Research Article

Probe of Radiant Flow on Temperature-Dependent Viscosity Models of Differential Type MHD Fluid

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This paper numerically investigates the combined effects of the radiation and MHD on the flow of a viscoelastic Walters’ B liquid fluid model past a porous plate with temperature-dependent variable viscosity. To study the effects of variable viscosity on the fluid model, the equations of continuity, momentum with magnetohydrodynamic term, and energy with radiation term have been expanded. To understand the phenomenon, Reynold’s model and Vogel’s model of variable viscosity are also incorporated. The dimensionless governing equations are two-dimensional coupled and highly nonlinear partial differential equations. The highly nonlinear PDEs are transferred into ODEs with the assistance of suitable transformations which are solved with the help of numerical techniques, namely, shooting technique coupled with Runge–Kutta method and BVP4c solution method for the numerical solutions of governing nonlinear problems. Viscosity is considered as a function of temperature. Skin friction coefficient and Nusselt number are investigated through tables and graphs in the present probe. The behavior of emerging parameters on the velocity and temperature profiles is studied with the help of graphs. For Reynold’s model, we have shrinking stream lines and increasing three-dimensional graphs. y and Pr are reduced for both models.

1. Introduction

Non-Newtonian fluids have been a subject of great interest to researchers recently because of their various applications in industry and engineering. This is due to distinctive characteristics of such fluids in nature. In general, the mathematical problems in non-Newtonian fluids are more complicated because they are nonlinear and higher order than those in viscous fluids. Despite their complexities, scientists and engineers are engaged in non-Newtonian fluid dynamics. The analysis of boundary layer flow of viscous and non-Newtonian fluids has been the locus of extensive research by various scientists due to its importance in continuous casting, glass blowing, paper production, polymer extrusion, aero-dynamic extrusion of plastic sheet, and several others. Rajagopal et al. [1] have focused their research towards non-Newtonian fluid flows due to stretching of a flat surface. As far as literature survey is concerned many researchers [2–23] have worked on MHD radiation effects of viscous fluids.

Effects of thermal diffusion and chemical reaction on MHD flow of a dusty viscoelastic fluid have been inspected by Prakash et al. [34]. Abdul Hakeem et al. [24] have found the effect of heat radiation in Walters’ B fluid over a stretching sheet with nonuniform heat source/sink and elastic deformation. Recently, unsteady free convection flow in Walters’ B fluid and heat transfer analysis have been presented by Khan et al. [24]. Wang and Ng [25] investigated a similar flow to the present study but for an electrically nonconducting fluid and outside a magnetic field.

Uddin et al. [26] studied MHD flow bounded by a nonlinearly stretching surface with radiation. Brownian motion and thermophoresis in magnetohydrodynamic (MHD) bioconvection flow of nanoliquid via nonlinear thermal radiation is addressed by Makinde and Animasun [27]. The hydromagnetic pivot flows of an Oldroyd-B fluid in a porous medium was discussed by Khan et al. [28]. Khan et al. [29] studied the heat and mass transfer of viscoelastic MHD flow over a porous magnifying sheet with
degeneration of energy and stress work. The flows of Walters’ B fluid for numerical or applicable results for both steady and transient at great length in a distinct range of geometries using broad scale of analytical or computational approaches have been studied [30–32]. Prakash et al. [33] inspected the effects of chemical reaction and thermal diffusion on the MHD flow of a dusty viscoelastic fluid. The effect of heat radiation in Walters’ B fluid over a magnifying sheet with nonuniform elastic deformation and heat source was found by Abdul Hakeem et al. [34]. Khan et al. [24] represented the unsteady free convection flow in heat transfer analysis and Walters’ B fluid. Under different pressure gradients the thermal effects of a dusty viscoelastic fluid on unsteady fluid between two parallel plates was studied by Madhurai and Kalpana [35]. The objective here is to study numerically the combined effects of the radiation and MHD on the flow of a viscoelastic fluid model past a porous plate with temperature dependent variable viscosity. The problem is divided into two different parts in which the first part explicates that the plate has greater temperature than fluids temperature. The second part describes that plate is insulated. To understand the phenomenon, Reynold’s model and Vogel’s model of variable viscosity with magnetohydrodynamic and radiation effects of viscoelastic Walters’ B non-Newtonian fluid flow are incorporated. The shooting technique is habituated to attain the numerical solution of arising governing equations and solved with BVP4 software of Maple program. Three-dimensional and stream lines graphs were enlarged and reduced, respectively. The behavior of emerging parameters on the velocity and temperature profiles is studied with the help of graphs.

2. Mathematical Equations

The Cauchy stress tensor for Walters’ B fluid is given by

$$ S = -pI + 2\eta_0 e - 2k_0 \frac{\delta e}{\delta t}, $$

where pressure of the fluid is \( p \) and

$$ \eta_0 = \int_0^\infty N(\tau) d\tau, $$

$$ k_0 = \int_0^\infty \tau N(\tau) d\tau. $$

Here, \( N(\tau) \) is distribution function with relaxation time \( \tau \).

3. Physical Modeling of the Problem

The current problem gives the flow of a Walters’ B fluid flow past an infinite spongy plate.

The temperature and velocity fields are

$$ \theta = \theta(y), $$

$$ v = \nu(y)i + w(y)k. $$

The governing equations are

$$ \frac{\partial \rho}{\partial t} + \text{div}(\rho v) = 0, $$

$$ \frac{\rho}{\partial t} \frac{\partial v}{\partial t} = \text{div}(S) + \rho b, $$

$$ \frac{\partial e}{\partial t} = S.L - \text{div}(q) + \rho r. $$

For an incompressible fluid, (4) takes the form

$$ \text{div} v = 0, $$

$$ u = -v_0 = \text{Constant}, $$

where \( v_0 > 0 \) is suction and \( v_0 < 0 \) represents blowing at the plate. Momentum equation under the effects of magnetohydrodynamics for the current problem is

$$ \frac{\partial \rho}{\partial z} = \rho \nu_0 \frac{\partial w}{\partial x} + \frac{\partial \nu_0 \frac{\partial w}{\partial y} + \eta_0 \frac{\partial^2 w}{\partial y^2} + k_0 \nu_0 \frac{\partial^3 w}{\partial y^3} - \sigma B_0^2 w, $$

where \( \nu_0 > 0 \) is suction and \( \nu_0 < 0 \) represents blowing at the plate.}

Equations (8)–(10) with magnetohydrodynamic effects formed as

$$ \frac{\partial \rho}{\partial z} = \rho \nu_0 \frac{\partial w}{\partial x} + \frac{\partial \nu_0 \frac{\partial w}{\partial y} + \eta_0 \frac{\partial^2 w}{\partial y^2} + k_0 \nu_0 \frac{\partial^3 w}{\partial y^3} - \sigma B_0^2 w, $$

$$ \frac{\partial \rho}{\partial y} = 0, $$

$$ \frac{\partial \rho}{\partial x} = 0. $$

Define pressure as modified as

$$ \tilde{P} = P - (2k_0) \left( \frac{\partial w}{\partial y} \right)^2. $$

Equations (8)–(10) can be written as

$$ \rho \nu_0 \frac{\partial w}{\partial y} + \frac{\partial \nu_0 \frac{\partial w}{\partial y} + \eta_0 \frac{\partial^2 w}{\partial y^2} + k_0 \nu_0 \frac{\partial^3 w}{\partial y^3} - \sigma B_0^2 w = \rho \nu_0 \frac{\partial w}{\partial y} + \frac{\partial \nu_0 \frac{\partial w}{\partial y} + \eta_0 \frac{\partial^2 w}{\partial y^2} + k_0 \nu_0 \frac{\partial^3 w}{\partial y^3} - \sigma B_0^2 w = L_1, $$

where

$$ \frac{\partial \rho}{\partial z} = L_1. $$

The boundary conditions are

$$ w(0) = 0, $$

$$ w(y) \rightarrow W_{co}, \text{ as } y \rightarrow \infty. $$

As we have 3rd-order equation (15), so we need another boundary condition. Therefore, in free stream,
\[ S_{yx}\big|_{y\rightarrow \infty} = \left[ \eta_0 \frac{d^2 w}{dy^2} + k_0 \nu_0 \frac{d^3 w}{dy^3} \right]_{y\rightarrow \infty} = 0. \tag{18} \]

We use the following conditions:
\[ \frac{d\omega}{dy} = 0, \quad \text{as} \quad y \rightarrow \infty, \tag{19} \]
and also take another assumption that
\[ L_1 = 0. \tag{20} \]

Now, we are going to discuss the heat transfer in (6).
\[ q = -k \text{grad} \theta, \tag{21} \]
where \( q \) is heat flux. The radiation parameter is
\[ q_r = \frac{4aR}{3k} \frac{\partial \theta^4}{\partial x}. \tag{22} \]

Then
\[ k \frac{d^2 \theta}{dy^2} + \rho C_p \nu_0 \frac{d\theta}{dy} + \eta_0 \left( \frac{d\omega}{dy} \right) \frac{d^2 w}{dy^2} + \frac{1}{\rho C_p} \frac{d\theta}{dy} = 0. \tag{23} \]

\( C_p \) is specific heat and the boundary conditions for (21) are given in two parts as follows:

Part 1:
This part gives conditions for constant wall temperature of the fluid:
\[ \begin{align*}
\theta(0) &= \theta_0, \\
\theta(x) &\rightarrow 0_\infty, \quad \text{as} \quad x \rightarrow \infty. \tag{24}
\end{align*} \]

Case 2:
This gives insulated wall of the fluid
\[ \frac{d\theta}{dy}\big|_{y\rightarrow 0} = 0, \quad \theta(y) \rightarrow \theta_\infty, \quad \text{as} \quad y \rightarrow \infty. \tag{25} \]

3.1. Solution for Constant Wall Temperature. The dimensionless parameters can be defined as
\[ \begin{align*}
\bar{Y} &= \frac{Y}{L} \\
\bar{w} &= \frac{w}{W_\infty} \\
\bar{\theta} &= \frac{\theta - \theta_\infty}{\theta_0 - \theta_\infty}, \tag{26}
\end{align*} \]
where
\[ L = \frac{k_0 \nu_0}{\eta_0^*}, \tag{27} \]
is the characteristic "length" and also
\[ \bar{\eta}_0 = \frac{\eta_0}{\eta_0^*}. \tag{28} \]

Using the above relations, (15) and (23) become
\[ \begin{align*}
\frac{d^3 w}{dy^3} + \frac{d^3 w}{dy^3} + \gamma \frac{d\bar{\theta}}{dy} + \lambda \eta_0 \left( \frac{d\bar{\theta}}{dy} \right)^2 + \lambda \left( \frac{d\bar{\theta}}{dy} \right) \frac{d^3 w}{dy^3} &= 0. \tag{29}
\end{align*} \]

For simplicity, the bars are removed from (29)–(30) and get
\[ \begin{align*}
\frac{d^3 w}{dy^3} + \eta_0 \frac{d^3 w}{dy^3} + \gamma \frac{d\omega}{dy} + \lambda \eta_0 \frac{d\omega}{dy} &= \gamma M w = L_2, \tag{31}
\end{align*} \]

\[ \begin{align*}
\left( 1 + \frac{4}{3} R \right) \frac{d^2 \theta}{dy^2} + \gamma \eta_0 \frac{d\omega}{dy} + \lambda \eta_0 \left( \frac{d\omega}{dy} \right)^2 + \lambda \left( \frac{d\omega}{dy} \right) \frac{d^3 w}{dy^3} &= 0. \tag{32}
\end{align*} \]
where

\[ y = \frac{\rho k_0 W^2}{\eta_0}, \]
\[ M = \frac{\sigma B_0^2 K_0}{\rho \eta_0}, \]
\[ R = \frac{4\sigma^* \theta_\infty^3}{K K^*}, \]
\[ Pr = \frac{\eta_0^* C_k}{k}, \]
\[ \lambda = \frac{W_\infty^2 \eta_0}{k(\theta_0 - \theta_\infty)}. \]

Here, \( c \) is dimensionless length, \( M \) is MHD term coefficient, \( R \) is radiation, \( Pr \) is Prandtl number and \( \lambda \) is dimensionless quantity. The dimensionless boundary conditions are

\[ w(0) = 0, \]
\[ w \rightarrow 1 \text{ as } y \rightarrow \infty, \]
\[ \frac{dw}{dy} \rightarrow 0 \text{ as } y \rightarrow \infty, \]
\[ \theta(0) = 1, \]
\[ \theta \rightarrow 1 \text{ as } y \rightarrow \infty. \]

3.2. Solution for Insulated Plate. Here, we introduce non-dimensional temperature parameter

\[ \theta^* = \frac{\theta - \theta_\infty}{\theta_b - \theta_\infty}, \]

where \( \theta_b \) is bulk temperature. Eckert number is

\[ E^* = \frac{W_\infty^2}{c(\theta_b - \theta_\infty)}. \]

The boundary conditions for dimensionless flow are

\[ \frac{d\theta}{dy}\bigg|_{y=0} = 0, \]
\[ \theta(y) \rightarrow \theta_\infty \text{ as } y \rightarrow \infty. \]  

The skin friction and Nusselt number [7] are expressed as

\[ C_f = \frac{\tau_w}{(1/2)\rho W^2}, \]
\[ Nu = \frac{\gamma q_w}{k(T_w - T_\infty)} \]

where

\[ Nu = -\theta'(0). \]

4. Reynold’s Model

The viscosity for this model is expressed as

\[ \eta_0 = e^{-N\theta}, \]

which can be solved by using Maclaurin’s series as

\[ \eta_0 = 1 - N\theta. \]

Using the value of \( \eta_0 \) in (31) and (32), we obtain

\[ \frac{\rho C_p}{\rho} \frac{\partial q}{\partial y} = 0. \]
5. Vogel’s Model

In this case,
\[\mu_0 = \eta_0^* \exp \left( \frac{D}{(E + \theta)} \right),\]  \tag{46}

which implies the following.

\[
\frac{d^3 w}{dy^3} + \frac{G}{G^*} \frac{G^*}{G^*} \frac{d^2 w}{dy^2} \frac{d^2 w}{dy^2} + \gamma \frac{d w}{dy} \frac{d w}{dy} - \frac{DG}{G^*} \frac{d \theta}{dy} \frac{d \theta}{dy} - \gamma M w = L_2, \tag{48}
\]

\[
\left( 1 + \frac{4}{3} R \right) \frac{d^3 \theta}{dy^3} + \gamma \frac{Pr}{Pr} \frac{d \theta}{dy} + \frac{G}{G^*} \frac{G^*}{G^*} \left( \frac{dw}{dy} \right)^2 - \lambda \frac{DG}{G^*} \frac{d \theta}{dy} \left( \frac{dw}{dy} \right)^2 + \lambda \left( \frac{dw}{dy} \right)^2 \frac{d^2 w}{dy^2} - \frac{1}{\rho C_p} \frac{\partial q_r}{\partial y} = 0. \tag{49}
\]

6. Numerical Solution

For the purpose of numerical investigation, we have made comparison of our current article with three previous publications, which shows our results in this study are better than the previous literature [4, 12, 20]. The solution for (44) and (45) and (48) and (49) is obtained by using shooting technique with Runge–Kutta method [23, 36–38].

6.1. Solution for Reynolds’s Model. Equations (44) and (45) are for the desired form

\[
w'' = -\gamma w' - (1 - N \theta) w'' + N \theta' w' - \gamma M w, \tag{50}
\]

\[
\theta' = -\frac{1}{(1 + (4/3)R)} \left( \gamma Pr \theta' - \lambda (w')^2 - \lambda w' w'' + N \theta (w')^2 - \frac{1}{\rho C_p} \frac{\partial q_r}{\partial y} \right). \tag{51}
\]

Now, we define new variables,

\[
\begin{align*}
    w &= s_1, \\
    w' &= s_2, \\
    w'' &= s_3, \\
    w''' &= s_4, \\
    \theta &= s_5, \\
    \theta' &= s_6, \\
    \theta'' &= s_7.
\end{align*}
\]

By using new variables, we get

\[
\begin{align*}
    s_1' &= s_2, \\
    s_2' &= s_3, \\
    s_3' &= s_4, \tag{52}
\end{align*}
\]

\[
\begin{align*}
    s_4' &= -\gamma w' - (1 - N \theta) w'' + N \theta' w' - \gamma M w, \\
    s_6' &= -\frac{1}{(1 + (4/3)R)} \left( \gamma Pr \theta' - \lambda (w')^2 - \lambda w' w'' + N \theta (w')^2 - \frac{1}{\rho C_p} \frac{\partial q_r}{\partial y} \right).
\end{align*}
\]
Figure 1: Physical geometry of the problem.

Figure 2: Effects of $N$ on temperature portray for Reynold’s model.

Figure 3: Influences of $R$ on temperature outline for Reynold’s model.

Figure 4: Impact of on velocity field for Reynold’s model.
Figure 5: Effects of $W$ on velocity profile for Reynold's model.

Figure 6: Impact of $D$ on temperature outline for Vogel's model.

Figure 7: Effect of $G^*$ on temperature field for Vogel's model.

Figure 8: Impact of $G$ on temperature profile for Vogel's model.
Figure 9: Influence of on velocity profile for Vogel’s model.

Figure 10: Effect of $y$ on temperature outline for Vogel’s model.

Figure 11: Influences of $N$ on velocity field for Vogel’s model.

Figure 12: Impact of $R$ on temperature outline for Vogel’s model.
$\gamma = 0.1, W = 0.2.$

Figure 13: Effects of $N$ on velocity profile for Reynold’s model.

$N = 1.1, \lambda = 1.5, Pr = 7.7, R = 0.4.$

Figure 14: Impact of $\gamma$ on temperature field for Reynold’s model.

$N = 0.1, N = 0.5, N = 1.0$

Figure 15: Effects of $\gamma$ and $N$ on Nusselt number for Reynold’s model.

$C_f Re/2 D = 0.25$

Figure 16: Effects of $\gamma$ and $D$ on skin friction for Vogel’s model.
Along with boundary conditions,

\[ s_1(0) = 0, \]
\[ s_1(\infty) = 1, \]
\[ s_2(0) = 1, \]
\[ s_2(\infty) = 0, \]
\[ s_4(0) = 1, \]
\[ s_4(\infty) = 0. \]  \hspace{1cm} (53)

6.2. Solution for Vogel’s Model. In this solution, (48) and (49) are

\[ w'' = -\gamma w' - \left(1 - \frac{D}{E^2}\right) \frac{G}{G^*} w'' + \frac{DG}{G^* E^2} \theta' w' - \gamma M w, \]

\[ \theta'' = -\frac{1}{1 + (4/3)R} \left(-\gamma Pr \theta' - \left(1 + \frac{D \theta}{E^3}\right) \frac{G}{G^*} \lambda (w')^2 - \lambda w' w'' - \frac{1}{\rho C_p} \frac{\partial q_r}{\partial y}\right). \]  \hspace{1cm} (54)

As previous case,

\[ s_3' = -\gamma w' - \left(1 - \frac{D}{E^2}\right) \frac{G}{G^*} w'' + \frac{DG}{G^* E^2} \theta' w' - \gamma M w, \]

\[ s_5' = -\frac{1}{1 + (4/3)R} \left(-\gamma Pr \theta' - \left(1 + \frac{D \theta}{E^3}\right) \frac{G}{G^*} \lambda (w')^2 - \lambda w' w'' - \frac{1}{\rho C_p} \frac{\partial q_r}{\partial y}\right). \]  \hspace{1cm} (55)

with the same boundary conditions as in (53).

7. Graphical Results and Discussion

In graphical portray, Figure 1 explains the physical geometry of the problem. Figure 2 gives portray of

\[ N = 0.1, 2.2, 4.3, 6.4 \] for Reynold’s model on temperature field. Figure 3 shows the behavior of \( R = 0.1, 0.3, 0.6, 1.0 \) for Reynold’s model on temperature portray. Figure 4 renders effects of \( \gamma = 0.1, 2.5, 3.0, 3.4 \) for Reynold’s model on velocity field. Figure 5 describes effects of \( W = 0.2, 3.2, 6.2, 9.4 \) on Reynold’s model for velocity
Figure 6 draws the consequences of $D = 0.3, 1.5, 2.5, 3.0$ on temperature distribution for Reynold’s model. Figure 7 limns the impact of $G^* = 0.1, 0.17, 0.25, 0.4$ on temperature profile for Vogel’s model. Figure 8 tells the influence of $G = 0.3, 1.4, 2.8, 4.5$ for Vogel’s model on temperature profile. Figure 9 represents $y = 0.1, 2.5, 4.5, 6.0$ on Vogel’s model for velocity outline. Figure 10 shows the effect of $\lambda = 0.1, 0.4, 0.7, 1.0$ on Vogel’s model for temperature. Figure 11 depicts impact of $N = 1.0, 3.0, 5.0, 7.0$ on Vogel’s model for velocity profile. Figure 12 shows the behavior of $R = 0.2, 0.6, 1.4, 4.0$ on Vogel’s model for temperature portray. Figure 13 gives the effects of $N = 0.1, 1.2, 2.3, 3.4$ on velocity profile for Reynold’s model. Figure 14 shows the impact of $y = 0.1, 0.2, 0.3, 0.4$ for temperature field of Reynold’s model. Figure 15 depicts the influence of $\gamma$ and $N = 0.1, 0.5, 1.0$ for Reynold’s model’s Nusselt number. Figure 16 represents the impact for Vogel’s model on $D = 0.1, 0.25, 0.5$ and $y$ for skin friction. Figure 17 shows effects for Vogel’s model on Nusselt number for $\gamma$ and
Figure 23: The portrait of 3-D graph of Reynold’s model for $\gamma = 0.9$

Table 1: The values of change in $\gamma$ for temperature of Reynold’s model at the wall.

| $\lambda$ | 1.5 | 2.0 | 2.5 | 3.0 |
|----------|-----|-----|-----|-----|
| $N = 0.1$ |     |     |     |     |
| $W = 0.2$ |     |     |     |     |
| $Pr = 7.7$ | $-0.8769141$ | $-0.9210312$ | $-0.9650590$ | $-1.0089976$ |
| $R = 1.5$ |     |     |     |     |
| $y = 0.1$ |     |     |     |     |

Table 2: The values of change in $W$ for temperature of Reynold’s model at the wall.

| $\lambda$ | 1.5 | 2.0 | 2.5 | 3.0 |
|----------|-----|-----|-----|-----|
| $N = 0.1$ |     |     |     |     |
| $R = 0.6$ |     |     |     |     |
| $Pr = 7.7$ | $-0.8718237$ | $-0.9246222$ | $-0.9818317$ | $-1.04378$ |
| $\lambda = 1.5$ |     |     |     |     |
| $y = 0.1$ |     |     |     |     |

Table 3: The values of change in $R$ for temperature of Reynold’s model at the wall.

| $\lambda$ | 0.7 | 3.3 | 5.9 | 8.5 |
|----------|-----|-----|-----|-----|
| $N = 1.3$ |     |     |     |     |
| $R = 0.1$ |     |     |     |     |
| $Pr = 7.7$ | $-4.2929048$ | $-6.9167525$ | $-9.1619449$ | $-11.076930$ |
| $\lambda = 1.5$ |     |     |     |     |
| $y = 0.1$ |     |     |     |     |

Table 4: The values of change in $\lambda$ for temperature of Reynold’s model at the wall.

| $\lambda$ | 1.5 | 2.0 | 2.5 | 3.0 |
|----------|-----|-----|-----|-----|
| $N = 0.1$ |     |     |     |     |
| $W = 0.2$ |     |     |     |     |
| $Pr = 7.7$ | $-0.8769141$ | $-0.9210312$ | $-0.9650590$ | $-1.0089976$ |
| $R = 1.5$ |     |     |     |     |
| $y = 0.1$ |     |     |     |     |

Table 5: The values of change in $Pr$ for temperature of Reynold’s model at the wall.

| $Pr$ | 7.7 | 8.8 | 9.9 | 10.10 |
|-----|-----|-----|-----|--------|
| $N = 0.1$ |     |     |     |     |
| $W = 0.2$ |     |     |     |     |
| $\lambda = 2.5$ | $-0.9650590$ | $-1.0069910$ | $-1.0499959$ | $-1.0579279$ |
| $R = 1.5$ |     |     |     |     |
| $y = 0.1$ |     |     |     |     |

Table 6: The values of change in $N$ for temperature of Reynold’s model at the wall.

| $N$ | 0.1 | 0.4 | 0.7 | 1.0 |
|-----|-----|-----|-----|-----|
| $Pr = 8.8$ |     |     |     |     |
| $W = 0.2$ |     |     |     |     |
| $\lambda = 2.5$ | $-1.0069910$ | $-1.0285733$ | $-1.0497506$ | $-1.0702968$ |
| $R = 1.5$ |     |     |     |     |
| $y = 0.1$ |     |     |     |     |

Table 7: The values of change in $y$ for temperature of Vogel’s model at the wall.

| $y$ | 0.1 | 0.3 | 0.5 | 0.7 |
|-----|-----|-----|-----|-----|
| $N = 1.3$ |     |     |     |     |
| $R = 0.1$ |     |     |     |     |
| $Pr = 7.7$ | $-2.9582914$ | $-4.075563$ | $-5.3502953$ | $-6.6967343$ |
| $\lambda = 1.7$ |     |     |     |     |
| $D = 1.9$ |     |     |     |     |
| $E = 2.0$ |     |     |     |     |
| $G^* = 1.8$ |     |     |     |     |
| $G = 1.1$ |     |     |     |     |

Table 8: The values of change in $\lambda$ for temperature of Vogel’s model at the wall.

| $\lambda$ | 0.7 | 3.3 | 5.9 | 8.5 |
|----------|-----|-----|-----|-----|
| $N = 1.3$ |     |     |     |     |
| $R = 0.1$ |     |     |     |     |
| $Pr = 7.7$ | $-4.2929048$ | $-6.9167525$ | $-9.1619449$ | $-11.076930$ |
| $\lambda = 1.5$ |     |     |     |     |
| $y = 0.5$ |     |     |     |     |
| $D = 1.9$ |     |     |     |     |
| $E = 2.0$ |     |     |     |     |
| $G = 1.1$ |     |     |     |     |

Figure 23: The portrait of 3-D graph of Reynold’s model for $\gamma = 0.9$
Table 9: The values of change in Pr for temperature of Vogel’s model at the wall.

| Pr   | 7.7 | 8.8 | 9.9 | 10.1 |
|------|-----|-----|-----|------|
| N = 1.3 | R = 0.1 | λ = 1.7 | γ = 0.5 | D = 1.9 |
| 5.3502950 | 5.8292081 | 6.3140881 | 6.4026828 |

Table 10: The values of change in N for temperature of Vogel’s model at the wall.

| N    | 1.4 | 1.5 | 1.6 | 1.7 |
|------|-----|-----|-----|-----|
| Pr = 9.9 | R = 0.1 | λ = 1.7 | γ = 0.5 | D = 1.9 |
| 6.3140881 | 6.3323518 | 6.3506975 | 6.3691254 |

Table 11: The values of change in R for temperature of Vogel’s model at the wall.

| R    | 0.4 | 0.5 | 0.6 | 0.7 |
|------|-----|-----|-----|-----|
| Pr = 9.9 | N = 1.6 | λ = 1.7 | γ = 0.5 | D = 1.9 |
| 6.3506975 | 5.6990981 | 5.1785360 | 4.7545397 |

Table 12: The values of change in E for temperature of Vogel’s model at the wall.

| E    | 2.0 | 2.4 | 2.8 | 3.2 |
|------|-----|-----|-----|-----|
| Pr = 9.9 | N = 1.6 | λ = 1.7 | γ = 0.5 | D = 1.9 |
| 5.1785360 | 5.2495374 | 5.2941906 | 5.3239363 |

Table 13: The values of change in D for temperature of Vogel’s model at the wall.

| D    | 1.9 | 2.4 | 2.9 | 3.4 |
|------|-----|-----|-----|-----|
| Pr = 9.9 | N = 1.6 | λ = 1.7 | γ = 0.5 | D = 1.9 |
| 5.2941906 | 5.2615104 | 5.2295760 | 5.1983724 |

Table 14: The values of change in G* for temperature of Vogel’s model at the wall.

| G*   | 1.8 | 2.4 | 3.0 | 3.6 |
|------|-----|-----|-----|-----|
| Pr = 9.9 | N = 1.6 | λ = 1.7 | γ = 0.5 | D = 1.9 |
| 5.2295760 | 5.3720192 | 5.4587700 | 5.5161120 |

Table 15: The values of change in G for temperature of Vogel’s model at the wall.

| G    | 1.1 | 2.6 | 3.1 | 3.6 |
|------|-----|-----|-----|-----|
| Pr = 9.9 | N = 1.6 | λ = 1.7 | γ = 0.5 | D = 1.9 |
| 5.4582770 | 5.3018863 | 5.1474593 | 4.9949967 |

Table 16: Values of Nusselt number for distinct parameters for Reynold’s model.

| y     | Pr   | λ    | N    | W    | R    | −θ′(0) |
|-------|------|------|------|------|------|--------|
| 0.1   | 7.7  | 1.5  | 0.1  | 0.2  | 0.6  | −0.8769141 |
| 0.12  | −0.9303180 |
| 0.13  | −0.9857684 |
| 0.1   | −0.8769141 |
| 7.8   | 0.15  | 0.2  | 0.9  | 0.3  | 0.1  | −0.8857447 |
| −0.8982711 |
| −0.9019798 |
| 1.6   | 0.7   | 1.7  | 0.8  | 1.5  | −0.8857447 |
| −0.8945717 |
| 0.2   | 0.3   | 0.1  | 0.3  | 0.1  | 0.1  | −0.8769141 |
| −0.8998446 |
| −0.9051285 |
| 0.1   | 0.03  | 0.01 | 0.03 | 0.2  | 0.2  | −0.8769141 |
| −0.9055561 |
| −0.9113143 |
| 0.7   | 0.8   | 0.6  | 0.8  | 0.6  | 0.6  | −0.8769141 |
| −0.8697710 |
| −0.8438659 |
| −0.8769141 |

D = 0.1, 0.5, 1.0. Figures 18–20 give the stream lines of y = 0.1, 0.5, 0.9 for Reynold’s model. Figures 21–23 tell 3 – D structures of y = 0.1, 0.5, 0.9 for Reynold’s model. Tables 1–6 give the change in temperature for Reynold’s model on N, W, R, y, λ and Pr at the wall. Tables 7–15 depict the change in temperature for Vogel’s model on G, D, E, G*, R, N, y, λ and Pr at the wall. Table 16 elucidates the inuence of N, W, R, y, λ and Pr on Nusselt
Table 17: Values of Nusselt number for distinct parameters for Vogel’s model.

| $\gamma$ | G | $G^*$ | D | E | N | R | $\lambda$ | Pr | $\theta$ (0)   |
|---------|---|------|---|---|---|---|---------|----|-------------|
| 0.1     | 1.1 | 1.8  | 1.9 | 2.0 | 1.3 | 0.1 | 1.7     | 7.7 | -2.9582914   |
| 0.12    |     |      |     |    |    |     |         |     | -3.0601193   |
| 0.13    |     |      |     |    |    |     |         |     | -3.1119549   |
| 0.1     |     |      |     |    |    |     |         |     | -2.9582914   |
| 1.2     |     |      |     |    |    |     |         |     | -2.9084736   |
| 1.3     |     |      |     |    |    |     |         |     | -2.8591143   |
| 1.1     |     |      |     |    |    |     |         |     | -2.9582914   |
| 1.9     |     |      |     |    |    |     |         |     | -2.9873442   |
| 2.0     |     |      |     |    |    |     |         |     | -3.0136246   |
| 1.8     |     |      |     |    |    |     |         |     | -2.9582914   |
| 2.1     |     |      |     |    |    |     |         |     | -2.9754999   |
| 2.2     |     |      |     |    |    |     |         |     | -2.9905879   |
| 2.0     |     |      |     |    |    |     |         |     | -2.9582914   |
| 1.4     |     |      |     |    |    |     |         |     | -2.9615122   |
| 1.5     |     |      |     |    |    |     |         |     | -2.9647360   |
| 1.3     |     |      |     |    |    |     |         |     | -2.9582914   |
| 0.2     |     |      |     |    |    |     |         |     | -2.7521710   |
| 0.3     |     |      |     |    |    |     |         |     | -2.5831617   |
| 0.1     |     |      |     |    |    |     |         |     | -2.9582914   |
| 1.8     |     |      |     |    |    |     |         |     | -2.8305255   |
| 1.9     |     |      |     |    |    |     |         |     | -2.9083449   |
| 1.7     |     |      |     |    |    |     |         |     | -2.9582914   |
| 7.8     |     |      |     |    |    |     |         |     | -2.6785432   |
| 7.9     |     |      |     |    |    |     |         |     | -2.8762390   |
| 7.7     |     |      |     |    |    |     |         |     | -2.9083449   |

Table 18: Values of skin friction for distinct parameters for Reynold’s model.

| N | $\lambda$ | Pr | $\gamma$ | W | R | $(1/2)C_f Re$ |
|---|---|---|---|---|---|----------------|
| 0.1 | 1.7 | 7.7 | 0.1 | 0.2 | 0.2 | -0.1616633   |
| 0.2 |     |     |     |    |    | -0.1683275   |
| 0.3 |     |     |     |    |    | -0.1749933   |
| 0.1 |     |     |     |    |    | -0.1616633   |
| 1.8 |     |     |     |    |    | -0.1615948   |
| 1.9 |     |     |     |    |    | -0.1615263   |
| 1.7 |     |     |     |    |    | -0.1616633   |
| 7.8 |     |     |     |    |    | -0.1616311   |
| 7.9 |     |     |     |    |    | -0.1615991   |
| 7.7 |     |     |     |    |    | -0.1616633   |
| 0.2 |     |     |     |    |    | -0.0867723   |
| 0.3 |     |     |     |    |    | -0.0124727   |
| 0.1 |     |     |     |    |    | -0.1616633   |
| 0.3 |     |     |     |    |    | -0.09055561  |
| 0.4 |     |     |     |    |    | -0.1709230   |
| 0.2 |     |     |     |    |    | -0.1802208   |
| 0.3 |     |     |     |    |    | -0.1620517   |
| 0.4 |     |     |     |    |    | -0.1623734   |
| 0.2 |     |     |     |    |    | -0.1616633   |

Table 19: Values of skin friction for distinct parameters for Vogel’s model.

| G | $G^*$ | D | $\lambda$ | $\gamma$ | Pr | N | R | $(1/2)C_f Re$ |
|---|------|---|---------|--------|----|---|---|----------------|
| 2.1 | 0.4  | 0.3 | 0.7     | 0.1    | 2.0 | 7.7 | 1.3 | 0.2 | -22.004282   |
| 2.2 |     |     |         |        |     |    |    |     | -20.506954   |
| 2.3 |     |     |         |        |     |    |    |     | -26.049557   |
| 2.1 |     |     |         |        |     |    |    |     | -22.004282   |
| 0.5 |     |     |         |        |     |    |    |     | -14.881911   |
| 0.6 |     |     |         |        |     |    |    |     | -11.134132   |
| 0.4 |     |     |         |        |     |    |    |     | -22.004282   |
| 0.4 |     |     |         |        |     |    |    |     | 0.41191405   |
| 0.5 |     |     |         |        |     |    |    |     | 0.53440194   |
| 0.3 |     |     |         |        |     |    |    |     | -22.004282   |
| 0.3 |     |     |         |        |     |    |    |     | -22.004911   |
| 0.2 |     |     |         |        |     |    |    |     | -22.005536   |
| 0.1 |     |     |         |        |     |    |    |     | -22.004282   |
| 0.1 |     |     |         |        |     |    |    |     | 0.24714771   |
| 0.2 |     |     |         |        |     |    |    |     | 0.21495112   |
| 0.3 |     |     |         |        |     |    |    |     | -22.004282   |
| 0.1 |     |     |         |        |     |    |    |     | -22.002875   |
| 0.7 |     |     |         |        |     |    |    |     | -22.001468   |
| 0.8 |     |     |         |        |     |    |    |     | -22.004282   |
| 0.2 |     |     |         |        |     |    |    |     | -22.023073   |
| 0.3 |     |     |         |        |     |    |    |     | -22.041869   |
| 0.1 |     |     |         |        |     |    |    |     | -22.004282   |
| 0.2 |     |     |         |        |     |    |    |     | 0.22098766   |
| 0.3 |     |     |         |        |     |    |    |     | 0.22567879   |
| 0.1 |     |     |         |        |     |    |    |     | -22.004282   |

G, D, $E$, $G^*$, N, $\gamma$, $\lambda$ and Pr for skin friction coefficient of Vogel’s model.

8. Concluding Remarks

In this inquisition, the numerical solution of Walters’ B fluid model with MHD and radiation effects of both time dependent viscosity models has been discussed. Influences of these parameters are presented with the help of graphs and tables. Some important points of the study of this problem are the following:

(1) A sensible growth is seen in the velocity portrait as increase in G and the velocity curve decreases with the enlargement of $\gamma$, D and N for both models

(2) $\gamma$ and Pr are decreases for Reynold’s as well as Vogel’s models

(3) The stream lines are sighted to shrink and the 3 – D graphs bended with the increase in $\gamma$ of Reynold’s model

(4) Skin friction curve increases with the increase in N, while Nusselt number graph decreases with the enlargement in $\lambda$

Data Availability

The data used to support the study are available from the corresponding author upon request.
Conflicts of Interest

The authors declare that they have no conflicts of interest.

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