ONE-PARTICLE AND COLLECTIVE ELECTRON SPECTRA IN HOT AND DENSE QED AND THEIR GAUGE DEPENDENCE

O.K. Kalashnikov

Theoretical Physics Division,
P.N. Lebedev Physical Institute,
Russian Academy of Sciences,
117924 Moscow, Russia.

Abstract

The one-particle electron spectrum is found for hot and dense QED and its properties are investigated in comparison with the collective spectrum. It is shown that the one-particle spectrum (in any case its zero momentum limit) is gauge invariant, but the collective spectrum, being qualitatively different, is always gauge dependent. The exception is the case $m, \mu = 0$ for which the collective spectrum long wavelength limit demonstrates the gauge invariance as well.
1 Introduction

At present a problem exists to establish whether the effective electron mass in statistical QED (the same as in QFT) is gauge invariant or this mass depends on the calculational schemes, in particular on the gauge fixed or on the chosen regularization procedure to treat the infrared and ultraviolet divergencies. This problem is rather complicated and up today is not solved finally although many attempts were made starting from the fiftieth years [1,2]. In these pioneer papers the gauge dependence of the electron Green function was investigated at first in QFT and then in statistical QED [3]. It was shown that the longitudinal part of the gauge field generated only the phase factor for the electron Green function and its gauge dependence was separated but, unfortunately, only in the coordinate space. In the momentum space where the physical information is extracted from this function the gauge dependence of many physical quantities (in particular, the gauge dependence of the electron spectra) remains completely unclear and each time the special investigations are necessary to display it. Moreover all transformations in [1-3] were made on the formal level ignoring the problems which encounter the real calculations (e.g. the problem of treating divergencies, in the first rate infrared divergencies). The real calculations which were done later in [4,5] have demonstrated that within this problem many questions have remained unsolved and require the more careful investigations. For QFT in the recent paper [6] a number of these questions were eliminated but the direct extension of its results to statistical QED is doubtful and requires the special revision: firstly the electron spectrum in statistical QED is completely different from QFT (besides one-particle excitations there are the collective ones with different effective masses); and secondly in statistics unlike QFT all possible degrees of freedom are revived, therefore the special requirements are necessary to eliminate their influence on the physical results when any regularization is made. Moreover all infrared divergencies are aggravated in statistics and, for example, any calculations in the arbitrary $\alpha$-gauge require many efforts to block up the additional divergencies from the new pole in $D^0$-function (the $1/p^4$-pole). Only the Feynman and Coulomb gauges remain enough reliable since these gauges can be exploited without any additional infrared regularization and there is a guarantee that all exact theory properties will be kept: in the first rate the gauge invariance which is very sensitive to any calculational disagreements.

In this paper both gauges (the Feynman and Coulomb ones) are used to determine the electron spectra in statistical QED and investigate their gauge dependence. Sometimes the arbitrary $\alpha$-gauge will be briefly discussed but only to illustrate the results found. Two different kinds of the electron spectra in statistical QED are found: the one-particle and collective excitations. The one-particle electron spectrum which results from perturbative calculations is similar to the bare electron one and to the spectrum found in QFT. The collective electron excitations are generated by nonperturbative calculations and in the leading order have not any analogy in QFT. This spectrum has four well-separated branches which present the quasi-particle and quasi-hole excitations [7,8,9]; all branches are always massive (there is a well-separated spectrum of four effective masses) and their properties are completely different from the bare one. To perform calculations the standard temperature Green function technique is used and the case of zero damping is only considered. We concentrate our attention to solve the fermion dispersion relation in a more complete form and to investigate the gauge dependence of all spectra found.
2 QED Lagrangian and electron self-energy

The QED Lagrangian in covariant gauges has the form
\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^2 + \bar{\psi} [\gamma_\mu (\partial_\mu - ie V_\mu) + m] \psi \\
- \mu \bar{\psi} \gamma_4 \psi + \frac{1}{2\alpha} (\partial_\mu V_\mu)^2
\] (1)

where \( F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu \) is the Abelian field strength; \( \psi \) and \( \bar{\psi} \) are the Dirac fields; \( \mu \) and \( m \) are the electron chemical potential and the bare electron mass, respectively; \( \alpha \) is the gauge fixing parameter (\( \alpha = 1 \) for the Feynman gauge). The metrics is chosen to be Euclidean, and \( \gamma_4^2 = 1 \).

To find the Fermi excitations in hot QED we start with the usual Schwinger-Dyson equation
\[
G^{-1}(q) = G_0^{-1}(q) + \Sigma(q)
\] (2)
and calculate the electron self-energy according to its exact representation [3]
\[
\Sigma(q) = e^2 \beta \sum_{\nu_{1}}^{R} \int \frac{d^3p}{(2\pi)^3} D_{\mu\nu}(p) \gamma_\mu G(p+q) \Gamma_{\nu}(p+q, q|p).
\] (3)

We study \( \Sigma(q) \) only in the one-loop approximation where the bare Green functions and the bare vertex are used to calculate Eq.(3). Using this expression for \( \Sigma(q) \) the electron excitations will be found in two cases: perturbatively, exploiting the bare mass shell (one-particle spectrum) and nonperturbatively when the spectrum and a new mass shell are determined simultaneously (collective excitations). Both these spectra (especially their long wavelength limits) will be considered in different gauges (in the Feynman and Coulomb ones) to understand their influence on the final result.

Within the one-loop approximation the exact decomposition for \( \Sigma(q) \) is given by
\[
\Sigma(q) = i\gamma_\mu K_\mu(q) + m Z(q)
\] (4)
and is used to find nonperturbatively the function \( G(q) \)
\[
G(q) = \frac{-i\gamma_\mu (\hat{q}_\mu + K_\mu) + m (1 + Z)}{(\hat{q}_\mu + K_\mu)^2 + m^2 (1 + Z)^2}.
\] (5)

This representation leads to the one-loop dispersion relation for the Fermi excitations which in any gauge has the form
\[
[ (iq_4 - \mu) - K_4]^2 = q^2 (1 + K)^2 + m^2(1 + Z)^2
\] (6)
and after the standard analytical continuation it can be solved analytically or numerically. Here \( K_4 = i\hat{K}_4 \) and \( \hat{q} = \{(q_4 + i\mu), q\} \).

The one-loop expression for \( \Sigma(q) \) in the Feynman gauge was used many times earlier and is known as follows [9]
\[
\Sigma^F(q) = -e^2 \int \frac{d^3p}{(2\pi)^3} \left\{ \frac{1}{i\epsilon_p} \frac{n^B_p \left[ (\gamma_4 \epsilon_p + (i\gamma \cdot p + 2m) \right]}{q_4 + i(\mu + |\epsilon_p|) + (p - q)^2} + \frac{n^B_p}{|p|} \left[ (p + \mu - iq_4)\gamma_4 - [i\gamma \cdot (p + q) + 2m]}{q_4 + i(\mu + |p|) + \epsilon^2_{p+q}} \right] - [h.c.] \right\} (m, \mu) \rightarrow -(m, \mu)
\] (7)
where $\epsilon_p = \sqrt{p^2 + m^2}$ is the bare electron energy; $n^B_p = \{\exp[\beta|p|] - 1\}^{-1}$ and $n^\pm_p = \{\exp[\beta(\epsilon_p \pm \mu)] + 1\}^{-1}$ are the Bose and Fermi occupation numbers, respectively.

The analogous expression for $\Sigma(q)$ in the Coulomb gauge is more complicated and can be calculated within Eq.(3) using for this case the appropriate $D^0$-function

$$D^0_{ij}(p) = \left(\delta_{ij} - \frac{p_ip_j}{p^2}\right)\frac{1}{p_i^2 + p^2}.$$  \hspace{1cm} (8)

The standard $\gamma$-matrix algebra is used to make the summation over spinor indices

$$\Sigma^C = \frac{2e^2}{\beta} \sum_{p_4} \frac{d^3p}{(2\pi)^3} \frac{m + i\gamma_4(p + \hat{q})_4 + i\gamma_4[p + q|p|/p^2]}{[p + \hat{q}]^2 + m^2} [\epsilon_p - m - i\gamma_4(p + \hat{q})_4 + i\gamma_4(p + q)]$$

and then the remaining summation over the Bose frequencies within Eq.(9) (or over the Fermi ones in its another form) should be made in the usual manner [3]. The result has the form

$$\Sigma^C(q) = -e^2 \int \frac{d^3p}{(2\pi)^3} \left\{ \left[ \frac{n^B_p \gamma_4\epsilon_p + i\gamma_4(p - q)}{\epsilon_p} \frac{(p - q)|p|}{[q_4 + i(\mu + \epsilon_p)]^2 + (p - q)^2} \right] + \frac{n^B_p |p|}{|p|} \frac{\gamma_4\epsilon_p - i\gamma_4(p - m)}{2(p - q)^2} \right\}$$

and along with Eq.(7) will be used to calculate the electron spectrum and its limits.

## 3 One-particle electron spectrum

Now we investigate the one-particle electron excitations in the different gauges: at first in the Feynman and Coulomb gauges and then in the arbitrary $\alpha$-gauge. The spectrum found (in any case its long wavelength limit) is gauge invariant and qualitatively is the same as the bare one. It has two branches $i\omega_1 = \mu_R \pm m_R$ but their chemical potential and effective mass are quantitatively improved due to interaction with the medium. To find this spectrum in the leading order the dispersion relation (6) is solved with $\Sigma(q_4, q)$ taken at once on the bare mass shell $i\omega_1 = \mu \pm \sqrt{q^2 + m^2}$ that is evidently correct to establish $e^2$-corrections. However this procedure requires the additional analysis to establish the next-to-leading term (e.g. the $e^4$-term) where the perturbative calculations are more complicated. In this case the mass shell should be shifted as well to make all calculations selfconsistently.

The one-particle electron spectrum in the leading order is:

$$i\omega_1 = (\mu + K_4) \pm \sqrt{m^2 \left(1 + Z(q)\right)^2 + q^2 \left(1 + K(q)\right)^2}$$  \hspace{1cm} (11)

since all functions being put at once on the bare mass shell are independent from $i\omega_1$.

Our task is to calculate (11) explicitly using the expressions for $\Sigma(q)$ found above in the
Feynman (F.G) and Coulomb (C.G) gauges. These calculations are rather lengthy since they require at first to extract the functions \( Z(q) \) and \( K_\mu(q) \) from \( \Sigma(q) \) (e.g. as it is done in the Coulomb gauge)

\[
Z_C(q) = -\frac{\epsilon^2}{|p|} \int \frac{d^3p}{(2\pi)^3} \left\{ \frac{n_\mu^+}{|q_4 + i(\mu + |p|)|^2 + \epsilon^2_{p+q}} \right\} + \left[ h.c.; (\mu \to -\mu) \right]
\]

\[
K_C^C(q) = -\frac{\epsilon^2}{|p|} \int \frac{d^3p}{(2\pi)^3} \left\{ \frac{n_\mu^+}{|q_4 + i(\mu + |p|)|^2 + \epsilon^2_{p+q}} \right\} - \left[ h.c.; (\mu \to -\mu) \right]
\]

and then, after the bare mass shell has been put into these expressions, to expand them in the powers of \( p \). However, in the leading order which is only considered here, the final result is very simple and can be presented as follows

\[
\bar{K}_4(0) = -2I_B \pm \frac{I_A}{m}, \quad Z(0) = -4I_Z \pm \frac{4I_B}{m} \quad (F.G)
\]

\[
\bar{K}_4(0) = \pm \frac{I_A}{m}, \quad Z(0) = -4I_Z \pm \frac{2I_B}{m} \quad (C.G)
\]

(13)

to demonstrate that these functions are gauge dependent (the same as \( \Sigma(q) \) taken on the bare mass shell [4,5]).

Nevertheless the long wavelength limit for the one-particle electron excitations (due to the algebraic transformations) has another form

\[
iq_4 = \mu_R \pm m_R = \mu \left( 1 + 2I_B \right) \pm \left[ m(1 - 4I_Z) + \frac{I_A}{m} \right]
\]

(14)

and this limit is the same in both gauges. Here \( I_B = \mu \tilde{I}_B \) and other abbreviations are:

\[
I_A = e^2 \int_0^\infty \frac{d|p|}{4\pi^2} \left[ \epsilon_\mu n_\mu^+ + n_\mu^- \right], \quad I_B = -e^2 \int_0^\infty \frac{d|p|}{4\pi^2} \frac{n_\mu^+ - n_\mu^-}{2\epsilon_p}, \quad I_Z = e^2 \int_0^\infty \frac{d|p|}{4\pi^2} \frac{n_\mu^+ + n_\mu^-}{2\epsilon_p}.
\]

(15)

The found invariance is very specific and, undoubtedly, is destroyed when the spectrum for all momenta is considered. However, the effective electron mass (the one-particle spectrum limit at zero momentum found perturbatively) is gauge invariant, and namely this fact is demonstrated above.

The obtained spectrum limit can be repeated in the arbitrary \( \alpha \)-gauge to demonstrate the difficulties which arise due to the additional pole \( 1/p^4 \) and their influence on the calculations made. In this gauge the \( D^0 \)-function is more complicated and has the additional tensor structure

\[
D^0_{ij}(p) = \frac{\delta_{\mu\nu}}{p^2} + (\alpha - 1) \frac{p_\mu p_\nu}{p^4}
\]

(16)
which introduces at once the $\alpha$-dependence into all calculations. Now the one-loop electron self-energy (3) is transformed to be

$$\Sigma(q) = \Sigma(q)^F + (\alpha - 1) \frac{e^2}{\beta} \sum_{p_4}^B \int \frac{d^3p}{(2\pi)^3} \frac{G(p + q)}{p^4} (\gamma_\nu p_\nu)$$

(17)

where the first term reproduces the standard result for $\Sigma(q)$ in the Feynman gauge and the last one gives its gauge dependence. This term can be calculated in the more explicit form using (for example) the Ward identity for statistical QED [3]

$$i q_\mu \Gamma_\mu(p, p + q|q) = G^{-1}(p + q) - G^{-1}(p)$$

(18)

and the simple algebra. The final result has the known form

$$\Sigma(q)^\alpha = (\alpha - 1) \frac{e^2}{\beta} \sum_{p_4}^B \int \frac{d^3p}{(2\pi)^3} \frac{G(p + q)}{p^4} (\gamma_\nu p_\nu) = G^{-1}(q) \Sigma'(q)$$

(19)

which is very convenient for our discussion if there is a possibility to regularize the $\Sigma'(q)$-quantity. Without any additional regularization this quantity is singular on the bare mass shell and $\alpha$-dependence is kept (as it was found above here and in [4,5]). However, if any regularization is done to make $\Sigma'(q)$ finite, the electron self-energy taken on the bare mass shell is gauge invariant and reproduces the spectrum found in the Feynman gauge. Moreover this spectrum will be gauge invariant for any momenta and, namely, this fact in comparison with the results obtained above, is very doubtful. The most probably that such regularization of $\Sigma'(q)$ is possible only for $q = 0$ to be in agreement with the results of calculations in the Feynman and Coulomb gauges. However, the more complicated situation is not excluded: the independence for $\alpha$ does not mean else the real gauge invariance. To be sure one needs to check this fact in another outstanding gauge which is free on any additional regularization (e.g. the Coulomb gauge attracts the special attention). Probably, within Eq.(19) namely this situation is taken place, but formally here any regularization is acceptable since its influence on the one-particle electron spectrum is blocked up due to the zero factor which is $G^{-1}(p)$ on the bare mass shell.

4 Collective electron spectrum in Coulomb gauge

This gauge along with the Feynman gauge (where $\alpha = 1$) is very convenient for the practice calculations since no additional infrared regularization is necessary to treat all perturbative diagrams. This is not the case for any $\alpha$-gauge where a new pole $(1/p^4$-pole in the $D_0$-function) introduces many additional difficulties and makes all calculations very complicated. Indeed in the $\alpha$-gauge the one-loop electron self-energy is given by

$$\Sigma^\alpha(q) = \frac{e^2}{\beta} \sum_{p_4}^B \int \frac{d^3p}{(2\pi)^3} \frac{(\alpha + 3)m + (\alpha + 1)i\gamma_\mu(p + \hat{q})_\mu}{(p + \hat{q})^2 + m^2}$$

$$- (\alpha - 1) \frac{e^2}{\beta} \sum_{p_4}^B \int \frac{d^3p}{(2\pi)^3} \frac{2i\gamma_\mu p_\mu(p + \hat{q}|p)}{(p + \hat{q})^2 + m^2}$$

(20)
and one can see that now the collective spectra which will be found exploiting this expression is surely gauge dependent. Moreover, since $Z^{\alpha} = Z^{F}(\alpha + 3)/4$, this dependence is kept for the long wavelength spectrum limit as well. Moreover the calculation of Eq.(20) requires the special procedure to treat its last integral: for example, one can consider that

$$
\frac{1}{p^4} = \lim_{\kappa^2 \to 0} \left( - \frac{\partial}{\partial \kappa^2} \right) \frac{1}{p^2 + \kappa^2}
$$

(21)

or introduce another way of calculation. It is not excluded that the additional infrared regularization (e.g. the dimensional regularization) will be necessary. In any case the result will be rather complicated and non-single value.

The situation is different in the Coulomb gauge where all calculations are standard and selfconsistent. Their result was done above and will be used here to solve the dispersions relation (6). Only the long wavelength spectrum limit (the case $q = 0$ in Eq.(6)) will be considered below and therefore only two functions $Z$ and $K_4$ should be extracted from Eq.(10). These functions are given by Eq.(12) and can be algebraically transformed to be

$$
Z^C(q_4, 0) = -e^2 \int_0^\infty \frac{dp}{2\pi^2} \left\{ \frac{4p^2}{4\epsilon_p^2 (i\mu_4 - \mu)^2 - [(i\mu_4 - \mu)^2 + m^2]^2} \left[ \frac{(i\mu_4 - \mu)^2 + m^2}{\epsilon_p} n^+_p + n^-_p \right] + n^+_p n^-_p \right\}
$$

(22)

and then analogously we find $K_4$-function

$$
iK^C_4(q_4, 0) = -e^2 \int_0^\infty \frac{dp}{2\pi^2} \left\{ \frac{4p^2 \epsilon_p}{4\epsilon_p^2 - (i\mu_4 - \mu)^2 [1 + \frac{m^2}{(i\mu_4 - \mu)^2}]^2} \left[ \frac{n^+_p + n^-_p}{2} + \frac{(i\mu_4 - \mu)}{2\epsilon_p} \left( 1 + \frac{m^2}{(i\mu_4 - \mu)^2} \right) \frac{n^+_p - n^-_p}{2} \right] \right. - \left. \frac{i\mu_4 - \mu}{2} \right\}
$$

(23)

Now our problem is to solve Eq.(6) explicitly. To this end we should put Eq.(22)-(23) on the new mass shell $(i\mu_4 - \mu) = \omega$ and in this form substitute these integrals into Eq.(6). However in this case the arisen equation is very complicated and can be used only for numerical calculations. On the other hand, keeping the perturbative accuracy the obtained integrals could be simplified using a condition $m << T$ and omitting all $m^2$ terms. If only the leading term for the small $m$ is kept, these functions have the form

$$
-iK^C_4(q_4, 0) = \frac{I_A}{\omega} + I_B
$$

(24)

$$
Z^C(q_4, 0) = -3I_Z + 2\frac{I_B}{\omega}
$$

(25)
where all integrals were earlier determined by Eq.(15). The dispersion equation is the simple quadratic equation

$$\omega^2 - \omega [ \eta m (1 - 3I_Z) + I_B ] - (I_A + 2\eta m I_B) = 0 \quad (26)$$

whose solution reproduces the final result. This result is given by

$$E^C(0) = \mu + \frac{1}{2} [ \eta m_R + I_B ] \pm \sqrt{\frac{[\eta m_R + I_B]^2}{4} + (I_A + 2\eta m I_B)}. \quad (27)$$

where \( m_R = m(1 - 3I_Z) \) and at once Eq.(27) demonstrates that the spectrum found in the Coulomb gauge is qualitatively the same as the one in the Feynman gauge. In particular, the spectrum branches are split even at zero momentum and have four well-separated effective masses: two of them are related to the quasi-particle excitations and two others present the new quasi-hole ones. This is evident from Eq.(27) where \( \eta = \pm 1 \). However, this spectrum is not coincident quantitatively with the one found in the Feynman gauge [9]

$$E^F(0) = \mu + \frac{1}{2} [ \eta m_R - I_B ] \pm \sqrt{\frac{[\eta m_R - I_B]^2}{4} + (I_A + 4\eta m I_B)}. \quad (28)$$

where \( m_R = m(1 - 2I_Z) \) and the \( \mu \)-dependence are different. Only for the case \( m, \mu \) these results are similar and gauge independent.

## 5 Conclusion

To summarize we have established two kinds of the electron excitations in statistical QED: the one-particle spectrum and collective one which in the leading order is absent in QFT. The one-particle spectrum is qualitatively the same as the bare one and can be gauge invariant, in any case its effective mass is always the gauge invariant quantity. It has two branches \( iq_4 = \mu_R \pm m_R \) and their chemical potential and effective mass are quantitatively improved due to the interaction with the medium. This is not the case for the collective electron spectrum which has nonperturbative nature and is completely different from the bare one. This spectrum is split and their branches develop the different dynamical masses (the gap at zero momentum) which are always nonzero in the medium. These dynamical masses are not connected with bare masses and are generated always even for the case \( m, \mu = 0 \) being in the last case the gauge invariant quantity. But if the bare electron mass is not equal to zero all parameters which determine the collective spectrum are gauge dependent and their connection with the real electron excitations should be investigated separately. However one can see (comparing our results in the Feynman and Coulomb gauges) that these changes are only quantitative and qualitatively the found spectrum is the same in any gauges and namely that, probably, demonstrates the physical sense of the gauge invariance principle. The same situation is not new and takes place, for example, with the infrared fictitious pole in statistical QCD [10] which can be found in any gauges but quantitatively its position depends on the gauges fixed. Many of the results found here are valid as well for statistical QCD in the leading order but in the non-Abelian theory the situation is more complicated and surely requires the understanding of the gauge invariance principle in a broader sense.
References

1.) L. D. Landay and I. M. Khalatnikov, Sov. Phys. JETP 2 (1955) 69.

2.) E. S. Fradkin, Sov. Phys. JETP 2 (1955) 258.

3.) E. S. Fradkin, The doctoral thesis from FIAN (1960); published in Trudy Fiz. Inst. 29 (1965) 7 (Proc. P. N. Lebedev Physical Inst. 29 (1967) 1).

4.) R. Jackiw and S. Templeton, Phys. Rev. D23 (1981) 2291.

5.) S. Deser, R. Jackiw and S. Templeton, Ann. Phys. (N.Y) 140 (1982) 372.

6.) I. V. Tyutin and Vad. Yu. Zeitlin, hep-th/9711137.

7.) V. V. Klimov, Yad. Fiz. 33 (1981) 1734 (Sov. J. Nucl. Phys. 33 (1981) 934); Zh. Eksp. Teor. Fiz. 82 (1982) 336 (Sov. Phys. JETP 55 (1982) 199).

8.) R. D. Pisarski, Nucl. Phys. A498 (1989) 423c.

9.) O. K. Kalashnikov, Mod. Phys. Lett. A12 (1997) 347 and also Pis’ma Zh. Eksp. Teor. Fiz. 67 (1998) 3 (JETP Lett. 67 (1998) 1).

10.) O. K. Kalashnikov, Phys. Lett. B 279 (1992) 367.