Cocycles in categories of fibrant objects

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Abstract

Dwyer and Kan developed a homotopical version of the calculus of fractions in order to get a handle on simplicial localisations of categories. We use this to show that, for a category of fibrant objects (in the sense of Brown), Jardine’s cocycle categories functorially compute the homotopy type of the hom-spaces in the simplicial localisation. As an application, we deduce a non-abelian version of Verdier’s hypercovering theorem suggested by Rezk.

Introduction

Given a category $\mathcal{C}$ and a subcategory $\mathcal{W} \subseteq \mathcal{C}$, the category $\mathcal{C}[\mathcal{W}^{-1}]$ obtained from $\mathcal{C}$ by freely inverting the morphisms in $\mathcal{W}$ is straightforward to construct: its objects are the objects in $\mathcal{C}$ and its morphisms are equivalence classes of zigzags of arrows in $\mathcal{C}$ where the backward-pointing arrows are in $\mathcal{W}$. The simplicial and hammock localisations introduced by Dwyer and Kan [1980a,b] are homotopy-theoretic versions of this construction and can be shown to have the appropriate universal property in the context of $(\infty, 1)$-categories.

Although the hom-spaces of the hammock localisation already have a fairly simple explicit description, just as in the case of ordinary localisation, one can sometimes obtain an even simpler description when the pair $(\mathcal{C}, \mathcal{W})$ has good properties. For instance, Dwyer and Kan [1980c] have shown that it suffices to consider only zigzags of the form

$$
\bullet \xleftarrow{\sim} \bullet \rightarrow \bullet \xleftarrow{\sim} \bullet
$$
Cocycles in categories of fibrant objects

when \( \mathcal{C} \) is a closed model category in the sense of Quillen [1967] and \( \mathcal{W} \) is its subcategory of weak equivalences. On the other hand, when \( \mathcal{C} \) is a category of fibrant objects in the sense of Brown [1973] (and \( \mathcal{W} \) is its subcategory of weak equivalences), it is well known that every morphism in \( \mathcal{C}[\mathcal{W}^{-1}] \) can be represented by what Jardine [2009] calls ‘cocycles’, i.e. zigzags of the form

\[
\bullet \xrightarrow{\sim} \bullet \quad \bullet
\]

and the main goal of this paper is to show that the hom-spaces of the hammock localisation of \((\mathcal{C}, \mathcal{W})\) have the homotopy type of the nerve of the cocycle categories.

The hardest part of the proof presented in this paper is showing that every category of fibrant objects admits a homotopical calculus of right fractions. In fact, we will see this is true for any homotopically replete full subcategory of any category of fibrant objects. It is not so hard to verify the claim when we have functorial path objects: indeed, this is a folklore result, and the closely related case of a Waldhausen category with a cylinder functor is described in [Weiss, 1999, §1]. One can then determine the homotopy type of the hom-spaces of the hammock localisation of a general category of fibrant objects by embedding it in a category of simplicial presheaves—this is the strategy employed in the proof of Proposition 3.23 in [Cisinski, 2010b]—but we will take a different approach to eliminating the hypothesis of functorial path objects. The key observation is that there is a contractible space parametrising certain special (section of trivial fibration, trivial fibration)-factorisations of weak equivalences. Almost everything else follows formally: indeed, we will treat this situation axiomatically by defining the notion of a homotopical calculus of cocycles.

As an application of the main result, we consider the category of (locally fibrant) simplicial presheaves on a site. From the point of view of homotopy theory, the central problem of sheaf theory is essentially the determination of the homotopy type the space of sections (over a given object in the site) of the hypersheaf associated with a given simplicial presheaf: for example, as Brown [1973, §3] observed, sheaf cohomology can be paraphrased in these terms via the formula below,

\[
R^n\Gamma(-, A) \cong \pi_{m-n}\Gamma(-, K(A, m))
\]

where \( A \) is an abelian (pre)sheaf, \( K(A, m) \) is the simplicial (pre)sheaf corresponding (under Dold–Kan) to the chain complex consisting of just \( A \) in degree \( m \), and \( m \geq n \). Verdier’s hypercovering theorem in its classical form is a colimit formula for sheaf cohomology in terms of generalised Čech cochain complexes, and following a suggestion of Rezk [2014], we derive a non-abelian version

2
that computes (up to weak homotopy equivalence) $R\Gamma(-, X)$ for any locally fibrant simplicial presheaf $X$ in terms of a homotopy colimit of simplicial sets of generalised sections of $X$.

**Outline**

- In §1, we collect some miscellaneous facts about homotopy colimits.
- In §2, we review the definitions and fundamental results regarding zigzags in relative categories.
- In §3, we introduce the notion of a homotopical calculus of cocycles, which is a sufficient condition for a category with weak equivalences to admit a homotopical calculus of right fractions.
- In §4, we prove that a category of fibrant objects with functorial path objects admits a homotopical calculus of cocycles.
- In §5, we define the notion of a simplicial category of fibrant objects and give a homotopy colimit formula for the hom-spaces of its hammock localisation.
- In §6, we apply this theory to the study of simplicial presheaves on a site.
- In §A, we show that a general category of fibrant objects (i.e. possibly without functorial path objects) admits a homotopical calculus of cocycles.

**Conventions**

It will be convenient to implicitly assume that categories are small, especially in §§2–5 and §A. Since the categories of interest are usually not small, it is not possible to apply these results as stated literally; one way to work around this to adopt a suitable universe axiom. Alternatively, because most of the categories under consideration in §6 are *essentially* small, one could just replace them with small skeletons where necessary, thereby avoiding the use of universes.

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Cocycles in categories of fibrant objects

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1 Homotopy colimits

The following definition is due to Bousfield and Kan [1972].

Definition 1.1. Let $X : \mathcal{C}^{\text{op}} \to \text{sSet}$ be a small simplicially enriched diagram. The homotopy colimit $\text{holim}_{\mathcal{C}^{\text{op}}} X$ is the diagonal of the bisimplicial set $B_*(X, \mathcal{C}, \Delta 1)$ defined below,

$$B_n(X, \mathcal{C}, \Delta 1) = \coprod_{(c_0, \ldots, c_n)} X(c_n) \times \mathcal{C}(c_{n-1}, c_n) \times \cdots \times \mathcal{C}(c_0, c_1)$$

where the disjoint union is indexed over $(n+1)$-tuples of objects in $\mathcal{C}$, with the evident face and degeneracy operators.

Example 1.2. Let $X : \mathcal{C}^{\text{op}} \to \text{Set}$ be a small diagram and let $\mathcal{D}$ be the category of elements of $X$, i.e. $(1 \downarrow X)^{\text{op}}$ where $(1 \downarrow X)$ is the comma category. Regarding $X$ as a diagram $\mathcal{C}^{\text{op}} \to \text{sSet}$, it is not hard to see that $\text{holim}_{\mathcal{C}} X$ is (isomorphic to) the nerve $N(D)$.

We will need some miscellaneous facts about homotopy cofinality. First, let us say that a weakly contractible category is a category $\mathcal{A}$ such that the unique morphism $N(\mathcal{A}) \to \Delta^0$ is a weak homotopy equivalence of simplicial sets. We then make the following definition:

Definition 1.3. A homotopy cofinal functor is a functor $F : \mathcal{C} \to \mathcal{D}$ with the following property: for all objects $d$ in $\mathcal{D}$, the comma category $(d \downarrow F)$ is weakly contractible.

Lemma 1.4. Let $P : \mathcal{E} \to \mathcal{B}$ be a Grothendieck fibration. The following are equivalent:

(i) The (strict) fibres of $P$ are weakly contractible.

(ii) $P$ is a homotopy cofinal functor.

Proof. Let $b$ be an object in $\mathcal{B}$. There is a functor $P^{-1}\{b\} \hookrightarrow (b \downarrow P)$ sending objects $e$ in $P^{-1}\{b\}$ to $(e, \text{id}_b)$ in $(b \downarrow P)$, and it is well known that this functor has a right adjoint when $P : \mathcal{E} \to \mathcal{B}$ is a Grothendieck fibration. Since adjoint functors induce homotopy equivalences of nerves, it follows that $P^{-1}\{b\}$ is weakly contractible if and only if $(b \downarrow P)$ is weakly contractible. ■
Lemma 1.5. Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ be functors. If $GF : \mathcal{C} \to \mathcal{E}$ is homotopy cofinal and $G : \mathcal{D} \to \mathcal{E}$ is fully faithful, then $F : \mathcal{C} \to \mathcal{D}$ is also homotopy cofinal.

Proof. Let $d$ be any object in $\mathcal{D}$. If $G : \mathcal{D} \to \mathcal{E}$ is fully faithful, then there is an isomorphism $(d \downarrow F) \cong (G(d) \downarrow GF)$, so $F : \mathcal{C} \to \mathcal{D}$ is homotopy cofinal when $GF : \mathcal{C} \to \mathcal{E}$ is. \hfill \blacksquare

Theorem 1.6 (Quillen’s Theorem A). Homotopy cofinal functors are weak homotopy equivalences of categories.

Proof. See [Quillen, 1973, §1]. \hfill \square

Lemma 1.7. Consider a pullback diagram in $\mathbf{Cat}$:

\[
\begin{array}{ccc}
\mathcal{E}' & \xrightarrow{L} & \mathcal{E} \\
\downarrow{P'} & & \downarrow{P} \\
\mathcal{B}' & \xrightarrow{F} & \mathcal{B}
\end{array}
\]

If $P : \mathcal{E} \to \mathcal{B}$ is a Grothendieck fibration and $F : \mathcal{B}' \to \mathcal{B}$ has a right adjoint, then $L : \mathcal{E}' \to \mathcal{E}$ also has a right adjoint.

Proof. Let $G : \mathcal{B} \to \mathcal{B'}$ be any right adjoint for $F : \mathcal{B}' \to \mathcal{B}$, let $e_1$ be an object in $\mathcal{E}$, let $e_0'$ be an object in $\mathcal{E}'$, let $b_1 = e_1$, let $b_0' = P'(e_0')$, choose a cartesian morphism $\tilde{\varepsilon}_{e_1} : (\varepsilon_{b_1})^* e_1 \to e_1$ in $\mathcal{E}$ such that $P(\tilde{\varepsilon}_{e_1}) = \varepsilon_{b_1}$, where $\varepsilon_{b_1} : F(G(b_1)) \to b_1$ is the counit component, and let $R(e_1)$ be the unique object in $\mathcal{E}'$ such that $P'(R(e_1)) = Gb_1$ and $L(R(e_1)) = (\varepsilon_{b_1})^* e_1$. We then have the following commutative diagram,

\[
\begin{array}{cccc}
\mathcal{E}'(e_0', R(e_1)) & \xrightarrow{L} & \mathcal{E}(L(e_0'), L(R(e_1))) & \xrightarrow{\mathcal{E}(L(e_0'), \varepsilon_{e_1})} & \mathcal{E}(L(e_0'), e_1) \\
\downarrow{P'} & & \downarrow{P} & & \downarrow{P} \\
\mathcal{B}'(b_0', G(b_1)) & \xrightarrow{F} & \mathcal{B}(F(b_0'), F(G(b_1))) & \xrightarrow{\mathcal{B}(F(b_0'), \varepsilon_{b_1})} & \mathcal{B}(F(b_0'), b_1)
\end{array}
\]

where both squares and the outer rectangle are pullback diagrams; but the composite of the bottom row is a bijection, so the composite of the top row is also a bijection. Thus, $L : \mathcal{E}' \to \mathcal{E}$ indeed has a right adjoint. \hfill \blacksquare

Lemma 1.8. Let $X : \mathcal{C}^{\text{op}} \to \mathbf{sSet}$ be a small simplicially enriched diagram. If $\mathcal{C}$ has cotensor products $\Delta^m \times c$ for every standard simplex $\Delta^m$ and every object $c$ in $\mathcal{C}$, then (regarding the underlying category $\mathcal{C}$ as a simplicially enriched category with discrete hom-spaces) the canonical comparison morphism

\[
\text{holim}_{\mathcal{C}^{\text{op}}} X \to \text{holim}_{\mathcal{C}^{\text{op}}} X
\]

is a weak homotopy equivalence.
Proof. Let $Y_\bullet$ and $Y'_\bullet$ be the transposes of the bisimplicial sets $B_\bullet(X, \mathcal{C}, \Delta^1)$ and $B_\bullet(X, \mathcal{C}, \Delta^1)$, respectively, and for each natural number $m$, let $\mathcal{C}_m$ be the $m$-th level of the simplicial category corresponding to $\mathcal{C}$. (Note that $\mathcal{C} = \mathcal{C}_0$.) Then,

$$Y_{m,n} = \coprod_{(c_0, \ldots, c_n)} X(c_n)_m \times \mathcal{C}_m(c_{n-1}, c_n) \times \cdots \times \mathcal{C}_m(c_0, c_1)$$

$$Y'_{m,n} = \coprod_{(c_0, \ldots, c_n)} X(c_n)_m \times \mathcal{C}_0(c_{n-1}, c_n) \times \cdots \times \mathcal{C}_0(c_0, c_1)$$

and the canonical comparison morphism $\text{holim}_C X \to \text{holim}_C X$ is simply the diagonal of the bisimplicial set morphism $Y'_\bullet \to Y_\bullet$ defined in degree $m$ by the $m$-fold iterated degeneracy $\mathcal{C}_0 \to \mathcal{C}_m$. But, for any $c$ and $c'$ in $\mathcal{C}$,

$$\mathcal{C}_0(c', \Delta^m \uplus c) \cong \mathcal{C}_m(c', c)$$

so the $m$-fold iterated degeneracy $\mathcal{C}_0 \to \mathcal{C}_m$ has a right adjoint. It follows by lemma 1.7 that the morphisms $Y'_m \to Y_m$ are nerves of left adjoint functors and hence are (simplicial) homotopy equivalences a fortiori. Thus, by the homotopy invariance of diagonals,[1] the induced morphism $\text{holim}_C X \to \text{holim}_C X$ is a weak homotopy equivalence.}

We will also need the following version of the Grothendieck construction:

Definition 1.9. Let $\mathcal{X} : \mathcal{C}^{\text{op}} \to \text{Cat}$ be a small diagram. The oplax colimit for $\mathcal{X}$ is the category $\text{lim}^{\text{Gr}}_{\mathcal{C}^{\text{op}}} \mathcal{X}$ defined below:

- The objects are pairs $(c, x)$ where $c$ is an object in $\mathcal{C}$ and $x$ is an object in $\mathcal{X}(c)$.
- The morphisms $(c', x') \to (c, x)$ are pairs $(f, g)$ where $f : c' \to c$ is a morphism in $\mathcal{C}$ and $g : x' \to \mathcal{X}(f)(x)$ is a morphism in $\mathcal{X}(c')$.
- Composition and identities are inherited from $\mathcal{C}$ and $\mathcal{X}$.

Example 1.10. Let $X : \mathcal{C}^{\text{op}} \to \text{Set}$ be a small diagram. Regarding $X$ as a diagram $\mathcal{C}^{\text{op}} \to \text{Cat}$, it is not hard to see that $\text{lim}^{\text{Gr}}_{\mathcal{C}^{\text{op}}} X$ is the category of elements of $X$, i.e. $(1 \downarrow X)^{\text{op}}$.

Theorem 1.11 (Thomason’s homotopy colimit theorem). Let $\mathcal{X} : \mathcal{C}^{\text{op}} \to \text{Cat}$ be a small diagram. There is a weak homotopy equivalence

$$\text{holim}_{\mathcal{C}^{\text{op}}} N \circ \mathcal{X} \to N\left(\text{lim}^{\text{Gr}}_{\mathcal{C}^{\text{op}}} \mathcal{X}\right)$$

which is moreover natural in $\mathcal{C}$ and $\mathcal{X}$.

Proof. See [Thomason, 1979].

[1] See e.g. Theorem 15.11.11 in [Hirschhorn, 2003].
2 Zigzags in relative categories

Recall the following definitions from [Barwick and Kan, 2012]:

Definition 2.1.

• A relative category is a pair $\mathcal{C} = (\text{und}\mathcal{C}, \text{weq}\mathcal{C})$ where $\text{und}\mathcal{C}$ is a category and $\text{weq}\mathcal{C}$ is a (usually non-full) subcategory of $\text{und}\mathcal{C}$ containing all the objects.

• Given a relative category $\mathcal{C}$, a weak equivalence in $\mathcal{C}$ is a morphism in $\text{weq}\mathcal{C}$.

• The homotopy category of a relative category $\mathcal{C}$ is the category $\text{Ho}\mathcal{C}$ obtained by freely inverting the weak equivalences in $\mathcal{C}$.

• Given relative categories $\mathcal{C}$ and $\mathcal{D}$, a relative functor $\mathcal{C} \to \mathcal{D}$ is a functor $\text{und}\mathcal{C} \to \text{und}\mathcal{D}$ that restricts to a functor $\text{weq}\mathcal{C} \to \text{weq}\mathcal{D}$, and the relative functor category $[\mathcal{C}, \mathcal{D}]_h$ is the relative category whose underlying category is the full subcategory of the ordinary functor category $[\text{und}\mathcal{C}, \text{und}\mathcal{D}]$ spanned by the relative functors, with the weak equivalences being the natural transformations whose components are weak equivalences in $\mathcal{D}$.

Remark 2.2. The 2-category of (small) categories admits several 2-fully faithful embeddings into the 2-category of (small) relative categories; unless otherwise stated, we will regard an ordinary category as minimal relative category where the only weak equivalences are the identity morphisms. In particular, given an ordinary category $\mathcal{C}$ and a relative category $\mathcal{D}$, we will often tacitly identify the ordinary functor category $[\mathcal{C}, \mathcal{D}]$ with the relative functor category $[\mathcal{C}, \mathcal{D}]_h$.

Definition 2.3.

• A zigzag type is a finite sequence of non-zero integers $(k_0, \ldots, k_n)$, where $n \geq 0$, such that for $0 \leq i < n$, the sign of $k_i$ is the opposite of the sign of $k_{i+1}$.

• Given a finite sequence of integers $(k_0, \ldots, k_n)$, $[k_0; \ldots; k_n]$ is the relative category whose underlying category is freely generated by the graph

$$
\begin{align*}
&0 \quad \cdots \quad |k_0| + \cdots + |k_n| \\
&\text{(where (counting from the left) the first $|k_0|$ arrows point rightward (resp. leftward) if $k_0 > 0$ (resp. $k_0 < 0$), the next $|k_2|$ arrows point rightward (resp. leftward) if $k_1 > 0$ (resp. $k_1 < 0$), etc., with the weak equivalences being generated by the leftward-pointing arrows.}}
\end{align*}
$$
A zigzag in a relative category $C$ of type $[k_0; \ldots; k_n]$ is a relative functor $[k_0; \ldots; k_n] \to C$; given a zigzag, its length is $m = |k_0| + \cdots + |k_n|$, its domain is the image of the object $0$, and its codomain is the image of the object $m$.

**Example 2.4.** For example, $[-1; 2]$ denotes the relative category generated by the following graph,

$$
\begin{array}{ccc}
0 & \xleftarrow{\sim} & 1 & \longrightarrow & 2 & \longrightarrow & 3 \\
\end{array}
$$

with $1 \to 0$ being the unique non-trivial weak equivalence.

**Remark 2.5.** For any $[k_0; \ldots; k_n]$, if $|k_0| + \cdots + |k_n| > 0$, then there is a unique zigzag type $(l_0, \ldots, l_m)$ such that $[k_0; \ldots; k_n] = [l_0; \ldots; l_m]$. However, it is convenient to allow unnormalised notation.

**Definition 2.6.** Let $X$ and $Y$ be objects in a relative category $C$ and let $(k_0, \ldots, k_n)$ be a finite sequence of integers. The category of zigzags in $C$ from $X$ to $Y$ of type $(k_0; \ldots; k_n)$ is the category $C[k_0; \ldots; k_n](X, Y)$ defined below:

- The objects are the zigzags in $C$ of type $[k_0; \ldots; k_n]$ whose domain is $X$ and whose codomain is $Y$.
- The morphisms are commutative diagrams in $C$ of the form

$$
\begin{array}{ccc}
X & \longrightarrow & \bullet & \longrightarrow & \cdots & \longrightarrow & \bullet & \longrightarrow & Y \\
\downarrow \simeq & & \downarrow & & & \downarrow \simeq \\
X & \longrightarrow & \bullet & \longrightarrow & \cdots & \longrightarrow & \bullet & \longrightarrow & Y \\
\end{array}
$$

where the top row is the domain, the bottom row is the codomain, and the vertical arrows are weak equivalences in $C$.

- Composition and identities are inherited from $C$.

**Remark.** In other words, the morphisms in $C[k_0; \ldots; k_n](X, Y)$ are certain hammocks of width 1, in the sense of Dwyer and Kan [1980b].

For brevity, let us say that a weak homotopy equivalence of categories is a functor $F : C \to D$ such that $N(F) : N(C) \to N(D)$ (i.e. the induced morphism of nerves) is a weak homotopy equivalence of simplicial sets. The following is a variation on the ‘homotopy calculus of right fractions’ introduced in [Dwyer and Kan, 1980b].

8
Definition 2.7. A relative category $C$ admits a **homotopical calculus of right fractions** if it satisfies the following condition:

- For all natural numbers $k$ and all objects $X$ and $Y$ in $C$, the evident functor
  \[
  C[-1;k](X,Y) \to C[-1;k;-1](X,Y)
  \]
  defined by inserting an identity morphism is a weak homotopy equivalence of categories.

**Remark 2.8.** Let $C$ be a relative category and let $W$ be $\text{weq} C$ considered as a relative category where all morphisms are weak equivalences. Then the following are equivalent:

(i) $C$ admits a homotopy calculus of right fractions in the sense of Dwyer and Kan [1980b].

(ii) Both $C$ and $W$ admit a homotopical calculus of right fractions in the sense of the above definition.

Moreover, if the weak equivalences in $C$ have the 2-out-of-3 property, then $W$ admits a homotopical calculus of right fractions if $C$ does.

**Remark 2.9.** If a relative category $C$ admits a homotopical calculus of right fractions, then $C$ also admits a homotopical three-arrow calculus. In particular, the results of [Low and Mazel-Gee, 2014] apply, i.e. any Reedy-fibrant replacement $\hat{N}(C)$ of the Rezk classification diagram $N(C)$ is a Segal space, and $\hat{N}(C)$ is a complete Segal space if $C$ is a saturated relative category.

**Theorem 2.10** (Dwyer and Kan). Let $C$ be a relative category and let $L^H C$ be the hammock localisation.

(i) If $C$ admits a homotopical calculus of right fractions, then the reduction morphism $N(C[-1;1](X,Y)) \to L^H C(X,Y)$ is a weak homotopy equivalence of simplicial sets.

(ii) The reduction morphism $N(C[-1;1](X,Y)) \to L^H C(X,Y)$ is natural in the following sense: given any weak equivalence $X \to X'$ and any morphism $Y \to Y'$ in $C$, the following diagram commutes in $\text{sSet}$,

\[
\begin{array}{ccc}
N(C[-1;1](X,Y)) & \longrightarrow & L^H C(X,Y) \\
\downarrow & & \downarrow \\
N(C[-1;1](X',Y')) & \longrightarrow & L^H C(X',Y')
\end{array}
\]

where the left vertical arrow is defined by composition and the right vertical arrow is defined by concatenation.
Proof. (i). This is Proposition 6.2 in [Dwyer and Kan, 1980b]. Note that the second half of the ‘homotopy calculus of right fractions’ condition is not used, so it does indeed suffice to have a homotopical calculus of right fractions.

(ii). Obvious. □

Corollary 2.11. Let \( \mathcal{C} \) be a relative category. If \( \mathcal{C} \) admits a homotopical calculus of right fractions, then for any weak equivalences \( X \rightarrow X' \) and \( Y \rightarrow Y' \) in \( \mathcal{C} \), the induced functor

\[
\mathcal{C}([-1;1])(X,Y) \rightarrow \mathcal{C}([-1;1])(X',Y')
\]

is a weak homotopy equivalence of categories.

Proof. Use naturality (as in theorem 2.10) and Proposition 3.3 in [Dwyer and Kan, 1980b]. □

3 The homotopical calculus of cocycles

The following notion of ‘cocycle’ is originally due to Jardine [2009].

Definition 3.1. Let \( \mathcal{C} \) be a relative category and let \( \mathcal{V} \subseteq \text{weq} \mathcal{C} \) be a subcategory that contains all identity morphisms.

- Given objects \( X \) and \( Y \) in \( \mathcal{C} \), a \( \mathcal{V} \)-cocycle \( (f, v) : X \rightarrow Y \) in \( \mathcal{C} \) is a diagram in \( \mathcal{C} \) of the form below,

\[
X \leftarrow v \quad \tilde{X} \xrightarrow{f} Y
\]

where \( v : \tilde{X} \rightarrow X \) is a morphism in \( \mathcal{V} \).

We write \( \mathcal{C}_{\mathcal{V}}([-1;1])(X,Y) \) for the full subcategory of \( \mathcal{C}([-1;1])(X,Y) \) spanned by the \( \mathcal{V} \)-cocycles.

- If \( \mathcal{V} = \text{weq} \mathcal{C} \), then we may simply say cocycle instead of ‘\( \mathcal{V} \)-cocycle’.

Remark 3.2. In other words, a cocycle in \( \mathcal{C} \) is a zigzag of type \([-1;1]\).

Proposition 3.3. Let \( \mathcal{C} \) be a relative category and let \( \gamma : \mathcal{C} \rightarrow \text{Ho} \mathcal{C} \) be the localising functor. If \( \mathcal{C} \) admits a homotopical calculus of right fractions, then:

(i) Every morphism \( X \rightarrow Y \) in \( \text{Ho} \mathcal{C} \) can be factored as \( \gamma(f) \circ \gamma(w)^{-1} \) for some cocycle \( (f, w) : X \rightarrow Y \) in \( \mathcal{C} \).

(ii) Two cocycles \( X \rightarrow Y \) represent the same morphism \( X \rightarrow Y \) in \( \text{Ho} \mathcal{C} \) if and only if they are in the same connected component of \( \mathcal{C}([-1;1])(X,Y) \).

10
Proof. This is an immediate consequence of Proposition 3.1 in [Dwyer and Kan, 1980b] and theorem 2.10. ■

Recall that a category with weak equivalences is a relative category in which the weak equivalences have the 2-out-of-3 property and include all isomorphisms.

Heuristically, a homotopical calculus of cocycles for a category with weak equivalences consists of three pieces of data: a class \( \mathcal{V} \) of “good” weak equivalences, a category of “enhanced” cocycles, and a forgetful functor from the category of “enhanced” cocycles to the category of cocycles, such that:

- \( \mathcal{V} \) is closed under composition and pullback.
- “Enhanced” cocycles can be pulled back along pairs of weak equivalences.
- The underlying cocycle of an “enhanced” cocycle is a \( \mathcal{V} \)-cocycle.
- Every cocycle can be replaced with an “enhanced” cocycle in a homotopically unique way.

More precisely, we make the following definition.

Definition 3.4. Let \( \mathcal{C} \) be a category with weak equivalences. A homotopical calculus of cocycles for \( \mathcal{C} \) consists of a subcategory \( \mathcal{V} \subseteq \text{weq} \mathcal{C} \) containing all isomorphisms, a category \( \mathcal{C}^{\text{fun}} \), and a functor \( U : \mathcal{C}^{\text{fun}} \to \text{weq} \mathcal{C} \) satisfying the following conditions:

- \( \mathcal{V} \) is closed under pullback in \( \mathcal{C} \) in the sense that, for any morphism \( v : X \to Y \) in \( \mathcal{V} \) and any morphism \( g : Y' \to Y \) in \( \mathcal{C} \), there is a pullback diagram in \( \mathcal{C} \) of the form below,

\[
\begin{array}{ccc}
X' & \to & X \\
\downarrow^v & & \downarrow^w \\
Y' & \to & Y \\
\end{array}
\]

and in any such pullback diagram, \( v' : X' \to Y' \) is also in \( \mathcal{V} \).

- The composite

\[
\mathcal{C}^{\text{fun}} \xrightarrow{U} \text{weq} \mathcal{C} \xrightarrow{(\text{dom, codom})} \text{weq} \mathcal{C} \times \text{weq} \mathcal{C}
\]

is a Grothendieck fibration, where \( \text{dom} \) (resp. \( \text{codom} \)) is the functor \( \text{weq} \mathcal{C} \to \text{weq} \mathcal{C} \) sending a cocycle \( X \to Y \) to \( X \) (resp. \( Y \)), and \( U : \mathcal{C}^{\text{fun}} \to \text{weq} \mathcal{C} \) preserves cartesian morphisms.
• For each object $E$ in $C^\text{fun}$, the leftward-pointing arrow of the cocycle $UE$ is a morphism in $V$.

• For each pair $(X, Y)$ of objects in $C$, writing $C^\text{fun}(X, Y)$ for the strict fibre of the above functor $C^\text{fun} \to \text{weq}C \times \text{weq}C$, the induced functor

$$U_{X, Y}: C^\text{fun}(X, Y) \to C^{[-1;1]}(X, Y)$$

is homotopy cofinal.

**Remark 3.5.** Morphisms in $V$ can be pulled back along arbitrary morphisms in $C$, so the $V$-cocycle category $C^{[-1;1]}_V(X, Y)$ is contravariantly pseudofunctorial in $X$ and strictly functorial in $Y$.

**Remark 3.6.** We do not require $(\text{dom}, \text{codom}) : \text{weq}[[[-1;1], C]_h \to \text{weq}C \times \text{weq}C$ to be a Grothendieck fibration. Nonetheless, it still makes sense to talk about cartesian morphisms in $\text{weq}[[[-1;1], C]_h$. For example, consider a cocycle in $C$,

$$X \leftarrow^v Z \rightarrow^f Y$$

where $v : Z \to X$ is in $V$; then, for any weak equivalence $w : X' \to X$ in $C$, we can form the following commutative diagram in $C$,

$$
\begin{array}{ccc}
X' & \leftarrow^{v'} & Z' \\
\downarrow^w & & \downarrow^f \\
X & \leftarrow_v & Z \\
& \downarrow_f & \downarrow \\
& Y & \\
\end{array}
$$

where the left square is a pullback diagram in $C$, and it is straightforward to verify that the corresponding morphism $(f', v') \to (f, v)$ is a cartesian morphism in $\text{weq}[[[-1;1], C]_h$.

The primary example of a homotopical calculus of cocycles is the case where $C$ is a category of fibrant objects, $V$ is the subcategory of trivial fibrations in $C$, $C^\text{fun}$ is a certain full subcategory of $\text{weq}[[[-1;1], C]_h$, and the functor $U : C^\text{fun} \to \text{weq}[[[-1;1], C]_h$ is the inclusion. The details of this are deferred to the following sections.

For the remainder of this section, let $C$ be a category with weak equivalences and let $V \subseteq \text{weq}C$, $C^\text{fun}$, and $U : C^\text{fun} \to \text{weq}[[[-1;1], C]_h$ be the data of a homotopical calculus of cocycles in $C$.

**Lemma 3.7.** Let $D$ be a full subcategory of $C$ (regarded as a relative category with the same weak equivalences) and let $U : D^\text{fun} \to \text{weq}[[[-1;1], D]_h$ be defined
by the following pullback diagram in \( \textbf{Cat} \):

\[
\begin{array}{ccc}
\mathcal{D}^{\text{fun}} & \xrightarrow{U} & \text{weq} \left[ [-1; 1], \mathcal{D} \right]_h \\
\downarrow & & \downarrow \\
\mathcal{C}^{\text{fun}} & \xrightarrow{U} & \text{weq} \left[ [-1; 1], \mathcal{C} \right]_h
\end{array}
\]

If \( \mathcal{D} \) is a homotopically replete in \( \mathcal{C} \), then \( \mathcal{V} \cap \mathcal{D} \subseteq \text{weq} \mathcal{D} \), \( \mathcal{D}^{\text{fun}} \), and \( U : \mathcal{D}^{\text{fun}} \to \text{weq} \left[ [-1; 1], \mathcal{D} \right]_h \) define a homotopical calculus of cocycles in \( \mathcal{D} \).

**Proof.** Since \( \mathcal{D} \) is a full and homotopically replete subcategory of \( \mathcal{C} \), \( \mathcal{V} \cap \mathcal{D} \) is closed under pullback in \( \mathcal{D} \). It is not hard to verify that the following diagram is a pullback square in \( \textbf{Cat} \),

\[
\begin{array}{ccc}
\text{weq} \left[ [-1; 1], \mathcal{D} \right]_h \xrightarrow{\text{(dom, codom)}} & \text{weq} \mathcal{D} \times \text{weq} \mathcal{D} \\
\downarrow & & \downarrow \\
\text{weq} \left[ [-1; 1], \mathcal{C} \right]_h \xrightarrow{\text{(dom, codom)}} & \text{weq} \mathcal{C} \times \text{weq} \mathcal{C}
\end{array}
\]

so by the pullback pasting lemma, the outer rectangle in the diagram below is also a pullback diagram in \( \textbf{Cat} \):

\[
\begin{array}{ccc}
\mathcal{D}^{\text{fun}} & \xrightarrow{U} & \text{weq} \left[ [-1; 1], \mathcal{D} \right]_h \xrightarrow{\text{(dom, codom)}} & \text{weq} \mathcal{D} \times \text{weq} \mathcal{D} \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{C}^{\text{fun}} & \xrightarrow{U} & \text{weq} \left[ [-1; 1], \mathcal{C} \right]_h \xrightarrow{\text{(dom, codom)}} & \text{weq} \mathcal{C} \times \text{weq} \mathcal{C}
\end{array}
\]

Recalling that the class of Grothendieck fibrations is closed under pullback in \( \textbf{Cat} \), we deduce that the composite of the top row is a Grothendieck fibration, as required. Moreover, any morphism in \( \text{weq} \left[ [-1; 1], \mathcal{D} \right]_h \) that is a cartesian morphism in \( \text{weq} \left[ [-1; 1], \mathcal{C} \right]_h \) is automatically a cartesian morphism in \( \text{weq} \left[ [-1; 1], \mathcal{D} \right]_h \), so \( U : \mathcal{D}^{\text{fun}} \to \text{weq} \left[ [-1; 1], \mathcal{D} \right]_h \) preserves cartesian morphisms. Since the remaining axioms can be checked fibrewise, this completes the proof. \( \blacksquare \)

**Lemma 3.8.** For each object \( Y \) in \( \mathcal{C} \), there exist an object \( I_Y \) in \( \mathcal{C}^{\text{fun}} \) and a commutative diagram in \( \mathcal{W} \) of the form below,

\[
\begin{array}{ccc}
Y & \xleftarrow{id} & Y \\
\downarrow & \downarrow \downarrow & \downarrow \downarrow \\
Y & \xleftarrow{\text{Path}(Y)} & Y \\
\end{array}
\]

where the bottom row is the cocycle \( UI_Y \).

---

[2] i.e. for any weak equivalence \( w : X \to Y \) in \( \mathcal{C} \), if either \( X \) or \( Y \) is in \( \mathcal{D} \), then \( X, Y, \) and \( w : X \to Y \) are all in \( \mathcal{D} \).
Proof. Let $Z$ be the cocycle $(\text{id}_Y, \text{id}_Y) : Y \to Y$ in $C$. By definition, $U_{Y, Y} : C^\text{fun}(Y, Y) \to C^{[-1;1]}(Y, Y)$ is a homotopy cofinal functor, so the comma category $(Z \downarrow U_{Y, Y})$ is weakly contractible. In particular, it is inhabited, so there indeed exist an object $I_Y$ in $C^\text{fun}$ and a commutative diagram of the required form.  

Lemma 3.9. For any pair $(X, Y)$ of objects in $C$, in the following commutative diagram,

\[
\begin{array}{ccc}
C^\text{fun}(X, Y) & \longrightarrow & C^{[-1;1]}_{Y}(X, Y) \\
\downarrow & & \downarrow \\
C^\text{fun}(X, Y) & \longrightarrow & C^{[-1;1]}(X, Y)
\end{array}
\]

every arrow is a weak homotopy equivalence of categories.

Proof. The bottom horizontal arrow is a homotopy cofinal functor and the right vertical arrow is fully faithful. By lemma 1.5, the top horizontal arrow is also a homotopy cofinal functor, so by Quillen’s Theorem A (1.6) and the 2-out-of-3 property, the inclusion is indeed a weak homotopy equivalence of categories.  

Lemma 3.10. Let $W = \text{weq} C$, let $Y$ be an object in $C$, and let $R$ be the following category:

- The objects are tuples $(E, w, u)$, where $E$ is an object in $C^\text{fun}$, and $w$ and $u$ are weak equivalences in $C$ making the diagram in $C$ shown below commute,

\[
\begin{array}{ccc}
Y & \longrightarrow & Y \\
\downarrow^w & & \downarrow^u \\
E & \longrightarrow & Y
\end{array}
\]

where the bottom row is the cocycle $UE$. (So $(\text{codom} \circ U)E = Y$.)

- The morphisms are morphisms $k$ in $C^\text{fun}$ such that the morphism $Uk$ in $\text{weq} [[-1;1], C]_h$ makes the evident diagram commute. (In particular, $(\text{codom} \circ U)k = \text{id}_Y$ in $C$.)

- Composition and identities are inherited from $C^\text{fun}$.

Then the functor $P : R \to Y/W$ defined by sending $(E, w, u)$ to $w$ is a Grothendieck fibration whose (strict) fibres are weakly contractible.
Proof. Let \((E, w, u)\) be an object in \(\mathcal{R}\) and suppose \(UE\) is the following cocycle:

\[
\begin{array}{ccc}
X & \overset{v}{\leftarrow} & Z \\
& & \overset{r}{\rightarrow} \\
& & Y
\end{array}
\]

Let \(w' : Y \to X'\) and \(f : X' \to X\) be weak equivalences in \(\mathcal{C}\) such that the diagram below commutes:

\[
\begin{array}{ccc}
Y & \cong & Y \\
\downarrow \scriptstyle w' & & \downarrow \scriptstyle w \\
X' & \overset{f}{\rightarrow} & X
\end{array}
\]

Since \(\langle \text{dom}, \text{codom} \rangle \circ U : \mathcal{C}^{\text{fun}} \to \mathcal{W} \times \mathcal{W}\) is a Grothendieck fibration, there is a cartesian morphism \(k : E' \to E\) in \(\mathcal{C}^{\text{fun}}\) such that \(Uk\) is of the form below:

\[
\begin{array}{ccc}
X' & \overset{v'}{\leftarrow} & Z' \\
& & \overset{r'}{\rightarrow} \\
& & Y
\end{array}
\]

Moreover, since \(Uk : UE' \to UE\) is a cartesian morphism in \(\text{weq} \left[[-1; 1], \mathcal{C} \right]_h\), there is a weak equivalence \(u' : Y \to Z'\) in \(\mathcal{C}\) making the following diagram commute:

\[
\begin{array}{ccc}
Y & \cong & Y & \cong & Y \\
\downarrow \scriptstyle w' & & \downarrow \scriptstyle w & & \downarrow \scriptstyle id \\
X' & \overset{v'}{\leftarrow} & Z' & \overset{u'}{\rightarrow} & Y \\
\downarrow \scriptstyle f & & \downarrow \scriptstyle g & & \downarrow \scriptstyle id \\
X & \overset{v}{\leftarrow} & Z & \overset{u}{\rightarrow} & Y \\
\downarrow \scriptstyle w & & \downarrow \scriptstyle r & & \downarrow \scriptstyle id
\end{array}
\]

Thus, \((E', w', u')\) is an object in \(\mathcal{R}\) and \(k : E' \to E\) defines a morphism \((E', w', u') \to (E, w, u)\) in \(\mathcal{R}\).

We now show that \(k : (E', w', u') \to (E, w, u)\) is a cartesian morphism in \(\mathcal{R}\). Suppose we have a morphism \(h : (E'', w'', u'') \to (E, w, u)\) in \(\mathcal{R}\) and a commutative diagram in \(\mathcal{W}\) of the form below,

\[
\begin{array}{ccc}
Y & \overset{w''}{\rightarrow} & Y \\
\downarrow \scriptstyle w' & & \downarrow \scriptstyle w \\
X'' & \overset{x'}{\rightarrow} & X'
\end{array}
\]
where \( f \circ x' : X'' \to X \) is the underlying morphism in \( \mathcal{C} \) of \( Ph : w'' \to w \). Since \( k : E' \to E \) is a cartesian morphism in \( \mathcal{C} \), there is a unique morphism \( h' : E'' \to E' \) such that \( k \circ h' = h \) with \((\text{dom} \circ U)h' = x' \) and \((\text{codom} \circ U)h' = \text{id}_Y\). A similar argument using the fact that \( Uk : UE' \to UE \) is a cartesian morphism in \( \text{weq}([-1; 1], \mathcal{C})_h \) shows that \( k \) defines a morphism \((E'', w'', u'') \to (E', w', u') \) in \( \mathcal{R} \). Hence, \( k : (E', w', u') \to (E, w, u) \) is indeed a cartesian morphism in \( \mathcal{R} \).

Finally, it remains to be shown that the (strict) fibres of \( P : \mathcal{R} \to Y/\mathcal{W} \) are weakly contractible. But for any object \( w : Y \to X \) in \( Y/\mathcal{W} \), the corresponding fibre of \( P \) is isomorphic to the comma category \(((\text{id}_Y, w) \downarrow U_{X,Y})\), and since \( U_{X,Y} : \mathcal{C}^\text{fun}(X, Y) \to \mathcal{C}([-1; 1])(X, Y) \) is a homotopy cofinal functor, \(((\text{id}_Y, w) \downarrow U_{X,Y})\) is weakly contractible, as required.

\[ \text{Lemma 3.11.} \text{ Let } \mathcal{W} = \text{weq} \mathcal{C} \text{ and let } (X, Y) \text{ be a pair of objects in } \mathcal{C}, \text{ let } k \text{ be a natural number, let } \mathcal{H}_0(X, Y) = \mathcal{C}([-1; k])(X, Y), \text{ let } \mathcal{H}_1(X, Y) = \mathcal{C}([-1; k; -1]), \text{ and let } \mathcal{H}_2(X, Y) \text{ be defined by the following pullback diagram in } \text{Cat}, \]

\[
\begin{array}{ccc}
\mathcal{H}_2(X, Y) & \xrightarrow{Q} & \mathcal{R} \\
\downarrow d & & \downarrow P \\
\mathcal{H}_1(X, Y) & \xrightarrow{} & Y/\mathcal{W}
\end{array}
\]

where \( \mathcal{H}_1(X, Y) \to Y/\mathcal{W} \) is the evident projection and \( P : \mathcal{R} \to Y/\mathcal{W} \) is the Grothendieck fibration defined in lemma 3.10.

(i) The functor \( d : \mathcal{H}_2(X, Y) \to \mathcal{H}_1(X, Y) \) is a weak homotopy equivalence of categories.

(ii) There is a weak homotopy equivalence \( s^2 : \mathcal{H}_0(X, Y) \to \mathcal{H}_2(X, Y) \) such that the composite \( d \circ s^2 : \mathcal{H}_0(X, Y) \to \mathcal{H}_1(X, Y) \) is the functor \( s : \mathcal{H}_0(X, Y) \to \mathcal{H}_1(X, Y) \) defined by inserting an identity morphism.

\[ \text{Proof.} \] (i) Lemma 3.10 says \( P : \mathcal{R} \to Y/\mathcal{W} \) is a Grothendieck fibration with weakly contractible (strict) fibres, and these properties are preserved by pullback, so \( d : \mathcal{H}_2(X, Y) \to \mathcal{H}_1(X, Y) \) is also a Grothendieck fibration with weakly contractible (strict) fibres. Hence, by lemma 1.4 and Quillen’s Theorem A (1.6), \( d : \mathcal{H}_2(X, Y) \to \mathcal{H}_1(X, Y) \) is a weak homotopy equivalence of categories.

(ii). Let \( s^2 : \mathcal{H}_0(X, Y) \to \mathcal{H}_2(X, Y) \) be the unique functor such that \( d \circ s^2 = s \) and \( Q \circ s^2 \) is the constant functor with value \( I_Y \), where \( I_Y \) is an object in \( \mathcal{R} \) as in lemma 3.8. We will construct a functor \( r^2 : \mathcal{H}_2(X, Y) \to \mathcal{H}_0(X, Y) \) making
the following diagram commute in \( \text{Ho sSet} \),

\[
N(H_0(X,Y)) \xrightarrow{N(s^2)} N(H_2(X,Y)) \xrightarrow{N(s)} N(H_0(X,Y)) \xrightarrow{N(r^2)} N(H_1(X,Y))
\]

and (by the 2-out-of-6 property) it will follow that \( s^2 : H_0(X,Y) \to H_2(X,Y) \) is indeed a weak homotopy equivalence of categories.

First, observe that every object in \( H_2(X,Y) \) has an underlying commutative diagram in \( C \) of the form below:

\[
\begin{array}{cccccc}
Y & \xleftarrow{\text{id}} & Y \\
\uparrow{q} & & \uparrow{q} \\
X'_k & \xleftarrow{u} & Y \\
\downarrow{v_k} & & \\
X_0 & \to & \cdots & \to & X_{k-1} & \to X_k \\
\end{array}
\]

For \( 0 \leq j < k \), write \( f_j \) for the morphism \( \tilde{X}_j \to \tilde{X}_{j+1} \) in the above diagram. Since \( v_k : X'_k \to \tilde{X}_k \) is in \( \mathcal{V} \), we may functorially construct the following commutative diagram in \( C \),

\[
\begin{array}{cccccc}
\tilde{X}'_0 & \to & \cdots & \to & \tilde{X}'_{k-1} & \to \tilde{X}'_k \\
\downarrow{v_0} & & & \downarrow{v_k} & & \downarrow{v_k} \\
X_0 & \to & \cdots & \to & X_{k-1} & \to X_k \\
\end{array}
\]

where each square is a pullback diagram in \( C \). We then obtain the diagram in \( C \) shown below,

\[
\begin{array}{cccccc}
\hat{X} & \xleftarrow{\text{id}} & X \\
\uparrow{q} & & \uparrow{q} \\
\hat{X}'_0 & \xleftarrow{u} & \hat{X}'_{k-1} & \xleftarrow{\text{id}} & \hat{X}'_k \\
\downarrow{v_0} & & \downarrow{v_k} & & \downarrow{v_k} \\
\hat{X}_0 & \to & \cdots & \to & \hat{X}_{k-1} & \to \hat{X}_k \\
\end{array}
\]

where every vertical arrow is a weak equivalence in \( C \). Omitting the rightmost arrow in the top row gives an object in \( H_0(X,Y) \), so this construction defines a functor \( r^2 : H_2(X,Y) \to H_0(X,Y) \) equipped with a zigzag of natural weak equivalences connecting \( d \) and \( s \circ r^2 \).
Now suppose $X_k = Y$ and $w = id_Y$. Then, for $0 \leq j < k$, there is a unique morphism $u_j : X_j \to X_j'$ in $C$ making the diagram below commute,

\[
\begin{array}{c}
\begin{array}{c}
\dot{X}_j \xrightarrow{f_j} \dot{X}_{j+1} \\
\downarrow^{u_j} \downarrow^{u_{j+1}} \\
\dot{X}'_j \xrightarrow{f'_j} \dot{X}'_{j+1} \\
\downarrow v_j \downarrow v_{j+1} \\
X_j \xrightarrow{f_j} X_{j+1}
\end{array}
\end{array}
\]

where (for convenience) we define $u_k = u$. Since $q \circ u = id_Y$, we obtain the following commutative diagram in $C$,

\[
\begin{array}{c}
\begin{array}{c}
X \xleftarrow{X_0} \xrightarrow{\ldots} \xrightarrow{X_{k-1}} Y \\
\downarrow^{u_0} \downarrow^{u_{k-1}} \downarrow^{q \circ f'_{k-1}} \\
X' \xleftarrow{X'_0} \xrightarrow{\ldots} \xrightarrow{X'_{k-1}} Y
\end{array}
\end{array}
\]

where every vertical arrow is weak equivalence in $C$. Thus, we have a natural weak equivalence $id_{Ho(X,Y)} \Rightarrow r^2 \circ s^2$. This completes the proof of the claim. ■

**Theorem 3.12.** Let $C$ be a category with weak equivalences, let $L^H_C$ be the hammock localisation, and let $X$ and $Y$ be objects in $C$. If $C$ admits a homotopical calculus of cocycles with distinguished subcategory $V \subseteq weq C$, then:

(i) The reduction morphism $N(C^{[-1;1]}_V(X,Y)) \to L^H_C(X,Y)$ is a weak homotopy equivalence.

(ii) The reduction morphism $N(C^{[-1;1]}_V(X,Y)) \to L^H_C(X,Y)$ is natural in the following sense: for any morphisms $X' \to X$ and $Y \to Y'$ in $C$, the following diagram commutes in $Ho sSet$,

\[
\begin{array}{c}
\begin{array}{c}
N(C^{[-1;1]}_V(X,Y)) \xrightarrow{N(C^{[-1;1]}_V(X,Y))} L^H_C(X,X) \\
\downarrow \downarrow \\
N(C^{[-1;1]}_V(X',Y')) \xrightarrow{N(C^{[-1;1]}_V(X',Y'))} L^H_C(X',Y')
\end{array}
\end{array}
\]

where the left vertical arrow is defined as in remark 3.5 and the right vertical arrow is defined by concatenation.

(iii) There is an isomorphism

\[
N(C^{[-1;1]}_V(-,-)) \cong L^H_C(-,-)
\]

of functors $Ho C^{op} \times Ho C \to Ho sSet$.  

18
Proof. (i). Combine theorem 2.10 with lemmas 3.9 and 3.11.

(ii). Straightforward.

(iii). This is an immediate consequence of (i) and (ii).

4 Categories of fibrant objects

The following definition is due to Brown [1973].

**Definition 4.1.** A **category of fibrant objects** is a category $\mathcal{C}$ with finite products and equipped with a pair $(\mathcal{W}, \mathcal{F})$ of subclasses of ${\text{mor}} \mathcal{C}$ satisfying these axioms:

(A) $(\mathcal{C}, \mathcal{W})$ is a category with weak equivalences.

(B) Every isomorphism is in $\mathcal{F}$, and $\mathcal{F}$ is closed under composition.

(C) Pullbacks along morphisms in $\mathcal{F}$ exist in $\mathcal{C}$, and the pullback of a morphism that is in $\mathcal{F}$ (resp. $\mathcal{W} \cap \mathcal{F}$) is also a morphism that is in $\mathcal{F}$ (resp. $\mathcal{W} \cap \mathcal{F}$).

(D) For each object $X$ in $\mathcal{C}$, there is a commutative diagram of the form below,

$$
\begin{array}{c}
X \\
\downarrow \Delta \\
\times X
\end{array}
\xrightarrow{i} \begin{array}{c}
\text{Path}(X) \\
\downarrow p
\end{array}

$$

where $\Delta : X \to X \times X$ is the diagonal morphism, $i : X \to \text{Path}(X)$ is in $\mathcal{W}$, and $\text{Path}(X) \to X \times X$ is in $\mathcal{F}$.

(E) For any object $X$ in $\mathcal{C}$, the unique morphism $X \to 1$ is in $\mathcal{F}$.

In a category of fibrant objects as above,

- a **weak equivalence** is a morphism in $\mathcal{W}$,
- a **fibration** is a morphism in $\mathcal{F}$, and
- a **trivial fibration** (or **acyclic fibration**) is a morphism in $\mathcal{W} \cap \mathcal{F}$.

**Example 4.2.** Of course, the full subcategory of fibrant objects in a model category is a category of fibrant objects, with weak equivalences and fibrations inherited from the model structure.
Example 4.3. Let $\mathcal{M}$ be a right proper model category, let $\mathcal{W}$ be the class of weak equivalences, and let $\mathcal{F}$ be the class of morphisms $p : X \to Y$ with the following property: every pullback square in $\mathcal{M}$ of the form below

$$
\begin{array}{ccc}
X' & \longrightarrow & X \\
\downarrow & & \downarrow^p \\
Y' & \longrightarrow & Y
\end{array}
$$

is also a homotopy pullback square in $\mathcal{M}$. (In other words, $\mathcal{F}$ is the class of sharp maps in $\mathcal{M}$ in the sense of Rezk [1998].) Let $\mathcal{E}$ be the full subcategory of $\mathcal{M}$ spanned by those objects $X$ such that the unique morphism $X \to 1$ is in $\mathcal{F}$. Then $\mathcal{E}$ is a category of fibrant objects, with weak equivalences $\mathcal{W}$ and fibrations $\mathcal{F}$.

Definition 4.4. Let $\mathcal{C}$ be a category of fibrant objects and let $(X, Y)$ be a pair of objects in $\mathcal{C}$. A functional correspondence $(p, v) : X \to Y$ is a cocycle,

$$
\begin{array}{ccc}
X & \leftarrow \tilde{X} & \longrightarrow \ Y \\
\downarrow^v & & \downarrow^p \\
\end{array}
$$

such that $(p, v) : \tilde{X} \to Y \times X$ is a fibration.

We write $\mathcal{C}_{\text{fun}}$ (resp. $\mathcal{C}^{\text{fun}}(X, Y)$) for the full subcategory of $\text{weq} \mathcal{C}$ (resp. $\mathcal{C}^{[-1;1]}(X, Y)$) spanned by the functional correspondences.

Remark 4.5. Since product projections in a category of fibrant objects are fibrations, it follows that $v : \tilde{X} \to X$ is a trivial fibration and $p : \tilde{X} \to Y$ is a fibration. However, the converse is not true: for instance, $(\text{id}_Y, \text{id}_Y) : Y \to Y$ is rarely a functional correspondence.

Lemma 4.6. Let $\mathcal{C}$ be a category of fibrant objects and let $\mathcal{W} = \text{weq} \mathcal{C}$.

(i) The functor $\mathcal{C}_{\text{fun}} \to \mathcal{W} \times \mathcal{W}$ sending functional correspondences $X \to Y$ to the pair $(X, Y)$ is a Grothendieck fibration.

(ii) The inclusion $\mathcal{C}_{\text{fun}} \to \text{weq} \mathcal{C}$ preserves cartesian morphisms.

Proof. Consider a functional correspondence in $\mathcal{C}$, say:

$$
\begin{array}{ccc}
X & \leftarrow \tilde{X} & \longrightarrow \ Y \\
\downarrow^v & & \downarrow^p \\
\end{array}
$$

Let $f : X' \to X$ and $g : Y' \to Y$ be weak equivalences in $\mathcal{C}$. By axiom C, we may form the following pullback diagram in $\mathcal{C}$:

$$
\begin{array}{ccc}
\tilde{X}' & \longrightarrow & \tilde{X} \\
\downarrow^{(p', v')} & & \downarrow^{(p, v)} \\
Y' \times X' & \longrightarrow & Y \times X
\end{array}
$$
By considering the cases \( f = \text{id} \) and \( g = \text{id} \) separately and using the pullback pasting lemma, we may deduce that \( v' : \tilde{X}' \to X' \) is a trivial fibration, and hence that \( (p', v') : X' \to Y' \) is a functional correspondence in \( C \). It is then straightforward to verify that the commutative diagram

\[
\begin{array}{ccc}
X' & \xleftarrow{v'} & \tilde{X}' \\
\downarrow{f} & & \downarrow{g} \\
X & \xleftarrow{v} & \tilde{X} \\
\end{array}
\]

defines a cartesian morphism in both \( C_{\text{fun}} \) and \( \text{weq} \{[-1; 1], C \}_h \).

It is convenient to slightly strengthen the axioms given earlier.

**Definition 4.7.** A **path object functor** for a category of fibrant objects \( C \) consists of the following data:

- A functor \( \text{Path} : C \to C \).
- Natural transformations \( i : \text{id}_C \Rightarrow \text{Path} \) and \( p_0, p_1 : \text{Path} \Rightarrow \text{id}_C \) such that \( (\text{Path}(C), i_X, (p_0)_X, (p_1)_X) \) is a path object for every object \( X \) in \( C \), i.e. \( (p_0)_X, (p_1)_X) : \text{Path}(X) \to X \times X \) is a fibration and \( (p_0)_X \circ i_X = (p_1)_X \circ i_X = \text{id}_X \).

We say \( C \) has **functorial path objects** if it admits a path object functor.

**Lemma 4.8** (Factorisation lemma). Let \( f : X \to Y \) be a morphism in a category of fibrant objects \( C \).

(i) There exists a commutative diagram in \( C \) of the form below,

\[
\begin{array}{ccc}
X & \xleftarrow{\text{id}} & X \\
\downarrow{u} & & \downarrow{f} \\
X & \xleftarrow{v} & E_f \\
\end{array}
\]

where the bottom row is a functional correspondence in \( C \).

(ii) Moreover, if \( C \) has functorial path objects, then \( u, v, \) and \( p \) can be chosen functorially (with respect to \( f \)).

**Proof.** See (the proof of) the factorisation lemma in [Brown, 1973].

**Lemma 4.9.** Let \( C \) be a category of fibrant objects and let \( (X, Y) \) be a pair of objects in \( C \). If \( C \) has functorial path objects, then the inclusion \( U_{X,Y} : C_{\text{fun}}(X, Y) \to C([-1; 1](X, Y) \) is homotopy cofinal.
Cocycles in categories of fibrant objects

Proof. Let \((f, w) : X \to Y\) be a cocycle in \(C\). We must show that the comma category \(((f, w) \downarrow U_{X,Y})\) is weakly contractible. By lemma 4.8, we may factor \(\langle f, w \rangle : \tilde{X} \to Y \times X\) as a weak equivalence followed by a fibration, yielding an object in \(((f, w) \downarrow U_{X,Y})\). We may then use the functoriality of this factorisation to construct a zigzag of natural weak equivalences between \(\text{id}_{((f, w) \downarrow U_{X,Y})}\) and a constant endofunctor, and it follows that \(((f, w) \downarrow U_{X,Y})\) is weakly contractible. 

Remark 4.10. The argument in the proof above is essentially the same as the proof of Theorem 14.6.2 in [Hirschhorn, 2003], but applied in a different context.

Theorem 4.11. Let \(C\) be a category of fibrant objects and let \(\mathcal{V}\) be the subcategory of trivial fibrations in \(C\). If \(C\) has functorial path objects, then \(\mathcal{V} \subseteq \text{weq} C\), \(C^{\text{fun}}\), and \(C^{\text{fun}} \hookrightarrow \text{weq } [[-1; 1], C]_h\) constitute a homotopical calculus of cocycles in \(C\).

Proof. Combine lemmas 4.6 and 4.9. 

To extend the above result to the case where \(C\) is not assumed to have functorial path objects, we would have to prove lemma 4.9 without using functorial factorisations. We will do this in the appendix.

5 Simplicial categories of fibrant objects

One way of getting a category of fibrant objects with functorial path objects is to take the full subcategory of fibrant objects in a simplicial closed model category. We may treat these axiomatically as follows:

Definition 5.1. A simplicial category of fibrant objects is a simplicially enriched category \(C\) with simplicially enriched finite products and equipped with a pair \((\mathcal{W}, \mathcal{F})\) of subclasses of \(\text{mor} C\) satisfying axioms A, B, C, E, and these additional axioms:

\[(C_\Delta)\] Simplicially enriched pullbacks along morphisms in \(\mathcal{F}\) exist in \(C\).

\[(F)\] For any finite simplicial set \(K\) and any object \(X\) in \(C\), there exists an object \(K \downarrow X\) in \(C\) equipped with a (simplicially enriched natural) isomorphism

\[\text{sSet}(K, C(\_, X)) \cong C(\_, K \downarrow X)\]

of simplicially enriched functors \(C^{\text{op}} \to \text{sSet}\).
(G) For any monomorphism $i : K \to L$ of finite simplicial sets and any fibration $p : X \to Y$ in $\mathcal{C}$, the morphisms

$$i \cap \id_Y : L \cap Y \to K \cap Y \quad \quad \text{id}_K \cap p : K \cap X \to K \cap Y$$

are fibrations in $\mathcal{C}$, the morphism

$$i \boxtimes p : L \cap X \to (K \cap X) \times_{K \cap Y} (L \cap Y)$$

induced by the commutative diagram in $\mathcal{C}$ shown below

$$
\begin{array}{ccc}
L \cap X & \xrightarrow{id_L \cap p} & L \cap Y \\
\downarrow{i \cap \id_X} & & \downarrow{i \cap \id_Y} \\
K \cap X & \xrightarrow{id_K \cap p} & K \cap Y
\end{array}
$$

is a fibration in $\mathcal{C}$, and if $i$ is an anodyne extension (resp. $p$ is a trivial fibration), then both $i \cap \id_Y$ (resp. $\text{id}_K \cap p$) and $i \boxtimes p$ are trivial fibrations.

**Example 5.2.** Of course, if $\mathcal{M}$ is a simplicial closed model category and $\mathcal{M}_f$ is the simplicially enriched full subcategory of fibrant objects, then $\mathcal{M}_f$ admits the structure of a simplicial category of fibrant objects with the weak equivalences and fibrations inherited from $\mathcal{M}$.

**Proposition 5.3.** Let $\mathcal{C}$ be a simplicial category of fibrant objects. Then the underlying ordinary category $\mathcal{C}$ (satisfies axiom D and) is a category of fibrant objects with functorial path objects.

**Proof.** It straightforward to verify that $\Delta^1 \cap (-)$ is (the functor part of) a path object functor for $\mathcal{C}$. ■

**Lemma 5.4.** Let $\mathcal{C}$ be a simplicial category of fibrant objects, let $X$ be an object in $\mathcal{C}$, let $\mathcal{Q}$ be the full subcategory of the simplicially enriched slice category $\mathcal{C}/X$ spanned by the trivial fibrations, and let $p : U \to X$ be an object in $\mathcal{Q}$, i.e. a trivial fibration in $\mathcal{C}$.

(i) For any finite simplicial set $K$, the cotensor product $K \cap_X p : K \cap_X U \to X$ exists in $\mathcal{Q}$.

(ii) For any monomorphism $i : K \to L$ of finite simplicial sets, the induced morphism $i \cap_X U : L \cap_X U \to K \cap_X U$ is a trivial fibration in $\mathcal{C}$.

**Proof.** (i). Define the object $K \cap_X p : K \cap_X U \to X$ in $\mathcal{C}/X$ by the following pullback diagram in $\mathcal{C}$,

$$
\begin{array}{ccc}
K \cap_X U & \to & K \cap U \\
\downarrow{i \cap_X U} & & \downarrow{\text{id}_K \cap p} \\
X & \to & K \cap X
\end{array}
$$
where the bottom arrow is the morphism induced by the unique morphism
$K \to \Delta^0$. By axiom G, $id_K \triangleleft p : K \triangleleft U \to K \triangleleft X$ is a trivial fibration in $C$, so by axiom C, $K \triangleleft_X p : K \triangleleft_X U \to X$ is also a trivial fibration in $C$, hence is an object in $Q$. It is straightforward to verify that $K \triangleleft_X p$ has the required simplicially enriched universal property in $Q$.

(ii). By axiom G, we have a trivial fibration

$$i \Box p : L \triangleleft U \to (K \triangleleft U) \times_{K \triangleleft_X} (L \triangleleft X)$$

induced by the commutative diagram in $C$ shown below:

\[
\begin{diagram}
\text{L} \triangleleft U & \rTo{id_L \triangleleft p} & \text{L} \triangleleft X \\
\text{i} \triangleleft \text{id}_U & \dTo & \text{i} \triangleleft \text{id}_X \\
\text{K} \triangleleft U & \rTo{id_K \triangleleft p} & \text{K} \triangleleft X
\end{diagram}
\]

Moreover, by the pullback pasting lemma, we have the following commutative diagram in $C$,

\[
\begin{diagram}
\text{L} \triangleleft_X U & \rTo{\text{i} \triangleleft_X} & \text{L} \triangleleft U \\
\text{i} \triangleleft_X & \dTo & \text{i} \Box \text{p} \\
\text{K} \triangleleft_X U & \rTo{(\text{K} \triangleleft X \text{U}) \times_{\text{K} \triangleleft_X} (\text{L} \triangleleft X)} & \text{K} \triangleleft X \\
\text{Z} \triangleleft_X \text{p} & \dTo & \text{i} \triangleleft \text{id}_X \\
\text{X} & \rTo{i \triangleleft \text{id}_X} & \text{K} \triangleleft X
\end{diagram}
\]

where every square and rectangle is a pullback diagram in $C$. Thus, by axiom C, $i \triangleleft_X U : L \triangleleft_X U \to K \triangleleft_X U$ is indeed a trivial fibration in $C$. ■

**Lemma 5.5.** With notation as in lemma 5.4:

(i) $Q$ has simplicially enriched finite products.

(ii) Given any monomorphism $i : K \to L$ of finite simplicial sets and any pair $(p', p)$ of objects in $Q$, for each morphism $f : K \to Q(p', p)$, there exist a morphism $v : p'' \to p'$ in $Q$ and a morphism $g : L \to Q(p'', p)$ making the following diagram commute:

\[
\begin{diagram}
\text{K} & \rTo{f} & Q(p', p) \\
\text{i} & \dTo & Q(v, p) \\
\text{L} & \rTo{g} & Q(p'', p)
\end{diagram}
\]
Proof. (i). It is clear that \( Q \) has a simplicially enriched terminal object, and the existence of simplicially enriched binary products is an immediate consequence of axioms C and C_\( \Delta \).

(ii). By lemma 5.4, \( f : K \to Q(p', p) \) corresponds to a morphism \( \tilde{f} : p' \to K \sqcap_X p \) in \( Q \), and by axiom C, we may form the following pullback diagram in \( Q \),

\[
\begin{array}{ccc}
p'' & \xrightarrow{\tilde{g}} & L \sqcap_X p \\
\downarrow v & & \downarrow i \sqcap_X p \\
p' & \xrightarrow{f} & K \sqcap_X p
\end{array}
\]

where (the underlying morphism of) \( v : p'' \to p' \) is a trivial fibration in \( C \). Then \( \tilde{g} : p'' \to L \sqcap_X p \) corresponds to a morphism \( g : L \to Q(p'', p) \), and it is straightforward to see that diagram in question commutes.

\[\blacksquare\]

Corollary 5.6. Let \( C \) be a simplicial category of fibrant objects, let \( X \) be an object in \( C \), let \( Q \) be the full subcategory of the simplicially enriched slice category \( C/\sqcap_X \) spanned by the trivial fibrations, and let \( \pi_0[Q] \) be the category obtained by applying \( \pi_0 \) to the hom-spaces of \( Q \). Then \( \pi_0[Q]^{\text{op}} \) is a filtered category.

Proof. Recalling lemma 5.5, it suffices to show that, for any parallel pair \( f_0, f_1 : p' \to p \) in \( Q \), there is a morphism \( v : p'' \to p' \) in \( Q \) such that \( f_0 \circ v = f_1 \circ v \) in \( \pi_0[Q] \). But \( (f_0, f_1) \) define a morphism \( f : \partial \Delta^1 \to Q(p', p) \), so the lemma implies there exist a morphism \( v : p'' \to p' \) in \( Q \) and a morphism \( g : \Delta^1 \to Q(p'', p) \) making the diagram below commute,

\[
\begin{array}{ccc}
\partial \Delta^1 & \xrightarrow{f} & Q(p', p) \\
\downarrow & & \downarrow Q(v, p) \\
\Delta^1 & \xrightarrow{g} & Q(p'', p)
\end{array}
\]

so we indeed have \( f_0 \circ v = f_1 \circ v \) in \( \pi_0[Q]^{\text{op}}(p'', p) \).

\[\blacksquare\]

The next result may be regarded as a homotopical version of Theorem 1 in [Brown, 1973], which describes the hom-sets in the homotopy category of a category of fibrant objects. Indeed, we will derive a closely related result as a corollary.

Theorem 5.7. Let \( C \) be a simplicial category of fibrant objects, let \( \Lambda^H C \) be the hammock localisation, let \( X \) be an object in \( C \), let \( Q \) be the full subcategory
of the simplicially enriched slice category $\mathcal{C}/X$ spanned by the trivial fibrations, and let $U : Q \to C$ be the evident projection. Then,

$$\text{holim}_{Q^{\text{op}}} \mathcal{C}(U, -) \simeq \mathbb{L}^I \mathcal{C}(X, -)$$

by a zigzag of weak equivalences of functors $\mathcal{C} \to \mathbf{sSet}$. In particular,

$$\text{holim}_{Q^{\text{op}}} \mathcal{C}(U, -) : \mathcal{C} \to \mathbf{sSet}$$

preserves weak equivalences.

**Proof.** Let $Q$ be the underlying ordinary category of $\mathcal{Q}$. By lemmas 1.8 and 5.4,

$$\text{holim}_{Q^{\text{op}}} \mathcal{C}(U, -) \simeq \text{holim}_{\mathcal{Q}^{\text{op}}} \mathcal{C}(U, -)$$

so it suffices to verify the following:

$$\text{holim}_{\mathcal{Q}^{\text{op}}} \mathcal{C}(U, -) \simeq \mathbb{L}^I \mathcal{C}(X, -)$$

Moreover, recalling theorem 4.11 and proposition 5.3, we may apply theorem 2.10 and lemma 3.9 to reduce the problem to showing that

$$\text{holim}_{\mathcal{Q}^{\text{op}}} \mathcal{C}(U, -) \simeq \mathbb{N}(\mathcal{C}([-1;1]) \Delta \cdot \dashv (-))$$

by a zigzag of weak equivalences of functors. By using Thomason’s homotopy colimit theorem (1.11), it is not hard to see that there is a weak equivalence

$$\text{holim}_{\mathcal{Q}^{\text{op}}} \text{disc} \mathcal{C}(U, -) \simeq \mathbb{N}(\mathcal{C}([-1;1]) \Delta \cdot \dashv (-))$$

of functors $\mathcal{C} \to \mathbf{sSet}$, where on the LHS we have the *ordinary* hom-functor. In particular,

$$\text{holim}_{\Delta^{\text{op}}} \text{holim}_{\mathcal{Q}^{\text{op}}} \text{disc} \mathcal{C}(U, \Delta \cdot \dashv (-)) \simeq \text{holim}_{\Delta^{\text{op}}} \mathbb{N}(\mathcal{C}([-1;1]) \Delta \cdot \dashv (-))$$

but on the one hand, by corollary 2.11,

$$\mathbb{N}(\mathcal{C}([-1;1]) \Delta \cdot \dashv (-)) \simeq \text{holim}_{\Delta^{\text{op}}} \mathbb{N}(\mathcal{C}([-1;1]) \Delta \cdot \dashv (-))$$

and on the other hand, by the Bousfield–Kan theorem,[3]

$$\text{holim}_{\Delta^{\text{op}}} \text{disc} \mathcal{C}(U, \Delta \cdot \dashv (-)) \simeq \mathcal{C}(U, -)$$

so by interchanging homotopy colimits, the claim follows. ■

[3] See paragraph 4.3 in [Bousfield and Kan, 1972, Ch. XII] or Theorem 18.7.4 in [Hirschhorn, 2003].
Corollary 5.8. With notation as above,
\[ \lim_{\pi_0[Q]^{op}} \pi_0 C(U, -) \cong \text{Ho} C(X, -) \]
as functors \( \mathcal{C} \to \text{Set} \).

Proof. Since \( \pi_0 : s\text{Set} \to \text{Set} \) is a simplicially enriched left Quillen functor, it takes homotopy colimits in \( s\text{Set} \) to homotopy colimits in \( \text{Set} \). Homotopy colimits in \( \text{Set} \) are the same as (simplicially enriched) colimits, thus,
\[ \lim_{Q^{op}} \pi_0 C(U, -) \cong \pi_0 \mathcal{L}^H C(X, -) \cong \text{Ho} C(X, -) \]
But the evident localising functor \( Q \to \pi_0[Q] \) induces an equivalence between the category of simplicially enriched diagrams \( Q^{op} \to \text{Set} \) and the category of (ordinary) diagrams \( \pi_0[Q]^{op} \to \text{Set} \), so
\[ \lim_{\pi_0[Q]^{op}} \pi_0 C(U, -) \cong \lim_{Q^{op}} \pi_0 C(U, -) \]
and we are done. \( \square \)

6 Verdier’s hypercovering theorem

Throughout this section, let \( \mathcal{C} \) be a small category with a Grothendieck topology \( J \), let \( \mathcal{M} \) be the category of simplicial presheaves on \( \mathcal{C} \), equipped with the \( J \)-local model structure of Joyal [1984] and Jardine [1987], and for each regular cardinal \( \kappa \), let \( \mathcal{M}_{<\kappa} \) be the full subcategory of \( \kappa \)-presentable objects in \( \mathcal{M} \).

Proposition 6.1. For arbitrarily large regular cardinals \( \kappa \), \( \mathcal{M}_{<\kappa} \) inherits the structure of a simplicial closed model category from \( \mathcal{M} \), including functorial factorisations.

Proof. Use Propositions 1.17, 5.9, and 5.20 in [Low, 2014a]. \( \square \)

Recall also the notion of a \( J \)-local fibration of simplicial presheaves on \( \mathcal{C} \): in the case where \( (\mathcal{C}, J) \) is a site with enough points, a morphism of simplicial presheaves on \( \mathcal{C} \) is a \( J \)-local fibration if and only if all its stalks are Kan fibrations. Let \( \mathcal{E} \) be the full subcategory of \( \mathcal{M} \) spanned by the \( J \)-locally fibrant simplicial presheaves on \( \mathcal{C} \) and let \( \mathcal{E}_{<\kappa} = \mathcal{E} \cap \mathcal{M}_{<\kappa} \).

Proposition 6.2. For arbitrarily large regular cardinals \( \kappa \), \( \mathcal{E}_{<\kappa} \) is a simplicial category of fibrant objects, with weak equivalences being the \( J \)-local weak equivalences and fibrations being the \( J \)-local fibrations.
Proof. Let $\kappa$ be any uncountable regular cardinal such that $|\text{mor } C| < \kappa$. It is clear that axioms A, B, and E are satisfied, and a cardinality argument can be used to verify axiom C and $C_\Delta$. (Under our hypothesis on $\kappa$, a simplicial presheaf on $C$ is in $\mathcal{M}_{<\kappa}$ if and only if it has $< \kappa$ elements.) A similar argument shows that the cotensor products $K \otimes X$ are in $\mathcal{M}_{<\kappa}$ if $K$ is a finite simplicial set and $X$ is in $\mathcal{M}_{<\kappa}$, so it suffices to verify that $\mathcal{E}$ satisfies axioms F and G; for this, we may use the same method as the proof of Lemma 1.15 in [Low, 2014b], i.e. first reduce to the case of simplicial sheaves, and then apply Barr’s embedding theorem to reduce to the case of simplicial sets, which is well known.[4]

Henceforth, fix an infinite regular cardinal $\kappa$ such that $\mathcal{M}_{<\kappa}$ and $\mathcal{E}_{<\kappa}$ satisfy the conclusions of propositions 6.1 and 6.2.

**Lemma 6.3.** Let $\Gamma : \mathcal{C}^{\text{op}} \times \mathcal{M} \to \mathbf{sSet}$ be the functor defined by the following formula:

$$\Gamma(C, X) = X(C)$$

Then, for each object $C$ in $C$:

(i) Let $h_C$ be the simplicial presheaf represented by $C$. There is an isomorphism

$$\Gamma(C, -) \cong \mathcal{M}(h_C, -)$$

of functors $\mathcal{M} \to \mathbf{sSet}$, where the RHS is the simplicial hom-functor.

(ii) $\Gamma(C, -) : \mathcal{M} \to \mathbf{sSet}$ is a right Quillen functor. In particular, it has a total right derived functor $R\Gamma(C, -) : Ho \mathcal{M} \to Ho \mathbf{sSet}$.

(iii) Let $L^H_\mathcal{M} \mathcal{M}_{<\kappa}$ be the hammock localisation of $\mathcal{M}_{<\kappa}$. There is an isomorphism

$$R\Gamma(C, -) \cong L^H_\mathcal{M} \mathcal{M}_{<\kappa}(h_C, -)$$

of functors $Ho \mathcal{M}_{<\kappa} \to Ho \mathbf{sSet}$.

**Proof.** (i). Use the Yoneda lemma.

(ii). $\mathcal{M}(h_C, -)$ is a right Quillen functor because $\mathcal{M}$ is a simplicial closed model category where all objects are cofibrant, so $\Gamma(C, -)$ is also a right Quillen functor. The existence of a total right derived functor is then a standard result.[5]

(iii). Apply either Remark 5.2.10 in [Hovey, 1999] or Proposition 16.6.23 in [Hirschhorn, 2003] to Theorem 3.8 in [Low, 2014c].

[4] See e.g. Theorem 3.3.1 in [Hovey, 1999].

[5] See e.g. Theorem 8.5.8 in [Hirschhorn, 2003].

28
Lemma 6.4. Let $X$ and $Y$ be objects in $\mathcal{E}_{<\kappa}$. Then the morphism
\[ \mathbf{L}^\infty \mathcal{E}_{<\kappa}(X, Y) \to \mathbf{L}^\infty \mathcal{M}_{<\kappa}(X, Y) \]
induced by the inclusion $\mathcal{E}_{<\kappa} \to \mathcal{M}_{<\kappa}$ is a weak homotopy equivalence of simplicial sets.

Proof. Use Proposition 3.5 in [Dwyer and Kan, 1980b].

The following version of Verdier’s hypercovering theorem is due to Jardine [2012] and Rezk [2014].

Proposition 6.5. Let $C$ be an object in $\mathcal{C}$ and let $\mathcal{V}$ be the subcategory of $J$-local trivial fibrations in $\mathcal{E}_{<\kappa}$. Then there are isomorphisms
\[ \mathbf{R} \Gamma(C, -) \cong \mathbf{L}^\infty \mathcal{E}_{<\kappa}(\mathbf{h}_C, -) \cong \mathbf{N} \left( (\mathcal{E}_{<\kappa})_{[-1:1]}(\mathbf{h}_C, -) \right) \cong \mathbf{N} \left( (\mathcal{E}_{<\kappa})_{\mathcal{V}}(\mathbf{h}_C, -) \right) \]
of functors $\mathbf{Ho} \mathcal{E}_{<\kappa} \to \mathbf{Ho} \mathbf{sSet}$.

Proof. Combine theorem 2.10 and lemmas 3.9, 6.3, and 6.4.

One may then derive a homotopy colimit formula for $\mathbf{R} \Gamma(C, -)$ analogous to Verdier’s original colimit formula (cf. Théorème 7.4.1 in [SGA 4b, Exposé V] or Theorem 8.16 in [Artin and Mazur, 1969]):

Proposition 6.6. Let $C$ be an object in $\mathcal{C}$, let $\mathcal{Q}$ be the simplicially enriched full subcategory of the simplicially enriched slice category $(\mathcal{E}_{<\kappa})/\mathbf{h}_C$ spanned by the $J$-local trivial fibrations, and let $U : \mathcal{Q} \to \mathcal{E}_{<\kappa}$ be the evident projection functor. Then there is an isomorphism
\[ \mathbf{R} \Gamma(C, -) \cong \mathbf{holim}_{\mathcal{Q}^{\mathbf{op}}} \mathcal{E}_{<\kappa}(U, -) \]
of functors $\mathbf{Ho} \mathcal{E}_{<\kappa} \to \mathbf{Ho} \mathbf{sSet}$.

Proof. Apply theorem 5.7 and proposition 6.5.

Corollary 6.7. Let $K$ be a Kan complex of cardinality $< \kappa$ and let $\Delta K$ be the constant simplicial presheaf on $\mathcal{C}$ with value $K$. With other notation as above, we have
\[ \mathbf{R} \Gamma(C, \Delta K) \cong \mathbf{holim}_{\mathcal{Q}^{\mathbf{op}}} \mathbf{sSet} \left( \lim_{\mathcal{Q}^{\mathbf{op}}} \circ U, K \right) \]
as objects in $\mathbf{Ho} \mathbf{sSet}$, and this is natural in $K$.

Proof. As usual, we have the following isomorphism of simplicially enriched functors $\mathcal{Q}^{\mathbf{op}} \to \mathbf{sSet}$:
\[ \mathbf{sSet} \left( \lim_{\mathcal{Q}^{\mathbf{op}}} \circ U, K \right) \cong \mathcal{E}_{<\kappa}(U, \Delta K) \]
The claim follows, by proposition 6.6.
A Categories of fibrant objects redux

The following is what Cisinski [2010a] calls a ‘catégorie dérivable à gauche’:

Definition A.1. A Cisinski fibration category is a category $\mathcal{C}$ equipped with a pair $(\mathcal{W}, \mathcal{F})$ of subclasses of mor $\mathcal{C}$ satisfying these axioms:

D0. $\mathcal{C}$ has a terminal object $1$. A fibrant object in $\mathcal{C}$ is an object $X$ such that the unique morphism $X \to 1$ in $\mathcal{C}$ is in $\mathcal{F}$. Any object isomorphic to a fibrant object is fibrant, and $1$ is fibrant.

D1. $(\mathcal{C}, \mathcal{W})$ is a category with weak equivalences.

D2. $\mathcal{F}$ is closed under composition and every isomorphism between fibrant objects in $\mathcal{C}$ is in $\mathcal{F}$. If $p : X \to Y$ is in $\mathcal{F}$ and $g : Y' \to Y$ a morphism between fibrant objects in $\mathcal{C}$, then the pullback of $p$ along $g$ exists in $\mathcal{C}$ and is a morphism that is in $\mathcal{F}$.

D3. If $p : X \to Y$ is in $\mathcal{W} \cap \mathcal{F}$ and $g : Y' \to Y$ is a morphism between fibrant objects in $\mathcal{C}$, then the pullback of $p$ along $f$ (exists in $\mathcal{C}$ and) is a morphism that is in $\mathcal{W} \cap \mathcal{F}$.

D4. If $f : X \to Y$ is a morphism in $\mathcal{C}$ and $Y$ is fibrant, then there exist a morphism $i : X \to \hat{X}$ in $\mathcal{W}$ and a morphism $p : \hat{X} \to Y$ in $\mathcal{F}$ such that $f = p \circ i$.

In a Cisinski fibration category as above,

- a weak equivalence is a morphism in $\mathcal{W}$,
- a fibration is a morphism in $\mathcal{F}$, and
- a trivial fibration (or acyclic fibration) is a morphism in $\mathcal{W} \cap \mathcal{F}$.

We will often abuse notation and say $\mathcal{C}$ is a Cisinski fibration category, without mentioning the data $\mathcal{W}$ and $\mathcal{F}$.

Example A.2. Every category of fibrant objects is a Cisinski fibration category in the obvious way. Moreover, if $\mathcal{C}$ is a category of fibrant objects and $Y$ is an object in $\mathcal{C}$, then the slice category $\mathcal{C}/Y$ is Cisinski fibration category where the fibrant objects are the fibrations with codomain $Y$.

The following result (due to Denis-Charles Cisinski) will appear in [BHH]; we thank Geoffroy Horel for sharing it with us.

Theorem A.3 (Cisinski). Let $\mathcal{C}$ be a Cisinski fibration category and let $\mathcal{C}^\circ$ be the full subcategory of $\mathcal{C}$ spanned by the fibrant objects. Then the inclusion weq $\mathcal{C}^\circ \hookrightarrow$ weq $\mathcal{C}$ is a homotopy cofinal functor.

30
Proof. Let $X$ be an object in $\mathcal{C}$. We must show that the comma category $(X \downarrow U)$ is weakly contractible. By the asphericity lemma (1.6) in [Cisinski, 2010b], it suffices to verify the following: for any finite poset $\mathcal{J}$ and any diagram $F : \mathcal{J} \to (X \downarrow \text{weq } C^o)$, there is a zigzag of natural transformations connecting $F$ to a constant diagram.

First, observe that diagrams $F : \mathcal{J} \to (X \downarrow \text{weq } C^o)$ are the same as diagrams functors $Y : \mathcal{J} \to \text{weq } C$ equipped with a cone $\varphi : \Delta X \Rightarrow Y$. By Théorème 1.30 in [Cisinski, 2010a], there is a natural weak equivalence $\theta : Y \Rightarrow \hat{Y}$ where $\hat{Y}$ is fibrant over the boundaries (‘fibrant sur les bords’), so by Proposition 1.18 in op. cit., the limit $\lim_{\mathcal{J}} \hat{Y}$ exists in $\mathcal{C}$ and fibrant. Thus, the cone $\theta \circ \varphi : \Delta X \Rightarrow \hat{Y}$ can be factored as a weak equivalence $i : X \to \hat{X}$ in $\mathcal{C}$ followed by a cone $\hat{\varphi} : \Delta \hat{X} \Rightarrow \hat{Y}$, where $\hat{X}$ is a fibrant object in $\mathcal{C}$, and hence, we have the following diagram in $[\mathcal{J}, \text{weq } C]$:

\[
\begin{array}{ccc}
\Delta X & : & \Delta X \\
\varphi \downarrow & & \downarrow \Delta i \\
Y & \overset{\theta}{\longrightarrow} & \hat{Y} & \overset{\hat{\varphi}}{\longleftarrow} & \Delta \hat{X}
\end{array}
\]

This shows that $F : \mathcal{J} \to (X \downarrow \text{weq } C^o)$ is indeed connected to a constant diagram. ■

Corollary A.4. Let $\mathcal{C}$ be a category of fibrant objects, let $(X, Y)$ be a pair of objects in $\mathcal{C}$, and let $\mathcal{C}^{\text{fun}}(X, Y)$ be the category of functional correspondences $X \rightarrow Y$. Then the inclusion $U_{X,Y} : \mathcal{C}^{\text{fun}}(X, Y) \hookrightarrow \mathcal{C}^{[-1;1]}(X, Y)$ is homotopy cofinal.

Proof. Let $(f, w) : X \to Y$ be a cocycle in $\mathcal{C}$. We must show that the comma category $((f, w) \downarrow U_{X,Y})$ is weakly contractible. Let $\mathcal{D}$ be the slice category $\mathcal{C}_{/Y \times X}$ considered as a Cisinski fibration category in the obvious way. It is not hard to see that $\mathcal{C}^{\text{fun}}(X, Y)$ is isomorphic to a full subcategory of $\text{weq } \mathcal{C}_{/Y \times X}$, contained in the full subcategory $\text{weq } (\mathcal{C}_{/Y \times X})^o$ spanned by the fibrant objects (i.e. fibrations in $\mathcal{C}$ with codomain $Y \times X$). Moreover, the comma category $((f, w) \downarrow U_{X,Y})$ is isomorphic to the comma category $((f, w) \downarrow \text{weq } (\mathcal{C}_{/Y \times X})^o)$, so by theorem A.3, $((f, w) \downarrow U_{X,Y})$ is weakly contractible. ■

As promised, we obtain the following generalisation of theorem 4.11:

Theorem A.5. Let $\mathcal{C}$ be a category of fibrant objects, let $\mathcal{V}$ be the subcategory of trivial fibrations in $\mathcal{C}$, and let $\mathcal{C}^{\text{fun}}$ be the category of all functional correspondences in $\mathcal{C}$. Then $\mathcal{V} \subseteq \text{weq } \mathcal{C}$, $\mathcal{C}^{\text{fun}}$, and $\mathcal{C}^{\text{fun}} \hookrightarrow \text{weq } [[-1;1], \mathcal{C}]_h$ constitute a homotopical calculus of cocycles in $\mathcal{C}$.

Proof. Combine lemma 4.6 and corollary A.4. ■
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