FOURTH-ORDER DISPERSIVE SYSTEMS ON THE ONE-DIMENSIONAL TORUS

HIROYUKI CHIHARA

ABSTRACT. We present the necessary and sufficient conditions of the well-posedness of the initial value problem for certain fourth-order linear dispersive systems on the one-dimensional torus. This system is related with a dispersive flow for closed curves into compact Riemann surfaces. For this reason, we study not only the general case but also the corresponding special case in detail. We apply our results on the linear systems to the fourth-order dispersive flows. We see that if the sectional curvature of the target Riemann surface is constant, then the equation of the dispersive flow satisfies our conditions of the well-posedness.

1. INTRODUCTION

We study the initial value problem for a system of fourth-order dispersive partial differential equations on the one-dimensional torus of the form

\[
L\vec{u} = \vec{f}(t, x) \quad \text{in} \quad \mathbb{R} \times \mathbb{T},
\]

\[
\vec{u}(0, x) = \vec{\phi}(x) \quad \text{in} \quad \mathbb{T},
\]

where

\[
L = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},
\]

\[
A(x) = \begin{bmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{bmatrix}, \quad B(x) = \begin{bmatrix} b_{11}(x) & b_{12}(x) \\ b_{21}(x) & b_{22}(x) \end{bmatrix},
\]

\[
C(x) = \begin{bmatrix} c_{11}(x) & c_{12}(x) \\ c_{21}(x) & c_{22}(x) \end{bmatrix}, \quad D(x) = \begin{bmatrix} d_{11}(x) & d_{12}(x) \\ d_{21}(x) & d_{22}(x) \end{bmatrix},
\]

\[
a_{jk}(x), b_{jk}(x), c_{jk}(x) \in C^\infty(\mathbb{T}), \quad \text{and} \quad C^\infty(\mathbb{T}) \text{ is the set of all complex-valued smooth functions on } \mathbb{T}.
\]

The present paper is mainly concerned with the well-posedness of the initial value problem \((1)-(2)\). A non-Kowalewskian is said to be dispersive-type if the initial value problem for it is expected to be well-posed in both directions in time. There are many papers studying the well-posedness of the initial value problem for dispersive equations. Unfortunately, however, the results on the necessary and sufficient conditions of the well-posedness are limited to the Schrödinger evolution equations on the torus, and one-dimensional cases. See \([2], [3], [12], [13], [21], [22]\) and \([23]\). Generally speaking, if there exists a trapped classical orbit generated by the principal symbol, then the local smoothing effect of solutions of dispersive equations breaks down. See \([7]\) for instance. In particular, if the domain of the space variables is compact, the smoothing effect does not occurs at all. For this reason,
in case of the torus, restrictions on the equations for the well-posedness becomes stronger, and it is
relatively easy to obtain the necessary and sufficient conditions.

Here we change the subject. In the last decade the geometric analytic studies on dispersive flows
between manifolds have been relatively attractive in mathematics. Most of the equations are originated
in classical mechanics. Some results were obtained by the geometric point of view, and another
ones were based on the analytic approach. In any case, most of the results are concerned with the
relationship between the geometric settings and the structure of the equations. See, e.g., [1], [4], [5],
[6], [8], [9], [10], [15], [16], [17], [18] and references therein.

The system (1) is related with a fourth-order dispersive flow for closed curves into compact Rie-
mann surfaces of the form

\[ u_t = a \tilde{J}(u) \nabla_x^2 u_x + \{1 + bg(u_x, u_x)\} \tilde{J}(u) \nabla_x u_x + cg(u_x u_x, u_x) \tilde{J}(u) u_x \quad \text{in} \quad \mathbb{R} \times T, \tag{3} \]

where \( \mathbb{R} \times T \ni (t, x) \mapsto u(t, x) \in N \). \( (N, \tilde{J}, g) \) is a compact Riemann surface with a complex structure \( \tilde{J} \) and a Kähler metric \( g \), \( u_t = du(\partial/\partial t), u_x = du(\partial/\partial x) \), \( du \) is the differential of the mapping \( u \), \( \nabla \)
is the induced connection for the Levi-Civita connection \( \nabla^N \) of \( (N, \tilde{J}, g) \), \( a \in \mathbb{R} \setminus \{0\} \) and \( b, c \in \mathbb{R} \)
are constants. Note that \( C(t) = \{u(t, x) \mid x \in \mathcal{T}\} \) is a closed curve on \( N \) for any fixed \( t \in \mathbb{R} \), and
\( u \) describes the motion of a closed curve subject to the equation (3). From a point of view of linear
partial differential equations, the equation (3) has a loss of derivative of order one, and the classical
energy estimates of solutions never work. Moreover, no smoothing effect of solutions can be expected
since the sauce of the mapping is a compact space \( \mathcal{T} \). Recently, in spite of this difficulty, Onodera ([19])
has been studying the initial value problem for (4), and succeeded in the construction of time-local
solutions. Unfortunately, however, his approach is based on massive and complicated computations,
and is not comprehensive. In other words, it is very difficult to understand how he can resolve the loss
of derivative of order one.

The purpose of this paper is to give the necessary and sufficient conditions of the well-posedness
of the initial value problem (1)-(2), and to have insight into the structure of (3). Indeed, we also study
the initial value problem for a special system of the form

\[ \mathcal{L} \tilde{w} = \tilde{h}(t, x) \quad \text{in} \quad \mathbb{R} \times T, \tag{4} \]
\[ \tilde{w}(0, x) = \tilde{w}_0(x) \quad \text{in} \quad T, \tag{5} \]

where

\[ \mathcal{L} = I \frac{\partial}{\partial t} + J \frac{\partial^4}{\partial x^4} + \beta(x) \frac{\partial^2}{\partial x^2} + \gamma(x) \frac{\partial}{\partial x}, \]

\( \tilde{w}(t, x) = \{w_1(t, x), w_2(t, x)\} \) is a \( \mathbb{R}^2 \)-valued unknown function of \( (t, x) \in \mathbb{R} \times T \), \( \tilde{w}_0(x) \) and \( \tilde{h}(t, x) \)
are given functions,

\[ J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \beta(x) = \begin{bmatrix} \beta_{11}(x) & \beta_{12}(x) \\ \beta_{21}(x) & \beta_{22}(x) \end{bmatrix}, \quad \gamma(x) = \begin{bmatrix} \gamma_{11}(x) & \gamma_{12}(x) \\ \gamma_{21}(x) & \gamma_{22}(x) \end{bmatrix}, \]

and \( \beta_{jk}(x) \) and \( \gamma_{jk}(x) \) are real-valued smooth functions on \( T \). The system (4) is a special case of (1),
and very close to (3). Obviously, the prospect of the analysis of the differential operator \( \mathcal{L} \) is bad
since its principal part has only the off-diagonal components. Generally speaking, however, studies
on such real-valued linear systems are useful for solving the initial value problem for dispersive flows
into almost Hermitian manifolds. Indeed, if the matrix like \( J \) is replaced by the almost complex
structure, the methods established for the systems are applicable to some dispersive flows only with
minor changes in many cases. The necessary and sufficient conditions of \( L^2 \)-well-posedness of the
initial value problem (4)-(5) are reduced to those of (1)-(2). Moreover, we give the direct proof of
the sufficiency of the well-posedness of (4)-(5) with application to (3) in mind. We remark that the
sufficiency of our conditions on the well-posedness of (1)-(2) work also if the coefficients depend on
\( t \). Finally, we introduce a moving frame along the curve described by \( u(t, \cdot) \) on \( N \) for each \( t \), and
obtain an $\mathbb{R}^2$-valued system from the equation of a higher order spatial derivative of $u$. We see that if the sectional curvature of the target Riemann surface is constant, then this system satisfies the the sufficient conditions of $L^2$-well-posedness. We believe that our approach in the present paper will give a perspective to [19].

To state our results on the well-posedness of (1)-(2), we introduce some function spaces. We denote by $L^2(T; \mathbb{C}^2)$ the set of $\mathbb{C}^2$-valued square integrable functions on $T$. We denote by $C(\mathbb{R}; L^2(T; \mathbb{C}^2))$ and $L^1_{\text{loc}}(\mathbb{R}; L^2(T; \mathbb{C}^2))$ the set of all $L^2(T; \mathbb{C}^2)$-valued continuous functions on $T$, and the set of all $L^2(T; \mathbb{C}^2)$-valued locally integrable functions on $T$ respectively. Our main results are the following.

**Theorem 1.** The following conditions (I) and (II) are mutually equivalent.

(I) The initial value problem (1)-(2) is $L^2$-well-posed, that is, for any $\tilde{\varphi} \in L^2(T; \mathbb{C}^2)$ and for any $f \in L^1_{\text{loc}}(\mathbb{R}; L^2(T; \mathbb{C}^2))$, (1)-(2) has a unique solution $\tilde{u} \in C(\mathbb{R}; L^2(T; \mathbb{C}^2))$.

(II) The coefficients $a_{jk}(x)$, $b_{jk}(x)$ and $c_{jk}(x)$ satisfy the following conditions

\begin{align}
\text{Im} \int_0^{2\pi} a_{11}(x) dx &= 0, \\
\text{Im} \int_0^{2\pi} a_{22}(x) dx &= 0, \\
\text{Im} \int_0^{2\pi} \left\{ b_{11}(x) - \frac{3a_{11}(x)^2 - 4a_{12}(x)a_{21}(x)}{8} \right\} dx &= 0, \\
\text{Im} \int_0^{2\pi} \left\{ b_{22}(x) + \frac{3a_{22}(x)^2 - 4a_{12}(x)a_{21}(x)}{8} \right\} dx &= 0, \\
\text{Im} \int_0^{2\pi} \left\{ c_{11}(x) + \frac{i}{2} a'_{12}(x)a_{21}(x) \\
&- \frac{a_{11}(x)b_{11}(x) - a_{12}(x)b_{21}(x) - a_{21}(x)b_{12}(x)}{2a_{11}(x)^3 + 4a_{11}(x)a_{12}(x)a_{21}(x) - 2a_{12}(x)a_{21}(x)a_{22}(x)} \right\} dx &= 0, \\
\text{Im} \int_0^{2\pi} \left\{ c_{22}(x) - \frac{i}{2} a_{12}(x)a'_{21}(x) \\
&+ \frac{a_{22}(x)b_{22}(x) - a_{12}(x)b_{21}(x) - a_{21}(x)b_{12}(x)}{2a_{22}(x)^3 - 4a_{12}(x)a_{21}(x)a_{22}(x) + 2a_{11}(x)a_{12}(x)a_{21}(x)} \right\} dx &= 0.
\end{align}

We essentially diagonalize our system (1) by an appropriate system of pseudodifferential operators, and the proof of Theorem 1 is reduced to Mizuhara’s results on single equations of the form

\[
\frac{\partial v}{\partial t} \pm iD_x^4 v + ia(x)D_x^3 v + ib(x)D_x^2 v + ic(x)D_x v + id(x)v = g(t, x) \quad \text{in} \quad \mathbb{R} \times T,
\]

where $v(t, x)$ is a complex-valued unknown function, $a(x)$, $b(x)$, $c(x)$, $d(x)$ and $g(t, x)$ are given functions. In [13] he proved the following.

**Theorem 2 (Mizuhara, [13]).** The initial value problem for (12) is $L^2$-well-posed if and only if

\[
\text{Im} \int_0^{2\pi} a(x) dx = 0,
\]
that the following properties: for all nonnegative integers

\[ \text{Im} \int_0^{2\pi} \left\{ b(x) + \frac{3}{8} a(x)^2 \right\} dx = 0, \tag{14} \]

\[ \text{Im} \int_0^{2\pi} \left\{ c(x) - \frac{a(x) b(x)}{2} + \frac{a(x)^3}{8} \right\} dx = 0. \tag{15} \]

Here we used the double-sign corresponds.

If we consider the system for \(|w_1 + iw_2, w_1 - iw_2|\) instead of \(\bar{u}\), Theorem 1 implies the necessary and sufficient conditions of \(L^2\)-well-posedness of (4)–(5).

**Theorem 3.** The initial value problem (4)–(5) is \(L^2\)-well-posed if and only if

\[ \text{Im} \int_0^{2\pi} \text{tr}(\beta(x)) dx = 0, \tag{16} \]

\[ \text{Im} \int_0^{2\pi} \text{tr}(J\gamma(x)) dx = 0, \tag{17} \]

where \(\text{tr}(\beta(x)) = \beta_{11}(x) + \beta_{22}(x)\) and \(\text{tr}(J\gamma(x)) = \gamma_{12}(x) - \gamma_{21}(x)\).

We will check that Theorem 1 implies Theorem 3 and prove the sufficiency in Theorem 3 directly with the applications to (3) in mind. Our direct proof of the sufficiency in Theorem 3 works also in the case that the coefficients \(\beta_{jk}\) and \(\gamma_{jk}\) are \(C^1\)-functions in time.

Finally we introduce a moving frame along the curve \(u(t, \cdot)\) on \(TN\), and consider \(\nabla^l_x u_x\), which is \(l\)-th order derivative of \(u\) in \(x\) for some large integer \(l\). The system (10) for the two components of \(\nabla^l_x u_x\) in the moving frame satisfies a fourth-order dispersive system like (4). We see that this system satisfies the conditions (16) and (17) in some sense provided that the sectional curvature of the Riemann surface \((N, J, g)\) is constant. Such geometric reductions originated from the pioneering work of Chang, Shatah and Uhlenbeck in [11]. They constructed a moving frame along solutions to the one-dimensional Schrödinger map equation \(u_t = J(u)\nabla_x u_x\) into compact Riemann surfaces, and obtained a complex-valued equation from the equation for \(u_x\). Being inspired with [11], Onodera studied the reduction of third and fourth-order one-dimensional dispersive flows in [16]. Unfortunately, it is not easy to understand the structure of the modified equations for \(u_x\) from the point of view of linear partial differential equations. In other words, it is hard to distinguish an unknown from coefficients both of them consist of the original unknown function \(u\). In the present paper we obtain the system for the higher order spatial derivative of \(u\) instead of \(u_x\). We believe that our reduction is more comprehensive to understand the relationship between the structure of the equation and the geometric settings.

The plan of the present paper is as follows. In Section 2 we shall prove Theorem 1. In Section 3 we shall prove Theorem 2. Finally, in Section 4 we shall introduce the moving frame, and study the system for higher order spatial derivatives of \(u\).

**2. Proof of Theorem 1**

In this section we prove Theorem 1. We modify the system (1) by using bounded pseudodifferential operators. Then, the system (1) becomes a pair of single equations essentially, and the proof of Theorem 1 is reduced to Theorem 2. We make use of elementary pseudodifferential calculus on \(\mathbb{R}\). See, e.g., [11], [14] and [24] for this.

Let \(m \in \mathbb{R}\). We denote by \(S^m(\mathbb{T})\) the set of all smooth functions \(q(x, \xi)\) of \((x, \xi) \in \mathbb{T} \times \mathbb{R}\) with the following properties: for all nonnegative integers \(\alpha\) and \(\beta\), there exists a positive constant \(C_{\alpha\beta}\) such that

\[ \left| \frac{\partial^{\alpha + \beta} q}{\partial x^\beta \partial \xi^\alpha} (x, \xi) \right| \leq C_{\alpha\beta} (1 + |\xi|)^{m - \alpha}. \]
holds for all \((x, \xi) \in \mathbb{T} \times \mathbb{R}\). Let \(\mathcal{S}(\mathbb{R})\) be the Schwartz class of rapidly decreasing functions on \(\mathbb{R}\). A linear operator \(Q : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})\) is said to be a pseudodifferential operator with a symbol \(q\) if

\[
Q u(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{i(x-y)\xi} q(x, \xi) u(y) dy d\xi
\]

for \(u \in \mathcal{S}(\mathbb{R})\). We remark that the operator \(Q\) can be extended as \(Q : C^\infty(\mathbb{T}) \to C^\infty(\mathbb{T})\) since \(q(x, \xi)\) is \(2\pi\)-periodic in \(x\). Conversely, if a linear operator \(Q : C^\infty(\mathbb{T}) \to C^\infty(\mathbb{T})\) is given, its symbol \(\sigma(Q)\) is computed by \(\sigma(Q)(x, \xi) = e^{-ix\xi} Q e^{ix\xi}\). In what follows we mainly deal with \(2 \times 2\) matrix-valued symbols. The classes of such symbols are denoted by \(S^m(\mathbb{T}; M(2))\).

**Proof of Theorem** We split the proof into several steps in order to make our pseudodifferential calculus simple and comprehensive. We mainly use pseudodifferential operators of negative orders \(-1, -2, -3\). We apply these operators to step by step. Let \(r\) be a sufficiently large positive number. Pick up a smooth function \(\varphi_r(\xi)\) on \(\mathbb{R}\) such that \(0 < \varphi_r(\xi) \leq 1\), \(\varphi_r(\xi) = 1\) for \(|\xi| \geq r + 1\), \(\varphi_r(\xi) = 0\) for \(|\xi| \leq r\) and \(\varphi_r(\xi) = \varphi(-\xi)\). The last property \(\varphi_r(\xi) = \varphi(-\xi)\) is not necessary in this section, but will be essentially used in the next section. For \(2 \times 2\) matrices, we use the following notation:

\[
A^\text{diag}(x) = \begin{bmatrix} a_{11}(x) & 0 \\ 0 & a_{22}(x) \end{bmatrix}, \quad A^\text{off}(x) = \begin{bmatrix} 0 & a_{12}(x) \\ a_{21}(x) & 0 \end{bmatrix}
\]

for

\[
A(x) = \begin{bmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{bmatrix}.
\]

The set of all bounded linear operators on \(L^2(\mathbb{T}; \mathbb{C}^2)\) is denoted by \(\mathcal{L}(L^2(\mathbb{T}; \mathbb{C}^2))\), and its norm is denoted by \(||\cdot||\).

**Step 1:** diagonalization of \(iA(x)D_x^2\). Note that \(EA^\text{off}(x) + A^\text{off}(x)E = 0\). We define a pseudodifferential operator \(\tilde{A}_1\) of order \(-1\) by

\[
\sigma(\tilde{A}_2)(x, \xi) = \frac{1}{2} EA^\text{off}(x) \frac{\varphi_r(\xi)}{\xi} = -\frac{1}{2} A^\text{off}(x) E \frac{\varphi_r(\xi)}{\xi}.
\]

Set \(A_1 = I + \tilde{A}_1\). Then \(\sigma(A_1)(x, \xi) \in S^{-1}(\mathbb{T}; M(2))\) and \(\sigma(A_1)(x, \xi) \in S^0(\mathbb{T}; M(2))\). Note that \(\|A_1\| = \mathcal{O}(1/r)\). If we take a sufficiently large \(r > 0\), then \(A_1\) is invertible on \(\mathcal{L}(L^2(\mathbb{T}; \mathbb{C}^2))\), and the inverse is given by a Neumann series

\[
A_1^{-1} = I + \sum_{l=1}^\infty (-\tilde{A}_1)^l = I - \tilde{A}_1 + \tilde{A}_1^2 - \tilde{A}_1^3 + \tilde{A}_1^4 A_1^{-1}.
\]

(18)

Set \(L_1 = A_1 L A_1^{-1}\) and \(P_1 = A_1 P A_1^{-1}\) for short. Then we have

\[
A_1 L = L_1 A_1 = \left\{ I \frac{\partial}{\partial t} + iP_1 \right\} A_1.
\]

We compute \(P_1 = A_1 P A_1^{-1}\). Using (18), we have

\[
P_1 = (I + \tilde{A}_1) P (I - \tilde{A}_1 + \tilde{A}_1^2 - \tilde{A}_1^3 + \tilde{A}_1^4 A_1^{-1})
\]

\[
= P + (-P \tilde{A}_1 + \tilde{A}_1 P) - (-P \tilde{A}_1 + \tilde{A}_1 P) \tilde{A}_1 + (-P \tilde{A}_1 + \tilde{A}_1 P) \tilde{A}_1^2
\]

(19)

modulo \(\mathcal{L}(L^2(\mathbb{T}; \mathbb{C}^2))\). We compute the common term \(-P \tilde{A}_1 + \tilde{A}_1 P\). Since \(\tilde{A}_1\) is a pseudodifferential operator of order \(-1\), we deduce that

\[
-P \tilde{A}_1 + \tilde{A}_1 P = -E D_x^4 \tilde{A}_1 + \tilde{A}_1 E D_x^4
\]

(20)

\[-A(x) D_x^3 \tilde{A}_1 + \tilde{A}_1 A(x) D_x^3
\]

(21)
We compute the above term by term. Here we recall the definition of $\tilde{\Lambda}_1$, and the identities $E^2 = I$ and $A^{\text{off}}(x)E + EA^{\text{off}}(x) = 0$. We deduce that
\[
\sigma(20)(x, \xi) \equiv 2 \sum_{k=0}^{2} \frac{(-i)^k}{k!} \left\{ -\frac{\partial^k}{\partial \xi^k} E^{\xi^4} \right\} \left\{ \frac{\partial^k}{\partial x^k} \frac{1}{2} E A^{\text{off}}(x) \varphi_r(\xi) \right\} \xi \\
+ \left\{ -\frac{\partial^k}{\partial \xi^k} A^{\text{off}}(x)E \varphi_r(\xi) \right\} \left\{ \frac{\partial^k}{\partial x^k} E^{\xi^4} \right\} \xi \\
= -\frac{1}{2} \sum_{k=0}^{2} \frac{(-i)^k}{k!} \left( \frac{\partial^k A^{\text{off}}(x)}{\partial x^k} (x) \varphi_r(\xi) \right) \left( \frac{\partial^k}{\partial x^k} \xi^4 \right) \\
+ A^{\text{off}}(x) \left( \frac{\partial^k \varphi_r(\xi)}{\partial x^k} \right) \left( \frac{\partial^k}{\partial x^k} \xi^4 \right) \xi \\
\equiv -A^{\text{off}}(x)\xi^3 + 2i \frac{\partial A^{\text{off}}(x)}{\partial x}(x)\xi^2 + 3 \frac{\partial^2 A^{\text{off}}(x)}{\partial x^2}(x)\xi,
\]
\[
\sigma(21)(x, \xi) \equiv \sum_{k=0}^{1} \frac{(-i)^k}{k!} \left\{ -\frac{\partial^k}{\partial \xi^k} A(x)\xi^3 \right\} \left\{ \frac{\partial^k}{\partial x^k} \frac{1}{2} E A^{\text{off}}(x) \varphi_r(\xi) \right\} \xi \\
+ \left\{ -\frac{\partial^k}{\partial \xi^k} \frac{1}{2} A^{\text{off}}(x)E \varphi_r(\xi) \right\} \left\{ \frac{\partial^k}{\partial x^k} A(x)\xi^3 \right\} \\
= \left\{ \frac{1}{2} A(x)E A^{\text{off}}(x) - \frac{1}{2} A^{\text{off}}(x)EA(x) \right\} \xi^2 \\
+ \left\{ \frac{3i}{2} A(x)E \frac{\partial A^{\text{off}}(x)}{\partial x} - \frac{i}{2} A^{\text{off}}(x)E \frac{\partial A}{\partial x}(x) \right\} \xi,
\]
\[
\sigma(22)(x, \xi) \equiv \left\{ -\frac{1}{2} B(x)E A^{\text{off}}(x) - \frac{1}{2} A^{\text{off}}(x)EB(x) \right\} \xi,
\]
modulo $S^0(\mathbb{T}; M(2))$. Combining the above, we obtain
\[
\sigma(-P\tilde{\Lambda}_1 + \tilde{\Lambda}_1 P)(x, \xi) \\
= -A^{\text{off}}(x)\xi^3 \\
+ \left\{ 2i \frac{\partial A^{\text{off}}(x)}{\partial x}(x) - \frac{1}{2} A(x)E A^{\text{off}}(x) - \frac{1}{2} A^{\text{off}}(x)EA(x) \right\} \xi^2 \\
+ \left\{ 3 \frac{\partial^2 A^{\text{off}}(x)}{\partial x^2}(x) + \frac{3i}{2} A(x)E \frac{\partial A^{\text{off}}(x)}{\partial x} - \frac{i}{2} A^{\text{off}}(x)E \frac{\partial A}{\partial x}(x) \right\} \xi \\
- \frac{1}{2} B(x)E A^{\text{off}}(x) - \frac{1}{2} A^{\text{off}}(x)EB(x) \right\} \xi,
\]
modulo $S^0(\mathbb{T}; M(2))$. By using (23), we deduce that
\[
\sigma\left(-P\tilde{\Lambda}_1 + \tilde{\Lambda}_1 P\right)(x, \xi) \\
\equiv \sum_{k=0}^{1} \frac{(-i)^k}{k!} \sigma\left(P\tilde{\Lambda}_1 \tilde{\Lambda}_1 P\right)(x, \xi) \left\{ \frac{\partial^k}{\partial x^k} \frac{1}{2} E A^{\text{off}}(x) \varphi_r(\xi) \right\} \\
\equiv \frac{1}{2} A^{\text{off}}(x)E A^{\text{off}}(x)\xi^2
\[
\begin{align*}
&+ \left\{-\frac{3i}{2} A^\text{off}(x)E \frac{\partial A^\text{off}}{\partial x}(x) - i \frac{\partial A^\text{off}}{\partial x}(x)EA^\text{off}(x) \right. \\
&\left. + \frac{1}{4} A(x)EA^\text{off}(x)E A^\text{off}(x) + \frac{1}{4} A^\text{off}(x)EA(x)E A^\text{off}(x) \right\}\xi, \\
&\sigma \left( -P\tilde{\Lambda}_1 + \tilde{\Lambda}_1 P \right) \tilde{\Lambda}_2^2 (x, \xi) \\
\equiv& \sigma \left( -P\tilde{\Lambda}_1 + \tilde{\Lambda}_1 P \right) (x, \xi) \left\{ \frac{1}{2} EA^\text{off}(x) \frac{\varphi_r(\xi)}{\xi} \right\}^2 \\
\equiv& -\frac{1}{4} A^\text{off}(x)EA^\text{off}(x)E A^\text{off}(x)\xi, 
\end{align*}
\]
modulo \( S^0(\mathbb{T}; M(2)) \). Substituting (23), (24) and (25) into (19), we obtain
\[
P_1 \equiv ED_x^4 + A^\text{diag}(x)D_x^3 + B_1(x)D_x^2 + C_1(x)D_x \mod \mathcal{L}(L^2(\mathbb{T}; \mathbb{C}^2)),
\]
\[
\begin{align*}
B_1(x) &= B(x) + 2i \frac{\partial A^\text{off}}{\partial x}(x) \\
&\quad + \frac{1}{2} A^\text{off}(x)EA^\text{off}(x) - \frac{1}{2} A(x)EA^\text{off}(x) - \frac{1}{2} A^\text{off}(x)EA(x), \\
C_1(x) &= C(x) + 3 \frac{\partial^2 A^\text{off}}{\partial x^2}(x) \\
&\quad + \frac{3i}{2} A(x)E \frac{\partial A^\text{off}}{\partial x}(x) - i \frac{\partial A^\text{off}}{\partial x}(x)E \frac{\partial A}{\partial x}(x) \\
&\quad - \frac{3i}{2} A^\text{off}(x)E \frac{\partial A^\text{off}}{\partial x}(x) - i \frac{\partial A^\text{off}}{\partial x}(x)E A^\text{off}(x) \\
&\quad + \frac{1}{4} A(x)EA^\text{off}(x)E A^\text{off}(x) + \frac{1}{4} A^\text{off}(x)EA(x)E A^\text{off}(x) \\
&\quad - \frac{1}{4} A^\text{off}(x)EA^\text{off}(x)E A^\text{off}(x) \\
&\quad - \frac{1}{2} B(x)E A^\text{off}(x) - \frac{1}{2} A^\text{off}(x)EB(x).
\end{align*}
\]

**Step 2:** diagonalization of \( iB_1(x)D_x^2 \). Note that \( EB^\text{off}(x) + B^\text{off}(x)E = 0 \). We define a pseudodifferential operator \( \tilde{\Lambda}_2 \) of order \(-2\) by
\[
\sigma(\tilde{\Lambda}_2)(x, \xi) = \frac{1}{2} EB^\text{off}(x)E \frac{\varphi_r(\xi)}{\xi^2} = -\frac{1}{2} B^\text{off}(x)E \frac{\varphi_r(\xi)}{\xi^2}.
\]
Set \( \Lambda_2 = I + \tilde{\Lambda}_2 \). Then \( \sigma(\tilde{\Lambda}_2)(x, \xi) \in S^{-2}(\mathbb{T}; M(2)) \) and \( \sigma(\Lambda_2)(x, \xi) \in S^0(\mathbb{T}; M(2)) \). Note that \( \|\tilde{\Lambda}_2\| = O(1/r^2) \). In the same way as Step 1, \( \Lambda_2 \) is invertible on \( \mathcal{L}(L^2(\mathbb{T}; \mathbb{C}^2)) \), and the inverse is given by \( \Lambda_2^{-1} = I - \tilde{\Lambda}_2 + \tilde{\Lambda}_2^2 \Lambda_2^{-1} \). Set \( L_2 = \Lambda_2 L_1 \Lambda_2^{-1} \) and \( P_2 = \Lambda_2 P_1 \Lambda_2^{-1} \) for short. Then we have
\[
\Lambda_2 L_1 = L_2 \Lambda_2 = \left\{ I \frac{\partial}{\partial t} + iP_2 \right\} \Lambda_2.
\]
We compute \( P_2 = \Lambda_2 P_1 \Lambda_2^{-1} = \Lambda_2 \Lambda_1 P \Lambda_1^{-1} \Lambda_2^{-1} \). By using \( \Lambda_2^{-1} = I - \tilde{\Lambda}_2 + \tilde{\Lambda}_2^2 \Lambda_2^{-1} \), we have
\[
P_2 = (I + \tilde{\Lambda}_2) P_1 (I - \tilde{\Lambda}_2 + \tilde{\Lambda}_2^2 \Lambda_2^{-1}) \\
\equiv P_1 - P_1 \tilde{\Lambda}_2 + \tilde{\Lambda}_2 P_1 \\
\equiv P_1 + (-ED_x^4 \tilde{\Lambda}_2 + \tilde{\Lambda}_2 ED_x^4) + (-A^\text{diag}(x)D_x^3 \tilde{\Lambda}_2 + \tilde{\Lambda}_2 A^\text{diag}(x)D_x^3) (29)
\]
modulo $\mathcal{L}(L^2(\mathbb{T}; \mathbb{C}^2))$. Since $E^2 = 1$, $EB^{\text{off}}(x) + B^{\text{off}}(x)E = 0$ and $\sigma(\tilde{\Lambda}_2)(x, \xi) \in S^{-2}(\mathbb{T}; M(2))$, the symbols of the second and third terms of the right hand side of (29) are

$$
\sigma(-ED_2^4 \tilde{\Lambda}_2 + \tilde{\Lambda}_2 ED_2^4)(x, \xi) \\
= \sum_{k=0}^{\infty} \frac{(-i)^k}{k!} \left\{ \left\{ \frac{\partial^k}{\partial \xi^k} E^{\xi^4} \right\} \left\{ \frac{1}{2} \frac{\partial}{\partial x^k} E^{\bar{B}^{\text{off}}}(x) \frac{\varphi_r(\xi)}{\xi} \right\} + \left\{ \frac{1}{2} \frac{\partial}{\partial x^k} B^{\text{off}}(x) \frac{\varphi_r(\xi)}{\xi^3} \right\} \right\}
$$

$$
\equiv -B^{\text{off}}(x)\xi^2 + 2i \frac{\partial B^{\text{off}}(x)}{\partial x}(\xi),
$$

$$
\sigma(-A^{\text{diag}}(x)D_2^3 \tilde{\Lambda}_2 + \tilde{\Lambda}_2 A^{\text{diag}}(x)D_2^3)(x, \xi)
\equiv \left\{ \frac{1}{2} A^{\text{diag}}(x) E^{B^{\text{off}}}(x) - \frac{1}{2} B^{\text{off}}(x) E A^{\text{diag}}(x) \right\} \xi,
$$

modulo $S^0(\mathbb{T}; M(2))$. Substituting these into (29), we obtain

$$P_2 \equiv ED_2^2 + A^{\text{diag}}(x)D_2^3 + B_1^{\text{diag}}(x)D_2^2 + C_2(x)D_x \mod \mathcal{L}(L^2(\mathbb{T}; \mathbb{C}^2)), \quad (30)
$$

$$C_2(x) = C_1(x) + 2i \frac{\partial B_1^{\text{off}}(x)}{\partial x} - \frac{1}{2} A^{\text{diag}}(x) E^{B^{\text{off}}}(x) - \frac{1}{2} B^{\text{off}}(x) E A^{\text{diag}}(x). \quad (31)
$$

**Step 3:** diagonalization of $iC_2(x)D_x$. Note that $EC_2^{\text{off}}(x) + C_2^{\text{off}}(x)E = 0$. We define a pseudodifferential operator $\tilde{\Lambda}_3$ of order $-3$ by

$$
\sigma(\tilde{\Lambda}_3)(x, \xi) = \frac{1}{2} EC_2^{\text{off}}(x) \frac{\varphi_r(\xi)}{\xi^3} = -\frac{1}{2} C_2^{\text{off}}(x) E \frac{\varphi_r(\xi)}{\xi^3}.
$$

Set $\Lambda_3 = I + \tilde{\Lambda}_3$. Then $\sigma(\tilde{\Lambda}_3)(x, \xi) \in S^{-3}(\mathbb{T}; M(2))$ and $\sigma(\Lambda_3)(x, \xi) \in S^0(\mathbb{T}; M(2))$. Note that $\|\tilde{\Lambda}_3\| = O(1/r^3)$. In the same way as Step 1, $\Lambda_3$ is invertible on $\mathcal{L}(L^2(\mathbb{T}; \mathbb{C}^2))$, and the inverse is given by $\Lambda_3^{-1} = I - \tilde{\Lambda}_3 + \tilde{\Lambda}_3^2 \Lambda_3^{-1}$. Set $L_3 = \Lambda_3 L_2 \Lambda_3^{-1}$ and $P_3 = \Lambda_3 P_2 \Lambda_3^{-1}$ for short. Then we have

$$
\Lambda_3 L_2 = L_2 \Lambda_3 = \left\{ I \frac{\partial}{\partial t} + iP_2 \right\} \Lambda_3.
$$

We compute $P_3 = \Lambda_3 P_2 \Lambda_3^{-1} = \Lambda_3 \Lambda_2 \Lambda_1 P \Lambda_1^{-1} \Lambda_2^{-1} \Lambda_3^{-1}$. By using $\Lambda_3^{-1} = I - \tilde{\Lambda}_3 + \tilde{\Lambda}_3^2 \Lambda_3^{-1}$, we have

$$P_3 = (I + \tilde{\Lambda}_3) P_2 (I - \tilde{\Lambda}_3 + \tilde{\Lambda}_3^2 \Lambda_3^{-1})
\equiv P_2 - P_2 \tilde{\Lambda}_3 + \tilde{\Lambda}_3 P_2
\equiv P_2 + (-ED_2^4 \tilde{\Lambda}_3 + \tilde{\Lambda}_3 ED_2^4) \mod \mathcal{L}(L^2(\mathbb{T}; \mathbb{C}^2)). \quad (32)
$$

Since $E^2 = 1$, $EC_2^{\text{off}}(x) + C_2^{\text{off}}(x)E = 0$ and $\sigma(\tilde{\Lambda}_3)(x, \xi) \in S^{-3}(\mathbb{T}; M(2))$, the symbol of the second term of the right hand side of (32) is

$$
\sigma(-ED_2^4 \tilde{\Lambda}_3 + \tilde{\Lambda}_3 ED_2^4)(x, \xi) \equiv -E \xi^4 \cdot \frac{1}{2} EC_2^{\text{off}}(x) \frac{\varphi_r(\xi)}{\xi^3} - \frac{1}{2} C_2^{\text{off}}(x) \frac{\varphi_r(\xi)}{\xi^3} E \cdot E \xi^4
\equiv -C_2^{\text{off}}(x) \xi \mod S^0(\mathbb{T}; M(2)).
$$

Substituting this into (32), we obtain

$$P_3 \equiv ED_2^4 + A^{\text{diag}}(x)D_2^3 + B_1^{\text{diag}}(x)D_2^2 + C_2^{\text{diag}}(x)D_x \mod \mathcal{L}(L^2(\mathbb{T}; \mathbb{C}^2)), \quad (33)
$$

This completes our diagonalization of $L$. 
**Step 4.** We shall complete the proof of Theorem 1. Set $\Lambda = \Lambda_3 \Lambda_2 \Lambda_1$ for short. Since $\Lambda$ is an invertible bounded linear operators on $L^2(T; \mathbb{C}^2)$ and $\Lambda L \equiv L_3 \Lambda \mod \mathcal{L}(L^2(T; \mathbb{C}^2))$, the $L^2$-well-posedness of (1)-(2) is equivalent to that of the initial value problem of the form

$$L_3 \vec{u} = \vec{f}(t, x), \quad \vec{u}(0, x) = \vec{\phi}(x).$$

The system $L_3 \vec{u} = \vec{f}(t, x)$ is exactly a pair of two single equations, and not a system essentially. Thus the $L^2$-well-posedness of (34) is reduced to Theorem 2. It suffices to check the conditions (13), (14) and (15). By using (33), (27), (28) and (31), we can obtain the concrete form of $L_3$. The first row of $L_3$ is

$$\frac{\partial}{\partial t} + iD^4_x + a_{11}(x)D^3_x + b_{1,11}(x)D^2_x + c_{2,11}(x)D_x,$$

$$b_{1,11}(x) = b_{11}(x) + \frac{1}{2} a_{12}(x)a_{21}(x),$$

$$c_{2,11}(x) = c_{11}(x) + \frac{i}{2} \{a_{12}(x)a_{21}(x)\}_x + \frac{i}{2} a'_{12}(x)a_{21}(x) - \frac{1}{4} a_{12}(x)a_{21}(x)\{a_{11}(x) - a_{22}(x)\} + \frac{1}{2} \{a_{12}(x)b_{21}(x) + a_{21}(x)b_{12}(x)\},$$

and the second row of $L_3$ is

$$\frac{\partial}{\partial t} - iD^4_x + a_{22}(x)D^3_x + b_{1,22}(x)D^2_x + c_{2,22}(x)D_x,$$

$$b_{1,22}(x) = b_{22}(x) - \frac{1}{2} a_{12}(x)a_{21}(x),$$

$$c_{2,22}(x) = c_{22}(x) - \frac{i}{2} \{a_{12}(x)a_{21}(x)\}_x + \frac{i}{2} a_{12}(x)a'_{21}(x) + \frac{1}{4} a_{12}(x)a_{21}(x)\{a_{11}(x) - a_{22}(x)\} - \frac{1}{2} \{a_{12}(x)b_{21}(x) + a_{21}(x)b_{12}(x)\}.$$

Checking the conditions (13), (14) and (15) for the above, we can obtain (6), (7), (8), (9), (10) and (11). We omit the detail. \[\square\]

### 3. Proof of Theorem 3

In this section we prove Theorem 3. Firstly we prove it by using Theorem 1. Secondly we give the direct proof of its sufficiency of $L^2$-well-posedness. We believe that this will be helpful for studying the initial value problem for (3).

**Proof of Theorem 3.** Let $\vec{w}$ be a solution to (4)-(5). Let $M$ be a $2 \times 2$ matrix defined by

$$M = \begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix}.$$

Then

$$M^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}.$$

Set $\vec{U} = M\vec{w}$, $\vec{F} = M\vec{h}$, $\vec{U}_0 = M\vec{w}_0$ and $\mathcal{L}_1 = M\mathcal{L}M^{-1}$. Then we have

$$\mathcal{L}_1 \vec{U} = \vec{F}, \quad \vec{U}(0, x) = \vec{U}_0(x).$$

(35)
The $L^2$-well-posedness of (4)-(5) is equivalent to that of the initial value problem for (35). We shall obtain the concrete form of $\mathcal{L}_1$. Simple computations give
\[
\mathcal{L}_1 = M \left\{ I \frac{\partial}{\partial t} + J \frac{\partial^4}{\partial x^4} + \beta(x) \frac{\partial^2}{\partial x^2} + \gamma(x) \frac{\partial}{\partial x} \right\} M^{-1} \\
= M \left\{ I \frac{\partial}{\partial t} + JD_x^4 - \beta(x) D_x^2 + i\gamma(x) D_x \right\} M^{-1} \\
= I \frac{\partial}{\partial t} + MJM^{-1} D_x^4 - M\beta(x)M^{-1} D_x^2 + iM\gamma(x)M^{-1} D_x \\
= I \frac{\partial}{\partial t} + i(-iJM^{-1}) D_x^4 + i(iM\beta)M^{-1} D_x^2 + iM\gamma(x)M^{-1} D_x \\
= I \frac{\partial}{\partial t} + iED_x^4 + i\tilde{B}(x) D_x^2 + i\tilde{C}(x) D_x,
\]
where
\[
\tilde{B}(x) = \frac{1}{2} \begin{bmatrix} b_{11}(x) & b_{12}(x) \\ b_{21}(x) & b_{22}(x) \end{bmatrix}, \quad \tilde{C}(x) = \frac{1}{2} \begin{bmatrix} c_{11}(x) & c_{12}(x) \\ c_{21}(x) & c_{22}(x) \end{bmatrix},
\]
\[
\tilde{b}_{11}(x) = \{\beta_{12}(x) - \beta_{21}(x)\} + i\{\beta_{11}(x) + \beta_{22}(x)\}, \\
\tilde{b}_{12}(x) = -\{\beta_{12}(x) + \beta_{21}(x)\} + i\{\beta_{11}(x) - \beta_{22}(x)\}, \\
\tilde{b}_{21}(x) = \{\beta_{12}(x) + \beta_{21}(x)\} + i\{\beta_{11}(x) - \beta_{22}(x)\}, \\
\tilde{b}_{22}(x) = -\{\beta_{12}(x) - \beta_{21}(x)\} + i\{\beta_{11}(x) + \beta_{22}(x)\}, \\
\tilde{c}_{11}(x) = \{\gamma_{11}(x) + \gamma_{22}(x)\} - i\{\gamma_{12}(x) - \gamma_{21}(x)\}, \\
\tilde{c}_{12}(x) = \{\gamma_{11}(x) - \gamma_{22}(x)\} + i\{\gamma_{12}(x) + \gamma_{21}(x)\}, \\
\tilde{c}_{21}(x) = \{\gamma_{11}(x) - \gamma_{22}(x)\} - i\{\gamma_{12}(x) + \gamma_{21}(x)\}, \\
\tilde{c}_{22}(x) = \{\gamma_{11}(x) + \gamma_{22}(x)\} + i\{\gamma_{12}(x) - \gamma_{21}(x)\}.
\]
Hence, Theorem 1 implies that the initial value problem for (35) is $L^2$-well-posed if and only if both (16) and (17) hold. This completes the proof. \[\square\]

We give the direct proof of the sufficiency of Theorem 3. We begin with studying Fourier multipliers mapping real-valued functions to real-valued functions. Recall the requirements of the smooth function $\varphi_r(\xi): 0 \leq \varphi_r(\xi) \leq 1, \varphi_r(\xi) = 1$ for $|\xi| \geq r+1, \varphi_r(\xi) = 0$ for $|\xi| \leq r$, and $\varphi_r(\xi) = \varphi(-\xi)$. Note that the last one $\varphi_r(\xi) = \varphi(-\xi)$ is crucial here. Let $l$ be a nonnegative integer. We consider a Fourier multiplier $p_l(D_x)$ whose symbol is $p_l(\xi) = \varphi_r(\xi)/(i\xi)^l$. We make use of the following properties of $p_l(D_x)$.

Lemmma 4. Let $v \in \mathcal{S}(\mathbb{R})$. If $\text{Im} \, v(x) = 0$, then $\text{Im} \, p_l(D_x) v(x) = 0$.

**Proof.** Suppose that $v \in \mathcal{S}(\mathbb{R})$ and $\text{Im} \, v(x) = 0$. By using change of variable $\xi \mapsto \eta = -\xi$, we deduce that
\[
p_l(D_x)v(x) - p_l(D_x)v(x) \\
= \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{i(x-y)\xi} \frac{\varphi_r(\xi)}{(i\xi)^l} v(y)dyd\xi - \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{i(x-y)\xi} \frac{\varphi_r(\xi)}{(i\xi)^l} v(y)dyd\xi \\
= \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{i(x-y)\xi} \frac{\varphi_r(\xi)}{(i\xi)^l} v(y)dyd\xi - \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{i(x-y)(-\xi)} \frac{\varphi_r(\xi)}{(-i\xi)^l} v(y)dyd\xi \\
= \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{i(x-y)\xi} \frac{\varphi_r(\xi)}{(i\xi)^l} v(y)dyd\xi - \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{i(x-y)\eta} \frac{\varphi_r(-\eta)}{(i\eta)^l} v(y)dyd\xi
\]
we take a sufficiently large $r > 0$, then \( \Lambda_4 \) is invertible on \( L^2(\mathbb{T}; \mathbb{R}^2) \), and the inverse satisfies 
\[
\Lambda_4^{-1} = I + \tilde{\Lambda}_4 + \tilde{\Lambda}_4^2 + \tilde{\Lambda}_4^3 + \tilde{\Lambda}_4^4 \Lambda_4^{-1}.
\]
This completes the proof. □

Finally we prove the sufficiency of both (16) and (17) directly. We denote by \( L^2(\mathbb{T}; \mathbb{R}^2) \) the set of all \( \mathbb{R}^2 \)-valued square-integrable functions on \( \mathbb{T} \), and by \( C^\infty(\mathbb{T}; \mathbb{R}) \) the set of all real-valued smooth functions on \( \mathbb{T} \).

**Direct Proof of the Sufficiency in Theorem 3.** Suppose that both (16) and (17) hold. We introduce a gauge transform on \( L^2(\mathbb{T}; \mathbb{R}^2) \), and modify the operator \( \mathcal{L} \) so that \( \beta(x) \) and \( \gamma(x) \) become skew-symmetric and symmetric respectively. For simpler computations, we construct the gauge transform by a product of three pseudodifferential operators of order zero. For this reason we split the proof into three steps. Here we explain more detail of this strategy. Let \( X = [x_{jk}] \) be a \( 2 \times 2 \) real matrix. We split \( X \) into three parts:

\[
X = \frac{\text{tr}(X)}{2} I + \frac{1}{2} \{ X + J X J \} + \frac{1}{2} \{ X - J X J - \text{tr}(X) I \}. \]

Note that both \( (\text{tr}(X)) I/2 \) and \(-\text{operator norm}(JX)J/2\) commute with \( J \), and that

\[
\frac{1}{2} \{ X + J X J \} = J \cdot \frac{1}{2} \{ X + J X J \} - \frac{1}{2} \{ X + J X J \} \cdot J,
\]

\[
\frac{1}{2} \{ X + J X J \} = \frac{\text{tr}(X)}{2} I + \frac{1}{2} \{ J X - X J \} J,
\]

\[
\frac{1}{2} \{ X - J X J - \text{tr}(X) I \} = -\frac{\text{tr}(JX)}{2} J.
\]

We eliminate

\[
\frac{1}{2} \{ \beta(x) + J \beta(x) \} = \frac{\text{tr}(\beta(x))}{2} I + \frac{1}{2} \{ J \beta(x) - \beta(x) J \} J
\]

from \( \mathcal{L} \) in first two steps. In the third step we eliminate

\[
\frac{1}{2} \{ \gamma(x) - J \gamma(x) J \} = -\frac{\text{tr}(J \gamma(x))}{2} J
\]

from \( \mathcal{L} \).

**Step 1:** We eliminate \( \text{tr}(\beta(x)) I/2 \). Set

\[
\Psi_4 (x) = \int_0^x \text{tr}(\beta(y)) dy.
\]

We deduce that \( \Psi_4 \in C^\infty(\mathbb{T}; \mathbb{R}) \) since \( \beta_{jk} \) is a real-valued smooth functions on \( \mathbb{T} \) and (16). Set

\[
\sigma(\tilde{\Lambda}_4)(x, \xi) = \frac{1}{8} \Psi_4(x) \frac{\varphi_r(\xi)}{i\xi} J, \quad \Lambda_4 = I - \tilde{\Lambda}_4.
\]

It follows that \( \sigma(\tilde{\Lambda}_4)(x, \xi) \in S^{-1}(\mathbb{T}; M(2)) \), \( \sigma(\Lambda_4)(x, \xi) \in S^0(\mathbb{T}; M(2)) \) and \( \| \tilde{\Lambda}_4 \| = O(1/r) \). If we take a sufficiently large \( r > 0 \), then \( \Lambda_4 \) is invertible on \( L^2(\mathbb{T}; \mathbb{R}^2) \), and the inverse satisfies

\[
\Lambda_4^{-1} = I + \tilde{\Lambda}_4 + \tilde{\Lambda}_4^2 + \tilde{\Lambda}_4^3 + \tilde{\Lambda}_4^4 \Lambda_4^{-1}.
\]
Set $\mathcal{L}_4 = \Lambda_4 \mathcal{L} \Lambda_4^{-1}$. Then $\Lambda_4 \mathcal{L} = \mathcal{L}_4 \Lambda_4$. We compute $\mathcal{L}_4$ in detail. Since $\tilde{\Lambda}_4$ is a pseudodifferential operator of order $-1$, we deduce that

\[
\mathcal{L}_4 = \Lambda_4 \left\{ I \frac{\partial}{\partial t} + J \frac{\partial^4}{\partial x^4} + \beta(x) \frac{\partial^2}{\partial x^2} + \gamma(x) \frac{\partial}{\partial x} \right\} \Lambda_4^{-1}
\approx I \frac{\partial}{\partial t} + \gamma(x) \frac{\partial}{\partial x}
\quad + (I - \tilde{\Lambda}_4) \left\{ J \frac{\partial^4}{\partial x^4} + \beta(x) \frac{\partial^2}{\partial x^2} \right\} (I + \tilde{\Lambda}_4 + \tilde{\Lambda}_4^2 + \tilde{\Lambda}_4^3 + \tilde{\Lambda}_4^4) \nonumber
\approx I \frac{\partial}{\partial t} + J \frac{\partial^4}{\partial x^4} + \beta(x) \frac{\partial^2}{\partial x^2} + \gamma(x) \frac{\partial}{\partial x}
\quad + \left( J \frac{\partial^4}{\partial x^4} \tilde{\Lambda}_4 - \tilde{\Lambda}_4 J \frac{\partial^4}{\partial x^4} \right) (I + \tilde{\Lambda}_4 + \tilde{\Lambda}_4^2) + \left( \beta(x) \frac{\partial^2}{\partial x^2} \tilde{\Lambda}_4 - \tilde{\Lambda}_4\beta(x) \frac{\partial^2}{\partial x^2} \right)
\]

modulo $\mathcal{L}(L^2(T; \mathbb{R}^2))$. We see the last two terms in detail. Since $J^2 = -I$, we deduce that

\[
\sigma \left( J \frac{\partial^4}{\partial x^4} \tilde{\Lambda}_4 - \tilde{\Lambda}_4 J \frac{\partial^4}{\partial x^4} \right) (x, \xi)
\approx 2 \sum_{k=0}^{\infty} \frac{(-i)^k}{k!} \left\{ \frac{\partial^k}{\partial \xi^k} \right\} \left\{ \frac{\partial^k}{\partial x^k} \right\} \frac{1}{8} \Psi_4(x) \left( \frac{\varphi_r(x)}{i \xi} \right) \right\} I
\]

\[
= -\sum_{k=1}^{\infty} \frac{(-i)^k}{k!} \left\{ \frac{\partial^k}{\partial \xi^k} \right\} \left\{ \frac{\partial^k}{\partial x^k} \right\} \frac{1}{8} \Psi_4(x) \left( \frac{\varphi_r(x)}{i \xi} \right) \right\} J
\]

\[
= -\frac{1}{2} \Psi_4'(x) (i \xi)^2 I - \frac{3}{4} \Psi_4''(x) (i \xi) J,
\]

\[
\sigma \left( J \frac{\partial^4}{\partial x^4} \tilde{\Lambda}_4 - \tilde{\Lambda}_4 J \frac{\partial^4}{\partial x^4} \right) (x, \xi)
\approx \left\{ \frac{1}{2} \Psi_4'(x) (i \xi)^2 \right\} \left\{ \frac{1}{8} \Psi_4(x) \left( \frac{\varphi_r(x)}{i \xi} \right) \right\} J
\]

\[
= \frac{1}{32} \left\{ \Psi_4(x)^2 \right\}_x (i \xi) J,
\]

\[
\sigma \left( \beta(x) \frac{\partial^2}{\partial x^2} \tilde{\Lambda}_4 - \tilde{\Lambda}_4 \beta(x) \frac{\partial^2}{\partial x^2} \right) (x, \xi)
\approx \beta(x) (i \xi)^2 \cdot \frac{1}{8} \Psi_4(x) \left( \frac{\varphi_r(x)}{i \xi} \right) J - \frac{1}{8} \Psi_4(x) \left( \frac{\varphi_r(x)}{i \xi} \right) J \cdot \beta(x) (i \xi)^2
\approx \frac{1}{8} \Psi_4(x) \left\{ \beta(x) J - J \beta(x) \right\} (i \xi),
\]
modulo $S^0(\mathbb{T}; M(2))$. Substituting (37), (38) and (39) into (36), we obtain

$$
\mathcal{L}_4 \equiv I \frac{\partial}{\partial t} + J \frac{\partial^4}{\partial x^4} + \beta_4(x) \frac{\partial^2}{\partial x^2} + \gamma_4(x) \frac{\partial}{\partial x} \quad \mod \mathcal{L}(L^2(\mathbb{T}; \mathbb{R}^2)),
$$

(40)

$$
\beta_4(x) = \beta(x) - \frac{1}{2} \Psi_4'(x) I = \frac{1}{2} \left[ \beta_{11}(x) - \beta_{22}(x) \begin{array}{c} 2 \beta_{12}(x) \\ -\beta_{11}(x) + \beta_{22}(x) \end{array} \right],
$$

(41)

$$
\gamma_4(x) = \gamma(x) - \frac{3}{4} \Psi_4'(x) I - \frac{1}{12} \left( \Psi_4(x)^2 \right)_x J + \frac{1}{8} \Psi_4(x) \{ \beta(x) J - J \beta(x) \}.
$$

(42)

**Step 2:** We eliminate $\{ J \beta(x) - \beta(x) J \} / 2$. Set

$$
\sigma(\tilde{\Lambda}_5)(x, \xi) = \frac{1}{2} \beta_4(x) J p_{2}(\xi) = \frac{1}{2} \beta_4(x) J \frac{\varphi_{r}(\xi)}{(i \xi)^2}, \quad \Lambda_5 = I + \tilde{\Lambda}_5.
$$

It follows that $\sigma(\tilde{\Lambda}_5)(x, \xi) \in S^{-2}(\mathbb{T}; M(2))$, $\sigma(\Lambda_5)(x, \xi) \in S^0(\mathbb{T}; M(2))$ and $\| \tilde{\Lambda}_5 \| = O(1/r^2)$. If we take a sufficiently large $r > 0$, then $\Lambda_5$ is invertible on $\mathcal{L}(L^2(\mathbb{T}; \mathbb{R}^2))$, and the inverse satisfies $\Lambda_5^{-1} = I - \tilde{\Lambda}_5 + \tilde{\Lambda}_5 \Lambda_5^{-1}$. Set $\mathcal{L}_5 = \Lambda_5 \mathcal{L}_4 \Lambda_5^{-1} = \Lambda_5 \mathcal{L}_4 \Lambda_4^{-1} \Lambda_5^{-1}$. Then $\Lambda_5 \mathcal{L}_4 = \mathcal{L}_5 \Lambda_5$. We compute $\mathcal{L}_5$ in detail. Since $\tilde{\Lambda}_5$ is a pseudodifferential operator of order $-2$, we deduce that

$$
\mathcal{L}_5 = \Lambda_5 \left\{ I \frac{\partial}{\partial t} + J \frac{\partial^4}{\partial x^4} + \beta_4(x) \frac{\partial^2}{\partial x^2} + \gamma_4(x) \frac{\partial}{\partial x} \right\} \Lambda_5^{-1}
$$

$$
\equiv I \frac{\partial}{\partial t} + J \frac{\partial^4}{\partial x^4} + \beta_4(x) \frac{\partial^2}{\partial x^2} + \gamma_4(x) \frac{\partial}{\partial x} + (I + \tilde{\Lambda}_5) J \frac{\partial^4}{\partial x^4} (I - \tilde{\Lambda}_5)
$$

$$
\equiv I \frac{\partial}{\partial t} + J \frac{\partial^4}{\partial x^4} + \beta_4(x) \frac{\partial^2}{\partial x^2} + \gamma_4(x) \frac{\partial}{\partial x} + \left( \Lambda_5 J \frac{\partial^4}{\partial x^4} - J \frac{\partial^4}{\partial x^4} \Lambda_5 \right)
$$

(43)

modulo $\mathcal{L}(L^2(\mathbb{T}; \mathbb{R}^2))$. We see the last term in detail. Since $\beta_4(x) J^2 = - \beta(x)$ and $J \beta_4(x) J = t \beta_4(x)$, we deduce that

$$
\sigma \left( \tilde{\Lambda}_5 \frac{\partial^4}{\partial x^4} - J \frac{\partial^4}{\partial x^4} \tilde{\Lambda}_5 \right) (x, \xi)
$$

$$
\equiv \sum_{k=0}^{1} \frac{(-i)^k}{k!} \left\{ \frac{\partial^k}{\partial \xi^k} \frac{1}{2} \beta_4(x) J p_2(\xi) \right\} \left\{ \frac{\partial^k}{\partial x^k} J (i \xi)^4 \right\}
$$

$$
- \left\{ \frac{\partial^k}{\partial \xi^k} J (i \xi)^4 \right\} \left\{ \frac{\partial^k}{\partial x^k} \frac{1}{2} \beta_4(x) J p_2(\xi) \right\}
$$

$$
= - \frac{1}{2} \sum_{k=0}^{1} \frac{(-i)^k}{k!} \left\{ \beta_4(x) \frac{\partial^k \varphi_r(\xi)}{(i \xi)^2} \right\} \left\{ \frac{\partial^k}{\partial x^k} (i \xi)^4 \right\}
$$

$$
+ \left\{ \frac{\partial^k}{\partial \xi^k} (i \xi)^4 \right\} \left\{ \frac{\partial^k}{\partial x^k} t \beta_4(x) \frac{\varphi_r(\xi)}{(i \xi)^2} \right\}
$$

$$
\equiv - \frac{1}{2} \left\{ \beta_4(x) + t \beta_4(x) \right\} (i \xi)^2 - 2 \frac{\partial^4 \beta_4}{\partial x^4}(x) (i \xi) \mod S^0(\mathbb{T}; M(2)).
$$

(44)

Substitute (44) into (43). By using (41) and (42), we deduce

$$
\mathcal{L}_5 \equiv I \frac{\partial}{\partial t} + J \frac{\partial^4}{\partial x^4} + \frac{1}{2} (\beta_4(x) - t \beta_4(x)) \frac{\partial^2}{\partial x^2} + \left\{ \gamma_4(x) - \frac{2}{2} \frac{\partial^4 \beta_4}{\partial x^4} \right\} \frac{\partial}{\partial x}
$$

$$
= I \frac{\partial}{\partial t} + J \frac{\partial^4}{\partial x^4} - \frac{\beta_{12}(x) - \beta_{21}(x)}{2} J \frac{\partial^2}{\partial x^2} + \left\{ \gamma_4(x) - \frac{2}{2} \frac{\partial^4 \beta_4}{\partial x^4} \right\} \frac{\partial}{\partial x}
$$
\[
\gamma_5 (x) = \gamma_4 (x) - 2 \frac{\partial \beta_4}{\partial x} (x) + \frac{\{ \text{tr} (J \beta (x)) \}}{2} x J
\]
\[
= \gamma (x) + \frac{1}{8} \Psi_4 (x) \{ \beta (x) J - J \beta (x) \} - \frac{3}{4} \Psi_4'' (x) I
\]
\[
- \frac{1}{32} \{ \Psi_4 (x)^2 \} x J - 2 \frac{\partial \beta_4}{\partial x} (x) + \frac{\{ \text{tr} (J \beta (x)) \}}{2} x J.
\]

Step 3: Skew-symmetric part of \( \gamma_5 (x) \). Set
\[
\gamma^\text{sym}_5 (x) = \frac{1}{2} \{ \gamma_5 (x) + t \gamma_5 (x) \}, \quad \frac{1}{2} \{ \gamma_5 (x) - t \gamma_5 (x) \} = \mu (x) J
\]
for short. Then \( \gamma_5 (x) = \gamma^\text{sym}_5 (x) + \mu (x) J \). Simple computations with (41) yield
\[
\beta (x) J - J \beta (x) = \left[ \begin{array}{cc}
\beta_{12} (x) + \beta_{21} (x) & -\beta_{11} (x) + \beta_{22} (x) \\
-\beta_{11} (x) + \beta_{22} (x) & -\beta_{12} (x) - \beta_{21} (x)
\end{array} \right],
\]
\[
\frac{1}{2} \left\{ -2 \frac{\partial \beta_4}{\partial x} (x) + 2 \frac{\partial \beta_4}{\partial x} (x) \right\} = \{ \beta_4 (x) - t \beta_4 (x) \}_x = -\{ \text{tr} (J \beta (x)) \}_x J.
\]
Substituting these into (46), we have
\[
\mu (x) = -\frac{\text{tr} (J \gamma (x))}{2} - \left\{ \frac{\text{tr} (J \beta (x))}{2} + \frac{\Psi_4 (x)^2}{32} \right\}_x.
\]
Set
\[
\Psi_6 (x) = -\frac{1}{2} \int_0^x \text{tr} (J \gamma (y)) dy - \frac{\text{tr} (J \beta (x))}{2} - \frac{\Psi_4 (x)^2}{32}.
\]
We deduce that \( \Psi_6 \in C^\infty (T; \mathbb{R}) \) since (17), and that \( \Psi_6 (x) = \mu (x) \).

Here we introduce a system of pseudodifferential operators \( \Lambda_6 = I + \tilde{\Lambda}_6 \) defined by
\[
\sigma (\tilde{\Lambda}_6) (x, \xi) = \frac{1}{4} \Psi_6 (x) I p_2 (\xi) = \frac{1}{4} \Psi_6 (x) I \frac{\varphi_r (\xi)}{(i \xi)^2}.
\]
It follows that \( \sigma (\tilde{\Lambda}_6) (x, \xi) \in S^{-2} (T; M (2)), \sigma (\Lambda_6) (x, \xi) \in S^0 (T; M (2)) \) and \( \| \tilde{\Lambda}_6 \| = O (1/r^2) \). If we take a sufficiently large \( r > 0 \), then \( \Lambda_6 \) is invertible on \( \mathcal{L} (L^2 (T; \mathbb{R}^2)) \), and the inverse satisfies \( \Lambda_6^{-1} = I - \tilde{\Lambda}_6 + \tilde{\Lambda}_6^2 \Lambda_6^{-1} \). Set \( \mathcal{L}_6 = \Lambda_6 \mathcal{L}_5 \Lambda_6^{-1} = \Lambda_6 \Lambda_5 \mathcal{L}_6 \Lambda_5^{-1} \Lambda_6^{-1}. \) Then \( \mathcal{L}_6 \mathcal{L}_5 = \mathcal{L}_6 \Lambda_6 \). We compute \( \mathcal{L}_6 \) in detail. Since \( \Lambda_6 \) is a pseudodifferential operator of order \(-2\), we deduce that
\[
\mathcal{L}_6 = \Lambda_6 \left\{ I \frac{\partial}{\partial t} + J \frac{\partial^4}{\partial x^4} - \frac{\partial}{\partial x} \frac{\text{tr} (J \beta (x))}{2} J \frac{\partial}{\partial x} + \{ \gamma_5^\text{sym} (x) + \mu (x) J \} \frac{\partial}{\partial x} \right\} \Lambda_6^{-1}
\]
\[
= I \frac{\partial}{\partial t} - \frac{\partial}{\partial x} \frac{\text{tr} (J \beta (x))}{2} J \frac{\partial}{\partial x}
\]
\[
+ \{ \gamma_5^\text{sym} (x) + \mu (x) J \} \frac{\partial}{\partial x} + (I + \tilde{\Lambda}_6) \frac{\partial^4}{\partial x^4} (I - \tilde{\Lambda}_6)
\]
\[
= I \frac{\partial}{\partial t} + J \frac{\partial^4}{\partial x^4} - \frac{\partial}{\partial x} \frac{\text{tr} (J \beta (x))}{2} J \frac{\partial}{\partial x}.
\]
Consider the initial value problem of the form
denote the set of all bounded continuous
and all the components in
Substituting (48) into (47), we obtain
modulo \( \mathcal{L}(L^2(\mathbb{T}; \mathbb{R}^2)) \). We see the last term in detail. By using \( \Psi'_0(x) = \mu(x) \), we deduce that

\[
\frac{\partial^4}{\partial x^4} \frac{\partial^4}{\partial x^4} (x, \xi) \equiv \sum_{k=0}^{1} \frac{(-i)^k}{k!} \left\{ \left\{ \frac{\partial^{k_1} I (i \xi)^{k_1}}{4} \right\} \left\{ \frac{\partial^{k_2} \Psi_0(x) I P_2(\xi)}{4} \right\} \right\} \left\{ \frac{\partial^{k_2} \Psi_0(x) I P_2(\xi)}{4} \right\} J \left\{ \Psi'_0(x) \phi_r(\xi) \right\} J \equiv -\mu(x)(i \xi) J \mod S^0(\mathbb{T}; M(2)).
\]

Substituting (48) into (47), we obtain

\[
\mathcal{L}_6 \equiv I \frac{\partial}{\partial t} + J \frac{\partial^4}{\partial x^4} - \frac{\partial}{\partial x} \frac{\partial (J \beta(x))}{2} J \frac{\partial}{\partial x} + \gamma_5^{\text{sym}}(x) \frac{\partial}{\partial x}
\]

modulo \( \mathcal{L}(L^2(\mathbb{T}; \mathbb{R}^2)) \). It is easy to check that the initial value problem for \( \mathcal{L}_6 \) is \( L^2 \)-well-posed. Hence (41)-(53) is also \( L^2 \)-well-posed since \( \Lambda_6 \Lambda_5 \Lambda_4 \) is automorphic on \( L^2(\mathbb{T}; \mathbb{R}^2) \). We omit the detail. This completes the proof.

Finally, we remark that the sufficiency of Theorem 5 holds also in case that all the coefficients depend on time variable \( t \in \mathbb{R} \). To state this precisely, we here introduce some function spaces. We denote the set of all bounded continuous \( C^\infty(\mathbb{T}; \mathbb{R}) \)-valued functions on \( \mathbb{R} \) by \( C_b(\mathbb{R}; C^\infty(\mathbb{T}; \mathbb{R})) \). Set

\[
C_b^1(\mathbb{R}; C^\infty(\mathbb{T}; \mathbb{R})) = \left\{ a(t, x) \in C^1(\mathbb{R}; C^\infty(\mathbb{T}; \mathbb{R})) \mid a(t, x), a_t(t, x) \in C_b(\mathbb{R}; C^\infty(\mathbb{T}; \mathbb{R})) \right\}.
\]

Consider the initial value problem of the form

\[
\mathcal{P} \bar{w} = \bar{h}(t, x) \quad \text{in} \quad \mathbb{R} \times \mathbb{T},
\]

\[
\bar{w}(0, x) = \bar{w}_0(x) \quad \text{in} \quad \mathbb{T},
\]

where

\[
\mathcal{P} = I \frac{\partial}{\partial t} + J \frac{\partial^4}{\partial x^4} + \bar{\beta}(t, x) \frac{\partial^2}{\partial x^2} + \bar{\gamma}(t, x) \frac{\partial}{\partial x},
\]

and all the components in \( \bar{\beta}(t, x) = [\bar{\beta}_{jk}(t, x)]_{j,k=1,2} \) and \( \bar{\gamma}(t, x) = [\bar{\gamma}_{jk}(t, x)]_{j,k=1,2} \) are supposed to belong to \( C_b^1(\mathbb{R}; C^\infty(\mathbb{T}; \mathbb{R})) \). In the same way as the direct proof of the sufficiency in Theorem 5, we can prove the following.

**Theorem 5.** If we assume that

\[
\text{Im} \int_0^{2\pi} \text{tr}(\bar{\beta}(t, x)) dx = 0,
\]

(52)
of TN by $u$ linear ordinary differential equations with an independent variable $e$ and is the parallel transport of the unit normal vector $\tilde{N}$ along $e$ be the moving frame along $e$ and is the unit tangent vector of the closed curve $C(t) = \{ u(t, x) \mid x \in \mathbb{T} \}$ at $u(t, 0)$ for any fixed $t \in \mathbb{R}$. Let $e(t, x)$ be the parallel transport of $e_0(t)$ along $C(t)$. In other words, $e(t, x)$ is the unique solution to the initial value problem of the system of linear ordinary differential equations with an independent variable $x \in \mathbb{T}$ and a parameter $t \in \mathbb{R}$ of the form

$$\nabla_x e(t, \cdot) = 0 \quad \text{in} \quad \mathbb{T}, \quad e(t, 0) = e_0(t).$$

Since $\nabla^N \tilde{J} = 0$, $\tilde{J} e$ solves the initial value problem

$$\nabla_x \tilde{J} e(t, \cdot) = 0 \quad \text{in} \quad \mathbb{T}, \quad \tilde{J} e(t, 0) = \tilde{J} e_0(t),$$

and is the parallel transport of the unit normal vector $\tilde{J} e_0(t)$ of $C(t)$ at $u(t, 0)$. It is easy to see that the pair of $e$ and $\tilde{J} e$ is a moving frame along $u$. We remark that $e(t, x)$ is not necessarily $2\pi$-periodic in $x$ since the coefficients of the lower order terms $\Gamma^\alpha_{\beta\gamma}(u) \partial u^\beta / \partial x$ do not necessarily have $2\pi$-periodic primirve. In other words, roughly speaking,

$$\int_0^{2\pi} \Gamma^\alpha_{\beta\gamma}(u) \partial u^\beta / \partial x \, dx$$

for any $t \in \mathbb{R}$, then the initial value problem (50)-(51) is $L^2$-well-posed.

Indeed, if $\Lambda_7(t) = I + \tilde{\Lambda}_7(t)$ is invertible and $\sigma(\tilde{\Lambda}_7(t))$ is an $S^{-1}(T; M(2))$-valued function of class $C_b^1$, then

$$\Lambda_7(t) I \frac{\partial}{\partial t} \Lambda_7(t)^{-1} = I \frac{\partial}{\partial t} \tilde{\Lambda}_7(t)^{-1} = I \frac{\partial}{\partial t}$$

modulo a class of all $\mathcal{L}(L^2(T; \mathbb{R}^2))$-valued bounded continuous functions in $t \in \mathbb{R}$. This is the only difference between the sufficiency of Theorem 3 and that of Theorem 5. The other parts of the proof of Theorem 5 are exactly the same as that of Theorem 3. We omit the detail.

### 4. Dispersive Flows and Moving Frames

We turn our attention to the dispersive flow (3). In this section we derive a fourth-order dispersive system like (4) from the equation of the dispersive flow (3). We see that if the sectional curvature of $\nabla_7$ is a moving frame along $e$ and is the unit tangent vector of the closed curve $C(t) = \{ u(t, x) \mid x \in \mathbb{T} \}$ at $u(t, 0)$ for any fixed $t \in \mathbb{R}$. Let $e(t, x)$ be the parallel transport of $e_0(t)$ along $C(t)$. In other words, $e(t, x)$ is the unique solution to the initial value problem of the system of linear ordinary differential equations with an independent variable $x \in \mathbb{T}$ and a parameter $t \in \mathbb{R}$ of the form

$$\nabla_x e(t, \cdot) = 0 \quad \text{in} \quad \mathbb{T}, \quad e(t, 0) = e_0(t).$$

Since $\nabla^N \tilde{J} = 0$, $\tilde{J} e$ solves the initial value problem

$$\nabla_x \tilde{J} e(t, \cdot) = 0 \quad \text{in} \quad \mathbb{T}, \quad \tilde{J} e(t, 0) = \tilde{J} e_0(t),$$

and is the parallel transport of the unit normal vector $\tilde{J} e_0(t)$ of $C(t)$ at $u(t, 0)$. It is easy to see that the pair of $e$ and $\tilde{J} e$ is a moving frame along $u$. We remark that $e(t, x)$ is not necessarily $2\pi$-periodic in $x$ since the coefficients of the lower order terms $\Gamma^\alpha_{\beta\gamma}(u) \partial u^\beta / \partial x$ do not necessarily have $2\pi$-periodic primirve. In other words, roughly speaking,
does not necessarily vanish.

Let \( R \) be the Riemann curvature tensor of \((N, \tilde{J}, g)\). The sectional curvature at \( u \in N \) is denoted by \( K(u) \). Here we summarize properties of \( R \) used in our computations below.

**Lemma 6.** We have the following properties.

(i) \( R(e, e)e = R(\tilde{J}e, \tilde{J}e)e = 0 \).

(ii) \( \tilde{J}R(\cdot, \cdot)\tilde{J}e = R(\cdot, \cdot)\tilde{J}e \).

(iii) \( R(\tilde{J}e, e)e = -R(e, \tilde{J}e)e = K(u)\tilde{J}e, R(\tilde{J}e, e)e = -R(e, \tilde{J}e)e = -K(u)e \).

**Proof.** Let \( X, Y, Z, W \) be vector fields on \( N \). The claim (i) follows from a basic property of Riemann curvature tensor \( R(X, Y)Z + R(Y, X)Z = 0 \). The claim (ii) follows from the definition of \( \tilde{J} \) and \( \tilde{J} \) is an orthonormal basis of each tangent space \( T_uN \), we have

\[
R(\tilde{J}e, e)e = g(R(\tilde{J}e, e)e, e) + g(R(\tilde{J}e, e)e, \tilde{J}e)\tilde{J}e.
\]

On one hand, combining (i) and a basic property of Riemann curvature tensor

\[
g(R(X, Y)Z, W) = g(R(Z, W)X, Y),
\]

we have \( g(R(\tilde{J}e, e)e, e) = 0 \). On the other hand, the definition of the sectional curvature at \( u \in N \) is

\[
K(u) = \frac{g_u(R(X, Y)X, Y)}{g_u(X, X)g_u(Y, Y) - g_u(X, Y)^2} \quad \text{for} \quad X, Y \in T_uN \setminus \{0\},
\]

and we have \( g(R(\tilde{J}e, e)e, \tilde{J}e) = K(u) \). Hence we obtain \( R(\tilde{J}e, e)e = K(u)\tilde{J}e \). \( \square \)

Here we begin with the derivation of a system of partial differential equations. Let \( l \) be an integer not smaller than four. Set \( u_x = \xi \tilde{J}e + \eta \tilde{J}e \) and \( U = \nabla_x^l u_x = V e + W \tilde{J}e \). In what follows we denote different functions of \( u, u_x, \ldots, \nabla_x^l u \) by the same notation “OK”, and we set \( g(\cdot, \cdot) = g_{u(t, x)}(\cdot, \cdot) \) and \( \tilde{J} = \tilde{J}(u(t, x)) \) for short. Apply \( \nabla_x^l \) to (3). Then we have

\[
\nabla_t U = a \tilde{J}\nabla_x^l U + \tilde{J}\nabla_x^l U \\
+ \sum_{\alpha=0}^{l-1} \nabla_x^{l-1-\alpha} \{ R(u_t, u_x) \nabla_x^2 u_x \} \\
+ b \sum_{\alpha+\beta+\gamma=l+1} \frac{(l+1)!}{\alpha!\beta!\gamma!} g(\nabla_x^\beta u_x, \nabla_x^\gamma u_x) \tilde{J}\nabla_x^{\alpha+1} u_x \\
+ c \sum_{\alpha+\beta+\gamma=l+1} \frac{(l+1)!}{\alpha!\beta!\gamma!} g(\nabla_x^{\beta+1} u_x, \nabla_x^\gamma u_x) \tilde{J}\nabla_x^{\alpha} u_x.
\]

We split each term in the above equation into main part and OK part. It follows from the definition of the covariant derivative \( \nabla_t \) that

\[
\nabla_t U = V_t e + W_t \tilde{J}e + V\nabla_t e + W \tilde{J}\nabla_t e = V_t e + W_t \tilde{J}e + \text{OK}.
\]

Since \( \nabla_x e = \nabla_x \tilde{J}e = 0 \), we have

\[
a \tilde{J}\nabla_x^l U + \tilde{J}\nabla_x^l U = a \tilde{J}\nabla_x^l (Ve + W \tilde{J}e) + \tilde{J}\nabla_x^l (Ve + W \tilde{J}e) \\
= -a \nabla_x^{l+1} e + a V_x e + \tilde{J} \nabla_x^{l+1} e - W_x e + V_{xx} \tilde{J}e.
\]
We compute (54). A simple computation yields

\[ \sum_{\alpha=0}^{l-1} \nabla_x^{l-1-\alpha} \{ R(aJ\nabla_x^2 u_x, u_x)\nabla_x^\alpha u_x \} + \text{OK} \]

\[ = \sum_{\alpha=0}^{1} \nabla_x^{l-1-\alpha} \{ R(aJ\nabla_x^3 u_x, u_x)\nabla_x^\alpha u_x \} + \text{OK}. \quad (59) \]

Set

\[ \xi^{(a)}(t, x) = \frac{\partial^a x}{\partial x^a}(t, x), \quad \eta^{(a)}(t, x) = \frac{\partial^a \eta}{\partial x^a}(t, x), \quad \alpha = 0, 1, 2, \ldots \]

for short. We applying Lemma 6 to the right hand side of (59). We deduce that

\[ R(aJ\nabla_x^3 u_x, u_x)\nabla_x^\alpha u_x = R( aJ(\xi^{(3)} e + \eta^{(3)} \tilde{J} e), \xi e + \eta \tilde{J} e)(\xi^{(a)} e + \eta^{(a)} \tilde{J} e) \]

\[ = aR(\xi^{(3)} \tilde{J} e - \eta^{(3)} \tilde{J} e, \xi e + \eta \tilde{J} e)(\xi^{(a)} e + \eta^{(a)} \tilde{J} e) \]

\[ = a\{\xi^{(3)} + \eta^{(3)}\}R(J e, e)(\xi^{(a)} e + \eta^{(a)} \tilde{J} e) \]

\[ = \{ -aK(u)\xi^{(a)} \xi^{(3)} - aK(u)\eta^{(a)} \eta^{(3)} \} e \]

\[ + \{ aK(u)\xi^{(a)} \xi^{(3)} + aK(u)\eta^{(a)} \eta^{(3)} \} \tilde{J} e. \quad (60) \]

Substitute (60), into (59). We obtain

\[ \sum_{\alpha=0}^{l-1} \nabla_x^{l-1-\alpha} \{ R(aJ\nabla_x^3 u_x, u_x)\nabla_x^\alpha u_x \} + \text{OK} \]

\[ = \{ -aK(u)\xi V_{xx} - aK(u)\eta^2 W_{xx} \} e \]

\[ + \{ aK(u)\xi^2 V_{xx} + aK(u)\xi \eta W_{xx} \} \tilde{J} e \]

\[ + \{ -a(l-1)(K(u)\xi \eta)x V_x - a(l-1)(K(u)\eta^2) W_x \} e \]

\[ + \{ a(l-1)(K(u)\xi^2)x V_x + a(l-1)(K(u)\xi \eta) W_x \} \tilde{J} e \]

\[ + \{ -aK(u)\xi \eta V_x - aK(u)\eta \eta W_x \} e \]

\[ + \{ aK(u)\xi \eta V_x + aK(u)\eta \xi W_x \} \tilde{J} e + \text{OK}. \quad (61) \]

We compute (55). Since \( \nabla^N g = 0 \), we deduce that

\[ (\alpha, \beta, \gamma) = (l+1, 0, 0), (l, 1, 0), (l, 0, 1), (0, l+1, 0), (0, 0, l+1) \]

+ \text{OK}

\[ = bg(u_x u_x)J\nabla_x^2 U + 2(l+1)bg(\nabla_x u_x, u_x)J\nabla_x U \]

\[ + 2bg(\nabla_x U, u_x)J\nabla_x u_x + \text{OK} \]

\[ = b\{\xi^2 + \eta^2\}J\{V_{xx} e + W_{xx} \tilde{J} e\} + (l+1)b\{\xi^2 + \eta^2\} x J\{V_x e + W_x \tilde{J} e\} \]

\[ + 2bg(V_x e + W_x \tilde{J} e, \xi e + \eta \tilde{J} e)J\{\xi e + \eta \tilde{J} e\} + \text{OK} \]

\[ = b\{\xi^2 + \eta^2\} \{-W_{xx} e + V_{xx} \tilde{J} e\} + (l+1)b\{\xi^2 + \eta^2\} x \{-W_x e + V_x \tilde{J} e\} \]

\[ + 2b\{\xi V_x + \eta W_x\} \{-\eta_x e + \eta \xi \tilde{J} e\} + \text{OK} \]

\[ = b\{\xi^2 + \eta^2\} \{-W_{xx} e + V_{xx} \tilde{J} e\} \]

\[ + b\{\xi \eta V_x - \{(l+1)\xi^2 + (l+2)\eta^2\} x W_x\} e \]

\[ + b\{(l+2)\xi^2 + (l+1)\eta^2\} x V_x + 2\xi \eta W_x \} \tilde{J} e + \text{OK} \quad (62) \]

We compute (56). In the same way as (62), we deduce that

\[ (\alpha, \beta, \gamma) = (l, 1, 0), (l, 0, 1), (0, l+1, 0), (0, 0, l+1) \]

+ \text{OK}
Combining (57), (58), (61), (62) and (63), we obtain

\[
\hat{\beta} = -\beta - \hat{c}c + c = c\beta \quad \text{OK}
\]

\[
\begin{align*}
+ c(l + 1)g(V_x e + W_x \tilde{J} e, \xi e + \eta \tilde{J} e) & \tilde{J} \{\xi e + \eta \tilde{J} e\} \\
+ c(l + 2)g(V_x e + W_x \tilde{J} e, \xi e + \eta \tilde{J} e) & \tilde{J} \{\xi e + \eta \tilde{J} e\} \\
+ \frac{c}{2} \{\xi^2 + \eta^2\}_x \tilde{J} \{V_x e + W_x \tilde{J} e\} & \text{OK}
\end{align*}
\]

\[
\begin{align*}
= c\xi x x + \eta W_x x \{ -\eta e + \xi \tilde{J} e \} \\
+ c(l + 1)\{\xi V_x + \eta W_x \} \{ -\eta e + \xi \tilde{J} e \} \\
+ c(l + 2)\{\xi x V_x + \eta x W_x \} \{ -\eta e + \xi \tilde{J} e \} \\
+ \frac{c}{2} \{\xi^2 + \eta^2\}_x \{ -W_x e + V_x \tilde{J} e \} & \text{OK}
\end{align*}
\]

\[
\begin{align*}
= c\xi x x - \eta^2 W_x x e + c\{\xi^2 V_x x + \eta x W_x x \} \tilde{J} e \\
+ c \left[ -\{l + 1\}(\xi \eta)_x + \xi \eta \} V_x - \left\{ \frac{1}{2} \xi^2 + (l + 2)\eta^2 \right\}_x \right] W_x \}
\]

\[
+ c \left\{ (l + 2)\xi^2 + \frac{1}{2} \eta^2 \right\}_x \{ V_x + (l + 1)(\xi \eta)_x + \xi \eta \} W_x \}
\tilde{J} e + \text{OK},
\end{align*}
\]

Equation (63)

Combining (57), (58), (61), (62) and (63), we obtain

\[
\left\{ \frac{1}{2} \frac{\partial}{\partial t} - aJ \frac{\partial^4}{\partial x^4} + \hat{\beta}(t, x) \frac{\partial}{\partial x^2} + \hat{\gamma}(t, x) \frac{\partial}{\partial x} \right\} \left[ \begin{array}{c} V \\ W \end{array} \right] = \text{OK},
\]

where

\[
\hat{\beta}(t, x) = \left[ \begin{array}{c} \hat{\beta}_{11}(t, x) \\ \hat{\beta}_{12}(t, x) \\ \hat{\beta}_{21}(t, x) \\ \hat{\beta}_{22}(t, x) \end{array} \right], \quad \hat{\gamma}(t, x) = \left[ \begin{array}{c} \hat{\gamma}_{11}(t, x) \\ \hat{\gamma}_{12}(t, x) \\ \hat{\gamma}_{21}(t, x) \\ \hat{\gamma}_{22}(t, x) \end{array} \right],
\]

\[
\begin{align*}
\hat{\beta}_{11} &= \{ aK(u) + c \} \xi \eta, \\
\hat{\beta}_{12} &= 1 + b \xi^2 + \{ aK(u) + b + c \} \eta^2, \\
\hat{\beta}_{21} &= -1 - \{ aK(u) + b + c \} \xi^2 - b \eta^2, \\
\hat{\beta}_{22} &= -\{ aK(u) + c \} \xi \eta,
\end{align*}
\]

\[
\begin{align*}
\hat{\gamma}_{11} &= \frac{\partial}{\partial x} \left\{ \{ a(l - 1)K(u) + c(l + 2) \} \xi \eta \right\} + \{ aK(u) + 2b - c \} \xi \eta_x, \\
\hat{\gamma}_{12} &= \frac{\partial}{\partial x} \left[ \left\{ b(l + 2) + \frac{c}{2} \right\} \xi^2 + \left\{ \frac{a}{2}(2l - 1)K(u) + (b + c)(l + 2) \right\} \eta^2 \right] \\
&- \frac{a}{2} \left\{ \frac{\partial}{\partial x} K(u) \right\} \eta^2, \\
\hat{\gamma}_{21} &= -\frac{\partial}{\partial x} \left[ + \left\{ \frac{a}{2}(2l - 1)K(u) + (b + c)(l + 2) \right\} \xi^2 + \left\{ b(l + 2) + \frac{c}{2} \right\} \eta^2 \right]
\end{align*}
\]
\[
\hat{\gamma}_{22} = -\frac{\partial}{\partial x} \{[a(l-1)K(u) + c(l+2)] \xi \eta \} - \{aK(u) + 2b - c\} \xi x \eta.
\]

It is easy to see that \(\hat{\beta}_1(t, x) + \hat{\beta}_2(t, x) \equiv 0\),
\[
\hat{\gamma}_{12}(t, x) - \hat{\gamma}_{21}(t, x) = \frac{\partial}{\partial x} \{H(u)g(u_x, u_x)\} - \frac{a}{2} \left\{ \frac{\partial}{\partial x} K(u) \right\} g(u_x, u_x),
\]
\[
H(u) = \frac{a}{2}(2l-1)K(u) + b(2l+3) + \frac{c}{2}(2l+5),
\]
\[
\int_0^{2\pi} \{\hat{\gamma}_{12}(t, x) - \hat{\gamma}_{21}(t, x)\} dx = -\frac{a}{2} \int_0^{2\pi} \left\{ \frac{\partial}{\partial x} K(u) \right\} g(u_x, u_x) dx, \quad t \in \mathbb{R}.
\]

Unfortunately, \(V(t, x)\) and \(W(t, x)\) are not \(2\pi\)-periodic functions in \(x\). We shall obtain a system for \(2\pi\)-periodic functions in \(x\) by correction. We denote by \(2\pi \theta(t)\) the correction angle for the closed curve \(C(t)\), which is the angle formed by \(e(t, 0)\) and \(e(t, 2\pi)\) in \(T_{u(t, 0)} N\), and said to be the holonomy angle of \(C(t)\) at \(u(t, 0) \in N\). If \(C(t)\) is the boundary enclosing a contractive domain \(D(t)\), then \(\theta(t)\) is given by
\[
\theta(t) = \frac{1}{2\pi} \int_{D(t)} K(u) du^1 \wedge du^2.
\]

See [20] Section 7.3 for this. Set
\[
\tilde{Z}(t, x) = P(\theta(t)x) \begin{bmatrix} V(t, x) \\ W(t, x) \end{bmatrix}, \quad P(s) = \begin{bmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{bmatrix}, \quad s \in \mathbb{R}.
\]

Then \(\tilde{Z}\) is \(2\pi\)-periodic in \(x\). The normalized angle \(\theta(t)\) is determined by \(u(t, x)\) and \(u_x(t, x)\), and \(\theta(t)\) is \(C^1\) provided that \(u, u_x, \nabla_x u_x, \nabla_x^2 u_x, \nabla_x^3 u_x, \text{ and } \nabla_x^4 u_x\) are continuous. Note that
\[
J = P \left(\frac{\pi}{2}, \frac{\partial}{\partial x} P(\theta(t)x) = \theta(t)P \left(\theta(t)x + \frac{\pi}{2}\right) = \theta(t)JP \left(\theta(t)x\right) = \theta(t)P \left(\theta(t)x\right) J.\right.
\]

Set \(P = P(\theta(t)x)\) for short. Multiply (64) by \(P\) from the left. Then
\[
P \left\{ I \frac{\partial}{\partial t} - a \frac{\partial}{\partial x} \frac{\partial^4}{\partial x^4} + \hat{\beta}(t, x) \frac{\partial}{\partial x^2} + \hat{\gamma}(t, x) \frac{\partial}{\partial x} \right\} t P \tilde{Z} = \tilde{P} \tilde{F}(t, x).
\]

We compute this in detail. Simple computations give
\[
P \frac{\partial}{\partial t} t P = I \frac{\partial}{\partial t} - \theta(t) J, \tag{66}
\]
\[
-\frac{\partial}{\partial x} \frac{\partial^4}{\partial x^4} \hat{P} = -a \frac{\partial^4}{\partial x^4} - 4a \theta(t) I \frac{\partial^3}{\partial x^3} + 6a \theta(t)^2 J \frac{\partial^2}{\partial x^2} + 4a \theta(t)^3 I \frac{\partial}{\partial x} - a \theta(t)^4 J, \tag{67}
\]
\[
P \frac{\partial}{\partial x^2} \frac{\partial^2}{\partial x^2} \frac{\partial}{\partial x} \tilde{P} = P \hat{\beta}(t, x)^t \frac{\partial^2}{\partial x^2} \frac{\partial}{\partial x} - 2 \theta(t) P \hat{\beta}(t, x)^t \frac{\partial}{\partial x} - \theta(t)^2 P \hat{\beta}(t, x)^t \frac{\partial}{\partial x} - P \hat{\gamma}(t, x)^t \frac{\partial}{\partial x} - \theta(t) P \hat{\gamma}(t, x)^t \frac{\partial}{\partial x} P J, \tag{68}
\]
\[
P \hat{\gamma}(t, x)^t \frac{\partial}{\partial x} \tilde{P} = P \hat{\gamma}(t, x)^t \frac{\partial}{\partial x} - \theta(t) P \hat{\gamma}(t, x)^t \frac{\partial}{\partial x} P J. \tag{69}
\]

Substitute (66), (67), (68), and (69) into (65). We obtain
\[
\left\{ I \frac{\partial}{\partial t} - a \frac{\partial^4}{\partial x^4} - a \theta(t) I \frac{\partial^3}{\partial x^3} + \hat{\beta}(t, x) \frac{\partial^2}{\partial x^2} + \hat{\gamma}(t, x) \frac{\partial}{\partial x} \right\} \tilde{Z} = \tilde{F}(t, x), \tag{70}
\]

where
\[
\hat{\beta}(t, x) = P \hat{\beta}(t, x)^t P + 6a \theta(t)^2 J,
\]
\[
\hat{\gamma}(t, x) = P \hat{\gamma}(t, x)^t P + 4a \theta(t)^3 I - 2 \theta(t) P \hat{\beta}(t, x)^t P J,
\]
and \( \vec{F}_1(t, x) \) is a function of \( u, u_x, \nabla_x u_x, \nabla_x^2 u_x, \nabla_x^3 u_x \) and \( \nabla_x^4 u_x \). It is easy to check that
\[
\text{tr}(\hat{\beta}(t, x)) = \text{tr}(\hat{\beta}_1(t, x)), \quad \text{tr}(\hat{J}_1(t, x)) = \text{tr}(\hat{J}_1(t, x)).
\]
We see (70) as a system of partial differential equations for \( \vec{Z} \). The third order term in (70) has no essential influence on the well-posedness of the initial value problem. Theorem 5 shows that if \( \mathcal{K}(u) \) is constant, then the initial value problem for (70) is \( L^2 \)-well-posed. In other words, if the sectional curvature of the target Riemann surface is constant, then the initial value problem for (5) is made to be solvable.

Acknowledgements. The author would like to thank Eiji Onodera for invaluable comments and helpful information on the holonomy.

REFERENCES

[1] N.-H. Chang, J. Shatah and K. Uhlenbeck, Schrödinger maps, Comm. Pure Appl. Math. 53 (2000), 590–602.
[2] H. Chihara, The initial value problem for Schrödinger equations on the torus, Int. Math. Res. Not. 2002:15 (2002), 789–820.
[3] H. Chihara, The initial value problem for a third order dispersive equation on the two dimensional torus, Proc. Amer. Math. Soc. 133 (2005), 2083–2090.
[4] H. Chihara, Schrödinger flow into almost Hermitian manifolds, Bull. Lond. Math. Soc. 45 (2013), 37–51.
[5] H. Chihara and E. Onodera, A third order dispersive flow for closed curves into almost Hermitian manifolds, J. Funct. Anal. 257 (2009), pp.388-404.
[6] H. Chihara and E. Onodera, A fourth-order dispersive flow into Kähler manifolds, arXiv:1308.5542.
[7] S.-I. Doi, Smoothing effects of Schrödinger evolution groups on Riemannian manifolds, Duke Math. J. 82 (1996), 679–706.
[8] N. Koiso, The vortex filament equation and a semilinear Schrödinger equation in a Hermitian symmetric space, Osaka J. Math. 34 (1997), 199–214.
[9] N. Koiso, Long time existence for vortex filament equation in a Riemannian manifold, Osaka J. Math. 45 (2008), 265–271.
[10] N. Koiso, Vortex filament equation in a Riemannian manifold, Tohoku Math. J. 55 (2003), 311–320.
[11] H. Kumano-go, “Pseudo-Differential Operators”, The MIT Press, 1981.
[12] S. Mizohata, “On the Cauchy Problem”, Academic Press, 1985.
[13] R. Mizuhara, The initial value problem for third and fourth order dispersive equations in one space dimension, Funkcial. Ekvac. 49 (2006), 1–38.
[14] F. Nicola and L. Rodino, “Global Pseudo-Differential Calculus on Euclidean Spaces”, Birkhäuser, 2010.
[15] E. Onodera, A third-order dispersive flow for closed curves into Kähler manifolds, J. Geom. Anal. 18 (2008), 889–918.
[16] E. Onodera, Generalized Hasimoto transform of one-dimensional dispersive flows into compact Riemann surfaces, SIGMA Symmetry Integrability Geom. Methods Appl. 4 (2008), article No. 044, 10 pages.
[17] E. Onodera, A remark on the global existence of a third order dispersive flow into locally Hermitian symmetric spaces, Comm. Partial Differential Equations 35 (2010), 1130–1144.
[18] E. Onodera, A curve flow on an almost Hermitian manifold evolved by a third order dispersive equation, Funkcial. Ekvac. 55 (2012), 137–156.
[19] E. Onodera, private communication.
[20] I. M. Singer and J. A. Thorpe, “Lecture Notes on Elementary Topology and Geometry”, Springer-Verlag, 1967.
[21] J. Takeuchi, A necessary condition for the well-posedness of the Cauchy problem for a certain class of evolution equations, Proc. Japan Acad. 50 (1974), 133–137.
[22] S. Tarama, Remarks on \( L^2 \)-wellposed Cauchy problem for some dispersive equations, J. Math. Kyoto Univ. 37 (1997), 757–765.
[23] S. Tarama, \( L^2 \)-well-posed Cauchy problem for fourth order dispersive equations on the line, Electron. J. Differential Equations 2011 (2011), 1–11.
[24] M. Taylor, “Pseudodifferential Operators”, Princeton University Press, 1981.

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, KAGOSHIMA UNIVERSITY, KAGOSHIMA 890-0065, JAPAN
E-mail address: chihara@sda.att.ne.jp