Complete analysis to minimum-error discrimination of mixed four qubit states with arbitrary prior probabilities

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In this work, we provide a complete analysis to minimum-error discrimination of mixed four qubit states with arbitrary prior probabilities. For the complete analysis, the most important work to do is to find the necessary and sufficient conditions for the existence of null measurement operator. From the geometric structure of qubit states, we obtain the analytic condition for deciding the existence of a null operator in minimum-error measurement for mixed four qubit states, which also gives the necessary and sufficient conditions for every optimal POVM to have non-zero elements. Using the condition, we completely analyze minimum-error discrimination of mixed four qubit states with arbitrary prior probabilities.

Quantum state discrimination[1–3] is one of the most fundamental tasks in quantum information theory. It identifies the fundamental limit of distinguishability of systems dictated by postulates of quantum theory, and is also useful in various quantum information applications[4]. When one of states $\{\rho_i\}_{i=1}^N$ is prepared with \textit{a priori} probability $q_i$ from $\{q_i\}_{i=1}^N$, we may denote the situation as $\{q_i, \rho_i\}_{i=1}^N$. Then quantum state discrimination is the process of verifying the unknown state out of the possibilities of $\{\rho_i\}_{i=1}^N$.

Minimum-error discrimination (MD)[5–7] means the process achieved by minimising the average error or equivalently maximising probability of making correct guesses in average. The latter is called the guessing probability. This seeks optimization of measurement, a set of positive-operator-valued-measures (POVMs), $\{M_i\}_{i=1}^N$, which is a non-negative resolution of the identity operator i.e. $\sum_i M_i = I$ with $M_i \geq 0$. Then, the guessing probability is expressed as,

$$p_{\text{guess}} = \max_{\{M_i\}_{i=1}^N} \sum_{i=1}^N q_i p(i|i) \tag{1}$$

where $p(i|j) = \text{tr}[\rho_j M_i]$ denotes the probability of obtaining outcome $i$ when $\rho_j$ has been prepared. So far, two-state minimum error discrimination, regardless of the dimension of quantum state, is the case that optimal discrimination is completely solved, for arbitrary states and prior probabilities, in an analytic form[5, 8]. Apart from the case, despite the fundamental and practical importance, little has been known as analytic formula for the guessing probability or also for finding optimal measurement[9–18].

Recently, much effort has been devoted to deriving analytical methods for optimal discrimination of qubit states. One of the approaches[14–17, 19] is to make use of semidefinite programming[20, 21] together with the Bloch sphere expression of qubit states. In fact, for a non-negative operator $A$, its Bloch vector is denoted by $\vec{r}_A$ such that

$$A = \frac{\text{tr}[A]}{2} (I + \vec{r}_A \cdot \vec{\sigma}),$$

where $\vec{\sigma}$ is the Pauli matrices ($\sigma_X, \sigma_Y, \sigma_Z$).

An optimization task in semidefinite programming can be generally formulated in the so-called linear complementarity program (LCP)[4, 15] that generalizes primal and dual problems. The LCP approach to optimal qubit state discrimination has also been formulated and presented in another form of geometric method. The LCP of qubit state discrimination is to use operators $K$ and $\{r_i, \rho_i\}_{i=1}^N$ such that for all $i = 1, \cdots, N$

$$K = q_i \rho_i + r_i \rho_i, \quad \text{and} \quad r_i \text{tr}[\rho_i M_i] = 0, \tag{2}$$

where $\{r_i\}_{i=1}^N$ are constants, not probabilities, and $\{M_i\}_{i=1}^N$ is a measurement. Once these operators $K$, $\{r_i, \rho_i\}_{i=1}^N$, and $\{M_i\}_{i=1}^N$ satisfying the above are found, they define optimal operators for optimal discrimination. Then the guessing probability corresponds to $\text{tr}[K]$ as it is shown in Eq. (3).

$$p_{\text{guess}} = \min_{K \geq q_i \rho_i, \forall i} \text{tr}[K]. \tag{3}$$

A set of POVMs satisfying the orthogonality with $\{\rho_i\}_{i=1}^N$ forms an optimal measurement. As the LCP approach deals with a list of equations that characterizes optimal parameters of the primal and dual problems, it could be considered more complicated in general. The advantage lies at the fact that the structure of the problem is exploited. In fact, the optimality conditions in Eq. (2) can find an underlying geometry of states and the optimal parameters in the state space.

If prior probabilities are not equal, the LCP approach has not yet found a geometric method for solving optimal discrimination. However, remarkably, the optimization was successfully achieved in three mixed qubit states
with arbitrary prior probabilities. A closed formula for three mixed qubit states could be derived. Note that it is a formula for any three qubit states given with arbitrary prior probabilities. As far as we are aware of, it is the only analytic form of optimal state discrimination other than the two-state discrimination when no prior information is assumed on given states but the dimension $d = 2$.

However, minimum error discrimination of four mixed qubit states with arbitrary prior probabilities is still an unsolved problem. And in this work, we will provide a complete analysis to minimum error discrimination of four mixed qubit states with arbitrary prior probabilities. For minimum error discrimination of four mixed qubit states with arbitrary prior probabilities, we will analyze the necessary and sufficient conditions for the existence of null measurement operator. Specifically, from the geometric structure of qubit states, we will obtain the analytic condition for deciding the existence of a null operator in minimum-error measurement for four qubit states. It equips us with the necessary and sufficient conditions for every optimal POVM to have non-zero elements. From the condition, we can completely analyze minimum-error discrimination of four mixed qubit states with arbitrary prior probabilities.

The purpose of MD is to minimize the guessing error, so that guessing the quantum state produced is done as accurately as possible. The dual problem to minimize $\rho$, so that guessing the quantum state produced is done with arbitrary prior probabilities.

Because our optimization problem satisfies these conditions, the optimal duality gap disappears. By this property, the optimal parameters of both the problems satisfy the complementary slackness $r_i \text{tr} [\tilde{\rho}_i M_i] = 0(\forall i)$. With this condition, the constraints of primal problem and dual problem are the necessary and sufficient conditions to optimality of two problems. In optimization problem, the optimality condition is Karus-Khun-Tucker (KKT) condition. And the problem of the approach to an optimal solution using these conditions is called LCP. Using this way, we obtain the optimal POVM and the set of complementary states, which provides optimal values to two problems.

When $\{M_i\}_{i=1}^N$ is an optimal POVM with null operators $M_j = 0(\forall j \notin \chi), \{M_i\}_{i \in \chi}$ is a POVM which produces $p_{\text{guess}} = \max_{\{\rho_i\}_{i=1}^N} \sum_{i \in \chi} q_i \text{tr} [\rho_i M_i]$ where $\chi$ is a subset of $\{1, 2, \cdots, N\}$. And if we can optimize $\{q_i, \rho_i\}_{i=1}^N$ through MD with less than $N$ quantum states, we have

$$p_{\text{guess}} = \max_{|x| = N-1} p_{\text{guess}}^x. \quad (4)$$

However if $\{q_i, \rho_i\}_{i=1}^N$ is not optimized with MD of less than $N$ quantum states, Eq. (4) cannot be used. At this time we have

$$p_{\text{guess}} > \max_{1 \leq i \leq N} q_i. \quad (5)$$

It is because if $p_{\text{guess}} = q_k, \{M_k = 1\}$ is an optimal POVM. Therefore we should know the condition to classify two cases and the optimization method for MD with $N$ quantum states. In this paper $q_1$ denotes the largest prior probability, which means max $q_i = q_1$. This assumption does not violate generality of our problem.

For qubit states ($d = 2$), POVM elements and qubit states can be represented by non-negative numbers $p_i$ and Bloch vectors $\vec{u}_i, \vec{v}_i, \vec{w}_i$:

$$M_i = p_i (I + \vec{u}_i \vec{u}_i^T), \rho_i = \frac{1}{2} (I + \vec{u}_i \vec{u}_i^T), \tilde{\rho}_i = \frac{1}{2} (I + \vec{w}_i \vec{w}_i^T). \quad (6)$$

When the representation is applied to KKT condition, the constraint to the primal problem (or POVM constraint) is transformed to

$$p_i \geq 0 \quad \forall i, \quad \sum_{i=1}^N p_i = 1, \quad \text{and} \quad \sum_{i=1}^N p_i \vec{u}_i = \vec{0}, \quad (7)$$

and the constraint to the dual problem is expressed to

$$\text{tr} [K] = q_i + r_i \quad \forall i \quad \text{and} \quad \text{tr} [K \cdot \vec{\sigma}] = q_i \vec{v}_i + r_i \vec{w}_i \quad \forall i. \quad (8)$$

And the complementary slackness condition is represented as

$$p_i r_i (1 + \vec{u}_i \cdot \vec{w}_i) = 0 \quad \forall i. \quad (9)$$

When every optimal POVM element is nonzero and its guessing probability is greater than the largest prior probability $q_1$, by Eqs. (6),(7),(8), we have $p_i, r_i > 0(\forall i)$. Then the complementary slackness condition is written as $\|\vec{u}_i\|_2 = 1, \vec{w}_i = -\vec{u}_i (\forall i)$. We should note that even in case that there exists such optimal POVM, there may exist another optimal POVM with null operators. The KKT condition (7),(8),(9) can be expressed using $p_i, r_i > 0(\forall i)$

(i) $p_i, r_i > 0 \quad \forall i, \quad \sum_{i=1}^N p_i = 1, \quad \sum_{i=1}^N p_i \vec{u}_i = \vec{0},$  
(ii) $r_i - r_i = e_i \quad \forall i, \quad r_i \vec{u}_i - r_i \vec{u}_i = \vec{s}_i \quad \forall i,$  
(iii) $\|\vec{u}_i\|_2 = 1 \quad \forall i, \quad \vec{w}_i = -\vec{u}_i \quad \forall i,$

where

$$e_i = q_1 - q_i \quad \text{and} \quad s_i = q_i \vec{v}_i - q_1 \vec{v}_1. \quad (11)$$

Reversely when $\{q_i, \vec{u}_i; r_i, \vec{v}_i, \vec{w}_i\}_{i=1}^N$ fulfills the above conditions, $\{M_i\}_{i=1}^N$ and $\{r_i, \vec{u}_i\}_{i=1}^N$ corresponding to Eq. (6)
are an optimal POVM with $N$ nonzero elements and a set of complementary states, which provides the same optimal value $p_{\text{guess}}$ greater than the largest prior probability $q_1$. Therefore Eq. (10) is a necessary and sufficient condition for every optimal POVM to have $N$ nonzero elements.

Now to consider the minimum error discrimination of four mixed qubit states with arbitrary prior probabilities, we explain geometric structure of $N$ qubit states.

**Geometric Structure of $N$ qubit states**

In a set $C$, we call the set of affine combinations of all points as the affine hull of $C$.

$$\text{aff} \ C = \{ \sum x_i c_i | c_i \in C \forall i, x_i \in \mathbb{R} \forall i, \sum x_i = 1 \}. \quad (12)$$

And the dimension of aff $C$ is called as the affine dimension of $C$. The geometric form of $\{q_i, \rho_i\}_{i=1}^N$ can be classified by affine dimension of $\{s_i\}_{i=1}^N$. We denote the dimension as $D$. Since Bloch vectors lie in three dimensional real space, we have $0 \leq D \leq 3$.

In $C$, the set of convex combinations of every point is called as the convex hull of $C$.

$$\text{conv} \ C = \{ \sum x_i c_i | c_i \in C \forall i, x_i \geq 0 \forall i, \sum x_i = 1 \}. \quad (13)$$

$\text{conv} \{s_i\}_{i=1}^N$ is a point in $D = 0$, a line segment in $D = 1$, a polygon in $D = 2$, and a polyhedron in $D = 3$, respectively.

When $\{p_i, \tilde{u}_i; r_i, \tilde{u}_i\}_{i=1}^N$ satisfies Eq. (10), by the second condition of (ii), the affine dimension of $\{r_i, \tilde{u}_i\}_{i=1}^N$ is $D$, and, by (i), the relative interior of $\text{conv} \{r_i, \tilde{u}_i\}_{i=1}^N$ contains $\vec{0}$ which is the origin of the Bloch ball. If $D < N - 1$, by Carathéodory’s theorem [22], there is $\{p_i\}_{i=1}^N$ with $D + 1$ or fewer nonzero elements such that $\{p_i, \tilde{u}_i\}_{i=1}^N$ satisfies $\tilde{p}_i \geq 0(\forall i)$, $\sum_{i=1}^N \tilde{p}_i = 1$, and $\sum_{i=1}^N \tilde{p}_i r_i \tilde{u}_i = \vec{0}$. Then $\{p_i = \tilde{p}_i r_i / \sum_{j=1}^N \tilde{p}_j r_j; \tilde{u}_i\}_{i=1}^N$ satisfies Eq. (7). This means that there exists an optimal POVM with $D + 1$ or fewer nonzero elements because $\{p_i, \tilde{u}_i; r_i, \tilde{u}_i\}_{i=1}^N$ satisfies KKT condition (7),(8),(9). Therefore, in the case of $D = 0$, by $N \geq 2$, we have $p_{\text{guess}} = q_1$ and in the case of $N > 4$, by $D \leq 3$, $\{q_i, \rho_i\}_{i=1}^N$ is optimized with MD of less than $N$ quantum states [23, 24]. Since $D > N - 1$ is impossible, if $D \neq N - 1$, we can obtain the guessing probability through MD with less than $N$ quantum states.

**Geometric optimality condition for $N$ qubit states**

Let us consider the MD of $\{q_i, \rho_i\}_{i=1}^N$ with $D = N - 1$. When $\{p_i, \tilde{u}_i; r_i, \tilde{u}_i\}_{i=1}^N$ fulfills Eq. (10), the polytope $\text{conv} \{s_i\}_{i=1}^N$ becomes a $D$-dimensional simplex, which is a line segment in $N = 2$ or a triangle in $N = 3$, or a tetrahedron in $N = 4$. Then the second condition of (ii) of Eq. (10) means that $\text{conv} \{r_i, \tilde{u}_i\}_{i=1}^N$ is congruent to $\text{conv} \{s_i\}_{i=1}^N$ and can be overlapped by parallel transport $\mathbf{e}^*$. This implies the following facts. Firstly, the distance from $r_1 \tilde{u}_1$ to $r_2 \tilde{u}_1$ is the same as $l_i$, which is the distance between $q_1 \tilde{u}_i$ and $q_1 \tilde{u}_i$, that is, $l_i = ||s_i||_2$. Secondly, the relative interior of $\text{conv} \{s_i\}_{i=1}^N$, $\Omega$, contains $\mathbf{e}^*$ because the relative interior of $\text{conv} \{r_i, \tilde{u}_i\}_{i=1}^N$ contains $\vec{0}$. Note that $\Omega$ has infinite numbers of elements. Thirdly, $l_i = ||s_i - \mathbf{e}^*||_2$, $\tilde{u}_i = \frac{s_i - \mathbf{e}^*}{||s_i - \mathbf{e}^*||_2} = -\tilde{w}_i$. \quad (14)

Since $\text{conv} \{s_i\}_{i=1}^N$ is a simplex, for any $\mathbf{e} \in \Omega$, there exists a unique $\{t_i(\mathbf{e})\}_{i=1}^N$ which satisfies the following relations.

$$t_i(\mathbf{e}) > 0 \forall i, \quad \sum_{i=1}^N t_i(\mathbf{e}) = 1, \quad \sum_{i=1}^N t_i(\mathbf{e})s_i = \mathbf{e}, \quad (15)$$

which implies that

$$p_i = \frac{t_i(\mathbf{e}^*) ||s_i - \mathbf{e}^*||_2}{\sum_{j=1}^N t_j(\mathbf{e}^*) ||s_j - \mathbf{e}^*||_2}. \quad (16)$$

In fact, for any $\mathbf{e} \in \Omega$, $\{p_i, \tilde{u}_i; r_i, \tilde{u}_i\}_{i=1}^N$ defined in the following way, satisfies (i) and (iii) of Eq. (10), and the second condition of (ii).

$$p_i = \frac{t_i(\mathbf{e}) ||s_i - \mathbf{e}||_2}{\sum_{j=1}^N t_j(\mathbf{e}) ||s_j - \mathbf{e}||_2}, \quad \tilde{u}_i = \frac{s_i - \mathbf{e}}{||s_i - \mathbf{e}||_2}, \quad \tilde{w}_i = \frac{s_i - \mathbf{e}}{||s_i - \mathbf{e}||_2}. \quad (17)$$

Therefore $\{p_i, \tilde{u}_i; r_i, \tilde{w}_i\}_{i=1}^N$ satisfying Eq. (10) except the first condition of (ii) are always innumerable. $\mathbf{e}^*$ is an optimal $\mathbf{e} \in \Omega$ which satisfies up to the first condition of (ii) of Eq. (10).

The meaning of the first condition of (ii) can be understood as $\mathbf{e}^* \in \bigcap_{i=1}^N C_i$ because $r_i = ||s_i - \mathbf{e}||_2$ in Eq. (17). Here $C_i$ is the hyperboloid made of points such that difference between the distance $s_i$ and $0$ is $e_i$. That is,

$$C_i = \{ \mathbf{e} \in \mathbb{R}^3 : ||s_i - \mathbf{e}||_2 - ||\mathbf{e}||_2 = e_i \} \forall i \in \mathcal{I}, \quad (18)$$

where

$$\mathcal{I} = \{ i \in \mathbb{Z} : 2 \leq i \leq N \}. \quad (19)$$

This implies that if $\mathbf{e} \in \bigcap_{i \in \mathcal{I}} C_i \cap \Omega$, $\{p_i, \tilde{u}_i; r_i, \tilde{u}_i\}_{i=1}^N$ of Eq. (17) satisfies every condition in Eq. (10), that is, $\mathbf{e} = \mathbf{e}^*$. Furthermore if $\bigcap_{i \in \mathcal{I}} C_i \cap \Omega$ is empty, there does not exist $\{p_i, \tilde{u}_i; r_i, \tilde{w}_i\}_{i=1}^N$ satisfying Eq. (10). In this case, $\{q_i, \rho_i\}_{i=1}^N$ is optimized with MD of less than $N$ quantum states. Therefore $\{q_i, \rho_i\}_{i=1}^N$ is not optimized with MD of less than $N$ quantum states if and only if

$$D = N - 1 \text{ and } \bigcap_{i \in \mathcal{I}} C_i \cap \Omega \neq \emptyset. \quad (20)$$

In addition, using $\Theta_i$ which is the angle between two segments $\text{conv}\{0, \mathbf{e}^*\}$ and $\text{conv}\{0, \mathbf{s}_i\}$, we can obtain the following relation by hyperbolic equation.

$$||\mathbf{e}^*||_2 = \frac{l_i^2 - e_i^2}{2(l_i \cos \Theta_i + e_i)} \forall i \in \mathcal{I}. \quad (21)$$
Here $\bar{c}^\circ$ represents an element of $\left(\bigcap_{i \in \mathcal{I}} C_i \right) \cap \Lambda$, and is a potential candidate of $c^\bullet$, where $\Lambda$ is the relative interior of convex cone whose apex base is $\bar{0}(\text{conv}(\{\bar{s}_i\}_{i \in \mathcal{I}}))$. Since $||c^\circ||_2$ is strictly increasing function of $\Theta_i$, $c^\circ$ and $\{\Theta_i\}_{i \in \mathcal{I}}$ are unique. By $\Omega \subset \Lambda$, if $c^\circ \in \Omega$, we obtain $c^\bullet = c^\circ$. This means that $c^\bullet$ is unique. Therefore if Eq. (20) holds, the optimal POVM is unique, and the guessing probability $p_{\text{guess}}$ is $q_1 + ||c^\circ||_2$ by $r_1 = ||c^\circ||_2$. The following theorem summarizes our results obtained in this section.

**Theorem 1** When $D = N - 1$, $N$ qubit states $\{q_i, \rho_i\}_{i = 1}^N$ is not optimized with MD of less than $N$ quantum states if and only if $\exists c^\circ \in \Omega$. Then the minimum-error measurement is unique, and the guessing probability is $q_1 + ||c^\circ||_2$. The optimal POVM and the set of complementary states are obtained by substituting Eq. (17) of $c = c^\circ$ into Eq. (6).

Note that the semidefinite programming of MD or the derivation of Theorem 1 is independent of the normalization of prior probabilities. In other words, Theorem 1 can be applied even to the case of un-normalized prior probabilities.

**OPTIMAL DISCRIMINATION OF FOUR QUBIT STATES**

Now, we will explain $\exists c^\circ \in \Omega$ in terms of analytic form. First of all, by the geometric condition $D = N - 1$, we classify our problem into the cases of $N = 2, 3,$ and $4$. In $N = 2$, $\text{conv}(\{\bar{s}_i\}_{i = 1}^N)$ is a line segment. In $N = 3$, it is a triangle. And in $N = 4$, it becomes a tetrahedron. To avoid the repetition of representation at $N = 3$ or 4, we use different indices $x, y, z$. That is, when $N = 3$, we use only $x, y \in \mathcal{I}$. However, when $N = 4$, we use $x, y, z \in \mathcal{I}$. The process to check the existence of $c^\bullet$ can be understood in three steps. At first, we confirm the existence of $\{C_i \cap \Lambda\}_{i \in \mathcal{I}}$. Secondly, we check the existence of the intersection point $c^\circ$. Thirdly, we verify $c^\circ \in \Omega$. Then we can see that $c^\circ$ becomes $c^\bullet$.

**Necessary and sufficient condition for no null measurement operator**

The first step can be represented, by Eq. (21), as the following analytic form:

$$l_i > e_i \quad \forall i \in \mathcal{I}. \quad (22)$$

In the case of $N = 2$, $\bigcap_{i \in \mathcal{I}} (C_i \cap \Lambda)$ becomes $C_2 \cap \Lambda$, and the element of the set is located in $\Omega$. Therefore $\exists c^\circ \in \Omega$ can be completely represented by Eq. (22). However it is not the cases when $N = 3$ or 4 since even in the case where $\{C_i \cap \Lambda\}_{i \in \mathcal{I}}$ may exist, $c^\circ$ may not exist and even when $c^\circ$ exist, it may not be the element of $\Omega$.

To express the second step in an analytic form, we have to find an analytic form of $\{\Theta_i\}_{i \in \mathcal{I}}$, satisfying Eq. (21). If $N = 2, \Theta_2 = 0$ holds trivially. However, if $N = 3$ or 4, we substitute the following relations into Eq. (21).

$$\cos \Theta_y = \cos \Theta_x \cos \theta_{xy} + \sin \Theta_x \sin \theta_{xy} \cos \Phi_{xy},$$

$$\sin^2 \phi_z = \cos^2 \Phi_{xz} + \cos^2 \Phi_{zy}$$

$$- 2 \cos \phi_z \cos \Phi_{xz} \cos \Phi_{zy}, \quad (23)$$

where $\theta_{xy}$ is the internal angle between two line segments $\text{conv}\{\bar{s}_1, \bar{s}_x\}$ and $\text{conv}\{\bar{s}_1, \bar{s}_y\}$, $\phi_z$ is the internal angle between two triangles $\text{conv}\{\bar{s}_1, \bar{s}_x, \bar{s}_z\}$ and $\text{conv}\{\bar{s}_1, \bar{s}_y, \bar{s}_z\}$, and $\Phi_{xy}$ is the internal angle between two triangles $\text{conv}\{\bar{s}_1, \bar{s}_x, c^\circ\}$ and $\text{conv}\{\bar{s}_1, \bar{s}_y, c^\circ\}$. When $N = 3$, we get $\Theta_i = \alpha_i (\forall i \in \mathcal{I})$:

$$\alpha_x = \arccos \left(\frac{-X_{xy}Y_{xy} + \sqrt{1 + X_{xy}^2 - Y_{xy}^2}}{1 + X_{xy}^2} \right), \quad (24)$$

where

$$X_{xy} = l_x (l_z^2 - c_x^2) - l_y (l_z^2 - c_y^2) \cos \theta_{xy},$$

$$Y_{xy} = c_x (l_z^2 - c_x^2) - c_y (l_z^2 - c_y^2). \quad (25)$$

In the case of $N = 4$, we have $\Theta_i = \beta_i (\forall i \in \mathcal{I})$:

$$\beta_z = \arccos \left(\frac{-Z_z + \sqrt{2Z_z^2 - (X_z + \sin^2 \phi_z)(Y_z - \sin^2 \phi_z)}}{X_z + \sin^2 \phi_z} \right), \quad (26)$$

where

$$\bar{X}_z = X_z^2 + Y_z^2 - 2X_zX_y \cos \phi_z,$$

$$\bar{Y}_z = Y_z^2 + X_z^2 - 2X_zY_z \cos \phi_z,$$

$$Z_z = X_zY_z + X_zY_z \cos \phi_z,$$

$$(-X_zY_z + Y_zX_z) \cos \phi_z. \quad (27)$$

Then, $\Phi_{xy}$ is given as the following $\gamma_{xy}$:

$$\gamma_{xy} = \arccos \left(\frac{X_{xy} \cos \beta_x + Y_{xy}}{\sin \beta_x} \right). \quad (28)$$

If Eq. (22) is fulfilled, the value inside the root of $\alpha_x$ is always positive however that of $\beta_x$ is not always positive. Therefore unlike $N = 3$ we should add the following condition in the case of $N = 4$:

$$Z_z^2 \geq (\bar{X}_z + \sin^2 \phi_z)(\bar{Y}_z - \sin^2 \phi_z) \quad \forall i \in \mathcal{I}. \quad (29)$$

In fact points corresponding to $\{\alpha_i\}_{i \in \mathcal{I}}$ at $N = 3$ and points corresponding to $\{\beta_i\}_{i \in \mathcal{I}}$ at $N = 4$ are located in $\bigcap_{i \in \mathcal{I}} (C_i \cap \text{aff}(\{\bar{s}_i\}_{i = 1})$. By $\Lambda \subset \text{aff}(\{\bar{s}_i\}_{i = 1})$, this implies that if one of them is included in $\Lambda$, the point is $c^\circ$, but if one of them is not included in $\Lambda$, $c^\circ$ does not exist. Therefore for $\alpha_x$ and $\alpha_y$ at $N = 3$, we have

$$\alpha_x, \alpha_y < \theta_{xy} \text{ and } \alpha_x + \alpha_y < \pi. \quad (30)$$
If we consider \( \{ (\gamma_{zx}, \gamma_{zy}) \}_z \) at \( N = 4 \), we find
\[
\gamma_{zx}, \gamma_{zy} < \varphi_2 \quad \text{and} \quad \gamma_{zx} + \gamma_{zy} < \pi \quad \forall z \in \mathcal{I}. \tag{31}
\]

Final stage can be represented as the relation between \( \| \mathbf{c}^o \|_2 \) and \( \| \mathbf{g} \|_2 \). \( \mathbf{g} \) denotes the intersection point of two sets of \( \{ s_1, \mathbf{c}^o \} \) and \( \text{conv\{} s_i \}_{i \in \mathcal{I}} \). The inequality of \( \| \mathbf{c}^o \|_2 < \| \mathbf{g} \|_2 \) implies \( \mathbf{c}^o \in \Omega \), and the inequality of \( \| \mathbf{c}^o \|_2 \geq \| \mathbf{g} \|_2 \) means \( \mathbf{c}^o \notin \Omega \). Since \( \text{conv\{} s_i \}_{i \in \mathcal{I}} \) is a simplex, \( \text{conv\{} s_i \}_{i \in \mathcal{I}} \) becomes a simplex. It means that there exists a unique \( \{ \tilde{t}_i \} \in \mathcal{I} \) satisfying the following relations
\[
\tilde{t}_i > 0 \quad \forall i \in \mathcal{I}, \quad \sum_{i \in \mathcal{I}} \tilde{t}_i = 1, \quad \sum_{i \in \mathcal{I}} \tilde{t}_i s_i = \mathbf{g}. \tag{32}
\]

Then \( \{ \tilde{t}_i \} \in \mathcal{I} \) and \( \| \mathbf{g} \|_2 \) are expressed as follows:
\[
\begin{align*}
N = 2 : & \quad \tilde{t}_2 = 1, \quad \| \mathbf{g} \|_2 = t_2, \\
N = 3 : & \quad \tilde{t}_x = \frac{t_x \sin \alpha_y}{\Delta_1 \sin \alpha_x + t_y \sin \alpha_y}, \quad \| \mathbf{g} \|_2 = \frac{\Delta_2 (\Gamma_1 \sin \alpha_x + \Gamma_2 \sin \alpha_y)}{\Delta_1 \sin \alpha_x + \Delta_2 \sin \alpha_y}, \\
N = 4 : & \quad \tilde{t}_2 = \frac{\Delta_1 \Gamma_1}{\Delta_2 \Gamma_1 + \Delta_3 \Gamma_3 + \Delta_4 \Gamma_4}, \quad \| \mathbf{g} \|_2 = \frac{\Delta_2 \Gamma_2 + \Delta_3 \Gamma_3 + \Delta_4 \Gamma_4}{\Delta_2 \Gamma_2 + \Delta_3 \Gamma_3 + \Delta_4 \Gamma_4},
\end{align*}
\]

where
\[
\Gamma_z = \sin \beta_z \sin \gamma_{xy}. \tag{33}
\]

Here \( \Delta_z \) is the area of triangle \( \text{conv\{} s_1, s_x, s_y \} \) and \( \Delta_{\text{tetra}} \) is the volume of tetrahedron \( \text{conv\{} s_i \}_{i = 1} \). Therefore if \( \exists \mathbf{c}^o \in \Omega \) holds, \( \mathbf{c}^* \) and \( \{ \tilde{t}_i (\mathbf{c}^*) \} \in \mathcal{I} \) can be expressed as follows:
\[
\mathbf{c}^* = \frac{\| \mathbf{c}^o \|_2 \cdot \sum_{i \in \mathcal{I}} \tilde{t}_i s_i}{\| \mathbf{g} \|_2}, \quad t_i (\mathbf{c}^*) = \frac{\tilde{t}_i (\mathbf{c}^*)}{\| \mathbf{g} \|_2} \quad \forall i \in \mathcal{I}. \tag{34}
\]

Figure 1 shows \( \mathbf{c}^o \) when \( D = N - 1 \) and \( \exists \mathbf{c}^o \in \Omega \). Now, we use \( m \) to denote the maximal number of null operators which can exist in optimal measurement. Table I in appendix provides the analytic form of necessary and sufficient condition for the case of \( m = 0 \) in \( D = N - 1 \) of N qubit states. Briefly, \( l_2 > e_2 \) is the condition of two qubit states in the case of \( N = 2 \) and \( D = 1 \). C1(C2) is the condition for \( N = 3 \) and \( D = 2 (N = 4 \) and \( D = 3 \) in three(four) qubit states.

The following theorem provides the complete analysis to the minimum error discrimination of four mixed qubit states with arbitrary prior probabilities.

**Theorem 2** When arbitrary four qubit states \( \{ q_i, p_i \} \) with \( D = 3 \) are given, every optimal POVM has four nonzero elements if and only if
\[
\begin{align*}
l_2 &> e_2, \quad l_3 > e_3, \quad l_4 > e_4, \\
Z^2_2 &\geq (X_2 + \sin^2 \varphi_2) (Y_2 - \sin^2 \varphi_2), \\
Z^2_3 &\geq (X_3 + \sin^2 \varphi_3) (Y_3 - \sin^2 \varphi_3), \\
Z^2_4 &\geq (X_4 + \sin^2 \varphi_4) (Y_4 - \sin^2 \varphi_4), \\
\gamma_{23} &< \varphi_2 < \gamma_{24} < \pi, \\
\gamma_{34} &< \varphi_3 < \gamma_{32} < \pi, \\
\gamma_{42} &< \varphi_4 < \gamma_{43} < \pi,
\end{align*}
\]

Then \( \{ p_i, \bar{u}_i \} \in \mathcal{I} \) of the unique optimal POVM and guessing probability are given by
\[
\begin{align*}
p_1 &= \frac{6 \Delta_{\text{tetra}} (l_2 \cos \beta_2 + e_2) - (l_2^2 - e_2^2) \sum_{i = 2}^4 \Gamma_i \Delta_i}{2(l_2 \cos \beta_2 + e_2)(3 \Delta_{\text{tetra}} + \sum_{i = 2}^4 \Gamma_i \Delta_i)} \quad \forall i \neq 1, \\
\bar{u}_1 &= \frac{\Delta_2 \Gamma_2 + \Delta_3 \Gamma_3 + \Delta_4 \Gamma_4}{3 \Delta_{\text{tetra}}}, \\
p_i &= \frac{\Delta_2 \Gamma_2 + \Delta_3 \Gamma_3 + \Delta_4 \Gamma_4}{2(l_2 \cos \beta_2 + e_2)} \quad \forall i \neq 1, \\
\bar{u}_i &= \frac{6 \Delta_{\text{tetra}} (l_i \cos \beta_i + e_i) - (l_i^2 - e_i^2) \sum_{j \neq i} \Gamma_j \Delta_j \bar{s}_j}{3 \Delta_{\text{tetra}} (l_i^2 + e_i^2 + 2l_i e_i \cos \beta_i)}, \quad \forall i \neq 1, \\
\text{and} \quad p_{\text{guess}} = q_1 + \frac{l_2^2 - e_2^2}{2(l_2 \cos \beta_2 + e_2)}. \tag{36}
\end{align*}
\]

Here let us consider an example of minimum error discrimination to mixed four qubit states with non-equal prior probabilities. To understand the behavior of minimum error discrimination according to prior probabilities, we consider non-fixed prior probabilities. The mixed
four qubit states and the prior probabilities are as follows:

\[
\begin{align*}
q_1 &= \frac{1+h}{4}, \quad \vec{v}_1 = \frac{1}{2} (1, 0, -1), \\
q_2 &= \frac{1}{4}, \quad \vec{v}_2 = \frac{1+h}{2} (-1, 0, -1), \\
q_3 &= \frac{1}{4}, \quad \vec{v}_3 = \frac{1-h}{2} (0, 1, 1), \\
q_4 &= \frac{1-h}{4}, \quad \vec{v}_4 = \frac{1}{2} (0, -1, 1),
\end{align*}
\]

(37)

When \( h = 0 \), the prior probabilities are identical and \( \{\vec{v}_i\}_{i=1}^4 \) forms a symmetric tetrahedron of which circumcenter and circumradius are \( 0 \) and \( \frac{1}{\sqrt{2}} \), respectively.

Then the guessing probability becomes \( \frac{1}{4} + \frac{1}{4\sqrt{2}} \). It is because if the purify \( f_i(= \|\vec{v}_i\|_2) \) of equiprobable four qubit states is \( f \), the origin \( 0 \) is included in relative interior of \( \text{conv}\{v\}_{i=1}^4 \) and the guessing probability becomes \( \frac{1}{4} + \frac{1}{4f[15]} \).

However, when \( 0 < h \leq \sqrt{2} - 1 \), we have \( q_1 > q_2 = q_3 > q_4 \) and \( \{\vec{v}_i\}_{i=1}^4 \) constructs a non-symmetric tetrahedron. In addition, purity of four qubit states cannot be the same, because of \( f_2 > f_1 = f_3 \). Therefore, without using the method proposed in this paper, guessing probability cannot be obtained. Once we use the result of section 3.2, we can obtain the guessing probability.

\[
p_{\text{guess}} = \begin{cases} 
\frac{1}{4} + \frac{h}{4} + \frac{1-h^2+2h^3+12h^4+4h^5-h^6+2h^7+2h^8}{4h(2-h-10h^2-2h^3+2h^4-2h^5-2h^6-2h^7-2h^8+4(1-h^2)\sqrt{2-10h^2+5h^4-2h^6})} & \text{if } h < h^*, \\
\frac{1}{4} + \frac{h}{4} + \frac{1-h^2+2h^3+12h^4+4h^5-h^6+2h^7+2h^8}{8h(7+10h-6h^2-2h^3-h^4)+8(1+h)(1+2h)(5-2h+h^2)} & \text{if } h \geq h^*.
\end{cases}
\]

(38)

\[\text{FIG. 2: Guessing probability of non-equiprobable non-symmetric four qubit states. If } 0 \leq h < h^* \approx 0.1440, \text{ we get four elements optimal POVM. However, if } h^* \leq h \leq \sqrt{2} - 1 \approx 0.4143, \text{ we obtain three elements optimal POVM.}\]

\( h^*(\approx 0.1440) \) is the value which is the boundary between four elements optimal POVM and three elements optimal POVM. If \( 0 \leq h < h^* \), we get four elements optimal POVM. However, if \( h^* \leq h \leq \sqrt{2} - 1 \), we obtain three elements optimal POVM. Figure 2 shows \( p_{\text{guess}}(h) \) intuitively.

In summary, we provided a complete analysis to minimum error discrimination of four mixed qubit states with arbitrary prior probabilities. For minimum error discrimination of four mixed qubit states with arbitrary prior probabilities, we analyzed the necessary and sufficient conditions for the existence of null measurement operator. Specifically, from the geometric structure of qubit states, we obtained the analytic condition for deciding the existence of a null operator in minimum-error measurement for four qubit states, which lets us figure out the necessary and sufficient conditions for every optimal POVM to have non-zero elements. From the condition, we completely analyzed minimum-error discrimination of four qubit states with arbitrary prior probabilities. In addition, we provided a relevant example.

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TABLE I: The necessary and sufficient condition that every optimal measurement has no null operator for $N$ qubit states in $D = N - 1$. $\exists \bar{x} \in \Omega$ has different analytic form for $N = 2, 3, 4$. Note that $q_1$ is the largest prior probability.

| Geometric form | Analytic form |
|----------------|---------------|
| $N = 2$        | $C_2 \cap \Lambda \neq \emptyset$ | $l_2 > e_2$ |

(C0)

| $N = 3$        | $C_2 \cap \Lambda, C_3 \cap \Lambda \neq \emptyset$ | $l_2 > e_2, l_3 > e_3$ |
|----------------|-------------------------------------------------|------------------------|
| $\exists \bar{x}$ when $C_2 \cap \Lambda, C_3 \cap \Lambda \neq \emptyset$ | $\alpha_2, \alpha_3 < \theta_23$ and $\alpha_2 + \alpha_3 < \pi$ |
| $\bar{x} \in \Omega$ when $\exists \bar{x}$ | $\frac{l_2^2 - e_2^2}{2(l_2 \cos \theta_23 + e_2)} < \frac{l_3^2 \sin \theta_23}{2l_2 \sin \alpha_2 + l_3 \sin \alpha_3}$ |

(C1)

| $N = 4$        | $C_2 \cap \Lambda, C_3 \cap \Lambda, C_4 \cap \Lambda \neq \emptyset$ | $l_2 > e_2, l_3 > e_3, l_4 > e_4$ |
|----------------|-------------------------------------------------|------------------------|
| $\exists \bar{x}$ when $C_2 \cap \Lambda, C_3 \cap \Lambda, C_4 \cap \Lambda \neq \emptyset$ | $Z_2^2 \geq (X_2 + \sin^2 \phi_2) (Y_2 - \sin^2 \phi_2)$ |
| $\bar{x} \in \Omega$ when $\exists \bar{x}$ | $Z_4^2 \geq (X_4 + \sin^2 \phi_4) (Y_4 - \sin^2 \phi_4)$ |

(C2)

\[
e_1, \bar{s}_1, l_i \quad e_i = q_i - q_1, \quad \bar{s}_i = q_i \bar{v}_i - q_1 \bar{v}_1, \text{ and } l_i = \|\bar{s}_i\|_2
\]

\[x, y, z \quad \{x, y, z\} = \{2, 3\} \text{ when } N = 3, \quad \{x, y, z\} = \{2, 3, 4\} \text{ when } N = 4
\]

\[\theta_{xy} \quad \text{angle between line segments } \text{conv}\{\bar{s}_1, \bar{s}_x\} \text{ and } \text{conv}\{\bar{s}_1, \bar{s}_y\} \text{ when } N = 3, 4
\]

\[\phi_z \quad \text{angle between triangles } \text{conv}\{\bar{s}_1, \bar{s}_z, \bar{s}_x\} \text{ and } \text{conv}\{\bar{s}_1, \bar{s}_z, \bar{s}_y\} \text{ when } N = 4
\]

\[\Delta_2, \Delta_{tetra} \quad \text{area of triangle } \text{conv}\{\bar{s}_1, \bar{s}_x, \bar{s}_y\}, \quad \text{volume of tetrahedron } \text{conv}\{\bar{s}_i\}^4_{i=1}
\]

\[X_{xy}, Y_{xy} \quad \frac{l_x(l_x^2 - e_x^2) - l_y(l_y^2 - e_y^2) \cos \theta_{xy}}{l_x(l_x^2 - e_x^2) \sin \theta_{xy}}, \quad \frac{e_x(l_x^2 - e_x^2) - e_y(l_y^2 - e_y^2)}{l_y(l_y^2 - e_y^2) \sin \theta_{xy}}
\]

\[X_{xy} \hat{x} \pm X_{xy} \hat{y} - 2X_{xy}X_{xy} \cos \phi_{xy}, \quad Y_{xy}^2 + Y_{xy}^2 - 2Y_{xy}Y_{xy} \cos \phi_{xy}
\]

\[\hat{Z}_z \quad X_{xx}X_{xy} + X_{xy}Y_{xy} - (X_{xx}Y_{xy} + X_{xx}X_{xy}) \cos \phi_{xy}
\]

\[\alpha_x, \beta_x \quad \text{arccos} \left(\frac{X_{xy}Y_{xx} + \sqrt{1+X_{xy}^2 - Y_{xx}^2}}{1+X_{xy}^2}, \quad \text{arccos} \left(\frac{-2X_{xy}Y_{xx} + \sqrt{2^2 - (X_{xx}+\sin^2 \phi_{xy})^2}(Y_{xx}-\sin^2 \phi_{xy})}{X_{xx}+\sin^2 \phi_{xy}}\right)\right)
\]

\[\gamma_{xy}, \Gamma_z \quad \text{arccos} \left(\frac{X_{xy} \cos \phi_{xy} + Y_{xy}}{\sin \beta_{xy}}\right), \quad \sin \beta_x \sin \gamma_{xy}
\]