Generalization of the Hamilton-Jacobi approach for higher order singular systems

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Abstract
We generalize the Hamilton-Jacobi formulation for higher order singular systems and obtain the equations of motion as total differential equations. To do this we first study the constrains structure present in such systems.
1 Introduction

The study of singular systems has reached a great status in physics since the development by Dirac [1, 2, 3] of the generalized Hamiltonian formulation. Since then, this formalism has found a wide range of applications in Field Theory [4, 5, 6, 7] and it is still the main tool for the analysis of singular systems. Despite the success, it is always interesting to apply different formalisms to the analysis of singular systems since they may show new features of the system under study, in a similar way to what happens in Classical Dynamics [8].

Recently an approach based on Hamilton-Jacobi formalism was developed to study singular first order systems [9, 10]. This approach consists in using Carathéodory’s equivalent Lagrangians method to write down the Hamilton-Jacobi equation for the system and make use of its singularity to write the equations of motion as total differential equations in many variables. This new approach, due to its very recent development, has been applied to very few examples [11, 12, 13, 14] and it is still necessary a better understanding of its features, its advantages and disadvantages in the study of singular systems when compared to Dirac’s Hamiltonian formalism.

Besides that, theories with higher order Lagrangians (or higher order wave equations) are important in the context of many physical problems. Examples range from Podolsky’s Generalized Electrodynamics [15] to tachyons (ref. [16] and references there in).

Our aim here is to make a formal generalization of Hamilton-Jacobi formalism for singular systems with arbitrarily higher order Lagrangians. This generalization is motivated by the attention that higher order systems have received in
literature \[17, 18, 19\]. A treatment for the case of second order Lagrangians has already been developed by the authors \[20\], but here we will make a more general treatment that will begin with the analysis of the constraints’ structure of such systems (Sect. 2). Next we will use Carathéodory’s equivalent Lagrangians method to derive the Hamilton-Jacobi equation for higher order systems (Sect. 3) and analyze the singular case (Sect. 4). The conclusions will be drawn in Sect. 5.

2 Constraints structure in higher order systems

We will analyze a system described by a Lagrangian dependent up to the \( K \)-th derivative of the \( N \) generalized coordinates \( q_i \), i.e.

\[
L \left( q_i, \dot{q}_i, \ldots, \left( ^{(K)} q_i \right) \right) ; \quad \left( ^{(s)} q_i \right) \equiv \frac{d^s q_i}{dt^s},
\]

were \( s = 0, 1, \ldots, K \) and \( i = 1, \ldots, N \). For such systems the Euler-Lagrange equations of motion, obtained through Hamilton’s principle of stationary action, will be

\[
\sum_{s=0}^{K} (-1)^s \frac{d^s}{dt^s} \left( \frac{\partial L}{\partial \left( ^{(s)} q_i \right)} \right) = 0.
\]

This is a system of \( N \) differential equations of \( 2K \)-th order so we need \( 2KN \) initial conditions to solve it. These conditions are the initial values of \( q_i, \dot{q}_i, \ldots, \left( ^{(2K-1)} q_i \right) \) that describe the velocity phase space (VPS).

The Hamiltonian formalism for theories with higher order derivatives, that has been first developed by Ostrogradski \[21\], treats the derivatives \( \left( ^{(s)} q_i \right) (s = 0, \ldots, K - 1) \) as coordinates. So we will indicate this writing them as \( \left( ^{(s)} q_i \right) \equiv q_{(s)i} \). In Ostrogradski’s formalism the momenta conjugated respectively to \( q_{(K-1)i} \) and
$q_{(s-1)i}$ ($s = 1, ..., K - 1$) are introduced as

$$p_{(K-1)i} \equiv \frac{\partial L}{\partial (K) q_i},$$

(1)

$$p_{(s-1)i} \equiv \frac{\partial L}{\partial (s) q_i} - \dot{p}_{(s)i}; \; s = 1, ..., K - 1.$$  

(2)

Notice that the momenta $p_{(s)i}$ ($s \geq 0$) will only be dependent on the derivatives up to $(2K - 1 - s)q_i$.

The Hamiltonian is defined as

$$H = \sum_{s=0}^{K-1} p_{(s)i}^{(s+1)}q_i - L \left( q_1, ..., (K) q_i \right),$$

(3)

where we use Einstein’s summation rule for repeated indexes as will be done throughout this paper. Anyway, we will write explicitly the summation over the index ($s$) inside the parenthesis for a question of intelligibility.

The Hamilton’s equations of motion will be written as

$$\dot{q}_{(s)i} = \frac{\partial H}{\partial p_{(s)i}} = \left\{ q_{(s)i}, H \right\},$$

(4)

$$\dot{p}_{(s)i} = -\frac{\partial H}{\partial q_{(s)i}} = \left\{ p_{(s)i}, H \right\},$$

(5)

were $\{ , \}$ is the Poisson bracket defined as

$$\{ A, B \} \equiv \sum_{s=0}^{K-1} \frac{\partial A}{\partial q_{(s)i}} \frac{\partial B}{\partial p_{(s)i}} - \frac{\partial B}{\partial q_{(s)i}} \frac{\partial A}{\partial p_{(s)i}}.$$

The fundamental Poisson brackets are

$$\left\{ q_{(s)i}, p_{(s')j} \right\} = \delta_{ss'}\delta_{ij}; \left\{ q_{(s)i}, q_{(s')j} \right\} = \left\{ p_{(s)i}, p_{(s')j} \right\} = 0,$$

were $i, j = 1, ..., N$ and $s, s' = 0, ..., K - 1$. With this procedure the phase space ($PS$) is described in terms of the canonical variables $q_{(s)i}$ and $p_{(s)i}$ (where $i = 1, ..., N$ and $s = 0, ..., K - 1$) obeying $2NK$ equations of motion (given by
equations (4) and (5)) which are first order differential equations. So we have to fix \(2KN\) initial conditions to solve these equations. These initial conditions are analogue to those needed in Euler-Lagrange equations, but now they are the initial values of the canonical variables.

However, this passage from \(VPS\) to \(PS\) is only possible if we can solve the momenta expressions (4) and (2) with respect to the derivatives \(\dot{q}_i, \ldots, (2K-1)q_i\) so that these can be expressed as functions of the canonical variables and eliminated from the theory. The necessity of expressing the derivatives \(\dot{q}_i\) as functions of the canonical variables is clear since these derivatives are present in the Hamiltonian definition (3) and must be eliminated from all equations in the Hamiltonian formulation. This procedure is completely analogous to the elimination of the velocities \(\dot{q}_i\) in the Hamiltonian formalism of a first order system [5, 22].

The same argument cannot be applied to the derivatives \((K+1)q_i, \ldots, (2K-1)q_i\) since derivatives higher than \(\dot{q}_i\) are not present in the Hamiltonian. But now, the necessity of expressing these derivatives as functions of canonical variables comes from the fact that they are present in the momenta expressions (2). So, fixing the initial conditions of these momenta in the Hamiltonian formulation is equivalent to fixing the initial conditions to the derivatives \((K+1)q_i, \ldots, (2K-1)q_i\) in the Lagrangian formulation. The same relation connects the momenta \(p_{(K-1)i}\) and the derivative \((K)\dot{q}_i\): fixing the initial conditions for the momenta \(p_{(K-1)i}\) in the Hamiltonian formulation is equivalent to fixing the initial condition for the derivatives \((K)\dot{q}_i\).

Then, it is necessary that all the momenta (4) and (2) be linearly independent functions of the derivatives \((K)\dot{q}_i, \ldots, (2K-1)\dot{q}_i\) so that the latter ones can be solved uniquely with respect to the former.

The expression (4) for the momenta \(p_{(K-1)i}\) shows that they are dependent
only on derivatives up to \( (K) q_i \), so these derivatives can be solved as functions

\[
(q)^{(K)}_i = f_{(K)i}(q^{(s)}_j; p^{(K-1)_j}); \ s = 0, ..., K - 1,
\]

if, and only if, the momenta \( p^{(K-1)_i} \) are linearly independent functions of the derivatives \( (K) q_i \). For this, it is necessary that the matrix

\[
H_{ij} \equiv \frac{\partial p^{(K-1)_i}}{\partial (K) q_j} = \frac{\partial^2 L}{\partial (K) q_i \partial (K) q_j}
\]

be non singular. This matrix \( H_{ij} \) is called the Hessian matrix of the system and is simply the Jacobian matrix of the change of variables \( (K) q_j \rightarrow p^{(K-1)_i} \).

Now, from definitions (1) and (2), we can see that the momenta \( p^{(K-2)_i} \) are dependent on derivatives up to \( (K+1) q_i \). In addition, the dependence on \( (K+1) q_i \) is linear and the coefficients are the elements \( H_{ij} \) of the Hessian matrix. Considering that the derivatives \( (K) q_i \) have already been eliminated from equations using the expression (6) the derivatives \( (K+1) q_i \) can be solved as functions

\[
(q)^{(K+1)}_i = f_{(K+1)i}(q^{(s)}_j; p^{(K-1)_j}, p^{(K-2)_j}); \ s = 0, ..., K - 1,
\]

if, and only if, the momenta \( p^{(K-2)_i} \) are linearly independent functions of the derivatives \( (K+1) q_i \). For that it will be necessary that the Jacobian matrix of the change of variables \( (K+1) q_j \rightarrow p^{(K-2)_i} \), with elements \( J_{ij} \) given by

\[
J_{ij} = \left. \frac{\partial p^{(K-2)_i}}{\partial (K+1) q_j} \right|_{\begin{array}{c}
q^{(K)} u = f^{(K)}(K)_u
\end{array}} = \left. \frac{\partial^2 L}{\partial (K) q_i \partial (K) q_j} \right|_{\begin{array}{c}
q^{(K)} u = f^{(K)}(K)_u
\end{array}} = -H_{ij}|^{(K+1)}_{q^{(K)} u = f^{(K)}(K)_u},
\]

be non singular.

Continuing this process, we can use the fact that the momenta \( p^{(s)_i} \) are dependent on derivatives up to \( (2K-1-s) q_i \) and, noticing that the highest derivatives appear linearly with coefficients that are the elements of the Hessian matrix, show that the derivatives \( (K+p) q_i \) can be solved as functions

\[
(q)^{(K+p)}_i = f_{(K+p)i}(q^{(s)}_j; p^{(K-1)_j}, ..., p^{(K-1-p)_j}); \ s, p = 0, ..., K - 1
\]
if, and only if, the Jacobian matrix of the change of variables \( q_j \rightarrow p(K-1-p)_i \), with elements \( J_{ij} \) given by

\[
J_{ij} = \left. \frac{\partial p(K-1-p)_i}{\partial (K+p)q_j} \right|_{q=\hat{f}(K+d)u} = (-1)^p \left. \frac{\partial^2 L}{\partial (K)q_i \partial (K)q_j} \right|_{q=\hat{f}(K+d)u}
\]

\[
J_{ij} = (-1)^p H_{ij}|_{q=\hat{f}(K+d)u}
\]

\((d = 0, \ldots, p - 1)\), is non singular. Consequently, it will be the non singularity of the Hessian matrix \( (7) \) that will determine if the passage from the VPS to PS is possible or not.

Let’s suppose now that the Hessian matrix has rank \( P = N - R \). In this case it will not be possible to express all derivatives \( \hat{q}_i, \ldots, q_{2K-1}_i \) in the form of equation \( (8) \). Without loss of generality, we can choose the order of coordinates in such a way that the \( P \times P \) sub-matrix in the bottom right corner of the Hessian matrix has nonvanishing determinant

\[
\det ||H_{ab}|| = \det \left| \frac{\partial^2 L}{\partial (K)q_a \partial (K)q_b} \right| \neq 0; \ a, b = R + 1, \ldots, N.
\]

With this condition, we can only solve the \( P = N - R \) derivatives \( q_a \) as functions of the coordinates \( q(s)_i \), the momenta \( p(K-1)_b \) and the unsolved derivatives \( q_\alpha \) \((\alpha = 1, \ldots, R)\) as follows

\[
q_a = \hat{f}_{(K)a} \left( q(s)_i; \ p(K-1)_b; \ q_\alpha \right).
\]

If we substitute this expression in the momenta definition \( (11) \) for \( p(K-1)_i \) we obtain

\[
p(K-1)_i = \left. \frac{\partial L}{\partial (K)q_i} \right|_{q_a=\hat{f}(K)a} = g(K-1)_i \left( q(s)_j; \ p(K-1)_a; \ q_\alpha \right).
\]
But, since we have \( p_{(K-1)\alpha} \equiv g_{(K-1)\alpha} \), the other \( R \) functions \( g_{(K-1)\alpha} \) can not contain the unsolved derivatives \( q_\alpha \) or we would be able to solve more of these derivatives as functions of the canonical variables, what contradicts the fact that the rank of the Hessian matrix is \( P \). So we have the expressions

\[
p_{(K-1)\alpha} = g_{(K-1)\alpha} \left( q(s)_j; p_{(K-1)\alpha} \right)
\]

which correspond to primary constraints

\[
\Phi_{(K-1)\alpha} = p_{(K-1)\alpha} - g_{(K-1)\alpha} \left( q(s)_j; p_{(K-1)\alpha} \right) \approx 0
\]

in Dirac’s Hamiltonian formalism for singular systems.

Analogously, we can only solve the \( P \) derivatives \( q_\alpha \) as functions of the coordinates \( q(s)_i \), the momenta \( p_{(K-1)\alpha} \) and \( p_{(K-2)\alpha} \) and the unsolved derivatives \( q_\alpha \) \((K)\) and \((K+1)\) \( q_\alpha \) \((\alpha = 1, \ldots, R)\) as follows

\[
q_{(K+1)\alpha} = f_{(K+1)\alpha} \left( q(s)_i; p_{(K-1)\alpha}; p_{(K-2)\alpha}; q_{(K)\alpha}; q_{(K+1)\alpha} \right).
\]

Substituting this expression in momenta definitions (2) and using the above argument relative to the rank of the Hessian matrix, we obtain for the momenta \( p_{(K-2)\alpha} \) the following expression

\[
p_{(K-2)\alpha} = g_{(K-2)\alpha} \left( q(s)_j; p_{(K-1)\alpha}; p_{(K-2)\alpha} \right)
\]

that corresponds to new primary constraints

\[
\Phi_{(K-2)\alpha} = p_{(K-2)\alpha} - g_{(K-2)\alpha} \left( q(s)_j; p_{(K-1)\alpha}; p_{(K-2)\alpha} \right) \approx 0.
\]

Continuing this process we find that there will be expressions

\[
p_{(K-1-p)\alpha} = g_{(K-1-p)\alpha} \left( q(s)_j; p_{(K-1)\alpha}; \ldots; p_{(K-1-p)\alpha} \right)
\]

\((p = 0, \ldots, K - 1)\) that correspond to primary constraints

\[
\Phi_{(K-1-p)\alpha} = p_{(K-1-p)\alpha} - g_{(K-1-p)\alpha} \left( q(s)_j; p_{(K-1)\alpha}; \ldots; p_{(K-1-p)\alpha} \right) \approx 0.
\]
As a result, in a higher order system, the existence of constraints involving a
given momentum $p_{(K-1)\alpha}$ will imply the existence of constraints involving all $p_{(s)\alpha}$
momenta conjugated to the derivatives $q_{(s)\alpha} = q_{\alpha}^{(s)} (s = 0, \ldots, K-1)$ due to the fact
that the derivatives $q_{\alpha}^{(K)}, \ldots, q_{\alpha}^{(2K-1)}$ can’t be expressed as functions of the canonical
variables. Consequently, if the Hessian matrix has rank $P = N - R$ there are
$KR$ expressions of the form $(11)$ that correspond to $KR$ primary constraints in
Dirac’s formalism as given by $(12)$.

The existence of such constraints’ structure in higher order systems has already
been noticed by other authors. Nesterenko [17] and Batlle et al. [18] discerned
the existence of such constraints structure in second order systems, while Saito et
al. [19] showed, by different arguments, that the constraints structure exhibited
above exists for arbitrarily higher order systems. Furthermore, it is important to
observe that the constraint structure showed above is different for a higher order
Lagrangian obtained from a lower order one by adding a total time derivative.
We will not discuss this case here but the reader can find a detailed analysis of
the constraint structure for such Lagrangians in reference [19].

3 Hamilton-Jacobi formalism

Now we will use Carathéodory’s equivalent Lagrangians method to extend the
Hamilton-Jacobi formalism to a general higher order Lagrangian. The procedure
described in the sequence can be applied to any higher order Lagrangian and is
not restricted to a singular one.

Carathéodory’s equivalent Lagrangians method [23] can be easily applied to
higher order Lagrangians. Given a Lagrangian $L(q_i, \dot{q}_i, \ldots, \dddot{q}_i)$, we can obtain
a completely equivalent one given by

\[ L' = L \left( q_i, \ldots, q_i^{(K)} \right) - \frac{dS \left( q_i, \ldots, q_i^{(K-1)}; t \right)}{dt}. \]

These Lagrangians are equivalent because the action integral given by them have simultaneous extremum. So we can choose the function \( S \left( q_i, \ldots, q_i^{(K-1)}; t \right) \) in such a way that we get an extremum of \( L' \) and consequently we will get an extremum of the Lagrangian \( L \).

To do this, it is enough to find a set of functions \( \beta_{(s)}(q_j, q_{(1)} j, \ldots, q_{(s-1)} j; t) \), \( s = 1, \ldots, K \), and \( S \left( q_i, \ldots, q_i^{(K-1)}; t \right) \) such that

\[ L' \left( q_i, q_{(1)} i = \beta_{(1)} i, \ldots, q_i^{(K)} i = \beta_{(K)} i; t \right) = 0 \tag{13} \]

and for all neighborhood of \( q_{(s)} i = \beta_{(s)} i(q_j, q_{(1)} j, \ldots, q_{(s-1)} j; t) \)

\[ L' \left( q_i, \ldots, q_i^{(K)} \right) > 0. \tag{14} \]

With these conditions satisfied, the Lagrangian \( L' \) will have a minimum in \( q_{(s)} i = \beta_{(s)} i \), so that the action integral will also have a minimum and the solutions of the differential equations given by

\[ q_{(s)} i = q_i = \frac{d^s q_i}{dt^s} = \beta_{(s)} i, \]

\( s = 1, \ldots, K \), will correspond to an extremum of the action integral.

From the definition of \( L' \) we have

\[ L' = L \left( q_j, \ldots, q_j^{(K)} \right) - \frac{\partial S \left( q_j, \ldots, q_j^{(K-1)}; t \right)}{\partial t} - \sum_{u=0}^{K-1} \frac{\partial S \left( q_j, \ldots, q_j^{(K-1)}; t \right)}{\partial q_{(u)} i} \frac{dq_{(u)} i}{dt}. \]

Using condition (13) we obtain

\[ \left[ \frac{\partial S \left( q_j, \ldots, q_j^{(K)}; t \right)}{\partial t} - \sum_{u=0}^{K-1} \frac{\partial S \left( q_j, \ldots, q_j^{(K)}; t \right)}{\partial q_{(u)} i} \frac{dq_{(u)} i}{dt} \right]_{q_{(s)} i = \beta_{(s)} i} = 0, \]
\[
\frac{\partial S}{\partial t} \bigg|_{q(u), \beta(u)} = \left[ L \left( q_j, \ldots, (K) q_j \right) - \sum_{u=0}^{K-1} \frac{\partial S}{\partial q(u)} q(u+1) \right] \bigg|_{q(u), \beta(u)}.
\]

Since \( q(s)_i = \beta(s)_i \) is a minimum point of \( L' \) we must have

\[
\left. \frac{\partial L'}{\partial q} \right|_{q(s)_i = \beta(s)_i} = 0 \Rightarrow \left. \left[ \frac{\partial L}{\partial (K) q(s)_i} - \frac{\partial}{\partial (K) q(s)_i} \left( \frac{dS}{dt} \right) \right] \right|_{q(s)_i = \beta(s)_i} = 0,
\]

or

\[
\left. \frac{\partial S}{\partial q(K-1)_i} \right|_{q(s)_i = \beta(s)_i} = \left. \frac{\partial L}{\partial (K) q(s)_i} \right|_{q(s)_i = \beta(s)_i}.
\]

For the same reason we must have

\[
\left. \frac{\partial L'}{\partial (K-1)_i} \right|_{q(s)_i = \beta(s)_i} = 0 \Rightarrow \left. \left[ \frac{\partial L}{\partial q(K-1)_i} - \frac{\partial}{\partial q(K-1)_i} \left( \frac{dS}{dt} \right) \right] \right|_{q(s)_i = \beta(s)_i} = 0,
\]

or

\[
\left. \frac{\partial S}{\partial q(K-2)_i} \right|_{q(s)_i = \beta(s)_i} = \left. \left[ \frac{\partial L}{\partial q(K-1)_i} - \frac{d}{dt} \frac{\partial S}{\partial q(K-1)_i} \right] \right|_{q(s)_i = \beta(s)_i}.
\]

Following this procedure we have the general expression

\[
\left. \frac{\partial S}{\partial q(u-1)_i} \right|_{q(s)_i = \beta(s)_i} = \left[ \frac{\partial L}{\partial q(u)_i} - \frac{d}{dt} \frac{\partial S}{\partial q(u)_i} \right] \bigg|_{q(s)_i = \beta(s)_i},
\]

were \( u = 1, \ldots, K - 1 \).
Now, using the definitions for the conjugated momenta given by equations (1) and (2) in the expressions (16) and (18) we obtain

\[ p_{(u)i} = \frac{\partial S}{\partial q_{(u)i}} , \ u = 0, ..., K - 1. \]  

(19)

So, we can see from equation (15) that, to obtain an extremum of the action, we must get a function \( S(q_i, ..., q_{(K-1)i}, t) \) such that

\[ \frac{\partial S}{\partial t} = -H_0 \]  

(20)

where \( H_0 \) is

\[ H_0 = \sum_{u=0}^{K-1} p_{(u)i} (u+1)q_i - L(q_i, ..., q_{(K)}^i) \]  

(21)

and the momenta \( p_{(u)i} \) are given by equation (19).

These are the fundamental equations of the equivalent Lagrangian method, and equation (20) is the Hamilton-Jacobi partial differential equation (HJPDE).

### 4 The singular case

We consider now the application of the formalism developed in the previous section to a system with a singular higher order Lagrangian. As we showed in Sect. 2, when the Hessian matrix has a rank \( P = N - R \) the momenta variables will not be independent among themselves and we will obtain expressions like equation (11). We will rewrite these expressions as

\[ p_{(u)\alpha} = -H_{(u)\alpha} (q_{(s)j}, p_{(s)a}) , \ u, \ s = 0, ..., K - 1 \]  

(22)

where we are supposing that the expression for the momentum \( p_{(u)\alpha} \) depends on all momenta \( p_{(s)a} \), although we have showed that the expression for the momentum \( p_{(u)\alpha} \) is not dependent on any momenta \( p_{(s)a} \) with \( s < u \). We do this for simplicity.
The Hamiltonian $H_0$, given by equation (21), becomes

$$H_0 = \sum_{u=0}^{K-2} p(u\alpha)^{(u+1)} q_a + p(K-1) a f(K) \alpha + \sum_{u=0}^{K-1} q_a p(u\alpha)^{(u)}|_{p(s)\beta = -H(s)\beta} - L \left( q(s)_{i \alpha}, q_{\alpha}, q_{\alpha} = f(K) a \right),$$

(23)

where $\alpha, \beta = 1, ..., R; a = R + 1, ..., N$. On the other hand we have

$$\frac{\partial H_0}{\partial (K)} = p(K-1) a \frac{\partial f(K) a}{\partial (K)} + p(K-1) a \frac{\partial L}{\partial (K)} - \frac{\partial L}{\partial (K)} \frac{\partial f(K) a}{\partial (K)} = 0,$$

so the Hamiltonian $H_0$ does not depend explicitly upon the derivatives $q_{\alpha}^{(K)}$.

Now we will adopt the following notation: the time parameter $t$ will be called $t(s)_0 \equiv q(s)_0$ (for any value of $s$); the coordinates $q(s)_\alpha$ will be called $t(s)_\alpha$; the momenta $p(s)_\alpha$ will be called $P(s)_\alpha$ and the momentum $p(s)_0 \equiv P(s)_0$ will be defined as

$$P(s)_0 \equiv \frac{\partial S}{\partial t},$$

(24)

while $H(s)_0 \equiv H_0$ for any value of $s$.

Then, to obtain an extremum of the action integral, we must find a function $S(t(c)_\alpha; q(c)_\alpha, t)$ ($c = 0, ..., K - 1$) that satisfies the following set of HJPDE

$$H'_0 \equiv H'_0(s)_0 \equiv P(s)_0 + H(s)_0 \left( t, t(u)_\alpha; q(u)_\alpha, p(u)_\alpha = \frac{\partial S}{\partial q(u)_\alpha} = 0, \right.$$

(25)

$$H'_{(s)\alpha} \equiv P(s)_\alpha + H(s)_\alpha \left( t(u)_\alpha; q(u)_\alpha, p(u)_\alpha = \frac{\partial S}{\partial q(u)_\alpha} = 0. \right.$$  

(26)

where $s, u = 0, ..., K - 1$ and $\alpha = 1, ..., R$. If we let the index $\alpha$ run from 0 to $R$ we can write both equations as

$$H'_0(s)_\alpha \equiv P(s)_{\alpha} + H(s)_{\alpha} \left( t(u)_\alpha, q(u)_\alpha, p(u)_\alpha = \frac{\partial S}{\partial q(u)_\alpha} = 0. \right.$$  

(27)

From the above definition above and equation (23) we have

$$\frac{\partial H'_0(s)_0}{\partial p(u)_b} = -\frac{\partial L}{\partial (K)} \frac{\partial f(K) a}{\partial p(u)_b} - \sum_{s=0}^{K-1} q_{\alpha} \frac{\partial H(s)_{\alpha}}{\partial p(u)_b} + p(K-1) a \frac{\partial f(K) a}{\partial p(u)_b} + q(u+1)_b,$$
\[ \frac{\partial H'(s)^0}{\partial p(u)b} = \dot{q}_b - \sum_{s=0}^{K-1} q_{s,\alpha} \frac{\partial H(s,\alpha)}{\partial p(u)b}, \]

where \( u = 0, ..., K - 1 \), \( \alpha = 1, ..., R \) and we used the fact that \( \sum_{c=0}^{c+1} q_{c+1} = \frac{dq(c)}{dt}; c = 0, ..., K - 1 \).

Multiplying this equation by \( dt = dt(s)_0 \) we have

\[ dq(u)_b = \frac{\partial H'(s)^0}{\partial p(u)_b} dt(s)_0 + \sum_{s=0}^{K-1} \frac{\partial H'(s,\alpha)}{\partial p(u)_b} dq(s)_\alpha. \]

Using \( t(s)_\alpha = q(s)_\alpha \) and making the index \( \alpha \) run from 0 to \( R \), we have

\[ dq(u)_b = \sum_{s=0}^{K-1} \frac{\partial H'(s,\alpha)}{\partial p(u)_b} dt(s)_\alpha. \]

We must call attention to the fact that in the above expression, for \( \alpha = 0 \), we have the term

\[ \sum_{s=0}^{K-1} \frac{\partial H'(s)^0}{\partial p(u)_b} dt(s)_0 = \frac{\partial H'_0}{\partial p(u)_b} dt \]

that should not be interpreted as

\[ \sum_{s=0}^{K-1} \frac{\partial H'(s)^0}{\partial p(u)_b} dt(s)_0 = K \cdot \frac{\partial H'_0}{\partial p(u)_b} dt. \]

This somewhat unusual choice of notation allows us to express the results in a compact way.

Noticing that we have the expressions

\[ dq(u)_\beta = \sum_{s=0}^{K-1} \frac{\partial H'(s,\alpha)}{\partial p(u)_\beta} dt(s)_\alpha = \delta_{su} \delta_{\alpha, \beta} dt(s)_\alpha \equiv dt(s)_\beta \]

identically satisfied for \( \alpha, \beta = 0, 1, ..., R \), we can write the expression (28) as

\[ dq(u)_i = \sum_{s=0}^{K-1} \frac{\partial H'(s,\alpha)}{\partial p(u)_i} dt(s)_\alpha; \ i = 1, ..., N. \]
If we consider that we have a solution $S(q_1, ..., q_{K-1}, t)$ of the set of HJPDE given by equation (27) then, differentiating that equation with respect to $q(u)_c$, we obtain

$$\frac{\partial H'_(s)\alpha}{\partial q(u)_c} + \sum_{d=0}^{K-1} \frac{\partial H'_(s)\alpha}{\partial p(d)_\beta} \frac{\partial^2 S}{\partial t(d)_\beta \partial q(u)_c} + \sum_{d=0}^{K-1} \frac{\partial H'_(s)\alpha}{\partial p(d)_a} \frac{\partial^2 S}{\partial q(d)_a \partial q(u)_c} = 0 \quad (30)$$

for $\alpha, \beta = 0, 1, ..., R$; $s, u, d = 0, 1, ..., K - 1$ and $c = 0, 1, ..., N$.

From the momenta definitions we can obtain

$$dp(u)_c = \sum_{d=0}^{K-1} \frac{\partial^2 S}{\partial q(u)_c \partial t(d)_\beta} dt(d)_\beta + \sum_{d=0}^{K-1} \frac{\partial^2 S}{\partial p(d)_a \partial q(d)_a} dq(d)_a. \quad (31)$$

Now, contracting equation (31) with $dt(s)_\alpha$ and adding the result to equation (31) we get

$$dp(u)_c = \sum_{s=0}^{K-1} \frac{\partial H'_(s)\alpha}{\partial q(u)_c} dt(s)_\alpha \quad (32)$$

were, as before, $u = 0, 1, ..., K - 1$; $c = 0, 1, ..., N$ and $\alpha = 0, 1, ..., R$.

Making $Z \equiv S(t(s)_\alpha; q(s)_a)$ and using the momenta definitions together with equation (29) we have

$$dZ = \sum_{d=0}^{K-1} \frac{\partial S}{\partial t(d)_\beta} dt(d)_\beta + \sum_{d=0}^{K-1} \frac{\partial S}{\partial q(d)_a} dq(d)_a,$$

$$dZ = - \sum_{d=0}^{K-1} H(d)_\beta dt(d)_\beta + \sum_{d=0}^{K-1} p(d)_a \left( \sum_{s=0}^{K-1} \frac{\partial H'_(s)\alpha}{\partial p(d)_\beta} dt(s)_\alpha \right).$$
With a little change of indexes we get
\[
\frac{dZ}{dt} = \sum_{d=0}^{K-1} \left( -H_{(d)\beta} + \sum_{s=0}^{K-1} p_{(s)\alpha} \frac{\partial H_{(d)\beta}'}{\partial p_{(s)\alpha}} \right) dt_{(d)\beta}.
\] (33)

This equation together with equations (29) and (32) are the total differential equations for the characteristics curves of the HJPDE given by equation (27) and, if they form a completely integrable set, their simultaneous solutions determine \(S\left(t_{(s)\alpha}; q_{(s)\alpha}\right)\) uniquely from the initial conditions. Besides that, equations (29) and (32) are the equations of motion of the system written as total differential equations.

5 Conclusions

We have obtained the equations of motion for the canonical variables of a singular higher order system as total differential equations. Each coordinate \(q_{(s)\alpha} \equiv t_{(s)\alpha}\) \((\alpha = 1, ..., R)\) is treated as a parameter that describes the system evolution. The Hamiltonians \(H'_{(s)\alpha}\) will be the generators of the canonical transformations parametrized by \(t_{(s)\alpha}\) in the same way the Hamiltonian \(H_0\) is the generator of time evolution. If we have \(K = 1\) the results obtained here will reduce to the case of first order systems showed in ref. [9]. For \(K = 2\) we have the same results obtained for a second order system of ref. [20], where the Hamilton-Jacobi formalism was applied to Podolsky generalized electrodynamics and the results were compared to Dirac’s Hamiltonian formalism.

The integrability conditions that have to be satisfied by equations (29), (32) and (33) are analogous to those that have to be satisfied in the first order case. These conditions have been derived in ref. [10] and can be easily applied to the higher order case developed here. These integrability conditions are equivalent to the consistence conditions in Dirac’s formalism.
We must point out that one of the reasons to consider the constraints’ structure described in Sect. 2 is the fact that if we had considered only the constraints containing the momenta \( p(K-1)_\alpha \), given by equations of the form (9), when developing the singular case in Sect. 4, the coordinate \( t_{(K-1)_\alpha} \equiv q_{(K-1)_\alpha} \) would be an arbitrary parameter in the formalism but the coordinate \( t_{(K-2)_\alpha} \equiv q_{(K-2)_\alpha} \) (that obeys \( q_{(K-1)_\alpha} \equiv \dot{q}_{(K-2)_\alpha} \)) would have a dynamics of its own. So, when we choose to deal with all constraints given by expression (11) we are avoiding such contradictions. Furthermore, if we had not made this choice, we would have an unnecessary extra work when analyzing the integrability conditions since the constraints involving momenta \( p(s)_\alpha \) with \( s < K - 1 \) would appear in this stage imposing extra integrability conditions.

As we mentioned in Introduction, Hamilton-Jacobi formalism is not well studied for singular systems yet. We still lack a complete analysis of the relation between the procedures in this new formalism for singular systems and traditional ones, specially the relation with Dirac’s Hamiltonian formalism. Besides, Hamilton-Jacobi formalism shall be applied to various physical systems so that we can get a better understanding of its potential to deal with specific problems.

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