One-Loop Effective Action: Nonlocal Form Factors and Renormalization Group

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Abstract—We review and present full details of the Feynman diagram based and the heat-kernel method based calculations of the simplest nonlocal form factors in the one-loop contributions of a massive scalar field. The paper has pedagogical and introductory purposes and is intended to help the reader in better understanding the existing literature on the subject. The functional calculations are based on the solution by Avramidi and Barvinsky and Vilkovisky for the heat kernel and are performed in curved space-time. One of the important points is that the main structure of nonlocalities is the same as in the flat background.

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1. INTRODUCTION

The main method of calculating quantum loop corrections in QFT (quantum field theory) is based on integration of the Feynman diagrams in the momentum representation. At the same time, to work in curved space (space-time), one has to go beyond this technique because the global Fourier transformation in curved space is impossible. There are three different main approaches to curved-space calculations. The first one is based on expanding the external metric on the flat background. \( g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \), and making the calculations in flat space, treating \( h_{\mu\nu} \) as an external flat-space field. The covariance and locality of the divergences make such an approach possible and in many cases useful [1, 2]. The same concerns, in many cases, the derivation of the finite nonlocal part of the diagrams [3, 4].

Another approach to the calculations in curved space is based on the use of normal coordinates and the local momentum representation [5]. One of the advantages of this method is an explicit covariance. In some cases, it provides serious technical benefits, e.g., for deriving the effective potential in the mass-dependent schemes of renormalization [6, 7]. At the same time, since the local momentum representation is essentially based on the expansion in the vicinity of a single space-time point, this method is not well suited for the nonlocal contributions.

Finally, the Schwinger-De Witt technique [8, 9] is the most efficient way to derive the one-loop divergences in a curved space background. About 25 years ago there was a significant progress in the development of the heat-kernel methods by Avramidi [10], Barvinsky and Vilkovisky [11]. As a result, the general expressions for the non-localities in curved space have been derived, and this opened the way for calculating the one-loop nonlocal form factors for different fields [3, 4, 12, 13] and models (see, e.g., [14]).

From the viewpoint of physical applications, the similarities and main differences between the standard Schwinger-De Witt technique and the new heat-kernel methods are as follows. In both cases, one deals with the first several terms in the derivative expansion of the covariant effective action in “curvatures.” In the case of gravity, due to the covariance, this expansion has the form of a power series in the curvature tensor and its contractions (curvatures). Also, for the operator of the standard form

\[
\hat{\mathcal{H}} = \hat{\Pi} + 2\hat{h}^\alpha \nabla_\alpha + \hat{\Pi},
\]

the expressions such as

\[
\hat{\mathcal{P}} = \hat{\Pi} + \frac{i}{6} R - \nabla_\alpha \hat{h}^\alpha - \hat{h}_\alpha \hat{h}^\alpha
\]

and

\[
\hat{S}_{\alpha\beta} = \left[ \nabla_\beta, \nabla_\alpha \right] \hat{\Pi} + \nabla_\beta \hat{h}_\alpha - \nabla_\alpha \hat{h}_\beta + \hat{h}_\beta \hat{h}_\alpha - \hat{h}_\alpha \hat{h}_\beta
\]

are also included in the list of curvatures.

Many physical applications are based on the terms which are quadratic and at most cubic in curvatures.
The main difference is that the standard Schwinger–De Witt technique deals with the high energy limit (namely, it is related to the limit $s \to 0$ in the proper-time representation). The corresponding terms are UV-divergent and hence local. As a result, they are usually irrelevant in the IR limit. Of course, for massless fields, there is a certain duality between UV and IR, hence one can always restore the most important part of the IR-relevant nonlocal terms, e.g., by integrating the conformal anomaly. However, in the case of a massive field such integration cannot be used or it has a very restricted physical sense [15, 16], because of the IR decoupling in gravity. In general, the decoupling is important since it enables one to separate the relevant and irrelevant degrees of freedom at low energies (in the IR), and thus it represents one of the main ingredients of the effective field theory approach.

At least one of the first papers on the gravitational decoupling was [2], where it was shown that in the $k^2 \ll m^2$ and $|R_{..}| \ll m^2$ limit the expression for the renormalized $(T_{\mu\nu})$ of a massive scalar field in curved space-time becomes local. The corresponding terms have mass dependence $\sim m^{-2}$ and no $\mu$-dependence since there is no direct relation to the UV divergences. The explicit expressions for the nonlocal form factors enable one to explore the details of the IR limit for the massive fields, and hence one can observe and explore such a phenomenon as the low-energy decoupling.

In what follows we present the details of deriving the gravitational form factors, which lead to the gravitational analog of the Appelquist and Carazzone decoupling theorem [17].

Indeed, the heat-kernel solution of [11] is known only for operators of the form (1), while in some cases we need to work with operators of a different form, where the solution for the heat-kernel is unknown. Thus it is very important to establish the relation between the Feynman diagrams based and heat-kernel based calculations of the nonlocal form factors. There are some calculations of this sort in particular cases [3, 13], but they deal with specific cases of free fields on a curved background and are technically complicated. For this reason, we present a very simple, pedagogical derivation of the nonlocal form factors in flat space using diagrams and compare it with the heat-kernel calculation, which just repeats (correcting some misprints) the one of [3, 12]. For the sake of generality, the diagram calculation is partly performed in dimensional regularization and in the covariant Euclidean cut-off regularization, demonstrating equivalence between the two regularizations for the logarithmic divergences, something being certainly well known in different contexts (see, e.g., [6, 9, 19, 20]). Our purpose is to present this known feature in a clear and simple form.

The paper is organized as follows. In the next Sec. 2 we discuss the calculation of divergences and nonlocal form factors in dimensional and the covariant cut-off regularizations. In Sec. 3 we demonstrate the general derivation of nonlocal form factors in curved space-time, in full detail. In Sec. 4 the example of a massive scalar field is elaborated. We do not go into similar detail for massive fermions and vector fields because it could be quite boring and, also because the reader can easily elaborate these two examples as exercises, e.g., following the papers [4, 12, 13]. Finally, in Sec. 5 we draw our conclusions and discuss some perspectives in this area.

2. TWO TYPES OF UV REGULARIZATION

There are many different regularization schemes, and in fact, it is not difficult to invent a new one. The most used examples are the cut-off regularizations (including three-dimensional cut-off in momentum space, four-dimensional Euclidean cut-off and the covariant cut-off in the proper-time integral), Pauli-Villars (conventional, covariant and higher-derivative covariant), analytic (different versions), zeta-regularization (which is not a regularization, properly speaking), point-splitting and dimensional regularization, which has the advantages of preserving the gauge symmetry and being the simplest one in many cases.

Since the dimensional regularization [21] preserves the gauge symmetry explicitly (unlike the cut-off and some others), it can be used not only in the one-loop order but also for the multi-loop diagrams. The disadvantages are that one cannot see quadratic divergences, also it is not really “physical,” such as the cut-off regularization, for instance. Anyway, dimensional regularization is one of the most used regularizations, hence let us describe its use in detail.

2.1. Mathematical Preliminaries

We shall need a few special mathematical tools, as reviewed below.

2.1.1. Analytic continuation. Consider two regions $D_1$ and $D_2$ on the complex plane. The analytic continuation theorem tells us that in some cases one can extend the analytic function from some set of points to the larger region, uniquely.

Consider two functions $F_1(z)$ and $F_2(z)$, defined and analytic on $D_1$ and $D_2$, respectively. Suppose $D_1 \cap D_2 = D$. Furthermore, we assume that $F_1(z) = F_2(z)$ on a set which belongs to $D$ and has at least

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1 There was also an earlier calculation of the gravitational form factors with temperature [18], but without an analysis of decoupling.
one accumulation point. Then, \( F_1(z) = F_2(z) \) on the whole \( D \).

Our strategy will be to define such a continuation for badly defined integrals (in Euclidean signature), such as, e.g.,

\[
I_4 = \int \frac{d^4p}{(2\pi)^4} \frac{1}{(p^2 + m^2)(p - k)^2 + m^2}.
\]

We assume this integral to be defined in Euclidean four-dimensional space, but our purpose is to make a continuation from dimension four to a complex dimension \( 2\omega \), \( I_4 \rightarrow I_{2\omega} \), such that \( I_{2\omega} \) be an analytic on the complex plane except some at most countable number of points. Then, in the vicinity of the point \( \omega = 2 \) we have

\[
I_{2\omega} = \left( \text{divergent} \sim \frac{1}{2 - \omega} \text{pole term} \right) + \text{finite terms} + \text{vanishing } O(2 - \omega) \text{ terms}.
\]

Our first purpose will be to establish the first, divergent term, with the pole at \( \omega = 2 \).

2.1.2. Gaussian integral. This integral in the dimension \( 2\omega \) reads

\[
\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} e^{-xk^2 + 2bk} = \frac{1}{(2\pi)^{2\omega}} \left( \frac{\pi}{x} \right) ^{\omega} e^{b^2/x}.
\]

For natural \( 2\omega = 1, 2, 3, 4, \ldots \), this integral can be easily derived. For complex values of \( \omega \), Eq. (5) should be seen as a definition, or as an analytic continuation. One can see the standard review [21] for a detailed explanation of the procedure of continuation \( 4 \rightarrow n \rightarrow 2\omega \).

A typical example of applying (5) is related to the representation

\[
\frac{1}{k^2 + m^2} = \int_0^\infty d\alpha e^{-\alpha(k^2 + m^2)}.
\]

Consider continuation of the integral (4) into the dimension \( n = 2\omega \),

\[
I_{2\omega} = \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \left( k^2 + m^2 \right) \left( k - p \right)^2 + m^2 \]

\[
= \int_0^\infty d\alpha_1 \int_0^\infty d\alpha_2 \int_0^\infty d\omega k \times e^{-\alpha_1(k^2 + m^2) - \alpha_2((k - p)^2 + m^2)}.
\]

Changing the order of integrations, it is easy to note that the integral over \( k \) is exactly of the type (5), hence we arrive at

\[
I_{2\omega} = \int_0^\omega d\alpha_1 \int_0^\omega d\alpha_2 \int_0^\omega \frac{d^{2\omega}k}{(2\pi)^{2\omega}}
\]

\[
\times e^{-k^2(\alpha_1 + \alpha_2) + 2\alpha_2 kp - (\alpha_1 + \alpha_2)m^2 - \alpha_2 p^2}
\]

\[
= \int_0^\omega d\alpha_1 \int_0^\omega d\alpha_2 \frac{1}{(2\pi)^2} \left( \frac{\pi}{\alpha_1 + \alpha_2} \right) ^\omega \times e^{\alpha_2^2 p^2 - \alpha_2(p^2 + m^2) - \alpha_2 m^2}.
\]

The last representation will prove useful, at some moment.

2.1.3. Some properties of the gamma function. The gamma function is defined as

\[
\Gamma(z) = \int_0^\infty dt e^{-t}.
\]

The main properties which concern us, are

\[
\Gamma(z + 1) = z\Gamma(z) \implies \Gamma(n + 1) = n!,
\]

\[
\Gamma \left( \frac{1}{2} \right) = \sqrt{\pi} \implies \Gamma \left( n + \frac{1}{2} \right) = \frac{1 \cdot 3 \cdot 5 \cdots}{2^{n-1}} \sqrt{\pi},
\]

\[
\Gamma(z) = \lim_{n \to \infty} n! e^{z - \frac{1}{1!} - \frac{z}{2!} + \frac{z^2}{3!} - \cdots} \left( z + 1 \right) \cdots \left( z + n \right).
\]

From the last representation it directly follows that \( \Gamma(z) \) has simple poles at zero and negative integer points, \( z = 0, -1, -2, \ldots \), and nowhere else. Another representation, where this fact can be seen explicitly, is the Weirstrass’s partial fraction expansion

\[
\Gamma(z) = \Gamma_n(z) = \sum_{n=0}^\infty \frac{(-1)^n}{n!(z + n)} + \int_1^\infty dt e^{-t}.
\]

It is clear that \( \Gamma(z) \) is analytic everywhere except at \( z = 0, -1, -2, \ldots \).

An explicit representation of \( \Gamma(1 - \omega) \) can be obtained from

\[
\Gamma(2 - \omega) = (1 - \omega)\Gamma(1 - \omega)
\]

and Eq. (10),

\[
\Gamma(2 - \omega) = \lim_{n \to \infty} J_\omega, \text{ where}
\]

\[
J_\omega = \frac{n! e^{\omega - 2\omega} \ln n \ln^2 \omega \cdot \cdots (n + 2 - 2\omega)}{(2 - \omega)(3 - \omega) \cdots (n + 2 - \omega)}.
\]

The expression under the limit can be transformed as

\[
J_\omega = \frac{n! e^{(2 - \omega)\ln n}}{(2 - \omega)(1 + 2 - \omega)(2 + 2 - \omega) \cdots (n + 2 - 2\omega)}.
\]

Obviously, the divergent part of \( J_\omega \), in the limit \( \omega \to 2 \), is

\[
J_\omega^{\text{div}} = \frac{n! \cdot 1 \cdot 2 \cdots n}{(2 - \omega) \cdot 1 \cdot 2 \cdots n} = \frac{1}{2 - \omega}.
\]
The finite part can be evaluated by means of the following transformations:

\[
\frac{1}{2 - \omega} e^{(2 - \omega) \ln n} = \frac{1}{2 - \omega} [1 + (2 - \omega) \ln n + \mathcal{O}((2 - \omega)^2)] \approx \frac{1}{2 - \omega} + \ln n + \mathcal{O}(2 - \omega),
\]

or

\[
\frac{1}{k - \omega} = \frac{1}{k - 2 + (2 - \omega)} = \frac{1}{k - 2} \cdot \frac{1 + \frac{2 - \omega}{k - 2}}{2 - \omega}.
\]

Therefore,

\[
J_\omega = \frac{1}{2 - \omega} + \ln n - \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) + \mathcal{O}(2 - \omega).
\]  

(14)

The sum of the finite terms is

\[
\gamma = \lim_{n \to \infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n\right),
\]

and its value is \(\gamma = 0.57721\ldots\) (the Euler-Mascheroni constant, or just Euler’s constant).

Although we shall keep it, the finite contribution has no much importance, because it sums up with the infinite term \(1/(2 - \omega)\). The conventional notation is

\[
\frac{1}{\varepsilon} = \frac{1}{(4\pi)^2(n - 4)}, \quad n - 4 = -2(2 - \omega).
\]

Finally, from (13) and (12) it follows

\[
\Gamma(2 - \omega) = \int_0^\infty e^{-t} dt \approx \frac{1}{2 - \omega} - \gamma + \mathcal{O}(2 - \omega),
\]

\[
\Gamma(1 - \omega) = \frac{1}{1 - \omega} \Gamma(2 - \omega)
\]

\[
= \frac{1}{1 + (2 - \omega) \Gamma(2 - \omega)} = \frac{1}{2 - \omega} - 1 + \gamma + \mathcal{O}(2 - \omega),
\]

\[
\Gamma(-\omega) = \frac{1}{2(2 - \omega)} + \frac{3}{4} \gamma + \mathcal{O}(2 - \omega).
\]

(17)

(18)

2.1.4. Volume of the sphere. Finally, let us calculate the volume of the \(m\)-dimensional sphere with the radius \(R = (x_1^2 + x_2^2 + \cdots + x_m^2)^{1/2}\). The dimensional arguments tell us that

\[
V_m = C_m R^m,
\]

where \(C_m\) are the coefficients which we need to calculate. For this sake, consider the Gaussian integral

\[
I = \int_{-\infty}^\infty dx_1 \int_{-\infty}^\infty dx_2 \cdots \int_{-\infty}^\infty dx_m e^{-ax_1^2 + ax_2^2 + \cdots + x_m^2}
\]

\[
\times \left[\int_{-\infty}^\infty dx e^{-x^2}\right]^m = \left(\frac{\pi}{a}\right)^{m/2}.
\]  

(20)

On the other hand, \(dV_m = mC_m R^{m-1} dR\), hence

\[
I = \int_{-\infty}^\infty e^{-aR^2} mC_m R^{m-1} dR.
\]

Making the change of variables \(z = aR^2\), we get

\[
dR = \frac{1}{2a} \left(\frac{a}{z}\right)^{1/2} dz, \quad R^{m-1} = \left(\frac{z}{a}\right)^{(m-1)/2},
\]

and therefore,

\[
I = mC_m \int_0^\infty e^{-z} \left(\frac{z}{a}\right)^{m/2} \cdot \frac{1}{2a} \left(\frac{a}{z}\right)^{1/2} dz
\]

\[
= mC_m \int_0^\infty e^{-z} \frac{z^{m-1}}{2a \Gamma(m/2)} dz = \frac{mC_m}{2a \Gamma(m/2)} \Gamma(m/2).
\]  

(21)

Since (20) and (21) is the same thing, we get

\[
C_m = \frac{\pi^{m/2}}{\Gamma(m/2)} \frac{\Gamma(m/2 + 1)}{\Gamma(m/2 + 1)}
\]

\[
= \frac{\pi^m}{m!} R^m.
\]  

(22)

The last relation is valid for any natural \(m\), but we can also continue it to an arbitrary complex dimension \(2\omega\).

2.2. The Simplest Loop Integral

Now we are in a position to start regularizing loop integrals. The general strategy will be to continue

\[
I_4 \rightarrow I_{2\omega} = \int \frac{d^{2\omega} x}{(2\pi)^{2\omega}} \cdots,
\]

such that \(I_{2\omega}\) is defined in the whole complex plane except for some points, including \(\omega = 2\). Typically,

\[
I_{2\omega} = (\text{pole at } \omega = 2) + \text{regular terms}.
\]

Consider a scalar theory with \(\lambda\phi^4\) interaction,

\[
S = \int d^4 z \left\{\frac{1}{2} (\partial \phi)^2 - \frac{m^2}{2} \phi^2 - \frac{\lambda}{4!} \phi^4\right\}.
\]  

(24)
Let us first rewrite it as a Euclidean action, by changing the variable \( z^0 = -iz^4 \). Then
\[
d^4 z = d^4 z^0 d^4 z = -id^4 z^4 d^4 z = -id^4 z_E,
\]
and \((\partial \varphi)^2 = (\partial \varphi_E)^2 - (\partial \varphi)^2 = -(\partial \varphi)^2\). (25)

Finally,
\[
S = -i \int d^4 z E \left\{ -\frac{1}{2} (\partial \varphi)^2_E - \frac{m^2}{2} \varphi^2 - \frac{\lambda}{4!} \varphi^4 \right\}. \tag{26}
\]

Consider the diagram
\[
\begin{array}{c}
\text{Diagram 1} \\
\frac{1}{2} = \frac{1}{2} I_{go} = -\frac{\lambda}{2} \int \frac{d^4 p}{(2\pi)^4} \frac{1}{p^2 + m^2}.
\end{array} \tag{27}
\]

First of all, consider the cut-off calculation of this simple diagram,
\[
\frac{1}{2} I_{go} = -\frac{\lambda}{16\pi^2} \int_0^\Omega \frac{4p^3 dp}{p^2 + m^2} = -\frac{\lambda}{32\pi^2} \left\{ \int_0^\Omega p^2 dp^2 - m^2 \int_0^\Omega \frac{p^2 dp^2}{p^2 + m^2} \right\} = -\frac{\lambda}{32\pi^2} \left\{ \Omega^2 - m^2 \ln \frac{\Omega^2}{m^2} \right\}, \tag{28}
\]

where the \( O(\Omega^{-1}) \) terms were omitted as irrelevant ones.

The dimensional regularization of this diagram is not equally simple, but will prove instructive for the future. We have
\[
I_4 \rightarrow I_{2\omega} = \int \frac{d^2 \omega}{(2\pi)^2} \frac{1}{(2\pi)^2 \Gamma(\omega + 1)} \int_0^\infty \frac{p^{2\omega - 1} dp}{p^2 + m^2}. \tag{29}
\]

Recall that \( \Gamma(\omega + 1) = \omega \Gamma(\omega) \), hence
\[
I_{2\omega} = \frac{2\pi^{\omega} \Gamma(\omega)}{(2\pi)^2 \Gamma(\omega + 1)} \int_0^\infty \frac{p^{2\omega - 1} dp}{p^2 + m^2}. \tag{30}
\]

This integral can be expressed via the Beta function
\[
B(x, y) = \frac{\Gamma(x) \Gamma(y)}{\Gamma(x + y)} = \int_0^\infty dtt^{x-1}(1 + t)^{-x-y}. \tag{31}
\]

In (30), we denote \( p^2 = tm^2 \) and obtain
\[
p^{2\omega - 1} dp = \frac{1}{2} m^2 \omega \, t^{\omega - 1} dt,
\]

Then,
\[
I_{2\omega} = \frac{\pi^\omega (m^2)^{\omega - 1}}{(4\pi^2)^{\omega}} \frac{\Gamma(\omega)}{\Gamma(\omega + 1)} \int_0^\infty dtt^{\omega - 1}(1 + t)^{-1}
\]
\[
= \frac{1}{(4\pi)^{\omega}} \frac{(m^2)^{\omega - 1}}{\Gamma(\omega)} B(\omega, 1 - \omega)
\]
\[
= \frac{(m^2)^{\omega - 1} \Gamma(\omega) \Gamma(1 - \omega)}{(4\pi)^{\omega} \Gamma(1)}
\]
\[
= \frac{(m^2)^{\omega - 1} \Gamma(1 - \omega)}{(4\pi)^{\omega}} \Gamma(1 - \omega), \tag{32}
\]

where we have identified \( x - 1 = \omega = 1 \) and \( -x - y = -1 \) as arguments of (31).

One can use Eq. (17) to rewrite the result (32):
\[
I_{2\omega} = (m^2)^{\omega - 1} \frac{\pi^\omega}{(4\pi)^{\omega}} \left( -\frac{1}{2 - \omega} + \gamma - 1 \right)
\]
\[
= \frac{m^2}{(4\pi)^{\omega} (m^2)^{\omega - 2}} \left( \frac{m^2}{4\pi \mu^2} \right)^{\omega - 2} \times \left( -\frac{1}{2 - \omega} + \gamma - 1 \right), \tag{33}
\]

where \( \mu \) is a renormalization parameter, with the mass dimension, \( [\mu] = [m] \). Furthermore,
\[
\left( \frac{m^2}{4\pi \mu^2} \right)^{\omega - 2} = e^{(\omega - 2) \ln \left( \frac{m^2}{4\pi \mu^2} \right)}
\]
\[
= 1 + (2 - \omega) \ln \left( \frac{4\pi \mu^2}{m^2} \right) + \cdots, \tag{34}
\]

and we finally arrive at
\[
I_{2\omega} = \frac{m^2}{(4\pi)^{\omega} (m^2)^{\omega - 2}} \left[ \frac{1}{2 - \omega} + \gamma - 1 - \ln \left( \frac{4\pi \mu^2}{m^2} \right) \right]
\]
\[
= m^2 (\mu^2)^{\omega - 2} \left[ \frac{1}{2 \varepsilon} + \frac{\gamma}{(4\pi)^2} - \frac{1}{(4\pi)^2} \right]
\]
\[
- \frac{1}{(4\pi)^2} \ln \left( \frac{4\pi \mu^2}{m^2} \right), \tag{35}
\]

which leads us to
\[
\frac{1}{2} I_{go} = -\frac{\lambda m^2}{2} (\mu^2)^{\omega - 2} \left[ \frac{1}{\varepsilon} + \frac{\gamma}{2(4\pi)^2} - \frac{1}{2(4\pi)^2} \right]
\]
\[
- \frac{1}{2(4\pi)^2} \ln \left( \frac{4\pi \mu^2}{m^2} \right). \tag{36}
\]

The next observation is that one can always redefine \( \mu \) and absorb the term \( \gamma - 1 \) into the \( \ln \mu \). Of course, this is not a compulsory operation.
The comparison between (36) and the result in the cut-off regularization (28) shows that in dimensional regularization there is nothing like quadratic divergences $\mathcal{O}(\Omega^2)$. On the other hand, there is a direct relationship between the leading logarithm $\ln \Omega^2$ term and the $1/\varepsilon$ term. The correspondence is given by the relation

$$\ln \frac{\Omega^2}{m^2} \longleftrightarrow -\frac{\mu^{n-4}}{\varepsilon}, \quad \varepsilon = (4\pi)^2(n-4),$$

which is universal and holds for all logarithmically divergent diagrams. Let us note that this is a particular manifestation of the general rule. The leading logarithms are the same in all regularization schemes [19].

The expression (36) has no dependence on the external momenta and therefore does not contribute to the nonlocal part. However, the situation is different for other loop integrals.

### 2.3. UV Divergence and the Nonlocal Form Factor

As a second example, consider the diagram

![Diagram](image)

$$I_4 = \frac{\lambda^2}{2} \int \frac{d^4p}{(2\pi)^4} \frac{1}{(p^2 + m^2)[(p-k)^2 + m^2]}. \quad (38)$$

As the first step, we derive the divergent part of (38) in the cut-off regularization. To this end we make the following transformation:

$$I_4 = \int \frac{d^4p}{(2\pi)^4} \frac{1}{(p^2 + m^2)[(p-k)^2 + m^2]}$$

$$= \int \frac{d^4p}{(2\pi)^4} \frac{1}{(p^2 + m^2)^2}$$

$$+ \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 + m^2}$$

$$\times \left[ \frac{1}{[(p-k)^2 + m^2]} - \frac{1}{p + m^2} \right]. \quad (39)$$

Recall that $d^4p = \pi^2 p^2 dp^2$ in $n = 4$. Therefore, the first integral is logarithmically divergent, and the second one is finite. Then,

$$I_4^{\text{div}} = \int \frac{p^2 dp^2}{(4\pi)^2(p^2 + m^2)}$$

$$= \frac{1}{(4\pi)^2} \ln \frac{\Omega^2}{m^2} + (\text{finite terms}).$$

and hence (38) is

$$\frac{\lambda^2}{2(4\pi)^2} \ln \frac{\Omega^2}{m^2} + (\text{finite terms}). \quad (40)$$

Let us now start with the dimensional regularization calculation. First we have to define

$$I_{2\omega} = \int \frac{d^{2\omega}p}{(2\pi)^{2\omega}} \frac{1}{((p^2 + m^2)(p-k)^2 + m^2)}.$$ 

Obviously, at $\omega = 2$ the integral $I_{2\omega}$ coincides with $I_4$, and also $I_{2\omega}$ is analytic on a complex plane in the vicinity of $\omega = 2$, where it has a pole (as we will see in brief).

We can use the Feynman formula (the simplest version)

$$\frac{1}{ab} = \int_0^1 \frac{da}{[a\alpha + b(1-\alpha)]^2}. \quad (42)$$

Using (42), one can cast (41) into the form

$$I_{2\omega} = \int_0^1 d\alpha \int \frac{d^{2\omega}p}{(2\pi)^{2\omega}} \frac{1}{(p^2 + a^2)^2}, \quad (43)$$

where

$$a^2 = m^2 + \alpha(1-\alpha)k^2. \quad (44)$$

Since (43) is convergent on the complex plane, one can make a shift of the integration variable, $p_\mu \rightarrow p_\mu - \alpha k_\mu$. A simple calculation gives us

$$I_{2\omega} = \int_0^1 d\alpha \int \frac{d^{2\omega}p}{(2\pi)^{2\omega}} \frac{1}{(p^2 + a^2)^2}. \quad (45)$$

The main advantage of (45) is that it does not depend on the angles. One can use the same steps that took us from (29) to (30), and the change of variable $p^2 = a^2 t$, to arrive at

$$I_{2\omega} = \int_0^1 d\alpha \int \frac{d^{2\omega}p}{(2\pi)^{2\omega}} \frac{1}{(p^2 + a^2)^2}$$

$$= \int_0^\infty dt \int_0^{(4\pi)^{2\omega}} \frac{d\Gamma(\omega)/\Gamma(2-\omega)\Gamma(2)}{(4\pi)^{2\omega}} a^{2\omega-2}(1+t)^{-2}. \quad (46)$$

Comparing this to (31), we identify $x = \omega$ and $y = 2 - \omega$. Then,

$$I_{2\omega} = \frac{1}{(4\pi)^{2\omega}} \int_0^\infty d\alpha \frac{\Gamma(\omega)\Gamma(2-\omega)\Gamma(2)\alpha^{2\omega-4}}{X}.$$
Recall that $\Gamma(2) = 1$ and $\alpha^2 = m^2 + \alpha(1 - \alpha) k^2$, while $\Gamma(2 - \omega) = \frac{1}{2 - \omega} - \gamma$. Then,

$$I_{2\omega} = \frac{1}{(4\pi)^{\omega}} \left( \frac{1}{2 - \omega} - \gamma \right) \times \int_0^1 \! d\alpha \left[ m^2 + \alpha(1 - \alpha) k^2 \right]^{\omega - 2}.$$  \hspace{1cm} (47)

Let us denote $\tau = k^2 / m^2$ and transform

$$\left[ m^2 + \alpha(1 - \alpha) k^2 \right]^{\omega - 2} = (m^2)^{\omega - 2} e^{(\omega - 2) \ln \left[ 1 + \alpha(1 - \alpha) \tau \right]}$$

$$= (m^2)^{\omega - 2} \left[ 1 - (2 - \omega) \ln \left\{ 1 + \alpha(1 - \alpha) \tau \right\} + O(\omega - 2)^2 \right].$$  \hspace{1cm} (48)

Substituting this expression into (47), we arrive at

$$I_{2\omega} = \frac{1}{(4\pi)^{\omega}} \left( \frac{1}{2 - \omega} - \gamma \right) \left( m^2 \right)^{\omega - 2} \left[ 1 - (2 - \omega) \right]$$

$$\times \int_0^1 \! d\alpha \ln \left\{ 1 + \alpha(1 - \alpha) \tau \right\}$$

$$= \frac{(m^2)^{\omega - 2}}{(4\pi)^{\omega}} \left[ \frac{1}{2 - \omega} \right]$$

$$- \gamma - \int_0^1 \! d\alpha \ln \left\{ 1 + \alpha(1 - \alpha) \tau \right\}.$$  \hspace{1cm} (49)

In the last formula, the first term is the divergence, and the integral over $\alpha$ represents the nonlocal form factor, which is the desired physical result. Also,

$$\frac{(m^2)^{\omega - 2}}{(4\pi)^{\omega}} = \left( \frac{\mu^2}{4\pi^2} \right)^{\omega - 2} \left( \frac{m^2}{4\pi \mu^2} \right)^{\omega - 2}$$

$$= \left( \frac{\mu^2}{4\pi^2} \right)^{\omega - 2} e^{(\omega - 2) \ln \left( \frac{4\pi \mu^2}{m^2} \right)}$$

$$= \left( \frac{\mu^2}{4\pi^2} \right)^{\omega - 2} \left[ 1 + (2 - \omega) \right]$$

$$\times \ln \left( \frac{4\pi \mu^2}{m^2} \right) + O(2 - \omega)^2.$$  \hspace{1cm} (50)

An elementary (albeit deserving to be checked by the reader) integration provides

$$\int_0^1 \! d\alpha \ln \left\{ 1 + \alpha(1 - \alpha) \tau \right\} = -2Y.$$  \hspace{1cm} (51)

where $a^2 = \frac{4\pi}{\tau + 4} = \frac{4k^2}{k^2 + 4m^2}$

and $Y = 1 - \frac{1}{a} \ln \left\{ \frac{2 + a}{2 - a} \right\}$.  \hspace{1cm} (52)

Replacing (50) and (51) in (49), we arrive at

$$I_{2\omega} = \frac{(\mu^2)^{\omega - 2}}{(4\pi)^2} \left[ \frac{1}{2 - \omega} - \gamma + \ln \left( \frac{4\pi \mu^2}{m^2} \right) + 2Y \right]$$

$$= (\mu^2)^{\omega - 2} \left(- \frac{2}{\varepsilon} - \frac{\gamma}{4\pi^2} + \frac{1}{4\pi^2} \ln \left( \frac{4\pi \mu^2}{m^2} \right) \right.$$  \hspace{1cm} (53)

$$+ \frac{2Y}{(4\pi)^2} \right).$$

The last thing to do is to explore the form factor $Y$ in the two extremes, namely, high- and low-energy limits,

1) UV, $k^2 \gg m^2$, that is, $\tau \gg 1$,

2) IR, $k^2 \ll m^2$, that is, $\tau \ll 1$.  \hspace{1cm} (54)

1) Consider the UV regime, that means $k^2 \gg m^2$ and $\tau \gg 1$. Thus,

$$a^2 = \frac{4k^2}{k^2 + 4m^2} = \frac{4}{1 + \frac{4m^2}{k^2}} = 4 \left( 1 - \frac{4m^2}{k^2} + \cdots \right),$$

hence $a \approx 2 - \frac{4m^2}{k^2}$. Then $2 + a \approx 4 - \frac{4m^2}{k^2}$, and $2 - a \approx \frac{4m^2}{k^2}$, such that

$$Y \approx 1 - \frac{1}{2} \ln \left( \frac{k^2}{m^2} \right).$$  \hspace{1cm} (55)

In this case, $I_{2\omega}$ in (53) includes the combination

$$I_{2\omega} = \frac{(\mu^2)^{\omega - 2}}{(4\pi)^2} \left[ \frac{1}{2 - \omega} + \ln \left( \frac{4\pi \mu^2}{m^2} \right) \right.$$  \hspace{1cm} (56)

$$+ 2 - \gamma - \ln \left( \frac{k^2}{m^2} \right) \right].$$

Let us stress that this is a very significant and important relation, as it shows two things at once. The first point is that the large-$k^2$ limit means large $\mu^2$ limit and v.v. Thus, it is sufficient to establish the large $\mu$ limit within the MS scheme of renormalization, to know the physical UV limit, that is, the behavior of the quantum system at high energies. Let us recall that we already know well how to explore the large-$\mu$ limit throughout the usual renormalization group, in particular, in curved space [22–24].

The second aspect is that one can always restore large-$\mu^2$ limit from the coefficient of the divergent term with the $1/\varepsilon$-factor. On the other hand, there is also a direct correspondence with the large cut-off
limit in (40). All in all, we can say that the UV limit is pretty well controlled by the leading logarithmic divergences, which can be derived easily within the heat-kernel methods, even without the use of Feynman diagrams.

2. Consider now the IR regime, when \( k^2 \ll m^2 \), or, equivalently, \( \tau \ll 1 \). Then,
\[
a^2 \approx \frac{k^2}{m^2} \ll 1
\]
and hence \( a \approx \frac{k}{m} \). As a consequence of this,
\[
\ln \frac{2 + a}{2 - a} \approx \ln \frac{2 + \frac{k}{m}}{2 - \frac{k}{m}} \approx \ln \left(1 + \frac{k}{m}\right) \approx \frac{k}{m},
\]
and therefore,
\[
Y = 1 - \frac{1}{a} \ln \left|\frac{2 + a}{2 - a}\right| \approx 1 - \frac{m}{k} \approx 0. \tag{57}
\]
In the zero-order approximation, there is no nonlocal form factor in the IR. This means that there is no \( \ln(k^2/m^2) \) that corresponds to the divergences \( 1/\varepsilon \), or \( \ln \Omega \). The next orders of expansion read
\[
Y = \frac{1}{12} \frac{k^2}{m^2} + \frac{1}{120} \left(\frac{k^2}{m^2}\right)^2 + \cdots. \tag{58}
\]
The first term is the evidence that the decoupling is quadratic in this case. The same quadratic dependence takes place in all cases when we can check it. In the IR limit, the divergences and momentum dependence do not correlate with each other. This phenomenon is called IR decoupling, or the “decoupling theorem.” It was discovered in QED in 1975 by Appelquist and Corrazzone [17].

As we already mentioned above, for the tadpole diagram (36) there is no nonlocal form factor. In this case, one may think that the UV divergence is “artificial,” but this is not a correct viewpoint, because in general the logarithmic form factor corresponds to the sum of all divergent contributions, including the ones of the tadpoles. This argument is necessary for the correct calculation in curved space-time [3], since only taking it into account one can establish the relationship between leading logs of momentum in the UV and the dependence on \( \mu \). Now we can better understand this point, since we know that the \( \mu \)-dependence in (36) appears together and in not separable from the divergent term.

3. NONLOCAL FORM FACTORS IN CURVED SPACE-TIME

Let us now turn around to the functional method and consider in full detail the derivation of the form factors using the heat-kernel solution of [11].

The one-loop contribution to the Euclidean Effective Action for a massive field is defined as the trace of the coincidence limit of the logarithm of determinant of the bilinear form of the action, or, equivalently, as an integral of the heat kernel over the proper time \( s \),
\[
\Gamma^{(1)} = \frac{1}{2} \text{Tr} \ln \left( -\hat{\square} + m^2 - \hat{P} + \frac{1}{6} \hat{R} \right)
= \frac{1}{2} \int_0^\infty \frac{ds}{s} \text{Tr} K(s). \tag{59}
\]
This formula is valid for bosonic fields in Euclidean space-time, while for fermions the overall sign in Eq. (59) must be changed. Here \( K(s) \) is the heat kernel of the bilinear form of the classical action of the theory, \( \square = \nabla^2 \) is the covariant Laplacian, and
\[
\text{Tr} K(s) = \frac{(\mu^2)^{2-\omega}}{(4\pi s)^\omega} \int d^4x \sqrt{g} e^{-sm^2}
\times \text{tr} \left\{ \hat{1} + s\hat{P} + s^2 \left[ \hat{1} R_{\mu\nu} f_1(\tau) R^{\mu\nu} + \hat{1} R f_2(\tau) R + \hat{P} f_3(\tau) R + \hat{P} f_4(\tau) \hat{P} + \hat{R}_{\mu\nu} f_5(\tau) \hat{R}^{\mu\nu} \right] \right\}. \tag{60}
\]
Here \( \tau = -s\square \), and we use notation \( \hat{R}_{\mu\nu} = [\nabla_\mu, \nabla_\nu] \) from [9]. Let us note that we use \( \square = g_{\mu\nu} \nabla_\mu \nabla_\nu \) even in Euclidean space. In Eq. (60), the terms between braces are matrices in the space of the fields (scalar, vector, or fermion). The zero-order term proportional to \( \text{Tr} \hat{1} \) corresponds to the quartic divergence, or to the coefficient \( a_0 \) in the Schwinger-DeWitt expansion. The term with \( \text{Tr} \hat{R} \) corresponds to the quadratic divergences, or to the \( a_1 \) coefficient, and all the rest corresponds to the logarithmic divergences and is related to the \( a_2 \) coefficient plus finite terms. The infinite tower of terms of third and higher orders in curvatures is omitted in this formula.

As we already saw in the diagrams-based approach, the \( a_0 \) and \( a_1 \) terms can be eliminated by the choice of the regularization scheme, so we shall mainly focus on \( a_2 \) and related terms. The functions \( f_{1,5} \) have the form \( [11] \)
\[
f_1(\tau) = \frac{f(\tau) - 1 + \tau/6}{\tau^2},
f_2(\tau) = \frac{f(\tau) - 1 - \tau^2}{24\tau} - \frac{f(\tau) - 1 + \tau/6}{8\tau^2},
f_3(\tau) = \frac{f(\tau)}{12} + \frac{f(\tau) - 1}{2\tau},
f_4(\tau) = \frac{f(\tau)}{2},
\]
2 Indeed, there are finite nonlocal surface terms related to \( a_1 \), and these are regularization-independent. One can learn about this aspect in [4, 13].
\[ f_5(\tau) = \frac{1 - f(\tau)}{2\tau}, \]  

(61)

where

\[ f(\tau) = \int_0^1 d\alpha \, e^{-\alpha(1-\alpha)\tau}, \quad \text{and} \quad \tau = -s\square. \]  

(62)

In what follows we shall describe the derivation of the integral in (59) for the particular case of a massive scalar field.

4. FORM FACTORS FOR THE MASSIVE SCALAR THEORY

The action for the theory with the general nonminimal coupling to the scalar curvature is

\[ S = \int d^4x \sqrt{g} \left\{ \frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi + \frac{1}{2} (m^2 + \xi R) \varphi^2 \right\}, \]  

(63)

and therefore,

\[ \hat{P} = -\left( \xi - \frac{1}{6} \right) R, \quad \text{and} \quad \hat{R}_{\alpha\beta} = 0. \]  

(64)

In the case under consideration \( \hat{I} = 1 \), also it is good to note that we did not include \( m^2 \) into \( \hat{P} \), as it was done with \( \mathcal{P} \) in (2).

According to (59) and (60), the bilinear in curvatures part of the effective action can be given by the proper-time integral of the heat kernel,

\[ \Gamma^{(1)} = \frac{1}{2} \int_0^\infty \frac{ds}{s} \frac{m^{2(2-\omega)}}{(4\pi s)^{\omega}} \int d^4x \sqrt{g} e^{-sm^2} \]

\[ \times \text{tr} \left\{ 1 + s\hat{P} + s^2 \left[ R_{\mu\nu} f_1(-s\nabla^2) R^{\mu\nu} + R f_2(-s\nabla^2) R + \hat{P} f_3(-s\nabla^2) R + \hat{R}_{\mu\nu} f_5(-s\nabla^2) \hat{R}^{\mu\nu} \right] \right\}. \]  

(65)

Let us derive the integrals over proper time in Eq. (65), starting from the simplest ones.

4.1. Zero-Order Term

Consider the term which corresponds to the \( a_0 \)-coefficient in the expression for the divergences,

\[ \Gamma_0^{(1)} = \frac{1}{2} \int_0^\infty \frac{ds}{s} \frac{m^{2(2-\omega)}}{(4\pi s)^{\omega}} \int d^4x \sqrt{g} e^{-sm^2} \]

\[ \times \int d^4x \sqrt{g} e^{-sm^2}. \]  

(66)

It proves useful to make a change of variables as follows:

\[ s = \frac{t}{m^2}, \quad ds = \frac{dt}{m^2}, \]

\[ \frac{ds}{s^{1+\omega}} = \frac{dt}{t^{1+\omega}}. \]  

(67)

Then the integral becomes

\[ \Gamma_0^{(1)} = \frac{1}{2} \int d^4x \sqrt{g} \frac{m^{4(2-\omega)}}{(4\pi)^{2\omega}} \int_0^\infty \frac{dt}{t^{1+\omega}} e^{-t} \]

\[ = \frac{1}{2} \int d^4x \sqrt{g} \frac{m^{4(2-\omega)}}{(4\pi)^{2\omega}} \left\{ \frac{m^2}{(2\pi)^2} \right\}^{\omega-2} \]

\[ \times \left[ \frac{1}{(2-\omega)} + \frac{3}{2} + O(2-\omega) \right] = \frac{1}{2(4\pi)^2} \int d^4x \sqrt{g} \]

\[ \times \left[ \frac{1}{2-\omega} + \ln \left( \frac{4\pi^2 m^2}{m^2} \right) + \frac{3}{2} \right]. \]  

(68)

In the calculation presented above we have used the relations (18) and (34). It proves useful to introduce the following new notation:

\[ \frac{1}{\varepsilon_{\omega,\mu}} = \frac{1}{2(4\pi)^2} \left[ \frac{1}{\omega-2} - \ln \left( \frac{4\pi^2 m^2}{m^2} \right) \right]. \]  

(69)

Then the one-loop contribution to the cosmological constant term (68) becomes

\[ \Gamma_0^{(1)} = \int d^4x \sqrt{g} \left[ -\frac{1}{\varepsilon_{\omega,\mu}} + \frac{3}{4(4\pi)^2} \right] \frac{m^4}{2}. \]  

(70)

We can observe that this expression consists of the UV divergence, corresponding to the -term hidden in \( 1/\varepsilon_{\omega,\mu} \) and the irrelevant constant term, which can be easily absorbed into \( 1/\varepsilon_{\omega,\mu} \) by changing \( \mu \). There is no non-local form factor in the expression (70). As we already explained above, this is a natural result since such a form factor should be constructed from \( \square \), which, when acting on \( m^4 \), gives zero.

4.2. First-Order Term

In the next order in \( s \), we meet

\[ \Gamma_1^{(1)} = \frac{1}{2} \int_0^\infty \frac{ds}{s} \frac{m^{2(2-\omega)}}{(4\pi s)^{\omega}} \int d^4x \sqrt{g} e^{-sm^2} \]

\[ \times \text{tr}(s\hat{P}) = -\frac{1}{2} \int_0^\infty \frac{ds}{s^{1+\omega}} e^{-sm^2} \]

\[ \times \int d^4x \sqrt{g} \frac{m^{4(2-\omega)}}{(4\pi)^{2\omega}} \left( \xi - \frac{1}{6} \right) R \]

\[ = -\frac{1}{2} \frac{m^{2(2-\omega)}}{(4\pi)^{2\omega}} m^{2(\omega-1)} \int d^4x \sqrt{g} \left( \xi - \frac{1}{6} \right). \]  

(71)
\[ \times \Gamma(1 - \omega)R = \left[ -\frac{1}{\varepsilon_{\omega,\mu}} + \frac{1}{2(4\pi)^2} \right] \]

\[ \times \left( \xi - \frac{1}{6} \right) \int d^4x \sqrt{\text{det} g} m^2 R, \quad (71) \]

where we have used the expansion (34), the definition (69), and the relation

\[ \text{tr} P = -\left( \xi - \frac{1}{6} \right) R. \quad (72) \]

Thus, without invoking the surface terms [13], the effective action is local, and the logarithmic dependence on the renormalization parameter \( \mu \) is completely controlled by the pole of \( 1/(2 - \omega) \). The results (70) and (71) enable one to construct the Minimal Subtraction based renormalization group equations for the cosmological constant density and the Newton constant, but they do not provide the nonlocal terms hidden behind the renormalization group.

### 4.3. Second-Order Terms

Working with the next-order terms is much more involved. We shall calculate them one by one, to find the coefficients \( l^i_{1,5} \) and \( l_{1,5} \), which determine the final form factors of the \( R_{\mu\nu} \cdot R^{\mu\nu} \) and \( R \cdot R \)-terms. The general expression in the second order in curvature can be cast into the form

\[ \bar{\Gamma}^{(2)}_2 = \bar{\Gamma}_{R_{\mu\nu}^{R}} + \bar{\Gamma}_{R^2} = \sum_{k=1}^{5} \bar{\Gamma}_k. \quad (73) \]

Finally, recalling that \( \bar{\nabla}_\alpha = 0 \), we have to evaluate

\[ \bar{\Gamma}^{(1)}_2 = \frac{1}{2} \int_0^\infty ds e^{-sm^2} s^{1-\omega} \left( \mu^2 \right)^{2-\omega} \frac{1}{(4\pi)^\omega} \]

\[ \times \left[ \int d^4x \sqrt{\text{det} g} \left\{ R_{\mu\nu} f_1 (-s) R_{\mu\nu}^{R} \right. \right. \]

\[ \left. + R \left[ f_2 (-s) - \left( \xi - \frac{1}{6} \right) f_3 (-s) \right) \right. \]

\[ \left. + \left( \xi - \frac{1}{6} \right)^2 f_4 (-s) \right\} \right]. \quad (74) \]

By replacing the relations (61) and (62) in (73) and, once again, \( -s \) by \( \tau \), we arrive at

\[ \bar{\Gamma}^{(1)}_2 = \frac{1}{2} \int_0^\infty ds e^{-sm^2} s^{1-\omega} \left( \mu^2 \right)^{2-\omega} \frac{1}{(4\pi)^\omega} \]

\[ \times \left[ \int d^4x \sqrt{\text{det} g} \left\{ \frac{f(\tau)}{\tau^2} - \frac{1}{\tau^2} + \frac{1}{6\tau} \right) R_{\mu\nu} \right. \]

\[ + R \left( \frac{1}{288} - \frac{\xi}{12} + \frac{\xi^2}{2} \right) f(\tau) \]

\[ + \left( \frac{1}{24} - \frac{\xi}{2} \right) \frac{f(\tau)}{\tau} - \frac{f(\tau)}{8\tau^2} \]

\[ + \left( \frac{\xi}{2} - \frac{1}{16} \right) \frac{1}{\tau} + \frac{1}{8\tau^2} \right] R \right\}, \quad (75) \]

where the condensed notation \( \bar{\xi} = \xi - 1/6 \) has been used.

It proves useful to introduce a new set of coefficients:

\[ l_1^* = 0, \quad l_2^* = 0, \quad l_3^* = 1, \quad \]

\[ l_4^* = \frac{1}{6}, \quad l_5^* = -1, \quad \text{and} \]

\[ l_1 = \frac{1}{288} - \frac{1}{12} \bar{\xi} + \frac{1}{2} \bar{\xi}^2, \quad l_2 = \frac{1}{24} - \frac{1}{2} \bar{\xi}, \quad \]

\[ l_3 = -\frac{1}{8} = -l_5, \quad l_4 = -\frac{1}{16} + \frac{1}{2} \bar{\xi}. \quad (76) \]

Furthermore, the basic integrals from Eq. (75) will be denoted as (recall that \( \tau = -s \))

\[ M_1 = \int_0^\infty \frac{ds}{(4\pi)^\omega} e^{-m^2 s} s^{1-\omega} f(\tau) \]

\[ \frac{\mu^{2(2-\omega)}}{(4\pi)^\omega} \int_0^\infty dt e^{-t} t^{1-\omega} f(\tau), \]

\[ M_2 = \int_0^\infty \frac{ds}{(4\pi)^\omega} e^{-m^2 s} s^{-\omega} f(\tau) \]

\[ \frac{\mu^{2(2-\omega)}}{(4\pi)^\omega} \int_0^\infty dt e^{-t} f(\tau) \]

\[ M_3 = \int_0^\infty \frac{ds}{(4\pi)^\omega} e^{-m^2 s} s^{1-\omega} f(\tau) \]

\[ \frac{\mu^{2(2-\omega)}}{(4\pi)^\omega} \int_0^\infty dt e^{-t} f(\tau) \]

\[ M_4 = \int_0^\infty \frac{ds}{(4\pi)^\omega} e^{-m^2 s} s^{-\omega} f(\tau) \]

\[ \frac{\mu^{2(2-\omega)}}{(4\pi)^\omega} \int_0^\infty dt e^{-t} \frac{1}{u^{2t+1}}, \]
To calculate the remaining three integrals, we introduce the form
\[ M_5 = \int_0^\infty \frac{ds}{(4\pi)^w} e^{-m^2 s} s^{1-\omega} \]

\[ = \frac{\mu^{2(2-\omega)}}{(4\pi)^w} m^{2(w-2)} \int_0^\infty dt e^{-t} \frac{1}{u^2 t^{1+\omega}}, \quad (77) \]

where we have already made change of variables (67) and also denoted (it is the Fourier conjugate of \( \tau \) in Eq. (48))

\[ u = -\frac{m^2}{\Box}. \quad (78) \]

A relevant observation is that all individual features of the present theory (like the scalar in the present case) are encoded in the coefficients (76), while the integrals (77) are universal in the sense that they will be the same for any theory which provides an operator of the form (1), at least with \( \hat{h}^\alpha = 0 \). Thus the derivation of the same set of integrals (77) enables one to derive the form factors in many interesting cases.

In the new notations (76) and (77), the second-order part of the one-loop effective action can be cast into the form

\[ \Gamma_{\alpha}^{(1)} = \Gamma_{\alpha}^{(1)} + \Gamma_{\alpha}^{(1)} R_{\mu\nu} + \Gamma_{\alpha}^{(1)} R = \frac{1}{2} \int d^4 x \sqrt{g} \]

\[ \times \sum_{k=1}^5 \left\{ R_{\alpha\mu l} M_k R_{\alpha\nu} + R l_k M_k R \right\}. \quad (79) \]

Let us now calculate the integrals (77). We will need the expansion (34) and the previous formulas (16), (18) for the gamma functions.

Taking these formulas into account, the derivation of \( M_4 \) and \( M_5 \) is an elementary exercise, and we just present the results:

\[ M_4 = -\frac{1}{(4\pi)^2 u} \left\{ 1 + \frac{1}{2 - \omega} + \ln \left( \frac{4\pi \mu^2}{m^2} \right) \right\}, \quad (80) \]

\[ M_5 = \frac{1}{(4\pi)^2 u^2} \left\{ \frac{3}{2} + \frac{1}{2 - \omega} + \ln \left( \frac{4\pi \mu^2}{m^2} \right) \right\}. \quad (81) \]

To calculate the remaining three integrals, we introduce the new notations

\[ a^2 = \frac{4u}{u + 4} = \frac{4\Box}{\Box - 4m^2}, \]

\[ \text{and} \quad \frac{1}{u} = \frac{1}{a^2} - \frac{1}{4}, \quad (82) \]

being exactly the Fourier image of (51) in the Euclidean space, with

\[ a^2 = \frac{4k^2}{k^2 + 4m^2} \geq 0, \text{ also } a^2 \leq 4. \quad (83) \]

We can assume, for definiteness, that \( a \) changes from \( a = 0 \) in the IR to \( a = 2 \) in the UV.

Furthermore, we will need the following integral:

\[ A = -\frac{1}{2} \int_0^1 d\alpha \ln \left( 1 + a(1 - \alpha) \right) \]

\[ = 1 - \frac{1}{a} \ln \left( \frac{2 + a}{2 - a} \right), \quad (84) \]

which is the same as \( Y \) from Eq. (52) in the coordinate space.

The remaining calculation of the first three integrals is not complicated, and we just give the final results in terms of \( a \) and \( A \):

\[ M_1 = \frac{2}{\epsilon_{\omega,\mu}} + \frac{2A}{(4\pi)^2}, \quad (85) \]

\[ M_2 = \left\{ \frac{2}{\epsilon_{\omega,\mu}} + \frac{1}{(4\pi)^2} \left( \frac{1}{12} - \frac{1}{a^2} \right) \right\} \]

\[ + \frac{1}{(4\pi)^2} \left\{ \frac{1}{18} - 4A \right\} \]

\[ = \frac{1}{(4\pi)^2} \left\{ \frac{1}{12} - \frac{1}{a^2} - \frac{4A}{3a^2} + \frac{1}{18} \right\}, \quad (86) \]

\[ M_3 = \frac{1}{(4\pi)^2} \left\{ \frac{3}{2} + \frac{1}{2 - \omega} + \ln \left( \frac{4\pi \mu^2}{m^2} \right) \right\} \]

\[ \times \left\{ \frac{1}{2a^4} - \frac{1}{12a^2} + \frac{1}{160} \right\} \]

\[ + \frac{8A}{15a^4} - \frac{7}{180a^2} + \frac{1}{400} \}. \quad (87) \]

Now one can construct a useful combinations for the scalar case, such as

\[ M_{R^2 \alpha \nu} = l_1^2 M_3 + l_4^2 M_4 + l_5^2 M_5 \]

\[ = M_3 + \frac{1}{6} M_4 - M_5 \]

\[ = \frac{1}{(4\pi)^2} \left\{ \frac{1}{2 - \omega} \left( \frac{1}{60} \right) + \ln \left( \frac{4\pi \mu^2}{m^2} \right) \left( \frac{1}{60} \right) \right\} \]

\[ + \frac{8A}{15a^4} + \frac{2}{45a^2} + \frac{1}{150} \}, \quad (88) \]

\[ M_{R^2} = l_1 M_1 + l_2 M_2 + l_3 M_3 + l_4 M_4 \]

\[ + l_5 M_5 = \left( \frac{1}{288} - \frac{1}{12} \xi + \frac{1}{2} \xi^2 \right) M_1 \]
Finally, we meet
\[ \bar{\Gamma}^{(1)}_2 = \frac{1}{2} \int d^4x \sqrt{g} \left\{ R_{\mu\nu} M_{R_\mu R_\nu} R^{\mu\nu} + RM_{R_\mu} R^\mu \right\}. \]  \
(90)

Let us note that there is a third term, related to the square of the Riemann tensor. However, for any integer \( N \) one can prove, utilizing the Bianchi identities and partial integrations, that (see, e.g., [25])
\[ E_{4,N} = R_{\mu\nu;\alpha\beta} \Box^N R^{\mu\nu;\alpha\beta} - 4 R_{\mu\nu} \Box^N R^{\mu\nu} + R \Box^N R = O(R^3) + \text{total derivatives}. \]  \
(91)

This means that in the bilinear in curvature approximation, such as the one we discuss here, one can safely use the reduction formula related to the Gauss-Bonnet term,
\[ R_{\mu\nu;\alpha\beta} f(\Box) R^{\mu\nu;\alpha\beta} = 4 R_{\mu\nu} f(\Box) R^{\mu\nu} - R f(\Box) R. \]  \
(92)

As a result, in the curvature-squared approximation, there is no way to see the nonlocalities associated to the Gauss-Bonnet combination. Hence, we can use either \( R^2_{\mu\nu} \) and \( R^2 \) terms, or some other equivalent basis. For various applications, the most useful basis consists of the square of the Weyl tensor instead of the square of the Ricci tensor. The transition can be done using the formulas
\[ C^2 = R^2_{\mu\nu;\alpha\beta} - 2 R^2_{\mu\nu} + \frac{1}{3} R^2 = E_4 + 2 W, \]  \
(93)
where \( W = R^2_{\mu\nu} - \frac{1}{4} R^2 \), \( E_4 \) is irrelevant, and
\[ \bar{M}_{R^2} = M_{R^2} + \frac{1}{3} M_{R^2_{\mu\nu}}. \]  \
(94)

Introducing the form factors \( k_W \) and \( k_R \),
\[ k_W = k_{R_{\mu\nu}} = \frac{8 A}{15 a^4} + \frac{2}{45 a^2} + \frac{1}{150}, \]  \
(95)
\[ k_R = A \xi^2 + \left( \frac{2 A}{3 a^2} + \frac{1}{18} - \frac{A}{6} \right) \xi - \frac{A}{18 a^2}, \]  \
(96)
and taking zero-, first-, and second-order terms together, one can write down the effective action up to the second order in curvatures:
\[ \bar{\Gamma}^{(1)}_2 = \frac{1}{2(4\pi)^2} \int d^4x \sqrt{g} \left\{ m^4 \left[ \frac{1}{2} \xi - \frac{1}{6} \right] \frac{A}{60(2-\xi)} + \ln \left( \frac{4\pi m^2}{m^2} \right) + \frac{1}{2} \right\} \]  
\[ + \frac{1}{2} C_{\mu\nu;\alpha\beta} f(\Box) R^{\mu\nu;\alpha\beta} + R \left[ \frac{1}{2} \left( \xi - \frac{1}{6} \right) \right] \left( \frac{1}{2} \right) \]  
\[ + \ln \left( \frac{4\pi m^2}{m^2} \right) + k_R \]  
\[ \left\{ \bar{R}_\mu \right\}. \]  \
(97)

Let us make some observations concerning the final result for the vacuum effective action for the scalar field (97), with the form factors (95) and (96). First of all, one has to stress that this action is essentially nonlocal in the higher derivative sector. The result is exact in the derivatives of the curvature tensor, but it is only of the second order in the curvatures themselves. On the other hand, the lower-derivative terms, namely, quantum corrections to the cosmological constant and to the term linear in curvature, do not have nonlocal parts.

The nonlocalities derived from the heat kernel and from the Feynman diagrams in dimensional regularization are the same. This is confirmed by the correspondence between the quantities \( Y \) and \( A \) in our two considerations, and also by the original calculations of [3] and [4] of the gravitational form factors.

For the massless (or UV, for the massive field) limit, we need to assume \(-\Box/m^2 \gg 1\), in the sense \( k^2/m^2 \gg 1 \) for the Euclidean momentum \( k \). Then the form factors \( k_W \) and \( k_R \) can be elaborated following the method of Eq. (56). According to (82), in this limit \( a \to 2 \). Then,
\[ k_W \sim -\frac{1}{120} \ln \left( \frac{\Box}{\mu^2} \right) \]  
\[ + \text{constant and vanishing terms}, \]  \
(98)
\[ k_R \sim -\frac{1}{2} \left( \xi - \frac{1}{6} \right) \frac{1}{2} \ln \left( \frac{\Box}{\mu^2} \right) \]  
\[ + \text{constant and vanishing terms}. \]  \
(99)
These relations show that in the UV limit one can restore the nonlocal terms from the logarithmic divergences. On the other hand, for the massive models out of the UV limit, the nonlocal terms have a complex structure, and there is no way to restore them from the divergences.

We can say that the logarithmic UV divergence controls the minimal subtraction scheme based renormalization group, covered by the $\mu$-dependence, and also agrees with the physical behavior of the theory in UV, that means the logarithmic dependence on the momenta $p$ in the regime when $(p/m) \to \infty$. The final observation over the form factor (95) is that the expression (98) enables one to find the Weyl-squared part of the conformal anomaly in the massless limit. To this end, one has to use the conformal parametrization of the metric $g_{\mu\nu} = g'_{\mu\nu} \exp\{2\sigma(x)\}$ and note that

$$\Box = e^{-2\sigma(x)}[\Box' + O(\sigma)].$$

Now, deriving the anomaly by the prescription

$$\langle T_{\mu}^{\nu} \rangle = -\frac{2}{\sqrt{-g}} \frac{\delta \Gamma[g_{\mu\nu}]}{\delta g_{\mu\nu}}$$

$$= \frac{1}{\sqrt{-g}} e^{-2\sigma} \frac{\delta \Gamma[g_{\mu\nu}e^{2\sigma}]}{\delta \sigma} \bigg|_{g_{\mu\nu}=g'_{\mu\nu}, \sigma \to 0},$$

we can immediately recover from (100) the $C^2$ part of the anomaly with the correct coefficient, identical to the one of the corresponding divergence [26].

Let us make one more observation concerning the form factor for the Weyl-squared term. The calculations described above have been done in the Euclidean signature. However, if performing the derivation with the Minkowski signature and $\Box = \Box_E \to \Box + i\varepsilon$ prescription, the UV form factor (98) gains an imaginary addition. The imaginary term is known to describe the creation of massless particles by the gravitational field, as discussed in [27] (see also [28] for a more detailed derivation). This result shows the relation between the conformal anomaly and the rate of particle creation, which is (in the leading approximation) proportional to the Weyl-squared and to the $R^2$-squared terms in the nonlocal effective action. It would be interesting to extend this result for the creation of massive particles in the very early Universe using the pseudo–Euclidean analog of the form factors (95) and (96), which can be also easily generalized to the massive fermions and vectors [12].

A similar derivation for the Gauss-Bonnet part of the anomaly is impossible, exactly because the corresponding form factor is of the third order in curvature, and therefore is beyond the scope of the present consideration. The calculation of this term in the strictly massless theory has been done in [29] and [30]. At the same time, the form factors are very helpful in better understanding the problem of ambiguity of the conformal anomaly, related to the local $R^2$ term in the anomaly-induced action and $\Box R$-term in the UV divergence [31]. In particular, using the covariant Pauli-Villars regularization, one can show that this ambiguity takes place not only in the dimensional [32, 33], but also in other regularizations. Moreover, if the conformal limit is achieved by taking the massless limit $\xi \to 1/6, m \to 0$ in the nonconformal model, the $R^2$ term remains nonlocal until the limit is taken, and then there is no discontinuity or ambiguity in the anomaly-induced result for the loop contribution.

In the IR limit, when $k^2 \ll m^2$, one can observe a very different situation. The asymptotic behavior of $Y$ and $k_W$ is, in this case, of the power-like form, e.g.,

$$Y = -\frac{1}{12} \frac{k^2}{m^2} \left(\frac{1}{10} \frac{k^2}{m^2}\right) +..., \quad (102)$$

$$k_W = -\frac{1}{840} \frac{k^2}{m^2} \left(\frac{1}{18} \frac{k^2}{m^2}\right) +... \quad (103)$$

One can see that there is no logarithmic “running” in the IR, and hence there is no direct relationship between the dependence on the momenta and on $\mu$ in this region. This is the gravitational decoupling, which can be also seen in the $\beta$ functions [3]. In other words, in the IR, nonlocal terms simply disappear, while the divergences remain the same. Thus the logarithmic divergences give important clues on the UV behavior of the theory but become noninformative in the IR.

5. CONCLUSIONS

We have presented a detailed derivation of the nonlocal form factor in the scalar massive theory in curved space-time. The logarithmically divergent part does not depend on the regularization, the same concerns the finite nonlocal part.

Let us briefly discuss some of the main unsolved problems related to the form factors.

In the recent work [34] the well-known result for the form factors from the mixed diagrams (see, e.g., [35]) has been generalized to the weak gravitational background. This type of calculations is interesting, as it enables one to explore the IR decoupling of the massive degrees of freedom in the renormalizable models of quantum gravity, which have higher derivatives and hence massive ghostlike degrees of freedom. Such a calculation would confirm that Einstein’s GR is the universal effective model of quantum gravity in the IR [36]. Indeed, this calculation meets serious technical difficulties, and it would be
more than useful to work with the functional methods instead of the diagrams.

Another interesting, albeit difficult problem, is to extend the integration of the heat kernel [3, 12] in the massive theories, to the third order in curvatures form factors, using the general result of [29]. Since in this case there are several external momenta, the study of decoupling may be more interesting, especially because the third-order terms may be relevant for creation of particles from vacuum.

In conclusion, both diagrammatic and functional derivations of the nonlocal parts of the effective action are, in general, well developed. However, there are new applications in this area, which may be challenging problems for the future.

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