Optimal Design for Synchronization of Kuramoto Oscillators in Tree Networks
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Abstract—In this paper, we develop an optimization framework to optimize phase synchronization in tree networks of coupled Kuramoto oscillators. We consider the phase cohesiveness metric, that accounts for the maximum phase difference across all the edges in the network, we address three different optimization problems: (i) The optimal critical coupling problem in which we design the natural frequency of each oscillator to minimize the required coupling for synchronization; (ii) the nodal-frequency design problem, in which we design the natural frequencies to optimize the phase cohesiveness; and (iii) the edge-weight design problem, in which we design the link weights. We assume that tuning the natural frequencies and/or modifying the link weights has an associated cost and develop an optimization framework to find the optimal allocation of resources under certain assumptions. We illustrate the effectiveness of our approach via numerical simulations.

I. INTRODUCTION

Synchronization in networks of coupled oscillators is one of the most fundamental problems in networked dynamical systems. Many real world phenomena can be modeled as a system of coupled oscillators, from biological systems (such as pacemaker cells in the heart) to technological systems (such as clock synchronization in computing networks and power grids). Network of coupled oscillators present a rich dynamic behavior, as reported in the vast literature on this topic [5]–[11]. In this paper, we analyze the following question: How can we design the network with uniform coupling strengths, in which we label each edge

in [12] as a condition of synchronization in networks of Kuramoto oscillators. We then consider three related problems: (i) Optimal critical coupling problem in which we design the natural frequency of each oscillator to minimize the required coupling for synchronization; (ii) the frequency design problem, in which we design the link weights. The rest of the paper is organized as follows. In Section II some preliminaries are covered and the problem is formally defined. Section III analyzes the dynamics and stability of the Kuramoto oscillators which is required for development of the main results. In Section IV the framework of the optimal synchronization problems is developed. Illustrative examples are highlighted in Section V. Concluding remarks are drawn in section VI.

II. BACKGROUND AND PROBLEM STATEMENT

A. Preliminaries and Notation

Notation: Let 1n and 0n be the n-dimensional vector of unit and zero entries. In denotes the n-dimensional identity matrix. Given n-tuple (x₁,...,xₙ), let x ∈ ℝⁿ be the associated vector. The infinity norm of x is denoted as ||x||∞ = max₁≤i≤n|xᵢ|. Define the vector-valued functions

\[ \sin(x) = (\sin(x₁), \cdots, \sin(xₙ)) \]

and

\[ \sin^{-1}(x) = (\sin^{-1}(x₁), \cdots, \sin^{-1}(xₙ)) \]

where the sin⁻¹ function is defined on the interval [−π/2, π/2]. For a given matrix A ∈ ℝᵐ×ⁿ, the kernel of A is the set ker(A) = {x ∈ ℝⁿ : Ax = 0ₙ}. We denote S as the unit circle.

Algebraic Graph Theory: A graph is defined as G := (V,E), where V is a set of n nodes (or vertices) and E is a set of m undirected edges (or links). We assume that the graph is connected and has no self-loops. We consider graphs with weights associated to edges and nodes. We denote the weight of an edge e = {i,j} ∈ E (respectively a node i ∈ V) as wₑ or wᵢj (respectively ωᵢ).

The weighted adjacency matrix of a directed graph G, denoted by A_G = [aᵢⱼ], is an n × n symmetric matrix defined entry-wise as aᵢj = wᵢj if {i,j} ∈ E, and aᵢj = 0, otherwise. The weighted Laplacian matrix of G is defined as L_G = diag(A_G1_n) − A_G. The (unweighted) incidence matrix of G is defined on an directed and labeled version of the undirected graph G, in which we label each edge

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in the graph with a unique label \( e \in \{1, \ldots, m\} \) and assign an arbitrary direction to it. In other words, we substitute each undirected edge \( \{i, j\} \in E \) for an ordered pair \( (i, j) \), in arbitrary order. For the ordered pair \( (i, j) \), we say that \( i \) is the source and \( j \) is the sink of the directed edge. For the resulting directed graph, the incidence matrix \( B = [b_{ie}] \in \mathbb{R}^{n \times m} \) is defined component-wise as \( b_{ie} = 1 \) if \( j \) is the sink node of edge \( e \), \( b_{ie} = -1 \) if \( i \) is the source node of edge \( e \), and \( b_{ie} = 0 \) otherwise. For \( x \in \mathbb{R}^n \), notice that \( (BT x)_e = x_j - x_i \) for a link \( (i, j) \) labeled \( e \).

For a weighted graph, we assign an arbitrary direction to it. In other words, we can define its algebraic connectivity, strictly positive \([13]\). Although the Laplacian matrix is noninvertible, we can define its Moore-Penrose pseudo-inverse as \( L' := (L + \frac{1}{n} \mathbf{1}_n \mathbf{1}_n^T)^{-1} - \frac{1}{n} \mathbf{1}_n \mathbf{1}_n^T \). If the eigenvalue decomposition of \( L \) is given by \( L = U \text{diag}(\{\lambda_1, \ldots, \lambda_n\}) U^T \), its Moore-Penrose pseudo inverse is given by \( L' = U \text{diag}(\{\lambda_1^{-1}, \ldots, \lambda_n^{-1}\}) U^T \) \([13]\). For any connected graph with \( n \) vertices, the identity \( LL' = L' L = I_n - \frac{1}{n} \mathbf{1}_n \mathbf{1}_n^T \) holds.

Elements of (Quasi) Convex Optimization: A set \( C \subseteq \mathbb{R}^n \) is convex if for any \( x_1, x_2 \in C \) and any \( 0 \leq \theta \leq 1 \), we have that \( \theta x_1 + (1 - \theta) x_2 \in C \). A function \( f : \mathbb{R}^n \to \mathbb{R} \) is called convex if its domain is convex, and if for all \( x, y \in \text{dom}(f) \), and \( \theta \in [0, 1] \), we have that \( f(\theta x + (1 - \theta) y) \leq \theta f(x) + (1 - \theta) f(y) \). A function \( f : \mathbb{R}^n \to \mathbb{R} \) is called quasi-convex if its domain is convex and all its sublevel sets, \( S_{\alpha} := \{x \in \text{dom}(f) : f(x) \leq \alpha\} \), are convex for \( \alpha \in \mathbb{R} \). An equivalent definition is that for any \( x, y \in \text{dom}(f) \) and \( \theta \in [0, 1] \), the inequality \( f(\theta x + (1 - \theta) y) \leq \max \{f(x), f(y)\} \) holds. An optimization problem with convex feasible set and (quasi) convex objective function is called a (quasi) convex optimization problem \([14]\).

B. Synchronization in Networks of Heterogeneous Oscillators

In this section, we describe the dynamics of the network of coupled oscillators under consideration and present a synchronization condition recently proposed in \([12]\). Although this synchronization condition applies to various models of coupled oscillators, we will use the Kuramoto model in our exposition, which can be described by the following set of nonlinear ODEs:

\[
\dot{\theta}_i(t) = \omega_i - \sum_{j=1}^{n} a_{ij} \sin(\theta_j(t) - \theta_i(t)), \quad \text{for } i \in V, \quad (1)
\]

where \( \theta_i(t) \in \mathbb{S} \) is the angular position of the \( i \)-th oscillator, \( \omega_i \) is its natural frequency, and \( a_{ij} \) is the weight of edge \( \{i, j\} \) in the network. We have the following definitions:

**Definition II.1.** A solution to the coupled oscillator model \([1]\) is said to be frequency-synchronized if the frequencies \( \theta_i(t) \) become asymptotically identical, i.e. \( \lim_{t \to \infty} |\dot{\theta}_i(t) - \dot{\theta}_j(t)| = 0, \forall \{i, j\} \in E \).

It is straightforward to show that the synchronized frequency of model \([1]\) is given by the average of the natural frequencies, \( \omega_{\text{sync}} := \frac{1}{n} \mathbf{1}_n \omega = \frac{1}{n} \sum_{i=1}^{n} \omega_i \), where \( \omega := (\omega_1, \ldots, \omega_n)^T \). In general, we can transform the Kuramoto dynamics using a rotating reference frame such that \( \omega_{\text{sync}} = 0 \). For a frequency-synchronized solution, phase differences \( |\theta_i(t) - \theta_j(t)| \) converge to a constant. Accordingly, we have the following definition:

**Definition II.2.** A frequency-synchronized solution to the coupled oscillator model \([1]\) is said to be phase-synchronized if all the phases become asymptotically identical, i.e. \( \lim_{t \to \infty} |\theta_i(t) - \theta_j(t)| = 0, \forall \{i, j\} \in E \).

By substituting the phase-synchronized solution \( \theta(t) = \theta_{\text{sync}}(t) \mathbf{1}_n \) into \([1]\), it is observed that phase-synchronization is achievable only when all the natural frequencies are identical; otherwise, one can only expect frequency-synchronization. In \([12]\), the synchronization condition is described in terms of the set \( \Delta_G(\gamma) := \{\{\theta_1, \ldots, \theta_n\} : |\theta_i - \theta_j| \leq \gamma \text{ for all } \{i, j\} \in E\} \), for \( \gamma \in [0, \pi/2] \). Also, let \( \Delta_G(\gamma) \) be the interior of \( \Delta_G(\gamma) \). Then, we have the following alternative definition:

**Definition II.3.** A solution \( \theta(t) := (\theta_1(t), \ldots, \theta_n(t))^T \) to the Kuramoto model in \([1]\) is said to be synchronized if there exists \( \gamma \in [0, \pi/2] \) and \( \omega_{\text{sync}} \in \mathbb{R}^n \) such that \( \lim_{t \to \infty} \theta(t) = \Delta_G(\gamma) \) and \( \lim_{t \to \infty} \dot{\theta}(t) = \omega_{\text{sync}} \mathbf{1}_n \). The minimum value of such \( \gamma \) is called phase cohesiveness.

**Remark II.1.** The restriction of phase cohesiveness to the first quadrant is required for stability of the synchronized solution, which is addressed in Section III.

C. Synchronization Criteria

Occurrence of synchronization in coupled oscillators depends on the competition between the coupling strength and heterogeneity of the natural frequencies. Roughly speaking, the oscillator model \([1]\) synchronizes if the network coupling dominates the heterogeneity in the natural frequencies.

One measure of synchronization is the so-called order parameter \( R \) of the ensemble of the oscillators which is defined as follows:

\[
Re^{j\psi} = \frac{1}{N} \sum_{j=1}^{N} e^{j \theta_j} \quad (2)
\]

The magnitude of \( R \) quantifies the degree of alignment. \( R = 1 \) corresponds to complete alignment, while \( R = 0 \) shows complete misalignment. In a related work in \([13]\), \( R \) is approximated in the strongly synchronized regime (where
the phases are closely clustered) in terms of the steady state solutions and the optimal network and/or the optimal set of natural frequencies are designed based on maximizing $R$.

In [12], the following synchronization criterion was proposed:

**Criterion 1.** Consider a network of oscillators coupled via a graph $G$ with incidence matrix $B$, link strengths $w = (w_1, \ldots, w_n)^T$, and natural frequencies $\omega = (\omega_1, \ldots, \omega_n)^T$. Then, if

$$\|L^T \omega\|_\infty := \|B^T L^T \omega\|_\infty \leq \sin(\gamma),$$  

(3)

the network achieves a level $\gamma$ of phase cohesiveness, for $\gamma \in [0, \pi/2)$.

Criterion 1 guarantees the existence of a unique and stable solution $\theta(t)$ with synchronized frequencies and cohesive phases $|\theta_i - \theta_j| \leq \gamma$ for every connected pair $\{i, j\} \in \mathcal{E}$. In this paper, we exploit criterion 1 to optimize the phase cohesiveness of the synchronized solution, treating the natural frequencies $\omega$ and the weights $w$ as optimization variables.

### III. SYNCHRONIZED SOLUTION AND STABILITY

#### A. Fixed-Points Characterization

In the following, we study the fixed points of the Kuramoto model corresponding to the frequency-synchronized solution. The model 1 can be cast in matrix form as:

$$\dot{\theta}(t) = f(\theta(t)) \triangleq \omega - BW \sin(B^T \theta(t)), \quad (4)$$

where $W := \text{diag}(w)$ is the diagonal matrix of the weights. The frequency-synchronized solution is $\theta(t) = \omega_{\text{sync}} 1_n$ or $\theta(t) = \omega_{\text{sync}} 1_n + \theta^*$ for some fixed $\theta^* \in \mathbb{R}^n$. Since the phase differences across the edges rather than their absolute values are important, we premultiply both sides of (4) by $B^T$ to express the dynamics in terms of edge phase difference vector $\varphi(t) := B^T \theta(t)$ as follows:

$$\dot{\varphi}(t) = g(\varphi(t)) \triangleq B^T \omega - B^T BW \sin(\varphi(t)), \quad (5)$$

The frequency-synchronized solution of (4) is mapped to the fixed points of (5). In other words, the synchronized solution can be stated as $\dot{\varphi}(t) = 0$, or $\varphi(t) = \varphi^*$ for some fixed $\varphi^* \in \mathbb{R}^m$. We have the following lemma:

**Lemma III.1.** All the fixed points of (5) can be expressed as

$$\sin(\varphi^*) = W^{r-1} B^T (BW^r B^T)^T \omega + W^{-1} F y, \quad r \in \mathbb{R},$$  

(6)

where $F \in \mathbb{R}^{m \times m-n+1}$ is a matrix whose columns span the null space of $B$, i.e. $BF = 0$, $y \in \mathbb{R}^{m-n+1}$, and $r$ is an arbitrary real number.

**Proof:** We start by equating $g(\varphi)$ in (5) to zero:

$$g(\varphi^*) = B^T (\omega - BW \sin(\varphi^*)) = 0.$$  

Noting that for connected graphs, $\ker(B^T) = \text{span}\{1_n\}$, and we get $BW \sin(\varphi) = \omega - \alpha 1_n$ for arbitrary $\alpha \in \mathbb{R}$.

By multiplying the both sides by $1_n^T$, we conclude that $\alpha = \omega_{\text{sync}}$. It follows that the fixed point satisfies the following

$$BW \sin(\varphi^*) = \omega - \omega_{\text{sync}} 1_n, \quad (7)$$

The complete solution of (7) is the sum of any particular solution and the homogenous solution resulted from the degeneracy of the incidence matrix $B$ (or, equivalently, the existence of cycles in the graph):

$$\sin(\varphi^*) = [\sin(\varphi^*)]_p + [\sin(\varphi^*)]_h$$

where in the second line, we have used the fact that $L_r L_n^T = I_n - \frac{1}{n} 1_n 1_n^T$ for the Laplacian matrix $L_r := BW^r B^T$. On the other hand, the homogeneous solution must satisfy $BW [\sin(\varphi^*)]_h = 0_n$ by definition. This implies that $W[\sin(\varphi^*)]_h \in \ker(B)$. Therefore, $W[\sin(\varphi^*)]_h$ lives in the null space of $B$, i.e. $W[\sin(\varphi^*)]_h = F y$ for some $y \in \mathbb{R}^{m-n+1}$, which leads to $[\sin(\varphi^*)]_h = W^{-1} F y$. The proof is complete.

Notice that the exponent $r$ in the solution (6) acts as a degree of freedom. Proper choice of $r$ will be helpful in the weight design problem for optimal cohesiveness which is addressed in Subsection IV-C.

**Remark III.1.** According to the constraint $\varphi^* = B^T \theta^*$, for $\theta^*$ to be realizable from (6), $y$ must be chosen such that

$$\varphi^* = \sin^{-1}\left(W^{r-1} B^T (BW^r B^T)^T \omega + W^{-1} F y\right) \in \text{Im}(B^T),$$

or, equivalently, since $\text{Im}(B^T) \perp \ker(B)$, we have that

$$F^T \sin^{-1}\left(W^{r-1} B^T (BW^r B^T)^T \omega + W^{-1} F y\right) = 0,$$

This corresponds to the geometric constraint that the sum of the phase differences along any cycle be zero.

**Corollary III.2.** For tree graphs, $F = 0$ and the fixed point $\varphi^*$ is unique. Accordingly, the particular solution $W^{r-1} B^T (BW^r B^T)^T \omega$ is independent of $r$. In the special case of $r = 0$, we have that

$$\sin(\varphi^*) = B^T (BW B^T)^T \omega = B^T L^T \omega, \quad (8)$$

provided that $\|B^T L^T \omega\|_\infty \leq 1$. 

B. Stability Analysis and Critical Coupling

In this section, we provide local stability analysis of the synchronized solution, which is directly related to the critical coupling below which the oscillators become incoherent [16].

**Lemma III.3.** Any fixed point of (5) is locally exponentially stable if \( \|\varphi^*\|_\infty < \frac{\pi}{2} \).

**Proof:** It is easier to analyze the stability of phase dynamics (4) rather than the phase difference dynamics (5). Without loss of generality, we assume that \( \omega_{\text{sync}} = 0 \). The Jacobian of \( f(\omega) \) at a fixed point \( \varphi^* = B^T \theta^* \) is given by

\[
J(\theta^*) = \frac{\partial f(\theta)}{\partial \theta} |_{\theta = \theta^*} = -BW \text{diag}(\cos(\varphi^*))B^T,
\]

The Jacobian has one trivial eigenvalue with \( 1_n \) as the associated eigenvector which is due to the rotational symmetry of the model [1]. If \( \|\varphi^*\|_\infty < \pi/2 \), we have that \( \text{diag}(\cos(\varphi^*)) \) has negative, implying local exponential stability.

In the case of uniform weights in trees, \( W = k1_{n-1} \) where \( k \) is the coupling strength, the fixed point can be written from (8) as

\[
\sin(\varphi^*) = k^{-1}B^T(BB^T)^{\dagger}\omega,
\]

At the critical coupling, the smallest nontrivial eigenvalue of the Jacobian becomes zero which happens when \( \cos(\varphi^*) = 0 \) for some \( e \in E \). This is equivalent to \( |\sin(\varphi^*)| = 1 \) or \( \|\sin(BB^T)\|_\infty = 1 \). Therefore, the critical coupling for tree graphs is given by:

\[
k_{cr} = \|B^T(BB^T)^{\dagger}\omega\|_\infty,
\]

Dekker et al. [17] have studied the critical coupling for various classes of tree and its statistics when frequencies are realizations of a probability density.

**Remark III.2.** For the case of uniform natural frequencies, we have that \( \omega = \alpha 1_n \) and since \((BB^T)^{\dagger}1_n = 0\), it follows from (11) that \( k_{cr} \rightarrow 0 \) as \( \omega \rightarrow \alpha 1_n \), meaning, the network always locally synchronizes as long as the underlying graph remains connected.

C. Optimal Network Design Problems

In this section, we state three design problems under study. We consider several combinations. Regarding the design variables, we consider two possibilities:

1) Design of natural frequencies: In this type of problem, we are allowed to tune the natural frequencies of the oscillators within a certain feasible set denoted by \( F_\omega \subset \mathbb{R}^n \). In our framework, we assume that tuning the natural frequencies has an associated cost. In particular, we assume a frequency-tuning cost function, denoted by \( g_\omega(\omega) : \mathbb{R}^n \rightarrow \mathbb{R}^+ \), which represents the cost of tuning the set of oscillators to a desired vector of natural frequencies \( \omega = (\omega_1, \ldots, \omega_n) \).

2) Design of link strengths: In this type of problem, we can also assume that link strengths can be modified by incurring a cost. We denote by \( f_\omega(\omega) : \mathbb{R}^m \rightarrow \mathbb{R}^+ \) the cost of modifying the vector of link weights to a desired profile vector \( w \in \mathbb{R}^m \), which we constrain to be in a feasible range \( F_w \subset \mathbb{R}^m_+ \).

Once the set of design variables is chosen, we consider the following three cases for the objective function:

1) Optimal Critical Coupling: In this case, we are given a tree graph with uniform coupling strengths, \( \omega = k1_m \), a synchronization frequency \( \omega_{\text{sync}} \), and a maximum budget \( C \) to invest on modifying the natural frequencies such that the optimized network can synchronize at \( \omega_{\text{sync}} \) with minimum possible coupling \( k \).

2) Optimal Cohesiveness: In this case, we assume that we are given a fixed set of natural frequencies \( \omega_0 \), and a maximum budget \( C \) to invest on modifying the link strengths. Alternatively, the natural frequencies are tuned given a fixed vector of link weights \( w_0 \). The objective is to find the optimal design to maximize the resulting phase cohesiveness.

3) Budget minimization: In this case, we assume that we are given a desired level of phase cohesiveness as a design specification, and we are asked to find the optimal allocation of link strengths (or natural frequencies) to achieve this cohesiveness with minimum budget.

In Section IV, we study these optimal design problems. We use tools from convex and quasi-convex analysis to provide efficient algorithms to solve these problems in polynomial time, as well as providing optimality guarantees, assuming that the cost functions are quasi-convex.

IV. OPTIMAL NETWORK DESIGN

A. Optimal Critical Coupling

For a tree graph with uniform weights on the edges, the minimum value of the coupling strength \( k \) to achieve synchronization is a function of the natural frequencies. This critical coupling can be optimized by tuning the natural frequencies as the following problem suggests:

**Problem 1** (Optimal Critical Coupling Frequency Design). Given (i) a tree graph with incidence matrix \( B \) and uniform coupling strengths \( w = k1_m \), (ii) a frequency tuning function \( g_\omega(\omega) : \mathbb{R}^n \rightarrow \mathbb{R}^+ \), (iii) a feasible design set \( F_\omega \subset \mathbb{R}^n \), and (iv) a fixed budget \( C > 0 \), (v) a synchronized frequency \( \omega_{\text{sync}} \), find the optimal vector of natural frequencies, denoted by \( \omega^* \), such that the network synchronizes at \( \omega_{\text{sync}} \) with minimum possible critical coupling \( k \). Problem 1 can be formulated as follows:

\[
\min_{\omega} \|B^T(BB^T)^{\dagger}\omega\|_\infty
\]

s.t. \( g_\omega(\omega) \leq C \),

\[
\omega \in F_\omega,
\]

\[
\frac{1}{n}1_n^T\omega = \omega_{\text{sync}},
\]

(12)
where the cost function is the critical coupling as in (11). As we show below, Problem 1, 2, and 3 are tractable under the following assumptions:

**Assumption 1.** The frequency tuning function, \( g_V(\omega) \), is quasi-convex.

**Assumption 2.** The frequency feasible design set, \( F_\omega \), is convex.

The frequency tuning function is naturally a nondecreasing function of frequencies, implying that quasi-convexity of \( g_V(\omega) \) is a non-restricting assumption. A reasonable choice of frequency design set is box constraint: \( \omega \leq \omega \leq \overline{\omega} \), where \( \omega := (\omega_1, \ldots, \omega_m) \) and \( \overline{\omega} := (\overline{\omega}_1, \ldots, \overline{\omega}_m) \) are lower and upper bounds on the frequencies.

**Proposition 1.** Problem 1 along with Assumption 1 and 2 defines a convex optimization problem.

Proof: The cost function of Problem 1 is the critical coupling, is the infinity norm of an affine transformation of \( \omega \). The first two constraints are convex by Assumptions 1 and 2. The last constraint is a linear equality. Therefore, Problem 1 is convex.

**B. Frequency Design for Optimal Cohesiveness**

Next, we address the frequency design problem regarding maximizing the cohesiveness of the synchronized solution. This problem is relevant, for example, in the design of decentralized computing networks for clock synchronization. For this purpose, we define the incohesion function \( \Phi_r : \mathbb{R}_+^m \times \mathbb{R}^n \rightarrow [0, 1] \) by setting \( F = 0 \) in (6) as follows:

\[
\Phi_r(w, \omega) = \| W^{-1} B^T (BW^T B^T)^T \omega \|_\infty, \tag{13}
\]

Here, \( \Phi_r(w, \omega) \) is the sine of the largest phase distance among the neighboring nodes, which is a function of the graph structure \( B \), the coupling strengths \( w \) and the natural frequencies \( \omega \). We state below the frequency design problems under consideration:

**Problem 2 (Budget-Constrained Frequency Design).** Given (i) a tree graph with incidence matrix \( B \), (ii) a fixed vector of link strengths \( w_0 \in \mathbb{R}_+^m \), (iii) a synchronized frequency \( \omega_{sync} \), (iv) a frequency tuning function \( g_V(\omega) : \mathbb{R}^n \rightarrow \mathbb{R}_+^n \), (v) a feasible design set \( F_\omega \subset \mathbb{R}^n \), and (vi) a fixed budget \( C > 0 \); find the optimal vector of natural frequencies, denoted by \( \omega^* \), that minimizes the incohesion of the synchronized phases with synchronized frequency \( \omega_{sync} \).

**Problem 2** can be mathematically formulated as follows:

\[
\begin{align*}
\min_{\omega} \quad & \Phi_r(w_0, \omega) \\
\text{s.t.} \quad & g_V(\omega) \leq C, \\
& \omega \in F_\omega, \\
& \frac{1}{n} 1^T_n \omega = \omega_{sync},
\end{align*} \tag{14}
\]

where \( \Phi_r(w_0, \omega) \) is defined as in (13).

Similarly, the problem of minimizing the budget to achieve a desired phase cohesiveness is stated as follows:

**Problem 3 (Cohesiveness-Constrained Frequency Design).** Given (i) a tree graph with incidence matrix \( B \), (ii) a fixed vector of link strengths \( w_0 \in \mathbb{R}_+^m \), (iii) a synchronized frequency \( \omega_{sync} \), (iv) a frequency tuning function \( g_V(\omega) : \mathbb{R}^n \rightarrow \mathbb{R}_+^n \), (v) a feasible design set \( F_\omega \subset \mathbb{R}^n \), and (vi) a desired level of phase cohesiveness \( \Phi^* \in (0, 1) \); find the optimal vector of natural frequencies \( \omega^* \) to achieve the cohesiveness \( \Phi^* \) at minimum possible cost and with synchronized frequency \( \omega_{sync} \). This problem can be mathematically expressed as follows:

\[
\begin{align*}
\min_{\omega} \quad & g_V(\omega) \\
\text{s.t.} \quad & \Phi_r(w_0, \omega) \leq \Phi^*, \\
& \omega \in F_\omega, \\
& \frac{1}{n} 1^T_n \omega = \omega_{sync},
\end{align*} \tag{15}
\]

**Proposition 2.** Problem 2 (respectively Problem 3) along with Assumption 1 and 2 define a convex (respectively quasi-convex) optimization problem.

Proof: Given a fixed set of coupling strengths \( w_0 \), the incohesion function \( \Phi_r(w_0, \omega) \) is the infinity norm of an affine transformation of \( \omega \), and hence is convex. Accordingly, under Assumption 1 and 2 the constraints of Problem 2 are convex. It follows that Problem 2 is convex. The objective function of Problem 3 is quasi-convex by Assumption 1 and the associated constrained are convex by convexity of \( \Phi_r(w_0, \omega) \) and Assumption 2. It follows that Problem 3 is quasi-convex.

**Remark IV.1.** For the case of separable cost functions, the set of constraints are \( \sum_{i \in V} g_i(\omega_i) \leq C \) and \( \omega \leq \omega \leq \overline{\omega} \), where \( \omega := (\omega_1, \ldots, \omega_n) \) and \( \overline{\omega} := (\overline{\omega}_1, \ldots, \overline{\omega}_m) \) are lower and upper bounds on the frequencies.

**C. Weight Design for Optimal Cohesiveness**

Finally, we address the problem of optimal design of weights for a tree graph with a fixed set of natural frequencies in order to minimize the incohesion of the synchronized phases.

**Problem 4 (Budget-Constrained Weight Design).** Given (i) a fixed vector of natural frequencies \( \omega_0 \), (ii) a link-strength cost function \( f_E(w) : \mathbb{R}_+^n \rightarrow \mathbb{R}_+^n \), (iii) a feasible design set \( F_w \subset \mathbb{R}_+^m \), and (iv) a fixed budget \( C > 0 \); find the optimal vector of link strengths, denoted by \( w^* \), that minimizes the incohesion of the synchronized phases. This problem can be mathematically expressed as follows:

\[
\begin{align*}
\min_{w} \quad & \Phi_r(w, \omega_0) \\
\text{s.t.} \quad & f_E(w) \leq C, \\
& w \in F_w,
\end{align*}
\]
Remark IV.2. For the case of separable cost functions, the set of constraints are \( \sum_{e \in \mathcal{E}} f_e(w_e) \leq C \) and \( 0_m < w \leq \omega \), where \( \omega := (\omega_1, \ldots, \omega_m) \) and \( w := (\omega_1, \ldots, \omega_m) \) are lower and upper bounds on the admissible weights.

We show below that Problem 4 is tractable under the following assumption \([18]\):

**Assumption 3.** The link cost function \( f_e(w_e) \) is twice differentiable and satisfies the following condition:

\[
f''_e(w_e) \geq -\frac{2}{w_e} f'_e(w_e), \quad \forall e \in \mathcal{E}
\]

for \( w_e \in [\omega_e, \bar{\omega}_e] \).

Notice that Assumption 3 is weaker (respectively stronger) than convexity for monotonically increasing (respectively decreasing) cost functions \( f_e(w_e), e \in \mathcal{E} \).

**Proposition 3.** Problem 4 along with Assumption 3 can be cast as a convex optimization problem.

**Proof:** Define the change of variable \( v_e = w_e^{-1} \). Then, it follows that \( g_e(v_e) := f_e(w_e^{-1}) \) is convex, since:

\[
\frac{d^2}{dv_e^2} g_e(v_e) = f''_e(w_e^{-1}) v_e^{-4} + f'_e(w_e^{-1})(2v_e^{-3}), \quad e \in \mathcal{E}
\]

\[
= f''_e(w_e) w_e^2 + f'_e(w_e) (2w_e^3) > 0,
\]

\( e \in \mathcal{E} \).

The feasible set \( \{ v_e \in \mathbb{R}_+ \mid \sum_{e \in \mathcal{E}} g_e(v_e) \leq C \} \) is mapped to the set \( \{ v_e \in \mathbb{R}_+ \mid \sum_{e \in \mathcal{E}} f_e(v_e) \leq C \} \) which is convex by the last result. Recall that \( r \) in \([13]\) is a degree of freedom. Setting \( r = 0 \) is the proper choice for the change of variable \( v_e = w_e^{-1} \). It follows that \( \Phi_0(w, \omega_0) = \| W^{-1} B^T (BB^T)^{-1} \omega_0 \|_{\infty} \). Now, Problem 4 can be rewritten in terms of new variables \( v_e, e \in \mathcal{E} \) as follows:

\[
\text{minimize} \quad \| V B^T (BB^T)^{-1} \omega_0 \|_{\infty}
\]

\[
\text{s.t.} \quad \sum_{e \in \mathcal{E}} g_e(v_e) \leq C,
\]

\[
\omega_e^{-1} \leq v_e \leq \bar{\omega}_e^{-1},
\]

where \( v = (v_1, \ldots, v_m) \), and \( V = \text{diag}(v) \). The cost function is the infinity norm of a linear function of \( v \). The constraints are convex by \([17]\). Therefore, Problem 4 becomes convex in the space of \( v \).

**V. NUMERICAL SIMULATIONS**

In this section, we provide some numerical examples to illustrate the design problems discussed in previous sections.

**Example V.1. (Frequency Design)** Consider a star graph with \( n = 10 \) nodes and a vector of natural frequencies \( \omega^0 = (\omega_1^0, \ldots, \omega_n^0)^T \), where each frequency is independently chosen from the standard normal distribution. We assume that the frequency tuning cost function is separable and quadratic centered at the frequency \( \omega^0 \), i.e., \( g_i(\omega_i) = (\omega_i - \omega_i^0)^2 \). We first solve Problem 7 for a range of budgets from 0 to 1.9. Figure 7 depicts the decrease in the optimal critical coupling \( k_{cr}^* \) as the frequency tuning budget \( C \) is increased. We also consider the following experiment: we fix a budget \( C = 2 \) and find the optimal critical coupling for 100 different random realizations of the natural frequency vector. In Fig. 2 we order these realizations in increasing values of their critical couplings (blue line). For each realization, we design the vector of natural frequencies to minimize the resulting critical coupling, given the budget constraint (red line).

We also run a second set of experiments to illustrate our solution to Problem 2. For the same range of budgets, 0 to 1.9, and the same star network, we find the optimal set of natural frequencies that minimizes the incohesion function. We plot the relationship between the optimal cohesiveness and the available budget in Fig. 7.

![Fig. 1. Optimal critical coupling \( k_{cr} \) vs. frequency tuning cost \( C \). Zero budget corresponds to the original critical coupling.](image)

![Fig. 2. Comparison of critical couplings before and after frequency design for various realizations of the natural frequency. The frequency tuning budget is set to \( C = 2 \).](image)
natural frequencies are linearly distributed according to \( \omega_i = 2\omega_{\text{sync}}/ (n+1) \). Assume that \( \omega_{\text{sync}} = 1 \), and the initial weights (before optimization) are uniform, i.e., \( w_0 = 101_{n-1} \). We assume that the cost of modifying each of the links is given by \( f_i(w_i) = (w_i - w_0^i)^2 \), and the available budget for each link is the same. Under these assumptions, we solve Problem 4 for a range of budgets \( C \in [1,20] \).

Fig. 5 represents the optimal value of cohesiveness versus the available budget. Fig. 5 illustrates the distribution of the optimal weights across the network. It is interesting to note that the optimal weights are distributed symmetrically on the chain.

VI. CONCLUSIONS

We have studied several network design problems for optimal synchronization in tree networks of coupled Kuramoto oscillators. The phase cohesiveness, which measures the maximum distance between neighboring phases in the synchronized regime was used as the design criterion. Three general design problems were addressed: (i) minimum critical coupling design problem, in which the optimal set of frequencies were found for achieving the minimum coupling strength; (ii) frequency design problem for optimal cohesiveness, and (iii) weight design problem for optimal cohesiveness. All the problems were shown to be convex under mild assumptions on the frequency and weight tuning cost functions. Illustrative figures were provided to show the effectiveness of the approach.

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