Gauge invariant response functions in Algebraic Fermi liquids

M. Franz, T. Pereg-Barnea, D. E. Sheehy and Z. Tešanović

1Department of Physics and Astronomy, University of British Columbia, Vancouver, BC, Canada V6T 1Z1
2Department of Physics and Astronomy, Johns Hopkins University, Baltimore, MD 21218

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A new method is developed that permits the simple evaluation of two-loop response functions for fermions coupled to a gauge field. We employ this method to study the gauge-invariant response functions in the Algebraic Fermi liquid, a non-Fermi liquid state proposed to describe the pseudogap phase in the QED$_3$ theory of cuprate superconductors. The staggered spin susceptibility is found to exhibit a characteristic anomalous dimension exponent $\eta_\mu$, while other correlators show behavior consistent with the conservation laws imposed by the symmetries of the underlying theory.

Low-energy effective theories of certain correlated electronic systems are known to be equivalent to (2+1)-dimensional quantum electrodynamics (QED$_3$). The latter describes $N$ species of massless 'relativistic' Dirac fermions coupled to a massless gauge field $a_{\mu}$. While the physics behind these different reincarnations of QED$_3$ varies from case to case, these theories are all of considerable general interest for the following reason: In the so-called symmetric phase of QED$_3$, long-range interactions mediated by the massless gauge field produce characteristic anomalous correlations between electrons which decay on long length- and time-scales as nontrivial power laws. This symmetric phase has been termed variously as the Algebraic Spin (AFL) or Algebraic Fermi (AF) liquid and embodies a unique realization of a non-Fermi liquid state of electronic matter in 2 spatial dimensions.

The power law correlations are encoded in the fermion propagator of the theory, $G(x) = \langle \Psi(x)\bar{\Psi}(0) \rangle \sim x^{-(2+\eta)}$, where $\eta = (4/3\pi^2N)(3\xi - 2)$ is the anomalous dimension exponent and $\xi \geq 0$ is a gauge fixing parameter. The above fermionic propagator is gauge-dependent and therefore it cannot represent the behavior of the physical electron in the underlying theory. Much effort has gone into constructing and calculating the proper gauge-invariant electron propagator ($\bar{\Psi}$) that prohibits the correlators of conserved operators from directly having a characteristic anomalous dimension and discuss this result in terms of a theorem that prohibits the correlators of conserved operators from acquiring an anomalous dimension.

In what follows we focus on the QED$_3$ theory of cuprate pseudogap as formulated in Refs. 6, 9, but our technique remains applicable to any other reincarnation of QED$_3$. In the former QED$_3$ emerges as a low-energy effective theory for the nodal topological fermions $\Psi_i(x)$, in a phase-disordered $d$-wave superconductor ($d$SC). The Lagrangian density is

$$L_D = \sum_{i=1}^{2} \bar{\Psi}_i(x) \gamma_{\mu} (i\partial_{\mu} - a_{\mu}) \Psi_i(x) + \frac{1}{2}K_{\mu}(\partial \times a)^2_{\mu},$$

where the gauge field $a_{\mu}$ encodes the topological frustration encountered by fermions as they propagate through the "soup" of fluctuating unbound vortex-antivortex pairs. $x = (\tau, r)$ denotes the space-time coordinate and $\gamma_{\mu}$ ($\mu = 0, 1, 2$) are $4 \times 4$ gamma matrices satisfying $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$. The bare dynamics of the gauge field is Maxwellian with stiffness $K_{\mu}$.

In the above theory $\Psi_i(x)$ is a four component spinor describing the fermionic excitations at the $l$-th pair of antipodal nodes of the underlying $d$SC. Its individual components are related to the original electron operators $c_{\sigma}$ through the singular gauge transformation

$$c_{\sigma}(r, \tau) = e^{i\varphi_{\sigma}(r, \tau)} \psi_{\sigma}(r, \tau),$$

detailed in Refs. 6, 8. The purpose of this transformation is to "unwind" the phase $\varphi_{\sigma}(r, \tau) = \varphi_{\sigma}(r) + \varphi_{\tau}(r, \tau)$ of the fluctuating SC order parameter $\Delta(r, \tau) = \Delta_0 e^{i\varphi(r, \tau)}$ in favor of coupling to the gauge field $a_{\mu}$ which is related to coarse-grained gradients of phases $\varphi_{\sigma}$. The difficulty in computing the gauge-invariant electron propagator $G(x - x')$ stems from the necessity to evaluate...
The susceptibilities are then given as

\[ q \text{clidean notation with 3-momenta} \]

sider in particular the spin susceptibility, continued by short length-scale physics that is not properly
described by the effective theory \[1\].

Here we focus on the charge and spin density correlators which do not suffer from the above problem. Consider in particular the spin susceptibility,

\[ \chi_s(\mathbf{q}, i\omega) = \int_0^\beta d\tau e^{i\omega\tau} \langle \Gamma_\tau S^z_\mathbf{q}(\tau)S^z_{-\mathbf{q}}(0) \rangle, \tag{3} \]

with \( \beta = 1/T \) and \( S^z_\mathbf{q} = \int d^2x e^{i\mathbf{q}\cdot\mathbf{x}} \sum_\sigma \sigma c^\dagger_\mathbf{q}\sigma(x)c_\mathbf{q}\sigma(x) \) the \( z \)-component of the electron spin operator at wavevector \( \mathbf{q} \). The charge susceptibility \( \chi_c \) is defined as in Eq. \(3\) but with \( S^z_\mathbf{q} \) replaced by the charge density operator \( \rho_\mathbf{q} = \int d^2x e^{i\mathbf{q}\cdot\mathbf{x}} \sum_\sigma c^\dagger_\mathbf{q}\sigma(x)c_\mathbf{q}\sigma(x) \). Both spin and charge densities are gauge singlets with respect to \( a_\mathbf{q} \), i.e., they do not pick up any phase factors under the transformation \( e^{i\mathbf{q}\cdot\mathbf{x}} \).

Following Ref. \[4\] we adopt the representation for the gamma matrices given by \( \gamma_\mu = \sigma_3 \otimes (\sigma_2, \sigma_1, -\sigma_3) \) with \( \sigma_1 \) the Pauli matrices. We further define \( \gamma_5 = \sigma_1 \otimes 1 \) and \( \gamma_5 = \sigma_3 \otimes 1 \) and note that these anticommute with all \( \gamma_\mu \)'s. With these conventions we may express the density operators as

\[ S^\nu_\mathbf{q}(\tau) = \int_p \bar{\Psi}_\mathbf{q}(p)\Gamma_\nu\Psi_\mathbf{q}(q-p), \tag{4} \]

where vertex \( \Gamma_\nu \) takes the form \( \Gamma^+_\nu = (i\gamma_0, i\gamma_0) \) and \( \Gamma^-_\nu = (-i\gamma_0\gamma_2, i\gamma_0\gamma_1) \) for spin and charge densities with momentum transfer close to \( \mathbf{q} = (0, 0) \), respectively, and \( \Gamma^+_5 = (\gamma_5, 0) \), \( \Gamma^-_5 = (-\gamma_5\gamma_5, 0) \) for momentum transfer near \( \mathbf{q} = Q_\perp \approx (\pi, \pi) \). We have switched to Euclidean notation with 3-momenta \( q = (q_0, \mathbf{q}) \), \( \int_p \) denotes \( \int d^3p/(2\pi)^3 \) and summation over \( l = 1, 2 \) is assumed. The susceptibilities are then given as

\[ \chi(q) = \int_p \int_{p'} \langle \bar{\Psi}_\mathbf{q}(p)\Gamma_\nu\Psi_\mathbf{q}(p+q)\bar{\Psi}_{\mathbf{q}'}(p')\Gamma_\nu\Psi_{\mathbf{q}'}(p'-q) \rangle. \tag{5} \]

We evaluate \( \chi(q) \) at \( T = 0 \) by formally considering \( N/2 \) identical copies of fields \( \Psi_1 \) and \( \Psi_2 \). Then to leading or-
der in \(1/N\) expansion \[11\] we require 4 diagrams depicted in Fig. \[1\](a–d). The wavy line represents the gauge field propagator that becomes universal at long wavelengths due to the screening by topological fermions \[3\].

\[ D_{\mu\nu}(q) = \frac{8}{qN} \left( \delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} (1 - \xi) \right), \tag{6} \]

in the sense that \( K_\mu \) drops out and only reappears as an upper cutoff of the theory, \( \Lambda \sim 1/K \).

The bare bubble Fig. \[1\](a) reads

\[ \chi_0(q) = \int_p \text{Tr}[G_0(p)\Gamma_1G_0(p-q)\Gamma_1] \tag{7} \]

where \( G_0(p) = p^\mu/p^2 \equiv p_\mu\gamma_\mu/p^2 \) is the free Dirac propagator, and can be evaluated by standard methods (see e.g. Appendix B of Ref. \[4\]). One obtains

\[ \chi_0(q) = -\text{Tr}[\gamma_\mu\Gamma_1\gamma_\nu\Gamma_1] \frac{q}{64} \left( \delta_{\mu\nu} + \frac{q_\mu q_\nu}{q^2} \right). \tag{8} \]

We observe that the structure of the bare bubble susceptibility is determined entirely by the commutation relation of the vertex with the \( \gamma \) matrices. We shall see that this property remains true for the higher order diagrams as well.

Diagrams (b), (c) and (d) represent the leading \(1/N\) corrections due to the gauge field and contain all the interesting physics. We note that their sum is \textit{gauge invariant}. This follows since they represent correlators of gauge invariant density operators. One can explicitly verify that this is so by employing the Ward identity \( G_0^{-1}(p+k) - G_0^{-1}(k) = p_\mu\gamma_\mu \). We are thus free to fix the gauge in what follows we adopt the Feynman gauge \( (\xi = 1) \) in which \( D_{\mu\nu}(q) = (8/N)\delta_{\mu\nu}/q \).

\[ \chi_{(b+c)}(q) = 2 \int_p \text{Tr}[G_0(p)\Gamma_1G_0(p-q)\Gamma_1G_0(p)\Sigma(p)] \tag{9} \]

with \( \Sigma(p) = \int_k D_{\mu\nu}(k)\gamma_\mu G_0(p-k)\gamma_\nu = -\eta_F \phi \ln(\Lambda/p) \), and

\[ \eta_F = \frac{4}{3\pi^2 N} \tag{10} \]

the fermion anomalous dimension in the Feynman gauge \[3\]. Inserting \( \Sigma(p) \) into Eq. \(9\) and combining with Eq. \(8\) one can express the divergent contribution as

\[ \chi_{(b+c)}(q) \simeq 2\eta_F \chi_0(q) \ln(\Lambda/q). \tag{11} \]

Diagram (d) contains the vertex correction and reads
As mentioned above the direct evaluation of this type of integral presents a significant challenge \cite{13, 14}. We are interested in the leading \( q \to 0 \) behavior of \( \chi(q) \). In this limit the \( p \) integral has singularities as \( p \to 0 \) and \( p \to k \), corresponding to divergences in the first and second term in the square brackets respectively. The main contribution to the integral therefore comes from the vicinity of these two points and we may evaluate it by expanding the regular part of the integrand at the singular point. Retaining only the leading term thus gives

\[
\chi(\delta)(q) \simeq \int_k \int_p \text{Tr} \left[ (G_0(p-q)\Gamma_l G_0(p)) \gamma_\mu (G_0(-k)\Gamma_l G_0(-k-q)) \gamma_\nu \right] D_{\mu\nu}(k).
\]

(12)

Performing a variable shift \( p \to p + k \) in the second term we may simplify the above expression as

\[
\chi(\delta)(q) = 2\text{Tr} \left[ \Omega_l(q) \int_p (G_0(p-q)\Gamma_l G_0(p)) \right]
\]

(14)

with \( \Omega_l(q) = \int_k \gamma_\mu (G_0(k)\Gamma_l G_0(k-q)) \gamma_\nu D_{\mu\nu}(k) \). The last integral is again easy to evaluate and gives

\[
\Omega_l(q) = [\gamma_\mu \gamma_\nu \Gamma_l \gamma_\nu \gamma_\mu] \eta_F \ln(\Lambda/q).
\]

(15)

We now notice that for vertices composed of products of gamma matrices (such as those entering the spin and charge densities defined above) it holds that

\[
\gamma_\mu \gamma_\nu \Gamma_l \gamma_\nu \gamma_\mu = \lambda \Gamma_l,
\]

(16)

where \( \lambda \) is a number. In particular we shall encounter two types of vertices. Type-I vertices commute or anticommute with all \( \gamma_\mu \)'s (e.g. \( \Gamma_l = 1, \gamma_3, \gamma_5, \gamma_3 \gamma_5 \)) and in this case \( \lambda = 9 \). Type-II vertices anticommute with one or two of \( \gamma_\nu \)'s and commute with the rest (e.g. \( \Gamma_l = \gamma_0, \gamma_0 \gamma_1, \ldots \)). In this case \( \lambda = 1 \). With this insight Eq. (14) becomes

\[
\chi(\delta)(q) = 2\eta_F \lambda \ln(\Lambda/q) \int_p \text{Tr} [G_0(p)\Gamma_l G_0(p-q)\Gamma_l].
\]

Combining this result with Eqs. (14) and (11) we can write the result for the full susceptibility to 1/N order,

\[
\chi(q) = \chi_0(q)(1 - 2\eta_F(1 - \lambda) \ln(\Lambda/q)).
\]

(17)

This correction may be interpreted as the leading term of a power law \cite{11}, so that we have

\[
\chi(q) \sim \chi_0(q) \left( \frac{\Lambda}{q} \right)^{\eta_4}, \quad \eta_4 = \frac{8(\lambda - 1)}{3\pi^2 N}.
\]

(18)

The anomalous dimension exponent \( \eta_4 \) is entirely determined by the algebraic properties of the vertex \( \Gamma_l \) through Eq. (15). In particular for type-I vertex (\( \lambda = 9 \))

\[
\eta_4 = \frac{64}{3\pi^2 N},
\]

(19)

while for type-II vertex \( \eta_4 = 0 \).

Returning back to physics we see that the vertex for staggered spin susceptibility is type-I and will therefore exhibit nontrivial anomalous dimension exponent Eq. (15), in agreement with the result of Ref. \cite{14} obtained by laborious direct evaluation of the vertex correction. Uniform spin susceptibility has type-II vertex and therefore does not acquire anomalous dimension from diagrams (b-d), again in agreement with \cite{12}. In addition both charge susceptibilities are type-II and will not exhibit anomalous dimension.

Finally we note that if \( \Gamma_l \) coincides with one of the \( \gamma_\mu \)'s then diagrams of the type shown in Fig. 1(e) are nonvanishing and must be included in the leading order. One can show that resummation of diagrams (e) modifies the \( \chi_0(q) \sim q \) behavior in Eq. (8) to \( q^2 \). This can be viewed as another type of anomalous dimension due to the coupling to the gauge field and we shall discuss it more fully elsewhere.

The formalism we have developed allows us to study spin and charge conductivities in the AFL. These can be calculated through the Kubo formula as \( \sigma_{ij}(\omega) = -\text{Im} \Pi_{ij}^{\text{ret}}(\omega)/\omega \), where \( \Pi_{ij}^{\text{ret}}(\omega) = \Pi_{ij}(i\omega \to \omega + i\delta) \) with

\[
\Pi_{ij}(i\omega) = -\int_0^\beta d\tau e^{i\omega \tau} \langle \Gamma_j(\tau)j_i(0) \rangle
\]

(20)

the current-current correlation function. Indices \( i, j = 1, 2 \) label the spatial components of the 3-current \( j_\mu \). The spin current is given as \( j_\mu^s(x) = \Psi(x)\gamma_\mu \Psi(x) \) while the electric current is given as \( j_\mu^e(x) = \hat{\Psi} \gamma_0 \gamma_\mu \Psi - \hat{\Psi} \gamma_0 \gamma_1 \Psi_2 - \hat{\Psi} \gamma_1 \gamma_2 \Psi_1 - \hat{\Psi} \gamma_2 \gamma_1 \Psi_2 \). To leading 1/N order the computation of the correlator (20) involves the same diagrams depicted in Fig. 1 and we can therefore simply adopt the result obtained for the susceptibility in
We see that the vertices involved in $j^e_\mu$ and $j^s_\mu$ are all type-II and therefore neither spin nor charge conductivity will exhibit anomalous dimension.

We may conclude that of all quantities considered only the staggered spin susceptibility will bear the unique signature of the AFL in that its frequency, momentum and temperature dependence will be controlled by the anomalous dimension exponent $\eta_4$ given in Eq. (19). The other quantities will behave essentially as if the theory was uncorrelated.

The fact that some quantities remain essentially unaffected by strong long range correlations can be traced to a field theoretic theorem [15, 16] which states that the correlator of conserved currents (i.e. currents satisfying $\partial_\mu j_\mu = 0$) must exhibit a scaling dimension which agrees with its engineering dimension. Physically this means that some quantities are constrained by their conservation laws to such a degree that correlations cannot alter their long-distance scaling behavior. As an example consider the spin current $j^s_\mu$. It is a conserved current, $\partial_\mu j^s_\mu = 0$ guaranteed by the gauge invariance of the theory [i.e. invariance under local symmetry $\Psi_i(x) \rightarrow e^{i\theta(x)}\Psi_i(x)$]. As we have verified by explicit calculation neither spin conductivity nor uniform spin susceptibility exhibit anomalous dimension ($\sim 1/N$) beyond the $\chi_0 \sim q^2$ behavior mentioned above, in accord with the theorem. Understanding the behavior of the charge conductivity is less straightforward as we do not expect the quasiparticle current to be conserved in a (phase-disordered) superconductor. Indeed there is no symmetry that would guarantee $j^e_\mu$ conservation. However, the theory is known to posses a “chiral” symmetry ($\Psi_i \rightarrow e^{i\gamma_3\gamma_5\phi}\Psi_i$) [18] which produces two conserved currents: $j^e_\mu = \bar{\Psi}_i(\gamma_3\gamma_5)\Psi_i$ (no sum over $l$). By virtue of the identity $\gamma_0\gamma_3\gamma_5 = -\gamma_1\gamma_2$ these are related to the spatial components of $j^s_\mu$, explaining the lack of anomalous dimension in the electrical conductivity.

In the $4 \times 4$ representation of Dirac gamma matrices there are 4 type-I vertices listed below Eq. (10). Thus there should be 3 other physical observables which exhibit anomalous scaling dimension. What are these?

In turns out that $\gamma_3$ corresponds to the staggered spin susceptibility at the wave vector $Q_1 = -Q_1$, while $\gamma_3\gamma_5$ correspond to susceptibilities to formation of subdominant (phase-incoherent) SC order in $s$ and $p$ channels, respectively. These quantities are members of the QED$_3$ chiral manifold [17, 18], which is the manifold of broken-symmetry states occurring for $N < N_c \approx 32/\pi^2 = 3.24$ at zero temperature. This phenomenon is known as spontaneous chiral symmetry breaking [19].

The anomalous dimension exponent $\eta_4$ that we find in the above susceptibilities appears to anticipate the transition into the chiral symmetry broken phase. Indeed if we combine $\chi_0(q)$ from Eq. (8) with Eq. (18) we find $\chi(q) \sim q^{1-\eta_4}$, implying divergent $q \rightarrow 0$ susceptibility for $N < 64/3\pi^2 = 4.64$. In the context of the QED$_3$ theory of cuprates [6] we expect the experimental manifestations of the nontrivial anomalous dimension to appear in the quantities inhabiting the QED$_3$ chiral manifold, most prominently the staggered spin susceptibility that is directly measurable by neutron scattering. In addition, a similar nontrivial response should obtain in the charge channel near $(0, \pi)$ as a consequence of enlarged chiral manifold discussed in [18]. This particular response is unique to the present theory and will not be present in the SU(2) theory of Ref. [11].

We conclude by observing that the physics of the Algebraic Fermi liquid bears distinct similarity to that of 1D Luttinger liquids. While the single particle properties are distinctly non-Fermi liquid like, only certain measurable physical responses exhibit the unique fingerprint of the AFL in the form of anomalous scaling dimension. The method developed here allows us to easily identify these quantities and helps in our search for situations where other quantities might develop anomalous behavior.

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