1. Introduction

In this paper, we mainly consider the following three-species lattice Lotka–Volterra competition system

\[
\begin{align*}
\tilde{u}_j(t) &= \tilde{d}_1 D_2[\tilde{u}_j](t) + r_1 \tilde{u}_j(t)[1 - \tilde{u}_j(t) - b_{12} \tilde{v}_j(t) - b_{13} \tilde{w}_j(t)], \\
\tilde{v}_j(t) &= \tilde{d}_2 D_2[\tilde{v}_j](t) + r_2 \tilde{v}_j(t)[1 - b_{21} \tilde{u}_j(t) - \tilde{v}_j(t)], \\
\tilde{w}_j(t) &= \tilde{d}_3 D_2[\tilde{w}_j](t) + r_3 \tilde{w}_j(t)[1 - b_{31} \tilde{u}_j(t) - \tilde{w}_j(t)],
\end{align*}
\]

where \( D_2[\tilde{u}_j](t) = \tilde{u}_{j+1}(t) - 2 \tilde{u}_j(t) + \tilde{u}_{j-1}(t), \) \( D_2[\tilde{v}_j](t) = \tilde{v}_{j+1}(t) - 2 \tilde{v}_j(t) + \tilde{v}_{j-1}(t), \) \( D_2[\tilde{w}_j](t) = \tilde{w}_{j+1}(t) - 2 \tilde{w}_j(t) + \tilde{w}_{j-1}(t). \) Here, \( \tilde{u}_j(t), \tilde{v}_j(t), \tilde{w}_j(t) \) are the population densities of three species at time \( t \) and niches \( j \), respectively; \( b_{1k}, b_{k1} > 0 \) \((k = 2, 3)\) are the competition coefficients between species \( \tilde{u}, \tilde{v}, \tilde{w} \); \( r_i > 0 \) \((i = 1, 2, 3)\) are the growth rates and \( \tilde{d}_i > 0 \) \((i = 1, 2, 3)\) are the diffusion coefficients of species \( \tilde{u}, \tilde{v}, \tilde{w} \), respectively. The model (1) is generally used to describe such a scenery that species \( \tilde{u} \) competes with the other two species \( \tilde{v} \) and \( \tilde{w} \). However, there is no direct effect between the latter two species.

A continuous revision of the system (1) is as below

\[
\begin{align*}
\tilde{u}_t &= \tilde{d}_1 \tilde{u}_{xx} + r_1 \tilde{u}(1 - \tilde{u} - b_{12} \tilde{v} - b_{13} \tilde{w}), \\
\tilde{v}_t &= \tilde{d}_2 \tilde{v}_{xx} + r_2 \tilde{v}(1 - b_{21} \tilde{u} - \tilde{v}), \\
\tilde{w}_t &= \tilde{d}_3 \tilde{w}_{xx} + r_3 \tilde{w}(1 - b_{31} \tilde{u} - \tilde{w}).
\end{align*}
\]
The study of dynamics of system (2) has been studied quite extensively. For this, we refer the reader to \[3,14\] for the stability of spatially inhomogenous positive equilibria and travelling waves, and to \[4,5\] for the propagation dynamics simulated by numerical simulations, as well as to \[6\] for the existence of some new entire solutions. The uniqueness of the travelling wavefront was investigated in \[18\]. The issue of competitive exclusion or competitor-mediated coexistence was established in \[19\]. Lately, via constructing a pair of upper and lower solutions and using the method of monotone iteration, Hu et al. \[13\] considered the existence of the forced travelling wave solution for a nonlocal dispersal Lotka–Volterra cooperation model under the assumption that the habitat shifts with a constant speed.

It is well known that lattice system sometimes is more suitable to describe the competition among species than continuous system for aggregation and dispersion. Indeed, the lattice dynamical systems have been widely used to simulate biological problems in past years. As to the spatial invasion of the species in unbounded domain, Guo et al. \[12\] studied the minimal wave speed determinacy and obtained some results of linear selection, with an idea by studying the corresponding lattice dynamical systems. A general approach is developed to deal with the asymptotic behaviour of travelling wave solutions in a class of three-component lattice dynamical systems in \[24\]. Also, Dong et al. \[9\] investigated the asymptotic behaviour of travelling wave solutions and established a new type of entire solutions for a three-component system with nonlocal dispersal. Su and Zhang \[21\] proved that the travelling wavefronts with large speed are exponentially asymptotically stable for three-species Lotka–Volterra competition system. Guo et al. \[11\] proved the stability and uniqueness of the travelling waves of the discrete bistable three-species competition system by constructing super-/sub-solutions. In addition, Chen et al. \[7\] further studied the nonlinear stability of the monostable travelling wave solution and improved the previous results about stability. Recently, Gao and Wu \[10\] proved the existence of minimal wave speed and established the asymptotic behaviour of the travelling wave solutions for three-component lattice system with delay. Zhang et al. \[26\] obtained the existence and monotonicity of bistable travelling wave solutions for a three-species competitive-cooperative lattice dynamical system with delay.

In this article, we are mainly interested in the issue of linear/nonlinear selection of the minimal wave speed for the lattice Lotka–Volterra competition system with three species and expect to get necessary and sufficient conditions of linear or nonlinear selection. To proceed, unitizing the coefficients of species \(\tilde{u}\) to compare with species \(\tilde{v}\) and \(\tilde{w}\), respectively, in the system (1), we first make the following transformations

\[
\sqrt{\frac{r_1}{d_1}} x \rightarrow x, \ r_1 t \rightarrow t, \ \frac{\tilde{d}_2}{d_1} = d_1, \ \frac{\tilde{d}_3}{d_1} = d_2, \ \frac{r_2}{r_1} = \alpha, \ \frac{r_3}{r_1} = \beta. \tag{3}
\]

Then we obtain a non-dimensional system

\[
\begin{align*}
\tilde{u}_j'(t) &= D_2[\tilde{u}_j](t) + \tilde{u}_j(t)[1 - \tilde{u}_j(t) - b_{12}\tilde{v}_j(t) - b_{13}\tilde{w}_j(t)], \\
\tilde{v}_j'(t) &= d_1 D_2[\tilde{v}_j](t) + \alpha \tilde{v}_j(t)[1 - b_{21}\tilde{u}_j(t) - \tilde{v}_j(t)], \\
\tilde{w}_j'(t) &= d_2 D_2[\tilde{w}_j](t) + \beta \tilde{w}_j(t)[1 - b_{31}\tilde{u}_j(t) - \tilde{w}_j(t)], \tag{4}
\end{align*}
\]
which can be converted to be a cooperative system

\[
\begin{align*}
\left\{ \begin{array}{l}
\dot{u}_j(t) &= D_2[u_j(t) + u_j(t)[1 - b_{12} - b_{13} - u_j(t) + b_{12}v_j(t) + b_{13}w_j(t)], \\
\dot{v}_j(t) &= d_1 D_2[v_j(t) + \alpha [1 - v_j(t)][b_{21}u_j(t) - v_j(t)], \\
\dot{w}_j(t) &= d_2 D_2[w_j(t) + \beta [1 - w_j(t)][b_{31}u_j(t) - w_j(t)]
\end{array} \right.
\end{align*}
\]

(5)

by taking the transformation \(\tilde{u}_j(t) = u_j(t), \tilde{v}_j(t) = 1 - v_j(t), \tilde{w}_j(t) = 1 - w_j(t)\) on (4).

Since we focus on the monostable travelling waves in the present paper, the parameters in (5) should obey the following so-called monostable conditions, that we assume to be satisfied

\[
b_{12} + b_{13} < 1, \ b_{21} > 1, \ b_{31} > 1.
\]

(6)

As a consequence of (6), system (5) possesses at least five equilibria: the unstable state \(e_0 := (0, 0, 0)\), the stable coexistence state \(e_1 := (1, 1, 1)\), as well as the other extinction states, including \(e_2 := (0, 1, 1), \ e_3 := (0, 1, 0), \ e_4 := (0, 0, 1)\). These properties are similar to the ones of the corresponding continuous system, see [12]. Traveling wave solutions play an important role in understanding the nonlinear biological and physical phenomena. Particularly, they usually can be applied to describe the spreading and invading behaviours in ecology [15,16,20,22,23,25]. The non-negative travelling wavefronts connecting \(e_1\) to \(e_0\) have the following form

\[
(u_j, v_j, w_j)(t) = (U, V, W)(z), \quad z = j - ct,
\]

where \(c\) is called the wave speed and \((U, V, W)\) is called the wave profile. Substituting \((U, V, W)(z)\) into (5) leads to

\[
\begin{align*}
D_2[U](z) + cU'(z) + U(z)[1 - b_{12} - b_{13} - U(z) + b_{12}V(z) + b_{13}W(z)] &= 0, \\
d_1 D_2[V](z) + cV'(z) + \alpha [1 - V(z)][b_{21}U(z) - V(z)] &= 0, \\
d_2 D_2[W](z) + cW'(z) + \beta [1 - W(z)][b_{31}U(z) - W(z)] &= 0,
\end{align*}
\]

(7)

System (7) can be treated as a discrete version of the continuous system and the existence of the monostable travelling wavefronts connecting \(e_1\) and \(e_0\) was established in [12] by using Schauder’s fixed point theorem. That is to say that there exists a non-negative wave speed \(c_{\text{min}} \geq 0\), which is known as minimal wave speed, so that (7) has a non-negative monotone solution for any \(c \geq c_{\text{min}}\), see [12]. For purpose of studying the stability of the travelling wave solution, Chen et al. [5] also investigated the existence and monotonicity of the travelling wave solutions for system (7).

Next, we shall discuss the speed determinacy of the travelling wavefronts. Firstly, one needs to analyse the local asymptotic behaviour for the positive wave profile of system (7) near the equilibrium point \(e_0\). Linearizing the system (7) around \(e_0\), we obtain the following system

\[
\begin{align*}
D_2[U](z) + cU'(z) + (1 - b_{12} - b_{13})U(z) &= 0, \\
d_1 D_2[V](z) + cV'(z) + \alpha [b_{21}U(z) - V(z)] &= 0, \\
d_2 D_2[W](z) + cW'(z) + \beta [b_{31}U(z) - W(z)] &= 0.
\end{align*}
\]

(8)
Letting \((U, V, W)(z) = (\xi_1, \xi_2, \xi_3)e^{-\mu z}\) for some positive constants \(\xi_1, \xi_2, \xi_3\) and \(\mu\) and substituting it into (8), then we get

\[
\begin{pmatrix}
\Gamma_1(\mu) & 0 & 0 \\
\alpha b_{21} & \Gamma_2(\mu) & 0 \\
\beta b_{31} & 0 & \Gamma_3(\mu)
\end{pmatrix}
\begin{pmatrix}
\xi_1 \\
\xi_2 \\
\xi_3
\end{pmatrix} =
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix},
\]

(9)

with

\[
\Gamma_1(\mu) = e^{-\mu} + e^\mu - 2 - c\mu + 1 - b_{12} - b_{13},
\]

\[
\Gamma_2(\mu) = d_1(e^{-\mu} + e^\mu - 2) - c\mu - \alpha,
\]

\[
\Gamma_3(\mu) = d_2(e^{-\mu} + e^\mu - 2) - c\mu - \beta.
\]

The first equation is decoupled. From \(\Gamma_1(\mu) = 0\), we can find a linear speed \(c_0\) defined by

\[
c_0 = \min_{\mu > 0} \frac{(e^{-\mu} + e^\mu - 2) + (1 - b_{12} - b_{13})}{\mu}.
\]

(10)

It is easy to know that for each \(c > c_0\), \(\Gamma_1(\mu) = 0\) has two solutions denoted by

\[
\mu_1 := \mu_1(c) \quad \text{and} \quad \mu_2 := \mu_2(c)
\]

(11)

with \(\mu_2 > \mu_1\). Moreover, \(\mu_1(c)\) is a decreasing function and \(\mu_2(c)\) is an increasing function with respect to \(c\), satisfying

\[
\bar{\mu} := \mu_1(c_0) = \mu_2(c_0).
\]

(12)

For clarity, we give the definitions of linear and nonlinear selection of the minimal wave speed.

**Definition 1.1:** If \(c_{\min} = c_0\), then we call the minimal wave speed is linearly selected; otherwise, if \(c_{\min} > c_0\), we call the minimal wave speed is nonlinearly selected.

In this paper, we aim at investigating the challenging problem on the linear and nonlinear selection of the minimal wave speed for system (7) by making use of the upper–lower method and the comparison principle in monostable case. Some fundamental lemmas and definitions are first listed to show the existence of the travelling wave solutions and the uniqueness of the wave speed. Then by constructing several upper and lower solutions, sufficient conditions of the linear and nonlinear selection are provided.

**2. Selection mechanism of the minimal wave speed**

In this section, we will discuss the speed selection mechanism of the minimal wave speed by applying the upper/lower solution method together with the comparison principle. To make it clear, we give the definitions of upper and lower solutions of (7) as follows.
Definition 2.1 (Upper and lower solutions): If a triple function \((U, V, W)(z)\) is continuous and differentiable on \(R\) except at finite points \(z_i (i = 1, 2, \ldots, n)\), such that
\[
\begin{align*}
- cU'(z) &\geq (\leq) D_2[U](z) + U(z)[1 - b_{12} - b_{13} - U(z) + b_{12}V(z) + b_{13}W(z)], \\
- cV'(z) &\geq (\leq) d_1D_2[V](z) + \alpha[1 - V(z)][b_1U(z) - V(z)], \\
- cW'(z) &\geq (\leq) d_2D_2[W](z) + \beta[1 - W(z)][b_3U(z) - W(z)], \\
(U, V, W)(-\infty) &\geq (\leq)(1, 1, 1), \quad (U, V, W)(\infty) \geq (\leq)(0, 0, 0),
\end{align*}
\]
for given \(c \geq c_0\), then \((U, V, W)(z)\) is called an upper (a lower) solution of system (7).

Next, we start by stating the sufficient condition for the linear selection.

Theorem 2.2 (Linear selection): The minimal wave speed of (7) is linearly selected provided that there is an upper solution \((\bar{U}, \bar{V}, \bar{W})(j - ct)\) for a wave speed \(c = c_0 + \varepsilon\) with \(\varepsilon\) being any a sufficiently small positive number.

\textbf{Proof:} Usually speaking, by using the upper and lower solution method, we need to find some appropriate upper and lower solutions to prove the existence of the travelling wave solution. However, the lower solution can be established by the ideas explored in [1,2,8] such that it always exists for \(c \geq c_0\) for almost any monostable nonlinear systems. This implies that one only needs to construct appropriate upper solution. By the assumption, for \(c = c_0 + \varepsilon\) with \(\varepsilon\) being any a sufficiently small positive number, there is an upper solution. Therefore, the travelling wave exists for \(c = c_0 + \varepsilon\). Letting \(\varepsilon \to 0\), then \(c_0 = c_{\text{min}}\). Thus, according to the definition (1.1), the minimal wave speed of (7) is linearly selected. \(\square\)

As for the nonlinear selection, we can employ the abstract results in [17, Theorem 2.4] to the system (1) to obtain a necessary and sufficient conditions as follows.

Theorem 2.3 (Nonlinear selection): Suppose that \((\underline{U}, \underline{V}, \underline{W})(j - c_1t) \geq 0\) is a lower solution to the system (1.7). Then the minimal wave speed is nonlinearly selected if and only if there exists a speed \(c = c_1 > c_0\) so that \(\underline{U}(z_1), z_1 := j - c_1t\), satisfies \(\limsup_{z_1 \to -\infty} \underline{U}(z_1) < 1\) with the behaviour \(\underline{U}(z_1) \sim e^{-\mu_2z_1}\), where \(\mu_2\) is defined in (11).

3. Explicit conditions

In this section, we show some sufficient conditions of the speed selection for the system (7). Through constructing somewhat delicate upper/lower solutions, new explicit conditions of the linear/nonlinear selection will be given as the formalizations of four theorems.

According to Theorem 2.2, we should seek an upper solution so that the minimal wave speed is linearly selected. To this end, we define \(\bar{U}(z)\) as
\[
\bar{U}(z) = \frac{1}{2} \left( 1 - \tanh \frac{\bar{\mu}z}{2} \right),
\]
where \(\bar{\mu}\) is defined in (12). A direct calculation yields
\[
\bar{U}'(z) = -\bar{\mu} \bar{U}(z)(1 - \bar{U}(z)),
\]
\[ D_2[\overline{U}](z) = \overline{U}(z)(1 - \overline{U}(z)} \frac{\tau(1 - 2\overline{U}(z))}{1 + \tau \overline{U}(z)(1 - \overline{U}(z))}, \]

where \( \tau = e^{-\overline{\mu}} + e^{\overline{\mu}} - 2 \). For convenience, we denote

\[ L_1(U(z), V(z), W(z)) := D_2[U](z) + cU'(z) + U(z) [1 - b_{12} - b_{13} - U(z) + b_{12}V(z) + b_{13}W(z)], \]

\[ L_2(U(z), V(z)) := d_1D_2[V](z) + cV'(z) + \alpha [1 - V(z)][b_{21}U(z) - V(z)], \]

\[ L_3(U(z), W(z)) := d_2D_2[W](z) + cW'(z) + \beta [1 - W(z)][b_{31}U(z) - W(z)]. \]

Substituting (14)–(16) into \( L_1(U(z), V(z), W(z)) \), we have

\[ L_1(\overline{U}(z), \overline{V}(z), \overline{W}(z)) \]

\[ = \overline{U}^2(z)(1 - \overline{U}(z)} \left[ -2\tau + \tau^2F(\overline{U}(z)) + \frac{b_{12}(\overline{V}(z) - \overline{U}(z)) + b_{13}(\overline{W}(z) - \overline{U}(z))}{\overline{U}(z)(1 - \overline{U}(z))} \right], \]

where \( F(\overline{U}(z)) := \frac{-2\overline{U}^2(z) + 3\overline{U}(z) - 1}{1 + \tau \overline{U}(z)(1 - \overline{U}(z))} \). It is easy to calculate that \( \max[F(\overline{U}(z))] = \frac{1}{\tau + 4 + 2\sqrt{\tau + 4}} \). Hence, we have the following estimation

\[ L_1(\overline{U}(z), \overline{V}(z), \overline{W}(z)) \leq \overline{U}^2(z)(1 - \overline{U}(z)} \left[ -\tau \left( 1 + \frac{2}{\sqrt{\tau + 4}} \right) \right. \]

\[ + \left. \frac{b_{12}(\overline{V}(z) - \overline{U}(z)) + b_{13}(\overline{W}(z) - \overline{U}(z))}{\overline{U}(z)(1 - \overline{U}(z))} \right]. \]

**Theorem 3.1:** The minimal wave speed of (7) is linearly selected, if

\[
\begin{align*}
0 < d_1 < 1 + \frac{1 - b_{12} - b_{13} + \alpha}{\tau}, \\
0 < d_2 < 1 + \frac{1 - b_{12} - b_{13} + \beta}{\tau}, \\
\max \left\{ 1, \frac{\alpha b_{21}}{(1 - d_1)\tau + 1 - b_{12} - b_{13} + \alpha}, \frac{\beta b_{31}}{(1 - d_2)\tau + 1 - b_{12} - b_{13} + \beta} \right\} < \min \left\{ b_{21}, b_{31}, \frac{\tau \left( 1 + \frac{2}{\sqrt{\tau + 4}} \right)}{b_{12} + b_{13}} \right\},
\end{align*}
\]

are satisfied.

**Proof:** Let \( \overline{U}(z) \) be the function defined in (14) and \( \overline{V}(z), \overline{W}(z) \) be the functions given by

\[ \overline{V}(z) = \min\{1, P_1 \overline{U}(z)\} = \begin{cases} 1, & z \leq z_1, \\ P_1 \overline{U}(z), & z > z_1, \end{cases} \]

\[ \overline{W}(z) = \min\{1, P_2 \overline{U}(z)\} = \begin{cases} 1, & z \leq z_2, \\ P_2 \overline{U}(z), & z > z_2. \end{cases} \]
Here, without lose of generality we choose $P_1$ and $P_2$ satisfying $P_1 \leq P_2$ and

$$\max \left\{ 1, \frac{\alpha b_{21}}{(1 - d_1)\tau + 1 - b_{12} - b_{13} + \alpha} \right\} < P_1 < \min \left\{ \frac{\tau (1 + \frac{2}{\sqrt{\tau^2 + 4}})}{b_{21}}, \frac{\tau (1 + \frac{2}{\sqrt{\tau^2 + 4}})}{b_{12} + b_{13}} \right\},$$

$$\max \left\{ 1, \frac{\beta b_{31}}{(1 - d_2)\tau + 1 - b_{12} - b_{13} + \beta} \right\} < P_2 < \min \left\{ \frac{\tau (1 + \frac{2}{\sqrt{\tau^2 + 4}})}{b_{31}}, \frac{\tau (1 + \frac{2}{\sqrt{\tau^2 + 4}})}{b_{12} + b_{13}} \right\}.$$  

(23)

We discuss the $V$-equation with four cases.

Case 1. When $z \leq z_1 - 1$, we have $\overline{V}(z + 1) = \overline{V}(z) = \overline{V}(z - 1) = 1$. It then follows that

$$L_2(\overline{U}(z), \overline{V}(z)) = 0.$$  

(24)

Case 2. When $z_1 - 1 < z \leq z_1$, we have $\overline{V}(z - 1) = \overline{V}(z) = 1, \overline{V}(z + 1) = P_1 \overline{U}(z + 1)$. Hence,

$$L_2(\overline{U}(z), \overline{V}(z)) = d_1[P_1 \overline{U}(z + 1) - 1] < 0.$$  

(25)

Case 3. When $z_1 < z \leq z_1 + 1$, it can be seen that $\overline{V}(z - 1) = 1, \overline{V}(z) = P_1 \overline{U}(z), \overline{V}(z + 1) = P_1 \overline{U}(z + 1)$. Consequently, we obtain

$$L_2(\overline{U}(z), \overline{V}(z)) = d_1P_1 \tau \overline{U}(z)(1 - \overline{U}(z))G(\overline{U}) + d_1[1 - P_1 \overline{U}(z - 1)]$$

$$- (\tau + 1 - b_{12} - b_{13})P_1 \overline{U}(z)(1 - \overline{U}(z))$$

$$+ \alpha(b_{21} - P_1)\overline{U}(z)(1 - P_1 \overline{U}(z))$$

$$\leq \overline{U}(z)(1 - \overline{U}(z)) \left[ d_1P_1 G(\overline{U}) - (\tau + 1 - b_{12} - b_{13})P_1 \right.$$

$$+ \alpha(b_{21} - P_1) \frac{1 - P_1 \overline{U}(z)}{1 - \overline{U}(z)} \left. \right]$$

$$\leq \overline{U}(z)(1 - \overline{U}(z)) \left\{ \alpha b_{21} - [(1 - d_1)\tau + 1 - b_{12} - b_{13} + \alpha] P_1 \right\},$$

(26)

where

$$G(\overline{U}(z)) := \frac{1 - 2\overline{U}(z)}{1 + \tau \overline{U}(z)(1 - \overline{U}(z)).}$$  

(27)

Recalling that $\frac{\alpha b_{21}}{(1 - d_1)\tau + 1 - b_{12} - b_{13} + \alpha} < P_1 < b_{21}$ and $0 < d_1 < 1 + \frac{1 - b_{12} - b_{13} + \alpha}{\tau}$, the value of the last function of (26) is no greater than zero, i.e. $L_2(\overline{U}(z), \overline{V}(z)) \leq 0$.

Case 4. When $z > z_1 + 1$, we have $\overline{V}(z + 1) = P_1 \overline{U}(z + 1), \overline{V}(z) = P_1 \overline{U}(z), \overline{V}(z - 1) = P_1 \overline{U}(z - 1)$. By use of these relations and conditions in (20), we know

$$L_2(\overline{U}(z), \overline{V}(z)) = \overline{U}(z)(1 - \overline{U}(z)) \left[ d_1P_1 G(\overline{U}) - (\tau + 1 - b_{12} - b_{13})P_1 \right.$$

$$\left. \right].$$
\[
+\alpha(b_{21} - P_1) \frac{1 - P_1 \overline{U}(z)}{1 - \overline{U}(z)} < 0
\]  

(28)

Similarly, by the assumptions \(0 < d_2 < 1 + \frac{1-b_{12}-b_{13}+\beta}{\tau} \) and \(\frac{\beta b_{31}}{\tau(1-d_2) + (1-b_{12}-b_{13}+\beta)} < P_2 < b_{31}, \) we have

\[
L_3(\overline{U}(z), \overline{W}(z)) < 0.
\]  

(29)

As for the \(U\)-equation, it follows from (19), (21) and (22) that

\[
L_1(\overline{U}(z), \overline{V}(z), \overline{W}(z))
\]

\[
\leq \left\{ \begin{array}{ll}
\overline{U}^2(z)(1 - \overline{U}(z)) \left[ -\tau \left( 1 + \frac{2}{\sqrt{\tau + 4}} \right) + (b_{12} + b_{13})P_1 \right], & \text{if } z \leq z_1, \\
\overline{U}^2(z)(1 - \overline{U}(z)) \left[ -\tau \left( 1 + \frac{2}{\sqrt{\tau + 4}} \right) + b_{12}P_1 + b_{13}P_2 \right], & \text{if } z_1 < z \leq z_2, \\
\overline{U}^2(z)(1 - \overline{U}(z)) \left[ -\tau \left( 1 + \frac{2}{\sqrt{\tau + 4}} \right) + \left( \frac{P_1 - 1}{P_2} \right) b_{12} + b_{13} \right] P_2, & \text{if } z > z_2,
\end{array} \right.
\]

(30)

Since \(1 < P_2 < \frac{\tau}{b_{12} + b_{13}}, \) we have

\[
L_1(\overline{U}(z), \overline{V}(z), \overline{W}(z)) < 0.
\]  

(31)

In conclusion, one can infer that \((\overline{U}(z), \overline{V}(z), \overline{W}(z))\) is an upper solution of system (7). As a result of Theorem 2.2, the linear selection is realized.

\begin{center}
\textbf{\textit{Theorem 3.2:}} The minimal wave speed of (7) is linearly selected, if
\end{center}

\[
\begin{aligned}
0 < d_1 < 1 - \frac{\alpha(b_{21} - 1) - (1 - b_{12} - b_{13})}{\tau}, \\
0 < d_2 < 1 - \frac{\beta(b_{31} - 1) - (1 - b_{12} - b_{13})}{\tau}, \\
\max \{0, 1 - \alpha(b_{21} - 1), 1 - \beta(b_{31} - 1)\} < b_{12} + b_{13} < \min \left\{1, \tau \left( 1 + \frac{2}{\sqrt{\tau + 4}} \right) \right\},
\end{aligned}
\]

(32)

or

\[
\begin{aligned}
0 < d_1 < 1 + \frac{1 - b_{12} - b_{13} - \alpha(b_{21} - 1)}{\tau}, \\
0 < d_2 < 1 + \frac{1 - b_{12} - b_{13} - \beta(b_{31} - 1)}{\tau}, \\
0 < b_{12} + b_{13} < \min \left\{1 - \alpha(b_{21} - 1), 1 - \beta(b_{31} - 1), \tau \left( 1 + \frac{2}{\sqrt{\tau + 4}} \right) \right\},
\end{aligned}
\]

(33)

are satisfied.

\begin{center}
\textbf{\textit{Proof:}} Again, let \(\overline{U}(z)\) be the function defined in (14) and choose \(\overline{V}(z), \overline{W}(z)\) as
\end{center}

\[
\overline{V}(z) = \overline{W}(z) = \overline{U}(z) [a + 1 - a \overline{U}(z)],
\]

(34)

where \(a\) is a constant and satisfies \(0 < a \leq 1\). It is straightforward to compute that

\[
\overline{V}'(z) = \overline{W}'(z) = -\mu(1 - \overline{U}(z))(a + 1 - 2a \overline{U}(z)),
\]

(35)
and
\[ D_2[\bar{V}](z) = \bar{U}(z)(1 - \bar{U}(z)) \left[ (a + 1 - 2a\bar{U}(z))G(\bar{U}(z)) - a\tau\bar{U}(z)(1 - \bar{U}(z))H(\bar{U}(z)) \right], \] (36)

with \( G(\bar{U}) \) as in (27) and
\[
H(\bar{U}(z)) := (\tau + 4) \left( \frac{1}{1 + \tau\bar{U}(z)(1 - \bar{U}(z))} - \frac{1}{\tau + 4} \right)^2 - \frac{1}{\tau + 4}. \] (37)

Plugging (34)–(36) into the \( V \)-equation, we have
\[
L_2(\bar{U}(z), \bar{V}(z)) = \bar{U}(z)(1 - \bar{U}(z))(a + 1 - 2a\bar{U}(z))
\times \left[ d_1\tau(G(\bar{U}(z)) - P(\bar{U}(z))H(\bar{U}(z))) - c\bar{\mu} + \alpha Q(\bar{U}(z)) \right] \] (38)

with
\[
P(\bar{U}(z)) := \frac{a\bar{U}(z)(1 - \bar{U}(z))}{a + 1 - 2a\bar{U}(z)},
\]
\[
Q(\bar{U}(z)) := \frac{(1 - a\bar{U}(z))(b_{21} - a - 1 + a\bar{U}(z))}{a + 1 - 2a\bar{U}(z)}. \] (39)

Notice that \( \bar{U}(z) \in [0, 1] \), one can check that \( \max\{G(\bar{U})\} = 1, \max\{P(\bar{U})\} = 0, \min\{H(\bar{U})\} = \frac{8}{\tau + 4}>0 \) and \( \max\{Q(\bar{U})\} = b_{21} - 1 \), which ensure that
\[
L_2(\bar{U}(z), \bar{V}(z)) \leq \bar{U}(z)(1 - \bar{U}(z))(a + 1 - 2a\bar{U}(z))
\times [(d_1 - 1)\tau - 1 + b_{12} + b_{13} + \alpha(b_{21} - 1)]. \] (40)

Obviously, the assumption \( 0 < d_1 < 1 - \frac{\alpha(b_{21} - 1) - (1 - b_{12} - b_{13})}{\tau} \) together with \( 1 > b_{12} + b_{13} > 1 - \alpha(b_{21} - 1) \) or the assumption \( 0 < d_1 < 1 + \frac{1 - b_{12} - b_{13} - \alpha(b_{21} - 1)}{\tau} \) together \( 0 < b_{12} + b_{13} < 1 - \alpha(b_{21} - 1) \) implies that
\[
L_2(\bar{U}(z), \bar{V}(z)) < 0. \] (41)

By taking a similar argument, we can show that
\[
L_3(\bar{U}(z), \bar{W}(z)) = d_2D_2[\bar{W}](z) + c\bar{W}'(z) + \beta[1 - \bar{W}(z)][b_{31}\bar{U}(z) - \bar{W}(z)] < 0, \] (42)

provided that \( 0 < d_2 < 1 - \frac{\beta(b_{31} - 1) - (1 - b_{12} - b_{13})}{\tau} \) combing with \( 1 > b_{12} + b_{13} > 1 - \beta(b_{31} - 1) \) or \( 0 < d_2 < 1 + \frac{1 - b_{12} - b_{13} - \beta(b_{31} - 1)}{\tau} \) combing with \( 0 < b_{12} + b_{13} < 1 - \beta(b_{31} - 1) \), which holds according to (32).

As far as the \( U \)-equation is concerned, one can substitute (34) into (19) to conclude that
\[
L_1(\bar{U}(z), \bar{V}(z), \bar{W}(z)) \leq \bar{U}^2(z)(1 - \bar{U}(z)) \left[ -\tau \left( 1 + \frac{2}{\sqrt{\tau + 4}} \right) + b_{12} + b_{13} \right] < 0. \] (43)

Here, we have used the inequality \( 0 < b_{12} + b_{13} < \min\{1, \tau(1 + \frac{2}{\sqrt{\tau + 4}})\} \).
In view of (41), (42) and (43), one can deduce that \((\overline{U}(z), \overline{V}(z), \overline{W}(z))\) is an upper solution of system (7). By Theorem 2.2, the proof is completed. ■

Next, we construct several special lower solutions to consider nonlinear selection of the minimal wave speed for the travelling waves.

**Theorem 3.3:** The minimal wave speed of (7) is nonlinearly selected provided that

\[
\begin{align*}
1 > b_{12} + b_{13} &> \tau^2 + 2\tau, \\
b_{21} &> 2 + \frac{(3\tau + \frac{1}{2}\tau^2)d_1 + 2\tau + 2(1 - b_{12} - b_{13})}{\alpha}, \\
b_{31} &> 2 + \frac{(3\tau + \frac{1}{2}\tau^2)d_2 + 2\tau + 2(1 - b_{12} - b_{13})}{\beta}.
\end{align*}
\]  

(44)

**Proof:** Define \(\overline{U}(z)\) as

\[
\overline{U}(z) = \frac{1}{2} \left(1 - \tanh \frac{\mu_2 z}{2}\right),
\]  

(45)

where \(\mu_2 := \mu_2(c)\) with \(c = c_0 + \epsilon\), and \(c_0\) is defined in (10), \(\epsilon > 0\) is an enough small number. A simple calculation gives

\[
\begin{align*}
\overline{U}'(z) &= -\mu_2 \overline{U}(z)(1 - \overline{U}(z)), \\
D_2[\overline{U}](z) &= \overline{U}(z)(1 - \overline{U}(z)) \frac{\lambda(1 - 2\overline{U}(z))}{1 + \lambda \overline{U}(z)(1 - \overline{U}(z))},
\end{align*}
\]  

(46) \hspace{1cm} (47)

where \(\lambda = e^{-\mu_2} + e^{\mu_2} - 2\). Substituting the formulas (45)–(47) into (17) leads to

\[
L_1(\overline{U}(z), \overline{V}(z), \overline{W}(z))
= \overline{U}^2(z)(1 - \overline{U}(z)) \left[-2\lambda + \lambda^2 F(\overline{U}(z)) + \frac{b_{12}(\overline{V}(z) - \overline{U}(z)) + b_{13}(\overline{W}(z) - \overline{U}(z))}{\overline{U}(z)(1 - \overline{U}(z))}\right],
\]  

(48)

where \(F(\overline{U}(z)) := \frac{-2\overline{U}(z)^2 + 3\overline{U}(z) - 1}{1 + \lambda \overline{U}(z)(1 - \overline{U}(z))}\).

By selecting

\[
\overline{V}(z) = \overline{W}(z) = \overline{U}(z)(2 - \overline{U}(z)),
\]  

(49)

we have

\[
\overline{V}'(z) = \overline{W}'(z) = -2\mu_2 \overline{U}(z)(1 - \overline{U}(z))^2,
\]  

(50)

and

\[
D_2[\overline{V}](z) = D_2[\overline{W}](z) = \overline{U}(z)(1 - \overline{U}(z))^2 [2\lambda G(\overline{U}(z)) - \lambda \overline{U}(z) H(\overline{U}(z))],
\]  

(51)

where

\[
G(\overline{U}(z)) = \frac{1 - 2\overline{U}(z)}{1 + \lambda \overline{U}(z)(1 - \overline{U}(z))},
\]
\[ H(U(z)) = (\lambda + 4) \left( \frac{1}{1 + \lambda U(z)(1 - U(z))} - \frac{1}{\lambda + 4} \right)^2 - \frac{1}{\lambda + 4}. \] (52)

For the \(U\)-equation, substituting (49) into (48) yields
\[
L_1(U(z), V(z), W(z)) = U^2(z)(1 - U(z)) \left[ -2\lambda + \lambda^2 F(U(z)) + b_{12} + b_{13} \right]
\geq U^2(z)(1 - U(z))[-2\lambda - \lambda^2 + b_{12} + b_{13}]
\rightarrow U^2(z)(1 - U(z))(-2\tau - \tau^2 + b_{12} + b_{13}),
\] (53)
as \(\epsilon \to 0^+\), where the first inequality in (44) ensures that
\[ L_1(U(z), V(z), W(z)) > 0. \] (54)

For the \(V\)-equation, by virtue of (49)–(51) and by letting \(\epsilon \to 0^+\), it follows that
\[
L_2(U(z), V(z)) = U(z)(1 - U(z))^2 \left[ d_1 \lambda (2G(U(z) - U(z)H(U(z))) - 2c\mu_2 
+ \alpha(b_{21} - 2 + U(z)) \right]
\rightarrow U(z)(1 - U(z))^2 \left[ - \left( 3\tau + \frac{1}{2}\tau^2 \right) d_1 - 2\tau - 2(1 - b_{12} - b_{13}) 
+ \alpha(b_{21} - 2) \right]
> 0.
\] (55)

where the second inequality in (3.3) has been used.

For the \(W\)-equation, by the third inequality in (3.3), we arrive at
\[
L_2(U(z), W(z)) \rightarrow U(z)(1 - U(z))^2 \left[ - \left( 3\tau + \frac{1}{2}\tau^2 \right) d_2 - 2\tau - 2(1 - b_{12} - b_{13}) 
+ \beta(b_{31} - 2) \right]
> 0.
\] (56)
The combination of (54), (55) and (56) implies that \((U(z), V(z), W(z))\) is a lower solution to the system (7). Then by Theorem 2.3, the proof is accomplished. \(\blacksquare\)

**Theorem 3.4:** The minimal wave speed of (7) is nonlinearly selected provided that
\[
1 - \tau^2 - 2\tau > \max \left\{ \frac{(d_1 + 1)\tau + \tau^2 + (1 - b_{12} - b_{13}) + \alpha}{\alpha b_{21}}, \frac{(d_2 + 1)\tau + \tau^2 + (1 - b_{12} - b_{13}) + \beta}{\beta b_{31}} \right\}.
\] (57)

**Proof:** Clearly, the condition (57) enables us to pick up a number \(k\) to satisfy
\[
1 - \tau^2 - 2\tau > k > \max \left\{ \frac{(d_1 + 1)\tau + \tau^2 + (1 - b_{12} - b_{13}) + \alpha}{\alpha b_{21}} \right\},
\]
\[(d_2 + 1)\tau + \tau^2 + (1 - b_{12} - b_{13}) + \beta \\bigg/ \beta b_{31}\]. \quad (58)

Now, for such a number \(k\), we redefine \(U(z)\) as
\[
U(z) = \frac{k}{2} \left( 1 - \tanh \frac{\mu_2 z}{2} \right)
\]
and \(V(z)\) and \(W(z)\) as
\[
V(z) = W(z) = \frac{U(z)}{k}.
\]

Through a direct calculation, we have
\[
U'(z) = -\mu_2 U(z) \left( 1 - \frac{U(z)}{k} \right)
\]
and
\[
V'(z) = W'(z) = -\mu_2 \frac{U(z)}{k} \left( 1 - \frac{U(z)}{k} \right).
\]

Moreover, we need to repress the second-order difference terms as
\[
D_2[U](z) = U(z) \left( 1 - \frac{U(z)}{k} \right) \frac{\lambda \left( 1 - \frac{2U(z)}{k} \right)}{1 + \lambda \frac{U(z)}{k} \left( 1 - \frac{U(z)}{k} \right)}
\]
and
\[
D_2[V](z) = D_2[W](z) = \frac{U(z)}{k} \left( 1 - \frac{U(z)}{k} \right) \frac{\lambda \left( 1 - \frac{2U(z)}{k} \right)}{1 + \lambda \frac{U(z)}{k} \left( 1 - \frac{U(z)}{k} \right)}.
\]

Substituting (59)–(63) into (17), we obtain
\[
L_1(U(z), V(z), W(z)) = \frac{U^2(z)}{k} \left( 1 - \frac{U(z)}{k} \right) \left[ -2\lambda + \lambda^2 \frac{U(z)}{k} \left( \frac{1 - k}{1 - \frac{U(z)}{k}} \right) \right] \geq \frac{U^2(z)}{k} \left( 1 - \frac{U(z)}{k} \right) \left( -2\lambda - \lambda^2 + 1 - k \right),
\]
\quad (65)
where

\[
F\left( \frac{U(z)}{k} \right) := -2 \frac{U^2(z)}{k} + 3 \frac{U(z)}{k} - 1 + \lambda \frac{U(z)}{k} (1 - \frac{U(z)}{k}).
\]

Due to the fact that \( \mu_2 \to \bar{\mu} \) and \( e^{-\mu} + e^{\mu} - 2 \to 0 \) as \( \epsilon \to 0^+ \), one can see from the left side part of (58) that

\[
L_1(U(z), V(z), W(z)) \to \frac{U^2(z)}{k} \left( 1 - \frac{U(z)}{k} \right) (-2\tau - \tau^2 + 1 - k) \geq 0.
\]

Inserting (60)–(64) into the \( V \)-equation and making use of the right part of (58), it follows by setting \( \epsilon \to 0^+ \) that

\[
L_2(U(z), V(z)) = \frac{U(z)}{k} \left( 1 - \frac{U(z)}{k} \right) \left[ d_1 \left( -\lambda - \lambda^2 F\left( \frac{U(z)}{k} \right) \right) - c\mu_2 + \alpha(b_{21}k - 1) \right]
\]

\[
\geq \frac{U(z)}{k} \left( 1 - \frac{U(z)}{k} \right) \left[ -(d_1 + 1)\tau - \tau^2 - (1 - b_{12} - b_{13}) + \alpha(b_{21}k - 1) \right] > 0.
\]

For the \( W \)-equation of system (7), we can deduce in a similar manner as the \( V \)-equation that

\[
L_3(U(z), W(z)) \geq \frac{U(z)}{k} \left( 1 - \frac{U(z)}{k} \right) \left[ -(d_2 + 1)\tau - \tau^2 - (1 - b_{12} - b_{13}) + \beta(b_{31}k - 1) \right] > 0.
\]

Therefore, \((U(z), V(z), W(z))\) is a lower solution of system (7). As a result, Theorem 2.3 indicates that the minimal wave speed is nonlinear selected. The proof is completed. ■

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