MAXIMAL ESTIMATES FOR THE BILINEAR RIESZ MEANS ON HEISENBERG GROUPS

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Abstract. In this article, we investigate the maximal bilinear Riesz means $S_\alpha^*$ associated to the sublaplacian on the Heisenberg group. We prove that the operator $S_\alpha^*$ is bounded from $L^{p_1} \times L^{p_2}$ into $L^p$ for $2 \leq p_1, p_2 \leq \infty$ and $1/p = 1/p_1 + 1/p_2$ when $\alpha$ is large than a suitable smoothness index $\alpha(p_1, p_2)$. For obtaining a lower index $\alpha(p_1, p_2)$, we define two important auxiliary operators and investigate their $L^p$ estimates, which play a key role in our proof.

1. Introduction

A classical problem in Fourier analysis is to make precise the sense in which the Fourier inversion formula

$$f(x) = \int_{\mathbb{R}^n} \hat{f}(\xi)e^{2\pi ix \cdot \xi} d\xi$$

holds when $f$ is a function on $\mathbb{R}^n$. A natural way of formulating this identity is in terms of a summability method. For instance, one may consider the convergence of the Bochner-Riesz means defined by

$$B_R^\delta(f)(x) = \int_{|\xi|<R} \hat{f}(\xi) \left(1 - \frac{|\xi|^2}{R^2}\right)^\delta e^{2\pi ix \cdot \xi} d\xi$$

as $R \to \infty$ for some suitable $\delta$. The almost everywhere convergence of the Bochner-Riesz is related to the maximal operator $B_R^\delta = \sup_{R > 0} |B_R^\delta f|$. When $n = 1$, Hunt showed that if $\delta = 0$ and $f \in L^p(\mathbb{R})$, $1 < p < \infty$, $B_R^\delta f$ converges to $f$ almost everywhere as $R \to \infty$. When $n \geq 2$ and $p \geq 2$, Carbery, Rubio de Francia and Vega \cite{11} showed that for all $f \in L^p(\mathbb{R}^n)$ with $2 \leq p \leq 2n/(n - 1 - 2\delta)$, $B_R^\delta f$ converges to $f$ almost everywhere as $R \to \infty$ by using the weighted $L^2$-estimates of the maximal operator $B_R^\delta = \sup_{R > 0} |B_R^\delta|$. When $n = 2$ and $1 < p < 2$, Tao \cite{10} proved that if $\delta > \max\left\{\frac{3}{4p} - \frac{2}{3}, \frac{7}{6p} - \frac{2}{3}\right\}$, $B_R^\delta$ is bounded from $L^p$ into $L^{p, \infty}$, i.e. $B_R^\delta f$ converges to $f$ almost everywhere as $R \to \infty$.

The bilinear Bochner-Riesz means is defined by

$$B_R^\delta(f, g)(x) = \int_{|\xi|<R} \hat{f}(\xi)\hat{g}(\eta) \left(1 - \frac{|\xi|^2 + |\eta|^2}{R^2}\right)^\delta e^{2\pi ix \cdot (\xi + \eta)} d\xi d\eta.$$

Its almost everywhere convergence is depended on the $L^{p_1} \times L^{p_2} \to L^p$ boundedness of the maximal operator $B_R^\alpha = \sup_{R > 0} |B_R^\alpha|$. Grafakos, He and Honzík \cite{5} showed that $B_R^\alpha$ is bounded from $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$ if $\alpha > \frac{2n+3}{4}$. Jeong and Lee \cite{4} gave a comprehensive study on this problem when $n \geq 2$ and $2 \leq p_1, p_2 \leq \infty$, $1/p = 1/p_1 + 1/p_2$. Inspired by their work, we

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shall investigated the $L^{p_1} \times L^{p_2} \to L^p$ boundedness of the maximal bilinear Riesz means on the Heisenberg group.

Strichartz [8, 9] developed the harmonic analysis on the Heisenberg group as the spectral theory of the sublaplacian. One may define the Riesz means in terms of the spectral decomposition of the sublaplacian. Let

$$\mathcal{L} f = \int_0^\infty \lambda P_\lambda f \, d\mu(\lambda)$$

be the spectral decomposition of the sublaplacian $\mathcal{L}$. The Riesz means associated to the sublaplacian $\mathcal{L}$ is defined by

$$S^\delta_\alpha f = \int_0^\infty (1 - r\lambda)^\delta\, P_\lambda f \, d\mu(\lambda).$$

Gorges and Müller [2] investigate the almost everywhere convergence of the Riesz means and obtained a similar result to that in [1]. They showed that $S^\delta_\alpha f \to f$ as $r \to 0$ for $\delta > 0$ and $f \in L^p(\mathbb{H}^n)$ provided that $\frac{Q-1}{Q} \left( \frac{\delta}{p} - \frac{\alpha}{p-1} \right) < \frac{1}{p} \leq \frac{1}{2}$.

The bilinear Riesz means associated to the sublaplacian $\mathcal{L}$ on the Heisenberg is defined by

$$S^\alpha_\delta(f, g) = \int_0^\infty \int_0^\infty (1 - r(\lambda_1 + \lambda_2))^{\alpha}\, P_{\lambda_1} f \, P_{\lambda_2} g \, d\mu(\lambda_1) \, d\mu(\lambda_2).$$

The corresponding maximal operator is denoted by $S^\alpha_\delta = \sup_{r>0} |S^\alpha_\delta|$. As same as the Euclidean case, we hope to obtain the smooth indices $\alpha(p_1, p_2)$ as low as possible so that $S^\alpha_\delta$ is bounded from $L^{p_1}(\mathbb{H}^n) \times L^{p_2}(\mathbb{H}^n)$ into $L^p(\mathbb{H}^n)$ when $\alpha > \alpha(p_1, p_2)$.

2. Preliminaries

First we recall some basic facts about the Heisenberg group. These facts are familiar and easy to find in many references. Let $\mathbb{H}^n$ denote the Heisenberg group whose underlying manifold is $\mathbb{C}^n \times \mathbb{R}$ and the group law is given by

$$(z, t)(w, s) = (z + w, t + s + \frac{1}{2} \text{Im}(z \cdot \overline{w})).$$

The Haar measure on $\mathbb{H}^n$ coincides with the Lebesgue measure on $\mathbb{C}^n \times \mathbb{R}$. A homogeneous structure on $\mathbb{H}^n$ is given by the non-isotropic dilations $\delta_r(z, t) = (rz, r^2 t)$. We define a homogeneous norm on $\mathbb{H}^n$ by

$$|x| = \left( \frac{1}{16} |z|^4 + t^2 \right)^{\frac{1}{4}}, \quad x = (z, t) \in \mathbb{H}^n.$$ 

This norm satisfies the triangle inequality and leads to a left-invariant distance $d(x, y) = |x^{-1}y|$. The ball of radius $r$ centered at $x$ is

$$B(x, r) = \{ y \in \mathbb{H}^n : |x^{-1}y| < r \}.$$ 

The Haar measure $dx$ satisfies $d\delta_r(x) = r^Q dx$ where $Q = 2n + 2$ is the homogeneous dimension of $\mathbb{H}^n$. If $f$ and $g$ are functions on $\mathbb{H}^n$, their convolution is defined by

$$(f * g)(x) = \int_{\mathbb{H}^n} f(xy^{-1}) \, g(y) \, dy, \quad x, y \in \mathbb{H}^n.$$ 

For each $\lambda \in \mathbb{R}^*$ and $f \in \mathcal{S}(\mathbb{H}^n)$, the inverse Fourier transform of $f$ in variable $t$ is defined by

$$f^\lambda(z) = \int_{-\infty}^{\infty} e^{i\lambda t} f(z, t) \, dt.$$
An easy calculation shows that
\[(f * g)^\lambda(z) = \int_{\mathbb{C}^n} f^\lambda(z - \omega) g^\lambda(\omega) e^{\frac{i}{2} \lambda \text{Im}(z \omega)} d\omega, \quad z, \omega \in \mathbb{C}^n.\]
Thus, we are led to the convolution of the form
\[f *_\lambda g = \int_{\mathbb{C}^n} f(z - \omega) g(\omega) e^{\frac{i}{2} \lambda \text{Im}(z \omega)} d\omega,\]
which are called the \(\lambda\)-twisted convolution.

The sublaplacian \(L\) is defined by
\[L = - \sum_{j=1}^{n} (X_j^2 + Y_j^2)\]
where
\[X_j = \frac{\partial}{\partial x_j} + \frac{1}{2} y_j \frac{\partial}{\partial t}, \quad j = 1, 2, \ldots, n,\]
\[Y_j = \frac{\partial}{\partial y_j} - \frac{1}{2} x_j \frac{\partial}{\partial t}, \quad j = 1, 2, \ldots, n,\]
are left invariant vector fields on \(\mathbb{H}^n\). Up to a constant multiple, \(L\) is the unique left invariant, rotation invariant differential operator that is homogeneous of degree two. Therefore, it is regarded as the counterpart of the Laplacian on \(\mathbb{R}^n\). The sublaplacian \(L\) is a positive and essentially self-adjoint operator. In the following, we state the spectral decomposition of \(L\) (cf. [11]).

Let \(\varphi_k\) be the Laguerre functions on \(\mathbb{C}^n\) given by
\[\varphi_k(z) = L_{n-1}^k \left( \frac{1}{2} |z|^2 \right) e^{-\frac{1}{4} |z|^2},\]
where \(L_{n-1}^k\) are the Laguerre polynomials of type \(n-1\) defined on \(\mathbb{R}\) by
\[L_{n-1}^k(t) e^{-t} t^{n-1} = \frac{1}{k!} \left( \frac{d}{dt} \right)^k (e^{-t} t^{k+n-1}).\]
Define functions
\[e_k^\lambda(z, t) = e^{-i \lambda t} \varphi_k^\lambda(z) = e^{-i \lambda t} \varphi_k(\sqrt{|\lambda|} z), \quad \lambda \in \mathbb{R}^n.\]
For \(f \in L^2(\mathbb{H}^n)\), we have the expansion
\[(2.1) \quad f(z, t) = \sum_{k=0}^{\infty} \int_{-\infty}^{\infty} f * e_k^\lambda(z, t) d\mu(\lambda)\]
where \(d\mu(\lambda) = (2\pi)^{-n-1} |\lambda|^n d\lambda\) is the Plancherel measure for \(\mathbb{H}^n\). Each \(f * e_k^\lambda\) is the eigenfunction of \(L\) with eigenvalue \((2k + n) |\lambda|\). We also have the Plancherel formula
\[(2.2) \quad \|f\|_2^2 = (2\pi)^{-2n-1} \sum_{k=0}^{\infty} \int_{-\infty}^{\infty} \int_{\mathbb{C}^n} \left| f^\lambda *_\lambda \varphi_k^\lambda(z) \right|^2 \lambda^{2n} dz d\lambda.\]
Defining
\[\overline{e_k^\lambda(z, t)} = \overline{e_k^{2n+\lambda}(z, t)},\]
we can rewrite the decomposition (2.1) as
\[ f(z, t) = \int_{-\infty}^{\infty} \sum_{k=0}^{\infty} (2k + n)^{-n-1} f \ast \tilde{e}_{k}^{\lambda}(z, t) \, d\mu(\lambda). \]

Let
\[ P_{\lambda} f(z, t) = \sum_{k=0}^{\infty} (2k + n)^{-n-1} f \ast (\tilde{e}_{k}^{\lambda} + \tilde{e}_{k}^{-\lambda})(z, t). \]
Then (2.1) can be written as
\[ f(z, t) = \int_{0}^{\infty} P_{\lambda} f(z, t) \, d\mu(\lambda). \]

It is clear that \( P_{\lambda} f \) is an eigenfunction of the \( \mathcal{L} \) with eigenvalue \( \lambda \) and we have the spectral decomposition
\[ \mathcal{L} f = \int_{0}^{\infty} \lambda P_{\lambda} f \, d\mu(\lambda). \]

Define the bilinear Riesz means associated to the sublaplacian \( \mathcal{L} \) for \( f, g \in \mathcal{S}(\mathbb{H}^n) \) by
\[ S_{\alpha}^\varphi(f, g) = \int_{0}^{\infty} \int_{0}^{\infty} (1 - r \lambda_1^2 - \lambda_2^2) P_{\lambda_1} f P_{\lambda_2} g \, d\mu(\lambda_1) \, d\mu(\lambda_2). \]
The corresponding maximal operator is defined by
\[ S_{\alpha}^\varphi(f, g)(x) = \sup_{r > 0} |S_{\alpha}^\varphi(f, g)(x)|. \]

3. \( L^p \)-estimate for auxiliary multiplier operator

In this Section, we shall define two auxiliary operators and investigate their \( L^p(l^\infty) \) estimates, which play a key role in the proof of our main Theorem.

Let \( I = [-1, 1] \) and consider a class of smooth function
\[ C^N(I) = \left\{ \varphi : \text{supp} \varphi \subset I, \| \varphi \|_{C^N(\mathbb{R})} = \max_{0 \leq n \leq N} \left\| \frac{d^n}{dt^n} \varphi \right\|_{L^\infty(\mathbb{R})} \leq 1 \right\}. \]
For \( \varphi \in C^N(I) \) and \( \rho, \delta, r > 0 \), we define the multiplier operator
\[ F_{\rho, \delta, r}^\varphi f(x) = \int_{0}^{\infty} \varphi \left( \frac{\rho - r |\lambda|}{\delta} \right) P_{\lambda} f \, d\mu(\lambda), \quad f \in \mathcal{S}(\mathbb{H}^n), \]
and define its corresponding operator for \( k \in \mathbb{Z} \) by
\[ D_{\rho, \delta, k}^\varphi f(x) = \left( \sum_{\rho \in \mathbb{Z} \cap [0, 2]} \int_{1}^{2} \left| F_{\rho, \delta, 2k, r}^\varphi f(x) \right|^2 \, dr \right)^{1/2}, \quad f \in \mathcal{S}(\mathbb{H}^n). \]
Write \( F_{\rho, \delta}^\varphi = F_{\rho, \delta, 1}^\varphi \) for simplicity. Since that \( P_{\lambda} \) is a convolution operator, then
\[ F_{\rho, \delta, r}^\varphi f(x) = \int_{\mathbb{H}^n} f(x \omega^{-1}) K_{\rho, \delta, r}^\varphi(\omega) \, d\omega \]
where the kernel \( K_{\rho, \delta, r}^\varphi \) is given by
\[ K_{\rho, \delta, r}^\varphi(\omega) = \sum_{k=0}^{\infty} (2k + n)^{-n-1} \int_{\mathbb{R}} \varphi \left( \frac{\rho - r |\lambda|}{\delta} \right) \tilde{e}_{k}^{\lambda}(\omega) \, d\mu(\lambda). \]
Notice that for any $t > 0$,
\[ K_{\rho,\delta,t}^\varphi(\omega) = t^{\frac{\varphi}{2}} K_{\rho,\delta,r}^\varphi\left(\delta \sqrt{t} \omega\right). \]

It is easy to verify that
\begin{equation}
F_{t \rho,\delta,r}^\varphi f(x) = F_{\rho,\delta,r}^\varphi f(\frac{x}{\sqrt{t}}) = F_{\rho,\delta,r}^\varphi f\left(\frac{x}{\sqrt{t}}\right).
\end{equation}

where $f_s = f(s \cdot)$ for any $s \in \mathbb{R}$, $s \neq 0$. Especially, if $r = 1$ and $t = 1/r$, we have
\begin{equation}
K_{\rho,\delta,1}^\varphi(\omega) = \left(\frac{1}{r}\right)^{\frac{\varphi}{2}} K_{\rho,\delta,r}^\varphi\left(\frac{1}{\sqrt{r}} \omega\right) \text{ and } F_{\rho,\delta,r}^\varphi f(x) = F_{\rho,\delta}^\varphi f\left(\frac{x}{\sqrt{r}}\right).
\end{equation}

**Proposition 1.** Let $2 \leq p \leq \infty$ and $0 < \delta \leq 1/4$. Suppose that $b > \frac{1}{2}(D - 1)$ where $D = 2n + 1$ is the topological dimension of $\mathbb{H}^n$. Then, we have that
\begin{equation}
\left\lVert \left(\frac{1}{r^{1/2}} \left| F_{\rho,\delta}^\varphi f \right|^2 d\rho \right)^{1/2} \right\rVert_{L_p(\mathbb{H}^n)} \leq C \delta^{-(b - \frac{1}{2})} \lVert f \rVert_{L_p(\mathbb{H}^n)}.
\end{equation}

It follows that for any $\varepsilon > 0$,
\begin{equation}
\left\lVert \sup_{k \in \mathbb{Z}} \left| D_{\delta,k}^\varphi f \right| \right\rVert_{L_p(\mathbb{H}^n)} = \left\lVert \sup_{k \in \mathbb{Z}} \left( \sum_{\rho \in 2\mathbb{Z}[0,2]} \left| F_{\rho,\delta,2^k,\rho}^\varphi f \right|^2 dr \right)^{1/2} \right\rVert_{L_p(\mathbb{H}^n)} \leq C \delta^{-(b - \frac{1}{2}) - \varepsilon} \lVert f \rVert_{L_p(\mathbb{H}^n)}.
\end{equation}

**Proof.** It is easy to prove (3.3). By Corollary 2.6 in [6], we know that the kernel $K_{\rho,\delta}^\varphi$ of multiplier operator $F_{\rho,\delta}^\varphi$ satisfies
\[ \int_{\mathbb{H}^n} \left| K_{\rho,\delta}^\varphi(\omega) \right| d\omega \leq \left\lVert \varphi\left(\frac{\rho - \delta \cdot}{\delta}\right) \right\rVert_{L_{b+\frac{1}{2}}} \leq C \delta^{-b} \]
for any $b > \frac{1}{2}(D - 1)$, where the Sobolev norm is defined by
\[ \lVert f \rVert_{L^2_{b}} = \left( \int_{\mathbb{R}} |x|^b \hat{f}(x)^2 \, dx \right)^{1/2}. \]

By Young's inequality, we get that for any $1 \leq p \leq \infty$,
\[ \left\lVert F_{\rho,\delta}^\varphi f \right\rVert_{L_p(\mathbb{H}^n)} \leq \left\lVert K_{\rho,\delta}^\varphi \right\rVert_{L^1(\mathbb{H}^n)} \lVert f \rVert_{L_p(\mathbb{H}^n)} \leq C \delta^{-b} \lVert f \rVert_{L_p(\mathbb{H}^n)}. \]

Then, using Minkowski’s inequality for $p \geq 2$, it follows that
\[ \left\lVert \left(\frac{1}{r^{1/2}} \left| F_{\rho,\delta}^\varphi f \right|^2 d\rho \right)^{1/2} \right\rVert_{L_p(\mathbb{H}^n)} \leq \left( \int_{\mathbb{H}^n} \left| F_{\rho,\delta}^\varphi f \right|^2 \frac{dx}{r^{2/2}} \right)^{1/2} \leq C \delta^{\frac{1}{2}} \left\lVert F_{\rho,\delta}^\varphi f \right\rVert_{L_p(\mathbb{H}^n)} \leq C \delta^{-(b - \frac{1}{2})} \lVert f \rVert_{L_p(\mathbb{H}^n)}. \]

To obtain (3.6), we decompose interval $[0, 2]$ into dyadic subintervals as follows:
\[ [0, 2] = [0, 4\delta] \cup [4\delta, 2], \quad [4\delta, 2] = \bigcup_{j=-1}^{j_0} I_j = \bigcup_{j=-1}^{j_0} [4\delta, 2] \cap [2^{-j-1}, 2^{-j}]. \]
where \( j_0 \) is the smallest integer satisfying \( 2^{-j_0-1} \leq 4\delta \). Then, by the triangle inequality, we have that

\[
\left\| \sup_{k \in \mathbb{Z}} |D_{\delta, k}^\rho f(x)| \right\|_{L^p(\mathbb{H}^n)} = \left\| \sup_{k \in \mathbb{Z}} \left( \sum_{\rho \in \delta \mathbb{Z} \cap [0, 2]} \int_1^2 |F_{\rho, \delta, 2^k r}^\rho f(x)|^2 \, dr \right) \right\|_{L^p(\mathbb{H}^n)}^{1/2}
\]

\[
\leq \left\| \left( \sup_{k \in \mathbb{Z}} \left( \sum_{\rho \in \delta \mathbb{Z} \cap [0, 4\delta]} \int_1^2 |F_{\rho, \delta, 2^k r}^\rho f(x)|^2 \, dr \right) \right) \right\|_{L^p(\mathbb{H}^n)}^{1/2}
\]

\[
+ \sum_{j=-1}^{j_0} \left\| \left( \sup_{k \in \mathbb{Z}} \left( \sum_{\rho \in \delta \mathbb{Z} \cap I_j} \int_1^2 |F_{\rho, \delta, 2^k r}^\rho f(x)|^2 \, dr \right) \right) \right\|_{L^p(\mathbb{H}^n)}^{1/2},
\]

Setting

\[
I_j = \left\| \left( \sup_{k \in \mathbb{Z}} \left( \sum_{\rho \in \delta \mathbb{Z} \cap I_j} \int_1^2 |F_{\rho, \delta, 2^k r}^\rho f(x)|^2 \, dr \right) \right) \right\|_{L^p(\mathbb{H}^n)}^{1/2},
\]

for \(-1 \leq j \leq j_0\) and

\[
II = \left\| \left( \sup_{k \in \mathbb{Z}} \left( \sum_{\rho \in \delta \mathbb{Z} \cap [0, 4\delta]} \int_1^2 |F_{\rho, \delta, 2^k r}^\rho f(x)|^2 \, dr \right) \right) \right\|_{L^p(\mathbb{H}^n)}^{1/2},
\]

it follows that

\[
(3.7) \quad \left\| \sup_{k \in \mathbb{Z}} |D_{\delta, k}^\rho f(x)| \right\|_{L^p(\mathbb{H}^n)} \leq \sum_{j=-1}^{j_0} I_j + II.
\]

By the first equality relation in (3.3), we notice that for any \(-1 \leq j \leq j_0\), \( j \neq 0 \),

\[
\left\| \left( \sup_{k \in \mathbb{Z}} \sum_{\rho \in \delta \mathbb{Z} \cap I_j} \int_1^2 |F_{\rho, \delta, 2^k r}^\rho f(x)|^2 \, dr \right) \right\|_{L^p(\mathbb{H}^n)}^{1/2} = \left\| \left( \sup_{k \in \mathbb{Z}} \sum_{\rho \in \delta \mathbb{Z} \cap [0, 2]} \int_1^2 |F_{\rho, \delta, 2^k r}^\rho f(x)|^2 \, dr \right) \right\|_{L^p(\mathbb{H}^n)}^{1/2},
\]

and \( 2^{-j} \geq 2^{-j_0} > 4\delta \) such that \( 2^j \delta < 1/4 \) for any \( 0 < \delta \leq 1/4 \). These imply that once \( I_0 \leq \delta^{-(b-\frac{1}{2})} \|f\|_{L^p(\mathbb{H}^n)} \), then \( I_j \leq (2^j \delta)^{-(b-\frac{1}{2})} \) and

\[
(3.8) \quad \sum_{j=-1}^{j_0} I_j \leq \sum_{j=-1}^{j_0} (2^j \delta)^{-(b-\frac{1}{2})} \leq C\delta^{-(b-\frac{1}{2})-\varepsilon} \|f\|_{L^p(\mathbb{H}^n)}
\]
for any $\varepsilon > 0$ since $j_0 = O(\log(1/\delta))$. Thus, to obtain (3.10), it suffices to show that

$$\max\{I_0, II\} \leq \delta^{-(b-\frac{r}{2})}.$$ 

To estimate $I_0$, we consider the Littlewood-Paley projection operator $P_m$, $m \in \mathbb{Z}$, defined by

$$P_m f = \int_0^{\infty} \beta(2^{-m}\lambda)P_\lambda f d\mu(\lambda),$$

where $\beta \in C_0^\infty(\frac{1}{2}, 2]$ satisfying $0 \leq \beta \leq 1$ and $\sum_{m \in \mathbb{Z}} \beta(2^{-m}t) = 1$ for each $t > 0$. Then, we have that

$$f = \int_0^{\infty} P_\lambda f d\mu(\lambda) = \sum_{m \in \mathbb{Z}} \int_0^{\infty} \beta(2^{-m}\lambda)P_\lambda f d\mu(\lambda) = \sum_{m \in \mathbb{Z}} P_m f.$$ 

Since $\text{supp} \varphi \subset [-1, 1]$, $\rho \in [1/2, 1]$, $r \in [1, 2]$ and $0 < \delta \leq 1/4$, we see that

$$\varphi\left(\frac{\rho - 2^k r \lambda}{\delta}\right) \beta(2^{-m}\lambda) \equiv 0 \text{ expect } -3 \leq k + m \leq 2.$$ 

Using this, we can get that for any $k \in \mathbb{Z}$,

$$F_{\rho, \delta, 2^k r}^\varphi f = F_{\rho, \delta, 2^k r}^\varphi \left(\sum_{m \in \mathbb{Z}} P_m f\right) = \sum_{m \in \mathbb{Z}} \int_0^{\infty} \varphi\left(\frac{\rho - 2^k r \lambda}{\delta}\right) \beta(2^{-m}\lambda)P_\lambda f d\mu(\lambda) = \sum_{m \in \mathbb{Z}, -3 \leq k + m \leq 2} F_{\rho, \delta, 2^k r}^\varphi (P_m f).$$

(3.9)

Since that $p \geq 2$, applying (3.9), (3.11) and Mikowski’s inequality, it follows that

$$I_0 \leq \left(\int_{\mathbb{R}^n} \left(\sup_{k \in \mathbb{Z}} \sum_{\rho \in \mathbb{Z} \cap I_0} \int_1^{2^2} \left|F_{\rho, \delta, 2^k r}^\varphi f(x)\right|^2 dr \right)^{\frac{q}{2}} dx\right)^{\frac{1}{q}}$$

$$= \left(\int_{\mathbb{R}^n} \left(\sup_{k \in \mathbb{Z}} \sum_{\rho \in \mathbb{Z} \cap I_0} \int_1^{2^2} \left|F_{\rho, \delta, 2^k r}^\varphi (P_m f)(x)\right|^2 dr \right)^{\frac{q}{2}} dx\right)^{\frac{1}{p}}$$

$$= \left(\int_{\mathbb{R}^n} \left(\sup_{k \in \mathbb{Z}} \sum_{\rho \in \mathbb{Z} \cap I_0} \int_1^{2^2} \left|F_{\rho, \delta, 2^k r}^\varphi (P_m f)\sqrt{2^k r}\left(\frac{\delta}{\sqrt{2^k r}} x\right)\right|^2 dr \right)^{\frac{q}{2}} dx\right)^{\frac{1}{p}}.$$
Inserting this into (3.10) and using the Littlewood-Paley theorem, we can obtain that

\[
\left( \sum_{k \in \mathbb{Z}} \sum_{m \in \mathbb{Z}, -3 \leq k + m \leq 2} \left( \int_1^2 \left( \frac{2^k r}{2^{k+1} r} \right)^{\frac{Q}{p}} \left( \sum_{\rho \in \delta \mathbb{Z} \cap I_0} |F_{\rho, \delta}^\varphi (P_m f) \sqrt{2^k r} |^2 \right)^{\frac{1}{2}} \left( \sum_{\rho \in \delta \mathbb{Z} \cap I_0} |F_{\rho, \delta}^\varphi (P_m f) \sqrt{2^k r} |^2 \right)^{\frac{1}{2}} dr \right)^{\frac{1}{2}} \right)
\]

\[=\sum_{k \in \mathbb{Z}} \sum_{m \in \mathbb{Z}, -3 \leq k + m \leq 2} \left( \int_1^2 \left( \frac{2^k r}{2^{k+1} r} \right)^{\frac{Q}{p}} \left( \sum_{\rho \in \delta \mathbb{Z} \cap I_0} |F_{\rho, \delta}^\varphi (P_m f) \sqrt{2^k r} |^2 \right)^{\frac{1}{2}} \left( \sum_{\rho \in \delta \mathbb{Z} \cap I_0} |F_{\rho, \delta}^\varphi (P_m f) \sqrt{2^k r} |^2 \right)^{\frac{1}{2}} dr \right)^{\frac{1}{2}} \]

(3.10)

Notice that the \(L^p\)-boundedness properties of the square function in (3.5) and the discretize square function in the above are essentially equivalent. Hence, we have that

\[
\left\| \left( \sum_{\rho \in \delta \mathbb{Z} \cap I_0} |F_{\rho, \delta}^\varphi (P_m f) \sqrt{2^k r} |^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{H}^n)} \leq C \delta^{-(b-\frac{1}{2})} \left\| (P_m f) \sqrt{2^k r} \right\|_{L^p(\mathbb{H}^n)}
\]

Inserting this into (3.10) and using the Littlewood-Paley theorem, we can obtain that

\[
I_0 \leq \sum_{k \in \mathbb{Z}} \sum_{m \in \mathbb{Z}, -3 \leq k + m \leq 2} \left( \int_1^2 \left( \frac{2^k r}{2^{k+1} r} \right)^{\frac{Q}{p}} \left( \sum_{\rho \in \delta \mathbb{Z} \cap I_0} |F_{\rho, \delta}^\varphi (P_m f) \sqrt{2^k r} |^2 \right)^{\frac{1}{2}} \left( \sum_{\rho \in \delta \mathbb{Z} \cap I_0} |F_{\rho, \delta}^\varphi (P_m f) \sqrt{2^k r} |^2 \right)^{\frac{1}{2}} dr \right)^{\frac{1}{2}} \]

\[\leq C \delta^{-(b-\frac{1}{2})} \left( \sum_{k \in \mathbb{Z}} \sum_{m \in \mathbb{Z}, -3 \leq k + m \leq 2} \left\| P_m f \right\|_{L^p(\mathbb{H}^n)}^2 \right)^{\frac{1}{2}} \]

\[\leq C \delta^{-(b-\frac{1}{2})} \left\| f \right\|_{L^p(\mathbb{H}^n)} \]

(3.11)

Next, we consider the estimate of \(II\). Notice that

\[
F_{\rho, \delta}^\varphi f (x) = \int_0^\infty \varphi \left( \frac{\rho - \lambda}{\delta} \right) P_\lambda f d\mu (\lambda) = \int_{\mathbb{H}^n} f (x \omega^{-1}) K_{\rho, \delta}(\omega) d\omega.
\]

Setting \(R_t(\omega)\) to be the kernel of the Riesz means \(\int_0^t (1 - \frac{t}{t'})^d P_\lambda f d\mu (\lambda)\), we see that

\[t \to R_t(\omega)\]
is a function of bounded variation. Then, the kernel $K^\varphi_{\rho,\delta}$ can be written as

$$K_{\rho,\delta}(\omega) = \int_0^\infty \varphi\left(\frac{\rho - \lambda}{\delta}\right) \frac{\partial}{\partial \lambda} R^0_{\lambda}(\omega) d\mu(\lambda).$$

Integration by parts and using the identity

$$(3.12) \quad \frac{\partial}{\partial t}(t^m R^m_t(\omega)) = mt^{m-1} R^{m-1}_t(\omega),$$

where $m$ is a positive integer, we get that

$$K_{\rho,\delta}(\omega) = c_m \int_0^\infty \left(\frac{\partial^{2m+2}}{\partial \lambda^{2m+2}} \varphi\left(\frac{\rho - \lambda}{\delta}\right)\right) \lambda^{2m+1} R^{2m+1}_\lambda(\omega) d\mu(\lambda).$$

It is known that (see Theorem 2.5.3 in [11])

$$(3.13) \quad |R^{2m+1}_\lambda(\omega)| \leq C \lambda^{\frac{2m}{2}} (1 + \lambda^{\frac{1}{2}} |\omega|)^{-2m}.$$

We let $m = \frac{Q}{2} + 1$ and $\varphi \in C^N(I)$ with $N = 2m + 2$. Then, we have that

$$|\frac{\partial^{2m+2}}{\partial \lambda^{2m+2}} \varphi\left(\frac{\rho - \lambda}{\delta}\right)| \leq \delta^{-(2m+2)}.$$

This together with (3.13) yield that for any $\rho \in \delta \mathbb{Z} \cap [0, 4\delta]$

$$|K_{\rho,\delta}(\omega)| \leq c_m (1 + |\omega|)^{-2m} \delta^{-(2m+2)} \int_0^{5\delta} \lambda^{2m+1-m-Q + n} d\lambda$$

and

$$c_m \delta^{-m+n+\frac{Q}{2}} (1 + |\omega|)^{-2m} \leq c_m (1 + |\omega|)^{-2m}.$$

Using (3.11) and Young’s inequality, it follows that for any $r \in [1, 2]$, $k \in \mathbb{Z}$ and $\rho \in \delta \mathbb{Z} \cap [0, 4\delta]$,

$$\left\| F^\varphi_{\rho,\delta,2^k,r} f \right\|_{L^p(\mathbb{H}^n)} = \left\| F^\varphi_{\rho,\delta} f \right\|_{L^p(\mathbb{H}^n)} \leq c_m \| f \|_{L^p(\mathbb{H}^n)} \int_{\mathbb{H}^n} (1 + |\omega|)^{-2m} d\omega \leq C \| f \|_{L^p(\mathbb{H}^n)}.$$

Hence, by Minkowski’s inequality, we can get that

$$II = \left\| \left( \sup_{k \in \mathbb{Z}} \left( \sum_{\rho \in \delta \mathbb{Z} \cap [0,4\delta]} \int_1^2 \left| F^\varphi_{\rho,\delta,2^k,r} f(x) \right|^2 dr \right)^\frac{1}{2} \right) \right\|_{L^p(\mathbb{H}^n)}$$

$$\leq \sup_{k \in \mathbb{Z}} \left( \int_1^2 \left\| \sum_{\rho \in \delta \mathbb{Z} \cap [0,4\delta]} \left| F^\varphi_{\rho,\delta,2^k,r} f \right|^2 \right\|_{L^\frac{p}{2}(\mathbb{H}^n)} dr \right)^\frac{1}{2}$$

$$\leq \sup_{k \in \mathbb{Z}} \left( \int_1^2 \sum_{\rho \in \delta \mathbb{Z} \cap [0,4\delta]} \left\| F^\varphi_{\rho,\delta,2^k,r} f \right\|_{L^p(\mathbb{H}^n)}^2 dr \right)^\frac{1}{2} \leq C \| f \|_{L^p}.$$

Applying (3.7), (3.8) and the above estimates (3.11), (3.14), we can conclude that for any $\varepsilon > 0$

$$\left\| \sup_{k \in \mathbb{Z}} D^\varepsilon_{\delta,k} f(x) \right\|_{L^p(\mathbb{H}^n)} \leq C \delta^{-N\left(k-\frac{1}{2}\right)} \| f \|_{L^p(\mathbb{H}^n)}.$$

The proof of (3.13) is complete.□
In [3], we proved that for any function \( m \in L^\infty(\mathbb{R}) \) and \( 0 \leq a < b \), the multiplier operator \( T_m f = \int_a^b m(\lambda) P_x f(\lambda) \, d\mu(\lambda) \) is bounded from \( L^p(\mathbb{H}^n) \) into \( L^2(\mathbb{H}^n) \) for any \( 1 \leq p \leq 2 \), i.e.

\[
\|T_m f\|_2 \leq C \|m\|_\infty \| (b - a)b^n \|^{(\frac{1}{p} - \frac{1}{2})} \|f\|_p.
\]

Using this estimate, it is easy to check that

\[
\left\| \left( \int_{1/2}^1 \left| F^x_{\rho,\delta} f(x) \right|^2 \, d\rho \right)^{1/2} \right\|_{L^2(\mathbb{H}^n)} \leq C \delta^{\frac{1}{2}} \|f\|_{L^2(\mathbb{H}^n)}.
\]

By the same argument of Proposition 1, we have that

\[
(3.15) \quad \left\| \sup_{k \in \mathbb{Z}} |D^x_{\delta,k} f| \right\|_{L^2(\mathbb{H}^n)} \leq C \delta^{\frac{1}{2} - \epsilon} \|f\|_{L^2(\mathbb{H}^n)}
\]

for any \( \epsilon > 0 \). Then, by interpolation between the estimates in (3.15) and (3.6) for \( p = \infty \), we can get the following result:

**Corollary 1.** Let \( 2 \leq p \leq \infty \) and \( 0 < \delta \leq 1/4 \). Suppose that \( b > \frac{1}{2}(D - 1) \) where \( D = 2n + 1 \) is the topological dimension of \( \mathbb{H}^n \). Then,

\[
\left\| \sup_{k \in \mathbb{Z}} |D^x_{\delta,k} f| \right\|_{L^p(\mathbb{H}^n)} \leq C \delta^{\frac{1}{2} - \epsilon - \frac{1}{2} p} \|f\|_{L^p(\mathbb{H}^n)}.
\]

**4. Boundedness of the maximal operator \( S^x_\alpha \)**

**Theorem 1.** Let \( 2 \leq p_1, p_2 \leq \infty \) and \( 1/p = 1/p_1 + 1/p_2 \). If \( \alpha > D(1 - \frac{1}{p}) + \frac{3}{p} \), then \( S^x_\alpha \) is bounded from \( L^{p_1}(\mathbb{H}^n) \times L^{p_2}(\mathbb{H}^n) \) into \( L^p(\mathbb{H}^n) \).

**Proof.** Fix \( \alpha > 0 \). Let us choose \( \psi \in C^\infty_0([\frac{1}{2}, 2]) \) and \( \psi_0 \in C^\infty_0([-\frac{3}{4}, \frac{3}{4}]) \) such that

\[
(1 - t)^\alpha = \sum_{\delta \in D} \delta^\alpha \psi \left( \frac{1 - t}{\delta} \right) + \psi_0(t), \quad 0 \leq t \leq 1,
\]

where \( D = \{ 2^k : k \in \mathbb{Z} \text{ and } k \leq -2 \} \). Using this, we can decompose

\[
S^x_\alpha = \sum_{\delta \in D} \delta^\alpha S^x_\delta + S^x_0
\]

where

\[
S^x_\delta(f, g) = \int_0^\infty \int_0^\infty \psi \left( \frac{1 - r \lambda_1 - r \lambda_2}{\delta} \right) P_{\lambda_1} f P_{\lambda_2} g \, d\mu(\lambda_1) d\mu(\lambda_2),
\]

and

\[
S^x_0(f, g) = \int_0^\infty \int_0^\infty \psi_0(r \lambda_1 + r \lambda_2) P_{\lambda_1} f P_{\lambda_2} g \, d\mu(\lambda_1) d\mu(\lambda_2).
\]

It follows that

\[
(4.1) \quad S^x_\alpha(f, g)(x) \leq \sum_{\delta \in D} \delta^\alpha \sup_{r > 0} |S^x_\delta(f, g)(x)| + \sup_{r > 0} |S^x_0(f, g)(x)|.
\]

Since that \( \psi_0 \in C^\infty_0([-\frac{3}{4}, \frac{3}{4}]) \), using Holder’s inequality, it is easy to see that for any \( 2 \leq p_1, p_2 \leq \infty \) and \( 1/p = 1/p_1 + 1/p_2 \),

\[
\|S^x_0(f, g)\|_{L^p(\mathbb{H}^n)} = \left\| \sup_{r > 0} |S^x_0(f, g)| \right\|_{L^p(\mathbb{H}^n)} \leq \|fg\|_{L^p(\mathbb{H}^n)} \leq \|f\|_{L^{p_1}(\mathbb{H}^n)} \|g\|_{L^{p_2}(\mathbb{H}^n)}.
\]
Therefore, to obtain Theorem 11 we have to focus on obtaining estimates for the maximal operator

\[ S_\delta^s(f, g)(x) = \sup_{r \geq 0} |S_r^\delta(f, g)(x)| \quad \text{for } 0 < \delta \leq \frac{1}{4}. \]

By the fundamental theorem of calculus, we see that \(|F(t)| \leq |F(s)| + \int_t^s |F'(\tau)| \, d\tau\) for any \(t, s \in [1, 2]\). This implies that

\[ S_\delta^s(f, g)(x) = \sup_{k \in \mathbb{Z}} \sup_{1 \leq r \leq 2} |S_{2^k r}^\delta(f, g)(x)| \leq \sup_{k \in \mathbb{Z}, r \in [1, 2]} \left( \int_0^1 \left| \frac{\partial}{\partial r} S_{2^k r}^\delta(f, g)(x) \right| \, dr \right) \leq \frac{2^k}{\delta} \int_0^\infty \int_0^\infty \left( \frac{1 - 2^k r (\lambda_1 + \lambda_2)}{\delta} \right) P_{\lambda_1} f P_{\lambda_2} g \, d\mu(\lambda_1) d\mu(\lambda_2), \]

we can conclude that \(\frac{\partial}{\partial r} S_{2^k r}^\delta(f, g)\) satisfies the same quantitative properties as \(\frac{1}{\delta} S_{2^k r}^\delta(f, g)\) when \(1 \leq r \leq 2\). Hence, to estimate \(S_\delta^s(f, g)\), it suffices to consider the operator

\[ (f, g) \mapsto \sup_{k \in \mathbb{Z}, r \in [1, 2]} \left( \int_0^1 |S_{2^k r}^\delta(f, g)(x)| \, dr \right). \]

To estimate this operator, we choose \(\varphi \in C_0^\infty(I)\) satisfying \(\sum_{l \in \mathbb{Z}} \varphi(t) = 1\) for all \(t \in \mathbb{R}\). Fix \(0 < \delta \leq 1/4\) and set \(\tilde{\delta} = \delta^{1+\kappa}\) with some \(\kappa > 0\). Then, for any \(r > 0\), we can write

\[ \psi \left( \frac{1 - r \lambda_1 - r \lambda_2}{\delta} \right) = \sum_{\sigma \in \delta \mathbb{Z}, \rho \in \delta \mathbb{Z} \cap [0, 2]} \varphi \left( \frac{\rho - r \lambda_1}{\delta} \right) \psi \left( \frac{\sigma - r \lambda_2}{\delta} \right) \psi \left( \frac{1 - r \lambda_1 - r \lambda_2}{\delta} \right). \]

Since \(\text{supp} \varphi \subset [-1, 1]\) and \(\text{supp} \psi \subset [1/2, 2]\), then

\[ \varphi \left( \frac{\rho - r \lambda_1}{\delta} \right) \psi \left( \frac{1 - r \lambda_1 - r \lambda_2}{\delta} \right) = 0 \quad \text{except} \quad 1 - 3\delta \leq r \lambda_2 + \rho \leq 1 + \delta. \]

It follows that

\[ \varphi \left( \frac{\rho - r \lambda_1}{\delta} \right) \psi \left( \frac{1 - r \lambda_1 - r \lambda_2}{\delta} \right) = 0 \quad \text{except } \sigma \in [1 - 4\delta, 1 + 2\delta]. \]

Thus,

\[
S_\delta^s(f, g) = \sum_{\sigma \in \delta \mathbb{Z} \cap [1 - 4\delta, 1 + 2\delta]} \sum_{\rho \in \delta \mathbb{Z} \cap [0, 2]} \int_0^\infty \int_0^\infty \varphi \left( \frac{\rho - r \lambda_1}{\delta} \right) \psi \left( \frac{1 - r \lambda_1 - r \lambda_2}{\delta} \right) P_{\lambda_1} f P_{\lambda_2} g d\mu(\lambda_1) d\mu(\lambda_2) \times \psi \left( \frac{1 - r \lambda_1 - r \lambda_2}{\delta} \right) P_{\lambda_1} f P_{\lambda_2} g d\mu(\lambda_1) d\mu(\lambda_2).
\]

At the same time, by the Fourier inversion formula, we have

\[
(4.3) \quad \psi \left( \frac{1 - r \lambda_1 - r \lambda_2}{\delta} \right) = \int_\mathbb{R} \hat{\psi}(\tau) e^{2\pi i \tau \left( \frac{1 - r \lambda_1 - r \lambda_2}{\delta} \right)} d\tau = \int_\mathbb{R} \hat{\psi}(\tau) e^{2\pi i \tau \left( \frac{\sigma - r \lambda_1 - r \lambda_2}{\delta} \right)} e^{2\pi i r \left( \frac{1 - \lambda_1 - \lambda_2}{\delta} \right)} d\tau.
\]

Applying Taylor’s theorem for \(e^{2\pi i (\frac{\sigma - r \lambda_1 - r \lambda_2}{\delta})}\), we get that for any \(\rho \in \delta \mathbb{Z} \cap [0, 2]\),

\[
e^{2\pi i (\frac{\sigma - r \lambda_1 - r \lambda_2}{\delta})} = \sum_{N=0}^{\infty} \frac{1}{N!} \left( \frac{\tau (\sigma - r \lambda_1 - r \lambda_2)}{\delta} \right)^N.
\]
Putting (4.4) into the right hand side of (4.3), it follows that
\[ r = \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{0 \leq a+b \leq N} c_{a,b} \left( \frac{\tau (\rho - r \lambda_1)}{\delta} \right)^a \left( \frac{\tau (\sigma - \rho - r \lambda_2)}{\delta} \right)^b. \]

Putting (4.4) into the right hand side of (4.3), it follows that
\[ \psi \left( \frac{1 - r \lambda_1 - r \lambda_2}{\delta} \right) = \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{0 \leq a+b \leq N} c_{a,b} \left( \int_{\mathbb{R}} \psi(\tau) \tau^{a+b} e^{2\pi i \frac{\delta}{\tau}} d\tau \right) \left( \frac{\rho - r \lambda_1}{\delta} \right)^a \left( \frac{\sigma - \rho - r \lambda_2}{\delta} \right)^b. \]

Insert this into (4.2) and let \( \varphi_\beta(t) = t^\beta \varphi(t). \) Notice that \( \varphi_\beta \in C^\infty_0(I) \) for any \( \beta \in \mathbb{N}. \) Then, we can obtain that
\[ S_t^\delta(f, g) = \sum_{\sigma \in \mathbb{Z} \cap [1-4\delta, 1+2\delta]} \sum_{\rho \in \mathbb{Z} \cap [0, 2]} \int_0^\infty \int_0^\infty \varphi \left( \frac{\rho - r \lambda_1}{\delta} \right) \varphi \left( \frac{\sigma - \rho - r \lambda_2}{\delta} \right) \times \psi \left( \frac{1 - r \lambda_1 - r \lambda_2}{\delta} \right) P_{\lambda_1} f P_{\lambda_2} g d\mu(\lambda_1) d\mu(\lambda_2) \]
\[ = \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{0 \leq a+b \leq N} c_{a,b} \left( \psi^{(a+b)} \left( \frac{1 - \sigma}{\delta} \right) \right) \left( \frac{\rho - r \lambda_1}{\delta} \right)^a \left( \frac{\sigma - \rho - r \lambda_2}{\delta} \right)^b P_{\lambda_1} f P_{\lambda_2} g d\mu(\lambda_1) d\mu(\lambda_2) \]
\[ = \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{0 \leq a+b \leq N} c_{a,b} \delta^{(a+b)} \int_{\mathbb{R} \cap [1-4\delta, 1+2\delta]} \left( \psi^{(a+b)} \left( \frac{1 - \sigma}{\delta} \right) \right) \times \sum_{\rho \in \mathbb{Z} \cap [0, 2]} \varphi \left( \frac{\rho - r \lambda_1}{\delta} \right) P_{\lambda_1} f d\mu(\lambda_1) \int_0^\infty \varphi \left( \frac{\sigma - \rho - r \lambda_2}{\delta} \right) P_{\lambda_2} g d\mu(\lambda_2). \]

Set \( r = 2^k r \) for \( k \in \mathbb{Z}, r \in [1, 2]. \) Applying the triangle inequality, Cauchy-Schwartz' inequality and Hölder's inequality, we get that
\[ \left\| \sup_{k \in \mathbb{Z} \cap [1, 2]} \left\| S_{2^k r}^\delta(f, g) \right\|_{L^p(\mathbb{R}^n)} \right\|_{L^p(\mathbb{R}^n)} \]
\[ \leq \left\| \sup_{k \in \mathbb{Z} \cap [1, 2]} \left\| \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{0 \leq a+b \leq N} c_{a,b} \delta^{(a+b)} \sum_{\rho \in \mathbb{Z} \cap [1-4\delta, 1+2\delta]} \left( \psi^{(a+b)} \left( \frac{1 - \sigma}{\delta} \right) \right) \times \sum_{\rho \in \mathbb{Z} \cap [0, 2]} \left| F_{\pi, 2^k r} \varphi \right| \right\|_{L^p(\mathbb{R}^n)} \]
Using (4.5) and Corollary 1, we see that

\[ \tilde{F}_{\gamma}(\delta) \leq \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{0 \leq a+b \leq N} c_{a,b} \delta^{c(a+b)} \sum_{\sigma \in \tilde{\mathbb{Z}}^n} |\psi(a+b)(\frac{1-\sigma}{\delta})| \]

\[ \times \left\| \sup_{k \in \mathbb{Z}} \int_{1}^{2} \sum_{\rho \in \tilde{\mathbb{Z}}^n(0,2)} |F_{\rho,\gamma,2k}^a f| \left| F_{\rho,\gamma,2k}^b g \right| dr \right\|_{L^p(\mathbb{H}^n)} \]

\[ \leq \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{0 \leq a+b \leq N} c_{a,b} \delta^{c(a+b)} \sum_{\sigma \in \tilde{\mathbb{Z}}^n(1-4\delta,1+2\delta)} |\psi(a+b)(\frac{1-\sigma}{\delta})| \]

\[ \times \left\| \sup_{k \in \mathbb{Z}} \int_{1}^{2} \sum_{\rho \in \tilde{\mathbb{Z}}^n(0,2)} |F_{\rho,\gamma,2k}^a f| \left| \sum_{\rho} F_{\rho,\gamma,2k}^b g \right| dr \right\|_{L^p(\mathbb{H}^n)} \]

Notice that \( \sigma - \rho \in \tilde{\mathbb{Z}} \cap [-4\delta - 1, 1 + 2\delta] \) for any \( \sigma \in \tilde{\mathbb{Z}} \cap [1-4\delta, 1+2\delta] \), \( \rho \in \tilde{\mathbb{Z}} \cap [0, 2] \) and \( F_{\gamma}^b = 0 \) if \( \sigma - \rho \in \tilde{\mathbb{Z}} \cap [-4\delta - 1, 1] \). So,

\[ \sup_{k \in \mathbb{Z}} \left( \sum_{\rho \in \tilde{\mathbb{Z}}^n(0,2)} \int_{1}^{2} \left| F_{\rho,\gamma,2k}^a f \right|^2 \left| \sum_{\rho} F_{\rho,\gamma,2k}^b g \right|^2 \left| \int_{1}^{2} \right| dr \right)^{1/2} = \sup_{k \in \mathbb{Z}} |D_{\delta,k}^\gamma f| \]

and

\[ \sup_{k \in \mathbb{Z}} \left( \sum_{\rho \in \tilde{\mathbb{Z}}^n(0,2)} \int_{1}^{2} \left| F_{\rho,\gamma,2k}^b g \right|^2 \left| \sum_{\rho} F_{\rho,\gamma,2k}^a f \right|^2 \left| \int_{1}^{2} \right| dr \right)^{1/2} \leq \sup_{k \in \mathbb{Z}} \left( \sum_{\rho \in \tilde{\mathbb{Z}}^n(0,2)} \int_{1}^{2} \left| F_{\rho,\gamma,2k}^b g \right|^2 \left| \sum_{\rho} F_{\rho,\gamma,2k}^a f \right|^2 \left| \int_{1}^{2} \right| dr \right)^{1/2} \]

Using (4.5) and Corollary 1, we see that

\[ \left\| \sup_{k \in \mathbb{Z}} \int_{1}^{2} \left| S_{\gamma,\gamma} f(g) \right| \right\|_{L^p(\mathbb{H}^n)} \]

\[ \leq \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{0 \leq a+b \leq N} c_{a,b} \delta^{c(a+b)} \sum_{\sigma \in \tilde{\mathbb{Z}}^n(1-4\delta,1+2\delta)} |\psi(a+b)(\frac{1-\sigma}{\delta})| \]

\[ \times \left\| \sup_{k \in \mathbb{Z}} |D_{\delta,k}^\gamma f| \right\|_{L^p(\mathbb{H}^n)} \right\|_{L^p(\mathbb{H}^n)} \]

\[ \leq C \delta^{-(2b-1)(1-\frac{1}{p}) - \frac{1}{p} - \varepsilon} \left\| f \right\|_{L^p(\mathbb{H}^n)} \left\| g \right\|_{L^p(\mathbb{H}^n)} \delta^{-1 - \kappa} \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{0 \leq a+b \leq N} c_{a,b} \delta^{c(a+b)} \sum_{\sigma \in \tilde{\mathbb{Z}}^n(1-4\delta,1+2\delta)} |\psi(a+b)(\frac{1-\sigma}{\delta})| \]

\[ \leq C \delta^{-(2b-1)(1-\frac{1}{p}) - \frac{1}{p} - \varepsilon} \left\| f \right\|_{L^p(\mathbb{H}^n)} \left\| g \right\|_{L^p(\mathbb{H}^n)} \delta^{-1 - \kappa} \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{0 \leq a+b \leq N} c_{a,b} \delta^{c(a+b)} \sum_{\sigma \in \tilde{\mathbb{Z}}^n(1-4\delta,1+2\delta)} |\psi(a+b)(\frac{1-\sigma}{\delta})| \]

\[ \leq C \delta^{-(2b-1)(1-\frac{1}{p}) - \frac{1}{p} - \varepsilon} \left\| f \right\|_{L^p(\mathbb{H}^n)} \left\| g \right\|_{L^p(\mathbb{H}^n)} \delta^{-1 - \kappa} \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{0 \leq a+b \leq N} c_{a,b} \delta^{c(a+b)} \sum_{\sigma \in \tilde{\mathbb{Z}}^n(1-4\delta,1+2\delta)} |\psi(a+b)(\frac{1-\sigma}{\delta})| \]

\[ \leq C \delta^{-(2b-1)(1-\frac{1}{p}) - \frac{1}{p} - \varepsilon} \left\| f \right\|_{L^p(\mathbb{H}^n)} \left\| g \right\|_{L^p(\mathbb{H}^n)} \delta^{-1 - \kappa} \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{0 \leq a+b \leq N} c_{a,b} \delta^{c(a+b)} \sum_{\sigma \in \tilde{\mathbb{Z}}^n(1-4\delta,1+2\delta)} |\psi(a+b)(\frac{1-\sigma}{\delta})| \]
holds for $\kappa = 1 + \varepsilon$ with any $\varepsilon > 0$ and $b > \frac{1}{2}(D-1)$ with $D = 2n+1$ is the topological dimension of $\mathbb{H}^n$. It follows that

$$\left\| S^\delta_{\kappa}(f,g) \right\|_{L^p(\mathbb{H}^n)} \leq \sup_{k \in \mathbb{Z}} \int_1^2 \left\| S^\delta_{2k+\varepsilon}(f,g) \right\|_{L^p(\mathbb{H}^n)} + \sup_{k \in \mathbb{Z}} \int_1^2 \left\| \frac{\partial}{\partial r} S^\delta_{2k+\varepsilon}(f,g) \right\|_{L^p(\mathbb{H}^n)}$$

$$\leq C \left( \delta^{-(2b-1)(1-\frac{1}{p})-\frac{1}{p} - 2 - \varepsilon} + \delta^{-(2b-1)(1-\frac{1}{p})-\frac{1}{p} - 2 - \varepsilon - 1} \right) \left\| f \right\|_{L^{p_1}(\mathbb{H}^n)} \left\| g \right\|_{L^{p_2}(\mathbb{H}^n)}$$

Therefore, whenever $\alpha > \alpha(p_1, p_2) = D(1 - \frac{1}{p}) + \frac{2}{p}$, we can choose $b > \frac{1}{2}(D-1)$ and $\varepsilon > 0$ such that $\alpha > (2b - 1)(1 - \frac{1}{p}) + \frac{1}{p} + 2 + \varepsilon$. Then

$$\left\| S^\delta_{\kappa}(f,g) \right\|_{L^p(\mathbb{H}^n)} \leq \sum_{\delta \in D} \delta^\alpha \left\| S^\delta_{\kappa}(f,g) \right\|_{L^p(\mathbb{H}^n)} + \left\| S^0_{\kappa}(f,g) \right\|_{L^p(\mathbb{H}^n)}$$

$$\leq \sum_{\delta \in D} \delta^\alpha \left( \frac{(2b-1)(1-\frac{1}{p}) + \frac{1}{p} + \varepsilon}{\delta} \right) \left\| f \right\|_{L^{p_1}(\mathbb{H}^n)} \left\| g \right\|_{L^{p_1}(\mathbb{H}^n)} + C \left\| f \right\|_{L^{p_1}(\mathbb{H}^n)} \left\| g \right\|_{L^{p_2}(\mathbb{H}^n)}$$

The proof of Theorem 1 is complete. \hfill $\square$

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