COSMETIC CROSSING CONJECTURE FOR GENUS ONE KNOTS WITH NON-TRIVIAL ALEXANDER POLYNOMIAL

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Abstract. We prove the cosmetic crossing conjecture for genus one knots with non-trivial Alexander polynomial. We also prove the conjecture for genus one knots with trivial Alexander polynomial, under some additional assumptions.

1. Introduction

A cosmetic crossing is a non-nugatory crossing such that the crossing change at the crossing preserves the knot. A cosmetic crossing conjecture [Kir, Problem 1.58] asserts there are no such crossings.

Conjecture 1 (Cosmetic crossing conjecture). An oriented knot $K$ in $S^3$ does not have cosmetic crossings.

Here a crossing $c$ of a knot diagram $D$ is nugatory if there is a circle $C$ on the projection plane that transverse to the diagram $D$ only at $c$. Obviously the crossing change at a nugatory crossing always preserves the knot, so the cosmetic crossing conjecture can be rephrased that when a crossing change at a crossing $c$ preserves the knot, then $c$ is nugatory.

In [BFKP] Balm-Friedl-Kalfagianni-Powell proved the following constraints for genus one knots to admit a cosmetic crossing.

Theorem 1.1. [BFKP] Theorem 1.1, Theorem 5.1] Let $K$ be a genus one knot that admits a cosmetic crossing. Then $K$ has the following properties.

- $K$ is algebraically slice.
- For the double branched covering $\Sigma_2(K)$ of $K$, $H_1(\Sigma_2(K); \mathbb{Z})$ is finite cyclic.
- If $K$ has a unique genus one Seifert surface, $\Delta_K(t) = 1$.

In this paper, by using the 2-loop part of the Kontsevich invariant, we prove the cosmetic crossing conjecture for genus one knot with non-trivial Alexander polynomial.

Theorem 1.2. Let $K$ be a genus one knot. If $\Delta_K(t) \neq 1$, then $K$ satisfies the cosmetic crossing conjecture.

For genus one knot $K$ with $\Delta_K(t) = 1$ we get an additional constraint for $K$ to admit a cosmetic crossing. Let $\lambda$ be the Casson invariant of integral homology spheres and let $w_3(K) = \frac{1}{36} V''''_K(1) + \frac{1}{12} V'''_K(1)$ be the primitive integer-valued degree 3 finite type invariant of $K$. Here $V_K(t)$ is the Jones polynomial of $K$.

Theorem 1.3. Let $K$ be a genus one knot with $\Delta_K(t) = 1$. If $\lambda(\Sigma_2(K)) - 2w_3(K) \not\equiv 0 \pmod{16}$, then $K$ satisfies the cosmetic crossing conjecture.
The cosmetic crossing conjecture has been confirmed for several cases: 2-bridge knots [Tor], fibered knots [Kal], knots whose double branched coverings are L-spaces with square-free 1st homology [LiMo], and some satellite knots [BaKa]. Except the last satellite cases and the unknot, all the knots mentioned so far, including knots treated in Theorem 1.1, has non-trivial Alexander polynomial.

Theorem 1.3 gives examples of non-satellite knots with trivial Alexander polynomial satisfying the cosmetic crossing conjecture. Let \( K = \text{pretzel}(p, q, r) \) be the pretzel knot for odd \( p, q, r \). Obviously, as long as \( K \) is non-trivial, \( g(K) = 1 \). The Alexander polynomial of \( K \) is

\[
\Delta_K(t) = \frac{pq + qr + rp + 1}{4} t + \frac{-2pq - 2qr - 2rp + 1}{2} + \frac{pq + qr + rp + 1}{4} t^{-1}.
\]

Hence, for example, the pretzel knot \( K = \text{pretzel}(4k + 1, 4k + 3, -(2k + 1)) \) has the trivial Alexander polynomial.

**Corollary 1.4.** If \( k \equiv 1, 2 \pmod{4} \), the pretzel knot \( K = \text{pretzel}(4k + 1, 4k + 3, -(2k + 1)) \) satisfies the cosmetic crossing conjecture.

### 2. Cosmetic crossing of genus one knot and Seifert surface

We review an argument of [BFKP, Section 2, Section 3] that relates a cosmetic crossing change and Seifert matrix.

A crossing disk \( D \) of an oriented knot \( K \) is an embedded disk having exactly one positive and one negative crossing with \( K \). A crossing change can be seen as \( \varepsilon = \pm 1 \) Dehn surgery on \( \partial D \) for an appropriate crossing disk \( D \), and the crossing is nugatory if and only if \( \partial D \) bounds an embedded disk in \( S^3 \setminus K \).

Assume that \( K \) admits a cosmetic crossing with the crossing disk \( D \). Then as is discussed in [BFKP, Section 2], there is a minimum genus Seifert surface \( S \) of \( K \) such that \( \alpha := D \cap S \) is a properly embedded, essential arc in \( S \).

If \( g(S) = 1 \), such an arc \( \alpha \) is non-separating. We take simple closed curves \( a_x, a_y \) of \( S \) so that

- \( a_x \) intersects \( \alpha \) exactly once.
- \( a_x \) and \( a_y \) form a symplectic basis of \( H_1(S; \mathbb{Z}) \).

Then we view \( K = \partial S \) as a neighborhood of \( a_x \cup a_y \) and express \( K \) by a framed 2-tangle \( T \) as depicted in Figure 1.

**Figure 1.** A spine tangle \( T \) adapted to the cosmetic crossing
We call the framed tangle \( T \) a spine tangle of \( K \) adapted to the cosmetic crossing of a genus one knot \( K \). Let \( M = \begin{pmatrix} n & \ell \\ \ell & m \end{pmatrix} \) be the linking matrix of \( T \), where \( n \) (resp. \( m \)) is the framing of the strand \( x \) (resp. \( y \)) and \( \ell \) is the linking number of two strands of \( T \).

Let \( K' \) be a knot obtained from \( K \) by crossing change along the crossing disk \( D \). Then \( S \) gives rise to a Seifert surface \( S' \) of \( K' \) ([BFKP, Proposition 2.1]). \( K' \) has a spine tangle presentation \( T' \), so that \( T' \) and \( T \) are the same as unframed tangles, and that the linking matrix of \( T' \) is \( M' = \begin{pmatrix} n \pm 1 & \ell \\ \ell & m \end{pmatrix} \).

With respect to the basis \( \{ a_x, a_y \} \), the Seifert matrix \( V \) of \( K \) and the Seifert matrix \( V' \) of \( K' \) are given by
\[
V = \begin{pmatrix} n & \ell \\ \ell & m \end{pmatrix}, \quad V' = \begin{pmatrix} n \pm 1 & \ell \\ \ell \pm 1 & m \end{pmatrix}
\]
respectively. Since \( K \) and \( K' \) are the same knot,
\[
\Delta_K(t) = \det(V - tV^T) = \Delta_K'(t).
\]
By direct computation, this implies that
\[
(2.1) \quad m = 0.
\]
In particular, \( K \) is algebraically slice.

### 3. 2-LOOP POLYNOMIAL OF GENUS ONE KNOT

Here we quickly review the 2-loop polynomial. For details, see [Oht]. Let \( \mathcal{B} \) be the space of open Jacobi diagram. For a knot \( K \) in \( S^3 \), let \( Z^\sigma(K) \in \mathcal{B} \) be the Kontsevich invariant of \( K \), viewed so that it takes value in \( \mathcal{B} \) by composing the inverse of the Poincaré-Birkhoff-Witt isomorphism \( \sigma : \mathcal{A}(S^1) \to \mathcal{B} \).

A Jacobi diagram whose edge is labeled by a power series \( f(h) = c_0 + c_1 h + c_2 h^2 + c_3 h^3 + \cdots \) represents the Jacobi diagram \( f(h) \). It is known that (the logarithm of) the Kontsevich invariant \( Z^\sigma(K) \) is written in the following form [GaKr, Kri].

\[
\log_\sqcup Z^\sigma(K) = \frac{1}{2} \left( \log \left( \frac{\sinh(\frac{\pi}{2} h)}{\sqrt{\pi} - h} \right) - \frac{1}{2} \log(\Delta_K(h)) \right) + \sum_{i: \text{finite}} \frac{\frac{p_{i,1}(h)/\Delta_K(h)}{p_{ii,1}(h)/\Delta_K(h)}}{p_{i,0}(h)/\Delta_K(h)} + (\ell > 2)\text{-loop parts}.
\]

Here
- \( \Delta_K(t) \) is the Alexander polynomial of \( K \), normalized so that \( \Delta_K(1) = 1 \) and \( \Delta_K(t) = \Delta_K(t^{-1}) \) hold.
- \( \log_\sqcup \) is the logarithm with respect to the disjoint union product \( \sqcup \) of \( \mathcal{B} \), given by
\[
\log_\sqcup(1 + D) = D - \frac{1}{2} D \sqcup D + \frac{1}{3} D \sqcup D \sqcup D + \cdots.
\]
• $p_{i,j}(e^h)$ is a polynomial of $e^h$.

Let

$$\Theta(t_1, t_2, t_3; K) = \sum_{\varepsilon \in \{\pm 1\}} \sum_{\sigma \in S_3} p_{i,1}(t_{\sigma(1)}^\varepsilon)p_{i,2}(t_{\sigma(2)}^\varepsilon)p_{i,3}(t_{\sigma(3)}^\varepsilon).$$

Here $S_3$ is the symmetric group of degree 3. The 2-loop polynomial $\Theta_K(t_1, t_2) \in \mathbb{Q}[t_1^\pm, t_2^\pm]$ of a knot $K$ is defined by

$$\Theta_K(t_1, t_2) = \Theta(t_1, t_2, t_3; K)|_{t_3 = t_1^{-1}t_2^{-1}}.$$

The reduced 2-loop polynomial is a reduction of the 2-loop polynomial defined by

$$\hat{\Theta}_K(t) = \frac{\Theta_K(t, 1)}{(t_2^+ - t_2^-)^2} \in \mathbb{Q}[t^\pm].$$

In general, although Ohtsuki developed fundamental techniques and machineries that enable us to compute $\Theta_K(t_1, t_2)$, the computation of the 2-loop polynomial is much more complicated than the computation of the 1-loop part (i.e., the Alexander polynomial). Fortunately, when the knot has genus one, Ohtsuki proved a direct formula of $\Theta_K(t_1, t_2)$ [Oht, Theorem 3.1]. Consequently he gave the following formula of the reduced 2-loop polynomial of genus one knots.

**Theorem 3.1.** [Oht] Corollary 3.5] Let $K$ be a genus one knot expressed by using a framed 2-tangle $T$ as in Figure 7 and let $M = \begin{pmatrix} n & \ell \\ \ell & m \end{pmatrix}$ be the linking matrix of $T$. Then

$$\hat{\Theta}_K(t) = \left(\frac{(n+m)(d - \frac{nm}{2}) - \ell(\ell + 1)(\ell + 1)}{2} + 12v_3\right)\left(-2 - \frac{2d + 1}{3}(t + t^{-1} - 2)\right)$$

$$- 4\left(mv_2^{xx} + mv_2^{yy} - (\ell + \frac{1}{2})v_2^{xy} + 3v_3\right)\Delta_K(t)$$

Here

• $d = nm - \ell^2 - \ell$. In particular, $\Delta_K(t) = dt + (1 - 2d) + dt^{-1}$.
• $v_2^{xx}$, $v_2^{yy}$, $v_2^{xy}$ (resp. $v_3$) are some integer-valued finite type invariant of $T$ whose degree is 2 (resp. 3), which do not depend on the framing.

4. CONSTRAINT FOR COSMETIC CROSSINGS

We prove the Theorem 1.2 and Theorem 1.3 at the same time.

**Theorem 4.1.** Let $K$ be a genus one knot. If $K$ admits a cosmetic crossing, then $\Delta_K(t) = 1$ and $\lambda(\Sigma_2(K)) - 2w_3(K) \equiv 0 \pmod{16}$.

**Proof.** Assume that $K$ is a genus one knot admitting a cosmetic crossing. We express $K$ using a spine tangle $T$ adapted to the cosmetic crossing. Then as we have seen (2.1), the linking matrix of $T$ is $M = \begin{pmatrix} n & \ell \\ \ell & 0 \end{pmatrix}$. Moreover, for the knot $K'$ obtained by the crossing change, $K'$ has a spine tangle $T'$ which is identical with $T$ as an unframed tangle with linking matrix is $M' = \begin{pmatrix} n \pm 1 & \ell \\ \ell & 0 \end{pmatrix}$. 
Since the finite type invariants \( v^{xy}_2, v^{yy}_2, v^{yy}_2 \) and \( v_3 \) do not depend on the framing, by Theorem 5.1
\[
0 = \hat{\Theta}_K(t) - \hat{\Theta}_{K'}(t)
\]
\[
= d(-2 - \frac{2d+1}{3}(t + t^{-1} - 2)) - 4v^{yy}_2(dt + (1 - 2d) + dt^{-1})
\]
\[
= d\left( -\frac{2d+1}{3} - 4v^{yy}_2 \right) t + \frac{d(4d-4)}{3} + 4v^{yy}_2(2d-1) + d\left( -\frac{2d+1}{3} - 4v^{yy}_2 \right) t^{-1}.
\]
Therefore
\[
(4.1) \quad d\left( -\frac{2d+1}{3} - 4v^{yy}_2 \right) = \frac{d(4d-4)}{3} + 4v^{yy}_2(2d-1) = 0.
\]

If \( d \neq 0 \), by (4.1) \( d = \frac{2}{3} \). Since \( d \in \mathbb{Z} \), this is a contradiction so we conclude \( d = 0 \) and \( \Delta_k(t) = 1 \).

Then by (4.1), \( d = 0 \) implies \( v^{yy}_2 = 0 \). Moreover, since \( d = n(\ell-2)-\ell = -\ell(\ell+1) \), we get \( \ell = 0, -1 \). Thus by Theorem 5.1 the reduced 2-loop polynomial is
\[
\hat{\Theta}_K(t) = 12v_3\left(-2 - \frac{1}{3}(t + t^{-1} - 2)\right) - 4\left( \ell + \frac{1}{2} \right) v^{xy}_2 - 3v_3)
\]
hence
\[
\hat{\Theta}_K(1) = -12v_3 + 4\left( \ell + \frac{1}{2} \right) v^{xy}_2, \quad \hat{\Theta}_K(-1) = 4v_3 + 4\left( \ell + \frac{1}{2} \right) v^{xy}_2.
\]

On the other hand, by [Oht] Proposition 1.1
\[
\hat{\Theta}_K(1) = 2w_3(K), \quad \hat{\Theta}_K(-1) = -\frac{1}{12} V'_K(-1)V_K(-1).
\]

Since \( \Delta_k(-1) = V'_K(-1) = 1 \), by Mullins’ formula of the Casson-Walker invariant \( \lambda_w \) of the double branched coverings [Mull],
\[
\lambda_w(\Sigma_2(K)) = \frac{\frac{V'_K(-1)}{6V_K(-1)} + \frac{\sigma(K)}{4}}{4}
\]
we get
\[
\hat{\Theta}_K(-1) = \frac{1}{2} \lambda_w(\Sigma_2(K)).
\]

For an integral homology sphere, the Casson invariant \( \lambda \) is twice of the Casson-Walker invariant \( \lambda_w \) hence we conclude
\[
\lambda(\Sigma_2(K)) - 2w_3(K) = \hat{\Theta}_K(-1) - \hat{\Theta}_K(1) = 16v_3.
\]

\( \square \)

Proof of Corollary 1.4. The reduced 2-loop polynomial of genus pretzel knots \( P(p, q, r) \) was given in [Oht] Example 3.6. In particular, for \( K = P(4k+1, 4k+3, -(2k+1)) \), \( \hat{\Theta}_K(1) \) and \( \hat{\Theta}_K(-1) \) are given by
\[
\hat{\Theta}_K(1) = -\frac{1}{8}(4k+2)(4k+4)(-2k), \quad \hat{\Theta}_K(-1) = -\frac{1}{24}(4k+2)(4k+4)(-2k)
\]
hence
\[
\lambda(\Sigma_2(K)) - 2w_3(K) = \hat{\Theta}_K(-1) - \hat{\Theta}_K(1) = \frac{1}{12}(4k+2)(4k+4)(-2k)
\]
\[
= -16\frac{(2k+1)(k+1)}{12}.
\]
When \( k \equiv 1, 2 \pmod{4} \), \( \frac{(2k+1)(k+1)}{12} \notin \mathbb{Z} \) hence \( K \) does not admit cosmetic crossing by Theorem 1.3. □

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