THE \( a \)-POINTS OF THE RIEMANN ZETA-FUNCTION AND THE FUNCTIONAL EQUATION

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Abstract. We prove an equivalent of the Riemann hypothesis in terms of the functional equation (in its asymmetrical form) and the \( a \)-points of the zeta-function, i.e., the roots of the equation \( \zeta(s) = a \), where \( a \) is an arbitrary fixed complex number.

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1. Motivation and Statement of the Main Results

Let \( s = \sigma + it \) be a complex variable. The Riemann zeta-function \( \zeta \) is for \( \sigma > 1 \) defined by

\[
\zeta(s) = \prod_p (1 - p^{-s})^{-1},
\]

where the product is taken over all prime numbers \( p \), and by analytic continuation elsewhere except for a simple pole at \( s = 1 \). The Euler product representation above indicates the relevance of \( \zeta \) for the distribution of prime numbers. The yet unproven Riemann hypothesis claims that all nontrivial (non-real) zeros \( \rho = \beta + i\gamma \) of the Riemann zeta-function \( \zeta \) lie on the critical line \( \frac{1}{2} + i\mathbb{R} \). This distribution of zeros would lead to the least possible error term in the prime number theorem.

We define the Chebyshev function by

\[
\psi(x) = \sum_{n \leq x} \Lambda(n),
\]

where \( \Lambda \) denotes the von Mangoldt-function that counts prime powers \( n = p^k \) with logarithmic weight \( \log p \). The prime number theorem is the asymptotic formula \( \psi(x) \sim x \), and the Riemann hypothesis is equivalent to

\[
(1) \quad \psi(x) = x + O(x^{1/2 + \epsilon}),
\]

where here and in the sequel \( \epsilon > 0 \) is arbitrary; this equivalence was first proved by Helge von Koch [12] and relies on the explicit formula from Bernhard Riemann’s path-breaking memoir [16].

Another more analytic aspect is the distribution of values of \( \zeta \). Given a complex number \( a \), the roots of the equation \( \zeta(s) = a \) are called \( a \)-points; we denote these roots in the right half-plane by \( \rho_a = \beta_a + i\gamma_a \) and their count is very similar to the number of nontrivial zeros (i.e., the case \( a = 0 \)). It was Edmund
Landau who suggested in his invited lecture \cite{13} at the International Congress of Mathematicians in Cambridge 1912 to study the distribution of \( a \)-points with the words: “The points at which an analytic function is equal to 0 are very important; but equally interesting are the points at which it assumes a certain value \( a \).”

The so-called Nevanlinna theory, or value distribution theory, developed by Rolf Nevanlinna a little later, takes up this idea.

In this note we investigate the \( a \)-points of \( \zeta \) in the context of the functional equation, that is

\[ \zeta(s) = \Delta(s) \zeta(1-s), \]

where

\[ \Delta(s) := 2(2\pi)^{1-s} \sin \frac{\pi s}{2} \Gamma(1-s). \]

Note that we could as well consider other functions satisfying a similar functional equation, e.g. Dirichlet \( L \)-functions. However, for the sake of simplicity, we restrict to the case of \( \zeta \).

Our first result deals with an asymptotic formula for \( \Delta \) at the \( a \)-points.

**Theorem 1.** Let \( a \) be an arbitrary fixed complex number. Then, as \( T \to \infty \),

\[ \sum_{a \geq 0, 0 < \gamma_a < T} \Delta(\rho_a) = a \frac{T}{2\pi} \log \frac{T}{2\pi e} - \psi \left( \frac{T}{2\pi} \right) + O \left( T^{1/2+\epsilon} \right), \]

Moreover, the Riemann hypothesis is true if, and only if,

\[ \sum_{a \geq 0, 0 < \gamma_a < T} \Delta(\rho_a) = a \frac{T}{2\pi} \log \frac{T}{2\pi e} - \frac{T}{2\pi} + O \left( T^{1/2+\epsilon} \right). \]

All implicit constants in the error terms here and below depend on \( a \). Here and in the sequel we consider only \( a \)-points in the right half-plane; in what follows, however, we do not write explicitly \( \beta_a \geq 0 \) anymore. Note that all but finitely many \( a \)-points in the left half-plane lie arbitrarily close to the trivial (real) \( \zeta \)-zeros (as an application of Rouché’s theorem implies).

Using the prime number theorem in combination with the best known zero-free region for \( \zeta \) (found by Nikolay Korobov and (independently) Ivan Vinogradov; cf. \cite{11} Theorem 12.2), leads to the unconditional asymptotic formula

\[ \sum_{a \geq 0, 0 < \gamma_a < T} \Delta(\rho_a) = a \frac{T}{2\pi} \log \frac{T}{2\pi e} - T \frac{T}{2\pi} + O \left( T^{1/2+\epsilon} \right), \]

where \( c > 0 \) is an absolute constant, independent of \( a \). Note that Eduard Wirsing \cite{20} gave an elementary proof of the prime number theorem with remainder term of order \( x/(\log x)^c \), where \( c > 0 \) is arbitrary; the same result was achieved around the same time by Enrico Bombieri \cite{6}. Although this is remarkable in terms of the used machinery, the result falls short of what analytic methods can achieve.

\footnote{The authors’ translation of the German original text: “Es sind be einer analytischen Funktion die Punkte, an denen sie 0 ist, zwar sehr wichtig; ebenso interessant sind aber die Punkte, an denen sie einen bestimmten Wert \( a \) annimmt.”}
The number $N(T)$ of nontrivial $\zeta$-zeros $\rho = \beta + i\gamma$ satisfying $0 < \gamma < T$ (counting multiplicities) is ruled by the von Mangoldt-formula, that is

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} + O(\log T).$$

For the number $N_a(T)$ of $a$-points $\rho_a = \beta_a + i\gamma_a$ with $0 < \gamma_a < T$ in the right half-plane (counting multiplicities) we have as well

$$N_a(T) = \frac{T}{2\pi} \log \frac{T}{2\pi c_a} + O(\log T) \sim \frac{T}{2\pi} \log T \sim N(T),$$

where $c_a = 1$ for $a \neq 1$ and $c_1 = 2$, which was first shown by Landau [5].

Hence, by Theorem 1 for any complex number $a$, the mean-value of $\Delta$ at the $a$-points of $\zeta$ in the right half-plane exists and

$$\lim_{T \to \infty} \frac{1}{N_a(T)} \sum_{0 < \gamma_a < T} \Delta(\rho_a) = a.$$

This limit is minimal in absolute value for $a = 0$, the case of $\zeta$-zeros.

This special case had been considered earlier (as the authors learned lately). It will be shown in the proof of Theorem 1 that, as $T \to \infty$, we have unconditionally

$$\psi(x) = -\sum_{0 < \gamma < T} \Delta(\rho) + O_x(T^{1/2+\epsilon}),$$

where $2\pi x = T$, from which we immediately deduce via (1) that the Riemann hypothesis is true if, and only if,

$$\sum_{0 < \gamma < T} \Delta(\rho) = -\frac{T}{2\pi} + O_x(T^{1/2+\epsilon}).$$

This result is essentially included in a paper of Johannes Schöfengeier [17] (by a different method) and, later, Juan Arias de Reyna [1] (by a similar reasoning).

Concerning the reflected arguments $1 - \rho_a$ in place of the $a$-points, we have a rather different picture:

**Theorem 2.** Let $a$ be a complex number $\neq 0$. Then, as $T \to \infty$,

$$\sum_{0 < \gamma_a < T} \Delta(1 - \rho_a) = \frac{1}{a} \sum_{0 < \gamma_a < T} \zeta(1 - \rho_a) = \Omega(1 - b_a),$$

where $b_a$ is the supremum of the real parts $\beta_a$.

The first equality follows directly immediately from the functional equation. For the second equality recall that we write $f(t) = \Omega(g(t))$ if $f(t) = o(g(t))$ does not hold as $t \to \infty$. The case $a = 0$ is, of course, different since the nontrivial $\zeta$-zeros are symmetrically distributed with respect to the critical line and the real axis. The case $a = 1$ is a little different for technical reasons since then the logarithmic derivative of $\zeta(s) - a$ (which is an important tool in our reasoning below) is not an ordinary Dirichlet series. For other values of $a$, however, one has $b_a > 1$, and $b_{|a|} \to \infty$ as $a$ tends to 1 (as follows from (6) in the subsequent section). Hence, it follows that the mean-value of $\Delta(1 - \rho_a)$ (resp. of $\Delta(\rho_a)^{-1}$) does not exist for uncountably
many $a$ (very likely for all $a \neq 0$), while the mean-value of $\Delta(\rho_a)$ always exists (by Theorem 1).

In the following section we collect some results that will be needed in the later proofs, namely the proof of Theorem 1 for the case $a = 0$ in Section 3, resp. the general case $a \neq 1$ in Section 4, and the proof of Theorem 2 in Section 5.

2. Preliminaries about $a$-points and Gonek’s lemma

In 1911, Harald Bohr [3] was the first to prove that $\zeta(s)$ takes any complex value different from zero in the right half-plane $\sigma > 1$. Another proof of this result was given by Landau [14] whose method was based on consideration of the reciprocals of Dirichlet series $f(s)$. Landau proved that if $f(s)$ is an ordinary Dirichlet series with non-zero constant term, then $1/f(s)$ can also be represented as an ordinary Dirichlet series in the zero-free right half-plane of $f(s)$, which always exists (see for example [18] §9.6). In fact, it can be shown that $\sup \{\sigma : f(s) = 0\}$ is the abscissa of convergence of $1/f(s)$.

Therefore, if $a \neq 0, 1$, then $(\zeta(s) - a)^{-1}$ can be represented as an ordinary Dirichlet series whose abscissa of convergence is $b_a = \sup \beta_a$. Based on the discussion above, we deduce that $1 < b_a < \infty$. Moreover, if $b_a$ denotes the abscissa of absolute convergence of $(\zeta(s) - a)^{-1}$, then the product

$$\frac{\zeta'(s)}{\zeta(s) - a} = \sum_{n \geq 1} \frac{\Lambda_a(n)}{n^s}$$

is also an ordinary Dirichlet series which is absolutely convergent for $\sigma > b_a$. The implicitly defined coefficients $\Lambda_a$ generalize the von Mangoldt-function $\Lambda = -\Lambda_0$.

These results and more can be found in Landau's article, as well as for example in the work of Bombieri & Amit Ghosh [7, Section 3] and Siegfried Baluyot & Steven Gonek [2].

In view of the inequalities $0 < |\zeta(s)| \leq \zeta(\sigma)$, valid for $\sigma > 1$, and $\zeta(s) = 1 + 2^{-s}(1 + o(1))$ as $\sigma \to +\infty$, it follows that the $a$-points cannot lie too far to the right for $a \neq 1$. More precisely, one has

$$b_{|a|} \sim \frac{1}{\log 2} \log \frac{1}{|a| - 1},$$

as $|a| \to \infty$. In the special case $a = 1$, when $a$ is equal to the constant term of the Dirichlet series for $\zeta$, one considers analogously the Dirichlet series $2^s \sum_{n \geq 2} n^{-s}$.

The following result about the value-distribution of the logarithm of the zeta-function belongs to Bohr & Børge Jessen [4].

**Lemma 1.** If $\log \zeta(s)$ comes arbitrarily near to a given number $c$ on a vertical line $\sigma_0 + iR$, where $\sigma_0 > 1$, then in every strip $\sigma_0 - \delta < \sigma < \sigma_0 + \delta$ the value $c$ is taken more than $\kappa(c, \sigma_0, \delta)T$ times, for large $T$, in $0 < t < T$, where $\kappa(c, \sigma_0, \delta)$ is a positive constant.

Moreover, there exist positive constants $\kappa_1(c) < \kappa_2(c)$, depending on $c$, such that the number $M_c(T)$ of zeros of $\log \zeta(s) - c$ in $\sigma > 1$ satisfies

$$\kappa_1(c)T < M_c(T) < \kappa_2(c)T.$$
For a proof see [4] and [19], §11.8, respectively. It is obvious how this applies to the \(a\)-points of \(\zeta\) via taking \(a = \exp(c)\).

In addition, assuming the Riemann hypothesis, already Landau [5] showed that the \(a\)-points are clustered around the critical line, and Norman Levinson [15] improved upon this by proving unconditionally that most of the \(a\)-points have real part arbitrarily close to \(1/2\) (in a quantitative way).

We need in addition the partial fraction decomposition of the logarithmic derivative of \(\zeta(s) - a\), namely

\[
\frac{\zeta'(s)}{\zeta(s) - a} = \sum_{|\tau - \gamma_a| \leq 1} \frac{1}{s - \rho_a} + O(\log(2 + |t|)),
\]

valid for \(|t| \geq 1\) and \(-1 \leq \sigma \leq \sigma_0\) with any fixed positive real number \(\sigma_0\). This is a straightforward generalization of the classical formula for \(a = 0\); a proof of (7) can be found in [8], Lemma 8. In view of (4) and (5) this implies the bound

\[
\frac{\zeta'(s)}{\zeta(s) - a} \ll \log(2 + |t|)^2,
\]

valid for the same range as (7).

We conclude with two standard results. First of all, a bound for \(\Delta\) (defined by (3)) that follows immediately from Stirling’s formula, i.e.,

\[
\Delta(\sigma + it) = \left( \frac{t}{2\pi} \right)^{1/2 - \sigma} \exp \left( -it \log \frac{t}{2\pi e} + \frac{\pi i}{4} \right) (1 + O(t^{-1}))
\]

for \(t \geq 1\), and

\[
\Delta(\sigma + it) \asymp |t|^{1/2 - \sigma}
\]

as \(|t| \to \infty\); see for example [19], (4.12.3).

Finally, we need a modified version of a lemma due to Gonek [9]:

**Lemma 2.** Let \((b_n)_{n \geq 1}\) be a sequence of complex numbers such that \(b_n \ll \epsilon n^{d+\epsilon}\) for some \(d \geq 0\). If \(c > d\) and \(m \geq 0\) is an integer, then as \(T \to \infty\)

\[
\frac{1}{2\pi i} \int_{c+i}^{c+iT} \Delta^{(m)}(1-s) \sum_{n \geq 1} b_n n^{-s} ds = \sum_{n \leq T/(2\pi)} b_n (\log n)^m + O \left( T^{c-1/2} (\log T)^{m+1} \right).
\]

This result is also our starting point for the proof of Theorem [1].

**3. Proof of the special case** \(a = 0\)

We apply Lemma 2 to the logarithmic derivative

\[
-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n \geq 2} \Lambda(n) n^{-s},
\]

\[\text{The additional log factor in the error term does not exist in the original article. However, having a closer look in the proof shows that it cannot be omitted.}\]
where the Dirichlet series on the right converges absolutely in the half-plane $\sigma > 1$ and uniformly in every compact subset. This yields

$$-\psi(x) = -\sum_{n \leq x} \Lambda(n) = \frac{1}{2\pi i} \int_{1+\delta+iT}^{1+\delta+iT} \Delta(1-s) \frac{\zeta'(s)}{\zeta(s)} ds + O_\delta(T^{1/2+2\delta}),$$

where $\delta > 0$ is fixed and $T = 2\pi x$.

We assume that $T$ is at a distance from any $\zeta$-zero; more precisely, $|T - \gamma| \gg (\log T)^{-1}$ by (4); this restriction can later be removed at the expense of an error term of size $O(T^{1/2+\delta} \log^2 T)$ by (8) and (9).

Shifting the path of integration to the left, it follows from Cauchy’s theorem that

$$\int_{1+\delta+iT}^{1+\delta+iT} \Delta(1-s) \frac{\zeta'(s)}{\zeta(s)} ds = \left\{ \int_{-\delta+iT}^{-\delta+iT} + \int_{-\delta+iT}^{-\delta+iT} + \int_{-\delta+iT}^{-\delta+iT} \right\} \ldots + 2\pi i \Sigma,$$

where $\Sigma$ is the sum of residues at the nontrivial $\zeta$-zeros in the interior of the rectangular contour; here we suppose that $\delta \in (0, 1)$ in order to avoid a contribution from trivial zeros. Notice that $\Delta(s)$ is regular except for simple poles at $s = 1 + 2n$ for $n \in \mathbb{N}$.

The lower horizontal integral is bounded by a constant. For estimating the other integrals we apply in a straightforward manner (9) as well as (8) in combination with (4), and get

$$\int_{-\delta+iT}^{-\delta+iT} \Delta(1-s) \frac{\zeta'(s)}{\zeta(s)} ds \ll \int_1^T t^{-1/2-\delta} (\log t)^2 dt \ll T^{1/2+\epsilon},$$

and

$$\int_{-\delta+iT}^{-\delta+iT} \Delta(1-s) \frac{\zeta'(s)}{\zeta(s)} ds \ll \int_{-\delta}^{1+\delta} T^{-1/2+\sigma} (\log t)^2 dt \ll T^{1/2+\epsilon}$$

by choosing $\delta = \epsilon/2$.

For the residue at a nontrivial $\zeta$-zero $\rho = \beta + i\gamma$ we find

$$\text{res}_{s=\rho} = \lim_{s \to \rho} (s - \rho) \frac{\zeta'(s)}{\zeta(s)} \Delta(1-s) = \Delta(1-\rho).$$

Taking into account the functional equation, with $\rho$ also $1 - \rho$ is a nontrivial $\zeta$-zero and $1 - \rho$ as well; since the sum over $\Lambda(n)$ is real and $\Delta$ on the real axis, there is no effect of conjugation. Hence, we arrive at

$$\Sigma = \sum_{0<\gamma<T} \Delta(\rho).$$

Substituting this together with the estimates above leads to the unconditional asymptotic formula

$$-\psi(x) = \sum_{0<\gamma<T} \Delta(\rho) + O_\epsilon(T^{1/2+\epsilon}).$$

In view of the equivalence of the Riemann hypothesis to (1) this proves Theorem 1 for $a = 0$. 

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The initial idea for this paper actually was to find a new proof of the prime number theorem. For this purpose we applied Gonek’s lemma (instead of Perron’s formula) to the logarithmic derivative of $\zeta$. This is indeed the beginning of the proof above.

Interestingly, the case $a \neq 0$ of Theorem 1 needs a different reasoning (to our knowledge).

4. Proof of Theorem 1 in case $a \neq 0$

Suppose that $a \neq 0$. Again we may assume that $T$ is at a distance from any $a$-point; more precisely, $|T - \gamma_a| \gg (\log T)^{-1}$, this time by (5); this restriction can be removed at the expense of an error term of size $O(T^{1/2} \log T)$ by (5) and (9).

By Cauchy’s theorem, we have
\[
\sum_{0 < \gamma_a < T} \Delta(\rho_a) = \frac{1}{2\pi i} \int_R \frac{\zeta'(s)}{\zeta(s) - a} \Delta(s) \, ds,
\]
where $R$ is the positive oriented rectangular contour with vertices $\kappa + i, \kappa + iT, -\delta + iT, -\delta + i$ with some small $\delta > 0$ and a sufficiently large $\kappa > 1$ such that there are no $a$-points to the right of $\kappa - \epsilon + i\mathbb{R}$, that is $\kappa = b_a + 2\epsilon$, which is possible since Dirichlet series have a right half-plane free of zeros and $a$-points; the left vertical line segment does not cause any trouble since all but finitely many $a$-points in the left half-plane are distant from the line of integration (as already mentioned).

We split the integral into four; the two horizontals ones can be treated as in the case $a = 0$ above with the estimate for $\zeta'(s)/(\zeta(s) - a)$. For the vertical intervals we proceed also quite similarly.

On the left we use the lower bound
\[
\zeta(\sigma + it) \gg t^{1/2 - \sigma - \epsilon},
\]
valid for $t \geq 2$ and $\sigma \leq 0$, which follows easily from (9) and the functional equation (2) (or see [8], Lemma 4). This allows to develop the logarithmic derivative into a convergent geometric series, namely
\[
\frac{\zeta'(s)}{\zeta(s) - a} = \frac{\zeta'(s)}{\zeta(s)} \cdot \frac{1}{1 - a/\zeta(s)} = \frac{\zeta'(s)}{\zeta(s)} \cdot \left\{ 1 + \frac{a}{\zeta(s)} + \sum_{k \geq 2} \left( \frac{a}{\zeta(s)} \right)^k \right\}.
\]
This leads to
\[
\int_{-\delta + iT}^{-\delta + i} \frac{\zeta'(s)}{\zeta(s) - a} \Delta(s) \, ds = I_1 + I_2 + I_3,
\]
say, where
\[
I_3 = \int_{-\delta + iT}^{-\delta + i} \frac{\zeta'(s)}{\zeta(s)} \sum_{k \geq 2} \left( \frac{a}{\zeta(s)} \right)^k \Delta(s) \, ds \ll \int_1^T (\log t)^2 t^{2\epsilon - \delta - 1/2} \, dt \ll T^{1/2 + 2\epsilon}.
\]
We continue with

\[ I_1 = \int_{-\delta}^{\delta+iT} \frac{\zeta'(s)}{\zeta(s)} \Delta(s) \, ds \]

which we may rewrite by using the functional equation as

\[ I_1 = \int_{1+\delta-iT}^{1+\delta+iT} \left( \frac{\Delta'(1-s)}{\Delta(s)} - \frac{\zeta'(s)}{\zeta(s)} \right) \Delta(1-s) \, ds = J_1 + J_2, \]

say. Using again (9), we find

\[ J_1 = -\int_{1+\delta-iT}^{1+\delta+iT} (\Delta(1-s))' \, ds \ll T^{1/2+\delta}. \]

Moreover, \( J_2 \) turns out to be the conjugate of

\[ J_2 = \int_{1+\delta-iT}^{1+\delta+iT} \left( \frac{\Delta'(1-s)}{\Delta(s)} - \frac{\zeta'(s)}{\zeta(s)} \right) \frac{1}{\zeta(s)} \, ds = H_1 + H_2, \]

say. The main term arises from \( H_1 \); to see that we apply the first derivative test (see, for example, [11, Lemma 2.1]) and get

\[ H_1 = ia \int_1^T \left( \log \frac{t}{2\pi} + O(t^{-1}) \right) \left( 1 + \sum_{n \geq 2} \mu(n)n^{-1-\delta-it} \right) \, dt = iaT \log \frac{T}{2\pi e} + O(\log T). \]

In a similar way one bounds

\[ H_2 = a \int_{1+\delta-iT}^{1+\delta+iT} \sum_{m \geq 2} \Lambda(m)m^{-s} \sum_{n \geq 1} \mu(n)n^{-s} \, ds \ll 1. \]

Thus,

\[ I_2 = iaT \log \frac{T}{2\pi e} + O(\log T). \]

It remains to evaluate the vertical integral on the right, that is

\[ \int_{\kappa+iT}^{\kappa+iT} \frac{\zeta'(s)}{\zeta(s) - a} \Delta(s) \, ds. \]

Using (8) and (9) this integral is bounded from above by \( T^{3/2-\kappa} \).

Collecting all results together, recalling that \( \kappa > 1 \) and setting \( \delta = \epsilon/2 \), we obtain that

\[ \sum_{0 < \gamma_0 < T} \Delta(\rho_a) = a \frac{T}{2\pi} \log \frac{T}{2\pi e} - \psi \left( \frac{T}{2\pi} \right) + O \left( T^{1/2+\epsilon} \right). \]

This implies Theorem 1 in case \( a \neq 0 \).
The statement follows from the distribution of values in the half-plane of absolute convergence of \( \zeta \) and Lemma [1] of Bohr & Jessen in particular.

To see that, let \( \epsilon > 0 \) be given. We assume that

\[
\sum_{0 < \gamma_n < T} \Delta(1 - \rho_a) = O(T^{b_a - 1/2 - \epsilon}),
\]

as \( T \to \infty \). By Lemma [1] there exist infinitely many \( a \)-points \( \rho_a^+ = \beta_a^+ + i\gamma_a^+ \) with \( |\beta_a^+ - b_a| < \epsilon \) and \( \gamma_a^+ \to \infty \). It follows then that

\[
|\Delta(1 - \rho_a^+)| = \left( \frac{\gamma_a^+}{2\pi} \right)^{\beta_a^+ - 1/2} \left( 1 + O \left( \frac{1}{\gamma_a^+} \right) \right) \geq \frac{1}{2} \left( \frac{\gamma_a^+}{2\pi} \right)^{b_a - 1/2 - \epsilon}
\]

for any sufficiently large \( \gamma_a^+ \). Hence, if we denote by \( \gamma'_a \) the term succeeding \( \gamma_a^+ \) in the increasing sequence \( (\gamma_a)_{\gamma_a > 0} \), we can obtain from the above relations that

\[
\left| \sum_{0 < \gamma_n < \gamma'_a} \Delta(1 - \rho_a) \right| \geq |\Delta(1 - \rho_a^+)| - \left| \sum_{0 < \gamma_n < \gamma_a^+} \Delta(1 - \rho_a) \right| \gg \epsilon \left( \gamma'_a \right)^{b_a - 1/2 - \epsilon}
\]

It is not difficult to see that \( \gamma'_a = \gamma_a^+ + O(1) \), \( \gamma_a^+ \to \infty \), or even \( o(1) \), since the methods for obtaining such results are in their majority similar to the results regarding gaps of \( \zeta \)-zeros (see for example [19], Chapter IX]). Therefore,

\[
\left( \gamma'_a \right)^{b_a - 1/2 - \epsilon} \ll \epsilon \left| \sum_{0 < \gamma_n < \gamma'_a} \Delta(1 - \rho_a) \right| + o(1) = o(\epsilon \left( \gamma'_a \right)^{b_a - 1/2 - \epsilon}),
\]

as \( \gamma'_a \to \infty \), which is a contradiction. This finishes the proof of Theorem [2].

One could be tempted to improve upon this result by more advanced methods. We assume that \( a \neq 1 \) as well. Again we may rewrite the sum in question as a contour integral,

\[
\sum_{0 < \gamma_n < T} \Delta(1 - \rho_a) = \frac{1}{2\pi i} \int_R \frac{\zeta'(s)}{\zeta(s) - a} \Delta(1 - s) \, ds,
\]

where \( R \) is a rectangular contour, similar to the one in the previous section with the difference that the right vertical segment has real part \( \Re = \Re(\epsilon) := b_a + \epsilon \).

The horizontal line segments can be treated as before; they can be bounded by \( O_{\epsilon}(T^{\pi - 1/2}) \) which is larger than their contributions in the previous cases.

For the integral over the left vertical line we have that

\[
\frac{1}{2\pi i} \int_{-\delta - iT}^{-\delta + iT} \frac{\zeta'(s)}{\zeta(s) - a} \Delta(1 - s) \, ds \ll \int_1^T (\log t)^2 t^{-1/2 + \delta} \, dt \ll T^{1/2 + \epsilon},
\]

where we set \( \delta = \epsilon/2 \).

For the integral over the right vertical line, we apply Lemma [2] and obtain

\[
\frac{1}{2\pi i} \int_{\pi + iT}^{\pi + iT} \frac{\zeta'(s)}{\zeta(s) - a} \Delta(1 - s) \, ds = \sum_{n \leq T/(2\pi)} \Lambda_a(n) + O_{\epsilon}(T^{\pi - 1/2}).
\]
Therefore,
\[
\sum_{0 < \gamma_n < T} \Delta(1 - \rho_n) = \sum_{n \leq T/(2\pi)} \Lambda_a(n) + O_\epsilon \left( T^{\pi-1/2} \right).
\]
This is obviously a more general formula than the one in Theorem 1 for the case of $a = 0$ since $\Lambda_0 = -\Lambda$ and $\overline{b}_0 = 1$. Also, for any $a \neq 1$ such that $\overline{b}_a - b_a < 1/2$, the formula above and Theorem 2 imply that
\[
\Omega_\epsilon \left( T^{b_a-1/2-\epsilon} \right) = \sum_{0 < \gamma_n < T} \Delta(1 - \rho_n) \ll_\epsilon T^{b_a+\epsilon}.
\]
Applying Perron’s formula (see [19 §3.12]), we have
\[
\sum_{n \leq x} \Lambda_a(n) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\zeta'(s)}{\zeta(s) - a} \frac{x^s}{s} \, ds + O_\epsilon \left( \frac{x^{\kappa+1}}{\tau} \right),
\]
where $\tau > 0$ is a parameter and $x = T/(2\pi)$ or any other positive quantity having a fixed distance from integers. Shifting the line of integration to the left, we get contributions from the residues of the integrand at the $a$-points $\rho_a = \beta_a + i\gamma_a$ of $\zeta$ as well as from the simple pole of $\zeta(s)$ at $s = 1$. The latter residue, however, turns out to be small compared to
\[
\text{res}_{s=\rho_a} = \lim_{s \to \rho_a} (s - \rho_a) \frac{\zeta'(s)}{\zeta(s) - a} \frac{x^s}{s} = \frac{x^{\rho_a}}{\rho_a},
\]
which is of size $T^{b_a}/\gamma_a$ in our situation.

If there is not too much cancelation for the sum of these residues, one could expect their contribution to be of order $T^{b_a} \sum_{0 < \gamma < T} \gamma_a^{-1}$ which, in view of Lemma \ref{lemma} is of size $T^{b_a} \log T$. Actually, a hypothetical $\zeta$-zero off the critical line (and also if there are more than just one), would have a similar effect on the error term in the prime number theorem (see [10 Chapter V]).

We have imposed above the condition $\overline{b}_a - b_a < 1/2$ to deduce an upper bound for the sum of $\Delta(1 - \rho_n)$. It is natural to ask for which complex numbers $a$ this condition holds, when in general we know that the maximum difference of the abscissa of convergence and absolute convergence can be 1 (see [18 §9.13]).

A straightforward way to describe such points is to bound $\overline{b}_a$ by $3/2$ since $\overline{b}_a > 1$ for $a \neq 1$. Let $\sigma > 1$ be sufficiently large such that $|\zeta(s) - 1| = 2^{-\sigma}(1 + o(1)) < |a - 1|$. Then
\[
\frac{1}{\zeta(s) - a} = \frac{1}{1 - a} \left( 1 - \frac{\zeta(s) - 1}{a - 1} \right)^{-1} = \frac{1}{1 - a} \sum_{k \geq 0} \left( \frac{\zeta(s) - 1}{a - 1} \right)^k = -\sum_{k \geq 0} \sum_{n \geq 1} \frac{d^*_k(n)}{(a - 1)^{k+1}n^s},
\]
where $d^*_k(n)$ denotes the number of decompositions of $n$ into a product of $k$ factors greater than 1. We observe that
\[
\sum_{k \geq 0} \sum_{n \geq 1} \left| \frac{d^*_k(n)}{(a - 1)^{k+1}n^s} \right| = \frac{1}{|a - 1|} \sum_{k \geq 0} \sum_{n \geq 1} \frac{d_k^*(n)}{|a - 1|^k n^{s\sigma}} = \frac{1}{|a - 1| - |\zeta(\sigma) - 1|}
\]
for any $\sigma > \sigma^*$ where $\zeta(\sigma^*) - 1 = |\zeta(\sigma^*) - 1| = |a - 1|$.

Thus, for any such $\sigma$ the double series converges absolutely and we can interchange summation to obtain a Dirichlet series representation of the reciprocal of $\zeta(s) - a$ and, consequently, of its logarithmic derivative for which we are going to have that $\overline{b_a} \leq \sigma^*$. Hence, for every complex number $a$ satisfying $|a - 1| > \zeta(3/2) - 1 \approx 1.6123...$ it follows that $\overline{b_a} < 3/2$. On the other hand, for such $a$ the Omega-theorem does not imply that the non-existence of the mean of $\Delta(1 - \rho_a)$. One can also see from the construction above that if $a > 1$ then $b_a = \overline{b_a} = \sigma^*$ which also serves our purpose.

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