ON THE LOCAL $C^{1,\alpha}$ SOLUTION OF IDEAL MAGNETO-HYDRODYNAMICAL EQUATIONS

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ABSTRACT. This paper is devoted to the study of the two-dimensional and three-dimensional ideal incompressible magneto-hydrodynamic (MHD) equations in which the Faraday law is inviscid. We consider the local existence and uniqueness of classical solutions for the MHD system in Hölder space when the general initial data belongs to $C^{1,\alpha}(\mathbb{R}^n)$ for $n = 2$ and $n = 3$.

1. Introduction. The magneto-hydrodynamic (MHD) flows describe the dynamics of electrically conducting fluids, such as the geomagnetic dynamo in geophysics, plasmas, liquid metals, solar winds and solar flares in astrophysics, and salt water or electrolytes (see, for instance, [1, 4, 12, 29]), which can be described as a combination of the Navier-Stokes equations of fluid dynamics and the Maxwell’s equations of electromagnetism.

For the general MHD system in $\mathbb{R}^n$, $\mathbf{u} = (u_1, \cdots, u_n)^T$ is the velocity of the flows, $\mathbf{b} = (b_1, \cdots, b_n)^T$ denotes the magnetic field, $\mathbf{E}$ is the electric field, $\mathbf{J}$ is the current density, $p$ is the scalar pressure of the fluid and $\mu \geq 0$ is the viscosity of the fluid which is the inverse of the Reynolds number. The momentum equation is

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \mu \Delta \mathbf{u} + \mathbf{J} \times \mathbf{b}. \quad (1)$$

The ideal Ohm’s law for a plasma is given by

$$\mathbf{E} + \mathbf{u} \times \mathbf{b} = 0.$$
Faraday's law is
\[ \partial_t b = -\nabla \times E. \]

In an imperfectly conducting fluid the magnetic field can generally move through the fluid following a diffusion law with the resistivity of the plasma serving as the diffusion constant. This includes an extra term in Ohm's Law and gives the resistive MHD. The Faraday’s law in this case reads
\[ \partial_t b = \nabla \times (u \times b) + \nu \Delta b. \]  \hspace{1cm} (2)

Here \( \nu \geq 0 \) is the resistivity constant which is inversely proportional to the electrical conductivity constant. The low-frequency Ampere’s law neglects displacement current and is given by
\[ J = \frac{1}{\mu_0} \nabla \times b \]  \hspace{1cm} (3)

where \( \mu_0 \geq 0 \) is the magnetic constant. The incompressibility and the magnetic divergence constraints are
\[ \nabla \cdot u = 0, \quad \nabla \cdot b = 0. \]  \hspace{1cm} (4)

The MHD system is the combination of (1), (2), (3) and (4) which is also stated as
\[ \begin{cases} \partial_t u + (u \cdot \nabla)u + \nabla p - \mu \Delta u = -\frac{1}{2} \nabla |b|^2 + (b \cdot \nabla)b, \\ \partial_t b + (u \cdot \nabla)b = b \cdot \nabla u + \nu \Delta b, \\ \nabla \cdot u = 0, \quad \nabla \cdot b = 0, \\ b_{|t=0} = b_0, \\ u_{|t=0} = u_0. \end{cases} \]  \hspace{1cm} (5)

If \( \mu = \nu = 0 \), (1)-(5) is the ideal MHD equations. For more details of the related background, see, for instance, [23], [5], [11], [20].

System (5) with zero diffusivity in the equation for the magnetic field can be applied to plasma models when the plasma is strongly collisional, or the resistivity due to these collisions is extremely small.

The purpose of this paper is to show the existence and uniqueness of a local classical solution for an ideal incompressible MHD equations with zero diffusivity in Hölder space for 2 and 3 dimensions:
\[ \begin{cases} \partial_t u + (u \cdot \nabla)u + \nabla p = -\frac{1}{2} \nabla |b|^2 + (b \cdot \nabla)b, \\ \partial_t b + (u \cdot \nabla)b = b \cdot \nabla u, \\ \nabla \cdot u = 0, \quad \nabla \cdot b = 0, \\ b_{|t=0} = b_0, \\ u_{|t=0} = u_0. \end{cases} \]  \hspace{1cm} (6)

Using the divergence free condition, we may replace \( p \) with \( p + \frac{1}{2} |b|^2 \) and obtain the following equations:
\[ \begin{cases} \partial_t u + (u \cdot \nabla)u + \nabla p = (b \cdot \nabla)b, \\ \partial_t b + (u \cdot \nabla)b = (b \cdot \nabla)u, \\ \nabla \cdot u = 0, \quad \nabla \cdot b = 0, \\ b_{|t=0} = b_0, \\ u_{|t=0} = u_0. \end{cases} \]  \hspace{1cm} (7)
In terms of the Elsässer variables
\[
\begin{cases}
Z^+ = u + b, \\
Z^- = u - b,
\end{cases}
\] (8)
these equations can be rewritten in the following form
\[
\begin{aligned}
\frac{\partial Z^+}{\partial t} + Z^- \cdot \nabla Z^+ &= -\nabla p, \\
\frac{\partial Z^-}{\partial t} + Z^+ \cdot \nabla Z^- &= -\nabla p, \\
\nabla \cdot Z^+ &= 0, \\
\nabla \cdot Z^- &= 0,
\end{aligned}
\] (9)
When equations (9) are linearized around the static solution with a constant magnetic field \( B_0 \), one obtains that the fluctuations \( z^\pm = Z^\pm \pm B_0 \) propagate along the \( B_0 \) magnetic field in opposite directions. This suggests that in the original nonlinear problem, a strong enough magnetic field will reduce the nonlinear interactions \cite{2} and inhibit formation of strong gradients. This effect was observed in direct numerical simulations of equations (9) with periodic boundary conditions \cite{14}. These calculations show that in the presence of a strong enough magnetic field, solutions remain analytic in a strip whose width is bounded from below.

The related problem of Euler equations in Hölder space has been studied for a long time, see, for instance, \cite{28}. Before going further, let us first briefly review some existence results of incompressible magneto-hydrodynamic equations from various aspects (we will not attempt to address exhaustive reference in this paper). It is well-known \cite{13}, \cite{30} that in the 2D case, system (5) with kinematic viscosity and magnetic resistivity has a unique global classical solution for every initial data \( u_0, b_0 \in H^s(\mathbb{R}^2), s \geq 2 \). In the 3D case, system (5) with kinematic viscosity and magnetic diffusion is locally well-posed for any given initial data \( u_0, b_0 \in H^s(\mathbb{R}^3), s \geq 3 \).

However all these results considered the full kinematic viscosity and magnetic diffusion. If any one of these parameters is zero, the global regularity issue has not been completely settled even in the 2D case. In \cite{10}, Cao and Wu gave a review about the 2D MHD equations with mixed partial dissipation and magnetic diffusion. For initial data close to any given non-zero equilibrium, Hu, Lei and Lin \cite{18} used the weak dissipative mechanism to prove the global regularity of magneto-hydrodynamic equations with partial magnetic dissipation near equilibrium. If the initial data belongs to \( H^2 \), in \cite{9}, Cao and Wu proved some global regularity results for the 2D MHD equations with mixed partial dissipation and magnetic diffusion. In the same paper, the authors also proved that the inviscid and resistive MHD system has a global weak solution with \( (u_0, b_0) \in \mathcal{H}^1(\mathbb{R}^2) \). If there is only kinematic viscosity, recently, using the Lagrangian transformation method combined with other methods such as the anisotropic Littlewood-Paley analysis techniques, in \cite{24, 25, 26, 16, 32, 17}, the authors obtained the global well-posedness of classical solutions for the 2D and 3D MHD equations under the assumption that the initial velocity field and the displacement of the magnetic field from a non-zero constant are sufficiently small in appropriate Sobolev spaces. However, the arguments involved, despite their general interests, were rather complicated. In \cite{27, 54}, the authors gave a new and simple
proof, which involves only the energy estimate method, interpolating inequalities and a couple of elementary observations.

In particular, it is worth mentioning that in [2], for the ideal MHD equations, Bardos, Sulem and Sulem [2] proved the global existence of the classical solution when the initial data \((u_0, b_0)\) is close to the equilibrium state \((0, B_0)\). It is also pointed out that the fluctuations \(u + b - B_0\) and \(u - b + B_0\) propagate along the \(B_0\) magnetic field in opposite directions. The method used by Bardos, Sulem and Sulem [2] plays an important role in our proof.

In the absence of the well-posedness theory, the development of blow-up theory is of major importance for both theoretical and practical purposes. There are a lot of papers about the blow-up criterion, see, for instance, [3, 7, 22, 33, 31]. Although many important results have been obtained in recent years, as far as we know, the global well-posedness or blow-up result of the following problems still remains widely open:

- Two-dimensional MHD equations with only magnetic diffusion.
- Two-dimensional ideal MHD equations.
- Three-dimensional incompressible MHD equations.

We will pay attention to these interesting topics. The main result of this short paper is stated as:

**Theorem 1.1.** For general initial data \(u_0, b_0 \in C^{1,\alpha}(\mathbb{R}^n)\), if \(T\) is sufficiently small, then the ideal incompressible magneto-hydrodynamic system (6) has a unique local solution \((u, b)\) in \(C([0, T), C^{1,\alpha}(\mathbb{R}^n))\), \(n = 2\) and 3.

This paper is organized as follows. In Section 2 we present the notation used in this paper, some Schauder-type estimates will be presented in Section 3, and in Section 4, we will complete the proof of Theorem 1.1.

2. **Preliminaries.** The fluctuations \(Z^\pm\) satisfy

\[
\begin{align*}
\frac{\partial Z^+}{\partial t} + Z^- \cdot \nabla Z^+ &= -\nabla p, \\
\frac{\partial Z^-}{\partial t} + Z^+ \cdot \nabla Z^- &= -\nabla p, \\
\nabla \cdot Z^+ &= \nabla \cdot Z^- = 0, \\
Z^+_|_{t=0} &= Z^+_0, \\
Z^-|_{t=0} &= Z^-_0.
\end{align*}
\] (10)

First, let

\[ j^\pm = \text{curl} Z^\pm \] (11)

then by (10), we have

\[
\begin{align*}
\frac{\partial j^+}{\partial t} + Z^- \cdot \nabla j^+ &= -\sum_k \nabla Z^-_k \wedge \partial_k Z^+, \\
\frac{\partial j^-}{\partial t} + Z^+ \cdot \nabla j^- &= -\sum_k \nabla Z^+_k \wedge \partial_k Z^-,
\end{align*}
\] (12)

where

\[ (u \wedge v)_i = \sum_{j, k} \varepsilon_{ijk} u_j v_k \]
and for each triad \((i, j, k), i, j, k \in \{1, 2, 3\}\),
\[
e_{ijk} = \begin{cases} 
1 & \text{if } (i, j, k) \text{ is an even permutation of } \{1, 2, 3\}, \\
-1 & \text{if } (i, j, k) \text{ is an odd permutation of } \{1, 2, 3\}, \\
0 & \text{otherwise}.
\end{cases}
\]

To avoid loss of derivatives, in this paper, we will integrate along the characteristics lines. Two families of characteristics \(x = x^\pm(t)\) are associated to equations (12):
\[
\begin{align*}
\dot{x}^- &= Z^-(x^-(t), t), \\
x^-(0) &= a^-.
\end{align*}
\]
and
\[
\begin{align*}
\dot{x}^+ &= Z^+(x^+(t), t), \\
x^+(0) &= a^+.
\end{align*}
\]

Along the characteristics direction \(x = x^-(t)\), we have
\[
\begin{align*}
\dot{Z}^-(x^-(t), t) &= -\nabla p(x^-(t), t), \\
\dot{j}^-(x^-(t), t) &= -\sum_k \nabla Z^+_k(x^-(t), t) \wedge \partial_k Z^+(x^-(t), t).
\end{align*}
\]

Thus
\[
\begin{align*}
Z^+(x^-(t), t) &= Z^+_0(a^-) - \int_0^t \nabla p(x^-(\tau), \tau)d\tau, \\
j^+(x^-(t), t) &= j^+_0(a^-) - \sum_k \int_0^t \nabla Z^-_k(x^-(\tau), \tau) \wedge \partial_k Z^+(x^-(\tau), \tau)d\tau.
\end{align*}
\]

For \(0 < \alpha < 1\), \(C^{0,\alpha}(\mathbb{R}^n)\) denotes the space of (scalar or vector) functions which are Hölder continuous. It is equipped with the norm
\[
|u|_{0,\alpha} = |u|_0 + \sup_{x,y \in \mathbb{R}^n \atop x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\alpha},
\]
where
\[
|u|_0 = \sup_{x \in \mathbb{R}^n} |u(x)|,
\]
and we also use the definition
\[
|u|_{C^\alpha} = \sup_{x,y \in \mathbb{R}^n \atop x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\alpha}.
\]

Similarly, \(C^{1,\alpha}(\mathbb{R}^n)\) is the space of functions which are Hölder continuous together with their derivatives, and is equipped with the norm
\[
|u|_{1,\alpha} = |u|_{0,\alpha} + \sum_{i=1}^n \left| \frac{\partial u}{\partial x_i} \right|_{0,\alpha}.
\]

If \(u \in C^{0,\alpha}(\mathbb{R}^n), \ v \in C^{0,\alpha}(\mathbb{R}^n)\), we have the following basic inequality (see, for instance, [15]),
\[
|uv|_{0,\alpha} \lesssim |u|_{0,\alpha}|v|_{0,\alpha}.
\]

In this paper, we write \(X \lesssim Y\) to indicate \(X \leq CY\) for some constant \(C > 0\).
3. A Schauder-type lemma. Denote
\[ \omega = \nabla \times u. \]
Using the divergence free property, we have
\[ \Delta u = -\nabla \times \omega. \]  
By the property of pseudo-differential operators, we have the following estimates
\[ |\nabla u|_0 \lesssim |u|_0 + |\omega|_{0, \alpha}, \]
and
\[ |\nabla u|_{C^\alpha} \lesssim |\omega|_{C^\alpha}. \]
see, for instance, [2] for a detailed proof.

Combining the above estimates and noticing that for a divergence free vector field \( u \), we have
\[ -\Delta u = \nabla \times (\nabla \times u). \]
Hence, we can obtain the following lemma:

**Lemma 3.1.** For \( Z^\pm \) satisfying \( \nabla \cdot Z^\pm = 0 \), and \( j^\pm = \text{curl} Z^\pm \), we have
\[ |\nabla Z^\pm|_0 \lesssim (|Z^\pm|_0 + |j^\pm|_{0, \alpha}), \]
and
\[ |\nabla Z^\pm|_{C^\alpha} \lesssim |j^\pm|_{C^\alpha}. \]

In order to prove Theorem 1.1, we first establish the following proposition:

**Proposition 1.** Let \( Z_0^+, Z_0^- \in C^{1, \alpha}(\mathbb{R}^3) \), then we have the following estimates
\[ |Z^+|_{0, \alpha} + |j^+|_{0, \alpha} \lesssim |Z_0^+|_{0, \alpha} + |j_0^+|_{0, \alpha} + TN^2(T), \]
\[ |Z^-|_{0, \alpha} + |j^-|_{0, \alpha} \lesssim |Z_0^-|_{0, \alpha} + |j_0^-|_{0, \alpha} + TN^2(T), \]
where
\[ N(t) = M^+(t) + M^-(t), \]
\[ M^+(t) = \sup_{0 \leq \tau \leq t} (|j^+(\cdot, \tau)|_{0, \alpha} + |Z^+(\cdot, \tau)|_{0, \alpha}), \]
\[ M^-(t) = \sup_{0 \leq \tau \leq t} (|j^-\cdot, \tau)|_{0, \alpha} + |Z^-\cdot, \tau)|_{0, \alpha}). \]

**Proof.** Along the characteristics line \( x = x^-(t) \), we have
\[ \frac{d}{dt} Z^+(x^-(t), t) = -\nabla p(x^-(t), t), \]
and
\[ \frac{d}{dt} j^+(x^-(t), t) = - \sum_k \nabla Z^+_k(x^-(t), t) \wedge \partial_k Z^+(x^-(t), t). \]
Thus
\[ Z^+(x^-(t), t) = Z_0^+(a^-) - \int_0^t \nabla p(x^-(\tau), \tau) d\tau, \]
and
\[ j^+(x^-(t), t) = j_0^+(a^-) - \sum_k \int_0^t \nabla Z^-_k(x^-(\tau), \tau) \wedge \partial_k Z^+(x^-(\tau), \tau) d\tau. \]
The pressure function $p$ satisfies
\[ \Delta p = -\text{div}(Z^- \cdot \nabla Z^+). \] (32)

Hence, by Lemma 3.1 and (21), we have
\[ |\nabla p|_0 \lesssim |Z^- \otimes Z^+_0|_0 + |Z^- \cdot \nabla Z^+_0|_{0,\alpha} \]
\[ \lesssim |Z^-|_0|Z^+_0|_0 + |Z^-|_{0,\alpha}(|Z^+_0|_0 + |j^+_0|_{0,\alpha}) \] (33)

Hence, we have
\[ |Z^+_0|_0 \lesssim |Z^+_0|_0 + \int_0^t N^2(s)ds \]
\[ \lesssim |Z^+_0|_0 + TN^2(T). \] (34)

To estimate the $\tilde{C}^\alpha$ norms, we consider the distance $\rho(\tau) = |x^-(\tau) - y^-(\tau)|$ between two characteristics starting from $x^-(0) = a^-$ and $y^-(0) = b^-$, we have
\[
\frac{d}{dt} \left[ \frac{Z^+(x^-(t), t) - Z^+(y^-(t), t)}{|x^-(t) - y^-(t)|^\alpha} \right] \\
\leq \alpha \frac{|\rho'(t)|}{\rho(t)^{\alpha+1}} |Z^+(x^-(t), t) - Z^+(y^-(t), t)| \\
+ \frac{1}{\rho^\alpha(t)} \left[ \frac{d}{dt} \left( Z^+(x^-(t), t) - Z^+(y^-(t), t) \right) \right].
\] (35)

Integrating from 0 to $t$ and using (28) we have
\[ \frac{Z^+(x^-(t), t) - Z^+(y^-(t), t)}{|x^-(t) - y^-(t)|^\alpha} \]
\[ \lesssim |Z^+_0|_{0,\alpha} + \int_0^t \left| \frac{|\rho'(\tau)|}{\rho(\tau)^{\alpha+1}} |Z^+(x^- (\tau), \tau) - Z^+(y^- (\tau), \tau)| \right| d\tau \] (36)

Integrating from 0 to $t$ and using (13) we have
\[ \int_0^t \left| \frac{|\rho'(\tau)|}{\rho(\tau)^{\alpha+1}} |Z^+(x^- (\tau), \tau) - Z^+(y^- (\tau), \tau)| \right| d\tau \]
\[ \lesssim \int_0^t \frac{|Z^-(x^- (\tau), \tau) - Z^-(y^- (\tau), \tau)|}{\rho(\tau)^{\alpha+1}} |Z^+(x^- (\tau), \tau) - Z^+(y^- (\tau), \tau)| d\tau \] (37)

\[ \lesssim \int_0^t N^2(s)ds \lesssim TN^2(T). \]

As
\[ |\nabla p|_{\tilde{C}^\alpha} \lesssim |Z^- \cdot \nabla Z^+|_{\tilde{C}^\alpha} \lesssim N^2(t), \] (38)

we have
\[ \int_0^t \left| \frac{1}{\rho^\alpha(\tau)} (\nabla p(x^- (\tau), \tau) - \nabla p(y^- (\tau), \tau)) \right| d\tau \]
\[ \lesssim \int_0^t N^2(s)ds \lesssim TN^2(T). \] (39)
Similarly, we also have
\begin{equation}
|j^+|_0 \lesssim |j_0^+|_0 + \int_0^t N^2(s) ds \\
\lesssim |j_0^+|_0 + TN^2(T),
\end{equation}
and
\begin{equation}
|j^+_\alpha| \lesssim |j_0^+_\alpha| + TN^2(T).
\end{equation}

So, we have
\begin{equation}
|Z^+|_{0,\alpha} + |j^+|_{0,\alpha} \lesssim |Z_0^+|_{0,\alpha} + |j_0^+|_{0,\alpha} + TN^2(T).
\end{equation}

Analogous inequality holds for
\begin{equation}
|Z^-|_{0,\alpha} + |j^-|_{0,\alpha} \lesssim |Z_0^-|_{0,\alpha} + |j_0^-|_{0,\alpha} + TN^2(T).
\end{equation}

Combining all these estimates, we finally have
\begin{equation}
N(T) \lesssim N(0) + TN^2(T).
\end{equation}

So, if $T$ is sufficiently small, we can show the local existence results. Details can be seen in the following section.

4. Proof of the main theorem.

Proof. Without loss of generality, we only consider the 3D case. Once the theorem is proved for $n = 3$, it is not difficult to get the 2D case by the same method.

We start the proof of Theorem 1.1 by the method of successive approximations. That is, we consider a successive approximation sequence \{(Z^+_{m+1}, Z^-_{m+1})\} solving
\begin{align*}
\frac{\partial Z^+_{m+1}}{\partial t} + Z^-_{m} \cdot \nabla Z^+_{m+1} &= -\nabla p_{m+1}, \\
\frac{\partial Z^-_{m+1}}{\partial t} + Z^+_{m} \cdot \nabla Z^-_{m+1} &= -\nabla p_{m+1}, \\
\nabla \cdot Z^+_{m+1} &= 0, \\
\nabla \cdot Z^-_{m+1} &= 0,
\end{align*}
\begin{align*}
Z^+_{m+1}(x, 0) &= Z_0^+(x), \\
Z^-_{m+1}(x, 0) &= Z_0^-(x).
\end{align*}

To show that \{(Z^+_{m+1}, Z^-_{m+1})\} converges to a solution of (9), it suffices to prove that \{(Z^+_{m+1}, Z^-_{m+1})\} obeys the following properties:

(I) There exists a time interval $[0, T_1]$ over which \{(Z^+_{m+1}, Z^-_{m+1})\} are bounded uniformly in $m$. More precisely, we show that
\begin{equation}
|Z^+_{m+1}|_{0,\alpha} + |j^+_{m+1}|_{0,\alpha} \leq C(T_1)
\end{equation}
and
\begin{equation}
|Z^-_{m+1}|_{0,\alpha} + |j^-_{m+1}|_{0,\alpha} \leq C(T_1)
\end{equation}
for all $m$, where $C(T_1)$ is a constant independent of $m$ and $j^\pm_{m+1} = \text{curl } Z^\pm_{m+1}$.

In fact, for the Euler equations in 3 dimensions, Majda and Bertozzi [28] proved the global classical solution in $C(0, T; C^{1+\alpha}(\mathbb{R}^3))$ and the existence of local solution in 2 dimensions. Using the same technique, we can show that $Z^+$ and $Z^-$ belong
to $C(0,T; C^{1+\alpha}(\mathbb{R}^3))$, which needs a complicated standard energy method, i.e., (I) holds similarly as the Euler case in [28], for all
\[(Z_0^+, Z_0^-) \in H^s(\mathbb{R}^d), \ s > n/2 + 1,\]
there exists $T > 0$ such that the Cauchy problem [45] has a unique local smooth solution $(Z^+(t,x), Z^-(t,x))$ satisfying
\[(Z^+, Z^-) \in C((0, T); H^s) \cap C^1([0, T); H^{s-1}).\]
Then from the embedding theorem $H^s \hookrightarrow C^0, \alpha$ with real index $s > n/2 + 1$, we can prove that the property (I) holds. For more details, we refer to [28].

**Remark 1.** In Lei [21], the author considered the axially symmetric 3D incompressible MHD system, and derived the unique global solution $(u, B)$ for the Cauchy problem $(u_0, B_0) \in H^s$ with $s \geq 2$ and $B_0^2/r \in L^\infty$ which satisfy
\[\|u\|_{H^2}^2 + \|B\|_{H^2}^2 + \int_0^T \|
abla u(s)\|_{H^2}^2 ds \leq C_0 e^{C_0(1+t)\frac{7}{4} e^{C_0 t}}.\] (46)

In Cai and Lei [6], the authors proved the global existence of a unique classical solution for the ideal MHD equations with or without viscosity in weighted Sobolev spaces. Using the same technique in [6] and [21], from equation (45), we can also show that
\[(Z^+_m, Z^-_m) \in L^\infty(0, T_1; C^{1+\alpha})\]
and
\[((Z^+_m)_t, (Z^-_m)_t) \in L^\infty(0, T_1; C^\alpha),\]
then by a compactness argument, we have
\[(Z^+_m, Z^-_m) \in C(0, T_1; C^\alpha) \cap L^\infty(0, T_1; C^{1+\alpha}).\]
From the interpolation inequality, we can prove property (I). For more details, one can see [6] and [21].

(II) There exists $T_2 > 0$ such that \{\(Z^+_{m+1} - Z^+_m\) and \(Z^-_{m+1} - Z^-_m\) are both Cauchy sequences in $C^{1,\alpha}$, namely
\[|Z^+_{m+1} - Z^+_m|_{0, \alpha} \leq C(T_2)2^{-m},\]
\[|j^+_{m+1} - j^+_m|_{0, \alpha} \leq C(T_2)2^{-m},\]
\[|Z^-_{m+1} - Z^-_m|_{0, \alpha} \leq C(T_2)2^{-m},\]
\[|j^-_{m+1} - j^-_m|_{0, \alpha} \leq C(T_2)2^{-m}.\] (47)

**Remark 2.** In Chen, Miao and Zhang [8], the authors proved the local existence and uniqueness of smooth solutions in Triebel-Lizorkin spaces for the ideal MHD equations. Moreover, they also presented a blow-up criteria in such spaces with respect to curl of $u$ and $b$.

Using similar techniques as in [8], i.e., by Littlewood-Paley decomposition and some uniform estimate (see step 2 of the proof in [8], pages 566-573), we can derive property (II). Here we omit the details.
With the properties stated in (I) and (II), we can then prove that there exists a positive time \( T_2 > 0 \) independent of \( m \) such that \( \{ (Z_m^+, Z_m^-) \} \) is a Cauchy sequence with limit \( \{ (Z^+, Z^-) \} \) satisfying

\[
Z^\pm \in C^{0, \alpha},
\]
\[
j^\pm \in C^{0, \alpha},
\]
\[
Z_m^\pm \to Z^\pm \text{ in } C^{0, \alpha},
\]
\[
j_m^\pm \to j^\pm \text{ in } C^{0, \alpha}.
\]

for any \( t \leq \min\{T_1, T_2\} \).

Here, using similar procedures as in step 3 of the proof in [8] (pages 570-573), by the compact embedding theorem, it is then easy to show that \( \{ (Z^+, Z^-) \} \) solves [9] and thus we obtain a local solution. Here we skip the details.

Similarly, we can prove the uniqueness.

This completes the proof of Theorem 1.1.

**Remark 3.** As the proofs of \( n = 2 \) and \( n = 3 \) are similar, we present here only the proof of the case \( n = 3 \). We leave the proof of \( n = 2 \) to interested readers.

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