A PROOF OF THE REFINED PRV CONJECTURE VIA THE CYCLIC CONVOLUTION VARIETY

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Abstract. In this brief note we illustrate the utility of the geometric Satake correspondence by employing the cyclic convolution variety to give a simple proof of the Parthasarathy-Ranga Rao-Varadarajan conjecture, along with Kumar’s refinement. The proof involves recognizing certain MV-cycles as orbit closures of a group action, which we make explicit by unique characterization. In an appendix, joint with P. Belkale, we discuss how this work fits in a more general framework.

1. Introduction

We give a short proof of the Parthasarathy-Ranga Rao-Varadarajan conjecture first proven independently by Kumar in [Kum88] and Mathieu in [Mat89]. Our method extends to give a proof of Verma’s refined conjecture which was first proven by Kumar in [Kum89].

Let $\hat{G}$ be a complex reductive group, whose representation theory we are interested in. (We reserve the symbol $G$ for the complex reductive Langlands dual group of $\hat{G}$ because $G$ will be used more prominently in the proof, which goes through the geometric Satake correspondence.) Fix a maximal torus $\hat{T}$ and Borel subgroup $\hat{B}$ of $\hat{G}$. Let $W$ be the Weyl group of $\hat{G}$ (equivalently, of $G$). The statement of the original theorem is

Theorem 1.1 (PRV conjecture). Let $\lambda, \mu$ be dominant weights for $\hat{G}$ with respect to $\hat{B}$, and let $w \in W$ be any Weyl group element. Find $v \in W$ so that $\nu := v(-\lambda - w\mu)$ is dominant. Then

$$(V(\lambda) \otimes V(\mu) \otimes V(\nu))^{\hat{G}} \neq (0).$$

Kumar proved a refinement of this theorem in [Kum88] regarding the dimensions of the spaces of invariants. Let $W_\delta$ for any weight $\delta$ denote the stabilizer subgroup of $\delta$ in $W$. The stronger theorem is

Theorem 1.2 (Refinement). Let $\lambda, \mu, \nu, w$ be as above. Let $m_{\lambda,\mu,w}$ count the number of distinct cosets $\hat{u} \in W_\lambda \backslash W/W_\mu$ such that $-\lambda - w\mu$ and $-\lambda - u\mu$ are $W$-conjugate (equivalently, $\nu$ can be written $q(-\lambda - u\mu)$ for some $q \in W$). Then

$$\dim (V(\lambda) \otimes V(\mu) \otimes V(\nu))^{\hat{G}} \geq m_{\lambda,\mu,w}.$$ In particular, since $m_{\lambda,\mu,w} \geq 1$ by definition, the second theorem implies the first.

We will use properties of a certain complex variety called the cyclic convolution variety, whose definition we recall; see [Hai03, §2], although our symmetric formulation is from [Kam07, §1]. Let $G$ be the Langlands dual group to $\hat{G}$ with dual torus $T$ and Borel subgroup $B$. Let $\lambda_i, i = 1, \ldots, s$ be a collection of dominant weights for $\hat{G}$ w.r.t. $\hat{B}$; these induce dominant coweights of $G$ w.r.t. $B$. Set $K = \mathbb{C}((t)), O = \mathbb{C}[[t]]$. Each cocharacter $\lambda : \mathbb{C}^\times \to T$ induces an element $t^\lambda$ of $G(K)$; denote by $[\lambda]$ its image in $G(K)/G(O)$. Recall that via the Chevalley decomposition any two points $L_1, L_2$ in $G(K)/G(O)$ give rise to a unique dominant coweight $\lambda$ of $T$ such that

$$(L_1, L_2) = g([0], [\lambda]).$$
for some \( g \in G(K) \); we write \( \lambda = d(L_1, L_2) \) to convey this information concisely.

The cyclic convolution variety is

\[
\text{Gr}_{G,c}(\vec{\lambda}) := \{(L_1, \ldots, L_s) \in (G(K)/G(O))^s \mid L_s = [0], d(L_{i-1}, L_i) = [\lambda_i] \ \forall i\},
\]

where we take \( L_0 \) to mean \( L_s \). The maximum possible dimension of \( \text{Gr}_{G,c}(\vec{\lambda}) \) is \( \langle \rho, \sum \lambda_i \rangle \), where \( \rho \) is the usual half-sum of positive roots for \( G \), and via the geometric Satake correspondence ([Lus83, Gin, BD, MV07]) the number of irreducible components of this dimension (if any) is equal to

\[
\dim (V(\lambda_1) \otimes \cdots \otimes V(\lambda_s))^G;
\]

see also [Hai03, Proposition 3.1].

Our task is therefore to produce irreducible components of the right dimension, which we find as \( G(O) \)-orbit closures of suitable points. These are in bijection with certain MV-cycles which we make explicit. As a corollary we obtain the following known result:

**Corollary 1.3.** Let \( \lambda, \mu \) be dominant and \( w \in W \) such that \( \nu := \lambda + w\mu \) is also dominant. Then the multiplicity of \( V(\nu) \) inside \( V(\lambda) \otimes V(\mu) \) is exactly 1.

This is already known from a multiplicity theorem of Kostant; see [Kos59, Lemma 4.1] and [Kum10, Corollary 3.8]. (It is also a consequence of Roth’s theorem [Rot11] where \( P_l = B \), \( \bar{G} = \{1\} \), and the Schubert calculus equation is

\[
[\Omega_{w^{-1}}] \odot_0 [\Omega_e] \odot_0 [X_{w^{-1}}] = 1,
\]

using the notation found there.)

Our technique of producing components of the right dimension is not limited to the PRV setting; we illustrate this by an explicit example in Section 5.

Our proof of Theorem 1.1 should be compared with the proof of [Ric14, Lemma 5.5], where a geometric analogue of PRV is proved. There a one-sided dimension estimate on a fibre of the convolution morphism provides existence of components of the correct dimension, but the fibre component is not realized as the (closure of an) orbit under a group action; nor is the specific MV-cycle mentioned. The lower bound on number of components (yielding the refined version) is not made there.

See also [Hai06, Theorem 6.1], where non-emptiness of the relevant variety (but not its dimension) is established, implying Theorem 1.1 only in the case where \( \lambda, \mu \) are sums of minuscule coweights.

In an appendix, joint with P. Belkale, we describe the relationship of this work to a more general question on the transfer of invariants between Langlands dual groups, with the PRV case corresponding to the inclusion of a maximal torus inside a reductive group.

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2. **Proof of the conjecture**

We will need some additional notation: let \( \Phi \) denote the set of roots of \( G \), and for \( \alpha \in \Phi \) let \( \alpha \geq 0 \) mean \( \alpha \) is a positive root w.r.t. \( B \) (likewise \( \alpha \leq 0 \) means \( -\alpha \geq 0 \)).
Proof. Step 1 We claim that the cyclic convolution variety $\text{Gr}_{G,c(\lambda,\mu,\nu)}$ is nonempty. Indeed, the point $x = ([\lambda], [\lambda + w\mu], [0])$ satisfies

$$
([0], [\lambda]) = 1([0], [\lambda])
$$

$$
([\lambda], [\lambda + w\mu]) = t^\lambda w([0], [\mu])
$$

$$
([\lambda + w\mu], [0]) = t^{\lambda + w\mu}v^{-1}([0], [\nu]).
$$

Step 2 Observe that any $\text{Gr}_{G,c(\lambda)}$ has a $G(\mathcal{O})$-diagonal action on the left. We claim that the orbit $G(\mathcal{O})x \subseteq \text{Gr}_{G,c(\lambda,\mu,\nu)}$ is a finite-dimensional subvariety and has dimension $\langle \rho, \lambda + \mu + \nu \rangle$; this will conclude the proof, since the connectedness of $G(\mathcal{O})$ means $G(\mathcal{O})x$ is contained in an irreducible component of $\text{Gr}_{G,c(\lambda)}$ necessarily of dimension $\langle \rho, \lambda + \mu + \nu \rangle$.

For any integer $N > 0$, let $K_N$ denote the kernel of the surjective group homomorphism

$$
G(\mathcal{O}) \rightarrow G(\mathcal{O}/(t^N)).
$$

Observe that, for high enough $N \gg 0$, $K_N$ stabilizes the point $x$ (it suffices to embed $G$ into some $GL_m$ and examine matrix entries). Therefore $G(\mathcal{O})x$ has a transitive action by the finite-dimensional linear algebraic group $G(\mathcal{O}/(t^N))$.

The stabilizer $\text{Stab}_{G(\mathcal{O}/(t^N))}(x)$ is the image of

$$
\text{Stab}_{G(\mathcal{O})}(x) = G(\mathcal{O}) \cap t^\lambda G(\mathcal{O}) t^{-\lambda} \cap t^{\lambda + w\mu} G(\mathcal{O}) t^{-\lambda - w\mu} \subseteq G(\mathcal{O})
$$

under the quotient; i.e., $\text{Stab}_{G(\mathcal{O}/(t^N))}(x) = \text{Stab}_{G(\mathcal{O})}(x)/K_N$.

By the orbit-stabilizer theorem, $G(\mathcal{O}/(t^N))x \simeq G(\mathcal{O}/(t^N))/\text{Stab}_{G(\mathcal{O}/(t^N))}(x)$. As $G(\mathcal{O}/(t^N))/\text{Stab}_{G(\mathcal{O}/(t^N))}(x)$ is a smooth finite-dimensional variety, we may calculate its dimension by the dimension of its tangent space at the origin. For an arbitrary group scheme $H$ over $\mathbb{C}$, one takes $\text{Lie}(H)$ to mean the kernel of $H(\mathbb{C}[[\varepsilon]](\varepsilon^2)) \xrightarrow{\varepsilon \rightarrow 0} H(\mathbb{C})$. Since Lie commutes with intersections (of subgroups of $G(\mathcal{K})$, see [Mil17, §10.c]), $\text{Lie}((\text{Stab}_{G(\mathcal{O})}(x)))$ is

$$
\mathfrak{g}(\mathcal{O}) \cap \text{Ad}_t^{\lambda} \mathfrak{g}(\mathcal{O}) \cap \text{Ad}_t^{\lambda + w\mu} \mathfrak{g}(\mathcal{O}) \simeq \mathfrak{h}(\mathcal{O}) \oplus \bigoplus_{\alpha \in \Phi} t^{\max(0, \langle \alpha, \lambda \rangle, \langle \alpha, \lambda + w\mu \rangle)} \mathfrak{g}_{\alpha}(\mathcal{O})
$$

Thus in the quotient

$$
\text{Lie}((\text{Stab}_{G(\mathcal{O}/(t^N))}(x))) \simeq \mathfrak{h}(\mathcal{O}/(t^N)) \oplus \bigoplus_{\alpha \in \Phi} t^{\max(0, \langle \alpha, \lambda \rangle, \langle \alpha, \lambda + w\mu \rangle)} \mathfrak{g}_{\alpha}(\mathcal{O}/(t^N));
$$

note that, for every $\alpha$, $\langle 0, \alpha \rangle \leq N$ and $\langle \alpha, \lambda + w\mu \rangle \leq N$ so that $K_N \subseteq \text{Stab}_{G(\mathcal{O})}(x)$. For the finite-dimensional affine group scheme $S := \text{Stab}_{G(\mathcal{O}/(t^N))}(x)$, $\text{Lie}(S)$ is naturally identified with the tangent space of $S$ at the identity. Therefore the $\mathbb{C}$-dimension of the tangent space

$$
\mathfrak{g}(\mathcal{O}/(t^N))/\text{Lie}((\text{Stab}_{G(\mathcal{O}/(t^N))}(x)))
$$

is

$$
\sum_{\alpha \in \Phi} \max(0, \langle \alpha, \lambda \rangle, \langle \alpha, \lambda + w\mu \rangle).
$$

The proof of the claim therefore reduces to the following calculation.

Step 3 We claim that $\langle \rho, \lambda + \mu + \nu \rangle = \sum_{\alpha \in \Phi} \max(0, \langle \alpha, \lambda \rangle, \langle \alpha, \lambda + w\mu \rangle)$. Let us examine the sum on the right in two parts, summing over $\alpha \leq 0$ and $\alpha \geq 0$ separately.
If $\alpha \geq 0$, then $\max(0, \langle \alpha, \lambda \rangle, \langle \alpha, \lambda + w\mu \rangle) = \max(\langle \alpha, \lambda \rangle, \langle \alpha, \lambda + w\mu \rangle)$ due to the dominance of $\lambda$. Furthermore, $\langle \alpha, \lambda \rangle$ will be the bigger of the two unless $\langle \alpha, w\mu \rangle \geq 0$. Therefore

$$\sum_{\alpha \geq 0} \max(0, \langle \alpha, \lambda \rangle, \langle \alpha, \lambda + w\mu \rangle) = \sum_{\alpha \geq 0} \langle \alpha, \lambda \rangle + \sum_{\alpha \geq 0} \langle \alpha, w\mu \rangle.$$

The first sum on the RHS is clearly equal to $\langle 2\rho, \lambda \rangle$. As for the second sum, observe that $\langle \alpha, w\mu \rangle \geq 0 \iff \langle w^{-1}\alpha, \mu \rangle \geq 0$. As $\mu$ is dominant, this happens only when $w^{-1}\alpha \geq 0$ or when $w^{-1}\alpha \leq 0$ and $\langle w^{-1}\alpha, \mu \rangle = 0$. The latter class of $\alpha$ doesn’t contribute to the sum, so that second RHS term is equal to

$$\sum_{\alpha \geq 0} \langle \alpha, w\mu \rangle = \sum_{\Phi^+ \cap w\Phi^+} \langle \alpha, w\mu \rangle,$$

where $\Phi^+$ denotes the set of positive roots. As is well known (see for example [Kum02, 1.3.22.3]), $\sum_{\Phi^+ \cap w\Phi^+} \alpha = \rho + w\rho$. Putting everything together so far, the original sum over $\alpha \geq 0$ yields $\langle 2\rho, \lambda \rangle + \langle \rho + w\rho, w\mu \rangle$.

If $\alpha \leq 0$, then $\max(0, \langle \alpha, \lambda \rangle, \langle \alpha, \lambda + w\mu \rangle) = \max(0, \langle \alpha, \lambda + w\mu \rangle)$. Recall that $\lambda + w\mu = -v^{-1}\nu$; therefore the sum over $\alpha \leq 0$ is

$$\sum_{\alpha \leq 0} \langle \alpha, -v^{-1}\nu \rangle = \sum_{\alpha \geq 0} \langle \alpha, v^{-1}\nu \rangle.$$

As before, this equals $\langle \rho + v^{-1}\rho, v^{-1}\nu \rangle$. Finally, we conclude as desired that the dimension of the space in question is

$$\langle 2\rho, \lambda \rangle + \langle \rho + w\rho, w\mu \rangle + \langle \rho + v^{-1}\rho, v^{-1}\nu \rangle$$

$$= \langle \rho, \lambda + w\mu + v^{-1}\nu \rangle + \langle \rho, \lambda \rangle + \langle w\rho, w\mu \rangle + \langle v^{-1}\rho, v^{-1}\nu \rangle$$

$$= 0 + \langle \rho, \lambda + \mu + \nu \rangle.$$

\[\square\]

3. PROOF OF THE REFINEMENT

Proof. Suppose $u \in W$ is such that $\nu = q(-\lambda - u\mu)$ for some $q \in W$. Then $x(u) := ([\lambda], [\lambda + u\mu], [0])$ satisfies

$$([0], [\lambda]) = 1([0], [\lambda])$$

$$([\lambda], [\lambda + u\mu]) = t^\lambda u([0], [\mu])$$

$$([\lambda + u\mu], [0]) = t^{\lambda + u\mu} q^{-1}([0], [\nu]);$$

therefore $x(u) \in \text{Gr}_{G,c(\lambda, \mu, \nu)}$ and $G(\mathcal{O})x(u)$ is a subvariety of $\text{Gr}_{G,c(\lambda, \mu, \nu)}$ of dimension $\langle \rho, \lambda + \mu + \nu \rangle$ for exactly the same reason as before.

Claim If $x(u) = gx(u')$ for some $g \in G(\mathcal{O})$, then $\bar{u} = \bar{u}' \in W_\lambda \setminus W/W_\mu$.

Proof Assume $x(u) = gx(u')$ for some $g \in G(\mathcal{O})$. Fix $q, q'$ satisfying $\nu = q(-\lambda - u\mu) = q'(-\lambda - u'\mu)$. We are given that $g[\lambda] = [\lambda]$ and $g[\lambda + u'\mu] = [\lambda + u\mu]$. First we demonstrate that we can replace $g$ with an element of $G$. Recall from [MV07] that there is a map

$$ev_0 : G\nu_\lambda \to G/P_\lambda,$$
where $P_\lambda$ is the smallest parabolic containing $B^-$ and $L_\lambda$, where $B^-$ the Borel opposite to $B$ and $L_\lambda$ is the centralizer of $t^\lambda$ in $G$. The map is given by $g(t)t^\lambda G(O) \mapsto g(0)P_\lambda$ and makes $Gr_\lambda$ an affine bundle over $G/P_\lambda$.

We find that $g(0)P_\lambda = P_\lambda$ by taking $ev_0$ of both sides of the equation $g[\lambda] = [\lambda]$; i.e., $g(0) \in P_\lambda$. From $\nu = q(-\lambda - u\mu)$ we have $-w_0\nu = w_0q(\lambda + u\mu)$. The second equation can be formulated as

$$gq^{-1}w_0^{-1}[-w_0\nu] = q^{-1}w_0^{-1}[-w_0\nu],$$

which under $ev_0$ gives $g(0)q^{-1}w_0^{-1}P_{-w_0\nu} = q^{-1}w_0^{-1}P_{-w_0\nu}$.

We now attempt to replace $g(0)$ with a Weyl group element, as follows. Since $g(0) \in P_\lambda$, the double cosets

$$P_\lambda q^{-1}w_0^{-1}P_{-w_0\nu} = P_\lambda q^{-1}w_0^{-1}P_{-w_0\nu}$$

agree, in which case

$$W_{\lambda}q^{-1}w_0^{-1}W_{-w_0\nu} = W_{\lambda}q^{-1}w_0^{-1}W_{-w_0\nu}$$

by [BT65, Corollaire 5.20] (see also [Kum89, Lemma 2.2]). Writing $rq^{-1}w_0^{-1}r' = q^{-1}w_0^{-1}$ for some $r \in W_{\lambda}, r' \in W_{-w_0\nu}$, observe that

$$\lambda + u\mu = q^{-1}w_0^{-1}(-w_0\nu) = rq^{-1}w_0^{-1}(-w_0\nu) = r(\lambda + u'\mu) = \lambda + ru'\mu;$$

therefore $ru'\mu = u\mu$ and thus $ruW_\mu = uW_\mu$. This gives $W_\lambda uW_\mu = W_\lambda uW_\mu$ as desired.

So for any pair $\bar{u}, \bar{u}'$ distinct in $W_{\lambda}\setminus W/W_\mu$ (such that $-\lambda - u\mu$ and $-\lambda - u'\mu$ are both conjugate to $\nu$), the orbits $G(O)x(\bar{u})$ and $G(O)x(\bar{u}')$ must be disjoint. Each orbit $G(O)x(\bar{u})$ is irreducible, so the closure $G(O)x(\bar{u})$ inside $Gr_{G,c(\lambda,\mu,\nu)}$ is an irreducible component of the same (top) dimension. Disjoint orbits necessarily give distinct (possibly not disjoint) irreducible components. Therefore the number of irreducible components of the top dimension of $Gr_{G,c(\lambda,\mu,\nu)}$ is at least $m_{\lambda,\mu,\nu}$, from which the theorem follows.

4. RELATION TO MV-CYCLICES

Here we recall the summary of the geometric Satake correspondence as presented in [And03]. Let $F_\nu = \pi^{-1}(1(\nu))$ be the fibre of the natural projection map

$$\overline{Gr_\lambda \times Gr_\mu} := \{(aG(O), bG(O)) \in \overline{Gr_\lambda \times Gr_{\lambda+\mu}} \mid a^{-1}bG(O) \in \overline{Gr_\mu}\} \xrightarrow{\pi} \overline{Gr_{\lambda+\mu}}$$

over $[\nu]$, where $\nu \geq \lambda + \mu$. Then the multiplicity of $V(\nu)$ inside $V(\lambda) \otimes V(\mu)$ is equal to the number of irreducible components of $F_\nu$ of dimension $\langle \rho, \lambda + \mu - \nu \rangle$, the maximal possible dimension. (There is a $1 - 1$ correspondence between these irreducible components and those of top dimension in $Gr_{G,c(\lambda,\mu,-w_0\nu)}$.) According to [And03, Theorem 8], the irreducible components of $F_\nu$ of dimension $\langle \rho, \lambda + \mu - \nu \rangle$ are exactly the Mirković-Vilonen cycles for $\overline{Gr_\lambda}$ at weight $\nu - \mu$ contained in $t^\nu \overline{Gr_\mu}$.

As observed in [And03], $\pi^{-1}(1(\nu)) = \overline{Gr_\lambda} \cap t^\nu \overline{Gr_\mu}$. Therefore $\pi^{-1}(1(\nu))$ carries a natural $H := G(O) \cap t^G(O)t^{-\nu}$-action on the left. Note that $H$ is connected for the following reason: any $x(t) \in H$ has a path $x(st)$ connecting it to $x(0)$ as $s$ varies from 1 to 0. This gives a retraction of $H$ onto $P_\nu \subset H$, and $P_\nu$ is path-connected (as before, $P_\nu$ is the parabolic subgroup of $G$ containing $B^-$ and $L_\nu$). Therefore each irreducible component of $\pi^{-1}(1(\nu))$ is $H$-stable.

**Theorem 4.1.** Let $\lambda, \mu$ be dominant coweights. If $\nu = v(\lambda + w\mu)$ is dominant for some $v, w \in W$, then $V(\nu)$ appears in $V(\lambda) \otimes V(\mu)$ with multiplicity at least 1. In fact, there is a unique MV-cycle $\overline{Gr_\lambda}$ at weight $\nu - \mu$ contained in $t^\nu \overline{Gr_\mu}$ which contains $[v\lambda]$ (equivalently, contains $[qv\lambda]$ for all $q \in W_\nu$).
Proof. The point \([v\lambda]\) is clearly contained in \(\pi^{-1}([\nu])\), since \([-v\lambda + \nu] = [v\mu] \in \text{Gr}_\mu\).

Claim \(H.[v\lambda]\) has dimension \(\langle \rho, \lambda + \mu - \nu \rangle\).

Proof. Exactly analogous to the previous dimension calculation.

Therefore the closure of \(H.[v\lambda]\) gives an irreducible component of \(\pi^{-1}([\nu])\) of the right dimension, so contributing to the multiplicity of \(V(\nu)\) in \(V(\lambda) \otimes V(\mu)\).

For uniqueness: if \(A\) is any other irreducible component, \([v\lambda] \in A\) implies \(H.[v\lambda] \subseteq A\), which forces \(A = \overline{H.[v\lambda]}\).

Notably, any lift of any \(q \in W_\nu\) to \(G\) satisfies \(q \in t^\nu G(O)t^{-\nu}\); therefore \([qv\lambda] \in H.[v\lambda]\). \(\square\)

In similar style, the \(m_{\lambda,\mu,w,\nu}\)-many components produced as in the refinement are simply the \(H\)-orbits of the \([r\lambda]s\), where \(\nu = r(\lambda + u\mu)\) as \(u\) varies in \(W_\lambda/W/W_\mu\).

Corollary 4.2. If \(\nu = \lambda + w\mu\) is dominant, the multiplicity of \(V(\nu)\) in \(V(\lambda) \otimes V(\mu)\) is exactly 1.

Proof. The cycle \(A = \overline{H.[\lambda]}\) contributes 1 to the multiplicity count. Since every MV-cycle of \(\overline{\text{Gr}_\lambda}\) at weight \(\nu - \mu\) contained in \(t^\nu \overline{\text{Gr}_{-\mu}}\) must contain \([\lambda]\) and be \(H\)-stable, \(A\) must be the only such cycle. \(\square\)

5. The converse fails

The entire basis of this work is a very strange phenomenon: for PRV triples \(\lambda, \mu, \nu\), there exist irreducible components of \(\text{Gr}_{G,c(\lambda,\mu,\nu)}\) containing a dense \(G(O)\)-orbit (equivalently, there exist MV-cycles in \(F_\nu\) containing a dense \(H\)-orbit). One could ask: given an irreducible top component of a cyclic convolution variety \(\text{Gr}_{G,c(\lambda,\mu,\nu)}\) that contains a dense \(G(O)\)-orbit, is it true that \(\lambda, \mu, \nu\) is a PRV triple? The answer turns out to be false:

Theorem 5.1. There exist \(G, \lambda, \mu, \nu\) and an irreducible component \(A \subset \text{Gr}_{G,c(\lambda,\mu,\nu)}\) of dimension \(\langle \rho, \lambda + \mu + \nu \rangle\) such that

1. \(A = \overline{G(O)x}\) for some \(x\);
2. there are no elements \(v, w \in W\) making \(\nu = v(-\lambda - w\mu)\) true.

Proof. Here is an example: take \(G = \text{SL}_2\), \(\lambda = \mu = \nu = \alpha^\vee\), the single positive coroot. Criterion (2) is easy to verify: \(w\mu = \pm \alpha^\vee\) for any \(w \in W\), and \(v^{-1}\nu = \pm \alpha^\vee\) for any \(v \in W\). But

\[\pm \alpha^\vee = -\alpha^\vee \pm \alpha^\vee\]

is not true for any choices of \(+, -\).

As for (1): let \(y = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} t^{\alpha^\vee}\).

Claim \(x := ([\alpha^\vee], [\bar{y}], [0]) \in \text{Gr}_{G,c(\lambda,\mu,\nu)}\).

Proof. We have

\[([0], [\alpha^\vee]) = 1([0], [\alpha^\vee])\]

\[([\alpha^\vee], \bar{y}) = t^{\alpha^\vee} \begin{bmatrix} 0 & 1 \\ -1 & t \end{bmatrix} ([0], [\alpha^\vee])\]

\[([\bar{y}, [0]) = t^{\alpha^\vee} \begin{bmatrix} 0 & 1 \\ -1 & t \end{bmatrix} t^{\alpha^\vee} \begin{bmatrix} 0 & 1 \\ -1 & t \end{bmatrix} ([0], [\alpha^\vee])\];

the second line follows from

\[y = \begin{bmatrix} t & 1 \\ 0 & t^{-1} \end{bmatrix} = \begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & t \end{bmatrix} \begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & t \end{bmatrix}\]
and the third from
\[
\begin{bmatrix}
t & 0 \\
0 & t^{-1}
\end{bmatrix}
\begin{bmatrix}
0 & 1 \\
-1 & t
\end{bmatrix}
\begin{bmatrix}
t & 0 \\
0 & t^{-1}
\end{bmatrix}
\begin{bmatrix}
0 & 1 \\
-1 & t
\end{bmatrix}
\begin{bmatrix}
t & 0 \\
0 & t^{-1}
\end{bmatrix}
= \begin{bmatrix}
-t & 1 \\
-1 & 0
\end{bmatrix}.
\]

**Claim** The dimension of $SL_2(\mathcal{O}).x$ is $\langle \rho, 3\alpha \rangle = \langle \alpha/2, 3\alpha \rangle = 3$.

**Proof** The stabilizer of $x$ has Lie algebra
\[
L = \mathfrak{sl}_2(\mathcal{O}) \cap \text{Ad}_{\rho \vee} \mathfrak{sl}_2(\mathcal{O}) \cap \text{Ad}_{\gamma} \mathfrak{sl}_2(\mathcal{O});
\]
we now try to express this vector space more explicitly.

Let $e, f, h$ be the standard basis of $\mathfrak{sl}_2(\mathbb{C})$; then
\[
\mathfrak{sl}_2(\mathcal{O}) = e(\mathcal{O}) \oplus h(\mathcal{O}) \oplus f(\mathcal{O})
\]
and
\[
\text{Ad}_{\rho \vee} \mathfrak{sl}_2(\mathcal{O}) = t^2e(\mathcal{O}) \oplus h(\mathcal{O}) \oplus t^{-2}f(\mathcal{O}).
\]

Of course, $\text{Ad}_h \mathfrak{sl}_2 = \text{Ad}_x \text{Ad}_\rho \mathfrak{sl}_2$, where $z = \begin{bmatrix} 1 & t \\
0 & 1
\end{bmatrix}$. One calculates
\[
\text{Ad}_{z^{-1}} e = e; \text{ Ad}_{z^{-1}} h = h + 2te; \text{ Ad}_{z^{-1}} f = f - th - t^2e.
\]

Let $X \in \mathfrak{sl}_2(\mathcal{O}) \cap \text{Ad}_{\rho \vee} \mathfrak{sl}_2(\mathcal{O})$ be arbitrary: $X = p_e e + p_h h + p_f f$, where $\text{val}_t(p_e) \geq 2, \text{val}_t(p_h) \geq 0$, and $\text{val}_t(p_f) \geq 0$ (as usual, $\text{val}(0) = \infty$).

Now $X \in \text{Ad}_h \mathfrak{sl}_2(\mathcal{O})$ if and only if $\text{Ad}_{z^{-1}} X \in \text{Ad}_{\rho \vee} \mathfrak{sl}_2(\mathcal{O})$. As
\[
\text{Ad}_{z^{-1}} X = (p_e + 2tp_h - t^2p_f)e + (p_h - tp_f)h + p_ff,
\]
this is if and only if $\text{val}_t(p_h) \geq 1$ (if $\text{val}_t(p_h) = 0$, then the $e$-coefficient has $t$-valuation $1$ since $\text{val}_t(p_e - t^2p_f) \geq 2$.)

Therefore $L = t^2e(\mathcal{O}) \oplus th(\mathcal{O}) \oplus f(\mathcal{O})$, in which case
\[
\mathfrak{sl}_2(\mathcal{O})/L \simeq \mathfrak{sl}_2(\mathcal{O})/t^2e(\mathcal{O}) \oplus th(\mathcal{O}) \oplus f(\mathcal{O}),
\]
and the latter has dimension $3$. So $\text{dim} SL_2(\mathcal{O}).x = 3$.

The usual arguments then apply: $SL_2(\mathcal{O}).x$ is irreducible of maximal dimension; therefore its closure is an irreducible component. \qed

**Appendix: A more general framework**

by Prakash Belkale and Joshua Kiers

Let $H \to G$ be an embedding of complex reductive algebraic groups, and assume maximal tori and Borel subgroups are chosen such that $T_H \subseteq T_G$ and $B_H \subseteq B_G$. A priori, there is not a map $H' \to G'$ of Langlands dual groups; i.e., taking Langlands dual is not functorial. However, for any collection of coweights $\lambda_1, \ldots, \lambda_s$ for $T_H$ dominant w.r.t. $B_H$, there is a morphism of cyclic convolution varieties
\[
\Phi : \text{Gr}_{H,c(\check{\lambda})} \to \text{Gr}_{G,c(\check{\lambda})},
\]
where for each $i$, the “transfer” $\chi'_i := w_i \lambda_i$ is the unique $G$-Weyl group translate of $\lambda_i$, viewed as a coweight of $T_G$, which is dominant w.r.t. $B_G$. The morphism is just the embedding $H(K)/H(\mathcal{O}) \to G(K)/G(\mathcal{O})$ in each factor; one easily verifies it is well-defined.

Therefore it is clear that $\text{Gr}_{H,c(\check{\lambda})} \neq \emptyset \implies \text{Gr}_{G,c(\check{\lambda})} \neq \emptyset$.

\footnote{We thank N. Fakhruddin and S. Kumar for useful discussions.}
Question 5.2. Under what conditions on $H, G$ is it true that
\[
(V(\lambda_1) \otimes \cdots \otimes V(\lambda_s))^{H^\vee} \neq (0) \implies (V(\lambda'_1) \otimes \cdots \otimes V(\lambda'_s))^{G^\vee} \neq (0)
\]
for every tuple $(\lambda_1, \ldots, \lambda_s)$?

Equivalently, under what conditions on $H, G$ is it the case that if $\text{Gr}_{H,e(\lambda)}$ has top-dimensional components then $\text{Gr}_{G,c(\vec{\lambda})}$ does, too?

We note that consideration of mappings of “dual groups” is an important theme in the Langlands program (cf. the functoriality conjecture [Gel84, Conjecture 3]).

The weaker implication
\[
\exists N \text{ s.t. } (V(N\lambda_1) \otimes \cdots \otimes V(N\lambda_s))^{H^\vee} \neq (0) \implies \exists N' \text{ s.t. } (V(N'\lambda'_1) \otimes \cdots \otimes V(N'\lambda'_s))^{G^\vee} \neq (0)
\]
does hold; this is because the Hermitian eigenvalue cones for $H^\vee$ and $H$ are isomorphic, as are those for $G^\vee$ and $G$, see [KLM03, Theorem 1.8], and there is a map between the Hermitian eigenvalue cones for $H$ and $G$ since there is a compatible mapping of maximal compact subgroups, see [BK10]. Therefore implication (5.1) always holds when $G$ is of type $A$ [KT99] or types $D_4, D_5, D_6$ [KKM09, Kie19] by saturation. Here we note that $\text{Gr}_{G,c(\vec{\lambda})} \neq \emptyset$ implies that $\sum \lambda'_i$ is in the coroot lattice for $G$ which equals the root lattice of $G^\vee$.

Setting $s = 3$, the PRV theorem can be phrased as a partial answer to this question: if $H = T_G$ is a maximal torus of $G$, then (under no further conditions) implication (5.1) always holds. Indeed, $(V(\lambda_1) \otimes V(\lambda_2) \otimes V(\lambda_3))^{T^\vee} \neq (0)$ if and only if $\lambda_1 + \lambda_2 + \lambda_3 = 0$; therefore the $\lambda_i$ satisfy $\lambda'_1 + w\lambda'_2 + v\lambda'_3 = 0$ for suitable $w, v \in W$ and PRV says that $(V(\lambda'_1) \otimes V(\lambda'_2) \otimes V(\lambda'_3))^{G^\vee} \neq (0)$.

A series of instances where the implication (5.1) holds can be found in [HS15, §2]. In these examples $H$ is the subgroup of fixed points of a group $G$ under a diagram automorphism. Further, in each of these situations $H$ is of adjoint type.

When $H = PSL(2)$ and $G$ is arbitrary, implication (5.1) holds with no conditions. This follows from the linearity of the map $(\lambda_i) \mapsto (\lambda'_i)$ when the $\lambda_i$ are each coweights of $SL(2)$ and from the special form of the Hilbert basis of the tensor cone for $SL(2)$: they are $(\omega, \omega, 0)$ and permutations, so their transfers are $(\lambda', \lambda', 0)$ for some $\lambda'$. Since $(N\lambda', N\lambda', 0)$ have invariants for some $N$ by (5.2), $N\lambda'$ is self-dual; therefore $\lambda'$ is also.

When $H = PSp(4)$ (type $C_2$) and $G = PSp(4m)$, we have checked that the transfer property (5.1) holds. To do this, we establish that the transfer map on dominant weights is linear. Then we identify a finite generating set for the tensor semigroup for $PSp(4)$, using a result of Kapovich and Millson [KM06]. Finally we check the transfer property on this set.

However, we can exhibit the failure of (5.1) when $H = SL(2)$ and $G = SO(5)$, the map being the standard $SL(2)$ embedding corresponding to the root $\alpha_1$. Therefore some conditions on $H, G$ must be necessary; perhaps is suffices to assume that $Z(H')$ maps into $Z(G)$ where $H' = [H, H]$ is the semisimple part of $H$, and $Z(\cdot)$ denotes the center. This includes the PRV case (since $H' = 1$), as well as any case where $H$ is of adjoint type; it furthermore excludes the counterexample with $SL(2) \subseteq SO(5)$.

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