AN $L^4$ ESTIMATE FOR A SINGULAR ENTANGLED QUADRILINEAR FORM

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Abstract. The twisted paraproduct can be viewed as a two-dimensional trilinear form which appeared in the work by Demeter and Thiele on the two-dimensional bilinear Hilbert transform. $L^p$ boundedness of the twisted paraproduct is due to Kovač, who in parallel established estimates for the dyadic model of a closely related quadrilinear form. We prove an $(L^4, L^4, L^4, L^4)$ bound for the continuous model of the latter by adapting the technique of Kovač to the continuous setting. The mentioned forms belong to a larger class of operators with general modulation invariance. Another instance of such is the triangular Hilbert transform, which controls issues related to two commuting transformations in ergodic theory, and for which $L^p$ bounds remain an open problem.

1. Introduction

For four functions $F_1, F_2, F_3, F_4$ on $\mathbb{R}^2$ we denote their "entangled product"

$$F_{(F_1,F_2,F_3,F_4)}(x,x',y,y') := F_1(x,y) F_2(x',y) F_3(x',y') F_4(x,y').$$

(1.1)

Let $m$ be a bounded function on $\mathbb{R}^2$, smooth away from the origin and satisfying

$\left| \partial^\alpha m(\xi, \eta) \right| \lesssim (|\xi| + |\eta|)^{-|\alpha|}$

(1.2)

for all multi-indices $\alpha$ up to some large finite order. To any such $m$ we associate a quadrilinear form $\Lambda = \Lambda_m$ defined as

$$\Lambda(F_1, F_2, F_3, F_4) := \int_{\mathbb{R}^2} \hat{F}(\xi, -\xi, \eta, -\eta) m(\xi, \eta) d\xi d\eta$$

for Schwartz functions $F_j \in \mathcal{S}(\mathbb{R}^2)$, where $F := F_{(F_1,F_2,F_3,F_4)}$. The object of this paper is to establish the following bound.

Theorem 1. The quadrilinear form $\Lambda$ satisfies the estimate

$$|\Lambda(F_1, F_2, F_3, F_4)| \lesssim \|F_1\|_{L^4(\mathbb{R}^2)} \|F_2\|_{L^4(\mathbb{R}^2)} \|F_3\|_{L^4(\mathbb{R}^2)} \|F_4\|_{L^4(\mathbb{R}^2)}.$$

(1.3)

When $m$ is identically one, $\Lambda$ corresponds to the pointwise product form

$$\Lambda(F_1, F_2, F_3, F_4) = \int_{\mathbb{R}^2} F_1(x,y) F_2(x,y) F_3(x,y) F_4(x,y) dxdy.$$

The bound (1.3) is then an immediate consequence of Hölder’s inequality and holds in a larger range of exponents. In general, we can formally write $\Lambda(F_1, F_2, F_3, F_4)$ as

$$\int_{\mathbb{R}^4} F_1(x,y) F_2(x',y) F_3(x',y') F_4(x,y') \kappa(x - x', y - y') dxdx'dydy',$$

(1.4)

where $\kappa$ is a two-dimensional Calderón-Zygmund kernel.

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1For two non-negative quantities $A$ and $B$ we write $A \lesssim B$ if there is an absolute constant $C > 0$ such that $A \leq CB$. We write $A \lesssim P B$ if the constant depends on a set of parameters $P$.

2The Fourier transform we use is defined in (2.4).
The motivation for these objects originates in the study of the twisted paraproduct [5]. We call the twisted paraproduct a trilinear form $T = T_m$ defined as

$$T(F_1, F_2, F_3) := \Lambda(F_1, F_2, F_3).$$

That is, the fourth function in the entangled product $F$ is the constant function one. The form $T$ was proposed by Demeter and Thiele [2] as the dual of a particular case of the two-dimensional bilinear Hilbert transform. This was the only case which could not be treated with the time-frequency techniques in [2]. Lack of applicability of the latter is closely related with general modulation symmetries that the operators $T$ and $\Lambda$ exhibit. An example of such a symmetry is that for any $g \in L^\infty(\mathbb{R})$ we have invariance

$$\Lambda((1 \otimes g)F_1, F_2, F_3, F_4) = \Lambda(F_1, (1 \otimes g)F_2, F_3, F_4),$$

where $(f \otimes g)(x, y) := f(x)g(y)$. This is evident from their "entangled" structure. One can informally say that the generalized modulation invariance is present since several functions depend on the same one-dimensional variable.

First bounds for $T$ are due to Kovač [5], who established

$$|T(F_1, F_2, F_3)| \lesssim \|F_1\|_{L^{p_1}(\mathbb{R}^2)}\|F_2\|_{L^{p_2}(\mathbb{R}^2)}\|F_3\|_{L^{p_3}(\mathbb{R}^2)}$$

whenever $1/p_1 + 1/p_2 + 1/p_3 = 1$ and $2 < p_1, p_2, p_3 < \infty$. His approach was the Bellman function technique. The fiber-wise Calderón-Zygmund decomposition of Bernicot [1] extended the range of exponents to $1 < p_1, p_3 < \infty$, $2 < p_2 \leq \infty$.

Kovač observed that adding the fourth function $F_4$ to $T$ completes the cyclic structure of the form and results in an object with a high degree of symmetry. For instance, for even kernels $\kappa$ one has $\Lambda(F_1, F_2, F_3, F_4) = \Lambda(F_3, F_4, F_1, F_2)$. Moreover, $T$ and $\Lambda$ can be seen as the smallest non-trivial examples of a family of entangled multilinear forms associated to bipartite graphs, whose dyadic models were studied in [4].

To prove (1.5), Kovač passed through a dyadic version of $\Lambda$, which we call $\Lambda_d$. More precisely, he considered (1.4) with $\kappa$ replaced by the perfect Calderón-Zygmund kernel

$$\sum_{I \times J} \varphi_I^d(x) \varphi_J^d(x') \psi_J^d(y) \psi_I^d(y'),$$

The sum in (1.6) runs over all dyadic squares $I \times J$ in $\mathbb{R}^2$. The scaling function and the Haar function are for a dyadic interval $I$ defined as

$$\varphi_I^d := |I|^{-1/2} \mathbf{1}_I$$

and

$$\psi_I^d := |I|^{-1/2} (\mathbf{1}_{I_{\text{left half}}} - \mathbf{1}_{I_{\text{right half}}}),$$

respectively. In [4] it is shown that the form $\Lambda_d$ maps $L^{p_1} \times L^{p_2} \times L^{p_3} \times L^{p_4} \to \mathbb{C}$ for all Hölder-type exponents satisfying $2 < p_j \leq \infty$. The large range of exponents was achieved by first proving a local bound with the summation in (1.6) running over a subset of dyadic squares called trees. The contributions of a single tree were then integrated into a global estimate. By defining the dyadic model $T_d$ of $T$ via the relation with $\Lambda_d$ in the obvious way, bounds for $T_d$ followed.

It remained to tackle $T$ for continuous kernels. This problem was via the so-called cone decomposition, see [6], first reduced to the case

$$\kappa(s, t) = \sum_{k \in \mathbb{Z}} 2^k \varphi(2^k s) 2^k \psi(2^k t),$$

A dyadic square is a product of two dyadic intervals of the same length. A dyadic interval is an interval of the form $[2^m s, 2^m (s + 1)]$, $k, m \in \mathbb{Z}$.

We write $\mathbf{1}_A$ for the characteristic function of a set $A \subseteq \mathbb{R}$. 

where \( \varphi, \psi \in S(\mathbb{R}) \) are two Schwartz functions and \( \hat{\psi} \) is supported on \( \{1 \leq |\xi| \leq 2\} \).

Their dilations by \( 2^k \) can be seen as continuous analogues of \( \varphi_I, \psi_I \). The bound (1.5) was in [5] finally established by relating the special case of \( T \), associated to (1.7), to the dyadic \( T_d \). This was done using the square functions of Jones, Seeger and Wright [3], which compare convolutions to the martingale averages.

A natural question is what we can say about \( \Lambda \) for any \( m \) satisfying (1.2), if the function \( I \) is replaced by other functions \( F \). The transition technique from the dyadic to the continuous case via the mentioned square functions does not apply.

In the present note we obtain an answer in this direction by adapting the techniques used to treat \( \Lambda_d \) in [5] to the continuous setting. We address the simplest \( L^4 \) case only. It is expected that suitable tree decompositions will eventually enable us to prove (1.3) for a larger range of exponents. However, for the considered quadrilinear form we cannot make use of the fiber-wise Calderón-Zygmund decomposition by Bernicot.

The core argument in [5] intertwines two applications of the Cauchy-Schwarz inequality, which gradually separates the functions \( F_j \), and two applications of an algebraic identity, which "interchanges" the functions \( \varphi^d \) and \( \psi^d \). This identity, involving a telescoping argument in the dyadic case, is now replaced by a differential equality combining the fundamental theorem of calculus and the Leibniz rule. The main issue in the continuous setup is that the mentioned algebraic trick can be applied twice if the functions \( \varphi, \psi \), decomposing the kernel, are sufficiently symmetric. For example, even functions would do. Moreover, they need to possess enough decay and have certain smoothness properties, which should be maintained throughout the process. Suitable candidates which fulfill the requirements are, for instance, the Gaussian exponential functions.

Although we cannot expect our functions \( \varphi, \psi \) to be even, or more, to be the Gaussian exponential functions, we are able to overcome the mentioned restrictions as follows.

First, the reduction to the case with a concrete kernel is done with a careful choice of the functions \( \varphi, \psi \). This way we obtain some of the required symmetry and regularity.

Second, after each application of the Cauchy-Schwarz inequality we dominate certain functions with a suitable superposition of dilated Gaussian exponential functions. This gradually reduces the two algebraic steps to the case of Gaussians, which most resembles the dyadic telescoping trick.

Beside extending the exponent range, it would be of interest to obtain boundedness results for the continuous models of the forms from [4], associated to bipartite graphs.

Let us briefly comment on another related open problem. There is a question of establishing \( L^p \) estimates for the akin trilinear form

\[
\Lambda_\triangle(F_1, F_2, F_3) := \int_\mathbb{R} \hat{F}(\xi, \xi, \xi)\text{sgn}(\xi)d\xi
\]

where the entangled product \( F \) is now given by

\[
F(x, y, z) := F_1(x, y)F_2(y, z)F_3(z, x).
\]

Passing to the spatial side, one has

\[
\Lambda_\triangle(F_1, F_2, F_3) = \int_{\mathbb{R}^3} F_1(x, y)F_2(y, z)F_3(z, x)\frac{1}{x + y + z}dxdydz.
\]

The structure of this form corresponds to the three-cycle and is for this reason called the triangular Hilbert transform. No \( L^p \) bounds for \( \Lambda_\triangle \) or for its dyadic model are known.

Lack of the bipartite structure prevents to approach it with the techniques from [4].

Boundedness of \( \Lambda_\triangle \) would imply boundedness for certain instances of the two-dimensional bilinear Hilbert transform and the twisted paraproduct. Further interests in
Λ_Δ arise in ergodic theory. It is proposed by Demeter and Thiele [2] to approach the open question of pointwise almost everywhere convergence for ergodic averages
\[
\frac{1}{N} \sum_{n=1}^{N} f(T^n x) g(S^n x),
\]
where \( S, T : X \to X \) are two commuting measure preserving transformations on a probability space \( X \), via an examination of the triangular Hilbert transform.

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2. Decomposition of the symbol

To begin, we reduce the general symbol to a particular function by decomposing \( m \) into pieces which are supported on parts of two double cones. We follow the main ideas discussed in [6]. However, we do not discretize, but rather keep continuum in the scale.

The Fourier transform we shall use throughout this note is defined as
\[
\hat{f}(\omega) := \int_{\mathbb{R}^n} f(\tau) e^{-2\pi i \tau \cdot \omega} \, d\tau.
\]
By a smooth partition of unity and symmetry in \( \xi, \eta \) we may assume that \( m \) is supported on the double cone
\[
\{(\xi, \eta) : |\eta| \geq 1.001|\xi|\}
\]
centered around the \( \eta \)-axis. Choosing double cones over single cones will allow us to use functions which are symmetric around the origin. We can choose the partition of unity such that (1.2) is preserved, possibly with a different constant.

![Figure 1. Decomposition of \( m \).](image)

Let \( \theta \) be a function on \( \mathbb{R}^2 \) such that \( \hat{\theta} \) is smooth, real, radial and supported in the annulus \( B(0, 2.7) \setminus B(0, 1.7) \). For every \((\xi, \eta) \neq 0\) we normalize
\[
\int_0^\infty \hat{\theta}(t\xi, t\eta) \frac{dt}{t} = 1.
\]
This can be achieved, since \( \hat{\theta} \) is radial and supported away from 0. Then we can write
\[
m(\xi, \eta) = \int_0^\infty m_t(\xi, \eta) \frac{dt}{t},
\]
where \( m_t(\xi, \eta) := m(\xi, \eta) \hat{\theta}(t\xi, t\eta) \).
In what comes we will be working with certain smooth bump functions, for which we need the following technical lemma. Its proof can be found in the appendix.

**Lemma 2.** Let \( \varepsilon := 0.001 \). There exists a non-negative real-valued function \( f \in C_0^\infty(\mathbb{R}) \) which is supported in \([1, 3]\), even about 2 and constantly equal 1 on \([1 + \varepsilon, 3 - \varepsilon]\), such that \( f^{1/2} \) and

\[
\left( \int_x^\infty \frac{f(t) + f(-t)}{t} dt \right)^{1/2}
\]

belong to \( C_0^\infty(\mathbb{R}) \).

Now consider \( m_1 \). Its support is contained in the union of the rectangles

\([-2, 2] \times [-3, -1] \quad \text{and} \quad [2, 2] \times [1, 3] \).

Let \( f \) be the function from Lemma 2 and let \( \vartheta_1, \vartheta_2 \in \mathcal{S}(\mathbb{R}) \) be such that \( \hat{\vartheta}_1(\xi) = f((\xi + 2)/2) \) and \( \hat{\vartheta}_2(\xi) = f(\xi) + f(-\xi) \). Then \( \vartheta_1 \otimes \vartheta_2 \) equals 1 on the support of \( m_1 \). Thus, by dilating \( \vartheta_1, \vartheta_2 \) in \( t \), for every \( t > 0 \) we can write

\[
m_t(\xi, \eta) = m_t(\xi, \eta) \hat{\vartheta}_1(t\xi) \hat{\vartheta}_2(t\eta).
\]

This can be rewritten further using the Fourier inversion formula on \( m_t \) as

\[
m_t(\xi, \eta) = \left( \int_{\mathbb{R}^2} m_t(u, v) e^{2\pi i ut\xi} e^{2\pi ivt\eta} du dv \right) \hat{\vartheta}_1(t\xi) \hat{\vartheta}_2(t\eta),
\]

where \( m_t := t^2 \hat{m}_t(t, t) \). Integrating by parts sufficiently many times, using that (1.2) holds for \( m(\xi/t, \eta/t) \) uniformly in \( t \) and considering the support of \( m_t \) we obtain

\[
|\mu_t(u, v)| = t^2 \left| \int_{\mathbb{R}^2} m_t(\xi, \eta) e^{-2\pi i (ut\xi + vt\eta)} d\xi d\eta \right| = \left| \int_{\mathbb{R}^2} m_t \left( \frac{\xi}{t}, \frac{\eta}{t} \right) e^{-2\pi i (u\xi + v\eta)} d\xi d\eta \right| \lesssim (1 + |u|)^{-12} (1 + |v|)^{-12}.
\]

Define \( \varphi^{(u)}, \psi^{(v)} \) by

\[
\tilde{\varphi}^{(u)}(\xi) := (1 + |u|)^{-5} (\hat{\vartheta}_1(\xi))^{1/2} e^{\pi i u \xi},
\]

\[
\tilde{\psi}^{(v)}(\eta) := (\tilde{\vartheta}_2(\eta))^{1/2} e^{\pi i v \eta}.
\]

By Lemma 2 we have \( (\hat{\vartheta}_1)^{1/2} \in C_0^\infty(\mathbb{R}) \), so the function \( \varphi^{(u)} \) satisfies the bound

\[
|\varphi^{(u)}(x)| \lesssim (1 + |x|)^{-5},
\]

which is uniform in \( u \). We will use this fact in the following section. Now we can write

\[
m(\xi, \eta) = \int_0^\infty \int_{\mathbb{R}^2} \tilde{\mu}_t(u, v) (\tilde{\varphi}^{(u)}(t\xi))^2 (\tilde{\psi}^{(v)}(t\eta))^2 du dv \frac{dt}{t},
\]

where the coefficients \( \tilde{\mu}_t \) are defined as

\[
\tilde{\mu}_t(u, v) := (1 + |u|)^{10} \mu_t(u, v).
\]

Note that \( (\hat{\vartheta}_1)^{1/2} \) and \( (\tilde{\vartheta}_2)^{1/2} \) are real-valued and even, so \( \varphi^{(u)}, \psi^{(v)} \) are multiples of translates of real-valued functions and thus real-valued.

To summarize, a double cone we have decomposed \( \Lambda(F_1, F_2, F_3, F_4) \) into

\[
\int_{\mathbb{R}^2} \tilde{\mu}_t(u, v) \int_0^\infty \int_{\mathbb{R}^2} \tilde{F}(\xi, -\xi, -\eta)(\tilde{\varphi}^{(u)}(t\xi))^2 (\tilde{\psi}^{(v)}(t\eta))^2 d\xi d\eta \frac{dt}{t} du dv.
\]
By the rapid decay of the coefficients \( \tilde{\mu}_t \) it will suffice to prove (1.3) for the form
\[
\int_0^\infty \int_{\mathbb{R}^2} \tilde{F}(\xi, -\xi, \eta, -\eta)(\tilde{\varphi}^{(u)}(t\xi))^2(\tilde{\psi}^{(v)}(t\eta))^2 d\xi d\eta \frac{dt}{t},
\]
provided that the estimate holds uniformly in the parameters \( u, v \).

From now on we shall assume that the functions \( F_j \in \mathcal{S}(\mathbb{R}^2) \) are real-valued, as otherwise we can split them into real and imaginary parts and use quadrilinearity of \( \Lambda \).

3. PROOF OF THEOREM 1

The proof proceeds with studying the special case (2.6). For the rest of this note we will in general consider forms associated to four functions \( \phi_i \in \mathcal{S}(\mathbb{R}) \), defined as
\[
\Lambda_{\phi_1, \phi_2, \phi_3, \phi_4}(F_1, F_2, F_3, F_4) := 
\int_0^\infty \int_{\mathbb{R}^2} \tilde{F}(\xi, -\xi, \eta, -\eta)\tilde{\varphi}_1(t\xi)\tilde{\varphi}_2(-t\xi)\tilde{\varphi}_3(t\eta)\tilde{\varphi}_4(-t\eta)d\xi d\eta \frac{dt}{t}.
\]
Note that the object (2.6), which we need to estimate, is obtained by choosing
\[
\phi_1 = \varphi^{(u)}, \quad \phi_3 = \psi^{(v)}, \\
\phi_2 = \varphi^{(-u)}, \quad \phi_4 = \psi^{(-v)}.
\]
This follows from the definition (2.4) and from the functions \( \tilde{\varphi}_1, \tilde{\varphi}_2 \) being even.

We need to express \( \Lambda_{\phi_1, \phi_2, \phi_3, \phi_4} \) on the spatial side. Let us denote by \([f]_t\) the \( L^1 \)-dilation of a function \( f \) by a parameter \( t > 0 \), i.e. \([f]_t(x) := t^{-1}f(t^{-1}x)\). Then, \([\tilde{f}]_t(\xi) = \tilde{f}(t\xi)\). Since the integral of the Fourier transform of a Schwartz function in \( \mathbb{R}^4 \) over the hyperplane
\[
\{(\xi, -\xi, \eta, -\eta) : \xi, \eta \in \mathbb{R}\}
\]
equals the integral of the function itself over the perpendicular hyperplane
\[
\{(p, p, q, q) : p, q \in \mathbb{R}\},
\]
we can write \( \Lambda_{\phi_1, \phi_2, \phi_3, \phi_4}(F_1, F_2, F_3, F_4) \) as
\[
\frac{1}{t} \int_0^\infty \int_{\mathbb{R}^2} F*([\varphi_1]_t \otimes [\varphi_2]_t \otimes [\varphi_3]_t \otimes [\varphi_4]_t)(t, p, q, q) dp dq dt.
\]
Expanding the convolution, the last display can be identified as
\[
\frac{1}{t} \int_0^\infty \int_{\mathbb{R}^6} F_1(x, y)F_2(x', y)F_3(x', y')F_4(x, y') [\varphi_1]_t(p-x)[\varphi_2]_t(p-x')[\varphi_3]_t(q-y)[\varphi_4]_t(q-y') dx dy dx' dy dy' dp dq dt.
\]
Now we are ready to start. The inequality (1.3) we want to establish is homogeneous, so we may normalize
\[
\|F_j\|_{L^4(\mathbb{R}^2)} = 1,
\]
for \( j = 1, 2, 3, 4 \). Thus, we are set to show
\[
|\Lambda_{\varphi^{(u)}, \varphi^{(-u)}, \psi^{(v)}, \psi^{(-v)}}(F_1, F_2, F_3, F_4)| \lesssim 1.
\]

The proof starts with an application of the Cauchy-Schwarz inequality. To preserve the mean zero property of \( \psi^{(v)}, \psi^{(-v)} \) we separate the involved functions according to
the variables $y, y'$ and estimate $|\Lambda_{\varphi(u), \varphi(-u), \psi(v), \psi(-v)}(F_1, F_2, F_3, F_4)|$ by

$$\int_0^\infty \int_{\mathbb{R}^2} \left| \int_{\mathbb{R}} F_1(x, y) F_2(x', y)[\psi(v)]_t (q-y)dy \right| \int_{\mathbb{R}} F_3(x', y') F_4(x, y')[\psi(-v)]_t (q-y')dy' \left| [(\varphi(u)]_t (p-x)] [\varphi(-u)]_t (p-x')dxdy'dpdt. \right.$$

Applying the Cauchy-Schwarz inequality bounds this expression by the product

$$\Lambda_{\varphi(u)|\varphi(-u), \psi(v), \psi(-v)}(F_1, F_2, F_3, F_4) \leq \Lambda_{\varphi(u)|\varphi(-u), \psi(v), \psi(-v)}(F_1, F_2, F_3, F_4)^{1/2} \Lambda_{\varphi(u), \varphi(-u), \psi(v), \psi(-v)}(F_1, F_2, F_3, F_4)^{1/2}. \right.$$ 

We estimate the first factor of the above display, the second is dealt with similarly.

To further separate the involved functions we would like to apply the Cauchy-Schwarz inequality again, which now needs to be done in the complementary variables. So we need to "switch" the functions $\varphi(u)$ and $\psi(v)$. This is where we make use of the following lemma, a continuous analogue of the telescoping identity from [5].

**Lemma 3.** Assume that we have two pairs of real-valued Schwartz functions $(\rho_i, \sigma_i)$, $i = 1, 2$, which satisfy

$$-t\partial_t |\hat{\rho}_i(t\tau)|^2 = |\hat{\sigma}_i(t\tau)|^2 \text{ for } i = 1, 2. \tag{3.1}$$

Then with $c := |\hat{\rho}_1(0)|^2 |\hat{\rho}_2(0)|^2$ we have

$$\Lambda_{\sigma_1, \rho_2}(F_1, F_2, F_3, F_4) + \Lambda_{\rho_1, \sigma_2}(F_1, F_2, F_3, F_4) = c \int_{\mathbb{R}^2} F_1 F_2 F_3 F_4, \tag{3.2}$$

where we have for short denoted $\Lambda_{\sigma, \rho} = \Lambda_{\sigma, \sigma, \rho, \rho}$.

**Proof.** By the fundamental theorem of calculus,

$$\int_0^\infty \partial_t (|\hat{\rho}_1(t\xi)|^2 |\hat{\rho}_2(t\eta)|^2)dt = -|\hat{\rho}_1(0)|^2 |\hat{\rho}_2(0)|^2. \tag{3.3}$$

The left hand-side of (3.3) equals

$$\int_0^\infty t \partial_t (|\hat{\rho}_1(t\xi)|^2 |\hat{\rho}_2(t\eta)|^2) \frac{dt}{t} + \int_0^\infty |\hat{\rho}_1(t\xi)|^2 t \partial_t (|\hat{\rho}_2(t\eta)|^2) \frac{dt}{t}. \tag{3.4}$$

The functions $\rho, \sigma$ are real-valued, so $\overline{\hat{\rho}(\eta)} = \hat{\rho}(-\eta)$, and analogously for $\sigma$. Together with (3.1) this shows that (3.4) can be written as

$$-\int_0^\infty \hat{\sigma}_1(t\xi) \hat{\sigma}_1(-t\xi) \hat{\rho}_2(t\eta) \hat{\rho}_2(-t\eta) \frac{dt}{t} - \int_0^\infty \hat{\rho}_1(t\xi) \hat{\rho}_1(-t\xi) \hat{\sigma}_2(t\eta) \hat{\sigma}_2(-t\eta) \frac{dt}{t}. \tag{3.5}$$

Now multiply (3.3) by $\overline{\hat{F}(\xi, -\xi, \eta, -\eta)}$ and integrate in the variables $\xi, \eta$. It remains to use (3.5) and to evaluate the right hand-side of (3.3) as $|\hat{\rho}_1(0)|^2 |\hat{\rho}_2(0)|^2$ times

$$\int_{\mathbb{R}^2} \overline{\hat{F}(\xi, -\xi, \eta, -\eta)} d\xi d\eta = \int_{\mathbb{R}^2} F(x, x, y, y) dxdy$$

$$= \int_{\mathbb{R}^2} F_1(x, y) F_2(x, y) F_3(x, y) F_4(x, y) dxdy.$$

This proves the claim. \hfill \Box
To apply Lemma 3.2 we would like to have \( \varphi^{(u)} = \varphi^{(-u)} \), as then we would have
\[
\Lambda_{|\varphi^{(u)}|,|\varphi^{(-u)}|,\psi^{(v)}_1,\psi^{(v)}_2} = \Lambda_{|\varphi^{(u)}|,\psi^{(v)}}.
\]
However, we do not have \( \varphi^{(u)} = \varphi^{(-u)} \) in general. For this and to circumvent possible lack of smoothness of \(|\varphi^{(\pm u)}|\), we dominate \(|\varphi^{(\pm u)}|\) with a superposition of the Gaussian exponential functions. Consider
\[
\Phi(x) := \int_1^\infty \frac{1}{\alpha^5} e^{-\left(\frac{x^2}{\alpha}\right)^2} d\alpha = \frac{1}{2\pi^4} (1 - e^{-x^2(x^2 + 1)}).
\]
The function \( \Phi \) is positive, continuous at zero and for large \( x \) comparable to \( x^{-4} \). Let us denote the \( L^1 \)-normalized Gaussian rescaled by a parameter \( \alpha > 0 \) by
\[
g_\alpha(x) := \frac{1}{\sqrt{\pi\alpha}} e^{-\left(\frac{x}{\alpha}\right)^2}.
\]
(3.6)
Then we can write
\[
\Phi(x) = \sqrt{\pi} \int_1^\infty \frac{1}{\alpha^4} g_\alpha \, d\alpha.
\]
Since \(|\varphi^{(\pm u)}|\) satisfies the decay estimate (2.5), we can bound it pointwise by \( \Phi \) multiplied by some positive constant which is uniform in \( u \). Positivity of the integrands in
\[
\Lambda_{|\varphi^{(u)}|,|\varphi^{(-u)}|,\psi^{(v)}_1,\psi^{(v)}_2}(F_1, F_2, F_2, F_1) = \int_0^\infty \int_\mathbb{R}^4 \left( \int_\mathbb{R} F_1(x, y) F_2(x', y)[\psi^{(v)}_1(q-y)dy \right)^2
\]
\[
\|\varphi^{(u)}\|_t[p-x] \|\varphi^{(-u)}\|_t[p-x'] \, dx \, dx' \, dq \, dt
\]
(3.7)
then allows us to dominate
\[
\Lambda_{|\varphi^{(u)}|,|\varphi^{(-u)}|,\psi^{(v)}_1,\psi^{(v)}_2}(F_1, F_2, F_2, F_1) \lesssim \int_1^\infty \int_1^\infty \Lambda_{g_\alpha, g_\beta, \psi^{(v)}_1, \psi^{(v)}_2}(F_1, F_2, F_2, F_1) \frac{d\alpha \, d\beta}{\alpha^4 \beta^4}.
\]
To reduce to only one scaling parameter in the last line we split the integration into the regions \( \alpha \geq \beta \) and \( \alpha < \beta \). By symmetry it suffices to estimate the region \( \alpha \geq \beta \) only, on which we bound \( \beta g_\beta \leq \alpha g_\alpha \) for \( \alpha, \beta \geq 1 \). This leaves us with estimating of
\[
\int_1^\infty \Lambda_{g_\alpha, \psi^{(v)}_1}(F_1, F_2, F_2, F_1) \frac{d\alpha}{\alpha^3}.
\]
We shall now apply Lemma 3 with \( (\rho_1, \sigma_1) = (g_\alpha, h_\alpha) \) and \( (\rho_2, \sigma_2) = (\phi, \psi^{(v)}) \), where we define \( h_\alpha(x) := \alpha(g_\alpha)'(x) \) and \( \phi \) is defined via
\[
\hat{\phi}(\xi) := \left( \int_\xi^\infty |\hat{\psi^{(v)}_2}(\tau)|^2 \frac{d\tau}{\tau} \right)^{1/2}.
\]
(3.8)
Since \(|\hat{\psi^{(v)}_2}|^2 = \hat{\varphi}_2 \), by Lemma 2 the function \( \hat{\phi} \) belongs to \( C^\infty_0(\mathbb{R}) \). Note that the two considered pairs of functions \( (\rho_1, \sigma_1) \) satisfy (3.2), which follows by a straightforward calculation. Lemma 3 now yields
\[
\Lambda_{g_\alpha, \psi^{(v)}_1}(F_1, F_2, F_2, F_1) = -\Lambda h_\alpha, \phi(F_1, F_2, F_2, F_1) + \hat{\phi}(0)^2 \int_{\mathbb{R}^2} F_1^2 F_2^2.
\]
(3.9)
By the Cauchy-Schwarz inequality we have
\[
\int_{\mathbb{R}^2} F_1^2 F_2^2 \leq \|F_1\|_{L^4(\mathbb{R}^2)}^2 \|F_2\|_{L^4(\mathbb{R}^2)}^2 = 1,
\]
so it remains to consider the first term on the right hand-side of (3.9).
To estimate it we repeat the just performed steps, which will further separate the functions \( F_1, F_2 \). The role of \( \varphi^{(\pm u)} \) is now taken over by \( \phi \) and the role of \( \psi^{(\pm v)} \) is taken
over by $h_{\alpha}$. Therefore we can group the integrals in $\Lambda_{h_{\alpha},\phi}$ according to the variables $x, x'$, and bound $|\Lambda_{h_{\alpha},\phi}(F_1, F_2, F_2, F_1)|$ by

$$\int_0^{\infty} \int_{\mathbb{R}^4} \left| \int_{\mathbb{R}} F_1(x, y) F_1(x', y') [h_{\alpha}]_t (p - x) dx \right| \int_{\mathbb{R}} F_2(x', y') F_2(x', y) [h_{\alpha}]_t (p - x') dx' \left| [\phi]_t (q - y) \right| [\phi]_t (q - y') dy dq dt \frac{dt}{t}.$$ 

Applying the Cauchy-Schwarz inequality we obtain

$$|\Lambda_{h_{\alpha},\phi}(F_1, F_2, F_2, F_1)| \leq |\Lambda_{h_{\alpha},\phi}(F_1, F_1, F_1)|^{1/2} |\Lambda_{h_{\alpha},\phi}(F_2, F_2, F_2)|^{1/2}.$$ 

Now we dominate the rapidly decaying function $|\phi|$ by a positive constant times $\Phi$, which gives for the first factor

$$\Lambda_{h_{\alpha},\phi}(F_1, F_1, F_1) \lesssim \int_1^{\infty} \int_1^{\infty} \Lambda_{h_{\alpha},h_{\alpha},g_{\gamma},g_3}(F_1, F_1, F_1) \frac{dy}{\gamma} \frac{d\delta}{\delta^4}. \tag{3.10}$$

By symmetry it again suffices to estimate

$$\int_1^{\infty} \Lambda_{h_{\alpha},g_{\gamma}}(F_1, F_1, F_1) \frac{dy}{\gamma^3}. \tag{3.11}$$

Lemma 3 with $(\rho_1, \sigma_1) = (g_{\alpha}, h_{\alpha})$ and $(\rho_2, \sigma_2) = (g_{\gamma}, h_{\gamma})$ gives

$$\Lambda_{h_{\alpha},g_{\gamma}}(F_1, F_1, F_1) = -\Lambda_{g_{\alpha},h_{\gamma}}(F_1, F_1, F_1) + \int_{\mathbb{R}^2} F_1^4.$$ 

The key gain we obtain from having reduced to a single function $F_1$ is that

$$\Lambda_{g_{\alpha},h_{\gamma}}(F_1, F_1, F_1) \geq 0, \tag{3.11}$$

which can be seen by writing the form in (3.11) in an analogous way as in (3.7) and using positivity of $g_{\alpha}$. By our normalization, $\int_{\mathbb{R}^2} F_1^4 = 1$. Thus,

$$\Lambda_{h_{\alpha},g_{\gamma}}(F_1, F_1, F_1) \leq 1.$$ 

This establishes the desired estimate for the form $\Lambda_{\phi^{(u)},\phi^{(-u)}},\psi^{(v)},\psi^{(-v)}$.

4. Appendix

In this appendix we give the following remaining proof.

Proof of Lemma 2. We construct a function $f$ which has a prescribed behaviour near the endpoints of its support, so that the considered square roots are evidently smooth. The construction essentially consists of algebraic manipulations of $\varphi(x) := e^{-\frac{1}{2}} 1_{(0,\infty)}(x)$.

Consider the function

$$g(x) := c \varphi_1(x) \varphi_2(2 - \frac{2}{\varepsilon} x),$$

where $\varphi_1$ and $\varphi_2$ are defined as

$$\varphi_1(x) := ((3 - x) \varphi'(x))^\prime \quad \text{and} \quad \varphi_2(x) := \frac{\varphi(x)}{\varphi(x) + \varphi(1 - x)}. \tag{3.12}$$

The constant $c > 0$ is chosen such that $\int_{\mathbb{R}} g = 1$. The function $g$ is smooth, non-negative and supported on $[0, \varepsilon]$. Since $\varphi_2$ equals 1 for $x \geq 1$, for $\delta := \varepsilon/2$ we have

$$g = c \varphi_1 \quad \text{on} \quad (-\infty, \delta).$$

The factor $(3 - x)$ in the definition of $\varphi_1$ will be convenient when investigating (22).
We consider the antiderivative
\[ f(x) := \int_{-\infty}^{x} g(t - 1) - g(3 - t)dt, \]
which is smooth and even about \( x = 2 \), i.e. \( f(x) = f(4 - x) \). Moreover, it is supported on \([1, 3]\), positive on \((1, 3)\) and constantly equals 1 on \([1 + \varepsilon, 3 - \varepsilon]\). We have
\[ f(x) = c(4 - x)\varphi'(x - 1) \text{ on } (-\infty, 1 + \delta). \] (4.1)
Thus, \( f^{1/2} \) is smooth at \( x = 1 \). Smoothness at \( x = 3 \) follows by symmetry.

Consider now (2.2), which is due to oddness of the integrand equal to
\[ h(x) := \int_{-\infty}^{x} -\frac{f(t) + f(-t)}{t} dt. \]
The function \( h \) is even, supported on \([-3, 3]\) and positive on \((-3, 3)\). Using \( f(-t) = f(t + 4) \) and (4.1) we see that
\[ h(x) = c\varphi(x + 3) \text{ on } (-\infty, -3 + \delta). \]
This shows smoothness of \( h^{1/2} \) at \( x = -3 \). By symmetry the same holds at \( x = 3 \), which establishes the claim of the lemma. □

References

[1] F. Bernicot, Fiber-wise Calderón-Zygmund decoposition and application to a bi-dimensional paraproduct, Illinois J. Math. 56 (2012), no. 2, 415-422.
[2] C. Demeter and C. Thiele, On the two-dimensional bilinear Hilbert transform, Amer. J. Math., 132 (2010), no. 1, 201-256.
[3] R. L. Jones, A. Seeger, J. Wright, Strong variational and jump inequalities in harmonic analysis, Trans. Amer. Math. Soc., 360 (2008), no. 12, 67116742.
[4] V. Kovač, Bellman function technique for multilinear estimates and an application to generalized paraproducts, Indiana Univ. Math. J., 60 (2011), no. 3, 813-846.
[5] V. Kovač, Boundedness of the twisted paraproduct, Rev. Mat. Iberoam., 28 (2012), no. 4, 1143-1164.
[6] C. Thiele, Wave packet analysis, CBMS Reg. Conf. Ser. Math., 105, AMS, Providence, RI, 2006.

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