Regular frames for spherically symmetric black holes revisited

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We consider a space-time of a spherically symmetric black hole with one simple horizon. As a standard coordinate frame fails in its vicinity, this requires continuation across the horizon and constructing frames which are regular there. Up to now, several standard frames of such a kind are known. It was shown in literature before, how some of them can be united in one picture as different limits of a general scheme. However, some types of frames (the Kruskal-Szekeres and Lemaître ones) and transformations to them from the original one remained completely disjoint. We show that the Kruskal-Szekeres and Lemaître frames stem from the same root. Overall, our approach in some sense completes the procedure and gives the most general scheme. We relate the parameter of transformation $e_0$ to the specific energy of fiducial observers and show that in the limit $e_0 \to 0$ a homogeneous metric under the horizon can be obtained by a smooth limiting transition.

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I. INTRODUCTION

The Schwarzschild black hole \cite{1} is the core object of general relativity, the properties of its space-time play a crucial role in understanding space-time of black holes. The standard coordinate system in which the Schwarzschild metric is written uses so-called curvature (Schwarzschildian) coordinates and fails on the event horizon. To repair this drawback, there are several "standard" transformations and corresponding coordinate systems - such as Eddington-Finkelstein coordinates (EF), Kruskal-Szekeres ones, Novikov systems, Gullstrand-Painlevé (GP) and Lemaître coordinates. All these coordinates and methods of their constructions look very different. Meanwhile, it turned out that all of these transformations (or at least their part) can be united, if one introduces an additional parameter in the coordinate transformation. This parameter $e_0$ has the meaning of the energy per unit mass for a reference (fiducial) observer. In this sense, particle dynamics is encoded in the typical transformations to the regular frame. Then, taking the limit $e_0 \to \infty$, one can recover some familiar coordinate systems and metrics \cite{2}, \cite{3}, \cite{4}. In this sense, previous metrics are contained as different limiting cases of some more general one.

The approach developed in \cite{2}, \cite{4} does not include the transformation to the synchronous frame. Meanwhile, this frame simplifies the whole picture and thus plays an important role. The construction of such a frame for the Schwarzschild metric was done by Lemaître \cite{5}. Quite recently, this was generalized to metrics other than the Schwarzschild one. To this end, two different procedures were suggested, \cite{6}, \cite{7}.

In spite of the fact that some frames were combined in a single general construction, the whole picture remains quite intricate and even the ways of particular unifications also look very different. Unifying particular approaches and metrics, we can separate the whole set of possible transformations to two kinds. The first one (A) envolves the parameter $e_0$ having the meaning of energy per unit mass of fiducial observers. This includes such systems in which fiducial observers (characterized by a constant value of a spatial coordinate) move not freely. The bright example is the Kruskal-Szekeres system. The second class (B) contains the transition to synchronous systems. For example, this concerns the Novikov system \cite{8} or more general Bronnikov - Dynnikova - Galaktionov (BDG) system \cite{6}. In appearance, classes A and B look completely separated, derived from different requirements and seem to be complementary to each other. However, we show that, as a matter of fact, there is deep
and very simple connection between both classes. We also consider the limit quite different from [2], [3], [4] where it was implied that \( \epsilon_0 > 1 \). This is the limit \( \epsilon_0 \to 0 \). Then, another synchronous system typical of the Kantowski-Sachs (KS) metric [9], [10] appears explicitly.

The most general regular frame is the one suggested by Fomin [11]. Wrongly, this paper was not noticed in due course and remained poorly cited. We use the approach of Fomin to show that all other ones can be obtained from it.

Our consideration applies to a class of metrics more general than the Schwarzschild one. It includes the Schwarzschild-de Sitter, Reissner-Nordström metrics, etc. Further generalization is straightforward. We use the geometric system of units in which fundamental constants \( G = c = 1 \).

\section{II. GENERALIZED GULLSTRAND-PAINLEVÉ FRAME}

Let us consider the metric

\[ ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2 d\omega^2, \]  

where \( d\omega^2 = d\theta^2 + d\phi^2 \sin^2 \theta \). It represents the spherically symmetric solution of the Einstein equations, provided the components of the stress-energy tensor obey the relation \( T^r_r = T^0_0 \).

Let \( r = r_+ \) correspond to the event horizon, so \( f(r_+) = 0 \). The original frame fails in the vicinity of the horizon. To repair the situation, one can introduce a new time variable

\[ d\tilde{t} = e_0 dt + e_0 \frac{dr}{f} V \equiv e_0 dt + P_0 \frac{dr}{f}, \]  

where initially we consider \( e_0 \) as a positive function of coordinates \( r \) and \( t \),

\[ P_0 = e_0 V = \sqrt{e_0^2 - f}, \]  

\[ V = \sqrt{1 - \frac{f}{e_0^2}}. \]  

After substitution in (1), one obtains

\[ ds^2 = -\frac{f}{e_0^2} d\tilde{t}^2 + \frac{2d\tilde{t}dr}{e_0} V + \frac{dr^2}{e_0^2} + r^2 d\omega^2. \]  

It can be also written in the form

\[ ds^2 = -d\tilde{t}^2 + \frac{1}{e_0^2} (dr + V e_0 d\tilde{t})^2 + r^2 d\omega^2. \]
Now,
\[ g^{\bar{t}\bar{t}} = -1, \quad g^{\bar{t}r} = V e_0, \quad g^{rr} = f. \] (7)

As \( d\bar{t} \) should be a total differential, the integrability conditions have to be fulfilled:
\[ (e_0)_r = \frac{e_0 (e_0)_t}{P_0 f}. \] (8)

Eq. (5) corresponds to eq. (3.19) of [3]. Let \( e_0 = const. \) If \( e_0 = 1, \) we arrive at the Gullstrand-Painlevé frame [12], [13], generalized to an arbitrary \( f [7]. \) If \( f = 1 - \frac{2m}{r} \) in (5), we return to the Schwarzschild version of the GP system [14], [2], [4].

For the GP system the cross term with \( d\bar{t}dr \) defines a coordinate flow velocity \( V \) which has a direct physical meaning - it is the 3-velocity of a free falling particle with the unit energy with respect to the static coordinate system (see below). From (5) we can see that this direct correspondence is lost since for \( e_0 \neq 1, \) the additional factor \( (1/e_0) \) appears. The reason is, however, rather clear - since the intervals of physical distance \( dl \) in \( \bar{t} = const \) sections are connected with the interval of coordinate distance \( dr \) via \( dl = dr/e_0, \) the physical 3-velocity of the generalized GP system with respect to the stationary system is still equal to \( V, \) as it should be. So, we still can think of the coordinate system with \( e_0 = const \) as realized by free falling particles with the energy \( e_0. \)

This can be confirmed by direct simple calculations. We can choose tetrads attached to an static observer. Then, in coordinates \((t, r, \theta, \phi)\)
\[ h^{(0)}_\mu = \sqrt{f} (-1, 0, 0, 0), \] (9)
\[ h^{(1)}_\mu = (0, \frac{1}{\sqrt{f}}, 0, 0). \] (10)

Let us introduce the three-velocity in a standard way [15]
\[ V^{(i)} = -h^{(i)}_\mu u^\mu \frac{h^{(0)}_\mu u^\mu}{h^{(0)}_\mu u^\mu}, \] (11)
where \( u^\mu \) is the four-velocity. Eq. (11) is valid in general. Now, we will apply it to motion of a particle moving freely with a constant specific energy \( e_0. \) For pure radial motion,
\[ \frac{dt}{d\tau} = \frac{e_0}{f}, \] (12)
\[ \frac{dr}{d\tau} = -P_0. \] (13)
where $\tau$ is the proper time along the trajectory, $P_0$ is given by (3). Then, $u^\mu = (\frac{e_0}{f}, -P_0, 0, 0)$. By substitution into (11), we obtain that

$$V^{(1)} = V.$$  \hspace{2cm} (14)

Thus $V$ has the meaning of a velocity measured by a static observer, $P_0 = e_0 V$ being the corresponding momentum. Then, $e_0$ can be thought of as the energy of some effective particle moving in the given background,

$$e_0 = \frac{\sqrt{f}}{\sqrt{1 - V^2}}, \quad P_0 = \frac{V \sqrt{f}}{\sqrt{1 - V^2}}. \hspace{2cm} (15)$$

If a particle moves not freely, $e_0$ ceases to be an integral of motion and depends on time. However, as stressed in Sec. 3 of [16], even in this case equations of motion retain their validity with $e_0 = e_0(t)$. Moreover, we can admit formally dependence of spatial coordinates as well in $e_0$ that enters transformation (2). Although $e_0$ is not an integral of motion in this case, formulas (15) show that this can be still considered as a pure local Lorentz transformation. Below we will see how this helps in constructing regular frames.

The case of $e_0 = 1$ is a special one since spatial sections $d\tilde{t} = 0$ are flat. In this case spatial intervals are simply differences $r_2 - r_1$. When $e_0 \neq 1$, the factor $1/e_0^2$ before $dr^2$ makes them non-flat. For $e_0 \geq 1$, the proper distance between points with fixed $r_2$ and $r_1$ measured along the hypersurface $\tilde{t} = \text{const}$ is less than in the case when $e_0 = 1$. This is some reminiscent of the Lorentz contraction in special relativity (SR). Indeed, in SR ($f = 1$) the proper distance along the hypersurface $t = \text{const}$ ($e_0 = 1, V = 0$) is equal to $\Delta r = r_2 - r_1$. If another observer passes by him with velocity $V$ and the specific energy $e_0 = \frac{1}{\sqrt{1 - V^2}} > 1$, the proper distance measured in its own frame ($\tilde{t} = \text{const}$) equals $\Delta l = \frac{\Delta r}{e_0} = \Delta r \sqrt{1 - V^2} < \Delta r$. However, if $e_0 < 1$, the corresponding situation has no analogue in SR since along the surface $\tilde{t} = \text{const}$ the proper distance $\Delta l > \Delta r$. This is due to the fact that the space-time is curved since in the flat one such observers are absent. Meanwhile, for any fixed $r$ there is a lower bound on possible $e_0$ which is equal to $f(r)$. Therefore, among all states with different $e_0$ and fixed $dr$, the maximum value of the proper distance $dl = dr/e_0$ is achieved for a minimum value of $e_0 = \sqrt{f(r)}$ that coincides with the distance in the static frame. This is natural since an observer with minimal possible $e_0$ for a given $r$ has zero flow velocity (4) and thus coincides with a stationary observer. In this sense, there is some analogy with SR again since the minimum of the proper distance is achieved in the rest frame.
If we rescale time according to
\[ \tilde{t} = \hat{t}e_0 \] (16)
with \( e_0 = \text{const} > 0 \),
\[ ds^2 = -f d\hat{t}^2 + 2d\hat{t}drV + \frac{dr^2}{e_0^2} + r^2 d\omega^2. \] (17)

It follows from (2), (16) that
\[ \hat{t} = t - \int \frac{dr}{f} V = t - \int dr^* V, \] (18)
where
\[ r^* = \int \frac{dr}{f} \] (19)
is the tortoise coordinate.

If \( f = 1 - \frac{2M}{r} \) and we write \( e_0 = \frac{1}{p^2} \), we return to the coordinates of Ref. [2] - see eq. (3.5) there. It is worth noting that eq. (5) is valid even if \( e_0 \) depends on both coordinates. Meanwhile, transformation from (5) to (17) implies that \( e_0 = \text{const} \).

If one takes the limit \( e_0 \to \infty \) in (17), the metric in the EF coordinates is reproduced:
\[ ds^2 = -f d\hat{t}^2 + 2d\hat{t}dr + r^2 d\omega^2. \] (20)

In doing so, \( V \to 1 \),
\[ \hat{t} = t - r^* = u \] (21)
is the EF coordinate.

It is worth noting that the transformation [2] somewhat generalizes that in [4]. However, in contrast to [4], we do not use the double GP coordinates and obtain the limit \( e_0 \to \infty \) directly from (5) after rescaling the time coordinate \( \tilde{t} \to \hat{t} \). If one starts with \( \hat{t} \) from the very beginning, the limiting transition from [2] is reproduced easily with their \( T \) equal to \( \hat{t} \). Meanwhile, as the transformation used in [2] does not include the parameter \( e_0 \), it is relatively restricted in that it is unable to describe the diversity of different approaches.

### III. DIAGONAL METRIC

We can consider transformation that can be interpreted as modification of the approach developed by Fomin [11]. For completeness, we present here the main features of this
approach, though in contrast to the original paper, we use our notations with the parameters $e_0$ and $P_0$.

Let in new coordinates $T$, $\rho$ the metric be regular and diagonal,

$$ds^2 = -F(T, \rho)dT^2 + G(T, \rho)d\rho^2 + r^2(\rho, T)d\omega^2. \quad (22)$$

We perform transformations according to which

$$dt = \frac{e_0}{f} \sqrt{F}dT - \frac{\sqrt{GP_0}}{f}d\rho, \quad (23)$$

$$dr = e_0 \sqrt{G}d\rho - P_0 \sqrt{F}dT, \quad (24)$$

where $P_0$ is given by eq. (3).

The inverse transformation reads

$$dT = \frac{1}{\sqrt{F}}(dte_0 + dr P_0), \quad (25)$$

$$d\rho = \frac{1}{\sqrt{G}} \left( dtP_0 + dre_0 f \right). \quad (26)$$

It is easy to check that in new coordinates the metric is indeed diagonal. If we put in

$$F = 1$$

and, instead of $\rho$, will use our previous coordinate $r$, this would correspond to
the transformation (2) that leads to the GP metric (5).

It follows from (23) - (26) that

$$\frac{r_T}{r_\rho} \sqrt{\frac{G}{F}} = -V, \quad (27)$$

$$F = t^2_T f(1 - V^2) = \frac{r^2_T (1 - V^2)}{fV^2}, \quad (28)$$

$$G = \frac{t^2_\rho f(1 - V^2)}{V^2} = \frac{r^2_\rho (1 - V^2)}{f}. \quad (29)$$

Eqs. (27), (28) correspond to eqs. (9), (10) of [11].

To ensure that the left hand side of (23) and (24) is the total differential, the integrability
conditions should be satisfied:

$$\left( \frac{e_0}{f} \sqrt{F} \right)_{,\rho} = - \left( \frac{\sqrt{GP_0}}{f} \right)_{,T}, \quad (30)$$

$$\left( e_0 \sqrt{G} \right)_{,T} = - \left( P_0 \sqrt{F} \right)_{,\rho}. \quad (31)$$
Here,

\[ f,\rho = e_0 \sqrt{G} f'(r), \quad (32) \]

\[ f,T = -P_0 \sqrt{F} f'(r) \quad (33) \]

After substitution into (30) we get

\[ -\sqrt{F}G f'(r) + \left( e_0 \sqrt{F} \right)_\rho + \left( P_0 \sqrt{G} \right)_T = 0 \quad (34) \]

If \( F = 1 \), (25) coincides with (2). In general, \( e_0 = e_0(\rho, T) \). Then, it cannot be interpreted as a conserved energy, although one can define the quantity \( V \) formally according to (15).

If we assume that \( e_0 \) is finite (at least, finite near the horizon), it follows from (15) the universal behavior of \( V \):

\[ V = \sqrt{1 - \frac{f}{e_0^2}} \approx 1 - \frac{f}{2e_0^2} \approx 1 - \frac{\kappa}{e_0^2}(r - r_+), \quad (35) \]

where we took into account that

\[ f \approx 2\kappa(r - r_+), \quad (36) \]

\( \kappa \) is the surface gravity that agrees with eqs. (21), (22) of [11].

It is instructive to analyze the example suggested by Fomin for the Schwarzschild metric, when \( V = \tanh \frac{t}{2r_+} \). In this case,

\[ e_0 = \sqrt{1 - \frac{r_+}{r} \cosh \frac{t}{2r_+}}. \quad (37) \]

We see that for a fixed \( r > r_+ \), \( \lim_{t \to \infty} e_0 = \infty \). If, instead, we fix \( t \), then \( \lim_{r \to r_+} e_0 = 0 \). For our purposes, it is important that \( e_0 \) remain finite and nonzero for an observer falling in a black hole. Then, we consider \( t \) and \( r \) as related by equations of motion. For a geodesic observer with some \( e \),

\[ \frac{dr}{dt} = \frac{f}{e} \sqrt{e^2 - f}, \quad (38) \]

whence near the horizon of the Schwarzschild black hole

\[ \frac{t}{r_+} \approx \ln \left( \frac{r - r_+}{r_+} \right). \quad (39) \]

As a result,

\[ e_0 \approx 1. \quad (40) \]
Then, the transformations (23), (24) acquire the meaning of the local Lorentz transformations and are equivalent to eqs. (7) of [11]. It follows from (23), (24) that
\[
\frac{t'}{r'} = -\frac{P_0}{f e_0},
\] (41)
where prime denotes derivative with respect to $\rho$. Eq. (41) corresponds to eq. (17) of [3]. It can be also rewritten in the form
\[
\frac{t'}{r'} = -\frac{V}{f}.
\] (42)

IV. SYNCHRONOUS SYSTEM AND RELATION TO BDG

Now, we will consider a particular case when $e_0$ does not depend on $T$. Then, it follows from (34) with (30) with (32), (33) that
\[
\frac{de_0}{d\rho} \sqrt{F} + e_0 \left( \sqrt{F} \right)_{,\rho} - \frac{1}{2} \sqrt{F} G f'(r) + P_0 \left( \sqrt{G} \right)_{,T} = 0.
\] (43)

And, (31) gives us
\[
e_0 \left( \sqrt{G} \right)_{,T} + P_0 \left( \sqrt{F} \right)_{,\rho} + \frac{1}{P_0} \sqrt{F} e_0 \frac{de_0}{d\rho} - \frac{1}{2} e_0 \sqrt{G} f'(r) = 0.
\] (44)

From these two equations we obtain that $F_{,\rho} = 0$, so eqs. (43) and (44) are equivalent. If $F = F(T)$, we can always rescale time to achieve $F = 1$. Then, the frame becomes synchronous. The function $G$ should obey the equation
\[
P_0 \left( \sqrt{G} \right)_{,T} = \left( \frac{f'}{2} \sqrt{G} - \frac{de_0}{d\rho} \right),
\] (45)
whence
\[
\sqrt{G} = \frac{P_0}{e_0} \mu(\rho, r).
\] (46)

It follows from (3) and (33) that
\[
\left( \sqrt{G} \right)_{,T} = \frac{f'}{2e_0}.
\] (47)

After substitution in (45) we obtain
\[
P_0^2 \mu_{,T} = -e_0^2 e_0.
\] (48)

Then, it is easy to find the solution with the help of the ansatz
\[
\mu = z(\rho) + e_0 e_0' \eta(r),
\] (49)
where \( \eta' = P_0^{-2} \), whence

\[
\eta = \int^r \frac{\bar{d}\bar{r}}{P_0^3(\bar{r})}
\]  

(50)

for a given \( \rho \). It follows from (46) that

\[
\sqrt{G} = \frac{P_0}{e_0} [z(\rho) + e_0 e_0' \eta(r)],
\]  

(51)

so

\[
ds^2 = -dT^2 + \left(\frac{P_0}{e_0}\right)^2 [z(\rho) + e_0 e_0' \eta(r)]^2 d\rho^2 + r^2 d\omega^2.
\]  

(52)

It can be written also in the form

\[
ds^2 = -dT^2 + \left(\frac{r_{,\rho}}{e_0}\right)^2 d\rho^2 + r^2 d\omega^2
\]  

(53)

where we used (24).

This exactly corresponds (in our notations) to eqs. (19), (20) of [6].

Thus we obtained the synchronous form of the metric from the local Lorentz transformation following the approach of [11]. Meanwhile, it was found in [6] due to analysis of equations of motion of geodesics with different energies.

If the requirement \( \frac{\partial e_0}{\partial T} \rho = 0 \) is relaxed, the metric function depends, in general, on both \( T \) and \( \rho \). Then, world lines of fiducial observers with \( \rho = const \) and \( \theta = const \), \( \phi = const \) are not geodesics. Indeed, in this case we have for the nonzero component of the four-acceleration \( a^\mu \):

\[
a^\rho = \frac{F_{,\rho}}{2FG},
\]  

(54)

\[
a^2 = a_\mu a^\mu = \frac{(F_{,\rho})^2}{4GF^2}.
\]  

(55)

As it is assumed, by construction, that \( F \) and \( G \) are finite and nonzero on the horizon, acceleration \( a \) remains finite there.

If, instead of \( T \) and \( \rho \), one uses \( T \) and \( r \), the generalization of the GP frame can be obtained. Indeed, it follows from (23), (24) that

\[
ds^2 = -dT^2 + 2 \frac{P_0}{e_0^2} FdTdr + \frac{Fdr^2}{e_0^2}.
\]  

(56)

Obviously, metric (56) is regular in the vicinity of the horizon. If \( e_0 = const, F = 1 \), we return to (17) after change \( T \rightarrow -T \).
V. DOUBLE GP COORDINATES

In Ref. [4], two coordinates \( \tilde{t} \) and \( \tau \) were used for constructing a regular Schwarzschild metric. These coordinates represent advanced and retarded GP coordinates. In this Section, we extend the corresponding procedure considering more general metrics (1). It is quite straightforward but, bearing in mind that corresponding formulas can be useful in further applications, we list them explicitly. Let us introduce the coordinate \( \tau \) according to

\[
d\tau = e_0 dt - e_0 \frac{dr}{f} V. \tag{57}
\]

Then,

\[
ds^2 = \frac{f}{4P_0^2 e_0^2} [f(d\tilde{t}^2 + d\tau^2)] - 2(2e_0^2 - f)d\tau d\tilde{t} + r^2 d\omega^2. \tag{58}
\]

This metric is still deficient near the horizon. To repair this, one can introduce Kruskal-type variables \( \tilde{t}' \) and \( \tau' \). Let, for simplicity, \( e_0 \) be constant. Then,

\[
\tilde{t}' = t_0 \exp(\frac{\kappa \tilde{t}}{e_0}) = t_0 \exp(\kappa t + \kappa \frac{\chi}{e_0}), \tag{59}
\]

\[
\tau' = -t_0 \exp(-\frac{\kappa \tilde{t}}{e_0}) = -t_0 \exp(-\kappa t + \kappa \frac{\chi}{e_0}), \tag{60}
\]

where \( t_0 \) is some constant,

\[
\chi = \int \frac{dr}{\sqrt{e_0^2 - f}} = \int dr^* \sqrt{\frac{e_0^2}{f} - f} = e_0 \int dr^* V, \tag{61}
\]

\( \kappa \) is the surface gravity, \( r^* \) is defined in [10]. Then, one can check that in variables \( \tilde{t}' \), \( \tau' \) the metric takes the form

\[
ds^2 = \frac{f}{4P_0^2 e_0^2 \kappa^2} [f(\frac{d\tilde{t}'^2}{\tilde{t}'^2} + \frac{d\tau'^2}{\tau'^2})] + 2(2e_0^2 - f) \frac{d\tau'}{\tau'} \frac{d\tilde{t}'}{\tilde{t}'} + r^2 d\omega^2. \tag{62}
\]

Near the horizon,

\[
f \approx 2\kappa (r - r_+), \quad \chi = \frac{e_0}{2\kappa} \ln(r - r_+) + \chi_{\text{reg}}, \tag{63}
\]

where \( \chi_{\text{reg}} \) is regular near \( r_+ \). Then,

\[
r^* \approx \frac{1}{2\kappa} \ln(r - r_+), \tag{64}
\]

\[
\tilde{t}' \approx t_0 \exp \kappa v, \tag{65}
\]

\[
\tau' \approx -t_0 \exp(-\kappa u), \tag{66}
\]
where

\[ u = t - r^*, \quad v = t + r^*. \]  

(67)

Then, near the horizon, taking into account (35), we have

\[ f \approx -2\kappa \left( \frac{\tilde{t} \tau'}{e_0^2} \right). \]  

(68)

As a result, the metric (62) is regular near the horizon. If \( t_0 = M \) and \( f = 1 - \frac{2M}{r} \), we return to the Schwarzschild case considered in [4]. The whole space-time splits to four regions, similarly to the Kruskal metric in the Schwarzschild case. Transformations (59), (60) correspond to the quadrant I in [4] and can be adjusted to other quadrants. We will not dwell upon on this.

If \( e_0 = 1 \), we return to the standard transformations that bring the metric into the Kruskal form. Now, we can also perform a limiting transition \( e_0 \to \infty \) and observe that

\[ u = \lim_{e_0 \to \infty} \frac{\tau}{e_0} = -\int \frac{dr}{f} = t - r^*, \]  

(69)

\[ v = \lim_{e_0 \to \infty} \frac{\tilde{t}}{e_0} = t + r^*. \]  

(70)

In this limit,

\[ ds^2 = -fdu dv + r^2 d\omega^2, \]  

(71)

so \( u \) and \( v \) have the meaning of the Eddington-Filkenstein coordinates. In doing so, \( \tilde{t}' \) and \( \tau' \) have the meaning of standard Kruskal coordinates.

It is worth noting an important scale property of coordinates \( \tilde{t}' \) and \( \tau' \). One can compare two limits: (i) \( e_0 \to \infty \) for any fixed \( r \geq r_+ \) and (ii) the horizon limit \( r \to r_+ \) for any fixed \( e_0 \). In both limits these coordinates behave in the same manner. We see that the value \( e_0 \) does not have a crucial influence on the coordinate frame, the metric remains regular on the horizon.

VI. SOME EXAMPLES

In this section we present some examples how different metrics, initially discovered using totally different approaches can be incorporated into the general scheme described in the present paper.
First of all, we can note that Eq. (53) is a generalization of the Lemaître - Tolman - Bondi solution (LBT) of Einstein equations valid for dust. To see this, it is sufficient to write

\[ e_0^2 = 1 + h(\rho). \]  (72)

Then, it corresponds to eq. (103.6) of [20], where we used \( h \) instead of \( f \) in [20] and \( \rho \) instead of \( R \). Meanwhile, we would like to stress that the metric (53) is more general and, in particular, its origination can have nothing to do with dust.

From the other hand, (53) can be considered as a generalization of the Novikov frame [8] used for the description of the Schwarzschild metric, if we identify \( e_0^2(\rho) = R^* \) instead of \( f \) in (31.12a) of [24].

Another interesting example appears if we put \( e_0 = 1 \) and

\[ f(r) = 1 - H^2r^2. \]  (73)

It is convenient to rescale \( \rho \) in such a way that \( z(\rho) = 1 \).

It is seen from (3) that

\[ P_0 = Hr. \]  (74)

Then, eq. (51) gives us

\[ \sqrt{G} = Hrz(\rho). \]  (75)

It is convenient to take \( z(\rho) = 1 \). It follows from (24) that

\[ r,\rho = Hr \]  (76)

and it follows from (31) that

\[ r,T = -r,\rho. \]  (77)

As a result, we can write

\[ r = r_0 \exp(H\rho - HT), \]  (78)

where \( r_0 \) is a constant. We see that the expression for \( r \) is factorized into a product of a function of \( \rho \) and a function of \( T \). This means that by appropriate redefinition of \( \rho \) in the form \( \chi = \exp(H\rho) \) we can kill all the dependence \( r \) upon \( \rho \) and obtain a metric with the dependence upon \( T \) only. It is convenient also to choose \( r_0 = H^{-1} \) and make redefinition \( \tilde{T} = -T \). Then,

\[ ds^2 = -dT^2 + \frac{\exp(2HT)}{H^2}(d\chi^2 + \chi^2d\omega^2), \]  (79)
where we omitted tilde. This is nothing else than the standard Friedmann form of the de Sitter flat metric. It is interesting that allowing for non-constant $e_0$ it is possible to get also positively and negatively curved de Sitter solutions, see [6].

As for GP form of the metric, it has the form

$$ds^2 = -(1 - H^2 r^2) dt^2 + 2H r dr dt + dr^2$$

(80)

from which we can extract the Hubble law for the velocity of the flow $V = H r$. It is known that this form is valid not only for de Sitter solution, but for an arbitrary Friedmann cosmology [26].

Note that the fact that the resulting diagonal metric (79) appears to be a homogeneous one explicitly is connected with a particular form of the function $f$ in eq. (3) and particular value $e_0 = 1$ which leads to factorizable expression for $r$. Meanwhile, in the next section we will see that there exists another family of homogeneous metrics existing for an arbitrary function $f$.

VII. THE LIMIT $e_0 \to 0$

Let us consider the limiting transition $e_0 \to 0$. It cannot be done in the metric (5) directly. In this limit, the axis $r$ and $T$ become collinear since in (24) the term with $d\rho$ drops out. As a result, these coordinates fail to be suitable for constructing a regular frame. Also, the limit under discussion cannot be taken in the form of metric (52), (53). Formally, the proper distance between two arbitrary points with different values of their radial coordinate $r$ grows like $1/e_0$ and the metric becomes degenerate.

However, for a synchronous metric the limit $e_0 = 0$ is allowed. To make a meaningful result, we need to rescale the spatial coordinate according to $\rho = e_0 \tilde{\rho}$ and take the limit under discussion only afterwards. Then, it follows from (24) with $e_0 = 0, F = 1$ that

$$T = - \int^r \frac{d\tilde{r}}{\sqrt{g(r)}},$$

(81)

where $g = -f > 0$. Thus this transformation is legitimate under the horizon only. It brings the metric in the form

$$ds^2 = -dT^2 + g(r(T)) d\rho^2 + r^2(T) d\omega^2.$$  

(82)
Schematically, the timelike geodesics with $e_0 = 0$ are depicted on Fig. 1 where a relevant part of the Kruskal diagram is depicted. It is worth noting that in synchronous form of the metric the variable $\tilde{\rho}$ is always a spatial one, and $T$ is always a temporal one. However, some peculiarities of the $e_0 = 0$ case lead to peculiar properties of the corresponding synchronous frame. It can be easily seen that the metric now depends on the temporal coordinate only, becoming an homogeneous one. This is however not surprising, since, as the $T = \text{const}$ hypersurface coincides now with the $r = \text{const}$ hypersurface, and any spatial dependence in a spherically symmetric metric is in fact the $r$-dependence. Therefore, it is clear that the hypersurface $T = \text{const}$ in the $e_0 = 0$ case has no spatial dependence at all. The central singularity is not present in any nonsingular $T = \text{const}$ plane, and instead, is present in the observer’s future.

This form of the metric can also be obtained directly from (1) if one interchanges the role of coordinates $r$ and $t$ and makes the coefficient $g_{\tilde{t}\tilde{t}} = -1$ by rescaling the time coordinate. This is just the form, first introduced by Novikov - see [21] and eqs. 2.4.8 and 2.4.9 in [19]. It can be considered as particular case of the cosmological Kantowski-Sachs metric.

The cosmological interpretation of this metrics gives a non-formal explanation of a curious fact about time needed to reach a singularity from a horizon. Indeed, the coordinate time before cosmological singularity $\Delta T = r_+$ obviously does not depend on a particular motion of an observer. As for the proper time from the horizon crossing to singularity hitting, it differs from $\Delta T$ by a Lorentz factor originating from the relative motion of the object in question with respect to the $e_0 = 0$ frame. As it is known from the SR, the Lorentz factor can only make the proper time shorter, so $\Delta T$ is the maximum possible proper time from a
horizon crossing to a singularity hitting, and it is achieved if the observer moves along the geodesic with $e_0 = 0$ – see [16] for detail, where other formal and informal treatments of this question have been given.

Returning to the GP metric (5), we can note that despite the original GP coordinate system has no smooth $e_0 \rightarrow 0$ limit, we can easily write down another coordinate system with a smooth limit at $e_0 = 0$. Indeed, if instead of $\tilde{t}$ and $r$, one uses coordinates $\tilde{t}$ and $t$, then, after substitution of (2) into (5), we obtain the metric in the form

$$ds^2 = -\frac{g}{P_0^2} dt^2 + \frac{g^2 dt^2}{P_0^2} + \frac{2 g e_0}{P_0^2} dt d\tilde{t} + r^2(\tilde{t}) d\omega^2,$$

(83)

$g = -f > 0$ under the horizon. It can be rewritten in the form

$$ds^2 = -d\tilde{t}^2 + \frac{g^2}{P_0^2} (dt + \frac{d\tilde{e}_0}{g})^2 + r^2(\tilde{t}) d\omega^2.$$  

(84)

As under the horizon the coordinate $t$ is space-like, the metric is expressed through one space-like and one time-like coordinate (in contrast to the original GP which has two time-like coordinates under the horizon). The non-diagonal term defines a coordinate “flow velocity” $-\frac{e_0}{g}$ which can be interpreted as a velocity with respect to $e_0 = 0$ frame. Indeed, in the $e_0 = 0$ limit it vanishes. It is known that the 3-velocity with respect to $e_0 = 0$ frame of a radially falling particle with the energy $e$ is equal to $-e/P$ (see eq. (97) in [25]). We get this value from the coordinate velocity if we remember that physical distance interval $dl$ is connected with the interval of the space-like coordinate $dt$ through $dl = (g/P)dt$.

So that, this metric, in some sense dual to GP, has better behavior under the horizon than the original GP and allows a smooth transition to $e_0 = 0$ limit.

VIII. SUMMARY

Thus we established the connection between two kinds of approaches, both of them being connected with the particle dynamics through the parameter $e_0$. In this sense, we revealed the meaning of main coordinate transformations from the original metric. Outside the horizon, some results are known but we extended corresponding interpretation, having considered the region inside the horizon.

Previous papers showed how to unify separate metrics and transformations. We made a next step and showed how one can unify the whole classes of unifying transformation.
Namely, if the parameter of coordinate transformation $e_0 = e_0(\rho)$, the Fomin metric (22) turns into the BDG frame. It is worth noting that the metric (22) is more general than the BDG one in that the coordinate lines of observers with $\rho = \text{const}$ are not necessarily geodesics.

It is also shown that, when $e_0 \to 0$, (22) turns smoothly into the metric considered by Novikov [21]. To the best of our knowledge, existence of this limit was not considered before in the context of black hole metric under the horizon. Thus the coordinates frame such as the Kruskal-Szekeres, homogeneous Kantowski-Sacks metric inside the horizon and Lemaitre ones, which look so differently, are now united as elements of a whole picture.

By contrary, the generalized GP metric has no smooth $e_0 \to 0$ limit. In a sense, we proposed a metric which can be considered as dual to GP. This new form of metric has a good behavior under the horizon, in particular, it is regular for $e_0 = 0$.

It is of interest to try extension of the approach under discussion to the rotating case. Especially interesting in this context is the possibility to build a general approach that would unite the coordinate transformations to regular frames with the the Janis-Newman algorithm [27] that relates static solutions and rotating metrics. Also, it is of interest to generalize the approach under discussion to higher dimensions. All this requires separate treatment.

IX. ACKNOWLEDGEMENT

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