Classification of Finite Highly Regular Vertex-Coloured Graphs*

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Abstract

A coloured graph is $k$-ultrahomogeneous if every isomorphism between two induced subgraphs of order at most $k$ extends to an automorphism. A coloured graph is $k$-tuple regular if the number of vertices adjacent to every vertex in a set $S$ of order at most $k$ depends only on the isomorphism type of the subgraph induced by $S$.

We classify the finite vertex-coloured $k$-ultrahomogeneous graphs and the finite vertex-coloured $k$-tuple regular graphs for $k \geq 4$. Our theorem in particular classifies finite vertex-coloured ultrahomogeneous graphs, where ultrahomogeneous means the graph is simultaneously $k$-ultrahomogeneous for all $k \in \mathbb{N}$.

1 Introduction

A structure is ultrahomogeneous\(^1\) if every isomorphism between two induced substructures within the structure extends to an automorphism of the entire structure. Ultrahomogeneous structures have been extensively studied in the literature (see related work below), in part because they find important applications in model theory and Ramsey theory in particular due to their construction as Fraïssé limits (see [Mac11]). In this paper we focus on finite simple graphs.

A graph is $k$-ultrahomogeneous if every isomorphism between two induced subgraphs of order at most $k$ extends to an automorphism. Note that a graph is 1-ultrahomogeneous if and only if it is transitive. Moreover, the 2-ultrahomogeneous graphs are precisely the rank 3 graphs, that is, transitive graphs with an automorphism group that acts transitively on pairs of adjacent vertices and also on pairs of non-adjacent vertices. Ultrahomogeneity is the same as simultaneous $k$-ultrahomogeneity for all $k \in \mathbb{N}$.

Conceptually, all these variants of homogeneity are a form of symmetry of the entire graph. In particular, they constitute a global property of the graph. While this makes the graph truly homogeneous, the property may also be difficult to check algorithmically. In contrast to this, a regularity condition is a local condition that does not necessarily imply global symmetries. Following Cameron (see [Cam04]), a graph $G$ is called $k$-tuple regular if for every pair of vertex sets $S$ and $S'$ on at most $k$ vertices that induce isomorphic subgraphs of $G$, the number of vertices adjacent to every vertex of $S$ is the same as the number of vertices adjacent to every vertex of $S'$.

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\(^1\)Some authors use the term homogeneous instead. We use the term ultrahomogeneous to highlight that every and not just some isomorphism between isomorphic substructures must extend to an automorphism.
Note that 1-tuple regularity is just ordinary regularity and that 2-tuple regularity of a graph precisely means that the graph is strongly regular. Also note that \( k \)-regularity is a local property and easily checkable algorithmically.

It is clear from the definition that \( k \)-ultrahomogeneity implies \( k \)-tuple regularity. Conversely, it is a priori not clear that high regularity should imply any form of ultrahomogeneity. In fact, of course not all graphs that are transitive (i.e., 1-ultrahomogeneous) are strongly regular (i.e., 2-tuple regular). However, it turns out that \((k+1)\)-tuple regularity is stronger than \( k \)-ultrahomogeneity for \( k \geq 3 \). We are not aware of a direct argument for this phenomenon. In fact the only available type of argument is via the classifications of the respective classes.

Such classifications are known for \( k \)-ultrahomogeneity and \( \ell \)-tuple regularity for \( k \geq 2 \) and \( \ell \geq 4 \), respectively (see Table 1). Introduced by Sheehan [She75], Gardiner [Gar76] and independently Gol’fand and Klin [GK78] classified all finite ultrahomogeneous simple graphs. They are, up to complementation, disjoint unions of isomorphic complete graphs, the complements of such graphs, the \( 3 \times 3 \) rook’s graph and the 5-cycle. It turns out that 5-tuple regular graphs are 5-ultrahomogeneous and even \( k \)-ultrahomogeneous for all \( k \geq 5 \) as shown by Cameron [Cam80] (and independently but unpublished by Gol’fand). They are therefore ultrahomogeneous. The only graphs that are 4-tuple regular but not 5-tuple regular are the Schläfi graph, the McLaughlin graph, and their complements, of which only the Schläfi graph and its complement are 4-ultrahomogeneous. The list of 3-ultrahomogeneous graphs that are not 4-tuple regular contains infinite families and exceptional graphs (see Table 1 and Section 2 for descriptions of the graphs). For the classification of 2-ultrahomogeneous graphs we refer to [LS86].

| 3-UH [CM85] | 4-TR [Buc80] | 4-UH [Buc80] | 5-TR [Cam80] \(^2\) | UH [Gar76] \(^2\) |
|----------------|----------------|----------------|----------------|----------------|
| \( 3 \times 3 \) rook’s graph \( R_3 \) | ✓ | ✓ | ✓ | ✓ |
| 5-cycle \( C_5 \) | ✓ | ✓ | ✓ | ✓ |
| \( sK_t \) \( (s, t \geq 1) \) | ✓ | ✓ | ✓ | ✓ |
| Schläfi graph | ✓ | ✓ | ✓ | ✓ |
| McLaughlin graph | ✓ | ✓ | ✓ | ✓ |
| Clebsch graph | | | | |
| Higman-Sims graph | | | | |
| \( m \times m \) rook’s graph \( R_m \) \( (m \geq 4) \) | | | ✓ | |
| aff. polar graph \( G^{\varepsilon}(F_2^{d}) \) \( (d \geq 3, \varepsilon \in \{+, -\}) \) | | | | |
| gen. quadrangle \( G(Q_5^d(q)) \) \( (q \) prime power) | | | | |

Table 1: Classification of finite \( k \)-UH graphs with \( k \geq 3 \) and finite \( \ell \)-TR graphs with \( \ell \geq 4 \) (up to complementation). For the classification of finite 2-UH graphs see [LS86].

In this paper we are interested in investigating the concepts of \( k \)-ultrahomogeneity and \( k \)-tuple regularity for finite vertex-coloured graphs. For these, all definitions are precisely the same except that all isomorphisms respect vertex colours. In model theoretic terms this is equivalent to \( \varepsilon \).

\(^2\)The classification of 5-TR was also given by Gol’fand (unpublished, see [Cam80]). The classification of UH was also given independently by [GK78].
to allowing an arbitrary number of unary relations.

Motivation. While applying the homogeneity and regularity concepts to vertex-coloured graphs may be interesting in its own right, our interest stems from an algorithmic perspective. It can be shown by induction on $k$ that $k$-tuple regularity is equivalent to requiring that for each ordered subset $S$ of $G$ of at most $k$ vertices the number of one-vertex extensions of $S$ of a certain isomorphism type (the type is determined by $N(v) \cap S$ when $S$ is extended by $v$) depends only on the ordered isomorphism type of the induced graph $G[S]$. From this it can be argued that $k$-tuple coloured graphs are precisely the graphs for which the $k$-dimensional Weisfeiler-Leman algorithm stabilizes already on the initial colouring. The Weisfeiler-Leman algorithm, with roots in algebraic graph theory, is a central tool for graph isomorphism testing and automorphism group computations [Kie20]. In recursive algorithms for the graph isomorphism problem, vertex colourings play an important role. Therefore vertex coloured graphs are the natural graph class to study. The colourings are used to introduce irregularities into the graph and thereby ensure recursive progress [MP14]. For example, in Babai’s quasipolynomial time graph isomorphism algorithm this is the case [Bab16]. The algorithm exploits a local to global approach that is based on the observation that when a graph’s regularity is sufficiently high, it will in fact be highly symmetric. This can then be exploited in the algorithm. For a strengthening of such local to global approaches, it is imperative to gain a better understanding under which conditions high regularity implies high symmetry in vertex coloured graphs.

Our results. In this paper we provide a classification for the $k$-ultrahomogeneous and the $\ell$-tuple regular finite vertex coloured graphs for $k \geq 4$ and $\ell \geq 4$, respectively.

For the classification result we exploit that there are four simple but generic operations that preserve the degree of regularity and homogeneity. Specifically these are complementation between two colour classes or within a colour class, colour disjoint union (i.e., disjoint union of graphs whose vertices do not share colours), homogeneous matching extension, and homogeneous blow up. In the homogeneous matching extension, a colour class that forms an independent set is duplicated. The duplicate vertices are all coloured with the same previously unused colour. The duplicate vertices are connected to their originals via a perfect matching. In a homogeneous blow-up, all vertices in a colour class that forms an independent set are replaced by cliques of the same size. A graph is reduced if it cannot be built from graphs of smaller order using these operations.

Our theorem shows that for $k \geq 4$ all $k$-tuple regular graphs are obtained from monochromatic and reduced bichromatic $k$-tuple regular graphs. The monochromatic graphs are shown in Table 1. For the bichromatic case we prove that the only irreducible graph that is $k$-tuple regular for some $k \geq 4$ is the extended Hadamard graph corresponding to the $2 \times 2$ Hadamard Matrix. This matrix is unique up to equivalence and a Sylvester matrix (see Theorem 6.5 and Table 4). The graph is ultrahomogeneous and thus $k$-tuple regular for all $k$. The underlying uncoloured graph is isomorphic to the Wagner graph, the M"obius ladder on 8 vertices.

Our theorem in particular provides the first classification of finite vertex-coloured ultrahomogeneous graphs.

Organization of the paper. After proving pointers to related literature (next subsection) and preliminaries (Section 2) we give a formal definition of the regularity concepts considered in the paper (Section 3). We investigate bichromatic highly regular graphs (Section 4) before arguing that in the trichromatic case under certain conditions one of the colour classes must be trivially connected (Section 5). We finally assemble the statements to our classification theorems for the multicoloured case (Section 6), leading to our discussion of future work (Section 7).
1.1 Related work

Regarding high regularity of graphs, the classifications for \( k \)-tuple regularity and \( k \)-homogeneity are summarized in Table 1. We should remark that some authors use the terms \( k \)-isoregular in place of \( k \)-tuple regularity. There is also a related concept called the \( t \)-vertex condition (see [Rei00]).

While the literature on \( k \)-ultrahomogeneity and \( k \)-tuple regularity of graphs is limited, there are numerous publications surrounding ultrahomogeneity. Extensive work has been done on directed graphs, for finite and infinite countable graphs. On top of that there are also various generalization of the homogeneity concept and classification results towards other structures as well as modified requirements on what precisely should be extendable to automorphisms. For a general survey with a focus on infinite graphs we refer to [Mac11].

Previous work on coloured graphs. There are several publications regarding ultrahomogeneity of edge-coloured graphs. Lockett and Truss [LT14a] have classified multipartite ultrahomogeneous graphs for which each colour class forms an independent set or a clique. They also allow multiple edge colours between different parts. Most of the work treats the infinite parts. Prior to that some special cases had been regarded. In [GGK96] infinite bipartite graphs are considered and in [JTS12] multipartite graphs without edge colourings are studied. In [Ros11] some infinite graphs with two vertex colours are considered, but a finite number of edge colours is allowed for edges with distinctly coloured end vertices.

Restricted homogeneity. Researchers have also considered various variants of homogeneity, where the requirement that isomorphism between substructures are to be extendable to automorphisms is only asked for certain substructures. For example distance transitive graphs are of this kind (see [vB07]). Related concepts more akin to tuple regularity and are \( k \)-transitivity (see [Cam80]). There are also classification results for connected homogeneity, where only isomorphisms between connected substructures need to extend. When restricted to substructures of order \( k \) we get \( k \)-connected homogeneous graph, for which there is a recent classification for various cases described in [DFPZ20]. We also refer to that paper for further pointers to literature. In fact our presentation of 3-ultrahomogeneous graphs follows the one provided there.

Other structures and other morphisms. Beyond graphs, there are numerous classifications of ultrahomogeneous objects. We refer to work of DeVillers [Dev02] and Macpherson’s survey [Mac11]. Another concept that has been studied is Homomorphism-homogeneity, where homomorphisms between substructures are required to extend to endomorphisms of the entire structure. More variants are obtained by considering epimorphisms and monomorphisms. We refer to [CN06,Loc08,RS10,LT14b] for pointers. Combinations of altered concepts are also studied. This leads, for example, to connected-homomorphism-homogeneity [Loc15].

2 Preliminaries

For a set \( A \) and a natural number \( k \in \mathbb{N} \), we set \( \binom{A}{k} := \{ B \subseteq A \mid |B| = k \} \). We denote the identity matrix of rank \( r \) by \( I_r \).

Graph basics. A (simple) graph \( G \) is a pair \((V,E)\) of sets, the vertices and edges, respectively, with \( E \subseteq \binom{V}{2} \) and \( V \cap E = \emptyset \). In this paper we are only concerned with finite non-empty graphs, that is, we assume \( V \) is finite and non-empty. We refer to the vertex set of \( G \) as \( V(G) \) and to its edge set as \( E(G) \). We call \( G \) edgeless if \( E(G) = \emptyset \). An edge \( \{u,v\} \) has end vertices \( u \) and \( v \). We denote the length of a shortest path from \( u \) to \( v \) by \( \text{dist}(u,v) \). By \( \overline{G} := (V, \binom{V}{2} - E) \) we denote the complement of \( G \).
We define the \textit{neighbourhood} of a vertex \( u \) in \( G \) as \( N_G(u) := \{ v \in V \mid uv \in E \} \). For \( U \subseteq V(G) \), we call the graph \( G[U] \) with vertex set \( U \) and edge set \( E(G) \cap \binom{U}{2} \) the \textit{subgraph induced by} \( U \).

Let \( G \) and \( G' \) be two graphs. A bijective map \( \varphi : V(G) \rightarrow V(G') \) is an \textit{isomorphism} from \( G \) to \( G' \) if for every pair of vertices \( u, v \in V(G) \) vertex \( u \) is adjacent to \( v \) in \( G \) if and only if \( \varphi(u) \) is adjacent to \( \varphi(v) \) in \( G' \). Should an isomorphism exist, then we say that \( G \) and \( G' \) are \textit{isomorphic} and write \( G \cong G' \). An isomorphism from \( G \) to itself is an \textit{automorphism}.

\textbf{Connection types of vertex sets.} Let \( R \) and \( B \) be disjoint vertex subsets of a graph \( G \). The sets \( R \) and \( B \) are \textit{homogeneously connected} if either every vertex of \( R \) is adjacent to every vertex of \( B \) or if no edge of \( G \) joins a vertex of \( R \) with a vertex of \( B \). Recall that a \textit{perfect matching} is a subset \( M \) of \( E(G) \) such that no two edges of \( M \) have a common end vertex and every vertex of \( V(G) \) is an end vertex of some edge in \( M \). We call \( R \) and \( B \) \textit{matching-connected} if the edges of \( G \) with one end vertex in \( R \) and one end vertex in \( B \) form a perfect matching of \( G[R \cup B] \) or the edges of \( G \) with one end vertex in \( R \) and one end vertex in \( B \) form a perfect matching \( G[R \cup B] \).

\textbf{Designs.} A \( t-(v, k, \lambda) \)-\textit{design} is a pair \((P, B)\) where \( P \) is a set of \textit{points} with \( |P| = v \) and \( B \subseteq \binom{P}{t} \) is a multiset of \textit{blocks} with the property that every set of \( t \) points is contained in precisely \( \lambda \) blocks. If \( k \in \{0, 1, v - 1, v\} \), then \((P, B)\) is \textit{degenerate}. Observe that \((P, B)\) is degenerate if and only if in the corresponding incidence graph the two sets \( P \) and \( B \) are homogeneously connected or matching-connected. We frequently use the classic result of Hughes [Hug65] that symmetric 3-designs are degenerate. Specifically,

\[ \text{if } |B| = v, \text{ then } t \in \{1, 2\} \text{ or } (P, B) \text{ is degenerate.} \quad (1) \]

We refer to [HP88] as a standard book on design theory.

\textbf{Regularity and strong regularity.} Let \( d \in \mathbb{N} \). A graph \( G \) of order \( n \) is \textit{(d-)regular} if every vertex of \( G \) is of degree \( d \). It is \textit{strongly regular} with parameters \((n, d, \lambda, \mu)\) if additionally every pair of adjacent vertices has \( \lambda \) common neighbours and every pair of non-adjacent vertices has \( \mu \) common neighbours. It is well known (see [GR01]) that a strongly regular graph with parameters \((n, d, \lambda, \mu)\) satisfies

\[ d(d - \lambda - 1) = (n - d - 1)\mu. \quad (2) \]

If both \( G \) and its complement are connected, then \( G \) is \textit{primitive}.\footnote{Some authors exclude imprimitive graphs in the definition of strong regularity.} Otherwise, \( G \) is \textit{imprimitive}.

\textbf{Lemma 2.1.} Let \( H \) be a strongly regular graph with parameter set \((n, d, \lambda, \mu)\). If \((V_1, V_2)\) is a partition of \( V(H) \) such that the induced subgraphs \( H[V_1] \) and \( H[V_2] \) are strongly regular with respective parameter sets \((n_1, d_1, \lambda_1, \mu_1)\) and \((n_2, d_2, \lambda_2, \mu_2)\), then the parameter set of \( H[V_1] \) determines the parameter set of \( H[V_2] \) as follows:

\[ n_2 = n - n_1, \quad (3) \]
\[ d_2 = d - \frac{(d - d_1)n_1}{n_2}, \quad (4) \]
\[ \lambda_2 = \lambda - \frac{\lambda(d - d_2)}{d_2} + \frac{(\lambda - \lambda_1)d_1n_1}{d_2n_2} \quad \text{if } d_2 \neq 0, \text{ and} \quad (5) \]
\[ \mu_2 = \frac{d_2(d_2 - \lambda_2 - 1)}{(n_2 - d_2 - 1)} \quad \text{if } d_2 \neq n_2 - 1. \quad (6) \]

If \( d_2 = 0 \) (respectively \( d_2 = n_2 - 1 \)), then \( \lambda_2 \) (respectively \( \mu_2 \)) can be chosen arbitrarily since \( H[V_2] \) is empty (respectively complete).
Proof. Since \((V_1, V_2)\) is a partition of \(V(H)\), Equation (3) follows.

None of the denominators in Equations (4) and (5) is zero since \(H\) is primitive and the sets \(V_1\) and \(V_2\) are non-empty. Regarding Equation (4) note that both \((d - d_1)n_1\) as well as \((d - d_2)n_2\) count the number of edges of \(H\) that have one end vertex in \(V_1\) and one in \(V_2\). We prove Equation (5). Double counting the triangles which contain vertices of \(V_1\) and \(V_2\) yields

\[
\frac{1}{2}(d - d_2)n_2\lambda = \frac{1}{2}n_1d_1(\lambda - \lambda_1) + \frac{1}{2}n_2d_2(\lambda - \lambda_2),
\]

where on the left side, all \((d - d_2)n_2\) edges with one end in \(V_1\) and the other end in \(V_2\) are considered. Such an edge is contained in \(\lambda\) triangles and each desired triangle is counted twice. The summands on the right side correspond to the triangles containing an edge with both ends in \(V_1\) or both ends in \(V_2\), respectively. Equation (5) is obtained by rearranging Equation (7). Observe that Equation (6) is a reformulation of Equation (2) applied to \(H[V_2]\). \(\square\)

Sporadic strongly regular graphs. The Clebsch graph, the Schläfi graph, the Higman-Sims graph, and the McLaughlin graph are the unique strongly regular graphs (up to isomorphism) with the parameters \((16, 5, 0, 2), (27, 16, 10, 8), (100, 22, 0, 6),\) and \((275, 112, 30, 56),\) respectively. See [GR01] for the uniqueness of the first two graphs, [Gew69] and [GS75] for the uniqueness of the Higman-Sims graph and the McLaughlin graph, respectively.

Graph families. We denote the complete graph of order \(t\) by \(K_t\) and the cycle of order \(t\) by \(C_t\). The disjoint union of \(s\) copies of \(K_t\) is denoted by \(sK_t\).

Let \(m \in \mathbb{N} \setminus \{0\}\). The \(m \times m\) rook’s graph \(R_m\) has vertex set \(\{v_{ij} \mid 1 \leq i, j \leq m\}\) and edge set \(\{v_{ij}v_{i'j'} \mid i = i' \text{ or } j = j' \text{ and } (i, j) \neq (i', j')\}\). For \(i \in \{1, \ldots, m\}\) we call \(v_{ij}\) \((1 \leq j \leq m)\) the \(i\)-th row of \(R_m\). (The \(j\)-th column is defined analogously.) If a graph \(G\) is isomorphic to \(R_m\), then we say that \(G\) is an \(R_m\). The graph \(R_m\) is strongly regular with parameters \((m^2, 2m - 2, m - 2, 2)\). Conversely, Shrikhande [Shr59] proved that for every \(m \neq 4\), a strongly regular graph with parameters \((m^2, 2m - 2, m - 2, 2)\) is an \(R_m\). For \(m = 4\) there exist exactly two types of strongly regular graphs with parameters \((16, 6, 2, 2)\), the \(R_4\) and the Shrikhande graph. Given a strongly regular graph \(G\) with parameter set \((16, 6, 2, 2)\), the following is an easy way to recognize the isomorphism type of \(G\): for \(v \in V(G)\), if \(G[N_G(v)] \cong 2K_3\), then \(G \cong R_4\). Otherwise, \(G[N_G(v)] \cong C_6\) and \(G\) is a Shrikhande graph.

Suppose \(q, d' \in \mathbb{N}_{\geq 1}\) and let \(\kappa\) be a quadratic form on \(F_q^{d'}\). A vector \(v \in F_q^{d'}\) is singular if \(\kappa(v) = 0\). A subspace \(W \subseteq F_q^{d'}\) is totally singular if every vector in \(W\) is singular. For even dimension \(d = 2d'\), with \(d \in \mathbb{N}_{\geq 1}\), there are up to isometry exactly two types quadratic forms. We say that \((F_q^{2d'}, \kappa)\) is of type + (respectively of type −) if the maximal totally singular subspaces of \(F_q^{2d'}\) are \(m\)-dimensional (respectively \(m - 1\)-dimensional). If \((F_q^{2d'}, \kappa)\) is of type \(\varepsilon \in \{+,-\}\), then the affine polar graph \(G^*(F_q^{2d'})\) has vertex set \(F_q^{2d}\), and two vectors \(u, v \in F_q^{2d}\) are adjacent if and only if \(\kappa(u - v) = 0\).

Let \((F_q^5, \kappa)\) be a quadratic space of type −. The bipartite graph \(G(Q_5, (q))\) has as vertex set all one-dimensional and two-dimensional totally singular subspaces of \(F_q^5\) with respect to \(\kappa\), where two subspaces are adjacent whenever one is a subspace of the other. These graphs are examples of generalized quadrangles.

Coloured graphs. A colouring of \(G\) is a map \(\chi : V(G) \to C\). The tuple \((G, \chi)\) is a coloured graph with colours from \(C\). Note that we allow adjacent vertices to have the same colour. If \(|\chi(V)| = \ell\), then \(G\) is \(\ell\)-coloured. We call \((G, \chi)\) monochromatic, bichromatic, or trichromatic if it is 1-coloured, 2-coloured, or 3-coloured, respectively. Let \((G, \chi)\) and \((G', \chi')\) be coloured graphs.

\[\text{6}\]
An isomorphism $\varphi: V(G) \to V(G')$ is colour-preserving if $\chi'(\varphi(u)) = \chi(u)$ for each $u \in V(G)$. For colored graphs, from now on all isomorphisms are required to be color-preserving. If there exists a (colour-preserving) isomorphism between $V(G)$ and $V(G')$, we say that $(G, \chi)$ and $(G', \chi')$ are isomorphic and write $(G, \chi) \cong (G', \chi')$. An automorphism of $(G, \chi)$ is a colour-preserving isomorphism from $V(G)$ to $V(G)$. Suppose $U \subseteq V(G)$. We call $(G, \chi)[U] := (G[U], \chi|_U)$ the (coloured) subgraph of $(G, \chi)$ induced by $U$. The complement of a coloured graph $(G, \chi)$ is $(\overline{G}, \chi)$. 

**Homogeneous blow-ups.** The homogeneous blow-up operation replaces a colour class forming an independent set by a union of cliques of the same size. More precisely, let $(H, \chi_H)$ and $(G, \chi_G)$ be coloured graphs. If there is an independent colour class $R$ of $(G, \chi_G)$, then we say that $(H, \chi_H)$ is a homogeneous blow-up of $(G, \chi_G)$ at $R$ if there exists an integer $t \geq 2$ so that $(H, \chi_H)$ is obtained by replacing each vertex $r$ of $R$ in $G$ by a $t$-clique $C_r$. The new vertices all obtain the colour in $\chi(R)$. Thus the neighbourhood of a new vertex $r' \in C_r$ in $H$ of colour $R$ is $N_H(r') = N_G(r) \cup (C_r \setminus \{r\})$.

**Hadamard matrices.** A Hadamard matrix is a matrix $H \in \{-1, 1\}^{s \times s}$ in which each two distinct rows are orthogonal. Such a matrix has rank $s$. If additionally all row sums of $H$ are equal, then $H$ is regular. Two Hadamard matrices $H_1, H_2 \in \{-1, 1\}^{s \times s}$ are equivalent if $H_2$ can be obtained from $H_1$ by a sequence of operations that multiply a subset of the rows and columns with $-1$ as well as row swaps and columns swaps. We write $H_1 \cong H_2$ to indicate that $H_1$ and $H_2$ are equivalent. The Kronecker product of two matrices $M \in \mathbb{R}^{m \times n}$ and $M' \in \mathbb{R}^{m' \times n'}$ is the matrix $M \otimes M' \in \mathbb{R}^{mm' \times nn'}$ with $(M \otimes M')_{i,j} = M_{i+m',j+n'}$. The Sylvester matrix of rank 2 is

$$\text{Slv}(2) := \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

For an integer $s \geq 2$, the Sylvester matrix of rank $2^s$ is $\text{Slv}(2^s) := \text{Slv}(2) \otimes \text{Slv}(2^{s-1})$. Specifically, this means that $\text{Slv}(2^s) = \begin{pmatrix} \text{Slv}(2^{s-1}) & \text{Slv}(2^{s-1}) \\ \text{Slv}(2^{s-1}) & -\text{Slv}(2^{s-1}) \end{pmatrix}$. Observe that Sylvester matrices are Hadamard matrices.

**Extended Hadamard graphs.** If $H$ is a Hadamard matrix, then the graph $G(H)$ with

$$V(G(H)) = \bigcup_{1 \leq i \leq n} \{r_i^+, r_i^-, c_i^+, c_i^-\},$$

$$E(G(H)) = \bigcup_{1 \leq i \leq n} \{r_i^+ c_i^+, r_i^- c_i^-\} \cup \bigcup_{1 \leq i,j \leq n} \{r_i^+ c_j^-, r_i^- c_j^+\}$$

is the Hadamard graph corresponding to $H$. It is well-known (cf. [BCN89]) that $G(H)$ is $s$-regular, and $|N_{G(H)}(u) \cap N_{G(H)}(v)| = s/2$ if $\text{dist}(u, v) = 2$. Moreover,

$$|N_{G(H)}(u) \cap N_{G(H)}(v) \cap N_{G(H)}(w)| = s/4$$

if $\text{dist}(u, v) = \text{dist}(u, w) = \text{dist}(v, w) = 2$. We extend the edge set of $G(H)$ by $\bigcup_{1 \leq i \leq n} \{r_i^+, r_i^-, c_i^+, c_i^-\}$ and equip the resulting graph $G'(H)$ with a 2-colouring $\chi_H: V(G'(H)) \to \{\text{red, blue}\}$ such that vertices of the form $r_i^+$ or $r_i^-$ are red and vertices of the form $c_i^+$ or $c_i^-$ are blue. The coloured graph $\text{XHd}(G(H)) := (G'(H), \chi_H)$ is
the extended Hadamard graph corresponding to $H$. See Figure 1 for illustrations of the extended Hadamard graph corresponding to $\text{Slv}(2)$. Let $(F, \chi_F)$ be a coloured graph. If there is a bijection $\varphi : \chi_F(V(F)) \to \{\text{red}, \text{blue}\}$ such that $(F, \varphi \circ \chi_F)$ is isomorphic to $(G'(H), \chi_H)$ for some Hadamard matrix $H$, then we say that $(F, \chi_F)$ is an extended Hadamard graph. It holds that $H_1 \cong H_2$ if and only if $\text{XHdG}(H_1) \cong \text{XHdG}(H_2)$. (10)

3 Homogeneity, tuple regularity, and regularity preserving operations

We formally define the two regularity concepts central for our paper. Both of them capture a form of high regularity in a graph.

**Definition 3.1** (Ultrahomogeneity and tuple regularity). A coloured graph $(G, \chi)$ is $k$-ultrahomogeneous, or $k$-UH, if every colour-preserving isomorphism between two induced coloured subgraphs of $(G, \chi)$ of order at most $k$ extends to a colour-preserving automorphism of $(G, \chi)$. Let $R$ be a colour class of $(G, \chi)$ and $U \subseteq V(G)$. We set

\[ N^R(U) := N^R_{(G, \chi)}(U) := \bigcap_{u \in U} N_G(u) \cap R, \text{ and } \]

\[ \lambda^R(U) := \lambda^R_{(G, \chi)}(U) := \left| N^R(U) \right|. \]

The coloured graph $(G, \chi)$ is $k$-tuple regular, or $k$-TR, if for every pair of subsets $U, U' \subseteq V(G)$ with $|U| = |U'| \leq t$ and $(G, \chi)[U] \cong (G, \chi)[U']$ and for every colour class $R$ it holds that $\lambda^R(U) = \lambda^R(U')$. A graph is ultrahomogeneous, or UH, if it is $k$-UH for every $k \in \mathbb{N}$.

Note that $(G, \chi)$ is $k$-TR if it is $k$-UH. Also note that a monochromatic graph $(G, \chi)$ is 2-TR if and only if $G$ is strongly regular.

**Lemma 3.2.** Let $R$ be a colour class of a graph $(G, \chi)$.

(i) If $(G, \chi)$ is $k$-TR (respectively $k$-UH) and $C$ is a union of colour classes of $(G, \chi)$, then $(G, \chi)[C]$ is $k$-TR (respectively $k$-UH).

(ii) If $(G, \chi)$ is $k$-TR for some $k \geq 2$, then $G[N^R(b)]$ and $G[R \setminus N^R(b)]$ are $(k-1)$-TR for each $b \in V(G) \setminus R$.

**Proof.** Claim (i) for ultrahomogeneity has already been observed in [JTS12]. It follows immediately from the respective definition of the properties $k$-TR and $k$-UH. We prove (ii). Let $b \in B$
and \( U, U' \subseteq V(G[N^R(b)]) \) with \(|U|, |U'| \leq k - 1 \) and \( G[N^R(b)][U] \cong G[N^R(b)][U'] \). It holds that \( G[U \cup \{b\}] \cong G[U' \cup \{b\}] \). Since \( G \) is \( k \)-TR,
\[
\lambda_{(G,\chi)[N^R_R(b)]}(U) = \lambda_{(G,\chi)[N^R_R(b)]}(U') = \lambda_{(G,\chi)[N^R_R(b)]}(U').
\]
Hence, \( G[N^R(b)] \) is \((k-1)\)-TR. The result for \( G[R \setminus N^R(b)] \) follows analogously. \( \square \)

4 Bicoloured graphs

We adhere to the following convention throughout this section. Unless stated otherwise, \((G, \chi)\) is a 2-coloured graph with \( \chi : V(G) \to \{\text{red}, \text{blue}\} \). We set \( R := \chi^{-1}(\text{red}) \) and \( B := \chi^{-1}(\text{blue}) \). A vertex \( v \) of \( G \) is red (respectively blue) if \( \chi(v) = \text{red} \) (respectively \( \chi(v) = \text{blue} \)).

Assume that \((G, \chi)\) is \( 3 \)-TR such that \( R \) and \( B \) induce \( 4 \)-TR subgraphs. In this section we prove that \( R \) and \( B \) are homogeneously connected if one of the colour classes induces a primitive graph. If both sets induce imprimitive graphs and \( R \) and \( B \) are not homogeneously connected, then either \((G, \chi)\) can be obtained by a homogeneous-blow-up, or \((G, \chi)\) is an extended Hadamard graph. We prove that the 2-coloured Wagner graph is the only extended Hadamard graph which is \( 4 \)-TR. Indeed, it is even ultrahomogeneous.

4.1 Colour classes inducing primitive graphs

The overall goal of this subsection is to prove the following theorem.

**Theorem 4.1.** Let \((G, \chi)\) be a \( 3 \)-TR graph. If \( G[R] \) is primitive and \( 4 \)-TR, then \( R \) and \( B \) are homogeneously connected.

Due to the classification of \( 4 \)-TR monochromatic primitive graphs (see Table 1), it suffices to prove that \( R \) and \( B \) are homogeneously connected whenever \( G[R] \) or its complement is the \( 3 \times 3 \) rook’s graph, the \( 5 \)-cycle, the Schläfli graph, or the McLaughlin graph.

**Lemma 4.2.** If \((G, \chi)\) is \( 2 \)-TR with \( G[R] \cong C_5 \), then \( R \) and \( B \) are homogeneously connected.

**Proof.** The \( C_5 \) does not allow for a partition into two non-empty sets such that both sets induce a regular graph. It follows with Lemma 3.2.(ii) that every \( b \in B \) satisfies \( N^R(b) \in \{\emptyset, R\} \). Since \((G, \chi)\) is \( 2 \)-TR, we obtain \( N^R(b) = N^R(b') \) for all \( b, b' \in B \). \( \square \)

**Lemma 4.3.** A strongly regular induced subgraph of a rook’s graph is an \( sK_t \) for some \( s, t \in \mathbb{N}_{\geq 1} \) or a rook’s graph.

**Proof.** Let \( m \in \mathbb{N}_{\geq 1} \) and let \( A \subseteq V(R_m) \) be a set of vertices such that \( G[A] \) is strongly regular with parameters \((n, d, \lambda, \mu)\). Assume that \( G[A] \) is not empty. If all neighbours of \( v_{i,j} \) in \( G[A] \) are in row \( i \) (respectively in column \( j \) ), then \( d = \lambda + 1 \), which implies \( G[A] \cong sK_{d} \) for some \( s \in \mathbb{N}_{\geq 1} \). Otherwise, there are \( i', j' \in \{1, \ldots, m\} \) with \( i' \neq i \) and \( j' \neq j \) such that \( v_{i',j}, v_{i,j'} \in A \). Every additional neighbour of \( v_{i,j} \) in \( G[A] \) is either in row \( i \) and, hence, a common neighbour of \( v_{i,j} \) and \( v_{i',j'} \) or in column \( j \) and, hence, a common neighbour of \( v_{i,j} \) and \( v_{i',j'} \). Consequently,
\[
d = |N_{G[A]}(v_{i,j}) \cap N_{G[A]}(v_{i',j'})| + |N_{G[A]}(v_{i,j}) \cap N_{G[A]}(v_{i,j'})| + 2 = 2\lambda + 2.
\]
Since \( G[A] \) is a subgraph of \( G \) we have \( \mu \in \{0, 1, 2\} \) and by Equation (2) \( \mu \neq 0 \). If \( \mu = 2 \), then \( n = (\lambda + 1)^2 \) and \( G[A] \) is a \( R_{\lambda+1} \) by Shrikhande’s Theorem [Shr59] (note that the graph induced
by six distinct neighbours of a vertex in a rook’s graph is of girth 3, whereas the neighbourhood of a vertex in the Shrinkhande graph induces a $C_6$. If $\mu = 1$, then $v_{i,j'} \notin A$ and hence, $v_{i',j''} \in A$ for some $j'' \notin \{j,j'\}$. Since $v_{i,j''}$ has a common neighbour with $v_{i,j'}$ in $A$, we obtain $v_{i,j''} \in A$. This is a contradiction since $v_{i,j}$ and $v_{i',j''}$ have two common neighbours in $G[A]$.  

**Corollary 4.4.** If $(G, \chi)$ is 3-TR and $G[R]$ is an $R_m$ for some $m \geq 3$, then $R$ and $B$ are homogeneously connected.

**Proof.** Towards a contradiction, suppose that there exists $b \in B$ with $\emptyset \subsetneq R \cap N_G(b) \subsetneq R$. By Part (ii) of Lemma 3.2 the red vertices can be partitioned into two sets $A$ and $B$ such that both graphs, $G[A]$ and $G[B]$, are 2-TR. By Lemma 4.3, $G[A]$ and $G[B]$ each are a disjoint union of cliques or a rook’s graph. However, a rook’s graph $R_m$ with $m \geq 3$ cannot be partitioned into two induced graphs of the form $sK_t$ and $s'K_{t'}$, respectively.

**Lemma 4.5.** If $H$ is a McLaughlin graph or a Schlafli graph and $(V_1, V_2)$ is a partition of $V(H)$, then one of the graphs $H[V_1]$ and $H[V_2]$ is not strongly regular.

**Proof sketch.** Assume that $H$ is the Schlafli graph. Suppose that $V(H)$ can be partitioned into two sets $V_1$ and $V_2$ such that both graphs $H[V_1]$ and $H[V_2]$ are strongly regular. We denote the respective parameter sets by $(n_1, d_1, \lambda_1, \mu_1)$ and $(n_2, d_2, \lambda_2, \mu_2)$. The Schlafli graph is of order 27, and hence, we may assume by symmetry that $n_1 \leq 13$.

According to Brouwer’s list (see [BH12, Subsection 9.9] and [Bro]) of feasible parameter sets for strongly regular graphs, there are five parameter sets of primitive strongly regular graphs of order at most 13, namely $(5, 2, 0, 1)$, $(9, 4, 1, 2)$, $(10, 3, 0, 1)$, $(10, 6, 3, 4)$, and $(13, 6, 2, 3)$. Choosing one of these for the parameters of $H[V_1]$ we can compute the parameters of $H[V_2]$ using Lemma 2.1. However, all possible choices for the parameters of $H[V_1]$ lead to a violation of the integrality conditions for the parameters of $H[V_2]$ (see also Appendix A). It remains to consider parameter sets $(n_1, d_1, \lambda_1, \mu_1)$ with $n_1 \leq 13$ which correspond to imprimitive graphs. Since the clique number of the Schlafli graph is 6 and its largest independent set is of cardinality 3, we may restrict ourselves to the following graphs:

- $sK_t$ with $s \leq 3$, $t \leq 6$, and $st \leq 13$,
- $sR_{t'}$ with $s' \leq 6$, $t' \leq 3$ and $s't' \leq 13$.

There are 24 of these imprimitive graphs in total. Again, all of possible parameter sets lead to a violation of the integrality conditions via Lemma 2.1 (see Appendix A). Altogether, there is no partition of the Schlafli into two strongly regular graphs.

With the same approach, we prove that the McLaughlin graph does not allow for such a partition either. We refer to Appendix A for the detailed calculations.  

**Corollary 4.6.** If $(G, \chi)$ is 3-TR and $G[R]$ is a McLaughlin graph or a Schlafli graph, then $R$ and $B$ are homogeneously connected.

**Proof.** Suppose that there exists some blue vertex $b \in B$ with $\emptyset \subsetneq N^R(b) \subsetneq R$. By Lemma 3.2(ii), both graphs $G[N^R(b)]$ and $G[R \setminus N^R(b)]$ are strongly regular. This contradicts Lemma 4.5.  

**Proof of Theorem 4.1.** Assume that $G[R]$ is primitive and 4-TR. According to the classification of 4-TR graphs (cf. [Cam04]), either $G[R]$ or its complement is isomorphic to $C_5$, $R_3$, the Schlafli graph, or the McLaughlin graph. The claim follows by applying Lemma 4.2, Corollary 4.4, or Corollary 4.6 to $(G, \chi)$ or its complement.
4.2 Colour classes inducing imprimitive graphs

In this subsection, we prove the following theorem.

**Theorem 4.7.** If \((G, \chi)\) is 3-TR with \(G[R] \cong s_R K_{t_R}\) and \(G[B] \cong s_B K_{t_B}\), then

(i) \(R\) and \(B\) are homogeneously connected, or

(ii) \(G[R]\) and \(G[B]\) each are edgeless or complete and \(R\) and \(B\) are matching-connected, or

(iii) \((G, \chi)\) is a homogeneous blow-up, or

(iv) \((G, \chi)\) is an extended Hadamard graph.

We first consider the case that both colour classes are edgeless or complete. This allows us to interpret the graph \((G, \chi)\) as an incidence graph of a structure.

**Lemma 4.8.** Let \((G, \chi)\) be 3-TR. If each of the graphs \(G[R]\) and \(G[B]\) is edgeless or complete, then \(R\) and \(B\) are either homogeneously connected or matching-connected.

**Proof.** Consider the incidence structure \(I\) with point set \(R\) and block multiset \(\{(N^R(b) \mid b \in B)\}\).

The 3-tuple regularity of \((G, \chi)\) implies that \(I\) is a symmetric 3-design. It follows with (1) that 
\[ k \in \{0,1,|R|-1,|R|\}, \]
where \(k\) denotes the size of a block of \(I\). If \(k \in \{0,|R|\}\), then \(R\) and \(B\) are homogeneously connected. Otherwise \(k \in \{1,|R|-1\}\), that is, the classes \(R\) and \(B\) are matching-connected.

Suppose \(b \in B\). If \(R\) and \(B\) are not homogeneously connected, then both graphs \(G[N^R(b)]\) and \(G[R \setminus N^R(b)]\) exist and are strongly regular by Lemma 3.2.(ii). It follows that \(G[N^R(b)]\) and \(G[R \setminus N^R(b)]\) are disjoint unions of cliques. In particular, either

- \(G[N^R(b)] \cong s_K K_t\) for some \(0 < s < s_R\). or
- \(G[N^R(b)] \cong s_R K_t\) for some \(0 < t < t_R\).

Note that the values \(s\) and \(t\) are independent of the choice of \(b\) since \((G, \chi)\) is 3-TR. In the following Lemma, we prove that Theorem 4.7 holds if we assume (11).

**Lemma 4.9.** Let \((G, \chi)\) be 3-TR with \(G[R] \cong s_R K_{t_R}\) and \(G[B] \cong s_B K_{t_B}\). Suppose \(b \in B\). If \(G[N^R(b)] \cong s_K K_t\) for some \(0 < s < s_R\), then \((G, \chi)\) can be obtained by a homogeneous blow-up of the independent red colour class of some graph \((H, \chi_H)\).

**Proof.** Since \((G, \chi)\) is 3-TR, it holds that \(G[N^R(b')] \cong s_K K_t\) for each \(b' \in B\). For each maximal red clique \(C\) of \((G, \chi)\) choose a vertex \(r_C \in C\). Set \(R_H := \{r_C \mid C\) is a maximal red clique\}. The graph \((H, \chi_H) := (G, \chi)[B \cup R_H]\) satisfies the required conditions.

Finally, we assume (12) and prove that \((G, \chi)\) is an extended Hadamard graph:

**Lemma 4.10.** Let \((G, \chi)\) be 3-TR with \(G[R] \cong s_R K_{t_R}\) and \(G[B] \cong s_B K_{t_B}\). If \(G[N^R(b)] \cong s_R K_t\) for some \(b \in B\) and \(0 < t < t_R\), then \((G, \chi)\) is an extended Hadamard graph.

**Proof.** For \(i \in \{1, \ldots, 8\}\) let \(U_i\) be a subset that induces a graph as depicted in Figure 2. Since \((G, \chi)\) is 3-TR, the values \(\lambda^R(U_i)\) and \(\lambda^B(U_i)\) are determined by the isomorphism type of \((G, \chi)[U_i]\).

**Claim 1:** For \(X \in \{B,R\}\) let \(C^X \subseteq X\) be a maximal clique in \(G[X]\). The design with points \(C^B\) and blocks \(\{(N(r) \cap C^B \mid r \in C^R)\}\) is a symmetric \(2-(t_R, t, \lambda^R(U_i))\)-design. In particular, \(t_B = t_R\) and \(G[N^B(r)] \cong s_B K_t\) for each \(r \in R\).
Finally, if \( \lambda \) contradiction since the red vertex in Claim 3:

Proof of Claim 3:

\[
\text{Let } t = |N_G(b_1) \cap C^R| = \lambda^R(U_2) + 1.
\]

Observe that \( N_G(b_1) \cap C^R \neq N_G(b_2) \cap C^R \). (Otherwise all blue vertices adjacent to \( b_1 \) have the exactly same neighbours in \( C^R \) which implies \( t \in \{0, t_R\} \). Consequently, there exists a red vertex \( r' \in C^R \) such that \( (G, \chi)[\{b_1, b_2, r'\}] \cong (G, \chi)[U_4] \). Hence

\[
|N_G(b_1) \cap N_G(b_2) \cap C^R| = \lambda^R(U_4).
\]

Interchanging the roles of \( R \) and \( B \), we obtain

\[
|N_G(r_1) \cap N_G(r_2) \cap C^B| = \lambda^B(U_3).
\]

Altogether, the considered structure is a symmetric \( 2-\tau_{tR} \)-design. It follows with Fisher’s inequality that \( t_B = |C^B| = |C^R| = t_R \) and \( G[N^B(r)] \cong s_B K_t \) for each \( r \in R \).

Claim 2: There does not exist a parameter \( \lambda_n \) such that every pair of non-adjacent blue vertices has exactly \( \lambda_n \) common neighbours in every maximal red clique.

Proof of Claim 2: Suppose towards a contradiction that every pair of non-adjacent blue vertices has \( \lambda_n \) common neighbours in every maximal red clique. Let \( C^B_1 \) and \( C^B_2 \) be two distinct maximal blue cliques and let \( C^R \) be a maximal red clique. Fix some \( b_1 \in C^B_1 \). Double-counting all vertex sets \( \{b_1, b_2, r\} \) that induce a triangle with \( b_2^I \in C^B_1, r \in C^R \) and all vertex sets \( \{b_1, r, b_2\} \) that induce a subgraph isomorphic to \( (G, \chi)[U_5] \) with \( r \in C^R \) and \( b_2 \in C^B_2 \) yields

\[
t_R \lambda_n = t^2 \text{ and } (t_R - 1) \lambda^R(U_4) = t(t - 1),
\]

respectively. This implies

\[
t = t_R(\lambda_n - \lambda^R(U_4)) + \lambda^R(U_4).
\]

If \( \lambda_n > \lambda^R(U_4) \), then \( t \geq t_R \) which contradicts the assumption \( t < t_R \). If \( \lambda_n = \lambda^R(U_4) \), then \( t = \lambda^R(U_4) \) which implies that the two blue vertices in \( U_4 \) are false twins. This is a contradiction since the red vertex in \( U_4 \) is adjacent to exactly one of the two blue vertices. Finally, if \( \lambda_n < \lambda^R(U_4) \), then \( t \leq 0 \) which contradicts the assumption \( t > 1 \).

Claim 3: \( t = \tau_{n/2} \).

Proof of Claim 3: If \( t < \frac{4}{t_R} \), then for each two non-adjacent blue vertices \( b_1 \) and \( b_2 \) and each maximal red clique \( C^R \) there exists \( r' \in C^R \) such that \( (G, \chi)[\{b_1, b_2, r'\}] \cong (G, \chi)[U_6] \) and,
hence, $|C^R \cap N_G(b_1) \cap N_G(b_2)| = \lambda^R(U_6)$ which contradicts Claim 2. If $t > \frac{t_R}{2}$, then for each two non-adjacent blue vertices $b_1$ and $b_2$ and each maximal red clique $C^R$ it holds that $|C^R \cap N_G(b_1) \cap N_G(b_2)| = \lambda^R(U_5) + 1$, which is a contradiction to Claim 2. ■

**Claim 4:** If $b_1$ and $b_2$ are two non-adjacent blue vertices and $C^R$ is a maximal red clique, then $|C^R \cap N_G(b_1) \cap N_G(b_2)| \in \{0, \frac{t_R}{2}\}$. 

**Proof of Claim 4:** It follows from Claim 1 that $N_G(b_1) \cap C^R \neq \emptyset$. For every $r' \in N_G(b_1) \cap C^R$ it holds that 

$$|C^R \cap N_G(b_1) \cap N_G(b_2)| = \begin{cases} \lambda^R(U_5) + 1 & \text{if } r' \text{ is adjacent to } b_2, \\ \lambda^R(U_5) & \text{otherwise.} \end{cases} (16)$$

By Claim 2 we may conclude that $$\lambda^R(U_6) \neq \lambda^R(U_5) + 1.$$ This implies, that all $r' \in N_G(b_1) \cap C^R$ satisfy the same condition on the right side of (16). Altogether, if $N_G(b_1) \cap N_G(b_2) \cap C^R \neq \emptyset$, then $N_G(b_1) \cap C^R = N_G(b_2) \cap C^R$. Claim 4 follows since $|N_G(b_1) \cap C^R| = \frac{t_R}{2}$ by Claim 3. ■

**Claim 5:** $t_R = 2$, $t = 1$.

**Proof of Claim 5:** Let $C^R$ be a maximal red clique and $C^B_1, C^B_2$ be two distinct maximal blue cliques. Further let $b_1 \in C^B_1$. By Claims 2 and 4, there exist vertices $b_2$ and $b_3$ in $C^B_2$ such that $C^R \cap N_G(b_1) = C^R \cap N_G(b_2)$ and $C^R \cap N_G(b_1) = C^R \cap N_G(b_3)$. It follows that $\lambda^R(U_4) = 0$. As argued in the proof of Claim 1 (see Equation (15)), we have $$0 = \lambda^R(U_4)(t_R - 1) = t(t - 1).$$ Hence, $t = 1$ and it follows with Claim 3 that $t_R = 2$. ■

We obtain that $G[R] \cong s_R K_2$ and $G[B] \cong s_B K_2$ and the edges that join a blue 2-clique with a red 2-clique form a perfect matching.

It remains to show that $(G, \chi)$ is an extended Hadamard graph. Choose an ordering $C^R_1, \ldots, C^R_{s_R}$ of the red 2-cliques and an ordering $C^B_1, \ldots, C^B_{2s_B}$ of the blue 2-cliques of $(G, \chi)$. Further, relabel the vertices of $G$ so that $C^R_i = \{r^+_i, r^-_i\}$ and $C^B_i = \{c^+_i, c^-_i\}$ for each $i \in \{1, \ldots, s\}$. Consider the matrix $H \in \{-1, 1\}^{s \times s}$ with 

$$H_{i,j} = \begin{cases} 1 & \text{if } c^+_i \text{ is adjacent to } r^+_j, \\ -1 & \text{otherwise} \end{cases}$$

for all $i, j \in \{1, \ldots, s\}$. Observe that $$|N_G(c^+_i) \cap N_G(c^+_j)| = \lambda^R(U_8) = |N_G(c^+_i) \cap N_G(c^-_j)|.$$ Hence, the rows of $H$ are pairwise orthogonal. Interchanging the roles of $R$ and $B$, we obtain that the columns of $H$ are pairwise orthogonal. In particular, $s_R = s_B$, matrix $H$ is a Hadamard matrix, and $(G, \chi)$ is isomorphic to the extended Hadamard graph of $H$. ■

**Proof of Theorem 4.7.** Let $(G, \chi)$ be 3-TR with $G[R] \cong s_R K_{t_R}$ and $G[B] \cong s_B K_{t_B}$. Assume that $R$ and $B$ are not homogeneously connected. If both induced subgraphs $G[R]$ and $G[B]$ are edgeless or complete, then Lemma 4.8 implies that $R$ and $B$ are matching-connected. Therefore, we may assume that $G[R]$ is neither edgeless nor complete, that is, $s_R, t_R \geq 2$. As argued above, either $G[N^R(b)]$ is isomorphic to $s_R K_{t_R}$ for some $0 < s < s_R$ or to $s_R K_{t_R}$ for some $0 < t < t_R$. In the first case, Lemma 4.9 implies that $(G, \chi)$ is a blow-up. In the second case, $(G, \chi)$ is an extended Hadamard graph by Lemma 4.10. ■
4.3 Extended Hadamard graphs

In this subsection we first classify the extended Hadamard graphs which are 3-UH (Theorem 4.16). Our strategy for this is as follows. We use the classification of Ó Catháin [ÓC11] of Hadamard graphs whose row permutation groups are 3-transitive. The 3-transitivity of the automorphism group’s action on rows (respectively columns) of the Hadamard matrix $H$ implies 3-transitivity of the induced action of $\text{Aut}(G(H))$ on the red (respectively blue) cliques of $G(H)$. However, we need to analyse the action on the vertices rather than on the cliques.

First, we need to show that every triple of independent red vertices can be mapped to every triple of independent red vertices. Due to the 3-transitive action on cliques, for this, we only need to show that within the subgroup of automorphisms fixing the first three cliques as sets, we can arbitrarily permute the vertices within each clique. It thus suffices to find for each of the first 3 cliques an automorphism that swaps the two vertices within the clique but fixes vertices within the other two cliques. (The action on cliques other than the first three can be arbitrary.)

For Hadamard matrices that are Sylvester matrices this is shown in Lemma 4.13 and for the matrix of rank 12 this is shown in Lemma 4.15(i).

Second, we need to show that for every pair of triples, each consisting of two non-adjacent red vertices and a blue vertex, there is an automorphism mapping one triple to the other. Once we have shown transitivity of the automorphism group’s action on pairs of non-adjacent red vertices in the first step, it suffices to show that among the subgroup of automorphisms that fix all points in the first two cliques pointwise, every blue vertex $b$ can be mapped to all blue vertices that have the same neighbourhood within the first two red cliques as $b$. For Sylvester matrices this is shown in Lemma 4.14 and for the Matrix of rank 12 this is shown in Lemma 4.15(ii).

We present our proof in terms of Hadamard matrices. A square matrix is monomial if each row and each column contains exactly one non-zero entry. We denote the set of all monomial matrices in $\{-1,1\}^{s \times s}$ by $\mathcal{M}_s$. Then $\mathcal{M}_s \times \mathcal{M}_s$ acts on the order-$s$ Hadamard matrices via $(A,B)H := AHB^{-1}$. If $H$ is a Hadamard matrix, then the automorphism group $\text{Aut}(H)$ of $H$ is the stabiliser of $H$ under this action.

An element of $\text{Aut}(H)$ is an automorphism of $H$. Observe that $A \in \mathcal{M}_s$ has a unique factorisation $A = D_A P_A$ into a diagonal matrix $D_A$ and a permutation matrix $P_A$. For a Hadamard matrix $H$ and $(A,B) \in \text{Aut}(H)$ set $\nu(A,B) = P_A$. Set $\mathcal{A}(H) := \nu(\text{Aut}(H))$ to be the induced row permutation group.

Theorem 4.11 ([ÓC11]). If $H \in \{-1,1\}^{s \times s}$ is a Hadamard matrix such that $\mathcal{A}(H)$ is 3-transitive, then $H$ is equivalent to a Sylvester matrix or $s = 12$.

By the theorem, we only need to consider graphs which are constructed from Hadamard matrices or from the (up to equivalence) unique Hadamard matrix of rank 12. By symmetry, these are also precisely the Hadamard whose column permutation groups act 3-transitively.

Lemma 4.12. If $H \in \{-1,1\}^{s \times s}$ is a Hadamard matrix and $(A,B) \in \text{Aut}(H)$, then

\[(i) \quad \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \otimes A, \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \otimes B \in \text{Aut}(\text{Slv}(2) \otimes H) \text{ and} \]

\[(ii) \quad \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) \otimes A, \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \otimes B \in \text{Aut}(\text{Slv}(2) \otimes H). \]

Proof. The statement follows with

\[
\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} H & H \\ H & -H \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix}^{-1} = \begin{pmatrix} AHB^{-1} & AHB^{-1} \\ AHB^{-1} & -AHB^{-1} \end{pmatrix} = \begin{pmatrix} H & H \\ H & -H \end{pmatrix}
\]

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and
\[
\begin{pmatrix}
A & 0 \\
0 & -A
\end{pmatrix}
\begin{pmatrix}
H & H \\
H & -H
\end{pmatrix}
\begin{pmatrix}
0 & B \\
0 & B
\end{pmatrix}^{-1} = \begin{pmatrix}
AHB^{-1} & AHB^{-1} \\
AHB^{-1} & -AHB^{-1}
\end{pmatrix} = \begin{pmatrix}
H & H \\
H & -H
\end{pmatrix}.
\]

\[\square\]

**Lemma 4.13.** Let \( H = \text{Slv}(2^t) \) for some \( t \geq 2 \). For each \( i \in \{1, 2, 3\} \), there exists an automorphism \((A, B) \in \text{Aut}(H)\) such that \( A_{i,i} = -1 \) and \( A_{j,j} = 1 \) for all \( j \in \{1, 2, 3\} \setminus \{i\} \).

**Proof.** It is easy to see that
\[
\left(\begin{array}{cc}
-1 & 0 \\
0 & -1
\end{array}\right), \left(\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right)\text{, and } (1_2, 1_2).
\]
are automorphisms of \( \text{Slv}(2) \). We prove the lemma by induction on \( t \). If \( t = 2 \), then by Lemma 4.12
\[
\left(\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right) \otimes \left(\begin{array}{cc}
-1 & 0 \\
0 & 1
\end{array}\right), \left(\begin{array}{cc}
1 & 0 \\
0 & 1
\end{array}\right) \otimes \left(\begin{array}{cc}
0 & 1 \\
1 & 0
\end{array}\right),\text{ and } (1_2 \otimes 1_2, 1_2 \otimes 1_2)
\]
are automorphisms of \( \text{Slv}(4) \) with the respective desired properties. Assume that the statement is satisfied for some \( t \geq 2 \) and for \( i \in \{1, 2, 3\} \), suppose \((A^{(i)}, B^{(i)}) \in \text{Aut}(\text{Slv}(2^t))\) such that \( A_{i,i}^{(i)} = -1 \) and \( A_{j,j}^{(i)} = 1 \) for each \( j \in \{1, 2, 3\} \setminus \{i\} \). By Lemma 4.12, we have \((1_2 \otimes A^{(i)}, 1_2 \otimes B^{(i)}) \in \text{Aut}(\text{Slv}(2^{t+1}))\) and \((1_2 \otimes A^{(i)})_{j,j} = -1\) whereas \((1_2 \otimes A^{(i)})_{j,j} = 1\).

Next, we search automorphisms for a Sylvester matrix that act on some of the columns in a particular manner while fixing the first and the second row. Additionally we want the coefficients with which the first two rows are multiplied to be 1.

Given a column \( c \) of a Hadamard matrix \( H \in \{-1, 1\}^{s \times s} \), let \( \omega_c(H) \) be the set of columns whose first two entries agree with the first two entries of \( c \) or which are their inverses. More formally, we define
\[
\omega_c(H) = \{ c' \in \{1, \ldots, s\} \mid (H_{1,c}, H_{2,c}) \in \{(H_{1,c}, H_{2,c}), (-H_{1,c}, -H_{2,c})\} \}.
\]

**Lemma 4.14.** Let \( t \in \mathbb{N}_{\geq 2} \) and let \( c \) be a column of \( \text{Slv}(2^t) \). For each \( j \in \omega_c(\text{Slv}(2^t)) \), there exists an automorphism \((A, B) \in \text{Aut}(\text{Slv}(2^t))\) such that \( A_{1,1} = A_{2,2} = 1 \) and \( B_{c,j} \in \{-1, 1\} \).

**Proof.** We prove the statement by induction on \( t \). If \( t = 2 \), then \( |\omega_c(\text{Slv}(2^t))| = 1 \) and the identity automorphism satisfies the claim. Assume from now on that the claim holds true for some \( t \geq 2 \). Let \( j \) be a column of \( \text{Slv}(2^{t+1}) \) with \( j \in \omega_c(\text{Slv}(2^{t+1})) \). Observe that \((j \mod 2^t + 1) \in \omega_{(c \mod 2^{t+1})}(\text{Slv}(2^t))\) by the construction of the Sylvester matrix. By assumption, there exists \((A, B) \in \text{Aut}(\text{Slv}(2^t))\) with \( A_{1,1} = A_{2,2} = 1 \) and \( B_{c \mod 2^{t+1}, j \mod 2^{t+1}} \in \{-1, 1\}\). If \( \max(c, j) \leq 2^t \) or \( \min(c \mod 2^t + 1, j \mod 2^t + 1) \geq 2^t + 1 \), then \((1_2 \otimes A, 1_2 \otimes B)\) is an automorphism of \( H \) which satisfies the claim. Otherwise, \((1_2 \otimes A, 1_2 \otimes B)\) is an automorphism of \( H \) which satisfies the desired property. \[\square\]
We now execute our proof strategy again for the Hadamard matrix $H^{12}$ of rank $12$. Up to equivalence this matrix is given by

$$H^{12} = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 \\
1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 \\
1 & 1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 \\
1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 \\
1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & -1 \\
1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 \\
1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 \\
1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 \\
1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & -1 \\
1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 \\
1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1
\end{pmatrix}.$$ 

**Lemma 4.15.**

(i) For each $i \in \{1, 2, 3\}$, there exists an automorphism $(A, B) \in \text{Aut}(H^{12})$ such that $A_{i,j} = -1$ and $A_{j,j} = 1$ for all $j \in \{1, 2, 3\} \setminus \{i\}$.

(ii) Let $c$ be a column. For each $j \in \omega_c(H^{12})$, there exists an automorphism $(A, B) \in \text{Aut}(H^{12})$ with $A_{1,j} = A_{2,j} = 1$ and $B_{c,j} \in \{1, -1\}$.

**Proof.** For notational brevity, we describe an automorphism $(A, B) \in \text{Aut}(H^{12})$ by a row permutation $\sigma^A$, a row inversion set $\mathcal{I}^A$, a column permutation $\sigma^B$, and a column inversion set $\mathcal{I}^B$ such that for $\Theta \in \{A, B\}$ we have

$$\Theta_{k,l} = \begin{cases}
-1, & \text{if } \sigma^\Theta(l) = k \text{ and } l \in \mathcal{I}^\Theta, \\
1, & \text{if } \sigma^\Theta(l) = k \text{ and } l \notin \mathcal{I}^\Theta, \\
0, & \text{otherwise.}
\end{cases}$$

For (i), the table below specifies for each $i \in \{1, 2, 3\}$ an automorphism $(A^{(i)}, B^{(i)})$ that maps the first column to the $j$-th column. Given $j \in \omega_c(H^{12})$ for some $c \in \{1, \ldots, 12\}$, the automorphism $(A^{(c)}(A^{(j)}, B^{(c)}(j)))$ satisfies the required conditions: if $j, c \notin \omega_1(H^{12})$, then we have $A^{(c)}_{2,2} = A^{(j)}_{2,2} = -1$ while $A^{(c)}_{1,1} = A^{(j)}_{1,1} = 1$ and $B^{(c)}_{c,1} = B^{(j)}_{c,1} = 1$ if $j, c \in \omega_1(H^{12})$. Thus, we have $A^{(c)}_{2,2} A^{(j)}_{1,1} = (A^{(c)}_{1,1} A^{(j)}_{2,2} = (B^{(c)}_{c,1} B^{(j)}_{c,1}) = 1$. A similar argument applies if $j, c \in \omega_2(H^{12})$.

| $i$ | $\sigma^{A(i)}$ | $\mathcal{I}^{A(i)}$ | $\sigma^{B(i)}$ | $\mathcal{I}^{B(i)}$ |
|-----|-----------------|-----------------|-----------------|-----------------|
| 1   | (4 1 2 5 9) (7 1 0 11 8) | \{1, 6, 7, 8, 10, 11\} | (1 6)(2 4 3 5) (7 1 0 8 11)(9 12) | \{1, \ldots, 12\} |
| 2   | (4 1 0 5 8) (7 9 11 12) | \{2, 6, 7, 9, 11, 12\} | (1 9)(2 7 3 8) (4 1 1 5 10)(6 12) | \{} |
| 3   | (4 1 1 5 7) (8 1 2 10 9) | \{3, 6, 8, 9, 10, 12\} | (1 12)(2 1 1 0 3 11) (4 7 5 8)(6 9) | \{} |

| $j$ | $\sigma^{A(j)}$ | $\mathcal{I}^{A(j)}$ | $\sigma^{B(j)}$ | $\mathcal{I}^{B(j)}$ |
|-----|-----------------|-----------------|-----------------|-----------------|
| 1   | (3 1 2 1 1) (4 9 6) (5 7 8) | \{} | (3 1 0 1 2)(4 7 8)(5 6 9) | \{} |
We first prove \( (λ, b) \{ A \) and \( \psi \H \) is a Sylvester matrix or \( H \) is equivalent to a Sylvestermatix or \( s = 12 \).

\[(ii) \ (G, χ) \) is 4-TR if and only if \( (G, χ) \cong \XHdG(\Slv(2)) \). Moreover, \( \XHdG(\Slv(2)) \) is UH.

**Proof.** We first prove (i). Suppose \( U \subseteq V(G) \). If \( U \) is not an independent set, then \( λ^R(U) = λ^B(U) = 0 \). Therefore, let \( U \) be independent with at most three vertices. We set \( r_U := |U \cap R| \) and \( b_U := |U \cap B| \). If \( r_U \geq 2 \), then \( λ^R(U) = 0 \). If \( r_U = 1 \), then the vertex \( u_v \in U \cap R \) has exactly one red neighbour \( v_r \) in \( (G, χ) \). Since \( U \) is independent and every blue vertex has a neighbour in \( \{u_v, v_r\} \), we obtain \( λ^R(U) = 1 \). For the remaining cases we claim that

\[
λ^R(U) = \frac{s}{2b_u - 1} \quad \text{if } r_U = 0 \text{ and } b_U \in \{1, 2, 3\}.
\]

If \( b_U \in \{1, 2\} \), then the equation above is a direct consequence of Equation (8). If \( b_U = 3 \), then this follows from Equation (9). (The \( λ^R \)-values can be obtained by interchanging the roles of the two colours.)

We prove (ii). If \( (G, χ) \) is 3-UH, then \( A(H) \) is 3-transitive. By Theorem 4.11 and Statement (10), we may assume that \( H \) is a Sylvester matrix or \( H^1 \). Suppose \( \{v_1, v_2, v_3\}, \{v_1', v_2', v_3'\} \subseteq V(G) \) and there is an isomorphism between \( (G, χ)[\{v_1, v_2, v_3\}] \) and \( (G, χ)[\{v_1', v_2', v_3'\}] \) mapping \( v_i \) to \( v_i' \). Due to unique neighbourhood within the 2-cliques, we may assume that \( vv' \notin E(G) \) for all vertices \( v, v' \in \{v_1', v_2', v_3'\} \) with \( χ(v) = χ(v') \).

First assume that \( \{v_1, v_2, v_3\}, \{v_1', v_2', v_3'\} \subseteq R \) and thus

\[
(G, χ)[\{v_1, v_2, v_3\}] \cong (G, χ)[\{v_1', v_2', v_3'\}] \cong \overline{K}_3.
\]

By the 3-transitivity of \( A(H) \) there exists automorphisms \( φ, φ' \in \text{Aut}((G, χ)) \) with \( φ(v_i), φ'(v_i') \in \{r_i^+, r_i^-\} \) for all \( i \in \{1, 2, 3\} \). Further, Lemma 4.13 (respectively Lemma 4.15(ii)) provides an automorphism \( ψ_j \) for each \( j \in \{1, 2, 3\} \) such that \( ψ_j(r_i^+) = r_j^{-} \) and \( ψ_j(r_i^-) = r_j^{+} \) for all \( j' \in \{1, 2, 3\} \setminus \{j\} \). Therefore, for every set \( J \subseteq \{1, 2, 3\} \) there exists an automorphism \( ψ_J \) with \( ψ_J(r_i^+) = r_j^{-} \) and \( ψ_J(r_i^-) = r_j^{+} \) for all \( j \in J \). This implies that there is a \( J \subseteq \{1, 2, 3\} \) such
that \( \varphi \circ \psi \circ \varphi^{-1} \) extends the colour-preserving isomorphism between \((G, \chi)[\{v_1, v_2, v_3\}]\) and \((G, \chi)[\{v'_1, v'_2, v'_3\}]\).

Now assume \(\{v_1, v_2, v_3\} \cap R = 2\). As in the previous case, there exists \(\varphi, \varphi' \in \text{Aut}((G, \chi))\) and \(J \subseteq \{1, 2\}\) such that \((\varphi \circ \psi)(v_i) = \varphi'(v'_i) \in \{r^+_i, r^-_i\}\) for all \(i \in \{1, 2\}\). Note that we might have \((\varphi \circ \psi)(v_3) \neq v_3\) and \(\varphi'(v'_3) \neq v'_3\). So from Lemma 4.14 (respectively Lemma 4.15(ii)) we obtain an automorphism \(\rho\) mapping \((\varphi \circ \psi)(v_3)\) to \(\varphi'(v'_3)\) while \((\varphi \circ \psi \circ \rho)(v_i) = \varphi'(v'_i)\) for each \(i \in \{1, 2\}\). Hence the automorphism \(\varphi \circ \psi \circ \rho \circ \varphi^{-1}\) is an extension of the colour-preserving isomorphism between \((G, \chi)[\{v_1, v_2, v_3\}]\) and \((G, \chi)[\{v'_1, v'_2, v'_3\}]\).

The remaining cases follow by interchanging the roles of the colours. Furthermore, by what we just argued, isomorphisms between induced subgraphs with fewer than 3 vertices also extend to automorphisms of \((G, \chi)\).

It remains to prove (iii). If \((G, \chi)\) is 4-TR, then there is a parameter \(\lambda^R (\lambda^B)\) such that any four independent blue (red) vertices have \(\lambda^R (\lambda^B)\) common red (blue) neighbours in \((G, \chi)\). Let \(r \in R\) and \(b \in B\) be adjacent in \(G\). Consider the incidence relation \(\mathcal{I}\) with point set \(p := N^R(b) \setminus \{r\}\), block set \(\{(r', b') \in E(G) \mid b' \in N^B(r) \setminus \{b\}\}\). Each three points of \(\mathcal{I}\) form together with \(r\) an independent 4-set of \(G\) and, hence, there exists \(\lambda^B - 1\) blocks containing all three points. Analogously, each three blocks intersect in \(\lambda^R - 1\) points. In particular, \(\mathcal{I}\) is a symmetric 3-design. We obtain with (1) that \(|N_G(b) \cap N_G(b')| \in \{0, 1, s-1, s\}\) for each block \(b'\) of \(\mathcal{I}\). Since \(G\) is an extended Hadamard graph, we have \(|N_G(b) \cap N_G(b')| = \gamma/2\). Altogether, we have \(s \in \{0, 2\}\). We obtain \(s = 2\) since there is no Hadamard matrix of rank 0. If \(H\) is a \(2 \times 2\)-Hadamard matrix, of which there is only one up to equivalence, then \((G, \chi) \cong \text{XHdG} (\text{Slv}(2))\).

For the “moreover” part, assume that \((G, \chi) \cong \text{XHdG} (\text{Slv}(2))\) (see Figure 1). Fix two non-adjacent vertices \(v_1\) and \(v_2\) in \(V(G)\). For each \(u \in V(G) \setminus \{v_1, v_2\}\) we call the triple \((\chi(u), \delta_{u \in N_G(v_1)}, \delta_{u \in N_G(v_2)})\) the type of \(u\), where \(\delta_A\) denotes the indicator function of a statement \(A\) (that is 1 if \(A\) is true and 0 otherwise).

**Observation:** No two distinct vertices in \(V(G) \setminus \{v_1, v_2\}\) are of the same type.

Let now \(S, S' \subseteq V(G)\) be vertex sets for which there exists an isomorphism \(\varphi\) between the two corresponding induced subgraphs of \((G, \chi)\). If \(|S| \leq 3\), then it follows with Part (ii) that \(\varphi\) extends to an automorphism of \((G, \chi)\). Therefore, suppose \(|S| \geq 4\). In particular, \(S\) contains two non-adjacent vertices \(w_1\) and \(w_2\). It follows with Part (ii) that the restriction of \(\varphi\) to \(\{w_1, w_2\}\) extends to an automorphism of \((G, \chi)\). It follows by the observation that this automorphism is an extension of \(\varphi\).

\[\square\]

5 Tricoloured graphs

**Theorem 5.1.** Let \((G, \chi)\) be a 3-TR 3-coloured graph with colour classes \(R, B,\) and \(Y\) such that each of the graphs \(G[R], G[B],\) and \(G[Y]\) is a disjoint union of complete graphs. If \((G, \chi)[R \cup B] \cong \text{XHdG}(H)\) for some Hadamard matrix \(H \in \{-1, 1\}^{s \times s}\), then \(R\) and \(Y\) are homogeneously connected and \(B\) and \(Y\) are homogeneously connected.

**Proof.** Throughout this proof, the unique equally-coloured neighbour of a vertex \(v\) in a Hadamard graph is denoted by \(\tilde{v}\).

Suppose that \((G, \chi)[R \cup Y]\) is an extended Hadamard graph. If \((G, \chi)[B \cup Y]\) is not an extended Hadamard graph, then by Theorem 4.7 the sets \(B\) and \(Y\) are either homogeneously connected or \(G[B \cup Y]\) can be obtained from a matching-connected graph \((G', \chi')\) by first applying a blow-up to the \(B\)-coloured vertices and then to the \(Y\)-coloured vertices of \(G'\). Suppose \(y \in Y\) and \(b \in B\). Observe that \(G'[\{y, b\}] \cong G[\{\hat{y}, \hat{b}\}]\) (independent of whether \(B\) and \(Y\) are homogeneously connected or \(G[B \cup Y]\) is a blow-up). With \(N^R(b) \cap N^R(\hat{b}) = \emptyset\) and \(\lambda^R(\{y\}) = s\) it follows that
\[ \lambda^R(\{b, y\}) = \lambda^R(\{\bar{b}, \bar{y}\}) = s/2. \] Hence, the \( s \)-dimensional vector \( v \) with \( v_i = 1 \) if \( b_i^+ \in N(y) \) and \( v_i = -1 \) otherwise is orthogonal to each of the \( s \) columns of \( H \), which is a contradiction since these form an orthogonal basis.

If \((G, \chi)[B \cup Y]\) is an extended Hadamard graph, then let \( y_1, y_2 \in Y \) be non-adjacent vertices. Consider the design with points \( N^R(y_1) \) and blocks \( \{N(b) \cap N^R(y_1) \mid b \in N^B(y_1)\} \). Since \((G, \chi)\) is 3-TR, we obtain that this is a symmetric 2-design. In particular, relabelling the vertices of \( G \) in such a way that \( N^R(y_1) = \{r_i^+ \mid 1 \leq i \leq s\} \) and \( N^B(y_1) = \{c_i^+ \mid 1 \leq i \leq s\} \) yields an extended Hadamard graph whose underlying Hadamard matrix is regular (that is, all rows have the same number of positive entries). This implies

\[ \lambda^R(\{b, y_1\}) = \lambda^R(\{\bar{b}, \bar{y}_1\}) = \frac{s}{2} \] for every \( b \in N^B(y_1) \). (17)

Let \( r \in N^R(y_1) \cap N^R(y_2) \). From the 3-tuple regularity of \((G, \chi)\) we obtain

\[ s/2 = \lambda^B(\{\bar{y}_1, y_2\}) = \lambda^B(\{\bar{y}_1, r, y_2\}) = \lambda^B(\{\bar{y}_1, r, \bar{y}_2\}) = \lambda^B(\{y_1, \bar{r}, \bar{y}_2\}) + \lambda^B(\{y_1, r, \bar{y}_2\}) \]

and, hence, \( \lambda^B(\{y_1, r, y_2\}) = \lambda^B(\{\bar{y}_1, r, \bar{y}_2\}) = s/2 - \lambda^B(\{y_1, r, \bar{y}_2\}) \). In particular,

\[ \lambda^B(\{y_1, r, \bar{y}_2\}) = \lambda^B(\{\bar{y}_1, r, y_2\}) = s/4. \] (18)

The equations (17) and (18) imply

\[ \frac{s}{4} \left( s + \sqrt{s} \right) = \sum_{b \in N^R(y_1) \cap N^R(y_2)} \left| N^R(b) \cap N^R(y_1) \right| \]

\[ = \sum_{r \in N^R(y_1)} \left| N^B(y_1) \cap N^B(\bar{y}_2) \cap N^B(r) \right| = \frac{s^2}{4} \]

which is a contradiction.

Therefore, we may assume that neither \( G[B \cup Y] \) nor \( G[R \cup Y] \) is an extended Hadamard graph. If \( G[B \cup Y] \) and \( G[R \cup Y] \) both arise from blow-ups, then there is a vertex in \( G \) formed by vertices \( r \in R, b \in B \) and \( y \in Y \). Let \( b' \in N^B(r) \setminus \{b\} \). Since \( G[R \cup B] \) is a Hadamard-graph, the vertices \( b \) and \( b' \) are non-adjacent. This implies that \( b' \notin N(y') \) for all \( y' \notin N^Y(y) \). Altogether, we have \( \lambda^Y(\{r, b\}) = |N^Y(y)| + 1 \) and \( \lambda^Y(\{r, b'\}) = 0 \) which contradicts that \((G, \chi)\) is 3-TR.

Hence, we may assume that \( G[R \cup Y] \) arises from blow-ups whereas \( B \) and \( Y \) are homogeneously connected. Since \((G[R \cup B])\) is an extended Hadamard graph, we know that there are two non-adjacent vertices \( b, b' \in B \) and two 2-cliques \( C_i^B, C_i^R \subseteq R \) such that \( |C_1^B \cap N^R(b) \cap N^R(b')| = 1 \) and \( |C_2^B \cap N^R(b) \cap N^R(b')| = 0 \). Choose \( y_i \in Y \) such that \( N^R(y_i) = C_i^B \) for \( i \in \{1, 2\} \). We obtain

\[ \lambda^R(\{y_1, b, b'\}) = 0 \] and \( \lambda^R(\{y_2, b, b'\}) = 2 \), which is a contradiction since \((G, \chi)[\{y_1, b, b'\}] \cong (G, \chi)[\{y_2, b, b'\}] \).

\[ \square \]

### 6 Reductions and classification theorems

Towards describing our classification theorem, we observe that there are four simple, generic operations that allow us to create highly regular graphs from other ones.

A graph \((H, \chi_H)\) is the colour complementation of another graph \((G, \chi_G)\) if \( H \) is obtained from \( G \) by replacing all edges with endpoints in two colour classes \( C \) and \( C' \) (possibly \( C = C' \)) with non-edges and vice versa. More formally, if \( V(G) = V(H) \) and \( \chi_H = \chi_G \) and there are possibly equal colour classes \( C, C' \) such that \( E(H) = (E(G) \setminus (C \times C')) \cup ((C \times C') \setminus (E(G) \cup C^2 \cup C'^2)) \).
Lemma 6.1. Suppose $HR \in \{UH, TR\}$ and $k \in \mathbb{N}_{\geq 1}$. If $(H, \chi_H)$ is the colour complementation of $(G, \chi_G)$, then $(H, \chi_H)$ is $k$-$HR$ if and only $(G, \chi_G)$ is $k$-$HR$.

Proof. The lemma follows since the isomorphisms between induced subgraphs of $(H, \chi_H)$ are precisely the isomorphisms between induced subgraphs of $(G, \chi_G)$.

The second operation we need to consider is a homogeneous blow-up.

Lemma 6.2. Suppose $HR \in \{UH, TR\}$ and $k \in \mathbb{N}_{\geq 1}$. If $(H, \chi_H)$ is a homogeneous blow-up of $(G, \chi_G)$, then $(H, \chi_H)$ is $k$-$HR$ if and only $(G, \chi_G)$ is $k$-$HR$.

Proof. Assume that $(G, \chi_G)$ has a colour class, and call the vertices within it red. Assume further that the red vertices in $(G, \chi_G)$ form an independent set and $(H, \chi_H)$ is obtained by blowing-up the red vertices in $(G, \chi_G)$ to $t$-cliques. For a vertex $w \in V(H)$ we denote by $\overline{w}$ the vertex of $V(G)$ from which it originated. (I.e., if $w$ is red, then it is the red vertex of $G$ that was blown up to create $w$. If $w$ is not red, then $\overline{w} = w$.)

We call two red vertices of $(H, \chi_H)$ equivalent if they are adjacent. Vertices of other colours are only equivalent to themselves. Every induced subgraph $X$ of $(H, \chi_H)$ is associated with an annotated induced subgraph $\overline{X}$ of $(G, \chi_G)$ on vertex set $\{\overline{x} | x \in V(X)\}$, where red vertices $w$ in $X$ are annotated with the number equivalent red vertices of $H$ they represent.

Observation 1: Two induced subgraphs of $(H, \chi_H)$ are isomorphic exactly if their annotated subgraphs of $(G, \chi_G)$ are isomorphic under an isomorphism that respects annotations.

Call a bijection between two sets of vertices in $(H, \chi_H)$ admissible if it maps equivalent vertices to equivalent vertices. Note that every isomorphism between two induced subgraphs of $H$ is admissible. Every admissible map $\varphi$ induces a bijection $\overline{\varphi}$ between vertex sets in $(G, \chi_G)$. An admissible map $\varphi$ is an isomorphism exactly if $\overline{\varphi}$ is an isomorphism respecting annotations.

Note that for every annotated induced subgraph $Y$ of $(G, \chi_G)$ whose red annotations are at most $t$ there is an induced subgraph $X$ of $(H, \chi_H)$ so that $\overline{X} = Y$.

Regarding ultrahomogeneity, suppose $(H, \chi_H)$ is $k$-$UH$ and suppose there is an isomorphism $\varphi'$ between two induced subgraphs $Y_1$ and $Y_2$ of $(G, \chi_G)$ of order at most $k$. Consider two induced subgraphs $X_1$ and $X_2$ of $(H, \chi_H)$ whose associated annotated graphs are $\overline{X_1} = Y_1$ and $\overline{X_2} = Y_2$ with all annotations of red vertices being 1. It follows that between $X_1$ and $X_2$ there is a isomorphism $\varphi$ with $\overline{\varphi} = \varphi'$. Since $(H, \chi_H)$ is $k$-$UH$ in $(H, \chi_H)$ there is an automorphism $\psi$ extending $\varphi$. This automorphism induces an automorphism $\overline{\psi}$ of $(G, \chi_G)$ extending $\varphi'$. Conversely if $(G, \chi_G)$ is $k$-$UH$ and there is an isomorphism $\varphi$ between two induced graphs $X_1$ and $X_2$ in $(H, \chi_H)$, this induces an isomorphism between the associated annotated subgraphs $\overline{X_1}$ and $\overline{X_2}$. If this isomorphism extends to an automorphism of $(G, \chi_G)$, then this automorphism lifts to an automorphism of $(H, \chi_H)$ extending $\varphi$.

Regarding tuple regularity, we use the following observation that says that the number of 1-vertex extensions of an induced subgraph $X$ of a particular isomorphism type can be deduced from the annotated graph.

Observation 2: Assume that $X$ is an induced subgraph of $(H, \chi_H)$ and let $\overline{X}$ be the associated annotated subgraph of $(G, \chi_G)$. For $v \in V(G)$, the number of 1-vertex extensions $X^\downarrow v$ of $X$ by a new vertex $w$ satisfying $\overline{w} = v$ is equal to

- $1$ if $v$ is not in $V(\overline{X})$ and not red,
- $t$ if $v$ is not in $V(\overline{X})$ and red,
- $0$ if $v$ is in $V(\overline{X})$ and not red, and
- $t - i$ if $v$ is in $V(\overline{X})$ and red with annotation $i$. 

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This implies that two isomorphic induced subgraphs in \((G, \chi_G)\) have the same number of 1-vertex extensions of each isomorphism type if and only if isomorphic lifts of the two graphs have the same number of 1-vertex extensions of each isomorphism type.

The third operation duplicates all vertices in an independent colour class and connects the duplicates via a matching as follows. Let \((G, \chi)\) and be \((G', \chi')\) coloured graphs. A \textit{(possibly) colour-permuting isomorphism} is a bijection \(\varphi: V(G) \to V(G')\) that is an isomorphism from \((G, \chi \circ \pi)\) to \((G', \chi')\) for some permutation \(\pi\) of the colours \(C\).

Assume there are distinct colour classes \(R\) and \(B\) of \((G, \chi)\) each inducing an independent set for which \(M := E(G[R \cup B])\) is a perfect matching of \(G[R \cup B]\). If the bijection from \(V(G) \setminus R\) to \(V(G) \setminus B\) that is induced by \(M\) and fixes all vertices in \(V(G) \setminus (R \cup B)\) is a colour-permuting isomorphism, then we say that \((G, \chi)\) is a \textit{homogeneous matching extension} of \((G, \chi)[V(G) \setminus R]\).

**Lemma 6.3.** Suppose \(HR \in \{UH, TR\}\) and \(k \in \mathbb{N}_{\geq 1}\). If \((H, \chi_H)\) is a homogeneous matching extension of \((G, \chi_G)\), then \((H, \chi_H^k)\) is \(k\)-HR if and only \((G, \chi_G)\) is \(k\)-HR.

**Proof.** The proof is similar to the previous one. For this note that a homogeneous matching extension is actually a homogeneous blow up with vertices in one colour class replaced be 2-cliques and where the two vertices in each clique obtaining distinct colours.

For ultrahomogeneity this lemma is also proven in [LT14a].

Finally, the fourth and last operation combines two graphs with disjoint colours. We say that \((H, \chi_H)\) is a \textit{colour disjoint union} of the graphs \((G, \chi_G)\) and \((G', \chi_G')\) if the colours of \((G, \chi_G)\) and \((G', \chi_G')\) are disjoint and \(H\) is the disjoint union of \(G\) and \(G'\), where vertices in \(H\) inherit their colour from the respective graph they originate from.

**Lemma 6.4.** Suppose \(HR \in \{UH, TR\}\) and \(k \in \mathbb{N}_{\geq 1}\). If \((H, \chi_H)\) is a colour disjoint union of \((G, \chi_G)\) and \((G', \chi_G')\), then \((H, \chi_H^k)\) is \(k\)-HR if and only \((G, \chi_G)\) and \((G', \chi_G')\) are both \(k\)-HR.

**Proof.** The lemma follows from the observation that an isomorphism between two induced subgraphs in \((H, \chi_H)\) naturally decomposes into two isomorphisms between induced subgraphs of \((G, \chi_G)\) and between induced subgraphs of \((G', \chi_G')\), respectively, and vice versa.

We call a graph \((G, \chi)\) \textit{reduced} if neither \((G, \chi)\) nor any of the graphs obtainable from \((G, \chi)\) by a sequence of colour complementations is a colour disjoint union, a homogeneous blow-up, or a homogeneous matching extension. Let us first summarize the bichromatic case using this terminology.

**Theorem 6.5.** Suppose \(HR \in \{UH, TR\}\) and \(k \in \mathbb{N}_{\geq 1}\). The only bichromatic reduced \(k\)-HR graph is the extended Hadamard graph corresponding to \(Slv(2)\), whose underlying uncoloured graph is the Wagner graph (Figure 1).

**Proof.** This follows directly by combining Theorems 4.1, 4.7, and 4.16.

We can now give a characterization of reduced highly regular graphs.

**Theorem 6.6.** Suppose \(HR \in \{UH, TR\}\) and \(k \in \mathbb{N}_{\geq 1}\). An irreducible graph is \(k\)-HR if and only if it is a monochromatic or bichromatic \(k\)-HR graph.

**Proof.** Suppose that \((G, \chi)\) is \(k\)-HR and irreducible. By Lemma 3.2(i), each colour class of \((G, \chi)\) induces a \(k\)-HR graph. If there is a colour class \(R\) in \((G, \chi)\) that induces a primitive strongly
regular graph, then by Lemma 4.2, Corollary 4.4, or Lemma 4.5 the colour class \( R \) is homogeneously connected to all other colour classes. Since \((G, \chi)\) is irreducible this implies that \((G, \chi)\) is monochromatic.

From Theorem 4.7 it follows that every pair of colour classes that is not homogeneously connected induces an extended Hadamard graph. If \((G, \chi)\) is not monochromatic, then there are two colour classes that are not homogeneously connected. But then Theorem 5.1 implies that the graph is bichromatic.

We have finally assembled all the tools required to state our classification.

**Corollary 6.7.** Suppose \( HR \in \{UH, TR\} \) and \( k \in \mathbb{N}_{\geq 4} \). A graph is \( k\)-HR if and only if it can be obtained from irreducible monochromatic and bichromatic \( k\)-HR graphs by an arbitrary combination of the following operations:

1. colour disjoint union,
2. homogeneous matching extension,
3. homogeneous blow-up, and
4. colour complementation.

When \( k = 3 \), we can also make a similar statement for graphs whose colour classes induce imprimitive graphs.

**Theorem 6.8.** Suppose \( HR \in \{UH, TR\} \). A reduced graph whose colour classes induce imprimitive graphs is 3-HR if and only if it is a 1-vertex graph or an extended Hadamard graph.

**Proof.** The proof is analogous to the proof of Theorem 6.6. □

7 Conclusion and further research

|                      | 3-TR | 3-UH | 4-TR | UH |
|----------------------|------|------|------|----|
| bichromatic Wagner graph \( XHdG(Slv(2)) \) | ✓    | ✓    | ✓    | ✓  |
| ext. Had. graph of rank 12 Hadamard matrix \( XHdG(H^{12}) \) | ✓    | ✓    | ✓    | ✓  |
| ext. Had. graph of Sylvester matrix \( XHdG(Slv(2^s)) \) \((s \geq 2)\) | ✓    | ✓    | ✓    | ✓  |
| possibly other graphs with a primitive colour class | ✓    | ✓    | ✓    | ✓  |
| ext. Had. graph \( XHdG(H) \) \((\text{not Sylvester matrix and } H \not\equiv H^{12})\) | ✓    | ✓    | ✓    | ✓  |
| possibly other graphs with a primitive colour class | ✓    | ✓    | ✓    | ✓  |

Table 4: Classification of finite, bichromatic, reduced \( k\)-TR and \( k\)-UH graphs for \( k \geq 4 \) and classification for finite, bichromatic, reduced 3-TR and 3-UH graphs with both colour classes inducing imprimitive graphs.

In this paper we classified, for \( k \geq 4 \) the \( k\)-ultrahomogeneous and the \( k\)-tuple regular finite graphs. Via several regularity-preserving operations, the classification boils down to analysing bichromatic graphs. A two-coloured version of the Wagner graph plays a prominent role in the characterization. For \( k = 3 \) we showed a classification when the colour classes do not induce primitive graphs. The results are also summarized in Table 4.

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An open problem that suggests itself is to extend the classification to 3-ultrahomogeneous vertex coloured graphs. For this, two things must be done. First one has to analyse whether primitive monochromatic colour classes can be non-homogeneously connected to other colour classes. One cannot rule this out in a straightforward manner as was done in this paper since some of the graphs of interest can actually be partitioned into two parts inducing strongly regular graphs. For example, if \( H \) is isomorphic to the Higman-Sims graph and \((V_1, V_2)\) is a partition of \( V(H) \) into two strongly regular graphs, then \( H[V_1] \) and \( H[V_2] \) are both isomorphic to the Hoffman-Singleton graph a strongly regular graph with parameters \((50, 7, 0, 1)\). So indeed, a partition into strongly regular graphs is possible. If non-trivial bichromatic connections are possible one would have to investigate whether irreducible trichromatic graphs exist.

However, in light of our algorithmic interest in local to global approaches, hinted at in the introduction, a non-obvious avenue of further research might be as follows. Note that both the concepts \( k \)-ultrahomogeneity and \( k \)-tuple regularity in some way require that among the induced subgraphs of a particular isomorphism type there is only one kind. Indeed, for ultrahomogeneity there should be at most one kind under the orbits of the automorphism group and for tuple regularity there should be at most one kind when considering the multiset of 1-vertex extensions. Can we classify or even quantify what happens when we allow several kinds but not too many, say a bounded number of them?

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Appendix

A Partitioning the Schlafli and the McLaughlin graph

The appendix details the calculations that have to be performed for the proof of Lemma 4.5 by hand. We include them for the convenience and overview for reviewers, but do not want to suggest that the tables should be published in the journal. We also checked the parameters using the computer. Here we used a brute force algorithm that not require any knowledge on possible parameter sets at all.

Let \((n, d, \lambda, \mu)\) be the parameter set of the Schlaffli graph (respectively McLaughlin graph). As explained in Lemma 4.5, the aim of the calculation is to check whether these two graphs can be partitioned into two strongly regular graphs \(H_1\) and \(H_2\) with the parameter sets \((n_1, d_1, \lambda_1, \mu_1)\) and \((n_2, d_2, \lambda_2, \mu_2)\), respectively. Iterating over Brouwer’s list of parameters of strongly regular graphs [Bro], we discard all \((n_1, d_1, \lambda_1, \mu_1)\) which do not satisfy following conditions: \(n_1 \leq \lfloor n/2 \rfloor\), \(d_1 \leq d\), \(\lambda_1 \leq \lambda\), and \(\mu_1 \leq \mu\). For the remaining possible parameter sets, we stepwise compute the parameter set \((n_2, d_2, \lambda_2, \mu_2)\) according to Lemma 2.1 in search for a violation. As the tables below display, in each of the cases at least one of the parameters \(d_2\), \(\lambda_2\), or \(\mu_2\) is not a natural number. Afterwards, we repeat this process for all parameter sets of the imprimitive strongly regular graphs \(sK_t\) and \(tK_s\). Here we can bound \(s\) by the independence number of the graph and \(t\) by its clique number.

Schlafli Graph The parameter set of the Schlaffli graph is \((27, 16, 10, 8)\), the clique number is 6, and the independence number is 3. Using Brouwer’s parameter list, 27 of 27 parameter combinations were pruned.

| \(n_1\) | \(d_1\) | \(\lambda_1\) | \(\mu_1\) | \(n_2\) | \(d_2\) | \(\lambda_2\) | \(\mu_2\) | Reason |
|---|---|---|---|---|---|---|---|---|
| 5 | 2 | 0 | 1 | 22 | 12.818 | \(d_2 \not\in \mathbb{N}\) |
| 9 | 4 | 1 | 2 | 18 | 10 | 5.800 | \(\lambda_2 \not\in \mathbb{N}\) |
| 10 | 3 | 0 | 1 | 17 | 8.353 | \(d_2 \not\in \mathbb{N}\) |
| 10 | 6 | 3 | 4 | 17 | 10.118 | \(d_2 \not\in \mathbb{N}\) |
| 13 | 6 | 2 | 3 | 14 | 6.714 | \(d_2 \not\in \mathbb{N}\) |
| \(K_1\) | 1 | 0 | - | - | 26 | 15.385 | \(d_2 \not\in \mathbb{N}\) |
| \(K_2\) | 2 | 1 | 0 | - | 25 | 14.800 | \(d_2 \not\in \mathbb{N}\) |
| \(K_3\) | 3 | 2 | 1 | - | 24 | 14.250 | \(d_2 \not\in \mathbb{N}\) |
| \(K_4\) | 4 | 3 | 2 | - | 23 | 13.739 | \(d_2 \not\in \mathbb{N}\) |
| \(K_5\) | 5 | 4 | 3 | - | 22 | 13.273 | \(d_2 \not\in \mathbb{N}\) |
| \(K_6\) | 6 | 5 | 4 | - | 21 | 12.857 | \(d_2 \not\in \mathbb{N}\) |
| \(K_2\) | 2 | 0 | - | 0 | 25 | 14.720 | \(d_2 \not\in \mathbb{N}\) |
| \(K_3\) | 3 | 0 | - | 0 | 24 | 14.571 | \(\lambda_2 \not\in \mathbb{N}\) |
| \(2K_2\) | 4 | 1 | 0 | 0 | 23 | 13.391 | \(d_2 \not\in \mathbb{N}\) |
| \(3K_2\) | 6 | 1 | 0 | 0 | 21 | 11.714 | \(d_2 \not\in \mathbb{N}\) |
| \(2K_3\) | 6 | 2 | 1 | 0 | 21 | 12 | 7.095 | \(\lambda_2 \not\in \mathbb{N}\) |
| \(3K_3\) | 9 | 2 | 1 | 0 | 18 | 9 | 3.222 | \(\lambda_2 \not\in \mathbb{N}\) |
| \(2K_4\) | 8 | 3 | 2 | 0 | 19 | 10.526 | \(d_2 \not\in \mathbb{N}\) |
| \(3K_4\) | 12 | 3 | 2 | 0 | 15 | 5.600 | \(d_2 \not\in \mathbb{N}\) |
| \(2K_5\) | 10 | 4 | 3 | 0 | 17 | 8.941 | \(d_2 \not\in \mathbb{N}\) |
| \(2K_6\) | 12 | 5 | 4 | 0 | 15 | 7.200 | \(d_2 \not\in \mathbb{N}\) |
McLaughlin Graph  The parameter set of the Schl"afli graph is (275, 112, 30, 56), the clique number is 5, and the independence number is 22. Using Brouwer’s parameter list, 372 of 372 parameter combinations were pruned.

| $2K_2$ | 4 2 0 2 | 23 13.565 | $d_2 \notin \mathbb{N}$ |
|---|---|---|---|
| $2K_3$ | 6 3 0 3 | 21 12.286 | $d_2 \notin \mathbb{N}$ |
| $3K_2$ | 6 4 2 4 | 21 12.571 | $d_2 \notin \mathbb{N}$ |
| $3K_3$ | 9 6 3 6 | 18 11 7.364 | $\lambda_2 \notin \mathbb{N}$ |
| $4K_2$ | 8 6 4 6 | 19 11.789 | $d_2 \notin \mathbb{N}$ |
| $5K_2$ | 10 8 6 8 | 17 11.294 | $d_2 \notin \mathbb{N}$ |

| $n_1$ | $d_1$ | $\lambda_1$ | $\mu_1$ | $n_2$ | $d_2$ | $\lambda_2$ | $\mu_2$ | Reason |
|---|---|---|---|---|---|---|---|---|
| 5 | 2 | 0 | 1 | 270 | 109.963 | $d_2 \notin \mathbb{N}$ |
| 9 | 4 | 1 | 2 | 266 | 108.346 | $d_2 \notin \mathbb{N}$ |
| 10 | 3 | 0 | 1 | 265 | 107.887 | $d_2 \notin \mathbb{N}$ |
| 10 | 6 | 3 | 4 | 265 | 108 | 28.945 | $\lambda_2 \notin \mathbb{N}$ |
| 13 | 6 | 2 | 3 | 262 | 106.740 | $d_2 \notin \mathbb{N}$ |
| 15 | 6 | 1 | 3 | 260 | 105.885 | $d_2 \notin \mathbb{N}$ |
| 15 | 8 | 4 | 4 | 260 | 106 | 28.415 | $\lambda_2 \notin \mathbb{N}$ |
| 16 | 5 | 0 | 2 | 259 | 105.390 | $d_2 \notin \mathbb{N}$ |
| 16 | 6 | 2 | 2 | 259 | 105.452 | $d_2 \notin \mathbb{N}$ |
| 16 | 9 | 4 | 6 | 259 | 105.637 | $d_2 \notin \mathbb{N}$ |
| 16 | 10 | 6 | 6 | 259 | 105.699 | $d_2 \notin \mathbb{N}$ |
| 17 | 8 | 3 | 4 | 258 | 105.147 | $d_2 \notin \mathbb{N}$ |
| 21 | 10 | 3 | 6 | 254 | 103.567 | $d_2 \notin \mathbb{N}$ |
| 21 | 10 | 4 | 5 | 254 | 103.567 | $d_2 \notin \mathbb{N}$ |
| 21 | 10 | 5 | 4 | 254 | 103.567 | $d_2 \notin \mathbb{N}$ |
| 25 | 8 | 3 | 2 | 250 | 101.600 | $d_2 \notin \mathbb{N}$ |
| 25 | 12 | 5 | 6 | 250 | 102 | 27.353 | $\lambda_2 \notin \mathbb{N}$ |
| 25 | 16 | 9 | 12 | 250 | 102.400 | $d_2 \notin \mathbb{N}$ |
| 26 | 10 | 3 | 4 | 249 | 101.349 | $d_2 \notin \mathbb{N}$ |
| 26 | 15 | 8 | 9 | 249 | 101.871 | $d_2 \notin \mathbb{N}$ |
| 27 | 10 | 1 | 5 | 248 | 100.895 | $d_2 \notin \mathbb{N}$ |
| 27 | 16 | 10 | 8 | 248 | 101.548 | $d_2 \notin \mathbb{N}$ |
| 28 | 9 | 0 | 4 | 247 | 100.324 | $d_2 \notin \mathbb{N}$ |
| 28 | 12 | 6 | 4 | 247 | 100.664 | $d_2 \notin \mathbb{N}$ |
| 28 | 15 | 6 | 10 | 247 | 101.004 | $d_2 \notin \mathbb{N}$ |
| 28 | 18 | 12 | 10 | 247 | 101.344 | $d_2 \notin \mathbb{N}$ |
| 29 | 14 | 6 | 7 | 246 | 100.447 | $d_2 \notin \mathbb{N}$ |
| 33 | 16 | 7 | 8 | 242 | 98.909 | $d_2 \notin \mathbb{N}$ |
| 35 | 16 | 6 | 8 | 240 | 98 | 26.286 | $\lambda_2 \notin \mathbb{N}$ |
| 35 | 18 | 9 | 9 | 240 | 98.292 | $d_2 \notin \mathbb{N}$ |
| 36 | 10 | 4 | 2 | 239 | 96.636 | $d_2 \notin \mathbb{N}$ |
| 36 | 14 | 4 | 6 | 239 | 97.238 | $d_2 \notin \mathbb{N}$ |
| 36 | 14 | 7 | 4 | 239 | 97.238 | $d_2 \notin \mathbb{N}$ |
| 36 | 15 | 6 | 6 | 239 | 97.389 | $d_2 \notin \mathbb{N}$ |
| 36 | 20 | 10 | 12 | 239 | 98.142 | $d_2 \notin \mathbb{N}$ |
| 36 | 21 | 10 | 15 | 239 | 98.293 | $d_2 \notin \mathbb{N}$ |
| 36 | 21 | 12 | 12 | 239 | 98.293 | $d_2 \notin \mathbb{N}$ |
| 36 | 25 | 16 | 20 | 239 | 98.895 | $d_2 \notin \mathbb{N}$ |
| 37 | 18 | 8  | 9  | 238 | 97.387 | $d_2 \notin \mathbb{N}$ |
| 40 | 12 | 2  | 4  | 235 | 94.979 | $d_2 \notin \mathbb{N}$ |
| 40 | 27 | 18 | 18 | 235 | 97.532 | $d_2 \notin \mathbb{N}$ |
| 41 | 20 | 9  | 10 | 234 | 95.880 | $d_2 \notin \mathbb{N}$ |
| 45 | 12 | 3  | 3  | 230 | 92.435 | $d_2 \notin \mathbb{N}$ |
| 45 | 16 | 8  | 4  | 230 | 93.217 | $d_2 \notin \mathbb{N}$ |
| 45 | 22 | 10 | 11 | 230 | 94.391 | $d_2 \notin \mathbb{N}$ |
| 45 | 28 | 15 | 21 | 230 | 95.565 | $d_2 \notin \mathbb{N}$ |
| 45 | 32 | 22 | 24 | 230 | 96.348 | $d_2 \notin \mathbb{N}$ |
| 49 | 12 | 5  | 2  | 226 | 90.319 | $d_2 \notin \mathbb{N}$ |
| 49 | 16 | 3  | 6  | 226 | 91.186 | $d_2 \notin \mathbb{N}$ |
| 49 | 18 | 7  | 6  | 226 | 91.619 | $d_2 \notin \mathbb{N}$ |
| 49 | 24 | 11 | 12 | 226 | 92.920 | $d_2 \notin \mathbb{N}$ |
| 49 | 30 | 17 | 20 | 226 | 94.221 | $d_2 \notin \mathbb{N}$ |
| 49 | 32 | 21 | 20 | 226 | 94.655 | $d_2 \notin \mathbb{N}$ |
| 49 | 36 | 25 | 30 | 226 | 95.522 | $d_2 \notin \mathbb{N}$ |
| 50 | 7  | 0  | 1  | 225 | 88.667 | $d_2 \notin \mathbb{N}$ |
| 50 | 21 | 4  | 12 | 225 | 91.778 | $d_2 \notin \mathbb{N}$ |
| 50 | 21 | 8  | 9  | 225 | 91.778 | $d_2 \notin \mathbb{N}$ |
| 50 | 28 | 15 | 16 | 225 | 93.333 | $d_2 \notin \mathbb{N}$ |
| 50 | 28 | 18 | 12 | 225 | 93.333 | $d_2 \notin \mathbb{N}$ |
| 53 | 26 | 12 | 13 | 222 | 91.468 | $d_2 \notin \mathbb{N}$ |
| 55 | 18 | 9  | 4  | 220 | 88.500 | $d_2 \notin \mathbb{N}$ |
| 55 | 36 | 21 | 28 | 220 | 93    | $24.742$ $\lambda_2 \notin \mathbb{N}$ |
| 56 | 10 | 0  | 2  | 219 | 85.918 | $d_2 \notin \mathbb{N}$ |
| 56 | 22 | 3  | 12 | 219 | 88.986 | $d_2 \notin \mathbb{N}$ |
| 56 | 33 | 22 | 15 | 219 | 91.799 | $d_2 \notin \mathbb{N}$ |
| 57 | 14 | 1  | 4  | 218 | 86.376 | $d_2 \notin \mathbb{N}$ |
| 57 | 24 | 11 | 9  | 218 | 88.991 | $d_2 \notin \mathbb{N}$ |
| 57 | 28 | 13 | 14 | 218 | 90.037 | $d_2 \notin \mathbb{N}$ |
| 57 | 32 | 16 | 20 | 218 | 91.083 | $d_2 \notin \mathbb{N}$ |
| 61 | 30 | 14 | 15 | 214 | 88.626 | $d_2 \notin \mathbb{N}$ |
| 63 | 22 | 1  | 11 | 212 | 85.255 | $d_2 \notin \mathbb{N}$ |
| 63 | 30 | 13 | 15 | 212 | 87.632 | $d_2 \notin \mathbb{N}$ |
| 63 | 32 | 16 | 16 | 212 | 88.226 | $d_2 \notin \mathbb{N}$ |
| 63 | 40 | 28 | 20 | 212 | 90.604 | $d_2 \notin \mathbb{N}$ |
| 64 | 14 | 6  | 2  | 211 | 82.275 | $d_2 \notin \mathbb{N}$ |
| 64 | 18 | 2  | 6  | 211 | 83.488 | $d_2 \notin \mathbb{N}$ |
| 64 | 21 | 0  | 10 | 211 | 84.398 | $d_2 \notin \mathbb{N}$ |
| 64 | 21 | 8  | 6  | 211 | 84.398 | $d_2 \notin \mathbb{N}$ |
| 64 | 27 | 10 | 12 | 211 | 86.218 | $d_2 \notin \mathbb{N}$ |
| 64 | 28 | 12 | 12 | 211 | 86.521 | $d_2 \notin \mathbb{N}$ |
| 64 | 30 | 18 | 10 | 211 | 87.128 | $d_2 \notin \mathbb{N}$ |
| 64 | 33 | 12 | 22 | 211 | 88.038 | $d_2 \notin \mathbb{N}$ |
| 64 | 35 | 18 | 20 | 211 | 88.645 | $d_2 \notin \mathbb{N}$ |
| 64 | 36 | 20 | 20 | 211 | 88.948 | $d_2 \notin \mathbb{N}$ |
| 64 | 42 | 26 | 30 | 211 | 90.768 | $d_2 \notin \mathbb{N}$ |
| 64 | 42 | 30 | 22 | 211 | 90.768 | \( d_2 \notin \mathbb{N} \) |
| 65 | 32 | 15 | 16 | 210 | 87.238 | \( d_2 \notin \mathbb{N} \) |
| 66 | 20 | 10 | 4  | 209 | 82.947  | \( d_2 \notin \mathbb{N} \) |
| 66 | 45 | 28 | 36 | 209 | 90.842  | \( d_2 \notin \mathbb{N} \) |
| 69 | 20 | 7  | 5  | 206 | 81.184  | \( d_2 \notin \mathbb{N} \) |
| 69 | 34 | 16 | 17 | 206 | 85.874  | \( d_2 \notin \mathbb{N} \) |
| 70 | 27 | 12 | 9  | 205 | 82.976  | \( d_2 \notin \mathbb{N} \) |
| 70 | 42 | 23 | 28 | 205 | 88.098  | \( d_2 \notin \mathbb{N} \) |
| 73 | 36 | 17 | 18 | 202 | 84.535  | \( d_2 \notin \mathbb{N} \) |
| 75 | 32 | 10 | 16 | 200 | 82      | \( \lambda_2 \notin \mathbb{N} \) | 21.951 |
| 75 | 42 | 25 | 21 | 200 | 85.750  | \( d_2 \notin \mathbb{N} \) |
| 76 | 21 | 2  | 7  | 199 | 77.246  | \( d_2 \notin \mathbb{N} \) |
| 76 | 30 | 8  | 14 | 199 | 80.683  | \( d_2 \notin \mathbb{N} \) |
| 76 | 35 | 18 | 14 | 199 | 82.593  | \( d_2 \notin \mathbb{N} \) |
| 76 | 40 | 18 | 24 | 199 | 84.503  | \( d_2 \notin \mathbb{N} \) |
| 76 | 45 | 28 | 24 | 199 | 86.412  | \( d_2 \notin \mathbb{N} \) |
| 77 | 16 | 0  | 4  | 198 | 74.667  | \( d_2 \notin \mathbb{N} \) |
| 77 | 38 | 18 | 19 | 198 | 83.222  | \( d_2 \notin \mathbb{N} \) |
| 78 | 22 | 11 | 4  | 197 | 76.365  | \( d_2 \notin \mathbb{N} \) |
| 81 | 16 | 7  | 2  | 194 | 71.918  | \( d_2 \notin \mathbb{N} \) |
| 81 | 20 | 1  | 6  | 194 | 73.588  | \( d_2 \notin \mathbb{N} \) |
| 81 | 24 | 9  | 6  | 194 | 75.258  | \( d_2 \notin \mathbb{N} \) |
| 81 | 30 | 9  | 12 | 194 | 77.763  | \( d_2 \notin \mathbb{N} \) |
| 81 | 32 | 13 | 12 | 194 | 78.598  | \( d_2 \notin \mathbb{N} \) |
| 81 | 40 | 13 | 26 | 194 | 81.938  | \( d_2 \notin \mathbb{N} \) |
| 81 | 40 | 19 | 20 | 194 | 81.938  | \( d_2 \notin \mathbb{N} \) |
| 81 | 40 | 25 | 14 | 194 | 81.938  | \( d_2 \notin \mathbb{N} \) |
| 81 | 48 | 27 | 30 | 194 | 85.278  | \( d_2 \notin \mathbb{N} \) |
| 82 | 36 | 15 | 16 | 193 | 79.710  | \( d_2 \notin \mathbb{N} \) |
| 82 | 45 | 24 | 25 | 193 | 83.534  | \( d_2 \notin \mathbb{N} \) |
| 85 | 14 | 3  | 2  | 190 | 68.158  | \( d_2 \notin \mathbb{N} \) |
| 85 | 20 | 3  | 5  | 190 | 70.842  | \( d_2 \notin \mathbb{N} \) |
| 85 | 30 | 11 | 10 | 190 | 75.316  | \( d_2 \notin \mathbb{N} \) |
| 85 | 42 | 20 | 21 | 190 | 80.684  | \( d_2 \notin \mathbb{N} \) |
| 88 | 27 | 6  | 9  | 187 | 72      | \( \lambda_2 \notin \mathbb{N} \) | 17.569 |
| 89 | 44 | 21 | 22 | 186 | 79.462  | \( d_2 \notin \mathbb{N} \) |
| 91 | 24 | 12 | 4  | 184 | 68.478  | \( d_2 \notin \mathbb{N} \) |
| 93 | 46 | 22 | 23 | 182 | 78.275  | \( d_2 \notin \mathbb{N} \) |
| 95 | 40 | 12 | 20 | 180 | 74      | \( \lambda_2 \notin \mathbb{N} \) | 19.730 |
| 96 | 19 | 2  | 4  | 179 | 62.123  | \( d_2 \notin \mathbb{N} \) |
| 96 | 20 | 4  | 4  | 179 | 62.659  | \( d_2 \notin \mathbb{N} \) |
| 96 | 35 | 10 | 14 | 179 | 70.704  | \( d_2 \notin \mathbb{N} \) |
| 96 | 38 | 10 | 18 | 179 | 72.313  | \( d_2 \notin \mathbb{N} \) |
| 96 | 45 | 24 | 18 | 179 | 76.067  | \( d_2 \notin \mathbb{N} \) |
| 96 | 50 | 22 | 30 | 179 | 78.749  | \( d_2 \notin \mathbb{N} \) |
| 97 | 48 | 23 | 24 | 178 | 77.124  | \( d_2 \notin \mathbb{N} \) |
| 99 | 14 | 1  | 2  | 176 | 56.875  | \( d_2 \notin \mathbb{N} \) |
| 99 | 42 | 21 | 15 | 176 | 72.625  | \( d_2 \notin \mathbb{N} \) |
| 99 | 48 | 22 | 24 | 176 | 76      | \( \lambda_2 \notin \mathbb{N} \) | 18.632 |
| 99 50 25 25 | 176 77.125 | $d_2 \notin \mathbb{N}$ \\
| 99 56 28 36 | 176 80.500 | $d_2 \notin \mathbb{N}$ \\
| 100 18 8 2  | 175 58.286 | $d_2 \notin \mathbb{N}$ \\
| 100 22 0 6   | 175 60.571 | $d_2 \notin \mathbb{N}$ \\
| 100 27 10 6  | 175 63.429 | $d_2 \notin \mathbb{N}$ \\
| 100 33 8 12  | 175 66.857 | $d_2 \notin \mathbb{N}$ \\
| 100 33 14 9  | 175 66.857 | $d_2 \notin \mathbb{N}$ \\
| 100 33 18 7  | 175 66.857 | $d_2 \notin \mathbb{N}$ \\
| 100 36 14 12 | 175 68.571 | $d_2 \notin \mathbb{N}$ \\
| 100 44 18 20 | 175 73.143 | $d_2 \notin \mathbb{N}$ \\
| 100 45 20 20 | 175 73.714 | $d_2 \notin \mathbb{N}$ \\
| 100 54 28 30 | 175 78.857 | $d_2 \notin \mathbb{N}$ \\
| 100 55 30 30 | 175 79.429 | $d_2 \notin \mathbb{N}$ \\
| 101 50 24 25 | 174 76.011 | $d_2 \notin \mathbb{N}$ \\
| 105 26 13 4  | 170 58.882 | $d_2 \notin \mathbb{N}$ \\
| 105 32 4 12  | 170 62.588 | $d_2 \notin \mathbb{N}$ \\
| 105 40 15 15 | 170 67.529 | $d_2 \notin \mathbb{N}$ \\
| 105 52 21 30 | 170 74.941 | $d_2 \notin \mathbb{N}$ \\
| 105 52 25 26 | 170 74.941 | $d_2 \notin \mathbb{N}$ \\
| 105 52 29 22 | 170 74.941 | $d_2 \notin \mathbb{N}$ \\
| 109 54 26 27 | 166 73.916 | $d_2 \notin \mathbb{N}$ \\
| 111 30 5 9   | 164 56.500 | $d_2 \notin \mathbb{N}$ \\
| 111 44 19 16 | 164 65.976 | $d_2 \notin \mathbb{N}$ \\
| 112 30 10 2  | 163 55.656 | $d_2 \notin \mathbb{N}$ \\
| 112 36 10 12 | 163 59.779 | $d_2 \notin \mathbb{N}$ \\
| 113 56 27 28 | 162 72.938 | $d_2 \notin \mathbb{N}$ \\
| 115 18 1 3   | 160 44.438 | $d_2 \notin \mathbb{N}$ \\
| 117 36 15 9  | 158 55.722 | $d_2 \notin \mathbb{N}$ \\
| 117 58 29 29 | 158 72.013 | $d_2 \notin \mathbb{N}$ \\
| 119 54 21 27 | 156 67.756 | $d_2 \notin \mathbb{N}$ \\
| 120 28 14 4  | 155 46.908 | $d_2 \notin \mathbb{N}$ \\
| 120 34 8 10  | 155 51.613 | $d_2 \notin \mathbb{N}$ \\
| 120 35 10 10 | 155 52.387 | $d_2 \notin \mathbb{N}$ \\
| 120 42 8 18  | 155 57.806 | $d_2 \notin \mathbb{N}$ \\
| 120 51 18 24 | 155 64.774 | $d_2 \notin \mathbb{N}$ \\
| 120 56 28 24 | 155 68.645 | $d_2 \notin \mathbb{N}$ \\
| 120 63 30 36 | 155 74.065 | $d_2 \notin \mathbb{N}$ \\
| 121 20 9 2   | 154 39.714 | $d_2 \notin \mathbb{N}$ \\
| 121 30 11 6  | 154 47.571 | $d_2 \notin \mathbb{N}$ \\
| 121 36 7 12  | 154 52.286 | $d_2 \notin \mathbb{N}$ \\
| 121 40 15 12 | 154 55.429 | $d_2 \notin \mathbb{N}$ \\
| 121 48 17 20 | 154 61.714 | $d_2 \notin \mathbb{N}$ \\
| 121 50 21 20 | 154 63.286 | $d_2 \notin \mathbb{N}$ \\
| 121 56 15 35 | 154 68 20.294 | $\lambda_2 \notin \mathbb{N}$ \\
| 121 60 29 30 | 154 71.143 | $d_0 \notin \mathbb{N}$ \\
| 122 55 24 25 | 153 66.549 | $d_2 \notin \mathbb{N}$ \\
| 125 28 3 7   | 150 42 -5 | $\lambda_2 \notin \mathbb{N}$ \\
| 125 48 28 12 | 150 58.667 | $d_2 \notin \mathbb{N}$ \\
| 125 52 15 26 | 150 62 16.290 | $\lambda_2 \notin \mathbb{N}$ \\

30
|   | 125 62 30 31 | 150 70.333 | d₂ \notin N |
|---|-------------|------------|-------------|
|   | 126 25 8 4 | 149 38.430 | d₂ \notin N |
|   | 126 45 12 18 | 149 55.342 | d₂ \notin N |
|   | 126 50 13 24 | 149 59.570 | d₂ \notin N |
|   | 126 65 28 39 | 149 72.255 | d₂ \notin N |
|   | 130 48 20 16 | 145 54.621 | d₂ \notin N |
|   | 133 24 5 4 | 142 29.537 | d₂ \notin N |
|   | 133 32 6 8 | 142 37.070 | d₂ \notin N |
|   | 133 44 15 14 | 142 48.310 | d₂ \notin N |
|   | 135 64 28 32 | 140 65.714 | d₂ \notin N |
|   | 136 30 8 6 | 139 31.770 | d₂ \notin N |
|   | 136 30 15 4 | 139 31.770 | d₂ \notin N |
|   | 136 60 24 28 | 139 61.122 | d₂ \notin N |
|   | 136 63 30 28 | 139 64.058 | d₂ \notin N |
|   | 125 62 30 31 | 150 70.333 | d₂ \notin N |
|   | 126 25 8 4 | 149 38.430 | d₂ \notin N |
|   | 126 45 12 18 | 149 55.342 | d₂ \notin N |
|   | 126 50 13 24 | 149 59.570 | d₂ \notin N |
|   | 126 65 28 39 | 149 72.255 | d₂ \notin N |
|   | 130 48 20 16 | 145 54.621 | d₂ \notin N |
|   | 133 24 5 4 | 142 29.537 | d₂ \notin N |
|   | 133 32 6 8 | 142 37.070 | d₂ \notin N |
|   | 133 44 15 14 | 142 48.310 | d₂ \notin N |
|   | 135 64 28 32 | 140 65.714 | d₂ \notin N |
|   | 136 30 8 6 | 139 31.770 | d₂ \notin N |
|   | 136 30 15 4 | 139 31.770 | d₂ \notin N |
|   | 136 60 24 28 | 139 61.122 | d₂ \notin N |
|   | 136 63 30 28 | 139 64.058 | d₂ \notin N |

| K₁ | 1 0 - - | 274 111.591 | d₂ \notin N |
| K₂ | 2 1 0 - | 273 111.187 | d₂ \notin N |
| K₃ | 3 2 1 - | 272 110.787 | d₂ \notin N |
| K₄ | 4 3 2 - | 271 110.391 | d₂ \notin N |
| K₅ | 5 4 3 - | 270 110 29.473 | λ₂ \notin N |
| K₁ | 2 0 - 0 | 273 111.179 | d₂ \notin N |
| K₂ | 3 0 - 0 | 272 110.765 | d₂ \notin N |
| K₃ | 4 0 - 0 | 271 110.347 | d₂ \notin N |
| K₄ | 5 0 - 0 | 270 109.926 | d₂ \notin N |
| K₅ | 6 0 - 0 | 269 109.502 | d₂ \notin N |
| K₆ | 7 0 - 0 | 268 109.075 | d₂ \notin N |
| K₇ | 8 0 - 0 | 267 108.644 | d₂ \notin N |
| K₈ | 9 0 - 0 | 266 108.211 | d₂ \notin N |
| K₉ | 10 0 - 0 | 265 107.774 | d₂ \notin N |
| K₁₀ | 11 0 - 0 | 264 107.333 | d₂ \notin N |
| K₁₁ | 12 0 - 0 | 263 106.890 | d₂ \notin N |
| K₁₂ | 13 0 - 0 | 262 106.443 | d₂ \notin N |
| K₁₃ | 14 0 - 0 | 261 105.992 | d₂ \notin N |
| K₁₄ | 15 0 - 0 | 260 105.538 | d₂ \notin N |
| K₁₅ | 16 0 - 0 | 259 105.081 | d₂ \notin N |
| K₁₆ | 17 0 - 0 | 258 104.620 | d₂ \notin N |
| K₁₇ | 18 0 - 0 | 257 104.156 | d₂ \notin N |
| K₁₈ | 19 0 - 0 | 256 103.688 | d₂ \notin N |
| K₁₉ | 20 0 - 0 | 255 103.216 | d₂ \notin N |
| K₂₀ | 21 0 - 0 | 254 102.740 | d₂ \notin N |
| K₂₁ | 22 0 - 0 | 253 102.261 | d₂ \notin N |

| 2K₂ | 4 1 0 0 | 271 110.362 | d₂ \notin N |
| 3K₂ | 6 1 0 0 | 269 109.524 | d₂ \notin N |
| 4K₂ | 8 1 0 0 | 267 108.674 | d₂ \notin N |
| 5K₂ | 10 1 0 0 | 265 107.811 | d₂ \notin N |
| 6K₂ | 12 1 0 0 | 263 106.935 | d₂ \notin N |
| 7K₂ | 14 1 0 0 | 261 106.046 | d₂ \notin N |
| 8K₂ | 16 1 0 0 | 259 105.143 | d₂ \notin N |
| 9K₂ | 18 1 0 0 | 257 104.226 | d₂ \notin N |
| $10K_2$ | 20 1 0 0 | 255 103.294 | $d_2 \notin \mathbb{N}$ |
| $11K_2$ | 22 1 0 0 | 253 102.348 | $d_2 \notin \mathbb{N}$ |
| $12K_2$ | 24 1 0 0 | 251 101.386 | $d_2 \notin \mathbb{N}$ |
| $13K_2$ | 26 1 0 0 | 249 100.410 | $d_2 \notin \mathbb{N}$ |
| $14K_2$ | 28 1 0 0 | 247 99.417 | $d_2 \notin \mathbb{N}$ |
| $15K_2$ | 30 1 0 0 | 245 98.408 | $d_2 \notin \mathbb{N}$ |
| $16K_2$ | 32 1 0 0 | 243 97.383 | $d_2 \notin \mathbb{N}$ |
| $17K_2$ | 34 1 0 0 | 241 96.340 | $d_2 \notin \mathbb{N}$ |
| $18K_2$ | 36 1 0 0 | 239 95.280 | $d_2 \notin \mathbb{N}$ |
| $19K_2$ | 38 1 0 0 | 237 94.203 | $d_2 \notin \mathbb{N}$ |
| $20K_2$ | 40 1 0 0 | 235 93.106 | $d_2 \notin \mathbb{N}$ |
| $21K_2$ | 42 1 0 0 | 233 91.991 | $d_2 \notin \mathbb{N}$ |
| $22K_2$ | 44 1 0 0 | 231 90.857 | $d_2 \notin \mathbb{N}$ |
| $2K_4$ | 6 2 1 0 | 269 109.546 | $d_2 \notin \mathbb{N}$ |
| $3K_4$ | 9 2 1 0 | 266 108.278 | $d_2 \notin \mathbb{N}$ |
| $4K_4$ | 12 2 1 0 | 263 106.981 | $d_2 \notin \mathbb{N}$ |
| $5K_4$ | 15 2 1 0 | 260 105.654 | $d_2 \notin \mathbb{N}$ |
| $6K_4$ | 18 2 1 0 | 257 104.296 | $d_2 \notin \mathbb{N}$ |
| $7K_4$ | 21 2 1 0 | 254 102.906 | $d_2 \notin \mathbb{N}$ |
| $8K_4$ | 24 2 1 0 | 251 101.482 | $d_2 \notin \mathbb{N}$ |
| $9K_4$ | 27 2 1 0 | 248 100.024 | $d_2 \notin \mathbb{N}$ |
| $10K_3$ | 30 2 1 0 | 245 98.531 | $d_2 \notin \mathbb{N}$ |
| $11K_3$ | 33 2 1 0 | 242 97.442 | $\lambda_3 \notin \mathbb{N}$ |
| $12K_3$ | 36 2 1 0 | 239 95.431 | $d_2 \notin \mathbb{N}$ |
| $13K_3$ | 39 2 1 0 | 236 93.822 | $d_2 \notin \mathbb{N}$ |
| $14K_3$ | 42 2 1 0 | 233 92.172 | $d_2 \notin \mathbb{N}$ |
| $15K_3$ | 45 2 1 0 | 230 90.478 | $d_2 \notin \mathbb{N}$ |
| $16K_3$ | 48 2 1 0 | 227 88.740 | $d_2 \notin \mathbb{N}$ |
| $17K_3$ | 51 2 1 0 | 224 86.955 | $d_2 \notin \mathbb{N}$ |
| $18K_3$ | 54 2 1 0 | 221 85.122 | $d_2 \notin \mathbb{N}$ |
| $19K_3$ | 57 2 1 0 | 218 83.239 | $d_2 \notin \mathbb{N}$ |
| $20K_3$ | 60 2 1 0 | 215 81.302 | $d_2 \notin \mathbb{N}$ |
| $21K_3$ | 63 2 1 0 | 212 79.311 | $d_2 \notin \mathbb{N}$ |
| $22K_3$ | 66 2 1 0 | 209 77.263 | $d_2 \notin \mathbb{N}$ |
| $2K_4$ | 8 3 2 0 | 267 108.734 | $d_2 \notin \mathbb{N}$ |
| $3K_4$ | 12 3 2 0 | 263 107.027 | $d_2 \notin \mathbb{N}$ |
| $4K_4$ | 16 3 2 0 | 259 105.266 | $d_2 \notin \mathbb{N}$ |
| $5K_4$ | 20 3 2 0 | 255 103.451 | $d_2 \notin \mathbb{N}$ |
| $6K_4$ | 24 3 2 0 | 251 101.578 | $d_2 \notin \mathbb{N}$ |
| $7K_4$ | 28 3 2 0 | 247 99.644 | $d_2 \notin \mathbb{N}$ |
| $8K_4$ | 32 3 2 0 | 243 97.646 | $d_2 \notin \mathbb{N}$ |
| $9K_4$ | 36 3 2 0 | 239 95.582 | $d_2 \notin \mathbb{N}$ |
| $10K_3$ | 40 3 2 0 | 235 93.447 | $d_2 \notin \mathbb{N}$ |
| $11K_3$ | 44 3 2 0 | 231 91.238 | $d_2 \notin \mathbb{N}$ |
| $12K_3$ | 48 3 2 0 | 227 88.952 | $d_2 \notin \mathbb{N}$ |
| $13K_3$ | 52 3 2 0 | 223 86.583 | $d_2 \notin \mathbb{N}$ |
| $14K_3$ | 56 3 2 0 | 219 84.128 | $d_2 \notin \mathbb{N}$ |
| $15K_4$ | 60 3 2 0 | 215 81.581 | $d_2 \notin \mathbb{N}$ |
| $16K_3$ | 64 3 2 0 | 211 78.938 | $d_2 \notin \mathbb{N}$ |
| $17K_4$ | 68 3 2 0 | 207 76.193 | $d_2 \notin \mathbb{N}$ |
|--------|---------|-----------|----------------|
| $18K_4$ | 72 3 2 0 | 203 73.340 | $d_2 \notin \mathbb{N}$ |
| $19K_4$ | 76 3 2 0 | 199 70.372 | $d_2 \notin \mathbb{N}$ |
| $20K_4$ | 80 3 2 0 | 195 67.282 | $d_2 \notin \mathbb{N}$ |
| $21K_4$ | 84 3 2 0 | 191 64.063 | $d_2 \notin \mathbb{N}$ |
| $22K_4$ | 88 3 2 0 | 187 60.706 | $d_2 \notin \mathbb{N}$ |
| $2K_5$ | 10 4 3 0 | 265 107.925 | $d_2 \notin \mathbb{N}$ |
| $3K_5$ | 15 4 3 0 | 260 105.769 | $d_2 \notin \mathbb{N}$ |
| $4K_5$ | 20 4 3 0 | 255 103.529 | $d_2 \notin \mathbb{N}$ |
| $5K_5$ | 25 4 3 0 | 250 101.200 | $d_2 \notin \mathbb{N}$ |
| $6K_5$ | 30 4 3 0 | 245 98.776 | $d_2 \notin \mathbb{N}$ |
| $7K_5$ | 35 4 3 0 | 240 96.250 | $d_2 \notin \mathbb{N}$ |
| $8K_5$ | 40 4 3 0 | 235 93.617 | $d_2 \notin \mathbb{N}$ |
| $9K_5$ | 45 4 3 0 | 230 90.870 | $d_2 \notin \mathbb{N}$ |
| $10K_5$ | 50 4 3 0 | 225 88 22.091 | $\lambda_2 \notin \mathbb{N}$ |
| $11K_5$ | 55 4 3 0 | 220 85 20.788 | $\lambda_2 \notin \mathbb{N}$ |
| $12K_5$ | 60 4 3 0 | 215 81.860 | $d_2 \notin \mathbb{N}$ |
| $13K_5$ | 65 4 3 0 | 210 78.571 | $d_2 \notin \mathbb{N}$ |
| $14K_5$ | 70 4 3 0 | 205 75.122 | $d_2 \notin \mathbb{N}$ |
| $15K_5$ | 75 4 3 0 | 200 71.500 | $d_2 \notin \mathbb{N}$ |
| $16K_5$ | 80 4 3 0 | 195 67.692 | $d_2 \notin \mathbb{N}$ |
| $17K_5$ | 85 4 3 0 | 190 63.684 | $d_2 \notin \mathbb{N}$ |
| $18K_5$ | 90 4 3 0 | 185 59.459 | $d_2 \notin \mathbb{N}$ |
| $19K_5$ | 95 4 3 0 | 180 55 -0.055 | $\lambda_2 \notin \mathbb{N}$ |
| $20K_5$ | 100 4 3 0 | 175 50.286 | $d_2 \notin \mathbb{N}$ |
| $21K_5$ | 105 4 3 0 | 170 45.294 | $d_2 \notin \mathbb{N}$ |
| $22K_5$ | 110 4 3 0 | 165 40 -22.200 | $\lambda_2 \notin \mathbb{N}$ |

| $2K_{12}$ | 4 2 0 2 | 271 110.376 | $d_2 \notin \mathbb{N}$ |
| $2K_{13}$ | 6 3 0 3 | 269 109.569 | $d_2 \notin \mathbb{N}$ |
| $2K_{14}$ | 8 4 0 4 | 267 108.764 | $d_2 \notin \mathbb{N}$ |
| $2K_{15}$ | 10 5 0 5 | 265 107.962 | $d_2 \notin \mathbb{N}$ |
| $2K_{16}$ | 12 6 0 6 | 263 107.163 | $d_2 \notin \mathbb{N}$ |
| $2K_{17}$ | 14 7 0 7 | 261 106.368 | $d_2 \notin \mathbb{N}$ |
| $2K_{18}$ | 16 8 0 8 | 259 105.575 | $d_2 \notin \mathbb{N}$ |
| $2K_{19}$ | 18 9 0 9 | 257 104.786 | $d_2 \notin \mathbb{N}$ |
| $2K_{20}$ | 20 10 0 10 | 255 104 27.919 | $\lambda_2 \notin \mathbb{N}$ |
| $2K_{21}$ | 22 11 0 11 | 253 103.217 | $d_2 \notin \mathbb{N}$ |
| $2K_{22}$ | 24 12 0 12 | 251 102.438 | $d_2 \notin \mathbb{N}$ |
| $2K_{23}$ | 26 13 0 13 | 249 101.663 | $d_2 \notin \mathbb{N}$ |
| $2K_{24}$ | 28 14 0 14 | 247 100.891 | $d_2 \notin \mathbb{N}$ |
| $2K_{25}$ | 30 15 0 15 | 245 100.122 | $d_2 \notin \mathbb{N}$ |
| $2K_{26}$ | 32 16 0 16 | 243 99.358 | $d_2 \notin \mathbb{N}$ |
| $2K_{27}$ | 34 17 0 17 | 241 98.598 | $d_2 \notin \mathbb{N}$ |
| $2K_{28}$ | 36 18 0 18 | 239 97.841 | $d_2 \notin \mathbb{N}$ |
| $2K_{29}$ | 38 19 0 19 | 237 97.089 | $d_2 \notin \mathbb{N}$ |
| $2K_{30}$ | 40 20 0 20 | 235 96.340 | $d_2 \notin \mathbb{N}$ |
| $2K_{31}$ | 42 21 0 21 | 233 95.597 | $d_2 \notin \mathbb{N}$ |
| $2K_{32}$ | 44 22 0 22 | 231 94.857 | $d_2 \notin \mathbb{N}$ |
| $3K_2$ | 6 | 4 | 2 | 4 | 269 | 109.591 | $d_2 \notin \mathbb{N}$ |
| $3K_3$ | 9 | 6 | 3 | 6 | 266 | 108.414 | $d_2 \notin \mathbb{N}$ |
| $3K_4$ | 12 | 8 | 4 | 8 | 263 | 107.256 | $d_2 \notin \mathbb{N}$ |
| $3K_5$ | 15 | 10 | 5 | 10 | 260 | 106.115 | $d_2 \notin \mathbb{N}$ |
| $3K_6$ | 18 | 12 | 6 | 12 | 257 | 104.996 | $d_2 \notin \mathbb{N}$ |
| $3K_7$ | 21 | 14 | 7 | 14 | 254 | 103.898 | $d_2 \notin \mathbb{N}$ |
| $3K_8$ | 24 | 16 | 8 | 16 | 251 | 102.821 | $d_2 \notin \mathbb{N}$ |
| $3K_9$ | 27 | 18 | 9 | 18 | 248 | 101.766 | $d_2 \notin \mathbb{N}$ |
| $3K_{10}$ | 30 | 20 | 10 | 20 | 245 | 100.735 | $d_2 \notin \mathbb{N}$ |
| $3K_{11}$ | 33 | 22 | 11 | 22 | 242 | 99.727 | $d_2 \notin \mathbb{N}$ |
| $3K_{12}$ | 36 | 24 | 12 | 24 | 239 | 98.745 | $d_2 \notin \mathbb{N}$ |
| $3K_{13}$ | 39 | 26 | 13 | 26 | 236 | 97.788 | $d_2 \notin \mathbb{N}$ |
| $3K_{14}$ | 42 | 28 | 14 | 28 | 233 | 96.858 | $d_2 \notin \mathbb{N}$ |
| $3K_{15}$ | 45 | 30 | 15 | 30 | 230 | 95.957 | $d_2 \notin \mathbb{N}$ |
| $3K_{16}$ | 48 | 32 | 16 | 32 | 227 | 95.084 | $d_2 \notin \mathbb{N}$ |
| $3K_{17}$ | 51 | 34 | 17 | 34 | 224 | 94.241 | $d_2 \notin \mathbb{N}$ |
| $3K_{18}$ | 54 | 36 | 18 | 36 | 221 | 93.430 | $d_2 \notin \mathbb{N}$ |
| $3K_{19}$ | 57 | 38 | 19 | 38 | 218 | 92.651 | $d_2 \notin \mathbb{N}$ |
| $3K_{20}$ | 60 | 40 | 20 | 40 | 215 | 91.907 | $d_2 \notin \mathbb{N}$ |
| $3K_{21}$ | 63 | 42 | 21 | 42 | 212 | 91.198 | $d_2 \notin \mathbb{N}$ |
| $3K_{22}$ | 66 | 44 | 22 | 44 | 209 | 90.526 | $d_2 \notin \mathbb{N}$ |
| $4K_2$ | 8 | 6 | 4 | 6 | 267 | 108.824 | $d_2 \notin \mathbb{N}$ |
| $4K_3$ | 12 | 9 | 6 | 9 | 263 | 107.300 | $d_2 \notin \mathbb{N}$ |
| $4K_4$ | 16 | 12 | 8 | 12 | 259 | 105.822 | $d_2 \notin \mathbb{N}$ |
| $4K_5$ | 20 | 15 | 10 | 15 | 255 | 104.392 | $d_2 \notin \mathbb{N}$ |
| $4K_6$ | 24 | 18 | 12 | 18 | 251 | 103.012 | $d_2 \notin \mathbb{N}$ |
| $4K_7$ | 28 | 21 | 14 | 21 | 247 | 101.684 | $d_2 \notin \mathbb{N}$ |
| $4K_8$ | 32 | 24 | 16 | 24 | 243 | 100.412 | $d_2 \notin \mathbb{N}$ |
| $4K_9$ | 36 | 27 | 18 | 27 | 239 | 99.197 | $d_2 \notin \mathbb{N}$ |
| $4K_{10}$ | 40 | 30 | 20 | 30 | 235 | 98.043 | $d_2 \notin \mathbb{N}$ |
| $4K_{11}$ | 44 | 33 | 22 | 33 | 231 | 96.952 | $d_2 \notin \mathbb{N}$ |
| $4K_{12}$ | 48 | 36 | 24 | 36 | 227 | 95.930 | $d_2 \notin \mathbb{N}$ |
| $4K_{13}$ | 52 | 39 | 26 | 39 | 223 | 94.978 | $d_2 \notin \mathbb{N}$ |
| $4K_{14}$ | 56 | 42 | 28 | 42 | 219 | 94.100 | $d_2 \notin \mathbb{N}$ |
| $4K_{15}$ | 60 | 45 | 30 | 45 | 215 | 93.302 | $d_2 \notin \mathbb{N}$ |
| $5K_2$ | 10 | 8 | 6 | 8 | 265 | 108.075 | $d_2 \notin \mathbb{N}$ |
| $5K_3$ | 15 | 12 | 9 | 12 | 260 | 106.231 | $d_2 \notin \mathbb{N}$ |
| $5K_4$ | 20 | 16 | 12 | 16 | 255 | 104.471 | $d_2 \notin \mathbb{N}$ |
| $5K_5$ | 25 | 20 | 15 | 20 | 250 | 102.800 | $d_2 \notin \mathbb{N}$ |
| $5K_6$ | 30 | 24 | 18 | 24 | 245 | 101.224 | $d_2 \notin \mathbb{N}$ |
| $5K_7$ | 35 | 28 | 21 | 28 | 240 | 99.750 | $d_2 \notin \mathbb{N}$ |
| $5K_8$ | 40 | 32 | 24 | 32 | 235 | 98.383 | $d_2 \notin \mathbb{N}$ |
| $5K_9$ | 45 | 36 | 27 | 36 | 230 | 97.130 | $d_2 \notin \mathbb{N}$ |
| $5K_{10}$ | 50 | 40 | 30 | 40 | 225 | 96 | 25 | 52.500 | $\mu_2 \notin \mathbb{N}$ |