A NOTE ON LARGE AUTOMORPHISM GROUPS
OF COMPACT RIEMANN SURFACES

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Abstract. Belolipetsky and Jones classified those compact Riemann surfaces of genus $g$ admitting a large group of automorphisms of order $\lambda(g-1)$, for each $\lambda > 6$, under the assumption that $g-1$ is a prime number. In this article we study the remaining large cases; namely, we classify Riemann surfaces admitting $5(g-1)$ and $6(g-1)$ automorphisms, with $g-1$ a prime number. As a consequence, we obtain the classification of Riemann surfaces admitting a group of automorphisms of order $3(g-1)$, with $g-1$ a prime number. We also provide isogeny decompositions of their Jacobian varieties.

1. Introduction and statement of the results

The classification of groups of automorphisms of compact Riemann surfaces is a stimulating subject of study, and has attracted a considerable interest since the nineteen century.

Let $S$ be a compact Riemann surface of genus $g \geq 2$. It is well-known that the full automorphism group of $S$ is finite, and that its order is bounded by $84(g-1)$.

A group of automorphisms $G$ of $S$ is said to be large if its order is strictly greater than $4(g-1)$; this bound arises naturally in the theory of Hurwitz spaces. In this case, it is known that $S$ is quasiplatonic (i.e. cannot be deformed non-trivially in the moduli space together with its automorphisms) or belong to a complex one-dimensional family. See [7, 11, 12, 24].

Compact Riemann surfaces with large groups of automorphisms have been considered from different points of view. For instance, the cyclic case was considered by Wiman [44], Kulkarni [24] and Singerman [42] (see also [19]), and the abelian case was classified by Lomuto in [26]. Riemann surfaces with $8(g+1)$ automorphisms were considered by Accola [1] and Maclachlan [27], and by Kulkarni in [23]. More recently, Riemann surfaces with $4g$ automorphisms were studied in [7] (see also [33]), and with $4(g+1)$ automorphisms in [12]. The maximal non-large case is considered in [34].

Belolipetsky and Jones [3] proved that under the assumption that $g-1$ is a prime number (sufficiently large for avoiding sporadic cases), a compact Riemann surface of genus $g$ admitting a large group of automorphisms of order $\lambda(g-1)$, where $\lambda > 6$, belongs to one of six infinite well-described sequences of Riemann surfaces.

In this article, we study and classify compact Riemann surfaces of genus $g \geq 8$ admitting a group of automorphisms of order $5(g-1)$ and $6(g-1)$, where $g-1$ a prime number; these cases were not considered in Belolipetsky-Jones’s article [3].

We also determine an isogeny decomposition of the corresponding Jacobian varieties.

The results of this paper are given in Theorem 1 and Theorem 2.

Theorem 1. Let $g \geq 8$ such that $g-1$ is prime. There exists a compact Riemann surface $S$ of genus $g$ with a group of automorphisms $G$ of order $5(g-1)$ if and only if $g \equiv 2 \mod 5$. Moreover, in this case:

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(1) the group $G$ is isomorphic to
$$C_{g-1} \rtimes_5 C_5 = \langle a, b : a^9 = b^5 = 1, bab^{-1} = a^r \rangle,$$
where $r$ is a 5-th primitive root of the unity in $\mathbb{F}_{g-1}$, and $G$ acts with signature $(0; 5, 5, 5)$,
(2) the action of $G$ extends to an action of a group $G'$ isomorphic to
$$C_{g-1} \rtimes_1 C_{10} = \langle a, c : a^9 = c^{10} = 1, cac^{-1} = a^{-r} \rangle,$$
with $r$ as before, and $G'$ acts with signature $(0; 2, 5, 10)$,
(3) there are exactly four pairwise non-isomorphic such Riemann surfaces $S$, and
(4) the Jacobian variety $JS$ of each $S$ decomposes, up to isogeny, as the product
$$JS \sim J(S/\langle a \rangle) \times (J(S/\langle c \rangle))^{10}.$$
Furthermore, possibly up to finitely many sporadic cases in small genera, the full automorphism group of $S$ is $G'$.

**Theorem 2.** Let $g \geq 8$ such that $g - 1$ is prime. There exists a compact Riemann surface of genus $g$ with a group of automorphisms of order $6(g - 1)$ if and only if $g \equiv 2 \mod 3$. Moreover, in this case:

(1) the Riemann surfaces form a closed one-dimensional equisymmetric family $\mathcal{F}_g$ of Riemann surfaces $S$ with a group of automorphisms $G$ isomorphic to
$$C_{g-1} \rtimes_6 C_6 = \langle a, c : a^9 = c^6 = 1, cac^{-1} = a^m \rangle,$$
where $m$ is a 6-th primitive root of the unity in $\mathbb{F}_{g-1}$, and $G$ acts with signature $(0; 2, 3, 3)$,
(2) the Jacobian variety $JS$ of each $S$ in $\mathcal{F}_g$ decomposes, up to isogeny, as the product
$$JS \sim J(S/\langle a \rangle) \times (J(S/\langle c \rangle))^{6},$$
(3) $\mathcal{F}_g$ contains two Riemann surfaces $X_1$ and $X_2$ with a group of automorphisms $G'$ of order $12(g - 1)$ isomorphic to $(C_{g-1} \rtimes_6 C_6) \cong C_2 = \langle c, z \rangle$ acting with signature $(0; 2, 3, 3)$, and
(4) the Jacobian variety $JX_i$ of each $X_i$ can be decomposed, up to isogeny, as
$$JX_i \sim J(X_i/\langle a \rangle) \times (J(X_i/\langle c, z \rangle))^{6}.$$
Furthermore, if $\mathcal{F}_g$ denotes the interior of $\mathcal{F}_g$ then:
(5) if $S \in \mathcal{F}_g$ then $G$ is the full automorphism group of $S$, and
(6) the boundary $\mathcal{F}_g \setminus \mathcal{F}_g$ of $\mathcal{F}_g$ is $\{X_1, X_2\}$, and the full automorphism group of $X_1$ and $X_2$ is $G'$.

As a consequence of the proof of the theorem above, we are able to easily derive a classification for the non-large case $\lambda = 3$.

**Corollary 1.** Let $g \geq 8$ such that $g - 1$ is prime. There exists a compact Riemann surface $S$ of genus $g$ with a group of automorphisms of order $3(g - 1)$ if and only if $g \equiv 2 \mod 3$. Furthermore, in this case $S$ belongs to the family $\mathcal{F}_g$ of Theorem 2. As a consequence, there is no compact Riemann surfaces of genus $g$ with full automorphism group of order $3(g - 1)$.

In Section 2 we will briefly review the background. The results will be proved in Sections 3, 4 and 5.

## 2. Preliminaries

### 2.1. Fuchsian groups.

Let $\mathbb{H}$ denote the upper-half plane, and let $\Gamma$ be a *cocompact Fuchsian group*; i.e. a discrete group of automorphisms of $\mathbb{H}$ with compact orbit space $\mathbb{H}/\Gamma$. The algebraic structure of $\Gamma$ is determined by its signature:
$$s(\Gamma) = (h; m_1, \ldots, m_l),$$
where $h$ denotes the topological genus of the surface $\mathbb{H}/\Gamma$, and $m_1, \ldots, m_l$ the branch indices in the (orbifold) universal covering $\mathbb{H} \to \mathbb{H}/\Gamma$. If $l = 0$, then $\Gamma$ is called a *surface Fuchsian group*.

Let $\Gamma$ be a Fuchsian group with signature (2.1). Then $\Gamma$ has a *canonical presentation* with generators $a_1, \ldots, a_h, b_1, \ldots, b_h, x_1, \ldots, x_l$ and relations
$$x_1m_1 = \cdots = x_l^{m_l} = \Pi_{i=1}^h a_i b_i a_i^{-1} b_i^{-1} \Pi_{i=1}^l = 1.$$  
(2.2)

The hyperbolic area of each fundamental region of $\Gamma$ is given by
\[ \mu(\Gamma) = 2\pi[2h - 2 + \sum_{j=1}^{l}(1 - \frac{1}{m_j})]. \]

Let \( \Gamma' \) be a group of automorphisms of \( \mathbb{H} \). If a Fuchsian group \( \Gamma \) is a finite index subgroup of \( \Gamma' \) then \( \Gamma' \) is also a Fuchsian group and their hyperbolic areas are related by the Riemann-Hurwitz formula
\[ \mu(\Gamma) = |\Gamma' : \Gamma| \cdot \mu(\Gamma'). \]

The complex dimension of the Teichmüller space associated to a Fuchsian group of signature \((2.1)\) is \( 3g - 3 + l \). See, for example, [15, 30, 41].

2.2. **Riemann surfaces and group actions.** Let \( S \) be a compact Riemann surface. We denote by \( \text{Aut}(S) \) the full automorphism group of \( S \), and say that a group \( G \) acts on \( S \) if there is a group monomorphism \( \psi : G \to \text{Aut}(S) \). The space of orbits \( S/G \) of the action of \( G \) induced by \( \psi(G) \) is naturally endowed with a Riemann surface structure such that the projection \( S \to S/G \) is holomorphic.

By the uniformization theorem, a Riemann surface \( S \) is conformally equivalent (isomorphic) to the quotient \( \mathbb{H}/\Gamma \), where \( \Gamma \) is a surface Fuchsian group. Lifting \( G \) to the universal covering \( \mathbb{H} \to \mathbb{H}/\Gamma \), the group \( G \) acts on \( S \) if and only if there is a Fuchsian group \( \Gamma' \) containing \( \Gamma \) and a group epimorphism
\[ \theta : \Gamma' \to G \] such that \( \ker(\theta) = \Gamma \)
(see [5, 15, 37, 41]). Such an epimorphism will be called a **surface epimorphism**. We say that the action of \( G \) on \( S \) is given or represented by the surface epimorphism \( \theta \). Note that the Riemann surface \( S/G \) is isomorphic to \( \mathbb{H}/\Gamma' \). We shall also say that \( G \) acts on \( S \) with signature \( s(\Gamma') \).

Let us assume that \( G \) is a subgroup of \( G_1 \). The action of \( G \) on \( S \cong \mathbb{H}/\Gamma \) is said to **extend** to an action of \( G_1 \) if and only if there is a Fuchsian group \( \Gamma'' \) containing \( \Gamma' \) together with a surface epimorphism
\[ \Theta : \Gamma'' \to G_1 \] in such a way that \( \Theta|_{\Gamma'} = \theta \), \( \ker(\Theta) = \ker(\theta) = \Gamma \), and \( \Gamma' \) and \( \Gamma'' \) have associated Teichmüller spaces of the same dimension. Singerman in [40] determined all those pairs of signatures \((s(\Gamma'), s(\Gamma''))\) for which it may be possible to extend actions. An action is called **maximal** if it cannot be extended in the aforementioned sense.

2.3. **Topologically equivalent actions.** Let \( S \) be a compact Riemann surfaces and let \( \text{Hom}^+(S) \) denote the group of orientation preserving homeomorphisms of \( S \). Two actions \( \psi_1 : G \to \text{Aut}(S) \) are said to be **topologically equivalent** if there exist \( \omega \in \text{Aut}(G) \) and \( h \in \text{Hom}^+(S) \) such that
\[ \psi_2(g) = h\psi_1(\omega(g))h^{-1} \quad \text{for all } g \in G. \] (2.3)

Note that topologically equivalent actions have the same signature. Each orientation preserving homeomorphism \( h \) satisfying (2.3) yields a group automorphism \( h^* \) of \( \Gamma' \) where \( \mathbb{H}/\Gamma' \cong S/G \). We shall denote the subgroup of \( \text{Aut}(\Gamma') \) consisting of the automorphisms \( h^* \) by \( \mathcal{B} \).

Two surface epimorphisms \( \theta_1, \theta_2 : \Gamma' \to G \) define topologically equivalent actions if and only there are \( \omega \in \text{Aut}(G) \) and \( h^* \in \mathcal{B} \) such that \( \theta_2 = \omega \circ \theta_1 \circ h^* \) (see [5, 17, 28]). We remark that if the genus of \( S/G \) is zero, then the group \( \mathcal{B} \) is generated by the **braid transformations** \( \Phi_{i,i+1} \in \text{Aut}(\Gamma') \) defined by:
\[ x_i \mapsto x_{i+1}, \quad x_{i+1} \mapsto x_{i+1}x_ix_i^{-1} \quad \text{and} \quad x_j \mapsto x_j \quad \text{when} \quad j \neq i, i+1 \]
for each \( i \in \{1, \ldots, l-1\} \). See, for example, [22, p. 31] and also [6, 20].

2.4. **Equisymmetric stratification.** Let \( \mathcal{M}_g \) denote the moduli space of compact Riemann surfaces of genus \( g \geq 2 \). It is well-known that \( \mathcal{M}_g \) is endowed with an orbifold structure and that its locus of orbifold-singular points, the so-called branch locus \( \mathcal{B}_g \), is formed by Riemann surfaces with non-trivial automorphisms for \( g \geq 3 \). For \( g = 2 \) the branch locus \( \mathcal{B}_2 \) consists of the Riemann surfaces admitting other automorphisms than the hyperelliptic involution. See, for example, [30].

It was proved in [6] that the branch locus \( \mathcal{B}_g \) admits an **equisymmetric stratification** \( \{\mathcal{M}_G^{G,\theta}\} \), where each equisymmetric stratum \( \mathcal{M}_G^{G,\theta} \), if non-empty, corresponds to one topological class of maximal actions. More precisely, \( \mathcal{B}_g \) can be written as
\[ \mathcal{B}_g = \cup_{G,\theta} \mathcal{M}_G^{G,\theta} \] (2.4)
where the closure $\bar{\mathcal{M}}_{g,\theta}$ of the stratum $\mathcal{M}_{g,\theta}$ consists of the Riemann surfaces of genus $g$ admitting an action of the group $G$ with fixed topological class given by $\theta$. We recall that $\mathcal{M}_{g,\theta}$ is a closed irreducible algebraic subvariety of $\mathcal{M}_g$. Observe that the union in (2.4) is taken over all possible actions of the non-trivial groups $G$ acting on a compact Riemann surface of genus $g$. See also [17].

In particular, in this work we shall use the following:

**Definition.** A closed family $\mathcal{F}$ of compact Riemann surfaces of genus $g$ whose members admit an action of a group $G$ will be called *equisymmetric* if its interior $\mathcal{F}$ consists of exactly one stratum.

2.5. **Decomposition of Jacobian varieties.** Let $S$ be a compact Riemann surface of genus $g \geq 2$. We denote by $JS$ the Jacobian variety (or simply the *Jacobians*) of $S$, and recall that $JS$ is an irreducible principally polarized abelian variety of dimension $g$. See [4].

The relevance of the Jacobian variety lies in the well-known Torelli’s theorem, which asserts that two compact Riemann surfaces are isomorphic if and only if their Jacobians are isomorphic as principally polarized abelian varieties.

If a finite group $G$ acts on $S$ then this action induces an isogeny decomposition

$$JS \sim J(S/G) \times A_2 \times \ldots \times A_r,$$

which is $G$-equivariant. The factors in (2.5) are in bijective correspondence with the rational irreducible representations of $G$; the factor $A_1 \sim J(S/G)$ is associated to the trivial representation (see [8, 25]).

The decomposition of Jacobians with group actions has been extensively studied; the simplest case was already noticed by Wirtinger in [45] and used by Schottky and Jung in [39]. For decompositions of Jacobians with respect to special groups, we refer to [9, 18, 20, 31, 32, 36].

Let $G$ be a finite group. For each complex representation $\rho : G \to \text{GL}(V)$ of $G$ we shall denote its *degree* by $d_V$; i.e. the dimension of $V$ as a complex vector space. If $H$ is a subgroup of $G$, then we shall denote the dimension of the vector subspace of $V$ fixed under the action $H$ by $d^H_V$. By abuse of notation, we shall write $V$ to refer to the representation $\rho$. See [38] for more details.

Let us assume that $G$ acts on a Riemann surface $S$ with signature (2.1), and that this action is determined by the surface epimorphism $\theta : \Gamma \to G$. Let $H_1, \ldots, H_i$ be groups of automorphisms of $S$ such that $G$ contains $H_i$ for each $i$. Following [35], the collection $\{H_1, \ldots, H_i\}$ is called *G-admissible* if

$$d^H_{V_1} + \ldots + d^H_{V_j} \leq d_V$$

for every complex irreducible representation $V$ of $G$ in $\mathfrak{Z}$, where the elements of $\mathfrak{Z}$ are characterized (by using [37, Theorem 5.12]) as follows:

1. the trivial representation belongs to $\mathfrak{Z}$ if and only if the genus of $S/G$ is different from zero, and
2. a non-trivial representation $V$ belongs to $\mathfrak{Z}$ if and only if $d_V(\gamma - 1) + \frac{1}{2} \sum_{i=1}^j (d_V - d^\theta_{V_i}(x_i)) \neq 0$,

where the $x_i$'s are canonical generators (2.2) of $\Gamma$.

The collection is called *admissible* if it is $G$-admissible for some group $G$. The main result of [35] ensures that if $\{H_1, \ldots, H_i\}$ is an admissible collection of groups of automorphism of a Riemann surface $S$ then

$$JS \sim \prod_{i=1}^j J(S/H_i) \times P$$

for some abelian subvariety $P$ of $JS$. See also [21].

**Notation.** Let $n \geq 2$ be an integer and let $q$ be a prime. Throughout this article we denote the cyclic group of order $n$ by $C_n$, the dihedral group of order $2n$ by $D_n$ and the field of $q$ elements by $\mathbb{F}_q$.

3. **Proof of Theorem 1**

Let $S$ be a compact Riemann surface of genus $g \geq 8$, where $q = g - 1$ is prime, and assume that $S$ has a group of automorphisms $G$ of order $5q$. By the Riemann-Hurwitz formula the signature of the action
of $G$ on $S$ is $(0; 5, 5, 5)$. By the classical Sylow’s theorems, if $q \not\equiv 1 \mod 5$ then $G$ is isomorphic to $C_{5q}$, and if $q \equiv 1 \mod 5$ then $G$ is isomorphic to either $C_{5q}$ or to $C_q \rtimes_5 C_5 = \langle a, b : a^q = b^5 = 1, bab^{-1} = a' \rangle$, where $r$ is a 5-th primitive root of the unity in $\mathbb{F}_q$. Note that since $C_{5q}$ cannot be generated by two elements of order five, if $q \not\equiv 1 \mod 5$, then there are no compact Riemann surfaces of genus $g$ with a group of automorphisms of order $5q$.

From now on we assume that $q \equiv 1 \mod 5$ and that $G \cong C_q \rtimes_5 C_5$.

Let $\Gamma$ be a Fuchsian group of signature $(0; 5, 5, 5)$ with canonical presentation

$$\Gamma = \langle x_1, x_2, x_3 : x_1^5 = x_2^5 = x_3^x = x_1 x_2 x_3 = 1 \rangle$$

and let $\theta : \Gamma \to G$ be a surface epimorphism representing the action of $G$ on $S$. We recall that $G$ has exactly four conjugacy classes of elements of order $5$; namely $\{a^l b^j : 1 \leq l \leq q \}$ for $j = 1, 2, 3, 4$.

If the epimorphism $\theta$ is defined by $\theta(x_1) = a^{l_1} b^{j_1}$, $\theta(x_2) = a^{l_2} b^{j_2}$ and $\theta(x_3) = a^{l_3} b^{j_3}$ where $l_1, l_2, l_3 \in \{1, \ldots, q\}$ and $i, j, k \in \{1, \ldots, 4\}$, then, after applying a suitable inner automorphism of $G$, we can assume $l_2 \equiv 0 \mod q$ and then $l_1 \equiv -r' l_2 \mod q$. As $l_2 \not\equiv 0 \mod q$ (otherwise $\theta$ is not surjective), we can consider the automorphism of $G$ given by $a \mapsto a^{l_2}$ and $b \mapsto b$, where $l_2 l_2 \equiv 1 \mod q$, to see that $\theta$ is equivalent to the epimorphism $\theta_{l,k}$ defined by

$$\theta_{l,k}(x_1) = a^{l-1} b^{k}, \quad \theta_{l,k}(x_2) = a b^{k} \quad \text{and} \quad \theta_{l,k}(x_3) = b^{k}.$$ 

Now, as the braid automorphisms act by permuting conjugacy classes of elements of $\Gamma$ and as $l + i + k \equiv 0 \mod 5$, there are at most four pairwise topologically non-equivalent actions of $G$ on $S$, represented by $\theta_1 = \theta_{1,2,2}$, $\theta_2 = \theta_{2,4,4}$, $\theta_3 = \theta_{1,1,3}$ and $\theta_4 = \theta_{3,3,4}$.

Following [40], the action given by each $\theta_n$ can be possibly extended to actions of signatures $(0; 3, 3, 5)$ and $(0; 2, 5, 10)$, and these actions, in turn, can be possibly extended to a maximal action of signature $(0; 2, 3, 10)$. Now, if an action of $G$ on $S$ extends to an action of signature $(0; 3, 3, 5)$ then $S$ would have 15$q$ automorphisms; however, as proved in [3], possibly up to finitely many sporadic cases in small genera this situation is not possible. Note that this fact also ensures that, possibly up to finitely many sporadic cases in small genera, none of the actions of $G$ extends to an action of signature $(0; 2, 3, 10)$.

Let us now consider a Fuchsian group $\Gamma_1$ of signature $(0; 2, 5, 10)$ with canonical presentation

$$\Gamma_1 = \langle y_1, y_2, y_3 : y_1^2 = y_2^5 = y_3^{10} = y_1 y_2 y_3 = 1 \rangle,$$

and a finite group group $G' = C_q \rtimes_10 C_{10}$ with presentation

$$\langle a, b, s : a^q = b^5 = s^2 = 1, bab^{-1} = a', sas = a^{-1}, [s, b] = 1 \rangle,$$

where $r$ is a 5-th primitive root of the unity in $\mathbb{F}_q$.

As proved in [3, Example (ii)] (see also [43, Theorem 3]), the surface epimorphisms $\Theta_n : \Gamma_1 \to G' \cong C_q \rtimes_10 C_{10}$ given by $\Theta_n(y_1) = a s$, $\Theta_n(y_2) = a b^{2n}$ and $\Theta_n(y_3) = b^{-2n} s$, $1 \leq n \leq 4$, define four pairwise non-isomorphic Riemann surfaces $X_1, \ldots, X_4$ of genus $g$ with full automorphism group $C_q \rtimes_10 C_{10}$.

Note that the subgroup of $\Gamma_1$ generated by $\bar{x}_1 = (y_1 y_3)^{-1}$, $\bar{x}_2 = y_2$ and $\bar{x}_3 = y_3^2$ is isomorphic to $G$, and that $\Theta_n(\bar{x}_1) = a^{-r^{2n}} b^{2n}$, $\Theta_n(\bar{x}_2) = a b^{2n}$ and $\Theta_n(\bar{x}_3) = b^{-4n}$. It follows that $\Theta_n|_{\Gamma_1} = \Theta_n$ for each $n \in \{1, 2, 3, 4\}$ and therefore each action of $G \cong C_q \rtimes_5 C_5$ on $S$ with signature $(0; 5, 5, 5)$ extends to an action of $G' \cong C_q \rtimes_10 C_{10}$ with signature $(0; 2, 5, 10)$; thus $S$ isomorphic to $X_i$ for some $i$. In particular, there does not exist a Riemann surface of genus $g$ with full automorphism group of order $5q$.

Finally, we decompose the Jacobian variety $JS$ of each $S$. If we set $c = bs$ and $m = -r$ then

$$\text{Aut}(S) \cong \langle a, c : a^q = c^{10} = 1, cac^{-1} = a^m \rangle = C_q \rtimes_10 C_{10}.$$ 

We shall use this presentation in the sequel. Set $\omega_i := \exp(\frac{2\pi i m}{10})$.

The group $C_q \rtimes_10 C_{10}$ has, up to equivalence, ten complex irreducible representations of degree 1, given by $U_i : a \mapsto 1, c \mapsto \omega_i^i$ for $0 \leq i \leq 9$.

Let $\alpha = \frac{2\pi i m}{10} \in \mathbb{N}$ and choose integers $k_1, \ldots, k_\alpha \in \{1, \ldots, q - 1\}$ in such a way that

$$\bigcup_{j=1}^\alpha \{k_j, k_j m, k_j m^2, \ldots, k_j m^9\} = \{1, \ldots, q - 1\},$$
where \( \sqcup \) stands for disjoint union. Then, the group \( C_q \times_{10} C_{10} \) has, up to equivalence, \( \alpha \) complex irreducible representations of degree 10, given by

\[
V_j : a \mapsto \text{diag}(\omega_q^{k_1}, \omega_q^{k_2}, \omega_q^{k_3}, \ldots, \omega_q^{k_{10}}), \quad c \mapsto \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix}
\]

for \( 1 \leq j \leq \alpha \).

Consider \( H = \langle a \rangle \) and \( H_t = \langle a^t, c \rangle \) for \( 1 \leq t \leq 10 \), and notice that \( d^{H_t}_U + \sum_{i=1}^{10} d^{H_t}_{V_i} = 1 = d_U \), and \( d^{H_t}_V + \sum_{i=1}^{10} d^{H_t}_{V_i} = 10 = d_{V_j} \) for each \( i \in \{1, \ldots, 9\} \) and for each \( j \in \{1, \ldots, \alpha\} \). Thereby, as explained in Subsection 2.5, the collection \( \{ H, H_1, \ldots, H_{10} \} \) is admissible and therefore, by [35], there is an abelian subvariety \( P \) of \( JS \) such that

\[
JS \sim J(S/H) \times \prod_{i=1}^{10} J(S/H_t) \times P \sim J(S/\langle a \rangle) \times (J(S/\langle c \rangle))^{10} \times P,
\]

where the second isogeny follows after noticing that, for each \( t \), the groups \( H_t \) and \( \langle c \rangle \) are conjugate.

Observe that the \( q \)-sheeted regular covering map \( S \rightarrow S/\langle a \rangle \) is unbranched, and that the regular covering map \( S \rightarrow S/\langle c \rangle \) ramifies over exactly three values, marked with 2, 5 and 10. Then, it follows from the Riemann-Hurwitz that the genera of \( S_1 \) or a 3-th primitive root of the unity in \( F \) is isomorphic to a semidirect product \( C_q \rtimes H \), where \( H \) is a group of order 6.

4. Proof of Theorem 2

Let \( S \) be a compact Riemann surface of genus \( g \geq 8 \), where \( q = g - 1 \) is prime, and assume that \( S \) has a group of automorphism \( G \) of order \( 6q \). By the Riemann-Hurwitz formula the possible signatures of the action of \( G \) on \( S \) are \( (0; 2, 2, 3, 3), (0; 2, 2, 3, 6) \) and \( (0; 3, 6, 6) \) for each genus and, in addition, the signature \( (0; 2, 7, 42) \) for \( g = 8 \).

First of all, the signature \( (0; 2, 7, 42) \) for \( g = 8 \) cannot be realized because there is no surface epimorphism from a Fuchsian group of signature \( (0; 2, 7, 42) \) to a (necessarily cyclic) group of order 42. In addition, by the classical Sylow’s theorems, \( G \) contains exactly one normal subgroup isomorphic to \( C_q \) and therefore \( G \) is isomorphic to a semidirect product \( C_q \rtimes H \), where \( H \) is a group of order 6.

Claim 1. \( H = C_6 \).

Let us assume that \( H = D_3 \) and therefore

\[
G \cong C_q \times D_3 = \langle a, b, s : a^9 = b^3 = s^2 = 1, (sb)^2 = 1, bab^{-1} = a^u, sas = a^v \rangle
\]

where \( u \) is either 1 or a 3-th primitive root of the unity in \( F_q \), and \( v = \pm 1 \).

1. If \( u = 1 \) and \( v = 1 \) then \( G \) is isomorphic to the direct product \( C_q \times D_3 \). However, as among every collection of generators of \( C_q \times D_3 \) there must be an element of order a multiple of \( q \), we see that there are no compact Riemann surfaces of genus \( g \) with a group of automorphisms isomorphic to \( C_q \times D_3 \) since the order of the generators of the Fuchsian groups are 2, 3 and 6.

2. If \( u = 1 \) and \( v = -1 \) then \( G \) is isomorphic to \( D_{3q} \) and therefore \( G \) has no elements of order 6. Moreover, the elements of order three are \( (ab)^q \) and \( (ab)^{2q} \), and the involutions are of the form \( s(ab)^l \) for \( 1 \leq l \leq 3q \). It can be checked that if the product of two involutions and two elements of order three is 1, then these elements generate \( D_6 \). All the above ensures that there are no Riemann surfaces of genus \( g \) with a group of automorphisms isomorphic to \( D_{3q} \).

3. Finally, if \( u \) is a 3-th primitive root of the unity in \( F_q \), then the equality \( (sb)a(sb)^{-1} = a^{2v} \) yields that the action of the involution \( sb \) on \( C_q \) has order three for \( v = 1 \) and order six for \( v = -1 \). This is not possible.

This proves Claim 1.

Thereby, \( G \cong C_q \rtimes C_6 = \langle a, b, s : a^9 = b^3 = s^2 = 1, [s, b] = 1, bab^{-1} = a^u, sas = a^v \rangle \) where \( u \) is either 1 or a 3-th primitive root of the unity in \( F_q \) and \( v = \pm 1 \).
Claim 2. $u$ is a 3-th primitive root of the unity in $\mathbb{F}_q$.
Assume $u=1$.

(1) If $v=1$, then $G \cong C_{6q}$ which is not generated by elements of order two and three. Thus, there
are no Riemann surfaces of genus $g$ with a group of automorphisms isomorphic to $C_{6q}$.

(2) If $v=-1$ then $G \cong C_q \rtimes_2 C_6$ where $C_6$ acts on $C_q$ with order two. The elements of order two
are of the form $a^l$, the elements of order six of the form $a^l b^s$ and $a^l b^2 s$ for $1 \leq l \leq q$, and the
elements of order three are $b$ and $b^2$. It can be seen that:
(a) $G$ cannot be generated by three elements, being two of them of order two and one of order
three, in such a way that their product has order three,
(b) the product of three elements of order two must have order two, and
(c) $G$ cannot be generated by two elements of order six whose product has order three.
All the above ensures that there are no Riemann surfaces of genus $g$ with a group of automor-
phisms isomorphic to $C_q \rtimes_2 C_6$.

This proves Claim 2.
Therefore, $G \cong C_q \times C_6$ with a presentation $(a, b, s : a^q = b^3 = s^2 = 1, [s, b] = 1, bab^{-1} = a^r, sas =
\theta)$, where $v = \pm 1$ and $r$ is a 3-th primitive root of the unity in $\mathbb{F}_q$. Consequently $g - 1 = q \equiv 1 \mod 3$.

We have two cases for the finite group $G$:

Case 1. If $v=1$ then $G$ is isomorphic to $C_q \times_3 C_6$ where $C_6$ acts on $C_q$ with order three. The ele-
ments of order 3 are of the form $a^l b$ and $a^l b^2$, the elements of order 6 of the form $a^l b s$ and $a^l b^2 s$ for $1 \leq l \leq q$, and $s$ is the unique element of order two.

Case 2. If $v=-1$ then $G$ is isomorphic to $C_q \times_6 C_6$ where $C_6$ acts on $C_q$ with order six. The elements
of order two are of the form $a^l s$, the elements of order three are of the form $a^l b$ and $a^l b^2$, and the
elements of order six of the form $a^l b s$ and $a^l b^2 s$ for $1 \leq l \leq q$.

We now study each possible signature separately.

Signature $(0; 2, 2, 2, 6)$. As in both groups $C_q \times_3 C_6$ and $C_q \times_6 C_6$ the product of three elements of
order two has order two, we see that there is no group of order 6$q$ acting on a Riemann surface of genus
$g$ with signature $(0; 2, 2, 2, 6)$. See also [12]

Signature $(0; 2, 2, 3, 3)$. We note that there are no compact Riemann surfaces of genus $g$ admitting
an action of $C_q \times_3 C_6$ with signature $(0; 2, 2, 3, 3)$; this follows from the fact that $s$ (which is the unique
involution) and an element of order three generate a group of order six.

By contrast, we show that there is a complex one-dimensional equisymmetric family $\mathcal{F}_q$ of Riemann
surfaces $S$ of genus $g$ with a group of automorphisms isomorphic to $C_q \times_6 C_6$ acting on $S$ with signature
$(0; 2, 2, 3, 3)$. Indeed, let $\Gamma_3$ be a Fuchsian group of signature $(0; 2, 2, 3, 3)$ with canonical presentation
\[ \Gamma_3 = \langle x_1, x_2, x_3, x_4 = x_1^2 = x_2^2 = x_3^2 = x_4^2 = x_1 x_2 x_3 x_4 = 1 \rangle. \]

Then the surface epimorphism $\theta_{3,0} : \Gamma_3 \to C_q \times_6 C_6$ defined by
\[ \theta_{3,0}(x_1) = s, \quad \theta_{3,0}(x_2) = as, \quad \theta_{3,0}(x_3) = ab^2 \quad \text{and} \quad \theta_{3,0}(x_4) = b, \]
provides the family $\mathcal{F}_q$ of Riemann surfaces admitting an action of $C_q \times_6 C_6$.

To prove that $\mathcal{F}_q$ is equisymmetric we notice that, up to a permutation of the generators of $\Gamma_3$, a
surface epimorphism $\theta_3 : \Gamma_3 \to C_q \times_6 C_6$ is of the form:
\[ \theta_3(x_1) = a^{l_1} s, \quad \theta_3(x_2) = a^{l_2} s, \quad \theta_3(x_3) = a^{l_3} b^2 \quad \text{and} \quad \theta_3(x_4) = a^{l_4} b, \]
for some $l_1, \ldots, l_4 \in \{1, \ldots, q\}$. Moreover, after applying a suitable automorphism of $G$ of the form
$a \mapsto a^u, b \mapsto a^v b$ we can suppose $l_1 \equiv 0 \mod q$ and $l_2 \equiv 1 \mod q$. Now, if we set $m = l_4$ then an
epimorphism $\theta_3$ is equivalent to one epimorphism $\theta_{3,m}$ given by
\[ \theta_{3,m}(x_1) = s, \quad \theta_{3,m}(x_2) = as, \quad \theta_{3,m}(x_3) = a^{l_3+(1+r)m} b^2, \quad \theta_{3,m}(x_4) = a^m b, \quad 1 \leq m \leq q \]
As $\Phi_{3,4}^n \theta_{3,m} = \theta_{3,2r+1+m}$, after iterating $\Phi_{3,4}$ a suitable number of times, we see that each epimorphism $\theta_{3,m}$ is equivalent to $\theta_{3,0}$, as desired.

We claim that the full automorphism group of a Riemann surface in the interior $\mathcal{F}_q$ of $\mathcal{F}_g$ is $G$. Indeed, otherwise by [40] the action would extend to an action of a group of order $12q$ of signature $(0; 2, 2, 3)$; however, this situation is not possible by [3, Theorem 2(a)] for $q \geq 19$ and by [10] for the remaining cases $q = 7$ and $q = 13$.

**Signature** $(0; 3, 6, 6)$. Let $\Gamma_1$ be a Fuchsian group of signature $(0; 3, 6, 6)$ and consider its canonical presentation

$$\Gamma_1 = \langle x_1, x_2, x_3 = x_1^3 = x_2^6 = x_3^3 \rangle = 1.\] Applying automorphisms of the finite group, we have that:

1. A surface epimorphism $\Gamma_1 \rightarrow C_q \times_3 C_6$ representing an action of $C_q \times_3 C_6$ on $S$ with signature $(0; 3, 6, 6)$ is equivalent to one defined by $\theta_1(x_{1,i}) = b^i$, $\theta_1(x_2) = a^{-r}b^s$ and $\theta_1(x_3) = a^i bs$ for $i = 1$ or $i = 2$.
2. A surface epimorphism $\Gamma_1 \rightarrow C_q \times_6 C_6$ representing an action of $C_q \times_6 C_6$ on $S$ with signature $(0; 3, 6, 6)$ is equivalent to the one defined by $\theta_2(x_1) = ab$, $\theta_2(x_2) = bs$ and $\theta_2(x_3) = a^{-r}bs$.

Using the results of [40], we can ensure that the action of $G$ on $S$ can be extended possibly only to actions with signatures $(0; 2, 6, 6)$ and $(0; 2, 4, 6)$.

Let $\Gamma_2$ be a Fuchsian group of signature $(0; 2, 6, 6)$ with canonical presentation

$$\Gamma_2 = \langle y_1, y_2, y_3 : y_1^2 = y_2^6 = y_3^6 = y_1 y_2 y_3 = 1.\] Following [3, Example (i)], there exist two non-isomorphic Riemann surfaces $X_1$ and $X_2$ of genus $g$ with a group of automorphisms of order $12q$ acting on it with signature $(0; 2, 6, 6)$. Furthermore,

$$\text{Aut}(X_i) \cong (C_q \times_6 C_6) \times C_2$$

for $i = 1, 2$.

with corresponding non-equivalent surface epimorphisms $\Theta_i : \Gamma_2 \rightarrow (C_q \times_6 C_6) \times C_2$ giving the actions of $\text{Aut}(X_i)$ on $X_i$ defined by:

$$\Theta_1(y_1) = as, \Theta_1(y_2) = bs, \Theta_1(y_3) = a^{-r} b^2 z \text{ and } \Theta_2(y_1) = as, \Theta_2(y_2) = b^2 sz, \Theta_2(y_3) = a^{-r} bz,$$

where $z$ generates the $C_2$ central factor.

**Claim 3.** If $S$ is a compact Riemann surface with an action of a group of order $6q$ with signature $(0; 3, 6, 6)$ then $S$ is isomorphic to either $X_1$ or $X_2$.

First of all, we have seen above that such an action is given by the surface epimorphisms $\theta_1$, and $\theta_2$. We see now that these actions extend. Setting $x_1' = y_1^2, x_2' = y_3$ and $x_3' = (y_2^3 y_3)^{-1}$, the subgroup of $\Gamma_2$ generated by $x_1', x_2', x_3'$ is isomorphic to $\Gamma_1$. Moreover, $\Theta_1(x_1') = b^2, \Theta_1(x_2') = a^{-r} b^2 z, \Theta_1(x_3') = ab^2 z$; $\Theta_2(x_1') = b, \Theta_2(x_2') = a^{-r} bz, \Theta_2(x_3') = abz$.

Note that $\langle a, b, z \rangle \cong C_q \times_3 C_6$ and that the restrictions

$$\Theta_1|_{\langle x_1', x_2', x_3' \rangle}, \Theta_2|_{\langle x_1', x_2', x_3' \rangle} : \Gamma_1 \cong \langle x_1', x_2', x_3' \rangle \rightarrow C_q \times_3 C_6$$

are precisely $\theta_{1,3}$ and $\theta_{1,1}$ respectively. It follows that the action $\theta_{1,1}$ and $\theta_{1,2}$ of $C_q \times_3 C_6$ on compact Riemann surfaces $S$ of genus $g$ with signature $(0; 3, 6, 6)$ extend to the action of $(C_q \times_6 C_6) \times C_2$ with signature $(0; 2, 6, 6)$ represented by $\Theta_2$ and $\Theta_1$ respectively; thus, $S$ isomorphic to $X_2$ in the first case, and $S$ isomorphic to $X_1$ in the second case.

Now, setting $x_1'' = y_1^2, x_2'' = y_2$ and $x_3'' = (y_3^3 y_2)^{-1}$, the subgroup of $\Gamma_2$ generated by $x_1'', x_2'', x_3''$ is isomorphic to $\Gamma_1$. Moreover, $\Theta_1(x_1'') = ab, \Theta_1(x_2'') = b(sz), \Theta_1(x_3'') = a^{-r}b(sz); \Theta_2(x_1'') = ab^2, \Theta_2(x_2'') = b^2(sz), \Theta_2(x_3'') = a^{-r}b^2(sz)$.

Note that $\langle a, b, sz \rangle \cong C_q \times_6 C_6$ and that the restrictions

$$\Theta_1|_{\langle x_1'', x_2'', x_3'' \rangle}, \Theta_2|_{\langle x_1'', x_2'', x_3'' \rangle} : \Gamma_1 \cong \langle x_1'', x_2'', x_3'' \rangle \rightarrow C_q \times_6 C_6$$
are equivalent to \( \theta_2 \). It follows that the action \( \theta_2 \) of \( C_q \rtimes_6 C_6 \) on a Riemann surface \( S \) with signature \( (0;3,6,6) \) extends to both actions of \( (C_q \rtimes_6 C_6) \times C_2 \) with signature \( (0;2,6,6) \) represented by \( \Theta_1 \) or by \( \Theta_2 \); thus, \( S \) is isomorphic to \( X_1 \) in the first case, and isomorphic to \( X_2 \) in the second case.

This proves Claim 3.

Note that \( \tilde{x}_1 = (y_1 y_2 y_3^{-1})^{-1}, \tilde{x}_2 = y_1, \tilde{x}_3 = y_2^{-1} \) and \( \tilde{x}_4 = y_3^{-1} \) generate a subgroup \( \tilde{\Gamma} \) of \( \Gamma_2 \) isomorphic to a Fuchsian group of signature \( (0;2,2,3,3) \). Furthermore, the restrictions \( \Theta_1|_{\tilde{\Gamma}} \) and \( \Theta_2|_{\tilde{\Gamma}} \) are epimorphisms equivalent to \( \theta_{3,10} \). This yields that \( X_1 \) and \( X_2 \) lie in the boundary of \( \bar{F}_S \) as desired.

Finally, as before, applying [3, Theorem 2(a)] for \( q \geq 19 \) and [10] for the remaining cases \( q = 7 \) and \( q = 13 \), we conclude that:

1. the Riemann surfaces \( X_1 \) and \( X_2 \) are the unique compact Riemann surfaces with a group of automorphisms of order \( 12q \).
2. there are no compact Riemann surfaces of genus \( g \) with \( 24q \) automorphisms (in particular, the action of \( G \) on \( S \) of signature \( (0;3,6,6) \) cannot be extended to an action of signature \( (0;2,4,6) \)), and therefore \( \{X_1, X_2\} = \bar{F}_S \setminus F_q \).

We now decompose the associated Jacobian varieties; to do that we proceed analogously as done in the proof of Theorem 1. Let \( S \in \bar{F}_q \) and set \( \omega := \exp(\frac{2\pi i}{6}) \).

Note that the group

\[ \text{Aut}(S) \cong C_q \rtimes_6 C_6 = \langle a, c : a^q = c^6 = 1, cac^{-1} = a^n \rangle \]

where \( n \) is a 6-th primitive root of the unity in \( \mathbb{F}_q \), has, up to equivalence, six complex irreducible representations of degree 1, given by \( U_i : a \mapsto 1, c \mapsto \omega^i_q \) for \( 0 \leq i \leq 5 \). In addition, \( C_q \rtimes_6 C_6 \) has \( \beta = \frac{2\pi i}{6} \in \mathbb{C} \) complex irreducible representations of degree 6, namely

\[ V_j : a \mapsto \text{diag}(\omega_q^{k_j}, \omega_q^{k_j n}, \omega_q^{k_j n^2}, \omega_q^{k_j n^3}, \omega_q^{k_j n^4}, \omega_q^{k_j n^5}) \] for \( 1 \leq j \leq \beta, \]

where \( k_j, \ldots, k_\beta \in \{1, \ldots, q - 1\} \) are integers chosen to satisfy that

\[ \bigcup_{j=1}^\beta \{k_j, k_j n, k_j n^2, \ldots, k_j n^5\} = \{1, \ldots, q - 1\}, \]

where \( \sqcup \) denotes the disjoint union.

Consider the subgroups \( H = \langle a \rangle \) and \( H_t = \langle a^t c \rangle \) for \( t \in \{1, \ldots, 6\} \). Note that

\[ d_{U_i}^H + \Sigma_{i=1}^6 d_{U_i}^H = 1 = d_{U_i}, \quad \text{and} \quad d_{V_j}^H + \Sigma_{i=1}^6 d_{V_i}^H = 6 = d_{V_j} \]

for each \( i \in \{1, \ldots, 6\} \) and for each \( j \in \{1, \ldots, \beta\} \). Thereby, the collection \( \{H, H_1, \ldots, H_6\} \) is admissible and therefore, by [35], there is an abelian subvariety \( Q \) of \( JS \) such that

\[ JS \sim J(S/H) \times \Pi_{i=1}^6 J(S/H_i) \times Q \sim J(S/\langle a \rangle) \times (J(S/\langle c \rangle))^6 \times Q, \]

where the second isogeny follows from the fact that, for each \( t \), the groups \( H_t \) and \( \langle c \rangle \) are conjugate.

The \( q \)-sheeted regular covering map \( S \rightarrow S/\langle a \rangle \) is unbranched, and the regular covering map \( S \rightarrow S/\langle c \rangle \) ramifies over exactly four values, two marked with 2 and two with marked 3. Thus, the Riemann-Hurwitz formula implies that the genera of \( S/\langle a \rangle \) and \( S/\langle c \rangle \) are 2 and \( \beta \) respectively; thus \( Q = 0 \).

Let \( S \) be one of the two non-isomorphic Riemann surfaces with \( 12q \) automorphisms. Each complex irreducible representation of \( \text{Aut}(S) \cong (C_q \rtimes_6 C_6) \times C_2 = \langle a, c \rangle \times \langle z \rangle \) coincides with the tensor product of a complex irreducible representation of \( C_q \rtimes_6 C_6 \) and one of \( C_2 \) (see, for example [38, p. 27]).

Thus, keeping the same notations as above, we see that the complex irreducible representations of \( (C_q \rtimes_6 C_6) \times C_2 \) are \( U_i^\pm : a \mapsto 1, c \mapsto \omega_q^i, z \mapsto \pm 1 \) for each \( 0 \leq i \leq 5 \), and

\[ V_j^\pm : a \mapsto \text{diag}(\omega_q^{k_j}, \omega_q^{k_j n}, \omega_q^{k_j n^2}, \omega_q^{k_j n^3}, \omega_q^{k_j n^4}, \omega_q^{k_j n^5}), \ c \mapsto \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \ z \mapsto \pm I_6 \]

for \( 1 \leq j \leq \beta \), where \( I_6 \) denotes the \( 6 \times 6 \) identity matrix.
If we write $N = \langle a \rangle$ and $N_t = \langle a^t c z \rangle$ for $t \in \{1, \ldots, 6\}$, then it can be checked that the collection $\{N, N_1, \ldots, N_6\}$ is admissible. In addition, as $N_t$ and $\langle c z \rangle$ are conjugate, we apply the result of [35] to ensure the existence of an abelian subvariety $R$ of $JS$ such that

$$JS \sim J(S/\langle a \rangle) \times (J(S/\langle c z \rangle))^6 \times R.$$ 

The $q$-sheeted covering regular map $S \to S/\langle a \rangle$ is unbranched and the regular covering map $S \to S/\langle c z \rangle$ ramifies over exactly three values, two marked with 2 and one with marked 3. The Riemann-Hurwitz formula implies that the genera of $S/\langle a \rangle$ and $S/\langle c z \rangle$ are 2 and $\beta$ respectively; thus $R = 0$.

This finishes the proof of Theorem 2.

5. PROOF OF COROLLARY 1

Let $S$ be a compact Riemann surface of genus $g \geq 8$, where $q = g - 1$ is prime, and assume that $S$ has a group of automorphisms $G$ of order $3q$. By the Riemann-Hurwitz formula the possible signatures for the action of $G$ on $S$ are (1; 3) and (0; 3, 3, 3) for each $g$ and, in addition, the signature (0; 7, 7, 21) for $g = 8$. The latter exceptional case for $g = 8$ can be disregarded because there are no surface epimorphisms from a Fuchsian group of signature (0; 7, 7, 21) to a (necessarily cyclic) group of order 21.

By the classical Sylow’s theorems if $q \not\equiv 1 \mod 3$ then $G$ is isomorphic to $C_{3q}$, and if $q \equiv 1 \mod 3$ then $G$ is isomorphic to either $C_{3q}$ or to $C_q \times_3 C_3 = \langle a, b : a^q = b^3 = 1, bab^{-1} = a^r \rangle$, where $r$ is a 3-th primitive root of the unity in $\mathbb{F}_q$.

As $C_{3q}$ is abelian, and as the commutator subgroup of $C_q \times_3 C_3$ does not have elements of order three, we see that there are no compact Riemann surfaces of genus $q$ with a group of automorphisms of order $3q$ acting with signature (1; 3). Furthermore, as $C_{3q}$ cannot be generated by elements of order three, we obtain that if $q \not\equiv 1 \mod 3$ there are no compact Riemann surfaces of genus $g$ with a group of automorphisms of order $3q$ acting with signature (0; 3, 3, 3).

Thus, from now on we assume that $g - 1 = q \equiv 1 \mod 3$ and that $G \cong C_q \times_3 C_3$.

Let $\Gamma'$ be a Fuchsian group of signature (0; 3, 3, 3) with canonical presentation

$$\Gamma' = \langle x_1, x_2, x_3, x_4 : x_1^3 = x_2^3 = x_3^3 = x_4 = x_1 x_2 x_3 x_4 = 1 \rangle$$

and let $\theta : \Gamma' \to G$ be a surface epimorphism representing the action of $G$ on $S$. We recall that $G$ has exactly two conjugacy classes of elements of order 3: $\mathcal{C}_1 = \{a^i b : 1 \leq i \leq q \}$ and $\mathcal{C}_2 = \{a^i b^2 : 1 \leq i \leq q \}$.

Note that among the elements $\theta(x_1), \ldots, \theta(x_4)$ of $G$ exactly two of them must belong to $\mathcal{C}_1$; otherwise their product is different from 1. Up to a permutation we can suppose that $\theta(x_i) = a^{i}b^2$ for $i = 1, 2$ and $\theta(x_i) = a^{i}b$ for $i = 3, 4$, for suitable $l_1, l_2, l_4$. Note that if $l_1 \equiv l_2 \mod q$, then $l_3 \equiv l_4 \mod q$ and $\theta$ is not surjective; thus, without lost of generality, we can assume $l_1 \not\equiv l_2 \mod q$. Now, by considering an automorphism of $G$ of the form $a \mapsto a^i, b \mapsto a^j b$, we can assume $l_1 \equiv 0 \mod q$ and $l_2 \equiv 1 \mod q$; therefore $\theta$ is equivalent to the epimorphism $\theta_n$ defined by

$$\theta_n(x_1) = b^2, \theta_n(x_2) = ab^2, \theta_n(x_3) = a^{-r(n+1)}b, \theta_n(x_4) = a^nb, \quad \text{with } 1 \leq n \leq q.$$ 

By [40], the action of $G$ on $S$ can possibly be extended to an action of signature (0; 2, 2, 3, 3). We shall prove that each action does extend to an action equivalent to the one given by the surface epimorphism $\theta_{3,0}$ and therefore the surfaces $S$ belong to the family $\mathcal{F}_9$ of Theorem 2.

Now, let us consider the Fuchsian group $\Gamma_3$ of signature (0; 2, 2, 3, 3) with canonical presentation

$$\Gamma_3 = \langle y_1, y_2, y_3, y_4 : y_1^2 = y_2^2 = y_3^2 = y_4^3 = y_1 y_2 y_3 y_4 = 1 \rangle$$

and let $G' \cong C_q \times_6 C_6$ be the finite group with presentation

$$\langle a, b, s : a^3 = b^6 = s^2 = 1, bab^{-1} = a^r, sas = a^{-1}, [s, b] = 1 \rangle,$$

as in Section 4. Following the proof of Theorem 2, each surface epimorphism $\Theta : \Gamma_1 \to C_q \times_6 C_6$ representing an action of $C_q \times_6 C_6$ is equivalent to the epimorphism $\Theta_m = \Theta_{3,m}$ in the notation of proof of Theorem 2) given by

$$\Theta_m(y_1) = s, \Theta_m(y_2) = as, \Theta_m(y_3) = a^{1+(1+r)m}b^2, \Theta_m(y_4) = a^mb \quad \text{for a suitable } 1 \leq m \leq q.$$
We notice that the group generated by $\tilde{x}_1 = y_3, \tilde{x}_2 = y_2^3, \tilde{x}_3 = y_4$ and $\tilde{x}_4 = y_3^2$, is isomorphic to $\Gamma'$ and 
$\Theta_m(\tilde{x}_1) = a^{1+(1+r)m}b^2$, $\Theta_m(\tilde{x}_2) = a^{(1+r)m}b^2$, $\Theta_m(\tilde{x}_3) = a^mb^2$, $\Theta_m(\tilde{x}_4) = a^{m-1}b$.

Now, after considering the automorphism of $G$ given by $a \mapsto a^{-1}, b \mapsto ab^j$ where $j = \frac{1+m(1+r)}{1-r}$, it follows that $\Theta_m|x = \theta(\tilde{x}^{m(1+r+1)})$ for each $m \in \{1, \ldots, q\}$. Thereby, each action of $C_q \times_3 C_3$ with signature $(0; 3, 3, 3)$ extends to an action of $C_q \rtimes_3 C_3$ with signature $(0; 2, 2, 3, 3)$, as desired.

As a consequence, there does not exist compact Riemann surfaces with full automorphism group of order $3q$, and the proof of Corollary 1 is complete.

Remark. The Riemann surfaces of genus $g = 3$ admitting the action of a group of order six or twelve are given and classified in [5]. There is no a Riemann surface of genus three admitting an automorphism of order five. The Riemann surfaces of genus $g = 4$ with a group of automorphisms of order fifteen or eighteen are given and classify in [2] and [13]. Among them there is the equisymmetric family of cyclic trigonal surfaces with two trigonal morphisms (see [14, 16]). Finally, the Riemann surfaces of genus $g = 6$ with twenty-five or thirty automorphisms are given in [29].

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