One-instanton predictions of Seiberg-Witten curves for product groups

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Abstract

One-instanton predictions for the prepotential are obtained from the Seiberg-Witten curve for the Coulomb branch of $\mathcal{N} = 2$ supersymmetric gauge theory for the product group $\prod_{n=1}^{m} SU(N_n)$ with a massless matter hypermultiplet in the bifundamental representation $(N_n, \bar{N}_{n+1})$ of $SU(N_n) \times SU(N_{n+1})$ for $n = 1$ to $m - 1$, together with $N_0$ and $N_{m+1}$ matter hypermultiplets in the fundamental representations of $SU(N_1)$ and $SU(N_m)$ respectively. The derivation uses a generalization of the systematic perturbation expansion about a hyperelliptic curve developed by us in earlier work.

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Spectacular advances have been made in our understanding of the non-perturbative behavior of supersymmetric gauge theories and string theories. In particular, the program of Seiberg and Witten [1] allows one to compute the exact behavior of low-energy four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories in various regions of moduli space from the following input: a Riemann surface or algebraic curve appropriate to the specific theory, and the Seiberg-Witten meromorphic one-form. When the curve in question is non-hyperelliptic, however, the explicit extraction of this information is a challenging technical problem. In this letter, we will use Seiberg-Witten theory to calculate the one-instanton predictions of an $\mathcal{N} = 2$ supersymmetric gauge theory based on the product group $\prod_{n=1}^{m} SU(N_n)$ by extending the methods of ref. [2]-[7].

The exact low-energy properties of $\mathcal{N} = 2$ theories are encapsulated in the form of the prepotential $\mathcal{F}(A)$, in terms of which the Wilson effective Lagrangian is

$$\mathcal{L} = \frac{1}{4\pi} \text{Im} \left[ \int d^4\theta \frac{\partial \mathcal{F}(A)}{\partial A^i} \bar{A}^i + \frac{1}{2} \int d^2\theta \frac{\partial^2 \mathcal{F}(A)}{\partial A^i \partial A^j} W^i W^j \right],$$

(1)

to lowest order in the momentum expansion, where $A^i$ are $\mathcal{N} = 1$ chiral superfields and $W^i$ are $\mathcal{N} = 1$ vector superfields. Holomorphy implies that the prepotential in the Coulomb phase has the form of an instanton expansion

$$\mathcal{F}(A) = \mathcal{F}_{\text{cl}}(A) + \mathcal{F}_{1-\text{loop}}(A) + \sum_{d=1}^{\infty} \mathcal{F}_{d-\text{inst}}(A).$$

(2)

Consider an $\mathcal{N} = 2$ supersymmetric gauge theory based on the gauge group $\prod_{n=1}^{m} SU(N_n)$. In addition to the chiral gauge multiplet in the adjoint representation of each of the factor groups, the theory we are considering contains a massless matter hypermultiplet in the bifundamental representation $(N_n, \bar{N}_{n+1})$ of $SU(N_n) \times SU(N_{n+1})$ for $n = 1, \ldots, m-1$; $N_0$ matter hypermultiplets in the fundamental representation of $SU(N_1)$ (whose masses we denote $e_k^{(0)}, 1 \leq k \leq N_0$); and $N_{m+1}$ matter hypermultiplets in the fundamental representation of $SU(N_m)$ (whose masses we denote $e_k^{(m+1)}, 1 \leq k \leq N_{m+1}$). The adjoint multiplets contain complex scalar fields $\phi^{(n)}$ for each of the factor groups. The Lagrangian has a potential with flat directions, along which the
symmetry is generically broken to $\prod_{n=1}^{m} U(1)^{N_n-1}$. The moduli space of the theory is therefore parametrized by the order parameters $e_k^{(n)}$ $(1 \leq k \leq N_n, 1 \leq n \leq m)$, which are the eigenvalues of the $\phi^{(n)}$ and satisfy the constraint $\sum_{k=1}^{N_n} e_k^{(n)} = 0$.

The curve for this product group theory was obtained using M-theory [8] and geometric engineering [9], and made more explicit in ref. [10]:

$$P_{N_0}(x) t_0^{n+1} - P_{N_1}(x) t_0^n + \sum_{j=0}^{m-1} (-)^{m-j+1} \left[ \prod_{\ell=1}^{m-j} \frac{L_{\ell}}{L_{\ell-j+1}} \right] P_{N_{m-j+1}}(x) t_j = 0, \quad (3)$$

where

$$P_{N_n}(x) = \prod_{i=1}^{N_n} (x - e_i^{(n)}), \quad (n = 0 \text{ to } m),$$

$$L_n^2 = \Lambda_n^{2N_n-N_{n-1}-N_{n+1}}, \quad (4)$$

with $\Lambda_n$ the quantum scale of the gauge group SU($N_n$). The requirement of asymptotic freedom, and restriction to the Coulomb phase, implies that $\Lambda_n$ appear with positive powers in (3).

The curve (3) describes a $(m+1)$-fold branched covering of the Riemann sphere, with sheets $n$ and $n+1$ connected by $N_n$ square-root branch-cuts centered about $x = e_k^{(n)}$ ($k = 1$ to $N_n$), and having endpoints $x_k^{(n)-}$ and $x_k^{(n)+}$. Following the approach of Seiberg-Witten, we will use this curve to compute the renormalized order parameters and their duals

$$2\pi i a_k^{(n)} = \oint_{A_k^{(n)}} \lambda \quad \text{and} \quad 2\pi i a_{D,k}^{(n)} = \oint_{B_k^{(n)}} \lambda, \quad (5)$$

where $\lambda$ is the Seiberg-Witten differential, and $A_k^{(n)}$ and $B_k^{(n)}$ ($2 \leq k \leq N_n$) are a set of canonical homology cycles for the Riemann surface. The cycle $A_k^{(n)}$ is chosen to be a simple contour on sheet $n$ enclosing the branch cut centered about $e_k^{(n)}$. The cycle $B_k^{(n)}$ goes from $x_1^{(n)-}$ to $x_k^{(n)-}$ on the $n$th sheet and from $x_k^{(n)-}$ to $x_1^{(n)-}$ on the $(n+1)$th [2]. Once we obtain $a_k^{(n)}$ and $a_{D,k}^{(n)}$, the prepotential can be computed by integrating

$$a_{D,k}^{(n)} = \frac{\partial F}{\partial a_k^{(n)}}, \quad (k = 2 \text{ to } N_n, \quad n = 1 \text{ to } m). \quad (6)$$
In our computation, we will perform a multiple perturbation expansion in the several quantum scales $L_n$, with the result for the prepotential given to one-instanton accuracy, i.e., $O(L_n^2)$ for all $n$. To calculate $a_k^{(n)}$ and $a_{D,k}^{(n)}$ for the group SU($N_n$), we define $t = \tilde{t} \prod_{\ell=1}^{n-1} L_\ell^2$ to recast the curve (3) as

$$\sum_{j=0}^{n} (-)^j \left[ \prod_{\ell=1}^{n-1} L_\ell^{2(j-\ell)} \right] P_{N_j}(x) \tilde{t}^{m+1-j} + \sum_{j=n+1}^{m+1} (-)^j \left[ \prod_{\ell=n}^{j-1} L_\ell^{2(j-\ell)} \right] P_{N_j}(x) \tilde{t}^{m+1-j} = 0,$$

or more explicitly

$$\cdots + (-)^{n-3} L_{n-2}^4 L_{n-1}^2 P_{N_{n-3}}(x) \tilde{t}^{m-n+4} + (-)^{n-2} L_{n-1}^2 P_{N_{n-2}}(x) \tilde{t}^{m-n+3} + (-)^{n-1} P_{N_{n-1}}(x) \tilde{t}^{m-n+2} + (-)^nP_{N_{n}}(x) \tilde{t}^{m-n+1} + (-)^{n+1} L_n^2 P_{N_{n+1}}(x) \tilde{t}^{m-n} + (-)^{n+2} L_n^2 L_{n+1} P_{N_{n+2}}(x) \tilde{t}^{m-n-1} + (-)^{n+3} L_n^4 L_{n+1}^2 P_{N_{n+3}}(x) \tilde{t}^{m-n-2} + \cdots = 0. \quad (8)$$

To obtain the one-loop, zero-instanton contribution, i.e., $O(\log L_n)$, to $a_k^{(n)}$ and $a_{D,k}^{(n)}$, one may set $L_\ell = 0$ for $\ell \neq n$, in which case the curve (8) reduces, after the change of variable $\tilde{t} = y/P_{N_{n-1}}(x)$, to the hyperelliptic curve

$$y^2 + 2A(x)y + B(x) = 0,$$

with

$$A(x) = -\frac{1}{2} P_{N_{n}}(x),$$

$$B(x) = L_n^2 P_{N_{n+1}}(x) P_{N_{n-1}}(x). \quad (10)$$

On one of the sheets, eq. (9) has the solution

$$y = -A - r \quad \text{where} \quad r = \sqrt{A^2 - B}, \quad (11)$$

from which we may compute the Seiberg-Witten differential $\lambda = x dy/y$ in the hyperelliptic approximation to be

$$\lambda_I = \frac{x \left( \frac{A'}{A} - \frac{B'}{2B} \right)}{\sqrt{1 - \frac{B}{A^2}}} dx. \quad (12)$$
(On the other sheet, the solution to eq. (9) is \( y = -A + r \)).

To obtain the one-instanton correction (i.e., \( O(L_n^2) \) for all \( \ell \)) to the order parameters, one again makes the change of variables \( \tilde{t} = y/P_{n-1}^2(x) \) in eq. (8), and keeps two more terms beyond those in eq. (9), obtaining the quartic curve

\[
\epsilon_1(x)y^4 + y^3 + 2A(x)y^2 + B(x)y + \epsilon_2(x) = 0,
\]

where

\[
\epsilon_1(x) = -L_{n-1}^2 P_{n-2}(x) P_{n-1}^2(x),
\]

\[
\epsilon_2(x) = -L_n^4 L_{n+1}^2 P_{n+2}(x) P_{n-1}^2(x).
\]

Rewriting eq. (13) as

\[
y + A + r = \frac{1}{y + A - r} \left[ -\epsilon_1 y^3 - \frac{\epsilon_2}{y} \right]
\]

and substituting \( y = -A - r \) into the right hand side, which is already first order in \( \epsilon \), we obtain

\[
y = -A - r - \frac{(A + r)^3}{2r} \epsilon_1 - \frac{1}{2r(A + r)} \epsilon_2 + \cdots,
\]

to first order in \( \epsilon_1 \) and \( \epsilon_2 \). The Seiberg-Witten differential is correspondingly modified to

\[
\lambda = \lambda_I + \lambda_{II} + \cdots,
\]

where \( \lambda_I \) is the hyperelliptic approximation (12) and

\[
\lambda_{II} = \left(-A \epsilon_1 - \frac{A}{B^2} \epsilon_2\right) dx = -L_{n-1}^2 P_{n-2}(x) P_{n-1}^2 dx - L_{n+1}^2 P_{n+2}(x) P_{n-1}^2 dx,
\]

obtained from a calculation similar to that in Appendix C of ref. [5].

One computes the order parameters (5) using the methods of refs. [2, 5, 6], obtaining

\[
a_k^{(n)} = e_k^{(n)} + \frac{1}{4} L_n^2 \frac{\partial S^{(n)}}{\partial e_k^{(n)}}(e_k^{(n)}) + \cdots, \quad (k = 1 \text{ to } N_n, \quad n = 1 \text{ to } m),
\]
where the residue functions \( S_k^{(n)}(x) \) are defined in terms of eq. (10) by

\[
\frac{S_k^{(n)}(x)}{(x - \epsilon_k^{(n)})^2} = \frac{B(x)}{A(x)^2}, \tag{20}
\]

Considerations analogous to those of Appendix D of ref. [5] give the identities

\[
\sum_{j=1}^{N_n} \frac{\partial S_j^{(n)}}{\partial x} \left( e_j^{(n)} \right) = 0, \tag{21}
\]

implying \( \sum_{i=1}^{N_n} a_i^{(n)} = \sum_{i=1}^{N_n} e_i^{(n)} \) to the order that we are working.

Next, the dual order parameters \( a_{D,k}^{(n)} \) are computed along the lines of sec. 5 of ref. [5], giving

\[
2\pi i a_{D,k}^{(n)} = \left[ 2N_n - N_{n+1} - N_{n-1} + 2 \log L_n + \text{const} \right] a_k^{(n)} - 2 \sum_{i \neq k}^{N_n} (a_k^{(n)} - a_i^{(n)}) \log (a_k^{(n)} - a_i^{(n)})
+ \sum_{i=1}^{N_n+1} (a_k^{(n)} - a_i^{(n+1)}) \log (a_k^{(n)} - a_i^{(n+1)})
+ \frac{1}{4} L_n^2 \frac{\partial S_k^{(n)}}{\partial x} (a_k^{(n)})
- \frac{1}{2} L_n^2 \sum_{i \neq k}^{N_n} \frac{S_i^{(n)}(a_k^{(n)})}{a_k^{(n)} - a_i^{(n)}}
+ \frac{1}{4} L_n^{n+1} \sum_{i=1}^{N_n} S_i^{(n+1)}(a_i^{(n+1)})
+ \sum_{i=1}^{N_n-1} \frac{S_i^{(n-1)}(a_i^{(n-1)})}{a_k^{(n)} - a_i^{(n-1)}}
+ \cdots, \tag{22}
\]

(\( k = 2 \) to \( N_n \), \( n = 1 \) to \( m \)). In eq. (22), we define \( a_k^{(0)} = \epsilon_k^{(0)} \) and \( a_k^{(m+1)} = \epsilon_k^{(m+1)} \) (the masses of the hypermultiplets in the fundamentals of \( SU(N_1) \) and \( SU(N_m) \) respectively), and \( L_0 = L_{m+1} = 0 \).

One then integrates eq. (6) using eq. (22) to obtain the prepotential (2) to one-instanton accuracy, finding

\[
\mathcal{F}_{1\text{-loop}} = \frac{i}{8\pi} \sum_{n=1}^{m} \sum_{i,j=1}^{N_n} (a_i^{(n)} - a_j^{(n)})^2 \log (a_i^{(n)} - a_j^{(n)})^2
- \frac{i}{8\pi} \sum_{n=0}^{m} \sum_{i=1}^{N_n} \sum_{j=1}^{N_n+1} (a_i^{(n)} - a_j^{(n+1)})^2 \log (a_i^{(n)} - a_j^{(n+1)})^2, \tag{23}
\]

and

\[
\mathcal{F}_{1\text{-inst}} = \frac{1}{8\pi i} \sum_{n=1}^{m} L_n^2 \sum_{k=1}^{N_n} S_k^{(n)}(a_k^{(n)}), \tag{24}
\]
where

\[ S_k^{(n)}(x) = \frac{4 \prod_{i=1}^{N_n+1} (x - a_i^{(n+1)}) \prod_{i=1}^{N_n-1} (x - a_i^{(n-1)})}{\prod_{i \neq k}^{N_n} (x - a_i^{(n)})^2}, \quad (k = 1 \text{ to } N_n, \quad n = 1 \text{ to } m). \tag{25} \]

Note that as \( S_k^{(n)}(a_k^{(n)}) \) depends on \( a_i^{(n+1)} \) and \( a_i^{(n-1)} \) as well as \( a_i^{(n)} \), eq. (24) is not just the naive sum of instanton contributions from each subgroup.

The one-loop prepotential (23) agrees with the perturbation theory result for a chiral gauge multiplet in the adjoint representation of each of the factor groups, a massless matter hypermultiplet in the bifundamental representation \((N_n, \bar{N}_{n+1})\) of \( SU(N_n) \times SU(N_{n+1}) \) for \( n = 1 \) to \( m - 1 \), \( N_0 \) matter hypermultiplets with masses \( a_k^{(0)} \) in the fundamental representation of \( SU(N_1) \), and \( N_{m+1} \) matter hypermultiplets with masses \( a_k^{(m+1)} \) in the fundamental representation of \( SU(N_m) \).

One check of the one-instanton correction (24) is provided by ref. [4], where various decoupling limits for \( \mathcal{N} = 2 \) \( SU(N) \) gauge theory with a massive hypermultiplet in the adjoint representation are considered. D’Hoker and Phong [4] obtain \( F_{1-\text{inst}} \) for the product group theory, but with restriction to a single quantum scale. We find agreement with their result when we restrict eqs. (24) and (25) to the special case of a single quantum scale, which therefore provides a test of the curve (3) obtained from M-theory.

In this paper, we showed that to compute the order parameters of the \( \mathcal{N} = 2 \) gauge theory for the product group \( \prod_{n=1}^{m} SU(N_n) \) to one-instanton accuracy, one need only consider the sequence of quartic curves (13), even though the complete curve for the theory (3) is of higher order (viz., \( m+1 \)) for \( m > 3 \), \( i.e., \) for products of three or more groups. (The case \( m = 2 \) was analyzed in ref. [7].) In the language of type IIA string theory, this means one need only consider all possible chains of four parallel neighboring NS 5-branes, among the total set of \( m+1 \) parallel NS 5-branes, to achieve one-instanton accuracy. For higher instanton accuracy, additional parallel 5-branes are required. An analogue of this result plays a crucial role in our analysis of the prepotential and Seiberg-Witten curve for \( SU(N) \) gauge theory with two antisymmetric and \( N_f \) fundamental hypermultiplets [11].
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