Stability analysis of heartbeat system with lyapunov’s direct method

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Abstract. The heart is a very vital organ. There are two main states in the heartbeat cycle, systole and diastole. The heartbeat system studied in this paper was taken from a study of Thanom and Loh in 2011. This paper is aimed to analyze the stability of the heartbeat system using Lyapunov’s Direct Method. This method requires a Lyapunov function. To construct the Lyapunov function, the Gradient Variable Method will be used. In this research, the Lyapunov function constructed \( V(x) = x_1^2 + x_2^2 ; \dot{V}(x) > 0 \) is obtained. This means \( \dot{V}(x) \) is said to be a positive definite and the equilibrium point \((0,0)\) of the heartbeat system is unstable.

1. Introduction

The dynamic system is a system of equations that is affected by changes in motion and time. One of the important studies in dynamic system problems is how the state of the system, whether the system is a stable system or not. Stability problems can be solved by various methods such as the eigenvalue method, the Routh-Horwitz criterion, and other methods.

At the end of the 19th century, a Russian mathematician, Aleksandr Mikhailovich Lyapunov, developed an approach to stability analysis that is now known as the Lyapunov direct method (Lyapunov’s direct method). The solution to the stability using the Lyapunov method directly requires a function called the Lyapunov function, [1] describe the direct method as one of the most powerful techniques for qualitative analysis of a dynamical system. The direct Lyapunov method is widely applied to analyze the stability of both linear and nonlinear systems, time-invariant or time-varying systems.

The human heart is a very vital organ. There are two states in the heartbeat cycle, namely diastole (relax) and systole (contraction). The heartbeat cycle begins when the heart is in a diastole state. The pacemaker or path opener at the upper right of the atrium triggers an electromagnetic wave slowly on the atrium. This wave causes the muscle fibers to contract and press the direction into the heart chambers. The same electromagnetic waves then spread rapidly in the chambers and cause all chambers to contract in a state of systole, pumping blood to the lungs and arteries. Then in this state of contraction, the muscle fibers will relax and return the heart to a state of diastole to complete one heartbeat cycle has provided a heartbeat system in a second-order differential equation system but there are still no journals discussing the stability of the system. [2]

2. Research Methods

This research method was carried out by studying the literature study on the system of nonlinear differential equation, the direct Lyapunov method. The method used for the construction of Lyapunov functions is to use the gradient variable method.
2.1 System of Nonlinear Differential Equations

System of Nonlinear differential equations can be written in the form of an equation [3]
\[
\dot{x} = f(x)
\]
(1)
where \( x \in \mathbb{R}^n \), \( f : D \subset \mathbb{R}^{n+1} \rightarrow \mathbb{R} \), and nonlinear function \( f \).

2.2 Heartbeat System

The human heartbeat system must have at least 3 basic cycles [4]:
1. The model must be based on a state of equilibrium with the rate of change in the length of the muscle fibers and the wave of electrochemical activity equal to zero.
2. There is a threshold that triggers electrochemical waves that cause the heart to contract.
3. The model is expected to return quickly in a balanced state.

The heartbeat system to be studied is given as follows [5]:
\[
\begin{align*}
\varepsilon \dot{x}_1 &= -(x_2^2 - Tx_1 + x_2), \quad T > 0 \\
x_2 &= x_1 - x_d
\end{align*}
\]
(2) \( (3) \)
where:
- \( x_1 \): length of muscle fibers
- \( x_2 \): electrochemical activity variable
- \( \varepsilon \): the parameter constant, which is a small positive value related to the eigenvalue
- \( x_d \): a scalar quantity that represents the length of muscle fibers in a diastole state
- \( T \): muscle tension

as \( T = 1 \), \( \varepsilon = 0.2 \), and \( x_d = 0 \) (refer to the literature taken)

2.3 Lyapunov’s Direct Method

The formed Lyapunov function can later be used to determine the stability of a system both linear and nonlinear [4]. In this study, Lyapunov’s function will be examined:
1. If \( V(x) > 0 \) for \( x \neq 0 \) and \( V(x) \leq 0 \) then \( V(x) \) is said to be a positive definite and the equilibrium point \( (0,0) \) of the system of differential equations in the equation (1) is stable.
2. If \( V(x) > 0 \) for \( x \neq 0 \) and \( V(x) < 0 \), then \( V(x) \) is said to be a positive definite and the equilibrium point \( (0,0) \) of the system of differential equations int the equation (1) is asymptotic stable.
3. If \( V(x) > 0 \) then \( V(x) \) is said to be a positive definite and the equilibrium point \( (0,0) \) of the system of differential equations in the equation (1) is unstable.

2.4 Gradient Variable Method

The Gradient Variable Method is an approach that is often used to construct Lyapunov functions. [6] in their journal explain some rules for building Lyapunov functions with a gradient variable approach as a way to determine the stability of a system. The relationship of a scalar function \( V(x) \) with its gradient is
\[
V(x) = \int_0^x \nabla V \, dx
\]
(4)
where \( \nabla V = \left( \frac{\partial V}{\partial x_1}, \ldots, \frac{\partial V}{\partial x_n} \right)^T \). For the scalar function \( V \) to be obtained to be single, the gradient function must meet the following curl condition.
\[
\frac{\partial V_i}{\partial x_j} = \frac{\partial V_j}{\partial x_i} \quad (i, j = 1, 2, \ldots, n)
\]
(5)
Curl conditions that are met result that the integration results do not depend on the integration path, but in general to get the function $V$ is enough to do integration along the parallel path, in other words

$$V(x) = \int_0^{x_1} \nabla V_1(x_1, 0, ..., 0) dx_1 + \int_0^{x_2} \nabla V_2(x_1, 0, ..., 0) dx_2 + \cdots + \int_0^{x_n} \nabla V_n(x_1, 0, ..., 0) dx_n$$

3. Result and Discussion

The first thing that needs to do is to figure out the system of Eq (2-3) given. Here, Fig. 1 depicts how a heartbeat system looks like

![Phase portrait of the 2nd-order heartbeat system](image)

Figure 1. Phase portrait of the 2nd-order heartbeat system

In Fig. 1, the cubic line (dashed curve) represents the steady-state of Eq (2). When $x_d = 0$ in Eq. (3), the equilibrium point of the system is at the origin. All trajectories initiated above the cubic line, that is, $x_1^2 - TX_1 + x_2 > 0$, direct downward toward the origin along the cubic line. Likewise, all trajectories started below the cubic line, i.e., $x_1^2 - TX_1 + x_2 < 0$, direct upward toward the origin along the cubic line. All trajectories end up at the limit cycle around the equilibrium point. It is obvious that the equilibrium point is unstable as the vector field inside the limit cycle directs away from the point.

By using the Gradient Variable method, the Lyapunov function construction is obtained $V(x) = x_1^2 + x_2^2 > 0$ dan $\dot{V}(x) > 0$, this means $V(x)$ is positive definite and the equilibrium point $(0,0)$ of the heartbeat system is unstable.

4. Conclusion

Based on the results of the equilibrium point is said to be unstable because the heart is an organ of the body that does not stop moving so it can be said that the heart is never in a stable condition. The system at the equilibrium point at $(0,0)$ is stable if it is local $\tau < 1$ and at the other two equilibrium points stable if $1 < \tau < \frac{\alpha(\alpha+b+c)}{\alpha-\beta-1}$. And for systems at two other equilibrium points, the Lyapunov function can be constructed using the gradient variable method.
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