Generalized V-line transforms in 2D vector tomography

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Abstract

We study the inverse problem of recovering a vector field in $\mathbb{R}^2$ from a set of new generalized V-line transforms in three different ways. First, we introduce the longitudinal and transverse V-line transforms for vector fields in $\mathbb{R}^2$. We then give an explicit characterisation of their respective kernels and show that they are complements of each other. We prove invertibility of each transform modulo their kernels and combine them to reconstruct explicitly the full vector field. In the second method, we combine the longitudinal and transverse V-line transforms with their corresponding first moment transforms and recover the full vector field from either pair. We show that the available data in each of these setups can be used to derive the signed V-line transform of both scalar component of the vector field, and use the known inversion of the latter. The final major result of this paper is the derivation of an exact closed form formula for reconstruction of the full vector field in $\mathbb{R}^2$ from its star transform with weights. We solve this problem by relating the star transform of the vector field to the ordinary Radon transform of the scalar components of the field.

1 Introduction

The primary task of integral geometry is reconstructing a scalar function or a vector field (or more generally a tensor field) from some kind of integral transform data. These types of reconstruction problems are often crucial parts of various non-invasive imaging techniques with applications in medicine, seismology, oceanography and many other areas.

A typical integral geometry problem can be formulated as follows. What information about a tensor field of rank $m$ can be recovered from its longitudinal ray transform (also known as ray transform)? It has been shown by several authors that a scalar function (corresponding to the case $m=0$) can be reconstructed uniquely from the knowledge of its ray transform (e.g. see [32]). For $m \geq 1$, this transform has a non-trivial kernel, which makes the full recovery of a tensor field impossible, when using only the ray transform data. In this case, only the solenoidal part of a tensor field can recovered. The latter problem has been studied in various settings by multiple authors, e.g. see [9, 10, 21, 22, 26, 31, 34, 35, 37, 38, 43] and the references therein.

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The non-injectivity of the ray transform raises a natural question: what kind of additional data is needed for full reconstruction of tensor fields? In this context, an injectivity result has been presented utilizing the so-called integral moment transforms over symmetric \( m \)-tensor fields in \( \mathbb{R}^n \), see [36]. We also point out some recent related works for the invertibility of these generalized transforms [2, 23, 24, 25, 27, 30].

Another approach to reconstruct the full tensor field is to work with the transverse ray transform (TRT) instead of (or in addition to) the longitudinal ray transform (LRT). For \( n = 2 \), TRT and LRT provide equivalent information, up to a linear transformation of the tensor field. In particular, TRT also has a non-trivial kernel, making the full recovery just from TRT impossible. However, one can combine the data from both transforms (TRT and LRT) in 2D to recover the vector field completely, see [11]. In contrast to the 2D case, when \( n \geq 3 \) it is known that a symmetric \( m \)-tensor field is completely determined by its transverse ray transform, see [1, 12, 19, 33, 39]. In addition to these injectivity results, there are also various reconstruction results for TRT in different settings \((n \geq 3)\), see [18, 28, 45] and reference therein.

In this article, we consider a full reconstruction of a vector field in \( \mathbb{R}^2 \) using a new set of integral transforms (see Tables 1 and 2). These operators are analogous to the ray transforms discussed above, but instead of integrating along straight lines they use V-shaped paths of integration, called V-lines.

The V-line transform (often also called broken ray transform) for scalar functions in \( \mathbb{R}^2 \) maps a function to its integrals along piecewise linear trajectories, which consist of two rays emanating from a common vertex. Two distinct classes of V-line transforms with some generalizations have been studied by various authors in recent past. The first class includes V-lines (and cones in higher dimensions) that have a vertex on the boundary of the image domain, i.e. outside of the support of the image function (see the review article [42] and the references there). These transforms often appear in image reconstruction problems using Compton cameras. The second class includes V-lines (as well as stars and cones) that have a vertex inside the image domain, and they appear in relation to single scattering tomography [3, 4, 5, 6, 7, 8, 13, 14, 15, 16, 17, 19, 20, 29, 40, 41, 44, 46]. The V-line transforms of vector fields discussed in this paper are a natural generalization of the second class of V-lines discussed above.

| Symbol | Name | Definition |
|--------|------|------------|
| \( \mathcal{L} \) | Longitudinal V-line transform | 2 |
| \( \mathcal{T} \) | Transverse V-line transform | 3 |
| \( \mathcal{I} \) | First moment longitudinal V-line transform | 5 |
| \( \mathcal{J} \) | First moment transverse V-line transform | 6 |
| \( \mathcal{S} \) | Vector-valued star transform | 7 |

Table 1: A list of integral operators discussed in the paper.
Reconstruction of $f$ from: 

| Knowledge of  | Theorem |
|--------------|---------|
| $Lf$ and $Tf$ | 3, 4    |
| $Lf$ and $I_f$ | 5       |
| $Tf$ and $J_f$ | 6       |
| $Sf$          | 7       |

Table 2: A list of reconstructions provided in the paper.

The rest of the paper is organized as follows. In Section 2 we introduce the notations and define the operators used in this article. In Section 3 we state the main results about the kernels of $L$ and $T$, as well as the relations of these transforms correspondingly to the curl and divergence of the vector field enabling its full recovery. In Section 4 we provide the proofs of theorems stated in the previous section. Section 5 describes the method of recovering the full vector field from either one of the pairs: $L$ with its first moment $I$, and $T$ with its first moment $J$. Section 6 presents an exact closed form formula for recovering the full vector field from its star transform. We finish the paper with some additional remarks listed in Section 7 and acknowledgements in Section 8.

2 Definitions and notations

In this section we introduce the notations and define the operators used in the article. Throughout the paper, we use bold font letters to denote vectors in $\mathbb{R}^2$ (e.g. $x$, $u$, $v$, $f$, etc), and regular font letters to denote scalars (e.g. $t$, $h$, $f_i$, etc). We denote by $x \cdot y$ the usual dot product between vectors.

For a scalar function $V(x_1,x_2)$ and a vector field $f = (f_1,f_2)$, we use the notations

$$\nabla V := \left( \frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2} \right), \quad \text{div} f := \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}, \quad \text{and} \quad \text{curl} f := \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}. \quad (1)$$

Let $u$ and $v$ be two linearly independent unit vectors in $\mathbb{R}^2$. For $x \in \mathbb{R}^2$, the rays emanating from $x$ in directions $u$ and $v$ are denoted by $L_u(x)$ and $L_v(x)$ respectively, i.e.

$$L_u(x) = \{ x + tu : 0 \leq t < \infty \} \quad \text{and} \quad L_v(x) = \{ x + tv : 0 \leq t < \infty \}.$$

A V-line with vertex $x$ is the union of rays $L_u(x)$ and $L_v(x)$. For the rest of the article we will assume that $u$ and $v$ are fixed, i.e. all V-lines have the same ray directions and can be parametrized simply by the coordinates $x$ of the vertex (see Figure 1a).

**Definition 1.** The **divergent beam transform** $X_u$ of function $h$ at $x \in \mathbb{R}^2$ in the direction $u$ is defined as:

$$X_u h(x) = \int_0^\infty h(x + tu) \, dt. \quad (2)$$

The directional derivative of a function in the direction $u$ is denoted by $D_u$, i.e.

$$D_u h = u \cdot \nabla h. \tag{3}$$

One can similarly define the divergent beam transform $\mathcal{X}_u$ and directional derivative $D_v$.

The goal of this paper is to recover a vector field from the knowledge of its various integral transforms, namely: $\mathcal{L}$, $\mathcal{T}$, $\mathcal{I}$, $\mathcal{J}$ and the star transform $\mathcal{S}$. These transforms are defined in analogy with the corresponding ray transforms of vector fields in $\mathbb{R}^2$, substituting the straight line trajectory of integration of the latter with a V-line or star trajectory for the former. In applications, V-lines correspond to flight paths of particles that scatter at some point in the medium. One can imagine a particle starting from a point at infinity traveling along direction $-u$ to $x$, where scattering happens, after which the particle goes to infinity in the direction of $v$. This discussion motivates the following:

**Definition 2.** Let $f = (f_1, f_2)$ be a vector field in $\mathbb{R}^2$ with components $f_i \in C^2_c(\mathbb{R}^2)$ for $i = 1, 2$. The **longitudinal V-line transform** of $f$ is defined as

$$\mathcal{L}_{u,v} f = -\mathcal{X}_u \left( f \cdot u \right) + \mathcal{X}_v \left( f \cdot v \right). \tag{4}$$

To define the second integral transform of interest, we need to make a choice for the normal unit vector corresponding to each branch of the V-line. We define the vector $\perp$ operation by $(x_1, x_2) \perp = (-x_2, x_1)$.

**Definition 3.** Let $f = (f_1, f_2)$ be a vector field in $\mathbb{R}^2$ with components $f_i \in C^2_c(\mathbb{R}^2)$ for $i = 1, 2$. The **transverse V-line transform** of $f$ is defined as

$$\mathcal{T}_{u,v} f = -\mathcal{X}_u \left( f \cdot u^\perp \right) + \mathcal{X}_v \left( f \cdot v^\perp \right). \tag{5}$$

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The orientation of normal vectors is chosen towards the same side of the path of the scattering particle. Hence, in the definition above the inner product of the unknown vector field is taken with the outward unit normal of the V-line at each point (see Figure 1a).

**Definition 4.** The **first moment divergent beam transform** of a function \( h \) in the direction \( u \) is defined as follows

\[
\mathcal{X}^1_u h = \int_0^\infty h(x + tu) t \, dt.
\]

Similarly \( \mathcal{X}^1_v h = \int_0^\infty h(x + tv) t \, dt \).

**Definition 5.** Let \( f = (f_1, f_2) \) be a vector field in \( \mathbb{R}^2 \) with components \( f_i \in \mathcal{C}^2_c(\mathbb{R}^2) \) for \( i = 1, 2 \). The **first moment longitudinal V-line transform** of \( f \) is defined as

\[
I_{u,v} f (x) = -\mathcal{X}^1_f (f \cdot u) + \mathcal{X}^1_v (f \cdot v).
\]

**Definition 6.** Let \( f = (f_1, f_2) \) be a vector field in \( \mathbb{R}^2 \) with components \( f_i \in \mathcal{C}^2_c(\mathbb{R}^2) \) for \( i = 1, 2 \). The **first moment transverse V-line transform** of \( f \) is defined as

\[
J_{u,v} f (x) = -\mathcal{X}^1_f (f \cdot u^\perp) + \mathcal{X}^1_v (f \cdot v^\perp).
\]

**Remark 1.** It is easy to verify that \( T_{u,v} f = -L_{u,v} f^\perp \) and \( J_{u,v} f = -\mathcal{I}_{u,v} f \).

**Remark 2.** Throughout the paper we assume that the linearly independent unit vectors \( u \) and \( v \) are fixed. Hence, to simplify the notations we will drop the indices \( u, v \) and refer to \( T_{u,v}, L_{u,v}, I_{u,v} \) and \( J_{u,v} \) simply as \( T, L, I \) and \( J \).

Let us assume that \( \text{supp} f \subseteq D_1 \), where \( D_1 \) is an open disc of radius \( r_1 \) centered at the origin. Then \( Lf, Tf, If \) and \( Jf \) are supported inside an unbounded domain \( D_2 \cup S_u \cup S_v \), where \( D_2 \) is a disc of some finite radius \( r_2 > r_1 \) centered at the origin, while \( S_u \) and \( S_v \) are semi-infinite strips (outside of \( D_2 \)) in the direction of \( u \) and \( v \) correspondingly (see Figure 1b). It is easy to notice that all three transforms \( LF, TF, IF \) and \( Jf \) are constant along the directions of rays \( u \) and \( v \) inside the corresponding strips \( S_u \) and \( S_v \). In other words, the restrictions of \( LF, TF, IF \) and \( Jf \) to \( \overline{D}_2 \) completely define them in \( \mathbb{R}^2 \).

**Remark 3.** Throughout the paper we assume that \( LF(x), TF(x), IF(x) \) and \( Jf(x) \) are known for all \( x \in \overline{D}_2 \).
3 Full field recovery using longitudinal and transverse VLT

The following two relations can be obtained by a simple calculation:

\[
\Delta f_1 = \frac{\partial}{\partial x_1} \text{div} f - \frac{\partial}{\partial x_2} \text{curl} f, \tag{8}
\]

\[
\Delta f_2 = \frac{\partial}{\partial x_2} \text{div} f + \frac{\partial}{\partial x_1} \text{curl} f. \tag{9}
\]

Therefore the Laplacian of each component of a vector field \( f \) can be computed explicitly if one knows \( \text{div} f \) and \( \text{curl} f \). These Laplacians together with the boundary information of \( f \) will determine \( f \) uniquely. Hence, we have the following

**Remark 4.** To recover a compactly supported vector field \( f \) uniquely, one only needs to reconstruct \( \text{div} f \) and \( \text{curl} f \) from the integral transforms under consideration.

The following two theorems show that there is a non-trivial kernel for each of the integral transforms \( \mathcal{L} \) and \( \mathcal{T} \). Moreover, the theorems explicitly characterize those kernels.

**Theorem 1.** The kernel of longitudinal V-line transform \( \mathcal{L} \) is the set of all potential vector fields \( f \). In other words,

\( \mathcal{L} f \equiv 0 \) if and only if \( f = \nabla V \), for some scalar function \( V \).

One can easily check that all potential vector fields \( f = \nabla V \) are curl-free (i.e. \( \text{curl} f = 0 \)) and vice versa. Thus, from the above theorem we conclude that all curl-free vector fields are in the kernel of \( \mathcal{L} \).

**Theorem 2.** The kernel of transverse V-line transform \( \mathcal{T} \) is the set of all divergence-free vector fields \( f \). In other words,

\( \mathcal{T} f \equiv 0 \) if and only if \( \text{div} f = 0 \).

Before moving on, we would like to recount here a crucial and well known theorem, which states that any vector field (with some boundary condition) can be decomposed uniquely into a divergence-free part and a curl-free part. The following decomposition result is true in more general settings, e.g. in arbitrary dimensions, as well as for tensor fields. But for our needs, it is sufficient to consider the statement just for vector fields in \( \mathbb{R}^2 \).

**Theorem** (Theorem 3.3.2, [37]). Let \( \Omega \) be a bounded domain in \( \mathbb{R}^2 \) and \( f \) be a vector field, whose support is contained in \( \Omega \). Then there exist a uniquely determined vector field \( f^s \) and a uniquely determined scalar function \( V \) satisfying

\[
f = f^s + \nabla V \quad \text{with} \quad \text{div} f^s = 0 \quad \text{and} \quad V|_{\partial \Omega} = 0. \tag{10}
\]
The fields \( f^s \) and \( \nabla V \) are known as the solenoidal part (divergence-free part) and the potential part (curl-free part) of \( f \) respectively.

From Theorems 1 and 2 we see that the solenoidal part \( f^s \) and the potential part \( \nabla V \) of \( f \) are always in the kernel of \( T \) and \( L \) respectively. Hence it is impossible to reconstruct the full vector field just from the knowledge of only one transform (\( L \) or \( T \)). Also, observe from the above decomposition that

\[
\text{curl} \ f = \text{curl} f^s \quad \text{and} \quad \text{div} \ f = \Delta V.
\]

This implies that the problem of recovering \( \text{div} \ f \) and \( \text{curl} \ f \) is reduced to the determination of \( \Delta V \) and \( \text{curl} f^s \). Our next two theorems state that it is indeed possible to reconstruct \( \Delta V \) and \( \text{curl} f^s \) explicitly from the knowledge of \( T f \) and \( L f \) respectively.

**Theorem 3.** Let \( f \) be a vector field in \( \mathbb{R}^2 \) with components in \( C_c^2(\mathbb{R}^2) \). Then \( \text{curl} \ f \) can be recovered from \( L f \) as follows:

\[
\text{curl} \ f = \frac{1}{\det(v, u)} D_u D_v L f. \tag{11}
\]

In particular, this implies that operator \( L \) is invertible over compactly supported divergence-free vector fields.

**Theorem 4.** Let \( f \) be a vector field in \( \mathbb{R}^2 \) with components in \( C_c^2(\mathbb{R}^2) \). Then \( \text{div} \ f \) can be recovered from \( T f \) as follows:

\[
\text{div} \ f = -\frac{1}{\det(v, u)} D_u D_v T f. \tag{12}
\]

In particular, this implies that operator \( T \) is invertible over compactly supported curl-free vector fields.

**Remark 5.** The quantity appearing in the denominator of expressions for \( \text{curl} \ f \) and \( \text{div} \ f \) is not zero, since \( u \) and \( v \) are linearly independent. In other words,

\[
\det(v, u) = v_1 u_2 - u_1 v_2 = u \cdot v^\perp \neq 0.
\]

In some cases one may be interested in an unknown scalar potential \( V \) supported in \( D_1 \), while only having measurements \( T f \) of its gradient \( f = \nabla V \). Since \( \text{div} \ f = \Delta V \), as a consequence of Theorem 4 we can recover the scalar function \( V \) explicitly by solving the following Dirichlet problem for the Poisson equation:

\[
\begin{aligned}
\Delta V(x) &= \frac{1}{\det(v, u)} D_u D_v T f(x) & \text{in} \ D_1, \\
V(x) &= 0 & \text{on} \ \partial D_1.
\end{aligned}
\]
Similarly, one may be interested in a compactly supported scalar function $W$, when the measurements $\mathcal{L}f$ are available only for $f = (\nabla W)\perp = \left(-\frac{\partial W}{\partial x}, \frac{\partial W}{\partial y}\right)$. In such cases, one may use the relation $\text{curl} f = \Delta W$ to get $W$ by solving the following Dirichlet boundary value problem

$$\begin{align*}
\Delta W(x) &= \frac{1}{\det(v, u)} D_u D_v \mathcal{L} f (x) \quad \text{in } D_1, \\
W(x) &= 0 \quad \text{on } \partial D_1.
\end{align*}$$

4 Proofs of Theorems 1, 2, 3, 4

In this section we prove all four previously stated theorems. We provide two proofs for each one of them: the first proof uses an analytic argument, while the second one presents a geometric explanation.

4.1 Proof of Theorem 1

For a given vector field $f \in C^2_c(\mathbb{R}^2)$, we want to show the existence of a scalar function $V$ satisfying the following:

$$\mathcal{L} f = 0 \quad \text{if and only if } f = \nabla V.$$

Analytic argument.

Using the definition of $\mathcal{L}$ and applying directional derivatives along $u$ and $v$ we get

$$\begin{align*}
\mathcal{L} f &= -\mathcal{X}_u (f \cdot u) + \mathcal{X}_v (f \cdot v) = 0 \iff \\
D_u D_v (-\mathcal{X}_u (f \cdot u) + \mathcal{X}_v (f \cdot v)) &= 0 \iff \\
D_v (f \cdot u) - D_u (f \cdot v) &= 0.
\end{align*}$$

Therefore

$$\mathcal{L} f = 0 \quad \text{if and only if } D_v (f \cdot u) - D_u (f \cdot v) = 0.$$

Hence to complete the proof of this theorem it suffices to show that

$$D_v (f \cdot u) - D_u (f \cdot v) = 0 \quad \text{if and only if } f = \nabla V, \text{ for some scalar function } V.$$
Consider,
\[ D_v(f \cdot u) - D_u(f \cdot v) = \left( v_1 \frac{\partial}{\partial x_1} + v_2 \frac{\partial}{\partial x_2} \right) (u_1 f_1 + u_2 f_2) - \left( u_1 \frac{\partial}{\partial x_1} + u_2 \frac{\partial}{\partial x_2} \right) (v_1 f_1 + v_2 f_2) \]
\[ = v_1 u_1 \frac{\partial f_1}{\partial x_1} + v_1 u_2 \frac{\partial f_2}{\partial x_1} + v_2 u_1 \frac{\partial f_1}{\partial x_2} + v_2 u_2 \frac{\partial f_2}{\partial x_2} - v_1 u_1 \frac{\partial f_1}{\partial x_1} - v_1 u_2 \frac{\partial f_2}{\partial x_1} - v_2 u_1 \frac{\partial f_1}{\partial x_2} - v_2 u_2 \frac{\partial f_2}{\partial x_2} \]
\[ = \det(v, u) \left( \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2} \right) = \det(v, u) \text{curl} f. \] (13)

Since \( u \) and \( v \) are linearly independent, we conclude
\[ D_v(f \cdot u) - D_u(f \cdot v) = 0 \text{ if and only if } \text{curl} f = 0. \]

It is known that for simply connected domains \( \text{curl} f = 0 \) if and only if \( f = \nabla V \) for some scalar function \( V \). This completes the proof of Theorem 1. ■

Geometric explanation.
(\(\Leftarrow\Rightarrow\)) Assume \( f = \nabla V \) for some scalar function \( V \), thus \( \text{curl} f = 0 \). One can think of transformation \( L \) as the integral of the tangent component of the vector field \( f \) along branches of the V-lines, i.e.
\[ Lf = \int_{L_u \cup L_v} f \cdot \tau dt, \]
where \( \tau \) is the unit tangent vector of the V-line (as shown in Figure 2a).

Consider a triangular closed contour defined by some finite intervals of the V-line and an additional "bridge" \( L_{uv} \) outside of \( D_1 \supseteq \text{supp} f \) (see Figure 2a). Let \( G \) denote the region enclosed by \( L_u \cup L_v \cup L_{uv} \). Using Green’s theorem and the fact that \( \text{curl} f = 0 \), we get:
\[ Lf = Lf + \int_{L_{uv}} f \cdot \tau dt = \int_{L_u \cup L_v \cup L_{uv}} f \cdot \tau dt = \int_G \text{curl} f ds = 0. \]

(\(\Rightarrow\)) The other direction of the statement in Theorem 1 is a consequence of Theorem 3. ■

4.2 Proof of Theorem 2
Recall, we want to prove that a vector field \( f \) is in the kernel of \( T \) if and only if the vector field \( f \) is divergence-free.
(a) A V-line $L_u \cup L_v$ and an additional line segment $L_{uv}$ outside of $\text{supp} f$ with unit tangent vectors $\tau$.

(b) A V-line $L_u \cup L_v$ and an additional line segment $L_{uv}$ outside of $\text{supp} f$ with unit normal vectors $n$.

Figure 2

**Analytic argument.**

Due to the following special relation between curl and divergence in $\mathbb{R}^2$:

$$\text{curl} f^\perp = \text{curl} (-f_2, f_1) = \frac{\partial f_1}{\partial x_1} - \frac{\partial (-f_2)}{\partial x_2} = \text{div} f,$$  \hspace{1cm} (14)

and the fact that $\mathcal{T} f = -\mathcal{L} f^\perp$, Theorem 1 implies

$$\mathcal{L} f^\perp = 0 \iff \text{curl} f^\perp = 0.$$  

Hence

$$\mathcal{T} f = 0 \iff \text{div} f = 0,$$

which ends the proof. ■

**Geometric explanation.**

($\iff$) Assume $\text{div} f = 0$. One can think of $\mathcal{T} f$ as the integral of the normal component of the vector field $f$ along branches of the V-lines, i.e.

$$\mathcal{T} f = \int_{L_u \cup L_v} f \cdot n \, dt,$$

where $n$ is the unit normal vector of the V-line (as shown in Figure 2b).

Consider a triangular closed contour defined by some finite intervals of the V-line and an additional “bridge” $L_{uv}$ outside of $D_1 \supseteq \text{supp} f$ (see Figure 2b). Let $G$ denote the region enclosed by $L_u \cup L_v \cup L_{uv}$. We have

$$\mathcal{T} f = \mathcal{T} f + \int_{L_{uv}} f \cdot n \, dt = \int_{L_u \cup L_v \cup L_{uv}} f \cdot n \, dt = \int_G \text{div} f \, ds = 0.$$  

($\implies$) The other direction of the statement in Theorem 2 is a consequence of Theorem 4. ■
4.3 Proof of Theorem 3

Analytic argument.
Recall the decomposition of a compactly supported vector field $f$ presented in formula (10):

$$ f = f^s + \nabla V, \quad \text{with } \text{div} f^s = 0 \quad \text{and} \quad V = 0 \quad \text{on } \partial D_1. $$

By applying $L$ to this decomposition and using the fact that $L(\nabla V) = 0$ (from Theorem 1), we get

$$ Lf(x) = Lf^s(x). $$

Taking the directional derivatives $D_u D_v$ of the above equation and using formula (13) we get

$$ D_u D_v Lf = D_u D_v [-X_u (f^s \cdot u) + X_v (f^s \cdot v)] = D_v (f^s \cdot u) - D_u (f^s \cdot v) $$

$$ = \det(v, u) \left( \frac{\partial f_2^s}{\partial x_1} - \frac{\partial f_1^s}{\partial x_2} \right). $$

Hence,

$$ \frac{\partial f_2^s}{\partial x_1} - \frac{\partial f_1^s}{\partial x_2} = \frac{1}{\det(v, u)} D_u D_v Lf. \quad (15) $$

Finally, we observe

$$ \frac{\partial f_2^s}{\partial x_1} - \frac{\partial f_1^s}{\partial x_2} = \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2} = \text{curl} f. $$

Combining the last relation with equation (15) we get the required expression for $\text{curl} f$:

$$ \text{curl} f = \frac{1}{\det(v, u)} D_u D_v Lf. $$

This completes the proof of Theorem 3. ■

Geometric explanation.
Consider the scalar function $h(x) := Lf(x)$ and the following finite difference of its values at the vertices of a rhombus (refer to Figure 3 for visualization):

$$ Cf(x, y, z, w) := [h(x) - h(y)] - [h(z) - h(w)] = h(x) - h(y) - h(z) + h(w). \quad (16) $$

If one sends the side length $\delta > 0$ of the rhombus to zero, then

$$ \lim_{\delta \to 0} \left[ \frac{1}{\delta^2} Cf(x, y, z, w) \right] = D_u D_v h(x). \quad (17) $$
Figure 3: A linear combination of $\mathbf{L}f$ at the vertices of a rhombus resulting in a contour integral of $f \cdot \tau$ along the boundary of the rhombus.

On the other hand, from Figure 3 it is easy to see that $Cf(x, y, z, w)$ is the clockwise contour integral of $f \cdot \tau$ along the boundary of the rhombus. At the same time, by definition of curl (equivalent to the one in (1)) for any infinitesimal region $P$ containing $x$ we have

$$\text{curl} f(x) = \lim_{|P| \to 0} \frac{1}{|P|} \oint_{\partial P} f \cdot \tau \, dt,$$

where the integral is taken along the contour traversed counterclockwise. Since the area of our infinitesimal rhombus is $-\delta^2 \det(v, u)$, formulas (17) and (18) imply that

$$\text{curl} f = \frac{1}{\det(v, u)} D_u D_v h,$$

which is what we wanted to show.

4.4 Proof of Theorem 4

Analytic argument.
Using the formula of Theorem 3 and relation (14) between divergence and curl we have

$$\text{curl} f = \frac{1}{\det(v, u)} D_u D_v \mathcal{L}f,$$

which translates into

$$\text{div} f = -\frac{1}{\det(v, u)} D_u D_v \mathcal{T}f,$$

and concludes the analytic proof of Theorem 4.

Geometric explanation.
The argument is very similar to that of Theorem 3, except $h(x) := \mathcal{T}f(x)$ here. Adding and
Figure 4: A linear combination of $Tf$ at the vertices of a rhombus resulting in a contour integral of $f \cdot n$ along the boundary of the rhombus.

subtracting the values of $h$ as before, we obtain the outward flux of the vector field from the boundary of the infinitesimal rhombus (see Figure 4).

At the same time, by definition of divergence (equivalent to the one in (1)) for any infinitesimal region $P$ containing $x$ we have

$$\text{div} f(x) = \lim_{|P| \to 0} \frac{1}{|P|} \int_{\partial P} f \cdot n \, dt$$

Hence,

$$\text{div} f = \frac{-1}{\det(v, u)}DuDv h.$$  

Taking $f = \nabla V$ completes the proof using $\Delta V = \text{div} \nabla V$.  

5 **Longitudinal and transverse VLT’s with their first moments**

In this section we show that the full vector field $f$ can be recovered from the knowledge of its longitudinal V-line transform $Lf$ and its first moment V-line transform $I_f$, or alternatively from the knowledge of its transverse V-line transform $Tf$ and its first moment V-line transform $J_f$.

The proofs of the theorems presented in this section use the signed V-line transform of a compactly supported scalar function $h$ (see [5, Definition 4]):

$$T_s h := X_u h - X_v h.$$  

(21)

This transform has an explicit inversion formula (e.g. see [5, Theorem 8]):

$$h(x) = \frac{1}{||v - u||} DuDv \int_0^{\infty} (T_s h)(x + wt) \, dt,$$

(22)

where

$$w = \frac{v - u}{||v - u||}.$$  

(23)

We can now state and prove the main results of this section.
\textbf{Theorem 5.} Let $f$ be a vector field in $\mathbb{R}^2$ with components in $C^2_c(\mathbb{R}^2)$. Then $f$ can be recovered explicitly from $\mathcal{L}f$ and $\mathcal{I}f$.

\textbf{Proof.} We know from Theorem 3 that $\text{curl } f$ can be expressed in terms of $\mathcal{L}f$ as follows:

$$\text{curl } f = \frac{1}{\det(v,u)} D_u D_v \mathcal{L}f.$$ 

To prove this theorem we show that the signed V-line transform for each component of $f$ can be computed explicitly in terms of $\text{curl } f$ and $\mathcal{I}f$. Indeed,

$$\frac{\partial T \mathcal{I}f}{\partial x_1} = - \int_0^\infty t \left( u_1 \frac{\partial f_1}{\partial x_1} + u_2 \frac{\partial f_2}{\partial x_1} \right) (x + tu) dt + \int_0^\infty t \left( v_1 \frac{\partial f_1}{\partial x_1} + v_2 \frac{\partial f_2}{\partial x_1} \right) (x + tv) dt$$

$$= - \int_0^\infty t \left( u_1 \frac{\partial f_1}{\partial x_1} + u_2 \frac{\partial f_1}{\partial x_2} \right) (x + tu) dt - u_2 \int_0^\infty t \left( \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2} \right) (x + tu) dt$$

$$+ \int_0^\infty t \left( v_1 \frac{\partial f_1}{\partial x_1} + v_2 \frac{\partial f_1}{\partial x_2} \right) (x + tv) dt + v_2 \int_0^\infty t \left( \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2} \right) (x + tv) dt$$

$$= - \int_0^\infty t \frac{df_1}{dt} (x + tu) dt + \int_0^\infty \frac{dt}{dt} f_1(x + tu) dt - u_2 \mathcal{X}_u^1(\text{curl } f) + v_2 \mathcal{X}_v^1(\text{curl } f)$$

$$= \mathcal{X}_u f_1 - \mathcal{X}_v f_1 = \frac{\partial T \mathcal{I}f}{\partial x_1} + u_2 \mathcal{X}_u^1(\text{curl } f) - v_2 \mathcal{X}_v^1(\text{curl } f). \tag{24}$$

Differentiating $T \mathcal{I}f$ with respect to $x_2$ and proceeding with a similar calculation, we get

$$\mathcal{X}_u f_2 - \mathcal{X}_v f_2 = \frac{\partial T \mathcal{I}f}{\partial x_2} = - \mathcal{X}_u(\text{curl } f)_1 + \mathcal{X}_v(\text{curl } f)_1 + u_2 \mathcal{X}_u^1(\text{curl } f) + v_1 \mathcal{X}_v^1(\text{curl } f). \tag{25}$$

Equations (24) and (25) express $T_s f_1$ and $T_s f_2$ in terms of known $\text{curl } f$ and $\mathcal{I}f$. Therefore, we can recover $f_1$ and $f_2$ explicitly by direct application of formula (22).

**Theorem 6.** Let $f$ be a vector field in $\mathbb{R}^2$ with components in $C^2_c(\mathbb{R}^2)$. Then $f$ can be recovered explicitly from $\mathcal{T}f$ and $\mathcal{J}f$.

\textbf{Proof.} From Theorem 4, we know that $\text{div } f$ can be expressed in terms of $\mathcal{T}f$ as follows:

$$\text{div } f = - \frac{1}{\det(v,u)} D_u D_v \mathcal{T}f.$$ 

In this case, we show that the signed V-line transform of each component of $f$ can be computed explicitly in terms of $\text{div } f$ and $\mathcal{J}f$. Indeed, since $\mathcal{J}f = -\mathcal{I}f^\perp$ we can use (24) to get

$$\frac{\partial \mathcal{J}f}{\partial x_1} = - \frac{\partial \mathcal{I}f^\perp}{\partial x_1} = - \mathcal{X}_u(f^\perp)_1 + \mathcal{X}_v(f^\perp)_1 + u_2 \mathcal{X}_u^1(\text{curl } f^\perp) - v_2 \mathcal{X}_v^1(\text{curl } f^\perp)$$

$$= \mathcal{X}_u f_2 - \mathcal{X}_v f_2 + u_2 \mathcal{X}_u^1(\text{div } f) - v_2 \mathcal{X}_v^1(\text{div } f),$$
where in the last equality we used the relations

\[(f^\perp)_1 = -f_2 \text{ and } \text{curl} f^\perp = \text{div} f.\]

Therefore, we have

\[\mathcal{X}_u f_2 - \mathcal{X}_v f_2 = \frac{\partial \mathcal{J} f}{\partial x_1} - u_2 \mathcal{X}_u^1(\text{div} f) + v_2 \mathcal{X}_v^1(\text{div} f).\]

Differentiating \(\mathcal{J} f\) with respect to \(x_2\) and proceeding in a similar way, we get

\[\mathcal{X}_u f_1 - \mathcal{X}_v f_1 = -\frac{\partial \mathcal{J} f}{\partial x_2} - u_1 \mathcal{X}_u^1(\text{div} f) + v_1 \mathcal{X}_v^1(\text{div} f).\]

The last two relations express \(T^s f_1\) and \(T^s f_2\) in terms of known \(\text{div} f\) and \(\mathcal{J} f\). Hence, we can recover \(f_1\) and \(f_2\) explicitly by direct application of formula (22).

### 6 Recovery of the full vector field from its star transform

In this section we derive an inversion formula for the star transform of vector-valued functions. Our reconstruction is analogous to the inversion of the star transform of scalar functions introduced in [6].

**Definition 7.** Let \(\gamma_1, \ldots, \gamma_m\) be a set of unit vectors in \(\mathbb{R}^2\). The corresponding **star transform** \(\mathcal{S} f\) of a vector field \(f\) is defined by

\[
\mathcal{S} f = \sum_{i=1}^m c_i \mathcal{X}_\gamma_i \begin{bmatrix} f \cdot \gamma_i \\ f \cdot \gamma_i^\perp \end{bmatrix}.
\]  

(26)

where \(c_1, \ldots, c_m\) is a set of non-zero weights in \(\mathbb{R}\).

Note that, in contrast with our definition of the V-line transform, the star transform data contains both the longitudinal and transverse components (this simplifies our discussion). Now, let \(\mathcal{R} h(\psi, s)\) denote the ordinary Radon transform of a scalar function \(h\) in \(\mathbb{R}^2\), along the line normal to the unit vector \(\psi\) and at signed distance \(s\) from the origin. Lemmas 1 and 2 in [6] provide the following identity:

\[
\frac{d}{ds} \mathcal{R}(\mathcal{X}_\psi, h)(\psi, s) = \frac{-1}{\psi \cdot \gamma_i} \mathcal{R} h(\psi, s). 
\]  

(27)

**Theorem 7.** Consider the vector-valued star transform \(\mathcal{S} f\) with branch directions \(\gamma_1, \ldots, \gamma_m\) and let

\[
\gamma(\psi) := -\sum_{i=1}^m \frac{c_i \gamma_i}{\psi \cdot \gamma_i} \in \mathbb{R}^2 \quad \text{and} \quad Q(\psi) := \left[ \gamma(\psi) \right]^{-1} \in \text{GL}(2, \mathbb{R}).
\]  

(28)
If the unit vector \( \psi \) is in the domain of \( Q(\psi) \), then

\[
Q(\psi) \frac{d}{ds} R(Sf)(\psi, s) = Rf(\psi, s),
\]

(29)

where \( Rf \) is the component-wise Radon transform of a vector field in \( \mathbb{R}^2 \). Hence, if \( Q(\psi) \) is defined almost everywhere, we can apply \( R^{-1} \) to recover \( f \).

**Proof.** From (27) we get

\[
\frac{d}{ds} R(Sf) = \sum_{i=1}^{m} c_i \frac{d}{ds} R\gamma_i \left[ \frac{f \cdot \gamma_i}{f \cdot \gamma_i^\perp} \right] = -\sum_{i=1}^{m} \frac{c_i}{\psi \cdot \gamma_i} R \left[ \frac{f \cdot \gamma_i}{f \cdot \gamma_i^\perp} \right],
\]

(30)

Using the linearity of \( R \) and inner product, we simplify the last expression further to obtain

\[
\frac{d}{ds} R(Sf) = \left[ \frac{Rf \cdot \gamma(\psi)}{Rf \cdot \gamma(\psi)^\perp} \right] = Q(\psi)^{-1} Rf.
\]

Finally,

\[
Q(\psi) \frac{d}{ds} R(Sf)(\psi, s) = Rf(\psi, s),
\]

which ends the proof. \( \blacksquare \)

It is easy to notice that matrix function \( Q(\psi) \) is undefined if and only if

\[
\gamma(\psi) = -\sum_{i=1}^{m} \frac{c_i \gamma_i}{\psi \cdot \gamma_i} = 0.
\]

(31)

Let us explore this condition in more detail. Bringing the fractions in the above sum to a common denominator, one can notice that the numerator is a vector function \( P(\psi) \), whose components are homogeneous polynomials in terms of components of \( \psi = (\psi_1, \psi_2) \), namely

\[
P(\psi_1, \psi_2) = \sum_{i=1}^{m} \left[ c_i \gamma_i \prod_{j \neq i} (\psi_1 a_j + \psi_2 b_j) \right] = 0,
\]

(32)

where \( \gamma_j = (a_j, b_j), j = 1, \ldots, m \).

Since \( \psi_1^2 + \psi_2^2 = 1 \), due to Bézout’s Theorem each (homogeneous) polynomial component of \( P(\psi_1, \psi_2) \) has either finitely many zeros on \( S^1 \), or is identically zero. Since the common denominator of expression for \( \gamma(\psi) \) also becomes zero only at finitely many values of \( \psi \), we conclude that:
Corollary 1. The star transform is invertible if and only if \( \gamma(\psi) \neq 0 \) for at least one \( \psi \).

This corollary help us to give a complete description of invertible star configurations.

Definition 8. We call a star transform \( S \mathbf{f} \) symmetric, if \( m = 2k \) for some \( k \in \mathbb{N} \) and (after possible re-indexing) \( \gamma_i = -\gamma_{k+i} \) with \( c_i = -c_{k+i} \) for all \( i = 1, \ldots, k \).

As a side note, the sign convention in \( c_i = -c_{k+i} \) is different from that of [6], which is due to the orientation that we are using in the definition of the star transform for vector fields.

Theorem 8. The star transform \( S \mathbf{f} \) is invertible if and only if it is not symmetric.

Proof. The argument follows closely the steps of the proof of Theorem 2 in [6], which we present here for completeness. Assume that for a fixed choice of \( \gamma_1, \ldots, \gamma_m \) there is no inversion for the corresponding star transform. By Corollary 1 we have \( \gamma(\psi) \equiv 0 \). Hence, the \( P(\psi_1, \psi_2) \equiv 0 \).

If we take \( \psi_1 = b_1 \) and \( \psi_2 = -a_1 \), then formula (32) implies the following

\[
0 = \prod_{j \neq 1} (\psi_1a_j + \psi_2b_j) = (a_2b_1 - a_1b_2) \cdots (a_mb_1 - a_1b_m).
\]

Hence, for some index \( \sigma(1) \) we are required to have \( a_{\sigma(1)}b_1 = a_1b_{\sigma(1)} \) or equivalently \( \frac{a_1}{b_1} = \frac{a_{\sigma(1)}}{b_{\sigma(1)}} \). Given the assumption that \( \gamma_i \)'s are distinct unit vectors, we conclude that \( \gamma_1 = -\gamma_{\sigma(1)} \). Applying this procedure with \( \psi_1 = a_j \) and \( \psi_2 = -a_j \) repeatedly for all \( j \), we conclude that in order for the star transform to be non-invertible, its ray directions have to come in opposite pairs.

Now, we prove the relation \( c_i = -c_{k+i} \) between the corresponding weights of each pair. Without loss of generality, let us assume that \( m = 2k \), \( \gamma_i = -\gamma_{k+i} \) for \( i = 1, \ldots, k \) and no pair of vectors \( \gamma_1, \ldots, \gamma_k \) are collinear. Then we can re-write formula (32) as

\[
P(\psi_1, \psi_2) = (-1)^k \prod_{j=1}^{k} (\psi_1a_j + \psi_2b_j) \sum_{i=1}^{k} \left( c_i + c_{k+i} \right) \gamma_i \prod_{j=1 \atop j \neq i}^{k} (\psi_1a_j + \psi_2b_j) = 0, \ \forall \ \psi_1, \psi_2 \in \mathbb{R}.
\]

Since \( \prod_{j=1}^{k} (\psi_1a_j + \psi_2b_j) = (\psi \cdot \gamma_1) \cdots (\psi \cdot \gamma_k) \), we have

\[
\gamma(\psi) = \frac{-P(\psi_1, \psi_2)}{(\psi \cdot \gamma_1) \cdots (\psi \cdot \gamma_m)} = \frac{-1}{(\psi \cdot \gamma_1) \cdots (\psi \cdot \gamma_k)} \sum_{i=1}^{k} \left( c_i + c_{k+i} \right) \gamma_i \prod_{j=1 \atop j \neq i}^{k} (\psi_1a_j + \psi_2b_j) \tag{33}
\]

for all \( \psi \in S^1 \) outside of the finite set \( \{ \psi : \psi \cdot \gamma_i = 0, \ i = 1, \ldots, m \} \).
Hence, in order for \( S \) to be non-invertible, we must have
\[
\sum_{i=1}^{k} \left( c_i + c_{k+i} \right) \gamma_i \prod_{\substack{j=1 \atop j \neq i}}^{k} (\psi_1 a_j + \psi_2 b_j) \equiv 0.
\]

Following the argument from the first part of the proof, if we take \( \psi_1 = b_1 \) and \( \psi_2 = a_1 \), then
\[
0 = (c_1 + c_{k+1}) \prod_{j=2}^{k} (\psi_2 a_j + \psi_2 b_j) = (c_1 + c_{k+1}) (a_2 b_1 - a_1 b_2) \ldots (a_k b_1 - a_1 b_k).
\]

Since all vectors \( \gamma_1, \ldots, \gamma_k \) are pairwise linearly independent, \( a_j b_1 - a_1 b_j \neq 0 \) for \( j = 2, \ldots, k \). Hence, the last equation implies \( c_{k+1} = -c_1 \).

Applying this procedure with \( \psi_1 = a_j \) and \( \psi_2 = -a_j \) repeatedly for all \( j \), we conclude that in order for the star transform to be non-invertible, we must have \( c_{k+i} = -c_i \) for all \( i = 1, \ldots, k \).

\( \blacksquare \)

Theorem 8 immediately implies the following.

Corollary 2. Any vector-valued star transform with an odd number of rays is invertible.

Remark 6. Similar to the case of the star transform of scalar functions studied in [6], in invertible configurations the function \( Q(\psi) \) (or equivalently the function \( \gamma(\psi) \)) contains information about stability of inversion of the star transform on vector fields. A comprehensive analysis of that function is a non-trivial task (e.g. see [6] for a similar problem in the scalar case) and the authors plan to address it in another publication.

Remark 7. When \( m = 2 \) and \( c_1 = -c_2 = 1 \), the star transform of vector field \( f \) corresponds to the vector function \( (Lf, Tf) \). Hence, Theorem 7 provides another approach to recovering the full vector field \( f \) from its longitudinal and transverse V-line transforms. In the special case when \( \gamma_1 = -\gamma_2 \) (and only in that case), the matrix \( Q(\psi) \) is undefined for any \( \psi \) and the corresponding transform is not invertible.

7 Additional remarks

1. There are some interesting similarities between the V-line vector tomography and classical (straight line) vector tomography, despite the differences in the concepts and techniques of deriving the results.

- The kernel descriptions for the longitudinal and transverse transforms are identical in the V-line case (obtained in this paper) and the straight line case (see [11]).
In article [11], the authors showed that the combination of LRT and TRT provides a unique reconstruction of a vector field in $\mathbb{R}^2$. We achieve the same result combining the V-line versions of those transforms.

The authors in [24, 30] used the combination of LRT and the first integral moment transform data to get the full vector field in $\mathbb{R}^2$. Our Theorem 5 achieves the same result in the case of V-line transforms.

2. We chose to restrict the statements of the paper to vector fields with components in $C^2_c(\Omega)$ for simplicity of proofs. However, this much of regularity is not necessary and the results of the article can be extended to more general classes of vector fields. For instance, the geometric versions of our proofs of Theorems 1, 2, 3, 4 hold for $\text{div } f$ and $\text{curl } f$ defined for $f$ with components in $L^1(\Omega)$. One may also use density arguments to extend our results to vector fields with components in $H^1_0(\Omega)$ and even to vector fields with components in the space of compactly supported distributions.

3. In dimensions $n \geq 3$, the longitudinal V-line transform can be defined in the same fashion as for $n = 2$, but the transverse V-line transform will require more details, since there is an $(n - 1)$-dimensional space of transverse directions. Once a proper choice for the transverse direction is made, techniques similar to the ones introduced in this paper can be used to study injectivity and invertibility for both transforms in higher dimensions. The authors plan to address these questions in a future publication.

4. Many of the results and techniques of this paper can be generalized naturally to a large class of Riemannian surfaces with well defined V-line transforms. In this general setting, we fix two branch directions $u$ and $v$ on the surface using a connection. The transformations are defined by integrating over the geodesics in these directions starting from any given point. The parallel vector fields defined by $u$ and $v$ will then play the rule of the directional derivatives $D_u$ and $D_v$ appearing in the inversion formulas.

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