Continuous Preparation of a Fractional Chern Insulator

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We present evidence of a direct, continuous quantum phase transition between a Bose superfluid and the 1/2 Laughlin state in a microscopic lattice model of optically dressed spin defects. In the process, we develop a detailed field theoretic description of this transition in terms of the low energy vortex dynamics, which takes into account the role of lattice symmetry breaking and half-filling. The continuity of this transition enables the quasi-adiabatic preparation and study of the topological state by optical techniques.

Quantum optical realizations of topological order promise to revolutionize our ability to harness and probe topological states. The canonical examples of topological order are provided by the fractional quantum Hall states, conventionally found in two-dimensional electron gases [11–12]. Their lattice cousins, the fractional Chern insulators (FCI), naturally arise when strongly interacting particles inhabit flat, topological band-structures [3–14]. Effective microscopic Hamiltonians whose ground states realize such phases have been discovered in synthetic quantum systems, ranging from ultracold gases in optical lattices to ensembles of solid-state defects [15–17].

Unlike typical condensed matter systems, quantum optical proposals of topological phases represent driven, non-equilibrium implementations in an effective Hamiltonian picture. Thus, even if an appropriate Hamiltonian can be realized, guiding the system to its ground-state is still a major challenge. Often, one cannot simply “cool” by decreasing the temperature of a surrounding bath. One approach to this problem is provided by quasi-adiabatic preparation, wherein the correlated ground state is reached from a simple initial state by slowly tuning the Hamiltonian parameters. In the case of FCIs, natural starting states include superfluids (SF) and charge-density wave (CDW) insulators, as these often arise in close proximity to the FCI state of interest [15].

Quasi-adiabatic preparation requires that any quantum phase transition between the initial and final state be continuous. A system tuned through a first order transition would need to be ramped exponentially slowly in system size to avoid being stuck in a metastable high energy state [15–19]. On the other hand, continuous quantum phase transitions allow for the possibility of either strictly adiabatic preparation with ramp time scaling as a power law in system size [20–22] or quasi-adiabatic preparation with a final state energy density scaling only as an inverse power law with the ramp time [23–25]. Unfortunately, there is relatively little known regarding quantum phase transitions between conventional and fractional phases as such transitions lie beyond the Ginzburg-Landau paradigm [26].

Field theories of possible critical points between Laughlin fractional quantum Hall states and Mott insulators were studied in [27–29], while the theory of superfluid to bosonic 1/2 Laughlin state was recently constructed in [30]. All of these theories assume that any additional lattice symmetries are preserved throughout the phase diagram. They thus require the bosons to be at integer filling of the lattice, and do not describe CDW insulators. Moreover, to date, none of these continuous transitions has been established in any microscopic model, as second order phase transitions are difficult to characterize in the small systems amenable to numerical study.

In this paper, we report two main advances. First, we establish the presence of a direct continuous transition between a superfluid and $\nu = 1/2$ FCI state in a microscopic model of interacting solid-state spin defects. We do this by showing that the direct superfluid - FCI transition splits into two transitions when we break inversion symmetry by adding periodic boundary conditions. We present evidence of a direct, continuous quantum phase transition between a Bose superfluid and the 1/2 Laughlin state in a microscopic lattice model of optically dressed spin defects. In the process, we develop a detailed field theoretic description of this transition in terms of the low energy vortex dynamics, which takes into account the role of lattice symmetry breaking and half-filling. The continuity of this transition enables the quasi-adiabatic preparation and study of the topological state by optical techniques.

![Figure 1](https://example.com/figure1.png)

**FIG. 1.** (a) Two parameter phase diagram of the driven NV model as determined by exact diagonalization of Eq. (2). (b) Phase diagram in the presence of microscopic inversion symmetry breaking parameter $g = 0.2$. The $(\pi, \pi)$ CDW insulator extends in two fingers which split the SF $\leftrightarrow$ FCI transition, showing that the underlying transition at $g = 0$ is continuous and protected by inversion symmetry. Spectra and structure factors collected on coarse grid sites; full diagnostics (see text) calculated on 1-D (red) cuts at spacing of 0.01. Markers with errorbars indicate regions where diagnostics were ambiguous. Markers without errorbars indicate ambiguous regions narrower than marker size.
symmetry in the microscopic model, as predicted by the effective field theory of the continuous transition. Since a first order phase transition would not be affected by a small change in parameters of the system, the splitting of the transition implies that it must be continuous. This qualitative signature thus provides a way to avoid the usual difficulty of observing finite-size scaling behavior in small systems. Second, we develop a detailed field theoretic description of this transition in terms of the low-energy vortex fields. This description naturally accommodates the spontaneous breaking of lattice symmetry of the Mott-insulating CDW state at half-filling, going beyond previous work.

Microscopic Model— The specific microscopic model whose phase diagram we study is a system of spin-1 impurities, Nitrogen-Vacancy (NV) centers in diamond, arranged in a two-dimensional square lattice. The model is closely related to those of the dipolar FCI states found numerically in systems of ultracold polar molecules [15]. We will briefly sketch the main ingredients below. See [17] and Appendix for more technical details of this realization.

To provide enough tunable parameters, we focus on $^{15}$N, which has nuclear spin $I = 1/2$ [31]. There are thus 6 states on each site: $|S_z, I_z\rangle$, where $S_z = \pm 1, 0, I_z = \pm 1/2$ are the possible electronic and nuclear spin states, respectively. Taking into account the zero-field splitting, an applied magnetic field and the hyperfine interaction, it is possible to arrange for each site to contain 4 states at low energy, the $S_z = 0, \pm 1$ states. By applying suitable optical dressing, the effective dynamics can be further restricted to a two-level system, with local dark states $|0\rangle = \beta |1, -\frac{1}{2}\rangle - \alpha |0, \frac{1}{2}\rangle$, and $|1\rangle = s |0, -\frac{1}{2}\rangle + v |1, \frac{1}{2}\rangle + w(\alpha |1, -\frac{1}{2}\rangle + \beta |0, \frac{1}{2}\rangle)$. The coefficients $\alpha, \beta$ are fixed by the hyperfine interaction and applied static fields while the coefficients $s, v, w$ are tunable by the optical dressing. In the rotating frame, the states $|0\rangle$ and $|1\rangle$ are split by an energy $\Delta$.

The defects interact via the magnetic dipole-dipole interaction,

$$H_{dd} = \frac{1}{2} \sum_{i \neq j} \frac{\kappa}{R_{ij}^3} \left[ \mathbf{S}_i \cdot \mathbf{S}_j - 3(\mathbf{S}_i \cdot \hat{\mathbf{R}}_{ij})(\mathbf{S}_j \cdot \hat{\mathbf{R}}_{ij}) \right],$$

(1)

where, $\kappa = \mu_0/(4\pi)$ and $\mathbf{R}_{ij}$ connects sites $i$ and $j$. The characteristic dipolar interaction strength is $\kappa/R_0^3 \ll \Delta$ where $R_0$ is the nearest-neighbor lattice spacing. Thus, while the dipolar interaction can ‘flip-flop’ $|10\rangle \leftrightarrow |01\rangle$ between sites resonantly, processes which change the total number of 1 sites are energetically suppressed. This emergent conservation law allows us to consider the system in terms of conserved hardcore bosonic operators, $a_i^\dagger = |1\rangle \langle 0|_i$, described by the Hamiltonian

$$H_B = -\sum_{ij} t_{ij} a_i^\dagger a_j + \frac{1}{2} \sum_{i \neq j} V_{ij} n_i n_j,$$

(2)

where $t_{ij} = \langle 1, 0| H_{dd} |0, 1\rangle$ and $V_{ij} = \langle 1, 1| H_{dd} |1, 1\rangle + \langle 0, 0| H_{dd} |0, 0\rangle - \langle 1, 0| H_{dd} |1, 0\rangle - \langle 0, 1| H_{dd} |0, 1\rangle$. The on-site energy $\Delta \sum_i a_i^\dagger a_i$ is a constant when particle number is fixed, and can be ignored. In addition to boson number conservation, $H_B$ is symmetric under lattice translations and spatial inversion, but not generically under any further lattice rotations unless the NV axis is perpendicular to the lattice plane.

An FCI can be realized in this system with two main kinetic ingredients: the single boson bands ought to be “flat”, such that their dispersion is small relative to the interactions, and they ought to carry a non-trivial Chern number. Such topological flat bands may be achieved by using different optical dressing parameters on the $a$ and $b$ sites of a two-site unit cell super lattice (red and blue, Fig. 2a); this amounts to defining $|1\rangle$ differently on the $a$ and $b$ sublattices [13]. Optimizing the dressing parameters reveals band structures with flatness ratios (bandgap to bandwidth) of order 10.

We now consider the many-body phases which arise at finite lattice filling fraction $\nu = 1/2$ per unit cell (i.e. 1/4 particle per site) in a regime with just such a flat underlying band structure. A phase diagram is shown in a two-parameter cut in Fig. 1a, calculated using exact diagonalization for sizes up to $N_{sites} = 36, N_{particles} = 9$. The parameters varied in this phase diagram are the optical dressing on $a$ sites ($\theta_a = \tan^{-1} (|v_a|/s_a)$) and the azimuth angle of the NV axis relative to the lattice plane $\Phi_0$. Roughly, $\theta_a$ controls the magnitude of the effective
dipolar interaction while $\Phi_0$ introduces additional band dispersion. These qualitative differences in the microscopics yield a rich phase diagram exhibiting both conventional and topological phases, as shown in Fig. 1a. A $\nu = 1/2$ bosonic Laughlin FCI arises where the dispersion is flattest and the dipolar tail of the interaction is weak. Turning up the interactions by varying $\theta_a$ causes the system to spontaneously break the lattice translational symmetry and form a commensurate CDW insulator at momentum $(\pi, \pi)$. Tuning away from the flat band regime by adjusting $\Phi_0$ leads to a phase transition into a superfluid, consistent with the microscopics being dominated by band dispersion. We identify these phases numerically with five diagnostics: i) ground-state degeneracy, ii) spectral flow under magnetic flux insertion (superfluid response), iii) real-space structure factor $(n(R)n(0))$, iv) the many-body Berry curvature $\sigma_{xy}$, and v) (for the FCI), the quasi-hole counting $\langle \bar{n} \rangle$. 

The above diagnostics unambiguously determine the phases deep within each phase. The phase boundaries sketched in Fig. 1a correspond to the regions where the diagnostics become ambiguous due to the finite size crossovers. These also correspond to the regions where the ground state energy varies most sharply as a function of the control parameters $\theta_a$ and $\Phi_0$, see Fig. 2a for a typical example. The error bars in the phase diagram correspond to the width of the crossover region as observed in the 5 diagnostics.

Whether the transition is continuous or first order is hard to extract directly by conventional methods from such small size numerics. So we use a trick: the known critical theories describing the direct SF$\leftrightarrow$FCI transition require a discrete symmetry, such as inversion, to protect them. If this symmetry is explicitly broken perturbatively, these theories predict that the direct transition splits into two transitions with an intervening Mott insulating state. On the other hand, were the transition critical theories describing the direct SF$\leftrightarrow$CDW insulator must stay on integer filling and thus need not break lattice translational symmetry. Here, we will present a different theory that applies directly to the case of interest in this paper, bosons at half-integer filling of a lattice, which takes into account the fact that the CDW insulator must spontaneously break lattice symmetry $\mathbb{Z}_2$. En passant, our new theory provides an alternative physical representation of the transition which emphasizes the role of vortex dynamics. A detailed account of this theory will appear in a longer companion work.

We begin by briefly reviewing the effect of half-filling on the vortices of a superfluid state on a rectangular lattice $\mathbb{Z}_4$. In the dual vortex theory, the vortices see the original particles as magnetic flux quanta $\mathbb{Z}_4$ and thus they feel on average a half flux quantum per plaquette of the dual lattice. This requires the translational symmetries of the vortex theory to be augmented by gauge transformations in order to commute with the Hamiltonian. The resulting $T_x$ and $T_y$ operators satisfy the “magnetic” translation algebra $T_x T_y = -T_y T_x$. From this it follows that the vortex bandstructure must have an even number of minima, protected by the translation symmetries. If these minima are not at inversion symmetric points in the magnetic Brillouin zone, then inversion symmetry $I$ requires that the number of minima be a multiple of 4, see Fig. 3a. The minimal case protected by this symmetry is that there are four such

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure3.png}
\caption{(a) Magnetic Brillouin zone for vortex fields $\phi_{i\alpha}^0$ in Landau gauge. Circles indicate dispersive minima and where the slow vortex fields are defined. (b) Two parameter phase diagram of theory (4) without inversion breaking. Slice in $r, v_1$ holding $v_2 < v_3 < 0$, $w_2 < 0$ and $w_1, v_3, v_2 > 0$ and $u > 0$ large enough to stabilize the potential, yields an inversion breaking CDW with $(\pi, \pi)$ ordering and a superfluid with $(\pi, \pi)$ current order. (c) Same phase diagram with $g \neq 0$ breaking inversion.}
\end{figure}
minima, which in Landau gauge we take to be at momenta $\pm k_0, \pm \pi k_0 + (0, \pi)$. We consider a soft-mode expansion of the vortex field near these minima, leading to four flavors of vortices which we label $\phi^\tau_{l\alpha}$ for $l = 0, 1$ and $\alpha = \uparrow, \downarrow$ with the following action of the symmetry operators:

$$
\begin{align*}
I : \phi^v \rightarrow & \tau_x \phi^v \\
T_x : \phi^v \rightarrow & e^{ik_0 x} \sigma_x \phi^v \\
T_y : \phi^v \rightarrow & e^{ik_0 y} \sigma_y \phi^v
\end{align*}
$$

where the $\tau$ ($\sigma$) Pauli matrices act on the $\alpha$ ($l$) index and $k_0$ is the momentum of the 0 $\uparrow$ field.

In the superfluid state, all of these vortices are undensified. When any combination of them condenses, the superfluid order is destroyed while the translation symmetry is broken, leading to insulating density wave states \[41\]. Remarkably, the 1/2 Laughlin FQH state can also be understood as a state where the vortices form an integer quantum Hall state \[38, 39\]. The above facts motivate the following vortex field theory which can interpolate between the FQH, superfluid, and CDW states:

$$
\begin{align*}
\mathcal{L} = & \frac{1}{2\pi} A_c \partial a + \frac{1}{2\pi} b^\dagger \partial b^\dagger - \frac{1}{2\pi} a \partial (b^\dagger + b^\dagger) \\
& + \sum_l |(\partial - ib\tau_x) \phi_l|^2 - V(\{\phi_l\}),
\end{align*}
$$

where the notation $a \partial b \equiv e^{i\mu\lambda} a_\mu \partial_\lambda b_\lambda$. Here, $a$ and $b^\dagger$ are internal $U(1)$ gauge fields; $A_c$ represents a background external gauge field used to probe the underlying boson current $j^\mu = 1/2\pi \epsilon_{\mu\nu\lambda\sigma} \partial_\nu a_\lambda \partial_\sigma b_\sigma$. The Chern-Simons terms bind 2$\pi$ units of flux of $b^\dagger + b^\dagger$ to $\phi_{l\alpha}$. These flux-$\phi_{l\alpha}$ composites represent the original vortex fields $\phi^\tau_{l\alpha}$. Under the action of the lattice symmetries, $\phi_l$ can be taken to be symmetric as $\phi^\tau_l$ in Eq. (3), while the gauge fields $b$ are invariant under $T_x, T_y$ and swap under $\mathcal{I}$.

The potential term $V = r\phi^\dagger \phi + V_4 + \cdots$ includes all other terms compatible with the physical and gauge symmetries. At quartic order, there are 7 couplings,

$$
\begin{align*}
V_4 = & u(\phi^\dagger \phi)^2 + v_1 \sum_l |\phi_{l\uparrow} \phi_{l\uparrow}|^2 + v_2 \sum_\alpha |\phi_{0\alpha} \phi_{1\alpha}|^2 \\
& + v_3 (|\phi_{0\uparrow} \phi_{1\uparrow}|^2 + |\phi_{1\downarrow} \phi_{0\uparrow}|^2) + w_1 \sum_\alpha \phi^*_{0\alpha} \phi_{1\alpha} \\
& + w_2 \phi^*_{0\uparrow} \phi_{1\uparrow} \phi_{0\downarrow} + w_3 \phi^*_{0\uparrow} \phi_{1\uparrow} \phi_{1\downarrow} \phi_{0\uparrow} + \text{c.c.}
\end{align*}
$$

This theory Eqs. (4) \[5\] is one of the central results of this paper. It is capable of describing all three phases found in the microscopic model: (1) When the $\phi_{l\alpha}$ are uncondensed, $\langle \phi_{l\alpha} \rangle = 0$, they can be integrated out to yield a pure gauge theory, which is the effective theory of the 1/2 Laughlin state \[26\]. (2) If one of the $\phi_{l\alpha}$ condenses, $b^\dagger$ is gapped by the Anderson-Higgs mechanism; the resulting theory describes a Mott insulator which, as shown below, breaks translation symmetry. (3) If both $b^\dagger$ gauge fields are Higgsed, the resulting theory $\mathcal{L} = 1/2\pi A_c \partial a + (\partial a)^2 + \cdots$ is the usual dual description of a superfluid.

The pattern of inversion and translation symmetry breaking in these phases follows from the behavior of the simplest gauge-invariant bilinears in the $\phi$ fields:

$$
\begin{align*}
\mathcal{O}_{0,0}^\sigma & = \phi_{1\uparrow}^\dagger \phi_{1\uparrow} & \mathcal{O}_{0,0}^\tau & = \phi_{1\uparrow}^\dagger \sigma^z \phi_{1\uparrow} \\
\mathcal{O}_{1,1}^\sigma & = \phi_{0\uparrow}^\dagger \sigma^x \phi_{0\uparrow} & \mathcal{O}_{1,1}^\tau & = \phi_{0\uparrow}^\dagger \sigma^y \phi_{0\uparrow}
\end{align*}
$$

where the $\sigma$ Pauli matrices act on the $l$ indices of the $\phi$ field. The operators $\mathcal{O}_{l \uparrow, k_y}^\sigma$ carry momentum $(k_x, k_y)$.

The linear combination $\mathcal{O}_{k_x, k_y}^\pm \equiv \mathcal{O}_{k_x, k_y}^\uparrow \pm \mathcal{O}_{k_x, k_y}^\downarrow$ is inversion even (odd). Depending on which $\mathcal{O}_{k_x, k_y}^\pm$ acquire expectation values, we can determine how translation and inversion symmetries are broken \[40\].

Figure 3b shows a particular 2-parameter slice of the mean-field phase diagram of \[41\] which shows direct continuous transitions between the $FCI \leftrightarrow SF$ and $FCI \leftrightarrow CDW$ phases, along with a continuous triple point terminating the first order line separating the $SF \leftrightarrow CDW$ phases. The CDW order is at momentum $(\pi, \pi)$, as seen in the numerics, while the superfluid has $(\pi, \pi)$ current order. Turning on the leading inversion breaking potential $V = g \phi^\dagger \pi \phi$, we find that the direct transition $FCI \leftrightarrow SF$ is split into two transitions by an intervening CDW state with width proportional to $g$ as in Fig. 3c. The topology of these phase diagrams, and in particular the splitting under inversion, matches that observed numerically in Fig. 1.

Other regions of the coupling phase space provide slightly different symmetry breaking patterns for the SF and CDW phases in similar phase diagram topologies – in all cases, the insulators exhibits commensurate density order and the SF breaks some lattice symmetry in addition to the broken $U(1)$. This is consistent with the observation that a superfluid in a band structure with non-inversion symmetric minima will likewise either condense into a standing wave or break inversion. The numerically determined dispersion for Eq. (2) indeed exhibits non-inversion symmetric minima, but the small sizes available to study the interacting superfluid prevent us from determining unequivocally whether this prediction is borne out.

Conclusions— We have provided the first concrete evidence that microscopic models exist which exhibit a continuous transition out of an FQH/FCI state. Surprisingly, our microscopic model arises directly from the physics of a lattice of NV centers. The transition lies well beyond the Ginzburg-Landau paradigm as the loss of topological order coincides with the emergence of symmetry breaking. Thus, our numerical results simultaneously validate the vortex field theory we present, along with previous field theoretic work \[30\]. The new description of the SF ↔ FCI transition also has implications
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I. Microscopics and Numerical Diagnostics

Here, we provide a description of the microscopic spin model underlying the numerics presented in the main text. To be specific, we consider Nitrogen-Vacancy defect centers in diamond. The electronic ground state of each NV center is a spin-1 triplet described by the Hamiltonian,

$$H_{NV} = D_0 S_z^2 + \mu_e BS_z, \quad (7)$$

where $D_0 = 2.87$ GHz is the zero field splitting, $\mu_e = -2.8$ MHz/Gauss is the electron spin gyromagnetic ratio, and $B$ is a magnetic field applied parallel to the NV axis. This electronic spin is coupled via hyperfine interactions to the $I = 1/2$ nuclear spin of the $^{15}$N impurity via

$$H_{HF} = A_I S_z I_z + A_\perp (S_x I_x + S_y I_y), \quad (8)$$

where $A_I \approx 3.0$ MHz and $A_\perp \approx 3.7$ MHz. We assume that the states $| -1, \pm \frac{1}{2} \rangle$ are far detuned by a dc magnetic field, and tune to the crossing of $| 0, -\frac{1}{2} \rangle$ and $| 1, \frac{1}{2} \rangle$, where states are labeled by $| S_z, I_z \rangle$. The $A_\perp$ term in (8) mixes the $| 0, \frac{1}{2} \rangle$ and $| 1, -\frac{1}{2} \rangle$ states, yielding the energy levels shown versus magnetic field in Fig. 1. We now define the states $| A \rangle = \beta | 1, -\frac{1}{2} \rangle - \alpha | 0, \frac{1}{2} \rangle$, $| B \rangle = | 0, -\frac{1}{2} \rangle$, $| C \rangle = \beta | 1, \frac{1}{2} \rangle + \alpha | 0, -\frac{1}{2} \rangle$, and $| D \rangle = | 0, \frac{1}{2} \rangle$. Then the field in (8) is given by $H_{HF} = A_\perp (| B \rangle \langle B | + | D \rangle \langle D |)$, which we use to describe the evolution of the states $| A \rangle$ and $| B \rangle$. If we assume that the $| A \rangle$ state is populated at $t = 0$, then the evolution of the system is given by

$$| A(t) \rangle = e^{-iH_{HF}t/\hbar} | A \rangle = e^{iA_\perp t/\hbar} | B(t) \rangle.$$
\(|\psi\rangle = |1, \frac{1}{2}\rangle\), and \(|D\rangle = \alpha |1, -\frac{1}{2}\rangle + \beta |0, \frac{1}{2}\rangle\). To allow for resonant hops of spin excitations we work at the point where states \(|B\rangle\) and \(|C\rangle\) are degenerate, which sets the coefficients \(\alpha = 0.56\) and \(\beta = 0.83\).

The effective states we use on each NV center are \(|\psi\rangle = |A\rangle\) and \(|\psi\rangle = |B\rangle + v|C\rangle + w|D\rangle\). The coefficients \(\alpha, v, w\) are determined via an optical “M” dressing scheme \([17]\) (see Fig. 4b) in which four lasers are used to couple in two electronic excited states, \(|\pm\rangle = |E_x\rangle \pm |A_2\rangle\), \(|E_x\rangle, |A_2\rangle\) are two specific excited electronic states of the NV. The state \(|\psi\rangle = |\Omega_2\Omega_3/\Omega, \psi = \Omega_1/\Omega|, w = -\Omega_1/\Omega|\Omega_4/\Omega|\) is a normalization. Note that lasers 1 and 3 must be linearly polarized, while lasers 2 and 4 are circularly polarized.

In the numerics presented in the main text, we use the parameterization

\[
|\psi\rangle = s_i \sin(\alpha_i) \sin(\theta_i), \quad v_i = \sin(\alpha_i) \cos(\theta_i) e^{i\phi_i}, \quad w_i = \cos(\alpha_i) e^{i\gamma_i}
\]

where \(i \in \{a, b\}\) (recall the square lattice is partitioned into \(a\) and \(b\) sites). The mixing angle \(\tan(\theta_i) = |s_i/v_i|\) characterizes the strength of the effective dipole moment of \(|\psi\rangle\), thereby determining the magnitude of the interactions. In the limit of \(\theta_i \to 0\) one finds that the spin-flip excitation carries minimal weight in \(|B\rangle = |0, -\frac{1}{2}\rangle\) and maximal weight in \(|C\rangle = |1, \frac{1}{2}\rangle\). Since the electronic spin dipole moment of \(|B\rangle\) is effectively zero, this implies that the dipolar interaction strength increases as \(\theta_i \to 0\). While topological flat-bands can be found for a variety of parameter regimes, we find that the closest numerics are obtained for: \(\theta_0 = 0.615, \Phi_0 = 5.32, \theta_a = 0.598, \theta_b = 1.051, \phi_a = 1.087, \phi_b = 3.402, \alpha_a = 2.844, \alpha_b = 2.258, \gamma_a = 4.089, \gamma_b = 4.047\). \((\theta_0, \Phi_0)\) are the polar angles describing the orientation of the NV axis relative to the 2D lattice plane. Here the bands exhibit a flatness ratio \(f \approx 8.8\) (Fig. 4c) and phase diagrams are subsequently obtained by varying \(\Phi_0\) and \(\theta_a\).

We now provide detailed examples of the diagnostics used to determine the many-body phases which arise at finite lattice filling fraction. The topological features of the \(\nu = 1/2\) FCI require the presence of a two-fold ground state degeneracy on a torus (Fig. 5a) as well as quasi-hole statistics which agree with a generalized Pauli principle (inset of Fig. 2 in maintext). Furthermore, the quantity analogous to the Hall conductance, \(\sigma_{xy} = \frac{e^2}{2\pi h} \int \int F(\theta_x, \theta_y) d\theta_x d\theta_y\), is non-ambiguous in the response of the system to boundary-condition twists \(\{\theta_x, \theta_y\}\). To diagnose the CDW, we require ground state degeneracy with \(\sigma_{xy} = 0\). Moreover, twisting the boundary condition in either the \(\hat{x}\) or \(\hat{y}\) direction does not affect the ground state energy suggesting an insulator. Finally, to diagnose a SF, we require a unique ground state. While our system sizes are too small to clearly observe the Goldstone mode of the SF, in contrast to the CDW, twisting the boundary condition dramatically alters the ground state energy; this is consistent with a SF which harbors long range phase coherence and hence, whose energies would naturally be affected by twists in the boundary condition.

To determine rough phase boundaries between the FCI, CDW and SF, we examine the change in the ground state energy as a function of \(\theta_a\) and \(\Phi_0\). In particular, we expect stable phases to occur as “smooth” plateaus of \(dE/d\theta_a\) \((dE/d\Phi_0)\), while phase transitions manifest as jumps in \(dE/d\theta_a\) \((dE/d\Phi_0)\). Figure 5b-h depicts examples of ground state energy cuts in both the horizontal and vertical direction of the phase diagram. Blue curves are obtained for no inversion breaking while red curves are obtained for \(g = 0.2\) and green curves for \(g = 0.4\).

For the horizontal cuts \(\theta_a = 0.75, 0.65, 0.55\) one indeed observes an extra kink between the FCI and SF phase upon the breaking of inversion. In this kink region, we find a two-fold degenerate ground state in momentum sectors \((0, 0)\) and \((0, \pi)\) with \(\sigma_{xy} = 0\). Moreover, we find that the real-space structure factor (see e.g. Fig. 3) is consistent with the intervening phase being a CDW checkerboard located on the B-site sublattice.
II. Field theory

Here, we provide some additional details regarding the analysis of the field theory, eq. (5) in the main text, reproduced here:

\[
\mathcal{L} = \frac{1}{2\pi} A_\alpha \partial_\alpha + \frac{1}{2\pi} b^\dagger \partial b^\dagger - \frac{1}{2\pi} a \partial (b^\dagger + b^\dagger) + \sum_l \left[ \left( \partial - i b^\dagger \gamma_x \right) \phi_l \right]^2 - V(\{\phi_l\}),
\]

(9)

As stated in the main text, this theory can simultaneously describe a superfluid, a CDW and a \(1/2\) Laughlin state, depending on whether the \(\phi\) fields are condensed such that they gap out both the \(b\) gauge fields by the Anderson-Higgs mechanism, one of the \(b\) fields or neither of the \(b\) fields. We describe the algebraic steps leading to these identifications below.

1/2 Laughlin state

When all scalar fields \(\phi_l\) are uncondensed (\(\langle \phi_l \rangle = 0\)) there is an energy gap to creating excitations associated with \(\phi_l\). Integrating them out yields only short-range interactions among the remaining fields. The resulting field theory is of the form

\[
\mathcal{L} = \frac{1}{2\pi} A_\alpha \partial_\alpha + \frac{1}{2\pi} b^\dagger \partial b^\dagger - \frac{1}{2\pi} a \partial (b^\dagger + b^\dagger) + \cdots,
\]

(10)

where \(\cdots\) include higher derivative terms for the gauge fields. These higher derivative terms are irrelevant compared to the Chern-Simons terms and so they can be ignored at long wavelengths. In this limit, we may integrate out \(b^\dagger\) to find the following constraint:

\[
\epsilon_{\mu\nu\lambda} \partial_\nu b^\lambda = \epsilon_{\mu\nu\lambda} \partial_\nu a_\lambda.
\]

(11)

Inserting this constraint back into (10) leads to

\[
\mathcal{L} = -\frac{2}{4\pi} a \partial a + \frac{1}{2\pi} A_\alpha \partial A_\alpha.
\]

(12)

This is the well-known effective Chern-Simons field theory for the \(1/2\) Laughlin state (see, e.g., [20]). To verify the Hall conductance, we can integrate out \(a\) and obtain the effective Lagrangian for the external probe field \(A_\alpha\),

\[
\mathcal{L} = \frac{1}{2\pi} A_\alpha \partial A_\alpha,
\]

(13)
which directly yields the 1/2 Hall conductance,
\[ j^\mu = \frac{\delta L}{\delta A_{e,\mu}} = \frac{1}{2} \frac{1}{2\pi} \epsilon^{\mu\nu\lambda} \partial_{\nu} A_{e,\lambda}. \] (14)

**Superfluid state**

Now we consider the case where both \( b^\dagger \) and \( b \) are gapped by the Anderson-Higgs mechanism. This occurs when \( \langle \phi_{l\dagger} \rangle \neq 0 \) and \( \langle \phi_{l} \rangle \neq 0 \), for some \( l, l' \). Upon integrating out \( b^\dagger \) and \( b \), which may be accomplished at long wavelengths by simply setting \( b^\mu = 0 \) in (9), we obtain the effective action
\[ L = \frac{1}{2\pi} A_e \partial a - \frac{1}{g} (\epsilon_{\mu\nu\lambda} \partial_{\nu} A_{e,\lambda})^2 + \cdots, \] (15)
where we have reinstated the leading higher order Maxwell term for \( a \). The \( \cdots \) include all other terms compatible with the gauge invariance of \( A_e \) and \( a \), and the lattice symmetries of the problem. In particular, this describes a theory of a massless 2+1 dimensional gauge field, where fluctuations of \( a \) physically correspond to particle density and current fluctuations, due to the coupling to the external probe field \( A_e \) in the first term. The above theory is precisely the dual action for a superfluid, where \( a \) is dual Goldstone mode of the superfluid.

As explained in the main text, in order to understand what additional lattice symmetries might be broken in this state, one must analyze the gauge invariant bilinears in \( \phi \), the \( O_{k_x,k_y}^{\pm} \) operators, that transform non-trivially under the inversion and lattice translations.

**Insulating state**

When \( \langle \phi_{l\alpha} \rangle \neq 0 \) for only one choice of \( \alpha \), then that \( b^\alpha \) is gapped by the Anderson-Higgs mechanism. For concreteness, we consider \( \alpha = \uparrow \). Setting \( b^\dagger = 0 \) then yields the following effective action:
\[ L = \frac{1}{2\pi} A_e \partial a - \frac{1}{2\pi} a \partial b^\dagger + \sum_l \left| \left( \partial - ib^\dagger \right) \phi_{l\uparrow} \right|^2 - \bar{V}(\{\phi_{l\uparrow}\}), \] (16)
where \( \bar{V} \) corresponds to the previous \( V \), but with the condensed bosons replaced by \( c \)-numbers. The remaining uncondensed bosons are massive. Integrating them out yields,
\[ L = \frac{1}{2\pi} A_e \partial a - \frac{1}{2\pi} a \partial b^\dagger + \cdots \] (17)

Now, we see that integrating out \( b^\dagger \) will enforce a constraint at long wavelengths:
\[ \epsilon_{\mu\nu\lambda} \partial_{\nu} a_{\lambda} = 0. \] (18)

This effectively Higgses \( a \). Reinstating the leading higher order terms for \( A_e \) gives the action
\[ L = -\frac{1}{g} (\epsilon_{\mu\nu\lambda} \partial_{\nu} A_{e,\lambda})^2 + \cdots, \] (19)
where \( \cdots \) include other terms compatible with the lattice symmetries and gauge invariance of \( A_e \). This is the effective response theory for an insulating state, as can be seen most simply by noting that the boson current \( j = \frac{\partial}{\partial t} A_e = 0 \) for uniform applied fields \( E = e\partial A_e \).

Moreover, all excitations of this phase are gapped, and there is no fractionalization, as expected for a topologically trivial insulator.

Again, in order to identify the type of symmetry-breaking order in this insulator, we need to analyze the fate of the gauge-invariant bilinears in \( \phi_{l\uparrow} \), which transform non-trivially under the symmetries. From this analysis, we conclude that the insulator necessarily breaks the lattice translation symmetries and is therefore properly identified as a CDW.

**Broken symmetry patterns**

The above states also lead to spontaneous breaking of the lattice symmetries. Here, we will provide some additional details of the analysis that allow us to determine the patterns of symmetry breaking for the superfluid and CDW state shown in Fig. 3 of the main text. A more exhaustive treatment for the full Ginzburg-Landau functional will appear in a future work.

In order to diagnose the patterns of broken symmetry, we use the gauge-invariant bilinear operators that transform non-trivially under the lattice translational and inversion symmetries. These were described in the main text. We reproduce them here for convenience:
\[ O_{0,0}^\alpha = \phi_{0,\alpha}^\dagger \phi_{0,\alpha}, \quad O_{\pi,0}^\alpha = \phi_{\pi,\alpha}^\dagger \sigma^x \phi_{\pi,\alpha}, \quad O_{\pi,\pi}^\alpha = \phi_{\pi,\alpha}^\dagger \sigma^y \phi_{\pi,\alpha} \] (20)

The linear combination \( O_{k_x,k_y}^{\pm} \equiv O_{k_x,k_y}^{\uparrow} \pm O_{k_x,k_y}^{\downarrow} \) is inversion even (odd).

The two-parameter slice of the phase diagram shown in Fig. 3 used the parameters \( v_2 < v_3 < 0, w_2 < 0, \) and \( w_1, v_3, v_3 > 0 \) and \( u > 0 \) large enough to stabilize the potential. We first consider the case where the inversion breaking parameter \( g = 0 \). In such a regime, one can verify that at mean-field level, when \( r > 0 \), all of the \( \phi \) are uncondensed, all lattice symmetries are preserved, and the system is in the FCI phase. When \( r < 0 \), then the system either realizes the superfluid phase (when \( v_1 < v_1^f \)) or the Mott insulating CDW phase (when \( v_1 > v_1^f \)). At the mean field level, the critical value is \( v_1^f = v_2 - 2u_1 - v_3 \).

In the Mott insulating CDW phase, in the parameter regime described above, the minimum of the Ginzburg-Landau functional requires \( |\phi_{0,1}| = |\phi_{1,1}| \neq 0, \phi_{0,1} = 0 \).
\( \phi_{1\downarrow} = 0 \), or vice versa \((|\phi_{0\downarrow}| = |\phi_{1\downarrow}| \neq 0, \phi_{0\uparrow} = \phi_{1\uparrow} = 0)\). Assuming the first case without loss of generality, we find that the fact that \( w_1 > 0 \) further implies in this regime that \( \phi_{0\uparrow} = \pm i\phi_{1\uparrow} \). Therefore, it is straightforward to verify:

\[
\langle O_{0,0} \rangle \neq 0, \quad \langle O_{\pi,0} \rangle = 0 \\
\langle O_{0,\pi} \rangle = 0, \quad \langle O_{\pi,\pi} \rangle \neq 0,
\]

while \( \langle O^I_{k_x,k_y} \rangle = 0 \). Therefore we see that the CDW phase in this parameter regime has \((\pi, \pi)\) ordering, as observed in the numerics.

In the superfluid phase, in the parameter regime described above, we find that the minimum of the Ginzburg-Landau functional requires \( |\phi_{0\downarrow}| = |\phi_{1\downarrow}| = |\phi_{0\uparrow}| = |\phi_{1\uparrow}| \), and \( \phi_{0\uparrow} = \pm i\phi_{1\uparrow}, \phi_{0\downarrow} = \mp \phi_{1\downarrow} \). From this, we can conclude that \( \langle O^a_{\pi,0} \rangle = 0, \quad \langle O^a_{0,\pi} \rangle = 0 \), and \( \langle O^T_{\pi,\pi} \rangle = -\langle O^T_{\pi,\pi} \rangle \neq 0 \). Therefore, \( \langle O^T_{\pi,\pi} \rangle = 0 \) and \( \langle O^T_{\pi,\pi} \rangle \neq 0 \). This implies that the superfluid phase has a non-zero order parameter with momentum \((\pi, \pi)\). Since this non-zero order parameter is inversion odd, it does not mix with the density, which is inversion even. It does, however, mix with the current, which is inversion odd. We conclude that the superfluid has a non-zero current order at \((\pi, \pi)\). Since the superfluid exists in the presence of strong time reversal symmetry breaking, it is reasonable that its ground state possesses non-zero average currents.

When \( g > 0 \), it is clear that the direct FCI to SF transition will be split into two transitions, with an intervening CDW state. This is because when \( g > 0 \), as we tune \( r \) from positive to negative, it is more favorable to first turn on the expectation value for \( \phi_{0\downarrow}, \phi_{1\downarrow} \) when \( r \sim g \), and then turn on the expectation value for the remaining fields when \( r \sim -g \).