Moments of Sectional Curvature

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Abstract

The sectional curvature of a compact Riemannian manifold $M$ can be seen as a random variable on the Grassmann bundle of 2–planes in $TM$ endowed with the Fubini–Study volume density. In this article we calculate the moments of this random variable by integrating suitable local Riemannian invariants and discuss the distribution of the sectional curvature of Riemannian products. Moreover we calculate the moments and the distribution of the sectional curvature for all compact symmetric spaces of rank 1 explicitly and derive a formula for the moments of general symmetric spaces. Interpolating the explicit values for the moments obtained we prove a weak version of the Hitchin–Thorpe Inequality.

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1 Introduction

Analyzing the sectional curvature is an indispensable tool in understanding the local geometry and the global topology of Riemannian manifolds. Given the importance of the sectional curvature we propose in this article to study the sectional curvature of a compact Riemannian manifold $M$ as a random variable defined on the Grassmann bundle of 2–planes in $TM$. Our main motivation for this study besides pure curiosity is to provide us with a large subalgebra of local Riemannian invariants, whose values are easily calculated for many interesting examples and can be interpreted directly in terms of the underlying geometry. The more explicit values of local Riemannian invariants are known the easier it is to find and prove general linear relations between these invariants.

The principal characteristic of the sectional curvature of a compact Riemannian manifold $M$ thought of as a random variable $\sec : \text{Gr}_2 TM \rightarrow \mathbb{R}$, $(p,\Sigma) \mapsto \sec_{R_p}(\Sigma)$, on the Grassmann bundle $\text{Gr}_2 TM$ of 2–planes in $TM$ is of course its distribution, a probability measure on the real line, whose support determines the minimum and maximum of the sectional curvature of $M$. Treating the sectional curvature as a random variable may thus be

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seen as an alternative route to the calculation of pinching constants compared for example to the work of Püttmann [P]. Determining the distribution of the sectional curvature directly however seems exceedingly difficult, for this reason we will focus instead on the moments of the sectional curvature with respect to a suitable volume density $\text{vol}_{FS}$ on $\text{Gr}_2 TM$. In essence the moments of the sectional curvature are integrated local Riemannian invariants:

**Theorem 2.4 (Moments of Sectional Curvature)**

Consider the category of euclidean vector spaces of dimension $m$ under isometries as morphisms and the functor, which associates to every such vector space $V$ with scalar product $g$ the vector space of algebraic curvature tensors $R : V \times V \times V \rightarrow V$, $(X, Y, Z) \mapsto R_{XYZ}$. The sequence $\{ \Psi_k \}_{k \in \mathbb{N}_0}$ of natural polynomials of degree $k$ on algebraic curvature tensors

$$\Psi_k^V(R) := \left( \frac{(-\Delta_g)^k}{[m + 2k - 2]_{2k}} \right)_{0} \left( X \mapsto \exp \left( \sum_{r>0} \frac{1}{2r} \text{tr}_V( R_{.,X}X^r ) \right) \right)$$

defined using the positive Laplace operator $\Delta_g$ on $C^\infty V$ associated to the scalar product $g$ and the falling factorial polynomial $[x]_s := x(x-1)\ldots(x-s+1)$ can be used to calculate the moments of the sectional curvature of every compact Riemannian manifold $M$ via:

$$E_M[\sec^k] := \frac{1}{\text{Vol} \text{Gr}_2 TM} \int_{\text{Gr}_2 TM} \sec^k_{Rp}(\Sigma) \text{vol}_{FS}(p, \Sigma) = \frac{1}{\text{Vol} M} \int_M \Psi_k^{T_p M}(R_p) \text{vol}_g(p)$$

Using this presentation we calculate the moments of the sectional curvature for all the compact symmetric spaces of rank 1 up to covering, besides the round spheres $S^m$ this family of examples comprises the complex and hyperbolic projective spaces $\mathbb{CP}^n$ and $\mathbb{HP}^n$ with $n \geq 2$ and the exceptional Cayley projective plane $\mathbb{OP}^2$. The sequence of moments for these symmetric spaces all correspond to very simple probability measures on $\mathbb{R}$, namely

$$E_{\mathbb{CP}^n}[F(\sec)] = \frac{1}{6} \left( n - \frac{3}{2} \right) \int_{1}^{4} \sqrt{\frac{s-1}{3}} \left( \frac{4-s}{3} \right)^{n-2} F(s) \, ds$$

$$E_{\mathbb{HP}^n}[F(\sec)] = \frac{3}{6} \left( 2n - \frac{3}{2} \right) \int_{1}^{4} \sqrt{\frac{s-1}{3}} \left( \frac{4-s}{3} \right)^{2n-3} F(s) \, ds$$

$$E_{\mathbb{OP}^2}[F(\sec)] = \frac{7}{6} \left( \frac{13}{2} \right) \int_{1}^{4} \sqrt{\frac{s-1}{3}} \left( \frac{4-s}{3} \right)^{3} F(s) \, ds$$

where in all cases the metric is normalized to have sectional curvatures in the interval $[1, 4]$. For general compact symmetric spaces the calculation of the moments of the sectional curvature reduces via a suitable variant of Weyl’s Integration Formula to an integration over spherical simplices, in this way we obtain closed formulas for the moments of the sectional curvature of the Fubini–Study metric on the Grassmannians of 2–planes in $\mathbb{R}^n$, $\mathbb{C}^n$ and $\mathbb{H}^n$. In order to augment this limited stock of examples we consider the behaviour of the moments of the sectional curvature under Riemannian products and obtain from Theorem 2.4:
Lemma 4.1 (Sectional Curvature Moments of Products)
Consider two compact Riemannian manifolds $M$ and $N$ of dimensions $m, n \geq 0$. The moments of the sectional curvature of the Riemannian product $M \times N$ can be calculated via

$$E_{M \times N}[\sec^k] = \sum_{r=0}^{k} \binom{k}{r} \frac{(m + 2r - 2)_{2r}(n + 2(k - r) - 2)_{2(k-r)}}{(m + n + 2k - 2)_{2k}} E_M[\sec^r] E_N[\sec^{k-r}]$$

where $[x]_s := x(x-1)\ldots(x-s+1)$ denotes the falling factorial polynomial as before.

From the point of view of combinatorics this product formula for the moments of the sectional curvature appears rather strange, nevertheless it has a very nice probabilistic model in form of a family of probability measures on the 2–simplex $\Delta_2 \subset \mathbb{R}^2$. This probabilistic model allows us to extend the product formula for moments to a product formula for the distribution of the sectional curvature of Riemannian products formulated in Corollary 4.2.

The preceding results indicate that the moments of the sectional curvature are best seen in the context of integrated local Riemannian invariants. Interestingly the algebra of all local Riemannian invariants allows for a graphical calculus similar to the graph algebras describing the Rozansky–Witten invariants [RW] of hyperkähler manifolds. According to Theorem 2.4 the moments of the sectional curvature generate a large and quite explicit subalgebra of local Riemannian invariants, which are easy to calculate and relate directly to Riemannian geometry. In a forthcoming publication [W] we will combine our knowledge of explicit moments of sectional curvature together with the graphical calculus mentioned above in order to prove linear relations between local Riemannian invariants by interpolation. Interpolating for example the first and second moments of the sectional curvature of the Riemannian manifolds $\mathbb{C}P^2, S^4, S^2 \times S^2$ and $S^1 \times S^3$ we easily obtain the following version

$$\frac{4\pi^2}{3} \chi(M) + \frac{4}{9} \int_M |\text{Ric}^\circ\|^2 \text{vol}_g(p)$$

$$= \int_M \Psi_2(R_p) \text{vol}_g(p) + 4 \int_M \left[ \Psi_2(R_p) - \Psi_1(R_p)^2 \right] \text{vol}_g(p)$$

of the Hitchin–Thorpe Inequality [Th] for compact 4-dimensional Riemannian manifolds $M$, where $\chi(M)$ is the Euler characteristic and $\text{Ric}^\circ := \text{Ric} - \frac{n}{4} \text{id}$ the trace free Ricci tensor.

Organisatorically this article is divided into three sections besides this introduction. In Section 2 we reformulate the integration over Grassmannians into a double integration over round spheres and use this reformulation together with an integration trick for integrating polynomials over round spheres to prove Theorem 2.4. Explicit examples for the moments and the distribution of sectional curvature are given in Section 3, while Section 4 studies Riemannian products and discusses the interpolation proof of the Hitchin–Thorpe Inequality. The author would like to thank U. Semmelmann, the students and the staff of the University of Stuttgart for their hospitality during several research stays in Stuttgart.
2 Integration over Grassmannians

In general the moments of a random variable are defined as integrals over the probability space in question, the moments of the sectional curvature of an algebraic curvature tensor are thus integrals over the Grassmannian of 2–planes in the underlying euclidean vector space. Integration is in itself nothing else but the evaluation of a very special linear functional on an infinite dimensional vector space, hence it is prudent to look out for integration tricks before embarking on an explicit integration in local coordinates or otherwise. In this section we discuss such an integration trick for the integration of polynomials over a Grassmannian of 2–planes, which we use in turn to encode the moments of the sectional curvature of an algebraic curvature tensor in a simple generating formal power series.

Recall that an algebraic curvature tensor on a euclidean vector space $V$ of dimension $m$ with scalar product $g \in \text{Sym}^{2}V^{*}$ is a trilinear map $R : V \times V \times V \to V, (X, Y, Z) \mapsto R_{X,Y}Z$, which is alternating $R_{X,Y}Z = -R_{Y,X}Z$ in its first two arguments, defines skew–symmetric endomorphisms $Z \mapsto R_{X,Y}Z$ of $V$ for all $X, Y \in V$ and satisfies the first Bianchi identity:
\[ R_{X,Y}Z + R_{Y,Z}X + R_{Z,X}Y = 0 \]

Every such algebraic curvature tensor $R$ is completely determined by is associated sectional curvature, which is a well–defined function on the Grassmannian $\text{Gr}_2V$ of 2–planes in $V$:
\[ \text{sec}_R : \text{Gr}_2V \to \mathbb{R}, \quad \text{span} \{ X, Y \} \mapsto \frac{g(R_{Y,X}X, Y)}{g(X, X)g(Y, Y) - g(X, Y)^2} \]

In the sequel we want to integrate the powers $\text{sec}_R^{k}$ of the sectional curvature of an algebraic curvature tensor $R$ against the volume density of the Grassmannian $\text{Gr}_2V$ endowed with the Fubini–Study metric $g_{\text{FS}}$. The principal idea to make this integration feasible is to convert integrals over $\text{Gr}_2V$ into double integrals over unit spheres, because integrals over unit spheres are relatively easy to calculate, compare for example Corollary 2.3. The part of the Cardan joint of this conversion is played by the Stiefel manifold of orthonormal 2–frames:
\[ \text{St}_2V := \{ (X, Y) \mid g(X, X) = 1 = g(Y, Y) \text{ and } g(X, Y) = 0 \} \]

Evidently $\text{St}_2V \subset V \times V$ is a submanifold with tangent space equal to
\[ T_{(X, Y)}\text{St}_2V = \{ (\dot{X}, \dot{Y}) \mid g(X, \dot{X}) = 0 = g(Y, \dot{Y}) \text{ and } g(\dot{X}, Y) + g(X, \dot{Y}) = 0 \} \quad (1) \]

in an orthonormal 2–frame $(X, Y) \in \text{St}_2V$. In order to have $\text{St}_2V$ acting as a Cardan joint we need to define a Riemannian metric $g_{\text{FS}}$ on $\text{St}_2V$ so that the canonical projections
\[ \text{pr}_{Gr} : \text{St}_2V \to \text{Gr}_2V, \quad (X, Y) \mapsto \text{span} \{ X, Y \} \]
\[ \text{pr}_{s} : \text{St}_2V \to S_V, \quad (X, Y) \mapsto X \]
to $\text{Gr}_2V$ and the unit sphere $S_V \subset V$ are Riemannian submersions with respect to the Fubini–Study metric on $\text{Gr}_2V$ and the round metric $g$ on $S_V$. In terms of the description (1) of the tangent space $T_{(X, Y)}\text{St}_2V$ in $(X, Y) \in \text{St}_2V$ the Riemannian metric of choice reads:
\[ g_{\text{FS}}((\dot{X}_1, \dot{Y}_1), (\dot{X}_2, \dot{Y}_2)) := g(\dot{X}_1, \dot{X}_2) + g(\dot{Y}_1, \dot{Y}_2) - g(\dot{Y}_1, \dot{X}_2)g(\dot{Y}_2, \dot{X}_1) \]
The interesting and rather unexpected final term in this definition can be written in a number of equivalent ways due to the relation \( g(Y, X) = -g(X, Y) \) valid for all tangent vectors \((\dot{X}, \dot{Y}) \in T_{(X,Y)}St_2V\). Of course we would get a Riemannian metric on \(St_2V\) even without this final term, the Fubini–Study metric \(g^{FS}\) however has the very convenient property that the tangent space \(T_{(X,Y)}St_2V\) decomposes orthogonally into the direct sums

\[
T_{(X,Y)}St_2V = \mathbb{R}(-Y, X) \oplus \{ \dot{X}, \dot{Y} \mid \dot{X}, \dot{Y} \text{ orthogonal to both } X \text{ and } Y \} = \{ (0, \dot{Y}) \mid \dot{Y} \text{ orthogonal to } X, Y \} \oplus \{ (\dot{X}, -g(\dot{X}, Y)) \mid \dot{X} \text{ orthogonal to } X \}
\]

where \(\mathbb{R}(-Y, X)\) is the vertical tangent space for the projection to the Grassmannian \(Gr_2V\), while \(\{ (0, \dot{Y}) \mid \dot{Y} \text{ orthogonal to } X, Y \}\) is the vertical tangent space for the projection \(pr_s\) to the unit sphere \(S_V\). In consequence the projection \(pr_s\) is a Riemannian submersion, because:

\[
g^{FS}_{(X,Y)}( (\dot{X}_1, -g(\dot{X}_1, Y) X), (\dot{X}_2, -g(\dot{X}_2, Y) X) ) = g(\dot{X}_1, \dot{X}_2) + g(\dot{X}_1, Y)g(\dot{X}_2, Y) - g(Y, \dot{X}_1)g(Y, \dot{X}_2) = g(\dot{X}_1, \dot{X}_2)
\]

For precisely this reason we had to include the final term into the definition of the Fubini–Study metric \(g^{FS}\) on the Stiefel manifold \(St_2V\). In passing we observe that the fiber of the projection \(pr_s\) to the unit sphere \(S_V\) in \(X \in S_V\) is the round unit sphere \(S_{\{X\}}\) in the orthogonal complement of \(X\) due to \(g^{FS}_{(X,Y)}((0, \dot{Y}_1), (0, \dot{Y}_2)) = g(\dot{Y}_1, \dot{Y}_2)\).

In order to verify that the projection \(pr_{Gr}\) to the Grassmannian \(Gr_2V\) of 2–planes is likewise a Riemannian submersion we recall that the Fubini–Study metric \(g^{FS}\) is defined by means of the embedding of \(Gr_2V\) into the vector space of symmetric endomorphisms of \(V\)

\[
Gr_2V \rightarrow \text{End}_{\text{sym}}V, \quad \Sigma \mapsto pr_{\Sigma}
\]

and the scalar product \(h(F, \tilde{F}) := \frac{1}{2} \text{tr}_V(F\tilde{F})\) on \(\text{End}_{\text{sym}}V\). In terms of the musical isomorphism \(\sharp : V \rightarrow V^*, \quad X \mapsto g(X, \cdot)\), we may write \(pr_{\Sigma} = X^\sharp \otimes X + Y^\sharp \otimes Y\) for some orthonormal 2–frame \((X, Y) \in St_2V\) representing \(\Sigma = \text{span}\{ X, Y \} \in Gr_2V\) and find

\[
pr_{\Sigma} = \dot{X}^\sharp \otimes X + \dot{X}^\sharp \otimes \dot{X} + \dot{Y}^\sharp \otimes Y + \dot{Y}^\sharp \otimes \dot{Y}
\]

for a tangent vector \((\dot{X}, \dot{Y}) \in T_{(X,Y)}St_2V\). Calculating the trace scalar product results in:

\[
\frac{1}{2} \text{tr}_V( pr_{\Sigma}^2 ) = g(\dot{X}, \dot{X}) + g(\dot{Y}, \dot{Y}) + 2g(Y, \dot{X})g(X, \dot{Y}) = g(pr^\sharp_{\Sigma} \dot{X}, \dot{X}) + g(pr^\sharp_{\Sigma} \dot{Y}, \dot{Y})
\]

In consequence we conclude that the projection \(pr_{Gr}\) from the Stiefel manifold \(St_2V\) of orthonormal 2–frames to the Grassmannian \(Gr_2V\) of 2–planes is a Riemannian submersion as well, its fibers are two disjoint circles of length \(2\pi\) due to \(g^{FS}_{(X,Y)}((-Y, X), (-Y, X)) = 1\).

The idea of using the Riemannian submersions \(pr_{Gr}\) and \(pr_s\) to convert integrals over the Grassmannian \(Gr_2V\) into double integrals over unit spheres is illustrated by the calculation

\[
\text{Vol } St_2V = \frac{2\pi^m}{\Gamma\left(\frac{m}{2}\right)} \frac{2\pi^{m-1}}{\Gamma\left(\frac{m-1}{2}\right)} = \frac{2^m \pi^{m-1}}{(m-2)!} \quad \text{Vol } Gr_2V = \frac{(2\pi)^{m-2}}{(m-2)!}
\]
of the volumes of the Stiefel manifold and the Grassmannian with respect to their Fubini–Study metrics, in which we use Euler’s formula $\text{Vol} S^V = \frac{2\pi^\frac{m}{2}}{\Gamma(\frac{m}{2})}$ for the volume of unit spheres and simply divide by the volume $4\pi$ of two disjoint circles to obtain $\text{Vol} \text{Gr}_2 V$.

Turning away from the Stiefel manifold and the Grassmannian we want to discuss a couple of integration tricks, which make the calculation of integrals over unit spheres feasible. As before we consider a euclidean vector space $V$ of dimension $m$ with scalar product $g \in \text{Sym}^2 V^\ast$. The dual scalar product $g^{-1}$ on $V^\ast$ is defined by declaring the mutually inverse musical isomorphisms $\sharp : V \rightarrow V^\ast$, $X \mapsto g(\cdot, X)$, and $\flat : V^\ast \rightarrow V$ to be isometries, this is

$$g^{-1}(\alpha, \beta) := g(\alpha^\flat, \beta^\flat) = \alpha(\beta^\flat) = \sum_{\mu = 1}^m \alpha(X_\mu) \beta(X_\mu)$$  \hspace{1cm} (2)

where $X_1, \ldots, X_m$ is some orthonormal basis for $V$. An orthonormal basis $x_1, \ldots, x_m \in V^\ast$ of $V^\ast$ with respect to the scalar product $g^{-1}$ can be thought of as a global orthonormal coordinate system for $V$ considered as a manifold endowed with the translation invariant Riemannian metric $g$, in particular the Laplace–Beltrami operator on $C^\infty V$ becomes:

$$\Delta_g := -\sum_{\mu = 1}^m \frac{\partial^2}{\partial x_\mu^2}$$  \hspace{1cm} (3)

Lemma 2.1 (Integration against Gaussian Probability Density)

Let $V$ be a euclidean vector space of dimension $m$ with scalar product $g \in \text{Sym}^2 V^\ast$ considered as a translation invariant Riemannian metric on $V$. The Laplace–Beltrami operator $\Delta_g$ associated to $g$ allows us integrate $p \in \text{Sym} V^\ast$ against the Gaussian probability density via:

$$e^{-\frac{1}{2} \Delta_g} \left|_0 p \right. = \frac{1}{(2\pi)^{\frac{m}{2}}} \int_V e^{-\frac{1}{2} g(X, X)} p(X) \text{vol}_g(X)$$

Proof: Consider the exponential function $X \mapsto e^{\alpha(X)}$ associated to a given $\alpha \in V^\ast$. Completing the square this function can be integrated easily against the Gaussian probability density associated to the translation invariant Riemannian metric $g$ on $V$

$$\frac{1}{(2\pi)^{\frac{m}{2}}} \int_V e^{-\frac{1}{2} g(X, X)} e^{\alpha(X)} \text{vol}_g(X) = \frac{1}{(2\pi)^{\frac{m}{2}}} \int_V e^{-\frac{1}{2} g(X - \alpha^\flat, X - \alpha^\flat)} e^{\frac{1}{2} g^{-1}(\alpha, \alpha)} \text{vol}_g(X)$$

$$= e^{\frac{1}{2} g^{-1}(\alpha, \alpha)} = e^{-\frac{1}{2} \Delta_g} \left|_0 e^{\alpha} \right.$$  \hspace{1cm} using the consequence $(-\Delta_g) e^\alpha = g^{-1}(\alpha, \alpha) e^\alpha$ of equation (2). The stated integration formula thus holds true for the homogeneous pieces $p = \frac{1}{k!} \alpha^k$ of the generating function $e^\alpha$ of degrees $k \geq 0$. By polarization and linearity this results extends to all of $\text{Sym} V^\ast$. □

In passing we remark that Lemma 2.1 reflects the characteristic property of the heat kernel of the Laplace–Beltrami operator on a flat Riemannian manifold, compare [BGV].
Consider a symmetric endomorphism $F : V \rightarrow V$ of a euclidean vector space $V$ of dimension $m$ with scalar product $g \in \text{Sym}^2 V^\ast$. The exponential function $X \mapsto \exp(t g(FX, X))$ can be integrated against the Gaussian probability density $(2\pi)^{-\frac{m}{2}} e^{-\frac{1}{2} g \text{vol}_g}$ on $V$ using:

$$
\frac{1}{(2\pi)^{\frac{m}{2}}} \int_V e^{-\frac{1}{2} g(X,X)} \exp(t g(FX, X)) \text{vol}_g(X) = \exp\left( \sum_{r \geq 0} \frac{(2t)^r}{2r} \text{tr}_V F^r \right)
$$

**Proof:** According to Sylvester’s Theorem of Inertia we can choose an orthonormal basis $X_1, \ldots, X_m$ of $V$ consisting of eigenvectors for the symmetric endomorphism $F$ with eigenvalues $f_1, \ldots, f_m \in \mathbb{R}$ respectively. In terms of the dual orthonormal basis $x_1, \ldots, x_m$ the function $X \mapsto g(FX, X)$ reads $\sum_{\mu=1}^m f_\mu x_\mu^2$, whereas the Laplace–Beltrami operator $\Delta_g$ is given by (3) as before. The integration formula of Lemma 2.1 thus becomes the formula:

$$
\frac{1}{(2\pi)^{\frac{m}{2}}} \int_V e^{-\frac{1}{2} g(X,X)} \exp\left( t g(FX, X) \right) \text{vol}_g(X) = e^{-\frac{1}{4} \Delta_g} \left| 0 \prod_{\mu=1}^m \exp(t f_\mu x_\mu^2) = \prod_{\mu=1}^m \left( e^{\frac{1}{2} \frac{\partial^2}{\partial x_\mu^2}} \right)_0 \exp(t f_\mu x_\mu^2) \right|
$$

The factors on the right hand can be evaluated explicitly using a little bit of combinatorics

$$
\left. e^{\frac{1}{2} \frac{\partial^2}{\partial x^2}} \right|_0 \exp(tf x^2) = \sum_{r \geq 0} \frac{(2t f)^r}{4^r r!^2} \frac{\partial^{2r}}{\partial x^{2r}} \bigg|_0 x^{2r} = \sum_{r \geq 0} \left( -\frac{1}{2} \right)^r (-2tf)^r
$$

$$
= (1 - 2tf)^{-\frac{1}{2}} = \exp\left( -\frac{1}{2} \log(1 - 2tf) \right) = \exp\left( \sum_{r \geq 0} \frac{(2t f)^r}{2r} f^r \right)
$$

besides the triviality $\frac{1}{4} \binom{2r}{r} = (-1)^r \binom{-\frac{1}{2}}{r}$ and Newton’s power series. With exp being multiplicative the integration formula thus follows from $\text{tr}_V F^r = \sum_{\mu=1}^m f_\mu^r$. \hfill \Box

**Corollary 2.3 (Integration of Polynomials over Spheres)**

The integral of a homogeneous polynomial $p \in \text{Sym}^k V^\ast$ of degree $k \in \mathbb{N}_0$ over the unit sphere $S_V \subset V$ of a euclidean vector space $V$ of dimension $m$ with scalar product $g$ can be
evaluated by using the Laplace–Beltrami operator $\Delta_g$ associated to the translation invariant Riemannian metric $g$ on $V$. More precisely this integration trick tells us for even $k = 2\kappa$:

$$\frac{1}{\Vol S_V} \int_{S_V} p(X) \text{vol}_g(X) = \frac{(-\Delta_g)^\kappa}{4^{\kappa} \kappa! \left[ \frac{m}{2} + \kappa - 1 \right] \kappa} \int_0^p$$

The antisymmetry of $p$ under the antipodal map makes the left hand side vanish for odd $k$.

**Proof:** The Laplace–Beltrami operator $\Delta_g$ maps homogeneous polynomials to homogeneous polynomials of course so that $(-\Delta)^\kappa |_0 p = 0$ for every homogeneous polynomial $p \in \text{Sym}^k V^*$ of degree $k$ unless $k = 2\kappa$ is even. By converting the integral over $V$ to polar coordinates we may thus rewrite Lemma 2.1 for a homogeneous polynomial $p$ of even degree

$$e^{-\frac{1}{2} \Delta_g} p \bigg|_0 = \frac{(-\Delta_g)^\kappa}{2^\kappa \kappa!} \bigg|_0 p = \frac{1}{(2\pi)^{\frac{m}{2}}} \int_V e^{-\frac{1}{2} g(X,X)} p(X) \text{vol}_g(X)$$

$$= \frac{1}{(2\pi)^{\frac{m}{2}}} \int_0^\infty e^{-\frac{1}{2} r^2} r^{m+k-1} dr \int_{S_V} p(X) \text{vol}_g(X)$$

$$= \frac{2^\kappa}{\Vol S_V} [\frac{m}{2} + \kappa - 1] \kappa \int_{S_V} p(X) \text{vol}_g(X)$$

using the substitution $\rho = \frac{1}{2} r^2$, $d\rho = r \, dr$, and the definition of the $\Gamma$–function to evaluate

$$\int_0^\infty e^{-\frac{1}{2} r^2} r^{m+k-1} dr = \int_0^\infty e^{-\rho} (2\rho)^{\frac{m+k-2}{2}} d\rho = 2^\kappa \Gamma \left( \frac{m}{2} + \kappa - 1 \right) \kappa \Gamma \left( \frac{m}{2} \right)$$

the radial integral as well as $\Gamma(x+1) = x \Gamma(x)$ and Euler’s formula $\Vol S_V = \frac{2\pi^\frac{m}{2}}{\Gamma \left( \frac{m}{2} \right)}$ for the volume of the unit sphere in dimension $m$.

In passing we want to point out a special consequence of the preceding proof, the integral of a homogeneous polynomial $p \in \text{Sym}^k V$ of even degree $k = 2\kappa$ against the volume density on $S_V$ agrees up to a conversion factor, which only depends on $\kappa$ and the dimension $m$, with the integral of $p$ against the Gaussian probability density $(2\pi)^{-\frac{m}{2}} e^{-\frac{1}{2} g} \text{vol}_g$ on $V$:

$$\frac{1}{(2\pi)^{\frac{m}{2}}} \int_V e^{-\frac{1}{2} g(X,X)} p(X) \text{vol}_g(X) = 2^\kappa \left[ \frac{m}{2} + \kappa - 1 \right] \kappa \left( \frac{1}{\Vol S_V} \int_{S_V} p(X) \text{vol}_g(X) \right)$$

This reformulation of Corollary 2.3 comes in handily in the proof of the following theorem:

**Theorem 2.4 (Moments of Sectional Curvatures)**

Let $R : V \times V \times V \to V$ be an algebraic curvature tensor on an euclidean vector space $V$ of dimension $m$ with scalar product $g$. The powers $\text{sec}_R^k$, $k \in \mathbb{N}_0$, of the sectional curvature function $\text{sec}_R : \text{St}_2 V \to \mathbb{R}$, $(X, Y) \mapsto g(R_{Y,X} X, Y)$, integrate over $\text{St}_2 V$ to the expression:

$$\frac{1}{\Vol \text{St}_2 V} \int_{\text{St}_2 V} g( R_{Y,X} X, Y )^k \text{vol}_F S( X, Y )$$

$$= \frac{(-\Delta_g)^k}{[m + 2k - 2]2k} \bigg|_0 \left( X \mapsto \exp \left( \sum_{r>0} \frac{1}{2r} \text{tr}_V ( R_{.,X} X )^r \right) \right)$$
In particular we obtain for the scalar curvature \( \kappa(R) \) of the algebraic curvature tensor \( R \):

\[
\frac{\kappa(R)}{m(m-1)} = \frac{1}{\text{Vol} \, \text{St}_2 V} \int_{\text{St}_2 V} g(R_{Y,X}X, Y) \, \text{vol}_{FS}(X, Y)
\]

Since the sectional curvature function \( \text{sec}_R \) is constant along the fibers of the projection \( \text{pr}_{\text{Gr}} \) to the Grassmannian both integral formulas still hold true with \( \text{St}_2 V \) replaced by \( \text{Gr}_2 V \).

**Proof:** Recall that the projection \( \text{St}_2 V \rightarrow S_V \), \( (X, Y) \mapsto X \), is a Riemannian submersion for the Fubini–Study metric \( g^{FS} \) on the Stiefel manifold \( \text{St}_2 V \) of orthonormal 2–frames in \( V \), hence we can replace the integral over \( \text{St}_2 V \) by a double integral over spheres. Somewhat more precisely the inner integral is over the unit sphere \( S\{X\}^\perp \) of the \((m-1)\)–dimensional orthogonal complement \( \{X\}^\perp \) to the integration variable \( X \in S_V \) of the outer integral over \( S_V \). Evidently inner and outer integration both integrate a homogeneous polynomial of degree \( 2k \) over a unit sphere so that Corollaries 2.2 and 2.3 together imply:

\[
\frac{1}{\text{Vol} \, \text{St}_2 V} \int_{\text{St}_2 V} g(R_{Y,X}X, Y)^k \, \text{vol}_{FS}(X, Y)
\]

\[
= \frac{1}{\text{Vol} \, S_V} \int_{S_V} \left[ \frac{1}{\text{Vol} \, S\{X\}^\perp} \int_{S\{X\}^\perp} g(R_{Y,X}X, Y)^k \, \text{vol}_g(Y) \right] \, \text{vol}_g(X)
\]

\[
= \left( -\Delta_g \right)^k \left. \frac{k!}{4^k k! \left( \frac{m}{2} + k - 1 \right)_k} \right|_0 \frac{2^k \left( \frac{m-1}{2} + k - 1 \right)_k}{k!} \left( X \mapsto \text{res}_{t=0} \left[ \frac{dt}{tk+1} \exp \left( \sum_{r>0} \frac{(2t)^r}{2r} \text{tr}_V(R_{.,X}X)^r \right) \right] \right)
\]

In this calculation the factor \( \frac{2^k \left( \frac{m-1}{2} + k - 1 \right)_k}{k!} \) is the conversion factor of the crucial observation (4), which relates the integration over \( S\{X\}^\perp \) with the integration over \( \{X\}^\perp \), the factor \( k! \) on the other hand simply reflects that we are integrating \( g(R_{Y,X}X,Y)^k \) over \( Y \) instead of \( \frac{dt}{t} g(R_{Y,X}X,Y)^k \) as in Corollary 2.2. In the resulting expression we have used a formal residue notation in order to pick up the coefficient of \( t^k \) in the generating function:

\[
\text{res}_{t=0} \left[ \frac{dt}{tk+1} \exp \left( \sum_{r>0} \frac{(2t)^r}{2r} \text{tr}_V(R_{.,X}X)^r \right) \right]
\]

Strictly speaking it is not necessary to single out the coefficient of \( t^k \) in this way, because every \( t^r \) comes along with a homogeneous polynomial in \( X \) of degree \( 2r \) so that the coefficient of \( t^k \) in the exponential is accompanied by a homogeneous polynomial in \( X \) of degree \( 2k \). On the other hand the operator \( (-\Delta_g)^k \mid_0 \) sees nothing else but homogeneous polynomials of degree \( 2k \) so that we may forget about picking up the coefficient of \( t^k \) and may simply set \( t = \frac{1}{2} \) multiplying the overall result by \( 2^k \) in compensation. In this way we obtain eventually:

\[
\frac{1}{\text{Vol} \, \text{St}_2 V} \int_{\text{St}_2 V} g(R_{Y,X}X, Y)^k \, \text{vol}_{FS}(X, Y)
\]
\[
\begin{align*}
&= \frac{(-\Delta_g)^k}{4^k \left[ \frac{m}{2} + k - 1 \right] \left[ \frac{m-1}{2} + k - 1 \right] k_0} \left( X \mapsto \exp \left( \sum_{r>0} \frac{1}{2r} \text{tr}_V(R_{\cdot, X} X)^r \right) \right) \\
&= \frac{(-\Delta_g)^k}{[m + 2k - 2]_{2k} k_0} \left( X \mapsto \exp \left( \sum_{r>0} \frac{1}{2r} \text{tr}_V(R_{\cdot, X} X)^r \right) \right)
\end{align*}
\]

In order to verify the statement about the scalar curvature of the algebraic curvature tensor \( R \) we choose an orthonormal basis \( X_1, \ldots, X_m \) of \( V \) so that we may rewrite the trace of the Jacobi operator \( \text{tr}_V(R_{\cdot, X} X) = \sum g(R_{X_{\mu}, X} X, X_{\mu}) \) in the evaluation of the expression:

\[
(-\Delta_g) \left( X \mapsto 1 + \frac{1}{2} \sum_{\mu=1}^m g(R_{X_{\mu}, X} X, X_{\mu}) + \ldots \right) = \sum_{\mu, \nu=1}^m g(R_{X_{\mu}, X} X_{\nu}, X_{\mu})
\]

Taking up the lead given in the preceding proof we may define the generating power series of the moments of the sectional curvature of an algebraic curvature tensor \( R \) on \( V \) by:

\[
\exp \left( \sum_{r>0} \frac{1}{2r} \text{tr}_V(R_{\cdot, X} X)^r \right) = \exp \left( -\frac{1}{2} \sum_{r>0} \frac{(-1)^{r+1}}{r} \text{tr}_V(-R_{\cdot, X} X)^r \right) = \exp \left( -\frac{1}{2} \text{tr}_V \log(\text{id} - R_{\cdot, X} X) \right)
\]

The standard matrix identity \( \exp(\text{tr}_V A) = \det_V(\exp A) \) converts this definition into:

**Remark 2.5 (Generating Power Series for Sectional Curvature Moments)**

Consider an algebraic curvature tensor \( R \) on a euclidean vector space \( V \). In the sense of Theorem 2.4 the sectional curvature moments of \( R \) have the following generating power series:

\[
\det_V^{-\frac{1}{2}} \left( \text{id} - R_{\cdot, X} X \right) := \exp \left( \sum_{r>0} \frac{1}{2r} \text{tr}_V(R_{\cdot, X} X)^r \right)
\]

At the end of section we want to reformulate Theorem 2.4 in a way useful for the calculation of the moments of the sectional curvature of a compact Riemannian manifold \( M \) of dimension \( m \) with Riemannian metric \( g \). The Grassmann bundle of 2–planes in \( TM \)

\[
\text{Gr}_2TM := \{ (p, \Sigma) \mid p \in M \text{ and } \Sigma \in \text{Gr}_2T_pM \}
\]

carries a natural Riemannian metric again denoted by \( g \), which makes the canonical projection \( \text{pr} : \text{Gr}_2TM \longrightarrow M \) a Riemannian submersion and induces the Fubini–Study metric \( g_{FS} \) on every fiber \( \text{Gr}_2T_pM \). The only thing we still have to specify is the orthogonal complement to the vertical tangent space, which we choose to be given by parallel transport of 2–planes with respect to the Levi–Civita connection on \( M \). In this way \( \text{Gr}_2TM \) becomes a compact Riemannian manifold endowed with a canonical function, namely the sectional curvature

\[
\text{sec} : \text{Gr}_2TM \longrightarrow \mathbb{R}, \quad (p, \Sigma) \mapsto \text{sec}_{R_p}(\Sigma)
\]
where \( R_p \) is the curvature tensor of \( M \) in the point \( p \). In turn we may consider the sectional curvature of a compact Riemannian manifold \( M \) as a random variable, whose moments and distribution form a legitimate object of study. The pinching constant for \( M \) for example equals the quotient of the minimum and maximum value of \( \text{sec} \) provided \( \text{sec}_{R_p}(\Sigma) > 0 \) in at least one 2–plane \((p, \Sigma) \in \text{Gr}_2 TM\).

In this context Theorem 2.4 tells us that the moments of the sectional curvature of a Riemannian manifold \( M \) can be calculated by integrating natural polynomials in the curvature tensor over \( M \). Naturality in this context refers to the fact that these polynomials are defined for all euclidean vector spaces of dimension \( m \) and are invariant under all isometries between these spaces. In particular natural polynomials are defined for all the tangent spaces \( T_p M \) of Riemannian manifolds \( M \) of dimension \( m \), in turn their integrals over \( M \) give rise to local Riemannian invariants for compact Riemannian manifolds in the sense of [W].

**Corollary 2.6 (Sectional Curvature Moments of Riemannian Manifolds)**

For every euclidean vector space \( V \) of dimension \( m \) with scalar product \( g \) there exist natural \( O(V, g) \)-invariant polynomials \( \Psi^V_k \) of degree \( k \in \mathbb{N}_0 \) on the vector space \( \mathcal{K} r V \) of algebraic curvature tensors on \( V \) such that the \( k \)-th moment of the sectional curvature equals

\[
\frac{1}{\text{Vol} \text{Gr}_2 TM} \int_{\text{Gr}_2 TM} \text{sec}^k_{R_p}(\Sigma) \text{vol}_g(p, \Sigma) = \frac{1}{\text{Vol} M} \int_M \Psi^{T_p M}_k(R_p) \text{vol}_g(p)
\]

for every compact Riemannian manifold \( M \) of dimension \( m \), to wit the polynomial \( \Psi^V_k \) reads:

\[
\Psi^V_k(R) := \left. \left( -\frac{(\Delta_g)^k}{[m+2k-2]2k} \right) \right|_0 \left( X \mapsto \exp \left( \sum_{r>0} \frac{1}{2r} \text{tr}_V(R \cdot X X)^r \right) \right)
\]

Perhaps the most interesting aspect of this corollary is that it is possible, at least in principle, to calculate the pinching constant of a compact Riemannian manifold \( M \) without calculating not even a single sectional curvature \( \text{sec}_{R_p}(\Sigma) \) explicitly. In fact it could turn out to be feasible in some examples to calculate all the higher moments of the random variable \( \text{sec} \) on \( \text{Gr}_2 TM \), the minimum and maximum of the sectional curvature can then be reconstructed from these moments. The maximum of the absolute value of \( \text{sec} : \text{Gr}_2 TM \to \mathbb{R} \) say is given by

\[
\max_{(p, \Sigma) \in \text{Gr}_2 TM} |\text{sec}_{R_p}(\Sigma)| = \lim_{k \to \infty} 2k \sqrt{\frac{1}{\text{Vol} \text{Gr}_2 TM} \int_{\text{Gr}_2 TM} \text{sec}^{2k}_{R_p}(\Sigma) \text{vol}_g(p, \Sigma)}
\]

and very similar formulas can be used to calculate the minimum and maximum of \( \text{sec} \).

**3 Examples of Sectional Curvature Distributions**

Explicit calculations in differential geometry are in general restricted to the relatively small class of Riemannian homogeneous spaces, in which all points look alike in the sense that the isometry group acts transitively. Homogeneous spaces are thus examples of choice to
illustrate the calculation of the moments of sectional curvature discussed in Theorem 2.4, not to the least because the eventual integration over the manifold in Corollary 2.6 makes no difference at all. In this section we will calculate the moments and the distribution of the sectional curvature for all compact rank one symmetric spaces collectively known under the acronym CROSS: the round spheres $S^m$, the complex and quaternionic projective spaces $CP^n$ and $HP^n$ with $n \geq 2$ and the exceptional Cayley projective plane $OP^2$. Moreover we will briefly discuss the problems arising in the case of Riemannian symmetric spaces of higher rank. A detailed introduction to the general theory of symmetric spaces including Cartan’s classification can be found in the classical reference [H].

Recall to begin with that the falling factorial polynomials $[x]_s := x(x-1) \ldots (x-s+1)$ with parameter $s \in \mathbb{Z}$ allow us to generalize the binomial coefficients via $\binom{x}{s} := \frac{[x]_s}{s!}$ to all $x \in \mathbb{R}$ and $s \in \mathbb{Z}$. One of the best–known of all the numerous identities satisfied by binomial coefficients extends directly to all $x, y \in \mathbb{R}$, simply because it depends polynomially on $x, y$:

$$\sum_{s=0}^{k} \binom{x}{s} \binom{y}{k-s} = \binom{x+y}{k}$$

In this form this classical binomial identity becomes important in the combinatorial argument

$$\sum_{s=0}^{k} \frac{(-1)^s}{x+s} \binom{k}{s} = \frac{k!}{[x+k]_{k+1}} \sum_{s=0}^{k} (-1)^s \frac{[x+s-1]_s [x+k]_{k-s}}{s! (k-s)!} \quad (5)$$

valid for all $k \in \mathbb{N}_0$, which allows us to evaluate the following integral for all $a, b, k \in \mathbb{N}_0$:

$$\int_0^1 (u^2)^a (1-u^2)^b (1+3u^2)^k \, du = \sum_{r=0}^{k} 4^r \binom{k}{r} \int_0^1 (u^2)^{a+r} (1-u^2)^{b+k-r} \, du$$

$$= \sum_{r=0}^{k} 4^r \binom{k}{r} \left( \sum_{s=0}^{b+k-r} \frac{(-1)^s}{2a+2r+2s+1} \binom{b+k-r}{s} \right)$$

$$= \frac{1}{2} \sum_{r=0}^{k} 4^r \binom{k}{r} \frac{[b+k-r]_{k-r} [b]_b [a+r-\frac{1}{2}]_r}{[a+b+k+\frac{1}{2}]_k [a+b+\frac{1}{2}]_{b+1}}$$

$$= \frac{1}{(2a+1)(a+b+\frac{1}{2})} \sum_{r=0}^{k} 4^r \frac{(-1)^r}{b} \frac{(-2^r a + b + \frac{1}{2})}{b-k-r}$$

In the last line we have used $[x+s]_s = (-1)^s \left[ -\frac{2x+2}{2} \right]_s$ thrice in order to simplify the result. Making the substitution $s = 1 + 3u^2$, $ds = 6u \, du$, in the integral we conclude:

$$\frac{(2a+1)}{6} \left( a + b + \frac{1}{2} \right) \int_1^4 \sqrt{\frac{s-1}{3}} \left( \frac{4-s}{3} \right)^b \frac{s^k \, ds}{6} = \sum_{r=0}^{k} 4^r \frac{(-1)^r}{b} \frac{(-2^r a + b + \frac{1}{2})}{b-k-r} \quad (6)$$
En nuce this formula can be seen as a formula for the \(k\)-th order moments of a family of probability measures on the interval \([1, 4]\) parametrized by \(a, b \in \mathbb{N}_0\).

The distribution of the sectional curvature on the round spheres \(S^m\) of dimension \(m\) and scalar curvature \(\kappa > 0\), the simplest examples of compact symmetric spaces of rank 1, is too simple to be of any interest, because \(S^m\) has constant sectional curvature \(\sec \equiv \frac{\kappa}{m(m-1)}\). Nevertheless it may be instructive to apply Lemma 2.4 in the special case \(\kappa = m(m-1)\) corresponding the round metric on \(S^m\) of constant sectional curvature \(\sec \equiv 1\) to verify that all higher moments turn out to be equal to 1. Instead of using the algebraic curvature tensor \(R^{S^m}\) of the sphere \(S^m\) of radius 1 directly we consider the so-called Jacobi operator

\[
R_{\cdot, X}^{S^m} : V \rightarrow V, \quad Y \mapsto R_{Y, X}^{S^m} = g(X, X)Y - g(X, Y)X
\]

for a fixed vector \(X \in V\), which evidently has eigenvalues 0 and \(g(X, X)\) on the eigenspaces \(\mathbb{R}X\) and \(\{X\}^\perp\) respectively. In turn the generating power series of Remark 2.5 reads

\[
\det \frac{1}{2} (\text{id} - R_{\cdot, X}^{S^m}) = \left( 1 - g(X, X) \right)^{-\frac{m-1}{2}} = \sum_{k \geq 0} \left( \frac{m-1}{k} \right)(-g(X, X))^k
\]

using once again Newton’s power series expansion \((1 + t)^x = \sum_{k \geq 0} \binom{x}{k} t^k\). On the other hand the polynomial \(X \mapsto g(X, X)^k\) equals 1 on the unit sphere \(S_V \subset V\) of a euclidean vector space \(V\) of dimension \(m\) and thus integrates over \(S_V\) to \(Vol S_V\) for all \(k \in \mathbb{N}_0\). Reading Corollary 2.3 in light of this trivial observation we obtain directly:

\[
\frac{(-\Delta_g)^k}{[m + 2k - 2]_{2k}} \left| X \mapsto g(X, X)^k \right| \quad = \quad \frac{4^k k! \left[ \frac{m}{2} + k - 1 \right]_{k}}{[m + 2k - 2]_{2k}} = \left( \frac{-1}{k} \right)^k 
\]

Comparing this result with equation (7) for the generating power series of sectional curvature moments we conclude that all moments of the round metric on \(S^m\) equal 1.

Turning from round spheres to more complicated examples we recall that the complex projective spaces \(\mathbb{C}P^n\), \(n \geq 2\), of dimensions \(m = 2n\) are the prototypical examples of Kähler manifolds. In particular all their tangent spaces \(T_p \mathbb{C}P^n\) carry an orthogonal complex structure \(I \in \text{End} T_p \mathbb{C}P^n\) satisfying \(I^2 = -\text{id}\) as well as \(g(IX, IY) = g(X, Y)\). According to [B] the algebraic curvature tensor of the unique symmetric Riemannian metric \(g\) on \(\mathbb{C}P^n\) of scalar curvature \(\kappa > 0\) can be written in terms of this orthogonal complex structure:

\[
R_{X, Y}^{\mathbb{C}P^n} = -\frac{\kappa}{4 n(n+1)} \left( g(X, Z)Y - g(Y, Z)X + g(IX, Z)IY - g(IY, Z)IX + 2g(IX, Y)IZ \right)
\]

In consequence the Jacobi operators \(R_{\cdot, X}^{\mathbb{C}P^n} : T_p \mathbb{C}P^n \rightarrow T_p \mathbb{C}P^n\) read for all \(X \in T_p \mathbb{C}P^n\):

\[
Y \mapsto R_{Y, X}^{\mathbb{C}P^n} = -\frac{\kappa}{4 n(n+1)} \left( g(Y, X)X - g(X, Y)Y - 3g(Y, IX)IX \right)
\]
With $\mathbb{C}P^n$ being a rank 1 symmetric space the spectrum of the Jacobi operator $R^{\mathbb{C}P^n}_{X,X}$ depends of course only on the norm square $g(X,X)$ of the argument vector $X \in T_p \mathbb{C}P^n$. Specifically we find for the scalar curvature $\kappa = 4n(n+1)$ the eigenvalues $0, 4g(X,X)$ and $g(X,X)$ on the eigenspaces $\mathbb{R}X$, $\mathbb{R}IX$ and $\{X, IX, KX\}^\perp$ respectively so that the sectional curvature for this value of $\kappa$ takes values in the interval $[1, 4]$. The generating power series of the moments of the sectional curvature introduced in Remark 2.5 thus reads:

$$\exp \left( \sum_{r>0} \frac{1}{2r} \text{tr}_V (R^{\mathbb{C}P^n}_{X,X} X)^r \right) = (1 - 4g(X,X))^{-\frac{1}{4}} (1 - g(X,X))^{-\frac{4n-2}{2}}$$

$$= \sum_{k \geq 0} \left[ \sum_{r=0}^{k} 4^r \left( \frac{-\frac{1}{2}}{r} \right) \left( \frac{-\frac{2n-2}{2}}{k-r} \right) \right] (-g(X,X))^k$$

Equation (8) and Theorem 2.4 tell us the $k$-th order moment of the sectional curvature

$$\frac{1}{\text{Vol} \text{Gr}_2 \mathbb{C}P^n} \int_{\text{Gr}_2 \mathbb{C}P^n} \sec^k \text{vol}_g = \sum_{k \geq 0} \sum_{r=0}^{k} 4^r \left( \frac{-\frac{1}{2}}{r} \right) \left( \frac{-\frac{2n-2}{2}}{k-r} \right) \left( \frac{-\frac{2n-2}{2}}{k-r} \right)$$

of $\mathbb{C}P^n$. Comparing this result with the explicit moments of the probability measures defined in equation (6) we find a match for $a = 0$, $b = n-2$, the corresponding probability measure thus describes the distribution of the sectional curvature on $\mathbb{C}P^n$.

Virtually nothing in this argument has to be changed to calculate the distribution of the sectional curvature on the quaternionic projective spaces $\mathbb{H}P^n$ of dimension $m = 4n$. Instead of one orthogonal complex structure $I$ the tangent spaces $T_p \mathbb{H}P^n$ of $\mathbb{H}P^n$ feature three such structures $I, J, K \in \text{End} \ T_p \mathbb{H}P^n$ satisfying $IJ = K$, the only difference to $\mathbb{C}P^n$ is that $I, J, K$ are neither globally defined nor do we have a means to distinguished them. Nevertheless the algebraic curvature tensor of the symmetric metric $g$ on $\mathbb{H}P^n$ of scalar curvature $\kappa > 0$ is simply the extension of $R^{\mathbb{C}P^n}$ to three orthogonal complex structures

$$R^{\mathbb{H}P^n}_{X,Y} = -\frac{\kappa}{16n(n+2)} \left( g(X,Z)Y - g(Y,Z)X + g(IX,Z)IY - g(IY,Z)IX + 2g(IX,Y)IZ + g(JX,Z)JY - g(JY,Z)JX + 2g(JX,Y)JZ + g(KX,Z)KY - g(KY,Z)KX + 2g(KX,Y)KZ \right)$$

compare [B]. Similar to the case of $\mathbb{C}P^n$ the Jacobi operator $R^{\mathbb{H}P^n}_{X,X}$ has eigenvalues $0, 4g(X,X)$ and $g(X,X)$ on $\mathbb{R}X$, span $\{IX, JX, KX\}$ and $\{X, IX, JX, KX\}^\perp$ respectively for the special value $\kappa = 16n(n+2)$ of the scalar curvature, for which the sectional curvature takes values in the interval $[1, 4]$. In consequence we find the generating function

$$\exp \left( \sum_{r>0} \frac{1}{2r} \text{tr}_V (R^{\mathbb{H}P^n}_{X,X} X)^r \right) = (1 - 4g(X,X))^{-\frac{1}{4}} (1 - g(X,X))^{-\frac{4n-4}{2}}$$

and conclude for the $k$-th order moment of the sectional curvature of $\mathbb{H}P^n$:

$$\frac{1}{\text{Vol} \text{Gr}_2 \mathbb{H}P^n} \int_{\text{Gr}_2 \mathbb{H}P^n} \sec^k \text{vol}_g = \sum_{k \geq 0} \sum_{r=0}^{k} 4^r \left( \frac{-\frac{1}{2}}{r} \right) \left( \frac{-\frac{2n-4}{2}}{k-r} \right) \left( \frac{-\frac{2n-4}{2}}{k-r} \right)$$

(10)
In terms of the probability measures defined in equation (6) these moments correspond to the probability measure with parameters \( a = 1 \) and \( b = 2n - 3 \).

The curvature tensor of the only remaining compact symmetric space of rank 1, the Cayley projective plane \( \mathbb{O}P^2 \) of dimension \( m = 16 \), is more difficult to describe, because we would need to discuss some properties of spinors in dimension 9 first. For the unique symmetric Riemannian metric \( g \) on \( \mathbb{O}P^2 \) of scalar curvature \( \kappa = 576 \) the Jacobi operators \( R_{X,Y}^\mathbb{O}P^2 \) have nevertheless the eigenvalues 0, 4\( g(X,Y) \) and \( g(X,Y) \) of multiplicities 1, 7 and 8 respectively. The generating function of sectional curvature moments with these multiplicities

\[
\exp \left( \sum_{r>0} \frac{1}{2r} \text{tr}_V ( R_{X,Y}^\mathbb{O}P^2 )^r \right) = (1 - 4 g(X,Y))^{-\frac{3}{2}} (1 - g(X,Y))^{-\frac{5}{2}}
\]
corresponds to the probability measure with parameters \( a = b = 3 \) and the moments:

\[
\frac{1}{\text{Vol} \ Gr_2 T \mathbb{O}P^2} \int_{\text{Gr}_2 T \mathbb{O}P^2} \text{sec}^k \ vol_g = \sum_{r=0}^k 4^r \frac{\binom{-\frac{3}{2}}{r}}{\binom{-\frac{5}{2}}{k-r}}
\]

(11)

**Corollary 3.1 (Examples of Sectional Curvature Densities)**

The sectional curvature \( \text{sec} : \text{Gr}_2 T\text{M} \rightarrow [\min, \max] \) of a Riemannian metric \( g \) on a compact Riemannian manifold \( M \) can be thought of as a random variable with associated probability measure on \([\min, \max]\). In this way the unique symmetric Riemannian metrics with sectional curvatures in \([1, 4]\) on the complex and quaternionic projective spaces \( \mathbb{C}P^n \) and \( \mathbb{H}P^n \) with \( n \geq 2 \) as well as the Cayley projective plane \( \mathbb{O}P^2 \) define the probability measures:

\[
\frac{1}{\text{Vol} \ Gr_2 T \mathbb{C}P^n} \int_{\text{Gr}_2 T \mathbb{C}P^n} F(\text{sec}) \ vol_g = \frac{1}{6} \binom{n - \frac{3}{2}}{n - 2} \int_1^4 \sqrt{s - 1}^{-1} \left( \frac{4 - s}{3} \right)^{n-2} F(s) \ ds
\]

\[
\frac{1}{\text{Vol} \ Gr_2 T \mathbb{H}P^n} \int_{\text{Gr}_2 T \mathbb{H}P^n} F(\text{sec}) \ vol_g = \frac{3}{6} \binom{2n - \frac{3}{2}}{2n - 3} \int_1^4 \sqrt{s - 1}^{-1} \left( \frac{4 - s}{3} \right)^{2n-3} F(s) \ ds
\]

\[
\frac{1}{\text{Vol} \ Gr_2 T \mathbb{O}P^2} \int_{\text{Gr}_2 T \mathbb{O}P^2} F(\text{sec}) \ vol_g = \frac{7}{6} \binom{15/2}{3} \int_1^4 \sqrt{s - 1}^{-5} \left( \frac{4 - s}{3} \right)^{3} F(s) \ ds
\]

The calculation of the sectional curvature densities of compact Riemannian symmetric spaces of higher rank is significantly more involved, however we want to discuss the general setup at the end of this section. In general every Killing field \( X \) for a Riemannian metric \( g \) on a connected manifold \( M \) is automatically an affine Killing field for the Levi–Civita connection

\[
\mathcal{L}_X \nabla \nabla = [X, \nabla Z] - \nabla_{[X,Y]} Z - \nabla_Y [X, Z] = \nabla^2_{Y, Z} X + R_{X,Y} Z
\]

and thus satisfies the extended Killing equation with the auxiliary section \( \mathcal{X} := \nabla X \)

\[
\nabla_Y X = \mathcal{X} Y \quad \nabla_Y \mathcal{X} = -R_{X,Y}
\]

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moments defined in Remark 2.5 can be replaced by $\det$ for all $X, Y$ of the involutive automorphism $\theta$ of a point $p$ and $T_p$ some and hence all its Lie algebra $g$.

the curvature of a Riemannian symmetric space is completely determined by the algebra structure of the sequence of the preceding discussion of symmetric spaces is certainly that the sectional curvature $\kappa$ is simply the pointwise commutator of endomorphisms, because

$$\nabla_Z [X, Y] = - \nabla_X (Y X - \theta Y X)$$

for every vector field $Z \in \Gamma(TM)$. In other words the vector bundle $\text{End}_{\text{skew}} TM \oplus TM$ becomes a bundle of algebras, but not in general a bundle of Lie algebras, under the pointwise bracket defined in equation (12) with the property that the evaluation in $p \in M$

$$t_p : g \mapsto \text{End}_{\text{skew}} T_p M \oplus T_p M, \quad X \mapsto (\nabla X)_p \oplus X_p$$

is an injective antihomomorphism of algebras, compare [BGV]. For Riemannian homogeneous spaces the composition of $t_p$ with the projection to $T_p M$ is surjective, what makes symmetric spaces rather special among homogeneous spaces however is that the image of $g$ is compatible

$$g \cong t_p(g) = \left( t_p(g) \cap \text{End}_{\text{skew}} T_p M \right) \oplus \left( t_p(g) \cap T_p M \right)$$

with the direct sum for all $p \in M$ and thus invariant under the involutive automorphism $\theta_p$ of the algebra $\text{End}_{\text{skew}} T_p M \oplus T_p M$ defined by $\theta_p(X \oplus X) := X \oplus (-X)$. In consequence the Lie algebra $g$ of Killing vector fields of a connected Riemannian symmetric space $M$ decomposes as a vector space into the direct sum $g = \xi_g \oplus p_g$ of the preimages $\xi_g$ and $p_g$ of $\text{End}_{\text{skew}} T_p M$ and $T_p M$ respectively under $t_p$, which agree with the $(+1)$ and $(-1)$–eigenspaces respectively of the involutive automorphism $\theta_p$ of $\text{End}_{\text{skew}} T_p M \oplus T_p M$ restricted to $g$.

For the purpose of calculating moments of sectional curvature the most interesting consequence of the preceding discussion of symmetric spaces is certainly that the sectional curvature of a Riemannian symmetric space is completely determined by the algebra structure of its Lie algebra $g$ of Killing vector fields. More precisely there exists a unique invariant scalar product on $g$ such that the evaluation map $\text{ev}_p : p_g \cong T_p M, X \mapsto X_p$, is an isometry for some and hence all $p \in M$, moreover this isometry identifies the curvature tensor $R^g/K$ in a point $p \in M$ with the so–called triple product $[H]$ defined on the subspace $p_g \subset g$

$$(\text{ad}^2 X) Y = [0 \oplus X, [0 \oplus X, 0 \oplus Y]] = 0 \oplus (R^g/K)_{X, Y} X$$

for all $X, Y \in p_g$. In particular the generating power series of the sectional curvature moments defined in Remark 2.5 can be replaced by $\det^{-\frac{1}{2}} p (\text{id} + \text{ad}^2 X)$ in the algebraic model $p_g \cong T_p M$ so that Theorem 2.4 implies the following generating formal power series

$$\sum_{k \geq 0} (-t)^k \left( -\frac{m-1}{2} \right) \Psi_k(R^g/K) = \frac{1}{\text{Vol} S_p} \int_{S_p} \det^{-\frac{1}{2}} p (\text{id} + t \text{ad}^2 X) \text{vol}_g(X)$$

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for the moments of the sectional curvature of the algebraic curvature tensor $R^G/K$ corresponding to the Riemannian symmetric space $M = G/K$. Combining this explicit description of the $k$–th order moments $\Psi_k(R^G/K)$ of the sectional curvature of $M$ with a suitable variant of the classical Integration Formula of Weyl for compact Lie groups [BGV] we obtain eventually the following formula for the moments of the sectional curvature of $M = G/K$:

**Lemma 3.2 (Weyl’s Integration Formula for Sectional Curvature Moments)**

Consider a compact Riemannian symmetric space $M = G/K$ of dimension $m$ with isometry group $G := \text{Isom} M$ and stabilizer $K \subset G$ of a given base point $p \in M$. The Lie algebra of Killing vector fields decomposes orthogonally $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ with respect to the unique invariant scalar product $g^{\text{ext}}$ on $\mathfrak{g}$, which makes the evaluation $\text{ev}_p : \mathfrak{p} \rightarrow T_p M$ in the base point an isometry. Choosing a maximal abelian subalgebra $\mathfrak{a} \subset \mathfrak{p}$ with centralizer and normalizer $\text{Zent} \mathfrak{a} \subset \text{Norm} \mathfrak{a} \subset K$ we may encode the sectional curvature moments of $M$ in the series

$$\sum_{k \geq 0} (-t)^k \left( -\frac{m-1}{2k} \right) \Psi_k(R^G/K)$$

$$= \frac{1}{|\mathcal{W}_a| \cdot \text{Vol}(K, g^{\text{ext}}) \cdot \text{Vol}(\text{Zent} \mathfrak{a}, g^{\text{ext}})} \int_{S_a} \det_p^{-\frac{1}{2}}(\text{id} + t \text{ad}^2 X) \sqrt{\det_p^*( -\text{ad}^2 X ) \text{vol}_g(X)}$$

where $\mathcal{W}_a := \text{Norm} \mathfrak{a} / \text{Zent} \mathfrak{a}$ is the Weyl group associated to $\mathfrak{a} \subset \mathfrak{p}$ and $\det_p^*(-\text{ad}^2 X)$ is the product of all non–zero and hence positive eigenvalues of $-\text{ad}^2 X$ on $\mathfrak{p}$.

In contrast to the standard Integration Formula for compact Lie groups the square root of the orbit volume ratio $\det_p^*(-\text{ad}^2 X)$ in the modified Integration Formula for compact Riemannian symmetric spaces cannot be replaced by a polynomial in $X$, because the integrand does not change sign under the Weyl group. More than a mere annoyance this unexpected twist is a serious obstacle in the explicit calculation of the moments of the sectional curvature, because the integration trick of Lemma 2.3 does no longer apply. At least for symmetric spaces of rank $\dim \mathfrak{a} = 2$ this problem can be circumvented by replacing the integration over $S_a$ by an integration over the circular arc $S_a/\mathcal{W}_a$. In this way one obtains for example the moments

$$\Psi_k(R^G_{\text{Gr}^2\mathbb{R}^n}) = \frac{1}{(-\frac{2n-5}{k^2})} \sum_{\mu, \nu \geq 0, \mu + \nu \leq \frac{k}{2}} \frac{(-1)^\nu}{4^\mu} \frac{n-3}{n+2\mu+2\nu-3} \left( -\frac{n-4}{2\mu} \right) \left( -\frac{1}{2\nu} \right) \left( -\frac{n+2\mu+4\nu-2}{2k-2\mu-2\nu} \right)$$

of the sectional curvature for the Fubini–Study metric on the Grassmannians of 2–planes in $\mathbb{R}^n$ with sectional curvatures in the interval $[0, 2]$ and scalar curvature $\kappa = 2(n-2)^2$, somewhat more complicated formulas hold true for the Grassmannians of 2–planes in $\mathbb{C}^n$ and $\mathbb{H}^n$. For the Grassmannian $\text{Gr}^2\mathbb{R}^4$ the explicit formula can be verified directly, because $S^2 \times S^2$ endowed with half the product metric is a 2–fold Riemannian cover of $\text{Gr}^2\mathbb{R}^4$. 

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4 Moments and Densities of Riemannian Products

Certainly the simplest way to produce new examples of Riemannian manifolds is to take products, hence it seems prudent to study this construction with a view on the moments of the sectional curvature in order to extend our stock of examples. Somewhat surprisingly it turns out that both the moments and the distribution of the sectional curvature behave very nicely under Riemannian products to the extent that there exists a simple probabilistic model for Riemannian products. The explicitly known values of the moments of sectional curvature for the examples $\mathbb{C}P^2$, $S^4$ together with the values for the products $S^2 \times S^2$ and $S^1 \times S^3$ calculated in this section will be used in due course to prove a weak version of a well-known theorem of Hitchin–Thorpe [Th] concerning 4-dimensional Einstein manifolds.

Recall that the canonical projections $\text{pr}_M : M \times N \longrightarrow M, (p, q) \longmapsto p$, and $\text{pr}_N$ from the Cartesian product $M \times N$ of two Riemannian manifolds $M$ and $N$ with Riemannian metrics $g$ and $h$ respectively to its factors induces a canonical isomorphism of tangent spaces

$$T_{(p, q)}(M \times N) \cong T_p M \oplus T_q N,$$

which turns the Cartesian into the Riemannian product $M \times N$ endowed with the metric:

$$(g^M \oplus g^N)_{(p, q)}(X, Y) := g^M_p(\text{pr}_M X, \text{pr}_M Y) + g^N_q(\text{pr}_N X, \text{pr}_N Y).$$

The curvature $R^{M \times N}_{(p, q)}$ of the Levi–Civita connection $\nabla$ associated to this product metric in a point $(p, q) \in M \times N$ decomposes likewise into the direct sum $R^M_p \oplus R^N_q$ in the sense

$$(R^{M \times N}_{(p, q)})_{X, Y, Z} = (R^M_p)_{\text{pr}_M X, \text{pr}_M Y, \text{pr}_M Z} \oplus (R^N_q)_{\text{pr}_N X, \text{pr}_N Y, \text{pr}_N Z}$$

for all $X, Y, Z \in T_{(p, q)}(M \times N)$. In turn the Jacobi operator associated to a tangent vector $X \in T_{(p, q)}(M \times N)$ reduces to the direct sum $R^{M \times N}_{-, -, X} : T_p M \oplus T_q N \longrightarrow T_p M \oplus T_q N$ of the Jacobi operators of the projections $\text{pr}_M X \in T_p M$ and $\text{pr}_N X \in T_q N$ so that the generating power series of sectional curvature moments of $M \times N$ introduced in Remark 2.5

$$\det^{-\frac{1}{2}}_{T_p M \oplus T_q N}\left(\text{id} - R^{M \times N}_{-, -, X} X\right)$$

$$= \det^{-\frac{1}{2}}_{T_p M}\left(\text{id} - R^{M}_{-, \text{pr}_M X, \text{pr}_M X} X\right) \cdot \det^{-\frac{1}{2}}_{T_q N}\left(\text{id} - R^{N}_{-, \text{pr}_N X, \text{pr}_N X}\right)$$

becomes simply the product. As a direct consequence we obtain from Theorem 2.4:

**Lemma 4.1 (Sectional Curvature Moments of Products)**

Let $M$ and $N$ be Riemannian manifolds of dimensions $m, n \geq 2$ respectively. The curvature tensor of the Riemannian product manifold $M \times N$ in a point $(p, q) \in M \times N$ decomposes into the direct sum $R^{M \times N}_{(p, q)} = R^M_p \oplus R^N_q$ of the curvature tensors of $M$ in $p$ and $N$ in $q$. Hence the pointwise moments of the sectional curvatures of $R^{M \times N}$ can be calculated via

$$\Psi_k(R^{M \times N}_{(p, q)}) = \sum_{r=0}^{k} \binom{k}{r} \frac{[m + 2r - 2]_{2r} [n + 2(k - r) - 2]_{2(k-r)}}{[m + n + 2k - 2]_{2k}} \Psi_r(R^M_p) \Psi_{k-r}(R^N_q)$$

and this formula still holds true for the integrated pointwise moments of Corollary 2.6.
From the point of view of combinatorics the formula for the sectional curvature moments of Riemannian products is rather strange, however it turns out that this formula has a very neat model in probability theory. In order to discuss this model we recall a simple integration trick for integrating polynomials over the standard simplex in dimension \( n \in \mathbb{N}_0 \):

\[
\Delta_n := \{ (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1, \ldots, x_n \geq 0 \text{ and } x_1 + \ldots + x_n \leq 1 \}
\]

Every polynomial in the coordinates \( x_1, \ldots, x_n \) and \( x_0 := 1 - x_1 - \ldots - x_n \) integrates to

\[
\int_{\Delta_n} x_0^{k_0} x_1^{k_1} x_2^{k_2} \ldots x_n^{k_n} \, dx_1 \, dx_2 \ldots \, dx_n = \frac{k_0! \, k_1! \, k_2! \ldots \, k_n!}{(n + k_0 + k_1 + k_2 + \ldots + k_n)!}
\]

against the Lebesgue measure on the standard simplex \( \Delta_n \subset \mathbb{R}^n \). Needless to say this integration trick can be proved by straightforward induction on \( n \) with the combinatorial argument (5) coming in handily in both the base and the induction step. In consequence the following density defines a probability measure on \( \Delta_2 := \{ (x, y) \mid x, y \geq 0 \text{ and } x+y \leq 1 \} \)

\[
\frac{(m+n-2)!}{(m-2)!(n-2)!} \int_{\Delta_2} x^{m-2} y^{n-2} \, dx \, dy
\]

for all integers \( m, n \geq 2 \) with the characteristic property that the moments of the random variable \( x^2 \mu + y^2 \nu \) with arbitrary constants \( \mu, \nu \in \mathbb{R} \) reproduce the formula of Lemma 4.1

\[
\frac{(m+n-2)!}{(m-2)!(n-2)!} \int_{\Delta_2} (x^2 \mu + y^2 \nu)^k \, x^{m-2} \, y^{n-2} \, dx \, dy = \sum_{r=0}^{k} \binom{k}{r} \frac{(m+n-2)!}{(m-2)!(n-2)!} \frac{(m+2r-2)!}{(m+n+2k-2)!} \mu^r \nu^{k-r}
\]

for all \( k \in \mathbb{N}_0 \). The proper reason for this striking appearance of the simplex \( \Delta_2 \) in the product formula of Lemma 4.1 is of course that \( \Delta_2 \) parametrizes the orbits of the natural action of \( O(V) \times O(W) \) on the Grassmannian \( Gr_2(V \oplus W) \) of 2–planes in \( V \oplus W \). Without discussing this rather interesting observation in more detail we use the simplicity of the relevant probability measure on \( \Delta_2 \) to obtain the following corollary of Lemma 4.1:

**Corollary 4.2 (Sectional Curvature Densities for Products)**

*For all \( m, n \geq 2 \) and all \( \mu, \nu \in \mathbb{R} \) there exists a unique probability measure \( \rho_{\mu, \nu}^{m,n}(s) \, ds \) on \( \mathbb{R} \) with support in the convex hull \( \min\{\mu, \nu, 0\}, \max\{\mu, \nu, 0\} \) of \( \mu, \nu, 0 \) such that the sectional curvature density of the product of two compact Riemannian manifolds \( M \) and \( N \) of dimensions \( m, n \) and sectional curvature densities \( \rho^M(\mu) \, d\mu \) and \( \rho^N(\nu) \, d\nu \) can be calculated via:

\[
\rho^{M \times N}(s) = \int_{\mathbb{R} \times \mathbb{R}} \rho_{\mu, \nu}^{m,n}(s) \, \rho^M(\mu) \, \rho^N(\nu) \, d\mu \, d\nu
\]

The probability density \( \rho_{\mu, 0}^{m,n}(s) \) for the value \( \nu = 0 \) say reads for all \( \mu \neq 0 \) and \( m, n \geq 2 \)

\[
\rho_{\mu, 0}^{m,n}(s) = \frac{(m+n-2)!}{(m-2)!(n-1)!} \sqrt{s}^{m-3} \frac{1}{\mu} \left( 1 - \sqrt{\frac{s}{\mu}} \right)^{n-1} \frac{ds}{2 |\mu|}
\]
with support in \([0, \mu]\) or \([\mu, 0]\) for positive or negative \(\mu\) respectively so that \(\frac{x}{\mu} > 0\) for almost all \(s \in \mathbb{R}\). Incidentally this measure can be used for \(n = 1\) as well in order to calculate the sectional curvature density for the Riemannian product with a circle \(N = S^1\).

**Proof:** For given constants \(\mu, \nu \in \mathbb{R}\) and dimensions \(m, n \geq 2\) let us consider the function \(s := x^2 \mu + y^2 \nu\) on the simplex \(\Delta_2\) as a random variable with respect to the probability measure \((14)\). The inequality \(x^2 + y^2 \leq x + y \leq 1\) valid for all \((x, y) \in \Delta_2\) implies that \(s\) is a convex linear combination of \(\mu, \nu\), \(0\), moreover all three values are evidently attained on \(\Delta_2\). In consequence the random variable \(s\) has a probability density \(\rho_{\mu, \nu}^m(n)(s)\) \(ds\) on the interval \([\min\{\mu, \nu, 0\}, \max\{\mu, \nu, 0\}\]\), whose moments reproduce the product formula of Lemma 4.1. The moments of the probability density \(\rho^{M \times N}(s)\) \(ds\) defined in the Corollary thus equal

\[
\int_{\mathbb{R}} \rho^{M \times N}(s)^k ds = \int_{\mathbb{R} \times \mathbb{R}} \left( \int_{\mathbb{R}} \rho_{\mu, \nu}^m(n)(s)^k ds \right) \rho^M(\mu) \rho^N(\nu) d\mu d\nu = \sum_{r=0}^{k} \binom{k}{r} \left[ \frac{m + 2r - 2}{2r} \frac{n + 2(k - r) - 2}{2(k-r)} \right] \left( \int_{\mathbb{R}} \rho^M(\mu)^{2r} d\mu \right) \left( \int_{\mathbb{R}} \rho^N(\nu)^{k-r} d\nu \right)
\]

the moments of the sectional curvature of the Riemannian product \(M \times N\) for all \(k \in \mathbb{N}_0\) so that the corresponding probability measures necessarily agree, too.

In the special case \(\nu = 0, \mu \neq 0\) the random variable \(s := x^2 \mu\) depends on \(x\) only so that we may integrate the probability density \((14)\) over \(y\) to obtain the distribution of \(x\). The subsequent substitution \(x = \sqrt{\frac{2}{\mu}}\) with \(xdx = \frac{ds}{2|\mu|}\) provides us with the distribution

\[
\frac{(m + n - 2)!}{(m - 2)! (n - 2)!} \int_{0}^{1-x} x^{m-2} y^{n-2} dy dx = \frac{(m + n - 2)!}{(m - 2)! (n - 1)!} \sqrt{s}^{m-3} \mu \frac{1}{(n - 1)!} \left( 1 - \sqrt{s} \right)^{n-1} \frac{ds}{2|\mu|}
\]

for the random variable \(s\) itself. By assumption \(m \geq 2\) so that the apparent pole of this density in \(s = 0\) is always integrable. A discussion of the special case \(n = 1\) along a similar line of argument is left to the reader. \(\square\)

Let us close this section with a small application illustrating the examples and formulas derived above, which exemplifies the use of the moments of the sectional curvature in the algebraic framework of local Riemannian invariants discussed in [W]. It is well-known that there exists a sequence \(\text{pf}_1, \text{pf}_2, \text{pf}_3, \ldots\) of natural polynomials of degree 1, 2, 3, \ldots respectively for algebraic curvature tensors over euclidean vector spaces such that the Euler characteristic of every compact Riemannian manifold \(M\) of even dimension \(m = 2n\) is given by:

\[
\chi(M) = \int_M \text{pf}_n^{T_p M} (R_p) \text{vol}_g(p)
\]

In passing we remark that \(\text{pf}_n\) is not the better-known Pfaffian \(\text{Pf}_n\), the latter is not even a natural polynomial in the sense discussed in the comments before Corollary 2.6, nevertheless
the two polynomials are very closely related. For essentially the same reason no other characteristic number of a compact manifold \( M \) can be written similarly as the integral of a natural polynomial on curvature tensors over \( M \), the Pontryagin numbers for example depend explicitly on the choice of orientation for \( M \).

Specifically for a homogeneous Riemannian metric \( g \) on a manifold \( M \) of even dimension \( m = 2n \) the value \( \text{pf}_2^{\mathbb{C}P^2}(R_p) \in \mathbb{R} \) of this natural polynomial on the curvature tensor \( R_p \) is independent of the point \( p \in M \) and thus equals the quotient of the Euler characteristic \( \chi(M) \) of \( M \) by its volume with respect to \( g \). In this way we obtain the explicit values

\[
\text{pf}_2(R_p^{S^4}) = \frac{\chi(S^4)}{\text{Vol}S^4} = \frac{3}{4\pi^2} \quad \text{pf}_2(R_p^{\mathbb{C}P^2}) = \frac{\chi(\mathbb{C}P^2)}{\text{Vol} \mathbb{C}P^2} = \frac{24}{4\pi^2}
\]

for the round sphere \( S^4 \) of radius 1 and volume \( \frac{8\pi^2}{3} \) and the complex projective plane \( \mathbb{C}P^2 \) endowed with the Fubini–Study metric \( g_{\mathbb{C}P^2} \) of volume \( \frac{1}{2\pi^2} \text{Vol}S^5 = \frac{8\pi^2}{3} \). A similar calculation for the Riemannian product \( S^2 \times S^2 \) of two round spheres of radius 1 and volume \( 4\pi \) leads to the value \( \text{pf}_2(R_p^{S^2 \times S^2}) = \frac{1}{4\pi^2} \). Eventually we consider the Riemannian product \( S^1 \times S^3 \) of two round spheres of radius 1 with vanishing Euler characteristic \( \text{pf}_2(R_p^{S^1 \times S^3}) = 0 \).

Incidentally three of our four examples, namely \( \mathbb{C}P^2, S^4 \) and \( S^2 \times S^2 \) are Einstein manifolds, whereas \( S^1 \times S^3 \) is not. In order to discuss this point we extend the scalar product \( g \) of a euclidean vector space \( V \) of dimension \( m \) to a scalar product \( g^{-1} \) on the space \( \text{Sym}^2V^* \) of symmetric 2–forms on \( V \) by choosing an orthonormal basis \( E_1, \ldots, E_m \) of \( V \) and setting:

\[
g^{-1}(h, \tilde{h}) := \frac{1}{2} \sum_{\mu, \nu=1}^m h(E_\mu, E_\nu) \tilde{h}(E_\mu, E_\nu) \quad (15)
\]

This specific normalization of the scalar product \( g^{-1} \) is characterized by \( g^{-1}(g, g) = \frac{m}{2} \) due to the more general \( g^{-1}(g, h) = \frac{1}{2} \text{tr}_g h := \frac{1}{2} \sum h(E_\mu, E_\mu) \) for all \( h \in \text{Sym}^2V^* \), of course other normalizations can and have been used in the literature. Regardless of this normalization question the scalar product \( g^{-1} \) can be used to characterize Einstein manifolds \( M \) of dimension \( m \) by the vanishing of the square norm of the so–called trace free Ricci tensor

\[
|\text{Ric}_{p}^\circ|^2 := g^{-1}(\text{Ric}_p - \frac{\kappa_p}{m} g, \text{Ric}_p - \frac{\kappa_p}{m} g) = |\text{Ric}_p|^2 - \frac{\kappa_p^2}{2m}
\]

where \( \text{Ric}_p \) is the Ricci and \( \kappa_p := \text{tr}_p \text{Ric}_p \) the scalar curvature of \( M \) in \( p \). Evidently the norm square of the Ricci tensor is additive \( |\text{Ric}_{p \times q}^{M \times N}|^2 = |\text{Ric}_p^M|^2 + |\text{Ric}_q^N|^2 \) under taking Riemannian products so that the norm square of its trace free part \( \text{Ric}_{(p,q)}^\circ \) is given by

\[
|\text{Ric}_{(p,q)}^\circ|^2 = \frac{\kappa_p^{M^2}}{2m} + \frac{\kappa_q^{N^2}}{2n} - \frac{(\kappa_p^M + \kappa_q^N)^2}{2(m+n)} = \frac{(n \kappa_p^M - m \kappa_q^N)^2}{2mn(m+n)}
\]

for two Einstein manifolds \( M, N \) of dimensions \( m, n \geq 0 \), in particular \( |\text{Ric}^\circ| = \frac{(-6)^2}{24} \neq 0 \) for the Riemannian product \( S^1 \times S^3 \) of two round spheres of radius 1. Completing the values of \( \text{pf}_2 \) and \( |\text{Ric}| \) of the four examples \( \mathbb{C}P^2, S^4, S^2 \times S^2 \) and \( S^1 \times S^3 \) by the values of their sectional curvature moments calculated in Section 3 and Lemma 4.1 respectively we obtain
which essentially proves the following theorem originally due to Hitchin–Thorpe [Th]:

**Theorem 4.3 (Euler Characteristic of 4–Dimensional Manifolds)**

The Euler characteristic of a 4–dimensional compact Riemannian manifold \( M \) can be written:

\[
\frac{4 \pi^2}{3} \chi(M) + \frac{4}{9} \int_M |\text{Ric}^o|^2 \text{vol}_g(p) = \int_M \Psi_2(R_p) \text{vol}_g(p) + 4 \int_M \left[ \Psi_2(R_p) - \Psi_1(R_p)^2 \right] \text{vol}_g(p)
\]

In particular \( \chi(M) \geq 0 \) for every compact 4–dimensional Einstein manifold \( M \) with equality, if and only if \( M \) is actually a flat Riemannian manifold.

**Proof:** It is well–known and easy to prove that the vector space of natural polynomials of degree 2 for algebraic curvature tensors over euclidean vector spaces has dimension 3 and is actually spanned by \( \kappa^2, |\text{Ric}|^2 \) and \( |R|^2 \). The table of explicit values for the four examples \( \mathbb{C}P^2, S^4, S^2 \times S^2 \) and \( S^1 \times S^3 \) shows that the three natural quadratic polynomials \( \Psi_1, \Psi_2 \) and \( |\text{Ric}^o|^2 \) are linearly independent and thus span the natural quadratic polynomials, in particular we find for the natural quadratic polynomial \( \frac{4 \pi^2}{3} \text{pf}_2 \) the linear combination

\[
\frac{4 \pi^2}{3} \text{pf}_2 = 5 \Psi_2 - 4 \Psi_1^2 - \frac{4}{9} |\text{Ric}^o|^2
\]

again by looking at the explicit values tabulated above. In the resulting identity

\[
\frac{4 \pi^2}{3} \text{pf}_2(R) + \frac{4}{9} |\text{Ric}^o|^2 = 5 \Psi_2(R) - 4 \Psi_1(R)^2 = \Psi_2(R) + 4 \left[ \Psi_2(R) - \Psi_1(R)^2 \right]
\]

valid for all algebraic curvature tensors \( R \) the two expressions \( \Psi_2(R) \) and \( \Psi_2(R) - \Psi_1(R)^2 \) are the second moment and the variance of the sectional curvature considered as a random variable on \( \text{Gr}_2V \). In particular both expressions are non–negative with equality if and only if the random variable takes the value 0 almost everywhere. \( \square \)
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