Fractional Action Cosmology: Some Dark Energy Models in Emergent, Logamediate and Intermediate Scenarios of the Universe

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Abstract

In the framework of Fractional Action Cosmology, we have reconstructed the scalar potentials and scalar fields, namely, quintessence, phantom, tachyon, k-essence, DBI-essence, Hessence, dilaton field and Yang-Mills field. To get more physical picture of the variation of the scalar field and potential with time, we express scale factor in emergent, logamediate and intermediate scenarios, under which the Universe expands differently.

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I. INTRODUCTION

Fractional action cosmology (FAC) is based on the principles and formalism of the fractional calculus applied to cosmology. The fractional derivative and fractional integrals are the main tools in fractional calculus, where the order of differentiation or integration is not an integer. The fractional calculus is immensely useful in various branches of mathematics, physics and engineering [1]. In doing FAC, one can proceed in two different ways [2]: the first one is quite easy as one has to replace the partial derivatives in the Einstein field equations with the corresponding fractional derivatives; the second technique involves deriving the field equations and geodesic equations from a more fundamental way, namely starting with the principle of least action and replacing the usual integral with a fractional integral. This later technique is more useful in giving extra features of the FAC [3]:

Rami introduced the FAC by introducing the fractional time integral,

\[ S = -\frac{m}{2\Gamma(\xi)} \int \dot{x}^\mu \dot{x}^\nu g_{\mu\nu}(x) (t - \tau)^{\xi-1} d\tau. \]  

(Ia)

Here \( \Gamma(\xi) = \int_0^\infty t^{\xi-1} e^{-t} dt \) is the Gamma function, \( 0 < \xi \leq 1, 0 < \tau < t, \) \( m \) = constant and \( \dot{x}^\mu = \frac{dx^\mu}{d\tau} \). The variation yields an extra term in the field equations which he termed as ‘variable gravitational constant \( G' \). Moreover, when the weight function in the fractional time integral is replaced with a sinusoidal function, then the solution of the corresponding field equations yield a variable cosmological constant and an oscillatory scale factor [4]

\[ S = \frac{m}{2} \int_0^\tau \dot{x}^\mu \dot{x}^\nu g_{\mu\nu}(x) e^{-\chi \sin(\beta t)} dt, \]  

(Ib)

where \( \chi = 0 \) reduces to the standard action. In [5], the authors extended the previous study by working out with a general weight function:

\[ S = \frac{m}{2} \int_0^\tau g_{\mu\nu}(x) \dot{x}^\mu \dot{x}^\nu \mu(\chi, t) dt, \]  

(Ic)

Several examples were studied and cosmological parameters were calculated in there. An interesting feature of FAC is that it yields an expanding Universe whose scale factor goes like power law form or exponential form depending on the choice of the weight function. Hence cosmic acceleration can be modeled in FAC.

Reconstruction of potentials has been done by several authors in various cases. Capozziello et al [8] considered scalar-tensor theories and reconstruct their potential and coupling by demanding a background \( \Lambda \)CDM cosmology. In the framework of phantom quintessence cosmology, [9] used the Noether Symmetry Approach to obtain general exact solutions for the cosmological equations. In this paper, we are going to reconstruct the potentials and scalar fields, namely, quintessence, phantom, tachyonic, k-essence, DBI-essence, Hessence, dilaton field and Yang-Mills field. Such reconstructions have been studied previously in other gravitational setups [6]. To get more
physical insight into the model, we express scale factor in three useful forms namely emergent, logamediate and intermediate scenarios, under which the Universe expands differently. Such expansion scenarios are consistent with the observations with some restrictions on their parameters.

II. FRACTIONAL ACTION COSMOLOGICAL MODEL

For a FRW spacetime, the line element is

\[ ds^2 = -dt^2 + a^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right], \tag{1} \]

where \( a(t) \) is the scale factor and \( k (= 0, \pm 1) \) is the curvature scalar. We consider the Universe contains normal matter and dark energy. From Eq. (1), the Einstein equations for the space-time given by equation (1) are

\[ H^2 + \frac{2(\xi - 1)}{T_1} H + \frac{k}{a^2} = \frac{8\pi G}{3} \rho, \tag{2} \]

\[ \dot{H} - \frac{(\xi - 1)}{T_1} H - \frac{k}{a^2} = -4\pi G (\rho + p), \tag{3} \]

where \( T_1 = t - \tau \), \( \rho = (\rho_m + \rho_\phi) \) and \( p = (p_m + p_\phi) \). Here \( \rho_m \) and \( p_m \) are the energy density and pressure of the normal matter connected by the equation of state

\[ p_m = w_m \rho_m \ , \ -1 \leq w_m \leq 1 \tag{4} \]

and \( \rho_\phi \) and \( p_\phi \) are the energy density and pressure due to the dark energy.

Now consider there is an interaction between normal matter and dark energy. Dark energy interacting with dark matter is a promising model to alleviate the cosmic coincidence problem. In Ref. [10], the authors studied the signature of such interaction on large scale cosmic microwave background (CMB) temperature anisotropies. Based on the detail analysis in perturbation equations of dark energy and dark matter when they are in interaction, they found that the large scale CMB, especially the late Integrated Sachs Wolfe effect, is a useful tool to measure the coupling between dark sectors. It was deduced that in the 1σ range, the constrained coupling between dark sectors can solve the coincidence problem. In Ref. [11], a general formalism to study the growth of dark matter perturbations when dark energy perturbations and interactions between dark sectors were presented. They showed that the dynamical stability on the growth of structure depends on the form of coupling between dark sectors. Moreover due to the influence of the interaction, the growth index can differ from the value without interaction by an amount up to the observational sensibility, which provides an opportunity to probe the interaction between dark sectors through future observations on the growth of structure.

Due to this interaction, the normal matter and dark energy are not separately conserved. The energy conservation equations for normal matter and dark energy are

\[ \dot{\rho}_m + 3H(p_m + \rho_m) = -3\delta H \rho_m, \tag{5} \]
and

\[ \dot{\rho}_\phi + 3H(p_\phi + \rho_\phi) = 3\delta H \rho_m, \tag{6} \]

where \( H = \dot{a}/a \) is the Hubble parameter.

From equation (5) we have the expression for energy density of matter as

\[ \rho_m = \rho_0 a^{-3(1+w_m+\delta)}, \tag{7} \]

where \( \rho_0 \) is the integration constant.

### III. EMERGENT, LOGAMEDIATE AND INTERMEDIATE SCENARIOS

**Emergent Scenario**: For emergent Universe, the scale factor can be chosen as

\[ a(T_1) = a_0 \left( \lambda + e^{\mu T_1} \right)^n \tag{8} \]

where \( a_0, \mu, \lambda \) and \( n \) are positive constants. (1) \( a_0 > 0 \) for the scale factor \( a \) to be positive; (2) \( \lambda > 0 \), to avoid any singularity at finite time (big-rip); (3) \( a > 0 \) or \( n > 0 \) for expanding model of the Universe; (4) \( a < 0 \) and \( n < 0 \) implies big bang singularity at \( t = -\infty \).

So the Hubble parameter and its derivatives are given by

\[ H = \frac{n \mu e^{\mu T_1}}{(\lambda + e^{\mu T_1})^2}, \quad \dot{H} = \frac{n \lambda \mu^2 e^{\mu T_1}}{(\lambda + e^{\mu T_1})^3}, \quad \ddot{H} = \frac{n \lambda \mu^3 e^{\mu T_1} (\lambda - e^{\mu T_1})}{(\lambda + e^{\mu T_1})^4} \tag{9} \]

Here \( H \) and \( \dot{H} \) are both positive, but \( \ddot{H} \) changes sign at \( T_1 = \frac{1}{\mu} \log \lambda \). Thus \( H, \dot{H} \) and \( \ddot{H} \) all tend to zero as \( t \to -\infty \). On the other hand as \( t \to \infty \) the solution gives asymptotically a de Sitter Universe.

**Logamediate Scenario**: Consider a particular form of Logamediate Scenario, where the form of the scale factor \( a(t) \) is defined as

\[ a(T_1) = e^{A(\ln T_1)^\alpha}, \tag{10} \]

where \( A \alpha > 0 \) and \( \alpha > 1 \). When \( \alpha = 1 \), this model reduces to power-law form. The logamediate form is motivated by considering a class of possible cosmological solutions with indefinite expansion which result from imposing weak general conditions on the cosmological model. Barrow has found in their model, the observational ranges of the parameters are as follows: \( 1.5 \times 10^{-92} \leq A \leq 2.1 \times 10^{-92} \) and \( 2 \leq \alpha \leq 50 \). The Hubble parameter \( H = \frac{\dot{a}}{a} \) and its derivative become,

\[ H = \frac{A \alpha}{T_1} (\ln T_1)^{\alpha-1}, \quad \dot{H} = \frac{A \alpha}{T_1^2} (\ln T_1)^{\alpha-2}(\alpha - 1 - \ln T_1) \tag{11} \]
• **Intermediate Scenario**: Consider a particular form of Intermediate Scenario, where the scale factor $a(t)$ of the Friedmann universe is described as

$$a(t) = e^{BT_1^\beta},$$

where $B\beta > 0$, $B > 0$ and $0 < \beta < 1$. Here the expansion of the Universe is faster than Power-Law form, where the scale factor is given as, $a(T_1) = T_1^n$, where $n > 1$ is a constant. Also, the expansion of the Universe is slower for Standard de-Sitter Scenario where $\beta = 1$. The Hubble parameter $H = \frac{a}{a}$ and its derivative become,

$$H = B\beta T_1^{\beta-1}, \quad \dot{H} = B\beta(\beta-1)T_1^{\beta-2}$$

**IV. VARIOUS CANDIDATES OF DARK ENERGY MODELS**

**A. Quintessence or Phantom field**

Quintessence is described by an ordinary time dependent and homogeneous scalar field $\phi$ which is minimally coupled to gravity, but with a particular potential $V(\phi)$ that leads to the accelerating Universe. The action for quintessence is given by

$$S = \int d^4x \sqrt{-g} \left[-\frac{1}{2}g^{ij}\partial_i \phi \partial_j \phi - V(\phi)\right].$$

The energy momentum tensor of the field is:

$$T_{ij} = -\frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{ij}},$$

which gives

$$T_{ij} = \partial_i \phi \partial_j \phi - g_{ij} \left[\frac{1}{2}g^{kl}\partial_k \phi \partial_l \phi + V(\phi)\right].$$

The energy density and pressure of the quintessence scalar field $\phi$ are as follows

$$\rho_\phi = -T^0_0 = \frac{1}{2} \dot{\phi}^2 + V(\phi),$$

$$p_\phi = T^i_i = \frac{1}{2} \dot{\phi}^2 - V(\phi).$$

The EoS parameter for the quintessence scalar field is given by

$$\omega_\phi = \frac{p_\phi}{\rho_\phi} = \frac{\dot{\phi}^2 - 2V(\phi)}{\dot{\phi}^2 + 2V(\phi)}.$$ 

For $\omega_\phi < -1/3$, we find that the Universe accelerates when $\dot{\phi}^2 < V(\phi)$. 
The energy density and the pressure of the quintessence (phantom field) can be represented by the minimally coupled spatially homogeneous and time dependent scalar field $\phi$ having positive (negative) kinetic energy term given by

$$\rho_\phi = \frac{\epsilon}{2} \dot{\phi}^2 + V(\phi)$$  \hspace{1cm} (14)

and

$$p_\phi = -\frac{\epsilon}{2} \dot{\phi}^2 - V(\phi)$$  \hspace{1cm} (15)

where $V(\phi)$ is the relevant potential for the scalar field $\phi$, $\epsilon = +1$ represents quintessence while $\epsilon = -1$ refers to phantom field.

Scalar field models of phantom energy indicate that it can behave as a long range repulsive force [12]. Moreover the phantom energy has few characteristics different from normal matter, for instance, the energy density $\rho(t)$ of the phantom field increases with the expansion of the Universe; it can be used as a source to form and stabilize traversable wormholes [14–17]; the phantom energy can disrupt all gravitationally bound structures i.e from galaxies to black holes [18–23]; it can produce infinite expansion of the Universe in a finite time thus causing the 'big rip' [24].

From above equations, we get

$$\dot{\phi}^2 = \frac{(1 + w_m)}{\epsilon} \rho_m + \frac{1}{4\pi\epsilon G} \left[ -\dot{H} + \frac{(\xi - 1)}{T_1} H + \frac{k a^{-2}}{a^2} \right],$$  \hspace{1cm} (16)

and

$$V = \frac{(w_m - 1)}{2} \rho_m + \frac{1}{8\pi G} \left[ \dot{H} + 3H^2 + \frac{5(\xi - 1)}{T_1} H + \frac{2k}{a^2} \right]$$  \hspace{1cm} (17)

• For emergent scenario, we get the expressions for $\phi$ and $V$ as

$$\phi = \int \sqrt{-\frac{(1 + w_m) \rho_0 a_0^{-3(1+w_m+\delta)}}{\epsilon (\lambda + e^{\mu T_1})^{3n(1+w_m+\delta)}} + \frac{1}{4\pi\epsilon G} \left\{ \frac{-n\lambda \mu^2 e^{\mu T_1}}{(\lambda + e^{\mu T_1})^2} + \frac{(\xi - 1)n\mu e^{\mu T_1}}{T_1(\lambda + e^{\mu T_1})} + \frac{k a_0^{-2}}{(\lambda + e^{\mu T_1})^{2n}} \right\} dT_1,$$  \hspace{1cm} (18)

and

$$V = \frac{(w_m - 1) \rho_0 a_0^{-3(1+w_m+\delta)}}{2(\lambda + e^{\mu T_1})^{3n(1+w_m+\delta)}} + \frac{1}{8\pi G} \left\{ \frac{n\mu^2 e^{\mu T_1}(\lambda + 3n e^{\mu T_1})}{(\lambda + e^{\mu T_1})^2} + \frac{5(\xi - 1)n\mu e^{\mu T_1}}{T_1(\lambda + e^{\mu T_1})} + \frac{2k a_0^{-2}}{(\lambda + e^{\mu T_1})^{2n}} \right\}.$$  \hspace{1cm} (19)

• For logarithmic scenario, we get the expressions for $\phi$ and $V$ as

$$\phi = \int \sqrt{-\frac{(1 + w_m) \rho_0 e^{-3A(1+w_m+\delta)(\ln T_1)^\alpha}}{\epsilon} + \frac{1}{4\pi\epsilon G} \left\{ \frac{A\alpha}{T_1^2} (\ln T_1)^{\alpha-2}(1 - \alpha + \xi \ln T_1) + k e^{-2A(\ln T_1)^\alpha} \right\} dT_1$$  \hspace{1cm} (20)
Figs. 1-3 show the variations of $V$ against quintessence or phantom field $\phi$ in the emergent, logamediate and intermediate scenarios respectively. Solid, dash and dotted lines represent $k = -1, +1, 0$ respectively. Blue and red lines represent quintessence field ($\epsilon = +1$) and phantom field ($\epsilon = -1$) respectively.

and

$$V = \frac{(w_m - 1)\rho_0}{2} e^{-3A(1+w_m+\delta)(\ln T_1)^\alpha} + \frac{A\alpha}{T_1^2} (\ln T_1)^{\alpha-2}\left\{\alpha - 1 + (5\xi - 6) \ln T_1 + 3A\alpha (\ln T_1)^\alpha \right\} + 2k e^{-2A(\ln T_1)^\alpha}. \quad (21)$$

For intermediate scenario, we get the expressions for $\phi$ and $V$ as

$$\phi = \int \sqrt{\frac{(1 + w_m)\rho_0}{\epsilon}} e^{-3B(1+w_m+\delta)T_1^\beta} + \frac{1}{4\pi G} \left\{B\beta(\xi - \beta)T_1^{\beta-2} + k e^{-2BT_1^\beta}\right\} \ dT_1, \quad (22)$$

and

$$V = \frac{(w_m - 1)\rho_0}{2} e^{-3B(1+w_m+\delta)T_1^\beta} + \frac{1}{8\pi G} \left[B^2T_1^{\beta-2}(5\xi + \beta + 3B\beta T_1^\beta) + 2k e^{-2BT_1^\beta}\right]. \quad (23)$$

In figures 1, 2 and 3, we have plotted the potentials against the scalar fields for the quintessence and phantom fields in emergent, logamediate and intermediate scenarios of the universe respectively in fractional action cosmology. It has been observed in figure 1 that after gradual decay, the potential starts increasing with scalar field for
quintessence as well as phantom field models of dark energy in the emergent scenario of the universe irrespective of its type of curvature. On the contrary, when logamediate scenario is considered, the figure 2 exhibits a continuous decay in the potential $V$ with increase in the scalar field $\phi$. A different behavior is observed in figure 3 that depicts the behavior of the potential $V$ against scalar field $\phi$ in the case of intermediate scenario of the universe. The blue lines in this figure show a continuous decay in $V$ with increase in $\phi$ for quintessence model. However, the red lines exhibit an increasing pattern of $V$ with scalar field $\phi$.

**B. Tachyonic field**

A rolling tachyon has an interesting equation of state whose state parameter smoothly interpolates between $-1$ and 0 [28]. Thus, tachyon can be realized as a suitable candidate for the inflation at high energy [29] as well as a source of dark energy depending on the form of the tachyon potential [30]. Therefore it becomes meaningful to reconstruct tachyon potential $V(\phi)$ from some dark energy models. An action for tachyon scalar $\phi$ is given by Born-Infeld like action

$$S = - \int d^4x \sqrt{-g} V(\phi) \sqrt{1 - g^{ij} \partial_i \phi \partial_j \phi} \quad (24)$$

where $V(\phi)$ is the tachyon potential. Energy-momentum tensor components for tachyon scalar $\phi$ are obtained as

$$T_{ij} = V(\phi) \left[ \frac{\partial_i \phi \partial_j \phi}{\sqrt{1 - g^{ij} \partial_i \phi \partial_j \phi}} + g_{ij} \sqrt{1 - g^{kl} \partial_k \phi \partial_l \phi} \right] \quad (25)$$

The energy density $\rho_{\phi}$ pressure $p_{\phi}$ due to the tachyonic field $\phi$ have the expressions

$$\rho_{\phi} = \frac{V(\phi)}{\sqrt{1 - \epsilon \dot{\phi}^2}}, \quad (26)$$

$$p_{\phi} = -V(\phi) \sqrt{1 - \epsilon \dot{\phi}^2}, \quad (27)$$

where $V(\phi)$ is the relevant potential for the tachyonic field $\phi$. It is to be seen that $\frac{p_{\phi}}{\rho_{\phi}} = -1 + \epsilon \dot{\phi}^2 > -1$ or $< -1$ accordingly as normal tachyon ($\epsilon = +1$) or phantom tachyon ($\epsilon = -1$).

From above, we get

$$\dot{\phi}^2 = \left[ -\frac{(1 + w_m)}{\epsilon} \rho_m + \frac{1}{4\pi\epsilon G} \left\{ -\ddot{H} + \frac{(\xi - 1)}{T_1} H + \frac{k}{a^2} \right\} \right]$$
\[ \left[ -\rho_m + \frac{3}{8\pi G} \left( H^2 + \frac{2(\xi - 1)}{T_1} H + \frac{k}{a^2} \right) \right]^{-1} \]

and

\[ V = \left[ \frac{w_m \rho_m}{8\pi G} \left( 2\dot{H} + 3H^2 + \frac{4(\xi - 1)}{T_1} H + \frac{k}{a^2} \right) \right]^{\frac{1}{2}} \]

For emergent scenario, we get the expressions for \( \phi \) and \( V \) as

\[
\phi = \int \left[ \frac{(1 + w_m) \rho_0 a_0^{-3(1+w_m+\delta)}}{\epsilon (\lambda + e^{\mu T_1})^3 n (1+w_m+\delta)} + \frac{1}{4\pi \epsilon G} \left\{ -\frac{n \lambda e^{\mu T_1}}{\lambda + e^{\mu T_1}} + \frac{(\xi - 1) n e^{\mu T_1}}{T_1 (\lambda + e^{\mu T_1})} + \frac{k a_0^{-2}}{(\lambda + e^{\mu T_1})^2} \right\} \right]^{\frac{1}{2}} dT_1
\]

and

\[
V = \left[ \frac{w_m \rho_0 a_0^{-3(1+w_m+\delta)}}{(\lambda + e^{\mu T_1})^3 n (1+w_m+\delta)} + \frac{1}{8\pi G} \left\{ n \mu e^{\mu T_1} (2\lambda + 3n e^{\mu T_1}) + \frac{4(\xi - 1) n e^{\mu T_1}}{T_1 (\lambda + e^{\mu T_1})} + \frac{k a_0^{-2}}{(\lambda + e^{\mu T_1})^2} \right\} \right]^{\frac{1}{2}}
\]

For logamediate scenario, we get the expressions for \( \phi \) and \( V \) as

\[
\phi = \int \left[ -\frac{(1 + w_m) \rho_0}{\lambda} e^{-3A(1+w_m+\delta)} (\ln T_1)^{\alpha} + \frac{1}{4\pi \epsilon G} \left\{ \frac{A_0 \alpha}{T_1^2} (\ln T_1)^{\alpha - 2} (1 - \alpha + \xi (\ln T_1)) + k e^{-2A(\ln T_1)^\alpha} \right\} \right]^{\frac{1}{2}} dT_1
\]

and

\[
V = \left[ -\rho_0 e^{-3A(1+w_m+\delta)} (\ln T_1)^{\alpha} + \frac{3}{8\pi G} \left\{ \frac{A_0 \alpha}{T_1^2} (\ln T_1)^{\alpha - 1} \{ A_0 (\ln T_1)^{\alpha - 1} + 2(\xi - 1) \} + k e^{-2A(\ln T_1)^\alpha} \right\} \right]^{\frac{1}{2}}
\]

For intermediate scenario, we get the expressions for \( \phi \) and \( V \) as

\[
\phi = \int \left[ -\frac{(1 + w_m) \rho_0}{\lambda} e^{-3B(1+w_m+\delta)} T_1^\beta + \frac{1}{4\pi \epsilon G} \left\{ B_0 \beta (\xi - \beta) T_1^{\beta - 2} + k e^{-2B T_1^\beta} \right\} \right]^{\frac{1}{2}}
\]
Figs. 4-6 show the variations of $V$ against tachyonic field $\phi$ in the emergent, logamediate and intermediate scenarios respectively. Solid, dash and dotted lines represent $k = -1, +1, 0$ respectively. Blue and red lines represent normal tachyonic field ($\epsilon = +1$) and phantom tachyonic field ($\epsilon = -1$) respectively.

\[
\times \left[ -\rho_0 e^{-3B(1+w_m+\delta)T_1} + \frac{3}{8\pi G} \left\{ B\beta T_1^{\beta-2} (2(\xi - 1) + B\beta T_1^{\beta}) + k e^{-2BT_1^{\beta}} \right\} \right]^{-\frac{1}{2}} dT_1 \quad (34)
\]

and

\[
V = \left[ -\rho_0 e^{-3B(1+w_m+\delta)T_1^\beta} + \frac{3}{8\pi G} \left\{ B\beta T_1^{\beta-2} (2(\xi - 1) + B\beta T_1^{\beta}) + k e^{-2BT_1^{\beta}} \right\} \right]^{\frac{1}{2}}
\]
\[ w_m \rho_0 e^{-3B(1+w_m+\delta)}T_1^\beta + \frac{1}{8\pi G} \left\{ B\beta T_1^\beta \left( -2(2\xi + \beta - 3) + 3B\beta T_1^\beta \right) + k e^{-2BT_1^\beta} \right\}^{\frac{1}{2}}. \]  

(35)

In figure 4, the \( V-\phi \) plot for normal tachyon and phantom tachyon models of dark energy is presented for emergent scenario of the universe. Potential of normal tachyon exhibits decaying pattern. However, it shows increasing pattern for phantom tachyonic field \( \phi \). It happens irrespective of the curvature of the universe. In the logamediate scenario (figure 5) the potentials for normal tachyon and phantom tachyon exhibit increasing and decreasing behavior respectively with increase in the scalar field \( \phi \). From figure 6 we see a continuous decay in the potential for normal tachyonic field in the intermediate scenario. However, in this scenario, the behavior of the potential varies with the curvature of the universe characterized by interacting phantom tachyonic field. For \( k = -1, 1 \), the potential increases with phantom tachyonic field and for \( k = 0 \), it decays after increasing initially.

C. k-essence

In the kinetically driven scalar field theory, we have non-canonical kinetic energy term with no potential. Scalars, modelling this theory, are popularly known as k-essence. Motivated by Born-Infeld action of String Theory, it was used as a source to explain the mechanism for producing the late time acceleration of the universe. This model is given by the action \([31]\)

\[ S = \int d^4x \sqrt{-g} \mathcal{L}(\phi, \dot{X}), \]  

(36)

with

\[ \mathcal{L}(\phi, \dot{X}) = K(\phi) \dot{X} + L(\phi) \dot{X}^2, \]  

(37)

ignoring higher order terms of

\[ \dot{X} = \frac{1}{2} g^{ij} \partial_i \phi \partial_j \phi. \]  

(38)

Using the following transformations, \( \phi = \int d\bar{\phi} \sqrt{|L(\bar{\phi})|/K(\bar{\phi})} \), \( X = \frac{|L|}{K} \dot{X} \) and \( V(\phi) = K^2/|L| \), the action can be rewritten as

\[ S = \int d^4x \sqrt{-g} V(\phi) \mathcal{L}(X), \]  

(39)

with

\[ \mathcal{L}(X) = X - X^2. \]  

(40)

From the action, the energy-momentum tensor components can be written as

\[ T_{ij} = V(\phi) \left[ \frac{d\mathcal{L}}{dX} \partial_i \phi \partial_j \phi - g_{ij} \mathcal{L} \right]. \]  

(41)
The energy density and pressure of k-essence scalar field $\phi$ are given by

$$\rho_k = V(\phi)(-X + 3X^2), \quad (42)$$

and

$$p_k = V(\phi)(-X + X^2), \quad (43)$$

where $\phi$ is the scalar field having kinetic energy $X = \frac{1}{2}\dot{\phi}^2$ and $V(\phi)$ is the k-essence potential.

From above, we get

$$\phi^2 = \left[ 2(w_m - 1)\rho_m + \frac{1}{2\pi G} \left\{ \dot{H} + 3H^2 + \frac{5(\xi - 1)}{T_1}H + \frac{2k}{a^2} \right\} \right]$$

$$\times \left[ (3w_m - 1)\rho_m + \frac{3}{4\pi G} \left\{ \dot{H} + 2H^2 + \frac{3(\xi - 1)}{T_1}H + \frac{k}{a^2} \right\} \right]^{-1}, \quad (44)$$

and

$$V = \left[ (3w_m - 1)\rho_m + \frac{3}{4\pi G} \left\{ \dot{H} + 2H^2 + \frac{3(\xi - 1)}{T_1}H + \frac{k}{a^2} \right\} \right]^2$$

$$\times \left[ 2(w_m - 1)\rho_m + \frac{1}{2\pi G} \left\{ \dot{H} + 3H^2 + \frac{5(\xi - 1)}{T_1}H + \frac{2k}{a^2} \right\} \right]^{-1}. \quad (45)$$

- For emergent scenario, we have

$$\phi = \int \left[ \frac{2(w_m - 1)\rho_0 a_0^{-3(1+w_m+\delta)}}{(\lambda + e\mu T_1)^{3n(1+w_m+\delta)}} + \frac{1}{2\pi G} \left\{ n\mu e^{\mu T_1}(\lambda + 3ne^{\mu T_1}) \right. \right.$$ 

$$\left. \left\{ \dot{H} + 3H^2 + \frac{5(\xi - 1)n\mu e^{\mu T_1}}{T_1(\lambda + e\mu T_1)} + \frac{2k a_0^{-2}}{(\lambda + e\mu T_1)^{2n}} \right\} \right]^\frac{1}{2}$$

$$\times \left[ \frac{(3w_m - 1)\rho_0 a_0^{-3(1+w_m+\delta)}}{(\lambda + e\mu T_1)^{3n(1+w_m+\delta)}} + \frac{3}{4\pi G} \left\{ n\mu e^{\mu T_1}(\lambda + 2ne^{\mu T_1}) \right. \right.$$ 

$$\left. \left\{ \dot{H} + 2H^2 + \frac{3(\xi - 1)n\mu e^{\mu T_1}}{T_1(\lambda + e\mu T_1)} + \frac{k a_0^{-2}}{(\lambda + e\mu T_1)^{2n}} \right\} \right]^\frac{1}{2}$$

$$dt. \quad (46)$$

and

$$V = \left[ \frac{(3w_m - 1)\rho_0 a_0^{-3(1+w_m+\delta)}}{(\lambda + e\mu T_1)^{3n(1+w_m+\delta)}} + \frac{3}{4\pi G} \left\{ n\mu e^{\mu T_1}(\lambda + 2ne^{\mu T_1}) \right. \right.$$ 

$$\left. \left\{ \dot{H} + 2H^2 + \frac{3(\xi - 1)n\mu e^{\mu T_1}}{T_1(\lambda + e\mu T_1)} + \frac{k a_0^{-2}}{(\lambda + e\mu T_1)^{2n}} \right\} \right]^2$$

$$\times \left[ \frac{2(w_m - 1)\rho_0 a_0^{-3(1+w_m+\delta)}}{(\lambda + e\mu T_1)^{3n(1+w_m+\delta)}} + \frac{1}{2\pi G} \left\{ n\mu e^{\mu T_1}(\lambda + 3ne^{\mu T_1}) \right. \right.$$ 

$$\left. \left\{ \dot{H} + 3H^2 + \frac{5(\xi - 1)n\mu e^{\mu T_1}}{T_1(\lambda + e\mu T_1)} + \frac{2k a_0^{-2}}{(\lambda + e\mu T_1)^{2n}} \right\} \right]^{-1}. \quad (47)$$
• For logamediate scenario, we get the expressions for $\phi$ and $V$ as

$$
\phi = \int \left[ 2(w_m - 1)\rho_0 e^{-3A(1+w_m+\delta)(\ln T_1)^{\alpha}} + \frac{1}{2\pi G} \left\{ \frac{A\alpha}{T_1^2}(\ln T_1)^{\alpha-2}(\alpha - 1 + (5\xi - 6) \ln T_1 + 3A\alpha (\ln T_1)^{\alpha}) + 2k e^{-2A(\ln T_1)^{\alpha}} \right\} \right] \frac{dT_1}{T_1}.
$$

and

$$
V = \left[ (3w_m - 1)\rho_0 e^{-3A(1+w_m+\delta)(\ln T_1)^{\alpha}} + \frac{3}{4\pi G} \left\{ \frac{A\alpha}{T_1^2}(\ln T_1)^{\alpha-2}(\alpha - 1 + (3\xi - 4) \ln T_1 + 2A\alpha (\ln T_1)^{\alpha}) + k e^{-2A(\ln T_1)^{\alpha}} \right\} \right]^2.
$$

• For intermediate scenario, we get the expressions for $\phi$ and $V$ as

$$
\phi = \int \left[ 2(w_m - 1)\rho_0 e^{-3B(1+w_m+\delta)T_1^\beta} + \frac{1}{2\pi G} \left\{ B\beta(5\xi + \beta - 6 + 3B\beta T_1^\beta)T_1^{\beta-2} + 2k e^{-2BT_1^\beta} \right\} \right]^{\frac{1}{2}} dT_1,
$$

and

$$
V = \left[ (3w_m - 1)\rho_0 e^{-3B(1+w_m+\delta)T_1^\beta} + \frac{3}{4\pi G} \left\{ B\beta(3\xi + \beta - 4 + 2B\beta T_1^\beta)T_1^{\beta-2} + k e^{-2BT_1^\beta} \right\} \right]^2.
$$

From figures 7, 8 and 9 we see that for interacting k-essence the potential $V$ always decreases with increase in the scalar field $\phi$ in all of the three scenarios and it happens for open, closed and flat universes.

**D. DBI-essence**

Consider that the dark energy scalar field is a Dirac-Born-Infeld (DBI) scalar field. In this case, the action of the field be written as

$$
S_D = -\int d^4x a^3(t) \left[ T(\phi) \sqrt{1 - \frac{\dot{\phi}^2}{T(\phi)}} + V(\phi) - T(\phi) \right],
$$

(52)
Figs. 7–9 show the variations of $V$ against k-essence field $\phi$ in the emergent, logamediate and intermediate scenarios respectively. Red, green and blue lines represent $k = -1, +1, 0$ respectively.

where $T(\phi)$ is the warped brane tension and $V(\phi)$ is the DBI potential. The energy density and pressure of the DBI-essence scalar field are respectively given by

$$\rho_D = (\gamma - 1)T(\phi) + V(\phi),$$  \hspace{1cm} (53)
and
\[ p_D = \frac{\gamma - 1}{\gamma} T(\phi) - V(\phi), \] (54)

where \( \gamma \) is given by
\[ \gamma = \frac{1}{\sqrt{1 - \frac{\phi^2}{\phi^2 T} \frac{\phi}{\phi T}}} \] (55)

Now we consider here particular case \( \gamma = \text{constant}. \) In this case, for simplicity, we assume \( T(\phi) = T_0 \phi^2 \) \((T_0 > 1).\) So we have \( \gamma = \frac{T_0}{T_0 - 1}. \) In this case the expressions for \( \phi, T(\phi) \) and \( V(\phi) \) are given by
\[ \dot{\phi}^2 = \frac{T_0 - 1}{T_0} \left[ -(1 + w_m) \rho_m + \frac{1}{4 \pi G} \left( -\dot{H} + \frac{\xi - 1}{T_1} H + \frac{k}{a^2} \right) \right], \] (56)
\[ T = \sqrt{T_0(T_0 - 1)} \left[ -(1 + w_m) \rho_m + \frac{1}{4 \pi G} \left( -\dot{H} + \frac{\xi - 1}{T_1} H + \frac{k}{a^2} \right) \right], \] (57)
and
\[ V = \left( T_0 - \sqrt{T_0(T_0 - 1)} \right) (1 + w_m) - w_m \rho_m - \frac{1}{8 \pi G} \left[ \left( 1 - T_0 + \sqrt{T_0(T_0 - 1)} \right) \dot{H} + 3H^2 \right. \]
\[ + 2 \left( T_0 - \sqrt{T_0(T_0 - 1)} + 2 \right) \frac{\xi - 1}{T_1} H + \left( 2T_0 - 2\sqrt{T_0(T_0 - 1)} + 1 \right) \frac{k}{a^2} \right]. \] (58)

• For emergent scenario, we get the expressions for \( \phi, T \) and \( V \) as
\[ \phi = \left( \frac{T_0 - 1}{T_0} \right)^{\frac{1}{2}} \int \left[ -\frac{(1 + w_m) \rho_0 a_0^{3(1 + w_m + \delta)}}{(\lambda + e^{\mu T_1})^{3n(1 + w_m + \delta)}} + \frac{1}{4 \pi G} \left\{ \frac{n \lambda T_0^{2e^{\mu T_1}}}{(\lambda + e^{\mu T_1})^2} + \frac{(\xi - 1) n \mu e^{\mu T_1}}{T_1(\lambda + e^{\mu T_1})} + \frac{k a_0^{-2}}{(\lambda + e^{\mu T_1})^{2n}} \right\} \right] \frac{1}{T_1} dT_1 \] (59)
\[ T = \sqrt{T_0(T_0 - 1)} \left[ -\frac{(1 + w_m) \rho_0 a_0^{3(1 + w_m + \delta)}}{(\lambda + e^{\mu T_1})^{3n(1 + w_m + \delta)}} + \frac{1}{4 \pi G} \left\{ -\frac{n \lambda T_0^{2e^{\mu T_1}}}{(\lambda + e^{\mu T_1})^2} + \frac{(\xi - 1) n \mu e^{\mu T_1}}{T_1(\lambda + e^{\mu T_1})} + \frac{k a_0^{-2}}{(\lambda + e^{\mu T_1})^{2n}} \right\} \right], \] (60)
and
\[ V = \left( T_0 - \sqrt{T_0(T_0 - 1)} \right) (1 + w_m) - w_m \rho_0 a_0^{3(1 + w_m + \delta)} \left( \frac{1}{(\lambda + e^{\mu T_1})^{3n(1 + w_m + \delta)}} \right) \left\{ -\frac{1}{8 \pi G} \left[ \left( 1 - T_0 + \sqrt{T_0(T_0 - 1)} \right) \frac{n \lambda T_0^{2e^{\mu T_1}}}{(\lambda + e^{\mu T_1})^2} \right. \right. \]
\[ + \left. \left. \frac{3n^2 \mu^2 e^{2\mu T_1}}{(\lambda + e^{\mu T_1})^2} + 2 \left( T_0 - \sqrt{T_0(T_0 - 1)} + 2 \right) \frac{(\xi - 1)}{T_1} \frac{n \mu e^{\mu T_1}}{(\lambda + e^{\mu T_1})} + \left( 2T_0 - 2\sqrt{T_0(T_0 - 1)} + 1 \right) \frac{k a_0^{-2}}{(\lambda + e^{\mu T_1})^{2n}} \right] \right\}. \] (61)
• For logamediate scenario, we get the expressions for $\phi$, $T$ and $V$ as

$$
\phi = \left( \frac{T_0 - 1}{T_0} \right)^{\frac{1}{2}} \int \left[ -(1 + w_m)\rho_0 e^{-3A(1+w_m+\delta)(\ln T_1)^\alpha} + \frac{1}{4\pi G} \left( \frac{A\alpha}{T_1^2} (\ln T_1)^{\alpha-2} (1 - \alpha + \xi \ln T_1) + k e^{-2A(\ln T_1)^\alpha} \right) \right]^{\frac{1}{2}} dT_1
$$

(62)

$$
T = \sqrt{T_0(T_0 - 1)} \left[ -(1 + w_m)\rho_0 e^{-3A(1+w_m+\delta)(\ln T_1)^\alpha} + \frac{1}{4\pi G} \left( \frac{A\alpha}{T_1^2} (\ln T_1)^{\alpha-2} (1 - \alpha + \xi \ln T_1) + k e^{-2A(\ln T_1)^\alpha} \right) \right],
$$

and

$$
V = \left[ (T_0 - \sqrt{T_0(T_0 - 1)}) (1 + w_m) - w_m \right] \rho_0 e^{-3A(1+w_m+\delta)(\ln T_1)^\alpha} - \frac{1}{8\pi G} \left[ 2 \left( T_0 - \sqrt{T_0(T_0 - 1)} \right) + 2 \left( \xi - 1 \right) \frac{A\alpha}{T_1^2} (\ln T_1)^{\alpha-1} \right. \\
+ \frac{3A^2\alpha^2}{T_1^2} (\ln T_1)^{2\alpha-2} + \left( 1 - T_0 + \sqrt{T_0(T_0 - 1)} \right) \frac{A\alpha}{T_1^2} (\ln T_1)^{\alpha-2} (\alpha - 1 - \ln T_1) + \left( 2T_0 - 2\sqrt{T_0(T_0 - 1)} + 1 \right) k e^{-2A(\ln T_1)^\alpha} \right].
$$

(63)

• For intermediate scenario, we get the expressions for $\phi$, $T$ and $V$ as

$$
\phi = \left( \frac{T_0 - 1}{T_0} \right)^{\frac{1}{2}} \int \left[ -(1 + w_m)\rho_0 e^{-3B(1+w_m+\delta)T_1^\beta} + \frac{1}{4\pi G} \left( B\beta(\xi - \beta)T_1^{\beta-2} + k e^{-2BT_1^\beta} \right) \right]^{\frac{1}{2}} dT_1.
$$

(65)

$$
T = \sqrt{T_0(T_0 - 1)} \left[ -(1 + w_m)\rho_0 e^{-3B(1+w_m+\delta)T_1^\beta} + \frac{1}{4\pi G} \left( B\beta(\xi - \beta)T_1^{\beta-2} + k e^{-2BT_1^\beta} \right) \right]
$$

(66)

and

$$
V = \left[ (T_0 - \sqrt{T_0(T_0 - 1)}) (1 + w_m) - w_m \right] \rho_0 e^{-3B(1+w_m+\delta)T_1^\beta} - \frac{1}{8\pi G} \left[ \left( 1 - T_0 + \sqrt{T_0(T_0 - 1)} \right) B\beta(\beta - 1)T_1^{\beta-2} \\
+ 3B^2\beta^2T_1^{2\beta-2} + 2 \left( T_0 - \sqrt{T_0(T_0 - 1)} + 2 \right) \frac{B\beta}{T_1^2} B\beta T_1^{\beta-1} + \left( 2T_0 - 2\sqrt{T_0(T_0 - 1)} + 1 \right) k e^{-2BT_1^\beta} \right]
$$

(67)

When we consider an interacting DBI-essence dark energy, we get decaying pattern in the $V$-$\phi$ plot for emergent and intermediate scenarios in the figures 10 and 12. However, from figure 11 we see an increasing plot of $V$-$\phi$ for for interacting DBI-essence in the logamediate scenario.

E. Hessence

Wei et al [32] proposed a novel non-canonical complex scalar field named “hessence” which plays the role of quintom. In the hessence model the so-called internal motion $\hat{\theta}$ where $\theta$ is the internal degree of freedom of
hessence plays a phantom like role and the phantom divide transitions is also possible. The Lagrangian density of the hessence is given by

$$\mathcal{L}_h = \frac{1}{2} [\partial_\mu \phi]^2 - \phi^2 (\partial_\mu \theta)^2 - V(\phi).$$

The pressure and energy density for the hessence model are given by

$$p_h = \frac{1}{2} (\dot{\phi}^2 - \phi^2 \dot{\theta}^2) - V(\phi),$$

Figs.10-12 show the variations of $V$ against DBI field $\phi$ in the emergent, logamediate and intermediate scenarios respectively. Solid, dash and dotted lines represent $k = -1, +1, 0$ respectively.
\[
\rho_h = \frac{1}{2}(\dot{\phi}^2 - \phi^2 \dot{\bar{\phi}}^2) + V(\phi),
\]
(70)

with
\[
Q = a^3 \phi^2 \dot{\bar{\phi}} = \text{constant},
\]
(71)

where \(Q\) is the total conserved charge, \(\phi\) is the hessence scalar field and \(V\) is the corresponding potential.

From above we get,
\[
\dot{\phi}^2 - \frac{Q^2}{\phi^2} = -(1 + w_m) \rho_m + \frac{1}{4\pi G} \left( -\dot{H} + \frac{\xi - 1}{T_1} H + \frac{k}{a^2} \right),
\]
(72)

and
\[
V = \frac{1}{2} (w_m - 1) \rho_m + \frac{1}{8\pi G} \left( \dot{H} + 3H^2 + \frac{5(\xi - 1)}{T_1} H + \frac{2k}{a^2} \right).
\]
(73)

- For emergent scenario, we get the expressions for \(\phi\) and \(V\) as
\[
\dot{\phi}^2 - \frac{Q^2}{\phi^2} = -(1 + w_m) \rho_0 \, e^{-3(1 + w_m + \delta)} \left( \frac{\lambda}{\lambda + e^{\mu T_1}} \right)^{3n(1 + w_m + \delta)} + \frac{1}{4\pi G} \left\{ -n \lambda \mu e^{\mu T_1} \left( \frac{\lambda + e^{\mu T_1}}{\lambda + e^{\mu T_1}} \right)^{2n} \left( 1 + \frac{\xi - 1}{T_1} - \frac{k}{a_0^2} \right) \right\},
\]
(74)

and
\[
V = \frac{(w_m - 1) \rho_0}{2} \, e^{-3(1 + w_m + \delta)} \left( \frac{\lambda}{\lambda + e^{\mu T_1}} \right)^{3n(1 + w_m + \delta)} + \frac{1}{8\pi G} \left\{ n \lambda \mu e^{\mu T_1} \left( \frac{\lambda + e^{\mu T_1}}{\lambda + e^{\mu T_1}} \right)^{2n} \left( 1 + \frac{\xi - 1}{T_1} - \frac{k}{a_0^2} \right) \right\}.
\]
(75)

- For logamediate scenario, we get the expressions for \(\phi\) and \(V\) as
\[
\dot{\phi}^2 - \frac{Q^2 e^{-6A (\ln T_1)^o}}{\phi^2} = -(1 + w_m) \rho_0 \, e^{-3A(1 + w_m + \delta)(\ln T_1)^o} + \frac{1}{4\pi G} \left\{ \frac{A_0}{T_1^2} (\ln T_1)^{\alpha - 2}(1 - \alpha + \xi \ln T_1) + k \, e^{-2A(\ln T_1)^o} \right\}
\]
(76)

and
\[
V = \frac{(w_m - 1) \rho_0}{2} \, e^{-3A(1 + w_m + \delta)(\ln T_1)^o} + \frac{1}{8\pi G} \left[ \frac{A_0}{T_1^2} (\ln T_1)^{\alpha - 2}\{\alpha - 1 + (5\xi - 6) \ln T_1 + 3A\alpha(\ln T_1)^o \} + 2k \, e^{-2A(\ln T_1)^o} \right].
\]
(77)

- For intermediate scenario, we get the expressions for \(\phi\) and \(V\) as
\[
\dot{\phi}^2 - \frac{Q^2 e^{-6B T_1^3}}{\phi^2} = -(1 + w_m) \rho_0 \, e^{-3B(1 + w_m + \delta)T_1^3} + \frac{1}{4\pi G} \left\{ B\beta(\xi - \beta)T_1^{\beta - 2} + k \, e^{-2B T_1^{3}} \right\},
\]
(78)

and
\[
V = \frac{(w_m - 1) \rho_0}{2} \, e^{-3B(1 + w_m + \delta)T_1^3} + \frac{1}{8\pi G} \left[ B\beta T_1^{\beta - 2}(5\xi + \beta + 3B \beta T_1^{3}) + 2k \, e^{-2B T_1^{3}} \right].
\]
(79)
Figs. 13-15 show the variations of $V$ against hessence field $\phi$ in the emergent, logamediate and intermediate scenarios respectively. Red, green and blue lines represent $k = -1, +1, 0$ respectively.

For interacting hessence dark energy, figure 13 shows increase in the potential with scalar field and figures 14 and 15 show decay in the potential with scalar field. This means the potential for interacting hessence increases in the emergent universe and decays in logamediate and intermediate scenarios.

### F. Dilaton Field

The energy density and pressure of the dilaton dark energy model are given by \[^{27}\]

\[
\rho_d = -X + 3Ce^{\lambda\phi}X^2, \tag{80}
\]

and

\[
p_d = -X + Ce^{\lambda\phi}X^2, \tag{81}
\]
where $\phi$ is the dilaton scalar field having kinetic energy $X = \frac{1}{2} \dot{\phi}^2$, $\lambda$ is the characteristic length which governs all non-gravitational interactions of the dilaton and $C$ is a positive constant.

We get,

$$\phi = \int \left[ \frac{1}{2} (3w_m - 1) \rho_m + \frac{3}{8\pi G} \left( \dot{H} + 2H^2 + \frac{3(\xi - 1)}{T_1} H + \frac{k}{a^2} \right) \right]^\frac{1}{2} dT_1. \tag{82}$$

• For emergent scenario, we have

$$\phi = \int \left[ \frac{(3w_m - 1)\rho_0 a_0^{-3(1+w_m+\delta)}}{2(\lambda + e^{\mu T_1})^{3n(1+w_m+\delta)}} + \frac{3}{8\pi G} \left\{ \frac{n\mu^2 e^{\mu T_1} (\lambda + 2 ne^{\mu T_1})}{(\lambda + e^{\mu T_1})^2} + \frac{3(\xi - 1)n \mu e^{\mu T_1}}{T_1 (\lambda + e^{\mu T_1})} + \frac{k a_0^{-2}}{(\lambda + e^{\mu T_1})^{2n}} \right\} \right]^\frac{1}{2} dT_1. \tag{83}$$

• For logamediate scenario, we get

$$\phi = \int \left[ \frac{3}{8\pi G} \left\{ \frac{A\alpha}{T_1^2} (\ln T_1)^{\alpha - 2} (\alpha - 1 + (3\xi - 4) \ln T_1 + 2A\alpha (\ln T_1)^\alpha) + k e^{-2A(\ln T_1)^\alpha} \right\} \right.$$

$$\left. + \frac{1}{2} (3w_m - 1) \rho_0 e^{-3A(1+w_m+\delta)(\ln T_1)^\alpha} \right]^\frac{1}{2} dT_1. \tag{84}$$

• For intermediate scenario, we get

$$\phi = \int \left[ \frac{1}{2} (3w_m - 1) \rho_0 e^{-3B(1+w_m+\delta)T_1^\beta} + \frac{3}{8\pi G} \left\{ B\beta (3\xi + \beta - 4 + 2B\beta T_1^\beta) T_1^{\beta - 2} + k e^{-2B T_1^\beta} \right\} \right]^\frac{1}{2} dT_1. \tag{85}$$

For interacting dilaton field, the scalar field $\phi$ always increases with cosmic time $T_1$ irrespective of the scenario of the universe we consider. This is displayed in figures 16, 17 and 18 for emergent, logamediate and intermediate scenarios respectively.

G. Yangs-Mills Dark Energy

Recent studies suggest that Yang-Mills field can be considered as a useful candidate to describe the dark energy as in the normal scalar models the connection of field to particle physics models has not been clear so far and the weak energy condition cannot be violated by the field. In the effective Yang Mills Condensate (YMC) dark energy model, the effective Yang-Mills field Lagrangian is given by [34],

$$\mathcal{L}_{YMC} = \frac{1}{2} bF(\ln \left| \frac{F}{K^2} \right| - 1), \tag{86}$$
Figs. 16-18 show the variations of dilaton field $\phi$ against time $T_1$ in the emergent, logamediate and intermediate scenarios respectively. Red, green and blue lines represent $k = -1, +1, 0$ respectively.

where $K$ is the re-normalization scale of dimension of squared mass, $F$ plays the role of the order parameter of the YMC where $F$ is given by, $F = -\frac{1}{2} F^{\alpha\mu} F_{\alpha\mu} = E^2 - B^2$. The pure electric case we have, $B = 0 \ i.e. F = E^2$.

From the above Lagrangian we can derive the energy density and the pressure of the YMC in the flat FRW spacetime as

$$\rho_y = \frac{1}{2} (y + 1) bE^2,$$  \hspace{1cm} (87)

and

$$p_y = \frac{1}{6} (y - 3) bE^2,$$  \hspace{1cm} (88)

where $y$ is defined as,

$$y = \ln \left| \frac{E^2}{K^2} \right|.$$  \hspace{1cm} (89)
We get,

\[ E^2 = \left[ \frac{1}{2b} (3w_m - 1) \rho_m + \frac{3}{8\pi G b} \left( \dot{H} + 2H^2 + \frac{3(\xi - 1)}{T_1} H + \frac{k}{a^2} \right) \right]. \]  

(90)

• For emergent scenario, we have

\[ E^2 = \left[ \frac{(3w_m - 1) \rho_0 a_0^{-3(1+w_m+\delta)}}{2b (\lambda + e^{\mu T_1})^{3n(1+w_m+\delta)}} + \frac{3}{8\pi b G} \left\{ \frac{n \mu^2 e^{\mu T_1} (\lambda + 2n e^{\mu T_1})}{(\lambda + e^{\mu T_1})^2} + \frac{3(\xi - 1) n \mu e^{\mu T_1}}{T_1 (\lambda + e^{\mu T_1})} + \frac{k a_0^{-2}}{(\lambda + e^{\mu T_1})^{2n}} \right\} \right]. \]  

(91)

• For logamediate scenario, we get

\[ E^2 = \left[ \frac{3}{8\pi b G} \left\{ \frac{A \alpha}{T_1^2} (\ln T_1)^{\alpha - 2} (\alpha - 1 + (3\xi - 4) \ln T_1 + 2A \alpha (\ln T_1)^\alpha) + k e^{-2A (\ln T_1)^\alpha} \right\} \right. 

+ \left. \frac{1}{2b} (3w_m - 1) \rho_0 e^{-3A (1+w_m+\delta)(\ln T_1)^\alpha} \right]. \]  

(92)

• For intermediate scenario, we get

\[ E^2 = \left[ \frac{1}{2b} (3w_m - 1) \rho_0 e^{-3B (1+w_m+\delta) T_1^\beta} + \frac{3}{8\pi b G} \left\{ B \beta (3\xi + \beta - 4 + 2B \beta T_1^\beta) T_1^{\beta - 2} + k e^{-2B T_1^\beta} \right\} \right]. \]  

(93)

When we consider Yang-Mills dark energy, we find that \( E^2 \) is always increasing with cosmic time \( T_1 \). This is displayed in figures 19, 20 and 21 for emergent, logamediate and intermediate scenarios respectively.

V. CONCLUSION

This paper is dedicated to the study of reconstruction of scalar fields and their potentials in a newly developed model of Fractional Action Cosmology by Rami [3]. The fields that we used are quintessence, phantom, tachyonic, k-essence, DBI-essence, Hessence, dilaton field and Yang-Mills field. We assumed that these fields interact with the matter. These fields are various options to model dark energy which is varying in density and pressure, so called variable dark energy. Different field models possess various advantages and disadvantages. The reconstruction of the field potential involves solving the Friedmann equations in the FAC model with the standard energy densities and pressures of the fields, thereby solving for the field and the potential. For simplicity, we expressed these complicated expressions explicitly in time dependent form. We plotted these expressions in various figures throughout the paper.

In plotting the figures for various scenarios, we choose the following values: Emergent scenario: \( \xi = .6, n = 4, \lambda = 8, \mu = .4, a_0 = .7, G = 1 \) (all DE models); Logamediate: \( \xi = .6, \alpha = 3, A = 5, G = 1 \) (all DE models);
Figs.19-21 show the variations of $E^2$ against time $T_1$ in the emergent, logamediate and intermediate scenarios respectively. Red, green and blue lines represent $k = -1, +1, 0$ respectively.

Intermediate: $\xi = .6$, $\beta = .4$, $B = 2$, $G = 1$ (all DE models). Moreover in all cases $\delta = .05$, $w_m = .01$. In figures 1 to 3, we show the variations of $V$ against $\phi$ in the emergent, logamediate and intermediate scenarios respectively for phantom and quintessence field. In the first two cases, the potential function is a decreasing function of the field. For the quintessence field, the potential is almost constant while for the phantom field, the potential increases for different field values. Figures (4-6) show the variations of $V$ against $\phi$ in the emergent, logamediate and intermediate scenarios respectively for the tachyonic field. In figure 4, the $V-\phi$ plot for normal tachyon and phantom tachyon models of dark energy is presented for emergent scenario of the universe. Potential of normal tachyon exhibits decaying pattern. However, it shows increasing pattern for phantom tachyonic field $\phi$. It happens irrespective of the curvature of the universe. In the logamediate scenario (figure 5) the potentials for normal tachyon and phantom tachyon exhibit increasing and decreasing behavior respectively with increase in the scalar field $\phi$. From figure 6 we see a continuous decay in the potential for normal tachyonic field in the intermediate scenario. However, in this scenario, the behavior of the potential varies with the curvature of
the universe characterized by interacting phantom tachyonic field. For $k = -1, 1$, the potential increases with phantom tachyonic field and for $k = 0$, it decays after increasing initially.

Similarly figures (7-9) show the reconstructed potentials for the k-essence field. We have seen that for interacting k-essence the potential $V$ always decreases with increase in the scalar field $\phi$ in all of the three scenarios and it happens for open, closed and flat universes. When we consider an interacting DBI-essence dark energy, we get decaying pattern in the $V-\phi$ plot for emergent and intermediate scenarios in the figures 10 and 12. However, from figure 11 we see an increasing plot of $V-\phi$ for for interacting DBI-essence in the logamediate scenario. For interacting hessence dark energy, figures 13 shows increase in the potential with scalar field and figures 14 and 15 show decay in the potential with scalar field. This means the potential for interacting hessence increases in the emergent universe and decays in logamediate and intermediate scenarios. Figures (16-18) discuss the dilaton field while figures (19-21) show the behavior of the Yang-Mills field in the FAC. For interacting dilaton field, the scalar field $\phi$ always increases with cosmic time $T_1$ irrespective of the scenario of the universe and when we consider Yang-Mills dark energy, we find that $E^2$ in always increasing with cosmic time $T_1$.

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