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Combinatorial 3-Manifolds with 10 Vertices

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Abstract

We give a complete enumeration of combinatorial 3-manifolds with 10 vertices: There are precisely 247,882 triangulated 3-spheres with 10 vertices as well as 518 vertex-minimal triangulations of the sphere product $S^2 \times S^1$ and 615 triangulations of the twisted sphere product $S^2 \times S^1$.

All the 3-spheres with up to 10 vertices are shellable, but there are 29 vertex-minimal non-shellable 3-balls with 9 vertices.

1 Introduction

Let $M$ be a triangulated 3-manifold with $n$ vertices and face vector $f = (n, f_1, f_2, f_3)$. By Euler’s equation, $n - f_1 + f_2 - f_3 = 0$, and by double counting the edges of the ridge-facet incidence graph, $2f_2 = 4f_3$, it follows that

$$f = (n, f_1, 2f_1 - 2n, f_1 - n).$$

A complete characterization of the $f$-vectors of the 3-sphere $S^3$, the sphere product $S^2 \times S^1$, the twisted sphere product (or 3-dimensional Klein bottle) $S^2 \times S^1$, and of the real projective 3-space $\mathbb{RP}^3$ was given by Walkup.

**Theorem 1** (Walkup [30]) For every 3-manifold $M$ there is an integer $\gamma(M)$ such that

$$f_1 \geq 4n + \gamma(M)$$

for every triangulation of $M$ with $n$ vertices and $f_1$ edges. Moreover there is an integer $\gamma^*(M) \geq \gamma(M)$ such that for every pair $(n, f_1)$ with $n \geq 0$ and

$$\left(\frac{n}{2}\right) \geq f_1 \geq 4n + \gamma^*(M)$$

there is a triangulation of $M$ with $n$ vertices and $f_1$ edges. In particular,

(a) $\gamma^* = \gamma = -10$ for $S^3$,

(b) $\gamma^* = \gamma = 0$ for $S^2 \times S^1$,

(c) $\gamma^* = 1$ and $\gamma = 0$ for $S^2 \times S^1$, where, with the exception $(9, 36)$, all pairs $(n, f_1)$ with $n \geq 0$ and $4n + \gamma(M) \leq f_1 \leq \left(\frac{n}{2}\right)$ occur,

(d) $\gamma^* = \gamma = 7$ for $\mathbb{RP}^3$, and

(e) $\gamma^*(M) \geq \gamma(M) \geq 8$ for all other 3-manifolds $M$. 

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Table 1: Combinatorial 3-manifolds with up to 10 vertices.

| Vertices | Types \ All | $S^3$ | $S^2 \times S^1$ | $S^2 \times S^1$ |
|----------|------------|-------|------------------|------------------|
| 5        | 1          | 1     | –                | –                |
| 6        | 2          | 2     | –                | –                |
| 7        | 5          | 5     | –                | –                |
| 8        | 39         | 39    | –                | –                |
| 9        | 1297       | 1296  | –                | 1                |
| 10       | 249015     | 247882| 518              | 615              |

By Walkup’s theorem, vertex-minimal triangulations of $S^2 \times S^1$, $S^2 \times S^1$, and $\mathbb{R}P^3$ have 9, 10, and 11 vertices, respectively. The 3-sphere can be triangulated vertex-minimally as the boundary of the 4-simplex with 5 vertices. But otherwise, rather little is known on vertex-minimal triangulations of 3- and higher-dimensional manifolds. See [24], [25], and [29] for a discussion, further references, and for various examples of small triangulations of 3-manifolds.

The exact numbers of different combinatorial types of triangulations of $S^3$, $S^2 \times S^1$, and $S^2 \times S^1$ with up to 9 vertices and of neighborly triangulations (i.e., triangulations with complete 1-skeleton) with 10 vertices were obtained by Grünbaum and Sreedharan [15] (simplicial 4-polytopes with 8 vertices), Barnette [8] (combinatorial 3-spheres with 8 vertices), Altshuler [2] (combinatorial 3-manifolds with up to 8 vertices), Altshuler and Steinberg [5] (neighborly 4-polytopes with 9 vertices), Altshuler and Steinberg [6] (neighborly 3-manifolds with 9 vertices), Altshuler and Steinberg [7] (combinatorial 3-manifolds with 9 vertices), Altshuler [3] (neighborly 3-manifolds with 10 vertices).

In this paper, the enumeration of 3-manifolds is continued: We completely classify triangulated 3-manifolds with 10 vertices. Moreover, we determine the combinatorial automorphism groups of all triangulations with up to 10 vertices, and we test for all 3-spheres (and all 3-balls) with up to 10 vertices (with up to 9 vertices) whether they are constructible, shellable, or vertex-decomposable.

2 Enumeration

We used a backtracking approach, described as mixed lexicographic enumeration in [26], to determine all triangulated 3-manifolds with 10 vertices: The vertex-links of a triangulated 3-manifold with 10 vertices are triangulated 2-spheres with up to 9 vertices. Altogether, there are 73 such 2-spheres, which are processed in decreasing size. As a first vertex-star of a 3-manifold that we are going to build we take the cone over one of the respective 2-spheres and then add further tetrahedra (in lexicographic order) as long as this is possible. If, for example, a triangle of a partial complex that we have built is contained in three tetrahedra, then this violates the pseudo-manifold property that in a triangulated 3-manifold every triangle is contained in exactly two tetrahedra. We backtrack, remove the last tetrahedron that we have added, and try to add to our partial complex the next tetrahedron (with respect to the lexicographic order). See [26] for further details on the enumeration.
Table 2: Combinatorial 3-manifolds with 10 vertices.

| f-vector\Types | All   | $S^3$ | $S^2 \times S^1$ | $S^2 \succeq S^1$ |
|----------------|-------|-------|------------------|-------------------|
| (10,30,40,20)  | 30    | 30    | –                | –                 |
| (10,31,42,21)  | 124   | 124   | –                | –                 |
| (10,32,44,22)  | 385   | 385   | –                | –                 |
| (10,33,46,23)  | 952   | 952   | –                | –                 |
| (10,34,48,24)  | 2142  | 2142  | –                | –                 |
| (10,35,50,25)  | 4340  | 4340  | –                | –                 |
| (10,36,52,26)  | 8106  | 8106  | –                | –                 |
| (10,37,54,27)  | 13853 | 13853 | –                | –                 |
| (10,38,56,28)  | 21702 | 21702 | –                | –                 |
| (10,39,58,29)  | 30526 | 30526 | –                | –                 |
| (10,40,60,30)  | 38575 | 38553 | 10               | 12                |
| (10,41,62,31)  | 42581 | 42498 | 37               | 46                |
| (10,42,64,32)  | 30526 | 39299 | 110              | 117               |
| (10,43,66,33)  | 28439 | 28087 | 162              | 190               |
| (10,44,68,34)  | 14057 | 13745 | 145              | 167               |
| (10,45,70,35)  | 3677  | 3540  | 54               | 83                |
| Total:         | 249015| 247882| 518              | 615               |

**Theorem 2** There are precisely 249015 triangulated 3-manifolds with 10 vertices: 247882 of these are triangulated 3-spheres, 518 are vertex-minimal triangulations of the sphere product $S^2 \times S^1$, and 615 are triangulations of the twisted sphere product $S^2 \succeq S^1$.

Table 1 gives the total numbers of all triangulations with up to 10 vertices. The numbers of 10-vertex triangulations are listed in detail in Table 2. All triangulations can be found online at [22]. The topological types were determined with the bistellar flip program BISTELLAR [23]; see [10] for a description.

For a given triangulation, it is a purely combinatorial task to determine its combinatorial symmetry group. We computed the respective groups with a program written in GAP [14].

**Corollary 3** There are exactly 1, 1, 5, 36, 408, and 7443 triangulated 3-manifolds with 5, 6, 7, 8, 9, and 10 vertices, respectively, that have a non-trivial combinatorial symmetry group.

The symmetry groups along with the numbers of combinatorial types of triangulations that correspond to a particular group are listed in Table 3. Altogether, there are 14 examples that have a vertex-transitive symmetry group; see [18].

All simplicial 3-spheres with up to 7 vertices are polytopal. However, there are two non-polytopal 3-spheres with 8 vertices, the Grünbaum and Sreedharan sphere [15] and the Barnette sphere [8]. The classification of triangulated 3-spheres with 9 vertices into polytopal and non-polytopal spheres was started by Altshuler and Steinberg [5], [6], [7] and completed by Altshuler, Bokowski, and Steinberg [4] and Engel [13]. For neighborly simplicial 3-spheres with 10 vertices the numbers of polytopal and non-polytopal spheres were determined by Altshuler [3], Bokowski and Garms [11], and Bokowski and Sturmfels [12].

**Problem 4** Classify all simplicial 3-spheres with 10 vertices into polytopal and non-polytopal spheres.
Table 3: Symmetry groups of triangulated 3-manifolds with up to 10 vertices.

| $n$ | Manifold | $|G|$ | $G$ | Types | $n$ | Manifold | $|G|$ | $G$ | Types |
|-----|----------|------|-----|--|-----|----------|------|-----|--|
| 5   | $S^3$    | 120  | $S_3$, | transitive 1 | 10  | $S^3$    | 1   | trivial | 240683 |
|     |          |      |        |            |     |          | 2   | $Z_2$    | 6675  |
|     |          |      |        |            |     |          | 3   | $Z_3$    | 10    |
| 6   | $S^3$    | 48   | $O^* = Z_2 \wr S_3$ | 1 | 4   | $Z_4$    | 53  |        |     |
|     |          | 72   | $S_3 \rtimes Z_2$, | transitive 1 | 5   | $Z_5$    | 1   |        |     |
|     |          |      |        |            |     |          | 6   | $Z_6$    | 1    |        |     |
| 7   | $S^3$    | 8    | $D_4$ | 2 | 12  | $S_3 \times Z_2$ | 15  |        |     |
|     |          | 12   | $D_6 \rtimes Z_2$ | 1 | 14  | $D_6 \rtimes Z_2$ | 31  |        |     |
|     |          | 14   | $D_7$  | transitive 1 | 10  | $Z_{10}$ | 1   |        |     |
|     |          | 48   | $D_4 \rtimes D_3$ | 1 |     |          |     |        |     |
| 8   | $S^3$    | 1    | trivial | 3 | 12  | $S_3 \times Z_2$ | 15  |        |     |
|     |          | 2    | $Z_2$   | 13 | 16  | $D_4 \rtimes Z_2$ | 3   |        |     |
|     |          | 4    | $Z_4$   | 1  | 20  | $D_{10}$ | 2   |        |     |
|     |          | 6    | $Z_2 \times Z_2$ | 9 | 24  | $T^* = S_4$ | 1   |        |     |
|     |          | 8    | $S_3$   | 1  |     |          |     |        |     |
|     |          | 8    | $Z_2^3$ | 1  |     |          |     |        |     |
|     |          | 12   | $S_3 \times Z_2$ | 4 |     |          |     |        |     |
|     |          | 16   | $D_4 \rtimes Z_2$ | 1 |     |          |     |        |     |
|     |          | 60   | $D_5 \rtimes D_3$ | 1 |     |          |     |        |     |
|     |          | 384  | $Z_2 \wr S_4$, | transitive 1 |     |          |     |        |     |
| 9   | $S^3$    | 1    | trivial | 889 | 1    | $S^2 \times S^1$ | 1   | trivial | 420  |
|     |          | 2    | $Z_2$   | 319 | 2    | $Z_2$    | 95  |        |     |
|     |          | 3    | $Z_3$   | 3   | 2    | $Z_2$    | 95  |        |     |
|     |          | 4    | $Z_4$   | 3   | 10   | $Z_{10}$ | 1   |        |     |
|     |          | 6    | $Z_2 \times Z_2$ | 46 |     |          |     |        |     |
|     |          | 6    | $Z_6$   | 1   | 16   | $(2, 2, 2)^2$ | 1   |        |     |
|     |          | 8    | $S_3$   | 8   | 20   | $D_{10}$ | 1   |        |     |
|     |          | 8    | $Z_2^3$ | 3   |     |          |     |        |     |
|     |          | 12   | $S_3 \times Z_2$ | 10 |     |          |     |        |     |
|     |          | 18   | $D_9$   | transitive 1 |     |          |     |        |     |
|     |          | 24   | $T^* = S_4$ | 3 |     |          |     |        |     |
|     |          | 72   | $D_6 \rtimes D_3$ | 1 |     |          |     |        |     |
|     |          | 80   | $D_5 \rtimes D_4$ | 1 |     |          |     |        |     |
|     |          | 18   | $D_9$   | transitive 1 |     |          |     |        |     |

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Table 4: Combinatorial 3-balls with up to 9 vertices.

| Vertices | Types | All | Non-Shellable | Not Vertex-Decomposable |
|----------|-------|-----|---------------|------------------------|
| 4        | 1     | –   | –             | –                      |
| 5        | 3     | –   | –             | –                      |
| 6        | 12    | –   | –             | –                      |
| 7        | 167   | –   | 2             | –                      |
| 8        | 1021  | –   | 628           | –                      |
| 9        | 2451305 | 29 | 623819        | –                      |

3 3-Balls

Along with the enumeration of triangulated 3-spheres with up to 10 vertices we implicitly have enumerated all triangulated 3-balls with up to 9 vertices: Let $B^3_{n-1}$ be a triangulated 3-ball with $n-1$ vertices and let $v_n$ be a new vertex. Then the union $B^3_{n-1} \cup (v_n \ast \partial B^3_{n-1})$ of $B^3_{n-1}$ with the cone $v_n \ast \partial B^3_{n-1}$ over the boundary $\partial B^3_{n-1}$ with respect to $v_n$ is a triangulated 3-sphere. Thus there are at most as many combinatorially distinct 3-spheres with $n$ vertices as there are combinatorially distinct 3-balls with $n-1$ vertices.

If, on the contrary, we delete the star of a vertex from a triangulated 3-sphere $S^3_n$ with $n$ vertices, then, obviously, we obtain a 3-ball with $n-1$ vertices. If we delete the star of a different vertex from $S^3_n$ then we might or might not obtain a combinatorially different ball. Let $\#B^3(n-1)$ and $\#S^3(n)$ be the numbers of combinatorially distinct 3-balls and 3-spheres with $n-1$ and $n$ vertices, respectively. Then

$$\#S^3(n) \leq \#B^3(n-1) \leq n \cdot \#S^3(n).$$

For the explicit numbers of simplicial 3-balls with up to 9 vertices see Table 4.

4 Vertex-Decomposability, Shellability, and Constructibility

The concepts of vertex-decomposability, shellability, and constructibility describe three particular ways to assemble a simplicial complex from the collection of its facets (cf. Björner [9] and see the surveys [16], [19], and [31]). The following implications are strict for (pure) simplicial complexes:

vertex decomposable $\implies$ shellable $\implies$ constructible.

It follows from Newman’s and Alexander’s fundamental works on the foundations of combinatorial and PL topology from 1926 [27] and 1930 [1] that a constructible $d$-dimensional simplicial complex in which every $(d-1)$-face is contained in exactly two or at most two $d$-dimensional facets is a PL $d$-sphere or a PL $d$-ball, respectively.

A shelling of a triangulated $d$-ball or $d$-sphere is a linear ordering of its $f_d$ facets $F_1, \ldots, F_{f_d}$ such that if we remove the facets from the ball or sphere in this order, then at every intermediate step the remaining simplicial complex is a simplicial ball. A simplicial ball or sphere is shellable if it has a shelling; it is extendably shellable if any partial shelling $F_1, \ldots, F_i$, $i < f_d$, can be extended to a shelling; and it is strongly non-shellable if it has no free facet that can be removed from the triangulation without loosing ballness.
A triangulated \(d\)-ball or \(d\)-sphere is **constructible** if it can be decomposed into two constructible balls of smaller size and if, in addition, the intersection of the two balls is a constructible ball of dimension \(d-1\); it is **vertex-decomposable** if we can remove the star of a vertex \(v\) and the remaining complex and the link of \(v\) are again vertex-decomposable balls.

We tested vertex-decomposability and shellability with a straightforward backtracking implementation.

**Corollary 5** All triangulated \(3\)-spheres with \(n \leq 10\) vertices are shellable and therefore constructible.

An example of a non-constructible and thus non-shellable \(3\)-sphere with 13 vertices has been constructed in [19]. It remains open whether there are non-shellable respectively non-constructible \(3\)-spheres with 11 and 12 vertices.

**Corollary 6** All triangulated \(3\)-balls with \(n \leq 8\) vertices are extendably shellable.

Examples of non-shellable \(3\)-balls can be found at various places in the literature (cf. the references in [19], [20], and [31]) with the smallest previously known non-shellable \(3\)-ball by Ziegler [31] with 10 vertices.

**Corollary 7** There are precisely twenty-nine vertex-minimal non-shellable simplicial \(3\)-balls with 9 vertices, ten of which are strongly non-shellable. The twenty-nine balls have between 18 and 22 facets, with one unique ball \(B_{3,9,18}\) having 18 facets and \(f\)-vector \((9, 33, 43, 18)\).

A list of facets and visualization of the ball \(B_{3,9,18}\) is given in [20].

The cone over a simplicial \(d\)-ball with respect to a new vertex is a \((d+1)\)-dimensional ball. It is shellable respectively vertex-decomposable if and only if the original ball is shellable respectively vertex-decomposable (cf. [28]).

**Corollary 8** There are non-shellable \(3\)-balls with \(d+6\) vertices and 18 facets for \(d \geq 3\).

Each of the 29 non-shellable \(3\)-balls with 9 vertices can be split into a pair of shellable balls.

**Corollary 9** All triangulated \(3\)-balls with \(n \leq 9\) vertices are constructible.

Klee and Kleinschmidt [17] showed that all simplicial \(d\)-balls with up to \(d+3\) vertices are vertex-decomposable.

**Corollary 10** There are not vertex-decomposable \(3\)-balls with \(d+4\) vertices and 10 facets for \(d \geq 3\).

In fact, there are exactly two not vertex-decomposable \(3\)-balls with 7 vertices; see [21] for a visualization of these two balls. One of the examples has 10 tetrahedra, the other has 11 tetrahedra.

For the numbers of not vertex-decomposable \(3\)-balls with up to 9 vertices see Table 4.

**Corollary 11** All triangulated \(3\)-spheres with \(n \leq 8\) vertices are vertex-decomposable.

Klee and Kleinschmidt [17] constructed an example of a not vertex-decomposable polytopal \(3\)-sphere with 10 vertices.
**Corollary 12** There are precisely 7 not vertex-decomposable 3-spheres with 9 vertices, which are all non-polytopal. Moreover, there are 14468 not vertex-decomposable 3-spheres with 10 vertices.

Four of the seven examples with 9 vertices are neighborly with 27 tetrahedra, the other three have 25, 26, and 26 tetrahedra, respectively. The 25 tetrahedra of the smallest example are:

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1234 1235 1246 1257 1268
1278 1345 1456 1567 1679
1689 2468 2579 3458 3568
2379 3689 3789 4568 5679.
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