Noncommutative Graphene

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Abstract

We consider a noncommutative description of graphene. This description consists of a Dirac equation for massless Dirac fermions plus noncommutative corrections, which are treated in the presence of an external magnetic field. We argue that, being a two-dimensional Dirac system, graphene is particularly interesting to test noncommutativity. We find that momentum noncommutativity affects the energy levels of graphene, but that it does not entail any kind of correction to the Hall conductivity.
I. INTRODUCTION

Graphene is a two-dimensional configuration of carbon atoms organized in a hexagonal honeycomb structure [1-3]. Often, a crystal lattice is a Bravais lattice, that is, an infinite array of discrete points with an organization and orientation that appears exactly the same, from whatever point the array is viewed. However, the graphene hexagonal lattice is non-Bravais, as only the next-to-nearest neighbor points appear with the same organization and orientation. In the case of graphene one has two triangular Bravais lattices, $A$ and $B$ which together form the non-Bravais graphene lattice, and the difference between them is a rotation of $\pi$. The hexagonal lattice belongs to the class of bipartite lattices, and so one can say that graphene is a bipartite non-Bravais lattice with two carbon atoms per unit cell. For the two sub-lattices one has the same primitive vectors, which depend explicitly on the distance between the two lattice points. With these primitive vectors one can characterize any space point as a linear combination of them. In the momentum space, one can obtain the reciprocal primitive vectors and the corresponding Brillouin zones, i.e. a uniquely defined primitive cell in the reciprocal space. The first Brillouin zone forms a hexagon, which is rotated by $\pi/12$ compared to the hexagonal structure in position space and the corners of the first Brillouin zone are usually organized in two sets, the Dirac points $K$ and $K'$. They are six Dirac points in the total, but only two are worth considering due to the periodicity of the momenta in the Brillouin zone [2].

It turns out that graphene’s low energy excitations are relativistic corresponding to massless, quasi-free fermions that can be theoretically described by the Dirac equation for these particles [3]. Thus, one considers the Dirac equation at the vicinity of the Dirac points $K$ and $K'$. Expanding the dispersion relation around these points, one has to a first order approximation a linear relation, which, for small energies, gives origin to the so-called Dirac cones. These cones imply that graphene can be seen as a conventional semiconductor, given that there is no gap between conduction and valence bands.

We start with Dirac equation,

$$ i\hbar \frac{\partial \psi}{\partial t} = H_D \psi , $$

where the wave function $\psi$ in the graphene case describes the electron states around the Dirac points $K$ and $K'$, and the Dirac Hamiltonian is given by [4]

$$ H_D = c[\vec{\alpha} \cdot \vec{P} + \beta mc] , $$

(1)
where $\bar{\alpha}$ and $\beta$ are the Dirac matrices and $\vec{P} = (-i\hbar \partial_x, -i\hbar \partial_y, 0)$. In the case of graphene, one has massless particles that move through the honeycomb lattice with a velocity $v_F \sim 10^6 \text{ms}^{-1}$, the so-called Fermi velocity. Thus, for instance, the Dirac Hamiltonian around the Dirac point $K$ reads
\[ H = v_F \vec{\sigma} \cdot \vec{P}. \] (3)

where, $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ and $\sigma_x$, $\sigma_y$ and $\sigma_z$ are the Pauli matrices. The same Hamiltonian can be written to the Dirac Point $K'$ with $\vec{\sigma}^*$ given by $\vec{\sigma}^* = (\sigma_x, -\sigma_y, \sigma_z)$. We can now write the Hamiltonian for the two Dirac points as 4,
\[ H_D = \begin{pmatrix} H_K & 0 \\ 0 & H_{K'} \end{pmatrix} = v_F \begin{pmatrix} 0 & p_x - ip_y & 0 & 0 \\ p_x + ip_y & 0 & 0 & 0 \\ 0 & 0 & 0 & p_x + ip_y \\ 0 & 0 & p_x - ip_y & 0 \end{pmatrix}. \] (4)

As mentioned, the wave function $\psi$ consists of two components, one describing $K$ and other $K'$. Moreover, for these two Dirac points one has an eigenvector that describes the probability of an electron state to be on sub-lattice A in the upper component, or on the sub-lattice B in the lower component of the eigenstate. Thus,
\[ \psi = \begin{pmatrix} \psi^K \\ \psi^{K'} \end{pmatrix}, \] (5)

where $\psi^K$ and $\psi^{K'}$ are two dimensional eigenstates,
\[ \psi^K = \begin{pmatrix} \phi^A \\ \phi^B \end{pmatrix}, \quad \psi^{K'} = \begin{pmatrix} \phi^{A'} \\ \phi^{B'} \end{pmatrix}. \] (6)

To obtain the dispersion relation for the energy one has to solve the following eigenvalue problem for each Dirac point,
\[ H_K \psi^K = E_K \psi^K, \]
\[ H_{K'} \psi^{K'} = E_{K'} \psi^{K'}. \] (7)

We have for the eigenvalues, for the two Dirac points $K$ and $K'$, 4
\[ E_{K,K'} = \pm \hbar v_F |\vec{k}| \] (8)

and to evaluate the eigenvectors of the system we consider Eq. (4) in the momentum representation,
\[ (\vec{\sigma} \cdot \hat{k}) \psi^K(\vec{k}) = \pm \psi^K(\vec{k}) \] (9)
\[ (\vec{\sigma}^* \cdot \hat{k}) \psi^{K'}(\vec{k}) = \pm \psi^{K'}(\vec{k}) \] (10)
where $\vec{\sigma}$ and $\vec{\sigma}^*$ are defined above. Moreover, these eigenvectors in the momentum space, for the two Dirac points, have the form

$$
\psi^K(\vec{k}) = \frac{1}{\sqrt{2}} \begin{pmatrix}
e^{-i\phi_k/2} \\
\pm e^{i\phi_k/2}
\end{pmatrix}, \quad \psi^{K'} = \frac{1}{\sqrt{2}} \begin{pmatrix}
e^{i\phi_k/2} \\
\pm e^{-i\phi_k/2}
\end{pmatrix},
$$

(11)

where $\phi_k$ is the polar angle of vector $\vec{k}$ in the momentum space. The signs $\pm$ correspond to the eigenvalues of the energy spectrum for each Dirac point, given by Eq. (8). One clearly sees that the pseudo spinor $\psi(\vec{k})$ has a definite pseudo-helicity. Furthermore, the existence of chiral symmetry, in the previous eigenvectors, allows for the observation of the anomalous quantum Hall effect [3–6]. This is a striking feature of graphene, which instead of the usual Landau levels for a semiconductor in a magnetic field, it exhibits a different degeneracy factor. This is because the fundamental energy level of the graphene has two valleys [5, 6].

It has been argued in various instances that noncommutativity should be considered as a fundamental feature of space-time at the Planck scale [7]. Various noncommutative field theory models [8–11] have been discussed as well as many extensions of quantum mechanics [12–21]. Of particular interest is the so-called phase-space noncommutativity which has been investigated in the context of quantum cosmology [22], black holes physics and the singularity problem [23–25]. The phase-space noncommutative algebra is given by

$$
[\hat{x}_i, \hat{x}_j] = i\theta_{ij}, \quad [\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}, \quad [\hat{p}_i, \hat{p}_j] = i\eta_{ij}, \quad i, j = 1, \ldots, d
$$

(12)

where $\eta_{ij}$ and $\theta_{ij}$ are antisymmetric real constant $(d \times d)$ matrices and $\delta_{ij}$ is the identity matrix. The key property of this extended algebra is that it is related to the standard Heisenberg-Weyl algebra:

$$
[\hat{x}'_i, \hat{x}'_j] = 0, \quad [\hat{x}'_i, \hat{p}'_j] = i\hbar\delta_{ij}, \quad [\hat{p}'_i, \hat{p}'_j] = 0, \quad i, j = 1, \ldots, d,
$$

(13)

by a class of linear (non-canonical) transformations:

$$
\hat{x}_i = \hat{x}_i \left( \hat{x}'_j, \hat{p}'_j \right) \quad \hat{p}_i = \hat{p}_i \left( \hat{x}'_j, \hat{p}'_j \right).
$$

(14)

More recently, a new representation of noncommutativity has been proposed [26]. This representation uses the Pauli matrices as the fundamental elements of the noncommutative algebra. It allows for a noncommutative extension of graphene by treating it as a quantum charged particle subjected to an electromagnetic field [27].
Since graphene is a two-dimensional system it is expected to supply an interesting model where to test noncommutativity. First, because it is a special system where quantum relativistic phenomena, typical of high-energy physics, arise at low-energies. Second because in non-relativistic quantum mechanics time is always a commutative variable and so noncommutativity, in this context, can only be applied to the spatial variables. Since any sympletic form in an odd dimension is always degenerate, after a linear transformation we can only have noncommutativity in two dimensions. Graphene provides a real two-dimensional physical system. Various bounds and inequalities, which appear in the context of two-dimensional noncommutative quantum mechanics, where derived in [19].

In this work, we extend the model of graphene to a noncommutative phase-space setting. We consider graphene in an external magnetic field and a noncommutative geometry. In section II A we review the concepts behind the graphene model in the presence of a magnetic field. Then, in section II B we present the noncommutative extension of the graphene in a constant magnetic field. In section II C we examine the effect of noncommutativity on graphene’s anomalous quantum Hall effect. In section III we compare the experimental values obtained for the energy levels of graphene with our theoretical predictions and obtain a bound for the noncommutative parameter $\eta$. Finally, in section IV, we summarize the main conclusions and results.

II. GRAPHENE IN AN EXTERNAL MAGNETIC FIELD

In this section, we start by obtaining the energy dispersion relation for a layer of graphene subjected to an external constant magnetic field. We then show how to extend the problem to a phase-space noncommutative setting, and consider the implications of this extension on the anomalous quantum Hall effect.

A. The Commutative Case

Let us consider a layer of graphene in a external constant magnetic field, $\vec{B} = B\hat{e}_z$. In our units $c = 1$. We introduce the $\vec{B}$-field through the minimal coupling to the vector potential, such that $\vec{B} = \vec{\nabla} \times \vec{A}$

$$\vec{P} \rightarrow \vec{P} - e\vec{A},$$

(15)
where
\[ \vec{A} = \frac{B}{2} (-y, x, 0) \, , \quad (16) \]
and \( e \) is the charge of the electron. Thus, for the two Dirac points \( K \) and \( K' \) the Hamiltonians read \[ \underline{2} \):
\[
H_K = v_F \begin{pmatrix} 0 & p_x - ip_y + \frac{eB}{2} (y + ix) \\ p_x + ip_y + \frac{eB}{2} (y - ix) & 0 \end{pmatrix} \, , \quad (17)
\]
\[
H_{K'} = v_F \begin{pmatrix} 0 & p_x + ip_y + \frac{eB}{2} (y - ix) \\ p_x - ip_y + \frac{eB}{2} (y + ix) & 0 \end{pmatrix} \, \quad (18)
\]
and the energy eigenvalue equation for the wave function Eq. (5), at the Dirac point \( K \), is then:
\[
\frac{\hbar v_F}{\xi} \left[ -i \left( \xi \partial_x - \frac{x}{\xi} \right) + \left( -\xi \partial_y + \frac{y}{\xi} \right) \right] \phi^B = E_K \phi^A \, ,
\]
\[
\frac{\hbar v_F}{\xi} \left[ -i \left( \xi \partial_x + \frac{x}{\xi} \right) + \left( \xi \partial_y + \frac{y}{\xi} \right) \right] \phi^A = E_K \phi^B \, ,
\]
where \( \xi = \sqrt{\frac{\hbar}{eB}} = \sqrt{2} l_B \) is an auxiliary variable and \( l_B \) is the so-called magnetic length \[ \underline{2} \] .

Redefining the variables as,
\[
\partial_{\bar{x}} = \xi \partial_x \, , \quad \partial_{\bar{y}} = \xi \partial_y \, ,
\]
\[
\bar{x} = \frac{x}{\xi} \, , \quad \bar{y} = \frac{y}{\xi} \, ,
\]
the system, Eq. (19), turns into
\[
\frac{\hbar v_F}{\xi} \left[ -i \left( \partial_{\bar{x}} - \bar{x} \right) + \left( -\partial_{\bar{y}} + \bar{y} \right) \right] \phi^B = E_K \phi^A \, ,
\](a)
\[
\frac{\hbar v_F}{\xi} \left[ -i \left( \partial_{\bar{x}} + \bar{x} \right) + \left( \partial_{\bar{y}} + \bar{y} \right) \right] \phi^A = E_K \phi^B \, .
\](b)

Solving Eq. (21b) for \( \phi^B \) and substituting into Eq. (21a), yields
\[
\left( \frac{\hbar v_F}{\xi} \right)^2 \left[ -i \left( \partial_{\bar{x}} - \bar{x} \right) + \left( -\partial_{\bar{y}} + \bar{y} \right) \right] \left[ -i \left( \partial_{\bar{x}} + \bar{x} \right) + \left( \partial_{\bar{y}} + \bar{y} \right) \right] \phi^A = E_K^2 \phi^A \, ,
\]
which is a second order equation analogous to the energy eigenvalue equation for the quantum harmonic oscillator in two dimensions. It can be solved using the set of annihilation and creation operators,
\[
a_x = \frac{1}{\sqrt{2}} (\bar{x} + \partial_{\bar{x}}) \, , \quad a_x^\dagger = \frac{1}{\sqrt{2}} (\bar{x} - \partial_{\bar{x}}) \, ,
\]
\[
a_y = \frac{1}{\sqrt{2}} (\bar{y} + \partial_{\bar{y}}) \, , \quad a_y^\dagger = \frac{1}{\sqrt{2}} (\bar{y} - \partial_{\bar{y}}) \, .
\]
However, it is more convenient to use left/right operators, as in the case of the quantum harmonic oscillator in two dimensions. These new operators are responsible not only for adding (subtracting) a quantum of energy, but also for adding (subtracting) a quantum of angular momentum, 院长, in the direct direction (right operators with subscript \( d \)) or in the inverse direction (left operators with subscript \( e \)). Furthermore they simplify considerably the calculations. Thus, one introduces the left/right operators as,

\[
a_d = \frac{1}{\sqrt{2}}(a_x - ia_y), \quad a_d^\dagger = \frac{1}{\sqrt{2}}(a_x^\dagger + ia_y^\dagger),
\]

\[
a_e = \frac{1}{\sqrt{2}}(a_x + ia_y), \quad a_e^\dagger = \frac{1}{\sqrt{2}}(a_x^\dagger - ia_y^\dagger).
\]

(24)

So, Eq. (22) can now be rewritten in terms of these left/right operators,

\[
4 \left( \frac{\hbar v_F}{\xi} \right)^2 (a_e^\dagger a_e) \phi^A = E_K^2 \phi^A.
\]

(25)

One sees that the Hamiltonian depends only on the left operators. Note that for the angular momentum one has \( l_z = \hbar (n_d - n_e) \equiv \hbar m \), where \( n_e \) and \( n_d \) are the eigenvalues associated to the left and right number operators \( (N_e = a_e^\dagger a_e, \ N_d = a_d^\dagger a_d) \) respectively; \( m \) is the eigenvalue associated to the angular momentum operator in the \( z \)-direction. Let us suppose that \( \phi^A = \phi^A_{(n_e, n_d)} \), then the energy spectrum for the Dirac point \( K \) is given by

\[
E_K = \pm 2\frac{\hbar v_F}{\xi} \sqrt{n_e} = \pm \sqrt{2\hbar v_F} \frac{l_B}{\xi} \sqrt{n_e},
\]

(26)

with \( n_e = 0, 1, 2, ... \).

Finally, if we consider the eigenvalue problem, Eq. (21b), and substitute the eigenvalue \( E_K \) obtained in Eq. (26), we get the eigenstates for the Dirac \( K \) point.

\[
\psi^{K}_{(n_e, n_d)} = \begin{pmatrix} \phi^A_{(n_e, n_d)} \\ \pm i\phi^B_{(n_e-1, n_d)} \end{pmatrix}, \quad \psi^{K'}_{(n_e, n_d)} = \begin{pmatrix} \phi^A_{(n_e-1, n_d)} \\ \pm i\phi^B_{(n_e, n_d)} \end{pmatrix}.
\]

(27)

We also included the eigenstates for the Dirac point \( K' \) which can be obtained using the same method and display a dispersion relation identical to the one given by Eq. (26).

### B. The Noncommutative Case

In this section we consider a phase-space noncommutative algebra as in Eq. (12). In order to relate the noncommutative variables \( (x_i, p_i) \) with the commutative ones \( (x'_i, p'_i) \) we
use the following SW map \(7, 17\) (see also \[18\])

\[
x = x' - \frac{\theta}{2\hbar} p'_y, \quad p_x = p'_x + \frac{\eta}{2\hbar} y', \\
y = y' + \frac{\theta}{2\hbar} p'_x, \quad p_y = p'_y - \frac{\eta}{2\hbar} x',
\]

\[\text{(28)}\]

where \(\theta\) and \(\eta\) are real constant parameters. The noncommutative variables then satisfy the algebra

\[
[x_i, x_j] = i\theta_{ij}, \quad [p_i, p_j] = i\eta_{ij}, \quad [x_i, p_j] = i\hbar_{\text{eff}} \delta_{ij} = i\hbar \delta_{ij} \left(1 + \frac{\theta \eta}{4\hbar^2}\right), \quad \text{(29)}
\]

\(i, j = 1, 2\). This is the noncommutative algebra Eq.(12 with an effective Planck constant \[17, 18\]. It reduces to Eq.(12 exactly when \(\theta = 0\), this being the case we are going to consider.

Using the same potential vector as in section IIA, Eq. (16), and substituting the noncommutative variables by the commutative ones, through the SW map, Eq. (28), we get,

\[
(p - eA)^{NC} = \begin{pmatrix}
p_x + \frac{eB}{2} y \\
p_y - \frac{eB}{2} x
\end{pmatrix} = \begin{pmatrix}
\lambda p'_x + \frac{eB}{2} \mu y' \\
\lambda p'_y - \frac{eB}{2} \mu x'
\end{pmatrix},
\]

where

\[
\lambda = \left(1 + \frac{eB\theta}{4\hbar}\right), \quad \mu = \left(1 + \frac{\eta}{eB\hbar}\right). \quad \text{(30)}
\]

Thus, for the Dirac point \(K\), one gets the following Hamiltonian:

\[
H_K = v_F \begin{pmatrix}
0 & \lambda(p'_x - ip'_y) + \frac{eB}{2} \mu(y' + ix') \\
\lambda(p'_x + ip'_y) + \frac{eB}{2} \mu(y' - ix') & 0
\end{pmatrix}. \quad \text{(31)}
\]

A straightforward comparison with Hamiltonian Eq. (17) shows that noncommutativity reveals itself through constants \(\lambda\) and \(\mu\). For future convenience one introduces the constant

\[
\gamma = \sqrt{\frac{2\hbar \lambda}{eB \mu}} = l_B \sqrt{\frac{2\lambda}{\mu}}. \quad \text{(32)}
\]

In what follows we shall consider only momenta noncommutativity, since in the general Dirac problem, configuration space noncommutativity leads to the breaking of gauge symmetry \[11\], a symmetry preserved in the graphene lattice \[29\]. Indeed, if one evaluates the velocity of a charged particle,

\[
v = \frac{i}{\hbar} [H_{NC}, r], \quad \text{(33)}
\]
we obtain an extra term depending on the $\theta$ parameter, and not the expected result $v = v_F \sigma$. Hence $\lambda = 1$ and

$$\gamma = \sqrt{\frac{2\hbar}{eB\mu}} = l_B \sqrt{\frac{2}{\mu}}. \quad (34)$$

Thus, the equations to be solved are now,

$$\frac{\hbar v_F}{\gamma} \left[-i \left(\gamma \partial_x - \frac{x}{\gamma}\right) + \left(-\gamma \partial_y + \frac{y}{\gamma}\right)\right] \phi^B = E_K \phi^A, \quad (35)$$

Following the strategy discussed in the last section, one obtains for the energy spectrum

$$E_{NC}^K = \pm 2\hbar v_F \sqrt{\frac{eB}{2\hbar}} \left(1 + \frac{\eta}{eB\hbar}\right) n_e$$

$$= \pm \frac{\hbar v_F}{l_B} 2 \left(1 + \frac{l_B^2\eta^2}{\hbar^2}\right) n_e, \quad (36)$$

where $n_e$ is a non-negative integer. We clearly see that the energy at the Dirac point $K$ depends explicitly on the noncommutative parameter associated with the momenta. Moreover, one concludes that the noncommutative effect is coupled with the magnetic field. Of course, the same dispersion relation for energy is found to the other Dirac point, $K'$. Furthermore, the eigenvectors for the two Dirac points can be evaluated. One has

$$\psi_K^{(n_e,n_d)} = \begin{pmatrix} \phi_A^{(n_e,n_d)} \\ \pm i\phi_B^{(n_e,n_d)} \end{pmatrix}, \quad \psi_{K'}^{(n_e,n_d)} = \begin{pmatrix} \phi_A^{(n_e-1,n_d)} \\ \pm i\phi_B^{(n_e,n_d)} \end{pmatrix}. \quad (37)$$

The eigenvectors can be written in polar coordinates [28]

$$\phi_L^{(n_e,n_d)} = \mathcal{F}(\rho)e^{imp\varphi}e^{-\sigma \rho^2}, \quad (38)$$

where $L = A, B, \sigma > 0$ is some constant, $\rho^2 = x^2 + y^2$, $\mathcal{F}(\rho)$ is some polynomial of $\rho$ and $m$ is the quantum number associated to the angular momentum $l_z$, which is the integer $(m = n_d - n_e)$.

Clearly, the zero-energy level for this system is $E_{NC}^K = E_{NC}^{K'} = 0$. However, the eigenvectors for the fundamental level are

$$\psi_K^{(0,n_d)} = \begin{pmatrix} \phi_A^{(0,n_d)} \\ 0 \end{pmatrix}, \quad \psi_{K'}^{(0,n_d)} = \begin{pmatrix} 0 \\ \pm i\phi_B^{(0,n_d)} \end{pmatrix}, \quad (39)$$

meaning that the zero energy for the two Dirac points are associated with two linearly independent electron states.
C. Quantum Hall effect and noncommutativity

We conclude from the discussion in the previous section that an external magnetic field perpendicular to a graphene sheet renders the energy spectrum discrete. It is known that a charged particle subjected to an electromagnetic field has a discrete energy spectrum, the Landau quantization. However, graphene is a special case, since it is actually a Dirac problem instead of a Schrödinger one [3]. The difference between each Landau level is considerable. This difference in the energy levels has implications for the quantum Hall effect (QHE).

The QHE is observable in two-dimensional metals, as for instance in bound low-temperature surfaces where electrons are constrained to two dimensions [2]. That occurs when the temperature is drastically reduced, the Hall resistivity becomes independent of the magnetic field, and a quantized Hall plateau is formed. Experimentally, the Hall conductivity is given by

\[ \sigma_{xy} = \frac{e^2}{h} \nu , \]  

(40)

where \( \nu \) is an integer number and one has the Integer quantum Hall effect (IQHE). However, the QHE observed in graphene is anomalous (AQHE) [5, 6]. The difference between the IQHE and the AQHE lies in the fundamental energy level, i.e. for the IQHE the first Landau level is observable at zero energy, but for AQHE the first Hall plateau appears when the lowest Landau level is half filled, and the conductivity takes the form,

\[ \sigma_{xy} = \pm 4 \left( \nu + \frac{1}{2} \right) \frac{e^2}{h} , \]  

(41)

where the factor \( 4(\nu + 1/2) \) is evaluated by taking into account the presence of a zero mode shared by two Dirac points, and that there are \( 4(\nu +1/2) \) occupied states that are transferred from one edge to another. Notice that this effect shows up at room temperature, contrasting with the usual QHE in semiconductors which is typically a low temperature effect, [2, 3].

Following Ref. [30], whenever the Fermi level lies in a gap, the Hall conductivity is given by

\[ \sigma_H = e \frac{\partial n(\epsilon_F)}{\partial B} , \]  

(42)

where \( n(\epsilon_F) \) is the density of states, and \( \epsilon_F \) is the energy levels of graphene given by Eq. (8). Thus, in the usual commutative case, the energy levels are given by Eq. (26) and so the density of states is

\[ \epsilon_F = \pm \hbar v_F \sqrt{2 \frac{eB}{h} n_e} \Rightarrow n(\epsilon_F) = \frac{eB}{\hbar} . \]  

(43)
The same strategy can be used to evaluate the Hall conductivity for the noncommutative case. In this case, the energy levels are given by Eq. (36), where the noncommutative correction is explicit. In what concerns the density of states, it seems rather logical that it is affected by the noncommutativity and the simplest way to incorporate this dependence is through the expression:

\[ n(\epsilon_F) = \frac{eB}{\hbar} \left( 1 + \frac{\eta}{eB\hbar} \right). \]  

(44)

Thus,

\[ \sigma_H = g_e e \left( \frac{e}{\hbar} \left( 1 + \frac{\eta}{eB\hbar} \right) - \frac{eB}{\hbar} \frac{\eta e\hbar}{(eB\hbar)^2} \right) = \frac{g_e e^2}{\hbar}, \]  

(45)

where \( g_e = 4(\nu + 1/2) \) is the degeneracy factor. One concludes that even though the momenta noncommutativity induces a change in the energy levels of the graphene electrons, it does not affect the Hall conductivity.

III. COMPARISON WITH EXPERIMENTAL RESULTS

In this section, a bound on the noncommutative parameter, \( \eta \), can be obtained using the available experimental results. In Ref. \[31\], infrared (IR) spectroscopy in the presence of a magnetic field was used to resolve the levels of the Landau spectrum for one single layer graphene. Two resonances were resolved for magnetic fields up to \( B = 18 \ T \), and their energy position was shown to scale as \( \sqrt{B} \) with a slope corresponding to a \( v_F = (1.12 \pm 0.02) \times 10^6 \ m s^{-1} \) for a particular energy \[31\]. This value for the Fermi velocity is related with the transition from \( n = -1 \) to \( n = 0 \) (in the case of holes) and \( n = 0 \) to \( n = 1 \) (in the case of electrons). These transitions correspond to the filling factor \( \nu = -2 \) in the IQHE, the lowest Landau level transitions possible in graphene.

The energy for the Landau levels \( n = 1 \) or \( n = -1 \),

\[ E_K = \pm \sqrt{2\hbar v_F B} = \pm \sqrt{2e\hbar v_F^2 B}, \]  

(46)

where \( + \) is for electrons and \( n = 1 \), and \( - \) is for holes and \( n = -1 \). For these levels one has \( E_K = \pm (172 \pm 3) \ meV \) \[31\]. Thus, if the noncommutative energy spectrum for graphene is given by Eq. (36), and considering that the uncertainty in the energy is at most 6 \( meV \),

\footnote{Notice that for magnetic field above 10 \( T \), the Zeeman energy \( g\mu_B B \) is negligible in comparison with the energy of the Landau levels \[2\].}
one can obtain a bound for the noncommutative parameter $\eta$. Using Eq. (36) it follows that:

$$\eta \frac{l_p^2}{\hbar^2} < 0.069.$$  

(47)

Thus, the noncommutative parameter $\eta$ satisfies

$$\eta < 2.1 \times 10^{-53} \text{ kg}^2\text{m}^2\text{s}^{-2}$$

$$\Rightarrow \sqrt{\eta} < 8.6 \text{ eV/c}.$$  

(48)

Naturally, this bound is not as stringent as the one arising from the hyperfine transition in the hydrogen atom, $\sqrt{\eta} \leq 2.26 \mu\text{eV/c}$, [11], one of the most accurate experimental results in the whole of physics. Despite of that, the above reasonings show that noncommutative effects are consistent with what is known about graphene physics.

In what concerns other bounds for the momentum noncommutative parameter, notice that the one arising from the gravitational quantum well [17] and from the equivalence principle [32] depend on an assumption about the configuration space noncommutative parameter, $\theta$, and cannot the compared with the above bound without fixing a value for $\theta$.

IV. CONCLUSIONS

In this work the phase-space noncommutative extension of the graphene in the presence of an external constant magnetic field was examined. More precisely, only momenta noncommutativity was considered since the noncommutativity associated with the configuration variables implies the breaking of gauge invariance. The introduction of momenta noncommutativity determines a correction of the energy spectrum of graphene in the presence of a magnetic field. Moreover, it was shown that this noncommutativity does not affect the graphene anomalous quantum Hall effect.

Finally, comparison with experimental data reveals that $\sqrt{\eta} \leq 8.6 \text{ eV/c}$, a bound that is not very stringent, but that indicates that there is no contradiction between noncommutative effects and graphene’s physics. These results show that momentum noncommutativity yields interesting results also at low-energies and that its implications are not restricted to quantum comology [22] and black holes physics [23, 25].
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