A UNIFORM BOUND FOR THE LAGRANGE POLYNOMIALS OF LEJA POINTS FOR THE UNIT DISK

ABSTRACT. We study uniform estimates for the family of fundamental Lagrange polynomials defined from any Leja sequence for the complex unit disk. The main result claims that all these polynomials are uniformly bounded on the disk, i.e. independently on the range $N$ of the associated $N$-Leja section. An essential and immediate application is an estimate of the associated Lebesgue constants as $\Lambda_N = O(N)$, $\forall N \geq 1$. We also deal with the special case of $N = 2^p - 1$ where we give an improvement of the main result.

CONTENTS

1. Introduction

1.1. Definition of Leja points for the unit disk. In this paper we deal with the estimates of the Lagrange polynomials of Leja points for the unit disk. We first remind the complex unit disk,

$$\mathbb{D} = \{ z \in \mathbb{C}, |z| < 1 \},$$

and $\overline{\mathbb{D}}$ is the closed one.

Next, for $N \geq 1$, for $z_1, \ldots, z_N$ all different complex numbers and $z \in \mathbb{C}$, we consider the fundamental Lagrange interpolation polynomial (FLIP) associated with $z_k$,

$$l_k^{(N)}(z) = \prod_{j=1, j \neq k}^{N} \frac{z - z_j}{z_k - z_j}, \quad k = 1, \ldots, N.$$  

The problem of finding good sets $\{ z_k \}_{k \geq 1}$ for Lagrange interpolation (i.e. for which we can have some control of the associated FLIPs) is a domain of big interest. One of them is called a Leja sequence and will be considered in the whole paper.

Definition 1. A Leja sequence $\mathcal{L}$ for $\overline{\mathbb{D}}$ is a sequence $\{ z_1, z_2, \ldots, z_k, \ldots \}$ that satisfies the following property:

$$|z_1| = 1,$$
and for all \( k \geq 2 \),

\[
\sup_{z \in \mathbb{D}} \left| \prod_{j=1}^{k-1} |z - z_j| \right| = \prod_{j=1}^{k-1} |z_k - z_j|.
\]  

(1.3)

For all \( N \geq 1 \), the \( N \)-Leja section \( \mathcal{L}_N \) of a Leja sequence \( \mathcal{L} \) is the finite sequence given by the first \( N \) points of \( \mathcal{L} \).

These sequences took their name from F. Leja (see [4]) but they were first considered by A. Edrei (see [4], p. 78). Of course, these sequences are not necessarily unique. Even if we fix the first \( k \) points \( z_j \), the choice for \( z_{k+1} \) can be multiple in general. On the other hand, it follows by the maximum principle that all the \( z_j \)'s lie on the unit circle.

Finally, these sets can be interpreted as one-dimensional Fekete sets (see [5]): a \( N \)-Fekete set for the compact subset \( \overline{\mathbb{D}} \) is a set of \( N \) elements \( z_1^*, \ldots, z_N^* \in \overline{\mathbb{D}} \) which maximize (in modulus) the Vandermonde determinant, i.e.

\[
|VdM (z_1^*, \ldots, z_N^*)| = \sup_{z_1, \ldots, z_N \in \overline{\mathbb{D}}} |VdM (z_1, \ldots, z_N)|
\]

\[
= \sup_{z_1, \ldots, z_N \in \overline{\mathbb{D}}} \prod_{1 \leq i < j \leq N} |z_j - z_i|.
\]  

(1.4)

One of the essential differences is that determining Fekete sets is an \( N \)-dimensional (with respect to \( \mathbb{C} \)) optimization problem while determining Leja sequences is just a 1-dimensional one. In addition, it follows from Definition 1 that the construction of Leja sequences is inductive (unlike any \( N \)-Fekete set which requires a new research of an \( N \)-tuple \( (z_1^*, \ldots, z_N^*) \) for every \( N \geq 1 \)).

1.2. Main result and some applications. The essential result that we will prove is the following.

**Theorem 1.** Let \( \mathcal{L} = \{z_1, z_2, \ldots\} \) be a Leja sequence for the unit disk. Then the FLIPs \( l_k^{(N)}(z) \) are uniformly bounded with respect to \( N \geq 1 \) and \( k = 1, \ldots, N \), i.e.

\[
\sup_{N \geq k \geq 1} \left( \sup_{z \in \mathbb{D}} \left| l_k^{(N)}(z) \right| \right) \leq \pi \exp(3\pi).
\]

First, the explicit bound shall not be optimal: indeed, this can be seen along the whole proof; on the other hand, some improvements for the bound will be given in a special case for \( N \) below (Subsection 1.3, Theorem 2).

Next, an important interpretation of this result is that any \( N \)-Leja section for the disk has essentially the same property than any \( N \)-Fekete set. Indeed, it is known that the FLIPs associated with any \( N \)-Fekete set are always bounded by 1: this can be shown by noticing that every FLIP can be written as follows for all \( z \in \mathbb{C} \),

\[
\left| l_k^{(N)}(z) \right| = \left| \frac{VdM (z_k^*, \ldots, z_{k-1}^*, z, z_{k+1}^*, \ldots, z_N^*)}{VdM (z_1^*, \ldots, z_N^*)} \right|
\]
(where we have replaced \( z_k^* \) by \( z \)). It follows by (1.4) (and from the fact that 
\[
\sup_{z \in \mathbb{D}} \left| I_{z_k}^{(N)}(z) \right| \geq \left| I_{z_k^*}^{(N)}(z_k^*) \right| = 1
\]
that
\[
\sup_{z \in \mathbb{D}} \left| I_{z_k}^{(N)}(z) \right| = 1 \quad \text{for all} \quad k = 1, \ldots, N.
\]

Thus, the Fekete sets are essentially the best ones for Lagrange interpolation and uniform stability of the associated FLIPs. Nevertheless, constructing them is generally a hard task. Therefore, a natural question is if there exist simpler sets with the same property. Theorem 1 gives an affirmative answer with the Leja sequences for the unit disk (with a bigger bound but still universal).

Finally, as another application we can immediately deduce an estimate analogous to (1.15) from [3] for the Lebesgue constant \( \Lambda_{L_N} \), of the \( N \)-section from any Leja sequence for the unit disk. We remind that the Lebesgue constant is defined for \( N \geq 1 \) by

\[
\Lambda_{L_N} = \sup_{z \in \mathbb{D}} \left( \sum_{k=1}^{N} \left| I_{z_k}^{(N)}(z) \right| \right).
\]

**Corollary 1.** For all \( N \geq 1 \),

\[
\Lambda_{L_N} \leq \pi \exp(3\pi) \times N = O(N).
\]

Of course, this result gives a weaker estimate than (1.15) from [3] where it is proved that \( \Lambda_{L_N} \leq 2N \) for all \( N \geq 1 \). We cannot \textit{a priori} hope any improvement of the bound for \( O(N) \) by this way since we crudely estimate

\[
\Lambda_{L_N} = \sup_{z \in \mathbb{D}} \left( \sum_{k=1}^{N} \left| I_{z_k}^{(N)}(z) \right| \right) \leq \sum_{k=1}^{N} \sup_{z \in \mathbb{D}} \left| I_{z_k}^{(N)}(z) \right|.
\]

Nevertheless, the consequence of this result still gives an improvement of Corollary 7 from [2] where it is proved that \( \Lambda_{L_N} = O(N \ln N) \).

1.3. **On the special case of** \( N = 2^p - 1 \). As it has been pointed out above, the bound from Theorem 1 shall not be optimal. That is why we want to consider in this subsection the special case of \( N = 2^p - 1 \) where the associated estimate can be considerably improved. Indeed, it is first proved (see Section 5, Proposition 1) that

\[
\sup_{z \in \mathbb{D}} \left| I_k^{(2^p-1)}(z) \right| \leq 2 \quad \text{for all} \quad k = 1, \ldots, 2^p - 1
\]

(1.8) and for exceptional values of \( k \), the bound cannot be better than \( 4/\pi \).

On the other hand, numerical simulations let us think that for \textit{almost} \( k = 1, \ldots, 2^p - 1 \), the associated bound for \( I_k^{(2^p-1)} \) is \textit{almost} 1, i.e. any \( (2^p - 1) \)-Leja section is \textit{almost} a \( (2^p - 1) \)-Fekete set for the unit disk. We shall explain what is meant by using \textit{almost} and this is specified by the following result.

**Theorem 2.** Let \( \mathcal{L} \) be any Leja sequence for the unit disk and let consider for all \( p \geq 1 \), \( \left\{ I_k^{(2^p-1)} \right\}_{1 \leq k \leq 2^p - 1} \) the associated family of FLIPs. Then

\[
\lim_{p \to +\infty} \left[ \frac{1}{2^p - 1} \sum_{k=1}^{2^p - 1} \sup_{z \in \mathbb{D}} \left| I_k^{(2^p-1)}(z) \right| \right] = 1.
\]
More precisely, for all \( p \geq 2 \),
\[
2^p - 1 < \sum_{k=1}^{2^p-1} \sup_{z \in \mathbb{D}} |l_k^{(2^p-1)}(z)| \leq (1 + \varepsilon(1/p)) \times (2^p - 1) ,
\]
where \( \lim_{p \to +\infty} \varepsilon(1/p) = 0 \).

First, this result means that, except for an asymptotic number of values for \( k \), the FLIPs are asymptotically bounded by 1.

Next, as an application it is a heuristic confirmation of Theorem 8 from \cite{2} where it is proved that \( \Lambda_{L_2} = 2^p - 1 \) for all \( p \geq 1 \). If we wanted to get this last result as an application of Theorem 2, we should then prove a better estimate, like for example
\[
\sum_{k=1}^{2^p-1} \sup_{z \in \mathbb{D}} |l_k^{(2^p-1)}(z)| \leq 2^p - 1 .
\]
Unfortunately, and as it could be suspected, this cannot be possible as specified by the left-hand side of (1.10).

Finally, this makes us think that this way will not allow us to prove the conjecture from \cite{2} that \( \Lambda_L \leq N \) for all \( N \geq 1 \) (although we think that the worst values of \( \Lambda_L \) appear for \( N = 2^p - 1 \), as it has been pointed out in the last part of \cite{3}, Numerical illustration, p. 198–199).

Acknowledgments

I would like to thank J. Ortega-Cerdà for having introduced me this interesting problem and for all the rewarding ideas about it. I would also like to thank J.-P. Calvi for his help and the discussions on this subject.

2. A couple of reminders and preliminary results

2.1. Reminders. First, we recall some properties and applications of the roots of the unity: for any \( m \geq 1 \), let
\[
\Omega_m = \left\{ \exp \left( \frac{2i\pi u}{m} \right), \ u = 0, \ldots, m - 1 \right\}
\]
be the set of the \( m \)-th roots of the unity. Then one has the following result.

Lemma 1. For all \( m \geq 1 \), \( z_k \in \Omega_m \) and \( z \in \mathbb{D} \) with \( z \neq z_k \), one has
\[
\prod_{z_j \in \Omega_m, z_j \neq z_k} (z - z_j) = \frac{z^m - 1}{z - z_k} = \sum_{j=0}^{m-1} z_k^{m-j-1} z^j .
\]

It follows that
\[
\prod_{z_j \in \Omega_m, z_j \neq z_k} |z_k - z_j| = m
\]
and for all \( z \neq z_k \),
\[
|l_k^{(m)}(z)| = \prod_{z_j \in \Omega_m, z_j \neq z_k} \left| \frac{z - z_j}{z_k - z_j} \right| = \frac{1}{m} \left| \frac{z^m - 1}{z - z_k} \right| = \frac{1}{m} \left| \sum_{j=0}^{m-1} z_k^{m-j-1} z^j \right| .
\]

In addition,
\[
\sup_{z \in \mathbb{D}} |l_k^{(m)}(z)| = |l_k^{(m)}(z_k)| = 1 .
\]
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Proof. First, since $\Omega_m$ is the set of the $m$-th roots of the unity, one has

$$\prod_{z_j \in \Omega_m, z_j \neq z_k} (z - z_j) = \frac{\prod_{z_j \in \Omega_m} (z - z_j)}{z - z_k} = \frac{z^m - 1}{z - z_k}.$$ 

On the other hand (as $z_k \in \Omega_m$),

$$\frac{z^m - 1}{z - z_k} = \frac{z^m - z_k^m}{z - z_k} = \sum_{j=0}^{m-1} z_k^{m-j-1} z^j,$$

and this proves (2.2).

Next, (2.3) follows since

$$\prod_{z_j \in \Omega_m, z_j \neq z_k} (z_k - z_j) = \lim_{z \to z_k, z \neq z_k} \prod_{z_j \in \Omega_m, z_j \neq z_k} (z - z_j) = \lim_{z \to z_k, z \neq z_k} \frac{z^m - 1}{z - z_k} = \frac{d}{dz} (z^m) \big|_{z = z_k} = mz_k^{m-1}.$$ 

Finally, (2.2) and (2.3) yield (2.4) for all $z \neq z_k$ since

$$\left| l_k^{(m)}(z) \right| = \prod_{z_j \in \Omega_m, z_j \neq z_k} \left| z - z_j \right| \leq \frac{1}{m} \prod_{z_j \in \Omega_m, z_j \neq z_k} \left| z_k - z_j \right| \leq \frac{1}{m} \sum_{j=0}^{m-1} |z_k|^{m-j-1} |z|^j \leq 1.$$

In particular, for all $z \in \overline{D}$,

$$\left| l_k^{(m)}(z) \right| = \frac{1}{m} \sum_{j=0}^{m-1} z_k^{m-j-1} z^j \leq \frac{1}{m} \sum_{j=0}^{m-1} |z_k|^{m-j-1} |z|^j \leq 1.$$ 

On the other hand,

$$\left| l_k^{(m)}(z_k) \right| = \prod_{z_j \in \Omega_m, z_j \neq z_k} \left| \frac{z_k - z_j}{z_k - z_j} \right| = 1$$

and the last assertion of the lemma follows.

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In all the following, we will assume that any considered Leja sequence $L = (z_1, z_2, \ldots)$ starts at 1, i.e.

(2.6) $z_1 = 1$.

Next, for any integer $N \geq 1$, $L_N$ will mean the $N$-section of $L$, i.e. the first $N$ indexed points

(2.7) $L_N := (z_1, z_2, \ldots, z_{N-1}, z_N)$.

We will also consider its binary decomposition

(2.8) $N = 2^{p_1} + 2^{p_2} + \cdots + 2^{p_n},$

where

(2.9) $n \geq 1$ and $p_1 > p_2 > \cdots > p_n \geq 0$

($n$ is the number of "ones" in this decomposition).
2.2. Preliminary results. In this subsection, we will give some preliminary results that will be useful in order to prove Theorem 1. We begin with the following lemma that is a rewriting of the FLIPs by using the binary decomposition of \( N \).

**Lemma 2.** Let remind the binary decomposition (2.8),

\[
N = 2p_1 + 2p_2 + \cdots + 2p_n, \tag{2.10}
\]

where \( n \geq 2 \) and \( p_1 \geq p_2 \geq \cdots \geq p_n \geq 0 \). Then for all \( k = 1, \ldots, 2p_1 \) and \( z \in \mathbb{C} \) with \( z \neq z_k \),

\[
|l_k^{(N)}(z)| = \frac{1}{2^{p_1}} \left| z^{2^{p_1}} - 1 \right| \times \prod_{q=2}^{n} \left| z^{2^{p_q}} - \alpha_q^{2^{p_q}} \right|, \tag{2.11}
\]

where for all \( q = 2, \ldots, n \),

\[
\alpha_q = \rho_1 \rho_2 \cdots \rho_{q-1} \tag{2.12}
\]

and for \( j = 1, 2, \ldots, n \), \( \rho_j \) is a \( 2^{p_j} \)-th root of \(-1\).

One also has for all \( k = 1, \ldots, 2p_1 \) and \( z \in \mathbb{C} \),

\[
|l_k^{(N)}(z)| = \frac{1}{2^{p_1}} \left| \sum_{j=0}^{2^{p_1}-1} z_{k}^{2^{p_1}-j-1}z \right| \times \prod_{q=2}^{n} \left| z^{2^{p_q}} - \alpha_q^{2^{p_q}} \right|, \tag{2.13}
\]

**Proof.** First, we know (see Theorem 5 from [1] or Theorem 1 from [2]) that

\[
\mathcal{L}_N = \left( \Omega_{2p_1}, \rho_1 \tilde{\mathcal{L}}_{N-2p_1} \right), \tag{2.14}
\]

where \( \Omega_{2p_1} \) is the set of the \( 2^{p_1} \)-th roots of the unity, \( \rho_1 \) is a \( 2^{p_1} \)-th root of \(-1\) and \( \tilde{\mathcal{L}}_{N-2p_1} \) is (maybe another) \( (N-2p_1) \)-Leja section (that also starts at \( 1 \)). Now \( N-2p_1 = 2p_2 + \cdots + 2p_n \) then we can follow the process with

\[
\tilde{\mathcal{L}}_{N-2p_1} = \left( \Omega_{2p_2}, \rho_2 \tilde{\mathcal{L}}_{N-2p_1-2p_2} \right)
\]

and so on. We also know by Theorem 5 from [1] that any \( 2^{p_n} \)-Leja section (that starts at \( 1 \)) will consist of the \( 2^{p_n} \)-th roots of the unity, \( \Omega_{2p_n} \), (where the equality is meant as sets). This finally gives

\[
\mathcal{L}_N = \left( \Omega_{2p_1}, \rho_1 \Omega_{2p_2}, \rho_1 \rho_2 \Omega_{2p_3}, \ldots, \rho_1 \cdots \rho_{n-1} \Omega_{2p_n} \right)
\]

\[
= \left( \Omega_{2p_1}, \alpha_2 \Omega_{2p_2}, \alpha_3 \Omega_{2p_3}, \ldots, \alpha_n \Omega_{2p_n} \right).
\]

In particular, \( z_k \in \Omega_{2p_1} \) for all \( 1 \leq k \leq 2p_1 \), then

\[
l_k^{(N)}(z) = \left( \prod_{z_j \in \Omega_{2p_1}, z_j \neq z_k} \frac{z - z_j}{z_k - z_j} \right) \times \prod_{q=2}^{n} \left( \prod_{z_j \in \Omega_{2p_q}, z_j \neq z_k} \frac{z - z_j}{z_k - z_j} \right).
\]

On the one hand, one has by (2.4) from Lemma 1 that

\[
\prod_{z_j \in \Omega_{2p_1}, z_j \neq z_k} \frac{z - z_j}{z_k - z_j} = \frac{1}{2^{p_1}} \left| z^{2^{p_1}} - 1 \right| = \frac{1}{2^{p_1}} \sum_{j=0}^{2^{p_1}-1} z_k^{2^{p_1}-j-1}z_j.
\]
On the other hand, for all \( q = 2, \ldots, n \),
\[
\prod_{z_j \in \alpha_q \Omega_{2^pq}} \frac{z - z_j}{z_k - z_j} = \prod_{z_j \in \Omega_{2^pq}} \frac{z - \alpha_q z_j}{z_k - \alpha_q z_j} = \prod_{z_j \in \Omega_{2^pq}} \frac{z/\alpha_q - z_j}{z_k/\alpha_q - z_j}
\]

(since \( \alpha_q = 1 \) for all \( q = 2, \ldots, n \)), and the lemma is proved.

As a consequence, we get the following result that will be useful in the next section.

**Lemma 3.** Under the same hypothesis for \( N \) as in Lemma 2, one has for all \( k = 1, \ldots, 2^p \) and \( z \in \mathbb{C} \) with \( z \neq z_k \),
\[
|f_k^{(N)}(z)| = \frac{1}{2^{p_1}} \frac{|z^{2^p q - 1}|}{|z - z_k|} \prod_{q=2}^{n} \frac{|z^{2^p q} + \omega_0^{2^p q}|}{|z_k^{2^p q} + \omega_0^{2^p q}|},
\]
where \( \omega_0 \) is a \( 2^{p_1} \)-th root of \(-1\). In fact,
\[
\omega_0 = \alpha_{n+1} = \rho_1 \rho_2 \cdots \rho_{n-1} \rho_n.
\]

Similarly, one also has for all \( k = 1, \ldots, 2^p \) and \( z \in \mathbb{C} \) (with the same \( \omega_0 \)),
\[
|f_k^{(N)}(z)| = \frac{1}{2^{p_1}} \sum_{j=0}^{2^p q - 1} z_k^{2^p q - j - 1} z^j \prod_{q=2}^{n} \frac{|z^{2^p q} + \omega_0^{2^p q}|}{|z_k^{2^p q} + \omega_0^{2^p q}|}.
\]

**Proof.** It suffices to show that, for all \( q = 1, \ldots, n \), one has \( \omega_0^{2^p q} = -\alpha_q^{2^p q} \). This is immediate from (2.12) since
\[
\omega_0^{2^p q} = (\rho_1 \cdots \rho_{q-1})^{2^p q} \times \rho_q^{2^p q} \times (\rho_{q+1} \cdots \rho_n)^{2^p q} = \alpha_q^{2^p q} \times \exp(i\pi) \times \exp[i\pi(2p_q - p_{q+1} + \cdots + 2p_n - p_n)]
\]
and \( 0 < p_q - p_{q+1} < \cdots < p_q - p_n \). Similarly,
\[
\omega_0^{2^p q} = \rho_1^{2^p r} \times (\rho_2 \cdots \rho_n)^{2^p q} = -1.
\]

**Remark 2.1.** \( \omega_0 \) is any of \( 2^p \) possible choices for the \((N+1)\)-th Leja point \( z_{N+1} \). Indeed,
\[
\prod_{j=1}^{N} |z - z_j| = \prod_{q=1}^{n} \left| z^{2^p q} + \omega_0^{2^p q} \right| \leq 2^n
\]
for all \( |z| \leq 1 \). In addition, this bound is reached if and only if for all \( q = 1, \ldots, n \),
\[
|z^{2^p q} + \omega_0^{2^p q}| = |(z/\omega_0)^{2^p q} + 1| = 2,
\]
then if and only if
\[
\left( \frac{z}{\omega_0} \right)^{2^p q} = 1, \quad \text{i.e.} \quad z_{N+1} \in \omega_0 \Omega_{2^p q}.
\]
On the other hand, notice that the data of $\Omega_{2p_1}$ and $\omega_0$ (any fixed $2p_1$-th root of $-1$), give a $N$-Leja section (that starts at $z_1 = 1$). Indeed, set

$$\rho_q = \exp \left( \frac{i\pi}{2p_q} \right) \quad \text{for all} \quad q = 2, \ldots, n, \quad \text{and} \quad \rho_1 := \frac{\omega_0}{\rho_2 \cdots \rho_n}.$$ 

Then $\rho_q$ is a $2p_q$-th root of $-1$ for all $q = 2, \ldots, n$, and $\rho_1$ is a $2p_1$-th root of $-1$ since $\omega_0$ is and $\rho_q \in \Omega_{2p_1}$ for all $q = 2, \ldots, n$ (recall from the hypothesis of Lemma 2 that $n \geq 2$ and $p_1 > p_2 > \cdots > p_n \geq 0$). Then the $N$-sequence defined by

$$(2.18) \quad \mathcal{L}_N := \left( \Omega_{2p_1}, \rho_1 \Omega_{2p_2}, \rho_1 \rho_2 \Omega_{2p_3}, \cdots, \rho_1 \cdots \rho_{n-1} \Omega_{2p_n} \right),$$

is the $N$-section of a Leja sequence (that starts at 1), and for all $k = 1, \ldots, 2p_1$, the function defined by $(2.15)$ or $(2.17)$ is the FLIP (at least in modulus) associated with the $k$-th point from $\Omega_{2p_1} \subset \mathcal{L}_N$.

Now we give the proof of the two following preliminary (and classical) lemmas.

**Lemma 4.** For all $x \in \left[0, \frac{\pi}{2}\right]$, one has the following estimates:

$$(2.19) \quad \frac{2}{\pi} x \leq \sin(x) \leq x.$$ 

One also has for all $x \in \mathbb{R}$,

$$(2.20) \quad |\sin(x)| \leq |x|.$$ 

**Proof.** The first estimate of $(2.19)$ follows from the concavity of the function $\sin(x)$ on $\left[0, \frac{\pi}{2}\right]$ (by writing $x = \left(1 - \frac{2}{\pi} x\right) \times 0 + \left(\frac{2}{\pi} x\right) \times \frac{\pi}{2}$). The second one can be deduced by considering the variations on $\left[0, \frac{\pi}{2}\right]$ of the function $x \mapsto x - \sin(x)$.

In particular, the estimate $(2.19)$ yields $(2.20)$ for all $x \in \left[0, \frac{\pi}{2}\right]$. If $x \geq \frac{\pi}{2}$, then

$$|\sin(x)| \leq 1 \leq \frac{\pi}{2} \leq x$$

and this proves $(2.20)$ for all $x \geq 0$. Finally, if $x \leq 0$, then applying the above estimate to $-x \geq 0$ leads to

$$|\sin(x)| = |-\sin(x)| = |\sin(-x)| \leq |-x| = |x|.$$ 

The following one can be proved by considering the variations of the function $x \in \mathbb{R} \mapsto \exp(x) - x - 1$.

**Lemma 5.** For all $x \in \mathbb{R}$, one has

$$\exp(x) \geq 1 + x.$$ 

Next, we give and prove the following trigonometric formula.

**Lemma 6.** For all $m \geq 0$ and $\alpha \notin \pi \mathbb{Z}$, one has

$$\prod_{j=1}^{m} \cos \left( \frac{\alpha}{2^j} \right) = \frac{\sin(\alpha)}{2^m \sin(\alpha/2^m)}.$$  

$$(2.21)$$
Proof. First, we claim that, for all $m \geq 0$, $\alpha/2^m \notin \pi\mathbb{Z}$.

Indeed, this assertion can be proved by induction on $m \geq 0$. For $m = 0$, $\alpha/2^0 = \alpha \notin \pi\mathbb{Z}$. Now if for $m \geq 1$ being given, one has that $\alpha/2^m \in \pi\mathbb{Z}$, then one also has that $\alpha/2^{m-1} = 2 \times \alpha/2^m \in \pi\mathbb{Z}$. This is impossible by induction hypothesis.

In particular, this implies that $\sin(\alpha/2^m) \neq 0$ for all $m \geq 0$, then the expressions which appear in (2.21) make sense. The formula (2.21) will be proved by induction on $m \geq 0$. For $m = 0$, one has that

$$\prod_{j=1}^{0} \cos \left( \frac{\alpha}{2^j} \right) = \prod_{j \in \emptyset} \cos \left( \frac{\alpha}{2^j} \right) = 1 = \frac{\sin(\alpha)}{2^0 \sin \left( \frac{\alpha}{2^0} \right)}.$$ 

If it is true for $m \geq 0$, then let consider $m+1$. First, $\cos \left( \frac{\alpha}{2^{m+1}} \right) \neq 0$, otherwise $\alpha/2^{m+1} \in \frac{\pi}{2} + \pi\mathbb{Z}$ then $\alpha/2^m = 2 \times \alpha/2^{m+1} \in \pi\mathbb{Z}$. This is impossible by the above claim. Next, we get by induction hypothesis

$$\prod_{j=1}^{m+1} \cos \left( \frac{\alpha}{2^j} \right) = \prod_{j=1}^{m} \cos \left( \frac{\alpha}{2^j} \right) \times \cos \left( \frac{\alpha}{2^{m+1}} \right) = \frac{\sin(\alpha)}{2^m \sin \left( \frac{\alpha}{2^m} \right)} \times \cos \left( \frac{\alpha}{2^{m+1}} \right)$$

$$= \frac{\sin(\alpha)}{2^{m+1} \sin \left( \frac{\alpha}{2^{m+1}} \right)},$$

and this proves the induction.

$\sqrt{\Box}$

In the next section, we will not need the binary decomposition of $l$, else the alternating binary one as specified by the following result.

Lemma 7. For all integer $l \geq 1$, one can write

$$\sum_{i=1}^{2L} (-1)^{i-1} 2^{s_i}, \text{ where } L \geq 1 \text{ and } s_1 > s_2 > \cdots > s_{2L} \geq 0. \quad (2.22)$$

Proof. First, $l$ being a nonzero integer can be written with a binary decomposition, i.e.

$$l = \sum_{j=1}^{K} 2^{r_j},$$

where $K \geq 1$ and $r_1 > r_2 > \cdots > r_K \geq 0$.

Next, the formula (2.22) will be proved by induction on $K \geq 1$. In addition, we will prove that

$$s_1 = r_1 + 1. \quad (2.23)$$

If $K = 1$, then

$$l = 2^{r_1} = 2^{r_1+1} - 2^{r_1},$$

and the assertion is true by setting $L = 2$, $s_1 = r_1 + 1$ and $s_2 = r_1$ (and we have $s_1 > s_2 \geq 0$).
Now let be \( K \geq 2 \). If \( r_i = r_1 - i + 1 \) for all \( i = 1, \ldots, K \) (this happens when \( l \) is a chain of consecutive ”ones” in its binary decomposition), then

\[
\begin{align*}
l & = \sum_{i=1}^{K} 2^{r_1-i+1} = 2^{r_1-K+1} \sum_{i=1}^{K} 2^{K-i} = 2^{r_1-K+1} \sum_{i=0}^{K-1} 2^i \\
& = 2^{r_1-K+1} \left( \frac{2^K - 1}{2 - 1} \right) = 2^{r_1+1} - 2^{r_K},
\end{align*}
\]

and the assertion is true by setting \( L = 2 \), \( s_1 = r_1 + 1 \) and \( s_2 = r_K \) (and we have \( s_1 > s_2 \geq 0 \)).

Otherwise, there is at least one interior ”zero” in the binary decomposition of \( l \). Let \( v \geq 1 \) be such that,

\[
\begin{align*}
\begin{cases}
r_i = r_1 - i + 1, & \text{for all } i = 1, \ldots, v; \\
r_v+1 \leq r_v - 2
\end{cases}
\end{align*}
\]

(counting the consecutive ”ones” from \( r_1 \) in the first chain from the binary decomposition of \( l \), \( r_v \) is the rank of the last ”one”). Then

\[
\sum_{i=1}^{v} 2^{r_i} = \sum_{i=1}^{v} 2^{r_1-i+1} = 2^{r_1-v+1} \sum_{i=1}^{v} 2^{v-i} = 2^{r_1-v+1} \sum_{i=0}^{v-1} 2^i
\]

\[
= 2^{r_1-v+1} \left( \frac{2^v - 1}{2 - 1} \right) = 2^{r_1+1} - 2^{r_v+1}.
\]

On the other hand, let consider the (nonzero) integer

\[
\sum_{i=1}^{K'} 2^{r_i'} := \sum_{i=v+1}^{K} 2^{r_i},
\]

where \( K' := K - v \), \( r_i' := r_{i+v} \) for all \( i = 1, \ldots, K' \). One still has that \( K' \geq 1 \) (since \( v + 1 \leq K \)) and \( r_1' > r_2' > \cdots > r_{K'}' = r_K \geq 0 \). Since \( K' = K - v \leq K - 1 \), the induction hypothesis can be applied to

\[
\sum_{i=1}^{K'} 2^{r_i'} = \sum_{i=1}^{2L'} (-1)^{i-1} 2^{r_i'},
\]

with \( L' \geq 1 \) and \( s_1' > s_2' > \cdots > s_{2L'}' \geq 0 \). It follows that

\[
l = \sum_{i=1}^{v} 2^{r_i} + \sum_{i=v+1}^{K} 2^{r_i} = 2^{r_1+1} - 2^{r_v+1} + \sum_{i=1}^{K'} 2^{r_i'}
\]

\[
= 2^{r_1+1} - 2^{r_v+1} + \sum_{i=1}^{2L'} (-1)^{i-1} 2^{r_i'} = \sum_{i=1}^{2L} (-1)^{i-1} 2^{s_i'},
\]

where we have set \( L := L' + 1 \), \( s_1 := r_1 + 1 \), \( s_2 := r_1 - v + 1 \) and \( s_i := s_{i-2} \) for all \( i = 3, \ldots, 2L = 2L' + 2 \). In particular, one still has that \( s_1 > s_2 \) and by (2.24), \( s_2 = r_1 - v + 1 = r_v \geq r_{v+1} + 1 = r_1' + 1 = s_1' = s_3 \) (since by the induction hypothesis applied to (2.23), one has \( s_1' = r_1' + 1 \)). The induction hypothesis applied to \( s_1' > s_2' > \cdots > s_{2L}' = s_{2L-2}' \geq 0 \), yields \( s_3 > s_4 > \cdots > s_{2L} \geq 0 \), and the proof is finished.

\[\checkmark\]
Remark 2.2. In the above formula, $L$ is the number of chains of consecutive "ones" in the binary decomposition of $l$.

Since we will deal with alternating sums in the next section, the following result will be useful.

Lemma 8. Let consider

$$\sum_{i=1}^{M} (-1)^i a_i$$ (respectively $\sum_{i=1}^{M} (-1)^{i-1} a_i$),

where $a_i > 0$ is decreasing. Then for all $J = 1, \ldots, M$,

$$(2.25) \quad \left| \sum_{i=J}^{M} (-1)^i a_i \right| = \sum_{i=J}^{M} (-1)^{i-J} a_i \leq a_J.$$

Proof. The proof is by descending induction on $J = M, \ldots, 1$. The assertion is obvious for $J = M$. If $1 \leq J \leq M - 1$, then the induction hypothesis yields

$$\sum_{i=J}^{M} (-1)^{i-J} a_i = a_J - \sum_{i=J+1}^{M} (-1)^{i-J-1} a_i \geq a_J - \sum_{i=J+1}^{M} (-1)^{i-J-1} a_i,$$

since $a_i$ is decreasing. In particular, $\sum_{i=J}^{M} (-1)^{i-J} a_i \geq 0$ then

$$\left| \sum_{i=J}^{M} (-1)^i a_i \right| = \left| \sum_{i=J}^{M} (-1)^{i-J} a_i \right| = \sum_{i=J}^{M} (-1)^{i-J} a_i.$$

On the other hand,

$$\sum_{i=J}^{M} (-1)^{i-J} a_i = a_J - \sum_{i=J+1}^{M} (-1)^{i-J-1} a_i \leq a_J,$$

since by the induction hypothesis, $\sum_{i=J+1}^{M} (-1)^{i-J-1} a_i = \left| \sum_{i=J+1}^{M} (-1)^{i-J-1} a_i \right| \geq 0$. The proof is finished.

\[\checkmark\]

3. Proof of Theorem 1 in a special but essential case

In this part, we will deal with the FLIP $f_1^{(N)}$ (the one that is associated with $z_1 = 1$), i.e. by recalling (2.14) and (2.17) from Lemma 3

$$\text{(3.1)} \quad \left| f_1^{(N)}(z) \right| = \frac{1}{2m} \left| z^{e_1} - 1 \right| \times \prod_{q=2}^{n} \left| z^{2^q e_1} + \omega_0^{2^q e_1} \right| \left| 1 + \omega_0^{2^q} \right| \quad \text{(for all } z \neq 1),$$

$$\text{(3.2)} \quad \left| f_1^{(N)}(z) \right| = \frac{1}{2m} \sum_{j=0}^{2^{e_1}-1} z^j \times \prod_{q=2}^{n} \left| z^{2^q e_1} + \omega_0^{2^q e_1} \right| \left| 1 + \omega_0^{2^q} \right| \quad \text{(for all } z \in \mathbb{C}),$$

with the following hypothesis:
Some preliminary results.

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- $N \geq 1$ and $N$ is not a pure power of 2, i.e. there is $p_1 \geq 1$ such that $2^{p_1} < N < 2^{p_1+1}$. We can then remind the binary decomposition of $N$ given by (2.8) with (2.9):

$$N = 2^{p_1} + \cdots + 2^{p_n} \quad \text{with} \quad p_1 > p_2 > \cdots > p_n \geq 0 \text{ and } n \geq 2;$$

- as it was defined in Lemma 3, $\omega_0$ is a $2^{p_1}$-th root of $-1$, then it can be written as $\exp((2l+1)i\pi/2^{p_1})$ with $0 \leq l \leq 2^{p_1} - 1$. In this section, we will not consider the special case of $\omega_0 = \exp(i\pi/2^{p_1})$, i.e.

$$\omega_0 = \exp\left(\frac{2l + 1}{2^{p_1}} i\pi\right) \quad \text{with} \quad 1 \leq l \leq 2^{p_1} - 1;$$

- we will deal with $|z| = 1$. It can be written as $\exp(i\theta)$ where $\theta \in ]-\pi, \pi]$, but the following writing will be more useful in this section:

$$z = \exp(i\pi\theta) \quad \text{with} \quad \theta \in ]-1, 1].$$

The goal of this section is to prove Theorem 1 for the FLIP $l_1^{(N)}$ under these above restrictions, i.e. that for all $|z| = 1$ and all $l = 1, \ldots, 2^{p_1} - 1$,

$$\left| l_1^{(N)}(z) \right| \leq \pi \exp(3\pi).$$

We begin with noticing that since $|\theta| \leq 1$, then one has that:

1. or $|\theta| \leq 1/2^{p_1}$, then this case will be handled in Subsection 3.2
2. or else $1/2^{p_1} < |\theta| \leq 1/2^{p_n}$, since by 3.3 one has $0 < 1/2^{p_1} < \cdots < 1/2^{p_n}$, it follows that

$$1/2^{p_{q_0} - 1} < |\theta| \leq 1/2^{p_{q_0}} \quad \text{with} \quad q_0 = 2, \ldots, n.$$

We will deal with this case in Subsection 3.3
3. otherwise $1/2^{p_n} < |\theta| \leq 1$, then this case will be handled in Subsection 3.4

3.1. Some preliminary results. Before dealing with the different cases, we need a couple of preliminary results which will be useful in this section. We begin with the first one.

**Lemma 9.** For all $q = 2, \ldots, n$, for all $l = 1, \ldots, 2^{p_1} - 1$, and all $|\theta| \leq 1$, one has

$$\left| \cos\left(\frac{2l + 1}{2^{p_1} - p_q + 1} \pi - \frac{2^{p_q} \pi \theta}{2}\right) \right| \leq 1 + \frac{2^{p_q} \pi |\theta|}{2} \left| \tan\left(\frac{2l + 1}{2^{p_1} - p_q + 1} \pi\right)\right|.$$

**Proof.** Indeed, one has that

$$\left| \cos\left(\frac{2l + 1}{2^{p_1} - p_q + 1} \pi - \frac{2^{p_q} \pi \theta}{2}\right) \right| \leq$$
the last estimate coming by (2.20) from Lemma 4.

Next, we give the following result that uses the alternating binary decomposition of $l$ that is guaranteed by Lemma 7 (since we have assumed that $l \geq 1$).

**Lemma 10.** Let $l \geq 1$ be defined by (3.4) and let consider its alternating binary decomposition

\[
(3.8) \quad l = \sum_{i=1}^{2L} (-1)^{i-1}2^{s_i},
\]

where $L \geq 1$ and

\[
(3.9) \quad s_1 < s_2 < \cdots < s_{2L} \geq 0.
\]

Then $s_1 \leq p_1$. In addition, if we set

\[
(3.10) \quad s_{2L+1} := -1
\]

and

\[
(3.11) \quad s_0 := p_1 + 1,
\]

then one still has that

\[
 s_0 > s_1 > \cdots > s_{2L} > s_{2L+1},
\]

and for all $q = 2, \ldots, n$, or

\[
(3.12) \quad p_q \in T := \bigcup_{j=0}^{2L} \{p_1 - s_j \leq i \leq p_1 - s_{j+1} - 2\}
\]

(with the convention that the subsets for which $s_{j+1} = s_j - 1$ are empty), or else

\[
(3.13) \quad p_q \in S := \{p_1 - s_j - 1, j = 1, \ldots, 2L\}.
\]

**Proof.** First, the decomposition (3.8) yields

\[
l = \sum_{i=1}^{2L} (-1)^{i-1}2^{s_i} \geq 2^{s_1} - \sum_{i=2}^{2L} (-1)^{i-1}2^{s_i} \geq 2^{s_1} - 2^{s_2},
\]

the last estimate being justified by Lemma 8 (because $s_j$ is decreasing). Since $s_2 \leq s_1 - 1$ by (3.9), it follows by (3.10) that

\[
2^{p_1} - 1 \geq l \geq 2^{s_1} - 2^{s_2-1} = 2^{s_1-1}.
\]

If we had that $s_1 \geq p_1 + 1$, this would yield

\[
2^{p_1} - 1 \geq 2^{(p_1+1)-1} = 2^{p_1}.
\]
and this is impossible. Then \( s_1 \leq p_1 \).

Now let fix \( q = 2, \ldots, n \). We know by (3.3) and (3.10) that
\[
p_q \leq p_2 \leq p_1 - 1 = p_1 - s_{2L+1} - 2.
\]
Similarly, we have by (3.3) and (3.11) that
\[
p_q \geq p_n \geq 0 > p_1 - s_0.
\]
It follows that \( p_1 - s_0 \leq p_q \leq p_1 - s_{2L+1} - 2 \), i.e. \( p_q \in T \cup S \) and the lemma is proved (since \( T \) and \( S \) are incompatible).

\[\square\]

Now that we have defined the sets \( T \) and \( S \), we will give auxiliary results which deal with these sets. We begin with \( T \).

**Lemma 11.** Let \( q = 2, \ldots, n \), and \( l = 1, \ldots, 2^{p_1} - 1 \). If \( p_q \in T \) then for all \( |\theta| \leq 1 \), one has
\[
\left| \frac{\cos \left( \frac{2l + 1}{2^{p_1 - p_q + 1} \pi} \right) - \cos \left( \frac{2l + 1}{2^{p_1 - p_q + 1} \pi} \right) \theta \right| \leq 1 + \frac{2^{p_0} \pi |\theta|}{2}.
\]

**Proof.** First, since \( p_q \in T \), by (3.12) there is \( j_q \) with \( 0 \leq j_q \leq 2L \) such that
\[
p_1 - s_{j_q} \leq p_q \leq p_1 - s_{j_q + 1} - 2,
\]
then (by (3.8) and the convention (3.10))
\[
\frac{2l + 1}{2^{p_1 - p_q + 1} \pi} = \frac{\pi}{2^{p_1 - p_q}} (l + 2^{2L+1}) = \frac{\pi}{2^{p_1 - p_q}} \sum_{i=1}^{2L+1} (-1)^{i-1} 2^{s_i}
\]
\[
= \frac{\pi}{2^{p_1 - p_q}} j_q \sum_{i=1}^{j_q} (-1)^{i-1} 2^{s_{i-p_1-p_q}} + \frac{\pi}{2^{p_1 - p_q}} \sum_{i=j_q+1}^{2L+1} (-1)^{i-1} 2^{s_i-p_1-p_q}.
\]
In the first sum, one has \( s_i - p_1 + p_q \geq s_{j_q} - p_1 + p_q \geq 0 \) by (3.9) and (3.11) for all \( i = 1, \ldots, j_q \), then
\[
b_q := \sum_{i=1}^{j_q} (-1)^{i-1} 2^{s_i-p_1-p_q} \in \mathbb{Z}
\]
(notice that the sum \( \sum_{i=1}^{j_q} (-1)^{i-1} 2^{s_i-p_1-p_q} \) may be empty if \( j_q = 0 \); even in this case, the above assertion holds true since \( b_q = \sum_{i \in \emptyset} (-1)^{i-1} 2^{s_i-p_1-p_q} = 0 \in \mathbb{Z} \)). In the second sum (that cannot be empty since \( j_q + 1 \leq 2L + 1 \), one has
\[
\pi \sum_{i=j_q+1}^{2L+1} (-1)^{i-1} 2^{s_i-p_1-p_q} = \frac{(-1)^{j_q} \pi}{2^{p_1 - p_q}} \sum_{i=j_q+1}^{2L+1} (-1)^{i-j_q-1} 2^{s_{j_q+1-s_i}}.
\]
It follows that

\[
\left| \tan \left( \frac{2l + 1}{2p_1 - p_q + 1} \pi \right) \right| = \left| \tan \left( b_q \pi + \frac{(-1)^{j_q} \pi}{2p_1 - p_q - s_{q+1}} \sum_{i = j_q + 1}^{2L+1} \frac{(-1)^{i-j_q-1}}{2^{s_{q+1}+i-1}} \right) \right|
\]

(3.16)

\[
= \left| \tan \left( \frac{\pi}{2p_1 - p_q - s_{q+1}} \sum_{i = j_q + 1}^{2L+1} \frac{(-1)^{i-j_q-1}}{2^{s_{q+1}+i-1}} \right) \right| .
\]

Next, we claim that

\[
0 < \frac{\pi}{2p_1 - p_q - s_{q+1}} \sum_{i = j_q + 1}^{2L+1} \frac{(-1)^{i-j_q-1}}{2^{s_{q+1}+i-1}} \leq \frac{\pi}{4}.
\]

Indeed, one has on the one hand that \( p_1 - p_q - s_{q+1} \geq 2 \) by (3.14), then

\[
0 < \frac{\pi}{2p_1 - p_q - s_{q+1}} \leq \frac{\pi}{4}.
\]

On the other hand, one has by Lemma 8 that

\[
\sum_{i = j_q + 1}^{2L+1} \frac{(-1)^{i-j_q-1}}{2^{s_{q+1}+i-1}} \leq \sum_{i = j_q + 1}^{2L+1} \frac{(-1)^{i-j_q-1}}{2^{s_{q+1}+i-1}} \leq \frac{1}{2^{s_{q+1}+s_{q+1}+1}} = 1,
\]

and

\[
\sum_{i = j_q + 1}^{2L+1} \frac{(-1)^{i-j_q-1}}{2^{s_{q+1}+i-1}} \geq 1 - \sum_{i = j_q + 2}^{2L+1} \frac{(-1)^{i-j_q}}{2^{s_{q+1}+i-1}} \geq 1 - \frac{1}{2^{s_{q+1}+s_{q+2}}} \geq 1 - \frac{1}{2} = \frac{1}{2},
\]

the last estimate coming from (3.9) (notice that the sum \( \sum_{i = j_q + 2}^{2L+1} (-1)^{i-j_q}/2^{s_{q+1}+i-1} \) may be empty if \( j_q = 2L \); even in this case, one still has that \( \sum_{i = j_q + 1}^{2L+1} (-1)^{i-j_q-1}/2^{s_{q+1}+i-1} = 1 - 0 \geq 1/2 \). Hence

\[
\frac{1}{2} \leq \sum_{i = j_q + 1}^{2L+1} \frac{(-1)^{i-j_q-1}}{2^{s_{q+1}+i-1}} \leq 1.
\]

The claim follows by (3.18) and (3.19).

In particular, \( \frac{\pi}{2p_1 - p_q - s_{q+1}} \sum_{i = j_q + 1}^{2L+1} (-1)^{i-j_q-1}/2^{s_{q+1}+i-1} \in \left] 0, \frac{\pi}{2} \right[ \) where the function \( \tan \) is positive and increasing. It follows by (3.16) that

\[
\left| \tan \left( \frac{2l + 1}{2p_1 - p_q + 1} \pi \right) \right| = \left| \tan \left( \frac{\pi}{2p_1 - p_q - s_{q+1}} \sum_{i = j_q + 1}^{2L+1} \frac{(-1)^{i-j_q-1}}{2^{s_{q+1}+i-1}} \right) \right|
\]

\[
= \left| \tan \left( \frac{\pi}{2p_1 - p_q - s_{q+1}} \sum_{i = j_q + 1}^{2L+1} \frac{(-1)^{i-j_q-1}}{2^{s_{q+1}+i-1}} \right) \right| \leq \tan \left( \frac{\pi}{4} \right) = 1,
\]
Then for all \( j \) by (3.13) from Lemma 10 there is
\[
\frac{\cos \left( \frac{2l + 1}{2p_1 - p_q + 1} \pi - \frac{2p_q \pi \theta}{2} \right)}{\cos \left( \frac{2l + 1}{2p_1 - p_q + 1} \pi \right)} \leq 1 + \frac{2p_q \pi |\theta|}{2} \left| \tan \left( \frac{2l + 1}{2p_1 - p_q + 1} \pi \right) \right| \leq 1 + \frac{2p_q \pi |\theta|}{2},
\]
and this proves the lemma.

Next, we deal with the set \( S \).

**Lemma 12.** Let be \( q = 2, \ldots, n \), and \( l = 1, \ldots, 2p_i - 1 \). Assume that \( p_q \in S \), i.e. by (3.13) from Lemma 10 there is \( j_q \) with \( 1 \leq j_q \leq 2L \) such that
\[
(3.20) \quad p_q = p_1 - s_q - 1.
\]
Then for all \( |\theta| \leq 1 \),
\[
\frac{\cos \left( \frac{2l + 1}{2p_1 - p_q + 1} \pi - \frac{2p_q \pi \theta}{2} \right)}{\cos \left( \frac{2l + 1}{2p_1 - p_q + 1} \pi \right)} \leq 1 + \frac{2p_q \pi |\theta|}{2^{1 + s_q \tilde{b}_q + 1}}
\]
(\( s_q \tilde{b}_q + 1 \) makes sense since \( 2 \leq j_q + 1 \leq 2L + 1 \)).

**Proof.** First, \( q \) and the associated \( j_q \) being fixed, one has by (3.8) and the convention \( (3.10) \) from Lemma 10 that
\[
\frac{2l + 1}{2p_1 - p_q + 1} \pi = \frac{\pi}{2p_i - p_q} \sum_{i=1}^{2L+1} (-1)^{i-1} 2^s_i,
\]
\[
= \left( \sum_{i=1}^{j_q} (-1)^{i-1} 2^s_i - p_i + p_q \right) + \left( -1 \right)^{j_q-1} \pi \frac{2L+1}{2p_1 - p_q - s_q} + \sum_{i=j_q+1}^{2L+1} (-1)^{i-1} \pi \frac{2^s_i}{2p_1 - p_q - s_i}.
\]
As before, if the first sum is not empty (otherwise it gives \( 0 \in \pi \mathbb{Z} \)), one has by (3.9) and (3.20) for all \( i = 1, \ldots, j_q - 1 \), that
\[
s_i - p_i + p_q \geq s_{j_q - 1} - p_1 + p_q \geq s_{j_q} + 1 - p_1 + p_q = -1 + 1 = 0,
\]
then
\[
\tilde{b}_q := \sum_{i=1}^{j_q-1} (-1)^{i-1} 2^s_i - p_i + p_q \in \mathbb{Z}.
\]
Similarly, for all \( i = j_q + 1, \ldots, 2L + 1 \),
\[
\pi \sum_{i=j_q+1}^{2L+1} \frac{(-1)^{i-1}}{2p_1 - p_q - s_i} = \frac{(-1)^{j_q} \pi}{2^{2p_1 - p_q - s_{j_q} + 1}} \sum_{i=j_q+1}^{2L+1} \frac{(-1)^{i-1} j_q - 1}{2^{s_{j_q+1} - s_i}}.
\]
Since by \(3.20\) again, \(p_1 - p_q - s_{j_q} = 1\), it follows that

\[
\left| \tan \left( \frac{2l + 1}{2p_1 - p_q + 1} \pi \right) \right| = \left| \tan \left( \frac{-b_q \pi + (-1)^{j_q-1} \pi}{2} + \frac{(-1)^{j_q} \pi}{2p_1 - p_q - s_{j_q} + 1} \sum_{i=j_q+1}^{2l+1} \frac{(-1)^{i-j_q-1}}{2^{s_{j_q}+1-s_i}} \right) \right|
\]

\[
= \left| \cot \left( \frac{\pi}{2p_1 - p_q - s_{j_q} + 1} \sum_{i=j_q+1}^{2l+1} \frac{(-1)^{i-j_q-1}}{2^{s_{j_q}+1-s_i}} \right) \right|
\]

\[
(3.21) \quad \leq \frac{1}{\sin \left( \frac{\pi}{2p_1 - p_q - s_{j_q} + 1} \sum_{i=j_q+1}^{2l+1} \frac{(-1)^{i-j_q-1}/2^{s_{j_q}+1-s_i}}{2} \right)}.
\]

Next, as in the proof of Lemma 11, we claim that

\[
(3.22) \quad \frac{\pi/2}{2p_1 - p_q - s_{j_q} + 1} \leq \frac{\pi}{2p_1 - p_q - s_{j_q} + 1} \sum_{i=j_q+1}^{2l+1} \frac{(-1)^{i-j_q-1}}{2^{s_{j_q}+1-s_i}} \leq \frac{\pi}{4}.
\]

Indeed, one has on the one hand by \(3.9\) and \(3.20\) that \(p_1 - p_q - s_{j_q+1} \geq p_1 - p_q - s_{j_q} + 1 = 1 + 1 = 2\) then

\[
(3.23) \quad \frac{\pi}{2p_1 - p_q - s_{j_q} + 1} \leq \frac{\pi}{4}.
\]

On the other hand, one has by Lemma 8 that

\[
\sum_{i=j_q+1}^{2l+1} \frac{(-1)^{i-j_q-1}}{2^{s_{j_q}+1-s_i}} \leq \left| \sum_{i=j_q+1}^{2l+1} \frac{(-1)^{i-j_q-1}}{2^{s_{j_q}+1-s_i}} \right| \leq \frac{1}{2^{s_{j_q}+1-s_{j_q}+2}} = 1,
\]

and

\[
\sum_{i=j_q+1}^{2l+1} \frac{(-1)^{i-j_q-1}}{2^{s_{j_q}+1-s_i}} = 1 - \sum_{i=j_q+2}^{2l+1} \frac{(-1)^{i-j_q}}{2^{s_{j_q}+1-s_i}} \geq 1 - \sum_{i=j_q+2}^{2l+1} \frac{(-1)^{i-j_q}}{2^{s_{j_q}+1-s_i}} \geq 1 - \frac{1}{2} = \frac{1}{2},
\]

the last estimate coming from \(3.9\). Once again, if the sum \(\sum_{i=j_q+2}^{2l+1} (-1)^{i-j_q}/2^{s_{j_q}+1-s_i}\) is empty (only if \(j_q = 2L\)), then one still has that \(\sum_{i=j_q+1}^{2l+1} (-1)^{i-j_q-1}/2^{s_{j_q}+1-s_i} = 1 \geq 1/2\). Hence

\[
(3.24) \quad \frac{1}{2} \leq \sum_{i=j_q+1}^{2l+1} \frac{(-1)^{i-j_q-1}}{2^{s_{j_q}+1-s_i}} \leq 1.
\]

The claim follows by \(3.23\) and \(3.24\).
Lemma 13. The following application (that is well-defined by (3.20))

\[
\begin{align*}
\tan \left( \frac{2l + 1}{2p_1 - p_q + 1} \pi \right) & \leq \sin \left( \frac{\pi}{2p_1 - p_q - s_{j_q} + 1} \sum_{i=j_q+1}^{2L+1} \frac{1}{(i-j_q-1)^{1/2}} \right) \\
& = \sin \left( \frac{\pi}{2p_1 - p_q - s_{j_q} + 1} \sum_{i=j_q+1}^{2L+1} (i-j_q-1)^{1/2} \right) \\
& \leq \frac{1}{\sin \left( \frac{\pi}{2p_1 - p_q - s_{j_q} + 1} \right)} \leq \frac{1}{2} \frac{\pi}{2p_1 - p_q - s_{j_q} + 1} = 2^{p_1 - p_q - s_{j_q} + 1},
\end{align*}
\]

the last estimate being justified by (2.19) from Lemma 13.

Finally, an application of Lemma 9 yields

\[
\begin{align*}
\left| \cos \left( \frac{2l + 1}{2p_1 - p_q + 1} \pi - \frac{2p_q \pi \theta}{2} \right) \right| & \leq 1 + \frac{2p_q |\theta|}{2} \left| \tan \left( \frac{2l + 1}{2p_1 - p_q + 1} \pi \right) \right| \\
& \leq 1 + \frac{2p_q |\theta|}{2} \times 2^{p_1 - p_q - s_{j_q} + 1} = 1 + \frac{2p_q \pi |\theta|}{2^{1+ s_{j_q} - 1}},
\end{align*}
\]

and the lemma is proved.

We finish the subsection with this result about the application \( q \mapsto j_q \).

**Lemma 13.** The following application (that is well-defined by (3.20))

\[
(3.25) \quad \{2 \leq q \leq n, p_q \in S\} \to \{1 \leq j \leq 2L\}
\]

\[
q \mapsto j_q,
\]

is strictly decreasing. It is in particular injective.

**Proof.** Indeed, let be \( q, q' \) with \( 2 \leq q < q' \leq n \) and such that \( p_q, p_{q'} \in S \). By (3.13), there are \( 1 \leq j_q, j_{q'} \leq 2L \) such that \( p_q = p_1 - s_{j_q} - 1 \) and \( p_{q'} = p_1 - s_{j_{q'}} - 1 \). Since one has by (3.3) that

\[
p_1 - s_{j_q} - 1 = p_q > p_{q'} = p_1 - s_{j_{q'}} - 1,
\]

it follows that \( j_q < j_{q'} \) and by (3.9) this gives

\[
j_q > j_{q'}.
\]

3.2. **First case:** \( |\theta| \leq 1/2p_1 \). In this subsection, we want to prove the estimate (3.20) when \( |\theta| \leq 1/2p_1 \). We can assume in all the following that \( \theta \neq 0 \) since

\[
\{1}^{(N)}(1) = \{1}^{(N)}(z_1) = 1.
\]

First, one has by (3.2), (3.4) and (3.5) that

\[
\left| \{1}^{(N)}(z) \right| = \left| \{1}^{(N)}(\exp(i\pi\theta)) \right| = \]
\[
\frac{1}{2^{p_1}} \left| \sum_{j=0}^{2^{p_1}-1} \exp(2^j i \pi \theta) \right| = \prod_{q=2}^{n} \frac{\exp(2^q \pi i \theta) + \exp\left(\frac{2l+1}{2^{p_1-p_q}} i \pi\right)}{1 + \exp\left(\frac{2l+1}{2^{p_1-p_q}} i \pi\right)} \leq \frac{1}{2^{p_1}} \left( \sum_{j=0}^{2^{p_1}-1} 1 \right) \prod_{q=2}^{n} \frac{1 + \exp\left(\frac{2l+1}{2^{p_1-p_q}} i \pi - 2^{p_q} i \pi \theta\right)}{1 + \exp\left(\frac{2l+1}{2^{p_1-p_q}} i \pi\right)} = \prod_{q=2}^{n} \frac{2 \cos\left(\frac{2l+1}{2^{p_1-p_q}}\right)}{2 \cos\left(\frac{2l+1}{2^{p_1-p_q}} \frac{\pi}{2}\right)} \prod_{q=2}^{n} \frac{\cos\left(\frac{2l+1}{2^{p_1-p_q+1}}\frac{2^{p_q} \pi \theta}{2}\right)}{\cos\left(\frac{2l+1}{2^{p_1-p_q}}\frac{\pi}{2}\right)}.
\]

Next, an application of Lemma 5 yields

\[
(3.26) \quad \left| f_1^{(N)}(z) \right| \leq \left[ \prod_{2 \leq q \leq n, p_q \in T} \prod_{2 \leq q \leq n, p_q \in S} \left| \frac{\cos\left(\frac{2l+1}{2^{p_1-p_q+1}}\frac{2^{p_q} \pi \theta}{2}\right)}{\cos\left(\frac{2l+1}{2^{p_1-p_q}}\frac{\pi}{2}\right)} \right| \right] \prod_{2 \leq q \leq n, p_q \in T} \left( 1 + \frac{2^{p_q} \pi |\theta|}{2^{p_1}} \right) \leq \prod_{q=2}^{n} \left( 1 + \frac{2^{p_q} \pi |\theta|}{2} \right) \leq \prod_{q=2}^{n} \left( 1 + \frac{\pi/2}{2^{q-1}} \right),
\]

the last estimate being justified by the condition that \(|\theta| \leq 1/2^{p_1}\). On the other hand, an immediate induction on \(q = 1, \ldots, n\), that uses \([3.3]\), gives that

\[
p_1 - p_q \geq q - 1.
\]

In addition, an application of Lemma 5 yields

\[
(3.27) \quad \prod_{2 \leq q \leq n, p_q \in T} \left| \frac{\cos\left(\frac{2l+1}{2^{p_1-p_q+1}}\frac{2^{p_q} \pi \theta}{2}\right)}{\cos\left(\frac{2l+1}{2^{p_1-p_q}}\frac{\pi}{2}\right)} \right| \leq \prod_{q=2}^{n} \left( 1 + \frac{\pi/2}{2^{q-1}} \right) \leq \prod_{q=2}^{n} \exp\left(\frac{\pi/2}{2^{q-1}}\right) \leq \exp\left(\sum_{q \geq 1} \frac{\pi/2}{2^q}\right) = \exp\left(\frac{\pi}{2}\right).
\]
Now we deal with the product associated with $S$. Lemma 12 leads to

$$\prod_{2\leq q\leq n, p_q \in S} \left| \cos \left( \frac{2l + 1}{2p_i - p_q} \pi - \frac{2^{p_i} \pi \theta}{2} \right) \right| \leq \prod_{2\leq q\leq n, p_q \in S} \left( 1 + \frac{2^{p_i} |\theta| \pi}{2^{1+s_q} + 1} \right)$$

the last estimate being justified by the condition that $|\theta| \leq 1/2^{p_i}$. On the other hand, the application $q \mapsto j_q$ is injective by Lemma 13, then so is

$$\{2 \leq q \leq n, p_q \in S\} \rightarrow \{2 \leq j \leq 2L + 1\}$$

$$q \mapsto j_q + 1.$$ It follows that

$$\prod_{2\leq q\leq n, p_q \in S} \left| \cos \left( \frac{2l + 1}{2p_i - p_q} \pi - \frac{2^{p_i} \pi \theta}{2} \right) \right| \leq \prod_{j=2}^{2L+1} \left( 1 + \frac{\pi}{2^{1+j_q} + 1} \right).$$

In addition, an immediate descending induction on $j = 2L + 1, \ldots, 2$, with (3.9) and the convention (3.11) together yield

$$1 + s_j \geq 2L + 1 - j.$$ This fact and an application of Lemma 13 lead to

$$\prod_{2\leq q\leq n, p_q \in S} \left| \cos \left( \frac{2l + 1}{2p_i - p_q} \pi - \frac{2^{p_i} \pi \theta}{2} \right) \right| \leq \prod_{j=2}^{2L+1} \left( 1 + \frac{\pi}{2^{2L+1-j}} \right) \leq \prod_{j=2}^{2L+1} \exp \left( \frac{\pi}{2^{2L+1-j}} \right)$$

(3.28)

$$\leq \exp \left( \sum_{j=0}^{\infty} \frac{\pi}{2^j} \right) = \exp (2\pi).$$

Finally, the estimates (3.26), (3.27) and (3.28) together yield

$$\left| t_1^{(N)} (z) \right| \leq \exp \left( \frac{\pi}{2} \right) \times \exp (2\pi) \leq \exp (3\pi),$$

and this proves the estimate (3.6) in the case $|\theta| \leq 1/2^{p_i}$.

3.3. **Second case**: $1/2^{p_i} \leq |\theta| \leq 1/2^{p_0}$. In this subsection we fix $q = q_0 = 2, \ldots, n$ (where we remind that $q_0$ is defined by (3.7) ) and

(3.29)

$$\frac{1}{2^{p_{q_0} - 1}} \leq |\theta| \leq \frac{1}{2^{p_0}}.$$ Let consider the following set

(3.30) $$S_\theta := \{ q = q_0, \ldots, n, p_q \in S \} = \{ q_i \}_{1 \leq i \leq m},$$

where (only if $S_\theta$ is non empty) $m = \text{card}(S_\theta) \geq 1$ and the numeration $(q_i)_{1 \leq i \leq m}$ is chosen so that

(3.31) $$q_0 \leq q_1 < \cdots < q_i < \cdots < q_m \leq n.$$
Before dealing with this case, we want to give a couple of auxiliary results which
will be useful in all the subsection. The first one deals with the \( q \)'s whose \( p_q \in T \)
(recall definition (3.12)).

**Lemma 14.** \( q_0 = 2, \ldots, n \) being fixed, one has for all \( \bar{q} \geq q_0 \) and all \( |\theta| \leq 1/2^{p_{q_0}} \),

\[
\prod_{\bar{q} \leq q \leq n, p_q \in T} \left| \cos \left( \frac{2l + 1}{2p_{q_1} - p_{q_2} + 1} \pi - \frac{2p_q \pi \theta}{2} \right) \right| \leq \exp(\pi).
\]

**Proof.** First, one can assume that \( q_0 \leq \bar{q} \leq n \) (indeed, if \( \bar{q} > n \), then the above
product is empty and the estimate holds true). For all \( q = \bar{q}, \ldots, n \) such that \( p_q \in T \), one can apply Lemma [14] to get (since \( \bar{q} \geq q_0 \))

\[
\prod_{\bar{q} \leq q \leq n, p_q \in T} \left| \cos \left( \frac{2l + 1}{2p_{q_1} - p_{q_2} + 1} \pi - \frac{2p_q \pi \theta}{2} \right) \right| \leq \prod_{q_0 \leq q \leq n, p_q \in T} \left( 1 + \frac{2p_q \pi |\theta|}{2} \right)
\]

\[
\leq \prod_{q \leq n, p_q \in T} \left( 1 + \frac{2p_q \pi |\theta|}{2} \right) \leq \prod_{q = q_0}^{n} \left( 1 + \frac{\pi/2}{2q_{q_0} - q_0} \right),
\]

the last estimate being justified by the condition that \( |\theta| \leq 1/2^{p_{q_0}} \). On the other hand, an immediate induction on \( q = q_0, \ldots, n \) that uses (3.3) yields

\[
p_{q_0} - p_q \geq q - q_0.
\]

It follows by applying Lemma [5] that

\[
\prod_{\bar{q} \leq q \leq n, p_q \in T} \left| \cos \left( \frac{2l + 1}{2p_{q_1} - p_{q_2} + 1} \pi - \frac{2p_q \pi \theta}{2} \right) \right| \leq \prod_{q = q_0}^{n} \left( 1 + \frac{\pi/2}{2q_{q_0} - q_0} \right) \leq \prod_{q = q_0}^{n} \exp \left( \frac{\pi/2}{2q_{q_0} - q_0} \right)
\]

\[
\leq \exp \left( \sum_{j = 0}^{n} \frac{\pi/2}{2^j} \right) = \exp(\pi),
\]

and the lemma is proved.

\( \sqrt{\ }

The next one deals with the set \( S_{\theta} \).

**Lemma 15.** \( q_0 = 2, \ldots, n \) being fixed and \( q_1 \) being defined by (3.30), one has for all \( |\theta| \leq 1/2^{p_{q_0}} \),

\[
\prod_{q \in S_{\theta}, q \neq q_1} \left| \cos \left( \frac{2l + 1}{2p_{q_1} - p_{q_2} + 1} \pi - \frac{2p_q \pi \theta}{2} \right) \right| \leq \exp(2\pi).
\]

**Proof.** First, one can assume that neither \( S_{\theta} \) nor \( S_{\theta} \setminus \{q_1\} \) are empty, i.e. \( m \geq 2 \)
in (3.30), otherwise the assertion is obvious since \( \prod_{q_0}(\cdot) = 1 \). We deduce from (3.30) that

\[
S_{\theta} \setminus \{q_1\} = \{q_i, i = 2, \ldots, m\}.
\]
On the other hand, we remind by (3.20) that for all \( i = 2, \ldots, m \) (since every \( q_i \in S_\theta \) then \( p_{q_i} \in S \),
\[
p_{q_i} = p_1 - s_{j_{q_i}} - 1.
\]
It follows by Lemma 12 that
\[
\prod_{q \in S_\theta, q \neq q_i} \left| \frac{\cos \left( \frac{2l + 1}{2p_1 - p_\theta + \pi} - \frac{2p_\| \pi \theta}{2} \right)}{\cos \left( \frac{2l + 1}{2p_1 - p_\theta + \pi} \right)} \right| = \prod_{i=2}^{m} \left| \frac{\cos \left( \frac{2l + 1}{2p_1 - p_{q_i} + 1} - \frac{2p_\| \pi \theta}{2} \right)}{\cos \left( \frac{2l + 1}{2p_1 - p_{q_i} + 1} \right)} \right|
\]
(3.32)
\[
\leq \prod_{i=2}^{m} \left( 1 + \frac{2p_\| \theta | \pi}{2^{1+s_{j_{q_i}}}} \right).
\]
On the other hand, we know by Lemma 13 that the application \( q \mapsto j_q \), is (strictly) decreasing. Since by (3.31), \( q_{i-1} < q_i \) for all \( i = 2, \ldots, m \), it follows that \( j_{q_{i-1}} > j_{q_i} \), i.e.
\[
\dot{j}_{q_{i-1}} \geq \dot{j}_{q_i} + 1.
\]
In addition, the application \( j \mapsto s_j \) being decreasing by (3.20), this yields
\[
(3.33)
\]
whose application at the estimate (3.32) leads to
\[
\prod_{q \in S_\theta, q \neq q_i} \left| \frac{\cos \left( \frac{2l + 1}{2p_1 - p_\theta + \pi} - \frac{2p_\| \pi \theta}{2} \right)}{\cos \left( \frac{2l + 1}{2p_1 - p_\theta + \pi} \right)} \right| \leq \prod_{i=2}^{m} \left( 1 + \frac{2p_\| \theta | \pi}{2^{1+s_{j_{q_i}}}} \right)
\]
(3.34)
\[
\leq \prod_{i=2}^{m} \left( 1 + \frac{2p_\| \theta | \pi}{2^{1+s_{j_{q_{i-1}}}}} \right) = \prod_{i=1}^{m-1} \left( 1 + \frac{2p_\| \theta | \pi}{2^{1+s_{j_{q_i}}}} \right).
\]
Since \( q_i \) satisfies (3.20) for all \( i = 1, \ldots, m - 1 \), it follows that
\[
\dot{s}_{j_{q_{i-1}}} + 1 = p_1 - p_{q_i},
\]
then the estimate (3.34) becomes by Lemma 8
\[
\prod_{q \in S_\theta, q \neq q_i} \left| \frac{\cos \left( \frac{2l + 1}{2p_1 - p_\theta + \pi} - \frac{2p_\| \pi \theta}{2} \right)}{\cos \left( \frac{2l + 1}{2p_1 - p_\theta + \pi} \right)} \right| \leq \prod_{i=1}^{m-1} \left( 1 + \frac{2p_\| \theta | \pi}{2p_1 - p_{q_i}} \right) \leq \prod_{i=1}^{m-1} \exp \left( 2p_\| \theta | \pi \right)
\]
(3.35)
\[
\leq \exp \left( \sum_{i=1}^{m-1} 2p_\| \theta | \pi \right) = \exp \left( \sum_{q \in S_\theta} 2p_q | \theta | \pi \right)
\]
the last estimate being justified by the condition that \( |\theta| \leq 1/2^{p_\theta} \). On the other hand, an immediate induction on \( q = q_\theta, \ldots, n, \) that uses (3.3) yields
\[
p_{q_{\theta}} - p_q \geq q - q_\theta.
\]
The estimate (3.35) then becomes
\[
\prod_{q \in S_\theta, q \neq q_0} \left| \cos \left( \frac{2l + 1}{2p_1 - p_q + 1} \pi - \frac{2p_q \pi \theta}{2} \right) \right| \leq \exp \left( \sum_{q=q_0}^{n} \frac{\pi}{2q-4q_0} \right) \leq \exp \left( \sum_{j \geq 0} \frac{\pi}{2^j} \right) = \exp (2\pi),
\]
and the lemma is proved.

Now we can deal with the required estimate (3.4) after fixing \( q_0 = 2, \ldots, n \) and \( \theta \) with \( 1/2^{p_{q_0}-1} \leq |\theta| \leq 1/2^{p_q} \). First (since \( \theta \neq 0 \) then \( z \neq 1 \)), one has by (3.1), (3.3) and (3.5) that
\[
|\ell_1^{(N)}(z)| = \frac{1}{2^{p_1}} \left| \frac{z^{2p_1} - 1}{z - 1} \right| \times \prod_{q=2}^{n} \left| \frac{z^{2p_q} + \omega_0^{2p_q}}{1 + \omega_0^{2p_q}} \right|
\]
\[
\leq \frac{1}{2^{p_1}} \left| \frac{z^{2p_1} + 1}{z - 1} \right| \times \prod_{q=2}^{q_0-1} \left| \frac{z^{2p_q} + \omega_0^{2p_q}}{1 + \omega_0^{2p_q}} \right| \times \prod_{q=q_0}^{n} \left| \frac{z^{2p_q} + \omega_0^{2p_q}}{1 + \omega_0^{2p_q}} \right|
\]
\[
= \frac{1}{2^{p_1}} \left| \frac{\exp(i\pi \theta) - 1}{1 + \exp \left( \frac{2l + 1}{2p_1 - p_q} i\pi \right)} \right| \prod_{q=q_0}^{n} \left| \frac{\exp \left( 2p_q i\pi \right) + \exp \left( \frac{2l + 1}{2p_1 - p_q} i\pi \right) \right|}{1 + \exp \left( \frac{2l + 1}{2p_1 - p_q} i\pi \right)}
\]
then
\[
|\ell_1^{(N)}(z)| \leq \frac{1}{2^{p_1}} \left| \frac{\sin(\pi|\theta|/2)}{\sin(\pi|\theta|/2)} \right| \prod_{q=2}^{q_0-1} \left| \frac{1}{\cos \left( \frac{2l + 1}{2p_1 - p_q} \pi \right)} \right| \prod_{q=q_0}^{n} \left| \frac{\cos \left( \frac{2l + 1}{2p_1 - p_q} \pi \right) - \frac{2p_q \pi \theta}{2}}{\cos \left( \frac{2l + 1}{2p_1 - p_q} \pi \right)} \right|
\]
\[
(3.36) \quad \leq \frac{1}{2^{p_1}} \frac{1}{|\theta|} \prod_{q=2}^{q_0-1} \left| \frac{1}{\cos \left( \frac{2l + 1}{2p_1 - p_q} \pi \right)} \right| \times \prod_{q=q_0}^{n} \left| \frac{\cos \left( \frac{2l + 1}{2p_1 - p_q} \pi \right) - \frac{2p_q \pi \theta}{2}}{\cos \left( \frac{2l + 1}{2p_1 - p_q} \pi \right)} \right|
\]
the last estimate following by (2.19) from Lemma 4 indeed, since \( 0 < |\theta| \leq 1 \), then \( \pi|\theta|/2 \in \left[ 0, \frac{\pi}{2} \right] \) where the function \( \sin \) is positive thus,
\[
|\sin \left( \frac{\pi|\theta|}{2} \right)| = \left| \sin \left( \frac{\pi|\theta|}{2} \right) \right| = \sin \left( \frac{\pi|\theta|}{2} \right) \geq \frac{2}{\pi} \times \frac{\pi|\theta|}{2} = |\theta|.
\]
Now if we assume that \( S_\theta = \emptyset \),
then \( \{ p_q, q_0 \leq q \leq n \} \subset T \) by Lemma [10] and

\[
|l_1^{(N)}(z)| \leq \frac{1}{2p_1|\theta|} \times \prod_{q=2}^{q_0-1} \left| \cos \left( \frac{2l + 1}{2p_1 - p_{q-1} + 1} \pi \right) \right| \times \prod_{q_\theta \leq q \leq n, p_q \in T} \left| \cos \left( \frac{2l + 1}{2p_1 - p_{q-1} + 1} \pi - \frac{2p_\theta \pi \theta}{2} \right) \right|.
\]  

(3.38)

On the other hand, one has

\[
\prod_{q=2}^{q_0-1} \left| \cos \left( \frac{2l + 1}{2p_1 - p_{q-1} + 1} \pi \right) \right| \geq \prod_{j=p_1 - p_{q_0 - 1} + 1}^{p_1 - p_{q_0 - 1} + 1} \left| \cos \left( \frac{2l + 1}{2j} \pi \right) \right| \geq \prod_{j=2}^{p_1 - p_{q_0 - 1}} \left| \cos \left( \frac{2l + 1}{2j} \pi \right) \right| = \prod_{j=1}^{p_1 - p_{q_0 - 1}} \cos \left( \frac{(2l + 1)\pi/2}{2j} \right),
\]

since \( p_1 - p_2 + 1 \geq 2 \) and any term of the involved products is not greater than 1. An application of Lemma [6] (with \( \alpha = (2l+1)\pi/2 \notin \pi\mathbb{Z} \) and \( m = p_1 - p_{q_0 - 1} \geq p_1 - p_1 = 0 \) since \( q_0 \geq 2 \)) yields

\[
\prod_{q=2}^{q_0-1} \left| \cos \left( \frac{2l + 1}{2p_1 - p_{q-1} + 1} \pi \right) \right| \geq \frac{|\sin ((2l + 1)\pi/2)|}{2^{p_1 - p_{q_0 - 1}} |\sin \left( \frac{(2l + 1)\pi/2}{2p_{q_0 - 1}} \right)|} \geq \frac{1}{2^{p_1 - p_{q_0 - 1}}}.
\]  

(3.39)

It follows that (3.38), (3.39) and Lemma [14] (with the choice of \( \tilde{q} = q_0 \)) together yield

\[
|l_1^{(N)}(z)| \leq \frac{1}{2p_1|\theta|} \times 2^{p_1 - p_{q_0 - 1}} \times \exp(\pi) = \frac{\exp(\pi)}{2^{p_{q_0 - 1} - 1}|\theta|} \leq \exp(\pi),
\]

the last estimate being justified by the condition that \( |\theta| \geq 1/2^{p_{q_0 - 1}} \), and this proves the required assertion in the case \( S_0 = \emptyset \).

Now we assume that \( S_0 \) is non-empty. In particular, we can deal with \( q_1 \), i.e.

\[
\left| \frac{\cos \left( \frac{2l + 1}{2p_1 - p_{q_1} + 1} \pi - \frac{2p_\theta \pi \theta}{2} \right)}{\cos \left( \frac{2l + 1}{2p_1 - p_{q_1} + 1} \pi \right)} \right| \leq
\]
Then the estimates (3.36), (3.39) and (3.42) together yield

\[ 2 \max \left[ \frac{\cos \left( \frac{2l + 1}{2p_{l_1} - p_{q_1} + 1} \pi \right)}{\cos \left( \frac{2l + 1}{2p_{l_1} - p_{q_1} + 1} \pi \right)}, \frac{2p_{q_1} \pi}{2} \right] \leq\]

\[ 2 \max \left[ \frac{\cos \left( \frac{2l + 1}{2p_{l_1} - p_{q_1} + 1} \pi \right)}{\cos \left( \frac{2l + 1}{2p_{l_1} - p_{q_1} + 1} \pi \right)}, \frac{2p_{q_1} \pi}{2} \right] \]

then by applying (2.20) from Lemma 4

\[ \frac{\cos \left( \frac{2l + 1}{2p_{l_1} - p_{q_1} + 1} \pi \right)}{\cos \left( \frac{2l + 1}{2p_{l_1} - p_{q_1} + 1} \pi \right)} \leq 2 \max \left[ \frac{\cos \left( \frac{2l + 1}{2p_{l_1} - p_{q_1} + 1} \pi \right)}{\cos \left( \frac{2l + 1}{2p_{l_1} - p_{q_1} + 1} \pi \right)}, \frac{2p_{q_1} \pi}{2} \right]. \]

Now, let assume that

\[ \max \left[ \frac{\cos \left( \frac{2l + 1}{2p_{l_1} - p_{q_1} + 1} \pi \right)}{\cos \left( \frac{2l + 1}{2p_{l_1} - p_{q_1} + 1} \pi \right)}, \frac{2p_{q_1} \pi}{2} \right] = \frac{\cos \left( \frac{2l + 1}{2p_{l_1} - p_{q_1} + 1} \pi \right)}{\cos \left( \frac{2l + 1}{2p_{l_1} - p_{q_1} + 1} \pi \right)}, \]

then (3.40) becomes

\[ \frac{\cos \left( \frac{2l + 1}{2p_{l_1} - p_{q_1} + 1} \pi \right)}{\cos \left( \frac{2l + 1}{2p_{l_1} - p_{q_1} + 1} \pi \right)} \leq 2. \]

It follows by Lemma 14 (with the choice of \( \bar{q} = q_0 \)) and Lemma 15 that

\[ \prod_{q = q_0}^{n} \frac{\cos \left( \frac{2l + 1}{2p_{l_1} - p_{q} + 1} \pi \right)}{\cos \left( \frac{2l + 1}{2p_{l_1} - p_{q} + 1} \pi \right)} \times \prod_{q_0 \leq q \leq n, p_q \in T} \prod_{q \in S_q, q \neq q_1} \frac{\cos \left( \frac{2l + 1}{2p_{l_1} - p_{q} + 1} \pi \right)}{\cos \left( \frac{2l + 1}{2p_{l_1} - p_{q} + 1} \pi \right)} \times \]

\[ \leq 2 \times \exp(\pi) \times \exp(2\pi) \leq \pi \exp(3\pi). \]

Then the estimates (3.39), (3.39) and (3.42) together yield

\[ \left| f_1^{(N)}(z) \right| \leq \frac{1}{2p_{\bar{q}}} \times 2p_{q_0 - 1} \times \pi \exp(3\pi) = \frac{\pi \exp(3\pi)}{2p_{q_0 - 1} |\bar{q}|} \leq \pi \exp(3\pi), \]

the last estimate being justified by the condition that \( |\bar{q}| \geq 1/2p_{q_0 - 1} \), and this proves the required assertion in this case.
The remaining case is the one for which

\[(3.43) \quad \max \left[ \left| \frac{z^{2^l+1} - 1}{2^p_1 z - 1} \right| \prod_{q=2}^{n} \left| \frac{z^{2^p_q} + \omega_0^{2^p_q}}{1 + \omega_0^{2^p_q}} \right| \leq \frac{2}{2^{p_1}} \prod_{q=2}^{n} \frac{2}{1 + \omega_0^{2^p_q}} \prod_{q=q_1}^{n} \frac{z^{2^p_q} + \omega_0^{2^p_q}}{1 + \omega_0^{2^p_q}} \right].
\]

We prove an estimate similar to (3.36) with \(q_0\) replaced by \(q_1\). Since \(z \neq 1\), one still has by (3.31), (3.34) and (3.35) that

\[
\left| l_1^{(N)}(z) \right| = \left| \frac{z^{2^p_1 - 1}}{2^p_1 |z - 1|} \prod_{q=2}^{n} \frac{z^{2^p_q} + \omega_0^{2^p_q}}{1 + \omega_0^{2^p_q}} \right| \leq \frac{2}{2^p_1 |z - 1|} \prod_{q=2}^{n} \frac{2}{1 + \omega_0^{2^p_q}} \prod_{q=q_1}^{n} \frac{z^{2^p_q} + \omega_0^{2^p_q}}{1 + \omega_0^{2^p_q}}
\]

\[
= \frac{2}{2^p_1} \left[ \exp(i\pi\theta) - 1 \right] \prod_{q=2}^{n} \left[ 1 + \exp \left( \frac{2l+1}{2^p_q - \omega_0^{2^p_q}} \right) \right]
\]

\[
\times \exp(2^p_1 i\pi\theta) + \exp \left( \frac{2l+1}{2^p_q - \omega_0^{2^p_q}} i\pi \right)
\]

\[
\times \prod_{q=q_1+1}^{n} \left[ 1 + \exp \left( \frac{2l+1}{2^p_q - \omega_0^{2^p_q}} \right) \right],
\]

then

\[
\left| l_1^{(N)}(z) \right| \leq \frac{1}{2^p_1} \left[ \sin(\pi\theta/2) \right] \prod_{q=2}^{n} \left[ \cos \left( \frac{2l+1}{2^p_q - \omega_0^{2^p_q}} - i\pi \theta \right) \right] \prod_{q=q_1+1}^{n} \left[ \cos \left( \frac{2l+1}{2^p_q - \omega_0^{2^p_q}} \right) \right]
\]

On the other hand, one has by (3.40) and (3.43) that

\[
\left| \cos \left( \frac{2l+1}{2^p_q - \omega_0^{2^p_q} + 1} \pi - i\pi \theta \right) \right| \leq \frac{2 \max \left[ \cos \left( \frac{2l+1}{2^p_q - \omega_0^{2^p_q} + 1} \pi \right) \right], \frac{2^p_q \pi \theta}{2}}{\cos \left( \frac{2l+1}{2^p_q - \omega_0^{2^p_q} + 1} \pi \right)}
\]

It follows by also applying (3.37) that

\[
\left| l_1^{(N)}(z) \right| \leq \frac{1}{2^p_1} \frac{1}{\left| \prod_{q=2}^{n} \cos \left( \frac{2l+1}{2^p_q - \omega_0^{2^p_q} + 1} \pi \right) \right|} \times
\]

\[
\times 2^p_q \pi \theta \times \prod_{q=q_1+1}^{n} \left| \cos \left( \frac{2l+1}{2^p_q - \omega_0^{2^p_q} + 1} \pi - i\pi \theta \right) \right|
\]

\[
(3.44)
\]
Now one has

\[
\prod_{q=2}^{q_1} \left| \cos \left( \frac{2l + 1}{2^{p_1-p_q+1}} \pi \right) \right| \geq \prod_{j=p_1-p_{q_1}+1}^{p_1-p_{q_1}+1} \left| \cos \left( \frac{2l + 1}{2r} \pi \right) \right| \geq \prod_{j=2}^{p_1-p_{q_1}+1} \left| \cos \left( \frac{2l + 1}{2^j} \pi \right) \right|
\]

\[= \prod_{j=1}^{p_1-p_{q_1}} \left| \cos \left( \frac{(2l + 1)\pi/2}{2^j} \right) \right| .\]

since \(p_1-p_{q_1}+1 \geq 2\) and any term of the involved products is not greater than 1. An application of Lemma 6 (with \(\alpha = (2l+1)\pi/2 \notin \pi\mathbb{Z}\) and \(m = p_1-p_{q_1} \geq p_1-p_2 > 0\) since \(q_1 \geq q_\theta \geq 2\)) yields

\[
(3.45) \quad \prod_{q=2}^{q_1} \left| \cos \left( \frac{2l + 1}{2^{p_1-p_q+1}} \pi \right) \right| \geq \frac{|\sin ((2l + 1)\pi/2)|}{2^{p_1-p_{q_1}}} \left| \sin \left( \frac{(2l + 1)\pi/2}{2^{p_1-p_{q_1}}} \right) \right| \geq \frac{1}{2^{p_1-p_{q_1}}}. \]

Next, since by \((3.30), \quad \{q_1 + 1 \leq q \leq n, \quad p_q \in S\} = \{q_i, \quad i = 2, \ldots, m\} = S_{\theta} \setminus \{q_1\}, \quad \) one can apply Lemma 10 (for all \(q = q_1 + 1, \ldots, n\)), Lemma 14 (with the choice of \(q = q_1 + 1 > q_1 \geq q_\theta\)) and Lemma 15 to get

\[
\prod_{q=q_1+1}^{n} \left| \cos \left( \frac{2l + 1}{2^{p_1-p_q+1}} \pi - \frac{2^{p_0} \pi \theta}{2} \right) \right| = \left[ \prod_{q_1+1 \leq q \leq n, \quad p_q \in T} \left| \cos \left( \frac{2l + 1}{2^{p_1-p_q+1}} \pi - \frac{2^{p_0} \pi \theta}{2} \right) \right| \right] \left( \prod_{q \in S_{\theta}, q \neq q_1} \left| \cos \left( \frac{2l + 1}{2^{p_1-p_q+1}} \pi - \frac{2^{p_0} \pi \theta}{2} \right) \right| \right)
\]

\[\leq \exp (\pi) \times \exp (2\pi) = \exp (3\pi) . \]

Finally, the estimates \((3.44), (3.45)\) and \((3.46)\) together yield

\[
\left| l_1^{(N)} (z) \right| \leq \frac{1}{2^{p_1} |\theta|} \times 2^{p_1-p_{q_1}} \times 2^{p_0} |\pi | \times \exp (3\pi) = \pi \exp (3\pi) ,
\]

and this proves the required estimate \((3.30)\) in the case \(1/2^{p_\theta-1} \leq |\theta| \leq 1/2^{p_\theta} . \)

The assertion being true for all \(\theta\) with \(1/2^{p_{q_\theta}-1} \leq |\theta| \leq 1/2^{p_{q_\theta}} , \) and all \(q_\theta = 2, \ldots, n\), the required estimate \((3.30)\) is then proved for all \(\theta\) with \(1/2^{p_1} \leq |\theta| \leq \) \(1/2^{p_\theta} . \)
3.4. Third case: $|\theta| \geq 1/2^{p_n}$. Now we fix $\theta$ with $1/2^{p_n} \leq |\theta| \leq 1$. In particular, $z \neq 1$ then one has by (3.3), (3.4) and (3.5) that

$$
\left| f^{(N)}_1(z) \right| = \frac{1}{2^{p_1}} \frac{|z^{2^{p_1}} - 1|}{|z - 1|} \prod_{q=2}^{n} \frac{|z^{2^{p_q}} + \omega_0^{2^{p_q}}|}{|1 + \omega_0^{2^{p_q}}|} \leq \frac{1}{2^{p_1}} \frac{|z^{2^{p_1}} - 1|}{|z - 1|} \prod_{q=2}^{n} \frac{|z^{2^{p_q}}|}{|1 + \omega_0^{2^{p_q}}|} 
$$

$$
= \frac{1}{2^{p_1}} |\exp(i\pi\theta) - 1| \prod_{q=2}^{n} \frac{2}{1 + \exp\left(\frac{2l + 1}{2^{p_1} - p_q} i\pi\right)}
$$

$$
= \frac{1}{2^{p_1}} \prod_{q=2}^{n} \frac{1}{\cos\left(\frac{2l + 1 + \pi}{2^{p_1} - p_q + \pi}\right)} \leq \frac{1}{2^{p_1}} |\theta| \prod_{q=2}^{n} \cos\left(\frac{2l + 1}{2^{p_1} - p_q + \pi}\right),
$$

the last estimate being an application of (3.37), i.e. $|\sin(\pi\theta/2)| \geq |\theta|$ (that is still valid since $\pi|\theta/2| \in [0, \pi/2]$). Since we have $|\theta| \geq 1/2^{p_n}$, it follows that

$$
(3.47) \quad \left| f^{(N)}_1(z) \right| \leq \frac{1}{2^{p_1} - p_n} \prod_{q=2}^{n} \frac{1}{\cos\left(\frac{2l + 1}{2^{p_1} - p_q + \pi}\right)}.
$$

On the other hand, for all $q = 2, \ldots, n$, one has by (3.3)

$$
2 \leq p_1 - p_2 + 1 \leq p_1 - p_q + 1 \leq p_1 - p_n + 1,
$$

then

$$
\prod_{q=2}^{n} \left| \cos\left(\frac{2l + 1}{2^{p_1} - p_q + \pi}\right) \right| \geq \prod_{j=2}^{p_1 - p_n + 1} \left| \cos\left(\frac{2l + 1}{2j}\right) \right| = \prod_{j=1}^{p_1 - p_n} \cos\left(\frac{(2l + 1)\pi/2}{2j}\right),
$$

since any term of the involved products is not greater than 1. An application of Lemma (with $\alpha = (2l + 1)\pi/2 \notin \pi\mathbb{Z}$ and $m = p_1 - p_n$) yields

$$
(3.48) \quad \prod_{q=2}^{n} \left| \cos\left(\frac{2l + 1}{2^{p_1} - p_q + \pi}\right) \right| \geq \frac{|\sin((2l + 1)\pi/2)|}{2^{p_1 - p_n} |\sin((2l + 1)\pi/2)|} \geq \frac{1}{2^{p_1} - p_n}.
$$

Thus, the estimates (3.47) and (3.48) together lead to

$$
\left| f^{(N)}_1(z) \right| \leq \frac{1}{2^{p_1} - p_n} \times 2^{p_1 - p_n} = 1,
$$

and this proves the required estimate (3.4) in this last case, and finally completes its whole proof for all $|\theta| \leq 1$.

4. Proof of Theorem 1 in the general case

4.1. A couple of auxiliary results for the proof of Theorem 1. In the previous section, we have considered the special case of $l^{(N)}_1$ (i.e. the FLIP associated with $z_1 = 1$) and $l = 1, \ldots, 2^{p_1} - 1$ for $\omega_0$. The following result gives a way to extend the required estimate (3.6) for every FLIP associated with $z_k$ where $k = 2, \ldots, 2^{p_1}$. We remind from (3.3) that, in all the subsection, we consider $N$ defined as follows:

$$
(4.1) \quad N = 2^{p_1} + \cdots + 2^{p_n} \quad \text{with} \quad p_1 > p_2 > \cdots > p_n \geq 0 \text{ and } n \geq 2.
$$
We will also use in all the following the simplified notation (for any function $f$ defined on the closed disk):

$$\|f\|_{\mathcal{D}} := \sup_{z \in \mathcal{D}} |f(z)| .$$

We can begin with the following result.

**Lemma 16.** Let $\mathcal{L}_N$ be the $N$-Leja section of any fixed Leja sequence $\mathcal{L}$ (that starts at $z_1 = 1$). For all $k = 1, \ldots, 2^p$, let consider $z_k \in \mathcal{L}$ (i.e. $z_k \in \Omega_{2^p}$, is any $2^p$-th root of the unity) and its associated FLIP $l_k^{(N)}$. Then for all $z \in \mathbb{C}$,

$$l_k^{(N)}(z_k z) = l_1^{(N)}(z),$$

where $l_1^{(N)}$ is the FLIP associated with $z_1 = 1$ of (possibly) another $N$-Leja section $\mathcal{L}_N$ (i.e. the $N$-Leja section of a possibly another Leja sequence that also starts at $z_1 = 1$).

In particular,

$$\|l_k^{(N)}\|_{\mathcal{D}} = \|l_1^{(N)}\|_{\mathcal{D}} .$$

**Proof.** Let be $z_k \in \Omega_{2^p}$ (i.e. a $2^p$-th root of the unity). We know by (2.15) from Lemma 3 that there is $\omega_0$ a $2^p$-th root of $-1$ such that, for all $z \in \mathbb{C}$ with $z \neq z_k$,

$$l_k^{(N)}(z) = \frac{1}{2^p} \left| z^{2^p} \right| \prod_{q=2}^{n} \left| \frac{z^{2^p} + \omega_0^{2^q}}{z^{2^p} + \frac{1}{\omega_0^{2^q}}} \right|,$$

then for all $z \neq 1$,

$$l_k^{(N)}(z_k z) = \frac{1}{2^p} \left| z_k^{2^p} \right| \prod_{q=2}^{n} \left| \frac{z_k^{2^p} + \omega_0^{2^q}}{z_k^{2^p} + \frac{1}{\omega_0^{2^q}}} \right|,$$

$$= \frac{1}{2^p} \left| z_k^{2^p} \right| \prod_{q=2}^{n} \left| \frac{z_k^{2^p} + \omega_0^{2^q}}{z_k^{2^p} + \frac{1}{\omega_0^{2^q}}} \right|,$$

It follows that for all $z \neq 1$,

$$l_k^{(N,\omega_0)}(z_k z) = l_k^{(N,\omega_1)}(z),$$

where

$$\omega_1 := \frac{\omega_0}{z_k}$$

is still a $2^p$-th root of $-1$ and $l_k^{(N,\omega_0)}$ (resp. $l_k^{(N,\omega_1)}$) is the FLIP associated with $z_k$ (resp. $z_1 = 1$) and the $2^p$-th root $\omega_0$ (resp. $\omega_1$). Remind as specified by (2.18) from Remark 2.4 that the data of $\Omega_{2^p}$ and $\omega_0$ conversely gives a $N$-Leja section (that starts at $z_1 = 1$) and whose first FLIP is exactly $l_k^{(N,\omega_1)}$. This proves (4.2) by setting $l_1^{(N)} := l_1^{(N,\omega_1)}$, and (4.3) follows since $|z_k| = 1$. 

$\sqrt{\Box}$
We finish this subsection with the following result that is the proof of the required estimate (3.6) for \( l_1^{(N)} \) and the unique case of \( \omega_0 \) that was not considered in the previous section.

**Lemma 17.** Let fix \( k = 1 \) (i.e. \( z_k = z_1 = 1 \)) and \( l = 0 \) in (3.4), i.e.

\[
\omega_0 = \exp \left( \frac{i\pi}{2p_1} \right).
\]

Then

\[
\| l_1^{(N)} \|_{\mathbb{T}} \leq \frac{\pi}{2}.
\]

In particular, the estimate (3.6) is still valid in this case.

**Proof.** For all \( z \in \mathbb{C} \) with \( |z| \leq 1 \), one has by (3.2) that

\[
\left| l_1^{(N)}(z) \right| = \frac{1}{2p_1} \sum_{j=0}^{2p_1-1} z_j \prod_{q=2}^{n} \left| \frac{z^{2^q} + \omega_0^{2^q}}{1 + \omega_0^{2^q}} \right| \leq \frac{1}{2p_1} \sum_{j=0}^{2p_1-1} |z|^j \prod_{q=2}^{n} \left| \frac{z^{2^q} + \omega_0^{2^q}}{1 + \omega_0^{2^q}} \right|
\]

\[
\leq \frac{1 \times \prod_{q=2}^{n} \left( 1 + \exp \left( i\pi/2p_1-p_q \right) \right)}{1 + \exp \left( 2p_1-p_q \right)} = \prod_{q=2}^{n} \left( 1 + \exp \left( i\pi/2p_1-p_q \right) \right) \cdot
\]

\[
\prod_{q=2}^{n} \left( 1 + \exp \left( i\pi/2p_1-p_q \right) \right) \geq \prod_{j=p_1-p_n+1}^{p_1} \left| \cos \left( \pi/2 \right) \right| \geq \prod_{j=2}^{p_1} \left| \cos \left( \pi/2 \right) \right| ,
\]

since any term in the involved products is not greater than 1. It follows by Lemma \( b \) with \( \alpha = \pi/2 \) (\( \not\in \pi\mathbb{Z} \)) and \( m = p_1 \) that

\[
\prod_{q=2}^{n} \left| \cos \left( \pi/2p_1-p_q \right) \right| \geq \prod_{j=p_1-p_n+1}^{p_1} \left| \cos \left( \pi/2 \right) \right| \geq \prod_{j=2}^{p_1} \left| \cos \left( \pi/2 \right) \right| ,
\]

where the last estimate is justified by (2.20) from Lemma \( a \) and the proof is finished by (4.4).
4.2. Proof of Theorem 1. Before giving the proof of Theorem 1 we need the following preliminary result in which we deal with the other FLIPs associated with \( z_k \in \mathcal{L}_N \setminus \Omega_{2^{p_1}} \).

**Lemma 18.** Let \( \mathcal{L}_N \) be a \( N \)-Leja section whose first point \( z_1 \) starts at 1. There is an \((N - 2^{p_1})\)-Leja section \( \tilde{\mathcal{L}}_{N-2^{p_1}} \) that also starts at \( \tilde{z}_1 = 1 \), with the following property: for all \( k = 2^{p_1} + 1, \ldots, N \), there is \( k' \) with \( 1 \leq k' \leq N - 2^{p_1} \) such that

\[
\left\| f_k^{(N)}(z) \right\|_{\mathfrak{F}} \leq \left\| f_{k'}^{(N-2^{p_1})}(z) \right\|_{\mathfrak{F}},
\]

where \( f_{k'}^{(N-2^{p_1})} \) is the FLIP associated with \( \tilde{z}_{k'} \in \tilde{\mathcal{L}}_{N-2^{p_1}} \).

In addition, the correspondence

\[
k, 2^{p_1} + 1 \leq k \leq N \rightarrow k', 1 \leq k' \leq N - 2^{p_1},
\]

is well-defined and one-to-one.

**Proof.** First, let consider \( k = 2^{p_1} + 1, \ldots, N \) and the FLIP \( f_k^{(N)} \) associated with \( z_k \). \( \mathcal{L}_N \) being a \( N \)-Leja section that starts at \( z_1 = 1 \), one necessarily has by Theorem 5 from [1] (or (2.14)) that \( z_k \notin \Omega_{2^{p_1}} \), i.e., \( z_k \) is a \( 2^{p_1} \)-th root of \( -1 \). It follows that

\[
\left| f_k^{(N)}(z) \right| = \left| \prod_{z_j \in \mathcal{L}_N \setminus \Omega_{2^{p_1}} \land z_j \neq z_k} \frac{z - z_j}{z_k - z_j} \right|
\]

\[(4.5) = \left| \prod_{z_j \in \Omega_{2^{p_1}}} \frac{z - z_j}{z_k - z_j} \right| \times \left| \prod_{z_j \in \mathcal{L}_N \setminus \Omega_{2^{p_1}} \setminus \Omega_{2^{p_1}} \land z_j \neq z_k} \frac{z - z_j}{z_k - z_j} \right|.\]

On the one hand, one has for all \( |z| \leq 1 \),

\[
\left| \prod_{z_j \in \Omega_{2^{p_1}}} \frac{z - z_j}{z_k - z_j} \right| = \frac{|z|^{2^{p_1}} - 1}{|z_k|^{2^{p_1}} - 1} \leq \frac{|z|^{2^{p_1}} + 1}{|z_k|^{2^{p_1}} - 1} = \frac{2}{2} = 1,
\]

since \( z_k^{2^{p_1}} = -1 \).

On the other hand, one has (again by Theorem 5 from [1], or (2.14)) that

\[
\mathcal{L}_N \setminus \Omega_{2^{p_1}} = \omega_1 \tilde{\mathcal{L}}_{N-2^{p_1}},
\]

where \( \omega_1 \) is a \( 2^{p_1} \)-th root of \( -1 \), \( \tilde{\mathcal{L}}_{N-2^{p_1}} \) is the \((N - 2^{p_1})\)-section of (maybe) another Leja sequence \( \tilde{\mathcal{L}} = \{ \tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_j, \ldots \} \) with \( \tilde{z}_1 = 1 \), and the above equality is meant as sets. In particular, \( z_k \in \omega_1 \tilde{\mathcal{L}}_{N-2^{p_1}} \) can be written as \( z_k = \omega_1 \tilde{z}_{k'} \) with \( 1 \leq k' \leq N - 2^{p_1} \). This proves that the correspondence:

\[
z_k, 2^{p_1} + 1 \leq k \leq N \rightarrow \tilde{z}_{k'}, 1 \leq k' \leq \ N - 2^{p_1}
\]

(then \( k \rightarrow k' \) as well), is well-defined and injective. Since \( \text{card} (\mathcal{L}_N \setminus \Omega_{2^{p_1}}) = N - 2^{p_1} = \text{card} (\omega_1 \tilde{\mathcal{L}}_{N-2^{p_1}}) \), it is also one-to-one.

In particular, this leads to

\[
\left| \prod_{z_j \in \mathcal{L}_N \setminus \Omega_{2^{p_1}} \setminus \Omega_{2^{p_1}} \land z_j \neq z_k} \frac{z - z_j}{z_k - z_j} \right| = 
\]
Section 3. The theorem is then true for $L_n$. We first prove the theorem for the FLIP (4.8) of Theorem 1 from [2], or (2.14). An application of (4.3) from Lemma 16 yields since (4.7)

$$\frac{z - z_j}{z_k - z_j} = \prod_{\tilde{z}_j \in \tilde{L}_{N-2^p_1}, \tilde{z}_j \neq \tilde{z}_k} \frac{z - \omega_1 \tilde{z}_j}{\omega_1 z_k - \omega_1 \tilde{z}_j},$$

(4.7)

where $\tilde{L}_{N-2^p_1}$ is the FLIP associated with $\tilde{z}_k$ from the $(N - 2^p_1)$-Leja section $\tilde{L}_{N-2^p_1}$.

Finally, the estimates (4.5), (4.6) and (4.7) together yield

$$\sup_{|z| \leq 1} \left| f_k^{(N)}(z) \right| \leq 1 \times \sup_{|z| \leq 1} \left| f_{k'}^{(N-2^p_1)} \left( \frac{z}{\omega_1} \right) \right|,$$

and the lemma is proved since $|\omega_1| = 1$.

$\square$

Now we can finally give the proof of Theorem 1. We then consider $L_N$ the $N$-section of any fixed Leja sequence, where

$$N = 2^p_1 + 2^p_2 + \cdots + 2^p_n \quad \text{with} \quad p_1 > p_2 > \cdots > p_n \geq 0 \quad \text{and} \quad n \geq 1.$$

**Proof.** First, by the symmetry of the disk, we can wlog assume that (the first Leja point) $z_1 = 1$. Next, by the maximum modulus principle, it suffices to prove the required estimate for all $|z| = 1$, i.e. $z = \exp(i\pi \theta)$ with $\theta \in [-1, 1]$. The proof is by induction on $n \geq 1$, where $n$ is defined from (4.8).

The special case of $n = 1$ means that $N = 2^p_1$ with $p_1 \geq 0$. Then $L_{2^p_1} = \Omega_{2^p_1}$ by Theorem 5 from [1], and (2.5) from Lemma 1 yields for all $k = 1, \ldots, 2^p_1$,

$$\sup_{z \in \mathbb{T}} \left| f_k^{(2^p_1)}(z) \right| = 1.$$  

An alternative argument is that the set $\Omega_{2^p_1}$ is a $2^p_1$-Fekete set for the unit disk (as specified by (1.5) from the Introduction, see also [5]). Thus, all the FLIPs are bounded by 1.

Now let be $N$ with $n \geq 2$ (i.e. $2^p_1 < N < 2^p_1 + 1$), let be $L_N$ and consider the associated $\omega_0$ defined from Lemma 3 and that can be written as follows:

$$\omega_0 = \exp \left( \frac{2l + 1}{2p_1} \pi i \right) \quad \text{with} \quad l = 0, \ldots, 2^p_1 - 1.$$  

We first prove the theorem for the FLIP $\tilde{f}_1^{(N)}$ associated with $z_1 = 1$, and we fix $|z| = 1$. If $l = 0$, then the theorem is a consequence of Lemma 17. Otherwise, $1 \leq l \leq 2^p_1 - 1$ then the assertion is a consequence of what has been proved in Section 4. The theorem is then true for $\tilde{f}_1^{(N)}$ with any $N$-Leja section $L_N$.

Next, if $2 \leq k \leq 2^p_1$, then $z_k \in \Omega_{2^p_1}$ (i.e. $z_k$ is a $2^p_1$-th root of the unity, see Theorem 1 from [2], or (2.14)). An application of (4.3) from Lemma 16 yields

$$\left\| f_k^{(N)} \right\| = \left\| f_1^{(N)} \right\|,$$

where $\tilde{f}_1^{(N)}$ is the FLIP associated with $\tilde{z}_1 = 1$ from (possibly) another $N$-Leja section $L_N$. Since the theorem is valid for $\tilde{f}_1^{(N)}$ and any $N$-Leja section, it follows
that it holds true for $l_k^{(N)}$. Thus, it is proved for all $k = 1, \ldots, 2^{p_1}$ (and for any $N$-Leja section).

Lastly, let consider $k = 2^{p_1} + 1, \ldots, N$ and the FLIP $l_k^{(N)}$ associated with $z_k$. $\mathcal{L}_N$ being a $N$-Leja section that starts at $z_1 = 1$, necessarily $z_k \notin \Omega_{2^{p_1}}$, i.e. $z_k$ is a $2^{p_1}$-th root of $-1$ (see Theorem 5 from [1]). An application of Lemma 18 gives that

$$
\left\| l_k^{(N)}(z) \right\|_D \leq \left\| l_k^{(N)}(z) \right\|_D,
$$

where $\tilde{l}_{k'}^{(N-2^{p_1})}$ is the FLIP associated with $\tilde{z}_{k'} \in \tilde{\mathcal{L}}_{N-2^{p_1}}$ and $\tilde{\mathcal{L}}_{N-2^{p_1}}$ is an $(N-2^{p_1})$ Leja section that also starts at $\tilde{z}_1 = 1$. Since by (4.8),

$$
N-2^{p_1} = 2^{p_2} + 2^{p_3} + \cdots + 2^{p_n} = \sum_{q=1}^{n-1} 2^{p_{q+1}},
$$

it follows that the induction hypothesis can be applied to $\tilde{\mathcal{L}}_{N-2^{p_1}}$ with $n-1$, and the above inequality becomes

$$
\left\| l_k^{(N)}(z) \right\|_D \leq \left\| l_k^{(N-2^{p_1})} \right\|_D \leq \pi \exp(3\pi).
$$

This finally achieves the induction and the whole proof of the theorem.

√

5. On the special case of $N = 2^p - 1$

In this part, we want to prove Theorem 2 that gives an additional result for the special case of $N = 2^{p_1+1} - 1 = 2^{p_1} + 2^{p_1-1} + \cdots + 2 + 1$, i.e.

$$
n = p_1 + 1 \quad \text{and} \quad p_q = p_1 - q + 1, \quad \text{for all} \quad q = 1, \ldots, p_1 + 1.
$$

We will first deal with a subfamily of FLIPs: for all $l = 0, \ldots, 2^{p_1} - 1$, we consider the $2^{p_1}$-th root of $-1$,

$$
\omega_l := \exp \left( \frac{2l + 1}{2^{p_1}} i\pi \right),
$$

and for all $z \neq 1, \omega_l$, we set

$$
l_{\omega_l}^{(N)}(z) := \frac{1 - \omega_l}{2^{p_1+1}} \times \frac{z^{2^{p_1}+1} - 1}{(z - 1) (z - \omega_l)}.
$$

As we will see in the following, $l_{\omega_l}^{(N)}$ is indeed a FLIP (at least in modulus) for all $l = 0, \ldots, 2^{p_1} - 1$. Before giving the proof of Theorem 2, we want to give preliminary results for the estimate of the $l_{\omega_l}^{(N)}$'s for almost every $l = 1, \ldots, 2^{p_1} - 1$ (as it will be specified below).

5.1. A preliminary estimate for $l_{\omega_l}^{(N)}$ and for almost every $l$. We begin with an estimate of $|1 - \omega_l|$.

Lemma 19. Let fix $\varepsilon_0 > 0$ small enough such that $\varepsilon_0 2^{p_1} \leq (1 - \varepsilon_0) 2^{p_1} - 1$ for all $p_1 \geq 1$ (for example, if $\varepsilon_0 \leq 1/4$). Then for all $l$ with $\varepsilon_0 2^{p_1} \leq l \leq (1 - \varepsilon_0) 2^{p_1} - 1$ (where $\omega_l = \exp ((2l + 1)i\pi/2^{p_1})$, one has

$$
|1 - \omega_l| \geq 4\varepsilon_0.
$$
Proof. We know from hypothesis about \( l \) that
\[
\frac{2l + 1}{2^{2p_1}} \geq \frac{2 \times \varepsilon_0 2^{p_1} + 1}{2^{2p_1}} \geq \frac{\varepsilon_0 2^{p_1 + 1}}{2^{2p_1}} = 2 \varepsilon_0.
\]
Similarly,
\[
\frac{2l + 1}{2^{2p_1}} \leq \frac{2 \times ((1 - \varepsilon_0) 2^{p_1} - 1) + 1}{2^{2p_1}} = \frac{(1 - \varepsilon_0) 2^{p_1 + 1} - 1}{2^{2p_1}} \leq 2 (1 - \varepsilon_0).
\]
Thus,
\[
0 < \pi \varepsilon_0 \leq \frac{2l + 1}{2^{2p_1}} \leq \pi (1 - \varepsilon_0) < \pi.
\]
In particular, \( \frac{2l + 1}{2^{2p_1}} \in [0, \pi] \) then
\[
\left| \sin \left( \frac{2l + 1}{2^{2p_1}} \frac{\pi}{2} \right) \right| = \sin \left( \frac{2l + 1}{2^{2p_1}} \frac{\pi}{2} \right) \geq \min \{ \sin (\pi \varepsilon_0), \sin (\pi - \pi \varepsilon_0) \} = \sin (\pi \varepsilon_0).
\]
It follows that
\[
|1 - \omega_l| = \left| 1 - \exp \left( \frac{2l + 1}{2^{2p_1}} i \pi \right) \right| = 2 \left| \sin \left( \frac{2l + 1}{2^{2p_1}} \frac{\pi}{2} \right) \right| \geq 2 \sin (\pi \varepsilon_0).
\]
On the other hand, since \( 0 < \pi \varepsilon_0 \leq \pi/4 < \pi/2 \), an application of \([2.19]\) from Lemma 4 yields
\[
|1 - \omega_l| \geq 2 \sin (\pi \varepsilon_0) \geq 2 \times \frac{2}{\pi} \times \pi \varepsilon_0 = 4 \varepsilon_0
\]
and this proves the lemma.

The following result gives an estimate of \( l_{\omega_l}^{(N)} \) for the \( z \)'s which are close to 1 or \( \omega_l \).

**Lemma 20.** Let \( \varepsilon_0 > \) small enough such that \( \varepsilon_0 2^{p_1} \leq (1 - \varepsilon_0) 2^{p_1} - 1 \) for all \( p_1 \geq 1 \) (for example, if \( \varepsilon_0 \leq 1/4 \)). Then for all \( l \) with \( \varepsilon_0 2^{p_1} \leq l \leq (1 - \varepsilon_0) 2^{p_1} - 1 \) and all \( z \in \mathbb{D} \) such that \( |z - 1| \leq \varepsilon_0^2 \) or \( |z - \omega_l| \leq \varepsilon_0^2 \) (where \( \omega_l = \exp \left( (2l + 1)i \pi/2^{p_1} \right) \), see \([5.2]\) ), one has
\[
\left| l_{\omega_l}^{(N)}(z) \right| \leq \frac{1}{1 - \varepsilon_0/4}.
\]

**Proof.** First, let notice that \( l_{\omega_l}^{(N)} \) is a polynomial since by \([5.3]\), for all \( z \in \mathbb{C} \),
\[
l_{\omega_l}^{(N)}(z) = \frac{1 - \omega_l}{2^{p_1 + 1}} \times \frac{(z^{2^{p_1}})^2 - 1}{(z - 1)(z - \omega_l)} = \frac{1 - \omega_l}{2^{p_1 + 1}} \times \frac{(z^{2^{p_1}} - 1) \times (z^{2^{p_1}} + 1)}{(z - 1)(z - \omega_l)} = \frac{1 - \omega_l}{2^{p_1 + 1}} \times \frac{z^{2^{p_1} - 1} - z^{2^{p_1}}}{z - 1} \times \frac{z^{2^{p_1} - 1} - z^{2^{p_1}}}{z - \omega_l} = \frac{1 - \omega_l}{2^{p_1 + 1}} \left( \sum_{j=0}^{2^{p_1} - 2} z^j \right) \left( \sum_{j=0}^{2^{p_1} - 1} \omega_l^{2^{p_1} - 1 - j} z^j \right)
\]
(because \( \omega_l^{2^{p_1}} = -1 \)). In particular, \( l_{\omega_l}^{(N)} \) is continuous then it suffices to prove the required estimate for all \( z \neq 1, \omega_l \).

Next, if \( |z - 1| \leq \varepsilon_0^2 \), then by Lemma 19
\[
\frac{|z - 1|}{|1 - \omega_l|} \leq \frac{\varepsilon_0^2}{4 \varepsilon_0} = \frac{\varepsilon_0}{4}.
\]
Lemma 21. Let fix $\varepsilon_0 > 0$ small enough. Then there is $P_1(\varepsilon_0) \geq 1$ (that only depends on $\varepsilon_0$) such that, for all $p_1 \geq P_1(\varepsilon_0)$ and all $l$ with $\varepsilon_0 2^{p_1} \leq l \leq (1 - \varepsilon_0) 2^{p_1} - 1$, one has
\[
\left\| \frac{I}{\omega_l} \right\|_{\mathbb{P}} \leq \frac{1}{1 - \varepsilon_0/4}.
\]
Proof. Let fix any \( z \in \mathbb{C} \) with \( |z| \leq 1 \). If \( |z-1| \geq \varepsilon_0^2 \) and \( |z-\omega_l| \geq \varepsilon_0^2 \), then in particular \( z \neq 1, \omega_l \), and one has that (since \( |\omega_l|=1 \) and \( |z| \leq 1 \))

\[
\left| f^{(N)}_{\omega_l}(z) \right| = \left| 1-\omega_l \right| \left| z^{2p_1+1}-1 \right| \leq \frac{2}{2^{p_1+1} \varepsilon_0} \leq \frac{1}{2^{p_1+1}}.
\]

Since \( \lim_{p_1 \to +\infty} \frac{2}{2^{p_1+1} \varepsilon_0} = 0 \), there is \( P_1(\varepsilon_0) \geq 1 \) such that \( \frac{2}{2^{p_1+1} \varepsilon_0} \leq 1 \) for all \( p_1 \geq P_1(\varepsilon_0) \), then

\[
\left| f^{(N)}_{\omega_l}(z) \right| \leq \frac{1}{1-\varepsilon_0/4}.
\]

Otherwise, \( |z-1| \leq \varepsilon_0^2 \) or \( |z-\omega_l| \leq \varepsilon_0^2 \) thus, Lemma 20 yields for all \( p_1 \geq 1 \) (then for all \( p_1 \geq P_1(\varepsilon_0) \) in particular),

\[
\left| f^{(N)}_{\omega_l}(z) \right| \leq \frac{1}{1-\varepsilon_0/4},
\]

and the lemma follows.

\( \Box \)

5.2. **Proof of Theorem** 2 for the first half of the \( f^{(N)}_k \)'s. In this subsection, we give a proof of the theorem in the special case of the \( f^{(N)}_k \)'s for \( k = 1, \ldots, 2^{p_1} \).

We begin with the following preliminary property for polynomials.

**Lemma 22.** For all \( m \geq 0 \), one has

\[
(X - Y) \prod_{j=0}^{m} (X^{2^j} + Y^{2^j}) = X^{2^{m+1}} - Y^{2^{m+1}}.
\]

**Proof.** The proof is by induction on \( m \geq 0 \). For \( m = 0 \), one has that

\[
(X - Y) \prod_{j=0}^{0} (X^{2^j} + Y^{2^j}) = (X - Y) (X^{2^0} + Y^{2^0}) = (X - Y)(X + Y) = X^{2^1} - Y^{2^1}.
\]

Now if \( m \geq 0 \), then the induction hypothesis yields

\[
(X - Y) \prod_{j=0}^{m+1} (X^{2^j} + Y^{2^j}) = (X - Y) \prod_{j=0}^{m} (X^{2^j} + Y^{2^j}) \times (X^{2^{m+1}} + Y^{2^{m+1}})
\]

\[
= \left( X^{2^{m+1}} - Y^{2^{m+1}} \right) \times \left( X^{2^{m+1}} + Y^{2^{m+1}} \right)
\]

\[
= \left( X^{2^{m+1}} \right)^2 - \left( Y^{2^{m+1}} \right)^2 = X^{2^{m+2}} - Y^{2^{m+2}},
\]

and the induction is achieved.

\( \Box \)

The following lemma gives the link between the \( f^{(N)}_k \)'s and the \( f^{(N)}_{\omega_l} \)'s introduced at the beginning of this section.

**Lemma 23.** Let \( \mathcal{L}_N \) be a \( N \)-Leja section for the unit disk (that starts at 1). For all \( k = 1, \ldots, 2^{p_1} \), let \( f^{(N)}_k \) be the FLIP associated with \( z_k \in \mathcal{L}_N \) (that is a \( 2^{p_1} \)-th
root of the unity since it belongs to $\Omega_{2^{p_1}}$). Then there is $l(k)$ with $0 \leq l(k) \leq 2^{p_1} - 1$ such that
\[
\| t_k^{(N)} \| = \| t_{\omega(l(k))}^{(N)} \|, \quad \text{with} \quad \omega(l(k)) = \exp \left( \frac{2l(k) + 1}{2^{p_1}} i \pi \right).
\]

In addition, the correspondence
\[ k, \ 1 \leq k \leq 2^{p_1} \quad \mapsto \quad l(k), \ 0 \leq l(k) \leq 2^{p_1} - 1, \]
is well-defined and one-to-one.

**Proof.** First, we know by Lemma 3 that there is $\omega$ a $2^{p_1}$-th root of $-1$ (we designate it by $\omega$ instead of $\omega_0$ to not confuse it with the notation from (5.2) for $l = 0$) such that for all $z \neq z_k$, $\omega$ (by also recalling (5.1)),
\[
\left| t_k^{(N)}(z) \right| = \frac{1}{2^{p_1}} \left| \frac{z^{2^{p_1}} - 1}{z - z_k} \right| \times \prod_{q=2}^{n} \left| \frac{z^{2^{p_q}} + \omega^{2^{p_q}}}{z - z_k} \right|
\]
\[
= \frac{1}{2^{p_1}} \left| \frac{z^{2^{p_1}} + \omega^{2^{p_1}}}{z - z_k} \right| \prod_{q=2}^{p_1+1} \left| \frac{z^{2^{p_q-1}+1} + \omega^{2^{p_q-1}+1}}{z^{2^{p_q-1}+1} + \omega^{2^{p_q-1}+1}} \right|
\]
\[
= \frac{1}{2^{p_1}} \left| \frac{z^{2^{p_1}} + \omega^{2^{p_1}}}{z - z_k} \right| \prod_{q=0}^{p_1-1} \left| \frac{z^{2^q} + \omega^{2^q}}{z^{2^q} + \omega^{2^q}} \right|.
\]

Now successive applications of Lemma 22 give (remind that $z_k$ is a $2^{p_1}$-th root of the unity, and since $\omega$ is a $2^{p_1}$-th root of $-1$, then it is a $2^{p_1+1}$-th root of the unity)
\[
\prod_{q=0}^{p_1-1} \left| \frac{z^{2^q} + \omega^{2^q}}{z - z_k} \right| = \frac{|z_k - \omega| \times \prod_{q=0}^{p_1-1} \left| \frac{z^{2^q} + \omega^{2^q}}{z - \omega} \right|}{|z_k - \omega|} = 2
\]
(notice that we cannot have $z_k = \omega$ since $\omega \notin \Omega_{2^{p_1}} \ni z_k$), and for all $z \neq \omega$,
\[
\left| \frac{z^{2^{p_1}} + \omega^{2^{p_1}}}{z - \omega} \right| \prod_{q=0}^{p_1-1} \left| \frac{z^{2^q} + \omega^{2^q}}{z - \omega} \right| = \frac{|z - \omega| \times \prod_{q=0}^{p_1-1} \left| \frac{z^{2^q} + \omega^{2^q}}{z - \omega} \right|}{|z - \omega|} = \frac{|z^{2^{p_1+1}} - \omega^{2^{p_1+1}}|}{|z - \omega|} = \frac{|z^{2^{p_1+1}} - 1|}{|z - \omega|}.
\]

It follows that for all $z \neq z_k$, $\omega$,
\[
\left| t_k^{(N)}(z) \right| = \frac{1}{2^{p_1}} \times \frac{1}{|z - z_k|} \times \frac{\left| \frac{z^{2^{p_1+1}} - 1}{z - \omega} \right|}{2} \times \frac{|z_k - \omega|}{2^{p_1+1}}
\]
\[
= \frac{1 - \omega/z_k}{2^{p_1+1}} \left| \frac{z/z_k}{z/z_k - 1} \right| \times \frac{1}{|z/z_k - \omega/z_k|} = \frac{1}{|\omega/z_k|} \left| \frac{z^{2^{p_1+1}} - 1}{z/z_k} \right|
\]
by (5.3) (and because $z_k^{2^{p_1+1}} = 1^2 = 1$). $\omega/z_k$ being still a $2^{p_1}$-th root of $-1$, there is $l(k)$ with $0 \leq l(k) \leq 2^{p_1} - 1$ such that $\omega/z_k = \omega(l(k))$. In particular,
\[
\sup_{|z| \leq 1} \left| t_k^{(N)}(z) \right| = \sup_{|z| \leq 1} \left| t_{\omega/z_k}^{(N)}(z/z_k) \right| = \left| t_{\omega/z_k}^{(N)}(z/z_k) \right|,
\]
and this proves the first assertion.
In order to prove the other one, we first remind from Remark 5.1 that $\omega$ is any of the $2^{p_n}$ different choices for $z_{N+1}$. Here by (5.1), $p_n = 0$ and $N + 1 = 2^{p_1} + 1$, then $\omega$ is the only possible choice for $z_{N+1}$ and $L_N \cup \{ \omega \} = L_{N+1} = \Omega_{2^{p_1}+1}$ by Theorem 5 from [1] (the equality being meant as sets). In particular, the uniqueness of $\omega$ implies that for any $\omega_i$, there is at most one $z_k$ such that $\omega_i = \omega/z_k$. This proves that the application $k \mapsto l(k)$ is injective and the one-to-one correspondence follows because $\text{card} \{ k, 1 \leq k \leq 2^{p_1} \} = 2^{p_1} = \text{card} \{ l, 0 \leq l \leq 2^{p_1} - 1 \}$.

Now we can prove a special case of Theorem 2 for the first half of the $l^{(N)}_k$'s, i.e. for $k = 1, \ldots, 2^{p_1}$.

**Lemma 24.** Let fix $\varepsilon_0 > 0$ small enough. Then there is $P_1(\varepsilon_0) \geq 1$ large enough such that for all $p_1 \geq P_1(\varepsilon_0)$, one has

$$\frac{1}{2^{p_1}} \sum_{k=1}^{2^{p_1}} \left\| l^{(N)}_k \right\|_\infty \leq 2\pi \exp(3\pi) \left( \varepsilon_0 + \frac{1}{2^{p_1}} \right) + \frac{1 - 2\varepsilon_0}{1 - \varepsilon_0/4}.$$ 

**Proof.** First, we know by Lemma 23 that

$$\sum_{k=1}^{2^{p_1}} \left\| l^{(N)}_k \right\|_\infty = \sum_{k=1}^{2^{p_1}} \left\| l^{(N)}_{\omega(k)} \right\|_\infty = \sum_{l=0}^{2^{p_1} - 1} \left\| l^{(N)}_{\omega_l} \right\|_\infty.$$ 

Next, an application of Lemma 21 gives $P_1(\varepsilon_0) \geq 1$ such that for all $p_1 \geq P_1(\varepsilon_0)$,

$$\sum_{l=0}^{2^{p_1} - 1} \left\| l^{(N)}_{\omega_l} \right\|_\infty \leq \sum_{0 \leq l < \varepsilon_0 2^{p_1}} + \varepsilon_0 2^{p_1} \leq (1 - \varepsilon_0) 2^{p_1 - 1} + (1 - \varepsilon_0) 2^{p_1 - 1} \leq 2^{p_1 - 1} + \varepsilon_0 2^{p_1} + 1 \times \pi \exp(3\pi),$$

the other estimates being applications of Theorem 1. Hence

$$\frac{1}{2^{p_1}} \sum_{k=1}^{2^{p_1}} \left\| l^{(N)}_k \right\|_\infty \leq \frac{1}{2^{p_1}} \left[ 2\pi \exp(3\pi) \times (\varepsilon_0 2^{p_1} + 1) + \frac{(1 - 2\varepsilon_0) 2^{p_1}}{1 - \varepsilon_0/4} \right] = 2\pi \exp(3\pi) \left( \varepsilon_0 + \frac{1}{2^{p_1}} \right) + \frac{1 - 2\varepsilon_0}{1 - \varepsilon_0/4},$$

and the lemma is proved. 

**Remark 5.2.** We could have used in the proof the sharper estimate from Proposition 1 below in order to replace the bound $\pi \exp(3\pi)$ by 2 (since $N = 2^{p_1+1} - 1$). However, we will see that it is useless for the proof of Theorem 2.

5.3. **Proof of Theorem 2.** In order to give the proof, we first want to begin with a preliminar result in which we deal with the FLIPs $l^{(N)}_k$ for $k = 2^{p_1} + 1, \ldots, N$.

**Lemma 25.** Let $L_N$ be any $N$-Leja section for the unit disk that starts at $z_1 = 1$. For all $k = 1, \ldots, N$, one has

$$\left\| l^{(N)}_k \right\|_\infty \leq \left\| l^{(2\pi - 1)}_k \right\|_\infty.$$
where $1 \leq q \leq p_1 + 1$, $1 \leq k' \leq 2q^{q-1}$ and $l_{k'}^{(2^q - 1)}$ is the FLIP associated with $\tilde{z}_{k'} \in \tilde{L}_{2q-1}$ and $\tilde{L}_{2q-1}$ is a $(2^q - 1)$-Leja section that also starts at $\tilde{z}_1 = 1$.

In addition, the correspondence between $k$ and $(q, k')$ is one-to-one: more precisely the application defined by

$$
(5.4) \quad \begin{cases} 
k, & 1 \leq k \leq 2^{p_1} \\
k, & 1 \leq k \leq 2^{p_1}, \\
2^{p_1} + 1 \leq k \leq N \rightarrow (q, k'), & 1 \leq q \leq p_1, 1 \leq k' \leq 2^{q-1},
\end{cases}
$$
is well-defined and is a one-to-one correspondence.

Proof. The proof is by induction on $p_1 \geq 0$ where $N = 2^{p_1+1} - 1$. If $p_1 = 0$, then $\mathcal{L}_N = \tilde{L}_{2^{p_1} + 1 - 1} = \mathcal{L}_1 = \left\{ z_1 \right\} = \{1\}$. Necessarily, $k = 1$ and the lemma is obvious by taking $q = 1$, the same $1$-Leja section $\mathcal{L}_1 = \{1\}$ and $k' = k = 1$ (and the correspondence (5.4) is of course well-defined and one-to-one).

Now let be $p_1 \geq 1$. First, if $1 \leq k \leq 2^{p_1}$, then we take $q = p_1 + 1$, the same $N$-Leja section $\mathcal{L}_N$ and $k' = k$. In addition, the application

$$
(5.5) \quad k, 1 \leq k \leq 2^{p_1} \rightarrow (p_1 + 1, k), 1 \leq k \leq 2^{p_1},
$$
is obviously well-defined and is a one-to-one correspondence.

Otherwise, $2^{p_1} + 1 \leq k \leq N$, then an application of Lemma [18] yields

$$
(5.6) \quad \left\| \tilde{l}_{k'}^{(N)} \right\|_{\mathcal{H}} \leq \left\| \tilde{l}_{k'}^{(N-2^{p_1})} \right\|_{\mathcal{H}},
$$
where $\tilde{l}_{k'}^{(N-2^{p_1})}$ is the FLIP associated with $\tilde{z}_{k'} \in \tilde{L}_{N-2^{p_1}}$ and $\tilde{L}_{N-2^{p_1}}$ is an $(N-2^{p_1})$-Leja section that also starts at $\tilde{z}_1 = 1$. In addition, the application

$$
(5.7) \quad k, 2^{p_1} + 1 \leq k \leq N \rightarrow k', 1 \leq k' \leq N - 2^{p_1},
$$
is well-defined and is a one-to-one correspondence.

Since $N - 2^{p_1} = 2^{p_1+1} - 1 - 2^{p_1} = 2^{p_1} - 1$, the induction hypothesis applied to $p_1 - 1$ and the $(2^{p_1} - 1)$-Leja section $\tilde{L}_{2^{p_1} - 1}$, leads to

$$
(5.8) \quad \left\| \tilde{l}_{k'}^{(N-2^{p_1})} \right\|_{\mathcal{H}} \leq \left\| \tilde{l}_{k''}^{(2^{q-1})} \right\|_{\mathcal{H}},
$$
where $1 \leq q \leq p_1 \leq p_1 + 1$, $1 \leq k' \leq 2^{q-1}$ and $l_{k''}^{(2^{q-1})}$ is the FLIP associated with $\tilde{z}_{k''} \in \tilde{L}_{2q-1}$, where $\tilde{L}_{2q-1}$ is a $(2^{q-1})$-Leja section that starts at $\tilde{z}_1 = 1$.

Moreover, the application defined by

$$
(5.9) \quad k, 1 \leq k' \leq 2^{p_1} - 1 \rightarrow (q, k''), 1 \leq q \leq (p_1 - 1) + 1, 1 \leq k'' \leq 2^{q-1},
$$
is well-defined and is a one-to-one correspondence, then so is the following one by composition of (5.7) and (5.9):

$$
(5.10) \quad k, 2^{p_1} + 1 \leq k \leq N \rightarrow (q, k''), 1 \leq q \leq p_1, 1 \leq k'' \leq 2^{q-1}.
$$
Finally, the partial applications (5.5) and (5.10) yield (5.4). On the other hand, the estimates (5.6) and (5.8) achieve the induction and the proof of the lemma.

As a first consequence, we have the following improvement for the bound of $l_{k}^{(N)}$ in Theorem [1].
Proposition 1. One has for the special case of $N = 2^{p_1+1} - 1$ that
\[
\frac{4}{\pi} (1 + \varepsilon(1/N)) \leq \max_{1 \leq k \leq N} \left| I_{\omega_l}^{(N)} \right|_{\mathbb{D}} \leq 2
\]
where $\lim_{N \to +\infty} \varepsilon(1/N) = 0$. In addition, $\varepsilon(1/N)$ can be chosen so that for all $N \geq 3$,
\[
(5.11) \quad \frac{4}{\pi} (1 + \varepsilon(1/N)) > 1.
\]

Of course, it can be numerically checked that the upper bound is not optimal: except for exceptional values of $k$, the bound of $l_k^{(N)}$ is almost 1. On the other hand, the lower bound is reached only for some of these exceptional values of $k$.

Proof. First, we can wlog assume that the Leja sequence $L$ starts at $z_1 = 1$.

We begin with the upper bound. The proof is by induction on $p_1 \geq 0$. If $p_1 = 0$, i.e. $N = 2^{p_1+1} - 1 = 2$, then necessarily $k = 1$ and the estimate is obvious since $\left| I_{\omega_l}^{(1)} \right|_{\mathbb{D}} = \left| I_{\omega_l}^{(2)} \right|_{\mathbb{D}} = 1$.

If $p_1 \geq 1$ and $N = 2^{p_1+1} - 1$, then Lemma [18] and the induction hypothesis applied to $2^{(p_1-1)+1} - 1 = 2^{p_1+1} - 2^{p_1} - 1 = N - 2^{p_1}$, yield
\[
\max_{2^{p_1+1} \leq k \leq N} \left| I_{\omega_l}^{(N)} \right|_{\mathbb{D}} \leq \max_{1 \leq k' \leq N - 2^{p_1}} \left| I_{\omega_l}^{(N-2^{p_1})} \right|_{\mathbb{D}} \leq 2.
\]
\[
(5.12)
\]
On the other hand, one has by Lemma [23] that
\[
\max_{1 \leq k \leq 2^{p_1}} \left| I_{\omega_l}^{(N)} \right|_{\mathbb{D}} = \max_{1 \leq k \leq 2^{p_1}} \left| I_{\omega_l}^{(N)} \right|_{\mathbb{D}} = \max_{0 \leq l \leq 2^{p_1} - 1} \left| I_{\omega_l}^{(N)} \right|_{\mathbb{D}}.
\]
\[
(5.13)
\]
Now for all $l = 0, \ldots, 2^{p_1} - 1$ and all $z \in \mathbb{D}$ with $z \neq 1, \omega_l$, one has by (5.3) that
\[
\left| I_{\omega_l}^{(N)} (z) \right| = \frac{|1 - \omega_l|}{2^{p_1} + 1} \cdot \frac{|z^{2^{p_1} + 1} - 1|}{|z - 1| |z - \omega_l|} \leq \frac{|1 - z| + |z - \omega_l|}{2^{p_1} + 1} \cdot \frac{|z^{2^{p_1} + 1} - 1|}{|z - 1| |z - \omega_l|} \\
= \frac{1}{2^{p_1} + 1} \left( 1 + |z^{2^{p_1} + 1} - 1| \right) \frac{|z^{2^{p_1} + 1} - 1|}{|z - 1| |z - \omega_l|} \\
\leq \frac{1}{2^{p_1} + 1} \left( 1 + |z^{2^{p_1} + 1} - 1| \right) \frac{|z^{2^{p_1} + 1} - 1|}{|z - 1| |z - \omega_l|} \\
= \frac{1}{2^{p_1} + 1} \left( 1 + |z^{2^{p_1} - 1} - 1| \right) \frac{|z^{2^{p_1} - 1} - 1|}{|z - 1| |z - \omega_l|} \\
= \frac{1}{2^{p_1} + 1} \left( 1 + \frac{|z^{2^{p_1} - 1} - 1|}{|z - 1|} \right) \frac{|z^{2^{p_1} - 1} - 1|}{|z - 1| |z - \omega_l|}.
\]

since $\omega_l$ is a $2^{p_1}$-th root of $-1$. The symmetry of the unit disk and Lemma [1] yield
\[
\left| I_{\omega_l}^{(N)} (z) \right| = \left| I_{\omega_l}^{(N)} \right|_{\mathbb{D}} = \left| I_{\omega_l}^{(N)} \right|_{\mathbb{D}} = \left| I_{\omega_l}^{(N)} \right|_{\mathbb{D}} = 1.
\]
\[
\text{It follows that}
\left| I_{\omega_l}^{(N)} \right|_{\mathbb{D}} \leq \left| I_{\omega_l}^{(N)} \right|_{\mathbb{D}} + \left| I_{\omega_l}^{(N)} \right|_{\mathbb{D}} = 2,
\]
and (5.13) becomes
\[
(5.14) \quad \max_{1 \leq k \leq 2^{p_1}} \left| I_{\omega_l}^{(N)} \right|_{\mathbb{D}} = \max_{0 \leq l \leq 2^{p_1} - 1} \left| I_{\omega_l}^{(N)} \right|_{\mathbb{D}} \leq 2.
\]
Finally, (5.12) and (5.14) prove the induction.

In order to prove the lower bound, we first have by (5.13) that for all $N \geq 1$,

$$\max_{1 \leq k \leq N} \left\| f_k^{(N)} \right\|_{\mathcal{D}} \geq \max_{1 \leq k \leq 2^{p_1}} \left\| f_k^{(N)} \right\|_{\mathcal{D}} = \max_{0 \leq \ell \leq 2^{p_1} - 1} \left\| f_{\omega_1}^{(N)} \right\|_{\mathcal{D}} \geq \left\| f_{\omega_0}^{(N)} \left( \exp \left( i\pi / 2^{p_1+1} \right) \right) \right\|_{\mathcal{D}}.$$  

(5.15)

On the other hand, one has by (5.15), (5.16) and the estimate (2.19) from Lemma 4 that for all $p_1 \geq 1$,

$$\left\| f_{\omega_0}^{(N)} \left( \exp \left( i\pi / 2^{p_1+1} \right) \right) \right\| = \frac{1 - \exp \left( i\pi / 2^{p_1} \right) \times \exp \left( 2^{p_1+1} \times \pi / 2^{p_1} \right) \times \exp \left( i\pi / 2^{p_1+1} \right) - 1}{2^{p_1} \times \exp \left( i\pi / 2^{p_1} \right) - 1 \times \exp \left( i\pi / 2^{p_1} \right) - \exp \left( i\pi / 2^{p_1} \right)}$$

$$= \frac{2 \sin \left( \pi / 2^{p_1+1} \right) \times 2}{2^{p_1} \times 2 \sin \left( \pi / 2^{p_1+2} \right) \times 2 \sin \left( \pi / 2^{p_1+2} \right)}$$

$$= \frac{2 \sin \left( \pi / 2^{p_1+2} \right) \cos \left( \pi / 2^{p_1+2} \right)}{2^{p_1+1} \sin \left( \pi / 2^{p_1+2} \right)}$$

(5.16)

$$\sim \frac{1}{2^{p_1} \times \pi / 2^{p_1+2} \rightarrow \pi / \pi} \rightarrow \frac{4}{\pi}.$$  

(5.17)

Since $N \rightarrow +\infty$ iff $p_1 \rightarrow +\infty$, it follows that (5.15) and (5.17) lead to

$$\max_{1 \leq k \leq N} \left\| f_k^{(N)} \right\|_{\mathcal{D}} \geq \left\| f_{\omega_0}^{(N)} \left( \exp \left( i\pi / 2^{p_1+1} \right) \right) \right\| = \frac{4}{\pi} \left( 1 + \varepsilon \left( 1/\pi \right) \right),$$

where $\lim_{N \rightarrow +\infty} \varepsilon \left( 1/\pi \right) = 0$, and this yields the required lower bound. In addition, one also has by (5.13), (5.16) and the estimate (2.19) from Lemma 4 that for all $p_1 \geq 1$,

$$\max_{1 \leq k \leq N} \left\| f_k^{(N)} \right\|_{\mathcal{D}} \geq \frac{\cos \left( \pi / 2^{p_1+2} \right)}{2^{p_1} \sin \left( \pi / 2^{p_1+2} \right)} \geq \frac{\cos \left( \pi / 2^{p_1+2} \right)}{2^{p_1} \times \pi / 2^{p_1+2}} \geq \frac{4 \cos (\pi / 8)}{\pi} > 1,$$

and this proves (5.11) (since $p_1 \geq 1$ iff $N = 2^{p_1+1} - 1 \geq 3$) and completes the whole proof of the proposition.

\[ \checkmark \]

Now we can give the proof of Theorem 2.

**Proof.** First, $N = 2^{p_1+1} - 1$ and $\mathcal{L}_N$ a $N$-Leja section being given, by symmetry of the unit disk, we can wlog assume that $\mathcal{L}_N$ starts at $z_1 = 1$. Next, let consider $k_N = 1, \ldots, N$, that satisfies

$$\left\| f_{k_N}^{(N)} \right\|_{\mathcal{D}} = \max_{1 \leq k \leq N} \left\| f_k^{(N)} \right\|_{\mathcal{D}}.$$  

We know by Proposition 3 and (5.11) that for all $N \geq 3$,

$$\left\| f_{k_N}^{(N)} \right\|_{\mathcal{D}} \geq \frac{4}{\pi} \left( 1 + \varepsilon \left( 1/\pi \right) \right) > 1,$$
where \( \lim_{N \to +\infty} \varepsilon(1/N) = 0 \). On the other hand, since for all \( k = 1, \ldots, N \), \( \| t_k^{(N)} \|_B \geq \| t_k^{(N)} \|_{L_k^2(z_k)} \) = 1, it follows that

\[
\sum_{k=1}^{N} \| t_k^{(N)} \|_B = \| t_{k_N}^{(N)} \|_B + \sum_{k=1, k \neq k_N}^{N} \| t_k^{(N)} \|_B \geq \frac{4}{\pi} (1 + \varepsilon(1/N)) + \sum_{k=1, k \neq k_N}^{N} 1
\]

(5.18)

\[
= \frac{4}{\pi} (1 + \varepsilon(1/N)) + N - 1 > 1 + (N - 1) = N.
\]

In particular, this last estimate yields

\[
\liminf_{N \to +\infty} \left[ \frac{1}{N} \sum_{k=1}^{N} \| t_k^{(N)} \|_B \right] \geq 1.
\]

(5.19)

The essential part of the proof is to deal with the other inequality. We first have that

\[
\frac{1}{N} \sum_{k=1}^{N} \| t_k^{(N)} \|_B = \frac{1}{2p_1+1-1} \sum_{k=1}^{2p_1+1-1} \| t_k^{(N)} \|_B
\]

\[
= \frac{2p_1}{2p_1+1-1} \times \frac{1}{2p_1} \sum_{k=1}^{2p_1} \| t_k^{(N)} \|_B + \frac{2p_1}{2p_1+1-1} \times \frac{1}{2p_1} \sum_{k=2p_1+1}^{2p_1+1-1} \| t_k^{(N)} \|_B,
\]

then (since \( N \to +\infty \) iff \( p_1 \to +\infty \))

\[
\limsup_{N \to +\infty} \left[ \frac{1}{N} \sum_{k=1}^{N} \| t_k^{(N)} \|_B \right] \leq \frac{1}{2} \times \limsup_{p_1 \to +\infty} \left[ \frac{2p_1}{2p_1+1} \sum_{k=1}^{2p_1} \| t_k^{(N)} \|_B \right] + \frac{1}{2} \times \limsup_{p_1 \to +\infty} \left[ \frac{2p_1}{2p_1+1} \sum_{k=2p_1+1}^{2p_1+1} \| t_k^{(N)} \|_B \right].
\]

(5.20)

Now let fix any \( \varepsilon > 0 \) small enough. On the one hand, by Lemma [21] there is \( P_1(\varepsilon) \geq 1 \) such that for all \( p_1 \geq P_1(\varepsilon) \),

\[
\frac{1}{2p_1} \sum_{k=1}^{2p_1} \| t_k^{(N)} \|_B \leq 2\pi \exp(3\pi) \left( \varepsilon + \frac{1}{2p_1} \right) + \frac{1 - 2\varepsilon}{1 - \varepsilon/4},
\]

then

\[
\limsup_{p_1 \to +\infty} \left[ \frac{1}{2p_1} \sum_{k=1}^{2p_1} \| t_k^{(N)} \|_B \right] \leq 2\pi \varepsilon \exp(3\pi) + \frac{1 - 2\varepsilon}{1 - \varepsilon/4}.
\]

(5.21)

On the other hand, the one-to-one correspondence defined by (5.4) from Lemma [25] and the associated estimate lead to

\[
\sum_{k=2p_1+1}^{2p_1+1} \| t_k^{(N)} \|_B \leq \sum_{q=0}^{p_1-1} \sum_{k'=1}^{2^q-1} \| t_{k'}^{(2^q-1)} \|_B = \frac{p_1-1}{2^q} \times \frac{1}{2} \sum_{k'=1}^{2^q} \| t_{2^q-k'}^{(2^q-1)} \|_B.
\]

(5.22)

Now for every \( q \) with \( P_1(\varepsilon) \leq q \leq p_1 - 1 \), we can still apply Lemma [21] for the \( (2^q-1) \)-Leja section \( \tilde{L}_{2^q-1} \) (with \( p_1 \) replaced by \( q \) and \( N = 2^q+1 - 1 \) replaced
by $2^q - 1$ to get
\[
\frac{1}{2^q} \sum_{k' = 1}^{2^q} \left\| f_{k'}^{(2^q+1-1)} \right\|_\infty \leq 2\pi \exp(3\pi) \left( \varepsilon + \frac{1}{2^q} \right) + \frac{1 - 2\varepsilon}{1 - \varepsilon/4},
\]
then
\[
\sum_{q = P_1(\varepsilon)}^{P_1(\varepsilon) - 1} 2^q \times \frac{1}{2^q} \sum_{k' = 1}^{2^q} \left\| f_{k'}^{(2^q+1-1)} \right\|_\infty \leq 2\pi \exp(3\pi) + \frac{1 - 2\varepsilon}{1 - \varepsilon/4}
\]
For all $q$ with $0 \leq q \leq P_1(\varepsilon) - 1$, we just have by Theorem 1 that
\[
\sum_{q = P_1(\varepsilon) - 1}^{P_1(\varepsilon) - 2^q} \sum_{k' = 1}^{2^q} \left\| f_{k'}^{(2^q+1-1)} \right\|_\infty \leq \pi \exp(3\pi) \sum_{q = P_1(\varepsilon) - 1}^{P_1(\varepsilon) - 2^q} \left( 2\pi \exp(3\pi) + \frac{1 - 2\varepsilon}{1 - \varepsilon/4} \right) 2^{P_1(\varepsilon)}
\]
It follows that (5.22), (5.23) and (5.24) together yield
\[
\sum_{k = 2^{P_1(\varepsilon) - 1}}^{2^{P_1(\varepsilon) - 1}} \left\| f_{k}^{(N)} \right\|_\infty \leq \pi \exp(3\pi) 2^{P_1(\varepsilon)} + \left( 2\pi \exp(3\pi) + \frac{1 - 2\varepsilon}{1 - \varepsilon/4} \right) 2^{P_1(\varepsilon)} + 2\pi \exp(3\pi) P_1(\varepsilon)
\]
thus,
\[
\limsup_{P_1(\varepsilon) \to +\infty} \left[ \frac{1}{2^{P_1(\varepsilon) - 1}} \sum_{k = 2^{P_1(\varepsilon) - 1}}^{2^{P_1(\varepsilon) - 1}} \left\| f_{k}^{(N)} \right\|_\infty \right] \leq 2\pi \exp(3\pi) + \frac{1 - 2\varepsilon}{1 - \varepsilon/4} + \lim_{P_1(\varepsilon) \to +\infty} \left( \frac{\pi \exp(3\pi) 2^{P_1(\varepsilon)}}{2^{P_1(\varepsilon)}} + 2\pi \exp(3\pi) \frac{P_1(\varepsilon)}{2^{P_1(\varepsilon)}} \right)
\]
Finally, the estimates (5.22), (5.24) and (5.26) together yield
\[
\limsup_{N \to +\infty} \left[ \frac{1}{N} \sum_{k=1}^{N} \left\| f_{k}^{(N)} \right\|_\infty \right] \leq \frac{1}{2} \left( 2\pi \exp(3\pi) + \frac{1 - 2\varepsilon}{1 - \varepsilon/4} \right) + \frac{1}{2} \left( 2\pi \exp(3\pi) + \frac{1 - 2\varepsilon}{1 - \varepsilon/4} \right)
\]
\[
= 2\pi \exp(3\pi) + \frac{1 - 2\varepsilon}{1 - \varepsilon/4},
\]
and \( \varepsilon > 0 \) being arbitrary, we can deduce that
\[
\limsup_{N \to +\infty} \left[ \frac{1}{N} \sum_{k=1}^{N} \left\| i_k^{(N)} \right\|_{\mathcal{F}} \right] \leq \lim_{\varepsilon \to 0, \varepsilon > 0} \left( 2\pi \varepsilon \exp(3\pi) + \frac{1 - 2\varepsilon}{1 - \varepsilon/4} \right) = 1.
\]

This last estimate and (5.19) lead to
\[
1 \leq \liminf_{N \to +\infty} \left[ \frac{1}{N} \sum_{k=1}^{N} \left\| i_k^{(N)} \right\|_{\mathcal{F}} \right] \leq \limsup_{N \to +\infty} \left[ \frac{1}{N} \sum_{k=1}^{N} \left\| i_k^{(N)} \right\|_{\mathcal{F}} \right] \leq 1
\]
and this proves (1.9). In addition, this leads with (5.18) for all \( N \geq 3 \) to
\[
N < \sum_{k=1}^{N} \left\| i_k^{(N)} \right\|_{\mathcal{F}} = N \left( 1 + \varepsilon(1/N) \right),
\]
and this proves (1.10) and completes the whole proof of the theorem (since \( N = 2^{p_1+1} - 1 \geq 3 \) iff \( p_1 + 1 \geq 2 \)).

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