leq (\lambda_1^{-1} + \alpha^2)^2 |A^{3/2} u_m|^2. \quad (3.23)$$

Hence we obtain

$$\frac{1}{2} \frac{d}{dt} |q_m|^2 + \frac{\nu}{2} |q_m|^2 \leq c\nu^{-3} \alpha^{-4} k_1^2 (\lambda_1^{-1} + \alpha^2)^2 |A^{3/2} u_m|^2 + c\nu^{-1} |AB_m|^2 + |B_m|^6. \quad (3.24)$$

In the following we denote by $c_i$ some constants depending on $\nu, \eta, \alpha, k_1, \lambda_1$. Integrating over $(s, t)$ and using (3.18) and (3.17) we have

$$|q_m(t)|^2 \leq |q_m(s)|^2 + c_0 \left( 2K_1 + \frac{1}{2s} k_1 \mu^{-1} \right) + 2 \int_s^t \left( K_1 + \frac{1}{2r} k_1 \mu^{-1} \right) \, dr. \quad (3.25)$$

We integrate this expression with respect to $s$ over $(\frac{t}{2}, t)$, $t > 0$ and use (3.18), (3.23) to obtain

$$|q_m(t)|^2 \leq \frac{1}{2} K_1^2 t + c_1 + \frac{c_2}{t} + \frac{c_3}{t^2} \quad (3.26)$$
For $t > \frac{1}{v \lambda_1}$ we integrate (3.25) with respect to $s$ over the interval $\left( t - \frac{1}{v \lambda_1}, t \right)$. Note that, by applying also (3.18) and (3.23), we have

$$
|q_m(t)|^2 \leq c_4 + c_5 \left( t - \frac{1}{v \lambda_1} \right)^{-1} + c_6 \ln \left( 1 - \frac{1}{v \lambda_1 t} \right)^{-1}.
$$

From (3.26) and (3.27) we have, for $t > 0$,

$$
|q_m(t)|^2 \leq k_3(t),
$$

where $k_3(t)$ has the following properties

(i) $k_3(t)$ is finite for all $t > 0$;

(ii) $k_3(t)$ is independent of $m$;

(iii) If either $u^{in} \not\in D(A^{3/2})$ or $B^{in} \not\in V$, then $k_3(t)$ depends on $\nu, \eta, \alpha, |u^{in}|, \|u^{in}\|, \|B^{in}\|$ and

$$
\lim_{t \to 0} k_3(t) = \infty;
$$

(iv) $\limsup_{t \to \infty} k_3(t) = R^2 < \infty$, $R^2$ depends on $\nu, \eta, \alpha$, but not on $u^{in}$ and $B^{in}$.

Returning to (3.24) and integrating over $(t, t + \tau)$, for $t > 0$, $\tau \geq 0$ and using (3.28) we obtain

$$
\nu \int_t^{t+\tau} \|q_m\|^2 \leq k_4(t, \tau),
$$

where $k_4(t, \tau)$ as a function of $t$ satisfies properties (i)-(iii) as $k_3(t)$ above.

**Remark 3.2.** If $B^{in} \in V$ and $u^{in} \in D(A)$, then by (3.16), Young’s and Poincaré inequalities we can bound (3.22) by

$$
|B(B_m, B_m, \nabla \times q_m)| \leq c\nu^{-1} \lambda_1^{-1/2} k_2 |AB_m|^2 + \frac{\nu}{4} \|q_m\|^2.
$$

Hence we have

$$
\frac{1}{2} \frac{d}{dt} |q_m|^2 + \frac{\nu}{2} \|q_m\|^2 \leq c\nu^{-4} \alpha^{-4} k_1^2 \left( \lambda_1^{-1} + \alpha^2 \right)^2 |A^{3/2} u_m|^2 + c\nu^{-1} \lambda_1^{-1/2} k_2 |AB_m|^2
$$

and by integrating over $(0, t)$ and using (3.16) we obtain

$$
|q_m(t)|^2 \leq |q_m(0)|^2 + c\nu^{-4} \alpha^{-6} k_1^2 \left( \lambda_1^{-1} + \alpha^2 \right)^2 (K_1 + k_2) + c\nu^{-1} \lambda_1^{-1/2} k_2 \eta^{-1} (K_1 + k_2).
$$

If, additionally, $u^{in} \in D(A^{3/2})$, then using (3.23), we obtain

$$
|q_m(t)|^2 \leq \left( \lambda_1^{-1} + \alpha^2 \right)^2 |A^{3/2} u^{in}|^2 + c\nu^{-4} \alpha^{-6} k_1^2 \left( \lambda_1^{-1} + \alpha^2 \right)^2 (K_1 + k_2) + c\nu^{-1} \lambda_1^{-1/2} k_2 \eta^{-1} (K_1 + k_2).
$$

### 3.4 Existence of weak solutions

Let us summarize our estimates. For any $T > 0$ we have

(i) From (3.6)

$$
\|u_m\|_{L^\infty([0, T]; H)}^2 \leq k_1, \quad \|u_m\|_{L^\infty([0, T]; V)}^2 \leq \frac{k_1}{\alpha^2} \quad \text{or} \quad \|v_m\|_{L^\infty([0, T]; V')}^2 \leq \frac{k_1}{\alpha^2} \left( \lambda_1^{-1} + \alpha^2 \right)^2,
$$

(ii) (3.31) and (3.32)
(ii) From (3.7) we have
\[ \|u_m\|_{L^2([0,T];V)}^2 \leq \frac{k_1}{2\nu}, \]  
(3.33)
\[ \|u_m\|_{L^2([0,T];D(A))}^2 \leq \frac{k_1}{2\nu\alpha^2}, \]  
(3.34)
or
\[ \|v_m\|_{L^2([0,T];H)}^2 \leq \frac{k_1}{2\nu\alpha^2} \left( \lambda_1^{-1} + \alpha^2 \right)^2, \]  
(3.35)
and
\[ \|B_m\|_{L^2([0,T];V)}^2 \leq \frac{k_1}{2\eta}, \]  
(3.36)

(iii) From (3.17) we have for any \( \tau \in (0,T) \)
\[ \|u_m\|_{L^\infty([\tau,T];D(A))}^2 \leq \frac{k_2(\tau)}{\alpha^2} \text{ or } \|v_m\|_{L^\infty([\tau,T];H)}^2 \leq \frac{k_2(\tau)}{\alpha^2} \left( \lambda_1^{-1} + \alpha^2 \right)^2 \]
and
\[ \|B_m\|_{L^\infty([\tau,T];V)}^2 \leq k_2(\tau), \]
where \( k_2(\tau) \to \infty \) as \( \tau \to 0^+ \).

Now we establish uniform estimates, in \( m \), for \( \frac{du_m}{dt}, \ \frac{dv_m}{dt} \). Let us recall (3.2a). We have, by (3.35),
\[ \|Av_m\|_{L^2([0,T];D(A)')}^2 \leq \frac{k_1}{2\nu\alpha^2} \left( \lambda_1^{-1} + \alpha^2 \right)^2. \]
Also, by (2.15),
\[ \left\| P_m \tilde{B} (u_m, v_m) \right\|_{D(A)'} \leq c \lambda_1^{-1/4} \|u_m\| \|v_m\|, \]
hence, applying (3.31) and (3.35),
\[ \left\| P_m \tilde{B} (u_m, v_m) \right\|_{L^2([0,T];D(A)')}^2 \leq c \frac{k_1^2}{\alpha^4 \nu \lambda_1^{1/2}} \left( \lambda_1^{-1} + \alpha^2 \right)^2. \]
Additionally, by (2.13), we have
\[ \|P_mB(B_m, B_m)\|_{D(A)'} \leq c \left( \lambda_1 \right)^{-1/4} \|B_m\| \|B_m\|, \]
therefore, using (3.32) and (3.36), we obtain
\[ \|P_mB(B_m, B_m)\|_{L^2([0,T];D(A)')}^2 \leq c \frac{k_1^2}{\eta \lambda_1^{1/2}}. \]

Consequently, by (3.2a) and the above
\[ \left\| \frac{du_m}{dt} \right\|_{L^2([0,T];D(A)')}^2 \leq c \frac{k_1^2}{\alpha^4 \nu \lambda_1^{1/2}} + \frac{k_1}{2\alpha^2} + c \frac{k_1^2}{\eta \lambda_1^{1/2}} := K \]  
(3.37)
and, in particular,
\[ \left\| \frac{du_m}{dt} \right\|_{L^2([0,T];H)}^2 \leq \frac{K}{\alpha^4}. \]  
(3.38)

Now we establish uniform estimates, in \( m \), for \( \frac{dB_m}{dt} \). Let us recall (3.2b). We have, by (3.36),
\[ \|AB_m\|_{L^2([0,T];V')}^2 \leq \frac{k_1}{2\eta}. \]
Also, by (2.4),

\[ \|P_m B(u_m, B_m)\|_{V'} \leq c (\lambda_1)^{-1/4} \|u_m\| \|B_m\|, \]

Hence, by (3.31) and (3.36),

\[ \|P_m B(u_m, B_m)\|_{L^2([0,T]; V')}^2 \leq c \frac{k_1^2}{2\alpha^2 \eta \lambda_1^{1/2}}. \]

Similarly

\[ \|P_m B(B_m, u_m)\|_{V'} \leq c (\lambda_1)^{-1/4} \|B_m\| \|u_m\| \]

and

\[ \|P_m B(B_m, u_m)\|_{L^2([0,T]; V')}^2 \leq c \frac{k_1^2}{2\alpha^2 \eta \lambda_1^{1/2}}. \]

Hence, from the above and (3.2b), we have

\[ \left\| \frac{dB_m}{dt} \right\|_{L^2([0,T]; V')}^2 \leq c \frac{k_1^2}{2\alpha^2 \eta \lambda_1^{1/2}} + \frac{k_1}{2} := \bar{K}. \tag{3.39} \]

From (3.34) and (3.38), using Aubin’s Compactness Lemma (see, for example, [9, Lemma 8.4], [34] or [46]), we may assume that there exists a subsequence \( u_{m'} \) of \( u_m \) and \( u \in L^2([0,T]; D(A)) \cap C([0,T]; H) \) such that

\[
\begin{align*}
  u_{m'} &\to u \quad \text{weakly in } L^2([0,T]; D(A)), \tag{3.40a} \\
  u_{m'} &\to u \quad \text{strongly in } L^2([0,T]; V) \quad \text{and} \tag{3.40b} \\
  u_{m'} &\to u \quad \text{strongly in } C([0,T]; H), \tag{3.40c}
\end{align*}
\]

as \( m' \to \infty \). Moreover, \((d/dt) u_{m'} \to (d/dt) u\) weakly in \( L^2([0,T]; H) \). Or equivalently, by (3.35) and (3.37), there exists a subsequence \( v_{m'} \) of \( v_m \) such that

\[
\begin{align*}
  v_{m'} &\to v \quad \text{weakly in } L^2([0,T]; H), \tag{3.41a} \\
  v_{m'} &\to v \quad \text{strongly in } L^2([0,T]; V'), \tag{3.41b} \\
  v_{m'} &\to v \quad \text{strongly in } C([0,T]; D(A)'), \tag{3.41c}
\end{align*}
\]

\((d/dt) v_{m'} \to (d/dt) v\) weakly in \( L^2([0,T]; D(A)'), \) as \( m' \to \infty, \) where \( v = u + \alpha^2 A u \) is in \( L^2([0,T]; H) \cap C([0,T]; D(A)'), \) Also, by (3.36) and (3.39), there exists a subsequence \( B_{m'} \) of \( B_m \) and \( B \in L^2([0,T]; V) \cap C([0,T]; V') \) such that

\[
\begin{align*}
  B_{m'} &\to B \quad \text{weakly in } L^2([0,T]; V), \tag{3.42a} \\
  B_{m'} &\to B \quad \text{strongly in } L^2([0,T]; H), \tag{3.42b} \\
  B_{m'} &\to B \quad \text{strongly in } C([0,T]; V'), \tag{3.42c}
\end{align*}
\]

and \((d/dt) B_{m'} \to (d/dt) B\) weakly in \( L^2([0,T]; V'), \) as \( m' \to \infty. \)

Since \( v_{m'} \to v \) weakly in \( L^2([0,T]; H) \) and strongly in \( L^2([0,T]; V') \) and \( B_{m'} \to B \) weakly in \( L^2([0,T]; V) \) and strongly in \( L^2([0,T]; H) \), then there exists a set \( E \subset [0,T] \) of Lebesgue measure zero and a subsequence of \( v_{m'}, B_{m'} \), which we relabel \( v_m, B_m \) respectively, such that \( v_m(s) \to v(s) \) weakly in \( H \) and strongly in \( V' \) for every \( s \in [0,T] \setminus E, \) and \( B_m(s) \to B(s) \) weakly in \( V \) and strongly in \( H \) for every \( s \in [0,T] \setminus E. \)

Let \( w \in D(A), \xi \in V, \) then by taking the inner product of (3.2a) with \( w, \) and of (3.2b) with \( \xi \) and integrating over the interval \([t_0, t], \) \( t, t_0 \in [0,T], \) we have

\[
\begin{align*}
  (v_m(t), w) - (v_m(t_0), w) + \int_{t_0}^{t} \left( B(u_m(s), v_m(s)) + P_m w \right) ds &\tag{3.43a} \\
  + \nu \int_{t_0}^{t} \left( v_m(s), Aw \right) ds = \int_{t_0}^{t} \left( B(B_m(s), B_m(s)) + P_m w \right) ds, \\
  (B_m(t), \xi) - (B_m(t_0), \xi) + \int_{t_0}^{t} \left( B(u_m(s), B_m(s)) + P_m \xi \right) ds &\tag{3.43b} \\
  - \int_{t_0}^{t} \left( B(B_m(s), u_m(s)) + P_m \xi \right) ds + \eta \int_{t_0}^{t} \left( (B_m(s), \xi) \right) ds = 0.
\end{align*}
\]
First we consider (3.43a). Since \( v_m(s) \to v(s) \) weakly in \( H \), then for \( t, t_0 \in [0, T] \setminus E \)

\[
(v_m(t), w) - (v_m(t_0), w) \to (v(t), w) - (v(t_0), w), \quad \text{as } m \to \infty
\]

and since \( w \in D(A) \) we also have

\[
\lim_{m \to \infty} \int_{t_0}^{t} (v_m(s), Aw) \, ds = \int_{t_0}^{t} (v(s), Aw) \, ds.
\]

Now

\[
\lim_{m \to \infty} |P_m Aw - Aw| = \lim_{m \to \infty} \|P_m w - w\| = \lim_{m \to \infty} \|P_m w - w\| = 0. \tag{3.44}
\]

For the nonlinear terms we have

\[
\left| \int_{t_0}^{t} \left( \tilde{B}(u_m(s), v_m(s)), P_m w \right) - \left( \tilde{B}(u(s), v(s)), w \right)_{D(A)^\prime} \, ds \right|
\]

\[
\leq \left| \int_{t_0}^{t} \left( \tilde{B}(u_m(s), v_m(s)), P_m w - w \right)_{D(A)^\prime} \, ds \right|
\]

\[
+ \left| \int_{t_0}^{t} \left( \tilde{B}(u_m(s) - u(s), v_m(s)), w \right)_{D(A)^\prime} \, ds \right|
\]

\[
+ \left| \int_{t_0}^{t} \left( \tilde{B}(u(s), v_m(s) - v(s)), w \right)_{D(A)^\prime} \, ds \right|
\]

\[
=: I^{(1)}_m + I^{(2)}_m + I^{(3)}_m
\]

By (2.15)

\[
I^{(1)}_m \leq c(\lambda_1)^{-1/4} \int_{t_0}^{t} \|u_m(s)\| \|v_m(s)\| \|P_m Aw - Aw\| \, ds,
\]

using Cauchy-Schwarz inequality we obtain

\[
I^{(1)}_m \leq c(\lambda_1)^{-1/4} \|P_m Aw - Aw\| \|u_m\|_{L^2([0,T];V)} \|v_m\|_{L^2([0,T];H)} ,
\]

hence by (3.33), (3.35) and (3.44) \( \lim_{m \to \infty} I^{(1)}_m = 0 \).

Again, by (2.15),

\[
I^{(2)}_m \leq c(\lambda_1)^{-1/4} \int_{t_0}^{t} \|u_m(s) - u(s)\| \|v_m(s)\| \|Aw\| \, ds,
\]

and by Cauchy-Schwarz and (3.35),

\[
I^{(2)}_m \leq c(\lambda_1)^{-1/4} \|Aw\| \frac{k_1}{2\alpha^2} (\lambda_1^{-1} + \alpha^2) \left( \int_{0}^{T} \|u_m(s) - u(s)\|^2 \, ds \right)^{1/2},
\]

hence \( \lim_{m \to \infty} I^{(2)}_m = 0 \), since \( u_m \to u \) in \( L^2([0,T];V) \).

Finally, we show that \( \lim_{m \to \infty} I^{(3)}_m = 0 \). We define a linear functional for \( h \in L^2([0,T];H) \) by

\[
\phi(h) = \int_{t_0}^{t} \left( \tilde{B}(u(s), h), w \right)_{D(A)^\prime} \, ds,
\]

by (2.15) and Cauchy-Schwarz

\[
|\phi(h)| \leq c(\lambda_1)^{-1/4} \|Aw\| \|u(s)\|_{L^2([0,T];V)} \|h(s)\|_{L^2([0,T];H)}
\]

hence, due to (3.33), \( \phi \) is a bounded linear functional, and thus, since \( v_m \to v \) weakly in \( L^2([0,T];H) \),

\[
\lim_{m \to \infty} \phi(v_m(s) - v(s)) = 0.
\]
and hence \( \lim_{m \to \infty} I_m^{(3)} = 0 \). It remains to pass to the limit in the right hand side element of (3.43a).

\[
\int_{t_0}^{t} \left( B(B_m(s), B_m(s)) - (B(u(s), B(s)) \right), \langle B(B_m(s) - B(s), w \rangle_{V^*} \), ds \right| 
\leq \left| \int_{t_0}^{t} \langle B(B_m(s), B_m(s)) - B(u(s), B(s)) \rangle_{V^*} \), ds \right| 
+ \left| \int_{t_0}^{t} \langle B(B_m(s) - B(s), w \rangle_{V^*} \), ds \right| 
+ \left| \int_{t_0}^{t} \langle B(B(s), B_m(s) - B(s)) \rangle_{V^*} \), ds \right| 
= : J_m^{(1)} + J_m^{(2)} + J_m^{(3)}.
\]

Now, by (2.4) and Poincaré inequality

\[
J_m^{(1)} \leq c (\lambda_1)^{-1/4} \| P_m w - w \| \| B_m \|_{L^2([0,T];V)}^2,
\]

hence, by (3.44), \( \lim_{m \to \infty} J_m^{(1)} = 0 \).

By (2.4) and Poincaré inequality

\[
J_m^{(2)} \leq c (\lambda_1)^{-1/4} \int_{t_0}^{t} \| B_m(s) - B(s) \| \| B_m(s) \| \| w \| ds
\]

and, applying Cauchy-Schwarz and (3.36),

\[
J_m^{(2)} \leq c (\lambda_1)^{-1/4} \| w \|_{L^2 ([0,T],H)} \left( \int_{t_0}^{t} \| B_m(s) - B(s) \|^2 ds \right)^{1/2},
\]

hence, since \( B_m \to B \) weakly in \( L^2 ([0,T],V) \), we have \( \lim_{m \to \infty} J_m^{(2)} = 0 \) (similarly to the argument given for \( I_m^{(3)} \)).

Similarly we can show that \( \lim_{m \to \infty} J_m^{(3)} = 0 \).

It remains to pass to the limit in (3.43b). Note that

\[
\lim_{m \to \infty} |P_m \xi - \xi| = 0. \tag{3.45}
\]

We recall that \( B_m(s) \to B(s) \) weakly in \( V \) and strongly in \( H \) for every \( s \in [0,T] \setminus E \), hence the convergence for the linear terms is easy. For the nonlinear terms we have

\[
\int_{t_0}^{t} \left( B(u_m(s), B_m(s)) - (B(u(s), B(s)), ds \right| 
\leq \left| \int_{t_0}^{t} \langle B(u_m(s), B_m(s)) - B(u(s), B(s)) \rangle_{V^*} \), ds \right| 
+ \left| \int_{t_0}^{t} \langle B(u_m(s) - u(s), B_m(s)) \rangle_{V^*} \), ds \right| 
+ \left| \int_{t_0}^{t} \langle B(u(s), B_m(s) - B(s)) \rangle_{V^*} \), ds \right| 
= : S_m^{(1)} + S_m^{(2)} + S_m^{(3)}.
\]

Now, by (2.9) and Cauchy-Schwarz inequality

\[
S_m^{(1)} \leq c (\lambda_1)^{-1/4} \| P_m \xi - \xi \| \| u_m \|_{L^2 ([0,T];D(A))} \| B_m \|_{L^2 ([0,T],V)}^2,
\]

hence, by (3.34),(3.36) and (3.45), \( \lim_{m \to \infty} S_m^{(1)} = 0 \).

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Again, by (2.9) and Cauchy-Schwarz inequality
\[ S_{m}^{(2)} \leq c (\lambda_{1})^{-1/4} \| \xi \| B_{m} \| L^{2}(0,T;V) \| |u_{m}(s) - u(s)|^{2} \| L^{2}(0,T;D(A)) \]
hence, since \( u_{m} \to u \) weakly in \( L^{2}([0,T];D(A)) \), we have \( \lim_{m \to \infty} S_{m}^{(2)} = 0 \) (similarly to the case for \( J_{m}^{(3)} \). By similar arguments, using that \( B_{m} \to B \) weakly in \( L^{2}([0,T];V) \), we obtain also that \( \lim_{m \to \infty} S_{m}^{(3)} = 0 \).

For the term \( \int_{t_{0}}^{t} (B(B_{m}(s),u_{m}(s)),P_{m}\xi) \) we can perform the same estimates using (2.10) to bound operator \( B \).

Hence, we can pass to the limit in (3.43) and we obtain that for every \( t, t_{0} \in [0,T] \setminus E \)
\begin{align*}
(v(t),w) - (v(t_{0}),w) + \int_{t_{0}}^{t} \left< B(u(s),v(s)),w \right>_{D(A)'} \, ds + \nu \int_{t_{0}}^{t} \left< v(s),Aw \right> \, ds & = \int_{t_{0}}^{t} \left< B(B(s),w) \right>_{V}, \, ds, \\
(B(t),\xi) - (B(t_{0}),\xi) + \int_{t_{0}}^{t} (B(u(s),B(s)),\xi) \, ds - \int_{t_{0}}^{t} (B(B(s),u(s)),\xi) \, ds & = \eta \int_{t_{0}}^{t} (\xi,\xi) \, ds = 0.
\end{align*}

for every \( w \in D(A), \xi \in V \).

Now we show that \( v \in C([0,T];V') \) (or equivalently \( u \in C([0,T];V) \) and \( B \in C([0,T];H) \).

Notice that since \( \|v_{m}\|_{L^{\infty}([0,T];V')} \leq \frac{\nu}{\delta_{m}} (\lambda_{1}^{-1} + \alpha^{2})^{2} \) and \( v_{m} \to v \) strongly in \( L^{2}([0,T];V') \) then \( \|v\|_{L^{\infty}([0,T];V')} \leq \frac{\nu}{\delta_{m}} (\lambda_{1}^{-1} + \alpha^{2})^{2} \). Hence (3.46a) implies that \( v(t) \in C_{w}([0,T];V') \) because \( D(A) \) is dense in \( V \). Since, also, for a fixed \( t_{0} \), \( \|v(t)\|_{V'} \to \|v(t_{0})\|_{V'} \), as \( t \to t_{0} \), then we have \( v \in C([0,T];V') \), or equivalently \( u \in C([0,T];V) \).

Similarly, since \( \|B_{m}\|_{L^{\infty}([0,T];H)} \leq k_1 \) and \( B_{m} \to B \) strongly in \( L^{2}([0,T];H) \) and \( D(A) \) is dense in \( H \) and because of (3.46b) we have \( B \in C([0,T];H) \).

### 3.5 Uniqueness and continuous dependence of weak solutions on the initial data

Next, we show the continuous dependence of weak solutions on the initial data and, in particular, the uniqueness of weak solutions.

Let \( u, B \) and \( \bar{u}, \bar{B} \) be any two weak solutions of (2.17) on the interval \([0,T]\) with initial values \( u(0) = u^{in}, \ B(0) = B^{in}, \ \bar{u}(0) = \bar{u}^{in}, \ \bar{B}(0) = \bar{B}^{in} \). We denote \( v = u + \alpha^{2}Au, \ \bar{v} = \bar{u} + \alpha^{2}A\bar{u}, \ \delta u = u - \bar{u}, \ \delta v = v - \bar{v} \) and \( \delta B = B - \bar{B} \). Then (2.17) implies
\begin{align*}
\frac{d}{dt} \delta v + \nu A \delta v + \bar{B} (\delta u, v) + \bar{B} (\delta v, \delta v) = B (\delta B, B) + B (\bar{B}, \delta B), \quad (3.47) \\
\frac{d}{dt} \delta B + \eta A \delta B = -B (\delta u, B) - B (\delta \bar{u}, \delta B) + B (\delta u, u) + B (\bar{B}, \delta u), \quad (3.48) \\
\delta u(0) = \delta u^{in} = u^{in} - \bar{u}^{in}, \quad (3.49) \\
\delta B(0) = \delta B^{in} = B^{in} - \bar{B}^{in}. \quad (3.50)
\end{align*}

Since \( dw/dt \in L^{2}([0,T];D(A)') \), \( \delta u \in L^{2}([0,T];D(A)) \) and \( dB/dt \in L^{2}([0,T];V') \), \( B, \ \bar{B}, \ \delta B \in L^{2}([0,T];V) \) and due to the identities (2.6) and (2.8), we have for almost every \( t \in [0,T] \)
\begin{align*}
\left< \frac{d}{dt} \delta v, \delta u \right>_{D(A)'} + \nu \left( \| \delta u \|^{2} + \alpha^{2} |A \delta u|^{2} \right) + \left< \bar{B} (\delta \bar{u}, \delta v), \delta u \right>_{D(A)'} & = \left< B (\delta B, B), \delta u \right>_{D(A)'} + \left< B (\bar{B}, \delta B), \delta u \right>_{V'}, \\
\left< \frac{d}{dt} \delta B, \delta B \right>_{V'} + \eta \| \delta B \|^{2} & = - \left< B (\delta u, B), \delta B \right>_{V'} + \left< B (\delta B, u), \delta B \right>_{V'} + \left< B (\bar{B}, \delta u), \delta B \right>_{V'}. \quad (3.51)
\end{align*}
Notice that by theorem of interpolation by Lions and Magenes, see, e.g., [46, Chap. III, Lemma 1.2],
\[
\left\langle \frac{d}{dt}\delta v, \delta u \right\rangle_{D(A)'} = \frac{d}{dt} (|\delta u|^2 + \alpha^2\|\delta u\|^2)
\]
and
\[
\left\langle \frac{d}{dt}\delta B, \delta B \right\rangle_{V'} = \frac{d}{dt}|\delta B|^2,
\]
thus we have
\[
\frac{d}{dt} (|\delta u|^2 + \alpha^2\|\delta u\|^2) + \nu (|\delta u|^2 + \alpha^2|A\delta u|^2) + \left\langle \tilde{B}(\bar{u}, \delta v), \delta u \right\rangle_{D(A)'} = \left\langle B(\delta B, \delta u), \delta u \right\rangle_{D(A)'} + \left\langle \tilde{B}(\bar{u}, \delta v), \delta u \right\rangle_{D(A)'} + \left\langle B(\tilde{B}, \delta u), \delta B \right\rangle_{V'}, \tag{3.51a}
\]
\[
\frac{d}{dt}|\delta B|^2 + \eta|\delta B|^2 = -\left\langle B(\delta u, B), \delta B \right\rangle_{V'} + \left\langle B(\delta u, B), \delta B \right\rangle_{V'} + \left\langle B(\delta u, B), \delta B \right\rangle_{V'}, \tag{3.51b}
\]

By summation of (3.51a) and (3.51b) we obtain
\[
\frac{d}{dt} (|\delta u|^2 + \alpha^2\|\delta u\|^2 + |\delta B|^2) + \nu (|\delta u|^2 + \alpha^2|A\delta u|^2) + \eta|\delta B|^2
\]
\[
= -\left\langle \tilde{B}(\bar{u}, \delta v), \delta u \right\rangle_{D(A)'} + \left\langle \tilde{B}(\bar{u}, \delta v), \delta u \right\rangle_{D(A)'} + \left\langle B(\tilde{B}, \delta u), \delta B \right\rangle_{V'}, \tag{3.51b}
\]

By (2.16) we get
\[
\left\langle \tilde{B}(\bar{u}, \delta v), \delta u \right\rangle_{V'} \leq c \left( |\tilde{u}|^{1/2} |A\bar{u}|^{1/2} |\delta v| |\delta u| + |A\bar{u}| |\delta v| |\delta u|^{1/2} |\delta u|^{1/2} \right),
\]
and by applying Young’s inequality
\[
\left\langle \tilde{B}(\bar{u}, \delta v), \delta u \right\rangle_{V'} \leq \frac{c}{\eta \lambda_1^{1/2}} |A\bar{u}|^2 (|\delta u|^2 + \alpha^2\|\delta u\|^2) + \frac{\nu}{2} |\delta u|^2 + \frac{\nu^2}{4} \alpha^2 |A\delta u|^2.
\]

By (2.13) and Young’s inequality we have
\[
\left\langle \tilde{B}(\delta B, \delta u), \delta u \right\rangle_{D(A)'} \leq c|\delta B|^2\|B\|^2 + \frac{1}{\eta \alpha^2}\|\delta u\|^2 + \frac{\nu}{4} \alpha^2 |A\delta u|.
\]

Also, by (2.10) and Young’s inequality we obtain
\[
|B(\delta u, B)| \leq \frac{c}{\eta \lambda_1^{1/2}} |A\bar{u}|^2 |\delta B|^2 + \frac{\nu}{2} \|\delta u\|^2
\]

Summing up we have
\[
\frac{d}{dt} \left( |\delta u|^2 + \alpha^2\|\delta u\|^2 + |\delta B|^2 \right) + \nu \left( |\delta u|^2 + \alpha^2|A\delta u|^2 \right) + \frac{\eta}{2} |\delta B|^2
\]
\[
\leq \left( \frac{c}{\nu \lambda_1^{1/2}} |A\bar{u}|^2 + \frac{1}{\nu \alpha^2} |B|^2 + \frac{c}{\eta \lambda_1^{1/2}} |A\bar{u}|^2 \right) \left( |\delta u|^2 + \alpha^2\|\delta u\|^2 + |\delta B|^2 \right).
\]

We denote
\[
z(s) = \frac{c}{\nu \lambda_1^{1/2}} |A\bar{u}|^2 + \frac{1}{\nu \alpha^2} |B|^2 + \frac{c}{\eta \lambda_1^{1/2}} |A\bar{u}|^2
\]
and use Gronwall’s inequality to obtain
\[
|\delta u(t)|^2 + \alpha^2\|\delta u(t)\|^2 + |\delta B(t)|^2 \leq \left( |\delta u(0)|^2 + \alpha^2\|\delta u(0)\|^2 + |\delta B(0)|^2 \right) \exp \left( \int_0^t z(s) ds \right), \tag{3.52}
\]
since \(u, \bar{u} \in L^2([0, T]; D(A))\) and \(B \in L^2([0, T]; V)\) the integral \(\int_0^t z(s) ds\) is finite. Hence (3.52) implies the continuous dependence of the weak solutions of (2.17) on the initial data in any bounded interval of time \([0, T]\). In particular, the solutions are unique.
3.6 Strong solutions

**Theorem 3.3.** Let $T > 0$, $\mathbf{u}^m \in V$, $B^m \in H$. Then there exists a unique solution $\mathbf{u}, B$ of (2.17) on $[0, T]$ satisfying

$$\mathbf{u} \in L^\infty_{loc} \left( [0, T] ; D(A^{3/2}) \right) \cap L^2_{loc} \left( [0, T] ; D(A^2) \right) \cap C \left( [0, T] ; V \right) \cap L^2 \left( [0, T] ; D(A) \right) \quad (3.53)$$

and

$$B \in L^\infty_{loc} \left( [0, T] ; V \right) \cap L^2_{loc} \left( [0, T] ; D(A) \right) \cap C \left( [0, T] ; H \right) \cap L^2 \left( [0, T] ; V \right). \quad (3.54)$$

If $B^m \in V$ and $\mathbf{u}^m \in D(A)$ then the solution is the strong solution

$$\mathbf{u} \in C \left( [0, T] ; D(A) \right) \cap L^2 \left( [0, T] ; D(A^{3/2}) \right),$$

$$B \in C \left( [0, T] ; V \right) \cap L^2 \left( [0, T] ; D(A) \right).$$

If, additionally, $\mathbf{u}^m \in D(A^{3/2})$ then

$$\mathbf{u} \in C \left( [0, T] ; D(A^{3/2}) \right) \cap L^2 \left( [0, T] ; D(A^2) \right).$$

**Remark 3.4.** Following the techniques presented in [16] (see also [13]) we can show that for any $t > 0$ the solution is analytic in time with values in a Gevrey class of regularity of spatial analytic functions. As a result, we have an exponentially fast convergence in the wave number $m$, as $m \to \infty$, in a certain sense, of the Galerkin approximation to the unique strong solution of (2.17), see, for instance, [10, 27]. This Gevrey regularity result also implies the exponential decay of large wavenumber modes in the dissipation range of turbulent flows [11].

**Proof.** We use the Galerkin estimates derived in the previous subsections and similar ideas and compactness theorems in the corresponding spaces to converge to the strong solution. For (3.53) and (3.54) we need the estimates (3.28), (3.29), (3.6), (3.7) and (3.17), (3.18), (3.6), (3.7). For $B^m \in V$, $\mathbf{u}^m \in D(A)$ we use the estimate (3.16) and if $\mathbf{u}^m \in D(A^{3/2})$ we use (3.30). Also, since the strong solutions are weak, by uniqueness of weak solutions the strong solutions are unique. 

\[\square\]

4 Convergence to the solutions of MHD equations as $\alpha \to 0^+$

We emphasize again that our point of view is that the alpha model is to be considered as a regularizing numerical scheme. The next theorem shows that using the a priori estimates established previously, one can extract subsequences of the weak solutions of system (2.17), which converge, as $\alpha \to 0^+$, (in the appropriate sense defined in the theorem) to a Leray-Hopf weak solution of the three-dimensional MHD equations on any time interval $[0, T]$. For the definition and existence of weak solutions of the 3D MHD equations, see, for instance, [12] and [42]. The notion of a Leray-Hopf weak solution of MHD that satisfies the energy inequality (4.1) is inspired from a Leray-Hopf solution of NSE and formulated in the theorem. Also, if the initial data is smooth we prove that a subsequence of the strong solutions of the MHD-$\alpha$ equations converges to the unique strong solution of the 3D MHD on an interval $[0, T]$, $(u^m, B^m)$, which is the interval of existence of the strong solution.

**Theorem 4.1.** Let $T > 0$, $\mathbf{u}^m \in V$, $B^m \in H$ and denote by $\mathbf{u}_\alpha$, $B_\alpha$ and $v_\alpha = u_\alpha + \alpha^2 A u_\alpha$ the weak solution of (2.17) on $[0, T]$. Then there are subsequences $\mathbf{u}_{\alpha_j}$, $v_{\alpha_j}$, $B_{\alpha_j}$ and a pair of functions $v, B \in L^\infty \left( [0, T] ; H \right) \cap L^2 \left( [0, T] ; V \right)$ such that, as $\alpha_j \to 0^+$,

(i) $\mathbf{u}_{\alpha_j} \to v$ and $B_{\alpha_j} \to B$ weakly in $L^2 \left( [0, T] ; V \right)$ and strongly in $L^2 \left( [0, T] ; H \right)$,

(ii) $v_{\alpha_j} \to v$ weakly in $L^2 \left( [0, T] ; H \right)$ and strongly in $L^2 \left( [0, T] ; V' \right)$ and

(iii) $\mathbf{u}_{\alpha_j} (t) \to v (t)$ and $B_{\alpha_j} (t) \to B (t)$ weakly in $H$ and uniformly on $[0, T]$.

Furthermore, the pair $v, B$ is a Leray-Hopf weak solution of the MHD equations

$$\frac{dv}{dt} + \dot{B} (v, v) + \nu Av = B (B, B),$$

$$\frac{dB}{dt} + B (v, B) - B (B, v) + \eta AB = 0.$$
with initial data \(v(0) = u^{in}, B(0) = B^{in}\), which satisfies the energy inequality

\[
|v(t)|^2 + |B(t)|^2 + 2 \int_{t_0}^t (\nu \|v(s)\|^2 + \eta \|B(s)\|^2) \, ds \leq |v(t_0)|^2 + |B(t_0)|^2
\]

(4.1)

for almost every \(t_0, 0 \leq t_0 \leq T\) and all \(t \in [t_0, T]\).

**Proof.** From estimates (3.6) and (3.7), by passing to the limit (using the proof of Theorem 3.1), we have that the solution of (2.17) satisfies

\[
|u_\alpha(t)|^2 + \alpha^2 \|u_\alpha(t)\|^2 + |B_\alpha(t)|^2 \leq k_1
\]

and

\[
2 \int_0^T (\nu \|u_\alpha(t)\|^2 + \alpha^2 \|A u_\alpha(t)\|^2 + \eta \|B_\alpha(t)\|^2) \, dt \leq k_1,
\]

notice that since \(\alpha \to 0^+\) we can assume that \(0 < \alpha \leq L\); consequently, we can bound the right hand side by \(k_1 := |u^{in}|^2 + L^2 \|u^{in}\|^2 + |B^{in}|^2\), which is independent of \(\alpha\), therefore we can extract subsequences \(u_{\alpha_j}, v_{\alpha_j}, B_{\alpha_j}\), such that

\[
u \alpha \to u \quad \text{weakly in} \quad L^2([0,T] ; V),
\]

\[
v_{\alpha_j} \to v \quad \text{weakly in} \quad L^2([0,T] ; H) \quad \text{and}
\]

\[
B_{\alpha_j} \to B \quad \text{weakly in} \quad L^2([0,T] ; V),
\]

as \(\alpha_j \to 0^+\).

Now we establish uniform estimates, independent of \(\alpha\), for \(dB_\alpha/dt\) and \(d u_\alpha/dt\). From (2.17b) we have

\[
\left\| A^{-1} \frac{d B_\alpha}{dt} \right\| \leq \left\| A^{-1} B(u_\alpha, B_\alpha) \right\| + \left\| A^{-1} B(B_\alpha, u_\alpha) \right\| + \eta \|B_\alpha\|,
\]

notice that by (2.13)

\[
|A^{-1} B(u_\alpha, B_\alpha)| \leq c \lambda_1^{-1/4} \|u_\alpha\| \|B_\alpha\|,
\]

hence

\[
\|B(u_\alpha, B_\alpha)\|_{L^2([0,T]; D(A)' \setminus D(A))} \leq c \lambda_1^{-1/2} \int_0^T \|u_\alpha(t)\|^2 \|B_\alpha(t)\|^2 \, dt
\]

\[
\leq c \lambda_1^{-1/2} k_1^2 \eta^{-1}
\]

and similarly

\[
\|B(B_\alpha, u_\alpha)\|_{L^2([0,T]; D(A)' \setminus D(A))} \leq c \lambda_1^{-1/2} k_1^2 \eta^{-1}.
\]

Hence

\[
\left\| \frac{d B_\alpha}{dt} \right\|_{L^2([0,T]; D(A)')} \leq K,
\]

where \(K\) is independent of \(\alpha\).

From (2.17a) we have

\[
\left\| A^{-1} \frac{d u_\alpha}{dt} \right\| \leq \left\| A^{-1} (I + \alpha^2 A)^{-1} \tilde{B}(u,v) \right\| + \nu \|u_\alpha\| + \left\| A^{-1} (I + \alpha^2 A)^{-1} B(B) \right\|,
\]

and using (2.15)

\[
|A^{-1} (I + \alpha^2 A)^{-1} \tilde{B}(u_\alpha, v_\alpha)| \leq |A^{-1} \tilde{B}(u_\alpha, v_\alpha)|
\]

\[
\leq c \lambda_1^{-1/4} \|u_\alpha\| \|v_\alpha\|
\]

\[
\leq c \lambda_1^{-1/4} \|u_\alpha\| (\|u_\alpha\| + \alpha^2 \|A u_\alpha\|),
\]

thus

\[
|A^{-1} (I + \alpha^2 A)^{-1} \tilde{B}(u_\alpha, v_\alpha)|^2 \leq 2c \lambda_1^{-1/2} k_1 (\|u_\alpha\|^2 + \alpha^2 \|A u_\alpha\|^2),
\]

20
and
\[
\int_0^T |A^{-1}(I + \alpha^2 A)^{-1} \tilde{B}(u_\alpha(t), v_\alpha(t))|^2 dt \leq c \lambda_1^{-1/2} k_1^2 \nu^{-1}.
\]
Also by (2.13)
\[
\|B(B_\alpha, B_\alpha)\|_{L^2([0,T]; D(A)'')} \leq c \lambda_1^{-1/2} k_1^2 \eta^{-1}.
\]
As a result we have
\[
\|v_\alpha - u_\alpha\|_{L^2([0,T];V')} = \alpha_j^2 \int_0^T \|u_{\alpha_j}(t)\| dt \leq \alpha_j^2 \nu^{-1} k_1,
\]
we obtain that \(v_\alpha \to u\) in \(L^2([0,T]; V')\), as \(\alpha_j \to 0^+\); and hence also that \(u(t) = v(t)\) almost everywhere on \([0,T]\).

Now, following the lines of the proof of Theorem 3.1, we can extract further subsequences (which we relabel by \(u_{\alpha_j}, v_{\alpha_j}\) and \(B_{\alpha_j}\)) and show that as \(\alpha_j \to 0^+\),
\[
\tilde{B}(u_{\alpha_j}, v_{\alpha_j}) \to \tilde{B}(v, v) = B(v, v)
\]
weakly in \(L^1([0,T]; D(A)'')\), and
\[
B(B_{\alpha_j}, B_{\alpha_j}) \to B(B, B), B(u_{\alpha_j}, B_{\alpha_j}) \to B(v, B), B(B_{\alpha_j}, u_{\alpha_j}) \to B(B, v)
\]
weakly in \(L^1([0,T]; V')\). Hence, we can pass to the limit (in the interpretation given by (2.19)) in
\[
\left\langle \frac{d}{dt} v_{\alpha_j}, w \right\rangle_{D(A)'} + \left\langle \tilde{B}(u_{\alpha_j}, v_{\alpha_j}), w \right\rangle_{D(A)'} + \nu(V, A w) = \left\langle B(B_{\alpha_j}, B_{\alpha_j}), w \right\rangle_{V'},
\]
\[
\left\langle \frac{d}{dt} B_{\alpha_j}, \xi \right\rangle_{V'} + (B(u_{\alpha_j}, B_{\alpha_j}), \xi) - (B(B_{\alpha_j}, u_{\alpha_j}), \xi) + \eta(V, (B, \xi)) = 0,
\]
\(w \in D(A), \xi \in V\) and we obtain that
\[
\left\langle \frac{d}{dt} v, w \right\rangle_{D(A)'} + (B(v, v), w)_{D(A)'} + \nu(V, (v, w)) = \left\langle B(B, B), w \right\rangle_{V'},
\]
\[
\left\langle \frac{d}{dt} B, \xi \right\rangle_{V'} + (B(v, B), \xi) - (B(B, v), \xi) + \eta(V, (B, \xi)) = 0,
\]
for every \(w \in D(A), \xi \in V\) and for almost every \(t \in [0,T]\).

Now, since \(v \in L^2([0,T]; V)\), one can show that \(B(v, v) \in L^1([0,T]; V')\) and then also that \((d/dt) v \in L^1([0,T]; V')\), and since \(w \in D(A)\), which is dense in \(V\), we obtain the weak formulation of the MHD equations
\[
\left\langle \frac{d}{dt} v, w \right\rangle_{V'} + (B(v, v), w)_{V'} + \nu(V, (v, w)) = \left\langle B(B, B), w \right\rangle_{V'},
\]
\[
\left\langle \frac{d}{dt} B, \xi \right\rangle_{V'} + (B(v, B), \xi) - (B(B, v), \xi) + \eta(V, (B, \xi)) = 0,
\]
for every \(w, \xi \in V\) and for almost every \(t \in [0,T]\).

We notice, that every weak solution of (2.17) satisfies the energy equality (3.1) and hence the energy inequality (4.1) follows by passing to the lim inf as \(\alpha \to 0^+\), using the fact that if \(x_\alpha \to x\) weakly in a Hilbert space \(X\), then \(\|x\| \leq \lim \inf \|x_\alpha\|\).  

Theorem 4.2. Let $T > 0$, $u^m \in D(A)$, $B^m \in V$ and denote by $u_\alpha, B_\alpha$ and $v_\alpha = u_\alpha + \alpha^2 Au_\alpha$ the strong solution of (2.17) on $[0, T]$. Then there exist $T_*, T_0, \nu, \eta, B^m, T_0)$, $0 < T_0 < T$, subsequences $u_{\alpha_j}, v_{\alpha_j}, B_{\alpha_j}$ and a pair of functions $v, B \in L^\infty(0, T_0; V) \cap L^2(0, T_0; D(A))$ such that, as $\alpha_j \to 0^+$,

(i) $u_{\alpha_j} \to v$ and $B_{\alpha_j} \to B$ weakly in $L^2(0, T_0; D(A))$ and strongly in $L^2(0, T_0; V)$,

(ii) $v_{\alpha_j} \to v$ weakly in $L^2(0, T_0; V)$ and strongly in $L^2(0, T_0; H)$ and

(iii) $u_{\alpha_j}(t) \to v(t)$ and $B_{\alpha_j}(t) \to B(t)$ weakly in $V$ and uniformly on $[0, T_0]$.

Furthermore, the pair $v, B$ is the unique strong solution of the 3D MHD equations on $[0, T_0]$ with initial data $v(0) = u^m, B(0) = B^m$. The strong solution of the 3D MHD equations satisfies the energy equality

$$|v(t)|^2 + |B(t)|^2 + 2 \int_{t_0}^t (\nu|v(s)|^2 + \eta|B(s)|^2) \, ds = |v(t_0)|^2 + |B(t_0)|^2, \quad 0 \leq t_0 \leq t \leq T_0.$$

Proof. To prove the theorem we need to show that there exists $T_*$ such that we have a uniform (independent of $\alpha$) bound on

$$\|u_\alpha(t)\|^2 + \alpha^2|Au_\alpha(t)|^2 + \|B_\alpha(t)\|^2 \leq c$$

and

$$\int_0^{T_*} (\nu|Au_\alpha(t)|^2 + \alpha^2|A^{3/2}u_\alpha(t)|^2 + \eta|AB_\alpha(t)|^2) \, dt \leq c$$

in $[0, T_0]$. Then we can continue similarly to the proof of the previous theorem, appropriately smoothing the data and replacing $T$ by $T_*$. Next we derive the formal estimates on (4.2) and (4.3) that can be proved rigorously using the Galerkin approximation scheme and then passing to the limit using the proof of Theorem 3.1.

Let us recall (3.13). By (2.9) and several applications of Young’s inequality we bound

$$\left| \left( B(u_\alpha, v_\alpha), Au_\alpha \right) \right| \leq c\|u_\alpha\|^{1/2} |Au_\alpha|^{1/2} (\|u_\alpha\| + \alpha^2|A^{3/2}u_\alpha|) |Au_\alpha|$$

$$\leq c\nu^{1/2} |u_\alpha|^{1/2} + \nu^{-1} \alpha^6 |Au_\alpha|^6 + \frac{\nu}{4} |Au_\alpha|^2 + \frac{\nu}{2} \alpha^{1/2} |A^{3/2}u_\alpha|^2.$$

By (2.9)

$$|(B(B_\alpha, B_\alpha), B_\alpha)| \leq c\|B_\alpha\|^{1/2} |AB_\alpha|^{1/2} |B_\alpha| |Au_\alpha|$$

$$\leq c\nu^{1/2} |B_\alpha|^{1/2} + \nu^{-1} |AB_\alpha|^6 + \frac{\eta}{4} |B_\alpha|^2 + \frac{\eta}{2} |AB_\alpha|^2.$$

By (2.9) we also have

$$|(B(B_\alpha, u_\alpha), AB_\alpha)| \leq c\|B_\alpha\|^{1/2} |u_\alpha| |AB_\alpha|^{3/2}$$

$$\leq c\nu^{-1} |B_\alpha|^{3/2} + \nu^{-1} |AB_\alpha|^6 + \frac{\eta}{8} |AB_\alpha|^2$$

and by (2.10)

$$|(B(u_\alpha, B_\alpha), AB_\alpha)| \leq c\|B_\alpha\|^{1/2} |u_\alpha| |AB_\alpha|^{3/2}.$$

Hence by (3.13) and the above estimates we have

$$\frac{d}{dt} (\|u_\alpha\|^2 + \alpha^2|Au_\alpha|^2 + |B_\alpha|^2) + \nu (|Au_\alpha|^2 + \alpha^2|A^{3/2}u_\alpha|^2) + \eta |AB_\alpha|^2 \leq c\mu^{-3} (\|u_\alpha\|^6 + \alpha^6 |Au_\alpha|^6 + |B_\alpha|^6)$$

Denote

$$y = \|u_\alpha\|^2 + \alpha^2|Au_\alpha|^2 + |B_\alpha|^2.$$

Now, if $y(0) = 0$, that is $u^m = B^m = 0$, then the solution is steady $u_\alpha(t) \equiv 0$, $v_\alpha(t) \equiv 0$, $B_\alpha(t) \equiv 0$ and $v(t) \equiv 0$, $B(t) \equiv 0$ exists for all $t \geq 0$. Otherwise, from (4.4) we have

$$\frac{d}{dt} y \leq c\mu^{-3} y^3.$$
and thus \( y(t) \leq 2y(0) \)

for \( 0 \leq t \leq \frac{3}{8}c^3\mu^3y(0)^{-2} \). We conclude that

\[
\|u_\alpha(t)\|^2 + \alpha^2|Au_\alpha(t)|^2 + \|B_\alpha(t)\|^2 \leq 2\left(\|u^{in}\|^2 + \alpha^2|Au^{in}|^2 + \|B^{in}\|^2\right)
\]

for \( 0 \leq t \leq T_* := \min\left(T, \frac{3}{8}c^3\mu^3y(0)^{-2}\right) \). Also, by integrating (4.4) over \((0, T_*)\), we obtain

\[
\int_0^{T_*} \left(\nu(|Au_\alpha(t)|^2 + \alpha^2 |A^{3/2}u_\alpha(t)|^2) + \eta|AB_\alpha(t)|^2\right) dt \\
\leq \|u^{in}\|^2 + \alpha^2|Au^{in}|^2 + \|B^{in}\|^2 + \frac{c\mu^{-3}}{T_*} \left(\|u^{in}\|^2 + \alpha^2|Au^{in}|^2 + \|B^{in}\|^2\right)^3.
\]

Assuming that \( 0 < \alpha \leq L \), the bounds are independent of \( \alpha \).

\[
\square \frac{\mathcal{C}}{\mathcal{C}}
\]

5 Discussion

We proved the well-posedness of the three-dimensional MHD-\( \alpha \) model (1.6) in the periodic boundary conditions. This model modifies the nonlinearity of the MHD equations (1.1) without enhancing dissipation. We showed that the model has a unique global weak (or strong, for smooth initial data) solution. Also, there is a subsequence of weak solutions of the MHD-\( \alpha \) equations that converge, as \( \alpha \to 0^+ \), (in the appropriate sense) to a Leray-Hopf weak solution (which satisfies the energy inequality (4.1)) of the MHD equations (1.1) on any time interval \([0, T]\). Also, if the initial data is smooth, a subsequence of solutions converges for a short interval of time, to the unique strong solution of the MHD equations on this interval. These properties are essential for the \( \alpha \) models to be regarded as regularizing numerical schemes. In a follow up paper, we intend to do the error estimates in which we will investigate the error in terms of \( m \) and \( \alpha \). Namely, the distance between the solution of the Galerkin MHD-\( \alpha \) model to that of the exact strong solution of the MHD equations, for smooth initial data.

There are many different \( \alpha \) models. For example, the global well-posedness can be shown for the 3D Modified-Leray-\( \alpha \)-MHD model (1.9). However, at the moment we are unable to find a conserved quantity in the ideal version of (1.9), which can be identified with a cross helicity, contrary to the MHD-\( \alpha \) model (1.6), where there exist the ideal invariants that could be identified with the three invariants of the original 3D MHD equations.

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