A Relation between $\Gamma$-Convergence of Functionals
and their Associated Gradient Flows*

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Abstract E. De Giorgi conjectured in 1979 that if a sequence of functionals converges in the sense of $\Gamma$-convergence to a limiting functionals, then the corresponding gradient flows will converge as well after changing timescale appropriately. In this paper, we will show this conjecture holds true for a rather wide kind of functionals.

Keywords $\Gamma$-convergence parabolic equations parabolic minima asymptotic behaviour

In 1979, E. De Giorgi [1] asked if there was a general relation between $\Gamma$-convergence of functionals and convergence of solutions to the associated parabolic equations. He further conjectured in the same paper that when a sequence of functionals converges in the sense of $\Gamma$-convergence to a limiting functional, then the corresponding gradient flows will converge as well (maybe after an appropriate change of timescale). Also see [2; P.216] and [3; P.507]. Although there is no any result, up to the author’s knowledge, confirming this conjecture, it was supported by the results of Bronsard and Kohn in [2], and Owen, Rubinstein and Sternberg in [3], respectively, where they studied the singular limit of Ginzburg-Landau dynamics (up to a $\varepsilon$-scaling time):

\[ u_t - \varepsilon^2 \Delta u + u^3 - u = 0 \quad (0.1) \]

which are the gradient flows of the following functionals:

\[ f_\varepsilon(u) = \int_\Omega \left( \frac{1}{4}(u^2 - 1)^2 + \frac{\varepsilon^2}{2} |\nabla u|^2 \right) dx, \quad \Omega \subset \mathbb{R}^n. \quad (0.2) \]

The $\Gamma$-limit of functionals (0.2) as $\varepsilon \to 0^+$ was derived by Modica in [4]. Combination of the results in [2, 3] with ones in [4] suggests that De Giorgi’s conjecture should be answered positively, at least for some special functionals.

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In this paper, we will confirm De Giorgi’s conjecture for a rather wide kind of functionals. Precisely speaking, we will establish a relation between Γ-convergence of functionals and the convergence of their parabolic-minima. The Γ-convergence may be proved by similar arguments in [5, 6]. Furthermore, we discover that parabolic-minima of rather many functionals are nothing but the solutions to the gradient flows of the corresponding functionals.

1 Main results

We begin with the following assumptions and notations: Ω denotes a bounded open set in \( R^n \), \( p > 1 \), \( T > 0 \) and \( m \) is a positive integer. Let \( \Omega_T = \Omega \times (0, T) \),

\[
V_p(\Omega_T, m) = L^p([0, T], W^{1,p}(\Omega, R^m)), \quad V^0_p(\Omega_T, m) = L^p([0, T], W^{1,0}_0(\Omega, R^m)), \tag{1.1}
\]

and

\[
Du(x, t) = \nabla u(x, t) = \left( \frac{\partial u^i(x, t)}{\partial x_j} \right) \quad (1 \leq i \leq m, 1 \leq j \leq n) \tag{1.2}
\]

for a vector valued function \( u \).

Suppose \( \Phi: R^m \to R^m \), and \( f(x, t, u, \lambda): \Omega_T \times R^m \times R^{mn} \to R \) such that for each \( v \in V_p(\Omega_T, m) \)

\[
\Phi(v) \in L^1(\Omega_T, R^m) \quad \text{and} \quad f(x, t, v, Dv) \in L^1(\Omega_T). \tag{1.3}
\]

Consider the parabolic functional

\[
F(v, \Omega_T) = \int_{\Omega_T} f(x, t, v, Dv) dx \quad v \in V_p(\Omega_T, m). \tag{1.4}
\]

Following the idea of the papers [7], we introduce the definition of parabolic-minima.

**Definition 1.1.** Assume that \( \Phi \) and \( f \) satisfy (1.3), \( u_0(x) \in L^1(\Omega, R^m) \). A function \( u \in V_p(\Omega_T, m) \) is called a parabolic-minimum of \( F \) (defined by (1.4)) with respect to the function couple \( (\Phi, u_0) \) if for all \( \eta \in C^\infty([0, T], C^\infty_0(\Omega, R^m)) \) with \( \eta(\cdot, T) = 0 \),

\[
- \int_{\Omega_T} \Phi(u) \frac{\partial \eta}{\partial t} dx dt + F(u, \Omega_T) \leq F(u - \eta, \Omega_T) + \int_{\Omega} u_0 \eta(x, 0) dx. \tag{1.5}
\]
**Definition 1.2.** \( \tau \) is called as the sw-topology of \( V_p(\Omega_T, m) \), if \( v^\varepsilon \) converges to \( v \) in \( V_p(\Omega_T, m) \) with the topology \( \tau \) if and only if \( v^\varepsilon \rightarrow v \) strongly in \( L^p(\Omega_T, R^m) \) and \( Dv^\varepsilon \rightharpoonup Dv \) weakly in \( L^p(\Omega_T, R^{mn}) \). We denote this convergence by \( v^\varepsilon \overset{\tau}{\rightharpoonup} v \).

Now consider a sequence of functionals defined in \( V_p(\Omega_T, m) \) by

\[
F^\varepsilon(v, \Omega_T) = \int_{\Omega_T} f^\varepsilon(x, t, v, Dv) dx dt, \quad (\varepsilon \rightarrow 0), \tag{1.6}
\]

where each \( f^\varepsilon: \Omega_T \times R^m \times R^{mn} \rightarrow R \) is a Caratheodory function satisfying

\[
0 \leq f^\varepsilon(x, t, u, \lambda) \leq C(1 + |u|^p + |\lambda|^p) \tag{1.7}
\]

and

\[
|f^\varepsilon(x, t, u_1, \lambda_1) - f^\varepsilon(x, t, u_2, \lambda_2)| \leq C(|u_1 - u_2|^\alpha + |\lambda_1 - \lambda_2|^\alpha)(1 + |u_1|^{p-\alpha} + |\lambda_1|^{p-\alpha} + |u_2|^{p-\alpha} + |\lambda_2|^{p-\alpha}) \tag{1.8}
\]

for some constants \( C > 0 \) and \( \alpha \in (0, 1) \).

The main result of this paper is the following theorem where we refer to [6, 8] for the definition of \( \Gamma \)-convergence.

**Theorem 1.1.** Suppose that the hypotheses (1.6), (1.7) and (1.8) hold true and that \( F^\varepsilon \) \( \Gamma \)-converges to \( F \) with sw-topology, i.e.,

\[
\Gamma(\tau) \lim_{\varepsilon \rightarrow 0} F^\varepsilon(v, \Omega_T) = F(v, \Omega_T), \forall v \in V_p(\Omega_T, m), \tag{1.9}
\]

where \( \tau \) is the sw-topology of \( V_p(\Omega_T, m) \). If for each \( \varepsilon > 0 \), \( u^\varepsilon \in V^0_p(\Omega_T, m) \) is a parabolic-minimum of \( F^\varepsilon \) with respect to \((\Phi, u^\varepsilon_0)\) such that as \( \varepsilon \rightarrow 0 \),

\[
u^\varepsilon_0 \rightarrow u_0 \quad \text{and} \quad \Phi(u^\varepsilon_0) \rightharpoonup \Phi(u_0) \quad \text{weakly in} \quad L^1(\Omega_T, R^m) \tag{1.10}
\]

and

\[
u^\varepsilon \overset{\tau}{\rightharpoonup} u \in V^0_p(\Omega_T), \quad \partial_t \Phi(u^\varepsilon) \rightharpoonup \partial_t \Phi(u) \quad \text{weakly in} \quad L^q(\Omega_T, R^m) \tag{1.11}
\]

with \( q = \frac{p}{p-1} \), then \( u \) is a parabolic-minimum of \( F(u) \) with respect to \((\Phi, u_0)\).
We will prove this theorem in section 3, while in next section, we will study the equivalence of parabolic-minima with some parabolic systems and discuss the justification for assumptions (1.10) and (1.11). We would like to point out that assumption (1.9) may checked by argument similar to those in [5, 6, 8]. In the case of \( f^\varepsilon(x, t, u, \cdot) = f(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}, \frac{u}{\varepsilon}, \cdot) \) with \( f \) being periodic in the first three variables, it was proved in [9].

2 Parabolic-minima and parabolic equations

Lemma 2.1. Suppose \( f \in C^1 \) in \( v \) and \( \lambda \) such that

\[
|f(x, t, v, \lambda)| + |f_v(x, t, v, \lambda)| + |f_\lambda(x, t, v, \lambda)| \leq C(1 + |v|^p + |\lambda|^p).
\]

(2.1)

If \( u \in V^0_p(\Omega_T) \) is a parabolic minimum of the functional \( F(v, \Omega_T) \) given by (1.3)-(1.4) with respect to \( (\Phi, u_0) \), then \( u \) is a weak solution to its gradient flow which is the following initial-boundary valued problem:

\[
\begin{cases}
\Phi(u) \in L^1(\Omega_T, R^m), \quad \text{and for } i = 1, 2, \cdots, m \\
\frac{\partial \Phi(u)}{\partial t} - \frac{\partial f^{\lambda_i}}{\partial x_\alpha}(x, t, u, Du) + f_u(x, t, u, Du) = 0 \quad \text{in } \Omega_T, \\
u = 0 \quad \text{on } \Sigma = \partial \Omega \times (0, T), \quad \Phi(u(x, 0)) = u_0(x)
\end{cases}
\]

(2.2)

Proof. For each \( \eta \in C^\infty([0, T], C^\infty_0(\Omega, R^m)) \) with \( \eta(\cdot, T) = 0 \) and any \( h \in R \setminus \{0\} \), replace \( \eta \) by \( h\eta \) in (1.5), use the mean value formula and then divide by \( h \). Letting \( h \rightarrow 0^+ \) and \( h \rightarrow 0^- \) respectively, we obtain the desired result.

Remark 2.1. If \( F(v, \Omega_T) \) has a parabolic minimum and (2.2) has a unique solution, then lemma 2.1 implies that the solution must be the parabolic-minimum. But we don’t know the existence for the parabolic-minimum. Nevertheless, the following examlples show that parabolic-minima of some functionals are nothing but weak solutions to their corresponding gradient flow equations.

Example 2.1. For each parameter \( \varepsilon \), let \( [a_{ij}^\varepsilon(x, t)] \) be a symmetric, measurable and uniformly bounded and positive-definite matrix function in \( \Omega_T \). Assume that \( \Phi = (\phi^1, \cdots, \phi^m) \) is a map from \( R^m \) to itself with monotonic components. Consider the asymptotic behaviour of weak solutions to the equations of general
Newtonian filtration:

\[
\begin{aligned}
\frac{\partial \phi^k(u)}{\partial t} &= \frac{\partial}{\partial x_i} \left( a_{ij}^\varepsilon(x, t) \frac{\partial u^k}{\partial x_j} \right) + h^k_\varepsilon(x, t) \quad \text{in} \quad \Omega_T, \; k = 1, 2, \ldots, m \\
u &= 0 \quad \text{on} \quad \partial \Omega \times (0, T) \\
u(x, 0) &= u_0(x).
\end{aligned}
\] (2.3)

It is well-known that under some additional assumptions, for instance, \( h^\varepsilon = (h_1^\varepsilon, \ldots, h_m^\varepsilon) \in L^2(\Omega_T, \mathbb{R}^m) \) and all \( \phi^k \) have bounded derivatives, the solutions \( u^\varepsilon \) to (2.3) belong to \( V_2(\Omega_T, m) \) and

\[
\| \frac{\partial t}{\partial t} \Phi(u^\varepsilon) \|_{L^2(\Omega_T)} + \| u^\varepsilon \|_{V_2(\Omega_T)} \leq C
\] (2.4)

for some constant \( C \) independent of \( \varepsilon \). See [10, 11]. Using this result, lemma B below and the compact imbedding theorem, we have a sequence of \( u^\varepsilon \) such that it satisfies (1.10) and (1.11) for \( p = 2 \). Moreover, we have the following conclusion.

**Lemma 2.2.** For each \( \varepsilon \), the solution \( u^\varepsilon \) to (2.3) is a parabolic-minimum of the functional

\[
F_1^\varepsilon(v) = \int_{\Omega_T} \frac{1}{2} a_{ij}^\varepsilon(x, t) \frac{\partial v^k}{\partial x_i} \frac{\partial v^k}{\partial x_j} - h^k_\varepsilon v^k \, dx \, dt \quad v \in V_2(\Omega_T, m)
\] (2.5)

with respect to \( (\Phi, u_0) \).

**Proof.** For simplicity, we denote \( u^\varepsilon \) by \( u \). Since \( u \) is a weak solution to (2.3), we see that for each \( \eta \in C^\infty([0, T], C_0^\infty(\Omega, \mathbb{R}^m)) \) with \( \eta(\cdot, T) = 0 \),

\[
-\int_{\Omega_T} \Phi(u) \frac{\partial \eta}{\partial t} \, dx \, dt + F_1^\varepsilon(u) = \frac{1}{2} \int_{\Omega_T} a_{ij}^\varepsilon \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \, dx \, dt - \int_{\Omega_T} a_{ij}^\varepsilon \frac{\partial u}{\partial x_i} \frac{\partial \eta}{\partial x_j} \, dx \, dt
- \int_{\Omega_T} h_\varepsilon(u - \eta) \, dx + \int_{\Omega} u_0 \eta(x, 0) \, dx.
\] (2.6)

As \( [a_{ij}^\varepsilon(x, t)] \) is positive-definite, the sum of first two terms on the right hand in (2.6) is no larger than

\[
\frac{1}{2} \int_{\Omega_T} a_{ij}^\varepsilon \frac{\partial (u - \eta)}{\partial x_i} \frac{\partial (u - \eta)}{\partial x_j} \, dx \, dt.
\]

Thus, (2.6) turns to

\[
-\int_{\Omega_T} \Phi(u) \frac{\partial \eta}{\partial t} \, dx \, dt + F_1^\varepsilon(u) \leq F_1^\varepsilon(u - \eta) + \int_{\Omega} u_0 \eta(x, 0) \, dx,
\]
as desired.

**Example 2.2.** Suppose that \( a^\varepsilon(x,t) \) is a family of measurable functions which are positive and bounded uniformly in parameter \( \varepsilon \). Let \( p > 1 \) and \( \Phi \) be the same as in example 2.1. If for each \( \varepsilon \), \( u^\varepsilon \) is the weak solution to the following P-Laplace equation:

\[
\begin{aligned}
\partial_t \phi^k - \partial_{x_a}(a^\varepsilon(x,t)|\nabla u|^{p-2}\partial_{x_a} u) &= 0 \quad \text{in} \quad \Omega_T, \; k = 1, 2, \ldots, m \\
u &= 0 \quad \text{on} \quad \partial \Omega \times (0,T), \quad u(x,0) = u_0(x).
\end{aligned}
\]  

(2.7)

then \( u^\varepsilon \) is a parabolic minimum of

\[
F_2^\varepsilon(v) = \frac{1}{p} \int_{\Omega_T} a^\varepsilon(x,t)|\nabla v|^p dxdt.
\]

The proof is similar to the arguments used in proving lemma 2.2. We omit the details.

**Example 2.3.** Let \( u_\varepsilon \) be the solutions to the Cauchy problem of equation (0.1) with initial data \( u_0^\varepsilon \) satisfying \( u_0^\varepsilon \geq 1 \). Obviously, \( |u_\varepsilon(x,t)| \geq 1 \), for all \((x,t) \in \mathbb{R}^n \times (0,\infty)\), by the maximum principle.

Since for each \( \eta \in C^\infty([0,T],C_0^\infty(\Omega,\mathbb{R}^m)) \) with \( \eta(\cdot,T) = 0 \), we obtain that

\[
-\int_0^T \int_{\mathbb{R}^n} u_\varepsilon \frac{\partial \eta}{\partial t} dxdt + \frac{1}{2} \int_0^T \int_{\mathbb{R}^n} \left[ \varepsilon^2|\nabla u_\varepsilon| + (u_\varepsilon^2 - 1)^2 \right] dxdt - \int_{\mathbb{R}^n} u_0^\varepsilon \eta(x,0) dx
\]

\[
= \frac{\varepsilon^2}{2} \int_0^T \int_{\mathbb{R}^n} \left[ |\nabla(u_\varepsilon - \eta)|^2 - |\nabla \eta|^2 \right] dxdt
\]

\[
+ \frac{1}{2} \int_0^T \int_{\mathbb{R}^n} \left[ (u_\varepsilon^2 - 1)((u_\varepsilon - \eta)^2 - 1) - \eta^2(u_\varepsilon^2 - 1) \right] dxdt
\]

\[
\leq \frac{\varepsilon^2}{2} \int_0^T \int_{\mathbb{R}^n} |\nabla(u_\varepsilon - \eta)|^2 dxdt
\]

\[
+ \frac{1}{4} \int_0^T \int_{\mathbb{R}^n} (u_\varepsilon^2 - 1)^2 dxdt + \frac{1}{4} \int_0^T \int_{\mathbb{R}^n} [(u_\varepsilon - \eta)^2 - 1]^2 dxdt,
\]

where we have used the fact that \( u_\varepsilon^2 - 1 \geq 0 \) and Young’s inequality. This immediately implies that \( u_\varepsilon \) is a parabolic-minimum of functionals of type (0.2), i.e., \( u_\varepsilon \) is a parabolic-minimum of

\[
F_3^\varepsilon(u) = \int_0^T \int_{\mathbb{R}^n} \left( \frac{1}{4}(u^2 - 1)^2 + \frac{\varepsilon^2}{2}|\nabla u|^2 \right) dx
\]

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with respect to \((I, u_0^\varepsilon)\), where \(I\) is the identity map.

**Remark 2.2.** For the initial-boundary value problem of (0.1), we also have a similar conclusion.

### 3 A proof of theorem 1.1

To prove theorem 1.1, we need two well-known results.

**Lemma A.** Suppose that \(f: \mathbb{R} \times \mathbb{R} \to \mathbb{R}\), then there exists a function \(\delta: \varepsilon \to \delta(\varepsilon)\) such that \(\varepsilon \to 0\) implies \(\delta(\varepsilon) \to 0\) and

\[
\lim_{\varepsilon \to 0} f(\delta(\varepsilon), \varepsilon) = \lim_{\delta \to 0} \lim_{\varepsilon \to 0} f(\delta, \varepsilon)
\]

**Proof.** See [5; P.32-33].

**Lemma B.** If \(p > 1\), then

1. \(\{u_h\}\) converges to \(u\) with respect to \(\text{sw}\)-topology of \(V_p(\Omega_T, m)\) if and only if
   \[u_h \to u \text{ strongly in } L^p(\Omega_T, \mathbb{R}^m) \text{ and } \sup_h \|\nabla u_h\|_{L^p(\Omega_T, \mathbb{R}^{mn})} < \infty;\]
2. the fact that \(\{u_h\}\) converges to \(u\) with respect to \(\text{sw}\)-topology of \(V_p(\Omega_T, m)\) implies that \(\sup_h \|u_h\|_{V_p(\Omega_T, m)} < \infty\).

**Proof.** See section 2 of chapter 1 in [12].

Now we are in the position to prove theorem 1.1. Let \(\tau\) be the \(\text{sw}\)-topology. For simplicity, denote \(V_p(\Omega_T, m)\) by \(V_p(\Omega_T)\).

Fix \(\eta \in C^\infty([0, T], C_0^\infty(\Omega, \mathbb{R}^m))\) arbitrarily. According to assumption (1.9) and the definition of \(\Gamma\)-convergence[8], we can choose \(\{w^\varepsilon\} \subset V_p(\Omega_T)\) satisfying

\[
w^\varepsilon \xrightarrow{\tau} u - \eta
\]  

such that

\[
F(u - \eta, \Omega_T) = \lim_{\varepsilon \to 0} F^\varepsilon(w^\varepsilon, \Omega_T).
\]
Since \( u - \eta \in V^0_p(\Omega_T) \), we can assert that there exist a sequence \( \{v^\varepsilon\} \subset V^0_p(\Omega_T) \) such that
\[
v^\varepsilon \xrightarrow{\tau} u - \eta \tag{3.3}
\]
and
\[
\lim_{\varepsilon \to 0} F^\varepsilon(v^\varepsilon, \Omega_T) = F(u - \eta, \Omega_T). \tag{3.4}
\]
Indeed, for each \( Q_0 \subset \subset \Omega_T \), let \( R = 2^{-1} \text{dist}(Q_0, \partial \Omega_T) \). For any \( \delta \in (0, 1) \), define
\[
Q_i = \{ x \in \Omega_T; \text{dist}(x, Q_0) < i\delta R \}, i = 1, 2, \ldots [\delta^{-1}].
\]
Choose \( \phi_i \in C_0^\infty(\Omega_T) \) such that \( 0 \leq \phi_i \leq 1, \phi_i = 1 \text{ in } Q_{i-1}, \phi_i = 0 \text{ in } \Omega_T \setminus Q_i \) and \( |D\phi_i| \leq C(\delta, n, R) \). let
\[
u^{\varepsilon, i} = u - \eta + \phi_i(w^\varepsilon - u + \eta).
\]
By (1.7) we have
\[
F^\varepsilon(u^{\varepsilon, i}, \Omega_T) = F^\varepsilon(u^{\varepsilon, i}, Q_{i-1}) + F^\varepsilon(u^{\varepsilon, i}, \Omega_T \setminus Q_i) + F^\varepsilon(u - \eta, \Omega_T \setminus Q_i)
\leq F^\varepsilon(w^\varepsilon, \Omega_T) + C\left[\|u - \eta\|_{V^p_p(\Omega_T \setminus Q_0)}^p + \|w^\varepsilon\|_{V_p^p(Q_i \setminus Q_{i-1})}^p + C(\delta, R)\|w^\varepsilon - u + \eta\|_{L_p^p(\Omega_T \setminus Q_0)} + |\Omega_T \setminus Q_0|\right]. \tag{3.5}
\]
Obviously, we can find \( i(\delta) \in \{1, 2, \ldots, [\delta^{-1}]\} \) such that
\[
F^\varepsilon(u^{\varepsilon, i(\delta)}, \Omega_T) = \min\{F^\varepsilon(u^{\varepsilon, i}, \Omega_T); i = 1, 2, \ldots [\delta^{-1}]\}.
\]
Furthermore, (3.1) implies
\[
u^{\varepsilon, i(\delta)} \xrightarrow{\tau} u - \eta \text{ for each } \delta. \tag{3.6}
\]
Summing (3.5) for \( i \) from 1 to \([\delta^{-1}]\), we arrive at the estimate
\[
F^\varepsilon(u^{\varepsilon, i(\delta)}, \Omega_T) \leq F^\varepsilon(w^\varepsilon, \Omega_T) + C\left[\|u - \eta\|_{V^p_p(\Omega_T \setminus Q_0)}^p + [\delta^{-1}]^{-1}\|w^\varepsilon\|_{V^p_p(\Omega_T)}^p + C(\delta, R)\|w^\varepsilon - u + \eta\|_{L_p^p(\Omega_T \setminus Q_0)} + |\Omega_T \setminus Q_0|\right].
\]
Applying this estimate, (3.6), (1.9), the definition of \( \Gamma \)-convergence[8], (3.2) and lemma B, we obtain that
\[
F(u - \eta, \Omega_T) \leq \liminf_{\delta \to 0} \liminf_{\varepsilon \to 0} F^\varepsilon(u^{\varepsilon, i(\delta)}, \Omega_T)
\leq \limsup_{\delta \to 0} \limsup_{\varepsilon \to 0} F^\varepsilon(u^{\varepsilon, i(\delta)}, \Omega_T)
\leq F(u - \eta, \Omega_T) + C\left[\|u - \eta\|_{V^p_p(T \setminus Q_0)}^p + |\Omega_T \setminus Q_0|\right].
\]
Letting $Q_0 \to \Omega_T$, we have
\[
\lim_{\delta \to 0} \lim_{\varepsilon \to 0} F^\varepsilon(u^{\varepsilon,i(\delta)}, \Omega_T) = F(u - \eta, \Omega_T).
\] (3.7)

Now define
\[
f(\delta, \varepsilon) = \left| F^\varepsilon(u^{\varepsilon,i(\delta)}, \Omega_T) - F(u - \eta, \Omega_T) \right|
+ \|u^{\varepsilon,i(\delta)} - u + \eta\|_{L^p(\Omega_T)} + \|D\phi^{i(\delta)}(u^\varepsilon - u + \eta)\|_{L^p(\Omega_T)}.
\]

Then (3.6), (3.7) and lemma B implies that
\[
\lim_{\delta \to 0} \lim_{\varepsilon \to 0} f(\delta, \varepsilon) = 0.
\]

By virtue of lemma A, we conclude that there exists a function $\delta(\varepsilon)$ such that
\[
\lim_{\varepsilon \to 0} f(\delta(\varepsilon), \varepsilon) = 0.
\]

Therefore, setting $i(\varepsilon) = i(\delta(\varepsilon))$ and $v^\varepsilon = u^{\varepsilon,i(\varepsilon)}$, we see that $\{v^\varepsilon\} \subset V^0_p(\Omega_T)$ satisfy (3.4). Moreover, we have that as $\varepsilon \to 0$,
\[
\|v^\varepsilon - u + \eta\|_{L^p(\Omega_T)} \to 0 \quad \text{and} \quad \|D\phi^{i(\varepsilon)}(u^\varepsilon - u + \eta)\|_{L^p(\Omega_T)} \to 0,
\]
which, together with lemma B, implies that $\{v^\varepsilon\}$ satisfy (3.3).

Now take $\{u^\varepsilon\}$ as in theorem 1.1. Then for each $\varepsilon$, $u^\varepsilon - v^\varepsilon \in V^0_p(\Omega_T)$. Thus we can choose $\eta^\varepsilon \in C^\infty([0,T], C^\infty_0(\Omega, \mathbb{R}^m))$ such that
\[
\lim_{\varepsilon \to 0} \|\eta^\varepsilon - (u^\varepsilon - v^\varepsilon)\|_{V^p_p(\Omega_T)} = 0.
\]

Furthermore, we may assume
\[
\eta^\varepsilon(x, 0) = \eta(x, 0), \quad \eta^\varepsilon(x, T) = 0.
\] (3.8)

By (1.8), we get that
\[
|F^\varepsilon(v^\varepsilon, \Omega_T) - F^\varepsilon(u^\varepsilon - \eta^\varepsilon, \Omega_T)| \leq C(m, n, p) \times
\|u^\varepsilon - v^\varepsilon - \eta^\varepsilon\|_{V^p_p(\Omega_T)}^p \left(1 + \|\eta^\varepsilon\|_{V^p_p(\Omega_T)} + \|u^\varepsilon\|_{V^p_p(\Omega_T)} + \|v^\varepsilon\|_{V^p_p(\Omega_T)}\right)^{p-\alpha}.
\]
Observing that the sequence \( \{ u^\varepsilon \} \) and \( \{ v^\varepsilon \} \) are bounded (see lemma B), we obtain

\[
\lim_{\varepsilon \to 0} F^\varepsilon(v^\varepsilon, \Omega_T) = \lim_{\varepsilon \to 0} F^\varepsilon(u^\varepsilon - \eta^\varepsilon, \Omega_T). \tag{3.9}
\]

Since for each \( \varepsilon \), \( u^\varepsilon \) is a parabolic-minimum of \( F^\varepsilon \), it follows from (3.4), (3.9) and (1.10) that

\[
F(u - \eta, \Omega_T) \geq \liminf_{\varepsilon \to 0} \left( F^\varepsilon(u^\varepsilon, \Omega_T) - \int_{\Omega_T} \Phi(u^\varepsilon) \frac{\partial \eta^\varepsilon}{\partial t} \, dx \, dt \right) - \int_{\Omega} u_0(x) \eta(x, 0) \, dx. \tag{3.10}
\]

By (1.11) and (3.3), \( u^\varepsilon - v^\varepsilon \rightharpoonup \eta \), so \( \eta^\varepsilon \rightharpoonup \eta \). Therefore, again by assumption (1.11), we have

\[
\lim_{\varepsilon \to 0} \int_{\Omega_T} \Phi(u^\varepsilon) \frac{\partial \eta^\varepsilon}{\partial t} \, dx \, dt = \int_{\Omega_T} \Phi(u) \frac{\partial \eta}{\partial t} \, dx \, dt.
\]

Thus (3.10) yields

\[
F(u - \eta, \Omega_T) \geq \liminf_{\varepsilon \to 0} F^\varepsilon(u^\varepsilon, \Omega_T) - \int_{\Omega_T} \Phi(u) \frac{\partial \eta}{\partial t} \, dx \, dt - \int_{\Omega} u_0(x) \eta(x, 0) \, dx.
\]

Noting \( u^\varepsilon \rightharpoonup u \) (by (1.11)) and using (1.9), definitions 1.1 and the definition of \( \Gamma \)-convergence[8], we have completed the proof of theorem 1.1.

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