Symplectic topology of K3 surfaces via mirror symmetry

NICK SHERIDAN AND IVAN SMITH

ABSTRACT: We study the symplectic topology of certain K3 surfaces (including the “mirror quartic” and “mirror double plane”), equipped with certain Kähler forms. In particular, we prove that the symplectic Torelli group may be infinitely generated, and derive new constraints on Lagrangian tori. The key input, via homological mirror symmetry, is a result of Bayer and Bridgeland on the autoequivalence group of the derived category of an algebraic K3 surface of Picard rank one.

1 Introduction

Let \((X, \omega)\) be a simply-connected closed symplectic 4-manifold. Let \(G(X, \omega) = \pi_0 \text{Symp}(X, \omega)\) denote the symplectic mapping class group of \(X\); elucidating the structure of \(G(X, \omega)\), and of the natural map \(G(X, \omega) \to \pi_0 \text{Diff}(X)\), are central aims in four-dimensional topology. This paper brings to bear insight from homological mirror symmetry, in the specific case of (certain symplectic forms on) K3 surfaces.

1.1 Context and questions

Many aspects of \(\pi_0 \text{Diff}(X)\) seem a priori out of reach, since we have no knowledge of \(\pi_0 \text{Diff}_c(\mathbb{R}^4)\) – there is therefore no smooth closed four-manifold for which the smooth mapping class group is known – but in the symplectic setting we have Gromov’s theorem \(\text{Symp}_c(\mathbb{C}^2, \omega_{\text{st}}) \simeq \{1\}\), via his spectacular determination that \(\text{Symp}(\mathbb{C}^2, \omega_{\text{st}}) \simeq \mathbb{P}U(3)\) and \(\text{Symp}(\mathbb{P}^1 \times \mathbb{P}^1, \omega + \omega) \simeq (SO(3) \times SO(3)) \rtimes \mathbb{Z}/2\).

Away from the setting of rational or ruled surfaces, much less is known. Results of Seidel [Sei99] and Tonkonog [Ton15] show that \(G(X, \omega) \to \pi_0 \text{Diff}(X)\) has infinite kernel for “most” even-dimensional hypersurfaces in projective space, and Seidel [Sei08b] has shown that for certain symplectic K3 surfaces this kernel contains a subgroup isomorphic to the pure braid group \(PB_m\) for \(m \leq 15\).

However, very little is known about the structure of \(G(X, \omega)\) itself. For instance, the following questions\(^1\) remain open in almost all cases:

(Q1) (Seidel) Is \(G(X, \omega)\) finitely generated?

(Q2) (Fukaya) Does \(G(X, \omega)\) have finitely many orbits on the set of Lagrangian spheres in \(X\)?

(Q3) (Donaldson) Is the symplectic Torelli group, i.e. the kernel \(I(X, \omega)\) of \(G(X, \omega) \to \text{Aut}(H^*(X; \mathbb{Z}))\), generated by squared Dehn twists in Lagrangian spheres?

Contrast with the (symplectic) mapping class group \(\Gamma_g\) of a closed oriented surface \(\Sigma_g\) of genus \(g\), which is finitely presented, generated by Dehn twists, and acts with finitely many orbits on the infinite set of isotopy classes of simple closed curves on \(\Sigma_g\), or the classical Torelli group, which is infinitely generated when \(g = 2\) and finitely generated when \(g > 2\).

One also has long-standing questions concerning the structure of Lagrangian submanifolds in a complex projective manifold, for example:

(Q4) (Donaldson [Don00, Question 4]) Do all Lagrangian spheres in a complex projective manifold arise from the vanishing cycles of deformations to singular varieties?

\(^1\)These attributions are folkloric.
More specific to a symplectic Calabi–Yau surface \((X, \omega)\), we have:

(Q5) (Seidel) Does every Lagrangian torus \(T^2 \subset X\) of Maslov class zero represent a non-zero class in \(H_2(X; \mathbb{Z})\)?

The corresponding result for the four-dimensional torus (with the standard flat Kähler form) has a positive answer, see [AS10, Corollary 9.2].

1.2 Sample results

We are able to give partial answers to some of these questions, in particular we prove:

**Theorem 1.1** There is a symplectic K3 surface \((X, \omega)\) for which the symplectic Torelli group \(I(X, \omega)\) surjects onto an infinite-rank free group, and in particular is infinitely generated.

To be more explicit, the result holds for generic ambient Kähler forms on either of the following two K3 hypersurfaces:

1. \(X\) is the “mirror quartic”, i.e. the crepant resolution of a quotient of the Fermat quartic hypersurface by \((\mathbb{Z}/4)^2\);
2. \(X\) is the “mirror double plane”, i.e. the crepant resolution of a quotient of the Fermat sextic hypersurface in \(\mathbb{P}(1,1,1,3)\) by the group \(\mathbb{Z}/6 \times \mathbb{Z}/2\).

In both cases, \(X \subset Y\) is the proper transform of an anticanonical hypersurface \(\bar{X} \subset \bar{Y}\) in the toric variety \(\bar{Y}\) associated to an appropriate reflexive simplex \(\Delta\), where \(Y \to \bar{Y}\) is a toric resolution of singularities that is chosen in such a way that \(X\) is a crepant resolution of \(\bar{X}\). “Ambient” Kähler forms on \(X\) are those restricted from \(Y\), and such a form is “generic” if \([\omega] \cap H^2(X; \mathbb{Z})\) is as small as possible amongst ambient Kähler forms.

**Remark 1.2** In fact we prove a stronger result, namely that the group \(K(X, \omega)\), which is the kernel of the natural map \(\pi_0 \text{Symp}(X, \omega) \to \pi_0 \text{Diff}(X)\), surjects onto an infinite-rank free group (cf. Corollary 5.9). This has the following consequence, suggested to us by Dietmar Salamon. Following [Kro97], if \(\text{Diff}_0(X)\) denotes the group of diffeomorphisms which are smoothly isotopic to the identity, then \(\text{Diff}_0(X)\) acts on

\[\Omega = \{a \in \Omega^2(X) \mid a \text{ is symplectic and cohomologous to } \omega\}\]

via the map \(f \mapsto (f^{-1})^* \omega\). Moser’s theorem shows the action is transitive on the connected component containing \(\omega\), with stabiliser \(\text{Symp}_0(X) = \text{Symp}(X, \omega) \cap \text{Diff}_0(X)\). The long exact sequence of homotopy groups for the associated Serre fibration yields a surjective homomorphism

\[\pi_1(\Omega, \omega) \to K(X, \omega)\].

It follows that the fundamental group of the space of symplectic forms on \(X\) isotopic to \(\omega\) surjects onto an infinite-rank free group. See also Lemma 5.10.

Seidel’s question (Q5) above (at least the part regarding non-vanishing of the homology class) was motivated by the following beautiful line of thought, explained to us by Seidel: suppose that \(L \subset (X, \omega)\) is a Maslov-zero Lagrangian torus with vanishing homology class, and \(X^\circ\) is a K3 mirror to \((X, \omega)\). There exist non-commutative deformations of \(D(X^\circ)\) which destroy all point-like objects (i.e., for which all point-like objects become obstructed); therefore the corresponding symplectic deformations...
of \((X, \omega)\) should destroy the point-like object \(L\). However if \([L] = 0\), then we can deform \(L\) to remain Lagrangian under any deformation of the symplectic form by a Moser-type argument: a contradiction, so such \(L\) could not have existed. We use a variation on this idea to prove:

**Theorem 1.3** Let \(X\) be a mirror quartic or mirror double plane, and \(\omega\) any ambient Kähler form on \(X\). Then every Maslov-zero Lagrangian torus \(L \subset (X, \omega)\) has non-trivial homology class.

The Maslov class hypothesis is obviously necessary, since there are Lagrangian tori in a Darboux chart. Under stronger hypotheses one can say more; in particular, if the symplectic form is generic in the same sense as before, then Maslov-zero Lagrangian tori are homologically primitive.

Some of our results do not concern the groups \(G(X, \omega)\) and \(I(X, \omega)\) themselves, but rather their homological algebraic cousin \(\text{Auteq} \mathcal{D} \mathcal{F}(X)\), where \(\mathcal{D} \mathcal{F}(X)\) is the split-closed derived Fukaya category. For instance, we show that for the particular symplectic \(K3\) surfaces considered in Theorem 1.1, the corresponding Torelli-type group \(\text{ker}[\text{Auteq} \mathcal{D} \mathcal{F}(X) \to \text{Aut} \mathcal{H}_* \mathcal{D} \mathcal{F}(X)]\) is indeed generated by squared Dehn twists, whilst \(\text{Auteq} \mathcal{D} \mathcal{F}(X)\) itself is finitely presented, but not generated by Dehn twists.

In the same vein, we give positive answers to weakened versions of questions (Q2) and (Q4) in certain circumstances:

**Theorem 1.4** Let \(X\) be a mirror quartic or mirror double plane, and \(\omega\) a generic ambient Kähler form on \(X\). Then:

1. Every Lagrangian sphere is Fukaya-isomorphic to a vanishing cycle.
2. \(G(X, \omega)\) acts transitively on the set of Fukaya-isomorphism classes of Lagrangian spheres.

**Remark 1.5** In the body of this paper we will only consider four-dimensional symplectic Calabi–Yau manifolds, for which a complete construction of the Fukaya category can be carried out using classical pseudoholomorphic curve theory. It is interesting to note that our results would have implications in higher dimensions, once the relevant foundational theory is established. For instance, let \((X, \omega)\) be a symplectic \(K3\) surface as in Theorem 1.1. Consider the product \(Z = (X \times S^2, \omega \oplus \omega_{FS})\). Conjecturally,

\[
\mathcal{D} \mathcal{F}(Z) \simeq \mathcal{D} \mathcal{F}(X) \oplus \mathcal{D} \mathcal{F}(X)
\]

splits as a direct product of two copies of \(\mathcal{D} \mathcal{F}(X)\), compare to [Sei14, Example 3.23]. Given this, one infers that the symplectic Torelli group \(I(Z)\) surjects onto an infinite-rank free group. By contrast, for simply-connected manifolds of dimension \(\geq 5\), Sullivan showed [Sul77, Theorem 10.3] that the differentiable Torelli group \(\text{ker}(\pi_0 \text{Diff}(Z) \to \text{Aut} \mathcal{H}^*(Z))\) is commensurable with an arithmetic group, in particular is finitely presented. Thus, such a stabilised version of Theorem 1.1 would be intrinsically symplectic in nature. (Our lack of knowledge of four-dimensional smooth mapping class groups makes it hard to draw precisely the same contrast in the setting of Theorem 1.1 itself.)

### 1.3 Outlines and further questions

The results above are obtained by combining two main ingredients:

- the proof of homological mirror symmetry for Greene–Plesser mirrors, cf. [SS17];
- Bayer and Bridgeland’s proof [BB17] of Bridgeland’s conjecture [Bri08] on the group of autoequivalences of the derived category \(\mathcal{D}(X^o)\) of a \(K3\) surface \(X^o\), in the case that \(X^o\) has Picard rank one.
Remark 1.6  An important point is that Bayer and Bridgeland prove Bridgeland’s conjecture for Picard rank one $K3$ surfaces, but the higher-rank case remains open. In particular, their proof does not apply to the mirror of the quartic surface (which has Picard rank 19). That is why we cannot use Seidel’s proof of homological mirror symmetry in that case [Sei15] as input for our results, but must instead use the more general results proved in [SS17].

Remark 1.7  Another slight delicacy is that the mirror to a symplectic $K3$ surface $(X, \omega)$, in the sense of homological mirror symmetry, is an algebraic $K3$ surface over the Novikov field $\Lambda$, whilst the work of Bayer and Bridgeland, at least as written, is for $K3$ surfaces over $\mathbb{C}$. We circumvent this using standard ideas around the “Lefschetz principle”, i.e. the fact that any algebraic $K3$ surface over a field of characteristic zero is in fact defined over a finitely generated extension field of the rationals, and such fields admit embeddings in $\mathbb{C}$.

We now explain how we combine these two ingredients to obtain information about symplectic mapping class groups. We study $G(X, \omega)$ via its action on the Fukaya category. It turns out always to act by “Calabi–Yau” autoequivalences, so that we have a homomorphism
\[ G(X, \omega) \to \text{Auteq}_{\text{CY}} \mathcal{D}\mathcal{F}(X, \omega)/[2]. \]
If $X^0$ is homologically mirror to $(X, \omega)$, then the autoequivalence group of $\mathcal{D}\mathcal{F}(X, \omega)$ can be identified with the autoequivalence group of the derived category of $\mathcal{D}(X^0)$. The Bayer–Bridgeland theorem identifies the subgroup of Calabi–Yau autoequivalences of $\mathcal{D}(X^0)$ with $\pi_1(M)$, where $M$ is a version of the ‘Kähler moduli space’ of $X^0$ defined via Bridgeland’s stability conditions (it is denoted $\mathbb{K}_M^d$ in the body of the paper).

We identify $M$ as the ‘complex moduli space’ of $X$, and construct a symplectic monodromy homomorphism $\pi_1(M) \to G(X, \omega)$. We show that the composition
\[ \pi_1(M) \to G(X, \omega) \to \text{Auteq}_{\text{CY}} \mathcal{D}\mathcal{F}(X, \omega)/[2] \cong \text{Auteq}_{\text{CY}} \mathcal{D}(X^0)/[2] \cong \pi_1(M) \]
is an isomorphism. Considering the respective Torelli subgroups, we have a similar composition
\[ \pi_1(M) \to I(X, \omega) \to \text{Auteq}_0 \mathcal{D}\mathcal{F}(X, \omega)/[2] \cong \text{Auteq}_0 \mathcal{D}(X^0)/[2] \cong \pi_1(M) \]
equal to an isomorphism, where $M$ is a certain cover of $M$ (denoted $D_M^d$ in the body of the paper).

It follows, in particular, that $I(X, \omega)$ surjects onto $\pi_1(M)$. Theorem 1.1 then follows from the fact that $M$ is the complement of a countably infinite discrete set of points in the upper half plane, and in particular its fundamental group is an infinite-rank free group.

More precisely, the fact that the compositions (1), (2) are isomorphisms implies that
\begin{align*}
(3) & \quad G(X, \omega) \cong \pi_1(M) \rtimes Z(X, \omega) \quad \text{and} \\
(4) & \quad I(X, \omega) \cong \pi_1(M) \rtimes Z(X, \omega),
\end{align*}
where
\[ Z(X, \omega) := \ker(G(X, \omega) \to \text{Auteq} \mathcal{D}\mathcal{F}(X, \omega)/[2]) \]
denotes the ‘Floer-theoretically trivial’ subgroup of the symplectic mapping class group (it is contained in $I(X, \omega)$). Thus, we have essentially reduced the problem of computing $G(X, \omega)$ and $I(X, \omega)$ to the problem of computing $Z(X, \omega)$. This raises the following:

(Q6)  Let $(X, \omega)$ be a symplectic $K3$ whose Fukaya category is non-degenerate (i.e., the open-closed map hits the unit). Can $Z(X, \omega)$ be non-trivial?
The caveat of non-degeneracy of the Fukaya category is added to avoid a negative answer due to the symplectic manifold ‘not having enough Lagrangians to detect symplectomorphisms’.

In a similar vein, Theorem 1.4 reduces questions (Q2) and (Q4) to the following general:

(Q7) Can there exist Fukaya-isomorphic but non-Hamiltonian-isotopic Lagrangian spheres in a symplectic $K3$?

**Acknowledgements** I.S. is indebted to Daniel Huybrechts for patient explanations of the material of Section 4.1. He is furthermore grateful to Arend Bayer, Tom Bridgeland, Aurel Page, Oscar Randal-Williams, Tony Scholl and Richard Thomas for helpful conversations and correspondence.

N.S. was partially supported by a Sloan Research Fellowship, and by the National Science Foundation through Grant number DMS-1310604 and under agreement number DMS-1128155. Any opinions, findings and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation. N.S. also acknowledges support from Princeton University and the Institute for Advanced Study. I.S. is partially supported by a Fellowship from EPSRC.

## 2 Complex $K3$ surfaces

General references are [BHPVdV04, Huy16, Dol96].

### 2.1 Basics

A $K3$ surface (over the complex numbers $\mathbb{C}$) is a compact complex surface $X$ with $H^1(X, \mathcal{O}_X) = 0$ and with trivial canonical sheaf $K_X \cong \mathcal{O}_X$. The Picard group is $\text{Pic}(X) = \text{NS}(X) = H^2(X, \mathbb{Z}) \cap H^{1,1}(X, \mathbb{C})$, equipped with the cupproduct pairing; this is a lattice of rank $0 \leq \rho(X) \leq 20$. While all $K3$ surfaces are Kähler, the fact that Picard rank zero can occur reflects the fact that not all are algebraic. In this paper, we will only consider algebraic $K3$ surfaces, so we assume henceforth that $\rho(X) > 0$. The Picard lattice then has signature $(1, \rho(X) - 1)$. The lattice $\text{NS}(X)^\perp$ is called the transcendental lattice.

The cohomology group

$$H^*(X, \mathbb{Z}) = H^0(X, \mathbb{Z}) \oplus H^2(X, \mathbb{Z}) \oplus H^4(X, \mathbb{Z}),$$

comes equipped with a polarized weight two Hodge structure, whose algebraic part is given by

$$\mathbb{N}(X) = H^0(X, \mathbb{Z}) \oplus \text{NS}(X) \oplus H^4(X, \mathbb{Z}),$$

and whose polarization is given by the Mukai symmetric form

$$\langle (r_1, D_1, s_1), (r_2, D_2, s_2) \rangle = D_1 \cdot D_2 - r_1 s_2 - r_2 s_1.$$

The lattice $H^*(X, \mathbb{Z})$ is isomorphic to $U^{\oplus 4} \oplus E_8^{\oplus 2}$, where $U$ is the hyperbolic lattice and our convention is that the $E_8$-lattice is negative definite. The subgroup $\mathbb{N}(X)$ has signature $(2, \rho(X))$. 
2.2 Lattices

Let \( L \) be a lattice, i.e. a free \( \mathbb{Z} \)-module with a non-degenerate but not necessarily unimodular pairing \((\cdot, \cdot)\). We will sometimes write \( x^2 \) for \((x, x)\). The \textit{period domain} associated to \( L \) is
\[
D(L) := \{ \Omega \in \mathbb{P}(L^\perp) \mid (\Omega, \Omega) = 0, \ (\Omega, \overline{\Omega}) > 0 \}.
\]
\( D(L) \) is an open complex submanifold of a quadric hypersurface in \( \mathbb{P}(L^\perp) \). The group of isometries \( O(L) \) acts on \( D(L) \) on the left.

An embedding \( M \hookrightarrow L \) of lattices is \textit{primitive} if \( L/M \) is torsion free. Two primitive embeddings \( \iota_1 : M \hookrightarrow L \) and \( \iota_2 : M \hookrightarrow L \) are isomorphic if there is an isometry \( \phi : L \rightarrow L \) for which \( \phi \circ \iota_1 = \iota_2 \). Any embedding of the hyperbolic lattice \( U \) into a lattice \( L \) is primitive, and induces a splitting \( L = U \oplus U^\perp \).

We will denote the set of \((-2)\)-classes in a lattice \( L \) by
\[
\Delta(L) := \{ \delta \in L \mid (\delta, \delta) = -2 \}.
\]

2.3 Lattice polarized \( K3 \) surfaces

For the rest of the paper, let \( L = U^{\oplus 3} \oplus E_8^{\oplus 2} \) be the “\( K3 \) lattice” (isomorphic to \( H^2(X; \mathbb{Z}) \) for any complex \( K3 \) surface \( X \)). \( L \) has signature \((3, 19)\).

Let \( M \) denote an even non-degenerate lattice of signature \((1, t)\). Fix a primitive lattice embedding \( M \hookrightarrow L \). We remark that such embeddings are unique up to isomorphism if \( t \leq 9 \) by [Dol96, Corollary 5.2], and in particular when \( t = 0 \), i.e. for rank one lattices. We will always assume that \( M^\perp \) contains a hyperbolic plane.

Choose a connected component \( C(M)^+ \) of
\[
\{ x \in M_{\mathbb{R}} : x^2 > 0, \ (x, \delta) \neq 0 \ \forall \delta \in \Delta(M) \}.
\]

**Definition 2.1** (Dolgachev) An \( M \)-polarized \( K3 \) surface is a pair \((X, j)\), with \( X \) a \( K3 \) surface and \( j : M \hookrightarrow \text{Pic}(X) \) a primitive lattice embedding. We say that \((X, j)\) is (pseudo-)ample if the image of \( C(M)^+ \) intersects the (pseudo-)ample cone. A \textit{marked} \( M \)-polarized \( K3 \) surface is a pair \((X, \phi)\) with \( X \) a \( K3 \) surface and \( \phi : H^2(X; \mathbb{Z}) \rightarrow L \) a lattice isometry with \( \phi^{-1}(M) \subset \text{Pic}(X) \). Observe that any marked \( M \)-polarized \( K3 \) surface \((X, \phi)\) defines an \( M \)-polarized \( K3 \) surface \((X, j_\phi)\), where \( j_\phi = \phi^{-1}|_M \).

Let
\[
\tilde{D}_M := D(M^\perp) = \{ \Omega \in \mathbb{P}(M^\perp_{\mathbb{C}}) : (\Omega, \Omega) = 0, \ (\Omega, \overline{\Omega}) > 0 \}
\]

denote the period domain associated to the orthogonal complement \( M^\perp \) of \( M \subset L \).

A marked \( M \)-polarized \( K3 \) surface \((X, \phi)\) has a period point
\[
\phi(H^{2,0}(X)) \in \tilde{D}_M \subset \mathbb{P}(M^\perp_{\mathbb{C}}) \subset \mathbb{P}(L_{\mathbb{C}}).
\]

There is a fine (non-separated) moduli stack \( \mathcal{K}_M \) of marked \( M \)-polarized \( K3 \) surfaces, with a surjective period map \( \text{per} : \mathcal{K}_M \rightarrow \tilde{D}_M \). The restriction of \( \text{per} \) to the subset \( \mathcal{K}_M^{\text{pa}} \) of pseudo-ample marked surfaces is also surjective, and the further restriction to the subset \( \mathcal{K}_M^{\text{pa}} \) of ample marked surfaces has image
\[
\tilde{D}^a_M := \tilde{D}_M \setminus \bigcup_{\delta \in \Delta(M^\perp)} \delta^\perp.
\]

However, neither of these restrictions need be injective.
Definition 2.2 We define $\mathcal{K}_M := D_M / \Gamma(M)$, where $\Gamma(M) := \{ \sigma \in O(L) : \sigma|_M = \text{id} \}$. It contains $\mathcal{K}_M^a := \mathcal{D}_M^a / \Gamma(M)$.

We observe that $\mathcal{K}_M$ is a quotient of (the homogeneous symmetric domain) $\mathcal{D}_M$ by an arithmetic subgroup of $O(2, 19 - t)$, which gives it the structure of a complex analytic orbifold. There is a fine (non-separated) moduli stack of $M$-polarized $K3$ surfaces such that the embedding $M \hookrightarrow \text{Pic}(X) \subset H^2(X; \mathbb{Z})$ is isometric to $M \hookrightarrow L$, which comes with a period map to $\mathcal{K}_M$.

We now observe that $\mathcal{K}_M$ has two connected components, which are distinguished by the orientation of the positive-definite two-plane in $M^\perp$ spanned by the real and imaginary parts of $\Omega$. They are interchanged by the complex conjugation map, $\Omega \mapsto \bar{\Omega}$. We label them $D_M$ and $\mathcal{D}_M$, arbitrarily (if we are given an $M$-polarized $K3$ surface, we can choose $D_M$ to be the component containing its period point). We define $\Gamma(M)^+ \subset \Gamma(M)$ to be the subgroup that preserves the connected components: it can have index one or two (see [Dol96, Proposition 5.6] for a sufficient condition for the index to be one), so $\mathcal{K}_M$ can have either one or two connected components. One of the connected components will always be $\mathcal{K}_M := D_M / \Gamma(M)^+$, and if there is a second it will be isomorphic to $\mathcal{D}_M$. Since we will be interested in the fundamental group of these moduli spaces, it will usually make more sense to deal with the connected $\mathcal{K}_M$ rather than the possibly disconnected $\mathcal{K}_M$.

Remark 2.3 The inclusion $M \hookrightarrow \text{Pic}(X)$ is an isomorphism unless the period point of $(X, \phi)$ lies in $\mathcal{D}_M^a$ for some $M' \supsetneq M$. Thus the fibre of the period map over a very general point in $\mathcal{D}_M$ (or, analogously, $\mathcal{K}_M$) consists of $M$-polarized $K3$ surfaces with $M \cong \text{Pic}(X)$, where we recall that “very general” means “away from a countable collection of positive-codimension subvarieties”.

Recall that by assumption there is a copy of the hyperbolic lattice $U$ in $M^\perp$. We fix one, and write (symmetrically)

$$M^\perp = U \oplus M^o; \quad (M^o)^\perp = U \oplus M$$

where $M^o$ has signature $(1, 18 - t)$. As an abstract lattice, $M^o$ does not depend on the choice of embedding $U \hookrightarrow M^\perp$, provided primitive lattice embeddings of $M$ into $L$ are unique up to isomorphism.

Definition 2.4 $\mathcal{K}_{M^o}$ is a Dolgachev mirror moduli space to $\mathcal{K}_M$.

Remark 2.5 Observe that $\text{dim}(\mathcal{K}_M) + \text{dim}(\mathcal{K}_{M^o}) = 20$. Mirror symmetry matches up $M$-polarized $K3$ surfaces $X$ equipped with complexified Kähler classes $[\omega_C] \in M_C$, and $M^o$-polarized $K3$ surfaces $X^o$ equipped with complexified Kähler classes $[\omega_C^o] \in M_C^o$. Since mirror symmetry sends Kähler structures on $X$ to complex structures on $X^o$ (and vice-versa), it is unsurprising that the dimension of the space of Kähler classes that we consider on $X$ is equal to

$$\text{rk}(M) = 1 + t = \text{dim} \mathcal{K}_{M^o},$$

the dimension of the space of complex structures that we consider on $X^o$.

Example 2.6 $M^o = \langle 2n \rangle$ admits a unique primitive embedding in $L$ up to $O(L)$, and in this case the mirror family is essentially unique. Dolgachev gives these families explicitly [Dol96, Section 7]. We have

$$\langle 2n \rangle^\perp = U^{\oplus 2} \oplus E_8^{\oplus 2} \oplus \langle -2n \rangle \quad \text{so the mirror lattice is} \quad M = U \oplus E_8^{\oplus 2} \oplus \langle -2n \rangle.$$
The family $\mathbb{K}_M$ is one-dimensional, and $D_M$ is isomorphic to the upper half-plane $\mathfrak{h}$. Dolgachev computes that $\Gamma(M)^+$ is isomorphic to the Fricke modular group $\Gamma_0(n)^+ \subset \mathbb{P}SL(2; \mathbb{R})$, which is generated by the matrices
\[
\left(\begin{array}{cc}
a & b \\
c & d
\end{array}\right) \in SL(2; \mathbb{Z}) \text{ with } n|c, \quad \text{and} \quad \left(\begin{array}{cc}0 & -1/\sqrt{n} \\
\sqrt{n} & 0
\end{array}\right).
\]
So we have $\mathbb{K}_M \cong \mathfrak{h}/\Gamma_0(n)^+$.

### 2.4 Monodromy

There is no universal family of pseudo-ample $M$-polarized K3 surfaces over $\mathbb{K}_M$, nor of ample $M$-polarized K3 surfaces over $\mathbb{K}_M^a$. However, $\mathbb{K}_M^a$ can be regarded as the fine moduli stack of ‘C-ample $M$-polarized K3 surfaces’ (following [BB17, Lemma 7.6]). Here, $\mathcal{C}$ is a connected component of the locus
\[
(8) \quad \{x \in C(M)^+ : (x, \delta) \neq 0, \forall \delta \in L\backslash M^+, \delta^2 = -2\}.
\]
A C-ample $M$-polarized K3 surface is defined to be an $M$-polarized K3 surface $(X, j)$ such that $j(C)$ is contained in the ample cone. If $\mathcal{K}_a^a \subset \mathcal{K}_M^a$ denotes the subspace of $C$-ample marked $M$-polarized K3 surfaces, then the restriction of the period map per : $\mathcal{K}_a^a \to \mathcal{D}_M^a$ is bijective. It follows that $\mathbb{K}_M^a$ carries a universal family $\mathcal{X}_C$ of C-ample $M$-polarized K3 surfaces, so is a fine moduli space of such (for any $C$, although the universal family is different for different $C$).

Each class $\alpha \in C$ defines a fibrewise Kähler class on $p : \mathcal{X}_C \to \mathbb{K}_M^a$, since $\alpha$ is invariant under $\Gamma(M)$. Thus we can construct a fibrewise Kähler form on $\mathcal{X}_C$, together with a symplectic connection by [GLS96, Theorem 1.2.5] (with minor modifications to take into account the fact that the base is an orbifold: the connection is constructed using a fibre-compatible two-form, which one must ‘average’ on orbifold charts to make it equivariant). Then parallel transport with respect to the connection yields a symplectic monodromy homomorphism
\[
(9) \quad \pi_1(\mathbb{K}_M^a) \longrightarrow G(X, \omega)
\]
(where by abuse of notation we write $\pi_1$ for the orbifold fundamental group of an orbifold).

#### Remark 2.7
Note that there is a natural isomorphism of the universal family of $C$-ample $M$-polarized K3 surfaces over $\mathbb{K}_M^a$ with the complex conjugate of the universal family of $(\neg C)$-ample $M$-polarized K3 surfaces over $\mathbb{K}_M^a$. This implies that the monodromy homomorphism does not depend on the choice of component $D_M$ of $\mathcal{D}_M^a$, in the case that $\mathbb{K}_M^a = \mathbb{K}_M^a \sqcup \mathbb{K}_M^a$ has two connected components: we can identify $\pi_1(\mathbb{K}_M^a) \cong \pi_1(\mathbb{K}_M^a')$ and $G(X, \omega) \cong G(\tilde{X}, -\omega)$ so that the monodromy homomorphisms get identified.

#### Remark 2.8
If $\alpha \in C(M)^+$ does not lie in the locus (8), then we do not have a symplectic monodromy map (9). Indeed, $\alpha$ is orthogonal to some class $\delta \in L\backslash M^+$ with $\delta^2 = -2$. Any period point in $\mathcal{D}_M^a$ belonging to $D(M \oplus \langle \delta \rangle)^+ \subset D(M^+) = \mathcal{D}_M$ corresponds to a K3 surface $X$ for which $M \oplus \langle \delta \rangle \subset \text{Pic}(X)$, whence by Riemann–Roch either $\delta$ or $-\delta$ is effective and represented by a rational curve. Thus, there is a period point $p \in \mathcal{D}_M^a$ such that $\alpha$ does not lie in the Kähler cone of any element $(X, \phi)$ of the fibre of the period map over $p$. There is then no way to construct a universal family over $\mathbb{K}_M^a$ with a fibrewise Kähler form with cohomology class $\alpha$, so we cannot construct our symplectic monodromy map. There may however be a monodromy map from $\pi_1(\mathbb{K}_M^a')$ for some sub-lattice $M' \subset M$. 

Theorem of 1.2.5] (with minor modifications to take into account the fact that the base is an orbifold: the connection is constructed using a fibre-compatible two-form, which one must ‘average’ on orbifold charts to make it equivariant). Then parallel transport with respect to the connection yields a symplectic monodromy homomorphism
\[
(9) \quad \pi_1(\mathbb{K}_M^a) \longrightarrow G(X, \omega)
\]
(where by abuse of notation we write $\pi_1$ for the orbifold fundamental group of an orbifold).
Remark 2.9 Consider the anticanonical linear system $| - K_Y | \cong \mathbb{P}^N$ on a Fano 3-fold $Y$, and let $M = \text{Pic}(Y)$. If $\Delta \subset \mathbb{P}^N$ is the discriminant of singular hypersurfaces, there is a natural family of $M$-polarized K3 surfaces over $\mathbb{P}^N \setminus \Delta$, and hence a classifying map $\rho : \mathbb{P}^N \setminus \Delta \to \mathbb{C}^3_M$ to the coarse moduli space of $M$-polarized K3 surfaces. Fix a smooth anticanonical hypersurface $X \subset Y$. Any choice of Kähler form $\Omega$ on $Y$ yields a monodromy homomorphism

$$\pi_1(\mathbb{P}^N \setminus \Delta) \to G(X, \Omega|_X).$$

If $\Omega|_X$ lies in the locus (8), then (10) factors through (9) via the map on fundamental groups defined by $\rho$. If $\rho$ is surjective, the restriction map from the Kähler cone of $Y$ to that of $X$ necessarily lands in (8). If $\rho$ is not surjective, then $\Omega|_X$ may not lie inside (8); if it does not, then the period point guaranteed by Remark 2.8 will lie outside $\text{im}(\rho)$ (i.e., it corresponds to a K3 surface which does not belong to the anticanonical system $| - K_Y |$). In that case, the family of Kähler K3-surfaces over $\text{im}(\rho)$ defined by $\Omega$ will not extend to a family over all of $\mathbb{C}^3_M$.

If $X$ is $M$-polarized, and $\omega$ is a Kähler form such that $[\omega]$ lies in (8), we now have a map of exact sequences

$$\begin{array}{cccccc}
1 & \to & \pi_1(D_M^q) & \to & \pi_1(\mathbb{C}^3_M) & \to \Gamma(M)^+ & \to 1 \\
\downarrow & & \downarrow & & \downarrow & \\
1 & \to & I(X, \omega) & \to & G(X, \omega) & \to & \text{Aut} H^2(X; \mathbb{Z}).
\end{array}$$

Definition 2.10 Suppose that $(X, j)$ is $M$-polarized and $\omega$ is a Kähler form with $[\omega] \in j(C(M)^+)$. We say $\omega$ is generic if $[\omega]^+ \cap H^2(X; \mathbb{Z}) \cong M^+$ is as small as possible.

Observe that the notion of genericity depends on the polarization $j$. Also observe that if $\omega$ is generic, then $j^{-1}([\omega])$ lies in the locus (8).

Remark 2.11 If $\omega$ is generic and $\phi \in G(X, \omega)$, it is clear that $\phi^*$ lies in $\Gamma(M)$. By a result of Donaldson [Don90, Proposition 6.2], it in fact lies in $\Gamma(M)^+$. By commutativity of (11), the map $G(X, \omega) \to \Gamma(M)^+$ is in fact surjective (this is [Don90, Proposition 6.1]). So for generic $\omega$, the ‘$\text{Aut} H^2(X; \mathbb{Z})$’ in the lower right-hand corner of (11) can be replaced by ‘$\Gamma(M)^+ \to 1$’, and we have a map of exact sequences in which the rightmost vertical arrow is the identity map.

2.5 The Bayer–Bridgeland theorem

Fix a triangulated category $D$ linear over a field $k$. Suppose $D$ is proper, i.e., that $\oplus_{i \in \mathbb{Z}} \text{Hom}_D(E, F[i])$ is finite-dimensional for all objects $E$, $F$. Let $K_{\text{num}}(D)$ denote the numerical Grothendieck group, i.e. the quotient of $K^0(D)$ by the kernel of the Euler form. We let $\text{Stab}(D)$ denote the set of stability conditions on $D$ which are numerical, full and locally-finite in the terminology of [Bri07, Bri08]. Recall that a numerical stability condition $\sigma = (Z, \mathcal{P})$ on $D$ comprises a group homomorphism $Z : K_{\text{num}}(D) \to \mathbb{C}$, called the central charge, and a collection of full additive subcategories $\mathcal{P}(\phi) \subset D$ for $\phi \in \mathbb{R}$ which satisfy various axioms, see [Bri07]. The main result of op. cit. asserts that the space $\text{Stab}(D)$ has the structure of a complex manifold, such that the forgetful map

$$\pi : \text{Stab}(D) \to \text{Hom}_\mathbb{C}(K_{\text{num}}(D), \mathbb{C}),$$

...
taking a stability condition to its central charge, is a local isomorphism. The group of triangulated autoequivalences $\text{Auteq}(D)$ acts on $\text{Stab}(D)$; an element $\Phi \in \text{Aut}(D)$ acts by
\[ \Phi : (Z, \mathcal{P}) \mapsto (Z', \mathcal{P}') , \quad Z'(E) = Z(\Phi^{-1}(E)) , \quad \mathcal{P}' = \Phi(\mathcal{P}) . \]

Suppose now $X$ is a complex algebraic K3 surface of Picard rank $\rho(X)$, and $D(X)$ its bounded derived category. Any object of $D(X)$ has a Mukai vector $v(E) = \text{ch}(E)\sqrt{\text{det}X} \in \mathbb{N}(X)$, and the association $E \mapsto v(E)$ sets up an isomorphism
\[ K_{\text{num}}(D(X)) \otimes \mathbb{C} \cong \mathbb{N}(X) \otimes \mathbb{C} . \]

Riemann–Roch takes the form
\[ \chi(E, F) = \sum_{i \in \mathbb{Z}} (-1)^i \dim_{\mathbb{C}} \text{Hom}_X^i(E, F) = -(v(E), v(F)) . \]

We denote $\text{Stab}(X) := \text{Stab}(D(X))$. We can identify $\text{Hom}_{\mathbb{Z}}(K_{\text{num}}(D(X)), \mathbb{C}) \cong \mathbb{N}(X) \otimes \mathbb{C}$ via the intersection pairing, so (12) becomes a local isomorphism
\[ (13) \quad \pi : \text{Stab}(X) \to \mathbb{N}(X) \otimes \mathbb{C} . \]

Since any autoequivalence is of Fourier–Mukai type, the Mukai vector of its kernel induces a correspondence; its action on cohomology preserves the Hodge filtration, the integral structure and the Mukai pairing, yielding a map
\[ \varpi : \text{Auteq}(D(X)) \to \text{Aut} H^*(X; \mathbb{Z}) \]

to the group of Hodge isometries. The image of $\varpi$ is the index-2 subgroup $\text{Aut}^+ H^*(X; \mathbb{Z})$ of Hodge isometries which preserve orientations of positive-definite 4-planes [Muk87, Orl97, HMS09]. We denote the kernel of $\varpi$ by $\text{Auteq}^0 D(X)$.

Recall that $\mathbb{N}(X)$ has signature $(2, \rho(X))$. Define the open subset
\[ \mathcal{P}(X) \subset \mathbb{N}(X) \otimes \mathbb{C} \]
consisting of vectors $\Omega \in \mathbb{N}(X) \otimes \mathbb{C}$ whose real and imaginary parts span a positive definite 2-plane in $\mathbb{N}(X) \otimes \mathbb{R}$. This subset has two connected components, distinguished by the orientation induced on this 2-plane; let $\mathcal{P}^+(X)$ be the component containing vectors of the form $(1, i\omega, -\frac{1}{2} \omega^2)$ for an ample class $\omega \in \text{NS}(X) \otimes \mathbb{R}$. Consider the root system $\Delta(X) := \Delta(\mathbb{N}(X))$, and the corresponding hyperplane complement
\[ \mathcal{P}_0^+(X) = \mathcal{P}^+(X) \setminus \bigcup_{\delta \in \Delta(X)} \delta^\perp . \]

Let $\text{Stab}^\dagger(X) \subset \text{Stab}(X)$ be the connected component containing the (non-empty) set of “geometric” stability conditions, for which all skyscraper sheaves $\mathcal{O}_x$ are stable of the same phase.

**Theorem 2.12** (Bridgeland, Bayer–Bridgeland) 
Suppose that $\rho(X) = 1$. Then $\text{Stab}^\dagger(X)$ is simply connected, and the forgetful map (13) defines the universal cover
\[ (14) \quad \pi : \text{Stab}^\dagger(X) \to \mathcal{P}_0^+(X) . \]

The action of $\text{Auteq}(D(X))$ on $\text{Stab}(X)$ preserves the connected component $\text{Stab}^\dagger(X)$, and the action of the subgroup $\text{Auteq}^0 D(X)$ is identified with the group of deck transformations of the cover (14). As a

\[ ^{\text{more precisely, [Bri07] shows that for each connected component of the space of numerical, locally-finite stability conditions, the forgetful map defines a local isomorphism to a linear subspace } V \text{ of } \text{Hom}_{\mathbb{Z}}(K_{\text{num}}(D), \mathbb{C}), \text{ and [Bri08] defines such a stability condition to be full if the subspace } V \text{ is all of } \text{Hom}_{\mathbb{Z}}(K_{\text{num}}(D), \mathbb{C}).} \]
consequence, there is an isomorphism of short exact sequences of groups

\[ 1 \longrightarrow \pi_1(\mathcal{P}^+_0(X)) \longrightarrow \pi_1(\mathcal{P}^+_0(X)/\text{Aut}^+ H^*(X)) \longrightarrow \text{Aut}^+ H^*(X; \mathbb{Z}) \longrightarrow 1 \]

\[ 1 \longrightarrow \text{Auteq}^0 D(X) \longrightarrow \text{Auteq} D(X) \longrightarrow \text{Aut}^+ H^*(X; \mathbb{Z}) \longrightarrow 1. \]

**Proof** This combines the main results of [Bri08] and [BB17].

“Bridgeland’s conjecture” asserts that the result holds without the hypothesis \( \rho(X) = 1 \).

**Corollary 2.13** If \( \rho(X) = 1 \), then there is an isomorphism of short exact sequences

\[ 1 \longrightarrow \pi_1(\mathcal{P}^+_0(X)/\mathbb{C}^*) \longrightarrow \pi_1(\mathcal{P}^+_0(X)/\mathbb{C}^* \times \text{Aut}_{\text{CY}}^+ H^*(X; \mathbb{Z})) \longrightarrow \text{Aut}_{\text{CY}}^+ H^*(X; \mathbb{Z}) \longrightarrow 1, \]

where \( M = \text{Pic}(X)^0 \), \( \text{Aut}_{\text{CY}}^+ H^*(X; \mathbb{Z}) \subset \text{Aut}^+ H^*(X; \mathbb{Z}) \) is the subgroup of Hodge isometries whose complexification acts trivially on \( H^{2,0}(X; \mathbb{C}) \), and \( \text{Auteq}_{\text{CY}} \) is the preimage of \( \text{Aut}_{\text{CY}}^+ \) under \( \phi \).

**Proof** This is explained in [BB17, Section 7], but we will make it more explicit for the reader’s convenience. There is a natural \( \mathbb{C}^* \) -action on \( \mathcal{P}^+_0(X) \), and the image of the corresponding generator of the fundamental group in \( \text{Auteq}^0 D(X) \) is the shift by 2. It follows by Theorem 2.12 that the bottom row of (15) is isomorphic to

\[ 1 \longrightarrow \pi_1(\mathcal{P}^+_0(X)/\mathbb{C}^*) \longrightarrow \pi_1(\mathcal{P}^+_0(X)/\mathbb{C}^* \times \text{Aut}_{\text{CY}}^+ H^*(X; \mathbb{Z})) \longrightarrow \text{Aut}_{\text{CY}}^+ H^*(X; \mathbb{Z}) \longrightarrow 1, \]

so it suffices for us to identify this with the top row.

Now the subset

\[ \Omega^+_0(X) := \{ \Omega \in \mathcal{P}^+_0(X) \mid (\Omega, \Omega) = 0 \} \subset \mathcal{P}^+_0(X) \]

is a deformation retract, so we may replace ‘\( \mathcal{P}^+ \)’ by ‘\( \mathcal{Q} \)’ in (16). It now suffices for us to identify \( \text{Aut}_{\text{CY}}^+ H^*(X; \mathbb{Z}) \) with \( \Gamma(M)^+ \), and \( \mathcal{Q}^+_0(X)/\mathbb{C}^* \) equivariantly with \( D_M^+ \). In fact, the latter is equivalent to identifying \( N(X) \) equivariantly with \( M^+ \): given an element \( \Omega \) of \( N(X)_{\mathbb{C}} \cong M^+_C \) which satisfies \( (\Omega, \Omega) = 0 \), it is easy to see that \( (\Omega, \Omega) > 0 \) if and only if the 2-plane spanned by the real and imaginary parts of \( \Omega \) is positive definite. Since we have freedom to choose \( D_M \), we can choose \( M \) to be the connected component corresponding to \( \mathcal{P}^+(X) \) under this identification.

Let us fix an identification \( H^2(X; \mathbb{Z}) \cong L \), identifying \( \text{Pic}(X) \) with a primitive sublattice \( \text{Pic} \subset L \) of rank 1. We choose a copy of \( U \) in \( \text{Pic}^+ \) (such \( U \) always exists), so \( \text{Pic}^+ = U \oplus M \). We now define an automorphism

\[ m : \text{L} \oplus U \rightarrow \text{L} \oplus U \]

by identifying \( L \oplus U \cong \text{U}^+ \oplus \text{U}^+ \oplus U \) (since \( U \) is unimodular), and letting \( m \) swap the two copies of \( U \) and fix \( U^+ \). We identify \( H^*(X; \mathbb{Z}) \cong H^2(X; \mathbb{Z}) \oplus H^0(X; \mathbb{Z}) \oplus H^4(X; \mathbb{Z}) \cong L \oplus U \); then the automorphism \( m \) sends \( N(X) \) isomorphically to \( M^+ \subset \text{L} \subset \text{L} \oplus U \).

It remains to check that this identification \( N(X) \cong M^+ \) is equivariant. Indeed, any \( \phi \in \text{Aut}^+ H^*(X; \mathbb{Z}) \) fixes the period point if and only if it fixes \( \text{Pic}^+ \subset H^2(X; \mathbb{Z}) \). Such \( \phi \) correspond, under \( m \), to automorphisms fixing \( M \oplus U \subset \text{L} \oplus U \), which are equivalent to automorphisms of \( L \) fixing \( M \). Thus,
$m$ induces an isomorphism $\text{Aut}_{\mathcal{CY}} H^*(X; \mathbb{Z}) \cong \Gamma(M)$. Finally, such a $\phi$ preserves orientations of positive-definite 4-planes if and only if the corresponding element of $\Gamma(M)$ preserves orientations of positive-definite 2-planes, if and only if it preserves the connected components of $\hat{D}_M$. So we have an identification of $\text{Aut}_{\mathcal{CY}}^+ (H^*(X; \mathbb{Z}))$ with $\Gamma(M)^+$, which identifies their respective actions on $\mathbb{N}(X)$ and $M^\perp$, as required.

\section{2.6 Two examples}

We will focus on two cases of Dolgachev’s construction, corresponding to the rank one lattices $M^0 = \langle 2 \rangle$ and $M^0 = \langle 4 \rangle$ (see Example 2.6). These arise geometrically as follows:

1. A very general quartic hypersurface $X \subset \mathbb{P}^3$ has $\text{Pic}(X) \cong \langle 4 \rangle$;
2. A very general sextic hypersurface $X \subset \mathbb{P}(1, 1, 1, 3)$ has $\text{Pic}(X) \cong \langle 2 \rangle$.

Note that in the second case, $\mathbb{P}(1, 1, 1, 3)$ has an isolated singularity, and the generic hypersurface is disjoint from that point and hence smooth.

\textbf{Example 2.14} Suppose $M^0 = \langle 4 \rangle$. Then
\[ \mathbb{K}^{a}_{M^0} = \{ \text{smooth quartics in } \mathbb{P}^3 \}. \]

We consider the “Dwork family”
\begin{equation}
Q_{\lambda} = \{ x_0^4 + x_1^4 + x_2^4 + x_3^4 + 4\lambda x_0 x_1 x_2 x_3 = 0 \} \subset \mathbb{P}^3
\end{equation}
of quartics, parametrized by $\lambda \in \mathbb{A}^1$. These are all invariant under the group $\Pi = \mathbb{Z}/4 \times \mathbb{Z}/4$ of diagonal projective transformations $[p^0, p^1, p^2, p^3]$ of $\mathbb{P}^3$ with $\sum a_j = 0$. If $\lambda^4 \neq 1$ then the quotient $Q_\lambda/\Pi$ has six $A_1$-singularities; the fibres over $\lambda^4 = 1$ have an additional nodal singularity. There is a simultaneous resolution of the Dwork family, which defines a family of K3’s over $\mathbb{A}^1 \setminus \{ \lambda^4 = 1 \}$. This is the Greene–Plesser mirror family to the family of smooth quartics in $\mathbb{P}^3$.

The fibres over $\lambda$ and $i\lambda$ are isomorphic, and so we obtain a family over $(\mathbb{A}^1 \setminus \{ \lambda^4 = 1 \})/(\mathbb{Z}/4)$. Dolgachev proves that this is in fact isomorphic to $\mathbb{K}^{a}_{M^0} \cong \mathfrak{h}/\Gamma_0(2)^+$ [Dol96, Theorem 8.2]. Precisely, the components of the toric boundary divisor of the ambient toric variety restrict to elements of $\text{Pic}(Q_\lambda/\Pi)$, which span a primitive sublattice isomorphic to $M$ (see also [Roh04]).

We can compactify $\mathfrak{h}/\Gamma_0(2)^+$ to an orbifold $\mathbb{P}^1$, by adding a point at $\lambda^4 = \infty$. The orbifold has 3 special points: the ‘cusp’ $\lambda^4 = \infty$ which must be removed to get $\mathbb{K}_M$, the nodal (or pseudo-ample) point $\lambda^4 = 1$ which must further be removed to get $\mathbb{K}^{a}_{M^0}$, and the order-4 orbifold point $\lambda^4 = 0$.

\textbf{Example 2.15} Suppose $M^0 = \langle 2 \rangle$. Then
\[ \mathbb{K}^{a}_{M^0} = \{ \text{double covers of } \mathbb{P}^2 \text{ branched along smooth sextics} \}. \]

The one-dimensional family of hypersurfaces
\begin{equation}
P_\lambda = \{ x_0^6 + x_1^6 + x_2^6 + x_3^6 + \lambda x_0 x_1 x_2 x_3 \} \subset \mathbb{P}(1, 1, 1, 3)
\end{equation}
are all invariant under a group $\Pi' = \mathbb{Z}/6 \times \mathbb{Z}/2$ of diagonal projective transformations. The simultaneous crepant resolution of the quotient defines the Greene–Plesser mirror family to the family of double planes, and Dolgachev shows that it yields a universal family of $M$-polarized surfaces, with the components of the toric boundary divisors yielding the embedding of $M$ in the Picard lattice as before (see [Dol96, Example 8.3] and [Roh04]).
The quotient \( \mathfrak{h}/\Gamma_0(1)^+ \) again compactifies to an orbifold \( \mathbb{P}^1 \) with 3 special points: the cusp, a nodal/pseudo-ample point, and an order 3 orbifold point.

We now record some consequences of the explicit descriptions of \( \mathbb{K}_M \) that were given in these two Examples. These can of course be proven more directly, by the techniques that Dolgachev uses to arrive at these explicit descriptions.

**Lemma 2.16** Suppose \( M^o = \langle 2 \rangle \) or \( \langle 4 \rangle \). Then \( \Gamma(M)^+ \) acts transitively on \( \Delta(U \oplus M^o)/\pm \text{id} \).

**Proof** There is a bijective correspondence between classes \([\delta] \in \Delta(U \oplus M^o)/\pm \text{id}\) and pseudo-ample points \( \delta^\perp \in D^a_M \) that must be removed to obtain \( D^a_M \). Therefore there is a bijective correspondence between orbits of the action of \( \Gamma(M)^+ \) on \( \Delta(U \oplus M^o)/\pm \text{id} \) and pseudo-ample points in \( \mathbb{K}_M \setminus \mathbb{K}_M^a \). We have seen in Examples 2.14 and 2.15 that \( \mathbb{K}_M \) contains a unique pseudo-ample point when \( M^o = \langle 2 \rangle \) or \( \langle 4 \rangle \).

**Lemma 2.17** Suppose \( M^o = \langle 2 \rangle \) (respectively, \( M^o = \langle 4 \rangle \)). Then we have identifications

\[
\pi_1(\mathbb{K}_M^a) \xrightarrow{(11)} \Gamma(M)^+ \\
\cong \frac{\mathbb{Z} \ast \mathbb{Z}}{\mathbb{Z}/p} \xrightarrow{\cong} \frac{\mathbb{Z}/2 \ast \mathbb{Z}}{\mathbb{Z}/p},
\]

where \( p = 3 \) (respectively, \( p = 4 \)) and the homomorphism on the bottom row is the obvious one.

**Proof** The identification \( \pi_1(\mathbb{K}_M^a) \cong \mathbb{Z} \ast \mathbb{Z}/p \) is immediate from the description of \( \mathbb{K}_M^a \) given in Examples 2.14 and 2.15: the factor \( \mathbb{Z} \) corresponds to loops around the pseudo-ample point, and the factor \( \mathbb{Z}/p \) to loops around the orbifold point. By the short exact sequence of groups (11), the kernel of the map \( \pi_1(\mathbb{K}_M^a) \to \Gamma(M)^+ \) is equal to the image of \( \pi_1(D^a_M) \). We recall that \( D^a_M \) is the complement of the infinite set of points \( \delta^\perp \in \mathfrak{h} \) corresponding to \( \delta \in \Delta(U \oplus M^o) \), so its fundamental group is generated by loops around these points. The covering group acts transitively on these points by Lemma 2.16, so the image of the fundamental group is generated by elements conjugate to the image of a loop around a single point \( \delta^\perp \). A loop around a single point \( \delta^\perp \in D^a_M \) maps to a loop going twice around the pseudo-ample point in \( \mathbb{K}_M^a \), which can be chosen to correspond to the element \( \mathbb{2} \in \mathbb{Z} \). Therefore the kernel of the map \( \mathbb{Z} \ast \mathbb{Z}/p \to \Gamma(M)^+ \) is generated by the elements conjugate to \( \mathbb{2} \in \mathbb{Z} \), which means the map can be identified with the projection \( \mathbb{Z} \ast \mathbb{Z}/p \to \mathbb{Z}/2 \ast \mathbb{Z}/p \) as required.

**Corollary 2.18** If \( M^o = \langle 2 \rangle \) or \( \langle 4 \rangle \) and \( \omega \) is generic, then \( G(X, \omega) \) is not generated by Dehn twists, although \( X \) contains a Lagrangian sphere.

**Proof** Considering the action of symplectic mapping classes on homology, and applying Remark 2.11 and Lemma 2.17, we obtain a surjective homomorphism

\[
G(X, \omega) \to \Gamma(M)^+ \cong \mathbb{Z}/2 \ast \mathbb{Z}/p.
\]

The Dehn twist in a given vanishing cycle can be arranged to map to \( 1 \in \mathbb{Z}/2 \). We know that the action of a Dehn twist in \( L \) on homology is given by the Picard–Lefschetz reflection in the corresponding homology class up to sign, \([L] \in \Delta(U \oplus M^o)/\pm \text{id}\). It follows by Lemma 2.16 that all Dehn twists map to elements conjugate to \( 1 \in \mathbb{Z}/2 \). Passing to abelianizations, it follows that the image of a Dehn twist under the map

\[
G(X, \omega) \to (\mathbb{Z}/2 \ast \mathbb{Z}/p)\text{ab} = \mathbb{Z}/2 \oplus \mathbb{Z}/p
\]

is \((1, 0)\), so Dehn twists can not generate \( G(X, \omega) \).
3 Homological mirror symmetry

A version of homological mirror symmetry was proved for certain $K3$ surfaces in [SS17]. In this section we recall the precise statement, and give some immediate formal consequences of it. These formal consequences are not specific to the setting of mirror symmetry for $K3$ surfaces: they should work the same for general compact Calabi–Yau mirror varieties. We will prove our main results in Section 5 by combining these formal consequences with geometric inputs specific to $K3$ surfaces.

3.1 Homological mirror symmetry statement

Let $\Lambda$ denote the universal Novikov field over $\mathbb{C}$:

\begin{equation}
\Lambda := \left\{ \sum_{j=0}^{\infty} c_j \cdot q^{\lambda_j} : c_j \in \mathbb{C}, \lambda_j \in \mathbb{R}, \lim_{j \to \infty} \lambda_j = +\infty \right\}.
\end{equation}

It is an algebraically closed field extension of $\mathbb{C}$, with a non-Archimedean valuation

\begin{equation}
\text{val} : \Lambda \to \mathbb{R} \cup \{\infty\}
\end{equation}

\begin{equation}
\text{val} \left( \sum_{j=0}^{\infty} c_j \cdot q^{\lambda_j} \right) := \min_j \{ \lambda_j : c_j \neq 0 \}.
\end{equation}

If $c_1(X) = 0$, then the Fukaya category $\mathcal{F}(X, \omega)$ is a $\Lambda$-linear $\mathbb{Z}$-graded (non-curved) $A_\infty$-category whose objects are Lagrangian branes: closed Lagrangian submanifolds $L \subset X$ equipped with orientations, gradings and Pin structures. A strictly unobstructed Lagrangian brane is a pair $(L, J_L)$ where $L$ is a Lagrangian brane and $J_L$ an $\omega$-compatible almost-complex structure such that there are no non-constant $J_L$-holomorphic spheres intersecting $L$, nor non-constant $J_L$-holomorphic discs with boundary on $L$. For the purposes of this paper we define $\mathcal{F}(X, \omega)$ to be the Fukaya category of strictly unobstructed Lagrangian branes, since this can be constructed by classical means [Sei15, Sei14]. This restriction on the objects is harmless when $X$ has real dimension $\leq 4$ (as is the case for $K3$ surfaces), because any Lagrangian brane $L$ admits a $J_L$ turning it into a strictly unobstructed Lagrangian brane by Lazzarini’s theorem [Laz00].

Let $(X, \omega)$ be as above, and $X^\circ$ a smooth Calabi–Yau algebraic surface over $\Lambda$. We say that $(X, \omega)$ is homologically mirror to $X^\circ$ if there is a $\mathbb{Z}$-graded $\Lambda$-linear $A_\infty$ quasi-equivalence

\begin{equation}
\mathcal{D}\mathcal{F}(X, \omega) \simeq \mathcal{D}(X^\circ)
\end{equation}

where the left-hand side denotes the split-closure of the category of twisted complexes on the Fukaya category (i.e., $\mathcal{D}\mathcal{C} := \Pi(Tw \mathcal{C})$ in the notation of [Sei08a]), and the right-hand side denotes a dg enhancement of the bounded derived category of coherent sheaves on $X^\circ$.

We recall Batyrev’s construction of mirror Calabi–Yau hypersurfaces in toric varieties [Bat94], in the two-dimensional case. Fix polar dual reflexive three-dimensional lattice polytopes $\Delta$ and $\Delta^\circ$. Let $\Xi_0$ denote the set of boundary lattice points of $\Delta^\circ$ which do not lie in the interior of a codimension-one facet. Fix a vector $\lambda \in (\mathbb{R}_{>0})^{\Xi_0}$. On the $A$-side, we will consider hypersurfaces in resolutions of the toric variety associated to $\Delta$. The elements of $\Xi_0$ index the toric divisors which intersect such a hypersurface, and the coefficients $\lambda_\kappa$ of $\lambda$ will determine the cohomology class of the Kähler form on the hypersurface.
The vector $\lambda$ determines a function $\psi_\lambda : \Delta^0 \to \mathbb{R}$ as the smallest convex function such that $\psi_\lambda(0) = 0$ and $\psi_\lambda(\kappa) \geq -\lambda_\kappa$. We assume:

\begin{itemize}
  \item[(*)] The decomposition of $\Delta^0$ into domains of linearity of $\psi_\lambda$ is a simplicial refinement $\Sigma$ of the normal fan $\overline{\Sigma}$ to $\Delta$, whose rays are generated by the elements of $\Xi_0$.
\end{itemize}

**Remark 3.1** In the language of [CK99, Section 6.2.3], condition (*) holds if and only if $\lambda$ lies in the interior of the cone $\text{cpl}(\Sigma)$ of the secondary fan (or Gelfand–Kapranov–Zelevinskij decomposition) of $\Xi_0 \subset \mathbb{R}^3$, where $\Sigma$ is a simplified projective subdivision of $\overline{\Sigma}$.

The morphism of fans $\Sigma \to \overline{\Sigma}$ induces a birational morphism of the corresponding toric varieties $Y \to \tilde{Y}$, and $Y$ is an orbifold. We will consider a smooth Calabi-Yau hypersurface $X \subset Y$ which avoids the orbifold points, and which is the proper transform of $\tilde{X} \subset \tilde{Y}$, a hypersurface with Newton polytope $\Delta$. There is a toric $\mathbb{R}$-Cartier divisor $D_\lambda = \sum_\kappa \lambda_\kappa \cdot D_\kappa$ with support function $\psi_\lambda$, which determines a toric Kähler form on $Y$, which restricts to a Kähler form $\omega_\lambda$ on $X$. Its cohomology class $[\omega_\lambda]$ is the restriction of $PD(D_\lambda)$ to $X$. The pair $(X, \omega_\lambda)$ depends up to symplectomorphism only on $(\Delta, \lambda)$.

On the $B$-side, the polytope $\Delta^0$ defines a line bundle over the toric variety $\tilde{Y}^0$, whose global sections have a basis indexed by the lattice points of $\Delta^0$. For any $d \in \Lambda \Xi_0$ we have a corresponding hypersurface

$$X_d^0 := \left\{ -\chi_0^0 + \sum_{\kappa \in \Xi_0} d_\kappa \cdot \chi_\kappa = 0 \right\} \subset \tilde{Y}^0.$$ 

**Theorem 3.2** Let $\Delta$ and $\Delta^0$ be polar dual reflexive three-dimensional simplices, and let $\lambda \in (\mathbb{R}_{>0})^{\Xi_0}$ satisfy (*). Then there exists a $d \in \Lambda \Xi_0$, with $\text{val}(d) = \lambda$, such that $(X, \omega_\lambda)$ is homologically mirror to $X_d^0$.

**Proof** This is [SS17, Theorem C]. We remark that the ‘embeddedness’ and ‘no bc’ conditions of op. cit. are automatic in this case. \hfill \Box

**Remark 3.3** The fact that $\text{val}(d) = \lambda$, with $\lambda$ satisfying (*), implies that $X_d^0$ is smooth, see [SS17, Proposition 4.4]. There is a precise conjectural description of the “mirror map” $\lambda \mapsto d(\lambda)$, but we shall not need it in this paper.

**Remark 3.4** Theorem 3.2 should continue to hold without the assumption that $\Delta$ and $\Delta^0$ are simplices, but it has not been proved.

3.2 Examples revisited

In the setting of Theorem 3.2, the relevant hypersurfaces can be understood as resolutions of quotients of hypersurfaces in (weighted) projective space, making a connection to the examples of Section 2.6.

\footnote{In general, this is a toric stack, and one should consider the stacky derived category of hypersurfaces in this stack; in the situations relevant to our applications, the hypersurfaces will be smooth as schemes, and the stacky derived category coincides with the derived category of the underlying scheme.}
Example 3.5  Let $\Delta$ be the reflexive 3-simplex with vertices $\{(1, 0, 0), (0, 1, 0), (0, 0, 1), (-1, -1, -1)\}$. Then $\nabla^\circ = \mathbb{P}^3$ is smooth, and the anticanonical hypersurface $X^\circ_{\nabla}$ is a quartic surface over $\Lambda$. The polar dual $\Delta^\circ$ has vertices $\{(3, -1, -1), (-1, 3, -1), (-1, -1, 3), (-1, -1, -1)\}$, and the toric variety associated to $\Delta^\circ$ is

$$\nabla = [\mathbb{P}^3 / \Pi] \cong \{x_0^4 - x_1 x_2 x_3 x_4 = 0\} \subset \mathbb{P}^4$$

where $\Pi = (\mathbb{Z}/4)^2$ is as in Example 2.14. Anticanonical hypersurfaces in $\nabla$ are hyperplane sections; these include the $\Pi$-invariant quartics $Q_0 \subset \mathbb{P}^3$ considered in (17), and in particular the Fermat quartic $Q_0$. $X$ is the crepant resolution of $Q_0 / \Pi$, resolving the 6 $A_3$-singularities. It supports a 19-dimensional family of ambient Kähler forms. Theorem 3.2 gives a quasi-equivalence

$$\mathcal{D}^f(X, \omega) \simeq \mathcal{D}(X^\circ_{\nabla}); \quad X^\circ_{\nabla} \subset \mathbb{P}^3$$

for ambient Kähler forms $\omega$ satisfying (*). The valuations of the coefficients $d_\kappa$ in 19 of the 20 quartic monomials defining $X^\circ_{\nabla}$ are determined by the $\omega$-areas of the exceptional resolution curves in $X$ and by its total volume, and the coefficient of the 20th monomial (corresponding to the coefficient of the point of $\Delta^\circ$) is $-1$.

Example 3.6  Let $\Delta$ be the reflexive 3-simplex given by the convex hull $\text{Conv}(0, 6e_1, 6e_2, 2e_3) - (1, 1, 1)$, so with vertices $\{(-1, -1, -1), (5, -1, -1), (-1, 5, -1), (-1, -1, 1)\}$. Then the toric variety $\nabla^\circ = \mathbb{P}(1, 1, 1, 3)$. The polar dual again defines a toric variety which is an orbifold quotient $[\mathbb{P}(1, 1, 1, 3) / \Pi']$ with $\Pi'$ as in Example 2.15. In the notation of that example, let $X$ denote the crepant resolution of $P_0 / \Pi'$, a hypersurface in the toric resolution of $[\mathbb{P}(1, 1, 1, 3) / \Pi']$. This supports a 19-dimensional family of ambient Kähler forms. Under hypothesis (*), Theorem 3.2 gives

$$\mathcal{D}^f(X, \omega) \simeq \mathcal{D}(X^\circ_{\nabla}); \quad X^\circ_{\nabla} \subset \mathbb{P}(1, 1, 1, 3)$$

a double plane, where $X^\circ_{\nabla} \to \mathbb{P}^2 \supset \Sigma^\circ_d$ is the double branched cover of $\mathbb{P}^2$ branched over a smooth sextic curve $\Sigma^\circ_d$ whose defining equation depends via the mirror map on $\omega$.

Remark 3.7  Fix a lattice $M$ and suppose one has proven mirror symmetry for $M$-polarized $K3$ surfaces $X$ (identifying the Fukaya category of the $M$-polarized $X$ with the derived category of the $M^\circ$-polarized $X^\circ$). The mirror map identifies the part of the Kähler moduli space of $X$ consisting of (complexified) Kähler forms whose cohomology class lies in $M^\vee$ with a neighbourhood of a maximally unipotent cusp in $\mathbb{H}$. Restricting the mirror map to the locus of $M'$-polarized surfaces shows that mirror symmetry for $M$-polarized surfaces implies that for $M'$-polarized surfaces, for any sublattice $M' \subset M$; so it is of most interest to address cases where the $A$-side lattice $M$ has rank as large as possible. If $X^\circ$ is to have a well-behaved bounded derived category then it should be projective, meaning $\text{rk}(M^\circ) \geq 1$; therefore the “hardest case” is $\text{rk}(M) = 19$. Of the infinitely many possible lattices $M^\circ = (2n)$, with $n \in \mathbb{Z}_{>0}$, exactly $n \in \{1, 2\}$ arise from the Greene–Plesser construction.

Remark 3.8  In [SS17, Section 1.7.2] we elaborate a version of homological mirror symmetry

\begin{equation}
\mathcal{D}^f(X_{20}, \omega)^{bc} \simeq \mathcal{A}_{d(\omega)}
\end{equation}

for (on the $A$-side) an $M$-polarized $K3$ surface $X_{20}$ where $\text{rk}(M) = 20$; the mirror is not a $K3$ surface (which would have Picard rank 0), but Kuznetsov’s $K3$-category $\mathcal{A}_{d}$ of a cubic four-fold [Kuz10]. $M$ contains rank 19 lattices $M_n$ with $M_n^\circ = (2n)$ for each $n$ satisfying [Huy17, Condition (***)], namely

\begin{equation}
n \equiv 0, 2 (6) \quad \text{and} \quad a_i \equiv 0 (2) \quad \text{for all primes} \quad p_i \equiv 2 (3) \quad \text{where} \quad 2n = \prod_i p_i^{a_i}.
\end{equation}
One could hope, via strengthenings of the results of [AT14] and [Huy17], to deduce from (24) homological mirror symmetry for (on the $B$-side) any K3 surface $X$ with $\langle 2n \rangle \leq \text{Pic}(X)$ for some $n$ satisfying (25). The $A$-side mirrors would then give further examples of symplectic K3 surfaces to which our main results (e.g., Theorem 1.1) apply.

### 3.3 Hochschild homology and Chern characters

We recall that the Hochschild homology $HH_\ast(D)$ of an $A_\infty$ category $D$ is a graded vector space. For the Fukaya category we have the open–closed map
\begin{equation}
\mathcal{O}_C : HH_\ast(D\mathcal{F}(X)) \to H^{n+}(X; \Lambda),
\end{equation}
which under certain hypotheses is an isomorphism. For example, this is a formal consequence of the existence of a homological mirror $X^\circ$ which is *maximally unipotent* (see [GPS15, Theorem 5.2]) or *smooth* (see [Gan16, Corollary 7]), and in particular holds in the context of Theorem 3.2. For the derived category, [Kel98] provides an isomorphism
\begin{equation}
HH_\ast(D(X^\circ)) \cong HH_\ast(X^\circ)
\end{equation}
where the right-hand side denotes the Hochschild homology of the variety $X^\circ$.

Thus, if $X$ and $X^\circ$ are homologically mirror in the sense of (22), we have an isomorphism of graded vector spaces
\begin{equation}
H^{n+}(X; \Lambda) \cong HH_\ast(X^\circ)
\end{equation}
as an immediate consequence.

We now recall the Chern character, which is a map
\[ Ch : K_0(D) \to HH_0(D) \]
defined for any $A_\infty$ category $D$, and the Mukai pairing, which is a graded bilinear pairing
\[ \langle \cdot, \cdot \rangle : HH_\ast(D) \otimes HH_\ast(D) \to k \]
defined for any proper $A_\infty$ category $D$. These are introduced by Shklyarov in the case of $dg$ categories [Shk13], where he proved that together they satisfy the *noncommutative Hirzebruch–Riemann–Roch theorem*:
\begin{equation}
\langle Ch(K), Ch(L) \rangle = -\chi(K, L).
\end{equation}

They extend immediately to the $A_\infty$ case, since any $A_\infty$ category is quasi-equivalent to a $dg$ category (the $A_\infty$ case is also treated explicitly in [She15]).

As a consequence, the isomorphism (28) intertwines Mukai pairings and Chern characters.

**Remark 3.9** In the case of the derived category, the composition
\[ K_0(D(X^\circ)) \xrightarrow{Ch} HH_0(D(X^\circ)) \xrightarrow{\mathcal{O}_C} HH_0(X^\circ) \xrightarrow{HKR} \bigoplus_H H^p (\Omega^H_{X^\circ}) \]
coincides with the usual Chern character [Câl05].

**Remark 3.10** In the case of the Fukaya category, the composition
\[ K_0(D\mathcal{F}(X)) \xrightarrow{Ch} HH_0(D\mathcal{F}(X)) \xrightarrow{\mathcal{O}_C} H^n(X; \Lambda) \]
takes a Lagrangian $L$ to $PD([L])$, the Poincaré dual of its homology class (we have arranged that $L$ bounds no non-constant holomorphic discs, and the constant discs sweep out the fundamental cycle of $L$).

### 3.4 Enhancements and autoequivalences

Bayer and Bridgeland’s theorem concerns autoequivalences of triangulated categories, but our version of homological mirror symmetry concerns $A_\infty$ categories. It will be necessary for us to use both notions, so we summarize some background material on the relationship between the two.

If $\mathcal{D}$ is an $A_\infty$ category, then we define $\text{Auteq}(\mathcal{D})$ to be the group of $A_\infty$ functors $F : \mathcal{D} \rightarrow \mathcal{D}$, such that $H^0(F)$ is an equivalence, considered up to isomorphism in $H^0(\text{nu-fun}(\mathcal{D}, \mathcal{D}))$. There is a natural homomorphism $\text{Auteq} \mathcal{D} \rightarrow \text{Aut} HH_*(\mathcal{D})$ by functoriality of Hochschild homology (where ‘Aut’ denotes graded linear automorphisms). Thus, if we have a quasi-equivalence $\mathcal{D} \mathcal{F}(X) \simeq \mathcal{D}(X^\circ)$, we obtain an isomorphism of exact sequences

$$1 \longrightarrow \text{Auteq}^0 \mathcal{D}(X) \longrightarrow \text{Auteq} \mathcal{D}(X) \longrightarrow \text{Aut} HH_*(\mathcal{D}(X)) \longrightarrow 1$$

where $\text{Auteq}^0$ is the subgroup of $\text{Auteq}$ consisting of autoequivalences acting trivially on $HH_*$. On the other hand, if $\mathcal{D}$ is a triangulated category, we define $\text{Auteq}(\mathcal{D})$ to be the group of exact (a.k.a. triangulated) autoequivalences of $\mathcal{D}$, considered up to isomorphism. If $\mathcal{D}$ is a triangulated $A_\infty$ category, then the cohomological category $H^0(\mathcal{D})$ has a natural triangulated structure, and there is a homomorphism $\text{Auteq} \mathcal{D} \rightarrow \text{Auteq} H^0(\mathcal{D})$ sending $F \mapsto H^0(F)$. This makes use of the following result: if $F$ is an $A_\infty$ functor between triangulated $A_\infty$ categories, then $H^0(F)$ is exact [Sei08a, Proposition 3.14].

The homomorphism $\text{Auteq} \mathcal{D} \rightarrow \text{Aut} HH_*(\mathcal{D})$ does not have an analogue in the general setting of triangulated categories; in the particular geometric setting that we consider, however, there is a substitute. Namely, we consider the derived category $\mathcal{D}(X^\circ)$ of a $K3$ surface $X^\circ$. Any autoequivalence of $\mathcal{D}(X^\circ)$ is of Fourier–Mukai type, and the Fourier–Mukai kernel induces an action on $HH_*(X^\circ)$. Thus we have a map $\text{Auteq} \mathcal{D}(X^\circ) \rightarrow \text{Aut} HH_*(X^\circ)$.

In this setting, $\mathcal{D}(X^\circ)$ has a unique dg (in particular, $A_\infty$) enhancement $\mathcal{D}(X^\circ)$ by [LO10], and the results of [Toë06, LS16] imply that $\text{Auteq} \mathcal{D}(X^\circ) \rightarrow \text{Auteq} \mathcal{D}(X^\circ)$ is in fact an isomorphism. There is an isomorphism $HH_*(X^\circ) \cong HH_*(\mathcal{D}(X^\circ))$ by [Kel98], and the homomorphism $\text{Auteq} \mathcal{D}(X^\circ) \rightarrow HH_*(\mathcal{D}(X^\circ))$ gets identified with $\text{Auteq} \mathcal{D}(X^\circ) \rightarrow \text{Aut} HH_*(X^\circ)$. Thus we can replace ‘$\mathcal{D}(X^\circ)$’ with ‘$\mathcal{D}(X^\circ)$’ and ‘$HH_*(\mathcal{D}(X^\circ))$’ with ‘$HH_*(X^\circ)$’ in (30).

Now we consider the subgroup $\text{Auteq}_{CY} \mathcal{D}(X^\circ) \subset \text{Auteq} \mathcal{D}(X^\circ)$ that appears in Corollary 2.13. Bayer and Bridgeland give an alternative characterization of this subgroup which will be useful to us. Recall that, since $X^\circ$ is a 2-dimensional Calabi–Yau variety, $\mathcal{D}(X^\circ)$ admits a 2-Calabi–Yau structure: in particular, there are functorial Serre duality pairings

$$\text{Hom}_{D(X^\circ)}(K, L) \otimes \text{Hom}_{D(X^\circ)}(L, K[2]) \rightarrow k$$

induced by a choice of holomorphic volume form. An autoequivalence is called $\text{Calabi–Yau}$ if it respects the Serre duality pairings. Bayer and Bridgeland show that $\text{Auteq}_{CY} \mathcal{D}(X^\circ) \subset \text{Auteq} \mathcal{D}(X^\circ)$
In the case at hand, the connected component of $\text{Symp}^\otimes(X)$ coincides with the subgroup of autoequivalences acting trivially on $HH_2(X^\circ)$ (see [BB17, Appendix A]).

**Definition 3.11** A weak proper $n$-Calabi–Yau structure on an $A_\infty$-category $\mathcal{D}$ over $k$ is an element $\phi \in HH_n(\mathcal{D})^\vee$ with the property that the pairing

$$\text{Hom}_\mathcal{D}^n(K, L) \otimes \text{Hom}_\mathcal{D}^{n-\ast}(L, K) \xrightarrow{\mu_\mathcal{D}} \text{Hom}_\mathcal{D}^n(L, L) \to HH_n(\mathcal{D}) \xrightarrow{\phi} k$$

is non-degenerate, for any objects $K, L$.

We say that $\psi \in \text{Auteq}\mathcal{D}$ is *Calabi–Yau* if it preserves $\phi$. We denote the subgroup of Calabi–Yau autoequivalences by $\text{Auteq}_{\text{CY}}\mathcal{D}$.

**Remark 3.12** Observe that if $HH_n(\mathcal{D})$ is one-dimensional, then all weak proper $n$-Calabi–Yau structures $\phi$ are scalar multiples of each other: it follows that the subgroup $\text{Auteq}_{\text{CY}}\mathcal{D} \subset \text{Auteq}\mathcal{D}$ coincides with the subgroup of autoequivalences which act trivially on $HH_n(\mathcal{D})$, and in particular does not depend on the choice of Calabi–Yau structure $\phi$. This holds, in particular, in the setting of (30), because $HH_2(\mathcal{D}(X^\circ)) \cong HH_2(X^\circ) \cong H^2(O_{X^\circ})$ has rank 1.

If $\mathcal{D}$ is a triangulated $A_\infty$ category, then a weak proper $n$-Calabi–Yau structure on $\mathcal{D}$ induces an $n$-Calabi–Yau structure on the triangulated category $H^0(\mathcal{D})$. If $F \in \text{Auteq}\mathcal{D}$ is Calabi–Yau, then $H^0(F) \in \text{Auteq} H^0(\mathcal{D})$ is Calabi–Yau. Thus we have a homomorphism $\text{Auteq}_{\text{CY}}\mathcal{D} \to \text{Auteq}_{\text{CY}} H^0(\mathcal{D})$.

In the case at hand, $HH_2(\mathcal{D}(X^\circ))$ has rank 1 by the HKR isomorphism, so $\text{Auteq}_{\text{CY}}\mathcal{D}(X^\circ)$ coincides with the subgroup of autoequivalences which act trivially on $HH_2(\mathcal{D}(X^\circ))$ by Remark 3.12. This corresponds to the subgroup of $\text{Auteq} D(X^\circ)$ consisting of autoequivalences which act trivially on $HH_2(X^\circ)$, which Bayer and Bridgeland prove is precisely $\text{Auteq}_{\text{CY}} D(X^\circ)$. As a result, we have an isomorphism of exact sequences:

$$\begin{array}{ccccccccc}
1 & \to & \text{Auteq}^0 D(X^\circ) & \to & \text{Auteq}_{\text{CY}} D(X^\circ) & \to & \text{Aut} HH_n(\mathcal{D}(X^\circ)) & \cong & \text{Auteq}^0 D(X^\circ) & \to & \text{Auteq}_{\text{CY}} D(X^\circ) & \to & \text{Aut} HH_n(X^\circ).
\end{array}$$

### 3.5 The symplectic mapping class group acts on the Fukaya category

Now recall [Sei00] that the group of graded symplectomorphisms $\text{Symp}^\otimes(X)$ is a central extension by $\mathbb{Z}$ of the group of all symplectomorphisms $\text{Symp}(X)$ of $X$. The Hamiltonian subgroup $\text{Ham}^\otimes(X)$ is the connected component of $\text{Symp}^\otimes(X)$ containing the identity, where we use the `Hamiltonian topology’ on $\text{Symp}(X)$ (see [Sei08b, Remark 0.1] for the definition: when $H^1(X; \mathbb{R}) = 0$, there is no distinction between symplectic and Hamiltonian isotopy, so the Hamiltonian topology coincides with the $C^\infty$ topology). In particular we have an isomorphism $\text{Symp}^\otimes(X)/\text{Ham}^\otimes(X) \cong \pi_0 \text{Symp}^\otimes(X)$.

**Lemma 3.13** There is a natural homomorphism

$$\pi_0 \text{Symp}^\otimes(X) \to \text{Auteq} \mathcal{D} \mathfrak{F}(X).$$
Proof. The construction of a homomorphism $\text{Symp}^g \rightarrow \text{Auteq} \mathcal{F}$ is explained in the setting of exact manifolds in [Sei08a, Section 10c], and the proof carries over to the strictly unobstructed setting in which we work. We compose this homomorphism with the homomorphism

$$\text{Auteq} \mathcal{F} \rightarrow \text{Auteq} \mathcal{DF},$$

which exists for any $A_{\infty}$ category $\mathcal{F}$.

We now recall that $\mathcal{F}(X)$ admits a canonical weak proper $n$-Calabi–Yau structure $\phi$, in the sense of Section 3.4 (where $n$ is the complex dimension of $X$). It is defined by

$$\phi(\alpha) := \langle OC(\alpha), e \rangle$$

(33)

where $OC : HH_*(\mathcal{F}(X)) \rightarrow H^{*+n}(X; \Lambda)$ is the open–closed string map, $\langle \cdot, \cdot \rangle$ is the intersection pairing, and $e \in H^0(X; \Lambda)$ is the unit (compare [She16, Section 2.8]). This induces a weak proper $n$-Calabi–Yau structure on $\mathcal{DF}(X)$, by Morita invariance of Hochschild homology.

Lemma 3.14. The map (32) lands in $\text{Auteq}_{\text{CY}} \mathcal{DF}(X)$, the subgroup of Calabi–Yau autoequivalences.

Proof. It suffices to show that any graded symplectomorphism $\psi$ induces an $A_{\infty}$ endofunctor $\psi_* : \text{Auteq} \mathcal{F}(X)$ whose action on Hochschild homology preserves the class $\phi \in HH_*(\mathcal{F}(X))^\Lambda$. This follows because $(\psi^{-1})^*$ acts on $H^*(X; \Lambda)$ preserving the unit $e$ and the pairing $\langle \cdot, \cdot \rangle$, and the open–closed map $OC$ is natural with respect to the action of $\psi$.

We denote the element of $\text{Symp}^g(X)$ corresponding to $k \in \mathbb{Z}$ by $[k]$. That is because the homomorphism (32) sends $[k]$ to the shift functor $[k]$. The central extension $\text{Symp}^g(X)/[2]$ of $\text{Symp}(X)$ by $\mathbb{Z}/2$ is canonically split (see, e.g., [She17, Section B.5]), so we obtain a homomorphism

$$\text{Symp}(X)/\text{Ham}(X) \rightarrow \text{Auteq}_{\text{CY}} \mathcal{DF}(X)/[2].$$

Corollary 3.15. If the open–closed map $OC : HH_*(\mathcal{F}(X)) \rightarrow H^{*+n}(X; \Lambda)$ is an isomorphism, then there is a natural homomorphism of exact sequences

$$1 \longrightarrow \text{I}(X, \omega) \longrightarrow \text{G}(X, \omega) \longrightarrow \text{Aut } H^*(X; \mathbb{Z}) \downarrow \downarrow \downarrow$$

$$1 \longrightarrow \text{Auteq}^0 \mathcal{DF}(X, \omega) \longrightarrow \text{Auteq}_{\text{CY}} \mathcal{DF}(X, \omega) \longrightarrow \text{Aut } HH_*(\mathcal{DF}(X, \omega)).$$

Here the rightmost vertical arrow is obtained by identifying $HH_*(\mathcal{DF}) \cong HH_*(\mathcal{F})$ via Morita invariance, and $HH_*(\mathcal{F}) \cong H^{*+n}(X; \mathbb{Z}) \otimes \Lambda$ via $OC$. Commutativity follows from the naturality of $OC$.

4. $K3$ surfaces over the Novikov field

A $K3$ surface over an arbitrary field $k$ is an algebraic surface $X$ over $k$ with $H^1(X, \mathcal{O}_X) = 0$ and $K_X \cong \mathcal{O}_X$. This again has a Picard group scheme $\text{Pic}(X)$; we let $\text{NS}(X)$ denote the Neron–Severi group, which is the quotient $\text{Pic}(X)/\text{Pic}^0(X)$ of the Picard group by its identity component. The fact that $H^1(X, \mathcal{O}_X) = 0$ implies that $\text{Pic}(X)$ consists of rigid isolated points, so $\text{Pic}(X) = \text{NS}(X)$.
4.1 Lefschetz principle

We will use the “Lefschetz principle” (see e.g. [Ekl73]) to translate results about complex $K3$ surfaces into corresponding results about $K3$ surfaces over an arbitrary field of characteristic zero (of course, we have the Novikov field $\Lambda$ in mind). The results we collect here are taken from [Huy16], see also [Huy10, Proposition 5.4], and will be standard in the relevant community. We include a discussion since the results may be less familiar to symplectic topologists.

Let $K$ be a field of characteristic zero, and $X$ a $K3$ surface over $K$. We can define $X$ using only a finite number of elements of $K$, so there exists a finitely-generated field $\mathbb{Q} \subset k_0 \subset K$ and a variety $X_0$ over $k_0$ such that $X = X_0 \times_{k_0} K$. We can embed $k_0$ in $\mathbb{C}$, so we obtain a variety $X_\mathbb{C} = X_0 \times_{k_0} \mathbb{C}$ (which depends on the choice of embedding). Using the fact that flat base change commutes with coherent cohomology, one can show that $X_0$, and therefore $X_\mathbb{C}$, are also $K3$ surfaces. We call $X_\mathbb{C}$ a “complex model” of $X$; the basic idea of the Lefschetz principle is to translate results about the complex $K3$ surface $X_\mathbb{C}$ into results about $X$.

**Lemma 4.1**  Let $X$ be a $K3$ surface over an algebraically closed field $k$, $K/k$ a field extension, and $X_K := X \times_k K$. Then the pull-back map $\text{Pic}(X) \to \text{Pic}(X_K)$ is a bijection.

**Sketch**  This is [Huy16, Chapter 17, Lemma 2.2]. To prove injectivity of the pull-back map, we observe that a line bundle $L \in \text{Pic}(X)$ is trivial if and only if the composition map

$$\text{Hom}(L, O_X) \otimes \text{Hom}(O_X, L) \to \text{Hom}(O_X, O_X) \cong k$$

is non-zero. Since coherent cohomology commutes with flat base change, this holds for $L$ if and only if it holds for the pull-back of $L$ to $X_K$ (note that this is true even if $k$ is not algebraically closed).

Surjectivity relies on the fact that $k$ is algebraically closed. Any line bundle on $X_K$ can be defined using finitely many elements of $K$, hence is defined over some finitely-generated extension of $k$, which is the quotient field of a finitely-generated $k$-algebra $A$. Localizing $A$ with respect to finitely many denominators, we may assume that the line bundle can in fact be defined over $A$, so it can be viewed as a family of line bundles on $X$ parametrized by $\text{Spec}(A)$. This family is classified by a morphism $\text{Spec}(A) \to \text{Pic}(X)$. The Picard scheme $\text{Pic}(X)$ is reduced and has dimension equal to the rank of $H^1(X, O_X)$, which for a $K3$ surface is 0. Since $k$ is algebraically closed, it follows that $\text{Pic}(X)$ is simply a disjoint union of points: so the classifying morphism must be constant, which means that the line bundle is pulled back from $X$ as required.

**Remark 4.2**  A crucial point in the proof of surjectivity in Lemma 4.1 was that line bundles on a $K3$ surface are rigid: they do not admit non-trivial deformations.

**Corollary 4.3**  Let $X$ be a $K3$ surface over an algebraically closed field of characteristic zero, and $X_\mathbb{C}$ a complex model. Then $\text{Pic}(X) \cong \text{Pic}(X_\mathbb{C})$.

**Proof**  We observe that the algebraic closure $\overline{k_0}$ of $k_0$ embeds into $\mathbb{C}$ and $K$, so the restriction maps

$$\text{Pic}(X) \to \text{Pic}(X_0 \times_{k_0} \overline{k_0}) \leftarrow \text{Pic}(X_\mathbb{C})$$

are isomorphisms by Lemma 4.1.

**Remark 4.4**  It follows immediately that any $K3$ surface over an algebraically closed field of characteristic zero has Picard rank $\leq 20$, since this is true of complex $K3$ surfaces. This is not true for $K3$ surfaces in finite characteristic.
**Remark 4.5** Although Corollary 4.3 shows that different complex models for $X$ have isomorphic Picard lattices, they may have different transcendental lattices.

The following two results are discussed in [Huy16, Chapter 16, Section 4.2]:

**Lemma 4.6** Let $X$ be a $K3$ surface over an algebraically closed field of characteristic zero, and $X_C$ a complex model. Then the set of isomorphism classes of spherical objects of $D(X)$ is in bijection with the set of isomorphism classes of spherical objects of $D(X_C)$.

**Proof** The proof follows that of Corollary 4.3, using the fact that spherical objects $E$ are rigid because $\text{Ext}^1(E, E) = 0$ by definition.

**Lemma 4.7** Let $X$ be a $K3$ surface over an algebraically closed field of characteristic zero, and $X_C$ a complex model. Then there is an isomorphism $\text{Auteq} D(X) \cong \text{Auteq} D(X_C)$, inducing isomorphisms:

\[
\begin{array}{c}
1 \to \text{Auteq}^0 D(X_C)/[2] \to \text{Auteq}_{CY} D(X_C)/[2] \to \text{Aut} HH_*(X_C)
\end{array}
\]

The dashed vertical arrow signifies an isomorphism between the images of the rightmost horizontal arrows. In other words, this diagram can be completed to an isomorphism of short exact sequences, by replacing the rightmost terms by the images of the rightmost horizontal arrows.

**Proof** The proof follows that of Corollary 4.3, using the fact that the Fourier–Mukai kernels defining autoequivalences are rigid (because $H^1(X, \mathcal{O}_X) = 0$). The action of autoequivalences on Hochschild homology is compatible with flat base change, so these isomorphisms preserve $\text{Auteq}^0$ (the subgroup acting trivially on $HH_*$) and $\text{Auteq}_{CY}$ (the subgroup acting trivially on $HH_2$). The dashed vertical arrow is induced by the inclusions $HH_*(X_C) \hookrightarrow HH_*(X_0 \times_{k_0} \overline{k}_0) \hookrightarrow HH_*(X)$, which are induced by base change by the inclusions $C \hookrightarrow \overline{k}_0 \hookrightarrow K$.

**Remark 4.8** The previous result does not hold for general varieties, since if $H^1(X; \mathcal{O}_X)$ is non-zero then one can tensor by (flat) line bundles with different structure group after extending scalars; the result relies on the fact that the Picard group is discrete.

Now we will consider point-like objects $E$ of $D(X)$, i.e. objects satisfying $\text{Ext}^*(E, E) \cong \wedge^*(K^{[2]})$. Given such an $E$, we may choose a finitely-generated $\mathbb{Q} \subset k_0 \subset K$ such that $X$ and the object $E$ are defined over $k_0$ (indeed, for any finite set of objects we can choose a finitely-generated field $k_0$ over which they are all defined). Thus we have an object $E_0$ of $D(X_0 \times_{k_0} \overline{k}_0)$ which pulls back to $E$ on $X$ and to $E_C$ on the complex model $X_C$. Since coherent cohomology commutes with flat base change, $E_0$ and $E_C$ are also point-like.

**Lemma 4.9** Let $X$ be a $K3$ surface over an algebraically closed field of characteristic zero, with $\rho(X) = 1$. Then any point-like object $E$ of $D(X)$ has non-zero Chern character $\text{Ch}(E) \neq 0 \in HH_0(X)$. 
Proof First we prove the result assuming $K = \mathbb{C}$. Bayer and Bridgeland show that any point-like (or spherical) object $E$ of $D(X)$ is quasi-stable for some stability condition, meaning that it is semistable and all its stable factors have positively proportional Mukai vector. Indeed, for a stability condition $\sigma$, let the $\sigma$-width of a point-like object be the difference between the phases of the maximal and minimal semistable factors in its Harder–Narasimhan filtration. Then [BB17, Lemma 6.3] shows that there exists a stability condition $\sigma$ such that $E$ has $\sigma$-width 0, and [BB17, Proposition 3.15] shows that either $E$ is $\sigma$-quasi-stable, or it is $\sigma'$-quasi-stable for a stability condition $\sigma'$ near $\sigma$. A quasi-stable object has non-trivial Mukai vector $\nu(E) \neq 0 \in \mathbb{N}(X)$, and therefore non-trivial Chern character $\text{Ch}(E) \neq 0 \in HH_0(X)$ as required.

Now we address the general case. Let $E$ be a point-like object of $D(X)$, and let us choose a complex model $X_{\mathbb{C}}$ for $X$ over which $E$ is defined. Taking Hochschild homology and Chern characters commute with flat base change, so $E$ has non-zero Chern character if and only if $E_{\mathbb{C}}$ does: since $\rho(X_{\mathbb{C}}) = \rho(X) = 1$ by Corollary 4.3, the result follows from the case $K = \mathbb{C}$ proved above.

Lemma 4.10 Let $X$ be a $K3$ surface over an algebraically closed field of characteristic zero, with $\text{Pic}(X) \cong (2n)$ where $n$ is square-free. Then for any point-like object $E$ of $D(X)$, there exists a spherical object $S$ with $\chi(E, S) = 1$.

Proof First we prove the result for $K = \mathbb{C}$. Let $E$ be a point-like object of $D(X)$: we will start by showing that there exists a $K3$ surface $Y$ and an equivalence $\eta : D(X) \tilde{\to} D(Y)$ taking $E$ to the skyscraper sheaf of a point. The set of stability conditions making $E$ quasi-stable is open [BB17, Proposition 2.10], and non-empty since $E$ is point-like, cf. the proof of Lemma 5.19. Thus we can pick a stability condition $\sigma$ which makes $E$ quasi-stable and which is generic in the sense of [BM14]. Suppose the Mukai vector of $E$ is $\nu = m \cdot v_0$ where $v_0$ is primitive and $m \in \mathbb{Z}_+$. The Mukai pairing $(v_0, v_0) = 0$, since $\chi(E, E) = 0$ using the fact that $E$ is point-like. Now, in [BM14, Section 6 and Proof of Lemma 7.2], Bayer and Macri show that there is a non-empty projective moduli stack of $\sigma$-semistable objects with the same Mukai vector as $E$. Let $M(v)$ be its coarse moduli space; then the coarse moduli space of $M(v_0)$ is again a $K3$ surface, which has a distinguished Brauer class $\alpha$. There is a Fourier–Mukai equivalence

$$\eta : D(X) \tilde{\to} D(M(v_0), \alpha)$$

(where the right-hand side denotes the derived category of twisted sheaves), which takes any complex in $D(X)$ defining a point of $M(v)$ to a torsion sheaf on the twisted $K3$ surface $M(v_0)$ of dimension 0 and length $m$. [BM14] proves that $\eta$ identifies $M(v)$ with the $m$-th symmetric product $\text{Sym}^m(M(v_0))$. It follows that the general point of $M(v)$ corresponds, under this identification, to the skyscraper sheaf of an $m$-tuple of pairwise distinct points, which has $\text{Ext}^1$ of rank $2m$. Since $\text{Ext}^1$ varies upper semicontinuously, it follows that $\text{Ext}^1(E, E)$ has rank at least $2m$; since $E$ is point-like, $m = 1$ and $v = v_0$. The derived equivalence $\eta$ then takes $E$ to the skyscraper sheaf of a point on the twisted $K3$ surface $(M(v), \alpha)$.

Recall that a $K3$ surface $Y$ equipped with a Brauer class $\beta$ is called a twisted Fourier–Mukai partner of $X$ if there is an equivalence $D(X) \simeq D(Y, \beta)$: so $(Y, \beta) := (M(v), \alpha)$ is a twisted Fourier–Mukai partner of $X$. Ma gives a formula for the number of twisted Fourier–Mukai partners of $X$ with Brauer class of a given order [Ma10, Proposition 5.1]. His result shows that if $\text{Pic}(X) \cong (2n)$ with $n$ square-free, then any twisted Fourier–Mukai partner of $X$ has trivial Brauer class; since this holds by hypothesis in our case, we must have $\beta = 0$. Now $O_Y$ is a spherical object of $D(Y)$ with $\chi(O_y, O_y) = 1$ for any $y \in Y$, so if we set $S := \eta^{-1}(O_Y)$ then $\chi(E, S) = 1$ as required.
Now we address the general case. Let \( E \) be a point-like object, and let us choose a complex model \( X_C \) for \( X \) over which \( E \) is defined. Then \( E_C \) is a point-like object of \( D(X_C) \), and \( \text{Pic}(X_C) \cong \text{Pic}(X) \) by Corollary 4.3, so there exists a spherical object \( S_C \) of \( D(X_C) \) for which \( \chi(E_C, S_C) = 1 \) by the previous argument. The spherical object descends to \( X_0 \times_{k_0} k_0 \) by Lemma 4.6, so we obtain a spherical object \( S \) of \( D(X) \) with \( \chi(E, S) = 1 \) as required.

### 4.2 Obtaining a Picard rank one mirror

Let \( K \) be an algebraically closed field of characteristic zero. Let \( \rho : X \to M \) be a family of \( K3 \) surfaces over \( K \), i.e. a proper smooth morphism of relative dimension 2 with both the relative dualising sheaf and \( R^1\rho_*\mathcal{O}_X \) being trivial.

**Theorem 4.11** Let \( \rho_0 \) be the Picard rank of the generic fibre \( X_\eta \). The locus \( \{ t \in M \mid \rho(X_t) > \rho_0 \} \) (called the Noether–Lefschetz locus) is a countable union of positive-codimension algebraic subvarieties.

**Proof** This is classical; references which explicitly deal with general algebraically closed fields include [CHM88] and [MP12].

Now we consider the family \( X^\circ \xrightarrow{d} M = \mathbb{A}^{\Xi_0} \) of hypersurfaces in \( \mathbb{Y}^\circ \) that is defined by (23) (it is defined over \( \mathbb{Z} \), hence over any field). Suppose that \( \mathbb{Y}^\circ \) is smooth as a scheme, away from its toric fixed points, so that the generic fibre of \( X^\circ \) is a smooth \( K3 \) surface. Suppose furthermore that, after base changing to \( \mathbb{C} \), the generic fibre has Picard rank \( \rho_0 \).

**Proposition 4.12** There exists a set \( \Upsilon \subset \mathbb{R}^{\Xi_0} \), a countable union of hyperplanes of rational slope passing through the origin, such that if \( d \in \Lambda^{\Xi_0} \) has valuation \( \text{val}(d) \notin \Upsilon \), then \( X^\circ_d \) is smooth with Picard rank \( \rho_0 \).

**Proof** Let \( \Delta \subset M \) denote the discriminant locus of the family \( X^\circ \). By Lemma 4.1, the generic fibre of the family after base changing to \( \Lambda \) has Picard rank \( \rho_0 \). By Theorem 4.11, the hypersurfaces of higher Picard rank are contained in a countable family of algebraic hypersurfaces in \( \Lambda^{\Xi_0} \setminus \Delta \).

We now restrict the family to \( (\Lambda^*)^{\Xi_0} \setminus \Delta \). The valuation image of any algebraic hypersurface in \( (\Lambda^*)^{\Xi_0} \) is a polyhedral complex of dimension \( |\Xi_0| - 1 \) called the tropical amoeba (see, e.g., [MS15, Proposition 3.1.6]). In particular it is contained in a finite union of affine hyperplanes of rational slope.

Now we observe that, for any \( a \in \mathbb{R}_{>0} \), the map \( q \mapsto q^a \) extends to an automorphism \( \psi_a \) of \( \Lambda \). We define a corresponding automorphism \( \Psi_a = (\psi_a, \ldots, \psi_a) \) of \( \Lambda^{\Xi_0} \), which satisfies \( \text{val}(\Psi_a(d)) = a \cdot \text{val}(d) \) and \( X^\circ_{\psi_a(d)} \cong \psi_a^*X^\circ_d \). It follows that the discriminant and Noether–Lefschetz loci are invariant under \( \Psi_a \), hence that their tropical amoebae are invariant under scaling by \( a \). Since \( a \in \mathbb{R}_{>0} \) was arbitrary, it follows that the affine hyperplanes making up the tropical amoebae pass through the origin. Therefore we can take \( \Upsilon \) to be the union of linear hyperplanes containing the tropical amoebae of the discriminant and Noether–Lefschetz loci.

**Proposition 4.13** In the situation of Theorem 3.2, suppose that \( \lambda \in (\mathbb{R}_{>0})^{\Xi_0} \) is irrational, i.e., does not lie on any rational hyperplane. Then \( X^\circ_\lambda \) has minimal Picard rank \( \rho_0 \).

**Proof** This follows from Proposition 4.12: since the hyperplanes making up \( \Upsilon \) have rational slope, they cannot contain \( \lambda = \text{val}(d) \).
5 Symplectic consequences

Let $X$ be an $M$-polarized $K3$ surface and $\omega$ a Kähler form with cohomology class $[\omega] \in C(M)^+$. Let $X^\circ$ be an $M^\circ$-polarized $K3$ surface over $\Lambda$ which is homologically mirror to $(X, \omega)$:

\[ \mathcal{DF}(X, \omega) \simeq \mathcal{D}(X^\circ). \]

We say that $\omega$ is generic if $[\omega]$ lies in the locus (8), and furthermore the embedding $M^\circ \hookrightarrow \text{Pic}(X^\circ)$ is an isomorphism.

Example 5.1 Theorem 3.2 shows that this version of homological mirror symmetry holds if $X^\circ = X_d^\circ$ is a quartic or double plane and $X$ is the Greene–Plesser mirror equipped with Kähler form $\omega = \omega_\lambda$, for appropriate $\lambda$. Note that $X$ is $M$-polarized, as was remarked in Section 2.6, and $X^\circ$ is $M^\circ$-polarized by the remarks in the same section combined with Lemma 4.1. If $[\omega]$ is irrational (i.e., does not lie in any rational hyperplane in $M_\mathbb{R}$), then it is generic: it lies in the locus (8) because all of the hyperplanes that are removed from the locus are rational; and we can choose $\lambda = \text{val}(d)$ to be irrational, so that $X_d^\circ$ has minimal Picard rank by Proposition 4.13.

For the purposes of this section we will pick a complex model $X^\circ_\mathbb{C}$ of $X^\circ$.

5.1 Spherical objects

If $\mathcal{D}$ is an $A_\infty$ category, we denote by $S(\mathcal{D})$ the set of spherical objects of $\mathcal{D}$ modulo quasi-isomorphism. We will consider the Chern character map

\[ \text{Ch} : S(\mathcal{D}) \to HH_0(\mathcal{D}). \]

For example, let us suppose that $M^\circ = \langle 2n \rangle$ and $\omega$ is generic, so that $X^\circ$ has Picard rank 1. We have a bijection $S(\mathcal{D}(X^\circ)) \cong S(\mathcal{D}(X^\circ_\mathbb{C}))$ by Lemma 4.6. The Chern character map sends

\[ \text{Ch} : S(\mathcal{D}(X^\circ_\mathbb{C})) \to \mathbb{N}(X^\circ_\mathbb{C}) \subset \bigoplus_p H^p(\Omega^p(X^\circ_\mathbb{C})). \]

In this case we have $\mathbb{N}(X^\circ_\mathbb{C}) \cong U \oplus M^\circ$, and the Chern character is equal to the Mukai vector: in particular its image is precisely the set of $(-2)$-classes $\Delta(U \oplus M^\circ)$.

Now we consider the symplectic side. One kind of spherical object of $\mathcal{DF}(X)$ is a Lagrangian sphere, and one kind of Lagrangian sphere is a vanishing cycle. Recall that

\[ D^\circ_M \cong \mathbb{H} \setminus \bigcup_{\delta \in \Delta(U \oplus M^\circ)} \delta^\perp \]

where $\mathbb{H}$ is the upper half-plane, and we have a vanishing cycle associated to a choice of vanishing path from a reference fibre to a nodal point $\delta^\perp \in \mathbb{H}$ for all $\pm \delta \in \Delta(U \oplus M^\circ)$, in homology class $\pm \delta$. Thus we have a map

\[ \Delta(U \oplus M^\circ) \xrightarrow{\text{van. cyc.}} S(\mathcal{D}(X)) \xrightarrow{\sim} S(\mathcal{D}(X^\circ)) \xrightarrow{\text{v}} \Delta(U \oplus M^\circ) \subset \mathbb{N}(X^\circ_\mathbb{C}), \]

where the first map depends on a choice of vanishing path for each $\delta^\perp$.

Lemma 5.2 The composition (38) is bijective. In particular, for every spherical object in $\mathcal{D}(X^\circ)$ there exists a vanishing cycle in $X$ with the corresponding Chern character.
Proof. For any category $\mathcal{D}$, we can consider the additive subspace of $HH_0(\mathcal{D})$ spanned by Chern characters of spherical objects. In fact the Mukai pairing on this subspace is integral by the noncommutative Hirzebruch–Riemann–Roch theorem, so this is a lattice associated to the category $\mathcal{D}$. We will denote this invariant of $\mathcal{D}$ by $S(\mathcal{D})$ for the purposes of the proof.

We have seen that $S(\mathcal{D}(X^o))$ is isomorphic to the subspace of $U \oplus M^o$ spanned by $\Delta(U \oplus M^o)$. The vanishing cycles span some sublattice of $S(\mathcal{D}(X))$, which by Remark 3.10 is isomorphic to the subspace of $H^2(X; \Lambda)$ spanned by their Poincaré duals; we know that this also is isomorphic to the sublattice of $U \oplus M^o$ spanned by $\Delta(U \oplus M^o)$. We also observe that the pairing on this lattice is the usual intersection pairing, because $\chi(HF^*(K, L)) = -\langle K, L \rangle$.

Thus we have an identification of $S(\mathcal{D}(X)) \cong S(\mathcal{D}(X^o))$ with the sublattice spanned by vanishing cycles. It is easy to check that this lattice is of full rank in $U \oplus M^o$ and in particular non-degenerate; a finite-rank non-degenerate lattice can not be properly embedded inside itself by discriminant considerations, so the vanishing cycles span all of $S$. It follows that the Chern characters of vanishing cycles are in bijection with Chern characters of spherical objects: both are in bijection with $\Delta(U \oplus M^o)$.

Remark 5.3. It may be possible to set up the equivalence (37) so that the vanishing cycle $L$ associated to $\delta$ gets mapped to the corresponding spherical sheaf in $Coh(X^o)$, but it would take more work to prove this.

5.2 Symplectic mapping class groups via HMS

Suppose $\text{rk}(M^o) = 1$, and $\omega$ is generic. We compose the various morphisms of short exact sequences of groups that we have constructed: the maps on the central terms are

$$
\pi_1(\mathbb{C}^2_M) \xrightarrow{(11)} G(X, \omega) \xrightarrow{(35)} \text{Auteq}_{CY} \mathcal{D}(X, \omega) \xrightarrow{(30),(31)} \text{Auteq}_{CY} D(X^o) \xrightarrow{(36),(15)} \pi_1(\mathbb{C}^2_M)
$$

(note that in some cases we wrote exact sequences without a ‘$\to 1$’ on the right: we turn these into short exact sequences in the canonical way by replacing the rightmost group with the image of the rightmost homomorphism in the exact sequence, so that all of the morphisms in (39) are morphisms of short exact sequences).

We recall the meanings of these morphisms:

- (11) comes from symplectic monodromy, and exists if $[\omega]$ lies in the locus (8): this holds by genericity of $\omega$.
- (35) comes from the action of the symplectic mapping class group on the Fukaya category; it relies on the open–closed map being an isomorphism, which we saw in Section 3.3 is a consequence of homological mirror symmetry.
- (30) exists by our assumption that $(X, \omega)$ and $X^o$ are homologically mirror.
- (31) maps Calabi–Yau autoequivalences of $\mathcal{D}(X^o)$ to Calabi–Yau autoequivalences of $D(X^o)$ by taking $H^0$, and always exists.
- (36) identifies the derived autoequivalence group of a $K3$ over $\Lambda$ with that of a $K3$ over $\mathbb{C}$, and always exists.
• (15) exists if $X^0_C$ has Picard rank 1, by the theorem of Bayer–Bridgeland. This is equivalent to $X^0$ having Picard rank 1 by Corollary 4.3, which holds by our assumptions that $\text{rk}(M^o) = 1$ and $\omega$ is generic. We have also replaced $HH_*(X^0_C)$ with $H^*(X^0_C; \mathbb{C})$ via the HKR isomorphism, which is valid by [MS09, Theorem 1.2].

**Remark 5.4** Considering the right-most terms in our exact sequences, we observe that (11) maps to $\text{Aut} H^2(X; \mathbb{Z})$ whereas (35) maps from $\text{Aut} H^*(X; \mathbb{Z})$. In order to be able to turn the composition into a morphism of short exact sequences, we use the fact that any symplectomorphism acts trivially on $H^0(X; \mathbb{Z}) \oplus H^4(X; \mathbb{Z})$, so the image of the map $G(X, \omega) \to \text{Aut} H^*(X; \mathbb{Z})$ is contained in the image of $\text{Aut} H^2(X; \mathbb{Z}) \hookrightarrow \text{Aut} H^*(X; \mathbb{Z})$.

**Proposition 5.5** Suppose $\text{rk}(M^o) = 1$, and $\omega$ is generic. Then the composition of morphisms of short exact sequences (39) is an isomorphism.

**Proof** By the 5-lemma, it suffices to prove the result on the initial and final terms of the short exact sequences. The maps between final terms are all isomorphisms, so it remains to prove the result for the initial terms. The composition of morphisms between initial terms has the form

$$\pi_1(D^a_M) \to I(X, \omega) \to \text{Auteq}^0 D(\mathcal{F}(X, \omega))/[2] \cong \ldots$$

$$\ldots \cong \text{Auteq}^0 D(X^0)/[2] \cong \text{Auteq}^0 D(X^0)/[2] \cong \text{Auteq}^0 D(X^0)/[2] \cong \pi_1(D^a_M).$$

We have

$$D^a_M \cong \mathfrak{h} \setminus \bigcup_{\delta \in \Delta(U \oplus M^o)} \delta^\perp,$$

where each $\delta^\perp$ is a single point in $\mathfrak{h}$, and $\pm \delta$ define the same point. Therefore $\pi_1(D^a_M)$ is a free group with generators indexed by $\Delta(U \oplus M^o)/\pm \text{id}$.

Choose paths from a basepoint in $D^a_M$ to each point $\delta^\perp$: these specify generators for the free group. The monodromy around $\delta^\perp$ is $\tau^2_\mathcal{L} \in I(X, \omega)$, a squared Dehn twist in the corresponding vanishing cycle $L$. The corresponding autoequivalence of $D(\mathcal{F}(X, \omega)$ is $\text{Tw}_3^2$, the squared algebraic twist in the corresponding spherical object by [Sei08a], which gets sent to the autoequivalence $\text{Tw}_3^2$ for the corresponding spherical object $S$ of $D(X^0_C)$.

Corollary 2.13 says that $\text{Auteq}^0 D(X^0_C)/[2] \cong \pi_1(D^a_M)$. By [BB17, Proposition 2.3], for every $\epsilon \in \Delta(U \oplus M^o)$ there exists a spherical vector bundle $S_\epsilon$, so that the spherical twist $\text{Tw}_3^2$ corresponds to a loop enclosing the hole $\epsilon^\perp$ under this isomorphism. Furthermore, by [BB17, Remark 6.10], $\text{Auteq}^0 D(X^0_C)$ acts transitively on the set of spherical objects $S$ with fixed Mukai vector $\nu(S) = \epsilon$. It follows that every squared spherical twist in a spherical object $S$ with Mukai vector $\nu(S) = \epsilon$ corresponds to a loop in $\pi_1(D^a_M)$ enclosing the hole $\epsilon^\perp$. By the Lefschetz principle, the analogues of these results hold for $D(X^0)$.

In particular, the composition (40) sends a loop enclosing $\delta^\perp$ to a loop enclosing $\epsilon^\perp$, where $\delta = [L]$ and $\epsilon = \nu(S)$ are Chern characters of spherical objects which correspond under mirror symmetry. The map $\delta \mapsto \epsilon$ is bijective by Lemma 5.2. Therefore the composition (40) sends one set of generators for the free group bijectively to another set of generators for the free group, so the map is an isomorphism as required. □

**Definition 5.6** Let

$$Z(X, \omega) := \text{ker}(G(X, \omega) \to \text{Auteq} D(\mathcal{F}(X, \omega))/[2])$$
denote the ‘Floer-theoretically trivial’ subgroup of the symplectic mapping class group. It is contained in \( I(X, \omega) \).

We have the following immediate corollary of Proposition 5.5:

**Corollary 5.7** Suppose \( \text{rk}(M^p) = 1 \), and \( \omega \) is generic. Then we have
\[
G(X, \omega) \cong \pi_1(\mathbb{Z}_M^0) \times Z(X, \omega) \quad \text{and} \\
I(X, \omega) \cong \pi_1(D_M^0) \times Z(X, \omega).
\]

**Definition 5.8** Let
\[
K(X, \omega) := \ker (\pi_0 \text{Symp}(X, \omega) \to \pi_0 \text{Diff}(X))
\]
denote the ‘smoothly trivial’ subgroup of the symplectic mapping class group. It is contained in \( I(X, \omega) \).

**Corollary 5.9** Suppose \( \text{rk}(M^p) = 1 \), and \( \omega \) is generic. Then we have homomorphisms
\[
\pi_1(D_M^0) \to K(X, \omega) \to \pi_1(D_M^0)
\]
whose composition is an isomorphism. In particular, \( K(X, \omega) \) is infinitely generated.

**Proof** We consider the composition \( \pi_1(D_M^0) \to I(X, \omega) \to \pi_1(D_M^0) \) appearing in (40), which is shown to be an isomorphism in Proposition 5.5. Observe that the image of the map \( \pi_1(D_M^0) \to I(X, \omega) \) is generated by squared Dehn twists, which are known to be smoothly trivial: so we can replace \( I(X, \omega) \) with the subgroup \( K(X, \omega) \).

Now we can refine Remark 1.2, which we recall concerned the space \( \Omega \) of symplectic forms on \( X \) cohomologous to \( \omega \). It essentially consisted of the observation that the composition
\[
\pi_1(\Omega) \to K(X, \omega) \to \pi_1(D_M^0)
\]
is surjective, where the first map is the connecting map in the long exact sequence associated to a Serre fibration, and the second was constructed in the proof of Corollary 5.9; this shows that \( \pi_1(\Omega) \) is infinitely generated.

**Lemma 5.10** There exists a map \( i : D_M^0 \to \Omega \), such that the composition
\[
\pi_1(D_M^0) \xrightarrow{i} \pi_1(\Omega) \xrightarrow{(42)} \pi_1(D_M^0)
\]
is an isomorphism.

**Proof** Consider the universaly family of \( C \)-ample marked \( M \)-polarized \( K3 \) surfaces over \( D_M^0 \). We saw in the proof of Corollary 5.9 that the monodromy homomorphisms are smoothly trivial; since \( D_M^0 \) is an Eilenberg–MacLane space, it follows that the family is smoothly trivial. Pulling back a choice of fibrewise Kähler form via a smooth trivialization induces a map \( D_M^0 \to \Omega \), and it is clear from the definitions that the composition
\[
\pi_1(D_M^0) \xrightarrow{i} \pi_1(\Omega) \to K(X, \omega)
\]
is precisely the symplectic monodromy homomorphism. It follows that the composition (43) is equal to the composition (41), which is an isomorphism by Corollary 5.9.
5.3 Lagrangian spheres

We now consider Lagrangian spheres $L \subset (X, \omega)$. We declare two Lagrangian spheres to be “Fukaya-isomorphic” if they can be equipped with a grading and Pin structure such that the corresponding objects of $\mathcal{F}(X, \omega)$ are quasi-isomorphic.

**Lemma 5.11** (= Theorem 1.4 (1)) Suppose $\text{rk}(M^0) = 1$, and $\omega$ is generic. Then any Lagrangian sphere $L \subset (X, \omega)$ is Fukaya-isomorphic to a vanishing cycle.

**Proof** Let $L$ be a Lagrangian sphere, and $\mathcal{E}_L$ the mirror spherical object of $D(X^\circ_C)$. By Lemma 5.2, there exists a vanishing cycle $V$ with $\text{Ch}(\mathcal{E}_V) = \text{Ch}(\mathcal{E}_L)$. We can apply [BB17, Remark 6.10] (which applies because $\rho(X^\circ_C) = 1$), which says that there exists $\Phi \in \text{Aut}^{0}_D D(X^\circ_C)$ with $\Phi(\mathcal{E}_V) \cong \mathcal{E}_L$. We have a homomorphism $\pi_1(D^+_L) \to \text{Aut}^{0}_D D(X^\circ_C)$ from the composition of short exact sequences (39), which is surjective by Proposition 5.5. Therefore, there exists $a \in \pi_1(D^+_L)$ such that $a \cdot V \cong L$. It is clear that $a \cdot V$ is the vanishing cycle whose vanishing path is the concatenation of the vanishing path of $V$ with the loop $a$, so we are done.

**Lemma 5.12** Suppose $\text{rk}(M^0) = 1$, and $\omega$ is generic. Then the orbits of $G(X, \omega)$ on the set of Fukaya-isomorphism classes of Lagrangian spheres in $(X, \omega)$ are in bijection with the orbits of $\Gamma(M)^+$ on $\Delta(U \oplus M^0)/\pm \text{id}$.

**Proof** Because the map $G(X, \omega) \to \text{Aut} H^2(X; \mathbb{Z})$ lands in $\Gamma(M)^+$ (see Remark 2.11), there is a well-defined map

$$\{ \text{Lagrangian spheres up to Fukaya-isomorphism} \}/G(X, \omega) \to (\Delta(U \oplus M^0)/\pm \text{id})/\Gamma(M)^+$$

(44) sending $L$ to its homology class (the choice of orientation of $L$ does not matter since we quotient by $\pm \text{id}$). This map is surjective by Lemma 5.2.

Suppose that $L_1$ and $L_2$ have the same image under (44). By Lemma 5.11, we may assume that $L_1$ and $L_2$ are vanishing cycles. Because the homomorphism $G(X, \omega) \to \Gamma(M)^+$ is surjective by Remark 2.11, we may assume that $[L_1] = [L_2] = \delta$ in $H_2(X; \mathbb{Z})$; so $L_1$ and $L_2$ are vanishing cycles from the same point $\delta^\perp$. It follows that their vanishing paths differ by an element of $\pi_1(D^+_M)$, so the vanishing cycles differ by its image under $\pi_1(D^+_M) \to I(X, \omega) \subset G(X, \omega)$. In particular, $L_1$ and $L_2$ represent the same class on the left-hand side of (44), so the map is injective as required.

**Corollary 5.13** If $\text{rk}(M^0) = 1$ and $\omega$ is generic, then $G(X, \omega)$ has finitely many orbits on the set of Fukaya-isomorphism classes of Lagrangian spheres.

**Proof** By Lemma 5.12, it suffices to check that $\Gamma(M)^+$ has finitely many orbits on $\Delta(U \oplus M^0)$. This follows from the following general result of Borel and Harish-Chandra [BHC62, Theorem 6.9]. Let $G$ be a reductive algebraic group over $\mathbb{Q}$, $V$ a rational finite-dimensional representation of $G$, $\Gamma \subset V_{\mathbb{Q}}$ a $G_{\mathbb{Q}}$-invariant lattice, and $Q$ a closed orbit of $G$. Then $Q \cap \Gamma$ consists of finitely many $G_{\mathbb{Q}}$-orbits.

**Corollary 5.14** (= Theorem 1.4 (2)) If $M^0 = \langle 2 \rangle$ or $\langle 4 \rangle$ and $\omega$ is generic, then $G(X, \omega)$ acts transitively on the set of Fukaya-isomorphism classes of Lagrangian spheres.

**Proof** Follows by Lemma 5.12 and Lemma 2.16.
Corollary 5.15 Suppose \( M^o = \langle 2 \rangle \) or \( \langle 4 \rangle \), and \( \omega \) is generic. If \( L \) and \( L' \) are Lagrangian spheres in \( X \), then the Dehn twists \( \tau_L \) and \( \tau_{L'} \) are conjugate in \( \text{Autoeq}_{CY} D\mathcal{F}(X) \).

Proof It is sufficient to prove that the algebraic twist functors \( \text{Tw}_L \) and \( \text{Tw}_{L'} \) are conjugate in \( \text{Autoeq}_{CY} D\mathcal{F}(X) \). The twist functors are conjugate provided the underlying spherical objects lie in the same orbit of the autoequivalence group, by \cite[Lemma 5.6]{Sei08a}.

The Dehn twist in a simple non-separating curve on a closed surface of genus \( g \geq 2 \) has non-trivial roots in the mapping class group \cite{MS09}. For \( g \geq 2 \), the image of a Dehn twist in the symplectic group \( \text{Sp}(2g; \mathbb{Z}) \) has a cube root (this fails when \( g = 1 \)).

Proposition 5.16 Suppose \( M^o = \langle 2 \rangle \) or \( \langle 4 \rangle \), and \( \omega \) is generic. Let \( L \subset X \) be a Lagrangian sphere. The Dehn twist \( \tau_L \) admits no cube root in \( G(X, \omega) \).

Proof By Proposition 5.5 and Lemma 2.17, we have a surjection \( G(X, \omega) \twoheadrightarrow \pi_1(\mathbb{K}^o_X) \cong \mathbb{Z} * \mathbb{Z}/p \). Furthermore the Dehn twist in a given vanishing cycle can be arranged to correspond to \( 1 \in \mathbb{Z} \). By Corollary 5.15, it follows that all Dehn twists map to elements conjugate to \( 1 \in \mathbb{Z} \). Now we pass to the abelianization \( (\mathbb{Z} * \mathbb{Z}/p)^{ab} \cong \mathbb{Z} \oplus \mathbb{Z}/p \): all Dehn twists map to \((1,0)\), and therefore have no cube root.

Note that since \( \tau_L^2 \) is smoothly trivial, \( \tau_L \) is its own cube root in \( \pi_0 \text{Diff}(X) \), so this is a symplectic and not smooth phenomenon.

5.4 Lagrangian tori

We now consider embedded Lagrangian tori \( L \subset (X, \omega) \) with Maslov class zero.

Lemma 5.17 Suppose that \( M^o = \langle 2n \rangle \) where \( n \) is square-free, and \( \omega \) is generic (observe that this is true in the situation of Example 5.1, where \( n = 1 \) or \( 2 \)). Then a Maslov-zero Lagrangian torus \( L \subset (X, \omega) \) represents a primitive homology class.

Proof The object \( L \) of \( D\mathcal{F}(X, \omega) \) corresponds to a point-like object \( \mathcal{E} \) of \( D(X^o) \). Therefore there exists a spherical object \( S \) of \( D(X^o) \) with \( \chi(\mathcal{E}, S) = 1 \), by Lemma 4.10 (using the fact that \( \text{Pic}(X^o) \cong \langle 2n \rangle \) because \( \omega \) is generic). By Lemma 5.2 there exists a vanishing cycle \( V \) which is mirror to a spherical object \( S' \) with the same Mukai vector as \( S \), so we have

\[-[L] \cdot [V] = \chi(\text{HF}^+(L, V)) = \chi(\mathcal{E}, S') = \chi(\mathcal{E}, S) = 1.\]

It follows that \([L]\) is primitive.

Remark 5.18 It would be nice to prove Lemma 5.17 by arguing that the Chern character of the mirror object \( \mathcal{E} \) to \( L \) is primitive, and that therefore the Chern character \([L]\) of \( L \) is primitive. However the relationship between the integral lattices on the two sides of mirror symmetry is not well-understood (compare \cite{KKP08}): that is why we circumvented this issue by appealing to Lemma 5.2.

For our final result, we change our assumptions on the symplectic \( K3 \) surface \((X, \omega)\). Rather than assuming that \((X, \omega)\) itself has a homological mirror satisfying certain hypotheses, we assume that \( \omega \) is a limit of \( \rho^o = 1 \) classes. We define this to mean that there exists a smooth one-parameter family of Kähler forms \( \omega(t) \) with \( \omega_0 = \omega \), having the following property: there exist times \( t_1, t_2, \ldots \) converging to 0 such that for all \( j \), \((X, \omega(t_j))\) admits a homological mirror \( X^o_j \) of Picard rank \( \rho(X^o_j) = 1 \).
**Lemma 5.19** Suppose \( \omega \) is a limit of \( \rho^o = 1 \) classes. If \( L \subset (X, \omega) \) is a Maslov-zero Lagrangian torus, then \([L] \not= 0 \in H_2(X; \mathbb{Z})\).

**Proof** Suppose \( L \subset (X, \omega) \) is a Maslov-zero Lagrangian torus with vanishing homology class. A Moser-type argument shows that \( L \) deforms to give a Maslov-zero Lagrangian torus \( L(t) \subset (X, \omega(t)) \) for \( t \in [0, \epsilon) \) sufficiently small that \( L(t) \) remains embedded. Thus there exists \( \tau \in (0, \epsilon) \) such that \( L(\tau) \subset (X, \omega(\tau)) \) is embedded and \((X, \omega(\tau)) \) admits a homological mirror \( X^o \) of Picard rank 1. \( L(\tau) \) defines a point-like object of \( \mathcal{D} \mathcal{F}(X, \omega(\tau)) \), and we denote the mirror point-like object by \( \mathcal{E} \in \mathcal{D}(X^o) \). Under the isomorphism

\[
H^2(X; \Lambda) \cong HH_0(\mathcal{D} \mathcal{F}(X, \omega(\tau))) \cong HH_0(\mathcal{D}(X^o)) \cong HH_0(X^o)
\]

we see that \( \mathcal{E} \) has trivial Chern character. However \( X^o \) has Picard rank 1, so this contradicts Lemma 4.9.

It is obvious that, if \( \text{rk}(M^o) = 1 \) and \( \omega \) is generic, then \( \omega \) is a limit of \( \rho^o = 1 \) classes. However there are more \( \rho^o = 1 \) classes, as we now describe.

Recall that we constructed the mirror quartic and mirror double plane as hypersurfaces \( X \subset Y \) depending on data \((\Delta, \lambda)\). The vector \( \lambda \in (\mathbb{R}_{>0})^{\Xi_0} \) determines a refinement \( \Sigma \) of \( \bar{\Sigma} \), and hence a toric resolution of singularities \( Y \rightarrow \bar{Y} \); and \( X \subset Y \) is the proper transform of \( \bar{X} \subset \bar{Y} \). The vector \( \lambda \) also determines a Kähler class \([\omega_{\lambda}]\) on \( X \).

The K3 surface \( X \) does not depend on the refinement \( \Sigma \), so long as the rays of \( \Sigma \) are generated by \( \Xi_0 \): changing \( \Sigma \) corresponds to changing \( Y \) by a birational modification in a region disjoint from \( X \) (the same is not true in higher dimensions). A convenient way of organizing the different refinements is provided by the “secondary fan” or “Gelfand–Kapranov–Zelevinskij decomposition” associated to \( \Xi_0 \subset \mathbb{R}^3 \), whose cones \( \text{cpl}(\Sigma) \) are indexed by fans \( \Sigma \) whose rays are generated by subsets of \( \Xi_0 \) (see [OP91] or [CK99, Section 6.2.3]). Our prescription for determining \( \Sigma \) from \( \lambda \) is equivalent to defining \( \Sigma \) to be the unique fan such that \( \lambda \) is contained in the interior of \( \text{cpl}(\Sigma) \).

For any \( \lambda \) lying in the interior of a cone \( \text{cpl}(\Sigma) \), where \( \Sigma \) is a refinement of \( \bar{\Sigma} \) whose rays are generated by \( \Xi_0 \), we obtain a Kähler class \( \omega_{\lambda} \) on \( X \). We call these Kähler classes the ambient Kähler classes on \( X \). Theorem 3.2 only provides a homological mirror to \((X, \omega_{\lambda})\) if \( \Sigma \) is furthermore simplicial (the corresponding cones \( \text{cpl}(\Sigma) \) are the top-dimensional ones). Nevertheless we have:

**Lemma 5.20** Let \( X \) be the mirror quartic or mirror double plane, and \( \omega \) be an ambient Kähler class. Then \( \omega \) is a limit of \( \rho^o = 1 \) classes.

**Proof** Suppose \( \omega = \omega_{\lambda} \); then \( \lambda \) lies in the closure of a top-dimensional cone \( \text{cpl}(\Sigma') \), where \( \Sigma' \) is a simplicial refinement of \( \bar{\Sigma} \) whose rays are generated by \( \Xi_0 \). Therefore we can choose a path \( \lambda(t) \) converging to \( \lambda(0) = \lambda \) such that \( \lambda(t) \) lies in the interior of \( \text{cpl}(\Sigma') \) for \( t \not= 0 \), and \( \lambda(t) \) is not constant for \( t \) close to 0. Then Theorem 3.2 provides a mirror \( X^o_{d(t)} \) to \((X, \omega_{\lambda(t)})\) for all \( t \not= 0 \). Because \( \lambda(t) \) is not constant for \( t \) close to 0, there is a sequence of times \( t_j \) converging to 0 at which \( \lambda(t_j) \) is irrational, so that the mirror \( X^o_{d(t)} \) has Picard rank 1 by Proposition 4.13.

Lemmas 5.20 and 5.19 combine to prove Theorem 1.3.
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N. SHERIDAN, CENTRE FOR MATHEMATICAL SCIENCES, UNIVERSITY OF CAMBRIDGE, ENGLAND.
Email: N.Sheridan@dpmms.cam.ac.uk

I. SMITH, CENTRE FOR MATHEMATICAL SCIENCES, UNIVERSITY OF CAMBRIDGE, ENGLAND.
Email: I.Smith@dpmms.cam.ac.uk