Accelerations As(MM) and Aco(MM) To Find Double Integrals Numerically

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Abstract:
The main objective of this research is to use methods of accelerating the first type of Al Tememe Acceleration, specifically triangular acceleration sine rule and triangular acceleration cosine rule which has the main error of the second level to find the continuous double integrations specified numerically with using the midpoint rule on both internal dimension \( t \) and exterior dimension \( z \) by both methods and we will code the first method with the symbol As(MM) and the second method the symbol Aco(MM) ; we assumed that the number of divisions on the internal dimension \( t \) was equal to the number of divisions on the external dimension \( z \), where we got in both methods a high accuracy in the results with relatively few partial periods and a short time with the use of the Matlab 2017 program.

1-Introduction:
There are many who don't know how much important calculus is in our life and this why Isaac Newton invented this complementary part of mathematics because he had found that algebra and geometry didn't solve many mathematical problems. At that time mathematicians could calculate the velocity of ships for example but failed to figure out the rate at which the ship was accelerating, so they needed a new mathematical method to solve the problems that involved changing variables; Newton spent about 18 months to form new theories in the science he called (the science of fluxions), which has evolved now for what we call (calculus) and which helped the engineer Apollo to chart the path between the earth and the moon [1,2].

In this days Medical Science Biologists use differential calculus to determine the exact rate of growth in a bacterial

in statistic Statisticians use calculus to evaluate survey data to help develop business plans for different companies, in Research Analysis it helps a company improve operating efficiency, increase production, and raise profits.[3]

Newton has helped many scientists in the development of the science of integration which interested in finding the function of the rate of its change that has extended to double integrations and finding its values for its importance in finding surface area for non-geometric shapes and unknown rules of its area, it can also calculate the inertial torque of flat surfaces and find the volume under the surface of double integrals [4].

and for the importance of double integrations the main objective in this research is to use a method that improves the results of numerical double integrations and increases velocity to reach the real value of integration by using the correction limits associated with the Newton-cotes rules. In 2009, Dheyaa used four composite methods of accelerating Romberg with the Simpsons rule and accelerating Romberg with the midpoint rule to give methods for calculating the values of double integral with continuous integrands, and
these methods are (RM(RS)), (RM(RM)), (RS(RS)) and (RS(RM)) and gave good results in terms of accuracy and number of partial periods used as well as the time spent calculating the integrations presented has shown that the four methods that adopted have given the same accuracy and the same number of partial periods for the integrations with continuous integrands into the integration intervals [5].

In 2019, Mohammed and Yasser used a series of new accelerated methods (AL-Tememe accelerations) to find specific unilateral integrations and the results were very good[1].

As for the numerical integration, it is defined as the study of how to find the approximate value of specific integration, and the history of applying numerical integration dating back to finding a circle area by (Greek Quadrature) method by dividing the circle into regular polygons, by this method (Archimedes) can find the upper and lower limits of pi ($\pi$).

In this research we will use the acceleration of AL-Tememe to find the approximate values of double integrations with continuous integrands and the limits of its correction $E(h)$, and this method is summarized by applying the rule of the mid point to both internal integration $t$ and external $z$ and where the number of divisions on the internal dimension $h_1$ is equal to the number of divisions on the external $h_2$ and then we improved the results using acceleration $As(MM)$ and $Aco(MM)$ then we got accurate results, and we will compare these methods in terms of accuracy and speed of approach to the analytical value of integration before and after using this acceleration.

Consider the integral $I$ defined as:

$$I = \int_{z_0}^{z_n} g(z)\,dz \quad \ldots (1)$$

$g(z)$ is a continuous function defined on $[z_0, z_n]$.

To compute the integral I approximately, we can write Newton-cotes formula as:

$$I = \int_{z_0}^{z_n} g(z)\,dz = G(k) + E(k) + R_G \quad \ldots (2)$$

Whereas, $G(k)$ is (Lagrangian- approximation) to the real value of the integral $I$, the letter $(G)$ refers to the rule type, $E(k)$ is the correction terms that can be added to $G(k)$, $k=\frac{z_n-z_0}{n}$; $n$ is the number of sub intervals we used and $(R_G)$ is the remainder.

The general formula of $G(k)$ is:

$$G(k) = k(w_0g_0 + w_1g_1 + w_2g_2 + \ldots + w_2g_n-2 + w_1g_n-1 + w_0g_n) \quad \ldots (3)$$

Where $g_m = g(z_m)$ and $z_m = z_0 + mk$; $m = 0,1,2,\ldots,n$ and weight coefficients $w_m$ take the sequence: $(w_0, w_1, w_2, \ldots, w_2, w_1, w_0)$.

In order to simplify the formula (3), we can write the weights by $w_0$ such that:

$$w_1 = 2(1-w_0) \quad \text{and} \quad w_2 = 2w_0,$$

we know that when $w_0 = 0$, we get midpoint rule and we symbolized by $M(k)$ where $M(k) = k(g_1 + g_3 + \ldots + g_{2j-1}); j=1,2,\ldots,n$.

The general form of $E(k)$ in the case of $M(k)$ is the following:

$$E(k) = \frac{1}{6} k^2(g_0' - g_0) + \frac{7}{360} k^4(g_0^{(3)} - g_0^{(3)}) + \frac{31}{15120} k^6(g_n^{(5)} - g_0^{(5)}) + \ldots \quad \ldots (4)$$

When the function of integrals and its derivatives are continuous in every point of integrations’ interval $[z_0, z_n]$.

The correction terms of $E(k)$ can be written as:

$$E(k) = I - M(k) = D_1k^2 + D_2k^4 + D_3k^6 + \ldots \quad \ldots (5)$$

Where $D_1, D_2, D_3, \ldots$ are constants that don’t depend on $k$ but on the values of integrands derivatives at the end point of interval $[z_0, z_n]$ [6].

Either in a case of double integral $I = \int_{z_0}^{z_n} \int_{t_0}^{t_n} f(t, z)\,dtdz \quad \ldots (6)$

The method MM (midpoint rule in both of external and internal dimension of integrals) is:

$$I = k\bar{k} \sum_{j=1}^{n} \sum_{i=1}^{m} f(t_0 + \frac{2i-1}{2}k, z_0 + \frac{2j-1}{2}k) \quad \ldots (7)$$
By division both of intervals \([z_0, z_n]\) and \([t_0, t_n]\) to the same number of subintervals
i.e. \(m=n\) then:
\[
k = k = n, \quad m = m = n, \quad n = 1, 2, 3, \ldots
\]
Therefore the integral I would be in the form:
\[
I = k^2 \sum_{j=1}^{n} \sum_{i=1}^{n} \int \left( t_0 + \frac{2i-1}{2} k, z_0 + \frac{2j-1}{2} k \right)
\]
\[
\ldots(8)[6]
\]
2- Accelerations As(MM) and Aco(MM):
To calculate the approximate value to the double integral (6) we will use (MM) method by using the formula
(8) we set \(n=1, 2, 3, \ldots\) to find the values of the second column after that we will implement “Triangular acceleration sine rule for Al-Tememe of the first kind”
\[
A_{\text{sin}}^\text{c}\approx \frac{k_1 \sin(k_2) M_1(k_1) - k_1 \sin(k_2) M_2(k_2)}{k_2 \sin(k_2) - k_1 \sin(k_1)} \quad \ldots(9)[1]
\]
Such that \(M_1(k_1)\) represents the value of the integration (6) numerically when \(k=k_1\), also, \(M_2(k_2)\) represents the value of same integration numerically when \(k=k_2\)
until we get a value with an error rate which is equal to or less than number named 
Eps (the absolute error of the difference between two consecutive values) this values representing the values of the third column.
As for the acceleration Aco(MM) we can get by the following:
We know that:
\[
E(k) = J - M(k) = A_1 k^2 + A_2 k^4 + A_3 k^6 + \ldots \approx \cos(k) - 1
\]
\[
J - M_1(k_1) \approx \cos(k_1) - 1 \quad \ldots(10)
\]
\[
J - M_2(k_2) \approx \cos(k_2) - 1 \quad \ldots(11)
\]
such that \(M_1(k_1)\) represents the value of the integration (6) numerically when \(k=k_1\), also, \(M_2(k_2)\) represents the value of same integration numerically when \(k=k_2\)
From the equations (10) and (11) we get:
\[
A_{\text{cos}}^\text{c} \approx \frac{M_2(k_2)(\cos(k_2) - 1) - M_1(k_1)(\cos(k_1) - 1)}{\cos(k_1) - \cos(k_2)} \quad \ldots(12)
\]
This acceleration get the same results of the triangular acceleration square sine law.
\[
A_{\text{sin}}^\text{c} \approx \frac{\sin^2(\frac{k_2}{2}) T_1(k_1) - \sin^2(\frac{k_2}{2}) T_2(k_2)}{\sin^2(\frac{k_2}{2}) - \sin^2(\frac{k_1}{2})} \quad [1]
\]
We implement the formula (12) on the values of (MM) method in the second column to get the values of fourth column until we get value with an error rate which is equal to or less than Eps.
3- Examples:
We will introduce some integrals which have continuous integrands on the internal and external integrals in the intervals \([t_0, t_n]\) and \([z_0, z_n]\) respectively and their numerical results by using (MM) method and their numerical results by using acceleration As(MM) and Aco(MM), and after that we will compare between the results before and after using acceleration.
3.1: \(\int_1^2 \int_0^1 te^{-(t+z)} dt dz\)
Its analytical value is 0.061447728197 and its rounded to 12 decimal.
3.2: \(\int_1^2 \int_0^1 \sqrt{t + zd} dt dz\)
Its analytical value is 1.40659671769 and its rounded to 12 decimal.
3.3: \(\int_1^2 \int_1^e z^2 dz dt\)
Its unknown analytical value.
4- The results:
In the example 3.1: \(\int_1^2 \int_0^1 te^{-(t+z)} dt dz\) the integrand is continuous with the intervals \([1,2] \times [0,1]\) and the
formula of the correction terms for the midpoint rules is similar to the formula in the equation (5).

We can notice from Table 1 when n=25 the values are correct for 9 decimal when we use As(MM) method
We can also notice when we use Aco(MM) method and when n=34 the values are correct for 9 decimal where 
Eps=10^-12, but we observed that the value by using (MM) method only without acceleration was correct for 3
decimal when n=25, and it was correct for 5 decimal when n=34.

In the example 3.2: \( \int_1^2 \int_0^1 \sqrt{1 + zdz} \) the integrand is continuous with the intervals \([1,2][0,1]\) and the formula of the correction terms for the midpoint rules is similar to the formula in the equation (5).

We notice from Table 2 when n=20 the values are correct for 8 decimal when we use As(MM) method, and we can also notice when we use Aco(MM) method when n=14 the values are correct for 8 decimal where Eps=10^-12, but we observed that the value by using (MM) method only without acceleration was correct for 4 decimal when n=16, 20 was correct for 4 decimal.

In the example 3.3: \( \int_1^2 \int_1^2 e^{zdz} \) the integrand is continuous with the intervals \([1,2][1,2]\) and the formula of the correction terms for the midpoint rules is similar to the formula in the equation (5).

We notice from Table 3 when we use As(MM) method, when n=61 the values are correct for 8 decimal, and we notice when we use Aco(MM) method when n=64 the values are correct for 8 decimal, and where Eps=10^-12, but we notice that the value by using (MM) method only without acceleration was correct for 4 decimal when n=61, 64.

5- Tables:

| n | MM          | As(MM)       | Aco(MM)      |
|---|-------------|--------------|--------------|
| 1 | 0.06166764416183 | 0.0613788962794 | 0.0615380957730 |
| 2 | 0.0631703930604 | 0.0614412252253 | 0.0614580262937 |
| 3 | 0.0622279647708 | 0.0614461569037 | 0.0614503173773 |
| 4 | 0.0618895221926 | 0.0614471697958 | 0.0614486623417 |
| 5 | 0.0617313426277 | 0.0614474815807 | 0.06144381888 |
| 6 | 0.0616450101815 | 0.0614476028200 | 0.0614479403780 |
| 7 | 0.0615928153372 | 0.0614476578283 | 0.06144785941 |
| 8 | 0.061558827852 | 0.0614476856908 | 0.061447800431 |
| 9 | 0.061535932012 | 0.0614477010210 | 0.061447744427 |
| 10 | 0.0615189215457 | 0.0614477100184 | 0.0614477591589 |
| 11 | 0.061365795566 | 0.0614477155801 | 0.0614477497005 |
| 12 | 0.0614971188560 | 0.0614477191675 | 0.0614477435942 |
| 13 | 0.0614899877832 | 0.0614477215654 | 0.0614477395100 |
| 14 | 0.0614840755239 | 0.0614477232174 | 0.0614477366948 |
| 15 | 0.0614793935874 | 0.0614477243854 | 0.061447734704 |
| 16 | 0.0614756611425 | 0.0614477252301 | 0.061447732628 |
| 17 | 0.0614723844809 | 0.0614477258532 | 0.0614477321998 |
| 18 | 0.0614697221108 | 0.0614477263209 | 0.0614477314016 |
| 19 | 0.0614674873572 | 0.0614477266776 | 0.0614477307928 |
| 20 | 0.0614655446872 | 0.06144772666776 | 0.0614477303220 |
| 21 | 0.0614638878989 | 0.0614477269534 | 0.0614477299533 |
| 22 | 0.0614624534375 | 0.0614477271694 | 0.0614477296612 |
| 23 | 0.0614612011397 | 0.0614477273404 | 0.0614477294274 |
| 24 | 0.0614601020444 | 0.0614477274774 | 0.0614477292385 |
| 25 | 0.0614591321458 | 0.0614477275880 | 0.06144772980845 |
| 26 | 0.0614582719617 | 0.06144772980845 | 0.0614477289579 |
| 27 | 0.061457055484 | 0.0614477288531 |
| 28 | 0.0614568197562 | 0.0614477287658 |
Table (1) for calculating integration $\int_1^2 \int_0^1 e^{-z^2} dtdz = 0.061447728197$ by using (MM) method, As(MM) acceleration and Aco(MM) acceleration

| n  | MM               | As(MM)               | Aco(MM)               |
|----|------------------|----------------------|-----------------------|
| 1  | 1.4142135623731  | 1.4065321464658      | 1.406596181816        |
| 2  | 1.4085777065555  | 1.4065745915453      | 1.4065940534395       |
| 3  | 1.4079996768585  | 1.4065983975163      | 1.406592342730        |
| 4  | 1.406921254332   | 1.4065975381690      | 1.406599234730        |
| 5  | 1.4068223777579  | 1.4065987311177      | 1.4065994809196       |
| 6  | 1.4067336760497  | 1.4065991940552      | 1.406599756897        |
| 7  | 1.406725038311   | 1.4065994038301      | 1.4065996181816       |
| 8  | 1.406697475473   | 1.4065995099947      | 1.4065996395370       |
| 9  | 1.4066799349045  | 1.406599683732      | 1.4065996512237       |
| 10 | 1.406660121861   | 1.4065996026217      | 1.406599680562        |
| 11 | 1.406654207637   | 1.4065996237855      | 1.4065996622675       |
| 12 | 1.4066471709340  | 1.4065996374336      | 1.406599649779        |
| 13 | 1.406640649517   | 1.4065996465543      | 1.4065996667864       |
| 14 | 1.4066353567266  | 1.4065996528368      | 1.4065996697686       |
| 15 | 1.4066310365619  | 1.4065996572783      | 1.406599691816        |
| 16 | 1.4066274558938  | 1.406599604901       | 1.4065996604901       |
| 17 | 1.4066244510461  | 1.406599628591       | 1.4065996628591       |
| 18 | 1.4066219154267  | 1.406599646374       | 1.4065996646374       |
| 19 | 1.4066197470110  | 1.406599659933       | 1.4065996697686       |

Table (2) for calculating integration $\int_1^2 \int_0^1 \sqrt{t + z^2} dtdz = 1.406599671769$ by using (MM) method, As(MM) acceleration and Aco(MM) acceleration

| n  | MM               | As(MM)               | Aco(MM)               |
|----|------------------|----------------------|-----------------------|
| 1  | 2.7182818284590  | 2.9494868542097      | 2.9436339024608       |
| 2  | 2.8836226730752  | 2.9570835130932      | 2.9563697474505       |
| 3  | 2.9236601828792  | 2.958711642191      | 2.958521141437        |
| 4  | 2.938833866972   | 2.959154934483      | 2.9590863689145       |
| 5  | 2.946104221533   | 2.9593096617864      | 2.9592788504710       |
| 6  | 2.9501178329543  | 2.959374052926      | 2.9593582175696       |
| 7  | 2.9525651913401  | 2.95940466621       | 2.959395256116        |
| 8  | 2.9541639654405  | 2.959420827077      | 2.9594148628278       |
| 9  | 2.955248714273   | 2.9594291505842      | 2.9594256719035       |
| 10 | 2.956054765372   | 2.959434420293      | 2.9594320881171       |
| 11 | 2.956640578316   | 2.95943770690636     | 2.9594360876697       |
| 12 | 2.9570867656082  | 2.9594398459188      | 2.959438640295        |
| 13 | 2.9574345109520  | 2.9594412823300      | 2.959440281119        |
| 14 | 2.9577107187750  | 2.959442763612      | 2.9594416343958       |
| 15 | 2.957933719292   | 2.9594429817489      | 2.9594424900241       |
| 16 | 2.9581163736263  | 2.9594434930424      | 2.9594431104303       |
| 17 | 2.9582678258369  | 2.9594434930424      | 2.9594431104303       |
|   |   |   |
|---|---|---|
| 18 | 2.9583948024678 | 2.9594438717621 |
| 19 | 2.9585023043118 | 2.959443737893 |
| 20 | 2.9585941178067 | 2.95944421483 |
| 21 | 2.9586731523152 | 2.959446741532 |
| 22 | 2.9587416730035 | 2.95944680200 |
| 23 | 2.9588014646710 | 2.95944762816 |
| 24 | 2.9588539486866 | 2.95944795624 |
| 25 | 2.9589002695633 | 2.95944851674 |
| 26 | 2.9589413548867 | 2.959448985107 |
| 27 | 2.9589779649632 | 2.959450312925 |
| 28 | 2.9590107767721 | 2.959450689649 |
| 29 | 2.9590416124131 | 2.959450103829 |
| 30 | 2.9590667040637 | 2.959451267515 |
| 31 | 2.9590907181972 | 2.959451490139 |
| 32 | 2.9591255085021 | 2.959451679141 |
| 33 | 2.9591327772303 | 2.959451840430 |
| 34 | 2.9591505056885 | 2.959451978742 |
| 35 | 2.9591671030895 | 2.959452097893 |
| 36 | 2.9591823374785 | 2.959452200986 |
| 37 | 2.9591963540954 | 2.959452290542 |
| 38 | 2.9592092793399 | 2.959452368645 |
| 39 | 2.9592212362977 | 2.959452437008 |
| 40 | 2.9592322837649 | 2.959452497046 |
| 41 | 2.9592425448947 | 2.959452549953 |
| 42 | 2.9592520821604 | 2.959452596719 |
| 43 | 2.9592690620743 | 2.959452638180 |
| 44 | 2.9592692436817 | 2.959452675042 |
| 45 | 2.9592769795450 | 2.959452707903 |
| 46 | 2.9592842165792 | 2.959452737274 |
| 47 | 2.959290996645 | 2.959452763590 |
| 48 | 2.9592973577656 | 2.959452787222 |
| 49 | 2.959303334100 | 2.959452808497 |
| 50 | 2.9593089542299 | 2.9594528270825 |
| 51 | 2.9593142477613 | 2.9594528731414 |
| 52 | 2.9593192389279 | 2.959452880369 |
| 53 | 2.9593239503251 | 2.959452881355 |
| 54 | 2.9593284024769 | 2.9594528860750 |
| 55 | 2.9593326140589 | 2.9594528979788 |
| 56 | 2.9593366020953 | 2.959452910555 |
| 57 | 2.959340821303 | 2.959452920404 |
| 58 | 2.9593439638048 | 2.95945292418 |
| 59 | 2.9593473736681 | 2.959452937679 |
| 60 | 2.9593506105401 | 2.959452945266 |
| 61 | 2.9593536893720 | 2.959452952239 |
| 62 | 2.9593566204620 | 2.959452931033 |
| 63 | 2.9593594113136 | 2.959452938799 |
| 64 | 2.9593620759091 | 2.959452952571 |

Table (3) for calculating integration $\int_1^2 \int_1^2 e^{ztdz} = \text{unknown analytical value}$ by using (MM) method, $A_{\text{st}}(\text{MM})$ acceleration and $A_{\text{co}}(\text{MM})$ acceleration

6-Conclusion:
The result of this study showing that, when calculating continuous double integrals by using the methods As(MM) and Aco(MM) (depending on the triangular functions of Al-Tememe acceleration methods of first kind) [1], was obtained accuracy in results with a speed of approaching the analytical value as well as with a few partial periods.

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