Thermodynamic Behavior of particular $f(R,T)$ Gravity Models

M. Sharif * and M. Zubair †

Department of Mathematics, University of the Punjab, Quaid-e-Azam Campus, Lahore-54590, Pakistan.

Abstract

We investigate the thermodynamics at the apparent horizon of the FRW universe in $f(R,T)$ theory under non-equilibrium description. The laws of thermodynamics have been discussed for two particular models of $f(R,T)$ theory. The first law of thermodynamics is expressed in the form of Clausius relation $T_h d \hat{S}_h = \delta Q$, where $\delta Q = -d\hat{E} + W d\mathcal{V} + T_h d_j \hat{S}$ is the energy flux across the horizon and $d_j \hat{S}$ is the entropy production term. Furthermore, the conditions to preserve the generalized second law of thermodynamics are established with the constraints of positive temperature and attractive gravity. We have illustrated our results for some concrete models in this theory.

Keywords: Modified Gravity; Dark Energy; Apparent Horizon; Thermodynamics.

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1 Introduction

Recent astrophysical observations indicate that expansion of the universe is presently in an accelerated epoch. The most compelling evidence for this

*msharif.math@pu.edu.pk
†mzubairkk@gmail.com
is found in measurements of supernovae type Ia (SNeIa) [1] which is supported by renowned observations [2]-[5]. The mysterious component of energy named as dark energy (DE) is often introduced to explain this behavior of the universe. However, the mechanism responsible for the accelerated expansion is still under debate.

Two approaches have been used to illustrate the issue of current cosmic acceleration. Introducing an “exotic cosmic fluid” in the framework of Einstein gravity [6]-[8] is one direction to deal such issue, however this approach did not fully explain current empirical data. The other way is to discuss the modified theories of gravity such as $f(R)$ [9,10], $f(T)$ [11], where “$T$” is the torsion scalar in teleparallel and $f(R,T)$, where $R$ and $T$ being the Ricci scalar and the trace of energy-momentum tensor [12]-[13] etc. The $f(R,T)$ theory modifies Einstein Lagrangian through the coupling of matter and geometry. In fact, this modified gravity generalizes the $f(R)$ theory and necessitates an arbitrary function of $R$ and $T$. Recently, Bamba et al. [14] presented a comprehensive review of the problem of DE and modified theories.

Black hole thermodynamics suggests that there is a fundamental connection between gravitation and thermodynamics [15]. Hawking radiations [16] together with; proportionality relation between temperature and surface gravity, also connection between horizon entropy and area of a black hole [17] further support this idea. Jacobson [18] was the first to deduce the Einstein field equations from the Clausius relation $T_h dS_h = \delta Q$ together with entropy proportional to the horizon area. In case of a general spherically symmetric spacetime, it was shown that the field equations can be constituted as the first law of thermodynamics (FLT) [19].

The relation between the FRW equations and the FLT was shown in [20] for $T_h = 1/2\pi r_A$, $S_h = \pi r_A^2/G$. The field equations for FRW background were also formulated in Gauss-Bonnet and Lovelock theories by employing the corresponding entropy relation of static spherically symmetric black holes. Eling et al. [21] shown that we cannot find the correct field equations simply by using the Clausius relation in nonlinear theories of gravity. They remarked that the non-equilibrium description of thermodynamics is needed, whereby the Clausius relation is modified to $T_h dS_h = \delta Q + d_jS$, where $d_jS$ is the entropy production term. In ref.[22]-[26] it was shown that the FRW field equations in general relativity (GR) and modified theories can be rewritten as $dE = T_h dS_h + WdV$ (unified FLT on the trapping horizon suggested by Hayward [23]) with work term $W = \frac{1}{2}(\rho - p)$.

Wu et al. [27] developed a generalized procedure to construct FLT and
the generalized second law of thermodynamics (GSLT) at the apparent horizon of Friedmann universe. The validity conditions of GSLT were studied in modified theories of gravity. Gong and Wang [28] showed that equilibrium thermodynamics is achievable for extended theories of gravity and entropy correction terms can be confined to mass-like functions. Other alternative approaches [29]-[33] have also been developed to reinterpret the non-equilibrium correction. In [34], we have explored the GSLT in $f(R, T)$ theory and found necessary conditions for its validity. It has been shown that the equilibrium description isn’t feasible by redefining the dark energy components in $f(R, T)$ theory.

In present work, the thermodynamics laws are examined for two particular models of $f(R, T)$ theory. We show that the FRW equations can be rewritten in a form of the FLT $T_h d\hat{S}_h + T_h d_j \hat{S}_h = -d\hat{E} + W_{tot} dV$. We formulate the GSLT and explore the conditions to validate this law. The paper is arranged in following format: In section 2, we present a brief introduction of $f(R, T)$ theory. Section 3 is devoted to discuss the FLT and GSLT corresponding to the Friedmann equations of particular $f(R, T)$ models. Finally, concluding remarks are given in section 4.

2 $f(R, T)$ Gravity: An Overview

The $f(R, T)$ modified gravity is described by the action [12]

$$I = \int dx^4 \sqrt{-g} \left[ \frac{f(R, T)}{2\kappa} + \mathcal{L}_m \right],$$

(1)

where $\kappa = 8\pi G$, $\mathcal{L}_m$ defines the matter substances of the universe. The matter energy-momentum tensor $T^{(m)}_{\alpha\beta}$ is defined as [35]

$$T^{(m)}_{\alpha\beta} = -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\alpha\beta}}.$$  

(2)

The field equations can be found by varying the action of $f(R, T)$ gravity with respect to the metric tensor

$$R_{\alpha\beta} f(R, T) - \frac{1}{2} g_{\alpha\beta} f(R, T) + (g_{\alpha\beta} \Box - \nabla_\alpha \nabla_\beta) f(R, T)$$

$$= 8\pi G T^{(m)}_{\alpha\beta} - f(T(R, T)) T^{(m)}_{\alpha\beta} - f(T(R, T)) \Theta_{\alpha\beta},$$

(3)
where \( f_R \) and \( f_T \) represent derivatives of \( f(R, T) \) with respect to \( R \) and \( T \), respectively. The field equations depend on the source term \( \Theta_{\mu\nu} \), hence every selection of \( L_m \) generates a particular set of field equations.

We consider the perfect fluid as matter source with matter Lagrangian \( L_m = p_m \) so that \( \Theta_{\alpha\beta} \) is given by

\[
\Theta_{\alpha\beta} = -2T^{(m)}_{\alpha\beta} + p_m g_{\alpha\beta}.
\]

Substituting this value in Eq. (3), it follows that

\[
\frac{1}{2}g_{\alpha\beta} f - \frac{1}{2}g_{\alpha\beta} f + (g_{\alpha\beta} \Box - \nabla_\alpha \nabla_\beta) f_R = 8\pi GT^{(m)}_{\alpha\beta} + T^{(m)}_{\alpha\beta} f_T - p_m g_{\alpha\beta} f_T.
\]

The spatially homogeneous and isotropic, \((n + 1)\)-dimensional FRW universe is defined as

\[
ds^2 = h_{\alpha\beta} dx^\alpha dx^\beta + \tilde{r}^2 d\tilde{\Omega}^2_{n-1},
\]

where \( h_{\alpha\beta} = diag(-1, a^2/(1 - kr^2)) \) is the 2-dimensional metric, \( a(t) \) is the scale factor and \( k \) is the cosmic curvature; \( \tilde{r} = a(t)r \), \( x^0 = t \), \( x^1 = r \) and \( d\tilde{\Omega}^2_{n-1} \) is the metric of a \((n - 1)\)-dimensional sphere. For \( n = 3 \), we have \((3 + 1)\)-dimensional FRW metric in Einstein gravity, while one can have \( n \geq 4 \) in other gravity theories.

### 3 Thermodynamics in \( f(R, T) \) gravity

Now, we discuss the laws of thermodynamics for two particular choices of \( f(R, T) \) gravity [12].

#### 3.1 \( f(R, T) = f_1(R) + f_2(T) \)

Let us consider the following \( f(R, T) \) model

\[
f(R, T) = f_1(R) + f_2(T),
\]

where \( f_1 \) and \( f_2 \) are arbitrary functions of \( R \) and \( T \), respectively. The corresponding field equations are

\[
R_{\alpha\beta} f_{1R}(R) - \frac{1}{2}g_{\alpha\beta} f_1(R) + (g_{\alpha\beta} \Box - \nabla_\alpha \nabla_\beta) f_{1R}(R) = 8\pi GT^{(m)}_{\alpha\beta} + T^{(m)}_{\alpha\beta} f_{2T}(T) + \frac{1}{2}g_{\alpha\beta} f_2(T),
\]
where \( f_{1R}(R) = df_{1}/dR \) and \( df_{2}/dT \). The choice of \( f_{2}(T) = 0 \) implies the field equation of \( f(R) \) gravity. In FRW background, the field equations will become

\[
\begin{align*}
\left(H^2 + \frac{k}{a^2}\right) &= \frac{16\pi G_{\text{Eff}}}{n(n-1)}(\rho_m + \rho_{dc}), \quad (9) \\
\dot{H} - \frac{k}{a^2} &= -\frac{8\pi G_{\text{Eff}}}{(n-1)}(\rho_m + \rho_{dc} + p_{dc}), \quad (10)
\end{align*}
\]

where \( G_{\text{Eff}} = \frac{1}{f_{1R}(G + \frac{f_{2R}}{8\pi})} \), and

\[
\rho_{dc} = \frac{1}{8\pi GD} \left[ \frac{1}{2}(Rf_{1R} - f_{1} - f_{2}) - nH\dot{R}f_{1RR} \right], \quad (11)
\]

\[
p_{dc} = \frac{1}{8\pi GD} \left[ -\frac{1}{2}(Rf_{1R} - f_{1} - f_{2}) + (n-1)H\dot{R}f_{1RR} + \ddot{R}f_{1RR} \right. \\
\left. + \dot{R}^2 f_{1RRR} \right] \quad (12)
\]

and \( D = \left(1 + \frac{f_{2R}(R,T)}{8\pi G}\right) \). Substituting Eqs. (11) and (12) in conservation equation [34], we obtain

\[
q_t = \frac{n(n-1)}{16\pi G} \left( H^2 + \frac{k}{a^2} \right) \partial_t \left( \frac{f_{1R}}{D} \right). \quad (13)
\]

Clearly, this reduces to the energy transfer relation for \( f(R) \) theory if \( f(R, T) = f_1(R) \) [32, 33]. If the effective gravitational coupling is constant, we obtain \( q_t = 0 \).

### 3.1.1 First Law of Thermodynamics

Now, the FLT is constructed for the above \( f(R, T) \) model. The condition, \( h^{\mu\lambda}\partial_{\mu}\tilde{r}\partial_{\lambda}\tilde{r} = 0 \), gives the radius \( \tilde{r}_A \) of the apparent horizon

\[
\tilde{r}_A = \left( H^2 + \frac{k}{a^2} \right)^{-1/2}.
\]

The associated temperature is \( T_h = \frac{\kappa_{sg}}{2\pi} \), where \( \kappa_{sg} = \frac{1}{2\sqrt{-h}}\partial_{\mu}(\sqrt{-h}h^{\mu\nu}\partial_{\nu}\tilde{r}_A) = -\frac{1}{\tilde{r}_A}(1 - \frac{1}{2H}\frac{d[\ln \tilde{r}_A]}{d\tilde{t}}) \) is the surface gravity [20]. The temperature \( T_h = \frac{1}{2\pi\tilde{r}_A}(1- \)
\( \eta \) is positive for \( \eta = \frac{1}{2H} \frac{d \ln \tilde{r}_A}{dt} < 1 \). Applying the definition of \( \tilde{r}_A \), the condition to keep \( T_h \) positive is expressed as

\[
(\dot{H} - \frac{k}{a^2}) > -2(H^2 + \frac{k}{a^2}).
\] (14)

In GR, the horizon entropy is defined as \( S_h = A/4G \) \([15]-[17]\), where \( A = n \hat{\Omega}_n \tilde{r}_A^{n-1} = n\pi^{n/2}[\Gamma(n/2 + 1)]^{-1} \tilde{r}_A^{n-1} \) represents area of the apparent horizon. Wald \([36]\) proposed that in modified gravitational theories, the horizon entropy is associated with a Noether charge entropy. Brustein et al. \([37]\) showed that Wald entropy is equivalent to \( S_h = A/4G_{\text{Eff}} \) where \( G_{\text{Eff}} \) being the effective gravitational coupling. We can define the Wald entropy in \( f(R,T) \) theory as \([34]\)

\[
\dot{S}_h = \frac{A}{4G_{\text{Eff}}},
\] (15)

where \( G_{\text{Eff}} = GD(R,T)/f_1R \) for the first \( f(R,T) \) model. Following ref.\([34]\), one can obtain the FLT in the following form

\[
T_h d\dot{S}_h = \delta Q,
\]

where the energy flux \( \delta Q \) is

\[
\delta Q = -d\dot{E} + \frac{n}{2} \tilde{r}_A^{n-1}(\rho_t - p_t)d\tilde{r}_A + \frac{n\hat{\Omega}_n(n-1)\tilde{r}_A^{n-2}}{16\pi G} d\left( \frac{f_1R}{F} \right)
\]

\[
= -d\dot{E} + W_t d\mathcal{V} + \mathcal{V}_q dt + T_h S_h d\left( \frac{f_1R}{D} \right).
\] (16)

\( W_t = -\frac{1}{2}T^{(\mu\nu)}h_{\mu\nu} = \frac{1}{2}(\rho_t - p_t) \) is the total work density \([23]\). Thus, the FLT can be expressed as

\[
T_h d\dot{S}_h + T_h d_j\dot{S}_h = -d\dot{E} + W_{\text{tot}} d\mathcal{V},
\] (17)

where

\[
d_j\dot{S}_h = -\frac{n\hat{\Omega}_n(H^2 + \frac{k}{a^2})^{\frac{n-1}{2}}((n+1)H^2 + \dot{H} + n\frac{k}{a^2})d(f_1R/D)}{4G(2H^2 + \dot{H} + \frac{k}{a^2})}
\]

is the entropy production term developed for this model. This characterizes non-equilibrium treatment of the thermodynamics. The FLT for a flat FRW universe in \( f(R) \) theory \([32,33]\) can be retrieved from this result. For \( f(R,T) = R \), the term \( d_j\dot{S}_h \) vanishes, which leads to the FLT in Einstein gravity.
3.1.2 Generalized Second Law of Thermodynamics

Now we investigate the validity of GSLT in \( f(R, T) \) theory for this model. The FLT determines the horizon entropy given by Eq. (17). The composition of the entire matter and energy fluids within the horizon is given by Gibb’s equation \[39\]

\[ T_t d\dot{S}_t = dE_t + p_t dV, \tag{18} \]

where \( T_t \) and \( \dot{S}_t \) represent the temperature and entropy of all contents within the horizon, respectively. There is a relation of temperature within the horizon to \( T_h \) \[27\] i.e., \( T_t = b T_h \), here \( 0 < b < 1 \) to assure that \( 0 < T_t < T_h \). Consider \( \dot{S} \) being the sum of matter entropy within the horizon, horizon entropy and the non-equilibrium entropy production term.

The GSLT states that the time derivative of total entropy is not decreasing with time i.e.,

\[ T_h \dot{\dot{S}} = T_h (\dot{\dot{S}}_h + d_j \dot{S}_h + \dot{S}_t) \geq 0, \tag{19} \]

where \( d_j \dot{S}_h = \partial_t (d_j \dot{S}_h) \). Inserting Eqs. (17) and (18) in the above inequality, we obtain

\[
\frac{n(n-1)\dot{\Omega}_n}{16\pi HG} \left[ 2H \dot{r}_A \left[ (b-1) + \dot{r}_A \dot{r}_A^{-1}(2-b) \right] \left( \frac{f_{1R}}{D} \right) + (1-b)H \dot{r}_A \right] \times \partial_t \left( \frac{f_{1R}}{D} \right) \geq 0, \quad \text{where} \quad \dot{r}_A = -\dot{r}_A^{-3} H (\dot{H} - \frac{k}{a^2}). \tag{20}
\]

We can impose the constraint \( D/f_{1R} > 0 \) so that \( G_{Eff} \) is positive. Applying the positive temperature condition \( (\dot{H} - \frac{k}{a^2}) > -2(H^2 + \frac{k}{a^2}) \) with the temperature parameter \( b < 1 \), this gives

\[
\frac{n(n-1)\dot{\Omega}_n(H^2 + \frac{k}{a^2})^{-\frac{1}{2}}}{16\pi G D} \left[ 4Hf_{1R} + (1-b)D \partial_t \left( \frac{f_{1R}}{D} \right) \right] \geq 0. \tag{21}
\]

Thus the GSLT can be satisfied, provided that \( \partial_t (f_{1R}/D) > 0 \). If \( \partial_t (f_{1R}/D) < 0 \), then the GSLT is protected only if \( \left| \frac{2\partial_t (f_{1R}/D)}{f_{1R}/D} \right| < \frac{4H}{1-b} \). If the gravitational coupling constant is indeed a constant i.e., \( \partial_t (f_{1R}/D) = 0 \), then the GSLT always holds. The condition to preserve the GSLT in \( f(R) \) theory can be reproduced if \( f(R, T) = f_1(R) \). For \( k = 0 \) and \( f_2(T) = 0 \), one get the inequality already constructed by Wu et al. \[27\] in non-linear gravity. In the
thermal-equilibrium limit $b \sim 1$, the constraint to protect the GSLT is
\[
\frac{n(n-1)\Omega_n(H^2 + \frac{k}{a^2})^{-(\frac{n}{2}+1)}}{16\pi GD} \left[ H(\dot{H} - \frac{k}{a^2})^2 f_{1R} \right] \geq 0. \tag{22}
\]
The relation (22) depends on the choice of $f(R, T)$, for instance $f_1(R) = R$ with $f_2(T) = 0$ results in
\[
\frac{n(n-1)\Omega_n(H^2 + \frac{k}{a^2})^{-(\frac{n}{2}+1)}}{16\pi G} \left[ H(\dot{H} - \frac{k}{a^2})^2 \right] \geq 0.
\]
which is the condition for validity of GSLT in Einstein gravity. Here, we discuss the validity of GSLT for some particular forms of $f(R, T)$ gravity namely,
(i) $f_1(R) = f(R)$, $f_2(T) = \lambda T$,
(ii) $f_1(R) = R$, $f_2(T) = 2f(T)$.
In the first case, we consider the $f(R, T)$ model corresponding to the power law solution $a(t) = a_0 t^m \tag{13}$
\[
f(R, T) = \alpha_{k\omega} (-R)^k + \lambda T, \tag{23}
\]
where
\[
\alpha_{k\omega} = \frac{2^{3-2k} 3^{k-1} k A (4k - 3(1 + \omega))^{1-k}(1 + \omega)^{2k-2}}{k^2(6\omega + 8) - k(9\omega + 13) + 3(\omega + 1)}.
\]
For this model, the Hubble and deceleration parameters are $H = \frac{2k}{3(1 + \omega)}$ and $q = -1 + \frac{3(1+\omega)}{2k}$. The validity of the GSLT in $(3+1)$-dimensional flat FRW universe for the model (23) requires the condition
\[
T_h \dot{S} = \frac{9(1 + \omega)^2 \alpha_{k\omega}}{8k^2 \tilde{G}} \left( \frac{4k[4k - 3(1 + \omega)]}{3(1 + \omega)^2 t^2} \right)^{k-1} \geq 0, \tag{24}
\]
where $\tilde{G} = G + \frac{A}{8\pi}$. Now we present some constraints for particular values of $k = -2, -1, 1, 2$.

- For $k = 1$ this solution represents $\Lambda$CDM model and constraint on GSLT is given by
  \[
  T_h \dot{S} = \frac{9(1+\omega)^2 A}{8\tilde{G}} \geq 0, \text{ which is true if } A > 0 \text{ with } \omega \leq 3.
  \]

- For $k = 2$, the GSLT is valid if $T_h \dot{S} = \frac{A(5-3\omega)^2}{2Gt^2} \geq 0$, which requires $A < 0$. 

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Figure 1: Evolution of GSLT for different values of parameters $k$ and $\lambda$ (a) for present epoch $z = 0$ and (b) for $z = -0.9$

- For $k = -1, -2$, we find $T_h \dot{\mathcal{S}} \geq 0$ if $A > 0$ with $w \geq 0$. This choice would favor the expanding universe since $q < -1$.

The higher powers of curvature can be made available for larger values of $k$ and one can examine the validity of GSLT. If one consider the dust case $\omega = 0$, then the possible role of $\lambda$ and $k$ can be seen from graphical description as shown in Figure 1.

In the second case, the GSLT for the $f(R, T) = R + 2f(T)$ model demands the following inequality to be fulfilled

$$T_h \dot{\mathcal{S}} = \frac{\dot{H}^2}{2H^4\dot{\mathcal{G}}} \geq 0, \text{ where } \dot{\mathcal{G}} = G + 2f(T)/8\pi.$$ 

Here we consider the power law solution of the form $f(T) = a_1T + a_2T^k$, where $a_1$ and $a_2$ are parameters. Following [13] for the dust case, we set $a_1 = 1$ and $a_2 = \frac{2^3 - 2k3^k - 1k^3 - 2k}{4 + 2k}$. Thus the above inequality takes the form

$$T_h \dot{\mathcal{S}} = \frac{9\pi(4 + 2k)}{k^2[(4 + 2k)(8\pi G + 1) + 2^3 - 2k3^k - 1k^4 - 2kT^{k-1}]} \geq 0.$$ 

This shows that the GSLT holds for the $f(T)$ power law model and its validity is shown in Figure 2.
3.2 $f(R, T) = f_1(R) + f_2(R)f_3(T)$

A more general $f(R, T)$ gravity model is of the form [12]

$$f(R, T) = f_1(R) + f_2(R)f_3(T), \quad (25)$$

where $f_i \ (i = 1, 2)$ are functions of $R$ and $f_3$ is function of $T$. For a dust matter source, the field equation is obtained as

$$R_{\alpha\beta}[f_1R + f_2Rf_3] - \frac{1}{2}g_{\alpha\beta}f_1 + (g_{\alpha\beta}\Box - \nabla_\alpha \nabla_\beta)\{f_1R + f_2Rf_3\}
= 8\pi GT^{(m)}_{\alpha\beta} + T^{(m)}_{\alpha\beta}f_2f_3T + \frac{1}{2}g_{\alpha\beta}f_2f_3. \quad (26)$$

An equivalent Einstein field equation can be obtained with $G_{Eff} = \frac{1}{f_1R + f_2Rf_3} \left(G + \frac{f_2f_3}{8\pi}\right)$, whereas

$$\tilde{T}_{\alpha\beta}^{(de)} = \frac{1}{f_1R + f_2Rf_3} \left[\frac{1}{2}g_{\alpha\beta}(f_1 + f_2f_3) - R(f_1R + f_2Rf_3)\right] + (\nabla_\alpha \nabla_\beta - g_{\alpha\beta}\Box)(f_1R + f_2Rf_3). \quad (27)$$

In the following discussion, we define $\mathcal{J}(R, T) = f_1R(R) + f_2R(R)f_3(T)$.

For this $f(R, T)$ model, the field equation are identical to Eqs.(9)-(10),

Figure 2: Evolution of GSLT for case (ii) versus $T$ and $k$. 

$T$ $k$ $GSLT$
whereas
\[ \hat{\rho}_{dc} = \frac{1}{8\pi GB} \left[ \frac{1}{2} (RJ - f_1 - f_2 f_3) - nH(\dot{R}J_R + \dot{T}J_T) \right], \] (28)
\[ \hat{p}_{dc} = \frac{1}{8\pi GB} \left[ -\frac{1}{2} (RJ - f_1 - f_2 f_3) + (n-1)H(\dot{R}J_R + \dot{T}J_T) + \ddot{R}J_R \right. \\
\left. + \dddot{R}J_{RR} + 2\dot{R}\dot{T}J_RT + \dot{T}J_T + \dddot{T}J_{TT} \right] \] (29)

and \[ B(R, T) = \left( 1 + \frac{f_2(R) f_2(T)}{8\pi G} \right) \] which includes contributions from both matter and geometry. The total energy exchange term for this model is given by
\[ q_t = \frac{n(n-1)}{16\pi G} (H^2 + \frac{k}{a^2}) \partial_t \left( \frac{J}{B} \right). \] (30)

Now we analyze the validity of the FLT and GSLT for the above model.

### 3.2.1 First Law of Thermodynamics

The Wald entropy \[ \hat{S}_h = A/4G_{Eff} \] for function (25) becomes
\[ \hat{S}_h = n\Omega_n \hat{\Omega}^{n-1} \frac{J}{4GB}. \] (31)

In this case, FLT involves the energy flux \( \delta Q \) and entropy production terms of the form \[ 34 \]
\[ \delta Q = -d\dot{E} + \frac{n}{2} \Omega_n \hat{\Omega}^{n-1} (\rho_{tot} - p_{tot}) d\hat{A} + \frac{n(n-1)}{16\pi G} \Omega_n \hat{\Omega}^{n-2} d \left( \frac{J}{B} \right) \]
\[ = -d\dot{E} + W_{tot} d\mathcal{V} + \mathcal{V} q_{tot} dt + T_h \dot{S}_h d \left( \frac{J}{B} \right). \] (32)
\[ dJ \dot{S}_h = -\frac{1}{T_h} \mathcal{V} q_{tot} dt - S_h d \left( \frac{J}{B} \right) \]
\[ = - \frac{n\Omega_n (H^2 + \frac{k}{a^2}) \frac{n+1}{2} ((n+1)H^2 + \dot{H} + n\frac{k}{a^2}) d(J/B)}{4G(2H^2 + H + \frac{k}{a^2})}. \] (33)

The \( f(R, T) \) gravity model, \( f(R, T) = f_1(R) + f_2(R)f_2(T) \) involves the explicit non-minimal gravitational coupling between matter and curvature. Results obtained using this theory would be different from other models such
as $f(R)$ theory. The coupling of matter and geometry reveals that the matter energy-momentum tensor is no longer conserved and there is an energy transfer between the two components. Due to this interaction, the energy exchange term $q_t$ is found to be non-zero, so the entropy production term would be an additional term in this modified gravity. Hence, the FLT is established in more general $f(R, T)$ gravity and entropy production is induced in a non-equilibrium treatment of thermodynamics \cite{21, 34}. In recent papers \cite{33}, Bamba et al. shown that the entropy production term can be incorporated by redefinition of the field equations. However, in this theory, such treatment is not useful as shown in \cite{34}.

### 3.2.2 Generalized Second Law of Thermodynamics

To develop the GSLT for the second model, we consider the Gibbs equation \cite{18}. The horizon entropy is determined from FLT. The necessary constraint for the validity of the GSLT is shown in Eq. (19). For the $f(R, T)$ model \cite{25}, we obtain

$$n(n-1)\Omega_0 \frac{2H\dot{r}_A[(b-1)+\dot{r}_A\tilde{r}_A^{-4}(2-b)]}{16\pi H G} + (1-b)H\tilde{r}_A$$

$$\times \partial_t \left( \frac{\mathcal{J}}{\mathcal{B}} \right) \geq 0,$$

where $\dot{r}_A = -\tilde{r}_A^{-3}H(\dot{H} - \frac{\kappa}{a^2}). \quad (34)$

The effective gravitational coupling constant for this model is $G_{Eff} = GB/J$. We can impose the condition $B/J > 0$ to keep $G_{Eff} > 0$. For the positive temperature condition $(\dot{H} - \frac{\kappa}{a^2}) > -2(H^2 + \frac{\kappa}{a^2})$ with $b < 1$, Eq. (34) is reduced to

$$n(n-1)\Omega_0 \frac{H^2 + \frac{k}{a^2} \cdot \tilde{r}_A^{-\frac{2}{2}+1}}{16\pi GB} \left[ 4H\mathcal{J} + (1-b)B\partial_t \left( \frac{\mathcal{J}}{\mathcal{B}} \right) \right] \geq 0. \quad (35)$$

This shows that the GSLT is valid only if $\partial_t(\mathcal{J}/\mathcal{B}) > 0$. If the gravitational coupling constant is indeed a constant, the GSLT is always protected. If $\partial_t(\mathcal{J}/\mathcal{B}) < 0$, then the GSLT can hold only if $|\frac{\partial_t(\mathcal{J}/\mathcal{B})}{\mathcal{J}/\mathcal{B}}| \leq \frac{4H}{1-b}$. The GSLT in $f(\tilde{R})$ theory can be retrieved for $f_3(T) = 0$. If $T_i \sim T_h$, then the condition \cite{35} becomes

$$n(n-1)\Omega_0 \frac{H^2 + \frac{k}{a^2}}{16\pi GB} \left[ H(\dot{H} - \frac{\kappa}{a^2}) \mathcal{J} \right] \geq 0.$$
Figure 3: Evolution of GSLT for the model (36) with $m = 2$. The curve with larger slope corresponds to $z = -0.9$ while the other represents the present value ($z = 0$).

Figure 4: One can choose the specific value of one parameter and vary the others. In left panel, we choose $p = 1$ and show the constraints on parameters $m$ and $q$. The right panel represents the parametric values of $m$ and $p$ for fixed $q = 1$. 
We consider the $f(R, T)$ model \(^{(25)}\) with $f_1(R) = R$, $f_2(R) = R^p$ and $f_3(R) = T^q$, $(p, q > 0)$ so that in 4-dimensional flat FRW metric, the GSLT becomes

$$T_h \dot{S} = \frac{\dot{H}^2 (1 + pR^{p-1}T^q)}{2H^4G \left(1 + \frac{R^{p-t}T^q}{8\pi G}\right)} \geq 0. \quad (36)$$

For the power law case $a(t) = a_0 t^m$ with $\rho = \rho_0 a^{-3}$, this can be written as

$$T_h \dot{S} = \frac{8\pi [1 + p(6m(2m-1)t^{-2})^{p-1}(\rho_0 t^{-3m})^q]}{2m^2[8\pi G + (6m(2m-1)t^{-2})^{p-1}(\rho_0 t^{-3m})^q]} \geq 0. \quad (37)$$

We have examined the validity of relation \(^{(37)}\) and developed constraints on the parameters $m$, $p$ and $q$. The results are shown in Figures 3 and 4.

### 4 Conclusions

In this paper, the thermodynamics properties have been discussed in more general $f(R, T)$ theory. The non-equilibrium treatment of thermodynamics is addressed for two particular models of $f(R, T)$ gravity. In this modified theory, accelerated expansion may produce not just from scalar curvature part to the entire energy density of universe, but may include a matter component as well. The consequences of $f(R, T)$ theory may contribute to significant results when compared to other modified gravitational theories, applicable to various problems of contemporary interest such as accelerated cosmic expansion, gravitational collapse, dark matter and the detection of gravitational waves \(^{40}\). The detection of gravitational waves could be an excellent way to test general relativity and modified theories of gravity. Corda \(^{40}\) has investigated the detection of gravitational waves in $f(R)$ theory and it would be appealing to explore this issue in $f(R, T)$ theory.

It is shown that representation of equilibrium thermodynamics is not executable in this theory \(^{34}\). Hence the non-equilibrium treatment of thermodynamics is employed to discuss the laws of thermodynamics. Here, we studied two particular models of $f(R, T)$ theory to show the consequences of explicit coupling of matter and geometry. The gravitational coupling between matter and higher derivatives terms of curvature describes a transfer of energy and momentum beyond that normally existing in curved spaces.
This interaction leads to the entropy production term in this modified gravity. The FLT is formulated by employing the Wald’s entropy relation. We remark that an entropy production term is produced in this work but no such term in present in GR, Gauss-Bonnet [22], Lovelock [22] and braneworld [24] theories of gravity.

The validity of GSLT has also been investigated in this work. We have found that the GSLT holds with the conditions namely, attractive nature of gravity and temperature being positive. In fact, it is natural to assume the relation $T_t = bT_h$ and proportionality constant $b$ can be considered as unity, implying that the system is in thermal-equilibrium. Generally, the horizon temperature cannot match the temperature of all energy sources within the horizon, and the two mechanisms must experience interaction for some interval of time ahead of achieving the thermal-equilibrium. Moreover, the gravitational curvature-matter coupling in $f(R, T)$ theory may produce the unscripted flow of energy between the horizon and fluid. Also, the energy fluid of dark components does not permit the effective gravitational constant to be an approximate constant. In the limiting choice of thermal-equilibrium, we assume that $T_t$ is very close to $T_h$. We find that the GSLT is fulfilled in both phantom and quintessence regimes of the universe which seems to be consistent with refs.[41]. Furthermore, we have also developed constraints on some concrete $f(R, T)$ models corresponding to power law solution. It is significant to remark that the equilibrium treatment of thermodynamics in $f(R, T)$ theory would benefit from further study.

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