Optimal estimates and asymptotics for the stress concentration between closely located stiff inclusions

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December 3, 2013

Abstract

If stiff inclusions are closely located, then the stress, which is the gradient of the solution, may become arbitrarily large as the distance between two inclusions tends to zero. In this paper we investigate the asymptotic behavior of the stress concentration factor, which is the normalized magnitude of the stress concentration, as the distance between two inclusions tends to zero. For that purpose we show that the gradient of the solution to the case when two inclusions are touching decays exponentially fast near the touching point. We also prove a similar result when two inclusions are closely located and there is no potential difference on boundaries of two inclusions. We then use these facts to show that the stress concentration factor converges to a certain integral of the solution to the touching case as the distance between two inclusions tends to zero. We then present an efficient way to compute this integral.

AMS subject classifications. 35J25, 73C40

Key words. conductivity equation; anti-plane elasticity; stress; gradient blow-up; extreme conductivity; stress concentration factor

1 Introduction and statement of results

In presence of closely located stiff inclusions embedded in the relatively weak matrix, high stress concentration occurs in the narrow region between two inclusions. Such a phenomenon typically occurs in fiber-reinforced materials and the stiff inclusions represent the cross section of fibers. Recently, much effort has been devoted to quantitative understanding of this stress concentration. In this paper we continue our investigation on this and establish an efficient method to compute the magnitude of the stress concentration that immediately yields an asymptotic formula for the stress distribution and an optimal estimate for the concentration.

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*This work is supported by Korean Ministry of Education, Sciences and Technology through NRF grants Nos. 2010-0017532, 2013R1A1A1A05009699 and 2011-0009671.
To describe the problem and results in a precise manner, let $D_0^1$ and $D_0^2$ be a pair of (touching) bounded domains with $C^2 \gamma$ ($\gamma > 0$) boundaries such that

$$D_0^1 \subset \{(x, y) \in \mathbb{R}^2 \mid x < 0\}, \quad D_0^2 \subset \{(x, y) \in \mathbb{R}^2 \mid x > 0\}, \quad (1.1)$$

$$\partial D_0^1 \cap \partial D_0^2 = \{(0, 0)\}, \quad (1.2)$$

and $D_0^1$ and $D_0^2$ are convex at $(0, 0)$. The domains $D_0^1$ and $D_0^2$ are strongly convex at $(0, 0)$ if both $D_0^1$ and $D_0^2$ have positive curvatures there. By translating $D_0^2$ by a positive number $\epsilon$ along $x$-axis, while $D_0^1$ is fixed, we obtain $D_\epsilon^2$, i.e.,

$$D_\epsilon^2 := D_0^2 + (\epsilon, 0). \quad (1.3)$$

When there is no possibility of confusion, we drop superscripts and denote

$$D_1 := D_0^1, \quad D_2 := D_\epsilon^2. \quad (1.4)$$

For a given harmonic function $h$ in $\mathbb{R}^2$, let $u_\epsilon$ be the solution to the problem

$$\begin{cases}
\Delta u_\epsilon = 0 & \text{in } \mathbb{R}^2 \setminus (D_1 \cup D_2), \\
u \epsilon = \lambda_j \text{ (constant)} & \text{on } \partial D_j, \ j = 1, 2, \\
u \epsilon(x) - h(x) = O(|x|^{-1}) & \text{as } |x| \to \infty,
\end{cases} \quad (1.5)$$

where the constants $\lambda_j$ are determined by the conditions

$$\int_{\partial D_1} \partial_n u_\epsilon ds = \int_{\partial D_2} \partial_n u_\epsilon ds = 0. \quad (1.6)$$

Here and throughout this paper $\partial_n u_\epsilon$ denotes the outward normal derivative of $u_\epsilon$ on $\partial D_j$ ($j = 1, 2$). It is worth emphasizing that the constants $\lambda_1$ and $\lambda_2$ may or may not be different depending on the given $h$.

As mentioned before, inclusions $D_1$ and $D_2$ represent the two dimensional cross-sections of two parallel elastic fibers embedded in an infinite elastic matrix and $\epsilon$ is the distance between them. The solution $u_\epsilon$ represents the out-of-plane elastic displacement, and $\nabla u_\epsilon$ is proportional to the shear stress. The problem (1.5) may also be regarded as two dimensional conductivity equation in which case $D_1$ and $D_2$ represent perfect conductors of infinite conductivity. It is worth mentioning that we consider the situation where there are only two inclusions since our interest lies in estimating local high concentration of stress in the narrow region between two inclusions. There is a study to estimate global stress in a composite (with many inclusions) using a network approximation. We refer to [8] and references therein for that. We also mention that the problem under consideration in this paper has some connection with effective properties of composites with highly conducting inclusion. See [25] Section 10.10 for this connection.

In general, $\nabla u_\epsilon$ becomes arbitrarily large as the distance $\epsilon$ between two inclusions tends to zero, and the problem is to derive pointwise estimates of $\nabla u_\epsilon$ in terms of $\epsilon$. This problem was raised in [5] and there has been significant progress on it. It has been proved that the generic blow-up rate of $\nabla u_\epsilon$ is $\frac{1}{\sqrt{\epsilon}}$ in two dimensions [2, 3, 4, 6, 10, 14, 22, 26, 27], and $|\epsilon \ln \epsilon|^{-1}$ in three dimensions [9, 17, 14, 13, 15, 17, 21]. We emphasize that the gradient may or may not blow up depending on the given background harmonic function $h$. For
example, in the configuration of this paper the gradient blows up if \( h(x, y) = x \) and it does not if \( h(x, y) = y \) for circular inclusions \( D_1 \) and \( D_2 \). It is worth while to mention that the insulating case in two dimensions can be treated by duality as done in [4] for example. But the insulating case in three dimensions is an open problem: it is not even clear if the gradient actually blows up in three dimensions. It is also worth while to mention that if the conductivity of the inclusions is finite (away from \( \infty \) and 0), \( \nabla u_\epsilon \) is bounded regardless of \( \epsilon \) [9, 19, 20].

Recently a better understanding of the stress concentration has been obtained: an asymptotic behavior of \( \nabla u_\epsilon \) has been characterized by the singular function associated with \( D_1 \) and \( D_2 \), as \( \epsilon \) tends to 0. The singular function, denoted by \( q_\epsilon \), is the solution to the following problem:

\[
\begin{aligned}
\Delta q_\epsilon &= 0 \quad \text{in } \mathbb{R}^2 \setminus (D_1 \cup D_2), \\
q_\epsilon &= \text{constant} \quad \text{on } \partial D_j, \ j = 1, 2, \\
q_\epsilon(x) &= O(|x|^{-1}) \quad \text{as } |x| \to \infty, \\
\int_{\partial D_1} \partial_\nu q_\epsilon ds &= -1, \\
\int_{\partial D_2} \partial_\nu q_\epsilon ds &= 1.
\end{aligned}
\]  

(1.7)

We emphasize that the constant values of \( q_\epsilon \) on \( \partial D_1 \) and \( \partial D_2 \) are different, so that \( \nabla q_\epsilon \) blows up as \( \epsilon \to 0 \).

Let us recall some important facts about \( q_\epsilon \): If \( D_1^0 \) and \( D_2^0 \) are disks, then \( q_\epsilon \) is given explicitly by

\[
q_\epsilon(x) := \frac{1}{2\pi} \left( \ln |x - p_1| - \ln |x - p_2| \right),
\]  

(1.8)

where \( p_1 \in D_1 \) is the fixed point of the mixed reflection \( R_1 R_2 \) where \( R_j \) is the reflection with respect to \( \partial D_j, \ j = 1, 2 \), and \( p_2 \in D_2 \) is that of \( R_2 R_1 \). We emphasize that these points can be computed easily (see (4.11)). More generally, if \( D_1^0 \) and \( D_2^0 \) are strongly convex at \((0, 0)\), then let \( B_1 \) and \( B_2 \) be disks osculating to \( D_1 \) and \( D_2 \) at \((0, 0)\) and \((\epsilon, 0)\), respectively, and let \( q_{B, \epsilon} \) be the singular function associated with \( B_1 \) and \( B_2 \) as given in (1.8). Then, it is proved in [1] that the behavior of \( \nabla q_\epsilon \) is almost explicitly described as

\[
\nabla q_\epsilon = \nabla q_{B, \epsilon}(1 + O(\epsilon^{\gamma/2})) + O(1)
\]  

(1.9)

when \( \partial D_j \) is \( C^2,\gamma \) and \( \gamma \in (0, 1) \). Thus it follows that

\[
\|\nabla q_\epsilon\|_\infty = \frac{\sqrt{\kappa_1 + \kappa_2}}{\sqrt{2\pi}} \frac{1}{\sqrt{\epsilon}} + O(1),
\]  

(1.10)

where \( \kappa_2 \) and \( \kappa_2 \) are the curvatures of \( D_1^0 \) and \( D_2^0 \) at \((0, 0)\), respectively.

Using the singular function \( q_\epsilon \), the solution \( u_\epsilon \) to (1.5) can be decomposed as

\[
u_\epsilon = \alpha_\epsilon q_\epsilon + r_\epsilon
\]  

(1.11)

where

\[
\alpha_\epsilon = \frac{u_\epsilon|\partial D_2 - u_\epsilon|\partial D_1}{q_\epsilon|\partial D_2 - q_\epsilon|\partial D_1}.
\]  

(1.12)
Observe that \( r \) is also constant on \( \partial D_1 \) and \( \partial D_2 \), and \( r \mid_{\partial D_1} = r \mid_{\partial D_2} \), so that \( \nabla r \) is bounded on bounded subsets of \( \mathbb{R}^2 \setminus (D_1 \cup D_2) \) (see [12]). It means that the term \( \alpha \nabla q \) is responsible for the blow-up of \( \nabla u_\epsilon \), or more precisely,

\[
\nabla u_\epsilon = \alpha \nabla q + O(1) \quad \text{as } \epsilon \to 0.
\]

(1.13)

In particular, \( \alpha \) represents the magnitude (normalized by \( |\nabla q| \)) of the blow-up. So, it is appropriate to call the constant \( \alpha \) the stress concentration factor.

The purpose of this paper is to analyze the stress concentration factor \( \alpha \) when \( D_0 \) and \( D_0^2 \) are convex at \((0,0)\). We are particularly interested in finding \( \lim_{\epsilon \to 0} \alpha \epsilon \) (existence of the limit is a part of the study).

There have been some work on the stress concentration factor. It is proved in [12] that if \( D_0^1 \) and \( D_0^2 \) are disks, then

\[
\alpha = \frac{2r_1r_2}{r_1 + r_2} (\mathbf{n} \cdot \nabla h)(0,0) + O(\epsilon) \quad \text{as } \epsilon \to 0,
\]

(1.14)

where \( r_j \) is the radius of \( D_j \), \( j = 1, 2 \), and \( \mathbf{n} \) is the outward unit normal vector to \( \partial D_1 \) at \((0,0)\). An estimate for \( \alpha \) in terms of curvatures, size and \( \epsilon \) was established in [22] under the assumption that an inclusion has a much higher curvature than its size. It is also proved in [1] that if \( D_0^1 \) and \( D_0^2 \) are strongly convex at \((0,0)\), then

\[
\alpha = \frac{\sqrt{2\pi}}{\sqrt{k_1 + k_2}} \frac{1}{\sqrt{\epsilon}} \int_{\partial D_1 \cup \partial D_2} h \partial_\nu q \kappa \left( 1 + O(\epsilon^{3/2}) \right),
\]

(1.15)

and, as a consequence, that \( \alpha \) is bounded regardless of \( \epsilon \).

Observe that even if (1.9) yields a good information of \( q \) on the narrow region in between two inclusions, it is still difficult to evaluate the integral on the righthand side of (1.15) since it requires global information of \( \partial_\nu q \). In this paper, we present a new efficient method for finding \( \lim_{\epsilon \to 0} \alpha \epsilon \). It turns out that the limit is given as a certain integral of the solution \( u_0 \) for the touching case, namely,

\[
\begin{align*}
\Delta u_0 &= 0 \quad \text{in } \Omega, \\
u_0 &= \lambda_0 \quad \text{on } \partial \Omega, \\
\left(u_0(x) - h(x)\right) &= O(|x|^{-1}) \quad \text{as } |x| \to \infty,
\end{align*}
\]

(1.16)

where \( \Omega := \mathbb{R}^2 \setminus (D_0^1 \cup D_0^2) \) and \( \lambda_0 \) is a constant determined by the additional condition

\[
\int_{\Omega} |\nabla(u_0 - h)|^2 \, dA < \infty.
\]

(1.17)

To obtain the main result of this paper (Theorem 1.1), we first consider the touching case problem (1.16). We show that there exists a unique solution \( u_0 \) to (1.16) and \( \nabla u_0(x, y) \) decays exponentially fast as \((x, y)\) approaches to \((0,0)\) in \( \Omega \) (Theorem 2.1). It is worth mentioning that \( \Omega \) has cusps at the origin. We also prove a similar theorem for the residual part \( r_\epsilon \) in (1.11) (Theorem 3.1). This result was also obtained in [18] in a more general context. However, we include a proof in this paper since it is completely different from that in the paper mentioned above.
We prove these results in somewhat more general setting: We assume that the domains $D_j^0$ and $D_j^0$ are convex at $(0,0)$ and their order of contact at the point is $2m$ for some positive integer $m$. Thus, if $\partial D_j^0$ near $(0,0)$ is given as the graph of $x = x_j(y)$ ($j = 1, 2$), then there are constants $\delta_0 > 0$ and $c_j > 0$, $j = 1, \ldots, 4$, such that

$$- c_1 y^{2m} \leq x_1(y) \leq - c_2 y^{2m} \quad \text{and} \quad c_3 y^{2m} \leq x_2(y) \leq c_4 y^{2m},$$

(1.18)

for $|y| < \delta_0$. If $D_j^0$ and $D_j^0$ are strongly convex at $(0,0)$, then $m = 1$.

In terms of the solution $u_0$ to (1.10) we obtain the following theorem regarding the asymptotic behavior of the stress concentration factor.

**Theorem 1.1** Suppose that $\partial D_j^0$ ($j = 1, 2$) are $C^{2,\gamma}$ for some $\gamma > 0$, $D_1^0$ and $D_2^0$ are convex at $(0,0)$, and their order of contact at $(0,0)$ is $2m$. Let $u_0$ be the solution to (1.10) and let

$$\alpha_0 := \int_{\partial D_j^0} \partial_{\nu} u_0 \, ds.$$  

(1.19)

Then,

$$\alpha = \alpha_0 + O(\epsilon \log \epsilon^{2m-1}) \quad \text{as} \quad \epsilon \to 0.$$  

(1.20)

As an immediate consequence of Theorem 1.1 and (1.13), we obtain

$$\nabla u_\epsilon = \alpha_0 \nabla q_\epsilon + O(1) \quad \text{as} \quad \epsilon \to 0.$$  

(1.21)

in any bounded subset of $\mathbb{R}^2 \setminus (D_1 \cup D_2)$. Thus, the limit $\alpha_0$ can be regarded as an alternative concentration factor. Moreover, if $D_1^0$ and $D_2^0$ are strongly convex at $(0,0)$, then we have from (1.8) and (1.9) that

$$\nabla u_\epsilon(x) = \frac{\alpha_0}{2\pi} \left( \frac{x - p_1}{|x - p_1|^2} - \frac{x - p_2}{|x - p_2|^2} \right) (1 + O(\epsilon^{\gamma/2})) + O(1).$$  

(1.22)

These formulas have some important consequences. As a first consequence, we have the following identity (see section 4 for a proof):

$$\lim_{\epsilon \to 0} \sqrt{\epsilon} |\nabla u_\epsilon(\epsilon/2, 0)| = \frac{\alpha_0 \sqrt{\kappa_1 + \kappa_2}}{\sqrt{2\pi}},$$  

(1.23)

where $\kappa_i$ is the curvature of $\partial D_i^0$ at $(0,0)$ for $i = 1, 2$. Note that $(\epsilon/2, 0)$ is a point where $|\nabla u_\epsilon|$ has a value close to the maximal concentration.

Another consequence of (1.21) and (1.22) is related to numerical computation of $\nabla u_\epsilon$. Since high concentration of the gradient occurs in the narrow region, fine meshes may be required to compute $\nabla u_\epsilon$. However, since (1.21) and (1.22) extract the major singular term in an explicit way, it suffices to compute the residual term $\nabla r_\epsilon$ for which only regular meshes are required. This idea was exploited in [12] in the special case when $D_j$’s are disks using (1.8), (1.11) and (1.13). Implementation of this idea for the general case of strongly convex domains will be the subject of the forthcoming work. It is worth mentioning that there are some other methods to compute the solution when $D_1$ and $D_2$ are disks. See, for examples, [11] [23].

The last subject of this paper is regarding computation of $\alpha_0$. It turns out that, thanks to exponentially decaying property of the solution to the touching case, $\alpha_0$ can
be computed numerically only using regular meshes by truncating the narrow region near 
(0, 0).

This paper is organized as follows. We investigate the touching case in section 2. In
section 3 we obtain an estimate for the gradient of the residual term $r$. Section 4 is to
prove Theorem 1.1. In the last section we present a way to compute good approximations
of $\alpha_0$.

2 The solution for the touching case

In this section we prove the following theorem regarding the problem (1.1).

Theorem 2.1 Suppose that $\partial D^0_j$ $(j = 1, 2)$ are $C^{2, \gamma}$ for some $\gamma > 0$, $D^0_1$ and $D^0_2$ are
convex at $(0, 0)$, and their order of contact at $(0, 0)$ is finite. Then, there is a unique
solution $u_0$ to (1.1), and there are positive constants $A$, $C$ and $\delta$ such that

$$|\nabla u_0(x, y)| \leq C \exp \left( -\frac{A}{|y|} \right)$$

(2.1)

for $|y| \leq \delta$ and $x_1(y) < x < x_2(y)$, where $x_1$ and $x_2$ are the defining functions of $\partial D^0_1$ and $\partial D^0_2$ near $(0, 0)$.

The estimate (2.1) follows from

$$|u_0(x, y) - \lambda_0| \leq C \exp \left( -\frac{A}{|y|} \right)$$

(2.2)

by a standard estimate for harmonic functions. Here the constants $A$ and $C$ may differ
at each occurrence. In fact, since $\partial D^0_j$ are $C^{2, \gamma}$ and $u_0 - \lambda_0 = 0$ on $\partial D^0_j$, one can show
that $u_0(x, y) - \lambda_0$ can be extended by reflection (after the conformal transformations to
outside a disk) as harmonic functions for $(x, y)$ satisfying

$$x_1(y) - sy^{2m} < x < x_2(y)$$

and

$$x_1(y) < x < x_2(y) + sy^{2m}$$

for some $s > 0$, and the same estimate (2.2) holds for the extended functions. Here $2m$ is
the order of contact at $(0, 0)$. So, for each $(x, y) \in \mathbb{R}^2 \setminus (D^0_1 \cup D^0_2)$, there is $r > 0$ such that
$r > ty^{2m}$ for some $t > 0$ and $u_0$ is harmonic in $B_r(x, y)$. ($B_r(c)$ denotes the disk of radius
$r$ with the center at $c$.) So, we have

$$|\nabla u_0(x, y)| \leq \frac{C}{r^3} \int_{B_r(x, y)} |u_0 - \lambda_0| \ dA \leq \frac{C}{y^{6m}} \exp \left( -\frac{A}{|y|} \right) \leq C' \exp \left( -\frac{A'}{|y|} \right)$$

(2.3)

for some constant $A'$ and $C'$, which is the desired estimate.

The rest of this section is devoted to proving existence and uniqueness of $u_0$, and (2.2).
To construct the solution to (1.1) we use the transformation $1/z$, following [24]. Identify
$x = (x, y)$ in the plane with $z = x + iy$ and let

$$\Phi(z) = \frac{1}{z}.$$
Define
\[ \tilde{\Omega} := \Phi(\Omega), \quad \Gamma_1 := \Phi(\partial D_0^1), \quad \Gamma_2 := \Phi(\partial D_0^2). \]
Note that \( \Gamma_1 \) and \( \Gamma_2 \) are simple curves lying in the left and right half spaces, respectively, and \( \tilde{\Omega} \) is the region enclosed by \( \Gamma_1 \) and \( \Gamma_2 \). Since \( \partial D_0^j \) is \( C^{2,\gamma}(\gamma > 0) \) and \( D_0^j \) is convex at \((0,0)\) for \( j = 1, 2 \), one can easily see that there are constant \( a < b \) such that
\[ \tilde{\Omega} \subset \{ w = \xi + i\eta \mid a < \xi < b \}. \quad (2.4) \]
Moreover, \( \Gamma_1 \) near \( \infty \) is given by \( \xi = \psi_1(\eta) \) for some function \( \psi_1 \) satisfying
\[ \psi_1(\eta) \leq -C_1|\eta|^{2-2m} \quad (2.5) \]
for some constant \( C_1 > 0 \), and \( \Gamma_2 \) near \( \infty \) is given by \( \xi = \psi_2(\eta) \) for some function \( \psi_2 \) satisfying
\[ \psi_2(\eta) \geq C_2|\eta|^{2-2m} \quad (2.6) \]
for some constant \( C_2 > 0 \). In fact, we have
\[ \xi = \frac{x_1(y)}{y^2 + x_1(y)^2} \quad \text{with} \quad \eta = \frac{-y}{y^2 + x_1(y)^2} \quad (2.7) \]
on \( \partial D_0^1 \) near \((0,0)\). Thanks to (1.18), we have
\[ a < \psi_1(\eta) \leq -\frac{c_1y^{2m}}{y^2 + x_1(y)^2} \leq -C_1|\eta|^{2-2m}. \]
Thus we have (2.4), (2.6) can be proved similarly.

We need the following lemma whose proof will be given after completing the proof of Theorem 2.1.

**Lemma 2.2** Let \( \psi_j (j = 1, 2) \) be as defined by (2.7), and let \( a \) and \( b \) be the constants such that
\[ a < \psi_1(\eta) < \psi_2(\eta) < b \quad (2.8) \]
for all \( \eta > L \), where \( L \) is a large number. Let \( R \) be a domain given by
\[ R := \{(\xi,\eta) \mid \eta > L, \ \psi_1(\eta) < \xi < \psi_2(\eta)\}, \quad (2.9) \]
and let \( U \) be the solution in \( H^1(R) \) to the problem
\[ \begin{cases} \Delta U = 0 & \text{in } R, \\ U = 0 & \text{on } \xi = \psi_j(\eta), \ j = 1, 2, \\ U = \varphi & \text{on } \Gamma := \{(\xi,\eta) \mid \psi_1(\eta) < \xi < \psi_2(\eta)\}, \end{cases} \quad (2.10) \]
where \( \varphi \) is a bounded function. Then there are positive constants \( A \) and \( C \) such that
\[ |U(\xi,\eta)| \leq Ce^{-A\eta} \quad (2.11) \]
for all \( \eta > L \).
Because of (2.4), the Poincaré inequality holds in $\tilde{\Omega}$: for all $\tilde{u} \in H^1_0(\tilde{\Omega})$ (the standard Sobolev space with the zero trace)

$$\|\tilde{u}\|_{L^2(\tilde{\Omega})} \leq C\|\nabla \tilde{u}\|_{L^2(\tilde{\Omega})}$$

for some constant $C$. So, one can apply the Lax-Milgram Theorem to show that for $f \in H^{-1}(\tilde{\Omega})$ there exists a unique solution $\tilde{v} \in H^1_0(\tilde{\Omega})$ to

$$\begin{cases}
\Delta \tilde{v} = f & \text{in } \tilde{\Omega}, \\
\tilde{v} = 0 & \text{on } \partial\tilde{\Omega}.
\end{cases}$$

(2.12)

We choose $r_0 > 0$ such that $D_{01}^0 \cup D_{02}^0 \subset B_{r_0/2}(0)$.

Let $\chi$ be a smooth function such that $\chi(z) = 1$ if $z \in B_{r_0}(0)$ and $\chi(z) = 0$ if $z \notin B_{2r_0}(0)$.

For $h$ given in (1.16), let

$$f(w) = \Delta_w \left( \chi \left( \frac{1}{w} \right) h \left( \frac{1}{w} \right) \right),$$

and let $\tilde{v}$ be the solution to (2.13) with this $f$. Then one can check that $u_0$ given by

$$u_0(z) = h(z) + \left( \tilde{v} \left( \frac{1}{z} \right) - \chi(z)h(z) - \tilde{v}(0) \right)$$

(2.14)

is the solution to (1.16) and the constant value $\lambda_0$ is given by $-\tilde{v}(0)$. The uniqueness of the solution follows easily from the maximum principle.

Now, we show (2.2). If $z \in B_{r_0}(0) \setminus D_{1}^0 \cup D_{2}^0$, then we have

$$u_0(z) - \lambda_0 = \tilde{v} \left( \frac{1}{z} \right).$$

(2.15)

Choose $L$ so large that the support of $f$ lies in between two lines $\eta = \pm L$. Let $\tilde{\Omega}_{\pm L} := \tilde{\Omega} \cap \{ \pm \eta > L \}$, respectively. The boundedness of $\tilde{v}(\xi \pm (L + 1)i)$ can be shown easily by a standard estimate for harmonic functions similarly to (2.3), since $\tilde{v} = 0$ on $\partial\tilde{\Omega}_{\pm L} \cap \partial\tilde{\Omega}$ and $\tilde{v} \in L^2(\tilde{\Omega})$. We thus apply Lemma 2.2 to obtain

$$|\tilde{v}(\xi + i\eta)| \leq C_1 e^{-A_1|\eta|} \quad \text{for } |\eta| > L + 1$$

(2.16)

for some positive constant $A_1$ and $C_1$. We may choose a small positive number $\delta_1$ so that $\Phi(x, y) \in \tilde{\Omega}_{+(L+1)} \cup \tilde{\Omega}_{-(L+1)}$ for all $x + iy$ satisfying $|y| < \delta_1$ and $x_1(y) < x < x_2(y)$. Then, by (2.15) and (2.16), we have

$$|u_0(x, y) + \tilde{v}(0)| = |\tilde{v}(\xi + i\eta)| \leq C_2 e^{-A_1|\eta|} \leq C_3 e^{-\frac{A_2}{|\eta|^2}},$$

(2.17)

for $|y| < \delta_1$. The last inequality follows from (2.7), since $|x_1(y)| \simeq |y|^{2m}$. Here and throughout this paper, $a \simeq b$ stands for $\frac{1}{C_1}a \leq b \leq Ca$ for some constant $C$ independent of $\epsilon$. This completes the proof of Theorem 2.1. 

□
Proof of Lemma 2.2. By translating and scaling if necessary, we may assume $a = 0$, $b = \pi$ and $L = 0$. Let
\[ \tilde{R} := \{(\xi, \eta) \mid \eta > 0, \ 0 < \xi < \pi\}. \]
Decompose $\varphi$ as $\varphi = \varphi_+ - \varphi_-$ where $\varphi_\pm$ are nonnegative and bounded, and then extend $\varphi_\pm$ to $[0, \pi] \times \{0\}$ by assigning 0 outside $\Gamma$, and denote them by $\tilde{\varphi}_\pm$. Let $V_\pm$ be a solution in $H^1(\tilde{R})$ to
\[
\begin{aligned}
\Delta V_\pm &= 0 \quad \text{in } \tilde{R}, \\
V_\pm(0, \eta) &= V_\pm(\pi, \eta) = 0, \quad \eta > 0, \\
V_\pm(\xi, 0) &= \tilde{\varphi}_\pm(\xi), \quad 0 \leq \xi \leq \pi.
\end{aligned}
\]
Since $\tilde{\varphi}_\pm \geq 0$, we have $V_\pm \geq 0$, and by the maximum principle, we have
\[ -V_- \leq U \leq V_+ \quad \text{in } \tilde{R}. \quad (2.18) \]
One can find the solutions $V_\pm$ by separation of variables. In fact, we have
\[ V_\pm(\xi, \eta) = \sum_{n=1}^{\infty} a_\pm^n \sin n\xi e^{-n\eta}, \]
where $a_\pm^n$ is the Fourier coefficients of $\varphi_\pm$. In particular, we have
\[ |V_\pm(\xi, \eta)| \leq \left( \sum_{n=1}^{\infty} |a_\pm^n|^2 \right)^{1/2} \left( \sum_{n=1}^{\infty} e^{-2n\eta} \right)^{1/2} \leq C e^{-\eta}. \quad (2.19) \]
for $\eta \geq 1$. Even for $0 < \eta < 1$, this inequality holds with another constant $C$ since $\tilde{\varphi}_\pm$ are bounded. Thus, (2.11) follows from (2.18). This completes the proof. □

3 The behavior of $\nabla r_\epsilon$ in the narrow region
In this section, we consider the behavior of the gradient of $r_\epsilon$ given in (1.11) in the narrow region between $D_1$ and $D_2$ which we denote by $N_\delta$ for $\delta > 0$, namely,
\[ N_\delta := \{(x, y) \mid x_1(y) < x < x_2(y) + \epsilon, \ |y| < \delta\}. \quad (3.1) \]
Recall that $r_\epsilon$ satisfies
\[
\begin{aligned}
\Delta r_\epsilon &= 0 \quad \text{in } \Omega := \mathbb{R}^2 \setminus \overline{D_1 \cup D_2}, \\
r_\epsilon|_{\partial D_1} = r_\epsilon|_{\partial D_2} &= \text{constant}, \\
r_\epsilon(x) - h(x) &= O(|x|^{-1}) \quad \text{as } |x| \to \infty.
\end{aligned}
\]
In the previous section, it has been shown that $\nabla u_0$ is decreasing exponentially near origin. The following theorem shows that $\nabla r_\epsilon$ has such a decay property. As mentioned in Introduction, this result was also obtained in [18] in a more general setting. But two proofs are completely different.
**Theorem 3.1** Suppose that $\varepsilon$ is sufficiently small. There are positive constants $A$, $C$ and $\delta$ independent of $\varepsilon > 0$ such that
\[
|\nabla r_{\varepsilon}(x,y)| \leq C \exp\left( -\frac{A}{\sqrt{\varepsilon} + |y|} \right)
\] (3.3)
for any $(x,y) \in N_\delta$.

We prove the following lemma from which Theorem 3.1 follows by a standard elliptic estimate as explained briefly in the previous section.

**Lemma 3.2** Suppose that $\varepsilon$ is sufficiently small. There are positive constants $A$, $C$ and $\delta$ independent of $\varepsilon$ such that
\[
|r_{\varepsilon}(x,y) - r_{\varepsilon}|_{\partial D_1} | \leq C \exp\left( -\frac{A}{\sqrt{\varepsilon} + |y|} \right)
\] (3.4)
for any $(x,y) \in N_\delta$.

**Proof.** As before we identify points $(x,y)$ in $\mathbb{R}^2$ with $z = x + iy$ in $\mathbb{C}$. Choose two disks $B_1$ and $B_2$ whose centers are on the real axis such that $B_j \subset D_j$, $j = 1, 2$, $\partial B_1 \cap \partial D_1 = \{0\}$, and $\partial B_2 \cap \partial D_2 = \{\varepsilon\}$.

Let $c_j$ and $\rho_j$ be the center and radius, respectively, of $B_j$ for $j = 1, 2$. It is convenient to assume that $c_2 = 1 + \varepsilon$ so that $\rho_2 = 1$. Let
\[
\Phi_1(z) = \frac{1}{z - (1 + \varepsilon)}
\] (3.5)
which is the reflection with respect to $\partial B_2$ (and translation), and
\[
\Omega_1 := \Phi_1(\Omega), \quad B_3 := \Phi_1(B_1), \quad B_4 := \Phi_1(B_2),
\] (3.6)
and let $c_j$ and $\rho_j$ be the center and radius of $B_j$. Then, $c_4 = 0$, $\rho_4 = 1$, and $\rho_3 = c_3 - \Phi_1(0)$.

Observe that
\[
\Omega_1 \subset B_4 \setminus B_3,
\] (3.7)
the reflected domain $\Omega_1$ touches $\partial B_3$ and $\partial B_4$ at $\Phi_1(0)$ and $\Phi_1(\varepsilon)$, respectively, and
\[
\text{dist}(\partial B_3, \partial B_4) = \Phi_1(0) - \Phi_1(\varepsilon) = -\frac{1}{1 + \varepsilon} + 1 = \varepsilon + O(\varepsilon^2).
\] (3.8)

Let
\[
S := \{w \mid w = \xi + i\eta \in B_4 \setminus B_3, \; \xi < c_3, \; |\eta| < \rho_3/2\}.
\] (3.9)
Then one can choose $\delta$ independently of $\varepsilon$ so that $\Phi_1(N_\delta) \subset S$. Let
\[
\tilde{r}_\varepsilon(w) := r_\varepsilon \circ \Phi^{-1}_1(w) - r_\varepsilon|_{\partial D_1}, \quad w \in \Omega_1.
\] (3.10)
If $z = x + iy \in N_\delta$ and $w = \xi + i\eta = \Phi_1(z)$, then $\eta \simeq y$. Thus in order to prove (3.4), it suffices to show
\[
|\tilde{r}_\varepsilon(w)| \leq C \exp\left( -\frac{A}{\sqrt{\varepsilon} + |\eta|} \right)
\] (3.11)
for any $w \in \Phi_1(N_\delta)$.

We now transform $B_3$ so that the transformed disk becomes concentric to $B_4$ ($B_4$ is the unit disc). For that purpose let us write a lemma which can be easily verified.
Lemma 3.3 Let $B_\rho(c)$ be a disk such that $\overline{B_\rho(c)} \subset B_1(0)$. Then there is $\alpha$ with $|\alpha| < 1$ and $\rho_\ast > 0$ such that the M"obius transform $\varphi_\alpha$ defined by

$$\varphi_\alpha(w) = \frac{w - \alpha}{1 - \bar{\alpha}w} \tag{3.12}$$

maps $B_\rho(c)$ onto $B_{\rho_\ast}(0)$. In fact, $\alpha$ is given by

$$\alpha = \left[(|c|^2 - \rho^2 + 1) - \sqrt{(|c|^2 - \rho^2 + 1)^2 - 4|c|^2}\right] \frac{c}{2|c|^2}. \tag{3.13}$$

It is worth mentioning that M"obius transforms are automorphisms on $B_1(0)$.

Let $\Phi_2$ be the M"obius transform defined by (3.12) and (3.13) with $c = c_3$ and $\rho = \rho_3$, and let $B_5 = B_{\rho_3}(c_5) := \Phi_2(B_3)$. Then $c_5 = 0$. Since $\rho_3 = c_3 + \frac{1}{1+\epsilon}$, one can see from (3.13) that $\alpha$ is real and satisfies

$$\alpha = -1 + \beta \sqrt{\epsilon} + (1 + \frac{1}{c})\epsilon + O(\epsilon)$$. \tag{3.14}

where

$$\beta = \sqrt{\frac{2(c_3 + 1)}{|c_3|}}.$$  

To compute $\rho_5$, we observe that $\Phi_2(-\frac{1}{1+\epsilon}) \in \partial B_5$, and from (3.14) that

$$\Phi_2(-\frac{1}{1+\epsilon}) = -\frac{1}{1+\epsilon} + \frac{\alpha}{1+\epsilon} = -1 + \gamma \sqrt{\epsilon} + O(\epsilon),$$

where $\gamma = 2/\beta$. So, we have

$$\rho_5 = 1 - \gamma \sqrt{\epsilon} + O(\epsilon). \tag{3.15}$$

We emphasize that (3.15) implies in particular that

$$\text{dist}(\partial B_5, \partial B_4) = \gamma \sqrt{\epsilon} + O(\epsilon), \tag{3.16}$$

since $B_4 = B_1(0)$.

The proof of the following lemma will be given later in this section. Here $\arg(z)$ for $z \neq 0$ is supposed to take a value in $[0, 2\pi)$.

Lemma 3.4 Suppose that $\epsilon$ is sufficiently small. There exists a constant $C > 0$ independent of $\epsilon$ such that

$$\arg(\Phi_2(w)) \geq \frac{C \sqrt{\epsilon}}{|\eta| + \sqrt{\epsilon}} \tag{3.17}$$

for $w = \xi + \eta i \in S$ with $\eta \geq 0$, and

$$2\pi - \arg(\Phi_2(w)) \geq \frac{C \sqrt{\epsilon}}{|\eta| + \sqrt{\epsilon}} \tag{3.18}$$

for $\eta \leq 0$. 

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Let us introduce one more transformation $\Phi_3$:
\[
\Phi_3(\zeta) = \log \zeta
\]  
(3.19)
with the branch cut on the positive real axis. Then $\Phi_3$ maps $(B_4 \setminus B_5) \setminus \{\text{positive real axis}\}$ onto the rectangle $(a_0, 0) \times (0, 2\pi)$ where $a_0 = \log \rho_5 < 0$. We emphasize that
\[
a_0 = -\gamma \sqrt{\epsilon} + O(\epsilon),
\]  
(3.20)
which is a consequence of (3.15).

Let $\theta_0$ be the constant on the righthand side of (3.17) with $\eta = \rho_3/2$, i.e.,
\[
\theta_0 := \frac{C \sqrt{\epsilon}}{\rho_3 \sqrt{\epsilon}}.
\]  
(3.21)

Define $\Phi := \Phi_3 \circ \Phi_2$, and $R_{\theta_0} := (a_0, 0) \times (0, 2\pi - \theta_0)$. Then $\Phi^{-1}(R_{\theta_0}) \cap \Omega_1$ is a bounded subset of $\Omega_1$. Define
\[
\Omega_{\theta_0} := \Phi(\Phi^{-1}(R_{\theta_0}) \cap \Omega_1).
\]  
(3.22)

Then $\Omega_{\theta_0}$ is a connected subset of $R_{\theta_0}$ and has two lateral boundaries denoted by $l_1$ and $l_2$. Let
\[
\tilde{r}_\epsilon(r, \theta) := (\tilde{r}_\epsilon \circ \Phi^{-1})(r, \theta), \quad (r, \theta) \in \Omega_{\theta_0},
\]  
(3.23)
where $\tilde{r}_\epsilon$ is given in (3.10). Then, $\tilde{r}_\epsilon$ satisfies
\[
\begin{align*}
\Delta \tilde{r}_\epsilon &= 0 \quad \text{in } \Omega_{\theta_0}, \\
\tilde{r}_\epsilon &= 0 \quad \text{on } l_1 \cup l_2.
\end{align*}
\]  
(3.24)

We have the following lemma whose proof will be given at the end of this section.

**Lemma 3.5** There is a constant $C$ such that for $(r, \theta) \in \Omega_{\theta_0}$
\[
|\tilde{r}_\epsilon(r, \theta)| \leq C \exp \left( -\frac{\pi}{|a_0|}(\theta - \theta_0) \right) \quad \text{if } \theta \leq \pi
\]  
(3.25)
and
\[
|\tilde{r}_\epsilon(r, \theta)| \leq C \exp \left( -\frac{\pi}{|a_0|}(2\pi - \theta_0 - \theta) \right) \quad \text{if } \theta \geq \pi.
\]  
(3.26)

The desired inequality (3.11) follows from (3.25) and (3.26). To see this, we first observe that if $r + i\theta = \Phi_3 \circ \Phi_2(w)$, then $e^{r + i\theta} = \Phi_2(w)$, in other words, $\theta = \arg \Phi_2(w)$. Because of (3.20) and (3.21), we have $\theta_0/|a_0| \leq C$ for some constant $C$ independent of $\epsilon$ provided that $\epsilon$ is sufficiently small. Observe that if $w = \xi + i\eta \in S$ and $\eta > 0$, then $\theta = \arg \Phi_2(w) < \pi$. So it follows from Lemma 3.4 and (3.25) that
\[
|\tilde{r}_\epsilon(w)| \leq C \exp \left( -\frac{\pi}{|a_0|}\theta \right) \leq C_1 \exp \left( -\frac{A}{\sqrt{\epsilon} + |\eta|} \right).
\]

For $w = u + iv \in S$ with $\eta \leq 0$, $\theta = \arg \Phi_2(w) \geq \pi$. Lemmas 3.4 and 3.5 also yield
\[
|\tilde{r}_\epsilon(w)| \leq C \exp \left( -\frac{\pi}{|a_0|}(2\pi - \theta) \right) \leq C_2 \exp \left( -\frac{A_1}{\sqrt{\epsilon} + |\eta|} \right).
\]
So we have (3.11) and the proof of Lemma 3.2 is completed. □

Let us now prove Lemma 3.4 and Lemma 3.5.

Proof of Lemma 3.4. In this proof, we shall consider the case when \( w = \xi + \eta i \in S \) with \( \eta \geq 0 \) only. We first note that

\[
\text{Im } \Phi_2(w) = \frac{\eta(1 - \alpha^2)}{(1 - \alpha^2)^2 + \alpha^2 \eta^2}.
\]

Using (3.14) one can see that

\[
1 - \alpha^2 \geq C\sqrt{\epsilon}
\]

if \( \epsilon \) is sufficiently small, since \( |\xi| \leq \frac{1}{2}\rho_3 \leq \frac{1}{2} \). We observe that for \( w = \xi + i\eta \in S \),

\[
1 + \xi \leq 1 + c_3 - \sqrt{\rho_3^2 - \eta^2} = 1 + c_3 - \rho_3 + \frac{\eta^2}{\rho_3 + \sqrt{\rho_3^2 - \eta^2}} \leq \epsilon + \frac{\eta^2}{\rho_3};
\]

since \( w \in B_4 \setminus B_3 \) and \( \text{dist}(\partial B_3, \partial B_4) = -\frac{1}{1+\epsilon} + 1 \leq \epsilon \) by (3.8).

If \( \eta \geq \sqrt{\epsilon} \), then

\[
\text{Im } \Phi_2(w) \geq C_1 \frac{\eta \sqrt{\epsilon}}{\epsilon + \eta^2} \geq C_2 \frac{\sqrt{\epsilon}}{\eta}
\]

by (3.14) and the property that \( \frac{\eta}{\rho_3} \leq 1 \). Since \( |\Phi_2(w)| \geq \frac{1}{2} \) by (3.15), we have

\[
\sin(\text{arg } \Phi_2(w)) = \frac{\text{Im } \Phi_2(w)}{|\Phi_2(w)|} \geq 2C_2 \frac{\sqrt{\epsilon}}{\eta}.
\]

Thus we have

\[
\text{arg } \Phi_2(w) \geq C_3 \frac{\sqrt{\epsilon}}{\eta}.
\]

(3.27)

If \( 0 \leq \eta < \sqrt{\epsilon} \), then there exists \( w_0 = \xi_0 + i\eta_0 \) with \( |\eta_0| = \sqrt{\epsilon} \) so that

\[
\text{arg } \Phi_2(w) \geq \text{arg } \Phi_2(w_0),
\]

so it follows from (3.27) that

\[
\text{arg } \Phi_2(w) \geq C_3.
\]

This proves (3.17). □

Proof of Lemma 3.5. By definition, \( \Omega_{\theta_0} \) is a subset of \( R_{\theta_0} \), and \( \partial \Omega_{\theta_0} \cap \partial R_{\theta_0} \) belongs to \( \theta = \theta_0 \) or \( 2\pi - \theta_0 \). We define functions \( \psi_{\pm} \) in \( R_{\theta_0} \) as the solutions to

\[
\begin{align*}
\Delta \psi_{\pm} &= 0 & \text{in } R_{\theta_0}, \\
\psi_{\pm}(r, \theta) &= 0 & \text{on } \partial R_{\theta_0} \setminus \partial \Omega_{\theta_0}, \\
\psi_{\pm}(r, \theta) &= \max \{ \pm \tilde{r}_\epsilon(r, \theta), 0 \} & \text{on } \partial R_{\theta_0} \cap \partial \Omega_{\theta_0}.
\end{align*}
\]

(3.28)

It was shown in [1] that \( ||r_\epsilon - h||_{L^\infty(\Omega)} \) is bounded independently of \( \epsilon \). So, there is a constant \( M \) independent of \( \epsilon > 0 \) such that

\[
|\psi_{\pm}(r, \theta)| \leq M \quad \text{for all } (r, \theta) \in R_{\theta_0},
\]

(3.29)
and it can be shown in the same way as (2.18) in the previous section that
\[ -\psi_- \leq \tilde{r}_e \leq \psi_+ \quad \text{in} \quad \Omega_{\theta_0}. \]
(3.30)
So to prove (3.26) it suffices to show
\[ |\psi_{\pm}(r, \theta)| \leq C \exp \left( -\frac{\pi}{|a_0|} (2\pi - \theta_0 - \theta) \right) \quad \text{for} \quad \theta \in [\pi, 2\pi - \theta_0], \]
(3.31)
We prove (3.31) only for $\psi_+$ since the proof for $\psi_-$ is identical. The solution $\psi_+$ can be found explicitly:
\[ \psi_+ = \psi_+^e + \psi_+^o \]
where $\psi_+^e$ and $\psi_+^o$ are the even and odd parts about $\theta = \pi$ given by
\[ \psi_+^e(r, \theta) = \sum_{n=1}^{\infty} \alpha_n \sin \left( \frac{n\pi}{|a_0|} r \right) \cosh \left( \frac{n\pi}{|a_0|} (\theta - \pi) \right) \]
\[ \psi_+^o(r, \theta) = \sum_{n=1}^{\infty} \beta_n \sin \left( \frac{n\pi}{|a_0|} r \right) \sinh \left( \frac{n\pi}{|a_0|} (\theta - \pi) \right) \]
for some constants $\alpha_n$ and $\beta_n$.
Suppose that $\theta \geq \pi$. Then we have
\[ |\psi_+^e(r, \theta)| \leq \sum_{n=1}^{\infty} |\alpha_n| \exp \left( \frac{n\pi}{|a_0|} (\theta - \pi) \right) \]
\[ \leq 2 \sum_{n=1}^{\infty} |\alpha_n| \cosh \left( \frac{n\pi}{|a_0|} (\pi - \theta_0) \right) \exp \left( -\frac{n\pi}{|a_0|} (2\pi - \theta_0 - \theta) \right) \]
Note that
\[ \left( \sum_{n=1}^{\infty} |\alpha_n|^2 \cosh^2 \left( \frac{n\pi}{|a_0|} (\pi - \theta_0) \right) \right)^{1/2} = \left( \frac{2}{|a_0|} \right)^{1/2} \|\psi_+^e(\cdot, 2\pi - \theta_0)\|_{L^2([a_0, 0])} \leq \sqrt{2} M, \]
since $\|\sin \left( \frac{n\pi}{|a_0|} r \right)\|_{L^2([a_0, 0])} = \left( \frac{1}{2} |a_0| \right)^{1/2}$. So it follows from the Cauchy-Schwarz inequality that
\[ |\psi_+^e(r, \theta)| \leq 2\sqrt{2} M \left( \sum_{n=1}^{\infty} \exp \left( -\frac{2n\pi}{|a_0|} (2\pi - \theta_0 - \theta) \right) \right)^{1/2}, \]
and hence
\[ |\psi_+^e(r, \theta)| \leq C \exp \left( -\frac{\pi}{|a_0|} (2\pi - \theta_0 - \theta) \right) \]
(3.32)
for some constant $C$. Since $\psi_+^e(r, \theta) = \psi_+^e(r, 2\pi - \theta)$,
\[ |\psi_+^e(r, \theta)| \leq C \exp \left( -\frac{\pi}{|a_0|} (\theta - \theta_0) \right) \]
when \( \theta < \pi \) as well.

Since \( \sinh B \leq \sinh A e^{B-A} \) if \( 0 < B < A \), we obtain, for \( \theta \geq \pi \),
\[
\left| \psi_+^\alpha(r, \theta) \right| \leq \sum_{n=1}^{\infty} |\beta_n| \sinh \left( \frac{n\pi}{|a_0|}(\theta - \pi) \right) 
\leq \sum_{n=1}^{\infty} |\beta_n| \sinh \left( \frac{n\pi}{|a_0|}(\pi - \theta_0) \right) \exp \left( -\frac{n\pi}{|a_0|}(2\pi - \theta_0 - \theta) \right),
\]
and hence
\[
\left| \psi_+^\alpha(r, \theta) \right| \leq C \exp \left( -\frac{\pi}{|a_0|}(2\pi - \theta_0 - \theta) \right). \tag{3.33}
\]
Because of symmetry of \( \psi_+^\alpha \), we have
\[
\left| \psi_+^\alpha(r, \theta) \right| \leq C \exp \left( -\frac{\pi}{|a_0|}(\theta - \theta_0) \right),
\]
when \( \theta < \pi \) as well. So we have (3.31) and the proof is complete.

\[\Box\]

4 Proofs of Theorem 1.1

In this section we prove Theorem 1.1 and (1.23).

Proof of Theorem 1.1 One can see from (1.6), (1.7) and (1.11) that
\[
\alpha = \int_{\partial D_1} \partial_\nu r_\epsilon \, ds. \tag{4.1}
\]
So, it is enough to prove
\[
\left| \int_{\partial D_1^0} \partial_\nu (r_\epsilon - u_0) \, ds \right| \leq C|\log \epsilon|^{2m-1} \tag{4.2}
\]
for some constant \( C \) independent of \( \epsilon \).

Let \( V := \mathbb{R}^2 \setminus (D_1^0 \cup D_2^0 \cup D_2^1) \), and let \( \Gamma_1 := \partial D_2^0 \setminus D_2^1 \) and \( \Gamma_2 := \partial D_2^1 \setminus D_2^0 \). Then, \( \partial D_1, \Gamma_1, \) and \( \Gamma_2 \) constitute the boundary of \( V \). Let
\[
\varphi_\epsilon(x) := r_\epsilon(x) - u_0(x) - (r_\epsilon(0,0) - u_0(0,0)). \tag{4.3}
\]
Then, \( \varphi_\epsilon \) is a bounded harmonic function in \( V \) and \( \varphi_\epsilon \equiv 0 \) on \( \partial D_1 \). We claim that
\[
|\varphi_\epsilon(x)| \leq C\epsilon, \quad x \in V. \tag{4.4}
\]
In fact, if \( x \in \Gamma_1 \), then \( u_0(x) - u_0(0,0) = 0 \) and \( r_\epsilon(x + \epsilon) - r_\epsilon(0,0) = 0 \). Therefore, since \( \nabla r_\epsilon \) is bounded on any bounded subset of \( \mathbb{R}^2 \setminus D_1 \cup D_2 \) (refer to (11)), we have
\[
|\varphi_\epsilon(x)| = |r_\epsilon(x) - r_\epsilon(0,0)| = |r_\epsilon(x) - r_\epsilon(x + \epsilon)| \leq C\epsilon. \tag{4.5}
\]
Likewise we have for \( x \in \Gamma_2 \)
\[
|\varphi_\epsilon(x)| \leq C\epsilon. \tag{4.6}
\]
Since \( \varphi_\epsilon \equiv 0 \) on \( \partial D_1 \) and \( \varphi_\epsilon(x) \) is bounded, we obtain (4.4) by the maximum principle. Choose \( M \) so large that

\[
D_1^0 \subset \left( -\frac{M}{2}, 0 \right) \times \left( -\frac{M}{2}, \frac{M}{2} \right),
\]

and let \( \omega = (-M, 0) \times (-M, M) \). Since \( \varphi_\epsilon \) is harmonic in \( V \), we have

\[
\int_{\partial D_1^0} \partial_\nu (r_\epsilon - u_0) ds = \int_{\partial D_1^0} \partial_\nu \varphi_\epsilon ds = \int_{\partial \omega} \partial_\nu \varphi_\epsilon ds.
\]

Divide \( \partial \omega \) into three pieces: \( \partial \omega = \gamma_1 \cup \gamma_2 \cup \gamma_3 \) where

\[
\gamma_1 := \left\{ (0, y) \mid |y| \leq A_0 \frac{1}{|\log \epsilon|} \right\}, \quad \gamma_2 := \left\{ (0, y) \mid \frac{A_0}{|\log \epsilon|} < |y| \leq M \right\}, \quad \gamma_3 := \partial \omega \setminus (\gamma_1 \cup \gamma_2)
\]

and write

\[
\int_{\partial \omega} \partial_\nu \varphi_\epsilon ds = \int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3} \partial_\nu \varphi_\epsilon ds := I + II + III.
\]

Here, the constant \( A_0 \) is given by Theorems 2.1 and 3.1 so that

\[
|\nabla u_0(x, y)| + |\nabla r_0(x, y)| \leq C \exp \left( -\frac{A_0}{|y| + \sqrt{\epsilon}} \right)
\]

for \( |y| < \delta \) and \( x \in (x_1(y), x_2(y)) \).

If \( -\frac{A_0}{|\log \epsilon|} \leq y \leq \frac{A_0}{|\log \epsilon|} \), then (4.7) implies that

\[
|\nabla \varphi_\epsilon(0, y)| \leq C \exp \left( -\frac{A_0}{|y| + \sqrt{\epsilon}} \right).
\]

Thus we have

\[
|I| \leq C \epsilon. \tag{4.8}
\]

If \( \frac{A_0}{|\log \epsilon|} < |y| \leq M \), there is \( r > Cy^{2m} \) for some \( C \) such that \( B_r(0, y) \subset V \). It then follows from a gradient estimate for harmonic functions that (4.4) that

\[
|\nabla \varphi_\epsilon(0, y)| \leq C \frac{\epsilon}{y^{2m}},
\]

and

\[
|II| \leq C \epsilon \int_{|y| < \frac{A_0}{|\log \epsilon|}} \frac{1}{y^{2m}} dy \leq C \epsilon |\log \epsilon|^{-2m-1}. \tag{4.9}
\]

There is a constant \( r > 0 \) such that \( B_r(x) \subset V \) for all \( x \in \gamma_3 \). So, we have from (4.4) that for any \( x \in \gamma_3 \),

\[
|\nabla \varphi_\epsilon(x)| \leq C \frac{\epsilon}{r} \leq C \epsilon,
\]

and

\[
|III| \leq C \epsilon. \tag{4.10}
\]

Now, (4.2) follows from (4.8), (4.9), and (4.10), and the proof is complete. \( \square \)
The formula (1.23) is an immediate consequence of (1.22). In fact, if \( r_1 \) and \( r_2 \) are radii of circles osculating to \( \partial D_1 \) and \( \partial D_2 \) at \((0,0)\) and \((\epsilon,0)\), respectively, then it is proved in \[21\] that \( p_1 \) and \( p_2 \) which are fixed points of mixed reflections are given by

\[
p_i = \left(\frac{(-1)^i}{\sqrt{2}} \sqrt{\frac{r_1 r_2}{r_1 + r_2}} \sqrt{\epsilon} + O(\epsilon), 0\right) \quad \text{as} \ \epsilon \to 0.
\]

So we obtain (1.23) from (1.22).

5 Approximations of \( \alpha_0 \)

The region outside \( D_0^1 \cup D_0^2 \) has cusps at \((0,0)\), and it may cause some problem in computing \( \alpha_0 \). To avoid this trouble, we show that by replacing the cusp with a neck a good approximation of \( \alpha_0 \) can be obtained.

For \( \rho > 0 \) let

\[
D_{\rho} = (D_0^1 \cup D_0^2) \cup ([-\rho, \rho] \times [-\rho, \rho])
\]

which is of dumbbell shape, and let \( u_{\rho} \) be the solution to

\[
\begin{align*}
\Delta u_{\rho} &= 0 \quad \text{in} \ \mathbb{R}^2 \setminus D_{\rho}, \\
u_{\rho} &= \lambda_{\rho} \quad \text{on} \ \partial D_{\rho}, \\
u_{\rho}(x) - h(x) &= O(|x|^{-1}) \quad \text{as} \ |x| \to \infty,
\end{align*}
\]

where the constant \( \lambda_{\rho} \) is determined by the additional condition

\[
\int_{\partial D_{\rho}} \frac{\partial u_{\rho}}{\partial \nu} \, ds = 0.
\]

We have the following theorem.

**Theorem 5.1** Let \( \delta \) be the number appearing in Theorem \[21\]. For \( \rho \in (0, \delta/2) \), let

\[
\alpha_{\rho} = \int_{\partial D_{\rho} \setminus [-\rho, \rho] \times [-\rho, \rho]} \partial_x u_{\rho} \, ds.
\]

Then, there are constants \( C \) and \( A \) such that

\[
|\alpha_0 - \alpha_{\rho}| \leq C \exp\left(-\frac{A}{\rho}\right).
\]

**Proof.** Choose a point \( z_0 \) on the common boundary of \( D_{\rho} \) and \( D_0^1 \cup D_0^2 \) and let

\[
\varphi(z) := u_{\rho}(z) - u_0(z) - (u_{\rho}(z_0) - u_0(z_0)).
\]

Since \( u_{\rho}(z) - u_{\rho}(z_0) = 0 \) for all \( z \in \partial D_{\rho} \) and \( u_0(z) - u_0(z_0) = 0 \) on \( \partial D_0^1 \cup \partial D_0^2 \), we have

\[
\varphi(z) = 0, \quad z \in \partial D_{\rho} \setminus ([-\rho, \rho] \times [-\rho, \rho]).
\]

On the other hand, if \( x_1(\rho) \leq x \leq x_2(\rho) \), then we have from \[22\]

\[
|u_0(x + i\rho) - u_0(z_0)| \leq C e^{-\frac{A}{\rho}},
\]

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and hence
\[ |\varphi(x + i\rho)| = |u_0(x, \rho) - u_0(z_0)| \leq Ce^{-\frac{A}{\rho}}. \]  
(5.7)

Similarly one can see that if \( x_1(-\rho) \leq x \leq x_2(-\rho) \), then
\[ |\varphi(x - i\rho)| \leq Ce^{-\frac{A}{\rho}}. \]  
(5.8)

It follows from (5.6), (5.7), and (5.8) that
\[ |\varphi(z)| \leq Ce^{-\frac{A}{\rho}} \]  
(5.9)
for all \( z \in \partial D(\rho) \), and hence for all \( z \in \mathbb{R}^2 \setminus \overline{D(\rho)} \) by the maximum principle. Note that we may apply the maximum principle since \( u(\rho)(z) - u_0(z) \to 0 \) as \( |z| \to \infty \).

We now estimate
\[ \nabla(u(\rho)(z) - u_0(z)) = \nabla \varphi(z) \text{ on } \partial D^0_1 \setminus [-2\rho, 2\rho] \times [-2\rho, 2\rho]. \]
Because of (5.6), one can apply the argument used right after of Theorem 2.1 to see that \( \varphi(z) \) can be extended across \( \partial D^0_1 \setminus [-2\rho, 2\rho] \times [-2\rho, 2\rho] \) so that the extended function is harmonic in \( B_r(z) \) for all \( z \in \partial D^0_1 \setminus [-2\rho, 2\rho] \times [-2\rho, 2\rho] \) where \( r = s\rho^{2m} \) for some \( s > 0 \) (independent of \( \rho \)). Then by the gradient estimate for harmonic functions we have
\[ |\nabla(u(\rho) - u_0)(z)| \leq C2e^{-\frac{A}{\rho}}, \quad z \in \partial D^0_1 \setminus [-2\rho, 2\rho] \times [-2\rho, 2\rho]. \]  
(5.10)

It then follows from (2.1) and (5.10) that
\[ |\alpha_0 - \alpha(\rho)| \leq \int_{\partial D^0_1 \setminus [-2\rho, 2\rho] \times [-2\rho, 2\rho]} |\partial_{\nu}(u(\rho) - u_0)| \, ds + \int_{\partial D^0_1 \setminus [-2\rho, 2\rho] \times [-2\rho, 2\rho]} |\partial_{\nu}u_0| \, ds \]
\[ \leq Ce^{-\frac{A}{\rho}}. \]

This completes the proof. \( \square \)

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