Lepton Mixing Patterns from $PSL_2(7)$ with a Generalized CP Symmetry

Shu-Jun Rong

College of Science, Guilin University of Technology, Guilin, Guangxi 541004, China

Correspondence should be addressed to Shu-Jun Rong; rongshj@glut.edu.cn

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1. Introduction

CP violation in the hadron sector was observed in 1964 [1]. Whether there is a counterpart in the lepton sector is still a mystery. Recent neutrino oscillation experiments show that the 1-3 mixing angle of leptons is nonzero [2–5]. It intrigues experiments to detect the Dirac CP-violating phase. Particularly, some fit results [6, 7] hint that this phase is around $-\pi/2$. In the theoretical respect, how to predict nontrivial lepton CP phases is interesting. In order to obtain mixing parameters of leptons, discrete flavor symmetries are widely used [8–41]. However, if no perturbation is considered, only finite groups of large orders could accommodate the results of new experiments [35]. Furthermore, they give a trivial Dirac CP-violating phase [35]. In order to improve predictions of flavor groups, some efforts have been made in generalizations of symmetries [42, 43]. Particularly, an intriguing method called generalized CP (GCP) symmetry was introduced [44–66]. In this scenario, the leptonic lagrangian satisfies both flavor and GCP symmetries. After spontaneous symmetries breaking, the residual flavor and GCP symmetries constrain the structures of mass matrices of leptons. Then, information on leptonic mixing angles and CP phases is obtained. From groups $S_4$ and $A_5$ with GCP symmetries, a trivial or maximal Dirac CP phase is obtained [46, 56]. The maximal Dirac phase satisfies the 1$\sigma$ constraint from the new recent global fit data in case of inverted mass ordering [67]. However, it is not in the 1$\sigma$ range for the normal mass ordering. $S_4$ and $A_5$ are small modular groups. We want to know whether a large one could give a more fit CP phase.

In this paper, we study the predictions of the modular group $PSL_2(7)$ with the GCP symmetry in the case of Majorana neutrinos. We suppose that residual symmetries in both charged lepton and neutrino sectors are $Z_2 \times CP$. Seven types of mixing patterns at the 3$\sigma$ level of the new global fit data are obtained. Among these patterns, three types of patterns can give the Dirac CP phase which is in the 1$\sigma$ range of the global fit data. The effective mass of neutrinoless double-beta decay for these patterns is also examined.
from examination of the residual symmetries are presented. Finally, a summary is made.

2. Framework

In this section, we describe the basic facts of the group $PSL_2(7)$ and introduce the method of deriving lepton mixing patterns from the residual flavor and GCP symmetries.

2.1. Group Theory of $PSL_2(7)$

2.1.1. Generic Facts. The group $PSL_2(7)$ is also named $\Sigma$ (168). It could be constructed with two generators which satisfy following relations [68]:

\[
S^2 = T^7 = E,
\]

\[
(ST)^3 = (ST^{-1}ST)^4 = E,
\]

where $E$ is the identity element. This group has 6 conjugacy classes listed as follows [68]:

\[
\begin{align*}
1C_1 &: E, \\
21C_2 &: S, \\
56C_3 &: ST, \\
42C_4 &: ST^3, \\
24C_5^1 &: T, \\
24C_5^2 &: T^3,
\end{align*}
\]

where $iC_j$ denotes that the class contains $i$ elements of order $j$. Accordingly, there are 6 irreducible representations, namely,

\[
1, 3, 3^*, 6, 7, 8.
\]

Without loss of generality, we consider the 3-dimensional representation $3$ in the following sections. Accordingly, the generators could be expressed as [68]

\[
S = \frac{2}{\sqrt{7}} \begin{pmatrix} s_1 & s_2 & s_3 \\ s_2 & -s_3 & s_1 \\ -s_3 & s_1 & -s_2 \end{pmatrix},
\]

\[
T = \begin{pmatrix} \varphi^2_7 & 0 & 0 \\ 0 & \varphi_7 & 0 \\ 0 & 0 & \varphi^3_7 \end{pmatrix},
\]

where $s_k = \sin (k\pi/7)$, $\varphi_k = e^{2\pi i k/7}$.

Resorting to the conjugacy classes, we can obtain abelian subgroups of $PSL_2(7)$. These groups are candidates of the residual symmetries for leptons. In this paper, we consider the residual symmetry $Z_2 \times CP$ for leptons. So $Z_2$ subgroups are relevant. There are 21 $Z_2$ subgroups which are identified with the generators of them [68], i.e.,

\[
\begin{align*}
A_1 &: S, \\
A_2 &: T^2ST^3ST, \\
A_3 &: TST^3ST^2, \\
A_4 &: T^4ST^3, \\
A_5 &: T^3ST^4, \\
A_6 &: T^2ST^4ST^2, \\
A_7 &: ST^2ST^4ST^2S, \\
A_8 &: ST^4ST^3S, \\
A_9 &: ST^3ST^4S, \\
A_{10} &: T^6ST^6, \\
A_{11} &: T^2ST^5, \\
A_{12} &: T^6ST, \\
A_{13} &: TST^6, \\
A_{14} &: ST^4ST^4, \\
A_{15} &: ST^3ST^3, \\
A_{16} &: ST^2ST, \\
A_{17} &: ST^5ST^6, \\
A_{18} &: (T^2ST^3S)^2, \\
A_{19} &: (T^5ST^4S)^2, \\
A_{20} &: (ST^3ST^4)^2, \\
A_{21} &: (ST^4ST^3)^2.
\end{align*}
\]

2.1.2. Automorphism of $PSL_2(7)$. An automorphism of a group is a transformation which permute elements of the group. These transformations form a group, namely, the automorphism group. For the group $PSL_2(7)$, the structure of the automorphism group is simple. It is listed as follows:

\[
\begin{align*}
Z(PSL_2(7)) &= Z_4, \\
\text{Aut}(PSL_2(7)) &= PSL(2, Z_7) \times Z_2, \\
\text{Inn}(PSL_2(7)) &= PSL_2(7), \\
\text{Out}(PSL_2(7)) &= Z_2 = \{id, u\},
\end{align*}
\]

where $Z$, Aut, Inn, and Out denote the centre, the automorphism group, the inner automorphism, and the outer automorphism group, respectively. In detail, the inner automorphism group is composed of permutations of elements in the same conjugacy class. The outer automorphism group swaps conjugacy classes and representations. So it reflects the symmetries of the character table shown.
Table 1: Character table of the group PSL(2) [68].

| Rep. | 1C1 | 2C2 | 56C3 | 42C4 | 24C4 | 24C7 |
|------|-----|-----|------|------|------|------|
| 1    | 1   | 1   | 1    | 1    | 1    | 1    |
| 3    | 3   | -1  | 0    | 1    | φ2  | φ4  |
| 3*   | 3   | -1  | 0    | 1    | φ4  | φ2  |
| 6    | 6   | 2   | 0    | 0    | -1  | -1  |
| 7    | 7   | -1  | 1    | -1   | 0    | 0    |
| 8    | 8   | 0   | -1   | 0    | 1    | 1    |

in Table 1. The unique nontrivial outer automorphism of the group PSL(2) is

\[ u : 24C17 \rightarrow 24C27, \]

\[ 3 \rightarrow 3^*. \]

The representation of \( u \) could be obtained from its action on the generators \( S, T \), i.e.,

\[ u : S \rightarrow S, \]

\[ T \rightarrow T^* = T^6. \]

In the 3-dimensional representation, the specific equations of the transformation read

\[ X(u)S^*X^{-1}(u) = S^{-1} = S, \]

\[ X(u)T^*X^{-1}(u) = T^{-1} = T^*. \]

The solution is

\[ X(u) = e^{i\alpha} \text{diag}(1, 1, 1). \]

Since the global phase is trivial for the lepton mixing patterns, we choose \( e^{i\alpha} = 1 \) in the following sections. A general automorphism is the product of the inner and the outer one. It could be expressed as

\[ X(g_i) = \rho_3(g_i)X(u) = \rho_3(g_i), \text{ with } g_i \in PSL_2(7), \]

where \( \rho_3(g_i) \) is the 3-dimensional representation of the group element.

2.2. Approach

2.2.1. GCP Compatible with PSL(2) (7). The GCP transformation acts on the flavor space as

\[ \Phi \rightarrow X\Phi^C, \]

where \( \Phi \) is a multiplet of fields, \( X \) is a unitary matrix, and \( \Phi^C \) is the CP conjugation of \( \Phi \). In contrast, the flavor group acts on the fields as

\[ \Phi \rightarrow \rho(g)\Phi, \text{ with } g_i \in PSL_2(7). \]

Accordingly, the consistence condition of GCP is [45]

\[ (X^{-1}\rho(g)X)^* = \rho\left(g^\dagger\right). \]

Therefore, \( X \) is an automorphism of the flavor group PSL(2) (7). These GCP transformations form an automorphism group CP. The general theory satisfies the symmetry PSL(2) (7) × CP. After fermions obtain masses from the vacuum expectation values of scalar fields, the original symmetry PSL(2) (7) × CP is broken to Gc × CP, in the charged lepton sector and Gv × CP, in the neutrino sector. Thus, the mass matrices of charged leptons and Majorana neutrinos satisfy the relations

\[ \rho^*(g_e)m_em_e^*\rho(g_e) = m_em_e^*, \text{ with } g_e \in G_e, \]

\[ \rho^*(g_v)m_vm_v^*\rho(g_v) = m_vm_v^*, \text{ with } g_v \in G_v. \]

The CP transformation \( X \) follows the relations

\[ X_e^*m_em_e^*X_e = (m_em_e^*)^*, \]

\[ (X_e^*-1\rho(g_e)X_e)^* = \rho\left(g_e^\dagger\right). \]

\[ X_v^*m_vm_v^*X_v = m_vm_v^*, \]

\[ (X_v^*-1\rho(g_v)X_v)^* = \rho\left(g_v^\dagger\right). \]

Since masses of leptons are nondegenerate, the CP transformation \( X \) should be a symmetric unitary matrix [56], i.e.,

\[ X_a = X_a^T. \]

\[ X_aX_a^* = E, \text{ with } a = e, v(\text{no sum}). \]

So \( X_e \) can be decomposed as \( X_e = Q_eQ_e^T \). This kind of CP transformations is called Bickerstaff-Damhus automorphism (BDA) [70, 71]. For the group PSL(2) (7), all BDAs in the 3-dimensional representation are listed as follows:

\[ T^3ST^3, TST^4ST, TST^3ST^3S, STST^4STS, STST^5STS, \]

\[ E, T^4, ST^4S, \text{ with } i = 1, 2, 3, 4, 5, 6, \]

\[ (T^2ST^2)^j, (ST^2ST^2S)^j, (T^2ST^5ST^2)^j, \text{ with } j = 1, 2, 3. \]

2.2.2. Mixing Patterns from Residual Symmetries \( Z_2 \times CP \). Once the residual symmetries are fixed, the lepton mixing pattern could be obtained up to permutations of rows and columns. In the direct method, the mixing matrix is completely determined by the symmetries. In the semidirect method, only several elements of the matrix are certain because of degeneracy of the eigenvalues of the residual symmetries. We concern on the semidirect method in this paper. The residual symmetry is \( Z_2 \times CP \), \( Z_2 \times CP \), \( Z_2 \times CP \), in charged lepton and neutrino sectors, respectively. The consistence equation is written as

\[ X\rho^*(g_{cv})X^* = \rho(g_{cv}), \text{ with } g_{cv} \in Z_2. \]
Accordingly, the lepton mixing matrix \( U_{\text{PMNS}} \equiv U^\dagger \nu U_e \) is obtained from the matrix [46]

\[
U_{ce} = \Omega_{ce} U(\theta_{ce}) P_{ce},
\]

where \( R_{ce} \) is a rotation matrix with an angle parameter \( \theta_{ce} \), and \( P_{ce} \) is a phase matrix, i.e.,

\[
P_{ce} = \text{diag} \left( 1, i^j, i^k \right), \text{with } j, k = 0, 1, 2, 3.
\]

Because \( P_e \) gives nonphysical phases, it is omitted in the following sections.

2.2.3. Similar Transformations. In order to obtain viable mixing patterns, all possible combinations of residual symmetries should be examined. However, if two combinations are connected by a similarity transformation, namely,

\[
\rho \left( \Omega_{ce} \right) = V \rho \left( \Omega_{ce} \right) V^+, \quad X'_{ce} = VX_{ce}V^T,
\]

they would correspond to the same mixing matrix. Therefore, we could just examine nonequivalent combinations. In the following sections, \( Z_{2e} \) is fixed on the subgroup \( Z_2^5 \) which is generated by the group element \( S \). The consistent GCP transformations for \( Z_2^5 \) are listed as follows:

\[
\begin{align*}
X_1 &= E, \\
X_2 &= S, \\
X_3 &= T^2 ST^5 ST^2, \\
X_4 &= \left( T^2 ST^5 ST^5 \right)^*,
\end{align*}
\]

where \( X_1 \) and \( X_2 \) correspond to the equivalent mixing patterns, so do \( X_3 \) and \( X_4 \). \( Z_{2e} \) and \( X_e \) can be obtained from the similar transformations. In detail, for generators of \( Z_2^5 \) subgroups, we have \( \rho \left( A_i \right) = V_i S V_i^+ \) with \( V_i \) listed as follows:

\[
\begin{align*}
V_5 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \phi_7^3 & 0 \\ 0 & 0 & -\phi_7^* \end{pmatrix}, \\
V_6 &= \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \\
V_7 &= \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}, \\
V_8 &= \begin{pmatrix} 0 & 0 & 1 \\ \phi_7 & 0 & 0 \\ 0 & -\phi_7^2 & 0 \end{pmatrix}, \\
V_9 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & \phi_7 & 0 \\ 0 & 0 & -\phi_7^*^2 & 0 \end{pmatrix}, \\
V_{10} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \phi_7^2 & 0 \\ 0 & 0 & \phi_7^3 \end{pmatrix}, \\
V_{11} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \phi_7^*^2 & 0 \\ 0 & 0 & \phi_7^*^3 \end{pmatrix}, \\
V_{12} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \phi_7 & 0 \\ 0 & 0 & \phi_7^*^2 \end{pmatrix}, \\
V_{13} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \phi_7^* & 0 \\ 0 & 0 & \phi_7^* \end{pmatrix}, \\
V_{14} &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -\phi_7^3 \\ \phi_7 & 0 & 0 \end{pmatrix}, \\
V_{15} &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -\phi_7^3 \\ \phi_7 & 0 & 0 \end{pmatrix}, \\
V_{16} &= \begin{pmatrix} 0 & 0 & 1 \\ \phi_7^2 & 0 & 0 \\ 0 & \phi_7^* & 0 \end{pmatrix}.
\end{align*}
\]
\[ V_{17} = \begin{pmatrix} 0 & 0 & 1 \\ \phi_7^2 & 0 & 0 \\ 0 & -\phi_7^3 & 0 \end{pmatrix}, \]
\[ V_{18} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -\phi_7^* \\ \phi_7^2 & 0 & 0 \end{pmatrix}, \]
\[ V_{19} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -\phi_7^* \\ \phi_7^2 & 0 & 0 \end{pmatrix}, \]
\[ V_{20} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -\phi_7^2 \\ \phi_7^3 & 0 & 0 \end{pmatrix}, \]
\[ V_{21} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -\phi_7^2 \\ \phi_7^3 & 0 & 0 \end{pmatrix}. \]  

Particularly, \( V_1 \) is the identity matrix. So a general combination of the residual symmetries is of the form
\[ (Z_{2e}^{Ai}, X_{j}(A_i), Z_{2\nu}^{S}, X_{\nu k}(S)), \]  
(25)

with \( j, k = 1, 3 \). The corresponding lepton mixing matrix is written as
\[ U_{\text{PMNS}} = O^T(\theta_e) \Omega_1^T \Omega_1 \Omega_2 O(\theta_e) P_e, \]  
(26)

where
\[ O(\theta_{e\nu}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta_{e\nu} & \sin \theta_{e\nu} \\ 0 & -\sin \theta_{e\nu} & \cos \theta_{e\nu} \end{pmatrix}, \]
\[ \Omega_1 = \begin{pmatrix} 1 & 0 & 0 \\ -\sin \theta_1 & -\cos \theta_1 \sin \theta_2 & \cos \theta_1 \\ \cos \theta_1 & -\sin \theta_1 \sin \theta_2 & \sin \theta_1 \end{pmatrix}, \]
\[ \Omega_2 = \Omega_1 \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\pi/4} \cos \phi & e^{i3\pi/4} \sin \phi \\ 0 & e^{i\pi/4} \sin \phi & e^{i3\pi/4} \cos \phi \end{pmatrix}, \]  
(27)

with
\[ r_1 = 2 \sqrt{\frac{2}{7}} \sin \frac{2\pi}{7}, \]
\[ r_2 = 2 \sqrt{\frac{2}{7}} \sin \frac{\pi}{7}, \]
\[ r_3 = 2 \sqrt{\frac{2}{7}} \sin \frac{3\pi}{7}, \]  
(28)

\[ \theta_1 = \arcsin \frac{r_1}{\sqrt{r_1^2 + 1}}, \]
\[ \theta_2 = \arcsin \frac{r_2 \cos \theta_1}{\sqrt{r_1^2 + r_2^2 \cos^2 \theta_1}}, \]
\[ \phi = \arcsin \frac{1}{\sqrt{1 + x_1^2}}, \]  
(29)

where \( x_1 \) is a real root of the equation
\[ x_1^2 - 48x_1^6 + 323x_6^8 - 608x_6^4 + 323x_4^4 - 48x_4^2 + 1 = 0, \]
\[ x_1 = \pm 0.449807. \]

3. Results

3.1. Viable Mixing Matrices from Combinations of Residual Symmetries. Given the recent global fit data of neutrino oscillations [67], we examine the predictions of combinations of residual symmetries of the form \((Z_{2e}^{Ai}, X_{j}(A_i), Z_{2\nu}^{S}, X_{\nu k}(S))\) with the \( \chi^2 \) function defined as
\[ \chi^2 = \sum_{ij=13,23,12} \left( \frac{\sin^2 \theta_{ij} - (\sin^2 \theta_{ij})^{\text{ex}}}{\sigma_{ij}} \right)^2, \]  
(30)

where \((\sin^2 \theta_{ij})^{\text{ex}}\) is the best fit data from Ref. [67] and \(\sigma_{ij}\) is the 1\(\sigma\) error. The viable combinations at the 3\(\sigma\) level (up to equivalent ones) are listed as follows.

Type Ia:
\[ \left(Z_{2e}^{STST'}, X_{1e} = V_{16} \left(T^2 S T^5 S T^2 \right) V_{16}^T, Z_{2\nu}^{S}, X_{1\nu} = E \right), \]  
(31)

Type Ib:
\[ \left(Z_{2e}^{STST'}, X_{1e} = V_{16} V_{16}^T, Z_{2\nu}^{S}, X_{3\nu} = T^2 S T^5 S T^2 \right), \]  
(32)

Type Ib*:
\[ \left(Z_{2e}^{STST'}, X_{1e} = V_{17} V_{17}^T, Z_{2\nu}^{S}, X_{3\nu} = T^2 S T^5 S T^2 \right), \]  
(33)

Type Ia:
\[ \left(Z_{2e}^{STST'}, X_{1e} = V_{14} V_{14}^T, Z_{2\nu}^{S}, X_{1\nu} = E \right), \]  
(34)
where the matrix which experiencing permutations of rows and columns. For every combination, the predictions of the matrices in a pair are identical except the signs of the CP phases. So we can consider $U_{Ia}$, $U_{Ib}$, $U_{Ia}^*$, and $U_{Ib}^*$ as representatives.

3.2. Mixing Angles and CP Invariants. Employing the parametrization of the form

$$ U_{PMNS} = \begin{pmatrix}
    c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{i\delta} \\
    -s_{12}c_{23} - c_{12}s_{23}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}e^{i\delta} & c_{13}e^{i\delta} \\
    s_{12}s_{23} - c_{12}s_{23}e^{i\delta} & -c_{12}c_{23} - s_{12}s_{23}e^{i\delta} & c_{13}e^{i\delta}
\end{pmatrix} \cdot \begin{pmatrix}
    1 & 0 & 0 \\
    0 & e^{i\alpha/2} & 0 \\
    0 & 0 & e^{i(\beta/2 + \delta)}
\end{pmatrix}, $$

(40)

where $s_{ij} = \sin \theta_{ij}$, $c_{ij} = \cos \theta_{ij}$, $\delta$ is the Dirac CP-violating phase, and $\alpha$ and $\beta$ are Majorana phases; we could obtain lepton mixing angles and the CP invariants $J_{\text{cp}}$ [72], $J_1$, and $J_2$ defined as

$$ J_{\text{cp}} = \text{Im} \left[ U_{Ia}^* U_{Ia}^T U_{Ia}^T U_{Ia} \right] = \frac{1}{8} \sin 2\theta_{13} \sin 2\theta_{23} \sin 2\theta_{12} \cos \theta_{13} \sin \delta, $$

$$ J_1 = \text{Im} \left[ (U_{Ia}^*)^2 U_{Ia}^T U_{Ia} \right] = \sin^2 \theta_{12} \cos^2 \theta_{13} \cos^2 \theta_{13} \sin \alpha, $$

$$ J_2 = \text{Im} \left[ (U_{Ia}^*)^2 U_{Ia} \right] = \sin^2 \theta_{13} \cos \theta_{13} \cos \theta_{12} \sin \beta. $$

(41)

Their specific forms are listed as follows.

$U_{Ia}^*$:

$$ U_{Ia}^* = S_{12} = \begin{pmatrix}
    0 & 1 & 0 \\
    1 & 0 & 0 \\
    0 & 0 & 1
\end{pmatrix}, $$

(39)
$U_{\text{Fla}}$:

$$\sin^2 \theta_{13}(\theta_e, \theta_e) = 0.2438 \cos^2 \theta_e \cos^2 \theta_e + 0.1871 \cos \theta_e \sin \theta_e \sin \theta_e \\
+ 0.01618 \cos^2 \theta_e \sin^2 \theta_e + 0.264 \sin \theta_e \cos^2 \theta_e \\
+ 0.1659 \sin \theta_e \sin \theta_e - 0.16566 \sin \theta_e \sin \theta_e \\
+ 0.712 \sin^2 \theta_e \cos^2 \theta_e - 0.187 \sin^2 \theta_e \sin \theta_e \\
+ 0.13245 \sin^2 \theta_e \sin^2 \theta_e,$$

$$\sin^2 \theta_{23}(\theta_e, \theta_e) \approx \frac{0.375 - 0.3307 \cos 2 \theta_e}{1 - \sin^2 \theta_{13}(\theta_e, \theta_e)},$$

$$\sin^2 \theta_{12}(\theta_e, \theta_e) \approx \frac{0.375 + 0.2194 \cos 2 \theta_e - 0.2475 \sin 2 \theta_e}{1 - \sin^2 \theta_{13}(\theta_e, \theta_e)},$$

$$J_1(\theta_e, \theta_e) = \pm 0.05167 - 0.00391 \sin 2 \theta_e,$$

$$J_2(\theta_e, \theta_e) = 0.0000726 \cos (2 \theta_e - 4 \theta_e) - 0.05787 \cos (2 \theta_e + 2 \theta_e) - 0.05079 \cos (2 \theta_e - 2 \theta_e) + 0.07308 \sin 2 \theta_e,$$

$$U_{\text{Fla}}:$$

$$\sin^2 \theta_{13}(\theta_e, \theta_e) = 0.616 \cos^2 \theta_e \cos^2 \theta_e - 0.133 \cos \theta_e \sin \theta_e \sin \theta_e + 0.0769 \sin^2 \theta_e \cos^2 \theta_e + 0.1328 \sin 2 \theta_e \sin \theta_e + 0.01762 \sin 2 \theta_e \sin \theta_e - 0.1322 \sin 2 \theta_e \sin \theta_e - 0.1322 \sin 2 \theta_e \sin \theta_e + 0.4155 \sin^2 \theta_e \sin^2 \theta_e,$$

$$\sin^2 \theta_{23}(\theta_e, \theta_e) = \frac{0.2963 + 0.0502 \cos 2 \theta_e - 0.1185 \cos (2 \theta_e - 2 \theta_e) - 0.0502 \cos 2 \theta_e - 0.101 \cos (2 \theta_e + 2 \theta_e) - 0.01325 \sin 2 \theta_e + 0.01325 \sin 2 \theta_e + 0.000294 \sin (2 \theta_e + 2 \theta_e)}{1 - \sin^2 \theta_{13}},$$

$$\sin^2 \theta_{12}(\theta_e, \theta_e) = \frac{\sin^2 \theta_{13}(\theta_e, \theta_e) + \pi(2 \theta_e + \pi/2)}{1 - \sin^2 \theta_{13}(\theta_e, \theta_e)},$$

$$J_1(\theta_e, \theta_e) = 0.00919 \cos (2 \theta_e - 2 \theta_e) - 0.01601 \cos (2 \theta_e + 2 \theta_e) + 0.06163 \sin (2 \theta_e + 2 \theta_e),$$

$$J_1(\theta_e, \theta_e) = \pm 0.00475 - 0.0474 \cos 2 \theta_e - 0.0144 \cos (2 \theta_e - 4 \theta_e) + 0.0203 \cos (2 \theta_e - 2 \theta_e) + 0.0541 \cos 2 \theta_e + 0.0461 \cos (2 \theta_e + 2 \theta_e) - 0.03285 \cos (2 \theta_e + 4 \theta_e) + 0.002649 \sin 2 \theta_e - 0.0101 \sin (2 \theta_e - 4 \theta_e) + 0.02699 \sin (2 \theta_e - 2 \theta_e) + 0.03249 \sin 2 \theta_e + 0.05424 \sin (2 \theta_e + 2 \theta_e) - 0.01937 \sin (2 \theta_e + 2 \theta_e) - 0.04581 \sin (2 \theta_e + 4 \theta_e),$$

$$J_2(\theta_e, \theta_e) = J_1(\theta_e, \theta_e + \pi/4).$$
Figure 1: Parameter space for the mixing patterns constrained by the global fit data [67] at the 3σ level in the normal mass ordering (NO). The parameter spaces for $\theta_{12}$ are the strips with blue boundaries, and those for $\theta_{23}$ are with green boundaries. The strips for $\theta_{13}$ are tiny, i.e., almost reduced to black curves. Their intersection areas are signed by the red dots. The parameter spaces in the case of inverted mass ordering (IO) are similar. So they are not shown here.
Table 2: Best fit data of lepton mixing angles and CP phases. "N" and "I" denote the normal ordering of neutrino masses and the inverted ordering, respectively.

| Patterns     | $(\theta_{ei}^{b}, \theta_{ei}^{f})$ | $x_{\text{min}}^2$ | sin$^2\theta_{13}$ | sin$^2\theta_{23}$ | sin$^2\theta_{12}$ | $|\sin \delta|$ | $|\sin \alpha|$ | $|\sin \beta|$ |
|--------------|--------------------------------------|----------------------|---------------------|-------------------|-------------------|----------------|----------------|----------------|
| Ia(N)        | (0.18725π, 0.16444π), (-0.0297π, 0.72267π) | 0.000016             | 0.0216             | 0.547            | 0.320             | 0              | 0              | 0              |
| Ia(I)        | (0.18596π, 0.16544π), (-0.0297π, 0.72172π) | 0.02                  | 0.0220             | 0.549            | 0.318             | 0              | 0              | 0              |
| Ib(N)        | (-0.4028π, 0.3393π), (-0.8663π, -0.3393π) | 2.33                 | 0.0217             | 0.563            | 0.345             | 0.813          | 0.952          | 0.43            |
| Ib(I)        | (-0.4037π, 0.340π), (-0.8654π, -0.340π)   | 1.97                 | 0.0221             | 0.565            | 0.343             | 0.817          | 0.953          | 0.42            |
| Ila(N)       | (0.3694π, 0.0767π)                        | 0.064               | 0.0216             | 0.549            | 0.316             | 0.997          | 0.769          | 0.588           |
| Ila(I)       | (0.3701π, 0.0745π)                        | 0.057               | 0.022              | 0.553            | 0.316             | 0.998          | 0.774          | 0.547           |
| IIb(N)       | (-0.2345π, 0.06297π)                      | 0.79                | 0.0215             | 0.5507           | 0.304             | 0.800          | 0.617          | 0.747           |
| IIb(I)       | (-0.2348π, 0.06145π)                      | 0.81                | 0.0220             | 0.5546           | 0.304             | 0.742          | 0.616          | 0.730           |

For $U_{\text{Ib}}$, these two points give the same magnitude of sin $\delta$ with different signs. For the mixing matrices $U_{\text{IIa}}$ and $U_{\text{Iib}}$, there is only one best fit point in a period. sin $\delta$ for them are both positive. Even so, $U_{\text{IIa}}$ and $U_{\text{Iib}}$ could give negative sin $\delta$ while the mixing angles are kept the same. (ii) The best fit value of $\theta_{13}$ from the global fit data [67] is in the second octant. Accordingly, our fit data is in the same octant. In the case of normal mass ordering (NO), the best fit value of $\delta$ in $U_{\text{lb}}$ and $U_{\text{Iib}}$ could be around $-0.3\pi$. It is in the $1\sigma$ range of the global fit data. In the case of inverted mass ordering (IO), the best fit value of $\delta$ in $U_{\text{IIa}}$ is around $-0.5\pi$. It is also in the $1\sigma$ range of the global fit data.

3.4. The Effective Mass of Neutrinoless Double-Beta Decay $<m_{ee}>$. Although the residual symmetries $(Z_{2v} \times CP, Z_{2v} \times CP_{v})$ cannot restrain the masses of neutrinos $m_i$ with $i=1,2,3$, they may affect the effective mass of neutrinoless double-beta decay $<m_{ee}>$ through the mixing angles and Majorana phases. Here, $<m_{ee}>$ is expressed as

$$<m_{ee}> \equiv |m_{11}U_{11}^2 + m_{22}U_{12}^2 + m_{33}U_{13}^2|.$$ (46)

Employing the lepton mixing matrix $U(\theta_{ei}^{b}, \theta_{ei}^{f})$ and the best fit data on $\Delta m_{21}^2$, $|\Delta m_{31}^2|$ [67], we plot $<m_{ee}>$ against the mass of the lightest neutrino $m_0$ in Figure 2. For $(\theta_{ei}, \theta_{ej})$ taken from the $3\sigma$ range around the best fit data, the curves of $<m_{ee}>$ in every pattern are shown in Figure 3. We make some comments on the main results shown in these figures.

(i) In the case of IO, these patterns give stringent constraints on the ranges of $<m_{ee}>$. Particularly, $<m_{ee}>$ for patterns with the indexes (1, 0) and (1, 1) is independent of the parameters $(\theta_{ei}, \theta_{ej})$.

In the range of $m_0$ favored by cosmology, $<m_{ee}>$ is around 0.045 eV for patterns Ia, Ila, and Ib. For pattern Ib, it is 0.04 eV. These values approximate the upper limit from the global fit data at the $3\sigma$ level. They are in the reach of future double-beta decay experiments [73]. Interestingly, similar observations still hold for the patterns from the group $S_4$ with GCP [62].
Figure 2: The effective mass of neutrinoless double-beta decay \( \langle m_{ee} \rangle \) against the mass of the lightest neutrino \( m_0 \) with the best fit data \((\theta^\nu_e, \theta^\nu_\nu)\). The bound on \( m_0 \) from cosmology is taken from Ref. [74]. The constraint on \( \langle m_{ee} \rangle \) is from Ref. [75]. The best fit data of \( \Delta m^2_{12} \) and \( \Delta m^2_{13} \) is from Ref. [67]. The indexes \((j, k)\) are defined in Equation (21). The best fit data \((\theta^\nu_e, \theta^\nu_\nu)\) for pattern Ia and that for pattern Ib take the second one in Table 2, respectively.

Figure 3: The effective mass of neutrinoless double-beta decay \( \langle m_{ee} \rangle \) against the mass of the lightest neutrino \( m_0 \) in the 3\( \sigma \) ranges of \((\theta^\nu_e, \theta^\nu_\nu)\). The conventions follow those in Figure 2. The dashed boundary lines at the 3\( \sigma \) level are obtained from the global fit data [67].
(ii) In the case of NO, the variance of $<m_{ee}>$ is noticeable for every pattern. Particularly, for patterns Ia and Ib, $<m_{ee}>$ with the indexes (1, 1) could reach the upper limit at the 3σ level. Even so, it is not accessible to near future experiments.

(iii) In both NO and IO case, without the precise constraint on the Dirac CP phase, these four mixing patterns cannot be discriminated by future double-beta decay experiments because of the overlaps of ranges of $<m_{ee}>$

4. Summary

For the group $PSL_2(7)$ with GCP symmetries, the predictions of the residual symmetries $Z_2 \times CP$ in both neutrino and charged lepton sectors are examined. Seven types of viable mixing patterns at the 3σ level of the global fit data are obtained. Among them, six types are paired through the complex conjugation. Three types of patterns can give the Dirac CP phase which is in the 1σ range of the global fit data. With the parameters $(\theta_1, \theta_2)$, the constraints of residual symmetries on the effective mass of neutrinoless double-beta decay are also examined. In the case of IO, every pattern can give the effective mass accessible to the future experiments.

Data Availability

The global fit data supporting this research paper are from previously reported studies, which have been cited. The processed data are freely available.

Conflicts of Interest

The author declares that there is no conflict of interest regarding the publication of this paper.

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