Quantum secret sharing using squeezing and almost any passive interferometer

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We consider the sharing of quantum secret states using continuous variable systems. Specifically, we introduce an encoding procedure where we mix the secret mode with several ancillary squeezed modes through a passive interferometer. We derive simple conditions on the interferometer for this encoding scheme to give a secret sharing protocol and we prove that they are satisfied by almost any interferometer with respect to the Haar measure. This implies that, if the interferometer is chosen uniformly at random, the probability that it may not be used to implement a quantum secret sharing protocol is zero. Furthermore, we show that the decoding operation can be obtained and implemented efficiently in linear optics with an additional squeezer per access party. We analyze the quality of the reconstructed state as a function of the input squeezing.

Introduction.— In quantum state sharing, also known as quantum secret sharing (QSS), a dealer shares a secret state among a network of players such that certain sets of players, the authorized sets, can access the secret, and others, unauthorized sets, cannot. The original classical version was first introduced by Shamir [1], and its quantum versions appeared in a series of works in discrete [2,3] and continuous variables (CV) [4–7]. Secret sharing is an important primitive for multiparty cryptography, for example in electronic voting [8], byzantine agreement [9], and secure multiparty computation [10]. It is known that any quantum secret sharing scheme can be reduced to (k, 2k−1) threshold schemes [11], i.e. protocols with 2k−1 players, and where the authorized sets are any group of at least k players. Hence, in this work we limit ourselves to the case (k, 2k−1).

In recent years much effort has gone into the implementation of quantum secret sharing [4,5,11–13]. A particular issue with QSS in larger size networks is that the size of the shares—in i.e. the systems held by the players—must grow with the number of players [14]. In other words, adding more players to the network forces the use of higher dimensional encodings. One great advantage brought by the infinite dimensionality of CV systems is the absence of this issue [4,7]. Another motivation for exploring CV is that CV technologies offer great potential in terms of scaling, with record size of entanglement [15,16].

A key question for any implementation of CV QSS—or indeed any quantum information protocol—is its optimization, in particular in terms of noise tolerance or required squeezing. To this end, it is interesting to understand the most general properties of such a scheme, and how they may be implemented, as we do in this work.

We define here a general CV QSS procedure using an interferometer and ancillary squeezed states; we then fully characterize the working cases. Surprisingly we find that almost any interferometer will do: those that do not are vanishing in the Haar measure. These interferometers are linear in depth and the decoding procedure can be found easily and implemented simply. Furthermore, the access party does not need any squeezer to perform an arbitrary homodyne measurement on the secret state, and needs a single squeezer to reconstruct it. In addition, given the deep connection between secret sharing and error correction [3,14], together with the broad use of random encoding in quantum information [17,18], these results may indeed have much broader consequences.

CV quantum optics.— A convenient way to study a n mode CV system is through the 2n-dimensional phase space. The 2n components of the quadrature vector \( \xi = (q^+, p^+)^T \) are the position and momentum operators. The canonical commutation relations are then \([\xi_i, \xi_j] = iJ^{(n)}_{jk}\), where \(J^{(n)}\) is the standard symplectic form \(J^{(n)} = \begin{pmatrix} 0_n & \mathbb{I}_n \\ -\mathbb{I}_n & 0_n \end{pmatrix} \), \(0_n\) and \(\mathbb{I}_n\) being zero and identity \(n \times n\) matrices.

The state of a n-mode system is characterized by its Wigner function \(W(q,p)\) [19], a quasi-probability distribution taking values over the 2n-dimensional phase-space [20]. Gaussian states are naturally defined as those whose Wigner function is Gaussian, and they are fully characterized by the mean and covariance matrix of the quadrature vector \(\xi\).

Gaussian transformations—processes preserving the Gaussian characters of Gaussian states—are an essential subset of physical transformations, since they can actually be implemented with today’s technologies. Unitary Gaussian transformations are elegantly described by the formalism of symplectic matrices. In the Heisenberg picture, the action of a unitary Gaussian operation \(U_G\) is a

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linear map

\[
U \xi U = S \xi + \eta,
\]

where \(S\) is a \(2n \times 2n\) real symplectic matrix and \(\eta\) is a vector of real numbers \([21]\). Symplectic matrices acting on \(n\) modes are the matrices \(S\) preserving the standard symplectic form:

\[
SJ^{(n)}S^T = J^{(n)}.
\]

Under matrix multiplication, they form the group \(Sp(2n, \mathbb{R})\).

Of specific interest are squeezing and passive operations. Squeezing operations do not conserve photon number \([20]\) and are usually realized through nonlinear \(\chi^{(2)}\) optical processes. Independent squeezing operations on each mode are represented by diagonal symplectic matrices of the form \(K = \text{diag} (e^{r_1}, \ldots, e^{r_n}, e^{-r_1}, \ldots, e^{-r_n})\), where the parameters \(r_i\) are the squeezing parameter of each mode \([22]\).

Passive operations are defined as photon-number preserving Gaussian unitaries and can be performed with linear optics. They correspond to the subgroup \(L(n) = Sp(2n, \mathbb{R}) \cap O(2n)\) of orthogonal, symplectic matrices \([21]\). Each matrix \(S_L \in L(n)\) corresponds to \(n \times n\) unitary matrix \(X + iY \in U(n)\) such that \(S_L = (X - Y, -X)^T\).

This group isomorphism allows us to speak interchangeably of passive interferometers or the corresponding symplectic and unitary matrices.

**Description of the protocol.** — We consider \((k, 2k - 1)\) threshold schemes in which the dealer couples a mode in a secret state \(\rho_s\) to \(2k - 1\) modes which are (finely) squeezed in the \(p\) quadrature in a passive interferometer. Assuming that the same quadrature is squeezed in all the modes simplifies the notation but implies no loss of generality, as local phase-space rotations aligning the squeezing directions correspond to linear optics operations that could be included in the interferometer.

We thus start with \(2k\) modes, of which the first \(2k - 1\) are squeezed and the last is in the secret state. We stress that the secret state may be an arbitrary single-mode state. We denote the vector containing all the input quadratures by

\[
\xi_{\text{in}} = \begin{pmatrix} q_{sx} & p_{sx} \\ q_{px} & p_{px} \end{pmatrix}
\]

where the quadratures of the \(j\)th squeezed mode is related to the vacuum quadratures by \(q_j^{\text{sqz}} = e^{r_j} q_j^{(0)}\) and \(p_j^{\text{sqz}} = e^{-r_j} p_j^{(0)}\). After the dealer sends the modes through the interferometer \(S_L\), the state is described by the quadratures

\[
\xi_{\text{net}} = \begin{pmatrix} q^{\text{net}}_d & p^{\text{net}}_d \\ q^{\text{net}}_s & p^{\text{net}}_s \end{pmatrix} = S_L \xi_{\text{in}} = \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \xi_{\text{in}}.
\]

The dealer then measures \(p_d\) via a homodyne measurement on the \(2k\)th mode. Assuming that the dealer measures the quadrature \(p_d\) comes with no loss of generality. The outcome of the measurement \(p_d\) is broadcast to all the players. Following the homodyne detection, we have

\[
p_{d} = \sum_{l=1}^{2k-1} Y_{2k,l} q_{sqz} + \sum_{l=1}^{2k-1} X_{2k,l} p_{sqz} + Y_{2k,2k} q_s + X_{2k,2k} p_s,
\]

where \(q_{sqz}\) is discarded and the secret quadratures \(q_s\) and \(p_s\) are explicitly separated. At this point, the players’ modes should contain all information about \(\rho_s\). The procedure is outlined in Fig. 1.

**Decodability conditions.** — We now investigate the conditions \(S_L\) must satisfy in order for any set of \(k\) or more players to be able to access the secret quadratures. Specifically, for each authorized set we look for two independent linear combinations of quadratures that do not involve the anti-squeezed quadratures \(q_j^{\text{asp}}\) and contain \(q_s\) and \(p_s\) respectively. If such linear combinations exist, all the information about the secret state is effectively contained in \(2k - 1\) unmeasured modes, and it can be extracted through the decoding procedures described below.

Consider a subset of \(k\) players \(A = \{a_1, a_2, \ldots, a_k\}\) who are given the modes with quadratures \(\xi_A\)

\[
\xi_A = \begin{pmatrix} q_{a_1}^{\text{net}} & p_{a_1}^{\text{net}} \\ q_{a_2}^{\text{net}} & p_{a_2}^{\text{net}} \\ \vdots & \vdots \\ q_{a_k}^{\text{net}} & p_{a_k}^{\text{net}} \end{pmatrix}, \quad P_A = \begin{pmatrix} p_{a_1}^{\text{net}} & p_{a_2}^{\text{net}} & \cdots & p_{a_k}^{\text{net}} \end{pmatrix}
\]

The access party needs to cancel the dependency to the anti-squeezed quadratures. The broadcast \(p_d\) allows it to cancel one of them as follows. Clearly, the secret state must be coupled to the squeezed states, otherwise the secret would be destroyed by the dealer’s measurement. More formally, there exists \(l \in \{1, 2, \ldots, 2k - 1\}\) such that \(S_{L,k,1} \neq 0\) or \(S_{L,2k,l} \neq 0\). Up to a simple relabelling, we can assume \(Y_{2k,1} \neq 0\). We then solve Eq. (5) for \(q_{sqz}^l\) and substitute the result in Eq. (3) to eliminate \(q_{sqz}^l\) from the remaining \(q_{sqz}^{\text{net}}\) and \(p_{sqz}^l\). As a result, we obtain

\[
\begin{pmatrix} Q_A \end{pmatrix} = M_A q^{\text{asp}} + N_A p^{\text{asp}} + h_A q_s + h_P p_s + h_P p_d, \quad (7)
\]

FIG. 1. A scheme of the encoding procedure.
where the entries of the matrices $M^A$ and $N^A$ and the vectors $h_q^A$, $h_p^A$, $h_s^A$ are defined by the coefficients of $S_L$ and $q_{sqz}$ denotes the vector of anti-squeezed quadratures without $q_{sqz}^T$.

The term $h_q^A p_j$ depending on the measurement result can either be canceled locally by the access party through optical displacements or only accounted for after the reconstruction. We will therefore omit it from now on.

Any linear combination of $Q^A$ and $P^A$ is obtained multiplying Eq. (7) on the left by a vector $v \in \mathbb{R}^{2k}$. The product $v^T C_A$ does not contain any contribution of the anti-squeezed quadratures amounts iff $v$ lies in the kernel of $(M^A)^T$: $v \in \ker (M^A)^T$. By construction, $M^A$ has $2k$ rows and $2k - 2$ columns, therefore dim $(\ker(M^A)) \geq 2$.

We can thus always find two linearly independent vectors $v, w \in \ker (M^A)^T$. Let $R$ be the matrix whose rows are $v^T$ and $w^T$. Applying it to $\xi^A$, we get

$$R \xi^A = RN^A p^sqz + (v \cdot h_q^A \ v \cdot h_p^A) (q_s \ p_s)$$

$$= RN^A p^sqz + T (q_s \ p_s)$$

where the last line defines the $4 \times 4$ matrix $T$, which contains the projections $v$ and $w$ on $h_q$ and $h_p$. The access party $A$ can then sample from the secret quadratures iff $T$ is invertible. Then, multiplying $T^{-1}$ by Eq. (9) and defining $D = T^{-1} R$, $B = T^{-1} R N^A$, leads to

$$D \xi^A = B p^sqz + (q_s \ p_s).$$

This equation tells us that when the access party $A$ measures one of the linear combinations of quadratures defined by $D$, the outcomes will follow the same probability distribution as either $q_s$ or $p_s$ apart from random displacements due to the term $B p^sqz$. These displacements decrease with increasing input squeezing, ultimately vanishing for infinite squeezing. In this limit, the access party can perfectly sample from the original secret state. Note that real linear combinations of the rows of $D$ are linear combinations of $q_s$ and $p_s$ plus the squeezed quadratures, so $A$ can also measure arbitrary quadratures of the secret (see below). In other words, the Wigner function of the state reconstructed by $A$ is a convolution of the Wigner function of the secret state with a Gaussian function that tends to a Dirac distribution in the infinite squeezing limit (see Appendix D).

In summary, $A$ can reconstruct the secret iff there exists at least one $l \in \{1, 2, \ldots, 2k - 1\}$ such that $Y_{2k,l} \neq 0$, and the matrix $T$ in Eq. (9) is not singular.

Given any linear optical network $S_L$, these conditions can be checked for each access party (all groups of $k$ players). If they are satisfied for all access parties, then $S_L$ can be used for a $(k, 2k - 1)$ quantum secret sharing scheme. Constructing the matrix $T$ to compute its determinant requires finding two vectors in the kernel of $(M^A)^T$. As shown in Appendix A, this condition is equivalent to

$$\det T \neq 0 \quad \text{iff} \quad \det (M^A | h_q^A \ h_p^A) \neq 0,$$

where $(M^A | h_q^A \ h_p^A)$ is the matrix obtained appending $h_q^A$ and $h_p^A$ to $M^A$ as columns. This condition explicitly involves the coefficients of $S_L$, which will be useful to prove our main result.

We now clarify in which sense the above conditions allow one access party to decode the secret. Consider an access party $A$ and suppose $\det T \neq 0$. Clearly, the access party can measure the linear combinations defined by $D \xi^A$ by combining the results of local homodyne detections. Indeed, Eq. (10) can be rewritten for the position operator

$$q_s + \sum_{l=1}^{n-1} \frac{B_{lj} q_{sqz}^l}{D_{lj}} = \sum_{j=1}^{j=k} \left(D_{1j} q_j^A + D_{1j+k} p_j^A\right)$$

$$= \sum_{j=1}^{j=k} \alpha_j \left(\cos \theta_j Q_j^A + \sin \theta_j P_j^A\right)$$

for appropriately chosen $\alpha_j, \theta_j \in \mathbb{R}$. The players of $A$ achieve their goal measuring the rotated quadratures with angles $\theta_j$ and summing their results multiplied by $\alpha_j$. Since the same reasoning applies to the momentum operator and any linear combination of the two, the access party $A$ can carry out an arbitrary homodyne measurement of the secret $\rho_s$. This possibility to sample from $\rho_s$ allows $A$ to simulate any protocol needing homodyne measurements of $\rho_s$, from quantum key distribution [23], to measurement based quantum computing [24], or, when provided several copies of $\rho_s$, to tomography.

Moreover, the players of $A$ can physically reconstruct the secret state by applying the Gaussian transformation, constructed as follows. Let us call $\xi_s^{\text{out}} = D \xi^A$ and $\xi_s = (q_s \ p_s)^T$. Evaluating the commutators $[\xi_s^{\text{out}}, \xi_s^{\text{out}}]$ and remembering that the secret quadratures are conjugated canonical operators we have

$$[\xi_s^{\text{out}}, \xi_s^{\text{out}}] = [\xi_s^A, \xi_s^A] = i J_{lm}^{(1)}.$$

Since $S_L$ is symplectic, the quadratures of the access party also satisfy

$$[\xi_s^A, \xi_s^A] = i J_{lm}^{(k)}.$$

Using $\xi_s^{\text{out}} = D \xi_A$ leads to

$$[\xi_s^{\text{out}}, \xi_s^{\text{out}}] = i \left(D J^{(k)} D^T\right)_{lm} = i J_{lm}^{(1)}.$$

In other words, the rows of $D$ are conjugated vectors of a symplectic basis of $\mathbb{R}^{2k}$. The other basis elements can be constructed through a Gram-Schmidt like procedure [25], explained in Appendix C.

Let us call $S^A_D$ the symplectic matrix whose first and $k + 1$st rows are the rows of $D$, while the others are constructed by the above mentioned procedure. Its action
on the $2k$ vector of quadratures of the access party $A$ corresponds to a unitary Gaussian transformation $U \bar{D}$ such that

$$(U \bar{D}^A)^T \xi^A U \bar{D}^A = S_{\bar{D}}^A \xi^A. \quad (17)$$

By construction, the first and $k+1$st entries of $S_{\bar{D}}^A \xi^A$ are the output quadratures $\xi_{\text{out}}$, so if the players of $A$ apply the physical evolution corresponding to $U \bar{D}$ and $S_{\bar{D}}^A$, they end up with a mode in the secret state, modulo the noise coming from the squeezed quadratures.

Note that, in the general case, $S_{\bar{D}}^A$ may require squeezing. However, remarkably, it is possible to construct $S_{\bar{D}}^A$ that can be realized with a passive interferometer acting on the $k$ modes of $A$, two independent single-mode squeezers and a final passive interferometer between the same two modes. In fact, the number of squeezers can be further reduced to one per access party by replacing the second one by a homodyne measurement followed by an optical displacement depending on the measurement result. Note that the number of squeezers per access party in the decoding does not scale with the number of players. This result generalizes the result of [5] to all passive interferometers, including the ones mixing positions and momenta. This generalization beyond orthogonal transformations of the position operators is essential for the result stated in the next section.

Almost any interferometer can be used for QSS. — We now formalize our main result, namely that a CV QSS $(2k - 1, k)$ threshold scheme can be realized mixing the secret state with $2k - 1$ squeezed modes in almost any passive interferometer, where “almost any” is intended in the sense of Haar measure. A sketch of the proof, detailed in Appendix [E], follows.

Let $\mathcal{B}$ be the set of matrices that cannot be used for secret sharing. For $S_L \in L(n)$ to be in $\mathcal{B}$, either $Y_{nl} = 0 \forall l = 1, 2, \ldots, n - 1$, in which case we say that $S_L \in \mathcal{B}$, or there is at least one access party $A$ for which $\det{(M | h_A^w | h_A^w)} = 0$, in which case we say that $S_L \in \mathcal{B}^A$. Because of positivity and countable additivity, the Haar measure of $\mathcal{B}$ cannot be larger than the sum of the measure of the sets $\mathcal{B}$ and $\mathcal{B}^A$, so we just need to show that each of them has zero measure. Now, each of them is defined by a polynomial equation of the coefficients of $S_L$, which, regarding $U(n)$ as a manifold, identifies a lower dimensional set $20$. In other words, since $L(n)$ is a Lie group of dimension $n^2$, it can be parametrized by $n^2$ real variables defined in an appropriate region $E \subseteq \mathbb{R}^{n^2}$. The entries of $S_L$ can be written as polynomials of trigonometric functions of $n^2$ angles $\lambda$, so the determinant in Eq. (11) is a real analytic function $27, 28$ of $\lambda$, whose zero set has necessarily null measure on $E$. Therefore $\mathcal{B}$ has zero Haar measure in $L(n)$.

Up to a normalization factor, Haar measure can be seen as a uniform probability distribution over the unitary group. It follows that if the Haar measure of $\mathcal{B}$ of the unitary group is zero, then if one chooses a unitary matrix $U$ uniformly at random, then the probability that $U \in \mathcal{B}$ is zero. As a consequence, CV QSS schemes are robust to unitary imperfections in the interferometer fabrication, as long as $S_L$ can be fully characterized.

Quality of the reconstructed state. — The protocol is essentially a teleportation protocol, where the choice of the reconstruction location can be delayed until an access party is chosen. We can thus characterize the quality of the reconstructed state using similar criteria as for a Gaussian teleportation channel [20]. As usual in CV, the channel will only be perfect in the infinite squeezing limit. Otherwise, Gaussian noise will be added to the initial state, and some information will be leaked to unauthorized sets. Some additional considerations can be made: first, we can relate the amount of input squeezing to the fidelity of the reconstructed state. For coherent state input, when all the input squeezed states have the same squeezing, corresponding to $\Delta^2 p_j^{\text{out}} = e^{-2\eta}/2 \equiv \sigma^2(\eta)$, the fidelity of the reconstructed state can be expressed as (see Appendix [E])

$$\mathcal{F}^A(\eta) = [1 + \sigma^2(\eta) \eta + \sigma^4(\eta) \eta^2]^{-\frac{1}{2}} \quad (18)$$

with $\eta = \text{Tr}{(BB^T)}$ and $\zeta = \det{(BB^T)}$, where $B$ is defined in Eq. (10). Clearly $\mathcal{F}^A(\eta) \to 1$ for $\eta \to \infty$. In the secret sharing scenario, we should also ensure that as little information as possible is leaked to the adversaries. If the fidelity of the recovered state is above the $2/3$ bound for optimal cloning, the fidelity of the best possible state reconstructed by any adversary set would necessarily be lower due to the no-cloning theorem. Moreover, we observe that similar equations as Eq. (7) may be written for groups of $k < k$ players. However, the kernel of $(M^A)^T$ would consist of only the null vector for almost any $S_L$, so that the adversary groups would not be able to eliminate the anti-squeezed contributions from their share of the secret. This implies that their information about the secret would go to zero for infinite input squeezing. Furthermore, considering noisy squeezed states with a non-minimal antisqueezed quadrature, at finite squeezing, the quality of the reconstruction of the authorized party is not affected, while a higher anti-squeezing further degrades the quality for the unauthorized party.

Conclusions. — Introducing a general scheme for sharing general quantum states using Gaussian resources we were able to show that a sharing infrastructure can be constructed mixing squeezed states in almost any passive interferometer. Moreover, the scheme is easily generalizable to share multi-mode states, although we leave the details for future work. This opens a wide range of possibilities, since it implies that potentially any experimental setup producing multi-mode squeezed states can be used for efficient QSS. On the technological side, this also means that there is plenty of room for optimization that could be exploited for practical applications of quantum networks. The interest of our finding is not limited to cryptography, but could be used to share quantum resources in server-client architectures for optical quantum computing [21, 32], which is an increasingly studied
interferometers and not the full set of unitaries.

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[19] Here, q and p are real-valued n dimensional vectors, not vectors of operators.
Appendix A: Equivalent condition for invertibility of the matrix $T$

The decodability conditions derived in the main text are readily computed once $S_L$ is known but they require the explicit calculation of two vectors in the kernel of $(M^A)^T$, which is not very practical. We prove here a condition equivalent to the invertibility of $T$ in the case that $M^A$ has full rank $\text{rank}(M^A) = 2k - 2$. The condition (Eq. (11) in the main text) results in a polynomial equation in the coefficients of $S_L$ and thus does not require computing the kernel of $(M^A)^T$ explicitly. This will be particularly useful for the proof of our main result.

Let us call $V = \text{Ker} \left((M^A)^T\right) \subset \mathbb{R}^{2k}$. If $M^A$ has full rank, then $\dim(V) = 2$, since $M^A$ always has $2k$ rows and $2k - 2$ columns. Let us denote by $h_q^A|_V$ and $h_p^A|_V$ the projections on $V$ of $h_q^A$ and $h_p^A$ respectively. We proved in the main text that $T^{-1}$ exists if and only if $h_q^A|_V$ and $h_p^A|_V$ are linearly independent. Suppose that $v$ and $w$ are a basis of $V$. Then

$$h_q^A = a + \alpha v + \beta w$$
$$h_p^A = b + \gamma v + \delta w$$

with $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ and $a, b \in V^\perp \subset \mathbb{R}^{2k}$. Then

$$\left\{h_q^A|_V, h_p^A|_V\right\} \text{ are l. i. iff } \det \left(\begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array}\right) \neq 0.$$  \hspace{1cm} (A3)

Consider now the square matrix obtained appending $h_q^A$ and $h_p^A$ to $M$ as columns. We denote this matrix by $(M^A | h_q^A | h_p^A)$. Since the determinant is a multilinear, alternating function of the columns we have

$$\det (M^A | h_q^A | h_p^A) = \det (M^A | a + \alpha v + \beta w | b + \gamma v + \delta w)$$
$$= \det (M^A | \alpha v + \beta w | \gamma v + \delta w)$$
$$= \alpha \delta \det (M^A | v | w) + \beta \gamma \det (M^A | v | w)$$
$$= (\alpha \delta - \beta \gamma) \det (M^A | v | w)$$
$$= \det \left(\begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array}\right) \det (M^A | v | w)$$ \hspace{1cm} (A4)

where the second line follows from the fact that, since $M^A$ is full rank, $V^\perp = \text{span}\left(\left\{M^A (i)\right\}\right)$, having denoted by $M^A (i)$ the columns of $M^A$ (in other words, $V$ is the space of the vectors orthogonal to all the rows of $(M^A)^T$), so terms of the form $\det (M^A | a | x)$ or $\det (M^A | y | b)$ are automatically zero. Since by hypotesis $\det (M^A | v | w) \neq 0$, it follows that

$$\left\{h_q^A|_V, h_p^A|_V\right\} \text{ are l. i. iff } \det (M^A | h_q^A | h_p^A) \neq 0.$$  \hspace{1cm} (A5)

Since $M^A$, $h_q^A$ and $h_p^A$ are defined in terms of the coefficients of $S_L$ and the determinant is a polynomial function thereof, this is the condition we were looking for.

Appendix B: Some notations

We introduce here some notations and concepts from symplectic geometry that will be useful in the following sections.

As stated in the main text, any unitary Gaussian transformation $U_G$, neglecting phase-space translations, can be represented in the Heisenberg picture by the application of a symplectic matrix $S_U$ to the vector of quadratures of the field $\xi$. By definition, $n \times n$ symplectic matrices satisfy

$$S^T J^{(n)} S = J^{(n)}$$  \hspace{1cm} (B1)

with

$$J^{(n)} = \begin{pmatrix} 0_n & 1_n \\ -1_n & 0_n \end{pmatrix}$$  \hspace{1cm} (B2)
with \(0_n\) and \(I_n\) the \(n \times n\) zero and identity matrices respectively. Clearly, real symplectic matrices form a group under matrix multiplication, which we denote \(\text{Sp}(2n, \mathbb{R})\). If displacements are included, one gets the so-called inhomogeneous symplectic group. The canonical commutation relations can be written compactly as
\[
[J_\xi, J_\eta] = iJ_{\xi \eta}^{\{n\}}.
\] (B3)

It follows that unitary Gaussian operations preserve the canonical commutation relations.

We introduce the symplectic product between two vectors: given two vectors \(x, y \in \mathbb{R}^{2n}\), their symplectic product is defined as
\[
\langle x, y \rangle \equiv x^T J y.
\] (B4)

We denote by \(x \cdot y\) the ordinary euclidean product \(x \cdot y = \sum_j x_j y_j\). Note that taking the dot product between a vector of real numbers and the vector of quadratures results in a linear combination of quadrature operators. The commutator between two such combinations is simply related to the symplectic product of the vectors
\[
[x \cdot \xi, y \cdot \xi] = i\langle x, y \rangle
\] (B5)
as is easily shown using Eq. (B3).

**Appendix C: Extending the matrix \(D\) to a symplectic matrix**

We outline here an algorithm that can be used to extend the matrix \(D\) in Eq. (9) of the main text for an access party \(A\) to a symplectic operation \(S_{D}^{A}\) corresponding to a physical, unitary Gaussian operation that the access party can implement to output a mode in the secret state.

Let us start from the first rows of \(D\). We can interpret the first line, that we denote by \(x\), as defining the position quadrature of the decoded mode, while the second, denoted \(y\), defines the momentum quadrature. Our goal will be to find a symplectic matrix \(S_{D}^{A}\) that, when applied to the quadratures of the access party \(A\), \(\xi^{A}\), sends the quadratures of the first mode to the secret quadratures plus a combination of the squeezed quadratures \(\rho^{\text{sqz}}\). This is achieved if the first and \((k+1)\)th lines of \(S_{D}^{A}\) correspond to \(x\) and \(y\) respectively. \(S_{D}^{A}\) can be constructed as follows.

First, define
\[
x_1 = \frac{1}{\|x\|} x
\] (C1)
and \(y_1 = -J^{(k)} x_1\). The vectors \(x_1\) and \(y_1\) are both normalized and their symplectic product is
\[
\langle x_1, y_1 \rangle = 1
\] (C2)
since \((J^{(k)})^2 = -I\). Using Eq. (B5) we immediatley see that the operators \(q_1 = x_1 \cdot \xi^{A}\) and \(p_1 = y_1 \cdot \xi^{A}\) have unit commutator \([q_1, p_1] = 1\).

Let us call \(V \equiv \text{span} \{x_1, y_1\}^\perp \subset \mathbb{R}^{2k}\). The vector \(y\) can be decomposed as a sum of its projections on \(V\) and its orthogonal complement \(V^\perp\) (with respect to the standard euclidean product)
\[
y = y|_{V_1} + y|_{V_1^\perp} = \alpha x_1 + \beta y_1 + \gamma x_2
\] (C3)
where we choose \(\gamma\) such that \(x_2\) is normalized \(\|x_2\| = 1\). Since \(x_2 \in V_1^\perp\) we have
\[
0 = x_2 \cdot x_1 = x_2 \cdot \left( J^{(k)} y_1 \right) = \langle x_2, y_1 \rangle
\] (C4)
\[
0 = x_2 \cdot y_1 = -x_2 \cdot \left( J^{(k)} x_1 \right) = -\langle x_2, x_1 \rangle
\] (C5)
so that \(q_2 = x_2 \cdot \xi^{A}\) is a linear combination of quadratures that commutes with \(q_1\) and \(p_1\). Note that this implies \(\beta = 1/\|x\|\), since by construction \(\langle x, y \rangle = 1\).

We can now use a Gram-Schmidt-like procedure to find \(k-2\) orthonormal vectors \(\{x_3, \ldots, x_k\}\) in \(V_2^\perp\), with \(V_2 \equiv \text{span} \{x_1, y_1, x_2\}\), that have null symplectic product with any vector in \(V\). In practice, suppose we already found \(l\) such vectors and call \(V_l \equiv \text{span} \{x_1, y_1, x_2, \ldots, x_l\}\). We can then pick \(u_{l+1} \in V_{l+1}^\perp\) and define
\[
\hat{u}_{l+1} = u - \sum_{j=1}^{l} \langle x, y_j \rangle x_j + \sum_{j=1}^{l} \langle x, x_j \rangle y_j.
\] (C6)
The vector
\[ x_{i+1} = \frac{1}{\|u_{i+1}\|} u_{i+1} \]  
(C7)
is then normalized and satisfies
\[
\langle x_{i+1}, x_j \rangle = 0 \quad \text{for } j = 1, \ldots, l \tag{C8}
\]
\[
\langle x_{i+1}, y_1 \rangle = 0 \tag{C9}
\]
Each linear combination \( x_j \cdot \xi^A \) can be taken as a new position operator, since they all commute by construction. We can construct their conjugated momenta operator as \( p_j = y_j \cdot \xi^A \) with \( y_j = -J^{(k)} x_j \). With a similar reasoning as above, it is easy to see that
\[
[q_j, p_i] = i \delta_{ji} \tag{C10}
\]
\[
[q_j, q_i] = [p_j, p_i] = 0. \tag{C11}
\]
In other words, the matrix \( O_1 \) whose first \( k \) rows are the vectors \( x_j^T \) and the last \( k \) rows are the vectors \( y_j^T \) is both symplectic and orthogonal, hence it describes the action of a passive interferometer on the quadratures of the access party. In particular, it leaves the position of the first mode in
\[
q_1 = x_1 : \xi^A \tag{C12}
\]
which, by definition of \( x_1 \) is the decoded quadrature, just rescaled by the norm of \( x \). The momentum quadrature is instead
\[
p_1 = y_1 : \xi^A = \|x\| y \cdot \xi^A - \alpha \|x\| q_1 - \|x\| \gamma q_2 \tag{C13}
\]
where we used Eq. (C3) and used the fact that the position of the second mode after \( O_1 \) is applied is \( q_2 = x_2 : \xi^A \). To achieve decoding, we should rescale the position of the first mode and remove the contributions from \( q_1 \) and \( q_2 \) from the momentum of the first mode. As we show in the following, the first can be achieved with a squeezing operation on the first mode, while the second requires a controlled operation between the first and second mode, commonly called \( C_Z \) gate, and a shear on the first mode \[24\].

The correct scaling for the position can be obtained applying a squeezing operation on the first mode, described by a matrix \( K_1 \) that acts as
\[
K_1 : \left( \begin{array}{c} q_1 \\ p_1 \end{array} \right) \mapsto \left( \begin{array}{c} q'_1 \\ p'_1 \end{array} \right) = \left( \begin{array}{c} \|x\| q_1 \\ \frac{1}{\|x\|} p_1 \end{array} \right) \tag{C14}
\]
on the first mode and as the identity on all other modes. By construction, for \( p'_1 \) we have
\[
p'_1 = \frac{1}{\|x\|} y_1 : \xi^A = y_1 : \xi^A = y \cdot \xi^A - \frac{\alpha}{\|x\|} q_1 - \gamma q_2. \tag{C15}
\]
The contribution from \( q_2 \) can be removed applying a \( C_Z \) gate of strength \(-\gamma\) between the first and second mode. This amounts to the unitary operator \( C_Z (-\gamma) = \exp (-i \gamma q_1 \otimes q_2) \), which transforms the momenta of the first and second modes as
\[
C_Z (-\gamma)^\dagger p'_1 C_Z (-\gamma) = p''_1 = p'_1 - \gamma q_2 = y \cdot \xi^A - \frac{\alpha}{\|x\|} q_1 \tag{C16}
\]
\[
C_Z (-\gamma)^\dagger p_2 C_Z (-\gamma) = p_2 - \gamma q'_1 \tag{C17}
\]
and leaves all other quadratures (including positions of the first and second modes) invariant. Finally, the contribution from \( q'_1 \) can be removed applying a shear of strength \(-\alpha\), corresponding to the unitary operator \( P (\alpha) = \exp \left( (q')^2 \right) \), that transforms \( p''_1 \) according to
\[
P (\alpha)^\dagger p''_1 P (\alpha) = p'''_1 = p''_1 + \alpha q'_1 = y \cdot \xi^A \tag{C18}
\]
which is the desired result. Both the \( C_Z \) and the shear are unitary Gaussian operators and thus can be described by symplectic matrices, which we denote by \( K_2 \) and \( K_3 \), respectively. The matrix \( K_3 K_2 K_1 \) only acts on the first two
modes, so we can decompose it as a passive interferometer, followed by two independent squeezers on the first two
modes followed by another passive interferometer

\[ K_3 K_2 K_1 = O_3 K O_2. \]  

(C19)

The decoding operation we were looking for is given by

\[ S^A_D = K_3 K_2 K_1 O_1 = O_3 K O_2 O_1 = \hat{O}_2 K \hat{O}_1 \]  

(C20)

where \( \hat{O}_1 \) represent a passive interferometer on all the \( k \) modes, \( \hat{O}_2 \) is a passive interferometer on the first two modes
only, and \( K \) represents two independent squeezers on the first two modes. By construction \( S^A_D \) maps the position
of the first mode od the access party to \( x \cdot \xi^A \) and the momentum of the first mode to \( y \cdot \xi^A \), thus achieving the decoding.

Note that the decoding operation is not unique. For example, applying the shear on the first mode, to remove
the \( q_1 \) contribution from \( p_1 \) and then the \( C_2 \) to remove the \( q_2 \) contribution would lead to a different \( S^A_D \). Another
possibility is to avoid the controlled operation between the first two modes, reducing the number of online squeezing
operations to just one. This can be achieved if the second player measures \( q_2 \). Denoting by \( \nu \) the measurement result,
from Eq. (C13) we see that after the measurement the momentum of the first mode would read

\[ p_1 = y_1 \cdot \xi^A = \| x \| y \cdot \xi^A - \alpha \cdot \| x \| q_1 - \| x \| \gamma \nu \]  

(C21)

where \( \| x \| \gamma \nu \) is a real number, that amounts to a phase-space translation which the first player could correct applying
an optical displacement. The controlled operation would then be replaced by a homodyne measurement, classical
communication and a displacement conditioned on the measurement outcome.

Appendix D: Effect of finite-squeezing noise on the decoded state

We show here that for general input states the Wigner function of the reconstructed state can be represented as
the Wigner function of the input state convoluted with a Gaussian filter depending on the input squeezing. We then
derive Eq. (18) of the main text for coherent input (secret) states.

1. General input states

We now show that for any input state \( \rho_s \), with Wigner function \( W_s(q, p) \), not necessarily Gaussian, the Wigner
function \( W_{\text{out}}(q, p) \) of the state reconstructed by an access party is given by a convolution of \( W_s \) with a Gaussian
filter function. The covariance matrix of said filter function is related to the input squeezing, the encoding \( S_L \) and
the decoding \( S^A_D \) and becomes narrower for larger squeezing, eventually converging to a Dirac delta. In this limit, the
convolution outputs the secret Wigner function \( W_s \), meaning that the reconstruction is perfect.

Let us start from Eq. (10) of the main text, which we recall here

\[ \xi^{\text{out}} = B p^{\text{sqz}} + \left( \frac{q_s}{p_s} \right) \]  

(D1)

where \( \xi^{\text{out}} \equiv D \xi^A \) as in the main text. If the matrix \( B \) was the zero matrix, then the outcomes of the measurement
of any quadrature of the output state would follow the same probability distribution as if the same measurement was
performed on the input state. It follows that the output Wigner function \( W_{\text{out}}(q, p) \) would be equal to the input Wigner function \( W_s(q, p) \). If the matrix \( B \) is not zero, the output state is obtained by tracing out all other modes.

This amounts to averaging over all possible measurement outcome for the squeezed quadratures \( p_j^{\text{sqz}} \). By assumption,
the input modes are independently squeezed, so each \( p_j^{\text{sqz}} \) will contribute with a random Gaussian shift, with zero
mean and variance \( \sigma_j^2 = e^{-2\nu j}/2 \). Since the map that associates a Wigner function to each density matrix is linear,
the output Wigner function is

\[ W_{\text{out}}(q, p) = \int \left( \prod_{j=1}^{2k-1} \frac{e^{-y_j^2/2\sigma_j^2}}{\sigma_j \sqrt{2\pi}} \right) W_{\text{in}} \left( q - \sum_{j=1}^{2k-1} B_{1j} y_j, p - \sum_{j=1}^{2k-1} B_{2j} y_j \right) \]  

(D2)
We can describe the parameters $y_j$ collectively as Gaussian random variables with covariance matrix $\Delta^2 = \text{diag}(\sigma_1^2, \ldots, \sigma_{2k-1}^2)$. Let us now introduce the variables

$$u = \sum_{j=1}^{2k-1} B_{1j} y_j$$

$$v = \sum_{j=1}^{2k-1} B_{2j} y_j.$$  \hspace{1cm} (D3)

Being the sum of Gaussian random variables, they are themselves Gaussian distributed with covariance matrix

$$\tilde{\Gamma} = B \Delta^2 B^T$$  \hspace{1cm} (D4)

so Eq. (D2) can be rewritten as

$$W_{\text{out}}(q, p) = \int du \, dv \, \frac{\exp\left(-\frac{1}{2} (u, v) \tilde{\Gamma}^{-1} \left(\begin{array}{c} u \\ v \end{array}\right)\right)}{2\pi \sqrt{\text{det}(\tilde{\Gamma})}} W_{\text{in}}(q - u, p - v)$$

$$= \left[W_{\text{in}} * G_{0, \tilde{\Gamma}}\right](q, p)$$  \hspace{1cm} (D5)

with $G_{0, \tilde{\Gamma}}$ a Gaussian function with zero mean and covariance matrix $\tilde{\Gamma}$.

**2. Output fidelity with coherent input state**

Since the protocol only involves Gaussian (squeezed) ancillary states, Gaussian operations (passive interferometers, squeezers) and Gaussian measurement (homodyne) the ensemble of encoding and decoding can be collectively described as a Gaussian channel acting as

$$\rho_{\text{in}} \mapsto \rho_{\text{out}}.$$  \hspace{1cm} (D6)

We focus here on the case in which the input states are coherent states. Since they are also Gaussian, they are fully specified by the first and second moments of the quadratures’ probability distributions and the action of a Gaussian channel can be described as [30]

$$\begin{cases} 
\xi \mapsto T\xi + d \\
\Gamma \mapsto \Gamma T \Gamma^T + N 
\end{cases}$$  \hspace{1cm} (D7)

where $d \in \mathbb{R}^{2k}$, $T$ and $N = N^T$ are $2k \times 2k$ real matrices such that $N + iJ^{(2k)} - iT J^{(2k)} T^T \geq 0$ and $\Gamma$ is the covariance matrix of the input state

$$\Gamma_{jk} = \langle\{\xi_j, \xi_k\}\rangle.$$  \hspace{1cm} (D8)

For single-mode coherent states the covariance matrix is proportional to the $2 \times 2$ identity matrix $\Gamma = I/2$.

Let us focus on a single access party $A$. By construction, the quadratures of the reconstructed mode are related to the secret quadratures by Eq. (D1) (Eq. (10) in the main text). We directly see that $T = I$ and $d = 0$. In order to characterize the channel defined by decoding and reconstruction by $A$ we just need to find $N$. This is easily accomplished remembering that the input squeezed and secret modes are not correlated, so that

$$\langle p_j^{\text{sup}}, p_k^{\text{sup}} \rangle = \langle q_j^{\text{sup}}, q_k^{\text{sup}} \rangle = \langle q_j, q_k \rangle = 0$$  \hspace{1cm} (D9)

for any $k$ and $j \neq k$. We can thus compute

$$\Delta^2 q_{\text{out}} = \Delta^2 q + \sum_k B_{1k}^2 \Delta^2 p_k^{\text{sup}}$$

$$\frac{1}{2} \langle\{q_{\text{out}}, p_{\text{out}}\}\rangle = \frac{1}{2} \sum_k B_{1k} B_{2k} \Delta^2 p_k^{\text{sup}} + \frac{1}{2} \langle\{q_s, q_s\}\rangle.$$  \hspace{1cm} (D10)
So, denoting $\Delta^2 = \text{Diag} \left( \Delta^2 p_{\text{sqz}}^1, \ldots, \Delta^2 p_{\text{sqz}}^{2k-1} \right)$ we have

$$N = B \Delta^2 B^T. \quad (D11)$$

For the rest of the calculation, we restrict for simplicity to the where all the modes are squeezed by the same parameter $r$, so that

$$N = N (r) = \frac{e^{-2r}}{2} BB^T. \quad (D12)$$

To compute the Fidelity $\mathcal{F} (\alpha, r)$ as a function of the squeezing parameter for an arbitrary input coherent state $|\alpha\rangle$ we use the fact that for a pure input state, the fidelity reduces to a trace, which is just an overlap integral, in our case between two Gaussian functions, in the Wigner function formalism $^{22}$

$$\mathcal{F} (\alpha, r) = \langle \alpha | \rho^{\text{out}} (r) | \alpha \rangle = 2\pi \int dq \; dp \; W_\alpha (q, p) W^{(r)}_{\text{out}} (q, p). \quad (D13)$$

The Wigner functions of the two states are given by

$$W_\alpha (q, p) = \frac{1}{\pi} \exp \left\{ - (\xi - \xi_0)^T (\xi - \xi_0) \right\} \quad (D14)$$

$$W^{(r)}_{\text{out}} (q, p) = \frac{\det (I + 2N (r))^{-\frac{1}{2}}}{\pi} \exp \left\{ - (\xi - \xi_0)^T (I + 2N (r))^{-1} (\xi - \xi_0) \right\} \quad (D15)$$

so that $\mathcal{F} (\alpha, r)$ reduces to the Gaussian integral

$$\mathcal{F} (\alpha, r) = \frac{2}{\pi} \det (I + 2N (r))^{-\frac{1}{2}} \int d^2 \xi \exp \left\{ - \xi^T \left[ I + (I + 2N (r))^{-1} \right] \xi \right\} \quad (D16)$$

where we used the fact that the integral does not change with the change of variable $\xi \rightarrow \xi + \xi_0$. Standard integration techniques give

$$\mathcal{F} (\alpha, r) = \frac{2}{\pi} 2\pi [\det (I + 2N (r))]^{-\frac{1}{2}} \{ \det [2 (I + (I + 2N (r)))] \}^{-\frac{1}{2}} \quad (D17)$$

$$= 4 \{ \det (I + 2N (r)) \} \det [(I + (I + 2N (r)))^{-1}]^{-\frac{1}{2}} \quad (D18)$$

$$= 2\det (I + 2N (r) + I)^{-\frac{1}{2}} = \frac{1}{\sqrt{\det (I + N (r))}} \quad (D19)$$

where we used the fact that for a real number $x$ and an $l \times l$ matrix $M$ one has $\det (xM) = x^l \det (xM)$ and Binet’s formula to go from the second to the third line.

Now, by construction $N (r) = N (r)^T \geq 0$ and $N (r) \rightarrow 0$ for $r \rightarrow \infty$, so $\mathcal{F} (\alpha, r) \rightarrow 1$ for $r \rightarrow \infty$. Moreover, we can derive the simple expression of Eq. (18) of the main text by noting that there always exist an orthogonal matrix $O$ such that $OBB^TO^T = \text{diag} (\mu, \nu)$ and since the determinant is invariant under orthogonal transformations (as well as the trace) we have after some simple algebra

$$\det (I + N (r)) = 4 + 2e^{-2r} (\mu + \nu) + e^{-4r} \mu \nu = 4 + 2e^{-2r} \text{Tr} (BB^T) + e^{-4r} \det (BB^T) \quad (D20)$$

which plugged into Eq. (D19) gives the desired expression.

**Appendix E: Proof that the Haar measure of $\mathcal{B}$ is zero**

We outline here a proof of the fact that the set $\mathcal{B}$ of matrices that cannot be used for secret sharing has zero Haar measure. We first note that integration with respect to the Haar measure of a function defined on $U (n)$ can be written as an ordinary integral over some real variables. We then recall a parametrization of $U (n)$ providing a realization of said variables. Finally, we conclude the proof linking the decodability conditions to the zero set of real analytic functions.
1. Haar measure in terms of real variables

Although the treatment could apply to more general situations, let us consider directly the case of $U(n)$. Since the unitary group is a Lie group of dimension $n^2$, we can find an atlas, that is, a family of pairs $\{(V_j, \gamma_j)\}$ such that the open sets $V_j \subseteq U(n)$ cover $U(n)$ and each map $\gamma : V_j \rightarrow \mathbb{R}^{n^2}$ is a homeomorphism. For any function $f$ defined on $U(n)$ we can define $g$ on $E = \bigcup \gamma(V_j) \subseteq \mathbb{R}^{n^2}$ as

$$g(\lambda) = f(\gamma^{-1}(\lambda))$$

(E1)

for any $\lambda \in E$. Using the theorem of change of variable, we can then find real valued functions $\Delta_i(\lambda)$ such that we can write any integral with respect to the Haar measure, which we denote by $d\mu$, as

$$\int_{V_i} f(\alpha) d\mu^H(\alpha) = \int_{\gamma_i(V_i)} f(\gamma^{-1}(\lambda)) \Delta_i(\lambda) d^{n^2}\lambda.$$  
(E2)

The integral over the whole unitary group can be defined appropriately glueing together the charts $\{(V_j, \gamma_j)\}$ [26].

2. Parametrization of $U(n)$

Instead of an atlas, we consider here a single chart which covers almost all if $U(n)$ (we will not prove this). This is sufficient for our goals.

In particular, we will consider the parametrization in terms of Euler angles that was used in [36] to numerically generate Haar distributed unitary matrices. It relies on the fact that any unitary matrix $\alpha \in U(n)$ can be obtained as the composition of rotations in two-dimensional subspaces. Each elementary rotation is represented by a $n \times n$ matrix $E^{(j,k)}$ whose entries are all zero except for

$$E_{il}^{(j,k)} = 1 \quad \text{for} \quad l = 1, 2, \ldots, n-1 \quad l \neq j, k$$
$$E_{ij}^{(j,k)} = \cos(\phi_{jk}) e^{i\psi_{jk}}$$
$$E_{jk}^{(j,k)} = \sin(\phi_{jk}) e^{i\chi_{jk}}$$
$$E_{jk}^{(j,k)} = -\sin(\phi_{jk}) e^{-i\psi_{jk}}$$
$$E_{kk}^{(j,k)} = \cos(\phi_{jk}) e^{-i\psi_{jk}}$$

(E3)

From these elementary rotations one can construct the $n-1$ composite rotations

$$E_1 = E^{(1,2)}(\phi_{12}, \psi_{12}, \chi_1)$$
$$E_2 = E^{(2,3)}(\phi_{23}, \psi_{23}, 0) E^{(1,3)}(\phi_{13}, \psi_{13}, \chi_2)$$
$$E_3 = E^{(3,4)}(\phi_{34}, \psi_{34}, 0) E^{(2,4)}(\phi_{24}, \psi_{24}, 0)$$
$$\quad \times E^{(1,4)}(\phi_{14}, \psi_{14}, \chi_3)$$

(E4)

and finally any matrix $\alpha \in U(n)$ can be written as

$$\alpha = e^{i\eta} E_1 E_2 \ldots E_{n-1}.$$  
(E5)

This can be seen as a function that takes $n^2$ angles

$$\{\phi_{jk} \text{ for } 1 \leq j < k \leq n\}, \{\psi_{jk} \text{ for } 1 \leq j < k \leq n\}, \{\chi_l \text{ for } 1 \leq l < n\}, \eta$$

(E6)

and outputs a $n \times n$ unitary matrix. The function is defined in the region $E \subset \mathbb{R}^{n^2}$ such that

$$0 \leq \phi_{jk} < \frac{\pi}{2}, \quad 0 \leq \psi_{jk} < 2\pi, \quad 0 \leq \psi_{jk} < 2\pi, \quad 0 \leq \chi_l < 2\pi.$$  
(E7)
In summary we defined a map $\gamma^{-1} : \mathcal{E} \to V \subset U(n)$ which is one-to-one and whose image is the whole $U(n)$. In practice, given any $\lambda \in \mathcal{E}$ we can construct the matrix $\alpha = \gamma^{-1}(\lambda)$. So for any function $f : U(n) \to \mathbb{R}$ we can define $g : \mathbb{R}^{n^2} \to \mathbb{R}$ such that $g(\lambda) = f(\gamma^{-1}(\lambda))$. If $f$ is measurable with respect to the Haar measure, we can write

$$\int_{U(n)} f(\alpha) \, d\mu^H(\alpha) = \int_V f(\alpha) \, d\mu^H(\alpha)$$

$$= \int_{\mathcal{E}} f(\gamma^{-1}(\lambda)) \, \Delta(\lambda) \, d^{n^2} \lambda$$

with

$$\Delta(\lambda) = \frac{1}{\prod_{k-1}^{n} \text{Vol}(S^{2k-1})} \left( \prod_{1 \leq j < k \leq n} \sin^{2j-1}(\phi_{jk}) \right)$$

(E9)

where $\text{Vol}(S^{2k-1})$ is the hypersurface of the $2k - 1$ dimensional sphere in $2k$ dimensions [37], and

$$d^{n^2} \lambda = \left( \prod_{1 \leq j < k \leq n} d\phi_{jk} \right) \left( \prod_{1 \leq j < k \leq n} d\psi_{jk} \right) \left( \prod_{1 \leq i < n} \chi_l \right) \, d\eta.$$  

(E10)

The normalization included in the function $\Delta$ ensures that

$$\int_V d\mu^H(\alpha) = \int_{\mathcal{E}} \Delta(\lambda) \, d^{n^2} \lambda = 1.$$  

(E11)

Now, since $0 \leq \Delta(\lambda) \leq 1 \forall \lambda \in \mathcal{E}$ we have

$$\int_{U(n)} f(\alpha) \, d\mu^H(\alpha) = \int_{\mathcal{E}} f(\gamma^{-1}(\lambda)) \, \Delta(\lambda) \, d^{n^2} \lambda$$

$$\leq \int_{\mathcal{E}} f(\gamma^{-1}(\lambda)) \, d^{n^2} \lambda.$$  

(E12)

What we want to prove is that the integral of the indicator function $I_B$ of $B$

$$I_B(\alpha) = \begin{cases} 1 & \alpha \in B \\ 0 & \alpha \notin B \end{cases}$$

(E13)

over $U(n)$ with respect to the Haar measure is equal to zero. This will be achieved if we manage to prove that

$$\int_{\mathcal{E}} I_B(\gamma^{-1}(\lambda)) \, d^{n^2} x = 0$$  

(E14)

which is equivalent to

$$\int_{\gamma(B)} d^{n^2} \lambda = 0$$  

(E15)

namely that the image of $B$ under $\gamma$ has zero measure in $\mathcal{E}$. This is proven in the next section leveraging the fact that through $\gamma^{-1}$ the coefficients of any unitary matrix are written as real analytic functions of the angles.

### 3. Real analytic functions

Our main result then follows from the observation that $B$ is the union of the zero sets of real analytic functions. Real analytic functions are defined analogously to their complex counterpart as functions defined in some open set of $\mathbb{R}^N$ that can be written as the sum of a power series [27]. As in the complex case, a real analytic function is either identically zero, or its zero set has zero measure [27, 28] (See also [29] for a self-contained proof).
The parametrization of unitary matrices introduced in the previous subsection gives the coefficients of any unitary matrix as a product of trigonometric functions and complex exponentials of the angles. The coefficients of any symplectic orthogonal matrix are real or imaginary part of a unitary matrix, so they are trigonometric functions of the angles. As it is well known, sine and cosine can always be written as power series. Since the set of real analytic functions $\mathcal{F}$ is closed under linear combinations with real coefficients and point-wise multiplication \[38\], the coefficients $Y_{nl}(\lambda)$ are real analytic functions defined on $\mathcal{E}$. It follows that $\gamma^{-1}(\bar{B})$ has zero Lebesgue measure on $\mathcal{E}$ and thus $\bar{B}$ has zero Haar measure in $U(n)$.

$\mathcal{F}$ is also closed under quotient as long as the denominator is not equal to zero \[39\]. As a consequence

$$\det (M \mid h_q^A \mid h_p^A)$$  \(\text{(E16)}\)

defines a real analytic function of the angles in $\mathcal{E} \setminus \gamma^{-1}(\bar{B})$, where there is at least one $l$ such that $Y_{nl} \neq 0$ and we can define $M^A, h_q^A$ and $h_p^A$. As for $\bar{B}$, this implies that the Haar measure of each $B^A$ is zero, and thus the Haar measure of $\bar{B}$ is also zero. This concludes the proof.