Abstract

In an early approach, we proposed a kinetic model with multiple translational temperature [K. Xu, H. Liu and J. Jiang, Phys. Fluids 19, 016101 (2007)], to simulate non-equilibrium flows. In this paper, instead of using three temperatures in \( x \), \( y \), and \( z \)-directions, we are going to further define the translational temperature as a second-order symmetric tensor. Based on a multiple stage BGK-type collision model and the Chapman-Enskog expansion, the corresponding macroscopic gas dynamics equations in three-dimensional space will be derived. The zeroth-order expansion gives the 10 moment closure equations of Levermore [C.D. Levermore, J. Stat. Phys 83, pp.1021 (1996)]. To the 1st-order expansion, the derived gas dynamic equations can be considered as a regularization of Levermore’s 10 moments equations. The new gas dynamic equations have the same structure as the Navier-Stokes equations, but the stress strain relationship in the Navier-Stokes equations is replaced by an algebraic equation with temperature differences. At the same time, the heat flux, which is absent in Levermore’s 10 moment closure, is recovered. As a result, both the viscous and the heat conduction terms are unified under a single anisotropic temperature concept. In the continuum flow regime, the new gas dynamic equations automatically recover the standard Navier-Stokes equations. The current gas dynamic equations are natural extension of the Navier-Stokes equations to the near continuum flow regime and can be used for microflow computations. Two examples, the force-driven Poiseuille flow and the Couette flow in the transition flow regime, are used to validate the model. Both analytical and numerical results are encouraging.

1 Introduction

The transport phenomena, i.e., mass, heat, and momentum transfer, in the different flow regime is of a great scientific and practical interest. The classification of various flow regimes is based on the dimensionless parameter, i.e., the Knudsen number, which is a measure of the degree of rarefaction of the medium. The Knudsen number \( Kn \) is defined as the ratio of the mean free path

*email: makxu@ust.hk
†email:zlguo@mail.hust.edu.cn
to a characteristic length scale of the system. In the continuum flow regime where $Kn < 0.001$, the Navier-Stokes equations with linear relations between stress and strain and the Fourier’s law for heat conduction are adequate to model the fluid behavior. For flows in the continuum-transition regime ($0.1 < Kn < 1$), the Navier-Stokes (NS) equations are known to be inadequate. This regime is important for many practical engineering problems. Hence, there is a strong desire and requirement for accurate models which give reliable solutions with low computational costs.

One of the alternative approach to simulate the non-equilibrium flow is those based on the moment closures. Grad’s 13 moment equations are one of the most important ones, which provide the time evolution of the non-equilibrium quantities, such as the stress and the heat flux [8]. However, due to its hyperbolic nature, these equations lead to a well known sub-shocks problem inside a shock layer as the Mach number is larger than a critical value. In order to improve the validity of the 13 moment equations, based on the Chapman-Enskog expansion, Struchtrup and Torrilhon introduced terms of super-Burnett order to the balance of pressure deviator and heat flux vector in the moment equations, and got the regularized 13 moment (R13) equations which have much better performance in the non-equilibrium flow regime [20]. Another well-known moment system is Levermore’s 10 moment closure, which follows his hierarchy of non-perturbative moment closures with many desirable mathematical properties [13]. For example, these equations don’t suffer from the closure-breakdown deficiencies, and they always give physically realizable solution due to non-negative gas distribution function. However, the 10 moment Gaussian closure has no heat flux even though it is proved that Navier-Stokes viscous terms can be recovered in the continuum flow regime. In an effort to extend the Gaussian closure to include higher-order effects, Groth et. al. formulated perturbative variants to the original moment closure with new extended fluid dynamic model [10]. And the most well studied of these closures is a 35-moment closure. Recently, McDonald and Groth took a Chapman-Enskog-type expansion about either the moment equations or the kinetic equation, and introduced the heat flux into Levermore’s 10 moment closures and obtained extended fluid dynamic equations for non-equilibrium flow simulation [17]. The new system present improved results in the transition flow regime where the heat transfer has a significant effect.

In recent years, we have concentrated on the development of numerical schemes for the near continuum flow simulation. In order to capture the non-equilibrium physics in the transitional flow regime, we have extended the gas-kinetic Navier-Stokes flow solver with the following developments [23]. First, a closed solution of the gas distribution function up to the NS order has been used to derive a generalized particle collision time, subsequently to obtain the extended viscosity and heat conduction coefficients [24]. Later, in order to describe the non-equilibrium flow related to the molecular rotational and vibrational degree of freedom, a multiple time relaxation kinetic scheme has been introduced for the shock structure calculations [28]. Recently, the gas-kinetic scheme has been further extended to study the multiple translational temperature non-equilibrium [30]. The schemes developed in the above study give reasonable results in the transitional flow regime, such as the capturing of shock structure at different Mach numbers and flow phenomena which cannot be described properly by the NS equations. In the above simulations, the underlying physical model is a generalized BGK (GBGK) model [30], where the multiple stage relaxation processes have been considered.

In the gas-kinetic schemes, the solutions are obtained without knowing the explicit macroscopic governing equations. In this paper, following the numerical procedure we are going to fill up the gap and derive the underlying macroscopic governing equations for the monatomic gas. In an early approach [30], we introduced the multiple translational temperature into the kinetic model, where the energy exchanges between $x$-, $y$-, and $z$-directions are modeled through the particle collision. Based on the above kinetic model, in one dimensional space the generalized NS equations
are derived, where the viscous term in the NS equations is replaced by the temperature relaxation term. In this paper, we will further develop such a model and regard the temperature as a second order tensor. Since the gas flow may settle to an equilibrium state through multiple stages \[26\], one of the reasonable assumption is to use a Gaussian distribution with multiple temperature as a middle state. Physically, this state corresponds to a gas with different temperature in different directions. Therefore, the thermal energy or particle random motion is represented through a symmetric temperature tensor \(T_{ij}\). Due to historical reason, the temperature is defined as a thermodynamical variable, where there is no bulk fluid velocity variations. So, the temperature becomes a scalar concept. However, in the transition flow regime the gas has large bulk velocity variation, and there is no enough particle collisions to equalize the random particle motion. In order to construct gas dynamic equations, it is justified to extend the temperature from a scalar concept to a tensor. The commutable property of the random particle velocity determines the temperature to be a symmetric tensor. Actually, this kind of non-equilibrium gas property has been routinely extracted from the DSMC solutions. So, based on the physical model with the Gaussian distribution as a middle state between the real gas distribution function \(f\) and the equilibrium Maxwellian \(f^{eq}\) and the strategy used in the construction of the kinetic scheme, we are going to derive the corresponding macroscopic governing equations. Surprisingly, the obtained gas dynamic equations become regularization of Levermore’s 10 moment Gaussian closure, where additional viscous and heat conduction terms are obtained. The structure of gas dynamic equations are almost identical to the Navier-Stokes equations, but the constitutive relationship of the NS one, i.e., \(\sigma_{ij} = -\rho RT^{eq} \delta_{ij} + \mu (\partial_i U_j + \partial_j U_i - \frac{2}{3} \partial_k U_k \delta_{ij})\), is replaced by the new one \(\sigma_{ij} = -\rho RT_{ij} + \rho R(T^{eq}_{ij} - T_{ij})\). At the same time, the heat flux depends on the gradient of the temperature \(T_{ij}\).

This paper is organized in the following. Section 2 is about the introduction of kinetic equation and the generalized particle collision model. At the same time, the 10 moment closure and the generalized gas dynamic equations will be presented. Section 3 is about the applications of the new gas dynamic equations to two flow problems in the near continuum regime. The last section is the conclusion.

2 Generalized Gas Dynamic Equations

In this section, we first review the particle collision model, introduce the Gaussian closure, and derive the new gas dynamic equations. A monatomic gas will be considered in this paper.

2.1 Two stage gas-kinetic collision model

The gas-kinetic Bhatnagar-Gross-Krook (BGK) model has the form \[3\],

\[
\partial_t f + u_i \partial_i f = (f^{eq} - f)/\tau,
\]

where the particle distribution function \(f\) is a function of time \(t\), spatial location \(x_i\), and particle velocity \(u_i\). The left hand side of the above equation represents the free streaming of molecules in space, and the right side denotes the simplified collision term of the Boltzmann equation. In the BGK model, the collision operator involves a single relaxation time \(\tau\) for a non-equilibrium state to evolve to an equilibrium one \(f^{eq}\), which is an isotropic Gaussian

\[
f^{eq} = \frac{\rho}{(2\pi R T^{eq})^{3/2}} \exp[-\frac{(u_i - U_i)(u_i - U_i)}{2RT^{eq}}],
\]
where \( \rho \) is the density, \( T^{eq} \) is the equilibrium temperature, and \( U_i \) is the averaged macroscopic fluid velocity. Traditionally, based on the above BGK model, the Navier-Stokes and higher-order equations, such as Burnett and Super-Burnett, can be derived [7, 19]. Unfortunately, these higher-order equations have intrinsic physical and mathematical problems in the transitional flow regime. In general, the BGK collision term is valid only for flows close to the thermal equilibrium one. In order to extend the capacity of the BGK model to the non-equilibrium flow regime, we can re-write the collision term of the BGK model into two physical sub-processes,

\[
\partial_t f + u_i \partial_i f = (g - f) / \tau + (f^{eq} - g) / \tau,
\]

where \( g \) is a middle state between \( f \) and \( f^{eq} \), see figure 1. The DSMC solutions show that the randomness of particle motion depends on the spatial direction. So, a nature assumption about the middle state is a state with multiple temperature. In the above equation, the term \((f^{eq} - g) / \tau\) has no direct connection with \( f \), therefore, we can consider it as a source term in the above generalized BGK (GBGK) model,

\[
\partial_t f + u_i \partial_i f = (g - f) / \tau + Q,
\]

where \( Q = (f^{eq} - g) / \tau \) for the monatomic gas.

In an early approach, we have assumed that the middle state \( g \) has individual temperature in \( x-, y- \) and \( z- \) directions, and it has the following form in a two dimensional case,

\[
g = \rho \left( \frac{1}{\pi} \right)^{1/2} \left( \frac{l_y}{\pi} \right)^{1/2} \left( \frac{l_z}{\pi} \right)^{1/2} \exp \left[ -l_x(u - U)^2 - l_y(v - V)^2 - l_z w^2 \right].
\]

Here \( l_x = 1/(2RT_x), l_y = 1/(2RT_y), \) and \( l_z = 1/(2RT_z) \) are related to the translational temperature \( T_x, T_y, \) and \( T_z \) in \( x, y, \) and \( z \) directions. Here \( R \) is the gas constant.

In order to solve the above kinetic equation numerically to determine the time evolution of the macroscopic physical quantities \( (\rho, U, V, T_x, T_y, T_z) \), we proposed the following scheme. First, we expand the gas distribution function around the multiple temperature state \( g \), such as \( f = g - \tau(g + u_i \partial_i g) \), from which the numerical fluxes across each cell interface are evaluated. Second, the source term \( Q \) is integrated in a time step inside each computational cell and is regarded as a source term. The physical effect of the source term is to equalize the temperature in different directions. The above processes are composed of the relaxations from the non-equilibrium state \( f \) to a multiple temperature state \( g \), then \( g \) converges to an equal temperature Maxwellian. These two processes may have different relaxation time scales, and the specific formulation of \( g \) depends on the flow problems. According to the above model, we have derived the hydrodynamic equations in one dimensional space [30], where the stress-strain relationship in the Navier-Stokes equations is replaced by the temperature relaxation. In the continuum flow regime, where the middle state is close to the Maxwellian, the standard Navier-Stokes equations can be recovered. The numerical tests presented in [30] verified the validity of the above model. In this paper, we are going to use multiple temperature Gaussian as the middle state, and according to the similar procedure to derive general gas dynamic equations in three dimensional space.

To approximate the Boltzmann particle collision term by multiple BGK-type sub-processes have been investigated before by many authors, such as Callaway [6], Gorban and Karlin [9], and Levermore [13]. For the DSMC method, to include the rotation and vibration modes has been done through the BGK-type relaxation model as well. As realized in [13], for Maxwellian molecules, the use of the multiple BGK-type collision term is correct even if the flow is far away from the equilibrium. This suggests that the generalized BGK operator may be a legitimate approximation to the collision term for use in the near continuum flow regime.
2.2 Generalized gas dynamic equations: zeroth order

For the rarefied flow simulation, a generalized middle state $g$ between $f$ and $f^{eq}$ can be a Gaussian distribution. The Gaussian distribution appears to have been derived in the early work by Maxwell [16], and then re-discovered by many researchers, such as Holway [12] and Levermore [13]. In this paper, we directly define the gas temperature as a tensor instead of a scalar. The traditional temperature concept is coming from thermodynamics, where the local bulk fluid velocity deviation is absent. However, for the transport equations, the temperature basically represents the degree of randomness of particle motion. In order to dynamically equalize the temperature, there should have enough particle collision. In the near continuum flow regime, the number of particle collision inside a microscale device, is limited. It is most likely that the particles will keep the non-isotropic particle randomness.

One of the middle state we can use between $f$ and $f^{eq}$ is the Gaussian distribution,

$$
g = \frac{\rho}{\sqrt{\text{det}(2\pi RT_{ij})}} \exp\left(-\frac{1}{2} (u_i - U_i)(RT_{ij})^{-1}(u_j - U_j)\right),$$

where $T_{ij}$ is the positive definite temperature matrix, which is related to the thermal energy of the particle motion $\rho RT_{ij} = \int (u_i - U_i)(u_j - U_j)gdu$. Due to the commutable property between particle randomness velocities, such as $(u_i - U_i)(u_j - U_j) = (u_j - U_j)(u_i - U_i)$, $T_{ij}$ must be a symmetric tensor. In the Levermore’s approach [13], he developed a non-perturbative method, where the gas distribution function $f$ is assumed to be equal to $g$. Based on the kinetic equation (3) with the assumption $f = g$, i.e.,

$$\partial_t g + u_i \partial_i g = Q,$$

taking the moments

$$\psi = (1, u_i, u_i u_j)^T,$$

on the above equation gives the following Gaussian closure,

$$(5) \quad \partial_t \rho + \partial_k (\rho U_k) = 0,$$

$$(6) \quad \partial_t (\rho U_i) + \partial_k [\rho (U_i U_k + RT_{ik})] = 0,$$

$$(7) \quad \partial_t [\rho (U_i U_j + RT_{ij})] + \partial_k [\rho (U_i U_j U_k + RU_k T_{ij} + RU_i T_{jk} + RU_j T_{ki})] = \frac{1}{\tau} \rho R (T^{eq} \delta_{ij} - T_{ij}).$$

These equations are actually the zeroth-order Chapman-Enskog expansion, i.e., $f = g$, for the generalized BGK model. In the above equations, the source term on the right hand side is coming from the term $Q$ in the generalized BGK model. The equilibrium temperature $T^{eq}$ is defined by

$$T^{eq} = \frac{1}{3} \text{Tr}(T_{ij}).$$

If the state $g$ is equal to the Maxwellian, i.e., $g = f^{eq}$, the above equations reduce to the Euler equations. Based on the above equations, Levermore and Morokoff shows that if the initial data of $T_{ij}$ is symmetric positive definite, then it remains so [14].

Even without heat conduction terms, Levermore showed that the above equations recover the Navier-Stokes viscous terms in the continuum flow regime [13]. For the above system (5)-(7), the left hand side equations have a complete real eigenvalues and eigenvectors [4, 11], and the system is strictly hyperbolic. The temperature difference in different direction, such as the $T_{ij}$, basically shows that the sound speed depends on the spatial direction, the so-called anisotropic wave propagation due to the non-isotropic gas property.
In order to further introduce heat flux into the above 10 moment closure, McDonald and Groth recently took a Chapman-Enskog-type expansion about the Gaussian moment equations \[17\]. The heat conduction is added for the thermal temperature equation,

\[ \partial_t T_{ij} + \partial_k (U_k T_{ij}) + T_{ik} \partial_k U_j + T_{jk} \partial_k U_i + \partial_k Q_{ijk} = \frac{1}{\tau} (T^{eq} \delta_{ij} - T_{ij}), \]

where the heat flux \( Q \) has the form

\[ Q_{ijk} = -\frac{\tau}{\Pr} [T_{ki} \partial_i T_{ij} + T_{ji} \partial_i T_{ik} + T_{il} \partial_i T_{jk}]. \]

### 2.3 Generalized gas dynamic equations: first order

The generalized BGK model includes two relaxation processes. One is from \( f \) to the Gaussian \( g \), and the other is from \( g \) to an equilibrium state \( f^{eq} \). In the last section, the distribution function \( f \) is assumed to be equal to \( g \), and the 10 moment closure equations are derived. However, as presented in figure 1, the real distribution function should be different from \( g \), and the process of relaxation from \( f \) to \( g \) has to be considered. In the past years, we have developed gas-kinetic schemes based on the generalized BGK model, where a gas distribution function \( f \) around \( g \) has been constructed and used to evaluate the numerical fluxes in a finite volume scheme \[30\]. The schemes present reasonable numerical solutions in the near continuum flow regime, such as the micro-channel flow computation \[26\]. In the following, we are going to derive the corresponding macroscopic governing equations underlying the gas-kinetic scheme. The method used here can be regarded as the Chapman-Enskog expansion or iterative expansion \[19\], which are equivalent to each other.

The solution \( f \) around the Gaussian \( g \) is constructed using the iterative expansion to the 1st-order order \[19\],

\[ f = g - \tau (\partial_t g + u_i \partial_i g) + \tau Q, \]

where in the kinetic scheme \( \partial_t g \) is determined using the compatibility condition

\[ \int \psi (\partial_t g + u_i \partial_i g) du = \int \psi Q du, \]

which is exactly the 10 moment closure \[5\]-\[7\]. Substituting the distribution function \( f \) in \[9\] into the BGK model \[3\], the equation becomes

\[ \partial_t g + u_i \partial_i g = \tau (\partial_t^2 g + 2u_i \partial_i g + u_i u_j \partial_i g) + Q - \tau (\partial_t Q + u_i \partial_i Q). \]

Taking the moments \( \psi \) to the above equation and using Eq.\[10\] to express the time derivative in terms of the spatial derivative, we can get the following macroscopic equations,

\[ \partial_t \rho + \partial_k (\rho U_k) = 0, \]

\[ \partial_t (\rho U_i) + \partial_k [\rho (U_i U_k + RT_{ik})] = \partial_k [\rho R (T^{eq} \delta_{ki} - T_{ki})], \]

\[ \partial_t [\rho (U_i U_j + RT_{ij})] + \partial_k [\rho (U_i U_j U_k + RU_k T_{ij} + RU_j T_{jk} + RU_j T_{ki})] \]

\[ = 2 \rho R (T^{eq} \delta_{ij} - T_{ij}) \]

\[ + \partial_k \{ \rho R [U_k (T^{eq} \delta_{ij} - T_{ij}) + U_i (T^{eq} \delta_{jk} - T_{jk}) + U_j (T^{eq} \delta_{ki} - T_{ki})] \} \]

\[ + \rho R^2 \frac{1}{\Pr} (T_{ki} \partial_k T_{ij} + T_{il} \partial_l T_{jk} + T_{jl} \partial_l T_{ki}). \]
The above equations have been written in a similar way as the Navier-Stokes equations. The differences between the above equations and the 10 moment closure are the additional terms appeared on the right hand sides of the momentum and energy equations. It is interesting to see that the corresponding heat conduction term derived above has the same form as that obtained by McDonald and Groth, even though they are obtained through different considerations [17]. Based on the BGK-type collision model, a unit Prandtl number is obtained for the heat conduction term. However, since we believe that up to the NS order the structure of the gas dynamic equations will not be changed due to the BGK collision term or the exact Boltzmann collision model, we add the Prandtl number in the above corresponding heat flux term. Since the viscosity and heat conduction coefficients are the concepts for the continuum flow, the particle collision time
\[ \tau = \mu/(\rho RT^e_{\text{eq}}), \]
where \( \mu \) is the dynamical viscosity coefficient. Certainly, in the rarefied flow regime, the corresponding viscosity coefficient has to be modified [24]. Also, attention should be paid on the relaxation term in the energy equation, where additional \( 2 \) appears.

The above equations can be re-arranged to get the time evolution equation for the thermal energy \( \rho RT_{ij} \),
\[
\partial_t[\rho T_{ij}] + \partial_k[\rho U_k T_{ij}] = 2 \frac{\tau}{\rho}(T^e_{\text{eq}} \delta_{ki} - T_{ij})
- \rho[T_{kj} \partial_k U_i + T_{ki} \partial_k U_j]
+ \partial_k[\rho U_k (T^e_{\text{eq}} \delta_{ij} - T_{ij})] + \rho(T^e_{\text{eq}} \delta_{jk} - T_{jk}) \partial_k U_i + \rho(T^e_{\text{eq}} \delta_{ki} - T_{ki}) \partial_k U_j
\]
\[(a)\]
+ \partial_k \{ \tau R \frac{R}{Pr} (T_{kl} \partial_k T_{ij} + T_{il} \partial_l T_{jk} + T_{jl} \partial_j T_{ki}) \}
\[(b)\]
\[(c)\].

In comparison with the Navier-Stokes equations, the right hand side of the equation [15] has a clear physical meaning. Here (a). the forces multiplied by fluid deformation, which is the heating and cooling of the fluid by compression or expansion and this term represents a reversible process. (b). viscous dissipation, which is responsible for heat generation, and it is always positive and produces internal energy. This is an irreversible process. (c). heat conduction term, which is also irreversible.

The relaxation parameter \( \tau \) controls the distance between the non-equilibrium state \( f \) and the equilibrium one \( f^{eq} \). The current method for the derivation of the gas dynamic equations can be also used to other system, such as these with a more complicated non-equilibrium middle state \( g \), such as the distributions with 14 or 26 moments. However, a distribution function \( g \) with higher-order terms may correspond to macroscopic governing equations without clear physical meanings for the higher-order terms.

The generalized gas dynamic equations [12]-[14] have the same left hand side as the 10 moment closure equations [5]-[7] [14]. And the fluxes on the left have a complete eigenvalues and eigenvectors [4] [11] [18]. In other words, the left hand side is the hyperbolic part. For the right hand side, besides the heat flux as recently derived by McDonald and Groth [17], additional dissipative terms, i.e., \( \rho R(T^e_{\text{eq}} \delta_{ij} - T_{ij}) \) in the momentum equations, and \( \rho R[U_k(T^e_{\text{eq}} \delta_{ij} - T_{ij}) + U_j(T^e_{\text{eq}} \delta_{jk} - T_{jk}) + U_{ij}(T^e_{\text{eq}} \delta_{ki} - T_{ki})] \) in the thermal energy equation, have been obtained. Furthermore, in the present model the relaxation time appearing in the first term on the right hand side of Eq. [14] is \( \tau/2 \), while the corresponding term in the model by McDonald and Groth is \( \tau \). In our two-stage collision model. If we compare the above equations [12]-[14] with the Navier-Stokes equations,
where the energy equation can be also separated in a direction-by-direction componentwise form, we can immediately realize that the corresponding viscous term is given by
\[ \sigma'_{ij} = \rho R(T^{eq}\delta_{ij} - T_{ij}). \]
Therefore, the constitutive relationship for the new gas dynamic equations is
\[ \sigma_{ij} = -\rho RT_{ij} + \rho R(T^{eq}\delta_{ij} - T_{ij}). \]
(16)

In the following, we are going to show that the new constitutive relationship can recover the standard NS formulation in the continuum flow regime.

In the continuum flow regime, the Gaussian distribution g will come to the same state as the equilibrium one \( f^{eq} \), see figure 1. In this case, the 1st-order expansion of \( f \) presented in this section will be expanded basically around \( f^{eq} \). To the leading order of small \( \tau \), Eq.(14) gives the temperature deviation,
\[ \rho R(T^{eq}\delta_{ij} - T_{ij}) \approx (\tau/2)[\partial_i[\rho(U_iU_j + RT_{ij})] + \partial_k[\rho(U_iU_jU_k + R U_kT_{ij} + R U_jT_{kj} + R U_jT_{ki})]]. \]
(17)
In the above equation, when applying the equilibrium condition \( T_{ii} = T^{eq} \) and \( T_{ij} = T^{eq}\delta_{ij} \), and using the Euler equations to replace the temporal derivative by spatial derivative, we can get
\[ \rho R(T^{eq}\delta_{ij} - T_{ij}) \approx (\tau/2)\rho R T^{eq} [\partial_iU_j + \partial_jU_i - \frac{2}{3}\partial_kU_k\delta_{ij}] = (\mu/2)[\partial_iU_j + \partial_jU_i - \frac{2}{3}\partial_kU_k\delta_{ij}]. \]
(18)
Therefore, the constitutive relationship (16) becomes
\[ \sigma_{ij} = -\rho R T^{eq}\delta_{ij} + 2\rho R(T^{eq}\delta_{ij} - T_{ij}) \]
\[ = -\rho \delta_{ij} + \mu [\partial_iU_j + \partial_jU_i - \frac{2}{3}\partial_kU_k\delta_{ij}], \]
(19)
which is exactly the Navier-Stokes stress-strain relationship. In the Navier-Stokes equations, the stress becomes symmetric tensor which is constructed through rational mechanical analysis. However, in the new gas dynamic equations, the ”stress” tensor is automatically a symmetric tensor due to the commutable property of random particle velocities, i.e., \( (u_i - U_i)(u_j - U_j) = (u_j - U_j)(u_i - U_i) \). In order words, the present formulation gives a microscopic interpretation of the origin of symmetric stress tensor. At the same time, the heat flux transported in \( k \)-direction for the thermal energy \( \rho RT_{ij} \) becomes
\[ q_{kij} = \frac{\tau \rho R^2}{Pr}(T_{kl}\partial_lT_{ij} + T_{il}\partial_lT_{jk} + T_{jil}\partial_lT_{ki}). \]
These results are consistent with the early analysis for the one-dimensional flow [30].

In summary, the generalized gas dynamic equations have been derived based on the multiple stage BGK model. With the added Prandtl number Pr and the introduction of dynamic viscosity in the determination of relaxation parameter \( \tau \), the gas dynamic equations derived in this section are a closed system, which is a natural extension of the Navier-Stokes equations. Theoretically, the new gas dynamic equations cover a wider flow regime than that of the Navier-Stokes equations. The new constitutive relationship now has a microscopic physical basis. The viscous term involves only first-order derivatives of the flow variables. The ability to treat non-equilibrium flow problems without evaluating higher than first order derivatives would prove very advantages numerically. The method should be relatively insensitive to irregularity of the grid, the straight-forward computing using Discontinuous Galerkin framework, and the easy implementation of boundary conditions. Due to the low order equations, the less communication between the computational cells (small stencil) makes its numerical scheme more efficient to implement on parallel computing architecture.
3 Solutions of the generalized gas dynamic equations

In this section, we are going to present two flow problems in the near continuum flow regime for the validation of the generalized gas dynamic equations derived in the last section. Both test problems are microchannel flows, but with different flow speed and non-equilibrium properties.

3.1 Analytic solution of force-driven Poiseuille flow in near continuum flow regime

In this subsection, we will apply the generalized gas dynamic equations to rarefied gas flows, such as the case of force driven Poiseuille flow between two parallel plates. Both Direct Simulation Monte Carlo (DSMC) and kinetic theory have shown that even with a small Kn, the pressure and temperature profiles in this flow exhibit a different qualitative behavior from those predicted by the Navier-Stokes equations [15, 21, 22, 2, 31, 29]. Therefore, this flow can serve as a good test problem for any extended hydrodynamic equations intended for non-continuum flow computation.

In our test, the two walls of the channel locate at \( y = \pm H/2 \), respectively, and the force \( \mathbf{a} = (a, 0, 0) \) is along the \( x \) direction. The flow is assumed to be unidirectional, i.e., \( \partial_x \phi = 0 \) for any variable \( \phi \), and the velocity has only an \( x \)-component in the laminar and stationary case, i.e., \( \mathbf{U}_m = (U, 0, 0) \). Under such conditions, the generalized gas dynamic equations (12)-(14) reduce to

\[
2 \frac{dP_{xy}}{dy} - \rho a = 0,
\]

\[
\frac{d}{dy}(2P_{yy} - P) = 0,
\]

\[
\frac{1}{Pr} \frac{d}{dy} \left( \frac{2\tau}{Pr} \frac{d\theta_{xy}}{dy} + \frac{\tau}{Pr} \frac{d\theta_{xx}}{dy} \right) = \frac{2(P_{xx} - P)}{\tau} + \frac{4P_{xy}}{dy},
\]

\[
\frac{3}{Pr} \frac{d}{dy} \left( \frac{\tau}{Pr} \frac{d\theta_{yy}}{dy} \right) = \frac{2(P_{yy} - P)}{\tau},
\]

\[
\frac{1}{Pr} \frac{d}{dy} \left( \frac{\tau}{Pr} \frac{d\theta_{zz}}{dy} \right) = \frac{2(P_{zz} - P)}{\tau},
\]

and

\[
\frac{1}{Pr} \frac{d}{dy} \left( 2\tau P_{yy} \frac{d\theta_{xy}}{dy} + \tau P_{xy} \frac{d\theta_{yy}}{dy} \right) = \frac{2P_{xy}}{\tau} + (2P_{yy} - P) \frac{dU}{dy},
\]

where \( P_{ij} = \rho \theta_{ij} \) is the pressure tensor, and \( P = \rho \theta \) is the pressure, with \( \theta_{ij} = RT_{ij} \) and \( \theta = RT^{eq} \).

It is difficult to obtain an analytical solution from the above nonlinear system. Here we will try to find an approximate solution using a perturbation method similar to that used in Ref. [22]. To this end, we first introduce the following dimensionless variables:

\[
\hat{y} = \frac{y}{H}, \quad \hat{\rho} = \frac{\rho}{\rho_0}, \quad \hat{P}_{ij} = \frac{P_{ij}}{P_0}, \quad \hat{\theta}_{ij} = \frac{\theta_{ij}}{\theta_0}, \quad \hat{U} = \frac{U}{\sqrt{\theta_0}} \quad \text{and} \quad \hat{\tau} = \frac{\tau}{\tau_0},
\]

where the variables with subscript 0 represent the corresponding reference quantities. With these dimensionless variables, the system can be rewritten as

\[
\frac{d \hat{P}_{xy}}{d \hat{y}} - \epsilon \hat{\rho} = 0,
\]

\[
2 \hat{P}_{yy} - \hat{P} = C,
\]

where \( \epsilon = \frac{a}{H} \).
\[
\frac{d}{dy} \left( 2\tau \hat{P}_{xy} \frac{d\theta_{xy}}{dy} + \hat{\theta}_{xy} \frac{d\tau_{xx}}{dy} \right) = 2\Pr \delta^2 \left( \hat{P}_{xx} - \frac{\hat{P}}{\tau} \right) + 4\Pr \delta \hat{P}_{xy} \frac{d\hat{U}}{dy},
\]
\[
\frac{d}{dy} \left( \hat{\theta}_{yy} \frac{d\tau_{yy}}{dy} \right) = 2\Pr \delta^2 \left( \hat{P}_{yy} - \frac{\hat{P}}{\tau} \right),
\]
\[
\frac{d}{dy} \left( \hat{\theta}_{yy} \frac{d\tau_{zz}}{dy} \right) = 2\Pr \delta^2 \left( \hat{P}_{zz} - \frac{\hat{P}}{\tau} \right),
\]
and
\[
\frac{d}{dy} \left( 2\tau \hat{P}_{yy} \frac{d\theta_{xy}}{dy} + \hat{\tau} \hat{P}_{yy} \frac{d\theta_{yy}}{dy} + \hat{\theta}_{yy} \frac{d\tau_{yy}}{dy} \right) = 2\Pr \delta^2 \frac{\hat{P}_{xy}}{\tau} + C\Pr \delta \frac{d\hat{U}}{dy},
\]
where \( C \) is a constant, and \( \epsilon \) and \( \delta \) are two dimensionless parameters given by
\[
\epsilon = \frac{a}{2\theta_0}, \quad \delta = \frac{H}{\tau_0 \sqrt{\theta_0}} = \sqrt{\frac{1}{2} \frac{1}{\Pr}}.
\]

Now we assume that the force acceleration \( a \) is small such that \( \epsilon \ll 1 \), then we can expand the dimensionless flow quantities in powers of \( \epsilon \) (we will omit the hat for simplicity hereafter):
\[
\rho = 1 + \epsilon^2 \rho^{(2)} + O(\epsilon^4), \quad U = \epsilon U^{(1)} + O(\epsilon^3), \quad C = 1 + \epsilon^2 C^{(2)},
\]
\[
P_{xy} = \epsilon P_{xy}^{(1)} + O(\epsilon^3), \quad \theta_{xy} = \epsilon \theta_{xy}^{(1)} + O(\epsilon^3),
\]
\[
P_{ii} = 1 + \epsilon^2 P_{ii}^{(2)} + O(\epsilon^4), \quad \theta_{ii} = 1 + \epsilon^2 \theta_{ii}^{(2)} + O(\epsilon^4), \quad \text{for} \quad i = x, y, z,
\]
and
\[
P = 1 + \epsilon^2 P^{(2)} + O(\epsilon^4), \quad \theta = 1 + \epsilon^2 \theta^{(2)} + O(\epsilon^4), \quad \tau = 1 + \epsilon^2 \tau^{(2)} + O(\epsilon^4).
\]

The odd or even properties of the variables as functions of \( \epsilon \) are based on their symmetric properties in terms of the acceleration \( a \). In general, the odd(even) velocity moments of the distribution function \( f \) are also odd (even) functions of \( \epsilon \). Furthermore, from the definition \( P_{ij} = \rho \theta_{ij} \) we can obtain some useful relation which will be used later:
\[
(26) \quad \rho^{(2)} = P^{(2)} - \theta^{(2)} = P_{ii}^{(2)} - \theta_{ii}^{(2)}, \quad P_{xy}^{(1)} = \theta_{xy}^{(1)}, \quad \text{for} \quad i = x, y, z.
\]

Substituting the above expansions into the nondimensional system, we can obtain the first order differential equation in \( \epsilon \). Eqs. (20) and (25) go to
\[
\frac{dP_{xy}^{(1)}}{dy} - 1 = 0,
\]
\[
\Pr \delta \frac{dU^{(1)}}{dy} = \frac{d^2 \theta_{xy}^{(1)}}{dy^2} - 2\Pr \delta^2 P_{xy}^{(1)},
\]
which give
\[
(27) \quad P_{xy}^{(1)} = \theta_{xy}^{(1)} = y,
\]
\[
(28) \quad U^{(1)} = -\delta y^2 + C',
\]
where \( C' \) is a constant. Here, we have made use of the symmetry property of the velocity profile about \( y = 0 \).
From Eqs. (21), (22), (23), and (24), we can obtain the second order equations in \( \epsilon \):

\[
2P_{yy}^{(2)} - P^{(2)} = C^{(2)},
\]

\[
\frac{d^2\theta_{xx}^{(2)}}{dy^2} = 2\Pr\delta^2 \left( \theta_{xx}^{(2)} - \theta^{(2)} \right) - 1 - 8\Pr\delta^2 y^2,
\]

\[
3\frac{d^2\theta_{yy}^{(2)}}{dy^2} = 2\Pr\delta^2 \left( \theta_{yy}^{(2)} - \theta^{(2)} \right)
\]

and

\[
\frac{d^2\theta_{zz}^{(2)}}{dy^2} = 2\Pr\delta^2 \left( \theta_{zz}^{(2)} - \theta^{(2)} \right)
\]

where the results up to the first order Eqs. (27) and (28) have been used. Summing up the above three equations we have

\[
\frac{d^2}{dy^2} \left( 3\theta^{(2)} + 2\theta_{yy}^{(2)} \right) = -1 - 8\Pr\delta^2 y^2.
\]

From Eqs. (31) and (33), we can get

\[
\theta^{(2)} = \frac{2}{3} A \cosh \left( \frac{\sqrt{10}}{3} K y \right) - \frac{2K^2}{15} y^4 + \frac{43}{50} y^2 + \frac{231}{125K^2} + B,
\]

\[
\theta_{yy}^{(2)} = A \cosh \left( \frac{\sqrt{10}}{3} K y \right) - \frac{2K^2}{15} y^4 - \frac{77}{50} y^2 - \frac{693}{250K^2} + B.
\]

Since \( \theta^{(2)} \) is obtained, the other two temperatures \( \theta_{xx}^{(2)} \) and \( \theta_{zz}^{(2)} \) from Eqs. (30) and (32) can be constructed, respectively.

Now let’s find the second order pressure \( P^{(2)} \). From Eqs. (26) and (29), we can obtain

\[
P^{(2)} = C^{(2)} + 2(\theta^{(2)} - \theta_{yy}^{(2)}) = C^{(2)} - \frac{10A}{3} \cosh \left( \frac{\sqrt{10}}{3} K y \right) + \frac{24}{5} y^2 + \frac{231}{25K^2}.
\]

Then, \( \rho^{(2)} \) can be determined from (26) as

\[
\rho^{(2)} = P^{(2)} - \theta^{(2)} = C^{(2)} - \frac{8A}{3} \cosh \left( \frac{\sqrt{10}}{3} K y \right) + \frac{2K^2}{15} y^4 + \frac{197}{50} y^2 + \frac{924}{125K^2} - B.
\]

With the above results, we finally get the approximate solutions of the problem:

\[
\frac{U}{\sqrt{RT_0}} = -\epsilon \delta \left( \frac{y}{H} \right)^2 + U_s,
\]

\[
\frac{P}{P_0} = 1 + \epsilon^2 P^{(2)},
\]

\[
\frac{T}{T_0} = 1 + \epsilon^2 \theta^{(2)}.
\]

With these results, we are able to make some discussions on the velocity, pressure, and temperature profiles for the force-driven Poiseuille flow problem. First, it is clear that the velocity profile
is parabolic. But for the pressure and temperature, their profiles may be complicated due to the presence of the hyperbolic cosine function.

It is interesting to compare the approximate solutions of the present gas dynamic equations to those of the Navier-Stokes-Fourier (NSF) equations. For the same flow problem, the NSF equations reduce to

\[
\frac{d}{dy} \left( \mu \frac{dU}{dy} \right) + \rho a = 0, \\
\frac{dP}{dy} = 0, \\
\frac{d}{dy} \left( k \frac{dU}{dy} \right) + \mu \left( \frac{dU}{dy} \right)^2 = 0.
\]  
(37)  
(38)  
(39)

Using the similar perturbation method, we can get the following approximate solutions for the NSF equations:

\[
U = -\epsilon \delta \left( \frac{y}{H} \right)^2 + U_s, \quad P = \text{const}, \quad \frac{T}{T_0} = 1 + \epsilon^2 \theta^{(2)}_{\text{nsf}},
\]  
(40)

where \( \theta^{(2)}_{\text{nsf}} = -(2/15)\Pr \delta^2 \hat{y}^4 + D \), and \( D \) is a constant. It is shown that the velocity profile of the NSF equations is also parabolic, which is the same as the that of the generalized hydrodynamic equations. However, the NSF equations give a constant pressure, which is qualitatively different from the predictions of the new proposed model. For the temperature, it is evident that \( \theta^{(2)}_{\text{nsf}} \) is only part of \( \theta^{(2)} \), and has one local maximum at \( y = 0 \). On the other hand, the presence of terms of \( \cosh(y) \) and \( y^2 \) in \( \theta^{(2)} \) of the new model makes it possible to have a local minimum at \( y = 0 \), due to \( \left[ \theta^{(2)} \right]_{y=0} = 0 \) and \( \left[ \theta^{(2)} \right]_{y=0}' = -K^2(100A + 216)/135 + 1.72 > 0 \) as \( A < 2.322K^{-2} - 2.16 \).

To see this more clearly, we present the pressure and temperature profiles for the case of \( \text{Kn}=0.1 \) which was studied extensively using DSMC method [31]. The constants \( A \) and \( B \) in \( \theta \) and \( P \) of the present hydrodynamic model are obtained by enforcing the values of \( \theta \) at \( y = 0 \) and \( -H/2 \) to be the same as the DSMC data, and then \( C \) is obtained by enforcing \( P(y=0) \) to be the DSMC value. The NSF solutions are obtained by enforcing their pressure and temperature values to be identical to the DSMC data at \( y = 0 \). In Fig.2, the reduced temperature and pressure variations, \( \theta^{(2)} = (T/T_0 - 1)/\epsilon^2 \) and \( P^{(2)} = (P/P_0 - 1)/\epsilon^2 \), are shown for both the NSF equations and the present model. It is seen that the temperature and pressure profiles of the present model are in qualitatively agreement with the DSMC data. For example, the temperature takes a bimodal shape and exhibits a local minimum at \( y = 0 \), and the pressure has two local maximums near the two walls. These critical flow behaviors are absent in the profiles of the NSF equations. For example, the temperature minimum only appears on the super-Burnett order if the traditional BGK collision model is used to construct the gas dynamic equations [7, 25]. These observations demonstrate the fundamental difference between the present hydrodynamic model and the NSF equations.

### 3.2 Numerical solution for Couette Flows

For the new gas dynamic equations, a corresponding gas-kinetic scheme can be developed [30]. In order to further test the validity of the new governing equations, we will study the planar Couette flow here [1]. The height of the Couette system is \( h_0 = 50nm \). The wall’s temperatures are fixed at \( T_0 = 273K \) and at equilibrium, the pressure of the gas is 1 atm. Given that the walls distance is less than a mean-free path and the relative wall speed is high, the gas system will be strongly out of equilibrium. Specifically, the velocity distribution for the particles is non-Maxwellian. Same as the case in [1], the temperature along the channel is defined as \( T_x \) and the one perpendicular
to walls is $T_w$. With the fixed Knudsen number $Kn=1.25$, and various wall velocities, such as the Mach number $M = 0.5, 1.0, \text{ and } 1.5$, the simulation results from the new gas dynamic equations are shown in Fig.3, where the DSMC solutions are also included [1]. Clearly, at high Knudsen number and Mach numbers, the temperature is anisotropic. The new gas dynamic equations basically are capable in capturing this kind of non-equilibrium flow phenomena. Also, it is fully necessary for any gas dynamic equations to consider the temperature as a tensor for the non-equilibrium system.

4 Conclusion

Based on the multiple stage particle collision BGK model and the Gaussian distribution function as the middle state, the generalized gas dynamic equations have been derived. Since the gas temperature basically represents the molecular random motion, the direct extension of the temperature concept from a scalar to a second-order symmetric tensor $T_{ij}$ is physically reasonable. In the non-equilibrium flow regime, the randomness of the particle distribution indeed depends on the spatial orientation. The new gas dynamic equations have the same structure as the Navier-Stokes equations, but the NS constitutive relationship,

$$\sigma_{ij} = -\rho R T^{eq} \delta_{ij} + \mu (\partial_i U_j + \partial_j U_i - \frac{2}{3} \partial_k U_k \delta_{ij})$$

is replaced by

$$\sigma_{ij} = -\rho R T_{ij} + \rho R (T^{eq} \delta_{ij} - T_{ij}).$$

At the same time, the heat flux in the $k$–direction for the transport of thermal energy $\rho R T_{ij}$ becomes

$$q_{kij} = \frac{\tau \rho R^2}{Pr} (T_{kl} \partial_l T_{ij} + T_{il} \partial_l T_{jk} + T_{jl} \partial_l T_{ki}).$$

In the continuum flow regime, the generalized constitutive relationship and the heat flux term go back to the corresponding Navier-Stokes formulations. The new gas dynamic equations can be regarded as a regularization of Levermore’s 10 moment closure [13]. The gas dynamic equations have a wider applicable flow regime than that of the NS equations. They capture the time evolution of the anisotropic non-equilibrium flow variables, as demonstrated in our examples.

The traditional temperature concept is coming from thermodynamics, where there is no anisotropic particle random motion in space. However, for the non-equilibrium flow transport, due to the inadequate of particle collision the random molecule motion can become easily anisotropic. To directly consider the temperature as a tensor rather than a scalar is a reasonable description for the non-equilibrium flow. For a dilute gas, due to the lack of long range particle interaction, the randomness particle motion is the only source for the dissipation in the system. Under the new definition of the temperature, all dissipative effects in a dilute gas system, such as the viscosity and heat conduction, can be unified under the same concept $T_{ij}$. The current gas dynamic equations can be useful in the study of microflows. The further development of the gas dynamic equations based on the kinetic model and its scheme in [27], which are valid for the compressible shock waves, will be conducted in the near future.

Acknowledgement

K. Xu would like to thank Prof. C. Groth and Mr. J.G. McDonald for helpful discussion, and Dr. H.W. Liu for his help in the numerical testing of Couette flow. This research was supported by Hong Kong Research Grant Council 621005. Z. Guo acknowledges the support of the National Natural Science Foundation of China (50606012).
Appendix: Gas Dynamic Equations in 2D Space

As a special application, we consider 2D case, where there are 7 unknowns \((\rho, U, V, T_{xx}, T_{xy}, T_{yy}, T_{zz})\). The generalized gas dynamic equations for those unknowns are

\[
\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} + \frac{\partial \mathbf{G}}{\partial y} = \frac{\partial \mathbf{E}_V}{\partial x} + \frac{\partial \mathbf{F}_V}{\partial y} + \mathbf{S},
\]

where \(\mathbf{U}\) is the vector macroscopic flow variables, and \(\mathbf{F}\) and \(\mathbf{G}\) are \(x\) and \(y\) direction flux vectors given by

\[
\mathbf{U} = \begin{pmatrix}
\rho \\
\rho U \\
\rho V \\
\rho(U^2 + RT_{xx}) \\
\rho(UV + RT_{xy}) \\
\rho(V^2 + RT_{yy}) \\
\rho RT_{zz}
\end{pmatrix},
\]

\[
\mathbf{F} = \begin{pmatrix}
\rho U \\
\rho(U^2 + RT_{xx}) \\
\rho(U^3 + 3URT_{xz}) \\
\rho(U^2V + 2URT_{xy} + VRT_{xx}) \\
\rho(UV^2 + URT_{yy} + 2VRT_{xy}) \\
\rho URT_{zz}
\end{pmatrix},
\]

\[
\mathbf{G} = \begin{pmatrix}
\rho V \\
\rho(UV + RT_{xy}) \\
\rho(U^2V + VRT_{xx} + 2URT_{xy}) \\
\rho(UV^2 + URT_{yy} + 2VRT_{xy}) \\
\rho(V^3 + 3VRT_{yy}) \\
\rho VRT_{zz}
\end{pmatrix},
\]

\[
\mathbf{E}_V = \begin{pmatrix}
\rho R(T_{eq}T_{xx} - T_{xx}) \\
\rho R(T_{eq}T_{xy} - T_{xy}) \\
3\rho R[UT_{eq}T_{xx} - T_{xx}] + \tau R(T_{xx}\partial_y T_{xx} + T_{xy}\partial_y T_{xx}) \\
\rho R[V(T_{eq}T_{xx} - T_{xx}) + 2UT_{eq}T_{xy} - T_{xy} + \tau R(T_{xx}\partial_y T_{xx} + T_{xy}\partial_y T_{xx} + 2T_{xx}\partial_y T_{xx} + 2T_{xy}\partial_y T_{yy} + 2T_{xx}\partial_x T_{yy} + 2T_{xy}\partial_x T_{yy})] \\
\rho R[U(T_{eq}T_{yy} - T_{yy}) + 2VT_{eq}T_{xy} - T_{xy} + \tau R(T_{xx}\partial_y T_{xx} + 2T_{xx}\partial_x T_{yy} + 2T_{xy}\partial_x T_{yy} + 2T_{xy}\partial_x T_{yy} + 2T_{xx}\partial_y T_{yy} + 2T_{xx}\partial_x T_{yy})]
\end{pmatrix},
\]

\[
\mathbf{F}_V = \begin{pmatrix}
\rho R(T_{eq}T_{xx} - T_{xx}) \\
\rho R(T_{eq}T_{xy} - T_{xy}) \\
3\rho R[VT_{eq}T_{xx} - T_{xx}] + \tau R(T_{xx}\partial_y T_{xx} + T_{xy}\partial_y T_{xx} + T_{xx}\partial_x T_{yy} + 2T_{xy}\partial_x T_{yy} + 2T_{xx}\partial_x T_{yy} + 2T_{xy}\partial_x T_{yy}] \\
\rho R[V(T_{eq}T_{yy} - T_{yy}) + 2VT_{eq}T_{xy} - T_{xy} + \tau R(T_{xx}\partial_y T_{xx} + 2T_{xx}\partial_x T_{yy} + 2T_{xy}\partial_x T_{yy} + 2T_{xy}\partial_x T_{yy} + 2T_{xx}\partial_y T_{yy} + 2T_{xx}\partial_x T_{yy})]
\end{pmatrix},
\]

and the source term is

\[
\mathbf{S} = \begin{pmatrix}
0 \\
0 \\
0 \\
2\rho R(T_{eq}T_{xx} - T_{xx})/\tau \\
2\rho R(T_{eq}T_{xy} - T_{xy})/\tau \\
2\rho R(T_{eq}T_{yy} - T_{yy})/\tau \\
2\rho R(T_{eq}T_{zz} - T_{zz})/\tau
\end{pmatrix}.
\]
In the above equations, the equilibrium temperature

\[ T^{eq} = T^{eq}_{xx} = T^{eq}_{yy} = T^{eq}_{zz} = \frac{1}{3} \text{tr}(T_{ij}), \]

and

\[ T^{eq}_{xy} = T^{eq}_{yx} = 0. \]

References

[1] F.J. Alexander, A.L. Garcia, and B.J. Alder, Direct simulation Monte Carlo for thin-film bearings, Phys. Fluids 6, no. 12, pp. 3854-3860 (1994).

[2] K. Aoki, S. Takata, and T. Nakanishi, Poiseuille-type flow of a rarefied gas between two parallel plates driven by a uniform external force, Phys. Rev. E 65, 026315 (2002).

[3] P. L. Bhatnagar, E. P. Gross, and M. Krook, A model for collision processes in gases. I: Small amplitude processes in charged and neutral one-component systems, Phys. Rev. 94, 511–525 (1954).

[4] Shawn L. Brown, Approximate Riemann solvers for moment models of dilute gases, PhD Thesis at Aerospace Engineering department, University of Michigan (1996).

[5] C.P. Cai, D. Liu, and K. Xu, A one-dimensional multiple-temperature gas-kinetic BGK scheme for shock wave computation, AIAA J, to appear (2008).

[6] J. Callaway, Model for lattice thermal conductivity at low temperature, Phy Rev 113 (1959), pp. 1046-1051.

[7] S. Chapman and T. G. Cowling, The Mathematical Theory of Non-Uniform Gases (Cambridge University Press, Cambridge, 1971).

[8] H. Grad, On the kinetic theory of rarefied gases, Communications on Pure and Applied Math. 2 (1949), pp. 331-407.

[9] A.N. Gorban and I.V. Karlin, General approach to constructing model of the Boltzmann equation, Phyica A 206 (1994), pp. 401-420.

[10] C.P.T. Groth, P.L. Roe, T.I. Gombosi, S.L. Brown, On the nonstationary wave structure of a 35-moment closure for rarefied gas dynamics, paper AIAA 95-2312, June (1995).

[11] J.A. Hittinger, Foundations for the generalization of the Godunov method to hyperbolic systems with stiff relaxation source terms, PhD Thesis at Aerospace Engineering department, University of Michigan (2000).

[12] L.H. Holway, New statistical models for kinetic theory: methods of construction, Physics of fluids 9 (1966), pp. 1658-1673.

[13] C.D. Levermore, Moment closure hierarchies for kinetic theories, J. Stat. Phys. 83 (1996), pp. 1021-1065.

[14] C.D. Levermore and W.J. Morokoff, The Gaussian moment closure for gas dynamics, SIAM J. Appl. Math. 59 (1998), pp. 72-96.
[15] M. M. Malek, F. Baras, and A. L. Garcia, On the validity of hydrodynamics in plane Poiseuille flows, Physica A 240, 255 (1997).

[16] J.C. Maxwell, On the dynamical theory of gases, Philosophical Transactions of Royal Society of London 157 (1867), pp. 49-88.

[17] J.G. McDonald and C.P.T. Groth, Extended fluid-dynamic model for micron-scale flows based on Gaussian moment closure, 46th AIAA Aerospace Sciences Meeting and Exhibit, 7-10 January 2008, Reno, Nevada, AIAA 2008-691.

[18] J.G. McDonald, Numerical modeling of micron-scale flows using the Gaussian moment closure, Master Thesis at Graduate Department of Aerospace Engineering, University of Toronto (2005).

[19] T. Ohwada and K. Xu, The kinetic scheme for full Burnett equations, J. Comput. Phys. 201 (2004), pp.315-332.

[20] H. Struchtrup and M. Torrilhon, Regularization of Grad’s 13 moment equations: derivation and linear analysis, Physics of Fluids 15 (2003), pp. 2668-2680.

[21] M. Tij and A. Santos, Perturbation analysis of a stationary nonequilibrium flow generated by an external force, J. Stat. Phys. 76, 1399 (1994).

[22] M. Tij, M. Sabbane, and A. Santos, Nonlinear Poiseuille flow in a gas, Phys. Fluids 10, 1021 (1998).

[23] K. Xu, A gas-kinetic BGK scheme for the Navier-Stokes equations and its connection with artificial dissipation and Godunov method, J. Comput. Phys. 171, 289–335 (2001).

[24] K. Xu, Regularization of the Chapman-Enskog expansion and its description of shock structure, Physics Fluids 14 (2002), L17.

[25] K. Xu, Super-Burnett solutions for Poiseuille flow, Physics of Fluids 15, No. 7, pp. 2077-2080 (2003).

[26] K. Xu, A Generalized Bhatnagar-Gross-Krook Model for non-equilibrium flows, Phys. Fluids 20, 026101 (2008).

[27] K. Xu and E. Josyula, A multiple translational temperature model and its shock structure solution, Physical Review E 71, 056308 (2005).

[28] K. Xu and E. Josyula, Continuum formulation for non-equilibrium shock structure calculation, Communications Comput. Phys. 1, 425-450 (2006).

[29] K. Xu and Z.H. Li, Microchannel flows in slip flow regime: BGK-Burnett solutions, J. Fluid Mech. 513, 87 (2004).

[30] K. Xu, H. Liu and J. Jiang, Multiple temperature kinetic model for continuum and near continuum flows, Phys. Fluids 19, 016101 (2007).

[31] Y. Zheng, A. L. Garcia, and B. J. Alder, Comparison of Kinetic Theory and Hydrodynamics for Poiseuille Flow, J. Stat. Phys. 109, 495 (2002).
Equilibrium state

Multiple-temperature state

Original BGK model

Non-equilibrium state

Figure 1: Schematic representation of particle collision relaxation processes.

Figure 2: Reduced temperature (up) and pressure (down) variations at $Kn = 0.1$, where the results from the DSMC [31], the current equations, and the Navier-Stokes equations are presented.
Figure 3: Ratio of the temperature components $T_x/T_z$ vs the vertical position $Z = z/h_0$ in planar Couette flow for Knudsen number $Kn=1.25$, and Mach numbers $M = 0.5, 1.0, \text{ and } 1.5$. The solid lines are the solutions of the new gas dynamic equations and the circles are the DSMC solutions [1].