CATEGORICAL CRYSTALS FOR QUANTUM AFFINE ALGEBRAS

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Abstract. A new categorical crystal structure for the quantum affine algebras is presented. We introduce the notion of extended crystals $\hat{B}_g(\infty)$ for an arbitrary quantum group $U_q(g)$, which is the product of infinite copies of the crystal $B(\infty)$. For a complete duality datum $D$ in the Hernandez-Leclerc category $\mathcal{C}_g^0$ of a quantum affine algebra $U'_q(g)$, we prove that the set $B_D(g)$ of the isomorphism classes of simple modules in $\mathcal{C}_g^0$ has an extended crystal structure isomorphic to $\hat{B}_{g_{\text{fin}}}(\infty)$. An explicit combinatorial description of the extended crystal $B_D(g)$ for affine type $A_{n}(1)$ is given in terms of affine highest weights.

Contents

1. Introduction 2
2. Preliminaries 9
2.1. Crystals 9
2.2. Quantum affine algebras 10
2.3. R-matrices and related invariants 12
3. PBW theory for $\mathcal{C}_g^0$ 17
4. Extended crystals 22
5. Categorical crystals for quantum affine algebras 26

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1. Introduction

Let $q$ be an indeterminate and let $\mathcal{C}_g$ be the category of finite-dimensional integrable modules over a quantum affine algebra $U'_q(g)$. Because of its rich and complicated structure, the category $\mathcal{C}_g$ has been actively studied in various research areas (see [1, 9, 14, 18, 30, 44, 45] for example). The simple modules in $\mathcal{C}_g$ are indexed by using Drinfeld polynomials ([8, 9, 10, 11]) and they can be obtained as the head of ordered tensor product of fundamental representations ([1, 30, 44, 51]). Thus the simple modules in $\mathcal{C}_g$ are parameterized by the affine height weights (see Theorem 2.2 (iv) for the definition).

The category $\mathcal{C}_g$ has distinguished subcategories called Hernandez-Leclerc categories. We assume that $U'_q(g)$ is of untwisted affine ADE type. In [18], Hernandez and Leclerc introduced the monoidal full subcategory $\mathcal{C}^0_g$, which consists of objects whose all simple subquotients are obtained by taking the heads of tensor products of certain fundamental modules. Since any simple module in $\mathcal{C}_g$ can be obtained as a tensor product of suitable parameter shifts of simple modules in $\mathcal{C}^0_g$, the subcategory $\mathcal{C}^0_g$ occupies an important place in $\mathcal{C}_g$. For a Dynkin quiver $Q$, Hernandez and Leclerc introduced the monoidal subcategory $\mathcal{C}_Q$ of $\mathcal{C}^0_g$ defined by using the Auslander-Reiten quiver of $Q$ ([19]). It was proved in [19] that the complexified Grothendieck ring $\mathbb{C} \otimes_\mathbb{Z} K(\mathcal{C}_Q)$ is isomorphic to the coordinate ring $\mathbb{C}[N]$ of the unipotent group $N$ associated with a certain finite-dimensional simple Lie subalgebra $g_0$ of $g$ (see §2.2), and the set of the isomorphism classes of simple
modules in \( \mathcal{C}_Q \) corresponds to the upper global basis (or dual canonical basis) of \( \mathbb{C}[N] \) under the categorification.

The notion of the categories \( \mathcal{C}_g^0 \) and \( \mathcal{C}_Q \) is extended to all untwisted and twisted quantum affine algebras by using the poles of R-matrices among fundamental modules and a Q-datum \( Q \) ([15, 22, 25, 36, 46, 47]). The monoidal subcategory \( \mathcal{C}_Q \) is defined by using fundamental modules determined by a Q-datum \( Q \). Let \( g_{\text{fin}} \) be the simple Lie algebra of the type \( X_g \) defined in (3.3). Note that \( g_{\text{fin}} \) is of simply-laced finite type, and it arises naturally from the categorical structure of \( \mathcal{C}_g^0 \) ([33]). It coincides with \( g_0 \) when \( g \) is of untwisted affine ADE type. Similarly to the case of untwisted affine ADE type, the complexified Grothendieck ring \( \mathbb{C} \otimes \mathbb{Z} K(\mathcal{C}_Q) \) is isomorphic to the coordinate ring \( \mathbb{C}[N_{g_{\text{fin}}}] \) of the unipotent group \( N_{g_{\text{fin}}} \) associated with \( g_{\text{fin}} \), and the set of the isomorphism classes of simple modules corresponds to the upper global basis of \( \mathbb{C}[N_{g_{\text{fin}}}] \).

On the other hand, the notion of quiver Hecke algebras (or Khovanov-Lauda-Rouquier algebras) was introduced independently by Khovanov-Lauda [40, 41] and Rouquier [48] for a categorification of the half of a quantum group \( U_q(\mathfrak{g}) \). Let \( R \) be the quiver Hecke algebra associated with the quantum group \( U_q(\mathfrak{g}) \). The category \( R\text{-gmod} \) of finite-dimensional graded \( R \)-modules categorifies the unipotent quantum coordinate ring \( A_q(\mathfrak{n}) \), which can be understood as a \( q \)-deformation of the coordinate ring \( \mathbb{C}[N] \) associated with \( \mathfrak{g} \). The quiver Hecke algebras therefore takes an important position in the study of categorification and various new features have been studied (see [5, 21, 23, 26, 37, 42, 49, 52] for example).

The categorical structure of \( R\text{-gmod} \) was studied from the viewpoint of the crystal basis theory. The crystal graph (or shortly crystal) is one of the most powerful combinatorial tools to study quantum groups and their representations, and there are numerous connections and applications to various research areas including representation theory, combinatorics and geometry (see [27, 28, 29], and see also [6, 20, 43] and the references therein). The crystal arises naturally from the categorification using quiver Hecke algebras. In [42], Lauda and Vazirani proved that \( R\text{-gmod} \) has a categorical crystal structure isomorphic to \( B(\infty) \), i.e., the set of the isomorphism classes of simple \( R \)-modules has a \( U_q(\mathfrak{g}) \)-crystal structure isomorphic to the crystal \( B(\infty) \). The crystal operators \( \tilde{f}_i \) and \( \tilde{f}_i^* \) are given by taking the head of the convolution product with the 1-dimensional \( R(\alpha_i) \)-module \( L(i) \)
(see (3.2) for details). This result can be understood as a quiver Hecke algebra analogue of the classical results on the categorical crystal structure for affine Hecke algebras and their variants (see [2, 4, 16, 38, 50, 53], and see also [3, 39] and the references therein). Thus the set of the isomorphism classes of simple $R$-modules can be parameterized by the crystal $B(\infty)$ and the categorical relations among simple $R$-modules can be interpreted in terms of the crystal operators. This led us to a new combinatorial approach to study on $R$-gmod from the viewpoint of the crystal basis theory.

The PBW theory for $\mathcal{C}_g^0$ developed in [34] reveals interesting connections between quantum affine algebras and quiver Hecke algebras, which can be summarized as follows.

(I) The notion of strong (resp. complete) duality datum for $\mathcal{C}_g^0$ was introduced by investigating root modules. The complete duality datum generalizes the notion of $Q$-datum one step further. Let $\mathcal{D} = \{L_i\}_{i \in J}$ be a strong duality datum associated with a finite ADE type Cartan matrix $\mathbb{C}$, and let

$$\mathcal{F}_D : R_{\mathbb{C}}\text{-gmod} \longrightarrow \mathcal{C}_g^0$$

be the quantum affine Schur-Weyl duality functor [23], where $R_{\mathbb{C}}$ is the symmetric quiver Hecke algebra associated with $\mathbb{C}$. Let $\mathcal{C}_D$ be the image of $\mathcal{F}_D$ (see Section 3 for the precise definition). It was proved in [34] that the duality functor $\mathcal{F}_D$ sends simple modules in $R_{\mathbb{C}}$-gmod to simple modules in $\mathcal{C}_D$ and it preserves the invariants $\Lambda$, $\Lambda^\infty$ and $\mathfrak{b}$ (see [32] for details on $\Lambda$, $\Lambda^\infty$ and $\mathfrak{b}$ for $\mathcal{C}_g^0$). The invariants for $\mathcal{C}_g^0$ take a crucial role in the recent developments about block decomposition, the PBW theory, and monoidal categorification for quantum affine algebras (see [33, 34, 35]). The duality functor $\mathcal{F}_D$ tells us that $\mathcal{C}_D$ has the categorical crystal structure induced from $R_{\mathbb{C}}$-gmod via $\mathcal{F}_D$.

(II) The notion of affine cuspidal modules for $\mathcal{C}_g^0$ was introduced. Let $\mathcal{D}$ be a complete duality datum. Let $w_0$ be a reduced expression of the longest element $w_0$ of the Weyl group of $\mathfrak{g}_{\text{fin}}$, and let $\{V_k\}_{k=1, \ldots, \ell}$ be the set of the cuspidal modules of the quiver Hecke algebra $R_{\mathbb{C}}$ associated with $w_0$. The affine cuspidal modules $\{S_k\}_{k \in \mathbb{Z}}$ for $\mathcal{C}_g^0$ are defined by using $\{V_k\}_{k=1, \ldots, \ell}$ via the duality functor $\mathcal{F}_D$ (see Section 3 for the precise definition). When $\mathcal{D}$ arises from a $Q$-datum, the affine cuspidal modules coincide with the fundamental modules in $\mathcal{C}_g^0$. It turned out that all simple modules in $\mathcal{C}_g^0$ can be obtained uniquely
as the simple heads of the ordered tensor products $P_{\mathcal{D}, \omega_0}(a)$ of affine cuspidal modules. Moreover, the module $P_{\mathcal{D}, \omega_0}(a)$, called standard module, has the unitriangularity property. This generalizes the classical simple module construction taking the head of ordered tensor products of fundamental representations ([1, 30, 44, 45, 51]). From the viewpoint of the PBW theory, it is natural that the Grothendieck ring $K(\mathcal{C}_g^0)$ can be viewed as a product of infinite copies of $K(\mathcal{C}_D)$. Note that, when $D$ arises from a Dynkin quiver of a finite ADE type, it was already observed by Hernandez-Leclerc (see [19]).

The purpose of this paper is to study a new categorical crystal structure of the category $\mathcal{C}_g^0$ by using the new invariants and the PBW theory. Our main results can be summarized as follows:

(i) We introduce the notion of extended crystals $\hat{B}_g^{\infty}$ for an arbitrary quantum group $U_q(g)$. The extended crystal $\hat{B}_g^{\infty}$ can be understood as a product of infinite copies of $B(\infty)$ and its crystal operators are defined by using the crystal structure of $B(\infty)$. Thus the extended crystal $\hat{B}_g^{\infty}$ may look like an affinization of the crystal $B(\infty)$. As a usual crystal, the extended crystal $\hat{B}_g^{\infty}$ has the colored graph structure induced from its crystal operators.

(ii) We prove that the category $\mathcal{C}_g^0$ has a new categorical crystal structure isomorphic to the extended crystal $\hat{B}_g^{\infty}$. Let $\mathcal{D} = \{L_i\}_{i \in I_{\text{fin}}}$ be a complete duality datum of $\mathcal{C}_g^0$ and let $\mathcal{B}_D(g)$ be the set of the isomorphism classes of simple modules in $\mathcal{C}_g^0$. The operators $\tilde{F}_{i,k}$, $\tilde{E}_{i,k}$, $\tilde{F}_{i,k}^*$, and $\tilde{E}_{i,k}^*$ defined in (5.9) gives a categorical crystal structure on $\mathcal{B}_D(g)$. This crystal structure turns out isomorphic to that of $\hat{B}_g^{\infty}$. Thus the simple modules in $\mathcal{C}_g^0$ are parameterized by the extended crystal $\hat{B}_g^{\infty}$ which leads us to a new combinatorial approach to study $\mathcal{C}_g^0$.

(iii) We provide explicit formulas to compute the invariants $\Lambda$ and $\delta$ between $\mathcal{D}^kL_i$ $(k \in \mathbb{Z})$ and an arbitrary simple module in terms of the extended crystal.

(iv) We give an explicit combinatorial description of the extended crystal $\mathcal{B}_D(g)$ for affine type $A_n^{(1)}$ in terms of affine highest weights.

Let us explain our results more precisely. Let $I$ be an index set and let $g$ be the Kac-Moody algebra associated with a symmetrizable generalized Cartan matrix. The extended
crystal is defined as

$$\hat{B}_g(\infty) := \left\{ (b_k)_{k \in \mathbb{Z}} \in \prod_{k \in \mathbb{Z}} B(\infty) \mid b_k = 1 \text{ for all but finitely many } k \right\},$$

where $B(\infty)$ is the crystal of $U_q^- (g)$. Let $\hat{I} := I \times \mathbb{Z}$. For $b = (b_k)_{k \in \mathbb{Z}} \in \hat{B}_g(\infty)$ and $(i, k) \in \hat{I}$, we define the extended crystal operator $\tilde{F}_{(i, k)} (b)$ (resp. $\tilde{E}_{(i, k)} (b)$) in terms of the usual crystal operators $\tilde{f}_i (b_k)$ and $\tilde{e}_i^*(b_{k+1})$ (resp. $\tilde{e}_i (b_k)$ and $\tilde{f}_i^*(b_{k+1})$) according to the values of $\varepsilon_i(b_k)$ and $\varepsilon_i^*(b_{k+1})$ (see (4.1)). We show that there exist natural injections $\iota_k : B(\infty) \rightarrow \hat{B}_g(\infty)$ ($k \in \mathbb{Z}$) and interesting bijections $\ast$ and $D$ on $\hat{B}_g(\infty)$ compatible with the extended crystal structure (see Lemma 4.2 and Lemma 4.3). Similarly to the usual crystals, the extended crystal $\hat{B}_g(\infty)$ has a connected $\hat{I}$-colored graph structure induced from the operators $\tilde{F}_{i, k}$ (see Lemma 4.4).

We next deal with the category $\mathcal{C}_g^0$ from the viewpoint of the extended crystal. Let $U'_q(g)$ be a quantum affine algebra of arbitrary type and let $\mathcal{D} = \{ L_i \}_{i \in \hat{I}_{\text{fin}}}$ be a complete duality datum of $\mathcal{C}_g^0$. We define $\mathcal{B}(g)$ to be the set of the isomorphism classes of simple modules in $\mathcal{C}_g^0$. For $b = (b_k)_{k \in \mathbb{Z}} \in \hat{B}_{\text{fin}}(\infty)$, we define

$$\mathcal{L}_D (b) := \text{hd}(\cdots \otimes D^2(\mathcal{L}_D(b_2)) \otimes D(\mathcal{L}_D(b_1)) \otimes \mathcal{L}_D(b_0) \otimes D^{-1}(\mathcal{L}_D(b_{-1})) \otimes \cdots),$$

where $\text{hd}(X)$ stands for the head of a module $X$, $D$ is the right dual functor and $\mathcal{L}_D(b_k)$ is the simple module in $\mathcal{C}_D$ corresponding to $b_k$ via the duality functor $\mathcal{F}_D$ (see Lemma 3.2 (i)). Proposition 5.5 says that $\mathcal{L}_D (b)$ is simple and the map

$$\Phi_D : \hat{B}_{\text{fin}}(\infty) \rightarrow \mathcal{B}(g), \quad b \mapsto \mathcal{L}_D (b)$$

is bijective. Moreover the bijection $D$ on $\hat{B}_{\text{fin}}(\infty)$ is compatible with the functor $D$ under the map $\Phi_D$. For $M \in \mathcal{B}(g)$ and $(i, k) \in \hat{I}_{\text{fin}}$, we define

$$\tilde{F}_{i, k} (M) := (D^{h} L_i) \nabla M \quad \text{and} \quad \tilde{E}_{i, k} (M) := M \nabla (D^{k+1} L_i),$$

where $X \nabla Y$ denotes the the head of $X \otimes Y$. Since the operators $\tilde{F}_{i, k}$ and $\tilde{E}_{i, k}$ depend on the choice of $\mathcal{D}$, we write $\mathcal{B}_D(g)$ instead of $\mathcal{B}(g)$ when we consider $\mathcal{B}(g)$ together with the operators $\tilde{F}_{i, k}$ and $\tilde{E}_{i, k}$. We prove that $\tilde{F}_{i, k}$ and $\tilde{E}_{i, k}$ are inverse to each other and the
bijection $\Phi_D$ has compatibility with the extended crystal operators, i.e.,

$$
\Phi_D(\tilde{F}_{i,k}(b)) = \tilde{F}_{i,k}(\Phi_D(b)), \quad \Phi_D(\tilde{E}_{i,k}(b)) = \tilde{E}_{i,k}(\Phi_D(b))
$$

(see Theorem 5.9). Therefore $B_D(g)$ has the same extended crystal structure as $\tilde{B}_{\text{fin}}(\infty)$ and, for any two quantum affine algebras $U_q'(g)$ and $U_q'(g')$, $B_D(g) \simeq B_D(g')$ as a colored graph if and only if $g_{\text{fin}} \simeq (g')_{\text{fin}}$ as a simple Lie algebra. In particular, the $\tilde{f}_{\text{fin}}$-colored graph structure of $B_D(g)$ does not depend on the choice of complete duality data $D$ (see Corollary 5.10).

Explicit formulas for computing the invariants $\Lambda$ and $\bar{b}$ between $D^kL_i$ ($k \in \mathbb{Z}$) and a simple module are given in Theorem 6.3. Let $M$ be a simple module in $\mathcal{C}^0_g$ and let $b = b_D(M) := \Phi_D^{-1}(M) \in \tilde{B}_{\text{fin}}(\infty)$. By investigating properties of root modules, we prove that

(a) $\Lambda(D^kL_i, M) = 2\max\{x, r\} + \sum_{t \in \mathbb{Z}} (-1)^{\delta(t > k)}(\alpha_i, \text{wt}_t(b))$,

(b) $\Lambda(M, D^kL_i) = 2\max\{y, s\} + \sum_{t \in \mathbb{Z}} (-1)^{\delta(t < k)}(\alpha_i, \text{wt}_t(b))$,

(c) $b(D^kL_i, M) = \max\{x, r\} + \max\{y, s\} + (\alpha_i, \text{wt}_{k}(b))$.

where $x := \varepsilon^*_{(i,k+1)}(b)$, $r := \varepsilon_{(i,k)}(b)$, $s := \varepsilon^*_{(i,k)}(b)$, and $y := \varepsilon_{(i,k-1)}(b)$. We remark that these formulas can be understood as a quantum affine algebra analogue of the ones given in [31, Corollary 3.8].

For affine type $A_n^{(1)}$, we provide an explicit combinatorial description of the extended crystal $B_D(g)$ in terms of affine highest weights. Let $U_q'(g)$ be the quantum affine algebra of type $A_n^{(1)}$ and let $\mathcal{C}^0_g$ be the Hernandez-Leclerc category corresponding to $\sigma_0(g) := \{(i, (-q)^a) \in \mathbb{Z}_0 \times \mathbb{Z} \mid a - i \equiv 1 \mod 2\}$. Let

$$
\mathcal{P}_n := (\mathbb{Z}_{\geq 0})^{\otimes \mathcal{I}_n},
$$

where $\mathcal{I}_n := \{(i, a) \in \mathbb{Z}_0 \times \mathbb{Z} \mid a - i \equiv 1 \mod 2\}$. Note that the set of fundamental modules in $\mathcal{C}^0_g$ is $\{V(\varpi_i)(-q)^a \mid (i, a) \in \mathcal{I}_n\}$. For any $\lambda = \sum_{(i, a) \in \mathcal{I}_n} c_{(i,a)}(i,a) \in \mathcal{P}_n$, we denote by $V(\lambda)$ the simple module in $\mathcal{C}^0_g$ with the affine highest weight $\sum_{(i,a) \in \mathcal{I}_n} c_{(i,a)}(i,(-q)^a)$ (see Theorem 2.2 (iv)), which gives the bijection

$$
\Psi_n : \mathcal{P}_n \rightarrow B_D(g), \quad \lambda \mapsto V(\lambda) \quad \text{for} \ \lambda \in \mathcal{P}_n.
$$
For \((i,k) \in \widehat{I}_0\), we define the crystal operators \(\widetilde{F}_{i,k}\) and \(\widetilde{E}_{i,k}\) on \(\mathcal{P}_n\) in a combinatorial way (see Section 7.3). We briefly explain the combinatorial rule for \(\widetilde{F}_{i,k}\). For a given \(\lambda \in \mathcal{P}_n\), we first choose a suitable subset \(S_{i,k} = \{v_{2n}, v_{2n-1}, \ldots, v_1\}\) of \(\mathcal{P}_n\) and make a sequence of + and − according to \(S_{i,k}\) and the coefficients of \(\lambda\). We then cancel out all possible \((+,-)\) pairs to obtain a sequence of −’s followed by +’s. Let \(v_t\) be the element of \(S_{i,k}\) corresponding to the leftmost + in the resulting sequence of \(\lambda\). If such an \(v_t\) exists, then we define

\[
\widetilde{F}_{i,k}(\lambda) := \lambda - v_t + v_{t+1}.
\]

Otherwise, we define

\[
\widetilde{F}_{i,k}(\lambda) := \lambda + v_1.
\]

We remark that this combinatorial rule is quite similar to the \((+,−)\)-signature rule for the tensor product of crystals. Let \(D := \{V(\varpi_1)(−q^{2i−2})\}_{i \in \widehat{I}_0}\). Theorem 7.5 tells us that there is a bijection \(\Upsilon_n: \widehat{B}_0(\infty) \overset{\sim}{\longrightarrow} \mathcal{P}_n\) which is compatible with the crystal operators \(\widetilde{F}_{i,k}\) and \(\widetilde{E}_{i,k}\) and, for \((i,k) \in \widehat{I}_0\) and \(\lambda \in \mathcal{P}_n\),

\[
\widetilde{F}_{i,k}(\Psi_n(\lambda)) = \Psi_n(\widetilde{F}_{i,k}(\lambda)), \quad \widetilde{E}_{i,k}(\Psi_n(\lambda)) = \Psi_n(\widetilde{E}_{i,k}(\lambda)).
\]

Hence the following diagram

\[
\begin{array}{ccc}
\widehat{B}_0(\infty) & \xrightarrow{\Phi_D} & \mathcal{B}_D(\mathfrak{g}) \\
\Upsilon_n \downarrow & & \uparrow \Psi_n \\
\mathcal{P}_n & \xrightarrow{\Psi_n} & \mathcal{B}_n
\end{array}
\]

commutes, and the arrows are \(\widehat{I}_0\)-colored graph isomorphisms. In the course of proofs, the multisegment realization for the crystal \(B(\infty)\) is used crucially (see Section 7.1).

The paper is organized as follows. In Section 2, we briefly review necessary background on crystals, quantum affine algebras, and the new invariants \(\Lambda, \Lambda^\infty\) and \(\mathfrak{b}\). In Section 3, we recall the PBW theory for \(\mathcal{C}_0^0\) developed in [34]. In Section 4, we introduce the notion of the extended crystals. In Section 5, we prove that the category \(\mathcal{C}_0^0\) has a categorical crystal structure isomorphic to the extended crystal \(\widehat{B}_{0\text{fin}}(\infty)\). In Section 6, we give formulas to
compute the invariants $\Lambda$ and $\nu$ in terms of the extended crystal. In Section 7, we give a combinatorial description of the extended crystal $B_D(\mathfrak{g})$ in terms of affine highest weights.

2. Preliminaries

Convention.

(i) For a statement $P$, $\delta(P)$ is 1 or 0 according that $P$ is true or not.

(ii) For a (semi-)ring $R$ and a set $A$, we denote by $R^{\oplus A}$ the direct sum of copies of $R$ indexed by $A$.

(iii) For a module $X$ of finite length, $\text{hd}(X)$ denotes the head of $X$ and $\text{soc}(X)$ denotes the socle of $X$.

2.1. Crystals

Let $I$ be an index set. A quintuple $(A, P, \Pi, P^\vee, \Pi^\vee)$ is called a (symmetrizable) Cartan datum if it consists of

(a) a generalized Cartan matrix $A = (a_{ij})_{i,j \in I}$,

(b) a free abelian group $P$, called the weight lattice,

(c) $\Pi = \{\alpha_i \mid i \in I\} \subset P$, called the set of simple roots,

(d) $P^\vee = \text{Hom}_\mathbb{Z}(P, \mathbb{Z})$, called the coweight lattice,

(e) $\Pi^\vee = \{h_i \in P^\vee \mid i \in I\}$, called the set of simple coroots,

which satisfy the following properties:

(i) $\langle h_i, \alpha_j \rangle = a_{ij}$ for $i, j \in I$,

(ii) $\Pi$ is linearly independent over $\mathbb{Q}$,

(iii) for each $i \in I$, there exists $\Lambda_i \in P$, called a fundamental weight, such that $\langle h_j, \Lambda_i \rangle = \delta_{j,i}$ for all $j \in I$.

(iv) there is a symmetric bilinear form $(\cdot, \cdot)$ on $P$ satisfying $(\alpha_i, \alpha_i) \in \mathbb{Q}_{>0}$ and $\langle h_i, \lambda \rangle = 2(\alpha_i, \lambda)/(\alpha_i, \alpha_i)$.

We set $Q := \bigoplus_{i \in I} \mathbb{Z} \alpha_i$ and $Q^+ := \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i \subset Q$ and define $\text{ht}(\beta) = \sum_{i \in I} k_i$ for $\beta = \sum_{i \in I} k_i \alpha_i \in Q^+$. We write $\Delta^+$ for the set of positive roots associated with $A$ and set $\Delta^- := -\Delta^+$. Denote by $W$ the Weyl group, which is the subgroup of $\text{Aut}(P)$ generated by $s_i(\lambda) := \lambda - \langle h_i, \lambda \rangle \alpha_i$ for $i \in I$. We denote by $U_q(\mathfrak{g})$ the quantum group associated
with \((A, P, P^\vee, \Pi, \Pi^\vee)\), which is a \(\mathbb{Q}(q)\)-algebra generated by \(f_i, e_i\ (i \in I)\) and \(q^h\ (h \in P^\vee)\) with certain defining relations (see [20, Chapter 3] for details). We denote by \(U_q^- (g)\) the subalgebra of \(U_q (g)\) generated by \(f_i\ (i \in I)\). Let us recall the notion of a crystal. We refer the reader to \([27, 28, 29]\) and [20, Chapter 4] for more details.

**Definition 2.1.** A crystal is a set \(B\) endowed with maps \(\text{wt}: B \to P, \varphi_i, \varepsilon_i: B \to \mathbb{Z} \cup \{\infty\}\) and \(\tilde{e}_i, \tilde{f}_i: B \to B \sqcup \{0\}\) for all \(i \in I\) which satisfy the following axioms:

(a) \(\varphi_i(b) = \varepsilon_i(b) + \langle h_i, \text{wt}(b) \rangle\),
(b) \(\text{wt}(\tilde{e}_i b) = \text{wt}(b) + \alpha_i\) if \(\tilde{e}_i b \in B\), and \(\text{wt}(\tilde{f}_i b) = \text{wt}(b) - \alpha_i\) if \(\tilde{f}_i b \in B\),
(c) for \(b, b' \in B\) and \(i \in I\), \(b' = \tilde{e}_i b\) if and only if \(b = \tilde{f}_i b'\),
(d) for \(b \in B\), if \(\varphi_i(b) = -\infty\), then \(\tilde{e}_i b = \tilde{f}_i b = 0\),
(e) if \(b \in B\) and \(\tilde{e}_i b \in B\), then \(\varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) - 1\) and \(\varphi_i(\tilde{e}_i b) = \varphi_i(b) + 1\),
(f) if \(b \in B\) and \(\tilde{f}_i b \in B\), then \(\varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b) + 1\) and \(\varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1\).

We denote by \(B_{\theta}(\infty)\) the crystal of \(U_q^- (g)\) and let \(1\) be the highest vector of \(B_{\theta}(\infty)\). We simply write \(B(\infty)\) instead of \(B_{\theta}(\infty)\) if no confusion arises. The \(\mathbb{Q}(q)\)-antiautomorphism \(\ast\) of \(U_q (g)\) defined by

\[
(e_i)^\ast = e_i, \quad (f_i)^\ast = f_i, \quad (q^h)^\ast = q^{-h},
\]

(2.1) gives an involution on \(B(\infty)\). This provides another crystal structure with \(\tilde{e}_i^\ast, \tilde{f}_i^\ast, \varepsilon_i^\ast, \varphi_i^\ast\) on \(U_q^- (g)\), which is denoted by \(B(\infty)^\ast\). We set

(2.2) \(\text{ht}(b) := \text{ht}(- \text{wt}(b))\) for \(b \in B(\infty)\).

2.2. **Quantum affine algebras.** We assume that \(A = (a_{i,j})_{i,j \in I}\) is an affine Cartan matrix. Note that the rank of \(P\) is \(|I| + 1\). We choose a \(\mathbb{Q}\)-valued non-degenerate symmetric bilinear form \((\ , \ )\) on \(P\) satisfying

\[
\langle h_i, \lambda \rangle = \frac{2(\alpha_i, \lambda)}{(\alpha_i, \alpha_i)} \quad \text{and} \quad \langle c, \lambda \rangle = (\delta, \lambda)
\]

for any \(i \in I\) and \(\lambda \in P\), where \(\delta\) is the imaginary root in \(P\) and \(c\) is the central element in \(P^\vee\). We take \(\rho \in P\) (resp. \(\rho^\vee \in P^\vee\)) such that \(\langle h_i, \rho \rangle = 1\) (resp. \(\langle \rho^\vee, \alpha_i \rangle = 1\)) for any \(i \in I\). Let \(g\) be the affine Kac-Moody algebra associated with \(A\) and set \(I_0 := I \setminus \{0\}\). Here we
refer the reader to [34, Section 2.3] for a choice of $0 \in I$. We denote by $\mathfrak{g}_0$ the subalgebra of $\mathfrak{g}$ generated by $e_i, f_i$ ($i \in I_0$).

Let $q$ be an indeterminate and $k$ the algebraic closure of the subfield $\mathbb{C}(q)$ in the algebraically closed field $\hat{k} := \bigcup_{m>0} \mathbb{C}((q^{1/m}))$. Let $U'_q(\mathfrak{g})$ be the $k$-subalgebra of the quantum group $U_q(\mathfrak{g})$ generated by $e_i, f_i, K_i := q_i^{\pm h_i}$ ($i \in I$), where $q_i := q^{(\alpha_i, \alpha_i)}/2$. Let $\mathcal{C}_g$ be the category of finite-dimensional integrable $U'_q(\mathfrak{g})$-modules, i.e., finite-dimensional modules $M$ with a weight decomposition

$$M = \bigoplus_{\lambda \in P_{cl}} M_\lambda$$

where $P_{cl} := P/(P \cap \mathbb{Q} \delta)$. The tensor product $\otimes$ gives a monoidal category structure on $\mathcal{C}_g$. We set $M^{\otimes k} := M \otimes \cdots \otimes M$ for $k \in \mathbb{Z}_{\geq 0}$. For $M, N \in \mathcal{C}_g$, we denote by $M \triangleright N$ the head of $M \otimes N$ and by $M \Delta N$ the socle of $M \otimes N$. We say that $M$ and $N$ commute if $M \otimes N \simeq N \otimes M$. We say that $M$ and $N$ strongly commute if $M \otimes N$ is simple. A simple $U'_q(\mathfrak{g})$-module $L$ is real if $L \otimes L$ is simple. For $M \in \mathcal{C}_g$, we denote by $\mathcal{D}M$ and $\mathcal{D}^{-1}M$ the right and the left dual of $M$, respectively. We extend this to $\mathcal{D}^k(M)$ for all $k \in \mathbb{Z}$.

A simple module $L$ in $\mathcal{C}_g$ contains a non-zero vector $u \in L$ of weight $\lambda \in P_{cl}$ such that (i) $\langle h_i, \lambda \rangle \geq 0$ for all $i \in I_0$, (ii) all the weight of $L$ are contained in $\lambda - \sum_{i \in I_0} Z_{\geq 0} \text{cl}(\alpha_i)$, where $\text{cl} : P \to P_{cl}$ is the canonical projection. Such a $\lambda$ is unique and $u$ is unique up to a constant multiple. We call $\lambda$ the dominant extremal weight of $L$ and $u$ a dominant extremal weight vector of $L$. For each $i \in I_0$, we set

$$\varpi_i := \gcd(c_0, c_i)^{-1} \text{cl}(c_0 \Lambda_i - c_i \Lambda_0) \in P_{cl},$$

where the central element $c$ is equal to $\sum_{i \in I} c_i h_i$. For any $i \in I_0$, we denote by $V(\varpi_i)$ the $i$-th fundamental representation. Note that the dominant extremal weight of $V(\varpi_i)$ is $\varpi_i$.

For a module $M \in \mathcal{C}_g$, we denote by $M^{\text{aff}}$ the affinization of $M$ and by $z_M : M^{\text{aff}} \to M^{\text{aff}}$ the $U'_q(\mathfrak{g})$-module automorphism of weight $\delta$. Note that $M^{\text{aff}} \simeq k[z^{\pm 1}] \otimes_k M$ with the action

$$e_i (a \otimes v) = z^{h_i a} a \otimes e_i v \quad \text{for } a \in k[z^{\pm 1}] \text{ and } v \in M.$$
We sometimes write $M_z$ instead of $M^{\text{aff}}$ to emphasize the endomorphism $z$. For $x \in k^\times$, we define

$$M_x := M^{\text{aff}}/(z_M - x)M^{\text{aff}}.$$ 

We call $x$ a spectral parameter (see [30, Section 4.2] for details).

For $i \in I_0$, let $m_i$ be a positive integer such that $W_{\pi_i} \cap (\pi_i + \mathbb{Z}\delta) = \pi_i + \mathbb{Z}m_i\delta,$

where $\pi_i$ is an element of $\mathfrak{p}$ such that $\text{cl}(\pi_i) = \varpi_i$. Then, $V(\varpi_i)_x \simeq V(\varpi_i)_y$ if and only if $x^{m_i} = y^{m_i}$ for $x, y \in k^\times$ (see [1, Section 1.3]). We define

$$\sigma(\mathfrak{g}) := I_0 \times k^\times/\sim,$$

where the equivalence relation $\sim$ is given by

$$(i, x) \sim (j, y) \iff V(\varpi_i)_x \simeq V(\varpi_j)_y \iff i = j \text{ and } x^{m_i} = y^{m_j}.$$ 

We denote by $[(i, a)]$ the equivalence class of $(i, a)$ in $\sigma(\mathfrak{g})$. When no confusion arises, we simply write $(i, a)$ for the equivalence class $[(i, a)]$. For $(i, x)$ and $(j, y) \in \sigma(\mathfrak{g})$, we put $d$ many arrows from $(i, x)$ to $(j, y)$, where $d$ is the order of zeros of the denominator $d_{V(\varpi_i), V(\varpi_j)}(z_{V(\varpi_i)}/z_{V(\varpi_i)})$ at $z_{V(\varpi_j)}/z_{V(\varpi_i)} = y/x$ (see §2.3). Since $\sigma(\mathfrak{g})$ has a quiver structure, we choose a connected component $\sigma_0(\mathfrak{g})$ of $\sigma(\mathfrak{g})$. Since a connected component of $\sigma(\mathfrak{g})$ is unique up to a spectral parameter shift, $\sigma_0(\mathfrak{g})$ is uniquely determined up to a quiver isomorphism. The Hernandez-Leclerc category $\mathcal{C}_0^{\mathfrak{g}}$ is the smallest full subcategory of $\mathcal{C}_0^{\mathfrak{g}}$ such that

(a) $\mathcal{C}_0^{\mathfrak{g}}$ contains $V(\varpi_i)_x$ for all $(i, x) \in \sigma_0(\mathfrak{g})$,

(b) $\mathcal{C}_0^{\mathfrak{g}}$ is stable by taking subquotients, extensions and tensor products.

2.3. R-matrices and related invariants. In this subsection, we briefly recall the new invariants $\Lambda, \Lambda^\infty$ and $\mathfrak{b}$ introduced in [32] and several results of [32, 34]. For the notion of (universal) $R$-matrices and related properties, we refer the reader to [13], [1, Appendices A and B], [30, Section 8] and [17].
For non-zero modules $M, N \in \mathscr{C}_q$, we denote by $R^\text{univ}_{M,N_z} : k((z)) \otimes_{k[z^\pm 1]} (M \otimes N_z) \to k((z)) \otimes_{k[z^\pm 1]} (N_z \otimes M)$ the universal $R$-matrix. We say that $R^\text{univ}_{M,N_z}$ is rationally renormalizable if there exists $f(z) \in k((z))^\times$ such that

$$f(z)R^\text{univ}_{M,N_z} (M \otimes N_z) \subset N_z \otimes M.$$ 

If $R^\text{univ}_{M,N_z}$ is rationally renormalizable, then we can choose $c_{M,N}(z) \in k((z))^\times$ such that $R^\text{ren}_{M,N_z} := c_{M,N}(z)R^\text{univ}_{M,N_z}$ sends $M \otimes N_z$ to $N_z \otimes M$ and its specialization

$$R^\text{ren}_{M,N_z} |_{z = x} : M \otimes N_x \to N_x \otimes M$$

does not vanish at any $z = x \in k^\times$. Note that $R^\text{ren}_{M,N_z}$ and $c_{M,N}(z)$ are unique up to a multiple of $k[z^\pm 1]^\times = \bigsqcup_{n \in \mathbb{Z}} k^x z^n$. We call $R^\text{ren}_{M,N_z}$ the renormalized $R$-matrix and $c_{M,N}(z)$ the renormalizing coefficient. We denote by $r_{M,N}$ the specialization at $z = 1$

$$(2.3) \quad r_{M,N} := R^\text{ren}_{M,N_z} |_{z = 1} : M \otimes N \to N \otimes M,$$

and call it the $R$-matrix. The $R$-matrix $r_{M,N}$ is well-defined up to a constant multiple whenever $R^\text{univ}_{M,N_z}$ is rationally renormalizable. By the definition, $r_{M,N}$ never vanishes.

Let $M$ and $N$ be simple modules in $\mathscr{C}_q$ and let $u$ and $v$ be dominant extremal weight vectors of $M$ and $N$, respectively. Then there exists $a_{M,N}(z) \in k[[z]]^\times$ such that

$$R^\text{univ}_{M,N_z}(u \otimes v_z) = a_{M,N}(z)(v_z \otimes u).$$

Thus we have a unique $k(z) \otimes U'_q(g)$-module isomorphism

$$R^\text{norm}_{M,N_z} := a_{M,N}(z)^{-1}R^\text{univ}_{M,N_z} |_{k(z) \otimes k[z^\pm 1](M \otimes N_z)}$$

from $k(z) \otimes_{k[z^\pm 1]} (M \otimes N_z)$ to $k(z) \otimes_{k[z^\pm 1]} (N_z \otimes M)$, which satisfies $R^\text{norm}_{M,N_z}(u \otimes v_z) = v_z \otimes u$. We call $a_{M,N}(z)$ the universal coefficient of $M$ and $N$, and $R^\text{norm}_{M,N_z}$ the normalized $R$-matrix.

Let $d_{M,N}(z) \in k[z]$ be a monic polynomial of the smallest degree such that the image of $d_{M,N}(z)R^\text{norm}_{M,N_z}(M \otimes N_z)$ is contained in $N_z \otimes M$, which is called the denominator of $R^\text{norm}_{M,N_z}$. Then we have $R^\text{ren}_{M,N_z} = d_{M,N}(z)R^\text{norm}_{M,N_z} : M \otimes N_z \to N_z \otimes M$ up to a multiple of $k[z^\pm 1]^\times$. Thus $R^\text{ren}_{M,N_z} = a_{M,N}(z)^{-1}d_{M,N}(z)R^\text{univ}_{M,N_z}$ and $c_{M,N}(z) = \frac{d_{M,N}(z)}{a_{M,N}(z)}$ up to a multiple of $k[z^\pm 1]^\times$. In particular, $R^\text{univ}_{M,N_z}$ is rationally renormalizable whenever $M$ and $N$ are simple.
In the following theorem, we refer [30] for the notion of good modules. Note that any good module is real simple and every fundamental module $V(\varpi_i)$ is a good module.

**Theorem 2.2 ([1, 7, 30, 24]).**

(i) For simple modules $M$ and $N$ such that one of them is real, $M_x$ and $N_y$ strongly commute to each other if and only if $d_{M,N}(z)d_{N,M}(1/z)$ does not vanish at $z = y/x$.

(ii) For good modules $M$ and $N$, the zeroes of $d_{M,N}(z)$ belong to $\mathbb{C}[q^{1/m}]q^{1/m}$ for some $m \in \mathbb{Z}_{>0}$.

(iii) Let $M_k$ be a good module with a dominant extremal vector $u_k$ of weight $\lambda_k$, and $a_k \in k^x$ for $k = 1, \ldots, t$. Assume that $a_j/a_i$ is not a zero of $d_{M_i,M_j}(z)$ for any $1 \leq i < j \leq t$. Then the following statements hold.

(a) $(M_1)_{a_1} \otimes \cdots \otimes (M_t)_{a_t}$ is generated by $u_1 \otimes \cdots \otimes u_t$.

(b) The head of $(M_1)_{a_1} \otimes \cdots \otimes (M_t)_{a_t}$ is simple.

(c) Any non-zero submodule of $(M_t)_{a_t} \otimes \cdots \otimes (M_1)_{a_1}$ contains the vector $u_t \otimes \cdots \otimes u_1$.

(d) The socle of $(M_t)_{a_t} \otimes \cdots \otimes (M_1)_{a_1}$ is simple.

(e) Let $r: (M_1)_{a_1} \otimes \cdots \otimes (M_t)_{a_t} \to (M_t)_{a_t} \otimes \cdots \otimes (M_1)_{a_1}$ be $r_{(M_1)_{a_1},\ldots,(M_t)_{a_t}} := \prod_{1 \leq j < k \leq t} r_{(M_j)_{a_j},(M_k)_{a_k}}$. Then the image of $r$ is simple and it coincides with the head of $(M_1)_{a_1} \otimes \cdots \otimes (M_t)_{a_t}$ and also with the socle of $(M_t)_{a_t} \otimes \cdots \otimes (M_1)_{a_1}$.

(iv) For any simple module $M \in \mathcal{C}_g$, there exists a finite sequence $\{(i_k,a_k)\}_{1 \leq k \leq t}$ in $\sigma(\mathfrak{g})$ such that $M$ has $\sum_{k=1}^t \varpi_{i_k}$ as a dominant extremal weight and it is isomorphic to a simple subquotient of $V(\varpi_{i_1})_{a_1} \otimes \cdots V(\varpi_{i_t})_{a_t}$. Moreover, such a sequence $\{(i_k,a_k)\}_{1 \leq k \leq t}$ is unique up to a permutation.

We call $\sum_{k=1}^t (i_k,a_k) \in \widehat{\mathbb{P}}^+ := \mathbb{Z}_{\geq 0}^{\oplus \sigma(\mathfrak{g})}$, the affine highest weight of $M$.

We define

$$\varphi(z) := \prod_{s=0}^{\infty} (1 - \tilde{p}^s z) = \sum_{n=0}^{\infty} \frac{(-1)^n \tilde{p}^{n(n-1)/2}}{\prod_{k=1}^{n} (1 - \tilde{p}^k)} z^n \in k[[z]],$$
where $p^* := (-1)^{\rho^\vee,\delta} q^{(c,\rho)}$ and $\tilde{p} := (p^*)^2 = q^{2(c,\rho)}$. We consider the subgroup

$$\mathcal{G} := \left\{ cz^m \prod_{a \in k^\times} \varphi(az)^{\eta_a} \mid c \in k^\times, m \in \mathbb{Z}, \eta_a \in \mathbb{Z} \text{ vanishes except finitely many } a \text{'s.} \right\} \subset k((z))^\times.$$

For a subset $S$ of $\mathbb{Z}$, let $\tilde{p}^S := \{ \tilde{p}^k \mid k \in S \}$, and define group homomorphisms $\text{Deg}: \mathcal{G} \to \mathbb{Z}$ and $\text{Deg}^\infty: \mathcal{G} \to \mathbb{Z}$ by

$$\text{Deg}(f(z)) = \sum_{a \in \tilde{p}^\leq 0} \eta_a - \sum_{a \in \tilde{p}^> 0} \eta_a \quad \text{and} \quad \text{Deg}^\infty(f(z)) = \sum_{a \in \tilde{p}^\leq} \eta_a$$

for $f(z) = cz^m \prod_{a \in k^\times} \varphi(az)^{\eta_a} \in \mathcal{G}$.

For non-zero modules $M$ and $N$ in $\mathcal{C}_g$ such that $R_{M,N}^{\text{univ}}$ is rationally renormalizable, we define

$$\Lambda(M, N) := \text{Deg}(c_{M,N}(z)), \quad \Lambda^\infty(M, N) := \text{Deg}^\infty(c_{M,N}(z)), \quad \mathfrak{b}(M, N) := \frac{1}{2}(\Lambda(M, N) + \Lambda(N, M)).$$

Note that $\Lambda(M, N) \equiv \Lambda^\infty(M, N) \mod 2$.

Let $M$ and $N$ be simple modules in $\mathcal{C}_g$. By [32, Proposition 3.22, Proposition 3.18] (see also [34, Proposition 2.16]), we have

$$\Lambda(M, N) = \sum_{k \in \mathbb{Z}} (-1)^{k+\delta(k<0)} \mathfrak{b}(M, \mathcal{D}^k N) = \sum_{k \in \mathbb{Z}} (-1)^{k+\delta(k>0)} \mathfrak{b}(\mathcal{D}^k M, N),$$

(2.4) $$\Lambda^\infty(M, N) = \sum_{k \in \mathbb{Z}} (-1)^k \mathfrak{b}(M, \mathcal{D}^k N),$$

$$\Lambda(M, N) = \Lambda(N, \mathcal{D} M).$$

If $L$ is a real simple module, then by [32, Lemma 4.3 and Corollary 4.4] we have

- $\Lambda(M \triangledown N, L) = \Lambda(M, L) + \Lambda(N, L)$ and $\Lambda(L, N \triangledown M) = \Lambda(L, N) + \Lambda(L, M)$ if $L$ and $N$ strongly commute,
- $\Lambda(L, M \triangledown N) = \Lambda(L, M) + \Lambda(L, N)$ if $\mathcal{D} L$ and $N$ strongly commute,
- $\Lambda(M \triangledown N, L) = \Lambda(M, L) + \Lambda(N, L)$ if $\mathcal{D}^{-1} L$ and $M$ strongly commute.
Lemma 2.3 ([24, Corollary 3.13]). Let $L$ be a real simple module. Then for any simple module $X$, we have
\[(L \triangledown X) \triangledown \mathcal{D}L \simeq X, \quad \mathcal{D}^{-1}L \triangledown (X \triangledown L) \simeq X, \quad L \triangledown (X \triangledown \mathcal{D}L) \simeq X, \quad (\mathcal{D}^{-1}L \triangledown X) \triangledown L \simeq X.\]

Lemma 2.4 ([34, Lemma 2.28, Lemma 2.29]). Let $M$ and $N$ be real simple modules such that $\mathfrak{b}(M,N) = 1$. Then, we have
\[(i) \ M \triangledown N \text{ is a real simple module, and it commutes with } M \text{ and } N, \]
\[(ii) \text{ for any } m, n \in \mathbb{Z}_{\geq 0}, \text{ we have} \]
\[M^\otimes m \triangledown N^\otimes n \simeq \begin{cases} (M \triangledown N)^\otimes m \otimes N^\otimes (n-m) & \text{if } m \leq n, \\ M^\otimes (m-n) \otimes (M \triangledown N)^\otimes n & \text{if } m \geq n. \end{cases}\]

In particular, $M^\otimes m \triangledown N^\otimes n$ is real simple.

We now recall the notion of normal sequences. Let $L_1, L_2, \ldots, L_r$ be simple modules. The sequence $(L_1, \ldots, L_r)$ is called a normal sequence if the composition of the $R$-matrices
\[
r_{L_1,\ldots,L_r} := \prod_{1 \leq i < j \leq r} r_{L_i,L_j}
\]
\[= (r_{L_{r-1},L_r}) \circ \cdots \circ (r_{L_2,L_r} \circ \cdots \circ r_{L_1,L_2}) \circ (r_{L_{r-1},L_{r-2}} \circ \cdots \circ r_{L_1,L_2})
\]
\[: L_1 \otimes L_2 \otimes \cdots \otimes L_r \to L_r \otimes \cdots \otimes L_2 \otimes L_1
\]
does not vanishes.

Lemma 2.5 ([34, Lemma 2.19], [32, Lemma 4.15]). If $(L_1, \ldots, L_r)$ is normal and they are real except for at most one, then $\text{Im}(r_{L_1,\ldots,L_r})$ is simple and it coincides with the head of $L_1 \otimes \cdots \otimes L_r$ and also with the socle of $L_r \otimes \cdots \otimes L_1$.

Thus we have the following lemma.

Lemma 2.6 ([34, Lemma 2.23]). Let $(L_1, L_2, \ldots, L_r)$ be a normal sequence of simple modules such that they are real except for at most one. Then, for any $m \in \mathbb{Z}$, we have
\[\mathcal{D}^m(\text{hd}(L_1 \otimes L_2 \otimes \cdots \otimes L_r)) \simeq \text{hd}(\mathcal{D}^mL_1 \otimes \mathcal{D}^mL_2 \otimes \cdots \otimes \mathcal{D}^mL_r).\]
Lemma 2.7 ([34, Lemma 2.21], [32, Lemma 4.3 and Lemma 4.17]). Let $L, M, N$ be three simple modules such that they are real except for at most one. If one of the following conditions

(a) $b(L, M) = 0$ and $L$ is real,
(b) $b(M, N) = 0$ and $N$ is real,
(c) $b(L, D^{-1}N) = b(DL, N) = 0$ and $L$ or $N$ is real

holds, then $(L, M, N)$ is a normal sequence, i.e.,

$$\Lambda(L, M \nabla N) = \Lambda(L, M) + \Lambda(L, N), \quad \Lambda(L \nabla M, N) = \Lambda(L, N) + \Lambda(M, N).$$

Lemma 2.8 ([34, Lemma 2.24]). Let $L, M, N$ be simple modules. Assume that $L$ is real and one of $M$ and $N$ is real. Then $b(L, M \nabla N) = b(L, M) + b(L, N)$ if and only if $(L, M, N)$ and $(M, N, L)$ are normal sequences.

Definition 2.9. Let $(M, N)$ be an ordered pair of simple modules in $C_g$. We call it unmixed if $b(D M, N) = 0$, and strongly unmixed if

$$b(D^k M, N) = 0 \quad \text{for any } k \in \mathbb{Z}_{\geq 1}.$$ 

By [34, Lemma 5.2], we have

$$(2.5) \quad \Lambda^\infty(M, N) = \Lambda(M, N) \quad \text{as soon as the pair } (M, N) \text{ is strongly unmixed.}$$

Let $L_1, L_2, \ldots, L_r$ be simple modules. We say that the sequence $(L_1, L_2, \ldots, L_r)$ is unmixed (resp. strongly unmixed) if $(L_a, L_b)$ is unmixed (resp. strongly unmixed) for any pair $(a, b)$ such that $1 \leq a < b \leq r$. If a sequence $(L_1, \ldots, L_r)$ of real simple modules is unmixed, then $(L_1, \ldots, L_r)$ is normal ([34, Lemma 5.3]).

3. PBW theory for $C^0_g$

In this section, we review the PBW theory for $C^0_g$ developed in [34].

A module $L \in C_g$ is called a root module if $L$ is a real simple module such that

$$(3.1) \quad b(L, D^k L) = \delta(k = \pm 1) \quad \text{for any } k \in \mathbb{Z}.$$
Let $J$ be an index set and let $C = (c_{i,j})_{i,j \in J}$ be a simply-laced finite Cartan matrix. We denote by $C_\theta$ the simple Lie algebra associated with $C$. A family $D := \{L_i\}_{i \in J}$ of simple modules in $C_\theta$ is called a duality datum associated with $C$ if it satisfies the following conditions:

(a) $L_i$ is a real simple module for any $i \in J$,
(b) $\delta(L_i, L_j) = -c_{i,j}$ for any $i, j \in J$ such that $i \neq j$.

The duality datum $D$ is strong if it satisfies the following conditions:

(a) $L_i$ is a root module for any $i \in J$,
(b) $\delta(L_i, D^k(L_j)) = -\delta(k = 0) c_{i,j}$ for any $k \in \mathbb{Z}$ and $i, j \in J$ with $i \neq j$.

Let $D := \{L_i\}_{i \in J} \subset C_\theta$ be a duality datum associated with $C$. One can construct a quantum affine Schur-Weyl duality functor (shortly a duality functor)

$$\mathcal{F}_D : R_C\text{-gmod} \rightarrow C_\theta$$

using the duality datum $D$, where $R_C$ is the symmetric quiver Hecke algebra associated with $C$ (see [23, 34]). Note that $\mathcal{F}_D(M \circ N) \simeq \mathcal{F}_D(M) \otimes \mathcal{F}_D(N)$ and $\mathcal{F}_D(L(i)) = L_i$ for $i \in J$, where $\circ$ denotes the convolution product in $R_C\text{-gmod}$ and $L(i)$ is the 1-dimensional simple $R_C(\alpha_i)$-module.

**Theorem 3.1** ([34, Section 4.4]). Let $D = \{L_i\}_{i \in J}$ be a strong duality datum associated with $C$.

(i) The duality functor $\mathcal{F}_D$ is exact.

(ii) The duality functor $\mathcal{F}_D$ sends simple modules to simple modules.

(iii) The duality functor $\mathcal{F}_D$ induces an injective ring homomorphism

$$K_{q=1}(R_C\text{-gmod}) \hookrightarrow K(C_\theta),$$

where $K_{q=1}(R_C\text{-gmod})$ is the specialization of the Grothendieck ring $K_q(R_C\text{-gmod})$ at $q = 1$.

(iv) For any simple modules $M$, $N$ in $R_C\text{-gmod}$, we have

(a) $\Lambda(M, N) = \Lambda(\mathcal{F}_D(M), \mathcal{F}_D(N))$,
(b) $\delta(M, N) = \delta(\mathcal{F}_D(M), \mathcal{F}_D(N))$,
(c) $(\text{wt } M, \text{wt } N) = -\Lambda^\infty(\mathcal{F}_D(M), \mathcal{F}_D(N))$. 


\[ \tilde{\Lambda}(M, N) = \mathfrak{v}(\mathcal{D} F_D(M), F_D(N)), \]
\[ \mathfrak{v}(\mathcal{D}^k F_D(M), F_D(N)) = 0 \text{ for any } k \neq 0, \pm 1. \]

The category \( \mathcal{C}_D \) is defined to be the smallest full subcategory of \( \mathcal{C}_g^0 \) such that

(a) it contains \( F_D(L) \) for any simple \( R_C \)-module \( L \),

(b) it is stable by taking subquotients, extensions, and tensor products.

It was proved in [42] that the set of the isomorphism classes of simple modules in \( R_C \)-gmod has a crystal structure, which is isomorphic to the crystal \( B_{g_C}(\infty) \) of \( U^-_q(g_C) \).

The corresponding crystal operators \( \tilde{f}_i \) and \( \tilde{f}_i^* \) are given by

\[ \tilde{f}_i(M) = \text{hd}(L(i) \circ M) \quad \text{and} \quad \tilde{f}_i^*(M) = \text{hd}(M \circ L(i)) \]

for a simple \( R_C \)-module \( M \) and \( i \in J \). For \( b \in B_{g_C}(\infty) \), we denote by \( L(b) \) the corresponding self-dual simple module in \( R_C \)-gmod.

Lemma 3.2. Let \( \mathcal{D} = \{ L_i \}_{i \in J} \) be a strong duality datum associated with \( C \), and let \( B_D \) be the set of the isomorphism classes of simple modules in \( \mathcal{C}_D \).

(i) The duality functor \( F_D \) gives a bijection \( \mathcal{L}_D: B_{g_C}(\infty) \xrightarrow{\sim} B_D \) sending \( b \) to \[ [F_D(L(b))] \text{ for } b \in B_{g_C}(\infty). \]

(ii) For any \( i \in J \) and \( b \in B_{g_C}(\infty) \), we have

(a) \( \mathcal{L}_D(\tilde{f}_i(b)) \simeq L_i \nabla \mathcal{L}_D(b) \) and \( \mathcal{L}_D(\tilde{f}_i^*(b)) \simeq \mathcal{L}_D(b) \nabla L_i \),

(b) \( \mathcal{L}_D(\tilde{e}_i(b)) \simeq \mathcal{L}_D(b) \nabla \mathcal{D} L_i \) if \( \tilde{e}_i(b) \neq 0 \),

(c) \( \mathcal{L}_D(\tilde{e}_i^*(b)) \simeq \mathcal{D}^{-1} L_i \nabla \mathcal{L}_D(b) \) if \( \tilde{e}_i^*(b) \neq 0 \),

(d) \( \varepsilon_i(b) = \mathfrak{v}(\mathcal{D} L_i, \mathcal{L}_D(b)) \) and \( \varepsilon_i^*(b) = \mathfrak{v}(\mathcal{D}^{-1} L_i, \mathcal{L}_D(b)) \).

Proof. (i) follows from Theorem 3.1.

(ii) Since \( F_D \) is exact and monoidal, (a) follows from (3.2) by applying the functor \( F_D \).

We have (b), (c) by Lemma 2.3 and (d) by [34, Corollary 4.13]. \( \square \)

It is easy to see that, for any duality datum \( \mathcal{D} = \{ L_i \}_{i \in J} \) and \( k \in \mathbb{Z} \), the family \( \mathcal{D}^k \mathcal{D} := \{ \mathcal{D}^k L_i \}_{i \in J} \) is also a duality datum. If \( \mathcal{D} \) is strong, then so is \( \mathcal{D}^k \mathcal{D} \).

Lemma 3.3. Let \( k \in \mathbb{Z} \).

(i) The right dual functor \( \mathcal{D} \) induces a ring automorphism of \( K(\mathcal{C}_g) \).
(ii) The following diagram commutes:

\[
\begin{array}{ccc}
K_{q=1}(R_{\mathcal{C}}\text{-gmod}) & \xrightarrow{\mathcal{F}_D} & K(\mathcal{E}_g) \\
\downarrow^{\mathcal{F}_D} & & \downarrow^{\mathcal{D}^k} \\
K(\mathcal{E}_g). & &
\end{array}
\]

In particular, we have \(\mathcal{L}_{\mathcal{D}^k}(b) = \mathcal{D}^k(\mathcal{L}_D(b))\) for any \(b \in B(\infty)\).

**Proof.** (i) follows from the fact that \(\mathcal{D}\) induces an anti-automorphism of \(K(\mathcal{E}_g)\) which is a commutative ring. (ii) immediately follows from (i) and Lemma 2.6. \(\square\)

**Definition 3.4.** A duality datum \(\mathcal{D}\) is called complete if it is strong and, for any simple module \(M \in \mathcal{C}_g\), there exists simple modules \(M_k \in \mathcal{C}_D\) \((k \in \mathbb{Z})\) such that

(a) \(M_k \simeq 1\) for all but finitely many \(k\),
(b) \(M \simeq \text{hd}(\cdots \otimes \mathcal{D}^2 M_2 \otimes \mathcal{D} M_1 \otimes M_0 \otimes \mathcal{D}^{-1} M_{-1} \otimes \cdots)\).

When \(\mathcal{D}\) is complete, the simple Lie algebra \(\mathfrak{g}_C\) is of the type \(X_g\) given in (3.3) ([34, Proposition 6.2]). In this case, we write \(\mathfrak{g}_{\text{fin}}, I_{\text{fin}},\) etc. instead of \(\mathfrak{g}_C, J,\) etc.

| Type of \(g\) | \(\mathfrak{a}_n^{(1)}\) | \(\mathfrak{b}_n^{(1)}\) | \(\mathfrak{c}_n^{(1)}\) | \(\mathfrak{d}_n^{(1)}\) | \(\mathfrak{a}_n^{(2)}\) | \(\mathfrak{a}_{2n-1}^{(2)}\) | \(\mathfrak{d}_{n+1}^{(2)}\) |
|--------------|-------------------|-------------------|-------------------|-------------------|-------------------|-------------------|-------------------|
| \(A_n\) \((n \geq 1)\) | \(B_n\) \((n \geq 2)\) | \(C_n\) \((n \geq 3)\) | \(D_n\) \((n \geq 4)\) | \(A_{2n}\) \((n \geq 1)\) | \(A_{2n-1}\) \((n \geq 2)\) | \(D_{n+1}\) \((n \geq 3)\) |
| \(\text{Type } X_g\) | \(A_n\) | \(A_{2n-1}\) | \(D_{n+1}\) | \(D_n\) | \(A_{2n}\) | \(A_{2n-1}\) | \(D_{n+1}\) |
| \(\text{Type } \mathfrak{g}\) | \(E_6^{(1)}\) | \(E_7^{(1)}\) | \(E_8^{(1)}\) | \(F_4^{(1)}\) | \(G_2^{(1)}\) | \(E_6^{(2)}\) | \(D_4^{(3)}\) |
| \(\text{Type } X_{\mathfrak{g}}\) | \(E_6\) | \(E_7\) | \(E_8\) | \(E_6\) | \(D_4\) | \(E_6\) | \(D_4\) |

(3.3)

From now on, we assume that \(\mathcal{D} = \{L_i\}_{i \in I_{\text{fin}}}\) is a complete duality datum of \(\mathcal{C}_g^0\).

Let \(\Delta^+_{\text{fin}}\) be the set of positive roots of \(\mathfrak{g}_{\text{fin}}\) and let \(W_{\text{fin}}\) be the Weyl group associated with \(\mathfrak{g}_{\text{fin}}\). Let \(w_0\) be the longest element of \(W_{\text{fin}}\), and \(\ell\) denotes the length of \(w_0\). We choose an arbitrary reduced expression \(w_0 = s_{i_1}s_{i_2}\cdots s_{i_\ell}\) of \(w_0\). We extend \(\{i_k\}_{1 \leq k \leq \ell}\) to \(\{i_k\}_{k \in \mathbb{Z}}\)
by \( i_{k,+\ell} = (i_k)^* \) for any \( k \in \mathbb{Z} \), where \( i^* \) is a unique element of \( \mathcal{I}_{\binfty} \) such that \( \alpha_{i^*} = -w_0 \alpha_i \) for \( i \in \mathcal{I}_{\binfty} \). Let

\[
\{V_k\}_{k=1,\ldots,\ell} \subset R_{\binfty}-\text{mod}
\]

be the\textit{ cuspidal modules} associated with the reduced expression \( w_0 \). Note that \( V_k \) corresponds to the dual PBW vector corresponding to \( \beta_k := s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}) \in \Delta^+_{\binfty} \) for \( k = 1, \ldots, \ell \) under the categorification. We define a sequence of simple modules \( \{S_k\}_{k \in \mathbb{Z}} \) in \( \mathcal{C}_\theta \) as follows:

(a) \( S_k = \mathcal{F}_\mathcal{D}(V_k) \) for any \( k = 1, \ldots, \ell \), and we extend its definition to all \( k \in \mathbb{Z} \) by

(b) \( S_{k+\ell} = \mathcal{D}(S_k) \) for any \( k \in \mathbb{Z} \).

The modules \( S_k \) are called the\textit{ affine cuspidal modules} corresponding to \( \mathcal{D} \) and \( w_0 \).

\textbf{Proposition 3.5 (}[34, Proposition 5.7]) \textit{The affine cuspidal modules satisfy the following properties.}

(i) \( S_a \) is a root module for any \( a \in \mathbb{Z} \).

(ii) For any \( a, b \in \mathbb{Z} \) with \( a > b \), the pair \( (S_a, S_b) \) is strongly unmixed.

(iii) Let \( k_1 > \cdots > k_t \) be decreasing integers and \( (a_1, \ldots, a_t) \in \mathbb{Z}^t_{\geq 0} \). Then

(a) the sequence \( (S_{a_1}^{k_1}, \ldots, S_{a_t}^{k_t}) \) is normal,

(b) the head of the tensor product \( S_{a_1}^{k_1} \otimes \cdots \otimes S_{a_t}^{k_t} \) is simple.

We define \( \mathcal{Z} := \mathbb{Z}_{\geq 0}^\infty = \{(a_k)_{k \in \mathbb{Z}} \in \mathbb{Z}_{\geq 0}^\infty | a_k = 0 \text{ except finitely many } k \text{'s}\} \). We denote by \( \prec \) the bi-lexicographic order on \( \mathcal{Z} \), i.e., for any \( a = (a_k)_{k \in \mathbb{Z}} \) and \( a' = (a'_k)_{k \in \mathbb{Z}} \) in \( \mathcal{Z} \), \( a \prec a' \) if and only if the following conditions hold:

\[
\left\{
\begin{array}{l}
\text{there exists } r \in \mathbb{Z} \text{ such that } a_k = a'_k \text{ for any } k < r \text{ and } a_r < a'_r, \\
\text{there exists } s \in \mathbb{Z} \text{ such that } a_k = a'_k \text{ for any } k > s \text{ and } a_s < a'_s.
\end{array}
\right.
\]

For \( a = (a_k)_{k \in \mathbb{Z}} \in \mathcal{Z} \), we define

\[
P_{\mathcal{D}, w_0}(a) := \bigotimes_{k=-\infty}^{a_k} S_k^{a_k} \otimes \cdots \otimes S_2^{a_2} \otimes S_1^{a_1} \otimes S_0^{a_0} \otimes S_{-1}^{a_{-1}} \otimes S_{-2}^{a_{-2}} \otimes \cdots.
\]

Here, \( P_{\mathcal{D}, w_0}(0) \) should be understood as the trivial module \( 1 \). We call the modules \( P_{\mathcal{D}, w_0}(a) \)\textit{ standard modules} with respect to the cuspidal modules \( \{S_k\}_{k \in \mathbb{Z}} \).
Theorem 3.6 ([34, Theorem 6.10]).

(i) For any $a \in \mathbb{Z}$, the head of $P_{D,w_0}(a)$ is simple. We denote the head by $V_{D,w_0}(a) := \text{hd}(P_{D,w_0}(a))$.

(ii) For any simple module $M \in \mathcal{C}_g^0$, there exists a unique $a \in \mathbb{Z}$ such that $M \simeq V_{D,w_0}(a)$.

Therefore, the set $\{V_{D,w_0}(a) \mid a \in \mathbb{Z}\}$ is a complete and irredundant set of simple modules of $\mathcal{C}_g^0$ up to isomorphisms.

The element $a \in \mathbb{Z}$ associated with a simple module $M$ in Theorem 3.6 (ii) is called the cuspidal decomposition of $M$ with respect to the cuspidal modules $\{S_k\}_{k \in \mathbb{Z}}$, and it is denoted by $a_{D,w_0}(M)$.

Theorem 3.7 ([34, Theorem 6.12]). Let $a$ be an element of $\mathbb{Z}$.

(i) The simple module $V_{D,w_0}(a)$ appears only once in $P_{D,w_0}(a)$.

(ii) If $V$ is a simple subquotient of $P_{D,w_0}(a)$ which is not isomorphic to $V_{D,w_0}(a)$, then we have $a_{D,w_0}(V) \prec a$.

(iii) In the Grothendieck ring, we have

$$[P_{D,w_0}(a)] = [V_{D,w_0}(a)] + \sum_{a' \prec a} c(a') [V_{D,w_0}(a')]$$

for some $c(a') \in \mathbb{Z}_{\geq 0}$.

---

4. Extended crystals

In this section, we introduce the notion of the extended crystal of $B(\infty)$. Let $I$ be an index set and let $\mathfrak{g}$ be the Kac-Moody algebra associated with a symmetrizable generalized Cartan matrix $A = (a_{i,j})_{i,j \in I}$. We set

$$\hat{B}_g(\infty) := \left\{(b_k)_{k \in \mathbb{Z}} \in \prod_{k \in \mathbb{Z}} B(\infty) \mid b_k = 1 \text{ for all but finitely many } k\right\},$$

where $B(\infty)$ is the crystal of the negative half $U_q^{-}(\mathfrak{g})$. We set $1 := (1)_{k \in \mathbb{Z}} \in \hat{B}_g(\infty)$. 
For any integer \( k \in \mathbb{Z} \), let \( \pi_k : \hat{B}_g(\infty) \to B(\infty) \) be the \( k \)-th projection defined by \( \pi_k(b) = b_k \) for any \( b = (b_k)_{k \in \mathbb{Z}} \in \hat{B}_g(\infty) \). We denote by \( \iota_k : B(\infty) \to \hat{B}_g(\infty) \) the section of \( \pi_k \) which is defined by

\[
\pi_{k'} \circ \iota_k(b) = \begin{cases} 
  b & \text{if } k = k', \\
  1 & \text{if } k \neq k'
\end{cases}
\]

for any \( k' \in \mathbb{Z} \) and \( b \in B(\infty) \). We set \( \hat{I} := I \times \mathbb{Z} \).

Let \( (i, k) \in \hat{I} \). The maps \( \wt_k : \hat{B}_g(\infty) \to \mathbb{P} \) and \( \varepsilon_{(i,k)}, \varepsilon^*_{(i,k)} : \hat{B}_g(\infty) \to \mathbb{Z} \) are defined by

\[
\wt_k(b) := (-1)^k \wt(b_k), \quad \varepsilon_{(i,k)}(b) := \varepsilon_i(b_k), \quad \varepsilon^*_{(i,k)}(b) := \varepsilon^*_i(b_k)
\]

for any \( b = (b_k)_{k \in \mathbb{Z}} \in \hat{B}_g(\infty) \), and we define

\[
\widehat{\wt}(b) := \sum_{k \in \mathbb{Z}} \wt_k(b), \quad \widehat{\varepsilon}_{(i,k)}(b) := \varepsilon_{(i,k)}(b) - \varepsilon^*_{(i,k+1)}(b).
\]

We now define the extended crystal operators

\( \widehat{\mathcal{F}}_{(i,k)} : \hat{B}_g(\infty) \to \hat{B}_g(\infty) \) and \( \widehat{\mathcal{E}}_{(i,k)} : \hat{B}_g(\infty) \to \hat{B}_g(\infty) \),

by

\[
\widehat{\mathcal{F}}_{(i,k)}(b) := \begin{cases} 
  (\ldots, b_{k+2}, b_{k+1}, \tilde{\mathcal{F}}_i(b_k), b_{k-1}, \ldots) & \text{if } \widehat{\varepsilon}_{(i,k)}(b) \geq 0, \\
  (\ldots, b_{k+2}, \tilde{\varepsilon}^*_i(b_{k+1}), b_k, b_{k-1}, \ldots) & \text{if } \widehat{\varepsilon}_{(i,k)}(b) < 0,
\end{cases}
\]

\[
\widehat{\mathcal{E}}_{(i,k)}(b) := \begin{cases} 
  (\ldots, b_{k+2}, b_{k+1}, \tilde{\mathcal{E}}_i(b_k), b_{k-1}, \ldots) & \text{if } \widehat{\varepsilon}_{(i,k)}(b) > 0, \\
  (\ldots, b_{k+2}, \tilde{\varepsilon}^*_i(b_{k+1}), b_k, b_{k-1}, \ldots) & \text{if } \widehat{\varepsilon}_{(i,k)}(b) \leq 0,
\end{cases}
\]

for any \( (i, k) \in \hat{I} \) and \( b = (b_k)_{k \in \mathbb{Z}} \in \hat{B}_g(\infty) \). Note that \( \widehat{\mathcal{E}}_{(i,k)}(b) \) is non-zero for any \( b \in \hat{B}_g(\infty) \). When no confusion arises, we simply write \( \mathcal{F}_{i,k} \), \( \mathcal{E}_{i,k} \), \( \varepsilon_{i,k} \), etc. for \( \widehat{\mathcal{F}}_{(i,k)} \), \( \widehat{\mathcal{E}}_{(i,k)} \), \( \varepsilon_{(i,k)} \), etc.

For any \( b = (b_k)_{k \in \mathbb{Z}} \in \hat{B}_g(\infty) \), we define \( \star(b) := (b'_k)_{k \in \mathbb{Z}} \) by

\[
b'_k = \star(b_k) \quad \text{for any } k \in \mathbb{Z},
\]
Lemma 4.1. For $b \in \hat{B}_g(\infty)$ and $(i, k) \in \hat{I}$, we define
\[
\hat{\varepsilon}_{i, k}^*(b) := \hat{\varepsilon}_{i, -k}(\ast(b)), \quad \tilde{F}_{i, k}^*(b) := \ast(\tilde{F}_{i, -k}(\ast(b))), \quad \tilde{E}_{i, k}^*(b) := \ast(\tilde{E}_{i, -k}(\ast(b))).
\]
Then we have the following.

(i) $\hat{\varepsilon}_{i, k}^*(b) = -\hat{\varepsilon}_{i, -k}(b)$.

(ii) $\tilde{F}_{i, k}^*(b) = \tilde{E}_{i, k-1}(b)$.

(iii) $\tilde{E}_{i, k}^*(b) = \tilde{F}_{i, k-1}(b)$.

Proof. Let $b = (b_k)_{k \in \mathbb{Z}} \in \hat{B}_g(\infty)$ and $(i, k) \in \hat{I}$. Since $\varepsilon_i^*(b) = \varepsilon_i(\ast b)$ for $b \in B(\infty)$, we have
\[
\varepsilon_{i, -k}(\ast(b)) = \varepsilon_i(\ast b) = \varepsilon_{i}^*(b_k) = \varepsilon_{i, k}^*(b)
\]
which implies that
\[
\hat{\varepsilon}_{i, k}^*(b) = \varepsilon_{i, -k}^*(b_k) - \varepsilon_{i, k-1}(b) = -\hat{\varepsilon}_{i, k}(b).
\]
Thus we have (i). Using the definitions of $\tilde{F}_{i, k}^*$ and $\tilde{E}_{i, k}^*$, one can obtain the following:
\[
\tilde{F}_{i, k}^*(b) = \begin{cases} 
(\cdots, b_{k+1}, \tilde{f}_i^*(b_k), b_{k-1}, b_{k-2}, \cdots) & \text{if } \hat{\varepsilon}_{i, k}^*(b) \geq 0, \\
(\cdots, b_{k+1}, b_k, \tilde{\varepsilon}_{i}(b_{k-1}), b_{k-2}, \cdots) & \text{if } \hat{\varepsilon}_{i, k}^*(b) < 0,
\end{cases}
\]
\[
\tilde{E}_{i, k}^*(b) = \begin{cases} 
(\cdots, b_{k+1}, \tilde{\varepsilon}_{i}^*(b_k), b_{k-1}, b_{k-2}, \cdots) & \text{if } \hat{\varepsilon}_{i, k}^*(b) > 0, \\
(\cdots, b_{k+1}, b_k, \tilde{f}_{i}(b_{k-1}), b_{k-2}, \cdots) & \text{if } \hat{\varepsilon}_{i, k}^*(b) \leq 0.
\end{cases}
\]
Therefore, (ii) and (iii) follows from (i) and (4.1). \hfill \Box

The lemma below follows from the definitions.

Lemma 4.2. For any $b \in \hat{B}_g(\infty)$ and $(i, k) \in \hat{I}$, we have the following.

(i) $\hat{\text{wt}}(\tilde{F}_{i, k}(b)) = \hat{\text{wt}}(b) + (-1)^{k+1} \alpha_i$ and $\hat{\text{wt}}(\tilde{E}_{i, k}(b)) = \hat{\text{wt}}(b) + (-1)^k \alpha_i$.

(ii) $\hat{\varepsilon}_{i, k}(\tilde{F}_{i, k}(b)) = \hat{\varepsilon}_{i, k}(b) + 1$ and $\hat{\varepsilon}_{i, k}(\tilde{E}_{i, k}(b)) = \hat{\varepsilon}_{i, k}(b) - 1$. 

where $\ast$ is the anti-involution defined in (2.1). Then it gives an involution
\[
\ast : \hat{B}_g(\infty) \longrightarrow \hat{B}_g(\infty)
\]
such that $\ast(1) = 1$ and $\hat{\text{wt}}(b) = \hat{\text{wt}}(\ast(b))$ for $b \in \hat{B}_g(\infty)$. 

The lemma below follows from the definitions.
(iii) $\xi^*_{i,k}(\tilde{F}^*_{i,k}(b)) = \xi^*_{i,k}(b) + 1$ and $\xi^*_{i,k}(\tilde{E}^*_{i,k}(b)) = \xi^*_{i,k}(b) - 1$.
(iv) $\tilde{F}_{i,k}$ and $\tilde{E}_{i,k}$ are inverse to each other.
(v) $\tilde{F}^*_{i,k}$ and $\tilde{E}^*_{i,k}$ are inverse to each other.
(vi) For $b \in \hat{B}(\infty)$, we have
\[
\tilde{F}_{i,k}(\iota_k(b)) = \iota_k(\tilde{f}_i(b)),
\tilde{E}_{i,k}(\iota_k(b)) = \iota_k(\tilde{e}_i(b)) \quad \text{if } \tilde{e}_i(b) \neq 0,
\tilde{F}^*_{i,k}(\iota_k(b)) = \iota_k(\tilde{f}^*_i(b)),
\tilde{E}^*_{i,k}(\iota_k(b)) = \iota_k(\tilde{e}^*_i(b)) \quad \text{if } \tilde{e}^*_i(b) \neq 0.
\]

**Lemma 4.3.** Let $t \in \mathbb{Z}$. For $b = (b_k)_{k \in \mathbb{Z}} \in \hat{B}g(\infty)$, we define $D^t(b) \in \hat{B}g(\infty)$ by
\[
\pi_k(D^t(b)) = b_{k-t} \quad \text{for any } k \in \mathbb{Z}.
\]
Then it gives a bijection
\[
D^t : \hat{B}g(\infty) \rightarrow \hat{B}g(\infty)
\]
such that
\[
D^t(\tilde{F}_{i,k}(b)) = \tilde{F}_{i,k+t}(D^t(b)) \quad \text{and} \quad D^t(\tilde{F}^*_{i,k}(b)) = \tilde{F}^*_{i,k+t}(D^t(b))
\]
for any $(i,k) \in \hat{I}$.

**Proof.** It follows from (4.1), (4.3) and the definition of $D^t$. 

As a usual crystal, the set $\hat{B}g(\infty)$ has the $\hat{I}$-colored graph structure induced by the operators $\tilde{F}_{i,k}$ for $(i,k) \in \hat{I}$. We take $\hat{B}g(\infty)$ as the set of vertices and define the $\hat{I}$-colored arrows on $\hat{B}g(\infty)$ by
\[
b \xrightarrow{(i,k)} b' \quad \text{if and only if} \quad b' = \tilde{F}_{i,k}b = \tilde{E}^*_{i,k+1}b \quad (i,k) \in \hat{I}.
\]
We call $\hat{B}g(\infty)$ the extended crystal of $B(\infty)$.

**Lemma 4.4.** As an $\hat{I}$-colored graph, $\hat{B}g(\infty)$ is connected.
Proof. For $b = (b_k)_{k \in \mathbb{Z}} \in \hat{B}_g(\infty)$, set $ht(b) := \sum_{k \in \mathbb{Z}} ht(b_k)$, where $ht(b)$ is defined in (2.2). When $b \neq 1$, let $l(b) := \max\{k \in \mathbb{Z} \mid b_k \neq 1\}$.

Let $b = (b_k)_{k \in \mathbb{Z}} \in \hat{B}_g(\infty)$. We shall prove that $b$ is connected to 1 in the graph $\hat{B}_g(\infty)$ by induction on $ht(b)$.

If $ht(b) = 0$, then $b = 1$. Thus it is trivial that $b$ is connected to 1.

Suppose that $ht(b) \neq 0$. Let $l = l(b)$. Since $b_l \neq 1$, there exists $i \in I$ such that $\tilde{e}_i(b_l) \neq 0$. Since $\tilde{e}_i(b_l) > 0$, we have

$$\tilde{E}_{i,l}(b) = (\cdot \cdot \cdot , b_l, b_l-1, b_l-2 \cdot \cdot \cdot),$$

which says that $ht(\tilde{E}_{i,l}(b)) = ht(b) - 1$. Since $\tilde{E}_{i,l}(b)$ is connected to 1 by the induction hypothesis, $b$ is also connected to 1. □

Example 4.5. Let $I = \{1\}$ and let $B(\infty)$ be the crystal of $U_q^-(\mathfrak{sl}_2)$. We identify $B(\infty)$ with $\mathbb{Z}_{\geq 0}$ and simply write $\widetilde{F}_k$ instead of $\widetilde{F}_{1,k}$ for $k \in \mathbb{Z}$. Then the extended crystal $\hat{B}(\infty)$ is equal to $(\mathbb{Z}_{\geq 0})^{\oplus \mathbb{Z}}$, and for any $k \in \mathbb{Z}$ and $b = (b_k)_{k \in \mathbb{Z}} \in \hat{B}(\infty)$, we have

$$\widetilde{F}_k(b) = \begin{cases} (\cdot \cdot \cdot , b_{k+2}, b_{k+1} - 1, b_k, b_{k-1}, \cdot \cdot \cdot) & \text{if } b_{k+1} > b_k, \\ (\cdot \cdot \cdot , b_{k+2}, b_{k+1}, b_k + 1, b_{k-1}, \cdot \cdot \cdot) & \text{if } b_{k+1} \leq b_k. \end{cases}$$

5. Categorical crystals for quantum affine algebras

Let $U_q'(g)$ be a quantum affine algebra of arbitrary type. In this section, we will prove that the set of the isomorphism classes of simple modules in $\mathcal{C}_0^0$ has an extended crystal structure isomorphic to $\hat{B}_{\text{fin}}(\infty)$.

5.1. Root modules.

In this subsection, we shall prove several lemmas for categorical crystals.
Lemma 5.1. Let \( L \) be a root module. For \( a, b \in \mathbb{Z}_{\geq 0} \), we have

\[
L \nabla ((\mathcal{D}L)^{\otimes a} \nabla L^{\otimes b}) \simeq \begin{cases} 
(\mathcal{D}L)^{\otimes(a-1)} \nabla L^{\otimes b} & \text{if } a > b, \\
(\mathcal{D}L)^{\otimes a} \nabla L^{\otimes(b+1)} & \text{if } a \leq b,
\end{cases}
\]

\[
((\mathcal{D}L)^{\otimes a} \nabla L^{\otimes b}) \otimes \mathcal{D}L \simeq \begin{cases} 
(\mathcal{D}L)^{\otimes(a+1)} \nabla L^{\otimes b} & \text{if } a \geq b, \\
(\mathcal{D}L)^{\otimes a} \nabla L^{\otimes(b-1)} & \text{if } a < b.
\end{cases}
\]

Proof. Since the second isomorphism can be proved similarly, we show only the first isomorphism. By Lemma 2.4, we have

\[
d((\mathcal{D}L, \mathcal{D}L \nabla L) = 0, \tag{5.1}
\]

\[
d(L, \mathcal{D}L \nabla L) = 0, \tag{5.2}
\]

which implies that, by Lemma 2.5 and Lemma 2.7,

\[
hd((L)_{\otimes x} \otimes (\mathcal{D}L)_{\otimes y} \otimes (\mathcal{D}L \nabla L)_{\otimes z}) \text{ and } hd((L)_{\otimes x} \otimes (\mathcal{D}L \nabla L)_{\otimes y} \otimes (L)_{\otimes z}) \tag{5.3}
\]

are simple modules for any \( x, y, z \in \mathbb{Z}_{\geq 0} \).

(a) We shall first treat the case \( a > b \). It follows from Lemma 2.4, (5.1) and (5.3) that

\[
L \nabla ((\mathcal{D}L)^{\otimes a} \nabla L^{\otimes b}) \simeq L \nabla ((\mathcal{D}L)^{\otimes(a-1)} \otimes (\mathcal{D}L \nabla L)^{\otimes b})
\]

\[
\simeq (L \nabla (\mathcal{D}L)^{\otimes(a-b)}) \nabla (\mathcal{D}L \nabla L)^{\otimes b}.
\]

Since \( L \nabla (\mathcal{D}L)^{\otimes(a-b)} \simeq (L \nabla \mathcal{D}L) \nabla (\mathcal{D}L)^{\otimes(a-b-1)} \simeq (\mathcal{D}L)^{\otimes(a-b-1)} \), Lemma 2.4 and (5.1) tell us that

\[
L \nabla ((\mathcal{D}L)^{\otimes a} \nabla L^{\otimes b}) \simeq (\mathcal{D}L)^{\otimes(a-b-1)} \otimes (\mathcal{D}L \nabla L)^{\otimes b}
\]

\[
\simeq (\mathcal{D}L)^{\otimes(a-1)} \nabla L^{\otimes b}.
\]
Next we shall treat the case $a \leq b$. By Lemma 2.4, (5.2) and (5.3), we have
\[
L \nabla ( (\mathcal{D} L)^{\otimes a} \nabla L^{\otimes b} ) \cong L \nabla ( (\mathcal{D} L \nabla L)^{\otimes a} \otimes L^{\otimes (b-a)} ) \\
\cong L \otimes (\mathcal{D} L \nabla L)^{\otimes a} \otimes L^{\otimes (b-a)} \\
\cong (\mathcal{D} L \nabla L)^{\otimes a} \otimes L^{\otimes (b-a+1)} \\
\cong (\mathcal{D} L)^{\otimes a} \nabla L^{\otimes (b+1)}. \quad \square
\]

**Lemma 5.2.** Let $L, M, N$ be simple modules in $C_g$. Suppose that
- (a) $L$ is real,
- (b) $\mathfrak{d}(L, M) = 0$,
- (c) $M \otimes N$ has a simple head.

Then we have
\[
L \nabla (M \nabla N) \cong M \nabla (L \nabla N).
\]

**Proof.** By the assumptions (a), (b) and [32, Lemma 4.3], the diagram
\[
\begin{array}{ccc}
L \otimes M \otimes N & \xrightarrow{r_{L,M,N}} & M \otimes N \otimes L \\
\downarrow & & \downarrow \\
L \otimes (M \nabla N) & \xrightarrow{r_{L,M \nabla N}} & (M \nabla N) \otimes L
\end{array}
\]
commutes. Then it follows from the assumptions (a), (c) and [32, Proposition 4.5 (ii)] that $L \otimes M \otimes N$ has a simple head. Thus we have
\[
L \nabla (M \nabla N) \cong \text{hd}(L \otimes M \otimes N) \cong \text{hd}(M \otimes L \otimes N) \cong M \nabla (L \nabla N).
\]
\[
\square
\]

**Lemma 5.3.** Let $L$, $M$ and $N$ be simple modules in $C_g$. Suppose that
- (a) $L$ is a root module,
- (b) $\mathfrak{b}(L, M) = \mathfrak{b}(\mathcal{D} L, N) = 0$,
- (c) $\text{hd}(M \otimes (\mathcal{D} L)^{\otimes s} \otimes L^{\otimes t} \otimes N)$ is simple for any $s, t \in \mathbb{Z}_{\geq 0}$. 
Then, for any $a, b \in \mathbb{Z}_{\geq 0}$, we have

$$L \triangledown \text{hd}(M \otimes (\mathcal{D}L)^{\otimes a} \otimes L^{\otimes b} \otimes N) \simeq \begin{cases} \text{hd}(M \otimes (\mathcal{D}L)^{(a-1)} \otimes L^{b} \otimes N) & \text{if } a > b, \\ \text{hd}(M \otimes (\mathcal{D}L)^{a} \otimes L^{b+1} \otimes N) & \text{if } a \leq b. \end{cases}$$

**Proof.** We set $N' := \text{hd}((\mathcal{D}L)^{\otimes a} \otimes L^{\otimes b} \otimes N)$. Since we have
- $N'$ is simple by (c),
- $\mathfrak{d}(M, L) = 0$,

we obtain

$$L \triangledown \text{hd}(M \otimes (\mathcal{D}L)^{\otimes a} \otimes L^{\otimes b} \otimes N) \simeq L \triangledown (M \triangledown N')$$

(5.4)

$$\simeq M \triangledown \left( L \triangledown \left( ((\mathcal{D}L)^{\otimes a} \triangledown L^{\otimes b}) \triangledown N \right) \right).$$

On the other hand, since
- $(\mathcal{D}L)^{\otimes a} \triangledown L^{\otimes b}$ is real by Lemma 2.4,
- $\mathfrak{d}(\mathcal{D}L, N) = 0$,

we have

$$L \triangledown \left( ((\mathcal{D}L)^{\otimes a} \triangledown L^{\otimes b}) \triangledown N \right) \simeq \left( L \triangledown \left( ((\mathcal{D}L)^{\otimes a} \triangledown L^{\otimes b}) \right) \right) \triangledown N$$

(5.5)

$$\simeq \begin{cases} ((\mathcal{D}L)^{(a-1)} \triangledown L^{b}) \triangledown N & \text{if } a > b, \\ ((\mathcal{D}L)^{a} \triangledown L^{b+1}) \triangledown N & \text{if } a \leq b. \end{cases}$$

Here, the first isomorphism follows from Lemma 2.7 and the second from Lemma 5.1. Therefore the assertion follows from (5.4), (5.5) and (c). \qed

**Lemma 5.4.** Let $L$ be a root module and let $M$ be a simple module. We set $m := \mathfrak{d}(\mathcal{D}L, M)$ and $n := \mathfrak{d}(\mathcal{D}^{-1}L, M)$. Then, we have

(i) $\mathfrak{d}(\mathcal{D}L, M') = 0$ and $M \simeq L^\otimes m \triangledown M'$ for some simple module $M'$,
(ii) $\mathfrak{d}(\mathcal{D}^{-1}L, M'') = 0$ and $M \simeq M'' \triangledown L^\otimes n$ for some simple module $M''$. 

Proof. Define
\[ M' := M \nabla (D \mathcal{L})^m \quad \text{and} \quad M'' := (D^{-1} \mathcal{L})^n \nabla M. \]

By Lemma 2.3, we have
\[ M \simeq L^m \nabla M' \simeq M'' \nabla L^n. \]

Moreover, by [34, Lemma 3.4], we have
\[ d(D L, M') = d(D L, M \nabla (D \mathcal{L})^m) = d(D L, M) - m = 0, \]
\[ d(D^{-1} L, M'') = d(D^{-1} L, (D^{-1} \mathcal{L})^n \nabla M) = d(L'', M) - n = 0 \]

by a standard induction argument. \(\square\)

5.2. Categorical crystals.
Throughout this subsection, let \(\mathcal{D} := \{L_i\}_{i \in \mathcal{I}} \subset \mathcal{C}_0\) be a complete duality datum of \(\mathcal{C}_0\).

Let \(\widehat{\mathcal{B}}_{\mathfrak{g}_{\mathfrak{fin}}}^{\infty}\) be the extended crystal of \(\mathcal{U}^{-} (\mathfrak{g}_{\mathfrak{fin}})\). For \(\mathbf{b} = (b_k)_{k \in \mathbb{Z}} \in \widehat{\mathcal{B}}_{\mathfrak{g}_{\mathfrak{fin}}}^{\infty}\), we define
\[ \mathcal{L}_{\mathcal{D}}(\mathbf{b}) := \text{hd}(\cdots \otimes \mathcal{D}^2 L_2 \otimes \mathcal{D} L_1 \otimes L_0 \otimes \mathcal{D}^{-1} L_{-1} \otimes \cdots), \]

where \(L_k = \mathcal{L}_{\mathcal{D}}(b_k)\) for \(k \in \mathbb{Z}\). Here, \(\mathcal{L}_{\mathcal{D}}\) is given in Lemma 3.2 (i).

Let \(\mathcal{B}(\mathfrak{g})\) be the set of the isomorphism classes of simple modules in \(\mathcal{C}_0\). For a simple module \(M\) in \(\mathcal{C}_0\), we denote by \([M]\) the corresponding element in \(\mathcal{B}(\mathfrak{g})\). When no confusion arises, we simply write \(M\) instead of \([M]\).

Proposition 5.5.
(i) Let \(a, b \in \mathbb{Z}\) with \(a \leq b\) and let \(L_k\) be simple modules in \(\mathcal{C}_0\) for \(k = a, a+1, \ldots, b\).
Then we have
(a) the sequence \((\mathcal{D}^b L_b, \mathcal{D}^{b-1} L_{b-1}, \ldots, \mathcal{D}^a L_a)\) is strongly unmixed,
(b) \(\text{hd}(\mathcal{D}^b L_b \otimes \mathcal{D}^{b-1} L_{b-1} \otimes \cdots \otimes \mathcal{D}^a L_a)\) is a simple module,
(c) for \(m \in \mathbb{Z}\),
\[ \mathcal{D}^m \left(\text{hd}(\mathcal{D}^b L_b \otimes \cdots \otimes \mathcal{D}^a L_a)\right) \simeq \text{hd}(\mathcal{D}^{b+m} L_b \otimes \cdots \otimes \mathcal{D}^{a+m} L_a). \]
(ii) For any \(\mathbf{b} = (b_k)_{k \in \mathbb{Z}} \in \widehat{\mathcal{B}}_{\mathfrak{g}_{\mathfrak{fin}}}^{\infty}\), the module \(\mathcal{L}_{\mathcal{D}}(\mathbf{b})\) is simple.
We define a map \( \Phi_D : \hat{B}_{\text{fin}}(\infty) \to B(g) \) by
\[
\Phi_D(b) := L_D(b)
\]
for any \( b \in \hat{B}_{\text{fin}}(\infty) \).
Then the map \( \Phi_D \) is bijective.

For any \( t \in \mathbb{Z} \) and \( b \in \hat{B}_{\text{fin}}(\infty) \), we have
\[
\Phi_D(D^t(b)) = D^t(\Phi_D(b)),
\]
where \( D \) is the bijection of \( \hat{B}_{\text{fin}}(\infty) \) defined in Lemma 4.3.

Proof. Let \( w_0 \) be a reduced expression of the longest element \( w_0 \) of \( W_{\text{fin}} \) and set \( \ell := \ell(w_0) \).

Let \( S_k \) (\( k \in \mathbb{Z} \)) be the affine cuspidal modules corresponding to \( D \) and \( w_0 \).

(i) Since \( L_k \in \mathcal{C}_D \) (\( a \leq k \leq b \)), the sequence \( (\mathcal{D}^bL_b, \mathcal{D}^{b-1}L_{b-1}, \ldots, \mathcal{D}^aL_a) \) is strongly unmixed by Theorem 3.1 and there exist \( b_1^k, \ldots, b_\ell^k \in \mathbb{Z} \geq 0 \) such that
\[
L_k \simeq \text{hd}(S_0^\otimes b_\ell^k \otimes \cdots \otimes S_1^\otimes b_1^k).
\]

(ii) follows from (i).

(iii) Let \( M \) be a simple module in \( B(g) \). By Theorem 3.6, there exists a unique \( a = (a_k)_{k \in \mathbb{Z}} \in \mathbb{Z} \) such that
\[
M \simeq V_{D,w_0}(a).
\]
For each \( k \in \mathbb{Z} \), we set
\[
L_k := \text{hd}(S_\ell^{a_{k+1}} \otimes \cdots \otimes S_2^{a_{k+2}} \otimes S_1^{a_{k+1}}).
\]
By Proposition 3.5, \( L_k \) is a simple module in \( \mathcal{G}_D \) and
\[
M \simeq \text{hd}(\cdots \otimes \mathcal{D}^2 L_2 \otimes \mathcal{D} L_1 \otimes L_0 \otimes \mathcal{D}^{-1} L_{-1} \otimes \cdots).
\]
For each \( k \in \mathbb{Z} \), Lemma 3.2 says that there exists a unique \( b_k \in B_{\text{fin}}(\infty) \) such that \( \mathcal{L}_D(b_k) = L_k \). Thus the correspondence \( M \mapsto b := (b_k)_{k \in \mathbb{Z}} \) gives a map
\[
\Theta_D : \mathcal{B}(g) \to \widehat{B}_{\text{fin}}(\infty).
\]
It is obvious that \( \Theta_D \) is the inverse of \( \Phi_D \).

(iv) Let \( b = (b_k)_{k \in \mathbb{Z}} \in \widehat{B}_{\text{fin}}(\infty) \) and let \( L_k := \mathcal{L}_D(b_k) \) for \( k \in \mathbb{Z} \). Then, we have
\[
\mathcal{D}^t(\Phi_D(b)) = \mathcal{D}^t(\text{hd}(\cdots \otimes \mathcal{D} L_1 \otimes L_0 \otimes \mathcal{D}^{-1} L_{-1} \otimes \mathcal{D}^{-2} L_{-2} \otimes \cdots)) = \text{hd}(\cdots \otimes \mathcal{D}^{t+1} L_1 \otimes \mathcal{D}^t L_0 \otimes \mathcal{D}^{t-1} L_{-1} \otimes \mathcal{D}^{t-2} L_{-2} \otimes \cdots)) = \Phi_D(D^t(b)).
\]

Note that \( \Phi_D(1) = 1 \) by the definition. For a simple module \( M \in \mathcal{G}_g^0 \), we denote by
\[
b_D(M) := \Phi_D^{-1}(M) \in \widehat{B}_{\text{fin}}(\infty).
\]
Let us recall the duality functor \( \mathcal{F}_D : R_{\text{fin}} \text{-gmod} \to \mathcal{G}_g^0 \). Let \( w_0 \) be the longest element of \( W_{\text{fin}} \) and let \( \ell := \ell(w_0) \). For \( a, b \in \mathbb{Z} \) with \( a \leq b \), we set \( [a, b] := \{a, a+1, \ldots, b\} \).

**Lemma 5.6.** Let \( a, b \in \mathbb{Z} \) with \( a \leq b \). For each \( p \in [a, b] \), we choose a reduced expression \( \underline{w}_{0,p} \) of \( w_0 \), and let \( \{V_{p,k}\}_{k=1}^\ell \) be the cuspidal modules in \( R_{\text{fin}} \text{-gmod} \) associated with \( \underline{w}_{0,p} \). We set \( S_{p,k} := \mathcal{F}_D(V_{p,k}) \), and define
\[
S_p(c_p) := (\mathcal{D}^p(S_{p,\ell})^{\otimes c_{p,\ell}}, \ldots, \mathcal{D}^p(S_{p,2}^{\otimes c_{p,2}}), \mathcal{D}^p(S_{p,1}^{\otimes c_{p,1}}))
\]
for \( c_p = (c_{p,\ell}, \ldots, c_{p,2}, c_{p,1}) \in \mathbb{Z}_{\geq 0}^\ell \). Then their concatenation
\[
S_b(c_b) \ast S_{b-1}(c_{b-1}) \ast \cdots \ast S_a(c_a) := (\mathcal{D}^b(S_{b,\ell}), \mathcal{D}^b(S_{b,\ell-1}), \ldots, \mathcal{D}^a(S_{a,2}), \mathcal{D}^a(S_{a,1}))
\]
is strongly unmixed and normal.
Proof. Consider the pair \( (\mathcal{D}^p(S_{p,k}^{\otimes c_{p,k}}), \mathcal{D}^{p'}(S_{p',k'}^{\otimes c_{p',k'}})) \). If \( p = p' \) and \( k > k' \), then it is strongly unmixed by Proposition 3.5. If \( p > p' \), then it is also strongly unmixed by Theorem 3.1 (iv). \( \square \)

**Lemma 5.7.** Let \( i \in I_{\text{fin}} \), \( a, b \in \mathbb{Z} \) with \( a \leq b \). Let \( M_k \) be a simple module in \( C_D \) for \( k \in [a, b] \) and \( N_k \) a module in \( C_B \). For each \( k \in [a, b] \), we assume one of the following conditions:

(a) \( N_k \simeq \mathcal{D}^k M_k \),
(b) \( N_k \simeq \mathcal{D}^k(L_i^{\otimes n_k}) \otimes \mathcal{D}^k(M'_k) \) for some \( n_k \in \mathbb{Z}_{\geq 0} \), where \( M'_k \) is a simple module in \( C_D \) such that \( M_k \simeq L_i^{\otimes n_k} \triangleright M'_k \) and \( b(\mathcal{D}L_i, M'_k) = 0 \),
(c) \( N_k \simeq \mathcal{D}^k(M''_k) \otimes \mathcal{D}^k(L_i^{\otimes n_k}) \) for some \( n_k \in \mathbb{Z}_{\geq 0} \), where \( M''_k \) is a simple module in \( C_D \) such that \( M_k \simeq M''_k \triangleright L_i^{\otimes n_k} \) and \( b(\mathcal{D}^{-1}L_i, M''_k) = 0 \).

Then, the head \( \text{hd}(N_b \otimes \cdots \otimes N_{a+1} \otimes N_a) \) is simple and it is isomorphic to \( \text{hd}(\mathcal{D}^b M_b \otimes \cdots \otimes \mathcal{D}^{a+1} M_{a+1} \otimes \mathcal{D}^a M_a) \).

**Proof.** We take a reduced expression \( w_0 = s_{i_1}s_{i_2} \cdots s_{i_\ell} \) with \( i_1 = i \) and set \( w_0' := s_{i_2} \cdots s_{i_\ell} s_{i_1} \), where \( \alpha_{i_1} = -w_0(\alpha_i) \). Let \( \{S_k\}_{k \in \mathbb{Z}} \) and \( \{S'_k\}_{k \in \mathbb{Z}} \) be the affine cuspidal modules corresponding to \( (\mathcal{D}, w_0) \) and \( (\mathcal{D}, w_0') \) respectively. By the construction, we have \( S_1 = S'_\ell = L_i \), and

\[
\begin{align*}
\mathcal{S}_k &\simeq \mathcal{S}_{k}^{-1} S_1 & \text{for } 2 \leq k \leq \ell, \\
\mathcal{S}'_k &\simeq \mathcal{S}'_{k} S_1 & \text{for } 1 \leq k \leq \ell - 1.
\end{align*}
\]

by Proposition 3.5. Since \( M_k \in C_D \) for all \( k \), Theorem 3.6 says that there exists non-negative integers \( a_{i}^k \) such that

\[
M_k \simeq \begin{cases} \\
\text{hd}((S'_\ell)^{\otimes a_{\ell}^k} \otimes \cdots \otimes (S'_1)^{\otimes a_{1}^k}) & \text{in case (b),} \\
\text{hd}(S_{\ell}^{\otimes a_{\ell}^k} \otimes \cdots \otimes S_{1}^{\otimes a_{1}^k}) & \text{otherwise.}
\end{cases}
\]

Hence, by Lemma 2.3 and (5.8),

- in case (b), \( M'_k \simeq \text{hd}((S'_{\ell-1})^{\otimes b_{\ell-1}^k} \otimes \cdots \otimes (S'_1)^{\otimes b_1^k}) \) and \( n_k = a_{\ell}^k \),
- in case (c), \( M''_k \simeq \text{hd}(S_{\ell}^{\otimes a_{\ell}^k} \otimes \cdots \otimes S_{2}^{\otimes a_2^k}) \) and \( n_k = a_{1}^k \).
For \(k \in [a, b]\), let
\[
P_k := \begin{cases} \mathcal{D}^k((S_i')^{\otimes a_i^k}) \otimes \cdots \otimes \mathcal{D}^k((S_i')^{\otimes a_i^k}) & \text{in case (b),} \\ \mathcal{D}^k(S_i^{\otimes a_i^k}) \otimes \cdots \otimes \mathcal{D}^k(S_i^{\otimes a_i^k}) & \text{otherwise.} \end{cases}
\]

It follows from [32, Lemma 4.15], [34, Lemma 5.3] and Lemma 5.6 that \(\text{hd}(P_b \otimes P_{b-1} \otimes \cdots \otimes P_a)\) is simple and
\[
\text{hd}(P_b \otimes P_{b-1} \otimes \cdots \otimes P_a) \simeq \text{hd}(N_b \otimes N_{b-1} \otimes \cdots \otimes N_a) \\
\simeq \text{hd}(\mathcal{D}^b M_b \otimes \cdots \otimes \mathcal{D}^{a+1} M_{a+1} \otimes \mathcal{D}^a M_a).
\]

For \(M \in \mathcal{B}(\mathfrak{g})\) and \((i, k) \in \hat{I}_{\text{fin}}\), we define
\[
\tilde{\mathcal{F}}_{i, k}(M) := (\mathcal{D}^k L_i) \nabla M \quad \text{and} \quad \tilde{\mathcal{E}}_{i, k}(M) := M \nabla (\mathcal{D}^{k+1} L_i),
\]
\[
(5.9) \quad \tilde{\mathcal{F}}^*_{i, k}(M) := M \nabla (\mathcal{D}^k L_i) \quad \text{and} \quad \tilde{\mathcal{E}}^*_{i, k}(M) := (\mathcal{D}^{k-1} L_i) \nabla M.
\]

**Lemma 5.8.** Let \((i, k) \in \hat{I}_{\text{fin}}\).

(i) \(\tilde{\mathcal{F}}^*_{i, k} \simeq \tilde{\mathcal{E}}_{i, k-1}^*\) and \(\tilde{\mathcal{E}}^*_{i, k} \simeq \tilde{\mathcal{F}}_{i, k-1}^*\).

(ii) For simple modules \(M\) and \(N\), \(\tilde{\mathcal{F}}_{i, k}(M) \simeq N\) if and only if \(M \simeq \tilde{\mathcal{E}}_{i, k}(N)\).

(iii) For a simple module \(M\) and \(t \in \mathbb{Z}\), we have
\[
\mathcal{D}^t \tilde{\mathcal{F}}_{i, k}(M) \simeq \tilde{\mathcal{F}}_{i, k+t}(\mathcal{D}^t M), \quad \mathcal{D}^t \tilde{\mathcal{E}}_{i, k}(M) \simeq \tilde{\mathcal{E}}_{i, k+t}(\mathcal{D}^t M),
\]
\[
\mathcal{D}^t \tilde{\mathcal{F}}^*_{i, k}(M) \simeq \tilde{\mathcal{F}}_{i, k+t}(\mathcal{D}^t M), \quad \mathcal{D}^t \tilde{\mathcal{E}}^*_{i, k}(M) \simeq \tilde{\mathcal{E}}_{i, k+t}(\mathcal{D}^t M).
\]

**Proof.** (i) and (iii) follow from the definition (5.9). (ii) follows from Lemma 2.3. \(\square\)

Since the operators \(\tilde{\mathcal{F}}_{i, k}, \tilde{\mathcal{E}}_{i, k}, \tilde{\mathcal{F}}^*_{i, k}, \) and \(\tilde{\mathcal{E}}^*_{i, k}\) depend on the choice of \(\mathcal{D}\), we write \(\mathcal{B}_D(\mathfrak{g})\) instead of \(\mathcal{B}(\mathfrak{g})\) when we consider \(\mathcal{B}(\mathfrak{g})\) together with the operators \(\tilde{\mathcal{F}}_{i, k}, \tilde{\mathcal{E}}_{i, k}, \tilde{\mathcal{F}}^*_{i, k}, \) and \(\tilde{\mathcal{E}}^*_{i, k}\). The set \(\mathcal{B}_D(\mathfrak{g})\) has the \(\hat{I}_{\text{fin}}\)-colored graph structure induced from \(\tilde{\mathcal{F}}_{i, k}\) as follows. We take \(\mathcal{B}_D(\mathfrak{g})\) as the set of vertices and define the \(\hat{I}_{\text{fin}}\)-colored arrows on \(\mathcal{B}_D(\mathfrak{g})\) by
\[
[M] \xrightarrow{(i, k)} [M'] \quad \text{if and only if} \quad M' \simeq \tilde{\mathcal{F}}_{i, k} M \quad (i, k) \in \hat{I}_{\text{fin}}.
\]

Then we have the main theorem.

**Theorem 5.9.** Let \(\mathcal{D} := \{L_i\}_{i \in \hat{I}_{\text{fin}}}\) be a complete duality datum of \(\mathcal{C}_0^0\).
(i) For \((i, k) \in \widehat{I}_{\text{fin}}\) and \(b \in \widehat{B}_{\text{fin}}(\infty)\), we have
\[
\Phi_D(\widetilde{F}_{i, k}(b)) = \widetilde{F}_{i, k}(\Phi_D(b)), \quad \Phi_D(\widetilde{E}_{i, k}(b)) = \widetilde{E}_{i, k}(\Phi_D(b)),
\]
where \(\Phi_D\) is given in Proposition 5.5.

(ii) The map \(\Phi_D\) induces an \(\widehat{I}_{\text{fin}}\)-colored graph isomorphism
\[
\widehat{B}_{\text{fin}}(\infty) \simeq B_D(g)
\]
sending \(1\) to \([1]\).

Proof. (i) Thanks to Lemma 5.8, it suffices to show that \(\Phi_D\) commutes with \(\widetilde{F}_{i, k}\) for \((i, k) \in \widehat{I}_{\text{fin}}\). By Proposition 5.5 (iv) and Lemma 5.8 (iii), we may assume that \(k = 0\) from the beginning.

Let \(b = (b_k)_{k \in \mathbb{Z}} \in \widehat{B}_{\text{fin}}(\infty)\). We set \(M := \Phi_D(b)\) and \(M_k := L_D(b_k)\) for \(k \in \mathbb{Z}\), where \(L_D\) is given in Lemma 3.2. By the construction, we have
\[
M \simeq \text{hd}(\cdots \otimes \mathcal{D}^2 M_2 \otimes \mathcal{D} M_1 \otimes M_0 \otimes \mathcal{D}^{-1} M_{-1} \otimes \cdots).
\]

Let \(a := \text{hd}(\mathcal{D}^{-1} L_i, M_1)\) and \(b := \text{hd}(\mathcal{D} L_i, M_0)\). By Lemma 5.4, there exist simple modules \(M''_i\) and \(M'_0\) such that
- \(M_1 \simeq M''_1 \nabla L_i^a\) and \(\text{hd}(L'_i, M_1) = \text{hd}(\mathcal{D}^{-1} L_i, M''_1) = 0\),
- \(M_0 \simeq L_i^b \nabla M_0'\) and \(\text{hd}(\mathcal{D} L_i, M_0') = 0\).

We now set \(K := \text{hd}(\cdots \otimes \mathcal{D}^2 M_2 \otimes \mathcal{D} M''_1)\) and \(N := \text{hd}(M'_0 \otimes \mathcal{D}^{-1} M_{-1} \otimes \cdots)\). By the construction, \((K, \mathcal{D} L_i, L_i, N)\) is strongly unmixed, and Lemma 5.7 implies that
\[
M \simeq \text{hd}(\cdots \otimes \mathcal{D}^2 M_2 \otimes \mathcal{D} M_1 \otimes M_0 \otimes \mathcal{D}^{-1} M_{-1} \otimes \cdots)
\]
\[
\simeq \text{hd}(\cdots \otimes \mathcal{D}^2 M_2 \otimes \mathcal{D} M''_1 \otimes (\mathcal{D} L_i)^a \otimes L_i^b \otimes M'_0 \otimes \mathcal{D}^{-1} M_{-1} \otimes \cdots)
\]
\[
\simeq \text{hd}(K \otimes (\mathcal{D} L_i)^a \otimes L_i^b \otimes N).
\]

Since
\[
\mathcal{D}^{-1}(L_i) \nabla M_1 \simeq M''_1 \nabla L_i^{(a-1)} \quad \text{if } a > 0,
\]
\[
L_i \nabla M_0 \simeq L_i^{(b+1)} \nabla M'_0,
\]
Lemma 3.2 and Lemma 5.3 imply the following.

(a) If $a > b$, then

$$
\tilde{F}_{i,0}(M) \simeq L_i \nabla \text{hd}(K \otimes (\mathcal{F} L_i) \otimes L_i^{\otimes b} \otimes N) \\
\simeq \text{hd}(K \otimes (\mathcal{F} L_i) \otimes L_i^{\otimes(b+1)} \otimes N) \\
\simeq \text{hd}(\cdots \otimes \mathcal{D}^2 M_2 \otimes (\mathcal{D} M_i) \otimes (L_i^{\otimes(a-1)} \otimes (L_i^{\otimes b} \nabla M_i) \otimes \mathcal{D}^{-1} M_{-1} \otimes \cdots) \\
\simeq \text{hd}(\cdots \otimes \mathcal{D}^2 M_2 \otimes (\mathcal{D} M_i \nabla M_i) \otimes M_0 \otimes \mathcal{D}^{-1} M_{-1} \otimes \cdots),
$$

which implies $\Phi_D(\tilde{F}_{i,0}(b)) = \tilde{F}_{i,0}(\Phi_D(b))$.

(b) If $a \leq b$, then

$$
\tilde{F}_{i,0}(M) \simeq L_i \nabla \text{hd}(K \otimes (\mathcal{F} L_i) \otimes L_i^{\otimes b} \otimes N) \\
\simeq \text{hd}(K \otimes (\mathcal{F} L_i) \otimes L_i^{\otimes(b+1)} \otimes N) \\
\simeq \text{hd}(\cdots \otimes \mathcal{D}^2 M_2 \otimes (\mathcal{D} M_i \nabla M_i) \otimes (L_i^{\otimes(b+1)} \nabla M_i) \otimes \mathcal{D}^{-1} M_{-1} \otimes \cdots) \\
\simeq \text{hd}(\cdots \otimes \mathcal{D}^2 M_2 \otimes (\mathcal{D} M_i \otimes M_0 \otimes \mathcal{D}^{-1} M_{-1} \otimes \cdots),
$$

which implies $\Phi_D(\tilde{F}_{i,0}(b)) = \tilde{F}_{i,0}(\Phi_D(b))$.

(ii) follows directly from (i).

The corollary below follows from Theorem 5.9.

**Corollary 5.10.**

(i) Let $U'_q(g)$ and $U'_{q'}(g')$ be quantum affine algebras, and let $D$ and $D'$ be complete duality data of $\mathcal{C}_0^g$ and $\mathcal{C}_0^{g'}$, respectively. Then the following are equivalent.

(a) $\mathcal{B}_D(g) \simeq \mathcal{B}_{D'}(g')$ as an $\mathcal{I}_{\text{fin}}$-colored graph.

(b) $g_{\text{fin}} \simeq (g')_{\text{fin}}$ as a simple Lie algebra.

(ii) In particular, the $\mathcal{I}_{\text{fin}}$-colored graph structure of $\mathcal{B}_D(g)$ does not depend on the choice of complete duality data $D$. 
CATEGORICAL CRYSTALS FOR QUANTUM AFFINE ALGEBRAS

6. Crystals and invariants

Throughout this section, we fix a complete duality datum \( D := \{ L_i \}_{i \in I_{\text{fin}}} \) of \( \mathcal{C}_\theta^0 \). We will provide formulas to compute the invariants \( \Lambda \) and \( \vartheta \) between \( \mathcal{D}^k L_i \) (\( k \in \mathbb{Z} \)) and an arbitrary simple module in terms of the extended crystal.

Lemma 6.1. Let \( L \) be a root module.

(i) \( \Lambda(L, \mathcal{D} L) = 0 \) and \( \Lambda(\mathcal{D} L, L) = \Lambda(L, \mathcal{D}^{-1} L) = 2 \).

(ii) \( \Lambda(L, \mathcal{D} L \nabla L) = -2 \) and \( \Lambda(\mathcal{D} L \nabla L, L) = 2 \).

(iii) \( \Lambda(L, L \nabla \mathcal{D}^{-1} L) = 2 \) and \( \Lambda(L \nabla \mathcal{D}^{-1} L, L) = -2 \).

(iv) For any \( a, b \in \mathbb{Z}_{\geq 0} \), we have

(a) \( \Lambda(L, \mathcal{D} L^{\otimes a} \nabla L^{\otimes b}) = -2 \min\{a, b\} \),

(b) \( \Lambda(\mathcal{D} L^{\otimes a} \nabla L^{\otimes b}, L) = 2a \),

(c) \( \Lambda(L, L^{\otimes a} \nabla \mathcal{D}^{-1} L^{\otimes b}) = 2b \),

(d) \( \Lambda(L^{\otimes a} \nabla \mathcal{D}^{-1} L^{\otimes b}, L) = -2 \min\{a, b\} \).

Proof. (i) By (2.4), we have

\[ \Lambda(L, \mathcal{D} L) = \Lambda(L, L) = 0. \]

Since \( \Lambda(L, \mathcal{D} L) + \Lambda(\mathcal{D} L, L) = 2 \vartheta(L, \mathcal{D} L) = 2 \), we have \( \Lambda(\mathcal{D} L, L) = 2 \).

(ii) By Lemma 2.7, we have

\[ \Lambda(\mathcal{D} L \nabla L, L) = \Lambda(\mathcal{D} L, L) = 2. \]

Since \( \vartheta(L, \mathcal{D} L \nabla L) = 0 \), we obtain \( \Lambda(L, \mathcal{D} L \nabla L) = -\Lambda(\mathcal{D} L \nabla L, L) = -2 \).

(iii) We have \( \Lambda(L, L \nabla \mathcal{D}^{-1} L) = \Lambda(L, \mathcal{D}^{-1} L) = \Lambda(\mathcal{D} L, L) = 2 \). As \( \vartheta(L, L \nabla \mathcal{D}^{-1} L) = 0 \), we obtain \( \Lambda(L \nabla \mathcal{D}^{-1} L, L) = -2 \).

(iv) follows from (i), (ii), (iii), [32, Lemma 4.3] and Lemma 2.4. \( \square \)

Lemma 6.2. Let \( i \in I \) and \( b \in B_{\text{fin}}(\infty) \). We write \( M := L_{\mathcal{D}}(b) \in \mathcal{C}_\mathcal{D} \) and

\[ r := \varepsilon_i(b) \quad \text{and} \quad s := \varepsilon_i^*(b). \]
Let \( x, y \in \mathbb{Z}_{\geq 0} \) and set
\[
N := \text{hd}((\mathcal{D}L_i)^{\otimes x} \otimes M \otimes (\mathcal{D}L_i)^{\otimes y}).
\]

Then \( N \) is simple and we have
\[
\begin{align*}
(i) \quad & \Lambda(L_i, N) = 2\max\{x, r\} + (\alpha_i, \text{wt}(b)) - 2x + 2y, \\
(ii) \quad & \Lambda(N, L_i) = 2\max\{y, s\} + (\alpha_i, \text{wt}(b)) + 2x - 2y, \\
(iii) \quad & \mathfrak{d}(L_i, N) = \max\{x, r\} + \max\{y, s\} + (\alpha_i, \text{wt}(b)).
\end{align*}
\]

**Proof.** Note that \( N \) is simple because the triple \((\mathcal{D}L_i, M, \mathcal{D}^{-1}L_i)\) is strongly unmixed by Theorem 3.1.

(i) Let \( M' := \mathcal{L}_D(\tilde{e}_i^r(b)) \). Lemma 3.2 says that \( M \cong L_i^{\otimes r} \nabla M' \) and \((L_i, M')\) is strongly unmixed because \( \mathfrak{d}(\mathcal{D}^kL_i, M') = 0 \) for \( k > 1 \) and \( \mathfrak{d}(\mathcal{D}L_i, M') = \varepsilon_i(\tilde{e}_i^r(b)) = 0 \). For \( k, l \in \mathbb{Z}_{\geq 0} \), we have the following by Lemma 2.7:

- \((\mathcal{D}L_i)^{\otimes k}, (L_i)^{\otimes l}, M')\) is normal,
- for any simple module \( X \), \((L_i)^{\otimes k}, X, (\mathcal{D}^{-1}L_i)^{\otimes l}\) is normal because \((L_i)^{\otimes k}, (\mathcal{D}^{-1}L_i)^{\otimes l}\) is unmixed,
- for a real simple module \( Y \), \((L_i)^{\otimes k}, Y, M'\) is normal because \((L_i)^{\otimes k}, M'\) is unmixed,
- \( \Lambda(L_i, M') = \Lambda^\infty(L_i, M') = -(-\alpha_i, \text{wt}(b) + r\alpha_i) \) by (2.5).

Using the above observations with Lemme 6.1, we have
\[
\begin{align*}
\Lambda(L_i, N) &= \Lambda(L_i, ((\mathcal{D}L_i)^{\otimes x} \nabla M) \nabla (\mathcal{D}^{-1}L_i)^{\otimes y}) \\
&= \Lambda(L_i, (\mathcal{D}L_i)^{\otimes x} \nabla M) + \Lambda(L_i, (\mathcal{D}^{-1}L_i)^{\otimes y}) \\
&= \Lambda(L_i, ((\mathcal{D}L_i)^{\otimes x} \nabla L_i^{\otimes r}) \nabla M') + 2y \\
&= \Lambda(L_i, (\mathcal{D}L_i)^{\otimes x} \nabla L_i^{\otimes r}) + \Lambda(L_i, M') + 2y \\
&= -2 \min\{x, r\} - (-\alpha_i, \text{wt}(b) + r\alpha_i) + 2y \\
&= 2x + 2r - 2\min\{x, r\} + (\alpha_i, \text{wt}(b)) - 2x + 2y \\
&= 2\max\{x, r\} + (\alpha_i, \text{wt}(b)) - 2x + 2y.
\end{align*}
\]

(ii) can be proved in the same manner as above.
(iii) follows from (i) and (ii).

□

**Theorem 6.3.** Let \((i, k) \in \hat{I}_{\text{fin}}\) and let \(M\) be a simple module in \(\mathcal{C}^0_{\theta}\). We set \(b := b_D(M)\) and

\[
x := \varepsilon^*_{(i,k+1)}(b), \quad r := \varepsilon_{(i,k)}(b), \quad s := \varepsilon^*_{(i,k)}(b), \quad y := \varepsilon_{(i,k-1)}(b).
\]

Then we have

(i) \(\Lambda(D^kL_i, M) = 2\max\{x, r\} + \sum_{t \in \mathbb{Z}} (-1)^t \delta(t > k)(\alpha_i, \text{wt}_t(b))\),

(ii) \(\Lambda(M, D^kL_i) = 2\max\{y, s\} + \sum_{t \in \mathbb{Z}} (-1)^t \delta(t < k)(\alpha_i, \text{wt}_t(b))\),

(iii) \(d(D^kL_i, M) = \max\{x, r\} + \max\{y, s\} + (\alpha_i, \text{wt}_k(b))\).

**Proof.** Thanks to Proposition 5.5 (iv), we may assume that \(k = 0\).

We write \(b = (b_k)_{k \in \mathbb{Z}}\) and set \(M_k := L_i \otimes D(b_k) \in \mathcal{C}_D\) for \(k \in \mathbb{Z}\). Note that \(x = \varepsilon^*_i(b_1), r = \varepsilon_i(b_0), s = \varepsilon^*_i(b_0),\) and \(y = \varepsilon_i(b_{-1})\) by the definition. Set \(b'_1 := \varepsilon^*_i(b_1), b'_{-1} := \varepsilon^*_i(b_{-1})\) and

\[
M'_1 := L_i \otimes D(b'_1) \quad \text{and} \quad M'_{-1} := L_i \otimes D(b'_{-1}).
\]

Note that \(M_1 \simeq M'_1 \nabla L_i^{\otimes x} \quad \text{and} \quad M_{-1} \simeq L_i^{\otimes y} \nabla M'_{-1}\). We define

\[
X := \text{hd}(\cdots \otimes D^2 M_2 \otimes D M'_1), \quad N := \text{hd}(D M'_1 \otimes M_0 \otimes D^{-1} L_i^{\otimes y}), \quad Y := \text{hd}(D^{-1} M'_{-1} \otimes D^{-2} M_{-2} \otimes \cdots).
\]

By Lemma 5.7, we have

\[
M \simeq \text{hd}(\cdots \otimes D M_1 \otimes M_0 \otimes D^{-1} M_{-1} \otimes \cdots) \\
\simeq \text{hd}(\cdots \otimes D M'_1 \otimes D L_i^{\otimes x} \otimes M_0 \otimes D^{-1} (L_i^{\otimes y} \otimes D^{-1} M'_{-1} \otimes \cdots) \\
\simeq \text{hd}(X \otimes N \otimes Y).
\]

Moreover, the triple \((X, N, Y)\) is strongly unmixed and

- \(b(D^iL_i, Y) = b(L_i, Y) = 0\) and \(b(L_i, X) = b(D^{-1}L_i, X) = 0\),
- \(\Lambda(L_i, Y) = \Lambda^\infty(L_i, Y)\) and \(\Lambda(L_i, X) = -\Lambda(X, L_i) = -\Lambda^\infty(X, L_i)\),
by (2.5) and Proposition 3.5. Thus, it follows from [32, Lemma 4.3, Corollary 4.4], and Lemma 6.2 that

\[
\Lambda(L_i, M) = \Lambda(L_i, (X \nabla N) \nabla Y) = \Lambda(L_i, X \nabla N) + \Lambda(L_i, Y) \\
= \Lambda(L_i, X) + \Lambda(L_i, N) + \Lambda(L_i, Y) \\
= -\Lambda^\infty(L_i, X) + \Lambda(L_i, N) + \Lambda^\infty(L_i, Y) \\
= \left(-\alpha_i, \sum_{t \geq 1} \mathrm{wt}_t(b) - x\alpha_i\right) + 2\max\{x, r\} + (\alpha_i, \mathrm{wt}_0(b)) - 2x + 2y \\
- \left(-\alpha_i, \sum_{t \leq -1} \mathrm{wt}_t(b) - y\alpha_i\right) \\
= 2\max\{x, r\} + \sum_{t \in \mathbb{Z}} (-1)^{\delta(t>0)}(\alpha_i, \mathrm{wt}_t(b)).
\]

(ii) can be proved in the same manner as above.

(iii) follows from (i) and (ii). \(\square\)

Remark 6.4. Theorem 6.3 can be understood as a quantum affine algebra analogue of [31, Corollary 3.8].

7. Crystal description of \(B_D(\mathfrak{g})\) for affine type \(A_n^{(1)}\)

In this section, we give a combinatorial description of the extended crystal \(B_D(\mathfrak{g})\) for affine type \(A_n^{(1)}\) in terms of affine highest weights.

Let \(U'_q(\mathfrak{g})\) be the quantum affine algebra of affine type \(A_n^{(1)}\), and let \(Q_0\) be the root lattice of \(\mathfrak{g}_0\). Note that \(I = \{0, 1, \ldots, n\}\), \(I_0 = I \setminus \{0\}\), and

\[
(\alpha_i, \alpha_j) = 2\delta(i = j) - \delta(i - j \equiv 1 \text{ mod } n + 1) - \delta(i - j \equiv -1 \text{ mod } n + 1) \quad \text{for } i, j \in I.
\]

For \(i \in I_0\) and \(x \in \mathbf{k}^\times\), we have

\[
(7.1) \quad \mathcal{D}(V(\varpi_i)_x) \cong V(\varpi_{n+1-i})_{x(-q)^{n+1}}.
\]

We take as \(\mathcal{C}_0^{0}\) the Hernandez-Leclerc category corresponding to

\[
\sigma_0(\mathfrak{g}) := \{(i, (-q)^a) \in I_0 \times \mathbf{k} \mid a - i \equiv 1 \text{ mod } 2\}.
\]
7.1. Multisegments.

A segment is an interval \([a, b]\) for \(1 \leq a \leq b \leq n\), and a multisegment is a finite multiset of segments. We set \([a, b] := \emptyset\) if \(a > b\). When \(a = b\), we simply write \([a] = [a, b]\). We set \(MS_n\) to be the set of multisegments. For a multisegment \(m = \{m_1, \ldots, m_k\}\), we sometimes write \(m = m_1 + m_2 + \cdots + m_k\).

It is well-known that \(MS_n\) has an \(A_n\)-crystal structure and it is isomorphic to \(B(\infty)\) as a crystal (see [53, 12] for example). Note that the simple \(R_{A_n}\)-module corresponding to the segment \([a, b]\) is the one-dimensional \(R(\alpha_{a,b})\)-module

\[
L[a, b] := \frac{R(\alpha_{a,b})e(a, a + 1, \ldots, b)}{\sum_{k=1}^{b-a+1} R(\alpha_{a,b}) x_k e(a, \ldots, b) + \sum_{k=1}^{b-a} R(\alpha_{a,b}) \tau_k e(a, \ldots, b)}
\]

where \(\alpha_{a,b} := \sum_{k=a}^{b} \alpha_k\).

We briefly review the crystal structure of \(MS_n\) following [53]. We set \(\text{wt}(\cdot) := -\alpha_{a,b}\) and define \(\text{wt}(m) := \sum_{t=1}^{k} \text{wt}(m_t)\) for \(m = \sum_{t=1}^{k} m_t\). We define total orders \(<\) and \(<'\) on segments by

\[
\text{(7.2)} \quad [a, b] < [c, d] \quad \text{if either}\ (b < d) \ \text{or}\ (b = d \ \text{and}\ a > c),
\]

\[
\text{(7.3)} \quad [a, b] <' [c, d] \quad \text{if either}\ (a < c) \ \text{or}\ (a = c \ \text{and}\ b > d).
\]

for segments \([a, b]\) and \([c, d]\). Note that \(<\) (resp. \(<'\)) is called the left order (resp. right order) in [53]. Note that for \(m \in MS_n\), the corresponding simple \(R_{A_n}\)-module is

\[
\text{hd}(L(m_1) \circ \cdots \circ L(m_k)) \simeq \text{hd}(L(m'_1) \circ \cdots \circ L(m'_k))
\]

where \(m = \sum_{t=1}^{k} m_t = \sum_{t=1}^{k} m'_t\) with \(m_k \leq \cdots \leq m_1\) and \(m'_k \leq' \cdots \leq' m'_1\).

Let \(i \in \{1, 2, \ldots, n\}\) and \(m \in MS_n\). Then the crystal operators \(\tilde{f}_i\) and \(\tilde{e}_i\) are defined as follows.

We rearrange the segments of \(m\) having the forms \([i, t]\) and \([i + 1, t]\) from left to right by largest to smallest with respect to the order \(<\) in (7.2). Then we put - at each segment \([i, t]\) and + at each segment \([i + 1, t]\). This sequence is called the left \(i\)-signature sequence of \(m\), which is denoted by \(S^<_i(m)\). We then cancel out all \((+, -)\) pairs. If there is no +, then we define \(\tilde{f}_i(m) := m \cup \{[i]\}\). Otherwise, \(\tilde{f}_i(m)\) is defined to be the multisegment obtained from \(m\) by replacing \([i + 1, t]\)
placed at the left most + by $[i, t]$. If there is no $-$, then we define $\tilde{e}_i(m) := 0$. Otherwise, $\tilde{e}_i(m)$ is defined to be the multisegment obtained from $m$ by replacing $[i, t]$ placed at the right most $-$ by $[i + 1, t]$. Furthermore, $\varepsilon_i(m)$ is the number of the remaining $-$’s.

The crystal operators $\tilde{f}_i^*$ and $\tilde{e}_i^*$ can be defined in a similar manner.

We rearrange the segments of $m$ having the form $[t, i]$ and $[t, i - 1]$ from left to right by largest to smallest with respect to the order $<'$ in (7.3). Then we put + at each segment $[t, i]$ and − at each segment $[t, i - 1]$. This sequence is called the right $i$-signature sequence of $m$, which is denoted by $S_{i}^{<'}(m)$. We then cancel out all $(+, -)$ pairs. If there is no $-$, then we define $\tilde{f}_i^*(m) := m \cup \{[i]\}$. Otherwise, $\tilde{f}_i^*(m)$ is defined to be the multisegment obtained from $m$ by replacing $[t, i - 1]$ placed at the right most $-$ by $[t, i]$. If there is no $+$, then we define $\tilde{e}_i^*(m) := 0$. Otherwise, $\tilde{e}_i^*(m)$ is defined to be the multisegment obtained from $m$ by replacing $[t, i]$ placed at the leftmost most $+$ by $[t, i - 1]$. In this case, $\varepsilon_i^*(m)$ is the number of the remaining ‘$+$’s.

Remark 7.1. Note that $\tilde{e}_i$ and $\tilde{e}_i^*$ in our paper correspond to $\hat{e}_i$ and $\check{e}_i$ in [53, Section 2.3], respectively. We remark also that we swap $+$ and $-$ in the signature rule in [53, Rule 1 in Section 2.3] in order to match it with the extended crystal signature rule given in Section 7.3.

7.2. Hernandez-Leclerc category $\mathcal{C}_Q$.

Let $Q$ be the Q-datum consisting of the Dynkin quiver

$$1 \leftarrow 2 \leftarrow \cdots \leftarrow n - 1 \leftarrow n$$

with the height function $\xi(i) := i - 1$ for $i \in I_0 = \{1, \ldots, n\}$, and let $D$ be the complete duality datum arising from $Q$:

$$(7.4) \quad D = \{L_i\}_{i \in I_0}, \quad \text{where } L_i := V(\varpi_1)(-q)^{2i-2} \text{ for } i \in I_0.$$

We consider the subcategory $\mathcal{C}_D$. Note that the category $\mathcal{C}_D$ coincides with the subcategory $\mathcal{C}_Q$ determined by $Q$ (see [35, Section 6] for details). For any $k \in \mathbb{Z}$, we denote by
\( \mathcal{D}^k(\mathcal{C}_q) \) the full subcategory of \( \mathcal{C}_0^q \) whose objects are \( \mathcal{D}^k(M) \) for all \( M \in \mathcal{C}_q \), and set

\[
\mathcal{D}^k(\sigma_q(\mathfrak{g})) := \{ (i, (q^{-a})) \in \sigma_0(\mathfrak{g}) \mid V(\varpi_i)(-q)^a \in \mathcal{D}^k(\mathcal{C}_q) \}.
\]

Note that

\[
\sigma_q(\mathfrak{g}) = \{ (i, (q^{-a})) \in \sigma_0(\mathfrak{g}) \mid i-1 \leq a \leq 2n-1-i \}.
\]

Let \( w_0 = (s_1)(s_2s_1)(s_3s_2s_1) \cdots (s_{n}s_{n-1} \cdots s_1) \) be a \( \mathcal{Q} \)-adapted reduced expression of the longest element \( w_0 \) in the symmetric group \( \mathfrak{S}_{n+1} \). Let \( \ell := n(n+1)/2 \) and write \( (i_1, i_2, \ldots, i_{\ell}) = (1,2,1,3,2,1,\ldots,n,n-1,\ldots,1) \). For \( k = 1, \ldots, \ell \), we denote by \( E^* (\beta_k) \) the dual PBW vector corresponding to \( w_0 \) and \( \beta_k := s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}) \).

The complexified Grothendieck ring \( \mathbb{C} \otimes_{\mathbb{Z}} K(\mathcal{C}_q) \) is isomorphic to the coordinate ring \( \mathbb{C}[N] \) of the unipotent group \( N \) associated with the simple Lie algebra of type \( A_n \) and, under this isomorphism, the set of the isomorphism classes of simple modules in \( \mathcal{C}_q \) corresponds to the upper global basis (or dual canonical basis) of \( \mathbb{C}[N] \) ([19]). Recall that \( \alpha_{a,b} \) is the positive root \( \sum_{k=a}^b \alpha_k \) for \( 1 \leq a \leq b \leq n \). As dual PBW vectors are contained in the upper global basis, under this isomorphism, the PBW vector \( E^* (\alpha_{a,b}) \) corresponds to the fundamental module \( V(\varpi_{b-a+1})(-q)^{b+a-2} \) in \( \mathcal{C}_q \). Since a segment \( [a,b] \) is in 1-1 correspondence with a positive root \( \alpha_{a,b} \), the correspondence

\[
[a,b] \mapsto V(\varpi_{b-a+1})(-q)^{b+a-2}
\]

gives a bijection between the set of segments and the set of fundamental modules in \( \mathcal{C}_q \). Thus the fundamental modules in \( \mathcal{C}_q \) give the dual PBW basis of \( \mathbb{C} \otimes_{\mathbb{Z}} K(\mathcal{C}_q) \) associated with \( w_0 \), and the set \( B_\mathcal{Q} \) of the isomorphism classes of simple modules in \( \mathcal{C}_q \) has a crystal structure which is isomorphic to \( \text{MS}_n \simeq B(\infty) \) (see Lemma 3.2). For any multisegment \( m = \sum_k [a_k, b_k] \), let \( V_m \) be the simple module with the affine highest weight \( \sum_{k=a_k+1}^{b_k}(-q)^{b_k+a_k-2} \) (see Theorem 2.2 (iv)). Then, the correspondence \( m \mapsto V_m \) gives a bijection

\[
\phi_0 : \text{MS}_n \sim \to B_\mathcal{Q}, \quad \text{which is a crystal isomorphism}.
\]
7.3. Extended crystal realization.

We use the same notations in the previous subsections. Let

\[ S_n := \{ (i, a) \in I_0 \times \mathbb{Z} \mid a - i \equiv 1 \mod 2 \} \]

and

\[ P_n := (\mathbb{Z}_{\geq 0})^{\oplus S_n}. \]

For an element \( \lambda \in P_n \), we write

\[ \lambda = \sum_{(i, a) \in S_n} c_{i,a}(\lambda)(i,a) \] with \( c_{i,a}(\lambda) \in \mathbb{Z}_{\geq 0} \).

We regard \( S_n \) as a subset of \( P_n \). For \( (i, a) \in S_n \), we define

\[ D(i, a) := (n + 1 - i, a + n + 1) \quad \text{and} \quad D^{-1}(i, a) := (n + 1 - i, a - n - 1). \]

We extend to \( D^k(i, a) \) for any \( k \in \mathbb{Z} \) and, for a subset \( A \subset S_n \), define \( D^k(A) := \{ D^k(i, a) \mid (i, a) \in A \} \). We set

\[ S_0^n := \{ (i, a) \in S_n \mid i - 1 \leq a \leq 2n - 1 - i \}, \]

and define

\[ S_k^n := D^k(S_0^n) \quad \text{and} \quad P_k^n := (\mathbb{Z}_{\geq 0})^{\oplus S_k^n} \] for any \( k \in \mathbb{Z} \).

Let us recall \( \hat{\mathcal{P}}^+ := \mathbb{Z}_{\geq 0}^{\oplus \sigma_0(\mathfrak{g})} \), the set of affine highest weights defined in Theorem 2.2 (iv). The following lemma can be proved easily by using (7.1), (7.6), (7.10) and the definitions of \( S_n \) and \( S_k^n \).

Lemma 7.2.

(i) The map \( \psi_n: S_n \xrightarrow{\sim} \sigma_0(\mathfrak{g}) \) defined by

\[ \psi_n(i, a) = (i, (-q)^a) \] for any \( (i, a) \in S_n \)

is bijective. We extend it to the bijection between \( P_n \) and \( \hat{\mathcal{P}}^+ \), which is denoted by the same notation \( \psi_n: P_n \xrightarrow{\sim} \hat{\mathcal{P}}^+ \).

(ii) For any \( k \in \mathbb{Z} \), the restriction of \( \psi_n \) to the subset \( S_k^n \) gives the bijection

\[ \psi_n: S_k^n \xrightarrow{\sim} D^k(\sigma_\mathcal{Q}(\mathfrak{g})). \]
where $D_k^{(\sigma_Q(g))}$ is defined in (7.5). We extend it to the bijection between $P_n^+$ and $\hat{\mathbb{P}}_{Q,k} := \mathbb{Z}_{\geq 0}^{\oplus D_k^{(\sigma_Q(g))}}$, which is denoted by the same notation $\psi_n : P_n^+ \rightarrow \hat{\mathbb{P}}_{Q,k}$. 

(iii) For any $k, l \in \mathbb{Z}$ with $k \neq l$, we have $S_k^k \cap P_n = \emptyset$ and 

\[
\mathcal{I}_n = \bigcup_{k \in \mathbb{Z}} \mathcal{I}_n^k \quad \text{and} \quad \mathcal{P}_n = \bigoplus_{k \in \mathbb{Z}} \mathcal{P}_n^k.
\]

Example 7.3. Let $n = 3$. Then we have 

\[
\mathcal{I}_3^{-1} = \{(3, -4), (3, -2), (3, 0), (2, -3), (2, -1), (1, -2)\}, \\
\mathcal{I}_3^0 = \{(1, 0), (1, 2), (1, 4), (2, 1), (2, 3), (3, 2)\}, \\
\mathcal{I}_3^1 = \{(3, 4), (3, 6), (3, 8), (2, 5), (2, 7), (1, 6)\}.
\]

Pictorially, the sets $\mathcal{I}_3^{-1}$, $\mathcal{I}_3^0$ and $\mathcal{I}_3^1$ can be drawn as in the following figure where the elements of $\mathcal{I}_3^{-1}$, $\mathcal{I}_3^0$ and $\mathcal{I}_3^1$ are denoted by $*$, $\bullet$ and $\triangle$, respectively.

\[
\begin{array}{cccccccccc}
& -5 & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & & * & & & & & & & & & & & & \\
2 & * & & * & & & & & & & & & & & & \\
3 & * & * & * & & & & & & & & & & & & \\
\end{array}
\]

Since Theorem 2.2 (iv) tells us that simple modules in $\mathcal{C}_0^0$ are parameterized by elements of $\hat{\mathbb{P}}$ as affine highest weights, they are also parameterized by $\mathcal{P}_n$ via the map $\psi_n$ defined in Lemma 7.2. For $\lambda \in \mathcal{P}_n$, we denote by $V(\lambda)$ the corresponding simple module in $\mathcal{C}_0^0$ whose affine highest weight is $\psi_n(\lambda)$. Thus the map 

\[
\Psi_n : \mathcal{P}_n \rightarrow \mathcal{B}_{\mathcal{D}}(g), \quad \lambda \mapsto V(\lambda) \quad \text{for} \ \lambda \in \mathcal{P}_n
\]

is bijective. Note that $V(i, a) := V(\omega_i(a))$ and $D_m(V(i, a)) \simeq V(D_m(i, a))$ for $m \in \mathbb{Z}$.

We are now ready to define an extended crystal structure on $\mathcal{P}_n$. 

First we define the weight $\hat{\text{wt}}$ on $\mathcal{P}_n$ as follows. For any $(i, a) \in \mathcal{I}_n^0$, define 

\[
\hat{\text{wt}}(i, a) := -\sum_{k=A}^{B} \alpha_k \in Q_0,
\]
where \( A = \frac{a-i+3}{2}, \) \( B = \frac{a+i+1}{2} \). By Lemma 7.2 (iii), for any \((i, a) \in \mathcal{I}_n\), there exists a unique \( k \in \mathbb{Z} \) such that \( D^k(i, a) \in \mathcal{I}^0_n \). Define

\[
(7.13) \quad \text{wt}(i, a) = (-1)^k \text{wt}(D^k(i, a)).
\]

We finally define \( \widetilde{\text{wt}} : \mathcal{P}_n \to \mathbb{Q}_0 \) by

\[
\widetilde{\text{wt}}(\lambda) := \sum_{(i, a) \in \mathcal{I}_n} c_{i,a}(\lambda) \text{wt}(i, a) \quad \text{for} \quad \lambda = \sum_{(i, a) \in \mathcal{I}_n} c_{i,a}(\lambda)(i, a) \in \mathcal{P}_n.
\]

We now explain the extended crystal operators on \( \mathcal{P}_n \). For any \( i \in I_0 \), we define \( u_i := (1, 2(i - 1)) \in \mathcal{I}_n \) and set

\[
(7.14) \quad D := \{u_i \mid i \in I_0\} \subset \mathcal{I}_n.
\]

For any \((i, a) \in \mathcal{I}_n\), we define

\[
(7.15) \quad S(i, a) := \{(j, b) \in \mathcal{I}_n \mid j - b = i - a\} \subset \mathcal{I}_n,
\]

\[
S'(i, a) := \{(j, b) \in \mathcal{I}_n \mid j + b = i + a\} \subset \mathcal{I}_n,
\]

and set \( \text{sh}_t(i, a) := (i, a + t) \) for \( t \in 2\mathbb{Z} \).

Let \( \lambda \in \mathcal{P}_n \) and \((i, k) \in \widehat{I}_0\). We shall define \( \widetilde{F}_{i,k}(\lambda) \) and \( \widetilde{E}_{i,k}(\lambda) \). They will correspond to the operators \( \widetilde{F}_{i,k} \) and \( \widetilde{E}_{i,k} \) on \( \mathcal{B}_D(\mathfrak{g}) \).

Set \( \lambda = \sum_{(i, a) \in \mathcal{I}_n} c_{i,a}(\lambda)(i, a) \in \mathcal{P}_n \) with \( c_{i,a}(\lambda) \in \mathbb{Z}_{\geq 0} \).

(Step 1) Set

\[
u := D^k(u_i) = \begin{cases} 
(1, 2(i - 1) + k(n + 1)) & \text{if } k \text{ is even}; \\
(n, 2(i - 1) + k(n + 1)) & \text{if } k \text{ is odd}.
\end{cases}
\]
We define $S_{-i,k}$ and $S_{+i,k}$ as follows:

$$S_{-i,k} := \begin{cases} S(u) & \text{if } k \text{ is even}, \\ S'(u) & \text{if } k \text{ is odd}, \end{cases}$$

$$S_{+i,k} := \begin{cases} S(sh2(u)) & \text{if } k \text{ is even}, \\ S'(sh2(u)) & \text{if } k \text{ is odd}, \end{cases}$$

Set $S_{i,k} := S_{-i,k} \cup S_{+i,k}$. Let $\succ_k$ be the total order on $\mathcal{S}_n$ defined by

$$(j, a) \succ_k (j', a') \iff \begin{cases} (j > j') & \text{if } k \text{ is even}, \\ (j < j') & \text{if } k \text{ is odd}, \end{cases}$$

Note that for $\mu = \sum_{t=1}^r (j_t, a_t) \in \mathcal{P}_n$ with $(j_t, a_t) \in S_{i,k}$ $(t = 1, \ldots, r)$ and $(j_1, a_1) \succ_k \cdots \succ_k (j_r, a_r)$, we have

$$V(\mu) \simeq \text{hd}(V(\varpi_{j_1}) \otimes \cdots \otimes V(\varpi_{j_r})).$$

For simplicity, we set $v_t = 0$ unless $1 \leq t \leq 2n$. Note that $v_t \in S_{-i,k}$ for any odd $t$ and $v_t \in S_{+i,k}$ for any even $t$.

(Step 2) For $t \in \mathbb{Z}_{>0}$, let $-^t := \underbrace{- \cdots -}_{t}$ and $+^t := \underbrace{+ \cdots +}_{t}$, where $+$ and $-$ are symbols.

We set $-^0 := \emptyset$ and $+^0 := \emptyset$. For $k = 1, 2, \ldots, 2n$, we define

$$s_k := \begin{cases} +c_{-k}(\lambda) & \text{if } v_k \in S_{+i,k}, \\ -c_{-k}(\lambda) & \text{if } v_k \in S_{-i,k}, \end{cases}$$

and define the sequence $s$ as the concatenation $s_{2n} * s_{2n-1} * \cdots * s_1$ of $s_k$’s. This sequence is called the $(i, k)$-signature sequence of $\lambda$. We cancel out all the possible $(+, -)$ pairs to
obtain a sequence of $-$’s followed by $+$’s. The resulting sequence is called the reduced $(i, k)$-signature sequence of $\lambda$.

(Step 3) Let $v_t$ be the element of $S_{i, k}$ corresponding to the leftmost $+$ in the reduced $(i, k)$-signature sequence of $\lambda$. If such an $v_t$ exists, then we define

$$\tilde{F}_{i, k}(\lambda) := \lambda - v_t + v_{t+1}.$$ 

Otherwise, we define

$$\tilde{F}_{i, k}(\lambda) := \lambda + v_1.$$ 

Similarly, let $v_s$ be the element of $S$ corresponding to the rightmost $-$ in the reduced $(i, k)$-signature sequence of $\lambda$. If such an $v_s$ exists, then we define

$$\tilde{E}_{i, k}(\lambda) := \lambda - v_s + v_{s-1}.$$ 

Otherwise, we define

$$\tilde{E}_{i, k}(\lambda) := \lambda + v_{2n}.$$ 

Note that these operators $\tilde{F}_{i, k}$ and $\tilde{E}_{i, k}$ are compatible with the duality operator $D$ as seen in the next lemma.

**Lemma 7.4.** We have

(i) $D(S(u)) = S'(Du)$ and $D(S'(u)) = S(Du)$ for any $u \in \mathcal{I}_n$,

(ii) $D(S_{i,k}^\pm) = S_{i,k+1}^\pm$ and $D(S_{i,k}) = S_{i,k+1}$,

(iii) for any $(j, a), (j', a') \in \mathcal{I}_n$, we have $D(j, a) \geq_{k+1} D(j', a')$ if and only if $(j, a) \geq_k (j', a')$,

(iv) $D \circ \tilde{F}_{i, k} = \tilde{F}_{i, k+1} \circ D$.

**Proof.** (i) and (iii) immediately follow from the definition (7.10).

(ii) follows from (i).

(iv) easily follows from (ii), (iii). \qed

We thus obtain the $\tilde{I}_0$-colored graph structure on $\mathcal{P}_n$ induced by the operators $\tilde{F}_{i, k}$ as follows: take $\mathcal{P}_n$ as the set of vertices and define the $\tilde{I}_0$-colored arrows on $\mathcal{P}_n$ by

$$\lambda \xrightarrow{(i, k)} \lambda' \text{ if and only if } \lambda' = \tilde{F}_{i, k}(\lambda) \quad (i, k) \in \tilde{I}_0).$$
The following theorem is the main theorem of this section whose proof is postponed until the next subsection.

**Theorem 7.5.** Let \( \mathcal{D} := \{ \mathcal{L}_i \}_{i \in I_0} \), where \( \mathcal{L}_i := V(\varpi_1)(-q)^{2i-2} \) for any \( i \in I_0 = \{1, \ldots, n\} \).

(i) There is a bijection \( \Upsilon_n : \hat{\mathcal{B}}_{g_0}(\infty) \sim \rightarrow \mathcal{P}_n \) such that

(a) \( \Upsilon_n(1) = 0 \),
(b) \( \tilde{F}_{i,k}(\Upsilon_n(b)) = \Upsilon_n(\tilde{F}_{i,k}(b)) \) and \( \tilde{E}_{i,k}(\Upsilon_n(b)) = \Upsilon_n(\tilde{E}_{i,k}(b)) \),
(c) \( \tilde{\text{wt}}(\Upsilon_n(b)) = \tilde{\text{wt}}(b) \).

(ii) For \( (i, k) \in \hat{I}_0 \) and \( \lambda \in \mathcal{P}_n \), we have

\[
\tilde{F}_{i,k}(\Psi_n(\lambda)) = \Psi_n(\tilde{F}_{i,k}(\lambda)), \quad \tilde{E}_{i,k}(\Psi_n(\lambda)) = \Psi_n(\tilde{E}_{i,k}(\lambda)),
\]

where \( \Psi_n : \mathcal{P}_n \sim \rightarrow \mathcal{B}_\mathcal{D}(\mathfrak{g}) \) is the bijection defined in (7.12). Hence the following diagram

\[
\begin{array}{ccc}
\hat{\mathcal{B}}_{g_0}(\infty) & \xrightarrow{\Phi_D} & \mathcal{B}_\mathcal{D}(\mathfrak{g}) \\
\Upsilon_n & \downarrow & \Rightarrow \\
\mathcal{P}_n & \xrightarrow{\Psi_n} & \mathcal{B}_\mathcal{D}(\mathfrak{g})
\end{array}
\]

commutes, and the arrows are \( \hat{I}_0 \)-colored graph isomorphisms.

**Example 7.6.** Let \( n = 3 \). In this case, we have \( \mathcal{D} = \{ u_1, u_2, u_3 \} \), where \( u_1 = (1, 0) \), \( u_2 = (1, 2) \), \( u_3 = (1, 4) \), and the corresponding duality datum is

\[
\mathcal{D} = \{ V(\varpi_1), V(\varpi_1)(-q)^2, V(\varpi_1)(-q)^4 \}.
\]

We choose \( \lambda \in \mathcal{P}_3 \) as follows:

\[
\lambda = (3, -4) + (3, -2) + 2(2, -1) + (1, -2) + (1, 2) + (2, 1) + (2, 3) + 2(3, 4) + (2, 5) + (2, 7).
\]
Pictorially, we write the coefficients of $\lambda$ as follows:

| $\bar{a}$ | -5 | -4 | -3 | -2 | -1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|------------|----|----|----|----|----|---|---|---|---|---|---|---|---|---|
| 1          | .  | 1  | .  | .  | .  |   |   |   |   |   |   |   |   |   |
| 2          | .  | .  | 2  | 1  | 1  | 1 | 1 |   |   |   |   |   |   |   |   |
| 3          | 1  | 1  | .  | .  | 2  | . |   |   |   |   |   |   |   |   |   |

(i) Let $(i, k) = (1, 0) \in \hat{I}_0$. Then, $u = D_0^0(u_1) = u_1$ and

$$S_{1,0}^- = S(u) = \{(1, 0), (2, 1), (3, 2)\},$$

$$S_{1,0}^+ = S(sh_2(u)) = \{(1, 2), (2, 3), (3, 4)\},$$

and

$$S_{1,0} = \{(3, 4) \succ (3, 2) \succ (2, 3) \succ (2, 1) \succ (1, 2) \succ (1, 0)\}.$$  

We use the notations $v_k$ for the element of $S_{1,0}$ as in (7.16). For $k = 1, 2, \ldots, 6$, we obtain

$$s_6 = ++, \quad s_5 = \emptyset, \quad s_4 = +, \quad s_3 = -, \quad s_2 = +, \quad s_1 = \emptyset,$$

which gives the sequence $s = s_6 * s_5 * s_4 * s_3 * s_2 * s_1$. Canceling out all $(+, -)$ pairs, we have the following reduced $(1, 0)$-signature sequence of $\lambda$:

\[
\begin{array}{c|ccccccc}
\text{reduced (1,0)-signature} & v_6 & v_5 & v_4 & v_3 & v_2 & v_1 \\
\hline
& ++ & \emptyset & \emptyset & \emptyset & + & \emptyset
\end{array}
\]

Since $v_6$ is located at the leftmost $+$ and $v_t = 0$ for $t > 6$, we have

$$\tilde{F}_{1,0}(\lambda) = \lambda - v_6 + v_7 = \lambda - (3, 4),$$

and by Theorem 7.5, we have

$$\tilde{F}_{1,0}(V(\lambda)) = V(\tilde{F}_{1,0}(\lambda)) = V(\lambda - (3, 4)).$$
(ii) Let \((i, k) = (1, -1) \in \hat{I}_0\). In this case, we have \(u = D^{-1}(u_1) = (3, -4)\) and
\[
S_{1, -1}^- = S'(u) = \{(1, -2), (2, -3), (3, -4)\},
\]
\[
S_{1, -1}^+ = S'(sh_2(u)) = \{(1, 0), (2, -1), (3, -2)\},
\]
and
\[
S_{1, 1} = \{(1, 0) \succ (1, -2) \succ (2, -1) \succ (2, -3) \succ (3, -2) \succ (3, -4)\}.
\]
We use the notations \(v_k\) for the element of \(S_{1, -1}\) as in (7.16). For \(k = 1, 2, \ldots, 6\), we obtain
\[
s_6 = \emptyset, \ s_5 = -, \ s_4 = ++, \ s_3 = \emptyset, \ s_2 = +, \ s_1 = -,
\]
which gives the sequence \(s = s_6 * s_5 * s_4 * s_3 * s_2 * s_1\). Canceling out all \((+, -)\) pairs, we have the following reduced \((1, -1)\)-signature sequence of \(\lambda\):

| \(v_6\) | \(v_5\) | \(v_4\) | \(v_3\) | \(v_2\) | \(v_1\) |
|---|---|---|---|---|---|
| \(\emptyset\) | \(-\) | \(++\) | \(\emptyset\) | \(\emptyset\) | \(\emptyset\) |

Since \(v_4\) is located at the leftmost \(+\), we have
\[
\tilde{F}_{1, -1}(\lambda) = \lambda - v_4 + v_5 = \lambda - (2, -1) + (1, -2).
\]
and by Theorem 7.5, we have
\[
\tilde{\mathcal{F}}_{1, -1}(V(\lambda)) = V(\tilde{F}_{1, -1}(\lambda)) = V(\lambda - (2, -1) + (1, -2)).
\]

7.4. **Proof of Theorem 7.5.**

In this subsection, we will prove Theorem 7.5. We employ the same notations introduced in the previous subsections.

Recall that the set \(\text{MS}_n\) of multisegments defined in Section 7.1 has a crystal structure which is isomorphic to the crystal \(B(\infty)\). For any \(k \in \mathbb{Z}\), the correspondence \([a, b] \mapsto D^k(b - a + 1, b + a - 2)\) gives a bijection between the set of segments and \(\mathcal{P}_n^k\). Thus this induces a bijection \(\gamma_k: \text{MS}_n \xrightarrow{\sim} \mathcal{P}_n^k\).

On the other hand, for any \(k \in \mathbb{Z}\), let \(B_Q^k\) be the set of the isomorphism classes of simple modules in \(\mathcal{D}^k(Q)\). Since \(B_Q^k\) is equal to \(B_{\mathcal{D}}^k\), \(B_Q^k\) has a crystal structure which
is isomorphic to $B(\infty)$ (see Lemma 3.2 and Lemma 3.3). Since the elements of $B^k_Q$ are parameterized by affine highest weights in $\hat{P}^+_Q$, the map

$$\Psi^k_n: \mathcal{P}^k_n \rightarrow B^k_Q, \quad \lambda \mapsto V(\lambda)$$

is bijective. Note that $\Psi^k_n = \Psi^k_n|_{\mathcal{P}^k_n}$, where $\Psi^k_n$ is defined in (7.12). Thus we have the bijection $\phi^k := \Psi^k_n \circ \gamma^k: MS_n \sim B^k_Q$, i.e., the diagram

$$B(\infty) \simeq MS_n \xrightarrow{\phi^k} B^k_Q \simeq B(\infty)$$

commutes. By the definition (7.13), we have

$$\text{wt}(\gamma^k([a,b])) = (-1)^k \text{wt}(b-a+1, b+a-2) = (-1)^{k+1} \sum_{k=a}^{b} \alpha_k$$

(7.17)

$$= (-1)^k \text{wt}([a,b]).$$

**Example 7.7.** We use the same notations given in Example 7.3. In the table below, we write the segments $\gamma^{-1}_k(i,a)$ (where $(i,a) \in \mathcal{S}^k$ for $k = -1, 0, 1$) at the position $(i,a)$:

| $i \backslash a$ | $-5$ | $-4$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ | $4$ | $5$ | $6$ | $7$ | $8$ |
|---------------|------|------|------|------|------|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 1             |      |      |      |      |      |     |     |     |     |     |     |     |     |     |
| 2             |      |      |      |      |      |     |     |     |     |     |     |     |     |     |
| 3             |      |      |      |      |      |     |     |     |     |     |     |     |     |     |

Here, the underlined (resp. double underlined) segments are in the image of $\mathcal{S}^1_3$ (resp. $\mathcal{S}^{-1}_3$) under the map $\gamma^{-1}_1$ (resp. $\gamma^{-1}_3$).

From now on, we identify $MS_n$ with $B(\infty)$ as a crystal. Similarly to the extended crystal $\bar{B}_{\theta^0}(\infty)$, we define $\bar{MS}_n$ by

$$\bar{MS}_n := \left\{ (m_k)_{k \in \mathbb{Z}} \in \prod_{k \in \mathbb{Z}} MS_n \mid m_k = \emptyset \text{ for all but finitely many } k's \right\}.$$
Define the map 
\[ \gamma : \hat{\mathcal{MS}}_n \rightarrow \mathcal{P}_n, \quad b = (m_k)_{k \in \mathbb{Z}} \mapsto \sum_{k \in \mathbb{Z}} \gamma_k(m_k). \]

Thanks to Lemma 7.2, the map \( \gamma \) is bijective. Let \( \phi = \Psi_n \circ \gamma \). Then we have the following commutative diagram

\[
\begin{array}{ccc}
\hat{\mathcal{MS}}_n & \xrightarrow{\phi} & B_D(g) \\
\downarrow{\gamma} & & \downarrow{\Psi_n} \\
\mathcal{P}_n & \xrightarrow{\quad} & \mathcal{B}(\mathfrak{g})
\end{array}
\]

in which all the arrows are bijective. For any \( b \in \hat{\mathcal{MS}}_n \), we have

\begin{align}
(7.18) & \quad \gamma \circ D(b) = D \circ \gamma(b), \\
(7.19) & \quad \hat{\text{wt}}(b) = \hat{\text{wt}} \circ \gamma(b), \\
(7.20) & \quad \gamma(\emptyset) = 0,
\end{align}

where \( D \) is defined by \( D((m_k)_{k \in \mathbb{Z}}) = (m_{k-1})_{k \in \mathbb{Z}} \) and \( D \) is defined by (7.10). Note that (7.19) follows directly from (7.17).

**Lemma 7.8.** Let \( b = (m_k)_{k \in \mathbb{Z}} \in \hat{\mathcal{MS}}_n \) and set \( V_k := \phi_k(m_k) \in B^k_Q \) for \( k \in \mathbb{Z} \). Then \( \phi(b) \simeq \text{hd}(\cdots \otimes V_2 \otimes V_1 \otimes V_0 \otimes V_{-1} \otimes \cdots) \).

Thus, if we identify \( \hat{\mathcal{MS}}_n \) with \( \hat{\mathcal{B}}_{g_0}(\infty) \) by extending the isomorphism \( \mathcal{MS}_n \simeq B(\infty) \), then \( \phi \) coincides with the map \( \Phi_D \) defined in Proposition 5.5.

**Proof.** Let us recall the set \( D = \{ u_i \mid i \in I_0 \} \) defined in (7.14). Since the duality datum \( \mathcal{D} := \{ V(u_i) \}_{i \in I_0} \) is equal to the complete duality datum defined in (7.4), the fundamental modules in \( \mathcal{C}_g^0 \) forms the affine cuspidal modules corresponding to \( \mathcal{D} \) and \( w_0 \) defined in (7.7).

Let \( b = (m_k)_{k \in \mathbb{Z}} \in \hat{\mathcal{MS}}_n \) and set \( \lambda = \gamma(b) \). By writing \( \lambda = \sum_{k \in \mathbb{Z}} \lambda_k \) with \( \lambda_k \in \mathcal{P}_n^k \) \( (k \in \mathbb{Z}) \), we have

\[
V(\lambda) \simeq \text{hd}(\cdots \otimes V(\lambda_2) \otimes V(\lambda_1) \otimes V(\lambda_0) \otimes V(\lambda_{-1}) \otimes \cdots).
\]

Thus, by the definition of \( \gamma \), we have \( \gamma_k(m_k) = \lambda_k \) for \( k \in \mathbb{Z} \), which gives the assertion. \( \square \)
By the isomorphism $\hat{\mathcal{M}}_n \simeq \hat{B}_0(\infty)$, the extended crystal operators act on $\hat{\mathcal{M}}_n$. We will prove that $\gamma$ commutes with the extended crystal operator action where the extended crystal operator action on $\mathcal{P}_n$ is described in Section 7.3.

In order to show this, we need a couple of lemmas below.

For $i \in I_0$, we define

\[ i_A := \{(i, t) \mid i \leq t \leq n\} \subset \mathcal{M}_n, \]

\[ A_i := \{(t, i) \mid 1 \leq t \leq i\} \subset \mathcal{M}_n. \]

**Lemma 7.9.** Let $i \in I_0$. Let $S_{i,0}^-, S_{i,0}^+$ and $S_{i,0} = \{v_{2n}, v_{2n-1}, \ldots, v_1\}$ be the sets defined in (Step 1) in §7.3.

(i) For $1 \leq k \leq n$, we have

\[ v_{2k-1} = (k, 2(i - 1) + k - 1). \]

\[ v_{2k} = (k, 2(i - 1) + k + 1). \]

(ii) We have

\[ S_{i,0} \cap \mathcal{S}_n^0 = \{v_k \mid 1 \leq k \leq 2(n - i) + 1\}, \]

\[ S_{i,0} \cap \mathcal{S}_n^1 = \{v_k \mid 2(n - i) + 2 \leq k \leq 2n\}. \]

Moreover, we have, for any $\epsilon \in \{0, 1\},$

(a) for $1 \leq k \leq n - i + \epsilon$, $v_{2k-\epsilon} \in \mathcal{S}_n^0$ and $\gamma_{0}^{-1}(v_{2k-\epsilon}) = [i + 1 - \epsilon, k + i - \epsilon]$,

(b) for $n - i + \epsilon < k \leq n$, $v_{2k-\epsilon} \in \mathcal{S}_n^1$ and $\gamma_{1}^{-1}(v_{2k-\epsilon}) = [i + k - n - \epsilon, i - \epsilon]$.

(iii) We rearrange

\[ i_A \cup i_A = \{m_1, m_2, \ldots, m_{2n-2i+1}\} \]

from left to right by largest to smallest in the order $<$ defined in (7.2), and rearrange

\[ A_{i-1} \cup A_i = \{m'_1, m'_2, \ldots, m'_{2i-1}\} \]

from left to right by largest to smallest in the order $<'$ defined in (7.3). Then the sequence

\[ \{\gamma_1(m'_1), \ldots, \gamma_1(m'_{2i-1}), \gamma_0(m_1), \ldots, \gamma_0(m_{2n-2i+1})\} \]
is equal to the sequence \( \{ v_{2n}, v_{2n-1}, \ldots, v_2, v_1 \} \).

**Proof.** Note that \( S_{i,0} = S_{i,0}^- \cup S_{i,0}^+ \) and
\[
\gamma_0([a,b]) = (b - a + 1, b + a - 2) \quad \text{and} \quad \gamma_1([a,b]) = (n - b + a, n + b + a - 1)
\]
for any segment \([a,b] \in MS_n\).

(i) follows from the definitions of \( S_{i,0}^\pm \).

(ii) follows from (i) together with
\[
\begin{align*}
S_{0,n} &= \{ (i,a) \in S_n \mid 1 \leq i \leq n, \ i - 1 \leq a \leq 2n - i - 1 \}, \\
S_{1,n} &= \{ (i,a) \in S_n \mid 1 \leq i \leq n, \ 2n - i + 1 \leq a \leq 2n + i - 1 \}.
\end{align*}
\]

(iii) follows from (i). \( \square \)

**Example 7.10.** We use the same notations given in Example 7.7 and Lemma 7.9.

(i) Let \( i = 1 \). In this case, \( S_{1,0}^- = \{ (1,0), (2,1), (3,2) \} \), \( S_{1,0}^+ = \{ (1,2), (2,3), (3,4) \} \) and
\[
S_{1,0} = \{ (3,4) \succ (3,2) \succ (2,3) \succ (2,1) \succ (1,2) \succ (1,0) \} \subset S_{0,n} \cup S_{1,n}.
\]

Note that \( u = (1,0) \) and \( D(u) = (3,4) \). Pictorially we write the segments \( \gamma^{-1}(i,a) \) \(((i,a) \in S_{1,0})\) at the position \((i,a)\) as follows:

| \( i \) | \( a \) | \(-5\) | \(-4\) | \(-3\) | \(-2\) | \(-1\) | \(0\) | \(1\) | \(2\) | \(3\) | \(4\) | \(5\) | \(6\) | \(7\) | \(8\) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | | | | | \( [1] \) | \( [2] \) | | | | | | | |
| 2 | | | | | \( [1,2] \) | \( [2,3] \) | | | | | | | |
| 3 | | | | | | \( [1,3] \) | \( [1] \) | | | | | | | |

Since \( 1A \cup 2A = \{ [1,3], [2,3], [1,2], [2], [1] \} \) and \( 0A \cup 1A = \{ [1] \} \), we have
\[
S_{1,0} = \{ (3,4), (3,2), (2,3), (2,1), (1,2), (1,0) \} = \{ \gamma_1([1]), \gamma_0([1,3]), \gamma_0([2,3]), \gamma_0([1,2]), \gamma_0([2]), \gamma_0([1]) \}.
\]
(ii) Let \( i = 2 \). In this case, \( S^-_{2,0} = \{ (1, 2), (2, 3), (3, 4) \} \), \( S^+_{2,0} = \{ (1, 4), (2, 5), (3, 6) \} \) and
\[
S_{2,0} = \{ (3, 6) \succ (3, 4) \succ (2, 5) \succ (2, 3) \succ (1, 4) \succ (1, 2) \} \subset \mathcal{S}_n^0 \cup \mathcal{S}_n^1.
\]
Note that \( u = (1, 2) \) and \( D(u) = (3, 6) \). Pictorially we write the segments \( \gamma^{-1}(i, a) \) \((i, a) \in S_{2,0}\) at the position \((i, a)\) as follows:

\[
\begin{array}{cccccccccc}
  i & a \\
  1 & -5 & -4 & -3 & -2 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
  1 & \cdot & \cdot & \cdot & [2] & [3] & \cdot \\
  2 & \cdot & \cdot & \cdot & [2, 3] & [1, 2] & \cdot \\
  3 & \cdot & \cdot & \cdot & \cdot & [1] & [2] & \cdot \\
\end{array}
\]

Since \( 2A \cup 3A = \{ [2, 3] \succ [3] \succ [2] \} \) and \( A_1 \cup A_2 = \{ [2] \succ' [1] \succ' [1, 2] \} \), we have
\[
S_{2,0} = \{ (3, 6), (3, 4), (2, 5), (2, 3), (1, 4), (1, 2) \}
= \{ \gamma_1([2]), \gamma_1([1]), \gamma_1([1, 2]), \gamma_0([2, 3]), \gamma_0([3]), \gamma_0([2]) \}.
\]

Recall that, for a multisegment \( m \), \( S^<_i(m) \) and \( S^{<'}_i(m) \) denote the left and right \( i \)-signature sequences of \( m \) with respect to \( < \) and \( <' \) respectively, as described in Section 7.1.

**Lemma 7.11.** Let \( \lambda \in \mathcal{P}_n \) and write \( \gamma^{-1}(\lambda) = (m_k)_{k \in \mathbb{Z}} \). For \( i \in I_0 \), the concatenation \( S^{<'}_i(m_1) * S^<_i(m_0) \) is equal to the \((i, 0)\)-signature sequence of \( \lambda \).

**Proof.** Let \( S_{i,0} = \{ v_{2n}, v_{2n-1}, \ldots, v_1 \} \) be the set defined in (Step 1) in Section 7.3. By Lemma 7.9 (ii), the signature of \( v_k \) is equal to the \( i \)-signature of \( \gamma^{-1}(v_k) \) for any \( k \). Thus the assertion follows from Lemma 7.9 (iii). \( \square \)

We are now ready to prove Theorem 7.5.

**Proof of Theorem 7.5.** Thanks to (7.19), (7.20) and Lemma 7.8, it suffices to show that the map \( \gamma \) commutes with the crystal operators \( \tilde{F}_{i,k} \).

Let \((i, k) \in \hat{I}_0, \lambda \in \mathcal{P}_n \) and set \( b := \gamma^{-1}(\lambda) \). We shall prove
\[
\gamma(\tilde{F}_{i,k}(b)) = \tilde{F}_{i,k}(\lambda).
\]
Thanks to (7.18), Lemma 4.3 and Lemma 7.4 (iv), we may assume that $k = 0$.

We write $b = (m_k)_{k \in \mathbb{Z}}$ for $m_k \in MS_n$. Let $S_{i,0} = \{v_{2n}, v_{2n-1}, \ldots, v_1\}$ be the set defined in (Step 1) in Section 7.3. Let $v_t$ be the element of $S_{i,0}$ corresponding to the leftmost $+$ in the reduced $(i,0)$-signature sequence of $\lambda$. If there is no such an $v_t$, then we set $v_t := 0$.

On the other hand, we set

$$S' := S_i^{<'}(m_1) \quad \text{and} \quad S := S_i^{<}(m_0)$$

and let $\overline{S}$ (resp. $\overline{S}'$) be the sequence obtained from $S$ (resp. $S'$) by canceling out all $(+, -)$ pairs. Let $\overline{S}' \ast \overline{S}$ be the sequence obtained from the concatenation $S' \ast S$ by canceling out all $(+, -)$ pairs. By the crystal rule for $MS_n$ described in Section 7.1, $\varepsilon_{i,0}(b)$ (resp. $\varepsilon_{i,1}^{*}(b)$) is equal to the number of $-$’s (resp. $+$’s) in $\overline{S}$ (resp. $\overline{S}'$). Pictorially, the concatenation of $\overline{S}' \ast \overline{S}$ can be written as follows:

\begin{equation}
(7.21)
\begin{array}{c}
\overline{S}' \ast \overline{S} \\
\varepsilon_{i,1}^{*}(b) \\
\varepsilon_{i,0}(b)
\end{array}
\end{equation}

(Case 1) Suppose that $\varepsilon_{i,1}^{*}(b) > \varepsilon_{i,0}(b)$. Let $[s, i]$ be the segment placed at the leftmost $+$ in the sequence $\overline{S}'$. By the definition of the extended crystal operator $\tilde{F}_{i,0}$, we have

$$\tilde{F}_{i,0}(b) = (m'_k)_{k \in \mathbb{Z}},$$

where $m'_k = m_k$ for $k \neq 1$ and

\begin{equation}
(7.22)
\begin{array}{c}
m'_1 = \tilde{\varepsilon}^{*}_i(m_1) = m_1 - [s, i] + [s, i - 1].
\end{array}
\end{equation}

Since $\varepsilon_{i,1}^{*}(b) > \varepsilon_{i,0}(b)$, considering the configuration (7.21), $[s, i]$ is equal to the segment placed at the the leftmost $+$ in $\overline{S}' \ast \overline{S}$. Thanks to Lemma 7.9 (iii) and Lemma 7.11, $\gamma_1([s, i])$ is equal to the element $v_t$ of $S_{i,0}$ corresponding to the leftmost $+$ in the reduced
$(i, 0)$-signature sequence of $\lambda$. Thus, by Lemma 7.9 and (7.22), we have
\[ \tilde{F}_{i,k}(\lambda) = \lambda - v_t + v_{t+1} \]
\[ = \gamma(b) - \gamma_1([s, i]) + \gamma_1([s, i - 1]) \]
\[ = \gamma(\ldots, m_2, m_1 - [s, i] + [s, i - 1], m_0, \ldots) \]
\[ = \gamma(\ldots, m_2, \tilde{e}_i^*(m_1), m_0, \ldots) \]
\[ = \gamma(\tilde{F}_{i,k}(b)). \]

(Case 2) Suppose that $\varepsilon_{i,1}^*(b) \leq \varepsilon_{i,0}(b)$. Let $\theta := [i + 1, s]$ be the segment placed at the leftmost $+$ in the sequence $S$ if it exists. Otherwise, we set $\theta := \emptyset$. By the definition of the extended crystal operator $\tilde{F}_{i,0}$, we have
\[ \tilde{F}_{i,0}(b) = (m'_k)_{k \in \mathbb{Z}}, \]
where $m'_k = m_k$ for $k \neq 0$ and
\[ m'_0 = \tilde{f}_i(m_0) = \begin{cases} m_0 - \theta + [i, s] & \text{if } \theta \neq \emptyset, \\ m_0 + [i] & \text{if } \theta = \emptyset. \end{cases} \]

(7.23)

Since $\varepsilon_{i,1}^*(b) \leq \varepsilon_{i,0}(b)$, when $\theta \neq \emptyset$, $\theta$ is equal to the segment placed at the the leftmost $+$ in $S^\star S$ as seen by the figure (7.21). By Lemma 7.9 (iii) and Lemma 7.11,
\[ \theta = \emptyset \iff v_t = 0, \]
and if $\theta \neq \emptyset$, then $\gamma_0([i + 1, s])$ is equal to the element $v_t$ of $S_{i,0}$ corresponding to the leftmost $+$ in the reduced $(i, 0)$-signature sequence of $\lambda$. Thus, Lemma 7.9 and (7.23) imply the following: if $\theta \neq \emptyset$, then
\[ \tilde{F}_{i,k}(\lambda) = \lambda - v_t + v_{t+1} \]
\[ = \gamma(b) - \gamma_0([i + 1, s]) + \gamma_0([i, s]) \]
\[ = \gamma(\ldots, m_1, m_0 - [i + 1, s] + [i, s], m_1, \ldots) \]
\[ = \gamma(\ldots, m_1, \tilde{f}_i(m_0), m_1, \ldots) \]
\[ = \gamma(\tilde{F}_{i,k}(b)). \]
and if $\theta = \emptyset$, then

$$\tilde{F}_{i,k}(\lambda) = \lambda + v_1$$

$$= \gamma(b) + \gamma_0([i])$$

$$= \gamma(\ldots, m_1, m_0 + [i], m_{-1}, \ldots)$$

$$= \gamma(\ldots, m_1, \tilde{f}_i(m_0), m_{-1}, \ldots)$$

$$= \gamma(\tilde{F}_{i,k}(b)).$$

□

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