Spinorial density matrix equation and gauge covariance

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Abstract

In this work we apply the Lie group representation method introduced in the real time formalism for finite-temperature quantum-field theory, thermofield dynamics, to derive a spinorial density matrix equation. Symmetry properties of such equation are analysed, and as a basic result it is shown that one solution is the generalised density matrix operator proposed by Heinz, to deal with gauge covariant kinetic equations. In the same context, preliminary aspects of a Lagrangian formalism to derive kinetic equations, as well as quantum density matrix equations in curved space-time, are discussed.
I. INTRODUCTION

This paper presents a group symmetry scheme in order to treat the relativistic density matrix including the gauge covariance. In a previous paper [1] (to be referred from now on as I), we have set forth a representation theory for Lie groups, which is based in the thermofield dynamics (TFD), a real-time finite-temperature field theory via an operator method [2–4]. In I the TFD Hilbert space has been used as the representation space for Lie algebras, and so, the equations of motion in quantum TFD have been derived from the Galilei and Poincaré symmetries [1]; a classical counterpart of TFD has been identified [6]; and a connection of TFD with the GNS (Gel’fand, Naimark and Segal) construction [8] has been analysed in particular in association with elements of Hopf algebras [9–12]. A density-matrix equation for the Klein-Gordon field has been introduced and basic aspects of the kinetic theory for bosons has been analysed [5]. This has indicated the possibility to extend this symmetry group formalism for other systems of the quantum kinetic field theory, in particular by considering gauge fields and curved space-time. Such a study can bring the relativistic transport equations to a more fundamental level of symmetry group, and such aspect find interest in the recent developments in high energy physics and in early universe theories.

One remarkable prediction of the quantum chromodynamics is the possibility of a transition from a confined hadronic matter to a deconfined quark-gluon plasma (QGP), in an ultra-relativistic collision of heavy ions, at an energy density of \( \sim 2 \text{ GeV/fm}^3 \) and a temperature of \( \sim 200 \text{ MeV} \). Experimentally, such conditions will be realized soon, and this fact makes it essential to elaborate on some aspects, such as the signature of the QGP in the pre-equilibrium stage [13]. These developments have provided a strong motivation for a revival of the kinetic theory in order to treat classical or quantum quark-gluon systems [13,14]. It is our contention that the symmetry properties can provide an important tool for such developments. Therefore, this paper explores the fundamental consequences of symmetry on the kinetic theory under gauge invariance as in the case of a quark-gluon plasma.
In order to deal with gauge kinetic-theory models, Heinz [15–22] has proposed a generalization of the concept of density matrix to take into account explicitly the gauge fields. Considering a field \( \phi(x) \), Heinz has written the density matrix in the following form

\[
\Psi(x, y) = e^{-yd(x)} \phi(x) \otimes \bar{\phi}(x) e^{yd(x)^\dagger},
\]

where \( d(x) \) is a covariant derivative. Such a theory has been successful in introducing gauge invariant Wigner operators in relativistic phase spaces, such that transport equations have been derived, though, the collision terms have not been treated in a full and consistent way (indeed, difficulties with the collision terms are a characteristic aspect of the present state of the art of the gauge kinetic theory). One way to proceed in order to derive a full-collision kinetic approach is, first of all, to understand the underlying structure attached to the Heinz density operator. One such answer is presented in this paper, where we derive (1) as a solution of a density matrix equation proposed here via the representation theory of \( I \) considering the spinorial field. This result suggests a generalization of the Heinz density operator (1) to deal with quantum fields in curved space-time.

An approach to derive transport equations for systems treated by a quantum field in curved space-time background was proposed by Calzetta, Habib and Hu [23] and Calzetta and Hu [24]. In such a method problems related to the notion of particles in curved manifold, averages, distribution functions and distance between two points, all of them necessary to build a kinetic theory, have been partially overcome through the use of the Riemann normal coordinates, in which, in the neighborhood of the origin of coordinates, some modes are close to plane waves. So the metric tensor \( g_{\mu\nu} \) is expanded as

\[
g_{\mu\nu}(x) = \eta_{\mu\nu} + \frac{1}{3} R_{\mu\tau\nu\rho} x^\tau x^\rho + \ldots,
\]

where \( \eta_{\mu\nu} \) is a flat-space metric and \( R_{\mu\tau\nu\rho} \) is the Riemann tensor. On the other hand, information about early universes may be obtained by the time evolution of the density matrix of the universe, by considering the interaction of the quantum fields and many-particles systems with the classical gravitational field. An important problem in this context is of the gauge quantum kinetic theory, which we are interested to consider at this stage.
We proceed in the following way. In section 2 some structural elements of TFD are presented in connection with the representation theory developed in I, but rederiving the basic method along with our proposals here. In section 3 the density matrix equation is introduced via a representation of the Poincaré group and the gauge invariance is analysed in this context. In section 4 we derive a collisionless Boltzmann equation for the spinorial field in an algebraic way, such that the approach proposed here can be better compared with others in the literature. A possible Lagrangian formalism for the kinetic theory is discussed and some comments are made about the extension to the curved space-time as suggested by Calzetta, Habib and Hu [23,24]. Final remarks and conclusions are presented in Section 5.

II. THERMOFIELD DYNAMICS AND REPRESENTATION THEORY FOR LIE GROUPS

The thermofield dynamics is based on two main ingredients: one, the dual (or tilde) conjugation rules, introduces a doubling in the dynamical variable; while the other, the Bogolioubov transformation, introduces thermal effects through a vacuum condensation process. The doubling, which is a very attractive algebraic aspect in TFD, is defined in the following way.

Let \( \mathcal{L} = \{ A_i, i = 1, 2, 3, ... n; a, b, c, ... \in \mathbb{C} \} \) be an associative algebra on a \( \mathbb{C} \)-field. Tilde conjugation is a mapping \( \tau : \mathcal{L} \to \mathcal{L} \), denoted as \( \tau A = \tilde{A} \), furnishing the following properties [4]:

\[
\begin{align*}
(A_i A_j) &= \tilde{A}_i \tilde{A}_j, \\
(c A_i + A_j) &= c^* \tilde{A}_i + \tilde{A}_j, \\
(A_i^\dagger) &= (\tilde{A}_i)^\dagger, \\
(\tilde{A}_i) &= A_i, \\
[A_i, \tilde{A}_j] &= 0,
\end{align*}
\]

where \( [ \ , \ ] \) is the commutator. Usually, in TFD, there are two more rules: \( |0(\theta)\rangle = |0(\theta)\rangle \)
and \( \tilde{0}(\theta) = \langle 0(\theta) \rangle \), which show that vacua in TFD are tilde-conjugation invariant \( \tilde{7} \).

Automorphisms in \( \mathcal{L} \) can then be introduced through unitary operators, say \( U(\xi) \), such that we require \( \tau U(\xi) \tau = U(\xi) \). With this result, writing \( U(\xi) \) as \( U(\xi) = \exp(i\xi \hat{A}) \), where \( \hat{A} \) is a transformation generator, we have \( \tau \hat{A} \tau = -\hat{A} \). Therefore, \( \hat{A} \) can be considered as an odd polynomial function of \( A - \tilde{A} \), i.e.,

\[
\hat{A} = f(A - \tilde{A}) = \sum_{n=0}^{\infty} c_n (A - \tilde{A})^{2n+1},
\]

where the coefficients \( c_n \) are c-numbers.

We consider here the simplest case with \( n = 0 \), that is,

\[
\hat{A} = A - \tilde{A}.
\]

Therefore, this, Eq.(8), equips the TFD algebra, \( \mathcal{L} \), with an hat-isomorphism, satisfying the properties:

\[
(cA + B)^\sim = c \hat{A} + \hat{B}, \quad c \in \mathbb{R},
\]

\[
(icA)^\sim = ic(2A - \hat{A}),
\]

\[
(AB)^\sim = A \hat{B} + \hat{A}B - \hat{A}\hat{B},
\]

\[
[A, B]^\sim = [A, \hat{B}] + [\hat{A}, B] - [\hat{A}, \hat{B}],
\]

\[
[A, B]_+^\sim = [A, \hat{B}]_+ + [\hat{A}, B]_+ - [\hat{A}, \hat{B}]_+,
\]

\[
\hat{A} = 2\hat{A}.
\]

The proof follows from the definition of tilde conjugation rules and Eq.(8).

Now we are going to explore these generators (\( \hat{A} \)), as elements of a Lie algebra. Consider \( \ell = \{a_i, i = 1, 2, 3, ... \} \) a Lie algebra over the (real) field \( \mathbb{R} \), characterized by the algebraic relations \( a_i \triangle a_j = C_{ij}^k a_k \), where \( C_{ij}^k \in \mathbb{R} \) are the structure constants and \( \triangle \) is the Lie product (we are using the convention of sum over repeated indices). Taking the TFD Hilbert space, say \( \mathcal{H}_w \), as the space of representations for \( \ell \), we write

\[
[A_i, \hat{A}_j] = iC_{ij}^k \hat{A}_k.
\]
\[ [\hat{A}_i, A_j] = iD^k_{ij}A_k, \]  \\
\[ [A_i, A_j] = iE^k_{ij}A_k, \]  \\
That is to say, physically Eq.(15) represents the Lie symmetries characterized by the structure constants \( C^k_{ij} \) (by definition, the hat operators \( \hat{A} \) are the symmetry generators, a class of dynamical observables). Since the non-hat operators \( (A) \) can be identified with the observables, the other possible class of dynamical variables, then Eq.(16) shows the way that generators of symmetries determine dynamical changes in the observables. The Abelian (or non-Abelian) nature of the observables in regard to the measurement process is established by Eq.(17) (this interpretation has been used to write Eq.(17) as \([A_i, A_j] = 0\) and so to derive a classical TFD structure \([6,7]\)).

In this procedure, we have split the twofold structure of \( H_w \), by introducing a representation for Lie symmetry, defined by the (TFD) algebra given by Eqs.(15)-(17), which we denote by \( \ast \ell \) (regarding the close connection of this structure with the \( w^\ast \)-algebra, see Ref. [5]). In the next section we use this representation theory to analyse a spinorial representation for the \( \ast \)-Poincaré group, \( \ast p \).

Closing this section, let us observe that for a gauge symmetry, a set of generators of the gauge group will be added to the space-time symmetry group; and, of course, these two set of generators commute with each other. However, the implications of gauge groups as additional quantum numbers, such as color, etc, will appear in defining the Hilbert space.

III. POINCARÉ GROUP AND DENSITY-MATRIX FIELD EQUATIONS

Here we build a \( \ast -p \) Lie algebra assuming that in Eqs.(15)-(17) we have \( D^k_{ij} = E^k_{ij} = C^k_{ij} \), where \( C^k_{ij} \) are the structure constants of the Poincaré Lie algebra, such that we can write

\[ [M_{\mu\nu}, P_\sigma] = i(g_{\nu\sigma}P_\mu - g_{\mu\sigma}P_\nu), \]  \\
\[ [P_\mu, P_\nu] = 0, \]  \\
\[ [M_{\mu\nu}, M_{\sigma\rho}] = -i(g_{\mu\rho}M_{\nu\sigma} - g_{\nu\rho}M_{\mu\sigma} + g_{\mu\sigma}M_{\rho\nu} - g_{\nu\sigma}M_{\rho\mu}), \]
\[ [\hat{M}_{\mu\nu}, P_\sigma] = [M_{\mu\nu}, \hat{P}_\sigma] = i(g_{\nu\sigma} P_\mu - g_{\sigma\mu} P_\nu), \]
\[ [\hat{P}_\mu, P_\nu] = 0, \]  
\[ [\hat{M}_{\mu\nu}, M_{\sigma\rho}] = -i(g_{\mu\rho} M_{\nu\sigma} - g_{\nu\rho} M_{\mu\sigma} + g_{\mu\sigma} M_{\rho\nu} - g_{\nu\sigma} M_{\rho\mu}), \]
\[ [\hat{M}_{\mu\nu}, \hat{P}_\sigma] = i(g_{\nu\sigma} \hat{P}_\mu - g_{\sigma\mu} \hat{P}_\nu), \]
\[ [\hat{P}_\mu, \hat{P}_\nu] = 0, \]
\[ [\hat{M}_{\mu\nu}, \hat{M}_{\sigma\rho}] = -i(g_{\mu\rho} \hat{M}_{\nu\sigma} - g_{\nu\rho} \hat{M}_{\mu\sigma} + g_{\mu\sigma} \hat{M}_{\rho\nu} - g_{\nu\sigma} \hat{M}_{\rho\mu}), \]

where \( \hat{M}_{\mu\nu} \) stands for the generators of rotations and \( \hat{P}_\mu \) the generators of translations; whilst \( M_{\mu\nu} \) and \( P_\mu \) are, respectively, the corresponding observables for rotations and translations. The metric tensor is such that \( \text{diag}(g_{\mu\nu}) = (1, -1, -1, -1) \), and \( g_{\mu\nu} = 0 \) for \( \mu \neq \nu; \quad \mu, \ \nu = 0, 1, 2, 3. \)

The invariants of \( *p \) are
\[ W = w_\mu w^\mu, \]
\[ P^2 = P_\mu P^\mu, \]
\[ \hat{W} = w_\mu w^\mu - \hat{w}_\mu \hat{w}^\mu = 2\hat{w}_\mu w^\mu - \hat{w}_\mu \hat{w}^\mu, \]
\[ \hat{P}^2 = (P^2) = P_\mu P^\mu - \hat{P}_\mu \hat{P}^\mu = 2\hat{P}_\mu P^\mu - \hat{P}_\mu \hat{P}^\mu; \]

where \( w_\mu = \frac{1}{2} \varepsilon_{\mu\rho\sigma} M^{\nu\sigma} P^\rho \) are the Pauli-Lubanski matrices; \( \varepsilon_{\mu\rho\sigma} \) is the Levi-Civita symbol; and
\[ \hat{w}_\mu = \frac{1}{2} \varepsilon_{\mu\rho\sigma} \hat{M}^{\nu\sigma} P^\rho + \frac{1}{2} \varepsilon_{\mu\rho\sigma} M^{\nu\sigma} \hat{P}^\rho - \frac{1}{2} \varepsilon_{\mu\rho\sigma} \hat{M}^{\nu\sigma} \hat{P}^\rho. \]

Notice that the tensor
\[ \overline{W}_\mu = \frac{1}{2} \varepsilon_{\mu\rho\sigma} \hat{M}^{\nu\sigma} \hat{P}^\rho \]

can be used to define the scalar \( \overline{W} = \overline{w}_\mu \overline{w}^\mu \), which is not an invariant of \( *p \) but rather of the subalgebra (of \( *p \)) given by Eqs.(24)-(26).

Denoting the vectors of the representation space of \( \mathcal{H}_w \) by \( |\Psi\rangle \) such that
where $|\phi\rangle$ is an element of a Hilbert space $\mathcal{H}$, such that $\mathcal{H}_w = \mathcal{H} \otimes \mathcal{H}^*$. We also assume that $\langle \phi | \phi \rangle = 1$. The action of the operators $O$ and $\tilde{O}$ in $\mathcal{H}_w$ is then

$$O|\Psi\rangle \rightarrow [O \otimes 1]|\Psi\rangle = [O \otimes 1]|\phi\rangle \otimes \langle \phi |,$$

and

$$\tilde{O}|\Psi\rangle \rightarrow [1 \otimes O]|\Psi\rangle = [1 \otimes O]|\phi\rangle \otimes \langle \phi | O^\dagger.$$

As a consequence, we show that the invariant $(P^2)^\wedge$ has a null constant value in this representation, that is

$$(P^2)^\wedge |\Psi\rangle = (2 P^\mu \hat{P}_\mu - \hat{P}^\mu \hat{P}_\mu)|\Psi\rangle = 0.$$

This is but the density matrix equation for the Klein-Gordon field derived in $I$.

In order to construct a spinor density matrix equation, we introduce an equation as

$$(\alpha^\mu P_\mu)^\wedge |\Psi\rangle = 0,$$

such that, differently from Eqs. (34), we write

$$(\alpha^\mu P_\mu)^\wedge (\alpha^\mu P_\mu)^\wedge = (P^2)^\wedge.$$

Using Eq. (8), a generic solution is immediately found to be

$$\alpha^\mu = \sigma \gamma^\mu,$$
where now $|\Psi\rangle$ is a 16-component spinor, and $|\phi\rangle$ is the 4- (dual) Dirac spinor.

Multiplying the rhs of Eq.(38) by $|\phi\rangle$, it results in $(\gamma^\mu p_\mu - m) |\phi\rangle = 0$, the Dirac equation. Now, multiplying the lhs of Eq.(38) by $\langle \bar{\phi} |$ it results in $\langle \bar{\phi} | (p_\mu \gamma^\mu - m) = 0$, the conjugated Dirac equation. In this sense, in fact, Eq.(38) is a Liouville-von Neumann-like equation for the Dirac Field.

At this point, we are able to explore some symmetry properties of Eq. (38), as the invariance under similarity transformations. That is, consider

$$ (\gamma^\mu \hat{P}_\mu)' = U (\gamma^\mu \hat{P}_\mu) U^{-1}, \quad (39) $$

and

$$ |\Psi\rangle' = U |\Psi\rangle. \quad (40) $$

Then Eq.(38) reads

$$ (\gamma^\mu \hat{P}_\mu)' |\Psi\rangle' = 0. \quad (41) $$

On the other hand, if $[U, (\gamma^\mu \hat{P}_\mu)] = 0$ then $|\Psi\rangle'$ given in Eq.(40) is a solution of Eq.(38). In this case, one suggestive example is provided by $U = U(y \hat{P})$ written in the form

$$ U = U(y \hat{P}) = \exp[-iy \hat{P}], \quad (42) $$

where $y$ is the transformation parameter. Accordingly, Eq.(41) reads

$$ |\Psi(y)\rangle' = \exp(-iyP) |\phi\rangle \otimes \langle \bar{\phi} | \exp(iyP^\dagger). \quad (44) $$

The gauge covariance can be considered if we write

$$ P \rightarrow -id_\mu = P_\mu + g A_\mu, \quad (43) $$

where $d_\mu$ is the usual covariant derivative. Then

$$ |\Psi(y)\rangle' = \exp(-yd) |\phi\rangle \otimes \langle \bar{\phi} | \exp(yd^\dagger). \quad (44) $$
where the operation $\otimes$ is defined by Eq.(31). This function $|\Psi(y)\rangle$ is a solution of the density matrix equation, which is derived from Eq.(38), but rather considering the gauge field, that is

$$[(\gamma^\mu d_\mu) \otimes 1 - 1 \otimes (\gamma^\mu d_\mu)]|\phi\rangle \otimes |\bar{\phi}\rangle = 0.$$  

(45)

If we use the definition given by Eq.(31), then we can write $\langle x, x'|\Psi \rangle = \Psi(x, x')$, such that Eq.(14) can be written as

$$\Psi(x, y) = \exp[-yd(x)] \phi(x) \otimes \bar{\phi}(x) \exp[yd(x)^\dagger].$$

(46)

which is the generalized Heinz density operator. It is to be emphasized that $|\phi\rangle$ (or $\phi(x)$) are elements of the Hilbert space. So the various quantum numbers either from the space-time group or from the gauge group are assumed to be attached to each element of the representation space.

As another example of the use of such symmetry approach, we derive the TFD spinor-electrodynamics equations. The fields are assumed to be the basic dynamical variables, so that the Lagrangian may be written as

$$L = \bar{\phi} \hat{D}\phi + \bar{\phi}(\hat{D} - \hat{D} - m)\phi$$

$$+ \phi(\hat{D} - \hat{D} - m)\phi - \bar{\phi}(D - \hat{D} - m)\phi$$

$$+ \frac{1}{2} F^{\mu\nu} \hat{F}_{\mu\nu} - \frac{1}{4} \hat{F}^{\mu\nu} \hat{F}_{\mu\nu},$$

which is derived from the usual spinor-electrodynamics Lagrangian and the hat mapping, and where

$$D = i\gamma^\mu (\partial_\mu + ieA_\mu).$$

Therefore, it follows that

$$\hat{D}\phi + (D + m)\phi - \hat{D}\phi = 0,$$

$$(D + m)\phi = 0,$$

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which are the quasi-particle TFD equations for the spinor field. The other equations can be derived accordingly.

In the next section we are going to present another application of the method developed here by deriving a phase space Wigner structure attached to the Dirac field. We use the more general form of Eq.(38) including \( \sigma \neq 1 \), in such a way that we can better compare our formalism with the one proposed by Bohm and Hiley [25] and Holland [26] to derive phase-space spinor fields under a geometric perspective.

IV. THE DIRAC FIELD AND A COLLISIONLESS BOLTZMANN EQUATION

Notice that, using Eq.(37), Eq.(35) can be written as

\[
\left( \sigma^l \gamma^\mu \frac{\partial}{\partial x^\mu} - \sigma^r \gamma^\mu \frac{\partial}{\partial x'^\mu} \right) \Psi_{ts}(x, x') = 0,
\]

such the \( \Psi_{ts}(x, x') = \phi(x)\bar{\phi}(x') \), where the subscripts \( t, s = 1, 2, ..., 4 \) refer to the two spinor indices.

Here \( \gamma^\mu = \gamma^\mu \otimes 1 \), \( \gamma^\mu = 1 \otimes \gamma^\mu \) are 16 \times 16 matrices and \( \gamma^\mu \) are the Dirac matrices; \( \sigma^l = \sigma \otimes 1 \) and \( \sigma^r = 1 \otimes \sigma \); with \( \sigma \), the arbitrary Lorentz invariant to be specified; besides the \( \gamma^{r,l} \)-matrices fulfil two Clifford algebras:

\[
\{ \gamma^\mu, \gamma^\nu \} = \{ \gamma^\mu, \gamma^\nu \} = 2g^{\mu\nu}; \quad \{ \gamma^\mu, \gamma^\nu \} = 0
\]

Multiplying Eq.(47) by \( \left( \sigma^l \right)^{-1} \), it follows that

\[
\Lambda(\kappa) \Psi_{ts}(x, x') = \left( \gamma^\mu \frac{\partial}{\partial x^\mu} - \kappa \gamma^\mu \frac{\partial}{\partial x'^\mu} \right) \Psi_{ts}(x, x') = 0,
\]

where \( \kappa = (\sigma^l)^{-1} \sigma^r \). Using the following transformation

\[
\frac{\partial}{\partial x^\mu} = \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial q^\mu} - p_\mu \right), \quad \frac{\partial}{\partial x'^\mu} = \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial q'^\mu} + p_\mu \right),
\]

the square of \( \Lambda(\kappa) \) can be written as

\[
\Lambda^2(\kappa) = \Lambda^\mu_1(\kappa) \frac{\partial^2}{\partial q^\mu \partial q'^\nu} + \Lambda^\mu_2(\kappa) \frac{\partial}{\partial q^\mu} p_\nu + \Lambda^\mu_3(\kappa) p_\mu \frac{\partial}{\partial q'^\mu} + \Lambda^\mu_4(\kappa) p_\mu p_\nu.
\]
where

\[
\begin{align*}
\Lambda^{i\mu\nu}(\kappa) &= \frac{1}{2}(\gamma^l_{\mu} - \kappa \gamma^r_{\mu})(\gamma^l_{\nu} - \kappa \gamma^r_{\nu}), \\
\Lambda^{\mu\nu}(\kappa) &= \frac{1}{2}(\gamma^l_{\mu} + \kappa \gamma^r_{\mu})(\gamma^l_{\nu} - \kappa \gamma^r_{\nu}), \\
\Lambda^{\mu\nu}(\kappa) &= \frac{1}{2}(\gamma^l_{\mu} - \kappa \gamma^r_{\mu})(\gamma^l_{\nu} + \kappa \gamma^r_{\nu}), \\
\Lambda^{\mu\nu}(\kappa) &= \frac{1}{2}(\gamma^l_{\mu} + \kappa \gamma^r_{\mu})(\gamma^l_{\nu} + \kappa \gamma^r_{\nu}).
\end{align*}
\]

From these expressions for \(\Lambda^{i\mu\nu}(\kappa), (i = 1, \ldots, 4)\), the following operators can be defined

\[
a^{\mu+}(\kappa) = \frac{1}{\sqrt{2}}(\gamma^l_{\mu} + \kappa \gamma^r_{\mu}) \quad a^{\mu}(\kappa) = \frac{1}{\sqrt{2}}(\gamma^l_{\mu} - \kappa \gamma^r_{\mu});
\]

then Eq.(51) is rewritten as

\[
\Lambda^2(\kappa) = a^{\mu}(\kappa)a^{\nu}(\kappa)\frac{\partial^2}{\partial q^\mu \partial q^\nu} + a^{\mu+}(\kappa)a^{\nu}(\kappa)\frac{\partial}{\partial q^\mu} p_\nu + a^{\mu}(\kappa)a^{\nu+}(\kappa) p_\mu p_\nu.
\]

Notice that if \(\kappa\) anticommutes with the matrices \(\gamma^l_{\mu}\) and \(\gamma^r_{\mu}\), and satisfies \(\kappa^2 = 1\) (which is compatible with Eq.(38)), it leads to

\[
\{a^{\mu}, a^{\nu}\} = \{a^{\mu+}, a^{\nu+}\} = 0
\]

\[
\{a^{\mu+}, a^{\nu}\} = g^{\mu\nu},
\]

Observe that the requirement of Lorentz invariance for \(\kappa\) can be achieved if we define \(\sigma\) in Eq.(37) as \(\sigma = \gamma^5\) such that \(\kappa = \gamma^5\gamma^r_5\), where \(\gamma^5 = i\gamma^1\gamma^2\gamma^3\gamma^4\). Using these results, we follow Bohm and Hiley \[25\] to write

\[
\Lambda^2 = a^{\mu+}a^{\nu} \frac{\partial}{\partial q^\mu} p_\nu + a^{\mu}a^{\nu+} p_\mu \frac{\partial}{\partial q^\mu} + a^{\mu+}a^{\nu+}(p_\mu p_\nu).
\]

Therefore, from Eq.(49), we obtain

\[
p^\mu \frac{\partial}{\partial q^\mu} \Psi(q, p) = 0; \quad (53)
\]
which is the Bohm and Hiley results \[25\], as developed by Holland \[26\], about the relativistic phase space. But here, we have derived Eq.(53) from first principles, so that we can go somewhat further with it. Observe, in particular, that Eq.(53) can be derived also for the Klein-Gordon field if we use the transformation given by Eq.(50) in Eq.(34). It should be stressed here that within an approach based entirely on representations of symmetry, only a collisionless Boltzmann equation can be obtained. The role of dynamics can be included by postulating the interaction Lagrangian, that will lead to collision terms. The result, then, will be model-dependant. Consistently, Eq.(53) describes only a drift term, that should be the same for any kind of systems, including the classical ones.

In particular, in this formalism we take advantage of the fact that the state is treated as an amplitude, to implement higher order terms in Eq.(53) looking for an invariant Lagrangian procedure, avoiding then the (usual) intricate way to do that. Indeed, observe that one simple way to derive a Vlasov term (the influence of a external field via \( F^\mu \)) in Eq.(53), considering the first order (classical) limit in the Heinz approach, is to write the Lorentz invariant Lagrangian as

\[
L(q,p) = \frac{1}{2} \Psi^*(q,p)p_\mu \partial^\mu \Psi(q,p) - \frac{1}{2} \Psi(q,p)p_\mu \partial^\mu \Psi^*(q,p) - \frac{1}{2} \Psi^*(q,p) m F^\mu_\nu \partial_\nu \Psi(q,p) - \frac{1}{2} \Psi(q,p) m F^\mu_\nu \partial_\nu \Psi^*(q,p).
\]

Then the energy-momentum tensor is defined as

\[
T^{\mu\nu} = \int d^3p \frac{1}{p^0} \Psi^*(q,p)p^\mu p^\nu \Psi(q,p),
\]

which is still compatible with the usual definition of \( T^{\mu\nu} \) since \( \Psi^*(q,p)\Psi(q,p) \) can be interpreted as the classical distribution function in the relativistic phase space.

Another aspect to be explored in the present formalism is the inclusion of gravitational effects in the kinetic theory. The treatment of transport theory in a curved space-time background is somewhat hindered by the fact that a definition of the Wigner function \( \Psi(q,p) \) depends on the use of the Fourier transform of a spacetime correlation function with a translated argument. Fourier transformation (nor translations) are globally available.
in general curved spacetime. However, as shown by Calzetta, Habib and Hu [23], the construction is still meaningful, if carried out in the tangent space of an event in spacetime and related to the neighbourhood of that event by an exponential map. Therefore, starting from Eq.(45), the Liouville-Vlasov equation for the Wigner function of a spinor field coupled to a gauge field with field strength tensor $F^{\mu\nu}$ is then

$$(-p_{\mu}F^{\mu\nu}D_{\nu} + p^\mu D_{\mu})\Psi(q,p) = 0,$$

where $D_{\mu}$ is the covariant derivative including the curved space-time spin connection.

V. FINAL CONCLUSIONS AND REMARKS

Summarizing, we have used here a representation formalism for Lie algebras, in which the elements of the representation space, say $\mathcal{H}_w$, are closely connected with the notion of density matrix through the thermofield dynamics approach. We have started with the Poicaré Lie algebra in the $\mathcal{H}_w$-space. Then we have derived a gauge invariant density matrix equation for the spinorial field, and shown that one particular solution for such an equation is the gauge invariant density operator as proposed by Heinz. This fact has set forth a solid structure to support the Heinz approach, since it has been derived here by an analysis of group theory, i.e. the fundamental symmetry properties of the system. In the same context, analysing a Lagrangian formalism, we have derived the TFD equations; and so, we have presented an association of the Heinz density method and TFD. The phase space structure of the spinorial field has also been studied as another application of our method. In this case, once again, we have been able to advocate the group theoretical bases to derive the early results of Bohm and Hiley [25] and Holland [26] about the connection of the Clifford algebra and phase-space. But here we have proceeded somewhat further, pointing out a possible Lagrangian formalism to take into account higher order terms in a kinetic equation. This last fact opens the possibility to deal with the kinetic theory as a genuine quantum or classical field formalism (observe that, as we are analysing representations of a Lie algebra,
the function $\Psi(p, q)$ in Eq.(53) can be considered as a quantum-like field defined in a Fock space, similar to what has been developed for Galilean classical systems [6]; and so, a pertubative scheme can be introduced by symmetry properties of the Lagrangian, as has been argued in writing Eq.(54). Besides, we have discussed the effect of the curved space-time background for the spinorial density matrix equation. Some of these aspects are treated in a more detailed way in another paper [27]).

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