A New Hybrid Algorithm of Bisection and Modified Newton’s Method for the nth root-finding of a Real Number

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Abstract. This paper presents a new algorithm to approximate the nth root of a given real number \( z \). The proposed algorithm is a hybrid algorithm between the Bisection method and the combination of the inverse of sine series and Newton’s method. The proposed algorithm will be tested to find the numerical results on Matlab programming. The results showed that if \( 1 \leq z \), the proposed method converges with the initial interval \([1, z]\) and if \(0 < z < 1\), it converges with initial interval \([0, 4]\). The comparison of the obtained result with Newton’s method and AS-Newton’s method shown that the efficiency of the proposed algorithm may better than both of them concerning the initial value.

1. Introduction
The root-finding problem is one of the most important computational problems and applications in physics, chemistry, engineering. The root-finding problem is to find a solution \( f(x) = 0 \); the solution is called the root of the equation. The bracketing method and open method are two major classes of methods for finding roots. They are available are distinguished by the type of initial guess. The Bracketing method is based on two initial guesses that bracket the root, such as Bisection, False position method. The open method can involve one or more initial guesses, but there is no need for them to bracket root; Newton-Rapson, Secant method. The bracketing methods always work but converges slowly. In contrast, the open method can diverge, but they usually converge quicker. Because of this, it is often used the bracketing method to obtain a rough approximation to a solution, which used as a starting point for more rapidly converging methods.

The bisection method is the simples root-finding algorithm, which based on the intermediate value theorem (IVT). Suppose that \( f \in C[a, b] \) and \( f(a)f(b) < 0 \). By the IVT, there exists a number \( x_0 \in [a, b] \) such that \( f(x_0) = 0 \). The method consists of repeatedly bisecting the interval defined by these values and then selecting the subinterval in which the function changes sign, and therefore must contain a root.

Newton’s method is an open method based on Taylor’s polynomials and used to improve the root obtained by one of the bracketing methods. Newton’s method used the concept of a tangent at the initial approximate point \( x_0 \). The next approximate roots are taken those values where tangent intersects the
x-axis. Suppose that \( f \in C^2[a,b] \). Let \( x_0 \in [a,b] \) be an approximation to \( x \) such that \( f'(x_0) \neq 0 \) and \( |x - x_0| \) is small. Consider the Taylor polynomial for \( f(x) \) expand about \( x_0 \) and evaluated at \( x = x_1 \)

\[
 f(x_i) = f(x_0) + (x_i - x_0)f'(x_0) + \frac{1}{2}(x_i - x_0)^2 f''(\xi(x_i))
\]

where \( \xi(x_i) \) lies between \( x_i \) and \( x \). Since \( f(x_i) = 0 \),

\[
 0 = f(x_0) + (x_i - x_0)f'(x_0) + \frac{1}{2}(x_i - x_0)^2 f''(\xi(x_i))
\]

then

\[
 x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} - \frac{(x_i - x_0)^2}{2f'(x_0)} f''(\xi(x_i))
\]

Since \( |x_i - x_0| \) is small, then (3) \( x_i \approx x_0 - \frac{f(x_0)}{f'(x_0)} \). The process is repeated as

\[
 x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} , \quad n = 0,1,2,...
\]

Where \( f'(x_n) \neq 0 \).

For Newton’s method, the most noticeable disadvantages need to find the formula for \( f'(x) \) and may not guaranteed root. There are many researchers have improved the algorithm to solve the disadvantages of Newton’s method. For example, H.H.H. Homeier [1] suggested a modified Newton’s method with cubic convergence. Ehiwario, J.C. and Aghamie, S.O. [2] compare the rate of performance, the rate of convergence of Bisection method, Newton’s method and the Secant method of root-finding. Hussein, Altaee, and Hoomod [3] suggested a hybrid algorithm to bisection method with Newton’s method. Also, Hussein, Altaee, and Hoomod [4] investigate a parallel hybrid algorithm to improve the simultaneous root-find algorithm. J.Kim and et.al.[5] suggested an improved Hybrid Algorithm to the bisection Method and Newton-Raphson Method that reliable than the hybrid algorithm in [6] and more efficient than the bisection method. S. Thota [7] presented a new algorithm based on the exponential series and in which Secant method is special case. The proposed algorithm produces better approximate root than the bisection method, Regula-Falsi method, Newton’s method, and the Secant method. V.K. Srivastav, S.Thota, and M Kumar [8] suggested a new algorithm by modified Newton’s method. It based on the combination of the inverse of sine series and Newton’s method names as AS-Newton’s method. The iterative formula proposed as

\[
 x_{n+1} = x_n \left[ 1 + \sin^{-1} \left( \frac{f(x_n)}{x_n f'(x_n)} \right) \right]
\]

where \( f'(x_n) \neq 0, n = 0,1,2,... \). The iteration formula in (5) produces a sequence \( \{x_n\} \) with quadratically convergent. This algorithm is having better convergence than Newton’s method.

Since the root-bracketing techniques such as the Bisection method can be used as a starter method for the open method, then we introduce a new algorithm to find the solution of problems such that based on the Bisection method with the combination of the inverse of sine series and Newton’s method. Hence, a new method is proposed names as the improve hybrid method. This proposed method will be used to find the \( n^{th} \) root of a real number and compared it with Newton’s method and AS-Newton’s method based on number iterations and accuracy. The rest of the paper is as follows: The \( n^{th} \) root of a real number problem in section 2. The new hybrid algorithm in section 3 describes the proposed method, calculation steps, and flow chart and Section 4 shows the numerical results for finding the \( n^{th} \) root of a positive real number and discuss to illustrate the proposed algorithm and comparisons with Newton’s method and AS-Newton’s method are made to show efficiency of the proposed algorithm.
2. The $n^{th}$ root of real number problem

In mathematics, a $n^{th}$ root of a real number $z$ is a number $x$ when $x^n = z$, where $n$ is usually assumed to be a positive integer, and we call $n$ is the degree of the root. The $n^{th}$ root of $x$ are usually denoted with $\sqrt[n]{x}$. In case $z = 0$ the $n^{th}$ root of $z$ is zero for all $n$. In particular, the $n^{th}$ root of $z$ being considered as following:

- If $n$ is even and $z \in \mathbb{R}^+$ then $x$ is a real number and one is positive, one is negative, and the other are complex number.
- If $n$ is even and $z \in \mathbb{R}^-$ then there are none of $n^{th}$ root is a real number.
- If $n$ is odd and $z \in \mathbb{R}$ then one is a real number and the same sign as $x$ and the other roots are a complex number.

For every the $n^{th}$ root of $z$ which a positive real number will call the principal $n^{th}$ root. The $n^{th}$ root of a real number problem will be considered into two cases as followed [9]:

Let $n$ be a positive integer and $z \in \mathbb{R}^+$

Case 1 $x = \sqrt[n]{z}$ then $x^2 = z$ that $y$ is the solution of $x^2 - z = 0$

Case 2 $x = \sqrt[n]{z}$ Let $r$ be a remainder when four is divided by $n$. we have

$$\sqrt[n]{z} = \begin{cases} 
-\sqrt[n]{z}, & r = 1, 3 \\
 i\sqrt[n]{z}, & r = 2 \\
 \sqrt[n]{-1} \sqrt[n]{z}, & r = 0, 4 
\end{cases}$$

3. The new a hybrid algorithm

This method is used the root-bracketing technique as a starter method for the open method. A new algorithm is a new approach for finding a root of the given equations by proposing hybrid algorithm between the bisection algorithm and Modified Newton’s algorithm. The modified Newton’s algorithm suggested in [7] named as AS-Newton’s method such that based on the combination of inverse of sine series and Newton’s method.

For finding the $n^{th}$ root of positive real number $z$ is also a root of $x^n - z = 0$. The proposed algorithm as follows:

Given $f(x) = x^n - z, [a, b], j = 0$, tolerance $\delta = 10^{-5}$; and maximum number of iterations $m = 1000$

Step 1 for $i = 1$ to $2$

Step 2 calculate $x_i = \frac{(a + b)}{2}$ and $c = f(a)f(x_i)$

Step 3 if $f(x_i) = 0$ then $x_i$ is the $n^{th}$ root of $z$

Step 4 if $c < 0$ then $b = x_i$ else $a = x_i$ and go to step 2

Step 5 end for

Step 6 let $x_1 = x_2$

Step 7 Calculate $x = x_1 \left[ 1 + \sin^{-1} \left( \frac{x^n - z}{nx^{n+1}} \right) \right], j = j + 1$ and $E_r = \frac{|x - x_j|}{x}$%

Step 8 if $j > m$ then the iteration diverges.

Step 9 if $f(x) = 0$ or $E_r < \delta$ then $x$ is the $n^{th}$ root of $z$

Step 10 else $x_i = x$ and go to step 7

Step 11 stop iteration.
Figure 1. Flowchart of the proposed algorithm

The flowchart of the proposed algorithm shows in Fig.1. It takes a first approximation by applied two times the bisection method and complete approximation by use AS-Newton’s method.

4. Research Result

We implement the numerical algorithm on Dell i5 core computer using Matlab version 7.5.0342 (R2007b) 64 bit. The $n^{th}$ root of a positive real number $z$ is also a solution of $x^n - z = 0$. In this paper, we consider the efficiency of the proposed method by the number of iterations and compare with Newton’s method and AS-Newton’s method with tolerance $10^{-5}$.

| No | (n,z) | Newton’s Method with $x_0 = z$ | AS-Newton’s method with $x_0 = z$ | The proposed method with $[a,b]=[1,z]$ |
|----|------|-----------------|-----------------|-----------------|
|    |      | $x$  | $i$  | $x$  | $i$  | $x$  | $i$  |
| 1  | (3,6) | 1.817120593 | 8   | 1.817120593 | 8   | 1.8171205928 | 5   |
| 2  | (7,27) | 1.601328886 | 23  | 1.601328886 | 23  | 1.6013288856 | 15  |
| 3  | (3,728) | 8.995882891 | 16  | 8.995882891 | 16  | 8.9958828906 | 12  |
| 4  | (6,142) | 2.28409795 | 27  | 2.28409795 | 27  | 2.2840979502 | 20  |
| 5  | (20,4.82) | 1.081813388 | 34  | 1.081813388 | 34  | 1.0818133884 | 16  |
| 6  | (20,5.25) | 1.086445549 | 35  | 1.086445549 | 35  | 1.0864455494 | 17  |
| 7  | (16,3.75) | 1.086117858 | 24  | 1.086117858 | 24  | 1.0861178575 | 12  |
| 8  | (4,1250) | 10.57476985 | 29  | 10.57476985 | 29  | 10.5747698471 | 24  |
| 9  | (7,711) | 2.555112008 | 41  | 2.555112008 | 41  | 2.5551120079 | 32  |
| 10 | (12,250) | 1.584266847 | 63  | 1.584266847 | 63  | 1.5842668469 | 47  |
The following Table 1 shows the $n^{th}$ root of a positive real number $z$ when $z > 1$ and the iteration obtained for Newton’s method, AS-Newton’s method with the initial value $x_0 = z$, and the proposed method with the initial interval $[a, b] = [1, z]$. Form this results in 10 examples, all of methods were convergent, and the proposed method was better than Newton’s method and AS-Newton’s method concerning a number of iterations.

Table 2 The $n^{th}$ root of $z$ $(0 < z < 1)$ and the number of iteration (i) by a different method

| No  | (n,z)     | Newton’s Method with $x_0 = z$ | AS-Newton’s method with $x_0 = z$ | The proposed method with $[a, b] = [0, 4]$ |
|-----|-----------|-------------------------------|---------------------------------|---------------------------------------|
|     |           | $x$  | $i$  | $x$  | $i$  | $x$  | $i$  |
| 1   | (9,0.999) | 0.9998888395 | 3   | 0.9998888395 | 3   | 0.9998888395 | 2   |
| 2   | (7,0.9234)| 0.9886798790 | 5   | 0.9886798790 | 5   | 0.9886798790 | 4   |
| 3   | (4,0.25) | 0.7071067812 | 12  | Failure      | -    | 0.7071067812 | 6   |
| 4   | (12,0.015)| 0.7047054089 | 464 | Failure      | -    | 0.7047054089 | 9   |
| 5   | (15,0.074)| 0.8406504465 | 460 | Failure      | -    | 0.8406504465 | 7   |
| 6   | (3,0.0085)| 0.2040827551 | 19  | Failure      | -    | 0.2040827551 | 9   |
| 7   | (5,0.195)| 0.7211189731 | 22  | Failure      | -    | 0.7211189731 | 6   |
| 8   | (102,0.525)| divergent | -    | Failure      | -    | 0.9937026861 | 5   |
| 9   | (56,0.723)| divergent | -    | Failure      | -    | 0.9942248468 | 4   |
| 10  | (400,0.72)| divergent | -    | divergent   | -    | 0.9991790770 | 4   |

The $n^{th}$ root of a positive real number $z$ when $0 < z < 1$ and the iteration obtained for Newton’s method, AS-Newton’s method with the initial value $x_0 = z$, and the proposed method with initial interval $[a, b] = [0, 4]$ shows in Table 2. Form this results, the propose method was convergent but Newton’s method and AS-Newton' method diverged in some case. Some cases had failure in AS-method.

Table 3 The $n^{th}$ root of $z$ $(z > 1)$ and the number of iteration (i) by a different method with $x_0 = 1$

| No. | (n,z)     | Newton’s Method with $x_0 = 1$ | AS-Newton’s method with $x_0 = 1$ | The proposed method with $[a, b] = [1, z]$ |
|-----|-----------|-------------------------------|---------------------------------|---------------------------------------|
|     |           | $x$  | $i$  | $x$  | $i$  | $x$  | $i$  |
| 1   | (3,6)     | 1.817120593 | 6   | Failure | -    | 1.817120592 | 5   |
| 2   | (7,27)    | 1.601328886 | 13  | Failure | -    | 1.6013288856 | 15  |
| 3   | (3,728)   | 8.995882891 | 14  | Failure | -    | 8.9958828906 | 12  |
| 4   | (6,142)   | 2.28409795  | 36  | Failure | -    | 2.2840979502 | 20  |
| 5   | (4,1250)  | 10.57476985 | 25  | Failure | -    | 10.5747698471 | 24  |
| 6   | (7,711)   | 2.55512008  | 30  | Failure | -    | 2.555120079 | 32  |
| 7   | (12,250)  | 1.584266847 | 8   | Failure | -    | 1.5842668469 | 47  |
| 8   | (20,4.82)| 1.081813388 | 7   | 1.081813388 | 8   | 1.0818133884 | 16  |
| 9   | (20,5.25)| 1.086445549 | 8   | 1.086445549 | 8   | 1.0864455494 | 17  |
| 10  | (16,3.75)| 1.086117858 | 7   | 1.086117858 | 7   | 1.0861178575 | 12  |

Comparing the results in Table 3 from Newton’s method, AS-Newton’s method with the initial value $x_0 = 1$, and the proposed method with initial interval $[a, b] = [1, z]$, we found that the $n^{th}$ root of a positive real number $z$ when $z > 1$ by Newton’s method and the proposed method were convergent while AS-Newton’s method had failed in some case. The convergence of the proposed method is slower than Newton’s method.
5. Conclusion
The numerical result for finding the nth root of a positive real number \( z \), we conclude that if \( z > 1 \), the proposed method converges with the initial interval \([1, z]\) and if \( z < 1 \), it converges with the initial interval \([0, 4]\). Comparing Newton’s method and the AS-Newton’s method, the results show that both of them with the initial value \( x_0 = z \) is convergence and no difference in the number of iterative when \( z > 1 \) but diverge in some problem when \( 0 < z < 1 \). When the initial value \( x_0 = 1 \), AS-Newton’s methods are divergence or failure in some problems while Newton’s method converges.

Newton’s method and AS-Newton’s method converges to provide a sufficiently accurate initial value is chosen. For the nth root-finding of a positive real number problems, we found that Newton’s method with \( x_0 = 1 \) converged faster than \( x_0 = z \) and AS-Newton’s method with \( x_0 = z \) failed when \( z < n \). Moreover, the smallest tolerance may affect to diverge of iterative. Therefore, we need to consider an appropriate initial value to produce convergence.

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