A unifying perspective on linear continuum equations prevalent in science. Part VI: rapidly converging series expansions for their solution

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Abstract

We obtain rapidly convergent series expansions of resolvents of operators taking the form $A = \Gamma_1 B \Gamma_1$ where $\Gamma_1(k)$ is a projection that acts locally in Fourier space and $B(x)$ is an operator that acts locally in real space. Such resolvents arise naturally when one wants to solve any of the large class of linear physical equations surveyed in Parts I, II, III, and IV that can be reformulated as problems in the extended abstract theory of composites. We show how the information about the spectrum of $A$ can be used to greatly improve the convergence rate.

1 Introduction

In Parts I, II, III, and IV \cite{10\,13} we established that an avalanche of equations in science can be rewritten in the form

$$J(x, t) = L(x, t)E(x, t) - s(x, t), \quad \Gamma_1 E = E, \quad \Gamma_2 J = 0,$$

(1.1)

as encountered in the extended abstract theory of composites, $\Gamma_1 = \Gamma(k)$ is a projection operator that acts locally in Fourier space, and $s(x)$ is the source term. In Part V \cite{14} we established the connection between solving these equations and computing resolvents of operators of the form $A = \Gamma_1 B \Gamma_1$ where $B = B(x)$ acts locally in real space.

Here in Part VI we are concerned with using rapidly converging series expansion for the solution of (1.1) to obtain rapidly converging series expansions for resolvents of the form

$$R_0 = (z_0 I - A)^{-1} = z_0 (I - A/z_0)^{-1},$$

(1.2)

where the operator $A$ takes the form $A = \Gamma_1 B \Gamma_1$, in which $B = B(x)$ acts locally in real space and typically has an inverse, and one that is easily computed. Thus if $\Gamma_1$ or $B$ act on a field $F$ to produce a field $G$ then we have, respectively, that $G(x) = B(x)F(x)$ or $\hat{G}(k) = \Gamma_1 (k) \hat{F}(k)$, in which $\hat{G}(k)$ and $\hat{F}(k)$ are the Fourier components of $G$ and $F$.

As in the previous parts we define the inner product of two fields $P_1(x)$ and $P_2(x)$ to be

$$(P_1, P_2) = \int_{\mathbb{R}^4} (P_1(x), P_2(x))_T \, dx,$$

(1.3)

where $(\cdot, \cdot)_T$ is a suitable inner product on the space $T$ such that the projection $\Gamma_1$ is self-adjoint with respect to this inner product, and thus the space $E$ onto which $\Gamma_1$ projects is orthogonal to the space $J$ onto which $\Gamma_2 = I - \Gamma_1$ projects. We define the norm of a field $P$ to be $|P| = (P, P)^{1/2}$, and given any operator $O$ we define its norm to be

$$\|O\| = \sup_{P, |P|=1} |OP|.$$

(1.4)

When we have periodic fields in periodic media the integral in (1.3) should be taken over the unit cell $\Omega$ of periodicity. If the fields depend on time $t$ then we should set $x_4 = t$ take the integral over $\mathbb{R}^4$ with the integral over the spatial variables restricted to $\Omega$ if the material and fields are spatially periodic.

The goal of this paper is to review iterative methods that have been developed to accelerate the solution of problems in the extended theory of composites, and to transfer this knowledge to develop rapidly convergent iterative
schemes for the calculation of resolvents, where \( \mathbf{A} = \Gamma_1 \mathbf{B} \Gamma_1 \). These iterative methods automatically apply to calculating the action of the inverse of a matrix \( \mathbf{B} \) on a vector subspace, when the inverse of \( \mathbf{B} \) on the whole vector space is easily computed. They were first introduced by Moulinec and Suquet [19] in the context of calculating the fields and effective moduli in the theory of composites, and subsequently accelerated algorithms were discovered: see [16] Chapter 8 of [16]. They have been the subject of increasing attention: see [23] and references therein.

The work presented is largely based on the articles [1, 9, 19, 20] and Chapter 8 of [16], but develops some of the ideas further.

Specifically, the convergence of the expansions that we develop is best illustrated if we further assume that

\[
\chi_i(x) = \begin{cases} 
1 & \text{in phase } i \\
0 & \text{elsewhere,} 
\end{cases} 
\]

satisfying \( \chi_1(x) + \chi_2(x) = 1 \), while \( \mathbf{L}_1 \) and \( \mathbf{L}_2 \) are the tensors of the two phases, representing their material properties, and the “reference parameter” \( z_0 \) can be freely chosen. This family will serve as model problems for our analysis. Specifically, the convergence of the expansions that we develop is best illustrated if we further assume that

\[
\mathbf{B}(x) = z_0 \mathbf{I} - z_1 \mathbf{L}_1 \chi_1(x) - z_2 \mathbf{L}_2 \chi_2(x) = (z_0 - z_2) \mathbf{I} - (z_1 - z_2) \mathbf{I} \chi_1(x), 
\]

where \( \mathbf{A} = \Gamma_1 (\mathbf{B} \Gamma_1) \). These iterative methods automatically apply to calculating the action of the inverse of a matrix \( \mathbf{B} \) on a vector subspace, when the inverse of \( \mathbf{B} \) on the whole vector space is easily computed. They were first introduced by Moulinec and Suquet [19] in the context of calculating the fields and effective moduli in the theory of composites, and subsequently accelerated algorithms were discovered: see [16] Chapter 8 of [16]. They have been the subject of increasing attention: see [23] and references therein.

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\]
where now, for example, \( z_1 \) and \( z_2 \) may represent the conductivities of the two phases and \( z_0 \) a reference conductivity. With the particular choice \( z_0 = z_2 \) the expression (1.2) reduces to
\[
R = z_2^{-1}\{I - (1 - z_1/z_2)\Gamma_1\chi_1\Gamma_1\}^{-1},
\]
which is now again a problem directly of the form (1.2) with \( B \) and \( z_0 \) now being identified as
\[
B = \chi_1I, \quad z_0 = z_2/(z_2 - z_1).
\]

We will assume that \( z_0, \Gamma_1 \) and \( B \) are fixed and known. So the analysis in this paper is really just about computing the inverse of operators of the form \( I - \Gamma_1 B \Gamma_1/z_0 \). The parameter \( z_0 \), even if fixed, is helpful as the rates of convergence of the series we investigate are conveniently expressed in terms of \( z_0 \).

### 2 Some elementary series expansions

We start by assuming that \( B \) is real and that we know some bounds on it:
\[
b^-I \leq B \leq b^+I, \quad \text{implying} \quad b^-\Gamma_1 \leq A \leq b^+\Gamma_1,
\]
where the last identity follows by projecting the first inequality on the subspace \( \mathcal{E} \). We may sometimes know tighter bounds on \( A \):
\[
a^-I \leq A \leq a^+\Gamma_1, \quad \text{where} \quad a^- \geq b^-, \quad a^+ \leq b^+.
\]

Some approaches to deriving such bounds have been given in Section 3 of [14].

One well known expansion of the resolvent is the Laurent series:
\[
R(z_0)/z_0 = (I - A/z_0)^{-1} = \sum_{n=0}^{\infty} (A/z_0)^n,
\]
better known as the Neumann expansion or Born expansion in the context of operators \( A \), which holds provided the series converges and this is the case if the matrix or operator \( A/z_0 \) has norm less than 1. From the bounds (2.2) it follows that
\[
|A/z_0| \leq r_0, \quad \text{where} \quad r_0 = \frac{\max\{a^+, a^-\}}{|z_0|} \leq \frac{\max\{b^+, b^-\}}{|z_0|},
\]
and convergence of the expansion is assured if \( r_0 < 1 \), i.e. for \( |z_0| > \max\{a^+, a^-\} \). With \( B \) and \( z_0 \) being given by (1.13) we can take \( b^- = 0 \) and \( b^+ = 1 \) and (2.3) naturally reduces to
\[
R = z_2 \sum_{n=0}^{\infty} (1 - z_1/z_2)^n (\Gamma_1 \chi_1)^n.
\]

As shown for example in Section 2 of [14] of Part V, the solution of (1.1) is \( E = Rs \) where \( R \) can be expressed in various equivalent forms including
\[
R = [\Gamma_1 L \Gamma_1]^{-1} = [I - \Gamma B]^{-1}\Gamma = L^{-1} - L^{-1}[I - \tilde{\Gamma} B]^{-1}\tilde{\Gamma} L^{-1}
\]
where
\[
B(x) = L_0 - L(x), \quad \tilde{B}(x) = M_0 - L^{-1}(x)
\]
are operators that are local in real space, in which \( L_0 \) and \( M_0 \) are constant reference tensors, and where
\[
\Gamma = \Gamma_1 (\Gamma_1 L_0 \Gamma_1)^{-1}\Gamma_1, \quad \tilde{\Gamma} = \Gamma_2 (\Gamma_2 M_0 \Gamma_2)^{-1}\Gamma_2
\]
act locally in Fourier space, the inverses being respectively on the spaces \( \mathcal{E} \) and \( \mathcal{J} \) onto which \( \Gamma_1 \) and \( \Gamma_2 = I - \Gamma_1 \) project. With \( L_0 = z_0 I \), we have that \( \Gamma = \Gamma_1/z_0 \) and then it is apparent that with \( A = \Gamma_1 B \Gamma_1 \) where \( B = z_0 I - L \), \( R \) given by (2.6) is in fact the resolvent (1.5) when we consider \( B \) to be fixed and \( L \) to be a function of \( z_0 \). Conversely, if we are interested in computing the resolvent in (1.2) or equivalently (1.5), then we can recast it as a problem in
the theory of composites with $L = z_0I - B$. Having established this connection with the resolvent we can now apply all the theory developed in extended abstract theory of composites to resolvents of the required form, and conversely.

For sufficiently small $B$ we get the series expansion

$$R = [\Gamma_1L\Gamma_1]^{-1} = [I - \Gamma B]^{-1}\Gamma = \sum_{n=0}^{\infty} [\Gamma B]^n\Gamma = \sum_{n=0}^{\infty} [\Gamma (L_0 - L)]^n\Gamma. \tag{2.9}$$

Although for fixed $L$ each term in this series depends on $L_0$ the sum is independent of $L_0$ when the series converges. The choice of $L_0$ influences the rate of convergence, and indeed whether the series converges or not. The series expansion \[2.9\] is well known in the theory of composites: see, for example Chapter 14 of \[8\], \[21\], and references therein.

Alternatively, if $\tilde{B}$ is sufficiently small we have the expansion

$$[I - \tilde{\Gamma} \tilde{B}]^{-1}\tilde{\Gamma} = \sum_{n=0}^{\infty} [\tilde{\Gamma} \tilde{B}]^n\tilde{\Gamma}, \tag{2.10}$$

which may be inserted in \[2.6\] to get a different series expansion for $R$.

For the special case of a two phase medium where $B(x)$ takes the form \[1.9\] we may take $L_0 = L_2$ giving $B(x) = \chi(x)(L_1 - L_1)$ and obtain the expansion

$$R = [I - \Gamma\chi(L_2 - L_1)]^{-1}\Gamma = \sum_{n=0}^{\infty} [\Gamma\chi(L_2 - L_1)]^n\Gamma, \tag{2.11}$$

that is convergent for $L_1$ that is sufficiently close to $L_2$. More precisely, if $L_2$ is real and positive definite, then using that $\chi$ and $(L_2)^{1/2}\Gamma(L_2)^{1/2}$ are selfadjoint projections, we have

$$\|\Gamma\chi(L_2 - L_1)\| = \|(L_2)^{1/2}\Gamma(L_2)^{1/2}\chi[I - (L_2)^{-1/2}L_1(L_2)^{-1/2}]\| \leq \|I - (L_2)^{-1/2}L_1(L_2)^{-1/2}\|, \tag{2.12}$$

so the series converges if $\|I - (L_2)^{-1/2}L_1(L_2)^{-1/2}\| < 1$.

We can now take rapidly converging iterative methods for the solution of \[1.1\] and apply them to obtain rapidly convergent series expansions for the resolvent. These expansions, that will be a major focus of the paper, give the action of the resolvent on a source field $s$ in the form

$$Rs = C_0 \sum_{j=0}^{\infty} W^j s, \tag{2.13}$$

for suitable operators $C_0$ and $W_j$ whose action is relatively easy to compute (typically just requiring two fast Fourier transforms: to Fourier space and back). The iterative procedure of obtaining the fields

$$q_{i+1} = W q_i + s, \quad q_0 = s, \tag{2.14}$$

gives $C_0 q_{n}$ as a good approximation to $Rs$ for large enough $n$. There is no need to keep the $q_j$, for $j \leq i$ once one has computed $q_{i+1}$. If one has a series expansion of the form

$$Rs = C_0 \sum_{j=0}^{\infty} c^j W^j s, \tag{2.15}$$

and one is interested in $Rs$ as a function of $c$ (which may in turn be a function of another variable of interest, such as $z_0$), then one can replace the iterative procedure in \[2.14\] with

$$q_{i+1} = W q_i, \quad q_0 = s, \tag{2.16}$$

storing the $q_i$ as one goes along. Then if the series converges rapidly, the approximation

$$Rs \approx C_0 \sum_{j=0}^{n} c^j q_j \tag{2.17}$$

holds for relatively small values of $n$. Of course as $c$ is increased the series converges more slowly, or perhaps not at all, and then the approximation becomes poor for small values of $n$. 

4
3 Improvements to the Neumann or Born Series

We start by reviewing a well known route for improving the convergence rate of the Neumann or Born Series, thus we note that $A$ can be split as $A = (A - cI) + cI$ and the resolvent can be re-expressed as

$$
R_0 = [(z_0 - c)I - (A - cI)]^{-1} = (z_0 - c)^{-1}[I - [(A - cI)/(z_0 - c)]]^{-1},
$$

(3.1)

where $c$ can be chosen to make the associated expansion

$$
R = (z_0 - c)^{-1}\sum_{n=0}^{\infty}(A - cI)^n/(z_0 - c)^n
$$

(3.2)

converge more rapidly than the expansion with $c = 0$: the basic idea here is to choose $c$ to shift $A$ to decrease the spectral radius. Such splittings are well known for accelerating convergence, the best known being the Jacobi and Gauss-Seidel splittings $^2$. The expansion can clearly be calculated iteratively. If $B$ satisfies the bound (2.1) then a natural choice of $c$ is

$$
c = \frac{1}{2}(b^+ + b^-),
$$

(3.3)

giving

$$
|A - c\Gamma_1| \leq \alpha, \quad \text{where} \quad \alpha = \frac{1}{2}(b^- - b^-),
$$

(3.4)

and the series (3.2) is guaranteed to converge if

$$
|(A - cI)/(z_0 - c)| \leq |(B - c\Gamma_1)/(z_0 - c)| \leq r_1,
$$

(3.5)

where

$$
r_1 = \left|\frac{b^+ - b^-}{b^+ + b^- - 2z_0}\right| = \left|\frac{(b^+ - z_0) - (b^- - z_0)}{(b^+ - z_0) + (b^- - z_0)}\right| = \left|\frac{q - 1}{q + 1}\right|
$$

in which $q = (z_0 - b^+)/(z_0 - b^-).$

(3.6)

Improved convergence can be obtained if we have bounds on $A$ itself that are tighter than the bounds (2.1).

We now draw upon rapidly converging iterative methods for the solution of (1.1) in the extended theory of composites and apply them to obtain rapidly convergent series expansions for the resolvent. This will be the focus of the rest of the paper.

In particular, as (2.6) holds for any choice of $L_0$ we can transform to an equivalent problem where $B$ is replaced with

$$
B' = L_0' - L = B - L_0 + L_0',
$$

(3.7)

and we have the identity

$$
[I - \Gamma B]^{-1} = [I - \Gamma' B']^{-1} \Gamma' \quad \text{with} \quad \Gamma' = \Gamma_1(\Gamma_1 L_0' \Gamma_1)^{-1}\Gamma_1,
$$

(3.8)

and $\Gamma$ being given by (2.8). The associated series expansion when $L_0 = z_0 I$ is

$$
[I - \Gamma B/z_0]^{-1} = \sum_{n=0}^{\infty}[\Gamma' (B - z_0 I + L_0')]^n \Gamma'.
$$

(3.9)

Let us suppose that $B$ is Hermitian and satisfies the bound (2.1). We take $L_0' = z_0' I$ with

$$
z_0' = z_0 - c, \quad \text{where} \quad c = \frac{1}{2}(b^+ + b^-),
$$

(3.10)

so that $B' = B - cI$ satisfies

$$
-\frac{1}{2}(b^+ - b^-) \leq B' \leq \frac{1}{2}(b^+ - b^-).
$$

(3.11)

This implies $|B'/z_0'| \leq r_1$ where

$$
r_1 = \left|\frac{b^+ - b^-}{b^+ + b^- - 2z_0}\right| = \left|\frac{(b^+ - z_0) - (b^- - z_0)}{(b^+ - z_0) + (b^- - z_0)}\right| = \left|\frac{q - 1}{q + 1}\right|
$$

in which $q = (z_0 - b^+)/(z_0 - b^-).$

(3.12)
and the series expansion

\[
[I - \Gamma B]^{-1} \Gamma = [I - \Gamma' B']^{-1} \Gamma' = \sum_{j=0}^{\infty} [\Gamma' B']^j \Gamma' = \sum_{n=0}^{\infty} [\Gamma_1 B'/z_0^n] \Gamma_1/z_0^n
\]  

(3.13)

will converge provided \( r_1 < 1 \), i.e. provided \( z_0 < b^- \) or \( z_0 > b^+ \). Moreover \( r_1 \) determines the minimum rate of convergence. In the field of composites the series expansion (2.9) and the independence of the resulting sum on \( L_0 \) (assuming the sum of the series converges) is well known. Moulinec and Suquet [19] realized that the series could be easily computed by an iterative process as in (2.14). The action of \( B \) (or \( B' \)) can be computed in real space while the action of \( \Gamma \) (or \( \Gamma' \)) can be computed in Fourier space and Fast Fourier transforms can be used to transform between real space and Fourier space. The choice (3.10) is motivated by their choice of a “reference medium”. Moreover, and importantly, their approach is easily extended to nonlinear media [19], and has successfully been used for studying elastoplasticity, elastoviscoplasticity, dislocations, shape memory polycrystals, and crack prediction in brittle materials: See [23] and references therein, where Zhou and Bhattacharya use a related augmented Lagrangian method, also introduced in the accelerated scheme of Michel, Moulinec, and Suquet [6], to study bifurcations and liquid crystal elastomers.

By substituting the formula \( B' = B - cI \) in (3.13) with \( c = \frac{1}{2}(b^+ + b^-) \) we obtain

\[
[I - \Gamma B]^{-1} \Gamma = \sum_{n=0}^{\infty} [\Gamma_1 B'/z_0^n] \Gamma_1 = \sum_{n=0}^{\infty} \frac{(A - cI)^n}{(z_0 - c)n} \Gamma_1,
\]

(3.14)

which is exactly the same expansion as in (3.2) in view of the identity (1.5). The advantage of the expansion (3.9) is that it allows more general choices of \( L_0 \) not necessarily proportional to \( I \).

In particular, for the resolvent (1.12) with \( B \) and \( z_0 \) being given by (1.13) so that \( b^+ = 1, b^- = 0 \) and \( c = 1/2 \) we obtain \( z_0' = \frac{1}{2}(z_2 + z_1)/(z_2 - z_1) \) and the expansion (3.13) becomes

\[
R = \frac{1}{2} \sum_{n=0}^{\infty} \Gamma_1 [(\chi_1 - \frac{1}{2} I)n/(z_0^n)] = \frac{1}{2} \sum_{n=0}^{\infty} r^n [\Gamma_1 (2\chi_1 - I)]^n, \quad \text{where} \quad r = \frac{z_2 - z_1}{z_2 + z_1}
\]

(3.15)

Comparing this with (2.5) we now have an expansion where \( \chi_1 \) is replaced by \( \chi_1 - \frac{1}{2} I \) which has half the spectral radius.

The expansion still converges for an appropriate value of \( z_0' \) when \( B \) is not Hermitian, but, for some \( z_0 \), \( L = L_0 - B = z_0 I - B \) is bounded and coercive in the sense that there is some \( \alpha > 0 \) and \( \beta > 0 \) such that

\[
\beta > \|L\|, \quad \text{Re}(LP, P) > \alpha|P|^2, \quad \text{for all} \quad P.
\]

Then, as proved in Section 2.4 of [16], with \( z_0' = \beta^2/\alpha \) one gets the bound

\[
\|B'/z_0'\| = \|I - L/z_0'\| \leq \sqrt{1 - (\alpha/\beta)^2} < 1,
\]

(3.17)

which ensures convergence of the series (3.13).

4 An accelerated convergence method

To obtain, in most cases, accelerated convergence we use the identity

\[
[I - \Gamma_1 B/z_0]^{-1} = [I - \Gamma'(L_0 - L)]^{-1} = [I + M(L - L_0) - (M - \Gamma')(L - L_0)]^{-1} = [I + M(L - L_0)]^{-1}[I - (M - \Gamma')(L - L_0)]^{-1}[I + M(L - L_0)]^{-1}
\]

(4.1)

where

\[
K = (L - L_0)[I + M(L - L_0)]^{-1}, \quad \Psi = M - \Gamma'.
\]

(4.2)

This identity had its genesis in formulae for the fields and effective tensors in laminated materials [7], later independently arrived at in [22]. Then it was further employed in representations for the effective conductivity of a
composite as a function of the component conductivities: see equation (5.20) in [15]. It was used in [1] to develop the fast numerical schemes that we generalize here (see also sections 14.9, 14.10, and 14.11 in [8]). It also has proved invaluable for the development of the theory of exact relations in composites [3, 5] (see also Chapter 17 in [8] and the book [4]), and in the affiliated development of exact identities satisfied by the Green’s function (fundamental solution) in certain classes of inhomogeneous media (not necessarily with microstructure) and the associated discovery of a wealth of new conservation laws, called boundary field equalities [9].

The rate of convergence of the series is enhanced when \( \mathbf{M} \) is chosen to make the norm of \( \mathbf{\Psi} \) small. When \( \mathbf{L}_0 \) is positive definite we have that \( \mathbf{\Gamma} \leq \mathbf{L}_0^{-1} \) and so a natural choice is \( \mathbf{M} = \frac{1}{2} \mathbf{L}_0^{-1} \). In this case

\[
\mathbf{K} = 2(\mathbf{L} - \mathbf{L}_0')(\mathbf{L} + \mathbf{L}_0')^{-1} \mathbf{L}_0', \quad \mathbf{\Psi} = (\mathbf{L}_0')^{-1}(\mathbf{I} - 2\mathbf{\Gamma}'\mathbf{L}_0')/2.
\] (4.3)

Further let us suppose that \( \mathbf{L}_0' = z_0'\mathbf{I} \). Then we obtain

\[
|\mathbf{L} - \mathbf{L}_0'(\mathbf{L} + \mathbf{L}_0')^{-1}| = |(\mathbf{L} - z_0'\mathbf{I})(\mathbf{L} + z_0'\mathbf{I})^{-1}| \leq r_2,
\] (4.4)

where

\[
r_2 = \max \left\{ \frac{|z_0 - b^+ - z_0'|}{|z_0 - b^+ + z_0'|}, \frac{|z_0 - b^- - z_0'|}{|z_0 - b^- + z_0'|} \right\}.
\] (4.5)

We choose

\[
z_0' = \sqrt{(z_0 - b^+)(z_0 - b^-)}
\] (4.6)

to minimize \( r_2 \), giving

\[
r_2 = \frac{\sqrt{q} - 1}{\sqrt{q} + 1},
\] (4.7)

where \( q = (z_0 - b^+)/(z_0 - b^-) \) is the same as that given in (3.12). The value of \( q \) is always greater than 1 when the series converges, i.e. provided \( z_0 < b^- \) or \( z_0 > b^+ \). Comparing this with the expression for \( r_1 \), we see that we get faster convergence since \( \sqrt{q} \) is smaller than \( q \) and significantly smaller when \( q \) is large.

Consider the case, relevant to two phase conducting composites, where \( \mathbf{L} = z_1\chi_1(\mathbf{x})\mathbf{I} + z_2\chi_2(\mathbf{x})\mathbf{I} \). With the choice \( z_0' = \sqrt{z_1z_2} \) one has

\[
\mathbf{K} = 2(\mathbf{L} - z_0'\mathbf{I})(\mathbf{L} + z_0'\mathbf{I})^{-1} \mathbf{L}_0' = 2z_0'\sqrt{z_1z_2} - 1 \sqrt{z_1z_2} + 1 (\chi_1 - \chi_2)\mathbf{I},
\] (4.8)

so that the expansion of (4.1) becomes

\[
\mathbf{R} = [\mathbf{I} - \mathbf{\Gamma}_1\mathbf{B}/z_0]^{-1} = (\mathbf{L} - \mathbf{L}_0')^{-1} \mathbf{K}(\mathbf{I} - \mathbf{\Psi}\mathbf{K})^{-1} = (\mathbf{L} - \mathbf{L}_0')^{-1} \mathbf{K} \sum_{n=0}^{\infty} (\mathbf{\Psi}\mathbf{K})^n
\]

\[
= 2\sqrt{z_1z_2}(\mathbf{L} + \mathbf{I}\sqrt{z_1z_2})^{-1} \sum_{n=0}^{\infty} [(2\chi_1 - 1)(\mathbf{I} - 2\mathbf{\Gamma}_1)]^n \left[ \frac{\sqrt{z_1z_2} - 1}{\sqrt{z_1z_2} + 1} \right]^n.
\] (4.9)

Comparing this with (2.5) we now have an expansion where effectively \( \chi_1 \) and \( \mathbf{\Gamma}_1 \) are replaced by \((\chi_1 - \frac{1}{2})\mathbf{I} \) and \( \mathbf{\Gamma}_1 - \frac{1}{2}\mathbf{I} \) thus having the spectral radius of both. Due to the appearance of the terms \( \sqrt{z_1z_2} \) in this expansion it is best suited to the case where \( \mathbf{B} \) and hence \( \mathbf{A} \) are Hermitian. However, the expansion still works if they are not self adjoint. For the case where \( \mathbf{B}(\mathbf{x}) \) takes the form

\[
\mathbf{B} = z_0\mathbf{I} - z_1\mathbf{P} - z_2\mathbf{I} - \mathbf{P},
\] (4.10)

where \( \mathbf{P} \) is a projection, but not a Hermitian one and not necessarily local in real space, one still has the expansion (4.9) but \( \mathbf{I} - 2\mathbf{\Gamma}_1 \) no longer has norm 1. Such expansions will be used in Section 5.

Note that we always have the freedom to rescale a selfadjoint bounded \( \mathbf{B} \) so that it is replaced by a positive semidefinite operator of norm less than 1. To do this we rewrite the formula for \( \mathbf{R} \) appearing at the end of the first line in (1.5) by

\[
\mathbf{R} = [\bar{z}_0\mathbf{\Gamma}_1 - \bar{\mathbf{B}}\mathbf{\Gamma}_1]^{-1}/\alpha, \quad \text{where} \quad \bar{z}_0 = (z_0 - c)/\alpha, \quad \bar{\mathbf{B}} = (\mathbf{B} - c\mathbf{\Gamma}_1)/\alpha,
\] (4.11)

in which we are free to choose \( c \) and \( \alpha \). Taking

\[
c = \frac{1}{2}(b^+ + b^-) \quad \text{and} \quad \alpha = \frac{1}{2}(b^+ - b^-)
\] (4.12)
then guarantees that the spectrum of $\tilde{B}$ is between 0 and 1.

We remark that this method does not always converge more rapidly than Moulinec and Suquet’s method. Generally it does, with a large factor of improvement. However, it depends on the spectrum of $A$, which we will consider in the next section. For example, for the conductivity of composites of two isotropic phases having conductivities $\sigma_1$ and $\sigma_2$, Moulinec and Suquet’s method can sometimes converge for negative ratios of $\sigma_1/\sigma_2$, this is never the case for the “accelerated” method, as the square roots in (4.8) induce singularities that prevent convergence when $\sigma_1/\sigma_2 < 0$. This is explored in more detail in [20]: see Figure 1.

![Figure 1: Rates of convergence in the $z_1/z_2$-plane and in the $r$-plane for the original Moulinec-Suquet scheme [19,20] where $r = (z_2 - z_1)/(z_2 + z_1)$. The intervals of possible singularities are marked by red lines. As concluded in [20], even without knowledge of $\alpha$ and $\beta$ (that are $\alpha = 0.35$ and $\beta = 0.8$ in this example) their scheme can converge for negative values of $z_1/z_2$ and outperform the “Eyre-Milton” scheme in certain regions of the complex $z_1/z_2$-plane. The convergence rates for the “Eyre-Milton” scheme, correspond to those in the $z_1/z_2$-plane in Figure 3. The contours reflect the number of iterations $m$ needed for convergence to a tolerance $\epsilon$. In the $r$-plane, for small enough $\epsilon$, one needs at radius $r$ for $m$ to be such that $c(r/r_0)^m \approx \epsilon$ for some constant $c$ where $r_0 > 1$ is the radius of convergence, i.e., $m \approx \log(\epsilon/c)/\log(r/r_0)$ so we have plotted the contours of $-1/\log(r/r_0)$ and their preimages in the $z_1/z_2$-plane.]

Other accelerated schemes that do not use information about the spectrum of $A$ include those of Michel, Moulinec, and Suquet [6] and Monchiet and Bonnet [17]. All three accelerated schemes are compared in [18].

5 Getting even faster convergence when we have bounds on the spectrum of $A$

So far in developing our expansions we have used bounds on the operator $B$, but using the tools presented in Section 3 of [14], or otherwise, we may have bounds on the spectrum of the operator $A = \Gamma_1 B \Gamma_1$ in the subspace $E$ and, as we will see now, this information can be used to obtain faster convergence. The route explored here is by no means obvious but has its foundations in the theory of superfunctions, including the ideas of nonorthogonal subspace collections, as developed in Chapter 7 of [16], and the idea of substituting of one subspace collection into another subspace collection (first introduced in Section 29.1 of [8]). The analysis here closely parallels that in Chapter 8 of [16], which also outlines the reasoning for following the steps here.

To start we consider the following linear algebra problem: given $h, s, p_1, p_2, p_3$ and $E_1$, solve the matrix equation,

$$
\begin{pmatrix}
J & 0 \\
0 & J_2
\end{pmatrix}
\begin{pmatrix}
E \\
E_1
\end{pmatrix} = z_0
\begin{pmatrix}
E \\
E_1
\end{pmatrix} - s
\begin{pmatrix}
p_1^2 & p_1 p_2 & p_1 p_3 \\
p_2 p_1 & p_2^2 & p_2 p_3 \\
p_3 p_1 & p_3 p_2 & p_3^2
\end{pmatrix}
\begin{pmatrix}
E \\
E_1 \\
0
\end{pmatrix} - \begin{pmatrix}
h \\
0 \\
0
\end{pmatrix},
$$

(5.1)
for \( J \) in terms of \( E \). We will ultimately allow for \( p_1, p_2 \) and \( p_3 \), that are either real or purely imaginary, chosen with
\[
p_3^2 = 1 - p_1^2 + p_2^2, \tag{5.2}
\]
to ensure that \( P \) is a projection matrix, though not selfadjoint in our application. The significance of (5.1) is that it corresponds to a problem in the abstract theory of composites: define \( \mathcal{U}, \mathcal{E}, \) and \( \mathcal{J} \) to be the three subspaces spanned by the three unit vectors
\[
w_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad w_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad w_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \tag{5.3}
\]
respectively, so that \( \Gamma_i = w_i \otimes w_i, \ i = 1, 2, 3 \), are the projections onto \( \mathcal{U}, \mathcal{E}, \) and \( \mathcal{J} \) respectively. The associated projections are
\[
\Gamma_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Gamma_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Gamma_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \tag{5.4}
\]

Then (5.1) reduces to
\[
Jw_0 + J_2w_1 = \mathbf{L}(Ew_0 + E_1w_1) - hw_0, \quad \mathbf{L} = z_0\mathbf{I} - sP = (z_0 - s)P + z_0(I - P), \tag{5.5}
\]
which is a problem in the abstract theory of composites, that more generally takes the form: given \( E_0 \in \mathcal{U} \), and a source term \( \mathbf{h} \) in \( \mathcal{H} = \mathcal{U} \oplus \mathcal{E} \oplus \mathcal{J} \), and an operator \( \mathbf{L} \) mapping \( \mathcal{H} \) to \( \mathcal{H} \), find \( \mathbf{J}_0 \in \mathcal{U}, \mathbf{E} \in \mathcal{E} \) and \( \mathbf{J} \in \mathcal{J} \) such that
\[
\mathbf{J}_0 + \mathbf{J} = \mathbf{L}(\mathbf{E}_0 + \mathbf{E}) - \mathbf{h}. \tag{5.6}
\]

In our case, the subspaces \( \mathcal{U}, \mathcal{E}, \) and \( \mathcal{J} \) are clearly orthogonal, but \( P \) and \( \mathbf{I} - P \) do not generally project onto orthogonal subspaces: we have a nonorthogonal subspace collection when \( p_1, p_2 \) and \( p_3 \) are not all real.

The motivation for considering this problem is that the abstract theory of composites applies to resistor networks with say resistors having resistances \( R_1 \) and \( R_0 \). We have the freedom to replace every resistor in the network having resistance \( R_1 \) by a circuit just containing two weighted resistances \( R_0 \) and \( R_2(R_1) \), where \( R_2(R_1) \) is chosen so the net resistance (effective resistance) of the circuit equals \( R_1 \) and, say, \( R_2(1) = 1 \). Then the resistance \( R_0(R_1, R_0) \) of the entire network as a function of \( R_1 \) and \( R_0 \) will be the same as the resistance \( R_0(R_2, R_0) \) of the new network, having resistances \( R_0 \) and \( R_2 \) when \( R_2 = R_2(R_1) \). In particular, we can take the circuit to consist of a weighted resistance \( q_1R_0 \) in series with weighted resistances \( q_2R_2/t_2 \) and \( q_2R_0/t_0 \) in parallel, where \( q_1 + q_2 = 1 \) and \( t_0 + t_2 = 1 \), giving
\[
R_1 = q_1R_0 + \frac{q_2}{t_0/R_0 + t_2/R_2}. \tag{5.7}
\]

Mathematically, this step of replacing every resistor in the network having resistance \( R_1 \) by a circuit containing the resistances \( R_2(R_1) \) and \( R_0 \) is an example of substitution in subspace collections. The linear algebra problem (5.1) is nothing other than the equations one solves to arrive at (5.7), allowing for a source term \( s \). A field is a three dimensional vector. The projection \( P \) is nothing other than the projection onto the one dimensional space of fields in the resistor \( R_2 \); \( \mathcal{U} \oplus \mathcal{E} \) is the two dimensional space of fields corresponding to electrical currents, meeting the Kirchoff condition that the net currents flowing into a node equates with the net currents flowing out of that node, \( \mathcal{U} \oplus \mathcal{J} \) is the two dimensional space of fields resulting from potential drops, \( \mathcal{U} \) is the one dimensional space of fields that arise in the circuit when \( R_0 = R_2 = 1 \). (The spaces \( \mathcal{E} \) and \( \mathcal{J} \) are perhaps the reverse of what one first expects, but that is because we have resistances rather than conductances).

To find the norm of \( P \) we consider its action on a possibly complex vector \( a \). We have
\[
|Pa| = |p(p \cdot a)| \leq |p|^2|a|, \tag{5.8}
\]
with equality when \( a = p \). Thus \( P \) has norm
\[
|P|^2 = |p_1|^2 + |p_2|^2 + |p_3|^2, \tag{5.9}
\]
and this will surely be greater than or equal to 1 if (5.2) holds and \( p_1, p_2 \) and \( p_3 \) are either real or purely imaginary. For example, if \( p_1 \) is purely imaginary while \( p_2 \) and \( p_3 \) are purely real then (5.9) implies
\[
1 = -|p_1|^2 + |p_2|^2 + |p_3|^2 = |P|^2 - 2|p_1|^2, \tag{5.10}
\]
Figure 2: The substitution of orthogonal subspace collections parallels that of substituting in a two resistor network (a), chosen to have four terminals, the subnetwork (b), to obtain the new network (c). If $R_2$ is chosen so the net resistance of the subnetwork is $R_1$ then the response of the four terminal network (c) will be the same as the four terminal network (a). Our substitution of nonorthogonal subspace collections corresponds to taking $t_0$ negative. This has a physical interpretation if we replace all resistors with positive resistance by capacitors and all resistors with negative resistance by inductors and subject the network to voltages oscillating with a given frequency $\omega$. Adapted from Figure 7.7 in [16].

which forces $|P|^2$ to be greater than or equal to 1.

The matrix equation (5.1) is clearly satisfied with

$$E_1 = \frac{sp_1p_2}{z_0 - sp_2} E,$$

$$J = (z_0 - sp_2^2)E - sp_1p_2E_1 - h = \left( z_0 - p_1^2s - \frac{s^2p_1^2p_2^2}{z_0 - sp_2^2} \right) E - h$$

$$= \left( z_0 - \frac{p_1^2z_0s}{z_0 - sp_2^2} \right) E - h = (z_0 - b)E - h,$$

(5.11)

where

$$b = \frac{p_1^2z_0}{z_0/s - p_2^2}.$$  

(5.12)

A correspondence with (5.7) can be made by making the substitutions:

$$s = z_0 - z_2, \quad b = z_0 - z_1, \quad p_2^2 = t_0 = 1 - t_2, \quad p_1^2 = q_2t_2.$$  

(5.13)

Solving (5.12) for $s$ in terms of $b$ gives

$$s = \frac{b_2z_0}{p_1^2z_0 + bp_2^2}.$$  

(5.14)

Suppose now that in the extended abstract theory of composites we are interested in solving the equations

$$J(x) = [z_0I - B(x)]E(x) - s(x), \quad \text{with} \quad \Gamma_1E = E, \quad \Gamma_1J = 0,$$  

(5.15)

or equivalently in finding the resolvent [1.2] with $A = \Gamma_1B\Gamma_1$. Setting

$$S(x) = z_0B(x)[p_1^2z_0I + p_2^2B(x)]^{-1},$$  

(5.16)
our preliminary linear algebra problem shows this is equivalent to solving

\[
\begin{pmatrix}
J(x) \\
0 \\
J_2(x)
\end{pmatrix} = \begin{pmatrix}
z_0 I - \begin{pmatrix}
p_1^2 I & p_1 p_2 I & p_1 p_3 I \\
p_1 p_2 I & p_2^2 I & p_2 p_3 I \\
p_1 p_3 I & p_2 p_3 I & p_3^2 I
\end{pmatrix} & S(x) & 0 & 0 \\
0 & S(x) & 0 & 0
\end{pmatrix}
\begin{pmatrix}
E(x) \\
0 \\
E_2(x) \\
E_3(x)
\end{pmatrix} - \begin{pmatrix}
0 \\
0 \\
0 \\
0
\end{pmatrix},
\]

(5.17)

with \( \Gamma_1 E = E \) and \( \Gamma_1 J \). We are back at an equivalent problem in the extended abstract theory of composites as both \( J \) and \( E \) lie in orthogonal spaces. Specifically, we have

\[
J(x) = L(x)E(x) - s(x), \quad \Gamma_1 E = E, \quad \Gamma_1 J = 0 \quad \text{with} \quad L(x) = z_0 I - B(x), \quad \Gamma_1 = \begin{pmatrix}
\Gamma_1 & 0 & 0 \\
0 & I & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

(5.18)

To see how this can improve convergence, let us consider the case where \( B(x) \) and \( z_0 \) are given by (5.13). Then

\[
S(x) = \frac{z_0 \chi(x)}{p_1^2 z_0 + p_2^2} = -\frac{z_0 \chi(x)(z_1 - z_2)}{p_1^2 z_2 - p_2^2(z_1 - z_2)},
\]

(5.19)

and associated with (5.17) is the resolvent

\[
[z_0 I - \Gamma_1 B]^{-1} = z_0^{-1} \{I - [(z_2 - z_1)/z_2] \Gamma_1 A\}^{-1},
\]

(5.20)

where

\[
\Lambda(x) = \chi(x)p \otimes p, \quad z_2 = 1, \quad z_1 = 1 + \frac{z_0(z_1 - z_2)}{p_1^2 z_2 - p_2^2(z_1 - z_2)}.
\]

(5.21)

Note that \( \Lambda \) is a projection operator because both \( p \) and \( \chi \) are projections and thus the operator inverse in (5.20) has exactly the same form as in (5.12) with \( z_1, z_2 \) and \( \chi_1 \) being replaced by \( \tilde{z}_1, \tilde{z}_2 \), and \( \Lambda \). Thus (5.20) can be thought of as the resolvent associated with some sort of “two phase composite” with moduli \( \tilde{z}_1 \) and \( \tilde{z}_2 \). Also \( \tilde{z}_1 \) can be re-expressed in the form

\[
\tilde{z}_1 = 1 + \frac{(z_1 - z_2)(\beta - \alpha)}{(z_1 + \beta z_2)(1 + \alpha)} = \frac{(z_1 + \alpha z_2)(1 + \beta)}{(z_1 + \beta z_2)(1 + \alpha)},
\]

(5.22)

with

\[
\alpha = -1 - \frac{p_1^2}{p_2^2}, \quad \beta = -1 - \frac{p_1^2}{p_2^2}.
\]

(5.23)

Note that \( -\alpha \) (respectively \( -\beta \)) is obtained by substituting \( t = 0 \) (respectively \( t = \infty \)) in (5.21). Given real \( \beta > \alpha > 0 \) we need to choose \( p_1 \) and \( p_2 \) so that these equations are satisfied. This will necessitate complex solutions for \( p_2 \) since otherwise \( \beta \) will be negative. Explicitly we have

\[
p_1^2 = \left[ \frac{1}{1 + \alpha} - \frac{1}{1 + \beta} \right]^{-1}, \quad p_2^2 = \left[ 1 - \frac{1 + \beta}{1 + \alpha} \right]^{-1}.
\]

(5.24)

With \( p_1 \) being real and \( p_2 \) being purely imaginary, and so \( \Lambda \) is no longer Hermitian, even though it is a projection. This translates to a problem in the extended abstract theory of composites with a nonorthogonal subspace collection, as introduced in Chapter 8 of [10].

Next, we follow the steps outlined in the previous section, though now \( \Lambda \) does not have norm 1. We end up with an expansion

\[
[I - \Gamma_1 B/z_0]^{-1} = (L - L_0)^{-1}K[I - \Psi K]^{-1} = (L - L_0)^{-1}K \sum_{n=0}^{\infty} (\Psi K)^n
\]

\[
= 2\sqrt{z_1 z_2}(L + I\sqrt{z_1 z_2})^{-1} \sum_{n=0}^{\infty} [(2\chi_1 - I)(I - 2\Gamma_1)]^n \left[ \frac{\sqrt{z_1/z_2} - 1}{\sqrt{z_1/z_2} + 1} \right]^n
\]

\[
= \sum_{n=0}^{\infty} n^n C_n
\]

\[
= \sum_{n=0}^{\infty} C_n \left( \frac{\sqrt{z_1/z_2} - 1}{\sqrt{z_1/z_2} + 1} \right)^n = \sum_{n=0}^{\infty} C_n \left( \frac{\sqrt{(z_1 + \alpha z_2)(1 + \beta)}}{(z_1 + \beta z_2)(1 + \alpha)} - 1 \right)^n,
\]

(5.25)
where
\[ C_n = 2\sqrt{\frac{1}{\alpha^2} + 1} (L + 1) \sum_{n=0}^{\infty} \left| 2\Lambda - I \right| \left( I - 2\Gamma_1 \right)^n, \tag{5.26} \]
and
\[ v = \frac{w - 1}{w + 1}, \quad w = \sqrt{\frac{z_1}{z_2}}, \quad \hat{z}_1 \hat{z}_2 = \frac{(z_1/z_2 + \alpha)(1 + \beta)}{(z_1/z_2 + \beta)(1 + \alpha)}. \tag{5.27} \]

We now obtain lower bounds on the rate of convergence of the series using bounds on the spectrum of A. We suppose that the spectrum of A = \( \Gamma_1 \chi_1 \) on the subspace E lies inside the interval between \( a^- \) and \( a^+ \) (i.e. A satisfies (2.2)) and we let \( \alpha = (1/a^-) - 1 \) and \( \beta = (1/a^+) - 1 \) so that the singularities of \( \Gamma_1 L \Gamma_1 \) lie between \( z_1/z_2 = -\alpha \) and \( z_1/z_2 = -\beta \). Now v is obtained from \( z_1/z_2 \) through a series of transformations \( z_1/z_2 \rightarrow \hat{z}_1/\hat{z}_2 \rightarrow w \rightarrow v \) as indicated in Figure 3, which also shows how the possible singularities of A transform under these changes of variable. The mappings transform the singularities between \( z_1/z_2 = -\alpha \) and \( z_1/z_2 = -\beta \) in the \( z_1/z_2 \)-plane to singularities around the edge of the unit disk in the v-plane. The radius of convergence of the series is dictated by the resolvent’s nearest singularity to the origin in the v-plane. By construction, all singularities lie outside the unit disk in the v-plane and the mapping from \( \hat{z}_1/\hat{z}_2 \) to \( w = \sqrt{\frac{z_1}{z_2}} \) will create a singularity at the origin in the w-plane corresponding to a singularity on the unit disk. Consequently we deduce that
\[ \| (2\Lambda - I)(I - 2\Gamma_1) \| = 1. \tag{5.28} \]

This is by no means obvious as A, like P in (5.1), has norm exceeding 1.

It is to be emphasized that \( a^+ \) and \( a^- \) can be replaced by estimates of \( a^+ \) and \( a^- \), such as obtained by Rayleigh Ritz methods, or by the power method as reviewed at the beginning of Section 3 in [14]. One can still apply the same transformations only now \( (2\Lambda - I)(I - 2\Gamma_1) \) will have norm greater than 1.

If B is selfadjoint but not a projection operator, it is not unclear how to choose \( \alpha \) and \( \beta \) and it is also unclear how to bound the norm of the operator \( \Psi K \). However, after normalizing B as in (4.11) and (4.12) to ensure its spectrum is between 0 and 1, then it would be natural to choose \( \alpha \) and \( \beta \) so that the spectrum of \( A \) lies inside the interval between \( 1/(1 + \alpha) \) and \( 1/(1 + \beta) \). To determine the success of such an approach requires further analysis and/or numerical investigations.

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References

[1] David J. Eyre and Graeme W. Milton. A fast numerical scheme for computing the response of composites using grid refinement. European Physical Journal. Applied Physics, 6(1):41–47, April 1999.

[2] Gene H. Golub and Charles F. Van Loan. Matrix Computations. John Hopkins University Press, Baltimore and London, third edition, 1996.

[3] Yury Grabovsky. Exact relations for effective tensors of polycrystals. I: Necessary conditions. Archive for Rational Mechanics and Analysis, 143(4):309–329, 1998.

[4] Yury Grabovsky. Composite Materials: Mathematical Theory and Exact Relations. IOP Publishing, Bristol, UK, 2016.

[5] Yury Grabovsky, Graeme W. Milton, and Daniel S. Sage. Exact relations for effective tensors of composites: Necessary conditions and sufficient conditions. Communications on Pure and Applied Mathematics (New York), 53(3):300–353, March 2000.

[6] J. C. Michel, H. Moulinec, and Pierre M. Suquet. A computational method based on augmented Lagrangians and Fast Fourier Transforms for composites with high contrast. Computer Modeling in Engineering and Sciences, 1(2):79–88, 2000.
Figure 3: Convergence rates in the different planes, extending the analysis in Chapter 8 of [16] and in [20]. The mappings transform singularities between $-\beta$ and $-\alpha$ (with $\alpha = 0.5$ and $\beta = 2$ in this example) in the complex $z_1/z_2$-plane to singularities around the edge of the unit disk in the $v$-plane. The possible range of singularities are marked in red, though in the last two figures one could have singularities in the analytic continued function outside the unit disk in the $v$-plane or in the left hand side of the $w$-plane. The contours, as in Figure 1, reflect the number of iterations $m$ needed for convergence. They are level curves of $-1/\log(r)$ in the $v$-plane and their preimages in the other planes. Here $-\beta$ and $-\alpha$ could be outerbounds on the spectrum, or they could be sharp bounds marking the endpoints of the spectrum. Note that the contours in the $z_1/z_2$-plane coincide with those for the accelerated "Eyre-Milton" scheme in the $z_1/z_2$-plane, corresponding to the case $\beta = \infty$ and $\alpha = 0$.

[7] Graeme W. Milton. On characterizing the set of possible effective tensors of composites: The variational method and the translation method. Communications on Pure and Applied Mathematics (New York), 43(1):63–125, 1990.

[8] Graeme W. Milton. The Theory of Composites, volume 6 of Cambridge Monographs on Applied and Computational Mathematics. Cambridge University Press, Cambridge, UK, 2002. Series editors: P. G. Ciarlet, A. Iserles, Robert V. Kohn, and M. H. Wright.

[9] Graeme W. Milton. A new route to finding bounds on the generalized spectrum of many physical operators. Journal of Mathematical Physics, 59(6):061508, jun 2018.
Figure 4: Convergence rates when one only has estimates of $a^+$ and $a^-$ obtained via the Rayleigh Ritz method, or by the power method. One can still use the same transformations. However, now there will be branch cuts extending (ideally slightly) within the unit disk in the $v$-plane, say a distance $d_l$ on the left side and a distance $d_r$ on the right side. As a consequence the radius of convergence $r_0 < 1$ of the series in the $v$-plane will be the minimum of $1 - d_l$ and $1 - d_r$, with a corresponding change in the rates of convergence of the series as indicated by the contours of $-1/\log(r/r_0)$ in the $v$-plane and their preimages in the other planes. As in the previous figure, the possible singularities are marked in red. The contours, as in Figure 1, reflect the number of iterations $m$ needed for convergence. Here $-\beta$ and $-\alpha$ are the estimates of the endpoints of the spectrum in the $z_1/z_2$ plane, in this example $\alpha = 1$ and $\beta = 2$. The actual endpoints are the endpoints of the redline.

[10] Graeme W. Milton. A unifying perspective on linear continuum equations prevalent in physics. part i: Canonical forms for static and quasistatic equations. Available as arXiv:2006.02215 [math.AP], 2020.

[11] Graeme W. Milton. A unifying perspective on linear continuum equations prevalent in physics. part ii: Canonical forms for time-harmonic equations. Available as arXiv:2006.02433 [math-ph], 2020.

[12] Graeme W. Milton. A unifying perspective on linear continuum equations prevalent in physics. part iii: Canonical forms for dynamic equations with moduli that may, or may not, vary with time. Available as arXiv:2006.02432 [math-ph], 2020.
[13] Graeme W. Milton. A unifying perspective on linear continuum equations prevalent in physics. part iv: Canonical forms for equations involving higher order gradients. Available as arXiv:2006.03161 [math-ph], 2020.

[14] Graeme W. Milton. A unifying perspective on linear continuum equations prevalent in science. part v: resolvents; bounds on their spectrum; and their stieltjes integral representations when the operator is not selfadjoint. Available as arXiv:2006.03162 [math-ph], 2020.

[15] Graeme W. Milton and Kenneth M. Golden. Representations for the conductivity functions of multicomponent composites. Communications on Pure and Applied Mathematics (New York), 43(5):647–671, 1990.

[16] Graeme W. Milton (editor). Extending the Theory of Composites to Other Areas of Science. Milton–Patton Publishers, P.O. Box 581077, Salt Lake City, UT 85148, USA, 2016.

[17] Vincent Monchiet and Guy Bonnet. A polarizationbased FFT iterative scheme for computing the effective properties of elastic composites with arbitrary contrast. International Journal for Numerical Methods in Engineering, 89(11):1419–1436, November 2011.

[18] H. Moulinec and F. Silva. Comparison of three accelerated FFT-based schemes for computing the mechanical response of composite materials. International Journal for Numerical Methods in Engineering, 97(13):960–985, March 2014.

[19] H. Moulinec and Pierre M. Suquet. A fast numerical method for computing the linear and non-linear properties of composites. Comptes rendus des Séances de l’Académie des sciences. Série II, 318(??):1417–1423, 1994.

[20] Hervé Moulinec, Pierre Suquet, and Graeme W. Milton. Convergence of iterative methods based on Neumann series for composite materials: theory and practice. International Journal for Numerical Methods in Engineering, 114(10):1103–1130, January 2018.

[21] John R. Willis. Variational and related methods for the overall properties of composites. Advances in Applied Mechanics, 21:1–78, 1981.

[22] V. V. Zhikov. Estimates for the homogenized matrix and the homogenized tensor. Uspekhi Matematicheskikh Nauk = Russian Mathematical Surveys, 46:49–109, 1991. English translation in Russ. Math. Surv. 46(3):65–136 (1991).

[23] Hao Zhou and Kaushik Bhattacharya. An operator split for accelerated computational micromechanics. 2020. Submitted.