Variations of selective separability II: Discrete sets and the influence of convergence and maximality

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\section*{A B S T R A C T}

A space \(X\) is called selectively separable (R-separable) if for every sequence of dense subspaces \((D_n; \ n \in \omega)\) one can pick finite (respectively, one-point) subsets \(F_n \subseteq D_n\) such that \(\bigcup_{n \in \omega} F_n\) is dense in \(X\). These properties are much stronger than separability, but are equivalent to it in the presence of certain convergence properties. For example, we show that every Hausdorff separable radial space is R-separable and note that neither separable sequential nor separable Whyburn spaces have to be selectively separable. A space is called d-separable if it has a dense \(\sigma\)-discrete subspace. We call a space \(X\) D-separable if for every sequence of dense subspaces \((D_n; \ n \in \omega)\) one can pick discrete subsets \(F_n \subseteq D_n\) such that \(\bigcup_{n \in \omega} F_n\) is dense in \(X\). Although d-separable spaces are often also D-separable (this is the case, for example, with linearly ordered d-separable or stratifiable spaces), we offer three examples of countable non-D-separable spaces. It is known that d-separability is preserved by arbitrary products, and that for every \(X\), the power \(X^{\omega}[X]\) is d-separable. We show that D-separability is not preserved even by finite products, and that for every infinite \(X\), the power \(X^{\omega}[X]\) is not D-separable. However, for every \(X\) there is a \(Y\) such that \(X \times Y\) is D-separable. Finally, we discuss selective and D-separability in the presence of maximality. For example, we show that (assuming \(\mathfrak{d} = \mathfrak{c}\)) there exists a maximal regular countable selectively separable space, and that (in ZFC) every maximal countable space is D-separable (while some of those are not selectively separable). However, no maximal space satisfies the natural game-theoretic strengthening of D-separability.

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1. Introduction

The area known as Selection principles in Mathematics deals with selective variations of classical topological notions like compactness or separability (see [56] or [51] for a survey and [55] for another survey concentrating on open problems in the field). Looking at the selective version of a certain property adds a combinatorial skeleton to it that often makes it easier to deal with. For example, Aurichi [8] has recently given one of the few known partial solutions to van Douwen’s elusive D-space problem (see [25]) by replacing the Lindelöf property with one of its selective strengthenings, the Menger property.

In this paper we will be concerned with selective separability and its variations. This notion has recently gained particular attention, as witnessed by the papers [50,42,14–16,9,10,48,31]. A space is called \( D \)-separable (also called \( M \)-separable or \( SS \)) if for every sequence of dense subspaces \( (D_n; \ n \in \omega) \) one can pick finite sets \( F_n \subset D_n \) so that \( \bigcup_{n \in \omega} F_n \) is dense in \( X \). X is \( H \)-separable if for every sequence of dense subspaces \( (D_n; \ n \in \omega) \) one can pick finite \( F_n \subset D_n \) so that every non-empty open subset of \( X \) intersects all but finitely many \( F_n \). X is \( R \)-separable if for every sequence of dense subspaces \( (D_n; \ n \in \omega) \) one can pick \( p_n \in D_n \) so that \( \{p_n; \ n \in \omega\} \) is dense in \( X \). X is \( GN \)-separable if \( X \) is crowded and for every sequence of dense subspaces \( (D_n; \ n \in \omega) \) one can pick \( p_n \in D_n \) so that \( \{p_n; \ n \in \omega\} \) is groupable. This means that one can find pairwise disjoint non-empty and finite sets \( A_m \) for \( m < \omega \) in such a way that \( \{p_n; \ n \in \omega\} = \bigcup_{m \in \omega} [A_m; \ m \in \omega] \) and every non-empty open set in \( X \) intersects all but finitely many \( A_m \).

X is \( SS^+ \) if Two has a winning strategy in the following game \( G^\ominus_{fin}(D, D) \). D is the collection of all dense subspaces of \( X \). One picks \( D_0 \in D \), then Two picks a finite \( F_0 \subset D_0 \), then One picks \( D_1 \in D \), etc. Two wins if \( \bigcup_{n \in \omega} F_n \) is dense in \( X \). (The term \( SS^+ \) is from [10] but the notion and the game \( G^\ominus_{fin}(D,D) \) were introduced in [50].) Barman and Dow discovered [10] that every separable Fréchet space is selectively separable. Gruenhage and Sakai [31] pointed out that separable Fréchet spaces are even \( R \)-separable, and if, in addition, they have no isolated points, they are \( GN \)-separable. In Section 3 we discuss the possibility to extend these results to spaces satisfying convergence-type conditions weaker than Fréchet. It turns out that every regular separable radius space is selectively separable while separable sequential spaces or separable Whyburn spaces need not be selectively separable. We also consider the special case of countably compact spaces.

In Section 4 we consider a weaker form of selective separability: the sets \( F_n \) are supposed to be discrete rather than finite; we call this property \( D \)-separability. This may be also viewed as a natural selective strengthening of the notion of \( d \)-separability. Recall that \( X \) is called \( d \)-separable [2] (see also [3,1,53,57,38]) if \( X \) has a \( \sigma \)-discrete dense subspace. The notion of \( d \)-separability is almost as old as separability and was introduced by Kurepa in [40], where it is called condition \( K_0 \).

It turns out that in some cases \( d \)-separable spaces are \( D \)-separable. However, the behavior of \( d \)-separability and \( D \)-separability, is quite different. This is particularly apparent if one looks at the product operation. Every product of \( d \)-separable spaces is \( d \)-separable [3]; for every \( T_1 \) space \( X \), a high enough power of \( X \) is \( d \)-separable [38] while we show that there are two \( D \)-separable spaces with a non-\( d \)-separable product, and every (Tychonoff) space has some power which is not \( D \)-separable.

In Section 5, we discuss selective separability and \( D \)-separability in maximal spaces. For example, we show that (assuming \( \diamond = \nabla \)) there exists a maximal countable selectively separable space, and that (in ZFC) every maximal regular countable space is \( D \)-separable (while some of those are not selectively separable). However, no maximal space is \( D^+ \)-separable (\( D^+ \)-separability is a property stronger than \( D \)-separability that is defined in terms of topological games, see Definition 16 below).

2. Terminology and preliminaries

For undefined topological notions we refer to [26], while for undefined set-theoretic notions we refer to [34]. The letter \( X \) always denotes a topological space. \( X \) is Fréchet if for every non-closed \( A \subset X \) and every \( p \in \overline{A} \setminus A \) there is a sequence from \( A \) converging to \( p \). \( X \) is sequential if whenever \( A \) is non-closed there are a \( p \in \overline{A} \setminus A \) and a sequence from \( A \) converging to \( p \). \( X \) has countable tightness if whenever \( p \in A \) there is a countable \( B \subset A \) such that \( x \in \overline{B} \). \( X \) has countable fan tightness [4] if whenever \( p \in \overline{A} \) for all \( n \in \omega \) one can pick finite \( F_n \subset A_n \) so that \( p \in \bigcup_{n \in \omega} F_n \). \( X \) has countable strong fan tightness [49] if whenever \( p \in \overline{A} \) for all \( n \in \omega \) one can pick \( p_n \in A_n \) so that \( p \in \{p_n; \ n \in \omega\} \). \( X \) has countable dense fan tightness at the point \( p \) if for every sequence \( (D_n; \ n < \omega) \) of dense sets we can find finite sets \( F_n \subset D_n \) such that \( p \in \bigcup_{n < \omega} F_n \). We will be using the following simple proposition without explicit mention.

**Proposition 1.** Let \( X \) be separable. Then:

1. [14] \( X \) is selectively separable iff \( X \) has countable dense fan-tightness at every point.
2. \( X \) is \( H \)-separable iff for every \( p \in X \) and every sequence \( (D_n; \ n \in \omega) \) of dense subspaces of \( X \) one can pick finite sets \( F_n \subset D_n \) so that every neighborhood of \( p \) meets all but finitely many \( F_n \).
3. \( X \) is \( R \)-separable iff for every \( p \in X \) and every sequence \( (D_n; \ n \in \omega) \) of dense subspaces of \( X \) one can pick points \( p_n \in D_n \) so that \( p \subset \{p_n; \ n \in \omega\} \).
4. \( X \) is \( GN \)-separable iff for every \( p \in X \) and every sequence \( (D_n; \ n \in \omega) \) of dense subspaces of \( X \) one can pick points \( p_n \in D_n \) and represent \( \{p_n; \ n \in \omega\} = \bigcup_{m \in \omega} A_m \) where the sets \( A_m \) are non-empty, finite, and pairwise disjoint, so that every neighborhood of \( p \) intersects all but finitely many \( A_m \).
\[ \delta(X) = \sup(d(D); D \text{ is dense in } X). \] If \( X \) is compact, then \( \delta(X) = \pi w(X) \) [35]. Obviously, \( \delta(X) = \omega \) for every selectively separable space \( X \).

\( X \) is radial if for every \( A \subset X \) and every \( p \in A \) there is a well-ordered net \( \{ x_\alpha : \alpha < \kappa \} \subset A \) which converges to \( p \). \( X \) is pseudoradial if for every non-closed \( A \subset X \) there is a \( p \in A \setminus A \) and a well-ordered net \( \{ x_\alpha : \alpha < \kappa \} \subset A \) which converges to \( p \). A set \( A \) is \( \kappa \)-closed (where \( \kappa \) is a cardinal) if \( \overline{B} \subset A \) whenever \( B \subset A \) and \( |B| \leq \kappa \). \( X \) is semiradial (see [13,20]) if for every \( \kappa \), every non-\( \kappa \)-closed set \( A \) contains a well-ordered net of length \( \leq \kappa \) converging to a point outside \( A \). Among the various subclasses of pseudoradial spaces considered in the literature, the class of semiradial spaces is the smallest one which includes all radial and all sequential spaces.

\( X \) has the Whyburn property if for every \( A \subset X \) and every \( p \in A \setminus A \) there exists \( B \subset A \) such that \( \overline{B} = A \cup \{ p \} \). Every Fréchet space is Whyburn and every compact Whyburn space is Fréchet (see [38]). For \( p \in \omega^\omega \), the space \( \omega^\omega \cup \{ p \} \) with the topology inherited from \( \beta \omega \) is a non-Fréchet Whyburn topological space. The space \( C_p([0,1]) \) of all continuous functions from \([0,1] \) to \( \mathbb{R} \) with the topology of pointwise convergence is a nice Whyburn topological group which is not Fréchet [21].

Recall that for \( M \subset X \), \( seqcl(M) = \{ x \in X : \) there is a sequence converging from \( M \) to \( x \} \) and \( seqclw(M) \) is defined inductively by \( seqclw(M) = seqcl(\bigcup_{\beta < \kappa} seqcl(M)) \). If \( X \) is sequential then there exists an ordinal \( \alpha^* \) called the sequential order of \( X \) such that \( seqclw(M) = \overline{M} \) for every \( M \subset X \). The sequential order of any sequential space is \( \leq \omega_1 \).

Let \( \pi \) be the set of all functions \( s : n = [0,1,\ldots,n-1] \rightarrow \omega \) and let \( Seq = \bigcup_{n \in \omega} \pi^n \). If \( s \in \pi^\omega \) and \( k \in \omega \), we write \( s \prec k = s \cup \{(n,k)\} \in \pi^{n+1} \). Given a free filter \( F \) on \( \omega \), we denote by \( Seq(F) \) the topological space having \( Seq \) as the underlying set and the topology obtained by declaring a set \( U \subset Seq \) open if and only if for any \( s \in U \) \( \{ n : s \prec n \in U \} \in F \). \( Seq(F) \) is always a Hausdorff zero-dimensional dense-in-itself space (see [60] for more information). Sometimes we will assume that \( F \) is the Frechét filter, that is the filter of all cofinite subsets of \( \omega \). In this case, the space \( Seq(F) \) is also known under the name \( S_\omega \) (see [7]), and is a sequential space of sequential order \( \omega_1 \).

\( X \) is crowded (also called dense in itself) if \( X \) does not have isolated points. \( X \) is maximal if \( X \) is crowded and no topology strictly stronger than the topology of \( X \) is crowded. \( X \) is resolvable (\( \omega \)-resolvable) if \( X \) contains two (respectively, a countably infinite family of pairwise) disjoint dense subspaces. \( X \) is submaximal if every subset is open in its closure, or, equivalently (see [6]) if the complement of every dense set is closed and discrete. Every maximal space is submaximal. Every crowded submaximal (hence every maximal) space is irresolvable (= not resolvable). \( X \) is Baire if no non-empty open set in \( X \) is representable as the union of countably many nowhere dense sets. \( X \) is strongly irresolvable [39] if all non-empty open sets are irresolvable. Strongly irresolvable Baire is irresolvable as SIB [39]. For any space \( X \), the dispersion character \( \Delta(X) \) of \( X \) is defined as the minimum cardinality of a non-empty subset of \( X \). A crowded space \( X \) is extra-resolvable if there is a family \( G \) of dense subspaces of \( X \) such that \( |G| > \Delta(X) \) and for every two distinct \( G, G' \in G \), \( G \cap G' \) is nowhere dense. \( X \) is discretely generated [24] if whenever \( p \in A \) there is a discrete \( D \subset A \) such that \( p \in D \).

\( X \) is a \( \sigma \)-space if \( X \) has a \( \sigma \)-discrete network.

\( X \) is monotonically normal if one can assign to every point \( x \in X \) and open set \( U \subset X \) an open set \( H(x,U) \subset U \) such that \( x \in H(x,U) \cap H(y,V) = \emptyset \) then either \( x \in V \) or \( y \in U \). The function \( H \) is called a monotone normality operator.

\( X \) is stratifiable if one can assign to every \( n \in \omega \) and every closed set \( H \subset X \) an open set \( \{ n \} \subset H \) so that \( H = \bigcap_{n \in \omega} \overline{\Omega(n,H)} \) and \( G(n,H) \subset G(n,K) \) whenever \( H \subset K \). Every stratifiable space is both monotonically normal and a \( \sigma \)-space [29].

A set \( D \subset X^2 \) is called slim [30] if the intersection of \( D \) with every cross-section \( (\{p\} \times X) \cup (X \times \{p\}) \) is nowhere dense (in this cross section). If \( B \subset A \) then \( \pi B : \{ \alpha \in A : \exists \beta \in B \text{ such that } x_\alpha \in \{ \beta \} \} \) denotes the projection of the product onto a subproduct.

\( cov(M) \) is the minimum cardinality of a family of nowhere dense subsets of \( \mathbb{R} \) that covers \( \mathbb{R} \). A function \( g \in \omega^\omega \) is said to guess the family of functions \( \Phi \subset \omega^\omega \) if for every \( f \in \Phi \), \( f(n) = g(n) \) for infinitely many \( n \). It is known that if \( |\Phi| < cov(M) \), then there is a function \( g \) that guesses \( \Phi \), see [12].

3. Convergence and selective separability

Gruenhage and Sakai [31] observed that separable Fréchet spaces are \( R \)-separable. Basically, there are three “natural ways” to try to strengthen this result: one is to move from Fréchet to radial, another is to move from Fréchet to sequential (or, more generally, to spaces of countable tightness), and yet another is from Fréchet to Whyburn.

3.1. Radial spaces

Proposition 2.

1. A Hausdorff separable radial (with respect to dense subspace) space \( X \) is \( R \)-separable.

2. If, in addition, \( X \) does not have isolated points, then \( X \) is \( GN \)-separable.

Proof. (1) It suffices to show that, given a point \( p \in X \) and a sequence of dense subspaces \( (D_n : n \in \omega) \) one can pick \( p_n \in D_n \) so that \( p \in [p_n : n \in \omega] \). Assume \( p \) is not isolated, otherwise \( p \) is contained in every dense set and the statement we want to prove becomes trivial. Let \( \mathcal{U} \) be a maximal pairwise disjoint family of non-empty open sets in \( X \) such that \( x \notin \mathcal{U} \) for every \( U \in \mathcal{U} \) (*). Since \( X \) is Hausdorff and \( p \) is not isolated, \( \bigcup \mathcal{U} \) is dense in \( X \). Since \( X \) is separable, \( \mathcal{U} \) is countable; enumerate it as \( (U_n : n \in \omega) \). Put \( Y = \bigcup \{ D_n \cap U_n : n \in \omega \} \). Then \( Y \) is dense in \( X \) and thus there is \( S \subseteq Y \) which can be enumerated as
a well-ordered net converging to \( p \). We can assume that \( S \) is of minimal cardinality among all well-ordered nets contained in \( Y \) which converge to \( p \), so \( |S| \) is regular, and then it follows from (\( * \)) and the countability of \( \mathcal{U} \) that \( S \) is a convergent sequence. Again from (\( * \)), \( S \) must have non-empty intersection with infinitely many sets \( D_n \cap U_n \); pick a subsequence \( S' \subseteq S \) that intersects each \( D_n \cap U_n \) at most one point.

For \( n \in \omega \), if \( S' \cap D_n \cap U_n \) is non-empty, let \( p_n \) be the unique point in this intersection. Otherwise pick \( p_n \) arbitrarily. Then the points \( p_n \) are as desired.

(2) Partition \( \{p_n: n \in \omega\} \) from part 1 into pairwise disjoint finite sets \( A_m \) so that each \( A_m \) contains at least one point of \( S' \) and apply Proposition 1, part 4. \[\square\]

The above proposition cannot be extended to pseudoradial spaces because, as we will see below, even separable sequential spaces need not be selectively separable.

**Corollary 3.** Every compact separable radial space has countable \( \pi \)-weight.

**Proof.** For a selectively separable \( X \), \( \delta(X) = \omega \), and for a compact \( X \), \( \delta(X) = \pi w(X) \) [35]. \[\square\]

**Corollary 4.** Every separable compact monotonically normal space has countable \( \pi \)-weight.

**Proof.** Monotonically normal spaces are radial [61]. \[\square\]

While we could not find a reference for Corollary 3, Corollary 4 can be also derived from [28, Corollary 19] (which says that density equals \( \pi \)-weight for monotonically normal compact spaces).

One can wonder whether Corollary 3 can be extended to semiradial spaces. Some mild evidence would be provided by the fact that every compact sequential separable space has countable \( \pi \)-weight (this follows easily from the inequality \( \pi \chi(X) \leq \tau(X) \), for every compact space \( X \) see [33]) and every sequential space is semiradial. However, the answer is consistently negative. Bella [13] showed that the space \( 2^{\omega_1} \) is semiradial if and only if \( p > \omega_1 \). One might hope at least for a consistency result. Indeed, Dow proved [23] that there are models in which every compact separable radial space is Fréchet, but it is still an open problem if there are models in which every compact separable semiradial space is sequential.

In view of Corollary 3, we suggest a weaker form of this problem.

**Question 5.** Is it consistent that every compact separable semiradial space has countable \( \pi \)-weight?

3.2. Sequential spaces and spaces of countable tightness

Strengthening the Barman-Dow result in this direction is in general not possible.

If \( K \) is an infinite subset of \( \omega \), then it is easy to check that the set \( D_K = \bigcup \{k \omega: k \in K\} \) is dense in \( \text{Seq}(\mathcal{F}) \). Moreover, it is also quite easy to realize that, for any choice of a finite set \( F_n \subseteq n \omega \), the set \( \bigcup \{F_n: n < \omega\} \) is closed and nowhere dense in \( \text{Seq}(\mathcal{F}) \). Taking this into account, we see that if \( H_n = \bigcup \{k \omega: n \leq k < \omega\} \), then the sequence of dense sets \( \{H_n: n < \omega\} \) witnesses that the space \( \text{Seq}(\mathcal{F}) \) is never selectively separable.

If as \( \mathcal{F} \) we take the filter of all cofinite subsets of \( \omega \), then \( \text{Seq}(\mathcal{F}) \) turns out to be sequential. Indeed, if \( A \) is a non-closed subset of \( \text{Seq}(\mathcal{F}) \), then \( \text{Seq}(\mathcal{F}) \setminus A \) is not open and so there is some \( s \in \text{Seq}(\mathcal{F}) \setminus A \) such that the set \( \{n: s \sim n \in \text{Seq}(\mathcal{F}) \setminus A\} \) has an infinite complement \( E \). Therefore, \( S = \{s \sim n: n \in E\} \subseteq A \) and we immediately check that \( S \) converges to \( s \).

Thus, there is a countable sequential space which is not selectively separable. However, this space has sequential order \( \omega \). So, what is left open is:

**Question 6.** Is every Hausdorff separable sequential space of finite or countable sequential order selectively separable?

That separable spaces of countable tightness need not be selectively separable is well known. There are many examples such as \( C_p(\text{Irrationals}) \) or even some countable spaces [14,15]. However, adding some restrictions on the character of points or some covering properties we can get positive results. Here are a few of the most interesting.

**Proposition 7.**

(1) [14] If a separable space \( X \) of countable tightness has a dense set of points of character less than \( \omega \), then \( X \) is selectively separable.

(2) A regular countably compact separable space of countable tightness is selectively separable.

(3) [31] More generally, let \( X \) be a regular separable space of countable tightness. If each point is contained in a countably compact set of countable character in \( X \), then \( X \) is selectively separable.

Item (2) follows directly from the fact that a regular countably compact space of countable tightness has countable fan tightness [5]. In order to slightly improve this result we prove a proposition which may be of some independent interest.
Proposition 8. A regular countably compact space $X$ of countable tightness has countable strong fan tightness.

Proof. Let $b \in X$, $A_n \subset X$, $b \in \overline{A_n}$ for all $n \in \omega$. Without loss of generality we assume that the sets $A_n$ are countable. Put $\tilde{X} = \bigcup_{n \in \omega} A_n$. Then $X$ is a regular separable countably compact space of countable tightness; $b \in X$. Fix a base $B$ of neighborhoods of $b$ in $X$ such that $|B| \leq c$. For every $U \in B$ fix an open in $\tilde{X}$ set $V(U)$ such that $b \in V(U) \subset \overline{V(U)} \subset U$. Let $\tilde{X}$ be a space and $\{x_{n,U}: n \in \omega\}$ a limit point, say $su$. Then $su \in V(U) \subset U$.

Let $U \in B$. For every $n \in \omega$ pick $x_{n,U} \in A_n \cap V(U)$. Since $X$ is countably compact the set $\{x_{n,U}: n \in \omega\}$ has a limit point, say $su$. Then $su \in V(U) \subset U$.

As an immediate corollary we get that regular countably compact separable spaces of countable tightness are $\mathcal{R}$-separable. However, we are going to prove a stronger result which is a simultaneous improvement of all parts of Proposition 7. Let $X$ be a space and $x \in X$. Weakening the definition of the cardinal function $h(x, X)$ [26] we denote by $h^*(x, X)$ the smallest cardinal number $k$ such that there exists a countably compact $H \subset X$ with $x \in H$ and $\chi(H, X) = k$.

Theorem 9. Let $X$ be a regular separable space of countable tightness.

(a) If the inequality $h^*(x, X) < \omega$ holds in a dense set of points, then $X$ is selectively separable.

(b) If the inequality $h^*(x, X) < \mathsf{cov} (\mathcal{M})$ holds in a dense set of points, then $X$ is $\mathcal{R}$-separable.

Proof. Without any loss of generality, we may suppose that $X$ does not have isolated points. Let $(D_n: n \in \omega)$ be a sequence of dense subsets of $X$. Since from the hypotheses it follows that any dense set is separable, we may assume that each $D_n$ is countable.

Part a. Let us begin by fixing a countable dense set $C$ such that $h^*(x, X) < \omega$ holds for each $x \in C$. Let $\{J_x: x \in C\}$ be a partition of $\omega$ in infinite sets and fix for each $x \in C$ a countably compact subspace $H_x$ satisfying $x \in H_x$ and $\chi(H_x, X) = \kappa_x < \omega$. Let $\Phi = \prod_{\alpha \in \kappa_x} [D_\alpha]^{<\alpha}$ and for each $\phi \in \Phi$ let $[\phi] = D(\bigcup_{n \in \kappa_x} \phi(n): n \in J_x)$ (here $D(S)$ indicates the derived set of $S$).

Claim a. $x \in \bigcup \{[\phi]: \phi \in \Phi\}$.

Proof of Claim a. Fix an open neighborhood $V$ of $x$. We are going to check that $\overline{V} \cap \bigcup \{[\phi]: \phi \in \Phi\} \neq \emptyset$. Case I. If there exists an infinite set $J \Subset J_x$ such that $V \cap D_n \cap H_x$ is infinite for each $n \in J$, then we may define a function $\psi : \Phi \rightarrow [0, 1]$. Case II. Let $\{J_x: x \in C\}$ be a partition of $\omega$ in infinite sets and fix for each $x \in C$ a countably compact subspace $H_x$ satisfying $x \in H_x$ and $\chi(H_x, X) = \kappa_x < \omega$. Let $\Phi = \prod_{\alpha \in \kappa_x} [D_\alpha]^{<\alpha}$ and for each $\phi \in \Phi$ let $[\phi] = D(\bigcup_{n \in \kappa_x} \phi(n): n \in J_x)$ (here $D(S)$ indicates the derived set of $S$).

Claim b. For every $\phi \in \Phi$ let $\psi : \Phi \rightarrow [0, 1]$ and $\{J_x: x \in C\}$ be a partition of $\omega$ in infinite sets and fix for each $x \in C$ a countably compact subspace $H_x$ satisfying $x \in H_x$ and $\chi(H_x, X) = \kappa_x < \omega$.

Proof of Claim b. In the case when there exists an infinite $J \subset N_{k,U}$ such that $V_U \cap D(n \cap H_k)$ is infinite for each $n \in J$ the argument repeats the similar one from part a. So assume no such $J$ exists and then without loss of generality we assume
that $V_U \cap D_n \cap H_k = \emptyset$ for all $n \in N_{k,U}$. Enumerate $D_n = \{x_k: k \in \omega\}$. Let $\{O_\alpha: \alpha < \kappa_k\}$ (where $\kappa_k < \text{cov}(M)$ be a local base of $H_k$ in $X$. For every $\alpha$ fix a function $f_\alpha: N_{k,U} \to \omega$ defined in such a way that $x_n, f_\alpha(n) \in V_U \cap O_\alpha$ for each $n \in N_{k,U}$. Since $\kappa_k < \text{cov}(M)$ there is a function $g: N_{k,U} \to \omega$ such that for each $\alpha$ the equation $f_\alpha(n) = g(n)$ holds for infinitely many $n$'s. Define $\phi$ by $\phi(n) = g(n)$ for each $n \in N_{k,U}$. This proves the claim. \[\Box\]

Now, using Claim b, for every $U \in B$ fix $\phi_U \in \prod_{n \in N_{k,U}} D_n$ and $z_U \in \overline{V_U} \cap D(\phi(n): n \in N_{k,U})$. Put $Z = \{z_U: U \in B\}$. Then $x \in Z$ and, since $X$ has countable tightness, there is a countable subset, say $Z_0 = \{z_U: k \in \omega\}$, such that $x \in Z_0$. For $k \in \omega$, put $N_{k,U} = N_{k,U_k} \setminus \bigcup_{\alpha < \kappa_k} N_{k,U_\alpha}$. Then the sets $N_{k,U_\alpha}$ ($k \in \omega$) are pairwise disjoint and differ from $N_{k,U_k}$ only by finitely many elements. Let $n \in J_k$. If $n$ belongs to some (then only to one) $N_{k,U_\alpha}$ then put $a_n = a_n(g(n))$. If not, choose $a_n \in D_n$ arbitrarily. Then $x \in [G_n: n \in J_k]$. \[\Box\]

A crucial role in the proof of Theorem 9 as well as in Gruenhage and Sakai's proof of Proposition 7, part 3 of [31] is played by Proposition 7, part 2 (and its variation, Proposition 8). We may then try and generalize it in the following two ways. In one direction we may try to weaken “regular” to Hausdorff and in the other to weaken countably compact to pseudocompact. Unfortunately, we have an answer only for the first case.

It is well known (see [60]) that if $U$ is a free ultrafilter on $\omega$ then the space $\text{Seq}(U)$ is extremally disconnected. So, $X = \text{Seq}(U)$ is a countable Hausdorff zero-dimensional extremally disconnected non-selectively separable space. Now consider its Čech–Stone compactification $\beta X$. Theorem 11 of [59] shows that there exists a strengthening of the topology of $\beta X$ in such a way that the resulting space $Y$ has the following properties:

1. $X$ is a dense subspace of $Y$;
2. $Y$ is locally countable;
3. each closed infinite subset of $Y$ has cardinality $2^\omega$.

So we get:

**Example 10.** There exists a separable countably compact Urysohn space of countable tightness which is not selectively separable.

Moving from countably compact to pseudocompact appears much harder. Indeed, with a lot of effort, Bella and Pavlov [18] constructed a Tychonoff pseudocompact space of countable tightness which does not have countable fan tightness. But such a space has a countable set of isolated points and so it is selectively separable. For these reasons, the next problem sounds very interesting:

**Problem 11.** Find a Tychonoff pseudocompact separable space of countable tightness which is not selectively separable.

We don’t even know the answer to the possibly easier problem which we obtain by dropping separability from the above one.

**Problem 12.** Find a Tychonoff pseudocompact space of countable tightness which does not have countable dense fan tightness.

3.3. Whyburn spaces

Barman and Dow constructed a countable regular maximal space which is not selectively separable [10]. On the other hand, the first author and Yaschenko showed in [21] that every regular maximal space has the Whyburn property. Therefore we get:

**Corollary 13.** There exists a countable regular Whyburn space which is not selectively separable.

Tkachuk and Yaschenko [58] proved that every countably compact Whyburn space is Fréchet. So countably compact Whyburn separable spaces are selectively separable. However pseudocompact Whyburn spaces need not be Fréchet [46], not even if they have countable tightness [19]. So, also in view of Problem 12, we have the following question.

**Question 14.** Suppose $X$ is a pseudocompact Whyburn separable space. Is $X$ selectively separable? What if $X$ has countable tightness?
4. D-separability

A space is called $d$-separable if it contains a $\sigma$-discrete dense subspace. We introduce some selective versions of this property.

**Definition 15.** $X$ is $D$-separable if for every sequence of dense subspaces $(D_n: n \in \omega)$ one can pick discrete sets $F_n \subset D_n$ so that $\bigcup_{n \in \omega} F_n$ is dense in $X$.

$X$ is $DH$-separable if for every sequence of dense subspaces $(D_n: n \in \omega)$ one can pick discrete $F_n \subset D_n$ so that every non-empty open set in $X$ intersects all but finitely many $F_n$'s.

Consider the following games on a space $X$ (as above, $D$ denotes the collection of all dense subspaces of $X$). In the game $G_{\text{dis}}(D,D)$, One picks $D_0 \in D$, then Two picks a discrete $F_0 \subset D_0$, then One picks $D_1 \in D$, etc. Two wins if $\bigcup_{n \in \omega} F_n$ is dense in $X$. The game $G_{\text{dis,H}}(D,D)$ is similar, only Two wins if every non-empty open set in $X$ intersects all but finitely many $F_n$'s.

**Definition 16.** $X$ is $D^+$-separable if Two has a winning strategy in $G_{\text{dis}}(D,D)$. Say that $X$ is $DH^+$-separable if Two has a winning strategy in $G_{\text{dis,H}}(D,D)$.

The following implications (where, for example, SS denotes selectively separable and D denotes D-separable) are straightforward.

```
SS^+  SS  H
   |   |
   |   |
   |   |
D^+  D  DH
```

We know that most of the arrows in the diagram cannot be reversed. To see that an arrow pointing from a selective separability-type property (top row of the diagram) to a selective $d$-separability-type property (bottom row and center of the diagram) cannot be reversed simply take any metric non-separable space. In some cases we will be able to improve this and obtain a separable counterexample. For instance, Example 25 is a countable space showing that $D \nrightarrow SS$. Any countable submaximal space shows that $D \nrightarrow D^+$ (see Corollary 69 and Theorem 76). However, the relationship between $D$ and $DH$ is not well-understood, and hence we have the following open problem

**Problem 17.**

(1) Find an example of a (countable) $D$-separable non-$DH$-separable space.
(2) Find an example of a (countable) $D^+$-separable non-$DH^+$-separable space.
(3) Find an example of a (countable) $DH$-separable, non-$DH^+$-separable space.

4.1. Which spaces are $D$-separable?

**Proposition 18.**

(1) Every space with a $\sigma$-disjoint $\pi$-base is $DH^+$-separable.
(2) Every space with a $\sigma$-locally finite $\pi$-base is $DH^+$-separable.
(3) Every $T_1$ space with a $\sigma$-closure preserving $\pi$-base is $DH^+$-separable.

**Lemma 19.** If $\mathcal{U}$ and $\mathcal{V}$ are pairwise disjoint families of non-empty sets in $X$, then there is a pairwise disjoint family $\mathcal{W}$ of non-empty sets in $X$ such that:

(1) Every element of $\mathcal{U}$ contains an element of $\mathcal{W}$;
(2) Every element of $\mathcal{V}$ contains an element of $\mathcal{W}$;
(3) Every element of $\mathcal{W}$ is contained in some element of $\mathcal{U} \cup \mathcal{V}$.

**Proof.** Put $\mathcal{W}_0 = \{ U \cap V : U \in \mathcal{U}, V \in \mathcal{V} \text{ and } U \cap V \neq \emptyset \}$ and $\mathcal{W} = \mathcal{W}_0 \cup \{ U \in \mathcal{U} : \text{ there is no } W \in \mathcal{W} \text{ with } U \supset W \} \cup \{ V \in \mathcal{V} : \text{ there is no } W \in \mathcal{W} \text{ with } V \supset W \}$. □
Lemma 20. If \( \mathcal{U} \) is a locally finite family of non-empty open sets in a space \( X \) and \( D \) is a dense subspace of \( X \), then there is a discrete set \( A \) such that \( A \cap U \neq \emptyset \) for every non-empty \( U \in \mathcal{U} \).

Proof. For every \( U \in \mathcal{U} \) pick a point \( p_U \in U \cap D \). The local finiteness of \( \mathcal{U} \) implies that every \( x \in B \) is contained in a set \( V_x \subset B \) which is finite and open in \( B \). We choose as \( V_x \) an open set of minimum size. Let us call a point \( x \in B \) good if for each \( y \in V_x \) we have \( V_y = V_x \). It is clear that for each \( x \in B \) there is a good point \( y \) such that \( V_y \subset V_x \). Moreover, if \( y \) and \( z \) are good, then either \( V_y = V_z \) or \( V_y \cap V_z = \emptyset \). now, fix a well ordering on \( B \) and for each good point \( y \in B \) let \( a(y) = \min V_y \). The set \( A \) of all such \( a(y) \) is discrete and \( A \) intersects every element of \( \mathcal{U} \). \( \square \)

Proof of Proposition 18. (1) Let \( \mathcal{U} = \bigcup_{n \in \omega} \mathcal{U}_n \) (where each \( \mathcal{U}_n \) is pairwise disjoint and consists of non-empty sets) be a \( \pi \)-base of \( X \). Applying Lemma 19 inductively one gets pairwise disjoint families \( \mathcal{V}_n \) of non-empty open sets such that whenever \( m \leq n \), every element of \( \mathcal{V}_m \) is contained in some element of \( \mathcal{U}_n \), and every element of \( \mathcal{U}_m \) contains an element of \( \mathcal{V}_n \). At the \( n \)th inning One chooses a dense subspace \( D_n \) and Two picks for every \( U \in \mathcal{U}_n \) a point \( p_{D_n,U} \in U \) and sets \( F_n = \{ p_{D_n,U} : U \in \mathcal{U}_n \} \).

(2) Let \( \mathcal{U} = \bigcup_{n \in \omega} \mathcal{U}_n \) (where each \( \mathcal{U}_n \) is locally finite) be a \( \pi \)-base of \( X \). Put \( \mathcal{V}_n = \bigcup_{m \leq n} \mathcal{U}_m \). At the \( n \)th inning One chooses a dense subspace \( D_n \) and Two uses Lemma 20 to find a discrete subspace \( F_n \subset D_n \) which meets every element of \( \mathcal{V}_n \).

(3) Let \( \mathcal{U} = \bigcup_{n \in \omega} \mathcal{U}_n \) (where each \( \mathcal{U}_n \) is closure preserving) be a \( \pi \)-base of \( X \). At the \( n \)th inning One chooses a dense subspace \( D_n \) and Two picks for every \( U \in \mathcal{U}_n \) a point \( p_{D_n,U} \in U \) and sets \( F_n = \{ p_{D_n,U} : U \in \mathcal{U}_n \} \). The family \( \{ p_{D_n,U} : U \in \mathcal{U}_n \} \) is closure preserving. So, since \( X \) is T1, the set \( F_n = \{ p_{D_n,U} : U \in \mathcal{U}_n \} \) is discrete. \( \square \)

In particular, every metrizable space (or, more generally, every T1 M1-space (= a space with a \( \sigma \)-closure preserving base) is DH\(^+\)-separable. Actually, more than that is true: every M3 (= stratifiable) space is DH\(^+\)-separable, as we will see below.

Shapirovskii showed [53] that every space with a \( \sigma \)-point finite base is d-separable.

Question 21. Is every space with a \( \sigma \)-point finite base D-separable? If so, does it even satisfy the stronger properties from Definitions 15 and 16?

Proposition 22. Let \( X \) be a collectionwise Hausdorff discretely generated space with a \( \sigma \)-closed discrete dense set. Then \( X \) is DH\(^+\)-separable.

Proof. Let \( H = \bigcup_{n \in \omega} H_n \) be dense in \( X \) (where each \( H_n \) is closed and discrete). Without loss of generality we assume that \( H_n \subset H_m \) whenever \( n < m \). For every \( n \), fix a pairwise disjoint open expansion \( \{U_{n,X} : x \in H_n\} \) of \( H_n \).

At the \( n \)th inning One picks a dense \( D_n \subset X \). Then Two, for every \( x \in H_n \), picks a discrete \( F_{n,X} \subset D_n \cap U_{n,X} \) such that \( x \in F_{n,X} \) and sets \( F_n = \bigcup_{x \in H_n} F_{n,X} \). \( \square \)

Corollary 23. Every monotonically normal \( \sigma \)-space is DH\(^+\)-separable.

Proof. Every \( \sigma \)-space has a \( \sigma \)-closed discrete dense set, and every monotonically normal space is both collectionwise Hausdorff (see [29]) and discretely generated [24]. \( \square \)

Corollary 24. Every stratifiable space is DH\(^+\)-separable.

Proof. Because a stratifiable space is both monotonically normal and a \( \sigma \)-space. \( \square \)

Tsaban asked us in private communication whether a separable D-separable space has to be selectively separable. This can be disproved by taking the space \( Seq(\mathcal{F}) \) where \( \mathcal{F} \) is any ultrafilter on \( \omega \). Indeed, this space is countable, and hence it is trivially a \( \sigma \)-space. Moreover, it is monotonically normal by Theorem 3.2 of [37]. If \( \mathcal{F} \) is a Ramsey ultrafilter (which exists, for example, if one assumes CH), then \( Seq(\mathcal{F}) \) is even a topological group (see [60]). So we arrive to the following theorem:

Theorem 25. There is a countable DH\(^+\)-separable space \( X \) which is not selectively separable. Under CH the space \( X \) can even be taken to be a topological group.

Yet the following is still unknown.

\[ ^2 \text{In the T1 case, the proof is trivial: just pick a point in every non-empty element of } \mathcal{U}. \text{But the statement is valid without any assumption on separation.} \]
Question 26. Is there a compact separable $D$-separable non-selectively separable space?

Monotone normality alone does not imply $D$-separability. Indeed, it suffices to consider a Suslin Line $\mathbb{L}$. $\mathbb{L}$ is monotone normal because it is linearly ordered, and it cannot even have a $\sigma$-discrete dense set because every discrete set in $\mathbb{L}$ is countable, but $\mathbb{L}$ is not separable. Moreover, it is easy to see that for linearly ordered spaces, the three properties: $d$-separability, $D$-separability and having a $\sigma$-discrete $\pi$-base, are equivalent. This motivates the following question:

Question 27. Is it true that a monotonically normal space is $D$-separable if and only if it is $d$-separable?

We conclude the section with a partial positive result. The principal tool in the proof of it is a theorem by Gartside stating that $\pi w(X) = d(X) = hd(X)$ for $X$ having a monotonically normal compactification [28]:

Theorem 28. Suppose a space $X$ has a monotonically normal compactification. Then the following conditions are equivalent:

1. $X$ is $d$-separable;
2. $X$ is $D$-separable;
3. $X$ has a $\sigma$-disjoint $\pi$-base.

Proof. Of course it is enough to prove $(1) \Rightarrow (3)$. Call a non-empty open set homogeneous in $\pi$-weight ($h\pi w$ for short) if for every non-empty open $V \subseteq U$, $\pi w(V) = \pi w(U)$. It is clear that every non-empty open set contains an $h\pi w$ set and thus in every topological space $X$ one can find a pairwise disjoint family of $h\pi w$ sets $\mathcal{U}$ such that $X = \bigcup \mathcal{U}$. Note that $X$ is $d$-separable, or is $D$-separable, or is a space with a $\sigma$-disjoint $\pi$-base iff so is every element of $\mathcal{U}$. Moreover, if $X$ has a monotonically normal compactification, then also every element of $\mathcal{U}$ does. So, without loss of generality we can assume that $X$ itself is $h\pi w$.

So let $X$ be an $h\pi w$ space with a monotonically normal compactification and monotone normality operator $H$, and let $D = \bigcup_{n \in \omega} D_n$ be dense in $X$ where each $D_n$ is discrete. Without loss of generality, we can assume that the sets $D_n$ are pairwise disjoint. Let $\mathcal{P}$ be a $\pi$-base of $X$ of cardinality $\pi w(X) = \kappa$. By Gartside’s theorem (applied to the subspace $U$) and the fact that $X$ is $h\pi w$, we have that $|U \cap D| = \kappa$ for every $U \in \mathcal{P}$. Then, enumerating $\mathcal{P}$ and $D$, one easily defines an injection $f : \mathcal{P} \rightarrow D$ such that $f(U) \in U$ for every $U \in \mathcal{P}$. For $n \in \omega$, put $\mathcal{P}_n = \{ U \in \mathcal{P} : f(U) \in D_n \}$ and let $\{ V(f(U)) : U \in \mathcal{P}_n \}$ be a family of open sets such that $V(f(U)) \cap D_n = \{ f(U) \}$ and $V(f(U)) \subset U$ for every $U \in \mathcal{P}_n$. For $U \in \mathcal{P}_n$, put $U^*' = H(f(U), V(f(U)))$. By the properties of the monotone normality operator $H$, the family $\mathcal{P}_n^* = \{ U^* : U \in \mathcal{P}_n \}$ is pairwise disjoint and hence $\bigcup_{n < \omega} \mathcal{P}_n^*$ is a $\sigma$-disjoint $\pi$-base for $X$. \qed

4.2. Subspaces, unions

The following proposition has a straightforward proof:

Proposition 29.

1. Every open subspace, as well as every dense subspace, of a space with one of the properties from Definitions 15 and 16 has the same property.
2. If $X$ has an open dense subspace with one of the properties from Definitions 15 and 16 then $X$ has the same property.
3. A discrete sum of spaces with one of the properties from Definitions 15 and 16 has the same property.

It was shown in [31] that selective separability, $R$-separability and $GN$-separability are preserved by finite unions (to see that this is not immediate, it might be enough to mention that the question about $H$-separability remains open, and that $SS^+$ is not finitely additive [11]).

Proposition 30. A locally finite union of $D$-separable spaces is $D$-separable.

We will start by proving that finite unions of $D$-separable spaces are $D$-separable. The proof is a modification of the proof that selective separability is preserved by finite unions (see [31]).

Lemma 31. The union of two $D$-separable spaces is $D$-separable.

Proof. Let $X = A \cup B$ where $A$ and $B$ are $D$-separable and let $\{ D_n : n \in \omega \} \subset X$ be a sequence of dense sets. Let $U_n = X \setminus (\bigcup_{i \geq n} D_i) \cap A \cup \bigcup_{j < n} U_j$ and $U = \bigcup_{n \in \omega} U_n$. Then the sets $U_n$ are open in $X$ and pairwise disjoint.

Claim 1. For each $i \geq n$ the set $D_i \cap B \cap U_n$ is dense in $U_n$. 

Proof of Claim 1. Because $U_n$ is open, $D_i$ is dense in $X$, $X = A \cup B$, and for $i \geq n$, $D_i \cap U_n \cap A = \emptyset$. □

Claim 2. There are discrete $G_n \subseteq D_n$ such that $\bigcup_{n \in \omega} G_n$ is dense in $U$.

Proof of Claim 2. By Proposition 29, part 1, the subspace $B \cap U_n$ is $D$-separable for every $n \in \omega$. Therefore there are discrete $G^n_i \subseteq D_i \cap B \cap U_n$ for every $i \geq n$ such that $\bigcup_{i \geq n} G^n_i$ is dense in $B \cap U_n$ and hence in $U_n$ (because $B \cap U_n$ is dense in $U_n$). Let now $G_i = \bigcup_{n \in \omega} G^n_i$. Since $G^n_i \subseteq U_n$ for every $n \leq i$ and $(U_n; n \leq i)$ is a family of pairwise disjoint open sets we have that $G_i$ is a discrete subset of $D_i$. Moreover $\bigcup_{i \in \omega} G_i$ is dense in $U_n$ for every $n \in \omega$, and hence in $U$. □

Let now $V = X \setminus U$. We claim that $(\bigcup_{i \in \omega} D_i) \cap A$ is dense in $V$, and hence also in $A \cap V$. Indeed, if $x \in V$, then $x \notin U$, so $x \notin U_n$ and $x \notin \bigcup_{i < n} U_i$ for every $n \in \omega$, which together imply that $x \in (\bigcup_{i \geq n} D_i) \cap A$ for every $n \in \omega$.

Now $A \cap V$ is $D$-separable, so there are discrete $H_n \subseteq (\bigcup_{i \geq n} D_i) \cap A$ so that $\bigcup_{n \in \omega} H_n$ is dense in $V \cap A$ and hence in $V$.

For each $x \in H_n$ let $i_n(x) \in \omega \setminus n$ be such that $x \in D_{i_n(x)}$. Let $K_i = \{x : \exists n \in \omega \ (x \in H_n \land i_n(x) = i)\}$. Then $K_i$ is a discrete subset of $D_i$ and $\bigcup_{i \in \omega} K_i = \bigcup_{n \in \omega} H_n$, hence it is dense in $V$. Thus, if $G_n$ is as in Claim 2, then $\bigcup_{n \in \omega} (G_n \cup K_n)$ is dense in $X$ and each $G_n \cup K_n$ is discrete since $G_n \subseteq U$, $K_n \subseteq X \setminus U$ and $G_n$ and $K_n$ are both discrete. So $X$ is $D$-separable. □

Proof of Proposition 30. First of all, by induction, Lemma 31 can be extended to any finite union.

Now, let $X = \bigcup Y$ be a locally finite union, and let each $Y \in \mathcal{Y}$ be $D$-separable. For $n \geq 1$ put $X_n = \{x \in X :$ there is a neighborhood $U$ of $x$ such that $|(Y \in \mathcal{Y} : Y \cap U \neq \emptyset)| \leq n\}$. Then the sets $X_n$ are open in $X$, and $X = \bigcup_{n \in \omega} X_n$.

Put $Z_1 = X_1$. For $n > 1$, put $Z_n = X_n \setminus \bigcup_{i=1}^{n-1} Z_i$. Then the sets $Z_n$ are open in $X$, pairwise disjoint, and $X = \bigcup_{n \geq 1} Z_n$. Further, each $Z_n$ is a discrete union $Z_n = \bigcup Z_n A; A \subseteq \mathcal{Y}; |A| = n$ where $Z_{nA} = \{x \in Z_n :$ there is a neighborhood $U$ of $x$ such that $|(Y \in \mathcal{Y} : Y \cap U \neq \emptyset)| = n\}$. Finally, $X = \bigcup Z_{nA}; n \geq 1, A \subseteq \mathcal{Y}; |A| = n$ where the sets $Z_{nA}$ are open in $X$ and pairwise disjoint.

By Lemma 31 (extended to arbitrary finite unions) each $Z_{nA}$ is $D$-separable; hence by Proposition 29, part 3, so is $\bigcup Z_{nA}; n \geq 1, A \subseteq \mathcal{Y}; |A| = n$, hence by part 2 of the same proposition so is $X$. □

Question 32. Is every (locally) finite union of $DH$-separable spaces again $DH$-separable?

4.3. Products

The following two beautiful results witness how well-behaved $d$-separability is with respect to products.

Theorem 33. (Arhangel’skii [3]) Any product of $d$-separable spaces is $d$-separable.

Theorem 34. (Juhász and Szentmiklóssy [38]) For every $T_1$-space $X$, $X^{\omega(X)}$ is $d$-separable.

Theorems 33 and 34 imply two obvious corollaries:

Corollary 35.

1. For every $T_1$-space $X$ there is $\kappa(X)$ such that for every $\kappa \geq \kappa(X)$, $X^{\kappa}$ is $d$-separable.

2. For every $T_1$-space $X$ there is a $T_1$-space $Z$ such that the product $X \times Z$ is $d$-separable. If $X$ is Tychonoff, then so is $Z$.

It is natural to ask if $D$-separability is (finitely or infinitely) productive and if the analogue of Corollary 35 is true for $D$-separability. It turns out that the product of two $D$-separable spaces does not have to be $D$-separable, and that for part 2 of Corollary 35 the answer is affirmative while for part 1 the situation is almost the opposite.

Theorem 36. For every Tychonoff space $X$ with $|X| > 1$ and every $\kappa$ there is $\kappa' \geq \kappa$ such that $X^{\kappa'}$ is not $D$-separable.

Theorem 37. For every space $X$ there is a Tychonoff space $Z$ such that $X \times Z$ is $DH^+$-separable.

But before proving the above theorems let’s examine the case of finite products.

Theorem 38. Let $X$ be a $D$-separable space (or a space having another property from Definitions 15 and 16) and $Y$ be a space having a $\sigma$-disjoint $\pi$-base. Then $X \times Y$ is $D$-separable (or has the corresponding property).

Proof. (For $D$-separability) Let $B = \bigcup_{k \in \omega} B_k$ be a $\sigma$-disjoint $\pi$-base for $Y$ and let $\{D_k : k < \omega\}$ be a countable sequence of dense subsets of $X \times Y$. Let $\{B^\alpha_n : \alpha < \tau_n\}$ enumerate $B_0$ and $\{A_n : n < \omega\}$ be a partition of $\omega$. Observe that the set $\pi_X(D_k \cap \pi_Y^{-1}(B^\alpha_n))$ is dense in $X$ for every $k \in A_n$ and for every $\alpha \in \tau_n$. Fix $\alpha \in \tau_n$. Then for every $k \in A_n$ we can find
a discrete set $E^a_k \subseteq \mathcal{P}(D_k \cap \pi_2^{-1}(B^a_k))$ such that $\bigcup_{c \in A_c} E^{a,c}_k$ is dense in $X$. For every $x \in E^a_k$ pick a point $f(x) \in \pi_2^{-1}(x)$ such that $\pi_2(f(x)) \in B^a_k$ and set $F_k = \{f(x) : x \in E^a_k\}$ and $F_k = \bigcup_{\alpha \in \omega^*} F^a_k$. Then $F_k$ is discrete. In fact, let $(x, y) \in F_k$. Then $(x, y) \in F^a_k$ for some $\alpha < k$, so $(x, y) = f(x)$ for some $y \in B^a_k$. Now, since $E^a_k$ is discrete in $X$ there is an open $V \subseteq X$ such that $V \cap E^a_k = \{x\}$. Finally, observe that, since $E^a_k$ is a disjoint family $(V \times B^a_k) \cap F_k = \{(x, y)\}$, which shows that $F_k$ is discrete. Moreover $F_k \subseteq D_k$ and $\bigcup_{\alpha \in \omega^*} F^a_k$ is dense in $X \times Y$, which proves that $X \times Y$ is $D$-separable.

The proofs for the other properties differ only by minor changes. □

Example 39. [CH] The product of two countable selectively separable spaces need not be $D$-separable.

Proof. Let $X$ be a selectively separable countable maximal regular crowded space such that $X^2$ has no dense slim set, see [31]. Let us check that the proof from [31] that $X^2$ is not selectively separable provides more: that $X^2$ is not $D$-separable. Enumerate $X = \{x_i : i \in \omega\}$. For every $n \in \omega$ let $D_n = \{(x, y) : x, y \notin \{x_i : i \leq n\}\}$. Then $\{D_i : i \in \omega\}$ is a sequence of dense sets in $X$. Let $E_n \subseteq D_n$ be a discrete set. Then $\bigcup_{n \in \omega} E_n$ meets every cross-section in a finite union of discrete sets. Now, in a crowded space, every discrete set is nowhere dense and finite unions of nowhere dense sets are nowhere dense. Therefore $\bigcup_{n \in \omega} E_n$ cannot be dense in $X^2$, which proves that $X^2$ is not $D$-separable. □

As a byproduct we get that under CH there exists a countable non-$D$-separable space. Every countable $D$-forced dense subspace of $2^\omega$ (see [36]) provides a ZFC example of a countable space which is not $D$-separable. We owe this observation to Juhász and Soukup. We will now offer a direct construction of such a space. Small modifications of our construction will provide an argument to show that any space has a non-$D$-separable power.

Example 40. There is a dense countable subset $X \subseteq 2^\omega$ such that $X$ is not $D$-separable.

Lemma 41.

(1) For every countable subset $S \subseteq 2^\omega$, there is $\alpha < \epsilon$ such that $\pi_{\{\alpha, \omega\}}|S$ is a bijection.

(2) If a countable subset $S \subseteq 2^\omega$ is $\alpha$-discrete, then this can be witnessed by a projection to some initial face in $2^\omega$. That is, if $S = \bigcup_{n \in \omega} S_n$ where each $S_n$ is countable and discrete, then there is $\alpha < \epsilon$ such that $\pi_{\{\alpha, \omega\}}(S_n)$ is discrete for each $n$, and $\pi_{\{\alpha, \omega\}}|S$ is injective.

Proof. (1) Pick countably many standard neighborhoods of points of $S$ separating points of $S$ and use the fact that $\text{cf}(\epsilon) > \omega$.

(2) Pick standard neighborhoods of points of $S$ witnessing $\sigma$-discreteness. □

Construction of Example 40. First, it is easy to construct pairwise disjoint dense countable subspaces $Y_n, n \in \omega$ in $2^\omega$ such that for every two distinct $y_1, y_2 \in \bigcup_{n \in \omega} Y_n$, the set $I_{y_1, y_2} = \{\alpha < \epsilon : y_1(\alpha) \neq y_2(\alpha)\}$ has cardinality $\epsilon$. Using this, one can partition $\epsilon$ as $\epsilon = \bigcup A_\alpha \cap \alpha \subseteq Y$ so that each $A_\alpha$ has cardinality $\epsilon$, and for every two distinct $y_1, y_2 \in Y, C_{y_1, y_2} \subseteq I_{y_1, y_2}$. Next, by induction on $0 < \alpha < \epsilon$, we will construct countable subspaces $Z_\alpha \subseteq 2^\omega$ that will take the form $Z_\alpha = \{y_\alpha : y \in Y\}$. We also denote $Z_{\alpha, n} = \{y_\alpha : y \in Y_n\}$, so $Z_\alpha = \bigcup_{n \in \omega} Z_{\alpha, n}$. The points of $Z_{\alpha, n}$ are going to have the following property: if $0 \leq \gamma < \alpha < \beta < \epsilon$, then for all $y \in Y, y_\beta(\gamma) = y_\alpha(\gamma)$ (**). To start the induction, we set $Z_0 = Y$, that is $y_0 = y$ for all $y \in Y$.

Now let $0 < \alpha < \epsilon$, and suppose $X_n$ has been defined for all $\gamma < \alpha$. Let $y \in Y$. To define the corresponding point $y_\alpha \in Z_\alpha$, we have to define $y_\alpha(\gamma)$ for all $\gamma, 0 \leq \gamma < \epsilon$. If $0 \leq \gamma < \alpha$, then we set $y_\alpha(\gamma) = y_\gamma(\gamma)$ (and thus condition (**) continues to hold).

To define $y_\alpha(\alpha)$ we need some auxiliary notation. By the previous, we have in fact defined $\pi_{\{\alpha, \omega\}}(y_\alpha)$ for all $y \in Y$. For a subset $B \subseteq Y$, set $B_{\leftarrow \alpha} = \{\pi_{\{\alpha, \omega\}}(y_\alpha) : y \in B\} \subseteq 2^{\{\alpha, \omega\}}$. We have $A \subseteq C_A$ for some $A \subseteq Y$. If all the following conditions hold:

• (1) $A$ is infinite,

• (2) the mapping $A \rightarrow A_{\leftarrow \alpha}$ given by $y \mapsto \pi_{\{\alpha, \omega\}}(y_\alpha)$ is a bijection,

• (3) for every $n \in \omega$, $(A \cap Y_n)_{\leftarrow \alpha}$ is discrete,

then we set $y_\alpha(\alpha) = 0$ for all $y \in A$. Otherwise we set $y_\alpha(\alpha) = y(\alpha)$.

Finally, for all $y$ with $\alpha < \gamma < \epsilon$, we set $y_\gamma(\gamma) = y(\gamma)$. This concludes the construction of $Z_\alpha$.

Now we define the countable subspace $X \subseteq 2^\omega, X = \{y : y \in Y \}$ by setting $\tilde{y}(\alpha) = y_\alpha(\alpha)$ for all $y \in Y$. It follows from (**), that $\tilde{y}(\alpha)$ is a bijection whenever $0 \leq \gamma < \alpha < \epsilon$. For $n \in \omega$, we set $X_n = \{\tilde{y} : y \in Y_n\}$, thus we have $X = \bigcup_{n \in \omega} X_n$.

Claim 1. The mapping $y \mapsto \tilde{y}$ from $Y$ onto $X$ is a bijection.

Proof of Claim 1. Indeed, if $y_1, y_2$ be distinct elements of $Y$, then by our construction, since by (**) $C_{y_1, y_2} \subseteq I_{y_1, y_2}$, we have $\tilde{y}_1(\alpha) = y_1(\alpha) \neq y_2(\alpha) = \tilde{y}_2(\alpha)$ for every $\alpha \in C_{y_1, y_2}$. □
Claim 2. Each $X_n$ is dense in $2^c$ (and thus in $X$).

Proof of Claim 2. Indeed, let $F \subset c$ be finite, and let $\varphi \in 2^F$. We have to find $\tilde{y} \in X_n$ such that $\tilde{y}|_F = \varphi$. For each $i \in F$, there is $A_i \subset Y$ such that $i \in C_{A_i}$. Put $A = \{A_i: i \in F \text{ and conditions (1), (2), (3) were satisfied when the } i\text{th coordinates of the points of } X_i \text{ were defined}\}$. Pick $\alpha^*$ with $\max(F) < \alpha^* < c$. Using Lemma 41(1), we can assume that $\pi|_{[0,\alpha^*]}$ is a bijection. Then $T = \pi|_{[0,\alpha^*]}(\bigcup (A_i \cap Y_n: A_i \in A))$ is a finite union of discrete subspaces of $2^{[0,\alpha^*]}$, and thus $T$ is nowhere dense in $2^{[0,\alpha^*]}$. So $T' = \pi|_{[0,\alpha^*]}(Y_n \setminus T)$ is dense in $2^{[0,\alpha^*]}$. Pick $t \in T'$ with $t|_F = \varphi$ and $y \in Y_n$ with $\pi|_{[0,\alpha^*]}(y) = t$. Then $y \in X_n$, and $\tilde{y}|_F = y|_F = t|_F = \varphi$. □

Claim 3. For any choice of discrete $S_n \subset X_n, n \in \omega$, the set $S = \bigcup_{n \in \omega} S_n$ is not dense in $2^c$ (and thus not dense in $X$).

Proof of Claim 3. By Lemma 41(2), there is $\alpha^+ < c$ such that $\pi|_{[0,\alpha^*]}(S_n)$ is discrete for each $n$, and $\pi|_{[0,\alpha^*]}|_S$ is injective. Put $A = \{y \in Y: y \in S\}$. Pick $\alpha^{**} \in C_A$ so that $\alpha^{**} \geq \alpha^+$. Then $\tilde{y}(\alpha^{**}) = 0$ for every $y \in S$, and thus $S$ is not dense in $2^c$. □

Corollary 42. $2^c$ is not $D$-separable.

Question 43. What is $\text{cds} = \min\{\tau: 2^\tau \text{ contains a dense countable subspace which is not } D\text{-separable}\}$?

Question 44. What is $\text{dss} = \min\{\tau: 2^\tau \text{ is not } D\text{-separable}\}$?

Question 45. Is $\text{cds} = \text{dss}$?

Question 46. Is it true that for every separable Tychonoff space $X$ there is a separable Tychonoff space $Y$ such that $X \times Y$ is $D$-separable?

4.3.1. Proof of Theorem 36

It suffices to show that ($\dag$) for every Tychonoff $X$ there is a $\tau$ such that $X^\tau$ is not $D$-separable. Indeed, applying ($\dag$) to $X' = X^\tau$ we get the original statement of the theorem. In the case of a finite $X$, $\tau = c$ works by an easy modification of the argument from Example 40, so we assume $\lambda = |X|$ is infinite. Next, if $D$ is dense in $X$ and $D^\tau$ is not $D$-separable, then neither is $X^\tau$. So we can pass from $X$ to a dense subspace of minimal cardinality and thus assume $|X| = d(X)$ when proving the following.

Theorem 47. For every Tychonoff space $X$, $2^{\kappa(X)}$ is not $D$-separable.

Proof. The argument is parallel to one from Example 40, so we will omit some details. Fix a point $x_0 \in X$.

Let $\tau = 2^\lambda$ (where $\lambda = |X| = d(X)$). Since $\text{cf}(\tau) > \lambda$ we get the following:

Lemma 48.

(1) For every subset $S \subset X^\tau$, such that $|S| \leq \lambda$ there is a $\alpha < \tau$ such that $\pi|_{[0,\alpha]}|_S$ is a bijection.
(2) If a subset $S \subset 2^\tau$ such that $|S| \leq \lambda$ is $\sigma$-discrete, then this can be witnessed by a projection to some initial face in $X^\tau$. That is, if $S = \bigcup_{n \in \omega} S_n$ where each $S_n$ is discrete, then there is a $\alpha < \tau$ such that $\pi|_{[0,\alpha]}(S_n)$ is discrete for each $n$, and $\pi|_{[0,\alpha]}|_S$ is injective.

The routine proof of the next lemma is omitted.

Lemma 49. There exist pairwise disjoint dense subspaces $Y_n, n \in \omega$ in $X^\tau$ such that $|Y_n| \leq \lambda$ and for every two distinct $y_1, y_2 \in Y = \bigcup_{n \in \omega} Y_n$, the set $I_{y_1, y_2} = \{\alpha < \tau: y_1(\alpha) \neq y_2(\alpha)\}$ has cardinality $\tau$.

Using this, one can partition $\tau$ as $\tau = \bigcup (C_A: A \subset Y)$ so that each $C_A$ has cardinality $\tau$, and for every two distinct $y_1, y_2 \in Y$, $C_{I_{y_1, y_2}} \subset I_{y_1, y_2}$. Next, by induction on $0 \leq \alpha < \tau$, we will construct $\lambda$-sized subspaces $Z_\alpha \subset X^\tau$ that will take the form $Z_\alpha = \{y_\alpha: y \in Y\}$. We also denote $Z_{\alpha \cdots} = \{y_\alpha: y \in Y\}$, so $Z_\alpha = \bigcup_{n \in \omega} Z_{\alpha \cdots}$. The points of $Z_\alpha$ are going to have the following property: if $0 \leq \gamma < \alpha < \beta < \tau$ then for all $y \in Z_\alpha$, $y_\beta(\gamma) = y_\alpha(\gamma)$ ($\ast\ast$).

To start the induction, we set $Z_0 = Y$, that is $y_0 = y$ for all $y \in Y$.

Now let $0 < \alpha < \tau$, and suppose $Y_\gamma$s have been defined for all $\gamma < \alpha$. Let $y \in Y$. To define the corresponding point $y_\alpha \in Z_\alpha$, we have to define $y_\alpha(\gamma)$ for all $\gamma, 0 \leq \gamma < \alpha$. If $0 \leq \gamma < \alpha$, then we set $y_\alpha(\gamma) = y_\gamma(\gamma)$ (and thus condition $\ast\ast$ continues to hold).
To define $y_\alpha(\alpha)$ we need some auxiliary notation. By the previous, we have in fact defined $\pi_{[0,\alpha)}(y_\alpha)$ for all $y \in Y$. For a subset $B \subseteq Y$, set $B_{<\alpha} = \{\pi_{[0,\alpha)}(y_\alpha): y \in B\} \subseteq X^{[0,\alpha)}$. We have $\alpha \in C_A$ for some $A \subseteq Y$. If all the following conditions hold:

- (1) $A$ is infinite,
- (2) the mapping $A \rightarrow A_{<\alpha}$ given by $y \mapsto \pi_{[0,\alpha)}(y_\alpha)$ is a bijection,
- (3) for every $n \in \omega$, $(A \cap Y_n)_{<\alpha}$ is discrete,

then we set $y_\alpha(\alpha) = x_0$ for all $y \in A$. Otherwise we set $y_\alpha(\alpha) = y(\alpha)$.

Finally, for all $y$ with $\alpha < \gamma < \tau$, we set $y(\gamma) = y(\gamma)$. This concludes the construction of $Z_\alpha$.

So we have $Z_\alpha$ satisfying (**) for all $\alpha < \tau$. Now we define the subspace $\tilde{Y} \subseteq X^\tau$ by $\tilde{Y} = \{\tilde{y}: y \in Y\}$ where $\tilde{y}(\alpha) = y_\alpha(\alpha)$ for all $y \in Y$. It follows from (**) that $\tilde{y}(\gamma) = y_\alpha(\gamma)$ whenever $0 \leq \gamma \leq \alpha < \tau$. For $n \in \omega$, we set $\tilde{Y}_n = \tilde{[y}: y \in Y_n\}$, thus we have $\tilde{Y} = \bigcup_{n \in \omega} \tilde{Y}_n$.

Claim 1. The mapping $y \mapsto \tilde{y}$ from $Y$ onto $\tilde{Y}$ is a bijection.

Proof of Claim 1. Indeed, if $y_1, y_2$ be distinct elements of $Y$, then by our construction, since by (**) $C_{\{y_1, y_2\}} \subseteq I_{y_1, y_2}$, we have $y_1(\alpha) = y_2(\alpha)$ for every $\alpha \in C_{\{y_1, y_2\}}$. □

Claim 2. Each $\tilde{Y}_n$ is dense in $X^\tau$ (and thus in $\tilde{Y}$).

Proof of Claim 2. Indeed, let $F \subseteq \tau$ be finite, and let $\varphi \in (T \setminus \{\emptyset\})^F$ (where $T$ is the topology of $X$). We have to find $\tilde{y} \in \tilde{Y}_n$ such that $\varphi$ and $\tilde{y}$ are compatible for all $i \in F$. For each $i \in F$, there is $A_i \subseteq Y$ such that $\varphi(i) = \emptyset$. Put $A = \{A_i: i \in F \text{ and conditions (1), (2), (3) were satisfied when the ith coordinates of the points of } X_i \text{ were defined}\}$. Pick $\alpha^* \in \omega$ with $\text{max}(F) < \alpha^* < \tau$. Using Lemma 48(1), we can assume that $\pi_{[0,\alpha^*)}(\varphi)$ is a bijection. Then $T = \pi_{[0,\alpha^*)}(\bigcup(A_i \cap Y_n: A_i \in A))$ is a finite union of discrete subspaces of $X^{[0,\alpha^*)}$, and thus $T$ is nowhere dense in $X^{[0,\alpha^*)}$. So $T = \pi_{[0,\alpha^*)}(Y_n) \setminus T$ is dense in $X^{[0,\alpha^*)}$. Pick $t \in T^\tau$ with $t(i) \in \varphi(i)$ for every $i \in F$. There is $y \in Y_n$ with $y(\alpha^*) = t$. Then $\tilde{y} \in \tilde{Y}_n$, and $\tilde{y}(\alpha) = t$, so $\tilde{y}$ satisfies (†). □

Claim 3. For any choice of discrete $S_n \subseteq \tilde{Y}_n, n \in \omega$, the set $S = \bigcup_{n \in \omega} S_n$ is not dense in $X^\tau$ (and thus not dense in $X$).

Proof of Claim 3. By Lemma 48(2), there is $\alpha^* < \tau$ such that $\pi_{[0,\alpha^*)}(S_n)$ is discrete for each $n$, and $\pi_{[0,\alpha^*)}(S) = \pi_{[0,\alpha^*)}(S_n)$ is injective. Put $A = \{y \in Y: \tilde{y} \in S\}$. Pick $\alpha^{**} \in C_A$ so that $\alpha^{**} \geq \alpha^*$. Then $\tilde{y}(\alpha^{**}) = x_0$ for every $\tilde{y} \in S$, and thus $S$ is not dense in $X^\tau$. □

Claims 2 and 3 show that $\tilde{Y}$ is not $D$-separable. Since $\tilde{Y}$ is dense in $X^\tau$ it follows that $X^\tau$ is not $D$-separable. □

Question 50. Is it true that for every Tychonoff space $X$ there is $\kappa$ such that for all $\kappa' \geq \kappa$, $X^{\kappa'}$ is not $D$-separable?

4.3.2. Proof of Theorem 37

More specifically, we will prove:

**Theorem 51.** Let $X$ be any space, and let $Y$ be any space such that $\pi w(X) \leq \pi w(Y) = \kappa$ and $Y$ contains a cellular family of size $\kappa$. Then $X \times Y^\omega$ is $DH^+$-separable.

(Then, for Theorem 37, one can take $Z = Y^\omega$. As $Y$, one can take the discrete space of size $\pi w(X)$ or a one-point compactification of such a space, so $Z$ in Theorem 37 can be in addition assumed compact.)

**Proof.** Let $U$ and $V$ be $\pi$-bases of $X$ and $Y$ having minimal size. Let $[C_\alpha: \alpha < \kappa]$ be a cellular family in $Y$. For $m \in \omega$, let $[e_\alpha^m: \alpha < \kappa]$ be an enumeration of $U \times Y^m$.

On the $\alpha$th move, one chooses a dense subspace $S_m \subseteq X \times Y^\omega$, and Two, for every $m \in \omega$ and $\alpha < \kappa$ selects $d_\alpha^m \in (\bigcap_{n < m} W_\alpha^m) \cap S_m$ where $W_{\alpha}^m = e_\alpha^m(n)$ for $n \leq m$, $W_{\alpha}^m = C_\alpha$ and $W_{\alpha}^m = Y$ for $n > m + 1$. Let $D = \{d_\alpha^m: \alpha < \kappa\}$. Then $D_m \subseteq S_m$, $D_m$ is discrete, and $D_m$ intersects every non-empty open set in $X \times Y^\omega$ that depends only on the first $m + 1$ coordinates. Thus every non-empty open set in $X \times Y^\omega$ intersects all but finitely many $D_m$s. □

A consequence of the above theorem is that there is no single cardinal $\kappa$ such that $X^\kappa$ is not $D$-separable for every space $X$.

**Corollary 52.** For every $\kappa$, and every $\lambda \leq \kappa$, $(D(\kappa))^\lambda$ is $D$-separable (where $D(\kappa)$ is the discrete space of cardinality $\kappa$).
Proof. This is trivial if $\kappa$ is finite. So assume $\kappa$ is infinite. Now set $X = Y = D(\kappa)^\omega$ in Theorem 51 and observe that $(D(\kappa)^\omega)^\omega$ and $D(\kappa)^\kappa$ are homeomorphic. □

Another notable consequence of Theorem 51 is the fact that the $\omega$-power of any linearly ordered space is $D$-separable. This follows from the following result of Petr Simon.

Lemma 53. ([54]) Let $X$ be a linearly ordered topological space. Then $X^2$ contains a cellular family of size $d(X)$.

Corollary 54. Let $X$ be a linearly ordered topological space. Then $X^\omega$ is $D^H+$-separable.

Proof. Since $\pi w(X) = d(X)$ in linearly ordered spaces $X^2$ contains a pairwise disjoint open family of size $\pi w(X)$. Now let $Y = X^2$ in Theorem 51. □

So, although a Suslin Line is not even $d$-separable, its $\omega$-power is $DH^+$-separable.

4.4. Some more open problems

Tkachuk presented a large collection of sufficient conditions and necessary conditions for $d$-separability of $C_p(X)$ in [57]. Tkachuk gave a CH example of a compact space $X$ with a non-$d$-separable $C_p(X)$ and asked for a ZFC example of a Tychonoff space or even a compact space $X$ with non-$d$-separable $C_p(X)$. A Tychonoff ZFC example was presented in [38].

Problem 55. (1) Characterize $X$ such that $C_p(X)$ is $D$-separable.

(2) More specifically, suppose $C_p(X)$ is $d$-separable. Under what additional conditions on $X$ is $C_p(X)$ $D$-separable?

In [14] the authors noted that a compact space is selectively separable if and only if it has a countable $\pi$-base. This is a consequence of the fact that a compact space $X$ has a countable $\pi$-base iff every dense subspace of $X$ is separable [35].

Let $dd(Y)$ be the least cardinal $\kappa$ such that $Y$ has a dense set which is the union of $\kappa$ many discrete sets. Let $d_\pi(X) = \sup\{dd(D) : D$ is dense in $X\}$. Let $d_\pi(X)$ be the least cardinal $\kappa$ such that $X$ has a $\pi$-base which is the union of $\kappa$ many disjoint collections.

Conjecture 56.

(1) A compact space $X$ is $D$-separable iff $X$ has a $\sigma$-disjoint $\pi$-base.

(2) Let $X$ be a compact space. Then $d_\pi(X) = d_\pi(X)$.

By Theorem 38 if Conjecture 56(1) is true then the answer to the following question is positive.

Question 57. Is the product of two compact $D$-separable spaces still $D$-separable?

Recall that a space is called an $L$-space if it is hereditarily Lindelöf but not separable. Tkachuk [57] constructed under CH an $L$-space $X$ such that $X^2$ is $d$-separable. Later on, Moore [45] showed that a slight modification of his ZFC example of an $L$-space provides a $ZFC$ example of an $L$-space with a $d$-separable square.

Question 58. Is there a non-$D$-separable space $X$ such that $X^2$ is $D$-separable? Is there even a non-$d$-separable space with this property?

Question 59. Is there (in any model of ZFC) an example of an $L$-space with a $D$-separable square?

Note that replacing $D$-separability with selective separability both questions have easily a negative answer. Also, the influence of convergence properties on $D$-separability is not clear yet.

Question 60.

(1) Is every Fréchet $d$-separable space $D$-separable? What about $S(2^\kappa)$?

(2) Is there a sequential $d$-separable (separable, countable) non-$D$-separable space?

(3) Is there a Whyburn $d$-separable (separable, countable) non-$D$-separable space?
5. More on maximal (and submaximal) spaces

We conclude with some remarks on the interesting case of maximal and submaximal spaces. In a submaximal space every dense set is open, so in some sense dense sets are "big". This implies "a lot of freedom" in choosing a finite set and this in turn could suggest that a maximal space can easily be selectively separable, but we will see below that often things go differently.

In [14] it was shown that assuming \( d = \omega_1 \) there is a maximal regular space which is not selectively separable, and it was asked (1) whether or not such an example is possible within ZFC, and (2) is it true (at least consistently) that every countable maximal regular space is not selectively separable? Here is the progress obtained since then:

**Theorem 61.**

1. (Barman and Dow, [10]) There is (within ZFC) a countable maximal regular space which is not selectively separable.
2. (Barman and Dow, [10], Repovš and Zdomskyy, [48]) Consistently, there is no submaximal SS space (specifically, the existence of such a space implies the existence of a separable P-set in \( \omega^* \) while the existence of a ccc P-set in \( \omega^* \) is known to be independent from ZFC from [27]).
3. (Barman and Dow, [10]) \([\text{MA}_\text{ctble}] \) There exists a maximal regular countably selectively separable space. (So the existence of a maximal regular selectively separable space is independent of ZFC.)
4. (Gruenhage and Sakai, [31]) [CH] There is a maximal space \( X \) such that \( X \) is R-separable but \( X^2 \) is not selectively separable.\footnote{The first author and Gruenhage obtained the same result under a weaker assumption \( \text{MA}_\text{ctble} \).}
5. (Barman and Dow, [10]) Every crowded SS\(^+\) space is resolvable. Hence no maximal space is SS\(^+\).\footnote{And one can see from the argument in [10] that, more generally, no crowded submaximal space is SS\(^+\).}

Many results on maximal regular spaces, in particular Barman and Dow’s construction from Theorem 61(1) above, are based on the following theorem of van Douwen.

**Theorem 62.** ([22]) For any countable regular crowded space \( (X, \tau) \) there is a stronger regular topology \( \sigma \) such that the space \( (X, \sigma) \) has a dense subset which is a maximal space.

Below we present an alternative proof of Theorem 61, part 1 (based on the space \( \text{Seq}(\mathcal{F}) \) as the starting point), and construct a maximal regular countable SS space using a weaker assumption than in Theorem 61, part 3, namely \( d = \omega \). Then we discuss maximal D-separable spaces.

**Theorem 63.** There exists a countable regular maximal space which is not selectively separable.

**Proof.** Start by letting \( (X, \tau) = \text{Seq}(\mathcal{F}) \) and fix the sequence of dense subsets \( \{H_n: n < \omega\} \), where \( H_n = \bigcup_{k=0}^n \{\omega: n \leq k < \omega\} \).

Step 1: use van Douwen’s theorem to find \( \sigma \supset \tau \) and a dense subset \( Z \) of \( (X, \sigma) \) which is a regular maximal space.

Step 2: since each \( H_n \) has a closed scattered complement in \( (X, \tau) \), it follows that \( H_n \) remains dense and open in \( (X, \sigma) \) and so the set \( D_n = Z \cap H_n \) is dense in \( Z \).

Step 3: the sequence \( \{D_n: n < \omega\} \) cannot have a “good selection” because it would be also a “good selection” for the sequence \( \{H_n: n < \omega\} \) in \( (X, \tau) \). \( \Box \)

Recall that first Gruenhage under \([\text{CH}] \) (later included in [31]) and then Barman and Dow under \( \text{MA}_\text{ctble} \) have shown the existence of a countable regular maximal selectively separable space. Gruenhage’s construction gives a stronger result: a maximal R-separable space whose square is not selectively separable. We are going to show that, with respect to the weaker task to have just a maximal selectively separable space, \( d = \omega \) suffices. The construction we present below follows the pattern of that of Gruenhage.

**Lemma 64.** Let \( X \) be a space and \( x \in X \). If \( t(x, X) = \omega \) and \( \chi(x, X) < d \), then \( X \) has countable fan tightness at \( x \).

**Proof.** Let \( \{A_n: n < \omega\} \) be a sequence of sets such that \( x \in \overline{A_n} \) for each \( n \). Since \( t(x, X) = \omega \), we may assume each \( A_n \) countable and write \( A_n = \{a_{n,k}: k < \omega\} \). Let \( \{U_\alpha: \alpha < \kappa\} \) be a local base at \( x \) with \( \kappa < d \). For any \( \alpha \) we may define a function \( f_\alpha: \omega \to \omega \) by letting \( f_\alpha(n) = \min(k: a_{n,k} \in U_\alpha \cap A_n) \). Since \( \kappa < d \), the family \( \{f_\alpha: \alpha < \kappa\} \) cannot be dominating and so there exists \( g \in \omega^\omega \) such that the set \( \{n: f_\alpha(n) \leq g(n)\} \) is infinite for each \( \alpha \). Now, by letting \( F_n = \{a_{n,k}: k \leq g(n)\} \), we may easily check that \( x \in \bigcup\{F_n: n < \omega\} \). \( \Box \)

In [17] it is shown that any crowded space of countable fan tightness is \( \omega \)-resolvable. So we have:
Corollary 65. A countable crowded space of weight less than \( \varnothing \) is \( \omega \)-resolvable.

The above corollary is the main ingredient in the proof of the following lemma:

Lemma 66. \([\varnothing = \varnothing] \) Let \((X, \tau)\) be a countable crowded regular space of weight \( \leq \kappa \) where \( \kappa < \mathfrak{c} \). Then:

1. If \( A \) is a dense subset of \((X, \tau)\), then there is an enlargement \( \sigma_1 \) of \( \tau \) such that \((X, \sigma_1)\) is a regular crowded space of weight \( \leq \kappa \), \( A \in \sigma_1 \) and each dense open set in \((X, \tau)\) remains dense in \((X, \sigma_1)\);
2. If \( A \) is a crowded subset in \((X, \tau)\), then there exists an enlargement \( \sigma_2 \) of \( \tau \) such that \((X, \sigma_2)\) is a regular crowded space of weight \( \leq \kappa \), \( A \) is either open in \((X, \sigma_2)\) or it has an isolated point in \((X, \sigma_2)\) and each dense open set in \((X, \tau)\) remains dense in \((X, \sigma_2)\).

Proof. Part 1. By Corollary 65 the subspace \( A \) is \( \omega \)-resolvable and we may write \( A = \bigcup \{ A_n: n < \omega \} \), where each \( A_n \) is dense and \( A_i \cap A_j = \emptyset \) whenever \( i \neq j \). \( \sigma_1 \) is the topology on \( X \) generated by \( \tau \cup \{ A_n: n < \omega \} \cup \{ X \setminus A_n: n < \omega \} \).

Part 2. If \( A \) is dense in \( X \), then we may argue as in part 1. If not, let \( V = \text{Int}(X \setminus A) \) and consider the topology \( \tau' \) generated by \( \tau \cup \{ V \} \). \( \tau' \) is regular and any dense open set in \( \tau \) remains dense in \( \tau' \). If \( \emptyset \neq V \cap A \neq \emptyset \), then pick a point \( p \in V \cap A \) and apply part 1 to the space \((X, \tau')\) and the dense set \( V \cap A \). In the resulting topology \( \sigma_2 \) the set \( A \) is open. If \( V \cap A = \emptyset \), then pick a point \( p \in A \) and apply part 1 to the space \((X, \tau')\) and the dense set \( X \setminus (V \cap A \setminus \{ p \}) \). In the resulting topology \( \sigma_2 \) the point \( p \) is isolated in \( A \).

Theorem 67. \([\varnothing = \varnothing] \) There exists a countable regular maximal selectively separable space.

Proof. Let \( t_0 \) be a regular crowded second countable topology on the set \( \omega \). List all infinite subsets of \( \omega \) as \( \{ A_\alpha: \alpha < \mathfrak{c} \} \) and all \( \omega \)-sequences of subsets of \( \omega \) as \( \{ \langle D_n^\alpha: n < \omega \rangle: \alpha < \mathfrak{c} \} \) (in the latter each element is listed \( \epsilon \)-many times). For any \( \alpha < \mathfrak{c} \) we will construct a crowded regular topology \( \tau_\alpha \) on \( \omega \) in such a way that:

1. if \( \beta < \alpha \) then \( \tau_\beta \subseteq \tau_\alpha \) and any dense open set in \( \tau_\beta \) remains dense in \( \tau_\alpha \);
2. the weight of \( \tau_\alpha \) is at most \( \alpha + \omega \);
3. if \( A_\alpha \) is dense in \( \tau_\alpha \), then \( A_\alpha \) is dense open in \( \tau_{\alpha + 1} \);
4. if \( A_\alpha \) is not dense but crowded in \( \tau_\alpha \), then either \( A_\alpha \) is open in \( \tau_{\alpha + 1} \) or \( A_\alpha \) has an isolated point if \( \tau_{\alpha + 1} \); 5. if \( \langle D_n^\alpha: n < \omega \rangle \) is a sequence of dense open sets in \( \tau_\alpha \), then there are finite sets \( F_n^\alpha \subseteq D_n^\alpha \) such that the set \( \bigcup \{ F_n^\alpha: n < \omega \} \) is dense open in \( \tau_{\alpha + 1} \).

Suppose you have already defined topologies \( \tau_\beta \) and sequences \( \{ F_n^\beta: n < \omega \} \) for \( \beta < \alpha \) satisfying the above conditions. If \( \alpha \) is a limit ordinal, then we take as \( \tau_\alpha \) the topology generated by \( \bigcup \{ \tau_\beta: \beta < \alpha \} \). In this case, only condition (2) needs to be checked. Now, assume \( \alpha = \gamma + 1 \). If \( A_\gamma \) is crowded, then apply Lemma 66 to get a topology \( \tau' \) (\( \tau' \) is either \( \sigma_1 \) or \( \sigma_2 \) from Lemma 66 according to the fact that \( A_\gamma \) is or is not dense in \( \tau_\gamma \). Next, if \( \langle D_n^\gamma: n < \omega \rangle \) is a sequence of dense open sets in \( \tau_\gamma \) (and so even in \( \tau' \)), we may use Proposition 7, part \( A \) to find finite sets \( F_n^\gamma \subseteq D_n^\gamma \) in such a way that the set \( B = \bigcup \{ F_n^\gamma: m < \omega \} \) is dense in \( \tau' \). To finish the construction, apply again part 1 of Lemma 66 to get a topology \( \tau_{\gamma + 1} \) which is the enlargement of \( \tau' \) where \( B \) is dense open.

Let \( \tau \) be the topology generated by \( \bigcup \{ \tau_\alpha: \alpha < \mathfrak{c} \} \). If the set \( A \) is crowded in \( \tau \) and \( A = A_\alpha \), then \( A \) is also crowded in \( \tau_\alpha \). By construction, \( A = A_\alpha \) is open in \( \tau_{\alpha + 1} \) and so even in \( \tau \) (the second possibility in condition (4) cannot occur because \( A \) cannot have isolated points in \( \tau_{\alpha + 1} \). The fact that every crowded subset of \( \tau \) is open ensures that \( \tau \) is a maximal topology [22]. If \( \langle D_n^\alpha: n < \omega \rangle \) is a sequence of dense sets in \( \tau \), then each \( D_n^\alpha \) is dense in each \( \tau_\alpha \) and so there is some \( \beta < \mathfrak{c} \) such that each \( D_n^\alpha \) is dense open in \( \tau_\beta \). Since every \( \omega \)-sequence of subsets of \( \omega \) is listed \( \epsilon \)-many times, there is an ordinal \( \gamma \geq \beta \) such that \( \langle D_n^\alpha: n < \omega \rangle = \langle D_n^{\gamma^*}: n < \omega \rangle \). By condition (5) we get finite sets \( F_n^{\gamma^*} \subseteq D_n^{\gamma^*} = D_n \) in such a way that the set \( \bigcup \{ F_n^{\gamma^*}: n < \gamma \} \) is dense open in \( \tau_{\gamma + 1} \) and so dense even in \( \tau \). This shows that the space \( \langle \omega, \tau \rangle \) is selectively separable.

Now we go back to \( D \)-separability. It turns out that, at least in the countable case, maximal spaces are always \( D \)-separable.

Theorem 68. Let \( X \) be submaximal. Then the following conditions are equivalent:

1. \( X \) is \( D \)-separable;
2. \( X \) is \( d \)-separable;
3. \( X \) is \( \sigma \)-discrete;
4. \( X \) is \( \sigma \)-closed discrete.

The following is immediate:
Corollary 69. A countable submaximal space is D-separable.

Proof of Theorem 68. First, we prove the theorem for the special case when $X$ is crowded. The implications $(1) \implies (2)$ and $(4) \implies (3) \implies (2)$ are obvious.

$(2) \implies (1)$ and $(4)$: Let $H = \bigcup_{n \in \omega} H_n$ be a dense subspace of $X$ where each $H_n$ is discrete. Further, let $(D_n: n \in \omega)$ be an arbitrary sequence of dense subsets of $X$. For $n \in \omega$, put $D'_n = D_n \cap (H \setminus H_n)$. Then $D'_n$ is dense in $X$. Next, put $G_n = (D'_0 \cap \cdots \cap D'_n) \setminus D'_{n+1}$. Each $G_n$ is closed discrete being a subset of the complement to the dense set $D'_{n+1}$. By construction, we have $G_n \subseteq D_n$ and $D'_0 = \bigcup_{n \in \omega} G_n$. Therefore, $\bigcup_{n \in \omega} G_n$ is dense in $X$. Last, put $G_\omega = X \setminus D'_0$. Then $X = \bigcup_{n \leq \omega} G_n$ is a countable union of closed discrete subspaces.

So we have proved the theorem for crowded $X$. It follows in particular that $(\ast)$ every countable crowded submaximal space is D-separable. Now let $X$ be an arbitrary submaximal space. Replace every isolated point of $X$ with a copy of a countable crowded regular maximal space. Call the resulting space $\tilde{X}$; $\tilde{X}$ is crowded. It is easy to deduce from $(\ast)$ that $X$ has one of the properties $(1)$ through $(4)$ iff so does $\tilde{X}$. This completes the proof.

We will see that submaximal spaces can never be $D^+$-separable (see Theorem 76). However, we don't know the answer to the following question.

Question 70. Is there a countable submaximal space which is not DH-separable?

Arhangel'skii and Collins asked in [6] if all submaximal spaces are $\sigma$-discrete. Schröder proved in [52] that assuming $V = L$ the answer is affirmative. Thus we get:

Corollary 71. $[V = L]$ Every submaximal space is D-separable.

On the other hand, Kunen, Szymanski and Tall showed [39] that the existence of a measurable cardinal is consistent with ZFC iff the existence of a Tychonoff crowded SIB space is consistent with ZFC. Further, Levy and Porter proved the following proposition:

Proposition 72. ([41], Proposition 3.1 and a remark after it) The following conditions are equivalent:

1. There exists a submaximal Hausdorff space which is not $\sigma$-discrete;
2. There exists a crowded submaximal Hausdorff space which is not $\sigma$-discrete;
3. There exists a maximal space which is not $\sigma$-discrete;
4. There exists a crowded submaximal Hausdorff space which is not strongly $\sigma$-discrete;
5. There exists a maximal space which is not strongly $\sigma$-discrete;
6. There exists a crowded SIB space;
7. There exists a crowded Hausdorff space $X$ such that every real-valued function defined on $X$ is continuous at some point.

It follows [41] that the existence of a crowded submaximal (or, equivalently, the existence of a maximal) Hausdorff space which is not $\sigma$-discrete is equiconsistent with a measurable cardinal. We see from Theorem 68 that the existence of a submaximal space which is not D-separable can be added to the list of conditions in Proposition 72. Therefore we get:

Corollary 73. The existence of a submaximal space which is not D-separable is equiconsistent with a measurable cardinal.

As there exist (in ZFC) countable regular maximal non-selectively separable spaces, Corollary 69 implies the existence of a space with some of the properties of Example 25.

Corollary 74. There exists a countable regular (maximal) D-separable non-selectively separable space.

Unlike Example 25 such a space can never be sequential. Indeed, maximal spaces contain no non-trivial convergent sequences. However, Malykhin [43] has shown that maximal spaces can carry a group structure, so we wonder if Corollary 74 can be improved in the following way:

Question 75. Is there a countable regular maximal non-selectively separable topological group?
Such an improvement can never be achieved in ZFC alone because Protasov [47] has shown that there are models of ZFC with no maximal topological groups.

We conclude by showing that a crowded submaximal space cannot be $D^+$-separable. It was shown in [10] that every crowded $SS^+$ space is resolvable (and hence cannot be submaximal). We note that the argument extends to $D^+$-separable spaces.

**Theorem 76.** Every crowded $D^+$-separable space is $\omega$-resolvable (and hence non-submaximal).

**Proof.** Let $\sigma$ be the winning strategy for Two in the game $G_{\text{dis}}(D, D)$ on the space $X$. It suffices to show that any dense $D \subset X$ contains two disjoint dense subsets. Let $S_0 = \sigma(D)$, $T_0 = \sigma(D \setminus S_0)$, $S_1 = \sigma(D, D \setminus (S_0 \cup T_0))$, $T_1 = \sigma(D, D \setminus (S_0 \cup T_0 \cup S_1))$, $S_2 = \sigma(D, D \setminus (S_0 \cup T_0), D \setminus (S_0 \cup T_0 \cup S_1 \cup T_1))$, etc.

Because $\sigma$ is winning the disjoint sets $\bigcup_{n \in \omega} S_n$ and $\bigcup_{n \in \omega} T_n$ are dense.

**Proposition 77.** If $X$ is $D$-separable and $\omega$-resolvable, then there is a family $\mathcal{G}$ of dense subspaces of $X$ such that $|\mathcal{G}| = \omega$ and $G \cap G' \neq \emptyset$ for every distinct $G, G' \in \mathcal{G}$.

**Proof.** Fix a family $\{Y_n; n \in \omega\}$ of pairwise disjoint dense subsets of $X$. Also fix an almost disjoint family $A$ of infinite subsets of $\omega$ such that $|A| = \omega$. For each $A \in A$ apply the definition of $D$-separability to the family of dense subspaces $\{Y_n; n \in A\}$ to get discrete $F_{A,n} \subset Y_n$ such that $Z_A = \bigcup_{n \in \omega} F_{A,n}$ is dense in $X$. It remains to note that whenever $A, A' \in A$ are distinct, $Z_A \cap Z_{A'}$ is the union of finitely many discrete sets and thus nowhere dense.

**Corollary 78.** If $X$ is $D$-separable, $\omega$-resolvable and $\Delta(X) < \omega$ (in particular, if $X$ is countable), then $X$ is extra-resolvable.

**Corollary 79.** (1) If $X$ is a crowded $D^+$-separable space then there is a family $\mathcal{G}$ of dense subspaces of $X$ such that $|\mathcal{G}| = \omega$ and $G \cap G' \neq \emptyset$ for every distinct $G, G' \in \mathcal{G}$.

(2) If $X$ is a crowded $D^+$-separable space and $\Delta(X) < \omega$ (in particular, if $X$ is countable), then $X$ is extra-resolvable.

It follows from Corollary 78 that one way of constructing a countable non-$D$-separable space is to construct a countable non-extra-resolvable $\omega$-resolvable space. García Ferreira and Hrušák in [32] and Juhász, Soukup and Szentmiklóssy in [36] have independently come up with ZFC constructions of such a space.

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