Anomalous Jacobian Factor in the Polyakov Measure for Abelian Gauge Field in Curved Spacetimes

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ABSTRACT

The Polyakov measure for the Abelian gauge field is considered in the Robertson-Walker spacetimes. The measure is concretely represented by adopting two kind of decompositions of the gauge field degrees of freedom which are most familiarly used in the covariant and canonical path integrals respectively. It is shown that the two representations are different by an anomalous Jacobian factor from each other and also that the factor has a direct relationship to an uncancellation factor of the contributions from the Faddeev-Popov ghost and the unphysical part of the gauge field to the covariant one-loop partition function.
1. Introduction

Euclidean path integrals are most popularly used as attractive and powerful tools in the investigation of field theories in curved spacetimes and quantum gravity. They enable us to have suggestive discussions with respect to not only gauge symmetries of systems if the classical actions have them, but also manifest coordinate invariance which matches the spirit of general relativity.

Recently it has been pointed out in gauge theories in curved spacetimes, including linearized gravity, that there is a difference between covariant path integrals and hamiltonian path integrals, i.e., that there is a probability that covariant path integrals give rise to uncancellation of one-loop contributions from the Faddeev-Popov ghosts and the unphysical degrees of freedom of gauge fields. In the gravitational case, the zeta-function calculation of the one-loop partition function about the $S^4$ saddle point gives the difference by an integer in the scaling behavior or $\zeta(0)$ [1]. In the case of the Abelian gauge field on this manifold the difference is more complicated [2], and it is also shown that this difference is not a mere problem of gauge choice, but it results from a difference of measure between the covariant and canonical path integrals [3].

The purpose of this paper is to more clearly show the difference of the covariant and canonical path-integral measures for vector fields in the Robertson-Walker spacetimes with $K = +1$. In order to do it, we study the Polyakov measure and represent it by using two kind of decompositions of the gauge field degrees of freedom, i.e., the so-called covariant and canonical decompositions, which may be regarded as suitable ones for the Lorentz gauge and the Coulomb gauge, respectively (see Eqs. (2.3), (2.7) and (3.4)). Then it may be naively suggested by a formal discussion, i.e., using the truth that the Polyakov measure is formally defined by the Gaussian integral, that the Jacobian factor under the change of variables between the two representations would take a trivial value 1. But in this paper, it is pointed out that the Jacobian factor may take unfortunately an anomalous value, which has a direct relationship to the uncancellation factor of all ghost contributions in
the covariant one-loop partition function.

On the other hand, it might be remembered that connections among path integrals in various gauge choices could be given by the Faddeev-Popov procedure [4-7]. However, the ordinary discussions are almost too naive. It is suggested in this paper that the rigorous discussions in curved spacetimes must refer to the concrete definition of the path-integral measure adopted in each of gauge choices, because the measures may be different from one another. Therefore it says that all path integrals may not be same.

This paper is organized as follows: in section 2 the one-loop partition function in the Robertson-Walker spacetimes is discussed not always to give the contribution of the only physical modes, differently from the case of flat spacetime in which it is well known that the cancellation of the Faddeev-Popov ghost and the redundant variables of the gauge field goes well. In section 3 we concretely construct the Polyakov measure, using the above mentioned two decompositions, and then find a nontrivial Jacobian factor under the transformation between two measures. Section 4 is devoted to conclusion and discussions, particularly we discuss with respect to the Faddeev-Popov procedure.

2. One-Loop Partition Function in Robertson-Walker Spacetimes

In order to discuss that the covariant path integral in curved spacetimes has a probability that the one-loop contribution of the Faddeev-Popov ghost does not cancel out with one from the unphysical gauge field, we study, in this section, the case of the Robertson-Walker spacetimes with actually setting up a special coordinate system.

First, let us start with the BRST Euclidean path integral in any $D$-dimensional curved spacetime;

$$Z \overset{\text{def}}{=} \int \mathcal{D}A \Delta_{FP} \exp \left[ -\frac{1}{\hbar}(I + I^{GF}) \right]$$ (2.1)
with

\[
I \equiv \int d^D x \frac{1}{4} \sqrt{g} g^{IJ} g^{JM} F_{IJ} F_{MN},
\]

\[
I_{GF} \equiv \int d^D x \frac{1}{2\alpha} \sqrt{g} (\nabla^A A_I)^2,
\]

\[
\Delta_{FP} \equiv \left| \frac{1}{\sqrt{\alpha}} \det \left\{ \Box(s) \right\} \right|,
\]

where \(s\) means that the covariant d’Alembertian \(\Box\) acts on scalar fields and the functional measure is defined by the same method as that used in Ref. 8 so as to have coordinate invariance. Now the gauge field \(A_I\) can be decomposed into a divergenceless vector \(A_I^d\) and a covariant derivative of a scalar field, \(A_I^s\):

\[
A_I = A_I^d + A_I^s, \quad \nabla^A A_I^d = 0, \quad A_I^s \equiv \nabla^A - \Box^{-1} S,
\]

and then these two parts are completely decoupled in \(I\) and \(I_{GF}\), because the gauge invariant action \(I\) is independent of \(A_I^s\) since \(S\) can be considered a parameter of the gauge transformation and \(I_{GF}\) is obviously independent of \(A_I^d\). Thus the one-loop determinant of \(A_I\) in (2.1) is made of completely separated contributions of these two parts, i.e., \(\mathcal{Z}^{(d)} \times \mathcal{Z}^{(s)}\) in which \(\mathcal{Z}^{(d)}\) and \(\mathcal{Z}^{(s)}\) are one-loop corrections from \(A_I^d\) and \(A_I^s\) respectively;

\[
\mathcal{Z}^{(d)} = \left| \det \left[ -\Box + R^{IJ} \right] (v^d) \right|^{-1/2}, \quad \mathcal{Z}^{(s)} = \left| \det -\alpha^{-1} \Box(s) \right|^{-1/2},
\]

where \(v^d\) denotes divergenceless vector fields. Therefore, putting them in (2.1), \(\mathcal{Z}\) is written in a simple form usable in any curved spacetime:

\[
\mathcal{Z} = \left| \det -\Box(s) \right|^{1/2} \times \mathcal{Z}^{(d)}.
\]

It is important to note here that the contribution of the gauge parameter, \(\mathcal{Z}^{(s)}\), always cancels out with one from a half degree of freedom of the Faddeev-Popov ghost field (of course, we must take the same boundary condition on the fermionic
Faddeev-Popov ghost as that on the bosonic gauge field, similarly to the case of finite temperature gauge theories [9,10], however, it is not obvious whether the contribution of the longitudinal part, which is a gauge invariant but unphysical part of the divergenceless vector, cancels out with the remaining contribution of the Faddeev-Popov ghost field and also whether the one-loop partition function (2.1) gives the contribution from the only physical modes, i.e., the transverse part of the divergenceless vector.

Next, in order to study this issue in detail, let us study the case of the Robertson-Walker spacetimes with \( K = +1 \), choosing coordinates so that the metric has the form

\[
\text{d}s^2 = \text{d}\tau^2 + a^2(\tau)\text{d}\Omega^2_{D-1}, \quad \text{d}\Omega^2_{D-1} \overset{\text{def}}{=} \tilde{g}_{ij}\text{d}x^i\text{d}x^j,
\]

(2.6)

where \( \tilde{g}_{ij} \) is the metric of the unit \((D - 1)\)-sphere \( S^{D-1} \) according to \( K = +1 \) [11]. In such coordinates, regarding \( \tau \) as a time coordinate, \( A^d_I \) can be split into transverse and longitudinal components;

\[
A^d_I = A^T_I + A^L_I, \quad A^T_D = \tilde{\nabla}^i A^T_i = 0, \quad A^L_I \overset{\text{def}}{=} -\tilde{\nabla}_i \tilde{\Box}^{-1} a^{-(D-3)} \partial_D (a^{D-1} A^L_D),
\]

(2.7)

where \( \tilde{\nabla}_i \) and \( \tilde{\Box} \) are the covariant derivative and the covariant Laplacian on the unit \( S^{D-1} \). We then find the separation of two parts in \( I^d \), which means that \( \mathcal{Z}^{(d)} \) becomes \( \mathcal{Z}^{(T)} \times \mathcal{Z}^{(L)} \) with

\[
\mathcal{Z}^{(T)} = \left| \text{det} \left[ -\tilde{\Box}^{IJ} + R^{IJ} \right] (v^T) \right|^{-1/2}, \\
\mathcal{Z}^{(L)} = \left| \text{det} \left[ -\tilde{\Box} + \frac{D-2}{D-1} R_{DD} \right] (\tilde{s}) \right|^{-1/2},
\]

(2.8)

where \( v^T \) and \( \tilde{s} \), respectively, are transverse vector fields and scalar fields having no zero-eigenvalue mode of \( \tilde{\Box} \), and \( \mathcal{Z}^{(L)} \) is easily obtained by the aid of the following
formulae satisfying under certain boundary conditions;

\[
\int d^Dx \sqrt{g} g^{IJ} A_I^L A_J^L = \int d^Dx \sqrt{g} (aA_D^L) F^{-2} (aA_D^L),
\]

(2.9)

\[
\int d^Dx \sqrt{g} A_I^L \left[ -\Box g^{IJ} + R^{IJ} \right] A_J^L = \int d^Dx \sqrt{g} (aA_D^L) (-\Box) F^{-2} F^{-2} (aA_D^L),
\]

(2.10)

where \( F \) is defined by

\[
F \overset{\text{def}}{=} \left( -\Box \left[ -\Box (s) + \frac{D-2}{D-1} R_{DD} \right]^{-1} \right)^{1/2}
\]

(2.11)

with \( R_{DD} = -(D-1)a^{-1}\dddot{a} \) and the d’Alembertian operator acting on scalar fields, \( \Box (s) \). It must be noted here that \( F \) is not commutable with \( a \) and \( \partial_D \). From Eqs. (2.5) and (2.8), therefore, we obtain

\[
Z = \left| \det -\Box (s) \right|^{1/2} \left| \det -\Box \left[ -\Box (s) + \frac{D-2}{D-1} R_{DD} \right]^{-1} \right|^{1/2} Z^{(T)}.
\]

(2.11)

As the conclusion in this section, the extra factor in Eq. (2.11), i.e.,

\[
\left| \det -\Box (s) \right|^{1/2} \left| \det -\Box \left[ -\Box (s) + \frac{D-2}{D-1} R_{DD} \right]^{-1} \right|^{1/2},
\]

(2.12)

means that \( Z \) in Robertson-Walker spacetimes is not the one-loop contribution which the only physical modes give if it does not become a trivial value 1 under a used regularization, so the factor (2.12) becomes the uncancellation factor which expresses the contributions from the Faddeev-Popov ghost and the unphysical gauge field. Indeed, we know that in de Sitter spacetime it happens, i.e., the zeta-function calculation does not make it 1 [2].
3. Anomalous Jacobian Factor in the Polyakov Measure

In this section we concretely construct the Polyakov measure for vector fields in the Robertson-Walker spacetimes, which is formally defined by

\[ \int \mathcal{D}A_I \exp \left[ -\frac{1}{2\hbar} < A, A > \right] \equiv 1 \]  

(3.1)

with

\[ < A, A' > \equiv \int d^Dx \sqrt{\bar{g}} g^{IJ} A_I A'_J. \]  

(3.2)

The decompositions used in the discussion here are the following two: the first decomposition, which is covariant one, is Eqs. (2.3) and (2.7) with

\[ A_D^L \overset{\text{def}}{=} a^{-1} F Y, \]  

(3.3)

where \( Y \) is introduced as a scalar field in the Robertson-Walker spacetimes, and the second decomposition, being so-called canonical one, is represented by

\[ A_I = A_I^{Ap} + A_I^\rho + A_T^I, \]

\[ A_D^{Ap} \overset{\text{def}}{=} A_D, \quad A_I^{Ap} \overset{\text{def}}{=} 0, \]  

(3.4)

\[ A_D^\rho \overset{\text{def}}{=} 0, \quad A_i^\rho \overset{\text{def}}{=} \tilde{\nabla}^{-1} \rho. \]

The Gaussian integral is \( \int_{-\infty}^{+\infty} dx e^{-\lambda x^2} = \sqrt{\pi/\lambda} \), thus using the covariant decomposition leads to the covariant measure;

\[ \int \frac{\mathcal{D}S}{\sqrt{2\pi\hbar}} \frac{\mathcal{D}Y}{\sqrt{2\pi\hbar}} \frac{\mathcal{D}A_T^I}{\sqrt{2\pi\hbar}} \exp \left[ -\frac{1}{2\hbar} < A, A > \right] = 1, \]  

(3.5)

where \( \mathcal{D}S, \mathcal{D}Y \) and \( \mathcal{D}A_T^I \) may be defined with expansion coefficients of them in an orthonormal and complete set in the \( D \)-dimensional spacetime, and if we use the
canonical decomposition the Polyakov measure is defined by the canonical one [3]:

$$
\int \prod_{\tau} \left[ \left( \frac{\Delta a_{D-3}(\tau)}{2\pi \bar{\hbar}} \right)^{1/2} \mathcal{D}A_{D}(\tau) \right] \prod_{\tau} \left[ \frac{\Delta a_{D-3}(\tau)}{2\pi \bar{\hbar}(-\Box_{\bar{s}})} \right]^{1/2} \mathcal{D}\rho(\tau) \\
\times \prod_{\tau} \left[ \left( \frac{\Delta a_{D-3}(\tau)}{2\pi \bar{\hbar}} \right)^{1/2} \mathcal{D}A_{i}^{T}(\tau) \right] \exp \left[ -\frac{1}{2\bar{\hbar}} < A, A > \right] = 1,
$$

(3.6)
in which the time $\tau$ is specialized from the other coordinates, and its product might be defined in the discrete time formulation with a finite distance and its zero limitation after integrations, and then the functional measures on each time, $\mathcal{D}A_{D}(\tau)$, $\mathcal{D}\rho(\tau)$ and $\mathcal{D}A_{i}^{T}(\tau)$, are defined with their expansion coefficients in the basis of eigenfunctions of $\bar{\Box}$ on the unit $S^{D-1}$. Furthermore, it is noted here that $\rho = \bar{\nabla}^{i}A_{i}$ has no zero mode owing to

$$
\int d^{D-1}x \sqrt{g} \bar{\nabla}^{i}A_{i} \bar{S}^{0\mu} = 0,
$$

(3.7)
where $\bar{S}^{0\mu}$ is an eigenfunctions with zero eigenvalue of $\bar{\Box}$.

As mentioned in section 1, Eqs. (3.5) and (3.6) naively mean that the Jacobian factor under the change of variables between their two measures would become trivially 1, since both measures are defined through the Gaussian integral whose integration value is 1. However, as discussed below, the factor is not 1 and take an anomalous value.

First, we study about the physical variable: its relation between the two decompositions tells us that

$$
\prod_{\tau} \left[ \left( \frac{\Delta a_{D-3}(\tau)}{2\pi \bar{\hbar}} \right)^{1/2} \mathcal{D}A_{i}^{T}(\tau) \right] = \frac{\mathcal{D}A_{i}^{T}}{\sqrt{2\pi \bar{\hbar}}},
$$

(3.8)
where we note that the time $\tau$ product in the l.h.s. is changed, in the r.h.s., into the product of modes along the time axis, because we may more expand $A_{i}^{T}(\tau)$, which is already expanded with eigenfunctions on $S^{D-1}$, by using eigenfunctions in
the Robertson-Walker spacetimes. Furthermore, when expressing eigenfunctions in the Robertson-Walker Spacetimes in terms of eigenfunctions on $S^{D-1}$ and certain functions of the time $\tau$, the factors, $\prod_\tau \left( \Delta a^{D-3}(\tau) \right)^{1/2}$, cancel out with determinants of the time functions (see Ref. 3 in the $S^D$ case). Eq. (3.8), therefore, means that, with respect to the physical part, no anomalous thing happens under the transformation between the covariant and canonical measures.

Next, as for the unphysical variables, their relations between the two decompositions become

\begin{align}
A_D &= \partial_D (-\square)^{-1/2} S + a^{-1} FY, \\
\rho &= \tilde{\rho} (-\square)^{-1/2} S - a^{-(D-3)} \partial_D (a^{D-2} FY), \\
S &= -(-\square)^{-1/2} a^{-(D-1)} \partial_D (a^{D-1} A_D) - (-\square)^{-1/2} a^{-2} \rho, \\
Y &= F a^{-1} A_D - F a^{-1} \tilde{\rho}^{-1} \partial_D \rho,
\end{align}

where the relation between the former two equations and the later two equations is that of the inverse transformation. Now, let us separate the unphysical variables into the zero-mode and nonzero-mode parts of $\tilde{\rho}$, and study the Jacobian factor in each part.

With respect to zero mode of $\tilde{\rho}$, using Eq. (3.9) gives

\begin{equation}
\prod_\tau \left[ \left( \frac{\Delta a^{D-1}(\tau)}{2\pi \hbar} \right)^{1/2} \mathcal{D} A_D^{(0)}(\tau) \right] = | \det[\partial_D(\tilde{0})]| \det -\square^{(0)} |^{-1/2} \mathcal{D} S^{(0)} \sqrt{2\pi \hbar}, \tag{3.13}
\end{equation}

where the symbol $(\tilde{0})$ means that the values are of the zero-mode part, and we used the truth that $\rho$ has no zero mode and the assumption that $Y$, too, does not have the mode to be consistent with the $A_L^1$ definition. From Eq. (3.13), thus, if the factor,

\begin{equation}
| \det[\partial_D(\tilde{0})]| \det -\square^{(0)} |^{-1/2}, \tag{3.14}
\end{equation}

is 1, which is of course satisfied in flat spacetime, the two measures, i.e., the covariant and canonical measures, are same in this zero-mode part of $\tilde{\rho}$. 
As for the nonzero-mode part of $\tilde{\mathbf{\square}}$, we may find a nontrivial Jacobian factor having a direct relationship to the uncancellation factor (2.12). In order to derive it, let us carry out two kind of calculations: first, using (3.9) and (3.11) at the following steps: $(A_D, \rho) \rightarrow (A_D, S) \rightarrow (Y, S)$, then we have

$$
\prod_\tau \left[ \left( \frac{\Delta \tau a^{D-1}(\tau)}{2\pi \hbar} \right)^{1/2} \mathcal{D}A_D(\tau) \right] \left[ \left( \frac{\Delta \tau a^{D-3}(\tau)}{2\pi \hbar} \right)^{1/2} \mathcal{D}\rho(\tau) \right] \nonumber
= \left| \det -\mathbf{\square}(\tilde{s}) \right|^{1/2} \left| \det \left[ -\mathbf{\square} + \frac{D-2}{D-1} R_{DD} \right](\tilde{s}) \right|^{-1/2} \sqrt{\frac{2\pi \hbar}{2\pi \hbar}} \sqrt{2\pi \hbar}, \tag{3.15}
$$

while if we use Eqs. (3.10) and (3.12) at the steps such that $(A_D, \rho) \rightarrow (Y, \rho) \rightarrow (Y, S)$, we obtain

$$
\prod_\tau \left[ \left( \frac{\Delta \tau a^{D-1}(\tau)}{2\pi \hbar} \right)^{1/2} \mathcal{D}A_D(\tau) \right] \left[ \left( \frac{\Delta \tau a^{D-3}(\tau)}{2\pi \hbar} \right)^{1/2} \mathcal{D}\rho(\tau) \right] \nonumber
= \left| \det -\mathbf{\square}(\tilde{s}) \right|^{-1/2} \left| \det \left[ -\mathbf{\square} + \frac{D-2}{D-1} R_{DD} \right](\tilde{s}) \right|^{1/2} \sqrt{\frac{2\pi \hbar}{2\pi \hbar}} \sqrt{2\pi \hbar}, \tag{3.16}
$$

From Eqs. (3.15) and (3.16), therefore, the factor,

$$
\left| \det -\mathbf{\square}(\tilde{s}) \right|^{1/2} \left| \det \left[ -\mathbf{\square} + \frac{D-2}{D-1} R_{DD} \right](\tilde{s}) \right|^{-1/2}, \tag{3.17}
$$

might be thought to become naively 1 and both measures would be concluded to equal each other also as to the non-zero part of $\tilde{\mathbf{\square}}$. But the factor is obviously the same with the uncancellation factor (2.12) in the covariant one-loop partition function excluding the zero-mode part, and in order to actually calculate the factor (3.17) we need a regularization because the arguments of the determinants are infinite matrices. Thus, some regularizations might fail to make its value 1. Indeed, in the case of de Sitter spacetime, the zeta-regularization calculation does not make it 1, as mentioned in section 2. Hence the factor (3.17) may be called an anomalous Jacobian factor under the change of variables between the covariant and canonical measure.
4. Conclusion and Discussions

In this paper it was shown that in the Robertson-Walker spacetimes the path integral formula defined with the covariant measure does not provide the one-loop partition function which is made of contributions of the only physical modes and there the cancellation of the one-loop contributions between the Faddeev-Popov ghost and the unphysical modes of the gauge field does not hold, similarly to the case of de Sitter spacetime. We also studied about the Polyakov measure by representing it in terms of two kind of decompositions, i.e., the covariant and canonical ones, and showed that an anomalous Jacobian factor, which might not be 1 in some cases of regularizations, exists under the transformation between the two measures and it is the same with the above uncancellation factor in the one-loop partition function.

Let us discuss the regularization dependence in the anomalous Jacobian factor. If we take the discrete time formulation throughout, the Jacobian factor might become 1. Since the number of variables along the time axis is finite in the formulation, although degrees of freedom with respect to the spatial part are infinite, so the change of variables between the two Gaussian integrals, (3.5) and (3.6), is well-defined. Hence the naive discussion can be adopted without any problem. Of course, the actual calculation of the Jacobian factor is very difficult in curved spacetimes, though the arguments of the determinants become finite matrices. This discussion may suggest that there is a regularization dependence in the Jacobian factor, in particular, we may note that the one-loop calculation with the discrete time formulation might have a difference from the zeta-function calculation.

It is known that, also in the linearized gravitational case about de Sitter spacetime, the complete cancellation of the one-loop contributions from the all redundant degrees of freedom does not go well. Furthermore, the uncancellation factor may be related with an anomalous Jacobian factor like the case of the Abelian gauge field [12].

Finally we turn our discussion to the Faddeev-Popov procedure. It might
be believed that their procedure can make all path-integral formulae defined in various gauge choices be connected. However, as being obvious from section 3, how to define the path-integral measure used in each gauge fixing is important to have a rigorous discussion with respect to the dependence of gauge choice in path integrals in curved spacetimes. Because in order to make delta-functions used in the Faddeev-Popov procedure be well-defined, we have to adopt the most suitable decomposition as their arguments. For example, in the case of the Coulomb gauge the canonical decomposition is best, on the other hand, the covariant decomposition is most suitable for the Lorentz gauge case. Therefore it may be concluded that we must pay attention to that what kind of the decomposition of the gauge field degrees of freedom is used to define the path-integral measure whenever the Faddeev-Popov procedure is applied to a rigorous discussion of the gauge dependence in path integrals.

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