An operational framework for nonlocality

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Due to the importance of entanglement for quantum information purposes, a framework has been developed for its characterization and quantification as a resource based on the following operational principle: entanglement among \( N \) parties cannot be created by local operations and classical communication, even when \( N−1 \) parties collaborate. More recently, nonlocality has been identified as another resource, alternative to entanglement and necessary for device-independent quantum information protocols. We introduce an operational framework for nonlocality based on a similar principle: nonlocality among \( N \) parties cannot be created by local operations and allowed classical communication even when \( N−1 \) parties collaborate. We then show that the standard definition of multipartite nonlocality, due to Svetlichny, is inconsistent with this operational approach: according to it, genuine tripartite nonlocality could be created by two collaborating parties. We finally discuss alternative definitions for which consistency is recovered.

Introduction. The fundamental importance of entanglement in Quantum Information Science has driven a strong theoretical effort devoted to its characterization, detection and quantification. The resulting entanglement theory [1] has produced new mathematical tools, such as entanglement witnesses or entanglement measures, which find application also beyond the quantum information scenario for which they were initially derived, e.g. in Condensed Matter Physics [2], Quantum Thermodynamics [3] or Biology [4].

The first step when deriving this theoretical formalism consists in identifying the relevant objects and set of operations, see Table I. The relevant objects in the entanglement scenario are quantum states in systems composed by \( N \) observers, labeled by \( A_i \) with \( i = 1, \ldots, N \). The relevant set of operations is the set of local operations and classical communication (LOCC). The whole formalism then relies on the following principle, which has a clear operational motivation: entanglement of a quantum state is a resource that cannot be created by LOCC. This implies that those states that can be created by LOCC are not entangled. These states are called separable and can be written as [1]

\[
\rho_{A_1 \ldots A_N} = \sum_j p_j \rho^{A_1}_{j} \otimes \cdots \otimes \rho^{A_N}_{j}. \tag{1}
\]

In turn, those states that cannot be created by LOCC are entangled and require a nonlocal quantum resource for the preparation. It is easy to see that LOCC protocols map separable states into separable states. Finally, entanglement witnesses are Hermitian operators \( W \) such that \( \text{tr}(W \rho_S) \geq 0 \) for all separable states \( \rho_S \) but (ii) there exist an entangled state \( \rho \) such that \( \text{tr}(W \rho) < 0 \).

The picture becomes richer when considering intermediate cases where only some of the \( N \) parties share entangled states. For simplicity we restrict our considerations in what follows to three parties. Consider an entangled state in which only two parties, say \( A_2 \) and \( A_3 \) are entangled. The corresponding state is called biseparable and can be written as [1]

\[
\rho_{A_1 A_2 A_3} = \sum_j p_j \rho^{A_1 A_2}_{j} \otimes \rho^{A_3}_{j}. \tag{2}
\]

This state is not genuine 3-partite entangled, as for its LOCC creation, it suffices that two of the parties act together. Similarly as above, (i) LOCC protocols where \( A_2 \) and \( A_3 \) act together map biseparable states into biseparable states along the bipartition \( A_1 \)−\( A_2 A_3 \) and (ii) these states do not violate any entanglement witness along this partition.

Recently, a novel paradigm in Quantum Information Theory has been introduced: device-independent quantum information processing [5, 6]. Here, the main goal is to achieve an information task without making any assumptions about the internal working of the devices used in the protocol. The device-independent property of these applications make them appealing, both from a theoretical and implementation viewpoint. In this scenario, the main objects are correlated systems distributed among \( N \) observers. Each observer \( i \) can introduce a classical input \( x_i \) in his system, which produces a classical output \( a_i \). The system is just seen as a black box and no assumption is made about the internal process producing the output given the input, except that it cannot contradict quantum theory. The corre-

| Resource   | Objects                      | Operations |
|------------|------------------------------|------------|
| Entanglement | Quantum states              | LOCC       |
| Nonlocality | Joint Probability Distributions | WCCPI      |

TABLE I. Comparison of entanglement and non-locality from an operational point of view. Once the basic ingredients of the theory have been identified, an operational framework is based on the following principle: the resource contained in the states cannot increase under the set of operations.
lations among the input/output processes in each system are described by the joint conditional probability distribution \( P(a_1, \ldots, a_N | x_1, \ldots, x_N) \) of getting results \( a_1, \ldots, a_N \) when using the inputs \( x_1, \ldots, x_N \). The main reason why device-independent applications are possible in the quantum regime is because of the existence of nonlocal quantum correlations. Intuitively, since these correlations do not have a classical analogue, they allow for novel protocols with no classical counterpart.

The advent of device-independent applications leads to the identification of nonlocality as a novel quantum information resource, alternative to entanglement. Despite the only known way of getting nonlocal quantum correlations among different observers is by measuring entangled states, it is a well-established fact that entanglement and nonlocality represent inequivalent quantum properties [7, 8]. Now, similar to what happened for entanglement, it would be desirable to have an operational framework for the characterization and quantification of nonlocality as a resource. This is precisely the main motivation for this work.

The operational framework. In a similar way as it is done for entanglement, the first step consists in identifying the relevant objects and set of operations, see Table I. The relevant objects are the joint probability distributions \( P(a_1, \ldots, a_N | x_1, \ldots, x_N) \). The corresponding set of operations should include local processing of the classical inputs and outputs. Communication is allowed only if it takes place before the inputs are known, otherwise it can be used to create nonlocal correlations. Such communication can be used either to generate shared randomness or to announce the outcomes of a sequence of measurements prior to the realization of the nonlocal experiment. A general protocol in the nonlocality scenario would thus begin with a preparation phase, where one of the parties would measure its system and broadcast the measurement outcome. On the basis of that result, a second party would measure its system, etc. At the end of the preparation phase, the parties exchange some shared randomness and announce that they are ready for the nonlocal experiment. Communication between them is forbidden from this point on. The second step is the measurement phase, where each party is given an input, or question, and they compute the outcome or answer by using the correlations resulting from the preparation phase and by processing the obtained classical information at will. The last process is commonly referred to as ‘wirings’. Thus, in the nonlocality framework, the set of relevant operations is Wirings & Classical Communication Prior to the Inputs (WCCPI).

Once these two ingredients are identified, it is straightforward to obtain an operational definition of nonlocality: nonlocality of correlations \( P(a_1, \ldots, a_N | x_1, \ldots, x_N) \) is a resource that cannot be created by WCCPI.

Not surprisingly, this operational definition leads to the standard definition of nonlocality due to Bell [9] when considering \( N \) distant parties. Indeed, it is easy to see that the correlations that can be created by WCCPI have the form, see Eq. (1),

\[
P_L(a_1, \ldots, a_n | x_1, \ldots, x_N) = \sum_{\lambda} p(\lambda) P_1(a_1 | x_1, \lambda) \ldots P_n(a_n | x_n, \lambda), \tag{3}
\]

in which the local maps \( P_i(a_i | x_i, \lambda) \) produce the classical output \( a_i \) depending on the input \( x_i \) and a shared classical random variable \( \lambda \). All correlations that admit a decomposition (3) are local, while they are nonlocal otherwise. WCCPI protocols map local correlations into local correlations. Finally, if we collect all the probabilities \( P(a_1, \ldots, a_N | x_1, \ldots, x_N) \) into a vector \( \vec{P} \), any Bell inequality can be seen as a vector of real coefficients \( \vec{c} \) such that (i) \( \vec{c} \cdot \vec{P}_L \geq 0 \) for all local correlations \( \vec{P}_L \) but (ii) there exist correlations \( \vec{P} \) such that \( \vec{c} \cdot \vec{P} < 0 \).

As for entanglement, the next step is to characterize genuine multipartite nonlocality. This question has already been studied and the standard definition of genuine multipartite nonlocality is due to Svetlichny [10]. We restrict our considerations again to three parties and the partition \( A_1 - A_2 A_3 \), without loss of generality. According to Svetlichny, correlations that can be written as, see Eq. (2),

\[
P(a_1, a_2, a_3 | x_1, x_2, x_3) = \sum_{\lambda} p(\lambda) P_1(a_1 | x_1, \lambda) P_{23}(a_2, a_3 | x_2, x_3, \lambda), \tag{4}
\]

do not contain any genuine tripartite nonlocality, as there is a local decomposition when parties \( A_2 \) and \( A_3 \) are together. Correlations admitting a decomposition like (4) are named in what follows bilocal (BL). As it happened for entanglement and LOCC, it is expected that under WCCPI protocols along the partition \( A_1 - A_2 A_3 \), bilocal correlations are mapped into bilocal correlations. Consequently, no bipartite Bell inequality between \( A_1 \) and \( A_2 A_3 \) can be violated. Remarkably, we prove here that this intuition is incorrect. This implies that the standard definition of genuine multipartite nonlocality, given by [4], is inconsistent with the operational approach. In the following we show examples of the inconsistencies and also provide and discuss alternative definitions of genuine multipartite nonlocality that are consistent with our operational framework.

First, we show the inconsistencies of the definition of BL by providing correlations that (i) have a decomposition of the form (4) and (ii) become nonlocal along the partition \( A_1 - A_2 A_3 \) when a WCCPI protocol, where \( A_2 \) and \( A_3 \) collaborate, is implemented. An example of these correlations with a quantum realization can be established in the simplest scenario consisting of two measurements of two outcomes for each of the three observers. The measurements are performed on the quantum state

\[
\text{P}(a_1, a_2, a_3 | x_1, x_2, x_3) = \sum_{\lambda} p(\lambda) P_1(a_1 | x_1, \lambda) P_{23}(a_2, a_3 | x_2, x_3, \lambda). \tag{4}
\]
The value obtained is 2. This bipartite probability distribution does not have a local model. This can be verified by checking that the new bipartite distribution reads

\[ P(a_1, a_3|x_1, x_2) = \sum_{a_2} P(a_1, a_2, a_3|x_1, x_2, a_2). \]

This bipartite probability distribution \( P(a_1, a_3|x_1, x_2) \) does not have a local model. This can be verified by calculating the value of the Clauser-Horne-Shimony-Holt (CHSH) polynomial

\[ \beta = C(0,0) + C(0,1) + C(1,1) - C(1,0), \]

with \( C(x_1, x_2) = P(a_1 = a_2|x_1, x_2) - P(a_1 \neq a_2|x_1, x_2) \). The value obtained is \( \beta = \frac{\sqrt{2}}{2} \approx 2.12 \). Local correlations fulfill \( \beta \leq 2 \), thus we conclude that the correlations are nonlocal along the partition \( A_1 - A_2 A_3 \).

Alternatively, one can assess the inconsistency of Svetlichny’s definition by noting that our tripartite example behaves non-locally if one of the parties broadcasts its measurement outcomes before the nonlocal experiment takes place. Indeed, suppose that, prior to the experiment, \( A_2 \) measures \( x_2 = 1 \). If the result is \( a_2 = 1 \), \( A_1 \) and \( A_2 \) are projected onto the distribution \( P'(a_1, a_3|x_1, x_3) = P(a_1, a_3|x_1, x_3, x_2 = 1, a_2 = 1) \). On the contrary, if \( A_2 \) reads \( a_2 = -1 \), \( A_1 \) receives the order of inverting her measurement outcomes for measurement \( x_1 = 1 \), and so the system is projected again into \( P'(a_1, a_3|x_1, x_3) = P(-a_1, a_3|x_1, x_3, x_2 = 1, a_2 = -1) \). It can be checked that the new bipartite distribution \( P'(a_1, a_3|x_1, x_3) \) violates the CHSH inequality maximally (\( \beta = \frac{\sqrt{2}}{2} \approx 2.82 \)).

As mentioned above, the existence of these correlations implies that the standard definition of genuine multipartite nonlocality is inconsistent with our operational approach, as it would imply that genuine tripartite nonlocality could be created by WCCPI when two parties collaborate. Thus, the concept of genuine multipartite nonlocality is not correctly captured by Eq. (4). We can come back to this point below.

In our protocol the output of one of the parties is used as the input of the other party, or it is broadcast prior to the nonlocality experience. This implicitly assumes a temporal order in the measurements which is inconsistent with such decompositions. Indeed, all the examples of distributions of the form (4) leading to a Bell violation under WCCPI have to be such that the bilocal decomposition requires terms displaying signalling in both directions. Whether the converse is true, that is, whether every decomposition with such terms can be mapped via LOCC into a nonlocal one is an interesting open question. We come back to this point below.

It is now clear that tripartite correlations with bilocal models (4) such that all the terms \( P_{23}(a_2, a_3|x_2, x_3, \lambda) \) satisfy the no-signalling principle, i.e. marginal distributions on \( A_2 (A_3) \) do not depend on the input by \( A_3 (A_2) \)
protocols involving two collaborating parties, say $A$ and $B$, are closed under wirings in the sense that LOCC correlations to correlations which are local with respect to the partition $A$. The set of general bilocal correlations (BL) however, contains correlations that can be obtained by collaborating parties sharing no-signalling resources. They are operationally understood as correlations obtained by protocols in which the collaborating parties communicate, which is perfectly valid within our framework. Indeed, we show next that NSBL correlations do not define the largest set of correlations compatible with our framework, see Figure 2.

Consider instead the set of tripartite no-signalling correlations (see the Appendix for the corresponding $N$-party generalization) that can be decomposed as

$$P(a_1a_2a_3|x_1x_2x_3) = \sum_\lambda p_\lambda P(a_1|x_1, \lambda) P(a_2a_3|x_2x_3, \lambda)$$

with the distributions $P_{2\to3}$ and $P_{2\to3}$ obeying the conditions

$$P_{2\to3}(a_2|x_2, \lambda) = \sum_{a_3} P_{2\to3}(a_2a_3|x_2x_3, \lambda),$$

$$P_{2\to3}(a_3|x_3, \lambda) = \sum_{a_2} P_{2\to3}(a_2a_3|x_2x_3, \lambda).$$

We say that these correlations admit a time-ordered bilocal (TOBL) model. As can be seen from relations (7) and (8) we impose the distributions $P_{2\to3}$ and $P_{2\to3}$ to allow for signaling at most in one direction, indicated by the arrow. Decomposition (6) has also been considered in [13], and has a clear operational meaning: $P(a_1a_2a_3|x_1x_2x_3)$ can be simulated by a classical random variable $\lambda$ with probability distribution $p_\lambda$ distributed between parts $A_1$ and the composite system $A_2A_3$. This variable, $A_1$ generates the output according to the distribution $P(a_1|x_1, \lambda)$; on the other side, the two outputs $a_2, a_3$ are generated using either $P_{2\to3}(a_2a_3|x_2x_3, \lambda)$ or $P_{2\to3}(a_2a_3|x_2x_3, \lambda)$, depending on which of the inputs $x_2$ or $x_3$ is used first. Likewise, the bipartite distributions generated by a prior postselection, say, outcome $\tilde{a}_2$ of measurement $\tilde{X}_2$ of system $B$ can be simulated locally by $P(a_1|x_1, \lambda), P_{2\to3}(a_3|x_3, x_2 = \tilde{x}_2, a_2 = \tilde{a}_2, \lambda)$ after an appropriate modification of the weights $\{p_\lambda\}$ in eq. (6).

TOBL correlations are thus consistent with our operational point of view, as any WCCPI protocol along the partition $A_1 - A_2A_3$ maps TOBL correlations into probability distributions with a local model along this partition [11]. Interestingly, this set is strictly larger than the set of NSBL correlations (see Figure 2).

To prove this result, we consider the ‘Guess Your Neighbor’s Input’ (GYNI) polynomial [14]

$$\beta_{\text{GYNI}}(P(a_1, a_2, a_3|x_1, x_2, x_3)) = P(000|000) + P(110|011) + P(011|101) + P(101|110).$$

(9)

The maximum of this quantity over the set of probabilities having a NSBL decomposition is equal to 1, that is $\beta_{\text{GYNI}}(P \in \text{NSBL}) \leq 1$. In fact, consider the terms in the NSBL decomposition $P_1(a_1|x_1, \lambda)P_{23}^{\text{NS}}(a_2, a_3|x_2, x_3, \lambda)$. Without loss of optimality, one can restrict the analysis to correlations where $P_1(a_1|x_1, \lambda)$ is deterministic, say $P_1(0|0, \lambda) = P_1(0|1, \lambda)$. Thus, the GYNI polynomial for this set of probabilities satisfies

$$\beta_{\text{GYNI}}(P(a_1, a_2, a_3|x_1, x_2, x_3, \lambda)) = P_{23}^{\text{NS}}(0, 0|0, 0, \lambda) + P_{23}^{\text{NS}}(1, 1|0, 1, \lambda) \leq P_2(0|0, \lambda) + P_2(1|0, \lambda) \leq 1$$

(10)

with $P_2(a_2|x_2, \lambda) = \sum_{a_3} P_{23}^{\text{NS}}(a_2, a_3|x_2, x_3, \lambda)$ being a well-defined distribution due to the no-signalling constraints. One can easily check that the bound holds for any other deterministic choice of $P_1(0|0, \lambda)$ and $P_1(0|1, \lambda)$. As the NSBL decomposition is a convex mixture of these points, the GYNI polynomial is also bounded by 1. Note, however, that in Ref. [11] it is shown that there is a set of probabilities in TOBL obtaining larger values of the GYNI polynomial. Hence, the set of NSBL is strictly contained in TOBL.

Conclusions. We have introduced a novel framework for the characterization of nonlocality which has an operational motivation and captures the role of nonlocality as a resource for device-independent quantum information processing. In spite of its simplicity, the framework questions the current understanding of genuine multiparty nonlocality, as the standard definition adopted by
the community is inconsistent with it. Similar conclusions are reached from another perspective in [13]. We provide alternative frameworks where consistency is recovered. The main open question is now to identify the largest set of correlations that remain consistent under WCCPI protocols when some of the parties collaborate. We conjecture that TOBL correlations constitute such a set and, therefore, that for any bilocal model requiring two-way signalling terms there is a valid WCCPI protocol detecting its inconsistency.

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[12] Actually, the last construction suggests another venue to generate non-locality, namely, to make use of Stochastic Wirings & Classical Communication Prior to the Inputs (SWCCPI). However, it is easy to see that, whenever non-locality can be generated probabilistically with SWCCPI protocols, it can be also activated deterministically via WCCPI. Imagine, for instance, that $P_1(a_1, a_3|x_1, x_3) \equiv P(a_1, a_3|x_1, x_3, x_2 = 1, a_2 = +1)$ is non-local, but $P_2(a_1, a_3|x_1, x_3) \equiv P(a_1, a_3|x_1, x_3, x_2 = 1, a_2 = -1)$ is not. Let $\vec{c}$ be such that $\vec{c} \cdot \vec{P} \geq 0$ for all bipartite local distributions $\vec{P}$ and $\vec{c} \cdot P_1 < 0$. In the event $a_2 = -1$, $A_1$ and $A_3$ receive the order of simulating a local box $P_2'(a_1, a_3|x_1, x_3)$ such that $\vec{c} \cdot P_2' = 0$. Then, it is clear that the so-constructed box $Q(a_1, a_3|x_1, x_3) \equiv p(a_2 = -1|x_2)P_2'(a_1, a_3|x_1, x_3) + p(a_2 = 1|x_2)P_1(a_1, a_3|x_1, x_3)$ satisfies $\vec{c} \cdot Q < 0$, and, consequently, is non-local.
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Appendix: TOBL model for arbitrary number of parties. Suppose that \( M + N \) parties share a no-signalling set of correlations \( P(a_1, ..., a_{M+N}|x_1, ..., x_{M+N}) \). We are interested in which restrictions we should enforce over such a distribution in order to make sure that it cannot be used to violate a bipartite Bell inequality when parties 1, ..., \( M \) and \( M + 1, ..., M + N \) group together, even when several of such boxes are initially distributed.

One possibility is to demand the new bipartite object to behave as a generic classical bipartite device would. Viewed as bipartite, the distribution \( P(a_1, ..., a_{M+N}|x_1, ..., x_{M+N}) \) is such that it allows each of the two virtual parties (call them Alice and Bob) to perform sequential measurements on their subsystems. If we assume that the outcomes Alice and Bob observe are generated by a classical machine, it follows that \( P(a_1, ..., a_{M+N}|x_1, ..., x_{M+N}) \) can be written as:

\[
P(a_1, ..., a_{M+N}|x_1, ..., x_{M+N}) = \sum_{\lambda} p_{A}^\lambda \cdot p_{B}^\lambda, \tag{11}
\]

where we can regard each \( p_{A}^\lambda \) as a collection of probability distributions

\[
p_{\sigma(1) \rightarrow \cdots \rightarrow \sigma(M)}^\lambda (a_1, ..., a_M|x_1, ..., x_M), \tag{12}
\]

one for each possible permutation \( \sigma \) of the \( M \) physical parties. Here \( \sigma(1) \rightarrow \cdots \rightarrow \sigma(M) \) would indicate the process in which the first party to measure is \( \sigma(1) \), followed by \( \sigma(2) \), etc.

If, during a communication protocol, Alice must measure, say, \( x_3 \), she only has to choose an arbitrary permutation \( \sigma \), with \( \sigma(1) = 3 \) and then generate \( a_3 \) according to the probability distribution \( P_{\sigma(1) \rightarrow \cdots \rightarrow \sigma(M)}^\lambda (a_1, ..., a_{M}|x_1, ..., x_M) \). If, at some time later, she needs to simulate the measurement of \( x_1 \) and \( \sigma(2) \neq 1 \), she would thus have to find a new permutation \( \sigma' \), with \( \sigma'(1) = 3, \sigma'(2) = 1 \), and generate \( a_1 \) from the conditional probability distribution \( P_{\sigma'(1) \rightarrow \cdots \rightarrow \sigma'(M)}^\lambda (a_1, a_2, a_4, ..., a_{M}|x_1, ..., x_{M}, a_3) \). By consistency, for any pair of permutations \( \sigma^1, \sigma^2 \) such that \( \sigma^1(j) = \sigma^2(j) \), for all \( j \in \{1, ..., m\} \), such distributions need to satisfy the condition:

\[
\sum_{a>m} P_{\sigma^1(1) \rightarrow \cdots \rightarrow \sigma^1(M)}^\lambda (a_1, ..., a_{M}|x_1, ..., x_{M}) = \sum_{a>m} P_{\sigma^2(1) \rightarrow \cdots \rightarrow \sigma^2(M)}^\lambda (a_1, ..., a_{M}|x_1, ..., x_{M}), \tag{13}
\]

where \( \sum_{a>m} \) denotes the sum over all variables \( a_{\sigma(j)} \) with \( j > m \). The same considerations apply for \( P_{B}^\lambda \).

Local postselections on a prior sequence of Alice’s and Bob’s outcomes would imply changing the probabilities \( p_{A}^\lambda \), but otherwise can be simulated in a similar fashion. Putting everything together, we have that WCCPI operations over a set of (possibly different) boxes generate bipartite classical correlations if each box \( P(a_1, ..., a_{N+M}|x_1, ..., x_{M+N}) \), distributed along the partition \( 1..M|M+1..M+N \), admits a decomposition of the form

\[
P(a_1, ..., a_{M+N}|x_1, ..., x_{M+N}) = \sum_{\lambda} p_{A}^\lambda \cdot P_{\sigma^1(1) \rightarrow \cdots \rightarrow \sigma^1(M)}^\lambda (a_{\sigma(1)}, ..., a_{\sigma(M)}|x_{\sigma(1)} \cdots x_{\sigma(M)}) \\
\cdot P_{\sigma'(M+1) \rightarrow \cdots \rightarrow \sigma'(M+N)}^\lambda (a_{\sigma'(M+1)}, ..., a_{\sigma'(N+M)}|x_{\sigma'(N+1)} \cdots x_{\sigma'(N+M)}), \tag{14}
\]

The reader can check that in the tripartite case the above description reduces to the TOBL definition given in the main text.