Remarks on the Fundamental Lemma for stable twisted Endoscopy of Classical Groups

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Introduction

These notes go back to the beginning of the century as the fundamental work of Ngo [Ngo] on the fundamental lemma was not known. There is some overlap with the paper of Waldspurger [Wal], which has been written later. We reproduce the paper here for historical reasons and to get it available for the public as a reference.

The original aim of these notes was to prove a fundamental lemma for the stable lift from $H = \text{Sp}_4$ to $\hat{G} = \widetilde{\text{PGL}}_5$ over a local non archimedean field $F$ with residue characteristic $\neq 2$. Here $\widetilde{\text{PGL}}_5 = \text{PGL}_5 \rtimes \langle \Theta \rangle$ is generated by its normal subgroup $\text{PGL}_5$ of index 2 and the involution $\Theta : g \mapsto J \cdot g^{-1} \cdot J^{-1}$, where $J$ is the antidiagonal matrix with entries $(1, -1, 1, -1, 1)$.

We will (Cor. 7.10) prove that if the semisimple elements $\gamma \Theta \in \widetilde{\text{PGL}}_5(F)$ and $\eta \in \text{Sp}_4(F)$ match (see 1.11 for a definition of matching) then the corresponding stable orbital integrals (see 5.1) for the unit elements in the Hecke algebra match:

\[(i)\quad O^{st}_{\gamma \Theta}(1, \widetilde{\text{PGL}}_5) = O^{st}_{\eta}(1, \text{Sp}_4).\]

This theorem will have important applications in the theory of automorphic representations of the group $\text{GSp}_4$ over a number field and for the $l$-adic Galois-representations on the corresponding Shimura varieties [Wei1], [Wei2], [Wes].

In analyzing (i) using the Kazhdan-trick (lemma 5.5 below) we recognized that all essential computations had already been done by Flicker in [Fl2], where the corresponding fundamental lemma for the lift from $\text{GSp}_4 \simeq \text{GSpin}_5$ to $\text{GL}_4 \times \mathbb{G}_m$ has been proved. This phenomenon seems to be known to the experts [Hal3].

More generally one can discuss the fundamental lemma for the stable lift from $H$ to a classical split group with outer automorphism $\hat{G}$, where $H$ is the stable endoscopic group for $\hat{G}$. This fundamental lemma describes a relationship between ordinary stable orbital integrals on the endoscopic group $H$ and $\Theta$-twisted orbital integrals on $\hat{G}$. We will discuss the following lifts in detail:

$$
\begin{align*}
H & \quad \hat{G} \\
\text{Sp}_{2n} & \quad \text{PGL}_{2n+1} \rtimes \langle g \mapsto J^t g^{-1} J^{-1} \rangle \\
\text{GSpin}_{2n+1} & \quad \text{GL}_{2n} \times \mathbb{G}_m \rtimes \langle (g, a) \mapsto (J^t g^{-1} J^{-1}, \det g \cdot a) \rangle \\
\text{Sp}_{2n} & \quad SO_{2n+2} \simeq O_{2n+2}.
\end{align*}
$$

In each case we will reduce the fundamental lemma using the Kazhdan trick and a lot of observations in linear algebra to a statement which we call the $BC$-conjecture and which seems to be proven only for $n = 1, 2$.
**Conjecture:** If the regular topologically unipotent and algebraically semisimple elements $u \in SO_{2n+1}(\mathcal{O}_F)$ and $v \in Sp_{2n}(\mathcal{O}_F)$ are $BC$-matching (see \text{[1.12]}) then

$$(BC_n) \quad O^\text{st}_u(1, SO_{2n+1}) = O^\text{st}_v(1, Sp_{2n}).$$

Thus the ($\Theta$-twisted) fundamental lemmas for the three series of endoscopy will be reduced to a fundamental lemma like statement for ordinary (i.e. untwisted) stable orbital integrals on the groups $SO_{2n+1}$ of type $B_n$ resp. $Sp_{2n}$ of type $C_n$. An outline of the proofs will be given in chapter 2.
1 Stable endoscopy and matching

(1.1) Notations. In this paper we will denote by $F$ a $p$-adic field with ring of integers $\mathcal{O}_F$, prime ideal $p$ and uniformizing element $\varpi = \varpi_F$. The residue field of characteristic $p$ is denoted $\kappa = \kappa_F = \mathcal{O}_F/p$. By $\bar{F}$ we denote an algebraic closure of $F$. In the whole paper we will assume that $p \neq 2$. Only in this chapter $R$ denotes an arbitrary integral domain.

(1.2) Split Groups with automorphism. Let $G/R$ be a connected reductive split group scheme. We fix some "splitting" i.e. a triple $(B, T, \{X_\alpha\}_{\alpha \in \Delta})$ where $T$ denotes a maximal split-torus inside a rational Borel $B$, $\Delta = \Delta_G = \Delta(G, B, T) \subset \Phi(G, T) \subset X^*(T)$ the set of simple roots inside the system of roots and the $X_\alpha$ are a system (nailing) of isomorphisms between the additive group scheme $F$ and $B$.

Let $\Theta$ be a connected reductive split group scheme. We fix some "splitting" i.e. a tripel $(B, T, \{X_\alpha\}_{\alpha \in \Delta})$ where $T$ denotes a maximal split-torus inside a rational Borel $B$, $\Delta = \Delta_G = \Delta(G, B, T) \subset \Phi(G, T) \subset X^*(T)$ the set of simple roots inside the system of roots and the $X_\alpha$ are a system (nailing) of isomorphisms between the additive group scheme $G$ and the unipotent root subgroups $B_\alpha$. Here $X^*(T) = Hom(T, \mathbb{G}_m)$ denotes the character module of $T$, while $X_*(T) = Hom(\mathbb{G}_m, T)$ will denote the cocharacter module of $T$. Let $\Theta \in Aut(G)$ be an automorphism of $G$ which fixes the splitting, i.e. stabilizes $B$ and $T$ and permutes the $X_\alpha$. We assume $\Theta$ to be of finite order $l$. We denote by

$$\hat{G} = G \rtimes \langle \Theta \rangle$$

the (nonconnected) semidirect product of $G$ with $\Theta$. $\Theta$ acts on the (co)character module via $X_*(T) \ni \alpha^\vee \mapsto \Theta \circ \alpha^\vee$ resp. $X^*(T) \ni \alpha \mapsto \alpha \circ \Theta^{-1}$.

(1.3) The dual group. Let $\hat{G} = \hat{G}(\mathbb{C})$ be the dual group of $G$. By definition $\hat{G}$ has a tripel $(\hat{B}, \hat{T}, \{\hat{X}_\alpha\})$ such that we have identifications $X^*(\hat{T}) = X_*(\hat{T}), \ X_*(\hat{T}) = X^*(\hat{T})$ which identifies the (simple) roots $\alpha \in X^*(\hat{T})$ with the (simple) coroots $\alpha^\vee \in X_*(\hat{T})$, and the (simple) coroots $\hat{\alpha}^\vee \in X_*(\hat{T})$ with the (simple) roots $\alpha \in X^*(\hat{T})$. There exists a unique automorphism $\hat{\Theta}$ of $\hat{G}$ which stabilizes $(\hat{B}, \hat{T}, \{\hat{X}_\alpha\})$ and induces on $(X_*(\hat{T}), X^*(\hat{T}))$ the same automorphism as $\Theta$ on $(X^*(T), X_*(T))$.

(1.4) The $\Theta$-invariant subgroup in $\hat{G}$. Let $\hat{H} = (\hat{G}^{\hat{\Theta}})^0$ be the connected component of the subgroup of $\Theta$-fixed elements in $\hat{G}$. It is a reductive split group with tripel $(\hat{B}_H, \hat{T}_H, \{\hat{X}_\beta\}_{\beta \in \Delta_H})$, where $\hat{B}_H = \hat{B}^{\hat{\Theta}}, \ \hat{T}_H = \hat{T}^{\hat{\Theta}}$ and the nailing will be explained soon. We have the inclusion of cocharacter modules $X_*(\hat{T}_H) = X_*(\hat{\Theta}) \subset X_*(\hat{T})$ and a projection for the character module

$$P_\Theta : X^*(\hat{T}) \rightarrow (X^*(\hat{T})_\Theta)_{free} = X^*(\hat{T}_H),$$

where $(X^*(\hat{T})_\Theta)_{free}$ denotes the maximal free quotient of the coinvariant module $X^*(\hat{T})_\Theta$. For a $\mathbb{Z}[\Theta]$-module $X$ we define a map

$$S_\Theta : X \rightarrow X^\Theta, \ x \mapsto \sum_{i=0}^{ord_\Theta(x)-1} \Theta^i(x)$$
where \( \text{ord}_x(\Theta) = \min\{i > 0 \mid \Theta^i(x) = 0\} \) is length of the orbit \( \langle \Theta \rangle(x) \), which may vary on \( X \).

For the roots \( \Phi \) and coroots \( \Phi^\vee \) of a given root datum \( (X^*, X_*, \Phi, \Phi^\vee) \) we have to introduce a modified map \( S'_\Theta \) by

\[
S'_\Theta(\alpha) = c(\alpha) \cdot S_\Theta(\alpha) \quad \text{where} \quad c(\alpha) = \frac{2}{\langle \alpha^\vee, S_\Theta(\alpha) \rangle}
\]

resp. by the formula where the roles of \( \alpha \) and \( \alpha^\vee \) are exchanged. For all simple root systems with automorphisms which are not of type \( A_{2n} \) we have \( \langle \alpha^\vee, S_\Theta(\alpha) \rangle = 0 \) for \( i = 1, \ldots, \text{ord}_x(\Theta) - 1 \) which implies \( c(\alpha) = 1 \) i.e. \( S'_\Theta(\alpha) = S_\Theta(\alpha) \). We furthermore introduce the subset of short-middle roots and the dual concept of long-middle coroots:

\[
\Phi(\hat{G}, \hat{T})^{sm} = \left\{ \alpha \in \Phi(\hat{G}, \hat{T}) \mid \frac{1}{2} \cdot P_\Theta(\alpha) \notin P_\Theta(\Phi(\hat{G}, \hat{T})) \right\}
\]

\[
\Phi^\vee(\hat{G}, \hat{T})^{lm} = \left\{ \alpha^\vee \mid \alpha \in \Phi(\hat{G}, \hat{T})^{sm} \right\}
\]

**Proposition 1.5.** With the above notations we have

(i) \( \Phi(\hat{H}, \hat{T}_H) = P_\Theta(\Phi(\hat{G}, \hat{T})^{sm}) \) for the roots

(ii) \( \Phi^\vee(\hat{H}, \hat{T}_H) = S'_\Theta(\Phi^\vee(\hat{G}, \hat{T})^{lm}) \) for the coroots

\( \Delta^\vee_H = \Delta^\vee(\hat{H}, \hat{B}_H, \hat{T}_H) = S'_\Theta(\Delta^\vee_G) \) for the simple coroots

\( \Delta_H = \Delta(\hat{H}, \hat{B}_H, \hat{T}_H) = P_\Theta(\Delta_G) \) for the simple roots

Proof: This follows from [St, 8.1], which is restated in Theorem 2.1 below. \( \square \)

**Definition 1.6** (stable \( \Theta \)-endoscopic group). In the above situation a connected reductive split group scheme \( H/R \) will be called a stable \( \Theta \)-endoscopic group for \( (G, \Theta) \) resp. \( \hat{G} \) if its dual group is together with the splitting isomorphic to the above \( (\hat{H}, \hat{B}_H, \hat{T}_H, \{X_\beta\}_{\beta \in \Delta_H}) \).

Remarks: Since \( H \) is unique up to isomorphism (up to unique isomorphism if we consider \( H \) together with a splitting) we can call \( H \) the stable \( \Theta \)-endoscopic group for \( (G, \Theta) \). For a maximal split-torus \( T_H \subset H \) we have:

(iii) \( X_*(T_H) = X_*(T)_{\Theta} \) for the cocharacter-module

\( X^*(T_H) = X^*(T)^{\Theta} \) for the character-module

(1.7) To get examples we use the following notations:

\( \text{diag}(a_1, \ldots, a_n) \in \text{GL}_n \) denotes the diagonal matrix \( (\delta_{i,j} \cdot a_i)_{ij} \) and
antidiag($a_1, \ldots, a_n$) $\in$ GL$_n$ the antidiagonal matrix $(\delta_{i,n+1-j} \cdot a_i)_{ij}$ with $a_1$ in the upper right corner. We introduce the following matrix

$$ J = J_n = (\delta_{i,n+1-j} (-1)^{i-1})_{1 \leq i,j \leq n} = \text{antidiag}(1, -1, \ldots, (-1)^{n-1}) \in$ GL$_n(R).$$

and its modification $J'_n = \text{antidiag}(1, -1, 1, \ldots, (-1)^{n-1}, (-1)^{n-1}, \ldots, 1, -1, 1).$ Since $t J_n = (-1)^{n-1} J_n$ and $J'_n$ is symmetric we can define the

| Standard Symplectic Group | Sp$_{2n}$ | $\text{Sp}(J_{2n})$ |
|---------------------------|-----------|----------------------|
| Standard Split Odd Orthogonal Group | SO$_{2n+1}$ | $\text{SO}(J_{2n+1})$. |
| Standard Split Even Orthogonal Group | SO$_{2n}$ | $\text{SO}(J'_{2n})$. |

We consider the groups GL$_n$, SL$_n$, PGL$_n$, Sp$_{2n}$, SO$_n$ with the splittings consisting of the diagonal torus, the Borel consisting of upper triangular matrices and the standard nailing. We remark that the following map defines an involution of GL$_n$, SL$_n$ and PGL$_n$: \[ \Theta = \Theta_n : g \mapsto J_n \cdot t g \cdot J_n^{-1}. \]

**Example 1.8** ($A_{2n} \leftrightarrow C_n$).

$G = \text{PGL}_{2n+1}, \ \Theta = \Theta_{2n+1} \quad \text{has dual} \quad \hat{G} = \text{SL}_{2n+1}(\mathbb{C}), \ \hat{\Theta} = \Theta_{2n+1}$

$H = \text{Sp}_{2n} \quad \text{has dual} \quad \hat{H} = \text{SO}_{2n+1}(\mathbb{C})$

**Example 1.9** ($A_{2n-1} \leftrightarrow B_n$). The group $G = \text{GL}_{2n} \times \mathbb{G}_m$ has the automorphism \[ \Theta : (g, a) \mapsto (\Theta_{2n}(g), \det(g) \cdot a) \]

which is an involution since $\det(\Theta_{2n}(g)) = \det g^{-1}$. The dual $\hat{\Theta} \in \text{Aut}(\hat{G})$ satisfies \[ \hat{\Theta}(g, b) = (\Theta_{2n}(g) \cdot b, b), \] so that we get

$G = \text{GL}_{2n} \times \mathbb{G}_m, \ \Theta \quad \text{has dual} \quad \hat{G} = \text{GL}_{2n}(\mathbb{C}) \times \mathbb{C}^\times, \ \hat{\Theta}$

$H = \text{GSpin}_{2n+1} \quad \text{has dual} \quad \hat{H} = \text{GSp}_{2n}(\mathbb{C})$.

Recall that GSpin$_{2n+1}$ can be realized as the quotient $(\mathbb{G}_m \times \text{Spin}_{2n+1}) / \mu_2$, where $\mu_2 \simeq \{ \pm 1 \}$ is embedded diagonally, so that we get an exact sequence \[ 1 \rightarrow \text{Spin}_{2n+1} \rightarrow \text{GSpin}_{2n+1} \overset{\mu}{\rightarrow} \mathbb{G}_m \rightarrow 1, \]

where the "multiplier" map $\mu$ is induced by the projection to the $\mathbb{G}_m$ factor followed by squaring. Thus the derived group of GSpin$_{2n+1}$ is Spin$_{2n+1}$, i.e. a connected, split and simply connected group.
Example 1.10 \((D_{n+1} \leftrightarrow C_n)\). We furthermore have the situation:

\[
G = \text{SO}_{2n+2}, \quad \Theta = \text{int}(s) \quad \text{has dual} \quad \hat{G} = \text{SO}_{2n+2}(\mathbb{C}), \quad \hat{\Theta} = \text{int}(\hat{s})
\]

\[
H = \text{Sp}_{2n} \quad \text{has dual} \quad \hat{H} = \text{SO}_{2n+1}(\mathbb{C})
\]

where \(s \in O_{2n+2}\) denotes the reflection which interchanges the standard basis vectors \(e_{n+1}\) and \(e_{n+2}\) and fixes all other basis elements \(e_i\). The \(\hat{s}\) is of the same shape.

(1.11) Matching elements Since each semisimple \(\Theta\)-conjugacy resp. conjugacy class in \(G(\bar{F})\) resp. \(H(\bar{F})\) meets \(T(\bar{F})\) resp. \(T_H(\bar{F})\) we have canonical bijections

\[
i_G : \frac{G(\bar{F})_{ss}}{\Theta - \text{conj}} \cong \frac{T(\bar{F})_\Theta}{(W_G)^\Theta}
\]

\[
i_H : \frac{H(\bar{F})_{ss}}{\text{conj}} \cong \frac{T_H(\bar{F})}{W_H}
\]

where \(W_G = \text{Norm}(T,G)/T\) and \(W_H = \text{Norm}(T_H,H)/T_H\) denote the Weyl-groups. We further have an isomorphism

\[
N_{KS} : T(\bar{F})_\Theta \simeq (X_*(T) \otimes \bar{F}^\times)_\Theta \simeq X_*(T_H) \otimes \bar{F}^\times \simeq T_H(\bar{F})
\]

and observe \(W_H \simeq (W_G)^\Theta\). Therefore we may define:

Two \((\Theta-)\)semisimple elements \(\gamma \Theta \in G(\bar{F})\Theta\) and \(h \in H(\bar{F})\) are called matching if their \((\Theta-)\)stable conjugacy classes in \(G(\bar{F})\) resp. \(H(\bar{F})\) correspond to each other via the isomorphism \(i_H^{-1} \circ N_{KS} \circ i_G\).

(1.12) BC-matching: We have an isomorphism between the standard diagonal tori:

\[
i_{BC} : T_{SO_{2n+1}} \rightarrow T_{Sp_{2n}},
\]

\[
diag(t_1, \ldots , t_n, 1, t_n^{-1}, \ldots , t_1^{-1}) \mapsto diag(t_1, \ldots , t_n, t_n^{-1}, \ldots , t_1^{-1}).
\]

We observe that \(i_{BC}\) induces an isomorphism of Weyl groups:

\[
W_{SO_{2n+1}} \simeq S_n \ltimes \{\pm 1\}^n \simeq W_{Sp_{2n}}
\]

Two semisimple elements \(h \in SO_{2n+1}(F)\) and \(b \in Sp_{2n}(F)\) are called BC-matching if their stable conjugacy classes correspond to each other under the isomorphism \(i_{Sp_{2n}}^{-1} \circ i_{BC} \circ i_{SO_{2n+1}}\).

Example 1.13. In example [LS] above the norm map \(N_{KS} : T \rightarrow T_\Theta \simeq T_H\) is given by

\[
(iv) \quad \gamma = diag(t_1, t_2, \ldots , t_n, t_n+1, t_{n+2}, \ldots , t_{2n+1}) \in T \subset \text{PGL}_{2n+1}
\]

\[
\mapsto h = diag(t_1/t_{2n+1}, t_2/t_{2n}, \ldots , t_n/t_n+2, t_{n+2}/t_n, \ldots , t_{2n+1}/t_1) \in T_H \subset \text{Sp}_{2n}.
\]
Proof: We identify $X_*(T) \simeq \mathbb{Z}^{2n+1}/\mathbb{Z} \simeq \bigoplus_{i=1}^{2n+1} \mathbb{Z}f_i/\mathbb{Z}\sum_{i=1}^{2n+1} f_i$ and $X^*(T) \simeq \{ \sum_{i=1}^{2n+1} \alpha_i e_i | \sum_{i=1}^{2n+1} \alpha_i = 0 \}$, such that

$$f_i : t \mapsto \text{diag}(1, \ldots, 1, t, 1, \ldots, 1) \in T \quad \text{and}$$

$$e_i - e_j : T \ni \text{diag}(t_1, \ldots, t_{2n+1}) \mapsto t_i/t_j.$$ 

Similarly we identify $T_H \simeq G_m^n$ via $\text{diag}(t_1, \ldots, t_n, t_1^{-1}, \ldots, t_n^{-1}) \mapsto (t_i)_{1 \leq i \leq n}$ and write $X_*(T_H) \simeq X_*(G_m^n) \simeq \bigoplus_{i=1}^{n} \mathbb{Z}f_i'$ resp. $X^*(T_H) \simeq X^*(T_H) \simeq \bigoplus_{i=1}^{n} \mathbb{Z}\epsilon_i'$. The involution $\Theta$ acts as

$$\Theta(f_i) = -f_{2n+2-i}, \quad \Theta(e_i - e_j) = e_{2n+2-j} - e_{2n+2-i}.$$ 

Now it is clear that we have an identification $P_\Theta : X_*(T)_\Theta \simeq X_*(T_H)$ given by $P_\Theta(f_i) = f_i'$ and $P_\Theta(f_{2n+2-i}) = -f_i'$ for $1 \leq i \leq n$. This forces $P_\Theta(f_{n+1}) = P_\Theta(-\sum_{i \neq n+1} f_i) = P_\Theta(\Theta(\sum_{i=1}^{n} f_i) - \sum_{i=1}^{n} f_i) = 0$. Dual to this we have an injection $\iota : X^*(T_H) \simeq X^*(T)^\Theta \subset X^*(T)$ such that $\iota(e_i') = e_i - e_{2n+2-i}$. It is furthermore clear that this $P_\Theta$ induces the map \hspace{1cm} \text{(v)} The claim now follows if show that $P_\Theta$ and $\iota$ correspond to the natural maps on the side of the dual groups which are characterized by the equations of Proposition \hspace{1cm} \text{(i)} and \hspace{1cm} \text{(ii)} for our explicitly given maps $P_\Theta$ and $\iota$, namely the equations

\begin{align*}
\text{(v)} & \quad \Phi^\vee(H, T_H) = P_\Theta(\Phi^\vee(G, T)^{\text{sm}}) \quad \text{for the coroots} \\
\text{(vi)} & \quad \iota(\Phi(H, T_H)) = S_\Theta(\Phi(G, T)^{\text{sm}}) \quad \text{for the roots}.
\end{align*}

But we have

\begin{align*}
\text{(vii)} & \quad P_\Theta(\pm (f_i - f_j)) = \pm (f_i' - f_j') = P_\Theta(\pm (f_{2n+2-j} - f_{2n+2-i})), \\
\text{(viii)} & \quad P_\Theta(\pm (f_i - f_{2n+2-j})) = \pm (f_i' + f_j'), \\
& \quad P_\Theta(\pm (f_i - f_{n+1})) = \pm f_i' = P_\Theta(\pm (f_{n+1} - f_{2n+2-i})),
\end{align*}

where $1 \leq i, j \leq n$ in all three equations, but where additionally $i \neq j$ in \textbf{(vii)} while $i = j$ is allowed in \textbf{(viii)}. Nevertheless $P_\Theta(\pm (f_i - f_{2n+2-i})) = 2 \cdot f_i'$ is not a member of the right hand side of \textbf{(v)} since $f_i' = P_\Theta(f_i - f_{n+1})$. By the well known description of $\Phi^\vee(\text{Sp}_{2n}, T_H)$ we get the equality \textbf{(v)}.

Similarly we get

\begin{align*}
S_\Theta(\pm (e_i - e_j)) & = \iota(\pm (e_i' - e_j')) = S_\Theta(\pm (e_{2n+2-j} - e_{2n+2-i})), \\
S_\Theta(\pm (e_i - e_{2n+2-j})) & = \iota(\pm (e_i' + e_j')), \\
S_\Theta(\pm (e_i - e_{n+1})) & = \pm (e_i - e_{2n+2-i}) = S_\Theta(\pm (e_{n+1} - e_{2n+2-i})), \\
& = \pm \iota(e_i') = S_\Theta(\pm (e_i - e_{2n+2-i})).
\end{align*}

where $1 \leq i \neq j \leq n$ in the first two equations and $1 \leq i \leq n$ in the last two. Since $\Phi(\text{Sp}_{2n}, T_H) = \{ e_i' - e_j' | 1 \leq i \neq j \leq n \} \cup \{ e_i' + e_j' | 1 \leq i \neq j \leq n \} \cup \{ 2 \cdot e_i' | 1 \leq
Let \( \mathfrak{g} \) be a simple Lie algebra and \( e \) be a basis for \( \mathfrak{g} \). We consider the following basis (dual basis of \( e \)) for the (co)root systems we get may define a surjection \( P_{\Theta} \) maps the set of simple roots \( \{ e_i' - e_i' | 1 \leq i \leq n - 1 \} \) of \( H \) and that \( P_{\Theta} \) maps the set of simple coroots \( \{ f_i' - f_i' | 1 \leq i \leq n \} \) of \( G \) to the set of simple coroots \( \{ f_i' - f_i' | 1 \leq i \leq n - 1 \} \) of \( G \) to the set of simple coroots \( \{ f_i' - f_i' | 1 \leq i \leq n - 1 \} \). \( \square \)

Example 1.14. In example 139 above we consider additionally the projection \( pr_{ad} : GSpin_{2n+1} \to SO_{2n+1} = \text{Spin}_{2n+1}/\{ \pm 1 \} \). Then the composite map \( pr_{ad} \circ N_{KS} : T \to T_{ad} \subset SO_{2n+1} \) is given by

\[
\begin{align*}
(i) \quad & \gamma = (\text{diag}(t_1, t_2, \ldots, t_n, t_{n+1}, \ldots, t_{2n}), t_0) \quad \in T \subset GL_{2n} \times \mathbb{G}_m \\
& \mapsto h = \text{diag}(t_1/t_2, \ldots, t_n/t_{n+1}, 1, t_{n+1}/t_n, \ldots, t_{2n}/t_1) \in T_{ad} \subset SO_{2n+1}.
\end{align*}
\]

Proof: We consider the following basis \((e_i)_{0 \leq i \leq 2n}\) of \( X^*(T) \):

\[
\begin{align*}
& e_i : \quad T \ni (\text{diag}(t_1, \ldots, t_{2n}), t_0) \mapsto t_i.
& \Theta(e_i) = -e_{2n+1-i} \quad \Theta(e_0) = e_0 + \sum_{i=1}^{2n} e_i
\end{align*}
\]

Now it is clear that \( X^*(T_H) = X^*(T)^{\Theta} \) has as basis \((e'_0, e'_1, \ldots, e'_n)\) where \( e'_i = e_i - e_{2n+1-i} \) for \( 1 \leq i \leq n \) and \( e'_0 = e_0 + \sum_{i=1}^{2n} e_i \). Let \((f'_0, f'_1, \ldots, f'_n)\) be the dual basis of \( X_*(T_H) = X_*(T)^{\Theta} \). Then the projection map \( P_{\Theta} : X_*(T) \to X_*(T)^{\Theta} \) satisfies: \( P_{\Theta}(f'_0) = f'_0 \), \( P_{\Theta}(f'_i) = f'_i \), \( P_{\Theta}(f_{2n+1-i}) = -f'_i + f'_0 \) for \( 1 \leq i \leq n \).

For the (co)root systems we get:

\[
\begin{align*}
& \Phi(GSpin_{2n+1}, T_{ad}) = P_{\Theta}(\Phi(GL_{2n} \times \mathbb{G}_m, T)) \\
& \quad = \{ \pm e'_i | 1 \leq i \leq j \leq n \} \cup \{ e'_i | 1 \leq i \leq n \} \\
& \Phi'(GSpin_{2n+1}, T_{ad}) = S_{\Theta}(\Phi'(GL_{2n} \times \mathbb{G}_m, T)) \\
& \quad = \{ \pm (f'_j - f'_i) | 1 \leq i < j \leq n \} \cup \{ \pm (f'_i + f'_j - f'_0) | 1 \leq i < j \leq n \}.
\end{align*}
\]

The cocharacter group of the center of \( GSpin_{2n+1} \) is then recognized as \( \mathbb{Z}f'_0 \). Thus we may define a surjection \( pr_{ad} : X_*(T_{ad}) \to X_*(T_{ad}) \) by \( f'_0 \mapsto 0 \) and \( f'_i \mapsto f'_i \) for \( 1 \leq i \leq n \). Dually one has the injection \( \iota : X^*(T_{ad}) \hookrightarrow X^*(T_H) \), \( e_i \mapsto e'_i \). Now it is clear that \( pr_{ad} \circ P_{\Theta} \) induces the map \( \Phi' \) and it remains to show that we have the following relations analogous to (v) and (vi):

\[
\begin{align*}
& (x) \quad \Phi'(SO_{2n+1}, T_{ad}) = pr_{ad} \circ P_{\Theta}(\Phi'(G, T)^{sm}) \quad \text{for the coroots} \\
& (xi) \quad \iota(\Phi(SO_{2n+1}, T_{ad})) = S_{\Theta}(\Phi(G, T)^{lm}) \quad \text{for the roots}.
\end{align*}
\]
But this follows immediately from the above description of $P_\Theta(\Phi(GL_{2n} \times \mathbb{G}_m, T))$ and $S'_\Theta(\Phi^\vee(GL_{2n} \times \mathbb{G}_m, T))$ in view of the very simple shape of $pr_{ad}$ and $i$ and the knowledge of the (co)root system of $SO_{2n+1}$. The relation for the bases of the (co)root systems is checked in a similar way.

**Example 1.15.** In example (1.9) above we now analyze the relation between the multiplier map $\mu$ and matching. We claim: If $(h, a) \in GL_{2n}(F) \times F^\times$ and $\eta \in GSpin_{2n+1}(F)$ match then we have:

$$\mu(\eta) = \det(h) \cdot a^2.$$ 

Proof: In the notations of (1.14) the element $e' = 2e'_0 + \sum_{i=1}^n e'_i = 2e_0 + \sum_{i=1}^{2n} e_i \in X^*(T_H) = X^*(T)^\Theta$ corresponds to the character $(h, a) \mapsto \det(h) \cdot a^2$. Since $e'$ is orthogonal to the coroots $\Phi^\vee(GSpin_{2n+1}, T_H)$ it has to correspond to a multiple of the multiplier $\mu$. Now it is easy to see that it corresponds in fact to $\mu$.

**Example 1.16.** In example (1.10) above the norm map $N_{KS}: T \to T_\Theta \simeq T_H$ is given by

(xii) $\gamma = \text{diag}(t_1, t_2, \ldots, t_n, t_{n+1}, t_{n+1}^{-1}, \ldots, t_1^{-1}) \in T \subset SO_{2n+2}$

$$\mapsto h = \text{diag}(t_1, \ldots, t_n, t_{n+1}^{-1}, \ldots, t_1^{-1}) \in T_H \subset \text{Sp}_{2n}.$$ 

Proof: We consider the following basis $(e_i)_{1 \leq i \leq n+1}$ of $X^*(T)$:

$$e_i: T \ni \text{diag}(t_1, \ldots, t_n, t_{n+1}, \ldots, t_1^{-1}) \mapsto t_i.$$ 

Let $(f_i)_{1 \leq i \leq n+1}$ be the dual basis of $X_*(T)$. The involution $\Theta$ acts via

$$\Theta(e_i) = e_i, \quad \Theta(f_i) = f_i \quad \text{for } 1 \leq i \leq n$$

$$\Theta(e_{n+1}) = -e_{n+1}, \quad \Theta(f_{n+1}) = -f_{n+1}.$$ 

We furthermore use the bases $(e'_i)_{1 \leq i \leq n}$ of $X^*(T_H)$ and $(f'_i)_{1 \leq i \leq n}$ of $X_*(T_H)$ from example (1.13). It is clear that we have an isomorphism $\iota: X^*(T_H) \cong X^*(T)^\Theta$ given by $e'_i \mapsto e_i$ and a dual isomorphism $P_\Theta : (X_*(T)^\Theta)_\text{free} \cong X_*(T_H)$ induced by the dual map $P_\Theta(f_i) \mapsto f_i'$ for $1 \leq i \leq n$. It is clear that this $P_\Theta$ induces the map (xii).

Recall the (co)root systems of $SO_{2n+2}$

$$\Phi(SO_{2n+2}, T) = \{\pm e_i \pm e_j \mid 1 \leq i < j \leq n + 1\}$$

$$\Phi^\vee(SO_{2n+2}, T) = \{\pm f_i \pm f_j \mid 1 \leq i < j \leq n + 1\}.$$ 

We have $P_\Theta(f_i \pm f_j) = f'_i \pm f'_j$, $S'_\Theta(e_i \pm e_j) = \iota(e'_i \pm e'_j)$ for $1 \leq i < j \leq n$ and $P_\Theta(f_i \pm f_{n+1}) = f'_i$, $S'_\Theta(e_i \pm e_{n+1}) = e_i \pm e_{n+1} + \Theta(e_i \pm e_{n+1}) = \iota(2 \cdot e'_i)$ for $1 \leq i \leq n$. The relations (vi) and (vii) now follow from the knowledge of the (co)root system of $\text{Sp}_{2n}$. It is clear that $S'_\Theta$ maps the simple roots $e_i - e_{i+1}, e_n \pm e_{n+1}$ of $SO_{2n+2}$ to the simple roots $e'_i - e'_{i+1}, 2 \cdot e'_n$ of $\text{Sp}_{2n}$ and $P_\Theta$ maps the simple coroots $f_i - f_{i+1}, f_n \pm f_{n+1}$ of $SO_{2n+2}$ to the simple coroots $f'_i - f'_{i+1}, f'_n$ of $\text{Sp}_{2n}$.

\qed
2 Centralizers

Theorem 2.1 (Steinberg). Let $T$ be a $\Theta$-stable, maximal subtorus of $G$ and $t \in T(\overline{F})$. The group $G^{\Theta}$ of fixed point in $G$ under $\text{int}(t) \circ \Theta$ is reductive. The root system of the connected component $(G^{\Theta})^\circ$ of 1, viewed as a subsystem of $\Phi_\Theta = P_\Theta(\Phi(T, G))$, which might be identified with $\Phi_{\text{res}} = \{\alpha | T^{\Theta_0} \mid \alpha \in \Phi\}$, is given by

$$\Phi(G^{\Theta}, T^{\Theta_0}) \simeq \left\{ P_\Theta(\alpha) \in \Phi_\Theta \mid S_\Theta(\alpha)(t) = \begin{cases} 1 & \text{if } \frac{1}{2} P_\Theta(\alpha) \not\in \Phi_\Theta \\ -1 & \text{if } \frac{1}{2} P_\Theta(\alpha) \in \Phi_\Theta \end{cases} \right\}.$$ 

Proof: [St, 8.1] \hfill \Box

(2.2) Define $N_\Theta(t) = \prod_{i=0}^{\text{ord}(\Theta)-1} \Theta^i(t)$. In case $\Phi(G)$ is of type $A_{2n}$ or one $\Theta$-Orbit of components of type $A_{2n}$ the root system $\Phi(G^{\Theta}, T^{\Theta_0})$ in 2.1 is a maximal reduced subsystem of

$$\{P_\Theta(\alpha) \mid \alpha(N_\Theta(t)) = 1\}.$$ 

(2.3) For an irreducible root system $\Phi$ with basis $\Delta$ and $\theta \in \text{Aut}(\Phi, \Delta)$ we denote by $\tilde{\alpha} \in \Phi^-$ the negative root such that $-\text{c}(\tilde{\alpha}) S_\Theta(\tilde{\alpha}) = -S'_\Theta(\tilde{\alpha})$ is the highest root in $S'_\Theta(\Phi^+)$ with respect to the basis $S'_\Theta(\Delta)$.

To $(\Phi, \Delta, \Theta)$ we associate the extended Dynkin diagram

$$\Delta_{\text{ext}}(\Phi, \Theta) := P_\Theta(\Delta) \cup \{c(\tilde{\alpha})P_\Theta(\tilde{\alpha})\}$$

Proposition 2.4 (Dynkin). Let $G$ and $\Theta$ be as in 1.2

(a) For every $t \in T$ there exists $w \in W^\Theta$, such that $\Phi((G^{w(t)}\Theta)^\circ)$ has a basis in $\Delta_{\text{ext}}(\Phi, \Theta)$.

(b) Every proper subsystem of $\Delta_{\text{ext}}(\Phi, \Theta)$ occurs as $\Phi((G^{\Theta})^\circ)$ for some $t \in T$.

Proof: For a detailed proof we refer to [Bal] [2.42]. We remark that in case $\Phi = A_{2n}$ the extended Dynkin diagram $(P_\Theta(\Phi), \Delta_{\text{ext}}(\Phi, \Theta))$ looks like

\[ \bullet \rightarrow \cdots \rightarrow \circ \rightarrow \cdots \rightarrow \circ \rightarrow \cdots \rightarrow \circ \rightarrow \cdots \rightarrow \bullet \]

Therefore (a) will follow in the case $G = \text{SL}_{2n+1}$ from the fact, proven in lemma 4.9 that the groups $G^{\Theta}$ are isomorphic to groups of the form

$$G^{\Theta} \simeq \text{Sp}(2m_k) \times \text{Gl}(m_{k-1}) \times \cdots \times \text{Gl}(m_1) \times \text{SO}(2m_0 + 1).$$
where \((m_k, m_{k-1}, \ldots, m_1, m_0)\) runs through all partitions of \(n\) with \(m_k, m_0 \geq 0\) and \(m_i \geq 1\) for \(0 < i < k\). This means that the \(G^\Theta\) are of type \(C_{m_k} \times A_{m_{k-1}} \times \cdots \times A_{m_1} \times B_{m_0}\) and the \(\Delta(G^\Theta)\) are subsets of \(P_\Theta(\Delta) \cup \{-P_\Theta(\alpha^+)\}\), where \(\alpha^+ = \varepsilon_1 - \varepsilon_{2n+1}\) is the highest root for \(G\).

\((2.5)\) In the following table we list all simple root systems \(\Phi = \Phi(G)\), such that the semisimple (simply connected split) group \(G\) has an outer automorphism \(\Theta\), together with the root systems \(\Phi(H)\) of the stable endoscopic groups \(H\) of \((G, \theta)\). The ordinary simple roots are marked by a \(\circ\), the additional root \(\tilde{\alpha}\) by a \(\bullet\). We get six blocks, separated by double lines, which contain the following information:

\[
\begin{array}{c|c|c|c}
\Phi(G) & \Delta_{\text{ext}}(\Phi(G), id) & \theta & \Delta_{\text{ext}}(\Phi(G), \theta) \\
\hline
\Phi(H) & \theta = id & \Delta_{\text{ext}}(\Phi(H), id) \\
\end{array}
\]

Here \(\Delta_{\text{ext}}(\Phi(G), id)\) is arranged such that the \(\theta\)-orbits of roots are in vertical order. We may think of \(\Delta(\Phi(G), \theta)\) as a quotient diagram of \(\Delta(\Phi(G), id)\), but their seems to be no rule which describes the additional root \(c(\tilde{\alpha})P_\Theta(\tilde{\alpha})\). To obtain \(\Delta_{\text{ext}}(\Phi(H), id)\) one observes at first that \(\Phi(\hat{G}) \simeq \Phi(G)\), since \(G\) is of type \(ADE\), then remarks \(\Delta(\Phi(H), id) = \Delta(\Phi(\hat{G}), \theta)\), reversing the arrows in this diagram one gets the diagram of \(\Delta(\Phi(H), id)\), which finally has to be extended to \(\Delta_{\text{ext}}(\Phi(H), id)\).
### Table 2.6

| $\Phi(G)$       | $\Delta_{ext}(\Phi, id)$ | $\theta$ | $\Delta_{ext}(\Phi, \theta)$ |
|----------------|--------------------------|----------|-------------------|
| $A_{2n-1}$ $n \geq 3$ | ![Diagram](image1) | $\text{Ord}(\theta) = 2$ | ![Diagram](image2) |
| $B_n$ $n \geq 3$ | $\theta = \text{id}$ | ![Diagram](image3) |
| $A_3$ | ![Diagram](image4) | $\text{Ord}(\theta) = 2$ | ![Diagram](image5) |
| $B_2$ | $\theta = \text{id}$ | ![Diagram](image6) |
| $A_{2n}$ $n \geq 1$ | ![Diagram](image7) | $\text{Ord}(\theta) = 2$ | ![Diagram](image8) |
| $C_n$ $n \geq 1$ | $\theta = \text{id}$ | ![Diagram](image9) |
| $D_{n+1}$ $n \geq 3$ | ![Diagram](image10) | $\text{Ord}(\theta) = 2$ | ![Diagram](image11) |
| $C_n$ $n \geq 3$ | $\theta = \text{id}$ | ![Diagram](image12) |
| $E_6$ | ![Diagram](image13) | $\text{Ord}(\theta) = 2$ | ![Diagram](image14) |
| $F_4$ | $\theta = \text{id}$ | ![Diagram](image15) |
| $D_4$ | ![Diagram](image16) | $\text{Ord}(\theta) = 3$ | ![Diagram](image17) |
| $G_2$ | $\theta = \text{id}$ | ![Diagram](image18) |
(2.7) Comparison of diagrams: By construction the diagrams \( \Delta_{\text{ext}}(\Phi(G), \theta) \) and \( \Delta_{\text{ext}}(\Phi(H), id) \) are arranged in vertical order on the right hand side of each of the six blocks, and the corresponding spherical diagrams are obtained from each other by reversing the arrows. By inspection we see that the same statement holds for the extended diagrams with the exception that in case \( A_{2n} \leftrightarrow C_n \) there is no reversal of the arrow joining the additional root with the standard diagram.

A similar phenomenon appears when we consider centralizers: If \( s\theta \in G(\overline{F}) \theta \) and \( \sigma \in H(\overline{F}) \) match they can be assumed to lie in the diagonal tori such that:

\[
\mathbb{T}(\overline{F}) \ni s \mapsto \sigma \in \mathbb{T}(H(\overline{F})) = \mathbb{T}(\Theta(\overline{F})).
\]

If we compute \( \Phi(G^{s\theta}) \) and \( \Phi(H^{\sigma}) \) using (2.1) and (2.4)(b) we see by inspection that they can be arranged in vertical order as subdiagrams (in the sense of (2.4)(b)) of the diagrams \( \Delta_{\text{ext}}(\Phi(G), \theta) \) and \( \Delta_{\text{ext}}(\Phi(H), id) \), but the oriented arrows get reversed with the exception mentioned above. If we write \( (G^{s\theta})_{ad} \) resp. \( (H^{\sigma})_{ad} \) as product of simple groups, we therefore get common factors of type \( A \) and \( D \) but factors of type \( B \) in general correspond to factors of type \( C \) and vice versa and factors of type \( G_2 \) and \( F_4 \) come in with reversed arrows.

(2.8) Our strategy to prove the fundamental lemma in the case of classical split groups now goes as follows: By the Kazhdan lemma 5.5 the (twisted) stable orbital integral of \( g\theta \in G(F) \theta \) in the group \( \tilde{G} \) can be replaced by the ordinary stable orbital integral in the group \( G_{s\theta} \) of the topologically unipotent part \( u \). Now \( G^{s\theta} \) resp. \( H^{\sigma} \) is isogenous to a product \( G^{s\theta}_+ \times G^{s\theta}_* \) resp. \( H^{\sigma}_+ \times H^{\sigma}_* \), such that \( G^{s\theta}_+ \) is of type \( B \) or \( C \), \( H^{\sigma}_+ \) is the other of these two types and \( G^{s\theta}_* \) is isogenous to \( H^{\sigma}_* \). Similarly the stable orbital integral of some \( \gamma \in H(F) \), which matches with \( g \), can be computed as the stable orbital integral of the topologically unipotent part \( v \) in the group \( H^{\sigma} \), where the residually semisimple part \( \sigma \) of \( \gamma \) matches with \( s\theta \). Decomposing \( u = (u_+, u_*) \) and \( v = (v_+, v_*) \) we get that \( u_+ \) and \( v_+ \) coincide up to stable conjugation und up to powering, so they have matching stable orbital integrals. The fundamental lemma for \( g\theta \) and \( \gamma \) will now follow if we can assume that the stable orbital integrals of \( u_+ \) and \( v_+ \) match, which is essentially the \( BC \) conjecture 5.3 stated already in the introduction.

(2.9) In case \( \Phi = A_{2n} \) consider the exact sequence:

\[
1 \rightarrow \mu_{2n+1} \rightarrow \text{SL}_{2n+1} \rightarrow \text{PGL}_{2n+1} \rightarrow 1.
\]

Since the involution \( \Theta \) acts as \(-1\) on \( \mu_{2n+1} \simeq \mathbb{Z}/(2n + 1)\mathbb{Z} \) one gets the long exact sequence

\[
1 \rightarrow \kappa(G)^{\Theta} \rightarrow G^{s\Theta} \rightarrow G^{t\Theta} \rightarrow H^1(\langle \Theta \rangle, \kappa(G)) \rightarrow ...
\]

for every \( t \in \text{Inn}(G) \).

Example 2.10. The case \( \Phi = A_4 \)
Consider \( s \in T(\overline{F}) \), where \( T \) is the diagonal torus in \( G = \text{PGL}_5 \). There are seven
possible types of groups \( G^{s\Theta} \) nonisomorphic over \( \bar{F} \). (\( \zeta_n \) denotes a (fixed) primitive \( n \)-th root of unity in \( \bar{F} \).)

1. Case:

\[
G^{s\Theta} = \operatorname{SO}(5) = \{ g \in \operatorname{Sl}(5) \mid {}^t g \cdot J \cdot g = J \} \text{ is of type } \Phi(G^{s\Theta}) = B_2 \text{ and } \pi_1(\Phi(G^{s\Theta})) \simeq \mathbb{Z}/2.
\]

2. Case:

\[
G^{s\Theta} = \operatorname{Sp}(2 \cdot 2) = \{ g \in \operatorname{Sl}(5) \mid {}^t g \cdot \tilde{J} \cdot g = \tilde{J} \} \text{, where } \tilde{J} = J \cdot \operatorname{Diag}(-1, -1, 1, 1, 1), \text{ is of type } \Phi(G^{s\Theta}) = C_2 \text{ and } \pi_1(\Phi(G^{s\Theta})) = 1.
\]

3. Case:

\[
G^{s\Theta} \simeq \operatorname{SO}(3) \times \operatorname{Sp}(2), \text{ is of type } \Phi(G^{s\Theta}) = B_1 \times C_1 = A_1^2 \text{ and } \pi_1(\Phi(G^{s\Theta})) \simeq \mathbb{Z}/2.
\]

4. Case:

\[
G^{s\Theta} \simeq \operatorname{SO}(3) \times \operatorname{Gl}(1)
\]

5. Case:

\[
G^{s\Theta} \simeq \operatorname{Gl}(2)
\]

6. Case:
\[ G^\Theta \simeq \text{Sp}(2) \times \text{Gl}(1) \]

7. Case:

\[
\begin{pmatrix}
\zeta_6 & \cdots & \cdots \\
\zeta_{12} & \cdots & \\
\vdots & \ddots & \vdots \\
\vdots & & \zeta_{12} \\
\vdots & & \zeta_{12} \\
\zeta_{-1} & \cdots & \zeta_{-1} \\
\zeta_{-1} & \cdots & \zeta_{-1} \\
\zeta_{-1} & \cdots & \zeta_{-1} \\
\zeta_6 & \cdots & \zeta_6
\end{pmatrix}
\]

\[ G^\Theta \simeq \text{Gl}(1) \times \text{Gl}(1) \]
3 Topological Jordan decomposition

Definition 3.1. Let $tJZ_p$ be the category, whose objects are topological groups, such that the neighborhood filter of 1 has a basis consisting of pro-$p$-groups, and whose morphisms are the continuous homomorphisms.

Definition 3.2. An element $g$ of $G \in \text{Ob}(tJZ_p)$ is called

- strongly compact, if $g$ lies in a compact subgroup of $G$.
- topologically unipotent, if $\lim_{n \to \infty} g^{p^n} = 1$.
- residually semisimple, if $g$ is of a finite order, which is prime to $p$.

(3.3) For an element $g$ it is equivalent to be topologically unipotent and to lie in a pro-$p$-subgroup of $G$. In the definition above one can replace the sequence $(p^n)$ by an arbitrary sequence $(p^{n_k})$ satisfying $\lim_{k \to \infty} n_k = \infty$.

Example 3.4. Let $F$ be a $p$-adic field and $\tilde{G}$ an affine linear algebraic group. Then $G(F)$ in an Object of $tJZ_p$.

An element $g \in \tilde{G}(F)$ in the example 3.4 is strongly compact, iff the set $g^{\mathbb{Z}}$ is bounded inside $\tilde{G}(F)$. A third equivalent formulation is, that the eigenvalues of $\rho(g)$ are units in $\tilde{F}^\times$ for one/for all faithful representation(s) $\rho: G \to GL(V)$.

Each pro-$p$-group $U$ has a unique structure as a topological $\mathbb{Z}_p$-module, which extends the canonical structure as $\mathbb{Z}$-module (comp. [Hasse, §15.2]).

Lemma 3.5 (topological Jordan decomposition). Let $G$ be an object of $tJZ_p$. Every strongly compact $g \in G$ has a unique decomposition

$$g = g_u \cdot g_s = g_s \cdot g_u,$$

where $g_u \in G$ is topologically unipotent and $g_s \in G$ is residually semisimple.

Proof: We consider the (abelian!) closure $\overline{\langle g^{\mathbb{Z}} \rangle}$ of the abelian group $\langle g^{\mathbb{Z}} \rangle$, which is contained in a compact subgroup of $G$ and is therefore itself compact. Since $G \in \text{Ob}(tJZ_p)$ there exists an open pro-$p$-subgroup of $\overline{\langle g^{\mathbb{Z}} \rangle}$. The set $U$ of all topologically unipotent elements in $\overline{\langle g^{\mathbb{Z}} \rangle}$ contains this open subgroup, is a group since $\overline{\langle g^{\mathbb{Z}} \rangle}$ is abelian, and is therefore an open pro-$p$-subgroup $U$. The compactness of $\overline{\langle g^{\mathbb{Z}} \rangle}$ implies that $U$ has a finite index $N$ in it, which has to be prime to $p$. Since $U$ is a $\mathbb{Z}_p$-module and $N \in \mathbb{Z}_p^\times$ there exists a (topologically unipotent!) element $g_u \in U$ such that $g^N = g_u^N$. Since $g_u g = gg_u$ the element $g_s = g \cdot g_u^{-1}$ satisfies $g_s^N = 1$, i.e. is residually semisimple, and we get $g = g_s g_u$ and $g = g \cdot g_u \cdot g_u^{-1} = g_u \cdot g \cdot g_u^{-1} = g_u g_s$, i.e the existence of the decomposition is proved.

If $g = g'_u g'_s = g'_s g'_u$ is a second topological Jordan decomposition with $(g'_s)^{N'} = 1$ we choose a $p$-power $Q = p^m$ such that $Q \equiv 1 \mod NN'$ and get

$$\lim_{a \to \infty} g^{Q^a} = \lim_{a \to \infty} (g'_u)^{Q^a} \cdot (g'_s)^{Q^a} = \lim_{a \to \infty} (g'_u)^{Q^a} \cdot g_s = g'_s,$$

and by the same argument:

$$\lim_{a \to \infty} g^{Q^a} = g_s,$$

i.e. $g'_s = g_s$ and therefore also $g_u = g'_u$ i.e. the uniqueness assertion. \hfill \square
Corollary 3.6 (Properties of the topological Jordan decomposition).

(1) Let $g \in G$ be strongly compact, $N \in \mathbb{N}$ be prime to $p$, such that $g^N$ lies in some pro-$p$-group, and let $Q$ be a $p$-power with $Q \equiv 1 \pmod{N}$. Then we have
\[
\lim_{m \to \infty} g^{Q^m} = g_s.
\]

(2) We have $g_u \in G^{g_s}$ and $G^{g_s} = \text{Cent}(g_u, G^{g_s})$.

(3) Residually semisimple elements are semisimple.

(4) Let $m$ be prime to $p$ and $u$ be topologically unipotent. Then there exists a unique topologically unipotent $u_1$ such that
\[
u_1^m = u.
\]

(5) The topological Jordan decomposition is functorial in the following sense: The strongly compact (resp. the topologically unipotent, resp. the residually semisimple) elements define functors from $tJZ_p$ to $\text{Set}$. For each morphism $\varphi$ in $tJZ_p$ we have $\varphi((\cdot)_s) = (\varphi(\cdot))_s$ und $\varphi((\cdot)_u) = (\varphi(\cdot))_u$.

Especially:

(5a) If $H$ is a closed subgroup of $G$ and $g \in H$, then also $g_s$ and $g_u$ are in $H$.

Corollary 3.7. Let $F$ and $\tilde{G}$ be as in example 3.4 and assume furthermore that $|\pi_0(\tilde{G})|$ is prime to $p$. Each strongly compact element $g \in \tilde{G}(F)$ has a unique topological Jordan decomposition:
\[
g = g_u \cdot g_s = g_s \cdot g_u,
\]
where $g_s \in \tilde{G}(F)$ is residually semisimple and $g_u \in (\tilde{G})^o(F)$ topologically unipotent.

The functoriality implies the following statements:

(1) Let $\rho : \tilde{G} \to \tilde{G}'$ be a morphism of (not necessarily connected) reductive groups, defined over a finite extension of $F$. Then we have $\rho(g)_s = \rho(g_s)$ and $\rho(g)_u = \rho(g_u)$.

(2) If $g \in \tilde{G}(\mathcal{O}_F)$, then the image of the topological Jordan decomposition under the reduction map is the Jordan decomposition in $\tilde{G}(\mathbb{F}_q)$.

(Topologically unipotent elements must lie in $(\tilde{G})^o$ since by assumption $p$ does not divide $|\pi_0(\tilde{G})|$.)
4 Classification of $\Theta$-conjugacy classes

(4.1) If $(G, \Theta)$ is as in the examples \[8\] or \[9\] the problem of determining the $\Theta$-conjugacy classes of elements $s\Theta \in G(F)$ is equivalent to determine the classes of $h = sJ$ under the transformations $h \mapsto g \cdot h \cdot g^\dagger$. Namely we have the following commutative diagram:

$$
\begin{array}{ccc}
GL_{2n} & \xrightarrow{h \mapsto hJ^{-1}\Theta} & \tilde{GL}_{2n} \\
\downarrow_{h \mapsto g \cdot h \cdot g^\dagger} & & \downarrow_{x \mapsto g \cdot x \cdot g^{-1}} \\
GL_{2n} & \xrightarrow{h \mapsto hJ^{-1}\Theta} & \tilde{GL}_{2n}
\end{array}
$$

If we decompose $h = q + p$ in its symplectic part $p$ and its symmetric part $q$ we thus have to consider the problem of simultaneous normal forms for a symplectic and a symmetric bilinear form. To obtain results for orbital integrals we have to deal with this problem also over the ring of integers $\mathcal{O}_F$. The problem can be attacked if we assume $s\Theta$ to be semisimple (resp. residually semisimple if we work over $\mathcal{O}_F$).

(4.2) Notations: In the following $R$ denotes either a field of characteristic 0 or the ring of integers $\mathcal{O}_F$ of a local $p$-adic field $F$, where $p \neq 2$. We denote by $\mathfrak{m}$ the maximal ideal of $R$ (i.e. $\mathfrak{m} = (0)$ if $R$ is a field) and by $\kappa = R/\mathfrak{m}$ the residue field in the case $R = \mathcal{O}_F$.

Let $M$ denote a free $R$-module of finite rank $r$ with basis $(b_i)_{1 \leq i \leq r}$. A bilinear form $q : M \times M \rightarrow R$ is called unimodular if $\Delta(q) := \det(q(b_i, b_j)) \in R^\times$. This definition is obviously independent of the chosen basis $(b_i)$ since $\Delta(q)$ is an invariant in $R/(R^\times)^2$.

For $h \in GL_n(R)$ we have the following bilinear forms $b_h$ and $b'_h$ on the module $M = R^n$ of column vectors: $b_h(m_1, m_2) = ^t m_1 \cdot h \cdot m_2$ and $b'_h(m_1, m_2) = ^t m_1 \cdot ^t h \cdot m_2$.

An element $g \in GL(M)$ is called $R$-semisimple iff

- $g$ is semisimple in the case $R$ is a field,
- $g$ is residually semisimple (i.e. has finite order prime to $\text{char}(\kappa)$) in the case $R = \mathcal{O}_F$.

For $h \in GL_n(R)$ we call $N(h) = h \cdot ^t h^{-1}$ the (right) norm of $h$ and $N_l(h) = ^t h^{-1} \cdot h$ the left norm of $h$. $N(h)$ and $N_l(h)$ are conjugate by $h$ in $GL_n(R)$. Then $h$ is called $R$-$\Theta$-semisimple if $N(h)$ (or equivalently $N_l(h)$) is $R$-semisimple.

We remark that $h$ is $R$-$\Theta$-semisimple if and only if $h \cdot J^{-1} \cdot \Theta$ is semisimple respectively residually semisimple as an element of $GL_n(R) \rtimes \langle \Theta \rangle$.

Lemma 4.3. If $p : M \times M \rightarrow R$ is a unimodular symplectic form, then there exists a basis $(e_1, \ldots, e_a, f_g, \ldots, f_1)$ of $M$, such that $p$ has standard form with respect to this basis, i.e. $p(e_i, e_j) = p(f_i, f_j) = 0$ and $p(e_i, f_j) = \delta_{ij}$.
Lemma 4.4. If \( q : M \times M \to R \) is a unimodular symmetric bilinear form and \( R = \mathcal{O}_F \), then there exists a basis \( (e_i)_{1 \leq i \leq r} \) of \( M \) such that \( q(e_i, e_j) = \delta_{ij} \) for \( (i,j) \neq (r,r) \) and \( q(e_r, e_r) \) is some given element in the class of \( \Delta(q) \) in \( R^\times/(R^\times)^2 \).

Proof: Consider the reductions \( \kappa = R/\mathfrak{m} \), \( \bar{M} = M/\mathfrak{m}M \) and \( \bar{q} : \bar{M} \times \bar{M} \to \kappa. \) Since quadratic forms over finite fields are classified by their discriminants, the analogous statement for \( \bar{q} \) holds. By lifting a basis from \( \bar{M} \) to \( M \) we can therefore assume that \( q(b_i, b_j) \cong \delta_{ij} \mod \mathfrak{m} \) for \( (i,j) \neq (r,r) \). But now we can apply the Gram-Schmidt-Orthogonalization procedure (observe that elements congruent to 1 modulo \( \mathfrak{m} \) are squares since \( p \neq 2 \)) to obtain the claim. \( \Box \)

Lemma 4.5.

(a) If \( g \in GL(M) \) is \( R \)-semisimple then there exists a finite étale galois extension \( R'/R \) such that \( M' = M \otimes_R R' \) decomposes into the direct sum of eigenspaces: \( M' = \bigoplus \lambda M'_{\lambda} \), where \( g \) acts on \( M'_{\lambda} \) as the scalar \( \lambda. \)

(b) If \( g = N_i(h) \) for an \( R-\Theta \)-semisimple \( h \in GL_n(R) \) (see 4.2) then \( b_h(M'_\lambda, M''_\mu) = 0 = b'_h(M'_\lambda, M''_\mu) \) unless \( \lambda \mu = 1. \)

(c) The restrictions of the forms \( b_h \) and \( b'_h \) to \( M'_1 \) and \( M'_{-1} \) are unimodular. For \( \lambda \neq \pm 1 \) also the restrictions of \( b_h, b'_h, b_h + b'_h \) and \( b_h - b'_h \) to the modules \( N'_\lambda = M'_\lambda \oplus M'_{-\lambda} \) are unimodular.

Proof: (a) The minimal polynomial \( \chi(X) \) of \( g \) decomposes in pairwise different linear factors \( \chi(X) = \prod_{i=1}^r (X - \lambda_i) \) over some extension ring of \( R \). The ring \( R' = R[\lambda_i]_{1 \leq i \leq r} \) is finite étale and galois over \( R \), since the \( \lambda_i \) are roots of unity of order prime to \( \text{char}(\kappa) \) in the case \( R = \mathcal{O}_F \). By the same reason we have

\[
\lambda_i - \lambda_j \in (R')^\times \quad \text{for} \quad i \neq j
\]

in both cases for \( R \). We remark for later use that this statement remains correct if we add \( \pm 1 \) to the set of the \( \lambda_i \) (if they are not already among them). Therefore \( \chi_i(X) = \prod_{j \neq i} (X - \lambda_j) \cdot (\lambda_i - \lambda_j)^{-1} \in R'[X]. \) We have \( \sum_{i=1}^r \chi_i(X) = 1 \) since the left hand side is a polynomial of degree \( r - 1 \) which has the value 1 at \( r \) different places. Therefore \( M' \) is the sum of the subspaces \( M'_{\lambda_i} = \chi_i(g)(M) \). Since \( (g - \lambda_i) \cdot \chi_i(g) \) equals \( \chi(g) \cdot \prod_{j \neq i} (\lambda_i - \lambda_j)^{-1} = 0 \), the spaces \( M'_{\lambda_i} \) are eigenspaces for \( g \) and the sum \( M' = \sum_{i=1} M'_{\lambda_i} \) is direct.

(b) For \( m \in M'_{\lambda_i} \) and \( n \in M'_{\mu} \) we have \( m = \lambda^{-1} \cdot gm \) and \( n = \mu \cdot g^{-1} n. \) The claims follow immediately from the relations \( b_h(m, n) = \lambda^{-1} \cdot b'_h(m, n) \) and

\[
b_h(m, n) = \mu \cdot b'_h(m, n).
\]
(c) In view of the orthogonality relations (b) and the unimodularity of \( h \) and \( ^t h \) the claims for the restrictions of \( b_h \) and \( b_h' \) follow immediately. By the formula (iii) above we have for \( m \in M, n \in M_A^{-1} \):

\[
(b_h \pm b_h')(m, n) = (1 \pm \lambda) b_h(m, n) \\
(b_h \pm b_h')(n, m) = (1 \pm \lambda^{-1}) b_h(n, m).
\]

Since \( 1 \pm \lambda, 1 \pm \lambda^{-1} \in (R')^\times \) by the remark following (i) above, the claim follows also for the restrictions of \( b_h \pm b_h' \).

**Lemma 4.6.** For an \( R-\Theta \) -semisimple \( h \in GL_n(R) \) with decomposition \( h = p + q \), where \( p \) is skew-symmetric and \( q \) symmetric, we have a direct sum decomposition for \( M = R^n \)

\[
M = M_+ \oplus M_- \oplus M_0,
\]

where \( M_+ = \ker p, M_- = \ker q \) and \( M_0 = (M_+)^{\perp_q} \cap (M_-)^{\perp_p} \) is the intersection of the orthogonal complement of \( M_+ \) with the symplectic orthogonal complement of \( M_- \). The restrictions

\[
q_+ = q \mid M_+ \times M_+,
\]

\[
p_- = p \mid M_- \times M_-,
\]

\[
q_0 = q \mid M_0 \times M_0,
\]

\[
p_0 = p \mid M_0 \times M_0
\]

are unimodular.

Proof: We identify the matrices \( p, q \in GL_n(R) \) with the forms \( b_p, b_q \). We take an extension \( R'/R \) as in Lemma 4.5(a) and compute

\[
M'_{\pm 1} = \{ m \in M' \mid {}^t h^{-1} \cdot h \cdot m = \pm m \} = \{ m \in M' \mid hm = \pm hm \} = \ker(h \mp {}^t h).
\]

This means \( M'_1 = \ker(p \mid M') \) and \( M'_{-1} = \ker(q \mid M') \) and implies \( M'_1 = M_+ \otimes_R R', M'_{-1} = M_- \otimes_R R' \). Since unimodularity can be checked after the extension \( R'/R \) and \( b_h \) restricts to \( q_+ \) resp. \( p_- \) on \( M_+ \) resp. \( M_- \), we conclude from Lemma 4.5(c) that \( q_+ \) and \( p_- \) are unimodular. Then it is clear that we have the claimed decomposition in (orthogonal and symplectic orthogonal) direct summands. By Lemma 4.5(b) we get \( M_0 \otimes_R R' = \bigoplus_{\lambda \neq \pm 1} M'_\lambda \). By Lemma 4.5(c) again we conclude that the restrictions of \( b_h + b_h' = 2q \) and \( b_h - b_h' = 2p \) to this module are unimodular. So \( p_0 \) and \( q_0 \) are unimodular.

**Lemma 4.7** (Cayley transformation). Let \( p \in GL_n(R) \) be a skew-symmetric matrix. Let \( \text{Sym}_n(R)_{p-\text{ess}} \) denote the set of symmetric matrices \( q \) such that \( q \pm p \in GL_n(R) \) and \( \text{Sp}(p, R)_{\text{ess}} \) the set of symplectic transformations \( b \) such that \( b - 1 \in GL_n(R) \). Then the following holds:

(a) We have a bijection

\[
C : \text{Sym}_n(R)_{p-\text{ess}} \to \text{Sp}(p, R)_{\text{ess}}, \quad q \mapsto (q - p)^{-1} \cdot (q + p) = N_t(p + q).
\]

The inverse map is \( C^{-1} : b \mapsto p \cdot (b + 1) \cdot (b - 1)^{-1} \).
Lemma 4.8. If $p$ is orthogonal direct sum decomposition $b \in N$ then the $R$-$\Theta$-semisimple, and the $R$-semisimple elements of $Sp(p, R)_{ess}$.

Proof: (a) For $q \in Sym_n(R)_{p-ess}$ we put $h = p + q$ and $b = b^h \cdot h$. We have

$$\begin{align*}
\text{(i)} & \quad t^h \cdot b \cdot h = t^h \cdot h \cdot (t^h)^{-1} \cdot h \quad \text{i.e.} \\
\text{(ii)} & \quad t^h \cdot b \cdot h = h \quad \text{and by transposing} \\
\text{(iv)} & \quad t^h \cdot b \cdot h = t^h.
\end{align*}$$

Subtracting the last two equations we get $\bar{t}b \cdot p \cdot b = p$, i.e. $b \in Sp(p, R)$. Furthermore $b - 1 = (p - p)^{-1} \cdot ((p + q) - (q - p)) = (q - p)^{-1} \cdot 2p \in GL_n(R)$ by the assumptions. The map $C$ is therefore defined.

Conversely we get for $b \in Sp(p, R)_{ess}$ and $q = p \cdot (b + 1) \cdot (b - 1)^{-1}$ the equivalences:

$$
q = t^h \qquad \Leftrightarrow \quad p \cdot (b + 1) \cdot (b - 1)^{-1} = t^h (b - 1)^{-1} \cdot t^h (b + 1) \cdot (p - 1)
\Leftrightarrow \quad (\bar{t}b - 1)p(b + 1) = (\bar{t}b + 1)p \quad \Leftrightarrow \quad \bar{t}bp - bp - p = -(\bar{t}bp + \bar{t}p - bp + p)
\Leftrightarrow \quad \bar{t}bp = p \quad \Leftrightarrow \quad b \in Sp(p, R).
$$

Furthermore $q + p = p \cdot (b + 1) \cdot (b - 1)^{-1} \cdot 2b \cdot p \in GL_n(R)$. Therefore the map $C$ is also well defined. An easy calculation (as in the case of the usual Cayley transform) shows that the maps $C$ and $C^{-1}$ are inverse to another in their domain of definition.

(b) Since $C(q) = N_1(p + q) = (p + q)^{-1} \cdot N(p + q) \cdot (p + q)$ this follows from the definition of $R$-$\Theta$-semisimplicity.

(c) We have $C(t^g q \cdot g) = (t^g q g - p)^{-1} (t^g q g + p) = g^{-1} (q - p)^{-1} \cdot t^g q (q + p)g = g^{-1} \cdot (q - p)^{-1} \cdot (q + p) \cdot g = g^{-1} \cdot C(q) \cdot g$ for $g \in Sp(p, R)$.

Lemma 4.8. If $p$ is a unimodular symplectic form on a free $R$-module $N$ and $b \in Sp(p, R)$ is $R$-semisimple then there exists a $b$-invariant and with respect to $p$ orthogonal direct sum decomposition $N = N_1 \oplus N_0$ such that $b$ acts as identity on $N_1$ and $b|N_0 \in Sp(p, R)_{ess}$, where $p_0$ is the restriction of $p$ to $N_0$.

Proof: We argue as before: By lemma 4.5(a) we have for some finite étale ring extension $R' = R$ a decomposition of $N' = N \otimes_R R'$ into eigenspaces of $b$: $N' = \bigoplus N'_\lambda$, where $b$ acts as the scalar $\lambda$ on $N'_\lambda$. As in lemma 4.5(b) we can see, that $p(N'_\lambda, N'_\mu) = 0$ unless $\lambda \cdot \mu = 1$. This implies that $p$ is unimodular on $N'_1$ and therefore on $N_1$, thus $N$ is the direct sum of $N_1$ and the $p$-orthogonal complement $N_0$ of $N_1$. Since $b$ is a symplectic transformation, it leaves $N_0$ invariant. By the orthogonality relations for the $N_\lambda$ we have $N_0 \otimes_R R' = \bigoplus_{\lambda \neq 1} N'_\lambda$. Since $\lambda - 1 \in (R')^\times$ for $\lambda \neq 1$ the endomorphism $b - 1$ of $N_0$ induces an automorphism of $N_0 \otimes_R R'$ and is therefore itself an automorphism of $N_0$.
Lemma 4.9. Let $h = p + q \in GL_n(R)$ be $R$-$\Theta$-semisimple. Let $G^{h,\Theta}(R) = \{g \in GL_n(R) \mid g \cdot h \cdot g = h\}$. Then the following holds:

(a) With the notations of lemma 4.6 and of lemma 4.7 we have

$$G^{h,\Theta}(R) = O(q_+, R) \times Sp(p_-, R) \times (Sp(p_0, R) \cap O(q_0, R))$$

$$\cong O(q_+, R) \times (Sp(p_- \oplus p_0, R) \cap O(q_- \oplus q_0, R))$$

$$\cong O(q_+, R) \times \text{Cent} (C(q_- \oplus q_0), Sp(p_- \oplus p_0, R)).$$

(b) In the situation and with the notations of lemma 4.5 we have moreover

$$(Sp(p_0, R') \cap O(q_0, R')) = \left\{ (\phi_\lambda) \in \prod_{\lambda \neq \pm 1} GL(M'_\lambda) \mid \phi_{\lambda^{-1}} = t\phi_\lambda \text{ for all } \lambda \right\}$$

$$\cong \prod_{\lambda \in \mathcal{L}} GL(M'_\lambda)$$

where $\phi_{\lambda^{-1}} = t\phi_\lambda$ means that $b_h(\phi_{\lambda^{-1}}m_{\lambda^{-1}}, \phi_{\lambda}m_\lambda) = b_h(m_{\lambda^{-1}}, m_\lambda)$ for all $m_{\lambda^{-1}} \in M'_{\lambda^{-1}}, m_\lambda \in M'_\lambda$ and where $\mathcal{L}$ denotes a subset of the set of all $\lambda \neq \pm 1$, which takes from every pair $\{\lambda, \lambda^{-1}\}$ exactly one member.

(c) $(Sp(p_- \oplus p_0) \cap O(q_- \oplus q_0)) \cong \text{Cent} (C(q_- \oplus q_0), Sp(p_- \oplus p_0))$ is a connected reductive smooth group scheme $/R$ with connected special fiber, which becomes split over the finite étale extension $R'/R$.

(d) We have in the situation of 6.3

$$\text{Cent}(N(h), Sp_{2n}) \cong Sp_{2(n-g)} \times \text{Cent} (C(q_- \oplus q_0), Sp(p_- \oplus p_0))$$

where $2g$ is the rank of $M_- \oplus M_0$.

(e) To obtain the intersections of $G^{h,\Theta}(R)$ with $SL_n(R)$ one has only to replace $O(q_+, R)$ by $SO(q_+, R)$ on the right hand sides of (a).

Proof: (a) Since every $g \in G^{h,\Theta}(R)$ stabilizes the decomposition of lemma 4.6 one immediately gets the first two isomorphisms. The last one follows from lemma 4.7(c).

(b) Every $g \in G^{h,\Theta}(R)$ centralizes $N_1(h)$ and therefore has to respect the decomposition of $M_0 \otimes_R R'$ in eigenspaces of $N_1(h)$. The first description of $Sp(p_0, R') \cap O(q_0, R')$ follows now from 4.3(b). Since $b_h$ is unimodular on $M'_{\lambda^{-1}} \oplus M'_\lambda$ it induces an identification of $M'_{\lambda^{-1}}$ with the dual space of $M'_\lambda$. This means that $\phi_\lambda$ can vary through the whole $GL(M'_\lambda)$, while $\phi_{\lambda^{-1}}$ is then uniquely determined as the inverse of its adjoint.

We remark that the condition $\phi_\lambda = t\phi_{\lambda^{-1}}$ is equivalent to the condition $\phi_{\lambda^{-1}} = t\phi_\lambda$ and gives no extra restrictions. This is clear since we have $b_h(m_\lambda, m_{\lambda^{-1}}) = $
$b'_h(m_{\lambda-1}, m_{\lambda}) = \lambda \cdot b_h(m_{\lambda-1}, m_{\lambda})$ for $m_{\lambda-1} \in M'_{\lambda-1}, m_{\lambda} \in M'_{\lambda}$ by (iii), so the two possible identifications of $M'_{\lambda-1}$ with the dual of $M'_{\lambda}$ differ by a scalar and create the same adjoint. The last isomorphism follows.

(c) This follows from (a) and (b).

(d) This follows from the definition of $\mathcal{N}$ by the remark, that an element of $\text{Cent}(b, \text{Sp}_{2n}(R))$ has to respect the decomposition of lemma 4.8.

(e) It is clear, since symplectic transformations have determinant 1.

\begin{proof}
This follows from [K3, Prop. 7.1.].
\end{proof}

Lemma 4.10. Let $G/R = \mathcal{O}_F$ be a connected reductive group with connected special fiber $G \times_{\mathcal{O}_F} \kappa$ and $b \in G(R)$ be R-semisimple.

If $b' = h_{F}^{-1} \cdot b \cdot h_{F} \in G(R)$ for some $h_{F} \in G(\bar{F})$ then there exists $h_{R} \in G(R)$ with $b' = h_{R}^{-1} \cdot b \cdot h_{R}$.

\begin{proof}
This follows from $\mathcal{O}_F$. Prop. 7.1.]
\end{proof}

Lemma 4.11. Let $R = \mathcal{O}_F$ and $h \in \text{GL}_n(R)$ be $R$-semisimple and $h' = t^* g_F \cdot h \cdot g_F \in \text{GL}_n(R)$ for some $g_F \in \text{GL}_n(\bar{F})$. Then we have:

(a) If additionally $\det(g_F) \in F^\times$ there exists $g_R \in \text{GL}_n(R)$ with $h' = t^* g_R \cdot h \cdot g_R$.

(b) If we only assume $g_F \in \text{GL}_n(\bar{F})$ and if $n$ is odd there exist $g_R \in \text{GL}_n(R)$ and $\epsilon \in \mathcal{O}_F^\times$ such that $h' = \epsilon \cdot t^* g_R \cdot h \cdot g_R$.

(c) We get the statement of (a) if we additionally assume that the discriminants of $q_+$ and $q'_+$ coincide in $R^\times/(R^\times)^2$.

(d) Under the additional conditions $h, h' \in \text{SL}_n(\mathcal{O}_F)$, $g_F \in \text{SL}_n(\bar{F})$ and $n$ odd we can find $g_R \in \text{SL}_n(R)$ with $h' = t^* g_R \cdot h \cdot g_R$.

\begin{proof}
We use the objects occurring in lemma 4.10 for $h$ and denote the corresponding objects for $h'$ by a '$$. We have $\text{rank}(M'_+) = \dim(M_+ \otimes_R F) = \dim(M'_+ \otimes_R F) = \text{rank}(M'_+)$. By transforming $h$ and $h'$ with elements of $\text{GL}_n(R)$ we can therefore assume (using lemma 4.4) that

\begin{align*}
\text{(v)} \quad M_+ = M'_+ = R^m, \quad M_0 \oplus M_- = M'_0 \oplus M'_-, \quad p_* := p_0 \oplus p_- = p'_0 \oplus p'_-.
\end{align*}

The assumption and lemma 4.7(c) (applied in the case $R = \bar{F}$) now imply that the elements $C(0 \oplus q_0)$ and $C(0 \oplus q'_0)$ of $\text{Sp}(p_*, R)$ are conjugate by an element of $\text{Sp}(p_*, \bar{F})$. By lemma 4.10 they are conjugate by an element $g_* \in \text{Sp}(p_*, R)$, hence we get from lemma 4.7(c) the equality $q'_0 = t^* g_0 \cdot g_* \cdot q_0$ and therefore $p'_0 \oplus p'_- + q'_0 = t^* g_0(p_0 \oplus p_- + q_0)g_* \in M_0 \oplus M_-$. We have $\det(q'_0) = \det(h') \cdot \det(p'_0 \oplus p'_- + q'_0)^{-1} = \det(g_F)^2 \cdot \det(h) \cdot \det(p_0 + p_- + q_0)^{-1} = \det(g_F)^2 \cdot \det(g_+) \cdot \det(g_+) \cdot \det(q_+) \cdot \det(q_+) = \det(g_*) = 1$.

If case (a) we conclude using $R^\times \cap (F^\times)^2 = (R^\times)^2$ and lemma 4.4 that $q'_+$ and $q_+$ are transformed via an element $g_+ \in \text{GL}_n(M'_+)$, a statement which has been an additional assumption in (c) in view of lemma 4.4. We put $g_*$ and $g_+$ together to $g_R \in \text{GL}_n(R)$ which does the required job in cases (a) and (c).
We prove (b) for $\epsilon = \det(q'_+/det(q_+)$: We have $h'' := \epsilon^{-1}h' = g'_F \cdot h \cdot g'_F$ for $g'_F = \sqrt{\epsilon^{-1}} \cdot g_F \in \text{GL}_n(\bar{F})$. If $2r + 1$ is the rank of $M'_+$ we have $\det(q''_+) = \det(q'_+) \cdot \epsilon^{2r+1} = \det(q_+) \cdot \epsilon^{2r}$. Thus the additional assumption of (c) is fulfilled and we get $g_R \in \text{GL}_n(R)$ with $h'' = g_R \cdot h \cdot g_R$.

To prove (d) observe at first that we can assume the matrices transforming $h$ and $h'$ into the standard form $[\mathbf{v}]$ being in $\text{SL}_n(R)$ since one can modify them by elements of $\text{GL}(M_\pm)$ and since $\text{rank}(M_\pm) \geq 1$. From $\det(g_F) = 1$ we furthermore get $\det(q'_+) = \det(q_+)$ and therefore $\det(g_+) = \pm 1$. Since we can replace $g_+$ by $-g_+$ if necessary and $\text{rank}(M_\pm)$ is odd we can achieve $\det(g_+) = 1$ and therefore $\det(g_R) = 1$. \qed
5 Orbital integrals

(5.1) For a (not necessarily connected) reductive group \( \tilde{G}/O_F \) with connected component \( G = \tilde{G}^0 \) and elements \( \gamma \in \tilde{G}(F), \) \( f \in \mathcal{C}_c^\infty(\tilde{G}(F)) \) we define the orbital integral by:

\[
O_\gamma(f, \tilde{G}(F)) = \int_{G(F)/G(F)^\gamma} f(x\gamma x^{-1})dx/dx^\gamma
\]

where \( G(F)^\gamma \) denotes the centralizer of \( \gamma \) in \( G(F) \) and where we have chosen Haar measures \( dx \) resp. \( dx^\gamma \) on \( G(F) \) resp. \( G(F)^\gamma \) such that

\[
\text{vol}(G(O_F)) = 1 \quad \text{and} \quad \text{vol}_{dx^\gamma}((G(F)^\gamma)^0(O_F)) = 1.
\]

If \( 1_K \) denotes the characteristic function of a compact open subset \( K \subset G(F) \), we will use the following abbreviation:

\[
O_\gamma(1, \tilde{G}) = O_\gamma(1_{\tilde{G}(O_F)}, \tilde{G}(F))
\]

We further introduce stable orbital integrals

\[
O^s_\gamma(f, \tilde{G}(F)) = \sum_{\gamma' \sim \gamma} O_{\gamma'}(f, \tilde{G}(F)) \quad \text{respectively}
\]

\[
O^s_\gamma(1, \tilde{G}) = \sum_{\gamma' \sim \gamma} O_{\gamma'}(1_{\tilde{G}(O_F)}, \tilde{G}(F))
\]

where \( \gamma' \) runs through a set of representatives for the conjugacy classes inside the stable conjugacy class of \( \gamma \).

(5.2) Recall the construction of the quotient measure \( dg/dh \) on \( G/H \) for totally disconnected locally compact groups \( H \subset G \), where \( G \) and \( H \) are unimodular (e.g. \( G \) and \( H \) are the sets of \( F \)-valued points of reductive groups). One defines

\[
\text{vol}(K\gamma H/H) = \int_{G/H} 1_{K\gamma H/H}(g)dg/dh := \frac{\text{vol}_{dg}(K)}{\text{vol}_{dh}(\gamma^{-1}K\gamma \cap H)},
\]

where \( K \subset G \) is any open compact subgroup, and extends this by linearity to the space of all locally constant compactly supported functions on \( G/H \).

Of course one has to prove a compatibility condition, if \( K' \subset K \) is another open compact subgroup: For \( \gamma \in G \) let

\[
K\gamma H = \bigcup_{j \in J} K' \cdot \gamma_j \cdot \gamma \cdot H
\]

be a disjoint double coset decomposition with \( \gamma_j \in K \). We have to prove:

\[
(\text{i}) \quad \text{vol}(K\gamma H/H) = \sum_{j \in J} \text{vol}(K'\gamma_j\gamma H/H).
\]
5. Orbital integrals

Define $C_j := \{ K'y \in K' \setminus K \mid K' y \gamma \subset K' \gamma_j \gamma H \} \subset K' \setminus K$ for $j \in J$. Then we have a disjoint decomposition

$$K' \setminus K = \bigcup_{j \in J} C_j$$

and isomorphisms

$$i_j : (\gamma^{-1} \gamma_j^{-1} K' \gamma_j \gamma \cap H) \setminus (\gamma^{-1} K \gamma \cap H) \xrightarrow{\sim} C_j \hspace{1cm} h \mapsto K' \gamma_j \gamma H \cdot h \cdot \gamma^{-1}.$$

This implies

$$\frac{vol_{dg}(K)}{vol_{dg}(K')} = [K : K'] = \sum_{j \in J} \left[ (\gamma^{-1} K \gamma \cap H) : (\gamma^{-1} \gamma_j^{-1} K' \gamma_j \gamma \cap H) \right]$$

or equivalently

$$\frac{vol_{dg}(K)}{vol_{dh}(\gamma^{-1} K \gamma \cap H)} = \sum_{j \in J} \frac{vol_{dg}(K')}{vol_{dh}(\gamma^{-1} \gamma_j^{-1} K' \gamma_j \gamma \cap H)} ,$$

which is the claim (i).

The crucial statement we need in the following is the following type of a fundamental lemma:

**Conjecture 5.3.** If the regular algebraically semisimple and topologically unipotent elements $u \in SO_{2n+1}(F)$ and $v \in Sp_{2n}(F)$ are BC-matching (see 1.12) then

$$(BC_n) \hspace{1cm} O^u_{st}(1, SO_{2n+1}) = O^v_{st}(1, Sp_{2n}).$$

The (easy) case $(BC_1)$ is proved in [FL1, Stable case I in Proof of Theorem]. The case $(BC_2)$ is essentially proved in [FL2, Part II], as will be explained in 7.9.

**Warning:** While $(BC_1)$ is an immediate consequence of the exceptional isogeny $i_2 : Sp_2 = SL_2 \to PGL_2 = SO_3$ and the fact, that $\gamma^2$ and $i_2(\gamma)$ are $BC$-matching for $\gamma \in SL_2(F)$, the statement $(BC_2)$ is much deeper, since the exceptional isogeny $i_4 : Sp_4 \to SO_5$ does not satisfy the analogous matching property.

**Remark 5.4.** It follows immediately from the construction in 1.12 that we have a bijection between $F$-rational conjugacy classes in $SO_{2n+1}(F)$ and in $Sp_{2n}(F)$. By the theorem of Steinberg each $F$-rational conjugacy class in $Sp_{2n}(F)$ contains a rational element, since $Sp_{2n}$ is quasisplit and simply connected. But the same statement holds for $F$-rational topologically unipotent conjugacy classes in $SO_{2n+1}(F)$ as well:
If \( u \in SO_{2n+1}(F) \) is topologically unipotent and represents an \( F \)-rational conjugacy class, consider its two preimages \( v_1 \) and \( v_2 = -v_1 \) in \( Spin_{2n+1} \). Since \( p \neq 2 \) we have \( \lim_{n \to \infty} v_2^n = -\lim_{n \to \infty} v_1^n \), so that exactly one of the elements \( v_1, v_2 \) is topologically unipotent, say \( v_1 \). Since the Galois group respects the property to be topologically unipotent, the conjugacy class of \( v_1 \) is \( F \)-rational and therefore contains an \( F \)-rational element \( v' \) by the theorem of Steinberg. The image of \( v' \) in \( SO_{2n+1}(F) \) is the desired \( F \)-rational representative of the conjugacy class of \( u \).

Thus to every topologically unipotent element in \( v \in Sp_{2n}(F) \) is associated at least one \( BC \)-matching \( u \in SO_{2n+1}(F) \) and vice versa.

**Lemma 5.5** (Kazhdan-Lemma).

(a) For \( \tilde{G} = G \times (\Theta) \) as in 1.2 let us assume that the following statement holds:

\((*)\) If \( s_1 \Theta \) and \( s_2 \Theta \) for \( s_1, s_2 \in G(\mathcal{O}_F) \) are residually semisimple and conjugate by an element of \( G(F) \) then they are also conjugate by an element of \( G(\mathcal{O}_F) \).

If \( \gamma \Theta = u \cdot s \Theta = s \Theta \cdot u \) is a topological Jordan decomposition, where \( \gamma \in G(\mathcal{O}_F) \), \( u \) is topologically unipotent and \( s \Theta \) residually semisimple, we have

\[
O_{\gamma \Theta}(1, \tilde{G}) = \frac{1}{[G^{s\Theta}(\mathcal{O}_F) : (G^{s\Theta})_0(\mathcal{O}_F)]} \cdot O_u(1, G^{s\Theta})
\]

(b) Let \( H/\mathcal{O}_F \) be connected reductive with connected special fiber. For \( h \in H(\mathcal{O}_F) \) with topological Jordan decomposition \( h = v \cdot b = b \cdot v \), where \( v \) is topologically unipotent and \( b \) residually semisimple, we have

\[
O_h(1, H) = \frac{1}{[H^b(\mathcal{O}_F) : (H^b)_0(\mathcal{O}_F)]} \cdot O_v(1, H^b)
\]

Proof: (a) We first prove:

\((**)\) We have \( g \gamma \Theta g^{-1} \in G(\mathcal{O}_F) \Theta \) if and only if \( g \) is of the form \( g = k \cdot x \) where \( k \in G(\mathcal{O}_F) \) and \( x \in G^{s\Theta}(F) \) satisfies \( xux^{-1} \in G^{s\Theta}(\mathcal{O}_F) \).

The direction "\( \Rightarrow \)" is easy: Under the hypothesis we have \( g \gamma \Theta g^{-1} = kxus \Theta x^{-1}k^{-1} = k(xux^{-1})(s \Theta)k^{-1} \in G(\mathcal{O}_F) \). For the converse direction "\( \Leftarrow \)" let us assume that \( g \gamma \Theta g^{-1} \in G(\mathcal{O}_F) \Theta \). The topological Jordan decomposition is \( g \gamma \Theta g^{-1} = (gug^{-1}) \cdot (gs \Theta g^{-1}) \). Since \( (G(\mathcal{O}_F), \Theta) \) is a closed subgroup of \( \tilde{G}(F) \) we conclude from 3.64 that \( gs \Theta g^{-1} \in G(\mathcal{O}_F) \Theta \) and \( gug^{-1} \in G(\mathcal{O}_F) \). By the first inclusion and assumption (*) we get an element \( k \in G(\mathcal{O}_F) \) such that \( g(s \Theta)g^{-1} = k(s \Theta)k^{-1} \), which implies \( x = k^{-1} \cdot g \in G^{s\Theta}(F) \), where \( G^{s\Theta} \) is the centralizer of \( s \Theta \) in \( G \). Using \( g = kx \) the inclusion \( gug^{-1} \in G(\mathcal{O}_F) \) is now equivalent to \( xux^{-1} \in G(\mathcal{O}_F) \), which proves (**).
Lemma 4.10.\[\] \(g\) Proof: Notice that of orbital integrals:

\[\{ g \in G(F) \mid g \gamma \Theta g^{-1} \in G(O_F) \Theta \} = \bigcup_{i \in I} G(O_F) \cdot g_i \cdot G^\gamma \Theta,\]

where we can assume \(g_i \in G^{s\Theta}(F)\) in view of (**) Again from (**) we get the double coset decomposition

\[\{ x \in G^{s\Theta}(F) \mid xux^{-1} \in G^{s\Theta}(O_F) \} = \bigcup_{i \in I} G^{s\Theta}(O_F) \cdot g_i \cdot G^{\gamma \Theta},\]

so that it remains to prove

\[(ii) \sum_{i \in I} \frac{\text{vol}_{dh}(G(O_F))}{\text{vol}_{dh}(g_i^{-1} \cdot G(O_F) \cdot g_i \cap G^{s\Theta}(F))} = \frac{1}{[G^{s\Theta}(O_F) : (G^{s\Theta})^\circ(O_F)]} \sum_{i \in I} \frac{\text{vol}_{dh}(G^{s\Theta}(O_F))}{\text{vol}_{dh}(g_i^{-1} \cdot G^{s\Theta}(O_F) \cdot g_i \cap G^{\gamma \Theta}(F))},\]

where \(dh\) is a Haar measure on \(G^{s\Theta}(F)\) satisfying \(\text{vol}_{dh}((G^{s\Theta})^\circ(O_F)) = 1\). This implies \(\text{vol}_{dh}(G^{s\Theta}(O_F)) = [G^{s\Theta}(O_F) : (G^{s\Theta})^\circ(O_F)]\). On the other hand we claim

\[(iii) g_i^{-1} \cdot G(O_F) \cdot g_i \cap G^{\gamma \Theta}(F) = g_i^{-1} \cdot G^{s\Theta}(O_F) \cdot g_i \cap G^{\gamma \Theta}(F).\]

The inclusion "\(\supset\)" being trivial let us assume that \(g = g_i^{-1} \cdot \sigma \cdot g_i\) is an element of the left hand side. But \(g_i \in G^{s\Theta}(F)\) and \(g \in G^{s\Theta}(F) \subset G^{s\Theta}(F)\) imply \(\sigma \in G^{s\Theta}(F)\). Since \(G^{s\Theta}(F) \cap G(O_F) = G^{s\Theta}(O_F)\) we get that \(g\) lies in the right hand side, i.e. (iii) is proved. (ii) now follows immediately.

(b) is now clear: We have \(\bar{G} = G\) i.e. \(\Theta = 1\) and the assumption (*) is satisfied by lemma 4.10. \(\Box\)

The following lemmas will be useful in later chapters.

Lemma 5.6. If \(N \in \mathbb{N}\) is prime to \(p\) then we have for a reductive group \(G/O_F\) and \(\gamma \in G(F)\)

\[O_{\gamma^N}(1, G) = O_{\gamma}(1, G).\]

Proof: Notice that \(g \cdot \gamma \cdot g^{-1}\) lies in the closure of \((g \cdot \gamma^N \cdot g^{-1})^Z\) if \(N \in \mathbb{Z}_p^\times\). This gives the equivalence

\[g \cdot \gamma^N \cdot g^{-1} \in G(O_F) \iff g \cdot \gamma \cdot g^{-1} \in G(O_F),\]

which implies the identity of orbital integrals. \(\Box\)

Lemma 5.7. If \(G/O_F\) is of the form \(G = G_1 \times Z\) with a reductive group \(G_1\) and a finite group \(Z \simeq Z(O_F)\) then we have for \(\gamma \in G_1(F) \subset G(F)\) the following identity of orbital integrals:

\[O_{\gamma}(1, G) = O_{\gamma}(1, G_1).\]
Proof: We have $G(F)/G(F)\gamma \simeq G_1(F)/G_1(F)^\gamma$ since $Z \subset G^\gamma$, and the normalized Haar-measures on $G_1(F)$ and $G_1^\gamma(F)$ are the restrictions of the normalized Haar-measures on $G(F)$ and $G^\gamma(F)$, since $G^\circ = G_1^\circ$ and $(G^\gamma)^\circ = (G_1^\gamma)^\circ$. The claim follows.

Lemma 5.8. Let $1 \to T \to G \to H \to 1$ be an exact sequence of algebraic groups over $\mathcal{O}_F$ where $T$ is a split torus. Then we have for $\gamma \in G(F)$ with image $\eta \in H(F)$:

$$O^{st}_\gamma(1, G) = O^{st}_\eta(1, H).$$

Proof: We use the fact that the image of $(G^\gamma)^\circ$ in $H$ is $(H^\eta)^\circ$. By Hilbert 90 we get exact sequences $1 \to T(F) \to G(F) \to H(F) \to 1$ and $1 \to T(F) \to (G^\gamma)^\circ(F) \to (H^\eta)^\circ(F) \to 1$, so that we have an isomorphism $G(F)/(G^\gamma)^\circ(F) \simeq H(F)/(H^\eta)^\circ(F)$. Since $(H^\eta)^\circ$ has finite index in $H^\eta$ we can compute $O^{st}_\eta(1, H)$ as $\int_{H(F)/(H^\eta)^\circ(F)} 1_{H(O_F)}(h\eta h^{-1})dh/d\eta$. Similarly

$$O^{st}_\gamma(1, G) = \int_{G(F)/(G^\gamma)^\circ(F)} 1_{G(O_F)}(g\gamma g^{-1})dg/d\gamma.$$ 

Now the quotient measures on $G(F)/(G^\gamma)^\circ(F)$ and $H(F)/(H^\eta)^\circ(F)$ coincide since $G(O_F) \to H(O_F)$ and $(G^\gamma)^\circ(O_F) \to (H^\eta)^\circ(O_F)$, and we conclude $O^{st}_\gamma(1, G) = O^{st}_\eta(1, H)$.

It remains to check that the set $St_\gamma$ of conjugacy classes inside the stable conjugacy class of $\gamma$ maps bijectively to the corresponding set $St_\eta$ associated to $\eta$. But in the following commutative diagram of abelian groups with exact rows and columns the map $\iota$ must be an isomorphism:

\[
\begin{array}{cccc}
H^1(F, T) & \longrightarrow & H^1(F, T) \\
\downarrow & & \downarrow \\
1 \longrightarrow & St_\gamma & \longrightarrow & H^1_{ab}(F, G^\gamma) \\
& \downarrow_{\iota} & \downarrow & \downarrow \\
1 \longrightarrow & St_\eta & \longrightarrow & H^1_{ab}(F, H^\gamma) \\
& & \downarrow & \downarrow \\
& & H^2(F, T) & \longrightarrow & H^2(F, T).
\end{array}
\]

Here $H^1_{ab}(F, \cdot)$ denotes the abelianized cohomology of $[Boro]$ which coincides for nonarchimedean $F$ as a pointed set with the usual cohomology.
6 Comparison between $\text{PGL}_{2n+1}$ and $\text{Sp}_{2n}$

Recall (see 4.2) that $R$ is either a field of characteristic 0 or the ring of integers $O_F$ of a local non archimedean field $F$ with residue characteristic $\neq 2$.

(6.1) The explicit norm map $\mathcal{N}$. Our final goal being the comparison of $\Theta$-twisted stable orbital integrals on $\text{PGL}_{2n+1}$ with stable orbital integrals on $\text{Sp}_{2n}$, we will represent elements of $\text{PGL}_{2n+1}$ by elements of the groups $\text{GL}_{2n+1}$ resp. $\text{SL}_{2n+1}$.

Let $\text{GL}_n(R)_{\Theta\text{ss}}/\text{traf}$ resp. $\text{SL}_n(R)_{\Theta\text{ss}}/\text{traf}$ be the set of transformation classes of $R$-$\Theta$-semisimple (see 4.2) elements of $h \in \text{GL}_n(R)$ resp. $h \in \text{SL}_n(R)$ under the transformations $h \mapsto \iota ghg$ for $g \in \text{GL}_n(R)$ resp $g \in \text{SL}_n(R)$. Similarly let $\text{Sp}_2(R)_{\Theta\text{ss}}/\text{conj}$ be the set of conjugacy classes of $R$-semisimple elements in $\text{Sp}_2(R)$.

We define a norm map

$$\mathcal{N} : \text{GL}_{2n+1}(R)_{\Theta\text{ss}}/\text{traf} \longrightarrow \text{Sp}_{2n}(R)_{\Theta\text{ss}}/\text{conj}$$

as follows: If $h = p + q \in \text{GL}_{2n+1}(R)$ represents a class of the left hand side, we decompose $M = R^{2n+1} = M_+ \oplus M_- \oplus M_0$ as in lemma 4.6. We consider $M_+$ as the degenerate part of $M$ with respect to $p$ and denote the non degenerate part by $M_- := M_- \oplus M_0$. Since $p_+ = p_- \oplus p_0$ is a unimodular form on $M_+$ we can find a basis $(e_1, \ldots, e_g, f_g, \ldots, f)$ of $M_+$ such that $p_+$ has standard form with respect to this basis by lemma 4.3. Let $P_+$ resp. $Q_+$ be the matrix describing the (skew-)symmetric bilinear form $p_+$ resp. $q_- \oplus q_0$ with respect to this basis ($q_-$ is the zero form). Thus $P_+ = J_{2g}$ and $\text{Sp}(P_+) = \text{Sp}_{2g}$. Now $\mathcal{N}(h)$ or more precisely the image of the class of $h$ under the norm map $\mathcal{N}$ is defined to be the $\text{Sp}_{2n}(R)$-conjugacy class of $1_{2(n-g)} \times C(Q_+) \in \text{Sp}_{2(n-g)}(R) \times \text{Sp}_{2g}(R) \subset \text{Sp}_{2n}(R)$, where we use the Cayley-transform-map $C$ from lemma 4.7.

Remark 6.2. In the situation where the decomposition $M = R^{2n+1} = M_+ \oplus M_-$ is of the form $M = R^{2(n-g)+1} \oplus R^{2g}$ the matrix $h$ splits into the blocks $h_+ \in \text{GL}_{2(n-g)+1}(R)$ and $h_- \in \text{GL}_{2g}(R)$ so that $N_1(h_-) = \iota h_-^{-1} \cdot h_+$ is a symplectic transformation with respect to the alternating part $p_-$ of $h_-$. Then $C(Q_+) \in \text{Sp}_{2g}(R)$ is the conjugate of $N_1(h_-)$ by a matrix, which transforms $p_-$ into the standard form $J_{2g}$.

Proposition 6.3. Let $R$ be as above. Then the following statements hold:

(a) The map $\mathcal{N} : \text{GL}_{2n+1}(R)_{\Theta\text{ss}}/\text{traf} \longrightarrow \text{Sp}_{2n}(R)_{\Theta\text{ss}}/\text{conj}$ is well defined and surjective. In the case $R = O_F$ its fibers are of order $2 = \#(R^\times/(R^\times)^2)$ and describe the two different classes of unimodular quadratic forms on $M_+$.

(b) The restriction $\mathcal{N}_{\text{SL}}$ of $\mathcal{N}$ to $\text{SL}_{2n+1}(R)_{\Theta\text{ss}}/\text{traf}$ is surjective as well. It is bijective if $R$ is an algebraically closed field or if $R = O_F$.

(c) If $h$ represents a class in $\text{GL}_{2n+1}(R)_{\Theta\text{ss}}/\text{traf}$ then the image of $h \cdot J^{-1}\Theta$ in $\text{PGL}_{2n+1}(R) \rtimes \langle \Theta \rangle$ matches with $\mathcal{N}(h)$ in the sense of $\Theta$-endoscopy.
Proof: (a) and (b) The choices made in constructing $N(h)$ only allow $Q_*$ to be replaced by some $t^g \cdot Q_* \cdot g$ for $g \in \text{Sp}(P_*, R)$. By lemma 4.7(c) this does not change the conjugacy class of $N(h)$. Therefore the map $N$ is well defined. To prove surjectivity first observe that each class in $\text{Sp}_{2n}(R)_{ess/\text{conj}}$ has a representative of the form $(1_{2(n-g)}, b)$ with $b \in \text{Sp}_{2g}(R)_{ess}$ by lemma 4.3 with a unique $g \leq n$. The $\text{Sp}_{2g}(R)$-conjugacy-class of $b$ is unique. The bijectivity of the Cayley-transform map and property 4.7(c) then imply that there is a $Q_* \in \text{Sym}_{2g}(R)$, which is unique up to transformations with elements of $\text{Sp}_{2g}(R) = \text{Sp}(P_*, R)$, such that $b = C(Q_*)$. Now we consider the unimodular bilinear form $h_* = P_* + Q_*$ on $R^{2g}$ and some unimodular symmetric bilinear form $q_+$ on $R^{2(n-g)+1}$. The form $q_+ \oplus h_*$ on $R^{2n+1}$ is then unimodular and $R$-$\Theta$-semisimple. Since we can choose $q_+$ in such a way that $\det(q_+ \oplus h'_*) = 1$ we get the surjectivity statements of (a) and (b).

Since the transformation class of $h'_*$ is unique by the considerations above and since $h = q_+ \oplus h_*$ we conclude that the fibers of $N$ correspond to the transformation classes of unimodular quadratic forms on $M_+$. The remaining statements of (a) and (b) now follow from lemma 4.4.

(c) By the definition of matching (1.11) we can work over $R = F$ and therefore may assume that $\gamma = h \cdot J_{2n+1}^{-1}$ has diagonal form $\gamma = \text{diag}(t_1, \ldots, t_{2n+1})$. After applying a permutation in $W_{\text{SO}_{2n+1}}$ we may assume

(i) \[ t_i \neq t_{2n+2-i} \text{ for } i \leq g \text{ and } t_i = t_{2n+2-i} \text{ for } g + 1 \leq i \leq 2n + 1 - g. \]

We have:

\[
\begin{align*}
  h &= \text{antidiag}(t_1, -t_2, t_3, \ldots, t_{2n+1}) \\
  h \pm t^g h' &= \text{antidiag}(t_1 \pm t_{2n+1}, -t_2 \mp t_{2n}, t_3 \pm t_{2n-1}, \ldots, t_{2n+1} \pm t_1) \\
  t^g h^{-1} h &= \text{diag}(t_{2n+1}/t_1, t_{2n}/t_2, \ldots, t_{n+2}/t_n, t_{n}/t_{n+2}, \ldots, t_1/t_{2n+1})
\end{align*}
\]

This means that $M_+ \simeq R^{2(n-g)+1}$ is spanned by the standard basis elements $e_{g+1}, \ldots, e_{2n+1-g}$ of $R^{2n+1}$, and $M_* = M_+ \oplus M_0$ by $e_1, \ldots, e_g, e_{2n+2-g}, \ldots, e_{2n+1}$. Since $h - t^g h$ is an antidiagonal matrix, its non degenerate part can be transformed by a diagonal matrix $d$ into the standard form $J_{2g}$. Now we use remark 6.2 to get the following representative for $N(h)$, observing that conjugation by $d$ does not change a diagonal matrix:

\[
\text{diag}(t_{2n+1}/t_1, t_{2n}/t_2, \ldots, t_{n+2}/t_n, t_{n}/t_{n+2}, \ldots, t_1/t_{2n+1}),
\]

which may be conjugated by an element of the Weylgroup into the form

\[
\text{diag}(t_1/t_{2n+1}, t_2/t_{2n}, \ldots, t_n/t_{n+2}, t_{n+2}/t_n, \ldots, t_{2n+1}/t_1).
\]

The claim now follows from example 1.13. \(\square\)

**Corollary 6.4.** For every semisimple $\bar{\gamma} \Theta \in \widetilde{\text{PGL}}_{2n+1}(F)$ there exists a semisimple $\eta \in \text{Sp}_{2n}(F)$ matching with $\eta$ in the sense of 1.11 and vice versa.
Proof: If $\gamma \in \text{GL}_{2n+1}(F)$ represents a given $\tilde{\gamma}$ one applies part (c) of the proposition to $h = \gamma \cdot J_{2n+1}$. If $\eta$ is given one applies (b) and (c). \hfill \square

**Proposition 6.5.** Let $Z = \text{Cent}(\text{GL}_{2n+1}) \cong \mathbb{G}_m$ denote the center of $\text{GL}_{2n+1}$. Let $\tilde{\gamma} \in \text{PGL}_{2n+1}(F)$ be represented by $\gamma \in \text{GL}_{2n+1}(F)$. Since $2n+1$ is odd we can achieve that $\det(\gamma)$ has even valuation. Then

$$(i) \quad O_{\gamma \Theta}(1, \text{PGL}_{2n+1}) = 2 \cdot O_{\gamma \Theta}(1, \text{GL}_{2n+1}).$$

If moreover $\gamma \Theta$ is strongly compact with topological Jordan decomposition $\gamma \Theta = u \cdot (s \Theta) = (s \Theta) \cdot u$ we have $u \in \text{SL}_{2n+1}(F)$ and get

$$(ii) \quad O_{\gamma \Theta}(1, \text{PGL}_{2n+1}) = O_u(1, \text{SL}_{2n+1}^{s \Theta}).$$

Proof: The relation $\tilde{g} \cdot \tilde{\gamma} \cdot \tilde{g}^{-1} \in \text{PGL}_{2n+1}(O_F)$ means $g \cdot \gamma \cdot \Theta(g)^{-1} = \zeta \cdot k$ with $\zeta \in Z(F) \simeq F^*$ and $k \in \text{GL}_{2n+1}(O_F)$. Since $\det(\Theta(g)) = \det(g)^{-1}$ the relation implies taking determinants

$$(iv) \quad \det(g)^2 \cdot \det(\gamma) \cdot \zeta^{-2n-1} \in O_F^*.$$  

This implies that $\zeta$ has even valuation $2m$ for $m \in \mathbb{Z}$, since the valuation of $\det(\gamma)$ was assumed to be even. If we replace $g$ by $g' = \zeta \cdot \varphi^{-m} \cdot g$ for $\zeta \Theta \in O_F^*$ we get $g' \cdot \gamma \cdot \Theta(g')^{-1} \in \text{GL}_{2n+1}(O_F)$. Conversely the equation (iv) implies that every $g' \in g \cdot Z(F)$ with this property must be of the stated form.

Next observe that the condition $g \in \text{PGL}_{2n+1}^{s \Theta}(F)$ means that we have for some representative $g \in \text{GL}_{2n+1}(F)$ of $\tilde{g}$ and some $\zeta \in Z(F) \simeq F^*$ the relation $g \gamma \Theta(g)^{-1} = \zeta \gamma$. This implies the determinant equation: $\det(g)^2 = \zeta^{2n+1}$. Putting $\rho = \det(g)/\zeta^n \in Z(F)$ this implies $\zeta = \rho^2$ and $\det(g) = \rho^{2n+1}$. If we replace $g$ by $\rho^{-1} \cdot g$ we get $g \gamma \Theta(g^{-1}) = \gamma$ and $\det(g) = 1$. The only other element in $g \cdot Z(F)$ having the first property is $-g$, but $\det(-g) = -1$. This means that we have isomorphisms

$$\text{GL}_{2n+1}(F)^{s \Theta} \cong \text{SL}_{2n+1}(F)^{s \Theta} \times \{ \pm 1 \} \quad \text{and} \quad \text{SL}_{2n+1}(F)^{s \Theta} \cong \text{PGL}_{2n+1}^{s \Theta}(F).$$

Since the normalized Haar measure on $\text{PGL}_{2n+1}(F)$ is the quotient of the normalized Haar measure on $\text{GL}_{2n+1}(F)$ by the normalized Haar measure on $Z(F)$ (i.e. $\text{vol}(Z(O_F)) = 1$) and since the normalized measure on $\text{GL}_{2n+1}(F)^{s \Theta}$ restricts to the normalized Haar measure on $\text{PGL}_{2n+1}^{s \Theta}(F) \simeq \text{SL}_{2n+1}^{s \Theta}(F)$, the above considerations imply the relation (i).

If $\gamma \Theta$ is strongly compact we can assume that $\gamma \in \text{GL}_{2n+1}(O_F)$ and apply lemma 5.5 to get

$$O_{\gamma \Theta}(1, \text{PGL}_{2n+1}) = O_u(1, \text{GL}_{2n+1}^{s \Theta})$$

observing that $[\text{GL}_{2n+1}^{s \Theta}(O_F) : (\text{GL}_{2n+1}^{s \Theta})^0(O_F)] = 2$. But since $\text{GL}_{2n+1}(F)^{s \Theta} \simeq \text{SL}_{2n+1}(F)^{s \Theta} \times \{ \pm 1 \}$ we can apply lemma 5.7 to conclude (iii). \hfill \square
Lemma 6.6. Let $h = sJ \in GL_{2n+1}(O_F)$ be $R$-$\Theta$-semisimple and $b = (1_{2(n-g)}, b_s) \in Sp_{2n}(O_F)$ a representing element of $N(h)$ with $b_s \in Sp_{2g}(O_F)_{ess}$. Since $M_+$ is of odd rank $2(n - g) + 1$ we can identify $(M_+, q_+)$ with $(O_E^{2(n-g)+1}, \epsilon_{q_{sp}})$ for some $\epsilon \in O_E^\times$ and the standard splitform $q_{sp}$. Assume that we have $BC$-matching algebraically semisimple and topologically unipotent elements

$$u_+ \in SO_{2(n-g)+1}(F) \simeq SO(q_+) \text{ and } v_+ \in Sp_{2(n-g)}(F) \simeq \ker(b-1)(F) \cap Sp_{2n}(F)$$

and an additional algebraically semisimple and topologically unipotent element

$$u_s \in SO(q_s)(F) \cap Sp(p_s)(F) \simeq Cent(b_s, Sp_{2g}(F)).$$

Then the elements $\gamma\Theta = s\Theta \cdot (u_+, u_s) = (u_+, u_s) \cdot s\Theta \in PGL_{2n+1}(F)\Theta$ and $\eta := (v_+^2, u_s^2) \cdot b = b \cdot (v_+^2, u_s^2) \in Sp_{2n}(F)$ match.

Proof: As in the proof of lemma 6.1(c) we work in the case $F = F$ and assume that $\gamma$ resp. $\eta$ lie in the diagonal tori. The same holds for the residually semisimple parts $s$ resp. $b$ and the topologically unipotent parts $u = (u_+, u_s)$ and $v = (v_+^2, u_s^2)$. As the matching of $s\Theta$ and $b$ is already proved in 6.1(c) we only have to examine the topologically unipotent elements. We can make the assumption (i) and write

$$\begin{align*}
u_+ &= \text{diag}(w_{g+1}, \ldots, w_n, 1, w_n^{-1}, \ldots, w_{g+1}^{-1}) \in SO_{2(n-g)+1}(F) \\
u_s &= \text{diag}(w_1, \ldots, w_g, w_g^{-1}, \ldots, w_1^{-1}) \in Cent(b_s, Sp_{2g}(F))
\end{align*}$$

By the definition of $BC$-matching we can assume

$$\begin{align*}
u_+ &= \text{diag}(w_{g+1}, \ldots, w_n, w_n^{-1}, \ldots, w_{g+1}^{-1}) \in Sp_{2(n-g)}(F)
\end{align*}$$

Taking everything together we get from the description of $M_+$ and $M_s$ in the proof of lemma 6.1(c):

$$\begin{align*}
u &= (w_1, \ldots, w_n, 1, w_n^{-1}, \ldots, w_1^{-1}) \\
u &= (w_1^2, \ldots, w_n^2, w_n^{-2}, \ldots, w_1^{-2})
\end{align*}$$

and the claim follows again from example 1.13.

The statement of the following theorem is the fundamental lemma for semisimple elements in the stable endoscopic situation $(Sp_{2n}, \widehat{PGL}_{2n+1})$. Recall that the fundamental lemma also predicts the vanishing of orbital integrals for those rational elements, which match with no rational elements on the other side. But in view of corollary 6.4 this case does not occur.

Theorem 6.7. If the semisimple elements $\gamma\Theta \in \widehat{PGL}_{2n+1}(F)$ and $\eta \in Sp_{2n}(F)$ match in the sense of 1.14 and if conjecture $(BC_m)$ is true for all $m \leq n$ then we have

$$O_{\gamma\Theta}^st(1, \widehat{PGL}_{2n+1}) = O_{\eta}^st(1, Sp_{2n}).$$
Proof: Step 1 (Reductions): In the first step we will prove that the nonvanishing of one side of (vi) implies the nonvanishing of the other side and that we can reduce to the following situation:

- \( \gamma \in \text{GL}_{2n+1}(\mathcal{O}_F) \)
- \( \eta \in \text{Sp}_{2n}(\mathcal{O}_F) \)
- the topological Jordan decompositions are of the form

\[
\gamma \Theta = (u_+, u_*) \cdot s \Theta \quad \text{and} \quad \eta = (v_+^2, v_*^2) \cdot b
\]

such that

- \( b \) lies in \( \mathcal{N}(h) \) where \( h = s \cdot J_{2n+1} \),
- \( u_+ \) and \( v_+ \) are BC-matching,
- \( u_* \) can be identified with \( v_* \) under an isomorphism \( \text{Cent}(b_*, \text{Sp}_{2g}(\mathcal{O}_F)) \simeq \text{Aut}(h_*) \)

So let us assume that the right hand side of (vi) does not vanish. Then there exists \( \eta' \in \text{Sp}_{2n}(F) \) stably conjugate to \( \eta \) which has a nonvanishing orbital integral, i.e. can be conjugated into \( \text{Sp}_{2n}(\mathcal{O}_F) \). We can assume that \( \eta' \in \text{Sp}_{2n}(\mathcal{O}_F) \) and that its topological Jordan decomposition satisfies \( \eta' = b' \cdot v' = v' \cdot b \) with residually semisimple \( b' = (1_{2(n-g)}, b_*) \in \text{Sp}_{2(n-g)}(\mathcal{O}_F) \times \text{Sp}_{2g}(\mathcal{O}_F) \) and topologically unipotent \( v' \). We write \( v' \) in the form \( v' = ((v'_+)^2, (u'_*)^2) \) with \( v'_+ \in \text{Sp}_{2(n-g)}(\mathcal{O}_F) \) and \( u'_* \in \text{Cent}(b_*, \text{Sp}_{2g}(\mathcal{O}_F)) \) using 3.6(4) and the general assumption \( p \neq 2 \). Thus we have nonvanishing \( O_{\eta'}(1, \text{Sp}_{2n}) \) and get from the Kazhdan-lemma 5.5 and lemma 5.6

\[
\text{(vi)} \quad O_{\eta'}(1, \text{Sp}_{2n}) = O_{v'}(1, \text{Cent}(b', \text{Sp}_{2n})) = O_{(v'_+)^2}(1, \text{Sp}_{2(n-g)}) \cdot O_{(u'_*)^2}(1, \text{Cent}(b_*, \text{Sp}_{2g})) = O_{v'_+}(1, \text{Sp}_{2(n-g)}) \cdot O_{u'_*}(1, \text{Cent}(b_*, \text{Sp}_{2g})).
\]

Hence the stable orbital integral \( O_{v'_+}^{st}(1, \text{Sp}_{2(n-g)}) \) (being the sum of integrals of nonnegative functions) is strictly positive.

By remark 5.4 there exists a BC-matching between \( v'_+ \) and some \( u'_+ \in \text{SO}_{2(n-g)+1}(F) \). Then the equation (BC\(_{n-g}\)) implies that there exists \( u_+ \in \text{SO}_{2(n-g)+1}(F) \) with strictly positive orbital integral and BC-matching with \( v'_+ \), i.e. we can assume \( u_+ \in \text{SO}_{2(n-g)+1}(\mathcal{O}_F) \).

Let \( h = sJ \in \text{GL}_{2n+1}(\mathcal{O}_F)_{\text{Res}} \) be a residually semisimple element with \( \mathcal{N}(h) = b' \) and define the element \( \gamma' \Theta = (u_+, u'_*) \cdot s \Theta = s \Theta \cdot (u_+, u'_*) \in \text{GL}_{2n+1}(\mathcal{O}_F) \). Here we identify the \( \text{Cent}(s \Theta, \text{GL}_{2n+1}) \simeq G_{h, \Theta} \simeq O(q_+, R) \times \text{Cent}(C(q_*), \text{Sp}(p_*)) \simeq \text{SO}_{2(n-g)+1} \times \text{Cent}(b_*, \text{Sp}_{2g}) \), so that \( (u_+, u'_*) \) can be viewed as an element of the left hand side. The element \( \gamma' \Theta \in \text{PGL}_{2n+1}(\mathcal{O}_F) \) matches with \( \eta' \) (and therefore also with \( \eta \)) by lemma 5.6 and therefore lies in the stable conjugacy class of \( \gamma \Theta \).
If the left hand side of (vi) does not vanish, it is immediate that there exists $\gamma' \Theta \in \widehat{GL_{2n+1}}(O_F)$ in the stable conjugacy class of $\gamma \Theta$. By reversing the above arguments we see that there exists $\eta' \in \text{Sp}_{2n}(O_F)$ in the stable class of $\eta$. So excluding the tautological case that (vi) means $0 = 0$ we may assume without loss of generality that $\gamma \in \text{GL}_{2n+1}(O_F)$ and $\eta \in \text{Sp}_{2n}(O_F)$. We may furthermore assume that $\gamma \Theta = (u_+, u_-) \cdot s \Theta$ and $\eta = (v_+, v_-) \cdot b$ are the topological Jordan decompositions with $BC$-matching $u_+$ and $v_+$ and matching residually semisimple $s \Theta$ and $b$.

Step 2 (Calculation of the symplectic orbital integral): If $\eta' \in \text{Sp}_{2n}(F)$ is stable conjugate to $\eta$ then the residually semisimple parts $b'$ and $b$ are stable conjugate as well. If $\eta'$ has nonvanishing orbital integral then $\eta'$ and therefore also $b'$ can be conjugated into $\text{Sp}_{2n}(O_F)$, i.e. we can assume $b' \in \text{Sp}_{2n}(O_F)$. By the Kottwitz lemma [110] $b'$ and $b$ are conjugate over $\text{Sp}_{2n}(O_F)$ i.e. we can assume $b' = b$. This means that we obtain all relevant conjugacy classes in the stable conjugacy class of $\eta$ if we let $v_+'$ vary through a set of representatives for the conjugacy classes inside the stable conjugacy class of $v_+$ in $\text{Sp}_{2(n-g)}(F)$ and $u_+'$ through a set of representatives for the conjugacy classes inside the stable conjugacy class of $u_+$ in $\text{Cent}(b_+, \text{Sp}_{2g})$. Then the corresponding $\eta'$ are of the form

$$\eta' = b \cdot ((v_+'), (u_+')).$$

We get using (vi) and lemma 5.6

(vii) $O_{\eta}(1, \text{Sp}_{2n}) = \sum_{v_+' \sim v_+} O(v_+')^2(1, \text{Sp}_{2(n-g)}) \cdot \sum_{u_+' \sim u_+} O(u_+')^2(1, \text{Cent}(b_+, \text{Sp}_{2g}))$

$$= \sum_{v_+' \sim v_+} O(v_+')(1, \text{Sp}_{2(n-g)}) \cdot \sum_{u_+' \sim u_+} O(u_+')(1, \text{Cent}(b_+, \text{Sp}_{2g})).$$

Step 3 (Calculation of the $\Theta$-twisted orbital integral): We can repeat this argument in the $\Theta$-twisted situation, since by lemma [111](b) the class of the residually semisimple part $s \Theta$ of $\gamma \Theta$ is the only $\text{PGL}_{2n+1}(F)$-conjugacy class inside the stable class of $s \Theta$, which meets $\text{PGL}_{2n+1}(O_F)$. If we denote by $u_+'$ a set of representatives for the $\text{SO}_{2(n-g)+1}(F)$-conjugacy classes in the stable class of $u_+ \in \text{SO}_{2(n-g)+1}(O_F)$ we therefore get using proposition 6.5

(viii) $O_{\gamma \Theta}(1, \text{PGL}_{2n+1}) = \sum_{(u_+', u_+') \sim (u_+)} O(u_+', u_+')(1, \text{SL}_{2n+1})$

$$= \sum_{u_+' \sim u_+} O(u_+')(1, \text{SO}_{2(n-g)+1}) \cdot \sum_{u_+' \sim u_+} O(u_+')(1, \text{Cent}(b_+, \text{Sp}_{2g})).$$

Step 4 (End of the proof): Since $v_+$ and $u_+$ are $BC$-matching it only remains to apply $(\text{BC}_{n-g})$ in order to identify

$$\sum_{v_+' \sim v_+} O(v_+')(1, \text{Sp}_{2(n-g)}) \quad \text{with} \quad \sum_{u_+' \sim u_+} O(u_+')(1, \text{SO}_{2(n-g)+1}).$$
Thus the right hand sides of (vii) and (viii) coincide, and the proof of the theorem is finished.

\[\square\]
7 Comparison between $GL_{2n} \times GL_1$ and $GSpin_{2n+1}$

**Lemma 7.1** (Cayley transformation again). For a symmetric matrix $q \in GL_n(R)$ the following holds:

(a) We have a bijection

$$\tilde{C} : Alt_n(R)_{q-ess} \to O(q, R)_{ess}, \quad p \mapsto (p - q)^{-1} \cdot (q + p) = -N_i(p + q)$$

between the set $Alt_n(R)_{q-ess}$ of skew-symmetric matrices $p$ such that $p \pm q \in GL_n(R)$ and the set $O(q, R)_{ess}$ of orthogonal transformations $b$ such that $b - 1 \in GL_n(R)$. The inverse map is $\tilde{C}^{-1} : b \mapsto q \cdot (b + 1) \cdot (b - 1)^{-1}$.

(b) $\tilde{C}$ induces a bijection between those elements $q$ of $Alt_n(R)_{q-ess}$, for which $p + q$ is $R$-$\Theta$-semisimple, and the $R$-semisimple elements of $O(q, R)_{ess}$.

(c) The map $\tilde{C}$ satisfies $\tilde{C}((q \cdot p \cdot g) = g^{-1} \cdot \tilde{C}(p) \cdot g$ for $g \in O(q, R)$.

(d) We have $\det(b) = (-1)^n$ for $b \in O(q, R)_{ess}$.

Proof: (a) For $p \in Alt_n(R)_{q-ess}$ we put $h = p + q$ and $b = (p - q)^{-1} \cdot (q + p) = -t^h \cdot h$. Adding the formulas (iii) and (iv) in the proof of lemma 4.7 we get $(-t^b) \cdot q \cdot (-b) = q$, i.e. $b \in O(q, R)$.

Furthermore $b - 1 = (p - q)^{-1} \cdot ((p + q) - (p - q)) = (p - q)^{-1} \cdot 2q \in GL_n(R)$ by the assumptions. The map $\tilde{C}$ is therefore defined.

Conversely we get for $b \in O(q, R)_{ess}$ and $p = q \cdot (b + 1) \cdot (b - 1)^{-1}$ the equivalences:

$$p = -t^p \iff q \cdot (b + 1) \cdot (b - 1)^{-1} = t^b \cdot (b - 1)^{-1} \cdot t^b \cdot (b + 1) \cdot (-q)$$

$$\iff t^b \cdot q \cdot (b + 1) = t^b \cdot q \cdot (1 - b)$$

$$\iff t^b q b + t^b q - q b - q = -t^b q b + t^b q - q b + q$$

$$\iff t^b q b = q \iff b \in O(q, R).$$

Furthermore $p \pm q = q \cdot ((b + 1) \pm (b - 1)) \cdot (b - 1)^{-1} \in GL_n(R)$ since $(b - 1)^{-1}, 2b, 2, q \in GL_n(R)$. Therefore the map $\tilde{C}^{-1}$ is also well defined. An easy calculation using the relation $(b + 1) \cdot (b - 1)^{-1} = (b - 1)^{-1} \cdot (b + 1)$ shows that the maps $\tilde{C}$ and $\tilde{C}^{-1}$ are inverse to another in their domain of definition.

(b) and (c) follow as in the proof of lemma 4.7.

(d) is clear since every $b \in O_n(R)$ with $\det(b) = (-1)^{n-1}$ has 1 as an eigenvalue. (Alternatively we can use (a) and the computation $\det(-t^h \cdot h) = (-1)^n$.)

**Lemma 7.2.** If $q$ is a unimodular symmetric bilinear form on a free $R$-module $N$ and $b \in O(q, R)$ is $R$-semisimple then there exists a $b$-invariant $q$-orthogonal direct sum decomposition $N = N_1 \oplus N_2$ such that $b$ acts as identity on $N_1$ and $b|N_2 \in O(q_*, R)_{ess}$, where $q_*$ is the restriction of $q$ to $N_2$. 

□
Proof: The proof of lemma 4.8 can be adapted with obvious modifications.

(7.3) The explicit norm map \( \mathcal{N} \). Let \((\text{GL}_{2n}(R) \times R^\times)_{\text{Oss}}/\text{traf}\) be the set of transformation classes of \( R\Theta \)-semisimple elements \((h, a) \in \text{GL}_{2n}(R) \times R^\times \) under the transformations \( (h, a) \mapsto (gh, \det g^{-1} \cdot a) \) for \( g \in \text{GL}_{2n}(R), a \in R^\times \). Similarly let \( \text{SO}_{2n+1}(R)_{\text{Oss}}/\text{conj} \) be the set of conjugacy classes of \( R \)-semisimple elements in \( \text{SO}_{2n+1}(R) \). We define a norm map

\[
\mathcal{N} : (\text{GL}_{2n}(R) \times R^\times)_{\text{Oss}}/\text{traf} \to \text{SO}_{2n+1}(R)_{\text{Oss}}/\text{conj}
\]

as follows: If \((h, a) \in \text{GL}_{2n}(R) \times R^\times \) represents a class of the left hand side and if \( h = p + q \) is the decomposition in the symmetric part \( q \) and the skew-symmetric part \( p \), we decompose \( M = R^{2n} = M_+ \oplus M_0 \oplus M_0 \) as in lemma 4.6. The form \( q_* = q_+ + q_0 \) on \( M_+ = M_+ \oplus M_0 \) is unimodular. Since the ranks of \( M_+ \) and \( M_- \) are even we have \( M_+ \cong R^{2r} \) for some \( r \in \mathbb{N}_0 \). Let \( p'_* \) and \( q'_* \) be the \( 2r \times 2r \)-matrices which describe \( p_* \) and \( q_* \) with respect to the standard basis on \( R^{2r} \). Let \( \tilde{q}_- \) be a symmetric bilinear form on \( \tilde{M}_- := R^{2(n-r)+1} \) such that \( \Delta(q'_*) \cdot \Delta(\tilde{q}_-) \in (R^\times)^2 \). By lemma 4.4 we have an isomorphism of quadratic spaces

\[
i : (M_*, q_*) \oplus (\tilde{M}_-, \tilde{q}_-) \to (R^{2n+1}, J_{2n+1})
\]

(observe \( \det(J_{2n+1}) = 1 \)) which induces an injection

\[
j : O(M_*, q_*) \times O(\tilde{M}_-, \tilde{q}_-) \to O(M_+ \oplus \tilde{M}_-, q_* \oplus \tilde{q}_-) \to O_{2n+1}.
\]

This injection is canonical (i.e. independent of the chosen isomorphism \( i \)) on the set of conjugacy classes.

Now \( \mathcal{N}(h) \), the image of the class of \( h \) under \( \mathcal{N} \), is defined to be the \( O_{2n+1}(R) \)-conjugacy class of \( j(\tilde{C}(p'_*), 1_{2(n-r)+1}) \in O_{2n+1}(R) \), where we use the Cayley-transform-map \( \tilde{C} \) with respect to \( q'_* \) from lemma 7.1. We observe that \( \det(\tilde{C}(p'_*)) = 1 \) by lemma 7.1(d) and therefore \( \mathcal{N}(h) \) lies in \( \text{SO}_{2n+1}(R) \). Since the centralizer of \( j(\tilde{C}(p'_*), 1_{2(n-r)+1}) \) in \( O_{2n+1}(R) \) contains \( \{1_{2r}\} \times O_{2(n-r)+1}(R) \) i.e. elements of determinant \(-1\), the \( O_{2n+1}(R) \)-conjugacy class is in fact a \( \text{SO}_{2n+1}(R) \)-conjugacy class.

Lemma 7.4. In the notations of 7.3 the spinor norm of \( \mathcal{N}(h) \) is the class of \( \det(h) \mod (R^\times)^2 \).

Proof: It is sufficient to consider the case \( R = F \), since we have an injection \( O_F^\times/(O_F^\times)^2 \to F^\times/(F^\times)^2 \). If \( \sigma \) denotes the spinor norm of \( \mathcal{N}(h) \) we have by a theorem of Zassenhaus (comp. [Zas]) in the version of [Mas]

\[
\sigma \equiv \det \left( id - \tilde{C}(p'_*) \right) \cdot \Delta(q'_*) \mod (R^\times)^2.
\]

But \( id - \tilde{C}(p'_*) = (q'_* - p'_*)^{-1} \cdot 2 \cdot q'_* \) so that we get \( \sigma \equiv \det(q'_* - p'_*)^{-1} \cdot 2^{2r} \equiv \det(q'_* - p'_*) \mod (R^\times)^2 \). Furthermore \( \det(q'_* - p'_*) = \det(q'_* - p'_*) = \det(p'_* + q'_*) \). Since the discriminant of \( p_- \) is a square we finally get \( \sigma \equiv \det(p'_* + q'_*) \cdot \det(p_-) \equiv \det(h) \mod (R^\times)^2 \).
Proposition 7.5.

(a) The map \( N : (GL_{2n}(R) \times R^\times)_{R\Theta ss/traf} \to SO_{2n+1}(R)_{Rss/conj} \) is well defined and surjective. Two classes lie in the same fiber iff they have representatives of the form \((h, a_1)\) and \((h, a_2)\).

(b) If \((h, a)\) represents a class in \((GL_{2n}(R) \times R^\times)_{R\Theta ss/traf}\) then \((h \cdot J^{-1}, a)\Theta \in (GL_{2n}(R) \times R^\times) \times (\Theta)\) matches in the sense of \(\Theta\)-endoscopy with some element \(\eta \in GSpin_{2n+1}(R)\), which maps to \(N(h)\) under the projection \(pr_{ad} : GSpin_{2n+1} \to SO_{2n+1}\).

Proof: (a) If we replace \(p'\) by some \(^tg \cdot p' \cdot g\) for \(g \in O(q', R)\), this does not change the conjugacy class of \(N(h)\) by lemma 4.3(c). Since the effect of the other choices has already been considered, the map \(N\) is well defined.

To prove surjectivity first observe that each class \(b \in SO_{2n+1}(R)_{Rss/conj}\) can be represented after some transformation of \(J_{2n+1}\) in the form \((b', 1_{2(n-r)+1})\) with \(b' \in SO(q'_s, R)_{ss}\) by lemma 7.2 with a unique \(r \leq n\) and some symmetric \(q'_s \in GL_{2r}(R)\). One should think of \((b', 1_{2(n-r)+1})\) as a block-matrix

\[
\begin{pmatrix}
B_{11} & 0 & B_{12} \\
0 & 1_{2(n-r)+1} & 0 \\
B_{21} & B_{22}
\end{pmatrix}
\]

with \(b' = \begin{pmatrix} B_{11} & B_{22} \\ B_{21} & B_{22} \end{pmatrix}\).

Since the class of \(\Delta(q'_s)\) in \(R^\times/(R^\times)^2\) is the inverse of the class of \(\Delta(J_{2n+1})/\ker(b-1)\), the transformation class of \(q'_s\) is unique by lemma 4.3. Up to this the \(SO(q'_s, R)\)-conjugacy-class of \(b'\) is unique. The bijectivity of the Cayley-transform map and property 7.1(c) then imply that there is a \(p' \in Alt_{2r}(R)\), which is unique up to transformations with elements of \(SO(q'_s, R)\), such that \(b = C(p')\). Now we consider the unimodular bilinear form \(h' = p' + q'\) on \(R^{2r}\), which is unique up to transformations with elements of \(GL_{2r}(R)\), and some unimodular skew symmetric form \(p_-\) on \(R^{2(n-r)}\). The form \(p_- \oplus h'\) on \(R^{2n}\) is then unimodular and \(R-\Theta\)-semisimple i.e. corresponds to a \(R-\Theta\)-semisimple transformation class \(h\). For every \(R\)-semisimple \(a \in R^\times\) we get \(N(h, a) = b\). Since the transformation class of \(h'\) is unique by the above considerations and by lemma 4.3 we conclude that the fibers of \(N\) correspond to the different choices for the \(R\)-semisimple element \(a \in R^\times\).

(b) At first we consider the case that \(R = \bar{F}\) is an algebraically closed field, so that we may assume that \(\gamma = h \cdot J_{2n+1}^{-1}\) has diagonal form \(\gamma = diag(t_1, \ldots, t_{2n})\). After applying a permutation in \(W_{Sp_{2n}}\) we may assume

\[
(i) \quad t_i \neq t_{2n+1-i} \text{ for } i \leq r \text{ and } t_i = t_{2n+1-i} \text{ for } r + 1 \leq i \leq 2n - r.
\]

We have:

\[
\begin{align*}
h &= \text{antidiag}(t_1, -t_2, t_3, \ldots, -t_{2n}) \\
h \pm t'h &= \text{antidiag}(t_1 - t_2, t_2 + t_{2n-1}, t_3 + t_{2n-2}, \ldots, -(t_{2n} + t_1)) \\
-t'h^{-1} \cdot h &= diag(t_2/t_1, t_{2n-1}/t_2, \ldots, t_1/t_{2n})
\end{align*}
\]
Thus $M_+ \simeq R^{2(n-r)}$ is generated by the standard basis elements $e_{r+1}, \ldots, e_{2n-r}$ of $R^{2n}$ and $M_- = M_+ \oplus M_0$ has the basis $e_1, e_2, \ldots, e_r, e_{2n+1-r}, \ldots, e_{2n}$. The matrix of the symmetric bilinear form given by $t' d \cdot (h + t' \cdot h) \cdot d$ with respect to this latter basis has the standard form $J_n'$, if we take $d = \text{diag}((t_1 - t_2), (t_2 - t_{n+1}), \ldots, (t_r - t_{2n+1-r}), 1, \ldots, 1)$. Since $d$ and $t' \cdot h \cdot h$ commute we get from the definition of $\mathcal{N}(h)$ that $\mathcal{N}(h)$ is represented by the diagonal matrix

$$\text{diag}(t_{2n}/t_1, t_{2n}/t_2, \ldots, t_{n+1}/t_n, 1, t_n/t_{n+1}, \ldots, t_1/t_{2n}),$$

which can be conjugated into the form of example 1.14. This proves the claim in the case that $R$ is an algebraically closed field.

In the case that $R$ is arbitrary we consider the commutative diagram with exact rows and columns and a connecting homomorphism marked with $\ldots$ (snake lemma):

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \{\pm 1\} & \longrightarrow & \ldots \\
\downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & \text{Spin}_{2n+1}(R) & \longrightarrow & \text{Spin}_{2n+1}(R) & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & R^x & \longrightarrow & \text{GSpin}_{2n+1}(R) & \longrightarrow & \text{SO}_{2n+1}(R) & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\ldots & \longrightarrow & R^x & \longrightarrow & R^x & \longrightarrow & R^x/(R^x)^2 & \longrightarrow & 1
\end{array}
$$

It follows from this diagram and lemma 7.4 that a matrix $\eta_0$ in the class $\mathcal{N}(h)$ has a preimage $\eta \in \text{GSpin}_{2n+1}(R)$ such that $\mu(\eta) = \det(h) \cdot a^2$ and that the set $\{x \in \text{GSpin}_{2n+1}(\bar{F}) | pr_{ad}(x) = \eta_0, \mu(x) = \det(h) \cdot a^2\}$ just consists of $\pm \eta$. On the other hand by example 1.13 an element $\eta' \in \text{GSpin}_{2n+1}(\bar{F})$ matching with $(h, a)$ satisfies $\mu(\eta') = \det(h) \cdot a^2$. From the validity of the proposition over $\bar{F}$ now follows that either $\eta$ or $-\eta$ matches with $(h, a)$. This element has all desired properties.

Remark: To get (b) it is even in the case $R = F$ not sufficient just to apply Steinbergs theorem on rational elements to the rational conjugacy class inside $\text{GSpin}_{2n+1}(\bar{F})$, which matches with $(h \cdot J^{-1}, a)\Theta$: If $\eta' \in \text{GSpin}_{2n+1}(F)$ denotes such an element, then we only know from the case $R = F$ that $pr_{ad}(\eta')$ and $\mathcal{N}(h)$ are stably conjugate elements in $\text{SO}_{2n+1}(F)$. But the Spinor norm is not invariant under stable conjugation. Thus it is not clear without the use of lemma 7.4 that $\mathcal{N}(h)$ can be lifted to a class in $\text{GSpin}_{2n+1}(F)$, on which the multiplier $\mu$ takes the correct value.
Corollary 7.6. For each semisimple $\eta \in GSpin_{2n+1}(F)$ there exists an $F$-$\Theta$-semisimple $(h \cdot J^{-1}, a)\Theta \in (GL_{2n}(F) \times F^\times) \rtimes \langle \Theta \rangle$ matching with $\eta$.

Proof: By [7.3](a) for $R = F$ there exists $(h, a_1) \in GL_{2n}(F) \times F^\times$ with $pr_{ad}(\eta) \in \mathcal{N}(h)$ and by (b) there exists $\eta_1 \in GSpin_{2n+1}(F)$ matching with $(h \cdot J^{-1}, a_1)\Theta$ such that $pr_{ad}(\eta_1) = pr_{ad}(\eta)$. It follows $\eta = \eta_1 \cdot b$ for some $b \in F^\times \simeq \text{Center}(GSpin_{2n+1}(F))$. Then $(h \cdot J^{-1}, a_1 \cdot b)\Theta$ matches with $\eta$.

Lemma 7.7. For $G = GL_{2n} \times \mathbb{G}_m$ let $\gamma_1, \gamma_2 \in G(O_F), g_F \in G(F)$ be such that $\gamma_2 \Theta = g_F \cdot \gamma_1 \Theta \cdot g_F^{-1}$ with $\Theta$ as in example [1.2]. Then there exists $g_R \in G(O_F)$ with $\gamma_2 \Theta = g_R \cdot \gamma_1 \Theta \cdot g_R^{-1}$.

Proof: Write $\gamma_i = (h_i \cdot J_{2n}^{-1}, a_i), g_F = (h_F, a)$. Then the assumption means: $h_2 = h_F \cdot h_1 \cdot h_{2F}$ and $a_2 = a \cdot a_1 \cdot \det(h_F)^{-1} \cdot a^{-1}$ which implies $\det(h_F) \in O_F^\times$. By lemma 4.11(a) there exists $h_R \in GL_{2n}(O_F)$ with $h_2 = h_R \cdot h_1 \cdot h_{2F}$. This implies $\det(h_R)^2 = \det(h_F)^2$. If $(h_R) = -\det(h_F)$ then $h_R^{-1} = h_R \in O(h_1)(F)$ where $O(h_1) = \{ h \in GL_{2n} \mid h_{1F} \cdot h = h_1 \} = O(q_{r+1})F \cap \text{O}(q_{r+1} \oplus q_{-1})$ has determinant $-1$. This implies $M_{+1} \neq 0$ so that we get an element $h_\xi \in O(h_1)(O_F)$ of determinant $-1$. Replacing $h_R$ by $h_R \cdot h_\xi$ we can now assume $\det(h_R) = \det(h_F)$. With $g_R = (h_R, 1)$ we now have $\gamma_2 \Theta = g_R \cdot \gamma_1 \Theta \cdot g_R^{-1}$.

Lemma 7.8. Let $(h, a) = (sJ, a) \in GL_{2n}(O_F) \times O_F^\times \text{be } O_F\text{-semisimple and } b = (1_{2(n-r)+1}, b_*) \in SO_{2n+1}(O_F)$ a representing element of $\mathcal{N}(h)$. With $p_*, q_*, p_-$ as in [7.3] assume that we have matching topologically unipotent elements $u_\_ \in Sp_{2(n-r)}(F)$ and $v_\_ \in SO_{2(n-r)+1}(F)$ and an additional topologically unipotent element $u_* \in SO(q_*)(F) \cap Sp(p_*)(F) \simeq \text{Cent}(b_*, SO(q_*)(F))$.

Then the element $\gamma \Theta = (s, a)\Theta \cdot (u_-, u_*) = (u_-, u_*) \cdot (s, a)\Theta \in (GL_{2n}(F) \times F^\times) \rtimes \langle \Theta \rangle$ matches with some element $\eta \in GSpin_{2n+1}(F)$, which projects to $\beta := (v_2^2, u_2^*) \cdot b = b \cdot (v_2^2, u_2^*) \in SO_{2n+1}(F)$.

Proof: We first prove the existence of $\eta \in GSpin_{2n+1}(F)$ with the desired properties and thus work over the algebraically closed field $F = \bar{F}$ as in the proof of lemma 6.3(c). Thus we may assume that $\gamma$ and $\beta$ lie in the corresponding diagonal tori. The same holds for the residually semisimple parts $s$ resp. $b$ and the topologically unipotent parts $u = (u_+, u_*)$ and $v = (v_2^2, u_2^*)$. As $(s, a)\Theta$ matches with some $\eta_s$ which projects to $b$ by [7.3](b) we only have to examine the topologically unipotent elements. We can make the assumption [3] and write

\[
    u_- = \text{diag}(w_{r+1}, \ldots, w_n, w_{n+1}^{-1}, \ldots, w_{r+1}^{-1}) \in SP_{2(n-r)}(\bar{F})
\]

\[
    u_* = \text{diag}(w_1, \ldots, w_r, w_r^{-1}, \ldots, w_1^{-1}) \in SO_{2r}(\bar{F})^b.
\]

By the definition of $BC$-matching we can assume

\[
    v_- = \text{diag}(w_{r+1}, \ldots, w_n, 1, w_n^{-1}, \ldots, w_{r+1}^{-1}) \in SO_{2(n-r)+1}(\bar{F}).
\]
Taking everything together we get by the description of $M_-$ and $M_+$ in the proof of lemma 7.5(b):

$$u = (w_1, \ldots, w_n, w_n^{-1}, \ldots, w_1^{-1})$$
$$v = (w_1^2, \ldots, w_n^2, 1, w_n^{-2}, \ldots, w_1^{-2})$$

and the matching between $\gamma \Theta$ and some $\eta \in G\text{Spin}_{2n+1}(F)$ which projects to $\beta$ follows from example 1.14.

To get $\eta$ as an element of $G\text{Spin}_{2n+1}(F)$ we observe that the determinant of $\gamma J_{2n}^{-1}$ equals the spinor norm of $\beta$ as an element of $F^*/(F^*)^2$: This is already clear by 7.4 for the residually semisimple parts, but both topologically unipotent parts lead to the neutral element in $F^*/(F^*)^2$, since $2 \neq p$ by assumption. Now one argues as in the proof of 7.3(b) to get $\eta$ as an $F$-rational element.

**Theorem 7.9.** $(BC_2)$ is true.

Proof: We observe that every pair of $BC$-matching (topologically unipotent) elements $\bar{\gamma} \in \text{SO}_5(F)$ and $\eta_1 \in \text{Sp}_4(F)$ can be obtained from a pair of (topologically unipotent) elements $\gamma \in G\text{Spin}_4(F) \simeq G\text{Spin}_5(F)$ and $\eta \Theta = \Theta \eta \in (G\text{Spin}_4 \times G\text{Spin}_5)(F)$ such that $\bar{\gamma} = \text{pr}_\text{ad}(\gamma)$ and $\eta = (\eta_1, a) \in (G\text{Spin}_4 \times G\text{Spin}_5)^\Theta(F) \simeq (\text{Sp}_4 \times G\text{Spin}_5)(F)$ and such that $\gamma^2$ matches with $\eta \Theta$ in the sense of 1.11. This follows immediately from the definition of $BC$-matching 1.12 and example 1.14.

If we apply lemma 5.8 in the case $G = G\text{Spin}_5 \simeq G\text{Spin}_4$, $T = \mathbb{G}_m$, $H = \text{SO}_5$ and lemma 5.6 we get

$$O_{\gamma_2}^{st}(1, G\text{Spin}_4) = O_{\gamma}^{st}(1, \text{SO}_5)$$

Since we have $O_{\gamma}^{st}(1, \text{Sp}_4 \times G\text{Spin}_5) = O_{\eta_1}^{st}(1, \text{Sp}_4)$ by lemma 5.8 the statement of $(BC_2)$ is equivalent to the identity

$$O_{\gamma}^{st}(1, G\text{Spin}_4) = O_{\eta_1}^{st}(1, \text{Sp}_4 \times G\text{Spin}_5)$$

for matching topologically unipotent $\gamma \in G\text{Spin}_4(F)$ and $\eta \Theta = \Theta \eta \in (G\text{Spin}_4 \times G\text{Spin}_5)(F)$. In the case that $\eta$ is strongly $\Theta$-regular this has been proved in [FL2] ch. II. The general case follows by the germ expansion principle as in [Ha3, Rog].

**Corollary 7.10** (Fundamental lemma for $G\text{Spin}_4 \leftrightarrow \hat{\text{PGL}}_5$). If $\gamma \Theta \in \hat{\text{PGL}}_5(F)$ and $h \in \text{Sp}_4(F)$ are matching semisimple elements then we have

$$O_{\gamma \Theta}^{st}(1, \hat{\text{PGL}}_5) = O_{\eta}^{st}(1, \text{Sp}_4).$$

Proof: This follows from theorem 7.9 (BC1) (compare 5.3) and theorem 6.7.

**Theorem 7.11.** Let $G = GL_{2n} \times \mathbb{G}_m$. If $\gamma \Theta \in \hat{G}(F)$ and $\eta \in G\text{Spin}_{2n+1}(F)$ are matching semisimple elements and if conjecture (BC_m) is true for all $m \leq n$ then we have

(i) $O_{\gamma \Theta}^{st}(1, \hat{G}) = O_{\eta}^{st}(1, G\text{Spin}_{2n+1})$. 

(ii) $O_{\gamma \Theta}^{st}(1, \hat{G}) = O_{\eta}^{st}(1, G\text{Spin}_{2n+1})$. 


Proof: Let $h = pr_{ad}(\eta) \in \text{SO}_{2n+1}(F)$. In view of lemma 5.8 we have to prove

(iii) $O_{\gamma \Theta}^{st}(1, \hat{G}) = O_h^{st}(1, \text{SO}_{2n+1})$.

The proof is now similar to the proof of Theorem 6.7.

Step 1: Let us assume that the right hand side of (iii) does not vanish. Then there exists $h' \in \text{SO}_{2n+1}(F)$ stably conjugate to $h$ which has a nonvanishing orbital integral, i.e. can be conjugated into $\text{SO}_{2n+1}(O_F)$. We can assume that $h' \in \text{SO}_{2n+1}(O_F)$.

Since $h' = pr_{ad}(\eta')$ for some $\eta' \in G\text{Spin}_{2n+1}(F)$ in the stable conjugacy class of $\eta$ we can assume without loss of generality that $\eta' = \eta$ and thus $h' = h$. Furthermore we can assume that the topological Jordan decomposition is of the form $h = b \cdot v = v \cdot b$ with residually semisimple $b = (1_{2(n-r)+1}, b_s) \in \text{SO}_{2(n-r)+1}(O_F) \times \text{SO}(q_s, O_F)_{\text{ess}}$ and topologically unipotent $v$, where $q_s$ denotes the restriction of $J_{2n+1}$ to the orthogonal complement of $\text{ker}(b - 1) \simeq O_F^{2(n-r)+1}$. Here we observe that the restriction of $J_{2n+1}$ to $\text{ker}(b - 1)$ can be assumed to be a multiple of the standard form $J_{2(n-r)+1}$ and that $b_s$ has determinant 1 as $1_{2(n-r)+1}$ has determinant 1. We can write $v$ in the form $v = ((v_-)^2, (u_*)^2)$ with $v_- \in \text{SO}_{2(n-r)+1}(O_F)$ and $u_* \in \text{Cent}(b_s, \text{SO}(q_s, O_F))$ since $2 \in \mathbb{Z}^*_p$.

We remark that condition (*) in the Kazhdan-lemma 5.5 is satisfied by lemma 7.1 so that we get nonvanishing

$$O_h(1, \text{SO}_{2n+1}) = O_v(1, \text{Cent}(b, \text{SO}_{2n+1}))$$

Observing that we have an isomorphism

$$\text{Cent}(b, \text{SO}_{2n+1}) \simeq \text{SO}_{2(n-r)+1} \times \text{Cent}(b_s, \text{SO}(q_s)) \times \{\pm 1\}$$

we can decompose the orbital integral on the right hand side using lemma 5.7

$$O_h(1, \text{SO}_{2n+1}) = O_{(v_-)^2}(1, \text{SO}_{2(n-r)+1}) \cdot O_{(u_*)^2}(1, \text{Cent}(b_s, \text{SO}(q_s)))$$

$$= O_{v_-}(1, \text{SO}_{2(n-r)+1}) \cdot O_{u_*}(1, \text{Cent}(b_s, \text{SO}(q_s))),$$

(use lemma 5.1 in the last step) i.e. $O^{st}_{v_-}(1, \text{SO}_{2(n-r)+1})$ (being the sum of integrals of nonnegative functions) is strictly positive. Since $v_-$ is $BC$-matching with some $u'_- \in \text{Sp}_{2(n-r)}(F)$ the equation (BC$_{n-r}$) implies that there exists $u_- \in \text{Sp}_{2(n-r)}(F)$ matching with $v_-$ and with strictly positive orbital integral, i.e. we can assume $u_- \in \text{Sp}_{2(n-r)}(O_F)$. Let $s \in \text{GL}_{2n}(O_F)$ be a residually semisimple element with $\mathcal{N}(s \cdot J_{2n}^{-1}, a) = b$ for some $a \in O_F^\times$, chosen such that we can identify the corresponding $q_s$ on $M_s$ with the above obtained $q_s$. By modifying $a$ we can assume that $(s, a)\Theta$ matches with the residually semisimple part $\eta_s$ of $\eta$. We define the element $\gamma'\Theta = (u_-, u_*) \cdot s\Theta = s\Theta \cdot (u_-, u_*) \in \hat{G}(O_F)$. The element $\gamma'\Theta \in \hat{G}(O_F)$ matches with $h$ by lemma 7.8 and therefore lies in the stable conjugacy class of $\gamma\Theta$.

If the left hand side of (iii) does not vanish, it is immediate that there exists $\gamma'\Theta \in \hat{G}(O_F)$ in the stable conjugacy class of $\gamma\Theta$. By reversing the above arguments we
see that there exists $h' \in SO_{2n+1}(O_F)$ in the stable class of $h$. So excluding the tautological case that $\gamma \in G(O_F)$ and $h \in SO_{2n+1}(O_F)$. We may furthermore assume that $\gamma \Theta = (u_+, u_*) \cdot s \Theta$ and $h = (v_+^2, u_2^2) \cdot b$ are the topological Jordan decompositions with BC-matching $u_+$ and $v_+$ and matching residually semisimple $s \Theta$ and $\eta$.

**Step 2:** As in the proof of theorem 6.7 we get from lemma 4.10 the fact that we obtain all relevant conjugacy classes in the stable conjugacy class of $h$ if we let $v_-$ vary through a set of representatives of the stable conjugacy class of $u_-$ in $SO_{2(n-r)+1}(F)$ and $u_-'$ through a set of representatives of the stable conjugacy class of $u_*$ in $Cent(b_*, SO(q_*))$ and then consider all $h' = b \cdot ((v_+')^2, (u_')^2)$. i.e.

\[
\begin{align*}
\text{(iv)} \quad & O^s_h(1, SO_{2n+1}) = \sum_{v_-'} O_{(v_-')^2}(1, SO_{2(n-r)+1}) \cdot \sum_{u_-'} O_{(u_-')^2}(1, Cent(b_*, SO(q_*))) \\
& = \sum_{v_-'} O_{v_-'}(1, SO_{2(n-r)+1}) \cdot \sum_{u_-'} O_{u_-'}(1, Cent(b_*, SO(q_*))).
\end{align*}
\]

**Step 3:** We can repeat this argument in the $\Theta$-twisted situation, since by lemma 7.7 the class of the residually semisimple part $(s, a)\Theta$ of $\gamma \Theta$ is the only $G(F)$-conjugacy class inside the stable class of $(s, a)\Theta$, which meets $G(O_F)$ and since the Kazhdan-Lemma 5.5 holds for $\tilde{G}$ by the same lemma. We remark that $G(s, a)\Theta \simeq Sp_{2(n-r)} \times Cent(b_*, SO(q_*)) \times G_m$ by the definition of $\Theta$ and lemma 4.10(e), so $G(s, a)\Theta$ is connected and we can use lemma 5.8 to get rid of the $G_m$ factors in the following orbital integrals. If we denote by $u_-'$ a set of representatives for the $Sp_{2(n-r)}(F)$-conjugacy classes in the stable class of $u_- \in Sp_{2(n-r)}(O_F)$ we get

\[
\begin{align*}
\text{(v)} \quad & O^s_{\gamma \Theta}(1, \tilde{G}) = \sum_{(u_-', u_-') \sim (u_-, u_*)} O_{(u_-', u_-')}(1, G^{s \Theta}) \\
& = \sum_{u_-'} O_{u_-'}(1, Sp_{2(n-r)}) \cdot \sum_{u_-'} O_{u_-'}(1, Cent(b_*, SO(q_*))).
\end{align*}
\]

**Step 4:** Since $v_-$ and $u_-$ are BC-matching we can apply (BC$_{n-r}$) to get that the right hand sides of (iv) and (v) coincide, which proves the theorem.
8 Comparison between $\text{SO}_{2n+2}$ and $\text{Sp}_{2n}$

Let $R$ be as in [4.2]

Lemma 8.1. Let $N$ be a free $R$-module.

(a) If $p$ is a unimodular symplectic form on $N$ and if $\beta \in \text{Sp}(p, R)$ is $R$-semisimple then there exists a $\beta$-invariant orthogonal (with respect to $p$) direct sum decomposition $N = N_+ \oplus N_- \oplus N_*$ such that $\beta$ acts as identity on $N_+$, as $-\text{id}$ on $N_-$ and $\beta_* = \beta|N_* \in \text{Sp}(p_*)$ satisfies $\beta_* - \beta_*^{-1} \in \text{GL}(N_*)$, where $p_*$ is the restriction of $p$ to $N_*$.

(b) If $q$ is a unimodular symmetric bilinear form on $N$ and $b \in O(q, R)$ is $R$-semisimple then there exists a $b$-invariant orthogonal (with respect to $q$) direct sum decomposition $N = N_+ \oplus N_- \oplus N_*$ such that $b$ acts as identity on $N_+$, as $-\text{id}$ on $N_-$ and $b_* = b|N_* \in O(q_*)$ satisfies $b_* - b_*^{-1} \in \text{GL}(N_*)$, where $q_*$ is the restriction of $q$ to $N_*$.

Proof: The proof of lemma [4.8] can be adapted with obvious modifications: We have $b - b^{-1} = b^{-1} \cdot (b - 1) \cdot (b + 1)$, so that $b - b^{-1} \in \text{GL}(N_*)$ is equivalent to $b - 1, b + 1 \in \text{GL}(N_*)$.

Lemma 8.2. Let $b \in \text{GL}_n(R)$ satisfy $b - b^{-1} \in \text{GL}_n(R)$. Then the following holds:

(a) If $q \in \text{GL}_n(R)$ is symmetric and $b \in O(q, R)$ then the matrix $p = q \cdot (b - b^{-1})$ is unimodular skew-symmetric and we have $b \in \text{Sp}(p, R)$.

(b) If $p \in \text{GL}_n(R)$ is skew-symmetric and $b \in \text{Sp}(p, R)$ then the matrix $q = p \cdot (b - b^{-1})^{-1}$ is unimodular symmetric and we have $b \in \text{SO}(q, R)$.

(c) Under the conditions of (a) and (b) we have:

$$\text{Cent}(b, O(q)) = \text{Cent}(b, \text{Sp}(p)) = \text{Cent}(b, \text{SO}(q)).$$

(d) The above statements and formulas are invariant under the substitutions $b \mapsto g^{-1}bg, q \mapsto {}^tggq, p \mapsto {}^tgp$ for $g \in \text{GL}_n(R)$.

Proof: (a) We have $p = q \cdot b - ({}^t b \cdot q \cdot b) \cdot b^{-1} = qb - ({}^t q)(gb)$ and $\cdot bpb - ({}^t b)(gb)b - ({}^t bgb^{-1}) \cdot b = qb - ({}^t bqb) \cdot b^{-1} = gb - q = p$. Unimodularity of $p$ follows from $q, (b - b^{-1}) \in \text{GL}_n(R)$.

(b) We have $q = q \Leftrightarrow - (b - b^{-1})^{-1} \cdot p = p \cdot (b - b^{-1})^{-1} \Leftrightarrow (b - b^{-1})^{-1} = (b + b^{-1}p) \Leftrightarrow (b - b^{-1})^{-1} = (b - b^{-1})^{-1} \cdot (b - b^{-1})^{-1} = (b - b^{-1})^{-1} = q$. As an element of a symplectic group $b$ has determinant $1$.

(c) For $x \in \text{Cent}(b, \text{GL}_n(R))$ we have $xb = bx$ and $b^{-1}x = xb^{-1}$ which imply $x(b - b^{-1}) = (b - b^{-1})x$, so that we get $\cdot xqx = q \Leftrightarrow \cdot xqx(b - b^{-1}) = q(b - b^{-1})$ \Leftrightarrow
\( \gamma \) can work over \( R \) (\( N \) does not depend on the choices we made (apply lemma 8.2(d) ) and is the desired

Let \( \text{Proposition 8.4.} \)

We define a norm map

\[
N : O_{2n+2}(R)_{\text{Rss}}/\text{conj} \rightarrow \text{Sp}_{2n}(R)_{\text{Rss}}/\text{conj}
\]

as follows: If \( b \in O_{2n+2}(R) \) represents a class of the left hand side, we decompose \( N = R^{2n+2} = N_+ \oplus N_- \oplus N_{\ast} \) as in lemma 8.1(b). Let \( b_+ = \text{id}_{N_+}, b_- = -\text{id}_{N_-} \) and \( b_\ast = b | N_{\ast} \). Let \( q_\ast \) be the restriction of the form \( J_{2n+2} \) to \( N_{\ast} \). We may think of \( q_\ast \) as a symmetric matrix after introducing a basis of \( N_{\ast} \). Since \( b_\ast \in \text{Sp}(p) \) for \( p_\ast = q_\ast \cdot (b_\ast - b_\ast^{-1}) \) by lemma 8.2(a) we have \( \det(b_\ast) = 1 \). Therefore \( -1 = \det(b) = \det(b_+) \cdot \det(b_-) \cdot \det(b_\ast) = 1 \cdot (-1)^{\text{rank}N_{\ast}} \cdot 1 \), i.e. \( \text{rank}(N_{\ast}) \) is odd = \( 1 + 2r_+ \). Since \( \text{rank}(N_{\ast}) \) is even by lemma 8.2 we have \( \text{rank}(N_{\ast}) = 1 + 2r_+ \) for some \( r_+ \in \mathbb{N} \). Now we equip the \( R \)-module \( M = M_+ \oplus M_- \oplus N_{\ast} \) where \( M_+ \simeq R^{2r_+}, M_- \simeq R^{2r_-} \) with the alternating form \( p = J_{2r_+} \oplus J_{2r_-} \oplus p_\ast \) and the linear automorphism \( \beta = \text{id}_{M_+} \times -\text{id}_{M_-} \times b_\ast \in \text{Sp}(p) \). Identifying the symplectic space \( (M, p) \simeq (R^{2n}, J_{2n}) \) we can think of \( \beta \) as an element of \( \text{Sp}_{2n}(R) \). The conjugacy class of \( \beta \) in \( \text{Sp}_{2n}(R) \) does not depend on the choices we made (apply lemma 8.2(d) ) and is the desired \( N(b) \). It is clear that \( N(b) \) is \( R \)-semisimple.

**Proposition 8.4.** Let \( R \) be as in \( 4.2 \).

(a) The map \( N : O_{2n+2}(R)_{\text{Rss}}/\text{conj} \rightarrow \text{Sp}_{2n}(R)_{\text{Rss}}/\text{conj} \) is well defined. Each \( b \in O_{2n+2}(R)_{\text{Rss}}/\text{conj} \) matches with \( N(b) \) in the sense of \( \Theta \)-endoscopy (compare examples 1.10, 4.10).

(b) The map \( N \) is surjective, if \( R = \mathcal{O}_F \). Its fibers are of order \( 2 = \#(R^x / (R^x)^2) \) and describe the two different pairs \( (q_+, q_-) \) of classes of unimodular quadratic forms on \( (M_+, M_-) \) such that \( \Delta(q_+) \cdot \Delta(q_-) \equiv \det(q_\ast)^{-1} \mod (R^x)^2 \).

Proof: (a)That \( N \) is well defined is already clear. By the definition of matching we can work over \( R = \mathcal{F} \), so that we may assume that \( \gamma = b \cdot s^{-1} \in \text{SO}_{2n+2}(R) \) has diagonal form \( \gamma = \text{diag}(t_1, \ldots, t_{n+1}, t_{n+1}^{-1}, \ldots, t_1^{-1}) \), where \( s \) is the reflection defined
We have:

\[
b = \begin{pmatrix}
\text{diag}(t_1, \ldots, t_n) & 0 & 0 \\
0 & t_{n+1} & t_{n+1} \\
t^{-1}_{n+1} & 0 & \text{diag}(t_{n+1}^{-1}, \ldots, t_1^{-1})
\end{pmatrix}
\]

\[
b - b^{-1} = \text{diag}(t_1 - t_1^{-1}, \ldots, t_n - t_n^{-1}, 0, t_n^{-1} - t_n, \ldots, t_1^{-1} - t_1)
\]

With the standard basis \((e_i)_{1 \leq i \leq 2n+2}\) of \(R^{2n+2}\) we get:

\[
\begin{align*}
M_+ &= \langle e_i, e_{2n+3-i} \mid t_i = 1, 1 \leq i \leq n \rangle \oplus \langle t_{n+1} \cdot e_{n+1} + e_{n+2} \rangle \\
M_- &= \langle e_i, e_{2n+3-i} \mid t_i = -1, 1 \leq i \leq n \rangle \oplus \langle t_{n+1} \cdot e_{n+1} - e_{n+2} \rangle \\
M_* &= \langle e_i, e_{2n+3-i} \mid t_i \neq \pm 1, 1 \leq i \leq n \rangle
\end{align*}
\]

The corresponding description of \(N = N_+ \oplus N_- \oplus N_*\) can be arranged such that:

\[
\begin{align*}
N_+ &= \langle e'_i, e'_{2n+3-i} \mid t_i = 1, 1 \leq i \leq n \rangle \\
N_- &= \langle e'_i, e'_{2n+3-i} \mid t_i = -1, 1 \leq i \leq n \rangle \\
M_* &= \langle e'_i, e'_{2n+3-i} \mid t_i \neq \pm 1, 1 \leq i \leq n \rangle
\end{align*}
\]

where \(e'_i = (-1)^i \cdot (t_i - t_i^{-1})^{-1} e_i\) if \(t_i \neq \pm 1\) and \(1 \leq i \leq n\) and \(e'_j = e_j\) else. With respect to this new basis of \(M_*\) the symplectic form given by \(p_* = q_* \cdot (b_* - b_*^{-1})\) has standard form \(J_{2q}\), so that the symplectic form \(p\) on \(R^{2n}\) can be assumed to be of standard form \(J_{2n}\) with respect to the basis \(e'_1, \ldots, e'_n, e'_{n+3}, \ldots, e'_{2n+2}\). The symplectic transformation \(\beta = id_{N_+} \times (-id_{N_-}) \times b_*\) in \(\mathcal{N}(b)\) has the diagonal form \(\text{diag}(t_1, \ldots, t_n, t_n^{-1}, \ldots, t_1^{-1})\) with respect to this basis. The claim now follows from example 1.16.

(b) Let \(\beta \in \text{Sp}_{2n}(R)_{Rss}\). We decompose \(N = R^{2n} = N_+ \oplus N_- \oplus N_*\) as in lemma 8.11(a). Since this decomposition is \(J_{2n}\)-orthogonal the restrictions \(p_+, p_-, p_*\) of the symplectic form \(J_{2n}\) to \(N_+, N_-\) and \(N_*\) are unimodular, so these spaces have even rank: \(N_+ \simeq R^{2r_+}, N_- \simeq R^{2r_-}, N_* \simeq R^{2q}\). If we view \(p_*\) as skew symmetric matrix and \(\beta_* \in \text{Sp}(p_*) \subset \text{SL}_{2q}\) we can form the symmetric matrix (bilinear form) \(q_* = p_* \cdot (\beta_* - \beta_*^{-1})^{-1}\) and get \(\beta_* \in \text{SO}(q_*)\). For \(\epsilon \in R^x/(R^x)^2\) we consider the symmetric bilinear forms \(q_+ = \epsilon_+ \cdot J_{1+2r_+}\) on \(M_+ = R^{1+2r_+}\) and \(q_- = \epsilon_- \cdot J_{1+2r_-}\) on \(M_- = R^{1+2r_-}\). By lemma 1.7 there are two different choices of pairs \((\epsilon_+, \epsilon_-)\) such that the quadratic space \((M, q) = (M_+, q_+) \oplus (M_-, q_-) \oplus (N_*, q_*)\) is isomorphic to the standard space \((R^{2n+2}, J_{2n+2})\). For these two choices the element \(b = id_{M_+} \times (-id_{M_-}) \times \beta_* \in O(q, R)\
\) can be viewed as an element of \(\text{O}_{2n+2}(R)_{Rss}\), which maps to \(\beta\) under \(\mathcal{N}\). It is clear from the constructions that the two classes just obtained are all \(\text{SO}_{2n+2}(R)\)-conjugacy classes in \(\text{O}_{2n+2}(R)_{Rss}\) mapping to \(\beta\) under \(\mathcal{N}\). \(\square\)
Lemma 8.5. Let $\gamma_1 \in O_{2n+2}(O_F)^-$ be $R$-$\Theta$-semisimple.

(a) If $\gamma_2 \coloneqq g_F^{-1} \cdot \gamma_1 \cdot g_F \in O_{2n+2}(O_F)^-$ for $g_F \in SO_{2n+2}(F)$ there exists $g_R \in SO_{2n+2}(O_F)$ with $\gamma_2 = g_R^{-1} \cdot \gamma_1 \cdot g_R$.

(b) There is a unique $SO_{2n+2}(O_F)$-conjugacy class $\{\gamma'_1\}$ in $SO_{2n+2}(O_F)$ different from $\{\gamma_1\}$ such that for every $g_F \in SO_{2n+2}(\bar{F})$ with $\gamma_2 \coloneqq g_F^{-1} \cdot \gamma_1 \cdot g_F \in O_{2n+2}(O_F)^-$ there either exists $g_R \in SO_{2n+2}(O_F)$ with $\gamma_2 = g_R^{-1} \cdot \gamma_1 \cdot g_R$ or $g_R' \in SO_{2n+2}(O_F)$ with $\gamma_2 = (g_R')^{-1} \cdot \gamma_1 \cdot g_R'$.

Proof: (a) Let $M = O_F^{2n+2} = M_{+,i} \oplus M_{-,i} \oplus M_{s,i}$ for $i = 1, 2$ be the orthogonal decompositions with respect to $\gamma_i$ as in lemma 8.1(b) and let $q_{\pm,i}, q_{s,i}$ denote the restrictions of the standard form $J_{2n+2}$ to these subspaces. Since $M_{\pm,i}$ are eigenspaces of $\gamma_i$ we have

$$g_F(M_{\pm,2} \otimes O_F F) = M_{\pm,1} \otimes O_F F$$

Since $g_F \in SO_{2n+2}(F)$ we get for the orthogonal complement:

$$g_F(M_{s,2} \otimes O_F F) = M_{s,1} \otimes O_F F.$$ 

In fact $g_F$ induces isomorphisms of quadratic spaces:

$$(M_{\pm,2} \otimes O_F F, q_{\pm,2}) \rightarrow (M_{\pm,1} \otimes O_F F, q_{\pm,1}) \quad \text{and} \quad g_{F,*} : (M_{s,2} \otimes O_F F, q_{s,2}) \rightarrow (M_{s,1} \otimes O_F F, q_{s,1}).$$

Since the quadratic spaces are defined over $O_F$ and become isomorphic over $F$ and since the forms are unimodular, the spaces are isomorphic over $O_F$ by lemma 4.3 i.e. there exists $g_R' \in SO_{2n+2}(O_F)$ inducing isomorphisms

$$(M_{\pm,2}, q_{\pm,2}) \rightarrow (M_{\pm,1}, q_{\pm,1}) \quad \text{and} \quad g_{R,*'} : (M_{s,2}, q_{s,2}) \rightarrow (M_{s,1}, q_{s,1}).$$

If $\gamma_{s,i}$ denotes the restriction of $\gamma_i$ to $M_{s,i}$ we get

(i) $\quad \gamma_{s,2} = g_{F,*}^{-1} \cdot \gamma_{s,1} \cdot g_{F,*}$ \quad \text{and} \quad

(ii) $\quad \gamma_{s,3} := g_{R,*'}^{-1} \cdot \gamma_{s,2} \cdot (g_{R,*'})^{-1} \in SO(q_{s,1}, O_F).$

We have $g_{F,*} \cdot (g_{R,*'})^{-1} \in SO(q_{s,1}, F)$. Now it follows from (ii), (iii) and lemma 4.10 that there exists $g_*$ $\in SO(M_{s,1}, q_{s,1})$ satisfying $g_* \cdot \gamma_{s,3} \cdot g_*^{-1} = \gamma_{s,1}$. Then $g_R := (id_{M_{s,1}} \times id_{M_{+,1}} \times g_*) \cdot g_R' \in SO_{2n+2}(O_F)$ satisfies $g_R \cdot \gamma_2 \cdot g_R^{-1} = \gamma_1$.

(b) Let us assume that $g_2 \in SO_{2n+2}(\bar{F})$ satisfies $\gamma_2 = g_F^{-1} \cdot \gamma_1 \cdot g_F \in O_{2n+2}(O_F)$. We only know that the quadratic spaces become isomorphic over $\bar{F}$, but we have the additional discriminant conditions $\Delta(q_{s,1}) \cdot \Delta(q_{s,1}) \cdot \Delta(q_{s,1}) = \Delta(q_{s,2}) \cdot \Delta(q_{s,2}) \cdot \Delta(q_{s,2})$ and $\Delta(q_{s,1}) = \Delta(p_{s,1}) \cdot \det(\gamma_{s,1} - \gamma_{s,1}^{-1}) = 1 \cdot \det(g_{F,*}(\gamma_{s,2} - \gamma_{s,2}^{-1})g_{F,*}^{-1}) = \Delta(p_{s,2}) \cdot \det(\gamma_{s,2} - \gamma_{s,2}^{-1}) = \Delta(q_{s,2})$ in $O_F^\times/(O_F)^2$, where we use the fact that the $p_{s,i} :=$
with respect to these modified forms on $M_\pm,1$ consider an isomorphism of quadratic spaces $R(M_{\pm,1},q_{\pm,1})$ being fixed this means that there are two choices for the equivalence class of $q_{\pm,2}$ but the isomorphy-typ of the other quadratic spaces $(M_{-2},q_{-2})$ and $(M_{s,2},q_{s,2})$ are then uniquely determined. To construct $\gamma'_1$ we change the quadratic forms $q_{+,1}$ on $M_{+,1}$ and $q_{-,1}$ on $M_{-,1}$ to the other isomorphy-typ but make no change for $M_{s,1}$, consider an isomorphism of quadratic spaces $\iota : R^{2n+2} \to M_{+,1} \oplus M_{-,1} \oplus M_{s,1}$ with respect to these modified forms on $M_{\pm,1},M_{s,1}$ and the standard form $J'_{2n+2}$ on $R^{2n+2}$, and put finally $\gamma'_1 = \iota^{-1} \circ \gamma_1 \circ \iota$. The statement of (b) now follows as in part (a).

Lemma 8.6. In the notations of 8.3 let $b \in O_{2n+2}(\mathcal{O}_F)^-$ be residually semisimple and $\beta = 1_{2r_+} \times (-1_{2r_-}) \times b_+ \in Sp_{2n}(\mathcal{O}_F)$ a representing element of $N(b)$ with $b_+ - b_-^{-1} \in GL_{2g}(\mathcal{O}_F)$.

Assume we have BC-matching topologically unipotent elements $u_+ \in SO_{2r_+ + 1}(F)$ and $v_+ \in Sp_{2r_+}(F)$ resp. $u_- \in SO_{2r_- + 1}(F)$ and $v_- \in Sp_{2r_-}(F)$ and an additional topologically unipotent element $u_\ast \in Cent(b_+,SO(q_+,F)) \cong Cent(b_-,Sp(q_-,F))$. We form the topologically unipotent elements $u = u_+ \times u_- \times u_\ast \in Cent(b,SO_{2n+2}(F))$ and $v = v_+ \times v_- \times u_\ast \in Cent(\beta,Sp_{2n}(F))$.

Then the elements $g := bu = ub \in O_{2n+2}(\mathcal{O}_F)^-$ and $\gamma := \beta v = v\beta \in Sp_{2n}(F)$ match.

Proof: As in the proof of lemma 6.6 we work in the case $F = \bar{F}$ and assume that $g$ resp. $\gamma$ lie in the diagonal tori. The same holds for the residually semisimple parts $b$ resp. $\beta$ and the topologically unipotent parts $u$ and $v$. As the matching of $b$ and $\beta$ is already proved in 8.4 we only have to examine the topologically unipotent elements. We can arrange the diagonal matrices $u_\pm \in SO(q_\pm,F)$ such that their middle entries 1 correspond to the eigenvectors $t_{n+1} \cdot e_{n+1} \pm e_{n+2} \in M_\pm$, which get lost by the construction of $N_\pm$. Then the claim follows immediately from the definition of BC-matching 1.12, example 1.13 and the constructions in the proof of proposition 8.4.

Remark 8.7. The surjectivity statement of Proposition 8.4(b) is not true if $R$ is a field, for example a $p$-adic field $F$: Let $\Delta \in F^*$ denote a non square and

$$\beta = \begin{pmatrix} a_1 & a_2 & b_1 \\ b_2 & b_2\Delta & a_2 \\ b_1\Delta & b_2 & a_1 \end{pmatrix} \in Sp_4(F)$$

where

$$a_i = \frac{\lambda^2_i + \Delta}{\lambda^2_i - \Delta}, \quad b_i = \frac{2 \cdot \lambda_i}{\lambda^2_i - \Delta} \quad \text{for } \lambda_i \in F^*, \quad i = 1, 2.$$

Then we have $N_\ast = N$ and $\beta_\ast = \beta$ for $N = F^4$ and can compute

$$q_\ast := p_\ast \cdot (\beta_\ast - \beta_\ast^{-1})^{-1} = J_4 \cdot \text{antidiag}(2b_1,2b_2,2\Delta b_2,2\Delta b_1)^{-1}$$

$$= \text{diag} \left( \frac{1}{2b_1}, -\frac{1}{2b_2}, \frac{1}{2\Delta b_2}, -\frac{1}{2\Delta b_1} \right) = -\frac{1}{2\Delta b_1} \cdot \text{diag} \left( -\Delta, \frac{b_1}{b_2}, -\frac{b_1}{b_2}, 1 \right).$$
8. Comparison between $SO_{2n+2}$ and $Sp_{2n}$

Thus the quadratic form $q_s$ on $N$ is anisotropic if $b_1 \cdot b_2$ is not a norm of the extension $F\sqrt{\Delta}/F$. In this case $(N, q_s)$ cannot be obtained as direct summand of the six dimensional quadratic split space $(F^6, J_6^o)$. The considerations of $8.3$ and $8.4(b)$ then show that the conjugacy class of $\beta$ is not in the image of $N$.

The following theorem is again the fundamental lemma for a stable endoscopic lift modulo the $BC$-conjecture. But the non surjectivity of $N$ in the case of local fields forces us to include the vanishing statement for orbital integrals of elements, that do not match.

**Theorem 8.8.** Assume that conjecture $(BC_m)$ is true for all $m \leq n$.

(a) If $g \in \widetilde{SO}_{2n+2}(F) = O_{2n+2}(F)$ with $\det(g) = -1$ and $\gamma \in Sp_{2n}(F)$ are matching semisimple elements then we have

(iii) $O^s_\gamma(1, \widetilde{SO}_{2n+2}) = O^s_\gamma(1, Sp_{2n})$.

(b) If the semisimple $\gamma \in Sp_{2n}(F)$ matches with no element of $\widetilde{SO}_{2n+2}(F)$, then we have $O^s_\gamma(1, Sp_{2n}) = 0$.

**Proof:** Since the proof of (a) is similar to the proofs of theorems $6.7$ and $7.11$ we will be sketchy in some steps. We remark that (b) is an immediate corollary of the considerations in Step 1. If the right hand side of (iii) does not vanish one can construct an element $g \in \widetilde{SO}_{2n+2}(F)$ matching with $\gamma$ using proposition $8.4$ and lemma $8.6$.

**Step 1** As in the cited proofs we may assume without loss of generality that $g \in O_{2n+2}(O_F)$ and $\gamma \in Sp_{2n}(O_F)$. We may furthermore assume that $g = (u_+, u_-, u_s) \cdot s$ and $\gamma = (v_+, v_-, v_s) \cdot \sigma$ are the topological Jordan decompositions with $BC$-matching topologically unipotent $u_+ \in SO_{2r_++1}(O_F)$ and $v_+ \in Sp_{2r_+}(O_F)$ respectively $u_- \in SO_{2r_-+1}(O_F)$ and $v_- \in Sp_{2r_-}(O_F)$, matching residually semisimple $s \in O_{2n+2}(O_F)$ and $\sigma \in Sp_{2n}(O_F)$ and topologically unipotent $u_s \in Cent(\sigma_s, Sp_{2g})(O_F)$.

**Step 2** As in $6.7$ we obtain all relevant conjugacy classes in the stable conjugacy class of $\gamma$ if we let $v_+'$ resp. $v_-'$ vary through a set of representatives for the conjugacy classes in the stable conjugacy class of $v_+$ resp. $v_-$ in $Sp_{2r_+}(F)$ resp. $Sp_{2r_-}(F)$ and $v_+'$ through a set of representatives for the conjugacy classes in the stable conjugacy class of $u_s$ in $Cent(\sigma_s, Sp_{2g})(F)$ and then consider all $\gamma' = \sigma \cdot (v_+', v_-, u_s')$ i.e.

(iv) $O^s_\gamma(1, Sp_{2n})$ = $\sum_{v_+' \sim v_+} O^s_{v_+}(1, Sp_{2r_+}) \cdot \sum_{v_-' \sim v_-} O^s_{v_-}(1, Sp_{2r_-}) \cdot \sum_{u_s' \sim u_s} O^s_{u_s'}(1, Cent(\sigma_s, Sp_{2g})).$

**Step 3** In the $\Theta$-twisted situation $O_{2n+2}$ all relevant $\Theta$-conjugacy classes are of the form $g' = s' \cdot (u_+', u_-', u_s')$ where $u_s'$ is as above, $u_+ \cdot v_-'$ vary through a set of representatives for the conjugacy classes in the stable class of $u_+ \in SO_{2r_++1}(F)$ and $s'$
is either $s$ or a representative for the corresponding other conjugacy class $s''$ as in lemma 8.5(b). Observe the centralizers $SO_{2n+2}^s$ and $SO_{2n+2}^{s''}$ can be identified, since the two equivalence classes of symmetric unimodular forms on a free $O_F$-module of odd rank have representatives which are scalar multiples of each other. Therefore we can use the same collections of $u'_\pm$ and $u'_\ast$ for $s$ as for $s''$. The appearance of $s''$ thus introduces just an additional factor 2 in the computation. But since the centralizers $SO_{2n+2}^s$ and $SO_{2n+2}^{s''}$ have two connected components, there appears an additional factor $1/2$ when we apply the Kazhdan-lemma 5.5. Thus we get:

$$(v) \quad O^s_{g}(1, O_{2n+2}) = \sum_{(u'_+, u'_-, u'_\ast) \sim (u_+, u_-, u_\ast)} O_{(u'_+, u'_-, u'_\ast)}(1, SO_{2n+2}^s) = \sum_{u'_+ \sim u_+} O_{u'_+}(1, SO_{2r_++1}) \cdot \sum_{u'_- \sim u_-} O_{u'_-}(1, SO_{2r_-+1}) \cdot \sum_{u'_\ast \sim u_\ast} O_{u'_\ast}(1, Cent(\sigma_* , Sp_2)).$$

In the last step we applied lemma 5.7 in the situation $G = SO_{2n+2}^s = G_1 \times \{ \pm 1_{2n+2} \}$ where $G_1 = SO_{2r_++1} \times SO_{2r_-+1} \times Cent(\sigma_* , Sp_2)$.

Step 4 Since $v_\pm$ and $u_\pm$ are $BC$-matching we can apply $(BC_{r_\pm})$ to get that the right hand sides of $(iv)$ and $(v)$ coincide, which proves the theorem. \qed
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