A NON-COMMUTATIVE DISCRETE HYPERGROUP
ASSOCIATED WITH $q$-DISK POLYNOMIALS

PAUL G.A. FLORIS
Leiden University
Department of Mathematics and Computer Science
P.O. Box 9512
2300 RA Leiden
The Netherlands
e-mail: floris@wi.leidenuniv.nl
fax: +31 - 71 - 27 69 85

Keywords and phrases: $q$-disk polynomials, linearization coefficients, DJS-hypergroup.

0. Introduction.

In connection with a given system of orthogonal polynomials $\{p_n\}$ it is of great interest to know if there exist positive measures $\mu_{x,y}(z)$ and non-negative numbers $c_{k,l}(m)$ such that

\begin{equation}
\label{eq:0.1}
p_n(x)p_n(y) = \int p_n(z)d\mu_{x,y}(z)
\end{equation}

and

\begin{equation}
\label{eq:0.2}
p_k(x)p_l(x) = \sum_m c_{k,l}(m)p_m(x).
\end{equation}

The first formula, called product formula, gives rise to a positive convolution structure and the second formula, called linearization formula, to a dual positive convolution structure associated with these orthogonal polynomials. It is a quite classical result that such positivity results hold for Gegenbauer polynomials $P_n^{(\alpha,\alpha)}$ ($\alpha \geq -1/2$). Around 1970 these results were also proved for more general Jacobi polynomials $P_n^{(\alpha,\beta)}$ with $(\alpha, \beta)$ in a set containing $\{(\alpha, \beta) \in \mathbb{R}^2 \mid \alpha \geq \beta \geq -1/2\}$ (see [G1] and [G2]). In case such polynomials have an interpretation as spherical functions on a compact symmetric space $G/K$, these positivity results and associated convolution structure follow immediately from analysis on the space of $K$-biinvariant functions on $G$. In connection with this, see also the survey paper by Gasper [G3]. In the seventies the essential aspects of such zonal analysis on groups were abstracted into the concept of a DJS-hypergroup by work of Dunkl, Jewett and Spector. This made it possible to conclude that the positivity results
\(\alpha, \beta\) give rise to two associated hypergroups, one discrete and one continuous, and dual to each other. For an elaborate exposition of the theory of hypergroups we refer the reader to the book by Bloom and Heyer [BH].

A formula like (0.1) can often be given in an explicit way. Then it may also be possible to extend it, using Carlson’s theorem, by analytic continuation from some discrete set of parameter values to a more general set. However, in (0.2) the coefficients are often not explicitly known. Thus the analytic continuation method will not work, and one must look for an alternative way of proving positivity for the more general parameter set. One way of achieving this uses an addition formula for the polynomials \(p_n\) for the general set of parameters, obtained from the discrete case by a continuation argument (see the method described in [Koo1]).

In the theory of quantum groups it is a natural question to ask whether one can obtain results similar to the ones we have in the classical situation. In his paper [Koo4] Koornwinder showed that it is possible to associate a discrete hypergroup with the ’double coset space’ of a Gel’fand pair of compact quantum groups, although the construction is somewhat more involved than the classical one. But as in the classical situation the basic ingredient is positivity of linearization coefficients for the related spherical functions. It is perhaps good to note that the hypergroups arising in this way need not be commutative.

The aim of this paper is to give an example of a non-commutative discrete hypergroup associated with \(q\)-disk polynomials. These are polynomials \(R_{l,m}^{(\alpha)}\) in two non-commuting variables which are expressed through little \(q\)-Jacobi polynomials and that appear, for the value \(\alpha = n - 2\), as zonal spherical functions on a quantum analogue of the homogeneous space \(U(n)/U(n - 1)\). This fact was first proved in [NYM] (see also [Fl]). In a previous paper [Fl] we proved an addition formula for these \(q\)-disk polynomials. It is this addition formula that will allow us to prove positivity of linearization coefficients in a manner similar to [Koo1], and to construct from it a DJS-hypergroup following [Koo4].

The paper is organized as follows. In section one we briefly recall the definition of \(q\)-disk polynomials and some of their properties. Furthermore we will state the addition formula which they satisfy. Section two merely deals with the proof of positivity, or rather non-negativity, of the linearization coefficients. The proof resembles the way of reasoning in [Koo1]. Finally, in section three we explicitly construct the non-commutative discrete hypergroup related to the \(q\)-disk polynomials.

We end by fixing the notation and recalling some well-known facts. In all that follows we will keep \(0 < q < 1\) fixed.

Recall the definition of the little \(q\)-Jacobi polynomials:

\[
p_m(x; a, b; q) = 2\varphi_1[\begin{array}{c}
q^{-m} \\
abla_{aq}
\end{array}; q, qx] = \sum_{k=0}^{m} \frac{(q^{-m}; q)_k(abq^{m+1}; q)_k}{(aq; q)_k(q; q)_k(qx)_k}.
\]

If \(0 < aq < 1\) and \(bq < 1\) they satisfy the orthogonality
Here
\[(a; q)_\infty = \prod_{j=0}^{\infty} (1 - q^j a)\]
\[(a; q)_\beta = \frac{(a; q)_\infty}{(aq^\beta; q)_\infty}\]
are q-shifted factorials \((\beta \in \mathbb{C})\). In particular, if we let
\[P_{m}^{(\alpha, \beta)}(x; q) = P_{m}(x; q^\alpha, q^\beta; q) \quad (\alpha, \beta > -1)\]
then the orthogonality reads
\[
\int_{0}^{1} P_{l}^{(\alpha, \beta)}(x; q)P_{m}^{(\alpha, \beta)}(x; q)x^{\alpha} \frac{(qx; q)_\infty}{(q^{\beta+1}x; q)_\infty} d_q x = \\
\delta_{lm} \frac{(1-q)q^{m(\alpha+1)}}{1-q^{\alpha+\beta+2m+1}} (q; q)_m(q; q)_{\beta+m}.
\]
Here we used Jackson’s q-integral
\[
\int_{0}^{c} f(x) d_q x = c(1-q) \sum_{k=0}^{\infty} f(cq^k)q^k.
\]
Finally, write \(\mathbb{Z}_+\) for the non-negative integers:
\[\mathbb{Z}_+ = \{0, 1, 2, \ldots\}\.
\]

**Acknowledgement**: The author thanks Professor Tom H. Koornwinder for his useful suggestions.

**1. q-Disk polynomials and their addition formula.**

Suppose we are given a complex unital *-algebra \(\mathcal{Z}\) generated by the elements \(z\) and \(z^*\), subject to the relation
\[
z^* z = q^2zz^* + 1 - q^2
\]
and with *-structure \((z)^* = z^*\). It is not hard to show that \(\mathcal{Z}\) has as a linear basis the set \(\{z^k(z^*)^l : k, l \in \mathbb{Z}_+\}\).

On this algebra we define the \(q\)-disk polynomials \(R_{l,m}^{(\alpha)}(z, z^*; q)\) for \(\alpha > -1\) and \(l, m \in \mathbb{Z}_+\) as follows:
\[
R_{l,m}^{(\alpha)}(z, z^*; q) = \begin{cases} 
  z^{1-m} P_{m}^{(\alpha, l-m)}(1-zz^*; q) & (l \geq m) \\
  P_{l}^{(\alpha, m-l)}(1-zz^*; q)(z^*)^{m-l} & (l \leq m).
\end{cases}
\]
Note that...
It is easily seen that one has

\[ R_{l,m}^{(\alpha)}(z, z^*; q) = \sum_{i=0}^{l+m} c_i z^{l-i}(z^*)^{m-i} \text{ with } c_0 \neq 0. \]

The orthogonality of these polynomials can be expressed through q-integrals:

\[
\frac{1}{2\pi} \int_0^{2\pi} R_{l,m}^{(\alpha)}(e^{i\theta} z, e^{-i\theta} z^*; q^2)^* R_{l',m'}^{(\alpha)}(e^{i\theta} z, e^{-i\theta} z^*; q^2) d\theta
\]

\[
(1 - z z^*)^\alpha d_q z (1 - z z^*)
\]

\[
= \delta_{ll'} \delta_{mm'} c_{l,m}^{(\alpha)}
\]

where

\[
c_{l,m}^{(\alpha)} = \frac{(1 - q^{2(\alpha+1)}) q^{2m(\alpha+1)}}{1 - q^{2(m+1)}} \frac{(q^2; q^2)_l (q^2; q^2)_m}{(q^{2(\alpha+1)}; q^2)_l (q^{2(\alpha+1)}; q^2)_m}.
\]

Note that (1.4) is well-defined, since after integrating with respect to \( \theta \) one obtains a polynomial which is invariant under the transformation \( z \mapsto e^{i\theta} z, \ z^* \mapsto e^{-i\theta} z^* \), and hence is a polynomial in the single variable \( 1 - zz^* \).

The same orthogonality can be achieved using the linear functional \( h_{(\alpha)} : \mathcal{Z} \to \mathbb{C} \) \((\alpha > -1)\) defined as

\[
h_{(\alpha)}(z^k(z^*)^l) = \delta_{kl} q^{2k(\alpha+1)} \frac{(q^2; q^2)_k}{(q^{2(\alpha+2)}; q^2)_k}
\]

and satisfying \( h_{(\alpha)}(p^*) = \overline{h_{(\alpha)}(p)} \) for all \( p \in \mathcal{Z} \). Then

\[
h_{(\alpha)}(R_{l,m}^{(\alpha)}(z, z^*; q) R_{l',m'}^{(\alpha)}(z, z^*; q^2)) = \delta_{ll'} \delta_{mm'} c_{l,m}^{(\alpha)}
\]

It follows that the \( R_{l,m}^{(\alpha)}(z, z^*; q) \) \((l, m \in \mathbb{Z}_+)\) form an orthogonal basis for \( \mathcal{Z} \) with respect to the inner product defined by \( h_{(\alpha)} \). For more details we refer the reader to [Fl].

In [Fl, Thm. 3.5.8] we proved the following addition formula for these q-disk polynomials:

**Theorem 1**: Suppose we are given the abstract complex \(*\)-algebras \( \mathcal{X} \) and \( \mathcal{Y} \) with generators \( X_1, X_2, X_1^*, X_2^* \) and \( Y_1, Y_2, Y_1^*, Y_2^* \) respectively, relations

\[
X_1 X_2 = q X_2 X_1
\]

\[
X_1^* X_2 = q X_2^* X_1
\]

\[
X_2^* X_2 = q^2 X_2 X_2^* + (1 - q^2)
\]

\[
X_1^* X_1 = q^2 X_1 X_1^* + (1 - q^2)(1 - X_2 X_2^*)
\]

\[
Y_1 Y_2 = q Y_2 Y_1
\]

\[
Y_1^* Y_2 = q Y_2^* Y_1
\]

\[
(1.7)
\]
Define the following map on $X$ where $\alpha > 1$.

\begin{align}
(X_1)^* &= X_1^* \\
(Y_1)^* &= Y_1^* \\
(X_2)^* &= X_2^* \\
(Y_2)^* &= Y_2^*.
\end{align}

Then, for arbitrary $\alpha > 0$ and arbitrary $l, m \in \mathbb{Z}_+$ we have the following addition formula for the $q$-disk polynomials:

\begin{align}
R_{l,m}^{(\alpha)}(-qX_1 \otimes Y_1^* + X_2 \otimes Y_1, -qX_1^* \otimes Y_1 + X_2^* \otimes Y_2^*; q^2) = \\
\sum_{r=0}^{l} \sum_{s=0}^{m} c_{l,m;r,s}^{(\alpha)} R_{l-r,m-s}^{(\alpha+r+s)}(X_2, X_2^*; q^2) R_{r,s}^{(\alpha-1)}(X_1, X_1^*; 1 - X_2X_2^*; q^2) \otimes (-q)^{r-s} R_{l-r,m-s}^{(\alpha+r+s)}(Y_2, Y_2^*; q^2) Y_1^s(Y_1^*)^r.
\end{align}

Here

\[ c_{l,m;r,s}^{(\alpha)} = \frac{1 - q^{2(\alpha+r+s)+1}}{1 - q^{2(\alpha+1)}} \frac{c_{l,m}^{(\alpha)}}{c_{l-r,m-s}^{(\alpha)} c_{r,s}^{(\alpha-1)}} \]

(cf. (5)), and

\begin{align}
R_{l,m}^{(\alpha)}(A, B, C; q) = \begin{cases} 
C^m A^{l-m} P_m^{(\alpha,l-m)} \left( \frac{C - AB}{C} ; q \right) & (l \geq m) \\
C^l P_l^{(\alpha,m-l)} \left( \frac{C - AB}{C} ; q \right) B^{m-l} & (l \leq m).
\end{cases}
\end{align}

With this notation we have that $R_{l,m}^{(\alpha)}(A, B, 1; q) = R_{l,m}^{(\alpha)}(A, B; q)$.

Define the following map on $X$:

\[ \tilde{h}_{(\alpha)}(p(X_2, X_2^*)X_1^k(X_1^*)^l) = \delta_{kl} p(X_2, X_2^*)(1 - X_2X_2^*)^k q^{2k\alpha} \frac{(q^2; q^2)_k}{(q^{2(\alpha+1)}; q^2)_k} \]

where $p(X_2, X_2^*)$ is any (ordered) polynomial in $X_2, X_2^*$, and $k, l \in \mathbb{Z}_+$.

**Lemma 2:** Let $p_1(X_2, X_2^*)$ and $p_2(X_2, X_2^*)$ be arbitrary ordered polynomials in $X_2, X_2^*$ and let $p_3(X_1, X_1^*, 1 - X_2X_2^*)$ be any ordered polynomial in $X_1, X_1^*, 1 - X_2X_2^*$, homogeneous of degree $k$, where we put $\deg(X_1) = \deg(X_1^*) = \frac{1}{2}$ and $\deg(1 - X_2X_2^*) = 1$. Then

\[ \tilde{h}_{(\alpha)}(p_1(X_2, X_2^*)p_3(X_1, X_1^*, 1 - X_2X_2^*)p_2(X_2, X_2^*)) = \]

\[ h_{(\alpha-1)}(p_3(X_1, X_1^*, 1)) p_1(X_2, X_2^*)(1 - X_2X_2^*)^k p_2(X_2, X_2^*). \]

**Proof:** In view of (7) we can write

\[ p_3(X_1, X_1^*, 1 - X_2X_2^*) = \sum_{i=0}^{k} \sum_{j=0}^{k-i} c_{ij}(1 - X_2X_2^*)^i X_1^j(X_1^*)^{-j+2(k-i)}. \]

The assertion now easily follows when one uses the first two lines of (1.7) and the relations $X_2(1 - X_2X_2^*) = q^{-2}(1 - X_2X_2^*)X_2$ and $X_1(1 - X_2X_2^*) = q^{2(1 - X_2X_2^*)}X_1^*$.
Corollary 3: We have
\[ \tilde{h}_{(\alpha)}(R_{l-i,m-j}^{(a+i+j)}(X_2, X_2^*; q^2)R_{i,j}^{(a-1)}(X_1, X_1^*, 1 - X_2X_2^*; q^2)) \times \]
\[ R_{p,r}^{(a-1)}(X_1, X_1^*, 1 - X_2X_2^*; q^2)^* \delta_{p,r}^{(a-1)}(X_2, X_2^*; q^2)^*) = \]
\[ \delta_{l,i} \delta_{j,i} R_{l-i,m-j}^{(a+i+j)}(X_2, X_2^*; q^2)(1 - X_2X_2^*)^{i+j} R_{l-p,m' - r}^{(a+p+r)}(X_2, X_2^*; q^2)^* \].

2. Positivity of linearization coefficients.

Since the \( R_{l,m}^{(a)}(z, z^*; q^2) \) \((l, m \in \mathbb{Z}_+\) form a basis for \( \mathcal{Z} \), we have the following expansion in \( \mathcal{Z} \):
\[ R_{l,m}^{(a)}(z, z^*; q^2)R_{l,m'}^{(a)}(z, z^*; q^2) = \]
(2.1)
\[ \sum_{l'' , m'' \in \mathbb{Z}_+} a(l, m; l', m'; l'', m'') R_{l'', m''}^{(a)}(z, z^*; q^2). \]

Here only finitely many of the coefficients \( a(l, m; l', m'; l'', m'') \) are non-zero if we fix \( l, m, l', m' \). In fact the sum ranges over those values of \( l'', m'' \) such that \( l'' - m'' = l - m - (l' - m') \). The \( a(l, m; l', m'; l'', m'') \) are called linearization coefficients.

Theorem 4: For all \( \alpha > 0 \) and all possible choices of \( (l, m) \), \( (l', m') \), \( (l'', m'') \) \( \in \mathbb{Z}_+^2 \), the linearization coefficients are non-negative:
\[ a(l, m; l', m'; l'', m'') \geq 0. \]

Proof: First note that both the pair \( X_2, X_2^* \) and the pair \( Y_2, Y_2^* \) satisfy (1.1). Let \( \Omega = -qX_1 \otimes Y_1^* + X_2 \otimes Y_2 \). It is straightforward to verify that \( \Omega^* \Omega = q^2 \Omega^* + 1 - q^2 \).

This means that we have an identity similar to (2.1) but with \( z, z^* \) replaced by \( \Omega, \Omega^* \):
\[ R_{l,m}^{(a)}(\Omega, \Omega^*; q^2)R_{l', m'}^{(a)}(\Omega, \Omega^*; q^2)^* = \]
(2.2)
\[ \sum_{l'' , m'' \in \mathbb{Z}_+} a(l, m; l', m'; l'', m'') R_{l'', m''}^{(a)}(\Omega, \Omega^*; q^2). \]

Now substitute (1.9) into the right-hand side as well as for both factors in the left-hand side of (2.2) and apply \( \tilde{h}_{(\alpha)} \otimes id \) to this. By Lemma 2 and (1.6) we obtain for the right-hand side
\[ \sum_{l'', m''} \sum_{r=0}^{l''} \sum_{s=0}^{m''} a(l, m; l', m'; l'', m'') c_{r,s}^{(a)} \times \]
\[ \tilde{h}_{(\alpha)} \left( R_{l'' - r,m'' - s}^{(a+r+s)}(X_2, X_2^*; q^2)R_{r,s}^{(a-1)}(X_1, X_1^*, 1 - X_2X_2^*; q^2) \right) \otimes \]
\[ (-q)^{l'' - r} R_{l'' - r,m'' - s}^{(a+r+s)}(Y_2, Y_2^*; q^2)Y_1^s(Y_1)^r \]
\[ \sum_{l'' , m''} a(l, m; l', m'; l'', m'') c_{l'', m''}^{(a)}(0, 0, 0) \times \]
\[ R_{l'', m''}^{(a)}(X_2, X_2^*; q^2) \otimes R_{l'', m''}^{(a)}(Y_2, Y_2^*; q^2) = \]
\[ \sum_{l'', m''} a(l, m; l', m'; l'', m'') \times \]
A Non-Commutative Discrete Hypergroup Associated with q-Disk Polynomials

On the left we get

\[
\sum_{i=0}^{l} \sum_{m=0}^{l'} \sum_{j=0}^{m'} \sum_{p=0}^{m''} \overline{c}_{l,m,i,j}^{(α)} \overline{c}_{l',m',p,r}^{(α)} \left( \widetilde{h}(α) \otimes id \right)
\]

\[
\left( R_{l-i,m-j}^{(α+i+j)}(X_2, X^*_2; q^2) R_{i,j}^{(α-1)}(X_1, X^*_1, 1 - X_2 X^*_2; q^2) \times \)
\]

\[
R_{p-r}^{(α-1)}(X_1, X^*_1, 1 - X_2 X^*_2; q^2)^* R_{l'-p,m'-r}^{(α+p+r)}(X_2, X^*_2; q^2)^* \otimes \]

\[
(-q)^{i-j} R_{l-i,m-j}^{(α+i+j)}(Y_2, Y^*_2, 1; q^2)(Y_1^*)^i Y_j^1 \times \]

\[
(-q)^{p-r}(Y_1^*)^p R_{l'-p,m'-r}^{(α+p+r)}(Y_2, Y^*_2; q^2)^* \right)
\]

in view of Corollary 3. So by now we have the identity

\[
\sum_{i=0}^{l} \sum_{m=0}^{l'} \sum_{j=0}^{m'} \overline{c}_{l,m,i,j}^{(α)} \overline{c}_{l',m',i,j}^{(α)} \overline{c}_{l',m',i,j}^{(α-1)} \times \]

\[
R_{l-i,m-j}^{(α+i+j)}(X_2, X^*_2; q^2)(1 - X_2 X^*_2)^{i+j} R_{l'-i,m'-j}^{(α+i+j)}(X_2, X^*_2; q^2)^* \otimes \]

\[
R_{l-i,m-j}^{(α+i+j)}(Y_2, Y^*_2; q^2)(1 - Y_2 Y^*_2)^{i+j} R_{l'-i,m'-j}^{(α+i+j)}(Y_2, Y^*_2; q^2)^* = \]

\[
\sum_{l'', m''} a(l, m; l', m'; l'', m'') R_{l'', m''}^{(α)}(X_2, X^*_2; q^2) \otimes R_{l', m'}^{(α)}(Y_2, Y^*_2; q^2). \]

Write \(σ\) for the \(\ast\)-algebra anti-automorphism \(σ : \mathcal{Y} \to \mathcal{Y}\) which interchanges \(Y_2\) and \(Y^*_2\) and fixes \(Y_1\) and \(Y^*_1\). Note that

\[
σ(R_{r,s}^{(α)}(Y_2, Y^*_2; q^2)) = R_{r,s}^{(α)}(Y_2, Y^*_2; q^2)^*. \]

Letting \(id \otimes σ\) act on (2.3) yields

\[
\sum_{l'', m''} a(l, m; l', m'; l'', m'') R_{l'', m''}^{(α)}(X_2, X^*_2; q^2) \otimes R_{l', m'}^{(α)}(Y_2, Y^*_2; q^2)^* = \]

\[
\sum_{i=0}^{l} \sum_{m=0}^{l'} \sum_{j=0}^{m'} \overline{c}_{l,m,i,j}^{(α)} \overline{c}_{l',m',i,j}^{(α)} \overline{c}_{l',m',i,j}^{(α-1)} \times \]

\[
R_{l-i,m-j}^{(α+i+j)}(X_2, X^*_2; q^2)(1 - X_2 X^*_2)^{i+j} R_{l'-i,m'-j}^{(α+i+j)}(X_2, X^*_2; q^2)^* \otimes \]

\[
R_{l'-i,m'-j}^{(α+i+j)}(Y_2, Y^*_2; q^2)(1 - Y_2 Y^*_2)^{i+j} R_{l-i,m-j}^{(α+i+j)}(Y_2, Y^*_2; q^2)^*. \]
wind up with:

\[ a(l, m; l', m'; l'', m'') (c_{l',m'\alpha})^2 = \sum_{i=0}^{l} \sum_{j=0}^{m} c_{l,m; i,j} (c_{l',m'; i,j})^2 \times \]

\[ h_\alpha \left( R_{l',m'}^{(\alpha)}(X_2, X_2^*; q^2) \right) R_{l-m-j}^{(\alpha+i+j)}(X_2, X_2^*; q^2) \]

\[ (1 - X_2X_2^*)^{i+j} R_{l-m-j}^{(\alpha+i+j)}(X_2, X_2^*; q^2) \]

because \( h_\alpha \) satisfies \( h_\alpha(p^*) = \overline{h_\alpha(p)} \). Since \( 0 < q < 1 \), this will imply that \( a(l, m; l', m'; l'', m'') \geq 0 \).

**Remark**: Considering the case where \( l = m \) we thus obtain non-negativity for the linearization coefficients of the little \( q \)-Jacobi polynomials \( P_m^{(\alpha,0)}(x) \) \( (\alpha > -1) \).

### 3. A discrete hypergroup structure associated with \( q \)-disk polynomials.

In this section we construct a so-called DJS-hypergroup from the linearization formula (2.1) in the way it was pointed out by Koornwinder [Koo]. First we define the proper setting.

Let \( K \) be a locally compact Hausdorff topological space and let \( M(K) \) be the space of all complex regular Borel measures on \( K \), and \( M^1(K) \) the subset of all probability measures. For \( x \in K \) we denote by \( \delta_x \) the corresponding point measure: \( \delta_x(x) = 1 \) (so \( \delta_x \in M^1(K) \)). Assume that in addition there exist

(a) **convolution**: a continuous map \( K \times K \to M^1(K) \), \( (x, y) \to \delta_x \ast \delta_y \) in the weak topology with respect to \( C_c(K) \)

(b) **involution**: an involutive homeomorphism \( K \to K \), \( x \to \overline{x} \)

(c) **unit element**: a distinguished element \( e \in K \).

Upon identifying \( x \) with \( \delta_x \), the map in (a) extends uniquely to a continuous bilinear map \( M(K) \times M(K) \to M(K) \), \( (\mu, \nu) \to \mu \ast \nu \). The involution of (b) induces an involution \( \mu \to \mu^* \) on \( M(K) \) as follows: \( \mu^*(E) = \overline{\mu(E)} \) \( (E \subset K \) a Borel subset).

**Definition**: The quadruple \( (K, \ast, -, e) \) is called a **DJS-hypergroup** if for all \( x, y, z \in K \) the following conditions are met:

1. \( \delta_x \ast (\delta_y \ast \delta_z) = (\delta_x \ast \delta_y) \ast \delta_z \)
2. \( \text{supp}(\delta_x \ast \delta_y) \) is compact
3. \( (\delta_x \ast \delta_y)^* = \overline{\delta_x} \ast \overline{\delta_y} \)
4. \( \delta_e \ast \delta_x = \delta_x = \delta_x \ast \delta_e \)
5. \( e \in \text{supp}(\overline{\delta_x} \ast \delta_y) \) if and only if \( x = y \)
6. the map of \( K \times K \) to the space of nonvoid compact subsets of \( K \) given by \( (x, y) \to \text{supp}(\delta_x \ast \delta_y) \) is continuous. Here the target space has the topology as defined in [Je, §2.5].

The hypergroup is called **commutative** if \( \delta_x \ast \delta_y = \delta_y \ast \delta_x \) for all \( x, y \in K \), otherwise it is called **non-commutative**.

**Theorem 5**: Put \( K = \mathbb{Z}_+^2 \), endowed with the discrete topology. For \( (l, m), (l', m') \), and \( (l'', m'') \in K \) define:

\[ a(l, m; l', m'; l'', m'') = \sum_{i=0}^{l} \sum_{j=0}^{m} c_{l,m; i,j} (c_{l',m'; i,j})^2 \times \]

\[ h_\alpha \left( R_{l',m'}^{(\alpha)}(X_2, X_2^*; q^2) \right) R_{l-m-j}^{(\alpha+i+j)}(X_2, X_2^*; q^2) \]

\[ (1 - X_2X_2^*)^{i+j} R_{l-m-j}^{(\alpha+i+j)}(X_2, X_2^*; q^2) \]

where \( h_\alpha \) satisfies \( h_\alpha(p^*) = \overline{h_\alpha(p)} \). Since \( 0 < q < 1 \), this will imply that \( a(l, m; l', m'; l'', m'') \geq 0 \).
A Non-Commutative Discrete Hypergroup Associated with $q$-Disk Polynomials

with $a(l, m; l', m'; l'', m'')$ as in (2.1). As an involution on $K$ take $(l, m)^- = (m, l)$. Furthermore write $e = (0, 0)$.

Then the quadruple $(K, \ast, -, e)$ forms a non-commutative discrete DJS-hypergroup.

Proof: We have to verify (a), (1)-(6). Let us abbreviate $R_{l,m}^{(\alpha)}(z, z^*; q^2)$ by $R_{l,m}^{(\alpha)}$.

(a): The convolution is continuous since we have given $K$ the discrete topology. Define the multiplicative linear functional $\varepsilon : Z \to \mathbb{C}$ by $\varepsilon(z) = 1 = \varepsilon(z^*)$ (so in fact $\varepsilon$ is identically 1 on $Z$). If we now apply $\varepsilon$ to both side of (2.1) we get

$$1 = \sum_{l'', m''} a(l, m; l', m'; l'', m'')$$

hence $\delta_{(l, m)} \ast \delta_{(l', m')} \in M^1(K)$.

(1): From $R_{l,m}^{(\alpha)}(R_{l',m'}^{(\alpha)}R_{l'',m''}^{(\alpha)}) = (R_{l,m}^{(\alpha)}R_{l',m'}^{(\alpha)})R_{l'',m''}^{(\alpha)}$ it follows that

$$\sum_{r,s} a(l, m; l', m'; r, s)a(r, s; l'', m'')u, v) = \sum_{r,s} a(l', m'; l'', m''; r, s)a(l, m; r, s; u, v),$$

whence $\delta_{(l, m)} \ast (\delta_{(l', m')} \ast \delta_{(l'', m'')}) = (\delta_{(l, m)} \ast \delta_{(l', m')}) \ast \delta_{(l'', m'')}$.

(2): Since only finitely many of the elements $a(l, m; l', m'; l'', m'')$ are non-zero when $(l, m), (l', m')$ are fixed, the support of $\delta_{(l, m)} \ast \delta_{(l', m')}$ is compact.

(3): Using (1.3) we see that $(R_{l,m}^{(\alpha)}R_{l',m'}^{(\alpha)})^* = R_{l',m'}^{(\alpha)} R_{m,l}^{(\alpha)}$ which gives

$$\sum_{l'', m''} a(l, m; l', m'; m'', l'')R_{l',m'}^{(\alpha)} = \sum_{l'', m''} a(m', l'; m, l; l'', m'')R_{l',m'}^{(\alpha)}R_{l,m}^{(\alpha)}.$$

From this we obtain that

$$\begin{align*}
(\delta_{(l, m)} \ast \delta_{(l', m')})^* &= \delta_{(m', l')} \ast \delta_{(m, l)} \\
&= \delta_{(l', m')} \ast \delta_{(l, m)}^-
\end{align*}$$

(4): Since $R_{0,0}^{(\alpha)} = 1$ we get that $a(l, m; 0, 0; m'', l'') = \delta_{l''0} \delta_{m''0}$.

(5): Note that

$$\begin{align*}
\delta_{l''0} \delta_{m''0} c_{l,m}^{(\alpha)} &= h_n(R_{l,m}^{(\alpha)}* R_{l',m'}^{(\alpha)}) \\
&= h_n(R_{m,l}^{(\alpha)} R_{l',m'}^{(\alpha)}) \\
&= \sum_{l'', m''} a(m, l; l', m'; m'', l'')h_n(R_{l',m'}^{(\alpha)}) \\
&= a(m, l; l', m'; 0, 0).
\end{align*}$$

So $e = (0, 0) \in \text{supp}(\delta_{l''0} \ast \delta_{l''0}) = \text{supp}(\delta_{l''0} \ast \delta_{l''0})$ if and only if one has
References

[BH] W.R. Bloom, H. Heyer, "Harmonic analysis of probability measures on hypergroups", De Gruyter, 1994.

[Fl] P.G.A. Floris, Addition formula for q-disk polynomials, preliminary report.

[Ga1] G. Gasper, Linearization of the product of Jacobi polynomials. I, Canad. J. Math. 22 (1970), 171-175.

[Ga2] ———, Linearization of the product of Jacobi polynomials. II, Canad. J. Math. 22 (1970), 582-593.

[Ga3] ———, Positivity and special functions, in "Theory and Application of Special Functions", R. Askey (ed.), Academic Press, 1975, pp. 375-433.

[Je] R.I. Jewett, Spaces with an abstract convolution of measures, Adv. in Math. 18 (1975), 1-101.

[Koo1] T.H. Koornwinder, Positivity proofs for linearization and connection coefficients of orthogonal polynomials satisfying an addition formula, J. London Math. Soc. (2) 18 (1978), 101-114.

[Koo2] ———, Orthogonal polynomials in connection with quantum groups, in "Orthogonal Polynomials: Theory and Practice", P. Nevai (ed.), NATO ASI Series, vol 294, Kluwer, 1990, pp. 257-292.

[Koo3] ———, Positive convolution structures associated with quantum groups, in "Probability Measures on Groups X", H. Heyer (ed.), Plenum, 1991, pp. 249-268.

[Koo4] ———, Discrete hypergroups associated with compact quantum Gelfand pairs, Report 94-05, Math. Preprint Series, Dept. of Math. and Comp. Sci., University of Amsterdam, 1994; to appear in "Applications of hypergroups and related measure algebras", W.C. Connett, M.-O. Gebuhrer & A.L. Schwartz (eds.), Contemporary Math., Amer. Math. Soc.

[La] R. Lasser, Orthogonal polynomials and hypergroups, Rend. Math. (7) 2 (1983), 185-200.

[NYM] M. Noumi, H. Yamada, K. Mimachi, Finite dimensional representations of the quantum group $GL_q(n,\mathbb{C})$ and the zonal spherical functions on $U_q(n-1)\backslash U_q(n)$, Japan. J. Math. 19 (1993), no. 1, 31-80.