Static traversable wormholes in Lyra manifold

A. Sayahian Jahromi\textsuperscript{1}, H. Moradpour\textsuperscript{2*}

\textsuperscript{1} Zarghan Branch, Islamic Azad University, Zarghan, Iran
\textsuperscript{2} Research Institute for Astronomy and Astrophysics of Maragha (RIAAM), P.O. Box 55134-441, Maragha, Iran

At first, considering the Einstein framework, we introduce some new static traversable wormholes, and study the effects of a dark energy-like source on them. Thereinafter, a brief review on Einstein field equations in Lyra manifold is presented, and we address some static traversable wormholes in the Lyra manifold which satisfy the energy-conditions. It is also shown that solutions introduced in the Einstein framework may also meet the energy conditions in the Lyra manifold. Finally, we focus on vacuum Lyra manifold and find some traversable asymptotically flat wormholes. In summary, our study shows that it is theoretically possible to find a Lyra displacement vector field in a manner in which traversable wormholes satisfy the energy conditions in a Lyra manifold.

I. INTRODUCTION

Gravitational theories open a window to the interstellar trips through the support of notion of traversable wormhole \[1–18\]. Besides, the physics of wormholes is so close to that of black holes \[19–24\], a point which also encourages us to study wormholes. Although, in the Einstein theory, traversable wormholes do not respect all energy conditions simultaneously \[8\], there are some gravitational theories in which a traversable wormhole may support the energy conditions \[7–11, 25–35\]. In fact, find various traversable wormholes in different gravitational theories is an important and attractive issue in theoretical physics \[35–52\].

In the Einstein theory of gravity, the spacetime is a Riemannian manifold, geometry and energy-momentum sources are coupled to each other in a minimal way, and the spacetime is curved by the energy-momentum source \[53\]. In one hand, the successes of Einstein theory confirm that the spacetime is not flat everywhere \[53\]. On the other hand, due to the failure of this theory to provide proper description for various phenomenons, physicists try to modify it \[54–60\].

Using the Lyra generalization of the Riemannian geometry \[61\], Sen has constructed a modified version for the Einstein theory \[62\], which attracts a lot of attention to itself \[63–70\]. Some conformal dynamic traversable wormholes have also been studied in this theory which violate the null and weak energy conditions \[71\].

Traversable wormholes should satisfy the flaring-out condition \[4, 6\]. In the Einstein framework, this condition leads to the fact that the sum of the energy density and the radial pressure is not positive everywhere, and thus the energy conditions are not met by traversable wormholes in the Einstein framework \[4, 6\]. Here, we are interested to investigate the possibility of satisfying the energy conditions by static spherically symmetric traversable wormholes in the Lyra manifold.

In order to achieve our goal, at first, the general mathematical properties of static traversable wormholes are addressed in the next section. In addition, we introduce some new static traversable wormholes in the Einstein framework, and study the effects of a dark energy-like source on the solutions in Sec. (III). In fact, the provisions of this section also help us in understanding some difficulties of traversable wormholes in the Einstein framework. In the fourth section, providing a brief review on the Einstein field equations in the ordinary gauge of the Lyra manifold, we show that some solutions obtained in Sec. (III) can meet the energy conditions. More solutions are also studied in this section. In Sec. (V), considering the empty Lyra manifold, the possibility of obtaining traversable wormholes as solutions of vacuum field equations is also investigated. Section (VI) is devoted to a summary.

II. TRAVERSABLE WORMHOLES, GENERAL REMARKS

The general spherically symmetric traversable wormhole metric is written as

\[ds^2 = -U(r)dt^2 + \frac{dr^2}{1 - \frac{b(r)}{r}} + r^2(d\theta^2 + \sin^2 \theta d\phi^2),\]

where \(b(r)\) and \(U(r)\) are the shape and redshift functions, respectively. In order to have a wormhole with a throat located at \(r_0\), \(b(r)\) should satisfy the \(b(r_0) = r_0\) condition. Besides, the metric has to preserve its signature for \(r > r_0\) meaning that we must have \(U(r) > 0\) and \(b(r) < r\) when \(r > r_0\). In addition, it is useful to note here that to avoid getting horizon instead of the wormhole’s throat, the redshift function should satisfy the \(U(r_0) > 0\) condition. Finally, the satisfaction of the flaring-out condition \((f(r_0) \equiv b'(r_0) - 1 < 0)\) is necessary for obtaining a traversable wormhole \[4\].

Now, bearing the Einstein field equations in mind \((G_{\mu\nu} = T_{\mu\nu})\), one reaches
\[ \rho = \frac{b'(r)}{r^2}, \]
\[ p_r = \frac{U'(r)}{rU(r)}(1 - \frac{b(r)}{r}) - \frac{b(r)}{r^3}, \]
\[ p_t = p_r + \frac{r}{2} \left[ p'_r + (\rho + p_r) \frac{U'(r)}{2U(r)} \right], \]

for the non-zero components of the energy-momentum source that supports the geometry which is a spherically symmetric spacetime. In the above equation and also in this paper, prime denotes the derivative with respect to \( r \). Considering the flaring-out as well as \( b(r_0) = r_0 \) conditions, one can use Eq. (2) to obtain \( \rho(r_0) + p_r(r_0) < 0 \) in general relativity (GR) \([53]\). In fact, the components of energy-momentum tensor should satisfy the

\[ WEC \rightarrow \rho \geq 0, \rho + p_t > 0, \]
\[ NEC \rightarrow \rho + p_t \geq 0, \]
\[ DEC \rightarrow \rho \geq 0, \rho \geq |p_t|, \]
\[ SEC \rightarrow \rho + p_t \geq 0, \rho + p_r + 2p_t \geq 0, \] (3)

conditions \([53]\), where WEC and NEC are the weak energy condition and the null energy condition, respectively. DEC also denotes the dominant energy condition which declares that the velocity of energy transfer cannot be more than that of light. SEC, used as the abbreviation of the strong energy condition, stems from the attractive nature of gravity, and its form is the direct result of considering a spherically symmetric metric in the GR framework \([53]\). Therefore, it is apparent that, since \( \rho(r_0) + p_r(r_0) < 0 \), traversable wormholes do not respect these conditions everywhere \([6]\).

Here, we should remind that the above form of SEC is the direct result of working in the GR framework, and if one uses another gravitational theory, then SEC differs from the above relation \([53]\). It is also useful to remind that the energy conditions may be obtained by traversable wormholes in some modified gravity \([7-11, 22, 52]\).

Since anisotropy is arisen in various astrophysical phenomena, such as different types of wormholes and stars, etc. \([78-92]\), we are going to investigate this quantity in our study. One can also find some features of isotropic static spherically symmetric wormholes in Ref. \([100]\). The anisotropy parameter is defined as

\[ \Delta = p_t - p_r, \] (4)

which finally leads to

\[ \Delta = \frac{r}{2} \left[ p'_r + (\rho + p_r) \frac{U'(r)}{2U(r)} \right], \] (5)

for the above metric. Geometry is attractive (repulsive) for \( p_t < p_r \) \( (p_t > p_r) \). The \( \Delta = 0 \) case also means that an isotropic fluid supports the geometry.

### III. TRAVERSABLE WORMHOLES WITH HYPERBOLIC SHAPE FUNCTION

Let us consider \( b(r) = a \tanh(r) \) where \( a \) is a constant. The \( b(r_0) = 0 \) and flaring-out conditions imply \( a = \frac{r_0}{\tanh(r_0)} \) and \( f(r_0) = \frac{r_0}{\tanh(r_0)}(1 - \tanh^2(r_0)) - 1 < 0 \), respectively. In this situation, we have \( b(r) < r \) for \( r > r_0 \) and \( b(r)/r \to 0 \) at the \( r \to \infty \) limit. As it is obvious from Fig. 1, the flaring-out condition is respected by wormholes with arbitrary values of \( r_0 \).

![Fig. 1: The plot depicts \( f(r_0) \) function.](image)

#### A. Wormholes with constant redshift function

As the first example, we set \( U(r) = V \). Therefore, without lose of generality, one can set \( U(r) = 1 \) meaning that the metric is asymptotically flat. Following Eq. (2), one reaches at

\[ \rho = \frac{r_0[1 - \tanh^2(r)]}{r^2}, \]
\[ p_r = \frac{b(r)}{r^3}, \] (6)
\[ w_r(r) = \frac{\tanh(r)}{r[\tanh^2(r) - 1]}, \]
\[ p_t = \frac{r_0[r(\tanh^2(r) - 1) + \tanh(r)]}{2r^3\tanh(r_0)} \]

for the energy density and pressure components of the fluid which supports this geometry. As we have already seen, due to the flaring-out condition, the sum of energy density and radial pressure is of key importance in studying traversable wormholes. Therefore, considering this point as well as the spherical symmetry of metric, we calculated the radial state parameter \( w_r(r)(= \frac{p_r}{\rho}) \) in the above equation which helps us in simplifying our analysis.

Now, defining the mass function as

\[ m(r) = \int_{r_0}^{r} \rho 4\pi r^2 dr, \] (7)
we easily obtain

\[
m(r) = \frac{4\pi r_0}{\tanh(r_0)}[\tanh(r) - \tanh(r_0)],
\]

for mass which is independent of the redshift function. \(m(r)\) as the function of \(r\) has been plotted in Fig. (2) for different values of \(r_0\). As it is apparent, at the \(r \to \infty\) limit, we have \(m \approx \frac{4\pi r_0}{\tanh(r_0)}[1 - \tanh(r_0)]\) which is clearly a positive confined quantity. Therefore, an observer, located at infinity, measures a confined and positive mass for wormhole satisfying Eq. (8).

![Image](FIG. 2: The function \(\frac{m(r)}{4\pi}\) for some values of \(r_0\).

For this solution, \(\Delta \geq 0\) for \(r \geq 0\) signalling us a repulsive geometry. In addition, we have \(\rho > 0, p_t \geq 0\) and thus \(\rho + p_t > 0\) for all values of \(r\). Besides, although \(\rho\) is positive, \(w_r(r) < -1\) for \(r > 0\) meaning that the \(\rho + p_t > 0\) condition is violated everywhere. Therefore, there is no radial interval in which this solution can satisfy the energy conditions.

\[\rho = \frac{r_0[1 - \tanh^2(r)]}{r^2 \tanh(r_0)},\]

\[p_r = \frac{cnr^{n-2} - b(r)}{r^3}[cnr^n + 1],\]

\[w_r(r) = \frac{[cnr^{n+1} - \left(\frac{r_0 \tanh(r)}{\tanh(r_0)}\right)(cnr^n + 1)] \tanh(r_0)}{r_0[1 - \tanh^2(r)]},\]

\[p_t = \frac{1}{4r^3 \tanh(r_0)}[(cnr^n)^2(r \tanh(r_0) - r_0 \tanh(r)) + cnr^n(r_0(r \tanh^2(r) + (1 - 2n) \tanh(r)) + r(2n \tanh(r_0) - r_0)) + 2r_0(r(\tanh^2(r) - 1) + \tanh(r))].\]

It is also easy to check that the result obtained in Sec. (III A) is recovered at the appropriate limit of \(c \to 0\). Moreover, as it is obvious from (2), Eq. (5) is also valid in this case meaning that they are solutions with bounded mass function. Using Eq. (9), one gets

\[w_r(r_0) = \frac{\tanh(r_0)}{r_0[\tanh^{-1}(r_0) - 1]},\]

which is a relation between the wormhole throat and the value of the radial state parameter at the wormhole throat. One can also get this result by using Eq. (6) which belongs to the \(U(r) = 1\) case. It is the direct result of this fact that, at throat, we have \(b(r_0) = r_0\) and thus \(p(r_0) + \rho(r_0) = \frac{b'(r_0)-1}{r_0^2}\) which is independent of the redshift function (2). \(w_r\) has been plotted as a function of \(r_0\) in Fig. (3) indicating that \(w_r(r_0) < -1\) for \(r_0 > 0\), in full agreement with the flaring-out condition.

![Image](FIG. 3: The plot depicts \(w_r(r_0)\).

**Asymptotically flat wormholes**

In order to have the asymptotically flat wormholes, we should have \(U(r) \to 1\) and \(b(r)/r \to 0\) at the \(r \to \infty\)
order to have positive radial pressure at radius \( r > R \) one finds that, independent of the value of \( c \), if \( n < 0 \) and \( v = 1 \), then the asymptotically flat condition is preserved by the redshift function. Moreover, since \( p_r(r_0) < 0 \), in order to have positive radial pressure at radius \( r > R \), the radial pressure should satisfy the \( p_r(R) = 0 \) condition leading to

\[
c = \frac{b(R)R^{-n-1}}{n(1 - \frac{b(R)}{R})},
\]

for \( c \). Based on this result, since \( n < 0 \), the positive radial pressure is accessible only for negative \( c \), and thus, we only consider this case. In addition, this equation shows that, at the \( R \to r_0 \) limit, if \( b(R) = r_0 + \epsilon \), then we have \( c \approx \frac{1}{n+1} \left[ 1 - (n+1)\frac{b}{R} \right] \). Therefore, choosing very small values for \( \epsilon \), one can miniaturize the radial interval wherein the radial pressure is negative, but it is impossible to eliminate this interval by putting \( c = 0 \) leading to \( c \to -\infty \) which is meaningless. This result is fully compatible with the flaring-out condition. In Fig. (4), density and pressure components as well as the weak, null and strong energy conditions have been plotted as the functions of \( r \) for \( n = -2, c = -10 \) and \( r_0 = 1 \). Choosing smaller values for \( r_0 \) and \( c \), one can minimize the interval in which the \( p + p_r \geq 0 \) condition is violated. It is also useful to note that due to the flaring-out condition \( (b'(r_0) - 1 < 0) \) as well as the \( b(r_0) = r_0 \) condition, leading to \( p_r(r_0) < 0 \), this interval cannot completely be disappeared. In addition, the \( p + p_r \geq 0 \) condition starts to be satisfied for \( r \geq \tilde{R} \), where \( \tilde{R} \) is the radius for which \( \rho + p_r = 0 \). This yields the

\[
c = \frac{\tilde{b}(\tilde{R})\tilde{R}^{-n-1} - \tilde{b}'(\tilde{R})\tilde{R}^{-n}}{n(1 - \frac{\tilde{b}(\tilde{R})}{\tilde{R}})},
\]

relation. Comparing this result with Eq. (11), one can obtain \( R \neq \tilde{R} \) consistent with Fig. (4). It is also interesting to mention here that if we considered the \( b(r) = r_0 \) case, leading to \( \rho = b'(r) = 0 \) (or equally \( b'(\tilde{R}) = 0 \)), then we faced \( R = \tilde{R} \) meaning that both the \( \rho + p_r \geq 0 \) and \( p_r \geq 0 \) conditions start to be met from the same radius.

Finally, it is useful to note that, unlike radial pressure, the energy density of wormholes studied in Secs. (III A) and (III B) is positive everywhere. This result is in full agreement with the confined positive mass obtained in these sections meaning that the mass content of the total energy-momentum source supporting such geometries is positive and confined. We also found out that the energy conditions are not met by the obtained wormholes.

**C. Isotropic wormholes**

Before addressing some isotropic traversable wormholes, we study the behavior of anisotropy parameter of the case studied in Sec. (III B). The anisotropy parameter is plotted as a function of radius for some values of \( r_0 \), when \( n = -2 \) and \( c = -10 \), in Fig. (4). As it is obvious from both Eq. (9) and Fig. (4), the anisotropy is vanished at the \( r \to \infty \) limit. Moreover, it is also apparent that there is a transition from a repulsive geometry to an attractive geometry in both studied cases.

From Eq. (4), in order to obtain the isotropic solutions \( (p_t = p_r) \), we should have \( \Delta = 0 \) that yields

\[
\frac{U'(r)}{2U(r)} = -\frac{p_r'}{(\rho + p_r)}.
\]

One may use this equation to find the redshift function, if the radial state parameter is identified. Generally, one needs to know the \( p_r(\rho) \) relation for solving this equation. Here, we only consider the \( w_r = constant \) case. For a fluid of constant \( w_r \), this equation leads to

\[
U(r) = U_0 \rho \frac{2w_r}{1 + w},
\]

and thus

\[
U(r) = U_0 \left( r_0 \left[ 1 - \tanh^2 \left( \frac{r}{r_0} \right) \right] \frac{2w_r}{1 + w} \right),
\]

where we have used Eq. (9) to obtain this equation. It is easy to check that Eq. (11) is also valid in this case meaning that we have \( w_r = -\frac{r_0 \tanh^2 (r_0)}{r_0 \tanh (r_0) - 1} < -1 \) for \( r_0 > 0 \). Therefore, independent of the value of \( r_0 \), we have
FIG. 5: The $\Delta$ parameter as a function of $r$ for some values of $r_0$, whenever $n = -2$, $c = -10$. In accordance with the asymptotically flat behavior of metric, we have $\Delta \rightarrow 0$ for $r \rightarrow \infty$.

$U(r \rightarrow \infty) \rightarrow \infty$ meaning that these isotropic traversable wormholes are not asymptotically flat.

Bearing Eq. (15) in mind, it is obvious that solutions with $U(r) = constant$ may be obtained if we have either $w_\rho = 0$ or $p_r = \frac{r}{w_\rho} = p_0 \equiv constant$. Based on the above argument, and in agreement with the flaring-out condition, the $w_\rho = 0$ case does not lead to traversable wormholes and thus it is forbidden. For the second case, using Eq. (2), one can easily get $b(r) = -p_0 r^3$ which does not clearly respect the $b(r) < r$ condition for $r > r_0$. In addition, since the throat condition ($b(r_0) = 0$) implies $p_0 = \frac{r_0}{r_0}$, it is apparent that this case does not respect the flaring-out condition. Therefore, this case does not lead to wormhole, a result in agreement with our study in Sec. (III A). Further details on isotropic wormholes in the GR framework can be found in Ref. [100].

D. Hyperbolic wormholes in the presence of a dark energy-like source

Dark energy, as a controversial problem in physics, may be modeled by an ideal prefect fluid with $\rho = -p = constant > 0$. Since the current accelerating universe is dominated by dark energy, we are going to study the effects of this cosmic fluid on solutions found out in Sec. (III B). Moreover, we also point out to the $\rho < 0$ case used to describe an anti de-Sitter universe.

In order to achieve this goal, we consider a universe with $U(r) = v \exp(\epsilon r^n)$ filled by two fluids. In our model, one of them ($\Lambda$ source) has constant energy density and pressure which satisfy the $\rho_\Lambda = -p_\Lambda = \Lambda$ relation, and the other has energy-momentum tensor $T^\mu_\nu = diag(-\rho^e, p_r^e, p_t^e, p^e)$. It is also worth to remind that the asymptotically flat condition implies $n < 0$. In this situation, the Einstein field equations are written as

$$G_{\alpha\beta} = T_{\alpha\beta} = T_{\alpha\beta}^\epsilon - \Lambda g_{\alpha\beta}. \quad (16)$$

Using Eq. (9), one reaches at

$$\rho^\epsilon = \rho - \Lambda = \frac{r_0 [1 - \tanh^2(r)]}{r^2 \tanh(r_0)} - \Lambda, \quad (17)$$

$$p_r^\epsilon = p_r + \Lambda = cnr^n - 2 \frac{b(r)}{r^3} [cnr^n + 1] + \Lambda,$n

$$p_t^\epsilon = p_t + \Lambda = \frac{1}{4r^3 \tanh(r_0)} [(cnr^n)^2 (r \tanh(r_0)) - r_0 \tanh(r)] + (1 - 2n) \tanh(r) + r(2n \tanh(r_0) - r_0)) + 2r_0 (r \tanh(r) - 1 + \tanh(r)) + \Lambda.$$

Therefore, at the wormhole throat, we have

$$w_r^\epsilon(r_0) = \frac{p_r^\epsilon(r_0)}{\rho^\epsilon} = \frac{\Lambda - \frac{1}{r_0 \tanh(r_0)}}{1 - \tanh^2(r_0)} - \Lambda. \quad (18)$$

which implies that $\rho^\epsilon(r_0) < 0$, $p_r^\epsilon(r_0) \leq 0$ and $w_r^\epsilon(r_0) \geq 0$ whenever $\frac{1 - \tanh^2(r_0)}{r_0 \tanh(r_0)} < \Lambda \leq \frac{1}{r_0}$. In addition, for this range of the $\Lambda$ parameter, we have $\rho^\epsilon(r) < 0$ meaning that WEC cannot be satisfied by the $T_{\alpha\beta}^\epsilon$ source. We should note that although the energy density of total energy-momentum tensor ($\rho$) is positive everywhere, the energy density of unknown fluid ($\rho^\epsilon$) may be negative depending on the value of $\Lambda$. Moreover, it was found out that unlike the total energy-momentum tensor, for which $w_r(r_0) \leq -1$, one gets $w_r^\epsilon(r_0) \geq 0$ for the unknown fluid. In fact these state parameters are in a mutual relation as

$$w_r(r) = w_r^\epsilon(r)(1 - \frac{\Lambda}{\rho}) + \frac{\Lambda}{\rho}, \quad (19)$$

indicating that, at the $\Lambda \rightarrow 0$ limit, we have $w_r(r) \rightarrow w_r^\epsilon(r)$ compatible with our previous results obtained in Sec. (III B).

It is also obvious that Eq. (9) holds for the total energy-momentum tensor with energy density $\rho$ which is a positive quantity (see Eq. (13)). Moreover, for the mass corresponding to the unknown fluid ($m^\epsilon(r)$), we get
\[ m^2(r) = \int_0^r \rho^2 4\pi r^2 dr = m(r) - \frac{4\pi \Lambda}{3} (r^3 - r_0^3) \]

\[ = \frac{4\pi r_0}{\tanh(r_0)} [\tanh(r) - \tanh(r_0)] - \frac{4\pi \Lambda}{3} (r^3 - r_0^3), \quad (20) \]

plotted for some values of \( \Lambda \) and \( r_0 \) in Fig. 6. As it is obvious from both Fig. 6 and Eq. (20), for distances away from the wormhole throat \((r \gg r_0)\), we have \( m^2(r) \approx -\frac{4\pi \Lambda}{3} r^3 \) which is negative (positive) for \( \Lambda > 0 \) (\( \Lambda < 0 \)). The latter is due to this fact that at the \( r \to \infty \) limit, we have \( m \simeq C \) in accordance with the \( b(r \to \infty) \simeq C' \) behavior, where \( C \) and \( C' \) are constant. Since the mass corresponding to the \( \Lambda \) source \((m^\Lambda(r) = \frac{4\pi \Lambda}{3} (r^3 - r_0^3))\) increases with increasing radius, at first glance, one could have expected the total mass \((m(r))\) to be divergent. But, Eq. (20) shows the unknown fluid cancels the effect of \( \Lambda \) on the total mass, indicating that the total mass satisfies Eq. (15). In fact, this result can also be obtained using Eq. (17) which implies \( m^2(r) = m(r) - m^\Lambda(r) \). Therefore, the infinity observer gets a positive and bounded mass for the wormholes embedded in a spacetime filled by a source as \( \rho_\Lambda = -p_\Lambda = \Lambda \) together with an unknown fluid which meets Eqs. (17) and (20).

Eq. (17) yields \( \rho^x + p^x_r = \rho + p_r \), meaning that \( \Lambda \) does not affect the \( \rho + p_r \) term, and hence, the result obtained in Eq. (12) is also valid here. In fact, this is the direct result of the \( \rho_\Lambda + p_\Lambda = 0 \) property of the \( \Lambda \) source. In addition, due to the isotropy of the \( \Lambda \) source, one can expect \( \Delta = p_t - p_r = p^x_t - p^x_r \) confirmed by using Eq. (17). Additionally, the radial pressure of the unknown fluid gets its positive values for \( r \geq R_1 \), where \( R_1 \) is evaluated from the \( p^x_r(R_1) = 0 \) condition leading to

\[ c = \frac{b(R_1) R_1^{-n-1} - \Lambda R_1^{3-n}}{n(1 - b(R_1)/R_1)}. \quad (21) \]

Comparing this result with Eq. (12), we find out that both the \( \rho^x + p^x_r = \rho + p_r \geq 0 \) and \( p^x_r \geq 0 \) conditions may simultaneously be obtained if we have \( R_1 = \tilde{R} \). In addition, a comparison between this result and Eq. (11) helps us in obtaining the effect of \( \Lambda \) on the interval for which \( p^x_r(r) < 0 \). Indeed, one can use Eq. (14) in order to see that, unlike the \( \rho + p_r \) expression, \( \Lambda \) directly affects the intervals for which \( p^x_r(r) < 0 \) and \( \rho^x_r(r) < 0 \). From Eq. (17), it is also obvious that \( \rho^x(r) \) is positive whenever \( \Lambda \) is negative. Moreover, if \( \rho(r_0) < \Lambda \), then we have \( \rho^x(r) < 0 \) for \( r \geq r_0 \). Finally, for the \( 0 < \Lambda < \rho(r_0) \) case, the energy density of unknown fluid is negative if \( r > \mathcal{R} \) in which \( \mathcal{R} \) is evaluated from the \( \rho^x(\mathcal{R}) = 0 \) condition as

\[ 1 - \frac{\tanh^2(\mathcal{R})}{\mathcal{R}^2} = \frac{\Lambda \tanh(r_0)}{r_0}. \quad (22) \]

Using the above results as well as Eqs. (12), (17) and (21), one obtains

\[ \frac{1 - \tanh^2(R_1)}{R_1^2} = \frac{\Lambda \tanh(r_0)}{r_0}. \quad (23) \]

This equation establishes a relation between the radius of wormhole throat, \( \Lambda \) and \( R_1 \), and therefore, in order to have a real solution for \( R_1 \), \( \Lambda \) should be positive. Therefore, the wormhole throat affects the value of \( R_1 \) and thus the interval in which both the radial pressure \( (p^x_r) \) and the \( \rho^x + p^x_r \) quantity are negative. One can also combine Eqs. (22) and (23) with each other to get a mutual relation between \( \mathcal{R} \) and \( R_1 \) whenever \( 0 < \Lambda < \rho(r_0) \).

This manner, if \( \mathcal{R} = R_1 \), which is available only for \( \mathcal{R} = R_1 = 1 \), then we have \( \rho^x + p^x_r = \rho + p_r \geq 0, p^x_r \geq 0 \) and \( p^x(r) < 0 \) for \( r > 1 \). As we saw, the existence of \( \Lambda \) does not solve the difficulty of violating energy conditions by obtained solutions, and it may at most affect the rate of behavior of solutions. Hence, we do not plot energy conditions in the presence of \( \Lambda \) in order to avoid prolongation of the article.

In summary, as we have seen in Sec. (III B), the total energy-momentum tensor of obtained solutions does not respect the energy conditions. Our study in this section also shows that the problem of violation of energy conditions by traversable wormholes cannot be solved by adding the \( \Lambda \) term. In fact, although the total mass of obtained solutions is positive, the behavior of assumed unknown source does not represent a baryonic source.
IV. LYRA MANIFOLD AND EINSTEIN FIELD EQUATIONS

Introducing the gauge function $H(x^\mu)$ into the structureless manifold, and defining the displacement vector between two neighboring points as $Q(x^\nu + dx^\nu) - q(x^\nu) = H(x^\mu)dx^\mu$, Lyra proposed a generalization to the Riemannian geometry in which metric is invariant under the gauge and coordinate transformations [61, 62]. Full details on this geometry and its relation with Weyl geometry can be found in [61, 63].

The Einstein gravitational field equations in the ordinary gauge of the Lyra manifold, a generalization of the Riemannian geometry [61], are written as [62, 67]

$$ G_{\mu\nu} = G_{\mu\nu} + \tilde{G}_{\mu\nu} = T_{\mu\nu}, $$

where

$$ \tilde{G}_{\mu\nu} = \frac{3}{3} \phi_\mu \phi_\nu - \frac{3}{4} \phi_\alpha \phi_\alpha g_{\mu\nu}, $$

and we considered $8\pi G = 1 = c$. Here, $\phi_\mu$ denotes the Lyra displacement vector field which is the direct result of introducing the gauge function $H(x^\mu)$ into the structureless manifold [61, 62, 65]. It is also useful to mention that for spacetimes with diagonal $G_{\mu\nu}$ supported by a fluid with energy-momentum tensor $T_{\mu\nu} = diag(-\rho, p_r, p_t, p_t)$, $\tilde{G}_{\mu\nu}$ should be diagonal meaning that only one component of the vector field $\phi_\mu$ may not be zero, and its three other components must be zero. It has been shown that the $\phi_\nu = (constant, 0, 0, 0)$ vector field may play the role of cosmological constant [63]. The more general case of $\phi_\nu = (\alpha(t), 0, 0, 0)$ has also interesting results in the cosmological studies [64, 67]. Recently, it has been addressed that if one considers a suitable Lyra displacement vector field, then some dynamics wormholes may satisfy Eq. (21) in the FRW background [77]. In addition, some exact solutions for the field equations in a Lyra manifold of $\phi_\nu = (\alpha, 0, 0, 0)$, where $\alpha$ is either a constant or a function of $t$, have also been introduced which do not include any wormhole [68, 69]. In fact, there is no study on properties of a static traversable wormhole in the Lyra geometry, and in most cases, physicists have focused on the cosmological features of this theory [62, 77].

As we have previously mentioned, the energy conditions are not met by a traversable wormhole in GR [8]. In previous section, we derived some new traversable wormholes with bounded positive total mass. Thereinafter, we studied the effects of a dark energy like source on the obtained solutions. In summary, our investigation confirms that, in GR, the energy conditions are violated by traversable wormholes, and a dark energy like source cannot solve the problem of the energy condition violation.

Now, since WEC, NEC and DEC are independent of the gravitational theory used to study the system, they lead to the same expression for the energy density and pressure components as those obtained in Eq. (8). In order to find SEC in the Lyra manifold, bearing the attractive nature of gravity in mind, we use the Frobenius's theorem [53] in order to find

$$ SECL \rightarrow \rho + p_r + 2p_t \geq -3\phi_0 \phi^0. \quad (26) $$

From now on, by SECL we mean the strong energy condition in the Lyra manifold.

Here, we also address some general remarks about the relation between the anisotropy parameter and the Lyra displacement vector. Using Eqs. (1), (24) and (25), one easily finds

$$ \Delta = G_2^2 - G_1^2 = (G_2^2 + \tilde{G}_2^2) - (G_1^2 + \tilde{G}_1^2) = (G_2^2 - G_1^2) + (\tilde{G}_2^2 - \tilde{G}_1^2) = \Delta_E + \Delta_\phi, \quad (27) $$

where $\Delta_E \equiv G_2^2 - G_1^2$ is the anisotropy parameter corresponding to the Einstein tensor, and $\Delta_\phi \equiv \frac{1}{2}[\phi_2 \phi^2 - \phi_1 \phi^1]$ is due to the Lyra displacement vector field. Therefore, for attractive geometries $\Delta_E < -\Delta_\phi$, and in repulsive geometries $\Delta_E > -\Delta_\phi$. It is also useful to note that whenever $\Delta_E = 0$, geometry is attractive (repulsive) if $\phi_2 \phi^2 < \phi_1 \phi^1$ ($\phi_2 \phi^2 > \phi_1 \phi^1$).

A. Asymptotically flat wormholes in Lyra manifold

Considering $T_{\mu\nu} = diag(-\rho, p_r, p_t, p_t)$ and by inserting metric (11) into Eq. (24), we reach at

$$ \rho = \frac{b(r)}{r^2} - \frac{3}{2} \phi_0 \phi^0 + \frac{3}{4} \phi_\alpha \phi^\alpha, \quad (28) $$
$$ p_r = \frac{U^2(r)}{rU(r)} \left[ (G_1^2)' + (G_1^2 - G_0^2) \frac{U^2(r)}{2U(r)} \right] + \frac{3}{2} \phi_2 \phi^2 - \frac{3}{4} \phi_\alpha \phi^\alpha, $$
$$ p_\theta = G_1^2 + \frac{r}{2} \left[ (G_1^2)' + (G_1^2 - G_0^2) \frac{U^2(r)}{2U(r)} \right] $$
$$ + \frac{3}{2} \phi_2 \phi^2 - \frac{3}{4} \phi_\alpha \phi^\alpha, $$
$$ p_\phi = G_1^2 + \frac{r}{2} \left[ (G_1^2)' + (G_1^2 - G_0^2) \frac{U^2(r)}{2U(r)} \right] $$
$$ + \frac{3}{2} \phi_2 \phi^2 - \frac{3}{4} \phi_\alpha \phi^\alpha, $$

for the energy density and pressure components of the supporter fluid. It is also useful to mention here that, due to the spherically symmetric static nature of metric (11) implying, $p_\theta = p_\phi \equiv p_t, \phi_2 \phi^2 = \phi_3 \phi^3 \equiv \gamma$, and $\phi_\mu$ should either be constant or a function of radius. Therefore, the above equations are written as
\[
\rho = \frac{b'(r)}{r^2} - \frac{3}{2} \rho_0 \phi^0 + \frac{3}{4} \phi_0 \phi^\alpha, \\
p_r = \frac{U'(r)}{rU(r)} (1 - \frac{b(r)}{r}) - \frac{b'(r)}{r^3} + \frac{3}{2} \phi_1 \phi^1 - \frac{3}{4} \phi_0 \phi^\alpha, \\
p_t = G_1' + \frac{r}{2} \left[ (G_1')' + (G_1' - G_3) \frac{U'(r)}{2U(r)} \right] + \frac{3}{2} \gamma - \frac{3}{4} \phi_0 \phi^\alpha. 
\]

As we saw in previous sections, radial pressure is negative at the wormhole throat in GR. Now, based on these equations, one can obtain that if \((\frac{3}{2} \phi_1 \phi^1 - \frac{3}{4} \phi_0 \phi^\alpha)_{r=r_0} \geq \frac{1}{r_0^2}\), then we have wormholes with \(p(r_0) \geq 0\). In this manner, if \(\rho(r_0) > 0\), then, unlike GR, the \(\rho + p_r > 0\) condition is met at the wormhole throat \((r = r_0)\). Additionally, as we have previously mentioned, since the Einstein and energy-momentum tensors are diagonal here, at least three components of the Lyra displacement vector field should be zero, a point will be considered later.

Finally, using Eq. (29), we easily reach at

\[
m(r) = \int_{r_0}^r \frac{b'(r)}{r^2} - \frac{3}{2} \phi_0 \phi^0 + \frac{3}{4} \phi_0 \phi^\alpha|4\pi r^2 dr \\
= 4\pi (b(r) - b(r_0) - \frac{3}{2} \int_{r_0}^r [\phi_0 \phi^0 - \frac{1}{2} \phi_0 \phi^\alpha]r^2 dr), 
\]

for the mass function.

In the following subsections, we find several traversable wormholes and study some physical and mathematical properties of the corresponding energy-momentum source in the Lyndorf manifold.

**B. The \(U(r) = 1\) case**

As the first example, let us consider the time-like Lyra displacement vector field of \(\phi_\mu = \beta \delta_\mu^t\) leading to \(\phi_\alpha \phi^\alpha = \phi_0 \phi^0 = -\beta^2\) and thus

\[
\rho = \frac{b'(r)}{r^2} + \frac{3}{4} \beta^2, \\
p_r = -\frac{b(r)}{r^3} + \frac{3}{4} \beta^2, \\
p_t = \frac{b(r) - rb'(r)}{2r^3} + \frac{3}{4} \beta^2, 
\]

for the energy-momentum tensor. In obtaining these results, one should remember that for \(U(r) = 1\), we have \(G_1' = -\frac{b(r)}{r}\) (see also Eq. (2)). Inserting these equations into Eq. (29), one can easily obtain \(\rho + p_t + 2p_r + 3\phi_0 \phi^0 = 0\) meaning that SECL is met. In order to simplify our calculations, we define the transverse state parameter as \(w_t = \frac{2}{3} \rho\). Now, bearing the radial and transverse state parameters in mind, and using the above equations, we reach at

\[
\beta^2 = \frac{4}{3r^3(1 - w_r)}[w_r rb'(r) + b(r)], \\
\rho = \frac{1}{r^3(1 - w_r)}[rb'(r) + b(r)], \\
p_r = \frac{w_r}{r^3(1 - w_r)}[rb'(r) + b(r)], \\
p_t = \frac{1}{2r^3(1 - w_r)}[(3w_r - 1)rb'(r) + (3 - w_r)b(r)], \\
w_t = \frac{1}{2}[\frac{3w_r rb'(r) + (4 - w_r)b(r)}{rb'(r) + b(r)} - 1]. 
\]

The isotropic solutions are obtainable by applying the \(w_r = w_t\) condition to the above equations leading to \(b(r) = b_0 r^3\), where \(b_0\) is the integration constant, which does not preserve the \(b(r) < r\) requirement for \(r > r_0\).

It is also apparent that this solution does not respect both the \(b(r_0) = r_0\) and the flaring-out conditions simultaneously, meaning that this isotropic solution is not a traversable wormhole.

Now, consider the \(b(r) = \frac{r_0}{\tanh(r_0)} \tanh(r)\) case which has extensively been studied in the GR framework in Sec. (III A). The evolution of the non-zero energy-momentum components as functions of \(r\) have been plotted for \(w_r = \frac{1}{3}\) and \(r_0 = 1\) in Fig. (7). As it is obvious from both Eq. (31) and Fig. (7), at the \(r \rightarrow \infty\) limit, the anisotropy of this repulsive geometry \((\Delta > 0)\) is disappeared which is fully compatible with the asymptotically flat behavior of the metric. It is also apparent that this solution does not meet DEC everywhere. Here, \(p_t\) is positive and if we have a fluid with \(0 \leq w_r < 1\), then the radial pressure is positive. Moreover, since \(r \geq r_0\), we have \(\beta^2 = \frac{2r_0}{3r \tanh(r_0)}[r(1 - \tanh^2(r)) + 3 \tanh(r)] > 0\) and thus \(\beta\) is a real quantity. Let us focus on the \(-1 < w_r < 1\) case for which the \(\rho + p_t > 0\) condition is automatically satisfied in our example (because \(\rho > 0\)). Now, one can check that if the radial state parameter meets the \(-1 < w_r < 1\) condition, then a traversable wormhole with \(b(r) = \frac{r_0}{\tanh(r_0)} \tanh(r)\) and \(U(r) = 1\) satisfies NEC and WEC expressed in Eq. (8). In fact, only DEC is not preserved by these solutions.

For the second example, considering the space-like Lyra displacement vector field of \(\phi_\mu = \phi_\phi^\mu\) which leads to \(\phi_\alpha \phi^\alpha = \phi_1 \phi^1 = \phi^2(1 - \frac{b(r)}{r})\), and following the recipe which yielded Eq. (32), one finds
apply the Lyra displacement vector, we finally yield

\[ \rho = \frac{b(r)}{r^3(2w + 1)}, \quad p = \frac{w b(r)}{r^3(2w + 1)}, \]

where \( p = p_t = p_r \) is the isotropic pressure. In order to evaluate the mass function of the obtained solution, one can use Eq. (30) to get

\[ m(r) = \frac{4\pi r_0}{3w} \left[ 1 - \left( \frac{r_0}{r} \right)^{\frac{3w}{2w+1}} \right], \]

for \( w \neq 0 \) and therefore, mass is positive and bounded for these solutions if we have \( w > 0 \). For the \( w = 0 \) case, simple calculation leads to \( m(r) = 4\pi r_0 \ln(\frac{r}{r_0}) \) meaning that mass increases as a function of \( r \), and therefore, it is a solution with unbounded mass. In fact, for \( w < -\frac{1}{2} \) and \( 0 < w \), mass is bounded (or equally \( m \to \frac{4\pi r_0}{3w} \) at the \( r \to \infty \) limit), and it is unbounded for \( -\frac{1}{2} < w < 0 \). Combining these results with Eq. (36), one can easily see that fluids with state parameter meeting the above condition satisfy energy conditions expressed in Eq. (3), and therefore, the energy conditions are satisfied by the baryonic energy-momentum source (or equally \( 0 \leq w \leq \frac{1}{2} \)).

In summary, we obtained that if a suitable space-like Lyra displacement vector is chosen, then the isotropic baryonic sources can theoretically support the traversable asymptotically flat wormholes in the Lyra manifold.

### C. The exponential redshift function

Here, we focus on the \( U(r) = \exp(cr^n) \) case, which led to interesting results in Sec. (III B). Moreover, since this case is a generalization of Sec. (IV B), we only consider the time-like Lyra displacement vector fields. As a reminder, we found that, in GR framework (III B), \( n < 0 \) leads to asymptotically flat wormholes that for them radial pressure may get positive values only if \( c < 0 \) (see Eq. (11)). Therefore, we only consider the \( n, c < 0 \) case. Considering the time-like vector of \( \phi_\mu = \Sigma b_\mu \) and using (29), one obtains
\[ \rho = \frac{p_r}{w_r} = \frac{1}{r^3(1 - w_r)}[r b'(r) + b(r) - cnr^{n+1}(1 - \frac{b(r)}{r})], \]
\[ w_t = \frac{w_r - 1}{4(rb'(r) + b(r) - cnr^{n+1}(1 - \frac{b(r)}{r}))} \]
\[ \times (2 + cnr^{n}) - cn^2r^{n+1}(1 - \frac{b(r)}{r})(2 + cnr^{n}) + 2b(r)(cnr^{n-1} - 1) + \frac{4}{w_r - 1}[w_r b'(r) + b(r) - cnr^{n+1}(1 - \frac{b(r)}{r})]], \]

whenever \( \Sigma^2 = \frac{4}{3(1 - w_r)\exp(cn^2)}[\frac{w_r b'(r)}{r^2} + \frac{b(r)}{r^2}(1 + cnr^{n}) - cnr^{n-2}]. \) Inserting \( w_t = w_r = w \) in the above equation to find the isotropic solutions, one reaches at \( w = 1 \) which is the state parameter of stiff matter. Now, let us focus on the \( b(r) = \frac{r_0 \tanh(r_0)}{r} \) case. In this manner, although we have \( \rho > 0 \) and \( p_r > 0 \) for \( 0 \leq w_r < 1 \), the \( \rho + p_t \geq 0 \) condition is not satisfied for all values of \( r \) bigger than of \( r_0 \).

For another example, considering the \( \phi_\mu = \Theta \delta^\mu_0 \) case, where \( \Theta^2 = \frac{4(\tau)^2}{(\tau + 1)^2} \), we can use Eq. (29) in order to get
\[ \rho = \frac{r_0(1 - \tanh^2(r))}{r^2 \tanh(r_0)} + \frac{r_0 \tanh(r)}{\tanh(r_0)r^3}, \]
\[ p_r = \frac{1}{4r^3 \tanh(r_0)}[(cnr^{n})^2(r \tanh(r_0) - r_0 \tanh(r)) + cnr^{n}(r_0(r \tanh^2(r) + (1 - 2n) \tanh(r))]
\]
\[ + r(2n \tanh(r_0) - r_0) \tanh(r) + 2r_0(r \tanh^2(r) - 1)] + \frac{r_0 \tanh(r)}{\tanh(r_0)r^3}. \]

As it is obvious, the radial pressure and energy density are positive quantities for \( r \geq r_0 \). Moreover, comparing these results with those obtained in Eq. (29), one can easily obtain that \( p_t \) is also positive for \( r \geq r_0 \). Therefore, while WEC and NEC are satisfied by this solution, DEC is not met everywhere. In addition, it is easy to check that SECL is also satisfied by this case. These results are similar to those obtained in Sec. IV.B for time-like case.

\[ G_{\mu\nu} = \frac{3}{4} \phi_\mu \phi^\alpha g_{\mu\nu} - \frac{3}{2} \phi_\mu \phi_\nu, \]
\[ R = -\frac{3}{2} \phi_\mu \phi^\alpha. \]

where \( R \) is the Ricci scalar. From this equation, it is obvious that the \( \phi_\mu = 0 \) case, yielding \( G_{\mu\nu} = 0 \), does not lead to a traversable wormhole. One can also abridge the above equations as
\[ R_{\mu\nu} = -\frac{3}{2} \phi_\mu \phi_\nu. \]

Here, \( R_{\mu\nu} \) denotes the Ricci tensor. Therefore, for spacetimes in which the Ricci tensor is diagonal (or equally \( R_{\mu\nu} \propto \delta_{\mu\nu} \)), only one component of the \( \phi_\mu \phi_\nu \) tensor can be non-zero. It means that only one component of \( R_{\mu\nu} \) can be non-zero unless we have \( \phi_\mu = 0 \) leading to \( R_{\mu\nu} = 0 \) and thus \( G_{\mu\nu} = 0 \).

For an observer with four velocity \( v_\mu \), WEC is defined as \( T_{\mu\nu}v^\mu v^\nu \geq 0 \) [53]. Now, using Eq. (24), one reaches at \( G_{\mu\nu}v^\mu v^\nu \geq 0 \) for WEC. In fact, it is the geometrical interpretation for WEC in the Lyra manifold. For an empty Lyra manifold, where \( G_{\mu\nu} = T_{\mu\nu} = 0 \), we have \( G_{\mu\nu}v^\mu v^\nu = 0 \) meaning that WEC is marginally satisfied in an empty Lyra manifold. One can use the definitions of NEC and DEC [53] to check that similar argument is also valid for them in an empty Lyra manifold. Therefore, independent of the Lyra displacement vector field, WEC, NEC and DEC are marginally satisfied in an empty Lyra manifold. Finally, bearing Eq. (24) in mind, we easily find that SECL is met if \( 3\phi_0 \phi_0^\beta \geq 0 \).

A. Time-like displacement vector field

In our signature \((-+,+,+,+)\), \( \phi_\mu \) is a time-like vector field if \( \phi_\mu \phi^\mu < 0 \). Therefore, a time-like vector as \( \phi_\mu = \zeta(r) \delta_\mu_0 \) should respect the \( \phi_\mu \phi^\mu = -\frac{\zeta^2(r)}{\zeta'(r)} < 0 \) condition. Now, considering \( \phi_\mu = \zeta(r) \delta_\mu_0 \) and using metric (1) as well as Eq. (11), we obtain
\[ \zeta(r) = \sqrt{\frac{2R_0^0 U(r)}{3}}, \]
\[ \frac{R_0^0}{4U'(r)^2}[-2rU(r)U''(r)(r - b(r)) \]
\[ +(rU'(r))^2(1 - \frac{b(r)}{r}) + U(r)U'(r)(3b(r) + rb'(r) - 4r)] \]
In addition, from Eq. (10), one also finds
\[ \zeta(r) = \sqrt{\frac{2RU(r)}{3}}, \]
equal with Eq. (12), only if we have \( R = R_0 \) and thus 
\( R_1 = R_2 = R_3 = 0 \). The \( R_2 = R_3 = 0 \) condition leads to

\[
\frac{U'(r)}{U(r)} = \frac{b(r) + rb'(r)}{r^2 - rb(r)},
\]

(45)

which can finally be written as

\[
\frac{U''(r)}{U(r)} = \frac{r^2(1 - b(r))b''(r) - 2(r + b'(r))b(r)}{r(r - b(r))^2}.
\]

(46)

Now, inserting Eqs. (15) and (16) into the \( R_1 = 0 \) condition, one can reach an equation for \( b(r) \) as

\[
b''(r) = \frac{rb''(r)(2 + b'(r)) + b(r)(2b(r) + rb'(r))}{r^3(1 - \frac{b(r)}{r})}.
\]

(47)

Therefore, if the shape function of a traversable wormhole satisfies this equation, then its redshift function may be evaluated from Eq. (15). In this situation, the time-like Lyra displacement vector, supporting the traversable wormhole in the empty Lyra geometry, meets Eq. (12). Inserting Eq. (17) as well as Eqs. (16) and (15) into Eq. (13), we finally reach at

\[
R_0^0 = \frac{-2b'(r)(1 - \frac{b(r)}{r})}{r^2(1 - \frac{b(r)}{r})}.
\]

(48)

Indeed, based on the above equations, if one of the \( b(r) \), \( U(r) \) and \( \zeta(r) \) functions is known, then the two remaining functions may be evaluated. The only important thing is that Eqs. (14) and (17) should be respected by the solutions. It is also useful to remind here that, in order to have a traversable wormhole, the obtained solutions for \( b(r) \) and \( U(r) \) should meet the requirements of a traversable wormhole expressed in Sec. (11).

As we have previously claimed, SECL is met if \( 3\phi \phi^0 \geq 0 \) which yields the condition \( \zeta^2(r) \leq 0 \). On the other hand, we have \( \zeta^2(r) > 0 \) for a time-like Lyra displacement vector field. Therefore, a time-like displacement vector cannot support the obtained geometries and SECL in an empty Lyra manifold simultaneously. In summary, if we consider the mentioned time-like vector field and use the above recipe to find some traversable wormholes, solutions will not respect SECL.

Considering \( U(r) = 1 \), and by using Eq. (15), we get

\[
b(r) = \frac{r^2}{\phi},
\]

and thus \( \zeta(r) = \sqrt{\frac{8n^2(r^2 - \frac{r^2}{\phi})}{3(r^2 - \frac{r^2}{\phi})}} \). In order to obtain the shape function, the \( b(r_0) = r_0 \) has also been used, and it is obvious that it meets the flaring-out condition. Moreover, for \( r \geq \sqrt{\frac{r_0}{\phi}} \) we have \( \zeta^2(r) > 0 \) and thus SECL is not met. We should also note that SECL is met (\( \zeta^2(r) < 0 \)) for \( r_0 \leq r < \sqrt{\frac{r_0}{\phi}} \), but in this manner, the Lyra displacement vector field is imaginary. Although the physical meaning of an imaginary field is not clear [65], some physical consequences of this field and its mathematical properties have been studied, found in Ref. [65] and references therein.

Now, let us consider the \( U(r) = \exp(c r^n) \) case. Using Eq. (15), we get

\[
b(r) = \frac{r^2}{\phi} \exp(-c r^n - r_0^n)\]

and thus \( \zeta(r) = \sqrt{\frac{8n^2(r^2 - \frac{r^2}{\phi})}{3(r^2 - \frac{r^2}{\phi})}} \), where the \( b(r_0) = r_0 \) condition have also been applied to the solution. The flaring-out condition is also met if we have \( c \geq -\frac{r_0^n + 1}{\sqrt{n}} \) leading to this fact that the \( n, c < 0 \) case does not respect the flaring-out condition. Moreover, the \( b(r) < r \) condition, for \( r > r_0 \), is available if we have \( n, c > 0 \) or \( c < 0 \) for \( n < 0 \) meaning that \( b(r) \) is a decreasing function of \( r \) with a maximum located at \( r = r_0 \). In addition, whenever \( r^2 \exp(c r^n) \geq \frac{5r^2}{4} \exp(c r_0^n) \), we have \( \zeta(r) \geq 0 \) and as a result, SECL is not preserved.

In summary, we found out that, in an empty Lyra manifold, a real time-like vector field, parallel to the \( R_0^0 > 0 \) case, cannot lead to a traversable wormhole which satisfies SECL.

### B. Space-like displacement vector fields

For the space-like displacement vector fields, whenever \( G_{\mu\nu} \) and \( T_{\mu\nu} \) are diagonal, only one spatial component of the Lyra displacement vector field is non-zero, and moreover, since \( \phi_0 = 0 \), SECL is automatically preserved.

#### Transverse space-like vector fields

For the \( \phi_\mu = \phi_\delta^0 \) case, on one hand, following the recipe of the previous subsection, we get the \( R_3^3 = 0 \) and \( \phi = \sqrt{-2\gamma R_3^3} \) conditions. On the other hand, due to the spherical symmetry of the assumed static metric (11), \( R_2^2 = R_3^3 \) and \( \phi \) should either be a constant or a function of \( r \). Therefore, we should have \( \phi(r) = 0 \) meaning that such vector field does not exist. Similar argument is also available for the \( \omega_\mu = \omega_\delta^0 \) case and thus, a Lyra manifold with \( \omega_\mu = \omega_\delta^0 \), where \( \omega \) is a constant or a function of \( r \), cannot satisfy the field equations.

#### Radial Space-like vector field

Now, let us consider the \( \phi_\mu = \xi(r)\delta_\mu^0 \) case. From Eq. (11), we have

\[
\xi(r) = \sqrt{-\frac{2R_1^1}{3(1 - \frac{b(r)}{r})}} = \sqrt{-\frac{2R}{3(1 - \frac{b(r)}{r})}},
\]

(49)

which is true only if \( R_0^0 = R_2^2 = R_3^3 = 0 \). The \( R_2^2 = R_3^3 = 0 \) condition leads again to Eqs. (13) and (14) combined with \( R_0^0 = 0 \) to reach at
\[ b''(r) = \frac{1}{r^3(r-b(r))}[b'(r)(r^2(b'(r) - 2) + b(r)(3r - 1)) + b(r)(b(r - 2r)). \] (50)

Finally, it is a matter of calculation to show
\[
\frac{U''(r)}{U(r)} = \frac{1}{r^3(r-b(r))^2}[b'(r)(r^2(b'(r) - 2) + b(r)(3r - 1)) + b(r)(b(r - 2r(1 + r))].
\]

In fact, the above results help us in finding proper \( \xi(r) \). Therefore, once the shape function is known, we can use the above equations to find the redshift function and the Lyra displacement vector field. Theoretically, if one of the \( \xi(r) \), \( b(r) \) or \( U(r) \) functions is known, then one can use the above equations in order to find the two remaining functions.

As an example, we consider the simple case of \( U(r) = 1 \). We only focus on this simple case because the exponential redshift function leads to boring functions for the shape function which make analysis hard and impossible in some steps. In fact, this simple case is enough to show that SECL can be satisfied by traversable wormholes in an empty Lyra manifold, if a proper radial space-like Lyra displacement vector has been chosen. In this manner, inserting metric \( \Pi \) into Eq. (11), one reaches at
\[
\frac{3}{2} \phi_0 \phi_0^0 = 0,
\]
\[
R_1^1 = -\frac{3}{2} \phi_1 \phi_1^1 = \frac{rb'(r) - b(r)}{r^3},
\]
\[
R_2^2 = R_3^3 = -\frac{3}{2} \phi_2 \phi_2^2 = -\frac{3}{2} \phi_3 \phi_3^3 = \frac{b(r) + rb'(r)}{2r^3}.
\]

It is obvious that a time-like vector as \( \phi_\mu = \xi \delta_\mu^t \), where \( \xi \neq 0 \), cannot satisfy the above equations, a result which is in full agreement with our results obtained in subsection (\textbf{VA}).

The above equations are valid only for \( b(r) = \frac{b_0}{r} \), where \( b_0 \) is the integration constant, a shape function obtained and studied in a dynamics background [71]. The \( b(r_0) = r_0 \) condition lead to \( b(r) = \frac{b(r_0)}{r} \) and thus
\[
\phi_1 \phi_1^1 = -\frac{2}{3} R = \frac{4r_0^2}{3r^4},
\]
\[
\phi_0 = \phi_2 = \phi_3 = 0.
\]

It is also easy to check that the flaring-out condition is available here \( b'(r_0) = -1 < 1 \). Therefore, the vacuum Lyra manifold with a space-like displacement vector filed as \( \phi_\mu = -\frac{2r_0}{r^4} \delta_\mu^t \) may support a traversable asymptotically flat wormhole. Finally, we should remind that since we have \( \phi_0 = 0 \), SECL is automatically satisfied here.

\section{Conclusion}

Firstly, we introduced some new traversable wormholes and studied their physical properties in the GR framework. In addition, the asymptotically flat case as well as the effects of considering a source with energy-momentum similar to the cosmological constant on the obtained solutions have also been addressed.

Thereinafter, providing a summary on the Einstein field equations in the ordinary gauge of the Lyra manifold as a generalization of the Reimannian geometry, we studied some properties of the wormhole’s structure in the Lyra geometry. In Sec. (\textbf{V.B}), we derived some isotropic traversable wormholes which are asymptotically flat and can meet energy conditions if a proper space-like Lyra displacement vector field is chosen. Therefore, our study shows that baryonic matters are allowed to support traversable wormholes and also the asymptotically flat cases in the Lyra manifold.

Finally, considering an empty spacetime, the possibility of having traversable wormholes in the empty Lyra manifold has been investigated. Results show that the empty Lyra manifold may support traversable wormholes and also the asymptotically flat cases. Additionally, focusing on the simple case of \( U(r) = 1 \) in Sec. (\textbf{V.B}), we derived a traversable asymptotically flat wormhole in an empty Lyra manifold for that energy conditions are marginally satisfied if a proper radial space-like Lyra displacement vector is chosen. Therefore, it is theoretically possible to choose a suitable Lyra displacement vector field which lets the empty Lyra manifold support traversable wormholes.

\section*{Acknowledgments}

We are so grateful to the anonymous referee for valuable comments. The work of H. Moradpour has been supported financially by Research Institute for Astronomy & Astrophysics of Maragha (RIAAM) under project No.1/4717-171.

\begin{thebibliography}{99}
\bibitem{1} J. A. Wheeler, Phys. Rev. 97, 511 (1955).
\bibitem{2} J. A. Wheeler, Ann. Phys. 2, 604 (1957).
\bibitem{3} M. S. Morris, K. S. Thorne, U. Yurtsever, Phys. Rev. Lett. 61, 1446 (1988).
\bibitem{4} M. S. Morris, K. S. Thorne, Am. J. Phys. 56, 395 (1988).
\bibitem{5} M. Visser, Phys. Rev. D 39, 3182 (1989).
\bibitem{6} M. Visser, Lorentzian Wormholes, (AIP Press, Woodbury, NY, USA 1996).
\end{thebibliography}
[84] M. K. Mak, T. Harko, Proc. R. Soc. A 459, 393 (2003).
[85] V. V. Usov, Phys. Rev. D 70, 067301 (2004).
[86] C. Cattoen, T. Faber, M. Visser, Class. Quant. Grav. 22 4189 (2005).
[87] F. Rahaman, M. Kalam, K. A. Rahman, Int. J. Theor. Phys. 48 471 (2009).
[88] V. Varela, F. Rahaman, S. Ray, K. Chakraborty, M. Kalam, Phys. Rev. D82 044052 (2010).
[89] V. Varela, F. Rahaman, S. Ray, K. Chakraborty, M. Kalam, Phys. Rev. D 82, 044052 (2010).
[90] F. Rahaman, S. Ray, A. K. Jafry, K. Chakraborty, Phys. Rev. D 82, 104055 (2010).
[91] D. Horvat, S. Ilijic, A. Marunovic, Class. Quant. Grav. 28 025009 (2011).
[92] A. DeBenedictis, Phys. Rev. D 84, 104030 (2011).
[93] F. Rahaman, P. K.F. Kuhfittig, M. Kalam, A. A. Usmani, S. Ray, Class. Quant. Grav. 28, 155021 (2011).
[94] F. Rahaman, R. Maulick, A. K. Yadav, S. Ray, R. Sharma, Gen. Relativ. Gravit. 44, 107 (2012).
[95] M. Kalam, F. Rahaman, S. Ray, Sk. Monowar Hossein, I. Karar, J. Naskar, Eur. Phys. J. C 72, 2248 (2012).
[96] P. H. Nguyen, J. F. Pedraza, Phys. Rev. D 88, 064020 (2013).
[97] P. H. Nguyen, M. Lingam, MNRAS, 436, 2014 (2013).
[98] P. Bhar, F. Rahaman, S. Ray, V. Chatterjee, Eur. Phys. J. C 75, 190 (2015).
[99] S. K. Maurya, Y. K. Gupta, S. Ray, B. Dayanandan, Eur. Phys. J. C 75, 225 (2015).
[100] M. Cataldo, L. Liempi, P. Rodriguez, Phys. Lett. B 757, 130 (2016).