TWO–TWISTOR DESCRIPTION OF MEMBRANE

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We describe $D = 4$ twistorial membrane in terms of two twistorial three–dimensional world–volume fields. We start with the $D$–dimensional $p$–brane generalizations of two phase space string formulations: the one with $p + 1$ vectorial fourmomenta, and the second with tensorial momenta of $(p + 1)$–th rank. Further we consider tensionful membrane case in $D = 4$. By using the membrane generalization of Cartan–Penrose formula we express the fourmomenta by spinorial fields and obtain the intermediate spinor–space–time formulation. Further by expressing the world–volume dreibein and the membrane space–time coordinate fields in terms of two twistor fields one obtains the purely twistorial formulation. It appears that the action is generated by a geometric three–form on two–twistor space. Finally we comment on higher–dimensional ($D > 4$) twistorial $p$–brane models and their superextensions.

PACS numbers: 11.25.-w, 11.10.Ef, 11.30.Pb

I. INTRODUCTION

Since long time the idea of twistor space (see e. g. [1]) as describing the basic geometric arena for the physical phenomena is tested in various ways. In particular it is known that massless relativistic particles can be equivalently described by free one–twistor particle model (see e. g. [2, 3, 4, 5]); further it can be shown that massive relativistic particles with spin require in twistorial approach the two–twistor space $\mathbb{CP}^3$ providing a breakthrough in extending the twistor construction to non–self–dual field theories. Important contribution to the twistor programme is also provided by the proof that large class of perturbative amplitudes in $N = 4$ $D = 4$ supersymmetric YM theory and conformal supergravity can be derived by using tensionless superstring moving in supertwistor space. In such approach (see e. g. [15, 16]) the twistorial classical string is described by two–dimensional $CP(3|4)$ $\sigma$–model with (twisted) $N = 2$ world sheet SUSY. The correspondence with the space–time picture was derived in [15] on the level of quantized super-twistor string, providing twistor superstring field theory. By assuming the topological CS action for the (Euclidean) twistor superstring field one can reduce all the superstring excitations to the massless spectrum describing SUSY YM and supergravity theories. On the other hand recently in $D = 4$ the close link between the tensionful string model (Nambu–Goto action) and twistor geometry has been proposed on the classical level, with target space–time string coordinates composite in terms of twistor string fields [17, 18]. It appears that in such a model the twistorial target space in bosonic $D = 4$ string model consists of a pair of twistor string coordinates.

Recently in $M$–theory the (super)strings have lost their privileged role as the candidates for the Theory of Everything, and higher–dimensional $p$–branes ($p > 1$), in particular membranes, were considered. In this paper we shall show that analogously to the result presented in [17] one can introduce in two–twistor target space the purely twistorial membrane action. In the intermediate spinor–space–time formulation with fundamental Lorentz spinors and space–time coordinates we shall get the membrane action which can be also linked with the $p$–brane description in the framework of spinorial harmonics ([19]; see also [20, 21]). It should be added that the presence of cosmological term in Polyakov type action for membrane permits to obtain the purely twistorial model without the use of gauge fixing procedure, which was a necessary step in string case [17].

For simplicity we shall describe our composite membrane models in detail in $D = 4$, and without supersymmetric extension. Taking however into consideration that the multidimensional and supersymmetric extensions of twistors applied to twistorial string formulations has been already considered (see for example [15, 16, 22, 23]), we believe that this paper can be also useful e. g. for the consideration of the composite “$M$–theoretic” $D = 11$ supermembrane.

In order to achieve our goal firstly in Sect. 2 we shall show how to extend from string to $p$–brane the two known phase space formulations of bosonic string theory: the one using vectorial momenta fields [24, 25] and the second using tensorial momenta fields (see e. g. [26, 27, 28, 29]). These two formulations are based on the following two Liouville ($p + 1$)–forms which define the $p$–brane momenta

∗Supported by KBN grant 1 P03B 01828
1 One should add that unconventional application of one–twistor geometry to massive particles has been proposed in [13].
fields:

a) Vectorial momenta model:

\[ \Theta^{(1)} = p_\mu \, dX^\mu \rightarrow \Theta^{(p+1)} = P_\mu \wedge dX^\mu \]

where \( P_\mu \) is the following \( p \)-form

\[ P_\mu = P_{\mu}^m d\xi^n \wedge \ldots \wedge d\xi^p. \]

It is easy to see that if \( p = 1 \) we obtain the Siegel formula \([21]\) for the string momenta one form

\[ P_\mu = P_p^m d\xi^n \wedge \ldots \wedge d\xi^p. \]

b) Tensorial momenta model:

\[ \Theta^{(1)} = p_\mu dX^\mu \rightarrow \Theta^{(p+1)} = P_{\mu_1 \ldots \mu_{p+1}} dX^{\mu_1} \ldots \wedge dX^{\mu_{p+1}}. \]

If \( p = 1 \) one obtains the tensorial string momenta

\[ P_{\mu \nu} = -P_{\nu \mu} \quad (24, 23); \text{ for arbitrary } p \text{ see } (24). \]

In Sect. 2 we shall consider \( p \)-branes with arbitrary \( p \) and in arbitrary space–time dimension \( D \). We shall show that both phase space formulations using the momenta fields \([2] \) or \([3]\) are equivalent to the \( p \)-brane Dirac–Nambu–Goto action \([31, 32]\) as well as to the \( \sigma \)-model action \((p\)-brane extension of Howe–Tucker–Polyakov action for strings and membranes \([32, 33, 34]\)). Further, in Sect. 3 we consider for the case \( p = 2 \) and \( D = 4 \) (four-dimensional membrane) the intermediate spinor–space–time models for both phase space formulations. It appears that in such a model the spinors are constrained. \(^2\) In Sect. 4 we shall consider purely twistorial action for \( D = 4 \) membrane. We show that in this action the Lagrangian density is described by a canonical twistorial three–form. In Sect. 5 we present an outlook: we comment on the description of twistorial membranes in higher dimensions, their supersymmetric extensions and present the remarks about the purely twistorial \( p \)-branes \((p > 2)\).

II. TWO PHASE SPACE FORMULATIONS OF THE TENSIONFUL \( p \)-BRANE IN \( D \)-DIMENSIONAL SPACE–TIME

The tensionful \( p \)-brane propagating in flat Minkowski space is described by the nonlinear Dirac–Nambu–Goto action \(^3\)

\[ S = -T \int d^{p+1} \xi \sqrt{-g} \]

where \( \xi^m = (\tau, \sigma^1, \ldots, \sigma^p) \) are the world–volume coordinates,

\[ g \equiv \det(g_{mn}) \]

and

\[ g_{mn} = \partial_m X^\mu \partial_n X_\mu \]

is the induced metric on the \((p+1)\)-dimensional \( p \)-brane volume, \( T \) is the \( p \)-brane tension. In the case \( p = 1 \) the action \([1]\) is the Nambu–Goto action for string \([31]\) whereas if \( p = 2 \) the action \([5]\) is the Dirac action for relativistic membrane \([30]\).

Because the twistor coordinates replace the standard phase space variables, the transition to twistorial formulation should be imposed on the Hamiltonian–like formulations. In the case of tensionful \( p \)-branes \([4]\) there are known two Hamiltonian descriptions.

A. Phase space formulation with vectorial momenta

Let us use firstly the vectorial momenta defined by the formula \([2]\). The corresponding action of the tensionful \( p \)-brane looks as follows \([22]\)

\[ S = \int d^{p+1} \xi \left[ P_\mu^m \partial_m X^\mu + \frac{1}{2 T} (h)^{-1/2} h_{mn} P_\mu^m P_\nu^n + \frac{T}{2} (p - 1) (-h)^{1/2} \right]. \]

We note that last ‘cosmological’ term in the action is absent if \( p = 1 \) (string case).

Let us write down the equation of motion obtained after varying the world–volume metric \( h_{mn} \). Using \( \delta h = hh^{mn} \delta h_{mn} = -h h_{mn} \delta h_{mn} \) one gets

\[ P_\mu^m P_\mu^n - \frac{1}{2} h^{mn} \left[ h_{kl} P^k_\mu P^l_\mu + (p - 1) h \right] = 0. \]

Further expressing \( P_\mu^m \) in the action \([8]\) by its equation of motion \((h_{mn} h_{kn} = \delta^m_n)\)

\[ P_\mu^m = -T (-h)^{1/2} h^{mn} \partial_n X_\mu \]

one obtains the \( \sigma \)-model action for the \( p \)-brane of the Howe–Tucker–Polyakov type \(^4\)

\[ S = \frac{-T}{2} \int d^{p+1} \xi (-h)^{1/2} \left[ h^{mn} \partial_m X^\mu \partial_n X_\mu - (p - 1) \right]. \]

\(^2\) We propose alternative way of generating constraints in comparison with the framework of Lorentz harmonic approach \([13, 24, 21]\).

\(^3\) The indices \( m, n = 0, 1, \ldots, p \) are vector world–sheet indices; \( \mu, \nu = 0, 1, \ldots, D - 1 \) is vector space–time ones. We use the following flat metrics: \( \eta^{ab} = (-, +, \ldots, +), \eta^{\mu \nu} = (-, +, \ldots, +) \).

\(^4\) The string actions as \( d = 2 \) world sheet gravity interacting with string coordinate fields were originally proposed in \([32]\), where as well the description of spinning string using interacting \( d = 2 \) supergravity is presented.
where the variables $h^{mn}$ can be treated as independent ones. The equations of motions for $h^{mn}$ give
\[
\partial_m X^\mu \partial_n X_\mu - \frac{1}{2} h_{mn} \left[ h^{kl} \partial_k X^\mu \partial_l X_\mu - (p - 1) \right] = 0
\]
and lead to (using $h^{mn} h_{mn} = p + 1$)
\[
h_{mn} = g_{mn}
\]
where $g_{mn}$ is the induced metric (11) on the $p$–brane volume.\(^5\) After substitution of (12) in (11) we obtain after simple algebraic calculation the action (9).

It can be shown that from the action (8) one can derive the $p + 1$ Virasoro constraints, generating the world–volume diffeomorphisms. We shall divide world–volume indices $m, n = 0, 1, \ldots, p$ into one with zero value and remaining $m, n = 1, \ldots, p$, that is $m = (0, \tilde{m})$. The equations (10) lead to expression of the ‘auxiliary space $h_{\mu \nu}$ and the last term in the Lagrangian (16) can be written as follows
\[
- \frac{T \sqrt{h}}{2} h_{\mu \nu} \left[ \partial_\mu X^\sigma \partial_\nu X_\sigma \right] = - \frac{T \sqrt{h}}{2} \nabla_p \det(g_{\mu \nu}) .
\]

If the equations (12) are valid we get
\[
\det(h_{\mu \nu}) \det(g_{\mu \nu}) = 1
\]
and the last term in the Lagrangian (16) can be written as follows
\[
\frac{T}{2} (p - 1) \sqrt{-h} = \frac{T}{2} (p - 1) \sqrt{-h} \det(h_{\mu \nu}) \det(g_{\mu \nu}) .
\]

Inserting the expressions (17), (19) in the Lagrangian (16) we obtain the following standard form of the Lagrangian density in first order formalizm
\[
\mathcal{L} = P_\mu X^\mu - \frac{\sqrt{h}}{2T} \det(h_{\mu \nu}) \left[ \frac{1}{T} P_\mu P^\mu + T \det(g_{\mu \nu}) \right] - h_{\mu \nu} h_{\mu \nu} \left( P_\mu \partial_\nu X^\mu \right).
\]

We see that the formula (20) describes the set of $p + 1$ Virasoro constraints
\[
H_0 = \frac{1}{T} P_\mu P^\mu + T \det(g_{\mu \nu}) \approx 0 , \quad (21)
\]
\[
H_{\mu \nu} = P_\mu \partial_\nu X^\mu \approx 0
\]
with the Lagrange multipliers which are some nonlinear functions of the world–volume metric $h_{\mu \nu}$.

Let us finally deduce from the relations (9) the $p$–brane mass–shell condition. Multiplying (9) by $h_{\mu \nu}$ we get for $p > 1$
\[
h_{\mu \nu} P_\mu P^\nu = -(p + 1) h T^2
\]
or
\[
\frac{1}{2T} (-h)^{-1/2} h_{\mu \nu} P_\mu P^\nu = \frac{1}{2} (p + 1) (-h)^{1/2} T.
\]
In string case ($p = 1$) the contraction of l. h. s. of (9)
with $h_{mn}$ is identically vanishing and in any space–time dimension the string condition [29] is absent.

### B. Phase space formulation with tensorial momenta

Other phase space formulation of the $p$–brane [5] is the model with tensorial momenta. It is obtained by the use [6] total antisymmetric tensors $\epsilon^{m_1 \cdots m_{p+1}}$, $\epsilon_{m_1 \cdots m_{p+1}}$ and $\epsilon^{a_1 \cdots a_{p+1}}$, $\epsilon_{a_1 \cdots a_{p+1}}$ have the components $\epsilon^{01 \cdots p + 1} = 1$, $\epsilon_{01 \cdots p + 1} = -1$.

The formula (29) is very useful if we wish to consider $p$–brane (5) is the $p$–brane counterpart of the Liouville $(p + 1)$–form [4]. Such a formulation is directly related with the interpretation of $p$–branes as describing the dynamical $(p + 1)$–dimensional world volume elements described by the following $(p + 1)$–forms [6].

The $p$–brane action with tensorial momenta looks as follows [29]

$$S = \frac{2}{\sqrt{(p+1)!}} \int d^{p+1} \xi \left[ P_{\mu_1 \cdots \mu_{p+1}} \Pi^{\mu_1 \cdots \mu_{p+1}} - \Lambda \left( P_{\mu_1 \cdots \mu_{p+1}} P^{\mu_1 \cdots \mu_{p+1}} + T^2 \right) \right].$$

(26)

where

$$\Pi^{\mu_1 \cdots \mu_{p+1}} \equiv \epsilon^{m_1 \cdots m_{p+1}} \partial_{m_1} X^{\mu_1} \cdots \partial_{m_{p+1}} X^{\mu_{p+1}}.$$  

(27)

Expressing $P_{\mu_1 \cdots \mu_{p+1}}$ by its equation of motion, we get

$$P_{\mu_1 \cdots \mu_{p+1}} = \frac{1}{2 \Lambda} \Pi^{\mu_1 \cdots \mu_{p+1}}.$$  

(28)

After substituting (28) in the action (26) we obtain the 2($p + 1$)-th order action

$$S = \frac{1}{2 \sqrt{(p+1)!}} \int d^{p+1} \xi \left[ \Lambda^{-1} \Pi^{\mu_1 \cdots \mu_{p+1}} \Pi_{\mu_1 \cdots \mu_{p+1}} - \Lambda T^2 \right].$$

(29)

Eliminating the auxiliary field $\Lambda$ we obtain

$$S = -\frac{T}{\sqrt{(p+1)!}} \int d^{p+1} \xi \sqrt{-\Pi^{\mu_1 \cdots \mu_{p+1}} \Pi_{\mu_1 \cdots \mu_{p+1}}}. \quad (30)$$

The determinant of the matrix (7) is given by the formula

$$\det(g_{mn}) = \frac{1}{(p+1)!} \epsilon^{m_1 \cdots m_{p+1} n_1 \cdots n_{p+1}} g_{m_1 n_1} \cdots g_{m_{p+1} n_{p+1}} = \frac{1}{(p+1)!} \Pi^{m_1 \cdots m_{p+1}} \Pi_{m_1 \cdots m_{p+1}}.$$  

(31)

We see that the action (30) is classically equivalent to the action (13).

The formula (26) is very useful if we wish to consider for any $p$ the tensionless limit $T \to 0$. We obtain the formula [29, 37, 38]

$$S_{T=0} = \frac{1}{2 \sqrt{(p+1)!}} \int d^{p+1} \xi \frac{1}{\Lambda} \Pi^{\mu_1 \cdots \mu_{p+1}} \Pi_{\mu_1 \cdots \mu_{p+1}}.$$  

(32)

The formula (32) describes the $p$–brane counterpart of Brink–Schwarz action for massless particle [39].
where we used Tr($\rho^m \rho^m$) = $2h_{mn}$.

Let us recall the condition (24) which is valid for $p > 1$ i.e. also for membrane. In order to get consistency of (24) and (33) we should introduce the following constraint on spinors $\lambda$

$$A \equiv (\lambda \lambda)(\ddot{\lambda} \dddot{\lambda}) - 2T^2 = 0 \quad (35)$$

(we use notations $(\lambda \lambda) \equiv (\lambda^a \lambda_a)$, $(\ddot{\lambda} \dddot{\lambda}) \equiv (\ddot{\lambda}^a \dddot{\lambda}_a)$; note that $\ddot{\lambda}^a \dddot{\lambda}_a = \dddot{\lambda}^a \dddot{\lambda}_a$). Putting (23) and (33) in (8) and imposing via Lagrange multiplier the constraint (35) we obtain the action

$$S = \int d^3 \xi \left[ e \left( \ddot{\lambda}^a \rho^m \lambda_a \partial_m X^{\alpha a} + 2T \right) + LA \right] \quad (36)$$

which provides the intermediate spinor–space–time formulation of the membrane. Let us observe that the action (36) is invariant under the following Abelian local gauge transformation

$$\lambda'_{ai} = e^{\gamma} \lambda_{ai} \quad (37)$$

with real local parameter $\gamma(\xi)$. By fixing the gauge (37) we can replace one real constraint (35) by the following pair of constraints $^9$

$$(\lambda \lambda) = (\ddot{\lambda} \dddot{\lambda}) = \sqrt{2} T . \quad (38)$$

B. Formulation with tensorial momenta

Let us consider the general action which has the following form

$$S = \int d^3 \xi \left[ e c^m Q^a_m + 2T \right] \quad (39)$$

where $Q^a_m = Q^a_m (X, \lambda)$ do not depend on $e^a_m$. Using the relations

$$e = -\frac{1}{3!} \epsilon_{mkn} e_{abc} e^a_m e^b_n e^c_k , \quad e c^m = -\frac{1}{2} \epsilon_{mkn} e_{abc} e^a_m e^b_n e^c_k$$

we obtain the following equation of motion for $e^m_a$

$$e^m_a = -\frac{1}{T} Q^a_m . \quad (40)$$

Subsequently the action (39) takes the following classically equivalent form

$$S = -\frac{1}{2\sqrt{6}} \int d^3 \xi \epsilon_{abc} \epsilon^{mkn} Q^a_m Q^b_n Q^c_k . \quad (41)$$

Choosing in the action (36)

$$Q^a_m = (\ddot{\lambda}^a \rho^m \lambda_a) \partial_m X^{\alpha a} \quad (42)$$

and after supplementing the constraint (35) one gets our membrane action (36). Inserting the formula (10) we obtain

$$S = \frac{2}{\sqrt{6}} \int d^3 \xi \left( P_{\alpha \dot{\alpha} \beta \beta \gamma \gamma} \epsilon^{mkn} \partial_m X^{\alpha a} \partial_n X^{\beta \beta} \partial_k X^{\gamma \gamma} + \Lambda A \right) \quad (43)$$

where tensorial momenta are composites in term of fundamental spinors $^11$

$$P_{\mu \nu \lambda} = P_{\alpha \dot{\alpha} \beta \beta \gamma \gamma} P^\alpha_{\alpha \dot{\alpha} \beta \beta \gamma \gamma} = \frac{2}{\sqrt{6}} \epsilon_{abc} \epsilon_{\alpha \beta \gamma} \epsilon_{\dot{\alpha} \dot{\beta} \dot{\gamma}} \epsilon_{\gamma \gamma} . \quad (44)$$

We see that the intermediate spinor–space–time action (43) with composite tensorial momenta is obtained after the elimination of dreibein variables $e^a_m$. Taking into account the relation

$$(\ddot{\lambda}^a \rho^m \lambda_a) (\ddot{\lambda}^b \rho^m \lambda_b) = \delta^a_b T^2 , \quad (45)$$

following from (35), and $\epsilon_{abc} \epsilon_{\alpha \beta \gamma} = -3!$ we can show easily that the tensor (44) satisfies the membrane mass shell condition (compare with the constraint in the action (20))

$$P_{\mu \nu \lambda} P_{\mu \nu \lambda} = P_{\alpha \dot{\alpha} \beta \beta \gamma \gamma} P^\alpha_{\alpha \dot{\alpha} \beta \beta \gamma \gamma} = -\frac{T^2}{4} . \quad (46)$$

We see that in the action (36) the condition (40) follows from the constraint (35).

IV. PURELY TWISTORIAL FORMULATION OF THE MEMBRANE ($p = 2$) IN $D = 4$ SPACE–TIME

Further we introduce second half of twistor coordinates $\mu^a$, $\bar{\mu}^a$ by postulating the Penrose incidence relations generalized for $D = 4$ membrane fields

$$\mu^a = X^{\hat{\alpha} \alpha} \lambda_{ai} , \quad \bar{\mu}^a = \bar{\lambda}^i \lambda^{\hat{a} \alpha} . \quad (47)$$

We shall rewrite the action (36) by taking into account the relations (17). Using the relations (45), (47) we obtain

$$P_{\alpha \dot{\alpha} \beta \beta \gamma \gamma} \partial_m X^{\alpha a} = \epsilon \ddot{\lambda}^a \rho^m \lambda_a \partial_m X^{\alpha a} \quad (48)$$

$$= \frac{1}{2} \epsilon^m_a \left( \ddot{\lambda}^a \rho^b \partial_m \mu^b - \bar{\mu}^a \rho^b \partial_m \lambda_b \right) + c.c.$$ 

If we introduce the four–component twistors ($A = 1, \cdots , 4$)

$$Z_{Ai} = (\lambda_{ai}, \mu^a), \quad Z^{Ai} = (\bar{\mu}^a, -\bar{\lambda}^a) . \quad (49)$$
the relations (18) takes the form
\[ P^{m}_{\alpha i} \partial_{m} X^{\hat{\alpha} \alpha} = e \tilde{\lambda}_{\hat{a}} \rho^{a} \lambda_{i} \partial_{m} X^{\hat{\alpha} \alpha} \]
= \[ \frac{1}{2} e e_{m}^{a} \left( \partial_{m} \tilde{Z}^{A} \rho^{a} Z_{A} - \bar{Z}^{A} \rho^{a} \partial_{m} Z_{A} \right) \] (50)

Incidence relations (17) with real space–time membrane position field \( X^{\hat{a} \alpha} \) imply that the twistor field variables satisfy the constraints
\[ V_{i}^{j} \equiv \lambda_{ai} \hat{a}^{i j} - \rho_{i}^{\hat{a}} \hat{A}_{\alpha} \approx 0 \] (51)
which can be rewritten equivalently
\[ V_{i}^{j} = Z_{A i} \tilde{Z}_{A j} \approx 0 . \] (52)

We obtain the following membrane action (36) in twistor formulation with dreibein
\[ S = \int d^{3} \xi \left[ \frac{1}{2} e e_{a}^{m} \left( \partial_{m} \tilde{Z}^{A} \rho^{a} Z_{A} - \bar{Z}^{A} \rho^{a} \partial_{m} Z_{A} \right) + \right. \]
+ \[ 2 e T + \Lambda A + \Lambda_{j}^{i} V_{i}^{j} \] (53)
where \( A \) and \( \Lambda_{j}^{i} \) are the Lagrange multipliers. If we define the asymptotic twistors \( [1, 0] \)
\[ I^{AB} = \left( \begin{array}{cc} \epsilon_{\alpha} \beta & 0 \\ 0 & 0 \end{array} \right) , \quad I_{AB} = \left( \begin{array}{cc} 0 & 0 \\ 0 & \epsilon_{\alpha} \beta \end{array} \right) \] (54)
one can introduce the following notation
\[ (\lambda \lambda) \equiv \chi_{\alpha i} \chi_{ai} = \epsilon_{ij} I^{AB} Z_{A i} Z_{B j} \equiv (ZZ) \] (55)
\[ (\tilde{\lambda} \tilde{\lambda}) \equiv \tilde{\lambda}_{\hat{a}}^{i} \tilde{\lambda}_{\hat{a}}^{j} = \epsilon_{ij} I_{AB} \tilde{Z}^{A i} \tilde{Z}_{B j} \equiv (\tilde{Z} \tilde{Z}) \] (56)
and write down the fourlinear constraint (55) in the following twistorial form:
\[ A \equiv (ZZ)(\tilde{Z} \tilde{Z}) - 2 T^{2} = 0 . \] (57)

We shall eliminate the dreibein \( e_{m}^{a} \) by employing the formula (10). The action (53) correspond to the choice
\[ Q_{m}^{a} = \frac{1}{2} \left( \partial_{m} \tilde{Z}_{A} \rho^{A} Z_{A} - \bar{Z}^{A} \rho^{a} \partial_{m} Z_{A} \right) . \] (58)

One gets the final action depending only on two twistorial fields \( Z_{A i}(\tau, \sigma^{1}, \sigma^{2}) \) and suitably rescaled (in comparison with (53)) the Lagrange multipliers \( \Lambda, \Lambda_{j}^{i} \):
\[ S = - \frac{1}{256 \pi} \int d^{3} \xi \left[ \epsilon_{abc} \epsilon_{m n k} \left( \partial_{n} \tilde{Z}^{A} \rho^{a} Z_{A} - \bar{Z}^{A} \rho^{a} \partial_{n} Z_{A} \right) \left( \partial_{m} \tilde{Z}^{B} \rho^{b} Z_{B} - \bar{Z}^{B} \rho^{b} \partial_{m} Z_{B} \right) \left( \partial_{k} \tilde{Z}^{C} \rho^{c} Z_{C} - \bar{Z}^{C} \rho^{c} \partial_{k} Z_{C} \right) + \right. \]
+ \[ \Lambda A + \Lambda_{j}^{i} V_{i}^{j} \] (59)

The model (59) describes the \( D = 4 \) membrane in purely twistorial formulation. Introducing three one–forms with world-volume–vectorial index
\[ \Theta_{(1)}^{a} \equiv d \tilde{Z}^{A} \rho^{a} Z_{A} - \bar{Z}^{A} \rho^{a} dZ_{A} \] (60)
one can obtain the action (59) as induced on the membrane world volume by the following three–form
\[ \Theta_{(3)} = \epsilon_{abc} \Theta_{(1)}^{a} \wedge \Theta_{(1)}^{b} \wedge \Theta_{(1)}^{c} . \] (61)

V. OUTLOOK

In this paper we presented the new description of the twistorial membrane in \( D = 4 \) space–time. We would like now to comment on two generalizations:

i) to \( p \)–branes with \( p > 2 \) in arbitrary \( D \)–dimensional \( (D > p + 1) \) space–time

ii) to super–\( p \)–branes in higher dimensions.
lementary spinor components $\lambda_{\alpha i}$ by imposing consistently Majorana-, Weyl- or Majorana–Weyl conditions.

Inserting (22) in the action (8) we shall obtain the intermediate spinor–space–time formulation. Due to the mass-shell for the vectorial momenta (see (23)) the elementary spinors $\lambda_{\alpha i}$ will be constrained (compare with (35) and (45)). In order to get the purely twistorial formulation of $p$–branes one has to introduce $D$–dimensional incidence relation (47) which provides the doubling of spinor components and lifts the Lorentz spinors to twistors. It should be stress however that in general case the $D$–dimensional incidence relation will introduce extended $D$–dimensional space–time. Only suitable use of the additional spinor structures (e. g. quaternionic in $D = 6$) and imposition of the algebraic constraints in twistor space (e. g. selecting only null twistors lying on null hyperplanes) permits to obtain the incidence relations just with the Minkowski space–time coordinates.

The extension of the twistorial formalism for bosonic $p$–branes to super–$p$–branes requires the introduction of $p$–brane supertwistors. The techniques of supersymmetrization of various twistorial $p$–brane models were already studied (see e. g. [22, 23]). It should be also mentioned that our construction can be linked to the analysis based on the use of Lorentz harmonics [19] as well as with the formalism using $d = 11$ BPS preons [40, 41] described by generalized $OSp(1|64)$ supertwistor fields.

Acknowledgments

S.F. would like to thank Institute for Theoretical Physics, Wroclaw University for kind hospitality and a very friendly creative atmosphere. He would like to thank Bogoliubov–Infeld program for financial support. The work of S.F. was partially supported also by the RFBR grant 06-02-16684 and the grant INTAS-05-7928.

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