On the symmetry of $T\bar{T}$ deformed CFT

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Abstract

We propose a symmetry of $T\bar{T}$ deformed 2D CFT, which preserves the trace relation. Once we consider the background metric runs with the deformation parameter $\mu$, the deformation contributes an additional term in conformal killing equation, which plays the role of renormalization group flow of metric. The conformal symmetry coincides with the fixed point. On the gravity side, this symmetry can be described by a new asymptotic symmetry boundary condition of AdS$_3$. In addition, we derive that the stress tensor of the deformed CFT equals to Brown-York's quasilocal stress tensor on a finite boundary with a counterterm. For a specific example, BTZ black hole, we obtain $T\bar{T}$ deformed conformal killing vectors. Based on this symmetry, conserved charges are also studied.

Keywords: Symmetries, AdS-CFT Correspondence, Boundary Quantum Field Theory
1 Introduction

In AdS/CFT correspondence, it is proposed that the AdS gravity is dual to a CFT living on the boundary at spatial infinity. It provides a powerful approach to explore quantum gravity and research strongly coupled system from gravity side. But whether the holographic correspondence can be extended to effective QFTs for which the UV behavior is not described by a CFT. Recently, Smirnov and Zamolodchikov discovered a class of exactly solvable irrelevant deformations of 2D QFTs called $T\bar{T}$ deformation [1], which may provide a route to explore this question. The actions are defined by $T\bar{T}$ flow, and some properties have been studied in [2]. The deformations generated by $T\bar{T}$ are solvable in a certain sense, even if the original QFTs are not integrable. It defines a continuous family of theories along a $T\bar{T}$ flow trajectory in the integrable quantum field theories space. Furthermore, the eigenvalue of energy and momentum of deformed theory are simply related to the expectation values of stress tensor, and the spectrum of the theories can be exactly obtained by solving a differential equation [3]. Flow equation induced by $T\bar{T}$ deformation allows us to get the closed form formula for deformed Lagrangian, which relate to Nambu-Goto Lagrangian [2, 4].
The deformation in general dimensions are also considered [5, 6]. Irrelevant deformations, by definition, do not affect the IR behavior but will generally affect UV physics. Conformal field theory, as a UV complete framework, it attracts a big interesting to explore the deformed CFT, and $\mathcal{T}\bar{T}$ deformed CFT might provide a route to extend the holographic duality not only in the frame of CFT. Some properties of $\mathcal{T}\bar{T}$ deformed CFT are also studied. Based on the spectrum of deformed CFT, the partition function on the torus was obtained and modular properties have been discussed [7, 8].

It is shown that the conserved charges remain conserved along the $\mathcal{T}\bar{T}$ flow [1]. In a sense, the integrable deformation implies that it preserves an infinite amount of conserved charges, especially for the CFT, which have a conformal symmetry. As is known, the $\mathcal{T}\bar{T}$ deformation breaks the conformal symmetry because the operator $\mathcal{T}\bar{T}$ has conformal dimension 4. However, it is reasonable to believe there still some other symmetries go beyond conformal symmetry for an integrable and solvable QFTs. From the renormalization group viewpoint, the $\mathcal{T}\bar{T}$ flow defines a family of theories, and conformal field theory constitutes the fixed point of RG flow. So the symmetry also changes following the $\mathcal{T}\bar{T}$ flow. In case of CFT, the conformal symmetry leads to the traceless of stress tensor. The deformation turns the traceless to a trace relation, which depends on the deformation parameter $\mu$. Therefore, the trace relation may imply a symmetry of deformed CFT. We will give a detail illustration in section 3, especially for the background also depends on $\mu$. So the $\mathcal{T}\bar{T}$ flow has an effect of changing the background of the boundary field theory, which is associated with different radial cutoff. This feature have some omens to associate $\mathcal{T}\bar{T}$ deformed CFT to holographic renormalization group.

More interesting about the $\mathcal{T}\bar{T}$ deformed CFT is the holographic aspects. McGough, Mezei and Verlinde proposed a novel idea, when study the $\mathcal{T}\bar{T}$ deformed CFT [9]. It related the $\mathcal{T}\bar{T}$ deformed CFT to AdS gravity, explicitely, the deformation corresponds to putting Dirichlet boundary condition in the bulk at finite radius. The deformation parameter is identified with inverse of cutoff radius square $\mu = \frac{24\pi}{c} \frac{1}{r^2}$. This means the deformation of CFT is equivalent to a radial cutoff in bulk, actually a UV/IR relation. Some holographic dictionaries have been established to support this relation, like the energy spectrum of deformed CFT is equal to the Brown-York’s quasilocal energy and the RG equation is identical to Wheeler-DeWitt equation in 3D AdS gravity. It is a significative and challenging work to explore more holographic aspects of $\mathcal{T}\bar{T}$ deformed CFT. Entanglement entropy [10–15] and complexity [16] have been studied. Path integral optimization about $\mathcal{T}\bar{T}$ deformation and its geometric duality were explored [17]. In AdS$_3$/CFT$_2$, the symmetry plays an essential role, and asymptotic symmetry of AdS$_3$ gives the Virasoro algebra [18]. However, we know little about the symmetry of $\mathcal{T}\bar{T}$ on both boundary field theory and gravity side. We mainly focus on the symmetry of the $\mathcal{T}\bar{T}$ deformed CFT and give a holographic interpretation in
The cutoff AdS$_3$ with Dirichlet boundary condition means boundary metric depends on cutoff radius. According to this proposal, the boundary metric is not a fixed one. It inspires us to consider the $T\bar{T}$ deformed CFT living on a dynamical background. But this feature is unintelligible from field theory side. In fact, Cardy has discovered the $T\bar{T}$ deformation is equivalent to the random metric [19], which can give a description of $T\bar{T}$ deformed CFT with a dynamical background. The flow equation of metric can also be obtained. In this paper, we consider the symmetry, which preserve the trace relation of $T\bar{T}$ deformed CFT. The deformed conformal killing equation is obtained. Based on the flow equation of metric, the deformation can also been expressed in terms of change rate of metric, and conformal killing equation coincides with the fixed point. On the gravity side, we restudy the Dirichlet boundary condition and find it gives a new asymptotic symmetry of AdS$_3$, which corresponds to the $T\bar{T}$ symmetry. Moreover, we derive that the stress tensor equals to Brown-York’s quasilocal stress tensor. As a specific realization, we consider a simple nontrivial case, the BTZ black hole. We get the $T\bar{T}$ deformed conformal killing vectors and the conserved charges are also calculated. However, we can only get two independent conserved charges.

This paper is organized as follows: We restudy the $T\bar{T}$ deformed CFT on dynamical background, which runs with deformation parameter in section 2. In section 3, we propose a symmetry of $T\bar{T}$ deformed CFT based on the trace relation. The symmetry can be described by a deformed conformal killing equation. We give a holographic interpretation of the symmetry on the gravity side in section 4, which coincide with an asymptotic symmetry of AdS$_3$. The holographic stress tensor is also considered. In section 5, a specific example, BTZ black hole, is studied. Based on this symmetry, we get the $T\bar{T}$ deformed conformal killing vector and conserved charges are also considered. Conclusion, discussion about symmetry algebra and Chern-Simons formalism are given in section 6.

## 2 $T\bar{T}$ deformed CFT with dynamical background

In this section, we review some main results about $T\bar{T}$ deformed CFT and its holographic aspects, which would be useful for discussing the symmetry in this paper. Especially, the background metric runs with the deformation parameter $\mu$, instead of a static one. We start from the definition of $T\bar{T}$ deformed theory living on an arbitrary background.

The action of a $T\bar{T}$ deformed QFT is defined by the $T\bar{T}$ flow

$$\frac{\partial S}{\partial \mu} = \frac{1}{2} \int d^2x (\sqrt{g} T\bar{T})_{\mu},$$

(2.1)

where the operator $T\bar{T} \equiv (T^{ij}T_{ij} - T^2) = -\frac{1}{2} \varepsilon_{ik} \varepsilon_{jl} T^{ij} T^{kl}$. The label $\mu$ denotes the quantity
depends on the deformation parameter $\mu$, especially the background metric $(g_{ij})_{\mu}$. Usually, one may consider the theory living on a Minkowski spacetime or a static curved background, but the metric is not fixed in our configuration. Dimensional analysis yields the parameter $\mu$ has dimension of $[M]^{-2}$. In general, for a theory with a single mass scale $\lambda$

$$\lambda \frac{\partial S}{\partial \lambda} = \frac{1}{2} \int d^2x \sqrt{g} T^i_i.$$  

(2.2)

By setting $\mu = 1/\lambda^2$, one can get an important trace relation of $T\bar{T}$ deformed CFT

$$T^i_i = -2\mu T\bar{T},$$

(2.3)

which would imply the symmetry of $T\bar{T}$ deformed CFT we will discuss in section 3. For $\mu = 0$, the trace relation reduces to traceless of stress tensor, and this is consistent with CFT case. So the trace relation is just the generalization of traceless along the $T\bar{T}$ flow.

For an infinitesimal variation of $\mu$, the change in action is

$$S(\mu + \delta \mu) - S(\mu) = \frac{\delta \mu}{2} \int \sqrt{g} \varepsilon_{ik} \varepsilon_{jl} T^{ij} T^{kl}. $$

(2.4)

This will lead to a variation of path integral by inserting a factor $e^{-\delta S}$. It can be expressed as a functional integral through Hubbard-Stratonovich transformation

$$e^{-\frac{\delta \mu}{2} \int \sqrt{g} \varepsilon_{ik} \varepsilon_{jl} T^{ij} T^{kl}} \propto \int [dh] e^{\frac{1}{2} \int \sqrt{g} \varepsilon_{ij} h_{ij} h_{kl} d^2x - \int \sqrt{g} h_{ij} T^{ij} d^2x}. $$

(2.5)

By the definition of $T_{ij}$, the second term is equivalent to an infinitesimal change in the metric $h_{ij} = \delta g_{ij}$. The value of functional integral can be obtained at saddle point approximate

$$T_{ij} = \frac{1}{2\delta \mu} \varepsilon_{ik} \varepsilon_{jl} h^{ij} = \frac{1}{2\delta \mu} (g_{ij} h - h_{ij})$$

$$\Rightarrow h_{ij} = \delta g_{ij} = 2\delta \mu (g_{ij} T - T_{ij}).$$

(2.6)

At last, we get the flow equation about background metric

$$\frac{\partial}{\partial \mu} g_{ij} = -2(T_{ij} - T g_{ij}).$$

(2.7)

This means that the background metric also runs with the parameter $\mu$. If the initial theory, i.e. $\mu = 0$, is a 2D CFT, this result holds for a small $\mu$ or large $c$ limit, which means $c$ is large enough but fix the $\mu c$. This conclusion was obtained by Cardy via the viewpoint of random geometry [19], and the correlation function have been calculated [14, 20]. The flow equation of metric can also be obtained by the variational principle. In brief, the variation of action can be written as.

$$\delta S = -\frac{1}{2} \int d^2x \sqrt{g} T^{ij} \delta g_{ij}. $$

(2.8)
Combining (2.1), (2.8) and $\delta \partial_\mu S = \partial_\mu \delta S$, one can also get the flow equation. This variational principle approach was proposed by Guica and Monten in [21]. In addition, through resolving the flow equations, the stress tensor of $T\bar{T}$ deformed CFT can be obtained, which is coincide with the Brown-York’s quasilocal stress tensor [22].

The deformed Lagrangian in closed form have been obtained in [4] by reformulating the $T\bar{T}$ deformation as a functional equation. We can not get a closed form of the deformed Lagrangian with a unfixed metric, because the $T\bar{T}$ flow can just determine the action. If the metric also varies with $\mu$, which means there is a varied integral measure. So we can not get a deformed Lagrangian formula at present.

Something more interesting is the holographic aspects of $T\bar{T}$ deformed CFT. There is a proposal that the $T\bar{T}$ deformed CFT is dual to a cutoff AdS at $r = r_c$ with Dirichlet boundary condition [9, 23]. The bulk cutoff $r = r_c$ relate to the deformation parameter by

$$\mu = \frac{24\pi}{c} \frac{1}{r_c^2}, \quad (2.9)$$

This means different cutoff in bulk is equivalent to different deformation parameter on the boundary field. But in this paper, we would not fix the parameter $\mu$, so we may write $\mu = \frac{24\pi}{c} \frac{1}{r^2}$. All the conclusions we get still hold when setting $r = r_c$. An important aspect is that the induced metric on the finity radial boundary also depends on $r$ (or $\mu$). This feature matches the random geometry we reviewed above. Therefore, in order to explore the holographic aspects, it is essential to consider the dynamical background.

We list some results here, which would be instructive for us when discuss the conserved charge in section 5. For detail please see [9, 23]. The spectrum of $T\bar{T}$ deformed CFT on a cylinder with circumference $L$ can be obtained by solving a differential equation

$$E_n \equiv L\langle n|T_{tt}|n\rangle = \frac{2L}{\mu} \left( 1 - \sqrt{1 - \frac{2\pi \mu}{L^2} M_n + \frac{\mu^2 J_n^2}{L^4}} \right), \quad (2.10)$$

where $M_n = \Delta_n + \bar{\Delta}_n - c/12$ and $J_n = \Delta_n - \bar{\Delta}_n$. On the gravity side, the quasilocal proper energy of BTZ black hole defined by Brown and York [22, 24]

$$E \equiv \int \sqrt{g} \phi \tau_{ij} u^i u^j d\phi = \frac{r}{4G} \left( 1 - \sqrt{1 - \frac{8GM}{r^2} + \frac{16G^2J^2}{r^4}} \right), \quad (2.11)$$

where $u^i$ is the unit normal to a time slice. The proper size of the spatial circle on the boundary is $2\pi r$, and the circumference of cylinder usually set to $L = 2\pi$. Moreover, the quasilocal energy tends to $M$, when $r \to \infty$. With the relation (2.9), one can identify

$$LE_n = 2\pi r E. \quad (2.12)$$
What makes more sense is that we can write it as

\[ E_n = r \int \sqrt{g} \phi \tau_{ij} u^i u^j d\phi. \]  

One can also verify that the angular momentum \( J_n \) can also be obtained from gravity side by the definition of quasilocal conserved charge

\[ J_n = J \equiv \int \sqrt{g} \phi \tau_{ij} u^i \xi^j d\phi, \]  

where the \( \xi^i \) is the killing vector associate with \( J \).

3 Symmetry of \( T\bar{T} \) deformed CFT

In classical field theory, the invariance of the action under an infinitesimal transformation implies a symmetry. For a conformal invariant theory, things are more intriguing. The conformal transformation leaves the metric invariant up to a scaling factor

\[ \delta g_{ij} = \omega(x) g_{ij}. \]  

Then, the variation of action can be written as

\[ \delta S_{\text{CFT}} = -\frac{1}{2} \int d^2 x \sqrt{g} T^{ij} \delta g_{ij} = -\frac{1}{2} \int d^2 x \sqrt{g} T^i_i \omega(x). \]  

Thus the conformal invariance leads to the traceless of stress tensor \( T^i_i = 0 \). Conversely, the traceless of stress tensor implies conformal symmetry of a quantum field theory. For \( T\bar{T} \) deformed CFT, the conformal symmetry is abandoned. However, it has another property analogous to the traceless, namely the trace relation. From the viewpoint of flow equation, the deformed CFT is defined by \( T\bar{T} \) flow, and the \( T\bar{T} \) flow turns the traceless into the trace relation along the parameter \( \mu \). In the case of CFT, that is \( \mu = 0 \), the trace relation reduced to traceless which leads to the conformal symmetry. We may conjecture the trace relation would imply the symmetry of \( T\bar{T} \) deformed CFT.

In fact, the trace relation could be expressed as

\[ T^i_i + 2 \mu T\bar{T} = T^{ij}[g_{ij} + 2 \mu (T_{ij} - g_{ij} T^k_k)] = 0. \]  

If the traceless is replaced by trace relation in the action variation, we can get

\[ \delta S(\mu) = -\frac{1}{2} \int d^2 x \sqrt{g} T^{ij} \delta g_{ij} \]

\[ = -\frac{1}{2} \int d^2 x \sqrt{g} T^{ij}[g_{ij} + 2 \mu (T_{ij} - g_{ij} T^k_k)] \omega(x) = 0. \]
It would imply a symmetry, because of the invariance of action. That is to say the variation of metric, under this symmetry transformation, would be

\[ \delta g_{ij} = [g_{ij} + 2\mu(T_{ij} - g_{ij}T^{k}_{k})]\omega(x). \]  
(3.5)

The second term on the right hand side comes from the \( T\bar{T} \) deformation of conformal symmetry. However, it is not clearly to understand what does the effect of \( T\bar{T} \) deformation on geometry from above equation, because the deformation expressed in terms of the stress tensor rather than the metric. Fortunately, the second term can be written in another form as we have mentioned in section 2. The basic idea is the background is not fixed, but runs with deformation parameter \( \mu \). The deformation term is just the variation of metric along parameter \( \mu \), according to (2.7). Therefore, the variation of metric in \( T\bar{T} \) deformation becomes

\[ \delta g_{ij} = (g_{ij} - \mu \frac{\partial g_{ij}}{\partial \mu})\omega(x), \]  
(3.6)

from which, we can easily see that the deformation lead to an additional term \( \mu \partial_{\mu}g_{ij} \) compared with conformal killing equation (3.1). This term is just the rate of change of the metric along the parameter \( \mu \), and the fixed point consists with the conformal symmetry.

One should note that we do not write the variation of metric in Lie derivative form, because it depends on stress tensor which may refer to the non-geometry quantities. Generally, this is a gauge transformation, which may not be realized by an infinitesimal diffmorphism \( x^{i} \rightarrow x^{i} + \xi^{i} \). However, we would see that there actually exist a coordinates transformation on the boundary from the bulk asymptotic symmetry to achieve the variation of metric. So one can replace \( \delta g_{ij} \) by \( L_{\xi}g_{ij} \), and above equation becomes a \( T\bar{T} \) deformed conformal killing equation. We will give a detail illustration in next section. In addition, this \( T\bar{T} \) deformed conformal killing equation is not covariant under a coordinate transformation refers to \( \mu \) because of the second term. In fact, for two dimensional case, one can always transform the background to Minkowski spacetime. But the equation (3.6) should also change. So it does not simplify the problem, when one tries to solve this equation.

### 4 Holographic interpretation

In this section, we explore holographic aspects on the symmetry of \( T\bar{T} \) deformed CFT. We will show that the \( T\bar{T} \) symmetry corresponds to a new asymptotic boundary condition of AdS_3. Moreover, the stress tensor of boundary field theory can also be obtained from the gravity side.
4.1 New boundary condition of AdS$_3$

It was proposed that the $T\bar{T}$ deformed 2D CFT corresponds to a cutoff AdS$_3$. So the symmetry of the $T\bar{T}$ should have a holographic interpretation on the gravity side. We begin with the most general solution of Einstein equation with a negative cosmological constant, which can be expressed in Fefferman-Graham expansion \[25\]

$$ds^2 = G_{\mu\nu} dx^\mu dx^\nu = \frac{l^2}{r^2} dr^2 + \gamma_{ij} dx^i dx^j,$$

$$\gamma_{ij} = r^2 (g^{(0)}_{ij}(x) + \frac{1}{r^2} g^{(2)}_{ij}(x) + \frac{1}{r^4} g^{(4)}_{ij}(x)). \quad (4.1)$$

The Einstein equation also gives a constraint\[4.2\]

$$g^{(4)}_{ij} = \frac{1}{4} g^{(2)}_{ik} g^{(2)}_{jl}.$$  

From holographic correspondence, the boundary metric $\gamma_{ij}$ is not well defined at spatial infinity $r \to \infty$, but one can choose a regularization factor $1/r^2$ to cancel the singularity \[26\], such that the boundary metric is well defined, that is $g^{(0)}_{ij}$ for a Poincaré patch. Generally, one can choose $g^{(0)}_{ij} = \eta_{ij}$, then there is a conformal field theory on the spatial infinity boundary. In the context of AdS$_3$/CFT$_2$, Brown and Henneaux found an asymptotic boundary condition of AdS$_3$, and the asymptotic symmetry algebra which is consistent with Virasoro algebra with a nontrivial central charge \[18\]. So the conformal symmetry of boundary field theory can be described by an asymptotic symmetry in bulk. Some other asymptotic boundary conditions have been also explored like BMS$_3$, $U(1)$ Kac-Moody-Virasoro algebra \[27–30\]. But now we focus on the theory at finity boundary, a $T\bar{T}$ deformed CFT, which is dual to gravitational theory restricted on a compact sub-region of AdS spacetime. We suppose the boundary field theory live on the regularized background metric

$$g_{ij} = \frac{\gamma_{ij}}{r^2} = g^{(0)}_{ij}(x) + \frac{1}{r^2} g^{(2)}_{ij}(x) + \frac{1}{r^4} g^{(4)}_{ij}(x), \quad (4.3)$$

which is also well defined even at spatial infinity. We will show that the symmetry of $T\bar{T}$ deformed CFT corresponds a new asymptotic boundary condition for AdS$_3$

$$\delta G_{rr} = 0, \quad \delta G_{ri} = O\left(\frac{1}{r}\right), \quad \delta G_{ij} = \delta \gamma_{ij} = 0. \quad (4.4)$$

We consider an infinitesimal diffeomorphism transformation in bulk $\zeta^\mu = (\zeta^r, \xi^i)$, where the $\xi^i$ represents a diffeomorphism on the boundary of constant $r$. With this infinitesimal transformation, the variation of metric could be expressed by Lie derivative

$$\delta G_{rr} = 0$$

$$\Rightarrow \mathcal{L}_\zeta G_{rr} = \zeta^r \partial_r \left(\frac{l^2}{r^2}\right) + 2 \left(\frac{l^2}{r^2}\right) \partial_r \zeta^r = r \partial_r \zeta^r G_{rr} = 0 \quad (4.5)$$

$$\Rightarrow \xi^r = \xi^r(x), \quad (4.6)$$
where we have set $\xi^r = 2l\zeta^r/r$. This means that $\xi^r$ is an arbitrary function which does not depend on the radial coordinate $r$. Then we turn to consider the variation of induced metric on the boundary

$$\delta\gamma_{ij} = 0 \Rightarrow \mathcal{L}_\xi \gamma_{ij} = \xi^\mu \partial_\mu \gamma_{ij} + \gamma_{\mu k} \partial_j \zeta^k + \gamma_{kj} \partial_i \zeta^k = \xi^r \frac{r}{2l} \partial_r \gamma_{ij} + \xi^k \partial_k \gamma_{ij} + \gamma_{\mu k} \partial_j \zeta^k + \gamma_{kj} \partial_i \zeta^k = -\xi^r K_{ij} + \mathcal{L}_\xi \gamma_{ij} = 0,$$

$$\Rightarrow \mathcal{L}_\xi \gamma_{ij} = \xi^r K_{ij}. \quad (4.7)$$

Here the definition of extrinsic curvature is used

$$K_{ij} = \frac{1}{2} \mathcal{L}_n \gamma_{ij} = -\frac{r}{2l} \partial_r \gamma_{ij}, \quad (4.8)$$

where $n = (r/l, 0, 0)$ is the normal vector to surfaces of constant $r$. The (4.6) and (4.7) imply the variation of induced metric equal the extrinsic curvature up to a scaling factor $\xi^r$, that just depends on the boundary coordinates $x^i$. The second condition $\delta G_{ri} = O(1/r)$ implies asymptotic symmetry. When $r \to \infty$, the metric (4.1) becomes Poincaré patch and the symmetry is an isometry of AdS$_3$. Therefore, the asymptotic killing equation (4.7) is obtained. The $T\bar{T}$ deformed conformal killing equation is also got in last section. We, then, will show that this asymptotic symmetry is just the symmetry of $T\bar{T}$ deformed CFT living on a surface of constant $r$ with metric $g_{ij}$.

In fact, we should note that the boundary metric is regularized induced metric by $g_{ij} = \gamma_{ij}/r^2$, and that is the essential difference between (3.6) and (4.7). The (4.7) can be written in terms of $g_{ij}$

$$\mathcal{L}_\xi \gamma_{ij} = \xi^r K_{ij} = -\xi^r \frac{r}{2l} \partial_r \gamma_{ij} = -\xi^r \frac{r}{2l} \partial_r (r^2 g_{ij}) = -\xi^r \frac{r}{2l} (2rg_{ij} + r^2 \partial_r g_{ij}) = -\xi^r \frac{r^2}{l} (g_{ij} - \mu \partial_\mu g_{ij}), \quad (4.9)$$

where we have used the relation $\mu \sim \frac{1}{r^2}$ in above steps, and the coefficient is dispensable. After dividing out $r^2$ on both sides, it ends up with

$$\mathcal{L}_\xi g_{ij} = -\xi^r \frac{1}{l} (g_{ij} - \mu \frac{\partial g_{ij}}{\partial \mu}) \quad (4.10)$$

which is same as (3.6) with weyl-like factor $\omega(x) = -\xi^r(x)/l$. The nontrivial term $\mu \partial_\mu g_{ij}$ comes from the $T\bar{T}$ deformation. It has a contribution to this deformed conformal killing
equation, if we consider the high order terms of the boundary metric, because the high order term depends on $\mu$ (or $r$). The leading term $g_{ij}^{(0)}$ does not depend on $\mu$, which implies the conformal symmetry. When we push the boundary to spatial infinity, the high order terms tend to vanish, and the symmetry reduce to conformal symmetry. So the symmetry of $TT$ deformed CFT is naturally obtained from AdS$_3$ with a radial cutoff boundary, because we cannot ignore the high order terms. However, the solution of Einstein equation terminates after fourth order in Fefferman-Graham expansion, thus we can get the explicitly symmetry transformation at finite radial. We will discuss these in section 5 for BTZ black hole. But there is a trivial case, the Poincaré patch of AdS$_3$, that is $g_{ij}^{(0)} = \eta_{ij}$ and $g_{ij}^{(2)} = g_{ij}^{(4)} = 0$ in the form of (4.1). The boundary metric always Minkowski spacetime, even though for different radial cutoff. In order to get a nontrivial symmetry, we must consider the high order correction of Fefferman-Graham expansion.

We now consider the Fefferman-Graham expansion of boundary metric $g_{ij}$. From the deformed conformal killing equation (4.3) and (4.10), one can get

$$L_\xi \left( g_{ij}^{(0)} + \frac{1}{r^2} g_{ij}^{(2)} + \frac{1}{r^4} g_{ij}^{(4)} \right) = \left( g_{ij}^{(0)} - \frac{1}{r^4} g_{ij}^{(4)} \right) \omega(x),$$

from which we can see that the usual conformal symmetry is just the leading term of above equation, namely $L_\xi g_{ij}^{(0)} = g_{ij}^{(0)} \omega(x)$. The Brown-Henneaux’s boundary condition actually preserves the leading order. The $TT$ symmetry includes the high order correction, and the correction terminates after fourth order in case of AdS$_3$. Therefore, this new boundary condition just replaces the asymptotic symmetry of $\gamma_{ij}$ and $G_{ri}$ by exactly killing equations, but keeps the asymptotic symmetry of $G_{ri}$. That is we have mentioned (4.4).

### 4.2 Holographic stress tensor

As the $TT$ deformed conformal killing equation relates to the stress tensor of boundary field theory and the symmetry also has a holographic description. So the stress tensor might be calculated from the gravity side. In this subsection, we would derive a holographic duality of stress tensor, that is the stress tensor of $TT$ deformed CFT equals to the Brown-York’s quasilocal stress tensor on the cutoff surface. The trace anomaly of $TT$ deformed CFT is also considered.

We begin with a $TT$ deformed CFT placing at a finite radial surface distance $r$ from the center of the bulk. So the background metric is not fixed but depends on $\mu$. From (2.7), we can get the stress tensor of $TT$ deformed CFT

$$\hat{T}_{ij} \equiv T_{ij} - T g_{ij} = -\frac{1}{2} \partial_\mu g_{ij}. \quad (4.12)$$
The $T_{ij}$ represents the stress tensor of boundary field theory and we will use $\tau_{ij}$ to denote a holographic stress tensor from gravity side. It is a little complicated to express out the stress tensor $T_{ij}$ in terms of metric. We here would calculate the similar quantity $\hat{\tau}_{ij} \equiv \tau_{ij} - \tau \gamma_{ij}$ on the gravity side. Actually, the Brown-York’s quasilocal stress tensor of gravity is defined as \cite{22, 24}

\begin{equation}
\tau_{ij} = \frac{2}{\sqrt{\gamma}} \frac{\delta S_{grav}}{\delta \gamma^{ij}} = -\frac{1}{8\pi G} (K_{ij} - K \gamma_{ij} - \frac{1}{l} \gamma_{ij}),
\end{equation}

where we have added the last term to cancel divergences on the spatial infinity boundary. Taking trace and substituting back the equation, we arrive at

\begin{equation}
\hat{\tau}_{ij} \equiv \tau_{ij} - \tau \gamma_{ij} = -\frac{1}{8\pi G} (K_{ij} + \frac{1}{l} \gamma_{ij}).
\end{equation}

There is still something different from (4.12). However, noting that the extrinsic curvature can be written in terms of $g_{ij}$

\begin{equation}
K_{ij} = -\frac{r}{2l} \partial_r \gamma_{ij} = -\frac{1}{l} \gamma_{ij} - \frac{r^3}{2l} \partial_r g_{ij},
\end{equation}

plugging into (4.14), one can get

\begin{equation}
\hat{\tau}_{ij} = \frac{r^3}{16\pi G l} \partial_r g_{ij}.
\end{equation}

If we identify $\mu \sim 1/r^2$, explicitly $\mu = 4\pi G l / r^2$, the stress tensors on both sides are equivalent

\begin{equation}
\hat{T}_{ij} = \hat{\tau}_{ij} \iff T_{ij} = \tau_{ij}.
\end{equation}

Therefore we derive a holographic dictionary about stress tensor. Hence, by applying the Brown-Henneaux relation $c = 3l/2G$ \cite{18}, we obtain

\begin{equation}
\mu = \frac{6\pi l^2}{c} \frac{1}{r^2},
\end{equation}

which is just the relation (2.9), up to an unessential constant, because we use the different convention of the metric form. In fact, the same quantity is marked as $\tilde{\mu}$ in \cite{9}.

If we consider the Fefferman-Graham expansion of the induced metric (4.1), the stress tensor can be calculated from gravity side

\begin{equation}
\hat{T}_{ij} = \hat{\tau}_{ij} = -\frac{1}{8\pi G l} \left( g_{ij}^{(2)} + \frac{2}{r^2} g_{ij}^{(4)} \right).
\end{equation}

According to the holographic dictionary of AdS$_3$/CFT$_2$ \cite{31, 32}, the leading order coincide with the CFT case. When we push the boundary to spatial infinity, the metric reduces to $g_{ij}^{(0)}$. Then we can get the trace of stress tensor by contracting with $g^{(0)ij}$

\begin{equation}
T^i_i = \frac{1}{r^2} \tau^i_i = \frac{1}{8\pi G l} \text{Tr}[(g^{(0)})^{-1} g^{(2)}].
\end{equation}
Einstein equation perturbatively gives [33]

$$\text{Tr}[g^{(0)}^{-1}g^{(2)}] = \frac{l^2}{2} R,$$  \hspace{1cm} (4.21)

combining with the Brown-Henneaux relation, (4.20) ends up with the well-known trace anomaly of CFT

$$T^i_i = \frac{c}{24\pi} R.$$  \hspace{1cm} (4.22)

As for the higher term, we should contract (4.19) with $g^{ij}$ instead of $g^{(0)ij}$. It is a complicated perturbation calculation, because the scalar curvature also changes with parameter $\mu$ or for different cutoff $r$. However, with the correspondence of stress tensor, we can get the trace anomaly exactly from gravity side.

In fact, the $T \bar{T}$ operator can be calculated from gravity side

$$T \bar{T} = \tau i j \bar{\tau} ^ {ij} = \frac{1}{(8\pi G)^2} \left( K^{ij} K_{ij} - K^2 - \frac{2}{l} K - \frac{2}{l^2} \right).$$  \hspace{1cm} (4.23)

Noting the Gauss-Codazzi relation, we obtain

$$T \bar{T} = - \frac{1}{(8\pi G)^2} \left( R + \frac{2}{l} K + \frac{4}{l^2} \right).$$  \hspace{1cm} (4.24)

Besides, the trace of stress tensor is

$$T^i_i = \frac{1}{r^2} \tau ^ i _ i = - \frac{1}{8\pi G r^2} \left( K + \frac{2}{l} \right).$$  \hspace{1cm} (4.25)

Combining these equation, we arrive at the relation

$$T^i_i = \frac{c}{24\pi} R - 2\mu T \bar{T},$$  \hspace{1cm} (4.26)

The trace anomaly of $T \bar{T}$ deformed CFT, which is also obtained in [21, 34] from the field theory side. According to the stress tensor correspondence, we can reproduce it from gravity side.

The trace anomaly, in a sense, is the exact RG equation. It corresponds to Wheeler-DeWitt equation in 3D AdS gravity. Actually, the Brown-York’s stress tensor is the canoical momentum in ADM formalism, up to a counterterm. In classical level, if one replaces the boundary stress tensor by Brown-York’s stress tensor, the trace relation exactly gives the Hamiltonian constraint. After quantization, one can get that holographic RG equation is equivalent to Wheeler-DeWitt equation [9, 15].
5 BTZ black hole

The BTZ black hole [35, 36] is a good realization of cutoff AdS and $T\bar{T}$ deformed 2D CFT correspondence. The spectrum can be obtained by considering the quasilocal energy defined on a surface at finite radial location. In this section, we would explore more about $T\bar{T}$ deformed CFT from BTZ black hole, including $T\bar{T}$ symmetry and conserved charges.

There is a useful coordinate system in Fefferman-Graham gauge for BTZ black hole [37]

\[ ds^2 = \frac{l^2}{r^2} dr^2 + r^2 d\tilde{z}d\bar{\tilde{z}} + \frac{L\bar{L}}{r^2} dzd\bar{z} + Ldz^2 + \bar{L}d\bar{z}^2, \]  

(5.1)

where $z = t - i\phi$, $\tilde{z} = t + i\phi$, $L$ and $\bar{L}$ are constants related to the parameters of BTZ black hole $M$ and $J$

\[ L = \frac{(r_+ + r_-)^2}{2l} = \frac{1}{2}(M + J), \quad \bar{L} = \frac{(r_+ - r_-)^2}{2l} = \frac{1}{2}(M - J). \]  

(5.2)

In this coordinate system, the induced metric on a surface at finite radial can be expressed as

\[ \gamma_{ij} = (r^2 + \frac{L\bar{L}}{r^2})\bar{\eta}_{ij} + \bar{\delta}_{ij}, \]  

(5.3)

where we have defined

\[ \bar{\eta}_{ij} = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}, \quad \bar{\delta}_{ij} = \begin{pmatrix} L & 0 \\ 0 & \bar{L} \end{pmatrix}. \]  

(5.4)

The extrinsic curvature of the boundary can be calculated

\[ K_{ij} = \frac{1}{l}(r^2 - \frac{L\bar{L}}{r^2})\bar{\eta}_{ij}, \]  

(5.5)

then the $T\bar{T}$ deformed conformal killing equation (4.7) becomes

\[ \mathcal{L}_\xi \gamma_{ij} = \partial_i \xi_j + \partial_j \xi_i \]

\[ = \xi^r(x) \left( r^2 - \frac{L\bar{L}}{r^2} \right) \bar{\eta}_{ij}. \]  

(5.6)

We use partial derivative here, because the Levi-Civita connection of $\gamma_{ij}$ vanishes. The solutions of this equation are

\[ \xi^z = \epsilon(z) - \beta \bar{\epsilon}(\bar{z}), \]  

(5.7)

\[ \xi^\bar{z} = -\alpha \epsilon(z) + \bar{\epsilon}(\bar{z}), \]  

(5.8)
where $\epsilon(z), \bar{\epsilon}(\bar{z})$ are arbitrary functions depend on $z, \bar{z}$ respectively and $\alpha, \beta$ are introduced for convenient, with

$$\alpha = 2L \left( r^2 + L \bar{L} \right)^{-1}, \quad \beta = 2\bar{L} \left( r^2 + \frac{L \bar{L}}{r^2} \right)^{-1}. \quad (5.9)$$

Actually, the nonvanishing of $\alpha, \beta$ is the characteristic of $T\bar{T}$ deformation. As long as we push the boundary to spatial infinity $r \to \infty$, $\alpha, \beta$ tend to vanishing. The infinitesimal transformations become holomorphic and antiholomorphic mapping, which are conformal transformations.

So the general $T\bar{T}$ deformed conformal killing vectors about BTZ black hole are

$$k = \xi^z \partial_z + \xi^{\bar{z}} \partial_{\bar{z}} = \epsilon(z)(\partial_z - \alpha \partial_{\bar{z}}) + \bar{\epsilon}(\bar{z})(\partial_{\bar{z}} - \beta \partial_z). \quad (5.10)$$

we get the infinitesimal transformation $\xi^i = (\xi^z, \xi^{\bar{z}})$. According to Noether theorem, there must exist a conserved quantity, which plays an important role in physics. With this symmetry transformation $x^i \to x^i + \xi^i$, we have a conserved current which can be written as

$$j_i = T_{ij} \xi^j. \quad (5.11)$$

It is easy to check this is actually a conserved current because of the trace relation

$$\partial^i j_i = \partial^i T_{ij} \xi^j + T_{ij} \partial^i \xi^j$$

$$= \partial^i T_{ij} \xi^j + (T^i_i + 2\mu T\bar{T}) = 0. \quad (5.12)$$

The conserved charge is defined as integral on a time slice $\Sigma_t$

$$Q = \int_{\Sigma_t} \sqrt{g} \phi \phi j_0 d\phi. \quad (5.13)$$

In complex coordinates $z, \bar{z}$, the conserved charges are

$$Q_{\epsilon, \bar{\epsilon}} = \int_{\Sigma_t} \sqrt{g} \phi \phi \left[ (T_{zz} + T_{\bar{z}\bar{z}}) - \alpha(T_{z\bar{z}} + T_{\bar{z}z}) \right] \epsilon(z) d\phi$$

$$+ \int_{\Sigma_t} \sqrt{g} \phi \phi \left[ (T_{\bar{z}\bar{z}} + T_{zz}) - \beta(T_{z\bar{z}} + T_{\bar{z}z}) \right] \bar{\epsilon}(\bar{z}) d\phi, \quad (5.14)$$

where

$$g_{\phi\phi} = -g_{zz} + 2g_{z\bar{z}} - g_{\bar{z}\bar{z}} = 1 + \frac{L \bar{L}}{r^4} - \frac{L + \bar{L}}{r^2}, \quad (5.15)$$

and on a time slice we have $d\phi = idz = -id\bar{z}$. The components of stress tensor in $z, \bar{z}$ coordinate system

$$T_{zz} = \frac{1}{4}(T_{tt} - T_{\phi\phi} - 2iT_{t\phi}),$$

$$T_{\bar{z}\bar{z}} = \frac{1}{4}(T_{tt} - T_{\phi\phi} + 2iT_{t\phi}),$$

$$T_{z\bar{z}} = \frac{1}{4}(T_{tt} + T_{\phi\phi}).$$
Then we can get the relations
\[ T_{zz} + T_{z\bar{z}} = \frac{1}{2}(T_{tt} - iT_{t\phi}) = \frac{1}{2}(\tau_{tt} - i\tau_{t\phi}), \quad (5.16) \]
\[ T_{\bar{z}z} + T_{z\bar{z}} = \frac{1}{2}(T_{tt} + iT_{t\phi}) = \frac{1}{2}(\tau_{tt} + i\tau_{t\phi}), \quad (5.17) \]
\[ T_{zz} - T_{\bar{z}\bar{z}} = -iT_{t\phi} = -i\tau_{t\phi}. \quad (5.18) \]

In the second step, we used \( T_{ij} = \tau_{ij} \), which has been derived in section 4.2. With these relations, we can express conserved charges as
\[ Q_{\epsilon, \bar{\epsilon}} = \frac{1}{2} \int_{\Sigma_t} \sqrt{g_{\phi\phi}} \left[ (1 - \alpha)\tau_{tt} - i(1 + \alpha)\tau_{t\phi} \right] \epsilon(z) d\phi \\
+ \frac{1}{2} \int_{\Sigma_t} \sqrt{g_{\phi\phi}} \left[ (1 - \beta)\tau_{tt} + i(1 + \beta)\tau_{t\phi} \right] \bar{\epsilon}(\bar{z}) d\phi. \quad (5.19) \]

The conserved charges can be expressed in terms of spectrum and angular momentum. In fact, according to (2.13) and (2.14), we can get the formula of \( E_n \) and \( J_n \) from Brown-York’s stress tensor. The normal vector of a time slice in complex coordinates, i.e. \( z + \bar{z} = C \), can be calculated
\[ u^i = (a, a), \quad a = \frac{r}{\sqrt{r^2 + \frac{L\bar{L}}{r^2} + L + \bar{L}}}. \quad (5.20) \]

Then the spectrum can be obtained from gravity side
\[ E_n = r \int \sqrt{g_{\phi\phi}} \tau_{ij} u^i u^j d\phi = r \int \sqrt{g_{\phi\phi}} a^2 (\tau_{zz} + 2\tau_{z\bar{z}} + \tau_{\bar{z}\bar{z}}) d\phi \\
= ra^2 \int \sqrt{g_{\phi\phi}} \tau_{tt} d\phi. \quad (5.21) \]

For angular momentum, which associate with the killing vector \( \xi^i = \partial_\phi \)
\[ \xi^i = (b, -b), \quad b = \frac{r}{\sqrt{-(r^2 + \frac{L\bar{L}}{r^2}) + L + \bar{L}}}, \quad (5.22) \]

it can be written as
\[ J_n = \int \sqrt{g_{\phi\phi}} \tau_{ij} u^i \xi^j d\phi = \int \sqrt{g_{\phi\phi}} ab (\tau_{zz} - \tau_{\bar{z}\bar{z}}) d\phi \\
= -abi \int \sqrt{g_{\phi\phi}} \tau_{t\phi} d\phi. \quad (5.23) \]

In case of BTZ black hole, the stress tensor \( \tau_{ij} \) and \( \sqrt{g_{\phi\phi}} \) do not depend on integration variable \( \phi \), with \( \phi \sim \phi + 2\pi \). Noting the relations above, we can get
\[ Q_{\epsilon, \bar{\epsilon}} = \frac{1}{2} \left[ \frac{(1 - \alpha)E_n}{2\pi ra^2} + \frac{(1 + \alpha)J_n}{2\pi ab} \right] \int \epsilon(z) d\phi \\
+ \frac{1}{2} \left[ \frac{(1 - \beta)E_n}{2\pi ra^2} - \frac{(1 + \beta)J_n}{2\pi ab} \right] \int \bar{\epsilon}(\bar{z}) d\phi. \quad (5.24) \]
where the $\epsilon(z), \bar{\epsilon}(\bar{z})$ are arbitrary functions and $z, \bar{z}$ are linearly depend on $\phi$ on a time slice. Following the technique of CFT, we can also perform a mode expansion for the conserved charges. The holomorphic part and antiholomorphy part can be separated by setting $\epsilon = 0$ and $\bar{\epsilon} = 0$, respectively

$$Q_\epsilon = \sum_n \epsilon_n Q_n, \quad \bar{Q}_\epsilon = \sum_m \bar{\epsilon}_m \bar{Q}_m,$$  

(5.25)

where

$$Q_n = \frac{1}{2} \left[ \frac{(1 - \alpha)E_n}{2\pi ra^2} + \frac{(1 + \alpha)J_n}{2\pi ab} \right] \int z^{n+1} d\phi,$$  

(5.26)

$$\bar{Q}_m = \frac{1}{2} \left[ \frac{(1 - \beta)E_n}{2\pi ra^2} - \frac{(1 + \beta)J_n}{2\pi ab} \right] \int \bar{z}^{m+1} d\phi.$$  

(5.27)

Therefore, we can get the charge for $-1$ mode easily

$$Q_{-1} = \frac{1}{2} \left[ \frac{(1 - \alpha)E_n}{ra^2} + \frac{(1 + \alpha)J_n}{ab} \right],$$  

(5.28)

$$\bar{Q}_{-1} = \frac{1}{2} \left[ \frac{(1 - \beta)E_n}{ra^2} - \frac{(1 + \beta)J_n}{ab} \right].$$  

(5.29)

If we take the CFT limit, that is $r \to \infty$, they reduce to

$$Q_{-1} \to \frac{1}{2} (M + J) = L,$$  

(5.30)

$$\bar{Q}_{-1} \to \frac{1}{2} (M - J) = \bar{L}.$$  

(5.31)

which coincide with the CFT case marked as $L_0, \bar{L}_0$ [36, 37]. The $-1$ mode cooresponds to zero mode of CFT, because the radial quantization is used in CFT and time translation becomes dilation.

The charges for other modes can also be calculated from (5.26), (5.27), which are proportional to $Q_{-1}$ or $\bar{Q}_{-1}$. So we can not get an infinite amount of independent conserved charges. The reason might be that we can only obtain the global charges by this method, or the aspects of BTZ balck hole whose stress tensor does not depend on $\phi$. Actually, BTZ black hole has only two charges which are non-zero $M = L_0 + \bar{L}_0, J = L_0 - \bar{L}_0$. However, $T\bar{T}$ deformed CFT as an integrable system, it is still not much clear about the infinite amount of conserved charges, even though we have a symmetry.

6 Conclusion and discussion

The purpose of this paper is to explore the symmetry of $T\bar{T}$ deformed CFT. We are based on the idea that $T\bar{T}$ flow may turn the conformal symmetry to others, because this integrable
deformation preserves an infinite amount of conserved charges. We restudy the $T\bar{T}$ deformed CFT, and find the traceless of CFT becomes a trace relation along the $T\bar{T}$ flow. The former one implies the conformal symmetry. Naturally, we get the symmetry by replacing the traceless with trace relation, and the deformed conformal killing equation is obtained. Moreover, the proposal that $T\bar{T}$ deformed CFT coresponds to a cutoff AdS, inspired us to consider the boundary theory placing on a dynamical background. The deformation term is just the variation of metric along parameter $\mu$, and conformal symmetry coinside with the fixed point. From a holographic perspective, this symmetry is an asymptotic symmetry of AdS$_3$ with a new boundary condition, which removes the asymptotic behavior of induced metric, that is the Dirichlet boundary condition. Brown-Henneaux’s boundary condition just preserves the leading order and allow an asymptotic symmetry of induced metric in this case.

As a result of the metric flow equation refer to the stress tensor of field theory, it provides an approach to calculate the stress tensor from gravity side. Actually, we argued the holographic duality of stress tensor, that is the stress tensor of $T\bar{T}$ equals to Brown-York’s quasilocal stress tensor with a counterterm. Based on the stress tensor correspondence, trace anomaly of $T\bar{T}$ deformed CFT was obtained from gravity side exactly.

In order to obtain a specific symmetry, we consider BTZ black hole. The $T\bar{T}$ deformed conformal killing equation gives the infinitesimal transformations. We also computed the conserved charges of this symmetry. The result is turned to be consistent with the CFT case, when we push the boundary to spatial infinity. However, we still cannot get an infinite set of conserved charges. The reason may be the trace relation does not include all symmetry of this system or maybe because of the special feature of BTZ black hole. All in all, we did not get an infinite amount of independent conserved charges. So we can not get a nontrivial symmetry algebra of conserved charges about $T\bar{T}$ deformed CFT by this method.

However, analogous to the conformal symmetry, we can get the an infinite set of generators about the infinitesimal symmetry tranformation from (5.10) by performing a modes expansion with respect to $\epsilon(z), \bar{\epsilon}(\bar{z})$. These generators can constitute an algebra formally. In fact, we can write down the generators

$$L_n = -z^{n+1}(\partial_z - \alpha \partial_{\bar{z}}) = l_n - \alpha j_n,$$  \hspace{1cm} (6.1)

$$\bar{L}_m = -\bar{z}^{m+1}(\partial_{\bar{z}} - \beta \partial_z) = \bar{l}_m - \beta \bar{j}_m,$$  \hspace{1cm} (6.2)

Here, $l_n, \bar{l}_m$ are the familiar conformal generators, which obey the commutation relations

$$[l_n, l_m] = (n - m) l_{n+m}, [l_n, \bar{l}_m] = 0, [\bar{l}_n, \bar{l}_m] = (n - m) \bar{l}_{n+m}.$$  \hspace{1cm} (6.3)

After performing a nontrivial central extension, it becomes Virasoro algebra with a central charge. More about $T\bar{T}$ deformed CFT is the presence of $j_n, \bar{j}_m$. These yield the commutaion
relations of $T\bar{T}$ symmetry generators $L_n, \bar{L}_m$

\[ [L_n, L_m] = (n - m)L_{n+m}, \quad (6.4) \]
\[ [\bar{L}_n, \bar{L}_m] = (n - m)\bar{L}_{n+m}, \quad (6.5) \]
\[ [L_n, \bar{L}_m] = -\beta(n+1)z^{m+1}L_{n-1} + \alpha(m+1)z^{n+1}\bar{L}_{m-1}, \quad (6.6) \]

from which, one can see that these generators form an algebra. Up to a central extension, the difference between this algebra and Virasoro algebra is the commutation relation $[L_n, \bar{L}_m]$, which is vanishing in case of the later one. Of course, the generators $L_n, \bar{L}_m$ are also different from conformal generator by definition, because the nonvanishing of $\alpha, \beta$. If we push the boundary to infinity, this algebra reduces to Virasoro algebra. From an algebraic viewpoint, $T\bar{T}$ deformation effect is coupling the two copies of Virasoro algebra. But we should point out that the conserved charges associate with these generators are not independent, which we have calculated in Section 5. So we still do not clear about this symmetry algebra in a deep level.

In addition, AdS$_3$ gravity can be formulated as Chern-Simons theory, and gauge transformations can produce the diffeomorphisms [38]. Global charges in Chern-Simons theory are related to a particular Virasoro algebra via a twisted Sugawara construction $\mathcal{L}_{\xi g_{\phi\phi}} = 0$ in [39], see also [40, 41]. Therefore, Chern-Simons theory may provide a route to study the symmetry and holographic aspects of $T\bar{T}$ deformed CFT. In fact, with the boundary condition of $T\bar{T}$, namely $\mathcal{L}_{\xi \gamma_{ij}} = 0$, we have get the diffeomorphisms in (5.7) and (5.8). In terms of Chern-Simons formula, the global charges can be calculated

\[ Q_0 = (1 - \beta)L, \quad (6.7) \]
\[ \bar{Q}_0 = (1 - \alpha)\bar{L}, \quad (6.8) \]

which are proportional to $L, \bar{L}$. But the effect of $T\bar{T}$ deformation make these different from $L, \bar{L}$ because of non-vanishing of $\alpha, \beta$. For the CFT case $\alpha, \beta \rightarrow 0$, these charges reduce to $L, \bar{L}$. However, the charges are different from the $Q_{-1}, \bar{Q}_{-1}$ at finite radial coordinate $r$, because they are defined by different ways. The later are quasilocal conserved charges defined by Hamilton-Jacobi method from gravity side, which also have holographic aspects. But global charges in Chern-Simons are defined by surface integral method in a gauge theory. Whether or not, it is a way to research the $T\bar{T}$ deformed CFT. The detail calculation and discussion about Chern-Simons formalism are affixed to the Appendix A. It would be interesting to understand and explore more about the $T\bar{T}$ deformed CFT and holographic properties from different theories.
Acknowledgments

We would like to thank Feng Qu for useful discussions. This work is supported by the National Natural Science Foundation of China (NSFC) with Grant No.11847612, No.11875082.

A Chern-Simons formalism

In this appendix, we treat in more detail to calculate the charges from Chern-Simons formalism. We start from writing $\text{AdS}_3$ in gauge fields form. Three dimensional Einstein gravity with a negative cosmological constant can be expressed as $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ Chern-Simons gauge theory, with the action

$$S_{\text{grav}} = I(A) - I(\bar{A})$$

(A.1)

where

$$I(A) = \frac{\kappa}{4\pi} \int_M \text{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right), \kappa = \frac{l}{4G}$$

(A.2)

and

$$A^a = \omega^a + \frac{1}{l} e^a, \bar{A}^a = \omega^a - \frac{1}{l} e^a.$$  

(A.3)

In terms of triads, the metric (5.1) can be formulated as gauge fields

$$A = \begin{pmatrix} dr/2r & Ldz/r \\ rdz/r & -dr/2r \end{pmatrix} = (r - \frac{L}{r}) J_0 dz + \frac{1}{r} J_1 dr + (r + \frac{L}{r}) J_2 dz,$$

(A.4)

$$\bar{A} = \begin{pmatrix} -dr/2r & rd\bar{z} \\ \bar{L}d\bar{z}/r & dr/2r \end{pmatrix} = -(r - \frac{\bar{L}}{r}) J_0 d\bar{z} - \frac{1}{r} J_1 dr + (r + \frac{\bar{L}}{r}) J_2 d\bar{z},$$

(A.5)

where the bases $J_a$ are generators of $SL(2, \mathbb{R})$

$$J_0 = \frac{1}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, J_1 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, J_2 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

(A.6)

with commutation relations and killing metric

$$[J_a, J_b] = \varepsilon_{ab}^c J_c, \quad g_{ab} = \text{Tr}(J_a J_b) = \frac{1}{2} \eta_{ab} = \frac{1}{2} \text{diag}(-1, 1, 1).$$

(A.7)

Global charges in Chern-Simons theory are defined as

$$Q(\eta) = \frac{\kappa}{4\pi} \int_{\partial \Sigma} \text{Tr}(\eta A),$$

(A.8)
where the parameter of gauge transformation $\eta$ are related to the diffeomorphisms $\xi^i$ by

$$\eta^a = \xi^i A^a_i.$$  \hspace{1cm} (A.9)

In [39–41], Bañados and Carlip reproduced the Virasoro algebra from the Poisson bracket of the global charges by setting the boundary condition $L_{\xi} g_{\phi \phi} = 0$. That is the twisted Sugawara construction $\xi^r = -C(r) \partial_\phi \xi^\phi$, and $\xi^\phi$ just depends on $\phi$. In Schwarzschild coordinates, we have $C(r) = r/lN(r)$, where $N(r)$ is lapse function of BTZ black hole. After a partial integral, the charges can be expressed as

$$Q(\eta) = \frac{\kappa}{4\pi} \int_{\partial \Sigma} g_{ab} \eta^a A^b_\phi d\phi = \frac{\kappa}{4\pi} \int_{\partial \Sigma} g_{ab} (2C(r) \xi^\phi A^a_\phi \partial_\phi A^b_\phi + \xi^a A^a_\phi A^b_\phi) d\phi.$$ \hspace{1cm} (A.10)

But in our coordinates, we make a radial decomposition and treat the radial as time. As the gauge fields do not depend on $z, \bar{z}$, we can get the charges easily in case of BTZ black hole

$$Q(\xi) = \frac{\kappa}{4\pi} \int_{\partial \Sigma} g_{ab} \xi^a A^a z A^b z d\phi = \frac{\kappa}{2\pi} \int_{\partial \Sigma} \xi^z L d\phi.$$ \hspace{1cm} (A.11)

Similarly, we can get

$$\bar{Q}(\xi) = \frac{\kappa}{4\pi} \int_{\partial \Sigma} g_{ab} \xi^a \bar{A}^a z \bar{A}^b z d\phi = \frac{\kappa}{2\pi} \int_{\partial \Sigma} \xi^\bar{z} \bar{L} d\phi.$$ \hspace{1cm} (A.12)

We can also put it in terms of Fourier modes. The non-zero charges is zero mode, which related to $M$ and $J$ by $Q_0 = L \equiv (M + J)/2, \bar{Q}_0 = \bar{L} \equiv (M - J)/2$ (up to an unessential coefficient). These charges are also hold for the boundary at finite $r$, because the boundary condition gives $\xi^r = -C(r) \partial_\phi \xi^\phi$ that holds for all radial coordinate. But $C(r)$ would lead to a shift in central charge of the Virasoro algebra for different $r$. The definition of charge in Chern-Simons inspires us to consider the charge of $T \bar{T}$ deformed CFT.

Now we will put the $T \bar{T}$ boundary condition of the induced metric $\gamma_{ij}$ of (4.4) in this formula. This boundary condition is stronger than $L_{\xi} g_{\phi \phi} = 0$ because it restricts the whole induced metric not only $g_{\phi \phi}$ component. We have got the solution of $L_{\xi} \gamma_{ij} = 0$ exactly, i.e. (5.7) and (5.8). Then the charges in Chern-Simons theory are

$$Q(\xi) = \frac{\kappa}{4\pi} \int_{\partial \Sigma} g_{ab} (\xi^z A^a_\phi + \xi^\phi A^a z) A^b z d\phi.$$ \hspace{1cm} (A.13)

Noting the relations

$$A^a_\phi = 0, \hspace{0.5cm} A^a z = 0, \hspace{0.5cm} g_{ab} A^a z A^b z = 2L, \hspace{0.5cm} g_{ab} A^a z A^b z = 2\bar{L},$$ \hspace{1cm} (A.14)

and conventions we used above, we arrive at

$$Q(\xi) = \frac{\kappa}{2\pi} \int_{\partial \Sigma} [\epsilon(z) - \beta \bar{\epsilon}(\bar{z})] L d\phi,$$ \hspace{1cm} (A.15)

$$Q(\xi) = \frac{\kappa}{2\pi} \int_{\partial \Sigma} [-\alpha \epsilon(z) + \bar{\alpha} \bar{\epsilon}(\bar{z})] \bar{L} d\phi.$$ \hspace{1cm} (A.16)
Here \( d\phi = idz = -id\bar{z} \) on a time slice \( \Sigma_t \), so we just write \( d\phi \). The charges are proportional to \( L, \bar{L} \). Modes expansion of \( \epsilon(z), \bar{\epsilon}(\bar{z}) \) allow us to write down the zero mode which can be compared with CFT case

\[
\mathcal{Q}_0 = (1 - \beta)L, \quad (A.17)
\]
\[
\bar{\mathcal{Q}}_0 = (1 - \alpha)\bar{L}. \quad (A.18)
\]

These charges are associated with the boundary condition of \( T\bar{T} \) deformation, which is different from the case of Bañados at finite \( r \) because of \( \alpha, \beta \). Pushing the \( T\bar{T} \) symmetry to spatial infinity, \( \alpha, \beta \) tend to vanishing, and the charges reduce to the CFT case.

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