THE ASSOCIATED GRADED OF THE TEST MODULE FILTRATION

AXEL STÄBLER

ABSTRACT. We show that the associated graded of the test module filtration \(\tau(M, f^\lambda)_{\lambda \geq 0}\) admits natural Cartier structures. If \(\lambda\) is an \(F\)-jumping number, then these Cartier structures are nilpotent on \(\tau(M, f^{\lambda-\varepsilon})/\tau(M, f^\lambda)\) if and only if the denominator of \(\lambda\) is divisible by \(p\). We also show that these Cartier structures coincide with certain Cartier structures that are obtained by considering certain \(D\)-modules associated to \(M\) that were used to construct Bernstein-Sato polynomials.

Moreover, we point out that the zeros of the Bernstein-Sato polynomial attached to an \(F\)-regular Cartier module correspond to its \(F\)-jumping numbers. This generalizes [BS16a, Theorem 5.4]. Finally, we use non-\(F\)-pure modules to prove a weaker connection between Bernstein-Sato polynomials of an arbitrary Cartier module \((M, \kappa)\) and certain jumping numbers attached to \(M\).

INTRODUCTION

To fix ideas let us consider a hypersurface \(f\) inside a polynomial ring \(R = \mathbb{F}_p[x_1, \ldots, x_n]\). In order to study the singularities of \(f\) one may consider the test ideal filtration \(\tau(R, f^\lambda)\) which is a decreasing \(\mathbb{Q}\)-indexed right-continuous filtration of ideals that is defined in terms of certain \(p^{-e}\)-linear maps. One has \(\tau(R, f^0) = R\) and the smallest \(\lambda\) for which \(\tau(R, f^\lambda) \neq R\) is called the \(F\)-pure threshold, \(\text{fpt}(f)\) for short. It has been known for a long time that if \(f \in \mathbb{Q}[x_1, \ldots, x_n]\) and one considers the various reductions \(f_p\) of \(f\) to positive prime characteristic, then \(\text{fpt}(f_p) \xrightarrow{p \to \infty} \text{lc}(f)\) and \(\text{fpt}(f_p) \leq \text{lc}(f)\) for almost all \(p\) ([HY03, Theorem 6.8]). Here \(\text{lc}(f)\) is the so-called log-canonical threshold which is a similar characteristic zero invariant that is defined using an embedded resolution of singularities. It is moreover conjectured that \(\text{fpt}(f_p) = \text{lc}(f)\) for infinitely many \(p\) ([MTW04, Conjecture 3.6]).

It has been observed for quasi-homogeneous hypersurfaces that if the log canonical threshold does not coincide with the \(F\)-pure threshold, then the denominator of the \(F\)-pure threshold is a \(p\)th power (see [HNuBWZ16, Theorem 3.5] or [Mül16, Lemma 3.7 (2)]). On the other hand, there are only two known (families of) examples where the \(F\)-pure threshold does not coincide with the log canonical threshold but the denominator of the \(F\)-pure threshold is not divisible by \(p\) (see [MTW04, Example 4.5] and [CHSW16, Proposition 2.7, Corollary 2.10]).

From the point of view of birational geometry \(F\)-jumping numbers with a denominator divisible by \(p\) are special in the sense that the correspondence between certain Cartier linear maps and certain \(\mathbb{Q}\)-divisors breaks down in this case (e.g.}

Date: October 9, 2018.

2010 Mathematics Subject Classification. Primary 13A35; Secondary 14F10, 14B05.
This correspondence is central to many applications of test ideals in birational geometry.

The purpose of this note is to further illustrate that $F$-pure thresholds whose denominators are divisible by $p$ are special in the following sense. We call a finitely generated $R$-module $M$ endowed with a $p^{-1}$-linear map, i.e. an $R$-linear map $κ : F_κM → M$ a Cartier module. One can associate to $M$ and $f ∈ R$ a test module filtration $τ(M, f^λ)_{λ ≥ 0}$ that has similar properties as in the case $M = R$. In particular, we can form the associated graded $Gr^λM = ∐_{λ > 0} τ(M, f^{λ−τ})/τ(M, f^λ)$. In Section 2 we will attach a natural Cartier module structure to these summands and show that such a Cartier module is nilpotent if and only if the denominator of $λ$ is divisible by $p$. Here nilpotent means that some power of the structural map $κ$ acts as zero on the module.

This notion of nilpotence is interesting for the following reason. Nilpotent Cartier modules form a Serre subcategory so that we may consider the attached localized category of Cartier crystals. This category is then equivalent ([BB11]) to the category of unit $R[Γ]$-modules of Emerton and Kisin ([EK04]) and anti-equivalent to the category of perverse constructible $ℱ_p$-sheaves on the étale site associated to $Spec R$.

The first main result that we obtain is

**Theorem** (Theorem 2.3). Let $R$ be essentially of finite type over an $F$-finite field, $(M, κ)$ be a Cartier module, $f ∈ R$ and $λ$ an $F$-jumping number of the test module filtration of $M$ along $f$. The Cartier structure on $Gr^λM$ defined by $κ f^{a_e(λ)}$ is not nilpotent if and only if $a_e(λ) = λ(p^e − 1)$ and this quantity is an integer. In particular, if the denominator of $λ$ is divisible by $p$, then all these Cartier structures are nilpotent.

Finally, we also show, extending and vastly simplifying some results of [BS16a, Section 4], that test modules admit a simple description, akin to the case of an ideal in a polynomial ring, in many cases. A special case is the following

**Theorem** (cf. Theorem 2.6). Let $R$ be essentially of finite type over an $F$-finite field, $(M, κ)$ an $F$-regular Cartier module and $f ∈ R$ an $M$-regular element, then one has $τ(M, f^λ) = κ f^{[λ/p]e}M$ for all $e ≥ 0$.

There is evidence that $F$-pure thresholds with denominator divisible by $p$ are related to certain arithmetic phenomena like non-ordinarity via the anti-equivalence mentioned above. If $X = V(f) ⊆ P_k^{n+1}$ defines a smooth quasi-homogeneous Calabi-Yau hypersurface then the $F$-pure threshold of $f$ in $k[x_0, . . . , x_{n+1}]$ is $1 − \frac{h}{p}$, where $0 ≤ h ≤ n$ is the order of vanishing of the Hasse-invariant associated to a certain deformation space of $X$ (see [Mil16] for details). In particular, $h ≠ 0$ if and only if $X$ is not ordinary. As a special case if $dim X = 1$, then non-ordinariness coincides with the case $H^1_{cris}(X)_0 = H^1_{ét}(X) = 0$, where the index 0 indicates the slope zero part of crystalline cohomology. In this case, the crystalline cohomology is concentrated at slope $\frac{1}{2}$.

Moreover, we will also relate these Cartier structures on $Gr^λM$ to certain Cartier structures obtained from eigenspaces of higher Euler operators that play a crucial role when constructing Bernstein-Sato polynomials in positive characteristic (see [BS16a]). Similar results were observed by Bitoun ([Bit15]) for the case that $M = R$ in the (equivalent) framework of unit $R[Γ]$-modules. His proof relies on formal properties of $p$-adic expansions. We will show that it is in fact also a formal
In this direction is Theorem 3.8. Since its formulation requires several $\mathcal{D}$-module theoretic we only state a partial result in the introduction and refer the reader to Section 3 for a detailed exposition.

Let $\mathcal{D}_{\mathbb{R}[\theta]}^e$ be the ring of differential operators of order $\leq p^e - 1$. We denote by $\theta_i \in \mathcal{D}_{\mathbb{R}[\theta]}^e$ so-called higher Euler operator $t^i \partial^i_i$. To the data $(M, \kappa, f, \lambda)$ one associates $\mathcal{D}_{\mathbb{R}[\theta]}^e$-modules (for varying $e$). In order to construct Bernstein-Sato polynomials one looks at certain eigenspaces of the higher Euler operators associated to these $\mathcal{D}_{\mathbb{R}[\theta]}^e$-modules.

**Theorem** (cf. Theorem 3.8). Let $R$ be regular and essentially of finite type over an $F$-finite field, $(M, \kappa)$ an $F$-regular Cartier module and $f$ an $M$-regular element. Then for $e \gg 0$ there is an isomorphism $F^e \text{Gr}^\lambda M \to E_i$, where $E_i$ is a certain eigenspace associated to a certain $\mathcal{D}_{\mathbb{R}[\theta]}^e[\theta_1, \ldots, \theta_{p^e - 1}]$-module. This isomorphism induces a transition map $F^e \text{Gr}^\lambda M \to F^e \text{Gr}^\lambda M$. This transition map is induced by a non-nilpotent morphism $C : \text{Gr}^\lambda M \to F^s \text{Gr}^\lambda M$ if and only if $\lambda(p^e - 1) \in \mathbb{Z}$ and $e, a$ are multiples of $s$. Moreover, in this case $C$ is the adjoint of $\kappa^e f^{\lambda(p^e - 1)}$.

We will obtain a partial generalization of the correspondence between $F$-jumping numbers and zero of Bernstein-Sato polynomials for the case that $(M, \kappa)$ is $F$-regular to arbitrary Cartier modules. The precise statement we prove is

**Theorem** (cf. Theorem 5.3). Let $R$ be an $F$-finite regular ring and $(M, \kappa)$ a Cartier module. Fix a rational number $\lambda$. If for some $e \gg 0$ such that $\lambda(p^e - 1) \in \mathbb{Z}$ one has that $\lambda - \frac{1}{p^e}$ is a zero of the Bernstein-Sato polynomial $b_{M,f}^e(s)$, then $\text{Gr}^\lambda M \neq 0$.

Here $\text{Gr}^\lambda M$ is the filtration of $M$ by so-called non-$F$-pure modules. We will study their basic properties in Section 4 and also explain how they are a generalization of non-$F$-pure ideals studied in [FST11]. We will also see that some of the pathologies that non-$F$-pure ideals exhibit in comparison to characteristic zero are constrained to the cases where the “jumps” have a denominator divisible by $p$.

We start with a short review of test modules in Section 1. In Section 2 we discuss (non)-nilpotence of Cartier structures defined on the associated graded of the test module filtration. Then after recalling the necessary setup we relate these Cartier structures to the construction one uses to obtain Bernstein-Sato polynomials in Section 3. In Section 4 we introduce the notion of non-$F$-pure module and study its basic properties. In the last section we use these to prove Theorem 5.3.

**Acknowledgements.** I thank Kevin Tucker and Mircea Mustață for a useful discussion. Part of this paper was conceived while the author was visiting the University of Utah. I thank Karl Schwede for inviting me and for inspiring discussions. The author is supported by grant STA 1478/1-1 of the Deutsche Forschungsgemeinschaft (DFG).

1. A brief review of test modules

In this section we review very briefly the necessary facts on test modules that we need. We refer the reader to [BS16b] for a detailed treatment.

Fix an $F$-finite ring $R$. A Cartier module is a pair $(M, \kappa)$, where $M$ is an $R$-module and $\kappa : F^e \rightarrow M$ is an $R$-linear map. In many cases we assume $e = 1$. It is however crucial to allow $e \geq 1$ in order to obtain meaningful Cartier structures on
certain quotients. If we assume that Spec $R$ is embeddable into a smooth scheme, then for given $e$ there is a contravariant functor to the category constructible $\mathbb{F}_{p^e}$-sheaves on the étale site. If we localize at nilpotent Cartier modules (to be defined below), call the resulting category Cartier crystals, then this functor induces an anti-equivalence between Cartier crystals and perverse constructible $\mathbb{F}_{p^e}$-sheaves on the étale site (see [Sch16] and references therein).

Given a Cartier module $(M, \kappa)$ and $f \in R$ and $\lambda$ a non-negative rational number we can form $R$-linear maps $\kappa^e f^{(\lambda p^e)} : F^e_* M \to M$ given by $m \mapsto \kappa^e (f^{(\lambda p^e)} m)$ for varying $e \geq 1$. The collection of these maps with addition induced by the one in $M$ and multiplication by composition form an $\mathbb{N}$-graded subring $C$ of $\bigoplus_{e \geq 0} \text{Hom}(F^e_* M, M)$, where we set $C_0 = R$. It has both a left and a right $R$-module structure that are related via $r \varphi = \varphi p^e$ for any homogeneous element $\varphi$ of degree $e$. This ring is a special case of a so-called Cartier algebra.

1.1. Definition. Let $R$ be an $F$-finite ring. A Cartier algebra is an $\mathbb{N}$-graded ring $\bigoplus_{e \geq 0} C_e$ with $C_0 = R$ satisfying the relation $r \varphi = \varphi p^e$ for any $\varphi \in C_e$ and $r \in R$.

As usual $C_+ = \bigoplus_{e \geq 1} C_e$. We will write $C^h_+$ for $(C_+)^h$.

1.2. Remark. We will mostly use Cartier algebras of the form $\kappa^e f^{(\lambda p^e)}$ in this article. The reader familiar with test ideals may notice that people also often use algebras of the form $\kappa^e f^{(\lambda p^e - 1)}$. If one computes test modules then both notions yield the same result (cf. [Stä16, Lemma 3.1]). However, the categories of crystals are not the same and this will play an important role later on (cf. Section 4).

A $C$-module means a left module over $C$. We will moreover always assume that it is finitely generated as an $R$-module. We call a $C$-module $M$ nilpotent if $C_+^a M = 0$ for some (equivalently all) $a \gg 0$. A morphism $\varphi : M \to N$ of $C$-modules is a nil-isomorphism if its kernel and cokernel are nilpotent.

If $M$ is a $C$-module then the descending chain $C_+ M \supseteq C_+^2 M \supseteq \ldots$ stabilizes (see [Bli13, Proposition 2.13]) and we denote its stable member by $M$.

1.3. Definition. The test module $\tau(M, f^\lambda)$ is the smallest $C$-submodule $N$ of $M$ such that the inclusion $H^0_\eta(N) \subseteq H^0_\eta(M)$ is a nil-isomorphism for every associated prime $\eta$ of the $R$-module $M$.

At this point we encourage the reader to take a look at [BS16b, Sections 1 and 2] for further discussion. It is proven in [BS16b, Theorems 3.4 and 3.6] that test modules exist if $R$ is essentially of finite type over an $F$-finite field. Moreover, in this case the test module filtration $\tau(M, f^\lambda)_{\lambda \geq 0}$ is a decreasing right-continuous discrete filtration. Many other formal properties like Briançon-Skoda also hold in this more general situation (see [BS16b, Section 4]). We call a number $\lambda$ such that $\tau(M, f^\lambda) \neq \tau(M, f^{\lambda - \varepsilon})$ for all $\varepsilon \leq \lambda$ an $F$-jumping number.

Finally, we say that a prime $\eta \in \text{Spec } R$ of a $C$-module $M$ is an associated prime of $M$ if $H^0_\eta(M)_\eta$ is not nilpotent. These form a subset of the associated primes of the underlying $R$-module.

We call a Cartier module $(M, \kappa)$ $\mathcal{F}$-regular if $\tau(M, f^0) = M$. The smallest $\lambda > 0$ such that $\tau(M, f^\lambda) \neq \tau(M, f^0)$ is called the $\mathcal{F}$-pure threshold of $f$ with respect to $M$. More generally, if we have a $C$-module $N$, where $C_e = \kappa^e f^{(\lambda p^e)}$, then we say that $N$ is $\mathcal{F}$-regular if $\tau(N, f^\lambda) = N$.

If $N$ is a $C$-module and $\eta_1, \ldots, \eta_n$ its associated primes then we call $c_1, \ldots, c_n$ a sequence of test elements if $c_i \notin \eta_i$ and the $H^0_\eta(N_{c_i})$ are $\mathcal{F}$-regular. If all associated
primes of $M$ are minimal then we only need a single test element and this condition simplifies: If $N$ is a $C$-module whose associated primes are minimal, then we call $c \in R$ a test element if $c$ is not contained in any minimal prime of $N$ and $N_c$ is $F$-regular.

With this notion one has

1.4. **Theorem.** Let $R$ be essentially of finite type over an $F$-finite field, $(M, \kappa)$ a Cartier module and $f \in R$. Then there exists a sequence of test elements $c_1, \ldots, c_n$ for the $C$-module $M$, where $C_c = \kappa f^{\lceil \lambda p^{-1} \rceil}$, and one has

$$
\tau(M, f^\lambda) = \sum_{c \geq c_0} \sum_{i=1}^n C_c c_i^a M.
$$

for any $c_0 \geq 0$ and any $a_i \geq 1$. Moreover, if $M$ only has minimal associated primes, then this simplifies to

$$
\tau(M, f^\lambda) = \sum_{c \geq c_0} C_c \kappa^a M.
$$

**Proof.** See [BS16b, Theorem 3.4, Theorem 3.6] for the general case and [Bli13, Theorem 3.11] for the special case.

It is mostly this presentation that we will be used in this article. Also note that we will prove shortly that, if $(M, \kappa)$ is $F$-regular, then one has in fact $\tau(M, f^\lambda) = \kappa f^{\lceil \lambda p^{-1} \rceil} M$ (Theorem 2.6 below).

## 2. Nilpotence of the associated graded

Throughout this section $R$ is a ring essentially of finite type over an $F$-finite field. This assumption is imposed to ensure existence of test modules (see [BS16b, Theorem 3.6]; in our setup this automatically implies existence of a sequence of test elements – cf. [BS16b, Remark 3.7]) and discreteness of the filtration. Granting these notions our arguments work for arbitrary $F$-finite rings.

Fix a Cartier module $(M, \kappa)$ and $f \in R$. In [Stä16, Proposition 4.5] the author defined a Cartier structure on the associated graded of the test module filtration. Namely, if $Gr^\lambda(M) = \tau(M, f^{\lambda-1})/\tau(M, f^{\lambda})$, then $\kappa f^{\lceil \lambda p^{-1} \rceil}$ operates on this quotient. In fact, more generally $\kappa f^{\lceil \lambda (p^{-1}) \rceil}$ operates on $Gr^\lambda(M)$ and one easily checks if $Gr^\lambda M$ is nilpotent with respect to this Cartier structure, then also with respect to the one above.

While this definition may seem ad hoc we point out that if $i : \text{Spec } R/(f) \to \text{Spec } R$ denotes the natural inclusion and $f$ is a non-zero-divisor on $R$ then for any Cartier module $(M, \kappa)$ on $\text{Spec } R$ the induced Cartier structure on $R^{i\ast} M$ is given by $\kappa f^{p^{-1}}$ (cf. [BB13, Example 3.3.12]). By Briançon-Skoda one easily sees that the support of $Gr^\lambda(M)$ is contained in $\text{Spec } R/(f)$. We will in fact see shortly that these Cartier structures are very natural.

The next lemma was already proven in [Stä16, Proposition 3.2] for the case that $M$ has only minimal primes. We give a simplified proof here.

2.1. **Lemma.** Let $(M, \kappa)$ be a Cartier module and $f \in R$. Then for all $\lambda \geq 0$ we have $\kappa(\tau(M, f^\lambda)) = \tau(M, f^{\lambda})$. 

Proof. By virtue of [BS16b, Theorem 3.4] we have
\[
\tau(M, f^\lambda) = \sum_{i=1}^{n} \sum_{e \geq e_0} \kappa^e f^{[\lambda p^e]} c_i H_{\eta_i}^0(M)
\]
for any \(e_0 \geq 0\), where the \(\eta_i\) are the associated primes of \(M\) and the \(c_i\) form a sequence of test elements in the sense of [BS16b, Definition 3.1]. We thus have
\[
\kappa(\tau(M, f^\lambda)) = \sum_{i=1}^{n} \sum_{e \geq e_0} \kappa^{e+1} f^{[\frac{1}{p} p^{e+1}]} c_i H_{\eta_i}^0(M)
\]
\[
= \sum_{i=1}^{n} \sum_{e \geq e_0+1} \kappa^e f^{[\frac{1}{p} p^e]} c_i H_{\eta_i}^0(M) = \tau(M, f^{\frac{1}{p}}).
\]
\(\Box\)

2.2. Lemma. Let \((M, \kappa)\) be a Cartier module and \(f \in R\). For any integer \(a_e(\lambda)\) the map given by \(\kappa^e f^{a_e(\lambda)}\) induces a Cartier structure on \(Gr^\lambda M\) if and only if \(a_e(\lambda) \geq [\lambda(p^e - 1)]\).

Proof. Using Lemma 2.1 and Briançon-Skoda ([BS16b, Proposition 4.1]) one has
\[
\kappa^e f^{a_e(\lambda)} \tau(M, f^\lambda) = \tau(M, f^{\frac{\lambda - a_e(\lambda)}{p^e}}).
\]

For this to induce a Cartier structure on the quotient we must have \(\frac{\lambda - a_e(\lambda)}{p^e} \geq \lambda\).

Equivalently, \(a_e(\lambda) \geq \lambda(p^e - 1)\) and since \(a_e(\lambda)\) is an integer this is equivalent to \(a_e(\lambda) \geq [\lambda(p^e - 1)]\). One similarly checks that in this case \(\kappa^e f^{a_e(\lambda)} \tau(M, f^{\lambda - \varepsilon}) \subseteq \tau(M, f^{\lambda - \varepsilon})\) using that the test module filtration is decreasing. 

The first main result of this section is

2.3. Theorem. Let \((M, \kappa)\) be a Cartier module, \(f \in R\) and \(\lambda\) an \(F\)-jumping number of the test module filtration of \(M\) along \(f\). The Cartier structure on \(Gr^\lambda M\) defined by \(\kappa^e f^{a_e(\lambda)}\) is not nilpotent if and only if \(a_e(\lambda) = \lambda(p^e - 1)\) and this quantity is an integer. In particular, if the denominator of \(\lambda\) is divisible by \(p\), then all these Cartier structures are nilpotent.

Proof. We may write \([\lambda(p^e - 1)] = \lambda(p^e - 1) + \delta\) with \(0 \leq \delta < 1\) and \(\delta = 0\) if and only if \(\lambda(p^e - 1)\) is an integer. By Briançon-Skoda ([BS16b, Proposition 4.1]) we have
\[
\kappa^e f^{\lambda(p^e - 1) + \delta} \tau(M, f^{\lambda - \varepsilon}) = \kappa^e \tau(M, f^{\lambda - \varepsilon + \delta}) = \kappa^e \tau(M, f^{\lambda p^e + \delta - \varepsilon}).
\]

Now we use Lemma 2.1 and obtain
\[
\kappa^e \tau(M, f^{\lambda p^e + \delta - \varepsilon}) = \tau(M, f^{\lambda + \frac{\delta}{p^e}}).
\]

Note in fact, that we may take any \(\varepsilon'\) such that \(\varepsilon > \varepsilon' > 0\) and still have \(\tau(M, f^{\lambda - \varepsilon'}) = \tau(M, f^{\lambda - \varepsilon})\). Hence, if \(\delta > 0\) this actually forces \(\varepsilon < \delta\). But then \(\tau(M, f^{\lambda + \frac{\delta}{p^e}}) \subseteq \tau(M, f^{\lambda})\) which shows that the Cartier structure is nilpotent. The same argument also shows nilpotence for any \(a_e(\lambda) > [\lambda(p^e - 1)]\). 

\(\Box\)

2.4. Remark. Note that with the notation of the proof of Theorem 2.3 if \(\mu < \lambda\) is the previous \(F\)-jumping number then necessarily \(\lambda - \mu \leq \delta\). In particular, if \(\lambda\) is the \(F\)-pure threshold, then it actually follows that if \(\delta \neq 0\) then \(\delta \geq \lambda\) since we can take any \(0 < \varepsilon < \lambda\) without changing \(\tau(M, f^{\lambda - \varepsilon})\). Still assuming that \(\lambda\) is the
F-pure threshold and writing $\delta = \lfloor \lambda(p^e - 1) \rfloor - \lambda(p^e - 1) \geq \lambda$ we obtain equivalently that $\lfloor \lambda(p^e - 1) \rfloor \geq \lambda p^e$. From this one easily deduces that $\lambda \notin \left( \frac{p^e - 1}{p}, \frac{p^e}{p^e - 1} \right)$ for any integer $0 \leq a \leq p^e - 1$. This is the analogue of [BMS09, Proposition 4.3 (ii)] (which is proved using similar techniques) for modules. Also note that the result of [BMS09] requires $R$ to be regular and $F$-finite while the above argument works for any $F$-finite ring.

2.5. Example. Consider the cusp $f = x^2 + y^3 \in \mathbb{F}_p[x, y] = R$. If $\lambda$ denotes the $F$-pure threshold then $\tau(R, f^\lambda) = (x, y)$ and for $p > 3$ one has (see [Mus09, Example 3.4])

$$\lambda = \begin{cases} \frac{2}{3} & p \equiv 1 \mod 3, \\ \frac{4}{5} & p \equiv 2 \mod 3. \end{cases}$$

Moreover, $\lambda = \frac{1}{2}$ for $p = 2$ and $\lambda = \frac{2}{3}$ if $p = 3$. So if $p \equiv 1 \mod 3$ we can take $e = 1$ and obtain $\frac{2}{3}(p - 1) \in \mathbb{Z}$. Since the quotient $\text{Gr}^\lambda$ is just $\mathbb{F}_p$ we know that the obtained Cartier structure is not nilpotent it has to be $\kappa = \text{id}$.

Of course, for any $p$ one has $\text{Gr}^\lambda = \mathbb{F}_p$, (as an $R$-module quotient) so that it admits the non-nilpotent Cartier structure $\kappa = \text{id}$.

We end this section by proving a simple description of $\tau(M, f^\lambda)$ in the case where $(M, \kappa)$ is $F$-regular. This is very useful for computations. We will use it in the next section to extend the relation of $F$-jumping numbers and zeros of Bernstein-Sato polynomials ([BS16a]) to $F$-regular Cartier modules.

2.6. Theorem. Let $(M, \kappa)$ be a Cartier module and $f$ an $M$-regular element. If $(M, \kappa)$ is $F$-regular, then $\tau(M, f^\lambda) = \kappa^e f^{[\lambda p^e]} M$ for all $e \gg 0$.

Proof. By $F$-regularity we have $M = \tau(M, f^0)$. Hence, by Briançon-Skoda ([BS16b, Proposition 4.1]) and Lemma 2.1 we get

$$\kappa^e f^{[\lambda p^e]} M = \kappa^e f^{[\lambda p^e]} \tau(M, f^0) = \kappa^e \tau(M, f^{[\lambda p^e]}) = \tau(M, f^{[\lambda p^e]}) M.$$ 

Since $\lambda \leq \frac{\lfloor \lambda p^e \rfloor}{p^e} \leq \lambda + \frac{1}{p^e}$ we conclude by right-continuity that $\tau(M, f^{[\lambda p^e]}) = \tau(M, f^{[\lambda p^e]}) M$ for all $e$ sufficiently large. □

2.7. Corollary. Let $(M, \kappa)$ be a Cartier module and $f$ an $M$-regular element. Then $\tau(M, f^\lambda) = \tau(\tau(M, f^0), f^\lambda)$. In particular, we have $\tau(M, f^\lambda) = \kappa^e f^{[\lambda p^e]} \tau(M, f^0)$.

Proof. Clearly, $\tau(M, f^0) \subseteq M$ so that $\tau(\tau(M, f^0), f^\lambda) \subseteq \tau(M, f^\lambda)$ by [BS16b, Proposition 1.15].

For the other inclusion, by definition $\tau(M, f^0)$ is the smallest submodule of $M$ for which the inclusions $H^\eta_0(\tau(M, f^0)) \subseteq H^\eta_0(M)$ are nil-isomorphisms with respect to $\kappa$ for all associated primes $\eta$ of $M$. Being a nil-isomorphism here just means that some power of $\kappa$ annihilates the cokernel. But then a fortiori some power of $\mathcal{C}_+ = \bigoplus_{\eta > 1} \kappa f^{[\lambda p^e]}$ acts as zero on this cokernel. By definition of $\tau(M, f^\lambda)$ this shows $\tau(M, f^\lambda) \subseteq \tau(M, f^0)$. Now we may apply $\tau(\cdot, f^\lambda)$ to this inclusion to obtain $\tau(M, f^\lambda) \subseteq \tau(\tau(M, f^0), f^\lambda)$, where we again use [BS16b, Proposition 1.15] and the fact that $\tau$ (for a fixed Cartier algebra) is idempotent.

The final claim is an immediate application of Theorem 2.6. □

2.8. Remark. The assumption on the $F$-regularity cannot be omitted, that is, if $M$ is not $F$-regular, then $\tau(M, f^\lambda) \neq \kappa^e f^{[\lambda p^e]} M$ in general. Consider for example
$R = \mathbb{F}_p[x, y]$ and the Cartier module $M = \mathbb{F}_p[x, y] \cdot y^{-1} \subseteq j_* \mathbb{F}_p[x, y, y^{-1}]$, where $j : D(y) \to \text{Spec } R$, with Cartier structure induced by localization.

Then $M$ is not $F$-regular since $\mathbb{F}_p[x, y]$ is a proper submodule that generically agrees with $M$. It is easy to see that $M$ is $F$-pure. Moreover, $y$ is a test element for $(M, \kappa)$ and therefore $\tau(M, x^\lambda) = x^{[\lambda]} R$ while $\kappa^{[e]} x^{[\lambda p^e]} M$ is not even contained in $R$ since $y^{-1}$ is a fixed point for the Cartier operation.

Similarly, if $M$ is not $F$-pure and has only minimal associated primes then $\tau(M, f^\lambda) \neq \kappa^e f^{[\lambda p^e]} \kappa^a c M$ in general. For example, consider the Cartier module $M = k[x] \cdot x^n \subseteq k[x]$ with $n \geq 2$ and take $\lambda = 0$ and $f = x$. Clearly, $c = x$ is a test element and $M = \kappa^a k[x] \cdot x^n = k[x] \cdot x^{-1}$. We see that $\tau(M, x^0) = k[x]$ by the theorem. However, $\kappa^a x M = k[x] \cdot x^{-1}$ for all $a \gg 0$ so that (since $\lambda = 0$) $\kappa^e x^0 k[x] \cdot x^{-1} = k[x] \cdot x^{-1}$.

3. Cartier structures on the associated graded induced by differential operators

Throughout this section we assume that $R$ is a regular ring essentially of finite type over an $F$-finite field. Regularity of $R$ is critical since we need that $F^e_* R$ is a flat $R$-module to ensure that $F^e_!$ is exact.

The goal of this section is to show that the Cartier structures defined on the associated graded of $\tau(M, f^\lambda)$ at the beginning of Section 2 correspond to the Cartier structures obtained on quotients of eigenspaces of certain $\mathcal{D}_R[\theta_1, \theta_p, \ldots, \theta_{p-1}]$-modules associated to $(M, f^\lambda)$ for varying $e$. In particular, we will recover and generalize results of Bitoun ([Bit15]).

We start by recalling the necessary $\mathcal{D}$-module theoretic notions (see [Mus09] and [BS16a] for more elaborate discussions). For any ring $R$ containing $\mathbb{F}_p$ the ring of $\mathbb{F}_p$-linear differential operators $\mathcal{D}_R \subseteq \text{End}_{\mathbb{F}_p}(R)$ in the sense of Grothendieck ([GD67, Definition 16.8.1]) admits the so-called $p$-filtration $\mathcal{D}_R = \bigcup_{e \geq 0} \mathcal{D}^e_R$, where $\mathcal{D}^e_R \cong \text{End}_R(F^e_* R)$ with $\text{colim}_{e \geq 0} \mathcal{D}^e_R = \mathcal{D}_R$ (see [Cha74]). In particular, we see that for any $R$-module $M$ there is a natural right action of $\mathcal{D}^e_R$ on $F^e_* M = \text{Hom}_R(F^e_* R, M)$.

3.1. Convention. Unless otherwise specified modules over rings of differential operators will always be right modules.

3.2. Example. If $R = k[t_1, \ldots, t_n]$ is a polynomial ring over a perfect field $k$, then $\mathcal{D}^e_R = k[\partial^{[1]}_{t_1}, \ldots, \partial^{[p^e]}_{t_n}] \mid i = 1, \ldots, n$, where the $\partial^{[j]}_{t_i}$ are divided power operators that act as follows
\[
\partial^{[j]}_{t_i} \bullet (t_1^{a_1} \cdots t_n^{a_n}) = \binom{a_i}{j} t_1^{a_1} \cdots t_{i-1}^{a_{i-1}} t_i^{a_i-j} t_{i+1}^{a_{i+1}} \cdots t_n^{a_n}.
\]
Moreover, one has $[\partial^{[a]}_{t_i}, \partial^{[b]}_{t_j}] = 0$ for $i \neq j$.

As an aside we also mention that since $k$ is perfect $\mathcal{D}_R$ coincides with $k$-linear differential operators.

Assume from now on that $R$ is a regular essentially of finite type over an $F$-finite field and $f \in R$. We consider the graph embedding $\gamma : \text{Spec } R \to \text{Spec } R[t], t \mapsto f$. Given a Cartier module $M$ on $R$ we obtain a Cartier structure on $\gamma_* M$ via the natural isomorphism $\gamma_* F^e_* \cong F^e_* \gamma_*$, which we will from now on suppress from notation. Note that $\mathcal{D}^e_{R[t]}$ contains the differential operator $\theta_a = t^a \partial^{[a]}_t$ for any $a < p^e$. In fact, $\mathcal{D}^e_{R[t]} = \mathcal{D}^e_R[t, \partial^a_t \mid a < p^e]$. We call the $\theta_a$ the (higher divided power) Euler operators.
Lemma. Let \((M, \kappa)\) be a Cartier module on \(R\) and \(\gamma : \text{Spec } R \to \text{Spec } R[t], t \mapsto f\) the graph embedding. Then the adjoint of the Cartier structure \(F^{e1} \gamma_* M \to F^{e+1} \gamma_* M\) is given by

\[
\text{Hom}_{R[t]}(F^e_* R[t], \gamma_* M) \to \text{Hom}_{R[t]}(F^{e+1}_* R[t], \gamma_* M)
\]

\[
\varphi \mapsto \kappa^e \circ F^e_* \varphi.
\]

Proof. First of all, note that duality of finite morphisms yields a map

\[
\gamma_* M \to F^{e1} \gamma_* M, \quad m \mapsto [s \mapsto \kappa^e(\gamma^#(s)m)],
\]

where \(\gamma^#\) denotes the induced map \(R[t] \to R, t \mapsto f\), corresponding to \(F^e_* \gamma_* M \to \gamma_* M\). Applying \(F^{e1}\) (which is just a hom-functor) we get a map

\[
F^{e1} \gamma_* M \to F^{e1} F^{e1} \gamma_* M, \quad \varphi \mapsto [r \mapsto [s \mapsto \kappa^e(\gamma^#(r)(\varphi(s)))]].
\]

Now by tensor-hom adjunction we have \(F^{e1} F^{e1} \cong F^{e+1}\) which sends a map as above to the map \(s \mapsto \kappa^e(\varphi(\gamma^#(s)))\). \(\square\)

Fix a Cartier module \(M\). Recall that \(F^{e1} \gamma_* M\) admits a \(\mathcal{D}^e_R[t]\)-module structure and hence in particular a \(\mathcal{D}^e_R[\theta_1, \theta_{p}, \ldots, \theta_{p-1}]\)-module structure. As such it admits a decomposition into generalized eigenspaces \(F^{e1} \gamma_* M = \bigoplus_{i \in \mathbb{Z}} E_i\), where \(\theta_{p}^{\pm}\) acts on \(E_i\) by multiplication with \(i\) (see [BS16a, Lemma 3.1] for details). As follows from Lucas’ Theorem (see [Luc78, §XXI]) the projection \(F^{e1} \gamma_* M \to E_i\) is induced by \(\pi : F^e_* R[t] \to F^e_* R[t], Rt^m \mapsto 0\) unless \(m = i_1 + i_2 p + \ldots + i_e p^{e-1}\) in which case \(r t^m \mapsto r t^m\).

Moreover, this eigenspace decomposition is preserved by morphisms. If we consider a \(\mathcal{D}^e_R[\theta_1, \theta_{p}, \ldots, \theta_{p-1}]\)-module as a \(\mathcal{D}^e_R^{-1}[\theta_1, \theta_{p}, \ldots, \theta_{p-2}]\)-module, then these eigenspace decompositions are compatible with each other in the sense that if \(M = \bigoplus_{i \in \mathbb{Z}} E_i\) is an eigenspace decomposition, then \(E_i = \bigoplus_{j \in \mathbb{Z}} E_{i_1, \ldots, i_{e-1}, j}\) are the eigenspaces as \(\mathcal{D}^e_R^{-1}[\theta_1, \theta_{p}, \ldots, \theta_{p-2}]\)-module.

Note that one has a natural map \(\gamma_* F^e1 M \to F^{e1} \gamma_* M, \varphi \mapsto \varphi(\gamma^#)\). By abuse of notation we will write \(\gamma_* C^e(M)\) for the image of \(C^e(M) = \{[r \mapsto \kappa^e(\gamma^#(r))] \mid m \in M\}\) under this natural map.

A key technical result ([BS16a, Corollary 5.3]) yields that for all \(e \gg 0\) the \((i_1, \ldots, i_e)\)-eigenspace of the quotient

\[
N_e := (\gamma_* C^e(M)) \mathcal{D}_R^e[\theta_1, \theta_{p}, \ldots, \theta_{p-1}]/(\gamma_* C^e(f M)) \mathcal{D}_R^e[\theta_1, \theta_{p}, \ldots, \theta_{p-1}]
\]

is isomorphic as a \(\mathcal{D}_R^e\)-module to

\[
P_e := C^e(f^{i_1+i_2 p + \ldots + i_e p^{e-1}}) \cdot \mathcal{D}_R^e/C^e(f^{1+i_1+i_2 p + \ldots + i_e p^{e-1}}) \cdot \mathcal{D}_R^e.
\]

This isomorphism is given as follows. Given \(\varphi \in N_e\) that is contained in the \((i_1, \ldots, i_e)\)-eigenspace one has \(\varphi(r t^m) = 0\) for all \(m < p^e\) if \(m \neq i_1 + i_2 p + \ldots + i_e p^{e-1}\). The image of \(\varphi\) under this isomorphism is the map \(r \mapsto \varphi(r f^{i_1+i_2 p + \ldots + i_e p^{e-1}})\).

With this notation and under the additional assumption that \(M\) is \(F\)-regular it follows that \(P_e\) and hence the \((i_1, \ldots, i_e)\)-eigenspace of \(N_e\) above is naturally isomorphic as a \(\mathcal{D}_R^e\)-module to

\[
F^{e1} G_{t^e} M := F^{e1}(\tau(M, f^{i_1+i_2 p + \ldots + i_e p^{e-1}})/\tau(M, f^{1+i_1+i_2 p + \ldots + i_e p^{e-1}}))
\]
for all $e \gg 0$. For the case that $M$ has only minimal associated primes, see [BS16a, Corollary 4.8] and note that $F^c e^1$ is exact since $R$ is regular. In the general $F$-regular case, we can use Theorem 2.6 above instead. In fact, for $n \in \mathbb{N}$ we have $C^e(f^n M) = \{ [r \mapsto \kappa^e(r f^n m)] | m \in M \} \subseteq \text{Hom}(F^e R, M)$ while $F^c e^1 \tau(M) \subseteq F^c e^1 M = \text{Hom}(F^e R, M)$ and the natural inclusion $C^e(f^n M) \to F^c e^1 M$ induces the isomorphism.

We stress that [BS16a, Corollary 4.8] is the only test module theoretic input of [BS16a]. Thus, by virtue of Theorem 2.6 the result [BS16a, Theorem 5.4] immediately generalizes to the more general framework of $F$-regular Cartier modules in the sense of [BS16b]. In fact, Theorem 2.6 is more precise, since it guarantees an equality $\kappa^e f^n M = \tau(M, f^n \pi^e)$ for any $m \in \mathbb{Z}$ and $e \geq 1$, thereby removing an inaccuracy in the proof of [BS16a, Theorem 5.4] – the application of [BS16a, Corollary 4.8] is not correct since one may have to enlarge the numerator.

We will prove a partial generalization of this result to arbitrary Cartier modules in the last section.

With these preliminaries we can prove the

3.4. Proposition. Let $M$ be a Cartier module on $R$ and $f \in R$ a non-zero-divisor on $M$. Then the transition map $\gamma_\ast F^c e^1 M \to F^{c+a} \gamma_\ast M$ induces a map $N_e \to N_{e+a}$. Furthermore, if $E_i$ is the $(i_1, \ldots, i_e)$-eigenspace of $N_e$ and $E_{i,j}$ the $(i_1, \ldots, i_e, j_{e+1}, \ldots, j_{e+a})$-eigenspace of $N_{e+a}$ then we get an induced map

$$\alpha : E_i \longrightarrow N_e \longrightarrow N_{e+a} \longrightarrow E_{i,j},$$

where $\pi$ is the projection onto the eigenspace and $i$ is the natural inclusion.

If, in addition, $M$ is $F$-regular, then via the isomorphisms $E_i \to F^c Gr^i M$, $E_{i,j} \to F^{c+a} Gr^{i,j} M$ and adjunction we get an induced Cartier structure $F^c e^1 Gr^i M \to F^c e^1 Gr^i M$ which is given by $\varphi \mapsto \kappa^a f^{i_{e+1} + j_{e+2} p + \cdots + j_{e+a} p^{e+a-1}} \circ \varphi$.

Proof. Recall that the isomorphism $E_i \to F^c e^1 Gr^i$ is given by

$$\varphi \mapsto [r \mapsto \varphi(r f^{i_1 + i_2 p + \cdots + i_e p^{e-1}})].$$

Similarly, the image of $\alpha(\varphi)$ in $F^{c+a} Gr^{i,j} M$ is given by

$$r \mapsto \pi \kappa^a \varphi(r f^{i_1 + i_2 p + \cdots + i_e p^{e-1}} r).$$

We claim that this map coincides with the map

$$\psi_j : r \mapsto \kappa^a \varphi(r f^{i_1 + i_2 p + \cdots + i_e p^{e-1} + l_{e+1} p^{e} + \cdots + j_{e+a} p^{e+a-1} - 1}).$$

Note that $\{rt \mid r \in R \text{ with } 0 \leq t \leq p^{e+a} - 1\}$ is a set of generators for $F^e_{c+a} R[t]$. Since $N_{c+a}$ is the direct sum of its eigenspaces the claim comes down to verifying that the equality

$$\sum_{j \in F^e p} \psi_j = \kappa^a \varphi$$

holds in $N_{c+a}$, where we lift the $\psi_j$ to $N_{c+a}$ by taking the natural section of the projection onto the eigenspace.

Hence, $\psi_j$ evaluated at

$$f^{i_1 + i_2 p + \cdots + i_e p^{e-1} + l_{e+1} p^{e} + \cdots + j_{e+a} p^{e+a-1} - 1}$$

coincides with

$$\kappa^a \varphi \left( f^{i_1 + i_2 p + \cdots + i_e p^{e-1} + l_{e+1} p^{e} + \cdots + j_{e+a} p^{e+a-1} } \right)$$
if \( j = l \) and is zero otherwise. In particular, we see that the claimed identity holds.

Now since we may view \( \varphi \) as a map \( \varphi : F^e_* R \to M \) we have that
\[
\psi_j(r) = \kappa^a f^{j_1+1 + j_2 + 2 + \cdots + j_{c-1} + p^{a-1}} (f^{j_1+1 + i_2 p + \cdots + i_{c-1} p^{a-1}} r)
\]
which shows the claim. \( \square \)

### 3.5. Remark
We note that taking the projection onto the corresponding eigenspace in Proposition 3.4 is necessary (see [Mus09, Example 6.15]).

Next, recall that any \( \lambda \in [0, 1) \) admits a unique non-terminating \( p \)-adic expansion \( \lambda = \sum_{i \geq 1} \frac{c_i(\lambda)}{p^i} \) with \( 0 \leq c_i(\lambda) \leq p - 1 \). Moreover, one obtains for any \( e \geq 1 \) that
\[
\sum_{i=1}^e c_i(\lambda) \cdot \frac{1}{p^i} = \lfloor \lambda p^e \rfloor - 1.
\]
Hence, if \( \lambda \in [0, 1) \), then we can identify for all \( e \gg 0 \) \( F^e \Gr^\lambda M \) as a \( \mathcal{D}_R \)-module with the \( (c_1(\lambda), \ldots, c_s(\lambda)) \)-eigenspace of \( N_e \). Moreover, since the test module filtration is discrete we may assume, by choosing a larger \( e \) if necessary, that
\[
\tau(M, f^{\frac{[\lambda]}{p^e}}) = \tau(M, f^\lambda) \text{ and } \tau(M, f^{\frac{[\lambda]}{p^e + 1}}) = \tau(M, f^{\lambda + \varepsilon}).
\]
In other words \( \Gr^{\frac{[\lambda]}{p^e}} M = \Gr^\lambda M \).

### 3.6. Lemma
For \( e \gg 0 \) the morphism
\[
F^e \Gr^\lambda M \xrightarrow{\cong} E_i \xrightarrow{\alpha} E_{i,j} \xrightarrow{\cong} F^{e+a} \Gr^\lambda M,
\]
where \( \alpha \) is the map in Proposition 3.4, is of the form \( F^e C \) for a morphism \( C : \Gr^\lambda M \to F^{a} \Gr^\lambda M \) if \( \lambda(p^e - 1) \) is an integer for some \( s \in \mathbb{Z} \) and both \( e, a \) are multiples of \( s \).

**Proof.** \( \lambda(p^e - 1) \) is an integer if and only if the denominator of \( \lambda \) is not divisible by \( p \). This is equivalent to the fact that the \( p \)-adic expansion of \( \lambda \) is strictly periodic with period length dividing \( s \). In this case, one has \( \lambda(p^e - 1) = p^i \sum_{i=1}^s c_i(\lambda) \) (see e.g. [HNnBWZ16, Lemma 2.6]). Write \( e = e' s \) and \( a = a' s \) Then by Proposition 3.4 and the previous observation the Cartier structure \( F^e \Gr^\lambda M \to F^{e+a} \Gr^\lambda M \) is given by the adjoint of
\[
\kappa^a f^{\lfloor \lambda(p^e - 1) \rfloor} = (\kappa^a f^{\lfloor \lambda(p^{e'} - 1) \rfloor})^{a'}
\]
and we may set
\[
C = [m \mapsto [r \mapsto (\kappa^a f^{\lfloor \lambda(p^{e'} - 1) \rfloor})^{a'}(rm)].
\]
\( \square \)

### 3.7. Remark
Note that the Cartier structure on \( F^e \Gr^\lambda M \) is only giving us information on \( \Gr^\lambda M \) if we have a map \( C \) as in Lemma 3.6. Quite generally, in this case, \( C \) is a nil-isomorphism (see e.g. [Stä17, Lemma 2.2]) so that as crystals \( \Gr^\lambda M \) and \( F^e \Gr^\lambda M \) coincide. Otherwise \( F^e \Gr^\lambda M \) does, at least a priori, not encode more information than any other faithful functor.

We come to the main result of this section:

### 3.8. Theorem
Let \( R \) be an \( F \)-finite regular ring, \((M, \kappa)\) an \( F \)-regular Cartier module and \( f \) an \( M \)-regular element. Assume that \( \lambda \in \mathbb{Q} \) is an \( F \)-jumping number of the test module filtration of \( M \) along \( f \). Then for \( e \gg 0 \) we consider the map \( C : F^e \Gr^\lambda M \to F^{e+a} \Gr^\lambda M \) obtained by the isomorphism of Lemma 3.6.
If $\lambda(p^s - 1) \in \mathbb{Z}$, then $C$ is induced by a morphism $\text{Gr}^\lambda \to F^{s!} \text{Gr}^\lambda M$ if and only if $s \mid e$ and $s \mid a$. In this case, the morphism is given by the adjoint of $\kappa^a f^{\lambda(p^s - 1)}$ and is nilpotent.

If $\lambda(p^s - 1) \notin \mathbb{Z}$, then there is a morphism $\text{Gr}^\lambda M \to F^{s!} \text{Gr}^\lambda M$ inducing $C$ if and only if $s \mid e$, $s = a$ and $[\lambda(p^s - 1)]$ is of the form $c_1 p^{s-1} + c_2 p^{s-2} + \ldots + c_s$, where $\frac{c_1}{p} + \frac{c_2}{p^2} + \ldots + \frac{c_s}{p^s}$ is the truncated $p$-adic expansion of $\lambda$. In this case, the morphism is nilpotent and given by the adjoint of $\kappa^a f^{\lambda(p^s - 1)}$.

**Proof.** If $\kappa : F^s_* M \to M$ is a Cartier structure then its adjoint is given by $C : M \to F^e_* M, m \mapsto [r \mapsto \kappa(r m)]$. In what follows we will not distinguish between $C$ and $\kappa$. The Cartier structure $C$ is of the form $\kappa^a f^{b(a)}$ for some $b(a) \in \mathbb{N}$. In particular, the Cartier structure $\text{Gr}^\lambda M \to F^{s!} \text{Gr}^\lambda M$ is necessarily also of the form $\kappa^a f^{\ell}$. But then, using Lemma 2.2, it is not nilpotent if and only if $\ell = \lambda(p^s - 1)$ with $\lambda(p^s - 1) \in \mathbb{Z}$. Clearly, if some power of $\kappa^a f^{\lambda(p^s - 1)}$ induces $C$ then we must have $s \mid e$ and $s \mid a$. The converse is the assertion of Lemma 3.6.

For the addendum note that it is clearly necessary that $s \mid e$ and $s \mid a$. If we write the $p$-adic expansion of $\lambda$ as $\lambda = \sum_{i=1}^{c} c_i p^{-i}$, then the existence of such a morphism $\kappa^a f^{u_s(\lambda)} : \text{Gr}^\lambda M \to F^{s!} \text{Gr}^\lambda M$ is equivalent to a commutative diagram of the form

\[
\begin{array}{ccc}
\text{Gr}^\lambda M & \xrightarrow{\kappa^a f^{u_s(\lambda)} \frac{p^e + a}{p^{e+1}}} & F^{e!} \text{Gr}^\lambda M \\
& & \xrightarrow{\kappa^a f^{\lambda(p^s - 1)}} & F^{e+1} \text{Gr}^\lambda M.
\end{array}
\]

By Lemma 2.2 we may write $u_s(\lambda) = [\lambda(p^s - 1)] + m$ for some non-negative integer $m$. Using this, the diagram is commutative if and only if we have the following equality

$$([\lambda(p^s - 1)] + m) \frac{p^e - 1}{p^e} p^{a-1} + c_a + c_{a-1} p + \ldots + c_1 p^{a-1} = ([\lambda(p^s - 1)] + m) \frac{p^{e+1} - 1}{p^e} - 1.$$ 

This simplifies to

$$c_a + c_{a-1} p + \ldots + c_1 p^{a-1} = (\lambda(p^s - 1)] + m) \frac{p^{a-1} - 1}{p^s - 1}.$$ 

Note that $[\lambda(p^s - 1)] = c_1 p^{s-1} + c_2 p^{s-2} + \ldots + c_s + \varepsilon$, where $\varepsilon \in \{0, 1\}$ depending on whether the fractional part is negative or positive. Using this formula and writing $a = s \bar{a}$ we may further expand the equation to

$$c_a + c_{a-1} p + \ldots + c_1 p^{a-1} = c_1 p^{a-1} + c_2 p^{a-2} + \ldots + c_s p^{a-s} + (m + \varepsilon) p^{a-s}$$

$$+ c_1 p^{a-s} + c_2 p^{a-2-s} + \ldots + c_s p^{a-2s} + (m + \varepsilon) p^{a-2s}$$

$$\vdots$$

$$+ c_1 p^{s-1} + c_2 p^{s-2} + \ldots + c_s + (m + \varepsilon)$$

where we have $\bar{a}$ rows. The summands in the first row of the right hand side except the last one all occur on the left hand side. Subtracting this from both sides we observe that the left hand side is $< p^{a-s}$. Hence, if $m + \varepsilon$ is positive, then equality cannot hold. Thus, we must have $m = \varepsilon = 0$. 

If $m = \varepsilon = 0$, then the last summand of each row vanishes. If moreover $a = s$, then equality holds. If $s < a$, then there must exist $0 \leq i \leq s$ with $c_i \neq c_{r+i}$ for some $1 \leq r < \bar{a}$. Hence, equality cannot hold.

3.9. **Example.** The second case of the theorem occurs for instance for any $F$-jumping number of the form $\frac{ab}{p^s}$ with $a < p$ (and necessarily $p > 2$).

If the denominator of $\lambda$ is divisible by $p$ then the second case of the theorem can only occur if $\lambda = \frac{a}{p^b}$ with $p \nmid b$ and $a > b$. Indeed, assume that $a \leq b$. Denote the coefficients of the $p$-adic expansion of $\lambda$ by $c_i$ and the coefficients of the expansion of $\frac{2}{d_i}$ by $d_i$. Then $c_1 = \ldots = c_n = 0$ and $c_{n+i} = d_i$ for $i \geq 1$. In particular, the fractional part of $\lambda(p^s - 1)$ is then non-negative since it is of the form

$$\sum_{i \geq 1} \frac{c_i}{p^{r+i}} - \sum_{i \geq 1} \frac{c_i}{p^i}.$$ 

Since $\lambda(p^s - 1)$ is not an integer by assumption its fractional part is strictly positive so that the second case of the theorem cannot occur.

3.10. **Question.** Are there $F$-pure thresholds which actually satisfy the second case of Theorem 3.8?

A possible strategy may be to consider quasi-homogeneous polynomials $f \in k[x_1, \ldots, x_n]$ whose Jacobian ideal coincides, up to radical, with the irrelevant ideal. For then a likely candidate for the $F$-pure threshold is the log canonical threshold

$$\sum \frac{\deg(x_i)}{\deg f}.$$ 

If $\text{fpt}(f) \neq \text{lct}(f)$ then the denominator of the $F$-threshold has to be a $p$th power provided that $p \geq (n-1) \deg f - \deg f - n + 1$. So one would need $f$ with $\text{fpt}(f) = \text{lct}(f)$ and $f$ having degree $pb$ (since $\text{lct}(f) = \sum_{i=1}^{n} \frac{\deg(x_i)}{\deg f}$).

Finally, we show that these results continue to hold if we work with the image of the natural map $\gamma_+M \to \colim_{e \geq 0} F^{\epsilon} \gamma_+M$ instead of $\gamma_+M \to F^{\epsilon} \gamma_+M$ for varying $\epsilon$. This is the setting of [Mus09] (for $M = R$). Indeed, note that by [BS16a, Proposition 3.5] one has $\colim_{e \geq 0} F^{\epsilon} M = \gamma_+ \colim_{e \geq 0} F^{\epsilon} M$ as right $D_{R[t]}$-modules. If $M = \omega_R$ then $\colim_{e \geq 0} F^{\epsilon} \omega_R \cong \omega_R$ and the equivalence of right and left $D_{R[t]}$-modules yields $\gamma_+ R = R[t]_{-f}/R[t]$ which is what is considered in [Mus09].

Before we proceed, we need to recall one more notion from the theory of Cartier crystals. If $(M, \kappa)$ is a Cartier module then we can consider the union of all nilpotent submodules $M_{\text{nil}}$. This is again a nilpotent Cartier module and we define $\overline{M} = M/M_{\text{nil}}$. Note, in particular, that the natural projection $M \to \overline{M}$ is a nil-isomorphism.

We note the following

3.11. **Lemma.** Let $i : \text{Spec} R/I \to \text{Spec} R$ be a closed immersion and $(M, \kappa)$ a Cartier module. Then $i_* \overline{M} = \overline{i_* M}$.

3.12. **Theorem.** Assume that $(M, \kappa)$ is an $F$-regular Cartier module satisfying $\overline{M} = M$. Write $\gamma_+ M = \colim_{e \geq 0} F^{\epsilon} \gamma_+ M$ and denote the natural maps $F^{\epsilon} \gamma_+ M \to \gamma_+ M$ by $\varphi_\epsilon$. Then for any $e$ the quotients

$$\varphi_0(\gamma_+ C^{e}(M))D_{R}[\theta_1, \theta_p, \ldots, \theta_{pe-1}]/\varphi_0(\gamma_+ C^{e}(fM))D_{R}[\theta_1, \theta_p, \ldots, \theta_{pe-1}]$$
and
\[ \gamma_\ast C^\ast (M)\mathcal{D}_R^\ast [\theta_1, \theta_p, \ldots, \theta_{p^s - 1}]/\gamma_\ast C^\ast (fM)\mathcal{D}_R^\ast [\theta_1, \theta_p, \ldots, \theta_{p^s - 1}] \]
are naturally isomorphic as \(\mathcal{D}_R^\ast [\theta_1, \theta_p, \ldots, \theta_{p^s - 1}]\)-modules. In particular, we obtain \(\mathcal{D}_R^\ast\)-isomorphisms of generalized eigenspaces.

**Proof.** First of all, note that the natural map \(\varphi_\ast : F^{e\ast}\gamma_\ast M \to \gamma_\ast M\) is \(\mathcal{D}_R^\ast\)-linear.

Next, observe that by Lemma 3.11 we have \(\gamma_\ast M = \gamma_\ast M\). It follows that the map \(\gamma_\ast M \to F^{e\ast}\gamma_\ast M\) is injective for all \(e\) (cf. [Stä16, Lemma 6.20]). Hence, the natural map \(\varphi_0\) is also injective. Note that \(\varphi_0\) factors as \(\gamma_\ast M \xrightarrow{\gamma_\ast C^\ast} F^{e\ast}\gamma_\ast M \xrightarrow{\varphi_0} \gamma_\ast M\) for any \(e \geq 1\). Since \(\varphi_0\) is \(\mathcal{D}_R^\ast\)-linear we have
\[ \varphi_0(\gamma_\ast C^\ast (M)\mathcal{D}_R^\ast [\theta_1, \theta_p, \ldots, \theta_{p^s - 1}]) = \varphi_0(\gamma_\ast C^\ast (M)\mathcal{D}_R^\ast [\theta_1, \theta_p, \ldots, \theta_{p^s - 1}], \] and a similar assertion holds for \(\gamma_\ast C^\ast (fM)\). We conclude that the quotients
\[ \varphi_0(\gamma_\ast C^\ast (M)\mathcal{D}_R^\ast [\theta_1, \theta_p, \ldots, \theta_{p^s - 1}])/\varphi_0(\gamma_\ast C^\ast (fM))\mathcal{D}_R^\ast [\theta_1, \theta_p, \ldots, \theta_{p^s - 1}] \]
and
\[ \gamma_\ast C^\ast (M)\mathcal{D}_R^\ast [\theta_1, \theta_p, \ldots, \theta_{p^s - 1}]/\gamma_\ast C^\ast (fM)\mathcal{D}_R^\ast [\theta_1, \theta_p, \ldots, \theta_{p^s - 1}] \]
are naturally isomorphic as \(\mathcal{D}_R^\ast [\theta_1, \theta_p, \ldots, \theta_{p^s - 1}]\)-modules since \(\varphi_0\) is injective. In particular, we obtain \(\mathcal{D}_R^\ast\) isomorphisms of generalized eigenspaces.

3.13. **Proposition.** Let \((M, \kappa_M), (N, \kappa_N)\) be Cartier modules and \(f \in R\) \(M\)-regular and \(N\)-regular. If \(\varphi : M \to N\) is a nil-isomorphism then we get an induced nil-isomorphism \(\text{Gr}^\lambda M \to \text{Gr}^\lambda N\) with respect to any of the Cartier structures \(\kappa_M, f^{a_\ast}(\lambda)\), where \(? = M\) or \(N\) and \(a_\ast(\lambda) \geq \lfloor \lambda(p^e - 1) \rfloor\).

**Proof.** By [Stä17, Theorem 2.8] the restriction of \(\varphi\) induces a surjective map \(\tau(\varphi) : \tau(M, f^\lambda) \to \tau(N, f^\lambda)\). By [Stä17, Lemma 2.1, proof of Theorem 2.8] we may assume that both \(M, N\) are \(F\)-pure. In particular, \(\varphi\) is surjective. Now the claim follows from [Stä16, Theorem 5.8].

In particular, given an \(F\)-regular Cartier module \(M\) we obtain nil-isomorphisms \(\text{Gr}^\lambda M \to \text{Gr}^\lambda M\). Since for \(M\) working on the \(e\)th level or in the colimit induces natural isomorphisms of the generalized eigenspaces of the quotients by Theorem 3.12 we see that we obtain the same nilpotence results if we construct the quotients by working with the colimit.

4. **Non-\(F\)-pure modules**

In this section we study a generalization of the non-\(F\)-pure ideal or \(\phi\)-fixed ideal to modules. These non-\(F\)-pure ideals were first introduced in [FST11] and are further developed and studied in [Sch14] and [HSZ14]. They are the characteristic \(p\) analog of the so-called non-lc ideal and have applications to birational geometry.

The importance of these for us is that we will prove a connection between zeros of Bernstein-Sato polynomials and certain non-\(F\)-pure modules. These non-\(F\)-pure modules will form a decreasing and discrete filtration of the ambient Cartier module. However, there will be no continuity properties even if \(M\) is \(F\)-regular.

The actual connection with Bernstein-Sato polynomials will be discussed in the next section. Here we just develop the basic theory of non-\(F\)-pure modules.
Let us however briefly motivate why it is interesting to consider non-F-pure modules in this context. In the situation of the previous section, the generalized eigenspaces always admit a description of the form

\[ C^e(f^a M) : \mathcal{D}_R^m / C^e(f^{m+1} M) : \mathcal{D}_R^m \cong F^e((\kappa^e f^m M / \kappa^e f^{m+1} M)). \]

A first naive guess (based on [BS16a, Lemma 4.3] which asserts that the statement is true if \( M \) is \( F \)-regular) may be that \( \kappa^e f^m M \) (for \( e \gg 0 \)) should correspond to \( \mathcal{D}_+ M = \mathcal{M} \), where \( \mathcal{D}_+ = \bigoplus_{i \geq 1} \kappa^e f^{[\lambda p^e]} R \) with \( m + 1 = [\lambda p^e] \).

In practice, we replace the algebra \( \mathcal{D} \) with the algebra \( \mathcal{C} = \bigoplus_{e \geq 0} \kappa^e f^{[\lambda(p^e - 1)]} R \). Both algebras, \( \mathcal{C} \) and \( \mathcal{D} \), yield the same notion of test module (see [Stä16, Lemma 3.1]).

Unless specified otherwise \( R \) in this section is an arbitrary \( F \)-finite ring.

4.1. Definition. Fix an \( F \)-finite ring \( R \). Given a Cartier module \((M, \kappa)\) and an \( M \)-regular element \( f \) we denote by \( \mathcal{C} \) the Cartier algebra generated by the \( \kappa^e f^{[\lambda p^e - 1]} \). We define the non-F-pure submodule \( \sigma(M, f^\lambda) \) as \( \mathcal{C}^h M \) for all \( h \gg 0 \).

Note that by [Bli13, Proposition 2.13] the descending chain \( \mathcal{C}^h M \supseteq \mathcal{C}^{h+1} M \supseteq \ldots \) stabilizes for arbitrary Cartier algebras in any \( F \)-finite ring.

4.2. Remark. (a) The reason for working with the algebra \( \mathcal{C} \) rather than \( \mathcal{D} \) is that Lemma 4.3 does not hold in this context (cf. Example 4.5 below).

(b) This is not the definition that is used in [FST11]. We will see in Corollary 4.9 below that these definitions agree.

4.3. Lemma. Let \( R \) be an \( F \)-finite ring and let \((M, \kappa)\) be a Cartier module. Given an \( M \)-regular element \( f \) and a rational number \( \lambda \) we may consider the Cartier algebra \( \mathcal{C}_+ = \bigoplus_{e \geq 0} \kappa^e f^{[\lambda(p^e - 1)]} R \) acting on \( M \). For all \( a \gg 0 \) such that \( \lambda(p^a - 1) \in \mathbb{Z} \) we have \( \sigma(M, f^\lambda) = \kappa^a f^{[\lambda(p^a - 1)]} M \).

Proof. Note that by definition \( \sigma(M, f^\lambda) = \mathcal{C}^h M = \mathcal{C}^{h+1} M \) for all \( h \gg 0 \). We first deal with the inclusion from right to left. Fix a natural number \( h \) such that \( \mathcal{C}^h M = \mathcal{C}^{h+1} M \). Take any \( e \) such that \( \lambda(p^e - 1) \in \mathbb{Z} \) and set \( a = eh \). Then

\[ \underbrace{\kappa^e f^{[\lambda(p^e - 1)]} \cdots \kappa^e f^{[\lambda(p^e - 1)]}}_{\text{\( h \) times}} M = \kappa^a f^{[\lambda(p^e - 1)]} M \]

which shows the desired inclusion. Also note that, say, taking \( e \) minimal and replacing \( h \) with \( h + 1 \) shows that \( e(h + 1) \) also works. So that the inclusion holds for all \( a \gg 0 \) such that \( \lambda(p^a - 1) \in \mathbb{Z} \).

Given \( a \) such that \( \lambda(p^a - 1) \in \mathbb{Z} \) we show that for any \( h \geq a \) one has \( \mathcal{C}^h M \subseteq \kappa^a f^{[\lambda(p^a - 1)]} M \). It suffices to show that

\[ \kappa^{e_1} f^{[\lambda(p^{e_1} - 1)]} \cdots \kappa^{e_h} f^{[\lambda(p^{e_h} - 1)]} M \subseteq \kappa^a f^{[\lambda(p^a - 1)]} M \]

for all \( e_1, \ldots, e_h \geq 1 \).

Write \( e = e_1 + \ldots + e_h \) and \( e = la + r \) (note that \( l \geq 1 \)). Then we have

\[ \kappa^{e_1} f^{[\lambda(p^{e_1} - 1)]} \cdots \kappa^{e_h} f^{[\lambda(p^{e_h} - 1)]} M \subseteq \kappa^e f^{[\lambda(p^e - 1)]} M \]

\[ = \kappa^{la} f^{[\lambda(p^a - 1)]} \kappa^r f^{[\lambda(p^r - 1)] + \delta} M \subseteq \kappa^{la} f^{[\lambda(p^a - 1)]} M = \kappa^a f^{[\lambda(p^a - 1)]} M, \]

where \( \delta = [\lambda(p^e - 1)] - \lambda(p^r - 1) \). \( \square \)
4.4. Example. In this example we point out that the assumption that \( \lambda(p^a-1) \) is an integer in Lemma 4.3 cannot be omitted. Let \( k \) be an \( F \)-finite field of characteristic 2. Consider the Cartier module \( M = k[x]^1 \) with Cartier structure induced from the usual one on \( k[x] \) (i.e., \( x^{p^i} \mapsto 1, x^i \mapsto 0 \) for \( 0 \leq i \leq p-2 \)). Explicitly, \( \kappa(\frac{1}{x}) = \frac{1}{x} \kappa(g x^{p-1}). \)

\( M \) is \( F \)-pure but not \( F \)-regular (a test element is given by \( x \)). We filter \( M \) along \( f = x^2 \) and take \( \lambda = \frac{1}{2} \). Then for any \( e \geq 1 \) we have

\[
\kappa^e f[\lambda(p^{e-1})] M = \kappa^e f^{2^{e-1}} \frac{1}{x} k[x] = \kappa^e x^{2^{e-1}} k[x] = k[x].
\]

On the other hand, for any \( e \geq 2 \) we have \( C^e_+ M \subseteq C^e_+ M \). Elements of \( C^e_+ M \) in turn are sums of elements of

\[
\kappa^a f[\lambda(p^{a-1})] \kappa^b f[\lambda(p^{b-1})] M = \kappa^{a+b} f^{2^{a+b-1}} f^{2^{b-1}} M \in \kappa^{a+b} x^{2^{a+b-1}} x^{2^{b-1}} k[x] \subseteq (x),
\]

where \( a, b \geq 1 \). We conclude that \( C^e_+ M \subseteq (x) \) for \( e \geq 2 \).

As remarked earlier, Lemma 4.3 does not hold if we use the Cartier algebra \( D = \bigoplus_{e \geq 1} \kappa^e f[\lambda p^e] R \oplus R \):

4.5. Example. Consider \( M = k[x]^1 \) endowed with its usual Cartier structure. Take \( f = x \) then if \( D = \bigoplus_{a \geq 1} \kappa^a f[\lambda p^a] R \oplus R \) and \( \lambda = 1 \) one has \( C^a_+ M \subseteq \kappa^a f[\lambda p^a] M \) for all \( a, h \gg 0 \). Indeed, for any \( a \geq 1 \) the right hand side coincides with \( \kappa^a x^{p^a-1} k[x] = k[x] \). For the left hand side note that by [BS16a, Lemma 4.1] one has \( D_+ M = \kappa^a f[\lambda p^a] M \) for all \( a \gg 0 \). Hence, for \( h \gg 0 \) we have

\[
D^h_+ M = D^{h+2}_+ M = D^h_+ \kappa^a x^{p^a} \kappa^b x^{p^b} k[x] \frac{1}{x} = D^h_+ \kappa^a x^{p^a} k[x] = D^h_+ (x).
\]

Since \( x \) is a test element for \( M \) one observes that \( \tau(M, x^1) = (x) \). In particular, \( (x) \) is \( F \)-pure with respect to the Cartier algebra \( D \). Hence, \( D^h_+ (x) = (x) \).

The following example further illustrates that the filtration obtained via \( D \) also has all the shortcomings of the one obtained via \( C \).

4.6. Example. We work in the situation of Example 4.4 but use \( D_+ \). Then \( D^h_+ M \neq \kappa^a f[\lambda p^a] M \) for all \( h, a \gg 0 \). One has \( \kappa^a f[\lambda p^a] M = \kappa^a x^{p^a-1} k[x] = k[x] \). If we had an equality above then in particular \( D_+ k[x] = D_+ \kappa^a f[\lambda p^a] M = \kappa^a f[\lambda p^a] M = k[x] \). But for any \( e \geq 1 \) we have \( \kappa^e f[\lambda p^e] k[x] = \kappa^e x^{p^e} k[x] = (x) \).

Furthermore, using this filtration there will be no continuity properties. For instance, if \( (M, \kappa) \) is any \( F \)-pure non-\( F \)-regular Cartier module whose associated primes are all minimal and \( f \) a test element then for \( \lambda = 0 \) one obtains \( D^h_+ M = M \) while for any \( 0 < \lambda < 1 \) one obtains \( D^h_+ M = \tau(M, f^0) \) using [BS16a, Lemma 4.1] and Corollary 2.7. So the filtration is not right-continuous. On the other hand, if \( (M, \kappa) \) is \( F \)-regular then the filtration coincides with the test module filtration and is thus right-continuous but not left-continuous.

4.7. Proposition. Let \( R \) be \( F \)-finite and \( (M, \kappa) \) a Cartier module. Fix an \( M \)-regular element \( f \in R \). Then the filtration \( \sigma(M, f^\lambda) \) is discrete and decreasing.

Proof. In order to show that the filtration is decreasing simply observe that for any \( e \) one has \( \prod_{i=1}^e \kappa^{e_i} f[\lambda(p^i-1)] M \subseteq \prod_{i=1}^e \kappa^{e_i} f[\lambda(p^i-1)] M \). Discreteness now follows by the same argument as in [Bli13, Theorem 4.18].
In [FST11] the authors introduce some other candidates for the non-$F$-pure ideal and ask whether they all coincide. In what follows we write $C$ for the Cartier algebra generated in degree $e$ by $\kappa^e f^{\lambda(p^e-1)}$. The following variants are considered:

(a) For a fixed $n \gg 0$ define $\sigma_n(M, f^\lambda) = (C_{\geq n})^h M$ for $h \gg 0$.
(b) Suppose that $\lambda(p^e - 1) \in \mathbb{Z}$ for some sufficiently large and sufficiently divisible $e$. Write $D$ for the algebra that is given by $\bigoplus_{a \geq 0} \kappa^{ea} f^{\lambda(p^e - 1)}$. Then set $\sigma'(M, f^\lambda) = D^h_n M$ for $h \gg 0$.

Once again note that by [Bli13, Proposition 2.13] all these notions are well-defined for any $F$-finite ring $R$.

We point out that if one studies non-$F$-pure ideals (i.e. if $f$ is replaced by an ideal $\mathfrak{a} \subseteq R$), then one works with the integral closures of the $\mathfrak{a}^{[\lambda(p^e-1)]}$ making things more subtle (if $R$ is normal then any principal ideal generated by a non-zero-divisor is integrally closed – see e.g. [BH98, Proposition 10.2.3]).

As an application of Lemma 4.3 we show that these notions all coincide in our situation of a principal ideal. First we record an elementary

4.8. Lemma. Let $R$ be $F$-finite and $(M, \kappa)$ a Cartier module. Given a rational number $\lambda$ such that $\lambda(p^e - 1) \in \mathbb{Z}$ we have that the $\kappa^{ea} f^{\lambda(p^e - 1)} M$ form a decreasing chain for varying $e$.

Proof. One has $\kappa^{2a} f^{\lambda(p^{2a} - 1)} M = \kappa^{a} f^{\lambda(p^{a} - 1)} \kappa^{a} f^{\lambda(p^{a} - 1)} M \subseteq \kappa^{a} f^{\lambda(p^{a} - 1)} M$. \hfill $\Box$

4.9. Corollary. Let $R$ be essentially of finite type over an $F$-finite field. If the denominator of $\lambda$ is not divisible by $p$, then one has

$$\sigma(M, f^\lambda) = \sigma_n(M, f^\lambda) = \sigma'(M, f^\lambda)$$

for $n \gg 0$. In general, one still has $\sigma(M, f^\lambda) = \sigma_n(M, f^\lambda)$ for all $n \gg 0$.

Proof. Note that one clearly has containments $\sigma'(M, f^\lambda) \subseteq \sigma_n(M, f^\lambda) \subseteq \sigma(M, f^\lambda)$ whenever these objects are defined.

Let us consider the case $\lambda(p^a - 1) \in \mathbb{Z}$ for some $a$. It suffices to show that $\sigma(M, f^\lambda) \subseteq \sigma'(M, f^\lambda)$. By Lemma 4.3 we have $\sigma(M, f^\lambda) = \kappa^b f^{\lambda(p^b - 1)} M$ for infinitely many $b$. In particular, if $\sigma'(M, f^\lambda) = D^h_n M$ for some fixed $h \gg 0$ and $D^+ = \bigoplus_{a \geq 1} \kappa^{ea} f^{\lambda(p^a - 1)} R$ then we may arrange for $a = ch$. Now use Lemma 4.8.

Let us now consider the case of an arbitrary $\lambda$. Fix $h \gg 0$ such that $\sigma_n(M, f^\lambda) = (C_{\geq n})^h M = C^h_\lambda M = \sigma(M, f^\lambda)$. An argument just as in the second part of the proof of Lemma 4.3 shows that for any $a \geq n$ one has $C^a_+ M \subseteq C_{\geq n}^h M$. We conclude that $C_+^h M = C^a_+ M \subseteq C_{\geq n}^h M$. \hfill $\Box$

As is well-known to experts one cannot expect any continuity properties for the filtration $\sigma(M, f^\lambda)$ (not even in the case $M = R$). For completeness we give an example:

4.10. Example. Consider the case of Example 4.4 (i.e. $M = k[x]_{\frac{1}{2}}, f = x^2$) but this time for a field $k$ of characteristic $\neq 2$. We claim that $C^\infty_\lambda M = M$ for any $\lambda \leq \frac{1}{2}$ but $C^\infty_\lambda M = (x)$ for any $\lambda > \frac{1}{2}$. Since 2 and $p$ are coprime we can use Lemma 4.3 and obtain

$$C^\infty_\lambda M = \kappa^e x^{p^e - 1} \frac{1}{x} k[x] = k[x]$$
for $\lambda = \frac{1}{2}$ and hence for all $\lambda \leq \frac{1}{2}$ since the filtration is decreasing. If $\lambda > \frac{1}{2}$ then $\lambda(p^e - 1) > \frac{1}{2}(p^e - 1)$. Choosing $e$ in such a way that the left-hand side is an integer we get that $\lambda(p^e - 1) \geq \frac{1}{2}(p^e - 1) + 1$. Using Lemma 4.3 once more we compute

$$k^\varepsilon f^{\lambda(p^e - 1)}M \supseteq k^\varepsilon x^{p^e + 1}M = (x).$$

Now assume that $\text{char } k = 2$. We claim that the filtration is not left-continuous at $\lambda = \frac{1}{2}$ in this case. In Example 4.4 we saw that $\sigma(M, f^\lambda) \subseteq (x)$. We will show that for all $\varepsilon > 0$ one has $\sigma(M, f^{\lambda-\varepsilon}) = R$. We may assume that $(\lambda-\varepsilon)(p^a - 1) \in \mathbb{Z}$ for some $a \gg 0$. Then using Lemma 4.3 we obtain

$$\sigma(M, f^{\lambda-\varepsilon}) = k^\varepsilon f^{(\lambda-\varepsilon)(p^a - 1)}M = k^\varepsilon x^{2^a - 2^a + \varepsilon - 1 + 2\varepsilon} \frac{1}{x} k[x].$$

Since the exponent is non-zero and $\leq 2^a - 1$ we get $\sigma(M, f^{\lambda-\varepsilon}) = R$.

Finally, note that the Cartier module $(M, \kappa)$ is isomorphic to the Cartier module $(k[x], \kappa x^{p-1})$ via the multiplication by $x$ map $k[x]_{\frac{1}{x}} \to k[x]$.

4.11. Proposition. Assume that $R$ is an $F$-finite ring and $(M, \kappa)$ a Cartier module. If the denominator of $\lambda$ is divisible by $p$ then $\sigma(M, f^\lambda) = \sigma(M, f^{\lambda+\varepsilon})$ for all $0 < \varepsilon \ll 1$.

Proof. Since the filtration is decreasing by Proposition 4.7 we only need to show the inclusion from left to right.

Next note that $C^h_{\geq} \subseteq C_{\geq h}$. By definition we have $\sigma(M, f^\lambda) = C^h_{\geq} M = C^{h+\varepsilon} M \subseteq C^h_{\geq} C_{\geq} M$ for all $h \gg 0$ and all $g$. Hence, any element of $C^h_{\geq} C_{\geq} M$ may be written as

$$\prod_{i=1}^{h} \kappa^e_i f^{\lfloor \lambda(p^e_i - 1) \rfloor} \kappa^0 f^{\lfloor \lambda(p^e_0 - 1) \rfloor} M$$

with $e_0 \geq g$. Fix $h$ such that $C^h_{\geq} M = \sigma(M, f^\lambda)$. If we write $[\lambda(p^e_i - 1)] = \lambda(p^e_i - 1) + \delta_i$ with $0 < \delta_i < 1$ for $0 \leq i \leq h$ and $e = \sum_{i=0}^{h} e_i$, then we obtain that any element of $C^h_{\geq} C_{\geq} M$ is of the form

$$\kappa^e f^{\lambda(p^e - 1)} + \sum_{i=0}^{h} \delta_i p^{\sum_{j=i+1}^{h} e_j}.$$

Now we set

$$\varepsilon' = \lim_{e \to \infty} \frac{\sum_{i=0}^{h} \delta_i p^{\sum_{j=i+1}^{h} e_j}}{p^e - 1}.$$

Note that this sequence converges to $\varepsilon'$ from above and that $\varepsilon' > 0$. Thus by perturbing a little bit we may choose $\varepsilon > \varepsilon'$ such that the denominator of $\lambda + \varepsilon$ is not divisible by $p$ and such that

$$\kappa^e f^{\lambda(p^e - 1)} + \sum_{i=0}^{h} \delta_i p^{\sum_{j=i+1}^{h} e_j} m \in \kappa^e f^{(\lambda+\varepsilon)(p^e - 1)} M = \sigma(M, f^{\lambda+\varepsilon})$$

holds for all $e_h$ sufficiently large and divisible (using Lemma 4.3). \qed

It is observed in [FST11, Remark 14.11] that, contrary to the situation in characteristic zero, one does not have $\sigma(R, f^\lambda) = \tau(R, f^{\lambda-\varepsilon})$ if $R$ is $(F)$-regular.

However, we have the

4.12. Proposition. Let $R$ be essentially of finite type over an $F$-finite field, $(M, \kappa)$ an $F$-regular Cartier module and $f$ an $M$-regular element. If the denominator of $\lambda$ is not divisible by $p$, then $\sigma(M, f^\lambda) = \sigma(M, f^{\lambda-\varepsilon}) = \tau(M, f^{\lambda-\varepsilon})$. \qed
4.13. Question. Is the filtration \( \sigma(M, f^\lambda) \) left-continuous if the \( F \)-jumping numbers of \( \tau(M, f^\lambda) \) all have a denominator that is not divisible by \( p \)?

4.14. Proposition. Let \( R \) be essentially of finite type over an \( F \)-finite field, \( (M, \kappa) \) an \( F \)-regular Cartier module and \( f \) an \( M \)-regular element. Then for all \( \lambda \geq 0 \) one has \( \sigma(M, f^{\lambda+\varepsilon}) = \tau(M, f^\lambda) \) for all \( 0 < \varepsilon \ll 1 \).

Proof. Since \( M \) is \( F \)-regular we have \( \tau(M, f^0) = M \). As the filtration \( \sigma \) is discrete and decreasing there exists \( 0 < \varepsilon \ll 1 \) such that \( \sigma \) is constant in \( (\lambda, \lambda + \varepsilon) \). In particular, we may assume that the denominator of \( \lambda + \varepsilon \) is not divisible by \( p \).

Hence, we can apply Lemma 4.3 and obtain

\[
\sigma(M, f^{\lambda+\varepsilon}) = \kappa^e f^{(\lambda+\varepsilon)(p^\varepsilon-1)} M = \tau(M, f^{\lambda+\varepsilon}) = \tau(M, f^\lambda),
\]

where we use Briançon-Skoda ([BS16b, Proposition 4.1]) and Lemma 2.1 for the second equality and right-continuity of \( \tau \) (and the fact that we may choose a larger \( e \)) for the last equality.

This correspondence breaks down however in the non-\( F \)-regular case.

4.15. Example. Take \( M = k[x, y]_{\frac{1}{2}} \) with the usual Cartier structure, \( k \) any \( F \)-finite field of positive characteristic. We filter along \( f = y \). Then \( \tau(M, f^0) = k[x, y] \) and, in particular, \( \tau(M, f^1) = (y) \).

For \( \sigma \) we obtain \( \sigma(M, f^\lambda) = k[x, y]_{\frac{1}{2}} \) for \( \lambda \leq 1 \) and \( \sigma(M, f^{\lambda+\varepsilon}) = (y)_{\frac{1}{2}} \).

5. Bernstein-Sato polynomials attached to Cartier modules and the filtration \( \sigma \)

5.1. Definition. Given an \( F \)-finite ring \( R \), a Cartier module \( (M, \kappa) \) and an \( M \)-regular element \( f \) we define \( \text{Gr}^\lambda \sigma M = \sigma(M, f^\lambda)/\sigma(M, f^{\lambda+\varepsilon}) \) for any \( 0 < \varepsilon \ll 1 \).

5.2. Lemma. Let \( R \) be an \( F \)-finite ring, \( (M, \kappa) \) a Cartier module and \( f \) an \( M \)-regular element. Then \( \kappa^e f^{[\lambda(p^\varepsilon-1)]} \) defines a Cartier structure on \( \text{Gr}^\lambda \sigma M \). If \( \text{Gr}^\lambda \sigma M \neq 0 \), then this Cartier structure is non-nilpotent if and only if \( \lambda(p^\varepsilon-1) \in \mathbb{Z} \).

Proof. We have \( \sigma(M, f^\lambda) = C^0_{\lambda} M = C^1_{\lambda+1} M \supseteq \kappa^e f^{[\lambda(p^\varepsilon-1)]} C^1_{\varepsilon-1} M \). We may assume that \( (\lambda + \varepsilon)(p^\varepsilon-1) \in \mathbb{Z} \) for some \( \varepsilon > 0 \) and \( \sigma(M, f^{\lambda+\varepsilon}) = \kappa^e f^{(\lambda+\varepsilon)(p^\varepsilon-1)} M \). Write \( [\lambda(p^\varepsilon-1)] = \lambda(p^\varepsilon-1) + \delta \). Then we compute

\[
\kappa^e f^{[\lambda(p^\varepsilon-1)]} \sigma(M, f^{\lambda+\varepsilon}) = \kappa^{e+\delta} f^{\lambda(p^{\varepsilon+\delta}+\varepsilon(p^\varepsilon-1)+\varepsilon(p^\varepsilon-1)+\delta p^\varepsilon) M}.
\]

Replacing \( \varepsilon \) by \( 2p^\varepsilon \varepsilon' \) and possibly choosing a larger \( e \) so that \( \kappa^e f^{(\lambda+\varepsilon')(p^\varepsilon-1)} M = \sigma(M, f^{\lambda+\varepsilon'}) = \sigma(M, f^{\lambda+\varepsilon}) \)

we obtain that \( \varepsilon' p^\varepsilon(p^\varepsilon-1) \geq \varepsilon'(p^{\varepsilon+\delta}+\delta p^\varepsilon-1) \). Hence, we get the desired containment

\[
\kappa^e f^{[\lambda(p^\varepsilon-1)]} \sigma(M, f^{\lambda+\varepsilon}) \subseteq \sigma(M, f^{\lambda+\varepsilon}).
\]
For the supplement, if $\lambda(p^e - 1) \in \mathbb{Z}$, then $\sigma(M, f^\lambda) = \kappa^e f^{\lambda(p^e - 1)} M$ for all $e \gg 0$ such that $\lambda(p^e - 1) \in \mathbb{Z}$. So $\kappa^e f^{\lambda(p^e - 1)} \sigma(M, f^\lambda) = \sigma(M, f^\lambda)$. Conversely, we may write $[ \lambda(p^e - 1)] = \lambda(p^e - 1) + \delta = (\lambda + \delta')(p^e - 1)$. By construction the denominator of $\lambda + \delta'$ is not divisible by $p$ so that we obtain for suitable $e \gg 0$

$$(\kappa^e f^{[\lambda(p^e - 1)]})^e \sigma(M, f^\lambda) = \kappa^e f^{(\lambda + \delta')} f^{(p^e - 1)} \sigma(M, f^\lambda)$$

$$\subseteq \kappa^e f^{(\lambda + \delta')(p^e - 1)} M = \sigma(M, f^{\lambda + \delta'})$$

where we use Lemma 4.3 for the last equality. As the filtration is decreasing this shows nilpotence.

Assume that $R$ is regular and $F$-finite. Let us now recall from [BS16a] the notion of Bernstein-Sato polynomial attached to a Cartier module $(M, \kappa)$ and an $M$-regular element $f$. As in Section 3 we denote by $\gamma : \text{Spec } R \to \text{Spec } \mathbb{R}[t]$ the graph embedding $t \mapsto f$ and may consider, for any $e \geq 1$, the right $D_R[t_1, \ldots, t_{p^e - 1}]$-module

$N_e := (\gamma_* C^e(M)) D_R^e[\theta_1, \theta_p, \ldots, \theta_{p^e - 1}] / (\gamma_* C^e(f M)) D_R^e[\theta_1, \theta_p, \ldots, \theta_{p^e - 1}]$.

Now let $\Gamma_f^e \subseteq \mathbb{F}_p$ be the set of those $i = (i_1, \ldots, i_e)$ for which the corresponding $i$-eigenspace with respect to the action of the $\theta_1, \theta_p, \ldots, \theta_{p^e - 1}$ is non-trivial. Then by lifting $\mathbb{F}_p$ to $\{0, \ldots, p - 1\} \subseteq \mathbb{Z}$ we define the eth Bernstein-Sato polynomial $b^e_{M,f}(s)$ as

$$\prod_{i \in \Gamma_f^e} (s - (\frac{i_1}{p} + \ldots + \frac{i_e}{p^{e-1}})).$$

5.3. Theorem. Let $R$ be an $F$-finite regular ring and $(M, \kappa)$ a Cartier module. Fix a rational number $\lambda$. If $\lambda(p^e - 1)$ is a zero of the Bernstein-Sato polynomial for $e \gg 0$ such that $\lambda(p^e - 1) \in \mathbb{Z}$, then $Gr^e_\lambda$ is non-trivial.

Proof. First of all, we may assume that $(M, \kappa)$ is $F$-pure (use [Stä17, Lemma 2.1]). By construction the zeros of the Bernstein-Sato polynomial are in $[0, 1]$. Fix $0 < \varepsilon < 1$ such that $\lambda + \varepsilon \in \mathbb{Z}(p)$ and take $e \gg 0$ such that (using Lemma 4.3) $\sigma(M, f^\lambda) = \kappa^e f^{\lambda(p^e - 1)} M$ and $\sigma(M, f^{\lambda + \varepsilon}) = \kappa^e f^{(\lambda + \varepsilon)(p^e - 1)} M$. Note that one has $\kappa^e f^{\lambda(p^e - 1)} f^{\varepsilon} M \supseteq \kappa^e f^{(\lambda + \varepsilon)(p^e - 1)} M$ by possibly choosing a larger $e$.

Now $Gr^e_\lambda M$ is non-trivial if and only if $\kappa^e f^{\lambda(p^e - 1)} M \neq \kappa^e f^{(\lambda + \varepsilon)(p^e - 1)} M$. Since $R$ is regular the functor $F^e(\cdot) = \text{Hom}(F^e_* R, \cdot)$ is fully faithful (i.e. Frobenius is finite flat – [Kum09]) so that this is further equivalent to

$F^e(\kappa^e f^{\lambda(p^e - 1)} M) \neq F^e(\kappa^e f^{(\lambda + \varepsilon)(p^e - 1)} M)$.

Using [BS16a, Lemma 4.6], one obtains

$F^e(\kappa^e f^{\lambda(p^e - 1)} M) = C^e(f^{\lambda(p^e - 1)} M) D_R^e$

and similarly for $F^e(\kappa^e f^{(\lambda + \varepsilon)(p^e - 1)} M)$. Hence, $Gr^e_\lambda M \neq 0$ if and only if

$C^e(f^{\lambda(p^e - 1)} M) D_R^e \neq C^e(f^{(\lambda + \varepsilon)(p^e - 1)} M) D_R^e$

Let us write $\lambda(p^e - 1) = i_1 + \ldots + i_1 p^{e-1}$, with $i_j \in \{0, \ldots, p - 1\}$, and assume that $\lambda(p^e - 1)$ is a zero of the Bernstein-Sato polynomial. Then, by (left)exactnes of $F^e$, we conclude that

$C^e(f^{i_1 + \ldots + i_1 p^{e-1}} M) D_R^e \neq C^e(f^{1+i_1 + \ldots + i_1 p^{e-1}} M) D_R^e \supseteq C^e(f^{(\lambda + \varepsilon)(p^e - 1)} M) D_R^e$.

In particular, $Gr^e_\lambda M$ is non-zero as claimed.
5.4. Example. The following example illustrates that the other implication of Theorem 5.3 is not true. Take $M = k[x]_{\frac{1}{p}}$, $f = x$ and $\lambda = 1$. Then $\kappa^a x^{p^a - 1} x M = \kappa^a x^{p^a - 1} M = k[x]$ while $\kappa^a x^{p^a - 1} x^{(p^a - 1)} M = (x)$ as soon as $\varepsilon(p^a - 1) \geq 2$.

5.5. Remark. (a) If $(M, \kappa)$ is $F$-regular then all non-nilpotent information is recovered. Indeed, for suitable $e \gg 0$ one has $\sigma(M, f^{\lambda + \varepsilon}) = \kappa^e f^{(\lambda + \varepsilon)(p^a - 1)} M = \tau(M, f^{\lambda + \varepsilon - \frac{\varepsilon}{pe}})$, and $\kappa^e f^{(p^a - 1)} f M = \tau(M, f^{\lambda + \varepsilon - \frac{\varepsilon}{pe}})$ using Theorem 2.6. By right-continuity of $\tau$ these two quantities coincide for $0 < \varepsilon \ll 1$ and $e \gg 0$ suitable.

(b) Note that in characteristic zero one has that if $\lambda$ is a jumping number of the multiplier ideal filtration then it is a zero of the Bernstein-Sato polynomial. Here the situation is the other way around.

References

[BB11] M. Blickle and G. Böckle, Cartier modules: Finiteness results, J. reine angew. Math. 661 (2011), 85–123.

[BB13] ——, Cartier crystals, arXiv:1309.1035v1 (2013).

[BBH98] W. Bruns and J. Herzog, Cohen-Macaulay rings, revised edition, Cambridge University Press, 1998.

[Bit15] T. Bitoun, On a theory of the $b$-function in positive characteristic, arXiv:1501.00185 (2015).

[Bli13] M. Blickle, Test ideals via $p^{-e}$-linear maps, J. Algebraic Geom. 22 (2013), no. 1, 49–83.

[BMS09] M. Blickle, M. Mustaţă, and K. Smith, $F$-thresholds of hypersurfaces, Trans. Amer. Math. Soc. 361 (2009), no. 12, 6549–6565.

[BS16a] M. Blickle and A. Stäbler, Bernstein-Sato polynomials and test modules in positive characteristic, Nagoya Math. J. 222 (2016), no. 1, 74–99.

[BS16b] ——, Functorial Test Modules, preprint, arXiv:1605.09517 (2016).

[Cha74] S. U. Chase, On the homological dimension of algebras of differential operators, Comm. Algebra 1 (1974), no. 5, 351–363.

[CHSW16] E. Canton, D. J. Hernandez, K. Schwede, and E. E. Witt, On the behavior of singularities at the $F$-pure threshold, arXiv:1508.05427 (2016).

[EK04] M. Emerton and M. Kisin, The Riemann-Hilbert correspondence for unit $F$-crystals, Asterisque 293 (2004).

[FST11] Osamu Fujino, Karl Schwede, and Shunsuke Takagi, Supplements to non-lc ideal sheaves, Higher dimensional algebraic geometry, RIMS Kôkyûroku Bessatsu, B24, Res. Inst. Math. Sci. (RIMS), Kyoto, 2011, pp. 1–46. MR 2809647

[GD67] A. Grothendieck and J. Dieudonné, Eléments de géométrie algébrique IV, Quatrième partie, Inst. Hautes Études Sci. Publ. Math., 1967.

[HhnBWZ16] D. J. Hernández, L. Núñez Betancourt, E. E. Witt, and W. Zhang, $F$-pure thresholds of homogeneous polynomials, Michigan Math. J. 65 (2016), no. 1, 57–87.

[HSZ14] Jen-Chieh Hsiao, Karl Schwede, and Wenhuan Zhang, Cartier modules on toric varieties, Trans. Amer. Math. Soc. 366 (2014), no. 4, 1773–1795. MR 3152712

[HY03] N. Hara and K. Yoshida, A generalization of tight closure and multiplier ideals, Trans. Amer. Math. Soc. 355 (2003), no. 8, 3143–3174.

[Kun69] E. Kunz, Characterizations of regular local rings of characteristic $p$, Amer. J. Math. 91 (1969), 772–784.

[Luc78] E. Lucas, Théorie des Fonctions Numériques Simplement Périodiques, Amer. J. Math. 1 (1878), no. 2, 197–240.

[MTW04] M. Mustaţă, S. Takagi, and K-I. Watanabe, $F$-thresholds and Bernstein-Sato polynomials, Proceedings of the 4th European congress of mathematics (ECM) (2004), 341–364.

[Müll16] S. Müller, The $F$-pure threshold of quasi-homogeneous polynomials, arXiv:1601.08086 (2016).
M. Mustaţă, *Bernstein-Sato polynomials in positive characteristic*, J. Algebra **321** (2009), no. 1, 128–151.

K. Schwede, *F-adjunction*, Algebra Number Theory **3** (2009), no. 8, 907–950. MR 2587408

Karl Schwede, *A canonical linear system associated to adjoint divisors in characteristic \( p > 0 \)*, J. Reine Angew. Math. **696** (2014), 69–87. MR 3276163

T. Schedlemeier, *Cartier crystals and perverse constructible étale \( p \)-torsion sheaves*, PhD thesis (2016).

A. Stäbler, *\( V \)-filtrations in positive characteristic and test modules*, Trans. Amer. Math. Soc. **368** (2016), no. 11, 7777–7808.

A. Stäbler, *Test module filtrations for unit \( F \)-modules*, to appear J. Algebra (2017).

AXEL STÄBLER, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MICHIGAN, ANN ARBOR, MI 48109, USA

E-mail address: staebler@uni-mainz.de