Exponential Cubic B-spline Collocation Method for Solving the One Dimensional Wave Equation Subject to an Integral Conservation Condition

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Abstract. In this paper, a collocation method is presented based on the exponential B-spline functions for the numerical solution of the one-dimensional wave equation subject to an integral conservation condition. The wave equation is fully-discretized by using the exponential cubic B-spline collocation for spatial discretization and the finite difference method for the time discretization. The difference scheme is analyzed to be unconditionally stable. The results of a numerical experiment show the stability and efficiency of the proposed method.

1. Introduction
Consider the one-dimensional nonlinear wave equation

\[ U_t(x,t) - U_{xx}(x,t) = f(x,t), \]  

with the following conditions

\[ U(x,0) = g_1(x), \quad a \leq x \leq b, \]  
\[ U_t(x,0) = g_2(x), \quad a \leq x \leq b, \]  
and

\[ U(a,t) = l_1(t), \quad 0 \leq t \leq T, \]  
\[ \int_a^b U(x,t)dx = l_2(t), \quad 0 \leq t \leq T, \]

where \( f(x,t), g_1(x), g_2(x), l_1(t) \) and \( l_2(t) \) are known functions.

It plays a very important role in studying plasma physics, thermoelasticity, chemical heterogeneity and etc. The existence and uniqueness of the solution of this problem was discussed in [1]. Many numerical methods of this problem, such as finite difference schemes [2], the method of lines [3], a second kind Chebyshev polynomial approach [4], a meshless method using radial basis functions [5], have been presented.

The exponential B-splines are piecewise functions, which have a free parameter \( p \). By means of this collocation method, some partial differential equations have been solved numerically [6-9]. In this paper, we apply the exponential cubic B-spline collocation method to solve the one-dimensional wave equation subject to an integral conservation condition.
2. Collocation Method

The domain \([a, b]\] is divided into an \(M\) mesh, with \(h=(b-a)/M\), and \(\tau=T/N\), respectively. \(x_j=a+jh\), for \(j=0, 1, 2, \ldots, M\) is the \(j\)th node. \(t_k=kt\), \(k=0, 1, \ldots, N\), is the time level for the \(k\)th step.

An approximate solution \(u(x,t)\) to the analytical solution \(U(x,t)\) of equation (1) is given by

\[
u(x,t) = \sum_{j=1}^{M+1} C_j(t)W_j(x),
\]

where \(C_j(t)\) are time-dependent unknowns to be determined. The exponential cubic B-spline functions \(W_j(x)\) are defined by

\[
W_j(x) = \begin{cases} 
  b_2((x_j - 2 - x) - \frac{1}{p} \sinh(p(x_j - 2 - x))), & x \in [x_{j-2}, x_{j-1}], \\
  a_1 + b_1(x - x_j) + c_1 \exp(p(x - x_j)) + d_1 \exp(-p(x - x_j)), & x \in [x_{j-1}, x_j], \\
  a_1 + b_1(x - x_j) + c_2 \exp(p(x - x_j)) + d_1 \exp(-p(x - x_j)), & x \in [x_{j}, x_{j+1}], \\
  b_2((x - x_{j+2}) - \frac{1}{p} \sinh(p(x - x_{j+2}))), & x \in [x_{j+1}, x_{j+2}], 
\end{cases}
\]

where \(p\) is a free parameter, and

\[
\begin{align*}
  a_1 &= \frac{p h c}{p h c - s}, \\
  b_1 &= \frac{p}{2} \left( c(c-1) + s^2 \right), \\
  c_1 &= \frac{1}{4} \frac{\exp(-p h)(1-c) + s^2 \exp(-p h) - 1}{(p h c - s)(1-c)}, \\
  d_1 &= \frac{1}{4} \frac{\exp(p h)(1-c) + s^2 \exp(-p h) - 1}{(p h c - s)(1-c)}, \\
  b_2 &= \frac{p}{2(p h c - s)}, s = \sinh(p h), c = \cosh(p h).
\end{align*}
\]

So the approximant \(u_j^k = u(x_j, t_k)\) at the point \((x_j, t_k)\) over interval \([x_j, x_{j+1}]\) can be defined as

\[
u_j^k = \sum_{i=j-1}^{j+1} C_i^k(t)W_i(x).
\]

Then we have

\[
\begin{align*}
  u_j^k &= \delta_1 C_{j-1}^k + C_j^k + \delta_3 C_{j+1}^k, \\
  (u_x)_j^k &= \delta_2 C_{j-1}^k - \delta_3 C_{j+1}^k, \\
  (u_{xx})_j^k &= \delta_3 C_{j-1}^k - 2\delta_3 C_j^k + \delta_3 C_{j+1}^k,
\end{align*}
\]

where

\[
\begin{align*}
  \delta_1 &= \frac{s - p h}{(2(p h c - s)),} \\
  \delta_2 &= \frac{p c - 1}{(2(p h c - s))}, \\
  \delta_3 &= \frac{p^2 s}{(2(p h c - s))}.
\end{align*}
\]

By means of the \(\theta\)-scheme, time variables of the unknown solution \(U\) in (1) can be discretized. Then we obtain

\[
\frac{u_j^{k+1} - 2u_j^k + u_j^{k-1}}{\tau^2} - (1-\theta)(u_{xx})_j^k + \theta (u_{xx})_j^{k+1} = f(x_j, t_k)
\]

At \(k=0\), the term \(u^{k-1}\) in (12) can be determined by (3), that is
\[ u_j^{-1} = u_j^i - 2g_j(x_j) \]  

(13)

If \( \theta = 0 \), the difference scheme will be explicit. If \( \theta = 1/2 \), it will be a Crank-Nicolson scheme. If \( \theta = 1 \), it will be implicit.

To obtain a unique solution, two additional equations are needed. Thus, by terms of (4) and (5), we have

\[ \delta_1 C_{j-1}^{k+1} + C_{j+1}^{k+1} + \delta_1 C_{1}^{k+1} = l_1(t_{k+1}) \]  

(14)

\[ -\delta_2 C_{N+1}^{k+1} + \delta_2 C_{N-1}^{k+1} + \delta_1 C_{1}^{k+1} - \delta_1 C_{-1}^{k+1} = l_1''(t_{k+1}) - \int_0^0 f(x,t_{k+1})dx \]  

(15)

To start iterations of this system (12), the initial parameters \( C_{-1}^0, \ldots, C_{M+1}^0 \) can be obtained from the following conditions:

(i) \( (u_j)_0^i = g_1'(x_0) \),

(ii) \( u_j^0 = g_1(x_j), \quad j = 0,1,\ldots,M \),

(iii) \( (u_j)_N^0 = g_1'(x_M) \).

3. Stability Analysis

The stability of the presented method is investigated by Von Neumann method, which is applicable to linear scheme. Hence, following [10], the homogeneous part of the nonlinear equation (1) is linearized by assuming all nonlinear terms equal zero. Substituting (10) into the linear scheme of (12), we obtain the following equation

\[ V_1 C_{j-1}^{k+1} + V_2 C_{j+1}^{k+1} + V_3 C_{j}^{k+1} = V_1 C_{j-1}^k + V_2 C_{j}^k + V_3 C_{j+1}^k - \delta_1 C_{j-1}^{k-1} - \delta_1 C_{j+1}^{k+1}, \]  

(16)

where

\[ p_1 = \delta_1 - \theta \mu P_2 = 1 + 2\theta \mu P_3 = 2\delta_1 + (1 - \theta) \mu, \]

\[ p_3 = 2 - 2(1 - \theta) \mu, \mu = \tau \delta_2. \]  

(17)

The trial solutions (one Fourier mode out of the full solution) at a given point \( (x_j, t_k) \) are considered,

\[ C_j^k = \xi^k \exp(i j \eta h), \]  

(18)

where \( i = \sqrt{-1} \), \( \eta \) is the mode number and \( h \) is the element.

Combining (16) with (18), we have

\[ A\xi^2 - B\xi + C = 0, \]  

(19)

where

\[ A = P_2 + 2P_1 \cos(\eta h), \]

\[ B = P_4 + 2P_3 \cos(\eta h), \]

\[ C = 1 + 2\delta_1 \cos(\eta h). \]

Hence

\[ \xi = \frac{B \pm \sqrt{B^2 - 4AC}}{2A} = \frac{B \pm \sqrt{-1 \sqrt{4AC - B^2}}}{2A}. \]  

(20)

Then the amplification factor

\[ |\xi| = \sqrt{\left(\frac{B}{2A}\right)^2 + \left(\frac{\sqrt{4AC - B^2}}{2A}\right)^2} = \sqrt{\frac{C}{A}}. \]
\[
\sqrt{1 + 2\delta_1 \cos(\eta h)} \\
\sqrt{1 + 2\delta_2 \cos(\eta h) + 2\theta r^2 \delta_3 (1 - \cos(\eta h))} \leq 1
\]

since \( \delta_1 > 0, \delta_2 > 0, \) and \( 0 \leq \theta \leq 1. \)

Thus the proposed numerical scheme is unconditionally stable.

4. A Numerical Experiment
Consider the problem (1)-(5) given in [3] with \( a=0, b=1, T=5, \) and
\[
\begin{align*}
  f(x,t) &= 0, \\
  g_1(x) &= \cos(\pi x), \\
  g_2(x) &= 0, \\
  l_1(t) &= \cos(\pi t), \\
  l_2(t) &= 0.
\end{align*}
\]

The analytical solution is obtained in [3] as
\[
U(x,t) = \frac{1}{2} (\cos(\pi(x + t)) + \cos(\pi(x - t))).
\]

The numerical results are obtained by the exponential cubic B-spline collocation method. The following relative error \( (L_2) \) norm is applied to measure the accuracy of the proposed method
\[
L_2 = \frac{\sqrt{\sum_j |U_j - u_j|^2}}{\sqrt{\sum_j |U_j|^2}},
\]

where \( U_j \) are analytical solutions and \( u_j \) are approximate solutions respectively.

The approximate solutions with \( M=100, N=500, p=0.001 \) for different values of \( \theta \) are given in Figure1.

The relative errors at \( T=1 \) with \( M=100, N=500, p=0.001 \) for different values of \( \theta \) are given in Figure2.
Figure 2. Relative errors at $T=1$ with $M=100$, $N=500$, $p=0.001$

5. Conclusion
For solving the one-dimensional wave equation subject to an integral conservation condition, the exponential cubic B-spline collocation method is presented. The numerical experiment is proposed to illustrate the effectiveness of this method. The accuracy of calculation can be improved by choosing appropriate parameters $\theta$ and $p$.

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