SHARP EQUIVALENT FOR THE BLOWUP PROFILE TO THE GRADIENT OF A SOLUTION TO THE SEMILINEAR HEAT EQUATION

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Abstract. In this paper, we consider the standard semilinear heat equation
\[ \partial_t u = \Delta u + |u|^{p-1}u, \quad p > 1. \]
The determination of the (believed to be) generic blowup profile is well-established in the literature, with the solution blowing up only at one point. Though the blow-up of the gradient of the solution is a direct consequence of the single-point blow-up property and the mean value theorem, there is no determination of the final blowup profile for the gradient in the literature, up to our knowledge.

In this paper, we refine the construction technique of Bricmont-Kupiainen [3] and Merle-Zaag [54], and derive the following profile for the gradient:
\[ \nabla u(x, T) \sim -\frac{\sqrt{2b}}{p-1} \frac{x}{|x| \sqrt{\ln|x|}} \left[ \frac{b|x|^2}{2|\ln|x||} \right]^{-\frac{p+1}{2(p-1)}} \] as \( x \to 0, \)
where \( b = \frac{(p-1)^2}{4p}, \) which is as expected the gradient of the well-known blowup profile of the solution.

1. Introduction

In this paper, we consider the following semilinear heat equation
\[
\begin{align*}
\begin{cases}
\partial_t u &= \Delta u + |u|^{p-1}u \\
u(0) &= u_0 \in L^\infty(\mathbb{R}^N)
\end{cases}
\end{align*}
\]
where \( u : (x, t) \in \mathbb{R}^N \times [0, T) \to \mathbb{R} \) and \( p > 1. \) The local in time Cauchy problem can be solved in \( L^\infty, \) thanks to a fixed point technique (see Quittner and Souplet [60] for instance). Roughly speaking, for each initial data \( u_0 \in L^\infty, \) one of the following cases holds:
- Either the solution is global.
- Or it blows up in finite time \( T \) i.e
\[ \limsup_{t \to T} \|u(t)\|_{L^\infty} \to +\infty. \]
In this case, $T$ is called the blowup time, and if for some $a \in \mathbb{R}^N$, there exists $(a_n, t_n) \to (a, T)$ as $n \to +\infty$ such that
\[ |u(a_n, t_n)| \to +\infty \quad \text{as } n \to +\infty, \]
then $a$ is called a blowup point.

The behavior of solutions at blowup has generated a huge literature. The book by Quittner and Souplet [60] is a good resource on the subject. According to Herrero and Velázquez, the generic blowup behavior corresponds to the situation where the solution blows up only at one blowup point (say, the origin, from invariance by translation in space of equation (1.1)) with the final blow-up profile
\[ u(x, T) \equiv \lim_{t \to T} u(x, t) \quad \text{(1.2)} \]
satisfying
\[ u(x, T) \sim u^*(x) \equiv \left[ \frac{(p - 1)^2}{8p} \frac{|x|^2}{\ln |x|} \right]^{-\frac{1}{p-1}} \quad \text{as } x \to 0. \quad \text{(1.3)} \]
Note that the genericity of this profile was published only in one space dimension in Herrero and Velázquez [37] (see also the note [39]). The higher dimensional case was also proved by the same authors, but never published. Several papers proved the existence of solutions obeying this generic behavior (see Galaktionov et al [68] for a formal approach, Berger and Kohn [2] for a formal and a numerical evidence, Herrero and Velázquez [38] for a rigorous proof, and also Bricmont and Kupiainen [3] together with Merle and Zaag [54]).

Since the origin is the unique blowup point in the considered behavior, we easily see from the mean value theorem that $\nabla u$ blows up at the origin as well. Accordingly, as the convergence in (1.2) holds uniformly on every compact set of $\mathbb{R}^N \setminus \{0\}$, one may wonder whether the estimate (1.3) holds after differentiation in space. Up to our knowledge, such a result is not available in the literature, and we only have an upper bound proved by Abdelhedi and Zaag in [1]:
\[ |\nabla u(x, T)| \leq C|x|^{\frac{p+1}{p-1}} |\ln |x||^{\frac{p+3}{4(p-1)}}. \]
In this paper, we sharply adapt the construction method of [3] and [54] and show that (1.3) holds after differentiation in space.

That method was first introduced by Bressan in [5] and [4] for the heat equation with an exponential source, then in Bricmont and Kupiainen [3] and Merle and Zaag [54] for equation (1.1). It consists in a formal approach where the profile is obtained through an inner/outer expansion with matching asymptotics, followed by a rigorous proof where the PDE is linearized around the profile candidate. Then, the negative part of the spectrum (which is infinite dimensional) is controlled thanks to the decaying properties of the Laplacian, whereas the nonnegative part (which is finite dimensional) is controlled thanks to the degree theory.

As a matter of fact, we mention that in [3] and [54], the authors constructed a blowup solution satisfying the so-called intermediate blowup profile, valid for $0 \leq t < T$:
\[ \left\| (T-t)^{\frac{1}{p-1}} u(., t) - \left( p - 1 + \frac{(p - 1)^2}{4p} \frac{|.|^2}{(T-t) \ln(T-t)} \right)^{-\frac{1}{p-1}} \right\|_{L^\infty} \leq \frac{C}{1 + \sqrt{|\ln(T-t)|}}. \quad \text{(1.4)} \]
The derivation of the so-called final profile (1.3), valid at $t = T$, was later done by Zaag in [69].

Note that the constructive method given in those works was efficiently used in a very large class of parabolic equations such as in Merle and Zaag [53] for quenching problems; in Duong et al [18], Nguyen and Zaag in [57], and Tayachi and Zaag [67] for perturbed nonlinear source terms; in Duong et al [19, 20], Masmoudi and Zaag [55] and Nouaili and Zaag [59] for the Complex Ginzburg-Landau
equation; and Duong [26, 27], and also in Nouaili and Zaag [56] for non-variational complex valued heat equations.

More generally, a large literature has been devoted in the last 20 years to the construction of solutions of PDEs with prescribed behavior, beyond the case of parabolic equations such as: Type I anisotropic heat equation by Merle et al [51]; Type II blowup for heat equation by del Pino et al [21, 22, 23, 24], Schwery [65], Collot [13], Merle et al [12], Harada [33, 34], Seki [66]; blowup for nonlinear Schrödinger equation by Merle [43], Martel and Raphaël [46], Merle et al [48, 50, 49], Raphaël and Szeftel [62]; Blowup for wave equations by Côte and Zaag [14], Ming et al [52], Collot [8], Hillairet and Raphaël [35], Krieger et al [41, 40], Ghoul et al [30], Raphaël and Rodnianski [61], Donninger and Schörkhuber [25]; Blowup for KdV and gKdV [42], Côte [6, 7]; Schrödinger map by Merle et al [47]; Heat flow map by Ghoul et al [31], Raphaël and Schwery [64], Dávila et al [15]; Keller Segel system by Ghoul et al [10, 11], Schwery and Raphaël [63]; Prandtl’s system by Collot et al [9]; Stefan problem by Hadzic and Raphaël [36]; 3-dimensional compressible fluids by Merle et al [49]; quenching phenomena for MEMS devices by Duong and Zaag [28]; the Gierer-Meinhardt system by Duong et al [16, 17].

Next, we would like to mention some papers where upper bounds on the profile of the gradient were obtained, though with no sharp equivalent. For example, in Tayachi and Zaag [67], the authors constructed a blowup solution to the following equation

$$\partial_t u = \Delta u + \mu |\nabla u|^q + |u|^{p-1}u,$$

where

$$\mu > 0, p > 3 \text{ and } q = \frac{2p}{p+1}.$$

In addition to that, they proved that the gradient also blows up and gave an upper bound on the gradient’s final profile near 0:

$$|\nabla u(x, T)| \leq \begin{cases} 
C \frac{|x|^\frac{p+1}{p-1}}{\ln|x|^{\frac{(p-1)p}{2} - \epsilon}} & \text{if } p \in (3, 7), \\
C \frac{|x|^\frac{p+1}{p-1}}{\ln|x|^{\frac{(p-1)p}{2} - \epsilon}} & \text{if } p > 7,
\end{cases}$$

for some small $\epsilon > 0$. Similarly, such upper bounds were proved in many situations such as for a perturbed nonlinear heat equations with a gradient and a non-local term as in Abdelhedi and Zaag [1]; and nonlinear heat equations involving a critical power nonlinear gradient term as in Ghoul et al [32].

In our opinion, it is exciting and important to obtain the exact profile of the gradient. By sharply adapting the previously mentioned construction method to handle the gradient estimates, we get the following result:

**Theorem 1.1** (Construction of a blowup solution with a sharp determination of the gradient behavior). There exist initial data $u_0 \in W^{1,\infty}(\mathbb{R}^N)$ such that equation (1.1) has a unique solution which blows up in finite time $T(u_0) > 0$ only at the origin. In particular, the following holds:

(i) (Intermediate profile): For all $t \in [0, T)$,

$$\left\| (T-t)^{-\frac{1}{4} + \frac{1}{2}} u(., t) - \varphi_0 (z) \right\|_{L^\infty} \leq \frac{C \ln(|\ln(T-t)|)}{1 + |\ln(T-t)|}, \quad (1.5)$$

and

$$\left\| (T-t)^{-\frac{1}{4} + \frac{1}{2}} \nabla u(., t) - \frac{\nabla \varphi_0 (z)}{\sqrt{\ln(T-t)}} \right\|_{L^\infty} \leq \frac{C \ln(|\ln(T-t)|)}{1 + |\ln(T-t)|}, \quad (1.6)$$

where
where
\[ z = \frac{x}{\sqrt{(T-t) \ln(T-t)}} \quad \varphi_0(z) = (p-1+ b|z|^2)^{\frac{1}{p-1}} \quad \text{and} \quad b = \frac{(p-1)^2}{4p}, \]

(ii) (Final profile): \( u(x,T) \equiv \lim_{t \to T} u(x,t) \) exists for all \( x \neq 0 \) and \( u(\cdot,T) \in C^2(\mathbb{R}^N \setminus \{0\}) \), with a convergence holding uniformly in \( C^2 \) of every compact set of \( \mathbb{R}^N \setminus \{0\} \). In particular, we have the following equivalents:
\[ u(x,T) \sim u^*(x) \equiv \left[ \frac{b}{2} \frac{|x|^2}{\ln |x|} \right]^{-\frac{1}{p-1}}, \]

and
\[ \nabla u(x,T) \sim \nabla u^*(x) = -\frac{\sqrt{2b}}{p-1} \frac{x}{|x| \sqrt{\ln |x|}} \left[ \frac{b|x|^2}{2|\ln |x||} \right]^{-\frac{p+1}{2(p-1)}} \text{ as } x \to 0. \]

**Remark 1.2.** The main contribution of the theorem is to discover (1.6) and its consequence (1.9). The key estimate is to show a more precise error estimate in (1.5), namely
\[ \frac{C \ln |\ln(T-t)|}{1 + |\ln(T-t)|}, \]
which is sharper than the bound of the previous works [3] and [54], namely
\[ \frac{C}{1 + \sqrt{|\ln(T-t)|}}, \]
as already mentioned in (1.4). Indeed, this latter estimate is not enough to derive a sharp estimate on the gradient. The corner stone of the proof lays in fact in the introduction of a new *Shrinking set* below in Definition 3.1, with sharper estimates. We also mention other situations where a clever adaptation of the shrinking set lead to the derivation of a sharper blowup behavior. This was in particular the case a complex-valued heat equation with no variational structure, where the behavior of the imaginary part was derived by Duong in [26, 27]. We also mention the case of the Complex Ginzburg-Landau equation (CGL) in some critical setting in Duong et al [19] and Nouaili and Zaag [59]; and in the subcritical range in Duong et al [20] too, where a higher order expansion was derived thanks to a good adaptation of the shrinking set.

**Remark 1.3** (Stability). As we explained right before the statement of this theorem, the rigorous proof goes through the reduction of the question to a finite-dimensional problem, then the solution of this finite-dimensional problem thanks to the degree theory. In fact, it is possible to make the interpretation of the parameters of the finite-dimensional problem in terms of the blowup time and the blowup point, as originally done in [54], in order to show that the behavior described by (1.5), (1.6), (1.8) and (1.9) is stable under perturbations of initial data. More precisely, let us denote by \( \tilde{u}_0 \) the initial data constructed in Theorem 1.1 and \( \tilde{T} \) its blowup time. Then, there exists a neighborhood \( \mathcal{N}(u_0) \) of \( u_0 \) in \( W^{1,\infty} \) such that for all \( u_0 \in \mathcal{N}_0 \), the corresponding solution \( u(x,t) \) of equation (1.1) blows up in finite time \( T(u_0) \) at some unique blowup point \( a(u_0) \), with the blowup profiles (1.5), (1.6), (1.8) and (1.9) which remain valid for \( u(x,t) \), just by replacing \( x \) by \( x - a(u_0) \). Moreover, we have the following limit
\[ (a(u_0), T(u_0)) \to (0, \tilde{T}) \text{ as } u_0 \to \tilde{u}_0. \]

**Remark 1.4.** This construction works also in bounded domains with homogenous Dirichlet or Neumann boundary conditions, with almost the same proof. The only difference with the case of the whole space lays in the use of some cut-off argument, which runs smoothly, as in Mahmoudi, Nouaili and Zaag [45] or in Duong and Zaag [28].
Throughout this paper, $C$ is a constant which depends only on $N$ and $p$, and whose value may change from line to line. If we need a constant depending on other parameters, we will specify its dependence on parameters. However, we may omit the dependence on $K$, the constant introduced in the cut-off (2.15 below, in order to make the notations lighter.

2. Mathematical setting

Let us consider $T > 0$ and introduce the following similarity variables

$$y = \frac{x}{\sqrt{T-t}}, \quad s = -\ln(T-t) \quad \text{and} \quad w(y, s) = (T-t)^{\frac{1}{p-1}} u(x, t).$$

(2.1)

With this transformation, $u(x, t)$ satisfies equation (1.1) for all $(x, t) \in \mathbb{R}^N \times [0, T)$ if and only if $w(y, s)$ satisfies the following equation for all $(y, s) \in \mathbb{R}^N \times [-\ln T, +\infty)$:

$$\partial_s w = \Delta w - \frac{1}{2} y \cdot \nabla w - \frac{w}{p-1} + |w|^{p-1}w.$$

(2.2)

Since our goal is to construct a solution $u$ to equation (1.1) that blows up in finite time $T$ like $(T-t)^{\frac{1}{p-1}}$, we may change it thanks to (2.1) to the construction of a global solution $w(y, s)$ to (2.2) such that

$$\epsilon_0 \leq \limsup_{s \to +\infty} ||w(s)||_{L^\infty} \leq \frac{1}{\epsilon_0}$$

for some $\epsilon_0 > 0$. In [54], the authors further specified this goal and succeeded in showing the existence of a solution to equation (2.2) with the following profile:

$$\varphi(y, s) = \left( p - 1 + b \frac{|y|^2}{s} \right)^{\frac{1}{p-1}} - \frac{\kappa N}{2ps}, \quad \text{with} \quad \kappa = (p-1)^{\frac{1}{p-1}},$$

(2.3)

in the sense that for all $s \geq -\ln T$ and $T > 0$ small enough, we have

$$||w(s) - \varphi(s)||_{L^\infty} \leq \frac{C}{\sqrt{s}}.$$

In this paper, we show that the techniques of [54] can be delicately improved to get a smaller error term, namely: for all $s \geq -\ln T$,

$$||w(s) - \varphi(s)||_{L^\infty(\mathbb{R}^N)} \leq \frac{C \ln s}{s}.$$

(2.4)

Introducing

$$q = w - \varphi,$$

(2.5)

we see from equation (2.2) that $q$ satisfies the following equation:

$$\partial_s q = (\mathcal{L} + V)q + B(q) + R(y, s),$$

(2.6)

where

$$\mathcal{L} = \Delta - \frac{1}{2} y \cdot \nabla + \text{Id},$$

(2.7)

$$V = p(\varphi^{p-1} - \frac{1}{p-1}),$$

(2.8)

$$B(q) = |q + \varphi|^{p-1}(q + \varphi) - \varphi^p - p\varphi^{p-1}q,$$

(2.9)

$$R(y, s) = -\partial_s \varphi + \Delta \varphi - \frac{1}{2} y \cdot \nabla \varphi - \frac{\varphi}{p-1} + |\varphi|^{p-1}\varphi.$$

(2.10)

Thus, from (2.4), our aim becomes to construct a solution of equation (2.6) such that for all $s \geq -\ln T$,

$$||q(s)||_{L^\infty} \leq \frac{C \ln s}{s}.$$

(2.11)
In order to understand the dynamics of equation (2.6), we need to make some comments on the linear, nonlinear and remainder terms in that equation:

- **Linear operator \( \mathcal{L} \):** It is self-adjoint in the space \( L^2_p(\mathbb{R}^N) \) where \( \rho(y) = e^{-|y|^2/4/(4\pi)^{N/2}} \), with explicit spectrum

  \[
  \text{Spec}(\mathcal{L}) = \{ \lambda_m = 1 - \frac{m}{2} |m \in \mathbb{N}| \}.
  \] (2.12)

  Corresponding to the eigenvalue \( \lambda_m \), we have the eigenspace \( \mathcal{E}_m \)

  \[
  \mathcal{E}_m = \langle h_{m_1}(y_1).h_{m_2}(y_2)....h_{m_N}(y_N) \mid m_1 + ... + m_N = m \rangle,
  \] (2.13)

  where \( h_\ell \) is the (rescaled) Hermite polynomial in one dimension, defined by

  \[
  h_\ell(\xi) = \sum_{j=0}^{[\frac{\ell}{2}]} (-1)^j \frac{\ell!}{j!(\ell - 2j)!} \xi^{\ell - 2j}.
  \]

- **Potential \( V \):** It has the following properties:

  (i) \( V(\cdot, s) \to 0 \) in \( L^2(\mathbb{R}^N) \) as \( s \to +\infty \), which implies in particular that its effect is negligible with respect to the effect of \( \mathcal{L} \) (except maybe on the zero eigenspace of \( \mathcal{L} \)).

  (ii) \( V(y, s) \) is almost a constant outside the blowup region, i.e. for \( |y| \geq K_0\sqrt{s} \) with \( K_0 > 0 \) large enough. In particular, we have the following estimate

  \[
  \sup_{s \geq s_0, \frac{y}{\sqrt{s}} \geq C_0} \left| V(y, s) - \left( -\frac{2}{p - 1} \right) \right| \leq \epsilon,
  \]

  for some \( \epsilon > 0, C_0 > m, \) and \( s_0 \) large enough. Since \( -\frac{2}{p - 1} < -1 \) and the largest eigenvalue of \( \mathcal{L} \) is 1, we can say that \( \mathcal{L} + V \) has a strictly negative spectrum outside the blowup region.

- **Nonlinear term \( B(q) \):** It is superlinear, in the sense that it satisfies the following estimate

  \[
  \|B(q)\| \leq C|q|^{\bar{p}} \text{ where } \bar{p} = \min(2, p) > 1.
  \]

- **Remainder term \( R \):** It is small, in the sense that

  \[
  \forall s \geq -\ln T, \quad \|R(s)\|_{L^\infty} \leq \frac{C}{s},
  \]

  which is natural by definition (2.10), since the profile \( \varphi(y, s) \) defined in (2.3) is in fact an approximate solution of equation (2.2). Following the decomposition in [54] together with the remark given in item (ii) above, we introduce the following decomposition, for any \( r \in L^\infty(\mathbb{R}^N) \):

  \[
  r(y) = \chi(y, s)r(y) + (1 - \chi(y, s))r(y) \equiv r_b(y, s) + r_e(y, s),
  \] (2.14)

  where \( \chi(y, s) \) defined by

  \[
  \chi(y, s) = \chi_0 \left( \frac{\sqrt{s}y}{K} \right),
  \] (2.15)

  \( \chi_0 \) being a one-dimensional cut-off satisfying

  \[
  \text{Supp} (\chi_0) \subset [0, 2], \quad 0 \leq \chi_0(\xi) \leq 1, \forall \xi \geq 0 \text{ and } \chi_0(\xi) = 1, \forall \xi \in [0, 1],
  \] (2.16)

  and constant \( K > 0 \) be chosen large enough so that various estimates hold in the proof. Let us remark that

  \[
  \text{Supp} (r_b(s)) \subset \{ |y| \leq 2K\sqrt{s} \},
  \]

  \[
  \text{Supp} (r_e(s)) \subset \{ |y| \geq K\sqrt{s} \}.
  \]

  In addition, since the set of eigenfunctions of \( \mathcal{L} \) makes a basis of \( L^2_p \), we can write \( r_b \) as follows

  \[
  r_b(y, s) = r_0(s) + r_1(s) \cdot y + y^T \cdot r_2(s) \cdot y - 2 \text{ Tr}(r_2(s)) + r_-(y, s),
  \] (2.17)
where
\[ r_i(s) = (P_\beta(r_b(s)))_\beta \in \mathbb{N}^\times, \forall i \geq 0, \] (2.18)
with \( P_\beta(r_b) \) being the projection of \( r_b \) on the eigenfunction \( h_\beta \) defined as follows:
\[ P_\beta(r_b(s)) = \int_{\mathbb{R}^N} r_b(y, s) \frac{h_\beta}{\|h_\beta\|_{L_2^2(\mathbb{R}^N)}} dy, \forall \beta \in \mathbb{N}^N, \] (2.19)
and
\[ r_-(y, s) = P_-(r_b) = \sum_{\beta \in \mathbb{N}^N, |\beta| \geq 3} P_\beta(r_b(s)) h_\beta(y). \] (2.20)

Note that in (2.17), we are isolating the nonnegative eigenvalues of \( L \) (see (2.12)), since they are more delicate to handle.

Finally, we have just defined the following expansion, which will be extremely useful in the proof:
\[ r(y) = r_0(s) + r_1(s) \cdot y + y^T \cdot r_2(s) \cdot y - 2 \text{Tr}(r_2(s)) + r_-(y, s) + r_e(y, s). \] (2.21)

Note that this decomposition depends on time \( s \) appearing in the cut-off (2.15). If ever \( r(y) = r(y, s) \), then this gives a natural choice for the parameter \( s \) in the cut-off. If not, then we will specify which time is selected (this is the case in particular in Lemma 5.2 below).

### 3. The major steps of the proof assuming some technical results

Our method is in fact a refinement of the one used by Merle and Zaag in [54]. For that reason, we will follow the same framework of that paper, improving the estimates (and the presentation tool!). In order to keep the paper in a reasonable length, we will insist only on the improvements, and go rather quickly when the estimates are the same as in [54].

In order to achieve our goal in (2.11), we will in fact construct a solution of equation (2.6) trapped in some shrinking set, which is an improved version of the sets already introduced by Bricmont and Kupiainen in [3] and Merle and Zaag in [54]. More precisely, this is our improved version:

**Definition 3.1** (Shrinking set). For any \( K > 0, A > 0 \) and \( s > 0 \), we define \( V_{K,A}(s) \) as a subset of \( L^\infty(\mathbb{R}^N) \) satisfying

\[ |q_0(s)| \leq \frac{A}{s^2}, \quad |q_1(s)| \leq \frac{A}{s^2} \quad \text{and} \quad |q_2(s)| \leq \frac{A^2 \ln s}{s^2}, \]

\[ \frac{\|q_-(\cdot, s)\|_{L^\infty(\mathbb{R}^N)}}{1 + |y|^3} \leq \frac{A^6 \ln s}{s^2}, \quad \text{and} \quad \|q_e(\cdot, s)\|_{L^\infty(\mathbb{R}^N)} \leq \frac{A^7 \ln s}{s}, \]

where \( q_3(s), q_-(\cdot, s), \) and \( q_e(\cdot, s) \) are the components of \( q(\cdot, s) \) as defined in (2.21).

**Remark 3.2.** Note that the parameter \( K > 0 \) is involved in the decomposition (2.21), through the cut-off in (2.15). Note also that by definition, we immediately derive that if \( q \in V_{K,A}(s) \), then
\[ \|q(s)\|_{L^\infty} \leq C \frac{A^7 \ln s}{s}, \]
(3.1)
for some \( C = C(K) > 0 \). Thus, as announced right before this definition, our goal becomes to find some positive \( K, A \) and initial time \( s_0 \) so we are able to construct a solution of equation (2.6) satisfying
\[ \forall s \geq s_0, \quad q(s) \in V_{K,A}(s). \]
Naturally, we first start by specifying the form of initial data we are taking. As in [54], our initial data will depend on a set of parameters \((d_0, d_1) \in \mathbb{R}^{1+N}\), and will have a slightly modified form (this form was first used by Masmoudi and Zaag in [55]):

\[
\psi_{K,A}(s_0, d_0, d_1, y) = \psi(s_0, d_0, d_1, y) = \frac{A}{s_0^2}(d_0 + d_1 \cdot y) \chi_0 \left( \frac{2|y|}{K \sqrt{s_0}} \right),
\]

where \(\chi_0\) is defined (2.16) (note that we may or may not keep the parameters in our notation for initial data). In the following, we identify a set for the parameters \(d_0\) and \(d_1\), so that at time \(s = s_0\), the solution is already in \(V_{K,A}(s_0)\). We also estimate the other components of the decomposition (2.21).

**Proposition 3.3** (Preparing initial data). For all \(K \geq 1\) and \(A \geq 1\), we can define \(s_1(K, A) \geq 1\) such that for all \(s_0 \geq s_1\) the following holds with \(\psi = \psi(s_0, d_0, d_1)\) defined in (3.2):

(i) There exits \(D_{s_0} \subset [-2, 2]^{1+N}\) such that the function

\[
\begin{align*}
\Gamma : \mathbb{R}^{1+N} & \to \mathbb{R}^{1+N} \\
(d_0, d_1) & \mapsto (\psi_0, \psi_1)(s_0),
\end{align*}
\]

is affine, one to one from \(D_{s_0}\) to \(\hat{V}_A(s_0)\), where

\[
\hat{V}_A(s) = \left[-\frac{A}{s^2}, \frac{A}{s^2}\right]^{1+N}.
\]

In addition, we have

\[
|\psi_0| \leq \frac{A}{s_0^2}, \quad |\psi_1| \leq \frac{A}{s_0^2}, \quad |\psi_2| \leq \frac{1}{s_0^2},
\]

and

\[
\|\psi(\cdot)\|_{W^{1,\infty}(\mathbb{R}^N)} \leq \frac{\ln s_0}{s_0^2}, \quad \psi_\varepsilon(\cdot) \equiv 0,
\]

where \(\psi_0, \psi_1, \psi_2, \psi_\varepsilon\) and \(\psi_\varepsilon\) are the components of \(\psi\) as in (2.17).

**Proof.** The proof is quite similar to [67, Proposition 4.5], except for the gradient estimate, which follows very easily by the same techniques. For that reason, we omit the proof. \(\square\)

Now, we would like to introduce the following parabolic regularity estimate, valid for solutions of equation (2.6) starting from initial data (3.2).

**Proposition 3.4** (Parabolic regularity). For all \(K \geq 1\) and \(A \geq 1\), there exists \(s_2(K, A) \geq 1\) such that for all \(s_0 \geq s_2\) the following holds:

Consider \(q\) is a solution to (2.6) on \([s_0, s^*]\) for some \(s^* > s_0\) with \(q(s_0) = \psi(s_0, d_0, d_1)\) defined as in (3.2) for some \((d_0, d_1) \in D_{s_0}\) given in Proposition 3.3. Assume in addition that \(q(s) \in V_{K,A}(s)\) for all \(s \in [s_0, s^*]\). Then, it follows that

\[
\|\nabla q(s)\|_{L^\infty} \leq \frac{\tilde{C} A^7 \ln s}{s}, \quad \forall s \in [s_0, s^*],
\]

where \(\tilde{C}\) depends only on \(K\).
Proof. We follow the technique first introduced by Ebde and Zaag [29] then used by Tayachi and Zaag [67]. Since this step is crucial for the improvement, we give the proof, with 2 cases, depending on the position of $s$ with respect to $s_0 + 1$.

- **Case 1:** $s \leq s_0 + 1$. We write equation (2.6) in its integral form

$$q(s) = e^{(s-s_0)\mathcal{L}}q(s_0) + \int_{s_0}^{s} e^{(s-\tau)\mathcal{L}}G(\tau)d\tau,$$

(3.7)

where

$$G(\tau) = V(\tau)q(\tau) + B(q(\tau)) + R(\tau),$$

(3.8)

and $e^{t\mathcal{L}}$ is the semi-group corresponding to $\mathcal{L}$ defined in (2.7) (see [67] for more details). We now consider $s_1' = \min(s_0 + 1, s^*)$ and $s \in [s_0, s_1']$. Taking $s_0 \geq 1$, we see that for all $\tau \in [s_0, s]$, we have

$$s_0 \leq \tau \leq s \leq s_0 + 1 \leq 2s_0,$$

which implies that

$$\frac{1}{s} \leq \frac{1}{\tau} \leq \frac{1}{s_0} \leq \frac{2}{s}.$$

(3.9)

Now, applying the operator $\nabla$ to (3.7), we write

$$\nabla q(s) = \nabla e^{(s-s_0)\mathcal{L}}q(s_0) + \int_{s_0}^{s} \nabla e^{(s-\tau)\mathcal{L}}G(\tau)d\tau.$$

(3.10)

Let us mention to item (ii) of Lemma 4.15 in [67] that

- if $f \in L^\infty(\mathbb{R}^N)$, then $\|\nabla e^{\theta\mathcal{L}}f\|_{L^\infty} \leq \frac{Ce^\theta}{\sqrt{1-e^{-\theta}}}\|f\|_{L^\infty}$,

(3.11)

and

- if $f \in W^{1,\infty}(\mathbb{R}^N)$, then $\|\nabla e^{\theta\mathcal{L}}f\|_{L^\infty} \leq Ce^{\frac{\theta}{2}}\|\nabla f\|_{L^\infty}$,

(3.12)

for some $\theta > 0$.

Using (3.5) and (3.12) with $\theta = s - s_0 \leq 1$, we obtain

$$\|\nabla e^{(s-s_0)\mathcal{L}}q(s_0)\|_{L^\infty} \leq Ce^{\frac{s-s_0}{2}}\|\nabla q(s_0)\|_{L^\infty} \leq \frac{C}{s_0} \leq \frac{2C}{s}.$$  

(3.13)

In addition to that, we see from (3.8) that for $\tau \in [s_0, s]$,

$$|G(\tau)| \leq |V(\tau)q(\tau)| + |B(q(\tau))| + |R(\tau)|.$$

Since $V$ is bounded by definition (2.8) and (3.1) holds from the fact that $q(s) \in V_{K,A}(s)$ for all $s \in [s_0, s^*]$, it follows that

$$|V(\tau)q(\tau)| \leq |V(\tau)||q(\tau)| \leq \frac{CA^7\ln \tau}{\tau},$$

(with $C = C(K)$ here). Using again (3.1) together with Lemma A.1, we write

$$|B(q(\tau))| \leq C|q(\tau)|^{\bar{p}} \leq C\left(\frac{CA^7\ln \tau}{\tau}\right)^{\bar{p}} \leq C\frac{\ln \tau}{\tau},$$

for $s_0$ large enough, since $\bar{p} = \min(p, 2) > 1$. We also have

$$|R(\tau)| \leq \frac{C}{\tau}.$$

Combining these estimates we deduce

$$\|G(\tau)\|_{L^\infty} \leq \frac{CA^7\ln \tau}{\tau} \leq \frac{2CA^7\ln s}{s},$$

(3.14)
thanks to (3.9). Consequently, applying (3.11) with \( \theta = s - \tau \in [0, s - s_0] \subset [0, 1] \), we get
\[
\| \nabla e^{(s-\tau)L}G(\tau) \|_{L^\infty} \leq \frac{C}{\sqrt{1 - e^{-(s-\tau)}}} \frac{A^7 \ln s}{s}. \tag{3.14}
\]
Finally, taking the \( L^\infty \) norm of (3.10) and using again (3.9), we get
\[
\| \nabla q(s) \|_{L^\infty} \leq \| \nabla e^{(s-s_0)L}q(s_0) \|_{L^\infty} + \int_{s_0}^{s} \| \nabla e^{(s-\tau)L}G(\tau) \|_{L^\infty} d\tau
\]
\[
\leq \frac{C}{s} + \frac{CA^7 \ln s}{s} \int_{s_0}^{s} \frac{1}{\sqrt{1 - e^{-(s-\tau)}}} d\tau
\]
\[
\leq \frac{C_1(K)A^7 \ln s}{s}.
\]
- **Case 2:** We consider the case \( s > s_0 + 1 \), hence \( s^* \geq s > s_0 + 1 \). Taking \( s_0 \geq 1 \), it follows that \( s \geq s_0 + 1 \geq 2 \) and \( s = s - 1 + 1 \leq 2(s - 1) \).

Note that for all \( \tau \in [s - 1, s] \), we have
\[
\frac{1}{s} \leq \frac{1}{\tau} \leq \frac{2}{s}. \tag{3.15}
\]
Using Duhamel’s principle with initial time \( s - 1 \), we have
\[
q(s) = e^Cq(s - 1) + \int_{s - 1}^{s} e^{(s-\tau)L}G(\tau) d\tau.
\]
As for (3.10), we get
\[
\nabla q(s) = \nabla e^{C}q(s - 1) + \int_{s - 1}^{s} \nabla e^{(s-\tau)L}G(\tau) d\tau. \tag{3.16}
\]
Proceeding as for (3.13) and (3.14), using in particular (3.11), (3.1) and (3.15), we get
\[
\| \nabla e^{C}q(s - 1) \|_{L^\infty} \leq \frac{C}{\sqrt{1 - e^{-\tau}}} \| q(s - 1) \|_{L^\infty} \leq \frac{C A^7 \log(s - 1)}{s - 1} \leq 2C \frac{A^7 \log s}{s},
\]
\[
\| \nabla e^{(s-\tau)L}G(\tau) \|_{L^\infty} \leq \frac{CA^7 \ln \tau}{\sqrt{1 - e^{-(s-\tau)}}} \leq \frac{2CA^7 \ln s}{\sqrt{1 - e^{-(s-\tau)}}}. \tag{3.17}
\]
Using the inequalities and (3.16), we obtain
\[
\| \nabla q(s) \|_{L^\infty} \leq \| \nabla e^{C}q(s - 1) \|_{L^\infty} + \int_{s - 1}^{s} \| \nabla e^{(s-\tau)L}G(\tau) \|_{L^\infty} d\tau
\]
\[
\leq \frac{2CA^7 \ln s}{s} + \frac{2CA^7 \ln s}{s} \int_{s - 1}^{s} \frac{1}{\sqrt{1 - e^{-(s-\tau)}}} d\tau \leq \frac{C_2(K)A^7 \ln s}{s}.
\]
Finally, introducing \( \tilde{C}(K) = \max(C_1, C_2) \), we obtain
\[
\| \nabla q(s) \|_{L^\infty} \leq \frac{\tilde{C}A^7 \ln s}{s}, \forall s \in [s_0, s^*],
\]
which concludes the proof.

Recalling what we wrote in the remark following Definition 3.1 and in (3.2), our goal is to construct initial data (at time \( s = s_0 \)) \( \psi(s_0, d_0, d_1) \), so that the solution of (2.6) exists for all \( s \in [s_0, \infty) \) and satisfies
\[
q(s) \in V_{K,A}(s), \forall s \geq s_0. \tag{3.17}
\]
This is possible, as we state in the following:
Proposition 3.5 (Existence of a solution to (2.6) trapped in $V_{A,K}$). There exist $A, K \geq 1$, $s_0(A,K) \geq 1$ large enough, and $(d_0, d_1) \in \mathbb{R}^{1+N}$ such that the solution to equation (2.6), corresponding to initial data (at $s = s_0$) $\psi(s_0, d_0, d_1)$ defined in (3.2), exists for all $s \in [s_0, +\infty)$ and satisfies

$$q(s) \in V_{K,A}(s), \forall s \in [s_0, +\infty),$$

where the shrinking set $V_{K,A}$ is introduced in Definition 3.1.

Proof. In fact, the proof completely follows the method used in [3] and [54] and based on two main arguments:

- **Reduction to a finite dimensional problem:** In this step, we show that the control of $q(s)$ in the shrinking set $V_{K,A}(s)$ reduces to the control of $(q_0(s), q_1(s))$, the $(N+1)$-dimensional variable corresponding to the projection of $q(s)$ on $1$ and $y$, the expanding directions of the linear operator $\mathcal{L}$ (2.7) involved in equation (2.6).

- **Topological argument:** The control of the $(N+1)$-dimensional variable is then performed thanks to a topological argument linked to Brouwer’s lemma, where we fine-tune the $(N+1)$-dimensional variable $(d_0, d_1)$ appearing in initial data $\psi(s_0, d_0, d_1, y)$ (3.2).

Let us now briefly present these two arguments. Consider $A \geq 1, K > 1$ and $s_0(A,K)$ large enough such that Propositions 3.3 and 3.4 hold. Other restrictions on these constants will appear below in Proposition 3.6. Proceeding by contradiction, we assume that for all $(d_0, d_1) \in \mathcal{D}_{s_0}$, there exists $s(d_0, d_1) \geq s_0$ such that

$$q(s(d_0, d_1)) \notin V_{K,A}(s(d_0, d_1)),$$

where $q_{d_0,d_1}$ (we write $q$ for simplicity) is the solution to equation (2.6) corresponding to initial data $\psi(s_0, d_0, d_1)$ given at (3.2). Since $q(s_0) \in V_{K,A}(s_0)$ by item (ii) in Proposition 3.3, we can define $s_*(d_0, d_1) < +\infty$ such that

$$s_*(d_0, d_1) = \sup\{s_1 \geq s_0 \text{ such that } q(s) \in V_{K,A}(s), \forall s \in [s_0, s_1]\}.$$  

From continuity in time of the solution, it follows that $q(s_*(d_0, d_1)) \in \partial V_{K,A}(s_*(d_0, d_1))$. Here comes our first argument where we reduce the problem to a finite-dimensional one. More precisely, we have the following statement:

Proposition 3.6 (Reduction in finite dimensions and transverse crossing). There exist parameters $K \geq 1, A \geq 1$ and $s_0(A,K) \geq 1$ such that the following property holds (in addition to Propositions 3.3 and 3.4): Assume that

(a) Initial data $q(s_0) = \psi(s_0, d_0, d_1)$, given (3.2) for some $(d_0, d_1) \in \mathcal{D}_{s_0}$,

(b) $q(s) \in V_{K,A}(s)$, for all $s \in [s_0, s_*]$ for some $s_* \geq s_0$ with

$$q(s_*) \in \partial V_{K,A}(s_*).$$

Then, we have:

(i) (Reduction to finite dimensions): It holds that $(q_0(s_*), q_1(s_*)) \in \partial \hat{V}_A(s_*)$ defined in (3.3).

(ii) (Transverse crossing): There exists $\nu_0 > 0$ such that

$$\forall \nu \in (0, \nu_0), (q_0, q_1)(s_* + \nu) \notin \hat{V}_A(s_* + \nu),$$

which implies that

$$q(s_* + \nu) \notin V_{K,A}(s_* + \nu).$$

Let us assume this result and continue the proof of Proposition 3.5. We postpone the proof of Proposition 3.6 later to page 19 in Section 5, since it is long and technical. Adjusting the constants $A, K$ and $s_0$ as suggested in Proposition 3.6, we see that

$$(q_0, q_1)(s_*(d_0, d_1)) \in \partial \hat{V}_A(s_*(d_0, d_1)).$$
Therefore, by definition (3.3) of $\hat{V}_A(s)$, we can define $\Gamma$ as follows:

$$
\Gamma : \mathcal{D}_{s_0} \to \partial[-1,1]^{1+N}
$$

$$(d_0, d_1) \mapsto \Gamma(d_0, d_1) = \frac{s^2(d_0, d_1)}{A} (q_0, q_1) (s^*(d_0, d_1)).$$

Note then we have the following properties:

(i) $\Gamma$ is continuous from $\mathcal{D}_{s_0}$ to $\partial[-1,1]^{1+N}$ thanks to the continuity in time of $q$ on the one hand, and the continuity of $s^*(d_0, d_1)$, which is a direct consequence of the transverse crossing given in item (ii) of Proposition 3.6 above.

(ii) If $(d_0, d_1) \in \partial\mathcal{D}_{s_0}$, then we see from item (i) in Proposition 3.3 that $(q_0(s_0), q_1(s_0)) \in \partial\hat{V}_A(s_0)$. Using again the transverse crossing property, we see that $s^*(d_0, d_1) = s_0$. Hence, $\Gamma|_{\partial\mathcal{D}_{s_0}}$ is equal to $\hat{\Gamma}|_{\partial\mathcal{D}_{s_0}}$, where $\hat{\Gamma}$ was introduced in item (i) of Proposition 3.3. In particular, $\Gamma$ has a non-zero degree on the boundary.

From a corollary of Brouwer’s lemma, this is a contradiction. This concludes the proof of Proposition 3.5, assuming that Proposition 3.6 holds. Once that proof is given in Section 5, the proof will be complete.

4. The proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1.

Proof.

(i) Consider the constants $K$, $A$ and $s_0$ given in Proposition 3.5, and $q(y, s)$ the solution of equation (2.6) constructed there so that (3.18) holds. Thanks to (3.1) and Proposition 3.4, it follows that

$$
\forall s \geq s_0, \|q(s)\|_{L^\infty} + \|\nabla q(s)\|_{L^\infty} \leq C(K, A) \frac{\ln s}{s}.
$$

Introducing $T = e^{-s_0}$ then defining $w(y, s)$ and $u(x, t)$ by (2.5) and (2.1), we easily see from Section 2 that $u(x, t)$ is a solution to equation (1.1) that exists for all $(x, t) \in \mathbb{R}^N \times [0, T)$ such that (1.5) and (1.6) hold. In particular, we see that

$$
\left|\nabla u(z_0 \sqrt{T - t} | \ln(T - t), t)\right| \to \infty \text{ as } t \to T,
$$

for any $z_0 \neq 0$, hence, the origin is a blow-up point both for $u$ and $\nabla u$. The fact that neither $u$ nor $\nabla u$ blows up outside the origin is in fact a consequence of item (ii).

(ii) The existence of $u^* \in C^2(\mathbb{R}^N \setminus \{0\})$ such that $u(\cdot, t) \to u^*(\cdot)$ uniformly in $C^2$ of any compact set of $\mathbb{R}^N \setminus \{0\}$ follows from the classical argument given in Merle [44]. It remains to give the proofs of (1.8) and (1.9). Let us consider $x_0 \neq 0, |x_0| \leq \varepsilon_0$ small enough. We then define

$$
\begin{align*}
\mathcal{U}(x_0, \xi, \tau) &= \left(T - t(x_0)\right)^{\frac{1}{p-1}} u(x_0 + \xi \sqrt{T - t(x_0)}, t(x_0) + \tau(T - t(x_0))), \\
\mathcal{V}(x_0, \xi, \tau) &= \nabla_\xi \mathcal{U}(x_0, \xi, \tau).
\end{align*}
$$

Consider some $K > 0$. We remark that once $\varepsilon_0$ is fixed small enough, there uniquely exists $t(x_0) < T$ such that

$$
|x_0| = K \sqrt{(T - t(x_0)) | \ln(T - t(x_0))|}.
$$

(4.2)

It is obvious that

$$
t(x_0) \to T \text{ as } x_0 \to 0.
$$

(4.3)
Applying \( t = t(x_0) \) to (1.5) and (1.6), we derive from the definition (1.7) of the profile \( \varphi_0 \) that
\[
|U(x_0, \xi, 0) - (p - 1 + bK^2)^{-\frac{1}{p-1}}| \leq \frac{C}{|\ln(T - t(x_0))|^{\frac{1}{4}}},
\]
(4.4)
\[
\left|V(x_0, \xi, 0) + \frac{x_0}{|x_0|} \cdot \frac{2bK}{(p - 1)\sqrt{|\ln(T - t(x_0))|}} (p - 1 + bK^2)^{-\frac{p}{p-1}}\right| \leq \frac{C}{|\ln(T - t(x_0))|^{\frac{1}{4}}},
\]
(4.5)
for all \( |\xi| \leq |\ln(T - t(x_0))|^{\frac{1}{4}} \). Next, let us define for \( t \in [0, 1) \)
\[
\dot{U}(\tau) = ((p - 1)(1 - \tau) + bK^2)^{-\frac{1}{p-1}},
\]
(4.6)
\[
\dot{V}(\tau) = -\frac{x_0}{|x_0|} \cdot \frac{2bK}{(p - 1)\ln(T - t(x_0))^{\frac{1}{2}}} ((p - 1)(1 - \tau) + bK^2)^{-\frac{p}{p-1}}.
\]
(4.7)
In particular, these functions solve the following system
\[
\dot{U}(\tau) = U^p(\tau),
\]
(4.8)
\[
\dot{V}(\tau) = pU^{p-1}(\tau)\dot{V}(\tau),
\]
\[
\sup_{|\xi| \leq \frac{1}{4}|\ln(T - t(x_0))|^{\frac{1}{4}}, \tau \in [0, 1)} |U(x_0, \xi, \tau) - \dot{U}(\tau)| \leq \frac{C}{1 + |\ln(T - t(x_0))|^{\frac{1}{4}}},
\]
(4.9)
\[
\sup_{|\xi| \leq \frac{1}{4}|\ln(T - t(x_0))|^{\frac{1}{4}}, \tau \in [0, 1)} |V(x_0, \xi, \tau) - \dot{V}(\tau)| \leq \frac{C}{1 + |\ln(T - t(x_0))|^{\frac{1}{4}}},
\]
(4.10)
- **Proof of (4.9):** We don’t give the details, since the result is quite the same as in [27, page 1564]. In particular, it is independent from (4.10). Roughly speaking, up to some intricate cut-off estimates, we are simply using the continuity with respect to initial data for solutions of equation (1.1), since \( U \) is indeed a solution of that equation, thanks to the scaling invariance, and so does \( \dot{U} \), thanks to (4.8).
- **Proof of (4.10):** This step is new, though the idea behind it is the same as for (4.9): the continuity with respect to initial data of the PDE obtained by differentiation of (1.1), up to some cut-off estimates. However, one should bear in mind that without the sharp improvement we have made in the definition of the shrinking set (see Definition 3.1 above), this step is impossible to carry on. As a matter of fact, this is the very place where one understands how crucial is our contribution in this paper.

Using (1.1) and definition (4.1), we have
\[
\partial_\tau U = \Delta U + |U|^{p-1}U,
\]
(4.11)
and since (4.9) ensures that \( U > 0 \) on \( \{ |\xi| \leq \frac{1}{4}|\ln(T - t(x_0))|^{\frac{1}{4}} \} \), then, we readily derive the following PDE satisfied by \( \dot{V} \):
\[
\partial_\tau \dot{V} = \Delta \dot{V} + pU^{p-1}\dot{V}.
\]
(4.12)
By using the parabolic regularity for equation (4.12), together with (4.5) and (4.9), we derive the following rough estimate:
\[
\sup_{|\xi| \leq \frac{1}{4}|\ln(T - t(x_0))|^{\frac{1}{4}}, \tau \in [0, 1)} |\dot{V}(x_0, \xi, \tau)| \leq \frac{C(K)}{\sqrt{|\ln(T - t(x_0))|}}.
\]
(4.13)
We now introduce
\[
\dot{V}(x_0, \xi, \tau) = V(x_0, \xi, \tau) - \dot{V}(\tau),
\]
(4.14)
\[
\left|\dot{V}(x_0, \xi, \tau)\right| \leq \frac{C(K)}{\sqrt{|\ln(T - t(x_0))|}}.
\]
(4.15)
and
\[ \hat{V}(x_0, \xi, \tau) = \chi_1(\xi) \hat{V}(x_0, \xi, \tau), \]
where \( \chi_1(\xi) = \chi_0 \left( \frac{16|\xi|}{\ln(T - t(x_0))^{\frac{3}{2}}} \right), \)
and \( \chi_0 \) defined in (2.16). Note that \( V \) is a vector with \( V = (V_1, \ldots, V_N) \). We can write the following equation satisfied by \( \hat{V}_2 \):
\[ \partial_t \hat{V}_2 = \Delta \hat{V}_2 + p\mathcal{U}^{p-1} \hat{V}_2 + G(\xi, \tau), \quad (4.14) \]
where
\[ G(\xi, \tau) = \Delta \chi_1 V + \Delta \chi_1 \hat{V} - 2(\text{div}(\nabla \chi_1 V_1), \ldots, \text{div}(\nabla \chi_1 V_N)) + p\chi_1 \hat{V}(\mathcal{U}^{p-1} - \hat{\mathcal{U}}^{p-1}(\tau)). \]
As a matter of fact, using (4.13) and \( \chi_1 \)'s definition, we have
\[ |\nabla \chi_1 V_j| + |\Delta \chi_1 V| \leq \frac{C}{\ln(T - t(x_0))^{\frac{3}{4}}}, \]
for any \( j = 1, \ldots, N \).

In addition to that, we derive from (4.9) and (4.7) that
\[ |\chi_1 \hat{V}(\mathcal{U}^{p-1} - \hat{\mathcal{U}}^{p-1}(\tau))| \leq \frac{C}{\ln(T - t(x_0))^{\frac{3}{4}}}. \]

Recall the facts that for all \( s - \sigma > 0, g \in L^\infty \) and \( \bar{h} = (h_1, \ldots, h_N), h_i \in L^\infty \), we have
\[ \|e^{(s-\sigma)\Delta} g\|_{L^\infty} \leq \|g\|_{L^\infty} \quad \text{and} \quad \|e^{(s-\sigma)\Delta} \text{div}(\bar{h})\|_{L^\infty} \leq \frac{C}{\sqrt{s - \sigma}} \|\bar{h}\|_{L^\infty}, \]
where \( \|\bar{h}\|_{L^\infty} = \sum_{i=1}^N \|h_i\|_{L^\infty} \). Then, it follows from (4.15) that
\[ \|e^{(s-\sigma)\Delta} G\|_{L^\infty} \leq \frac{C}{\ln(T - t(x_0))^{\frac{3}{4}}} \left( 1 + \frac{1}{\sqrt{s - \sigma}} \right) \text{ for all } s - \sigma > 0. \]

Now, we rewrite (4.14) under integral form
\[ \hat{V}_2(\tau) = e^{\tau \Delta} \hat{V}_2(0) + \int_0^\tau e^{(s-\sigma)\Delta} (p\mathcal{U}^{p-1} \hat{V}_2 + G) d\sigma. \]
Taking the \( L^\infty \) norm, then using (4.5) and (4.15), together with the definitions of \( \hat{V}_2 \) and \( \hat{V} \), we obtain
\[ \|V_2(\tau)\|_{L^\infty} \leq \frac{C}{\ln(T - t(x_0))^{\frac{3}{4}}} + C \int_0^\tau \|V_2(\sigma)\|_{L^\infty} d\sigma. \]

Thanks to Grönwall’s lemma, we derive that
\[ \|\hat{V}_2(\tau)\|_{L^\infty} \leq \frac{C(K_0)}{\ln(T - t(x_0))^{\frac{3}{4}}}, \text{ for all } \tau \in [0, 1), \]
which concludes (4.10).
In addition to that, we apply a parabolic regularity technique to (4.11) and (4.12) (see more in [67]) to get
\[ \forall \tau \in \left[ \frac{1}{2}, 1 \right) \text{ and } |\xi| \leq \frac{1}{16} \ln(T - t(x_0))^{\frac{3}{4}}, |\partial_\tau \mathcal{U}(x_0, \xi, \tau)| + |\partial_\tau \mathcal{V}(x_0, \xi, \tau)| \leq C, \]
which ensures the existence of \( \lim_{\tau \to 1} \mathcal{U}(x_0, 0, \tau) \) and \( \lim_{\tau \to 1} \mathcal{V}(x_0, 0, \tau) \). We now give the final asymptotic behaviors to \( u^* \) and \( \nabla_x u^* \): From (4.2) and (4.3), we get
\[ \ln(T - t(x_0)) \sim 2 \ln |x_0| \quad \text{and} \quad T - t(x_0) \sim \frac{1}{K^2 2} \frac{|x_0|^2}{\ln |x_0|}, \]
as \( x_0 \to 0 \). Taking \( \xi = 0 \) and making \( \tau \to 1 \) in (4.9) and (4.10), we obtain
\[
\begin{align*}
    u(x_0,T) &= (T-t(x_0))^{-\frac{1}{\nu-1}} \lim_{\tau \to 1} \mathcal{U}(x_0,0,\tau) \sim \left[ b \frac{|x_0|^2}{2 \ln |x_0|} \right]^{-\frac{1}{\nu-1}}, \\
    \nabla u(x_0,T) &= (T-t(x_0))^{-\frac{1}{\nu-1}} \frac{1}{\nu} \lim_{\tau \to 1} \mathcal{V}(x_0,0,\tau) \sim -\frac{\sqrt{2b}}{p-1} \frac{x_0}{|x_0|} \frac{1}{\sqrt{\ln |x_0|}} \left[ b \frac{|x_0|^2}{2 \ln |x_0|} \right]^{-\frac{1}{\nu-1} - \frac{1}{2}},
\end{align*}
\]
as \( x_0 \to 0 \). This concludes the proof of Theorem 1.1.

\[ \square \]

5. Reduction in finite dimensions and transverse crossing

We prove Proposition 3.6 in this section. We proceed in two parts:
- In Part 1, we understand the dynamics of equation (2.6) inside the new shrinking set given in Definition 3.1. This is in fact the heart of our argument.
- In Part 2, we conclude the proof of Proposition 3.6.

**Part 1: Dynamics of equation (2.6)**

In this part, we follow the dynamics of each component of the solution, as introduced in the expansion (2.21). This is our statement:

**Proposition 5.1** (Dynamics of equation (2.6)). There exist \( K_3 \geq 1 \) and \( A_3 \geq 1 \) such that for all \( K \geq K_3, A \geq A_3 \) and \( \ell^* > 0 \), there exists \( s_3(A,K,\ell^*) > 1 \) such that for all \( \sigma \geq s_0 \geq s_3 \), the following property is valid:

Consider \( q(y,s) \) a solution of equation (2.6) such that \( q(s) \in V_{K,A}(s) \), for all \( s \in [\sigma,\sigma + \ell] \) for some \( \ell \in [0,\ell^*] \). If \( \sigma = s_0 \), we further assume that \( q(y,s_0) = \psi(s_0,d_0,d_1,y) \) defined in (3.2), for some \( (d_0,d_1) \in D_{s_0} \) defined in Proposition 3.3.

Then, for all \( s \in [\sigma,\sigma + \ell] \), the following properties hold:

(i) (Unstable modes \( q_0 \) and \( q_1 \)):
\[
|q'_0(s) - q_0(s)| \leq \frac{\bar{C}}{s^2} \quad \text{and} \quad |q'_1(s) - \frac{1}{2}q_1(s)| \leq \frac{\bar{C}}{s^2},
\]
for some constant \( \bar{C} > 0 \).

(ii) (Neutral mode \( q_2 \)):
\[
|q'_2(s) + \frac{2}{s}q_2(s)| \leq \frac{CA}{s^3}, \tag{5.1}
\]

(iii) (Control of negative spectrum part \( q_- \) and outer part \( q_e \)):

- If \( \sigma > s_0 \), then we have
\[
\|q_-(-,s)\| \leq \frac{C \ln s}{s^{\frac{5}{2}}} \left( A^6 e^{-\frac{s-\sigma}{2}} + A^7 e^{-(s-\sigma)^2} + A^2 e^{(s-\sigma)(s-\sigma)^2} + 1 + s - \sigma \right), \tag{5.2}
\]
\[
\|q_e(-,s)\| \leq \frac{C \ln s}{s} \left( A^7 e^{-\frac{s-\sigma}{2}} + A^6 e^{s-\sigma} + s - \sigma \right). \tag{5.3}
\]

- If \( \sigma = s_0 \), then we have
\[
\|q_-(-,s)\| \leq \frac{C \ln s}{s^{\frac{5}{2}}} \left( 1 + s - s_0 \right), \tag{5.4}
\]
\[
\|q_e(-,s)\| \leq \frac{C \ln s}{s} \left( 1 + s - s_0 \right). \tag{5.5}
\]
Proof. The proof is quite the same as \cite[Lemma 4.1]{18}, except for the bounds involving \( q_- \) and \( q_e \) which are substantially improved. Note that the improvement allows us to describe the final profile of the gradient in Theorem 1.1. Let us mention in addition that the result on \( q_- \) and \( q_e \) follows from the result in \cite[Lemma 2.7]{58} (see below) combined with the new bounds in the shrinking set \( V_{K,A}(s) \). For that reason, we omit the details of the proofs relating to to the finite-dimensional modes \( q_0, q_1 \) and \( q_2 \), and focus on the proof of the estimates on \( q_- \) and \( q_e \). By Duhamel’s principle, equation (2.6) leads to the following integral form
\[
q(s) = K(s,\sigma)(q(\sigma)) + \int_\sigma^s K(s,\tau) (B(q) + R)(\tau) d\tau,
\]
where \( K(s,\sigma) \) is the fundamental solution associated to the linear operator \( L + V \) defined in (2.7) and (2.8).

Let us recall the following fundamental result whose proof goes back to Bricmont and Kupiainen \cite{3}, though it was not stated there in this form. In fact, we owe the following form to Nguyen and Zaag \cite{58} (see Lemma 2.7 in that paper):

**Lemma 5.2** (Bricmont-Kupiainen \cite{3} ; Dynamics of the linear operator \( L + V \)). There exists \( K^* \geq 1 \) such that for all \( K \geq K^* \) and \( \ell^* > 0 \), there exists \( s^*(K,\ell^*) \) such that for all \( \sigma \geq s^* \) and \( v \in L^\infty \) satisfying
\[
\sum_{j=0}^2 |v_j(\sigma)| + \left| \frac{v_- (\sigma)}{1 + |y|^3} \right|_{L^\infty} + \| v_e(\sigma) \|_{L^\infty} < +\infty,
\]
where the components defined as in (2.21) (with \( \chi(y,\sigma) \) as a cut-off function in (2.15)), it holds that for any \( s \in [\sigma, \sigma + \ell^*] \),
\[
\left| \frac{\theta_- (y,s)}{1 + |y|^3} \right|_{L^\infty} \leq Ce^{s-\sigma}((s-\sigma)^2 + 1) \left( |v_0(\sigma)| + |v_1(\sigma)| + \sqrt{s}|v_2(\sigma)| \right)
\]
\[
+ Ce^{-s-\sigma} \left| \frac{v_- (\sigma)}{1 + |y|^3} \right|_{L^\infty} + C e^{-s-\sigma} \left| v_e(\sigma) \right|_{L^\infty},
\]
and
\[
\| \theta_e(s) \|_{L^\infty} \leq Ce^{s-\sigma} \left( 2 \sum_{j=0}^2 s^{\frac{3}{2}} |v_j(\sigma)| + s^{\frac{3}{2}} \left| \frac{v_- (\sigma)}{1 + |y|^3} \right|_{L^\infty} \right) + Ce^{-s-\sigma} \left| v_e(\sigma) \right|_{L^\infty},
\]
where \( \theta(s) = K(s,\sigma)v \).

**Proof.** See Lemma 2.7 in Nguyen and Zaag \cite{58}, \( \square \)

Since \( q(s) \) appears in (5.6) as a sum of 2 terms, we proceed in 3 steps, with step 1 and 2 dedicated to each term. We then conclude the proof in Step 3. As the fundamental solution \( K(s,\sigma) \) is involved in both of them, Lemma 5.2 will be crucial here.

Let us first take \( K \geq K^* \) and \( \sigma \geq s_3(K,\ell^*) \geq s^*(K,\ell^*) \) large enough (where \( K^* \) and \( s^* \) are defined in Lemma 5.2), so that
\[
\text{if } \sigma \leq \tau \leq s \leq \sigma + \ell^*, \text{ then } \frac{1}{s} \leq \frac{1}{\tau} \leq \frac{1}{\sigma} \leq \frac{2}{s}, \tag{5.9}
\]

**Step 1: The estimate for \( K(s,\sigma)q(\sigma) \)**

Using the fact that \( q(\sigma) \in V_{K,A}(\sigma) \) and (5.9), we get
\[
|q_0(\sigma)| \leq \frac{A}{\sigma^2} \leq \frac{CA}{s^2}, \quad |q_1(\sigma)| \leq \frac{A}{\sigma^2} \leq \frac{CA}{s^2}, \quad |q_2(\sigma)| \leq \frac{A^2 \ln \sigma}{\sigma^2} \leq \frac{CA^2 \ln s}{s^2}, \quad \text{and} \quad \| q_e(\sigma) \|_{L^\infty} \leq \frac{A^7 \ln \sigma}{\sigma} \leq \frac{CA^7 \ln s}{s}.
\]
Then, applying (5.7) and (5.8) with \( v = q(\sigma) \) and \( \theta(s) = K(s, \sigma)v \), we see that
\[
\left\| \frac{\theta_-(s)}{1 + |y|^3} \right\|_{L^\infty} \leq \frac{C \ln s}{s^{\frac{7}{2}}} \left( A^6 e^{-\frac{5}{2}} + A^7 e^{-(s-\sigma)^2} + A^2 e^{(s-\sigma)^2}((s-\sigma)^2 + 1) \right),
\]
(5.10)
and
\[
\left\| \theta_e(s) \right\|_{L^\infty} \leq \frac{C \ln s}{s} \left( A^7 e^{-\frac{5}{2}} + A^6 e^{(s-\sigma)} \right).
\]
(5.11)

Now, if \( \sigma = s_0 \), we get better bounds for \( q(s_0) \) from item (ii) of Proposition 3.3, improving the estimates as follows:
\[
\left\| \frac{\theta_-(s)}{1 + |y|^3} \right\|_{L^\infty} \leq \frac{C \ln s}{s^{\frac{7}{2}}} \quad \text{and} \quad \left\| \theta_e(s) \right\|_{L^\infty} \leq \frac{C \ln s}{s}.
\]
(5.12)

**Step 2:** The estimate on \( \int_0^T K(s, \tau) (B(q) + R)(\tau) d\tau \)

Here, we need some technical and straightforward estimates on \( B(q) \) and \( R \), which are stated in Lemma A.1. For that reason, we will take \( \sigma \geq s_3'(K, A, \ell^s) \) for some \( s_3' = \max(s^*(\ell^s), s_3(\ell^s), s_5(K, A)) \), where \( s^*, s_3 \) and \( s_5 \) are defined respectively in Lemma 5.2, right before (5.9) and in Lemma A.1. Then, we consider \( \ell \in [0, \ell^s] \), \( s \in [\sigma, \sigma + \ell] \) and \( \tau \in [\sigma, s] \). Note in particular that (5.9) holds here. Applying item (ii) of Lemma A.1 together with straightforward standard techniques, we see that
\[
\left\| R(\cdot, \tau) \right\|_{L^\infty} \leq \frac{2C}{\tau} \leq \frac{4C}{s}, \quad \left\| R_e(\cdot, \tau) \right\|_{L^\infty} \leq \frac{4C}{s}.
\]
(5.13)

\[
\left| R(y, \tau) - \frac{a_0}{\tau^2} \right| \leq \frac{C(1 + |y|^3)}{s^3}, \forall |y| \leq 2K\sqrt{s}, \left| R(y, \tau) \right| \leq \frac{C(1 + |y|^3)}{\tau^2} \leq \frac{C(1 + |y|^3)}{s^{\frac{7}{2}}}, \forall |y| \geq K\sqrt{s},
\]
(5.14)

which yields
\[
\left| R_e(y, \tau) \right| \leq \frac{C(1 + |y|^3)}{s^{\frac{7}{2}}}, \forall y \in \mathbb{R}^N.
\]

Now, we focus on the estimate for \( B(q) \). Using the fact that \( q(\tau) \in V_{K,A}(\tau) \) by hypothesis, we see from (3.1) together with straightforward computations based on Definition 3.1 of \( V_{K,A} \) that
\[
\left\| q(\cdot, \tau) \right\|_{L^\infty} \leq \frac{CA^7 \ln s}{s},
\]
and
\[
|q(y, \tau)| \leq C \left( \frac{A}{s^2} (1 + |y|) + \frac{A^2 \ln s}{s^2} (1 + |y|^2) + \frac{A^6 \ln s}{s^5} (1 + |y|^3) + 1_{\{|y| \geq K\sqrt{s}\}} \right).
\]

Let us define \( f = B(q) \). Applying item (ii) of Lemma A.1 with \( \bar{p} = \min(p, 2) > 1 \), we see that
\[
|f| \leq C |q|^\bar{p} \leq C \left( \frac{A^7 \ln s}{s} \right)^\bar{p} \leq \frac{C}{s}
\]
(5.15)
for \( s \) large enough. In addition to that, for all \( |y| \leq 2K\sqrt{s} \), (ii) of Lemma A.1 ensures that
\[
|f(y)| \leq C(K) |q(y, \tau)|^2
\]
\[
\leq C(K) \left( \frac{A^2}{s^4} (1 + |y|^2) + \frac{A^4 \ln^2 s}{s^4} (1 + |y|^4) + \frac{A^6 \ln^2 s}{s^5} (1 + |y|^6) + 1_{\{|y| \geq K\sqrt{s}\}} \left( \frac{A^7 \ln s}{s} \right)^2 \right)
\]
\[
\leq \frac{C(K)(1 + |y|^3)}{s^2}
\]
(5.16)
for \( s \) large enough, since
\[
\frac{|y|}{\sqrt{s}} \leq \frac{|y|}{\sqrt{T}} \leq 2K \text{ if } |y| \leq 2K\sqrt{T} \text{ and } \frac{|y|}{\sqrt{s}} \geq \frac{K}{\sqrt{2}} \text{ if } |y| \geq K\sqrt{s}. \]
On the other hand, when $|y| \geq K\sqrt{\tau}$, we derive from (5.9) that
\[ \frac{1 + |y|^3}{s^2} \geq C^{-1} K^3, \]
hence, using (5.15), we see that
\[ |f(y)| \leq \frac{C(1 + |y|^3)}{s^2} \leq \frac{C(1 + |y|^3)}{s^2}. \] \hspace{1cm} (5.17)
Finally, using (5.15), (5.16) and (5.17), we see that
\[ |B(q)(y, \tau)| \leq \frac{C(1 + |y|^3)}{s^2} \quad \text{and} \quad |B(q)(y)| \leq \frac{C}{s}. \] \hspace{1cm} (5.18)
Let us define $v(\tau) = B(q)(\tau) + R(\tau)$ and $\theta(s, \tau) = K(s, \tau)v(\tau)$, using (5.13), (5.14) and (5.18) together with (5.9), we see that
\[ |v(y, \tau) - \frac{a_0}{\tau^2}| \leq \frac{C(1 + |y|^3)}{s^2}, \quad \forall |y| \leq 2K\sqrt{\tau}, |v(y, \tau)| \leq \frac{C(1 + |y|^3)}{s^2}, \quad \forall |y| \geq K\sqrt{\tau}, \quad \text{and} \quad \|v(\tau)\|_{L^\infty} \leq \frac{C}{s}. \]
Then, by definition (2.21) of the components of $v$, we have the following estimates:
\[ |v_0(\tau)| \leq \frac{C}{s^2}, \quad |v_m(\tau)| \leq \frac{C}{s^2}, \quad m = 1, 2 \quad \text{and} \quad |v_-(y, \tau)| \leq \frac{C(1 + |y|^3)}{s^2} \quad \text{and} \quad |v_e(y, \tau)| \leq \frac{C}{s}. \] \hspace{1cm} (5.19)
Recalling that $s - \sigma \leq s - \sigma \leq \ell \leq \ell^*$, then, applying (5.7) and (5.8), we see that
\[ |\theta_-(y, s, \tau)| \leq \frac{C\ln s}{s^2}(1 + |y|^3) \quad \text{and} \quad \|\theta_e(\cdot, s, \tau)\|_{L^\infty} \leq \frac{C\ln s}{s}. \] \hspace{1cm} (5.20)
for $s$ large enough.
Now, by definition (2.20) of the negative component, we write
\[
\left( \int_{\sigma}^{s} K(s, \tau) (B(q) + R)(\tau) d\tau \right)_-
\]
\[ = P_-(\chi(\cdot, s) \int_{\sigma}^{s} K(s, \tau)(B(q) + R)(\tau) d\tau) = P_-(\int_{\sigma}^{s} \chi(s)K(s, \tau)v(\tau) d\tau) \]
\[ = P_-(\int_{\sigma}^{s} \left[ \sum_{j=0}^{2} \theta_j(s, \tau)h_j + \theta_-(s, \tau) \right] d\tau) = P_-(\int_{\sigma}^{s} \theta_-(s, \tau) d\tau), \]
since
\[ P_-(\int_{\sigma}^{s} \left[ \sum_{j=0}^{2} \theta_j(s, \tau)h_j \right] d\tau) = 0 \]
by definition (2.20) of the projector $P_-$.
Let us recall the following obvious estimate : if $|g(y)| \leq m(1 + |y|^3)$, then, we have
\[ |P_-(g)(y)| \leq Cm(1 + |y|^3). \]
Using (5.20), we obtain
\[ \left| \int_{\sigma}^{s} \theta_-(s, \tau) d\tau \right| \leq \frac{C(s - \sigma)\ln s(1 + |y|^3)}{s^2}, \]
therefore,
\[ \left| \left( \int_{\sigma}^{s} K(s, \tau) (B(q) + R)(\tau) d\tau \right)_- \right| \leq \frac{C(s - \sigma)\ln s}{s^2}(1 + |y|^3). \] \hspace{1cm} (5.21)
Similarly, we have
\[
\left( \int_{\sigma}^{s} K(s, \tau) (B(q) + R)(\tau)d\tau \right)e = (1 - \chi(s)) \int_{\sigma}^{s} K(s, \tau) (B(q) + R)(\tau)d\tau = \int_{\sigma}^{s} (1 - \chi(s))K(s, \tau)v(\tau)d\tau = \int_{\sigma}^{s} \theta_e(s, \tau)d\tau,
\]
and from (5.20), we obtain
\[
\left\| \left( \int_{\sigma}^{s} K(s, \tau) (B(q) + R)(\tau)d\tau \right)e \right\|_{L^\infty} \leq \frac{C(s - \sigma) \ln s}{s}. \tag{5.22}
\]

**Step 3: Conclusion of the proof of Proposition 5.1**
Since the solution \( q(s) \) appears in (5.6) as a sum of 2 terms handled in Step 1 and Step 2, one has simply to add the appropriate estimates from those steps (namely (5.10), (5.11), (5.12), (5.21) and (5.22)) in order to get the desired estimates and conclude the proof of Proposition 5.1.

\[\square\]

**Part 2: Proof of Proposition 3.6**
We claim that Proposition 3.6 is a consequence of the following:

**Proposition 5.3** (Improvement on the non-expanding directions). There exists \( K_4 \geq 1 \) and \( A_4 \geq 1 \) such that for all \( K \geq K_4 \) and \( A \geq A_4 \), there exists \( s_4(K, A) \geq 1 \) such that for all \( s_0 \geq s_4 \), the following property holds: Assume that

(a) Initial data \( \psi_{d_0,d_1} \) defined by (3.2), for some \( (d_0,d_1) \in D_{s_0} \) given in Proposition 3.3.
(b) \( q(s) \in V_{K,A}(s) \), for all \( s \in [s_0, s^*] \), for some \( s^* \geq s_0 \).

Then, we have the following estimates for all \( s \in [s_0, s^*] \):
\[
|q_2(s)| \leq \frac{A^2 \ln s}{s^2} - s^{-3}, \quad \left\| \frac{\psi - (., s)}{1 + |\psi|^3} \right\|_{L^\infty} \leq \frac{A^6 \ln s}{2s^7} \quad \text{and} \quad \|q_e(., s)\|_{L^\infty(\mathbb{R}^N)} \leq \frac{A^7 \ln s}{2s} \tag{5.23}
\]

Let us first use this proposition to derive Proposition 3.6, then we will proof Proposition 5.3.

**Proof of Proposition 3.6 assuming that Proposition 5.3 holds**. Let us take \( K = K_4, A = \max(A_4, 2\bar{C}) \), where \( K_4 \) and \( A_4 \) appear in Proposition 5.3 and \( \bar{C} \) in item (i) of Proposition 5.1, \( s_0 = s_4(K, A) \), and assume hypotheses (a) and (b) stated in Proposition 3.6. Clearly, the hypotheses of Proposition 5.3 are satisfied. Thus, estimate (5.23) holds.

- **Proof of item (i):** Since \( q(s_\ast) \in \partial V_{K,A}(s_\ast) \) by hypothesis (b), using the Definition 3.1 of \( \tilde{V}_{K,A}(s_\ast) \), we see from (5.23) that only \( q_0(s_\ast) \) and \( q_1(s_\ast) \) may touch the edges of the interval \( [\frac{-A}{s^2}, \frac{A}{s^2}] \). By definition (3.3) of \( \tilde{V}_{A}(s_\ast) \), the conclusion follows.

- **Proof of item (ii):** Since \( (q_0(s_\ast), q_1(s_\ast)) \in \partial \tilde{V}_{A}(s_\ast) \) by item (i), it follows that either \( |q_0(s_\ast)| = \frac{A}{s^2} \) or \( |q_1(s_\ast)| = \frac{A}{s^2} \). Without loss of generality, we may assume that \( |q_0(s_\ast)| = \frac{A}{s^2} \) (the other case follows similarly). Hence, there exists \( \sigma_0 = 1 \) or \( -1 \) such that \( q_0(s_\ast) = \sigma_0 \frac{A}{s^2} \). From item (i) in Lemma 5.1, recalling that \( A \geq 2\bar{C} \), we see that
\[
\sigma_0 \frac{d}{ds} \left( q_0(s) - \sigma_0 \frac{A}{s^2} \right) \bigg|_{s = s_\ast} > 0,
\]
which shows that the flow of \( q_0 \) is transverse outgoing on the curve of \( \sigma_0 \frac{A}{s^2} \) at \( s = s_\ast \). This implies in fact that \( q_0(s) \) will indeed cross that curve, which concludes the proof of (3.19). This concludes also the proof of Proposition 3.6, assuming that Proposition 5.3 holds.

\[\square\]

It remains now to prove Proposition 5.3 in order to finish the proof of Proposition 3.6.
Proof of Proposition 5.3. The proof is quite the same as the proof of [54, Proposition 3.7], thanks to Proposition 5.1. For instance, the estimate for $q_2$ follows similarly as in that work, and the proof techniques for $q_-$ and $q_e$ are also the same. For these reasons, we will only give the proof for $q_-$. Let us consider $K \geq K_3$, $A \geq A_3$, $\ell^* = \ln A$ and $s_0 \geq s_3(A, K, \ell^*)$, where the constants $K_3$, $A_3$ and $s_3$ are defined in Proposition 5.1. Then, let us assume that hypotheses (a) and (b) stated in Proposition 5.3 hold, for some $s_* \geq s_0$.

Consider then $s \in [s_0, s_*)$ and let us prove the estimate on $q_-$ stated in (5.23). Clearly, we are allowed to apply Proposition 5.1. Two cases then arise:

+ If $s - s_0 \leq \ln A$, then, we clearly can apply estimate (5.4) in Proposition 5.1, with $\ell = s - s_0$ and $\sigma = s_0$, and derive that

$$\begin{align*}
\|q_-(\cdot, s)\|_{L^\infty} & \leq \frac{C \ln s}{s^2} (1 + \ln A) \leq \frac{A^6 \ln s}{2s^2},
\end{align*}$$

provided that $A \geq A_{4,1}$, for some $A_{4,1} \geq 1$.

+ If $s - s_0 \geq \ln A$, then, we can apply estimate (5.2) with $\sigma = s - \ln A$ and $\ell = \ln A (= \ell^*)$ and derive that

$$\begin{align*}
\|q_-(\cdot, s)\|_{L^\infty} & \leq C \frac{\ln s}{s^{3/2}} \left( A^6 e^{-\ln A} + A^7 e^{-(\ln A)^2} + A^2 e^{\ln A} (1 + (\ln A)^2) + \ln A \right) \leq \frac{A^6 \ln s}{2s^2},
\end{align*}$$

provided that $A \geq A_{4,2}$, for some $A_{4,2} \geq 1$. Finally, fixing $A_4 = \max(A_3, A_{4,1}, A_{4,2})$ then taking $A \geq A_4$, the result follows and conclude the proof of Proposition 5.3. Since we have already proved that Proposition 3.6 follows from Proposition 5.3, this concludes the proof of Proposition 3.6 too. \hfill \Box

A. Some technical estimates

In this section, we recall from previous literature some useful and straightforward estimates on $V$, $B(q)$ and $R$ defined in (2.8), (2.9) and (2.10). This is our statement:

Lemma A.1. For any $K \geq 1$ and $A \geq 1$, there exists $s_5(K, A)$ such that for any $s \geq s_5(K, A)$, if $q(s) \in \mathcal{V}_{K,A}(s)$, then, the following hold:

(i) Potential $V$ defined in (2.8):

\begin{align*}
\left| V(y, s) + \frac{1}{4s} (|y|^2 - 2N) \right| & \leq \frac{C(K)(1 + |y|^4)}{s^2}, \text{ whenever } |y| \leq 2K \sqrt{s}, \\
\text{and} \\
|V(y, s)| & \leq C, \forall y \in \mathbb{R}^N.
\end{align*}

(ii) Nonlinear term $B(q)$ defined in (2.9):

\begin{align*}
|B(q)| & \leq C(K)|q|^2, \forall |y| \leq 2K \sqrt{s}, \\
\text{and} \\
|B(q)| & \leq C|q|^\bar{p}, \bar{p} = \min(2, p).
\end{align*}

(iii) Remainder term $R$ defined in (2.10):

\begin{align*}
\left| R(y, s) - \frac{a_0}{s^2} \right| & \leq \frac{C(K)(1 + |y|^4)}{s^3}, \forall |y| \leq 2K \sqrt{s}, \\
\|R(\cdot, s)\|_{L^\infty} & \leq \frac{C}{s},
\end{align*}

for some $a_0 \geq 0$.

Proof. The proof can be found in [69, Lemma B.5], and [18, Lemma A.9 for $\alpha = 0$]. \hfill \Box
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