VANISHING THEOREMS FOR THE COHOMOLOGY GROUPS OF FREE BOUNDARY HYPERSURFACES

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Abstract. In this paper we prove that there exists a universal constant $C$, depending on integers $n \geq 3$ and $1 \leq p \leq n-1$, such that if $M^n$ is a compact free boundary submanifold immersed in the unit ball $\mathbb{B}^{n+k}$ whose size of the traceless second fundamental form is less than $C$, then the $p$th cohomology group of $M$ vanishes. Employing a different technique we also obtain a rigidity result for compact free-boundary surfaces in the unit ball $\mathbb{B}^{2+k}$.

1. Introduction

In 1968, Simons proved in [19] that if $M^n$ is a compact manifold minimally immersed in the unit sphere $S^{n+k}$ whose the second fundamental form $A$ satisfies $\|A\|^2 \leq \frac{nk}{2}$, then either $\|A\|^2 = 0$ (i.e. $M$ is totally geodesic) or $\|A\|^2 = \frac{nk}{2}$. Later, Lawson [13] and Chern, do Carmo and Kobayashi [7] classified all minimal submanifolds in $S^{n+k}$ satisfying $\|A\|^2 = \frac{nk}{2}$. Such submanifolds are either the Veronese surface in $S^4$ or a family of product of two spheres with appropriated radii, now known as minimal Clifford tori. In particular, in codimension one, only Clifford tori occur. These important results say that there exists a gap in the space of minimal submanifolds in $S^{n+k}$ in terms of the size of the second fundamental form and the dimension. This kind of behavior has been observed in many other cases as we can see for instance in [1], [4], [12], [14], [15], [16], [18], [20]

We point out here the important contribution that was done by Lawson and Simons in [14] were they proved a topological gap result without assuming the minimality of the submanifold. The result is the following.

**Theorem 1.1** (Lawson-Simons). Let $M$ be a compact $n$-dimensional manifold, immersed in the unit sphere $S^{n+k}$ with second fundamental form $A$ satisfying $\|A\|^2 < \min\{p(n-p), 2\sqrt{p(n-p)}\}$, where $p \leq n-1$ is a positive integer. Then for any finitely generated abelian group $G$,

$$H_p(M; G) = H_{n-p}(M; G) = 0.$$

In particular, if $\|A\|^2 < \min\{n-1, 2\sqrt{n-1}\}$, then $M$ is a homotopy sphere.

This result actually follows from the remarkable fact that there is no stable currents (or stable varifolds) in the unit sphere, also proved in [14].

Taking a new perspective, Ambrozio and Nunes [3] obtained recently a geometric gap type theorem for free boundary minimal surfaces in the unit 3-ball. They proved that if $\|A\|^2(x, N(x)) \leq 2$, where $N(x)$ is the unit normal vector at $x \in M$, then $M$ is either an equatorial disk or a critical catenoid.

We recall that a submanifold $M^n$ with non empty boundary and minimally immersed in the unit ball $\mathbb{B}^{n+k}$ such way that $M \cap \partial \mathbb{B} = \partial M$ is called free boundary if $M$ intersects $\partial \mathbb{B}^{n+k} = S^{n+k-1}$ in a right angle along the boundary. Such submanifolds are critical points for the area functional for those variations that keep
Theorem 1.2. Let \( \Sigma \) be a free boundary compact surface immersed in \( \mathbb{B}^{2+k} \), for any positive integer \( k \). If \( \|\Phi\|^2 \leq 2 \), then \( \Sigma \) is topologically a disk.

If \( \Sigma \) is minimal, then we can improve the constant, and in virtue of a result due to A. Fraser and R. Schoen \([10]\) we have the following rigidity result.

Corollary 1.3. Let \( \Sigma \) be a free boundary compact surface minimally immersed in \( \mathbb{B}^{2+k} \), for any positive integer \( k \). If \( \|A\|^2 < 4 \), then \( \Sigma \) is an equatorial disk.

In higher dimensions, we prove that there is no non trivial harmonic \( p \)-forms on \( M \) with either Neumann or Dirichlet condition on the boundary. So, using the Hodge-de Rham’s theorem we actually have

Theorem 1.4. Let \( M^n \) be a compact oriented submanifold immersed in \( \mathbb{B}^{n+k} \), \( n \geq 3 \), which is free boundary and has flat normal bundle. If \( \|\Phi\|^2 \leq \frac{n(n-2)^2}{n(n-p)} \) for some \( 1 \leq p \leq \left\lfloor \frac{n}{2} \right\rfloor \), then the \( p \)th cohomology group of \( M^n \) with real coefficients vanishes, that is, \( H^p(M; \mathbb{R}) = H^{n-p}(M; \mathbb{R}) = 0 \). In particular, if \( \|\Phi\|^2 \leq \frac{(n-2)^2}{n(n-1)} \), then all cohomology groups of \( M \) vanish and \( M \) has only one boundary component.

To prove this theorem we employ a well-established Bochner technique with appropriated estimates of the curvature tensor in terms of the extrinsic geometry. To deal with the boundary term we use a Hardy type inequality for submanifolds, recently discovered by Batista, Mirandola and the third author \([5]\).

We also obtain similar results with improved constants in the case that \( M \) is minimal. More precisely:

Theorem 1.5. Let \( M^n \) be a compact oriented submanifold minimally immersed in \( \mathbb{B}^{n+1} \), \( n \geq 3 \), which is free boundary and has flat normal bundle. Given \( 1 \leq p \leq \left\lfloor \frac{n}{2} \right\rfloor \) we have the following assertions:

(a) If \( \|A(x)\|^2 \leq \frac{n(n-p+1)}{3(n-p)^2} \), then \( H^p_T(M) = \{0\} \). If additionally \( p = \left\lfloor \frac{n}{2} \right\rfloor \), then we also have \( H^p_N(M) = \{0\} \), whenever \( n \) is even or \( n = 3 \).

(b) If \( \|A(x)\|^2 < \frac{n(n-2)}{2(n-p)} \), then \( H^p_N(M) = \{0\} \), whenever \( p \leq \left\lfloor \frac{n}{2} \right\rfloor - 1 \) or \( p = \left\lfloor \frac{n}{2} \right\rfloor \) and \( n \geq 5 \) is odd.

In particular, if \( \|A\| \leq \frac{1}{2} \), then all cohomology groups of \( M \) vanish and \( M \) has only one boundary component.

Our theorems lead us to the following questions:

**Open question 1:** Do Theorems 1.4 and 1.5 hold without the condition on the flatness of the normal bundle?

And more generally:
**Open question 2:** Given $d > 0$, prove that there exists a positive constant $C = C(n, p) > 0$ such that if $|A|^2 < C$, then $\dim H_p(M) \leq d$, for all compact free boundary submanifold $M^n$ of $\mathbb{R}^{n+k}$.

**Acknowledgements:** The authors are grateful to Professor Levi Lima for fruitful conversations about this work. The first and the third authors were partially supported by CNPq-Brazil, CAPES-Brazil and FAPEAL-Brazil. The second author was partially supported by CAPES-Brazil and FAPEAL-Brazil.

2. Preliminaries

Let $M^n$ be a compact Riemannian manifold with non empty boundary. Let denote by $\Omega^p(M)$ the space of differential $p$-forms on $M$, $d : \Omega^p(M) \to \Omega^{p+1}(M)$ the exterior derivative, and $d^* : \Omega^p(M) \to \Omega^{p-1}(M)$ the codifferential, which can be viewed in terms of the Hodge star operator on $M$ as $d^* = (-1)^{n(p+1)+1} \ast d \ast$. We say that $\omega \in \Omega^p(M)$ is harmonic if $d\omega = 0$ and $d^* \omega = 0$ on $M$, that is, $\omega$ is closed and coclosed. A harmonic $p$-form $\omega$ on $M$ is called tangential if $i_\nu \omega = 0$ on $\partial M$ and normal if $\nu \wedge \omega = 0$ on $\partial M$.

We can consider the following subspaces of $\Omega^p(M)$:

\[ H^p_{N}(M) = \{ \omega \in \Omega^p(M) ; \omega \text{ is harmonic and tangential} \} \]

and

\[ H^p_{T}(M) = \{ \omega \in \Omega^p(M) ; \omega \text{ is harmonic and normal} \}. \]

It is well-known that the Hodge star operator and Hodge-de Rham theory applied for manifolds with boundary give the following isomorphisms (c.f. [2, Theorem 3])

\[ H^p_{N}(M) \cong H^{n-p}_{T}(M) \cong H^p(M; \mathbb{R}). \]

Moreover, $\dim H^1_{T}(M) \geq r - 1$, where $r$ is the number of boundary components of $M$ (c.f. [2, Lemma 4]).

Now, let us present some tools which are going to be of use. We start by recalling the integral version of the Weitzenböck formula on manifolds with umbilical boundary (see for instance [21] or [8]). Note that this is exactly the case when $M$ is a free boundary submanifold in the ball.

**Lemma 2.1 (Weitzenböck formula).** If $\partial M$ is totally umbilical in $M^n$ with second fundamental form $B = I$, then

\[ \int_M |\nabla \omega|^2 + \langle R_p(\omega), \omega \rangle = -\alpha \int_{\partial M} |\omega|^2, \]

where $\alpha = p$ or $\alpha = n - p$, depending whether $\omega \in H^p_{N}(M)$ or $\omega \in H^p_{T}(M)$, respectively.

Another useful result is the following refined Kato inequality for harmonics forms that can be found in [9] and [11].

**Lemma 2.2.** If $\omega$ is a harmonic $p$-form on $M^n$, with $1 \leq p \leq \left\lfloor \frac{n}{2} \right\rfloor$, then

\[ |\nabla \omega|^2 \geq \frac{n - p + 1}{n - p} |\nabla \omega|^{2}. \]
Next, we present two results in the context of submanifolds. Let $B^{n+k}$ be the closed unit ball in $\mathbb{R}^{n+k}$ centered at the origin. Consider a compact oriented immersed submanifold $M^n$ in $B^{n+k}$ with nonempty boundary $\partial M$ and denote by $X$ the unit vector normal to $\partial B^{n+k}$ which is outward pointing. Denote by $\nu$ the conormal vector field of $\partial M$, that is, the unit vector normal to $\partial M$ and tangent to $M$ which points to the outside of $M$. In this setting, the condition of $M^n$ to be free boundary in $B^{n+k}$ is equivalent to say that $X = \nu$ along $\partial M$.

Denote by $A$ the second fundamental form of $M$, by $\bar{H}$ the mean curvature vector of $M$ with respect to $A$ and by $\Phi$ the traceless part of $A$, i.e.,

$$\Phi(u, v) = A(u, v) - \langle u, v \rangle \bar{H}, \quad u, v \in T_x M, \ x \in M.$$

If $M^n$ is a compact oriented immersed submanifold in $B^{n+k}$ which is free boundary and has dimension $n \geq 3$, then it holds a Hardy type inequality on $M$. In fact, from a result due to M. Batista, H. Mirandola and the third author (see [5, Theorem 3.2]) we have

$$\frac{(n - \gamma)^p}{p^p} \int_M u^p \gamma^p \int_M \frac{u^p}{r^\gamma} |\nabla r|^2 \int_{\partial M} u^p,$$

for all nonnegative function $u \in C^1(M)$, $p \in [1, \infty)$, and $\gamma \in (-\infty, n)$, where $r$ is the distance function to the origin in $\mathbb{R}^{n+k}$, $\nabla r$ denotes the gradient of $r$ in $\mathbb{R}^{n+k}$, and $\nabla u$ is the gradient of $u$ in $M$.

Taking $p = \gamma = 2$ and observing that $1/r \geq 1$ on $M^n \subset B^{n+k}$ and $r = 1$ on $\partial M \subset S^n$, we have the following

**Lemma 2.3** (Batista-Mirandola-Vitório, [5]). If $M^n$ is a compact oriented immersed submanifold in $B^{n+k}$ which is free boundary and has dimension $n \geq 3$, then

$$\int_M u^2 \leq \left( \frac{2}{n - 2} \right)^2 \int_M |\nabla u|^2 + \left( \frac{n}{n - 2} \right)^2 \int_M u^2 |\bar{H}|^2 + \frac{2}{n - 2} \int_{\partial M} u^2$$

for all nonnegative function $u \in C^1(M)$.

Finally, we need an extrinsic estimate of the curvature term that appears in the Weitzenböck formula.

**Lemma 2.4** (Lin, see [17]). If $M^n$ is immersed in $\mathbb{R}^{n+k}$ with flat normal bundle, then

$$\langle \mathcal{R}_p(\omega), \omega \rangle \geq \left( p(n - p) |\bar{H}|^2 - \frac{p(n - p)}{n} |\Phi|^2 - |n - 2p| \sqrt{\frac{p(n - p)}{n}} |\bar{H}| |\Phi| \right) |\omega|^2.$$

3. **Proof of Theorem 1.3**

Fix $1 \leq p \leq \begin{pmatrix} n \\ 2 \end{pmatrix}$ and $\omega \in H^p_N(M)$ or $\omega \in H^p_r(M)$, and define $u = |\omega|$. It follows from the Weitzenböck formula and the refined Kato inequality that

$$- \alpha \int_{\partial M} u^2 = \int_M |\nabla u|^2 + \int_M \langle \mathcal{R}_p(\omega), \omega \rangle \geq \frac{n - p + 1}{n - p} \int_M |\nabla u|^2 + \int_M \langle \mathcal{R}_p(\omega), \omega \rangle,$$
where \( \alpha = p \) or \( \alpha = n - p \), depending whether \( \omega \in H^p_N(M) \) or \( \omega \in H^p_T(M) \), respectively. Using Lemma 2.4, we obtain
\[
-\alpha \int_{\partial M} u^2 \geq \frac{n - p + 1}{n - p} \int_M |\nabla u|^2 + p(n - p) \int_M u^2|\vec{H}|^2 - \frac{p(n - p)}{n} \int_M u^2|\Phi|^2
\]
(1)
\[-(n - 2p)\sqrt{\frac{p(n - p)}{n}} \int_M u^2|\vec{H}| \parallel \Phi \parallel.
\]
Fix \( \varepsilon > 0 \) and observe that
\[|\vec{H}| \parallel \Phi \parallel \leq \frac{\varepsilon}{2}|\vec{H}|^2 + \frac{1}{2\varepsilon}|\Phi|^2.\]
Then,
\[-\alpha \int_{\partial M} u^2 \geq \frac{n - p + 1}{n - p} \int_M |\nabla u|^2 + A \int_M u^2|\vec{H}|^2 - B\varphi^2 \int_M u^2,\]
where
\[A = A(n, p, \varepsilon) = p(n - p) - \frac{(n - 2p)}{2} \sqrt{\frac{p(n - p)}{\varepsilon}},\]
\[B = B(n, p, \varepsilon) = \frac{p(n - p)}{n} + \frac{(n - 2p)}{2\varepsilon} \sqrt{\frac{p(n - p)}{n}},\]
and \( \varphi = \sup_{M} \|\Phi\| \). Therefore, using Lemma 2.3, we have
\[0 \geq \left( \alpha - \frac{2B}{n - 2\varphi^2} \right) \int_{\partial M} u^2 + \left( \frac{n - p + 1}{n - p} - \frac{4B}{(n - 2)^2 \varphi^2} \right) \int_M |\nabla u|^2
\]
\[+ \left( A - \frac{n^2B}{(n - 2)^2 \varphi^2} \right) \int_M u^2|\vec{H}|^2.\]
It follows from the above inequality that if

- \( \varphi^2 < \frac{\alpha(n - 2)}{2B} = \phi_1 \),
- \( \varphi^2 \leq \frac{(n - p + 1)(n - 2)^2}{4(n - p)B} = \phi_2 \), and
- \( \varphi^2 \leq \frac{(n - 2)^2A}{n^2B} = \phi_3 \),

then \( u = |\omega| = 0 \) on \( \partial M \), which implies \( \omega = 0 \) since \( \omega \) is harmonic.

Now, to finish the proof of Theorem 1.4, we are going to prove that
\[\frac{p(n - 2)^2}{n(n - p)} = \phi_3(\varepsilon_1) = \max_{\varepsilon > 0} \phi_3 < \min\{\phi_1(\varepsilon_1), \phi_2(\varepsilon_1)\},\]
for
\[\varepsilon_1 = \sqrt{\frac{np}{n - p}}.\]

**Claim 3.1.** \[\frac{p(n - 2)^2}{n(n - p)} = \phi_3(\varepsilon_1) = \max_{\varepsilon > 0} \phi_3.\]

First, suppose that \( 2p < n \). It is not difficult to see that there exists a unique \( a = a(n, p) \) such that \( A(n, p, a) = 0 \). In fact,
\[a = \frac{2\sqrt{p\varepsilon(n - p)}}{n - 2p}.\]
Furthermore, \( A(n, p, \varepsilon) > 0 \) for \( \varepsilon \in (0, a) \) and \( A(n, p, \varepsilon) < 0 \) for \( \varepsilon > a \). Also, \( \lim_{\varepsilon \to 0^+} \phi_1 = 0 \). Then, to calculate
\[\max_{\varepsilon > 0} \phi_3 = \max_{\varepsilon \in (0, a)} \phi_3,\]
it is sufficient to find the critical points of $\varepsilon \mapsto \phi_3(n, p, \varepsilon)$ on the interval $(0, a)$. Define
\[ b = \frac{(n - 2p)}{2} \sqrt{\frac{p(n - p)}{n}} \]
and
\[ c = \frac{p(n - p)}{n}, \]
and observe that
\[ A = \frac{nc \varepsilon - b \varepsilon^2}{ce + b}. \]
A straightforward calculation gives that the unique critical point of $A/B$ on the interval $(0, +\infty)$ is given by
\[ \varepsilon_1 = \frac{-b + \sqrt{b^2 + nc^2}}{c}. \]
Observing that
\[ b^2 + nc^2 = \frac{p(n - p)(n - 2p)^2}{4n} + \frac{p^2(n - p)^2}{n} = \frac{p(n - p)}{4n}(n^2 - 4pn + 4p^2 + 4pn - 4p^2) = \frac{pm(n - p)}{4}, \]
we have
\[ \varepsilon_1 = \frac{-b + \sqrt{b^2 + nc^2}}{c} = \frac{n}{p(n - p)} \left( \frac{(n - 2p)}{2} \sqrt{\frac{p(n - p)}{n}} + \sqrt{\frac{pm(n - p)}{2}} \right) = \frac{n}{2\sqrt{p(n - p)}} \left( \frac{n - 2p}{\sqrt{n}} + \sqrt{n} \right) = \frac{\sqrt{np}}{n - p}. \]
Then, evaluating $A$ and $B$ at $\varepsilon = \varepsilon_1$, we obtain
\[ A(n, p, \varepsilon_1) = \frac{pm}{2} \]
and
\[ B(n, p, \varepsilon_1) = \frac{n - p}{2}. \]
Thus,
\[ \max_{\varepsilon > 0} \phi_3 = \phi_3(\varepsilon_1) = \frac{p(n - 2)^2}{n(n - p)}. \]
If $2p = n$, then $A = p(n - p) = p^2$ and $B = \frac{p(n - p)}{n} = \frac{p}{2}$. Therefore, $\phi_3$ is constant equal to
\[ \frac{(n - 2)^2 A}{n^2 B} = \frac{2p(n - 2)^2}{n^2} = \frac{p(n - 2)^2}{n(n - p)}. \]
Claim 3.2. $\phi_1(\varepsilon_1) > \phi_3(\varepsilon_1)$. 
Observe that \(2B(n, p, \varepsilon_1) = n - p\). Then,
\[
\phi_1(\varepsilon_1) > \phi_3(\varepsilon_1) = \frac{p(n - 2)^2}{n(n - p)}
\]
is equivalent to
\[
(2) \quad n\alpha > p(n - 2)
\]
If \(\alpha = p\), then \((2)\) is clearly true.
If \(\alpha = n - p\), then \((2)\) is equivalent to
\[
n^2 - 2pn + 2p > 0.
\]
Consider the polynomial \(p(x) = x^2 - 2px + 2p\). In order to \(p(n) > 0\), it is sufficient that one of the following alternatives occurs: the discriminant \(\Delta\) of \(p\) is negative or, when \(\Delta \geq 0\), \(n\) is greater than the largest root of \(p\). Observe that \(\Delta = 4p^2 - 8p\). Then, \(\Delta < 0\) when \(p = 1\). On the other hand, when \(2 \leq p \leq \left\lfloor \frac{n}{2} \right\rfloor\), we have \(\Delta \geq 0\) and the largest root of \(p\) satisfies
\[
x_1 = \frac{2p + \sqrt{\Delta}}{2} < \frac{4p}{2} = 2p \leq n.
\]
Claim 3.3. \(\phi_2(\varepsilon_1) > \phi_3(\varepsilon_1)\).

Using that \(2B(n, p, \varepsilon_1) = n - p\), we have
\[
\phi_2(\varepsilon_1) = \frac{(n - p + 1)(n - 2)^2}{2(n - p)^2} > \frac{(n - 2)^2}{2(n - p)} \geq \frac{p(n - 2)^2}{n(n - p)} = \phi_3(\varepsilon_1),
\]
where above we have used that
\[
\frac{n - p + 1}{n - p} > 1 \text{ and } n \geq 2p.
\]

4. Proof of Theorem 1.5

We start from inequality \((1)\) assuming \(\vec{H} = 0\), that is,
\[
-\alpha \int_{\partial M} u^2 \geq \frac{n - p + 1}{n - p} \int_M |\nabla u|^2 - \frac{p(n - p)}{n} \int_M u^2 |\Phi|^2.
\]
Using Lemma 2.3 in this case we get
\[
0 \geq \left(\alpha - \frac{2p(n - p)}{n(n - 2)} \phi^2\right) \int_{\partial M} u^2 + \left(\frac{n + 1 - p}{n - p} - \frac{4p(n - p)}{n(n - 2)}\right) \int_M |\nabla u|^2;
\]
and again, the proof follows if
\[
\bullet \quad \phi^2 < \frac{\alpha n(n - 2)}{2p(n - p)} = \phi_4, \text{ and }
\]
\[
\bullet \quad \phi^2 \leq \frac{n(n - p + 1)(n - 2)^2}{4p(n - p)^2} = \phi_5.
\]
Studying the solutions of the algebraic equation \(\phi_4 = \phi_5\) we obtain that:
\[
(1) \text{ If } \alpha = n - p, \text{ then } \phi_5 < \phi_4, \text{ and it corresponds to the first part of assertion (a).}
\]
\[
(2) \text{ If } \alpha = p = \left\lfloor \frac{n}{2} \right\rfloor \text{ and } n \text{ is even or } n = 3, \text{ then we also have } \phi_5 < \phi_4. \text{ It implies the second part of assertion (a).}
\]
\[
(3) \text{ In the remaining cases we have } \phi_4 < \phi_5 \text{ and they correspond to assertion (b).}
\]
5. Proofs of Theorem 1.2 and Corollary 1.3

Let $\Sigma$ be a free boundary surface in the Euclidean unit ball $B^{2+k}$. Assume that $\Sigma$ has genus $g$ and $r$ boundary components. So, the Euler characteristic of $\Sigma$, $\chi(\Sigma)$, is given by $\chi(\Sigma) = 2 - 2g - r$. Thus, the Gauss-Bonnet Theorem says that

$$\int_{\Sigma} K + \int_{\partial \Sigma} \kappa_g = 2\pi(2 - 2g - r),$$

where $K$ is the Gaussian curvature of $\Sigma$ and $\kappa_g$ is the geodesic curvature of the boundary. From the assumption that $\Sigma$ is free boundary in the unit ball, we have that

- $\kappa_g \equiv 1$;
- $|\partial \Sigma| = 2 \int_{\Sigma} \left(1 + \langle \vec{H}, x \rangle \right)$. 

On the other hand, the Gauss formula yields $K = |\vec{H}|^2 - \frac{1}{2} |\Phi|^2$. Using those properties above and the Gauss-Bonnet Theorem, a straightforward computation give us

$$\int_{\Sigma} \left(2 + 2\langle \vec{H}, x \rangle + |\vec{H}|^2 - \frac{1}{2} |\Phi|^2 \right) = 2\pi(2 - 2g - r),$$

which we can rewrite in the form

$$\int_{\Sigma} \left(1 - |x|^2 + |x + \vec{H}|^2 + 1 - \frac{1}{2} |\Phi|^2 \right) = 2\pi(2 - 2g - r).$$

Note that, if $|\Phi|^2 \leq 2$ then the left hand side of (4) is positive, then we have that $g = 0$ and $r = 1$. This concludes the proof of Theorem 1.2.

Finally, in the minimal case, the equation (3) simplifies to

$$\int_{\Sigma} \left(2 - \frac{1}{2} \|A\|^2 \right) = 2\pi(2 - 2g - r).$$

Therefore, if $\|A\|^2 \leq 4$, then $\Sigma$ is topologically a disk and the result follows from the Fraser-Schoen’s Theorem [10].

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