Abstract. We provide \( L^p \rightarrow L^q \) refinements on some Fourier restriction estimates obtained using polynomial partitioning. Let \( S \subset \mathbb{R}^3 \) be a compact \( C^\infty \) surface with strictly positive second fundamental form. We derive sharp \( L^p(S) \rightarrow L^q(\mathbb{R}^3) \) estimates for the associated Fourier extension operator for \( q > 3.25 \) and \( q \geq 2p' \) from an estimate of Guth that was used to obtain \( L^\infty(S) \rightarrow L^q(\mathbb{R}^3) \) bounds for \( q > 3.25 \). We present a slightly weaker result when \( S \) is the hyperbolic paraboloid in \( \mathbb{R}^3 \) based on the work of Cho and Lee. Finally, we give some refinements for the truncated paraboloid in higher dimensions.

1. Introduction

Let \( S \subset \mathbb{R}^d \) be a compact \( C^\infty \) hypersurface. The Fourier transform of a function \( f \in L^1(\mathbb{R}^d) \) is continuous, hence the restriction operator \( R_S f = \hat{f}|_S \) is well-defined on \( L^1(\mathbb{R}^d) \). However, it is impossible to restrict \( \hat{f} \) to a set of zero Lebesgue measure for \( f \in L^2(\mathbb{R}^d) \) since \( \hat{f} \) is merely in \( L^2(\mathbb{R}^d) \) in general. In the late 1960’s, Stein observed that the restriction operator \( R_S \) may still be defined on \( L^p(\mathbb{R}^d) \) for some \( 1 < p < 2 \) provided that the surface \( S \) is appropriately curved; see [F] and [S]. This type of results have been obtained from a priori restriction estimates

\[
\| \hat{f}|_S \|_{L^q(S, d\sigma)} \leq C \| f \|_{L^p(\mathbb{R}^d)},
\]

where \( d\sigma \) is the induced Lebesgue measure on \( S \).

However, for a given hypersurface \( S \), it is a difficult problem to determine optimal ranges of exponents \( p \) and \( q \). By duality, one may reformulate restriction estimates as extension estimates

\[
\| E_S f \|_{L^q(\mathbb{R}^d)} \leq C \| f \|_{L^p(S)},
\]

where \( E_S \) is the extension operator

\[
E_S f(x) = \int_S e^{2\pi i x \cdot \xi} f(\xi) d\sigma(\xi).
\]

When \( S \) is the sphere \( S^{d-1} \), or more generally a compact \( C^\infty \) hypersurface with nonvanishing Gaussian curvature, it is conjectured that [L] holds if and only if \( q > \frac{2d}{d-1} \) and \( q \geq \frac{d+1}{d+2} p' \), where \( p' = p/(p-1) \). This conjecture is related to many other conjectures, including the Bochner-Riesz and the Kakeya conjectures; see, for instance, [F], [B1] and [T1]. While many deep results have been obtained on the restriction conjecture, it remains open in the full \( p, q \) range for \( d \geq 3 \).

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Recently, Guth [G1] made further progress on this problem for positively curved surfaces in $\mathbb{R}^3$ using polynomial partitioning, a divide and conquer technique developed by Guth-Katz [GK] for the Erdős distinct distances problem.

**Theorem 1.1** (Guth). If $S \subset \mathbb{R}^3$ is a compact $C^\infty$ surface with strictly positive second fundamental form, then for all $q > 3.25$,

$$\|E_S f\|_{L^q(\mathbb{R}^3)} \leq C\|f\|_{L^\infty(S)}.$$  

In particular, Theorem 1.1 improves a previous result for $q > 56/17$ due to Bourgain-Guth [BG]. When $S$ is the sphere or the truncated paraboloid in $\mathbb{R}^3$, Theorem 1.1 yields $L^q(S) \rightarrow L^q(\mathbb{R}^3)$ estimates for the extension operator $E_S$ for all $q > 3.25$; see a remark after Theorem 1 in [BG] or Section 19.3 in [M].

We refine Theorem 1.1 by replacing $L^\infty(S)$ with $L^p(S)$ for $p \geq q/(q - 2)$, or equivalently $q \geq 2p'$. This range of exponents $p$ is sharp.

**Theorem 1.2.** If $S \subset \mathbb{R}^3$ is a compact $C^\infty$ surface with strictly positive second fundamental form, then for $q > 3.25$ and $q \geq 2p'$,

$$\|E_S f\|_{L^q(\mathbb{R}^3)} \leq C\|f\|_{L^{p'}(S)}.$$  

**Remark.** After submitting the previous version of this article to arXiv, we learned from Marina Iliopoulou that Bassam Shayya already obtained nearly sharp result for $q > 3.25$ and $q > 2p'$; see [IS]. That result is a consequence of more general weighted restriction estimates, which are of independent interest. From his nearly sharp result, it is not hard to obtain the sharp line $q = 2p'$ by using a bilinear interpolation argument from [TVV]; see Section 5.3. In addition, we find that our proof is similar to that in [IS], although the proof given here appears to be more straightforward. In this regard, Theorem 1.2 is essentially due to Bassam Shayya and we do not claim any originality for Theorem 1.2.

Sharp $L^p(S) \rightarrow L^q(\mathbb{R}^d)$ estimates were known in the bilinear range $q > 2(d+2)/d$ by the work of Tao-Vargas-Vega [TVV] and Tao [T2]; see also [W]. More recent $L^q(S) \rightarrow L^q(\mathbb{R}^d)$ bounds from [CG], [G1], and [G2] extend this range of $q$; see [LRS, Section 5.2]. In particular, when $S \subset \mathbb{R}^3$ is the sphere or the truncated paraboloid, Theorem 1.1 yields, combined with Tao’s bilinear estimate [T2], sharp $L^p(S) \rightarrow L^q(\mathbb{R}^3)$ estimates for a slightly smaller range of $q$: $q > 23/7 = 3.28\cdots$.

The main ingredient of Theorem 1.2 is an estimate of Guth [G1, Theorem 2.4] for the “broad” contribution to $E_S f$; see Theorem 2.1 below. Here is an overview of the proof. When $q > 2p'$, Theorem 1.2 follows from a variation of the proof from [G1] that Theorem 2.1 implies Theorem 1.1. Our refinement comes from the use of a parabolic rescaling argument which involves both $L^2(S)$ and $L^\infty(S)$ norms. This modification is natural in view of Theorem 2.1. As a result, we obtain

$$\|E_S f\|_{L^3(\mathbb{R}^3)} \leq C_\epsilon R^{\epsilon}\|f\|_{L^2(S)}^{10/13}\|f\|_{L^\infty(S)}^{3/13}$$

for any $\epsilon > 0$ and any ball $B_R$ of radius $R$, which implies, by real interpolation, $L^p(S) \rightarrow L^q(B_R)$ estimates for $q > 3.25$ and $q \geq 2p'$ with the epsilon loss $R^\epsilon$. This yields Theorem 1.2 for $q > 2p'$ by an epsilon removal lemma; see Theorem 5.3 in Appendix. For the case $q = 2p'$, we use a bilinear interpolation argument from [TVV].

It is worth noting that Cho and Lee [CL] obtained an analogue of Theorem 1.1 for negatively curved quadratic surfaces; see Theorem 1.1. Using their “broad” estimate, [CL, Theorem 3.3], we obtain
Theorem 1.3. Let \( S \) be a compact quadratic surface with strictly negative Gaussian curvature in \( \mathbb{R}^3 \). Then, for all \( q > 3.25 \) and \( q > 2p' \),

\[
\| E_S f \|_{L^q(\mathbb{R}^3)} \leq C \| f \|_{L^p(S)}.
\]

Lee [L] and Vargas [V] obtained (1.3) for \( q > 10/3 \) and \( q > 2p' \) using bilinear estimates; see also [TV]. Unlike in the case of positively curved surfaces, the end point \( q = 2p' \) remains open in Theorem 1.3. This is due to the fact that bilinear estimates for negatively curved surfaces require a stronger separation condition, which results in some loss in deriving linear estimates from bilinear ones; see [L] and [V]. Sharp estimates at \( q = 2p' \) seem to be known only in the Stein-Tomas range for \( q \geq 4 \); see [To], [St], [Gr], and [S].

In higher dimensions, Bourgain-Guth [BG] introduced a technique to derive linear restriction estimates from the multilinear restriction estimate of Bennett-Carbery-Tao [BCT]. Assume that \( S_1, S_2, \ldots, S_k \) are transverse caps on the truncated paraboloid \( S = \{ (\omega, |\omega|^2) \in \mathbb{R}^d : |\omega| \leq 1 \} \) for some \( 2 \leq k \leq d \) and that \( f_j \) is supported on \( S_j \) for each \( 1 \leq j \leq k \). The \( k \)-linear restriction estimate takes the following form

\[
\left\| \prod_{j=1}^{k} |E_{S}f_j|^{1/k} \right\|_{L^p(B_R)} \leq C_R R^{d+1} \prod_{j=1}^{k} \| f_j \|_{L^q(S_j)}^{1/k}.
\]

It is conjectured that (1.4) holds for \( p > 2 \cdot \frac{d+k}{d+k-2} \), which is already known when \( k = 2 \) [T2] and \( k = d \) [BCT]. See also [He1] and [He2] for certain sharp estimates for a class of surfaces.

Guth [G2] formulated a weaker variant of (1.4) called \( k \)-broad inequality and completely settled the question of optimal range of exponents \( p \) for all \( 2 \leq k \leq d \); see Theorem 5.1. Adapting the Bourgain-Guth induction on scale argument [BG], he derived new \( L^p(S) \rightarrow L^q(\mathbb{R}^d) \) estimates for \( E_S \) from the \( k \)-broad inequality. We remark that a part of his proof can be modified so that one obtains \( L^p(S) \rightarrow L^q(\mathbb{R}^d) \) estimates for some \( 2 \leq p < q \) for each \( k \)-linearity \( 2 \leq k \leq \frac{d}{2} + 1 \).

Theorem 1.4. Let \( d \geq 4 \) and \( S \) be the truncated paraboloid. For each integer \( 2 \leq k \leq \frac{d}{2} + 1 \), the operator \( E_S \) obeys the estimate

\[
\| E_S f \|_{L^q(\mathbb{R}^d)} \leq C \| f \|_{L^p(S)}
\]

for all

\[ q > q(k, d) = \frac{2(d+k)}{d+k-2} \quad \text{and} \quad p \geq p(k, d) = \frac{2(d-k+1)(d+k)}{(d-k+1)(d+k-2)(k-1)}. \]

When \( k = 2 \), Theorem 1.4 recovers sharp extension estimates in the bilinear range \( q > \frac{2(d+2)}{d+4} \) from [TVV] [T2]. When \( d \) is even and \( k = \frac{d}{2} + 1 \), then \( q(k, d) = p(k, d) = 2 \cdot \frac{1}{3d+2} \) and Theorem 1.4 recovers the result in [G2]. We note that \( 2 \leq p(k, d) < q(k, d) \) when \( 3 \leq k < \frac{d}{2} + 1 \) and Theorem 1.4 seems to be new in this range of \( k \). In particular, when \( d \) is odd and \( k = \frac{d+1}{2} \), then \( q(k, d) = 2 \cdot \frac{d+1}{3d-2} \) and \( p(k, d) < q(k, d) \), thus slightly extending the range of exponents \( p \).

It is expected that a better understanding on the Kakeya conjecture may lead to some further progress on the restriction problem; see, for example, [BG] and [D].
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2. Preliminaries

In this section, we prepare for the proof of Theorem 1.2. We recall the estimate on broad points [GI, Theorem 2.4] and the parabolic rescaling argument in [GI].

2.1. Estimate on broad points. Let $\epsilon > 0$ and $B^2_\epsilon(\omega) = \{x \in \mathbb{R}^2 : |x - \omega| \leq \epsilon \}$. Consider a surface $S \subset \mathbb{R}^3$ given as the graph of a function $h : B^2_\epsilon(0) \to \mathbb{R}$ satisfying the following conditions for some large $L = L(\epsilon)$, say $10^6 \epsilon^{-2}$.

**Conditions 2.1.**

1. $0 < 1/2 \leq \partial^2 h \leq 2$.
2. $0 = h(0) = \partial h(0)$.
3. $h$ is $C^L$, and for $3 \leq l \leq L$, $\|\partial^l h\|_{C^0} \leq 10^{-9}$.

Let $K = K(\epsilon)$ be a large number. Partition $S$ into $\sim K^2$ caps $\tau$ of diameter $\sim K^{-1}$. Then we may write $f = \sum_\tau f_\tau$, where $f_\tau = f \chi_\tau$.

We now introduce the concept of broad points. For $\alpha \in (0,1)$, $x$ is said to be $\alpha$-broad for $E_S f$ if

$$\max_\tau |E_S f_\tau(x)| \leq \alpha |E_S f(x)|.$$  

Define $\text{Br}_\alpha E_S f(x)$ to be $|E_S f(x)|$ if $x$ is $\alpha$-broad, and zero otherwise. With $\alpha = K^{-\epsilon}$, we see that

$$|E_S f(x)| \leq \max_\tau (\text{Br}_{K^{-\epsilon}} E_S f(x), K^\epsilon \max_\tau |E_S f_\tau(x)|).$$  

The term $\max_\tau |E_S f_\tau(x)|$ can be controlled by an induction argument using parabolic rescaling. The main difficulty lies in the estimation of $\text{Br}_{K^{-\epsilon}} E_S f$.

**Theorem 2.1** (Guth). For any $\epsilon > 0$, there exists $K = K(\epsilon)$ and $L = L(\epsilon)$ so that if $S$ obeys Conditions [2.7] with $L$ derivatives, then for any radius $R$,

$$\|\text{Br}_{K^{-\epsilon}} E_S f\|_{L^3(2R)} \leq C_\epsilon R^\epsilon \|f\|_2^{12/13} \|f\|_{L^\infty}^{1/13}.$$  

In fact, we may take $K(\epsilon) = e^{\epsilon^{-10}}$.

This is [GI] Theorem 2.4. It is sharp in the sense that given the right-hand side, the exponent 3.25 in the inequality may not be decreased. The proof of Theorem 2.1 involves polynomial partitioning, inductions on $R$ and $\|f\|_{L^2}$, bilinear estimates, and geometry of tubes and algebraic surfaces.

2.2. Parabolic rescaling. We summarize a scaling argument from [GI] Section 2.3 as a lemma. In what follows, we identify a function on the graph of $h : U \to \mathbb{R}$ with a function on $U \subset \mathbb{R}^2$.

**Lemma 2.2.** Assume that $h$ satisfies Conditions [2.7] with $L$ derivatives. Let $0 < r < 1$ and $S_0 \subset S$ be the graph of $h$ over $B^2_\epsilon(\omega_0)$ for some $\omega_0 \in B^2_\epsilon(0)$. Then there exists $h_1 : B^2_\epsilon(0) \to \mathbb{R}$ satisfying Conditions [2.7] with $L$ derivatives such that if $S_1$ is the graph of $h_1$, then

$$|E_{S_0} f(x)| = |E_{S_1} g(\Phi(x))|,$$
where
\[ g(\eta) = f(\omega_0 + r\eta)r^2|Jh||Jh_1|^{-1}, \]
\[ \Phi(x) = (rx_1 + r\partial_1 h(\omega_0)x_3, rx_2 + r\partial_2 h(\omega_0)x_3, r^2x_3). \]
Here, $|Jh|$ and $|Jh_1|$ are Jacobian factors bounded by $\sqrt{r}$. Moreover, $h_1$ satisfies $\partial_1^2 h_1(\eta) = \partial_1^2 h(\omega_0 + r\eta)$ and $\|\partial_1^2 h_1\|_{C^0} \leq r^{1-2}\|\partial_1^2 h\|_{C^0}$ for $3 \leq l \leq L$.

We shall use parabolic rescaling which involves both $L^2$ and $L^\infty$ norms. The following is a version of \cite{G1} Lemma 2.5.

**Lemma 2.3.** Let $S_0$ and $S_1$ as in Lemma 2.2. Assume that
\[ \|E_{S_1}g\|_{L^s(B_{10r}R)} \leq M\|g\|_{L^\infty(S_1)}^{1-\theta}\|g\|_{L^\infty(S_0)}^{\theta} \]
for some $0 \leq \theta \leq 1$.

**Proof.** Let $g$ be as in Lemma 2.2. Since $\det(\Phi) = r^4$ and $\Phi(B_R) \subset B_{10r}R$, we have
\[ \|E_{S_0}f\|_{L^s(B_R)} \leq r^{-4/q}\|E_{S_1}g\|_{L^s(B_{10r}R)} \leq r^{-4/q}M\|g\|_{L^\infty(S_1)}^{1-\theta}\|g\|_{L^\infty(S_1)}^{\theta}. \]

where we used $\|g\|_{L^\infty(S_1)} \leq 10r^{2/\theta}\|g\|_{L^\infty(S_0)}$. \hfill \qed

**3. PROOF OF THEOREM 1.2**

We shall first prove Theorem 1.2 when $q > 3.25$ and $q > 2p$. In Section 3.3 we extend the result to the scaling line $q = 2p$ by an interpolation argument.

### 3.1. An extension estimate implied by Theorem 2.1

The main ingredient of the proof is the following extension estimate (cf. \cite{G1} Theorem 2.2).

**Theorem 3.1.** For any $\epsilon > 0$, there exists $L = L(\epsilon)$ so that if $S$ obeys Conditions 2.1 with $L$ derivatives, then for any radius $R$, the extension operator $E_S$ obeys the inequality
\[ \|E_Sf\|_{L^3(B_R)} \leq C_S, R^{1/13}\|f\|_{L^2(S)}^{10/13}\|f\|_{L^\infty(S)}^{3/13}. \]

**Proof.** The proof is similar to the proof of \cite{G1} Theorem 2.2 for the local $L^\infty(S) \to L^{3.25}(B_R)$ estimate. We use not only $L^\infty(S)$ but also some $L^2(S)$ norm, which is suggested by Theorem 2.1.

We may assume that $0 < \epsilon < 1$ and $R \geq 1$. It will be useful to use the scale $\epsilon/2$ as well. Let $K = K(\epsilon/2) = \epsilon(\epsilon/2)^{10}$ and assume that $S$ obeys Conditions 2.1 with $L(\epsilon/2)$ derivatives, where $K(\epsilon)$ and $L(\epsilon)$ are the parameters in Theorem 2.1. Using 2.1 with $\epsilon/2$ instead of $\epsilon$, we bound $\int_{B_R}[E_Sf]^{3.25}$ by
\[ \int_{B_R}\|K_{-\epsilon/2}E_Sf\|^{3.25} + \sum_{\tau}\int_{B_R}\|K_{-\epsilon/2}E_Sf\|^{3.25}. \]

By Theorem 2.1 the first term in (3.2) is bounded by
\[ (C_{\epsilon/2}R^{1/2}\|f\|_{L^2(S)}^{12/13}\|f\|_{L^\infty(S)}^{1/13})^{3.25} \leq (C_{\epsilon/2}^{2/13}C_{\epsilon/2}R^{1/2}\|f\|_{L^\infty(S)}^{10/13}\|f\|_{L^\infty(S)}^{3/13})^{3.25} \]

1The use of $\epsilon/2$ is not necessary for estimates weaker than \cite{G1} where $\|f\|_{L^2(S)}^{10/13}\|f\|_{L^\infty(S)}^{3/13}$ is replaced by $\|f\|_{L^2(S)}^{1-\theta}\|f\|_{L^\infty(S)}^{\theta}$ for any $3/13 < \theta \leq 1$
Theorem 1.2 when $\epsilon > 0$.

To handle the second term in (3.2), we use an induction on $R$. Since Theorem 3.1 is trivial for $R = O(1)$, we shall assume that it holds for all radii less than $R/2$ with some constant $C_{S,R} \geq 2C_{S}^{2/3}C_{r/2}$, and then deduce that (3.1) holds for the radius $R$ with the same constant. Since the first term in (3.2) is bounded by $\frac{1}{2}C_{S}R^2 \|f\|^{10/13}_{L^2(S)} \|f\|^{3/13}_{L^\infty(S)}$, the induction closes if the second term in (3.2) is bounded by the same expression.

Recall that $\tau$ is a cap of diameter $\sim K$. Therefore, we may assume that $\tau$ is contained in the graph $S_0$ of $h$ over some ball $B^2_0(\omega_0)$ of radius $r = K^{-1}$. Let $S_1$ be the surface as in Lemma 2.2. As $10rR < R/2$, the induction hypothesis implies

$$\|E_{S_1}g\|_{L^{3,25}(B_{10rR})} \leq C_{S,R}(10rR)\|g\|_{L^2(S_1)}^{10/13}\|g\|_{L^\infty(S_1)}^{3/13},$$

which yields

$$\int_{B_R} |E_S f_\tau|^{3,25} \leq \left(10^{1+\epsilon}K^{-\epsilon}C_{S,R}R^2 \|f_\tau\|^{10/13}_{L^2(S)} \|f_\tau\|^{3/13}_{L^\infty(S)}\right)^{3,25}$$

by Lemma 2.3 and the fact that $E_S f_\tau = E_{S_1}f_\tau$.

We bound $\|f_\tau\|_{L^\infty(S)}$ by $\|f\|_{L^\infty(S)}$ and then sum (3.3) over $\tau$ using the embedding $l^2 \hookrightarrow l^{2,5}$. Then we get

$$\sum_{\tau} \int_{B_R} |K^{1/2}E_S f_\tau|^{3,25} \leq \left(10^{1+\epsilon}K^{-\epsilon/2}C_{S,R}R^2 \|f\|^{10/13}_{L^2(S)} \|f\|^{3/13}_{L^\infty(S)}\right)^{3,25}.$$

Therefore, the induction closes since $10^{1+\epsilon}K^{-\epsilon/2} = 10^{1+\epsilon}e^{-(\epsilon/2)^{-9}} \leq 1/2$. \hfill \Box

3.2. Theorem 1.2 when $q > 2p'$. From Theorem 3.1 we deduce the following result by a standard argument.

Theorem 3.2. If $S \subset \mathbb{R}^3$ is a compact $C^\infty$ surface with strictly positive second fundamental form, then for all $\epsilon > 0$ and any radius $R$, the extension operator $E_S$ obeys the inequality

$$\|E_S f\|_{L^{3,25}(B_R)} \leq C_{S,R}R^2 \|f\|^{10/13}_{L^2(S)} \|f\|^{3/13}_{L^\infty(S)}.$$

For the convenience of the reader, we sketch the standard argument here following [GI] Section 2.3. In the paper [GI], that argument was used for the global $L^\infty(S) \to L^2(\mathbb{R}^d)$ estimates, but it would also work for our situation. By a finite decomposition of $S$ and choosing an appropriate coordinate, we may assume that $S$ is contained in the graph of a smooth function $h : B^2_0(0) \to \mathbb{R}$ satisfying $h(0) = \partial h(0) = 0$. By the assumption, $\partial^2 h$ is positive definite and satisfies $\lambda^{-1} \leq \partial^2 h \leq \lambda$ for some $\lambda = \lambda_S > 1$. Given $L = L(\epsilon)$, we decompose $S$ into caps of diameter $r = r(\lambda, \|h\|_{C^\ell})$. Since the number of the caps depends only on $S$ and $\epsilon$, it suffices to prove the extension estimate associated with a fixed cap. We may choose $r$ sufficiently small, so that, after parabolic rescaling, $\|\partial^l h_1\|_{C^0} \leq 10^{-10} \lambda^{-l}$ for all $3 \leq l \leq L$. Then we do a change of variable so that $\partial^2 h_1(0)$ is the identity matrix. This may increase the size of the support of $h_1$, but by a further parabolic rescaling, $h_1$ can be made to satisfy Conditions 2.1 with $L$ derivatives. The ball $B_R$ may be dilated during these change of variables, but is contained in a ball of radius $CR$ for some constant $C = C_S$. By applying Theorem 3.1 we obtain Theorem 3.2.
We are now ready to deduce Theorem 1.2 for \( q > 2p' \). First, Theorem 3.2 immediately yields the restricted strong type \((p_0, q_0) = (13/5, 13/4)\) estimate
\[
\|E \chi_E\|_{L^3(B_R)} \leq C_{S,E} R \|\chi_E\|_{L^{13/5}(S)}
\]
for any measurable set \( E \subset S \). Observe that \( q_0 = 2p'_0 \). By real interpolation with the trivial \( L^1 \to L^\infty \) estimate, we obtain strong type estimates
\[
\|E f\|_{L^q(B_R)} \leq C_r R \|f\|_{L^p(S)}
\]
whenever \( q > 3.25 \) and \( q \geq 2p' \). Finally, we apply the epsilon removal lemma, Theorem 5.3 which gives Theorem 1.2 for \( q > 2p' \).

3.3. Bilinear argument for the case \( q = 2p' \). Following [TVV], but restricting only to smooth phases, we say that a function \( h : B^2_T(0) \to \mathbb{R} \) is elliptic if \( h \) is smooth, \( \partial h(0) = \partial^2 h(0) = 0 \), and the eigenvalues of \( \partial^2 h(x) \) lie in \([1 - \epsilon_0, 1 + \epsilon_0] \) for some \( 0 < \epsilon_0 \ll 1 \) for all \( x \in B^2_T(0) \). We say that a surface \( S \) is elliptic if \( S \) is contained in the graph of an elliptic defining function \( h \).

For the proof of Theorem 1.2 it is enough to work with elliptic surfaces by the parabolic rescaling argument in Section 3.2. Therefore, our goal is to prove that if \( S \) is an elliptic surface, then
\[
\|E S f\|_{L^q(\mathbb{R}^3)} \leq C\|f\|_{L^p(S)}
\]
for \( q > 3.25 \) and \( q = 2p' \). For this, we employ a bilinear interpolation argument as in the proof of [TVV] Theorem 4.1; see also [LRS] Section 5.2.

Assume that \( f_1 \) and \( f_2 \) are supported in \( O(1) \)-separated caps \( S_1 \) and \( S_2 \), respectively, contained in an elliptic surface \( S \). Note that (3.4) implies bilinear estimates by Cauchy-Schwarz;
\[
\|E S f_1 E S f_2\|_{L^{q/2}(\mathbb{R}^3)} \leq C\|f_1\|_{L^p(S)}\|f_2\|_{L^p(S)}.
\]
We say that \( (1/p, 1/q) \) is a bilinear pair if (3.5) holds for all elliptic surfaces. Let
\[
Q = \{(1/p, 1/q) \in [0, 1]^2 : q > 3.25, q > 2p' \}.
\]
In Section 3.2 we verified (3.4) for \( (1/p, 1/q) \in Q \) for any compact \( C^\infty \) surface \( S \) with strictly positive second fundamental form. Therefore, we know that each \( (1/p, 1/q) \in Q \) is a bilinear pair.

Fix \( q > 3.25 \) and \( q = 2p' \). In order to prove the linear estimate (3.4) with this pair of exponents, it is enough to verify that there exists \( \delta > 0 \) such that \( (1/p_1, 1/q) \) is a bilinear pair whenever \( (1/p_1, 1/q) \in B^2_T(1/p, 1/q) \); see [TVV] Theorem 2.2.

Note that the transversality of the caps \( S_1 \) and \( S_2 \) allows bilinear pairs \((1/p, 1/q)\) even for some \( q < 2p' \). In particular, we may take a bilinear pair \( (1/p_0, 1/q_0) = (7/12, 1/4) \) from [TVV] Theorem 2.3]; see also [MVV]. We can choose a sufficiently small \( \delta_0 = \delta_q \) so that, for each \((1/p_0, 1/q_0) \in B^2_T(1/p, 1/q)\), the line through \((1/p_0, 1/q_0)\) and \((1/p_0, 1/q_0)\) intersects \( Q \). In other words, there is a bilinear pair \((1/p_1, 1/q_1) \in Q\) such that
\[
(1/p_1, 1/q_1) = (1 - \theta)(1/p_0, 1/q_0) + \theta(1/p_1, 1/q_1)
\]
for some \( \theta \in (0, 1) \). Thus, bilinear interpolation (see e.g. [BL]) implies that \((1/p_1, 1/q_1)\) is a bilinear pair, which completes the proof of Theorem 1.2.
4. Sketch of the proof of Theorem 1.3

Cho and Lee [CL] obtained the following result based on the polynomial partitioning techniques from [G1].

Theorem 4.1 (Cho and Lee). Let $S$ be a compact quadratic surface with strictly negative Gaussian curvature in $\mathbb{R}^3$. Then, for all $q > 3.25$ and $p = q$,

$$
\|E_S f\|_{L^q(\mathbb{R}^3)} \leq C \|f\|_{L^p(S)}.
$$

Theorem 1.3 slightly improves the range of $p$ of Theorem 4.1. It requires a few modifications of the proof of Theorem 4.1 analogous to those made in Section 3. In fact, a further minor modification is necessary since the definition of broad points in [CL] is slightly different due to the need of a stronger separation condition for bilinear estimates. In particular, when doing an induction on $R$, one needs to perform an additional scaling associated with thin strips of dimensions $1 \times K^{-1}$. Nevertheless, arguing as in Section 3, it can be shown that [CL, Theorem 3.3], an estimate on broad points, yields

Theorem 4.2. Let $S \subset \mathbb{R}^3$ be the graph of $h(\omega_1, \omega_2) = \omega_1 \omega_2$ over the unit cube centered at the origin and $3/13 < \theta \leq 1$. Then for all $q > 0$ and radius $R$, the extension operator $E_S$ obeys the inequality

$$
\|E_S f\|_{L^q(S)} \leq C_{\theta, q} R^\theta \|f\|_{L^q(S)}.
$$

This is an analogue of Theorem 3.4. Note that the limiting case $\theta = 3/13$ is excluded. This is due to the additional scaling which does not shrink a ball $B_R$ to a ball of much smaller radius. However, Theorem 4.2 is strong enough to imply Theorem 1.3.

Currently, we do not know how to extend the result to the scaling line $q = 2p'$. The situation is somewhat different from the case of elliptic surfaces. In particular, when $S$ is the hyperbolic paraboloid, the bilinear estimate (4.5) fails to hold for any $q < 2p'$ without a stronger separation condition on $f_1$ and $f_2$: see [L] and [V]. This is related to the fact that the hyperbolic paraboloid contains line segments.

5. Some refinements in higher dimensions

Let $d \geq 2$. Following [G2], we consider $L^q(B^{d-1}) \rightarrow L^p(\mathbb{R}^d)$ extension estimates (note the change of the role of $p$ and $q$) for the operator

$$
Ef(x) = \int_{B^{d-1}} e^{i(x_1 \omega_1 + \cdots + x_d \omega_d - |x_\omega|^2)} f(\omega) d\omega,
$$

where $B^{d-1}$ is the unit ball in $\mathbb{R}^{d-1}$. The study of the operator $E_S$ for the truncated paraboloid in $\mathbb{R}^d$ reduces to the study of the operator $E$, and vice versa.

Here is the basic setup for the $k$-broad inequality in [G2] (see also [BG]). Consider a covering of the unit ball $B^{d-1}$ by a collection of finitely many overlapping balls $\tau$ of radius $K^{-1}$ for some $1 \ll K \ll R$. Then decompose $f$ as $f = \sum_\tau f_\tau$ where $f_\tau$ is supported on $\tau$. Let $n(\omega) \in S^{d-1}$ be a normal vector for the paraboloid in $\mathbb{R}^d$ at the point $(\omega, |\omega|^2)$. For a given subspace $V \subset \mathbb{R}^d$, we write $f \notin V$ if $\angle(n(\omega), v) > K^{-1}$ for all $\omega \in \tau$ and non-zero vectors $v \in V$. Otherwise, we write $\tau \in V$.

Next, consider a covering of $B_R$ by a collection of finitely many overlapping balls $B_{K^2}$, and then study $\int_{B_{K^2}} |\sum_\tau Ef_\tau|^p$ for each fixed $B_{K^2}$. Let $V \subset \mathbb{R}^d$ be a $(k-1)$-dimensional subspace. Then one may consider the “broad” part $\int_{B_{K^2}} |\sum_{\tau \notin V} Ef_\tau|^p$.
and the “narrow” part \( \int_{B_{R^2}} | \sum_{\tau \in V} Ef_{\tau} |^p \) separately. More precisely, Guth \cite{G1} defined the \( k \)-broad part of \( \| Ef \|_{L^p(B_R)} \) by

\[
\| Ef \|_{BL^p_{k,A}(B_R)} := \sum_{B_{K^2} \subset B_R} \min_{V_1, V_2, \ldots, V_{2^k}} \max_{\tau \in V_a} \text{ for all } a \int_{B_{K^2}} | Ef_{\tau} |^p
\]

for a parameter \( A \) and proved the following using polynomial partitioning.

**Theorem 5.1** (Guth). For any \( 2 \leq k \leq d \), and any \( \epsilon > 0 \), there is a constant \( A \) so that the following holds (for any value of \( K \)):

\[
\| Ef \|_{BL^p_{k,A}(B_R)} \lesssim_{K, \epsilon} R^d \| f \|_{L^q(B^{d-1})},
\]

for \( p \geq \frac{2(d+k)}{d+k-2} \) and \( q \geq 2 \).

We state a version of \cite{G2} Proposition 9.1 that derives extension estimates from the \( k \)-broad inequalities. We consider the regime \( q \leq p \) which seems to be more natural in view of the restriction conjecture.

**Proposition 5.2.** Suppose that for all \( K, \epsilon \), the \( k \)-broad inequality \eqref{5.1} holds for some \( 2 \leq q \leq p \leq 2 \cdot \frac{d+1}{d-1} \). If \( p \) is in the range

\[
p \geq \frac{d+1}{2} - \frac{d+1}{q},
\]

then \( E \) obeys

\[
\| Ef \|_{L^p(B_R)} \lesssim_{\epsilon} R^d \| f \|_{L^q}.
\]

Proposition \ref{5.2} follows from a minor modification of the proof of \cite{G2} Proposition 9.1 for the regime \( q \geq p \). Therefore, we shall focus only on the part that we need to modify. Let us first sketch the proof of \cite{G2} Proposition 9.1. The \( k \)-broad inequality allows one to reduce the problem to the estimation of the “\( k \)-narrow” part of \( \| Ef \|_{L^p(B_R)} \) where only \( O(K^{k-2}) \) many balls \( \tau \) contribute to the sum \( \sum_{\tau} Ef_{\tau} \).

After applying the \( l^2 \)-decoupling inequality due to Bourgain \cite{B3} to this narrow contribution (see also \cite{BD}), Hölder’s inequality is used to replace the \( l^2 \)-norm by the \( l^p \)-norm in order to facilitate the summation of

\[
\left( \sum_{\tau \in V_a} \left( \int W_{B_{K^2}} | Ef_{\tau} |^p \right)^{2/p} \right)^{p/2}
\]

over those balls \( B_{K^2} \subset B_R \). Here, \( W_{B_{K^2}} \) is a weight which is roughly the characteristic function of the ball \( B_{K^2} \).

Our modification for the proof of Proposition \ref{5.2} lies on the “\( k \)-narrow” part. After using \( l^2 \)-decoupling, we replace the \( l^2 \)-norm by the \( l^q \)-norm, which is suggested by the \( L^q \to L^p \) statement. This replaces \cite{G2} Equation (9.7) with

\[
\int_{B_{K^2}} | \sum_{\tau \in V_a} Ef_{\tau} |^p \leq C_\delta K^d K^{(k-2)(\frac{d}{2}-\frac{1}{q})} \left( \sum_{\tau} \left( \int W_{B_{K^2}} | Ef_{\tau} |^p \right)^{q/p} \right)^{p/q}
\]

for some \( 0 < \delta < \epsilon \). After the summation over \( 1 \leq a \leq A \), we sum the above expression over those balls \( B_{K^2} \subset B_R \) using Minkowski’s inequality. The remainder of the proof involves the induction on scale argument using parabolic rescaling. The induction closes when \eqref{5.2} is satisfied.
Let us put the condition (5.2) in context. When \( q = 2 \), the condition becomes the familiar Stein-Tomas range \( p \geq \frac{2(d + 1)}{d - 1} \). When \( k = 2 \), the condition (5.2) is equivalent to the necessary condition \( p \geq \frac{d + 1}{d - 1} q' \) for the \( L^q(B^{d-1}) \to L^p(\mathbb{R}^d) \) estimate for the extension operator \( E \). When \( q = p \), the condition (5.2) is identical to that in [G2] for the regime \( p \leq q \leq \infty \).

For \( d \geq 2 \) and each integer \( 2 \leq k \leq \frac{d}{2} + 1 \), define

\[
\bar{q}(k, d) := \frac{2(d - k + 1)(d + k)}{(d - k + 1)(d + k) - 2(k - 1)}.
\]

This \( \bar{q}(k, d) \) is found by setting the right hand side of (5.2) equal to \( \bar{p}(k, d) := \frac{2(d + k)}{d + k - 2} \) and then solving the equation for \( q \). Note that \( 2 \leq \bar{q}(k, d) < \bar{p}(k, d) \) if \( 2 \leq k < \frac{d}{2} + 1 \) and \( \bar{q}(k, d) = \bar{p}(k, d) \) if \( k = \frac{d}{2} + 1 \). Therefore, the \( k \)-broad inequality [G1] Theorem 1.5) and Proposition 5.3 yield local extension estimates

\[
\|Ef\|_{L^p(k,d)\mathbf{(B_R)}} \leq C_\alpha R^\alpha \|f\|_{L^q(k,d)(B^{n-1})},
\]

which implies Theorem 1.4 by the epsilon removal lemma, Theorem 5.3.

**APPENDIX: Epsilon removal for Fourier restriction estimates**

Let \( S \) be a compact \( C^\infty \) hypersurface in \( \mathbb{R}^d \). We shall assume that \( S \) is curved in the sense that the surface measure \( d\sigma \) on \( S \) satisfies the Fourier decay condition

(5.3)

\[
|\hat{d}\sigma(\xi)| \leq C(1 + |\xi|)^{-\rho}
\]

for some \( \rho > 0 \). For surfaces with non-vanishing Gaussian curvature, it is well-known that (5.3) holds with the maximum decay rate \( \rho = (d - 1)/2 \).

Tao’s epsilon removal lemma [T1] Theorem 1.2) allows one to obtain global restriction estimates from local restriction estimates of the form

(5.4)

\[
\|\hat{f}|_S\|_{L^p(S,d\sigma)} \leq C_\alpha R^\alpha \|f\|_{L^p(B_R)}
\]

at the expense of decreasing the exponent \( p \); see also [B2] and [TV]. Note that (5.4) is the dual of the local extension estimate

\[
\|Ef\|_{L^p'(B_R)} \leq C_\alpha R^\alpha \|f\|_{L^q'(S)}.
\]

Tao’s result was stated in the case \( p = q \) in [T1], but the argument works for \( p < q \) as well. We record this observation as a theorem.

**Theorem 5.3.** Let \( 1 \leq p \leq q \leq 2 \) and \( 0 < \alpha \ll 1 \). Assume that we have the local restriction estimate (5.4) for any ball \( B_R \) of radius \( R \) and any smooth function \( f \) supported in \( B_R \). Then there is a constant \( C_{d,\rho} > 0 \) such that we have

(5.5)

\[
\|\hat{f}|_S\|_{L^s(S,d\sigma)} \leq C\|f\|_{L^p(\mathbb{R}^d)} \quad \text{for} \quad \frac{1}{s} > \frac{1}{p} + \frac{C_{d,\rho}}{\ln \alpha}.
\]

In fact, we may take \( C_{d,\rho} = 5 \ln \left((d - 1)/\rho\right) \) in Theorem 5.3, but this is by no means optimal. Theorem 5.3 says, in particular, that if the local estimate (5.4) holds for any \( \alpha > 0 \), then the global estimate (5.5) holds for all \( 1 \leq s < p \).

It seems worth pointing out that Bourgain-Guth [BG] obtained and utilized an epsilon removal result for the case \( 1 = q < p < 2 \). Their result involves an additional ingredient: the Maurey-Nikishin factorization theorem.

A main step toward Theorem 5.3 is an extension of the local estimate (5.4) to a local estimate for a union of sparse balls.
Sparse Balls. Let $C(d,\rho) = (d - 1)/\rho$, where $\rho$ is as in \[5.3\]. We say that a collection of balls $\{B_R(x_i)\}_{i=1}^N$ in $\mathbb{R}^d$ is sparse if $|x_i - x_j| \geq (NR)^{C(d,\rho)}$ for $i \neq j$.

Given the extension of \[5.3\] for sparse balls, Theorem 5.3 can be obtained exactly as in \[T1\] or \[BG\]. Therefore, we shall be content with proving the following lemma, which is basically \[T1\] Lemma 3.2. In what follows, we write $A \lesssim B$ if $A \leq CB$ for some constant $C > 0$, which may vary from line to line.

**Lemma 5.4.** Assume that the local estimate \[5.4\] holds for some $1 \leq p \leq q \leq 2$.

Then we have

$$\left\| \hat{f} \right\|_{L^q(d\sigma)} \lesssim R^q \|f\|_{L^p(\mathbb{R}^d)},$$

whenever $f$ is supported in the union of a sparse collection of balls $\{B_R(x_i)\}_{i=1}^N$.

**Proof.** As in \[T1\], we use of the fact that \[5.4\] implies

\[5.6\]

$$\left\| \hat{f} |_{N_{R-1} S} \right\|_{L^q(\mathbb{R}^d)} \lesssim R^{-1/q} \|f\|_{L^p(\mathbb{R}^d)},$$

where $N_{R-1} S$ is the $R^{-1}$ neighborhood of $S$.

By the support assumption, we may write $f$ as $f(x) = \sum_{i=1}^N f_i(x - x_i)$ for some $f_i$ supported in $B_R(0)$. Let $\varphi$ be a smooth function such that $|\varphi| = 1$ on $B_1(0)$ and $\hat{\varphi}$ is supported in $B_1(0)$. Let $\varphi_R = \varphi(\cdot/R)$ and $f_i = f_i/\varphi_R$. We can write $f(x)$ as $\sum (f_i \varphi_R)(x - x_i)$. Let $e(t) = e^{-2\pi it}$. We claim that

\[5.7\]

$$\left\| \sum_i e(x_i \cdot \xi) g_i \ast \varphi_R |_{S} \right\|_{L^q_{\xi}(d\sigma)} \lesssim R^{1/q} \left( \sum_i \|g_i\|_{L^q(\mathbb{R}^d)}^q \right)^{1/q}$$

for all $g_i \in L^q(\mathbb{R}^d)$ and $1 \leq q \leq 2$.

Assume \[5.7\] for the moment. Note that for $\xi \in S$,

$$\hat{f}(\xi) = \sum_i e(x_i \cdot \xi) \hat{f}_i \ast \varphi_R(\xi) = \sum_i e(x_i \cdot \xi) |\hat{f}_i|_{N_{R-1} S} \ast \varphi_R(\xi)$$

since $\varphi_R$ is supported in $B_{R-1}(0)$. Therefore, the proof is completed by applying \[5.7\] with $g_i = |\hat{f}_i|_{N_{R-1} S}$ followed by \[5.6\] and the embedding $L^p \hookrightarrow L^q$;

$$\left( \sum_i \|f_i\|_{L^p(\mathbb{R}^d)}^q \right)^{1/q} \lesssim \left( \sum_i \|f_i\|_{L^q(\mathbb{R}^d)}^q \right)^{1/q} \leq \left( \sum_i \|f_i\|_{L^p(\mathbb{R}^d)}^p \right)^{1/p} = \|f\|_{L^p(\mathbb{R}^d)}.$$

The estimate \[5.7\] can be found in \[T1\] in a slightly different form. We give a proof for the convenience of the reader, incorporating a simplified $L^2$ estimate from \[BG\]. It is enough to establish \[5.7\] for $q = 1$ and $q = 2$ by interpolation. Consider the case $q = 1$. Note that $|\varphi_R| \lesssim R^d \chi_{B_{R-1}(0)}$. This gives that for any $y \in \mathbb{R}^d$

$$\int_S |\varphi_R(\xi - y)|d\sigma(\xi) \lesssim R^d |S \cap B_{R-1}(y)| \lesssim R.$$

This finishes the proof for $q = 1$ by the triangle inequality and Fubini’s theorem.

When $q = 2$, we shall prove \[5.7\] with $\varphi$ replaced by $\eta$, where $\eta$ is a smooth function supported in $B_2(0)$. Then the original statement follows by writing $\varphi$ as
a sum of compactly supported functions and using the rapid decay of \( \varphi \) away from \( B_2(0) \). Following \[BG\], we write \( \| \sum \epsilon(x_i \cdot \xi) g_i \ast \tilde{\eta}_R |S|^{2} L^2(\sigma) \) as

\[
(5.8) \quad \sum_i \left\| \mathcal{G}_i |S| \right\|_{L^2(d\sigma)}^2 + \sum_{i \neq j} \int S e((x_i - x_j) \cdot \xi) \mathcal{G}_i(\xi) \mathcal{G}_j(\xi) d\sigma(\xi),
\]

where \( \mathcal{G}_i = \tilde{g}_i \eta_R \).

We recall the standard \( L^2 \) estimate (see, for example, \[BG2 \] Lemma 3.2])

\[
\left\| \mathcal{G} \right\|_{L^2(B_R)} \lesssim R^{1/2} \| f \|_{L^2(B_R)}.
\]

Using this estimate and Plancherel’s theorem, we bound the first term in \[BGN \] by

\[
R \sum_i \left\| \mathcal{G}_i \right\|_{L^2(\mathbb{R}^d)}^2 \lesssim R \sum_i \left\| g_i \right\|_{L^2(\mathbb{R}^d)}^2.
\]

The integral in the second term in \( (5.8) \) is \( \tilde{G}_i \ast \tilde{G}_j \ast \tilde{\sigma}(x_i - x_j) \), where \( \tilde{G}_i(x) = \mathcal{G}_i(-x) \). We use the decay of \( \tilde{\sigma} \) and the sparsity assumption together with the fact that \( \tilde{G}_i \ast \tilde{G}_j \) is supported in \( B_{2R}(0) \) to obtain

\[
\left\| \tilde{G}_i \ast \tilde{G}_j \ast \tilde{\sigma}(x_i - x_j) \right\| \lesssim \left\| x_i - x_j \right\|^{-\rho} \left\| \mathcal{G}_i \right\|_{L^1(\mathbb{R}^d)} \left\| \mathcal{G}_j \right\|_{L^1(\mathbb{R}^d)} \lesssim (RN)^{-C(d,\rho)} R^d \left\| g_i \right\|_{L^2(\mathbb{R}^d)} \left\| g_j \right\|_{L^2(\mathbb{R}^d)}
\]

by the Cauchy-Schwarz inequality and Plancherel’s theorem. Recall that \( C(d,\rho) = (d - 1). \) Summing over \( i, j \) using Cauchy-Schwarz, we bound \( (5.8) \) by

\[
R \sum_i \left\| g_i \right\|_{L^2(\mathbb{R}^d)}^2 + (RN)^{-(d-1)} R^d N \sum_i \left\| g_i \right\|_{L^2(\mathbb{R}^d)}^2 \lesssim R \sum_i \left\| g_i \right\|_{L^2(\mathbb{R}^d)}^2,
\]

which completes the proof of \( (5.7) \) for the case \( q = 2 \). We remark that the proof, in fact, required a weaker sparseness condition \( \left| x_i - x_j \right| \geq (RN^{1/(d-1)})C(d,\rho) \) for \( i \neq j \).

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN-MADISON, MADISON, WI 53706 USA

E-mail address: jkim@math.wisc.edu