Coordinate Dual Averaging for Decentralized Online Optimization with Nonseparable Global Objectives*

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Abstract—We consider a decentralized online convex optimization problem in a network of agents, where each agent controls only a coordinate (or a part) of the global decision vector. For such a problem, we propose two decentralized variants (ODA-C and ODA-PS) of Nesterov’s primal-dual algorithm with dual averaging. In ODA-C, to mitigate the disagreements on the primal-vector updates, the agents implement a generalization of the local information-exchange dynamics recently proposed by Li and Marden [1] over a static undirected graph. In ODA-PS, the agents implement the broadcast-based push-sum dynamics [2] over a time-varying sequence of uniformly connected digraphs. We show that the regret bounds in both cases have sublinear growth in time, and the objective functions are Lipschitz-continuous convex functions with Lipschitz gradients. We also implement the proposed algorithms on a sensor network to complement our theoretical analysis.

I. INTRODUCTION

Decentralized optimization has recently been receiving significant attention due to the emergence of large-scale distributed algorithms in machine learning, signal processing, and control applications for wireless communication networks, power networks, and sensor networks; see, for example, [3]–[8]. A central generic problem in such applications is decentralized resource allocation for a multiagent system, where the agents collectively solve an optimization problem in the absence of full knowledge about the overall problem structure. In such settings, the agents are allowed to communicate to each other some relevant estimates so as to learn the information needed for an efficient global resource allocation. The decentralized structure of the problem is reflected in the agents’ local view of the underlying communication network, where each agent exchanges messages only with its neighbors.

In recent literature on control and optimization, an extensively studied decentralized resource allocation problem is one where the system objective function $f(x)$ is given as a sum of local objective functions, i.e., $f(x) = \sum_{i=1}^{n} f_i(x)$ where $f_i$ is known only to agent $i$; see, for example [9]–[25]. In this case, the objective function is separable across the agents, but the agents are coupled through the resource allocation vector $x$. Each agent maintains and updates its own copy of the allocation/decision vector $x$, while trying to estimate an optimal decision for the system problem. The vector $x$ is assumed to lie in (a subset of) $\mathbb{R}^d$, where $d$ may or may not coincide with the number of agents $n$.

Another decentralized resource allocation problem is the one where the system objective function $f(x)$ may not admit a natural decomposition of the form $\sum_{i=1}^{n} f_i(x)$, and the resource allocation vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ is distributed among the agents, where each agent $i$ is responsible for maintaining and updating only a coordinate (or a part) $x_i$ of the whole vector $x$. Such decentralized problems have been considered in [26]–[30] (see also the textbook [31]). In the preceding work, decentralized approaches converge when the agents are using weighted averaging, or when certain contraction conditions are satisfied. Recently, Li and Marden [1] have proposed a different algorithm with local updates, where each agent $i$ keeps estimates for the variables $x_j$, $j \neq i$, that are controlled by all the other agents in the network. The convergence of this algorithm relies on some contraction properties of the iterates. Note that all the aforementioned algorithms were developed for offline optimization problems.

Our work in this paper is motivated by the ideas of Li and Marden [1] and also by the broadcast-based subgradient push, which was originally developed by Kempe et al. [2] and later extended in [32] and in [15], [16] to distributed optimization. Specifically, we use the local information exchange model of [1] and [2], [15], [16], [32], but employ a different online decentralized algorithm motivated by the work of Nesterov [33]. We call these algorithms ODA-C (Online Dual Averaging with Circulation-based communication) and ODA-PS (Online Dual Averaging with Push-Sum based communication), respectively.

In contrast with existing methods, our algorithms have the following distinctive features: (1) We consider an online convex optimization problem with nondecomposable system objectives, which are functions of a distributed resource allocation vector. (2) In our algorithms, each agent maintains and updates its private estimate of the best global allocation vector at each time, but contributes only one coordinate to the network-wide decision vector. (3) We provide regret bounds in terms of the true global resource allocation vector $x$ (rather than some estimate on $x$ by a single agent). For both ODA-C and ODA-PS, we show that the regret has sublinear growth of $O(\sqrt{T})$ in time $T$ with the stepsize of the form $1/\sqrt{\tau + 1}$.

Our proposed algorithm ODA-PS is closest to recent papers [34], [35]. The papers proposed a decentralized algorithm for

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online convex optimization which is very similar to ODA-PS in a sense that they also introduce online subgradient estimations in primal [34] or dual [35] space into information aggregation using push-sum. In these papers, the agents share a common decision set in \( \mathbb{R}^d \), the objective functions are separable across the agents at each time (i.e., \( f_i(x) = \sum_{i=1}^n f'_i(x) \) for all \( t \)), and the regret is analyzed in terms of each agent’s own copy of the whole decision vector \( x \in \mathbb{R}^d \). Moreover, an additional assumption is made in [34] that the objective functions are strongly convex.

The paper is organized as follows. In Section II, we formalize the problem and describe how the agents interact. In Section III, we provide an online decentralized dual-averaging algorithm in a generic form and establish a basic regret bound which can be used later for particular instantiations, namely, for the two algorithms ODA-C and ODA-PS. These algorithms are analyzed in Sections IV where we establish \( O(\sqrt{T}) \) regret bounds under mild assumptions. In Section VI we demonstrate our analysis by simulations on a sensor network. We conclude the paper with some comments in Section VII.

**Notation:** All vectors are column vectors. For vectors associated with agent \( i \), we use a subscript \( i \) such as, for example, \( x_i, z_i \), etc. We will write \( x_i^k \) to denote the \( k \)th coordinate value of a vector \( x_i \). We will work with the Euclidean norm, denoted by \( \| \cdot \| \). We will use \( e_1, \ldots, e_n \) to denote the unit vectors in the standard Euclidean basis of \( \mathbb{R}^n \). We use 1 to denote a vector with all entries equal to 1, while \( I \) is reserved for an identity matrix of a proper size. For any \( n \geq 1 \), the set of integers \( \{1, \ldots, n\} \) is denoted by \( [n] \). We use \( \sigma_2(A) \) to denote the second largest singular value of a matrix \( A \).

**II. PROBLEM FORMULATION**

Consider a multiagent system (network) consisting of \( n \) agents, indexed by elements of the set \( \mathcal{V} = [n] \). Each agent \( i \in \mathcal{V} \) takes actions in an action space \( X \), which is a closed and bounded interval of the real line. At each time, the multiagent system incurs a time-varying cost \( f_i \), which comes from a fixed class \( \mathcal{F} \) of convex functions \( f : X^n \rightarrow \mathbb{R} \).

The communication among agents in the network is governed by either one of the following two models:

(G1) An undirected connected graph \( \mathcal{G} = (\mathcal{V}, \mathcal{E}) \): If agents \( i \) and \( j \) are connected by an edge (which we denote by \( i \leftrightarrow j \)), then they may exchange information with one another. Thus, each agent \( i \in \mathcal{V} \) may directly communicate only with the agents in its neighborhood \( \mathcal{N}_i \triangleq \{ j \in \mathcal{V} : i \leftrightarrow j \} \cup \{ i \} \). Note that agent \( i \) is always contained in its own neighborhood.

(G2) Time-varying digraphs \( \mathcal{G}(t) = (\mathcal{V}, \mathcal{E}(t)) \), for \( t \geq 1 \): If there exists a directed link from agent \( j \) to \( i \) at time \( t \) (which we denote by \( (j, i) \)), agent \( j \) may send its information to agent \( i \). We use the notation \( \mathcal{N}^\text{in}(t) \) and \( \mathcal{N}^\text{out}(t) \) to denote the in and out neighbors of agent \( i \) at time \( t \), respectively. That is,

\[
\mathcal{N}^\text{in}(t) \triangleq \{ j \mid (j, i) \in \mathcal{E}(t) \} \cup \{ i \},
\]

\[
\mathcal{N}^\text{out}(t) \triangleq \{ j \mid (i, j) \in \mathcal{E}(t) \} \cup \{ i \}.
\]

1Everything easily generalizes to \( X \) being a compact convex subset of a multidimensional space \( \mathbb{R}^d \); we mainly stick to the scalar case for simplicity.

In this case, we assume that there always exists a self-loop \((i, i)\) for all agent \( i \in \mathcal{V} \). Therefore, agent \( i \) is always contained in its own neighborhood. Also, we use \( d_i(t) \) to denote the out degree of node \( i \) at time \( t \), i.e.,

\[
d_i(t) \triangleq |\mathcal{N}^\text{out}(t)|.
\]

We assume \( B \)-strong connectivity of the graphs \( \mathcal{G}(t) \) with some scalar \( B > 0 \), i.e., a graph with the following edge set

\[
\mathcal{E}_B(t) = \bigcup_{i=(t-1)B+1}^{tB} \mathcal{E}(i)
\]

is strongly connected for every \( t \geq 1 \). In other words, the union of the edges appearing for \( B \) consecutive time instances periodically constructs a strongly connected graph. This assumption is required to ensure that there exists a path from one node to every other node infinitely often even if the underlying network topology is time-varying.

The network interacts with an environment according to the protocol shown in Figure 1. We leave the details of the signal generation process vague for the moment, except to note that the signals received by all agents at time \( t \) may depend on all the information available up to time \( t \) (including \( f_1, \ldots, f_t \), as well as all of the local information exchanged in the network). Moreover, the environment may be adaptive, i.e., the choice of the function \( f_t \) may depend on all of the data generated by the network up to time \( t \).

### Parameters

- **base action space** \( X \); network graph \( \mathcal{G} = (\mathcal{V}, \mathcal{E}) \); function class \( \mathcal{F} \)
- For each round \( t = 1, 2, \ldots \):
  1. Each agent \( i \in \mathcal{V} \) selects an action \( x^t_i \in X \)
  2. Each agent \( i \in \mathcal{V} \) exchanges local information with its neighbors \( \mathcal{N}_i \)
  3. The environment selects the current objective \( f_t \in \mathcal{F} \), and each agent receives a signal about \( f_t \)

Fig. 1. Online optimization with global objectives and local information.

Let us denote the network action at time \( t \) by

\[
x(t) = (x^1(t), \ldots, x^n(t)) \in X^n.
\]  

We consider the network regret \( R(T) \) at an arbitrary time horizon \( T \geq 1 \):

\[
R(T) \triangleq \sum_{t=1}^{T} f_t(x(t)) - \inf_{y \in X^n} \sum_{t=1}^{T} f_t(y).
\]

Thus, \( R(T) \) is the difference between the total cost incurred by the network at time \( T \) and the smallest total cost that could have been achieved with a single action in \( X^n \) in hindsight (i.e., with perfect advance knowledge of the sequence \( f_1, \ldots, f_T \)) and without any restriction on the communication between the agents. The problem is to design the rule (or policy) each agent \( i \in \mathcal{V} \) should use to determine its action \( x^t_i \) based on the
local information available to it at time $t$, such that the regret in (2) is (a) sublinear as a function of the time horizon $T$ and (b) exhibits “reasonable” dependence on the number of agents $n$ and on the topology of the communication graphs.

The regret in (2) is defined over the true network actions of individual agents, i.e., $x^t(i)$’s, rather than in terms of some estimates of $x(t)$ by individual agents. This notion of regret, which, to the best of our knowledge has been first introduced in [33], is inspired by the literature on team decision theory and decentralized control problems: The online optimization is performed by a team of cooperating agents facing a time-varying sequence of global objective functions $f_t$, which are nondecomposable (in contrast to decomposable objectives $\sum_i f_i(x)$, where $f_i$ is only revealed to agent $i$). Communication among agents is local, as dictated by the network topology, so no agent has all the information in order to compute a good global decision vector $x(t)$. By comparing the cumulative performance of the decentralized system to the best centralized decision achievable in hindsight, the regret in (2) captures the effect of decentralization. It also calls for analysis techniques that are different from existing methods in the literature.

III. THE BASIC ALGORITHM AND REGRET BOUND

We now introduce a generic algorithm for solving the decentralized online optimization problem defined in Section II. The algorithm uses the dual-averaging subgradient method of Nesterov [33] as an optimization subroutine.

Each agent $i \in V$ generates a sequence $\{x_i(t), z_i(t)\}_{t=1}^{\infty}$ in $X^n \times \mathbb{R}^n$, where the primal iterates

$$x_i(t) = (x_i^1(t), \ldots, x_i^n(t)) \in X^n$$

and the dual iterates

$$z_i(t) = (z_i^1(t), \ldots, z_i^n(t)) \in \mathbb{R}^n$$

are updated recursively as follows:

$$z_i^k(t+1) = \frac{1}{r_i} \delta_i u_i(t) + F_i^k(m_i(t)), \ k \in [n] \tag{3a}$$

$$x_i(t+1) = \Pi_{X^n}^\psi(G_i(t)(z_i(t+1)), \alpha(t)) \tag{3b}$$

with the initial condition $z_i(0) = 0$ for all $i \in V$. In the dual update (3a), $\delta_i^k$ is the Kronecker delta symbol, $r_i > 0$ is a positive weight parameter, $u_i(t) \in \mathbb{R}$ is a local update computed by agent $i$ at time $t$ based on the received signal about $f_i$, $m_i(t)$ are the messages received by agent $i$ at time $t$ [from agents in $N_i$ under the model (G1) or from $N_i^{\text{out}}(t)$ under the model (G2)], and $F_i^k, k \in [n]$, are real-valued mappings that perform local averaging of $m_i(t)$. In the primal update (3b), $G_i : \mathbb{R}^n \to \mathbb{R}^n$ is a mapping on dual iterates, $\{\alpha(t)\}_{t=1}^{\infty}$ is a nonincreasing sequence of positive step sizes, and the mapping $\Pi_{X^n}^\psi : \mathbb{R}^n \times (0, \infty) \to X^n$ is defined by

$$\Pi_{X^n}^\psi(z, \alpha) \triangleq \arg \min_{x \in X^n} \left\{ \langle z, x \rangle + \frac{1}{\alpha} \psi(x) \right\}, \tag{4}$$

where $\psi : X^n \to \mathbb{R}^+$ is a nonnegative proximal function.

We assume that $\psi$ is 1-strongly convex with respect to the Euclidean norm $\| \cdot \|$, i.e., for any $x, y \in X^n$ we have

$$\psi(y) \geq \psi(x) + \langle \nabla \psi(x), y - x \rangle + \frac{1}{2} \| x - y \|^2, \tag{5}$$

where $\nabla \psi$ denotes an arbitrary subgradient of $\psi$.

The dual iterate $z_i(t)$ computed by agent $i$ at time $t$ will be an estimate of the “running average of the subgradients” as seen by agent $i$, and will constitute an approximation of the true centralized dual-averaging subgradient update of Nesterov’s algorithm. The messages from $N_i$ entering into the dual-space dynamics are crucial for mitigating any disagreement between the agents’ local estimates of what the network action should be. The primal iterate $x_i(t)$ of agent $i$ at time $t$ is an approximation of the true centralized primal point for the subgradient evaluation.

Note that in (3a) the local update $u_i(t)$ based on the signal about $f_i$ affects only the $i$th coordinate of the dual iterate $z_i(t+1)$, while all other coordinates with $k \neq i$ remain untouched except for the averaging. The action of agent $i$ at time $t$ is then given by

$$x_i(t) = x_i^i(t),$$

i.e., by the $i$th component of the vector $x_i(t)$.

A concrete realization of the algorithm (3a)-(3b) requires specification of the rules for computing the local update $u_i(t)$, the messages exchanged by the agents, and the mappings $F_i^k$ and $G_i(t)$. In this paper, we present two different instantiations of this algorithm, namely, the circulation-based method inspired by [1] and the push-sum based method inspired by [2], [15], [16], [32]. We call these algorithms ODA-C (Online Dual Averaging with Circulation-based communication) and ODA-PS (Online Dual Averaging with Push-Sum based communication) and detail them in Section IV and V, respectively.

We now present a basic regret bound that can be used for any generic algorithm of the form (3a)-(3b) under the following assumption:

**Assumption 1:** All functions $f \in \mathcal{F}$ are Lipschitz continuous with a constant $L$:

$$\|f(x) - f(y)\| \leq L \|x - y\| \quad \text{for all } x, y \in X^n.$$  

**Theorem 1:** Let $\{x_i(t)\}_{t=1}^{\infty} \subset X^n$, $i \in V$, be the sequences of the agents’ primal iterates, let $\{u(t)\}_{t=1}^{\infty}$ with $u(t) = (u_1(t), \ldots, u_n(t))$ be the sequence of the agents’ local updates, and let $\{\bar{x}(t)\}_{t=1}^{\infty} \subset X^n$ be generated as

$$\bar{x}(t+1) = \Pi_{X^n}^\psi \left( \sum_{s=0}^t u(s), \alpha(t) \right). \tag{6}$$

Then, under Assumption 1, the network regret $R(T)$ in (2) can be upper-bounded in terms of $u(t)$ and $\bar{x}(t)$ as follows: for each $T \geq 1$,

$$R(T) \leq \frac{1}{\alpha(T)} \sum_{t=1}^{T} \alpha(t-1) \|u(t)\|^2 + \frac{C}{\alpha(T)}$$

$$+ L \sum_{t=1}^{T} \sum_{i=1}^{n} \|x_i(t) - \bar{x}(t)\| + \sqrt{n}D_x \sum_{t=1}^{T} \|\nabla f_i(\bar{x}(t)) - u(t)\|, \tag{E1} \tag{E2} \tag{E3}$$


where $D_X = \sup_{x,y \in X} |x - y|$ is the diameter of the set $X$, and $C = \sup_{x \in X^n} |\psi(x)|$.

**Remark** Since $\psi$ is a continuous function on the compact set $X^n$, $C < \infty$ by the Weierstrass theorem.

**Proof:** For any $t$ and any $y \in X^n$ we can write
\[
    f_t(x(t)) - f_t(y) = f_t(x(t)) - f_t(\bar{x}(t)) + f_t(\bar{x}(t)) - f_t(y) \\
    \leq \langle \nabla f_t(\bar{x}(t)), x(t) - \bar{x}(t) \rangle + \langle \nabla f_t(\bar{x}(t)), \bar{x}(t) - y \rangle \\
    \leq L\|x(t) - \bar{x}(t)\| + \|\nabla f_t(\bar{x}(t))\| \cdot \|x(t) - y\|,
\]
where the second step follows from convexity of $f_t$, while the last step uses the fact that all $f \in F$ are $L$-Lipschitz. Recalling that $x(t)$ is the network action vector (see [1]), we have the following for the first term in (7):
\[
    \|x(t) - \bar{x}(t)\| = \|\sum_{i=1}^{n} (x^i(t) - \bar{x}^i(t)) e_i\| \\
    \leq \sum_{i=1}^{n} \|x^i(t) - \bar{x}^i(t)\|,
\]
where the equality follows from the definition of $x(t)$ in [1] and $\bar{x}(t) = (\bar{x}^1(t), \ldots, \bar{x}^n(t))$. The second term in (7) can be further expanded as
\[
    \langle \nabla f_t(\bar{x}(t)), x(t) - \bar{x}(t) \rangle = \langle u(t), \bar{x}(t) - y \rangle + \langle \nabla f_t(\bar{x}(t)) - u(t), \bar{x}(t) - y \rangle.
\]
Now, from relation (6) we obtain
\[
    \bar{x}(t) = \arg\min_{x \in X^n} \left\{ \frac{t}{\alpha(t)} \|u(s) - x\| + \psi(x) \right\}.
\]
Therefore, by [36, Lemma 3], we can write
\[
    \sum_{t=1}^{T} \|u(t) - \bar{x}(t) - y\| \leq \frac{1}{2} \sum_{t=1}^{T} \alpha(t - 1) \|u(t)\|^2 + \frac{\psi(y)}{\alpha(T)}.
\]
For the second term on the right-hand side of (9), we have
\[
    \langle \nabla f_t(\bar{x}(t)) - u(t), \bar{x}(t) - y \rangle \\
    \leq \|\bar{x}(t) - y\| \|\nabla f_t(\bar{x}(t)) - u(t)\| \\
    \leq \sqrt{nD_X} \|\nabla f_t(\bar{x}(t)) - u(t)\|.
\]
Combining the estimates in Eqs. (7)-(11) and taking the supremum over all $y \in X^n$, we get the desired result. \qed

**IV. ODA-C AND ITS REGRET BOUND**

We now introduce a decentralized online optimization algorithm which uses a circulation-based framework for its dual update rule (3a). We refer to this algorithm as ODA-C (Online Dual Averaging with Circulation-based communication). ODA-C uses the network model (G1) for its communication.

**A. ODA-C**

Fix a vector $r = (r_1, \ldots, r_n)$ of positive weights and a nonnegative $n \times n$ matrix $M$, such that $M_{ij} \neq 0$ only if $j \in N_i$, satisfying the following symmetry condition:
\[
    r_i M_{ij} = r_j M_{ji}, \quad i, j \in \mathcal{V}.
\]
Then, ODA-C uses the following instantiation of the update rules in (3a)-(3b):
\[
    z^k_i(t + 1) = \frac{1}{r_i} \delta^k_i u_i(t) + z^k_i(t) \\
    + \sum_{j=1}^{n} M_{ij} (v^k_{j \rightarrow i}(t) - v^k_{i \rightarrow j}(t)), \quad k \in [n]
\]
where $\{v^k_{j \rightarrow i}(t), \ldots, v^k_{n \rightarrow i}(t)\} \in \mathbb{R}^n$ represents a vector of messages transmitted by agent $j$ to agent $i$, provided that $j \in N_i$. Since $i \in N_i$, we may include the previous dual iterate $z^k_i(t)$ and the outgoing messages $v^k_{i \rightarrow j}(t)$ in $m_i(t)$. The dual update rule (13a) is inspired by the state dynamics proposed by Li and Marden [1], whereas the primal update rule (13b) is exactly what one has in Nesterov’s scheme [33].

To complete the description of the algorithm, we must specify the update policies $\{u_i(t)\}$ and the messages $\{v^k_{i \rightarrow j}(t)\}$. We assume that all agents receive a complete description of $f_t$. Agent $i$ then computes
\[
    u_i(t) = \langle \nabla f_t(\bar{x}_i(t)), e_i \rangle, \quad i \in [n], \quad t \geq 0.
\]
and feeds this signal back into the dynamics (13a). Note, however, that the execution of the algorithm will not change if the agents never directly learn the full function $f_t$, nor even the full gradient $\nabla f_t(\bar{x}_i(t))$, but instead receive the local gradient signal $\nabla f_t(\bar{x}_i(t))$. The messages $v^k_{i \rightarrow j}(t)$ take the form
\[
    v^k_{i \rightarrow j}(t) = z^k_i(t)
\]
for all $t$ and all agents $i, j \in \mathcal{V}$ with $j \in N_i$.

**B. Regret of ODA-C with local gradient signals**

Let $\bar{z}(t) = (\bar{z}^1(t), \ldots, \bar{z}^n(t))$. Our regret analysis rests on the following simple but important fact:

**Lemma 1:** The weighted sum $\bar{z}(t) = \sum_{i=1}^{n} r_i z_i(t)$ evolves according to the linear dynamics
\[
    \bar{z}(t + 1) = \bar{z}(t) + \psi(t),
\]
where $\psi(t) = (u_1(t), \ldots, u_n(t))$. 

Remark: We observe that the relation in (16) holds regardless of the choices of decisions $v_j^{k-1}$ and $v_{j'}^{k-1}$. Moreover, we point out that if $u(t) = \nabla f_t(x(t))$, then the combination of (16) and (13b) will reduce to a centralized online variant of Nesterov’s scheme [37].

Proof: Let $V^k(t)$ denote the $n \times n$ matrix with entries $[V^k(t)]_{ij} = v^k_{j-1} - v^k_{j'}$. Then

$$\tilde{z}^k(t + 1) = \sum_{i=1}^{n} r_i \tilde{z}^k_i(t + 1)$$

$$= \sum_{i=1}^{n} r_i \left\{ \tilde{z}^k_i(t) + \frac{1}{r_i} \tilde{\delta}_i u_i(t) + \sum_{j=1}^{n} M_{ij} [V^k(t)]_{ij} \right\}$$

$$= \tilde{z}^k(t) + u_k(t) + \text{tr}[\hat{M}V^k(t)],$$

where $\hat{M}$ is an $n \times n$ matrix with entries $\hat{M}_{ij} = r_i M_{ij}$. Since $\hat{M}$ is a symmetric matrix, by (12), and $V^k(t)$ is skew-symmetric, tr[$\hat{M}V^k(t)$] = 0, so we obtain (16). □

Lemma 1 indicates that the vector $\tilde{z}(t)$ can be seen as a “mean field” of the local dual iterates $z_i(t)$ for $i \in \mathcal{V}$ at time $t$. Also, if we define

$$\bar{x}(t + 1) \triangleq \Pi_{\mathcal{X}^n}(\tilde{z}(t + 1), \alpha(t)),$$

then from relation (16) we have

$$\bar{x}(t + 1) = \Pi_{\mathcal{X}^n} \left( \sum_{s=0}^{t} u(s), \alpha(t) \right),$$

which coincides with relation (6) in Theorem 1. This allows us to make use of Theorem 1 in analyzing the regret of this algorithm. Furthermore, the definition of $\bar{x}(t)$ and relation (14) indicate that $u(t)$ will stay close to the centralized gradient $\nabla f_t(\bar{x}(t))$, and as a consequence, the errors (E1) and (E3) in Theorem 1 will remain small.

We now particularize the bound in Theorem 1 to this scenario under the following additional assumption:

Assumption 2: All functions $f \in \mathcal{F}$ are differentiable and have Lipschitz continuous gradients with constant $G$:

$$\|\nabla f(x) - \nabla f(y)\| \leq G\|x - y\|, \quad \forall f \in \mathcal{F}; x, y \in \mathcal{X}^n.$$  

Theorem 2: Under Assumptions 1, 2 the regret of any algorithm of the form (13a)-(13b), with $u(t)$ computed according to (14), can be upper-bounded as follows:

$$R(T) \leq \frac{nL^2}{2} T \sum_{t=1}^{T} \alpha(t - 1) + \frac{C}{\alpha(T)}$$

$$+ \left( L + \sqrt{nGDx} \right) T \sum_{t=1}^{T} \alpha(t - 1) \sum_{i=1}^{n} \|z_i(t) - \bar{z}(t)\|.$$  

Proof: The terms on the right-hand side of the bound in Theorem 1 can be further estimated as follows. Since each $f_t \in \mathcal{F}$ is $L$-Lipschitz,

$$\|u(t)\|^2 = \sum_{i=1}^{n} |\langle \nabla f_t(x_t(t)), e_i \rangle|^2$$

$$\leq \sum_{i=1}^{n} \|\nabla f_t(x_t(t))\|^2$$

$$\leq nL^2.$$  

It remains to estimate term (E3) in Theorem 1. To that end, we write

$$\|\nabla f_t(\bar{x}(t)) - u(t)\|$$

$$= \left\| \sum_{i=1}^{n} (\nabla f_t(\bar{x}(t)) - \nabla f_t(x_i(t)), e_i) e_i \right\|$$

$$\leq \sum_{i=1}^{n} \|\nabla f_t(\bar{x}(t)) - \nabla f_t(x_i(t))\|$$

$$\leq G \sum_{i=1}^{n} \|x_i(t) - \bar{x}(t)\|,$$

where we have exploited the fact that the gradients of all $f \in \mathcal{F}$ are $G$-Lipschitz.

Now, by construction,

$$\|x(t) - x_i(t)\|$$

$$= \left\| \Pi_{\mathcal{X}^n}(\bar{z}(t), \alpha(t - 1)) - \Pi_{\mathcal{X}^n}(z_i(t), \alpha(t - 1)) \right\|$$

$$\leq \alpha(t - 1) \|\bar{z}(t) - z_i(t)\|,$$

where the last step follows from the fact that the map $z \mapsto \Pi_{\mathcal{X}^n}(z, \alpha)$ is $\alpha$-Lipschitz (see, e.g., [33] Lemma 1). Substituting these estimates into the bound in Theorem 1 we get the result. □

This bound indicates that, if the network-wide disagreement term behaves nicely, the regret $R(T)$ will be sublinear in $T$ with a proper choice of the step size $\alpha(t)$. We illustrate this more specifically in the following corollary.

Corollary 1: Suppose that the policies for computing $\{u_i(t)\}$ and $\{v_j^{k-1}(t)\}$ are such that, for all $t$ and for any sequence $f_1, \ldots, f_T \in \mathcal{F}$,

$$\sum_{i=1}^{n} \|z_i(t) - \bar{z}(t)\| \leq K$$

for some finite constant $K > 0$ (which may depend on $n$ and on other problem parameters). Then, the regret of the algorithm (13a)-(13b) is bounded by

$$R(T) \leq \left[ \frac{nL^2}{2} + K \left( L + \sqrt{nGDx} \right) \right] T \sum_{t=1}^{T} \alpha(t - 1) + \frac{C}{\alpha(T)}.$$  

In particular, if we choose $\alpha(t) = \frac{1}{\sqrt{T + 1}}$ for $t \geq 0$, then the regret is of the order $O(\sqrt{T})$:

$$R(T) \leq \left[ nL^2 + 2K \left( L + \sqrt{nGDx} \right) \right] \sqrt{T} + C\sqrt{T + 1}.$$  

C. Full regret analysis

We now show that the network-wide disagreement term is indeed upper-bounded by some constant. We recall that $M_{ij} \neq 0$ only if $j \in N_i$. In addition to this, we posit the following assumptions on the pair $(r, M)$.

Assumption 3: The positive weights $r_1, \ldots, r_n$ sum to one:

$$\sum_{i=1}^{n} r_i = 1$$

and $r_i > 0$ for each $i \in [n]$.  

The matrix $M$ is row-stochastic, i.e.,
\[ \sum_{j=1}^{n} M_{ij} = 1 \text{ for each } i \in [n]. \]

The conditions we have imposed on the pair $(r, M)$ are equivalent to saying that $M$ is the transition probability matrix of a reversible random walk on $G$ with invariant distribution $r = (r_1, \ldots, r_n)$ [38]. Let
\[ z^k(t) = (z^k_1(t), \ldots, z^k_n(t)), \quad k \in [n], \quad t \geq 0, \quad (17) \]
and $r_* \triangleq \min_{1 \leq i \leq n} r_i$. We state the following bound for
\[ \sum_{i=1}^{n} \| z_i(t) - \bar{z}(t) \|^2: \]

**Lemma 2:** Under Assumptions 1 and 3, for the policy in (14)-(15) we have
\[ \sum_{i=1}^{n} \| z_i(t) - \bar{z}(t) \|^2 \leq \frac{nL^2}{r_*^2(1 - \sqrt{T - \lambda})^2} \]
for every $t \geq 1$, where
\[ \| f \|_r \triangleq \sqrt{\sum_{i=1}^{n} r_i f_i^2} \]
is the $r$-weighted $\ell_2$-norm of the vector $f \in \mathbb{R}^n$, and where $\lambda$ denotes the spectral gap of $M$ [38], i.e.,
\[ \lambda = \inf_{f \in \mathbb{R}^n, \langle f, r \rangle = 0} \frac{\| f \|_r^2 - \| Mf \|_r^2}{\| f \|_r^2}. \]

**Proof:** From the definitions of $z_i(t)$, $\bar{z}(t)$, and $z^k(t)$, we have
\[ \sum_{i=1}^{n} \| z_i(t) - \bar{z}(t) \|^2 \leq \sum_{k=1}^{n} \| z^k(t) - \bar{z}^k(t) \|^2. \quad (18) \]

Thus, we upper-bound the quantity on the right-hand side.

From (15), we can rewrite the dynamics (13a) as follows:
\[ z^{k}(t+1) = Mz^k(t) + \frac{1}{r_k} u_k(t) e_k, \quad (19) \]
where $z^k(t)$ is defined in (17). By unrolling the dynamics (19) and (16) from time 0 to $t$ and recalling that $z_i(0) = 0$ for all $i$, we obtain:
\[ z^k(t) = \frac{1}{r_k} \sum_{s=0}^{t-1} M^{t-s-1} u_k(s) e_k. \quad (20) \]

Moreover, by the definition of $\bar{z}(t)$ in Eq. (16), we have
\[ \bar{z}^k(t) = \frac{1}{r_k} \sum_{s=0}^{t-1} r_k u_k(s). \quad (21) \]

Note that $r_k = \langle r, e_k \rangle$. From (20) and (21), we have
\[ \| z^k(t) - \bar{z}^k(t) \| \leq \frac{1}{r_k} \sum_{s=0}^{t-1} \| M^{t-s-1} e_k - \langle r, e_k \rangle e_k \| \| u_k(s) \|. \quad (22) \]

By the properties of Markov matrices [38], for any $f \in \mathbb{R}^n$,
\[ \| M^t f - \langle r, f \rangle 1 \|^2 \leq \frac{1}{r_*} \| M^t f - \langle r, f \rangle 1 \|^2 \]
Therefore,
\[ \| M^{t-s-1} e_k - \langle r, e_k \rangle 1 \|^2 \leq \frac{1}{r_*} \| M^{t-s-1} f - \langle r, f \rangle 1 \|^2 \]
\[ = \frac{1}{r_*} (1 - \lambda)^{t-s-1} \| e_k - \langle r, e_k \rangle 1 \|^2 \]
\[ \leq \frac{(1 - \lambda)^{t-s-1}}{r_*}. \quad (23) \]

From relations (22) and (23), we obtain
\[ \| z^k(t) - \bar{z}^k(t) \|^2 \leq \frac{1}{r_*^3} \sum_{s=0}^{t-1} (1 - \lambda)^{t-s-1} \| u_k(s) \|^2 \]
\[ \leq \frac{nL^2}{r_*^2(1 - \sqrt{T - \lambda})^2}, \]
which proves the stated result. □

Lemma 2 captures the effect of the underlying network topology via the spectral gap $\lambda$ (also known as the Fiedler value), which captures the algebraic connectivity of the network. Since $G$ is assumed to be connected, $\lambda > 0$.

By combining Theorem 2 and Lemma 2 we can now provide a regret bound for $\text{ODA-C}$:

**Theorem 3:** Let Assumptions 1-3 hold. With the choice $\alpha(t) = \frac{1}{r_*^3/2(1 - \sqrt{T - \lambda})}$ for all $t \geq 0$, and under the policy $(14)-(15)$, the distributed algorithm $\text{ODA-C}$ achieves the following regret:
\[ R(T) \leq nL^2 \left( 1 + \frac{2}{r_*^3/2(1 - \sqrt{T - \lambda})} \right) \left( 1 + \frac{\sqrt{nGD}}{L} \right) \sqrt{T} \]
\[ + C \sqrt{T} + 1, \]

**Proof:** By Lemma 2 the averaging policy (15) satisfies
\[ \sum_{i=1}^{n} \| z_i(t) - \bar{z}(t) \|^2 \leq \frac{nL^2}{r_*^2(1 - \sqrt{T - \lambda})^2}. \]

Hence, by Jensen’s inequality,
\[ \sum_{i=1}^{n} \| z_i(t) - \bar{z}(t) \|^2 \leq \frac{nL^2}{r_*^2(1 - \sqrt{T - \lambda})}. \]

Therefore, the conditions of Corollary 2 hold with
\[ K = \frac{nL}{r_*^{3/2}(1 - \sqrt{T - \lambda})}, \]
and the stated result follows. □

This shows that, for any fixed communication network $G$ satisfying Assumption 3, the worst-case regret is bounded by $O(\sqrt{T})$. The constants also capture the dependence on the algebraic connectivity of the network via the spectral gap $\lambda$, as well as on the network size $n$. 

V. ODA-PS AND ITS REGRET BOUND

We now introduce another decentralized online optimization algorithm which uses the push-sum communication protocol for its dual update rule (3a). We refer to this algorithm as ODA-PS (Online Dual Averaging with Push-Sum based communication). ODA-PS uses the network model (G2) for its communication.

A. ODA-PS

For ODA-PS, each agent \( i \) maintains an additional scalar sequence \( \{w_i(t)\}_{t=1}^{\infty} \subset \mathbb{R} \). Then, this algorithm particularizes the update rule in (3a) as

\[
\begin{align*}
w_i(t + 1) &= \sum_{j=1}^{n} A(t)_{ij} w_j(t) \quad (24a) \\
z^k_i(t + 1) &= n\delta^k_i u_i(t) + \sum_{j=1}^{n} A(t)_{ij} z^k_j(t), \quad k \in [n] \quad (24b) \\
x_i(t + 1) &= \Pi_{X^i}^\psi \left( \frac{z_i(t + 1)}{w_i(t + 1)}, \alpha(t) \right) \quad (24c)
\end{align*}
\]

where the weight matrix \( A(t) \) is defined by the out-degrees of the in-neighbors, i.e.,

\[
[A(t)]_{ij} = \begin{cases} 
1/d_j(t) & \text{whenever } j \in N^i_{\text{out}}(t) \\
0 & \text{otherwise.}
\end{cases}
\]

The matrix \( A(t) \) is column stochastic by construction.

Note that the above update rules are based on a simple broadcast communication. Each agent \( i \) broadcasts (or pushes) the quantities \( w_i(t)/d_i(t) \) and \( z_i(t)/d_i(t) \) to all of the nodes in its out-neighborhood \( N^i_{\text{out}}(t) \). Then, in (24a)-(24b) each agent simply sums all the received messages to obtain \( w_i(t + 1) \) and \( z_i(t + 1) \). The update rule (24c) can be executed locally. Unlike ODA-C, the averaging matrix \( A(t) \) in ODA-PS does not require symmetry due to this broadcast-based nature of the push-sum protocol. However, the asymmetry requires uniformity of the positive weights \( r_i \) across all agents (cf. Eq. (5a)). Here we simply use \( r_i = 1/n \).

To complete the description of the algorithm, we must specify the update policies \( \{u_i(t)\} \). As in ODA-C, we assume that the signal agent \( i \) gets from the environment at time \( t \) is simply the \( i \)-th coordinate of the gradient of \( f_i \) at the agents primal variable \( x_i(t) \). Thus, we define:

\[
u_i(t) = \langle \nabla f_i(x_i(t)), e_i \rangle, \quad i \in [n], \quad t \geq 0,
\]

i.e., the update performed by agent \( i \) at time \( t \) is the simply the \( i \)-th coordinate of the gradient of \( f_i \) at the agent’s primal variable \( x_i(t) \).

We assume that each agent \( i \) initializes its updates with \( w_i(0) = 1 \) and \( z_i(0) = 0 \), while \( u_i(0) \) can be any arbitrary value in \( X \). We also recall that the local action of agent \( i \) at time \( t \) is given by the \( i \)-th coordinate of \( x_i(t) \), i.e.,

\[
x^i(t) = x^i_1(t).
\]

For notational convenience, let us denote the products of the weight matrices \( A(t), \ldots, A(s) \) by \( A(t : s) \), i.e.,

\[
A(t : s) \triangleq A(t) \cdots A(s) \quad \text{for all } t \geq s \geq 0.
\]

Also, we denote

\[
A(t - 1 : t) \triangleq I, \quad \text{for all } t \geq 1.
\]

B. Regret of ODA-PS with local gradient signals

For the regret analysis, we first study the dynamics of the dual iterates \( z_i(t) \) and its “mean field” \( \bar{z}(t) \) in the following lemma. We remind that \( \bar{z}(t) = (\bar{z}_1(t), \ldots, \bar{z}_n(t)) \) and

\[
\bar{z}^k(t) = (\bar{z}_1^k(t), \ldots, \bar{z}_n^k(t)), \quad k \in [n].
\]

Lemma 3: Let \( z_i(0) = 0 \) for all \( i \in \mathcal{V} \).

(a) The weighted sum

\[
\bar{z}(t) = \frac{1}{n} \sum_{i=1}^{n} z_i(t)
\]

evolves according to the linear dynamics

\[
\bar{z}(t + 1) = \bar{z}(t) + u(t),
\]

where \( u(t) = (u_1(t), \ldots, u_n(t)) \).

(b) For any \( i, k \in [n] \), the iterates in (24b) evolve according to the following dynamics

\[
z^k_i(t) = n \sum_{s=0}^{t-1} [A(t - 1 : s + 1)]_{ik} u_k(s).
\]

Proof:

(a) From relation (24b), we have for all \( k \in [n] \)

\[
\bar{z}^k(t + 1) = \frac{1}{n} \sum_{i=1}^{n} z^k_i(t + 1)
\]

\[
= \frac{1}{n} \sum_{i=1}^{n} \left[ n\delta^k_i u_i(t) + \sum_{j=1}^{n} [A(t)]_{ij} z^k_j(t) \right]
\]

\[
= u_k(t) + \frac{1}{n} \sum_{j=1}^{n} z^k_j(t) \sum_{i=1}^{n} [A(t)]_{ij}
\]

\[
= u_k(t) + \bar{z}^k(t),
\]

where the last equality follows from the column-stochasticity of the matrix \( A(t) \). The desired result follows by stacking up the scalar relation above over \( k \).

(b) By stacking up the equation (24b) over \( i \), we have for all \( t \geq 1 \) and \( k \in [n] \)

\[
z^k(t + 1) = A(t) \bar{z}^k(t) + nu_k(t) e_k.
\]

By unrolling this equation from time 0 to \( t \), we obtain

\[
z^k(t) = A(t - 1 : 0) \bar{z}^k(0)
\]

\[+ n \sum_{s=0}^{t-1} u_k(s) A(t - 1 : s + 1) e_k
\]

\[= n \sum_{s=0}^{t-1} u_k(s) A(t - 1 : s + 1) e_k,
\]

where the equalities follows from \( A(t - 1 : t) = I \) and the initial condition \( z_i(0) = 0 \) for all \( i \in \mathcal{V} \). We get the desired result by taking the \( i \)-th component of this vector. \( \square \)
Lemma 3 tells us that the vector $\bar{z}(t)$ acts as a “mean field” of the dual iterates $u_i(t)$. Also, if we define

$$\bar{x}(t+1) \triangleq \Pi^\psi_{X_u} (\bar{z}(t+1), \alpha(t)),$$

then from Lemma 3(a) we can see that

$$\bar{x}(t+1) \triangleq \Pi^\psi_{X_u} \left( \sum_{s=1}^{t} u(s), \alpha(t) \right),$$

which coincides with relation (9) in Theorem 1.

We now particularize the bound in Theorem 1 in this scenario under the additional assumption on the Lipschitz continuous gradients (Assumption 2 in Section IV).

Theorem 4: Under Assumptions 1-2, the regret of the algorithm (24a)-(24c) with the local update $u_i(t)$ of agent $i$ computed according to (26) can be upper-bounded as follows: for all $T \geq 1$,

$$R(T) \leq \frac{nL^2}{2} \sum_{t=1}^{T} \alpha(t-1) + \frac{C}{\alpha(T)}$$

$$+ (L + \sqrt{nGD_x}) \sum_{t=1}^{T} \alpha(t-1) \sum_{i=1}^{n} \left\| \frac{z_i(t)}{w_i(t)} - \tilde{z}(t) \right\|,$$

where the last inequality follows from the $\alpha$-Lipschitzian property of the map $\mathbf{z} \mapsto \Pi^\psi_{X_u} (\mathbf{z}, \alpha)$ [3, Lemma 1]. □

This bound tells us that the regret $R(T)$ will be sublinear in $T$ with proper choice of the step size $\alpha(t)$ if the network-wide disagreement term behaves nicely. Note that we can also make use of Corollary 1 here if we can show

$$\sum_{i=1}^{n} \left\| \bar{z}(t) - \frac{z_i(t)}{w_i(t)} \right\| \leq K,$$

for some constant $K > 0$.

C. Full regret analysis

We now show that the network-wide disagreement term in Theorem 4 is indeed upper-bounded by some constant. For doing this, we first restate a lemma from [16].

Lemma 4: Let the graph sequence $\{G(t)\}$ be $B$-strongly connected. Then the following statements are valid:

(a) There is a sequence $\{\phi(t)\} \subseteq \mathbb{R}^n$ of stochastic vectors such that the matrix difference $A(t : s) - \phi(t)1^n$ for $t \geq s$ decays geometrically, i.e., for all $i, j \in [n],$

$$\left\| A(t : s)_{ij} - \phi_i(t) \right\| \leq \beta \theta^{t-s} \quad \text{for all } t \geq s \geq 0,$$

where we can always choose

$$\beta = 4, \quad \theta = (1 - 1/n^{nB})^{1/B}.$$ 

If in addition each $G(t)$ is regular, we may choose

$$\beta = 2\sqrt{2}, \quad \theta = (1 - 1/4n^3)^{1/B},$$

or

$$\beta = \sqrt{2}, \quad \theta = \max_{t \geq 0} \sigma_2(A(t)),$$

whenever $\sup_{t \geq 0} \sigma_2(A(t)) < 1$.

(b) The quantity

$$\gamma = \inf_{t \geq 0} \left( \min_{1 \leq i \leq n} |A(t : 0)\mathbf{1}_i| \right)$$

satisfies

$$\gamma \geq \frac{1}{n^{nB}}.$$ 

Moreover, if the graphs $G(t)$ are regular, we have $\gamma = 1$. The next lemma provides an upper-bound for

$$\sum_{i=1}^{n} \left\| \frac{z_i(t)}{w_i(t)} - \bar{z}(t) \right\|^2.$$

Lemma 5: Let the sequences $\{z_i(t)\}$ and $\{w_i(t)\}$ be generated according to the algorithm (24a)-(24b). Recall that $\bar{z}(t) = \frac{1}{n} \sum_{i=1}^{n} z_i(t).$ Then, we have for all $T \geq 1$,

$$\sum_{i=1}^{n} \left\| \frac{z_i(t)}{w_i(t)} - \bar{z}(t) \right\|^2 \leq n \left\| \frac{z_k(t)}{w_k(t)} - \bar{z}(t) \right\|^2.$$

Thus, we can upper-bound the quantity on the right-hand side. By inspecting equation (24a), it is easy to see that for any $i \in V$ and $t \geq 1$, we have

$$w_i(t) = \sum_{\ell=1}^{t} |A(t - 1 : 0)|_{i\ell} w_i(0) = \sum_{\ell=1}^{t} |A(t - 1 : 0)|_{i\ell}.$$

From this and Lemma 4 we have the following chain of relations:

$$\bar{z}_k(t) - \bar{z}(t)$$

$$= n \sum_{s=0}^{t-1} |A(t - 1 : s + 1)|_{i\ell} u_k(s) - \sum_{s=0}^{t-1} u_k(s)$$

$$= \sum_{s=0}^{t-1} u_k(s) \left( \sum_{\ell=1}^{n} |A(t - 1 : s + 1)|_{i\ell} - \sum_{\ell=1}^{n} |A(t - 1 : 0)|_{i\ell} \right)$$

$$\leq \sum_{s=0}^{t-1} u_k(s) \left( \sum_{\ell=1}^{t} |A(t - 1 : s + 1)|_{i\ell} - \sum_{\ell=1}^{t} |A(t - 1 : 0)|_{i\ell} \right)$$

$$\leq \sum_{s=0}^{t-1} u_k(s) \frac{\beta \theta^{t-s} \gamma}{\theta^{t-1}}.$$  

(29)
where the inequalities follow from adding and subtracting $\phi_i(t-1)$ and from Lemma 4. From relation (26), we have
\[ |u_k(s)|^2 = \|\nabla f_s(x_k(s))\|^2 \leq \|\nabla f_s(x_k(s))\|^2 \leq L^2. \]

Combining this and the fact that $\beta \theta^t-s-2 \geq \beta \theta^t-1$ for all $s = 0, \ldots, t-1$, we further have
\[ \left| z^k(t) - z^k(t) \right| \leq \sum_{s=0}^{t-1} |u_k(s)| \frac{2\beta \theta^t-s-2}{\gamma} \leq \frac{2\beta L}{\gamma(\theta-1)}. \]

Substituting this estimate in relation (28), we get the desired result. □

By combining Theorems 4 and Lemma 5, we can now provide the regret bound of ODA-PS:

**Theorem 5:** Let Assumptions 1 and 2 hold. With the choice $\alpha(t) = \frac{1}{\sqrt{T}}$ for all $t \geq 0$, and under the policy (26), the distributed algorithm ODA-PS achieves the following regret:
\[ R(T) \leq nL^2 \left( 1 + \left( 1 + \frac{\sqrt{nGD\kappa}}{L} \right) \frac{4\beta \sqrt{n}}{\gamma(\theta(1-\theta))} \right) \sqrt{T} + C \sqrt{T+1}, \]
where the constants $\beta$, $\gamma$ and $\theta$ are as defined in Lemma 4.

**Proof:** By Jensen’s inequality, we have
\[ \sum_{t=1}^{n} \left| z_i(t) \right| w_i(t) - \tilde{z}(t) \right| \leq \sum_{t=1}^{n} \left| z_i(t) \right| w_i(t) - \tilde{z}(t) \right|^2. \]

Hence, using Lemma 5, we can estimate the network-wide disagreement term as follows:
\[ \sum_{t=1}^{n} \left| z_i(t) \right| w_i(t) - \tilde{z}(t) \right| \leq \sqrt{n} \sum_{t=1}^{n} \left| z_i(t) \right| w_i(t) - \tilde{z}(t) \right|^2 \cdot \frac{2\beta L}{\gamma(\theta-1)}. \]

Thus, the conditions of Corollary 1 with this modified network-wide agreement hold with
\[ K = n\sqrt{n} \frac{2\beta L}{\gamma(\theta-1)}. \]
and the stated result follows. □

The bound shows that, for any time-varying sequence of B-strongly connected digraphs, the worst-case regret of ODA-PS is of order $O(\sqrt{T})$. The constants also capture the dependence on the properties of the underlying network, i.e., the number of nodes $n$ and as well as the connectivity period $B$.

**VI. SIMULATION RESULTS**

Consider the problem of estimating some target vector $x \in \mathbb{R}^p$ using measurements from a network of $n$ sensors. Each sensor $i$ is in charge of estimating a subvector $x_i \in \mathbb{R}^{p_i}$ of $x$, where $p_i \ll p$ and $p = \sum_{i=1}^{n} p_i$ is some very large number. An example includes the localization of multiple targets, where in this case $x \in \mathbb{R}^p$ becomes a stacked vector of all target locations. When there are a number of spatially dispersed targets, we can certainly benefit from distributed sensing.

**VII. CONCLUSION**

We have studied an online optimization problem in a multiagent network. We proposed two decentralized variants of Nesterov’s primal-dual algorithm, namely, ODA-C using circulation-based dynamics for time-invariant networks and

...
ODA-PS using broadcast-based push-sum protocol for time-varying networks. We have established a generic regret bound and provided its refinements for certain information exchange policies. The regret is shown to grow as $O(\sqrt{T})$ when the step size is $\alpha(t) = 1/\sqrt{T} + t$. For ODA-C, the bound is valid for a static connectivity graph and a row-stochastic matrix $M = [M_{ij}]$ which is reversible with respect to a strictly positive probability vector $r$. For ODA-PS, the bound is valid for a uniformly strongly connected sequence of digraphs and column-stochastic matrices of weights $\lambda(t)$ whose components are based on the out-degrees of neighbors. Simulation results on a sensor network exhibit the desired theoretical properties of the two algorithms.

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