Sharp inequalities which generalize the divergence theorem

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Abstract

Subject to suitable boundary conditions being imposed, sharp inequalities are obtained on integrals over a region $\Omega$ of certain special quadratic functions $f(E)$ where $E(x)$ derives from a potential $U(x)$. With $E = \nabla U$ it is known that such sharp inequalities can be obtained when $f(E)$ is a quasiconvex function and when $U$ satisfies affine boundary conditions (i.e., for some matrix $D$, $U = Dx$ on $\partial \Omega$). Here we allow for other boundary conditions and for fields $E$ that involve derivatives of a variety orders of $U$. We also treat integrals over $\Omega$ of special quadratic functions $g(J)$ where $J(x)$ satisfies a differential constraint involving derivatives with, possibly, a variety of orders. The results generalize an example of Kang and the author in three spatial dimensions where $J(x)$ is a $3 \times 3$ matrix valued field satisfying $\nabla \cdot J = 0$. We also present an algorithm for generating extremal quasiconvex quadratic functions.

Keywords: Quasiconvexity, $Q^*$-convexity, Null-Lagrangians

1 Introduction

The divergence theorem

$$\int_{\Omega} \nabla \cdot U(x) \, dx = \int_{\partial \Omega} U \cdot n \, dS,$$

of Lagrange, Gauss, Ostrogradsky, and Green, is one of the most important theorems in mathematics, particularly in applied mathematics. (Here $n$ is the outward normal to the surface $\partial \Omega$ of the region $\Omega$). In one dimension it reduces to the fundamental theorem of calculus and by setting $U = wV$ one obtains the multidimensional version of integration by parts:

$$\int_{\Omega} V(x) \cdot \nabla w \, dx = \int_{\partial \Omega} wV \cdot n \, dS - \int_{\Omega} w \nabla \cdot V(x) \, dx,$$

(1.2)
which with $V = \nabla v$ yields Green’s first identity,

$$\int_{\Omega} \nabla v \cdot \nabla w \, dx = \int_{\partial \Omega} w \nabla v \cdot n \, dS - \int_{\Omega} w \Delta v(x) \, dx.$$  \hspace{1cm} (1.3)

Note that the left hand side of (1.1) can be reexpressed as

$$\int_{\Omega} \text{Tr } E \, dx,$$  \hspace{1cm} (1.4)

where $E = \nabla U$ is the gradient of the potential $U$.

This immediately leads to the question: for what functions $f$ can one express

$$\int_{\Omega} f(E(x)) \, dx$$  \hspace{1cm} (1.5)

in terms of boundary data, when $E$ derives from a potential? Such functions are known as null-Lagrangians. One example in two-dimensions of a null-Lagrangian is the determinant of $E$ with $E = \nabla U$ and $U$ being a two-component vector: one has

$$f(E) = \det E = \frac{\partial U_1}{\partial x_1} \frac{\partial U_2}{\partial x_2} - \frac{\partial U_1}{\partial x_2} \frac{\partial U_2}{\partial x_1} = \frac{\partial}{\partial x_1} \left[ U_1 \frac{\partial U_2}{\partial x_2} \right] + \frac{\partial}{\partial x_2} \left[ -U_1 \frac{\partial U_2}{\partial x_1} \right],$$  \hspace{1cm} (1.6)

and we directly see that the quantity on the right is a divergence. With

$$E = \begin{pmatrix} E_1 \\ E_2 \end{pmatrix} = \begin{pmatrix} \nabla U \\ U \end{pmatrix},$$  \hspace{1cm} (1.7)

another null-Lagrangian is

$$f(E) = (E_2 \cdot E_2) \text{Tr}(E_1) + 2E_2 \cdot \left[ (E_1 E_2) \right] = (U \cdot U) \nabla \cdot U + 2U \cdot [\nabla U U]$$

$$= \nabla \cdot \left[ U(U \cdot U) \right],$$  \hspace{1cm} (1.8)

which again is a divergence. It has been shown by Ball, Currie and Olver [1] that any $C^1$ null-Lagrangian can be expressed as a divergence, so that the evaluation of (1.5) reduces to an application of the divergence theorem. Necessary and essentially sufficient algebraic conditions to determine whether a (quadratic or nonquadratic) function is a null Lagrangian have been given by Murat [2, 3, 4] (see also [5]). In the important case where $E = \nabla U$, $f(E)$ is a null Lagrangian if and only if it is a linear combination of the subdeterminants (minors) of any order of $E$. For references see the paper of Ball, Currie, and Olver, who also show that when $E(x) = \nabla^k U(x)$ there are no new null Lagrangians $f(E)$ beyond those obtained by applying the result for $k = 1$ to the $d^{k-1}$-component potential $\nabla^{k-1} U(x)$.

Instead of seeking equalities it makes sense to search for sharp inequalities and try to find functions $f$ and boundary data for which one can obtain sharp bounds on integrals of the form (1.5). If $E = \nabla U$ and affine boundary conditions $U = Dx$ on $\partial \Omega$ are
imposed, then one possible value of $E$ in the interior of $\Omega$ is of course $E = D^T$ (we take
the convention that $\nabla U_1, \nabla U_2, \ldots, \nabla U_\ell$ are the columns, not rows, of $\nabla U$, so that $E$ is
a $d \times \ell$ matrix), and functions $f$ that satisfy the inequality

$$\int_{\Omega} f(E(x)) \, dx \geq \int_{\Omega} f(D^T) \, dx = |\Omega| f(D^T)$$

are called quasiconvex and for such functions the inequality is obviously sharp. (Here
$|\Omega|$ denotes the volume of $\Omega$). There is a large literature on examples of quasiconvex
functions which include convex and polyconvex functions. So far only the quadratic
quasiconvex functions have been completely characterized: see Tartar and Murat $[6, 7, 8]$.
In particular, the quadratic function $f(\nabla U)$ is quasiconvex if and only if $f(H)$ is non-
negative for all rank-one $d \times \ell$ matrices $H$. One of contributions of this paper is to show
that for quadratic quasiconvex functions that are sharply quasic onvex, sharp bounds on
the integral (1.5) can be obtained for a wide variety of boundary conditions, and not
just affine ones. We emphasize (see p.26 of $[9]$ and $[10]$) that it follows from an example
of Terpstra $[11]$ and has shown more simply by Serre $[12, 13]$ that there are quadratic
quasiconvex functions that are not the sum of a convex quadratic function and a null-
Lagrangian. (Terpstra’s results imply that such quadratic forms exist only if $\ell \geq 3$ and
d $\geq 3$). This indicates that, in general, the sharp bounds we obtain cannot be obtained
using null-Lagrangians.

We allow for more general fields $E(x)$, namely those with $m$ components $E_r(x)$, $r =
1, 2, \ldots, m$, that derive from some real or complex potential $U(x)$, with $\ell$ components
$U_1(x), \ldots, U_\ell(x)$ through the equations

$$E_r(x) = \sum_{q=1}^\ell L_{rq} U_q(x),$$

for $r = 1, 2, \ldots, m$, where $L_{rq}$ is the differential operator

$$L_{rq} = A_{rq} + \sum_{h=1}^t \sum_{a_1, \ldots, a_h = 1}^d A_{rqh}^{a_1 \cdots a_h} \frac{\partial}{\partial x_{a_1}} \frac{\partial}{\partial x_{a_2}} \cdots \frac{\partial}{\partial x_{a_h}}$$

of order $t$ in a space of dimension $d$ with real or complex valued constant coefficients $A_{rq}$
and $A_{rqh}^{a_1 \cdots a_h}$, some of which may be zero.

We will find that sharp inequalities on the integral (1.5) can be obtained for certain
boundary conditions when $f$ is a sharply $Q^*$-convex function that is quadratic. A $Q^*$-
convex function is defined as a function that satisfies

$$\langle f(E) \rangle \geq f(\langle E \rangle),$$

for all periodic functions $E = LU$ that derive from a potential $U(x)$ that is the sum of
a polynomial $U^0(x)$ and a periodic potential $U^1(x)$. Here the angular brackets denote
volume averages over the unit cell of periodicity. (By all periodic functions we also mean for all primitive unit cells of periodicity, including parallelopiped shaped ones). In the degenerate case, as pointed out to me by Marc Briane (private communication), there may be potentials that give rise to periodic fields, but are not expressible as the sum of a periodic part and a polynomial part, e.g. with \( L = \partial/\partial x_1 \) the potential \( U = x_1 \cos(x_2) \) is not so expressible. However such potentials are not of interest to us.

The function \( f \) is sharply \( Q^* \)-convex if in addition one has the equality

\[
\langle f(E) \rangle = f(\langle E \rangle),
\]

for some non-constant periodic function \( E = LU \) that derives from a potential \( U(x) \) that is the sum of a polynomial \( U_0(x) \) and a periodic potential \( U_1(x) \). We call the fields \( E \) and its potential \( U(x) \) \( Q^* \)-special fields. When the elements \( A_{rq} \) vanish and the coefficients \( A_{rqh}^{a_1...a_h} \) are zero for all \( h \neq t \) then \( Q^* \)-convexity may be equivalent to quasiconvexity, but otherwise they are not equivalent: see [1] for a precise definition of quasiconvexity.

Note that there could be a sequence of functions \( E_1(x), E_2(x), E_3(x), \ldots, E_n(x) \ldots \) each having the same average value \( E_0 \), but not converging to \( E_0 \), such that

\[
\lim_{n \to \infty} \langle f(E_n) \rangle = f(E_0)
\]

in which case we would call \( f \) marginally \( Q^* \)-convex, but not sharply \( Q^* \)-convex unless there existed a field \( E \) such that the equality (1.13) held. In this paper we are not interested in marginally \( Q^* \)-convex functions which are not sharply \( Q^* \)-convex.

An associated problem, which we are also interested in, is to obtain sharp bounds on integrals of the form

\[
\int_{\Omega} g(J(x)) \, dx
\]

for fields \( J(x) \) with \( m \) real or complex components \( J_r(x) \), \( r = 1, 2, \ldots, m \), that satisfy the differential constraints

\[
\sum_{r=1}^{m} L^\dagger_{qr} J_r(x) = 0
\]

for \( q = 1, 2, \ldots, \ell \), where

\[
L^\dagger_{qr} = \overline{A_{rq}} + \sum_{h=1}^{t} \sum_{a_1,...,a_h=1}^{d} (-1)^h A_{rqh}^{a_1...a_h} \frac{\partial}{\partial x_{a_1}} \frac{\partial}{\partial x_{a_2}} \ldots \frac{\partial}{\partial x_{a_h}},
\]

in which the bar denotes complex conjugation. Observe that the operators in (1.11) and (1.17) are formal adjoints.

Again we will find that sharp inequalities on the integral (1.15) can be obtained for certain boundary conditions when \( f \) is a sharply \( Q^* \)-convex function that is quadratic. In this setting a \( Q^* \)-convex function \( g \) is defined as a function that satisfies

\[
\langle g(J) \rangle \geq g(\langle J \rangle),
\]

for
for all periodic functions $\mathbf{J}$ satisfying $\mathbf{L}^\dagger \mathbf{J} = 0$. The function is sharply $Q^*$-convex if in addition one has the equality
\begin{equation}
\langle g(\mathbf{J}) \rangle = g(\langle \mathbf{J} \rangle), \tag{1.19}
\end{equation}
for some non-constant periodic function $\mathbf{J}(x)$ satisfying $\mathbf{L}^\dagger \mathbf{J} = 0$. We will call $\mathbf{J}(x)$ a $Q^*$-special field.

Presumably, the assumption that $f$ and $g$ are quadratic is not essential, but if they are not it becomes a difficult task to find ones which are sharply $Q^*$-convex. Also there should be some generalization of the theory presented here to allow for $Q^*$-special fields $\mathbf{E}$ and $\mathbf{J}(x)$ which are not periodic, but we do not investigate this here.

The theory developed here generalizes an example given in [14], reviewed in the next section, that itself stemmed from developments in the calculus of variations, the theory of topology optimization, and the theory of composites: see the books [15] [16] [17] [18]. A key component of this is the Fourier space methods developed by Tartar and Murat [6] [7] [8] in their theory of compensated compactness for determining the quasiconvexity of quadratic forms.

2 An example

To make the general analysis easier to follow we first review an example given in [14], which was used to obtain sharp estimates of the volume occupied by an inclusion in a body from electrical impedance tomography measurements made at the surface of the body.

Let us consider in dimension $d = 3$ a $3 \times 3$ real valued matrix valued field $\mathbf{J}(x)$ satisfying $\nabla \cdot \mathbf{J} = 0$. (Thus its three columns, not rows, are each divergence free). Alternatively $\mathbf{J}(x)$ could be regarded as a 9 component vector which corresponds to the case $m = 9$ and $t = 1$ in (1.16) and (1.17). We want to show, that for certain fluxes $\mathbf{q} = \mathbf{J}^T \mathbf{n}$ at the boundary $\partial \Omega$ one can obtain sharp lower bounds on the integral (1.15) when
\begin{equation}
g(\mathbf{J}) = \text{Tr}(\mathbf{J}^2) + \text{Tr}(\mathbf{J}^T \mathbf{J}) - [\text{Tr}(\mathbf{J})]^2. \tag{2.1}
\end{equation}
This special function, introduced in section 25.7 of [17], is known to be quasiconvex (which in this case is equivalent to $Q^*$ convexity), in the sense that the inequality
\begin{equation}
\langle g(\mathbf{J}) \rangle - g(\langle \mathbf{J} \rangle) \geq 0 \tag{2.2}
\end{equation}
holds for all periodic functions $\mathbf{J}(x)$, where the angular brackets denote volume averages over the unit cell of periodicity. Following the ideas of Tartar and Murat [6] [7] [8] the reason can be seen in Fourier space where the inequality (by Parseval’s theorem, since the function $g$ is real and quadratic) takes the form
\begin{equation}
\sum_{\mathbf{k} \neq 0} g(\text{Re}\{\hat{\mathbf{J}}(\mathbf{k})\}) + g(\text{Im}\{\hat{\mathbf{J}}(\mathbf{k})\}) \geq 0, \tag{2.3}
\end{equation}
which holds if \( g(H) \) is non-negative for all rank two real matrices \( H \), and in particular for the matrices \( \text{Re}[\hat{J}(k)] \) and \( \text{Im}[\hat{J}(k)] \) which are at most rank two since \( \nabla \cdot J = 0 \) implies \( k \cdot \hat{J}(k) = 0 \). To prove that \( g(H) \) is non-negative for all rank two real matrices, one first notes that \( g(H) \) is rotationally invariant in the sense that

\[
g(R^T H R) = g(H) \tag{2.4}
\]

holds for all rotations \( R \). This implies it suffices to check non-negativity with matrices \( H \) such that \([0,0,1]^T \cdot H = 0\), i.e. of the form

\[
H = \begin{pmatrix}
h_{11} & h_{12} & h_{13} \\
h_{21} & h_{22} & h_{23} \\
0 & 0 & 0
\end{pmatrix}, \tag{2.5}
\]

in which case

\[
g(H) = (h_{11} - h_{22})^2 + (h_{12} + h_{21})^2 + h_{13}^2 + h_{23}^2 \tag{2.6}
\]

is clearly non-negative and zero only when \( h_{11} = h_{22}, h_{12} = -h_{21}, \) and \( h_{13} = h_{23} = 0 \). More generally, \( g(H) \) will be zero if and only if the rank two matrix \( H \) has matrix elements of the form

\[
H_{ij} = \alpha_0(k_i k_j - \delta_{ij} k \cdot k) + \beta_0 \epsilon_{ijm} k_m, \tag{2.7}
\]

for some constants \( \alpha_0 \) and \( \beta_0 \), and for some vector \( k \) which is a vector such that \( k \cdot H = 0 \), where \( \epsilon_{ijm} \) is the completely antisymmetric Levi-Civita tensor taking the value +1 when \( ijm \) is an even permutation of 123, −1 when it is an odd permutation, and 0 otherwise.

This establishes (2.2) and shows one has the equality

\[
\langle g(J) \rangle = g(\langle J \rangle) = 0, \tag{2.8}
\]

for fields \( J(x) \) of the form

\[
J_{k\ell} = J_{k\ell}^0 + \frac{\partial^2 \alpha}{\partial x_k \partial x_\ell} - \delta_{k\ell} \Delta \alpha + \epsilon_{k\ell m} \frac{\partial \beta}{\partial x_m}, \tag{2.9}
\]

where the \( J_{k\ell}^0 \) are elements of a constant matrix \( J^0 \) and \( \alpha(x) \) and \( \beta(x) \) are arbitrary periodic functions (in this case \( J(x) \) has Fourier components of the form (2.7)). This can be established directly by noting that

\[
g(J) = \text{Tr}(J^T \mathcal{E}), \tag{2.10}
\]

where

\[
\mathcal{E} = \mathcal{J} + \mathcal{J}^T - \text{Tr}(\mathcal{J}) \mathcal{I} = \mathcal{E}^0 + 2 \nabla \nabla \alpha, \quad \mathcal{E}^0 = J^0 + (J^0)^T - \text{Tr}(J^0) \mathcal{I}, \tag{2.11}
\]

and so (2.8) follows by integration by parts:

\[
\langle g(J) \rangle = \langle \text{Tr}(J^T \mathcal{E}) \rangle = \text{Tr}\{ (J^0)^T [J^0 + (J^0)^T - \text{Tr}(J^0) \mathcal{I}] \} + 2 \langle \text{Tr}[ (J^0)^T \nabla \nabla \alpha] \rangle = g(J^0) - 2 \langle [\nabla \cdot J^0]^T \nabla \alpha] \rangle = g(\langle J \rangle). \tag{2.12}
\]
Now suppose \( \mathbf{J}(x) \) defined within \( \Omega \) satisfies the boundary condition \( \mathbf{q} = \mathbf{J}^T \mathbf{n} \) at the boundary \( \partial \Omega \) where \( \mathbf{q}_\ell \) has components

\[
q_\ell = n_k \left( J_{k\ell}^0 + \frac{\partial^2 \alpha}{\partial x_k \partial x_\ell} - \delta_{kl} \Delta \alpha + \epsilon_{k\ell m} \frac{\partial \beta}{\partial x_m} \right),
\]

(2.13)

for some functions \( \alpha(x) \) and \( \beta(x) \) defined in the neighborhood of \( \partial \Omega \), and for some constants \( J_{k\ell}^0 \). Then one can extend \( \alpha(x) \) and \( \beta(x) \) beyond this boundary, so that they are periodic (with a unit cell of periodicity \( C \) containing \( \Omega \)) and define a field \( \mathbf{J}(x) \) given by (2.9). One can also extend the function \( \mathbf{J}(x) \) so that it is periodic equals the function \( \mathbf{J}(x) \) outside \( \Omega \) and all its periodic copies. The boundary conditions (2.13) ensure that this periodic function satisfies \( \nabla \cdot \mathbf{J}(x) \) in a weak sense and therefore (2.2) implies

\[
\int_{C \setminus \Omega} g(\mathbf{J}(x)) \, dx + \int_{\Omega} g(\mathbf{J}(x)) \, dx \geq |C| g(\langle \mathbf{J} \rangle),
\]

(2.14)

where \( |C| \) is the volume of \( C \). Also (2.8) implies

\[
\int_{C \setminus \Omega} g(\mathbf{J}(x)) \, dx + \int_{\Omega} g(\mathbf{J}(x)) \, dx = |C| g(\langle \mathbf{J} \rangle).
\]

(2.15)

Furthermore since \( \nabla \cdot \mathbf{J} = 0 \) weakly, integration by parts implies that

\[
\int_C J_{k\ell} \, dx = \int_C \sum_{i=1}^3 \frac{\partial x_k}{\partial x_i} J_{i\ell} \, dx = \int_{\partial C} \sum_{i=1}^3 x_k n_i J_{i\ell} \, dS = \int_C \mathbf{J}_{k\ell} \, dx,
\]

(2.16)

and thus it follows that \( \langle \mathbf{J} \rangle = \langle \mathbf{J} \rangle \). So subtracting (2.15) from (2.14) gives

\[
\int_{\Omega} g(\mathbf{J}(x)) \, dx \geq \int_{\Omega} g(\mathbf{J}(x)) \, dx.
\]

(2.17)

Using (2.10) and integrating by parts, allows us to evaluate the right hand side:

\[
\int_{\Omega} g(\mathbf{J}(x)) \, dx = \int_{\Omega} \text{Tr}[(\mathbf{J})^T \nabla (\mathbf{x}^T \mathbf{E}_0 + 2 \nabla \alpha)] \, dx = \int_{\partial \Omega} \mathbf{n} \cdot \mathbf{J}(\mathbf{x}^T \mathbf{E}_0 + 2 \nabla \alpha) \, dS
\]

\[= \int_{\partial \Omega} \mathbf{q} \cdot (\mathbf{x}^T \mathbf{E}_0 + 2 \nabla \alpha) \, dS.
\]

(2.18)

In summary, for boundary fluxes of the form (2.13) we obtain the inequality

\[
\int_{\Omega} g(\mathbf{J}(x)) \, dx \geq \int_{\partial \Omega} \mathbf{q} \cdot (\mathbf{x}^T \mathbf{E}_0 + 2 \nabla \alpha) \, dS,
\]

(2.19)

which is sharp, being satisfied as an equality when \( \mathbf{J}(x) = \mathbf{J}(x) \) within \( \Omega \). In fact there is a huge range of fields for which one has equality since one is free to change \( \alpha(x) \) and \( \beta(x) \) in the interior of \( \Omega \) so long as the right side of (2.13) remains unchanged at the boundary \( \partial \Omega \).
3 Conditions for $Q^*$-convexity

Here we solve the problem of characterizing those quadratic functions $f$ or $g$ which are $Q^*$-convex. This is a straightforward extension of the ideas developed by Tartar and Murat [6, 7, 8] in their theory of compensated compactness for characterizing quadratic quasiconvex functions. Since $f$ and $g$ are quadratic we can let

$$f(E(x)) = E \cdot SE \equiv \sum_{r=1}^{m} \sum_{s=1}^{m} \mathcal{T}_r(x) S_{rs} E_s(x),$$

$$g(J(x)) = J \cdot TJ \equiv \sum_{r=1}^{m} \sum_{s=1}^{m} \mathcal{T}_r(x) T_{rs} J_s(x).$$

(3.1)

where the $S_{rs}$ or $T_{rs}$ are the real or complex valued elements of some Hermitian $m \times m$ matrix $S$ or $T$.

We first consider periodic functions $E(x)$ and $J(x)$ and potentials $U(x)$ that can be expressed in the form

$$U(x) = U^0(x) + U^1(x),$$

(3.2)

where $U^1(x)$ is periodic with zero average value and $U^0(x)$ is a polynomial with elements

$$U^0_q = B_q + \sum_{h=1}^{t} \sum_{a_1,...,a_h=1}^{d} B_{a_1,...,a_h} x_{a_1} x_{a_2} \ldots x_{a_h},$$

(3.3)

where the coefficients $B_q$ and $B_{a_1,...,a_h}$ are chosen so that

$$E^0_r = \sum_{q=1}^{\ell} L_{rq} U^0_q(x)$$

(3.4)

is a constant for $r = 1, 2, \ldots, m$. We let $\mathcal{E}^0$ denote the vector space spanned by all constant fields $E^0$ with elements $E^0_r$ expressible in the form (3.4).

By expanding $E(x)$, $U^1(x)$ and $J(x)$ in a Fourier series

$$E(x) = \langle E \rangle + \sum_{k \neq 0} \mathcal{E}(k) e^{ik \cdot x}$$

$$U^1(x) = \sum_{k \neq 0} \mathcal{U}^1(k) e^{ik \cdot x}$$

$$J(x) = \langle J \rangle + \sum_{k \neq 0} \mathcal{J}(k) e^{ik \cdot x},$$

(3.5)

where the sum is over all $k$ in the reciprocal lattice space we see that the differential constraints (1.10) or (1.16) imply that for all $k \neq 0$ in the reciprocal lattice

$$\mathcal{E}_r(k) = \sum_{q=1}^{\ell} \mathcal{L}_{rq}(k) \mathcal{U}^1_q(k), \quad \sum_{r=1}^{m} \mathcal{L}^\dagger_{qr}(k) \mathcal{J}_r(k) = 0,$$

(3.6)
where $\hat{L}_{qr}(k)$ and $\hat{L}_{qr}^\dagger(k)$ are the matrix elements

\[
\hat{L}_{rq}(k) = A_{rq} + \sum_{h=1}^t \sum_{a_1, \ldots, a_h=1}^d i^h A_{rqh}^{a_1 \ldots a_h} k_{a_1} k_{a_2} \ldots k_{a_h},
\]
\[
\hat{L}_{qr}^\dagger(k) = A_{rq} + \sum_{h=1}^t \sum_{a_1, \ldots, a_h=1}^d (-i)^h A_{rqh}^{a_1 \ldots a_h} k_{a_1} k_{a_2} \ldots k_{a_h}
\]

(3.7)
of the $m \times m$ matrix $\hat{L}(k)$ and its adjoint $\hat{L}^\dagger(k)$, defined for all $k \in \mathbb{R}^d$. In other words, the differential constraints imply that $\hat{E}(k)$ is in the range $E_k$ of $\hat{L}(k)$ and that $\hat{J}(k)$ is in the null-space $J_k$ of $\hat{L}^\dagger(k)$, where again these spaces are defined for all $k \in \mathbb{R}^d$ and not just those $k$ in the reciprocal lattice. We let $\Gamma_1(k)$ denote the projection onto $E_k$ and $\Gamma_2(k)$ denote the projection onto $J_k$. Since these spaces are orthogonal complements it follows that

\[
\Gamma_1(k) + \Gamma_2(k) = I, \quad \Gamma_1(k) \Gamma_2(k) = \Gamma_2(k) \Gamma_1(k) = 0, \quad \forall k \in \mathbb{R}^d
\]

(3.8)

where $I$ is the $m \times m$ identity matrix.

By substituting the Fourier expansion (3.5) for $E(x)$ in (1.12) we see that (1.12) holds if and only if the expression

\[
\langle E \cdot S E \rangle - \langle E \rangle \cdot S \langle E \rangle = \sum_{k \neq 0} \hat{E}(k) \cdot S \hat{E}(k) = \sum_{k \neq 0} f(\hat{E}(k))
\]

(3.9)
is non-negative. A necessary and sufficient condition for this to hold for all possible primitive unit cells is that for all non-zero $k \in \mathbb{R}^d$

\[
f(H) \geq 0 \quad \text{for all } H \in E_k,
\]

(3.10)
or equivalently that

\[
\Gamma_1(k) S \Gamma_1(k) \quad \forall k \in \mathbb{R}^d, \; k \neq 0,
\]

(3.11)

which is an algebraic condition which can be checked numerically, and in some cases analytically.

If $S$ is real (and hence symmetric), then (3.9) reduces to

\[
\langle E \cdot S E \rangle - \langle E \rangle \cdot S \langle E \rangle = \sum_{k \neq 0} \Re[\hat{E}(k)] \cdot S \Re[\hat{E}(k)] + \sum_{k \neq 0} \Im[\hat{E}(k)] \cdot S \Im[\hat{E}(k)].
\]

(3.12)

If in addition the coefficients $A_{rq}$ are real and the coefficients $A_{rqh}^{a_1 \ldots a_h}$ are real when $h$ is even and purely imaginary when $h$ is odd then $\hat{L}(k)$ is real and $H \in E_k$ if and only if the real and imaginary parts of $H$ lie in $E_k$. Thus, in this case, to guarantee (1.12) it suffices that (3.10) holds for real $H$, for all $k \neq 0$. 

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Similarly we look for Hermitian matrices $T$ which are $Q^*$-convex in the sense that (1.18) holds and a necessary and sufficient condition for this is that

$$g(H) \geq 0 \quad \text{for all } H \in \mathcal{J}_k,$$

or equivalently that

$$\Gamma_2(k)T\Gamma_2(k) \quad \forall k \in \mathbb{R}^d, \ k \neq 0. \quad (3.14)$$

If $T$ is real and the coefficients $A_{rq}$ are real and the coefficients $A_{rgh}^{a_1...a_h}$ are real when $h$ is even and purely imaginary when $h$ is odd, then it suffices to check (3.13) holds for real $H$.

4 Sharply $Q^*$-convex quadratic functions and their associated $Q^*$-special fields

The $Q^*$-convexity conditions (3.11) and (3.14) are clearly satisfied when $S$ and $T$ are positive definite, but in this case one only has equality in (3.10) and (3.13) when $H = 0$. For any $k \neq 0$ let $S_k$ denote the subspace of all $m$ dimensional vectors $H \in \mathcal{E}_k$ for which one has equality in (3.10), and let $\mathcal{U}_k$ denote the subspace of all $\ell$ dimensional vectors $G$ such that $\mathcal{L}(k)G \in S_k$. We call $f$ sharply $Q^*$-convex if, for some $k \neq 0$, $S_k$ contains at least one non-zero vector $H$. Note that if $H \in S_k$ then

$$H \cdot \Gamma_1 S \Gamma_1 H = 0, \quad (4.1)$$

and since from (3.11) $\Gamma_1 S \Gamma_1$ is positive semi-definite we deduce that $H$ must in fact be a null-vector of this matrix, which implies that

$$\Gamma_1 SH = 0, \quad \text{i.e. } SH \in \mathcal{J}_k, \quad (4.2)$$

and hence that

$$S\mathcal{L}(k)G \in \mathcal{J}_k, \quad \forall G \in \mathcal{U}_k. \quad (4.3)$$

Associated with a sharply $Q^*$-convex $f$ are periodic potentials $\mathcal{U}^1(x)$ expressible in the form

$$\mathcal{U}^1(x) = \sum_{k \neq 0} \mathcal{U}^1(k)e^{ikx}, \quad (4.4)$$

where $\mathcal{U}^1(k) \in \mathcal{U}_k$ for all $k \neq 0$. The unit cell of periodicity has to be chosen so the $k$ in the reciprocal lattice include some $k$ such that $S_k$ contains at least one non-zero vector.

Let us introduce the companion $Q^*$-special fields

$$E(x) = E^0 + \sum_{k \neq 0} \mathcal{E}(k)e^{ikx},$$

$$\mathcal{J}(x) = SE^0 + \sum_{k \neq 0} \mathcal{J}(k)e^{ikx}, \quad (4.5)$$
in which $\mathbf{E}^0 \in \mathcal{E}^0$ so that $\mathbf{E}^0 = L \mathbf{U}^0(x)$ for some polynomial potential $\mathbf{U}^0(x)$ and where

$$\hat{\mathbf{E}}(k) = \hat{L}(k)\hat{\mathbf{U}}^1(k), \quad \hat{\mathbf{J}}(k) = S\hat{\mathbf{E}}(k). \tag{4.6}$$

With $\mathbf{U}(x) = \mathbf{U}^0(x) + \mathbf{U}^1(x)$ these $Q^*$-special fields satisfy

$$\hat{\mathbf{J}}(k) = S\hat{\mathbf{E}}(k), \quad \mathbf{E} = L\mathbf{U}, \quad L^J = 0, \tag{4.7}$$

and

$$\langle \mathbf{E} \cdot S\mathbf{E} \rangle - \langle \mathbf{E} \rangle \cdot S\langle \mathbf{E} \rangle = \sum_{k \neq 0} \hat{\mathbf{E}}(k) \cdot S\hat{\mathbf{E}}(k) = 0, \tag{4.8}$$

so that (1.13) is satisfied.

Similarly for all $k \neq 0$ let $\mathcal{T}_k$ denote the subspace of all $m$ dimensional vectors $\mathbf{H} \in \mathcal{J}_k$ for which one has equality in (3.13). We call $g$ sharply $Q^*$-convex if, for some $k \neq 0$, $\mathcal{T}_k$ contains at least one non-zero vector $\mathbf{H}$. Analogous to (4.2), (3.14) implies

$$\Gamma^2 T\mathbf{H} = 0, \quad \text{i.e.} \quad T\mathbf{H} \in \mathcal{E}_k. \tag{4.9}$$

Since $\mathcal{E}_k$ is the range of $\hat{L}(k)$, there exists a (possibly non-unique) $\ell$ dimensional vector $\mathbf{G}$ such that

$$T\mathbf{H} = \hat{L}(k)\mathbf{G}. \tag{4.10}$$

Now associated with a sharply $Q^*$-convex $g$ are the $Q^*$-special fields

$$\mathbf{J}(x) = \mathbf{J}^0 + \sum_{k \neq 0} \hat{\mathbf{J}}(k)e^{ikx},$$

$$\mathbf{E}(x) = T\mathbf{J}^0 + \sum_{k \neq 0} \hat{\mathbf{E}}(k)e^{ikx}, \tag{4.11}$$

where

$$\hat{\mathbf{J}}(k) \in \mathcal{T}_k, \quad \hat{\mathbf{E}}(k) = T\hat{\mathbf{J}}(k) \in \mathcal{E}_k, \tag{4.12}$$

and $T\mathbf{J}^0 \in \mathcal{E}^0$ so that $T\mathbf{J}^0 = L\mathbf{U}^0(x)$ for some polynomial potential $\mathbf{U}^0(x)$. The unit cell of periodicity has to be chosen so the $k$ in the reciprocal lattice include some $k$ such that $\mathcal{T}_k$ contains at least one non-zero vector. We choose the periodic potential

$$\mathbf{U}^1(x) = \sum_{k \neq 0} \hat{\mathbf{U}}^1(k)e^{ikx}, \tag{4.13}$$

so that its Fourier coefficients satisfy

$$\hat{\mathbf{E}}(k) = \hat{L}(k)\hat{\mathbf{U}}^1(k), \tag{4.14}$$
for \( k \neq 0 \). With \( U(x) = U^0(x) + U^1(x) \) these \( Q^* \)-special fields satisfy
\[
E(x) = T(J(x), \quad E = LU, \quad L^\dagger J = 0,
\]
and
\[
\langle J \cdot T(J) \rangle - \langle J \rangle \cdot T\langle J \rangle = \sum_{k \neq 0} \hat{J}(k) \cdot T\hat{J}(k) = 0,
\]
so that (1.19) is satisfied.

5 An algorithm for generating sharply \( Q^* \)-convex quadratic functions and their associated \( Q^* \)-special fields and for generating extremal quasiconvex functions

To begin with let us suppose \( S \) can be expressed in the form
\[
S = V - vv^\dagger/\alpha,
\]
where \( V \) is Hermitian and positive definite, and \( \alpha \) is real and positive. The \( Q^* \)-convexity of \( f \) is equivalent to the inequality
\[
H \cdot VH \geq \overline{\gamma}(H \cdot v) + \gamma(H \cdot v) - \alpha|\gamma|^2
\]
holding for all complex \( \gamma \) and for all \( H \in E_k \), for all non-zero \( k \in \mathbb{R}^d \), as can be established by taking the maximum of the right hand side over \( \gamma \) thereby recovering (3.10). Rewriting this inequality as
\[
\alpha|\gamma|^2 \geq \overline{\gamma}(H \cdot v) + \gamma(H \cdot v) - H \cdot VH,
\]
and taking the maximum of the right hand side over \( H \in E_k \) we see that an equivalent condition is that
\[
\alpha \geq v \cdot \Gamma(k)v, \quad \forall k \in \mathbb{R}^d, \quad k \neq 0,
\]
where, assuming the \( \ell \times \ell \) matrix \( \hat{L}^\dagger(k)V\hat{L}(k) \) is non-singular for all \( k 
eq 0 \),
\[
\Gamma(k) = \hat{L}(k)(\hat{L}^\dagger(k)V\hat{L}(k))^{-1}\hat{L}^\dagger(k).
\]
If \( V = I \) then \( \Gamma(k) \) is equal to \( \Gamma_1(k) \), the projection onto the range \( \mathcal{E}_k \) of \( \hat{L}(k) \). More generally \( \Gamma(k) \) is defined to be the matrix such that for all vectors \( v \),
\[
\Gamma(k)v \in \mathcal{E}_k \quad \text{and} \quad \Gamma(k)v - Vv \in \mathcal{J}_k,
\]
implying
\[
\Gamma(k) = \Gamma_1(k)[\Gamma_1(k)V\Gamma_1(k)]^{-1}\Gamma_1(k),
\]
where the inverse is to be taken on the space \( \mathcal{E}_k \). In the context of quasiconvexity, the condition (5.4) was first derived in [19], stimulated by a result of Kohn and Lipton [20].
The simpler derivation presented here was suggested by an anonymous referee of a later paper (see also section 24.9 in [17]).

Clearly (5.4) is just satisfied if
\[ \alpha = \sup_{k \in \mathbb{R}^d, k \neq 0} v \cdot \Gamma(k)v, \]  
(5.8)

and in this case \( f \) is marginally \( Q^* \)-convex. However if this supremum is attained for some \( k \neq 0 \) then \( f \) is sharply \( Q^* \)-convex.

As an example, consider fields of the form
\[ E = \left( \begin{array}{c} \nabla U \\ U \end{array} \right), \]
(5.9)

where \( U(x) \) is a scalar potential. Suppose \( V = I \) and

\[ v = \left( \begin{array}{c} t \\ 1 \end{array} \right), \]
(5.10)

where \( t \) is a real \( d \)-component unit vector. Then \( \Gamma(k) \) is the projection

\[ \Gamma(k) = \Gamma_1(k) = \frac{1}{k^2 + 1} \left( \begin{array}{c} k \\ 1 \end{array} \right) \left( k^T \right)^T \left( \begin{array}{c} k \\ 1 \end{array} \right), \]
(5.11)

in which \( k^2 = k \cdot k \) and so

\[ v \cdot \Gamma(k)v = \frac{(t \cdot k + 1)^2}{k^2 + 1} \]
(5.12)

takes its maximum value \( \alpha = 2 \) when \( k = t \). (It is clear that the maximum occurs when \( t \) and \( k \) are parallel and then it is just a matter of simple algebra to show (5.12) is less than or equal to 2.) This example shows that the supremum can be attained, and in some cases may be attained only at one non-zero value of \( k \).

Returning to the general case, let \( \mathcal{K} \) denote the set of those \( k \neq 0 \) which attain the supremum in (5.8). Then for \( k \in \mathcal{K} \), \( \alpha = v \cdot \Gamma(k)v \) and the set \( \mathcal{U}_k \) consists of those \( G \) of the form

\[ G = a(\hat{L}^\dagger(k)V\hat{L}(k))^{-1}\hat{L}^\dagger(k)v, \]
(5.13)

where \( a \) is an arbitrary complex constant. Thus

\[ H \equiv \hat{L}(k)G = a\hat{L}(k)(\hat{L}^\dagger(k)V\hat{L}(k))^{-1}\hat{L}^\dagger(k)v = a\Gamma(k)v \]
(5.14)

satisfies

\[ SH = VH - av = -a\Delta(k)V^{-1}v, \]
(5.15)

where

\[ \Delta(k) \equiv V - V\Gamma(k)V \]
(5.16)
has the property that
\[ \hat{L}^\dagger(k) \Delta(k) = \hat{L}^\dagger(k) V - \hat{L}^\dagger(k) V \hat{L}(k) (\hat{L}^\dagger(k) V \hat{L}(k))^{-1} \hat{L}^\dagger(k) V = 0, \] (5.17)
implying \( SH \in J_k \). So the potential \( U^1(x) \) and the \( Q^* \)-special fields \( E(x) \) and \( J(x) \) take the form
\[ U^1(x) = \sum_{k \neq 0} a(k) [(\hat{L}^\dagger(k) V \hat{L}(k))^{-1} \hat{L}^\dagger(k)] V e^{i k \cdot x}, \] \[ E(x) = E^0 + \sum_{k \neq 0} a(k) \Gamma(k) V e^{i k \cdot x}, \] \[ J(x) = SE(x) = SE^0 - \sum_{k \neq 0} a(k) \Delta(k) V^{-1} V e^{i k \cdot x}, \] (5.18)
where \( E^0 \in \mathcal{E}^0 \) and
\[ a(k) = 0 \quad \text{if} \quad k \notin \mathcal{K}. \] (5.19)
The reciprocal lattice needs to be chosen so that it includes some \( k \in \mathcal{K} \) to ensure the \( a(k) \) in (5.18) are not all zero.

When \( V = I \) then \( \Delta(k) \) is equal to the projection \( \Gamma_2(k) \) onto the null-space \( J_k \) of \( \hat{L}^\dagger(k) \). More generally \( \Delta(k) \) is defined to be the matrix such that for all vectors \( w \),
\[ \Delta(k) w \in J_k \quad \text{and} \quad \Delta(k) w - V^{-1} w \in \mathcal{E}_k, \] (5.20)
implying
\[ \Delta(k) = \Gamma_2(k) [\Gamma_2(k) V^{-1} \Gamma_2(k)]^{-1} \Gamma_2(k), \] (5.21)
where the inverse is to be taken on the space \( J_k \).

We now use the fact that for an appropriate choice of the scalar polynomial \( q(k) \),
\[ P(k) = q(k) (\hat{L}^\dagger(k) V \hat{L}(k))^{-1} \hat{L}^\dagger(k) V \] (5.22)
is also polynomial in \( k \): to see this it suffices to take
\[ q(k) = \det [\hat{L}^\dagger(k) V \hat{L}(k)], \] (5.23)
although it is better to take the lowest degree polynomial which works. Then setting
\[ a(k) = q(k) \hat{c}(k), \] (5.24)
where the \( \hat{c}(k) \) are the Fourier components of some scalar periodic function \( c(x) \), we see that
\[ U^1(x) = P(\partial/\partial x_1, \partial/\partial x_2, \ldots, \partial/\partial x_d) c(x), \] (5.25)
which also gives the \( Q^* \)-special fields \( E = E^0 + L U^1 \) and \( J = S E \). The function \( c(x) \) has to be chosen so that \( \hat{c}(k) \) is zero if \( k \notin \mathcal{K} \). (We do not have to worry about this latter constraint if it happens that \( \mathcal{K} \) consists of all non-zero vectors in \( \mathbb{R}^d \)).
We assumed $\mathbf{V}$ was positive definite but in fact it is easy to check that the derivation goes through if the quadratic form associated with $\mathbf{V}$ is $Q^*$-convex. It could also be marginally $Q^*$-convex but in this case one needs to choose $\mathbf{v}$ so that
\[
\hat{\mathbf{L}}^\dagger(k)\mathbf{v} \text{ lies in the range of } \hat{\mathbf{L}}^\dagger(k)\hat{\mathbf{L}}(k) \quad \forall k \neq 0,
\]
(5.26)
since otherwise the supremum in (5.8) is surely infinite. Thus by induction, successively setting
\[
\mathbf{V} = \mathbf{V}_0 - \sum_{i=1}^{j-1} \mathbf{v}_i\mathbf{v}_i^\dagger/\alpha_i, \quad \mathbf{v} = \mathbf{v}_j, \quad \alpha = \alpha_j,
\]
(5.27)
for $j = 1, 2, \ldots, n$ (where $\mathbf{V}_0$ has an associated $Q^*$-convex quadratic form, $\mathbf{v} = \mathbf{v}_j$ has to be chosen so that (5.26) is satisfied, and $\alpha = \alpha_j$ is given by (5.8)) one finally reaches a point such that the quadratic function $f$ associated with
\[
\mathbf{S} = \mathbf{V}_0 - \sum_{i=1}^{n} \mathbf{v}_i\mathbf{v}_i^\dagger/\alpha_i,
\]
(5.28)
is an extremal quadratic $Q^*$-convex function in the sense that one cannot subtract any non-zero positive semi-definite tensor from $\mathbf{S}$ and retain its $Q^*$-convexity. In this case, with $\mathbf{V} = \mathbf{S}$ the supremum in (5.8) is infinite for all non-zero $\mathbf{v} \in \mathbb{C}^m$. Incidentally, this solves the problem of generating extremal quadratic quasiconvex functions (or equivalently ”extremal translations”): see [19] and chapters 24 and 25 in [17] for other insights into this problem.

In the end one obtains $Q^*$-special potentials of the form
\[
\mathbf{U}^\dagger(\mathbf{x}) = \sum_{j=1}^{n} \mathbf{P}_j(\partial/\partial x_1, \partial/\partial x_2, \ldots, \partial/\partial x_d)c_j(\mathbf{x}).
\]
(5.29)
Here the $n$ scalar periodic potentials $c_j(\mathbf{x})$ have Fourier components satisfying
\[
\hat{c}_j(\mathbf{k}) = 0 \quad \text{if } \mathbf{k} \notin \mathcal{K}_j,
\]
(5.30)
and, when one has made the substitutions (5.27), the $\mathbf{P}_j$ are polynomials given by (5.22) and the $\mathcal{K}_j$ consist of those $\mathbf{k}$ which attain the supremum in (5.8).

Similarly, supposing $\mathbf{T}$ can be expressed in the form
\[
\mathbf{T} = \mathbf{V}^{-1} - \mathbf{w}\mathbf{w}^\dagger/\beta,
\]
(5.31)
where $\mathbf{V}$ is positive definite, we see that $g$ is $Q^*$-convex if and only if
\[
\beta \geq \mathbf{w} \cdot \Delta(\mathbf{k})\mathbf{w}, \quad \forall \mathbf{k} \in \mathbb{R}^d, \; \mathbf{k} \neq 0.
\]
(5.32)
If we choose
\[
\beta = \sup_{\mathbf{k} \in \mathbb{R}^d, \; \mathbf{k} \neq 0} \mathbf{w} \cdot \Delta(\mathbf{k})\mathbf{w}
\]
(5.33)
then $g$ will be marginally $Q^*$-convex and sharply $Q^*$-convex if this supremum is attained for some $\mathbf{k} \neq 0$. Let $\mathcal{M}$ denote the set of those $\mathbf{k} \neq 0$ which attain the supremum. Then for $\mathbf{k} \in \mathcal{M}$, $\beta = \mathbf{w} \cdot \Delta(\mathbf{k})\mathbf{w}$ and the set $\mathcal{J}_k$ consists of those $\mathbf{H}$ of the form

$$H = b\Delta(\mathbf{k})\mathbf{w}, \quad (5.34)$$

where $b$ is an arbitrary complex constant. In this case we have

$$TH = b\mathbf{V}^{-1}\Delta(\mathbf{k})\mathbf{w} - b\mathbf{w} = -b\Gamma(\mathbf{k})\mathbf{V}\mathbf{w} = \hat{L}^\dagger(\mathbf{k})\mathbf{G}, \quad (5.35)$$

where

$$\mathbf{G} = -b(\hat{L}^\dagger(\mathbf{k})\mathbf{V}\hat{L}(\mathbf{k}))^{-1}\hat{L}^\dagger(\mathbf{k})\mathbf{V}\mathbf{w}. \quad (5.36)$$

So the potential $\mathbf{U}^1(\mathbf{x})$ and the $Q^*$-special fields $\mathbf{E}(\mathbf{x})$ and $\mathbf{J}(\mathbf{x})$ take the form

$$\mathbf{U}^1(\mathbf{x}) = \sum_{\mathbf{k} \neq 0} b(\mathbf{k})[(\hat{L}^\dagger(\mathbf{k})\mathbf{V}\hat{L}(\mathbf{k}))^{-1}\hat{L}^\dagger(\mathbf{k})]\mathbf{V}\mathbf{w} \mathbf{e}^{i\mathbf{k} \cdot \mathbf{x}},$$

$$\mathbf{E}(\mathbf{x}) = \mathbf{TJ}(\mathbf{x}) = \mathbf{TJ}^0 - \sum_{\mathbf{k} \neq 0} b(\mathbf{k})\Gamma(\mathbf{k})\mathbf{V}\mathbf{w} \mathbf{e}^{i\mathbf{k} \cdot \mathbf{x}},$$

$$\mathbf{J}(\mathbf{x}) = \mathbf{J}^0 + \sum_{\mathbf{k} \neq 0} b(\mathbf{k})\Delta(\mathbf{k})\mathbf{w} \mathbf{e}^{i\mathbf{k} \cdot \mathbf{x}}, \quad (5.37)$$

where $\mathbf{TJ}^0 \in \mathcal{E}^0$ and

$$b(\mathbf{k}) = 0 \quad \text{if } \mathbf{k} \notin \mathcal{M}. \quad (5.38)$$

We next look for the polynomial $q(\mathbf{k})$ of lowest degree such that

$$\mathbf{P}(\mathbf{k}) = -q(\mathbf{k})(\hat{L}^\dagger(\mathbf{k})\mathbf{V}\hat{L}(\mathbf{k}))^{-1}\hat{L}^\dagger(\mathbf{k})\mathbf{V}\mathbf{w} \quad (5.39)$$

is also polynomial in $\mathbf{k}$. Set

$$b(\mathbf{k}) = q(\mathbf{k})\hat{c}(\mathbf{k}), \quad (5.40)$$

where the $\hat{c}(\mathbf{k})$ are the Fourier components of some scalar periodic function $c(\mathbf{x})$ such that now $\hat{c}(\mathbf{k})$ is zero if $\mathbf{k} \notin \mathcal{M}$. Then we see that

$$\mathbf{U}^1(\mathbf{x}) = \mathbf{P}(\partial/\partial x_1, \partial/\partial x_2, \ldots, \partial/\partial x_d)c(\mathbf{x}), \quad (5.41)$$

which also gives the $Q^*$-special fields $\mathbf{E} = \mathbf{TJ}^0 + \mathbf{LU}^1$ and $\mathbf{J} = \mathbf{T}^{-1}\mathbf{E}$ (in the case where $\mathbf{T}$ is non-singular).

The derivation still goes through if the quadratic form associated with $\mathbf{V}^{-1}$ is $Q^*$-convex or marginally $Q^*$-convex. In the latter case one needs to choose $\mathbf{w}$ so that

$$\Gamma_2(\mathbf{k})\mathbf{w} \quad \text{lies in the range of } \Gamma_2(\mathbf{k})\mathbf{V}^{-1}\Gamma_2(\mathbf{k}) \quad \forall \mathbf{k} \neq 0, \quad (5.42)$$
since otherwise, from (5.21), the supremum in (5.33) is surely infinite. Thus by induction, successively setting

\[ V^{-1} = W_0 - \sum_{i=1}^{j-1} w_i w_i^\dagger / \beta_i, \quad w = w_j, \quad \beta = \beta_j, \] (5.43)

for \( j = 1, 2, \ldots, n \) (where \( W_0 \) has an associated \( Q^* \)-convex quadratic form, \( w = w_j \) has to be chosen so that (5.26) is satisfied, and \( \beta = \beta_j \) is given by (5.33)) one finally reaches a point such that the quadratic function \( g \) associated with

\[ T = W_0 - \sum_{i=1}^{n} w_i w_i^\dagger / \alpha_i \] (5.44)

is an extremal quadratic \( Q^* \)-convex function in the sense that one cannot substract any non-zero positive semi-definite tensor from \( T \) and retain its \( Q^* \)-convexity. In this case, with \( V^{-1} = T \) the supremum in (5.33) is infinite for all non-zero \( w \in \mathbb{C}^m \).

6 Sharp inequalities on the integrals over \( \Omega \)

Since the operators \( L \) and \( L^\dagger \) are formal adjoints the quantities

\[ f_0 = \int_\Omega f(E(x)) \, dx = \int_\Omega E(x) \cdot SE(x) = \int_\Omega E(x) \cdot J(x), \]

\[ g_0 = \int_\Omega g(J(x)) \, dx = \int_\Omega J(x) \cdot SJ(x) = \int_\Omega J(x) \cdot E(x) \] (6.1)

can be computed in terms of boundary terms using integration by parts. Now we show that

\[ \int_\Omega f(E(x)) \, dx \geq f_0, \quad \int_\Omega g(J(x)) \, dx \geq g_0, \] (6.2)

for all fields \( E(x) \) deriving from a potential \( U(x) \) that matches the appropriate boundary data of the potential \( \underline{U}(x) \) or for all fields \( J(x) \) that match the appropriate boundary data of \( \underline{J}(x) \), provided certain further ancillary conditions hold. First note that these inequalities are clearly sharp, being attained when \( E(x) = \underline{E}(x) \) or when \( J(x) = \underline{J}(x) \).

To establish the first inequality in (6.2) we first find a parallelopiped \( C \) that contains \( \Omega \) and that is formed from an integer number of the primitive unit cells of \( E(x) \). (If \( \Omega \) lies inside a primitive unit cell of \( E(x) \), then we can take \( C \) as this primitive cell, but otherwise we need to join a set of these primitive cells together to obtain a parallelopiped that covers \( \Omega \).) We extend \( E(x) \) outside \( \Omega \) so that it is periodic with \( C \) as a unit cell. In this cell, but outside \( \Omega \), it \( E(x) \) equals \( \underline{E}(x) \). Since the potential \( U(x) \) matches the boundary data of the potential \( \underline{U}(x) \) at the boundary \( \partial \Omega \), the appropriate jump conditions are satisfied across this interface. We extend the potential \( U(x) \) outside \( C \) so
that $U^1(x) \equiv U(x) - U^0(x)$ is $C$-periodic: if $x_0$ is at the center of another unit cell then for $x \in C$ we set

$$U(x + x_0) = U(x) + U^0(x + x_0) - U^0(x), \quad (6.3)$$

which in particular gives the potential inside the copies of $\Omega$ outside $C$. Defined in this way the relation $E = LU$ holds in a weak sense, and so the inequality (1.12) is satisfied, which we rewrite as

$$\int_{\Omega} f(E) \, dx + \int_{C \setminus \Omega} f(E) \, dx \geq |C| f(\langle E \rangle), \quad (6.4)$$

where $|C|$ is the volume of $C$. Also (4.8) holds, which we rewrite as

$$f_0 + \int_{C \setminus \Omega} f(E) \, dx = |C| f(\langle E \rangle), \quad (6.5)$$

So subtracting these equations gives

$$\int_{\Omega} f(E) \, dx \geq f_0 + |C| [f(\langle E \rangle) - f(\langle E \rangle)]. \quad (6.6)$$

This inequality in general requires us to know $\langle E \rangle$ and $\langle E \rangle$. Given a $m$-dimensional constant vector $J^0$ we have

$$\int_{C} J^0 \cdot (E - \langle E \rangle) \, dx = \int_{C} J^0 \cdot L(U - \langle U \rangle) \, dx = \int_{C} J^0 \cdot A(U - \langle U \rangle) \, dx, \quad (6.7)$$

where $A$ is the matrix with elements $A_{rq}$. (In establishing the last equality in (6.7) we have used integration by parts and the fact that $U$ and $\langle U \rangle$ are equal in the vicinity of the boundary of $C$). Since this holds for all $J^0$ we deduce that

$$\langle E \rangle = \langle E \rangle + \frac{1}{|C|} \int_{C} A(U - \langle U \rangle) \, dx. \quad (6.8)$$

So a sufficient condition for $f(\langle E \rangle)$ to be equal to $f(\langle E \rangle)$ is that

$$SA = 0, \quad (6.9)$$

so that the range of $A$ is in the null space of $S$ (which if $S$ is non-singular requires that $A = 0$). When this ancillary condition holds then clearly (6.6) implies the first inequality in (6.2).

To establish the second inequality in (6.2) we first find a parallelepiped $C$ that contains $\Omega$ and that is formed from an integer number of the primitive unit cells of $J(x)$. Then we extend $J(x)$ outside $\Omega$ so that it is $C$-periodic, and within the unit cell $C$ equals $J(x)$ outside $\Omega$. Since the field $J(x)$ matches the appropriate boundary data of the field $\langle J(x) \rangle$ at the boundary $\partial \Omega$, the appropriate jump conditions are satisfied across this interface.
Defined in this way the relation \( \mathbf{L}^\dagger \mathbf{J} = 0 \) holds in a weak sense, and so the inequality (1.18) is satisfied, which we rewrite as

\[
\int_{\Omega} g(\mathbf{J}) \, d\mathbf{x} + \int_{C \setminus \Omega} g(\mathbf{J}) \, d\mathbf{x} \geq |C| g(\langle \mathbf{J} \rangle). \tag{6.10}
\]

Also (4.10) holds, which we rewrite as

\[
g_0 + \int_{C \setminus \Omega} g(\mathbf{J}) \, d\mathbf{x} = |C| g(\langle \mathbf{J} \rangle), \tag{6.11}
\]

and subtracting these equations gives

\[
\int_{\Omega} g(\mathbf{J}) \, d\mathbf{x} \geq g_0 + |C| |g(\langle \mathbf{J} \rangle) - g(\langle \mathbf{J} \rangle)|. \tag{6.12}
\]

This inequality in general requires us to know \( \langle \mathbf{J} \rangle \) and \( \langle \mathbf{E} \rangle \). Given a \( m \) dimensional constant vector \( \mathbf{E}^0 \in \mathcal{E}^0 \) so that \( \mathbf{E}^0 = \mathbf{L}U^0(\mathbf{x}) \) for some polynomial potential \( U^0(\mathbf{x}) \) we have

\[
\int_C (\mathbf{J} - \mathbf{J}) \cdot \mathbf{E}^0 \, d\mathbf{x} = \int_C (\mathbf{J} - \mathbf{J}) \cdot \mathbf{L}U^0 \, d\mathbf{x} = 0, \tag{6.13}
\]

where the last equality follows from integration by parts, using the fact that \( \mathbf{J} = \mathbf{J} \) in a vicinity of the boundary of \( C \). Since this holds for all \( \mathbf{E}^0 \in \mathcal{E}^0 \) we deduce that a sufficient condition for \( g(\langle \mathbf{E} \rangle) \) to be equal to \( g(\langle \mathbf{E} \rangle) \) is that the range of \( \mathbf{T} \) be a subset of \( \mathcal{E}^0 \) and if this ancillary condition holds then clearly (6.12) implies the second inequality in (6.2).

7 A Generalization

Following the ideas of [21] (see also sections 24.1 and 30.4 of [17]) the arguments present here extend directly to integrals of the form

\[
\int_{\Omega} h(\widetilde{\mathbf{E}}(\mathbf{x}), \mathbf{J}(\mathbf{x})) \, d\mathbf{x}, \tag{7.1}
\]

where \( h \) is quadratic and the \( m \) component field \( \mathbf{J}(\mathbf{x}) \) still satisfies \( \mathbf{L}^\dagger \mathbf{J} = 0 \) while the \( n \)-component field \( \widetilde{\mathbf{E}}(\mathbf{x}) \) derives from a \( p \) component potential \( \widetilde{\mathbf{U}}(\mathbf{x}) \) according to the relations

\[
\widetilde{E}_r(\mathbf{x}) = \sum_{q=1}^{p} \tilde{L}_{rq} \widetilde{U}_q(\mathbf{x}), \tag{7.2}
\]

for \( r = 1, 2, \ldots, n \), where \( \tilde{L}_{rq} \) is the differential operator

\[
\tilde{L}_{rq} = \tilde{A}_{rq} + \sum_{h=1}^{v} \sum_{a_1, \ldots, a_h=1}^{d} \tilde{A}_{a_1 \ldots a_h} \frac{\partial}{\partial x_{a_1}} \frac{\partial}{\partial x_{a_2}} \ldots \frac{\partial}{\partial x_{a_h}} \tag{7.3}
\]
of order \( v \) in a space of dimension \( d \) with real or complex valued constant coefficients \( \tilde{A}_{rq} \) and \( \tilde{A}_{rqh}^{a_1 \ldots a_h} \).

The function \( h(\tilde{E}, J) \) can be expressed in the form

\[
h(\tilde{E}, J) = \begin{pmatrix} \tilde{E} \\ J \end{pmatrix} \cdot M \begin{pmatrix} \tilde{E} \\ J \end{pmatrix},
\]

(7.4)

where the matrix \( M \) is Hermitian. It is chosen to be sharply \( Q^* \)-convex so that for periodic fields \( \tilde{E}(x) \) and \( J(x) \) satisfying the differential constraints one has

\[
\langle h(\tilde{E}, J) \rangle \geq h(\langle \tilde{E} \rangle, \langle J \rangle),
\]

(7.5)

with equality for some non-constant \( Q^* \)-special fields \( \tilde{E} = \tilde{E} \) and \( J = J \).

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