A Morse type uniqueness theorem for non-parametric minimizing hypersurfaces

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Abstract

A classical result about minimal geodesics on $\mathbb{R}^2$ with $\mathbb{Z}^2$ periodic metric that goes back to H.M. Morse’s paper [10] asserts that a minimal geodesic that is asymptotic to a periodic minimal geodesic cannot intersect any periodic minimal geodesic of the same period. This paper treats a similar theorem for nonparametric minimizing hypersurfaces without selfintersections – as were studied by J. Moser, V. Bangert, P.H. Rabinowitz, E. Stredulinsky and others.

1 Introduction

The first progress to generalize the results of Morse [10] and G.A. Hedlund [5] – who studied the case of $\mathbb{R}^2$ with $\mathbb{Z}^2$-periodic metric – on minimal geodesics on surfaces to higher dimension was made by Moser [11]. He observed that the key features of minimal geodesics on $T^2$ are that they separate space and that they do not have selfintersections when projected to $T^2 = \mathbb{R}^2/D\mathbb{Z}^2$. We point out that this last property is not contained in the classical text and was proven in [2].

Amongst other theorems some of the classical results were generalized by Moser to graphs of functions $u : \mathbb{R}^n \to \mathbb{R}$, which are minimizers of a $\mathbb{Z}^{n+1}$-periodic variational problem and are without selfintersections. Below the setting is described precisely. Moser obtained an a priori estimate that

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asserts that any such graph stays within universally bounded Hausdorff distance to a plane, and he proved first existence results, namely that for any given unit vector $\alpha \in \mathbb{R}^{n+1}$ there exists such a graph that is within finite Hausdorff distance to a plane with unit normal $\alpha$. H. Koch, R. de la Llave and C. Radin, cf. [9], obtain results of this type for functions on lattices. A. Candel and de la Llave provide versions for functions on sets with more general group actions in [4]. In the framework of Moser, Bangert proves a fundamental uniqueness result in [1] and he carries out a detailed investigation of the minimal solutions in this framework in [3]. These result are considered as a codimension one version of Aubry-Mather Theory. Together with E. Valdinoci we observed in [8] that the results in [3] are related to a famous conjecture of E. de Giorgi. P.H. Rabinowitz and E. Sredulinsky also investigated the Moser framework in [12], [14] and [13]. They utilize a renormalized functional and find more complicated extremals – so called multibump solutions.

A central point in [3] is Theorem 2.1, cf. [3, Theorem (6.6)], however the proof given there is incomplete. With minor variations we adopt the notation of [3] and give a completion of the proof. Our strategy is inspired by Morse’s proof. In [6] we proved a version of this theorem for parametric minimizing hypersurfaces, cf. also [7]. Although it is possible to prove the parametric result carrying over the method used here, it is simpler and more natural to use the theory of (weak) calibrations. It is an open question whether there exists a suitable concept of calibration calibrating a given totally ordered family of nonselfintersecting minimizing graphs. It would be desirable to find a calibration that is $\mathbb{Z}^n$-invariant.

1.1 Moser’s variational problem and basic results

Given an integrand $F : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, periodic in the first $n+1$ variables, we study functions $u : \mathbb{R}^n \to \mathbb{R}$ that minimize the integral $\int F(x, u, u_x) \, dx$ w.r.t. compactly supported variations. We assume $F \in C^{2,\varepsilon}(\mathbb{R}^{2n+1})$ and that $F$ satisfies appropriate growth conditions, cf. [11, (3.1)], ensuring the ellipticity of the corresponding Euler-Lagrange equation. Under these conditions minimizers inherit regularity from $F$ and are of class $C^{2,\varepsilon}(\mathbb{R}^n)$. For $u : \mathbb{R}^{n+1} \to \mathbb{R}$ and $\bar{k} = (k, k') \in \mathbb{Z}^{n+1}$, define $T_{\bar{k}}u : \mathbb{R}^n \to \mathbb{R}$ as

$$T_{\bar{k}}u(x) = u(x - k) + k'. $$

Since $F$ is $\mathbb{Z}^{n+1}$-periodic, $T$ determines a $\mathbb{Z}^{n+1}$-action on the set of minimizers.

We look at minimizers $u$ without self-intersections, i.e. for all $\bar{k} \in \mathbb{Z}^{n+1}$ either $T_{\bar{k}}u < u$ or $T_{\bar{k}}u = u$ or $T_{\bar{k}}u > u$. Equivalently one can require that the
hypersurface \( \text{graph}(u) \subset \mathbb{R}^{n+1} \) has no self-intersections when projected into \( T^{n+1} = \mathbb{R}^{n+1}/\mathbb{Z}^{n+1} \).

We call minimizers without self-intersections shortly solutions and denote the set of all solutions by \( \mathcal{M} \). On \( \mathcal{M} \) we consider the \( C^1_{\text{loc}} \)-topology. For every \( u \in \mathcal{M} \) [11, Theorem 2.1] shows that \( \text{graph}(u) \) lies within universally bounded distance from a hyperplane. We define the rotation vector of \( u \) is as the unit normal \( \bar{a}_1(u) \in \mathbb{R}^{n+1} \) to this hyperplane, which has positive inner product \( \bar{a}_1 \cdot \bar{e}_{n+1} \) with the \((n+1)\)st standard coordinate vector.

Another fundamental result of Moser, cf. [11, Theorem 3.1], implies that every \( u \in \mathcal{M} \) is Lipschitz with constant depending only on \( \bar{a}_1(u) \) (and \( F \)).

If \( \bar{k} \cdot \bar{a}_1 > 0 \) (< 0), then \( T_{\bar{k}} u > u \) (< u). If \( \bar{k} \cdot \bar{a}_1 = 0 \), both cases are possible. There is a complete description in [3, (3.3)-(3.7)], that we subsume in Proposition 1.1.

For every \( u \in \mathcal{M} \) there exists an integer \( t = t(u) \in \{1, \ldots, n+1\} \) and unit vectors \( \bar{a}_1 = \bar{a}_1(u), \ldots, \bar{a}_t = \bar{a}_t(u) \), such that for \( 1 \leq s \leq t \) we have

\[
\bar{a}_s \in \text{span} \; \bar{\Gamma}_s, \quad \text{where} \; \bar{\Gamma} = \bar{\Gamma}_1 = \mathbb{Z}^{n+1} \quad \text{and} \quad \bar{\Gamma}_s = \bar{\Gamma}_s(u) := \mathbb{Z}^{n+1} \cap \text{span} \{\bar{a}_1, \ldots, \bar{a}_{s-1}\}^\perp,
\]

(1)

and the \( \bar{a}_1, \ldots, \bar{a}_t \) are uniquely determined by the following properties:

(i) \( T_{\bar{k}} u > u \) if and only if there exists \( 1 \leq s \leq t \) such that \( \bar{k} \in \bar{\Gamma}_s \) and \( \bar{k} \cdot \bar{a}_s > 0 \).

(ii) \( T_{\bar{k}} u = u \) if and only if \( \bar{k} \in \bar{\Gamma}_{t+1} \).

Moser proved in [11] that, if \( |\bar{a}_1| = 1 \) and \( \bar{a}_1 \cdot \bar{e}_{n+1} > 0 \), there exist functions \( u \in \mathcal{M} \) with \( \bar{a}_1(u) = \bar{a}_1 \). A system of unit vectors \( (\bar{a}_1, \ldots, \bar{a}_t) \) is called admissible if \( \bar{a}_1 \cdot \bar{e}_{n+1} > 0 \) and relation (1) is satisfied. For an admissible system \( (\bar{a}_1, \ldots, \bar{a}_t) \) we write

\[
\mathcal{M}(\bar{a}_1, \ldots, \bar{a}_t) = \{ u \in \mathcal{M} \mid t(u) = t \quad \text{and} \quad \bar{a}_s(u) = \bar{a}_s \; \text{for} \; 1 \leq s \leq t \}.
\]

The following observation describes the action of subgroups of \( \bar{\Gamma} \) on solutions.

Proposition 1.2. If \( u \in \mathcal{M}(\bar{a}_1, \ldots, \bar{a}_t) \), \( t > 1 \), then there exist functions \( u^- \) and \( u^+ \) in \( \mathcal{M}(\bar{a}_1, \ldots, \bar{a}_{t-1}) \) with the following properties:

(a) If \( \bar{k}_i \in \bar{\Gamma}_t \) and \( \lim_{i \to \infty} \bar{k}_i \cdot \bar{a}_t = \pm \infty \) then \( \lim_{i \to \infty} T_{\bar{k}_i} u = u^\pm \).

1We remark that our notion of rotation vector differs slightly from this notion in [3].
(b) $u^- < u < u^+$ and $T_k u^- \geq u^+$ if $k \in \overline{\Gamma}_s$ and $\overline{k} \cdot \overline{a}_s > 0$ for some $1 \leq s < t$.

Proof. [3 Proposition (4.2)].

Besides the fact that Theorem 2.1 below is of independent interest as uniqueness theorem, it is a central point in the proof of the following uniqueness and existence results, cf. [3 Sections 6 and 7]:

If $(\overline{a}_1, \ldots, \overline{a}_t)$ is admissible, then $\mathcal{M}(\overline{a}_1) \cup \mathcal{M}(\overline{a}_1, \overline{a}_2) \cup \ldots \cup \mathcal{M}(\overline{a}_1, \ldots, \overline{a}_t)$ are totally ordered. If $u_1, u_2 \in \mathcal{M}(\overline{a}_1, \ldots, \overline{a}_{t-1})$ satisfy $u_1 < u_2$ and are neighbouring, i.e. there exists no $u \in \mathcal{M}(\overline{a}_1, \ldots, \overline{a}_{t-1})$ with $u_1 < u < u_2$, then there exists $v \in \mathcal{M}(\overline{a}_1, \ldots, \overline{a}_t)$ with $u_1 < v < u_2$.

2 The Uniqueness Theorem

Theorem 2.1. Suppose $u \in \mathcal{M}(\overline{a}_1, \ldots, \overline{a}_t)$ and $t > 1$. Then there is no $v \in \mathcal{M}(\overline{a}_1, \ldots, \overline{a}_{t-1})$ with $u^- < v < u^+$.

For economical reasons it makes sense to use the following abbreviations for functions $u \in W^{1,2}_{\text{loc}}(\mathbb{R}^n)$ and $\varphi \in W^{1,2}_0(\mathbb{R}^n)$ and measurable sets $A \subset \mathbb{R}^n$ (cf. [11] and [3]):

$$I(u, A) := \int_A F(x, u, u_x) \, dx \quad \text{if this integral exists in } \mathbb{R} \cup \{\pm \infty\},$$

$$\Delta(u, \varphi, A) := \int_A \left( F(x, u + \varphi, u_x + \varphi_x) - F(x, u, u_x) \right) \, dx.$$

In order to prove the Theorem we will imitate Morse’s proof of [10 Theorem 13]. This is not straightforward because of several reasons:

The proof is based on comparison arguments for which we need to find “short” connections between solutions which are close (in $C^1_{\text{loc}}$). In the parametric case “slicing” from Geometric Measure Theory provides such short connections. In the non-parametric case we need connecting graphs, for which we can control the slope, because our variational problem punishes steepness. We extend the idea of [1, Lemma (6.8)] of constructing such connections.

In higher dimensions, we have to cope with two additional difficulties: Solutions could show different behaviour in different directions in view of Proposition [1]. A solution $u$ might be recurrent in some directions, periodic in some directions and heteroclinic in some directions (cf. [1] and [3]). Furthermore we can, in general, say nothing about how the hypersurfaces under consideration do intersect.
Proof of Theorem 2.1 for $n = 1$

In case $n = 1$ we carry over Morse’s technique to the non-parametric case. The proof in this case also serves as a guideline for the proof in case $n \geq 2$.

Suppose there exists a function $v \in \mathcal{M}(\bar{a}_1)$ with $u^- < v < u^+$. Following [3, proof of Theorem (6.6)], we choose the generator $\bar{k}_0 = (k_0, k'_0)$ of $\bar{\Gamma}_2 = \bar{\Gamma}_2(u)$ with $\bar{k}_0 \cdot \bar{a}_2(u) > 0$ and define

$$w = \max \left( u, \min(v, T_{\bar{k}_0} u) \right),$$

cf. figure [1] on page [8]. Clearly $k_0 \neq 0$. Without loss of generality we assume that $k_0 < 0$.

Remark 2.2. Why the proofs for $n = 1$ and $n \geq 2$ are different: The function $w$ (also in the higher dimensional case) is defined using $T_{\bar{k}_0} u$ and $k_0$ determines a one dimensional subspace $\mathbb{R}k_0 \subset \mathbb{R}^n$. We have to compare the energies of the functions $u$ and $w$ on domains that feature some periodicity in this direction. In case $n = 1$ we can use intervals, but in case $n \geq 2$ round balls are not suitable and, in view of Lemma [2.10], cuboids are also not suitable. We use cylinders with caps (the sets $\mathcal{Z}(r,t)$ below). Also the fact that $\mathbb{R}k_0 \subset \mathbb{R}^n$ for $n \geq 2$ makes a finer investigation necessary, cf. (12).

The Maximum Principle, cf. e.g. [11, Lemma 4.2], implies that $w$ is not minimizing. So we can save energy by a compactly supported variation. This observation is contained in the following lemma, which is a special case of Lemma 2.8 and proven in [3, (6.8)]:

**Lemma 2.3.** There exist $\delta > 0$ and $r_0 > 0$ and a function $\psi \in W^{1,2}_0(\mathbb{R})$ with $\text{spt} \psi \subset (-r_0, r_0)$ such that

$$\Delta \left( w, \psi, (-r_0, r_0) \right) < -\delta.$$

What is missing in the proof of [3, Theorem (6.6)] is the construction of a variation $u + \varphi$ of $u$ (with $\text{spt} \varphi$ contained in a compact interval $K$), that coincides with $w$ on $(-r_0, r_0)$ such that $I(u + \varphi, K) - I(u, K)$ is smaller than the gain $\delta$ provided by Lemma 2.3 say smaller than $\frac{\delta}{2}$:

**Lemma 2.4.** For $\delta > 0$ and $r_0 > 0$ from Lemma 2.3 there exist a compact set $K \supset (-r_0, r_0)$ and a function $\varphi \in W^{1,2}_0(\mathbb{R})$ with $\text{spt} \varphi \subset K$ such that $(u + \varphi) \big|_{(-r_0, r_0)} = w \big|_{(-r_0, r_0)}$ and

$$\Delta(u, \varphi, K) < \frac{\delta}{2}. \quad (2)$$

The corresponding result for $n \geq 2$ is Lemma 2.9. Once this is established one easily gives the
Proof of Theorem 2.1 for \( n = 1 \), assuming Lemma 2.4. If there would exist such a function \( v \), we could construct the function \( w \), and the two lemmas above yield compactly supported functions \( \psi \) and \( \varphi \) such that

\[
\Delta(u, \varphi + \psi, K) = \Delta(u, \varphi + \psi, (-r_0, r_0)) + \Delta(u, \varphi + \psi, K \setminus (-r_0, r_0)) \\
= \Delta(u + \varphi, \psi, (-r_0, r_0)) + \Delta(u, \varphi, (-r_0, r_0)) + \Delta(u, \varphi, K \setminus (-r_0, r_0)) \\
= \Delta(w, \psi, (-r_0, r_0)) + \Delta(u, \varphi, K) < -\delta + \frac{\delta}{2} = -\frac{\delta}{2} < 0,
\]

and this contradicts the minimality of \( u \).

For the proof of Lemma 2.4 we shall need two results: The first of these, Lemma 2.5, is a special case of \cite{1}, Lemma (6.8) and Lemma (6.9), or Lemma 2.10 below. If \( \varepsilon > 0 \) and \( t > 0 \) are given, it allows us to construct the function \( \varphi \) such that \( (u + \varphi)\big|_{(-t,t)} = w\big|_{(-t,t)} \) and \( |\Delta(u, \varphi, \text{spt } \varphi \setminus (-t,t))| < \varepsilon \), i.e. it is indeed what one would call a “short connection”. The second one is the non-parametric analogue of another result of Morse, cf. \cite{10}, Theorem 12, and asserts that the integral of a periodic solution over one period equals the energy of any other periodic solution over one period.

**Lemma 2.5.** Consider \( u_1, u_2 : \mathbb{R} \to \mathbb{R} \) with Lipschitz constant \( L \) and \( t \in \mathbb{R}^+ \) and suppose \( 0 \leq u_2 - u_1 \leq C \) for some \( C > 0 \). Then there exists a function \( g : \mathbb{R} \to \mathbb{R} \) such that

(a) \( g \) is Lipschitz with constant \( 2L + 1 \),

(b) \( g\big|_{[-t,t]} = u_2\big|_{[-t,t]} \),

(c) \( g\big|_{\mathbb{R}\setminus[-t-C,t+C]} = u_1\big|_{\mathbb{R}\setminus[-t-C,t+C]} \),

(d) \( \mathcal{L}^1\left( \{x \in \mathbb{R} \mid |x| \geq t, g(x) \neq u_1(x) \} \right) \leq (u_2 - u_1)(-t) + (u_2 - u_1)(t) \),

(e) there exists a constant \( \tilde{A} = \tilde{A}(C, L, F) \) such that

\[
\left| \int_{\mathbb{R}\setminus[-t,t]} \left( F(x, g, g_x) - F(x, u_1, (u_1)_x) \right) dx \right| \leq \tilde{A}\left( (u_2 - u_1)(-t) + (u_2 - u_1)(t) \right).
\]

**Remark 2.6.** Analogous statements are true if \( 0 \leq u_1 - u_2 \leq C \).
Proof. Let $pr : \mathbb{R} \to [-t, t]$ be the nearest point projection and define

$$g(x) := \max \{ u_2(pr(x)) - (L + 1)d(x, [-t, t]), u_1(x) \}.$$ 

One readily verifies that $g$ satisfies (a)–(d). Since $F(x, h(x), h_x(x))$ is uniformly bounded for all $x \in \mathbb{R}$ and all $h \in \text{Lip}(2L + 1)$, also (e) follows. \qed

Lemma 2.7. Consider the action $T'$ of $\mathbb{Z}k_0$ on $\mathbb{R}$, given by $T'_k x = x + k$ for every $k \in \mathbb{Z}k_0$. If $u_1, u_2 \in \mathcal{M}(\bar{u}_1)$ and $u_1 \leq u_2$ and $H_1, H_2$ are fundamental domains of $T'$, then $I(u_1, H_1) = I(u_2, H_2)$.

Proof. Let $\varepsilon > 0$ be given. By the assumed periodicity of $u_1$ and $u_2$ and the $\mathbb{Z}^2$-periodicity of $F$, we may assume without loss of generality that

$$H_1 = H_2 = \{ x \in \mathbb{R} | 0 \leq x < |k_0| \} =: H_0.$$ 

By periodicity of $u_1$ and $u_2$ there exists a constant $C > 0$ such that $u_2 - u_1 \leq C$. Let $n \in \mathbb{N}$ be such that $\frac{1}{n} \tilde{A}C < \varepsilon$ and set $t = n|k_0|$. Let $g$ be the function provided by Lemma 2.5. For $\varphi = g - u_1$ we have $u_1 + \varphi = u_2$ on $(-t, t)$ and, by minimality of $u_1$,

$$I(u_1,(-(t+C), t+C)) \leq I(u_1 + \varphi, (-(t+C), t+C)).$$

Using Remark 2.6 and Lemma 2.5(e), we obtain

$$|I(u_1, (-t,t)) - I(u_2, (-t,t))| \leq 2\tilde{A}C.$$ 

Then, by the assumed periodicity of $u_1$ and $u_2,$

$$2n|I(u_1, H_0) - I(u_2, H_0)| = |I(u_1, (-t,t)) - I(u_2, (-t,t))| \leq 2\tilde{A}C,$$

and thus $|I(u_1, H_0) - I(u_2, H_0)| < \varepsilon$. \qed

Proof of Lemma 2.4. According to Proposition 1.2(a) it is true that $T_{nk_0}u \to u^\pm$ in $C^1_{\text{loc}}$ as $n \to \pm \infty$. Thence

$$(w - u)(-t) + (w - u)(t) \to 0 \quad \text{as } t \to \infty.$$ 

Let $g_t$ be the functions provided by Lemma 2.5 for $u_1 = u$, $u_2 = w$ and $t > r_0$ for $r_0$ from Lemma 2.3. Set $\varphi_t := g_t - u$ and $K_t = \text{spt } \varphi_t$. Then, by Lemma 2.5(e), we may choose $t_0$ so large that for $t \geq t_0$

$$|\Delta(u, \varphi_t, K_t \setminus (-t,t))| < \frac{\delta}{4}. \quad (3)$$

This estimates the “cost of energy by short connections” outside $(-t,t)$. 

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Now we have to compare the energy of $u$ and $w$ inside $(-t, t)$. We will have to consider the following fundamental domains of $T'$ (recall that we assume $k_0 < 0$, and cf. Figure 1):

$$H'_t := (-t, -t - k_0]$$
$$H''_t := [t + k_0, t)$$
$$H := \{ x \in \mathbb{R} \mid u(x) < v(x) \leq T_k u(x) \} .$$

By continuity of $F$, the $C^1_{\text{loc}}$-convergence provided by Proposition 1.2(a) implies $|I(u, H'_t) - I(u, H'_t)| \to 0$ as $t \to \infty$. Hence by Lemma 2.7 we may choose $t' \geq t_0$ so large that for every $t \geq t'$ we have

$$|I(v, H) - I(u, H'_t)| < \frac{\delta}{4} . \quad (4)$$

By periodicity of $v$ and $u^-$ and by the above-mentioned $C^1_{\text{loc}}$, and hence $C^0_{\text{loc}}$-convergence, there exists $t'' \geq t'$ such that

$$H'_t \cap \{ u \geq v \} = \emptyset, \ H''_t \cap \{ T_k u < v \} = \emptyset \text{ and } H \cap (-t, t) = H' \quad (5)$$

for all $t \geq t''$. Consequently, for $t \geq t''$, there is the decomposition

$$w \cdot \chi(-t, t) = v \cdot \chi_H + u \cdot \chi(-t, t) \cap \{ u \geq v \} + T_k u \cdot \chi(-t, t) \cap \{ T_k u < v \} \quad (6)$$

Furthermore periodicity of $F$ and $v$ yields

$$I(T_k u, ((-t, t) \setminus H'_t) \cap \{ T_k u < v \}) = I(u, ((-t, t) \setminus H'_t) \cap \{ u < v \}) . \quad (7)$$
From the decomposition (6) for $w$ we deduce for $t \geq t''$, using (4), (5) and (7):

$$I(w, (-t, t)) < I(u, H'_t) + \frac{\delta}{4} + 0 + I(u, ((-t, t) \setminus H'_t) \cap \{u \geq v\})$$
$$+ 0 + I(u, ((-t, t) \setminus H'_t) \cap \{u < v\})$$
$$= I(u, (-t, t)) + \frac{\delta}{4}.$$  

Together with (3) this gives $\Delta(u, \varphi, K_t) < \frac{\delta}{2}$.

**Proof of Theorem 2.1 for $n \geq 2$**

We assume the existence of $v \in \mathcal{M}(\bar{a}_1, \ldots, \bar{a}_{t-1})$ with $u^- < v < u^+$. As in the one-dimensional case we follow [3] and define the function $w$ as follows: Choose $\bar{k}_0 = (k_0, (k_0')') \in \bar{\Gamma}_t$ with $\bar{k}_0 \cdot \bar{a}_t > 0$, and set

$$w = \max (u, \min(v, T_{k_0} u)).$$

Let us write $j = \text{rk} \bar{\Gamma}_t$. By [3, (6.8)] we have the following

**Lemma 2.8.** There exist $\delta > 0$ and $r_0 > 0$ such that for every $r > r_0$ there exists a function $\psi = \psi_r \in W^{1,2}(\mathbb{R}^n)$ with $\text{spt} \psi \subset B(0, r)$ such that

$$\Delta(w, \psi, B(0, r)) < -\delta r^{j-1}.$$

Here we will prove

**Lemma 2.9.** For every $r > 0$ there exists $s \geq r$, a compact set $K = K_s \supset B(0, s)$ and a function $\varphi = \varphi_s \in W^{1,2}_0(\mathbb{R}^n)$ with $\text{spt} \varphi \subset K$, $(u + \varphi)|_{B(0, s)} = w|_{B(0, s)}$ such that for $\delta > 0$ from Lemma 2.8 we have

$$\Delta(u, \varphi, K) < \frac{\delta}{2}s^{j-1}. \quad (8)$$

**Proof of Theorem 2.1, assuming Lemma 2.9.** If there existed such a function $v$, we construct the function $w$, and the two Lemmas above yield compactly supported functions $\psi = \psi_s$ and $\varphi = \varphi_s$, $s > r_0$, such that analogously to the case $n = 1$

$$\Delta(u, \varphi + \psi, K) < (-\delta + \frac{\delta}{2})s^{j-1} = -\frac{\delta}{2}s^{j-1} < 0,$$

and this contradicts the minimality of $u$.\qed
We shall need a modification of the “Slicing-Lemma” \cite{1} Lemmas (6.8) and (6.9)]. This is necessary since we need this result not only for balls but also for sets featuring some periodicity in the direction of \( k \) and (6.9)]. This is necessary since we need this result not only for balls but also for sets featuring some periodicity in the direction of \( k \), namely for the full “cylinder with caps”

\[
Z(r, t) := \{ x \in \mathbb{R}^n \mid d(x, \{ \lambda k_0 \mid |\lambda| \leq t \}) \leq r \}, \quad r > 0, t \in \mathbb{R}^+ \cup \{ \infty \}.
\]

Let \( C_t(r) \) denote the cylinder \( \{ x \in \mathbb{R}^n \mid |x \cdot k_0| \leq t \} \cap \partial Z(r, t) \) of radius \( r \) and height \( 2t \) with “soul” \( \mathbb{R}^n \). Let \( D_t(r) \) denote the set \( \partial Z(r, t) \setminus C_t(r) \) that consists of two open \( (n - 1) \)-half-spheres for \( t < \infty \), and is empty if \( t = \infty \). Note that \( \partial Z(r, t) = C_t(r) \cup D_t(r) \) for every \( r \in \mathbb{R}^+, t \in \mathbb{R}^+ \cup \{ \infty \} \).

By \( d\sigma \) we denote the \((n - 1)\)-dimensional area-element.

**Lemma 2.10.** Let \( u_1, u_2 : \mathbb{R}^n \to \mathbb{R} \) have Lipschitz constant \( L \) and suppose \( 0 \leq u_2 - u_1 \leq C \) and \( r \geq 1, t \in \mathbb{R}^+ \cup \{ \infty \} \). Then there exists a function \( g : \mathbb{R}^n \to \mathbb{R} \) such that

(a) \( g \) is Lipschitz with constant \( 2L + 1 \),

(b) \( g = u_2 \) inside \( Z(r, t) \),

(c) \( g = u_1 \) outside \( Z(r + C, t) \), which is compact if \( t < \infty \),

(d) \( \text{vol}_n \left( \{ x \in Z(r, t)^C \mid g(x) \neq u_1(x) \} \right) \leq (1 + C)^{n-2} \int_{C_t(r)} (u_2 - u_1)(x) \, d\sigma(x) 
+ (1 + C)^{n-1} \int_{D_t(r)} (u_2 - u_1)(x) \, d\sigma(x) \),

(e) there exists a constant \( \tilde{A} = \tilde{A}(n, C, L, F) \) such that

\[
\left| \int_{\mathbb{R}^n \setminus Z(r, t)} (F(x, g, g_x) - F(x, u_1, (u_1)_x)) \, dx \right| 
\leq \tilde{A} \int_{C_t(r)} (u_2 - u_1)(x) \, d\sigma(x) + \tilde{A} \int_{D_t(r)} (u_2 - u_1)(x) \, d\sigma(x).
\]

**Remark 2.11.** Analogous statements are true if \( 0 \leq u_1 - u_2 \leq C \).

**Proof.** We modify Bangert’s proof. Let \( \text{pr} : \mathbb{R}^n \to Z(r, t) \) be the nearest point projection and define

\[
g(x) := \max \left\{ u_2(\text{pr}(x)) - (L + 1)d(x, Z(r, t)), u_1(x) \right\}.
\]

Hence \( g \) satisfies (a) and (b). Since \( u_1 \) has Lipschitz constant \( L \) we have

\[
u_1(x) \geq u_2(\text{pr}(x)) + (u_1(\text{pr}(x)) - u_2(\text{pr}(x))) - Ld(x, Z(r, t)),
\]
and therefore \( g(x) = u_1(x) \) if \( d(x, Z(r, t)) \geq u_2(\text{pr}(x)) - u_1(\text{pr}(x)) \) and \( g \) satisfies (c).

If \( \nu_x \) denotes the outer unit normal to \( \partial Z(r, t) \) we consider the transformation maps
\[
\tilde{\tau} : C_t(r) \times \mathbb{R}^+ \to \mathbb{R}^n, (x, s) \mapsto x + s \nu_x \quad \text{and} \quad \\
\tau : D_t(r) \times \mathbb{R}^+ \to \mathbb{R}^n, (x, s) \mapsto x + s \nu_x,
\]
which occur in the following integration in cylindric and polar coordinates. Let \( J_{\tilde{\tau}} \) and \( J_\tau \) be the corresponding Jacobians.

\[
\text{vol}_n \left( \left\{ x \in Z(r, t)^C \mid g(x) \neq u_1(x) \right\} \right) \leq \int_{C_t(r)} \int_r^{r+(u_2-u_1)(x)} |J_{\tilde{\tau}}(x, s)| \, ds \, d\sigma(x) \]
\[
+ \int_{D_t(r)} \int_r^{r+(u_2-u_1)(x)} |J_\tau(x, s)| \, ds \, d\sigma(x) \]
\[
\leq (1 + C)^{n-2} \int_{C_t(r)} (u_2 - u_1)(x) \, d\sigma(x) \]
\[
+ (1 + C)^{n-1} \int_{D_t(r)} (u_2 - u_1)(x) \, d\sigma(x) \]

which is estimate (d). Since \( F(x, h(x), h_2(x)) \) is uniformly bounded for all \( x \in \mathbb{R}^n \) and all \( h \in \text{Lip}(2L + 1) \), we obtain (e).

we will need the following simple observation:

**Lemma 2.12.** Suppose \( j \in \{0\} \cup \mathbb{N} \) and \( f : \mathbb{R}^+ \to [0, \infty) \) is a measurable function, \( r_0 > 0 \) and \( \int_0^r f(s) \, ds \leq cr^j \) for a constant \( c > 0 \) and every \( r > r_0 \). Then, if \( i \in \mathbb{N} \) is such that \( 2^{i+1} \geq r_0 \), we obtain for every \( k \in \mathbb{N} \)

\[
\mathcal{L}^1 \left( \left\{ f(s) > 2^{i+1}ck s^{j-1} \right\} \cap [2^i, 2^{i+1}) \right) < \frac{1}{k} 2^i.
\]

Especially there exists a constant \( \tilde{c} > 0 \) and a sequence \( (s_i)_{i \in \mathbb{N}} \) with \( s_i \to \infty \) as \( i \to \infty \) such that \( f(s_i) < \tilde{c} s_i^{j-1} \).

**Proof.** \((j = 0)\): If for \( i \in \mathbb{N} \) with \( 2^{i+1} \geq r_0 \) the estimate was false, then
\[
\int_{2^i}^{2^{i+1}} f(s) \, ds > \frac{1}{k} 2^i \cdot 2 \cdot ck 2^{-(i+1)} = c,
\]
which contradicts \( \int_0^{2^{i+1}} f(s) \, ds \leq c \).
(\(j \geq 1\)): If for \(i \in \mathbb{N}\) the estimate was not true, we calculate

\[
c \cdot 2^{i+1} \geq \int_0^{2^{i+1}} f(s) \, ds \geq \int_{2^i}^{2^{i+1}} f(s) \, ds > \frac{1}{k} 2^i \cdot 2^{i+1} c k \cdot 2^{j-1} = c \cdot 2^{(i+1)j+1}.
\]

Division by \(2^{(i+1)j}\) yields the contradiction \(c > 2c\).

Lemma 2.13. Consider the action \(T'\) of \(\mathbb{Z}k_0\) on \(\mathbb{R}^n\), given by \(T'_k x = x + k\) for every \(k \in \mathbb{Z}k_0\). Consider \(u_1, u_2 \in \mathcal{M}(\bar{a}_1, \ldots, \bar{a}_{t-1})\) with \(u_1 \leq u_2\). Suppose \(T_k u_1 \geq u_2\) whenever there exists \(s \in \{1, \ldots, t-1\}\) such that \(\bar{k} \in \Gamma_s\) and \(\bar{k} \cdot \bar{a}_s > 0\), and let \(H_1, H_2\) be fundamental domains of \(T'\). Then there exists a sequence \(s_i \to \infty\) and a constant \(c_0 > 0\) such that

\[
|I(u_1, Z(s_i, \infty) \cap H_1) - I(u_2, Z(s_i, \infty) \cap H_2)| < c_0 s_i^{j-2}.
\]

Proof. For every \(v \in \mathcal{M}(\bar{a}_1, \ldots, \bar{a}_{t-1})\) and every \(r > 0\) and any two fundamental domains \(H_1, H_2\) of \(T'\) we have \(I(v, Z(r, \infty) \cap H_1) = I(v, Z(r, \infty) \cap H_2)\). Thus, it suffices to give the proof for

\[
H_1 = H_2 = \{x \in \mathbb{R}^n \mid 0 \leq x \cdot k_0 < |k_0|\} =: H_0.
\]

The idea is as follows: \(\text{vol}(Z(r, t))\) grows like \(ts^{j-1}\) and \(\text{vol}(Z(r, t) \cap H_0)\) grows like \(s^{j-1}\). By “short connections” and minimality of \(u_1\) and \(u_2\) we obtain the desired estimate.

For \(n \in \mathbb{N}\) we set \(t_n := n|k_0|\). For every \(r, n > 0\) we let \(g_{r,n}\) be the functions provided by Lemma 2.10 and set \(\varphi_{r,n} = g_{r,n} - u_1\). Minimality of \(u_1\) implies

\[
I(u_1, Z(r, t_n)) + I(u_1, \text{spt} \varphi_{r,n} \setminus Z(r, t_n)) = I(u_1, Z(r, t_n) \cup \text{spt} \varphi_{r,n}) \leq I(u_1 + \varphi_{r,n}, Z(r, t_n) \cup \text{spt} \varphi_{r,n}) = I(u_2, Z(r, t_n)) + I(u_1 + \varphi_{r,n}, \text{spt} \varphi_{r,n} \setminus Z(r, t_n)).
\]

Hence

\[
I(u_1, Z(r, t_n)) - I(u_2, Z(r, t_n)) \leq |I(u_1 + \varphi_{r,n}, \text{spt} \varphi_{r,n} \setminus Z(r, t_n)) - I(u_1, \text{spt} \varphi_{r,n} \setminus Z(r, t_n))|.
\]

By the assumption that \(T_k u_1 \geq u_2\) whenever there exists \(s \in \{1, \ldots, t-1\}\) such that \(\bar{k} \in \Gamma_s\) and \(\bar{k} \cdot \bar{a}_s > 0\), the set

\[
W := \{(x, x_{n+1}) \in \mathbb{R}^{n+1} \mid u_1(x) < x_{n+1} < u_2(x)\}
\]

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projects injectively into $\mathbb{R}^{n+1}/\bar{\Gamma}_t$. Furthermore, $W$ is $\mathbb{Z}k_0$-invariant and we obtain the following volume-growth estimate: There is a constant $\tilde{c} > 0$, independent of $n \in \mathbb{N}$, such that
\begin{equation}
\text{vol} \left( W \cap (Z(r, t_n) \times \mathbb{R}) \right) \leq \tilde{c} n r^{j-1} + \tilde{c} r^j. \tag{10}
\end{equation}
Since the left hand side of this estimate equals the integral
\[
\int_0^r \left( \int_{C_{i,n}(s)} (u_2 - u_1)(x) \, d\sigma(x) + \int_{D_{i,n}(s)} (u_2 - u_1) \, d\sigma(x) \right) \, ds,
\]
Lemma 2.12 yields a sequence $s_i \to \infty$ and a constant $c' > 0$ such that
\[
\int_{C_{i,n}(s_i)} (u_2 - u_1) \, d\sigma(x) \leq c' n s_i^{j-2} \quad \text{and}
\int_{D_{i,n}(s_i)} (u_2 - u_1) \, d\sigma(x) \leq c' s_i^{j-1}
\]
for every $n \in \mathbb{N}$. By Lemma 2.10 (e) there is a constant $c'' > 0$ such that
\[
\left| I(u_1 + \varphi_{s_i,n}, \text{spt} \varphi_{s_i,n} \setminus Z(s_i, t_n)) - I(u_1, \text{spt} \varphi_{s_i,n} \setminus Z(s_i, t_n)) \right| \leq c'' n s_i^{j-2} + c'' s_i^{j-1}.
\]
Together with estimate (9) this implies
\[
I(u_1, Z(s_i, t_n)) - I(u_2, Z(s_i, t_n)) \leq c'' n s_i^{j-2} + c'' s_i^{j-1}.
\]
Using Remark 2.11 we infer
\begin{equation}
\left| I(u_1, Z(s_i, t_n)) - I(u_2, Z(s_i, t_n)) \right| \leq c'' n s_i^{j-2} + c'' s_i^{j-1}. \tag{11}
\end{equation}
Consider a fixed $i \in \mathbb{N}$. By the $\mathbb{Z}k_0$-invariance of $u_1$ and $u_2$ we obtain for $j = 1, 2$
\[
I(u_j, Z(s_i, t_n)) = 2n I(u_j, Z(s_i, \infty) \cap H_0) + 2I(u_j, Z(s_i, t_n) \setminus \{ x \mid x \cdot k_0 \leq t_n \}).
\]
The modulus of the second term on the right hand side equals a constant $c^j$ depending on $s_i$ but not on $n$. Set $c''' = 5 \max\{ |c^1|, |c^2|, c'' s_i^{j-1} \}$, and infer from (11)
\[
c'' n s_i^{j-2} + c''' \geq 2n \left| I(u_1, Z(s_i, t_n) \cap H_0) - I(u_2, Z(s_i, t_n) \cap H_0) \right|.
\]
Considering $n \to \infty$, we infer $\left| I(u_1, Z(s_i, \infty) \cap H_0) - I(u_2, Z(s_i, \infty) \cap H_0) \right| \leq c_0 s_i^{j-2}$. \hfill \square
Proof of Lemma 2.9. We define the sets 
\[ W' := \{(x, x_{n+1}) \in \mathbb{R}^{n+1} \mid u(x) < x_{n+1} < w(x)\} \]
\[ W'' := \{(x, x_{n+1}) \in \mathbb{R}^{n+1} \mid u(x) < x_{n+1} < T_{k_0} u(x)\} \],
and consider the coverings
\[ \mathbb{R}^{n+1} \xrightarrow{\pi} \mathbb{R}^{n+1}/\mathbb{Z} \xrightarrow{p'} \mathbb{R}^{n+1}/\hat{\Gamma} \xrightarrow{\hat{p}} T^{n+1}. \] (12)

By Proposition 1.2(b) \( \hat{p} \) maps \( p'(\pi(W'')) \) injectively into \( T^{n+1} \). The group of deck transformations of \( p' \) is of rank \( j - 1 \), thence
\[ \operatorname{vol}_{n+1} \left( \pi(W'' \cap (Z(r, \infty) \times \mathbb{R})) \right) \leq cr^{j-1} \]
for some constant \( c > 0 \). Since \( \pi|_{W''} \) is injective and \( W' \subset W'' \), we have
\[ \operatorname{vol}_{n+1} \left( W' \cap (Z(r, \infty) \times \mathbb{R}) \right) \leq cr^{j-1}. \] (13)

Now we fix the radius \( s \) of \( Z(s, t) \): Integration in cylindric coordinates and Lemma 2.12 implies that there exists a sequence \( s_i \to \infty \), and a constant \( c > 0 \) such that
\[ \int_{C_{\infty}(s_i)} (w - u)(x) d\sigma(x) \leq cs_i^{j-2}. \] (14)

Remark 2.14. Lemma 2.12 allows us to choose the same sequence \( s_i \to \infty \) here and in Lemma 2.13, and we do so.

From now on let \( i \) be fixed (but arbitrarily large) such that
\[ s := s_i > \max \left\{ \frac{8c\bar{A}}{\delta}, \frac{8c_0}{\delta} \right\}, \] (15)
where \( c_0 \) is the constant from Lemma 2.13 and \( \delta \) from Lemma 2.8. Then
\[ c_0 s^{j-2} < \frac{\delta}{8}s^{j-1}. \] (16)

We fix the height \( t \) of \( Z(s, t) \): By (13), \( \operatorname{vol}_{n+1} \left( W' \cap (Z(s, \infty) \times \mathbb{R}) \right) < \infty \), Lemma 2.12 yields a sequence \( t_i \to \infty \) and a constant \( \hat{c} > 0 \) with
\[ \int_{D_{t_i}(s)} (w - u)(x) d\sigma(x) \leq \frac{\hat{c}}{t_i}. \]
This estimate together with (14) and Lemma 2.10 (e) yield functions $\varphi_{i,t}$ with $(u + \varphi_{i,t})|_{Z(s,t_i)} = w|_{Z(s,t_i)}$ and $\Delta(u, \varphi_{i,t}, Z(s,t_i)^c) < \epsilon A s^{-1} + \frac{\epsilon A}{t_i}$. We choose $l_0$ so large that for every $l \geq l_0$ we have $\frac{\epsilon A}{t_i} < \frac{\delta}{8}s^{j-1}$. Together with (15) we infer

$$|\Delta(u, \varphi_{i,t}, Z(s,t_i)^c)| < \frac{\delta}{4}s^{j-1}. \quad (17)$$

This estimates the “energy costs of the short connections” outside $Z(s,t_i)$.

Now we will compare the energies of $u$ and $w$ inside $Z(s,t_i)$. The set

$$H := \{ x \in \mathbb{R}^n \mid u(x) < v(x) \leq T_{k_0} u \}$$

is a measurable fundamental domain of the action $T'$ of $\mathbb{Z}k_0$ on $\mathbb{R}^n$ and we consider two more measurable fundamental domains $H'_l, H''_l$ that satisfy

$$Z(s,t_i) \setminus T_{k_0} Z(s,t_i) \subset H'_l \quad \text{and} \quad Z(s,t_i) \setminus T_{k_0} Z(s,t_i) \subset H''_l.$$ 

By the convergence provided by Proposition 1.2(a) and by continuity of $F$ there exists an integer $l_1 \geq l_0$ such that for every $l \geq l_1$:

$$|I(u^-, Z(s,t_i) \cap H'_l) - I(u, Z(s,t_i) \cap H'_l)| < \frac{\delta}{8}s^{j-1}. \quad (18)$$

Together with Lemma 2.13 and (16), this implies that

$$|I(v, Z(s,t_i) \cap H) - I(u, Z(s,t_i) \cap H'_l)| < \frac{\delta}{4}s^{j-1}. \quad (18')$$

By the assumed periodicity of $u^\pm$ and $v$, there exists a constant $\delta' > 0$ such that $|u^\pm(x) - v(x)| > \delta'$ on $Z(s,\infty) \cap H'_l$ for every $l \in \mathbb{N}$. Thus the above-mentioned convergence result implies that there exists an integer $l_2 \geq l_1$ such that for all $l \geq l_2$

$$Z(s,t_i) \cap \{ u \geq v \} = \emptyset = Z(s,t_i) \cap H''_l \cap \{ T_{k_0} u < v \}. \quad (19)$$

Set $K := Z(s,t_{l_2})$ and $\varphi = \varphi_s = \varphi_{i,l_2}$ and observe

$$w \cdot \chi_K = v \cdot \chi_K \cap H + u \cdot \chi_K \cap \{ u \geq v \} + T_{k_0} u \cdot \chi_K \setminus \{ T_{k_0} u < v \} = v \cdot \chi_K \cap H + u \cdot \chi_K \cap H'_l \cap \{ u \geq v \} + u \cdot \chi(K \setminus H'_l) \cap \{ u \geq v \}$$

$$+ T_{k_0} u \cdot \chi_K \cap H''_l \cap \{ T_{k_0} u < v \} + T_{k_0} u \cdot \chi(K \setminus H''_l) \cap \{ T_{k_0} u < v \}.$$ 

Furthermore periodicity of $F$ yields

$$I(T_{k_0} u, (K \setminus H''_l) \cap \{ T_{k_0} u < v \}) = I(u, (K \setminus H'_l) \cap \{ u < v \}) \quad (20)$$
The above decomposition of $w \cdot \chi_K$ and (18), (19) and (20) gives

$$I(w, K) < I(u, K \cap H'_i) + \frac{\delta}{4}s^{j-1} + 0 + I(u, (K \setminus H'_i) \cap \{u \geq v\}$$

$$+ 0 + I(u, K \setminus H'_i) \cap \{u < v\})$$

$$= I(u, K) + \frac{\delta}{4}s^{j-1}.$$ 

Together with $(u + \varphi) \cdot \chi_{K \cup spt \varphi} = w \cdot \chi_K + (u + \varphi) \cdot \chi_{spt \varphi \setminus K}$ and (17), this implies $I(u + \varphi, K \cup spt \varphi) < I(u, K \cup spt \varphi) + \frac{\delta}{2}s^{j-1}$.  

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