Stokes’ first problem for some non-Newtonian fluids: Results and mistakes

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Abstract
The well-known problem of unidirectional plane flow of a fluid in a half-space due to the impulsive motion of the plate it rests upon is discussed in the context of the second-grade and the Oldroyd-B non-Newtonian fluids. The governing equations are derived from the conservation laws of mass and momentum and three correct known representations of their exact solutions given. Common mistakes made in the literature are identified. Simple numerical schemes that corroborate the analytical solutions are constructed.

Keywords: Stokes’ first problem, Second-grade fluid, Oldroyd-B fluid, Integral transform methods, Finite-difference scheme

1. Introduction
Following Truesdell and Rajagopal (2000), consider a fluid of density \( \rho(x, t) \) and velocity field \( \mathbf{u}(x, t) \), where \( x \) is the spatial coordinate and \( t \) the temporal one. Conservation of mass dictates that

\[
\dot{\rho} + \nabla \cdot \mathbf{u} = 0,
\]

where a superimposed dot denotes the material derivative: \( \dot{\mathbf{u}} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \). Cauchy’s first law of continuum mechanics supplies the additional conservation of momentum equation:

\[
\rho \dot{\mathbf{u}} = \nabla \cdot \mathbf{T} + \mathbf{b},
\]

where \( \mathbf{T}(x, t) \) is the stress tensor, and \( \mathbf{b}(x, t) \) represents the body force(s).

We suppose that the fluid is incompressible and homogeneous so that \( \dot{\rho} = \nabla \cdot \mathbf{u} = 0 \Rightarrow \rho = \text{const.} =: \rho_0 > 0 \) for all \( x \) and \( t \). Then, Eq. (1) implies that \( \nabla \cdot \mathbf{u} = 0 \), meaning that such a fluid can only undergo isochoric motions. Assuming a Cartesian coordinate system, where \( x = x\hat{i} + y\hat{j} + z\hat{k} \) with unit vectors in the three coordinate directions \( \hat{i}, \hat{j} \) and \( \hat{k} \), one such motion is the unidirectional plane flow \( \mathbf{u} = u(y, t)\hat{i} \), which clearly satisfies \( \nabla \cdot \mathbf{u} = 0 \). Finally, we assume no external forces act on the fluid: \( \mathbf{b} = 0 \). All this means that the fluid fills the half-space \( y > 0 \) with a solid plate lying in the \( x,z \) plane (i.e., at \( y = 0 \)). The motion is uniform (translation-invariant) in the \( x \) and \( z \) directions.

In 1851, Stokes considered a specific case of such a unidirectional plane flow.\(^1\) He was interested in the case wherein the plate at \( y = 0 \) is set into motion suddenly at time \( t = 0^+ \). In other words, the plate’s velocity is given by \( U(t) = \dot{U}(t)H(t) \), where \( H(\cdot) \) denotes the Heaviside unit step function, and \( \dot{U}(t) \) is some smooth function, i.e., it possesses as many continuous derivatives with respect to \( t \) on \((-\infty, +\infty)\) as needed, that we are free to specify. Stokes himself made the distinction between \( U(t) \) and \( \dot{U}(t) \) clear (Stokes, 1851, p. 101), yet a bewildering array of papers from the 1990s and 2000s fail to take this into account. Thanks to the no-slip boundary condition, the fluid near the boundary assumes the velocity of the the plate, i.e., \( u(0, t) = \dot{U}(t)H(t) \). Consequently, if one needs to compute the acceleration of the fluid at the plate (e.g., when applying the Fourier sine transform to a mixed derivative), the correct expression is \( \frac{\partial u}{\partial t}(0, t) = \ddot{U}(t)H(t) + \dot{U}(t)\delta(t) \), where \( \delta(\cdot) \) denotes the Dirac delta function. Of course, such equalities are meant in the sense of distributions as the generalized functions \( H(t) \) and \( \delta(t) \) fail to have point-values everywhere (Kolmogorov and Fomin, 1975, §21). An ubiquitous but inexcusable mistake is to drop the \( \dot{U}(t)\delta(t) \) term.

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\(^1\)Some authors misattribute this problem to Rayleigh though Stokes was the first to solve it (see, e.g., Schlichting, 1979, Chap. V §4).
1.1. Second-grade fluid

The Rivlin–Ericksen fluids, also known as fluids of grade \(n\) (Truesdell and Rajagopal, 2000, Chap. 6), are a model of isotropic simple fluids of the differential type. Their constitutive relation can be written as an expansion in terms of the Rivlin–Ericksen tensors \(A_k\). For the incompressible second-grade fluid it takes the form

\[
T = -\rho I + S, \quad S = \mu_0 A_1 + \alpha_1 A_2 + \alpha_2 A_1, \quad \text{tr} D = 0, \quad (3)
\]

where \(p\) is the isotropic (indeterminate) stress, \(I\) is the identity tensor, \(S\) is the extra (determinate) stress, \(A_1 = 2D\), \(A_{k+1} = A_k + A_k \text{ grad} u + (\text{grad} u)^\top A_k\) \((k \geq 1)\), \(D \equiv \frac{1}{2} \left[ \text{grad} u + (\text{grad} u)^\top \right]\) is the symmetric part of the velocity gradient (sometimes referred to as the infinitesimal rate of strain) tensor, and a \(\top\) superscript denotes the transpose. The constant \(\mu_0(>0)\) is understood in the usual sense of fluid viscosity from Navier–Stokes theory, and the second-grade (constant) parameters \(\alpha_1\) and \(\alpha_2\) will be discussed shortly.

Ting (1963) was the first to consider plane flows of the second-grade fluid and give correct solutions using the Laplace transform and the Bromwich integral inversion formula. However, he did not consider Stokes’ first problem. Following his derivation, for which he is rarely given credit, the stress tensor for our unidirectional plane flow has the Laplace transform and the Bromwich integral inversion formula. However, he did not consider Stokes’ first problem. Following his derivation, for which he is rarely given credit, the stress tensor for our unidirectional plane flow has the component representation

\[
[T] = \begin{pmatrix}
-p + \alpha_2 \left( \frac{\partial u}{\partial y} \right)^2 & \mu_0 \frac{\partial u}{\partial y} + \alpha_1 \frac{\partial^2 u}{\partial y^2} & \mu_0 \frac{\partial u}{\partial y} + \alpha_1 \frac{\partial^2 u}{\partial y^2} \\
0 & -p + (\alpha_1 + 2\alpha_2) \left( \frac{\partial u}{\partial y} \right)^2 & 0 \\
0 & 0 & -p
\end{pmatrix}.
\]

Substituting Eq. (4) into Eq. (2), we obtain

\[
\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \mu_0 \frac{\partial^2 u}{\partial y^2} + \alpha_1 \frac{\partial^3 u}{\partial y^2 \partial y}, \quad 0 = -\frac{\partial p}{\partial y} + 2(\alpha_1 + 2\alpha_2) \frac{\partial u}{\partial y} \frac{\partial^2 u}{\partial y^2}, \quad 0 = -\frac{\partial p}{\partial z}.
\]

Since the plate is infinite, translational invariance in the \(x-z\) plane implies that the pressure cannot depend on \(x\) or \(z\) (Fetter and Walecka, 2003, §61), i.e., \(\partial p/\partial x = \partial p/\partial z = 0\). This leads to \(p = p(y, t) = p_\infty + (\alpha_1 + 2\alpha_2)(\partial u/\partial y)^2\) from the second equation in Eq. (5), where \(p_\infty\) is the ambient pressure and at most a function of \(t\). The determination of the pressure is always overlooked in papers on this topic, yet it is a fundamental part of the solution to this problem.

Finally, we note the following thermodynamic restrictions: \(\alpha_1 \geq 0\) and \(\alpha_1 + \alpha_2 = 0\) (Dunn and Rajagopal, 1995). When \(\alpha_1 < 0\), the problem becomes ill-posed in the sense of Hadamard (Coleman et al., 1965).

1.2. Oldroyd-B fluid

Oldroyd (1950) proposed a number of constitutive relations for incompressible fluids with fading strain memory (retardation) exhibiting stress relaxation. The so-called incompressible Oldroyd-B fluid is the one with

\[
T = -\rho I + S, \quad S + \lambda_1 \tilde{S} = \mu_0 (A_1 + \lambda_2 \dot{A}_1), \quad \text{tr} D = 0, \quad (6)
\]

where the upper-convected time derivative (Oldroyd, 1950) Sec. 3(a)) is given by

\[
\tilde{S} = S - (\text{grad} u)^\top S - S \text{ grad} u + (\text{div} u) S.
\]

Here, \(\lambda_1\) and \(\lambda_2\) are the relaxation time and retardation time, respectively, and \(\mu_0\) is the fluid’s viscosity (as before). Noting that \(S = S(y, t)\) due to translation-invariance in the \(x-z\) plane, the second equation in Eq. (6) has the component form:

\[
\begin{bmatrix}
S_{xx} + \lambda_1 \frac{\partial S_{xx}}{\partial t} - \lambda_1 S_{yx} \frac{\partial u}{\partial y} - \lambda_1 S_{xy} \frac{\partial u}{\partial y} & S_{xy} + \lambda_1 \frac{\partial S_{xy}}{\partial t} - \lambda_1 S_{yx} \frac{\partial u}{\partial y} \frac{\partial u}{\partial y} & S_{xz} + \lambda_1 \frac{\partial S_{xz}}{\partial t} - \lambda_1 S_{zx} \frac{\partial u}{\partial y} \\
S_{yx} + \lambda_1 \frac{\partial S_{yx}}{\partial t} - \lambda_1 S_{xy} \frac{\partial u}{\partial y} & S_{yy} + \lambda_1 \frac{\partial S_{yy}}{\partial t} - \lambda_1 S_{yx} \frac{\partial u}{\partial y} & S_{yz} + \lambda_1 \frac{\partial S_{yz}}{\partial t} - \lambda_1 S_{zx} \frac{\partial u}{\partial y} \\
S_{zx} + \lambda_1 \frac{\partial S_{zx}}{\partial t} - \lambda_1 S_{xy} \frac{\partial u}{\partial y} & S_{zy} + \lambda_1 \frac{\partial S_{zy}}{\partial t} - \lambda_1 S_{yx} \frac{\partial u}{\partial y} & S_{zz} + \lambda_1 \frac{\partial S_{zz}}{\partial t}
\end{bmatrix}
= \begin{bmatrix}
\frac{\partial u}{\partial y}^2 & \mu_0 \frac{\partial u}{\partial y} + \mu_0 \lambda_2 \frac{\partial^2 u}{\partial y^2} & 0 \\
\mu_0 \frac{\partial u}{\partial y} + \mu_0 \lambda_2 \frac{\partial^2 u}{\partial y^2} & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

(8)
From the assumptions that prior to start-up the fluid is at rest, we have that \( S(y, 0) \equiv 0 \), whence the equations for the components \( S_{xx}, S_{zx}, S_{xy}, S_{yz} \) and \( S_{zz} \) give
\[
S_{xx} = S_{zx} = S_{yy} = S_{yz} = S_{cy} = S_{cz} \equiv 0 \quad \forall t \geq 0, \quad y \geq 0. \tag{9}
\]
Since the stress tensor must be symmetric, i.e., \( \mathbf{T} = \mathbf{T}^T \) \((\Rightarrow S_{xy} = S_{yx}, \text{in particular})\), the remaining equations are
\[
S_{xy} + \lambda_1 \frac{\partial S_{xy}}{\partial t} = \mu_0 \frac{\partial u}{\partial y} + \mu_0 \lambda_2 \frac{\partial^2 u}{\partial y^2}, \quad S_{xx} + \lambda_1 \frac{\partial S_{xx}}{\partial t} - 2\lambda_1 S_{xy} \frac{\partial u}{\partial y} = -2\mu_0 \lambda_2 \left( \frac{\partial u}{\partial y} \right)^2. \tag{10}
\]
This derivation was first given by Tannen (1962), though many authors employ a clumsy and abbreviated version of it without giving him any credit. Oldroyd (1950, Sec. 4) presents a similar derivation for a problem in polar coordinates.

Substituting the first equation in Eq. (8) into Eq. (2) and recalling that \( \mathbf{S} = \mathbf{S}(y, t) \), we obtain
\[
\frac{\partial u}{\partial t} = \frac{\partial p}{\partial x} + \frac{\partial S_{xy}}{\partial y}, \quad 0 = \frac{\partial p}{\partial y} + \frac{\partial S_{yy}}{\partial y}, \quad 0 = \frac{\partial p}{\partial z}. \tag{11}
\]
As in Sec. 1.1, translation invariance in the \( x-z \) plane implies \( \partial p/\partial x = \partial p/\partial z = 0 \). Then, \( p = p(y, t) = p_\infty + S_{yy}(y, t) \). Substituting \( p_\infty \) thanks to Eq. (9) and the second equation in Eq. (11). Next, we can eliminate \( S_{xy} \) from the first equation in Eqs. (10) and (11) by taking the \( y \) derivative of the former and the \( t \) derivative of the latter. Thus, we arrive at
\[
\frac{\partial u}{\partial y} + \frac{\partial \lambda_1}{\partial t} \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial t^2} + \mu_0 \lambda_2 \frac{\partial^3 u}{\partial y^3}. \tag{12}
\]
The two non-trivial components \( S_{xx} \) and \( S_{yy} \) of the determinate stress can be calculated from Eq. (10) once \( u \) is found from Eq. (12). This completes the formulation of the problem.

Considerations from thermodynamics (Rajagopal and Srinivasa, 2000) restrict the values of the relaxation and retardation times to be such that \( \lambda_2 < \lambda_1 \), though here we present some solutions also valid for \( \lambda_2 \geq \lambda_1 \). Causality requires that \( \lambda_1 > 0 \). Then, for the problem to be well-posed in the sense of Hadamard, a necessary (but not sufficient) condition is that \( \lambda_2 > 0 \). Experimental observations support all of these restrictions (Toms and Strawbridge, 1953; Oldroyd, 1958).

2. Exact solutions by integral transform methods

2.1. Second-grade fluid

Defining \( \nu := \mu_0/\lambda_0 \) and \( \alpha := \alpha_1/\lambda_0 \) and supplying Eq. (5) with the boundary condition discussed in Sec. 1 and a proper decay condition as \( y \to \infty \), we have the following initial-boundary-value problem (IBVP):
\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial y^2} + \alpha \frac{\partial^3 u}{\partial y^3}, \quad (y, t) \in (0, \infty) \times (0, \infty); \tag{13a}
\]
\[
u(0, t) = U_0 H(t), \quad u \to 0 \text{ as } y \to \infty, \quad t > 0; \tag{13b}
\]
\[
u(0, 0) = 0, \quad y > 0. \tag{13c}
\]

Second-grade solution representation 1 (Christov and Christov, in press). Assume \( \alpha > 0 \). Using first the Fourier sine transform in \( y \) and solving the resulting ordinary differential equation in \( t \) with the Laplace transform, one obtains
\[
u(y, t) = U_0 H(t) \left[ 1 - \frac{2}{\pi} \int_0^{\infty} \frac{\sin(\xi y)}{\xi} \exp \left( -\frac{\nu^2 \xi^2 t}{1 + \alpha \xi^2} \right) d\xi + \frac{2\alpha}{\pi} \int_0^{\infty} \frac{\xi \sin(\xi y)}{1 + \alpha \xi^2} \exp \left( -\frac{\nu^2 \xi^2 t}{1 + \alpha \xi^2} \right) d\xi \right]. \tag{14}
\]

While many authors have attempted to obtain this solution, Christov and Christov (in press) show that all of them make the mistake of dropping the \( \hat{U}(t)\bar{H}(t) \) term (recall the discussion in Sec. 2) in the expression for the plate’s acceleration when applying the Fourier sine transform. A series of papers (Erdogan, 2003; Erdogan and Imrek, 2005, 2007a,b) promulgates this error, while incorrectly claiming that there is a “deeper” mathematical reason that
their erroneous solution does not agree with the correct Laplace transform solution. Meanwhile, three recent papers (Zierep and Fetecău, 2007; Zierep et al., 2007; Zierep and Bohnig, 2008) make use of the incorrect version of Eq. (14) to perform some manipulations rendering their results erroneous. The transform error is also committed in (Fetecău and Fetecău, 2002; Shen et al., 2006), wherein an incorrect solution of Stokes’ first problem for a second-grade fluid over a heated plate is obtained. Similarly, Khan et al. (2008) obtain erroneous solutions for MHD second-grade fluid flows.

**Second-grade solution representation 2** (Puri, 1984). Assume \( \alpha > 0 \). Using the Laplace transform in \( t \) and the Bromwich integral inversion formula, one obtains

\[
u(y, t) = U_0 H(t) \left[ 1 - \frac{1}{\pi} \int_0^\infty e^{-\nu t} \sin \left( \frac{y}{\sqrt{\alpha - \eta}} \sqrt{\alpha - \eta} \right) \frac{dn}{\eta} \right].
\] (15)

Note that Eq. (14) can be transformed into Eq. (15) by the substitution \( \eta = \xi^2 / (1 + \alpha^2 \xi^2) \). This solution representation is correctly generalized to the case of a porous half-space by Jordan and Puri (2003), while an erroneous version of the porous-half-space problem for the related Burgers fluid is corrected in Jordan (2010).

**Second-grade solution representation 3** (Bandelli, Rajagopal and Galdi, 1995). Assume \( \alpha > 0 \). Using the Laplace transform in \( t \), the quotient splitting technique of Morrison (1954) and standard Laplace inversion tables, one obtains

\[
u(y, t) = U_0 H(t) \left[ e^{-\nu t / \alpha} \int_0^\infty e^{-\xi^2 I_0(2 \sqrt{\xi t / \alpha})} \text{erfc} \left( \frac{y}{2 \sqrt{\alpha \xi}} \right) d\xi \right].
\] (16)

where \( \text{erfc}(\cdot) \) is the complementary error function and \( I_0(\cdot) \) the modified Bessel function of the first kind of order zero.

Note that both Bandelli et al. (1995) and Puri (1984) omit that the \( H(t) \) pre-factor. This is not a triviality because, as the following theorem shows, \( \lim_{t \to 0^+} \nu(y, t) \neq 0 = \nu(y, 0) \) for this problem. This is known as the *start-up jump* and its physical significance is gracefully explained in Jordan and Puri (2003; Jordan, 2010).

**Theorem 1** (Bandelli, Rajagopal and Galdi, 1995). Let \( \nu(y, t) \) be a function for which \( \partial \nu / \partial y, \partial^2 \nu / \partial t \) and \( \partial^2 \nu / \partial t \partial y \) are all integrable at \( (y, t) = (0, 0) \), then \( |u(\epsilon, 0) - u(0, \epsilon)| \to 0 \) as \( \epsilon \to 0^+ \).

The contrapositive of this theorem states that if \( |u(\epsilon, 0) - u(0, \epsilon)| \not\to 0 \) as \( \epsilon \to 0^+ \) (i.e., the initial and boundary data are *incompatible*), the solution to the IBVP may fail to have integrable either first and/or mixed second derivatives at \( (y, t) = (0, 0) \). This means that the solution itself is ill-behaved (singular) there. Indeed, for Stokes’ first problem we have incompatible initial and boundary data. Therefore, from Bandelli et al. (1995, Eq. (3.10)), we have \( |u(y, t) - u(y, 0)| \not\to 0 \) as \( t \to 0^+ \), whence \( \lim_{t \to 0^+} u(y, t) \neq u(y, t) \) \( \alpha > 0 \). This fact about limits of singular functions is completely unrelated to integral transform methods. In the recent literature, one can find baffling statements such as “it was shown that the previous attempts to solve the problem by using the Laplace transform technique are erroneous and that the method of the Laplace transform does not work for this problem” (Erdogan, 2003). However, such a statement is patently false as the two Laplace and one Fourier sine transform solutions above are all correct, they satisfy all conditions imposed, and they are equivalent.

2.2. Oldroyd-B fluid

Again, defining \( \nu := \mu_2 / \nu_0 \) and supplying Eq. (12) with the boundary condition discussed in Sec. 1 and a proper decay condition as \( y \to \infty \), we have the following IBVP:

\[egin{align*}
\frac{\partial \nu}{\partial t} + \lambda_1 \nu \frac{\partial^2 \nu}{\partial y^2} &= \nu \frac{\partial^2 \nu}{\partial y^2} + \nu L_2 \frac{\partial^2 \nu}{\partial t \partial y}, & (y, t) \in (0, 0) \times (0, 0); \\
u(0, t) &= U_0 H(t), & u \to 0 \text{ as } y \to \infty, & t > 0; \\
u(y, 0) &= 0, & \frac{\partial \nu}{\partial t}(y, 0) = 0, & y > 0;
\end{align*}
\] (17a, 17b, 17c)

It appears that Tanner (1962) was the first one to realize this for the Oldroyd-B fluid. He notes that “the integral form ... does not satisfy the ... conditions at \( t = 0 \), there being no derivative at that point.” Yet, worries about this fact prevented Amos (1960) from obtaining the solution in Eq. (13), though he had all the “pieces” of it. Meanwhile, Timoshenko (1963) shows his solutions for flows of second-grade fluids have continuous derivatives at \( t = 0 \), however, all of these are for compatible initial and boundary conditions.
Oldroyd-B solution representation 1 (Christov and Jordan, 2009). Define $\kappa := \lambda_2/\lambda_1$ and assume $\lambda_{1,2} > 0$. Using the Fourier sine transform in $y$, solving the resulting ordinary differential equation in $t$ with the Laplace transform, one obtains

$$u(y, t) = U_0 H(t) \left\{ 1 - \frac{2}{\pi} \int_0^\infty \mathcal{U}_s(\xi, t) \sin \left( \frac{\xi y}{\sqrt{\lambda_1}} \right) d\xi + \int_0^\infty \mathcal{U}_v(\xi, t) \sin \left( \frac{\xi y}{\sqrt{\lambda_1}} \right) d\xi + \int_0^\infty \mathcal{U}_r(\xi, t) \sin \left( \frac{\xi y}{\sqrt{\lambda_1}} \right) d\xi \right\}, \quad \kappa < 1,$$

$$u(y, t) = U_0 H(t) \left\{ 1 - \frac{2}{\pi} \int_0^\infty \mathcal{U}_s(\xi, t) \sin \left( \frac{\xi y}{\sqrt{\lambda_1}} \right) d\xi \right\}, \quad \kappa = 1,$$

$$u(y, t) = U_0 H(t) \left\{ 1 - \frac{2}{\pi} \int_0^\infty \mathcal{U}_s(\xi, t) \sin \left( \frac{\xi y}{\sqrt{\lambda_1}} \right) d\xi \right\}, \quad \kappa > 1,$$

where $\mathcal{U}_s(\xi, t)$ correspond to $f(\xi) \geq 0$, respectively:

$$\mathcal{U}_s(\xi, t) = \exp \left[ -g(\xi) t/\lambda_1 \right] \left\{ \sqrt{\xi(\xi)} \cosh \left( \xi y / \sqrt{\lambda_1} \right) + g(\xi) \sinh \left( \xi y / \sqrt{\lambda_1} \right) \right\} \left[ \kappa = 1 \right]$$

and

$$\xi_{1,2}^* = \kappa^{-1} \sqrt{2 - \kappa \pm \sqrt{1 - \kappa}}, \quad f(\xi) = \frac{1}{2} \left[ \kappa^2 \xi^4 - 2(2 - \kappa) \xi^2 + 1 \right], \quad g(\xi) = \kappa^2 / (1 + \kappa \xi^2) > 0.$$

Oldroyd-B solution representation 2 (Tanner, 1962). Define $\kappa := \lambda_2/\lambda_1$ and assume $\lambda_{1,2} > 0$. Using the Laplace transform in $t$ and the Bromwich integral inversion formula, one obtains

$$u(y, t) = U_0 H(t) \left\{ \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \exp \left\{ -\frac{y}{\sqrt{\lambda_1}} \sqrt{\eta} M(\eta) \left[ \cos \theta(\eta) - \sin \theta(\eta) \right] \right\} \times \sin \left\{ \frac{\eta}{\lambda_1} - \frac{y}{\sqrt{\lambda_1}} \sqrt{\eta} M(\eta) \left[ \cos \theta(\eta) + \sin \theta(\eta) \right] \right\} \frac{d\eta}{\eta} \right\}, \quad (19)$$

where

$$M(\eta) = \sqrt{\frac{1 + \eta^2}{1 + \kappa^2 \eta^2}}, \quad \theta(\eta) = \frac{1}{2} \left[ \tan^{-1} \eta - \tan^{-1}(\kappa \eta) \right].$$

Oldroyd-B solution representation 3 (Morrison, 1956). Define $\kappa := \lambda_2/\lambda_1$ and assume $\lambda_{1,2} > 0$ and $\kappa \leq 1$. Using the Laplace transform in $t$ and a special splitting of the resulting quotient, one can invert the transform-domain solution using standard tables of inverses to obtain

$$u(y, t) = U_0 H(t) \exp \left\{ \left[ 1 - \frac{1 - \kappa}{\kappa} \right] \frac{t}{\lambda_1} \right\} \left\{ \text{erfc} \left[ \frac{y}{2 \sqrt{\lambda_1} \kappa} - \frac{1 - \kappa}{\kappa} \right] \int_0^{\sqrt{\lambda_1}} \left\{ J_0 \left[ \frac{2 \sqrt{1 - \kappa}}{\kappa} \sqrt{\xi \left( \frac{1}{\lambda_1} - \xi \right)} \right] \right\} \exp \left\{ -\frac{2 - \kappa}{\kappa} \left( \frac{t}{\lambda_1} - \xi \right) \text{erfc} \left[ \frac{y}{2 \sqrt{\lambda_1} \xi \kappa} \right] \right\} d\xi \right\}, \quad (20)$$

where $J_p(\cdot)$ is the Bessel function of the first kind of order $p$.

Morrison (1956) and Tanner (1962) both neglect to multiply their solutions by $H(t)$. Though Morrison (1956) was not studying the Oldroyd-B fluid specifically, he obtained Eq. (17a) for the velocity in a viscoelastic rod whose stress response is modeled by a dashpot in series with an element consisting of another dashpot and a string in parallel.

Vieru et al. (2008) claim a “new” solution to the IBVP in Eq. (17) is obtained in their paper, though they use the Laplace transform and a quotient splitting very similar to the one in Morrison (1956). Additionally, it is claimed that “the diagrams of the solutions” from Fetecau and Fetecau (2003) “are identical” to those in Vieru et al. (2008).
Though it remains unclear why a diagram (rather than an accurate plot of the solution) is relevant, the solution in (Vieru et al., 2008) is suspect because it appears to agree with the incorrect solution in (Fetecau and Fetecau, 2003). The latter along with the solution for the porous half-space and porous half-space over a heat plate versions of this problem presented in (Tan and Masuoka, 2005b,a) were shown to be wrong by Christov and Jordan (2009). Other studies of Oldroyd-B (Fetecau, 2002; Fetecau et al., 2008; Khan, 2009; Fetecau et al., 2009), and Burgers (Khan et al., 2010) fluid flows are also erroneous because the mistake in applying the Fourier sine transform is made.

3. Numerical solutions by finite-difference methods

To provide an independent check on the transform solutions given in Sec. 2 we also solve the corresponding IBVPs numerically. The (uniform) spatial and temporal step sizes are defined as \( \Delta y := L/(M - 1) \) and \( \Delta t := t_f/(K - 1) \), where \( M \geq 2 \) and \( K \geq 2 \) are integers and now \( (y, t) \in (0, L) \times (0, t_f) \). Also, we let \( u^0_j \approx u(y_j, t) \) be the approximation to the exact solution on the grid, where \( y_j := j\Delta y (0 \leq j \leq M - 1) \) and \( t^p := n\Delta t (0 \leq n \leq K - 1) \). For appropriately chosen \( L \gg 1 \), the front does not reach the \( y = L \) boundary for any \( t \in (0, t_f) \), and so this is the "numerical infinity."

For the computations shown below, we use \( M = K = 5000 \) and \( L = 20 \) to obtain highly-accurate solutions. Mathematica’s built-in Gaussian elimination algorithm is used to invert the symmetric tridiagonal matrices resulting from the spatial discretizations. Additionally, the integral representations of the analytical solutions are evaluated using the high-precision numerical integration routine \texttt{NIntegrate} of the software package \textsc{Mathematica} (ver. 7.0.1).

3.1. Second-grade fluid

As shown in (Christov and Christov, in press), we may discretize Eq. (18a) as follows:

\[
\delta_t u_j^n = \nu \delta_y \delta_y [ \frac{1}{2} (u_j^{n+1} + u_j^n) ] + \alpha \delta_y \delta_y \delta_t u_j^n. \tag{21}
\]

Here, \( \delta_t \) is the forward temporal difference operator and \( \delta_y \) and \( \delta_y \) are, respectively, the forward and backward spatial difference operators (Strikwerda, 2004, §3.3). The boundary conditions from Eq. (13a) are implemented as

\[
u_0^n = \begin{cases} 0, & n = 0, \\ U_0, & 1 \leq n \leq K - 1; \end{cases} \quad u_{M-1}^n = 0, & 0 \leq n \leq K - 1. \tag{22}
\]

The initial condition is \( u_0^n = 0 \) (0 \( \leq j \leq M - 1 \)) owing to Eq. (13b). It is a straightforward, though lengthy, calculation (see, e.g., Strikwerda, 2004) to show that this implicit, two-level Crank–Nicolson-type discretization is unconditionally stable and has truncation error \( O(\Delta t^2 + (\Delta y)^2) \). Amos (1969) gives some explicit schemes for Eq. (13a), however, the present one is superior in both its accuracy and stability.

Since there exist dimensionless variables \( \bar{t} = (t \nu/\alpha) \) and \( \bar{y} = y / \sqrt{\nu} \) such that the problem and solution no longer depend on \( \nu \) and \( \alpha \) (Coleman et al., 1963; Bandelli et al., 1995), the qualitative shape of the solution is invariant and varying the parameters is not enlightening in any way. Therefore, in Fig. 11 we have made use of these dimensionless variables by showing \( u(y, \bar{t})/U_0 \) rather than \( u(y, t) \). It is clear that the three analytical solutions to the IBVP in Eq. (13) obtained by integral transforms presented in Sec. 2.1 agree identically with its numerical solution.

3.2. Oldroyd-B fluid

Constructing a reliable finite-difference scheme directly for Eq. (17a), subject to Eq. (17b), turns out to be a difficult task. An easier approach is to introduce the fluid’s acceleration \( w \equiv \partial u/\partial t \) with \( w_j^n = w(y_j, t^n) \), then Eq. (17a) can be trivially rewritten as a system that we discretize by the following semi-implicit Crank–Nicolson-type procedure:

\[
\delta_t w_j^n = \nu \delta_y \delta_y [ \frac{1}{2} (w_j^{n+1} + w_j^n) ] + \alpha \delta_y \delta_y \delta_t w_j^n. \tag{23}
\]

Noting that \( w(0, t) = U_0 \delta(t) \), the boundary conditions from Eq. (13b) are implemented as

\[
u_0^{-1/2} = \begin{cases} 0, & n = 0, \\ U_0, & 1 \leq n \leq K - 1; \end{cases} \quad w_0^n = \begin{cases} \frac{U_0}{\sqrt{n}}, & n = 0, \\ 1 \leq n \leq n^*, \end{cases} \quad w_{M-1}^n = w_{M-1}^n = 0, & 0 \leq n \leq K - 1. \tag{24}
\]
The initial conditions are $u^{-1/2}_j = w^0_j = 0 (0 \leq j \leq M - 1)$ owing to Eq. (17c). The scheme is not sensitive to the parameter $n^*$, so we take $n^* = 10$ in our calculations.

Unfortunately, due to the implementation of the δ-function boundary condition, it is no longer straightforward to show stability. Nevertheless, numerical experiments show the scheme is stable for $\Delta t = O(\Delta y)$. It is easy to establish the truncation error is $O((\Delta t)^2 + (\Delta y)^2)$. Townsend (1973) gives a $O((\Delta t)^2 + \Delta y)$ fully-implicit scheme for the system of Eqs. (10) and (11), however, the present scheme is simpler and more accurate. Other modern numerical approaches to one-dimensional viscoelastic flows can be found in (Amoreira and Oliveira, 2010) and the references therein.

By using the dimensionless variables $\tilde{t} = t/\lambda_1$ and $\tilde{y} = y/\sqrt{\nu \lambda_1}$ (Tanner, 1962), it can be shown that qualitative differences in the shape of the solution result only from having $\kappa < 1$, $\kappa = 1$ or $\kappa > 1$. Experiments (Toms and Strawbridge, 1953) suggest that $\kappa (\equiv \lambda_2/\lambda_1)$ is in the range 0.05 to 0.4. Therefore, in Fig. 2 we take $\kappa = 0.2$ and use the dimensionless variables above to show that the three analytical solutions to the IBVP in Eq. (17) obtained by integral transforms agree identically with its numerical solution.

4. Conclusion

In the last decade, a disturbing trend has emerged in non-Newtonian fluid mechanics. Many authors have been re-deriving known results, specifically those from the 1950s through 1980s on the simple flows of certain non-Newtonian fluids. Unfortunately, many errors have been made in these “new” derivations. When the problems considered are
actually novel, it is usually a minute change (e.g., in a boundary condition) that distinguishes them from the classical works. Even when free of mathematical errors, these make little (if any) contribution to the mechanics of fluids.

Here, we presented three known correct solutions (by the Fourier sine transform, by the Laplace transform with the Bromwich integral inversion formula and by Laplace transform with quotient splitting and tables of inverses) to Stokes’ first problem for the second-grade and Oldroyd-B fluids. Additionally, we presented a representative list of the papers in which the so-called “new solutions” are wrong. A complete list would be far too long to attempt here. The numerical schemes constructed in Sec. 3 provide a wholly independent check on the classical Laplace transform solutions and the corrected Fourier sine transform solutions.

Given the astonishing amount of misinformation on this topic in the literature, we offer some advice to future researchers based on the present work:

- The Laplace transform always works on a well-posed linear IBVP. The solution of Puri (1984) remains the first correct exact solution to Stokes’ first problem for the second-grade fluid, despite the denigrating remarks many authors, whose work is erroneous, make about this solution.
- The assumption that the fluid is initially at rest dictates that \( u(y, 0) \equiv 0 \) (identically), whence \( \frac{\partial u}{\partial t}(y, 0) \equiv 0 \) for any \( k \). Using the latter equality for \( k \geq 1 \) does not constitute an “additional” or “unphysical” assumption as incorrectly claimed in (e.g., Tan and Masuoka, 2005b; Vieru et al, 2008) and many derivative works thereof.
- For partial differential equations with mixed derivatives and incompatible initial and boundary data, the solution does not have to satisfy the initial condition in backward time, i.e., as \( t \to 0^+ \).
- One has to be very careful in applying the Fourier sine (or cosine) transform to partial differential equations with mixed derivatives and incompatible initial and boundary data because distributional derivatives of the Heaviside function will inevitably have to be taken for Stokes’ problems. Note that the same error exposed in (Christov and Jordan, 2009; Christov and Christov, in press) is committed when solving the unsteady version of Stokes’ second problem for non-Newtonian fluids. However, when \( \hat{U}(t) \sim t^k \) as \( t \to 0 \) (\( k \geq 1 \)), e.g., \( \hat{U}(t) = \sin(\omega t) \), the singularity is ameliorated thereby allowing an erroneous derivation to produce a correct solution. The same is true for the “constantly accelerating plate” problem in which \( \hat{U}(t) = At \).
- The decay boundary condition that \( u \to 0 \) “sufficiently fast” as \( y \to \infty \) is typically enough to guarantee a solution by the Laplace transform. In the case of the Fourier sine transform, keeping in mind we seek a classical solution of the partial differential equation, the implicit assumptions are made that \( u \) is twice continuously differentiable in \( y \), and that \( u, \frac{\partial u}{\partial y} \) and \( \frac{\partial^2 u}{\partial y^2} \) are all integrable for \( y \in (0, \infty) \). (Note that this presupposes nothing about the regularity of \( u \) in \( t \), which we saw above is quite low.) In fact, it is known from the theory of integration (see Theorem 33.7 and Proposition 23.8 in Priestley, 1997) that these (implicit) assumptions not only guarantee that the Fourier sine transform can be applied but also that \( u \) and \( \frac{\partial u}{\partial y} \to 0 \) as \( y \to \infty \). Confusion about this has led some authors to impose the additional (unnecessary) condition that \( \frac{\partial u}{\partial y} \to 0 \) as \( y \to \infty \).
- Taking the limit as the non-Newtonian parameter(s) go to zero should reduce any solution to the known Newtonian one. However, this a necessary but not sufficient condition. Consequently, this exercise provides any insight only if one fails to recover the Newtonian solution, meaning the non-Newtonian one is wrong. For the problems considered in the present work, it happens that both the correct and erroneous non-Newtonian solutions reduce to the correct Newtonian one, which is unaffected by the error in computing \( \frac{\partial}{\partial y}(0, t) \) as its governing equation does not have a mixed third derivative, hence one never has to compute \( \frac{\partial}{\partial y}(0, t) \) when applying the Fourier sine transform.
- The signs, thermodynamic restrictions on, and orders of magnitude of the non-Newtonian parameters cannot be ignored. Introducing dimensionless variables is an easy remedy to the lack of precise measurements of, e.g., the second-grade fluid’s parameter \( \alpha_1 \). Then, plots can be made with ease and without loss of generality.

Another important class of non-Newtonian fluids are those of the Maxwell type, which exhibit stress relaxation but have no strain memory (retardation), i.e., \( \lambda_2 = 0 \) in Eq. (6). A thorough overview of the various (correct) solution representations is given by Jordan et al (2004). And, a positivity-preserving numerical scheme, which can be
used as an independent check on any suspicious “new” solutions that may appear in the literature, was constructed by Mickens and Jordan (2004). Another approach to ascertaining the correctness of various solutions to Stokes’ first problem is to consider the asymptotic scalings of the velocity and shear stress with time (Muzychka and Yovanovich 2010) that they predict. Finally, we note that Preziosi and Joseph (1987) also provide correct solutions to Stokes’ first problem for viscoelastic fluids with a variety of memory kernels using the Laplace transform, while Phan-Thien and Chew (1988) present numerical solutions to Stokes’ first problem for a class of viscoelastic fluids that reduce to the Oldroyd-B and Maxwell models in certain distinguished limits.

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