NON-DENTABLE SETS IN BANACH SPACES

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Abstract. In his study of the Radon Nikodým property of Banach spaces, Bourgain showed (among other things) that in any closed, bounded, convex set \( A \) that is not dentable, one can find a separated, weakly closed bush. In this note, we prove a generalization of Bourgain’s result: in any bounded, non dentable set \( A \) (not necessarily closed or convex) one can find a separated, weakly closed approximate bush. Similarly, we obtain as corollaries the existence of \( A \)-valued quasimartingales with sharply divergent behavior.

1. Introduction

We were motivated by the question of whether using the Kuratowski measure of noncompactness in place of diameter leads to a different notion of dentability of (not necessarily closed or convex) subsets of \( X \). Proposition 3.1 shows that they do not. This generalizes results from Chapitre 4 of [Bou79] where \( A \) is assumed to be closed, bounded, and convex. In Section 3, we obtain as corollaries \( A \)-valued quasimartingales and \( \overline{\text{co}}(A) \)-valued martingales with sharply divergent behavior (Corollaries 3.3 and 3.4) whenever \( A \) is non-\( \varepsilon \)-dentable. In Section 4, we improve the results of Section 3 by showing that the range of the quasimartingale can be made weakly closed. As a further corollary, we show that one can find a countable set \( F \) with \( \lim_{F_{\geq f}} d(f, A) \to 0 \) such that \( \overline{\text{co}}(F) \cap \text{Ext}(\overline{\text{co}}^{w^*}(F)) = \emptyset \) (Corollary 4.9).

2. Preliminaries

For any topological vector space \( V \) over \( \mathbb{R} \) and \( E \subseteq V \), let \( \text{co}(E) \) denote the convex hull of \( E \), and \( \overline{\text{co}}(E) \) the closure of \( \text{co}(E) \) in \( V \). Henceforth, let \( (X, \| \cdot \|) \) be a Banach space over \( \mathbb{R} \).

Definition 2.1. For any \( A \subseteq X \), let \( \alpha(A) \) be the infimum over all \( \varepsilon > 0 \) so that \( A \) can be covered by finitely many sets of diameter at most \( \varepsilon \). \( \alpha(A) \) is called the Kuratowski measure of noncompactness of \( A \).

Definition 2.2. For any bounded, nonempty \( A \subseteq X \), \( f \in B_{X^*} \) (unit ball of \( X^* \)), and \( \delta > 0 \), we define the slice \( S(f, A, \delta) \), to be the set \( \{ a \in A : f(a) > \sup f(A) - \delta \} \). A slice of \( A \) is a set \( S(f, A, \delta) \) for some \( f \in B_{X^*} \) and \( \delta > 0 \).

Remark 2.3. Geometrically, a slice of \( A \) is a nonempty intersection of \( A \) with an open half-plane. Note that if \( S(f, \overline{\text{co}}(A), \delta) \) is a slice of \( \overline{\text{co}}(A) \), then \( S(f, \overline{\text{co}}(A), \delta) \cap A = S(f, A, \delta) \) is a slice of \( A \). This is due to the fact that

\[
\sup(f(\overline{\text{co}}(A))) = \sup(f(\overline{\text{co}}(A))) = \sup(f(\overline{\text{co}}(A))) = \sup(f(A))
\]

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Definition 2.4. A set $A \subseteq X$ is called $\epsilon$-dentable if there exists a slice of $A$ with $\text{diam}(A) \leq \epsilon$, and non-$\epsilon$-dentable otherwise. $A$ is dentable if it is $\epsilon$-dentable for every $\epsilon > 0$, and nondentable otherwise.

Remark 2.5. By Remark 2.3 if $\overline{A}$ is $\epsilon$-dentable, $A$ is $\epsilon$-dentable.

Definition 2.6. If $V$ is a topological vector space, $E \subseteq V$ and $e \in E$, $e$ is called a denting point of $E$ if $e \not\in \overline{e}(E \setminus U)$ for every neighborhood $U$ of $e$. Special cases are when $V$ is a Banach space equipped with the weak topology, or a dual Banach space equipped with the weak* topology, in which case we call $e$ a weak denting point or a weak* denting point, respectively.

Definition 2.7. Given a filtration $(\mathcal{A}_n)_{n \geq 0}$ and a positive sequence $\delta_n$, we say that a sequence of $X$-valued, $(\mathcal{A}_n)_{n \geq 0}$-adapted random variables $(M_n)_{n \geq 0}$ is a $\delta_n$-quasimartingale if

$$\|E(M_{n+1}|\mathcal{A}_n) - M_n\|_\infty \leq \delta_n$$

for all $n \geq 0$.

The following proposition can be found in Lemme 4.2 from [Bon79]. For the sake of self-containment, we include our own proof here.

Proposition 2.8. Let $\epsilon > 0$ and $\delta > 0$. Suppose that $C$ and $C_1$ are closed, bounded, convex sets with $C_1$ properly contained in $C$. If $C = \overline{\text{co}}(C_1 \cup C_2)$, where $C_2$ is a convex subset of $C$ and $\text{diam}(C_2) < \epsilon$, then there exists a slice $S$ of $C$ with $S \subseteq C_2 + B_\delta(0)$. In particular, $C$ is $\epsilon$-dentable.

Proof. We may assume that $\text{diam}(C) \leq 1$. Since $C_1$ is a proper convex subset of $C$, by Hahn-Banach separation there exists $f \in B_X^*$ such that

$$\sup f(C_1) < M := \sup f(C)$$

Hence $C_1 \subseteq C \setminus S(f, C, \alpha)$ for some $\alpha > 0$. So

$$C = \overline{\text{co}}((C \setminus S(f, C, \alpha)) \cup C_2)$$

For $\gamma > 0$, let $S_\gamma = S(f, C, \gamma)$. Consider $y \in S_\gamma$. There exist $\lambda \in [0, 1]$, $z_1 \in \text{co}(C \setminus S(f, C, \alpha))$, and $z_2 \in C_2$ such that $\|y - \lambda z_1 - (1 - \lambda)z_2\| < \gamma$. Hence

$$M - \gamma < f(y) \leq f((\lambda z_1 + (1 - \lambda)z_2) + \|y - \lambda z_1 - (1 - \lambda)z_2\| < \lambda f(z_1) + (1 - \lambda)f(z_2) + \gamma \leq \lambda(M - \alpha) + (1 - \lambda)M + \gamma$$

Hence $\lambda < 2\gamma/\alpha$. So

$$\|y - z_2\| < \lambda\|z_1 - z_2\| + \gamma \leq (2\gamma/\alpha) \text{diam}(C) + \gamma \leq \gamma(2/\alpha + 1)$$

So, setting $\gamma := \frac{\delta}{2\alpha + 1}$, we get $S := S_{\gamma} \subseteq C_2 + B_\delta(0)$. Note that $\text{diam}(S) \leq \text{diam}(C_2) + 2\delta < \epsilon$ for $\delta$ sufficiently small. So $C$ is $\epsilon$-dentable. $\square$

We now derive a corollary of this proposition that will play a crucial role in the proof of Lemma 1.3

Corollary 2.9. For any closed, bounded, convex, non-$\epsilon$-dentable $C \subseteq X$, any closed, convex $C' \subseteq C$, and any $D \subseteq C$ with $\alpha(D) < \epsilon$, if $C = \overline{\text{co}}(C' \cup D)$, then $C = C'$.
Proof. Let $C$, $C'$, and $D$ be as above. Assume $C = \overline{C}(C' \cup D)$. Since $\alpha(D) < \epsilon$, $D = B_1 \cup B_2 \cup \ldots B_n$ for some $B_i \subseteq D$ with $\text{diam}(B_i) < \epsilon$. Let $C_i = \overline{C}(B_i)$. Then $\text{diam}(C_i) = \text{diam}(B_i) < \epsilon$, and $C = \overline{C}(C' \cup C_1 \cup C_2 \cup \ldots C_n)$.

Since $C$ is closed, bounded, convex, and not $\epsilon$-dentable, and since $C_n \subseteq C$ is closed, convex with $\text{diam}(C_n) < \epsilon$, Proposition 2.8 (with $C_\epsilon = C_n$ and $C' = \overline{C}(C' \cup C_1 \cup C_2 \cup \ldots C_{n-1})$) implies that $C = \overline{C}(C' \cup C_1 \cup C_2 \cup \ldots C_{n-1})$. Since $\text{diam}(C_{n-1}) < \epsilon$, we may apply Proposition 2.8 again to obtain $C = \overline{C}(C' \cup C_1 \cup C_2 \cup \ldots C_{n-2})$. Iterating, we get $C = C'$.

\[\] 3. $\delta$-Separated Martingales and Bushes

Proposition 3.1. Let $A \subseteq X$ be bounded, and let $\epsilon > 0$. The following are equivalent:

\begin{enumerate}
  \item \(\alpha(S) \geq \epsilon\) for every slice $S \subseteq A$.
  \item \(\text{diam}(S) \geq \epsilon\) for every slice $S$ of $A$ ($A$ is $\epsilon$-dentable).
  \item \(\text{diam}(S) \geq \epsilon\) for every slice $S$ of $\overline{C}(A)$ ($\overline{C}(A)$ is $\epsilon$-dentable).
\end{enumerate}

Proof. Let $A$, $\epsilon$ be as above. (1) $\rightarrow$ (2) is clear from definition of $\alpha$. (2) $\rightarrow$ (3) follows from the fact that every slice of $\overline{C}(A)$ contains a slice of $A$. We now show (3) $\rightarrow$ (1) by contradiction. Let $C = \overline{C}(A)$, assume that $C$ is non-$\epsilon$-dentable and that there exists a slice $S = S(f,A,\delta)$ of $A$ with $\alpha(S) < \epsilon$. Set $S_C = S(f,C,\delta)$. Then $C \setminus S_C$ is a closed convex subset of $C$ and $C = \overline{C}(C \setminus S_C \cup S)$. Then Corollary 2.9 implies $C = C \setminus S_C$, a contradiction since $S_C \subseteq C$ and $S_C$ is nonempty. \hfill $\square$

As in Chapitre 4 of [Bon79], we obtain several corollaries.

Corollary 3.2. For any $A \subseteq X$ bounded and $\epsilon > 0$, if $A$ is $\epsilon$-dentable, then for all $\delta < \frac{\epsilon}{2}$ and all $a_1, a_2, \ldots a_n \in A$, $\overline{C}(A) = \overline{C}(A \setminus (B_\delta(a_1) \cup B_\delta(a_2) \cup \ldots B_\delta(a_n)))$.

Proof. Let $A$, $\epsilon$, $\delta$, and $a_1, a_2, \ldots a_n$ be as above. Suppose there exists $x \in \overline{C}(A) \setminus \overline{C}(A \setminus (B_\delta(a_1) \cup B_\delta(a_2) \cup \ldots B_\delta(a_n)))$. By Hahn-Banach separation, we can pick a slice $S$ of $\overline{C}(A)$ containing $x$ and disjoint from $\overline{C}(A \setminus (B_\delta(a_1) \cup B_\delta(a_2) \cup \ldots B_\delta(a_n)))$. Then $S \cap A$ is a slice of $A$ disjoint from $A \setminus (B_\delta(a_1) \cup B_\delta(a_2) \cup \ldots B_\delta(a_n))$, and thus $S \cap A \subseteq B_\delta(a_1) \cup B_\delta(a_2) \cup \ldots B_\delta(a_n)$, which implies $\alpha(S \cap A) \leq 2\delta < \epsilon$, contradicting Proposition 3.1. \hfill $\square$

We can use Corollary 3.2 to construct $A$-valued quasimartingales and $\overline{C}(A)$-valued martingales that diverge in a sharp manner.

Corollary 3.3. For any nonempty, bounded, non-$\epsilon$-dentable $A \subseteq X$, any $\delta < \frac{\epsilon}{2}$, and any positive, summable sequence $(\delta_n)_{n \geq 0}$, there exists a filtration of finite $\sigma$-algebras $(A_n)_{n \geq 0}$ on $[0,1]$, each of whose atoms are intervals, and an $(A_n)_{n \geq 0}$-adapted quasimartingale $(M_n)_{n \geq 0}$ such that, for all $s, t \in [0,1]$ and $m \neq n \geq 0$,

\begin{enumerate}
  \item $M_n$ takes values in $A$.
  \item $\|M_m(s) - M_m(t)\| > \delta$.
  \item $\|\mathbb{E}(M_{n+1} | A_n) - M_n\|_{\infty} < \delta_n$.
\end{enumerate}

Proof. Let $A \subseteq X$ and $\delta > 0$ be as above. We construct the martingale inductively. Let $x_0$ be any point of $A$, $A_0$ the trivial $\sigma$-algebra, and $M_0 \equiv x_0$. Suppose that, for some $N \in \mathbb{N}$, $A_n$ and $M_n$ have been constructed for all $n \leq N$ and satisfy the conclusion of the Corollary 3.3. Let $J$ be an atom of $A_N$, and let $x_J$ be the value of $M_N$ on $J$. Let $\{a_1, a_2, \ldots a_k\} \subseteq A$ be the set of all elements in the image of any one
of the $M_n$, $n \leq N$. By Corollary 3.2, $x_j \in \mathfrak{c}(A \setminus (B_3(a_1) \cup B_3(a_2) \cup \ldots \cup B_3(a_k)))$. Thus, there exists $z_j \in \mathfrak{c}(A \setminus (B_3(a_1) \cup B_3(a_2) \cup \ldots \cup B_3(a_k)))$ such that $\|x_j - z_j\| < \delta_N$. Since $z_j \in \mathfrak{c}(A \setminus (B_3(a_1) \cup B_3(a_2) \cup \ldots \cup B_3(a_k)))$, $z_j = \lambda_1 z_j^I + \lambda_2 z_j^2 + \ldots \lambda_m z_j^m$ for some $z_j^I, z_j^2, \ldots z_j^m \in A$ and $\lambda_1, \lambda_2, \ldots \lambda_m \in (0,1)$ with $\lambda_1 + \lambda_2 + \ldots \lambda_m = 1$ and $\|z_j^I - a_j\| > \delta$ for all $i \leq m$ and $j \leq k$. Now we subdivide the interval $J$ into $m$ pairwise disjoint subintervals, $J_1, J_2, \ldots J_m$, with $|J_i| = \lambda_i|J|$ for each $i$. Repeating this process for each atom $J \in \mathcal{A}_N$ gives us a collection of pairwise disjoint intervals, and we define $\mathcal{A}_{N+1}$ to be the $\sigma$-algebra that they generate. On each $J_i$, we define $M_{N+1}$ to be $z_j^I$. Then conclusions (1) and (2) hold, and (3) holds since $\|E(M_{N+1}|A_N) - M_N\|_{\infty} = \sup_{J,i} \|z_j^I - z_j^I\| < \delta_N$. □

Corollary 3.4. For any nonempty, bounded, non-$\epsilon$-determinate $A \subseteq X$, any $\delta < \frac{\epsilon}{4}$, and any positive, summable sequence $(\delta_n)_{n \geq 0}$, there exists a filtration of finite $\sigma$-algebras $(\mathcal{A}_n)_{n \geq 0}$ on $[0,1]$, each of whose atoms are intervals, an $(\mathcal{A}_n)_{n \geq 0}$-adapted quasimartingale $(M_n)_{n \geq 0}$, and an $(\mathcal{A}_n)_{n \geq 0}$-adapted martingale $(\overline{M}_n)_{n \geq 0}$ such that, for all $s, t \in [0,1]$ and $m \neq n \geq 0$,

1. $M_n$ takes values in $A$.
2. $\overline{M}_n$ takes values in $\mathfrak{c}(A)$.
3. $\|M_n - \overline{M}_n\|_{\infty} < \delta_n$.
4. $\|M_n(s) - M_n(t)\|, \|\overline{M}_n(s) - \overline{M}_n(t)\| > \delta$.

Proof. Let $A \subseteq X$, and $\delta > 0$ be as above. Choose $\delta' \in (\delta, \frac{\epsilon}{4})$ and assume $\sum_{n=0}^{\infty} \delta_n < \delta' - \delta$. Choose a positive sequence $(\gamma_k)_{k \geq 0}$ such that $\sum_{k=n}^{\infty} \gamma_k < \delta_n$, and note that this implies $\sum_{n=0}^{\infty} \gamma_n < \sum_{n=0}^{\infty} \delta_n < \delta' - \delta$. By Corollary 3.3, there is a filtration $(\mathcal{A}_n)_{n \geq 0}$ and an $A$-valued $(\mathcal{A}_n)_{n \geq 0}$-adapted quasimartingale $(M_n)_{n \geq 0}$ such that $\|M_n(s) - M_n(t)\| > \delta'$ for all $s, t \in [0,1]$, $m \neq n$, and $\|E(M_{n+1}|A_n) - M_n\|_{\infty} < \gamma_n$. This inequality, together with the fact that $(\delta_n)_{n \geq 0}$ is summable (and thus convergent to 0), implies, for each $n \geq 0$, the sequence $(E(M_k|A_n))_{k \geq n}$ is Cauchy in $L^\infty(I; X)$. Indeed, for $k > j \geq n$,

$$\|E(M_k - M_j|A_n)\|_{L^\infty(I; X)} \leq \sum_{r=j}^{k-1} \|E(M_{r+1} - M_r|A_n)\|_{L^\infty(I; X)} \leq \sum_{r=j}^{k-1} \|E(M_{r+1} - M_r|A_r)\|_{L^\infty(I; X)} = \sum_{r=j}^{k-1} \|E(M_{r+1}|A_r) - M_r\|_{L^\infty(I; X)} \leq \sum_{r=j}^{k-1} \gamma_r \leq \delta_j$$

Thus we may set $\overline{M}_n := \lim_{k \to \infty} E(M_k|A_n)$. Clearly, $(\overline{M}_n)_{n \geq 0}$ is adapted to $(\mathcal{A}_n)_{n \geq 0}$ and takes values in $\mathfrak{c}(A)$, showing (2). Let us check the martingale property:

$$E(\overline{M}_{n+1}|A_n) = E(\lim_{k \to \infty} E(M_k|A_{n+1})|A_n) = \lim_{k \to \infty} E(E(M_k|A_{n+1})|A_n) = \lim_{k \to \infty} E(M_k|A_n) = \overline{M}_{n+1}$$

showing (1). Next,

$$\|\overline{M}_n - M_n\|_{\infty} \leq \sum_{k=n}^{\infty} \|E(M_{k+1} - M_k|A_n)\|_{\infty} \leq \sum_{k=n}^{\infty} \|E(M_{k+1} - M_k|A_k)\|_{\infty}$$
Let $M_k = \sum_{k=0}^{n} \gamma_k < \delta_n$ showing (3). We then use (3) to show (4):

$$\|M_n(s) - M_m(t)\| \geq \|M_n(s) - M_m(t)\| - \delta_n - \delta_m > \delta' - (\delta' - \delta) = \delta$$

Remark 3.5. The union over $n$ of the image of $M_n$ forms a $\delta$-separated bush in $\overline{\co}(A)$. It is norm closed and lacks extreme points.

4. Weakly Closed $\delta$-separated Martingales and Bushes

In this section, we sharpen our results from the previous section by constructing an $A$-valued $\delta$-separated approximate bush that is weakly closed. The argument is more involved than those of the previous section. This again extends results from Bourgin in [Bou79]. $A$ is not assumed to be closed or convex in our case.

Definition 4.1. Let $A \subseteq B_X$ and let $C = \overline{\co}(A)$. For any $\gamma \in (0,1)$ and slice $S = S(f, C, \delta)$ of $C$, we define $S^\gamma = S(\frac{f}{\gamma}, C, \frac{\delta}{\gamma})$. $S^\gamma$ is called a $\gamma$-shallow parallel of $S$.

Lemma 4.2. For any $C \subseteq B_X$ closed and convex, any $\gamma \in (0,1)$, and any slice $S$ of $C$, $S^\gamma \subseteq S$. For any $E \subseteq C$ for which $C = \overline{\co}((C \setminus S) \cup E)$, $S^\gamma \subseteq \overline{\co}(E) + B_\alpha(0) \subseteq co(E) + B_\alpha(0)$.\]

Proof. Let $\gamma \in (0,1)$ and $S = S(f, C, \delta)$ a slice of $C$. Since $\gamma \in (0,1)$, $\frac{\delta}{\gamma} < \delta$ implying $S^\gamma = S(\frac{f}{\gamma}, C, \frac{\delta}{\gamma}) \subseteq S(f, C, \delta) = S$. For the second part, let $E \subseteq C$ such that $C = \overline{\co}((C \setminus S) \cup E)$. Let $y \in S^\gamma$, $\epsilon > 0$, and $M : = sup(f(C))$. Since $y \in C = \overline{\co}((C \setminus S) \cup E)$, there exist $\lambda \in [0,1]$, $z_1 \in (C \setminus S)$, $z_2 \in \overline{\co}(E)$, and $u \in E$ with $\|u\| < \epsilon$ such that $y = \lambda z_1 + (1 - \lambda)z_2 + u$. Then we have

$$M - \frac{\delta}{\gamma} < f(y) = \lambda f(z_1) + (1 - \lambda)f(z_2) + f(u) < \lambda(M - \delta) + (1 - \lambda)M + \epsilon$$

implying $\lambda < \frac{\epsilon}{\delta} + \frac{\epsilon}{\gamma}$. Since $\epsilon > 0$ was arbitrary, we must have $\lambda \leq \frac{\epsilon}{\gamma}$. Hence,

$$\|y - z_2\| \leq \|y - (1 - \lambda)z_2\| + \|(1 - \lambda)z_2 - z_2\| = \|\lambda z_1\| + \|\lambda z_2\| \leq 2\lambda \leq \gamma$$

This shows $y \in B_\gamma(z_2) \subseteq \overline{\co}(E) + B_\gamma(0)$. The final containment $\overline{\co}(E) + B_\gamma(0) \subseteq co(E) + B_\gamma(0)$ obviously holds.\]

Lemma 4.3. Let $A \subseteq X$ be nonempty and non-$\epsilon$-dental, and let $C = \overline{\co}(A)$ (by Remark 2.2, $C$ is non-$\epsilon$-dental). For any slice $S_0$ of $C$, $D \subseteq C$ with $\alpha(D) < \epsilon$, and $\gamma \in (0,1)$, let $S(S_0, D)$ be the collection of all slices $S$ of $C$ with $S \subseteq S_0 \setminus D$ and $S_\gamma(S_0, D) = \{S^\gamma\}_{S \in S(S_0, D)}$. Let $\Lambda = \Lambda(S_0, D, \gamma) \subseteq C$ denote the union of all sets in $S_\gamma(S_0, D)$. Then $C = \overline{\co}((C \setminus S_0) \cup (\Lambda \cap A))$.

Proof. Let $S_0$, $D$, $\gamma$, and $\Lambda$ be as above. By Corollary 2.9 (with $C' = \overline{\co}((C \setminus S_0) \cup (\Lambda \cap A))$ and $D = D$), it suffices to prove $C = \overline{\co}((C \setminus S_0) \cup D \cup (\Lambda \cap A))$. Assume $C \neq \overline{\co}((C \setminus S_0) \cup D \cup (\Lambda \cap A))$. Then by Hahn-Banach separation, there exists a slice $S$ of $C$ such that $S \subseteq \overline{\co}(A)$. It is norm closed and lacks extreme points.
This implies $S \subseteq S_0$, $S \cap D = \emptyset$, and $S \cap (\Lambda \cap A) = \emptyset$. Then $S \subseteq S_0 \setminus D$. Thus, $S \in S(S_0, D)$, so $S^\gamma \in S^\gamma(S_0, D)$, and finally $S^\gamma \subseteq A$. But since we also have $S^\gamma \subseteq S$ and $S \cap (\Lambda \cap A) = \emptyset$, $(S^\gamma \cap A) = S^\gamma \cap (\Lambda \cap A) = \emptyset$, a contradiction since $S^\gamma \cap A$ is a slice of $A$ (since $S^\gamma$ is a slice of $C = \mathfrak{A}(A)$) and slices of nonempty sets are nonempty. □

4.1. The Construction.

**Theorem 4.4.** Let $A \subseteq B_X$ be nonempty and non-$\epsilon$-dentable (not necessarily closed or convex), and $C = \mathfrak{A}(A)$ so that $C$ is also non-$\epsilon$-dentable. Fix $\delta < \frac{\epsilon}{2}$, and assume that $A$ is separable. Then $C$ is separable as well, so $C = \bigcup_{n=0}^{\infty} B_1$ for some open $B_1$ (relative to $C$) with $\text{diam}(B_1) < \epsilon$. Let $(\delta_n)_{n \geq 0}$ be a sequence of numbers in $(0, 1)$. There exist a finitely branching tree $\mathbb{T} \subseteq \mathbb{N}^{<\omega}$, an “approximate” bush $(x_b)_{b \in \mathbb{T}} \subseteq A$, and slices $(S_b)_{b \in \mathbb{T}}$ of $C$ such that, for all $n \in \mathbb{N}$,

1. For all $b \in \mathbb{T}_n$, $x_b \in S_b^n \cap A \subseteq S_0$.
2. If $n \geq 1$, then for all $b \in \mathbb{T}_n$, $S_b \cap B_{n-1} = \emptyset$ and $S_b \cap \left(\bigcup_{|\beta| \leq n-1} B_\beta(x_\beta)\right) = \emptyset$.
3. If $n \geq 1$, then for all $b \in \mathbb{T}_{n-1}$, if $(b, 1), \ldots, (b, q)$ are the immediate successors of $b$, then $S_{(b, i)} \subseteq S_b$ and $x_b \in \text{co}(x_{(b, 1)}, \ldots, x_{(b, q)}) + B_{2\delta_{n-1}}(0)$.

**Proof.** The proof is by induction on $n$. For the base case, let $S_0 = C$ and let $x_0$ be any element of $S_0$. For the inductive step, let $n \geq 0$ and assume $\mathbb{T}_{\leq n}, (x_b)_{b \in \mathbb{T}_{\leq n}} \subseteq A$, and $(S_b)_{b \in \mathbb{T}_{\leq n}} \subseteq C$ have been constructed, and satisfy (1)-(3). Let $b \in \mathbb{T}_n$. Let $D := B_n \cup \bigcup_{|\beta| \leq n-1} B_\beta(x_\beta)$, so that $\alpha(D) < 2\delta < \epsilon$. As in Lemma 4.3 let $S(S_b, D)$ be the collection of all slices $S$ of $C$ such that $S \subseteq S_b \setminus D$, $S_{b+1}(S_b, D) = \{S_{b+1} \mid S \in S(S_b, D)\}$, and $\Lambda = \bigcup S_{b+1}(S_b, D)$. By Lemma 4.3 $C = \mathfrak{A}(C \setminus S_b) \cup (\Lambda \cap A)$. Then by Lemma 4.2 $S_b^\delta \subseteq \text{co}(\Lambda \cap A) + B_{2\delta}(0)$. Then since $x_b \in S_b^\delta$, there exists $z \in \text{co}(\Lambda \cap A)$ such that $\|x_b - z\| < 2\delta_n$. Let $z_1, \ldots, z_q \in \Lambda \cap A$ and $\lambda_1^b, \ldots, \lambda_q^b \in [0, 1]$ such that $z = \lambda_1^bz_1 + \ldots + \lambda_q^bz_q$. For each $i = 1, \ldots, q$, since $z_i \in A$, there are slices $S_{z_i} \subseteq S(S_b, D)$ of $C$ with $z_i \in S_{z_i}^{\delta_{n+1}}$. By definition of $\Lambda$, we now define the children of $b$ to be $(b, 1), \ldots, (b, q)$, $x_{(b, i)}$ to be $z_i$, and $S_{(b, i)}$ to be $S_{z_i}$. Repeating this process for each $b \in \mathbb{T}_n$ gives us $\mathbb{T}_{n+1}$, $(x_b)_{b \in \mathbb{T}_{n+1}} \subseteq A$, and $(S_b)_{b \in \mathbb{T}_{n+1}} \subseteq C$.

(1) and (3) hold immediately by construction. It is also clear that (2) holds by recalling that $S_{(b, i)} \subseteq S(S_b, D)$, and then examining the definition of $D$ and $S(S_b, D)$.

**Remark 4.5.** The assumption that $A$ is separable can be removed (at the penalty of replacing $\epsilon$ by $\epsilon/2$) because of the following result: under the hypothesis of Theorem 4.4, $A$ contains a countable subset that is non-$\epsilon/2$-dentable. This is essentially proved in Lemma 2.2 of [May73], but we’ll include the argument here. Since $\text{diam}(S) > \epsilon$ for every slice $S$ of $A$, it follows that no slice is contained in a closed ball $B_{\epsilon/2}(x)$. Hence, if $a \in A$, then $a \in \mathfrak{A}(A \setminus B_{\epsilon/2}(a))$. So there exists a countable set $T(a) \subseteq A \setminus B_{\epsilon/2}(a)$ such that $a \in \mathfrak{A}(T(a))$. By applying this fact iteratively as in Lemma 2.2 of [May73], we can construct a countable $A_0 \subseteq A$ such that for every $a \in A_0$, we have $a \in \mathfrak{A}(A_0 \setminus B_{\epsilon/2}(a))$. Hence every slice $S$ of $A_0$ satisfies $\text{diam}(S) > \epsilon/2$. Hence $A_0$ is not $\epsilon/2$-dentable.

**Corollary 4.6.** For any separable $A \subseteq B_X$ nonempty and non-$\epsilon$-dentable, any $\delta < \frac{\epsilon}{2}$, and any positive $(\delta_n)_{n \geq 0}$, there exists a completely $\delta$-separated, $(\delta_n)_{n \geq 0}$-approximate bush $(x_b)_{b \in \mathbb{T}}$ in $A$ such that any other set $(y_b)_{b \in \mathbb{T}} \subseteq C = \mathfrak{A}(A)$,
with \( \sup_{t \in \mathbb{T}_n} \| y_b - x_b \| < \gamma_n \) for some \( \gamma_n \to 0 \), is weakly closed and discrete. In particular, \((x_b)_{b \in \mathbb{T}_n}\) is weakly closed and discrete.

Proof. Let \( A, \delta, (\delta_n)_{n \geq 0} \) be as above. Applying the construction of Theorem 4.3 with \((\delta_n/2)_{n \geq 0}\) in place of \((\delta_n)_{n \geq 0}\), yields a bush \((x_b)_{b \in \mathbb{T}_n}\). By Theorem 4.3(1), \( x_b \in A \) for all \( b \in \mathbb{T} \). Suppose \( b_1, b_2 \in \mathbb{T} \) with \( |b_2| > |b_1| \). Then by Theorem 4.3(1), \( x_{b_2} \in S_{b_2} \), and by Theorem 4.3(2), \( S_{b_2} \cap B_3(x_{b_1}) = \emptyset \), so \( \| x_{b_2} - x_{b_1} \| > \delta \). This means the bush is completely \( \delta \)-separated. By Theorem 4.3(3), if \( b \in \mathbb{T} \) and \( (b, 1), \ldots, (b, q) \) are the immediate successors of \( b \), then \( x_b \in \text{co}(x_{(b, 1)}, \ldots, x_{(b, q)}) + B_{\delta_n}(0) \). This means the bush is \((\delta_n)_{n \geq 0}\)-approximate.

Finally, let \((y_b)_{b \in \mathbb{T}_n} \subseteq C\), with \( \sup_{b \in \mathbb{T}_n} \| y_b - x_b \| < \gamma_n \) for some \( \gamma_n \to 0 \), and let \( z \) belong to the weak closure of \((z_b)_{b \in \mathbb{T}}\). Since \( C \) is norm closed and convex, it is weakly closed, and thus \( z \in C \). Then \( z \in B_b \) for some \( i \). Consider \( S_b \) for \( |b| = i + 1 \). Then \( S_b = S(f_b, C, \alpha_b) \) for some \( f_b \in B_X \) and \( \alpha_b > 0 \). Hence

\[
z \in B_i \subseteq C \setminus S_b = \{ x \in C : f_b(x) \leq \sup f(C) - \alpha_b \}.
\]

Since \( B_i \) is open in the norm topology relative in \( C \) and \( C \) is convex, it follows that \( B_i \subseteq \{ x \in C : f_b(x) < \sup f_b(C) - \alpha_b \} \). Since \( \gamma_n \to 0 \), we can find \( \gamma > 0 \) and \( N \) large enough so that \( B_i \subseteq \{ x \in C : f_b(x) < \sup f_b(C) - \alpha_b - \gamma \} \), \( N \geq i + 1 \), and \( \gamma_n < \gamma \) for all \( n \geq N \). Then we set \( U_b := \{ x \in C : f_b(x) < \sup f_b(C) - \alpha_b - \gamma \} \) and observe that it is a weak neighborhood of \( z \) in \( C \). Hence \( U := \cap_{|b| = i + 1} U_b \) is a weak neighborhood of \( z \) in \( C \). Now we wish to show the set \( U \cap (y_b)_{b \in \mathbb{T}_n} \) is finite, which will imply our desired conclusion that \((y_b)_{b \in \mathbb{T}_n}\) is weakly closed and discrete. We will show that \( U \cap (y_b)_{b \in \mathbb{T}_n} \) is finite by showing that \( U \cap (y_b)_{b \in \mathbb{T}_2^N} = \emptyset \).

Consider \( b \in \mathbb{T} \) with \( |b| \geq N \). Then \( \| y_b - x_b \| < \gamma_{|b|} < \gamma \). Let \( b_{i+1} \in \mathbb{T} \) denote the unique predecessor of \( b \) with \( |b_{i+1}| = i + 1 \). Then \( x_{b_{i+1}} \in S_{b_{i+1}} \), and hence \( f_{b_{i+1}}(x_b) > \sup f_{b_{i+1}}(C) - \alpha_{b_{i+1}} \). Since \( f_{b_{i+1}} \in B_X \), and \( \| y_b - x_b \| < \gamma \), this implies \( f_{b_{i+1}}(y_b) > \sup f_{b_{i+1}}(C) - \alpha_{b_{i+1}} - \gamma \). Thus, by definition of \( U_{b_{i+1}}, y_b \notin U_{b_{i+1}} \). By definition of \( U \) this proves \( U \cap (y_b)_{b \in \mathbb{T}_2^N} = \emptyset \).

**Corollary 4.7.** For any \( A \subseteq B_X \) nonempty and non-\( c \)-dentable, any \( \delta \leq \frac{\varepsilon}{2} \), and any positive sequence \((\delta_n)_{n \geq 0}\), there exists a filtration of finite \( \sigma \)-algebras \((A_n)_{n \geq 0}\), an \( A \)-valued, \((\mathcal{A}_n)_{n \geq 0}\)-adapted \((\delta_n)_{n \geq 0}\)-quasimartingale \((M_n)_{n \geq 0}\) with \( \| M_n(t) - M_m(t) \| > \delta \) for all \( n \geq m \geq 0 \) and \( s, t \in [0, 1] \), and the range of this quasimartingale is weakly closed and discrete.

**Proof.** Let \( A, \delta, (\delta_n)_{n \geq 0} \) be as above, and apply Corollary 4.6 to obtain a \((\delta_n)_{n \geq 0}\)-approximate bush \((x_b)_{b \in \mathbb{T}_n}\) which is weakly closed and discrete. We define the filtration \((A_n)_{n \geq 0}\) on \([0, 1]\) recursively: Let \( A_0 \) be the trivial \( \sigma \)-algebra. Suppose \( A_n \) has been defined as a finite whose atoms are intervals, the atoms are in bijection with \( T_n \) via \( b \mapsto I_b \), and for any \( b \in T_{n} \) and child \( (b, i) \in T_n \), \( \mathcal{L}(I_{(b, i)}) = I_b \chi^i_b \). Then for any \( b' \in T_n \) with children \((b', 1), \ldots, (b', q) \), we pick any subdivision of \( I_b \) into intervals \( I_{(b', 1)}, \ldots, I_{(b', q)} \) so that \( \mathcal{L}(I_{(b', i)}) = I_{b'} \chi^i_{b'} \). Take \( A_n \) to be the \( \sigma \)-algebra generated by these intervals. Then we define \( M_n \) to be \( \sum_{|b|=n} x_b \chi B_b \). We then have \( \| E(M_{n+1} A_n) - M_n \| \leq \sup_{t \in T_n} \| x_b - \chi x_{(b, 1)} - \ldots - \chi x_{(b, q)} \| < \delta_n \). The range of this quasimartingale is exactly the bush, and thus weakly closed and discrete.

**Corollary 4.8.** For any \( A \subseteq B_X \) nonempty and non-\( c \)-dentable, \( \delta < \frac{\varepsilon}{2} \), and positive, summable sequence \((\delta_n)_{n \geq 0}\), there exist a filtration of finite \( \sigma \)-algebras
\((A_n)_{n \geq 0}\), an \(A\)-valued, \((A_n)_{n \geq 0}\)-adapted \((\delta_n)_{n \geq 0}\)-quasimartingale \((M_n)_{n \geq 0}\) and \(\overline{\sigma}(A)\)-valued, \((A_n)_{n \geq 0}\)-adapted martingale \((\overline{M}_n)_{n \geq 0}\) with, for all \(n \neq m \geq 0\) and \(s, t \in [0, 1]\),

\begin{align*}
(1) \ |\overline{M}_n(s) - \overline{M}_n(t)| &> \delta. \\
(2) \ |M_n - \overline{M}_n|_{\infty} &< \delta_n. \\
(3) \ The \ range \ of \ (\overline{M}_n)_{n \geq 0} \ is \ weakly \ closed \ and \ discrete.
\end{align*}

Proof. Let \(A, \delta, (\delta_n)_{n \geq 0}\) be as above, and apply Corollary 4.8 to obtain the \(\sigma\)-algebra \((A_n)_{n \geq 0}\) and \(A\)-valued, \((\delta_n)_{n \geq 0}\)-quasimartingale \((M_n)_{n \geq 0}\) with weakly closed and discrete range. Construct \((\overline{M}_n)_{n \geq 0}\) from \((M_n)_{n \geq 0}\) just as in the proof of Corollary 3.3, so that \((\overline{M}_n)_{n \geq 0}\) is \(\overline{\sigma}(A)\)-valued and \((1)\) and \((2)\) hold. To see \((3)\), again note that the range of \((M_n)_{n \geq 0}\) is exactly \((x_n)_{n \in \mathbb{T}_n}\) from Corollary 4.6.

Since \((\overline{M}_n)_{n \geq 0}\) is adapted to the same finite filtration as \((M_n)_{n \geq 0}\), \((2)\) implies that the range of \(\overline{M}_n\) equals \((y_n)_{n \in \mathbb{T}_n}\) for some \(y_n \in \overline{\sigma}(A)\) and \(\sup_{n \in \mathbb{T}_n} \|y_n - x_n\| < \delta_n\). Then Corollary 4.6 implies \((3)\).

\[\blacksquare\]

Corollary 4.9. For any \(A \subseteq B_X\) nonempty and non-\(\epsilon\)-dentable and any \(\delta < \frac{1}{2}\), there exists a countable set \(F \subseteq \overline{\sigma}(A)\) such that

\begin{align*}
(1) \ \lim_{F \ni f \to \infty} d(f, A) &= 0 \\
(2) \ F \ is \ weakly \ closed \ and \ discrete \ and \ Ext(F) = \emptyset. \\
(3) \ \overline{\sigma}(F) \ has \ no \ weak \ denting \ point. \\
(4) \ \overline{\sigma}(F) \cap Ext(\overline{\sigma}^{**}(F)) &= \emptyset.
\end{align*}

Proof. Let \(A, \delta\) be as above. Let \(\delta_n\) be any positive, summable sequence, and let \((A_n)_{n \geq 0}, (M_n)_{n \geq 0}, \) and \((\overline{M}_n)_{n \geq 0}\) be the filtration, \((\delta_n)_{n \geq 0}\)-quasimartingale, and martingale afforded to us by Corollary 4.8. Let \(F \subseteq \overline{\sigma}(A)\) be the range of the martingale. Since \((M_n)_{n \geq 0}\) is \(A\)-valued and \(|M_n - \overline{M}_n|_{\infty} < \delta_n, \lim_{F \ni f \to \infty} d(f, A) = 0, \) showing \((1)\).

By Corollary 4.8 \(F\) is weakly closed and discrete and clearly has no extreme point since it is a \(\delta\)-separated bush, showing \((2)\).

Since weak denting points of \(\overline{\sigma}(F)\) are extreme points, and since \(F\) has no extreme points, the set of weak denting points of \(\overline{\sigma}(F)\) is contained in \(\overline{\sigma}(F) \setminus F\). But since \(\overline{\sigma}(F) \setminus F\) is weakly open in \(\overline{\sigma}(F)\), it follows that \(\overline{\sigma}(F) \setminus F\) contains no weak denting point. This shows \((3)\).

For \((4)\), we first observe that the converse of the Krein-Milman theorem (Lemma 8.5 [DSS58]) implies that every extreme point of \(\overline{\sigma}^{**}(F)\) is a weak* denting point of \(\overline{\sigma}^{**}(F)\). To see this, let \(x\) be an extreme point of \(\overline{\sigma}^{**}(F)\) and assume \(x\) is not a weak* denting point. Then there is an open neighborhood \(U \subseteq X^{**} \) of \(x\) such that \(x \in \overline{\sigma}^{**}((\overline{\sigma}^{**}(F) \setminus U)\). Then since \(\overline{\sigma}^{**}(F) \setminus U\) is weak* compact, the converse to Krein-Milman implies every extreme point of \(\overline{\sigma}^{**}(\overline{\sigma}^{**}(F) \setminus U)\), in particular \(x\), is contained in \(\overline{\sigma}^{**}(F) \setminus U\), a contradiction. Then \((4)\) follows from \((3)\) since weak* denting points of \(\overline{\sigma}(F) \cap \overline{\sigma}^{**}(F) \subseteq X^{**}\) are the same as weak denting points of \(\overline{\sigma}(F) \cap \overline{\sigma}^{**}(F) \subseteq X\).

\[\blacksquare\]

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