Some Liouville Theorems on Finsler Manifolds

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Abstract: We give some Liouville type theorems of $L^p$ harmonic (resp. subharmonic, superharmonic) functions on a complete noncompact Finsler manifold. Using the geometric relationship between a Finsler metric and its reverse metric, we remove some restrictions on the reversibility. These improve the recent literature (Zhang and Xia, 2014).

Keywords: harmonic function; Finsler manifold; Liouville theorem; reversibility

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1. Introduction

As is well known, Liouville theorems play an important role in analyzing the underlying manifolds. The classical Liouville theorem states that any nonnegative (or bounded) harmonic function on $\mathbb{R}^n$ must be constant. Up to now, there are many generalizations studied on complete Riemannian manifolds. Yau [1,2] proved that any positive (or bounded) harmonic function on a complete Riemannian manifold with nonnegative Ricci curvature must be constant and there are no nonnegative $L^p$ subharmonic functions on such a Riemannian manifold for $p \in (1,\infty)$. Yau’s results were then generalized by Sturm and Schoen, etc. See [3–5] and references therein. For general symmetric diffusion operators, Li [6] extended various Liouville theorems as above.

Recently, Zhang-Xia [7], Yin-He [8] and Yin-Zhang [9] extended the above Liouville theorems in the Finsler setting. Notice that, in [7,8], the Finsler manifolds discussed must have finite reversibility. In this paper, we show that this restriction can be removed. Specifically, we obtain the following results.

Theorem 1. Let $(M, F, d\mu)$ be an $n$-dimensional forward complete noncompact Finsler manifold. If a positive function $u \in W^{2,2}_{\text{loc}}(M) \cap C^1(M) \cap C^\infty(M_u)$ on $M$ satisfies $\Delta \log u \geq 0$ on $M_u$ and

$$\limsup_{r \to \infty} \frac{r^2}{V_1(r)} = \infty,$$

where $M_u = \{ x \in M | du(x) \neq 0 \}$, then $u$ is a constant. In particular, if $u \in L^1(M)$ and $\Delta \log u \geq 0$ on $M_u$, then $u$ is a constant.

Theorem 2. Let $(M, F, d\mu)$ be an $n$-dimensional complete noncompact Finsler manifold. Assume that

$$\int_1^\infty \frac{r}{V_p(r)} \, dr = \infty.$$
1. If \( p \in (\infty, 1) \) and \( u \in W^{2,2}_{\text{loc}}(M) \cap C^{1,\alpha}(M) \cap C^\infty(M) \) is a nonnegative superharmonic function on \( M \), then \( u \) is a constant.

2. If \( p \in (1, \infty) \) and \( u \in W^{2,2}_{\text{loc}}(M) \cap C^{1,\alpha}(M) \cap C^\infty(M) \) is a nonnegative subharmonic function on \( M \), then \( u \) is a constant.

Here, \( V_p(r), p \in \mathbb{R} \) is defined in (2) below, and some important concepts such as Finsler metric, Finsler Laplacian and harmonic (resp. subharmonic, superharmonic) functions will be given in Section 2, respectively.

**Remark 1.** If the Finsler manifold is compact, then, by the divergence theorem, we can prove all harmonic (resp. subharmonic, superharmonic) functions are constant. Theorem 1 can be regarded as a generalization of Theorem 1 in [2] when \( p = 1 \). If \((M, F, d\mu)\) is a Riemannian metric measure space, then Theorem 2 is exactly Theorem 1 in [5] or Theorem 13.1 in [10].

**Remark 2.** In comparison with [7], the condition on the reversibility is deleted in theorems above. There are many Finsler manifolds with infinity reversibility. Consider the Randers metric in \( \mathbb{B}^3(1) \)

\[
F(x, y) = \sqrt{(y^1)^2 + (y^2)^2 + (y^3)^2 + |x|y^1}, \quad |x| < 1,
\]

where \( |\cdot| \) denotes the standard Euclid norm. Then the reversibility

\[
\lambda_F(x) = \frac{1 + |x|}{1 - |x|} \to +\infty, \quad \text{if } x \to \partial \mathbb{B}^3(1).
\]

The geometric quantities between \( F \) and its reverse metric \( \overleftarrow{F} \) have many important relationships. For example,

1. \( \overleftarrow{F}(-df) = \overleftarrow{F}(df) \), \( \overleftarrow{\nabla} f = -\nabla (-f) \), \( \overleftarrow{\Delta} f = -\Delta (-f) \).

2. A forward (backward) distance function w.r.t. \( F \) is a backward (forward) distance function w.r.t. \( \overleftarrow{F} \), and vice versa.

3. A forward (backward) geodesic ball w.r.t. \( F \) is a backward (forward) geodesic ball w.r.t. \( \overleftarrow{F} \), and vice versa.

4. If \( f \) is a superharmonic (subharmonic) function w.r.t. \( \Delta \), then \( -f \) is a subharmonic (superharmonic) function w.r.t. \( \overleftarrow{\Delta} \), and vice versa.

To give a more elaborate improvement, we use these relations and thus avoid employing the reversibility. The remainder of the approaches adopted are similar to Zhang-Xia’s paper [7]. See also in [2,5,10] for the Riemannian case.

The contents of the paper are arranged as follows. In Section 2, some fundamental concepts which are necessary for the present paper are given, and some lemmas are contained. In Section 3, we prove the main theorems and give some corollaries.

**2. Preliminaries**

Let \( M \) be an \( n \)-dimensional smooth manifold and \( \pi : TM \to M \) be the natural projection from the tangent bundle \( TM \). Let \((x, y)\) be a point of \( TM \) with \( x \in M, y \in T_x M \), and let \((x', y')\) be the local coordinates on \( TM \) with \( y = y\partial/\partial x' \). A Finsler metric on \( M \) is a function \( F : TM \to [0, +\infty) \) satisfying the following properties:

(i) Regularity: \( F(x, y) \) is smooth in \( TM \setminus 0 \);

(ii) Positive homogeneity: \( F(x, \lambda y) = \lambda F(x, y) \) for \( \lambda > 0 \);

(iii) Strong convexity: The fundamental quadratic form
\[ g := g_{ij}(x, y)dx^i \otimes dx^j, \quad g_{ij} := \frac{1}{2}[F^2]_{ij}'y' \]
is positive definite.

Let \( X = X^i \frac{\partial}{\partial x^i} \) be a vector field. Then, the covariant derivative of \( X \) by \( v \in T_yM \) with reference vector \( w \in T_xM \setminus 0 \) is defined by
\[ D^v_x X(x) := \left\{ v^j \frac{\partial X^i}{\partial x^j}(x) + \Gamma^i_{jk}(w)v^jX^k(x) \right\} \frac{\partial}{\partial x^i}, \]
where \( \Gamma^i_{jk} \) denote the coefficients of the Chern connection.

For a smooth function \( u \), the gradient vector of \( u \) is
\[ \nabla u := \ell^{-1}(du), \]
where \( \ell : TM \to T^*M \) is Legendre transformation defined as
\[ \ell(y) := \begin{cases} g_y(y, \cdot), & y \in TM \setminus 0, \\ 0, & y = 0. \end{cases} \]

Let \( V = V^i \frac{\partial}{\partial x^i} \) be a smooth vector field on \( M \). The divergence of \( V \) with respect to an arbitrary smooth volume form \( d\mu \) is defined by
\[ \text{div} V := \sum_{i=1}^n \left( \frac{\partial V^i}{\partial x^i} + V^i \frac{\partial \Phi}{\partial x^i} \right), \]
where \( d\mu = e^\Phi dx \). Then, the Finsler Laplacian of \( u \) can be defined by
\[ \Delta u := \text{div}(\nabla u). \]

Since \( \Delta u \) is undefined at \( x \) where \( du(x) = 0 \), the definition can be viewed in distributional sense. That is, for \( u \in W^{1,2}(M) \),
\[ \int_M \varphi \Delta u d\mu = - \int_M d\varphi(\nabla u)d\mu, \quad \forall \varphi \in C_0^\infty(M). \tag{1} \]

We note here that since the gradient operator \( \nabla \) is not linear operator in general, the Finsler Laplacian is quite a bit different from the Riemannian Laplacian. Given a vector field \( V \) such that \( V \neq 0 \) on \( M_u \), where \( M_u := \{ x \in M | du(x) \neq 0 \} \), the weighted gradient vector and the weighted Laplacian on the weighted Riemannian manifold \((M, g_V)\) are defined by
\[ \nabla^V u := \begin{cases} g^{ij}(V) \frac{\partial u}{\partial x^i} \frac{\partial}{\partial x^j}, & \text{on } M_u, \\ 0, & \text{on } M \setminus M_u, \end{cases} \quad \Delta^V u = \text{div}(\nabla^V u). \]

It follows that \( \nabla^V u = \nabla u, \quad \Delta^V u = \Delta u. \)

Let \( u \) be a positive harmonic function on \( M, \Delta u = 0. \) It was proved that \( u \in W^{2,2}_{\text{loc}}(M) \cap C^{1,\alpha}(M) \cap C^\infty(M_u) \) (see [11]). We say that \( u \in W^{2,2}_{\text{loc}}(M) \cap C^{1,\alpha}(M) \cap C^\infty(M_u) \) is a subharmonic (resp. superharmonic) function on \( M \) if \( \Delta u \geq 0 \) (resp. \( \Delta u \leq 0 \)). In a weak sense, \( u \) is a subharmonic (resp. superharmonic) function in \( M \) if, for any positive function \( \varphi \in C_0^\infty(M) \), it holds
\[ \int_M \varphi \Delta u d\mu \geq (\text{resp.} \leq)0. \]
Let \((M, F)\) be a Finsler \(n\)-manifold. Fix a point \(x_0 \in M\). We denote a forward (resp. backward) geodesic ball of radius \(r\) with center at \(x_0\) by \(B^+_{x_0}(r)\) (resp. \(B^-_{x_0}(r)\)).

**Lemma 1.** Let \((M, F)\) be a Finsler \(n\)-manifold and \(x_0 \in M\). Then, there exists a function defined by

\[
\varphi(x) = \begin{cases} 
1, & x \in B^+_{x_0}(R); \\
0, & x \in M \setminus B^+_{x_0}(2R)
\end{cases}
\]

such that

\[
F^*(-d\varphi) \leq \frac{C}{R},
\]

where \(C\) is a positive constant.

**Proof.** Let \(\omega(t)\) be a smooth function on the real line with \(0 \leq \omega(t) \leq 1\) and \(\omega'(t) \leq 0\) such that

\[
\omega(t) = \begin{cases} 
1, & 0 \leq t \leq 1; \\
0, & t \geq 2.
\end{cases}
\]

Clearly, \(|\omega'(t)| \leq C\), where \(C\) is some positive constant. Define

\[
\varphi(x) = \omega\left(\frac{r(x)}{R}\right),
\]

where \(r(x) = d_F(x_0, x)\) is the distance function from \(x_0\). Then,

\[
F^*(-d\varphi) = \left|\frac{\omega'}{R}\right| F^*(dr) = \frac{|\omega'|}{R} F^*(dr) \leq \frac{C}{R} \text{ a.e. on } M.
\]

\[\square\]

Notice that \(\varphi \in C^\infty_0(B^+_{x_0}(2R) \setminus \{x_0\} \cup \text{cut}(x_0))\) and it is differentiable almost everywhere on \(B^+_{x_0}(2R)\) with bounded differential. Since a subharmonic (resp. superharmonic) function \(u\) belongs to \(W^{2,2}_{loc}(M) \cap C^{1,\alpha}(M) \cap C^\infty(M_u)\), and \(\Delta u = 0\) a.e. on \(M \setminus M_u\) (Lemma 3.5 in [12]), we find the Formula (1) still holds for this \(\varphi\).

3. Proof of the Main Theorems

For any nonnegative function \(u\), set

\[
V_p(r) = \begin{cases} 
\int_{B^+_{x_0}(r)} u^p d\mu, & p \geq 1; \\
\int_{B^-_{x_0}(r)} u^p d\mu, & p < 1.
\end{cases}
\]

(2)

Note that \(V_p(r) = \text{vol}^d\mu(B^-_{x_0}(r)) := V(r)\) if \(p = 0\).

**Proof of Theorem 1.** Set \(v = \sqrt{u}\). Then, in \(M_u = M_v\), one obtains

\[
0 \leq \frac{1}{2} \Delta \log u = \Delta \log v = \text{div} \left(\frac{\nabla v}{v}\right) = \frac{\Delta v}{v} - \frac{F(\nabla v)^2}{v^2},
\]

which gives

\[
v \Delta v \geq F(\nabla v)^2 \geq 0.
\]
Let \( \varphi \) be the function defined in Lemma 1. Then, it is differentiable almost everywhere on \( (B_{x_0}^+(2R)) \) with a bounded differential. Note that \( \Delta v = 0 \) a.e. on \( M \setminus M_\partial \) from Lemma 3.5 in [12]. Thus, by the divergence theorem, we have

\[
\int_{B_{x_0}^+(2R) \setminus M_\partial} \varphi^2 v \Delta v d\mu = \int_{B_{x_0}^+(2R)} \varphi^2 v \Delta v d\mu = -\int_{B_{x_0}^+(2R)} d(\varphi^2 v)(\nabla v) d\mu
\]

\[
= -\int_{B_{x_0}^+(2R)} \varphi^2 F(\nabla v)^2 d\mu - 2 \int_{B_{x_0}^+(2R)} \varphi v \varphi(\nabla v) d\mu
\]

\[
\leq -\int_{B_{x_0}^+(2R)} \varphi^2 F(\nabla v)^2 d\mu + 2 \int_{B_{x_0}^+(2R)} \varphi \varphi^* (\nabla v) d\mu.
\]

Therefore,

\[
\int_{B_{x_0}^+(2R)} \varphi^2 F(\nabla v)^2 d\mu \leq 2 \int_{B_{x_0}^+(2R)} \varphi v \varphi^*(\nabla v) d\mu
\]

\[
\leq 2 \left( \int_{B_{x_0}^+(2R)} \varphi^2 F(\nabla v)^2 d\mu \right)^{1/2} \left( \int_{B_{x_0}^+(2R)} v^2 F^*(\nabla \varphi)^2 d\mu \right)^{1/2},
\]

which implies that

\[
\int_{B_{x_0}^+(2R)} \varphi^2 F(\nabla v)^2 d\mu \leq 4 \int_{B_{x_0}^+(2R)} v^2 F^*(\nabla \varphi)^2 d\mu.
\]

By using Lemma 1 and the definition of \( \varphi \), we deduce

\[
\int_{B_{x_0}^+(R)} \varphi^2 F(\nabla v)^2 d\mu \leq \frac{4C^2}{R^2} \int_{B_{x_0}^+(2R)} v^2 d\mu = \frac{16C^2}{(2R)^2} \int_{B_{x_0}^+(2R)} u d\mu = 16C^2 \frac{V_v(2R)}{(2R)^2}.
\]

Letting \( R \to \infty \), it follows from \( \limsup_{R \to \infty} \frac{1}{V_v(R)} = \infty \) that \( F(\nabla v) = 0 \) everywhere. Since \( M \) is connected, \( v \) is a constant on \( M \) and so is \( u \). \( \Box \)

**Proof of Theorem 2.** Without loss of generality, we might as well assume \( u > 0 \). Otherwise, we can replace it by \( \tilde{u} = u + \epsilon > 0 \) for some positive number \( \epsilon \). We first prove (2) in Theorem 2. Let \( x_0 \) be a fixed point in \( M \) and \( r_0 \) be a number with \( 0 < r_0 < R \). Define

\[
\psi(x) = \begin{cases} 
1, & x \in B_{x_0}^+(r_0); \\
0, & x \in M \setminus B_{x_0}^+(R),
\end{cases}
\]

with \( \psi(x) \in C^\infty_0(B_{x_0}^+(R) \setminus \{x_0\} \cup \text{cut}(x_0)) \) satisfying

\[
F^* (-d\psi) \leq \frac{C}{R} \leq \frac{C}{r_0} := \tilde{C}.
\] (3)
Using the divergence theorem and similar arguments above, we have

\[
(p - 1) \int_{B_{r_0}^+(R) \cap M_0} u^{p-1} \Delta u \, d\mu \\
= (p - 1) \int_{B_{r_0}^+(R)} u^{p-1} \Delta u \, d\mu \\
= - (p - 1) \int_{B_{r_0}^+(R)} d(u^{p-1} \psi^2)(\nabla u) \, d\mu \\
= - (p - 1)^2 \int_{B_{r_0}^+(R)} u^{p-2} \psi^2 F(\nabla u)^2 \, d\mu - 2(p - 1) \int_{B_{r_0}^+(R)} u^{p-1} \psi d\psi(\nabla u) \, d\mu.
\]  

(4)

Set \( v = u^\frac{p}{2} \). Then, (4) becomes

\[
(p - 1) \int_{B_{r_0}^+(R)} u^{p-1} \psi^2 \Delta u \, d\mu \\
= - 4(1 - \frac{1}{p})^2 \int_{B_{r_0}^+(R) \setminus B_{r_0}^+(r_0)} \psi^2 F(\nabla v)^2 \, d\mu - 4(1 - \frac{1}{p}) \int_{B_{r_0}^+(R)} v \psi d\psi(\nabla v) \, d\mu.
\]

From the conditions in Theorem 2 and (3), it follows that

\[
(1 - \frac{1}{p})^2 \int_{B_{r_0}^+(R)} \psi^2 F(\nabla v)^2 \, d\mu \\
\leq - (1 - \frac{1}{p}) \int_{B_{r_0}^+(R) \setminus B_{r_0}^+(r_0)} v \psi d\psi(\nabla v) \, d\mu \\
\leq \left( \int_{B_{r_0}^+(R) \setminus B_{r_0}^+(r_0)} \sigma^2 F^2 (-d\psi)^2 \, d\mu \right)^{\frac{1}{2}} \left( 1 - \frac{1}{p} \right)^2 \int_{B_{r_0}^+(R) \setminus B_{r_0}^+(r_0)} \psi^2 F(\nabla v)^2 \, d\mu \right)^{\frac{1}{2}} \\
\leq \tilde{C} \left( \int_{B_{r_0}^+(R) \setminus B_{r_0}^+(r_0)} u^p \, d\mu \right)^{\frac{1}{2}} \left( 1 - \frac{1}{p} \right)^2 \int_{B_{r_0}^+(R) \setminus B_{r_0}^+(r_0)} \psi^2 F(\nabla v)^2 \, d\mu \right)^{\frac{1}{2}} \\
= \tilde{C} (V_p(R) - V_p(r_0))^\frac{1}{2} \left( 1 - \frac{1}{p} \right)^2 \int_{B_{r_0}^+(R) \setminus B_{r_0}^+(r_0)} \psi^2 F(\nabla v)^2 \, d\mu \right)^{\frac{1}{2}}.
\]

Let

\[
G(r) = (1 - \frac{1}{p})^2 \int_{B_{r_0}^+(r)} F(\nabla v)^2 \, d\mu.
\]

Then, by similar arguments as in [7], we can also reach

\[
\frac{1}{G(r_0)} - \frac{1}{G(R)} \geq \frac{1}{C^2} \frac{(R - r_0)^2}{V_p(R) - V_p(r_0)}.
\]

For fixed \( r_0 \), taking \( R_k = 2^k r_0, k \in \mathbb{N}^+ \), we have

\[
\frac{1}{G(r_0)} \geq \frac{1}{G(R_n)} + \frac{1}{C^2} \sum_{k=1}^{n} \frac{(R_k - R_{k-1})^2}{V_p(R_k) - V_p(R_{k-1})} \geq \frac{1}{4C^2} \sum_{k=1}^{n} \frac{(R_k)^2}{V_p(R_k)} \\
= \frac{1}{8C^2} \sum_{k=1}^{n} \frac{2R_k^2}{V_p(R_k)} = \frac{1}{8C^2} \sum_{k=1}^{n} \frac{2^{k+1}r_0}{V_p(2^k r_0)} \times 2^k r_0 \\
\geq \frac{1}{8C^2} \sum_{k=1}^{n} \int_{2^k r_0}^{2^{k+1} r_0} \frac{r}{V_p(r)} \, dr.
\]
Letting $n \to \infty$, we have
\[
\frac{1}{G(r_0)} \geq \frac{1}{8C^2} \int_{2r_0}^{\infty} \frac{r}{V_p(r)} \, dr = \infty,
\]
which means that
\[
\int_{B_{\hat{r}_n}(r_0)} F(\nabla v)^2 \, d\mu = 0.
\]
Therefore, by arbitrariness of $r_0$, we conclude that $v$ must be constant on $M$ and so is $u$.
Now, we are to prove (1) according to the cases $0 < p < 1$, $p < 0$ and $p = 0$, respectively.

**Case I**: $0 < p < 1$.
Let $x_0$ and $r_0$ be as above. Define
\[
\psi(x) = \begin{cases} 
1, & x \in B_{\hat{r}_n}(r_0); \\
0, & x \in M \setminus B_{\hat{r}_n}(R),
\end{cases}
\]
with $\psi(x) \in C_0^\infty(B_{\hat{r}_n}(R) \setminus (\{x_0\} \cup \text{cut}(x_0)))$ satisfying
\[
F^*(d\psi) \leq \frac{C}{R} \leq \frac{C}{r_0} := \hat{C}.
\]

By similar arguments, we also obtain (5) for the backward geodesic ball. The remainder of the proof is the same as above.

**Case II**: $p < 0$.
Set $v = -u^{\frac{2}{p-1}}$. Then,
\[
F(\nabla v) = -\frac{p}{2} u^{\frac{2}{p-1}} F(\nabla u).
\]
Let $\psi$ be a function as in Case I. We can also obtain
\[
0 \leq (p - 1) \int_{B_{\hat{r}_n}(R)} u^{p-1} \psi^2 \Delta u \, d\mu
\]
\[
= - (p - 1)^2 \int_{B_{\hat{r}_n}(R)} u^{p-2} \psi^2 F(\nabla u)^2 \, d\mu - 2(p - 1) \int_{B_{\hat{r}_n}(R)} u^{p-1} \psi \Delta \psi \, d\mu.
\]
\[
= - 4(1 - \frac{1}{p})^2 \int_{B_{\hat{r}_n}(R)} \psi^2 F(\nabla v)^2 \, d\mu - 4(1 - \frac{1}{p}) \int_{B_{\hat{r}_n}(R)} v \psi d\psi(\nabla v) \, d\mu.
\]
Therefore, we have
\[
(1 - \frac{1}{p})^2 \int_{B_{\hat{r}_n}(R)} \psi^2 F(\nabla v)^2 \, d\mu
\]
\[
\leq - (1 - \frac{1}{p}) \int_{B_{\hat{r}_n}(R) \setminus B_{\hat{r}_n}(r_0)} \nabla \psi d\psi(\nabla v) \, d\mu
\]
\[
\leq \left( \int_{B_{\hat{r}_n}(R) \setminus B_{\hat{r}_n}(r_0)} \nabla F^*(d\psi)^2 \, d\mu \right)^{\frac{1}{2}} \left( (1 - \frac{1}{p})^2 \int_{B_{\hat{r}_n}(R) \setminus B_{\hat{r}_n}(r_0)} \psi^2 F(\nabla v)^2 \, d\mu \right)^{\frac{1}{2}}
\]
\[
= \hat{C}(V_p(R) - V_p(r_0))^{\frac{1}{2}} \left( (1 - \frac{1}{p})^2 \int_{B_{\hat{r}_n}(R) \setminus B_{\hat{r}_n}(r_0)} \psi^2 F(\nabla v)^2 \, d\mu \right)^{\frac{1}{2}}.
\]
Then, by the same argument as above, one obtains that $u$ is constant.
Case III: \( p = 0 \).

For every \( k \in \mathbb{R}^+ \), set

\[
u_k = \begin{cases} 
  k, & u \geq k; \\
  u, & u < k.
\end{cases}
\]

Then, \( \nu_k \) is a nonnegative superharmonic function in a weak sense. We will prove this by following the arguments in ([10], p. 178). Let \( \beta \) be a symmetric, concave and bounded smooth function with \( |\beta'| \leq 1 \) and \( \beta'' \geq 0 \) (think of a smooth approximation of \( x \mapsto |x| \)). Define

\[
\hat{\nu}_k = \frac{u + k}{2} - \frac{\beta(u-k)}{2}
\]

(think of a smooth approximation of \( \nu_k \)). Then, \( d\hat{\nu}_k = \frac{1}{2}(1 - \beta')du \). Notice that \( 1 - \beta' \geq 0 \). By Legendre transformation, we have \( \nabla \hat{\nu}_k = \frac{1}{2}(1 - \beta')\nabla u \). Since \( \Delta \hat{\nu}_k = 0 \) a.e. on \( M \setminus M_{\hat{\nu}_k} \), for \( \psi \) defined in Case I, we have

\[
2 \int_{B_{\alpha}(R)} \psi \Delta \hat{\nu}_k d\mu = -2 \int_{B_{\alpha}(R)} d\psi(\nabla \hat{\nu}_k) d\mu = - \int_{B_{\alpha}(R)} (1 - \beta') d\psi(\nabla u) d\mu
\]

\[
= - \int_{B_{\alpha}(R)} d\{(1 - \beta')\psi\}(\nabla u) d\mu - \int_{B_{\alpha}(R)} \beta'' \phi(\nabla u)^2 d\mu
\]

\[
\leq - \int_{B_{\alpha}(R)} d\{(1 - \beta')\psi\}(\nabla u) d\mu = \int_{B_{\alpha}(R)} (1 - \beta') \psi \Delta ud\mu
\]

\[
\leq 0.
\]

The last step holds because \( (1 - \beta') \psi \) is differentiable almost everywhere on \( B_{\alpha}(R) \) with bounded differential and \( u \) is superharmonic. Thus, the claim follows by approximation.

It is shown that \( \nabla \nu_k = 0 \) a.e. on \( \{u \geq k\} \), \( \nabla \nu_k = \nabla u \) on \( \{u < k\} \) and \( \mu(\{x \in M | u(x) = k, du(x) \neq 0\}) = 0 \) w.r.t. the measure \( d\mu \) (see [7]). Thus, \( \Delta \nu_k = 0 \) a.e. on \( \{u \geq k\} \) and \( \Delta \nu_k = \Delta u \) on \( \{u < k\} \). Notice that \( \psi \) is differentiable almost everywhere on \( B_{\alpha}(R) \) with a bounded differential. Therefore, by similar arguments, we can also obtain (4) for \( \nu_k \) on \( B_{\alpha}(R) \). Set \( v_k = u_k^q \) for any \( q \in (0,1) \).

Then, we have (5) as follows:

\[
(1 - \frac{1}{q})^2 \int_{B_{\alpha}(R)} \psi^2 F(\nabla \nu_k)^2 d\mu
\]

\[
\leq \tilde{C} \left( \int_{B_{\alpha}(R) \setminus B_{\alpha}(r_0)} \frac{\nu_k^q}{\tilde{C}} \right)^{\frac{1}{2}} \left( (1 - \frac{1}{q})^2 \int_{B_{\alpha}(R) \setminus B_{\alpha}(r_0)} \psi^2 F(\nabla \nu_k)^2 d\mu \right)^{\frac{1}{2}}
\]

\[
= \tilde{C} (V_q(R) - V_q(r_0))^\frac{1}{2} \left( (1 - \frac{1}{q})^2 \int_{B_{\alpha}(R) \setminus B_{\alpha}(r_0)} \psi^2 F(\nabla \nu_k)^2 d\mu \right)^{\frac{1}{2}}.
\]

On the other hand, note that

\[
\int_{B_{\alpha}(R)} \chi_k^q d\mu \leq \int_{B_{\alpha}(R)} k^q d\mu = k^q V(R),
\]

which implies that

\[
\int_1^\infty \frac{r}{V_q(r)} dr = \infty.
\]

Then, by the same discussion in the proof of (2) and Case I of (1), we show that this \( \nu_k \) is constant. Since \( k \) is arbitrary, the function \( u \) is also constant.
Using Theorem 2, we can reach the following corollaries which extend Theorem 3 in [2] and Corollary 1 in [13], respectively.

**Corollary 1.** Let \((M, F, d\mu)\) and \(u\) be as in Theorem 2.

1. If \(p \in (-\infty, 1)\), then every nonnegative superharmonic function \(u \in L^p(M)\) is a constant. In particular, if \(\text{vol}^d\mu(M) < \infty\), then every nonnegative superharmonic function on \(M\) is a constant.
2. If \(p \in (1, \infty)\), then every nonnegative subharmonic function \(u \in L^p(M)\) is a constant.

**Corollary 2.** Let \((M, F, d\mu)\) and \(u\) be as in Theorem 2 and \(u\) be a nonnegative superharmonic function. If, for a sequence \(r_k \to +\infty\),

\[ V(r_k) \leq Cr_k^2, \]

then \(u\) is a constant, where \(C\) is a positive constant.

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