A CONVENIENT COORDINATIZATION OF SIEGEL-JACOBI DOMAINS

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ABSTRACT. We determine the homogeneous Kähler diffeomorphism $F_C$ which expresses the Kähler two-form on the Siegel-Jacobi ball $D^J_n = \mathbb{C}^n \times D_n$ as the sum of the Kähler two-form on $\mathbb{C}^n$ and the one on the Siegel ball $D_n$. The classical motion and quantum evolution on $D^J_n$ determined by a hermitian linear Hamiltonian in the generators of the Jacobi group $G^J_n = H_n \rtimes \text{Sp}(n, \mathbb{R})_\mathbb{C}$ are described by a matrix Riccati equation on $D_n$ and a linear first order differential equation in $z \in \mathbb{C}^n$, with coefficients depending also on $W \in D_n$. $H_n$ denotes the $(2n+1)$-dimensional Heisenberg group. The system of linear differential equations attached to the matrix Riccati equation is a linear Hamiltonian system on $D_n$. When the transform $F_C : (\eta, W) \to (z, W)$ is applied, the first order differential equation in the variable $\eta = (I_n - WW^{-1})(z + \bar{W}z) \in \mathbb{C}^n$ becomes decoupled from the motion on the Siegel ball. Similar considerations are presented for the Siegel-Jacobi upper half plane $X^J_n = \mathbb{C}^n \times X_n$, where $X_n$ denotes the Siegel upper half plane.

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1. Introduction

The Jacobi groups \([26]\) are semidirect products of appropriate semisimple real algebraic groups of hermitian type with Heisenberg groups \([59\ [11]\]. The Jacobi groups are unimodular, nonreductive, algebraic groups of Harish-Chandra type \([50]\). The Siegel-Jacobi domains are nonsymmetric domains associated to the Jacobi groups by the generalized Harish-Chandra embedding \([50\ [11]\], \([64\ -\ 66]\). The holomorphic irreducible unitary representations of the Jacobi groups based on Siegel-Jacobi domains have been constructed by Berndt, Böcherer, Schmidt, and Takase \([19\ [20]\, \([57\ -\ 59]\). Some coherent state systems based on Siegel-Jacobi domains have been investigated in the framework of quantum mechanics, geometric quantization, dequantization, quantum optics, nuclear structure, and signal processing \([39\ [49\ [53\ [8\ [10]\ [11]\ [12]\]. The Jacobi group was investigated by physicists under other names as Hagen \([30]\), Schrödinger \([17]\, or Weyl-symplectic group \([62]\). The Jacobi group is responsible for the squeezed states \([38\ [50\ [43\ [69\ [33]\] in quantum optics \([23\ [44\ [2\ [55\ [25]\].

The Jacobi group has been studied in the papers \([8\ [10]\] in connection with the group-theoretic approach to coherent states \([48]\). We have attached to the Jacobi group \(G_n^J = H_n \ltimes \text{Sp}(n, \mathbb{R})_C\) coherent states based on Siegel-Jacobi ball \(D_n^J\) \([10]\), which, as set, consists of the points of \(\mathbb{C}^n \times \mathcal{D}_n\). \(H_n\) denotes the \((2n+1)\)-dimensional Heisenberg group and \(\mathcal{D}_n\) denotes the Siegel ball. The case \(G_1^J\) was studied in \([8]\). We have determined the Kähler two-form \(\omega_n\) on \(D_n^J\) from the Kähler potential \([10]\) – the logarithm of the scalar product of two coherent states \([7]\) – and, via the partial Cayley transform, we have determined the Kähler two-form \(\omega_n^J\) on the Siegel-Jacobi upper-half plane \(\mathcal{X}_n^J = \mathbb{C}^n \times \mathcal{X}_n\), where \(\mathcal{X}_n\) is the Siegel upper half plane \([9]\). The Kähler two-form \(\omega_n^J\) was investigated by Berndt \([18]\) and Kähler \([36\ [37]\), while \(\omega_n\) and \(\omega_n^J\) have been investigated also by Yang \([65\ -\ 68]\). \(\omega_n\) is written compactly as the sum of two terms, one describing the Kähler two-form on \(\mathcal{D}_n\), \(\omega_{\mathcal{D}_n}\), the other one is \(\text{Tr}(A'(\mathbb{I}_n - \bar{W}W)^{-1} \wedge A)\), where \(A = dz + dW \bar{\eta}\), and \(\eta = (\mathbb{I}_n - \bar{W}W)^{-1}(z + W \bar{z}), z \in \mathbb{C}^n, W \in \mathcal{D}_n \ [9\ [10]\). Let us denote by \(FC\) the change of variables \(FC : \mathbb{C}^n \times \mathcal{D}_n \ni (\eta, W) \rightarrow (z, W) \in \mathcal{D}_n, z = \eta - W \bar{\eta}\). We put this change of variables in connection with the celebrated fundamental conjecture \([60\ [24]\] on the Siegel-Jacobi ball \(D_n^J\), as we did in \([13\ [14]\] for the Siegel-Jacobi disk \(D_1^J\). We also make similar considerations for the Siegel-Jacobi upper half plane \(\mathcal{X}_n^J\).

In \([4\ [5]\] we have considered the problem of dequantization of a dynamical system with Lie group of symmetry \(G\) on a Hilbert space \(\mathcal{H}\) in Berezin’s approach \([15\ [17]\] in the simple case of linear Hamiltonians. Linear Hamiltonians in generators of the Jacobi group appear in many physical problems of quantum mechanics, as in the case of the quantum oscillator acted on by a variable external force \([27\ [51\ [35]\].

What we find out is that the classical motion and quantum evolution on \(D_n^J\) determined by linear Hamiltonians in the generators of the Jacobi group \(G_n^J\) are described by a matrix Riccati equation on \(\mathcal{D}_n\) and a first order coupled linear differential equation for \(z \in \mathbb{C}^n\). The nice thing is that via the \(FC\) transform, the differential equation for \(\eta\) does not depend on \(W \in \mathcal{D}_n\). The variables \((\eta, W)\) appear to offer a convenient parametrization of the Siegel-Jacobi ball. Similar considerations are presented for the equations of motion on \(\mathcal{X}_n^J\).

In the present paper we extend to \(G_n^J\) our results established in \([14]\ for \(G_1^J\).
The paper is laid out as follows. The notation for the Jacobi algebra $\mathfrak{g}_\lambda^J$ is fixed in [2]. Starting with some notation on coherent states [18, 7, 33] deals with coherent states based on $D_n^J$ [10]. The holomorphic representation [6, 7] of the generators of the Jacobi group as first order differential operators with polynomial coefficients defined on $D_n^J$ is given in [41]. It is verified that the differential realization of the generators of $\mathfrak{sp}(n, \mathbb{R})_C$ has the right hermiticity properties with respect to the scalar product of functions on $D_n$ - Lemma [2] and similarly for the generators of $\mathfrak{g}_\lambda^J$ - Lemma [3]. In [4] we recall the expression of $\omega_n^J$ and in Proposition [3] we show that the partial Cayley transform -the transform which connects $D_n^J$ and $X_n^J$ - is a Kähler homogenous diffeomorphism. In Proposition [4] of [6] we determine the $FC$-transform for the Siegel-Jacobi domains $D_n^J$ and $X_n^J$. The proof is inspired by the paper [36] of Kähler. Corollary [1] expresses the reproducing kernel and the scalar product in the variables $(\eta, W) \in \mathbb{C}^n \times D_n$. [7] is devoted to classical motion and quantum evolution on the Siegel-Jacobi domains determined by hermitian linear Hamiltonians in the generators of the Jacobi group $G_n^J$. The equations of motion are written down explicitly in Proposition [6] in [7] and their integration is discussed in [7, 2]. Use is made of the methods of [12] to integrate the matrix Riccati differential equation on manifolds by linearization, previously applied in [5] in the case of hermitian symmetric spaces. The last two paragraphs in [8] refer to the Berry phase [52] on $D_n^J$ and the energy function associated to the Hamiltonian linear in the generators of the Jacobi group $G_n^J$ expressed in the variables $(\eta, W) \in \mathbb{C}^n \times D_n$. In these variables the energy function is written down as the sum of a real function in $\eta$ and one in $W$.

**Notation.** We denote by $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}$, and $\mathbb{N}$ the field of real numbers, the field of complex numbers, the ring of integers, and the set of non-negative integers, respectively. $M_{nm}(\mathbb{F}) \cong \mathbb{F}^{nm}$ denotes the set of all $m \times n$ matrices with entries in the field $\mathbb{F}$. $M_{n1}(\mathbb{F})$ is identified with $\mathbb{F}^n$. Set $M(n, \mathbb{F}) = M_{nn}(\mathbb{F})$. For any $A \in M(n, \mathbb{F})$, $A^t$ denotes the transpose matrix of $A$, $A^* = (A + A^t)/2$ and $A^\alpha = (A - A^t)/2$. For $A \in M_\alpha(\mathbb{C})$, $\bar{A}$ denotes the conjugate matrix of $A$ and $A^* = \bar{A}^t$. For $A \in M_n(\mathbb{C})$, the inequality $A > 0$ means that $A$ is positive definite. The identity matrix of degree $n$ is denoted by $I_n$ and $O_n$ denotes the $M_n(\mathbb{F})$-matrix with all entries 0. We denote by $\text{diag}(\alpha_1, \ldots, \alpha_n)$ the matrix which has the elements $\alpha_1, \ldots, \alpha_n$ on the diagonal and all the other elements 0. If $A$ is a linear operator, we denote by $A^\dagger$ its adjoint. We consider a complex separable Hilbert space $\mathcal{H}$ endowed with a scalar product which is antilinear in the first argument, $\langle \lambda x, y \rangle = \bar{\lambda} \langle x, y \rangle$, $x, y \in \mathcal{H}$, $\lambda \in \mathbb{C} \setminus 0$. A complex analytic manifold is Kählerian if it is endowed with a Hermitian metric whose imaginary part $\omega$ has $d\omega = 0$ [32]. A coset space $M = G/H$ is homogenous Kählerian if it carries a Kählerian structure invariant under the group $G$ [21]. By a Kähler homogeneous diffeomorphism we mean a diffeomorphism $\phi : M \to N$ of homogeneous Kähler manifolds such that $\phi^* \omega_N = \omega_M$. Ham($M$) denotes the Hamiltonian vector fields on the manifold $M$. We use Einstein convention that repeated indices are implicitly summed over. If $W = (w)_{ij}$ is a symmetric matrix, we introduce the symbols $\nabla_{ij} = \nabla_{ji} = \chi_{ij} \frac{\partial}{\partial w_{ij}}$, where $\chi_{ij} = \frac{1+\delta_{ij}}{2}$. In the expression $\chi_{ij} \frac{\partial}{\partial w_{ij}}$ of $\nabla_{ij}$ no summation is assumed.
2. The Jacobi algebra \( \mathfrak{g}_n^J \)

Let \( \mathfrak{h}_n \) denotes the \((2n+1)\)-dimensional Heisenberg algebra, isomorphic to the algebra

\[
\mathfrak{h}_n = \langle is1 + \sum_{i=1}^{n} (x_i a_i^\dagger - x_i a_i) >_{s \in \mathbb{R}, x_i \in \mathbb{C}},
\]

where \( a_i^\dagger \) (\( a_i \)) are the boson creation (respectively, annihilation) operators, which verify the canonical commutation relations

\[
[a_i, a_j^\dagger] = \delta_{ij}; \ [a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0.
\]

The displacement operator

\[
D(\alpha) := \exp(aa^\dagger - \bar{a}a) = \exp(-\frac{1}{2} |\alpha|^2) \exp(aa^\dagger) \exp(-\bar{a}a),
\]

verifies the composition rule:

\[
D(\alpha_2)D(\alpha_1) = e^{i\theta_\alpha(\alpha_2, \alpha_1)} D(\alpha_2 + \alpha_1), \ \theta_\alpha(\alpha_2, \alpha_1) := \Im(\alpha_2 \alpha_1^*).
\]

Here we have used the notation \( \alpha \beta = \alpha_i \beta_i \), where \( \alpha = (\alpha_i)_{i=1, \ldots, n} \in \mathbb{C}^n \).

The composition law of the Heisenberg group \( H_n \) is:

\[
(\alpha_2, t_2) \circ (\alpha_1, t_1) = (\alpha_2 + \alpha_1, t_2 + t_1 + \Im(\alpha_2 \alpha_1^*)).
\]

If we identify \( \mathbb{R}^n \) with \( \mathbb{C}^n \), \( (p, q) \mapsto \alpha \):

\[
\alpha = p + iq, \ p, q \in \mathbb{R}^n,
\]

then

\[
\Im(\alpha_2 \alpha_1^*) = (p_1^t, q_1^t) J \begin{pmatrix} p_2 \\ q_2 \end{pmatrix}, \ \text{where} \ J = \begin{pmatrix} \mathbb{O}_n & \mathbb{I}_n \\ -\mathbb{I}_n & \mathbb{O}_n \end{pmatrix}.
\]

The Jacobi algebra is the the semi-direct sum \( \mathfrak{g}_n^J := \mathfrak{h}_n \ltimes \mathfrak{sp}(n, \mathbb{R})_C \), where \( \mathfrak{h}_n \) is an ideal in \( \mathfrak{g}_n^J \), i.e. \([\mathfrak{h}_n, \mathfrak{g}_n^J] = \mathfrak{h}_n\), determined by the commutation relations:

\[
[a_k^\dagger, K_{ij}^-] = [a_k, K_{ij}^+] = 0,
\]

\[
[a_i, K_{k,j}^+] = \frac{1}{2} \delta_{ik} a_j^\dagger + \frac{1}{2} \delta_{ij} a_k^\dagger, \ [K_{k,j}^-, a_i^\dagger] = \frac{1}{2} \delta_{ik} a_j + \frac{1}{2} \delta_{ij} a_k,
\]

\[
[K_{0,j}^0, a_k^\dagger] = \frac{1}{2} \delta_{jk} a_i^\dagger, \ [a_k, K_{0,j}^0] = \frac{1}{2} \delta_{ik} a_j.
\]

The generators \( K_{0,\pm,}^\pm \) of \( \mathfrak{sp}(n, \mathbb{R})_C \) verify the commutation relations

\[
[K_{ij}^-, K_{kl}^-] = [K_{ij}^+, K_{kl}^+] = 0, \ 2[K_{ij}^-, K_{kl}^0] = K_{0}^0 K_{kl}^0 - K_{ij}^0 K_{kl}^0 - K_{ij}^0 K_{kl}^0,
\]

\[
2[K_{ij}^+, K_{kl}^0] = K_{0}^0 K_{kl}^0 + K_{ij}^0 K_{kl}^0 + K_{kl}^0 K_{ij}^0, \ 2[K_{ij}^0, K_{kl}^0] = K_{kl}^0 K_{ij}^0 - K_{ij}^0 K_{kl}^0.
\]

Now we briefly fix the notation concerning the symplectic group. For \( A \in \mathrm{GL}(2n, \mathbb{F}) \), we have (here \( \mathbb{F} \) is any of the fields \( \mathbb{R}, \mathbb{C} \)): \( A \in \mathrm{Sp}(n, \mathbb{F}) \iff A^t J A = J \). Consequently, a matrix \( X \in \mathrm{gl}(2n, \mathbb{F}) \) is in \( \mathfrak{sp}(n, \mathbb{F}) \) iff \( X^t J + JX = 0 \). The matrices from \( \mathfrak{sp}(n, \mathbb{R}) \) are also called infinitesimally symplectic or Hamiltonians \([16]\).

We recall also that \( g \in U(n, n) \) iff \( gKg^* = K \), where

\[
K = \begin{pmatrix} \mathbb{I}_n & \mathbb{O}_n \\ \mathbb{O}_n & -\mathbb{I}_n \end{pmatrix}.
\]
We summarize some properties of symplectic and Hamiltonian matrices (cf. \[54, 3, 28, 46\]; for the characterization of the eigenvalues, see \[16, 1, 40, 63\]):

**Remark 1.** a) \(X\) is a Hamiltonian matrix iff one of the following equivalent conditions are fulfilled:
1) \(X^t J + JX = 0\);
2) \(X = JR\), where \(R \in M(2n, \mathbb{R})\) is a symmetric matrix;
3) \(JX\) is symmetric;
4) \(X \in M(2n, \mathbb{R})\) has the form

\[
(2.9) \quad X = \begin{pmatrix} a & b \\ c & -a^t \end{pmatrix} \in \mathfrak{sp}(n, \mathbb{R}), \quad b = b^t, \quad c = c^t, \quad a, b, c \in M(n, \mathbb{R});
\]

b) If \(M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Sp}(2n, \mathbb{R})\), then the matrices \(a, b, c, d \in M(n, \mathbb{R})\) have the properties

\[
(2.10a) \quad a^tc = c^ta, \quad b^td = d^tb, \quad a^td - c^tb = \mathbb{I}_n;
\]
\[
(2.10b) \quad ab^t = ba^t, \quad cd^t = dc^t, \quad ad^t - bc^t = \mathbb{I}_n.
\]

c) Under the identification \((2.6)\) of \(\mathbb{R}^{2n}\) with \(\mathbb{C}^n\), we have the correspondence

\[
(2.11) \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M(2n, \mathbb{R}) \leftrightarrow M_C = \mathbb{C}^{-1} M \mathbb{C} = \begin{pmatrix} p & q \\ \bar{q} & \bar{p} \end{pmatrix}, \quad p, q \in M(n, \mathbb{C}),
\]

where

\[
(2.12) \quad \mathbb{C} = \begin{pmatrix} i\mathbb{I}_n & i\mathbb{I}_n \\ -i\mathbb{I}_n & i\mathbb{I}_n \end{pmatrix}, \quad \mathbb{C}^{-1} = \frac{1}{2} \begin{pmatrix} -i\mathbb{I}_n & -i\mathbb{I}_n \\ i\mathbb{I}_n & i\mathbb{I}_n \end{pmatrix};
\]

\[
(2.13a) \quad 2a = p + q + \bar{p} + \bar{q}, \quad 2b = i(\bar{p} - \bar{q} - p + q),
\]
\[
(2.13b) \quad 2c = i(p + q - \bar{p} - \bar{q}), \quad 2d = p - q + \bar{p} - \bar{q};
\]
\[
(2.14) \quad 2p = a + d + i(b - c), \quad 2q = a - d - i(b + c).
\]

In particular, to the Hamiltonian matrix \((2.9)\) \(X\) we associate \(X_C = \mathbb{C}^{-1} X \mathbb{C} \in \mathfrak{sp}(n, \mathbb{R})_\mathbb{C} = \mathfrak{sp}(n, \mathbb{C}) \cap \mathfrak{u}(n, n),\)

\[
(2.15) \quad X_C = \begin{pmatrix} p & q \\ \bar{q} & \bar{p} \end{pmatrix}, \quad p^* = -p, \quad q^* = q,
\]

where

\[
(2.16) \quad 2p = a - a^t + i(b - c); \quad 2q = a + a^t - i(b + c).
\]

The relations inverse to \((2.16)\) are

\[
(2.17) \quad 2a = p + \bar{p} + q + \bar{q}; 2b = i(\bar{p} - p + q - \bar{q}); 2c = i(p + q - \bar{p} - \bar{q}).
\]

Also, we have

\[
(2.18) \quad X \in \mathfrak{sp}(n, \mathbb{R})_\mathbb{C} \iff X = iKH, \quad H^* = H, \quad H \in M(2n, \mathbb{C}).
\]
d) To every $g \in \text{Sp}(n, \mathbb{R})$, we associate via (2.11) $g \mapsto g_\mathbb{C} \in \text{Sp}(n, \mathbb{R})_\mathbb{C} \equiv \text{Sp}(n, \mathbb{C}) \cap U(n, n)$

\begin{equation}
(2.19) \quad g_\mathbb{C} = \begin{pmatrix}
p & q \\
\overline{q} & \overline{p}
\end{pmatrix},
\end{equation}

where the matrices $p, q \in M(n, \mathbb{C})$ have the properties

\begin{align}
(2.20a) & \quad pp^* - qq^* = I_n, \quad pq^t = qp^t; \\
(2.20b) & \quad p^* p - q^* q = I_n, \quad p^t \overline{q} = q^* p.
\end{align}

e) The characteristic polynomial of a real Hamiltonian matrix is an even polynomial. If $\lambda$ is an eigenvalue of a Hamiltonian matrix with multiplicity $k$, so are $-\lambda, \bar{\lambda}, -\bar{\lambda}$ with the same multiplicity. Moreover, 0, if it occurs, has even multiplicity. If $A \in \mathfrak{sp}(n, \mathbb{R})$ has distinct eigenvalues $\lambda_1, \ldots, \lambda_m$, $-\lambda_1, \ldots, -\lambda_m$, there exists a symplectic matrix $S$ (possibly complex) such as $S^{-1}AS = \text{diag}(\lambda_1, \ldots, \lambda_m, -\lambda_1, \ldots, -\lambda_m)$.

The characteristic polynomial of a symplectic matrix is a reciprocal polynomial. If $\lambda$ is an eigenvalue of a real symplectic matrix with multiplicity $k$, so are $\lambda^{-1}, \bar{\lambda}, \bar{\lambda}^{-1}$ with the same multiplicity. Moreover, the multiplicities of the eigenvalues +1 and −1, if they occur, are even.

3. Coherent states on $\mathfrak{F}_n^J$

In order to fix the notation on coherent states [18], let us consider the triplet $(G, \pi, \mathfrak{F})$, where $\pi$ is a continuous, unitary representation of the Lie group $G$ on the separable complex Hilbert space $\mathfrak{F}$.

For $X \in \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of the Lie group $G$, let us define the (unbounded) operator $d \pi(X)$ on $\mathfrak{F}$ by $d \pi(X).v := d/dt|_{t=0} \pi(\exp tX).v$, whenever the limit on the right hand side exists. We obtain a representation of the Lie algebra $\mathfrak{g}$ on $\mathfrak{F}^\infty (\mathfrak{F}^\infty$ denotes the smooth vectors of $\mathfrak{F})$, the derived representation, and we denote $X.v := d \pi(X).v$ for $X \in \mathfrak{g}, v \in \mathfrak{F}^\infty$.

Let us now denote by $H$ the isotropy group. We consider (generalized) coherent states on complex homogeneous manifolds $M \cong G/H$ [18]. The coherent vector mapping is defined locally, on a coordinate neighborhood $\mathcal{V}_0$, $\varphi : M \to \mathfrak{F}$, $\varphi(z) = e_z$ (cf. [6] [7]), where $\mathfrak{F}$ denotes the Hilbert space conjugate to $\mathfrak{F}$. The vectors $e_z \in \mathfrak{F}$ indexed by the points $z \in M$ are called Perelomov’s coherent state vectors. Explicitly, $e_z = \exp(\sum_{\alpha \in \Delta_+} z_\alpha X_\alpha) e_0$, where $e_0$ is the extremal weight vector of the representation $\pi$, $\Delta_+$ are the positive roots of the Lie algebra $\mathfrak{g}$ of $G$, and $X_\alpha, \alpha \in \Delta$, are the generators [18].

The space $\mathfrak{F}_\mathfrak{F}$ of holomorphic functions is defined as the set of square integrable functions with respect to the scalar product

\begin{equation}
(3.1) \quad (f, g)_{\mathfrak{F}_\mathfrak{F}} = \int_M \overline{f(z)} g(z) \, d\nu_M(z, \bar{z}), \quad d\nu_M(z, \bar{z}) = \frac{\Omega_M(z, \bar{z})}{(e_z, e_\bar{z})}.
\end{equation}

Here $\Omega_M$ is the normalized $G$-invariant volume form

\begin{equation}
(3.2) \quad \Omega_M := (-1)^{\frac{n}{2}} \frac{1}{n!} (\omega \wedge \ldots \wedge \omega)_{n \text{ times}},
\end{equation}
and the $G$-invariant Kähler two-form $\omega$ on the $2n$-dimensional manifold $M$ is given by

$$\omega(z) = i \sum_{\alpha \in \Delta_+} g_{\alpha,\beta} dz_\alpha \wedge d \bar{z}_\beta,$$

(3.3)

$$g_{\alpha,\beta} = \frac{\partial^2}{\partial z_\alpha \partial \bar{z}_\beta} \log <e_z, e_\bar{z}> , \quad g_{\alpha,\beta} = \bar{g}_{\beta,\alpha}.$$

(3.4)

(3.1) is nothing else but Parseval overcompleteness identity [16].

Let us now introduce the map $\Phi : \mathfrak{H}^* \to \mathcal{F}_0$,

$$\Phi(\psi) := f_\psi, f_\psi(z) = \Phi(\psi)(z) = (\varphi(z), \psi)_g = (e_z, \psi)_g, \ z \in \mathcal{V}_0, \ \mathcal{V}_0 \subset M,$$

where we have identified the space $\mathfrak{H}$ complex conjugate to $\mathfrak{H}$ with the dual space $\mathfrak{H}^*$ of $\mathfrak{H}$.

Perelomov’s coherent state vectors [48] associated to the group $G_n^J$ with Lie algebra the Jacobi algebra $\mathfrak{g}_n^J$, based on the complex $N$-dimensional $(N = \frac{n(n+\beta)}{2})$ manifold - the Siegel-Jacobi ball $D_n^J := H_n/\mathbb{R} \times \text{Sp}(n, \mathbb{R})/U(n) = \mathbb{C}^n \times D_n$ - are defined as [10, 48]

$$e_{z,W} = \exp(X)e_0, \ X := \sum_{i=1}^n z_i a_i^+ + \sum_{i,j=1}^n w_{ij} K_{ij}^+, \ z \in \mathbb{C}^n; W \in D_n.$$

(3.6)

The non-compact hermitian symmetric space $\text{Sp}(n, \mathbb{R})/U(n)$ admits a matrix realization as a bounded homogeneous domain, the Siegel ball $D_n$

$$D_n := \{ W \in M(n, \mathbb{C}) : W = W^t, \mathbb{I}_n - W\bar{W} > 0 \}.$$

(3.7)

$D_n$ is a hermitian symmetric space of type CI (cf. Table V, p. 518, in [32]), identified with the symmetric bounded domain of type II, $\mathfrak{K}_H$ in Hua’s notation [34].

The vector $e_0$ appearing in (3.6) verifies the relations

$$a_i e_0 = 0, \ i = 1, \ldots, n; \ K_{ij}^+ e_0 \neq 0, \ K_{ij}^0 e_0 = 0, \ K_{ij}^0 e_0 = \frac{k_i}{4} \delta_{ij} e_0, \ i, j = 1, \ldots, n.$$

(3.8)

In (3.8), $e_0 = e_0^H \otimes e_0^K$, where $e_0^H$ is the minimum weight vector (vacuum) for the Heisenberg group $H_n$ with respect to the representation (2.3), while $e_0^K$ is the extremal weight vector for $\text{Sp}(n, \mathbb{R})$ corresponding to the weight $k$ in (3.8) with respect to a unitary representation $S$, see details in [10].

The following proposition describes the holomorphic action of the Jacobi group on the Siegel-Jacobi ball and some geometric properties of $D_n^J$ (cf. [10]):

**Proposition 1.** Let us consider the action $S(g)D(\alpha)e_{z,W}$, where $g \in \text{Sp}(n, \mathbb{R})$ is of the form (2.19), (2.20), $D(\alpha)$ is given by (2.3) and the coherent state vector is defined in (3.6). Then formulas (3.9) hold:

(3.9a) $S(g)D(\alpha)e_{z,W} = \lambda e_{z_1, W_1}, \ \lambda = \lambda(g, \alpha; z, W)$,

(3.9b) $W_1 = g \cdot W = (pW + q)(\bar{q}W + \bar{p})^{-1} = (Wq^* + p^*)^{-1}(q^* + Wp^*)$,

(3.9c) $z_1 = (Wq^* + p^*)^{-1}(z + \alpha - W\bar{\alpha})$,

(3.9d) $\lambda = \det(Wq^* + p^*)^{-k/2} \exp\left(\frac{k}{2}z - \frac{\bar{k}}{2}z_1\right) \exp(i\theta(\alpha, \eta))$,

(3.9e) $\eta = (\mathbb{I}_n - W\bar{W})^{-1}(z + W\bar{z})$,

(3.9f) $\eta_1 = p(\alpha + \eta) + q(\bar{\alpha} + \bar{\eta})$. 

The action of the Jacobi group \( G^J_n \) on the manifold \( \mathcal{D}^J_n \) is given by equations (3.9b), (3.9c). The composition law is
\[
(g_1, \alpha_1, t_1) \circ (g_2, \alpha_2, t_2) = (g_1 \circ g_2, g_2^{-1} \cdot \alpha_1 + \alpha_2, t_1 + t_2 + \mathfrak{S}(g_2^{-1} \cdot \alpha_1 \tilde{\alpha}_2)),
\]
and if \( g \) is as in (2.19), then \( g \cdot \alpha := \alpha_g \) is given by \( \alpha_g = p \alpha + q \tilde{\alpha} \), and \( g^{-1} \cdot \alpha = p^* \alpha - q^\dagger \tilde{\alpha} \).

The manifold \( \mathcal{D}^J_n \) has the Kähler potential (3.11), \( f = \log K \), with \( K \) given by (3.14),
\[
f = -\frac{k}{2} \log \det(\mathbb{I}_n - W\bar{W}) + \bar{z}'(\mathbb{I}_n - W\bar{W})^{-1} z
\]
\[+ \frac{1}{2} z' [\bar{W}(\mathbb{I}_n - W\bar{W})^{-1}] z + \frac{1}{2} z'[\mathbb{I}_n - W\bar{W}] z.
\]

The Kähler two-form \( \omega_n \), deduced as in (3.3), \( G^J_n \)-invariant to the action (3.9b), (3.9c) is
\[
-i \omega_n = \frac{k}{2} \text{Tr}(B \wedge \bar{B}) + \text{Tr}(A^t \bar{M} \wedge \bar{A}), A = dz + dW\eta,
\]
\[B = M dW, M = (\mathbb{I}_n - W\bar{W})^{-1}, \eta = M(z + W\bar{z}).
\]

The scalar product \( K : M \times \bar{M} \to \mathbb{C} \), \( K(\bar{x}, \bar{V}; y, W) = (e_{x,V}, e_{y,W})_k \) is:
\[
(e_{x,V}, e_{y,W})_k = \det(U)^{k/2} \exp F(\bar{x}, \bar{V}; y, W), \ U = (\mathbb{I}_n - W\bar{V})^{-1};\]
\[2F(\bar{x}, \bar{V}; y, W) = 2(\bar{x}, U\bar{y}) + (\bar{V}\bar{y}, U\bar{y}) + (\bar{x}, W\bar{U}\bar{x}).
\]

In particular, the reproducing kernel \( K = (e_{z,W}, e_{z,W}) \) is
\[
K = \det(M) \frac{k}{2} \exp F, M = (\mathbb{I}_n - W\bar{W})^{-1},
\]
\[2F = 2\bar{z}'Mz + \bar{z}'\bar{W}Mz + \bar{z}'MWz.
\]

The Hilbert space of holomorphic functions \( \mathcal{F}_K \) associated to the holomorphic kernel \( K \) given by (3.14) is endowed with the scalar product of the type (3.11)
\[
(\phi, \psi) = \Lambda_n \int_{z \in \mathbb{C}^n; z \cdot W = 0} \bar{f}_\phi(z, W) f_\psi(z, W) QK^{-1} \, dz \, dW,
\]
where the normalization constant \( \Lambda_n \) is given by (3.17)
\[
\Lambda_n = \frac{k - 3}{2\pi^{n(n+1)/2}} \prod_{i=1}^{n-1} \frac{(k-3-2n+i)\Gamma(k+i-2)}{\Gamma[k+2(i-n-1)]}.\]

Comparatively with the case of the symplectic group, a shift of \( p \) to \( p - 1/2 \) in the normalization constant (4.12) \( \Lambda_n = \pi^{-n} J^{-1}(p) \) is obtained. We write down the scalar product (3.16) also as \( (p = k/2 - n - 2) \)
\[
(\phi, \psi) = \Lambda_n \int_{z \in \mathbb{C}^n; W \in \mathcal{D}_n} \bar{f}_\phi(z, W) f_\psi(z, W) \rho_1 \, dz \, dW, \rho_1 = \det(\mathbb{I}_n - W\bar{W})^p \exp -F.
4. The differential action

Let us consider again the triplet $\langle G, \pi, \mathcal{F} \rangle$ introduced at the beginning of §3. The unitarity and the continuity of the representation $\pi$ imply that $\text{id} \pi(X)|_{\mathcal{F}^0}$ is essentially selfadjoint. Let us denote his image in $B_0(\mathcal{F}^\infty)$ by $A_M := \text{id} \pi(\mathcal{U}(\mathfrak{g}_C))$, where $B_0$ denotes the set of linear operators with formal adjoint, and $\mathcal{U}(\mathfrak{g}_C)$ denotes the universal covering algebra. If $\Phi : \mathcal{F}^* \to \mathcal{F}_A$ is the isometry \textit{(3.5)}, we are interested in the study of the image of $A_M$ via $\Phi$ as subset in the algebra of holomorphic, linear differential operators, $\Phi A_M \Phi^{-1} := A_M \subset \mathfrak{D}_M$.

The set $\mathfrak{D}_M$ (or simply $\mathfrak{D}$) of holomorphic, finite order, linear differential operators on $M$ is a subalgebra of homomorphisms $\mathcal{Hom}_{\mathbb{C}}(\mathcal{O}_M, \mathcal{O}_M)$ generated by the set $\mathcal{O}_M$ of germs of holomorphic functions of $M$ and the vector fields. We consider also the subalgebra $\mathfrak{A}_M$ of $A_M$ of differential operators with holomorphic polynomial coefficients. Let $U := \mathcal{V}_0 \subset M$, endowed with the local coordinates $(z_1, z_2, \cdots, z_n)$. We set $\partial_i := \frac{\partial}{\partial z_i}$ and $\partial^\alpha := \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_n^{\alpha_n}$, $\alpha := (\alpha_1, \alpha_2, \cdots, \alpha_n) \in \mathbb{N}^n$. The sections of $\mathfrak{D}_M$ on $U$ are $A : f \mapsto \sum a_\alpha \partial^\alpha f$, $a_\alpha \in \Gamma(U, \mathcal{O})$, $a_\alpha$-s being zero except a finite number.

For $k \in \mathbb{N}$, let us denote by $\mathfrak{D}_k$ the subset of differential operators of degree $\leq k$. The filtration of $\mathfrak{D}$ induces a filtration on $\mathfrak{A}$.

Summarizing, we have a correspondence between the following three objects \textit{[6, 7]}:

\begin{equation}
\mathfrak{g}_C \ni X \mapsto X \in A_M \mapsto \mathfrak{X} \in A_M \subset \mathfrak{D}_M, \text{ differential operator on } \mathcal{F}_A
\end{equation}

Moreover, it is easy to see \textit{[6]} that if $\Phi$ is the isometry \textit{(3.5)}, then $\Phi d\pi(\mathfrak{g}_C) \Phi^{-1} \subset \mathfrak{D}_1$ and we have

\begin{equation}
\mathfrak{g}_C \ni X \mapsto \mathfrak{X} \in \mathfrak{D}_1; \mathfrak{X}_z(f_\psi(z)) = \mathfrak{X}_z(e_\bar{z}, \psi) = (e_\bar{z}, X \psi),
\end{equation}

where

\begin{equation}
\mathfrak{X}_z(f_\psi(z)) = \left( P_X(z) + \sum Q_X(z) \frac{\partial}{\partial z_i} \right) f_\psi(z).
\end{equation}

In \textit{[6, 7]} we have advanced the hypothesis that for coherent state groups the holomorphic functions $P$ and $Q$ in \textit{(1.3)} are polynomials, i.e. $\mathfrak{A} \subset \mathfrak{A}_1 \subset \mathfrak{D}_1$.

In particular, the Jacobi algebra $\mathfrak{g}_J^*$ admits a realization in the space $\mathfrak{D}_1$ of holomorphic first order differential operators with polynomial coefficients, defined on the Siegel-Jacobi ball $\mathcal{D}_J^\prime$. The space of holomorphic functions on which the differential operators act is the space denoted $\mathcal{F}_K$ in Proposition \textit{[1]}.

For explicit realization of the representation \textit{[1, 5, 7]}, use is made of the formula $\text{Ad}(\exp X) = \exp(\text{ad}_X)$, i.e.

\begin{equation}
A e^X = e^X(A - [X, A] + \frac{1}{2} [X, [X, A]] + \ldots ),
\end{equation}

In order to take into account the symmetry of the matrix $W$ appearing in \textit{(3.6)}, we use the derivation formula:

\begin{equation}
\frac{\partial w_{ij}}{\partial w_{pq}} = \delta_{ip} \delta_{jq} + \delta_{iq} \delta_{jp} - \delta_{ij} \delta_{pq} \delta_{ip}, \ w_{ij} = w_{ji}.
\end{equation}

With \textit{(4.4)} and taking into account the commutation relations \textit{(2.2)}, \textit{(2.7)} and \textit{(2.8)} of the generators of the Jacobi algebra, we get the relations \textit{(4.0)} (see also §2.4, 3.3 in
where $\chi$ has been omitted, and \([49]\):

\begin{align}
(4.6a) \quad a^+_k e_{z,W} &= \frac{\partial}{\partial z_k} e_{z,W}, \quad a_k e_{z,W} = \left( z_k + w_{ki} \frac{\partial}{\partial z_i} \right) e_{z,W}; \\
(4.6b) \quad K^0_{kl} e_{z,W} &= \left( \frac{k_k}{4} \delta_{kl} + \frac{z_k}{2} \frac{\partial}{\partial z_k} + w_{kl} \nabla_{ik} \right) e_{z,W}, \quad K^+_k e_{z,W} = \nabla_{kl} e_{z,W}; \\
(4.6c) \quad K^-_{kl} e_{z,W} &= \left[ \frac{k_k + k_l}{4} w_{kl} + \frac{z_k z_l}{2} + \frac{1}{2} (z_i w_{ik} + z_k w_{il}) \frac{\partial}{\partial z_i} + w_{al} w_{kl} \nabla_{ia} \right] e_{z,W}.
\end{align}

Now we briefly recall some general considerations (for details see \([6, 8]\) and Lemma 1 in \([7]\)). For $X \in \mathfrak{g}$, let $X_e z := X z e_z$. Then $X \cdot z = X z e_z$.

But $(e_z, X_e e_z') = (X^+ e_z, e_z')$ and finally, with equation \((4.2)\), we have

\begin{equation}
X_e z'(e_z, e_z') = \mathcal{X}_1 (e_z, e_z')
\end{equation}

Using \((4.6)\) and the relation expressed in \((4.7)\), we have

**Lemma 1.** The Jacobi algebra $\mathfrak{g}^J_n$ admits a realization in the space $\mathcal{D}_1$ of differential operators on $\mathcal{D}_n$.

\begin{align}
(4.8a) \quad a_k &= \frac{\partial}{\partial z_k}, \quad a^+_k = z_k + w_{ki} \frac{\partial}{\partial z_i}; \\
(4.8b) \quad K^0_{kl} &= \frac{k_k}{4} \delta_{kl} + \frac{z_k}{2} \frac{\partial}{\partial z_k} + w_{kl} \nabla_{ik}, \quad K^+_k = \nabla_{kl}; \\
(4.8c) \quad K^-_{kl} &= \frac{k_k + k_l}{4} w_{kl} + \frac{z_k z_l}{2} + \frac{1}{2} (z_i w_{ik} + z_k w_{il}) \frac{\partial}{\partial z_i} + w_{al} w_{kl} \nabla_{ia}.
\end{align}

In the formulae above, $k, l = 1, \ldots, n, w_{kl} = w_{lk}, \frac{\partial}{\partial w_{kl}} = \frac{\partial}{\partial w_{lk}}$, and the dummy summation is on all indexes $1, \ldots, n$.

With the convention $\nabla = (\nabla_{ij})_{i,j=1,\ldots,n} = (\chi_{ij} \frac{\partial}{\partial w_{ij}})_{i,j=1,\ldots,n}$, formulae \((4.8)\) can be also written as

\begin{align}
(4.9a) \quad a &= \frac{\partial}{\partial z}, \quad a^+ = z + W \frac{\partial}{\partial z}; \\
(4.9b) \quad \mathbb{K}^- = \nabla_W, \quad \mathbb{K}^0 = \frac{k_k}{4} + \frac{1}{2} \frac{\partial}{\partial z} \otimes z + \nabla_W W; \\
(4.9c) \quad \mathbb{K}^+ = W' + \frac{1}{2} z \otimes z + \frac{1}{2} (W \frac{\partial}{\partial z} \otimes z + z \otimes \frac{\partial}{\partial z} W) + W \nabla_W W.
\end{align}

In \((4.9)\) $A \otimes B$ denotes the Kronecker product of matrices, here $(A \otimes B)_{kl} = a_k b_l$, $A = (a_k)$, $B = (b_l)$, $k = \text{diag}(k_1, \ldots, k_n)$, $w_{kl}' = (k_k + k_l) w_{kl}$, $k, l = 1, \ldots, n$.

We particularize the values of the operators given in Lemma 1 in the case of the action of $\text{Sp}(n, \mathbb{R}) \subseteq \mathcal{D}_n$ and we have

**Lemma 2.** The algebra $\mathfrak{sp}(n, \mathbb{R})$ defined by the commutations relations \((2.8)\) admits the realization in differential operators on the Siegel ball $\mathcal{D}_n$.

\begin{equation}
K^0_{kl} = \frac{k_k}{4} \delta_{kl} + w_{kl} \nabla_{il}, \quad K^-_{kl} = \nabla_{kl}, \quad K^+_k = \frac{k_k + k_l}{4} w_{kl} + w_{al} w_{kl} \nabla_{ia}.
\end{equation}
The generators of \( \text{Sp}(n, \mathbb{R})_{\mathbb{C}} \) have the hermiticity properties

\[
(4.11) \quad (K^+)^t_{kl} = K_{k'l()}, \quad (K^0)^t_{kl} = K_{k'l}^0
\]

with respect to the scalar product \((10)\).

\[
(4.12) \quad (\phi, \psi)_{\mathcal{H}} = \Lambda_n' \int_{1-W\bar{W}>0} \bar{f}_\phi(W) f_\psi(W) \rho \, dW,
\]

where \( k_i = k, i = 1, \ldots, n \) in \((3.8)\).

\[
(4.13) \quad \rho = \det(\mathbb{I}_n - W\bar{W})^q, \quad q = \frac{k}{2} - n + 1
\]

and \( \Lambda_n' = J_n^{-1}(q) \), with \( J_n(p) \) given by \((4.13)\), \( p > -1 \).

\[
(4.14) \quad J_n(p) = 2^n \pi^{\frac{n(n+1)}{2}} \prod_{i=1}^{n} \frac{\Gamma(2p + 2i)}{\Gamma(2p + n + i + 1)} = \frac{\pi^{n(n+1)/2}}{p+n} \prod_{i=1}^{n} \frac{\Gamma[2(i+p) - (i+2p+n-1)]}{\Gamma[i+2(p+n-1)]}.
\]

**Proof.** Firstly we have to check up that the operators \((4.10)\) verify the commutation relations \((2.8)\). This is a long calculation, based on the formula \((4.5)\).

Then we verify the relation \((K^+_k, f, g)_{\mathcal{H}} = (f, K^-_k, g)_{\mathcal{H}}\), which imposes to the reproducing kernel \( \rho \) the condition

\[
(4.15) \quad (\nabla_{kl} + q\bar{w}_{kl} - \bar{w}_{kl}\bar{w}_{ij}\nabla_{ij}) \rho = 0.
\]

The hermiticity condition \((K^0)^+_kl = K^0_{k'l}\) of the operators given by the first formula \((4.10)\) with respect to the scalar product \((4.12)\), with \( \rho \) given by \((4.13)\), imposes to the kernel function \( \rho \) the condition:

\[
(4.16) \quad (\bar{w}_{kp}\nabla_{lp} - w_{lp}\nabla_{kp}) \rho = 0.
\]

The conditions \((4.15)\) and \((4.16)\) are verified by the kernel \((4.13)\) using the relation

\[
(4.17) \quad \frac{\partial}{\partial w_{ik}} A = (-2X_{ik} + X_{ik}\delta_{ik})A, \quad \text{or} \quad \nabla A = -X, \quad \text{where} \quad A = \det(\mathbb{I}_n - W\bar{W}),
\]

which implies

\[
(4.18) \quad \frac{\partial \rho}{\partial w_{ab}} = q(-2X_{ab} + X_{ab}\delta_{ab})\rho, \quad \text{or} \quad \nabla \rho = -qX\rho, \quad \text{where} \quad X = X^t = \bar{W}(\mathbb{I}_n - W\bar{W})^{-1}.
\]

Indeed, with \((4.18)\), the condition \((4.15)\) reads \(-X + \bar{W} + \bar{W}X\bar{W} = 0\), while \((4.16)\) reads \((\bar{X}\bar{W})^t = \bar{XW}\). The last two conditions are verified because of the symmetry of the matrices \(X\) and \(W\).

\[\square\]

**Lemma 3.** The pairs of operators \(a^\dagger\) and \(a\), \(K^+_kl\) and \(K^-_{k'l}\), \(K^0_{k'l}\) and \(K^0_{k'l}\) are respectively hermitian conjugate with respect the scalar product \((3.19)\) for \( k_i = k \) in \((3.8)\).

**Proof.** We take the derivative of \((3.15)\) with respect with \(z_i\) and we find successively

\[
\frac{\partial F}{\partial z_i} = \bar{z}_p M_{pi} + \frac{1}{2}[(\bar{W}M)_{iq}z_q + z_p(\bar{W}M)_{pi}]
\]

\[= [M^t\bar{z} + \frac{1}{2}(\bar{W}Mz + \bar{M}Wz)]_i,
\]

\[= [M^t(\bar{z} + \bar{W}z)]_i,\]
and we introduce $\eta = M(z + W\bar{z})$, $M = (I_n - W\bar{W})^{-1}$. We get

$$\frac{\partial F}{\partial z_i} = \bar{\eta}_i,$$

$$\frac{1}{\rho_1} \frac{\partial \rho_1}{\partial z_i} = -\bar{\eta}_i.$$ 

We look for the derivative of $\rho_1$ from (3.19) with respect to the $w_{ik}$, and we have

$$\frac{\partial \rho_1}{\partial w_{ik}} = \bar{\eta}_i \bar{\eta}_k - \frac{1}{2} \bar{\eta}_i \bar{\eta}_k \delta_{ik}, \quad \text{or} \quad \nabla \rho_1 = \frac{1}{2} \bar{\eta} \otimes \bar{\eta}.$$ 

Indeed, with formula (4.5), we get

$$\frac{\partial M_{ab}}{\partial w_{ik}} = M_{ai} X_{kb} + M_{ak} X_{ib} - M_{ai} X_{ib} \delta_{ik},$$

$$\frac{\partial X_{ab}}{\partial w_{ik}} = X_{ai} X_{bk} + X_{ak} X_{ib} - X_{ai} X_{ib} \delta_{ik},$$

$$\frac{\partial \bar{X}_{ab}}{\partial w_{ik}} = M_{ai} M_{bk} + M_{ak} M_{bi} - M_{ai} M_{bk} \delta_{ik}.$$ 

In order to prove (4.23c), we use the fact that $\bar{X} = MW$, then we use (4.23a), (4.5), and the formula $I_n + XW = M = M^t$.

We write (3.15) as

$$2F = 2\bar{z}^t M z + \bar{z}^t X z + \bar{z}^t \bar{X} \bar{z}$$

and, with (4.23), we get

$$2\frac{\partial F}{\partial w_{ik}} = 2[(\bar{z}^t M)_i (X z)_k + (\bar{z}^t M)_k (X z)_i - (\bar{z}^t M)_i (X z)_k \delta_{ik}] +$$

$$+ (\bar{z}^t X)_i (X z)_k + (\bar{z}^t X)_k (X z)_i - (\bar{z}^t X)_i (X z)_k \delta_{ik} +$$

$$+ (\bar{z}^t M)_i (\bar{z}^t M)_k (\bar{z}^t M)_i - (\bar{z}^t M)_i (\bar{z}^t M)_k \delta_{ik}.$$ 

Then we use twice the relations $\bar{z}^t M + \bar{z}^t X = \bar{\eta}$, and (4.22) get proved.

With (4.17) and (4.22), we have for (4.21) the expression

$$-\frac{1}{\rho_1} \frac{\partial \rho_1}{\partial w_{ik}} = p(2X_{ik} - X_{ik} \delta_{ik}) + \bar{\eta}_i \bar{\eta}_k - \frac{1}{2} \bar{\eta}_i \bar{\eta}_k \delta_{ik}, \quad \text{and} \quad -\frac{1}{\rho_1} \nabla \rho_1 = pX + \frac{1}{2} \bar{\eta} \otimes \bar{\eta}.$$ 

We have to verify $(a_k f, g) = (f, a_k^t g)$ with respect to the scalar product (3.19), i.e.

$$\frac{\partial F}{\partial z_k} = \frac{\partial F}{\partial z_i} = z_k + w_{ki} \frac{\partial F}{\partial z_i}.$$ 

With (4.20), the last condition reads $\eta = z + W \bar{\eta}$, which is true.

In order to verify $\langle K^0_{kl}, f, g \rangle = (f, K^0_{kl} g)$ for the case $k_i = k$ in formula (3.8) with respect to the scalar product (3.19), we use the differential action for $K^0_{kl}$ in Lemma 1.
If we denote the integrant in the second term by \( f_{kl} \), using \((4.22), (4.23), (1.3) \) and the formula \( z = \eta - W\bar{\eta} \), we find
\[
\frac{f_{kl}}{\rho_1} = \frac{p}{2} \delta_{kl} + \frac{1}{2} \bar{\eta}\eta_{kl} + p(WMW)_{lk},
\]
and \( f_{kl} = f_{lk} \), because the symmetry of the matrices \( W \) and \( X \).

We also find for the integrant of \((K_{kl}^l f, g) = (f, K_{kl}^l g)\) the common value \( [p\bar{X}_{kl} + \frac{1}{2} \bar{\eta} \eta_{kl}] \rho_1 ]\).

5. The real Jacobi group \( G^J_n(\mathbb{R}) \)

We consider the real Jacobi group \( G^J_n(\mathbb{R}) = \text{Sp}(n, \mathbb{R}) \ltimes H_n \), where \( H_n \) is now the real Heisenberg group of real dimension \((2n + 1)\). Let \( g = (M, X, k), g' = (M', X', k') \in G^J_n(\mathbb{R}) \), where \( X = (\lambda, \mu) \in \mathbb{R}^{2n} \) and \( (X, k) \in H_n \). Then the composition law in \( G^J_n(\mathbb{R}) \)
is
\[
(5.1) \quad gg' = (MM', XM' + X', k + k' + XM'JX'^t).
\]
We shall also consider the restricted real Jacobi group \( G^J_n(\mathbb{R})_0 \), consisting only of elements of the form above, but \( g = (M, X) \).

We consider also the Siegel-Jacobi upper half plane \( \mathcal{X}_n^J := \mathcal{X}_n \times \mathbb{R}^{2n} \), where \( \mathcal{X}_n = \text{Sp}(n, \mathbb{R}) / U(n) \) is Siegel upper half plane realized as
\[
\mathcal{X}_n := \{ v \in M(n, \mathbb{C}) | v = s + iv, s, r \in M(n, \mathbb{R}), r > 0, s^t = s; r^t = r \}.
\]

Let us consider an element \( h = (g, l) \) in \( G^J_n(\mathbb{R})_0 \), i.e.
\[
(5.2) \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Sp}(n, \mathbb{R}), \ l = (n, m) \in \mathbb{R}^{2n},
\]
and \( v \in \mathcal{X}_n, \ u \in \mathbb{C}^n = \mathbb{R}^{2n} \).

Now we consider the partial Cayley transform \([10] \) \( \Phi : \mathcal{X}_n^J \rightarrow \mathcal{D}_n^J, \ \Phi(v, u) = (W, z) \)
\[
(5.3a) \quad W = (v - iI_n)(v + iI_n)^{-1},
(5.3b) \quad z = 2i(v + iI_n)^{-1}u,
\]
with the inverse partial Cayley transform \( \Phi^{-1} : \mathcal{D}_n^J \rightarrow \mathcal{X}_n^J, \ \Phi^{-1}(W, z) = (v, u) \)
\[
(5.4a) \quad v = i(I_n - W)^{-1}(I_n + W),
(5.4b) \quad u = (I_n - W)^{-1}z.
\]

Let us now define \( \Theta : G^J_n(\mathbb{R})_0 \rightarrow G^J_n, \ \Theta(h) = h_*, \ h = (g, n, m), \ h_* = (g_\mathbb{C}, \alpha) \). We shall verify that (see also \([66] [12] \))

**Proposition 2.** \( \Theta \) is an group isomorphism and the action of \( G^J_n(\mathbb{R})_0 \) on \( \mathcal{X}_n^J \) through the biholomorphic partial Cayley transform \([5.3] \), i.e. if \( \Theta(h) = h_* \), then \( \Phi h = h_\Phi \). More exactly, if the action of \( G^J_n(\mathbb{R})_0 \) on \( \mathcal{D}_n^J \) is given by \((3.9b), (3.9c) \), then the action of \( G^J_n(\mathbb{R})_0 \) on \( \mathcal{X}_n^J \) is given by \((g, l) \times (v, u) \rightarrow (v_1, u_1) \in \mathcal{X}_n^J \), where
\[
(5.5a) \quad v_1 = (av + b)(cv + d)^{-1} = (vc' + d')^{-1}(va' + b');
(5.5b) \quad u_1 = (vc' + d')^{-1}(u + vn + m).
\]
The matrices $g$ in (5.2) and $g_C$ in (2.19) are related by (2.11), (2.13), while $\alpha = m + in$, $m, n \in \mathbb{R}^n$.

**Proof.** We introduce for $W$ in (3.9b) its expression from (5.3a), and we get

$$W_1 = [(p + q)v + i(q - p)][(\bar{q} + \bar{p})v + i(\bar{p} - \bar{q})]^{-1}. $$

The expression of $W_1$ is introduced in the inverse (5.4a) of (5.3a) for $v_1$. Use is made of (2.13) and the first equality in (5.5a) is obtained. The second one is a consequence of the symmetry of $v$, and it can be directly checked up with equations (2.10).

For the second assertion, we start with (5.3b), $2i u_1 = (v_1 + i\mathbb{I}_n)z_1$, in which we introduce the expression (3.9c) for $z_1$. But with (5.4a), $v_1 + i\mathbb{I}_n = 2i(\mathbb{I}_n - W_1)^{-1}$, so we get

$$u_1 = (\mathbb{I}_n - W_1)^{-1}(Wq^* + p^*)[2i(v + i\mathbb{I}_n)^{-1}u + \alpha - W\alpha].$$

In the above expression we write $W_1$ as function of $W$ with the linear fractional transform (3.9b) and express $W$ as function of $v$ with the Cayley transform (5.3a). We replace $\alpha = m + in$, $m, n \in \mathbb{R}^n$, and express the matrix elements of $g_C$ in function of the matrix elements of $g$ via the relations (2.13) and we get also formula (5.5b). \[\square\]

**Proposition 3.** The partial Cayley transform is a Kähler homogeneous diffeomorphism, $\Phi^* \omega_n = \omega_n' = \omega_n \circ \Phi$, i.e. under the transform (5.3), the Kähler two-form (3.12) on $D_n^j$, $G_n^j$-invariant under the action (3.9b), (3.9c), becomes the Kähler two-form $\omega_n'$ (5.6) on $X_n^j$, $G_n^j(\mathbb{R})$-invariant to the action (5.6)

$$-i\omega_n' = \frac{k}{2}\text{Tr}(H \wedge \bar{H}) + \frac{2}{1}\text{Tr}(G' D \wedge \bar{G}),$$

where

$$D = (\bar{v} - v)^{-1}, H = D d v; G = d u - d v D(\bar{u} - u).$$

**Proof.** The expression (5.6) was deduced firstly in [9]. Here we shortly indicate how to check up the group invariance. We calculate only the second term in the sum (5.6) because the first one is well known.

Differentiating (5.5b), we get

$$d u_1 = (vc^t + d^t)^{-1}[d u + d v(cv + d)^{-1}(dn - c(u + m))].$$

Now we calculate $\Psi = (v_1 - \bar{v}_1)^{-1}(u_1 - \bar{u}_1)$. Starting from (5.3a) and taking into account (2.10), we have

$$v_1 - \bar{v}_1 = (\bar{v}c^t + d^t)^{-1}(v - \bar{v})(cv + d)^{-1}.$$  

Using

$$(\bar{v}c^t + d^t)(vc^t + d^t)^{-1} = \mathbb{I}_n = (\bar{v} - v)c'(vc^t + d^t)^{-1},$$

$$(\bar{v}c^t + d^t)(vc^t + d^t)^{-1}v - \bar{v} = (v - \bar{v})[\mathbb{I}_n + c'(d^t)^{-1}v]^{-1},$$

we get

$$\Psi = dn - cm + (cv + d)(v - \bar{v})^{-1}[(\bar{v}c^t + d^t)(vc^t + d^t)^{-1}u - \bar{u}].$$

Taking into account (2.10a) in the differential of (5.5b), we get

$$d v_1 = (vc^t + d^t) d v(cv + d)^{-1},$$

and we find

$$G_1 = (vc^t + d^t)^{-1}(d u + d v\Xi),$$

where $\Xi = 2i(T_c + T_d)$. \[\square\]
where

\[ \Xi = -(v - \bar{v})^{-1}Y + (v - \bar{v})^{-1}\bar{u}, \]

\[ Y = (v - \bar{v})(cv + d)^{-1}c + (\bar{v}c^t + \bar{d}^t)(vc^t + d^t)^{-1}. \]

Using relations of the type \( v(cv + d)^{-1}c = (\mathbb{I}_n + vd^{-1}c)^{-1}vd^{-1}c, \) it can be shown that \( Y = \mathbb{I}_n, \) and we get \( G_1 = (vc^t + d^t)^{-1}G, \) and the invariance of the second term in formula (5.6) is proved. Then the invariance of \( \omega'_n \) under the action of (5.3) follows.

\( \omega'_n \) given by (5.6) is the “\( n \)”-dimensional generalization of Berndt-Kähler two-form \( \omega'_1. \) In §37 in [36] Kähler calls \( \mathcal{X}^J_n \) Phasenraum der Materie, \( v \) is Pneuma, \( u \) is Soma.

### 5.1. Comparison with Yang’s results

J.-H. Yang [65]-[68] considers the Siegel-Jacobi space of degree \( n \) and order \( m, \) \( H_{n,m} = \mathcal{X}_n \times \mathbb{C}^m, \) the Heisenberg group

\[ H_{\mathbb{R}}^{(n,m)} = \{(\lambda, \mu, \kappa) | \lambda, \mu \in M_{mn}(\mathbb{R}), \kappa \in M_m(\mathbb{R})\}, \]

and the Jacobi group \( G^J = Sp(n, \mathbb{R}) \times H_{\mathbb{R}}^{(n,m)}, \) with the multiplication law

\[ (M_0, (\lambda_0, \mu_0, \kappa_0)) \cdot (M, (\lambda, \mu, \kappa)) = (M_0M, (\lambda_0 + \lambda, \mu_0 + \mu, \kappa_0 + \kappa + \lambda_0\mu^t - \mu_0\lambda^t)), \]

where \((\lambda_0, \mu_0) = (\lambda_0, \mu_0)M. \) \( G^J \) acts transitively on the Siegel-Jacobi space \( H_{n,m} \) by

\[ (M, (\lambda, \mu, \kappa))(\Omega, Z) = (M \circ \Omega, (Z + \lambda\Omega + \mu)(C\Omega + D)^{-1}), \]

and \( G^J/K^J \cong H_{n,m} \) is a nonreductive complex manifold, where \( K^J = U(n) \times \text{Sym}(n, \mathbb{R}). \)

Now we identify \((v, u)\) in our partial Cayley transform (11) in [67],

\[ \Omega = i(\mathbb{I}_n + W)(\mathbb{I}_n - W)^{-1}; Z = 2i\eta(\mathbb{I}_n - W)^{-1}; \]

\[ (v, u) \leftrightarrow (\Omega, \frac{Z^t}{2i}) \quad \text{and} \quad (W, z) \leftrightarrow (W, \eta'). \]

**Remark 2.** The Kähler two-form in the case \( m = 1 \) in Theorem 1 in [67] is the Kähler two-form on \( \mathcal{X}^J_n \) (5.6), while the Kähler two-form on \( \mathcal{D}^J_n \) (3.12) is the corresponding one given in Theorem 6 in [67].

**Proof.** We use also Yang’s notation \( \Omega = X + iY; Z = U + iV \) and we express our (5.6) in Yang’s notation as

\[ D = -\frac{1}{2i}Y^{-1}; \quad H = (-2iY)^{-1}d\Omega; \quad G = \frac{dZ^t}{2i} - d\Omega Y^{-1}V^t, \]

\[ -iw'_n = \frac{k}{8} \text{Tr}(Y^{-1}d\Omega \wedge Y^{-1}d\Omega) + \frac{1}{8} \text{Tr}[(dZ - VY^{-1}d\Omega)Y^{-1} \wedge (dZ^t - d\Omega Y^{-1}V^t)], \]

\[ -iw'_n = \frac{k}{8} \text{Tr}(Y^{-1}d\Omega \wedge Y^{-1}d\Omega) + \frac{1}{8} \text{Tr}(dZY^{-1} \wedge d\bar{Z}^t) + \frac{1}{8} \text{Tr}(VY^{-1}d\Omega Y^{-1} \wedge d\bar{Z}^t) \]

\[ -\frac{1}{8} \text{Tr}(dZY^{-1} \wedge d\bar{O}Y^{-1}V^t) - \frac{1}{8} \text{Tr}(VY^{-1}d\Omega Y^{-1} \wedge d\bar{Z}^t). \]

The second term in the sum above reads

\[ dZ_{ij}Y^{-1} \wedge d\bar{Z}_{ji} = Y^{-1}dZ_{ij} \wedge d\bar{Z}_{ik} \]

i.e. the corresponding term in Yang’s formula and similarly for the other 3 terms in the sum. \( \square \)
6. The fundamental conjecture for the Siegel-Jacobi domains

Let us remind the fundamental conjecture for homogeneous Kähler manifolds (Gindikin -Vinberg): every homogenous Kähler manifold is a holomorphic fiber bundle over a homogenous bounded domain in which the fiber is the product of a flat homogenous Kähler manifold and a compact simply connected homogenous Kähler manifold. The compact case was considered by Wang [61]; Borel [21] and Matsushima [45] have considered the case of a transitive reductive group of automorphisms, while Gindikin and Vinberg [60] considered a transitive automorphism group. We mention also the essential contribution of Piatetski-Shapiro in this field [22]. The complex version, in the formulation of Dorfmeister and Nakajima [24], essentially asserts that: every homogenous Kähler manifold, as a complex manifold, is the product of a compact simply connected homogenous manifold (generalized flag manifold), a homogenous bounded domain, and \( \mathbb{C}^n/\Gamma \), where \( \Gamma \) denotes a discrete subgroup of translations of \( \mathbb{C}^n \).

**Proposition 4.** Under the homogeneous Kähler transform \( FC \) (6.1),

\[
C^n \times D_n \ni (\eta, W) \xrightarrow{FC} (z, W) \in D'_n, \quad FC(\eta, W) = (z, W), \quad z = \eta - W\bar{\eta},
\]

(6.2) \( FC^{-1} : \eta = (\bar{I}_n - W\bar{W})^{-1}(z + W\bar{z}). \)

the Kähler two-form (3.12) on \( D'_n \), \( G'_n \)-invariant to the action (3.9b), (3.9c), becomes the Kähler two-form on \( D_n \times \mathbb{C}^n \), \( FC^*_n \omega_n = \omega_{n,0} \),

\[
- i\omega_{n,0} = \frac{k}{2} \text{Tr}(B \wedge B) + \text{Tr}(d\eta^2 \wedge d\bar{\eta}),
\]

invariant to the \( G'_n \)-action on \( D_n \times \mathbb{C}^n \), \( (g, \alpha) \cdot (\eta, W) \rightarrow (\eta_1, W_1) \), with \( W_1 \) given in (3.9c) and

\[
\eta_1 = p(\eta + \alpha) + q(\bar{\eta} + \bar{\alpha}).
\]

Under the homogeneous Kähler transform

\[
FC^{-1}_1 : \eta = (\bar{v} - i\bar{I}_n)(\bar{v} - v)^{-1}(v - i\bar{I}_n)^{-1}u - (\bar{v} - i\bar{I}_n)^{-1}\bar{u}.
\]

the Kähler two-form (5.6) becomes a Kähler two-form on \( \mathcal{X}_n \times \mathbb{C}^n \), \( FC^*_1 \omega'_n = \omega'_{n,0} \),

\[
- i\omega'_{n,0} = \frac{k}{2} \text{Tr}(H \wedge \bar{H}) + \text{Tr}(d\eta^2 \wedge d\bar{\eta}), \quad H = (\bar{v} - v)^{-1}d\bar{v}.
\]

The inverse transform of (6.5) is

\[
FC_1 : u = \frac{1}{2i}[(v + i\bar{I}_n)\eta - (v - i\bar{I}_n)\bar{\eta}].
\]

The Kähler two-form (6.6) is invariant to the action \( G'_n(\mathbb{R})_0 \) on \( \mathcal{X}_n \times \mathbb{C}^n \), \( (g, \alpha) \times (v, \eta) \rightarrow (v_1, \eta_1) \), where \( g \) has the form (5.2), \( v_1 \) is given by (5.5a), while

\[
\eta_1 = \frac{1}{2}(\eta + \alpha)[a + d + i(b - c)] + \frac{1}{2}(\bar{\eta} + \bar{\alpha})[a - d - i(b + c)].
\]

**Proof.** Following [36], [26], we introduce the variables \( P, Q \in \mathbb{R}^n \) such that \( u = vP + Q \), where \((u, v) \in \mathbb{C}^n \times \mathcal{X}_n \) are local coordinates on the Siegel-Jacobi upper-half plane \( \mathcal{X}_n \). Using (5.4b), we have

\[
u = vP + Q = (\bar{I}_n - W\bar{W})^{-1}z
\]
and we introduce in formula above for \( v \) the expression given by (5.4a). We get \( z = \eta - W\bar{\eta} \), where \( \eta = P + iQ \) has appeared already in (3.9e) and (3.12). For \( A \) in (3.12) we get \( A = d\eta - W d \bar{\eta} \). In (3.12), we make the transform (6.1). Also, from (6.1) and (5.3a), we have (3.9e), i.e. (6.2).

We use the relation \( M - W\bar{M} = I_n \) for the terms of the type \( d\eta_i^t \wedge d\bar{\eta}_j \), the symmetry of the matrices \( \bar{M}W \) for the terms of the type \( d\bar{\eta}_i^t \wedge d\eta_j \) and \( MW \) for the terms of the type \( d\eta_i^t \wedge d\eta_j \), and we get for \( \omega_{n,0} = \omega_n \circ FC \) the expression given by (6.3).

Now we calculate the action of \( G^{\prime}J_n \) on \( C_n \times D_n \) induced from the action (3.9b), (3.9c) on \( D_n \) applying the FC transform (6.1). We want to find \( \eta_1 \) from (6.2) with \( (W_1, z_1) \) given by (3.9b), (3.9c), (6.9) \( (I_n - W_1\bar{W}_1)\eta_1 = z_1 + W_1\bar{z}_1 \).

Firstly, with (2.20), we calculate the lhs of (6.9) as

\[
(I_n - W_1\bar{W}_1) = (Wq^* + p^*)^{-1}(I_n - W\bar{W})(q\bar{W} + p)^{-1},
\]

where \( p, q \) are components of the matrix \( g \in Sp(n,\mathbb{R}) \) as in (2.19).

Now we introduce the value \( z_1 = (Wq^* + p^*)^{-1}[\eta + \alpha - W(\bar{\eta} + \bar{\alpha})] \), in rhs of (6.9), and we write it as

\[
z_1 + W_1\bar{z}_1 = (Wq^* + p^*)^{-1}Y, \quad \text{where}
\]

\[
Y = P(\eta + \alpha) + Q(\bar{\eta} + \bar{\alpha}),
\]

\[
P = I_n - (q^t + Wp^t)(Wb^t + a^t)^{-1}\bar{W},
\]

\[
Q = -W + (q^t + Wp^t)(\bar{W}b^t + a^t)^{-1}.
\]

Using the successively the second relation in (2.20a), after an easy but long calculation, we find

\[
P = (I_n - W\bar{W})(p + q\bar{W})p,
\]

\[
Q = (I_n - W\bar{W})(p + q\bar{W})q.
\]

Combining (6.11)-(6.12), we get for \( \eta_1 \) the value given in (6.4).

In order to verify the invariance of the Kähler two-form (6.3) to the action (6.4), we have to check up that

\[
\text{Tr}(d\eta_i^t \wedge d\bar{\eta}_1) = \text{Tr}(d\eta_i^t \wedge d\bar{\eta}),
\]

which is true because of the first relation in (2.20b).

With (5.4a), we get

\[
(I_n - W\bar{W})^{-1} = \frac{1}{2i}(\bar{v} - iI_n)(\bar{v} - v)^{-1}(v + iI_n).
\]

We introduce (6.14), (5.3b) and (5.3a) in (3.9e) and we get (6.5).

(6.7) is obtained introducing in (5.3b) the expression (6.1) with \( W \) given by (5.3a).

Finally, (6.8) is a consequence of (6.4) and (2.14). Alternatively, the invariance (6.13), where \( \eta_1 \) has the expression (6.8), can be checked up directly, taking into account that
the matrices $a, b, c, d$ appearing in (6.8) are the components of $g \in \text{Sp}(n, \mathbb{R})$ as in (5.2), and consequently verify the conditions (2.10).

We recall that in the case $n = 1$, (6.6) appears in the paper [36] of Erich Kähler as equation (3) in §38.

**Corollary 1.** Under the FC-change of coordinates (6.1), $x = \eta - V \bar{\eta}$, $y = \xi - W \bar{\xi}$, the reproducing kernel (3.13) becomes

$$K(\bar{\eta}, \bar{V}; \xi, W) = (\det U)^{k/2} \exp F,$$

where

$$2F = F_0 + \Delta F;$$

$$F_0 = \xi^t \xi + \bar{\eta}^t \eta - \bar{\xi}^t \bar{V} \bar{\xi} - \eta^t \bar{V} \eta,$$

$$\Delta F = (\bar{\zeta}^t - \zeta^t \bar{V})U(\zeta - W \bar{\xi}) + (\bar{\eta}^t - \eta^t \bar{V})U(-\zeta + W \bar{\xi}); \zeta = \eta - \xi.$$

In particular, for $\xi = \eta, V = W$, we have $\Delta F = 0$, and

$$K = \det(M)^{k/2} \exp(\mathcal{F}), \quad \text{where} \quad \mathcal{F} = \bar{\eta}^t \eta - \frac{1}{2} \bar{\eta}^t W \eta - \frac{1}{2} \bar{\eta}^t W \bar{\eta},$$

and the scalar product (3.19) becomes

$$\langle \phi, \psi \rangle = \Lambda_n \int_{\eta \in \mathbb{C}^n; I_n-W \bar{W}>0} \bar{f}_\phi(\eta, W) f_\psi(\eta, W) \rho_2 \, d\eta \, dW,$$

$$\rho_2 = \det(I_n-W \bar{W})^q \exp(-\mathcal{F}), \, q = k/2 - n - 1,$$

with $\mathcal{F}$ given by (6.16).

Also we have the relation

$$-\frac{\partial F}{\partial w_{ij}} = \frac{\partial \mathcal{F}}{\partial w_{ij}} = -\bar{\eta}_i \bar{\eta}_j + \frac{1}{2} \bar{\eta}_i \bar{\eta}_j \delta_{ij}, \quad \text{or} \quad \nabla F = -\nabla \mathcal{F} = \frac{1}{2} \bar{\eta} \otimes \bar{\eta}.$$

**Proof.** Formula (6.15) is obtained by brute force calculation, using the relation $UW \bar{V} = U - I_n$.

In the expression (3.14), we make the change of variables (6.1), and we get easily the expression of $\mathcal{F}$ given in (6.16).

In order to get the factor $\rho_2$ in the expression (6.17), we firstly note that

$$(z^t, \bar{z}^t) = (\eta^t, \bar{\eta}^t) \begin{pmatrix} I_n & \bar{W} \\ -W & I_n \end{pmatrix}.$$}

Now, we observe that if we have a linear transformation $y = Cx$ of the column $n$-vectors $y$ and $x$, then

$$y_1 \wedge \cdots \wedge y_n = \det(C)x_1 \wedge \cdots \wedge x_n.$$ 

We apply the formula

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det A \det(D - CA^{-1}B).$$
7. CLASSICAL MOTION AND QUANTUM EVOLUTION ON SIEGEL-JACOBI DOMAINS

Let $M = G/H$ be a homogeneous manifold with the $G$-invariant Kähler two-form $\omega$ (3.3). The energy function $\mathcal{H}$ (the classical Hamiltonian, or the covariant symbol) attached to the quantum Hamiltonian $H$ is [16]

$$\mathcal{H}(z, \bar{z}) = <e_\bar{z}, e_z>^{-1} <e_\bar{z}|H|e_z> .$$

Passing on from the dynamical system problem in the Hilbert space $\mathcal{H}$ to the corresponding one on $M$ is called sometimes dequantization, and the dynamical system on $M$ is a classical one [4, 5]. Following Berezin [15, 17], the motion on the classical phase space can be described by the local equations of motion

$$\dot{z}_\alpha = i \{\mathcal{H}, z_\alpha\}, \quad \alpha \in \Delta_+,$$

where the Poisson bracket is defined as

$$\{f, g\} = \sum_{\alpha, \beta \in \Delta_+} (\mathcal{G}^{-1})_{\alpha, \beta} \left( \frac{\partial f}{\partial z_\alpha} \cdot \frac{\partial g}{\partial \bar{z}_\beta} - \frac{\partial f}{\partial \bar{z}_\alpha} \cdot \frac{\partial g}{\partial z_\beta} \right),$$

$f, g \in C^\infty(M)$ and $(\mathcal{G}^{-1})_{\alpha, \beta}$ are the matrix elements of the inverse of the matrix $\mathcal{G}$ defined in (3.4).

The classical equations of motion (7.1) on the manifold $M$ can be written down as

$$i \left( \begin{array}{cc} \mathcal{O}_n & \mathcal{G} \\ -\mathcal{G} & \mathcal{O}_n \end{array} \right) \left( \begin{array}{c} \dot{z} \\ \dot{\bar{z}} \end{array} \right) = - \left( \begin{array}{c} \partial / \partial z \\ \partial / \partial \bar{z} \end{array} \right) \mathcal{H}.$$
and the classical motion generated by the linear Hamiltonian (7.3) is given by the equations of motion on \( M = G/H \) [1, 5]:

\[
(7.5) \quad i \dot{z}_\alpha = \sum_{\lambda \in \Delta} \epsilon_\lambda Q_{\lambda, \alpha}, \quad \alpha \in \Delta_+.
\]

Similarly, the equations of motion on \( M = G/H \) determined by a Hamiltonian

\[
(7.6) \quad H = \sum_{\lambda \in \Delta} \epsilon_\lambda X^\dagger_\lambda,
\]

are

\[
(7.7) \quad i \dot{\tilde{z}}_\alpha = \sum_{\lambda \in \Delta} \epsilon_\lambda \tilde{Q}_{\lambda, \alpha}, \quad \alpha \in \Delta_+.
\]

We look also for the solutions of the Schrödinger equation attached to the Hamiltonian \( H \) (7.3)

\[
(7.8) \quad H \psi = i \dot{\psi}, \quad \text{where} \quad \psi = e^{i\varphi} < e_{\bar{z}}, e_z >^{-1/2} e_{\bar{z}}.
\]

Summarizing, in the conventions at the beginning of §4 and with the observation (4.7), we formulate the following (see also [5], [52]):

**Proposition 5.** On the homogenous manifold \( M = G/H \) on which the generators \( X_\lambda \in \mathfrak{g} \) admit a holomorphic representation as in (7.4), the classical motion and the quantum evolution generated by the linear Hamiltonian (7.3) are given by the same equation of motion (7.5). The phase \( \varphi \) in (7.8) is given by the sum \( \varphi = \varphi_D + \varphi_B \) of the dynamical and Berry phase,

\[
(7.9a) \quad \varphi_D = - \int_0^t \mathcal{H}(t) \, dt, \quad \text{where}
\]

\[
(7.9b) \quad \mathcal{H}(t) = \sum_{\lambda \in \Delta} \epsilon_\lambda \frac{(e_\bar{z}, X_\lambda e_z)}{(e_\bar{z}, e_z)}
\]

\[
(7.9c) \quad = \sum_{\lambda \in \Delta} \epsilon_\lambda (\tilde{P}_\lambda + \sum_{\beta \in \Delta_+} \tilde{Q}_{\lambda, \beta} \partial_\beta \ln < e_\bar{z}, e_z >)
\]

\[
(7.9d) \quad = \sum_{\lambda \in \Delta} \epsilon_\lambda \tilde{P}_\lambda + i \sum_{\beta \in \Delta_+} \dot{\tilde{z}}_\beta \partial_\beta \ln < e_\bar{z}, e_z > ;
\]

\[
(7.9e) \quad \varphi_B = -3 \int_0^t < e_\bar{z}, e_z >^{-1} < e_\bar{z}, e_z > \, dt
\]

\[
= \frac{i}{2} \int_0^t \sum_{\alpha \in \Delta_+} (\dot{z}_\alpha \partial_\alpha - \tilde{z}_\alpha \tilde{\partial}_\alpha) \ln < e_\bar{z}, e_z > .
\]

7.1. **Equations of motion.** Now we consider a Hamiltonian linear in the generators of the group \( G_n^J \)

\[
(7.10) \quad H = \epsilon_i a_i + \epsilon^\dagger_i a_i^\dagger + \epsilon_{ij}^0 K_{ij}^0 + \epsilon_{ij}^- K_{ij}^- + \epsilon_{ij}^+ K_{ij}^+.
\]
The hermiticity condition imposes to the matrices of coefficients \( \epsilon_{0,\pm} = (\epsilon_{0,\pm}^{i,j})_{i,j=1,...,n} \) the restrictions

\[
(7.11) \quad \epsilon_0^\dagger = \epsilon_0; \quad \epsilon_- = \epsilon_-^\dagger; \quad \epsilon_+ = \epsilon_+^\dagger; \quad \epsilon_-^\dagger = \epsilon_+.
\]

It is useful to introduce the matrices \( m, n, p, q \in M(n, \mathbb{R}) \) such that

\[
(7.12) \quad \epsilon_- = m + iq, \quad \epsilon_0/2 = p + iq; \quad p^t = p; \quad m^t = m; \quad n = n; \quad q^t = -q.
\]

We shall describe the dynamics on the Siegel-Jacobi ball (space) \( \mathcal{D}_n \) (respectively, \( \mathcal{X}_n \)) determined by the linear Hamiltonian (7.10), (7.11) and study the effect of the FC (FC\(_1\)) transform on the equations of motion. We introduce some notation. Let \( W \in M(n, \mathbb{C}) \) be coordinates on a homogenous manifolds \( M = G/H \) below \( M \) will be one of the Siegel-Jacobi domains \( \mathcal{D}_n \) or \( \mathcal{X}_n \) and let \( z \in \mathbb{C}^n \). We consider a matrix Riccati equation on the manifold \( M \) and a linear differential equation in \( z \)

\[
(7.13a) \quad \dot{W} = AW + WD + B + WCW, \quad A, B, C, D \in M(n, \mathbb{C});
\]

\[
(7.13b) \quad \dot{z} = M + Nz; \quad M = E + WF; \quad N = A + WC, \quad E, F \in C^n.
\]

**Proposition 6.** The classical motion and quantum evolution generated by the linear hermitian Hamiltonian (7.10), (7.11) are described by first order differential equations:

- **a)** on \( \mathcal{D}_n \), \((z, W) \in \mathbb{C}^n \times \mathcal{D}_n \) verifies (7.13), with coefficients

\[
(7.14a) \quad A_c = -\frac{i}{2} \epsilon_0^t, \quad B_c = -i \epsilon_-^t, \quad C_c = -i \epsilon_+^t, \quad D_c = A_c^t;
\]

\[
(7.14b) \quad E_c = -i \epsilon, \quad F_c = -i \epsilon^t.
\]

- **b)** on \( \mathcal{X}_n \), \((u, v) \in \mathbb{C}^n \times \mathcal{X}_n \), verifies (7.13), with coefficients

\[
(7.15a) \quad A_r = n + q, \quad B_r = m - p, \quad C_r = -(m + p), \quad D_r = n - q;
\]

\[
(7.15b) \quad E_r = 3 \epsilon; \quad F_r = -3 \epsilon^t.
\]

- **c)** under the FC transform (6.1), the differential equations in the variables \( \eta \in \mathbb{C}^n \), \( W \in \mathcal{D}_n \) become independent: \( W \) verifies (7.13) with coefficients (7.14a) and \( \eta \) verifies

\[
(7.16) \quad i \dot{\eta} = \epsilon + \epsilon_- \eta + \frac{1}{2} \epsilon_0^t \eta, \quad \eta \in \mathbb{C}^n.
\]

- **d)** under the FC\(_1\) transform, the equations in the variables \( \eta \in \mathbb{C}^n \), \( v \in \mathcal{X}_n \) become independent: \( \eta \) verifies (7.16), while \( v \) verifies (7.13a) with coefficients (7.15a).

**Proof.** Firstly, we proof (7.14). With (4.8a) in Lemma 1 we get from (7.5) the equations of motion for \( z \in \mathbb{C}^n \):

\[
i \dot{z}_\alpha = \epsilon_\alpha \delta_{i\alpha} + \epsilon_i w_{i\alpha} + \frac{\epsilon^+_{kl}}{2} (z_{k} w_{i\ell} + z_{\ell} w_{ik}) \delta_{i\alpha} + \frac{1}{2} \epsilon_0^t z_k \delta_{i\alpha}.
\]

The equations of motion (7.5) for \( w_{pq} \), \( W = (w_{pq})_{p,q=1,...,n} \), can be written down as

\[
i \dot{w}_{pq} = \epsilon_{kl} Q_{kl,pq}.
\]
With (4.8b), (4.8c) in Lemma 1 and (4.5), we have for $Q_{k,l,pq}$ the expressions:

$$K_0^{k,l} \rightarrow Q_{k,l,pq}^0 = w_{kp}w_{lq} + w_{lp}w_{qk} - w_{kl}^{pq} - w_{kl}^{qp},$$

$$K_\perp \rightarrow Q_{k,l,pq}^\perp = w_{kp}w_{lq} + w_{lp}w_{qk} - w_{kl}^{pq} - w_{kl}^{qp},$$

$$K_\parallel \rightarrow Q_{k,l,pq}^\parallel = \chi_{kl}(\delta_{kq} - \delta_{lp} - \delta_{kl}^{pq}\delta_{kp}).$$

Explicitly, the differential equations for $(W, z) \in \mathcal{D}_n$ are

$$(7.17a) \quad i\dot{W} = \epsilon_+ (W\epsilon_0^t) + W\epsilon_+ W, \quad W \in \mathcal{D}_n,$$

$$(7.17b) \quad i\dot{z} = \epsilon + W\bar{\epsilon}_+ + \frac{1}{2}\epsilon_0^t z + W\epsilon_+ z, \quad z \in \mathbb{C}_n,$$

and we have proved a).

Now we prove b). Firstly, we prove (7.15a).

With the Cayley transform (5.3a), we find for $\dot{W}$ the value

$$2i(v + iI_n)^{-1}\dot{v}(v + iI_n)^{-1} = A_c W + WD_c + B_c + WC_c W.$$

In the expression above we replace the value of $W$ as function of $v$ as given by (5.3a), and we find a matrix Riccati equation on $X_n$ in $v$ of the form (7.13a), where the matrix coefficients $A_c, D_c \in M(n, \mathbb{R})$ are expressed in function of the coefficients $A_c, D_c$ as

$$A_c = \frac{1}{2}(A_c - D_c + B_c + C_c), \quad B_c = \frac{1}{2}(A_c + D_c - B_c - C_c),$$

$$C_r = \frac{1}{2}(A_c + D_c + B_c + C_c), \quad D_r = \frac{1}{2}(-A_c + D_c - B_c - C_c).$$

With (7.12), we find the values given by (7.15a).

Now we prove (7.15b). We differentiate (5.3b) and, with (7.14b), we get successively

$$-2\dot{u} = \dot{v}(iz) + (v + iI_n)(i\dot{z})$$

$$= -2\dot{v}(v + iI_n)^{-1}u + (v + iI_n)\times$$

$$\times \{\epsilon + (v + iI_n)^{-1}(v - iI_n)\dot{\epsilon} + \frac{\epsilon_0^t}{2} + (v + iI_n)^{-1}(v - iI_n)\epsilon_+^t\}u$$

$$= (v + iI_n)\epsilon + (v - iI_n)\dot{\epsilon} + T(v + iI_n)^{-1}u,$$

where

$$T = -2\dot{v} + i(v + iI_n)\epsilon_0^t + 2i(v - iI_n)\epsilon_+. $$

With (7.15a) we get for $T$ the value

$$T = [v(\epsilon_0^t + \epsilon_+ + \epsilon_) + i(-\epsilon_0^t + \epsilon_+ - \epsilon_+)](v + iI_n),$$

and we obtain (7.15b). Explicitly, b) can be write down as

$$(7.19a) \quad -2\dot{v} = \epsilon_0^t - (\epsilon_+ + \epsilon_+) + iv(\epsilon_0^t + \epsilon_+ - \epsilon_+) +$$

$$i(-\epsilon_0^t + \epsilon_+ - \epsilon_+)v + v(\epsilon_0^t + \epsilon_+ - \epsilon_+)v;$$

$$(7.19b) \quad -2\dot{u} = v(\epsilon + \bar{\epsilon}) + i(\epsilon - \bar{\epsilon}) + [v(\epsilon_0^t + \epsilon_+ + \epsilon_+) + i(-\epsilon_0^t + \epsilon_+ - \epsilon_+)]u.$$

Below we prove (7.16). We take the derivative of $\eta$ in the $FC^{-1}$ transform (6.2), and we get

$$\dot{\eta} = (I_n - W\bar{W})^{-1}(\bar{W}W + W\bar{W})\eta + (I_n - W\bar{W})^{-1}(\dot{z} + \bar{W}\bar{z} + \bar{W}\dot{z}).$$
Then we introduce for $\dot{W}$ and $\dot{z}$ the values from (7.17a) and respectively (7.17b) and we pass from $z$ to $\eta$ with (6.1):

$$i(\eta_{n} - W\dot{W})\dot{\eta} = \{[\epsilon_{-} + (W\epsilon_{0})^{s} + W\epsilon_{+}W]\dot{W} - W[\epsilon_{-} + (\dot{W}\epsilon_{0})^{s} + \dot{W}\epsilon_{+}W]\}\eta + \epsilon - (W\epsilon_{0})^{s} + W\epsilon_{+}W\}[\dot{\eta} - W\eta] + \epsilon + W\epsilon + (\frac{\epsilon_{0}^{t}}{2} + \dot{W}\epsilon_{+})(\eta - W\eta).$$

We calculate the coefficient of $\eta$ as $\frac{1}{2}(\eta_{n} - W\dot{W})\epsilon_{0}$, those of $\dot{\eta}$ is $(\eta_{n} - W\dot{W})\epsilon_{-}$, and we get (7.16).

In order to avoid the longer calculation of introducing (7.15b) in (6.5), the motion in $(\eta, v) \in (C^n, \mathcal{X}_n)$ at d) is obtained putting together (7.16) and (7.15a). □

Note that starting with the Hamiltonian (7.10), linear in the generators of the Jacobi group $G_{n}^{J}$, the equation of motion (7.14a) (7.15a) on the Siegel upper half plane $\mathcal{X}_n$ depends only on the generators of de group $\text{Sp}(n, \mathbb{R})_{C}$ (respectively, $\text{Sp}(n, \mathbb{R})$).

### 7.2. Solution of the equations of motion.

We solve the system of differential equations on the Siegel-Jacobi domains appearing in Proposition 6.

a. Firstly, we recall how to solve the matrix Riccati equation (7.13a) by linearization. If we proceed to the homogenous coordinates $W = X Y^{-1}$, $X, Y \in M(n, C)$, a linear system of ordinary differential equations is attached to the matrix Riccati equation (7.13a) (cf. [42], see also [5])

$$\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = h \begin{pmatrix} X \\ Y \end{pmatrix}, \quad h = \begin{pmatrix} A & B \\ -C & -D \end{pmatrix}. \tag{7.20}$$

Every solution of (7.20) is a solution of (7.13a), whenever $\det(Y) \neq 0$.

For the motion on $\mathcal{D}_n$, the matrix elements of $h$ in (7.20), denoted $h_{c}$, are given in (7.14a), and the linear system of differential equations attached to the matrix Riccati equation (7.17a) is

$$\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = h_{c} \begin{pmatrix} X \\ Y \end{pmatrix}, \quad h_{c} = \begin{pmatrix} -i(\frac{\epsilon_{0}^{t}}{2}) & -i\epsilon_{-} \\ i\epsilon_{+} & i\frac{\epsilon_{0}^{t}}{2} \end{pmatrix}, \quad W = X/Y \in \mathcal{D}_n. \tag{7.21}$$

Due to the conditions (7.11), the matrix $h_{c}$ has the form (2.13), and we conclude that $h_{c} \in \text{sp}(n, \mathbb{R})_{C}$. The action of the element $g \in \text{Sp}(n, \mathbb{R})_{C}$ (2.19) on $W \in \mathcal{D}_n$ is given by the linear fractional transformation (3.9b).

Now we look at the matrix Riccati equation (7.13a) on the Siegel upper half plane $\mathcal{X}_n$ in the symmetric variables $v$ with coefficients $A_{r} - D_{r}$ given by (7.18) – or equation (7.19a). We associate to the matrix Riccati equation (7.19a) a linear system of first order differential equations of the type (7.20) with a matrix coefficients $h_{r}$

$$\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = h_{r} \begin{pmatrix} X \\ Y \end{pmatrix}, \quad h_{r} = \begin{pmatrix} A_{r} & B_{r} \\ -C_{r} & -D_{r} \end{pmatrix} = \begin{pmatrix} n + q & m - p \\ m + p & -n + q \end{pmatrix}, v = X/Y \in \mathcal{X}_n. \tag{7.22}$$
Due to the hermiticity conditions (7.11), the matrix $h_r$ verifies the conditions (2.9), because of (7.18), and we see that $h_r \in \mathfrak{sp}(n, \mathbb{R})$. The matrices $h_r$ and $h_c$ are related by relations of the type (2.16), (2.17), i.e. $h_c = (h_r)_c$ and have the same eigenvalues.

Using general considerations of solving systems of first order linear differential equations [31] applied to $\mathfrak{sp}(n, \mathbb{R})$ [46] and the considerations above, we formulate

**Remark 3.** The linear system of first order differential equations (7.21) (7.22) associated to the matrix Riccati equations (7.17a) (7.19a) on $\mathcal{D}_n$ ($X_n$) describes the time-dependent vector field induced by the infinitesimal action of the group $\text{Sp}(n, \mathbb{R})_C$ (respectively, $\text{Sp}(n, \mathbb{R})$). (7.21) (7.22) is a linear Hamiltonian system in the meaning of Meyer [46] or a canonical system in the sense of Yakulovich [63] (Hamiltonian system) (respectively, in the sense Yakulovich [63]) and $h_c = (h_r)_C$.

The infinitesimal group action of $\mathfrak{sp}(n, \mathbb{R})_C$ (respectively, $\mathfrak{sp}(n, \mathbb{R})$) is given by the Lie algebras homomorphism

\[ \nu_c : \mathfrak{sp}(n, \mathbb{R})_C \rightarrow \text{Ham}(\mathcal{D}_n), \quad \nu_r : \mathfrak{sp}(n, \mathbb{R}) \rightarrow \text{Ham}(X_n). \]

The infinitesimal group action associated to (7.20) is given by

\[ \nu \left( \begin{array}{cc} A & B \\ -C & -D \end{array} \right) = -(B + AZ + ZD + ZC Z)_{im} \frac{\partial}{\partial w_{im}}. \]

Let

\[ U(t, t_0) = \left( \begin{array}{cccc} U_1(t, t_0) & U_2(t, t_0) & \ldots & U_n(t, t_0) \\ U_1(t, t_0) & U_2(t, t_0) & \ldots & U_n(t, t_0) \end{array} \right) \]

be the fundamental matrix of the ordinary differential equation (7.20), i.e. $\dot{U} = hU$ such that $U(t_0, t_0) = 1$, where $h = h_c$ ($h = h_r$) for the motion on $\mathcal{D}_n$ (respectively, $X_n$).

Then the fundamental solution $U_c(t, t_0)$ ($U_r(t, t_0)$) corresponding to $h = h_c$ ($h = h_r$) is a $\text{Sp}(n, \mathbb{R})_C$ (respectively $\text{Sp}(n, \mathbb{R})$)-matrix and

\[ W(t, t_0) = [U_1(t, t_0)W(t_0) + U_2(t, t_0)][U_3(t, t_0)W(t_0) + U_4(t, t_0)]^{-1} \]

is the solution of equation (7.13a) with the initial condition $W(t_0, t_0) = W(t_0)$.

For selfcontainedness, we recall the terminology used in Remark 3. The systems of linear ordinary differential equations $\dot{z} = Az$, $A \in \mathfrak{sp}(n, \mathbb{R})$ appear in the context of *linear Hamiltonian systems* [46]. The eigenvalues of the Hamiltonian matrices are described in Remark 1(e). The Hamiltonian equations can be written as the system of ordinary differential equations

\[ \dot{z} = J \nabla H, \]

where $z^t = (q^t, p^t)$, $q, p \in \mathbb{R}^n$, $H = H(t, q, p)$ is the Hamiltonian, and here $\nabla H = (\frac{\partial H}{\partial q_1}, \ldots, \frac{\partial H}{\partial q_n})$. The Hamiltonian system (7.26) is called a *Hamiltonian linear system* (cf [46], also called canonical, cf [63] p. 110), if the Hamiltonian $H$ has the form

\[ H = \frac{1}{2} z^t S z, \quad S \in M(2n, \mathbb{R}), \quad S = S^t, \]

and the system (7.26) is written as the system of linear ordinary differential equations

\[ \dot{z} = J S z = A z, \quad A \in \mathfrak{sp}(n, \mathbb{R}), \quad S = S^t. \]
The equation \( \dot{z} = Az, A \in \text{sp}(n, \mathbb{R}) \), where \( A \) is expressed as in (2.18), is also called Hamiltonian, cf. [63] p. 111.

b. Now we discuss the linear system of the type (7.28) associated to the matrix Riccati equation (7.19a) in the case of \( T \)-periodic coefficients. Let \( \Delta(t) \) be the fundamental solution of the equation (7.28) with periodic coefficients such \( \Delta(0) = I_{2n} \). Then \( \Delta(t + T) = \Delta(t)\Delta(T), \forall t \in \mathbb{R} \). We take over Floquet-Lyapunov Theorem and Krein Gel’fand Theorem, (cf. Proposition 4.2.1, Theorem 3.4.2 and Corollary 3.4.1 in [46]; see also Ch II in [63]), which we formulate as:

Remark 4. The monodromy matrix \( \Delta(T) \) is a nonsingular, symplectic matrix. The fundamental matrix solution of Hamiltonian equation (7.28) that satisfies \( \Delta(0) = I_{2n} \) is of the form \( \Delta(t) = X(t) \exp(Kt) \), where \( X(t) \) is symplectic and \( T \)-periodic and \( K \) is Hamiltonian. Real \( X(t) \) and \( K \) can be found by taking \( X(t) \) to be \( 2T \)-periodic if necessary. The change of variables \( z = X(t)w \) transforms the periodic Hamiltonian system (7.28) to the constant Hamiltonian system

\[
\dot{w} = Kw, \quad K \in \text{sp}(n, \mathbb{R}).
\]

The linear autonomous Hamiltonian system (7.29) is stable iff: (i) \( K \) it has only pure imaginary eigenvalues and (ii) \( K \) is diagonalizable (over the complex numbers). The system (7.29) is parametrically stable if and only if: (i) All the eigenvalues of \( K \) are pure imaginary \( \pm \imath \alpha_i \); (ii) \( K \) is nonsingular; (iii) The matrix \( K \) is diagonalizable over the complex numbers; (iv) The restriction of the Hamiltonian \( H \) to the linear space generated by \( \pm \imath \alpha_j \) is positive or negative definite for each \( j \).

c. Let us discuss the solution of the matrix Riccati equation (7.13a) in the case of constant coefficients. Let \( \lambda_1, \ldots, \lambda_{2n} \) be the eigenvalues of characteristic equation associated to the matrix \( h \) defined in (7.20)

\[
\text{det}(h - \lambda I_{2n}) = 0.
\]

Let

\[
\Lambda = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix},
\]

where \( A_1 \) (\( A_2 \)) is a diagonal matrix with entries \( \lambda_i, i = 1, \ldots, n \) (respectively, \( i = n + 1, \ldots, 2n \)).

Let us suppose that the matrix \( h \) has a simple structure (cf. Ch. III in [29] or \( h \) has simple elementary divisors cf. [63]) and let \( V \) be the fundamental matrix [29] of \( h \) (7.20), i.e.

\[
Vh = \Lambda V.
\]

Let us consider a partition of \( V \) into block form

\[
V = \begin{pmatrix} V_1 & V_2 \\ V_3 & V_4 \end{pmatrix},
\]

where \( V_i, i = 1, \ldots, 4 \in M(n, \mathbb{C}) \).

Let \( Q = Q(W, A, B, C, D) \) the matrix which appears in the rhs of (7.13a), and let us denote by \( Q_0 \) the value of \( Q \) at \( W_0 = W(t = 0) \).
Remark 5. The autonomous linear system (7.20) associated to the matrix Riccati equation (7.13a) has the standard solution

\[(7.34) \quad \begin{pmatrix} X \\ Y \end{pmatrix} = e^{th} \begin{pmatrix} X_0 \\ Y_0 \end{pmatrix}.\]

The solution \(W\) of the matrix Riccati equation (7.13a) has the Taylor expansion

\[W(t) = W_0 + tQ_0 + \frac{t^2}{2}[(A + W_0C)Q_0 + Q_0(D + CW_0)] + \ldots.\]

If the matrix \(h\) (7.20) has a simple structure, then the solution of the matrix Riccati equation (7.13a) with constant coefficients with the initial condition \(W(0) = W_0\) is

\[(7.35a) \quad W = (V_1 - W'V_5)^{-1}(W'V_4 - V_2),\]
\[(7.35b) \quad W' = W'(t, W_0') = e^{tA_1}W_0'e^{-tA_2},\]
\[(7.35c) \quad W_0' = (V_1W_0 + V_2)(V_3W_0 + V_4)^{-1}.\]

Closed paths on the Siegel ball \(D_n\) \((W(T) = W(0))\) can be obtained if the imaginary eigenvalues \(\lambda_i\) of the matrix \(h\) are rationally commensurable and different and \(T\) is the least common multiple of \(2\pi|\lambda_m - \lambda_i|, i = 1, \ldots, n, m = n + 1, \ldots, 2n.\)

Suppose now we are in the case when the autonomous Hamiltonian system (7.21) is stable: the matrix \(h_r\) has \(2n\) purely imaginary distinct eigenvalues and the matrix \(h_r\) can be put in the form

\[(7.36) \quad h_r = \begin{pmatrix} \mathbb{O}_n & d \\ -d & \mathbb{O}_n \end{pmatrix}, \quad d = \text{diag}(\alpha_1, \ldots, \alpha_n).\]

Correspondingly, in (7.31) we get

\[(7.37) \quad \Lambda_1 = \text{id}, \quad \Lambda_2 = -\text{id}\]

which have to be introduced in (7.35) for the motion on \(X_n\) in the case of the constant coefficients in (7.15a).

We also are in the case where the matrix \(h_c\) has \(2n\) distinct pure imaginary eigenvalues and \(h_c\) can be put into the form

\[(7.38) \quad h_c = \begin{pmatrix} \text{id} & \mathbb{O}_n \\ \mathbb{O}_n & -\text{id} \end{pmatrix}, \quad d = \text{diag}(\alpha_1, \ldots, \alpha_n),\]

with the same eigenvalues as in (7.37).

Proof. As was mentioned in Remark 3 a), the linear system of first order differential equations (7.20) is associated with the matrix Riccati equation (7.13a). A solution to (7.20) projects to a solution to (7.13a) via the map \(\Psi(X, Y) = XY^{-1}\). The autonomous linear system (7.20) has the standard solution (7.34) [31]. We recall that if \(A \in M(n, \mathbb{F})\), \((\mathbb{F} = \mathbb{R}\) or \(\mathbb{C}\)) has the minimal polynomial \(\psi(\lambda) = (\lambda - \lambda_1)^{m_1}(\lambda - \lambda_2)^{m_2} \cdots (\lambda - \lambda_s)^{m_s}\), where \(\lambda_1, \ldots, \lambda_s\) are characteristic roots of \(A\), then

\[e^{At} = \sum_{k=1}^{s} [Z_{k1} + Z_{k2}t + \cdots + Z_{km_k}t^{m_k-1}]e^{\lambda_k t}.\]
The matrices $Z_{kj}$ are completely determined by $A$ \cite{29}. In particular, if the minimal polynomial has only simple roots, the Lagrange-Sylvester interpolation formula gives for $A \in M(n, \mathbb{C})$ \cite{29}

$$e^{At} = \sum_{k=1}^{n} \frac{(A - \lambda_1 I_n) \cdots (A - \lambda_{k-1} I_n) \cdots (A - \lambda_{k+1} I_n) \cdots (A - \lambda_n I_n)}{(\lambda_k - \lambda_1) \cdots (\lambda_k - \lambda_{k-1}) \cdots (\lambda_k - \lambda_{k+1}) \cdots (\lambda_k - \lambda_n)} e^{\lambda_k t}.$$ 

In the situation of Remark 4, we diagonalize the matrix $h$ via (7.32) and we make a change of variables (7.39)

$$\begin{pmatrix} X' \\ Y' \end{pmatrix} = \begin{pmatrix} V_1 & V_2 \\ V_3 & V_4 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}.$$ 

The system (7.20) in the new variables reads

$$\dot{X}' = \Lambda_1 X'; \quad \dot{Y}' = \Lambda_1 Y',$$

with the solution

$$X' = e^{t\Lambda_1} X_0'; \quad Y' = e^{t\Lambda_2} Y_0'.$$

We calculate $W' = X'Y'^{-1}$ and get the expression (7.35b), then we calculate $W_0' = X_0'(Y_0')^{-1}$ and we obtain (7.35c), and finally, the equation (7.35a). In the situation of Remark 4, $\Lambda_1 = i\theta, \Lambda_2 = -i\theta$.

The assumption contained in (7.36) with the consequence (7.37) is expressed in Proposition 3.1.18 in [1] and (7.38) appears in Remark 1, e). Then we apply the transform (2.11) and the assertion for $\mathfrak{sp}(n, \mathbb{R})_{\mathbb{C}}$ follows.

\section*{d. Finally, we discuss how to solve the decoupled system at c) in Proposition 6} In (7.16) we introduce $\eta = \xi - i\zeta, \xi, \zeta \in \mathbb{R}^n$, we put $\epsilon = b + ia$, where $a, b \in \mathbb{R}^n$. The first order complex differential equation equation (7.16) is equivalent with a system of first order real differential equations with real coefficients, which we write as

(7.40) \[ \dot{Z} = h_r Z + F, \quad Z = \begin{pmatrix} \xi \\ \zeta \end{pmatrix}, \quad F = \begin{pmatrix} a \\ b \end{pmatrix}. \]

Because of (7.11), the matrices $p, m, n$ are symmetric, while $q$ is antisymmetric, as in (7.12). This means that the matrix $h_r$ has the form (2.9), i.e. $h_r \in \mathfrak{sp}(n, \mathbb{R})$, as was already underlined in Remark 3. With general considerations on solving first order linear systems of nonhomogenous equations \cite{31} particularized to the case of Hamiltonian matrices \cite{46}, we get

\begin{remark}
Let $\Delta(t, t_0)$ be the fundamental solution of the homogeneous equation $\dot{Z} = h_r Z$ associated with (7.40), where $h_r \in \mathfrak{sp}(n, \mathbb{R})$. The fundamental matrix solution $\Delta(t, t_0)$ of the linear Hamiltonian system is symplectic. The solution of the inhomogeneous equation (7.40) is

$$Z(t) = \Delta(t, t_0)Z(t_0) + \int_{t_0}^{t} \Delta(t, \tau)F(\tau) \, d\tau.$$ 

In the case of constant coefficients, the solution of (7.40) is

$$Z = e^{h_r(t-t_0)} Z_0 + \int_{t_0}^{t} e^{h_r(t-\tau)} F(\tau) \, d\tau.$$ 
\end{remark}
So, solving the equation (7.40), we find \( \eta = \xi - i\zeta \), and we find the solution of (7.16). The solution of (7.13b) with coefficients (7.15), i.e. (7.17b), is obtained via the FC transform \( z = \eta - W\bar{\eta} \).

8. Phases

8.1. Berry phase for \( \mathcal{D}_n^\perp \). Formula (7.9c) for the Berry phase on the Siegel-Jacobi ball \( \mathcal{D}_n^\perp \) in the variables \((\bar{W}, z) \in \mathcal{D}_n \times \mathbb{C}^n\) reads:

\[
\frac{2}{i} \frac{d \varphi_B}{d \varphi_B} = (\sum w_{ij} \frac{\partial}{\partial w_{ij}} - cc)f + (\sum z_i \frac{\partial}{\partial z_i} - cc)f,
\]

where \((X - cc)\) means \((X - \bar{X})\). Above \( f \) is the Kähler potential (3.11) written as

\[
f = -k^2 \log(\det(I_n - W\bar{W})) + F.
\]

With (4.19), (4.22) and (4.17), we have

\[
\frac{\partial f}{\partial z_i} = \bar{\eta}_i,
\]

\[
\frac{\partial f}{\partial w_{ik}} = k^2 (2X_{ik} - X_{ik}\delta_{ik}) + \bar{\eta}_i\bar{\eta}_k - \frac{1}{2} \bar{\eta}_i\bar{\eta}_k\delta_{ik}, \quad \text{or} \quad \nabla f = \frac{1}{2}(kX + \bar{\eta} \otimes \bar{\eta}).
\]

With the FC-transform (6.1) \( z_i = \eta_i - w_{ij}\bar{\eta}_j \), we find

Remark 7. The Berry phase on the Siegel-Jacobi ball \( \mathcal{D}_n^\perp \) is expressed in the variables \((W, \eta) \in \mathcal{D}_n \times \mathbb{C}^n\) as

\[
\frac{2}{i} \frac{d \varphi_B}{d \varphi_B} = \left\{ \sum \frac{k}{2} (2X_{ij} - X_{ij}\delta_{ij}) - \frac{1}{2} \bar{\eta}_i\bar{\eta}_j\delta_{ij} \right\} d w_{ij} - cc + \left[ (\bar{\eta}_i + \bar{w}_{ij}\eta_j) d \eta_i - cc \right]
\]

\[
= \left\{ \frac{k}{2} \left[ 2\text{Tr}(X dW) - \text{Tr}(\text{diag}(X)\text{diag}(dW)) \right] - \frac{1}{2} \bar{\eta}_i\text{diag}(dW)\bar{\eta}_i - cc \right\} + \left[ d \bar{\eta}_i(\bar{\eta}_i + \bar{W}\eta) - cc \right].
\]

8.2. Dynamical phase. We calculate the energy function attached to the Hamiltonian (7.10) using the formula (7.9b). We write down \( \mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2 \), where \( \mathcal{H}_1 \) is the first term \( \mathcal{H}_1 = \sum_{\lambda \in \Delta} \epsilon_\lambda \tilde{\mathcal{P}}_\lambda \) in (7.9c), while \( \mathcal{H}_2 \) is the rest in (7.9c), which corresponds to \( i \sum_{\beta \in \Delta^+} \tilde{\mathcal{J}}_{\beta} \partial f \), but we prefer the brute force calculation with the formulae (4.6). With (4.6) and (8.1a), we get

\[
\mathcal{H}_1 = \epsilon^t z + \frac{k}{4} \text{Tr}(\epsilon_0) + \frac{k}{2} \text{Tr}(\epsilon W) + \frac{1}{2} z^t \epsilon z.
\]

For \( \mathcal{H}_2 \), we get with (4.6)

\[
\mathcal{H}_2 = \epsilon^t W\bar{\eta} + \bar{\epsilon}^t \bar{\eta} + \frac{1}{2} z^t \epsilon_0^t \bar{\eta} + z^t \epsilon_{-W}\bar{\eta} + \mathcal{H}_3,
\]

\[
\mathcal{H}_3 = (\epsilon^t w_{il} \nabla_{ik} + \epsilon^t_{kl}\nabla_{kl} + \epsilon_{-W}\bar{\eta} w_{il} \nabla_{ik}) f,
\]

where \( \nabla f \) has the value (8.1b).
Let us denote by $\Lambda = \Lambda(W, \epsilon_-, \epsilon_+, \epsilon_0)$ the matrix appearing in the rhs of (7.17a), i.e. $\Lambda = iQ$. We get

$$(8.4) \quad \mathcal{H}_3 = \frac{1}{2}[k \text{Tr}(\Delta X) + \eta^A \Delta \eta],$$

where $\Delta = \Lambda(W, \epsilon_+, \epsilon_-, \epsilon_0) = \epsilon_+ + (\epsilon_0 W)^s + W \epsilon_- W$.

Now we apply the FC-transform (6.1) in (8.2) and (8.3) and we express the energy function as sum of two separated terms in the independent variable $s \in \mathbb{C}^n$ and $W \in \mathcal{D}_n$

$$(8.5a) \quad \mathcal{H} = \mathcal{H}_\eta + \mathcal{H}_w,$$

$$(8.5b) \quad \mathcal{H}_\eta = \epsilon^A \eta + \bar{\epsilon}^B \bar{\eta} + \frac{1}{2}(\eta^A \epsilon_- \eta + \bar{\eta}^A \epsilon_+ \bar{\eta} + \bar{\eta}^A \epsilon_0 \eta),$$

$$(8.5c) \quad \mathcal{H}_w = \frac{k}{2} \text{Tr}((\epsilon_0)^s + [W \epsilon_- + \epsilon_+ \bar{W} + (\epsilon_0 W)^s \bar{W}] (\mathbb{I}_n - W \bar{W})^{-1}).$$

We calculate the critical points of the energy function (8.5) attached to linear hermitian Hamiltonian (7.10). We get

$$(8.6a) \quad \nabla \mathcal{H}_w = 2(\mathbb{I}_n - \bar{W} W)^{-1} \bar{\eta} (\mathbb{I}_n - W \bar{W})^{-1},$$

$$(8.6b) \quad \frac{\partial \mathcal{H}_\eta}{\partial \eta} = \epsilon + \epsilon_- \eta + \frac{1}{2} \epsilon_0 \bar{\eta}.$$
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