PSEUDO-HARMONIC HERMITIAN STRUCTURES ON WEYL MANIFOLDS

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Abstract. We find geometric conditions on a Hermitian-Weyl manifold under which the complex structure is a pseudo-harmonic map in the sense of G. Kokarev [18] from the manifold into its twistor space. This is done under the assumption that the dimension of the manifold is four or the Hermitian-Weyl structure is locally conformally Kähler.

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1. Introduction

Recall that an almost-complex structure on a Riemannian manifold is called almost-Hermitian or compatible if it is an orthogonal endomorphism of the tangent bundle of the manifold. It is well-known that if a Riemannian manifold admits an almost-Hermitian structure, it possesses many such structures, see, for example, [6]. Thus, it is natural to look for “reasonable” criteria that distinguish some of these structures. One way to obtain such criteria is to consider the almost-Hermitian structures on a Riemannian manifold \((M, g)\) as sections of its twistor bundle that parametrizes the almost-Hermitian structures on \((M, g)\). This is the bundle \(\pi : \mathcal{Z} \rightarrow M\) whose fibre at a point \(p \in M\) consists of all \(g\)-orthogonal complex structures \(I_p : T_p M \rightarrow T_p M\) \((I_p^2 = -Id)\) on the tangent space of \(M\) at \(p\).

The fibre of the bundle \(Z\) is the compact Hermitian symmetric space \(O(2m)/U(m)\), \(2m = \text{dim} M\), and its standard metric \(G = -\frac{1}{4} \text{Trace} (I_1 \circ I_2)\) is Kähler-Einstein. The Levi-Civita connection of \((M, g)\) gives rise to a splitting \(T\mathcal{Z} = H \oplus V\) of the tangent bundle of \(Z\) into horizontal and vertical parts. This decomposition allows one to define a 1-parameter family of Riemannian metrics \(\tilde{g}_t = \pi^* g + tG\), \(t > 0\), for which the projection map \(\pi : (\mathcal{Z}, \tilde{g}_t) \rightarrow (M, g)\) is a Riemannian submersion with totally geodesic fibres. In the terminology of [2, Definition 9.7], the family \(\tilde{g}_t\) is the canonical variation of the metric \(g\). Motivated by the harmonic map theory, C. M. Wood [24, 25] has suggested to consider as “optimal” those almost-Hermitian structures \(J : (M, g) \rightarrow (\mathcal{Z}, \tilde{g}_t)\) that are critical points of the energy functional under variations through sections of \(\mathcal{Z}\). In general, these critical points are not harmonic maps, but, by analogy, they are referred to as “harmonic almost-complex structures” in [25]; they are also called “harmonic sections” in [24], a term which seems more appropriate. Forgetting the bundle structure of \(\mathcal{Z}\), we can consider the almost-Hermitian structures that are critical points of the energy functional under variations through arbitrary maps \(M \rightarrow \mathcal{Z}\), not just sections. These structures are genuine harmonic maps from \((M, g)\) into \((\mathcal{Z}, \tilde{g}_t)\), and we refer to [11] for basic facts about harmonic maps. This point of view is taken in [8, 9] where the problem of
when an almost-Hermitian structure on a Riemannian four-manifold is a harmonic map from the manifold into its twistor space is discussed. In [8], the metric \( \tilde{g}_t \) is defined via the Levi-Civita connection; it is defined by means of a metric connection with totally skew-symmetric torsion in [9]. In [7] (cf. also [6]), the Riemannian 4-manifolds \((M, g)\) for which the Atiyah-Hitchin-Singer [1] or the Eells-Salamon almost-complex structure [12] is a harmonic map from the twistor space \((Z, \tilde{g}_t)\) of \((M, g)\) into the twistor space of \((Z, \tilde{g}_t)\) are described.

If \( g \) and \( g^1 = e^f g \) are conformal metrics, then clearly every \( g \)-orthogonal endomorphism of the tangent bundle \( TM \) is \( g^1 \)-orthogonal and vice versa. Hence the Riemannian manifolds \((M, g)\) and \((M, g^1)\) have the same twistor space, i.e. the twistor space depends on the conformal class of \( g \) rather than the metric \( g \) itself. Thus, it is natural to consider the twistor spaces in the context of conformal geometry, see, for example, [1, 13]. The harmonic map techniques has a useful extension in the conformal geometry introduced by G. Kokarev [18]. Recall that a smooth map \( \varphi : (M, g) \to (M', g') \) between Riemannian manifolds is harmonic exactly when the trace of its second fundamental form defined by means of the Levi-Civita connections vanishes. If \( M \) and \( M' \) are endowed with torsion-free connections \( D \) and \( D' \) respectively, a smooth map \( \varphi : M \to M' \) is called pseudo-harmonic in [18] if the trace of its second fundamental form defined by means of \( D \) and \( D' \) vanishes. The pseudo-harmonic maps share some important properties with the harmonic maps [18], to mention here only the unique continuation one.

Given a conformal manifold \((M, \mathfrak{c})\), it is natural to consider connections preserving the conformal class \( \mathfrak{c} \). Of special interest are those of them that have vanishing torsion. These are the Weyl connections studied in different aspects by many authors. Any Weyl connection \( D \) gives rise to a splitting of the tangent bundle \( TZ \) into horizontal and vertical parts. Then, taking a metric \( g \in \mathfrak{c} \), one can define a 1-parameter family \( \tilde{g}_t \) of Riemannian metrics on \( Z \) as above. We modify the Levi-Civita connection of \( \tilde{g}_t \) to a new torsion-free connection \( D' \) on \( Z \), and find the conditions on \( M \) under which the map \( J : M \to Z \) defined by an integrable almost-Hermitian structure \( J \) on \( M \) is pseudo-harmonic with respect to \( D \) and \( D' \). This is done in the case when the dimension of \( M \) is four, and in higher dimension when the structure \((g, J)\) is locally conformally Kähler, i.e. the Lee form \( \theta \) of \((M, g, J)\) satisfies the identity \( d\Omega = \theta \wedge \Omega \), where \( \Omega(X, Y) = g(JX, Y) \) is the fundamental 2-form of \((M, g, J)\). In dimension 4, this identity is automatically satisfied. If \( \dim M \geq 6 \), it implies \( d\theta = 0 \). Note also that, in any dimension, a Hermitian structure is locally conformally Kähler if and only if \( d\Omega = \theta \wedge \Omega \) and \( d\theta = 0 \), see, for example, [10, 23]. If \( D \) is the Weyl connection determined by \( g \) and \( \theta \), the map \( J \) is always pseudo-harmonic. Two examples illustrating the obtained result are given, one of them showing that there can be many Weyl connections for which \( J \) is a pseudo-harmonic map.

2. Basics about twistor spaces

In this section, we recall some basic facts about twistor spaces. For more details, see, for example, [4, 6].

2.1. The manifold of compatible linear complex structures. Let \( V \) be a real vector space of even dimension \( n = 2m \) endowed with an Euclidean metric \( g \). Denote by \( F(V) \) the set of all complex structures on \( V \) compatible with the metric \( g \), i.e. \( g \)-orthogonal. This set has the structure of an imbedded submanifold of the
vector space \( \mathfrak{so}(V) \) of skew-symmetric endomorphisms of \((V,g)\). The tangent space of \(F(V)\) at a point \(J\) consists of all endomorphisms \(Q \in \mathfrak{so}(V)\) anti-commuting with \(J\), and we have the decomposition \( \mathfrak{so}(V) = T_J F(V) \oplus \{ S \in \mathfrak{so}(V) : SJ - JS = 0 \} \) which is orthogonal with respect to the metric \( G(S,T) = -\frac{1}{2} \text{Trace}(S \circ T) \) of \( \mathfrak{so}(V) \).

The group \( O(V) \cong O(2m) \) of orthogonal transformations of \( V \) acts on \( F(V) \) by conjugation and the isotropy subgroup at a fixed complex structure \( J_0 \) is isomorphic to the unitary group \( U(m) \). Therefore \( F(V) \) can be identified with the homogeneous space \( O(2m)/U(m) \). In particular, \( \dim F(V) = m^2 - m \). Note also that the manifold \( F(V) \) has two connected components \( F_{\pm}(V) \): if we fix an orientation on \( V \), these components consists of all complex structures on \( V \) compatible with the metric \( g \) and inducing \( \pm \) the orientation of \( V \); each of them has the homogeneous representation \( SO(2m)/U(m) \).

The metric \( g \) of \( V \) induces a metric on \( \Lambda^2 V \) given by

\[
(1) \quad g(v_1 \wedge v_2, v_3 \wedge v_4) = \frac{1}{2} [g(v_1, v_3)g(v_2, v_4) - g(v_1, v_4)g(v_2, v_3)].
\]

Then we have an isomorphisms \( \mathfrak{so}(V) \cong \Lambda^2 V \), which sends \( S \in \mathfrak{so}(V) \) to the 2-vector \( S^\wedge \) for which

\[
2g(S^\wedge, u \wedge v) = g(Su, v), \quad u, v \in V.
\]

This isomorphism is an isometry with respect to the metric \( \frac{1}{2}G \) on \( \mathfrak{so}(V) \) and the metric \( g \) on \( \Lambda^2 V \).

### 2.2. The twistor space of a conformal manifold.

The twistor space of a conformal manifold \((M,\xi)\) of even dimension \( n = 2m \) is the bundle \( \pi : Z \to M \) whose fibre at a point \( p \in M \) consists of all complex structures of the tangent space \( T_p M \) that are compatible with the metrics of the conformal class \( \xi \). If \( M \) is oriented, we can consider the bundles over \( M \) whose fibre at \( p \) is the manifold of compatible complex structures yielding the positive and, respectively, the negative orientation of \( T_p M \). The latter bundles are disjoint open subsets of \( Z \) and are frequently called the positive and the negative twistor spaces of \((M,\xi)\), respectively. If \( M \) is connected and oriented, they are the connected components of the manifold \( Z \).

The bundle \( Z \) can be considered as a subbundle of the bundle \( A(TM) \) of the endomorphisms of \( TM \) that are skew-symmetric with respect to one Riemannian metric in \( \xi \), hence with respect to all such metrics. Let \( D \) be a connection on \((M,\xi)\) preserving the conformal class in the sense that for every Riemannian metric \( g \) in the conformal class \( \xi \), there exists a 1-form \( \varphi_g \) such that \( DG = \varphi_g \otimes g \); obviously, such a form is unique. The connection on the vector bundle \( \text{Hom}(TM, TM) \) induced by \( D \) will also be denoted by \( D \). Endow the bundle \( \text{Hom}(TM, TM) \) with the metric \( G(a,b) = \frac{1}{2} \text{Trace}_g \{ TM \ni X \to g(aX, bX) \} \), where \( g \) is a Riemannian metric in \( \xi \). Clearly the metric \( G \) does not depend on the particular choice of \( g \).

**Lemma 1.** The induced connection \( D \) preserves the metric \( G \) and the bundle \( A(TM) \).

**Proof.** Fix a point \( p \in M \), and take a metric \( g \in \xi \). Let \( e_1, \ldots, e_n \) be a \( g \)-orthonormal basis of \( T_p M \). We can extend this basis to a frame of vector fields \( E_1, \ldots, E_n \) in a (geodesically convex) neighbourhood of the point \( p \) such that \( (E_i)_p = e_i, \ DE_i |_p = 0 \) and \( g(E_i, E_j) = f \delta_{ij}, i, j = 1, \ldots, n \), where \( f = f_g \) is a positive smooth function
depending on $g$. Let $a, b$ be sections of $\text{Hom}(TM, TM)$ and $X \in T_pM$. Setting \( \varphi = \varphi_g \), we have

\[
(D_X G)(a, b) = \frac{1}{2} \sum_{i=1}^{n} [X(1/f)(g(aE_i, bE_i)) - \frac{1}{f(p)}g(D_X (aE_i), bE_i)]
\]

\[
= X(1/f)G(a, b)_p + \frac{1}{2} \frac{1}{f(p)} \sum_{i=1}^{n} [X(g(aE_i, bE_i)) - g(D_X (aE_i), bE_i)],
\]

Clearly, \( f(p) = g(E_i, E_i)_p = g(e_i, e_i) = 1 \) and \( X(f) = X(g(E_i, E_i)) = \varphi(X)g(e_i, e_i) = \varphi(X) \). Therefore, \( (D_X G)(a, b) = 0 \).

Let $S$ be a section of $A(TM)$ and $X, Y, Z$ vector fields on $M$. Then

\[
0 = X(g(SY, Z) + g(Y, SZ))
\]

\[
= \varphi(X)g(SY, Z) + g(D_X SY, Z) + g(SY, D_X Z)
\]

\[
+ \varphi(X)g(Y, SZ) + g(D_X Y, SZ) + g(Y, D_X SZ)
\]

\[
= g(D_X S)Y, Z) + g(Y, (D_X S)Z).
\]

Thus, the endomorphism $D_X S$ of $TM$ is $c$-skew-symmetric. 

3. Basics about Weyl connections

Let $M$ be a manifold of dimension $n = 2m$ endowed with a conformal class $c$ of Riemannian metrics. Recall that a Weyl connection on $(M, c)$ is a torsion-free connection $D$ on $M$ that preserves the conformal structure $c$. A conformal manifold equipped with a Weyl connection is called a Weyl manifold.

For $g \in c$, denote the Levi-Civita connection of $g$ by $\nabla^g$. Then

\[
D_X Y = \nabla^g_X Y - \frac{1}{2}[\varphi^g(X)Y + \varphi_g(Y)X - g(X, Y)\varphi^g],
\]

where the 1-form $\varphi_g$ is determined by $Dg = \varphi_g \otimes g$ and $\varphi^g$ is the dual vector field of the form $\varphi_g$ with respect to the metric $g$, i.e. $g(\varphi^g, Z) = \varphi_g(Z)$. If $g^1 = e^f g$ is another Riemannian metric in the conformal class, $f$ being a smooth function, then

\[
\nabla^{g^1}_X Y = \nabla^g_X Y + \frac{1}{2}[X(f)Y + Y(f)X - g(X, Y)\nabla^g f],
\]

\[
\varphi^{g^1} = df + \varphi_g, \quad \varphi^{g^1}_g = e^{-f}(\varphi^g + \nabla^gf),
\]

where $\nabla^gf$ is the gradient of $f$ with respect to $g$. In particular, the condition \( d\varphi_g = 0 \) does not depend on the choice of the metric $g$ in $c$. In this case, we say that $D$ determines a closed Weyl structure. If $d\varphi_g = 0$, then locally $\varphi_g = df$ for a smooth function $f$, so $D$ coincides locally with the Levi-Civita connection of the metric $e^{-f}g$. The condition that the form $\varphi_g$ is exact also does not depend on the choice of $g$, and a Weyl structure with exact $\varphi_g$ is called exact.

If $g$ is a Riemannian metric and $\varphi$ is a 1-form on a manifold, there is a unique Weyl connection for the conformal class $c$ of $g$ such that $\varphi_g = \varphi$. Indeed, for any conformal metric $g^1 = e^f g$, set $\varphi^{g^1} = df + \varphi_g$. Then, by \eqref{eq:3}, the right-hand side of \eqref{eq:2} does not depend on the choice of a metric in $c$, so identity \eqref{eq:2} defines a Weyl connection.
Now, let $D$ be a Weyl connection on $(M, \xi)$. Let $R^D$ and $R^\theta$ be the curvature tensors of the connections $D$ and $\nabla^\theta$, $g \in \xi$, respectively.

**Convention.** For the curvature tensor of a connection $\nabla$, we adopt the following definition $R(X,Y) = \nabla_X Y - [\nabla X, Y]$. A straightforward computation gives the following relation between $R^D$ and $R^\theta$ for $g \in \xi$. As in [21], it is convenient to set

$$R^D_g(X,Y) = (\nabla^\theta_X g)(Y) + \frac{1}{2}\varphi_g(X)\varphi_g(Y) - \frac{1}{4}g(\varphi_g^2, \varphi_g^2)g(X,Y).$$

Define an endomorphism $\Phi_g$ of $TM$ by $g(\Phi_g X, Y) = \Phi_g(X, Y)$. Then

$$R^D(X,Y)Z = R^\theta(X,Y)Z + \frac{1}{2}\{\Phi_g(X,Y)Z - \Phi_g(Y,X)Z + \Phi_g(X,Z)Y - \Phi_g(Y,Z)X + g(X,Z)\Phi_g Y - g(Y,Z)\Phi_g X\}.$$  

This formula implies the identities

$$g(R^D(X,Y)Z,T) + g(R^D(X,Y)T,Z) = d\varphi_g(X,Y)g(Z,T).$$

$$2g(R^D(X,Y)Z,T) - 2g(R^D(Z,T)X,Y) = d\varphi_g(X,Y)g(Z,T) - d\varphi_g(Z,T)g(X,Y) + d\varphi_g(X,Z)g(Y,T) + d\varphi_g(Y,T)g(X,Z) - d\varphi_g(Y,Z)g(X,T) - d\varphi_g(X,T)g(Y,Z).$$

$$R^D(X,Y)Z + R^D(Y,Z)X + R^D(Z,X)Y = 0.$$  

**Notation.** The Ricci tensor of the Weyl structure $(\xi, D)$ is defined by

$$\rho_D(X,Z) = \text{Trace}_g\{Y \rightarrow g(R^D(X,Y)Z,Y)\},$$

where $g \in \xi$.

If $J$ is a $\xi$-compatible almost-complex structure on $M$, the $*$-Ricci tensor of the almost-Hermitian Weyl structure $(\xi, D, J)$ is defined by

$$\rho^*_D(X,Z) = \text{Trace}_g\{Y \rightarrow g(R^D(JY,X)JZ,Y)\}.$$  

Clearly, $\rho_D$ and $\rho^*_D$ do not depend on the particular choice of the metric $g$.

Identity (6) implies the following formulas.

**Proposition 1.** Let $\rho_g$ and $\rho^*_g$ be the Ricci tensor and the $*$-Ricci tensor with respect to a Riemannian metric $g \in \xi$. Then

$$\rho_D(X,Z) = \rho_g(X,Z) + \frac{n}{2} - \frac{1}{2}\nabla^\theta_X g)(Z) - \frac{1}{2}(\nabla^\theta_Z g)(X) - \frac{n}{4}||\varphi_g||^2g(X,Z) - \frac{1}{4}g(\varphi_g^2, \varphi_g^2)g(X,Z)$$.  

$$\rho^*_D(X,Z) = \rho^*_g(X,Z) + \Delta^\theta_X g)(Z) - \frac{1}{2}[(\nabla^\theta_Z g)(X) - (\nabla^\theta_J X \varphi_g)(JZ)]$$

$$+ \frac{1}{2}[\varphi_g(X)\varphi_g(Z) + \varphi_g(JX)\varphi_g(JZ) - ||\varphi_g||^2g(X,Z)]$$

$$+ \frac{1}{2}[\delta^g(J^* \varphi_g) - \varphi_g(\delta^g J)]g(X,JZ),$$

where the norm and the codifferential are taken with respect to the metric $g$.

As is well-known, $\rho_g(X,Z) = \rho_g(Z,X)$ and $\rho^*_g(X,Z) = \rho^*_g(JZ,JX)$. Propositions 1 and these identities imply the following.
Corollary 1. \( \rho_D(X, Z) - \rho_D(Z, X) = \frac{n}{2} d\varphi_g(X, Z). \)

\( \rho^*_D(X, Z) - \rho^*_D(JZ, JX) = d\varphi_g(X, Z) + d\varphi_g(JX, JZ). \)

Corollary 2. (1) The Ricci tensor \( \rho_D \) is symmetric if and only if \( d\varphi_g = 0. \)

(2) The \( \ast \)-Ricci tensor \( \rho^*_D \) satisfies the identity \( \rho^*_D(X, Z) = \rho^*_D(JZ, JX) \) for all \( X, Z \in TM \) if and only if the \( (1, 1) \)-component of \( d\varphi_g \) w.r.t. \( J \) vanishes.

Claim (1) is well-known. Note also that, by (3), the 2-form \( d\varphi_g \) does not depend on the choice of \( g \in \mathfrak{c}. \)

Further, given a Riemannian metric \( g \) on \( M \), we shall often make use of the isomorphism \( A(TM) \cong \Lambda^2 TM \) that assigns to each \( a \in A(T_p M) \) the 2-vector \( a^\wedge \) determined by the identity

\[
2g(a^\wedge, X \wedge Y) = g(aX, Y), \quad X, Y \in T_p M,
\]

the metric on \( \Lambda^2 TM \) being defined by

\[
g(X_1 \wedge X_2, X_3 \wedge X_4) = \frac{1}{2} [g(X_1, X_3)g(X_2, X_4) - g(X_1, X_4)g(X_2, X_3)].
\]

Let \( \tilde{g} = e^f g, \) \( f \) being a smooth function, and denote by \( \tilde{a}^\wedge \) the 2-vector for which \( \tilde{g}(\tilde{a}^\wedge, X \wedge Y) = g(aX, Y). \) Then \( a^\wedge = e^{-f} a^\wedge. \)

Identity \([4, \text{formula (11)}]\) suggests the following.

Lemma 2. Let \( g \in \mathfrak{c}. \) Then, for every \( a, b \in A(T_p M) \) and \( X, Y \in T_p M, \)

\[
G(R^D(X, Y)a, b) = g(R^D([a, b]^\wedge)X, Y)
\]

\[
- \frac{1}{2} [d\varphi_g([a, b]^\wedge)g(X, Y) + d\varphi_g([a, b]X, Y) + d\varphi_g(X, [a, b]Y)].
\]

Proof. Let \( E_1, ..., E_n \) be a \( g \)-orthonormal basis of \( T_p M. \) Then

\[
[a, b]^\wedge = \frac{1}{2} \sum_{i,j=1}^n g([a, b]E_i, E_j) E_i \wedge E_j.
\]

Therefore, by \([7,\text{formula (8)}]\),

\[
g(R^D([a, b]^\wedge)X, Y) = \frac{1}{2} \sum_{i,j=1}^n g(R^D(E_i, E_j)X, Y) g([a, b]E_i, E_j)
\]

\[
= \frac{1}{2} \sum_{i,j=1}^n g(R^D(X, Y)E_i, E_j) g([a, b]E_i, E_j)
\]

\[
+ \frac{1}{2} [d\varphi_g([a, b]^\wedge)g(X, Y) + d\varphi_g([a, b]X, Y) + d\varphi_g(X, [a, b]Y)].
\]
Moreover,
\[
\sum_{i,j=1}^{n} g(R^D(X,Y)E_i, E_j)g([a, b]E_i, E_j) \\
= \sum_{i,j=1}^{n} g(R^D(X,Y)E_i, E_j)[g(abE_i, E_j) + g(aE_i, bE_j)], \\
= \sum_{i=1}^{n} g(R^D(X,Y)E_i, abE_i) \\
+ \sum_{i,j,k=1}^{n} g(R^D(X,Y)E_i, E_j)g(E_i, aE_k)g(E_j, bE_k) \\
= -\sum_{i=1}^{n} g(a(R^D(X,Y)E_i), bE_i) + \sum_{k=1}^{n} g(R^D(X,Y)aE_k, bE_k), \\
= 2G(R^D(X,Y)a, b).
\]

This proves the lemma. \(\square\)

4. The induced connection \(D'\) on the twistor space by a Weyl connection on the base manifold

Let \(I \in \mathcal{Z}\), and let \(\mathcal{H}_I\) be the horizontal subspace of \(T_I A(TM)\) with respect to a Weyl connection \(D\) on the bundle \(A(TM)\). Take a basis \(e_1, \ldots, e_{2m}\) of \(T_p M\), \(p = \pi(I)\), such that \(I e_{2k-1} = e_{2k}, \ I e_{2k} = -e_{2k-1}, \ k = 1, \ldots, m\), and \(g(e_k, e_i) = \lambda_i \delta_{kl}\) for every \(g \in \mathfrak{c}\), where \(\lambda_i\) is a positive constant depending on \(g\). We can extend this basis to a frame of vector fields \(E_1, \ldots, E_{2m}\) in a (geodesically convex) neighbourhood of the point \(p\) such that \((E_i)_p = e_i, \ DE_i|_p = 0\) and \(g(E_i, E_j) = f_g \delta_{ij}, \ i, j = 1, \ldots, 2m\), for every \(g \in \mathfrak{c}\), where \(f_g\) is a positive smooth function. Define a section \(K\) of \(A(TM)\) by \(KE_{2k-1} = E_{2k}, \ KE_{2k} = -E_{2k-1}\). Obviously, \(K\) takes values in \(\mathcal{Z}\), \(K(p) = I\), and \(DK|_p = 0\). Hence the horizontal space \(\mathcal{H}_I = K_*(T_p M)\) is tangent to \(\mathcal{Z}\). Thus, we have the decomposition \(TZ = \mathcal{H} \oplus \mathcal{V}\) of the tangent bundle of \(\mathcal{Z}\), where \(\mathcal{V} = Ker(\pi_\ast)\) is the vertical subbundle of \(TZ\). The vertical space \(\mathcal{V}_I\) at a point \(I \in \mathcal{Z}\) is the tangent space at \(I\) of the fibre of \(\mathcal{Z}\) through \(I\).

This fibre is the manifold of \(\mathfrak{c}\)-compatible complex structures on the vector space \(T_{\pi(I)} M\). Thus, the vertical space \(\mathcal{V}_I\) consists of skew-symmetric endomorphisms of \(T_{\pi(I)} M\) anti-commuting with \(I\). In particular, if \(V \in \mathcal{V}_I\) and \(g \in \mathfrak{c}\), then \(g(IX, VX) = 0\) for every \(X \in T_{\pi(I)} M\). This fact and identity \(\square\) imply the following.

Lemma 3. If \(I \in \mathcal{Z}\) and \(V \in \mathcal{V}_I\),
\[
G(R^D(X,Y)I, V) = -G(R^D(X,Y)V, I).
\]

Let \((\mathfrak{u}, x_1, \ldots, x_n)\) be a local coordinate system of \(M\), and let \(E_1, \ldots, E_n\) be a frame of \(TM\) on \(\mathfrak{u}\) such that, for every metric \(g \in \mathfrak{c}\), \(g(E_i, E_j) = f_g \delta_{ij}, \ i, j = 1, \ldots, n\), where \(f_g\) is a positive smooth function. Define sections \(S_{ij}\) of \(A(TM)\) by the formula
\[
S_{ij} E_l = \delta_{il} E_j - \delta_{jl} E_i, \quad i, j, l = 1, \ldots, n.
\]

These sections do not depend on the choice of the metric \(g \in \mathfrak{c}\) and \(S_{ij}, i < j\), form a \(G\)-orthonormal frame of \(A(TM)\). Set
\[
\hat{x}_i(a) = x_i \circ \pi(a), \quad y_{ji}(a) = G(a, S_{ji}), \ j < l,
\]
for \(a \in A(TM)\). Then \((\hat{x}_i, y_{ji})\) is a local coordinate system of the manifold \(A(TM)\).

For each vector field
\[
X = \sum_{i=1}^{n} X^i \frac{\partial}{\partial x_i}
\]
on \( \mathfrak{U} \), the horizontal lift \( X^h \) on \( \pi^{-1}(\mathfrak{U}) \) is given by

\[
X^h = \sum_{i=1}^{n} (X^i \circ \pi) \frac{\partial}{\partial x_i} - \sum_{j<l} \sum_{r<s} y_{rs} \left( G(D_X S_{rs}, S_{jl}) \circ \pi \right) \frac{\partial}{\partial y_{jl}}.
\]

(11) \[ \]

Let \( a \in A(TM) \) and \( p = \pi(a) \). Then (11) implies that, under the standard identification \( T_pA(T_pM) \cong A(T_pM) \) (=the skew-symmetric endomorphisms of \( T_pM \)), we have the well-known formula

\[
[X^h, Y^h]_a = [X, Y]^h_a + R^D(X, Y)a.
\]

(12) \[
\]

Any (local) section \( a \) of the bundle \( A(TM) \) determines a (local) vertical vector field \( \tilde{a} \) on \( \mathcal{Z} \) defined by

\[
\tilde{a}_I = \frac{1}{2} (a(q) + I \circ a(q) \circ I), \quad I \in \mathcal{Z}, \quad q = \pi(I).
\]

Thus, if \( aE_j = \sum_{l=1}^{n} a_{jl} E_l \),

\[
\tilde{a} = \sum_{j<l} \tilde{a}_{jl} \frac{\partial}{\partial y_{jl}},
\]

where

\[
\tilde{a}_{jl} = \frac{1}{2} [a_{jl} \circ \pi + \sum_{r,s=1}^{n} y_{rs} (a_{rs} \circ \pi) y_{sl}] .
\]

This formula and (11) imply the following formula, which is a kind of folklore appearing in different contexts (see, for example, [4, Lemma 1], [13, Appendix A]).

**Lemma 4.** If \( I \in \mathcal{Z} \) and \( X \) is a vector field on a neighbourhood of the point \( p = \pi(I) \), then

\[
[X^h, \tilde{a}]_I = (\tilde{D}Xa)_I.
\]

**Proof.** Take a frame \( E_1, \ldots, E_n \) in a neighbourhood of the point \( p = \pi(I) \) such that, for every metric \( g \in \mathfrak{c} \), \( g(E_i, E_j) = f_g \delta_{ij} \), \( i, j = 1, \ldots, n \), where \( f_g \) is a positive smooth function, and \( DE_i|_p = 0 \) for \( i = 1, \ldots, n \). Then \( (DXa)(E_i) = \sum_{l=1}^{n} X_p(a_{il})(E_l)p \) and the result follows by a trivial computation of \([X^h, \tilde{a}]_I \) in the coordinates \((\tilde{e}_i, y_{jl})\) introduced above. \( \square \)

Using the decomposition \( T\mathcal{Z} = \mathcal{H} \oplus \mathcal{V} \), we can define a 1-parameter family of Riemannian metrics on \( \mathcal{Z} \) for every \( g \in \mathfrak{c} \) setting

\[
\hat{g}_t(X^h + V, Y^h + W) = g(X, Y) + tG(V, W), \quad t > 0,
\]

(13) \[
\]

where \( X^h, Y^h \) are the horizontal lifts of \( X, Y \in TM \) (with respect to \( D \)) and \( V, W \) are vertical vectors. Then the projection map \( \pi : (\mathcal{Z}, \hat{g}_t) \to (M, g) \) is a Riemannian submersion.

**Notation.** Fix a metric \( g \in \mathfrak{c} \) and denote \( \nabla^g \), \( \varphi_g \) and \( \delta^g \) simply by \( \nabla \), \( \varphi \) and \( \delta \). Let \( \tilde{D} = \tilde{D}_{g,t} \) be the Levi-Civita connection of the metric \( \hat{g}_t \).

**Lemma 5.** Let \( I \in \mathcal{Z} \) and let \( a, b \) be sections of \( A(TM) \) in a neighbourhood of the point \( p = \pi(I) \) such that \( a(p) \in \mathcal{V}_I \) and \( b(p) \in \mathcal{V}_I \). Then

\[
(\tilde{D}_a \tilde{b})_I = 0.
\]
Proof. In the notation of the preceding lemma and its proof, if \( X \in T_p M \), then

\[
X^h I G(\tilde{a}, \tilde{b}) = \sum_{j l = 1}^n X^h I (\tilde{a}_j \tilde{b}_l) = G((\tilde{D}_X \tilde{a})_I, \tilde{b}_l) + G(\tilde{a}_I, (\tilde{D}_X \tilde{b})_l).
\]

It follows from the latter identity, Lemma \([4]\) and the Koszul formula for the Levi-Civita connection that \((\tilde{D}_X \tilde{b})_I \) is orthogonal to every \( X^h I \), so it is a vertical vector. Next, the Lie bracket of \( \tilde{a} \) and \( \tilde{b} \) is

\[
[\tilde{a}, \tilde{b}]_I = \frac{1}{4} \{ [a(p), b(p)] I - I [a(p), b(p)] \}, \quad p = \pi(I),
\]

where \([a(p), b(p)] \) is the commutator of the endomorphisms \( a(p) \) and \( b(p) \) of \( T_p M \) and, as usual, juxtaposition means composition of maps. Since the endomorphisms \( a(p) \) and \( b(p) \) anti-commute with \( I \),

\[
[a(p), b(p)] I - I [a(p), b(p)] = 0.
\]

Let \( W \in \mathcal{V} \) and let \( c \) be a local section of \( A(TM) \) such that \( c(p) = W \). Then

\[
\tilde{c}_I (G(\tilde{a}, \tilde{b})) = \sum_{j l = 1}^n W(\tilde{a}_j \tilde{b}_l) = \frac{1}{4} \{ G([W, a(p)] I - I [W, a(p)], b(p)) + G([W, b(p)] I - I [W, b(p)], a(p)) \}.
\]

Hence

\[
\tilde{c}_I (G(\tilde{a}, \tilde{b})) = 0.
\]

Now, it follows from the Koszul formula that \( \tilde{g}_I ((\tilde{D}_a \tilde{b})_I, \tilde{c}_I) = 0 \). This proves the lemma. \( \square \)

Let \( \{ \{ a \} \}, \quad \alpha = 1, \ldots, m^2 - m \), be a basis of a vertical space \( \mathcal{V} \). Take local sections \( a_{\alpha} \) of \( A(TM) \) such that \( a_{\alpha}(p) = V_{\alpha}, \quad p = \pi(I) \). Then the vertical vector fields \( \{ \tilde{a}_{\alpha} \} \) constitute a frame in a neighbourhood of \( I \). It follows from Lemma \([5]\) that the covariant derivative of a vertical vector field in a vertical direction is a vertical vector. Thus, the fibres of the bundle \( \pi : (Z, \tilde{g}_I) \rightarrow (M, g) \) are totally geodesic submanifolds. This, of course, follows also from the Vilms theorem (see, for example, \([2]\) Theorem 9.59).

The Koszul formula, identities \((12)\) and \((2)\), and the fact that the fibres of the twistor bundle are totally geodesic submanifolds imply the following formulas.

**Lemma 6.** If \( X, Y \) are vector fields on \( M \) and \( V \) is a vertical vector field on \( Z \), then at \( I \in Z \)

\[
(\tilde{D}_X Y^h)_I = (DX)_I + \frac{1}{2} [\varphi(X) Y^h_I + \varphi(Y) X^h_I - g(X, Y) (\varphi^h)_I] + \frac{1}{2} R^D(X, Y)_I.
\]

Also, \( \tilde{D}_V X^h = \mathcal{H} \tilde{D}_X V \), where \( \mathcal{H} \) means the horizontal component. Moreover,

\[
\tilde{g}_I ((\tilde{D}_V X^h, Y^h)_I = -\frac{1}{2} G(R^D(X, Y)_I, V).
\]

**Notation.** The lemma above suggests defining a new torsion-free connection \( D' = D'_{g, I} \) on \( Z \) as follows. Let \( I \in Z \). Let \( X, Y \) be vector fields on \( M \) in a neighbourhood of \( p = \pi(I) \), and let \( V, W \) be vertical vector fields on \( Z \) in a neighbourhood of \( I \). Set

\[
(D'_{X^h Y^h})_I = (DX)_I + \frac{1}{2} R^D(X, Y)_I,
\]

\[
D'_{X^h} X^h = \tilde{D}_X X^h, \quad D'_{X^h} V = \tilde{D}_X V, \quad D'_{V^h} W = \tilde{D}_V W.
\]
Recall that $R^D(X,Y)I$ in the first formula is a vertical vector at $I$. Clearly, the fibres of the bundle $\pi : Z \to M$ are still totally geodesic submanifolds when $Z$ is considered with the connection $D'$.

5. The second fundamental form w.r.t. $D$ and $D'$ of an almost-Hermitian structure as a map into the twistor space

Let $J$ be a $c$-compatible almost-complex structure. We endow $M$ with the orientation yielded by $J$. Then $J$ can be considered as a section of the positive twistor space of $(M,c)$.

**Notation.** Henceforth, $\pi : Z \to M$ will denote the positive twistor space of $(M,c)$.

Let $J^*TZ \to M$ be the pull-back of the bundle $TZ \to Z$ under the map $J : M \to Z$. Then we can consider the differential $J_* : TM \to TZ$ as a section of the bundle $\text{Hom}(TM,J^*TZ) \to M$.

**Notation.** Denote by $D^*$ the connection on $J^*TZ$ induced by the torsion-free connection $D'$ on $TZ$ defined in the preceding section. The Weyl connection $D$ on $TM$ and the connection $D^*$ on $J^*TZ$ induce a connection $D$ on the bundle $\text{Hom}(TM,J^*TZ)$. By definition, the second fundamental form of the map $J : M \to Z$ with respect to the connections $D$ and $D'$ is the bilinear form defined as

\[ II_J(D,D')(X,Y) = (\hat{D}_X J_*)(Y), \quad X,Y \in TM. \]

This form is symmetric since $D$ and $D'$ are torsion-free. Following [12], we say that the map $J : M \to Z$ is $(D,D')$-pseudo harmonic if

\[ \text{Trace}_g II_J(D,D') = 0, \]

where $g \in c$ and the trace clearly does not depend on the choice of $g$.

To compute this trace, we need the following lemma.

**Lemma 7.** For every $p \in M$, there exists a $\tilde{g}_t$-orthonormal frame of vertical vector fields $\{V_\alpha : \alpha = 1, \ldots, m^2 - m\}$, $2m = \text{dim} M$, in a neighbourhood of the point $J(p)$ such that

1. $(D_{V_\alpha} V_\beta)_{J(p)} = 0$, $\alpha, \beta = 1, \ldots, m^2 - m$.
2. If $X$ is a vector field near the point $p$, $[X^h, V_\alpha]_{J(p)} = 0$.
3. $D^h_{X^h}(V_\alpha \circ J) \perp V_{J(p)}$, where $V_\alpha \circ J$ is considered as a section of $A(TM)$.

The proof goes along the same lines as the proof of [6] Lemma 8, so it will be skipped.

**Proposition 2.** For every $X,Y,Z \in T_pM$, $p \in M$, and $W \in V_{J(p)}$

\[ 2\tilde{g}_t((\hat{D}_X J_*)(Y), Z^h_{J(p)}) = -tg(R^D(Y,Z)J(p), D_X J) - tG(R^D(X,Z)J(p), D_Y J), \]

\[ 2\tilde{g}_t((\hat{D}_X J_*)(Y), W) = tG(D^h_{XY} J + D^h_{YX} J, W), \]

where $D^h_{XY} J = D_X D_Y J - D_{D_X Y} J$ is the second covariant derivative of $J$ considered as a section of $A(TM)$.

**Proof.** Extend $X$ and $Y$ to vector fields in a neighbourhood of the point $p$. Let $V_1, \ldots, V_{m^2 - m}$ be a $\tilde{g}_t$-orthonormal frame of vertical vector fields with the properties (1) - (3) stated in Lemma [7].
Thus, hence

\[ D_X^* (J_* \circ Y) = (D'_{J_* X} Y^h) \circ J + \sum_{\alpha=1}^{m^2-m} g_t(D_Y J, V_\alpha \circ J)(V_\alpha \circ J). \]

By (14) and (16), By Lemma 7,

\[ D_X^* (J_* \circ Y) = (D'_{J_* X} Y^h) \circ J + \frac{1}{2} R^D (X \wedge Y) J(p) + (D'_{D_X Y} X^h) J(p). \]

By Lemma 7

\[ (D'_{J_* X} V_\alpha) J(p) = D'_{X_{J(p)}}^* V_\alpha + D'_{D_X Y} J V_\alpha, \]

\[ = [X^h, V_\alpha] J(p) + (D'_{V_\alpha} X^h) J(p) + D'_{D_X Y} J V_\alpha = (D'_{V_\alpha} X^h) J(p). \]

Thus,

\[ D_X^* (J_* \circ Y) = (D_X Y)_{J(p)}^h + (D'_{D_X Y} X^h) J(p) + (D'_{D_Y X} J^h) J(p) \]

\[ + \frac{1}{2} R^D (X \wedge Y) J(p) + \sum_{\alpha=1}^{m^2-m} G(D_X D_Y J, V_\alpha \circ J)(V_\alpha \circ J). \]

where

\[ R^D (X \wedge Y) J(p) = D_{D_X Y} J - D_{D_Y X} J - D_{X_Y} D_X J \]

by the definition we have adopted, and

\[ R^D (X \wedge Y) J(p) = \sum_{\alpha=1}^{m^2-m} G(R^D (X, Y) J(p), (V_\alpha) J(p))(V_\alpha) J(p). \]

It follows that

\[ (D_X J_*) (Y) = D_X^* (J_* \circ Y) - J_* (D_X Y), \]

\[ = D_X^* (J_* \circ Y) - (D_X Y)_{J(p)}^h - D_{D_X Y} J, \]

\[ = (D'_{D_X Y} J^h) J(p) + (D'_{D_Y X} J^h) J(p) \]

\[ + \frac{1}{2} V(D_X Y - D_{D_X Y} J + D_{Y_p} D_X J - D_{D_Y X} J). \]

Now, the proposition follows from (15). □
Lemma 8. For every $X, Y, Z \in T_p M$,

\[ G(R^D(X, Z)J, D_Y J) = 2g(R^D((J \nabla Y)\wedge X, Z) - g(R^D(\varphi^\sharp \wedge Y - J\varphi^\sharp \wedge Y)X, Z) \]

\[ -d\varphi((J \nabla Y)\wedge)g(X, Z) - d\varphi((J \nabla Y)(X), Z) - d\varphi(X, (J \nabla Y)(Z)) \]

\[ + \frac{1}{2}d\varphi(\varphi^\sharp \wedge Y - J\varphi^\sharp \wedge Y)g(X, Z) \]

\[ + \frac{1}{2}[\varphi(JX)d\varphi(JY, Z) + \varphi(X)d\varphi(Y, Z) - g(Y, JX)d\varphi(J\varphi^\sharp, Z) - g(Y, X)d\varphi(\varphi^\sharp, Z)] \]

\[ + \frac{1}{2}[\varphi(JZ)d\varphi(X, JY) + \varphi(Z)d\varphi(X, Y) - g(Y, JZ)d\varphi(X, J\varphi^\sharp) - g(Y, Z)d\varphi(X, \varphi^\sharp)]. \]

Proof. By Lemma 2,

\[ G(R^D(X, Z)J, D_Y J) = g(R^D([J, D_Y J]\wedge)X, Z) \]

\[ -\frac{1}{2}[d\varphi([J, D_Y J]\wedge)g(X, Z) + d\varphi([J, D_Y J]X, Z) + d\varphi(X, [J, D_Y J]Z)], \]

\[ = 2g(R^D((J \nabla Y)\wedge)X, Z) \]

\[ -d\varphi((J \nabla Y)\wedge)g(X, Z) - d\varphi((J \nabla Y)(X), Z) - d\varphi(X, (J \nabla Y)(Z)). \]

By 2,

\[ (J \nabla Y)(E) = J(\nabla Y)(E) - \frac{1}{2}[\varphi(E)JY + \varphi(E)Y - g(Y, J)\varphi^\sharp - g(Y, E)\varphi^\sharp], \]

for every $Y, E \in T_p M$. It follows that

\[ (J \nabla Y)\wedge = (J \nabla Y)\wedge - \frac{1}{2}(\varphi^\sharp \wedge Y - J\varphi^\sharp \wedge Y). \]

Now, the lemma follows easily. \qed

Proposition 2 Lemma 8 and identity 11 imply the following.

Corollary 3. For every tangent vector $Z \in T_p M$,

\[ -\frac{1}{2}g_\flat(\mathcal{H}(\text{Trace}_g \vec{D}J_\mu), Z^h_{(\mu)}) \]

\[ = 2\text{Trace}_g\{T_p M \ni X \to g(R^D((J \nabla X)\wedge)X, Z)\} + \rho_D(\varphi^\sharp, Z) - \rho_D(J\varphi^\sharp, JZ) \]

\[ -d\varphi((J \nabla Z)\wedge) + d\varphi(J \delta J, Z) - \text{Trace}_g\{T_p M \ni X \to d\varphi(X, (J \nabla X)(Z)) \}

\[ + \varphi(JZ)d\varphi(J\wedge) \]

\[ - \left(\frac{n}{2} - 1\right)d\varphi(\varphi^\sharp, Z) + d\varphi(J\varphi^\sharp, JZ). \]

Lemma 9. For every $Z, U \in T_p M$,

\[ g(\text{Trace}\{X \to (D^2_{X X} J)(Z)\}, U) = g(\text{Trace}\{X \to (\nabla^2_{X X} J)(Z)\}, U) \]

\[ + \frac{1}{2}(2 - n)g((\nabla \varphi, J)(Z), U) - \varphi(Z)g(\delta J, U) + \varphi(U)g(\delta J, Z) \]

\[ + g((\nabla Z)(J), \varphi^\sharp) - g((\nabla U)(J), \varphi^\sharp) + \frac{1}{2}d\varphi(JZ \wedge U + Z \wedge JU) \]

\[ + \frac{1}{4}(n - 3)[\varphi(Z)\varphi(JU) - \varphi(JZ)\varphi(U)] + \frac{1}{2}||\varphi^\sharp||^2g(Z, JU). \]
Proof. We have
\[
g((D^2_{XY}-J)(Z), U) = g((D_X(D_Y-J))(Z), U) - g((D_Y J)(D_X Z), U) \\
- g((D_{D_X Y} J)(Z), U),
\]
\[
= X.g((D_Y J)(Z), U) - g((D_Y J)(Z), D_X U) - \phi(X)g((D_Y J)(Z), U) \\
- g((D_Y J)(D_X Z), U) - g((D_{D_X Y} J)(Z), U).
\]
Also, by (2),
\[
g((D_Y J)(Z), U) = g((\nabla_Y J)(Z), U) - g(JY \wedge \varphi^2 + Y \wedge J\varphi^2, Z \wedge U).
\]
Applying these identities and identity (2), after a simple computation, we obtain
\[
g((D^2_{XY}-J)(Z), U) = g((\nabla^2_{XY}-J)(Z), U)
\]
\[
+ \frac{1}{2}g(\nabla_X J)(Z), U) + \varphi(Y)g((\nabla_X J)(Z), U) \\
+ \varphi(Z)g((\nabla_Y J)(X), U) - \varphi(U)g((\nabla_Y J)(X), Z)]
\]
\[
+ \frac{1}{2}g(X, U)g((\nabla_Y J)(\varphi^2), Z) - g(X, Z)g((\nabla_Y J)(\varphi^2), U)
\]
\[
- g((\nabla_X J)(Y) \wedge \varphi^2 + Y \wedge (\nabla_X J)(\varphi^2) + JY \wedge \nabla_X \varphi^2 \\
+ Y \wedge J\nabla_X \varphi^2, Z \wedge U)
\]
\[
+ \frac{1}{2}g(\varphi(JY) X \wedge \varphi^2 + ||\varphi^2||^2 JY \wedge X + \varphi(X) Y \wedge J\varphi^2 - \varphi(Y) JX \wedge \varphi^2 \\
- g(X, JY) Y \wedge \varphi^2 + g(X, Y) J\varphi^2 \wedge \varphi^2, Z \wedge U).
\]
Finally, note that
\[
g(\nabla_JZ, \varphi^2, U) - g(\nabla_JU, \varphi^2, Z) + g(\nabla_Z, \varphi^2, JU) - g(\nabla_U, \varphi^2, JZ)
\]
\[
= (\nabla_JZ, \varphi^2)(U) - (\nabla_JU, \varphi^2)(Z) + (\nabla_Z, \varphi^2)(JU) - (\nabla_U, \varphi^2)(JZ),
\]
\[
= d\varphi(JZ \wedge U) + d\varphi(Z \wedge JU).
\]
Let \(I \in {\mathfrak{Z}}\). Take a metric \(g_0 \in {\mathfrak{c}}\). Let \(e_1, \ldots, e_n\) be a \(g_0\)-orthonormal basis of \(T_p M, p = \pi(I)\), such that \(I e_{2k-1} = e_{2k}, k = 1, \ldots, m\). Extend this basis to a frame of vector fields \(E_1, \ldots, E_n\) in a neighbourhood of \(p\) such that \((E_i)_p = e_i, DE_i|_p = 0, i = 1, \ldots, n\), and, for every \(g \in {\mathfrak{c}}\), \(g(E_k, E_l) = f_g \delta_{kl}, k, l = 1, \ldots, n\), where \(f_g\) is a smooth positive function. As above, define sections \(S_{ij}, 1 \leq i, j \leq n\), of \(A(TM)\), by
\[
S_{ij} E_l = \delta_{il} E_j - \delta_{jl} E_i, \quad i, j, l = 1, \ldots, n.
\]
Set
\[
A_{r,s} = \frac{1}{\sqrt{2}}(S_{2r-1,2s-1} - S_{2r,2s}), \quad B_{r,s} = \frac{1}{\sqrt{2}}(S_{2r-1,2s} + S_{2r,2s-1}),
\]
\[
r = 1, \ldots, m - 1, s = r + 1, \ldots, m.
\]
The endomorphisms \(\{(A_{r,s})_p, (B_{r,s})_p\}\) of \(T_p M, r = 1, \ldots, m - 1, s = r + 1, \ldots, m\), constitute a \(G\)-orthonormal basis of the vertical space \(\mathcal{V}_I\). It follows that identifying
A(T_pM) with \( \Lambda^2 T_pM \) by means of the isomorphism \( a \to a^\wedge \) given by (9), every vector of \( \mathcal{V}_l \) is a linear combination of vectors of the form \( Z \wedge U - JZ \wedge JU \), \( Z, U \in T_pM \). Thus, \( \text{Trace}\{X \to g(D^2_XJ, W)\} = 0 \) for every vertical vector \( W \) of \( \mathcal{V}_l \) if and only if \( \text{Trace}\{X \to g((D^2_XJ)\wedge, Z \wedge U - JZ \wedge JU)\} = 0 \) for every tangent vectors \( Z, U \) of \( M \). In view of this remark and Proposition 2, we state the following.

**Corollary 4.** If \( X, Z, U \in T_pM \),

\[
g((\text{Trace} D^2_XJ)(Z), U) - g((\text{Trace} D^2_XJ)(JZ), JU) = g(\langle J\nabla^2_XJ \rangle(Z), U) - g(\langle J\nabla^2_XJ \rangle(JZ), JU) + (2 - n)g((\nabla\varphi)J)(Z, U)
\]

\[
- \varphi(Z)g(\delta J, U) + \varphi(JZ)g(\delta J, JU) + \varphi(U)g(\delta J, Z) - \varphi(JU)g(\delta J, JZ)
\]

\[
- g((\nabla_ZJ)(\varphi^\sharp), U) + g((\nabla_UJ)(\varphi^\sharp), Z) + g((\nabla_JZJ)(\varphi^\sharp), JU)
\]

\[
- g((\nabla_JU)(\varphi^\sharp), JZ) + d\varphi(JZ \wedge U + Z \wedge JU).
\]

6. **Hermitian structures on Weyl manifolds yielding pseudo-harmonic maps into the twistor space**

**Notation.** Denote by \( N \) the Nijenhuis tensor of \( J \):

\[
N(Y, Z) = [Y, Z] + [JY, JZ] - J[Y, JZ] - J[JY, Z].
\]

Let \( \Omega(X, Y) = g(JX, Y) \) be the fundamental 2-form of the almost-Hermitian manifold \((M, g, J) \) (\( g \in \mathfrak{c} \) being a fixed metric as in the preceding sections).

It is well-known (and easy to check) that

\[
2g((\nabla_XJ)(Y), Z) = d\Omega(X, Y, Z) - d\Omega(X, JY, JZ) + g(N(Y, Z), JX),
\]

for all \( X, Y, Z \in TM \). The condition that \( J \) is integrable \((N = 0) \) is equivalent to \((\nabla_XJ)(Y) = (\nabla_JX)(JY) \), \( X, Y \in TM \) [14 Corollary 4.2].

Suppose that the almost-complex structure \( J \) is integrable. Let

\[
\theta = -\frac{2}{n - 2} \delta\Omega \circ J
\]

be the Lee form of the Hermitian structure \((g, J) \). If \( n = 4 \), the Lee form satisfies the identity \( d\Omega = \theta \wedge \Omega \). For \( n \geq 6 \), this identity is satisfied if and only \((g, J) \) is a locally conformally Kähler structure. As is well-known, it implies \( d\theta = 0 \) when \( n \geq 6 \). Note also that, in any dimension, an Hermitian structure \((g, J) \) is locally conformally Kähler if and only if \( d\Omega = \theta \wedge \Omega \) and \( d\theta = 0 \), see, for example, [10] [23]. Clearly, if \((g, J) \) is such a structure, then for any metric \( g^1 \) conformal to \( g \), the structure \((g^1, J) \) is also locally conformally Kähler.

**Assumption.** We assume that the almost-complex structure \( J \) is integrable and the Lee from satisfies the identity

\[
d\Omega = \theta \wedge \Omega.
\]

**Remark.** If the latter identity is satisfied by an almost-Hermitian structure \((g, J) \), it is also satisfied by any almost-Hermitian structure \((g^1, J) \) with \( g^1 = e^f g \), \( f \) being a smooth function, since \( \Omega^1 = e^f \Omega \) and \( \theta^1 = \theta + df \) by the first identity of [3].
Let $B$ be the vector field on $M$ dual to the Lee form $\theta = -\delta \Omega \circ J$ with respect to the metric $g$. Thus

$$B = \frac{2}{n-2} J\delta J.$$ 

Identities (18) and (19) imply the following well-known formula

$$2(\nabla_X J)(Y) = g(JX, Y)B - g(B, Y)JX + g(X, Y)JB - g(JB, Y)X. \tag{20}$$

Now, we are going to apply Corollary 3.

Formula (20) implies that, for $X, Y, Z \in T_p M$,

$$g((J\nabla_X J)(Y), Z) = \frac{1}{2} g((J\nabla_X J)(Y), Z) = \frac{1}{2} g(B \wedge X - JB \wedge JX, Y \wedge Z).$$

So,

$$(J\nabla_X J)^\wedge = \frac{1}{2} (B \wedge X - JB \wedge JX),$$

and applying (20), we get

$$2\text{Trace}\{TM \ni X \to g(R^D((J\nabla_X J)^\wedge), X, Z)\}$$

$$= -\rho_D(B, Z) + \rho^*_D(JB, JZ) + d\varphi(B, Z) - d\varphi(JB, JZ).$$

Moreover, by (20),

$$\text{Trace}\{TM \ni X \to d\varphi(X, (J\nabla_X J)(Z))\}$$

$$= \frac{1}{2} [d\varphi(B, Z) + d\varphi(JB, JZ)] + \theta(JZ)d\varphi(J^\wedge).$$

It follows from Corollary 3 that

$$-\frac{1}{2} B_\ast (\mathcal{H}(\text{Trace} D^2 J\ast), Z^\ast_{J(p)})$$

$$= (\frac{n}{2} - 1)d\varphi((\theta - \varphi)^2, Z) - d\varphi(J(\theta - \varphi)^2, JZ) - (\theta - \varphi)(JZ)d\varphi(J^\wedge)$$

$$-\rho_D((\theta - \varphi)^2, Z) + \rho^*_D(J(\theta - \varphi)^2, JZ).$$

Thus, we have the following statement.

**Lemma 10.** Suppose that $J$ is a Hermitian structure whose Lee form satisfies identity (19). Then $\mathcal{H}(\text{Trace} D^2 J\ast) = 0$ if and only if for every $Z \in TM$

$$(\frac{n}{2} - 1)d\varphi((\theta - \varphi)^2, Z) - d\varphi(J(\theta - \varphi)^2, JZ) - (\theta - \varphi)(JZ)d\varphi(J^\wedge)$$

$$-\rho_D((\theta - \varphi)^2, Z) + \rho^*_D(J(\theta - \varphi)^2, JZ) = 0.$$ 

Next, by Corollary 3 and identity (20),

$$g((\text{Trace} D^2 X J)(Z), U) - g((\text{Trace} D^2 X J)(JZ), JU)$$

$$= g((\text{Trace} \nabla^2 X J)(Z), U) - g((\text{Trace} \nabla^2 X J)(JZ), JU)$$

$$+ d\varphi(JZ \wedge U + Z \wedge JU).$$

It follows from (20) that
two conditions are satisfied.

Theorem 1. Let $D$ be of type $(1,0)$, then the identity (19) implies the following.

Lemma 11. Suppose that $J$ is a Hermitian structure whose Lee form satisfies identity (19). Then $\nabla \text{Trace} D^2 J = 0$ if and only if $g((\text{Trace} D^2 J)(Z), U) = 0$ for all $Z, U$. Note also that a 2-form $\alpha$ is of type $(1,1)$ with respect to $J$ exactly when $\alpha(JZ \wedge U + Z \wedge JU) = 0$ for every $Z, U \in TM$. Thus, we have the following.

Lemma 11. Suppose that $J$ is a Hermitian structure whose Lee form satisfies identity (19). Then $\nabla \text{Trace} D^2 J = 0$ if and only if $g((\text{Trace} D^2 J)(Z), U) = 0$ for all $Z, U$. Note also that a 2-form $\alpha$ is of type $(1,1)$ with respect to $J$ exactly when $\alpha(JZ \wedge U + Z \wedge JU) = 0$ for every $Z, U \in TM$. Thus, we have the following.

Proposition 2. Lemmas 10 and 11 imply the following.

Theorem 1. Let $(M, \varphi)$ be a conformal $n$-dimensional manifold with a Weyl connection $D$, let $g \in \mathcal{C}$ and set $\varphi = \varphi_g$. Suppose that $J$ is a Hermitian structure on $(M, \varphi)$ such that Lee form $\theta$ of $(g, J)$ satisfies identity (19). Let $D'$ be the torsion-free connection on $Z$ defined by (19). Then $J$ determines a $(D, D')$-pseudo-harmonic map from $M$ into the positive twistor space $Z$ of $(M, \varphi)$ if and only if the following two conditions are satisfied.

(i) The 2-form

$$d(\varphi - \theta) + \frac{n(n - 4)}{2(n - 2)} \varphi \wedge \theta$$

is of type $(1,1)$ with respect to $J$. 

(ii) Proposition 2. Lemmas 10 and 11 imply the following.
is of type $(1,1)$ with respect to $J$.

(ii) For every tangent vector $Z \in TM$,

$$\left(\frac{n}{2} - 1\right) d\varphi((\theta - \varphi)^2, Z) - d\varphi(J(\theta - \varphi)^2, JZ) - (\theta - \varphi)(JZ)d\varphi(J^\perp) \nonumber$$

$$- \rho_D((\theta - \varphi)^2, Z) + \rho_D^*(J(\theta - \varphi)^2, JZ) = 0,$$

where $\sharp : T^*M \rightarrow TM$ is the isomorphism defined by means the metric $g$.

**Corollary 5.** In the notation of Theorem [1] if $\dim M = 4$, $J$ determines a $(D, D')$-pseudo-harmonic map from $M$ into the positive twistor space $Z$ of $(M, c)$ if and only if the 2-form $d(\theta - \varphi)$ is of type $(1,1)$ with respect to $J$ and

$$(\theta - \varphi)(JZ)d\varphi(J^\perp) + \rho_D((\theta - \varphi)^2, Z) - \rho_D^*(J(\theta - \varphi)^2, JZ) = 0 \text{ for } Z \in TM.$$

**Remark.** If $c$ consists of a single metric $g$ and we set $\varphi_g = 0$, then $D$ is the Levi-Civita connection of $g$ and $D'$ is the Levi-Civita connection of $\tilde{g}_{\theta}$. In this case, Corollary [5] coincides with [8, Theorem 1].

If $D$ is the Weyl connection determined by $g$ and $\varphi = \theta$, then conditions (i) and (ii) of Theorem [1] are trivially satisfied, hence $J : M \rightarrow Z$ is a pseudo-harmonic map. In fact, this is a consequence of Proposition [2] since $DJ = 0$ by [2] and (20). The latter identities implies that $D$ is the unique Weyl connection preserving the conformal class of $g$ and such that $J$ is parallel ([23]).

**Example 1.** It has been observed in [22] that every Inoue surface $M$ of type $S^0$ admits a locally conformally Kähler metric $g$ for which the Lee form $\theta$ is nowhere vanishing. Define the metric $\tilde{g}_{\theta}$ on the twistor space by means of the Levi-Civita connection of the metric $g$. It is shown in [3] that in this case the map $J : (M, g) \rightarrow (Z, \tilde{g}_{\theta})$ is not harmonic. But, $J$ is pseudo-harmonic if we use the Weyl connection $D$ defined by means of $g$ and $\varphi_g = \theta$.

Now, recall the construction of the Inoue surfaces of type $S^0$ ([10]). Let $A \in SL(3, \mathbb{Z})$ be a matrix with a real eigenvalue $\alpha > 1$ and two complex eigenvalues $\beta$ and $\overline{\beta}$, $\beta \neq \overline{\beta}$. Choose eigenvectors $(a_1, a_2, a_3) \in \mathbb{R}^3$ and $(b_1, b_2, b_3) \in \mathbb{C}^3$ of $A$ corresponding to $\alpha$ and $\beta$, respectively. Then the vectors $(a_1, a_2, a_3), (b_1, b_2, b_3), (\overline{a_1}, \overline{a_2}, \overline{a_3})$ are $\mathbb{C}$-linearly independent. Denote the upper-half plane in $\mathbb{C}$ by $\mathbb{H}$ and let $\Gamma$ be the group of holomorphic automorphisms of $\mathbb{H} \times \mathbb{C}$ generated by

$$g_0 : (w, z) \rightarrow (\alpha w, \beta z), \quad g_i : (w, z) \rightarrow (w + a_i, z + b_i), \quad i = 1, 2, 3.$$

The group $\Gamma$ acts on $\mathbb{H} \times \mathbb{C}$ freely and properly discontinuously. Then $M = (\mathbb{H} \times \mathbb{C})/\Gamma$ is a compact complex surface known as Inoue surface of type $S^0$.

Following [22], consider on $\mathbb{H} \times \mathbb{C}$ the Hermitian metric

$$g = \frac{1}{\nu^2} (du \otimes du + dv \otimes dv) + v(dx \otimes dx + dy \otimes dy), \quad u + iv \in \mathbb{H}, \quad x + iy \in \mathbb{C}.$$

This metric is invariant under the action of the group $\Gamma$, so it descends to a Hermitian metric on $M$ which we denote again by $g$. Instead on $M$, we work with $\Gamma$-invariant objects on $\mathbb{H} \times \mathbb{C}$. Let $\Omega$ be the fundamental 2-form of the Hermitian structure $(g, J)$ on $\mathbb{H} \times \mathbb{C}$, $J$ being the standard complex structure. Then

$$d\Omega = \frac{1}{v} dv \wedge \Omega.$$
Hence the Lee form is $\theta = d \ln v$. In particular, $d\theta = 0$, i.e. $(g, J)$ is a locally conformally Kähler structure. Set

$$E_1 = v \frac{\partial}{\partial u}, \quad E_2 = v \frac{\partial}{\partial v}, \quad E_3 = \frac{1}{\sqrt{v}} \frac{\partial}{\partial x}, \quad E_4 = \frac{1}{\sqrt{v}} \frac{\partial}{\partial y}.$$ 

These are $\Gamma$-invariant vector fields constituting an orthonormal basis such that $JE_1 = E_2$, $JE_3 = E_4$. The non-zero Lie brackets of $E_1, ..., E_4$ are

$$[E_1, E_2] = -E_1, \quad [E_2, E_3] = -\frac{1}{2} E_3, \quad [E_2, E_4] = -\frac{1}{2} E_4.$$

Denote the dual basis of $E_1, ..., E_4$ by $\eta_1, ..., \eta_4$. The 1-forms $\eta_1, ..., \eta_4$ are $\Gamma$-invariant. Take a $\Gamma$-invariant 1-form $\phi = a_1 \eta_1 + ... + a_4 \eta_4$, where $a_1, ..., a_4$ are real constants. The Levi-Civita connection $\nabla$ of the metric $g$ is $\Gamma$-invariant, hence the connection $D$ on $H \times C$ defined via (2) by means the metric $g$ and the form $\phi$ yields a Weyl connection on $M$.

We have $d\eta_1 = \eta_1 \wedge \eta_2$, $d\eta_2 = 0$, $d\eta_3 = \frac{1}{2} \eta_2 \wedge \eta_3$, $d\eta_4 = \frac{1}{2} \eta_2 \wedge \eta_4$ (the definition of the exterior product being so that $\eta_1 \wedge \eta_2 (E_1 \wedge E_2) = 1$, etc.). Note also that $\theta = \eta_2$. Then the form

$$d(\phi - \theta) = a_1 \eta_1 \wedge \eta_2 + \frac{1}{2} a_3 \eta_2 \wedge \eta_3 + \frac{1}{2} a_4 \eta_2 \wedge \eta_4$$

is of type $(1,1)$ w.r.t. $J$ exactly when $a_3 = a_4 = 0$. Thus $\phi = a_1 \eta_1 + a_2 \eta_2$ and $d\phi = a_1 \eta_1 \wedge \eta_2$.

The Levi-Civita connection $\nabla$ of $g$ is given by the following table (8):

$$\nabla_{E_i} E_2 = \frac{1}{2} E_3, \quad \nabla_{E_i} E_3 = -\frac{1}{2} E_2, \quad \nabla_{E_i} E_4 = \frac{1}{2} E_4, \quad \nabla_{E_4} E_4 = -\frac{1}{2} E_2,$$

all other $\nabla_{E_i} E_j = 0$.

Now, since $d\phi = a_1 \eta_1 \wedge \eta_2$ is of type $(1,1)$ with respect to $J$, condition (ii) of Theorem 1 takes the form

$$\nabla (\phi - \theta)(JZ) d\phi(J^\wedge Z) + \rho_D J(\phi - \theta)^\wedge Z - \rho_D^* (J(\phi - \theta)^\wedge Z) = 0.$$  

We have $d\phi(J^\wedge Z) = d\phi(E_1 \wedge E_2) + d\phi(E_3 \wedge E_4) = a_1$. Using (21), we can compute the curvature of the Weyl connection by means of identity (5). As a result, the non-zero components of $\rho_{D_{ij}} = \rho_D(E_i, E_j)$ are

$$\rho_{D_{11}} = -\frac{a_2 (a_2 + 2)}{2}, \quad \rho_{D_{12}} = \frac{a_1 (a_2 + 3)}{2}, \quad \rho_{D_{21}} = \frac{a_1 a_2}{2}, \quad \rho_{D_{22}} = -\frac{a_1^2 + 3}{2}, \quad \rho_{D_{33}} = \rho_{D_{44}} = -\frac{a_1^2 + a_2 (a_2 - 1)}{2}.$$ 

Also, the non-zero components of the $\ast$-Ricci tensor of $(g, J)$ $\rho_{D_{ij}}^* = \rho_D^*(E_i, E_j)$ are the following:

$$\rho_{D_{11}}^* = \rho_{D_{22}}^* = -\frac{a_2 + 2}{2}, \quad \rho_{D_{12}}^* = -\rho_{D_{21}}^* = -\rho_{D_{34}}^* = \rho_{D_{43}}^* = \frac{a_1}{2}, \quad \rho_{D_{33}}^* = \rho_{D_{44}}^* = -\frac{a_1^2 + (a_2 - 1)^2}{4}.$$ 

Now, it is easy to see that identity (22) holds for every $Z$ if and only if

$$a_1 (a_2 - 1) = 0, \quad a_1^2 + (a_2 - 1)^2 = 0.$$
Obviously, the solution of the latter system is $a_1 = 0$, $a_2 = 1$. Thus, $J$ is a pseudo-harmonic map from $M$ into its twistor space only when $\varphi = \theta$, i.e. $D$ is the Weyl connection determined by $g$ and the Lee form $\theta$ of $(g, J)$.

**Example 2.** Recall that a primary Kodaira surface $M$ is the quotient of $\mathbb{C}^2$ by a group of transformations acting freely and properly discontinuously [17, p. 787]. This group is generated by the affine transformations $\varphi_k(z, w) = (z + a_k, w + \overline{a}_k z + b_k)$, where $a_k, b_k, k = 1, 2, 3, 4$, are complex numbers such that $a_1 = a_2 = 0$, $b_2 \neq 0$, $\text{Im}(a_3 \overline{a}_4) = mb_1 \neq 0$ for some integer $m > 0$. The quotient space is compact.

It is well-known that $M$ can also be described as the quotient of $\mathbb{C}^2$ endowed with a group structure by a discrete subgroup $\Gamma$. The multiplication on $\mathbb{C}^2$ is defined by $$(a, b), (z, w) = (z + a, w + \overline{a} z + b), \quad (a, b), (z, w) \in \mathbb{C}^2,$$
and $\Gamma$ is the subgroup generated by $(a_k, b_k)$, $k = 1, \ldots, 4$ (see, for example, [3]). Considering $M$ as the quotient $\mathbb{C}^2/\Gamma$, every left-invariant object on $\mathbb{C}^2$ descends to a globally defined object on $M$.

As in [5, 8, 9], take a frame of left-invariant vector fields $A_1, \ldots, A_4$ such that
$$[A_1, A_2] = -2A_4, \quad [A_i, A_j] = 0 \text{ otherwise}.$$ 
These identities are satisfied, for example, by the following left-invariant frame
$$A_1 = -\frac{\partial}{\partial x} - x \frac{\partial}{\partial u} + y \frac{\partial}{\partial v}, \quad A_2 = \frac{\partial}{\partial y} + y \frac{\partial}{\partial u} + x \frac{\partial}{\partial v}, \quad A_3 = \frac{\partial}{\partial u}, \quad A_4 = \frac{\partial}{\partial v},$$
where $x + iy = z$, $u + iv = w$.

Denote by $g$ the left-invariant Riemannian metric on $M$ for which the frame $A_1, \ldots, A_4$ is orthonormal.

By [15], every complex structure on $M$ is induced by a left-invariant complex structure on $\mathbb{C}^2$. It is easy to see that every such a structure is given by ([19, 8])
$$JA_1 = \epsilon_1 A_2, \quad JA_3 = \epsilon_2 A_4, \quad \epsilon_1, \epsilon_2 = \pm 1.$$ 
Denote the complex structure defined by these identity by $J_{\epsilon_1, \epsilon_2}$.

The non-zero covariant derivatives $\nabla A_i A_j$ are ([8])
$$\nabla A_1 A_2 = -\nabla A_2 A_1 = -A_4, \quad \nabla A_1 A_4 = \nabla A_4 A_1 = A_2, \quad \nabla A_2 A_4 = \nabla A_4 A_2 = -A_1.$$ 
This implies that the Lee form is
$$\theta(X) = -2\epsilon_1 g(X, A_3).$$
Therefore $\theta = -2\epsilon_1 A_3$, $\nabla \theta = 0$.

Denote the dual basis of $A_1, \ldots, A_4$ by $\alpha_1, \ldots, \alpha_4$. Thus, $\theta = -2\epsilon_1 \alpha_3$. Any left-invariant 1-form $\varphi$ is of the form $\varphi = a_1 \alpha_1 + \ldots + a_4 \alpha_4$, where $a_1, \ldots, a_4$ are real constants. The connection $D$ on $\mathbb{C}^2$ defined via (2) by means the metric $g$ and the form $\varphi$ yields a Weyl connection on $M$.

We have $d\alpha_1 = d\alpha_2 = d\alpha_3 = 0$, $d\alpha_4 = 2\alpha_1 \wedge \alpha_2$. Hence the form $d(\varphi - \theta) = d\varphi - 2\alpha_4 \alpha_3 \wedge \alpha_2$ is of type $(1, 1)$ w.r.t. $J = J_{\epsilon_1, \epsilon_2}$.

Let $R$ be the curvature tensor of the Levi-Civita connection $\nabla$. Set for short $R_{ijk} = R(A_i, A_j) A_k$. Then the non-zero $R_{ijk}$ are ([8])
$$R_{121} = -3A_2, \quad R_{122} = 3A_1, \quad R_{141} = A_4,$$
$$R_{144} = -A_1, \quad R_{242} = A_4, \quad R_{244} = -A_2.$$
The non-zero covariant derivatives of the form \( \varphi = a_1 \alpha_1 + \ldots + a_4 \alpha_4 \) are
\[
(\nabla_{A_1} \varphi)(A_2) = -(\nabla_{A_2} \varphi)(A_1) = a_4, \quad (\nabla_{A_1} \varphi)(A_4) = (\nabla_{A_4} \varphi)(A_1) = -a_2,
(\nabla_{A_2} \varphi)(A_4) = \nabla_{A_4} \varphi)(A_2) = a_1.
\]

Now, using \([24]\), we can compute the curvature of the Weyl connection \( D \) by means of identity \([5]\), then the components \( \rho_{D_{ij}} = \rho_{D}(A_i, A_j) \) and \( \rho^*_{D_{ij}} = \rho^*_D(A_i, A_j) \) of the Ricci and *-Ricci tensors. The latter are given in the following tables.

\[
\begin{align*}
\rho_{D_{11}} &= -\frac{a_2^2 + a_3^2 + a_4^2 + 4}{2}, \quad \rho_{D_{12}} = 2a_4 + \frac{1}{2}a_1a_2, \quad \rho_{D_{13}} = \rho_{D_{31}} = \frac{1}{2}a_1a_3, \\
\rho_{D_{14}} &= -a_2 + \frac{1}{2}a_1a_4, \quad \rho_{D_{21}} = -2a_4 + \frac{1}{2}a_1a_2, \quad \rho_{D_{22}} = -\frac{a_1^2 + a_3^2 + a_4^2 + 4}{2}, \\
\rho_{D_{23}} &= \rho_{D_{32}} = \frac{1}{2}a_2a_3, \quad \rho_{D_{24}} = \rho_{D_{42}} = a_1 + \frac{1}{2}a_2a_4, \quad \rho_{D_{33}} = -\frac{a_1^2 + a_2^2 + a_3^2}{2}, \\
\rho_{D_{34}} &= \rho_{D_{43}} = \frac{1}{2}a_3a_4, \quad \rho_{D_{41}} = -a_2 + \frac{1}{2}a_1a_4, \quad \rho_{D_{44}} = -\frac{a_1^2 + a_2^2 + a_4^2 - 4}{2}, \\
\rho^*_{D_{11}} &= \rho^*_{D_{22}} = -\frac{a_2^2 + a_4^2 + 12}{4}, \quad \rho^*_{D_{12}} = -\rho^*_{D_{21}} = a_4, \\
\rho^*_{D_{13}} &= \rho^*_{D_{31}} = \varepsilon_1\varepsilon_2\rho^*_{D_{24}} = \varepsilon_1\varepsilon_2\rho^*_{D_{42}} = \frac{a_1a_3 + \varepsilon_1\varepsilon_2(2a_4 + a_2a_4)}{4}, \\
\rho^*_{D_{14}} &= \rho^*_{D_{41}} = -\varepsilon_1\varepsilon_2\rho^*_{D_{23}} = -\varepsilon_1\varepsilon_2\rho^*_{D_{32}} = \frac{-2a_2 + a_1a_4 - \varepsilon_1\varepsilon_2a_2a_3}{4}, \\
\rho^*_{D_{34}} &= -\rho^*_{D_{43}} = -\varepsilon_1\varepsilon_2a_4, \quad \rho^*_{D_{33}} = \rho^*_{D_{44}} = -\frac{a_1^2 + a_2^2}{4}.
\end{align*}
\]

Condition (ii) of Theorem \([1]\) reduces to the identity
\[
2a_4\varepsilon_1(\theta - \varphi)(JZ) + \rho_D((\theta - \varphi)^t, Z) - \rho^*_D(J(\theta - \varphi)^t, JZ) = 0.
\]

This identity is satisfied for every \( Z \in TM \) if and only if
\[
(1 - \varepsilon_2)a_2a_4 + (1 - \varepsilon_2)(2 + \varepsilon_1a_3)a_1 = 0,
(1 - \varepsilon_2)a_1a_4 - (1 - \varepsilon_2)(2 + \varepsilon_1a_3)a_2 = 0,
(1 + \varepsilon_2)(a_1^2 + a_2^2) + 2(1 - \varepsilon_2)a_1^2 = 0,
(1 - \varepsilon_2)(2 + \varepsilon_1a_3)a_4 = 0.
\]

Clearly, if \( \varepsilon_2 = 1 \), the solution of this system is \( a_1 = a_2 = 0, a_3 \) and \( a_4 \) arbitrary. For \( \varepsilon_2 = -1 \), the solutions are \( a_1, a_2 \) arbitrary, \( a_3 = -2\varepsilon_1, a_4 = 0 \) and \( a_1 = a_2 = a_4 = 0, a_3 \neq -2\varepsilon_1 \). Thus, on a Kodaira surface, there are many Weyl connections for which the complex structures \( J_{\varepsilon_1, \varepsilon_2} \) are pseudo-harmonic maps from \( M \) into its twistor space \( \mathcal{Z} \).

The next statement is a weaker version of \([20]\) Proposition 1.3).

**Corollary 6.** Let \( J_1 \) and \( J_2 \) be two Hermitian structures on a (connected) Riemannian manifold \((M, g)\). Suppose that \( J_1 \) and \( J_2 \) have the same Lee form \( \theta \) satisfying identity \([17]\). If \( J_1 \) and \( J_2 \) coincide on an open subset or, more-generally, if they coincide to infinite order at a point, they coincide on the whole manifold \( M \).

**Proof.** The complex structures \( J_1 \) and \( J_2 \) determine the same orientation on \( M \), hence the same positive twistor space \( \mathcal{Z} \). Let \( D \) be the Weyl connection determined
$g$ and $\theta$. If $\tilde{g}_t$ is the metric on $Z$ defined by means of $D$, and let $D'$ be the torsion-free connection on $Z$ defined by (16). Then the maps $J_1$ and $J_2$ of $M$ into $Z$ are $(D, D')$-pseudo-harmonic by Theorem 1. Thus, the result follows from the uniqueness theorem for pseudo-harmonic maps [18]. □

**Remark.** In fact, M. Pontecorvo [20] has proved that the conclusion of the above corollary holds without the assumption on the Lee forms. The idea of his proof is inspired by the theory of pseudo-holomorphic curves. It makes use of the observation in [12] that an almost-Hermitian structure $J$ on a Riemannian manifold $M$ is integrable if and only if the corresponding map $J: M \rightarrow Z$ is pseudo-holomorphic with respect to the almost-complex structure $J$ on $M$ and the Atiyah-Hitchin-Singer [1] almost-complex structure on $Z$.

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