FORECAST HORIZON OF DYNAMIC LOT SIZE MODEL FOR 
PERISHABLE INVENTORY WITH MINIMUM ORDER 
QUANTITIES

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ABSTRACT. We consider the dynamic lot size problem for perishable inventory 
under minimum order quantities. The stock deterioration rates and inventory 
costs depend on both the age of the stocks and their periods of order. Based 
on two structural properties of the optimal solution, we develop a dynamic 
programming algorithm to solve the problem without backlogging. We also 
extend the model by considering backlogging. By establishing the regeneration 
set, we give a sufficient condition for obtaining forecast horizon under without 
and with backlogging. Finally, based on a detailed test bed of instance, we 
obtain useful managerial insights on the impact of minimum order quantities 
and perishability of product and the costs on the length of forecast horizon.

1. Introduction. In a multi-period dynamic decision making problems, operation 
managers need to forecast the future commercial data (e.g., demand, cost) for deter-
mining current decisions in a rolling planning process. Operation managers utilize 
forecast data for the first certain periods to evaluate current decisions. The opera-
tion managers can make accurate decisions with more valid data when they forecast 
longer future horizon’s information. However, information further into the future 
has diminishing effect on the current decisions as well as less reliable and higher 
computational and forecasting expenses. Here, the extent of the impact of future 
data on current decisions is a foundational issue. In this context, the thought of 
forecast horizon has been extensively researched in the literature of operations re-
search and operations management [7]. More formally, denote \((t, T)\) as a pair of 
integers with \(1 \leq t \leq T\), if the optimal decisions in the periods \([1, t]\) are not affected 
by the parameters (e.g., demand, cost) of the model in the periods \([T + 1, +\infty]\), 
then \(T\) is defined forecast horizon and \(t\) is the homologous decision horizon. The 
significance of forecast horizon \(T\), if exits, decisions for the first \(t\) periods of some 
optimum solutions of the \(T\)-period problem are also optimum for each instance of 
the problem with larger than period \(T\). Resultantly, it is not necessary to forecast 
data beyond period \(T\).

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The idea of forecast horizon in operations management dates back to the works of Wagner and Whitin [48], who analyze a dynamic lot sizing (DLS) problem. They study a $T$-period production decision problem in which the known demands in each period must be satisfied, backloggings and stockouts are prohibited. The production costs function is fixed-plus-linear (variable setup costs plus constant unit production costs in each period). The inventory costs are linear (variable unit production costs in each period). The production and inventory capacities in each period are large enough. This is well known as W-W model. In their paper, there are two important contributions: (i) demonstrate the optimality of Zero Inventory Property (ZIP), using the property, develop an efficient forward dynamic programming (DP) to solve the problem; (ii) establish the forecast horizon theorem. Zabel [51] extends the study by Wagner and Whitin [48], points out ZIP still holds in the situation of variable unit ordering costs. While he investigates the forecast horizon results in this case of constant unit ordering costs. Eppen et al. [19] establish the forecast horizon results under the situation of variable unit ordering costs. Blackburn and Kunreuther [4] generalize the DLS model by permitting backlogging. They develop forecast horizon results under the assumption of no speculative motive for holding inventory and backlogging. Both constant and nonincreasing unit ordering costs are special cases of no speculative motive for holding inventory. Ludin and Morton [33] extend the forecast horizon research and introduce the concept of regeneration set. They demonstrate that finding a regeneration set is the key for finding the forecast horizon. Sethi and Chand [42] study the DLS model for the case of multiple finite production rates and develop an efficient forecast horizon procedure for this model. Chand and Morton [8] improve Ludin-Morton theorem [33] and give a procedure to find the minimum forecast horizon in this case of no speculative motive for holding inventory. Sandbothe and Thompson [39] extend the DLS model by permitting stockout. Using the theory of regeneration set, they establish the forecast horizon results. Chand et al. [9, 10] consider an undiscounted/discounted DLS model in which demand and cost parameters are constant for an initial few periods. Moreover, existence results of forecast horizon in their problem are obtained. Bylka and Sethi [5] relax some of the conditions that Chand et al. [9] assumed would guarantee the existence of forecast horizon. Federgruen and Tzur [20, 21, 22, 23] develop a simple forward algorithm which solves the DLS model without/with backlogging in $O(T \log T)$ time. Furthermore, they establish the minimum forecast horizon results under the situation of variable unit ordering costs. Chand and Sethi [11] consider a DLS problem with learning in setup. They develop an efficient DP algorithm and provide forecast horizon procedures. Tzur [46] generalizes the forecast horizon results by Chand and Sethi [11] and give a procedure which finds a minimum forecast horizon. Smith and Zhang [43] consider a DLS problem with time-varying convex production and inventory holding costs. They give a sufficient condition to compute the forecast horizon in the infinite horizon problem. For a comprehensive classified bibliography of the literature on the theory and application on forecast horizon, see Chand et al. [7]. Since 2002, the studies on the theory and applications of forecast horizon mainly include the following areas: game theory [24]; operations management [3, 6, 13, 15, 16, 17, 18, 25, 45]; Markov Decision Processes [12]; reservoir operations [2, 49, 50, 53].

We consider a single item DLS for perishable products with minimum order quantities (MOQ). Perishable products are omnipresent and obbligato to our life,
such as fruit, vegetable, meat. Perishable products also include pharmaceutics, blood, consumer electronics, season’s fashion, and so on. A report by Food Market Institute (2017) perishables account for 53% of the total supermarket sales of about 501 billion for the year 2016. Therefore, both practitioners and academic researchers have always been of tremendous interest to find effective inventory control policies for perishable products. There are some realistic reasons for including minimum lot size with ordering (production) planning. For example, some papers provide real world cases where MOQ is used. Suerie [44] think that there can be the need to reach a critical mass to initiate chemical reaction in a chemical production process. Small lot size is not reasonable because significant costs are required to switching in production [47]. Park and Klabjan [36] think that MOQ is business requirement such as the product required to be shipped in containers or pallets. In a word, the operation manager strives to achieve the economies of scale and guarantee full utilization of resources by adopting MOQ [14].

Hsu [30] first considers a DLS model for perishable products where an inventory stock deterioration rates and its carrying costs in each period depend on the age of the stock, then discusses situations where the traditional DLS models are not applicable, and proposes a new model with general concave production and inventory cost functions. Since new DLS model does not satisfy the famous zero-inventory property (ZIP), he explores two structural properties of the optimal solution and uses them to develop a dynamic programming algorithm which solves the problem in polynomial time. Hsu [31] generalizes the study by Hsu [30] on a DLS model for perishable products but without backlogging. He develops a polynomial time DP algorithm which solves two special instances, one with non-decreasing demands and the other with non-decreasing marginal backlogging costs with respect to the age of the backlogging. Sargut and Isik [40] extend the research of Hsu [31] by considering finite production capacity. They prove the problem is NP-hard and suggest a heuristic algorithm to find a good production and distribution plan, and then discuss the performance of the heuristic algorithm. Albeit these studies develop the efficient algorithm to solve DLS problems with perishable product, they do not analyze the forecast and decision horizons under the multi-period decision-making.

For the research of DLS models with MOQ, Anderson and Cheah [1] consider a multi-item, capacitated DLS problem with minimum batch sizes. They propose a heuristic algorithm to solve the problem and report the performance of the algorithm. Porras and Dekker [37] study the joint replenishment problem for multi-items under the MOQ constraint. They derive bounds on the basic cycle time and propose an efficient global optimization procedure to solve the problem. Okhrin and Richter [34] explore a single-item capacitated lot size problem with MOQ. Based on some structural properties of the optimal solution, they work out the DP algorithm to solve the problem in $O(T^3)$ time. Okhrin and Richter [35] continue the analysis of the uncapacitated single-item DLS problem with MOQ constraint. Hellion et al. [27] deal with the single-item capacitated lot size problem with MOQ under concave production and inventory costs. To solve this problem, they propose a polynomial time algorithm in $O(T^5)$. Hellion et al. [28] correct the mistake. The time complexity of their algorithm is $O(T^6)$. Hellion et al. [29] deal with the single-item capacitated DLS problem with MOQ and time windows under concave production and inventory costs. They present an optimal algorithm in $O(T^7)$ time. Park and Klabjan [36] consider the single-item DLS problem with MOQ and devise the algorithm to solve the problem by defining the valid inequalities for the associated
convex hull. Goerler and Voß [26] consider a capacitated DLS model with rework of defective items and minimum production quantities. Based on detailed numerical experiments, they give several managerial insights. None of these studies allows backlogging. But backlogging is an effective tool to reduce the operation costs when the minimum order quantity is larger in practice. The backlogging policy can maximize the reduction of the stock deterioration and the waste. In addition, these DLS studies on minimum order quantities do not consider the perishability of the product.

Before we proceed with the forecast horizon analysis of the DLS problem for perishable products with MOQ, we summarize the major contributions of this paper:

(1) We develop an efficient procedure to solve the DLS problem for perishable products with MOQ under without and with backlogging.
(2) Based on the monotonicity of order point (and regeneration point), we characterize a sufficient condition for a period \( t \) to be forecast horizon.
(3) Using a detailed test bed of instances, we obtain useful managerial insights on the impact of minimum order quantities and perishability of product and the costs on the length of forecast horizon.

The rest of the article is organized as follows. Section 2 describes the DLS problem under MOQ for perishable products. Section 3 derives some structural properties of the optimal order planning, and then develops a DP algorithm to solve the problem for the case without backlogging. Section 4 establishes forecast horizon results under without backlogging. Section 5 extends the results in Section 3 by considering the backlogging case. Section 6 presents computational experience and managerial insights. Section 7 concludes the paper.

2. Problem statement and formulation. The following notion is used in the DLS model for perishable products under MOQ.

\( n \): the problem horizon, i.e., the number of periods;
\( P(t) \): \( t \)-period problem, \( 1 \leq t \leq n \);
\( x_t \): quantity of the perishable product ordered (produced) in period \( t \), \( 1 \leq t \leq n \);
\( z_{it} \): the quantity of the demand in period \( t \) to be satisfied from order (production) in period \( i \), \( 1 \leq i \leq t \leq n \);
\( y_{it} \): the quantity ordered in period \( i \) and held at the beginning of period \( t \), which excludes the quantity \( z_{it} \) used to satisfy the demand in period \( t \), \( 1 \leq i \leq t \leq n \);
\( \sigma_t \): the setup (fixed) cost for the order in period \( t \), \( 1 \leq t \leq n \);
\( c_t \): the unit order cost in period \( t \), \( 1 \leq t \leq n \);
\( h_{it} \): the unit holding inventory cost in period \( t \), which is ordered in period \( i \), \( 1 \leq i \leq t \leq n \);
\( \alpha_{it} \): the fraction of \( y_{it} \) which is lost during period \( t \), \( 1 \leq i \leq t \leq n \);
\( d_t \): the demand in period \( t \), \( 1 \leq t \leq n \);
\( S \): the minimum order quantities.

\[
\delta(x) = \begin{cases} 
1; & x > 0 \\
0; & x = 0 
\end{cases}
\]

As Hsu [30] point out, the longer a unit is carried in stock, the faster it may deteriorate and the higher its inventory carrying cost in most practical applications. Hence, we have the following assumptions.

Assumption 1. \( \alpha_{it} \geq \alpha_{jt} \), for \( 1 \leq i \leq j \leq t \leq n \);

Assumption 2. \( h_{it} \geq h_{jt} \), for \( 1 \leq i \leq j \leq t \leq n \).
Without loss of generality, we assume zero inventories at the beginning of period 1. Based on the above notation and assumptions, the problem can be formulated as the following mathematical program:

Problem (P)

\[
\min_{\mathcal{I}} \sum_{t=1}^{n} (\sigma_{t}\delta(x_{t}) + c_{t}x_{t} + \sum_{i=1}^{t} h_{i}y_{it})
\]

Subject to: \(x_{t} - z_{it} = y_{it}, 1 \leq t \leq n\) \hspace{1cm} (1)

\((1 - \alpha_{i,t-1})y_{i,t-1} - z_{it} = y_{it}, \hspace{0.5cm} 1 \leq i < t \leq n\) \hspace{1cm} (2)

\[
\sum_{i=1}^{t} z_{it} = d_{t}, \hspace{0.5cm} 1 \leq t \leq n
\]

\[
x_{t}, \hspace{0.5cm} y_{it}, \hspace{0.5cm} z_{it} \geq 0, \hspace{0.5cm} 1 \leq i \leq t \leq n
\]

\[
x_{t} \geq S, \hspace{0.5cm} 1 \leq t \leq n
\]

The meanings of constraints (2)-(5) can be seen in Hsu \cite{30}. Constraint (6) is the minimum order quantities constraint. In the rest of the paper, we will denote \(\Omega\) as a feasible solution of (P).

For \(1 \leq i \leq k < t \leq n\), define \(A_{kt}^{i} = \frac{1}{\Pi_{i \leq j < t} (1 - \alpha_{ij})}\) and \(A_{kt}^{i} = 1\). By the definition, we have \(A_{kt}^{i} = A_{kq}^{i}A_{q,t}^{i}\), for \(k < q < t\). By Assumption 1, it is easy to see that \(A_{kt}^{i} \geq A_{kt}^{j}\), for \(1 \leq i < j \leq k < t \leq n\). We also note that, to satisfy one unit demand in period \(t\) by order in period \(i (i < t)\), we need to order \(A_{it}^{i}\) units in period \(i\) and carry \(A_{kt}^{i}\) units of inventory in each period \(k\), where \(i \leq k \leq t - 1\).

If \(0 \leq \alpha_{i,t-1} < 1\) and \(\alpha_{ii} = 1\), \(t \geq i + 1\), we define the lifetime of product is \(t - i + 1\) periods, denote \(m\) as the lifetime of product, so \(m = t - i + 1\), if \(\alpha_{ii} = 1\), we say the lifetime of product is \(1\) period.

A period \(t\) is called an order point (period) if \(x_{t} > 0\).

3. Properties of the optimal solution. In this section, we first show that the first-in-first-out policy to manage stocks of perishable products still holds in our problem.

Property 1. There exists an optimal solution \(\Omega^{*}\) to Problem (P), where if \(i < j\) are two order periods and \(z_{jk}^{*} > 0\) for some \(k \geq j\), then \(z_{it}^{*} = 0\) for all \(t, k < t \leq n\).

Proof. Suppose that we have an optimal solution \(\Omega^{+}\) with \(z_{jk}^{*} > 0\) and \(z_{it}^{*} > 0\), \(i < j \leq k < t\). We define \(\varepsilon = \min\{z_{it}, z_{jk}/A_{kt}^{i}\}\). We do the following changes and obtain a feasible solution \(\Omega^{*}\). Set:

\[
z_{it}^{*} = z_{it}^{+} - \varepsilon, \hspace{0.5cm} z_{jt}^{*} = z_{jt}^{+} + \varepsilon,
\]

\[
z_{ik}^{*} = z_{ik}^{+} + A_{kt}^{i}\varepsilon, \hspace{0.5cm} z_{jk}^{*} = z_{jk}^{+} - A_{kt}^{i}\varepsilon,
\]

\[
x_{i}^{*} = x_{i}^{+} - A_{ik}^{i}(A_{kt}^{i} - A_{kt}^{j})\varepsilon, \hspace{0.5cm} x_{j}^{*} = x_{j}^{+},
\]

\[
y_{it}^{*} = y_{it}^{+} - A_{ik}^{i}(A_{kt}^{i} - A_{kt}^{j})\varepsilon, \hspace{0.5cm} y_{jt}^{*} = y_{jt}^{+}, \hspace{0.5cm} \text{for } i \leq l \leq k - 1,
\]

\[
y_{it}^{*} = y_{it}^{+} - A_{kt}^{j}\varepsilon, \hspace{0.5cm} y_{jt}^{*} = y_{jt}^{+} + A_{lt}^{j}\varepsilon, \hspace{0.5cm} \text{for } k \leq l \leq t - 1.
\]
From assumption 1 and 2, we know that $\Omega^*$ keep the same as in $\Omega^+$. It is easy to see that $z_{it}^* = 0$ if $\varepsilon = z_{it}^+$, and $z_{jk}^* = 0$ if $\varepsilon = z_{jk}^+/A_{kt}^i$. Let $V(\Omega^+)$ and $V(\Omega^*)$ denote the optimal value of solution $\Omega^+$ and $\Omega^*$. Next we show that $V(\Omega^*) \leq V(\Omega^+)$. 

$$V(\Omega^+)-V(\Omega^*)=c_iA_{ik}^i(A_{kt}^i-A_{kt}^l)\varepsilon+\sum_{l=1}^{k-1}h_{il}A_{lk}^i(A_{kt}^i-A_{kt}^l)\varepsilon$$

$$+\sum_{l=k}^{l-1}h_{il}A_{lk}^i\varepsilon-\sum_{l=k}^{l-1}h_{jl}A_{jl}^i\varepsilon$$

$$\geq\sum_{l=k}^{l-1}h_{il}A_{lk}^i\varepsilon-\sum_{l=k}^{l-1}h_{jl}A_{jl}^i\varepsilon$$

From assumption 1 and 2, we know $(\sum_{l=k}^{l-1}h_{il}A_{lk}^i-\sum_{l=k}^{l-1}h_{jl}A_{jl}^i)\cdot \varepsilon \geq 0$, So we have $V(\Omega^*) \leq V(\Omega^+)$. 

Under no minimum order quantities constraint, Hsu [11] gives an important structural property that there is an optimal solution where the demands in a period are satisfied by order in exactly one of the periods $i, 1 \leq i \leq t$. While the property is not hold in our problem with no minimum order quantities constraint. We introduce the following new results which allow us to characterize the optimal order plan.

**Property 2.** In an optimal solution $\Omega^*$ to Problem (P), for all $i (1 \leq i \leq t \leq n)$, there exists:

$$(x_i^*-S)\cdot (1-\alpha_{s,i-1})y_{s,i-1}^*+x_i^*+t\sum_{k=i}^{t}A_{ik}^i d_k)\cdot x_i^*=0,$$

where $s$ is the largest order point that is less than or equal to period $i-1$.

**Proof.** According to the minimum order quantities constraint and first-in-first-out policy, we have that the demands in a period are satisfied by at most two order points. By constraint (6), we have $x_i \geq S$. Suppose in an optimal solution $\Omega^*$, $(x_i^*-S)\cdot (1-\alpha_{s,i-1})y_{s,i-1}^*+x_i^*+t\sum_{k=i}^{t}A_{ik}^i d_k)\cdot x_i^* \neq 0$, then there exists a period $i$ such that $x_i^* \neq S$; $x_i^* \neq 0;(1-\alpha_{s,i-1})y_{s,i-1}^*+x_i^*+t\sum_{k=i}^{t}A_{ik}^i d_k \neq 0$. Since $x_i^*$ is feasible, the following inequalities must hold: $x_i^*>0$; $x_i^*>S$; $(1-\alpha_{s,i-1})y_{s,i-1}^*+x_i^*>t\sum_{k=i}^{t}A_{ik}^i d_k$.

We define $\Delta = \min\{x_i^*-S,(1-\alpha_{s,i-1})y_{s,i-1}^*+x_i^*-\sum_{k=i}^{t}A_{ik}^i d_k\}$, then $\Delta > 0$. Assume $i < j$ are two order points, we construct the following feasible solutions $\Omega^{**}$ and $\Omega^{***}$. Set:

$$z_{it}^* = z_{it}^{**} + \Delta/A_{it}^i, \quad z_{jt}^* = z_{jt}^{**} - \Delta/A_{it}^j,$$

$$x_i^* = x_i^{**} + \Delta, \quad x_j^* = x_j^{**} - (\Delta/A_{it}^i) \cdot A_{jt}^j,$$

$$y_{ik}^* = y_{ik}^{**} + \Delta/A_{ik}^i, \quad i \leq k \leq t,$$

$$y_{jk}^* = y_{jk}^{**} - (\Delta/A_{it}^i) \cdot A_{jt}^j, \quad j \leq k \leq t.$$
Next, we show how to compute the minimum costs of structural properties, we can describe the DP recursion on optimal costs. Since the equation always holds, we have \( \Delta = 0 \), which contradicts the assumption \( \Delta > 0 \). Hence the theorem is true.

Jing and Lan [32] study a multi-item DLS problem for perishable products where stock deterioration rates and inventory costs are age-dependent. They explore structural properties in an optimal solution under two cost structures and develop a dynamic programming algorithm to solve the problem in polynomial time when the number of products is fixed, and establish forecast horizon results. While they do not consider minimum order quantities constraint. In this paper, we consider a DLS problem with perishable inventory and MOQ constraint, based on the two structural properties, we can describe the DP recursion on optimal costs.

For \( 1 \leq i \leq q \leq r \leq n \), define \( P(i,q,r) \) as the Problem \( (P) \) restricted to periods 1 through \( r \), where the last setup occurs in period \( i \) to satisfy the demands from period \( q \) through \( r \). Let period \( q \) denote the next-to-last order point. Denote \( V(i,q,r) \) as the minimum costs of \( P(i,q,r) \), \( V(r) \) as the optimum objective function value of \( r \)-period problem, by the above definitions, we have

\[
V(r) = \min_{1 \leq i \leq q \leq r \leq n} V(i,q,r)
\]  

Next, we show how to compute \( V(i,q,r) \).

If \( \sum_{k=q}^{r} A_{ik} d_k - (1 - \alpha_{g,q-1}) y_{g,q-1} \geq S \), then

\[
V(i,q,r) = V(q - 1) + \sigma_i + c_i \left( \sum_{k=q}^{r} A_{ik} d_k - (1 - \alpha_{g,q-1}) y_{g,q-1} \right)
\]

\[
+ \sum_{l=1}^{q-1} h_{il} \left( \sum_{k=q}^{r} A_{ik} d_k - (1 - \alpha_{g,q-1}) y_{g,q-1} \right) + \sum_{l=q}^{r-1} h_{il} \sum_{k=l+1}^{r} A_{ik} d_k;
\]

If \( \sum_{k=q}^{r} A_{ik} d_k - (1 - \alpha_{g,q-1}) y_{g,q-1} < S \), then
That is $\sigma$ period Forecast horizon results.

4. Forecast horizon results. In this section, we give a sufficient condition for period $t$ to be a forecast horizon under constant setup cost and unit order cost. That is $\sigma_t = \sigma$ and $c_t = c$.

**Definition 4.1.** Given an optimal solution for $P(n)$, assume period $s$ is the largest production point that is less than or equal to period $t$, period $t$ is referred to as a regeneration point (regeneration period) if $(1 - \alpha_{st})y_{st} < d_{t+1}$.

Under constant setup cost and unit order cost, there exists the following property for the optimal solution.

**Property 3.** There exists an optimal solution $\Omega^*$ to Problem $(P)$, assume period $s$ is the largest production point that is less than or equal to period $i - 1$, if $(1 - \alpha_{s,i-1})y_{s,i-1} \geq d_i$, then $x_i = 0, 1 \leq i \leq n$. (Or equivalently the contrapositive assertion: if period $i$ is an order point, then period $i - 1$ must be a regeneration point.)

**Proof.** Suppose there exists $(1 - \alpha_{s,i-1})y_{s,i-1} \geq d_i$ in an optimal solution of $P(n)$, a cheaper feasible solution can also construct as follows. We consider two cases:

1. If $(1 - \alpha_{s,i-1})y_{s,i-1} = \sum_{i=1}^{n} A_{i}^{l}d_{l}$, then reduce the order in period $i$ to zero, this alteration saves the setup costs and variable production costs and does not incur any additional costs; 2. if $\sum_{i=1}^{i'} A_{i}^{l}d_{l} \leq (1 - \alpha_{s,i-1})y_{s,i-1} < \sum_{i=1}^{i'} A_{i}^{l}d_{l}$, $i + 1 \leq i' \leq n$, then transfer the production quantities from period $i$ to $i'$, this alteration saves the inventory costs and does not incur any additional costs. Thus, if $x_i > 0$, then $(1 - \alpha_{s,i-1})y_{s,i-1}$ can not be greater than $d_i$.

For $1 \leq i \leq r \leq n$, define $P(i, r)$ as the Problem $(P)$ restricted to periods 1 through $r$, where the last setup occurs in period $i$ to satisfy the demands from period $i$ through $r$. Let period $g$ denote the next-to-last order point. Denote $V(i, r)$ as the minimum costs of $P(i, r)$, by the above definitions, we have

$$V(r) = \min_{1 \leq i \leq r \leq n} V(i, r).$$

Next, we show how to compute $V(i, r)$. Based on Property 3, we have

$$V(i, r) = \begin{cases} V(i - 1) + \sigma + cS + \sum_{l=1}^{r} h_{il}(\frac{S}{A_{il}^l} - \sum_{k=1}^{l} d_{k}), & \text{if } \sum_{l=1}^{r} A_{i}^{l}d_{l} \leq S + (1 - \alpha_{g,i-1})y_{g,i-1}; \\ V(i - 1) + \sigma + c(\sum_{l=1}^{r} A_{i}^{l}d_{l} - (1 - \alpha_{g,i-1})y_{g,i-1}) + \sum_{l=1}^{r-1} h_{il} \sum_{k=l+1}^{r} A_{ik}^{l}d_{k}, & \text{if } \sum_{l=1}^{r} A_{i}^{l}d_{l} > S + (1 - \alpha_{g,i-1})y_{g,i-1} \end{cases}$$

Thus, if $r = n$, then $V(n) = V(n - 1) + \sigma + cS + \sum_{l=1}^{r} h_{il}(\frac{S}{A_{il}^{l}} - \sum_{k=1}^{l} d_{k})$.
Let \( i(t) \) denote the last order point in an optimal solution to \( P(t) \). By the definitions, we have
\[
V(t) = V(i(t), t) = \min_{1 \leq i \leq t} V(i, t) \tag{12}
\]

Lemma 4.2. For any \( t^* \), \( t^* \geq t + 1 \), there exists \( i(t^*) \geq i(t) \) in the optimal solution to \( P(t^*) \)-problem.

Proof. It is sufficient to prove the lemma for \( t^* = t + 1 \), then its repeated application yields the final result. Consider \( P(t + 1) \)-problem, if \( i(t + 1) = t + 1 \), then we have

\[
i(t + 1) > i(t). \quad \text{If } i(t + 1) \leq t, \text{ and } x_{i(t)} = S, \text{ and } \sum_{l=1}^{i(t)-1} y_{i(t)} + x_{i(t)} \geq \sum_{l=i(t)}^{t+1} A_{i(t)}^{i(t)} d_l
\]

(or \( \sum_{l=i(t)}^{t} A_{i(t)}^{i(t)} d_l \leq \sum_{l=i(t)}^{t+1} y_{i(t)} + x_{i(t)} \leq \sum_{l=i(t)}^{t+1} A_{i(t)}^{i(t)} d_l \)), then we have \( i(t + 1) = i(t) \).

If \( i(t + 1) \leq t \), and \( x_{i(t)} > S \), assume \( i(t + 1) < i(t) \), by equation (11), we have
\[
V(i, t + 1) = V(i, t) + cA_{i,t+1}^i d_{t+1} + \sum_{l=1}^{t} h_{i(t)} A_{i,t+1}^i d_{t+1}. \tag{13}
\]

By equation (12) and (13), we have
\[
V(i(t), t + 1) = V(i(t), t) + cA_{i(t),t+1}^{i(t)} d_{t+1} + \sum_{l=i(t)}^{t} h_{i(t)} A_{i,t+1}^{i(t)} d_{t+1},
\]
\[
V(i(t + 1), t + 1) = V(i(t + 1), t) + cA_{i(t+1),t+1}^{i(t+1)} d_{t+1} + \sum_{l=i(t+1)}^{t} h_{i(t+1)} A_{i,t+1}^{i(t+1)} d_{t+1}.
\]

By the optimality of \( V(t + 1) \), we have \( V(t + 1) = V(i(t + 1), t + 1) < V(i(t), t + 1) \), that is
\[
V(i(t + 1), t) + cA_{i(t+1),t+1}^{i(t+1)} d_{t+1} + \sum_{l=i(t+1)}^{t} h_{i(t+1)} A_{i,t+1}^{i(t+1)} d_{t+1}
\]

\[
< V(i(t), t) + cA_{i(t),t+1}^{i(t)} d_{t+1} + \sum_{l=i(t)}^{t} h_{i(t)} A_{i,t+1}^{i(t)} d_{t+1}. \tag{14}
\]

Simplifying equation (14), we have
\[
V(i(t + 1), t) + cA_{i(t+1),t+1}^{i(t+1)} d_{t+1} - \sum_{l=i(t)}^{t} h_{i(t)} A_{i,t+1}^{i(t)} d_{t+1} < V(i(t), t).
\]

By the definition of \( A_{ki}^{it} \) and the assumption \( i(t + 1) < i(t) \), we have \( A_{i(t+1),t+1}^{i(t+1)} = A_{i(t),t+1}^{i(t)} = 0 \) and \( \sum_{l=i(t+1)}^{t} h_{i(t+1)} A_{i,t+1}^{i(t+1)} = \sum_{l=i(t)}^{t} h_{i(t)} A_{i,t+1}^{i(t)} \geq 0 \). Hence, we have
\[
V(i(t + 1), t) < V(i(t), t) = V(t), \quad \text{which contradicts the optimality of } V(t). \quad \square
\]
Theorem 4.3. Consider the optimal solution to the \( P(t) \)-period problem, if there are common first \( \tau \) periods' decisions \( \Theta \) such that \( \Theta \) are optimal for all \( P(r) \) with \( r \) in the set \( \{i(t) - 1, i(t), \ldots, t - 1\} \), then period \( t \) is forecast horizon, period \( \tau \) is the corresponding decision horizon. That is the first \( \tau \) periods' decisions are optimal for all \( P(t*) \) with \( t + 1 \leq t* \leq n \) and arbitrary demands and costs in period \( t + 1 \) and beyond.

Proof. By the monotonicity of the last optimal order point of Lemma 1, any \( P(t*) \) with \( t* \geq t \) has an optimal solution with at least one order point in the set \( \{i(t), i(t) + 1, \ldots, t\} \). By the property 3, we have any \( P(t*) \) with \( t* \geq t \) has an optimal solution with at least one regeneration point in the set \( \{i(t) - 1, i(t), \ldots, t - 1\} \). The set \( \{i(t) - 1, i(t), \ldots, t - 1\} \) is referred to as regeneration set \([33]\). By Ludin-Morton regeneration set theorem, solution to at least one of \( P(r) \) with \( r \) in the set \( \{i(t) - 1, i(t), \ldots, t - 1\} \) provides segmental solution to \( P(t*) \). If all these segmental solutions have common first \( \tau \) periods' decisions, the first \( \tau \) periods' decisions must be optimal for \( P(t*) \).

5. Model extension to allow backlogging. As mentioned before, the backlogging policy can maximize the reduction of the stock deterioration and the waste for perishable products when there exists the minimum order quantity constraint in practice. Now consider the extension to the problem where backlogging is permitted. Let \( w_{kt} \) denote the quantities of unsatisfied period \( k \) demand at the end of period \( t \) and \( b_{kt} \) as the unit backlogging cost of leaving the demands of period \( k \) unsatisfied in period \( t \). In most practical applications, the longer a certain quantity of demand is not satisfied, the higher the backlogging cost. Thus, we have the following assumption:

Assumption 3. \( b_{it} \geq b_{jt} \), for \( 1 \leq i \leq j \leq t \leq n \).

Since backlogging is allowed, denote \( z_{it} \) as the quantity of order in period \( i \) used to satisfy the demand in period \( t \), \( 1 \leq i \leq n \), \( 1 \leq t \leq n \). Without loss of generality, we assume that there is no backlogging at the beginning of period 1 and at the end of period \( n \). The problem with backlogging can be formulated as the following mathematical program:

Problem 1 (P1)

\[
\min \sum_{t=1}^{n} (\sigma_t x_t + c_t x_t + \sum_{i=1}^{t} h_{it} y_{it} + \sum_{k=1}^{t} b_{kt} w_{kt})
\]  
\[(15)\]

Subject to:

\[
x_t - \sum_{i=1}^{t} z_{it} = y_{it}, 1 \leq t \leq n
\]  
\[(16)\]

\[
(1 - \alpha_{i,t-1}) y_{i,t-1} - z_{it} = y_{it}, 1 \leq i < t \leq n
\]  
\[(17)\]

\[
w_{k,t-1} - z_{tk} = w_{kt}, 1 \leq k < t \leq n
\]  
\[(18)\]

\[
\sum_{i=1}^{n} z_{it} = d_t, 1 \leq t \leq n
\]  
\[(19)\]

\[
x_t, \ y_{it}, \ w_{it} \geq 0, 1 \leq i \leq t \leq n
\]  
\[(20)\]

\[
z_{it} \geq 0, 1 \leq i \leq n, 1 \leq t \leq n
\]  
\[(21)\]

\[
x_t \geq S, 1 \leq t \leq n
\]  
\[(22)\]
We call the above problem with backlogging Problem 1 (P1). Note that if \( b_{kt} = +\infty \), Problem 1 (P1) is equivalent to Problem (P). For Problem 1 (P1), we have the following important orthogonality condition.

### 5.1. Properties of the optimal solution with backlogging.

**Property 4.** In an optimal solution \( \Omega^* \) to Problem 1 (P1), there exists \( y_{ik} \cdot w_{kt} = 0 \) for all \( k, i, t, 1 \leq i \leq k \leq t \leq n \).

**Proof.** This orthogonality condition is a modification of a property proved in Zangwill [52]. We omit the details.

**Property 5.** There exists an optimal solution \( \Omega^* \) to problem, where if \( i < j \) are two order periods and \( z_{jk}^* > 0 \) for some \( k, 1 \leq k \leq n \), then \( z_{ik}^* = 0 \) for all \( t, k < t \leq n \).

**Proof.** From Property 4, we will not need to consider the following cases:

\[
i < k \leq j < t, \quad i \leq k < j \leq t, \quad i < k \leq t < j, \quad k < i \leq t < j,
\]

\[
k \leq i < t \leq j, \quad k \leq i < j \leq t, \quad k < i \leq t < j.
\]

To prove this property, we only need to consider two cases: \( i < j \leq k < t \) and \( k < t \leq i < j \). The case \( i < j \leq k < t \) is considered in the property 2. For the case \( k < t \leq i < j \), we also assume that there exists an optimal solution \( \Omega^+ \) with \( z_{jk} > 0 \) and \( z_{it} > 0 \), then modify the solution \( \Omega^+ \) to obtain a new feasible solution \( \Omega^* \) with \( z_{jk} = 0 \) or \( z_{it} = 0 \). The new feasible solution \( \Omega^* \) will not increase the total costs. Hence, we have a contradiction.

Based on Property 4 and 5, we present the DP algorithm to solve the problem with backlogging. Note that \( 1 \leq q \leq r \) for the backlogging case. Hence,

\[
V(r) = \min_{1 \leq i \leq n; 1 \leq q \leq r \leq n} V(i, q, r)
\]

In Section 3, we have shown how to compute \( V(i, q, r) \) for \( i \leq q \leq r \). Next, we will show how to compute \( V(i, q, r) \) for \( 1 \leq q \leq i - 1 \).

If \( g \leq q - 1 \) and \( \sum_{k=q}^{r} A_{ik}^t d_k - (1 - \alpha_{g,q-1})y_{g,q-1} \geq S \), then

\[
V(i, q, r) = V(q - 1) + \sum_{l=q}^{i-1} b_{l,i-1}(d_q - (1 - \alpha_{g,q-1})y_{g,q-1}) + \sum_{l=q+1}^{i-1} \sum_{k=q+1}^{t} b_{l,i-1} d_k
\]

\[
+ \sigma_i + c_i \sum_{k=q}^{r} A_{ik}^t d_k - (1 - \alpha_{g,q-1})y_{g,q-1} + \sum_{l=i}^{r-1} h_{il} \sum_{k=i+1}^{r} A_{il}^t d_k;
\]

\[(24)\]

If \( g \leq q - 1 \) and \( \sum_{k=q}^{r} A_{ik}^t d_k - (1 - \alpha_{g,q-1})y_{g,q-1} < S \), then

\[
V(i, q, r) = V(q - 1) + \sum_{l=q}^{i-1} b_{l,i-1}(d_q - (1 - \alpha_{g,q-1})y_{g,q-1})
\]

\[
+ \sum_{l=q+1}^{i-1} \sum_{k=q+1}^{t} b_{l,i-1} d_k + \sigma_i + c_i S + \sum_{l=i}^{q-1} h_{il} \frac{S}{A_{il}^t};
\]

\[(25)\]
Let \( j \) by the definitions, we have

\[
\frac{S + (1 - \alpha_{g,q-1})y_{g,q-1} - \sum_{u=q}^{i-1} d_u}{A_{ik}^j} - \sum_{k=q}^{l} A_{kl}^j
\]

Note that if \( q - 1 \leq g \leq i - 1 \), then \( \alpha_{g,q-1} = 0 \).

\[\square\]

5.2. Forecast horizon results with backlogging. Under the backlogging case, we need to redefine the regeneration point.

**Definition 5.1.** Given an optimal solution for \( P(n) \), assume period \( s \) is the largest production point that is less than or equal to period \( t \), period \( t \) is referred to as a regeneration point (regeneration period) if \( (1 - \alpha_{s,t})y_{s,t} < d_{t+1} \) and \( b_{k,t} = 0 \) for all \( k, 1 \leq k \leq t \).

There are the following properties besides Property 3 under the constant cost structure and backlogging.

**Property 6.** There exists an optimal solution \( \Omega^* \) to Problem (P1), assume period \( s \) is the largest production point that is less than or equal to period \( i - 1 \),

(a) if \( x_i \geq S \), then \( w_{it} = 0 \) for all \( t, 1 \leq i \leq n, i \leq t \leq n; \)

(b) if \( (1 - \alpha_{s,i-1})y_{s,i-1} \geq d_i \), then \( w_{kt} = 0 \) for all \( k, 1 \leq i \leq n, s \leq k \leq i \).

This property can be derived as a generation of property (b) and (c) in Zangwill [51]. We omit the details.

For \( 1 \leq i \leq r \leq n \), define \( P(j, i, r) \) as the Problem (P1) restricted to periods 1 through \( r \), where period \( j \) is next-to-last regeneration point and the last setup occurs in period \( i \) to satisfy the demands from period \( j + 1 \) through \( r \). Let period \( g \) denote the next-to-last order point. Denote \( V(j, i, r) \) as the minimum costs of \( P(j, i, r) \), by the above definitions, we have

\[
V(r) = \min_{0 \leq j \leq i \leq r \leq n} V(j, i, r)
\]

Next, we show how to compute \( V(j, i, r) \). Based on Property 3 and 6, we have

\[
V(j, i, r) = \begin{cases}
V(j) + \sum_{l=j+1}^{i-1} b_{l,i-1}(d_{l+1} - \sum_{k=1}^{j} y_{k,l+1}) + \sum_{l=j+1}^{i-1} \sum_{l=1}^{i} b_{l,i-1}d_k + \sigma + cS \\
+ \sum_{l=1}^{r} h_{il}(\frac{S}{A_{il}^j} - \sum_{k=i}^{l} A_{kl}^j), \\ 
\text{if } \sum_{l=1}^{r} d_l + \sum_{l=i}^{r} A_{il}^j d_l \leq S + (1 - \alpha_{g,i-1})y_{g,i-1} \\
V(j) + \sum_{l=j+1}^{i-1} b_{l,i-1}(d_{l+1} - \sum_{k=1}^{j} y_{k,l+1}) + \sum_{l=j+1}^{i-1} \sum_{l=1}^{i} b_{l,i-1}d_k + \sigma \\
+ c(\sum_{l=1}^{r} d_l + \sum_{l=i}^{r} A_{il}^j d_l - \sum_{l=1}^{i-1} y_{li}) + \sum_{l=1}^{r} h_{il} \sum_{k=1}^{i} A_{lk}^j d_k, \\
\text{if } \sum_{l=1}^{r} d_l + \sum_{l=i}^{r} A_{il}^j d_l > S + (1 - \alpha_{g,i-1})y_{g,i-1}
\end{cases}
\]

Let \( j(t) \) denote the next-to-last regeneration point in an optimal solution to \( P(t) \). By the definitions, we have

\[
V(t) = V(j(t), i(t), t) = \min_{1 \leq j \leq i \leq t \leq n} V(j, i, t)
\]

(27)
Lemma 5.2. For any \( t^*, t^* \geq t + 1 \), there exists \( j(t^*) \geq j(t) \) and \( i(t^*) \geq i(t) \) in the optimal solution to \( P(t^*) \)-problem.

Proof. It is sufficient to prove the lemma for \( t^* = t + 1 \), then its repeated application yields the final result. Consider \( P(t+1) \)-problem, if for \( t=1 \):

\[
S + \sum_{j=1}^{j(t)} y_{j,t(t)+1} \text{ and } \sum_{l=1}^{i(t)} d_l + \sum_{l=i(t)}^{t} A_{i(t)}^l d_l \leq S + \sum_{j=1}^{j(t)} y_{j,t(t)+1} \text{ or } \sum_{l=1}^{i(t)} d_l + \sum_{l=i(t)}^{t} A_{i(t)}^l d_l \leq S + j(t) \text{, we have } j(t+1) = j(t)
\]

and \( i(t+1) = i(t) \). If \( \sum_{l=1}^{i(t)} d_l + \sum_{l=i(t)}^{t} A_{i(t)}^l d_l > S + \sum_{j=1}^{j(t)} y_{j,t(t)+1} \), suppose \( j(t+1) < j(t) \), we discuss two cases: (I) \( i(t+1) < i(t) \) and (II) \( i(t+1) \geq i(t) \). For the case (I) \( i(t+1) < i(t) \), by the definition of \( V(t) \), we have

\[
V(t) = V(j(t), i(t), t) = V(j(t)) + c \sum_{l=j(t)+1}^{i(t)-1} d_l + \sum_{l=i(t)}^{t} A_{i(t)}^l d_l + \sum_{l=i(t)}^{t} \sum_{k=j(t)+1}^{j(t)-1} y_{k,l(t)+1} + \sum_{k=j(t)+1}^{j(t)+2} b_{l,i(t)-1} d_k + \sigma
\]

(29)

According to \( h_{it} \geq h_{jt} \) for \( 1 \leq i \leq j \leq t \leq n \), and \( A_{kt}^i \geq A_{kt}^j \) for \( 1 \leq i < j \leq k < t \leq n \), we have

\[
\sum_{l=j(t)+1}^{i(t)+1} h_{i(l(t))} \sum_{k=l(t)+1}^{i(l(t))+1} A_{k}^{i(l(t))} d_k - \sum_{l=i(t)}^{i(t)+1} h_{i(l(t))} \sum_{k=l(t)+1}^{i(l(t))+1} A_{k}^{i(l(t))} d_k \leq \sum_{l=j(t)+1}^{i(t)+1} h_{i(l(t))} \sum_{k=l(t)+1}^{i(l(t))+1} A_{k}^{i(l(t))} d_k - \sum_{l=i(t)}^{i(t)+1} h_{i(l(t))} \sum_{k=l(t)+1}^{i(l(t))+1} A_{k}^{i(l(t))} d_k
\]

(30)

and

\[
c(\sum_{l=j(t)+1}^{i(t)+1} d_l + \sum_{l=i(t)}^{i(t)+1} A_{i(t)}^l d_l - \sum_{l=1}^{i(t)-1} y_{i(t)} + c(\sum_{l=j(t)+1}^{i(t)+1} d_l + \sum_{l=i(t)}^{i(t)+1} A_{i(t)}^l d_l - \sum_{l=1}^{i(t)-1} y_{i(t)})
\]
\begin{align*}
\text{Adding the equations (29), (30) and (31), we have} \\
V(j(t)) + \sum_{l=j(t)+1}^{i(t)-1} b_{l,i(t)-1}(d_{j(t)+1} - \sum_{k=1}^{j(t)} y_{k,j(t)+1}) + \sum_{l=j(t)+2}^{i(t)-1} \sum_{k=j(t)+2}^{l} b_{l,i(t)-1} d_k \\
+ \sigma + c(\sum_{l=j(t)+1}^{i(t)-1} d_l + \sum_{l=i(t)}^{i(t)+1} A_{i(t)}^{(i(t)+1)} d_l - \sum_{l=1}^{i(t)-1} y_{i(i(t))}) + \sum_{l=i(t)}^{i(t)+1} h_{i(t)} + \sum_{k=i(t)}^{i(t)+1} A_{i(k)}^{(i(t)+1)} d_k \\
\leq V(j(t+1)) + \sum_{l=j(t)+1}^{i(t+1)-1} b_{l,i(t+1)-1}(d_{j(t)+1} - \sum_{k=1}^{j(t+1)} y_{k,j(t)+1}) \\
+ \sum_{l=j(t)+2}^{i(t+1)-1} \sum_{k=j(t)+2}^{l} b_{l,i(t+1)-1} d_k \\
+ \sigma + c(\sum_{l=j(t)+1}^{i(t+1)-1} d_l + \sum_{l=i(t)+1}^{i(t+1)} A_{i(t+1)}^{(i(t)+1)} d_l - \sum_{l=1}^{i(t+1)-1} y_{i(i(t)+1))}) + \sum_{l=i(t)}^{i(t)+1} h_{i(t)+1} + \sum_{k=i(t)+1}^{i(t)+1} A_{i(k)}^{(i(t)+1)} d_k \\
= F(j(t+1), i(t+1), t+1) = F(t+1)
\end{align*}

Therefore, we have an alternative optimal policy of \( P(t+1) \)-problem such that \( j(t+1) = j(t) \) and \( i(t+1) = i(t) \).

For the case (II) \( i(t+1) \geq i(t) \), we have

\begin{align*}
V(t) = V(j(t), i(t), t) = V(j(t)) \\
+ \sum_{l=j(t)+1}^{i(t)-1} b_{l,i(t)-1}(d_{j(t)+1} - \sum_{k=1}^{j(t)} y_{k,j(t)+1}) + \sum_{l=j(t)+2}^{i(t)-1} \sum_{k=j(t)+2}^{l} b_{l,i(t)-1} d_k \\
+ \sigma + c(\sum_{l=j(t)+1}^{i(t)-1} d_l + \sum_{l=i(t)}^{i(t)+1} A_{i(t)}^{(i(t)+1)} d_l - \sum_{l=1}^{i(t)-1} y_{i(i(t))}) + \sum_{l=i(t)}^{i(t)+1} h_{i(t)} + \sum_{k=i(t)+1}^{i(t)+1} A_{i(k)}^{(i(t)+1)} d_k \\
\leq V(j(t+1)) + \sum_{l=j(t)+1}^{i(t+1)-1} b_{l,i(t+1)-1}(d_{j(t)+1} - \sum_{k=1}^{j(t+1)} y_{k,j(t)+1}) \\
+ \sum_{l=j(t)+2}^{i(t+1)-1} \sum_{k=j(t)+2}^{l} b_{l,i(t+1)-1} d_k \\
+ \sigma + c(\sum_{l=j(t)+1}^{i(t+1)-1} d_l + \sum_{l=i(t)+1}^{i(t+1)} A_{i(t+1)}^{(i(t)+1)} d_l - \sum_{l=1}^{i(t+1)-1} y_{i(i(t)+1))}) + \sum_{l=i(t)}^{i(t)+1} h_{i(t)+1} + \sum_{k=i(t)+1}^{i(t)+1} A_{i(k)}^{(i(t)+1)} d_k,
\end{align*}
By equation (34), we have

\[ V(j(t)) + \sum_{l=j(t)+1}^{i(t)-1} b_{l,i(t)-1}(d_{j(t)+1} - \sum_{k=1}^{j(t)} y_{k,j(t)+1}) + \sum_{l=j(t)+2}^{i(t)-1} \sum_{k=j(t)+2}^{l} b_{l,i(t)-1} d_{k} \]

\[ \leq V(j(t+1)) + \sum_{l=j(t)+1}^{i(t)-1} b_{l,i(t)-1}(d_{j(t)+1} - \sum_{k=1}^{j(t)+1} y_{k,j(t)+1}) \]

\[ + \sum_{l=j(t)+2}^{i(t)-1} \sum_{k=j(t)+2}^{l} b_{l,i(t)-1} d_{k} \]

According to \( b_{lt} \geq b_{jt} \) for \( 1 \leq l \leq j \leq t \leq n \), we have

\[ \sum_{l=j(t)+1}^{i(t)-1} b_{l,i(t)+1}(d_{j(t)+1} - \sum_{k=1}^{j(t)} y_{k,j(t)+1}) + \sum_{l=j(t)+2}^{i(t)-1} \sum_{k=j(t)+2}^{l} b_{l,i(t)+1} d_{k} \]

\[ \leq \sum_{l=j(t)+1}^{i(t)-1} b_{l,i(t)+1}(d_{j(t)+1} - \sum_{k=1}^{j(t)+1} y_{k,j(t)+1}) \]

\[ + \sum_{l=j(t)+2}^{i(t)-1} \sum_{k=j(t)+2}^{l} b_{l,i(t)+1} d_{k} \]

Adding the equations (32) and (33), we have

\[ V(j(t)) + \sum_{l=j(t)+1}^{i(t)-1} b_{l,i(t)+1}(d_{j(t)+1} - \sum_{k=1}^{j(t)} y_{k,j(t)+1}) + \sum_{l=j(t)+2}^{i(t)-1} \sum_{k=j(t)+2}^{l} b_{l,i(t)+1} d_{k} \]

\[ \leq V(j(t+1)) + \sum_{l=j(t)+1}^{i(t)-1} b_{l,i(t)+1}(d_{j(t)+1} - \sum_{k=1}^{j(t)+1} y_{k,j(t)+1}) \]

\[ + \sum_{l=j(t)+2}^{i(t)-1} \sum_{k=j(t)+2}^{l} b_{l,i(t)+1} d_{k} \]

By equation (34), we have

\[ V(j(t)) + \sum_{l=j(t)+1}^{i(t)-1} b_{l,i(t)+1}(d_{j(t)+1} - \sum_{k=1}^{j(t)} y_{k,j(t)+1}) + \sum_{l=j(t)+2}^{i(t)-1} \sum_{k=j(t)+2}^{l} b_{l,i(t)+1} d_{k} + \sigma + c\left( \sum_{l=j(t)+1}^{i(t)-1} d_{l} + \sum_{l=i(t)+1}^{i(t)+1} A_{l}^{i(t)+1} d_{l} - \sum_{l=1}^{i(t)+1} y_{l(t)+1} \right) \]
\[
\sum_{l=i(t+1)}^{t} h_{i(t+1)} l \sum_{k=l+1}^{i(t+1)} A_{lk}^{i(t+1)} d_k
\]

\[
\leq V(j(t+1)) + \sum_{l=j(t+1)+1}^{i(t+1)-1} b_{l,i(t+1)} (d_{j(t)}+1) - \sum_{k=1}^{j(t)+1} y_{k,j(t)+1} + \sum_{l=j(t)+1+2}^{i(t)+1} b_{l,i(t+1)+2} d_k
\]

\[
\sigma + c(\sum_{l=j(t)+1+1}^{i(t)+1} d_l + \sum_{l=i(t)+1}^{i(t)+1} A_{l,i(t+1)} d_l - \sum_{l=1}^{i(t)+1-1} y_{l,i(t+1)})
\]

\[
+ \sum_{l=i(t+1)}^{t} h_{i(t+1)} l \sum_{k=l+1}^{i(t+1)} A_{lk}^{i(t+1)} d_k
\]

\[
= V(j(t+1), i(t+1), t+1) = V(t+1)
\]

Equation (35) violates the optimality of \( V(t+1) \). The hypothesis \( j(t+1) < j(t) \) has led to a contradiction. Hence, we have \( j(t+1) \geq j(t) \).

The proof of \( i(t+1) \geq i(t) \) can follow from the case (I).

The following forecast horizon theorem is a variation of Theorem 1.

**Theorem 5.3.** Consider the optimal solution to the \( P(t) \)-period problem, if there are common first \( \tau \) periods’ decisions \( \Theta \) such that \( \Theta \) are optimal for all \( P(r) \) with \( r \) in the set \( \{j(t), j(t)+1, \ldots, t-1\} \), then period \( t \) is forecast horizon, period \( \tau \) is the corresponding decision horizon.

6. Computational experience.

**Example 1.** Consider an instance with seven periods. Let the minimum order quantities \( S \) be 25. Let the demands vectors be \( (6, 8, 9, 12, 15, 7, 26) \). Assume the lifetime of the product to be 4 periods and there is no stock deterioration within the lifetime. The cost parameters are as follows: \( \sigma = 100; c = 5; h_{il} = 2, 1 \leq i \leq t \leq 7; b_{kt} = +\infty, 1 \leq k \leq t \leq 7 \).

The detailed results are shown in Table 1. Since \( i(7) = 7 \), period 7 is the forecast horizon and period 6 is the corresponding decision horizon.

| Table 1. Summary of Computations of Example 1 |
|---|---|---|---|---|---|---|---|
| \( t \) | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| \( d_t \) | 6 | 8 | 9 | 12 | 11 | 7 | 26 |
| \( x_t^* \) | 25 | 25 | 25 | 25 | 25 | 26 |
| \( x_t^* \) | 25 | 0 | 0 | 0 | 0 | 0 |
| \( x_t^* \) | 25 | 0 | 0 | 0 | 0 | 0 |
| \( x_t^* \) | 25 | 0 | 0 | 0 | 0 | 0 |
| \( x_t^* \) | 25 | 0 | 0 | 32 | 0 | 0 |
| \( x_t^* \) | 25 | 0 | 0 | 32 | 0 | 0 |
| \( C(t) \) | 263 | 285 | 289 | 399 | 544 | 607 | 837 |
We now show that how the parameters affect the forecast horizon by a detailed computational study. For \( t = 1, 2, \ldots, n \), the setup cost \( \sigma \) is set at 100; the unit order cost \( c \) is set at 5. We assume the demands are normally distributed with standard deviation 1.5 and mean 10. We generate 11 instances to enable the easy identification of the median forecast horizon.

Text 1. The minimum order quantities take seven values: 15, 20, 25, 30, 35, 40, and 50. The lifetime of product is set to 6. In order to simplify the calculation, the deterioration rates are set to zero within the product’s lifetime. We set the unit inventory holding costs as follows: if \( k + 1 < 6 \), \( h_{i,i+k} = 1.5 \), if \( k + 1 \geq 6 \), \( h_{i,i+k} = +\infty \); \( b_{kt} = +\infty \), \( 1 \leq k \leq t \leq n \).

Figure 1 is a plot of the median forecast horizon as a function of minimum order quantities. The median forecast horizon increases with the minimum order quantities. When minimum order quantities are larger, the number of periods covered by the optimal order quantities is longer, consequently larger forecast horizon.

Text 2. The lifetime \( m \) takes seven values: 3, 4, 5, 6, 7, 8, and 9. The deterioration rates are set to zero within the product’s lifetime. We set the unit inventory holding costs as follows: if \( k + 1 < m \), \( k \) is integer, then \( h_{i,i+k} = 1.5 \), if \( k + 1 \geq m \), then \( h_{i,i+k} = +\infty \); \( b_{kt} = +\infty \), \( 1 \leq k \leq t \leq n \).

Figure 2 is a plot of the median forecast horizon as a function of lifetime of product. The median forecast horizon increases with the lifetime, then remains constant. In general, a bigger lifetime of product results in bigger order quantities, and consequently, a larger forecast horizon. When the lifetime of product is very large and other parameters are fixed, the optimal order quantities do not change, so the forecast horizon can not be affected by the lifetime.

Text 3. The backlogging costs take eight values: 1, 1.5, 2, 2.5, 3, 10, 15, and 20. The lifetime of product is set to infinite and the deterioration rates are set to zero. We set the unit inventory holding costs as follows: \( h_{it} = 2 \), \( 1 \leq i \leq t \leq 7 \).

Figure 3 is a plot of the median forecast horizon as a function of backlogging cost. The median forecast horizon decreases with the backlogging cost, then remains constant. Because when the backlogging cost is smaller, the number of periods covered by the optimal order quantities is larger, consequently, a larger forecast horizon. When the backlogging cost is enough higher, the problem with backlogging reduces to the problem that backlogging is prohibited, so forecast horizon remains invariant.
Text 4. The lifetime of product is set at 6 and the deterioration rates are set to zero within the product’s lifetime. The unit inventory holding costs take eight values within the product’s lifetime: 1, 1.5, 2, 2.5, 3, 3.5, 4, and 5. Beyond the product’s lifetime, the unit inventory holding costs are set to infinite. We also set the infinite backlogging cost, that is, $b_{kt} = +\infty$, $1 \leq k \leq t \leq n$.

Figure 4 is a plot of the median forecast horizon as a function of inventory holding cost. For a given inventory holding cost, the median forecast horizon decreases with the inventory holding cost, then remains constant. Because the higher inventory holding cost leads to smaller order quantities, consequently, a smaller forecast horizon. When the inventory holding cost is enough higher, the optimal order quantities do not change, so forecast horizon remains invariant.

Above all, if inventory holding costs are higher, only the data information in the short term shall be taken into consideration when the operation manager builds order plans. On the contrary, the data information in the long term shall be handled in order to determine more accurate order lot size. If the minimum order quantities are larger or the product has the long lifetime or backlogging costs are smaller, the data information in the relatively long term shall be analyzed when the operation manager builds order plans. Conversely, the future data information in the relatively short term shall be analyzed. In conclusion, the current order decision of
the enterprise will be affected by so many factors such as product lifetime, minimum order quantities, backlogging costs and inventory holding costs. Therefore, when making order plans, the operation manager shall take various factors into consideration in order to achieve the reasonable decision results.

7. Conclusion. This paper investigates the DLS problem for perishable products with age-dependent stock deterioration rates and inventory costs under minimum order quantities. In a real world application, when the minimum order quantity is larger, backlogging is an effective tool to reduce the operation costs. Therefore, we extend the basic model to the backlogging case. Based on the structural properties of the optimal solution, we give an efficient procedure to solve the problem with backlogging and no backlogging. Consequently, we give a sufficient condition to obtain forecast horizon by establishing the monotonicity of the order point (and regeneration point). Finally, using a detailed test bed of instances, useful managerial insights are obtained on the impact of minimum order quantities and perishability of product and the costs on the length of forecast horizon.

Further research need to consider more realistic factors such as time windows. Another interesting extension of our work would be to consider more general cost structures studied in Sargut and Romeijn [41]. It would also be beneficial to extend the analysis to include remanufacturing options [37].

Notes. https://www.fmi.org/docs/default-source/facts-figures/supermarket-sales-by-department-2016.pdf?sfvrsn=ea23716e_2

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