The Schwarzschild Solution in the 4-Dimensional Kaluza-Klein Description of The Einstein’s Equations

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Abstract

The Kaluza-Klein formalism of the Einstein’s theory, based on the (2,2)-fibration of a generic 4-dimensional spacetime, describes general relativity as a Yang-Mills gauge theory on the 2-dimensional base manifold, where the local gauge symmetry is the group of the diffeomorphisms of the 2-dimensional fibre manifold. As a way of illustrating how to use this formalism in finding exact solutions, we apply this formalism to the spherically symmetric case, and obtain the Schwarzschild solution by solving the field equations.

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In spite of those efforts that have been made over the decades trying to understand general relativity as a local gauge theory such as the Maxwell or a Yang-Mills theory, it seems fair to say that the proper gauge theory formulation of general relativity is still lacking. Such a gauge theory formulation, if feasible at all, would allow us to understand general relativity in terms of familiar notions of standard gauge theories. Recently, we have proposed a Kaluza-Klein formalism \[1,2\] of general relativity, based on the (2,2)-fibration \[3,4\] of a generic 4-dimensional spacetime. In this (2,2)-fibration, the 4-dimensional spacetime is regarded as a local product of a 2-dimensional base manifold and a 2-dimensional fibre manifold. Introducing the Kaluza-Klein variables adapted to this fibration, we found that general relativity of 4-dimensional spacetimes can be written as a Yang-Mills gauge theory defined on the 2-dimensional base manifold, where the local gauge symmetry is the group of the diffeomorphisms of the 2-dimensional fibre manifold. The appearance of the diffeomorphisms of the 2-dimensional fibre space as the Yang-Mills gauge symmetry, among others, is the most distinguished feature of this formalism, which is valid for a generic spacetime that does not possess any isometries whatsoever.

In this Letter, as an application of this formalism, we shall obtain the Schwarzschild solution by solving the Einstein’s equations written in the Kaluza-Klein variables. After a short introduction to the general formalism \[5\], we shall present the Einstein’s equations written in the Kaluza-Klein variables. Then we shall solve the field equations, using the spherical symmetry. Discussions on possible applications of this formalism will follow.

Let us start by recalling that any metric of a 4-dimensional spacetime of the Lorentzian signature may be put to the following form \[5\]

\[
ds^2 = -2dudv - 2hdu^2 + e^\sigma \rho_{ab} \left(dy^a + A^a_+ du + A^a_- dv\right) \left(dy^b + A^b_+ du + A^b_- dv\right),
\]

at least locally, where \(\rho_{ab}(a, b = 2, 3)\) is the conformal 2-geometry of the transverse surface on the hypersurface \(u = \text{constant}\), satisfying the condition

\[
\det \rho_{ab} = 1.
\]

The geometry represented by the above metric can be best understood in terms of the bundle geometry; \((u, v)\) are the coordinates of the 2-dimensional base manifold denoted by \(M_{1+1}\), and \(y^a\) are the coordinates of the spacelike 2-dimensional fibre space denoted by \(N_2\). From \[1\] we find that the covariant metric of \(M_{1+1}\) is given by

\[
\begin{pmatrix}
-2h & -1 \\
-1 & 0
\end{pmatrix}.
\]

The field \(\sigma\) is a measure of the area of \(N_2\), and the fields \(A_{\pm}^a\) are the connecting vector fields that define the horizontal lift vector fields orthogonal to the fibre space. Notice that \(h, \sigma, \rho_{ab}\), and \(A_{\pm}^a\) in \[1\] are functions of all the coordinates \((u, v, y^a)\), as we assume no spacetime isometries.

The equations of motion of \(h, \sigma, \rho_{ab}\), and \(A_{\pm}^a\) can be obtained by varying \(I_4\), the 4-volume integral of the scalar curvature \(R_4\) of the spacetime represented by the metric \[1\], which is given by
\[ I_4 = \int du \, dv \, d^2y \, e^\sigma R_4 \]
\[ = \int du \, dv \, d^2y \left[ -\frac{1}{2} e^2 \rho_{ab} F_{++}^a F_{--}^b + e^\sigma (D_+ \sigma)(D_- \sigma) - \frac{1}{2} e^\sigma \rho^{ab} \rho^{cd} (D_+ \rho_{ac})(D_- \rho_{bd}) \right. \]
\[ + e^\sigma R_2 + 2he^\sigma \left\{ D_+^2 \sigma + \frac{1}{2} (D_- \sigma)^2 + \frac{1}{4} \rho^{ab} \rho^{cd} (D_- \rho_{ac})(D_- \rho_{bd}) \right\} \]
\[ + \text{surface terms.} \]  
\hspace{1cm} (4)

Here + and − stands for \( u \) and \( v \), respectively, and \( R_2 \) is the scalar curvature of the fibre space \( N_2 \). We summarize the notations below;

\[ F_{++}^a = \partial_+ A_+^a - \partial_- A_-^a - [A_+, A_-]^a \]
\[ = \partial_+ A_+^a - \partial_- A_-^a - A_+^c \partial_c A_-^a + A_-^c \partial_c A_+^a, \]  
\hspace{1cm} (5)
\[ D_+ \sigma = \partial_+ \sigma - [A_+, \sigma]_L \]
\[ = \partial_+ \sigma - A_+^a \partial_a \sigma - \partial_a A_+^a, \]  
\hspace{1cm} (6)
\[ D_\pm \rho_{ab} = \partial_\pm \rho_{ab} - [A_\pm, \rho]_{Lb} \]
\[ = \partial_\pm \rho_{ab} - A_\pm^c \partial_c \rho_{ab} - (\partial_a A_\pm^b) \rho_{cb} - (\partial_b A_\pm^a) \rho_{ac} + (\partial_c A_\pm^a) \rho_{ab}, \]  
\hspace{1cm} (7)
where \([A_\pm, \ast]_L\) is the Lie derivative of \( \ast \) along the vector fields \( A_\pm := A_\pm^a \partial_a \). Each term in \( I_4 \) strongly suggests that the integral should be interpreted as an action integral of a Yang-Mills type gauge theory defined on the 2-dimensional base manifold \( M_{1+1} \), interacting with the 2-dimensional field \( \sigma \) and non-linear sigma field \( \rho_{ab} \). The associated local gauge symmetry is the built-in diff\( N_2 \) symmetry, the group of the diffeomorphisms of the fibre space \( N_2 \). It must be mentioned here that each term in (4) is manifestly diff\( N_2 \)-invariant, and that the \( y^a \)-dependence is completely hidden in the Lie derivatives. In this sense we may regard the fibre space \( N_2 \) as a kind of an internal space as in Yang-Mills theories [3].

Apart from the eight equations of motion that follow from (4) by the variations, however, there are two additional equations we have to consider, which are associated with the gauge fixing of the 2-dimensional metric to the form (3). These equations that follow by varying the Einstein-Hilbert action before we impose the gauge fixing condition, turn out to be two of the four Einstein’s constraints [3]. They are found to be

\[ (a) \quad e^\sigma D_+ D_- \sigma + e^\sigma D_- D_+ \sigma + 2e^\sigma (D_+ \sigma)(D_- \sigma) - 2e^\sigma (D_- h)(D_+ \sigma) \]
\[ - \frac{1}{2} e^2 \rho_{ab} F_{++}^a F_{--}^b - e^\sigma R_2 - he^\sigma \left\{ (D_- \sigma)^2 - \frac{1}{2} \rho^{ab} \rho^{cd} (D_- \rho_{ac})(D_- \rho_{bd}) \right\} = 0, \]  
\hspace{1cm} (8)

\[ (b) \quad -e^\sigma D_+^2 \sigma - \frac{1}{2} e^\sigma (D_+ \sigma)^2 - e^\sigma (D_- h)(D_+ \sigma) + e^\sigma (D_+ h)(D_- \sigma) \]
\[ + 2he^\sigma (D_- h)(D_- \sigma) + e^\sigma F_{++}^a \partial_a h - \frac{1}{4} e^\sigma \rho^{ab} \rho^{cd} (D_+ \rho_{ac})(D_+ \rho_{bd}) + \partial_a (\rho^{ab} \partial_b h) \]
\[ + h \left\{ - e^\sigma (D_+ \sigma)(D_- \sigma) + \frac{1}{2} e^\sigma \rho^{ab} \rho^{cd} (D_+ \rho_{ac})(D_- \rho_{bd}) + \frac{1}{2} e^2 \rho_{ab} F_{++}^a F_{--}^b + e^\sigma R_2 \right\} \]
\[ + h^2 e^\sigma \left\{ (D_- \sigma)^2 - \frac{1}{2} \rho^{ab} \rho^{cd} (D_- \rho_{ac})(D_- \rho_{bd}) \right\} = 0. \]  
\hspace{1cm} (9)

Together with the above equations, the ten Einstein’s equations are given by
which are the equations of motion of $h$ point on the worldline symmetric line element in the form (1) [7]. Let us recall that the spherical symmetry with solutions by solving the above equations. For this purpose we need to write down the spherically symmetric vacuum solution of the Einstein’s equations by setting $D_{+}h = 0$ in (15), we can write the metric

\begin{equation}
2e^{\sigma}(D_{-}\sigma) + e^{\sigma}(D_{-}\sigma)^{2} + \frac{1}{2}e^{\sigma}\rho^{ab}\rho^{cd}(D_{-}\rho_{bc})(D_{-}\rho_{da}) = 0,
\end{equation}

\begin{equation}
D_{-}(e^{2\sigma}\rho_{ab}F_{ab}^{+}) - e^{\sigma}\partial_{a}(D_{-}\sigma) - \frac{1}{2}e^{\sigma}\rho^{bc}\rho^{de}(D_{-}\rho_{bd})(\partial_{a}\rho_{ce}) + \partial_{b}(e^{\sigma}\rho^{bc}D_{-}\rho_{ac}) = 0,
\end{equation}

\begin{equation}
-D_{+}(e^{2\sigma}\rho_{ab}F_{ab}^{+}) - e^{\sigma}\partial_{a}(D_{+}\sigma) - \frac{1}{2}e^{\sigma}\rho^{bc}\rho^{de}(D_{+}\rho_{bd})(\partial_{a}\rho_{ce}) + \partial_{b}(e^{\sigma}\rho^{bc}D_{+}\rho_{ac}) = 0,
\end{equation}

\begin{equation}
-2e^{\sigma}D_{h}^{2}h - 2e^{\sigma}(D_{-}h)(D_{-}\sigma) + e^{\sigma}D_{+}D_{-}\sigma + e^{\sigma}D_{+}D_{-}\sigma = 0,
\end{equation}

\begin{equation}
h\left\{e^{\sigma}D_{-}\rho_{ab} + e^{\sigma}\rho^{cd}(D_{-}\rho_{bc})(D_{-}\rho_{da}) + e^{\sigma}(D_{-}\rho_{ab})(D_{-}\sigma)\right\} - \frac{1}{2}e^{\sigma}(D_{-}D_{-}\rho_{ab} + D_{+}D_{+}\rho_{ab}) + \frac{1}{2}e^{\sigma}\rho^{cd}\{(D_{-}\rho_{bc})(D_{+}\rho_{ad}) + (D_{+}\rho_{bc})(D_{-}\rho_{ad})\} - \frac{1}{2}e^{\sigma}\{(D_{-}\rho_{ab})(D_{+}\sigma) + (D_{+}\rho_{ab})(D_{-}\sigma)\} + e^{\sigma}(D_{-}\rho_{ab})(D_{-}h) + \frac{1}{2}e^{2\sigma}\rho_{ac}\rho_{bd}F_{+}^{+}F_{+}^{+} - \frac{1}{4}e^{2\sigma}\rho_{ab}\rho_{cd}F_{+}^{+}F_{+}^{+} = 0,
\end{equation}

which are the equations of motion of $h$, $A_{a}^{+}$, $A_{a}^{-}$, $\sigma$, and $\rho_{ab}$, respectively.

Now we shall obtain the spherically symmetric vacuum solution of the Einstein’s equations by solving the above equations. For this purpose we need to write down the spherically symmetric line element in the form [1] [7]. Let us recall that the spherical symmetry with respect to a given observer means that the metric is independent of the orientation at each point on the worldline $C$ of that observer (see Fig.1). Let $\vartheta$ and $\varphi$ be the angular coordinates that define the orientation at that point. Then, due to the spherical symmetry, it suffices to consider the 2-dimensional subspace defined by $\vartheta = \text{constant}$ and $\varphi = \text{constant}$. Let $(u, v)$ be the coordinates of an arbitrary event $E$ in this subspace, which we introduce as follows; (a) Given an event $E$, draw a past-directed null geodesic from $E$ cutting the worldline $C$ at $P$. The coordinate $v$ is defined as the affine distance of the event $E$ from $P$ along the null geodesic. (b) The coordinate $u$ measures the location of the event $P$ from a certain reference point $O$ along the worldline $C$. The affine parameter $v$ has the coordinate freedom

\begin{equation}
v \longrightarrow v' = A(u)v + B(u),
\end{equation}

on each null hypersurface defined by $u = \text{constant}$. Also there is a reparametrization invariance

\begin{equation}
u \longrightarrow u' = f(u),
\end{equation}

where $f(u)$ is an arbitrary function of $u$. Notice that the equation $du = 0$ defines a null geodesic in the $(u, v)$-subspace. Choosing $A = 1$ and $B = 0$ in (15), we can write the metric
of this subspace as a product of $du$ and $dv + h(u,v)du$, where $h$ is an arbitrary function of $(u,v)$. Therefore the metric of the spherically symmetric spacetime is given by

$$ds^2 = -2dudv - 2h(u,v)du^2 + H(u,v)(d\vartheta^2 + \sin^2 \vartheta d\varphi^2),$$

(17)

which we recognize as the spherically symmetric line element written in the form (1), if we identify $y^a = (\vartheta, \varphi)$. Here the fibre space $N_2$ is a two sphere $S_2$ of radius $H^{1/2}(H > 0)$, whose scalar curvature is given by

$$R_2 = -\frac{2}{H}.$$ (18)

If we compare (17) with (1), we find that

$$A_{\vartheta} = A_{\varphi} = 0,$$
$$\rho_{\vartheta \vartheta} = \frac{1}{\sin \vartheta}, \quad \rho_{\varphi \varphi} = \sin \vartheta, \quad \rho_{\vartheta \varphi} = 0,$$
$$e^\sigma = H \sin \vartheta.$$ (19)

Notice that the diff$N_2$-covariant derivatives $D_\pm$ reduce to $\partial_\pm$ since $A_{\pm a}$ become zero. Then the Einstein’s equations (8), · · · , (14) become

(a) $\partial_\pm \partial_- \sigma + \partial_- \partial_+ \sigma + 2(\partial_+ \sigma)(\partial_- \sigma) - 2(\partial_- \sigma)(\partial_- h) + \frac{2}{H} - h(\partial_- \sigma)^2 = 0,$

(20)

(b) $\partial_+^2 \sigma + \frac{1}{2}(\partial_+ \sigma)^2 + (\partial_+ \sigma)(\partial_- h) - (\partial_- \sigma)(\partial_+ h) - 2h(\partial_- \sigma)(\partial_- h) + h\left\{\left(\partial_+ \sigma\right)(\partial_- \sigma) + \frac{2}{H}\right\} - h^2(\partial_- \sigma)^2 = 0,$

(21)

(c) $2\partial_-^2 \sigma + (\partial_- \sigma)^2 = 0,$

(22)

(d) $\partial_\alpha \partial_- \sigma = 0,$

(23)

(e) $\partial_\alpha \partial_+ \sigma - 2h\partial_\alpha \partial_- \sigma - 2\partial_\alpha \partial_- h = 0,$

(24)

(f) $\partial_+ \partial_- \sigma + \partial_- \partial_+ \sigma + (\partial_+ \sigma)(\partial_- \sigma) - 2\partial_-^2 h - 2(\partial_- h)(\partial_- \sigma) = 0,$

(25)

(g) $0 = 0.$

(26)

respectively. Let us integrate Eq. (22) first. It can be written as

$$2\partial_- X + X^2 = 0,$$

(27)

where $X := \partial_- \sigma$. Solving this equation, we find that

$$X = \frac{2}{v + 2F},$$

(28)

where $F$ is an arbitrary function of $(u, \vartheta, \varphi)$. Therefore $\sigma$ becomes

$$\sigma = 2\ln (v + 2F) + G$$
$$= \ln H + \ln \sin \vartheta,$$

(29)
where $G$ is another arbitrary function of $(u, \vartheta, \varphi)$. Choosing $F = 0$ and $G = \ln \sin \vartheta$, we find that
\[ \sigma = 2 \ln v + \ln \sin \vartheta, \]
\[ H = v^2, \] (30)
from which it follows that
\[ \partial_- \sigma = \frac{2}{v}, \quad \partial_+ \sigma = 0. \] (31)
Then Eqs. (23) and (24) are trivially satisfied, and the remaining equations become
\[ (a) \quad 2 \partial_- h + \frac{2}{v} \frac{h}{v} - \frac{1}{v} = 0, \] (32)
\[ (f) \quad \partial_-^2 h + \frac{2}{v} (\partial_- h) = 0. \] (33)
Notice that the Eq. (21) becomes identical to Eq. (32), and that Eq. (33) is a “trivial” equation since it results by taking a derivative of Eq. (32) with respect to $v$. Therefore we need to solve Eq. (32) only. Assuming the asymptotic flatness at the null infinity $v \to \infty$, we find that $h$ is given by
\[ 2h = 1 - \frac{2m}{v}, \] (34)
where $m$ is a constant. Plugging (30) and (34) into (17), the spherically symmetric solution of the vacuum Einstein’s equations is given by
\[ ds^2 = -2dudv - (1 - \frac{2m}{v})du^2 + v^2(d\vartheta^2 + \sin^2 \vartheta d\varphi^2). \] (35)
Thus we found the Schwarzschild solution using the Kaluza-Klein variables adapted to the (2,2)-fibration of 4-dimensional spacetimes. Notice that the metric (35) is independent of $u$ (as well as $\vartheta$ and $\varphi$), which tells us that $u$ is the Killing time, as implied by the Birkhoff’s theorem.

There are a few possible applications of this formalism. First, this formalism provides a natural 2-dimensional framework for a conventional gauge theory description of general relativity of 4-dimensional spacetimes, where the local gauge symmetry is $\text{diff}\mathbb{N}_2$, the infinite dimensional group of the diffeomorphisms of the 2-dimensional “auxiliary” space. This enables us to explore certain canonical aspects of the theory, such as constructing physical observables for instance, using the gauge invariant quantities. Probably one could also use relevant 2-dimensional field theoretic methods in studying 4-dimensional spacetime physics in this formalism [8].

Second, we expect this formalism to fit most naturally the studies of gravitational waves, since the physical degrees of freedom of gravitational waves reside precisely in the non-linear sigma field $\rho_{ab}$ [9]. It is also an interesting problem to examine exact solutions of the Einstein’s equations in the light of this formalism, and interpret them from the 2-dimensional gauge theory perspective. For instance, the Schwarzschild spacetime in this
Letter corresponds to the “vacuum” configuration, in the sense that the gauge fields $A_{\pm}^a$ are identically zero.

Third, this formalism should be compared with the lightcone cut formalism [10] of C. Kozameh and E.T. Newman, where the lightcone cuts are the master fields defined at the null infinity of asymptotically flat spacetimes. In that formalism the lightcone cuts at the null infinity are constructed using the Bondi coordinates, in which the metric assumes the form (1), and $N_2$ becomes $S_2$. The difference is that our formalism is valid for a generic spacetime, at least locally, whereas the lightcone cut formalism depends heavily on the asymptotic structure at infinity.

Finally, the self-dual Einstein’s equations have been studied extensively from diverse points of view, and each approach has its own advantage. Surely this Kaluza-Klein formalism will add one more to the list.

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FIG. 1. The construction of the coordinates \((u, v)\) in the line element \((17)\) assuming the spherical symmetry about an observer. Here the angular coordinates \((\vartheta, \varphi)\) are suppressed.