General Relativistic Analog Solutions for Yang-Mills Theory *

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Abstract

Finding solutions to non-linear field theories, such as Yang-Mills theories or general relativity, is usually difficult. The field equations of Yang-Mills theories and general relativity are known to share some mathematical similarities, and this connection can be used to find solutions to one theory using known solutions of the other theory. For example, the Schwarzschild solution of general relativity can be shown to have a mathematically similar counterpart in Yang-Mills theory. In this article we will discuss several solutions to the Yang-Mills equations which can be found using this connection between general relativity and Yang-Mills theory. Some comments about the possible physical meaning of these solutions will be discussed. In particular it will be argued that some of these analog solutions of Yang-Mills theory may have some connection with the confinement phenomenon. To this end we will briefly look at the motion of test particles moving in the background potential of the Schwarzschild analog solution.

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I. INTRODUCTION

Yang-Mills theories are non-Abelian gauge theories which have found a central role in particle physics in describing both the electroweak and strong interactions. The non-Abelian nature of Yang-Mills theories make the field equations non-linear, and therefore much more difficult to handle compared to Abelian gauge theories such as pure electromagnetism. For example, at the classical level (and also approximately at the quantum level if the quantum corrections are not too large – see the Introduction of Ref. [2]) one can use superposition for Abelian gauge theories, while even at the classical level, superposition is not valid for Yang-Mills theories. This non-linear nature of the Yang-Mills field equations makes finding solutions difficult. There are some well known and interesting solutions to the Yang-Mills field equations, such as the t ’Hooft-Polyakov monopole, the Julia-Zee dyon, the BPS dyon, and the instanton, but there is no systematic way of arriving at solutions to the Yang-Mills field equations.

General relativity can also in some sense be considered a non-Abelian gauge theory, and a mathematical connection between the two theories can be made. Using this connection between the two theories one can ask if the solutions to the field equations of one theory could provide a starting point to look for solutions in the other theory. This is in fact possible, and one can find a host of solutions in this manner. In this paper we will give a review of the various solutions found in this way and discuss some of their properties.

All of the solutions discovered in this way have the apparent weak point that they have an infinite field energy, i.e., there are singularities in the fields of the solutions which make the field energy infinite. This is to be contrasted with the finite field energy solutions of Refs. However, aside from the mathematical interest in studying all types of solutions that occur in such non-linear field theories, we present some ideas concerning the possible physical uses of such singular solutions. One speculation is that some of these solutions may be connected with the confinement phenomenon of the strong interaction. Just as the various black hole solutions (Schwarzschild or Kerr black holes) exhibit a type of
confinement for any particle which crosses the event horizon, so too the Yang-Mills analogs of these solutions may exhibit a confining behaviour.

The outline of the paper is as follows: First we will discuss the spherically symmetric solutions of the SU(2) Yang-Mills equations coupled to a scalar field (these are usually called the Yang-Mills-Higgs equations). Second we will discuss solutions for gauge groups other than SU(2). Finally we will examine the behaviour of a test particle which is placed in the potential of the Schwarzschild analog solution. We will see that under certain conditions this analog solution can confine the test particle, and that this system has a half-integer angular momentum even though all the fields involved are of integer angular momentum.

II. SU(2) YANG-MILLS FIELD EQUATIONS FOR SPHERICALLY SYMMETRIC FIELD CONFIGURATIONS

The system studied in this section is an SU(2) gauge theory coupled to a scalar field, $\phi^a$, in the triplet representation. The scalar field is taken to have no mass or self interaction. The Lagrangian for this system is

$$\mathcal{L} = -\frac{1}{4} G^a_{\mu\nu} G^a_{\mu\nu} + \frac{1}{2} (D_\mu \phi_a)(D^\mu \phi^a)$$

(1)

where $G^a_{\mu\nu}$ is the field strength tensor of the SU(2) gauge fields, which is defined in terms of the gauge fields $W^a_\mu$, as

$$G^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g \epsilon_{abc} W^b_\mu W^c_\nu$$

(2)

and $D_\mu$ is the covariant derivative of the scalar field which is given by

$$D_\mu \phi^a = \partial_\mu \phi^a + g \epsilon_{abc} W^b_\mu \phi^c$$

(3)

The general equations of motion for this system are

$$\partial^\nu G^a_{\mu\nu} = g \epsilon_{abc} [G^b_{\mu\nu} W^c_\nu - (D_\mu \phi_b) \phi_c]$$

$$\partial^\mu D_\mu \phi^a = g \epsilon_{abc} (D_\mu \phi_b) W^c_\mu$$

(4)
these field equations can be simplified through the use of a generalized Wu-Yang ansatz [12] which was used by Witten [13] to study multi-instanton solutions

\[ W_i^a = \epsilon_{aij} \frac{r^j}{gr^2} [1 - K(r)] + \left( \frac{r_i r_a}{r^2} - \delta_{ia} \right) \frac{G(r)}{gr} \]

\[ W_0^a = \frac{r^a}{gr^2} J(r) \]

\[ \phi^a = \frac{r^a}{gr^2} H(r) \]

(5)

\[ K(r), G(r), J(r), \text{ and } H(r) \] are the ansatz functions to be determined by the equations of motion. Inserting these expressions into the field equations in Eq. (4) we find the following set of coupled, non-linear equations,

\[ r^2 K'' = K(K^2 + G^2 + H^2 - J^2 - 1) \]

\[ r^2 G'' = G(K^2 + G^2 + H^2 - J^2 - 1) \]

\[ r^2 J'' = 2J(K^2 + G^2) \]

\[ r^2 H'' = 2H(K^2 + G^2) \]

(6)

where the primes denote differentiation with respect to \( r \). The most well known solutions to these equations are those discovered by Prasad and Sommerfield [6] and independently by Bogomolnyi [5]. They are

\[ K(r) = \cos(\theta)Cr \text{csch}(Cr) \]

\[ G(r) = \sin(\theta)Cr \text{csch}(Cr) \]

\[ J(r) = \sinh(\gamma)[1 - Cr \text{coth}(Cr)] \]

\[ H(r) = \cosh(\gamma)[1 - Cr \text{coth}(Cr)] \]

(7)

where \( C, \theta \) and \( \gamma \) are arbitrary constants. One of the nice properties of this solution is that it has finite field energy. In terms of the ansatz functions the energy density of the fields is

\[ T^{00} = \frac{1}{g^2 r^2} \left( K'^2 + G'^2 + \frac{(K^2 + G^2 - 1)^2}{2r^2} + \frac{J^2(K^2 + G^2)}{r^2} + \frac{(rJ' - J)^2}{2r^2} \right) \]

\[ + \frac{H^2(K^2 + G^2)}{r^2} + \frac{(rH' - H)^2}{2r^2} \]

(8)

For the solution in Eq. (7) this gives a non-singular energy density, which when integrated over all space yields a finite field energy of \( E = 4\pi C \text{cosh}^2(\gamma)/g^2 \). This finite energy property
of the BPS solution is one of the main reasons for the interest in this classical solution. We now examine the general relativistic analog solutions of the Yang-Mills equations.

A. Solutions with Spherical Singularities

To find the general relativistic analog solutions to the Yang-Mills field equations we begin by examining the Schwarzschild solution of general relativity. The Schwarzschild solution in Schwarzschild coordinates has two non-trivial components to the metric tensor: \( g_{00} \) and \( g_{rr} \). The non-trivial spatial element has the form \( g_{rr} = \frac{K}{1 - Kr} \) and \( g_{00} = -\frac{1}{g_{rr}} \) where \( K = \frac{1}{2GM} \). Trying this form of \( g_{rr} \) in Eq. (6) one immediately finds the following solution

\[
K(r) = \frac{\pm \cos \theta C r}{1 \pm Cr}, \quad G(r) = \frac{\pm \sin \theta C r}{1 \pm Cr},
\]

\[
J(r) = \frac{\sinh \gamma}{1 \pm Cr}, \quad H(r) = \frac{\cosh \gamma}{1 \pm Cr},
\]

(9)

where \( C, \gamma \) and \( \theta \) are arbitrary constants. The solution with the minus sign in the denominator (which we call the Schwarzschild-like solution) develops a singularity in the gauge fields \( (W^a_\mu) \) and scalar fields \( (\phi^a) \) on a spherical surface of radius \( r = r_0 = 1/C \). Both the Schwarzschild-like solution and the solution with the plus sign in the denominator develop singularities in the fields at \( r = 0 \). These field singularities lead to the field energy of these solutions being infinite, as can be seen by inserting the ansatz functions from Eq. (9) into Eq. (8) and trying to integrate over all space. The investigation of such infinite energy solutions to the Yang-Mills equations has been discussed by several authors [14] [15] [16] [17] [18] [19], and the earliest discussion actually pre-dates [14] the study of the finite energy solutions such as the t ’Hooft-Polyakov monopole or the BPS dyon.

Although the infinite field energy of these solutions could be seen as a drawback as compared to the finite energy solutions, there are other classical field theory solutions which nevertheless have a physical importance. The prime example is the Coulomb solution in electromagnetism which has a field singularity at \( r = 0 \) that is similar to the \( r = 0 \) singularities of the solutions in Eq. (9). The Schwarzschild-like solution, with its singular spherical
surface, has been speculated to have some connection with the confinement phenomenon for
quarks \[15\] \[17\] \[18\] \[20\] \[21\]. By studying the motion of a test particle which moves in the
potentials given by the minus sign solution in Eq. (9) one finds that the spherical singularity
in the fields represents a barrier which can trap the test particle inside the sphere. This is
similar in spirit to bag models of hadron structure where one looks at test particles moving
in some confining potential (such as an infinite spherical well). Also it is interesting that
this Schwarzschild-like solution was arrived at from the general relativistic solution for a
non-rotating black hole, which exhibits its own type of “confinement”: any particle which
passes within the event horizon becomes permanently trapped. One should be cautious
about pushing this analogy too far, since the nature of the spherical singularity in general
relativity and Yang-Mills theory are different. The singularity at the event horizon of the
general relativistic Schwarzschild solution is not a physical singularity, but a coordinate sin-
gularity as can be seen by writing the Schwarzschild solution in Kruskal coordinates, where
the only singularity is the one at \( r = 0 \). Both singularities in the Yang-Mills analog of the
Schwarzschild solution are true singularities in the fields.

The existence of singular solutions for certain field theories is not new (e.g. the singu-
larities in the Coulomb solution of electromagnetism, the Wu-Yang monopole solution \[12\],
or the meron solutions \[22\]). Even the appearance of a singularity in the gauge fields on a
spherical surface, such as occurs in the Schwarzschild-like solution of Eq. (9), which may at
first appear unique, can be found in other infinite energy solutions. These other solutions
possess an infinite set of concentric spherical surfaces on which the fields develop a singu-
larity. This could be taken as evidence that such spherical surfaces with singularities are
not uncommon features in classical solutions to the Yang-Mills field equations. The first of
these solutions can be obtained by exchanging the hyperbolic functions of the BPS solution
in Eq. (7) with their trigonometric counterparts

\[
K(r) = \cos(\theta)C r \csc(Cr) \quad \quad G(r) = \sin(\theta)C r \csc(Cr) \quad \quad J(r) = \sinh(\gamma)[1 - Cr \cot(Cr)] \quad \quad H(r) = \cosh(\gamma)[1 - Cr \cot(Cr)]
\] (10)
This solution was briefly discussed by Hsu and Mac \cite{23} in their derivation of the BPS solution \textit{(i.e.} Hsu and Mac start with a solution like that in Eq. (11) and apply the transformation $C \to iC$ to arrive at the BPS solution). This solution exhibits a series of concentric spherical surfaces on which the gauge and scalar fields become singular. These singularities are located on the spherical surfaces $Cr = n\pi$ where $n = 1, 2, 3, 4, \ldots$. Inserting the ansatz functions of Eq. (11) in Eq. (8) we find that the energy density of this solution is

$$T^{00} = \frac{2 \cosh^2(\gamma)}{r^2 g^2} \left[ C^2 \csc^2(Cr) \left( 1 - Cr \cot(Cr) \right)^2 + \frac{(C^2 r^2 \csc^2(Cr) - 1)^2}{2 r^2} \right].$$

(11)

The energy density becomes singular on the same spherical surfaces as the gauge and scalar fields. These spherical shells, on which the energy density becomes infinite, cause the total field energy of this solution to diverge.

To obtain the next solution we simply try the complementary trigonometric functions for the solution in Eq. (11). Doing this shows that the following is also a solution \cite{24} to the Eq. (11)

$$K(r) = \cos(\theta)Cr \sec(Cr) \quad G(r) = \sin(\theta)Cr \sec(Cr)$$

$$J(r) = \sinh(\gamma)[1 + Cr \tan(Cr)] \quad H(r) = \cosh(\gamma)[1 + Cr \tan(Cr)]$$

(12)

It should be noted that due to the linear $Cr$ term in each solution, one can not obtain the solution in Eq. (12) from the other trigonometric solution in Eq. (11) by simply letting $Cr \to Cr - \pi/2$. Although these two trigonometric solutions are in this sense distinct \textit{(i.e.} they are not simply related by the transformation $Cr \to Cr - \pi/2$) they are physically similar since most of the comments concerning the solution in Eq. (11) apply here as well. Most obviously the ansatz functions, and therefore the gauge and scalar fields, become singular when $Cr = n\pi/2$ where $n = 1, 3, 5, 7, \ldots$ and at $r = 0$. Thus this solution exhibits a series of concentric spherical surfaces on which its fields become singular as well as a point singularity at the origin. These singularities also show up in the energy density of this solution as they did for the solution in Eq. (11). The point singularity at $r = 0$ and the spherical singular
surfaces of the solutions in Eqs. (11), (12) are similar to that of the solutions from Eq. (9). However, the solutions in Eq. (9) only possessed one spherical surface on which the fields and energy density diverged. One conjectured use for the Schwarzschild-like solution is as a possible explanation of the confinement mechanism. When the Schwarzschild-like solution of Ref. [19] is treated as a background potential in which a test particle is placed it is found that the spherical singularity can act as an impenetrable barrier which traps the test particle either in the interior or the exterior of the sphere [20], giving a classical type of confinement. Similar results have been found for other singular solutions [15] [17] [18]. In addition Ref. [15] points out that such a classical type of confinement is only possible with infinite energy solutions. Treating the trigonometric solutions as a background potential would also trap test particles between any two of the concentric spherical singularities. These trigonometric solutions could possibly be used to solve the field equations in some limited range of \( r \), and then it could be patched to one of the other solutions which would solve the field equations for the remaining range of \( r \). This is similar to what is sometimes done in general relativity where one tries to patch an exterior solution with some interior solution.

Finally one can obtain a third solution to Eq. (6) by applying the transformation \( C \rightarrow iC \) to the solution [21] in Eq. (12). This yields
\[
K(r) = i \cos(\theta) Cr \sech(Cr) \quad G(r) = i \sin(\theta) Cr \sech(Cr)
\]
\[
J(r) = \sinh(\gamma)[1 - Cr \tanh(Cr)] \quad H(r) = \cosh(\gamma)[1 - Cr \tanh(Cr)]
\]
(13)
Since the ansatz functions \( K(r) \) and \( G(r) \) are imaginary, the space components of the gauge fields will be complex. Despite this all the physical quantities associated with this complex solution, such as energy density, are real. Inserting the ansatz functions of Eq. (13) into Eq. (8) we find that the field energy density is
\[
T^{00} = \frac{2 \cosh^2(\gamma)}{r^2 g^2} \left[ -C^2 \sech^2(Cr) \left(1 - Cr \tanh(Cr)\right)^2 + \frac{(C^2 r^2 \sech^2(Cr) + 1)^2}{2 r^2} \right]
\]
(14)
This energy density is real, but the total field energy is infinite due to the singularity at \( r = 0 \). Thus the above solution is more like a Wu-Yang monopole [12] or a charged point particle, as opposed to a finite energy BPS dyon.
In addition to the preceding infinite energy solutions which have gauge and scalar fields that become singular on some spherical surface, there are other types of infinite energy, general relativistic analog solutions. In general relativity if one allows for a nonzero cosmological constant, \( \Lambda \), then the time-time component of the metric tensor for the Schwarzschild solution becomes

\[
g_{00} = 1 - \frac{2GM}{r} - \frac{\Lambda r^2}{3}
\]

The Newtonian potential for this solution is

\[
\Phi = \frac{g_{00} - 1}{2} = \frac{-GM}{r} - \frac{\Lambda r^2}{6}
\]

Using Eq. (16) as a starting point one finds the following simple solution to Eq. (6)

\[
K(r) = \cos \theta \quad G(r) = \sin \theta \quad J(r) = H(r) = \frac{B}{r} + Ar^2
\]

where \( a, B \) and \( \theta \) are arbitrary constants. If one sets \( A = 0 \) then it can be seen the Schwarzschild-like solutions of Eq. (9) and those above in Eq. (17) are of a similar form in the limit \( C \rightarrow \infty \) and \( e^\gamma/C \rightarrow 2B \). Inserting the ansatz functions of Eq. (17) into the gauge and scalar fields of Eq. (5) one finds that the time component of the gauge field \( (W^a_0) \) and the scalar field \( (\phi^a) \) behave like \( Ar + B/r^2 \). The space part of the gauge fields \( (W^a_i) \) have a \( 1/r \) dependence. This classical solution exhibits a linear confining potential similar to those used in some phenomenological studies of hadronic spectra [24]. In addition lattice gauge theory arguments [28] seem to indicate that the confining potential between quarks should be linear. Classical solutions similar to those in Eq. (17) were also discussed in Ref. [29] in connection with the confinement problem.

This solution also has an infinite field energy. Inserting the ansatz functions of Eq. (17) into the energy density of Eq. (8) and integrating to get the total field energy one finds

\[
E = \int T^{00} d^3x = \frac{4\pi}{g^2} \int_{r_a}^{r_b} T^{00} r^2 dr = \frac{4\pi A^2}{g^2} \left( r_b^3 - r_a^3 \right) - \frac{8\pi B^2}{g^2} \left( \frac{1}{r_b^3} - \frac{1}{r_a^3} \right)
\]
where we have introduced an upper \((r_b)\) and lower \((r_a)\) cutoff in the radial coordinate. If one lets \(r_b \rightarrow \infty\) then the field energy becomes infinite due to the linear part of the gauge and scalar fields, while if one lets \(r_a \rightarrow 0\) then the field energy becomes infinite due to the singularity at \(r = 0\). Compared to the solutions in Eq. (9), which had infinite field energy from local singularities (either at \(r = 0\) or \(r = 1/C\)), the solution in Eq. (17) can have a infinite field energy from the point singularity at \(r = 0\) and/or the linearly increasing gauge and scalar fields as \(r \rightarrow \infty\). Again, although this classical solution has some undesirable characteristics, it also exhibits features which are found in some phenomenological studies of hadronic bound states.

\section*{C. SU(3) Solutions}

Up to this point we have discussed classical solutions to the Yang-Mills field equations for SU(2) fields. Since QCD involves the SU(3) gauge group it is natural to ask if there are any SU(3) or even SU(N) generalizations of the above solutions. One possibility is to embed the above SU(2) solutions into an SU(N) gauge theory \cite{30}. Recently \cite{31} a Schwarzschild-like classical solution was found which is not a simple embedding of the previous SU(2) solutions into an SU(N) gauge theory, but is a true SU(3) solution. To arrive at the SU(3) solution one makes the following generalization \cite{31} of the Wu-Wu ansatz \cite{32,33,34}

\begin{align}
W_0 &= \frac{-i}{gr^2} \left( \lambda^7 x - \lambda^5 y + \lambda^2 z \right) + \frac{1}{2} \lambda^a \left( \lambda^a_{ij} + \lambda^a_{ji} \right) \frac{x^i x^j}{gr^3} w(r) \\
W_a^i &= \left( \lambda^a_{ij} - \lambda^a_{ji} \right) \frac{i x^j}{gr^2} (1 - f(r)) + \lambda^a_{jk} \left( \epsilon_{ijk} x^k + \epsilon_{ikl} x^l \right) \frac{x^i}{gr^3} v(r)
\end{align}

where \(\lambda^a\) are the Gell-Mann matrices. Using this ansatz in the Yang-Mills field equations yields the following set of coupled differential equations for the functions \(f(r), v(r), \phi(r)\) and \(w(r)\)

\begin{align*}
  r^2 f'' &= f^3 - f + 7fv^2 + 2vw\phi - f(w^2 + \phi^2) \\
  r^2 v'' &= v^3 - v + 7vf^2 + 2fw\phi - v(w^2 + \phi^2)
\end{align*}
\[ r^2 w'' = 6w(f^2 + v^2) - 12fv\phi \]
\[ r^2 \phi'' = 2\phi(f^2 + v^2) - 4fvw \]  
where the primes denote differentiation with respect to \( r \). The nonlinear, coupled differential equations in Eq. (20) are the SU(3) equivalents of the equations in Eq. (3). In Ref. [31] several simplifying assumptions were made in order to make the problem more tractable. First, taking \( w = \phi = 0 \), reduces Eq. (20) to
\[ r^2 f'' = f(f^2 - 1 + 7v^2) \]
\[ r^2 v'' = v(v^2 - 1 + 7f^2) \]  
Then further simplifying by letting \( f^2 = v^2 = q^2/8 \) one finds that Eq. (21) reduces to the Wu-Yang [12] equation for \( q(r) \)
\[ r^2 q'' = q(q^2 - 1) \]  
This equation has been shown [14] [18] [31] to have a solution which is singular at some radius \( r = r_1 \). In other words near \( r = r_1 \) the solution will be of the form
\[ q(r) \approx \frac{A}{r_1 - r} \]  
where \( A \) and \( r_1 \) are constant. Thus, even with the scalar field absent one can find solutions to the pure gauge field theory equations which will tend to trap test particles behind a spherical barrier in much the same way as the Schwarzschild-like solution of Eq. (3). It is also possible to find closed form solutions to a special case of the system in Eq. (20) Taking \( v = w = 0 \) then the equations of Eq. (21) become
\[ r^2 f'' = f(f^2 - \phi^2 - 1) \]
\[ r^2 \phi'' = 2\phi f^2 \]  
which has the following simple closed form solution
\[ f(r) = \pm \frac{Cr}{1 \pm Cr} \]
\[ \phi(r) = \pm \frac{i}{1 \pm Cr} \]  
(25)
Other, similar solutions can be found by making different simplifying assumptions such as \( f = w = 0 \). Thus solutions with singular fields on a spherical surface are not unique to SU(2) gauge theories, but can also be found for SU(3) \([31]\) and in general for SU(N) \([30]\). The interesting aspect of the solutions given by Dzhunushaliev in Ref. \([31]\) is that these solutions are true SU(3) solutions rather than embeddings of the SU(2) solution into the SU(N) gauge group as in Ref. \([30]\). Also the SU(3) solutions presented here are pure gauge field solutions, as opposed to the general SU(2) solutions for the system given in Eq. (1), which involves scalar fields. In some sense the role of the scalar field of the SU(2) system is taken up by the time component of the gauge field in the SU(3) system. This can be seen by comparing the system of equations of Eq. (6) with the system of equations of Eq. (20): the equations for \( f(r), v(r) \) are similar to those for \( K(r), G(r) \) while the equations for \( w(r), \phi(r) \) are similar to those for \( J(r), H(r) \).

### III. ELECTROMAGNETIC PROPERTIES OF THE SU(2) SOLUTIONS

All of the SU(2) solutions to the Yang-Mills field equations have interesting “electromagnetic” features. To investigate these properties we will use ’t Hooft’s definition of a generalized, gauge invariant, U(1) field strength tensor \([3]\)

\[
F_{\mu\nu} = \partial_\mu (\hat{\phi}^a W^a_\nu) - \partial_\nu (\hat{\phi}^a W^a_\mu) - \frac{1}{g} \epsilon_{abc} \hat{\phi}^a (\partial_\mu \hat{\phi}^b)(\partial_\nu \hat{\phi}^c) \tag{26}
\]

where \( \hat{\phi}^a = \phi^a (\phi^b \phi^b)^{-1/2} \). This generalized U(1) field strength tensor reduces to the usual expression for the field strength tensor if one performs a gauge transformation to the Abelian gauge where the scalar field only points in one direction in isospin space \( i.e. \phi^a = \delta^a_3 \nu \) \([35]\). If one associates this U(1) field with the photon of electromagnetism then the solutions in Eqs. (11), (12), (13), (17) carry magnetic and/or electric charges. In general the electric and magnetic fields associated with these solutions are

\[
E_i = F_{i0} = \frac{r_i}{gr} \frac{d}{dr} \left( \frac{J(r)}{r} \right)
\]

\[
B_i = \frac{1}{2} \varepsilon_{ijk} F_{jk} = -\frac{r_i}{gr^3} \tag{27}
\]
The magnetic field of all the solutions is that of a point monopole of strength $-4\pi/g$. The reason for this will be discussed at the end of this section.

The electric field of the Schwarzschild-like solutions of Eq. (9) is easily found by inserting the ansatz function $J(r)$ from Eq. (9) into Eq. (27). Doing this gives

$$E_i = -\frac{r_i\sinh\gamma(1 \pm 2Cr)}{gr^3(1 \pm Cr)^2}$$

As $r \to \infty$ this electric field goes as $1/r^3$ which indicates that the net electric charge of this solution is zero, although there appears to be some kind of dipole charge distribution.

The electric fields of both of the trigonometric solutions presented in Eqs. (10) (12) are similar to each other in that they indicate that these solutions carry an infinite electric charge. Taking the ansatz function $J(r)$ from the trigonometric solution given in Eq. (10) and inserting it into Eq. (27) yields the following electric field

$$E_i = -\frac{\sinh(\gamma)r_i}{gr^3}(1 - C^2r^2\csc^2(Cr))$$

The electric field does not fall off for large $r$, but behaves like $r_i\csc^2(Cr)/r$. This electric field also becomes singular on the spherical surfaces defined by $Cr = n\pi$ where $n = 1, 2, 3, 4, \ldots$.

The trigonometric solution of Eq. (12) exhibits the same type of electric field except that it becomes singular on the spherical shells given by $Cr = n\pi/2$ (with $n$ odd) and at $r = 0$. The electric charge of this solution is also infinite since the electric field from Eq. (29) does not fall off as $r \to \infty$. For the special case where $\gamma = 0$, one finds that the solution carries no electric charge, but only a magnetic charge. Even in this case the energy density becomes singular on the concentric spherical surfaces and at the origin. Both the BPS solution and the solutions from Eqs. (10) (12) have the same finite magnetic strength of $-4\pi/g$. Although this solution is a dyon in the sense that it carries both magnetic and electric charge it is probably not correct to view it as a particle-like solution, since the electric field does not fall off, thus implying that these solutions have an infinite, spread out electric charge.

The electric field associated with the complex solution in Eq. (13) can be found in the same way as for the other solutions. Inserting the ansatz function $J(r)$ from Eq. (13) into Eq. (27) yields
\[ E_i = \frac{-\sinh(\gamma) r_i}{g r^3} \left( C^2 r^2 \text{sech}^2(Cr) + 1 \right) \]  

(30)

As with all the other solutions, the complex solution carries a magnetic charge of strength \(-4\pi/g\). In addition, by examining the behaviour of the electric field in Eq. (30) as \( r \to \infty \) one finds that this complex solution carries an electric charge of \(-4\pi \sinh(\gamma)/g\), which is the same as that carried by the BPS solution. One interesting feature of the solution from Eq. (13) is that even though the space components of the gauge fields are complex all the physical quantities (e.g. field energy, magnetic charge, electric charge) calculated from it are real. Also, unlike the solutions of Eqs. (10) (12), this complex solution can be viewed as a point-like dyon since it has a localized electric charge. The main difference between this solution and the BPS solution is the infinite field energy of the complex solution due to the field singularity at \( r = 0 \).

While many of the physical characteristics of these various solutions are substantially different in each case, the magnetic charge of all the solutions is the same. This comes about since the magnetic charge of each solution is a topological charge which carries the same value for each field configuration. A topological current, \( k_\mu \), can be defined as

\[ k_\mu = \frac{1}{8\pi} \epsilon_{\mu \nu \alpha \beta} \epsilon_{abc} \partial^\nu \hat{\phi}^a \partial^\alpha \hat{\phi}^b \partial^\beta \hat{\phi}^c \]  

(31)

The topological charge of this field configuration is then

\[ q = \int k_0 d^3x = \frac{1}{8\pi} \int (\epsilon_{ijk} \epsilon_{abc} \partial^i \hat{\phi}^a \partial^j \hat{\phi}^b \partial^k \hat{\phi}^c) d^3x \]

\[ = \frac{1}{8\pi} \int \epsilon_{ijk} \epsilon_{abc} \partial^i \hat{\phi}^a \partial^j \hat{\phi}^b \partial^k \hat{\phi}^c) d^3x \]  

(32)

For all the solutions one finds that \( \hat{\phi}^a = r^a/r \), which is the same regardless of the ansatz function \( H(r) \). In all cases we find that the topological charge is \( q = 1 \). In the next section when we examine the motion of a test particle in the background field of the Schwarzschild-like solution we will find that there is a field angular momentum due to the interaction of the test particle with the field configuration of the Schwarzschild-like solution. This field angular momentum can be seen to arise from the interaction of the topological magnetic
charge with the charge of the test particle, in much the same way that the configuration
of a normal magnetic charge and an electric charge lead to a field angular momentum.

IV. MOTION OF TEST PARTICLES IN SCHWARZSCHILD-LIKE POTENTIAL

We would now like to study the motion of a test particle in the background potential of
the Schwarzschild-like solution of Eq. (9). We will make several assumptions in doing this.
First we will take our test particle to be a scalar particle as in Refs. [18, 20]. One reason
for making this choice is to illustrate the spin from isospin [38] effect that occurs with these
solutions. As discussed in the preceding section all of these solutions carry a magnetic
charge. Many researchers have remarked on the fact [36, 37] that the composite system of
a magnetic charge and electric charge carry an angular momentum due to the configuration
of electric and magnetic fields. Even when the magnetic charge is topological in character,
as is the case with ’t Hooft-Polyakov monopoles, one finds [38] a similar effect whereby the
composite system of a topological magnetic charge and a particle with the charge of the
gauge group, will carry an angular momentum in the gauge fields. This has the interesting
consequence that if one wants to construct fermionic objects from the singular solutions
one should use scalar particle which are in the fundamental representation of the gauge
group – SU(2) for the solutions considered here. (Fermionic test particles in the adjoint
representation would also give a net fermionic bound state [21]). Our second assumption is
that the test particle will be coupled to the scalar field part of the solution of Eq. (9) via the
following substitution $m^2 \rightarrow (m + \lambda \sigma^a \phi^a/2)^2$ where $\lambda$ is an arbitrary coupling constant. Our
final assumption is to set $\theta = 0$ in Eq. (9) in order to not have to take the ansatz function
$G(r)$ into account. In this way the scalar particle $\Phi^A$ moving in the background field of the
Schwarzschild-like solution given in Eq. (9) becomes

\[
\left( \partial_0 - \frac{i q}{2} \sigma^a W^a_0 \right) \left( \partial_0 - \frac{i q}{2} \sigma^a W^a_0 \right)_B^A \Phi^B(x, t) - \left( \partial_i - \frac{i q}{2} \sigma^a W^a_i \right) \left( \partial_i - \frac{i q}{2} \sigma^a W^a_i \right)_B^A \Phi^B(x, t)
\]
\[= - \left( m + \frac{\lambda}{2} \sigma^a \phi^a \right)^2 \Phi^A(x, t) \tag{33} \]

Where \( \sigma^a \) are the standard Pauli matrices, and \( A, B \) are SU(2) group indices which can take on the values 1 or 2. Taking \( \Phi^A(x, t) = \Phi^A(x)e^{-iEt} \), using \((\sigma^a v^a)^2 = v^a v^a\) and expanding we find that Eq. (33) becomes

\[
\left[ -E^2 - g \sigma^a W^a_0 E - \frac{g^2}{4} (W^a_0)^2 - \nabla^2 + ig \sigma^a W^a_i \partial_i + \frac{g^2}{4} (W^a_i)^2 \right]^A_B \Phi^B(x)
\]

\[
= - \left[ m^2 + \lambda m \sigma^a \phi^a + \frac{\lambda^2}{4} (\phi^a)^2 \right]^A_B \Phi^B(x) \tag{34} \]

Inserting the ansatz form of the gauge and scalar fields from Eq. (5) into Eq. (34) then yields

\[
\left[ \nabla^2 - \frac{(1 - K(r))}{r^2} \sigma^a l^a - \frac{(1 - K(r))^2}{2r^2} + \frac{\sigma^a r^a}{r^2} E J(r) + \frac{J(r)^2}{4r^2} \right]^A_B \Phi^B(x)
\]

\[
= \left( E^2 - m^2 - \frac{\lambda m}{gr^2} \sigma^a r^a H(r) - \frac{\lambda^2}{4g^2r^2} H(r)^2 \right)^A_B \Phi^B(x) \tag{35} \]

where we have used \( ig \sigma^a W^a_i \partial_i = -i(1 - K(r))\sigma^a \epsilon_{aij} r^j \partial^i / r^2 = (1 - K(r))\sigma^a l^a / r^2 \) (\( l^a \) is the standard orbital angular momentum operator), and \((W^a_i)^2 = \epsilon_{aij} \epsilon_{aik} r^j r^k (1 - K(r))^2 / g^2 r^4 = 2(1 - K(r))^2 / g^2 r^2\). Since \( \sigma^aq^a \) does not commute with \( \sigma^a r^a \) Eq. (35) is difficult to handle.

By taking advantage of the free parameter \( \gamma \) which occurs in the ansatz functions \( J(r), H(r) \) one can choose \( \gamma \) such that \( E \sinh \gamma = \lambda mcosh \gamma / g \). With this choice the two \( \sigma^r l^a \) terms in Eq. (35) cancel one another. In order to handle the \( \sigma^a l^a \) term it is necessary to define the total angular momentum operator as

\[
J^a = l^a + S^a = l^a + \frac{1}{2} \sigma^a \tag{36} \]

Thus the total angular momentum comes not only from the orbital angular momentum, but has a contribution that looks like a spin angular momentum. The \( \sigma \) matrices in the last term of Eq. (33) are, however, connected with the isospin of the system rather than with spin. This is just the spin from isospin effect [38], and is connected with the fact that the Schwarzschild-like solution of Eq. (5) carries a topological magnetic charge. Thus even
though our system involves only integer spin fields (i.e. $W^a_\mu, \phi^a, \Phi^A$) the combined system is a spin 1/2 object. Using Eq. (36) the $\sigma^a l^a$ term can be expanded in the usual way as $\sigma^a l^a = 2 S^a l^a = J^2_{op} - l^2_{op} - S^2_{op}$ except now $S_{op}$ is the isospin operator rather than the spin operator. Finally, we make the simplifying assumption that $\lambda = g$ so that the $J(r)^2$ and $H(r)^2$ terms may be more easily combined. At this point there seems to be nothing special in this choice, but we will see that according to the arguments of Ref. [18] and [39], the barrier at $r = 1/C$ will only absolutely confine a particle if $\lambda \geq g$ while for $\lambda < g$ there will be some probability for the test particle to tunnel through the barrier. Combining all the preceding assumptions we find

\[
- \left[ \nabla^2 - \frac{1 - K(r)}{r^2} (J^2_{op} - l^2_{op} - S^2_{op}) - \frac{(1 - K(r))^2}{2r^2} + \frac{1}{4r^2} (J(r)^2 - H(r)^2) \right]^A_B \Phi^B(x) = \left( E^2 - m^2 \right)^A_B \Phi^B(x) \tag{37}
\]

In order to separate out the radial equation from Eq. (37) we take

\[
\Phi^A(x) = \frac{1}{r} f_{Jl}(r) Y^A_{JlM}(\theta, \phi) \tag{38}
\]

where the $Y^A_{JlM}$ are the standard spinor spherical harmonics that one gets from adding an orbital angular momenta $l^a$ to a spin 1/2. Here spin is replaced by isospin, but the math, and the spinor spherical harmonics, are exactly the same. Now inserting Eq. (38) into Eq. (37) yields

\[
- \left[ \frac{d^2}{dx^2} - \frac{D}{x^2} \right] f_{Jl}(r) = \left( E^2 - m^2 \right) f_{Jl}(r) \tag{39}
\]

where we have defined the constants $D = l(l + 1)$, $F = J(J + 1) - l(l + 1) - 3/4$. Then setting $x = Cr$ and inserting the ansatz functions $K(r), J(r), H(r)$ from Eq. (9) into Eq. (39) turns the problem into an effective one-dimensional Schrödinger equation

\[
- \frac{d^2}{dx^2} + \frac{D}{x^2} + \frac{F(1 - 2x)}{x^2(1 - x)} + \frac{(1 - 2x)^2}{2x^2(1 - x)^2} + \frac{1}{4x^2(1 - x)^2} \right] f_{Jl}(x) = \frac{\left( E^2 - m^2 \right)}{C^2} f_{Jl}(x) \tag{40}
\]
where all the non-derivative terms on the left hand side are the effective potential. The key feature of this effective potential are the singularities at \( x = 0 \) and \( x = 1 \). Now as \( x \to 1 \) the leading term in the effective potential goes like

\[
V_{\text{eff}}(x) = \frac{D}{x^2} + \frac{F(1 - 2x)}{x^2(1 - x)} + \frac{(1 - 2x)^2}{2x^2(1 - x)^2} + \frac{1}{4x^2(1 - x)^2} \to \frac{3}{4(1 - x)^2} \quad (41)
\]

It was argued in Refs. \[39\] \[18\] that such a singularity would only present a true barrier to the test particle (i.e. the probability of the test particle tunneling through the barrier would be zero) if the coefficient in Eq. (41) were greater than or equal to \( 3/4 \). Thus the effective potential of Eq. (40) just confines the test particle to remain in the range \( 0 \leq x \leq 1 \). The fact that the effective potential is just able to confine the test particle stems from our choice of \( \lambda = g \) for the coupling of the scalar potential \( \phi^a \) to the test particle \( \Phi^4 \). If we had taken \( \lambda < g \) then the coefficient in the limiting form of the effective potential from Eq. (41) would have been less than \( 3/4 \) and the test particle would no longer be confined (e.g. if one took \( \lambda = 0 \) it is straightforward, starting from Eq. (35), to show that one gets a coefficient of \( 1/2 \)). Conversely, when \( \lambda > g \) then the coefficient in Eq. (41) becomes greater than \( 3/4 \) and the test particle becomes confined. This has the interesting implication that the scalar potential plays an important role in this confinement mechanism. Although, generally confinement is thought to be just the result of the gauge interaction, there are phenomenological studies \[40\] \[41\] \[42\] which indicate that an effective scalar potential is involved in the confinement mechanism.

To get more detailed in the solution of Eq. (40) one must pick particular values of \( J \) and \( l \) (which determine the constants \( D \) and \( F \) in Eq. (40)), and solve for the eigenfunctions, \( f_{n,l}(x) \) and eigenvalues \( (E^2 - m^2)/C^2 \). In general this must be done numerically \[21\] \[20\], however, the key features of the effective one-dimensional potential of Eq. (41) (i.e. the singularities in the potential at \( x = 0 \) and \( x = 1 \)) make this potential similar to the Pöschl-Teller potential \[13\].

\[
V(x) = \frac{1}{2} V_0 \left[ \frac{\alpha(\alpha - 1)}{\sin^2(\pi x/2)} + \frac{\beta(\beta - 1)}{\cos^2(\pi x/2)} \right] \quad (42)
\]
where $\alpha, \beta, V_0$ are constants. By choosing $\alpha, \beta$ and $V_0$ correctly the Pöschl-Teller potential can be made similar to the effective potential from Eq. (41). Then the known eigenfunctions and eigenvalues of the Pöschl-Teller potential should give a good approximation to the eigenvalues and eigenfunctions of the potential from Eq. (41). The eigenfunctions for the Pöschl-Teller potential are

$$f_n(x) = K \sin^{\alpha}(\pi x/2) \cos^{\beta}(\pi x/2) \, _2F_1 \left( -n, \alpha + \beta + n, \alpha + \frac{1}{2}; \sin^2(\pi x/2) \right)$$

(43)

where $K$ is a constant fixed by normalization, $n$ is the radial quantum number which takes on values of $n = 0, 1, 2, 3, \ldots$, and $_2F_1(a, b; c; x)$ is the hypergeometric function. The eigenenergies for the Pöschl-Teller potential are

$$E_n = \frac{1}{2} V_0 (\alpha + \beta + 2n)^2$$

(44)

From the shape of both the Pöschl-Teller potential and the effective potential in Eq. (41) this is exactly the kind of dependence one would expect for the energy eigenvalues. For small energies (i.e. $\alpha + \beta > 2n$) both potentials behave like a harmonic oscillator potential and so one would expect that the leading term in $E_n$ should go like $2V_0(\alpha + \beta)n \propto n$. For large energies (i.e. $2n > \alpha + \beta$) both potentials behave like infinite spherical wells, and so one would expect that the leading term in $E_n$ should go like $2V_0 n^2 \propto n^2$. As a simple example we will consider the $l = 0$ case for the potential in Eq. (41). For $l = 0$ we find $J = 1/2$, $D = 0$ $F = 0$ and the potential in Eq. (41) becomes

$$V_{eff}(x) = \frac{3 - 8x + 8x^2}{4x^2(1-x)^2}$$

(45)

This potential approaches $3/(4(1-x)^2)$ as $x \to 1$ so the test particle is just confined to the range $0 < x < 1$. In this range $V_{eff}(x)$ of Eq. (45) reaches its minimum value of 4 at $x = 1/2$, and the potential is symmetric about this point. In order for the Pöschl-Teller potential to also be symmetric about $x = 1/2$, and to also take a value of 4 at this point we can choose $V_0 = 1$ and $\alpha = \beta = 2$. Now inserting these into Eq. (44) and remembering that our eigenvalue from Eq. (40) is $(E^2 - m^2)/C^2$ we find that the approximate energy of the bound states for this case with $l = 0$ is
\[ E_n^2 = m^2 + C^2(2 + n)^2 \]  

Note that this energy depends on the arbitrary constant \( C \), which sets the radius of the confining sphere \( r = 1/C \). As \( C \) increases the radius of the spherical shell decreases and from Eq. (46) the energy of the state increases as would be expected. Although in this \( l = 0 \) case it was particularly easy to determine \( V_0, \alpha, \beta \), the form of the bound state energy given by Eq. (46) will be similar even when \( l \neq 0 \).

## V. DISCUSSION AND CONCLUSIONS

In this article we have presented a variety of solutions to the field equations of Yang-Mills theory. Although finding exact solutions to non-linear field theories is in general difficult, many of the present solutions were found by using the mathematical connection which exists between Yang-Mills theory and general relativity. Since general relativity has been studied for a longer time than Yang-Mills theory there exists a body of known solutions which can serve as guides for finding solutions to the Yang-Mills or Yang-Mills-Higgs field equations. The Schwarzschild solution of general relativity, both without and with a cosmological term, gave rise to the solution with a spherical singularity in Eq. (9) and the linearly increasing solution of Eq. (17). Although both of these solutions suffered from the apparent drawback of having an infinite field energy, they also exhibited some possible connection with the confinement phenomenon. The linear solution of Eq. (17) is of the form of phenomenological potentials [27] that are often used in studies of heavy quark bound states. In addition lattice gauge theory arguments [28] favour a linear type of confining potential. The Schwarzschild-like solution of Eq. (8) has some similarities to bag models for quark bound states. Spherical singularities, similar to those of the Schwarzschild-like solution, were also found to occur in several other solutions as given in Eqs. (10) (12). Actually, the solutions given in Eqs. (10) (12) possessed an infinite set of concentric spheres on which the gauge and scalar fields became infinite. Thus, such spherically singular surfaces may not be uncommon features of Yang-Mills field theories. The SU(2) Schwarzschild-like solution can easily be generalized
to SU(N) by simply embedding the SU(2) solutions into an SU(N) gauge theory. It has also recently been found that true SU(3) solutions, which are not simply embeddings of the SU(2) solutions, can be given.

In the previous section the behaviour of a scalar test particle placed inside the background potential presented by the Schwarzschild-like solution was examined. In order for the Schwarzschild-like potential to confine the test particle, $\Phi^A$, that it was necessary to couple, $\Phi^A$, to the scalar part of the Schwarzschild-like solution, $\phi^a$, via the coupling $m^2 \rightarrow (m + \lambda \sigma^a \phi^a / 2)^2$, where $\lambda$ is the strength of the coupling between $\Phi^A$ and $\phi^a$. Even with this coupling it was found that confinement occurred when $\lambda \geq g$, while for $\lambda < g$ there would be some finite probability for $\Phi^A$ to tunnel out of the spherical well. Although normally it is thought that the confinement phenomenon is the result of only gauge interactions, there has been some work which indicates that an effective scalar interaction may be needed to completely explain confinement. Another interesting aspect of the bound state system studied in the previous section is that the total system was a fermion even though only integer spin fields were involved. The spin 1/2 nature of the bound state system resulted from the fact that the isospin 1/2 of the test particle $\Phi^A$ was converted into spin 1/2 when it was placed inside the Schwarzschild-like solution. Another way of arriving at this result is to note that almost all of the solutions presented here could be shown to carry a topological magnetic charge. Thus, in the same way that a standard magnetic charge - electric charge system carries a field angular momentum of 1/2 in their combined electromagnetic fields, so too the combined charges of the Schwarzschild-like solution and $\Phi^A$ carried a field angular momentum of 1/2 in their combined non-Abelian fields. If a realistic model of hadronic bound states can be constructed from these classical field theory solutions, then the fact that the net angular momentum of these states does not come entirely from the constituent particles, may offer a possible explanation of the EMC effect, which shows that a large part of the net spin of the proton does not come from the valence quarks.

In addition to the Schwarzschild-like solutions presented here it is also possible to take more complex solutions from general relativity to find other Yang-Mills solutions. In Ref.
the general relativistic Kerr solution was used to construct a new Yang-Mills solution. Although the final form of the Yang-Mills Kerr-like solution was not as simple as the Schwarzschild-like solutions, it did share the common feature of having confining surfaces on which the fields became singular. Finally it is also possible to use this method for finding solutions to non-linear field equations in reverse: starting from known solutions to the Yang-Mills equations one can obtain solutions to the general relativistic field equations.

VI. DEDICATION

This article is dedicated to the memory of Professor Fyodor Lunev.
REFERENCES

[1] C.N. Yang and R.L. Mills, Phys. Rev. 96, 191 (1954)

[2] J.D. Jackson, Classical Electrodynamics 2nd Edition, (John Wiley &Sons, 1975) pg. 251

[3] G. 't Hooft, Nucl. Phys. B79, 276 (1974); A.M. Polyakov, JETP 41, 988 (1975)

[4] B. Julia and A. Zee, Phys. Rev. D 11, 2227 (1975)

[5] E.B. Bogomolnyi, Sov. J. Nucl. Phys. 24, 449 (1976)

[6] M.K. Prasad and C.M. Sommerfield, Phys. Rev. Letts. 35, 760 (1975)

[7] A.A. Belavin, A.M. Polyakov, A.S. Schwartz, and Yu. S. Tyupkin, Phys. Lett. B59, 85 (1975)

[8] R. Utiyama, Phys. Rev. 101, 1597 (1956)

[9] M. Carmeli, Classical fields : General relativity and gauge theory, (Wiley, New York, 1982)

[10] F.A. Lunev, Phys. Lett. B295, 99 (1992); Theor. Math. Phys. 94, 48 (1993)

[11] F.A. Lunev, J. Math. Phys. 37, 5351 (1996)

[12] C.N. Yang and T.T. Wu, in Properties of Matter under Unusual Conditions, Ed. H. Mark and S. Fernbach, (Interscience, New York (1968))

[13] E. Witten, Phys. Rev. Lett. 38, 121 (1977)

[14] G. Rosen, J. Math. Phys. 13, 595 (1972)

[15] J.H. Swank, L.J. Swank and T. Dereli, Phys. Rev. D12, 1096 (1975)

[16] A.P. Protogenov, Phys. Lett. B67, 62 (1977)

[17] S.M. Mahajan and P.M. Valanju, Phys. Rev. D35, 2543; S.M. Mahajan and P.M. Valanju, Phys. Rev. D36, 1500 (1987)
[18] F.A. Lunev, Phys. Lett. B311 273 (1993)

[19] D. Singleton, Phys. Rev. D 51, 5911 (1995); D. Singleton, Nuovo Cimento A 109, 169 (1996)

[20] D. Singleton and A. Yoshida [hep-th 9505160]

[21] F.A. Lunev and O. Pavlovsky [hep-ph 9609452]

[22] V. De Alfaro, S. Fubini, and G. Furlan, Phys. Lett. B65, 163 (1976)

[23] J.P. Hsu and E. Mac, J. Math. Phys. 18, 100 (1977)

[24] D. Singleton, Int. J. Theo. Phys. 36, 1857 (1997)

[25] H. Ohanian, Gravitation and Space-time (W.W. Norton & Company, 1976), pg. 286

[26] D. Singleton and A. Yoshida, Int. J. Mod. Phys. A, 12, 4823 (1997)

[27] E. Eichten et. al., Phys. Rev. D17, 3090 (1978)

[28] K. Wilson, Phys. Rev. D10, 2445 (1974)

[29] D. Sivers and J. Ralston, Phys. Rev. D28, 953 (1983)

[30] D. Singleton, Z. Phys. C, 72, 525 (1996)

[31] V. Dzhunushaliev, [hep-th 9611096]; V. Dzhunushaliev, [hep-th 9707039]

[32] A.C.T. Wu and T.T. Wu, J. Math. Phys. 15, 53 (1974)

[33] W.J. Marciano and H. Pagels, Phys. Rev. D12, 1093 (1975)

[34] D.V. Gal'tsov and M.S. Volkov, Phys. Lett. B274, 173 (1992)

[35] J. Arafune, P.G.O. Freund, and C.J. Goebel, J. Math. Phys. 16, 433 (1975)

[36] J.J. Thomson, Elements of the Mathematical Theory of Electricity and Magnetism, 3rd Ed., Cambridge University Press, (1904) Section 284
[37] M.N. Saha, Ind. J. Phys. 10, 145 (1936); M.N. Saha, Phys. Rev. 75, 1968 (1949); H.A. Wilson, Phys. Rev. 75, 309 (1949)

[38] R. Jackiw and C. Rebbi, Phys. Rev. Letts. 36, 1116 (1976); P. Hasenfratz and G. ’t Hooft, Phys. Rev. Letts. 36 1119 (1976)

[39] J. Dittrich and P. Exner, J. Math. Phys. 26, 2000 (1985); V.B. Gostev and A.R. Frenkin, Theor. Math. Phys. 74, 247 (1988)

[40] C. Goebel, D. LaCourse, and M.G. Olsson, Phys. Rev. D41, 2917 (1990)

[41] Y. Shibata and H. Tezuka, Z. Phys. C, 62, 533 (1994)

[42] B. Ram, Am. J. Phys. 50, 549 (1982)

[43] S. Flügge, Practical Quantum Mechanics I, Springer-Verlag (1971), pg. 89

[44] J. Ashman et. al., Phys. Lett. B206, 364 (1988); Nucl. Phys. B328, 1 (1989)

[45] D. Singleton, J. Math. Phys. 37, 4574 (1996)

[46] D. Singleton, Phys. Lett. A223, 12 (1996)