Global existence and boundedness in a fully parabolic attraction-repulsion chemotaxis system with signal-dependent sensitivities without logistic source

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Abstract. This paper deals with the fully parabolic attraction-repulsion chemotaxis system with signal-dependent sensitivities,

\[
\begin{align*}
  u_t &= \Delta u - \nabla \cdot (u \chi(v) \nabla v) + \nabla \cdot (u \xi(w) \nabla w), \quad x \in \Omega, \ t > 0, \\
  v_t &= \Delta v - v + u, \quad x \in \Omega, \ t > 0, \\
  w_t &= \Delta w - w + u, \quad x \in \Omega, \ t > 0
\end{align*}
\]

under homogeneous Neumann boundary conditions and initial conditions, where \( \Omega \subset \mathbb{R}^n \) (\( n \geq 2 \)) is a bounded domain with smooth boundary, \( \chi, \xi \) are functions satisfying some conditions. Global existence and boundedness of classical solutions to the system with logistic source have already been obtained by taking advantage of the effect of logistic dampening (J. Math. Anal. Appl.; 2020;489;124153). This paper establishes existence of global bounded classical solutions despite the loss of logistic dampening.

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1. Introduction

Recently, in [1], we studied the fully parabolic attraction-repulsion chemotaxis system with signal-dependent sensitivities and logistic source,

\begin{equation}
\begin{aligned}
  u_t &= \Delta u - \nabla \cdot (u\chi(v)\nabla v) + \nabla \cdot (u\xi(w)\nabla w) + \mu u(1 - u), \\
  v_t &= \Delta v - v + u, \\
  w_t &= \Delta w - w + u,
\end{aligned}
\end{equation}

where \( \chi, \xi \) are some decreasing functions and \( \mu > 0 \). In the literature we obtained global existence and boundedness in (1.1) by taking advantage of the effect of the logistic term. In light of this result, the following question is raised:

Does boundedness of solutions still hold without logistic term?

In order to provide an answer to the above question, this paper focuses on the fully parabolic attraction-repulsion chemotaxis system with signal-dependent sensitivities,

\begin{equation}
\begin{aligned}
  u_t &= \Delta u - \nabla \cdot (u\chi(v)\nabla v) + \nabla \cdot (u\xi(w)\nabla w), & x \in \Omega, \ t > 0, \\
  v_t &= \Delta v - v + u, & x \in \Omega, \ t > 0, \\
  w_t &= \Delta w - w + u, & x \in \Omega, \ t > 0, \\
  \nabla u \cdot \nu &= \nabla v \cdot \nu = \nabla w \cdot \nu = 0, & x \in \partial \Omega, \ t > 0, \\
  u(x, 0) &= u_0(x), \ v(x, 0) = v_0(x), \ w(x, 0) = w_0(x), & x \in \Omega,
\end{aligned}
\end{equation}

where \( \Omega \subset \mathbb{R}^n (n \geq 2) \) is a bounded domain with smooth boundary \( \partial \Omega \); \( \nu \) is the outward normal vector to \( \partial \Omega \); \( \chi, \xi \) are positive known functions; \( u, v, w \) are unknown functions. The initial data \( u_0, v_0, w_0 \) are supposed to be nonnegative functions satisfying

\begin{equation}
\begin{aligned}
  u_0 &\in C^0(\overline{\Omega}) \quad \text{and} \quad u_0 \neq 0, \\
  v_0, w_0 &\in W^{1,\infty}(\Omega).
\end{aligned}
\end{equation}

We now explain the background of (1.2). Chemotaxis is a property of cells to move in response to the concentration gradient of a chemical substance produced by the cells. The origin of the problem of describing such biological phenomena is the chemotaxis model proposed by Keller–Segel [7]. The above systems (1.1) and (1.2) describe a process by which cells move in response to a chemotactant and a chemorepellent produced by the cells themselves. There are a lot of studies for such attraction-repulsion chemotaxis systems, and we introduce some of them, by reducing parameters to 1, as follows.

As to the system (1.1) with constant sensitivities (i.e., \( \chi(v) \equiv \chi, \xi(w) \equiv \xi \), where \( \chi, \xi > 0 \) are constants), global existence and boundedness were obtained in [3, 4, 5, 6]. More precisely, Jin–Wang [5] investigated the one-dimensional case. Also, when \( \chi = \xi \), Jin–Liu [4] studied the two- and three-dimensional cases. Recently, Jin–Wang [6] derived global existence and boundedness, and stabilization under the condition \( \xi \geq C \) with some \( C > 0 \) in the two-dimensional setting. In this way, boundedness is well established in the system (1.1) with logistic term.
On the other hand, the system (1.2) with \( \chi(v) \equiv \chi, \, \xi(w) \equiv \xi \) (positive constants) has been studied in [8, 9]. In the two-dimensional setting, Liu–Tao [9] established global existence and boundedness under the condition \( \chi < \xi \). In contrast, Lankeit [8] showed finite-time blow-up in the three-dimensional radial setting when \( \chi > \xi \). Moreover, some related works which deal with the parabolic–elliptic–elliptic version of (1.1) can be found in [11, 12, 13, 14]. Specifically, Tao–Wang [12] derived global existence and boundedness under the condition \( \chi < \xi \) in two or more space dimensions, while finite-time blow-up was proved in the two-dimensional setting when \( \chi > \xi \) and the initial data satisfy the conditions that \( \int_{\Omega} u_0 > \frac{8\pi}{\chi - \xi} \) and that \( \int_{\Omega} u(x)|x - x_0|^2 \, dx \) \((x_0 \in \Omega)\) is sufficiently small. Salako–Shen [11] obtained global existence and boundedness when \( \chi = 0 \) (or \( \mu > \chi - \xi + M \) with some \( M > 0 \) in (1.1)). Whereas in the two-dimensional setting, finite-time blow-up was shown by Yu–Guo–Zheng [14] and lower bound of blow-up time was given by Viglialoro [13] under the condition \( \chi > \xi \). Thus we can see that if there is no logistic term, boundedness breaks down in some cases.

As mentioned above, the logistic term seems helpful to derive boundedness in (1.1), whereas it is not clear whether boundedness in (1.2) without logistic term holds or not. The purpose of this paper is to establish a result on boundedness in (1.2) with extra information about the outcome by our previous work.

We now introduce conditions for the functions \( \chi, \xi \) in order to state the main theorem. We assume throughout this paper that \( \chi, \xi \) satisfy the following conditions:

\[
\begin{align*}
\chi & \in C^{1+\theta_1}([\eta_1, \infty)) \cap L^1(\eta_1, \infty) \quad (0 < \exists \theta_1 < 1), \quad \chi > 0, \quad (1.5) \\
\xi & \in C^{1+\theta_2}([\eta_2, \infty)) \cap L^1(\eta_2, \infty) \quad (0 < \exists \theta_2 < 1), \quad \xi > 0, \quad (1.6) \\
\exists \chi_0 > 0; & \quad s\chi(s) \leq \chi_0 \quad \text{for all } s \geq \eta_1, \quad (1.7) \\
\exists \xi_0 > 0; & \quad s\xi(s) \leq \xi_0 \quad \text{for all } s \geq \eta_2, \quad (1.8) \\
\exists \alpha > 0; & \quad \chi'(s) + \alpha|\chi(s)|^2 \leq 0 \quad \text{for all } s \geq \eta_1, \quad (1.9) \\
\exists \beta > 0; & \quad \xi'(s) + \beta|\xi(s)|^2 \leq 0 \quad \text{for all } s \geq \eta_2, \quad (1.10)
\end{align*}
\]

where \( \eta_1, \eta_2 \geq 0 \) are constants which will be fixed in Lemma 2.2; note that if \( v_0, w_0 > 0 \), then we can take \( \eta_1, \eta_2 > 0 \) (see also (2.1) with \( z_0 > 0 \)). Moreover, we suppose that \( \alpha, \beta \) appearing in the conditions (1.9), (1.10) satisfy

\[
\alpha > \frac{n}{2}(2\delta + 1)(n - 1)\delta + n + \sqrt{D}, \quad \beta > n + \sqrt{\frac{n}{2}}, \quad (1.11)
\]

for some \( \delta \in J := (\beta - n - \sqrt{(\beta - n)^2 - \frac{n}{2}}, \beta - n + \sqrt{(\beta - n)^2 - \frac{n}{2}}) \), where

\[
D := \frac{n\delta}{2}
\left(2\beta + (2n - 1)\delta \left[2\delta(\beta - n) + \frac{n}{2} \left(2n(\delta + 1)^2 - (2\delta + 1)^2\right)\right]\right).
\]

Then the main result reads as follows.
Theorem 1.1. Let $\Omega \subset \mathbb{R}^n (n \geq 2)$ be a bounded domain with smooth boundary. Assume that $(u_0, v_0, w_0)$ satisfy (1.3), (1.4). Suppose that $\chi, \xi$ fulfill (1.5)–(1.10) with $\alpha, \beta$ which satisfy (1.11) for some $\delta \in J$. Then there exists a unique triplet $(u, v, w)$ of nonnegative functions

$$u \in C^0(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty)),
$$
$$v, w \in C^0(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty)) \cap L^\infty(0, \infty; W^{1,\infty}(\Omega)),$$

which solves (1.2) in the classical sense. Moreover, the solution $(u, v, w)$ is bounded uniformly-in-time:

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{W^{1,\infty}(\Omega)} + \|w(\cdot, t)\|_{W^{1,\infty}(\Omega)} \leq C$$

for all $t > 0$ with some $C > 0$.

Remark 1.1. The condition (1.11) is identical to that in the previous literature, although the logistic term disappears.

The strategy for the proof of Theorem 1.1 is to show $L^p$-boundedness of $u$ with some $p > \frac{n}{2}$. The key is to derive the differential inequality

$$\frac{d}{dt} \int_\Omega u^p f(x, t) \, dx \leq c_1 \int_\Omega u^p f(x, t) \, dx - c_2 \left( \int_\Omega u^p f(x, t) \, dx \right)^{\frac{1}{\theta}} + c_3$$

with some constants $c_1, c_2, c_3 > 0$, $\theta \in (0, 1)$ and some function $f$ defined by using $v, w$.

In our previous work including the logistic term $\mu u(1 - u)$, the differential inequality

$$\frac{d}{dt} \int_\Omega u^p f(x, t) \, dx \leq c_1 \int_\Omega u^p f(x, t) \, dx - \mu p |\Omega|^{-\frac{1}{p}} \left( \int_\Omega u^p f(x, t) \, dx \right)^{1 + \frac{1}{p}}$$

was established. The second term on the right-hand side of this inequality, which is important in proving boundedness, is derived from the effect of the logistic term. In other words, in the absence of a logistic source, the proof in the previous work fails and we cannot obtain boundedness. Thus we will show the differential inequality (1.12) with the help of the diffusion term rather than the logistic term.

This paper is organized as follows. In Section 2 we collect some preliminary facts about local existence of classical solutions to (1.2) and a lemma such that an $L^p$-estimate for $u$ with some $p > \frac{n}{2}$ implies an $L^\infty$-estimate for $u$. Section 3 is devoted to the proof of global existence and boundedness (Theorem 1.1).

2. Preliminaries

In this section we give some lemmas which will be used later. We first present the result obtained by a similar argument in [2, Lemma 2.2] (see also [10, Lemma 2.1 and Remark 2.2]), which will be applied to the second and third equations in (1.2).
Lemma 2.1. Let $T > 0$. Let $u \in C^0(\overline{\Omega} \times [0, T))$ be a nonnegative function such that, with some $m > 0$, $\int_\Omega u(\cdot, t) = m$ for all $t \in [0, T)$. If $z_0 \in C^0(\overline{\Omega})$, $z_0 \geq 0$ in $\overline{\Omega}$ and $z \in C^0(\overline{\Omega} \times [0, T)) \cap C^{2,1}(\overline{\Omega} \times (0, T))$ is a classical solution of
\[
\begin{aligned}
  z_t &= \Delta z - z + u, \quad x \in \Omega, \ t > 0, \\
  \nabla z \cdot \nu &= 0, \quad x \in \partial \Omega, \ t > 0, \\
  z(x, 0) &= z_0(x), \quad x \in \Omega,
\end{aligned}
\]
then for all $t \in (0, T)$,
\[
\inf_{x \in \Omega} z(x, t) \geq \eta
\]
with
\[
\eta := \sup_{\tau > 0} \left( \min_{x \in \Omega} \left\{ e^{-2\tau} \min_{x \in \Omega} z_0(x), c_0 m (1 - e^{-\tau}) \right\} \right) \geq 0,
\]
where $c_0 > 0$ is a lower bound for the fundamental solution of $\varphi_t = \Delta \varphi - \varphi$ with Neumann boundary condition.

We next introduce a result on local existence of classical solutions to (1.2).

Lemma 2.2. Let $n \geq 1$ and let $(u_0, v_0, w_0)$ fulfill (1.3), (1.4). Put $m_0 := \int_\Omega u_0$ and let $\eta_1, \eta_2 \geq 0$ be constants given by (2.1) with $(z_0, m) = (v_0, m_0)$ and $(z_0, m) = (w_0, m_0)$, respectively. Assume that $\chi \in C^{1+\theta_1}(\overline{\eta_1, \infty}), \xi \in C^{1+\theta_2}(\overline{\eta_2, \infty})$ with some $\theta_1, \theta_2 \in (0, 1)$. Then there exists $T_{\text{max}} \in (0, \infty]$ such that (1.2) admits a unique classical solution $(u, v, w)$ such that
\[
\begin{aligned}
  u &\in C^0(\overline{\Omega} \times [0, T_{\text{max}})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\text{max}})), \\
  v, w &\in C^0(\overline{\Omega} \times [0, T_{\text{max}})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\text{max}})) \cap L^\infty_{\text{loc}}([0, T_{\text{max}}); W^{1,\infty}(\Omega))
\end{aligned}
\]
and $u$ has positivity as well as the mass conservation property
\[
\int_\Omega u(\cdot, t) = \int_\Omega u_0
\]
for all $t \in (0, T_{\text{max}})$, whereas $v$ and $w$ satisfy the lower estimates
\[
\inf_{x \in \Omega} v(x, t) \geq \eta_1, \quad \inf_{x \in \Omega} w(x, t) \geq \eta_2
\]
for all $t \in (0, T_{\text{max}})$. Moreover,
\[
\text{if } T_{\text{max}} < \infty, \text{ then } \limsup_{t \nearrow T_{\text{max}}} \left( \|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{W^{1,\infty}(\Omega)} + \|w(\cdot, t)\|_{W^{1,\infty}(\Omega)} \right) = \infty.
\]

Proof. Using a standard argument based on the contraction mapping principle as in [12, Lemma 3.1], we can show local existence and blow-up criterion (2.4). Note that the mass conservation property (2.2) can be obtained by integrating the first equation in (1.2) over $\Omega \times (0, t)$ for $t \in (0, T_{\text{max}})$, and that the lower estimates (2.3) follow from Lemma 2.1. \qed
In the following we assume that $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) is a bounded domain with smooth boundary, $\chi, \xi$ fulfill (1.5), (1.6), respectively, $(u_0, v_0, w_0)\) satisfies (1.3), (1.4). Then we denote by $(u, v, w)$ the local classical solution of (1.2) given in Lemma 2.2 and by $T_{\text{max}}$ its maximal existence time.

We next give the following lemma which tells us a strategy to prove global existence and boundedness.

**Lemma 2.3.** Assume that $\chi, \xi$ fulfill that $\chi(s) \leq K_1$, $\xi(s) \leq K_2$ for all $s \geq 0$ with some $K_1, K_2 > 0$, respectively. If there exist $K_3 > 0$ and $p > \frac{n}{2}$ satisfying

$$\|u(\cdot, t)\|_{L^p(\Omega)} \leq K_3$$

for all $t \in (0, T_{\text{max}})$, then we have

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{W^{1, \infty}(\Omega)} + \|w(\cdot, t)\|_{W^{1, \infty}(\Omega)} \leq C$$

for all $t \in (0, T_{\text{max}})$ with some $C > 0$.

*Proof.* See [1, Lemma 2.3]; note that it is rather easier to show without logistic term. \qed

### 3. Proof of Theorem 1.1

Thanks to Lemma 2.3, it is sufficient to derive an $L^p$-estimate for $u$ with some $p > \frac{n}{2}$. In order to establish the estimate for $u$ we introduce the function $f = f(x, t)$ by

$$f(x, t) := \exp \left( -r \int_{\eta_1}^{v(x,t)} \chi(s) \, ds - \sigma \int_{\eta_2}^{w(x,t)} \xi(s) \, ds \right), \quad (3.1)$$

where $r, \sigma > 0$ are some constants which will be fixed later. Here the function $f$ is finite valued, because integrability in (3.1) is assured by (1.5) and (1.6) together with (2.3).

Then we give the following lemma which was proved in [1, Lemma 3.2] with $\mu = 0$. Although in the literature we used the function $f$ with $\eta_1 = \eta_2 = 0$, the conclusion of the following lemma does not depend on the choice of $(\eta_1, \eta_2)$.

**Lemma 3.1.** Let $r, \sigma > 0$. Then for all $p > 1$, we have

$$\frac{d}{dt} \int_\Omega u^p f = I_1 + I_2 + I_3 - r \int_\Omega u^p f \chi(v)(-v + u) - \sigma \int_\Omega u^p f \xi(w)(-w + u) \quad (3.2)$$

for all $t \in (0, T_{\text{max}})$, where

$$I_1 := p \int_\Omega u^{p-1} f \nabla \cdot \left( \nabla u - u \chi(v) \nabla v + u \xi(w) \nabla w \right),$$

$$I_2 := -r \int_\Omega u^p f \chi(v) \Delta v,$$

$$I_3 := -\sigma \int_\Omega u^p f \xi(w) \Delta w.$$
We next state an estimate for $I_1 + I_2 + I_3$ in the following lemma.

**Lemma 3.2.** Let $r, \sigma > 0$, $\varepsilon \in [0, 1)$ and put

$$x := u^{-1} |\nabla u|, \quad y := \chi(v) |\nabla v|, \quad z := \xi(w) |\nabla w|.$$  

Then for all $p > 1$, the following estimate holds:

$$I_1 + I_2 + I_3 \leq -\varepsilon p(p - 1) \int_{\Omega} u^p f x^2$$  

$$+ \int_{\Omega} u^p f \cdot (a_1(\varepsilon)x^2 + a_2 xy + a_3xz + a_4 y^2 + a_5yz + a_6 z^2) \quad (3.3)$$

for all $t \in (0, T_{\text{max}})$, where

$$a_1(\varepsilon) := -(1 - \varepsilon)p(p - 1), \quad a_2 := p(p + 2r - 1),$$  

$$a_3 := p(p + 2\sigma - 1), \quad a_4 := -r(p + r + \alpha),$$  

$$a_5 := pr + p\sigma + 2r\sigma, \quad a_6 := -\sigma(-p + \sigma + \beta).$$

**Proof.** Noting that $-\varepsilon p(p - 1) + a_1(\varepsilon) = -p(p - 1)$ for all $\varepsilon \in (0, 1)$, we see that the estimate (3.3) is almost all the same as that in the case $\varepsilon = 0$ except multiplication by constants and is proved in [1, p. 10]. □

We next give the following lemma, which is useful to show that the second term on the right-hand side of (3.3) is nonpositive.

**Lemma 3.3.** Assume that $\alpha, \beta$ satisfy (1.11). Then there exist $p > \frac{n}{2}$ and $r, \sigma > 0$ such that

$$A_1 := \begin{vmatrix} a_1(0) & \frac{a_3}{2} & a_6 \\ \frac{a_1}{2} & a_6 & \frac{a_2}{2} \\ \frac{a_3}{2} & \frac{a_2}{2} & a_4 \end{vmatrix} > 0 \quad \text{and} \quad A_2 := \begin{vmatrix} a_1(0) & \frac{a_3}{2} & a_6 \\ \frac{a_1}{2} & a_6 & \frac{a_2}{2} \\ \frac{a_3}{2} & \frac{a_2}{2} & a_4 \end{vmatrix} < 0. \quad (3.4)$$

**Remark 3.1.** In [1. Proof of Lemma 3.3], we showed that there exist $p > \frac{n}{2}$ and $r, \sigma > 0$ such that $A_1 > 0$, $A_2 \leq 0$. Here, $A_2 \leq 0$ can be refined as $A_2 < 0$ for some $p > \frac{n}{2}$ and $r, \sigma > 0$. More precisely, in the literature we set

$$\varphi_2(r) := c_1 r^2 + c_2 r + c_3$$

with some $c_1, c_2, c_3 > 0$ and find $r > 0$ such that $\varphi_2(r) \leq 0$ by the following two conditions:

$$r_0 > 0, \quad \text{where} \ r_0 \text{ is the axis of the parabola } \varphi_2,$$

$$D_2 > 0, \quad \text{where} \ D_2 \text{ is the discriminant of } \varphi_2.$$

These conditions, indeed, show that $\varphi_2(r) < 0$ for some $r > 0$ which implies to $A_2 < 0$.

Combining the above three lemmas, we can derive the following important inequality which leads to the $L^p$-estimate for $u$.  

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Lemma 3.4. Assume that \( \chi, \xi \) satisfy (1.5)–(1.10) with \( \alpha, \beta \) which fulfill (1.11). Then there exist \( p > \frac{n}{2} \) and \( r, \sigma > 0 \) such that
\[
\frac{d}{dt} \int_{\Omega} u^p f + \varepsilon_0 p (p - 1) \int_{\Omega} u^{p-2} f \vert \nabla u \vert^2 \leq -r \int_{\Omega} u^p \chi (v - u) - \sigma \int_{\Omega} u^p \xi (w - u)
\]
for all \( t \in (0, T_{\text{max}}) \) with some \( \varepsilon_0 \in (0, 1) \).

**Proof.** We put
\[
A_1(\varepsilon) := \begin{vmatrix} a_1(\varepsilon) & a_2 \frac{a_3}{2} & a_4 \\ \frac{a_5}{2} & a_6 \end{vmatrix} \quad \text{and} \quad A_2(\varepsilon) := \begin{vmatrix} a_1(\varepsilon) & a_2 \frac{a_3}{2} & a_4 \\ \frac{a_5}{2} & a_6 \frac{a_7}{2} & a_8 \\ \frac{a_9}{2} & \frac{a_{10}}{2} & a_{12} \end{vmatrix}
\]
for \( \varepsilon \in [0, 1) \). Since \( A_1(0) > 0 \) and \( A_2(0) < 0 \) hold in view of (3.4) and the function \( a_1 : \varepsilon \mapsto -(1 - \varepsilon) p (p - 1) \) is continuous at \( \varepsilon = 0 \), we can find \( \varepsilon_0 \in (0, 1) \) such that \( A_1(\varepsilon_0) > 0 \) and \( A_2(\varepsilon_0) < 0 \). By using the Sylvester criterion, we have
\[
a_1(\varepsilon_0)x^2 + a_2 x y + a_3 x z + a_4 y^2 + a_5 y z + a_6 z^2 \leq 0.
\]
Combining (3.3) and (3.5) with (3.2), we arrive at the conclusion. \( \square \)

We now show the desired \( L^p \)-estimate for \( u \) with some \( p > \frac{n}{2} \).

**Lemma 3.5.** Let \( p > \frac{n}{2} \). Assume that \( \chi, \xi \) satisfy (1.5)–(1.10) with \( \alpha, \beta \) which fulfill (1.11). Then there exists \( C > 0 \) such that
\[
\| u(\cdot, t) \|_{L^p(\Omega)} \leq C
\]
for all \( t \in (0, T_{\text{max}}) \).

**Proof.** By Lemma 3.4, we see from the positivity of \( \chi, \xi \) and (1.7), (1.8) that
\[
\frac{d}{dt} \int_{\Omega} u^p f + \varepsilon_0 p (p - 1) \int_{\Omega} u^{p-2} f \vert \nabla u \vert^2 \leq -r \int_{\Omega} u^p \chi (v - u) - \sigma \int_{\Omega} u^p \xi (w - u)
\]
\[
\leq r \chi_0 \int_{\Omega} u^p f + \sigma \xi_0 \int_{\Omega} u^p f
\]
\[
= (r \chi_0 + \sigma \xi_0) \int_{\Omega} u^p f
\]
for all \( t \in (0, T_{\text{max}}) \) with some \( \varepsilon_0 \in (0, 1) \). Noting \( f \leq 1 \) in view of (3.1) and then using the Gagliardo–Nirenberg inequality together with the mass conservation property (2.2), we have
\[
\int_{\Omega} u^p f \leq \int_{\Omega} u^p = \| u^\frac{p}{2} \|^2_{L^2(\Omega)}
\]
\[
\leq c_1 \left( \| \nabla u \|^2_{L^2(\Omega)} + \| u^\frac{p}{2} \|^2_{L^p(\Omega)} \right)^{2\theta} \| u^\frac{p}{2} \|^2_{L^p(\Omega)}
\]
\[
= c_1 \left( \| \nabla u \|^2_{L^2(\Omega)} + \| u_0 \|^2_{L^1(\Omega)} \right)^{2\theta} \| u_0 \|^2_{L^1(\Omega)}
\]
\[
\leq c_2 \| \nabla u \|^2_{L^2(\Omega)} + c_3
\]
(3.7)
for all \( t \in (0, T_{\text{max}}) \) with some \( c_1, c_2, c_3 > 0 \), where \( \theta := \frac{\eta_1}{\eta_1 + 1 - \frac{n}{2}} \in (0, 1) \). Also, noticing from \( \chi \in L^1(\eta_1, \infty) \), \( \xi \in L^1(\eta_2, \infty) \) (see (1.5), (1.6)) that
\[
f \geq c_4 := \exp \left( -r \int_{\eta_1}^{\infty} \chi(s) \, ds - \sigma \int_{\eta_2}^{\infty} \xi(s) \, ds \right) > 0 \quad \text{on} \quad \Omega \times (0, T_{\text{max}}),
\]
we obtain
\[
\frac{4c_4}{p^2} \| \nabla u^p \|^2_{L^2(\Omega)} = \frac{4c_4}{p^2} \int_{\Omega} |\nabla u^p|^2 \leq \int_{\Omega} u^{p-2} f |\nabla u|^2. \tag{3.9}
\]
for all \( t \in (0, T_{\text{max}}) \). Combining (3.7), (3.9) with (3.6), we see that
\[
\frac{d}{dt} \int_{\Omega} u^p \leq c_5 \int_{\Omega} u^p - c_6 \left( \int_{\Omega} u^p \right)^{\frac{1}{p}} + c_7
\]
for all \( t \in (0, T_{\text{max}}) \) with some \( c_5, c_6, c_7 > 0 \). This provides a constant \( c_8 > 0 \) such that
\[
\int_{\Omega} u^p \leq c_8,
\]
which again by (3.8) implies
\[
\int_{\Omega} u^p \leq \frac{p^2 c_8}{4c_4}
\]
for all \( t \in (0, T_{\text{max}}) \) and thereby precisely arrive at the conclusion.

We are in a position to complete the proof of Theorem 1.1. If \( \chi, \xi \) satisfy (1.5)–(1.10) with \( \alpha, \beta \) fulfilling (1.11), then, according to the relations that \( \chi(s) \leq \chi(\eta_1) \) for all \( s \geq \eta_1 \) and \( \xi(s) \leq \xi(\eta_2) \) for all \( s \geq \eta_2 \) (see (1.9) and (1.10)), a combination of Lemmas 2.3 and 3.5, along with (2.4), leads to the end of the proof.
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