Simple Hyper-heuristics Optimise LEADINGONES in the Best Runtime Achievable Using Randomised Local Search Low-Level Heuristics

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Abstract

Selection hyper-heuristics are randomised search methodologies which choose and execute heuristics from a set of low-level heuristics. Recent research for the LEADINGONES benchmark function has shown that the standard Simple Random, Permutation, Random Gradient, Greedy and Reinforcement Learning selection mechanisms show no effects of learning. The idea behind the learning mechanisms is to continue to exploit the currently selected heuristic as long as it is successful. However, the probability that a promising heuristic is successful in the next step is relatively low when perturbing a reasonable solution to a combinatorial optimisation problem. In this paper we generalise the ‘simple’ selection-perturbation mechanisms so success

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can be measured over some fixed period of time $\tau$, rather than in a single iteration. For LEADINGONES we prove that the Generalised Random Gradient hyper-heuristic has the best possible performance achievable with the low-level heuristics (Randomised Local Search with different neighbourhood sizes), up to lower order terms. The performance of the hyper-heuristic improves as the number of low-level heuristics to choose from increases. In particular, with access to $k$ low-level heuristics, it outperforms the best-possible algorithm using less than $k$. Experimental analyses confirm these results for different problem sizes (up to $n = 10^8$) and shed some light on the best choices of the parameter $\tau$ in various situations.

1 Introduction

Many successful applications of randomised search heuristics to real-world optimisation problems have been reported. Despite these successes, it is still difficult to decide which particular search heuristic is a good choice for the problem at hand and what parameter settings should be used. In particular, while it is well understood that each heuristic will be efficient on some classes of problems and inefficient on others (Wolpert and Macready, 1997), very little guidance is available explaining how to choose an algorithm for a given problem. The high level idea behind the field of hyper-heuristics (HHs) is to overcome this difficulty by evolving the search heuristic for the problem rather than choosing one in advance (or applying several arbitrary ones until a satisfactory solution is found). The overall goal is to automate the design and the tuning of the algorithm and parameters for the optimisation problem, hence achieving a more generally applicable system.

Hyper-heuristics are usually classified into two categories: selection hyper-heuristics and generation hyper-heuristics (Burke et al., 2013). The former repeatedly select from a set of low-level heuristics to create new solutions for the problem, while the latter generate new heuristics from components of existing heuristics. The low-level heuristics can be further classified as either construction heuristics, which build a solution incrementally, or perturbation heuristics, which start with a complete solution and try to iteratively improve the current solution. While hyper-heuristics are general algorithm generation and selection methodologies, they may also be used as parameter control methods (see (Doerr and Doerr, 2018) for a comprehensive review of parameter control techniques). For instance, the HH may be asked to select between different heuristics that only differ by one parameter value.
However, while the aim of parameter control mechanisms is to essentially choose between different parameter values for a given algorithm, selection HHs work at a higher level since their aim is to learn which low-level heuristics (i.e., algorithms) work best at different stages of the optimisation process.

Despite the numerous successes for NP-hard optimisation problems, including scheduling (Cowling et al., 2001, 2002; Gibbs et al., 2010), timetabling (Özcan et al., 2010), vehicle routing (Asta and Özcan, 2014), cutting and packing (López-Camacho et al., 2014), the theoretical understanding of selection hyper-heuristics is very limited. Some insights into the behaviour of selection heuristics have been achieved via landscape analyses (Maden et al., 2009; Ochoa et al., 2009a,b). Concerning their performance, Lehre and Özcan (2013) presented the first runtime analysis of selection-perturbation hyper-heuristics. They considered a Simple Random hyper-heuristic (Cowling et al., 2001) that at each step randomly chooses between low-level heuristics and presented an example benchmark function class, called GAPPATH, where it is necessary to use more than one low-level heuristic to optimise the problem. Similar example functions have been constructed by He et al. (2012).

A comparative time-complexity analysis of selection hyper-heuristics has recently been presented by Alanazi and Lehre (2014). They considered several common selection mechanisms, namely Simple Random, Permutation, Random Gradient and Greedy (Cowling et al., 2001, 2002) and analysed their performance on the standard LEADINGONES benchmark function when using a low-level heuristic set consisting of a 1-bit flip and a 2-bit flip operator (i.e., the same set previously considered by Lehre and Özcan (2013)). The runtime analysis results show that the four ‘simple’ selection mechanisms have the same asymptotic expected runtime, while an experimental evaluation demonstrates that the runtimes are indeed equivalent already for small problem dimensions. Recently, additive Reinforcement Learning selection was, under mild assumptions, also shown to often have asymptotically equivalent performance to Simple Random selection, including for the same problem setting (i.e., LEADINGONES selecting between 1-bit and 2-bit mutation operators) (Alanazi and Lehre, 2016). In particular, the results indicate that selection mechanisms such as additive Reinforcement Learning and Random Gradient do not learn to exploit the more successful low-level heuristics and end up having the same performance as Simple Random selection.

The main idea behind the learning mechanisms considered above is to continue to exploit the currently selected heuristic as long as it is successful. Unlike construction heuristics, where iterating a greedy move on a currently successful heuristic may work for several consecutive construction steps,
the probability that a promising heuristic is successful in the next step is relatively low when perturbing a reasonable solution to a combinatorial optimisation problem.

In this paper, we generalise the ‘simple’ Random Gradient selection-perturbation learning mechanism analysed by Alanazi and Lehre (2014) so success can be measured over some fixed period of time \( \tau \) (i.e., Exploitation Stage), rather than in a single iteration. We use the LEADINGONES (LO) benchmark function to show that the generalised hyper-heuristic can be surprisingly fast. We first give exact leading constants in the expected runtimes for the ‘simple’ mechanisms on LO, proving that they all have expected runtime \( \frac{\ln(3)}{2} n^2 + o(n^2) \), confirming what was implied by the experimental analysis of Alanazi and Lehre (2014). This result indicates that all the ‘simple’ mechanisms essentially choose operators at random in each iteration, and perform worse than the single operator that always flips one bit (i.e., RLS) which in expectation takes \( \frac{1}{2} n^2 \) iterations to optimise LO. We then prove upper bounds on the same function for the Generalised Random Gradient (GRG) hyper-heuristic. We rigorously prove that the generalised hyper-heuristic has a better expected running time than RLS for appropriately chosen values for the parameter \( \tau \). Furthermore, the GRG hyper-heuristic can achieve the best possible expected runtime \( \left( \frac{1+\ln(2)}{4} n^2 + o(n^2) \right) \approx 0.42329 n^2 \), up to lower order terms, for an algorithm using the two mutation operators when \( \tau \) satisfies both \( \tau = \omega(n) \) and \( \tau \leq \left( \frac{1}{2} - \epsilon \right) n \ln(n) \), for some constant \( \epsilon > 0 \).

Since all the presented results consider hyper-heuristics that choose between only two operators, we present an analysis of the hyper-heuristics for LEADINGONES when they are allowed to choose between \( k = \Theta(1) \) operators as in practical applications. We first show that including more operators is detrimental to the performance of the ‘simple’ mechanisms. On the other hand, we prove that the performance of the GRG hyper-heuristic improves as the number of available operators to choose from increases. In particular, when choosing from \( k \) operators as low-level heuristics, GRG is, in expectation, faster than the best possible performance achievable of any algorithm using \( m < k \) operators, for \( k = \Theta(1) \).

We conclude the paper with an experimental analysis of the hyper-heuristics for different problem sizes (up to \( n = 10^8 \)). The experiments confirm that the Generalised Random Gradient hyper-heuristic outperforms its low-level heuristics. For two operators, it is shown how proper choices for the parameter \( \tau \) can lead to the near optimal performance already for these problem sizes, while a relationship between the performance of the hyper-heuristics and the problem size \( n \) is suggested. For hyper-heuristics that are able to choose between more than two operators, the experiments
Algorithm 1 Simple Selection Hyper-heuristic (Cowling et al. [2001, 2002; Alanazi and Lehre, 2014])

1: Choose $s \in S$ uniformly at random
2: while stopping conditions not satisfied do
3:     Choose $h \in H$ according to the learning mechanism
4:     $s' \leftarrow h(s)$
5:     if $f(s') > f(s)$ then
6:         $s \leftarrow s'$

show that the performance of GRG improves with more operators with appropriate choices of the learning period $\tau$.

The rest of the paper is structured as follows. In Section 2 we formally introduce the hyper-heuristic framework together with the ‘simple’ selection hyper-heuristics and the Generalised Random Gradient hyper-heuristic. In Section 3 we analyse the ‘simple’ and generalised hyper-heuristics on the LEADINGONES benchmark function. In Section 4 we present the results for hyper-heuristics that choose between more than two operators. Section 5 presents the experimental analysis. In the conclusion we present a discussion and some avenues for future work.

Compared to its conference version (Lissovoi et al. [2017]) this manuscript has been considerably extended. The results of Section 3 have been generalised to hold for any values of $\tau$ less than $(\frac{1}{2} - \epsilon) n \ln n$, for some constant $\epsilon > 0$. In general, the results have been considerably strengthened. In particular, compared to the extended abstract, in the present manuscript we prove that GRG optimises LO in the best possible expected runtime achievable, up to lower order terms. Section 4 is a completely new addition, while a more comprehensive set of experiments is included in Section 5.

2 Preliminaries

Let $S$ be a finite search space, $H$ a set of low-level heuristics and $f : S \rightarrow \mathbb{R}^+$ a cost function. Algorithm 1 shows the pseudocode representation for a ‘simple’ selection hyper-heuristic as used in previous experimental and theoretical work (Cowling et al. [2001, 2002; Alanazi and Lehre, 2014]).

The following learning mechanisms have been commonly used in the literature to solve combinatorial optimisation problems (Cowling et al., 2001, 2002):

- **Simple Random**, which selects a low-level heuristic independently with probability $p_h$ in each iteration (usually $p_h = \frac{1}{|H|}$, i.e., uniformly
at random);

- **Permutation**, which generates a random ordering of low-level heuristics and returns them in that sequence when called by the hyper-heuristic;

- **Greedy**, which applies all available low-level heuristics in parallel and returns the best found solution;

- **Random Gradient**, which randomly selects a low-level heuristic, and keeps using it as long as it obtains improvements.

Alanazi and Lehre (2014) derived upper and lower bounds on the expected runtime of these four ‘simple’ mechanisms (throughout the paper, bounds on the expected runtimes of the considered ‘simple’ mechanisms refer to the number of fitness evaluations performed by Algorithm 1 using these mechanisms) for the LEADINGONES benchmark function, choosing between flipping either one or two randomly chosen bits of the bit string with replacement \( H = \{1\text{BitFlip}, 2\text{BitFlip}\} \). It was shown experimentally by Alanazi and Lehre (2014) that all mechanisms have the same performance as just choosing low-level heuristics at random. We prove this by deriving the exact expected runtimes of the ‘simple’ mechanisms in Section 3. By making a heuristic selection decision in every iteration without taking past performance into account, the ‘simple’ mechanisms do not have enough time to learn which operator is preferable at the current optimisation stage.

In this paper we generalise the ‘simple’ Random Gradient mechanism to allow a longer time period to decide whether a low-level heuristic is currently successful or not, aiming to maintain the intrinsic ideas of the ‘simple’ learning mechanism while generalising sufficiently to allow learning to take place. In particular, the hyper-heuristic we present works as follows:

- **Generalised Random Gradient (GRG)**: A low-level heuristic is chosen uniformly at random (Decision Stage) and run for a period

1Throughout this paper, when discussing the MBitFlip operator, we refer to the mutation operator which flips \( m \) bits on the bit-string with replacement; that is, it is possible to flip and re-flip the same bit within the same mutation step. Since this has been used in previous literature on the topic (Lehre and Özcan, 2013; Alanazi and Lehre, 2014, 2016), we naturally continue with this choice. We note, without proof, that the results would hold for an operator flipping \( m \) bits without replacement (i.e., an operator that selects a new bit-string with Hamming distance \( m \) from the original bit-string); any performance differences would be limited to lower order terms. The latter operators are well known Randomised Local Search algorithms with neighbourhood size \( m \) (i.e., RLS\(_m\)). In particular, when flipping (with replacement) \( k = \Theta(1) \) bits, at least one bit is re-flipped with probability at most \( k/n = o(1) \); this will only affect lower order terms in the expected runtime results that we present.
Algorithm 2 Generalised Random Gradient Hyper-Heuristic

1: Choose $x \in S$ uniformly at random
2: while stopping conditions not satisfied do
3: \quad Choose $h \in H$ uniformly at random
4: \quad $c_t \leftarrow 0$
5: \quad while $c_t < \tau$ do
6: \quad \quad $c_t \leftarrow c_t + 1; x' \leftarrow h(x)$
7: \quad \quad if $f(x') > f(x)$ then
8: \quad \quad \quad $c_t \leftarrow 0; x \leftarrow x'$

of fixed time $\tau$. If an improvement is found before the end of the period, then a new period of time $\tau$ is immediately initialised (i.e., $c_t$ in Algorithm 2 is set to 0 immediately) (Exploitation Stage). If the chosen operator fails to provide an improvement in $\tau$ iterations, a new operator is chosen at random. The pseudocode for this hyper-heuristic is shown in Algorithm 2.

2.1 The Pseudo-Boolean Benchmark Function LEADINGONES

The LEADINGONES (LO) function counts the number of consecutive one-bits in a bit string before the first zero-bit:

$$LO(x) := \sum_{i=1}^{n} \prod_{j=1}^{i} x_j.$$ 

The unbiased black box complexity of LEADINGONES is $\Theta(n^2)$ (Lehre and Witt, 2012). This bound is matched by the performance of simple well studied heuristics. Randomised Local Search has an expected runtime of $0.5n^2$ fitness function evaluations (Buzdalov and Buzdalova, 2015) and it is well known that the standard (1+1) EA (with mutation rate $\frac{1}{n}$) has an expected runtime of $\frac{e-1}{3}n^2 - o(n^2) \approx 0.85914n^2$ (Böttcher et al., 2010). Böttcher et al. (2014) also showed that the best static mutation rate for the (1+1) EA is $\approx \frac{1.5696}{n}$, which improves the expected runtime to $\frac{e}{5}n^2 - O(n) \approx 0.77201n^2$. Furthermore, they showed that the (1+1) EA with an appropriately chosen dynamic mutation rate (i.e., $\frac{1}{LO(x)+1}$) can outperform any static choice giving an expected runtime of $\approx 0.68n^2$. If biased mutation operators are allowed but truncation selection is imposed then no asymptotic improvement may be achieved over the unbiased black box complexity. Indeed, Doerr and Lengler
(2013) recently proved that the best possible asymptotic performance of any (1+1) elitist black-box algorithm on LEADINGONES is \( \Omega(n^2) \).

Nevertheless, the unrestricted black box complexity of LO is \( O(n \log \log n) \) (Afshani et al., 2013) and there exist randomised search heuristics with expected runtimes of \( o(n^2) \). Recently some Estimation of Distribution Algorithms (EDAs) have been presented which have surprisingly good performance on the problem. Doerr and Krejca (2018) introduced a modified compact Genetic Algorithm (cGA) called sig-cGA that, rather than updating the frequency vector every generation, does so only once it notices a significance in its history of samples. They proved that the algorithm optimises LO, ONEMAX and BINVAL in \( O(n \log n) \) expected function evaluations. Two other algorithms have been proven to optimise LO in the same asymptotic expected time: a stable compact Genetic Algorithm (scGA) that biases updates that favour frequencies that move towards \( \frac{1}{2} \) (Friedrich et al., 2016) and a Convex Search Algorithm (CSA) using binary uniform convex hull recombination (for sufficiently large populations and with an appropriate restart strategy) (Moraglio and Sudholt, 2017). While the latter two algorithms have exceptional performance for LO, their runtime is very poor for ONEMAX, respectively providing runtimes at least exponential and super-polynomial in the problem size with high probability (Doerr and Krejca, 2018).

In this paper we aim to prove that a simple hyper-heuristic (GRG) runs in the best possible expected runtime achievable using RLS with different neighbourhood sizes (i.e., RLS\(_k\) with \( k\text{BITFLIP} \) mutation) as low-level heuristics. In particular, we will first derive the best possible expected runtime achievable by applying the RLS\(_k\) algorithms in any order and then prove that GRG matches it up to lower order terms.

### 2.2 Mathematical Techniques

We now introduce some important tools which will be used in the analyses.

**Theorem 1 Wald’s Equation** (Wald, 1944; Doerr and Künnemann, 2013)

Let \( T \) be a random variable with bounded expectation and let \( X_1, X_2, \ldots \) be non-negative random variables with \( E(X_i \mid T \geq i) \leq C \). Then

\[
E\left( \sum_{i=1}^{T} X_i \right) \leq E(T) \cdot C.
\]

**Theorem 2 Additive Drift Theorem** (He and Yao, 2001; Oliveto and Yao, 2011)

Let \( \{X_t\}_{t \geq 0} \) be a Markov process over a finite set of states \( S \), and \( g : S \rightarrow \)
function that assigns a non-negative real number to every state. Let the time to reach the optimum be
\[ T := \min \{ t \geq 0 \mid g(X_t) = 0 \} \]. If there exists \( \delta > 0 \) such that at any time step \( t \geq 0 \) and at any state \( X_t \geq 0 \) the following condition holds:

\[ E(\Delta(t) \mid g(X_t) > 0) = E(g(X_t) - g(X_{t+1}) \mid g(X_t) > 0) \geq \delta, \]

then

\[ E(T \mid X_0) \leq \frac{g(X_0)}{\delta} \]

and

\[ E(T) \leq \frac{E(g(X_0))}{\delta}. \]

**Theorem 3 Variable Drift Theorem (Johannsen, 2010)** Let \( (X_t)_{t \geq 0} \) be a stochastic process over some state space \( S \subseteq \{0\} \cup [x_{\text{min}}, x_{\text{max}}] \), where \( x_{\text{min}} > 0 \). Let \( h(x) \) be an integrable, monotone increasing function on \( [x_{\text{min}}, x_{\text{max}}] \) such that

\[ E(X_t - X_{t+1} \mid X_t \geq x_{\text{min}}) \geq h(X_t). \]

Then it holds for the first hitting time \( T := \min \{ t \mid X_t = 0 \} \) that

\[ E(T \mid X_0) \leq \frac{x_{\text{min}}}{h(x_{\text{min}})} + \int_{x_{\text{min}}}^{X_0} \frac{1}{h(x)} \, dx. \]

### 3 Hyper-Heuristics Are Faster Than Their Low-level Heuristics

In this section we show that the Generalised Random Gradient hyper-heuristic can outperform its constituent low-level heuristics, even when the latter are efficient for the problem at hand. We first introduce a lower bound on the expected runtime of all algorithms which use only the 1BITFLIP and 2BITFLIP operators on LEADINGONES. To achieve this result we will rely on the following theorem.

**Theorem 4 (Böttcher et al. (2010); Buzdalov and Buzdalova (2015))** The expected time needed to find the optimum of LEADINGONES given random initialisation is

\[ \frac{1}{2} \sum_{i=1}^{n} A_{n-i}, \]

where \( A_{n-i} \) is the expected time needed to find an improvement given a solution with fitness \( i \).
Theorem 4 states that for the LEADINGONES benchmark function maximising the probability of success implies minimising the expected waiting times $A_{n-i}$. The 1BitFlip operator has a success probability of $\frac{1}{n}$ while the 2BitFlip operator of $\frac{1}{n} \cdot \frac{n-i-1}{n} \cdot 2 = \frac{2n-2i-2}{n^2}$ where $i = LO(x)$. Hence, for $i \leq \frac{n}{2} - 1$ the 2BitFlip operator is more effective (i.e., has a higher probability of success and hence a lower expected waiting time for each improvement), while the 1BitFlip operator is preferable afterwards. Thus, the expected runtime of an algorithm using these operators in such a way gives a lower bound on all stochastic unbiased algorithms using only the same two operators. The expected runtime of such an algorithm can be proved following the approach used for the (1+1)-EA by Böttcher et al. (2010) using Theorem 4, which was shown to work for Randomised Local Search (RLS) by Buzdalov and Buzdalova (2015). Doerr (2018) proved later that Theorem 4 holds for any unbiased (1+1) black box algorithm on LEADINGONES. Since our analysis fits into the (1+1) algorithm framework (we analyse (1+1) algorithms, each differing by their stochastic mutation operator), we can apply Theorem 4 also in our analysis.

**Theorem 5** The best-possible expected runtime on LEADINGONES for an algorithm using only the 1BitFlip and 2BitFlip operators is $\frac{1 + \ln(2)}{4} - n^2 + O(n) \approx 0.42329n^2$.

**Proof:** We partition the analysis into two phases: Phase 1, where the 2BitFlip operator is used for the first half of the search, and Phase 2, where the 1BitFlip operator is used for the second half of the search. We consider their expected runtimes ($E(T_1)$ and $E(T_2)$) separately and sum them to find the final expected runtime.

**Phase 1.** The 2BitFlip operator keeps the non-leading bits randomly distributed (i.e., the bits after the first 0-bit all have an equal probability of being 1 or 0 independently of each other), so the result from Theorem 4 holds for this phase and can be applied in the same way as done by Buzdalov and Buzdalova (2015).

To find the expected runtime for this phase, we can consider every point $LO(x) = i \geq \frac{n}{2}$ to be optimal. Since the probability of improvement of the 2BitFlip operator is $\frac{2n-2i-2}{n^2}$, the expected time to make an improvement for $0 \leq i < \frac{n}{2}$ is $A_{n-i} = \frac{n^2}{2n-2i-2}$. We also consider for $i \geq \frac{n}{2}$, $A_{n-i} = 0$. 

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Hence, for Phase 1,

\[ E(T_1) = \frac{1}{2} \sum_{i=0}^{n-1} A_{n-i} = \frac{1}{2} \sum_{i=0}^{\frac{n}{2}-1} \frac{n^2}{2n - 2i - 2} = \frac{1}{2} \sum_{k=\frac{n}{4}}^{n-1} \frac{n^2}{2k} \]

\[ = \frac{n^2}{4} \sum_{k=\frac{n}{4}}^{n-1} \frac{1}{k} = \frac{n^2}{4} \left( H_{n-1} - H_{\frac{n}{2}-1} \right), \]

where \( H_x \) is the \( x^{th} \) Harmonic number, which can be bounded by

\[ \ln(n + 1) + \gamma - \frac{1}{n + 1} \leq H_n \leq \ln(n + 1) + \gamma - \frac{1}{2(n + 1)}, \]

and \( \gamma \) is the Euler-Mascheroni constant \((\gamma \approx 0.57722 \ldots)\). Hence, we can bound the term \( H_{n-1} - H_{\frac{n}{2}-1} \) from below by \( \ln(2) \), and from above by \( \ln(2) + \frac{3}{2n} \), which gives the following bounds on \( E(T_1) \):

\[ \ln(2) \cdot \frac{n^2}{4} \leq E(T_1) \leq \left( \ln(2) + \frac{3}{2n} \right) \cdot \frac{n^2}{4}. \]

Since \( E(T_1) \) is at most \( \frac{2n}{8} = O(n) \) greater than the lower bound of \( \ln(2) \cdot \frac{n^2}{4} \), we have:

\[ \frac{\ln(2)}{4} n^2 \leq E(T_1) \leq \frac{\ln(2)}{4} n^2 + \frac{3n}{8}. \]

**Phase 2** The expected runtime for Phase 2 can be calculated using Theorem 4 with \( A_{n-i} = n \) for \( i \geq \frac{n}{2} \). Similarly to Phase 1, we consider \( A_{n-i} = 0 \) for \( i < \frac{n}{2} \).

\[ E(T_2) = \frac{1}{2} \sum_{i=0}^{\frac{n}{2}-1} A_{n-i} = \frac{1}{2} \sum_{j=1}^{\frac{n}{2}} n = \frac{1}{2} \cdot n \cdot \frac{n}{2} = \frac{n^2}{4}. \]

Hence, the total expected runtime is:

\[ \frac{1 + \ln(2)}{4} n^2 \leq E(T_1) + E(T_2) \leq \frac{1 + \ln(2)}{4} n^2 + \frac{3n}{8}. \]

Thus, we have proved a theoretical lower bound of \( \frac{1 + \ln(2)}{4} n^2 + O(n) \approx 0.42329n^2 \) for the expected runtime of any algorithm using the 1BITFLIP and 2BITFLIP operator, which is better than the expected runtime of RLS (i.e., \( 0.5n^2 \)).
The rest of the section is structured as follows. In Subsection 3.1 we show that the ‘simple’ hyper-heuristic mechanisms all perform equivalently to Simple Random for LEADINGONES, up to lower order terms. In Subsection 3.2 we show that the Generalised Random Gradient hyper-heuristic is much faster and achieves this ‘optimal’ expected runtime (from Theorem 5), up to lower order terms.

3.1 ‘Simple’ Mechanisms

In this section we show that the standard ‘simple’ selection mechanisms all have the same expected runtime on LEADINGONES up to lower order terms. The following theorem derives the expected runtime for the Simple Random mechanism. The subsequent corollary extends the result to all the other ‘simple’ mechanisms.

**Theorem 6** Let p be the probability of choosing the 1BITFLIP mutation operator, and 1 − p the probability of choosing the 2BITFLIP mutation operator. Then, the expected runtime of the Simple Random mechanism on LEADINGONES for \( p \in (0, 1) \) is

\[
\frac{1}{4(1-p)} \ln \left( \frac{2-p}{p} \right) n^2 + o(n^2). 
\]

If \( p = 0 \) the expected runtime is infinite. If \( p = 1 \), the expected runtime is \( 0.5n^2 \).

**Proof:** If \( p = 0 \), only the 2BITFLIP operator is used. There is a non-zero probability of reaching the point \( 1^n - 10 \), which cannot be improved by using the 2BITFLIP operator. Hence, by the law of total probability, the expected runtime is infinite.

If \( p = 1 \), only the 1BITFLIP operator is used, resulting in exactly RLS, which has expected runtime \( 0.5n^2 \) (Buzdalov and Buzdalova, 2015).

If \( p \in (0, 1) \), we have an expected runtime of

\[
E(T) = \frac{1}{2} \sum_{i=1}^{n} A_{n-i},
\]

where \( A_{n-i} \) is the expected time needed to find an improvement given a solution with fitness \( i \), and the algorithm is initialised with a search point chosen uniformly at random (Theorem 4). In each iteration, the 1BITFLIP operator is chosen with probability \( p \) and leads to a fitness improvement with probability \( \frac{1}{n} \); the 2BITFLIP operator is chosen with probability \( 1-p \) and leads to a fitness improvement with probability \( \frac{2n-2i-2}{n^2} \). Hence,

\[
(A_{n-i})^{-1} = p \cdot \frac{1}{n} + (1-p) \cdot \frac{2n-2i-2}{n^2},
\]

and

\[
A_{n-i} = \frac{1}{n} + \frac{(1-p)(2n-2i-2)}{n^2} = \frac{n^2}{2(1-p)(n-i-1) + np}.
\]

Since the use of the 1BITFLIP and 2BITFLIP operators will keep the distribution of the non-leading bits the same, we can use Theorem 4. Hence, the total
expected runtime is

$$\frac{1}{2} \sum_{i=0}^{n-1} \frac{n^2}{2(1-p)(n-i-1) + np} = \frac{n^2}{2} \sum_{k=1}^{n} \frac{1}{(2p-2)k + (2-p)n}$$

(1)

$$= \frac{n^2}{2(2p-2)} \left( \ln \left( 1 + \frac{(2-p)}{2p-2} \right) + o(1) \right) = \frac{n^2}{4(1-p)} \ln \left( \frac{2-p}{p} \right) + o(n^2),$$

(2)

where Equation 1 becomes Equation 2 via the following simplification:

$$\sum_{k=1}^{n} \frac{1}{a \cdot k + b \cdot n} = \frac{1}{a} \sum_{k=1}^{n} \frac{1}{k + \frac{b}{a} \cdot n} = \frac{1}{a} \left( \sum_{k=1}^{(1+\frac{b}{a})n} \frac{1}{k} - \sum_{k=1}^{\frac{bn}{a}} \frac{1}{k} \right)$$

$$= \frac{1}{a} \left( \ln \left( 1 + \frac{b}{a} \right) + o(1) \right)$$

with $a = 2p - 2$ and $b = 2 - p$. $\square$

When $p = 0.5$ (i.e., equal chance of choosing each operator in each iteration), the standard Simple Random mechanism has an expected improvement time of $A_{n-1} = \frac{2n^2}{3n-2i-2}$, and has expected runtime $\frac{\ln(3)}{2} n^2 + o(n^2) \approx 0.54931n^2$. The expected runtime improves with increasing $p$, hence the optimal choice is $p = 1$.

**Corollary 7** The expected runtime of the Permutation, Greedy and Random Gradient mechanisms on LeadingOnes is $\frac{\ln(3)}{2} n^2 + o(n^2) \approx 0.54931n^2$.

**Proof:** Let $p_i$ be the probability that at least one improvement is constructed within two fitness function evaluations. For the Greedy and Permutation mechanisms, we have $p_i = \frac{1}{n} + (1 - \frac{1}{n}) \cdot \frac{2n-2i-2}{n^2}$ as either the 1BitFlip operator can succeed in one iteration, or it can fail and then the 2BitFlip operator can succeed in the next iteration. To upper bound the expected optimisation time, we note that the difference between $p_i$ (i.e., the expected waiting time for an improvement to be constructed in terms of fitness evaluations) and the $A_{n-1}$ waiting times for the standard Simple Random mechanism with $p = 0.5$ ($A_{n-1} = \frac{2n^2}{3n-2i-2}$), and thus the difference between the expected runtimes of these mechanisms and Simple Random is limited to lower-order terms. We note that with probability at most $\frac{2}{n^2}$, both mutations (the two mutations performed in parallel by the Greedy mechanism, or the two mutations performed sequentially by the Permutation mechanism) considered are improvements. As this occurs at most a constant number
of times in expectation (i.e., $O(n^2) \cdot \frac{2}{n^2} = O(1)$), and the maximum expected waiting time for any improving step is $O(n)$, the lower bound differs from the upper bound by at most an $O(1) \cdot O(n) = O(n)$ term. Thus, the expected runtime of these mechanisms is also $\frac{\ln(3)}{2} \cdot n^2 + o(n^2)$.

For the Random Gradient mechanism, we note that the probability that an operator, when repeated following a success, is successful again is at most $\frac{2}{n}$ (an upper bound on the success probability of the 2BITFLIP operator). Since there are at most $n$ improvements to be made throughout the search space, the expected number of repeats which produce an improvement is at most 2. If the chosen operator is not successful, the Random Gradient mechanism behaves identically to the Simple Random mechanism. Its expected runtime is therefore at least the expected runtime of the Simple Random mechanism less an $O(n)$ term, and at most the expected runtime of the Simple Random mechanism plus $n$ (as there are at most $n$ iterations where the mechanisms differ in operator selection). Thus, its expected runtime is also $\frac{\ln(3)}{2} \cdot n^2 + o(n^2)$. □

We point out that the lower bound for the Random Gradient hyper-heuristic contradicts the lower bound of $\frac{n^2}{9} (4 + 3 \ln \left(\frac{10}{3}\right)) \approx 0.846 n^2$ found by Alanazi and Lehre (2014). However, their bound results from a small mistake in their proof, and they should have found a lower bound of $\frac{n^2}{9} (3 \ln \left(\frac{10}{3}\right)) + o(n^2) \approx 0.401 n^2$, which agrees with our result of $\frac{\ln(3)}{2} n^2 + o(n^2) \approx 0.549 n^2$.

### 3.2 Generalised Random Gradient

In this subsection we present a rigorous theoretical analysis of the Generalised Random Gradient hyper-heuristic on LEADINGONES. The main result of this subsection is that the Generalised Random Gradient hyper-heuristic is able to match, up to lower order terms, the best-possible performance of any algorithm using the 1BITFLIP and 2BITFLIP operators on the LEADINGONES benchmark function, as presented in Theorem 5. We present the main result in Corollary 8 now, with the proof at the end of the subsection.

**Corollary 8** (Of Theorem 5) The expected runtime of the Generalised Random Gradient hyper-heuristic on LEADINGONES, with $\tau$ that satisfies both $\tau = \omega(n)$ and $\tau \leq (\frac{1}{2} - \epsilon) n \ln(n)$, for some constant $\epsilon > 0$, is at most $\frac{1 + \ln(2)}{4} n^2 + o(n^2) \approx 0.42329 n^2$.

This improves significantly upon the result of Lissovoi et al. (2017), which only considered setting $\tau = cn$ for constant $c$, and hence could not prove optimal expected runtime.
In order to derive this result, we first present the necessary prerequisite results. The following theorem is very general as it provides an upper bound on the expected runtime of the Generalised Random Gradient hyper-heuristic for any value of \( \tau \) smaller than \( \frac{1}{2} n \ln n \). In particular, the theorem allows us to identify values of \( \tau \) for which the expected runtime of the hyper-heuristic for the function is the optimal expected runtime that may be achieved by using two operators; this result is highlighted in Corollary 8 and depicted in Figure 1. Corollary 10 gives a tight upper bound on the expected runtime of the simple Random Gradient mechanism obtained by setting \( \tau = 1 \) in Theorem 9. Our proof technique partitions the search space into \( w = \log^2(n) \) chunks, each representing an equal range of fitness values.

**Theorem 9** The expected runtime of the Generalised Random Gradient hyper-heuristic on \( \text{LEADINGONES} \) with \( \tau \leq (\frac{1}{2} - \epsilon) n \ln(n) \), for some constant \( \epsilon > 0 \), is at most

\[
\frac{n^2}{2} \left( \sum_{j=1}^{w} \frac{\frac{\tau}{n} + e^{2\frac{\tau}{n}}(1 - \frac{j-1}{w}) M_2(j, w)}{(e^{\frac{\tau}{n}} + e^{2\frac{\tau}{n}}(1 - \frac{j}{w}) - 2)w} \right) + o(n^2)
\]

where

\[
M_2(j, w) := \begin{cases} \frac{\tau}{n} & \text{if } j = w \text{ or } \frac{1}{2(1 - \frac{j}{w})} > \frac{\tau}{n} \\ \frac{1}{2(1 - \frac{j}{w})} & \text{otherwise} \end{cases}
\]

and \( M_1(x) := \min\{\frac{\tau}{n}, x\} \), with \( w = \log^2(n) \).

Theorem 9 is a general theorem for any \( \tau \leq (\frac{1}{2} - \epsilon) n \ln(n) \), for some constant \( \epsilon > 0 \). Figure 1 presents the theoretical upper bounds from Theorem 9 for a range of linear \( \tau \) values. For \( \tau = 5n \), already the GRG hyper-heuristic outperforms RLS, giving an expected runtime of 0.46493\(n^2\). For \( \tau = 100n \), the performance improves to 0.42368\(n^2\), matching the best possible performance from Theorem 5 up to 3 decimal places. We have seen that GRG is able to exactly match this best possible performance, up to lower order terms, for \( \tau = \omega(n) \) in Corollary 8.

**Proof:** [Of Theorem 9]

For the purpose of this proof, we partition the optimisation process into \( w = \log^2(n) \) chunks based on the value of the \( \text{LEADINGONES} \) fitness function: during chunk \( j \), the LO value of the current solution is at least \( \frac{j-1}{w} n \) and less than \( \frac{j}{w} n \). After all the \( w \) chunks have been completed, the global optimum, with a LO value of \( n \), will have been found.

We upper bound the expectation of the runtime \( T \) of the Generalised Random Gradient hyper-heuristic on \( \text{LO} \) by the sum of the expected values
of $T_j$ (the expected time spent in each chunk of the optimisation process) plus the times taken to begin a chunk (that is, the time before the first random operator choice occurs in each chunk). As our analysis of $E(T_j)$ requires the chunk to start with a random choice of mutation operator, we bound $E(T) \leq \sum_{j=1}^{w} (E(T_j) + E(S_{j+1}))$ where $S_{j+1}$ is a random variable denoting the expected number of iterations between the first solution in chunk $j+1$ being constructed and the first random operator choice occurring in that chunk. We will later show that with proper parameter choices, the contribution of the $S_{j+1}$ terms can be bounded by $o(n^2)$, and therefore they do not affect the leading constant in the overall bound.

Let us now consider $T_j$, the number of iterations the hyper-heuristic spends in chunk $j$. Recall that a mutation operator is selected uniformly at random, and is allowed to run until it fails to produce an improvement within $\tau$ sequential iterations. Let $N_j$ be a random variable denoting the number of random operator choices the hyper-heuristic performs during chunk $j$, and $X_{j,1}, \ldots, X_{j,N_j}$ be the number of iterations the hyper-heuristic runs each chosen operator for. We note that $T_j = \sum_{k=1}^{N_j} X_{j,k}$ is a sum of a number of non-negative variables, and we will later show that $E(N_j)$ is bounded; thus, $E(T_j)$ can be bounded by applying Wald’s equation (Theorem [1]):

$$E(T_j) = E(\sum_{k=1}^{N_j} X_{j,k}) \leq E(N_j)E(X_j),$$

(3)
where \( E(X_j) \) denotes an upper bound on all \( E(X_{j,k}) \) in chunk \( j \).

To bound \( E(N_j) \), the expected number of times the random operator selection is performed during chunk \( j \), we lower bound the expected number of improvements found following the operator selection, and apply the Additive Drift Theorem (Theorem 2) to find the expected number of random operator selections occurring before a sufficient number of improvements have been found to enter the next chunk.

Let \( F_1 \) and \( F_2 \) denote the events that the 1BitFlip and 2BitFlip operators fail to find an improvement during \( \tau \) iterations. For the 1BitFlip operator (with probability of fitness improvement in one iteration \( 1/n \)), this event occurs with probability \( P(F_1) = (1 - \frac{1}{n})^\tau \) throughout the process, which is within \( (e^{-\pi} - \frac{1}{n}, e^{-\pi}) \) (since \( (1 - \frac{1}{n})^n \leq \frac{1}{e} \leq (1 - \frac{1}{n})^{n-1} \)). For the 2BitFlip operator (with probability of fitness improvement in one iteration \( \frac{2n-2i}{n^2} \)), recall that during chunk \( j \), the ancestor individual has at most \( i = \frac{m}{w} - 1 \), and at least \( i = (j-1)\frac{n}{w} - 1 \), one bits, and thus:

\[
P(F_2) \leq \left( 1 - 2 \cdot \frac{1}{n} \cdot n - \frac{(\frac{m}{w} - 1)}{n} \right)^\tau \leq e^{-2\pi(1-\frac{1}{n})}
\]

\[
P(F_2) \geq \left( 1 - 2 \cdot \frac{1}{n} \cdot n - \frac{(j-1)\frac{n}{w} - 1}{n} \right)^\tau
\]

\[
> \left( 1 - 2 \cdot \frac{1}{n} \cdot \frac{(1 - \frac{j-1}{w})}{n} \right)^\tau \geq e^{-2\pi(1-\frac{j-1}{w}) - \frac{1}{n}}
\]

We note that a geometric distribution with parameter \( p = P(F_1) \) (or \( p = P(F_2) \)) can be used to model the number of improvements that the 1BitFlip (or 2BitFlip) operator finds prior to failing to find an improvement for \( \tau \) iterations; the expectation of this distribution is \( \frac{1-p}{p} = \frac{1}{p} - 1 \). We are able to model these events as a geometric random variable since the GRG hyper-heuristic immediately restarts a period of \( \tau \) iterations with the chosen heuristic when a successful mutation occurs (i.e., it sets \( c_t = 0 \)). Hence, there are no periods of \( \tau \) consecutive unsuccessful mutations until the operator has finally failed. Combined over both operators, the expected number of improvements \( (D_j) \) produced following a single random operator selection during chunk \( j \), is:

\[
E(D_j) = \frac{1}{2} \left( \frac{1}{P(F_1)} - 1 \right) + \frac{1}{2} \left( \frac{1}{P(F_2)} - 1 \right) \geq \frac{e^\tau}{2} + \frac{e^{2\pi(1-\frac{j-1}{w})}}{2} - 1,
\]
by inserting the upper bounds on \( P(F_1) \) and \( P(F_2) \). We use this expectation as the drift on the progress of the randomly chosen operator in the Additive Drift Theorem to upper bound \( E(N_j) \). Recall that each chunk consists of advancing through at most \( n/w \) fitness values. Since bits beyond the leading ones prefix and the first zero bit remain uniformly distributed, at most \( \frac{n}{2w} \) improvements by mutation are required in expectation. If each step of a random process in expectation contributes \( E(D_j) \) improvements by mutation, then the expected number of steps required to complete chunk \( j \) is at most

\[
E(N_j) \leq \frac{n}{2w} E(D_j) \leq \frac{n}{\left( e^{\frac{1}{w}} + e^{\frac{2}{w}(1 - \frac{1}{w})} - 2 \right) w}, \tag{4}
\]

by the Additive Drift Theorem (Theorem 2).

To bound \( E(X_j) \), the expected number of iterations before a selected mutation operator fails to produce an improvement for \( \tau \) iterations, we apply Wald’s equation: let \( S \) be the number of improvements found by the operator before it fails, and \( W_1, \ldots, W_S \) be the number of iterations it took to find each of those improvements; then, once selected, the \texttt{1BitFlip} operator runs for:

\[
E(X_j \mid \texttt{1BitFlip}) = \tau + \sum_{k=1}^{S} E(W_k) = \tau + E(S) E(W_1 \mid S \geq 1, \texttt{1BitFlip})
\]

where \( \tau \) accounts for the iterations immediately before failure, and the sum for the iterations preceding each constructed improvement. Recall that \( E(S) = \frac{1}{P(F_1)} - 1 \) by the properties of the geometric distribution, and observe that \( E(W_1 \mid S \geq 1) = E(W_1 \mid W_1 \leq \tau) \leq \min \{ \tau, E(W_1) \} \). Using a waiting time argument (i.e., considering the expectation of a geometric random variable) gives \( E(W_1) = \frac{1}{P(F_1)} \) where \( F_1 \) is the event that the \texttt{1BitFlip} operator fails to find an improvement during \( \tau \) iterations, and, we get (with a similar argument for the \texttt{2BitFlip} operator, with \( E(W_2) \leq \frac{n^2}{2(n - \frac{n^2}{w} - 1)} = \frac{n}{2(1 - \frac{1}{w})} \)): \[
E(X_j \mid \texttt{1BitFlip}) \leq \tau + \left( \frac{1}{P(F_1)} - 1 \right) \min \{ \tau, n \},
\]

\[
E(X_j \mid \texttt{2BitFlip}) \leq \tau + \left( \frac{1}{P(F_2)} - 1 \right) \min \left\{ \tau, \frac{n}{2 (1 - \frac{1}{w})} \right\}.
\]
Combining these conditional expectations with lower bounds on $P(F_1)$ and $P(F_2)$ yields

\[ E(X_j) \leq \frac{1}{2} \cdot E(X_j \mid 1\text{BIT\_FLIP}) + \frac{1}{2} \cdot E(X_j \mid 2\text{BIT\_FLIP}) \]

\[ \leq \tau + \frac{\min\{\tau, n\}}{2 \left(e^{-\tau n} - \frac{1}{n}\right)} + \frac{\min\left\{\frac{\tau}{n}, \frac{n}{2(1-\frac{1}{w})}\right\}}{2 \left(e^{-\frac{2\tau}{n}(1-\frac{2}{w})} - \frac{1}{n}\right)}, \]

which can be simplified to:

\[ E(X_j) \leq \frac{n}{2} \left( \frac{2\tau}{n} + \frac{\min\{\tau, 1\}}{e^{-\tau n} - \frac{1}{n}} + \frac{\min\left\{\frac{\tau}{n}, \frac{1}{2(1-\frac{1}{w})}\right\}}{e^{-\frac{2\tau}{n}(1-\frac{1}{w})} - \frac{1}{n}} \right) \]

\[ \leq \frac{n}{2} \left( \frac{2\tau}{n} + e^{\frac{\tau}{n}} \min\{\frac{\tau}{n}, 1\} + e^{\frac{2\tau}{n}(1-\frac{1}{w})} \min\left\{\frac{\tau}{n}, \frac{1}{2(1-\frac{1}{w})}\right\} \right) + O(1) \]

using $\frac{a}{b-c} = \frac{a}{b} + \frac{ac}{b^2 - bc} = \frac{a}{b} + O\left(\frac{1}{n}\right)$, where $a$, $b$, and $c$ are constants with respect to $n$, to limit the contributions of the $-\frac{1}{n}$ terms in denominators to lower-order terms.

If $j = w$, then $\min\left\{\frac{\tau}{n}, \frac{1}{2(1-\frac{1}{w})}\right\}$ is undefined. However, we know that the chosen operator would only be applied for a maximum of $\tau$ steps. We thus refer to the functions $M_1(x) = \min\{\frac{\tau}{n}, x\}$ and $M_2(j, w)$ from now on, where

\[ M_2(j, w) := \begin{cases} \frac{\tau}{n} & \text{if } j = w \text{ or } \frac{1}{2(1-\frac{1}{w})} > \frac{\tau}{n} \\ \frac{1}{2(1-\frac{1}{w})} & \text{otherwise.} \end{cases} \]

Finally, we can bound the overall expected runtime of the hyper-heuristic. Recall that $S_j$ denotes the number of iterations the hyper-heuristic spends in chunk $j$ while using the random operator chosen during chunk $j - 1$, and as $E(S_j) \leq \max(E(X_j \mid 1\text{BIT\_FLIP}), E(X_j \mid 2\text{BIT\_FLIP})) < 2E(X_j) = O\left(n \cdot \exp\left(\frac{2\tau}{n}\right)\right)$, and since $w = \log^2(n) = o\left(\frac{n}{\exp\left(\frac{2\tau}{n}\right)}\right)$, \[ \sum_{j=1}^{w} E(S_{j+1}) = \]
Substituting the bounds (4) and (5) into (3) yields the theorem statement:

\[
E(T) \leq \sum_{j=1}^{w} (E(T_j) + E(S_{j+1})) \leq \left( \sum_{j=1}^{w} E(N_j)E(X_j) \right) + o(n^2) \\
\leq o(n^2) + \frac{n^2}{2} \times \sum_{j=1}^{w} \frac{2\pi + e^{2\pi(1-\frac{1}{w})} \cdot \frac{1}{w}}{\left( e^{\pi} + e^{\frac{2\pi}{w} (1-\frac{1}{w})} - 2 \right) w}.
\]

We can use the result of Theorem 9 to get a tight upper bound on the expected runtime of the simple Random Gradient mechanism by setting \( \tau = 1 \). We know the bound is tight by Theorem 6.

**Corollary 10** The expected runtime of the Generalised Random Gradient hyper-heuristic on LEADING ONES, with \( \tau = 1 \), is at most \( \frac{\ln(3)}{2} n^2 + o(n^2) \approx 0.54931 n^2 \).

**Proof:** Note that for \( \tau = 1 \), we have \( M_1(1) = M_2(j, w) = \frac{1}{n} \). Since \( w = \log^2 n \), we have \( e^{\frac{2\pi}{w}} = 1 + o(1) \). Inserting \( \tau = 1 \) and these expressions into the result from Theorem 9 gives the following result:

\[
E(T) \leq \frac{n^2}{2} \left( \sum_{j=1}^{w} \frac{\frac{2\pi}{n} + e^{\frac{\pi}{n}} \cdot \frac{1}{n} + e^{2\pi(1-\frac{1}{w})} \cdot \frac{4}{w}}{\left( e^{\pi} + e^{\frac{2\pi}{n} (1-\frac{1}{w})} - 2 \right) w} \right) + o(n^2) \\
= \frac{n}{2} \left( \sum_{j=1}^{w} \frac{e^{\frac{\pi}{n}} + e^{\frac{\pi}{n}} \cdot e^{\frac{2\pi}{n} (1-\frac{1}{w})} - 2 + 4}{e^{\frac{\pi}{n}} + e^{\frac{2\pi}{n} (1-\frac{1}{w})} - 2} \right) + o(n^2) \\
\leq \frac{n}{2} \left( \sum_{j=1}^{w} \frac{1 - \frac{4}{w}}{e^{\frac{\pi}{n}} + e^{\frac{2\pi}{n} (1-\frac{1}{w})} - 2} \right) + o(n^2) \\
\leq \frac{n}{2} \left( \sum_{j=1}^{w} \frac{4}{w} \cdot \frac{1}{e^{\frac{\pi}{n}} + e^{\frac{2\pi}{n} (1-\frac{1}{w})} - 2} \right) + o(n^2) \leq \left( \frac{\ln \left( \frac{\exp \left( \frac{1}{n} \right) + 2}{\exp \left( \frac{1}{n} \right)} \right)}{2 - \exp \left( \frac{1}{n} \right)} \right) n^2 + o(n^2) \\
= \frac{\ln(3)}{2} n^2 + o(n^2).
\]

We can now prove the main result of this section, i.e., Corollary 8.
Proof: [Of Corollary 6]
Consider first the terms $M_1(1)$ and $M_2(j, w)$. Since $\tau = \omega(n)$, we have $M_1(1) = 1$. For $M_2(j, w)$, it is important to note that for the first half of the search, $M_2(j, w) = \frac{1}{2(1 - \frac{1}{w})}$, and in the second half of the search, $M_2(j, w) = O(w) = n^{o(1)}$ (excluding the case when $j = w$ in which $M_2(j, w) = \frac{\tau}{n} = n^{o(1)}$). Note that $\frac{\tau}{n} = \omega(1)$ and $e^{2\tau/w} = o\left(n^{1/w}\right) = 1 + o(1)$. Hence:

$$E(T) \leq \frac{n^2}{2} \left( \sum_{j=1}^{w} \frac{2\pi}{n} + e^{\frac{2\pi}{n}} M_1(1) + e^{\frac{2\pi}{n}(1 - \frac{j}{w})} M_2[j, w] \right) + o(n^2)$$

$$= \frac{n^2}{2} \left( \sum_{j=1}^{w} \frac{2\pi}{n} + e^{\frac{2\pi}{n}} + e^{\frac{2\pi}{n}(1 - \frac{j}{w})} \cdot M_2[j, w] \right) + o(n^2)$$

$$= \frac{n^2}{2} \left( \sum_{j=1}^{w} \frac{2\pi}{n} + e^{\frac{2\pi}{n}} + e^{\frac{2\pi}{n}(1 - \frac{j}{w})} \cdot M_2[j, w] \right) + o(n^2)$$

$$= \frac{n^2}{2} \left( \sum_{j=1}^{w} \frac{1}{w} \left( e^{\frac{2\pi}{n}} + e^{\frac{2\pi}{n}(1 - \frac{j}{w})} \cdot M_2[j, w] - 2 \right) \right) + o(n^2)$$

The exponential terms will dominate the summation term (since $\frac{\tau}{n} = \omega(1)$). When $j > \frac{w}{2}$, we note that $\exp\left(\frac{\tau}{n}\right) > \exp\left(\frac{2\pi}{n} (1 - \frac{j}{w})\right)$ and this term will dominate the other. Vice versa for $j < \frac{w}{2}$, and at $j = \frac{w}{2}$, the two values are equal. Hence, we can split the sum into two sections:

$$E(T) \leq \frac{n^2}{2} \left( \sum_{j=1}^{w/2} \frac{1}{w} \left( e^{\frac{2\pi}{n}} + e^{\frac{2\pi}{n}(1 - \frac{j}{w})} \cdot M_2[j, w] - 2 \right) \right) + o(n^2)$$

$$= \frac{n^2}{2} \left( \sum_{j=1}^{w/2-1} \frac{1}{w} \left( e^{\frac{2\pi}{n}(1 - \frac{j}{w})} \cdot \frac{1}{2(1 - \frac{1}{w})} \right) \right) + \frac{n^2}{2w} + n^2 \left( \frac{1}{w} \left( e^{\frac{\pi}{n}} \right) \right) + o(n^2)$$

$$= \frac{n^2}{4} \left( \sum_{j=1}^{w/2} \frac{1}{w - j} \right) + n^2 + o(n^2) = \left(1 + \ln(2)\right) \frac{n^2}{4} = n^2 + o(n^2).$$
Note that at \( j = w \), the respective ‘dominating terms’ are \( \exp \left( \frac{1}{n} \tau n \right) \) and \( \exp \left( \frac{3}{n} (1 - \frac{3}{n}) \right) = 1 \), and \( M(j, w) = \frac{\tau}{n} < \frac{1}{2} \ln(n) \). Hence, \( \exp \left( \frac{1}{n} \tau n \right) \) will dominate. 

\[ \square \]

4 Increasing the Choice of Low-level Heuristics Leads to Improved Performance

Runtime analyses of hyper-heuristics are easier if the algorithms can only choose between two operators. However, in realistic contexts, hyper-heuristics have to choose between many more operators.

In this section, we consider the previously analysed Generalised Random Gradient hyper-heuristic, yet extend the set \( H \) of low-level heuristics to be of size \( k \geq 2 \) i.e., \( |H| = k = \Theta(1) \). We extend upon the previous analysis by considering the \( k \) operators as \( k \) different mutation operators, each flipping between 1 and \( k \) bits with replacement uniformly at random respectively. This more accurately represents hyper-heuristic approaches employed for real-world problems.

As before, we consider the LEADINGONES benchmark function. We will rigorously show that the performance of the ‘simple’ mechanisms deteriorates as the number (\( |H| = k = \Theta(1) \)) of operators increases. In addition, we prove decreasing upper bounds on the expected runtime of the Generalised Random Gradient hyper-heuristic as the number of operators increases.

The main result of this section is the following theorem, stating that a hyper-heuristic that chooses between \( k \) stochastic mutation operators has better expected performance on LEADINGONES than any algorithm, including the best-possible, using less than \( k \) stochastic mutation operators. We present the statement of Theorem 11 now, and the proof at the end of this section.

**Theorem 11** Let \( \tau \) satisfy both \( \tau = \omega(n) \) and \( \tau \leq \left( \frac{1}{k} - \epsilon \right) n \ln(n) \), for some constant \( \epsilon > 0 \). Then the expected runtime for the Generalised Random Gradient hyper-heuristic on LEADINGONES using \( (1, \ldots, k)\text{BITFLIP} \), where \( k = \Theta(1) \), is less than the best-possible expected runtime on LEADINGONES for any unary unbiased algorithm using any strict subset of \( (1, \ldots, k)\text{BITFLIP} \).

This result highlights the power of hyper-heuristics as general-purpose problem solvers. The inclusion of more heuristics to the set of low-level heuristics is implied to be preferable, showcasing the impressive learning capabilities of even simple hyper-heuristics.
We now present the necessary prerequisite analysis to derive this result. We first derive the best-possible expected runtime achievable by any hyper-heuristic using \( k = o(n) \) operators. Before we prove this, we introduce the following two helpful lemmata.

**Lemma 12** The probability of improvement of an operator which flips \( m \) indistinct bits in a bit-string on the LEADINGONES benchmark function, at the state \( \text{LO}(x) = i \), is

\[
P(\text{Imp}_m) = m \cdot \frac{1}{n} \cdot \left( \frac{n - i - 1}{n} \right)^{m-1} + O(n^{-2}).
\]

**Proof:** In the context of the LEADINGONES benchmark function, the way to make an improvement is to flip the first 0-bit after the leading 1-bits, whilst keeping the prefix leading 1-bits unflipped. Although it is possible to make improvements by flipping and re-flipping the same bit in some combination after flipping the first 0-bit into a 1-bit (with probability \( O(n^{-2}) \)), the operators are likely to make a fitness improvement by flipping the first 0-bit (with probability \( \frac{1}{n} \), which can occur within any of the \( m \) bit-flips), and using the remaining \( m - 1 \) flips within the suffix \( n - i - 1 \) bits (with probability \( \left( \frac{n - i - 1}{n} \right)^{m-1} \)). Hence, we can say that the probability of an improvement when flipping \( k \) bits is

\[
P(\text{Imp}_m) = m \cdot \frac{1}{n} \cdot \left( \frac{n - i - 1}{n} \right)^{m-1} + O(n^{-2}).
\]

\( \square \)

We note that through a simple calculation, the 1BitFlip operator is always the best choice in the second half of the search space \( \frac{n}{2} \leq i < n \). We can thus leave the formula in this state as the \( \frac{1}{n} \) success probability of the 1BitFlip operator will always dominate any \( O(n^{-2}) \) terms, and the \( P(\text{Imp}_m) \) terms are all \( O \left( \frac{1}{n^2} \right) \) in the first half of the search space.

**Lemma 13** Consider two bitflip operators flipping \( a \) and \( b \) bits respectively with replacement (1BitFlip and BBitFlip), with \( b > a \). Then 1BitFlip has a lower expected time to find a success than BBitFlip (excluding \( i = n - 1 \)) when

\[
i < n \left( 1 - \left( \frac{b}{a} \right)^{\frac{1}{n-a}} \right) - 1.
\]

This follows from Lemma 12. In particular, an operator flipping \( m \) bits outperforms, in expectation, an operator flipping \( m - 1 \) bits when \( i < \frac{n}{m} - 1 \).
It is worth noting that only an operator which flips an odd number of bits can make progress when \( \text{LO}(x) = n - 1 \), and the 1BITFLIP operator has the best success probability at this point (of \( \frac{1}{n} \)).

In Theorem 14 we present the result for the best-possible expected runtime on LEADINGONES for any unary (1+1) algorithm using (1,\ldots,k)BITFLIP as mutation operators. Similar results have recently been presented by Doerr (2018) and Doerr and Wagner (2018) for mutation operators flipping bits with replacement (note that the results are the same up to lower order terms).

**Theorem 14** The best-possible expected runtime on LEADINGONES for any unary (1+1) algorithm using (1,\ldots,k)BITFLIP, where \( k = o(n) \), is

\[
E(T_{k,\text{Opt}}) = \frac{1}{2} \left( \sum_{i=0}^{n-1} k \cdot \frac{1}{n} \cdot \left(\frac{n-i-1}{n}\right)^{k-1} + \sum_{m=1}^{k-1} \sum_{i=m+1}^{n-1} m \cdot \frac{1}{n} \cdot \left(\frac{n-i-1}{n}\right)^{m-1} \right) \pm o(n^2).
\]

**Proof:** The best-possible algorithm uses the mutation operator with the highest probability of success based on the current LO value at each LO \( x \); we refer to such an operator as ‘optimal’. From Lemma 13 we can say the MBITFLIP operator is optimal during the time when \( \frac{n}{m+1} \leq i \leq \frac{n}{m} - 1 \) (unless \( m = k \), in which case it is optimal for \( 0 \leq i \leq \frac{n}{k} - 1 \)). Since the 1BITFLIP operator has success probability \( O(n^{-1}) \), and is chosen with the same probability as all other operators, the leading term in any success probability is \( O((nk)^{-1}) = \omega(n^{-2}) \) and any \( O(n^{-2}) \) terms are insignificant. Thus, they can be grouped into a lower order \( o(n^2) \) term.

Hence, by standard waiting time arguments, coupled with Theorem 4 we have that the optimal expected runtime when using (1,\ldots,k)BITFLIP is

\[
E(T_{k,\text{Opt}}) = \frac{1}{2} \left( \sum_{i=0}^{n-1} k \cdot \frac{1}{n} \cdot \left(\frac{n-i-1}{n}\right)^{k-1} + \sum_{m=1}^{k-1} \sum_{i=m+1}^{n-1} m \cdot \frac{1}{n} \cdot \left(\frac{n-i-1}{n}\right)^{m-1} \right) \pm o(n^2).
\]

In particular, taking limits as \( n \to \infty \), we have \( E(T_{1,\text{Opt}}) = \frac{1}{2} n^2 \), \( E(T_{2,\text{Opt}}) = \frac{1+\ln(2)}{4} n^2 \approx 0.42329n^2 \), \( E(T_{3,\text{Opt}}) = \left( \frac{1}{3} + \frac{\ln(2)}{2} - \frac{\ln(3)}{4} \right) n^2 \approx 0.40525n^2 \), \( E(T_{5,\text{Opt}}) = \left( \frac{3721}{11520} + \frac{\ln(2)}{2} - \frac{\ln(3)}{4} \right) n^2 \approx 0.39492n^2 \). A simplified result for \( E(T_{k,\text{Opt}}) \) is difficult to find, as is the limit for the best-possible expected runtime as \( k \to \infty \); a numerical analysis by Doerr and Wagner (2018) suggests a result of \( \approx 0.388n^2 \).
Theorem 14 gives a theoretical lower bound on the expected runtime of any algorithm using the $k$ operators on LEADINGONES. We will see that including further operators can improve the expected runtime if they are used appropriately.

To emphasise the practical importance of our result, however, we first prove that the ‘simple’ mechanisms considered in the literature (Cowling et al., 2001, 2002; Alanazi and Lehre, 2014) exhibit worse performance when given access to larger sets of low-level heuristics.

4.1 ‘Simple’ Mechanisms

We will now see how the simple learning mechanisms (Simple Random, Permutation, Greedy and Random Gradient) perform when having to choose between $k$ operators (i.e., $(1, \ldots, k)$BitFlip). We will show that incorporating more operators is detrimental to the performance of the ‘simple’ mechanisms on LEADINGONES.

We again start by stating the expected runtime of the Simple Random mechanism, and use this as a basis for the other three mechanisms. Recall that the standard Simple Random mechanism chooses each operator uniformly at random in each iteration (i.e., with probability $\frac{1}{k}$ when using $k$ operators).

**Theorem 15** The expected runtime of the Simple Random mechanism on LEADINGONES using $(1, \ldots, k)$BitFlip, with $k = o(n)$, is

$$E(T_{SR,k}) = \frac{k}{2} \cdot \sum_{i=0}^{n-1} \sum_{m=1}^{k} m \cdot \frac{1}{n} \cdot \left(\frac{n-i-1}{n}\right)^{m-1} - o(n^2).$$

Note that, as a result of Theorem 15, the expected runtimes of the Simple Random mechanism increase as $k$ increases, implying that incorporating more operators is detrimental to the performance of the ‘simple’ mechanisms. In particular, the results for using 1, 2 and 3 operators respectively are $\frac{1}{2}n^2$, $\frac{ln(3)}{2}n^2 \approx 0.54931n^2$ and $(\frac{3\sqrt{2}}{4} \arctan \left(\frac{\sqrt{2}}{2}\right)) n^2 \approx 0.65281n^2$.

**Proof:** [Of Theorem 15]

We have seen previously in Lemma 12 that the probability of an improvement when flipping $m$ indistinct bits is

$$P(\text{Imp}_{m}) = m \cdot \frac{1}{n} \cdot \left(\frac{n-i-1}{n}\right)^{m-1} + O(n^{-2}).$$
Since we have \( k \) operators, the probability of an improvement in a single iteration is

\[
\frac{1}{k} \cdot \sum_{m=1}^{k} \left( m \cdot \frac{1}{n} \cdot \left( \frac{n - i - 1}{n} \right)^{m-1} + O(n^{-2}) \right)
\]

\[
= \frac{1}{kn} + \frac{1}{k} \cdot \sum_{m=2}^{k} \left( m \cdot \frac{1}{n} \cdot \left( \frac{n - i - 1}{n} \right)^{m-1} + O(n^{-2}) \right) = \Omega \left( \frac{1}{n} \right)
\]

and hence, using the same nomenclature as in Theorem 4, the expected time for an improvement given a distance of \( n - i \) to the global optimum, is

\[
A_{n-i} = \frac{1}{k} \cdot \sum_{m=1}^{k} m \cdot \frac{1}{n} \cdot \left( \frac{n - i - 1}{n} \right)^{m-1} - o(n).
\]

We can use Theorem 4 to give the result:

\[
E(T_{SR,k}) = \frac{1}{2} \sum_{i=0}^{n-1} A_{n-i} = \frac{1}{2} \sum_{i=0}^{n-1} \frac{1}{k} \cdot \sum_{m=1}^{k} m \cdot \frac{1}{n} \cdot \left( \frac{n - i - 1}{n} \right)^{m-1} - o(n^2).
\]

\[
= \frac{k}{2} \sum_{i=0}^{n-1} \sum_{m=1}^{k} m \cdot \frac{1}{n} \cdot \left( \frac{n - i - 1}{n} \right)^{m-1} - o(n^2).
\]

We now prove the same deteriorating performance for the other ‘simple’ hyper-heuristics.

**Corollary 16** The expected runtime of the Permutation, Greedy and Random Gradient mechanisms on **LeadingOnes** with \((1, \ldots , k)\text{BitFlip}, k = o(n)\), is

\[
E(T_k) = \frac{k}{2} \sum_{i=0}^{n-1} \sum_{m=1}^{k} m \cdot \frac{1}{n} \cdot \left( \frac{n - i - 1}{n} \right)^{m-1} \pm o(n^2).
\]

**Proof:** Let \( p_i \) be the probability that one fitness improvement is constructed within \( k \) fitness function evaluations. For the Greedy and Permutation
mechanisms, we have

\[ p_i = P(\text{Imp}_1) + (1 - P(\text{Imp}_1)) \cdot P(\text{Imp}_2) + \cdots + \left( \prod_{j=1}^{k-1} (1 - P(\text{Imp}_j)) \right) \cdot p_k \]

\[ = \sum_{m=1}^{k} \left( \prod_{j=1}^{m-1} \left( 1 - j \cdot \frac{1}{n} \cdot \left( \frac{n - i - 1}{n} \right)^{j-1} \right) \right) \cdot m \cdot \frac{1}{n} \cdot \left( \frac{n - i - 1}{n} \right)^{m-1} \]

\[ = \sum_{m=1}^{k} \left( (1 - o(1)) \cdot m \cdot \frac{1}{n} \cdot \left( \frac{n - i - 1}{n} \right)^{m-1} \right). \]

To move from the second equation to the third, we note that \( P_{\text{Imp}_j} = O \left( \frac{1}{n} \right) \) and \( \prod_{j=1}^{m-1} (1 - P_{\text{Imp}_j}) = (1 - O \left( \frac{1}{n} \right))^m = 1 - o(1) \). To upper bound the expected optimisation time, we note that the difference between \( \frac{k}{p_k} \) (i.e., the expected waiting time for an improvement to be constructed in terms of fitness evaluations) and the \( A_{n-i} \) expected waiting times used to prove Theorem 15 is at most constant, and thus the difference between the expected runtimes of these mechanisms and Simple Random is limited to lower-order terms. We note that with probability at most \( \frac{k(k-1)}{n^2} \), at least two mutations (of the \( k \)) considered are improvements (two parallel successes of the Greedy mechanism, or two mutation performed sequentially by the Permutation mechanism. As this occurs at most a sublinear number of times in expectation, and the maximum expected waiting time for any improving step is \( O(n) \), the lower bound differs from the upper bound by at most an \( o(n^2) \) term. Thus, the expected runtime of these mechanisms is also \( \frac{k}{2} \sum_{i=0}^{n-1} \sum_{m=1}^{k} m \cdot \frac{1}{n} \left( \frac{n - i - 1}{n} \right)^{m-1} \pm o(n^2) \).

For the Random Gradient mechanism, we note that the probability that an operator, when repeated following a success, is successful again is at most \( \frac{k}{n} \) (an upper bound on the success probability of the kBitFlip operator by Lemma 12). Since there are at most \( n \) improvements to be made throughout the search space, the expected number of repeats which produce an improvement is at most \( k = \Theta(1) \). If the chosen operator is not successful, the Random Gradient mechanism behaves identically to the Simple Random mechanism. Its expected runtime is therefore at least the expected runtime of the Simple Random mechanism less an \( o(n^2) \) term, and at most the expected runtime of the Simple Random mechanism plus \( n \) (as there are at most \( n \) iterations where the mechanisms differ in operator selection). Thus, its expected runtime is also \( \frac{k}{2} \sum_{i=0}^{n-1} \sum_{m=1}^{k} m \cdot \frac{1}{n} \left( \frac{n - i - 1}{n} \right)^{m-1} \pm o(n^2) \). □
In the next subsection, we will see that the Generalised Random Gradient hyper-heuristic has the best possible performance achievable on LEADINGONES, up to lower order terms, with the low-level heuristics available.

4.2 The Generalised Random Gradient Hyper-heuristic Has the Best Possible Performance Achievable

In this subsection, we present a rigorous theoretical analysis of the expected runtime of the Generalised Random Gradient hyper-heuristic on LEADINGONES using $k \geq 2$ operators, $k = \Theta(1)$. The derived upper bound on the expected runtime decreases with $k$.

The following general theorem bounds the expected runtime of the Generalised Random Gradient hyper-heuristic using $k$ low-level stochastic mutation heuristics of different neighbourhood size for any value of $\tau$ smaller than $\frac{1}{k} n \ln n$ for any values of $k = \Theta(1)$. The theorem allows us to identify values of $\tau$ for which the expected runtime of the hyper-heuristic for the function is the optimal expected runtime that may be achieved by using $k$ operators. This result will be highlighted in Corollary 18 for values of $\tau = \omega(n)$. The main result of this section has been presented in Theorem 11, which rigorously proves that increasing the number of operators (to $k$) leads to improved expected runtimes on the best-possible algorithm using $m < k$ operators. Similarly to the proof of Theorem 9, we partition the search space into $w = \log^2(n)$ chunks, each representing an equal range of fitness values.

**Theorem 17** The expected runtime of the Generalised Random Gradient hyper-heuristic with access to $k$ low-level stochastic mutation operators of different neighbourhood size on LEADINGONES for $\tau \leq \left(\frac{1}{k} - \epsilon\right) n \ln(n)$, for some constant $\epsilon > 0$ with $(1, \ldots, k)\text{BitFlip}, k = \Theta(1)$, is at most:

$$\frac{n^2}{2} \cdot \sum_{j=1}^{w} \left( \frac{k \cdot \frac{\tau}{n} + \left[ \sum_{m=1}^{k} e^{m \frac{\tau}{n} \left(1 - \frac{1-j}{w}\right)^m} \cdot M_m(j, w) \right]}{w \cdot \left[ \sum_{m=1}^{k} e^{m \frac{\tau}{n} \left(1 - \frac{1-j}{w}\right)^m} - k \right]} \right) \pm o(n^2)$$

where for $m \geq 2$,

$$M_m(j, w) := \begin{cases} \frac{\tau}{n} & \text{if } j = w \text{ or } \frac{1}{m(1-j/w)^{m-1}} > \frac{\tau}{n} \\ \frac{1}{m(1-j/w)^{m-1}} & \text{otherwise} \end{cases}$$

and $M_1(j, w) = \min\left\{ \frac{\tau}{n}, 1 \right\}$, with $w = \log^2(n)$.
Table 1: The upper bounds in the leading constants found from various parameter combinations of the number of operators $k$ and the learning period $\tau$ in Theorem 17 with $w = 100,000$.

Theorem 17 presents the general result of the expected runtime of the Generalised Random Gradient hyper-heuristic, with access to $k = \Theta(1)$ low-level stochastic mutation heuristics of different neighbourhood size, on LEADINGONES for $\tau \leq \left(\frac{1}{k} - \epsilon\right) n \ln(n)$, for some constant $\epsilon > 0$. In Table 1 we present some of the most interesting parameter combinations of $k$ and $\tau$ that arise from this result. When $\tau = 1$, the GRG hyper-heuristic acts the same as the Simple Random mechanism; when $\tau = 0.1 n \ln(n)$ GRG is able to find the best possible runtime achievable with the low-level heuristics, up to lower order terms, as will be presented in Theorem 18.

Proof: [Of Theorem 17]

We follow the proof idea for the Generalised Random Gradient hyper-heuristic for two operators (Theorem 9). That is, we partition the optimisation process into $w = \log^2(n)$ chunks based on the value of the LEADINGONES fitness function: during chunk $j$, the LO value of the current solution is at least $\left(\frac{j-1}{w}\right)n$ and less than $\frac{jn}{w}$. After all $w$ chunks have been completed, the global optimum, with a LO value of $n$, will have been found.

We upper bound the expectation of the runtime $T$ of the Generalised Random Gradient hyper-heuristic on LO by the sum of the expected values of $T_j$ (the expected time spent in each chunk of the optimisation process) plus the expected times taken to begin a chunk (that is, the time before the first random operator choice occurs in each chunk). As our analysis of $E(T_j)$ requires the chunk to start with a random choice of mutation operator, we bound $E(T) \leq \sum_{j=1}^{w} (E(T_j) + E(S_{j+1}))$ where $S_{j+1}$ is a random variable denoting the expected number of iterations between the first solution in chunk $j + 1$ being constructed and the first random operator choice occurring in that chunk. We will later show that with proper parameter
choices, the contribution of $S_{j+1}$ terms can be bounded by $o(n^2)$, and therefore they do not affect the leading constant in the overall bound.

Let us now consider $T_j$, the number of iterations the hyper-heuristic spends in chunk $j$. Recall that a mutation operator is selected uniformly at random, and is allowed to run until it fails to produce an improvement within $\tau$ sequential iterations. Let $N_j$ be a random variable denoting the number of random operator choices the hyper-heuristic performs during chunk $j$, and $X_{j,1}, \ldots, X_{j,N_j}$ be the number of iterations the hyper-heuristic runs each chosen operator for. We note that $T_j = \sum_{k=1}^{N_j} X_{j,k}$ is a sum of a number of non-negative variables, and we will later show that $E(N_j)$ is bounded; thus, $E(T_j)$ can be bounded by applying Wald's equation (Theorem I):

$$E(T_j) = E(\sum_{k=1}^{N_j} X_{j,k}) \leq E(N_j)E(X_j), \quad (6)$$

where $E(X_j)$ denotes an upper bound on all $E(X_{j,k})$ in chunk $j$.

To bound $E(N_j)$, the expected number of times the random operator selection is performed during chunk $j$, we lower bound the expected number of improvements found following operator selection, and apply the Additive Drift Theorem (He and Yao, 2001) to find the expected number of random operator selections occurring before a sufficient number of improvements have been found to enter the next chunk.

Recall from Lemma[12] that the improvement probability for a MBITFLIP operator is

$$P(\text{Imp}_m) = m \cdot \frac{1}{n} \cdot \left( \frac{n-i-1}{n} \right)^{m-1} + O(n^{-2}).$$

Since either the $m \cdot \frac{1}{n} \cdot \left( \frac{n-i-1}{n} \right)^{m-1}$ terms will asymptotically dominate, or the $P(\text{Imp}_m) = \frac{1}{n}$ term will asymptotically dominate, the lower order terms will be insignificant. In particular, we can combine the impact of the $O(n^{-2})$ terms into the $o(n^2)$ in the final bound on the expected runtime and omit them from the $P(\text{Imp}_m)$ terms in the calculations.

Let $F_m$ denote the event that the MBITFLIP operator fails to find an improvement during $\tau$ iterations. This will occur with probability $P(F_m) = (1 - P(\text{Imp}_m))^\tau$. Within chunk $j$, the ancestor individual has at most $\frac{jn}{w} - 1$, and at least $(j-1)n$, leading 1-bits. Hence:

$$P(F_m) \leq \left( 1 - m \cdot \frac{1}{n} \left( \frac{n - (\frac{jn}{w} - 1)}{n} \right)^{m-1} \right)^\tau \leq \left( 1 - m \cdot \frac{1}{n} \left( \frac{1 - (\frac{j}{w})}{1} \right)^{m-1} \right)^\tau \leq e^{-\frac{m}{n}(1 - \frac{j}{w})^{m-1}};$$

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\[
P(F_m) \geq \left(1 - m \cdot \frac{1}{n} \left(n - \left(\frac{(j-1)n}{w} - 1\right)^{m-1}\right)\right)^\tau \geq \left(1 - m \cdot \frac{1}{n} \left(1 - \frac{j-1}{w}\right)^{m-1}\right)^\tau
\]

We note that a geometric distribution with parameter \( p = P(F_m) \) can be used to model the number of improvements that the MBITFLIP operator finds prior to failing to find an improvement for \( \tau \) iterations; the expectation of this distribution is \( \frac{1}{1 - p} = \frac{1}{p} - 1 \). We are able to model these events as a geometric random variable since the GRG hyper-heuristic immediately restarts a period of \( \tau \) iterations with the chosen heuristic when a successful mutation occurs (i.e., it sets \( c_t = 0 \)); hence, there are no periods of \( \tau \) consecutive unsuccessful mutations until the operator has finally failed. Combined over all \( k \) operators, the expected number of improvements \( D_j \) produced following a single random operator selection during chunk \( j \), \( E(D_j) \), is greater than:

\[
E(D_j) \geq \sum_{m=1}^{k} \frac{1}{k} \left(1 - \frac{1}{P(F_m)^m} - 1\right) \geq \frac{1}{k} \left(\sum_{m=1}^{k} e^{\frac{m}{n}(1-\frac{j}{w})^{m-1}}\right) - 1
\]

by inserting the upper bounds for \( P(F_m) \). We use this expectation as the drift in the Additive Drift Theorem to upper bound \( E(N_j) \). Recall that each chunk consists of advancing through at most \( \frac{n}{w} \) fitness values. Since bits beyond the leading ones prefix and the first zero bit remain uniformly distributed, in expectation \( \frac{n}{2w} \) improvements by mutation are required. If each step of a random process in expectation contributes \( E(D_j) \) improvements by mutation, the expected number of steps required to complete chunk \( j \) is at most

\[
E(N_j) \leq \frac{\frac{n}{2w}}{E(D_j)} \leq \frac{k}{2} \cdot \frac{n}{w} \cdot \frac{1}{\sum_{m=1}^{k} e^{\frac{m}{n}(1-\frac{j}{w})^{m-1}}} - k
\]

by the Additive Drift Theorem.

To bound \( E(X_j) \), the expected number of iterations before a selected mutation operator fails to produce an improvement for \( \tau \) iterations, we apply Wald’s equation: let \( S \) be the number of improvements found by the operator before it fails, and \( \sum_{s=1}^{k} E(W_s) \) be the number of iterations it took to find each of those improvements. Then, once selected, the MBITFLIP operator runs for:

\[
E(X_j | \text{MBITFLIP}) = \tau + \sum_{s=1}^{S} E(W_s) = E(S)E(W_m | S \geq 1)
\]

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where $\tau$ accounts for the iterations immediately before failure, and the sum for the iterations preceding each constructed improvement. Recall that $E(S) = \frac{1}{P(F_m)} - 1$ by the properties of the geometric distribution, and observe that $E(W_m \mid S \geq 1) = E(W_m \mid W_m \leq \tau) \leq \min\{\tau, E(W_m)\}$. Using a waiting time argument (i.e., considering the expectation of a geometric random variable) for $E(W_m)$ being equal to the reciprocal of the improvement probability for the MBITFLIP operator, we get:

\[
E(X_j \mid m) \leq \tau + \left(\frac{1}{P(F_m)} - 1\right) \cdot \min\left\{\tau, \frac{1}{m \cdot \frac{1}{n} \cdot \left(\frac{n - (\frac{j}{w} - 1)}{n}\right)^{m-1}}\right\}
\]

\[
= \tau + \left(\frac{1}{e^{\frac{m \tau}{n}} \left(1 - \frac{j}{w}\right)^{m-1}} - 1\right) \cdot \min\left\{\tau, \frac{n}{m \left(1 - \frac{j}{w}\right)^{m-1}}\right\}
\]

\[
\leq \tau + e^{\frac{m \tau}{n}} \left(1 - \frac{j}{w}\right)^{m-1} \cdot \min\left\{\tau, \frac{n}{m \left(1 - \frac{j}{w}\right)^{m-1}}\right\} + O(1).
\]

Combining these conditional expectations yields

\[
E(X_j) \leq \sum_{m=1}^{k} \frac{1}{k} \cdot E(X_j \mid m) \leq \tau + \sum_{m=1}^{k} \left[\frac{1}{k} \cdot e^{\frac{m \tau}{n}} \left(1 - \frac{j}{w}\right)^{m-1} \cdot \min\left\{\tau, \frac{n}{m \left(1 - \frac{j}{w}\right)^{m-1}}\right\}\right] + O(k)
\]

\[
\leq n \left(\frac{\tau}{n} + \sum_{m=1}^{k} \left[\frac{1}{k} \cdot e^{\frac{m \tau}{n}} \left(1 - \frac{j}{w}\right)^{m-1} \cdot \min\left\{\tau, \frac{n}{m \left(1 - \frac{j}{w}\right)^{m-1}}\right\}\right]\right) + O(k)
\]

\[
\leq n \left(\frac{k \tau}{n} + \sum_{m=1}^{k} \left[e^{\frac{m \tau}{n}} \left(1 - \frac{j}{w}\right)^{m-1} \cdot \min\left\{\tau, \frac{n}{m \left(1 - \frac{j}{w}\right)^{m-1}}\right\}\right]\right) + O(k).
\]

(8)

At $j = w$, we have that $\min\left\{\tau, \frac{1}{m \left(1 - \frac{j}{w}\right)^{m-1}}\right\}$ is undefined. However, we know that the chosen operator would only be applied for a maximum of $\tau$ steps. Hence, we define $M_1(j, w) = \min\left\{\frac{\tau}{n}, 1\right\}$ and for $m \geq 2$,

\[
M_m(j, w) := \begin{cases} \frac{\tau}{n} & \text{if } j = w \text{ or } \frac{1}{m \left(1 - \frac{j}{w}\right)^{m-1}} > \frac{\tau}{n} \\ \frac{1}{m \left(1 - \frac{j}{w}\right)^{m-1}} & \text{otherwise.} \end{cases}
\]
Finally, we return to bounding the overall expected runtime of the hyper-heuristic.

Recall that $S_j$ denoted the number of iterations the algorithm spends in
chunk $j$ while using the random operator chosen during chunk $j - 1$, and as
$E(S_j) \leq \max(E(X_j | \text{MBITFLIP})_{m=1,...,k} < kE(X_j) = O(n \cdot \exp \left(\frac{k\tau}{n}\right))$, and since $w = \log^2(n) = o \left(\frac{n}{\exp(\frac{k\tau}{n})}\right), \sum_{j=1}^{w} E(S_{j+1}) = o(n^2)$. Substituting the bounds (7) and (8) into (6) yields the theorem statement:

$$E(T) \leq \sum_{j=1}^{w} (E(T_j) + E(S_{j+1})) \leq \sum_{j=1}^{w} E(N_j)E(X_j) + o(n^2)$$

$$\leq o(n^2) + \frac{n^2}{2} \cdot \sum_{j=1}^{w} \frac{k\tau}{w} + \sum_{m=1}^{k} \frac{e^{m\tau(1 - \frac{1}{w})^{m-1}} \cdot M_m(j, w)}{\left(\sum_{m=1}^{k} e^{m\tau(1 - \frac{1}{w})^{m-1}}\right) - k}.$$

We now show that for appropriate values of the parameter $\tau$, the $k$-operator
Generalised Random Gradient hyper-heuristic can achieve the best-possible expected runtime available for a mechanism using $k$-operators, up to lower order terms. The following Corollary provides an upper bound on the expected runtime for sufficiently large learning periods (i.e., $\tau = \omega(n)$).

**Corollary 18** The expected runtime of the Generalised Random Gradient hyper-heuristic on **LEADINGONES** using $(1, \ldots, k)$BITFLIP, where $k = \Theta(1)$, with $\tau$ that satisfies both $\tau = \omega(n)$ and $\tau \leq \left(\frac{1}{k} - \epsilon\right)n \ln(n)$, for some constant $\epsilon > 0$, is at most

$$\frac{1}{2} \left(\sum_{i=0}^{\frac{n}{k} - 1} \frac{1}{k \cdot \frac{1}{n} \cdot (\frac{n}{k} - 1)^{k-1}} + \sum_{m=1}^{k-1} \sum_{i=\frac{m}{n}}^{\frac{n}{k} - 1} \frac{1}{m \cdot \frac{1}{n} \cdot (\frac{n}{k} - 1)^{m-1}}\right) \pm o(n^2).$$

**Proof:** This proof follows a similar structure to that of Corollary 8.

We want to simplify the sum from the statement of Theorem 17 which states that

$$E(T) \leq \frac{n^2}{2} \cdot \sum_{j=1}^{w} \left(\frac{k \cdot \frac{1}{n} + \sum_{m=1}^{k} e^{m\tau(1 - \frac{1}{w})^{m-1}} \cdot M_m(j, w)}{w \cdot \left(\sum_{m=1}^{k} e^{m\tau(1 - \frac{1}{w})^{m-1}}\right) - k}\right) \pm o(n^2).$$

Since $\tau = \omega(n)$, we have that $\frac{1}{n} = \omega(1)$. Hence, we can simplify the $M$ terms. Firstly, we have that $M_1(j, w) = \min\{\frac{1}{n}, 1\} = 1$. Recall that, for
\( m \geq 2, \)
\[
M_m(j, w) := \begin{cases} 
\frac{\tau_n}{1} & \text{if } j = w \text{ or } \frac{1}{m(1 - \frac{j}{w})^{m-1}} > \frac{\tau_n}{n} \\
\frac{1}{m(1 - \frac{j}{w})^{m-1}} & \text{otherwise.} 
\end{cases}
\]

Since \( \frac{\tau_n}{n} = \omega(1) \), we have that \( \frac{\tau_n}{n} > \frac{1}{m(1 - \frac{j}{w})^{m-1}} \). Hence, we can simplify:
\[
M_m(j, w) := \begin{cases} 
\frac{\tau_n}{n} & \text{if } j = w \\
\frac{1}{m(1 - \frac{j}{w})^{m-1}} & \text{otherwise.} 
\end{cases}
\]

In particular, for \( j \neq w \), we have that \( M_1(j, w) = \Theta(1) \) and, for \( m \geq 2 \), \( M_m(j, w) = \Theta(1) \).

We can further simplify the sum from Theorem 17 by noting that, since \( \frac{\tau_n}{n} = \omega(1) \), the exponential term in each summand will be asymptotically dominant. In particular, the \( \frac{\tau_n}{n} \) term in the numerator, and the \( -k \) term in the denominator, will be asymptotically dominated. Hence, we can relegate these terms into the lower order \( o(n^2) \) term.

Furthermore, the difference between the exponential terms in the numerator and denominator in the sum from Theorem 17 differ only by a small multiplicative term, \( 1 + o(1) \). Hence, we can consider both as \( e^{m\frac{\tau_n}{n} (1 - \frac{j}{w})^{m-1}} \), with the difference again being relegated into the \( o(n^2) \) term.

With these modifications, we can simplify the sum:
\[
E(T) \leq \frac{n^2}{2} \cdot \sum_{j=1}^{w} \left( \frac{\left( k \cdot \frac{\tau_n}{n} + \left[ \sum_{m=1}^{k} e^{m\frac{\tau_n}{n} (1 - \frac{j}{w})^{m-1}} \cdot M_m(j, w) \right] \right)}{w \cdot \left( \left[ \sum_{m=1}^{k} e^{m\frac{\tau_n}{n} (1 - \frac{j}{w})^{m-1}} \right] - k \right)} + o(n^2) \right) 
\leq \frac{n^2}{2} \sum_{j=1}^{w} \left( \frac{\left( \sum_{m=1}^{k} e^{m\frac{\tau_n}{n} (1 - \frac{j}{w})^{m-1}} \cdot M_m(j, w) \right)}{w \cdot \left( \left[ \sum_{m=1}^{k} e^{m\frac{\tau_n}{n} (1 - \frac{j}{w})^{m-1}} \right] \right)} + o(n^2) \right).
\]

To further simplify the sum, we consider when each of the \( k \) exponential summand terms will be asymptotically dominant in the numerator. Since \( \frac{\tau_n}{n} = \omega(1) \), whichever exponential term has the largest exponent will asymptotically dominate the other terms. In particular, the \( m^{th} \) term will dominate when \( \frac{m \tau_n}{n} (1 - \frac{j}{w})^{m-1} \) is the largest amongst \( 1 \leq m \leq k \). Comparing the terms at \( m \) and \( m + 1 \), we see that the \( m^{th} \) term dominates the \( (m + 1)^{th} \) term when \( j \geq \frac{w}{m+1} \). Continuing these calculations, we have that the \( m^{th} \) term will dominate all others (i.e., have the largest exponent) when \( \frac{w}{m+1} \leq j \leq \frac{w}{m} \).
\[ \frac{w}{m} - 1 \] for \( m < k \), and the \( k^{th} \) term will dominate when \( j \leq \frac{w}{k} - 1 \). If a term is asymptotically dominated, we can relegate it into the \( o(n^2) \) lower order term. Hence, the only terms remaining in the numerator in the period when the \( m^{th} \) operator dominates will be the \( m^{th} \) exponential term multiplied by \( M_m(j, w) \); the denominator will similarly only contain the \( m^{th} \) exponential term, multiplied by \( w \). The sum will hence simplify to a sum of \( M_m(j, w) \) terms:

\[
E(T) \leq \frac{n^2}{2} \cdot \left( \sum_{j=1}^{\frac{w}{m}} M_k(j, w) + \sum_{m=1}^{k-1} \sum_{j=\frac{w}{m}+1}^{\frac{w}{m}} M_m(j, w) \right) \pm o(n^2).
\]

Recall that, for \( i = LO(x) \), the chunk \( j \) refers to when \( \frac{(j-1)n}{w} \leq i \leq \frac{wn}{n} - 1 \). In particular, if we substitute \( i \geq \frac{(j-1)n}{w} \), (or \( j \leq \frac{wn}{n} + 1 \)) into each summand, we will find

\[
\frac{1}{m \cdot \frac{1}{n}} \cdot \left( \frac{1}{n} \right)^{m-1} \leq \frac{1}{m \cdot \frac{1}{n}} \cdot \left( \frac{n-i-1}{n} \right)^{m-1}.
\]

Furthermore, using the upper and lower bounds on \( i \) to adjust the bounds on the sums (which will cancel out the multiplicative \( n \) term) gives the result:

\[
E(T) \leq \frac{n^2}{2} \cdot \left( \sum_{j=1}^{\frac{w}{m}} \frac{1}{m \cdot \frac{1}{n}} \cdot \left( \frac{1}{n} \right)^{m-1} + \sum_{m=1}^{k-1} \sum_{j=\frac{w}{m}+1}^{\frac{w}{m}} \frac{1}{m \cdot \frac{1}{n}} \cdot \left( \frac{1}{n} \right)^{m-1} \right) \pm o(n^2)
\]

Corollary [18] presents the result that the generalised random gradient hyper-heuristic with suitably chosen \( \tau \) runs in the best-possible expected runtime achievable on \texttt{LEADINGONES} with its low-level heuristics, up to lower order terms. Doerr and Wagner (2018) suggested that the theoretical performance limit for any algorithm using such mutation operators on
LEADINGONES is $\approx 0.388n^2$. Corollary 18 shows that GRG can match this theoretical performance limit up to one decimal place with 4 low-level heuristics, up to two decimal places with 11 low-level heuristics and up to three decimal places with 18 low-level heuristics.

We can now prove the main result of this section, i.e., Theorem 11.

**Proof:** [Of Theorem 11]

Corollary 18 shows that the GRG hyper-heuristic with $k$ mutation operators can match the best-possible performance for an algorithm with $k$ mutation operators from Theorem 14 up to lower order terms. The results from Corollary 18 and Theorem 14 imply that the only difference in the best case expected runtime for the GRG hyper-heuristic with $k$ operators ($(1, \ldots, k)\text{BITFLIP}$), and the best-possible expected runtime for an algorithm with $k-1$ operators (i.e., using $(1, \ldots, k-1)\text{BITFLIP}$), occurs when $0 \leq i \leq \frac{n}{k} - 1$. In this region, the best case expected performance of the $k$-operator variant matches the expected performance of only using the $k\text{BITFLIP}$ operator in this area, up to lower order terms, while the expected performance of the best-possible $k-1$-operator algorithm matches the expected performance of using the $(k-1)\text{BITFLIP}$ operator in this area.

We know from Lemma 13 that the expected runtime to optimise this area when applying the $k\text{BITFLIP}$ operator will be less than the expected runtime when applying the $(k-1)\text{BITFLIP}$ operator, and thus using the $k\text{BITFLIP}$ operator will be faster. Since in the rest of the search space the best case $k$-operator GRG hyper-heuristic and the best-possible $k-1$-operator algorithm will have the same expected performance (up to lower order terms), the best case expected runtime of the GRG hyper-heuristic with $k$ operators will be faster than the best-possible expected runtime of an algorithm $k-1$-operators.

Furthermore, continuing the argument for the best-possible performance of an algorithm with $(1, \ldots, m)\text{BITFLIP}$ operators ($m < k - 1$) implies that the best case expected runtime of the $k$-operator Generalised Random Gradient hyper-heuristic is faster than the best-possible performance of any algorithm with $m < k$ operators. In particular, GRG with $k$ operators ($(1, \ldots, k)\text{BITFLIP}$) is faster than the best-possible algorithm using any strict subset of $(1, \ldots, k)\text{BITFLIP}$. The result from Lemma 13 (and a similar argument as used above with $k-1$ operators) implies that GRG will outperform the best-possible algorithm in the area of the search space corresponding to where the best-possible algorithm is missing any of the $(1, \ldots, k)\text{BITFLIP}$ operators (i.e., if the best-possible algorithm is missing the $M\text{BITFLIP}$ operator, GRG will outperform it in the region where the LO fitness value satisfies
Figure 2: The leading constants in the theoretical upper bounds on the average number of fitness function evaluations required by the $k$-operator Generalised Random Gradient hyper-heuristic to find the \textsc{LeadingOnes} optimum (we have used a value of $w = 100,000$ in the result of Theorem 17).

\[
\frac{n}{m+1} \leq \text{LO}(x) \leq \frac{n}{m} - 1,
\]

leading to a faster expected runtime. □

Figure 2 shows the relevant theoretical upper bounds for the $k$-operator variant of the Generalised Random Gradient hyper-heuristic from Theorem 17. The upper bounds found by the hyper-heuristics with more operators are better than the ones with less operators, as given by Corollary 14. In particular, the upper bounds for any hyper-heuristic with $k$-operators outperforms the best possible expected runtime for any hyper-heuristic with less than $k$ operators, as implied by Theorem 11. We depict this result explicitly for $k = 3$ and $k = 5$ in Figure 3.

5 Complementary Experimental Analysis

In the previous sections we have proved that the Generalised Random Gradient hyper-heuristic performs efficiently for the \textsc{LeadingOnes} benchmark function for large enough problem sizes $n$. In this section we present some experimental results for the Generalised Random Gradient hyper-heuristic on \textsc{LeadingOnes} in order to shed light on its performance for different problem sizes up to $n = 10^8$. All parameter combinations have been simulated 10,000 times.

In order to efficiently handle larger problem dimensions experimentally,
Figure 3: A comparison of the optimal expected runtimes of the Generalised Random Gradient hyper-heuristic with \( k \)-operators against the leading constant in the theoretical upper bound of the expected runtime of the Generalised Random Gradient hyper-heuristic with \( k + 1 \)-operators; for \( k = 2 \) and \( k = 4 \). \( 2_{\text{Opt}} \approx 0.42329n^2 \) and \( 4_{\text{Opt}} \approx 0.39830n^2 \) are the best possible runtimes achievable by the GRG hyper-heuristic with access to 2 and 4 low-level heuristics respectively, from Corollary 18.

we do not simulate each individual mutation performed by the hyper-heuristic, but rather sample the waiting times for a fitness-improving mutation to occur using a geometric distribution (with the success probability \( p \) depending on the current operator and LO value of the current solution). Specifically, suppose \( T_{\text{im}} \) is a random variable denoting the number of mutations required to get one fitness-improving mutation; as \( T_{\text{im}} \) counts the number of independent trials each with probability \( p \) of success, \( P(T_{\text{im}} \leq k) = 1 - (1 - p)^k \) by the properties of the geometric distribution. Given access to a uniform(0, 1)-distributed random variable \( U \), \( T_{\text{im}} \) can be sampled by computing \( \left\lceil \frac{\log(1-U)}{\log(1-p)} \right\rceil \).

5.1 Two Low-level Heuristics (\( k = 2 \))

We first consider the Generalised Random Gradient hyper-heuristic using two operators only (i.e., 1B\text{IT}F and 2B\text{IT}F\text{LIP}), and look at the impact of the parameter \( \tau \) and problem size \( n \).

Figure 4 shows the runtimes for the generalised hyper-heuristics on \texttt{LEADINGONES} for \( n = 10,000 \) and \( n = 50,000 \), illustrating the effect \( \tau \) has on the runtime.

The Generalised Random Gradient hyper-heuristic’s performance clearly depends on the choice of \( \tau \). It is worth noting that as the problem size
increases, for $\tau \approx 0.55n \ln(n)$, the runtime seems to be approaching the optimal performance proved in Corollary 8 (i.e., $\frac{1+\ln(2)}{4}n^2 \approx 0.42329n^2$).

For well chosen $\tau$ values, the hyper-heuristic beats the expected runtime of RLS and also the experimental runtime for the recently presented $k$-bit mutation algorithm with self-adjusting $k$, of $0.450n^2$, for the parameter choices used and presented by Doerr et al. (2016). As $\tau$ increases past $0.6n^2$, we see a detriment in the performance of the hyper-heuristic. It is worth noting, however, that for $n = 50,000$, it is required that $\tau > 1.5n \ln(n) = 811,483$ to be worse than the RLS expected runtime of $0.5n^2$, indicating that the parameter is very robust.

Figure 5 shows the effects of increasing the problem size $n$ for a variety of fixed $\tau$ values. We can see that an increased problem size leads to faster runtimes. In particular, the hyper-heuristic requires a problem size of at least 200 before it outperforms RLS. The performance difference between the $\tau$ values decreases with increased $n$, indicating that further increasing $n$ would lead to similar, optimal performance for a large range of values, as implied by Corollary 8. For $n = 10^8$, the runtime for $\tau = 0.6n \ln n$ is $\approx 0.42716n^2$, only slightly deviated from the optimal value of $\approx 0.42329n^2$. 

Figure 4: Average number of fitness function evaluations required by the hyper-heuristics with $k = 2$ operators to find the LEADINGONES optimum for $n = 10,000$ (solid), $n = 50,000$ (dashed).
5.2 More Than Two Low-level Heuristics ($k \geq 2$)

We now consider the generalised hyper-heuristics using $k$ operators, $(1, \ldots, K)\text{BITFLIP}$, and look at the impact of the parameter $\tau$.

Figure 5 shows the runtimes for the Generalised Random Gradient hyper-heuristic when choosing from 2 to 5 separate mutation operators on LEADINGONES, for a problem size of $n = 100,000$.

We see that incorporating more operators can be beneficial to the performance of the GRG hyper-heuristic. Whilst the 2-operator hyper-heuristic achieves a best performance of $\approx 0.43742n^2$, the 5-operator hyper-heuristic achieves a best performance of $\approx 0.42906n^2$. It is, however, important to set $\tau$ optimally to achieve the best performance. For $0 \leq 0.35n \ln(n) \leq \tau$ and $\tau \geq 1.5n \ln(n)$, the 2-operator hyper-heuristic outperforms the others. However, it is unable to achieve a better optimal performance. The results imply that the more operators that are incorporated in the hyper-heuristic, the shorter the range of $\tau$ values for which it performs the best in comparison with the hyper-heuristics with fewer operators.

We know from Corollary 18 that for a sufficiently large problem size, GRG with 5-operators will tend towards the theoretical optimal result of $\approx 0.39492n^2$. Furthermore, we know from Theorem 11 that GRG equipped with more operators will give improved performance.

Recently Doerr and Wagner (2018) analysed a (1+1) EA on LEADINGONES,
where the mutation rate is updated on-the-fly. The number of bits to be flipped is sampled from a binomial distribution $B(n, p)$ without allowing a 0BITFLIP to occur, differently from standard bit mutation arguments by resampling until a non-zero bitflip occurs. Then a multiplicative comparison-based update rule, similar to the 1/5th rule from combinatorial optimisation (Auger, 2009) is applied to update the parameter $p$ (i.e., the mutation rate). A successful mutation increases the mutation rate by a multiplicative factor $A > 1$, while an unsuccessful mutation multiplies the mutation rate by a multiplicative factor $b < 1$. An experimental analysis is performed to tentatively identify the leading constants in the expected runtime for the best combinations of $A$ and $b$. The best leading constant that has been identified (with $A = 1.2$ and $b = 0.85$) is 0.4063. Assuming that their identified values indeed scale with the problem sizes (they only test problem sizes up to $n = 1500$), such leading constants are worse than those achieved by the Generalised Random Gradient hyper-heuristic.

\footnote{Note that while not allowing 0-bit flips is a reasonable choice for an extremely simplified EA such as the (1+1) EA, it is doubtful that it is a good idea in general. For instance, we conjecture that the expected runtime of the $(\mu+1)$ EA (Witt, 2006) and $(\mu+1)$ GA (Corus and Oliveto, 2018) would deteriorate for ONEMAX and LEADINGONES, while non-elitist EAs and GAs would require exponential time to optimise any function with a unique optimum. An alternative implementation that allows 0-bit flips to occur, but simply avoids evaluating offspring that are identical to their parents, would solve the issue while at the same time producing the results reported by Doerr and Wagner (2018). Such an idea was suggested by Doerr and Carvalho Pinto (2017).}
6 Conclusion

The field analysing theoretically the performance of hyper-heuristics is growing. Previous results have shown that the ‘simple’ learning mechanisms applied in the literature to combinatorial optimisation problems show no signs of learning on LEADINGONES and perform equivalently to choosing operators at random, up to lower order terms. We generalised the existing learning mechanisms to allow success to be measured over a longer period of time, \( \tau \), rather than in a single iteration. We proved that these generalised hyper-heuristics are faster than standard unary unbiased evolutionary and local-search algorithms for the LEADINGONES benchmark function.

To apply the generalised hyper-heuristics, a value for the learning period \( \tau \) is required. Although our results indicate that \( \tau \) is a fairly robust parameter (i.e., for \( n = 10,000 \) the Generalised Random Gradient hyper-heuristic achieved faster experimental runtimes than that of the \((1+1)\) EA for all tested values of \( \tau \) between 1 and \( 10^6 \), and faster than RLS for all tested values of \( \tau \) between 28,000 and 120,000), setting it appropriately will lead to optimal performance. Clearly \( \tau \) must be large enough to have at least a constant expected number of successes within \( \tau \) steps, if the hyper-heuristic has to learn about the operator performance. Naturally, setting large values of \( \tau \) may lead to large expected runtimes, since switching operators requires \( \Omega(\tau) \) steps.

We also have rigorously shown that the performance of the ‘simple’ mechanisms deteriorates as the choice of operators (choosing from \( k \geq 2 \)) increases, while the performance of the Generalised Random Gradient hyper-heuristic improves with a larger choice, as desired for practical applications.

Recently, Doerr et al. (2018) have equipped the Random Gradient hyper-heuristic with an adaptive update rule to automatically adapt the parameter \( \tau \) throughout the run (i.e., the learning period can change its duration during the optimisation process). They proved that the hyper-heuristic is able to achieve the same optimal performance, up to lower order terms, on the LEADINGONES benchmark function. This so called Adaptive Random Gradient hyper-heuristic, equipped with two operators, experimentally outperforms the best setting of the Generalised Random Gradient, confirming that \( \tau \) should not be fixed throughout the optimisation process. Several further directions can be explored in future work. Firstly, the hyper-heuristic’s performance on a broader class of problems, including classical ones from combinatorial optimisation, should be rigorously studied. Secondly, more sophisticated hyper-heuristics that have shown superior performance in practical applications should be analysed, such as the choice function approach (Cowling et al., 2001) which keeps track of the historical performance of each low-level heuristic and uses this
information to decide which one to be applied next. Finally, considering more general low-level heuristics (e.g. with different population sizes) will bring a greater understanding of the performance of selection hyper-heuristics.

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