Viscosity Solutions of Systems of PDEs with Interconnected Obstacles and Switching Problem

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Abstract This paper deals with existence and uniqueness of a solution in viscosity sense, for a system of $m$ variational partial differential inequalities with interconnected obstacles. A particular case is the Hamilton-Jacobi-Bellmann system of the Markovian stochastic optimal $m$-states switching problem. The switching cost functions depend on $(t, x)$. The main tool is the notion of systems of reflected backward stochastic differential equations with oblique reflection.

Keywords Real options · Backward stochastic differential equations · HJB system · Switching strategy · Viscosity solution of PDEs · Variational inequalities

1 Introduction

The multi-modes switching problem is by now well documented both in the economics and mathematical literature (see e.g. [3–5, 9, 10, 15, 16, 18, 19, 23–25], etc. and the references therein). The pioneering work of Brennan and Schwarz [3] deals with a two-modes switching problem describing the life cycle of an investment in the natural resource industry. A major switching problem of interest is related to investment of a capital in the most profitable economy in the globalization. More precisely, consider an agent (which can be e.g. a hedge fund) that aims at investing a capital in one of several economies denoted by $\epsilon_1, \ldots, \epsilon_m$. Her objective is to obtain the best return for the investment. The capital is invested in the economy $\epsilon_i$ up to the time when the agent makes the decision to switch it from $\epsilon_i$ to $\epsilon_j$ ($j \neq i$) because, e.g., there is no longer enough profitability in $\epsilon_i$. Moving the capital from $\epsilon_i$ to $\epsilon_j$ incures
expenditures which amount to $g_{ij}$. Therefore the agent faces two main problems, which are:

(i) What are the optimal successive times to move the capital?
(ii) When the decision to switch from the current economy is made, in which new economy will the capital be invested?

It is well-known that the stochastic optimal switching problem is related to systems of reflected backward stochastic differential equations (RBSDEs for short) with inter-connected obstacles or oblique reflection (see e.g. [4, 10, 16, 18, 19, 23]) of the following type: \( \forall i \in J := \{1, \ldots, m\}, \)
\[
\begin{align*}
Y^i_s &= h^i_T(\omega) + \int_s^T f^i(\omega, r) \, dr + K^i_T - K^i_s - \int_s^T Z^i_r \, dB_r, \quad \forall s \leq T \\
Y^i_s &\geq \max_{j \in J^-i} \{ Y^j_s - g_{ij}(\omega, s) \}, \quad \forall s \leq T \\
\int_0^T (Y^i_s - \max_{j \in J^-i} \{ Y^j_s - g_{ij}(\omega, s) \}) \, dK^i_s = 0,
\end{align*}
\]
(1.1)
where \( J^-i = J - \{i\} \) for any \( i \in J \). The solution \( (Y^i)_{i=1,m} \) of this system provides the optimal profit and an optimal strategy as well (for more details, the reader is referred to [10, 16, 18, 19]).

In the case of Markovian setting, i.e. when randomness stems from an exogenous process \((X^t,x^s)_{s \leq T}, (t,x) \in [0,T] \times \mathbb{R}^k\), solution of the following standard differential equation:
\[
\begin{align*}
dX^t,x^s &= b(s, X^t,x^s) \, ds + \sigma(s, X^t,x^s) \, dB_s, \quad s \in [t,T]; \\
X^t,x^s &= x \quad \text{for } s \leq t,
\end{align*}
\]
(1.2)
and the processes \((f^i(s,\omega))_{s \leq T}, (g^i(s,\omega))_{s \leq T}\) (resp. random variables \(h^i(T,\omega)\)) of (1.1) are equal to \((f^i(s, X^t,x^s(\omega)))_{s \leq T}, (g^i(s, X^t,x^s(\omega)))_{s \leq T}\) respectively (resp. \(h^i(X^T,x^s(\omega))\)) with deterministic functions \(f^i, g^i\) and \(h^i, i, j \in J\), the stochastic optimal switching problem is also related to the following system of variational inequalities with inter-connected obstacles: \( \forall i \in J \)
\[
\begin{align*}
\min \{ v^i(t,x) - \max_{j \in J^-i} (-g^i_j(t,x) + v^j(t,x)) - \partial_t v^i(t,x) - \mathcal{L}v^i(t,x) \\
- f^i(t,x) \} &= 0; \\
v^i(T,x) &= h^i(x),
\end{align*}
\]
(1.3)
where \( \mathcal{L} \) is the infinitesimal generator associated with \(X^{t,x}\). This system is nothing else but the Hamilton-Jacobi-Bellmann one associated with the stochastic optimal switching problem. In the example above, the process \(X^{t,x}\) can stand for the dynamics of the performance of the various economies.

In [15], the authors have proved that if the switching costs satisfy \(g^i_j(t,x) \geq \gamma_0 > 0, \forall (t,x)\), then system (1.3) has a unique continuous solution \((v^1, \ldots, v^m)\) whose relationship with the solution \((Y^i)_{i \in J}\) of system (1.1) in the Markovian setting, is given by the so-called Feynman-Kac formula:
\[
\forall i \in J, (t,x) \in [0,T] \times \mathbb{R}^k, \forall s \in [t,T], Y^i_s = v^i(s, X^t,x^s).
\]
(1.4)
The main objective of this paper is to deal with the general form of (1.3) i.e. the following system: \( \forall i \in J \)

\[
\begin{align*}
\min \left\{ v_i(t,x) - \max_{j \in J \setminus i} \left( -g_{ij}(t,x) + v_j(t,x) \right), -\partial_t v_i(t,x) - \mathcal{L} v_i(t,x) \\
- f_i\left(t, x, v^1(t,x), \ldots, v^m(t,x), \sigma^T(t,x) D_x v^i(t,x) \right) \right\} = 0; \\
v_i(T,x) = h_i(x).
\end{align*}
\]

(1.5)

We prove existence and uniqueness of a continuous solution in viscosity sense for (1.5). There are mainly two difficulties to handle in this study:

(i) The first one is related to the assumption that the switching costs \( (g_{ij}(t,x))_{i,j \in J} \)
only satisfy the non-free loop property (see (H3) below) and then it is not possible

(ii) The fact that the functions \( f_i \) may also depend both on \((y_i)_{i=1,...,m}\) and \(z^i\) makes
the problem not easy to deal with.

One motivation of our work is that the study of the risk-sensitive switching prob-
lem under Knightian uncertainty (see e.g. [17]), in the Markovian framework, turns
 into the study of a system of variational inequalities with interconnected obstacles of
type (1.5) as the functions \( (f_i)_{i=1,...,m} \), depend on both \( y_i \) and \( z_i \).

More precisely, our main contribution lies in the fact that the proof of existence
and uniqueness of a continuous viscosity solution of (1.5) is obtained under the two
following hypotheses on \( f_i \) and \( g_{ij}, i,j \in J \):

(i) the switching costs are only supposed to be non-negative and satisfy the non free
loop property (see (H3)-(ii) below);

(ii) for any \( i \in J \), \( f_i \) depends on \((y^1, \ldots, y^m, z^i)\) and satisfy either:
(a) \( \forall k \in J \setminus i \), the mappings \( y_k \rightarrow f_i(t, x, y^1, \ldots, y^{k-1}, y^k, y^{k+1}, \ldots, y^m, z) \)
are non-decreasing; or
(b) \( \forall k \in J \setminus i \), the mappings \( y_k \rightarrow f_i(t, x, y^1, \ldots, y^{k-1}, y^k, y^{k+1}, \ldots, y^m, z) \)
are non-increasing.

In order to study system (1.5), we shall rely on formula (1.4) to relate it to the
unique solution of the following system of reflected BSDEs with oblique reflection:
\( \forall i \in J, \forall s \leq T, \)

\[
\begin{align*}
Y^i_s & = h^i(X^i_T) + \int_s^T f_i(r, X^i_r, Y^1_r, \ldots, Y^m_r, Z^{i,1}_r, \ldots, Z^{i,m}_r, K^{i,1}_r, \ldots, K^{i,m}_r) \, dr + K^{i,1}_T - K^{i,1}_s \\
& - \int_s^T Z^{i,1}_r \, dB_r, \\
Y^i_s & \geq \max_{j \in J \setminus i} \{ Y^j_s - g_{ij}(s, X^j_s) \}; \\
\int_0^T (Y^{i,1}_s - \max_{j \in J \setminus i} \{ Y^{j,1}_s - g_{ij}(s, X^{j,1}_s) \}) \, dK^{i,1}_s = 0.
\end{align*}
\]

(1.6)

Note that there are several works on this system (1.6) (see e.g. [6, 11, 18, 19], etc.).
However, to match the weak assumptions we made on the data (i.e. \((f_i, h_i, g_{i,j})\)) we
need to update the existence and uniqueness results. This is the reason for which
system (1.6) is again considered and the main results are collected in the Appendix.
We mention that our results are out of the scope of those available in the existing literature [2, 11, 15] etc. and to our best knowledge they have not been stated yet. In particular the general case where the functions $f_i, i \in J$, are only Lipschitz in $(y^1, \ldots, y^m, z^l)$ is still open. In that latter case, we face two main difficulties. The first one is related to the obtention of the comparison of sub- and super-solutions of system (1.5) which plays an important role in our study. The second one is that, without the above assumptions on monotonicity, there is a lack of regularity of the functions $(v^i)_{i \in J}$, which we get from system (1.6) via Feynman-Kac’s formula. This is somehow difficult to fill in especially since we do not assume the existence of a positive and constant lower bound for the functions $(g_{ij})_{i,j \in J}$ as in [7] or [15].

This paper is organized as follows. In Sect. 2, we introduce the problem and make the assumptions we need, and therefore we give the precise definition of viscosity so- lution we use. Section 3 is devoted to the uniqueness result for the solution of system (1.5), when for any $i \in J$, the functions $f_i$ are non-decreasing w.r.t. to the coordinates $y_1, \ldots, y_i-1, y_i+1, \ldots, y_m$ i.e. when $(f_i)_{i \in J}$ satisfies (H2)(iv) (see Sect. 2). We prove it by establishing the classical comparison result. In Sect. 4, we deal with uniqueness in the case when $(-f_i)_{i \in J}$ satisfies (H2)(iv) as well as existence of solutions for both cases (i.e. when either $(f_i)_{i \in J}$ or $(-f_i)_{i \in J}$ satisfies (H2)(iv)). Note that, in the second case, our comparison result cannot be applied and the proof of uniqueness relies deeply on the existence/uniqueness result in the first case, the existence/uniqueness of a solution for system of reflected BSDEs (1.6) and on Feynman-Kac representation (1.4). In the Appendix, we collect and give sketches of the main results concerning existence and uniqueness for solutions for system of reflected BSDEs (1.6).

2 Preliminaries

Let $T$ (resp. $k$) be a fixed real (resp. integer) positive constant and recall that $J := \{1, \ldots, m\}$ ($J^{-i} = J - \{i\}$). Let us now consider the following functions: for $i, j \in J$,

\begin{align*}
\quad b : (t, x) \in [0, T] \times \mathbb{R}^k & \mapsto b(t, x) \in \mathbb{R}^k; \\
\quad \sigma : (t, x) \in [0, T] \times \mathbb{R}^k & \mapsto \sigma(t, x) \in \mathbb{R}^{k \times d}; \\
\quad f_i : (t, x, y^1, \ldots, y^m, z) \in [0, T] \times \mathbb{R}^{k+m+d} & \mapsto f_i(t, x, y^1, \ldots, y^m, z) \in \mathbb{R}; \\
\quad g_{ij} : (t, x) \in [0, T] \times \mathbb{R}^k & \mapsto g_{ij}(t, x) \in \mathbb{R} (i \neq j); \\
\quad h_i : x \in \mathbb{R}^k & \mapsto h_i(x) \in \mathbb{R}.
\end{align*}

Next let $\phi : (t, x) \in [0, T] \times \mathbb{R}^k \mapsto \phi(t, x) \in \mathbb{R}$ be a function. It is called of polynomial growth if there exist two non negative real constants $C$ and $\gamma$ such that:

$$
|\phi(t, x)| \leq C(1 + |x|^\gamma), \quad \forall (t, x) \in [0, T] \times \mathbb{R}^k.
$$

We denote by $\Pi^g$ the class of functions with polynomial growth and by $C^{1,2}([0, T] \times \mathbb{R}^k)$ (or simply $C^{1,2}$) the set of functions defined on $[0, T] \times \mathbb{R}^k$ with values in $\mathbb{R}$ which are $C^1$ in $t$ and $C^2$ in $x$. 

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Next let us introduce the following hypotheses:

(H1): The functions $b$ and $\sigma$ are jointly continuous, of linear growth in $(t,x)$ and Lipschitz continuous w.r.t. $x$, i.e., there exists a constant $C \geq 0$ such that for any $t \in [0,T]$ and $x, x' \in \mathbb{R}^k$

\[
|b(t,x)| + |\sigma(t,x)| \leq C(1 + |x|) \quad \text{and} \\
|\sigma(t,x) - \sigma(t,x')| + |b(t,x) - b(t,x')| \leq C|x - x'|.
\]

(2.1)

(H2): For any $i \in \mathcal{I}$,

(i) $(t,x) \mapsto f_i(t,x,y^1,\ldots,y^m,z)$ is continuous uniformly w.r.t. $(\vec{y},z) := (y^1,\ldots,y^m,z)$;

(ii) $f_i$ is uniformly Lipschitz continuous with respect to $(\vec{y},z)$, i.e., for some $C \geq 0$,

\[
|f_i(t,x,y^1,\ldots,y^m,z) - f_i(t,x,\bar{y}^1,\ldots,\bar{y}^m,\bar{z})| \leq C(|y^1 - \bar{y}^1| + \cdots + |y^m - \bar{y}^m| + |z - \bar{z}|), \quad \forall t, x, \vec{y}, z, \vec{\bar{y}}, \bar{z};
\]

(iii) the mapping $(t,x) \mapsto f_i(t,x,0,\ldots,0)$ is Borel and of polynomial growth.

(iv) Monotonicity: $\forall k \in \mathcal{I}^{-i}$, the mapping $y_k \in \mathbb{R} \mapsto f_i(t,x,y_1,\ldots,y_{k-1},y_k,y_{k+1},\ldots,y_m)$ is non-decreasing whenever the other components $(t,x,y_1,\ldots,y_{k-1},y_{k+1},\ldots,y_m)$ are fixed.

(H3): (i) For any $i, j \in \mathcal{I}$, $g_{ij}$ is jointly continuous in $(t,x)$, non-negative, i.e., $g_{ij}(t,x) \geq 0, \forall (t,x) \in [0,T] \times \mathbb{R}^k$ and are of polynomial growth;

(ii) The non-free loop property is satisfied, i.e., for any $(t,x) \in [0,T] \times \mathbb{R}^k$ and for any sequence of indices $i_1,\ldots,i_k$ such that $i_1 \neq i_2, i_1 = i_k$ and $\text{card}\{i_1,\ldots,i_k\} = k - 1$ we have:

\[
g_{i_1i_2}(t,x) + g_{i_2i_3}(t,x) + \cdots + g_{i_{k-1}i_k}(t,x) > 0, \quad \forall (t,x) \in [0,T] \times \mathbb{R}^k.
\]

As a convention we assume hereafter that $g_{ii} = 0$ for any $i \in \mathcal{I}$.

(H4): For each $i$, $h_i : x \mapsto h_i(x)$ is continuous of polynomial growth and satisfies

\[
\forall x \in \mathbb{R}, \quad h_i(x) \geq \max_{j \in \mathcal{I}^{-i}} (h_j(x) - g_{ij}(T,x)).
\]

Next let us introduce the following infinitesimal generator:

\[
\mathcal{L} \varphi(t,x) = \frac{1}{2} \text{Tr}[(\sigma\sigma^\top)(t,x)D^2_{xx}\varphi(t,x)] + b(t,x)^\top D_x \varphi(t,x)
\]

(2.2)

for a function $\varphi$ which belongs to $C^{1,2}([0,T] \times \mathbb{R}^k, \mathbb{R})$ ($\text{Tr}(\cdot)$ is the trace of a square matrix and $(\cdot)^\top$ stands for the transpose). It is associated with the diffusion process $X^{t,x}$ of (1.2).

In this paper we are concerned with the existence and uniqueness in viscosity sense of the solution $(v_1,\ldots,v_m) : (t,x) \in [0,T] \times \mathbb{R}^k \mapsto (v_1(t,x),\ldots,
\[ v_m(t,x) \in \mathbb{R}^m \] of the following system of \( m \) partial differential equations with interconnected obstacles: \( \forall i \in \mathcal{J}, \)

\[
\begin{align*}
\min \{ v_i(t,x) - \max_{j \in \mathcal{J}^{-i}} \left(-g_{ij}(t,x) + v_j(t,x)\right); \\
- \partial_t v_i(t,x) - \mathcal{L} v_i(t,x) \\
- f_i(t,x, v_1(t,x), \ldots, v_m(t,x), \sigma^\top(t,x), D_x v_i(t,x)) \} = 0;
\end{align*}
\]

(ii) For a function \( u \):

\[
\begin{align*}
&\text{(i) A function } u(t,x) \in [0, T] \times \mathbb{R}^k + \hookrightarrow u(t,x) \in \mathbb{R} \\
&\text{(ii) For a function } u : (t,x) \in [0, T] \times \mathbb{R}^k + \hookrightarrow u(t,x) \in \mathbb{R} \\
&\text{we now give the definition of a viscosity solution for the system (2.3) which will be done in terms of sub- and super-jets. So for any locally bounded function } u : (t,x) \in [0, T] \times \mathbb{R}^k + \hookrightarrow u(t,x) \in \mathbb{R}, \text{ we define its lower semicontinuous (lsc for short) envelope } u_* \text{ and upper semicontinuous (usc for short) envelope } u^* \text{ as follows:}
\end{align*}
\]

**Definition 1** (Subjects and superjets)

(i) For a function \( u : (t,x) \in [0, T] \times \mathbb{R}^k + \hookrightarrow \mathbb{R} \), lsc (resp. usc), the parabolic subj \( J^- u(t,x) \) (resp. the parabolic superjet \( J'^+ u(t,x) \)) of \( u \) at \( (t,x) \) in \( [0, T] \times \mathbb{R}^k \) consists of all triples \((p,q,M) \in \mathbb{R} \times \mathbb{R}^k \times \mathbb{S}^k \) satisfying:

\[
\begin{align*}
u(t',x') &\geq (\text{resp. } \leq) u(t,x) + p(t' - t) + \langle q, x' - x \rangle \\
&\quad + \frac{1}{2} \langle x' - x, M(x' - x) \rangle + o(|t' - t| + |x' - x|^2)
\end{align*}
\]

where \( \mathbb{S}^k \) is the set of symmetric real matrices of dimension \( k \).

(ii) For a function \( u : (t,x) \in [0, T] \times \mathbb{R}^k + \hookrightarrow \mathbb{R} \), lsc (resp. usc), the parabolic limiting subj \( J^- u(t,x) \) (resp. the limiting superjet \( J'^+ u(t,x) \)) of \( u \) at \( (t,x) \) in \( [0, T] \times \mathbb{R}^k \) consists of all triples \((p,q,M) \in \mathbb{R} \times \mathbb{R}^k \times \mathbb{S}^k \) for which there exist sequences \(((t_n,x_n))_{n \geq 0} \) and \(((p_n,q_n,M_n))_{n \geq 0} \) such that for any \( n \geq 0 \), \((p_n,q_n,M_n) \in J^- u(t_n,x_n) \) (resp. \( J'^+ u(t_n,x_n) \)), \((p_n,q_n,M_n) \rightarrow n \rightarrow \infty (p,q,M), (t_n,x_n) \rightarrow n \rightarrow \infty (t,x) \) and finally \( u(t_n,x_n) \rightarrow n \rightarrow \infty u(t,x) \).

We now give the definition of a viscosity solution for the system (2.3).

**Definition 2** Viscosity solution to (2.3)

(i) A function \((v_1, \ldots, v_m) : [0, T] \times \mathbb{R}^k + \hookrightarrow \mathbb{R}^m \) such that for any \( i \in \mathcal{J} \), \( v_i \) is lsc (resp. usc, is called a viscosity supersolution (resp. subsolution) to (2.3) if for any \( i \in \mathcal{J} \), for any \((t,x) \in [0, T] \times \mathbb{R}^k \) and any \((p,q,M) \in J^- v_i(t,x) \) (resp. \( J'^+ v_i(t,x) \)) we have:

\[
\begin{align*}
\min \{ v_i(t,x) - \max_{j \in \mathcal{J}^{-i}} \left(-g_{ij}(t,x) + v_j(t,x)\right); \\
- p - b(t,x)^\top q - \frac{1}{2} Tr \left[ (\sigma \sigma^\top)(t,x)M \right] \\
- f_i(t,x, v_1(t,x), \ldots, v_m(t,x), \sigma(t,x)^\top q) \} \geq 0 \text{ (resp. } \leq) h_i(x).
\end{align*}
\]
(ii) A locally bounded function \((v_1, \ldots, v_m) : [0, T] \times \mathbb{R}^k \to \mathbb{R}^m\) is called a viscosity solution to (2.3) if \((v_{1*}, \ldots, v_{m*})\) (resp. \((v_{1*}^*, \ldots, v_{m*}^*)\)) is a viscosity supersolution (resp. subsolution) of (2.3).

**Remark 2.1** In the previous definition, one can easily check that it is equivalent to require that (2.4) holds only for \((p, q, M)\) elements of subjets (resp. superjets) when assumptions (H2) on the functions \((f_i)_{i=1,m}\) are satisfied.

### 3 Uniqueness of the Solution of System (2.3)

In this section we deal with the issue of uniqueness of the solution of system (2.3) and to do so, we first establish an auxiliary result which is a classical one in viscosity solutions literature (see e.g. [1], p. 76).

**Lemma 3.1** Let (H1)–(H2) be fulfilled. Let \((v_i(t,x))_{i=1,m}\) be a supersolution of the system (2.3), then for any \(\gamma \geq 0\) there exists \(\lambda_0 > 0\) which does not depend on \(\theta\) such that for any \(\lambda \geq \lambda_0\) and \(\theta > 0\), the \(m\)-tuple \((v_i(t,x) + \theta e^{-\lambda t} |x|^{2\gamma+2})_{i=1,m}\) is a supersolution for (2.3).

**Proof** Without loss of generality we assume that the functions \(v_1, \ldots, v_m\) are lsc. By taking into account Remark 2.1 and for sake of convenience, we do not use the previous definition of a supersolution but an equivalent one (see e.g. [8]). Let \(i \in J\) be fixed and let \(\varphi \in C^{1,2}\) be such that the function \(\varphi - (v_i + \theta e^{-\lambda t} |x|^{2\gamma+2})\) has a local maximum in \((t,x)\) which is equal to 0. As \((v_i)_{i=1,m}\) is a supersolution for (2.3), then we have: \(\forall i \in J,\)

\[
\min \left\{ v_i(t,x) - \max_{j \in J^{-i}} (-g_{ij}(t,x) + v_j(t,x)); \right. \\
- \partial_t (\varphi(t,x) - \theta e^{-\lambda t} |x|^{2\gamma+2}) \\
- \frac{1}{2} \text{Tr} (\sigma.\sigma^\top (t,x) D^2_{xx} (\varphi(t,x) - \theta e^{-\lambda t} |x|^{2\gamma+2})) \\
- b(t,x)^\top D_x (\varphi(t,x) - \theta e^{-\lambda t} |x|^{2\gamma+2}) \\
- f_i (t,x, (v_1, \ldots, v_m)(t,x), \sigma^\top (t,x) D_x (\varphi(t,x) - \theta e^{-\lambda t} |x|^{2\gamma+2})) \right. \\
\left. \geq 0 \right. \\
\]

which implies that

\[
(v_i(t,x) + \theta e^{-\lambda t} |x|^{2\gamma+2}) - \max_{j \in J^{-i}} (-g_{ij}(t,x) + (v_j(t,x) + \theta e^{-\lambda t} |x|^{2\gamma+2})) \\
= v_i(t,x) - \max_{j \in J^{-i}} (-g_{ij}(t,x) + v_j(t,x)) \geq 0. \tag{3.1}
\]
On the other hand:
\[-\partial_i\left(\varphi(t, x) - \theta e^{-\lambda t} |x|^{2\gamma + 2}\right) - \frac{1}{2} \text{Tr}\left(\sigma(\sigma^\top(t, x) D_{xx}^2 \left(\varphi(t, x) - \theta e^{-\lambda t} |x|^{2\gamma + 2}\right)\right)\]
\[- D_x \left(\varphi(t, x) - \theta e^{-\lambda t} |x|^{2\gamma + 2}\right)^\top b(t, x)\]
\[- f_i(t, x, (v_1, \ldots, v_m)(t, x), \sigma^\top(t, x) D_x \left(\varphi(t, x) - \theta e^{-\lambda t} |x|^{2\gamma + 2}\right)\geq 0\]

and then
\[-\partial_i \varphi(t, x) - \frac{1}{2} \text{Tr}\left(\sigma(\sigma^\top(t, x) D_{xx}^2 \varphi(t, x)\right) - b(t, x)^\top D_x \varphi(t, x)\]
\[- f_i(t, x, (v_1(t, x) + \theta e^{-\lambda t} |x|^{2\gamma + 2})_{i=1, m}, \sigma^\top(t, x) D_x \varphi(t, x)\]
\[\geq \theta \lambda e^{-\lambda t} |x|^{2\gamma + 2} - \frac{1}{2} \theta e^{-\lambda t} \text{Tr}\left(\sigma(\sigma^\top(t, x) D_{xx}^2 |x|^{2\gamma + 2}\right)\]
\[- \theta e^{-\lambda t} b(t, x)^\top D_x \left(|x|^{2\gamma + 2}\right)\]
\[+ \left[ f_i(t, x, (v_1(t, x) + \theta e^{-\lambda t} |x|^{2\gamma + 2})_{i=1, m}, \sigma^\top(t, x) D_x \varphi(t, x)\right] \geq 0.\]

But the last term in the right-hand side of this latter inequality is equal to
\[\sum_{k=1, m} \theta e^{-\lambda t} C_{k}^{i, \lambda, \varphi} |x|^{2\gamma + 2} + \theta e^{-\lambda t} C_{i}^{\varphi, \lambda, \theta, \varphi} \sigma^\top(t, x) D_x \left(|x|^{2\gamma + 2}\right)\]

where $C_{k}^{i, \lambda, \varphi}$, $k = 1, \ldots, m$, and $C_{i}^{\varphi, \lambda, \theta, \varphi}$ are bounded by the Lipschitz constant of $f_i$ w.r.t. $y^k$, $k = 1, m$ and $z^i$, which is independent of $\theta$. Therefore, taking into account the growth conditions on $b$ and $\sigma$, there exists a constant $\lambda_0 \in \mathbb{R}^+$ which does not depend on $\theta$ such that if $\lambda \geq \lambda_0$, the right-hand side of (3.2) is non-negative. Henceforth, noting that $i$ is arbitrary in $\mathcal{J}$ together with (3.1), we obtain that $(v_1 + \theta e^{-\lambda t} |x|^{2\gamma + 2})_{i=1, m}$ is a viscosity supersolution for (2.3).

We now establish the comparison property between supersolutions and subsolutions of (2.3).

**Proposition 3.1** Assume that Assumptions (H1), (H2), (H3) and (H4) are fulfilled. Let $(u_1, \ldots, u_m)$ (resp. $(v^1, \ldots, v^m)$) be a subsolution (resp. supersolution) of the system of variational inequalities with inter-connected obstacles (2.3) such that for any $i = 1, \ldots, m, u_i$ and $v^i$ belong to $\Pi^\delta$. Then
\[\forall i = 1, \ldots, m, \quad u_i \leq v^i.\]

**Proof** It will be done in two steps.

**Step 1:** We assume that there exists a constant $\lambda$ small enough ($\lambda < -m. \max\{C_{k}^{j}, j = 1, \ldots, m\}$, $C_{k}^{j}$ being the Lipschitz constant of $f_j$ involved in (H2)-(ii)) such that: $\forall i \in \mathcal{J}, \forall t, x, y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_m, y, \tilde{y}, z,$ if $y \geq \tilde{y}$ then
\[f_i(t, x, y_1, \ldots, y_{i-1}, y, y_{i+1}, \ldots, y_m, z) - f_i(t, x, y_1, \ldots, y_{i-1}, \tilde{y}, y_{i+1}, \ldots, y_m, z) \leq \lambda(y - \tilde{y}).\]
Let \((u_1, \ldots, u_m)\) (resp. \((v^1, \ldots, v^m)\)) be a family of \(usc\) functions (resp. a family of \(lsc\) functions) subsolution (resp. supersolution) of \((2.3)\) and such that, for any \(i \in \mathcal{J}\), both \(u_i\) and \(v^i\) belong to \(\Pi^\delta\). By definition, there exist both \(\gamma > 0\) and \(C\) such that for any \(i \in \mathcal{J}\),

\[
|u_i(t, x)| + |v^i(t, x)| \leq C (1 + |x|^\gamma), \quad \forall (t, x) \in [0, T] \times \mathbb{R}^k.
\]

Next thanks to Lemma 3.1, there exist \(\nu\) large enough such that for any \(\theta > 0\),

\[
(v^i(t, x) + \theta e^{-\nu t} |x|^{2\gamma + 2})_i = 1, m\text{ is also a supersolution for (2.3). Therefore it is enough to show that for any } i \in \mathcal{J},
\]

\[
\forall (t, x) \in [0, T] \times \mathbb{R}^k, \quad u_i(t, x) \leq v^i(t, x) + \theta e^{-\nu t} |x|^{2\gamma + 2},
\]

and after taking the limit as \(\theta \to 0\) we obtain the desired result. So let us set

\[
\tilde{w}_{i, \theta, \nu}(t, x) = v^i(t, x) + \theta e^{-\nu t} |x|^{2\gamma + 2}, \quad (t, x) \in [0, T] \times \mathbb{R}^k
\]

which is denoted by \(\tilde{w}_i\) for simplicity.

To establish the comparison result, we proceed by contradiction assuming that \(\exists (\bar{t}, \bar{x}) \in [0, T] \times \mathbb{R}^k, \text{ s.t. } \max_{i \in J} (u_i(\bar{t}, \bar{x}) - \tilde{w}_i(\bar{t}, \bar{x})) > 0\).

Due to the growth condition on \(u_i\) and \(w_i\), there exists \(R > 0\) such that:

\[
\forall i \in \mathcal{J}, \forall (t, x) \in [0, T] \times \mathbb{R}^k \text{ s.t. } |x| \geq R, \quad u_i(t, x) - w_i(t, x) < 0.
\]

Taking into account the values of the subsolution and the supersolution at \(T\), it implies that

\[
0 < \max_{(t, x) \in [0, T] \times \mathbb{R}^k} \max_{i \in \mathcal{J}} (u_i(t, x) - w^i(t, x))
\]

\[
= \max_{(t, x) \in [0, T] \times B(0, R)} \max_{i \in \mathcal{J}} (u_i(t, x) - w^i(t, x))
\]

\[
= \max_{i \in \mathcal{J}} (u_i(t^*, x^*) - w^i(t^*, x^*)), \quad (3.4)
\]

where \((t^*, x^*) \in [0, T] \times B(0, R)\) and \(R > 0\) being now fixed.

We now define \(\tilde{\mathcal{J}}\) as follows:

\[
\tilde{\mathcal{J}} := \{ j \in \mathcal{J}, u_j(t^*, x^*) - w^j(t^*, x^*) = \max_{k \in \tilde{\mathcal{J}}} (u_k(t^*, x^*) - w^k(t^*, x^*)) \} . \quad (3.5)
\]

As in [20], let us show by contradiction that for some \(k \in \tilde{\mathcal{J}}\) we have:

\[
u_k(t^*, x^*) > \max_{j \in \tilde{\mathcal{J}} \setminus k} (u_j(t^*, x^*) - g_{kj}(t^*, x^*)). \quad (3.6)
\]

Suppose that for any \(k \in \tilde{\mathcal{J}}\) we have:

\[
u_k(t^*, x^*) \leq \max_{j \in \tilde{\mathcal{J}} \setminus k} (u_j(t^*, x^*) - g_{kj}(t^*, x^*)).
\]
then there exists \( l \in \mathcal{J}^{-k} \) such that

\[
    u_k(t^*, x^*) - u_l(t^*, x^*) \leq -g_{kl}(t^*, x^*).
\]

But \( w^k \) is a supersolution of (2.3), therefore we have

\[
    w^k(t^*, x^*) \geq w^l(t^*, x^*) - g_{kl}(t^*, x^*),
\]

and then

\[
    u_k(t^*, x^*) - u_l(t^*, x^*) \leq -g_{kl}(t^*, x^*) \leq w^k(t^*, x^*) - w^l(t^*, x^*).
\]

It implies that:

\[
    w^k(t^*, x^*) - u_k(t^*, x^*) \leq u_l(t^*, x^*) - u_l(t^*, x^*) \leq g_{kl}(t^*, x^*).
\]

which by definition of \( \tilde{\mathcal{J}} \) yields that the previous inequality is an equality. Therefore, \( l \) also belongs to \( \tilde{\mathcal{J}} \) and

\[
    u_k(t^*, x^*) - u_l(t^*, x^*) = -g_{kl}(t^*, x^*).
\]

Repeating this procedure as many times as necessary and since \( \tilde{\mathcal{J}} \) is finite we construct a loop of indices \( i_1, \ldots, i_p, i_{p+1} \) such that \( i_{p+1} = i_1 \) (\( i_1 \neq i_2 \)) and

\[
    \forall q \in \{1, \ldots, p\}, \quad u_{iq}(t^*, x^*) - u_{iq+1}(t^*, x^*) = -g_{iq, i_{q+1}}(t^*, x^*).
\]

Summing these equalities yields

\[
    g_{i_1, i_2}(t^*, x^*) + \cdots + g_{i_p, i_{p+1}}(t^*, x^*) = 0,
\]

which contradicts Assumption (H3) on \( (g_{ij})_{i, j \in \mathcal{J}} \), whence the desired result.

Let us now fix \( j \in \tilde{\mathcal{J}} \) that satisfies (3.6). For \( n \geq 1 \) let us define:

\[
    \Phi^j_n(t, x, y) := u_j(t, x) - w^j(t, y) - \varphi_n(t, x, y), \quad (t, x, y) \in [0, T] \times \mathbb{R}^{2k}, (3.7)
\]

where \( \varphi_n(t, x, y) := n|x - y|^{2\gamma + 2} + |x - x^*|^2 + |t - t^*|^2 \). Now let \( (t_n, x_n, y_n) \in [0, T] \times B'(0, R) \) be such that

\[
    \Phi^j_n(t_n, x_n, y_n) = \max_{(t, x, y) \in [0, T] \times B'(0, R)^2} \Phi^j_n(t, x, y),
\]

which exists since \( \Phi^j_n \) is usc (\( B'(0, R) \) is the closure of \( B(0, R) \)). Then we have:

\[
    \Phi^j_n(t^*, x^*, x^*) = u_j(t^*, x^*) - w^j(t^*, x^*)
    \leq u_j(t^*, x^*) - w^j(t^*, x^*) + \varphi_n(t_n, x_n, y_n)
    \leq u_j(t_n, x_n) - w^j(t_n, y_n). \quad (3.8)
\]
The definition of \( \varphi_n \) together with the growth condition of \( u_j \) and \( w^j \) implies that 
\( (\varphi_n(t_n,x_n,y_n)) \) is bounded and hence \( (x_n - y_n)_{n \geq 1} \) converges to 0. Next for any subsequence \( ((t_{n_l},x_{n_l},y_{n_l})) \) which converges to \((\tilde{t}, \tilde{x}, \tilde{x})\) we deduce from (3.8) that 
\[
 u_j(t^*,x^*) - w^j(t^*,x^*) \leq u_j(\tilde{t}, \tilde{x}) - w^j(\tilde{t}, \tilde{x}),
\]
since \( u_j \) is usc and \( w^j \) is lsc. As the maximum of \( u_j - w^j \) on \([0, T] \times B'(0, R)\) is reached in \((t^*, x^*)\) then this last inequality is an equality. Using both the definition of \( \varphi_n \) and (3.8), it implies that the sequence \( ((t_n,x_n,y_n))_n \) converges to \((t^*, x^*, x^*)\) from which we deduce:
\[
 n|x_n - y_n|^{\gamma + 2} \to 0 \quad \text{and} \quad \left( u_j(t_n,x_n), w^j(t_n,y_n) \right) \to_n \left( u_j(t^*,x^*), w^j(t^*,y^*) \right).
\]
Indeed this latter convergence holds since from (3.8) and the fact that \( u_j \) (resp. \( w^j \)) is usc (resp. lsc) we have:
\[
 u_j(t^*,x^*) - w^j(t^*,x^*) \leq \lim_n (u_j(t_n,x_n) - w^j(t_n,y_n)) \leq \lim_n u_j(t_n,x_n) - \lim_n w^j(t_n,y_n) \leq u_j(t^*,x^*) - w^j(t^*,x^*).
\]

Thus the sequence \( (u_j(t_n,x_n) - w^j(t_n,y_n))_{n \geq 0} \) converges to \( u_j(t^*,x^*) - w^j(t^*,x^*) \) as \( n \to \infty \). Then
\[
 \lim_n u_j(t_n,x_n) = u_j(t^*,x^*) - w^j(t^*,x^*) + \lim_n w^j(t_n,y_n) \geq u_j(t^*,x^*)
\]
\[
 \geq \lim_n u_j(t_n,x_n).
\]

It follows that the sequence \( (u_j(t_n,x_n))_n \) converges to \( u_j(t^*,x^*) \) and then \( (w^j(t_n,y_n))_n \) converges also to \( w^j(t^*,x^*) \).

Next as the functions \( (u_k)_{k \in \mathcal{K}} \) are usc and \( (g_{jk})_{j,k \in \mathcal{J}} \) are continuous, and since the index \( j \) satisfies (3.6), then there exists \( \rho > 0 \) such that for \((t,x) \in B((t^*,x^*), \rho)\) we have 
\[
 u_j(t,x) > \max_{k \in \mathcal{K}^{-j}} (u_k(t,x) - g_{jk}(t,x)).
\]
But by construction we can have \((t_n,x_n,u_j(t_n,x_n))_n \to_n (t^*,x^*,u_j(t^*,x^*))\) and once more since \( u_j \) is usc then for \( n \) large enough we have:
\[
 u_j(t_n,x_n) > \max_{k \in \mathcal{K}^{-j}} (u_k(t_n,x_n) - g_{jk}(t_n,x_n)). \quad (3.9)
\]

We can now apply Crandall-Ishii-Lions’s Lemma (see e.g. [8] or [14], p. 216) with 
\( \varphi_n^j, u_j, w^j \) and \( \varphi_n \) at \((t_n,x_n,y_n)\). This yields the existence of both \( (p^n u, q^n u, M^n u) \in \mathcal{J}^{2,+}(u_j)(t_n,x_n) \) and \( (p^n w, q^n w, M^n w) \in \mathcal{J}^{2,-}(w^j)(t_n,y_n) \) such that:
\[
 p^n u - p^n w = \partial_\nu \varphi_n(t_n,x_n,y_n) = 2(t_n - t^*), \quad q^n u_j = \partial_\nu \varphi_n(t_n,x_n,y_n),
\]
\[
 q^n w = -\partial_\nu \varphi_n(t_n,x_n,y_n) \quad \text{and} \quad \begin{pmatrix} M^n u \\ 0 \\ -N^n w \end{pmatrix} \leq A_n + \frac{1}{2n} A^2_n \quad \text{with} \ A_n = D^2_{(x,y)} \varphi_n(t_n,x_n,y_n). \quad (3.10)
\]
Next taking into account that \((u_i)_{i \in \mathcal{I}}\) (resp. \((w^i)_{i \in \mathcal{I}}\)) is a subsolution (resp. supersolution) of (2.3) and once more (3.9) we deduce:

\[
-p^u_n - b(t_n, x_n)^{\top} q^u_n - \frac{1}{2} \text{Tr}
\left[
(\sigma \sigma^{\top})(t_n, x_n) M^u_n
\right]
- f_j(t_n, x_n, (u_i(t_n, x_n))_{i \in \mathcal{I}}, \sigma(t_n, x_n)^{\top} q^u_n) \leq 0
\]

and

\[
-p^w_n - b(t_n, y_n)^{\top} q^w_n - \frac{1}{2} \text{Tr}
\left[
(\sigma \sigma^{\top})(t_n, y_n) M^w_n
\right]
- f_j(t_n, y_n, (w^i(t_n, y_n))_{i \in \mathcal{I}}, \sigma(t_n, y_n)^{\top} q^w_n) \geq 0.
\]

Making the difference between those two inequalities yields:

\[
-\left(p^u_n - p^w_n\right) - \left(b(t_n, x_n)^{\top} q^u_n - b(t_n, y_n)^{\top} q^w_n\right)
- \frac{1}{2} \text{Tr}
\left[
\left\{\sigma \sigma^{\top}(t_n, x_n) M^u_n - \sigma \sigma^{\top}(t_n, y_n) M^w_n\right\}
\right]
- \left\{f_j(t_n, x_n, (u_i(t_n, x_n))_{i \in \mathcal{I}}, \sigma(t_n, x_n)^{\top} q^u_n)\right\}
- \left\{f_j(t_n, y_n, (w^i(t_n, y_n))_{i \in \mathcal{I}}, \sigma(t_n, y_n)^{\top} q^w_n)\right\} \leq 0
\]

(3.11)

and then

\[
-\left\{f_j(t_n, x_n, (u_i(t_n, x_n))_{i \in \mathcal{I}}, \sigma(t_n, x_n)^{\top} q^u_n)\right\}
- \left\{f_j(t_n, y_n, (w^i(t_n, y_n))_{i \in \mathcal{I}}, \sigma(t_n, y_n)^{\top} q^w_n)\right\} \leq \Sigma_n,
\]

where, by using the Lipschitz assumption on \(b\) and \(\sigma\) and the convergence to zero of \((x_n - y_n)_n\), we deduce \(\lim_{n \to \infty} \Sigma_n \leq 0\). Next linearizing \(f_j\), which is Lipschitz w.r.t. \((y^i)_{i \in \mathcal{I}}\), and using hypothesis (3.3) we obtain:

\[
-\lambda\left(u_j(t_n, x_n) - w^j(t_n, y_n)\right) - \sum_{k \in \mathcal{I} - j} \Theta^{j,k}_n \left(u_k(t_n, x_n) - w^k(t_n, y_n)\right) \leq \Sigma_n,
\]

where \(\Theta^{j,k}_n\) is the increment rate of \(f_j\) w.r.t. \(y_k\) \((k \neq j)\). Thanks now to the monotonicity assumption [H2-(iv)], \(\Theta^{j,k}_n\) is nonnegative and bounded by \(C^j_f\) the Lipschitz constant of \(f_j\). Thus

\[
-\lambda\left(u_j(t_n, x_n) - w^j(t_n, y_n)\right) \leq \sum_{k \in \mathcal{I} - j} \Theta^{j,k}_n \left(u_k(t_n, x_n) - w^k(t_n, y_n)\right)^+ + \Sigma_n
\]

\[
\leq C^j_f \sum_{k \in \mathcal{I} - j} \left(u_k(t_n, x_n) - w^k(t_n, y_n)\right)^+ + \Sigma_n.
\]
Next taking the limit superior in both hand sides as \( n \to \infty \), using that \( u_k \) (resp. \( w^k \)) is usc (resp. lsc) and finally \( j \in \tilde{J} \) to obtain:

\[
-\lambda (u_j(t^*, x^*) - w^j(t^*, x^*)) \leq C_f^j \sum_{k \in J^{-j}} (u_k(t^*, x^*) - w^k(t^*, x^*))^+ \\
\leq (m - 1)C_f^j (u_j(t^*, x^*) - w^j(t^*, x^*))
\]

which is contradictory since \( u_j(t^*, x^*) - w^j(t^*, x^*) > 0 \) and \(-\lambda > mC_f^j\). Thus for any \( i \in J \), \( u_i \leq w^i \).

**Step 2**: The general case.

For arbitrary \( \lambda \in \mathbb{R} \), if \((u_j)_{j \in \mathcal{J}}\) (resp. \((v_j)_{j \in \mathcal{J}}\)) is a subsolution (resp. supersolution) of (2.3) then \( \tilde{u}_j(t, x) = e^{\lambda t}u_j(t, x) \) and \( \tilde{v}^j(t, x) = e^{\lambda t}v^j(t, x) \) is a subsolution (resp. supersolution) of the following system of variational inequalities with oblique reflection: \( \forall i \in \mathcal{J} \),

\[
\begin{cases}
\min \{ \tilde{v}_i(t, x) - \max_{j \in J^{-i}} (-e^{\lambda t}g_{ij}(t, x) + \tilde{v}_j(t, x)) \\
- \partial_t \tilde{v}_i(t, x) - L\tilde{v}_i(t, x) + \lambda \tilde{v}_i(t, x) \\
- e^{\lambda t} f_i(t, x, (e^{-\lambda t} \tilde{u}_i(t, x))_{i \in \mathcal{J}}, e^{-\lambda t} \sigma^\top(t, x).D_x \tilde{v}_i(t, x)) \} = 0 \\
\tilde{v}_i(T, x) = e^{\lambda T} h_i(x).
\end{cases}
\]

By choosing \( \lambda \) small enough we deduce that the function \( F_i \) defined as follows

\[
F_i(t, x, (u_k)_{k \in \mathcal{J}}, z) = -\lambda u_i + e^{\lambda t} f_i(t, x, (e^{-\lambda t} u_k)_{k \in \mathcal{J}}, e^{-\lambda t} z), \quad \forall i \in \mathcal{J},
\]

satisfies condition (3.3). Thanks to the result stated in Step 1, we obtain \( \tilde{u}_i \leq \tilde{v}^i, i \in \mathcal{J} \) from which it follows that \( u_i \leq v^i \), for any \( i \in \mathcal{J} \), which is the desired result. \( \square \)

As a by-product of the previous Proposition 3.1, we classically obtain:

**Corollary 1** Under (H1)–(H4), we have:

(i) The system of variational inequalities with inter-connected obstacles (2.3) has at most one solution in the class \( \Pi_8^g \);

(ii) If the solution in \( \Pi_8^g \) exists, it is necessarily continuous.

### 4 Existence and Uniqueness of a Solution for System (2.3)

#### 4.1 First Result

**Theorem 1** Assume that Assumptions (H1), (H2), (H3) and (H4) are fulfilled. Then the system of variational inequalities with inter-connected obstacles (2.3) has a unique continuous solution \((v^i)_{i \in \mathcal{J}}\) in the class \( \Pi_8^g \).
Proof. Let \((v^i)_{i \in J}\) be the functions constructed in Proposition 5.2 (Appendix, p. 186) and associated with the solution of the system of reflected BSDEs with interconnected obstacles associated with \(((f^i)_{i \in J}, (h^i)_{i \in J}, (g^i)_{i,j} \in J)\). Under (H1)–(H4), the solution of this system of RBSDEs exists and is unique. The functions \(v^i, i \in J\) belong to \(\Pi^g\) and are thus locally bounded. So we have to show that they are viscosity solutions of system (2.3). Finally, taking into account Corollary 1 and Proposition 3.1, we then have continuity and uniqueness. The proof will be given in two parts.

Part A: Supersolution property

First due to Proposition 5.2, for any \(i \in J\), \(v^i\) is lsc, i.e. \(v^i = v^i_*\). So we need to prove that the \(m\)-tuple \((v^i)_{i \in J}\) is a viscosity supersolution to (2.3). By construction and for any \(i \in J\) it holds

\[
v^i = \lim_{n \to \infty} v^{i,n},
\]

where \(v^{i,n}\), for \(n \geq 1\), is defined in (5.8) in Appendix. By El-Karoui et al.'s result ([12], Theorem 8.5), \(v^{i,n}\) is a viscosity solution of the following variational inequality or PDE with obstacle:

\[
\begin{align*}
\min \{ & v^{i,n}(t,x) - \max_{j \in J - \{i\}} [v^{j,n-1}(t,x) - g_{ij}(t,x)]; \partial_t v^{i,n} - \mathcal{L} v^{i,n}(t,x) \\
& - f^i(t,x, v^{1,n-1}, \ldots, v^{i-1,n-1}, v^{i,n}, v^{i+1,n-1}, \ldots, v^{m,n-1})(t,x), \sigma(t,x)^\top D_x v^{i,n}(t,x) \} = 0; \\
v^{i,n}(T,x) &= h_i(T,x).
\end{align*}
\]

Now let us fix \(i \in J\), let \((t,x) \in [0, T] \times \mathbb{R}^k\) and \((p,q,M) \in \bar{J}^* v^i(t,x)\). By (4.1) and Lemma 6.1 in [8], there exist sequences

\[
n_j \to \infty, \quad (p_j, q_j, M_j) \in \bar{J}^* v^{i,n_j}(t_j, x_j)
\]

such that:

\[
(t_j, x_j, v^{i,n_j}(t_j, x_j), p_j, q_j, M_j) \to j \to \infty (t, x, v^i(t, x), p, q, M).
\]

Next from the viscosity supersolution property for \(v^{i,n_j}\) we have:

\[
-p_j - b(t_j, x_j)^\top q_j - \frac{1}{2} \mathrm{Tr}(\sigma \sigma^\top(t_j, x_j) M_j) \\
- f^i(t_j, x_j, v^{1,n_j-1}, \ldots, v^{i-1,n_j-1}, v^{i,n_j}, v^{i+1,n_j-1}, \ldots, v^{m,n_j-1})(t_j, x_j), \\
\sigma(t_j, x_j)^\top q_j \geq 0
\]

which implies that:

\[
-p - b(t, x)^\top q - \frac{1}{2} \mathrm{Tr}(\sigma \sigma^\top(t,x) M) \\
\geq \lim_{j \to \infty} f^i(t_j, x_j, v^{1,n_j-1}, \ldots, v^{i-1,n_j-1}, v^{i,n_j}, v^{i+1,n_j-1}, \ldots, v^{m,n_j-1})(t_j, x_j), \\
\sigma(t_j, x_j)^\top q_j
\]

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\[
\lim_{l \to \infty} f_i(t_{j_l}, x_{j_l}, (v^{1,n_{j_l}-1}, \ldots, v^{i-1,n_{j_l}-1}, v^{i,n_{j_l}}, v^{i+1,n_{j_l}-1}, \ldots, v^{m,n_{j_l}-1}))(t_{j_l}, x_{j_l}), \sigma(t_{j_l}, x_{j_l})^\top q_{j_l}),
\]

(4.3)

for a subsequence \((j_l)_{l \geq 0}\). Using the uniform polynomial growth of \(v_{k,n}^i, k \in J\), there exists a subsequence of \((j_l)_{l \geq 0}\) which we still denote \((j_l)_{l \geq 0}\) such that for any \(k \in J\) the sequence \((v^{k,n_{j_l}-1}(t_{j_l}, x_{j_l}))_{l \geq 0}\) is convergent. Next, using that \(v_{k,n}^i\) is continuous, \(v_{k,n}^i \leq v_{k,n}^{i+1}\) and due to the \(lsc\) property of \(v_k^i\) then for any \((t, x) \in [0, T] \times \mathbb{R}^k\) and \(k \in J\) (see e.g. [1], p. 91),

\[
v_k^i(t, x) = v_{*k}(t, x) = \lim_{(t', x') \to (t, x), n \to \infty} v_{k,n}^i(t', x')
\]

which yields, for any \(k \in J\),

\[
\lim_{l \to \infty} v^{k,n_{j_l}-1}(t_{j_l}, x_{j_l}) \geq v_k^i(t, x).
\]

Going back to (4.3) and since \(f_i\) is continuous and for any \(k \in J\), the mapping \(y^k \mapsto f_i(t, x, y^1, \ldots, y^m, z)\) is non decreasing when all the other variables \((t, x, (y_j))\) are fixed from which we deduce

\[
-p - b(t, x)\top q - \frac{1}{2} Tr(\sigma \sigma^\top (t, x) M) \geq f_i(t, x, (v_k^i(t, x)))_{k=1,m}, \sigma(t, x)\top q).
\]

On the other hand we know that \(v_i^i(t, x) \geq \max_{j \in J-i}(v_j^i(t, x) - gij(T, x))\) and \(v_i^i(T, x) = hi(x)\) then \(v_i^i\) is a viscosity supersolution for the following PDE with obstacle:

\[
\begin{cases}
\min\{v_i^i(t, x) - \max_{j \in J-i}[v_j^i(t, x) - gij(t, x)];
- \partial_tv_i^i - L v_i^i(t, x) - f_i(t, x, (v_k^i(t, x)))_{k=1,m}, \sigma(t, x)\top D_x v_i^i(t, x))\} = 0, \\
v_i^i(T, x) = h_i(x).
\end{cases}
\]

Finally as \(i\) is arbitrary in \(J\) then the \(m\)-tuple \((v^1, \ldots, v^m)\) is a viscosity supersolution for the system of variational inequalities (2.3).

**Part B: Subsolution property**

We are going now to prove that \((v_i^*)_{i \in J}\) is a subsolution for (2.3). For sake of clarity, this will be done in two steps.

**Step 1:** Let us first prove that \(v_i^*(T, x) = h_i(x), \forall i \in J\).

For this and to begin with, we are going to show that: \(\forall i \in J\),

\[
\min\{v_i^*(T, x) - h_i(x); v_i^*(T, x) - \max_{j \in J-i}(v_j^*(T, x) - gij(T, x))\} = 0.
\]

Let \(i \in J\) be fixed. By definition we know that,

\[
v_i^*(T, x) = \lim_{(t', x') \to (T, x), t' < T} v_i^i(t', x') \geq \lim_{(t', x') \to (T, x), t' < T} v_{i,n}^i(t', x'), \quad \text{for any } n \geq 0
\]
then
\[ v^{i,*}(T, x) \geq v^{i,n}(T, x) = h_i(x) \]  
(4.4)
since \( v^{i,n} \) is continuous and at \( t = T \) it is equal to \( h_i(x) \). On the other hand we have,
\[ v^j(t, x) \geq \max\{v^j(t, x) - g_{ij}(t, x)\} \quad \forall (t, x), \]
then
\[ v^{i,*}(T, x) \geq \max_{j \in J^{-i}} (v^{j,*}(T, x) - g_{ij}(T, x)), \]  
(4.5)
which with (4.4) imply that:
\[ \min\{v^{i,*}(T, x) - h_i(x); v^{i,*}(T, x) - \max_{j \in J^{-i}} (v^{j,*}(T, x) - g_{ij}(T, x))\} \geq 0. \]  
(4.6)

Let us now show that the left-hand side of (4.6) cannot be positive. We follow here the same idea as in [2]. Let us suppose that for some \( x_0 \), there is \( \varepsilon > 0 \) such that:
\[ \min\{v^{i,*}(T, x_0) - h_i(x_0); v^{i,*}(T, x_0) - \max_{j \in J^{-i}} (v^{j,*}(T, x_0) - g_{ij}(T, x_0))\} = 2\varepsilon, \]  
(4.7)
and let us construct a contradiction. Let \((t_k, x_k)_{k \geq 1}\) be a sequence in \([0, T] \times \mathbb{R}^k\) such that:
\[ (t_k, x_k) \rightarrow (T, x_0) \quad \text{and} \quad v^j(t_k, x_k) \rightarrow v^{i,*}(T, x_0) \quad \text{as} \quad k \rightarrow \infty. \]
Since \( v^{i,*} \) is \( usc \) and of polynomial growth and taking into account (4.1), we can find a sequence \((\varrho^n)_{n \geq 0}\) of functions of \( C^{1,2}([0, T] \times \mathbb{R}^k) \) such that \( \varrho^n \rightarrow v^{i,*} \) and, on some neighborhood \( B_n \) of \((T, x_0)\) we have:
\[ \min\{\varrho^n(t, x) - h_i(x); \varrho^n(t, x) - \max_{j \in J^{-i}} (v^{j,*}(t, x) - g_{ij}(t, x))\} \geq \varepsilon, \quad \forall (t, x) \in B_n. \]  
(4.7)
After possibly passing to a subsequence of \((t_k, x_k)_{k \geq 1}\) we can assume that (4.7) holds on \( B^n_k := [t_k, T] \times B(x_k, \delta^n_k) \) for some \( \delta^n_k \in (0, 1) \) small enough in such a way that \( B^n_k \subset B_n \). Now since \( v^{i,*} \) is locally bounded then there exists \( \zeta > 0 \) such that \( |v^{i,*}| \leq \zeta \) on \( B_n \). We can then assume that \( \varrho^n \geq -2\zeta \) on \( B_n \). Next let us define:
\[ \tilde{\varrho}^n_k(t, x) := \varrho^n(t, x) + \frac{4\zeta|x - x_k|^2}{(\delta^n_k)^2} + \sqrt{T - t}. \]
Note that \( \tilde{\varrho}^n_k \geq \varrho^n \) and
\[ \left( v^{i,*} - \tilde{\varrho}^n_k \right)(t, x) \leq -\zeta \quad \text{for} \quad (t, x) \in [t_k, T] \times \partial B(x_k, \delta^n_k). \]  
(4.8)
As \( \partial_t(\sqrt{T - t}) \rightarrow -\infty \) as \( t \rightarrow T \), we can choose \( t_k \) large enough in front of \( \delta^n_k \) and the derivatives of \( \varrho^n \) to ensure that
\[ -\mathcal{L} \tilde{\varrho}^n_k(t, x) \geq 0 \quad \text{on} \quad B^n_k. \]  
(4.9)
Next let us consider the following stopping time \( \theta_n^k := \inf\{ s \geq t_k, (s, X_s^{t_k, x_k}) \in B_n^{k^c} \} \land T \) where \( B_n^{k^c} \) is the complement of \( B_n^k \), and \( \theta_k := \inf\{ s \geq t_k, v^i(s, X_s^{t_k, x_k}) = \max_{j \in \mathcal{J}^{-i}} (u^j(s, X_s^{t_k, x_k}) - g_{ij}(s, X_s^{t_k, x_k})) \} \land T \). Applying now Itô's formula to the process \((\tilde{\theta}_n^k(s, X_s))_s\) stopped at time \( \theta_k^n \land \theta_k \) and taking into account (4.7), (4.8) and (4.9) to obtain:

\[
\tilde{\theta}_n^k(t_k, x_k) = \mathbb{E}\left[\tilde{\theta}_n^k(\theta_k^n \land \theta_k, X_{\theta_k^n \land \theta_k}^{t_k, x_k}) - \int_{t_k}^{\theta_k^n \land \theta_k} \mathcal{L}\tilde{\theta}_n^k(r, X_r^{t_k, x_k}) \, dr\right] 
\geq \mathbb{E}\left[\tilde{\theta}_n^k(\theta_k^n, X_{\theta_k^n}^{t_k, x_k}) \mathbb{I}_{[\theta_k^n \leq \theta_k]} + \tilde{\theta}_n^k(\theta_k^n, X_{\theta_k^n}^{t_k, x_k}) \mathbb{I}_{[\theta_k^n > \theta_k]}\right] 
= \mathbb{E}\left[\tilde{\theta}_n^k(\theta_k^n, X_{\theta_k^n}^{t_k, x_k}) \mathbb{I}_{[\theta_k^n \leq \theta_k]} + \tilde{\theta}_n^k(T, X_T^{t_k, x_k}) \mathbb{I}_{[\theta_k^n = T]} \right] 
\geq \mathbb{E}\left[\tilde{\theta}_n^k(\theta_k^n, X_{\theta_k^n}^{t_k, x_k}) \mathbb{I}_{[\theta_k^n \leq \theta_k]} + \tilde{\theta}_n^k(T, X_T^{t_k, x_k}) \mathbb{I}_{[\theta_k^n = T]} \right] 
\geq \mathbb{E}\left[\{v^i(\theta_k^n, X_{\theta_k^n}^{t_k, x_k}) + \zeta\} \mathbb{I}_{[\theta_k^n \leq \theta_k]} + \left(\varepsilon + h_i(T, X_T^{t_k, x_k})\right) \mathbb{I}_{[\theta_k^n = T]} \right] 
\geq \mathbb{E}\left[\{v^i(\theta_k^n, X_{\theta_k^n}^{t_k, x_k}) + \zeta\} \mathbb{I}_{[\theta_k^n \leq \theta_k]} + \left(\varepsilon + h_i(T, X_T^{t_k, x_k})\right) \mathbb{I}_{[\theta_k^n = T]} \right] 
\geq \mathbb{E}\left[\{v^i(\theta_k^n, X_{\theta_k^n}^{t_k, x_k}) + \zeta\} \mathbb{I}_{[\theta_k^n \leq \theta_k]} + \left(\varepsilon + h_i(T, X_T^{t_k, x_k})\right) \mathbb{I}_{[\theta_k^n = T]} \right] 
= \mathbb{E}\left[v^i(t_k, x_k) - \int_{t_k}^{\theta_k^n \land \theta_k} f_i(s, X_s^{t_k, x_k}, (v^\ell(s, X_s^{t_k, x_k}))_{\ell=1,m}, Z_s^{t_k, x_k}) \, ds\right] + \zeta \land \varepsilon.
\]

To get the last inequality, we have used that the process \( Y^i = v^i(\cdot, X) \) stopped at time \( \theta_k^n \land \theta_k \) solves an explicit RBSDE system with triple of data given by \(((f_i), (h_i), (g_{ij}))_{i,j \in \mathcal{J}}\). In addition, by definition of \( \theta_k^n \land \theta_k \), we have \( dK^{i,t,x} = 0 \) on \([t_k, \theta_k]\). Next, and referring to Proposition 5.2 and Remark 2 in Appendix, we have that both \((v^\ell)_{\ell=1,m}\) and \(\|Z^i\|_{\mathcal{H}^2, d}(t, x)\) belong to \(\mathcal{P}^8\). Taking into account (5.3) and Assumption (H2)-(iii), we deduce that

\[
\lim_{k \to \infty} \mathbb{E}\left[\int_{t_k}^{\theta_k^n \land \theta_k} f_i(s, X_s^{t_k, x_k}, (v^\ell(s, X_s^{t_k, x_k}))_{\ell=1,m}, Z_s^{t_k, x_k}) \, ds\right] = 0.
\]

Therefore taking the limit in the previous inequalities yields:

\[
\lim_{k \to \infty} \tilde{\theta}_n^k(t_k, x_k) = \lim_{k \to \infty} \theta_n^k(t_k, x_k) + \sqrt{T - t_k} = \theta_n^k(T, x_0) 
\geq \lim_{k \to \infty} v^i(t_k, x_k) + \zeta \land \varepsilon = v^{*i}(T, x_0) + \zeta \land \varepsilon.
\]
But this is a contradiction since \( \varrho_n \rightarrow v_i^* \) pointwisely as \( n \rightarrow \infty \). Thus for any \( x \in \mathbb{R}^k \) we have:

\[
\forall i \in \mathcal{J}, \quad \min\left\{ v_i^*(T,x) - h_i(x); v_i^*(T,x) - \max_{j \in \mathcal{J}^{-i}} \left( v_j^*(T,x) - g_{ij}(T,x) \right) \right\} = 0.
\]

Let us now show that \( v_i^*(T,x) = h_i(x) \), \( \forall i \in \mathcal{J} \). So suppose that for some \( i \), \( v_i^*(T,x) > h_i(x) \), then from the previous equality there exists \( j \in \mathcal{J}^{-i} \) such that:

\[
v_i^*(T,x) = v_j^*(T,x) - g_{ij}(T,x).
\]

(4.10)

But once more we have \( v_j^*(T,x) > h_j(T,x) \). Otherwise, i.e. if \( v_j^*(T,x) = h_j(x) \), we would have from (4.10):

\[
h_i(x) < v_i^*(T,x) = v_j^*(T,x) - g_{ij}(T,x) = h_j(x) - g_{ij}(T,x),
\]

which is contradictory with (H4). Therefore there exists \( \ell \in \mathcal{J}^{-j} \) such that:

\[
v_j^*(T,x) = v_\ell^*(T,x) - g_{j\ell}(T,x) \quad \text{and then} \quad v_i^*(T,x) = v_\ell^*(T,x) - g_{j\ell}(T,x) - g_{ij}(T,x).
\]

Repeating this reasoning as many times as necessary we obtain a sequence of different indices \( i_1, \ldots, i_l \) such that

\[
v_{i_1}^*(T,x) = v_{i_1}^*(T,x) - \left( g_{i_1i_2}(T,x) + \cdots + g_{i_{l-1}i_l}(T,x) + g_{i_1i_l}(T,x) \right),
\]

which is contradictory with (H3)-(ii). Thus for any \( i \in \mathcal{J} \) we have:

\[
\forall x \in \mathbb{R}^k, \quad v_i^*(T,x) = h_i(x).
\]

(4.11)

**Step 2:** Let us now show that \( (v_i^*)_i \in \mathcal{J} \) is a subsolution to system (2.3) at any point \( (t,x) \in [0,T[ \times \mathbb{R}^k \).

First note that since \( v_i^{*,n} \nrightarrow v_i^* \) and \( v_i^{*,n} \) is continuous then we have (see e.g. [1], p. 91)

\[
v_i^*(t,x) = \lim_{n \rightarrow \infty} \sup v_i^{*,n}(t,x) = \lim_{n \rightarrow \infty, t' \rightarrow t, x' \rightarrow x} v_i^{*,n}(t',x').
\]

(4.12)

Besides for any \( i \in \mathcal{J} \) and \( n \geq 0 \) we deduce from the construction of \( v_i^{*,n} \) in (5.8) that:

\[
v_i^{*,n}(t,x) \geq \max_{\ell \in \mathcal{J}^{-i}} \left( v_{\ell}^{*,n}(t,x) - g_{i\ell}(t,x) \right),
\]

which implies in taking the limit:

\[
\forall i \in \mathcal{J}, \forall x \in \mathbb{R}^k, \quad v_i^*(t,x) \geq \max_{\ell \in \mathcal{J}^{-i}} \left( v_{\ell}^*(t,x) - g_{i\ell}(t,x) \right).
\]

(4.13)

Let us now fix \( i \in \mathcal{J} \) and let \( (t,x) \in [0,T[ \times \mathbb{R}^k \) be such that:

\[
v_i^*(t,x) - \max_{\ell \in \mathcal{J}^{-i}} \left( v_{\ell}^*(t,x) - g_{i\ell}(t,x) \right) > 0.
\]

(4.14)
Let \((p, q, M) \in \bar{J}^+ v^i(t, x)\). By (4.12) and Lemma 6.1 in [8], there exist sequences
\[ n_j \to \infty, \quad (p_j, q_j, M_j) \in J^+ v^{i,n_j}(t, x) \]
such that:
\[
\lim_{j \to \infty} (t_j, x_j, v^{i,n_j}(t_j, x_j), p_j, q_j, M_j) = (t, x, v^i(t, x), p, q, M).
\]
From the viscosity subsolution property for \(v^{i,n_j}\) at \((t_j, x_j)\) (see (4.2)), for any \(j \geq 0\), we have:
\[
\min \left\{ v^{i,n_j}(t_j, x_j) - \max_{\ell \in J} (v^{\ell,n_j}(t_j, x_j) - g_{i\ell}(t_j, x_j)), -p_j - b(t_j, x_j)^\top q_j - \frac{1}{2} \text{Tr}(\sigma\sigma^\top(t_j, x_j)M_j) - f_i(t_j, x_j, (v^{1,n_j-1}, \ldots, v^{i-1,n_j-1}, v^{i,n_j}, v^{i+1,n_j-1}, \ldots, v^{m,n_j-1})(t_j, x_j), \sigma(t_j, x_j)^\top q_j) \right\} \leq 0. \tag{4.15}
\]
Next the definition of \(v^{\ell^*}\) implies that
\[
v^{\ell^*}(t, x) \geq \limsup_{j \to \infty} v^{\ell,n_j}(t_j, x_j),
\]
therefore by (4.14), there exists \(j_0 \geq 0\), such that if \(j \geq j_0\) we have
\[
v^{i,n_j}(t_j, x_j) > \max_{\ell \in J} (v^{\ell,n_j}(t_j, x_j) - g_{i\ell}(t_j, x_j)).
\]
Then (4.15) implies that, for any \(j \geq j_0\),
\[
-p_j - b(t_j, x_j)^\top q_j - \frac{1}{2} \text{Tr}(\sigma\sigma^\top(t_j, x_j)M_j) - f_i(t_j, x_j, (v^{1,n_j-1}, \ldots, v^{i-1,n_j-1}, v^{i,n_j}, v^{i+1,n_j-1}, \ldots, v^{m,n_j-1})(t_j, x_j), \sigma(t_j, x_j)^\top q_j) \leq 0
\]
which implies that
\[
-p - b(t, x)^\top q - \frac{1}{2} \text{Tr}(\sigma\sigma^\top(t, x)M) \leq \lim_{j \to \infty} f_i(t_j, x_j, (v^{1,n_j-1}, \ldots, v^{i-1,n_j-1}, v^{i,n_j}, v^{i+1,n_j-1}, \ldots, v^{m,n_j-1})(t_j, x_j), \sigma(t_j, x_j)^\top q_j).
\]
We can now mimic the proof to show the supersolution property (see (4.3)) by working with an appropriate subsequence, to deduce that
\[
-p - b(t, x)^\top q - \frac{1}{2} \text{Tr}(\sigma\sigma^\top(t, x)M) - f_i(t, x, (v^{k^*}(t, x))_{k=1,m}, \sigma(t, x)^\top q) \leq 0.
\]
Then
\[
\min \left\{ v^i_0(t,x) - \max_{\ell \in J^{-i}} (v^\ell_0(t,x) - g_\ell(t,x)); \right. \\
- p - b(t,x)^\top q - \frac{1}{2} \text{Tr}(\sigma(t,x)M) - f_i(t,x, (v^{k*}_0(t,x))_{k=1,m}, \\
\sigma(t,x)^\top q) \right\} \leq 0
\]
which, in combination with (4.11), (4.13) and since \( i \) is arbitrary, means that \((v^i)_{i \in J}\) is a viscosity subsolution for (2.3). Thus the \( m \)-tuple \((v^i)_{i \in J}\) is a solution for (2.3) and Corollary 1 implies that it is continuous and unique. \( \square \)

As a by-product of Theorem 1 we obtain:

**Corollary 2** Under (H1)–(H4), there exist deterministic continuous functions \((v^i(t,x))_{i \in J}\) which belong to \(\Pi^g\) unique solution in viscosity sense of (2.3). Moreover the unique solution of the system of reflected BSDEs with inter-connected obstacles associated with \(((f_i)_{i \in J}, (h_i)_{i \in J}, (g_{ij})_{i,j \in J})\) (see Proposition 5.1 in Appendix) has the following representation:

\[
\forall i \in J, \forall (t,x) \in [0, T] \times \mathbb{R}^k, \forall s \in [t, T], \quad Y^{i;t,x}_s = v^i(s, X^{i;t,x}_s). \quad (4.16)
\]

4.2 Second Existence and Uniqueness Result: \((-f_i)_{i \in J}\) satisfy (H2)

In this section we are going to show that system (2.3) has a unique continuous solution when \((-f_i)_{i \in J}\) verify (H2) as well as Assumptions (H1), (H3) and (H4) satisfied. The main difference between this case and the previous one is located at the level of hypothesis (H2)-(iv) where we now require that for any \( i \in J \), for any \( k \in J^{-i} \), the mapping \( y^k \mapsto f_i(t,x,y^1,\ldots,y^{k-1},y^k,y^{k+1},\ldots,y^m,z) \) all variables \((t,x,(y_j)_{j \neq k})\) are fixed.

**Theorem 2** Assume that Assumptions (H1), (H3) and (H4) are fulfilled and the functions \((-f_i)_{i \in J}\) verify (H2). Then the system of variational inequalities with inter-connected obstacles (2.3) has a unique solution \((v^1,\ldots,v^m)\) in the class of continuous function belonging to \(\Pi^g\).

**Proof** (i) **Existence**: The candidate solution will be related here to the unique solution of the multidimensional reflected BSDE (5.4) (see Appendix).

**Step 1**: Construction of the candidate

Let \( \lambda \in \mathbb{R} \) and for \( i \in J \), let \( F_i \) be the function defined by:

\[
F_i(t,x,y^1,\ldots,y^m,z) = e^{\lambda t} f_i(t,x,e^{-\lambda t}y^1,\ldots,e^{-\lambda t}y^m,e^{-\lambda t}z_i) - \lambda y_i. \quad (4.17)
\]
Since \( f_i \) is uniformly Lipschitz w.r.t. \( y_i \) then \( F_i \) is so and for \( \lambda \) large enough, we obtain that for any \( k \in \mathcal{J} \), the mapping

\[
y_k \mapsto F_i(t, x, y_1, \ldots, y_{k-1}, y_k, y_{k+1}, \ldots, y_m, z)
\]

is non-increasing when \((t, x, y_1, \ldots, y_{k-1}, y_k, y_{k+1}, \ldots, y_m, z)\) are fixed, since \(-f_i\) satisfies (H2)-(iv). Next let us consider the following iterative Picard scheme: for any \( i \in \mathcal{J} \), \( Y^{i,0} = 0 \) and for \( n \geq 1 \), we define \((Y^{i,n})_{i \in \mathcal{J}}\) by:

\[
(Y^{1,n}, \ldots, Y^{m,n}) = \Phi((Y^{1,n-1}, \ldots, Y^{m,n-1}))
\]

where \( \Phi \) is defined in (5.15) in Appendix. Using Proposition 5.4-(ii.b), the sequence \((Y^{i,n})_{i \in \mathcal{J}}\) converges in \( \mathcal{H}^{2,m} \) to the unique solution \((Y^i)_{i \in \mathcal{J}}\) of the system of reflected BSDEs with oblique reflection associated with \(((F_i(s, X^{t,x}_s, y_1, \ldots, y_m, z))_{i \in \mathcal{J}}, (e^{\lambda T} h_i(X^{t,x}_T))_{i \in \mathcal{J}}, (e^{\lambda T} g_{ij}(s, X^{t,x}_s))_{i,j \in \mathcal{J}})\).

Next by using an induction argument on \( n \) and Corollary 2 of Sect. 4.1, we deduce the existence of deterministic continuous with polynomial growth functions \((v^{i,n})_{i \in \mathcal{J}}\) such that:

\[
\forall n \geq 0, \forall i \in \mathcal{J}, \forall (t, x) \in [0, T] \times \mathbb{R}^k, \forall s \in [t, T], \quad Y^{i,n}_{s,x} = v^{i,n}(s, X^{t,x}_s).
\]

But thanks to Proposition 5.4-(a) we have: for any \( i, n, m \) and \( s \leq T \),

\[
\begin{align*}
\mathbb{E}[|Y^{i,n}_{s,x} - Y^{i,m}_{s,x}|^2] & \leq C \|Y^{i,n-1}_{i \in \mathcal{J}} - Y^{i,m-1}_{i \in \mathcal{J}}\|_{\mathcal{H}^{2,m}}^2 \\
\mathbb{E}[|Y^{i}_{s,x} - Y^{i,m}_{s,x}|^2] & \leq C \|Y^{i}_{i \in \mathcal{J}} - Y^{i,m}_{i \in \mathcal{J}}\|_{\mathcal{H}^{2,m}}^2.
\end{align*}
\]

As the sequence \(((Y^{i,n}_{i \in \mathcal{J}}))_{n \geq 0}\) is convergent in \( \mathcal{H}^{2,m} \) then it is of Cauchy type which implies that \((v^{i,n})_{n \geq 0}\) is so and then converges pointwisely to a deterministic function \(v^i\), for any \( i \in \mathcal{J} \). Thus going back to (4.18) we deduce that:

\[
\forall i \in \mathcal{J}, \forall s \in [t, T], \mathbb{P}\text{-a.s.,} \quad Y^i_s = v^i(s, X^{t,x}_s).
\]

Let us now show that \(v^i, i \in \mathcal{J}\), belongs to \( \Pi^g \). Since \( \Phi \) is a contraction in \( (\mathcal{H}^{2,m}, \|\cdot\|_{\beta_0})\) (see Proposition 5.4 in Appendix) then we deduce:

\[
\forall n, q \geq 0, \quad \|Y^{i,n+q}_{i \in \mathcal{J}} - Y^{i,n}_{i \in \mathcal{J}}\|_{\beta_0} \leq \frac{C^n \Phi}{1 - C\Phi} \|Y^{i,1}_{i \in \mathcal{J}}\|_{\beta_0};
\]

with \( C\Phi \in ]0,1[ \) is the constant of contraction of the mapping \( \Phi \) which is also independent of \((t, x)\). As the norms \( \|\cdot\| \) and \( \|\cdot\|_{\beta_0} \) are equivalent then there exists a constant \( C_1 \) such that:

\[
\forall n, q \geq 0, \quad \|Y^{i,n+q}_{i \in \mathcal{J}} - Y^{i,n}_{i \in \mathcal{J}}\| \leq C_1 C^n \Phi \|Y^{i,1}_{i \in \mathcal{J}}\|_{\beta_0}.
\]
Taking now the limit as \( q \) goes to \( +\infty \) and in view of (4.18) and (4.19), if we take \( s = t \) we deduce that:

\[
\forall (t, x) \in [0, T] \times \mathbb{R}^k, \quad \left| v^i(t, x) - v^{i,n}(t, x) \right| \leq C_2 \left\| \left( Y^{i,1} \right)_{i \in J} \right\|.
\]

Finally one can check easily that \( \| (Y^{i,1})_{i \in J} \| (t, x) \) is of polynomial growth (since \( \mathbb{E}[\sup_{s \leq T} |X^{i,x}_s|^\gamma] \) belongs to \( \Pi^k \) for any \( \gamma \geq 0 \) (see (5.3))) and since \( v^{i,n} \) is so, then \( v^i \) also belongs to \( \Pi^k \) for any \( i \in J \).

**Step 2**: Continuity of the functions \( v^i, i \in J \).

We again rely on the convergence result of any sequence \( ((Y^{i,n})_{i \in J} = \Phi((Y^{i,n-1})_{i \in J}))_{n \geq 1} \) constructed via the Picard scheme (see (5.15)). So let us initialize the scheme as follows:

\[
\forall i \in J, \forall s \leq T, \quad Y^{i,0}_s = C \left( 1 + |X^{i,x}_s|^p \right), \quad (4.20)
\]

where the constants \( C \) and \( p \) are related to the growth of \( (v^i)_{i \in J} \), i.e.,

\[
\forall i \in J, \quad \left| v^i(t, x) \right| \leq C \left( 1 + |x|^p \right), \quad \forall (t, x) \in [0, T] \times \mathbb{R}^k.
\]

Next let \( (Y^{i,t,x})_{i \in \{1, \ldots, m\}} \) be the unique solution of the multidimensional RB-SDE associated with \( ((F_i(s, X^{i,x}_s, y^1_s, \ldots, y^m_s, z))_{i \in J}, (e^{\lambda T} h_i(X^{i,x}_T))_{i \in J}, (e^{\lambda t} g_{ij}(s, X^{i,x}_s))_{i,j \in J}) \) then we have \( Y^{i,t,x}_s = v^i(s, X^{i,x}_s) \) for any \( s \in [t, T], i \in J \), and where \( (v^i)_{i \in J} \) are the ones of Step 1. By definition of \( Y^{i,0} \) in (4.20), we have

\[
\mathbb{P}\text{-a.s., } \forall i \in J, \forall s \in [t, T], \quad Y^{i,t,x}_s \leq Y^{i,0}_s.
\]

Let us now prove by induction that for any \( n \geq 0 \) we have:

\[
\forall n \in \mathbb{N}, \forall i \in J, \forall s \in [t, T], \quad Y^{i,2n+1}_s \leq Y^{i}_s \leq Y^{i,2n}_s. \quad (4.21)
\]

For \( n = 0 \), the inequality of the right-hand side is already true. Let us consider the left-hand side. Recalling that by construction we have \( (Y^{i,1})_{i \in J} = \Phi((Y^{i,0})_{i \in J}) \) and \( (Y^i)_{i \in J} = \Phi((Y^i)_{i \in J}) \), we get the following comparison result for the two drivers of the RBSDE satisfied by \( Y^i \) and \( Y^{i,1} \):

\[
F_i(s, X^{i,x}_s, Y^{1}_s, \ldots, Y^{m}_s, z) \geq F_i(s, X^{i,x}_s, Y^{1,0}_s, \ldots, Y^{m,0}_s, z),
\]

since for any \( j \in J \) the mapping

\[
y_j \mapsto F_i(t, x, y_1, \ldots, y_{j-1}, y_j, y_{j+1}, \ldots, y_m, z)
\]

is non-increasing when \( (t, x, y_1, \ldots, y_{j-1}, y_{j+1}, \ldots, y_m, z) \) are fixed. Next using the comparison results of Remark 1-(i), (ii) we deduce that \( Y^{i,1} \leq Y^i \) for any \( i \in J \). Thus the property (4.21) is valid for \( n = 0 \). Now if it is satisfied for some \( n \) and repeating the same argument, it also holds for \( n + 1 \), whence the claim.
Next relying once more on Corollary 2 and an induction argument then, for any $i \in \mathcal{J}$ and $n \geq 0$, there exists a deterministic continuous function $\tilde{v}^{i,n}$ of $\Pi^g$ such that:

$$\forall (t, x) \in [0, T] \times \mathbb{R}^k, \forall s \in [t, T], \quad Y^{i,n}_s = \tilde{v}^{i,n}(s, X^{i,x}_s).$$

As in Step 1, for any $i \in \mathcal{J}$, the sequence $(\tilde{v}^{i,n})_{n \geq 0}$ is of Cauchy type and converges pointwisely to $v^i$. Besides the inequalities (4.21) imply that: $\forall n \geq 0$ and $i \in \mathcal{J}$, $\tilde{v}^{i,2n+1} \leq v^i \leq \tilde{v}^{i,2n}$. This finally yields

$$v^i = \lim_{n \to \infty} \tilde{v}^{i,2n+1} = \lim_{n \to \infty} \tilde{v}^{i,2n},$$

and therefore, for any $i \in \mathcal{J}$, $v^i$ is both usc and lsc and thus continuous.

Next, since $(\bar{Y}^i, Z^i, K^i)_{i \in \mathcal{J}}$ is the unique solution of the system of reflected BSDEs associated with the following triple of data $((F_i(s, X^{i,x}_s, y^1, \ldots, y^m, z))_{i \in \mathcal{J}}, (e^{\lambda T} h_i(X^{i,x}_T))_{i \in \mathcal{J}}, (e^{\lambda t} g_{ij}(s, X^{i,x}_s))_{i,j \in \mathcal{J}})$ then $((e^{-\lambda t} v^i)_{i \in \mathcal{J}}$ is the unique solution of the system of reflected BSDEs associated with $((f_i)_{i \in \mathcal{J}}, (h_i)_{i \in \mathcal{J}}, (g_{ij})_{i,j \in \mathcal{J}})$. Due to representation (4.19) of $Y^i$, $i \in \mathcal{J}$, and using once again the result by El-Karoui et al. ([13], Theorem 8.5) we get that $(e^{-\lambda t} v^i)_{i \in \mathcal{J}}$ is a continuous with polynomial growth solution of the system of variational inequalities with inter-connected obstacles (2.3). The proof of existence is now complete.

(ii) **Uniqueness:** Let $(\bar{u}^i)_{i \in \mathcal{J}}$ be another solution of (2.3) which is continuous and belongs to $\Pi^g$. Let $(\bar{Y}^i)_{i \in \mathcal{J}}$ be the continuous processes of $S^2$ (and also of $H^{2,m}$) such that for any $i \in \mathcal{J}$ and $s \leq T$, $\bar{Y}^{i,t,x}_s = \bar{u}^i(s \vee t, X^{i,x}_{s \vee t})$.

Next let $(\tilde{Y}^{i,t,x})_{i \in \mathcal{J}}$ be the processes defined by

$$(\tilde{Y}^{i,t,x})_{i \in \mathcal{J}} = \Phi((\bar{Y}^{i,t,x})_{i \in \mathcal{J}}).$$

Using then the RBSDE characterization of $\tilde{Y}^{i,t,x}$ and the Feynman-Kac representation (4.16) in Corollary 2, there exist a family $(\tilde{v}^{i})_{i \in \mathcal{J}}$ of deterministic continuous functions of $\Pi^g$ such that:

$$\forall i \in \mathcal{J}, \forall s \in [t, T], \quad \tilde{Y}^{i,t,x}_s = \tilde{v}^i(s, X^{i,x}_s).$$

Moreover $(\tilde{v}^{i})_{i \in \mathcal{J}}$ is the unique solution of the following system of variational inequalities with inter-connected obstacles: $\forall i \in \mathcal{J}$,

$$\left\{ \begin{array}{l}
\min \left\{ \tilde{v}^i(t, x) - \max_{j \in \mathcal{J} - i} \left( -g_{ij}(t, x) + \tilde{v}^j(t, x) \right); \\
- \partial_t \tilde{v}^i(t, x) - \mathcal{L} \tilde{v}^i(t, x) - f_i(t, x, (\tilde{u}^i(t, x))_{i \in \mathcal{J}}, \sigma^\top (t, x).D_x \tilde{v}^i(t, x)) \right\} = 0;
\tilde{v}^i(T, x) = h_i(x).
\end{array} \right.$$
As $(\bar{u}^i)_{i \in \mathcal{J}}$ is also a solution of (4.22), then by uniqueness result (Corollary 1) we have that for any $i \in \mathcal{J}$, $\bar{v}^i = \bar{u}^i$. It follows that $(\bar{Y}^{i,t,x})_{i \in \mathcal{J}}$ verify

$$\forall s \in [t, T], \quad \left(\left(\bar{Y}^{i,t,x}_s\right)_{s \in [t,T]}\right)_{i \in \mathcal{J}} = \Phi\left(\left(\left(\bar{Y}^{i,t,x}_s\right)_{s \in [t,T]}\right)_{i \in \mathcal{J}}\right),$$

i.e., $(\bar{Y}^{i,t,x})_{i \in \mathcal{J}}$ is a fixed point for $\Phi$ in $\mathcal{H}^{2,m}$ on $[t, T]$. But, using the result stated in Proposition 5.4, $(Y^i)_{i \in \mathcal{J}}$ of (4.19) is the unique fixed point of $\Phi$ in $\mathcal{H}^{2,m}$. Thus for any $i \in \mathcal{J}$ we have:

$$\forall s \in [t, T], \quad \bar{Y}^{i,t,x}_s = Y^i_s.$$  

As we know that $Y^i$ has the Feynman-Kac’s representation (4.19), it follows that for $i \in \mathcal{J}$, $\bar{u}^i = v^i$, and then the solution of (2.3) is unique in the class of continuous functions of $\Pi^g$. □

**Remark 4.1** Under Assumptions of Theorem 2, we have a similar representation result as in Corollary 2.

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**Appendix: Auxiliary Results on Systems of Reflected BSDES**

**5.1 The Multi-states Stochastic Optimal Switching Problem**

**5.1.1 Presentation of the Problem**

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a fixed probability space on which is defined a standard $d$-dimensional Brownian motion $B = (B_t)_{0 \leq t \leq T}$ whose natural filtration is $(\mathcal{F}^0_t := \sigma\{B_s, s \leq t\})_{0 \leq t \leq T}$. Let $\mathcal{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ be the completed filtration of $(\mathcal{F}^0_t)_{0 \leq t \leq T}$ with all $\mathbb{P}$-null sets contained in $\mathcal{F}_0$, hence $(\mathcal{F}_t)_{0 \leq t \leq T}$ satisfies the usual conditions, i.e., it is right continuous and complete. Furthermore, let:

- $\mathcal{P}$ be the $\sigma$-algebra on $[0, T] \times \Omega$ of $\mathcal{F}$-progressively measurable sets;
- $\mathcal{H}^{2,k}$ be the set of $\mathcal{P}$-measurable, $\mathbb{R}^k$-valued processes $w = (w_t)_{t \leq T}$ such that $\mathbb{E}[\int_0^T |w_s|^2 ds] < \infty$;
- $\mathcal{S}^2$ be the set of $\mathcal{P}$-measurable, continuous, $\mathbb{R}$-valued processes $w = (w_t)_{t \leq T}$ such that $\mathbb{E}[\sup_{t \leq T} |w_t|^2] < \infty$.

Problems of multi-states switching are usually encountered in the economic sphere (financial markets, energy, etc.). A strategy of switching is a pair $(\delta, \xi) := ((\tau_n)_{n \geq 1}, (\xi_n)_{n \geq 1})$ such that:

(i) $(\tau_n)_{n \geq 1}$ is a non-decreasing sequence of $\mathcal{F}$-stopping times (i.e. $\tau_n \leq \tau_{n+1}$ and $\tau_0 = 0$); $\tau_n$ is the $n$th time where the decision to switch is made;
(ii) $(\xi_n)_{n \geq 0}$ are random variables with values in $\mathcal{J}$ and for any $n \geq 0$, $\xi_n$ is $\mathcal{F}_{\tau_n}$-measurable ($\xi_0$ is the initial state which is assumed to be state 1).
A strategy of switching \( (\delta, \xi) \) := \(((\tau_n)_{n \geq 1}, (\xi_n)_{n \geq 1})\) is called \textit{admissible} if \( \mathbb{P}[\tau_n < T, \forall n \geq 0] = 0 \); we denote by \( \mathcal{D} \) the set of admissible strategies. Next, with a given \textit{admissible} strategy \( (\delta, \xi) \), we associate a stochastic process \((\alpha_t)_{t \leq T}\) which is the indicator of the system at time \( t \) and which is given by:

\[
\alpha_t = \mathbb{1}_{[0, \tau_1]}(t) + \sum_{n \geq 1} \xi_{n-1, \xi_n}(\tau_n) \mathbb{1}_{[\tau_n, \tau_{n+1}]}(t), \quad \forall t \in [0, T] .
\] (5.1)

We point out that when \( \tau_n = T \), the value of \( \xi_n \) is irrelevant since the horizon of the switching problem is already reached.

For \( i \in J \), let \((\psi_i(t, \omega))_{t \leq T}\) be a process of \( \mathcal{H}^{2,1} \) which stands for the instantaneous profit when the system is in state \( i \). Next for \( i \neq j \in J \), let \((g_{ij}(t, \omega))_{t \leq T}\) be a process of \( \mathcal{S}^2 \) which stands for the switching cost at time \( t \) from the current state \( i \) to another one \( j \).

Finally when the admissible strategy \( (\delta, \xi) = ((\tau_n)_{n \geq 1}, (\xi_n)_{n \geq 1}) \) is implemented, the expected total profit is given by:

\[
J(\delta, \xi) := \mathbb{E} \left[ \int_0^T \psi_{\alpha_t}(s) \, ds - \sum_{n \geq 1} g_{\xi_{n-1}, \xi_n}(\tau_n) \mathbb{1}_{[\tau_n < T]} \right] .
\]

In several works, the main objective is to find an optimal strategy, \textit{i.e.}, a strategy \((\delta^*, \xi^*)\) such that

\[
J(\delta^*, \xi^*) = \sup_{(\delta, \xi) \in \mathcal{D}} J(\delta, \xi) \tag{5.2}
\]

(see e.g. [10, 19]) or at least to characterize (see e.g. [4, 15]) the optimal payoff, i.e., the right-hand side of (5.2).

### 5.1.2 Connection with Systems of Reflected BSDEs with Oblique Reflection

In order to tackle the switching problem described above, we usually relate it to systems of reflected BSDEs with oblique reflection which we introduce below in the case we need in order to deal with system of variational inequalities (2.3). So let \((t, x) \in [0, T] \times \mathbb{R}^k \) and \((X^{t,x}_s)_{s \leq T}\) be the solution of the following stochastic differential equation:

\[
dX^{t,x}_s = b(s, X^{t,x}_s) \, ds + \sigma(s, X^{t,x}_s) \, dB_s, \quad s \in [t, T] \quad \text{and} \quad X^{t,x}_s = x \quad \text{for} \quad s \in [0, t].
\]

Since \( b \) and \( \sigma \) verify (H1), the solution of this equation exists, is unique and satisfies:

\[
\forall p \geq 1, \quad \mathbb{E} \left[ \sup_{s \leq T} |X^{t,x}_s|^p \right] \leq C (1 + |x|^p) . \tag{5.3}
\]

Next let us introduce the solution of the system of reflected BSDEs with oblique reflection associated with the deterministic functions \(((f_i)_{i \in J}, (g_{ij})_{i,j \in J}, (h_i)_{i \in J})\). A solution consists of \( m \) triples of processes \(((Y^{i,t,x}_t, Z^{i,t,x}_t, K^{i,t,x}_t))_{i \in J}\), denoted by...
\[((Y^i, Z^i, K^i))_{i \in J}, \text{ that satisfy: } \forall i \in J\]

\[
\begin{cases}
Y^i, K^i \in S^2, \ Z^i \in \mathcal{H}^{2,d} \text{ and } K^i \text{ non-decreasing and } K^i_0 = 0; \\
Y^i_s = h_i(X^i_T) + \int_s^T f_i(r, X^i_r, Y^1_r, \ldots, Y^m_r, Z^i_r) \, dr + K^i_T - K^i_s - \int_s^T Z^i_r \, dB_r, \\
\forall s \leq T \\
Y^i_s \geq \max_{j \in J \setminus i} \left\{ Y^j_s - g_{ij}(s, X^j_s) \right\}, \quad \forall s \leq T \\
\int_0^T (Y^i_s - \max_{j \in J \setminus i} \left\{ Y^j_s - g_{ij}(s, X^j_s) \right\}) \, dK^i_s = 0.
\end{cases}
\]

(5.4)

We first provide an existence result of the solution of system (5.4) and some of its properties as well.

**Proposition 5.1** Assume that:

1. the functions \((f_i)_{i \in J}\) satisfy (H2)-(ii), (iii) and (iv);
2. For any \(i, j \in J\), the functions \(g_{ij}\) (resp. \(h_i\)) verify (H3) (resp. (H4)).

Then the system (5.4) has a solution \(((Y^i, Z^i, K^i))_{i=1,m}\).

**Proof** Since the above assumptions are not exactly the same as the ones of Theorem 3.2 in [18] then, for sake of completeness, we give its main steps. So let us consider the following standard BSDEs:

\[
\begin{cases}
\bar{Y} \in S^2, \ \bar{Z} \in \mathcal{H}^{2,d} \\
\bar{Y}_s = \max_{i=1,m} h_i(X^i_T) + \int_s^T \left[ \max_{i=1,m} f_i \right](r, X^i_r, \bar{Y}_r, \ldots, \bar{Y}_r, \bar{Z}_r) \, dr - \int_s^T \bar{Z}_r \, dB_r, \quad s \leq T,
\end{cases}
\]

(5.5)

and

\[
\begin{cases}
Y \in S^2, \ Z \in \mathcal{H}^{2,d} \\
Y_s = \min_{i=1,m} h_i(X^i_T) + \int_s^T \left[ \min_{i=1,m} f_i \right](r, X^i_r, Y_r, \ldots, Y_r, Z_r) \, dr - \int_s^T Z_r \, dB_r, \quad s \leq T.
\end{cases}
\]

(5.6)

Thanks to the result by Pardoux-Peng [21], the solutions of both (5.5) and (5.6) exist and are unique. We next introduce the following sequences of BSDEs defined recur-
sively by: for any \( i \in \mathcal{J} \), \( Y_{i,0} = Y \) and for \( n \geq 1 \) and \( s \leq T \),

\[
\begin{aligned}
Y_{i,n} - K_{i,n}^i & \in S^2, \ Z_{i,n} \in H^2, \ d \text{ and } K_{i,n}^i \text{ non-decreasing};
Y_{i,n} = h_i(X_{T}^i) + \int_0^T f_i(r, X_{r}^{i,x}, Y_{r}^{i,n-1}, \ldots, Y_{r}^{i-1,n-1}, Y_{r}^{i,n}, Y_{r}^{i+1,n-1}, \ldots, Y_{r}^{m,n-1}, Z_{r}^i) \, dr \\
+ \int_0^T Z_{r}^{i,n} \, dB_r;
Y_{i,n}^s \geq \max_{j \in \mathcal{J} - i} \{ Y_{j,n}^{s-1} - g_{ij}(s, X_{s}^{i,x}) \};
\int_0^T (Y_{i,n}^s - \max_{j \in \mathcal{J} - i} \{ Y_{j,n}^{s-1} - g_{ij}(s, X_{s}^{i,x}) \}) \, dK_{i,n}^s = 0.
\end{aligned}
\]

(5.7)

By an induction argument and the result by El-Karoui et al. ([12], Theorem 5.2), we claim that the processes \((Y_{i,n}, Z_{i,n}, K_{i,n}^i)\) exist for any \( n \geq 1 \). Next using the comparison theorem of solutions of BSDEs (see e.g. Theorem 2.2 in [13]) we deduce that for any \( i \in \mathcal{J} \), \( Y_{i,0} \leq Y_{i,1} \). On the other hand, \( f_i \) satisfies the monotonicity property (H2)-(iv) and using once more the comparison of solutions of reflected BSDEs (see e.g. Theorem 4.1 in [12]) we obtain by induction that:

\[
\forall n \geq 0 \text{ and } i \in \mathcal{J}, \quad Y_{i,n} \leq Y_{i,n+1}.
\]

But the processes \(((\tilde{Y}, \tilde{Z}, 0))_{i \in \mathcal{J}}\) is a solution for the system of obliquely reflected BSDEs associated with \(((\max_{i=1,m} f_i)(s, X_{s}^{i,x}, y_1, \ldots, y^m, z))_{i \in \mathcal{J}}, (\max_{i=1,m} h_i(X_{T}^{i,x}))(i \in \mathcal{J}), (g_{ij}(s, X_{s}^{i,x}))_{i,j \in \mathcal{J}}\). Then an induction procedure and the repeated use of comparison theorem, which is justified in taking into account that \( f_i \) satisfies the monotonicity property (H2)-(iv), leads to

\[
\forall n \geq 0, \forall i \in \mathcal{J}, \quad Y_{i,n} \leq \tilde{Y}.
\]

Using now Peng’s monotonic limit theorem (see Theorem 2.1 in [22]), we deduce that for any \( i \in \mathcal{J} \), there exist:

(i) a càdlàg (for right continuous with left limits) process \( Y^i \) such that \( Y_{i,n} \nearrow Y^i \) pointwisely;
(ii) a process \( Z^i \) of \( H^{2,d} \) such that, at least for a subsequence, \((Z_{i,n}^i)_{n \geq 0}\) converges weakly to \( Z^i \) in \( H^{2,d} \) and strongly in \( L^p(dt \otimes dP) \) for any \( p \in [1, 2[ \)
(iii) a càdlàg non decreasing process \( K^i \) such that for any stopping time \( \tau \), \((K_{\tau}^{i,n})_{n \geq 0}\) converges to \( K^i_{\tau} \) in \( L^p(dP) \) for any \( p \in [1, 2[ \).

The remaining of the proof is the same as the one of Theorem 3.2 in [18]. We therefore leave it to the care of the reader.

\[\square\]

We now give a result related to comparison of the solutions of system (5.4) constructed in the previous proposition. Its proof is rather easy since an induction argument allows to compare the solutions of the approximating schemes and then to deduce the same property for the limiting processes (see e.g. [18] for more details).

Remark 1 (i) Let \((f_i^i)_{i \in \mathcal{J}}\) (resp. \((g_{ij}^i)_{i,j \in \mathcal{J}}\), resp. \((h_i^i)_{i \in \mathcal{J}}\)) be functions that satisfy (H2)-(ii), (iii), (iv) (resp. (H3), resp. (H4)) and let \(((Y_{i}^{\iota}, Z_{i}^{\iota}, K_{i}^{\iota}))_{i \in \mathcal{J}}\) be the solu-
tion of the system of reflected BSDEs associated with \(( (f_i'_{i})_{i \in J}, (g_{ij}'_{i,j})_{i,j \in J}, (h_i'_{i})_{i \in J} )\) constructed as in Proposition 5.1. If for any \(i, j \in J\) we have:

\[
f_i \leq f_i', \quad h_i \leq h_i' \quad \text{and} \quad g_{ij} \geq g_{ij}'
\]

then for any \(i \in J\),

\[
Y^i \leq Y'^i.
\]

(ii) In case of uniqueness of the solutions of those systems, this result reduces to the comparison of the solutions.

We next focus on the regularity properties of the solution of system (5.4) constructed in Proposition 5.1.

**Proposition 5.2** Assume that the assumptions of Proposition 5.1 are fulfilled. Then there exist lsc deterministic functions \((v_{i})_{i \in J}\), defined on \([0, T] \times \mathbb{R}^k\), \(\mathbb{R}\)-valued and belonging to \(\Pi^g\) such that:

\[
\forall i \in J, \forall s \in [t, T], Y^s_i = v^i(s, X^{t,x}_s),
\]

where \(((Y^i, Z^i, K^i))_{i \in J}\) is the solution of (5.4) constructed in Proposition 5.1.

**Proof** Under the hypotheses of Proposition 5.1, there exist deterministic continuous with polynomial growth functions \(\bar{v}(t,x)\) and \(v(t,x)\) with values in \(\mathbb{R}\) such that for any \(s \in [t, T]\), \(\bar{Y}_s = \bar{v}(s, X^{t,x}_s)\) and \(Y^s = v(s, X^{t,x}_s)\) ((13), Theorem 4.1).

Next by induction and thanks to the result by El-Karoui et al. ([12], p. 729), there exist deterministic continuous functions \(v^i, n(t,x)\) in the class \(\Pi^g\) such that for any \(i \in J\) and \(n \geq 0\),

\[
Y^i_{s} = v^i, n_{s}(s, X^{t,x}_s), \quad \forall s \in [t, T];
\]

where \(Y^i, n\) is the solution of (5.7) (see Step 1, Proposition 5.1). As \(Y^i, n \leq Y^{i, n+1} \leq \bar{Y}\) then, for fixed \(i\), the sequence \((v^i, n\)) \(n \geq 0\) is non-decreasing and such that \(v^i, n \leq \bar{v}\). Therefore it converges pointwisely to \(v^i\) which is lower semi-continuous on \([0, T] \times \mathbb{R}^k\), of polynomial growth since \(v \leq v^i \leq \bar{v}\) and finally for any \(s \in [t, T]\), \(Y^i_{s} = v^i(s, X^{t,x}_s)\).

\[\blacksquare\]

**Remark 2** As, for each \(i \in J\), \(v^i\) belongs to \(\Pi^g\), then classically (see e.g. [12]) one can show that for any \(i \in J\), \(\|Z^i\|_{H^{2,m}}(t, x)\) is also of polynomial growth.

We now provide a representation result for the solutions of system (5.4) and, as a by product, we obtain a uniqueness result in some specific cases. For later use, let us fix \(\overrightarrow{u} := (u^i)_{i=1,m}\) in \(H^{2,m}\) and let us consider the following system of reflected
BSDEs with oblique reflection: \( \forall i \in J, \forall s \leq T, \)

\[
\begin{align*}
Y_{u,i}^s, K_{u,i}^s & \in \mathcal{S}^2, Z_{u,i}^s \in \mathcal{H}^{2,d}, \text{and } K_{0,i}^s \text{ non-decreasing (} K_{0,i}^s = 0) ; \\
Y_{u,i}^T & = h_i(X_{T,x}^t) + \int_s^T f_i(r, X_{r,x}^r, \overrightarrow{u_r}, Z_{r,i}^r) dr + K_{T,i}^s - K_{s,i}^s - \int_s^T Z_{r,i}^r dB_r ; \\
Y_{u,i}^s & \geq \max_{j \in J \setminus i} \{ Y_{u,j}^s - g_{ij}(s, X_{s,x}^r) \} \\
\int_0^T (Y_{u,i}^s - \max_{j \in J \setminus i} \{ Y_{u,j}^s - g_{ij}(s, X_{s,x}^r) \}) ds & = 0.
\end{align*}
\]

Let \( s \leq T \) be fixed, \( i \in J \) and let \( \mathcal{D}_s^i \) be the following set of admissible strategies:

\[
\mathcal{D}_s^i := \{ \alpha = ( (\tau_n)_{n \geq 0}, (\xi_n)_{n \geq 0} ) \in \mathcal{D}, \xi_0 = i, \tau_0 = 0, \tau_1 \geq s \text{ and } E[ (A_\alpha^T) ] < \infty \}
\]

where \( A_\alpha^r, r \leq T, \) is the cumulative switching costs up to time \( r, \) i.e.,

\[
A_\alpha^r := \sum_{n \geq 1} g_{\xi_{n-1}, \xi_n}(\tau_n, X_{\tau_n,x}^r)1_{[\tau_n \leq r]} \text{ for } r < T \text{ and } A_T = \lim_{r \to T} A_\alpha^r, \ \mathbb{P}\text{-a.s.}
\]

Therefore and for any admissible strategy \( \alpha \) we have:

\[
A_\alpha^T = \sum_{n \geq 1} g_{\xi_{n-1}, \xi_n}(\tau_n, X_{\tau_n,x}^T)1_{[\tau_n < T]}.
\]

Let us now consider a strategy \( \alpha = ( (\tau_n)_{n \geq 0}, (\xi_n)_{n \geq 0} ) \in \mathcal{D}_s^i \) and let \( (P_\alpha, Q_\alpha) := (P_s^\alpha, Q_s^\alpha)_{s \leq T} \) be the solution of the following BSDE (which is not of standard type):

\[
\begin{align*}
P_\alpha^s & = h_\alpha(X_{T,x}^t) + \int_s^T f_\alpha(r, X_{r,x}^r, \overrightarrow{u_r}, Q_r^\alpha) dr \\
& \quad - \int_s^T Q_r^\alpha dB_r - (A_{\alpha}^T - A_s^\alpha), \ \forall s \leq T,
\end{align*}
\]

with

\[
h_\alpha(x) = \sum_{n \geq 0} h_{\xi_n}(x)1_{[\tau_n \leq T < \tau_{n+1}]} \quad \text{and}\]

\[
f_\alpha(t, x, (\xi_i^j)_{i \in J}, z) := \sum_{n \geq 0} f_{\xi_n}(r, x, (\xi_i^j)_{i \in J}, z)1_{[\tau_n \leq r < \tau_{n+1}]}.
\]

In setting up \( \tilde{P}_\alpha := P_\alpha - A_\alpha \), we easily deduce the existence and uniqueness of the process \( (P_\alpha^s, Q_\alpha^s) \), since \( A_\alpha^s \) is adapted and \( E[ (A_\alpha^T) ] < \infty \), and the generator as well as the terminal value of the transformed BSDE are standard.

We then have the following representation for the solution of \((5.9)\) which is the main relationship between the value function of the stochastic optimal switching problem and solutions of systems of reflected BSDEs with oblique reflection. This result usually referred as the verification theorem is not new and has been already shown in several contexts and under various assumptions.
Proposition 5.3 Assume that for any $i, j \in J$:
(i) $f_i$ satisfies (H2)-(ii), (iii);
(ii) $g_{ij}$ (resp. $h_i$) satisfies (H3) (resp. (H4)).

Then the solution of system of BSDEs (5.9) exists and satisfies:

$$\forall s \leq T, \forall i \in J, \quad Y_{s}^{u,i} = \operatorname{ess sup}_{\alpha \in \mathcal{D}_{s}^{i}} P_{s}^{\alpha}.$$  \hspace{1cm} (5.11)

Thus the solution of (5.9) is unique.

Proof Noting that the generator $f_i$ in system (5.9) trivially satisfies (H2)-(iv) (since it does not depend on variable $\overrightarrow{y}$) and relying on Proposition 5.1, we may only consider all the other assumptions on the functions $(f_i, g_{ij}, h_i)_{i,j \in J}$. Since $Y_{s}^{u,i}$ solves system (5.9) and following the strategy $\alpha \in \mathcal{D}_{s}^{i}$ in (5.4), we obtain:

$$Y_{s}^{u,i} \geq h_{a}(X_{T}^{t,x}) + \int_{s}^{T} f_{a}(r, X_{r}^{t,x}, \overrightarrow{u_{r}}, Z_{r}^{\alpha}) \, dr - \int_{s}^{T} Z_{r}^{\alpha} \, dB_{r} - (A_{T}^{\alpha} - A_{s}^{\alpha}) + \tilde{K}_{T}^{\alpha},$$ \hspace{1cm} (5.12)

where

$$\tilde{K}_{T}^{\alpha} = (K_{\tau_{1}}^{u,i} - K_{s}^{u,i}) + \sum_{n \geq 1} (K_{\tau_{n+1}}^{u,\xi_{n}} - K_{\tau_{n}}^{u,\xi_{n}}) \quad \text{and}$$

$$Z^{\alpha}_{r} = \sum_{n \geq 0} Z_{r}^{u,\xi_{n}} 1_{[\tau_{n} \leq r < \tau_{n+1}]} \quad \forall r \leq T.$$

As $\tilde{K}_{T}^{\alpha} \geq 0$ then we have:

$$Y_{s}^{u,i} \geq P_{s}^{\alpha}, \quad \forall \alpha \in \mathcal{D}_{s}^{i}.$$ 

Note that the right-hand side in (5.12) is not a BSDE, therefore we will rather consider the equation satisfied by $Y_{s}^{u,i} - P_{s}^{\alpha}$ where the pair $(P_{s}^{\alpha}, Q_{s}^{\alpha})$ satisfies (5.10). Then using an equivalent change of probability we deduce the previous inequality.

Next let $\alpha^{*} = (\tau_{n}^{*}, \xi_{n}^{*})_{n \geq 0}$ be the strategy defined recursively as follows: $\tau_{0}^{*} = 0$, $\xi_{0}^{*} = i$ and for $n \geq 0$,

$$\tau_{n+1}^{*} = \inf \left\{ r \geq \tau_{n}^{*}, Y_{r}^{u,\xi_{n}^{*}} = \max_{j \in J_{\xi_{n}^{*}}} \left( Y_{r}^{u,j} - g_{\xi_{n}^{*},j}(r, X_{r}^{t,x}) \right) \right\} \wedge T$$

and

$$\xi_{n+1}^{*} = \arg \max_{j \in J_{\xi_{n}^{*}}^{+}} \left\{ Y_{\tau_{n+1}^{*}}^{u,j} - g_{\xi_{n+1}^{*},j}(\tau_{n+1}^{*}, X_{\tau_{n+1}^{*}}^{t,x}) \right\}.$$

Let us show that $\alpha^{*} \in \mathcal{D}_{s}^{i}$ and then let us first prove that $P[\tau_{n}^{*} < T, \forall n \geq 0] = 0$. We proceed by contradiction assuming that $P[\tau_{n}^{*} < T, \forall n \geq 0] > 0$. By definition of $\tau_{n}^{*}$, we have:

$$P[ Y_{\tau_{n+1}^{*}}^{u,\xi_{n}^{*}} = Y_{\tau_{n+1}^{*}}^{u,\xi_{n+1}^{*}} - g_{\xi_{n}^{*},\xi_{n+1}^{*}}(\tau_{n+1}^{*}, X_{\tau_{n+1}^{*}}^{t,x}), \xi_{n}^{*} \in J_{\xi_{n+1}^{*}}^{-}, \forall n \geq 1 ] > 0.$$
As $\mathcal{J}$ is finite then there is a state $i_0 \in \mathcal{J}$ and a loop $i_0, i_1, \ldots, i_k, i_0$ of elements of $\mathcal{J}$ where $i_1 \neq i_0$ and finally a subsequence $(n_q)_{q \geq 0}$ such that:

$$P\left[ Y_{\tau_{n_q}^*}^{u,i} = Y_{\tau_{n_q}^*}^{u,i_{l+1}} - g_{i,l,i_{l+1}}(\tau_{n_q}^*, X_{\tau_{n_q}^*}^{l,x}) \right] > 0, \quad l = 0, \ldots, k, \ (i_{k+1} = i_0), \ \forall q \geq 0.$$

Therefore defining $\tau := \lim_{q \to \infty} \tau_{n_q}^*$ and taking the limit w.r.t. $q$ to obtain:

$$P\left[ Y_{\tau}^{u,i} = Y_{\tau}^{u,i_{l+1}} - g_{i,l,i_{l+1}}(\tau, X_{\tau}^{l,x}) \right] > 0, \quad l = 0, \ldots, k, \ (i_{k+1} = i_0).$$

Then

$$P\left[ g_{i_0,i_1}(\tau, X_{\tau}^{l,x}) + \cdots + g_{i_k,i_0}(\tau, X_{\tau}^{l,x}) = 0 \right] > 0,$$

which contradicts the no-loop property. We therefore have $P[\tau^*_n < T, \forall n \geq 0] = 0$.

Next it only remains to prove that $E[(A_{\alpha^*}^*)^2] < \infty$ and $\alpha^*$ is optimal in $\mathcal{D}_s^i$ for the switching problem (5.11). Following the strategy $\alpha^*$ and since $(Y_{\tau}^{u,i})_{i \in \mathcal{J}}$ solves the reflected BSDE (5.9), it yields:

$$Y_{\tau_{n}}^{u,i} = h_{\alpha^*}(X_{\tau_{n}}^{l,x}) + \int_{s}^{\tau_{n}} f_{\alpha^*}(r, X_{r}^{l,x}, \overline{u}_r, Z_{r}^{\alpha^*}) \, dr - \int_{s}^{\tau_{n}} Z_{r}^{\alpha^*} \, dB_r - A_{\tau_{n}}^{\alpha^*}, \quad (5.13)$$

noting that $K_{\tau_{r}}^{u,i} - K_{\tau_{n}}^{u,i} = 0$ holds for any $r, \tau_{n}^* \leq r \leq \tau_{n+1}^*$. Taking now the limit w.r.t. $n$ in (5.13) to obtain:

$$Y_{\tau}^{u,i} = h_{\alpha^*}(X_{\tau}^{l,x}) + \int_{s}^{T} f_{\alpha^*}(r, X_{r}^{l,x}, \overline{u}_r, Z_{r}^{\alpha^*}) \, dr - \int_{s}^{T} Z_{r}^{\alpha^*} \, dB_r - A_{T}^{\alpha^*}, \quad (5.14)$$

But using the assumptions (H4) and (H2)-(ii),(iii) satisfied by $h_i$ and $f_i$ respectively and since $\overline{u} \in \mathcal{H}_2^i$ and $Z_{\tau}^\alpha \in \mathcal{H}_2^d$ and $(Y_{\tau}^{i})_{i \in \mathcal{J}} \in (S^2)^m$, we deduce from (5.14) that $E[(A_{\alpha^*}^*)^2] < \infty$. It follows that $\alpha^* \in \mathcal{D}_s^i$ and $Y_{\tau}^{u,i} = P_{\tau}^{\alpha^*}$, thus (5.11) holds and the solution of (5.9) is unique. \hfill \Box

Next for $\overline{u} := (u^i)_{i=1,m} \in \mathcal{H}_2^m$ let us define

$$\Phi(\overline{u}) := (Y_{\tau}^{u,i})_{i=1,m}, \quad (5.15)$$

where $((Y_{\tau}^{u,i}, Z_{\tau}^{u,i}, K_{\tau}^{u,i}))_{i=1,m}$ is the solution of system (5.9) which exists and is unique under the assumptions of Proposition 5.3. Note that when the processes $(Y_{\tau}^{u,i})_{i=1,m}$ exist they belong to $(S^2)^m$ and then $\Phi$ is a mapping from $\mathcal{H}_2^m$ to $\mathcal{H}_2^m$.

The following result, established by Chassagneux et al. [6], shows that $\Phi$ is a contraction in $\mathcal{H}_2^m$ when endowed with an appropriate equivalent norm.

**Proposition 5.4** Assume that for any $i, j \in \mathcal{J}$ the following hypotheses are in force:

(i) $f_i$ verifies (H2)-(ii), (iii);
(ii) $g_{ij}$ (resp. $h_i$) verifies (H3) (resp. (H4)).
Then we have:

(a) For any \( \overrightarrow{u} = (u_i)_{i=1,m}, \overrightarrow{v} = (v_i)_{i=1,m} \in H^{2,m} \),

\[
\forall i \in \mathcal{J}, \forall s \leq T,
\mathbb{E}[|Y^{u,i}_s - Y^{v,i}_s|^2] \leq C \left( \|\overrightarrow{u} - \overrightarrow{v}\|_{H^{2,m}}^2 := \mathbb{E} \left[ \int_0^T |\overrightarrow{u}_r - \overrightarrow{v}_r|^2 \, dr \right] \right). \tag{5.16}
\]

(b) There exists \( \beta_0 \in \mathbb{R} \) such that the mapping \( \Phi \) is a contraction when \( H^{2,m} \) is endowed with the following equivalent norm:

\[
\| (u^i)_{i=1,m} \|_{\beta_0} := \left\{ \mathbb{E} \left[ \int_0^T e^{\beta_0 s} \left( \sum_{i=1,m} |u^i_s|^2 \right) \, ds \right] \right\}^{1/2},
\]

where \( \overrightarrow{u} = (u^i)_{i=1,m} \in H^{2,m} \).

Therefore \( \Phi \) has a fixed point \((Y^i)_{i \in \mathcal{J}} \) which belongs to \((S^2)^m \) and which provides a unique solution for system (5.4).

**Proof** We provide only its main steps since it has been already given in [6]. For \( i \in \mathcal{J} \), \( \overrightarrow{u} \) and \( \overrightarrow{v} \in H^{2,m} \) let us set

\[
\varphi_i(r, X^l,x_r, z) = f_i(r, X^l,x_r, u_r, z) \vee f_i(r, X^l,x_r, v_r, z), \quad r \leq T,
\]

and let us consider the solution, denoted by \((\tilde{Y}^i, \tilde{Z}^i, \tilde{K}^i)_{i \in \mathcal{J}} \), of the system of obliquely reflected BSDEs associated with \((\varphi_i(r, X^l,x_r, z))_{i \in \mathcal{J}}, (h_i)_{i \in \mathcal{J}} \) and \((g_{ij})_{i,j \in \mathcal{J}} \) which exists and is unique by Proposition 5.1. By Proposition 5.3, the following representation holds true:

\[
\forall s \leq T, \quad \tilde{Y}^i_s = \text{ess sup}_{a \in D^i_s} \tilde{P}^a_s
\]

where for \( a \in D^i_s \) the pair of processes \((\tilde{P}^a, \tilde{Q}^a)\) verifies:

\[
\begin{cases}
\tilde{P}^a \text{ is RCLL and } E[\sup_{\eta \leq T} |\tilde{P}^a_{\eta}|^2] < \infty, \quad \tilde{Q}^a \in \mathcal{H}^{2,d}; \\
\tilde{P}^a_{\eta} = h_a(X^l,x_T) + \int_0^T \varphi_a(r, X^l,x_r, \tilde{Q}^a_r) \, dr - \int_0^T \tilde{Q}^a_r dBr - (A^a_T - A^a_{\eta}), \quad \forall \eta \leq T.
\end{cases}
\]

Additionally an optimal strategy \( \tilde{a} \) exists i.e. \( \tilde{Y}^i_s = \tilde{P}_{\tilde{a}}^a \). Note here that the dependence of \( P^a_s \) on \( i \) is made through the strategy \( \tilde{a} \) which belongs to \( D^i_s \). Now since for any \( r \leq T \) and \( z \in \mathbb{R}^d \), \( \varphi_i(r, X^l,x_r, z) \geq f_i(r, X^l,x_r, \overrightarrow{u}_r, z) \) and \( \varphi_i(r, X^l,x_r, z) \geq f_i(r, X^l,x_r, \overrightarrow{v}_r, z) \) then by comparison and uniqueness (see Remark 1) we have:

\[
Y^{u,i}_s \leq \tilde{Y}^i_s \quad \text{and} \quad Y^{v,i}_s \leq \tilde{Y}^i_s. \tag{5.17}
\]

Next for \( a \in D^i_s \), let \( (P^a_s, Q^a_r)_{r \leq T} \) be the solution of the non-standard BSDE (5.10) and let \( (P'^a_s, Q'^a_r)_{r \leq T} \) be the solution of the same non-standard BSDE with generator \( f_a(r, X^l,x_r, \overrightarrow{v}_r, z) \). Then we have:

\[
P^a_s \leq Y^{u,i}_s \leq \tilde{Y}^i_s = \tilde{P}^a_s \quad \text{and} \quad P'^a_s \leq Y^{v,i}_s \leq \tilde{Y}^i_s = \tilde{P}^a_s
\]
which implies:

$$|Y_s^{u,i} - Y_s^{v,i}| \leq |\tilde{P}_s^{\tilde{a}} - P_s^{\tilde{a}}| + |\tilde{P}_s^{\tilde{a}} - P_s^{\tilde{a}}|.$$  \hfill (5.18)

But for any $\eta \leq T$ we also have:

$$\tilde{P}_\eta^{\tilde{a}} - P_\eta^{\tilde{a}}$$

$$= \int_\eta^T \left\{ f_\eta(r, X_r^{i,x}, \tilde{u}_r, \tilde{Q}_r) \vee f_\eta(r, X_r^{i,x}, \tilde{v}_r, \tilde{Q}_r) - f_\eta(\tilde{r}_r, X_r^{i,x}, \tilde{u}_r, \tilde{Q}_r) \right\} dr$$

and a similar equation is valid for $\tilde{P}_\eta^{\tilde{a}} - P_\eta^{\tilde{a}}$. Using Itô’s formula with $|\tilde{P}_\eta^{\tilde{a}} - P_\eta^{\tilde{a}}|^2$ and the inequality $|x \vee y - y| \leq |x - y|$, for any $x, y \in \mathbb{R}$, to obtain:

$$|\tilde{P}_\eta^{\tilde{a}} - P_\eta^{\tilde{a}}|^2 + \int_\eta^T |\tilde{Q}_r - Q_r|^2 dr \leq -2 \int_\eta^T (\tilde{P}_\eta^{\tilde{a}} - P_\eta^{\tilde{a}}) \{ \tilde{Q}_r - Q_r \} dB_r$$

$$+ 2 \int_\eta^T |\tilde{P}_r^{\tilde{a}} - P_r^{\tilde{a}}| f_\eta(r, X_r^{i,x}, \tilde{u}_r, \tilde{Q}_r) - f_\eta(r, X_r^{i,x}, \tilde{v}_r, \tilde{Q}_r) dr.$$  

We now classically show the existence of a real constant $C \geq 0$ such that:

$$\mathbb{E} \left[ \sup_{\eta \leq T} |\tilde{P}_\eta^{\tilde{a}} - P_\eta^{\tilde{a}}|^2 \right] \leq C \mathbb{E} \left[ \int_0^T \|\tilde{u}_r - \tilde{v}_r\|^2 dr \right].$$  \hfill (5.19)

In the same way considering $|\tilde{P}_\eta^{\tilde{a}} - P_\eta^{\tilde{a}}|^2$ we obtain a similar inequality as (5.19) where $\tilde{P}_\eta^{\tilde{a}}$ is replaced by $P_\eta^{\tilde{a}}$. Finally going back to (5.18), squaring and taking the expectation, we get (5.16).

Next in considering $e^{\beta\eta}|\tilde{P}_\eta^{\tilde{a}} - P_\eta^{\tilde{a}}|^2$ ($\beta > 0$) and arguing as previously we obtain a similar inequality as (5.16) where the term $e^{\beta\eta}$ appears. Finally it is enough to integrate w.r.t. $\eta$ and to choose $\beta = \eta_0$ appropriately in order to get that $\Phi$ is a contraction in the Banach space $(\mathcal{H}^{2,m}, \|\|_0)$. Therefore it has a fixed point $(Y_i^{i})_{i=1,m}$ which can be chosen continuous since $\Phi((Y_i^{i})_{i=1,m}) \in (\mathcal{S}^{2,m})$. Thus the system of reflected BSDEs with interconnected obstacles (5.4) has a unique solution. \hfill \Box

Remark 3 Let $(Y_i^{i,0})_{i \in \mathcal{I}}$ be fixed processes of $\mathcal{H}^{2,m}$ and for $n \geq 1$ let us set $(Y_i^{i,n})_{i \in \mathcal{I}} = \Phi((Y_i^{i,n-1})_{i \in \mathcal{I}})$. Then the sequence $((Y_i^{i,n})_{i \in \mathcal{I}})_{n \geq 0}$ converges in $(\mathcal{H}^{2,m}, \|\|_0)$ to the unique solution of the system of reflected BSDEs associated with $((f_i)_{i \in \mathcal{I}}, (g_{ij})_{i,j \in \mathcal{I}}, (h_i)_{i \in \mathcal{I}})$ since $\Phi$ is a contraction in $(\mathcal{H}^{2,m}, \|\|_0)$ and the norms $\|\|_0$ and $\|\|$ are equivalent.

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