Estimation of entanglement negativity of a two-qubit quantum system with two measurements

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Abstract – Numerous work had been done to quantify the entanglement of a two-qubit quantum state, but it can be seen that previous works were based on joint measurements on two copies or more than two copies of the quantum state under consideration. In this work, we show that a single copy and two measurements are enough to estimate the entanglement quantifier like entanglement negativity and concurrence. To achieve our goal, we establish a relationship between the entanglement negativity and the minimum eigenvalue of the structural physical approximation of the partial transpose of an arbitrary two-qubit state. The derived relation makes it possible to estimate entanglement negativity experimentally by Hong-Ou-Mandel interferometry with only two detectors. Also, we derive the upper bound of the concurrence of an arbitrary two-qubit state and have shown that the upper bound can be realized in experiment. We will further show that the concurrence of i) an arbitrary pure two-qubit state and ii) a particular class of mixed states, namely, rank-2 quasi-distillable mixed states, can be exactly estimated with two measurements.

Introduction. – Entanglement lies at the heart of quantum mechanics [1] and acts as an essential ingredient in quantum computing [2], quantum communication [3] and quantum cryptography [4]. The amount of entanglement in a given entangled system is directly proportional to the efficiency of a given entangled state in carrying out some quantum information processing task. This implies that if the system is a maximally entangled or even non-maximally entangled two-qubit system, then these entangled systems are able to perform better than separable states in a quantum information processing task. Therefore, it is necessary to determine whether the generated quantum state is entangled or not. This problem is known as the entanglement detection problem and it is studied by many authors applying different techniques [5–13]. The entanglement can also be detected by the structural physical approximation method (SPA) [14,15]. SPA is a physical means by which positive maps can be approximated by completely positive maps [16].

Like the entanglement detection problem, entanglement quantification is also equally important. To develop the quantum technology [17] and perform a quantum communication task [3], it is important to know the exact amount of entanglement in an entangled state. Great effort has been devoted to quantify the amount of entanglement by defining different entanglement measures such as entanglement of formation [18], entanglement negativity [19], logarithmic negativity [20], relative entropy of entanglement [21]. Entanglement of formation is a function of a quantity called concurrence and the same optimal pure state decomposition can be used to calculate concurrence and entanglement of formation [18,22]. Multiple efforts had been done for the estimation of entanglement of a two-qubit system without quantum state tomography (QST) [23–25]. In a similar fashion, numerous efforts had been given to determine concurrence for a two-qubit system without quantum state tomography [26–29]. The estimation of entanglement has been studied for a multipartite higher-dimensional system, too [30–32]. Walborn et al. [33] performed an experiment to quantify entanglement using a linear optics setup and determine the concurrence of a pure state by performing only a single measurement on two copies. Horodecki [34] provided a protocol that uses collective measurements on at most eight copies to determine the concurrence and negativity of a two-qubit system. Zhang et al. [35] presented protocols for the direct measurement of the concurrence for two-photon polarization entangled pure and...
mixed states without tomography. Their protocols need two copies of the state in each detection round.

It is clear from the above discussion that the methods proposed earlier for the estimation of either entanglement negativity or concurrence or both for a two-qubit system require either more than a single copy of the quantum state under investigation or to estimate more than two parameters. To the best of our knowledge, we find that there does not exist any single method that relies on a single copy and two measurements for the estimation of entanglement negativity and concurrence of a two-qubit system. This motivates us to present a method that needs one parameter estimation and single copy of the quantum state under investigation. In the present work, we will apply SPA of the partial transpose (PT) for the estimation of entanglement negativity and concurrence for a two-qubit system.

This paper is organized as follows: in the next section, we discuss about the entanglement negativity and point out the difficulty in realizing it in the experiment. Also, we derive the lower bound of the entanglement negativity using a mathematical inequality involving the trace of the product of two Hermitian matrices and the product of the eigenvalues of the same Hermitian matrices. In the third section, we provide the exact value of the entanglement negativity in terms of the minimum eigenvalue of the SPA PT of an arbitrary two-qubit mixed state $\rho_{AB}$. In the fourth section, we illustrate our results with examples. In the fifth section, we show that it is possible to express the exact value of the concurrence of the pure entangled state and quasi-distillable mixed state in terms of the minimum eigenvalue of the SPA PT of the two-qubit pure entangled state $\ket{\psi}_{AB}$ and quasi-distillable mixed state $\rho^{\text{quasi}}_{AB}$. In the sixth section, we determine the upper bound of the concurrence of an arbitrary two-qubit density matrix. In the last section, we summarize our results.

**Lower bound of entanglement negativity.** — Entanglement negativity $N(\rho)$ for the density matrix $\rho$ is defined as [36,37]

$$ N^{D}(\rho) = 2 \sum_{i} \max(0, -\nu_{i}), $$

where $\nu_{i}$’s are the negative eigenvalues of the partial transpose $\rho^{T}$ of the density matrix $\rho$.

Entanglement negativity was introduced by Vidal and Werner who have shown that it is indeed an entanglement monotone [19]. The number of negative eigenvalues of $\rho^{T}$ is at most two with $\rho$ denoting a two-qubit system [38]. A point to be noted is that, we restrict ourselves in this work to a two-qubit system. Thus, we need to determine only one negative eigenvalue of the partial transpose of a two-qubit state provided the quantum state is entangled. Although it is one of the important measures of entanglement for a bipartite as well as a multi-partite system, it involves PT, which is a positive but not completely positive map. This means that it is very difficult to realize PT in the laboratory and hence restrict the determination of entanglement negativity experimentally. To overcome this difficulty, we will apply the method of SPA of PT map. Due to this approximation, the PT map reduces to a completely positive map that can be realized in the experiment. The experimental demonstration of SPA PT for a two-qubit photonic system using single-photon polarization qubits and linear optical devices has been given by Lim et al. [39]. We note that an important application of the SPA PT method has been discussed recently in [40], where it has been shown that the SPA PT estimated the optimal singlet fraction with only two measurements.

In order to estimate the exact value of $N(\rho)$ experimentally, we first derive the lower bound of entanglement negativity in terms of a quantity that can be realized in the experiment. To proceed in this direction, let us first consider two subsystems $A$ and $B$ described by the Hilbert spaces $H_{A}$ and $H_{B}$, respectively, which are the part of the composite system described by the Hilbert space $H_{AB}$. Consider any two-qubit entangled state $\rho_{AB}$ in the composite Hilbert space $H_{AB}$. Now our task is to derive the lower bound of entanglement negativity $N(\rho_{AB})$, which quantifies the amount of entanglement between two subsystems $A$ and $B$ and show that the lower bound can be realized in the experiment with only two measurements.

To achieve this task, we start with the statement of a lemma.

**Lemma** [41]: For any two Hermitian $4 \times 4$ matrices $F_{1}$ and $F_{2}$, the inequality given below holds true:

$$ \sum_{i=1}^{4} \lambda_{i}(F_{1}) \lambda_{5-i}(F_{2}) \leq \text{Tr}(F_{1}F_{2}) \leq \sum_{i=1}^{4} \lambda_{i}(F_{1}) \lambda_{i}(F_{2}), $$

where $\lambda_{i}(F_{1})$ and $\lambda_{i}(F_{2})$ denote the eigenvalues of the matrices $F_{1}$ and $F_{2}$, respectively. The eigenvalues are arranged in descending order, i.e., $\lambda_{1}(\cdot) > \lambda_{2}(\cdot) > \lambda_{3}(\cdot) > \lambda_{4}(\cdot)$.

We use the above stated lemma for $F_{1} = \ket{\phi}\bra{\phi}$ and $F_{2} = \rho_{AB}^{T_{B}}$, where $\ket{\phi} = \alpha|00\rangle + \beta|11\rangle$, $T_{B}$ denotes the partial transposition with respect to the subsystem $B$ and $\rho_{AB}$ denotes the two-qubit density operator expressed in the computational basis as

$$ \rho_{AB} = \begin{pmatrix} t_{11} & t_{12} & t_{13} & t_{14} \\ t_{12}^{*} & t_{22} & t_{23} & t_{24} \\ t_{13}^{*} & t_{23}^{*} & t_{33} & t_{34} \\ t_{14}^{*} & t_{24}^{*} & t_{34}^{*} & t_{44} \end{pmatrix}, \sum_{i=1}^{4} t_{ii} = 1, $$

where $* \text{ denotes the complex conjugate.}$

As a result, the left-hand inequality of (2) will become

$$ \lambda_{4}(\rho_{AB}^{T_{B}}) \leq \text{Tr}(\ket{\phi}\bra{\phi}\rho_{AB}^{T_{B}}). $$

The amount of entanglement contained in $\rho_{AB}$ can be determined by entanglement negativity $N(\rho_{AB})$ defined in (1). It is a positive real number and is given by

$$ N(\rho_{AB}) = -2\lambda_{4}(\rho_{AB}^{T_{B}}) $$

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Inserting (5) in (4), we have
\[ -\frac{N(\rho_{AB})}{2} \leq \text{Tr}(\phi \rho T_{\rho_{AB}}) \] (6)

Note that the value of the R.H.S of inequality (6) cannot be determined experimentally because the partial transposition operation is not a physical operation. To obtain the value of R.H.S of inequality (6) experimentally, we use the structural physical approximation method to approximate the non-physical partial transposition operation by a completely positive operation. Let the structural physical approximation of \( \rho_{AB} \) be \( \bar{\rho}_{AB} \) and it is given by
\[
\bar{\rho}_{AB} = \left[ \frac{1}{3}(I \otimes \bar{T}) + \frac{2}{3}(\hat{\Theta} \otimes D) \right] \rho_{12}
= \begin{pmatrix}
E_{11} & E_{12} & E_{13} & E_{14} \\
E_{12}^* & E_{22} & E_{23} & E_{24} \\
E_{13}^* & E_{23}^* & E_{33} & E_{34} \\
E_{14}^* & E_{24}^* & E_{34}^* & E_{44}
\end{pmatrix},
\] (7)

where
\[
E_{11} = \frac{1}{9}(2 + t_{11}), \quad E_{12} = \frac{1}{9}(-it_{12} + t_{13}), \\
E_{13} = \frac{1}{9}(t_{13} - i(t_{13} + t_{24})), \quad E_{14} = \frac{1}{9}(-it_{14} + t_{23}), \\
E_{22} = \frac{1}{9}(2 + t_{22}), \quad E_{23} = \frac{1}{9}(t_{14} + it_{23}), \\
E_{24} = \frac{-i}{9}(t_{13}^* + t_{24}), \quad E_{33} = \frac{1}{9}(2 + t_{33}), \\
E_{34} = \frac{1}{9}(-it_{34} + t_{34}), \quad E_{44} = \frac{1}{9}(2 + t_{44}).
\] (8)

Let \( I \otimes \bar{T} \) and \( \hat{\Theta} \otimes D \) be local operators and are completely positive operators. The SPA transpose \( \bar{T} \) for a density operator \( \rho \) is given by
\[
\hat{T}(\rho) = \sum_{k=1}^{4} \text{Tr}(M_k \rho) |s_k \rangle \langle s_k|,
\] (9)

where \( \{ M_k = \frac{1}{2} b_1^* s_k^* s_k \}_{k=1}^4 \) is a complete measurement and \( |s_1^*\rangle = \frac{1}{\sqrt{1 + b_1^* b_1}} (|0\rangle + b_1^* |1\rangle) \), \( |s_2^*\rangle = \frac{1}{\sqrt{1 + b_2^* b_2}} (|0\rangle - b_2^* |1\rangle) \), and \( b_1 = \frac{i e^{2\pi i}}{1 + e^{2\pi i}}, \quad b_2 = \frac{e^{2\pi i}}{1 + e^{2\pi i}} \).

The other local operators \( \hat{\Theta} \) and \( D \) are defined as
\[
\hat{\Theta}(\cdot) = \sigma_y \hat{T}(\cdot) \sigma_y, \\
D(\cdot) = \frac{1}{4} \sum_{i=0, x, y, z} \sigma_i(\cdot) \sigma_i,
\] (10) (11)

where \( \sigma_0 = I \) and \( \sigma_x, \sigma_y, \sigma_z \) denote the Pauli matrices.

The quantum circuit for the realization of the SPA PT operation has been designed in [15].

The relation between \( \rho_{AB}^{T_{\rho_{AB}}} \) and \( \bar{\rho}_{AB} \) is given by [40]
\[
\text{Tr}(\phi \rho T_{\rho_{AB}}) = 9 \text{Tr}(\phi \rho_{AB}) - 2.
\] (12)

Using (6) and (12), we get
\[
\text{Tr}(\phi \rho_{AB}) \geq \frac{1}{18} [4 - N(\rho_{AB})].
\] (13)

Let \( \mu_{\text{min}} \) be the minimum eigenvalue of \( \bar{\rho}_{AB} \), then the eigenvalue equation is given by
\[
\bar{\rho}_{AB} \phi = \mu_{\text{min}} \phi.
\] (14)

Using the eigenvalue equation (14) in (13), we get
\[
N(\rho_{AB}) \geq 4 - 18 \mu_{\text{min}}.
\] (15)

The minimum eigenvalue \( \mu_{\text{min}} \) of the quantum state \( \bar{\rho}_{AB} \) can be determined by the formula [40]
\[
\mu_{\text{min}} = \frac{15}{8} F_{\text{avg}}(W, \bar{\rho}_{AB}) = \frac{47}{72},
\] (16)

where \( F_{\text{avg}}(W, \bar{\rho}_{AB}) \) and \( W \) denote the average fidelity and approximated entanglement witness operator, respectively. The approximated entanglement witness operator can be expressed in terms of entanglement witness operator \( W = |\phi \rangle \langle \phi| - \frac{1}{2} \mathbb{I} \) as [40]
\[
W = \frac{2}{9} W + \frac{7}{36} I.
\] (17)

The average fidelity \( F_{\text{avg}}(W, \bar{\rho}_{AB}) \) can be determined experimentally with only two measurements by using Hong-Ou-Mandel interferometry [40,42] and hence the minimum eigenvalue \( \mu_{\text{min}} \). Thus, we can determine the value of the lower bound of entanglement negativity experimentally with only two measurements.

**Determination of the exact value of entanglement negativity.** - We have already obtained the analytical lower bound of entanglement negativity and it is given by (15). Also we have shown that the analytic lower bound can be achieved experimentally. Now the problem is that the inequality (15) only tells us that the entanglement negativity can take values greater than the bound obtained but it does not determine the actual amount of entanglement in an arbitrary two-qubit state. We are now interested in obtaining the actual value of the entanglement negativity contained in an arbitrary two-qubit state. Note that the quantity \( 4 - 18 \mu_{\text{min}} \) is less than or equal to \( N(\rho_{AB}) \). This suggests that if we add a positive quantity \( Q \) to the quantity \( 4 - 18 \mu_{\text{min}} \) then it may be equal to \( N(\rho_{AB}) \). By adding a positive quantity \( Q \) to the R.H.S of (15), we get
\[
N(\rho_{AB}) = 4 - 18 \mu_{\text{min}} + Q, \quad Q > 0.
\] (18)

To search for the quantity \( Q \), we keep in mind the following facts: i) the inequality \( \mu_{\text{min}} < \frac{7}{9} \) holds for all entangled states \( \rho_{AB} \) [16] and ii) \( \text{Tr}(I - |\phi \rangle \langle \phi| \bar{\rho}_{AB}) = 1 - \mu_{\text{min}} > 0 \). Using these two facts, we can always choose \( Q = (\frac{7}{9} - \mu_{\text{min}}) \text{Tr}(I - |\phi \rangle \langle \phi| \bar{\rho}_{AB}) \). With
this choice of $Q$, (18) can be re-written as

$$N(\rho_{AB}) = 4 - 18\mu_{\text{min}} + \left(\frac{2}{9} - \mu_{\text{min}}\right)(1 - \mu_{\text{min}})$$

$$= \left(\frac{2}{9} - \mu_{\text{min}}\right)(19 - \mu_{\text{min}}), \quad \frac{1}{6} \leq \mu_{\text{min}} < \frac{2}{9}. \quad (19)$$

Here, we observe that $N(\rho_{AB})$ given in (19) is not normalized. The normalized $N(\rho_{AB})$ is then given by

$$N^N(\rho_{AB}) = K\left(\frac{2}{9} - \mu_{\text{min}}\right)(19 - \mu_{\text{min}}), \quad \frac{1}{6} \leq \mu_{\text{min}} < \frac{2}{9}. \quad (20)$$

Here $K$ is a normalization constant. $K$ can be determined by using the fact that $\mu_{\text{min}} = \frac{5}{9}$ for a maximally entangled state. Thus, the normalized entanglement negativity is given by

$$N^N(\rho_{AB}) = \frac{108}{113}\left(\frac{2}{9} - \mu_{\text{min}}\right)(19 - \mu_{\text{min}}), \quad \frac{1}{6} \leq \mu_{\text{min}} < \frac{2}{9}. \quad (21)$$

Since $N^N(\rho_{AB})$ is expressed in terms of $\mu_{\text{min}}$, the normalized entanglement negativity can be determined experimentally with two measurements using Hong-Ou-Mandel interferometry.

Thus, we have presented an entanglement estimation protocol in which we need only a single copy of the quantum state and no requirement of QST. This result contradicts the fact that it is impossible to detect the value of the entanglement with only single-copy measurements without QST [43,44]. The above contradiction can be explained by observing the fact that there does not exist any quantum operation that can achieve a non-physical operation such as the partial transpose map with unit fidelity in an experiment. Fidelity measures how close the approximate quantum operation is to the actual impossible operation [45]. By seeing this fact, it is necessary to have an approximate quantum operation that can approximate the partial transpose for the possible realization in the experiment. SPA is such an approximation of the PT operation that can be realized in the experiment with fidelity less than unity. Since our protocol for the estimation of entanglement is based on SPA PT, the estimation is an approximate estimation. Further, the parameter needed for the estimation of entanglement is the minimum eigenvalue of the SPA of the given state and this parameter is related with the average fidelity. So, the minimum eigenvalue can be estimated approximately as well as the negativity.

**Examples.** – We now illustrate with examples that the derived expression of normalized entanglement negativity given in (21) is indeed correct and its correctness can be shown by matching it with the expression of entanglement negativity obtained via definition (1).

**Example-I:** Let us first consider a pure entangled state described by the density matrix

$$\rho = M|01\rangle\langle 01| + \sqrt{M(1-M)}(|01\rangle\langle 10| + |10\rangle\langle 01|) + (1-M)|10\rangle\langle 10|.$$ \quad (22)

The entanglement negativity using the definition (1) is given by

$$N^D(\rho_{M}) = 2\sqrt{M(1-M)}. \quad (23)$$

The SPA-PT of $\rho_{M}$ is given by

$$\bar{\rho}_M = \left(\frac{2}{9} - \mu_{\text{min}}\right)\bar{W}, \quad \frac{1}{6} \leq \mu_{\text{min}} < \frac{2}{9}. \quad (24)$$

where $a = \sqrt{M(1-M)}$.

The minimum eigenvalue of $\bar{\rho}_M$ is given by

$$\mu_{\text{min}} = \frac{2}{9} - \frac{\sqrt{M(1-M)}}{9} \quad = \frac{15}{8} F_{\text{avg}}(\bar{W}, \bar{\rho}_M) - \frac{47}{72}, \quad (25)$$

where $F_{\text{avg}}(\bar{W}, \bar{\rho}_M)$ denotes the average fidelity between two quantum states $\bar{W}$ and $\bar{\rho}_M$.

The value of $\mu_{\text{min}}$ given by the second equality in (25) can be estimated experimentally by the Hong-Ou-Mandel interferometer setup with only two detectors [40,42]. Substituting $\mu_{\text{min}} = \frac{2}{9} - \frac{\sqrt{M(1-M)}}{9}$ in (21), we get the reduced expression of the normalized entanglement negativity as

$$N^N(\rho_{M}) = \frac{54}{9153} N^D(\rho_{M}) \left[169 + \frac{N^D(\rho_{M})}{2}\right]. \quad (26)$$

When we compare the expressions given in (23) and (26), then we find that the two curves for $N^D(\rho_{M})$ and $N^N(\rho_{M})$ almost coincide and this can be verified by fig. 1, too.
The entanglement negativity using the definition (1) is given by

\[ N = \text{minimize eigenvalue of} \rho \]

The SPA PT of \( N \) is given by

\[ \tilde{\rho}_H = \begin{pmatrix} 3-p/9 & 0 & 0 & p/18 \\ 0 & 2p/9 & 2p/9 & 0 \\ 0 & 0 & 2p/9 & 0 \\ p/18 & 0 & 0 & 2p/9 \end{pmatrix} \]

The minimum eigenvalue of \( \tilde{\rho}_H \) is given by

\[ \mu_{\text{min}} = \frac{5}{18} - \frac{p}{18} - \frac{\sqrt{1-2p+2p^2}}{18} \]

\[ = \frac{15}{8} F_{\text{avg}}(W, \tilde{\rho}_H) - \frac{47}{72} \]  

In this example also, we can follow the same procedure as explained in the previous example to estimate the minimum eigenvalue experimentally. Using (30) in (21), the expression of the normalized entanglement negativity reduced to

\[ N^N(\rho_H) = \frac{N^D(\rho_H)}{339}[338 + N^D(\rho_H)]. \]

Again if we compare the expressions given in (28) and (31), then we can see that the two curves for \( N^D(\rho_H) \) and \( N^N(\rho_H) \) almost overlap with each other (see fig. 2).

**Determination of the exact value of concurrence.**

- The first entanglement measure popularly known as concurrence was introduced by Wootters [18] and it is defined for the two-qubit state \( \rho \) as

\[ C(\rho) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\} \]

where \( \lambda_1, \lambda_2, \lambda_3, \lambda_4 \) denote the square root of the eigenvalues of the operator \( \rho (\sigma_y \otimes \sigma_y) \rho^*(\sigma_y \otimes \sigma_y) \) and these eigenvalues are arranged in descending order, \( \sigma_y \) represents the Pauli spin matrix and the complex conjugation is denoted by an asterisk.

We find that it is not possible to express always the exact value of concurrence in terms of a parameter that can be realized experimentally. In this section, we express the exact formula of the concurrence in terms of an experimentally accessible parameter for two classes of entangled states, viz., for pure entangled states and for quasi-distillable mixed entangled states.

**Pure entangled states.** It has been shown that the entanglement negativity and concurrence are equal for a pure two-qubit state [48]. Therefore, the concurrence for any pure two-qubit state \( |\psi\rangle_{AB} \) can be expressed in terms of the minimum eigenvalue \( \mu_{\text{min}} \) of \( \rho_{AB} \), which is a SPA-PT of a two-qubit state \( |\psi\rangle_{AB} \). Thus, the concurrence for the pure state \( |\psi\rangle_{AB} \) is given by

\[ C(|\psi\rangle_{AB}) = \frac{108}{113} \left( \frac{2}{9} - \mu_{\text{min}} \right) (19 - \mu_{\text{min}}), \]

\[ \frac{1}{6} \leq \mu_{\text{min}} < \frac{2}{9} \]

Therefore, the value of the concurrence for any two-qubit pure state can be realized experimentally.

**Quasi-distillable mixed states.** Quasi-distillable states are those non-maximally entangled states which cannot be distilled to a perfect singlet state but can be distilled to a state with arbitrarily high singlet fraction and this distillation procedure can be implemented with non-zero probability [49]. The class of quasi-distillable states is denoted by an asterisk.

Verstraete et al. [50] have shown that for any two-qubit entangled state \( \rho_{AB} \), entanglement negativity \( N(\rho_{AB}) \) and concurrence \( C(\rho_{AB}) \) are related by the inequality

\[ N(\rho_{AB}) \geq (1 - C(\rho_{AB}))^2 + C(\rho_{AB})^2 - (1 - C(\rho_{AB})). \]

The lower bound of (35) is achieved iff the state is a rank-2 quasi-distillable state [50]. Therefore, the entanglement negativity and concurrence for a rank-2 quasi-distillable state \( \rho_{AB}^{\text{qasi}} \) are related as

\[ N(\rho_{AB}^{\text{qasi}}) = \sqrt{(1 - C(\rho_{AB}^{\text{qasi}}))^2 + C(\rho_{AB}^{\text{qasi}})^2} - (1 - C(\rho_{AB}^{\text{qasi}})). \]

We note that the entanglement negativity that appeared in eq. (36) is normalized and hence the concurrence for a
rank-2 quasi-distillable state $\rho_{AB}^{\text{quasi}}$ is explicitly expressed in terms of normalized entanglement negativity as

$$C(\rho_{AB}^{\text{quasi}}) = -N(\rho_{AB}^{\text{quasi}}) + \sqrt{2N(\rho_{AB}^{\text{quasi}})(N(\rho_{AB}^{\text{quasi}}) + 1)},$$  \hspace{1cm} (37)

where $N(\rho_{AB}^{\text{quasi}})$ is given by eq. (21). Since the value of $N(\rho_{AB}^{\text{quasi}})$ can be estimated experimentally, the concurrence for a rank-2 quasi-distillable state can be estimated experimentally.

**Upper bound of the concurrence for the general two-qubit mixed states.** – Numerous efforts have been made to calculate the lower and upper bounds of concurrence theoretically or experimentally [51–55]. All these methods rely on the joint measurement on two copies of the given quantum state. In this section, we will show that the upper bound of the concurrence can be estimated using only a single copy of the given quantum state.

We now derive the upper bound of the concurrence using the Lewenstein-Sanpera decomposition [56] and the convexity property of concurrence. An arbitrary two-qubit density matrix $\rho$ can be decomposed as [56]

$$\rho = \lambda \rho_s + (1 - \lambda) |\psi\rangle_\rho \langle \psi|, \quad \lambda \in [0, 1],$$  \hspace{1cm} (38)

where $\rho_s$ denote two-qubit separable state and $|\psi\rangle_\rho$ represents two-qubit pure entangled state. We note here that the decomposition (38) is unique.

The concurrence of an arbitrary two-qubit quantum state $\rho$ is given by

$$C(\rho) = C(\lambda \rho_s + (1 - \lambda) |\psi\rangle_\rho \langle \psi|).$$  \hspace{1cm} (39)

Using the convexity property of concurrence [21,22], we have

$$C(\rho) \leq (1 - \lambda) C(|\psi\rangle_\rho \langle \psi|) = \frac{108}{113} (1 - \lambda) \left(2 - \frac{\mu_{\text{min}}}{9}\right) (19 - \mu_{\text{min}}),$$

$$\frac{1}{6} \leq \mu_{\text{min}} < \frac{2}{9}$$  \hspace{1cm} (40)

In the first line, we have used $C(\rho_s) = 0$ and eq. (33) is used in the second line. The equality holds when $\lambda = 0$ and $\lambda = 1$.

**Conclusion.** – To summarize, in this work we have studied the two most important measures of entanglement such as entanglement negativity and concurrence to quantify the entanglement in a two-qubit system. As the partial transposition (PT) of a two-qubit density matrix is not a physical operation, we apply the structural physical approximation (SPA) method to realize the partial transposition operation experimentally. It has already been shown that the minimum eigenvalue of a SPA PT state can be estimated experimentally [40]. Interestingly, we show that there is a connection between the entanglement negativity and the minimum eigenvalue of the SPA PT state and hence entanglement negativity can be estimated experimentally. We have provided some illustrations to show that the value of the entanglement negativity which would be obtained experimentally matches with the theoretical result. Further, we have shown that the value of the concurrence can be estimated experimentally for i) any arbitrary two-qubit pure states and ii) a particular class of mixed states known as rank-2 quasi-distillable states. Also we have obtained the upper bound of the concurrence of an arbitrary two-qubit quantum state. We can estimate the value of entanglement negativity and concurrence using the Hong-Ou-Mandel setup that requires only two measurements, which are much lesser in comparison to state tomography of an unknown state, thus signifying the practical utility of our work. Since we restrict ourselves in this work only to a two-qubit system, it would be interesting to apply the idea of this work to obtain the experimental value of entanglement negativity and concurrence for a higher-dimensional system or for a multi-partite system.

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