Navier-Stokes Hamiltonian

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(Dated: 2014-09-29)

Abstract

The Navier-Stokes Hamiltonian is derived from first principles. Its Hamilton equations are shown to be equivalent to the Navier-Stokes, continuity, and energy conservation equations of standard fluid mechanics. The derivations of the Navier-Stokes and Euler Hamiltonians are compared, with the latter having identical dynamics to the Euler equation with the viscosity terms dropped from the beginning. The two Hamiltonians have the same number of degrees of freedom in three spatial dimensions: six independent scalar potentials (although in the Navier-Stokes case the potentials are two vector fields), but their dynamical fields are necessarily different due to a theory with dissipation not mapping smoothly onto one without. Mass, momentum, and energy conservation give rise to the standard nonholonomic constraint on entropy which is used to construct the Navier-Stokes Hamiltonian. The Newton, Euler-Lagrange, and Hamilton sets of equations are shown to be equivalent for both of these Euler and Navier-Stokes fluids. The dynamical coordinate field of a dissipative fluid is a vector field that stores the initial position of all the fluid particles. Thus these appear to be natural coordinates for studying arbitrary separations of fluid particles over time. The final section discusses energy conservation of the Navier-Stokes Hamiltonian and derives its Poisson bracket with a general classical dissipative observable to set up later work with the similarity renormalization group.

PACS numbers: 47.10.ad, 47.10.Df, 05.10.Cc, 47.11.St

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I. INTRODUCTION

The Navier-Stokes Hamiltonian ($H_{NS}$) is derived from first principles. The goal is to come up with useful dynamical degrees of freedom in order to systematically integrate the equations of motion of a fluid one scale at a time with the similarity renormalization group using convenient approximations set up for a Hamiltonian. These renormalization group studies are saved for later work. This paper motivates the program and derives $H_{NS}$ from first principles along with its Poisson bracket with a general classical dissipative observable. First, we back up to the beginning in order to better know where $H_{NS}$ truly came from.

We start in a nonrelativistic framework (with all speeds of interest much less than the speed of light) and further assume that the mass density (hereafter just “density”) is high enough (and collisions strong enough) such that the continuum approximation is valid, yet low enough (and temperature high enough) such that quantum effects are negligible. This implies that the mean free path is very small with respect to the Kolmogorov and all other macroscopic scales (see Fig. 1) [1]:

\[ \ell_{3d} \sim \frac{1}{n\sigma_{tot}} \ll \lambda_{Kol} \] [then continuum approx. valid]

(where $n$ is the number density and $\sigma_{tot}$ is the total molecular cross section) and that the chemical potential $\mu \to -\infty$ as holds for classical physics. Then, for example for a classical ideal gas we have

\[ n\lambda_T^3 = e^{\frac{\mu}{k_B T}} \mu \to -\infty \ll 1 \] [then classical physics holds],

where $k_B$ is Boltzmann’s constant, $T$ is temperature and $\lambda_T$ is the thermal wavelength:

\[ \lambda_T \sim \frac{2\pi \hbar}{p_T} \sim \frac{2\pi \hbar}{\sqrt{2\pi m k_B T}}, \]

with $m$ the mass of a single molecule of the fluid and the quantum $\hbar$ making this quantity tiny except for very small temperatures which we assume do not occur. In summary: we start with a classical nonrelativistic fluid (e.g. air or water).

Our primary goal is to discuss fluid mechanics from a hamiltonian field theory point of view, however first we start with the newtonian equations of motion in order to better connect with what is already known and to get the correct starting definition of our canonical Hamiltonian.
FIG. 1: A typical fluid eddy of size $L$ with its macroscopic and microscopic substructure. A Kolmogorov microscale eddy of size $\eta_{Kol}$ is the smallest macroscopic structure. The successive microscopic substructure is the mean free path $\ell_{3d}$, the average molecular separation $d$, the molecular impact parameter $b$ and the thermal wavelength $\lambda_T$. For air at room temperature, all of these scales are separated by at least an order of magnitude as shown. For water, the story is similar but then $b$, $d$, and $\ell_{3d}$ are all of the same order of magnitude with the microscopic degrees of freedom strongly coupled. Nevertheless in both cases (air and water) $\lambda_T$ is much smaller than the average separation between the molecules themselves and their quantum nature is therefore inaccessible. In addition, in both cases, the mean free path is much smaller than dissipation scale $\eta_{Kol}$ and macroscale $L$; therefore the continuum approximation is valid and the Navier-Stokes equation becomes the paradigm of interest.
II. CONSERVATION EQUATIONS

Using Landau and Lifshitz [2] as a guide, in this section we start with mass and momentum conservation, include the principle of local thermodynamic equilibrium, and show that this leads to an energy-entropy equation which upon assuming energy conservation leads to the standard entropy constraint equation for a dissipative fluid. This introduces the notation for the rest of the paper, but more importantly, this entropy constraint equation is the nonholonomic constraint that is the key to the derivation of $H_{NS}$.

Mass and momentum conservation of a nonrelativistic fluid are given by [2]

\begin{align}
\text{mass} & \quad \partial_t \rho + \partial_i (\rho v_i) = 0 \, , \\
\text{momentum} & \quad \partial_t (\rho v_i) + \partial_j (\rho v_i v_j) = \partial_j \sigma_{ij} \, ,
\end{align}

where $\rho$ is density, $v_i$ is velocity and $\sigma_{ij}$ is the stress tensor defined next. Repeat indices are summed over the three spatial dimensions, $\partial_t$ is a shorthand for $\partial/\partial t$, and $\partial_i$ is a shorthand for $\partial/\partial x_i$. $\sigma_{ij}$ is the stress tensor given by

$$\sigma_{ij} = -p \delta_{ij} + \sigma'_{ij} \, ,$$

where $p$ is pressure, $\delta_{ij}$ is the Kronecker delta and $\sigma'_{ij}$ is the viscous stress tensor which for a Navier-Stokes (Newtonian) fluid is defined as

\begin{align}
\sigma'_{ij} & \equiv 2\eta e_{ij} + \zeta' \Delta \delta_{ij} \, , \\
e_{ij} & \equiv \frac{1}{2} (\partial_i v_j + \partial_j v_i) \, , \\
\Delta & \equiv \partial_i v_i = \nabla \cdot \mathbf{v} \, ,
\end{align}

where $e_{ij}$ is the strain-rate tensor and $\Delta$ is the rate of expansion [3]. $\eta$ and $\zeta$ are the shear and bulk viscosity respectively and $\zeta' \equiv \zeta - 2\eta/3$, defined such that the trace of the stress tensor is independent of shear viscosity.

Expanding out Eq. (2) and using Eq. (1) leads to the Navier-Stokes equation, the dynamical equation for vector field $\mathbf{v}$:

$$\partial_t v_i + v_j \partial_j v_i = \frac{\partial_j \sigma_{ij}}{\rho} \, .$$

In summary, momentum conservation is equivalent to mass conservation and the Navier-Stokes equation being simultaneously satisfied. Pick any two and you have the other.
Onto energy conservation which implies the entropy constraint of primary interest for the derivation of $H_{NS}$. Following Landau and Lifshitz [2], first take a partial time derivative of the kinetic energy density and then use mass and momentum conservation to rearrange the expression. This gives

$$
\partial_t \left( \frac{1}{2} \rho v^2 \right) = \frac{1}{2} (\partial_t \rho) v_i v_i + \rho v_i (\partial_i v_i)
$$

$$
= -\frac{v^2}{2} \partial_j (\rho v_j) + v_i (-\rho v_j \partial_j v_i + \partial_j \sigma_{ij})
$$

$$
= -\frac{v^2}{2} \partial_j (\rho v_j) - \rho v_j \partial_j (\frac{v^2}{2}) + v_i \partial_j \sigma_{ij}
$$

$$
= -\partial_j (\frac{1}{2} \rho v^2 v_j) + v_i \partial_j \sigma_{ij}.
$$

Moving the first term on the right to the left leaves the kinetic energy identity\(^1\):

$$
\partial_t \left( \frac{1}{2} \rho v^2 \right) + \partial_i \left( \frac{1}{2} \rho v^2 v_i \right) = v_i \partial_j \sigma_{ij}.
$$

Recall that this followed simply from the product rule of partial differentiation and the mass and momentum conservation equations. In words, the kinetic energy identity can be described by

The net difference in a fluid between the change of kinetic energy (per unit volume) over time and the kinetic energy (per unit time per unit area) that flows through a surface is given by the work done on the system (per unit time per unit volume) by the pressure and viscous forces of the fluid.

But there is more, we have not included thermal energy yet—the heat part of internal energy. Some of the viscous work is converted to heat just like rubbing your hands together. This leads to entropy through the second law of thermodynamics as discussed next.

Now we complement this kinetic energy identity with the internal energy conservation equation which follows from mass conservation and the first and second laws of thermodynamics. Strictly speaking the system is not in equilibrium, but locally it is assumed to be. Thus as long as we use differentials, everything still follows the textbook equations of equilibrium thermodynamics. Energy is always conserved, even with dissipation allowed,

\(^1\) We call this an ‘identity’ here because it is not an independent conservation equation beyond that of mass and momentum.
but it changes form as it flows about from work to heat and kinetic energy and back. The irreversible part of the work increases the entropy according to the second law. This is the physical basis of the relations that follow. Using scalar variable $v$ temporarily for specific volume (the inverse of density), from the first and second laws of thermodynamics, locally (spatial and temporal) we have

$$du = Tds - pdv = Tds + \frac{p}{\rho^2}d\rho ,$$

$$\Rightarrow dU \equiv d(\rho u) = \rho du + ud\rho$$

$$= \rho Tds + (u + \frac{p}{\rho})d\rho$$

$$= \rho Tds + h d\rho ,$$

where $U(\rho, s)$ is the internal energy density, in general a function of density and specific entropy, $s$. The specific enthalpy is $h = u + pv = u + p/\rho$, with $u$ being the specific internal energy (the qualifier ‘specific’ implies ‘per unit mass’ for all quantities). The above relation for $dU$ implies the following partial derivatives which will be used often in this paper:

$$\left( \frac{\partial U}{\partial s} \right)_\rho = \rho T,$$

$$\left( \frac{\partial U}{\partial \rho} \right)_s = h = u + \frac{p}{\rho} .$$

Rearranging these relations simply, the first one can be used as a definition for temperature and the second one as a definition for pressure. This is where the bulk of the temperature and pressure effects implicitly lie in the Hamiltonians that follow. Before leaving this part of the argument, note the following standard relation that will be used later as well:

$$dh = d(u + pv) = Tds + vdp = Tds + dp/\rho .$$

For any instant in time, locally in space, Eq. (11) implies that the “$\nabla p/\rho$” term of fluid mechanics can be replaced by

$$\nabla \frac{p}{\rho} = \nabla h - T \nabla s .$$

Continuing like above with the kinetic energy, but now for the internal energy: Take a partial time derivative of the internal energy density and use mass conservation as well as

\footnote{Notation warning: ‘$h$’ in fluid mechanics is often helicity or helicity density, however in this paper $h = u + p/\rho$ is the specific enthalpy.}
the thermodynamic relations just discussed to rearrange the expression leaving the internal
energy conservation equation. We have

\[ \partial_t U = \rho T \partial_t s + h \partial_t \rho \]
\[ = \rho T \partial_t s - h \partial_i (\rho v_i) \]
\[ = \rho T \partial_t s + \rho v_i \partial_i h - \partial_i (\rho h v_i) \]
\[ = \rho T \partial_t s + \rho v_i \partial_i h - \partial_i (U v_i) - \partial_i (pv_i) . \]

Moving the internal energy flux term to the left and arranging further gives

\[ \partial_t U + \partial_i (U v_i) = \rho T \partial_t s + \rho v_i \partial_i h - \partial_i (pv_i) \]
\[ = \rho T \partial_t s + \rho T v_i \partial_i s + v_i \partial_i p - \partial_i (pv_i) \]
\[ = \rho T D_t s - p \Delta , \]

where

\[ D_t \equiv \partial_t + \mathbf{v} \cdot \nabla \] (13)

is the standard material derivative and recall \( \Delta \) is Batchelor’s shorthand for the divergence of the velocity. In summary, internal energy conservation is given by

\[ \text{internal energy} \quad \partial_t U + \partial_i (U v_i) = \rho T D_t s - p \Delta . \] (14)

This followed from mass conservation and the local laws of thermodynamics. It is an independent dynamical equation for the internal energy density and is an expression of the first and second laws of thermodynamics applied to a fluid as it flows about. To help with interpretation of the quantities, note that Eq. (14) can be exactly rearranged by expanding out its second term on the left (the flux term). This gives

\[ D_t U = \rho T D_t s - \rho h \Delta , \]

where \( \rho h = p + U \) is the enthalpy density. Thus we see this really is just “\( du = T ds - pdv \)” (with mass conservation included) along a pathline of a fluid. Enthalpy density acts like the pressure “\( p \)” here and \( \Delta \) acts like the “\( dv \)” which is as it should be; e.g. on page 75 of [3] from the divergence theorem, \( \Delta \) is the rate of relative volume change:

\[ \Delta = \nabla \cdot \mathbf{v} = \lim_{\delta Vol \to 0} \frac{1}{\delta Vol} \frac{dVol}{dt} , \]
or the "local rate of expansion" as Batchelor calls it.

Now we have the two required pieces for the energy-entropy equation. Putting it all together: Add the kinetic energy identity, Eq. (6), and the internal energy conservation equation, Eq. (14). The left-hand sides add simply as written in Eq. (16) below. The right-hand sides are exactly rearranged as

\[
RHS_{KE+U} = v_i \partial_j \sigma_{ij} + \rho T D_t s - p \Delta
\]

\[
= -v_i \partial_i p + v_i \partial_j \sigma'_{ij} + \rho T D_t s - p \partial_i v_i
\]

\[
(same\ p)\quad -\partial_i (v_i p) + v_i \partial_j \sigma'_{ij} + \rho T D_t s
\]

\[
= -\partial_i (v_i p) + \partial_j (v_i \sigma'_{ij}) - \sigma'_{ij} \partial_j v_i + \rho T D_t s
\]

\[
\partial_j (v_i \sigma_{ij}) - \sigma'_{ij} \partial_j v_i + \rho T D_t s.
\]

Note carefully that \( \sigma'_{ij} \) is the viscous stress tensor, whereas \( \sigma_{ij} \) includes the pressure too according to Eq. (3). The point of obtaining this last equation, Eq. (15), is that this first term on the right can be physically identified with an energy flux term as discussed next.

Thus, altogether combining the kinetic energy identity and the internal energy conservation equation, and noting that the right-hand side can be rearranged as in Eq. (15), we are left with the energy-entropy equation:

\[
\partial_t (U + \frac{1}{2} \rho v^2) + \partial_i (U v_i + \frac{1}{2} \rho v^2 v_i) = \partial_j (v_i \sigma_{ij}) + [\rho T D_t s - \sigma'_{ij} \partial_j v_i].
\]

The square brackets group the entropy related terms and are convenient for later discussions. Note that Eq. (16) is the same as in Landau and Lifshitz near the bottom of p. 193 [2] (except that the heat flux term has not been added and subtracted yet—we do this below). However, Landau and Lifshitz did not provide separate kinetic and internal energy relations as we have done above with Eqs. (6) and (14). We find these separate kinetic and internal energy relations instructive and thus have included them here. Recall how it all came together above: this energy-entropy equation came from mass and momentum conservation and the first and second laws of local thermodynamics. It is one further dynamical equation beyond mass and momentum conservation because thermodynamics (thermal energy) has now been consistently taken into account.

As hinted by its joint name, the energy-entropy equation contains two physical principles within it: energy conservation (temporal translational invariance) and the increase of entropy due to irreversible work (second law of thermodynamics). The energy conservation terms in
Eq. (16) must be supplemented by a heat flux term that accounts for thermal conduction in the absence of motion but with inhomogeneities in the fluid due to temperature or chemical potential gradients. Thus, following Landau and Lifshitz [2], a “\( \nabla \cdot q \)” term is added and subtracted to the energy-entropy equation. \( q \) is the heat flux which is the heat energy per unit time per unit area from heat conduction in the fluid. In the case without diffusion (our initial interest) \( q = -\kappa \nabla T \) where \( \kappa \) is the thermal conductivity. Given this, as motivated from Eq. (16) itself, the energy conservation equation for a nonrelativistic viscous fluid is defined to be [2]

\[
\text{energy} \quad \partial_t (U + \frac{1}{2} \rho v^2) + \partial_i (\mathcal{U} v_i + \frac{1}{2} \rho v^2 v_i) \equiv \partial_j (v_i \sigma_{ij}) - \nabla \cdot q ,
\]

which in words can be stated as

*The net difference in a fluid between the change of internal plus kinetic energy (per unit volume) over time and the internal plus kinetic energy (per unit time per unit area) that flows through a surface is given by the net work done by the pressure and viscous forces (per unit time per unit area) acting on the surface of this fluid and the heat that is conducted through the surface.*

Bringing everything consistently to the left-hand side, as described on p. 229 of [2], \( q_j - v_i \sigma_{ij}' \) is the irreversible energy flux of a fluid whereas the remaining terms in the divergence part of Eq. (17)—including the pressure term—make up the reversible energy flux. Interestingly, note that the energy density term of Eq. (17), \( U + \frac{1}{2} \rho v^2 \), ends up being the final form of \( \mathcal{H}_E \) and \( \mathcal{H}_{NS} \), the very Hamiltonians we are seeking. But the point is that \( \mathbf{v} \) has not been defined yet in terms of hamiltonian variables. This will be done in the respective sections that follow.

Given energy conservation, Eq. (17), then the energy-entropy equation, Eq. (16), becomes the entropy terms in square brackets along with the \( \nabla \cdot q \) heat flux term with the correct sign. As was to be derived, we are left with the

\[
\rho T D_t s = \sigma_{ij}' \partial_j v_i - \nabla \cdot q .
\]

Summarizing the story that led up to this entropy constraint equation: it came simply from mass, momentum, and energy conservation applied to a classical nonrelativistic viscous
fluid where the concept of energy was necessarily enlarged to include heat according to the standard laws of local equilibrium thermodynamics. In the words of Landau and Lifshitz [2]: the left-hand side of Eq. (18) is the “amount of heat gained by unit volume of the fluid,” the first term on the right is the “energy dissipated into heat by viscosity” and the last term is the “heat conducted into the volume concerned.”

Before leaving this section we have two further notes: First, as can be seen by perusing the explicit first-principle proofs above, nowhere in these derivations was the actual form of the viscous stress tensor, $\sigma_{ij}'$, used.\(^3\) Thus, *the above entropy constraint equation also holds for a non-Newtonian fluid.* However, we will still call our results of the final section the “Navier-Stokes Hamiltonian” (defined by the Newtonian viscous stress tensor of Eq. (4)) because we think that is already interesting in itself, but keep in mind that the final $H_{NS}$ is actually more generally applicable. Second, as seen in the middle line with the “same p” comment above the equals sign in the block of equations ending with Eq. (15), the following two pressures where assumed to be equivalent: The one of

$$du = Tds - pdv$$

(the so-called equilibrium pressure [3]) and the one of

$$\sigma_{ij} \equiv -p \delta_{ij} + \sigma_{ij}'$$

(the pressure of the Navier-Stokes equation). In other words we are assuming that the pressure of the “$\nabla p/\rho$” term in the Navier-Stokes equation is equivalent to the local thermodynamic equilibrium pressure. For clarity note however that we are including a nonzero bulk viscosity $\zeta$. Therefore our pressure $p$ is not the same as the normal pressure acting on an arbitrary surface. This follows by noting (recall definitions of Eq. (4))

$$p_{\text{normal}} \equiv -\frac{1}{3} \text{tr}(\sigma) = -\frac{1}{3}(-3p + 2\eta \Delta + 3\zeta' \Delta)$$

$$= -\frac{1}{3}(-3p + 2\eta \Delta + 3\zeta \Delta - 3\frac{2}{3} \eta \Delta)$$

$$= p - \zeta \Delta. \quad (19)$$

So the pressure in the Navier-Stokes equation and the pressure normal to an arbitrary surface in the fluid differ by a term first order in $\zeta \Delta$; this includes a velocity derivative and a factor\(^3\) e.g. we did not have to assume that $\sigma_{ij}'$ was symmetric—although with angular momentum conservation it is [7]—or more importantly we did not assume that its form was necessarily that of a Newtonian fluid.
of the bulk viscosity. In nearly-incompressible fluids such as water it may be hard to discern a difference between $p$ and $p_{\text{normal}}$ and so we will not worry about this distinction for now, but note $\zeta$ and $\Delta$ remain arbitrary fields at this point and therefore the physics of Eq. (19) is contained in what follows.

III. EULER HAMILTONIAN

As a warm up for deriving $H_{NS}$ without the complication of dissipation, but also because the “no viscosity” and “small viscosity” theories really are different beasts, in this section we derive the Hamiltonian corresponding to the Euler equation of a general ideal fluid. By ‘general’ we mean the internal energy density depends on both density and specific entropy, and by ‘ideal’ we mean the usual of the viscosity terms of the Navier-Stokes equation being absent.

The procedure for deriving the Euler Hamiltonian $H_E$ starts with a definition of the lagrangian density for a general ideal fluid according to Zakharov and Kuznetsov [4]:

$$\mathcal{L}_E = \frac{1}{2} \rho \mathbf{v}^2 - \mathcal{U}(\rho, s) + \phi \left[ \partial_t \rho + \partial_i (\rho \mathbf{v}_i) \right] + \alpha \left[ \partial_t \beta + \mathbf{v}_i \partial_i \beta \right] + \lambda \left[ \partial_t s + \mathbf{v}_i \partial_i s \right].$$ (20)

The variables of $\mathcal{L}_E$ are described next, but first note that compared to [4] we change the names of some of the variables as well as the sign of the $\alpha$ and $\lambda$ lagrange multiplier scalar fields so that they are related by a plus sign to the conjugate momentum field that they represent ($\pi_\beta = +\alpha$ and $\pi_s = +\lambda$). The variables of $\mathcal{L}_E$ are

- $\rho$ : density
- $\mathbf{v}$ : constrained velocity vector
- $\mathcal{U}$ : internal energy density
- $s$ : specific entropy
- $\phi$ : velocity potential
- $(\alpha, \beta)$ : Clebsch potential pair
- $\lambda$ : lagrange multiplier field for the ideal entropy constraint

Now we derive the Euler-Lagrange equations for $\mathcal{L}_E$ and then show that they are equivalent to the equations of motion of an ideal fluid: the Euler, continuity and ideal entropy
constraint equations. Then the section concludes with a derivation of \( H_E \) and shows that its Hamilton equations satisfy these same dynamics.

### A. Euler-Lagrange equations

The Euler-Lagrange equation, from varying scalar field \( \phi \) for example, of a lagrangian density with at most first order derivatives is given by

\[
\partial_t \left( \frac{\partial L}{\partial (\partial_t \phi)} \right) + \partial_i \left( \frac{\partial L}{\partial (\partial_i \phi)} \right) = \frac{\partial L}{\partial \phi} .
\]  

(21)

Thus the Euler-Lagrange equations from varying all the fields of \( L_E \) are as follows.

**Varying \( \rho \):**

\[
\partial_t \left( \frac{\partial L_E}{\partial (\partial_t \rho)} \right) + \partial_i \left( \frac{\partial L_E}{\partial (\partial_i \rho)} \right) = \frac{\partial L_E}{\partial \rho} ,
\]

\[
\Rightarrow \partial_t \phi + \partial_i (\phi v_i) = \frac{v^2}{2} - \frac{\partial U}{\partial \rho} + \phi \partial_i v_i
\]

\[
\Rightarrow D_t \phi - \frac{v^2}{2} + h = 0 ,
\]

(22)

where \( D_t \) is the material derivative of Eq. (13). This correct dynamical equation for \( \phi \) may look puzzling for the moment, but it makes more sense once the \( v \) constraint has been worked out:

**Varying \( v \):**

\[
\partial_t \left( \frac{\partial L_E}{\partial (\partial_t v_i)} \right) + \partial_j \left( \frac{\partial L_E}{\partial (\partial_j v_i)} \right) = \frac{\partial L_E}{\partial v_i} ,
\]

\[
\Rightarrow \partial_t (0) + \partial_j (\rho \phi \delta_{ij}) = \rho v_i + \phi \partial_i \rho + \alpha \partial_i \beta + \lambda \partial_i \gamma
\]

\[
\Rightarrow \mathbf{v} = \nabla \phi - \frac{\phi}{\rho} \nabla \beta - \frac{\lambda}{\rho} \nabla \gamma
\]

(23)

The time derivative of \( \mathbf{v} \) is absent here and thus this is a constraint equation for \( \mathbf{v} \). Often in what follows we keep writing "\( \mathbf{v} \)" for simplicity, but what is really meant by this is the right-hand side of this constraint equation. This includes the dynamical equation for \( \phi \) that was just derived: Eq. (22), including of course the \( \mathbf{v} \) inside the material derivative. Now varying the remaining fields of \( L_E \) gives

**Varying \( \phi \):**

\[
\Rightarrow \partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0 ;
\]

(24)

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Varying $\alpha$:

\[ D_t \beta = 0; \quad (25) \]

Varying $\beta$:

\[ \Rightarrow \partial_t \alpha + \partial_i (\alpha v_i) = 0, \]
\[ \Rightarrow \partial_t \alpha + \nabla \cdot (\alpha v) = 0; \quad (26) \]

Varying $\lambda$:

\[ D_t s = 0; \quad (27) \]

Varying $s$:

\[ \Rightarrow \partial_t \lambda + \partial_i (\lambda v_i) = -\frac{\partial U}{\partial s} - \rho T, \]
\[ \Rightarrow \partial_t \lambda + \nabla \cdot (\lambda v) + \rho T = 0. \quad (28) \]

In summary, varying $\mathcal{L}_E$ of Eq. (20) produces one constraint equation for vector field $v$ and six equations of motion for scalar fields $\rho, \phi, \alpha, \beta, \lambda,$ and $s$: Eqs. (22)–(28). Now we show that these scalar equations of motion are equivalent to the dynamics contained in the Euler equation. Then we move on to the Euler Hamiltonian and show how its resulting Hamilton equations reproduce these same equations of motion.

### B. Euler equation equivalence proof

The exact equivalence between the Euler-Lagrange equations of $\mathcal{L}_E$ and the Euler equation (with mass conservation and basic thermodynamics included as well) is shown here. Starting with the momentum conservation equation of an ideal fluid, we insert velocity constraint Eq. (23), use thermodynamic relation Eq. (12), and show that the result can be exactly rearranged into a form proportional to the six dynamical equations of motion just discussed for $\rho, \phi, \alpha, \beta, \lambda,$ and $s$. 

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1. **Explicit proof**

Start with momentum conservation operator Eq. (2) with all terms moved to the left and the viscosity terms dropped:

\[
\mathcal{O}_E \equiv \partial_i (\rho v_i) + \partial_j (\rho v_i v_j) + \partial_i p ;
\] (29)

then substitute velocity constraint Eq. (23) written as

\[
\rho v_i = \rho \partial_i \phi - \alpha \partial_i \beta - \lambda \partial_i s ,
\] (30)

and use thermodynamic relation Eq. (12):

\[
\partial_i p = \rho \partial_i h - \rho T \partial_i s ;
\] (31)

finally, exactly rearrange the result into a form proportional to the dynamical Euler-Lagrange equations. This follows as shown below. Note that we are not setting \( \mathcal{O}_E \) to zero at this point, but rather are seeing how its operator form varies with these six scalar potentials from the pathlines (\( \mathbf{v} \) constraint) implied by \( \mathcal{L}_E \). After this \( \mathbf{v} \) constraint and thermodynamic \( \nabla p \) substitution has been made, the above \( \mathcal{O}_E \) becomes

\[
\mathcal{O}_E \rightarrow \partial_i (\rho \partial_i \phi - \alpha \partial_i \beta - \lambda \partial_i s) + \partial_j [v_j (\rho \partial_i \phi - \alpha \partial_i \beta - \lambda \partial_i s)] + \rho \partial_i h - \rho T \partial_i s .
\] (32)

This looks a little unwieldy, but we see with the product rule of partial differentiation this will only produce 17 total terms. Continuing, expand out the differential products, collect common factors, and exactly rearrange by adding and subtracting identical terms to get the \( D_t \beta \) and \( D_t s \) terms to work out right; then note three cancellations and the remaining dust
becomes the \( v^2/2 \) term because of the \( v \) constraint consistently upheld. Thus we obtain

\[
\mathcal{O}_E = \rho \partial_t h - \rho T \partial_s s +
+ (\partial_j \rho) \partial_j \phi + \rho \partial_t \partial_t \phi - (\partial_j \alpha) \partial_j \beta - \alpha \partial_t \partial_t \beta - (\partial_t \lambda) \partial_t s - \lambda \partial_t \partial_t s
+ (\partial_j v_j)(\rho \partial_t \phi - \alpha \partial_t \beta - \lambda \partial_t s) + v_j(\partial_j \rho) \partial_t \phi + v_j \rho \partial_t \partial_j \phi
- v_j(\partial_j \alpha) \partial_t \beta - v_j \alpha \partial_t \partial_j \beta - v_j(\partial_j \lambda) \partial_t s - v_j \lambda \partial_t \partial_j s
= \quad (\partial_t \phi) [\partial_t \rho + \nabla \cdot (\rho \mathbf{v})]
- (\partial_t \beta) [\partial_t \alpha + \nabla \cdot (\alpha \mathbf{v})]
- (\partial_t s) [\partial_t \lambda + \nabla \cdot (\lambda \mathbf{v}) + \rho T]
- \alpha \partial_t [D_t \beta]
- \lambda \partial_t [D_t s]
+ \rho \partial_t \left[D_t \phi - \frac{\mathbf{v}^2}{2} + h\right].
\tag{33}
\]

Seemingly almost like magic (symmetry being the underlying magician), we see \( \mathcal{O}_E \) is proportional to all six of these dynamical Euler-Lagrange equations under discussion. Finally note the following simple result

\[
\mathcal{O}_E \equiv \partial_t (\rho v_i) + \partial_j (\rho v_i v_j) + \partial_t p
= v_i [\partial_t \rho + \nabla \cdot (\rho \mathbf{v})] + \rho \left[ \partial_t v_i + v_j \partial_j v_i + \frac{\partial_t p}{\rho} \right],
\tag{34}
\]

which as mentioned earlier, shows that the momentum conservation operator is proportional to the continuity and in this case Euler equation operator. Thus, if all six of these independent Euler-Lagrange equations are satisfied (with respective operator vanishing), then \( \mathcal{O}_E = 0 \) and the Euler and continuity equations must therefore also both be satisfied and the two approaches are therefore dynamically equivalent. q.e.d. A final remark is to recall that in order to obtain this exact equivalence, local equilibrium thermodynamics was used too through Eq. [12]. In short, \( \mathcal{L}_E \) of Eq. [20] has the same dynamics as the general Euler equation where pressure depends on density and entropy. Now we use \( \mathcal{L}_E \) to derive the Euler Hamiltonian.

C. Hamilton equations

This section completes the derivation of the Euler Hamiltonian, \( H_E \), and shows that its Hamilton equations are the same six dynamical equations of motion as from the Euler-
Lagrange equations just discussed. These results are well known but the final section with
the Navier-Stokes Hamiltonian is perhaps not as well known and the two procedures are
compared in the end to highlight the differences and similarities between obtaining a Hamil-
tonian with and without dissipation.

To derive \( H_E \), first we need to determine its nonzero conjugate momentum fields. These
follow simply from \( L_E \) of Eq. (20):

\[
\pi_\rho \equiv \frac{\partial L_E}{\partial (\partial_t \rho)} = \phi(x, t), \tag{35}
\]

\[
\pi_\beta \equiv \frac{\partial L_E}{\partial (\partial_t \beta)} = \alpha(x, t), \tag{36}
\]

\[
\pi_s \equiv \frac{\partial L_E}{\partial (\partial_t s)} = \lambda(x, t). \tag{37}
\]

In the following, since it is unambiguous and leaves a cleaner notation, often we will con-
tinue to write ‘\( \phi \)’, ‘\( \alpha \)’, and ‘\( \lambda \)’ for the conjugate momentum fields, although this is actually
a shorthand for the conjugate momentum fields they represent: ‘\( \pi_\rho(x, t) \)’, ‘\( \pi_\beta(x, t) \)’, and ‘\( \pi_s(x, t) \)’ respectively.

Given these conjugate momenta, \( H_E \) follows from the canonical procedure\footnote{This “canonical procedure” for deriving a Hamiltonian is shown in the Navier-Stokes section of this paper.}:

\[
H_E \equiv \int d^3x H_E, \tag{38}
\]

\[
H_E = \pi_\rho \partial_t \rho + \pi_\beta \partial_t \beta + \pi_s \partial_t s - L_E
= \phi \partial_t \rho + \alpha \partial_t \beta + \lambda \partial_t s - L_E
= -\frac{1}{2} \rho \nabla v^2 + U(\rho, s) - \phi \nabla \cdot (\rho \nabla) - \alpha (\nabla \cdot \nabla) \beta - \lambda (\nabla \cdot \nabla) s. \tag{39}
\]

Note \( v \) satisfies the same constraint that came from varying \( L_E \) which for emphasis is
rewritten here. ‘\( v \)’ in Eq. \( (39) \) is really a shorthand for

\[
v \rightarrow \nabla \phi - \frac{\alpha}{\rho} \nabla \beta - \frac{\lambda}{\rho} \nabla s. \tag{40}
\]

Thus, these right three terms of Eq. \( (39) \) are seen to be related to a \( v^2 \) term; with a slight
rearrangement of the third to last term the above becomes

\[ H_E = -\frac{1}{2} \rho v^2 + U(\rho, s) - \partial_i (\rho \phi v_i) + \rho v_i \partial_i \phi - \alpha v_i \partial_i \beta - \lambda v_i \partial_i s \]

\[ = -\frac{1}{2} \rho v^2 + U(\rho, s) - \partial_i (\rho \phi v_i) - \partial_i v_i \partial_i \phi \]

\[ = -\frac{1}{2} \rho v^2 + U(\rho, s) - \rho v^2 - \nabla \cdot (\rho \phi v) \]

\[ = -\frac{1}{2} \rho v^2 + U(\rho, s) - \rho v^2 - \nabla \cdot (\rho \phi v) . \]  \hspace{1cm} (41)

Upon performing the integration required by Eq. (38), the last term of Eq. (41) is seen to be a surface integral at spatial infinity which we assume vanishes on physical grounds of all finite-energy fields vanishing there. Thus the Euler Hamiltonian for a nonrelativistic inviscid fluid is given by

\[ H_E[\rho, \phi; \beta, \alpha; s, \lambda] = \int d^3 x \left[ \frac{1}{2} \rho v^2 + U(\rho, s) \right] \text{ with constraint } 40 \hspace{1cm} (42) \]

Note that these functional arguments of \( H_E \) on the left are written in terms of its three dynamical coordinate fields: \( \rho \), \( \beta \), and \( s \); and their respective conjugate momentum fields: \( \phi \), \( \alpha \), and \( \lambda \).

Now onto Hamilton’s equations. Out of necessity, in the Navier-Stokes Hamiltonian section that follows, we will carefully go through the derivation of Hamilton’s equations because they are modified due to dissipation. However for this Euler fluid let us just quote the standard field theory result for Hamilton’s equations of a Hamiltonian with coordinate scalar field \( \phi \), for example, and conjugate momentum field \( \pi_\phi \) [5]:

\[ \partial_t \phi = \frac{\delta H_E}{\delta \pi_\phi} = \frac{\partial H_E}{\partial \pi_\phi} - \partial_i \left( \frac{\partial H_E}{\partial (\partial_i \pi_\phi)} \right) , \]  \hspace{1cm} (43a)

\[ \partial_t \pi_\phi = -\frac{\delta H_E}{\delta \phi} = -\frac{\partial H_E}{\partial \phi} + \partial_i \left( \frac{\partial H_E}{\partial (\partial_i \phi)} \right) . \]  \hspace{1cm} (43b)

The opposite sign of the last terms on the right follows from a spatial integration by parts and the dropping of spatial boundary terms at infinity because all field variations \( \delta \phi \) are assumed to vanish there. Note also that these Hamilton equations are based on variations of the field \( \delta \phi \) vanishing at the temporal endpoints (required for the Euler-Lagrange equation derivation itself which is used under-the-hood in defining a Hamiltonian [6]). We mention these details here because this vanishing of these field variation boundary terms causes no conceptual problem for the Euler fluid, but because of dissipation necessitates introducing
lagrangian vector field $X(x,t)$ for the Navier-Stokes fluid. Thus, here is a point in the
procedure where the Euler and Navier-Stokes fluids differ immensely.

Continuing, the Euler-fluid Hamilton equations for the three coordinate pairs of $H_E$
become

**coordinate $\rho$:**

\[
\partial_t \rho = \frac{\delta H_E}{\delta \pi_\rho} = \frac{\partial H_E}{\partial \rho} = \partial_i \left( \frac{\partial H_E}{\partial (\partial_i \phi)} \right),
\]

$\Rightarrow \partial_t \rho = -\partial_i (\rho v_i)$,

$\Rightarrow \partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0; \quad (44)$

**momentum $\pi_\rho = \phi$:**

\[
\partial_t \pi_\rho = \partial_t \phi = -\frac{\delta H_E}{\delta \rho} = -\frac{\partial H_E}{\partial \rho} + \partial_i \left( \frac{\partial H_E}{\partial (\partial_i \rho)} \right),
\]

$\Rightarrow \partial_t \phi = -\frac{1}{2} \mathbf{v}^2 - \frac{\rho}{2} v_i \left[ \frac{1}{\rho^2} \left( \alpha \partial_i \beta + \lambda \partial_i s \right) \right] - \frac{\partial U}{\partial \rho}$

$+ \frac{1}{2} \mathbf{v}^2 - \mathbf{v} \cdot \nabla \phi - \frac{\partial U}{\partial \rho}$

$\Rightarrow \frac{1}{2} \mathbf{v}^2 - \mathbf{v} \cdot \nabla \phi - h$

$\Rightarrow \partial_t \phi - \mathbf{v}^2 + h = 0; \quad (45)$

**coordinate $\beta$:**

\[
\partial_t \beta = \frac{\delta H_E}{\delta \pi_\beta} = \frac{\partial H_E}{\partial \beta} = \partial_i \left( \frac{\partial H_E}{\partial (\partial_i \alpha)} \right),
\]

$\Rightarrow \partial_t \beta = v_i (-\partial_i \beta)$,

$\Rightarrow \partial_t \beta = 0; \quad (46)$

**momentum $\pi_\beta = \alpha$:**

\[
\partial_t \pi_\beta = \partial_t \alpha = -\frac{\delta H_E}{\delta \beta} = -\frac{\partial H_E}{\partial \beta} + \partial_i \left( \frac{\partial H_E}{\partial (\partial_i \beta)} \right),
\]

$\Rightarrow \partial_t \alpha = \partial_i [v_i (-\alpha)]$

$\Rightarrow \partial_t \alpha + \nabla \cdot (\alpha \mathbf{v}) = 0; \quad (47)$

**coordinate $s$:**

\[
\partial_t s = \frac{\delta H_E}{\delta \pi_s} = \frac{\partial H_E}{\partial \lambda} = \partial_i \left( \frac{\partial H_E}{\partial (\partial_i \lambda)} \right),
\]

$\Rightarrow \partial_t s = v_i (-\partial_i s)$,

$\Rightarrow \partial_t s = 0; \quad (48)$
momentum $\pi_s = \lambda$:

$$
\partial_t \pi_s = \partial_t \lambda = -\frac{\delta H_E}{\delta s} = -\frac{\partial H_E}{\partial s} + \partial_t \left( \frac{\partial H_E}{\partial (\partial_t s)} \right),
$$

$$
\Rightarrow \partial_t \lambda = -\frac{\partial U}{\partial s} + \partial_t [v_i (-\lambda)]
$$

$$
\Rightarrow \rho T - \nabla \cdot (\lambda \mathbf{v}),
$$

$$
\Rightarrow \partial_t \lambda + \nabla \cdot (\lambda \mathbf{v}) + \rho T = 0. \quad (49)
$$

These are the same as the above Euler-Lagrange equations. q.e.d.

IV. NAVIER-STOKES HAMILTONIAN

This section in a sense is simply a repeat of the previous section with viscosity included. However, this one simple fact leads to dissipation and complicates the derivation of a Hamiltonian. Nevertheless, it is straightforward and this section derives the canonical Navier-Stokes Hamiltonian, $H_{NS}$, from first principles. In order to include dissipation, entropy constraint Eq. (18) must be accounted for consistently. As emphasized by Fukagawa and Fujitani [7], the viscosity dependence of $H_{NS}$ must enter through variations of the entropy and this necessarily introduces a nonholonomic constraint [8] on the system. In order to treat this constraint properly, with variations in entropy being proportional to variations in a coordinate field, one must introduce lagrangian vector field $X(x, t)$ [7] which can be thought of as a label to all of the fluid particles [4] although physically it is their initial positions at some arbitrary time $t_0$. Even though $X(x, t)$ is a lagrangian coordinate, we are still in the Euler description of a fluid and are not explicitly following all these fluid particles around. The standard field theory point of view is maintained, although as shown below, $X(x, t)$ turns out to be the main dynamical coordinate field of $H_{NS}$ and is assumed to be a one-to-one map to all of their initial positions [4]. Although some of phase space could be missed by this one-to-one assumption, in three dimensions especially, the missed part may turn out to be a set of measure zero—a starting ansatz of this approach. $H_{NS}$ derived below seems to be written with the natural dynamical coordinates for studying fluid particle separations over time. Future papers will study this further with the similarity renormalization group and Poisson bracket structure derived in this paper.
A. Nonholonomic constraint

The equations of motion follow from Hamilton’s principle that the variation of an action vanishes:

$$\delta S = \int d^4x \delta \mathcal{L} = \int dt \int d^3x \delta \mathcal{L} = 0,$$

(50)

where $\mathcal{L}$ is a lagrangian density and the integral is over all space and an interval of time between arbitrary endpoint initial and final times for the dynamics of interest. Interactions of interest are simply added to $\mathcal{L}$. This includes constraints of interest such as the entropy constraint of Eq. (18). Requiring it to be satisfied over all space gives, with some exact rearrangements,

$$0 = \int d^3x \left( \rho T D_t s - \sigma'_ij \partial_j v_i + \partial_j q_j \right)$$

$$= \int d^3x \left[ \rho T \partial_t s + \rho T v_i \partial_i s + v_i \partial_j \sigma'_{ij} + \partial_j (q_j - v_i \sigma'_{ij}) \right]$$

$$= \int d^3x \left[ \rho T \partial_t s + \rho T v_i \partial_i s + v_i \partial_j \sigma'_{ij} \right] + \int d^2x \hat{n}_j (q_j - v_i \sigma'_{ij}),$$

using the divergence theorem once again to convert the volume integral into a surface integral with unit outward normal $\hat{n}$. Note that the integrand of this surface integral in the last term is the exact same irreversible energy flux discussed below the energy conservation equation, Eq. (17). Since heat flux $q$ is proportional to temperature (and in general chemical potential) gradients and $\sigma'_ij$ is proportional to velocity gradients, and all field gradients are assumed to vanish at spatial infinity, this term can be safely dropped. Thus the constraint to be satisfied becomes

$$0 = \int d^3x \left[ \rho T \partial_t s + v_i \left( \rho T \partial_i s + \partial_j \sigma'_{ij} \right) \right],$$

(51)

where we have rearranged it for convenience as described next. Eq. (51) is a nonholonomic path-dependent constraint (with pathlines given by $v_i$) due to the irreversible dynamics of heat being added to the volume through viscosity. Thus the Euler Lagrangian, Eq. (20), can not be simply modified by just changing its last entropy constraint term to the above. Rather we must introduce lagrangian coordinate vector field $X(x, t)$ as discussed in the opening paragraph of this section.

Nonholonomic constraints such as Eq. (51) must be expressed as “a linear relation connecting the differentials of the [coordinates]” in field theory the meaning of coordinate

\[5\] with the usual minus sign: $L = T - V$. 

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naturally generalizes to coordinate field. Thus it is natural to introduce a lagrangian coordinate vector field $X(x,t)$, which is just a coordinate transformation, and to seek a linear relation connecting its variations, $\delta X(x,t)$, with those of the other coordinate field pertinent to any discussion of heat dissipation: specific entropy variation $\delta s(x,t)$. As detailed in [4, 7], $X(x,t)$ denotes the initial position of all the fluid particles that happen to be at $x$ at time $t$. At initial time $t_0$ we have

$$X(x,t_0) = x.$$ (52)

In other words, $X$ is just another $x$—but it is the initial one and an assumed one-to-one map to all of the fluid particles. Consult [4] for further discussions including the relabeling (gauge) symmetry of fluid particles that $X$ introduces. So $X(x,t)$ is just a coordinate transformation and we assume that it is a non-singular one, i.e. its Jacobian never vanishes:

$$J \equiv \frac{\partial(X_1, X_2, X_3)}{\partial(x_1, x_2, x_3)} \equiv \det(\hat{J}_{ij}) \equiv \det \left( \frac{\partial X_i}{\partial x_j} \right) \equiv \det (\partial_j X_i) \neq 0,$$ (53)

where like [4] the hat on $\hat{J}$ implies “matrix” and $\partial_i = \partial / \partial x_i$ is the same shorthand of (euclidean) field theory that we have been using throughout this paper.\(^6\)

Although this is not the most general choice [9], it is nevertheless intuitive and leads to the standard Navier-Stokes dynamics as shown below. Therefore we assume that $X$ is conserved along all fluid path lines:

$$\partial_t X + (v \cdot \nabla) X = D_t X = 0.$$ (54)

This is a very important dynamical assumption to what follows: Through it deriving the Navier-Stokes Hamiltonian becomes soluble. Given Eq. (53), inverting Eq. (54) for the pathlines, $v_i$, is easy and follows from the chain rule of partial differentiation:

$$v_i = -\frac{\partial x_i}{\partial X_j} \partial_t X_j;$$ (55)

just multiply this with $\partial_j X_k = \frac{\partial X_k}{\partial x_i}$, use $\frac{\partial X_k}{\partial X_j} = \delta_{jk}$, and it gives back Eq. (54). Substituting Eq. (55) into Eq. (51) changes the nonholonomic constraint into

$$0 = \int d^3 x \left[ \rho T \partial_t s - (\rho T \partial_i s + \partial_k \sigma_{ik}^' \left( \frac{\partial x_i}{\partial X_j} \right) \partial_t X_j \right].$$ (56)

\(^6\) Although “$X$ is just another $x$,” to manage our notation and in keeping with the Euler description of a fluid, we will always write out the derivatives with respect to $X$ in full such as $\partial / \partial X_i$; but whenever we come across $\partial / \partial x_i$ we will continue to replace it by the convenient $\partial_i$. 
Now, just like was done with some of the thermodynamic differential relations earlier, these partial time derivatives in Eq. (56) can be generalized to arbitrary field variations. The point is that the two field variations ($\delta X$ and $\delta s$) are proportional to each other and they can be either spatial or temporal or general variations as given by a differentiable manifold. Thus the final form of the nonholonomic constraint is [7]

\[
\rho T \delta s = \left( \rho T \partial_i s + \partial_k \sigma'_{ik} \right) \left( \frac{\partial x_i}{\partial X_j} \right) \delta X_j .
\]

This is the linear relation of Goldstein’s [8] that we were seeking between the coordinate field variations. It may be instructive to note that this factor $\frac{\partial x_i}{\partial X_j}$ is just the matrix inverse to the gradient of vector field $X$:

\[
\left( \frac{\partial x_i}{\partial X_j} \right) \nabla_k X_j = \frac{\partial x_i}{\partial X_j} \frac{\partial X_j}{\partial x_k} = \frac{\partial x_i}{\partial x_k} = \delta_{ik} .
\]

This is the same as saying that $\frac{\partial x_i}{\partial X_j}$ is the matrix inverse of Jacobian matrix $\hat{J}_{ij}$ defined in Eq. (53) above. Thus, $\frac{\partial x_i}{\partial X_j}$ is a nonlocal object with derivatives of vector field $X$ in the denominator.

Before leaving this section we discuss mass conservation in these new coordinates. Mass conservation equivalent to Eq. (1) is now simply given by the following kinematic constraint [7]

\[
\rho(x, t) = J \rho_0 = \rho(\nabla X) \rho_0(X) ,
\]

where $J$ is the Jacobian of Eq. (53) and all of the field dependencies are written out on the right for clarity.\footnote{Notation warning: what is written here as $J$ for the Jacobian is written as the inverse $J^{-1} = 1/J$ in [7] and [4] respectively.} As shown below in the Navier-Stokes equation section, taking a partial time derivative of Eq. (59) exactly reproduces the continuity equation, Eq. (1), as an operator and thus they are equivalent, but here we close this section with a simpler proof. Integrating Eq. (59) over all space and using the usual rules of a Jacobian shows that the total mass of the fluid:

\[
\int d^3x \rho(x, t) = \int d^3x J \rho_0 = \int d^3X \rho_0(X) ,
\]

is the same in both coordinate systems, i.e. mass is conserved. Now we have all the pieces for $H_{NS}$ and we move on to its derivation.
B. Navier-Stokes Lagrangian

Just like in the Euler Hamiltonian section above, the derivation of the canonical Navier-Stokes Hamiltonian, $H_{NS}$, begins with a definition of its lagrangian density. As motivated by earlier discussions, this time we use the form of Fukagawa and Fujitani [7]:

$$L_{NS}(\rho, v, s, K, X, P) = \frac{1}{2} \rho v^2 - U(\rho, s) + P \cdot D_t X + K [\rho - J(\nabla X) \rho_0(X)].$$  \hspace{1cm} (61)

Most of the variables have been defined earlier, however vector field $P$ and scalar field $K$ are new. They are simply the lagrange multiplier fields for the respective constraints of $X$ being conserved with the flow and mass be conserved generally. Vector field $P$ also ends up being the conjugate momentum field to $X$ itself, so these two fields together end up being the only dynamical fields of $L_{NS}$ and consequently $H_{NS}$ as well. Finally, it is important to note that $L_{NS}$ is implied to be augmented with the nonholonomic constraint of Eq. (57) that relates the $\delta s(x, t)$ variations to those of $\delta X(x, t)$. Actually, as emphasized by [7], because of dissipation, $L_{NS}$ (and therefore $H_{NS}$ too) is not a function of the specific entropy field $s(x, t)$, but rather can only be specified in terms of its variations. This is a consequence of having a nonholonomic constraint. This causes no problems in principle because the general dynamics of a theory come from the Hamilton equations and Poisson brackets (which evolve any other observables of interest besides just dynamical fields $X$ and $P$) and these follow from variations of $H_{NS}$ and the observables themselves as shown in the last section of this paper.

Now we move on to the Euler-Lagrange equations for $L_{NS}$ and show that they are equivalent to the Navier-Stokes equation (with mass and energy conservation). Then $H_{NS}$ is derived and its Hamilton equations are shown to be these same Euler-Lagrange equations.

C. Euler-Lagrange equations

The Euler-Lagrange equations of $L_{NS}$ with nonholonomic constraint Eq. (57) follow straightforwardly from Hamilton’s principle as shown next. To start, note that the following fields of $L_{NS}$ all have a simple Euler-Lagrange equation (with no spatial or temporal partial derivatives of the respective fields in $L_{NS}$): $v$, $P$, $\rho$, and $K$. Thus for simplicity of notation we define the following temporary collection of fields by

$$\chi_\alpha = (v, P, \rho, K)$$  \hspace{1cm} (62)
for $\alpha = 1, \ldots, 8$; this is just a shorthand for $\chi_1 = v_1, \chi_2 = v_2, \ldots, \chi_8 = K$. Then Hamilton’s principle on the Navier-Stokes action with constraint Eq. (57) becomes

$$0 = \delta S_{NS} = \int d^4x \, \delta L_{NS} =$$

$$\int d^4x \left[ \frac{\partial L_{NS}}{\partial \chi_\alpha} \delta \chi_\alpha + \frac{\partial L_{NS}}{\partial s} \frac{\delta s}{\delta X_i} \delta X_i + \frac{\partial L_{NS}}{\partial X_i} \delta X_i + \frac{\partial L_{NS}}{\partial (\partial_t X_i)} \delta (\partial_t X_i) + \frac{\partial L_{NS}}{\partial (\partial_j X_i)} \delta (\partial_j X_i) \right]$$

(63)

The integrand of the second term from Eqs. (9) and (61) becomes

$$\delta L^{(2)}_{NS} \equiv \frac{\partial L_{NS}}{\partial s} \frac{\delta s}{\delta X_i} \delta X_i = -\frac{\partial U}{\partial s} \frac{\delta s}{\delta X_i} \delta X_i = -\rho T \frac{\delta s}{\delta X_i} \delta X_i$$

(64)

which will often be left in this form for simplicity; however it is implied by this last expression that the remaining variational derivative is to be replaced by the nonholonomic constraint of Eq. (57) leaving

$$\frac{\delta L^{(2)}_{NS}}{\delta X_i} = -\rho T \frac{\delta s}{\delta X_i} = -\left( \rho T (j_2 s + \partial_k \sigma'_{jk}) \right) \frac{\delta x_j}{\delta X_i}.$$ 

(65)

This term comes up often in what follows. One way to state the results of varying $L_{NS}$ is to say that all of its functional derivatives follow the standard procedures of field theory except for the ones with respect to vector field $X$: because of dissipation, these variations, $\delta L_{NS}/\delta X_i$, are appended with this boxed term, $\delta L^{(2)}_{NS}/\delta X_i$.

We continue with a few standard looking steps deriving the Euler-Lagrange equations that result from Eq. (63) for two reasons: (1) it will be useful in the Hamiltonian and Poisson bracket sections below and (2) because we would like to emphasize that these steps would not follow so simply except for the fact that coordinate transformation $X(x,t)$ has been introduced. As emphasized by [7], the spatial and temporal boundary conditions of the surface terms could not be met if it were not for the specific entropy variations $\delta s$ being replaced by those of $\delta X$. Handling $\delta L^{(2)}_{NS}$ as above, continuing with Eq. (63) in the standard way gives two surface terms and two sign flips:

$$0 = \delta S_{NS} = \int d^4x \, \delta L_{NS} =$$

$$\int d^4x \left. \frac{\partial L_{NS}}{\partial (\partial_t X_i)} \delta X_i \right|_{\text{temporal endpoints}} + \int dt \, d^2x \hat{n}_j \left. \frac{\partial L_{NS}}{\partial (\partial_j X_i)} \delta X_i \right|_{\text{spatial boundary}} +$$

$$\int d^4x \left\{ \frac{\partial L_{NS}}{\partial \chi_\alpha} \delta X_\alpha + \left[ \frac{\partial L_{NS}}{\partial X_i} - \partial_t \left( \frac{\partial L_{NS}}{\partial (\partial_t X_i)} \right) - \partial_j \left( \frac{\partial L_{NS}}{\partial (\partial_j X_i)} \right) - \rho T \frac{\delta s}{\delta X_i} \right] \delta X_i \right\}.$$ 

(66)

Since $X$ has been introduced, these surface terms vanish by the same physical ansatz used for the Euler fluid: that coordinate field variations vanish at spatial infinity and the temporal
endpoints. $\delta s(x,t)$ cannot be made to vanish like this, its variations depend on the path taken \[7\]. The remaining fields are independent and their variations are arbitrary, thus this implies the following Euler-Lagrange equations for the Navier-Stokes fluid:

$$\frac{\partial \mathcal{L}_{\text{NS}}}{\partial \chi_\alpha} = 0 \quad ,$$

(67)

$$\partial_t \left( \frac{\partial \mathcal{L}_{\text{NS}}}{\partial (\partial_t X_i)} \right) = \frac{\partial \mathcal{L}_{\text{NS}}}{\partial X_i} - \partial_j \left( \frac{\partial \mathcal{L}_{\text{NS}}}{\partial (\partial_j X_i)} \right) - \rho T \frac{\delta s}{\delta X_i} \quad .$$

(68)

where recall $\chi_\alpha$ is just a convenient shorthand for the two vector and two scalar fields of Eq. \[62\].

Before carrying out these Euler-Lagrange equations, there are two math results to discuss. First, recall the definition of Jacobian $J$ discussed above and defined by Eq. \[53\]. Note that it depends on the gradient of $X$ and that $\mathcal{L}_{\text{NS}}$ of Eq. \[61\] contains a factor of $J$ in the mass conservation constraint. Thus, for the Euler-Lagrange equations, we need to know how to take a derivative of this Jacobian with respect to $\partial_j X_i$. This follows from the so-called method of cofactors \[7\ 10\]:

$$\frac{\partial J}{\partial (\partial_j X_i)} = J \frac{\partial x_j}{\partial X_i} \quad .$$

(69)

Second, note that a further derivative of this result vanishes \[7\]:

$$\partial_j \left( \frac{\partial J}{\partial (\partial_j X_i)} \right) = \partial_j \left( J \frac{\partial x_j}{\partial X_i} \right) = 0 \quad .$$

(70)

These two relations will be used in deriving the equations of motion below. The second of these equations, Eq. \[70\], is useful in canceling terms between the messes that arise. It follows from straight-forward algebra, with the partial derivatives of the two factors of the product exactly canceling, however to obtain the result note the following more general equation for the method of cofactors from Dirac’s book again \[10\]:

$$\frac{\partial J}{\partial x_k} \equiv \partial_k J \equiv \partial_k [\det (\partial_j X_i)] = J \frac{\partial x_j}{\partial X_i} \partial_k (\partial_j X_i) \quad .$$

(71)

At first sight the final result on the right looks a little unwieldy, but with all the factors explicitly shown in the determinant, it follows straightforwardly from the chain rule of calculus.

Back to the Euler-Lagrange equations: Respectively varying the independent fields of $\mathcal{L}_{\text{NS}}$, according to Eqs. \[67\] and \[68\], gives
Varying $v$:

$$\frac{\partial L_{NS}}{\partial v_i} = 0 ,$$

$$\Rightarrow \rho v_i + P_j \partial_i X_j = 0 ,$$

$$\Rightarrow [\rho v = -P_i \nabla X_i] . \quad (72)$$

This is the velocity constraint for the Navier-Stokes fluid. Note how written this way in terms of the momentum density, $\rho v$, the right-hand side does not depend on density. Once again, like with the velocity constraint for the Euler fluid, after these Euler-Lagrange equations have been derived, when ‘$\rho v$’ is written, ‘$-P_i \nabla X_i$’ is implied. Continuing with the remaining fields of $L_{NS}$ gives

Varying $P$:

$$\Rightarrow D_t X = 0 ; \quad (73)$$

Varying $\rho$:

$$\frac{\partial L_{NS}}{\partial \rho} = 0 ,$$

$$\Rightarrow \frac{v^2}{2} - \frac{\partial U}{\partial \rho} + K = 0 ,$$

$\Rightarrow \frac{v^2}{2} - h + K = 0 ,$

$$\Rightarrow [K = h - \frac{v^2}{2} ] ; \quad (74)$$

Varying $K$:

$$\frac{\partial L_{NS}}{\partial K} = 0 ,$$

$$\Rightarrow [\rho = J \rho_0 ] \quad (75)$$

where $\rho_0(\mathbf{X}(\mathbf{x}, t))$ is the initial density set by the physics of the problem and $J$ is the Jacobian of Eq. (53), a function of the gradient of $\mathbf{X}$ given explicitly by

$$J(\nabla \mathbf{X}) \equiv \frac{\partial (X_1, X_2, X_3)}{\partial (x_1, x_2, x_3)} = \epsilon_{ijk} (\partial_i X_1)(\partial_j X_2)(\partial_k X_3) = \frac{1}{6} \epsilon_{ijk} \epsilon_{lmn} (\partial_i X_l)(\partial_j X_m)(\partial_k X_n) , \quad (76)$$

with six terms in total; $\epsilon_{ijk}$ is the standard antisymmetric Levi-Civita symbol with $\epsilon_{123} \equiv 1$. Thus we see that Jacobian $J$ is a quite complex nonlocality (ending up in the denominator of
the Hamiltonian $H_{NS}$ that ensues), however the physics of the problem often allows $\rho = J\rho_0$ to be approximated as a constant or near-constant mean and then to perturb about this mean. Finally, as already mentioned, recall that below we show how a partial time derivative of this mass constraint, $\rho = J\rho_0$, is exactly equivalent to the continuity equation.

The final equation of motion for vector field $X$ is the hard one, but it is straightforward given the above setup. Thus, the final Euler-Lagrange equation comes from varying $X$ (and $s$ with implicit $X$ dependence):

\[
\partial_t \left( \frac{\partial L_{NS}}{\partial \partial_t X_i} \right) = \frac{\partial L_{NS}}{\partial X_i} - \partial_j \left( \frac{\partial L_{NS}}{\partial \partial_j X_i} \right) - \rho T \frac{\delta s}{\delta X_i}
\]

\[
\Rightarrow \partial_t P_i &= -KJ \frac{\partial \rho_0}{\partial X_i} - \partial_j \left( P_i v_j - K\rho_0 \frac{\partial J}{\partial \partial_j X_i} \right) - \rho T \frac{\delta s}{\delta X_i}
\]

\[
- KJ \frac{\partial \rho_0}{\partial X_i} - \partial_j \left( P_i v_j - K\rho_0 J \frac{\partial x_j}{\partial X_i} \right) - \rho T \frac{\delta s}{\delta X_i}
\]

\[
- KJ \frac{\partial \rho_0}{\partial X_i} - \partial_j \left( P_i v_j - K\rho_0 J \frac{\partial x_j}{\partial X_i} \right) - \left( \rho T \partial_j s + \partial_k \sigma'_{jk} \right) \frac{\partial x_j}{\partial X_i}
\]

\[
\text{(chain rule)}
\]

\[
- \partial_j \left( P_i v_j \right) + \partial_j \left( K\rho_0 \right) J \frac{\partial x_j}{\partial X_i} - \left( \rho T \partial_j s + \partial_k \sigma'_{jk} \right) \frac{\partial x_j}{\partial X_i}
\]

\[
\partial_t P_i + \nabla \cdot (P_i \mathbf{v}) = \left[ \rho \partial_j K - \rho T \partial_j s - \partial_k \sigma'_{jk} \right] \frac{\partial x_j}{\partial X_i}.
\]

Note that by using Eqs. (12) and (74) a perhaps more physical form of this last equation follows by noting the non-density pieces of the first two terms on the right are equal to the following:

\[
\nabla K - T \nabla s = \frac{\nabla p}{\rho} - \nabla \left( \frac{v^2}{2} \right),
\]

with a static and dynamic pressure contribution to this acceleration. Thus, all the equations of motion of $L_{NS}$ have been derived and we now move onto the proof that they are equivalent to the Navier-Stokes equation (along with mass conservation and the entropy constraint which are automatically maintained as constraints in this approach).

D. Navier-Stokes equation equivalence proof

Here we show that the equations of motion of $L_{NS}$ derived in the last subsection are equivalent to the Navier-Stokes equation, or more generally to the mass, momentum, and
energy conservation equations of Section II. First, note that mass and entropy are “automatically” maintained as constraints in this approach defined by lagrangian density $\mathcal{L}_{NS}$ of Eq. (61) and entropy constraint Eq. (57). Thus, one way to complete the proof is the following: (1) show that $\rho = J\rho_0$ is equivalent to the continuity equation, Eq. (1), and (2) show that the Navier-Stokes momentum conservation equation, Eq. (2), is equivalent to the Euler-Lagrange equations of $\mathcal{L}_{NS}$ from the previous subsection. We perform these two steps in order just below, but first for reference, here we list all of the equations of motion that came from these Euler-Lagrange equations of $\mathcal{L}_{NS}$ (including the entropy constraint for completeness):

$$\delta s = \left( \partial_j s + \frac{\partial k\sigma'_{jk}}{\rho T} \right) \frac{\partial x_j}{\partial X_i}, \quad (78a)$$

$$\rho = J\rho_0, \quad (78b)$$

$$\rho v = -P_i\nabla X_i, \quad (78c)$$

$$K = h - \frac{v^2}{2}, \quad (78d)$$

$$D_t X = 0, \quad (78e)$$

$$\partial_t P_i + \nabla \cdot (P_i v) = \left[ \rho \partial_j K - \rho T \partial_j s - \partial_k \sigma'_{jk} \right] \frac{\partial x_j}{\partial X_i}. \quad (78f)$$

The $X$ and $P$ equations are dynamical—the remaining equations are constraints. Next we show that these equations contain the same dynamics as that of the Navier-Stokes equation.

1. Explicit mass conservation equivalence proof

Here we take a partial time derivative of the so-called mass operator:

$$\mathcal{O}_m \equiv \rho - J\rho_0, \quad (79)$$

and show that it exactly reproduces the continuity equation, Eq. (1). Using the previously derived relations as referenced here above the equals signs, taking this time partial using
the product rule of differentiation gives

\[
\frac{\partial}{\partial t} O_m = \frac{\partial}{\partial t} \rho - \rho_0 \left( \frac{\partial}{\partial t} J \right) - J \left( \frac{\partial}{\partial t} \rho_0 \right)
\]

(temporal rule) \quad (71)

\[
\frac{\partial}{\partial t} \rho - \rho_0 \left( \frac{\partial}{\partial t} J \right) - J \left( \frac{\partial}{\partial t} \rho_0 \right)
\]

(chain rule) \quad (70)

\[
\frac{\partial}{\partial t} \rho - \rho_0 \left( \frac{\partial}{\partial X} \frac{\partial}{\partial t} X_i \right)
\]

(78a)

\[
\frac{\partial}{\partial t} \rho - \rho_0 \left( \frac{\partial}{\partial X} \frac{\partial}{\partial t} X_i \right)
\]

(78b)

\[
\frac{\partial}{\partial t} \rho + \frac{\partial}{\partial j} (\rho v_j)
\]

(80)

with the sign flipping exactly as required (symmetry magic again) in this last line according to Eq. (55). Thus the operators are exactly equivalent, and if one vanishes then so must the other—mass is conserved, q.e.d.

2. Explicit momentum conservation equivalence proof

Here we start with the Navier-Stokes momentum operator of Eq. (2) with all the terms on the left-hand side:

\[
O_{NS} \equiv \frac{\partial}{\partial t} (\rho v_i) + \frac{\partial}{\partial j} (\rho v_i v_j) - \frac{\partial}{\partial j} \sigma_{ij}
\]

(81)

then insert the velocity constraint of Eq. (78c) and the thermodynamic law of Eq. (12), and then physically rearrange to show that \(O_{NS}\) is proportional to the equation of motion operators of Eq. (78). With the equivalence of mass conservation as shown in the previous subsection, this exactly proves the equivalence of all of Eq. (78) with the Navier-Stokes equation at the operator level.

First, we need one preliminary math result. The equation of motion for \(P\), Eq. (78f), has a matrix \(\frac{\partial x_j}{\partial X_i}\) on the right-hand side that that needs to be inverted. Therefore, multiplying Eq. (78f) by \(\frac{\partial X_i}{\partial x_m}\), performing the implied sum over \(i\), using the chain rule, and rearranging dummy indices, produces an equivalent form of the equation of motion for \(P\):

\[
[\partial_t P_j + \nabla \cdot (P_j v)] \frac{\partial}{\partial t} X_j = \rho \partial_t K - \rho T \partial_s - \partial_j \sigma_{ij}
\]

(82)

We mention this because this is the form that appears in the proof below and this shows that it is equivalent to Eq. (78f).
Recall Eq. (3) for the relation between the stress tensor and the viscous stress tensor. Then starting with \( O_{NS} \) above, insert Eqs. (78c) and (12). Then use the product rule and rearrange the operator adding and subtracting the same term in two spots, and massage it into its final form proportional to the Euler-Lagrange equations as shown here:

\[
O_{NS} \equiv \partial_t (\rho v_i) + \partial_j (\rho v_i v_j) - \partial_j \sigma_{ij},
\]

\[
= \partial_t [-P_j \partial_i X_j] + \partial_k [-P_j (\partial_i X_j) v_k] + \rho \partial_i h - \rho T \partial_i s - \partial_j \sigma'_{ij},
\]

\[
= \rho \partial_i h - \rho T \partial_i s - \partial_j \sigma'_{ij} - (\partial_i P_j) \partial_i X_j - P_j \partial_i (\partial_i X_j)
\]

\[
- (\partial_k P_j) (\partial_i X_j v_k) - P_j (\partial_i \partial_k X_j) v_k - P_j (\partial_i X_j) \partial_k v_k,
\]

\[
(\text{exact rearrange}) \equiv \rho \partial_i h - \rho T \partial_i s - \partial_j \sigma'_{ij} - (\partial_i X_j) [\partial_i P_j + \partial_k (P_j v_k)]
\]

\[
- P_j \partial_i (D_t X_j) + P_j (\partial_i v_k) (\partial_k X_j),
\]

\[
(+/- \rho \partial_i K) \equiv - P_j \partial_i \{D_t X_j\}
\]

\[
+ \{ \rho \partial_i K - \rho T \partial_i s - \partial_j \sigma'_{ij} - (\partial_i X_j) [\partial_i P_j + \partial_k (P_j v_k)] \}
\]

\[
- \rho \partial_i K + \rho \partial_i h + P_j (\partial_i v_k) (\partial_k X_j),
\]

\[
(+/- \rho v_j \partial_i v_j) \equiv - P_j \partial_i \{D_t X_j\}
\]

\[
+ \{ \rho \partial_i K - \rho T \partial_i s - \partial_j \sigma'_{ij} - (\partial_i X_j) [\partial_i P_j + \partial_k (P_j v_k)] \}
\]

\[
+ \rho \partial_i \left\{ h - K - \frac{v^2}{2} \right\}
\]

\[
+(\partial_i v_j) \{\rho v_j + P_k \partial_j X_k\}.
\]

The last four lines are the form of the equation that we were seeking. Each factor in curly braces is the Euler-Lagrange equation operator for the Navier-Stokes fluid that we were seeking. They are the equations of motion for respective fields

(1) dynamical \( X \), Eq. (78c);

(2) dynamical \( P \), Eq. (82) (equiv. to Eq. (78f));

(3) constrained \( K \), Eq. (78d); and

(4) constrained \( v \), Eq. (78c).

As discussed earlier, but to close out this proof: the entropy constraint Eq. (78a) is built in and maintained for these Euler-Lagrange equations, and from the previous subsection, the mass constraint Eq. (78b) is independently equivalent to the continuity equation, Eq. (1); therefore it is all consistent and the Euler-Lagrange equations for \( L_{NS} \) of Eq. (61) are exactly equivalent to the Navier-Stokes equation, mass conservation, and energy conservation (or the
entropy constraint however one wants to state it as discussed in Section II). Interestingly, unlike the Euler fluid, note here with the Navier-Stokes fluid that mass and momentum conservation decouple; so for example violation of the mass constraint equation $\rho = J \rho_0$ only affects mass conservation, but does not directly couple with the dynamical Euler-Lagrange equations as shown by it not being present in Eq. (83) above—which came from momentum conservation alone. But with the Euler fluid, mass and momentum conservation are coupled as given by the final form of Eq. (33) which contains the continuity equation operator as its first term. The difference lies in the fact that with the Euler fluid, $\rho$ is a dynamical field, but with the Navier-Stokes fluid, $\rho$ is constrained (as appears to be required in order to introduce dissipation consistently which led to vector field $X$ being introduced in a fashion at least reminiscent of gauge invariance).

E. Hamilton equations

Now we continue with the $H_{NS}$ derivation to express the problem in terms of its canonical coordinates so as to see its flow through phase space. In a sense the hard part of this paper is done—equivalence with the Navier-Stokes equation has been proven—and now we just put the remaining pieces together and set up the field theory “Hamiltonian algebra” that will be used in subsequent papers. As is probably no surprise at this point, the Hamilton equations of $H_{NS}$ reproduce the same dynamical equations to those that came from the Euler-Lagrange equations of the previous subsection. However, just like earlier in deriving the Euler-Lagrange equations of $\mathcal{L}_{NS}$, here too the functional derivative of $X$ has an extra term to the standard ones and so for completeness we will start from first principles and derive the Hamilton equations with dissipation.

As before, derivation of a Hamiltonian starts with a discussion of the conjugate momenta. For lagrangian density $\mathcal{L}_{NS}$ of Eq. (61) the nonzero conjugate momentum fields are

$$\pi_{x_i} \equiv \frac{\partial \mathcal{L}_{NS}}{\partial \frac{\partial x_i}{\partial t}} = P_i(x, t),$$  \hspace{1cm} (84)

where $i = (1, 2, 3)$ as always in this paper with three spatial dimensions (recall $X$ is a position vector of the same type as $x$); so this is really three independent conjugate momentum fields.

At this point, because of the complexity that dissipation adds, we go through the derivation of the Hamilton equations which necessarily involves the definition of the Hamiltonian.
itself. Just like in Landau and Lifshitz’s jewel [6] but for field theory, start back with the expression used in deriving the Euler-Lagrange equations, but this time rearrange things so as to make the conjugate momenta come to the forefront via a so-called Legendre transformation. So back up to Eq. (63), drop the spatial surface terms like before (which necessarily required us to introduce vector field $\mathbf{X}(\mathbf{x}, t)$), and put in the definition of the conjugate momenta from Eq. (84). This gives

$$\delta \mathcal{L}_{NS} = \frac{\partial \mathcal{L}_{NS}}{\partial \chi^\alpha} \delta \chi^\alpha + \pi_X \delta (\partial_t X_i) + \left[ \frac{\partial \mathcal{L}_{NS}}{\partial X_i} - \partial_j \left( \frac{\partial \mathcal{L}_{NS}}{\partial (\partial_j X_i)} \right) - \rho T \frac{\delta s}{\delta X_i} \right] \delta X_i ,$$

where recall $\chi^\alpha$ is defined as in Eq. (62)—it is just a convenient shorthand because all of its fields do not have a conjugate momentum field and so their Euler-Lagrange equations are all the simple $\partial \mathcal{L} / \partial \chi^\alpha = 0$. Now, at this point the Euler-Lagrange equations of Eqs. (67) and (68) are enacted and the following identification is made:

$$\partial_t \left( \frac{\partial \mathcal{L}_{NS}}{\partial (\partial_t X_i)} \right) \rightarrow \partial_t \pi_{X_i} ,$$

which follows simply from definition Eq. (84) itself. Thus, Eq. (85) becomes

$$\delta \mathcal{L}_{NS} = \dot{\pi}_{X_i} \delta X_i + \pi_{X_i} \delta \dot{X}_i ,$$

where for cleanness of notation we use the dot notation for the temporal partial derivative here. This is the point where we leave the Lagrangian framework and move over to the Hamiltonian one: Rearranging terms gives simply

$$\delta \left( \pi_{X_i} \dot{X}_i - \mathcal{L}_{NS} \right) = -\dot{\pi}_{X_i} \delta X_i + \dot{X}_i \delta \pi_{X_i} .$$

Define the variational argument of the left-hand side as the hamiltonian density itself:

$$\mathcal{H}_{NS} \equiv \pi_{X_i} \dot{X}_i - \mathcal{L}_{NS} ,$$

and this variation becomes

$$\delta \mathcal{H}_{NS} = -\dot{\pi}_{X_i} \delta X_i + \dot{X}_i \delta \pi_{X_i} .$$

This implies the following Hamilton equations for the Navier-Stokes fluid:

$$\dot{X}_i = \frac{\delta \mathcal{H}_{NS}}{\delta \pi_{X_i}} ,$$

$$\dot{\pi}_{X_i} = -\frac{\delta \mathcal{H}_{NS}}{\delta X_i} .$$

---

8 This is the point where the temporal endpoint boundary term is also dropped since that ansatz is used in deriving the Euler-Lagrange equations themselves.
We will derive these equations more explicitly just below, but first we write the Hamiltonian that Eq. (89) implies. Thus the Naiver-Stokes Hamiltonian is given by

\[
H_{NS} \equiv \int d^3 x \mathcal{H}_{NS},
\]

\[
\mathcal{H}_{NS} = \pi X_i \dot{X}_i - L_{NS} \equiv P_i \dot{X}_i - L_{NS}
\]

\[
= \frac{1}{2} \rho \mathbf{v}^2 + \mathcal{U}(\rho, s) - P_i v_j \partial_j X_i - K (\rho - J \rho_0),
\]

which is not the final form of \( \mathcal{H}_{NS} \). There are two points to be made: (1) We will use \( P_i \) and \( \pi X_i \) interchangeably at this point because it is not ambiguous and \( P_i \) seems cleaner (but remember \( P(\mathbf{x}, t) \) is really a conjugate momentum field, precisely the one conjugate to vector field \( \mathbf{X}(\mathbf{x}, t) \)) and (2) In this derivation of \( \mathcal{H}_{NS} \), the dynamical field equations of \( \mathbf{X} \) and \( \mathbf{P} \) are being rearranged, but the constraint equations that followed from \( \partial L_{NS} / \partial \chi_\alpha = 0 \)—recall Eq. (85) and the discussion after it—are still being upheld. Specifically the following constraints are satisfied:

\[
\frac{\delta s}{\delta X_i} (\partial_j s + \frac{\partial k \sigma'_{jk}}{\rho T}) \frac{\partial x_j}{\partial X_i},
\]

\[
\rho = J \rho_0.
\]

\[
\rho \mathbf{v} - P_i \nabla X_i,
\]

\[
K = h - \frac{v^2}{2}.
\]

Thus, at first blush seemingly paradoxically (but its effects are still in there as shown below), the last term of Eq. (93) vanishes and we are left with the following equation which with one slight rearrangement from the \( \mathbf{v} \) constraint leaves the final form of \( \mathcal{H}_{NS} \) that we were trying to derive:

\[
\mathcal{H}_{NS} = -\frac{1}{2} \rho \mathbf{v}^2 + \mathcal{U}(\rho, s) - P_i v_j \partial_j X_i
\]

\[
= -\frac{1}{2} \rho \mathbf{v}^2 + \mathcal{U}(\rho, s) + v_j (\rho v_j) \]

\[
= \frac{(\rho \mathbf{v})^2}{2 \rho} + \mathcal{U}(\rho, s) \text{ with constraints (94).}
\]

This is the Navier-Stokes Hamiltonian (density) that has been consistently derived for a nonrelativistic viscous fluid. Writing it in this final “\( p^2/(2m) \)” form is seen to be convenient.
when one recalls the form of the Navier-Stokes momentum density given by Eq. (94c).\footnote{So this \((\rho v)^2 = (P_i \nabla X_i)^2\) numerator actually does not depend on density field \(\rho\), only the denominator does. Thus, the \(\rho = J \rho_0\) in the denominator gives rise to nonlocal effects from the Jacobian factor. See Eq. (76) for explicit expressions for Jacobian \(J\).}

Now onto the dynamical equations that the Hamilton equations imply. First writing the hamiltonian density explicitly in terms of its dynamical fields \(X\) and \(P\) gives

\[
\mathcal{H}_{NS} (X, P, s(X)) = \frac{(P_i \nabla X_i)^2}{2 J(\nabla X)} \rho_0(X) + U(J(\nabla X) \rho_0(X), s(X)) \tag{96}
\]

This starts out quartic in the fields in the numerator of the first term and there is no standard quadratic term of field theory \[12\]. Recall that \(J(\nabla X)\) is the Jacobian explicitly given by Eq. (76) with six terms in total and each term being cubic in \(\partial_j X_i\, and for this first \(\frac{p^2}{(2m)}\) term it is in the denominator which makes its contributions nonlocal. See the previous discussion after Eq. (76) where possible approximate physical starting points are discussed regarding this Jacobian and initial density \(\rho_0(X)\).

Now we start with \(\mathcal{H}_{NS}\) of Eq. (96), finish the derivation of its Hamilton equations, and then show that they are equivalent to the Euler-Lagrange equations of the Navier-Stokes fluid derived earlier and summarized by the six lines of Eq. (78) (which have already been shown to be equivalent to the Navier-Stokes equation itself). Varying \(H_{NS}\) using the explicit form of Eq. (96) gives the following equation. Using thermodynamic relation Eq. (9) and then integrating by parts leads to

\[
\delta H_{NS} = \int d^3 x \delta \mathcal{H}_{NS} (X, P, s(X)) = \int d^3 x \left[ \frac{\partial \mathcal{H}_{NS}}{\partial X_i} \delta X_i + \frac{\partial \mathcal{H}_{NS}}{\partial (\partial_j X_i)} \delta (\partial_j X_i) + \frac{\partial \mathcal{H}_{NS}}{\partial s} \delta s X_i + \frac{\partial \mathcal{H}_{NS}}{\partial P_i} \delta P_i + \frac{\partial \mathcal{H}_{NS}}{\partial (\partial_j P_i)} \delta (\partial_j P_i) \right]
\]

\[
= \int d^3 x \hat{n}_j \left[ \frac{\partial \mathcal{H}_{NS}}{\partial (\partial_j X_i)} \delta X_i + \frac{\partial \mathcal{H}_{NS}}{\partial (\partial_j P_i)} \delta P_i \right]
+ \int d^3 x \left\{ \left[ \frac{\partial \mathcal{H}_{NS}}{\partial X_i} - \partial_j \left( \frac{\partial \mathcal{H}_{NS}}{\partial (\partial_j X_i)} \right) + \rho T \frac{\delta s}{\delta X_i} \right] \delta X_i + \left[ \frac{\partial \mathcal{H}_{NS}}{\partial P_i} - \partial_j \left( \frac{\partial \mathcal{H}_{NS}}{\partial (\partial_j P_i)} \right) \right] \delta P_i \right\} \tag{97}
\]

which is in terms of \(\delta X\) and \(\delta P\) variations alone as required. The \(\delta s\) variations have been replaced with the correct \(\delta X\) ones given by nonholonomic constraint Eq. (94a). Dropping these spatial boundary terms on the physical grounds that \(\delta X\) and \(\delta P\) vanish at spatial infinity, and then comparing with the general form of \(\delta \mathcal{H}_{NS}\) already derived in Eq. (90),
gives the final operational form of the Hamilton equations for the Navier-Stokes fluid:

$$\frac{\partial_t X_i}{\partial P_i} = \frac{\delta H_{NS}}{\delta P_i} - \frac{\partial}{\partial j} \left( \frac{\partial H_{NS}}{\partial (\partial_j X_i)} \right), \quad (98a)$$

$$\frac{\partial_t P_i}{\partial X_i} = -\frac{\delta H_{NS}}{\delta X_i} + \frac{\partial}{\partial j} \left( \frac{\partial H_{NS}}{\partial (\partial_j X_i)} \right) - \rho T \frac{\delta s}{\delta X_i}. \quad (98b)$$

These followed straightforwardly even given dissipation.

Now explicitly carrying out these derivatives of $H_{NS}$ on the right-hand side of Eq. (98), we show that the Hamilton and Euler-Lagrange equations are equivalent for the Navier-Stokes fluid; and that therefore as shown by Section IV D they are also equivalent to the Navier-Stokes equation itself supplemented with mass and energy conservation. Clearly showing this was the main purpose of this paper.

1. Hamilton and Euler-Lagrange equations equivalence proof

The Hamilton and Euler-Lagrange equations are shown to be the same for the Navier-Stokes fluid. Earlier it was shown that the Euler-Lagrange equations, summarized by Eq. (78), are equivalent to the Navier-Stokes equation. Thus this will complete the proof of showing that the Hamilton, Euler-Lagrange and conservation equations of the Navier-Stokes fluid are all equivalent.

As discussed above, the Euler-Lagrange constraint equations are maintained in the Hamiltonian approach. So equivalence of the hamiltonian constraints, Eq. (94), with those of Eq. (78) is already “proven” and only the dynamical Hamilton equations of $X$ and $P$ remain to be proven equivalent with their respective Euler-Lagrange equation.

Thus, explicitly carrying out the derivatives of Eq. (98) on Eq. (96) gives for $X$’s dynamical equation:

$$\frac{\partial_t X_i}{\partial P_i} = \frac{\partial H_{NS}}{\partial P_i} - \frac{\partial}{\partial j} \left( \frac{\partial H_{NS}}{\partial (\partial_j P_i)} \right)\bigg|_{j=0} = P_k (\partial_j X_k)(\partial_j X_i) \quad \rho \quad (94b)$$

$$\frac{\partial_t P_i}{\partial X_i} = -\frac{\partial H_{NS}}{\partial X_i} + \frac{\partial}{\partial j} \left( \frac{\partial H_{NS}}{\partial (\partial_j X_i)} \right) - \rho T \frac{\delta s}{\delta X_i} \quad \rho \quad (94c)$$

$$= -\rho v_j (\partial_j X_i) \quad \rho$$

$$= -v_j (\partial_j X_i)$$

$$\Rightarrow \boxed{D_t X = 0}. \quad (99)$$
And now the hard one: the dynamical equation for $P$ as follows from Eq. (98b). One has to carefully take into account every $X$ dependent piece of Eq. (96), including the functional dependence of $\delta s(X)$ on $\delta X$; but note that this latter dependence has already been taken into account (as derived in Eq. (97)) and is given by the final term on the right of the first line below. Thus, given the previous setup, $P$’s Hamilton equation follows simply from the product and chain rules of calculus:

$$
\partial_t P_i = -\frac{\partial H_{NS}}{\partial X_i} + \partial_j \left( \frac{\partial H_{NS}}{\partial (\partial_j X_i)} \right) - \rho T \frac{\delta s}{\delta X_i} \\
= \frac{(P_j \nabla X_j)^2}{2 J(\nabla X) [\rho_0(X)]^2 \partial X_i} - \frac{\partial U}{\partial \rho} J(\nabla X) \frac{\partial \rho_0}{\partial X_i} - \rho T \frac{\delta s}{\delta X_i} \\
+ \partial_j \left( \frac{P_k P_l \partial_j X_k}{J(\nabla X) \rho_0(X)} \right) - \partial_j \left( \frac{(P_k \nabla X_k)^2}{2 \rho_0(X) [J(\nabla X)]^2} \frac{\partial J}{\partial \rho} \right) \\
+ \partial_j \left( \frac{\partial U}{\partial \rho} \rho_0(X) \frac{\partial J}{\partial \rho} \rho \frac{\partial x_j}{\partial X_i} \right) \\
= \frac{1}{2} \nabla \cdot (P \nabla) - h J \frac{\partial \rho_0}{\partial X_i} - \rho T \frac{\delta s}{\delta X_i} \\
- \partial_j (P_j v_j) - \partial_j \left( \frac{1}{2} \rho v^2 \nabla^2 \frac{\partial x_j}{\partial X_i} \right) + \partial_j \left( h \rho_0 J \frac{\partial x_j}{\partial X_i} \right) \\
\Rightarrow \partial_t P_i + \nabla \cdot (P \nabla) = \left[ \rho \partial_j K - \rho T \partial_j s - \partial_k \sigma'_{jk} \frac{\partial x_j}{\partial X_i} \right], \quad (100)
$$

which is the same as Eq. (78f) as was to be shown. In summary, the Lagrangian, Eq. (61), and Hamiltonian, Eq. (96), both with nonholonomic entropy constraint, Eq. (94a), satisfy the same equations of motion, those of a nonrelativistic viscous fluid given by the Navier-Stokes, continuity, and energy conservation equations of standard fluid mechanics. Also, if no approximations on any of the factors in $L_{NS}$ or $H_{NS}$ are made (such as numerical error in the process of integration etc.) then $L_{NS}$ implies $H_{NS}$ and vice versa. With mass constraint $\rho = J \rho_0$ and entropy constraint Eq. (57) understood to be satisfied as discussed amply above, a nice physical way to write the Navier-Stokes Hamiltonian is

$$
H_{NS} = \int d^3 x \left[ \frac{(P_i \nabla X_i)^2}{2 \rho} + U(\rho, s) \right]. \quad (101)
$$
F. Canonical Poisson brackets with dissipation

We close the paper by showing that $dH_{NS}/dt = 0$ even though $H_{NS}$ includes dissipation, and then derive the Poisson bracket for a general classical dissipative observable of a nonrelativistic viscous fluid. First the time derivative of the Hamiltonian. Actually, the hard part here has already been done when deriving the Hamilton equations: From the general Navier-Stokes Hamiltonian variation derived earlier and given by Eq. (90), combined with the explicit result of Eq. (97), the Hamilton equations and $X$ and $P$ field functional derivatives followed as Eqs. (98a) and (98b). Thus the time derivative of $H_{NS}$ is given by

$$\frac{dH_{NS}}{dt} = \int d^3x \left( \delta H_{NS} \frac{\partial X_i}{\partial t} + \frac{\delta H_{NS}}{\delta P_i} \frac{\partial P_i}{\partial t} \right)$$

which implies the following for observable $O$’s functional derivatives (dropping surface terms like discussed above since independent variations $\delta X$ and $\delta P$ vanish at spatial infinity)

$$\frac{\delta O}{\delta X_i} = \frac{\partial O}{\partial X_i} - \partial_j \left( \frac{\partial O}{\partial (\partial_j X_i)} \right) + \frac{\partial O}{\partial s} \frac{\partial s}{\partial X_i}$$

$$\frac{\delta O}{\delta P_i} = \frac{\partial O}{\partial P_i} - \partial_j \left( \frac{\partial O}{\partial (\partial_j P_i)} \right).$$
Allowing the variation to be a time derivative (see discussion below Eq. (56)), the above becomes
\[
\frac{dO}{dt}[\mathbf{X}, \mathbf{P}, s(\mathbf{X})] = \int d^3 x \left\{ \left[ \frac{\partial O}{\partial X_i} - \partial_j \left( \frac{\partial O}{\partial \partial_j X_i} \right) \right] + \frac{\partial O}{\partial s} \frac{\delta s}{\delta X_i} \right\} \frac{\partial X_i}{\partial t} + \left[ \frac{\partial O}{\partial P_i} - \partial_j \left( \frac{\partial O}{\partial \partial_j P_i} \right) \right] \frac{\partial P_i}{\partial t} \right\} \tag{105}
\]
which upon inserting Eqs. (104) and (98) becomes finally
\[
\frac{dO}{dt}[\mathbf{X}, \mathbf{P}, s(\mathbf{X})] = \int d^3 x \left\{ \delta O \frac{\delta H_{NS}}{\delta X_i} \delta P_j - \delta O \frac{\delta H_{NS}}{\delta P_i} \delta X_j \right\} \tag{106}
\equiv [O, H_{NS}] \tag{107}
\]
This last equation defines the Poisson bracket of a dissipative observable, in short
\[
\frac{dO}{dt} \equiv [O, H_{NS}] \tag{108}
\]
This looks like the standard result but note carefully that the functional derivatives \(\delta O/\delta X_i\) and \(\delta H_{NS}/\delta X_i\) of this Poisson bracket, explicitly defined by Eq. (106), are dissipative and given by Eqs. (104a) and (98b) respectively and \(\delta s/\delta X_i\) is the same entropy constraint Eq. (94a) used throughout this paper.

We end by working out the Poisson bracket between the Navier-Stokes Hamiltonian coordinate field \(\mathbf{X}\) and its conjugate momentum field \(\mathbf{P}\). The canonical result is a delta function and we show that it follows here too (a main point of introducing \(\mathbf{X}\) a la Fukagawa and Fujitani \[7\]). The canonical Poisson bracket for the dynamical fields of \(H_{NS}\) is
\[
\left[ X_i(\mathbf{x}'), \pi_{X_j}(\mathbf{x}'') \right] \equiv \left[ X_i(\mathbf{x}'), P_j(\mathbf{x}'') \right] \tag{106}
\int d^3 x \left[ \delta X_i(\mathbf{x}') \delta P_j(\mathbf{x}'') - \delta X_i(\mathbf{x}') \delta P_j(\mathbf{x}'') \right] \delta X_k(\mathbf{x}) \delta P_k(\mathbf{x}) \]
\[
= \int d^3 x \left[ \delta_i k \delta^3(x - x') \delta_j k \delta^3(x - x'') - 0 \right] \delta x' - x'' , \tag{109}
\]
where the third line is from the standard rules of functional differentiation. In some sense, both the numerator and denominator of factor \(\delta X_i(\mathbf{x}')/\delta X_k(\mathbf{x})\) dissipate as above and the standard result follows.

V. SUMMARY AND DISCUSSION

The Navier-Stokes Hamiltonian was derived from first principles. The mass, momentum and energy conservation equations of a nonrelativistic viscous fluid were shown to lead to the
standard entropy constraint. Viscous dissipation in a fluid leads to heat and the entropy of the fluid necessarily increases according to the laws of local thermodynamics. The entropy constraint became a nonholonomic constraint for both the Lagrangian and Hamiltonian of the Navier-Stokes fluid. The Euler-Lagrange and Hamilton equations of a viscous fluid were shown to be equivalent to the Navier-Stokes equation. The Lagrangian and Hamiltonian of the inviscid fluid were also derived and shown to give equivalent dynamics to that of the Euler equation. This was done to compare the viscous and inviscid theories since they are not smoothly related as the viscosity vanishes (discussed further in the next paragraph). The Poisson bracket structure of the Navier-Stokes fluid with dissipative functional derivatives was derived. This dissipative Hamiltonian algebra sets up future work with similarity renormalization group methods applied to viscous (and inviscid) fluids so as to resolve the respective dynamical scales in a systematic fashion. Hamiltonian methods allow convenient approximations through the variational principle and renormalization group transformations.

Note that the Euler and Navier-Stokes fluids have quite different pathlines as given by the \( \mathbf{v} \) constraints that came from both procedures:

\[
\mathbf{v}_E = \nabla \phi - \frac{\alpha}{\rho} \nabla \beta - \frac{\lambda}{\rho} \nabla S ,
\]

\[
\mathbf{v}_{NS} = -\frac{\rho}{\rho_0} \nabla X_i = -\frac{P_i \nabla X_i}{J(\nabla \mathbf{X}) \rho_0(\mathbf{X})} ,
\]

from Eqs. (23), (72) and (75) respectively.\(^\text{10}\) There are two points to be made both highlighting the differences between the Euler and Navier-Stokes fluids even though on a first sight, these \( \mathbf{v}_E \) and \( \mathbf{v}_{NS} \) decompositions look similar. First, for \( \mathbf{v}_E \) note how the \( \nabla \phi \) term has a plus sign\(^\text{11}\) and also this first term does not have any density dependence. \( \phi \) is of course the standard velocity potential of fluid mechanics and \( \mathbf{v}_E \) is the so-called Clebsch

---

\(^\text{10}\) To be clear: Everywhere in this paper, repeat indices are assumed to be summed over three spatial dimensions. Also, as explained in the body of the paper, vector field \( \mathbf{X} \) is just another coordinate like \( \mathbf{x} \), so the fact that there are three scalar potential pairs in the Navier-Stokes fluid (\( X_1, P_1; X_2, P_2; X_3, P_3 \)) is directly related to the choice of working in three spatial dimensions. The Navier-Stokes fluid seems to be perfectly coupled to three spatial dimensions.

\(^\text{11}\) Obviously \( \mathcal{L}_E \) of Eq. (20) could be defined so that this is a minus sign or vice versa on the other two terms of \( \mathbf{v}_E \), but the point is that then the conjugate momentum field for one of the \( \mathbf{v}_E \) terms has to have an opposite sign to the “shorthand” field that it represents. This does not occur for the Navier-Stokes fluid. There the signs of all three terms can be made to be the same in \( \mathbf{v}_{NS} \) and the conjugate momenta of \( \mathcal{L}_{NS} \) are related by a positive sign: \( \pi_{X_i} = +P_i \).
decomposition of the velocity field with Gauss potential \[13\] pairs \((\alpha, \beta)\) and \((\lambda, s)\). Interestingly, in order to obtain the Navier-Stokes Hamiltonian one was required to introduce coordinate transformation \(X(x, t)\) in order to handle the dissipation constraint properly and this already gave enough degrees of freedom and so a velocity potential was not required. The Euler and Navier-Stokes fluids are very different. Second, \(v_E\) and \(v_{NS}\) being different is even more readily apparent if we recall that \(\rho\) is a dynamical field for the Euler fluid (satisfying the standard continuity equation), whereas for the Navier-Stokes fluid, \(\rho = J\rho_0\) is a constraint which in terms of vector field \(X\) is quite complex: recall Eq. (76) for an explicit expression for \(J(\nabla X)\). Also note that since \(J\) (with derivatives of fields) is in the denominator it is a nonlocal operator in \(H_{NS}\). These differences should not come as a huge surprise since the Euler and Navier-Stokes fluid problems do not map smoothly onto each other: the zero viscosity and infinitesimal viscosity fluids are not limits of the same theory. Perhaps these variational principle forms of the theories help to make this clearer.

\textbf{Acknowledgments}

The author would like to thank Hiroki Fukagawa (of Keio and Kyushu University) and Robert Perry (of The Ohio State University) for enlightening discussions at University of Washington and Ohio State respectively. This work was supported in part by the Applied Physics Laboratory, University of Washington under Project No. APL-UW-900088.
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