Cohomology of group theoretic Dehn fillings II: A spectral sequence

Bin Sun

Abstract

This is the second paper in a series of three papers aiming to study cohomology of group theoretic Dehn fillings. In the present paper, we derive a spectral sequence for Cohen-Lyndon triples which can be thought of as a refined version of the classical Lyndon-Hochschild-Serre spectral sequence in the settings of group theoretic Dehn fillings. In the next paper [Sun19], we will apply this spectral sequence to study cohomological finiteness properties of Dehn fillings of acylindrically hyperbolic groups.

1 Introduction

1.1 Dehn surgery in 3-manifolds. In 3-dimensional topology, Dehn surgery is a process of cutting off a solid torus from a 3-manifold and then gluing the torus back in a different way. The Lickorish-Wallace theorem reveals an interesting and useful aspect of Dehn surgery: every closed connected orientable 3-manifold can be obtained from the 3-sphere by applying finitely many Dehn surgeries.

The second step of the surgery, called Dehn filling, begins with a 3-manifold $M$ with toral boundary and glues a solid torus to $M$ by identifying their boundaries, resulting in a new manifold. Topologically distinct ways of gluing a solid torus are parametrized by free homotopy classes of essential simple closed curves on $\partial M$, called slopes. Dehn filling interacts nicely with hyperbolicity:

Theorem 1.1 ([Thu82, Theorem [TH1]]). Let $M$ be a compact orientable 3-manifold with boundary a torus, such that $M \setminus \partial M$ admits a complete finite volume hyperbolic structure. Then for all but finitely many slopes on $\partial M$, the corresponding Dehn filling results in a hyperbolic 3-manifold.

1.2 Group theoretic Dehn filling. There is also a group theoretic version of Dehn fillings. Let $G$ be a group with a subgroup $H$ and let $N$ be a normal subgroup of $H$. The group theoretic Dehn filling corresponding to the triple $(G, H, N)$ is the quotient group $G/\langle \langle N \rangle \rangle$, where $\langle \langle N \rangle \rangle$ is the normal closure of $N$ in $G$.

Relations of these two kinds of Dehn fillings can be seen via the following example: Under the assumptions of Theorem 1.1, the natural map $\pi_1(\partial M) \to \pi_1(M)$ is injective and thus $\pi_1(\partial M)$ can be naturally thought of as a subgroup of $\pi_1(M)$. Let $G = \pi_1(M)$ and
$H = \pi_1(\partial M)$. Every slope $s$ on $\partial M$ generates a normal subgroup $N_s = \langle s \rangle \trianglelefteq H$. Let $M_s$ be the Dehn filling corresponding to a slope $s$, then $\pi_1(M_s) = G/\langle \langle N_s \rangle \rangle$ by the Seifert-van Kampen theorem.

Dehn filling is a useful tool in group theory. The solution of the virtually Haken conjecture makes use of Dehn fillings in word-hyperbolic groups [AGM13]. For certain relatively hyperbolic groups, Dehn fillings are used to study the Farrel-Jones conjecture and isomorphism problem [ACG18, DG18]. For mapping class groups of surfaces, [DGO17] constructs purely pseudo-Anosov normal subgroups by applying Dehn fillings. Other applications of Dehn fillings can be found in [AGM16, GMS16].

Analogs of Theorem 1.1 can be proved for groups satisfying certain negative curvature conditions. The first such result is due to Osin [Osi07] and independently to Groves-Manning [GM08] in the settings of relatively hyperbolic groups. Later, a generalization of relative hyperbolicity based on the notion of a hyperbolically embedded subgroup was proposed and a generalization of the results of [GM08, Osi07] was obtained [DGO17]. We refer to [DGO17, Osi18] for the corresponding definitions and results.

1.3 Motivation: a question on cohomology. If a manifold $M$ satisfies the assumptions of Theorem 1.1 and $M_s$ is the Dehn filling corresponding to a slope $s$ on $\partial M$, then Theorem 1.1 can be applied to compute the group cohomology $H^*(\pi_1(M_s);\cdot)$. Indeed, Theorem 1.1 asserts that in most cases, $M_s$ is a hyperbolic 3-manifold and thus the universal cover of $M_s$, namely $\mathbb{H}^3$, is contractible. It follows that $M_s$ is a model of the classifying space of $\pi_1(M_s)$. However, in the more general settings of hyperbolically embedded subgroups, no geometric construction is involved and it is unclear how the group cohomology of the Dehn filling quotient $H^*(G/\langle \langle N \rangle \rangle;\cdot)$ can be computed. Therefore, it is natural to ask the following.

**Question A.** Let $G$ be a group, let $H$ be a hyperbolically embedded subgroup of $G$ (denoted as $H \hookrightarrow_h G$), and let $N$ be a normal subgroup of $H$. What can be said about the group cohomology $H^*(G/\langle \langle N \rangle \rangle;\cdot)$?

In this series of three papers, we answer this question and obtain several applications. The first paper [Sun18] proves a Cohen-Lyndon type theorem for Dehn filling kernels. In the present paper, we derive a spectral sequence for Cohen-Lyndon triples, which can be regarded a refined version of the classical Lyndon-Hochschild-Serre spectral sequence, to compute cohomology of the corresponding Dehn filling quotients. With the aid of this spectral sequence, the third paper [Sun19] will study cohomological properties of Dehn fillings and obtain applications on acylindrically hyperbolic groups.

**Acknowledgement.** I would like to thank my supervisor, Professor Denis Osin, for the valuable discussions. This paper would not have been written without his help. I would also like to thank Professor Anna Marie Bohmann, who gave me many useful suggestions on an early version of this paper.
2 Main results

2.1 Cohen-Lyndon type theorems. Recall that in the first paper [Sun18] of this series, we obtained a free product structure, called the Cohen-Lyndon property, for sufficiently deep Dehn fillings of hyperbolically embedded subgroups.

Definition 2.1. Let $G$ be a group with a subgroup $H$. We say that a property $P$ holds for all sufficiently deep normal subgroups $N \triangleleft H$ if there exists a finite set $F \subset H \setminus \{1\}$ such that $P$ holds for all normal subgroups $N \triangleleft H$ with $N \cap F = \emptyset$.

Definition 2.2. Let $G$ be a group, let $H$ be a subgroup of $G$, and let $N$ be a normal subgroup of $H$. $(G, H, N)$ is called a Cohen-Lyndon triple if there exists a left transversal $T$ of $H \langle \langle N \rangle \rangle$ in $G$ such that $\langle \langle N \rangle \rangle$ is the free product of the groups $tNt^{-1}, t \in T$, denoted as $\langle \langle N \rangle \rangle = \prod_{t \in T} tNt^{-1}$.

Theorem 2.3 ([Sun18, Theorem 2.5]). Suppose that $G$ is a group with a subgroup $H \rightarrowtail G$. Then for all sufficiently deep normal subgroups $N \triangleleft H$, $(G, H, N)$ is a Cohen-Lyndon triple.

Cohen-Lyndon type theorems were first considered by Cohen-Lyndon [CL63], hence the name “Cohen-Lyndon triple”. The result of [CL63] was later generalized by [EH87, Theorem 1.1] and [GMS16, Theorem 4.8].

Remark 2.4. The result of [Sun18] is more general than Theorem 2.3 as it deals with the more general setting of a family of weakly hyperbolically embedded subgroups. Applications of the general result can be found in graphs of groups, e.g., amalgated free products and HNN extensions (see [Sun18, Corollary 6.8]).

2.2 A spectral sequence for Dehn fillings. Given a Cohen-Lyndon triple $(G, H, N)$, we introduce the following notation

$\overline{G} = G/\langle \langle N \rangle \rangle$, $\overline{H} = H/N$.

The main result of this paper is the following.

Theorem 2.5. If $(G, H, N)$ is a Cohen-Lyndon triple, then for every $\mathbb{Z}\overline{G}$-module $A$, there exists a spectral sequence of cohomological type

$$E_2^{p,q} = \begin{cases} H^p(\overline{H}; H^q(N; A)) & \text{if } q \neq 0 \\ H^p(\overline{G}; A) & \text{if } q = 0 \end{cases} \Rightarrow H^{p+q}(G; A). \quad (1)$$

Here, the action of $G$ on $A$ factors through $\overline{G}$. In particular, the action of $N$ on $A$ fixes every element.

Combining Theorems 2.3 and 2.5, we obtain:
Corollary 2.6. Suppose that $G$ is a group with a subgroup $H \hookrightarrow G$. Then for all sufficiently deep normal subgroups $N \triangleleft H$ and every $\mathbb{Z}G$-module $A$, there exists a spectral sequence of cohomological type (1).

Historically, spectral sequences were introduced by Leray [Ler46] in his attempt to compute cohomology of sheaves. In the proof of Theorem 2.5 we make use of the Lyndon-Hochschild-Serre spectral sequence, which was discovered by Lyndon [Lyn48] and then put into its current form by Hochschild-Serre [HS53].

In next paper [Sun19], we will run spectral sequence (1) backwards, compute $H^\ast(\overline{G}; A)$ from information about $H^\ast(G; A)$ and $H^\ast(H; H^\ast(N; A))$, and answer Question A. To enhance our answer, we supplement Theorem 2.5 by relating the differentials of (1) to the differentials of the standard Lyndon-Hochschild-Serre spectral sequence of the extension $1 \to N \to H \to \overline{H} \to 1$ (see Remark 4.3).

2.3 Applications and remarks. Recall that the cohomological dimension of a group $G$ is

$$cd(G) = \sup \{n \in \mathbb{N} \mid H^n(G; A) = \{0\} \text{ for some } \mathbb{Z}G\text{-module } A\}.$$ 

It follows immediately from Theorem 2.5 that if $(G, H, N)$ is a Cohen-Lyndon triple, then $cd(\overline{G}) \leq \max\{cd(G), cd(H) + cd(N) + 1\}$. In the next paper [Sun19], we will obtain a finer estimate of $cd(\overline{G})$. Moreover, we will provide conditions that guarantee $\overline{G}$ to be of type $FP_k$ for $1 \leq k \leq \infty$. As a further application, we will construct useful quotients of acylindrically hyperbolic groups that inherit certain cohomological finiteness properties of the mother groups.

Remark 2.7. In fact, we deal with a general version of Cohen-Lyndon triples which is defined for a family of subgroups and normal subgroups. The corresponding generalized version of Theorem 2.5 turns out to be useful in the next paper [Sun19] when we construct particular quotients of acylindrically hyperbolic groups.

Remark 2.8. If $G$ is a group hyperbolic relative to its subgroup $H$ (in particular, $H$ is a hyperbolically embedded subgroup of $G$ [DGO17, Proposition 2.4]) and both $G$ and $H$ are of type $FP_\infty$, then for sufficiently deep Dehn fillings, [Wan18, Theorem 1.1] provides a spectral sequence of homological type which computes the relative group cohomology $H^\ast(\overline{G}, H; \mathbb{Z}G)$ from certain combination of homology and cohomology. The spectral sequence (1) should not be confused with the spectral sequence of [Wan18], as there is no homology involved in (1). It is worth noting that (1) can also be applied to compute relative cohomology, as we will see in the next paper [Sun19].

2.4 Outline of the proof. The main idea of the proof of Theorem 2.5 is the following. The Lyndon-Hochschild-Serre spectral sequence for a $\mathbb{Z}G$-module $A$ and the group extension

$$1 \to \langle \langle N \rangle \rangle \to G \to \overline{G} \to 1$$

takes the form

$$E_2^{p,q} = H^p(\overline{G}; H^q(\langle \langle N \rangle \rangle; A)) \Rightarrow H^{p+q}(G; A).$$ (2)
Theorem 2.5 follows from a computation of $H^q(\overline{G}; H^q(\langle \langle N \rangle \rangle; A))$, which relies on the proposition below. To state it, we first note that if $(G, H, N)$ is a Cohen-Lyndon triple, then the natural map $\overline{\Pi} \to \overline{G}$ is injective [Sun18, Lemma 6.4], identifying $\overline{\Pi}$ with a subgroup of $\overline{G}$. Therefore, it makes sense to talk about $\text{CoInd}_{\overline{H}}^{\overline{G}}$, the co-induction from $\overline{H}$-modules to $\overline{G}$-modules.

**Proposition 2.9.** If $(G, H, N)$ is a Cohen-Lyndon triple, then for all $q \in \mathbb{Z} \setminus \{0\}$ and every $\overline{G}$-module $A$, there is a $\overline{G}$-module isomorphism

$$H^q(\langle \langle N \rangle \rangle; A) \cong \text{CoInd}_{\overline{H}}^{\overline{G}} H^q(N; A).$$

(3)

Thus, Shapiro’s lemma implies

**Corollary 2.10.** If $(G, H, N)$ is a Cohen-Lyndon triple, then for all $q \in \mathbb{Z} \setminus \{0\}$ and every $\overline{G}$-module $A$, there is an abelian group isomorphism

$$H^*(\overline{G}; H^q(\langle \langle N \rangle \rangle; A)) \cong H^*(\overline{\Pi}; H^q(N; A)).$$

(4)

Notice that for $q = 0$, we have

$$E_2^{0,0} = H^*(\overline{G}; H^0(\langle \langle N \rangle \rangle; A)) \cong H^*(\overline{G}; A^{\langle N \rangle}) \cong H^*(\overline{G}; A),$$

(5)

where $A^{\langle N \rangle}$ is the $\langle \langle N \rangle \rangle$-fixed-points of $A$. As $A$ is a $\overline{G}$-module, the $\langle \langle N \rangle \rangle$-action on $A$ fixes every point and thus $A^{\langle N \rangle} = A$.

The spectral sequence (1) is obtained by substituting terms of (2) for the terms on the right-hand sides of (3) and (4).

A natural way to prove Proposition 2.9 is to decompose $H^q(\langle \langle N \rangle \rangle; A)$ into a direct product $\prod_{t \in T} H^q(tNt^{-1}; A)$, which can be achieved by starting with a model $X$ of the classifying space of $N$ and then taking wedge sum of copies of $X$ to obtain a model of the classifying space of $\langle \langle N \rangle \rangle$. The problem with this approach is that one loses the information about the action $\overline{G} \rtimes H^q(\langle \langle N \rangle \rangle; A)$ and deriving Proposition 2.9 becomes impossible. Therefore, we must take another approach. We first prove that $\text{CoInd}_{\overline{H}}^{\overline{G}} H^*(N; A) \cong \text{Ext}_{\overline{Z}[\langle \langle N \rangle \rangle]}^* (\mathbb{Z}[G/H], A)$ as $\overline{G}$-modules. And then we show that there exist projective resolutions $P \to \mathbb{Z}[G/H]$ and $R \to \mathbb{Z}$ over $\mathbb{Z}[\langle \langle N \rangle \rangle]$ such that $P$ and $R$ coincide at dimension 1 and beyond. It follows that for all $q \neq 0$, we have

$$H^q(\langle \langle N \rangle \rangle; A) \cong \text{Ext}_{\overline{Z}[\langle \langle N \rangle \rangle]}^q (\mathbb{Z}[G/H], A) \cong \text{CoInd}_{\overline{H}}^{\overline{G}} H^q(N; A)$$

as $\overline{G}$-modules.

This paper is organized as follows. We begin by recalling necessary definitions in Section 3, where we define Cohen-Lyndon triples and introduce useful notations. We prove Theorem 2.5 in Section 4, where we first assume Proposition 2.9 and show Corollary 2.10 and Theorem 2.5, and then we prove Proposition 2.9 in Section 4.2.
3 Preliminaries

We begin by introducing conventions and notations. Throughout this paper, all modules (resp. group actions) are left modules (resp. actions). We write $\otimes$ for $\otimes\mathbb{Z}$, the tensor product over $\mathbb{Z}$. Given a group $G$, we write $\text{Hom}_G, \text{Ext}_G$ for $\text{Hom}_{\mathbb{Z}G}, \text{Ext}_{\mathbb{Z}G}$, respectively. If $G$ is the free product of its subgroups $G_\lambda, \lambda \in \Lambda$, then we write $G = \prod_{\lambda \in \Lambda} G_\lambda$. If $S$ is a subset of $G$, then $\langle \langle S \rangle \rangle$ is the normal closure of $S$ in $G$. If $H$ is a subgroup of $G$, then $\text{LT}(H, G)$ is the set of left transversals of $H$ in $G$, and we write $\text{CoInd}_H^G$ for the co-induction from $\mathbb{Z}H$-modules to $\mathbb{Z}G$-modules, i.e.,

$$\text{CoInd}_H^G A = \text{Hom}_H(\mathbb{Z}G, A)$$

for all $\mathbb{Z}H$-module $A$.

We briefly recall several actions and refer to [Bro94] for details. For every $\mathbb{Z}H$-module $A$, there is a $G$-action on $\text{CoInd}_H^G A$ given by

$$(g \circ f)(x) = f(x \cdot g) \text{ for all } g \in G, x \in \mathbb{Z}G, f \in \text{CoInd}_H^G A.$$  

Suppose that $H$ is a normal subgroup of $G$ and $B, C$ are $\mathbb{Z}G$-modules. Then there is a $G/H$-action on $\text{Hom}_H(B, C)$ induced by the following $G$-action

$$(g \circ f)(x) = g \cdot f(g^{-1} \cdot x) \text{ for all } g \in G, x \in B, f \in \text{Hom}_H(B, C).  \quad (6)$$

Let $P \to B$ be a projective resolution over $\mathbb{Z}G$. Then formula (6) with $P$ in place of $B$ induces a $G/H$-action on $\text{Ext}_H^*(B, C)$. In case $B = \mathbb{Z}$, (6) gives rise to an action of $G/H$ on $H^*(H; C)$.

3.1 Spectral sequences

Our main result involves spectral sequences. The reader is referred to [Rot09, Wei94] for an exposition of this algebraic object. In this section, we only clarify notations and terminologies.

Throughout this paper, all spectral sequences are first quadrant spectral sequences of cohomological type. We denote such a spectral sequence as $E = (E_r, d_r)_{r \geq a}$, where $E_r$ is the $E_r$-page of $E$ and $d_r$ is the differential on $E_r$. The notation $E_{a,q}^p \Rightarrow H^{p+q}$ indicates that $E$ converges to a graded abelian group $H = \bigoplus_{\ell \geq 0} H^\ell$.

A morphism between spectral sequences $E_1 = (E_{1,r}, d_{1,r})_{r \geq a}, E_2 = (E_{2,r}, d_{2,r})_{r \geq a}$ is denoted as

$$\phi = (\phi_r)_{r \geq b} : E_1 \to E_2,$$

where $\phi_r : E_{1,r} \to E_{2,r}$ is the restriction of $\phi$ and it is understood that $b \geq a$ and $\phi_r$ is only defined for $r \geq b$. For $p, q \in \mathbb{Z}$, we denote by

$$\phi_{r}^{p,q} : E_{1,r}^{p,q} \to E_{2,r}^{p,q}, \quad \phi_{r}^{p,*} : E_{1,r}^{p,*} \to E_{2,r}^{p,*}, \quad \phi_{r}^{*,q} : E_{1,r}^{*,q} \to E_{2,r}^{*,q}$$

the maps induced by $\phi_r$.

Also recall the notion of a Lyndon-Hochschild-Serre spectral sequence.
Definition 3.1. Let $G$ be a group, let $K$ be a normal subgroup of $G$, and let $A$ be a $\mathbb{Z}G$-module. The Lyndon-Hochschild-Serre (LHS) spectral sequence associated with the data $(G, K, A)$ is a spectral sequence $E = (E_r, d_r)_{r \geq 1}$ such that

$$E_2^{p,q} = H^p(G/K; H^q(K; A)) \Rightarrow H^{p+q}(G; A).$$

3.2 Cohen-Lyndon triples

Suppose that $G$ is a group with a family of subgroups $\{H_\lambda\}_{\lambda \in \Lambda}$ and for every $\lambda \in \Lambda$, $N_\lambda$ is a normal subgroup of $H_\lambda$. For future reference, we specify the following notations.

Notation 3.2. Let

$$\mathcal{N} = \bigcup_{\lambda \in \Lambda} N_\lambda, \quad \overline{G} = G/\langle \langle \mathcal{N} \rangle \rangle, \quad \overline{\Pi}_\lambda = H_\lambda/N_\lambda.$$ 

If $A$ is a $\mathbb{Z}\overline{G}$-module, then there is a natural action of $G$ on $A$ which factors through $\overline{G}$. In particular, the action of $N_\lambda$ on $A$ is trivial (i.e., fixes every element of $A$). We will compute the cohomology groups $H^*(G; A), H^*(H_\lambda; A)$, etc using this natural action.

For $\lambda \in \Lambda$, let

$$r_{H_\lambda} : H^*(G; A) \to H^*(H_\lambda; A), \quad r_{N_\lambda} : H^*(\langle \langle \mathcal{N} \rangle \rangle; A) \to H^*(N_\lambda; A)$$

be the restriction maps induced by the inclusions $H_\lambda \leq G$ and $N_\lambda \leq \langle \langle \mathcal{N} \rangle \rangle$, respectively. Let

$$r_{G} : H^*(G; A) \to \prod_{\lambda \in \Lambda} H^*(H_\lambda; A)$$

be the map induced by the maps $r_{H_\lambda}$. For $\lambda \in \Lambda$, the map $r_{N_\lambda}$ and the natural homomorphism $\overline{\Pi}_\lambda \to \overline{G}$ induce a cohomology map

$$\psi_\lambda : H^*(\overline{G}; H^*(\langle \langle \mathcal{N} \rangle \rangle; A)) \to H^*(\overline{\Pi}_\lambda; H^*(N_\lambda; A)),$$

where the $\overline{G}$ (resp. $\overline{\Pi}_\lambda$) cohomology is computed using the $\overline{G}$-action (resp. $\overline{\Pi}_\lambda$-action) on $H^*(\langle \langle \mathcal{N} \rangle \rangle; A)$ (resp. $H^*(N_\lambda; A)$) defined at the beginning of Section 3.

Let

$$\psi : H^*(\overline{G}; H^*(\langle \langle \mathcal{N} \rangle \rangle; A)) \to \prod_{\lambda \in \Lambda} H^*(\overline{\Pi}_\lambda; H^*(N_\lambda; A))$$

be the map induced by the maps $\psi_\lambda$.

Definition 3.3. We call $(G, \{H_\lambda\}_{\lambda \in \Lambda}, \{N_\lambda\}_{\lambda \in \Lambda})$ a Cohen-Lyndon triple if there exists a left transversal $T_\lambda \in LT(H_\lambda\langle \langle \mathcal{N} \rangle \rangle, G)$ for every $\lambda \in \Lambda$ such that

$$\langle \langle \mathcal{N} \rangle \rangle = \prod_{\lambda \in \Lambda, t \in T_\lambda} tN_\lambda t^{-1}.$$
Remark 3.4. If \((G, \{H_{\lambda}\}_{\lambda \in \Lambda}, \{N_{\lambda}\}_{\lambda \in \Lambda})\) is a Cohen-Lyndon triple and \(N_{\lambda} \neq \{1\}\) for every 
\(\lambda \in \Lambda\), then for all \(\lambda \in \Lambda\), we have \(H_{\lambda} \cap \langle N \rangle = N_{\lambda}\) \cite{Sun18} Lemma 6.4] and thus the natural homomorphism \(\overline{\pi}_{\lambda} : \overline{G} \to \overrightarrow{G}\) is injective, identifying \(\overline{\pi}_{\lambda}\) with a subgroup of \(\overrightarrow{G}\).

Sufficiently deep Dehn fillings of weakly hyperbolically embedded subgroups are studied in \cite{Sun18}, which proves the following.

**Theorem 3.5** \cite{Sun18 Theorem 5.1}. If the family \(\{H_{\lambda}\}_{\lambda \in \Lambda}\) weakly hyperbolically embeds into \((G, X)\) for some \(X \subset G\). Then for all sufficiently deep normal subgroups \(N_{\lambda} \triangleleft H_{\lambda}, (G, \{H_{\lambda}\}_{\lambda \in \Lambda}, \{N_{\lambda}\}_{\lambda \in \Lambda})\) is a Cohen-Lyndon triple.

### 3.3 A corollary to Shapiro’s lemma

As indicated in Section 2, we will prove Theorem 2.5 in the more general setting of a family of subgroups. It is therefore necessary to slightly generalize Shapiro’s lemma to a family of subgroups. However, we would rather call the following generalization a corollary as it actually follows directly from Shapiro’s lemma.

Let \(G\) be a group, let \(\{H_{\lambda}\}_{\lambda \in \Lambda}\) be a family of subgroups of \(G\), and let \(A_{\lambda}\) be a \(\mathbb{Z}H_{\lambda}\)-module for \(\lambda \in \Lambda\). There is a map

\[
\pi_{\lambda} : \text{CoInd}_{H_{\lambda}}^{G} A_{\lambda} \to A_{\lambda}, \quad \pi_{\lambda}(f) = f(1).
\]

The well-known Shapiro’s lemma asserts that for every \(\lambda \in \Lambda\), the inclusion \(H_{\lambda} \leq G\) and the map \(\pi_{\lambda}\) induce an isomorphism

\[
\phi_{\lambda} : H^{\ast}(G; \text{CoInd}_{H_{\lambda}}^{G} A_{\lambda}) \to H^{\ast}(H_{\lambda}; A_{\lambda}).
\]

Let

\[
\phi : \prod_{\lambda \in \Lambda} H^{\ast}(G; \text{CoInd}_{H_{\lambda}}^{G} A_{\lambda}) \to \prod_{\lambda \in \Lambda} H^{\ast}(H_{\lambda}; A_{\lambda})
\]

be the map on products induced by the maps \(\phi_{\lambda}\). Then \(\phi\) is an isomorphism. As \(\prod_{\lambda \in \Lambda} H^{\ast}(G; \text{CoInd}_{H_{\lambda}}^{G} A_{\lambda})\) is naturally isomorphic to \(H^{\ast}(G; \prod_{\lambda \in \Lambda} \text{CoInd}_{H_{\lambda}}^{G} A_{\lambda})\), we can view \(\phi\) as a map from \(H^{\ast}(G; \prod_{\lambda \in \Lambda} \text{CoInd}_{H_{\lambda}}^{G} A_{\lambda})\) to \(\prod_{\lambda \in \Lambda} H^{\ast}(H_{\lambda}; A_{\lambda})\) and we have the following.

**Lemma 3.6.** Let \(G\) be a group, let \(\{H_{\lambda}\}_{\lambda \in \Lambda}\) be a family of subgroups of \(G\), and let \(A_{\lambda}\) be a \(\mathbb{Z}H_{\lambda}\)-module for \(\lambda \in \Lambda\). Then the map \(\phi\) defined above is an isomorphism.

### 4 Proof of the main result

In the sequel, we will study cohomology of Cohen-Lyndon triples. For simplicity, we will be frequently using notations defined in Notation 3.2. Our goal is the following more general and precise version of Theorem 2.5.
Theorem 4.1. Let \((G, \{H_\lambda\}_{\lambda \in \Lambda}, \{N_\lambda\}_{\lambda \in \Lambda})\) be a Cohen-Lyndon triple, let \(A\) be a \(ZG\)-module, let \(E_G = (E_{G,r}, d_{G,r})_{r \geq 1}\) (resp. \(E_H = (E_{H,r}, d_{H,r})_{r \geq 1}\)) be the LHS spectral sequence associated with the data \((G, \langle \langle N \rangle \rangle, A)\) (resp. \((H_\lambda, N_\lambda, A)\)), and let \(E_H = (E_{H,r}, d_{H,r})_{r \geq 1}\) be the product of the spectral sequences \(E_{H_\lambda}\).

Then there exists a morphism of spectral sequences \(\phi = (\phi_r)_{r \geq 2} : E_G \to E_H\) which satisfies the following.

(a) The maps \(\phi \) and \(r_G\) are compatible.

(b) The map \(\phi_2\) can be identified with the map \(\psi\). Moreover, \(\phi_2\) induces an isomorphism \(\phi^*_q : E^*_{G,2} \to E^*_{H,2}\) for every \(q \neq 0\).

Combining Theorems 3.5 and 4.1, we obtain:

Corollary 4.2. Let \(G\) be a group with a family of subgroups \(\{H_\lambda\}_{\lambda \in \Lambda}\) weakly hyperbolically embedded into \((G, X)\) for some \(X \subset G\). Then for sufficiently deep normal subgroups \(N_\lambda \triangleleft H_\lambda\) and every \(ZG\)-module \(A\), there exists a morphism of spectral sequences \(\phi : E_G \to E_H\) which satisfies (a) and (b) of Theorem 4.1.

Assuming Theorem 4.1, we prove Theorem 2.5.

Proof of Theorem 2.5. Let \(E_G\) (resp. \(E_H\)) be the LHS spectral sequence associated with the data \((G, \langle \langle N \rangle \rangle, A)\) (resp. \((H, N, A)\)). Then

\[ E^p,q_{G,2} = H^p(G; H^q(\langle \langle N \rangle \rangle; A)) \Rightarrow H^{p+q}(G; A). \]

By Theorem 4.1 there exists a morphism \(\phi : E_G \to E_H\) such that \(\phi^*_q : E^*_{G,2} \to E^*_{H,2}\) is an isomorphism for all \(q \in \mathbb{Z} \setminus \{0\}\). Replace \(E^*_{G,2}\) with \(E^*_{H,2} = H^*(\overline{G}; H^q(N; A))\) for all \(q \in \mathbb{Z} \setminus \{0\}\). For \(q = 0\), equation (5) tells us that we can replace \(E^*_{G,2}\) with \(H^*(\overline{G}; A)\). After these replacements, we obtain the spectral sequence (1).

Remark 4.3. We can describe the differentials of (1) as follows. Let \(d_r\) (resp. \(d_{H,r}\)) be the differential on the \(E_r\)-page of the spectral sequence (1) (resp. the LHS spectral sequence \(E_H\)). Then \(d_r\) is induced by \(d_{H,r}\). More precisely, as (1) results from replacing certain terms of \(E_G\), we think of \(\phi : E_G \to E_H\) as a morphism from the spectral sequence (1) to \(E_H\) and we have a commutative diagram

\[
\begin{array}{ccc}
E_r & \xrightarrow{\phi} & E_{H,r} \\
\downarrow d_r & & \downarrow d_{H,r} \\
E_r & \xrightarrow{\phi} & E_{H,r}
\end{array}
\]

The key to the proof of Theorem 4.1 is the following generalization of Proposition 2.9.
Proposition 4.4. Suppose that \((G, \{H_\lambda\}_{\lambda \in \Lambda}, \{N_\lambda\}_{\lambda \in \Lambda})\) is a Cohen-Lyndon triple and \(N_\lambda \neq \{1\}\) for all \(\lambda \in \Lambda\). As Remark 3.4, think of the groups \(H_\lambda\) as subgroups of \(G\). Then for every \(\mathbb{Z}G\)-module \(A\), there exists a \(\mathbb{Z}G\)-module homomorphism (where the \(G\)-actions are defined at the beginning of Section 3)

\[
\eta : H^*(\langle N \rangle; A) \longrightarrow \prod_{\lambda \in \Lambda} \text{CoInd}_{H_\lambda}^G H^*(N_\lambda; A)
\]

satisfying the following.

(a) For all \(q \geq 1\), \(\eta\) maps \(H^q(\langle N \rangle; A)\) isomorphically onto \(\prod_{\lambda \in \Lambda} \text{CoInd}_{H_\lambda}^G H^q(N_\lambda; A)\).

(b) Let

\[
p_\mu : \prod_{\lambda \in \Lambda} \text{CoInd}_{H_\lambda}^G H^*(N_\lambda; A) \rightarrow \text{CoInd}_{H_\mu}^G H^*(N_\mu; A)
\]

be the coordinate projection, and define a map

\[
\pi_\mu : \text{CoInd}_{H_\mu}^G H^*(N_\mu; A) \rightarrow H^*(N_\mu; A) \quad \text{by} \quad \pi_\mu(f) = f(1).
\]

Then \(r_{N_\mu} = \pi_\mu \circ p_\mu \circ \eta\).

The proof of Proposition 4.4 is the content of Section 4.2. For the moment, we assume Proposition 4.4 and prove Theorem 4.1. We first show the following corollary to Proposition 4.4. Recall that Notation 3.2 defines a map \(\psi\).

Corollary 4.5. Let \((G, \{H_\lambda\}_{\lambda \in \Lambda}, \{N_\lambda\}_{\lambda \in \Lambda})\) be a Cohen-Lyndon triple. Then for all \(q \in \mathbb{Z} \setminus \{0\}\) and every \(\mathbb{Z}G\)-module \(A\), \(\psi\) maps \(H^q(G; H^q(\langle N \rangle; A))\) isomorphically onto \(\prod_{\lambda \in \Lambda} H^*(H_\lambda; H^q(N_\lambda; A))\).

Proof. We first deal with the special case where \(N_\lambda \neq \{1\}\) for all \(\lambda \in \Lambda\). In this case, Corollary 4.5 follows from Proposition 4.4 and Lemma 3.6.

Now we reduce the general case to the special case. Let

\[
\Lambda' = \{\lambda \in \Lambda \mid N_\lambda \neq \{1\}\}.
\]

Note that \((G, \{H_\lambda\}_{\lambda \in \Lambda'}, \{N_\lambda\}_{\lambda \in \Lambda'})\) is again a Cohen-Lyndon triple and, by the previous proved special case, Corollary 4.5 holds for \((G, \{H_\lambda\}_{\lambda \in \Lambda'}, \{N_\lambda\}_{\lambda \in \Lambda'})\), which implies that Corollary 4.5 also holds for \((G, \{H_\lambda\}_{\lambda \in \Lambda}, \{N_\lambda\}_{\lambda \in \Lambda})\). \(\square\)

Lemma 4.6. Suppose that \(G\) is a group, \(H\) is a subgroup of \(G\), \(K\) is a normal subgroup of \(G\), \(N = K \cap H\), and \(A\) is a \(\mathbb{Z}[G/K]\)-module. Let \(E_{G,r} = (E_{G,r}, d_{G,r})_{r \geq 1}\) (resp. \(E_H = (E_{H,r}, d_{H,r})_{r \geq 1}\)) be the LHS spectral sequence associated with the data \((G, K, A)\) (resp. \((H, N, A)\)). Then there exists a morphism of spectral sequences

\[
\phi = (\phi_r)_{r \geq 1} : E_G \rightarrow E_H
\]

which satisfies the following.
(a) \( \phi \) is compatible with \( r : \text{H}^*(G; A) \to \text{H}^*(H; A) \), the restriction map induced by the inclusion \( H \leq G \).

(b) Let

\[
\psi : E_{G,2} = \text{H}^*(G/K; \text{H}^*(K; A)) \to E_{H,2} = \text{H}^*(H/N; \text{H}^*(N; A))
\]

be the cohomology map induced by the inclusion \( N \leq K \) and the natural injection \( H/N \hookrightarrow G/K \). Then \( \psi \) can be identified with \( \phi_2 \).

Sketch of the proof. The lemma is well-known and its proof follows from standard diagram tracing, so we only sketch the proof. The LHS spectral sequence \( E_G \) is constructed as follows (for details, see [Rot09 Theorem 11.38]). Start with an injective resolution \( A \to I \) of \( A \) over the ring \( ZG \). Applying \( \text{Hom}_K \) to this resolution produces a cochain complex \( C_G \). And then one takes an injective Cartan-Eilenberg resolution \( C_G \to J_G \) over the ring \( Z[G/K] \). By applying \( \text{Hom}_{G/K} \) to \( J_G \), one gets a double complex \( D_G \). \( E_G \) arises from the row filtration of \( D_G \). The column filtration of \( D_G \) also induces a spectral sequence \( F_G = (F_{G,r}, \delta_{G,r})_{r \geq 1} \) such that \( F_{G,2} \) can be identified with \( \text{H}^*(G; A) \), telling us where \( E_G \) converges to.

The construction for \( E_H \) is similar. We need to start with an injective resolution of \( A \) over \( ZH \), but as \( H \leq G \), every injective \( ZG \)-module is automatically an injective \( ZH \)-module and thus we can view \( A \to I \) as an injective resolution over \( ZH \). Applying \( \text{Hom}_N \) to this resolution produces a cochain complex \( C_H \). And then there is an injective Cartan-Eilenberg resolution \( C_H \to J_H \) over \( Z[H/N] \). As \( H/N \) naturally embeds into \( G/K \), every injective \( Z[G/K] \)-module is automatically an injective \( Z[H/N] \)-module and thus \( C_G \to J_G \) can be regarded as a Cartan-Eilenberg resolution over \( Z[H/N] \). The obvious inclusion \( C_G \to C_H \) then induces a map \( J_G \to J_H \) [Wei94, Exercise 5.7.2], which further induces a map \( \phi_D : D_G \to D_H \), where \( D_H \) is the double complex resulting from applying \( \text{Hom}_{H/N} \) to \( J_H \). Once again, \( E_H \) arises from the row filtration of \( D_H \) and we let \( F_H = (F_{H,r}, \delta_{H,r})_{r \geq 1} \) be the spectral sequence arising from the column filtration of \( D_H \).

\( \phi_D \) clearly respects the column and row filtrations of \( D_G \) and \( D_H \) and thus induces morphisms of spectral sequences

\[
\phi = (\phi_r)_{r \geq 1} : E_G \to E_H, \quad \phi_F = (\phi_{F,r})_{r \geq 1} : F_G \to F_H.
\]

The reason why \( E_{G,2} \) (resp. \( E_{H,2}, F_{G,2}, F_{H,2} \)) can be identified with \( \text{H}^*(G/K; \text{H}^*(K; A)) \) (resp. \( \text{H}^*(H/N; \text{H}^*(N; A)), \text{H}^*(G; A), \text{H}^*(H; A) \)) is that \( E_{G,1} \) (resp. \( E_{H,1}, F_{G,1}, F_{H,1} \)) is a cochain complex \( \text{Hom}_{G/K}(Z, I_K) \) (resp. \( \text{Hom}_{H/N}(Z, I_N), \text{Hom}_G(Z, I), \text{Hom}_H(Z, I) \)). Here, \( I_K \) (resp. \( I_N \)) is an injective resolution of \( \text{H}^*(K; A) \) (resp. \( \text{H}^*(N; A) \)) over \( Z[G/K] \) (resp. \( Z[H/N] \)). Moreover, \( \phi_1 \) (resp. \( \phi_{F,1} \)) is a map from \( \text{Hom}_{G/K}(Z, I_K) \) (resp. \( \text{Hom}_G(Z, I) \)) to \( \text{Hom}_{H/N}(Z, I_N) \) (resp. \( \text{Hom}_H(Z, I) \)), which can be seen, via a diagram tracing, to induce the cohomology map \( \psi \) (resp. \( r \)), from which statement (b) (resp. (a)) follows.

Proof Theorem 4.4. Under the assumptions of Theorem 4.4, let us first construct, for every \( \lambda \in \Lambda \), a morphism \( \phi_\lambda = (\phi_{\lambda,r})_{r \geq 2} : E_G \to E_{H\lambda} \) of spectral sequences.

Let

\[
\Lambda' = \{ \lambda \in \Lambda \mid N_\lambda \neq \{1\} \}.
\]
Note that \((G, \{H_\lambda\}_{\lambda \in \Lambda'}, \{N_\lambda\}_{\lambda \in \Lambda'})\) is a Cohen-Lyndon triple. By Remark 3.4, for \(\lambda \in \Lambda'\), we have \(H_\lambda \cap \langle \langle N \rangle \rangle = N_\lambda\). Apply Lemma 4.6 and let
\[
\phi_\lambda : E_G \to E_{H_\lambda}
\]
be the morphism given by that lemma.

Let \(\lambda \in \Lambda \setminus \Lambda'\). Then Lemma 4.6 is not applicable, but we can construct a morphism by hand. For \(r > 2\), as \(H^0(\{1\}; A)\) is naturally isomorphic to \(A\), we have
\[
E^{p,q}_{H_\lambda,r} = \begin{cases} 
H^p(H_\lambda; A) & \text{if } q = 0 \\
0 & \text{if } q \neq 0
\end{cases}
\]

We define a family of maps \(\phi^{*,q}_{\lambda,r} : E^{*,q}_{G,r} \to E^{*,q}_{H_\lambda,r}\), \(q \in \mathbb{Z}\), by the following.

1. \(\phi^{*,q}_{\lambda,r} : E^{*,q}_{G,r} \to E^{*,q}_{H_\lambda,r}\) is the zero map for all \(q \in \mathbb{Z} \setminus \{0\}\).

2. For \(p \in \mathbb{Z}\), let \(R > r\) be sufficiently large such that \(E^{p,0}_{G,R}\) naturally embeds into \(H^p(G; A)\) (such an \(R\) exists as \(E^{k,\ell}_{G,2} \Rightarrow H^{k+\ell}(G; A)\)). By the definition of spectral sequences, there is a quotient map \(E^{p,0}_{G,r} \to E^{p,0}_{G,R}\). Let \(\phi^{p,0}_{\lambda,r}\) be the composition
\[
E^{p,0}_{G,r} \to E^{p,0}_{G,R} \to H^p(G; A) \overset{r_{H_\lambda}}{\longrightarrow} H^p(H_\lambda; A) = E^{p,0}_{H_\lambda,r}
\]
(the definition of \(\phi^{p,0}_{\lambda,r}\) does not depend on the choice of \(R\)).

The maps \(\phi^{*,q}_{\lambda,r}, q \in \mathbb{Z}, r \geq 2\), form a morphism \(\phi_\lambda : E_G \to E_{H_\lambda}\) between spectral sequences.

Claim. For every \(\lambda \in \Lambda\),

(i) \(\phi_\lambda\) is compatible with the map \(r_{H_\lambda}\);

(ii) \(\phi_{\lambda,2}\) can be identified with the map \(\psi_\lambda\).

Proof of the claim. If \(\lambda \in \Lambda'\), then statements (i) and (ii) follow from Lemma 4.6. If \(\lambda \in \Lambda \setminus \Lambda'\), then (i) and (ii) follow directly from the construction of \(\phi_\lambda\). \(\square\)

It follows from the claim that \(\phi\) is compatible with \(r_G\) and \(\phi_2\) can be identified with \(\psi\). For all \(q \in \mathbb{Z} \setminus \{0\}\), Corollary 4.5 implies that \(\psi\) maps \(H^*(\overline{G}; H^q(\langle \langle N \rangle \rangle; A))\) isomorphically onto \(\prod_{\lambda \in \Lambda} H^*(\overline{H_\lambda}; H^q(N_\lambda; A))\) and thus \(\phi^{*,q}_{2}\) is an isomorphism for all \(q \in \mathbb{Z} \setminus \{0\}\). \(\square\)

To finish this paper, it remains to prove Proposition 4.4.
4.1 Ext-groups of coset rings

Our approach to Proposition 4.4 goes as follows. Under the assumptions of Proposition 4.4, we show that for every \( q \geq 1 \), there is a sequence of \( \mathbb{ZG} \)-module isomorphisms

\[
\text{H}^q(\langle \langle N \rangle \rangle; A) \cong \prod_{\lambda \in \Lambda} \text{Ext}^q_{\langle \langle N \rangle \rangle} (\mathbb{Z}[G/H_\lambda], A) \cong \prod_{\lambda \in \Lambda} \text{CoInd}_{H_\lambda}^{G} \text{H}^q(N_\lambda; A)
\]

(7)

whose composition is the map \( \eta \) claimed by Proposition 4.4.

Let us start with the second isomorphism of (7). It suffices to prove

\[
\text{Ext}^*_{\langle \langle N \rangle \rangle} (\mathbb{Z}[G/H_\lambda], A) \cong \text{CoInd}_{H_\lambda}^{G} \text{H}^*(N_\lambda; A)
\]

(8)
as \( \mathbb{ZG} \)-modules.

Fix \( \lambda \). Let \( A \to I \) be an injective resolution over \( \mathbb{Z} \langle \langle N \rangle \rangle \). Then \( \text{Ext}^*_ {\langle \langle N \rangle \rangle} (\mathbb{Z}[G/H_\lambda], A) \) is the cohomology group of the cochain complex

\[
\text{Hom}_{\langle \langle N \rangle \rangle} (\mathbb{Z}[G/H_\lambda], I).
\]

(9)

As \( N_\lambda \leq \langle \langle N \rangle \rangle \), every injective \( \mathbb{Z} \langle \langle N \rangle \rangle \)-module is automatically an injective \( \mathbb{ZN}_\lambda \)-module and thus \( A \to I \) is also an injective resolution over \( \mathbb{ZN}_\lambda \). It is easy to see that \( \text{CoInd}_{H_\lambda}^{G} \text{H}^*(N_\lambda; A) \) is naturally isomorphic to the cohomology group of the cochain complex

\[
\text{CoInd}_{H_\lambda}^{G} \text{Hom}_{N_\lambda}(\mathbb{Z}, I),
\]

(10)

whose differential is induced by the differential of \( I \). Indeed, given a function \( f \in \text{CoInd}_{H_\lambda}^{G} \text{H}^*(N_\lambda; A) \), lift \( f \) to a function \( \tilde{f} \in \text{CoInd}_{H_\lambda}^{G} \text{Hom}_{N_\lambda}(\mathbb{Z}, I) \). It is easy to check that \( \tilde{f} \) represents an element \( [\tilde{f}] \) of the cohomology group of (10). The correspondence \( f \leftrightarrow [\tilde{f}] \) then provides the desired isomorphism.

Now we construct a chain map from (9) to (10). Suppose that a function \( f \in \text{Hom}_{\langle \langle N \rangle \rangle} (\mathbb{Z}[G/H_\lambda], I) \) is given. Define a function \( \tilde{f} \in \text{CoInd}_{H_\lambda}^{G} \text{Hom}_{N_\lambda}(\mathbb{Z}, I) \) by the following rule. Let \( T_\lambda \) be a left transversal in \( LT(H_\lambda \langle \langle N \rangle \rangle, G) \). Denote the image of elements \( t \in T_\lambda \) under the quotient map \( G \to G \) by \( \overline{t} \) and denote the image of \( \overline{t} \) under \( \overline{t} \) by \( \overline{t}^{-1} \). Then \( \overline{t}^{-1} \) is a function from \( \mathbb{Z} \) to \( I \), and we demand that \( \overline{t}^{-1}(1) = t^{-1} f(t H_\lambda) \). This uniquely determines the function \( \overline{f} \).

Let

\[
\phi_\lambda : \text{Hom}_{\langle \langle N \rangle \rangle} (\mathbb{Z}[G/H_\lambda], I) \to \text{CoInd}_{H_\lambda}^{G} \text{Hom}_{N_\lambda}(\mathbb{Z}, I)
\]

be the function sending every \( f \in \text{Hom}_{\langle \langle N \rangle \rangle} (\mathbb{Z}[G/H_\lambda], I) \) to the corresponding \( \overline{f} \). Direct computation shows that \( \phi_\lambda \) is a chain map and an isomorphism of abelian groups. Direct computation also shows that \( \phi_\lambda \) is \( \overline{G} \)-equivariant. It follows that \( \phi_\lambda \) is also a \( \mathbb{ZG} \)-module isomorphism and thus is a chain isomorphism. We conclude with:

**Lemma 4.7.** The map \( \phi_\lambda \) defined above induces a \( \mathbb{ZG} \)-module isomorphism (8).
4.2 Proof of Proposition 4.4

Consider the ring $\bigoplus_{\lambda \in \Lambda} \mathbb{Z}[G/H_{\lambda}]$. There is an augmentation homomorphism

$$\epsilon : \bigoplus_{\lambda \in \Lambda} \mathbb{Z}[G/H_{\lambda}] \to \mathbb{Z}, \quad \epsilon(gH_{\lambda}) = 1 \text{ for all } g \in G \text{ and } \lambda \in \Lambda.$$

$\epsilon$ induces a $\mathbb{Z}[G]$-module homomorphism of Ext-groups

$$\epsilon^* : H^*(\langle N \rangle; A) \to \text{Ext}^*_{\langle N \rangle}(\bigoplus_{\lambda \in \Lambda} \mathbb{Z}[G/H_{\lambda}], A).$$

As $\text{Ext}^*_\langle N \rangle(\bigoplus_{\lambda \in \Lambda} \mathbb{Z}[G/H_{\lambda}], A)$ is naturally isomorphic to $\prod_{\lambda \in \Lambda} \text{Ext}^*_{\langle N \rangle}(\mathbb{Z}[G/H_{\lambda}], A)$, $\epsilon^*$ can be regarded as a map from $\text{Hom}^*(\langle N \rangle; A)$ to $\prod_{\lambda \in \Lambda} \text{Ext}^*_{\langle N \rangle}(\mathbb{Z}[G/H_{\lambda}], A)$.

For every $\lambda \in \Lambda$, the map $\phi_\lambda$ constructed in Section 4.1 induces an isomorphism (still denoted by)

$$\phi_\lambda : \text{Ext}^*_\langle N \rangle(\mathbb{Z}[G/H_{\lambda}], A) \to \text{CoInd}_{G/H_{\lambda}}^G H^*(N_{\lambda}; A).$$

Let

$$\phi : \prod_{\lambda \in \Lambda} \text{Ext}^*_\langle N \rangle(\mathbb{Z}[G/H_{\lambda}], A) \to \prod_{\lambda \in \Lambda} \text{CoInd}_{G/H_{\lambda}}^G H^*(N_{\lambda}; A)$$

be the map on products induced by the maps $\phi_\lambda$.

Let $\eta = \phi \circ \epsilon^*$. Then diagram tracing shows $r_{N_\mu} = \pi_\mu \circ p_\mu \circ \eta$ for all $\mu \in \Lambda$. To finish the proof of Proposition 4.4 it suffices to show

**Lemma 4.8.** $\phi$ maps $H^q(\langle N \rangle; A)$ isomorphically onto $\text{Ext}^q_{\langle N \rangle}(\bigoplus_{\lambda \in \Lambda} \mathbb{Z}[G/H_{\lambda}], A)$ for all $q \geq 1$.

We approach Lemma 4.8 via specific free resolutions. We construct a free resolution $P \to \bigoplus_{\lambda \in \Lambda} \mathbb{Z}[G/H_{\lambda}]$ over $\mathbb{Z}[G]$ and show that $P$ is almost a free resolution of $\mathbb{Z}$ over $\mathbb{Z}[[N]]$. Formally, we construct a free resolution $R \to \mathbb{Z}$ over $\mathbb{Z}[[N]]$ and prove that $P$ and $R$ coincide at dimension 1 and beyond, which will imply Lemma 4.8.

As $(G, \{H_{\lambda}\}_{\lambda \in \Lambda}, \{N_{\lambda}\}_{\lambda \in \Lambda})$ is a Cohen-Lyndon triple, there exist transversals $T_\lambda \in LT(H_{\lambda}[N], G)$ such that

$$\langle N \rangle = \prod_{\lambda \in \Lambda, \tau \in T_\lambda}^* t_{N_{\lambda}} t_{-1}.$$

(11)

The resolution $P \to \bigoplus_{\lambda \in \Lambda} \mathbb{Z}[G/H_{\lambda}]$ results from picking one free resolution $P_{\lambda} \to \mathbb{Z}[G/H_{\lambda}]$ for each $\lambda \in \Lambda$ and then taking direct sum of these resolutions. Formally, let $F \to \mathbb{Z}$ be the bar resolution of $G$. For $\lambda \in \Lambda$, consider the abelian group tensor product $F \otimes \mathbb{Z}[G/H_{\lambda}]$. The actions of $G$ on $F$ and $F \otimes \mathbb{Z}[G/H_{\lambda}]$ are diagonal:

$$g \cdot (g_0, \ldots, g_n) = (gg_0, \ldots, gg_n), \quad g \cdot ((g_0, \ldots, g_n) \otimes g_{n+1}H_{\lambda}) = (gg_0, \ldots, gg_n) \otimes gg_{n+1}H_{\lambda}.$$
Let $P_\lambda$ be the $\mathbb{Z}\langle\langle N \rangle\rangle$-submodule of $F \otimes \mathbb{Z}[G/H_\lambda]$ generated by elements of the form $p \otimes tH_\lambda$, where $t$ ranges over elements of $T_\lambda$ and $p$ ranges over all tuples of elements of the set $tN_\lambda$.

As a $\mathbb{Z}G$-module, $F$ is freely generated by the set

$$\{(1, g_1, \ldots, g_q) \mid q \geq 0, g_1, \ldots, g_q \in G\}.$$

Therefore, $F \otimes \mathbb{Z}[G/H_\lambda]$, as a $\mathbb{Z}G$-module, is freely generated by the set

$$\{(1, g_1, \ldots, g_q) \otimes gH_\lambda \mid q \geq 0, g_1, \ldots, g_q \in G, gH_\lambda \text{ ranges over left cosets of } H_\lambda\}.$$

Thus,

**Lemma 4.9.** $P_\lambda$, as a $\mathbb{Z}\langle\langle N \rangle\rangle$-module, is freely generated by the set

$$\{(t, tn_1, \ldots, tn_q) \otimes tH_\lambda \mid q \geq 0, n_1, \ldots, n_q \in N_\lambda, t \in T_\lambda\}.$$

The boundary operator of $P \otimes \mathbb{Z}[G/H_\lambda]$ restricts to a boundary operator on $P_\lambda$, which turns $P_\lambda$ into a resolution. Moreover, the augmentation map of $F \otimes \mathbb{Z}[G/H_\lambda]$ maps $P_\lambda$ onto $\mathbb{Z}\langle\langle N \rangle\rangle T_\lambda H_\lambda / H_\lambda$. As $T_\lambda \in LT(H_\lambda \langle\langle N \rangle\rangle, G)$, we have $\langle\langle N \rangle\rangle T_\lambda H_\lambda = G$. It follows that $P_\lambda \to \mathbb{Z}[G/H_\lambda]$ is a free resolution over $\mathbb{Z}\langle\langle N \rangle\rangle$. Let

$$P = \bigoplus_{\lambda \in \Lambda} P_\lambda.$$

Then $P \to \bigoplus_{\lambda \in \Lambda} \mathbb{Z}[G/H_\lambda]$ is a free resolution over $\mathbb{Z}\langle\langle N \rangle\rangle$.

The resolution $R \to \mathbb{Z}$ results from modifying $P \to \bigoplus_{\lambda \in \Lambda} \mathbb{Z}[G/H_\lambda]$ at dimension 0. Let $R = \bigoplus_{q \geq 0} R_q$ be the graded abelian group such that

(a) $R_0 = \mathbb{Z}\langle\langle N \rangle\rangle$;

(b) $R_q = P_q$ for all $q \geq 1$, where $P_q$ is the component of $P$ at dimension $q$.

For $q \geq 1$, we demand that the boundary map $R_{q+1} \to R_q$ is the same as the boundary map $P_{q+1} \to P_q$. To define the boundary map $\partial_R^1 : R_1 \to R_0$, it suffices to specify the values of $\partial_R^1$ on the set

$$\{(t, tn) \otimes tH_\lambda \mid \lambda \in \Lambda, n \in N_\lambda, t \in T_\lambda\}$$

by Lemma 4.9 and we require that

$$\partial_R^1 ((t, tn) \otimes tH_\lambda) = 1 - tnt^{-1} \text{ for all } \lambda \in \Lambda, n \in N_\lambda, t \in T_\lambda.$$

Let $\epsilon : \mathbb{Z}\langle\langle N \rangle\rangle \to \mathbb{Z}$ be the augmentation map of $\mathbb{Z}\langle\langle N \rangle\rangle$. Our goal is the following.

**Lemma 4.10.** $R \xrightarrow{\epsilon} \mathbb{Z}$ is a free resolution over $\mathbb{Z}\langle\langle N \rangle\rangle$. Moreover, the augmentation map $\epsilon : \bigoplus_{\lambda \in \Lambda} \mathbb{Z}[G/H_\lambda] \to \mathbb{Z}$ induces a chain map
\[ \cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow \bigoplus_{\lambda \in \Lambda} \mathbb{Z}[G/H_{\lambda}] \longrightarrow 0 \]

\[ \cdots \longrightarrow R_2 \longrightarrow R_1 \longrightarrow R_0 \longrightarrow \mathbb{Z} \longrightarrow 0 \]  \hspace{1cm} (12)

such that for all \( q \geq 1 \), \( \epsilon_q \) is the identity map.

Lemma 4.10 clearly implies that \( \epsilon^* \) maps \( H^q(\langle \langle N \rangle \rangle; A) \) isomorphically onto \( \text{Ext}^q_{\langle \langle N \rangle \rangle} \left( \bigoplus_{\lambda \in \Lambda} \mathbb{Z}[G/H_{\lambda}], A \right) \) for all \( q \geq 1 \) and thus finishes the proof of Proposition 4.4.

To prove Lemma 4.10 for \( q \geq 1 \), let \( \epsilon_q \) be the identity map. To define \( \epsilon_0 \), it suffices to specify the values of \( \epsilon_0 \) on the set

\[ \{ t \otimes tH_\lambda \mid \lambda \in \Lambda, t \in T_\lambda \} \]

by Lemma 4.9 and we demand that

\[ \epsilon_0 (t \otimes tH_\lambda) = 1 \text{ for all } \lambda \in \Lambda \text{ and } t \in T_\lambda. \]

Clearly, \( (\epsilon_q)_{q \geq 0} \) is a chain map from \( P \) to \( R \).

Note that the \( \mathbb{Z}[\langle \langle N \rangle \rangle] \)-module \( \text{im}(\partial^R_1) \) is generated by the set

\[ \{ 1 - tnt^{-1} \mid \lambda \in \Lambda, n \in N_{\lambda}, t \in T_\lambda \} \]

and hence is the augmentation ideal of \( R_0 = \mathbb{Z}[\langle \langle N \rangle \rangle] \). To finish the prove of Lemma 4.10 it suffices to show the following.

**Lemma 4.11.** \( \ker(\partial^R_1) = \ker(\partial^P_1) \), where \( \partial^P_1 \) is the boundary map from \( P_1 \) to \( P_0 \).

Lemma 4.11 implies that the second row of (12) is exact and thus \( R \rightarrow \mathbb{Z} \) is a free resolution.

**Proof.** As \( \partial^R_1 = \epsilon_0 \circ \partial^P_1 \), we have \( \ker(\partial^P_1) \subset \ker(\partial^R_1) \). In order to prove the converse containment, we factorize elements of \( \langle \langle N \rangle \rangle \). By equation (13), every element \( m \neq 1 \) of \( \langle \langle N \rangle \rangle \) can be uniquely factorized as

\[ m = \prod_{i=1}^{k} t_i n_i t_i^{-1}, \]  \hspace{1cm} (13)

where

(i) \( t_i \in T_{\lambda_i}, n_i \in N_{\lambda_i} \setminus \{1\} \), and \( \lambda_1, \ldots, \lambda_k \in \Lambda \);

(ii) for \( i = 1, \ldots, k - 1 \), either \( \lambda_i \neq \lambda_{i+1} \) or \( t_i \neq t_{i+1} \).
Equation (13) is called the factorization of \( m \). The number of factors of \( m \), denoted as \( \omega(m) \), is the number \( k \) in (13). The factorization of \( 1 \in \langle N \rangle \) is just \( 1 = 1 \) and we let \( \omega(1) = 0 \).

For every \( \lambda \in \Lambda \) and every \( t \in T_\lambda \), let \( X_{\lambda,t} \) be the subset of \( \langle N \rangle \) consisting of elements whose factorizations do not end with a factor from \( tN_\lambda t^{-1} \).

We construct a basis for the abelian group \( P_1 \). For \( \lambda \in \Lambda \), let \( E_\lambda \) be a set of pairs of elements of \( N_\lambda \) such that

(a) for every \( n \in N_\lambda \), the pair \((n,n)\) belongs to \( E_\lambda \);

(b) if \( n_1, n_2 \) are distinct elements of \( N_\lambda \), then \( E_\lambda \) contains exactly one of \((n_1,n_2)\) and \((n_2,n_1)\).

Let

\[
S = \{(xtn_1, xtn_2) \otimes xtH_\lambda \mid \lambda \in \Lambda, t \in T_\lambda, x \in X_{\lambda,t}, (n_1, n_2) \in E_\lambda \} \subset R_1.
\]

Then \( S \) is a basis for the abelian group \( P_1 = R_1 \).

For every 
\[
s = (xtn_1, xtn_2) \otimes xtH_\lambda \in S,
\]
let

\[
\Omega(s) = \max\{\omega(xtn_1t^{-1}), \omega(xtn_2t^{-1})\}.
\]

Every \( r \in R_1 \) can be uniquely written in the form

\[
r = \sum_{s \in S} C_{r,s} s
\]

where \( C_{r,s} \in \mathbb{Z} \). The above sum makes sense as there are only finitely many non-zero terms.

We call the number \( C_{r,s} \) in the above equation the coefficient of \( r \) with respect to \( s \). Let \( \text{rank} : R_1 \to \mathbb{N} \) be the function summing the absolute values of the coefficients:

\[
\text{rank}(r) = \sum_{s \in S} |C_{r,s}|.
\]

Suppose \( r \in \ker(\partial^R_1) \). We prove \( r \in \ker(\partial^P_1) \) by an induction on \( \text{rank}(r) \). The base case \( \text{rank}(r) = 0 \) implies \( r = 0 \) and thus \( r \in \ker(\partial^P_1) \). So let us suppose that \( \text{rank}(r) > 0 \) and that, for all \( r' \in \ker(\partial^R_1) \) with \( \text{rank}(r') < \text{rank}(r) \), we have \( r' \in \ker(\partial^P_1) \).

Let

\[
s_0 = (x_0t_0n_1, x_0t_0n_2) \otimes x_0t_0H_{\lambda_0} \in S
\]

such that \( C_{r,s_0} \neq 0 \) and

\[
(\max \Omega) \text{ if } s \in S \text{ satisfying } C_{r,s} \neq 0, \text{ then } \Omega(s) \geq \Omega(s).
\]
If $n_1 = n_2$, consider the element $r' \in R_1$ such that $C_{r',s} = C_{r,s}$ for $s \in S \setminus \{s_0\}$ and $C_{r',s_0} = 0$. Direct computation shows
\[
\text{rank}(r') < \text{rank}(r), \quad \partial_1^P(r - r') = \partial_1^R(r - r') = 0.
\]
Thus, $\partial_1^R(r') = 0$ and the induction hypothesis implies $\partial_1^P(r') = 0$. It follows that
\[
\partial_1^P(r) = \partial_1^P(r - r') + \partial_1^P(r') = 0.
\]
Therefore, $r \in \ker(\partial_1^P)$.

Thus, without loss of generality, let us assume $n_1 \neq n_2$. It follows that at least one of $n_1$ and $n_2$ is not the identity of $G$. Without loss of generality, we may further assume $n_1 \neq 1$ (the case $n_2 \neq 1$ is similar), in which case
\[
\Omega(s_0) = \omega(x_0t_0n_1t_0^{-1}).
\]

Let us also assume $C_{r,s} > 0$ (otherwise, consider $-r$). Note that
\[
\partial_1^R(r) = \sum_{s \in S} C_{r,s} \partial_1^R(s). \tag{14}
\]

On the right-hand side of (14),
\[
C_{r,s_0} \partial_1^R(s_0) = C_{r,s_0} (x_0t_0n_2t_0^{-1} - x_0t_0n_1t_0^{-1}).
\]
Thus, $s_0$ contributes a negative number of $x_0t_0n_1t_0^{-1}$ to $\partial_1^R(r)$. As $\partial_1^R(r) = 0$, there exists some $s_1 \in S$ which contributes a positive number of $x_0t_0n_1t_0^{-1}$ to $\partial_1^R(r)$. In other words, one of the following cases happens
\begin{enumerate}
  \item $s_1 = (x_1t_1n_3, x_1t_1n_4) \otimes x_1t_1 H_{\lambda_1}$ with $C_{r,s_1} < 0$, $n_3 \neq n_4$, and $x_1t_1n_3t_1^{-1} = x_0t_0n_1t_0^{-1}$.
  \item $s_1 = (x_1t_1n_3, x_1t_1n_4) \otimes x_1t_1 H_{\lambda_1}$ with $C_{r,s_1} > 0$, $n_3 \neq n_4$, and $x_1t_1n_4t_1^{-1} = x_0t_0n_1t_0^{-1}$.
\end{enumerate}

Let us suppose that Case (1) happens (Case (2) can be treated in the same manner). Note that $n_3 \neq 1$ in this case. Indeed, if $n_3 = 1$, then $n_4 \neq 1$ since $n_4 \neq n_3$. It follows that
\[
\begin{align*}
\Omega(s_1) &> \omega(x_1t_1n_3t_1^{-1}) &\quad \text{as } n_3 = 1, n_4 \neq 1, x_1 \in X_{\lambda_1,t_1} \\
&= \omega(x_0t_0n_1t_0^{-1}) &\quad \text{as } x_1t_1n_3t_1^{-1} = x_0t_0n_1t_0^{-1} \\
&= \Omega(s_0),
\end{align*}
\]
which contradicts condition (max $\Omega$). Thus, $n_3 \neq 1$, which, together with the assumption $x_1 \in X_{\lambda_1,t_1}$, implies that the factorization of $x_1t_1n_3t_1^{-1}$ ends with $t_1n_3t_1^{-1}$.

As $x_0 \in X_{\lambda_0,t_0}$ and $n_1 \neq 1$, the factorization of $x_0t_0n_1t_0^{-1}$ ends with $t_0n_1t_0^{-1}$. Since $x_1t_1n_3t_1^{-1} = x_0t_0n_1t_0^{-1}$, we have
\[
t_0n_1t_0^{-1} = t_1n_3t_1^{-1} \in t_0N_{\lambda_0}t_0^{-1} \cap t_1N_{\lambda_1}t_1^{-1}.
\]
As $n_1 \neq 1$, we have
\[ t_0N_{\lambda_0}t_0^{-1} \cap t_1N_{\lambda_1}t_1^{-1} \neq \{1\}. \quad (15) \]

Equations (11) and (15) imply $\lambda_1 = \lambda_0, t_1 = t_0$, which, together with $x_1t_1n_3t_1^{-1} = x_0t_0n_1t_0^{-1}$, imply $n_1 = n_0, x_1 = x_0$ and thus
\[
s_1 = (x_0t_0n_1, x_0t_0n_4) \otimes x_0t_0H_{\lambda_0}.
\]

Exactly one of $(n_2, n_4)$ and $(n_4, n_2)$ is in $E_{\lambda_0}$. Without loss of generality, we assume that $(n_2, n_4) \in E_{\lambda_0}$ (the other case is similar). Let
\[
s_2 = (x_0t_0n_2, x_0t_0n_4) \otimes x_0t_0H_{\lambda_0},
\]
and let $r' \in R_1$ such that $C_{r', s} = C_{r, s}$ for $s \in S \setminus \{s_0, s_1, s_2\}$ and
\[
C_{r', s_0} = C_{r, s_0} - 1, \quad C_{r', s_1} = C_{r, s_1} + 1, \quad C_{r', s_2} = C_{r, s_2} - 1.
\]
As $C_{r, s_0} > 0, C_{r, s_1} < 0$, and $\partial_1^R(r) = 0$, direct computation shows
\[
\text{rank}(r') < \text{rank}(r), \quad \partial_1^P(r - r') = \partial_1^R(r - r') = 0.
\]
Thus, $\partial_1^R(r') = 0$ and the induction hypothesis implies $\partial_1^P(r') = 0$. It follows that
\[
\partial_1^P(r) = \partial_1^P(r - r') + \partial_1^P(r') = 0,
\]
that is, $r \in \ker(\partial_1^P)$. \qed

References

[ACG18] Y. Antolín, R. Coulon, and G. Gandini. Farrell-Jones via Dehn fillings. *J. Topol. Anal.*, 10(4):873–895, 2018.

[AGM13] I. Agol, D. Groves, and J. Manning. The virtual Haken conjecture. *Doc. Math.*, 18:1045–1087, 2013.

[AGM16] I. Agol, D. Groves, and J. Manning. An alternate proof of Wise’s malnormal special quotient theorem. *Forum Math. Pi*, 4:e1, 54, 2016.

[Bro94] K. Brown. *Cohomology of groups*, volume 87 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1994. Corrected reprint of the 1982 original.

[CL63] D. Cohen and R. Lyndon. Free bases for normal subgroups of free groups. *Trans. Amer. Math. Soc.*, 108:526–537, 1963.

[DG18] F. Dahmani and V. Guirardel. Recognizing a relatively hyperbolic group by its Dehn fillings. *Duke Math. J.*, 167(12):2189–2241, 2018.
