The flame displacement speed: A key quantity for turbulent combustion and combustion instability

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Abstract
Future combustion power and propulsion systems may operate in premixed regime enabling reduced fuel burn and reduced pollutant emissions. The turbulent premixed regime in those future combustion systems is likely to be in the corrugated regime where modeling the flame as a thin interface propagating into the fresh gas is made possible. The flame displacement speed is thus a key quantity for turbulent combustion in this regime. This quantity is also important for combustion instabilities. Indeed, the flame displacement speed \( S_d \) combined to the flow speed \( \nu \) determines the flame surface speed \( \nu_s \). The flame surface location has shown to play a major role on combustion instabilities. Research work have also demonstrated the role of the flame displacement speed on the flame response which is used for subsequent combustion instability prediction. In this context, the derivation of flame speed models and flame transfer function models based on this quantity are required. This paper presents the theoretical derivation of flame transfer function coefficients for swirling premixed flames in this context. The derivation is based on the definition of the flame speed for turbulent flame, its perturbed form for oscillating flow, and the kinematic flame-flow speed budget. The obtained results are compared to previous literature data and discussed. The effect of the flame angle, id est the effect of the swirl number on the flame response is also investigated. This works motivates detailed local measurements and simulations to evaluate flow-flame speed budget terms.

Keywords
Flow speed, flame speed, flame surface speed, flame transfer function, swirl, premixed

Introduction: Context and State of the Art
Turbulent combustion is an intrinsical part of flame stabilization where the flame can be seen as an interface propagating at a given local flame displacement speed. Turbulence and chemical reactions thus turbulent combustion are affected by the operating condition. Typical laboratory scale combustors operate at atmospheric conditions while jet engine operates at different conditions, typically higher pressure and inlet temperature. This affects turbulence and combustion processes, particularly the characteristics Kolmogorov and flame thicknesses scales. Calculation of estimates indicate that the premixed combustion regime

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will fall in the corrugated region of the premixed combustion regime diagram for those operating conditions. In that regime, the flame can be defined with its displacement speed $S_d$ throughout the flame structure. It is now important to review the various flame displacement speed definitions in the literature and their underlying assumptions. Throughout the manuscript, the terms flame displacement speed and flame speed are used to define the same quantity. For laminar flames, the unstretched laminar flame speed $S_d^0$ characterizes the propagation of a flame in a laminar unstrained flow which is in practice difficult to achieve. Various experimental setups are used to characterize this quantity. Multiple analytical forms of the unstretched laminar flame speed have been derived with asymptotic methods. When submitted to flow strain and flame stretch such as curvature effects, the stretched laminar flame $S_d$ speed can take the following form for low flame stretch level: $S_d = S_d^0 - M \kappa \delta$ where the Markstein number is defined by $M = L/\delta$ and $\delta$ is the laminar flame thickness. And where the Markstein length $L$ is the variation of flame speed with respect to the flame stretch.

These previous results are very important to the understanding and definition of premixed turbulent combustion flame speed in the flamelets regimes, corrugated or wrinkled regimes. The flame displacement speed $S_d$ in the turbulent premixed regimes where the flame can be seen as wrinkled or corrugated can be modeled with the following expression that include a thermal diffusion and a reacting term:

$$S_d = \frac{\nabla \cdot \lambda/c_p \nabla T + \omega_T/c_p}{\rho |
abla T|}$$

(1)

The first term of this expression for the displacement flame speed includes the thermal diffusion effects whereas the second term includes the heat release effects. The first term has a key role in determining the local unstretched ($S_d^0$) and stretched laminar ($S_d$) unburnt gas flame displacement speed whereas the second term becomes dominant within the internal turbulent flame structure. This expression can be indeed evaluated throughout the flame front.

To illustrate this point, one can provide order of magnitudes for each terms of Eq. 1 which results in: $S_d = S_d^0 + \delta \omega_T/(T_b - T_u) \rho c_p$, where $S_d^0 = \lambda/(\rho c_p \delta)$.

In addition to this expression, the so-called turbulent flame speed quantity $S_T$ correlation has been proposed for many experiments at various conditions (turbulence intensity, fuels...). In general, these correlations can take the following canonical form: $S_T/S_d^0 = 1 + a u'/S_d^0 b$ where $a$ and $b$ are experimentally obtained. In the present article, additional expressions are derived for turbulent premixed combustion regime by making use of (i) the kinematic relationship between the flame surface speed, the flow speed and the flame speed, and (ii) the static/dynamic splitting of Eq. (1). Throughout the manuscript, a quantity $\phi_b(x)$ stands for a static quantity and a quantity $\phi'_b(x, t)$ stands for a dynamic quantity.

There are two main methodologies to model and predict combustion instability (dynamic stability). The first one relies on the numerical simulation of the full coupled system (flow, acoustics, and combustion) with relevant boundary conditions while the second one relies on determining the so-called flame transfer function by coupling this latter to the acoustics network description of the system investigated. The flame response of premixed swirling flames is thus an important component of the prediction of combustion instabilities in power and propulsion systems. Consequently, it is required to obtain this flame response with experimental, numerical or theoretical tools. The theoretical determination of the flame transfer function for premixed swirling flames has been the topic of a few studies. In this article, two expressions of the local flame displacement speed are used to derive estimates of the coefficients of existing theoretical premixed swirling flame transfer function. In addition, the effect of the flame angle (id est the swirl number) on the flame transfer function is carried out.

Premixed swirling flame transfer function has been investigated in several research work because of their central role in combustion instability prediction. These studies have been experimental, numerical and theoretical. The main findings of these studies have been on the description and the identification of the mechanisms responsible of the flame response shape as a function of the frequency. These mechanisms are respectively the swirl number oscillation and the vortex rollup. These have been characterized and the flow mode conversion at the swirler responsible of the flame dynamics discovered and documented. A component that has received less attention is the theoretical swirling flame transfer function whereas such model enable to take into account key physical effects. It has been shown that the analytical model response of turbulent swirling premixed flame submitted to upstream flow modulation can be derived by making use of a specific expression of the relative

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### Table 1. List of flame displacement speed definitions.

| Name                        | Expression for $S_d$ | Reference |
|-----------------------------|----------------------|-----------|
| Laminar unstretched flame speed $S_d^0$ | $S_d^0$            | 8         |
| Laminar stretched flame speed $S_L$ | $S_L = S_d^0 - M \kappa \delta$ | 2         |
| Local displacement speed $S_d$ | $S_d = \frac{\nabla \cdot \lambda/c_p \nabla T + \omega_T/c_p}{\rho |
abla T|}$ | 13        |
| Local displacement speed $S_d$ | $(w^3 - v) \cdot n$ | 2         |
| Static flame speed $S_{d,0}$ | $S_{d,0} = (u_0 + v_0 + w_0)^2$ | 14        |
| Turbulent flame speed $S_T$ | $S_T = S_d^0 (1 + a u'/S_d^0 b)$ | 15        |
fluctuating flame speed ratio. The obtained expression is reproduced in the present paper, see Eq. 59. It has been observed that this response is a function of the non-dimensionalized pulsation \( \omega_x \), the flame angle \( \alpha \), the phasing \( \phi \) between axial and azimuthal velocity perturbations and the coefficients \( \zeta \) and \( \chi \), specific to swirling flames. The determination of these two coefficients, as of today, has been undertaken experimentally17 or numerically24 without theoretical attempt such as proposed in the present paper. In addition, the effect of swirling flame angle has been studied and documented in a few studies only. Therefore, the flame angle parameter is varied over a wide range and the obtained trend is discussed with respect to reported data29—31 documenting the effect of swirl number on the flame response or on combustion instability amplitude.

In the present paper, the focus is on attempting to unify the two existing definitions of the local displacement flame speed \( S_d \) with emphasis on turbulent premixed swirling flames. This attempt enables also to lead to the determination of estimates for the local expressions of the premixed swirling flame transfer function coefficients. This process allows to unite the definition of flame speed in turbulent combustion with expressions used in combustion instabilities model prediction. The first section of the paper presents the configuration. In the second section of this paper, the flame speed is derived based on a kinematic relationship, in the third it is derived based on the energy equations. In the fourth section, the two expressions are matched to derive the flame transfer function coefficients. The last section presents an application of the results on the swirling flame response.

**Geometry**

Figure 1 presents the geometry considered in this paper and the cartesian and cylindrical coordinates used. The figure represents a typical conical laminar flame configuration (left) and a turbulent premixed flame configuration (right). The presence of the swirler generates the swirling flame.

**Derivation of the fluctuating flame speed based on kinematic relationship**

*Expressions for the static component of stabilized flame*

The general expression linking the premixed flame displacement speed along the normal at the flame front \( S_d \), the flame surface speed \( w_s \), and the flow speed \( v \) is:

\[
w_s = v + S_d n \tag{2}
\]

Static components of stabilized flame configurations are analyzed and derived in the present section. The starting point is the expression Eq. 2 where the flame surface speed is zero as the static flame component is fixed in the laboratory frame of reference. This is reflected by:

\[
v_0 = -S_d n_0 \tag{3}
\]

The static flame front normal vector is obtained with the \( G \) field normalized gradient where a given value of \( G \) corresponds to a flame isosurface:

\[
n_0 = -\nabla G_0/||\nabla G_0|| = -\nabla T_0/||\nabla T_0|| \tag{4}
\]

From Eq. 4, one obtains the static normal vector components:

\[
\begin{align*}
    n_x,0 &= -\frac{\partial G_0}{\partial x} \left[ \left( \frac{\partial G_0}{\partial x} \right)^2 + \left( \frac{\partial G_0}{\partial y} \right)^2 + \left( \frac{\partial G_0}{\partial z} \right)^2 \right]^{1/2} \\
    n_y,0 &= -\frac{\partial G_0}{\partial y} \left[ \left( \frac{\partial G_0}{\partial x} \right)^2 + \left( \frac{\partial G_0}{\partial y} \right)^2 + \left( \frac{\partial G_0}{\partial z} \right)^2 \right]^{1/2} \\
    n_z,0 &= -\frac{\partial G_0}{\partial z} \left[ \left( \frac{\partial G_0}{\partial x} \right)^2 + \left( \frac{\partial G_0}{\partial y} \right)^2 + \left( \frac{\partial G_0}{\partial z} \right)^2 \right]^{1/2}
\end{align*}
\]

\[
\tag{5}
\]
Making use of those vector components into Eq. 3, one has:

\[
\begin{align*}
    u_0 &= -S_{d,0} n_{x,0} = S_{d,0} \frac{\partial G_0}{\partial x} a_0 \\
    v_0 &= -S_{d,0} n_{y,0} = S_{d,0} \frac{\partial G_0}{\partial y} a_0 \\
    w_0 &= -S_{d,0} n_{z,0} = S_{d,0} \frac{\partial G_0}{\partial z} a_0
\end{align*}
\]  

(6)

where \( a_0 \) is defined as:

\[
    a_0 = \left[ \left( \frac{\partial G_0}{\partial x} \right)^2 + \left( \frac{\partial G_0}{\partial y} \right)^2 + \left( \frac{\partial G_0}{\partial z} \right)^2 \right]^{-1/2}
\]  

(7)

Combining equations two by two in the system Eqs. 6 leads to:

\[
\begin{align*}
    &u_0 = -S_{d,0} \frac{\partial G_0}{\partial x} = v_0 \frac{\partial G_0}{\partial y} = w_0 \frac{\partial G_0}{\partial z} \\
    &v_0 = w_0 \frac{\partial G_0}{\partial y} = u_0 \frac{\partial G_0}{\partial z} \\
    &w_0 = u_0 \frac{\partial G_0}{\partial z} = v_0 \frac{\partial G_0}{\partial y}
\end{align*}
\]  

(8)

It is now possible to explicitly link the flame speed and the flow speeds. It is conducted by making use of the systems of equations Eqs. 6 and Eqs. 8. One has:

\[
    u_0 = S_{d,0} \frac{\partial G_0}{\partial x} a_0 = S_{d,0} \left[ \frac{\partial G_0}{\partial x} \right] \left[ \left( \frac{\partial G_0}{\partial x} \right)^2 + \left( \frac{\partial G_0}{\partial y} \right)^2 + \left( \frac{\partial G_0}{\partial z} \right)^2 \right]^{-1/2}
\]  

(9)

By factorizing the axial component of the gradient of \( G_0 \), one obtains:

\[
    u_0 = S_{d,0} \frac{\partial G_0}{\partial x} \left[ \left( \frac{\partial G_0}{\partial x} \right)^2 + \left( \frac{\partial G_0}{\partial y} \right)^2 + \left( \frac{\partial G_0}{\partial z} \right)^2 \right]^{-1/2}
\]  

(10)

This last expression becomes:

\[
    u_0 = S_{d,0} \left[ 1 + \left( \frac{v_0}{u_0} \right)^2 + \left( \frac{w_0}{u_0} \right)^2 \right]^{-1/2}
\]  

(11)

The same operations lead to the following expressions for the other vector components:

\[
    v_0 = S_{d,0} \left[ 1 + \left( \frac{u_0}{v_0} \right)^2 + \left( \frac{w_0}{v_0} \right)^2 \right]^{-1/2}
\]  

(12)

and:

\[
    w_0 = S_{d,0} \left[ 1 + \left( \frac{u_0}{w_0} \right)^2 + \left( \frac{v_0}{w_0} \right)^2 \right]^{-1/2}
\]  

(13)

Any of these expressions leads to the following result for a configuration where the static flame surface speed is zero:

\[
    S_{d,0}^2 = u_0^2 + v_0^2 + w_0^2
\]  

(14)

This expression states the balance between the static flame speed and the components of the static velocity vector. For example, for a 1D laminar, one has directly: \( S_{d,0} = u_0 \). For a 3D turbulent flame, this expression indicates that the local static flame speed is directly related to the three components of velocity at the flame front, which is important for turbulent combustion. Indeed, the flame speed \( S_d \) is a well defined known quantity for laminar flame and this quantity may hold for certain regimes (flamelet, corrugated) of turbulent premixed combustion corresponding to low Karlovitz number where the flame front wrinkled by the upstream turbulence propagates at that speed \( S_d \) locally. Knowing the estimates of the flow speeds in a combustor and the 1D laminar flame speed of characteristic fuels premixture, it is possible to indicate that the previous expression implies two consequences. The first one is that the static flame component is fully determined by the components of the static velocity vector. As it is expected that the static flame speed \( S_{d,0} \) is equivalent to the laminar flame speed (including flow strain, flame dilation and curvature), this implies that the flame stabilize in low velocity region, regions satisfying expression Eq. 14. The second consequence is that for turbulent flame as well, the previous expression of \( S_{d,0} \) can be seen as a surrogate of the turbulent flame speed \( S_T \).

**Expressions for the dynamic component of stabilized flame**

*Canonical form.* The method to derive the link between flow, flame and flame surface speeds for the dynamic component of stabilized flame is equivalent to that of the previous section but now includes the fluctuating terms. Starting with Eq. 2, one obtains:

\[
\begin{align*}
    u &= w_s^f + S_d \frac{\partial G}{\partial x} a \\
    v &= \omega_s^f + S_d \frac{\partial G}{\partial y} a \\
    w &= \omega_s^f + S_d \frac{\partial G}{\partial z} a
\end{align*}
\]  

(15)

where \( \alpha \) is defined as:

\[
    \alpha = \left[ \left( \frac{\partial G}{\partial x} \right)^2 + \left( \frac{\partial G}{\partial y} \right)^2 + \left( \frac{\partial G}{\partial z} \right)^2 \right]^{-1/2}
\]  

(16)

Similarly to the previous section, the calculations lead to the following system of equations:

\[
\begin{align*}
    (u - w_s^f) \frac{\partial G}{\partial y} &= (v - w_s^f) \frac{\partial G}{\partial x} \\
    (v - w_s^f) \frac{\partial G}{\partial z} &= (w - w_s^f) \frac{\partial G}{\partial y} \\
    (w - w_s^f) \frac{\partial G}{\partial x} &= (u - w_s^f) \frac{\partial G}{\partial z}
\end{align*}
\]  

(17)
Combining the system of equations Eqs. 17 and Eqs. 15, the following system is determined:

$$\begin{align*}
\frac{u - w^s_x}{S_d} &= 1 + \left( \frac{v - w^y_y}{u - w^s_x} \right)^2 + \left( \frac{w - w^z_z}{u - w^s_x} \right)^2 \\
\frac{v - w^y_y}{S_d} &= 1 + \left( \frac{u - w^s_x}{v - w^y_y} \right)^2 + \left( \frac{w - w^z_z}{v - w^y_y} \right)^2 \\
\frac{w - w^z_z}{S_d} &= 1 + \left( \frac{u - w^s_x}{w - w^z_z} \right)^2 + \left( \frac{v - w^y_y}{w - w^z_z} \right)^2 \\
\end{align*}$$

Those expressions can also be written as:

$$\begin{align*}
\frac{u - w^s_x}{S_d} &= \left[ 1 + \left( \frac{v - w^y_y}{u - w^s_x} \right)^2 + \left( \frac{w - w^z_z}{u - w^s_x} \right)^2 \right]^{-1/2} \\
\frac{v - w^y_y}{S_d} &= \left[ 1 + \left( \frac{u - w^s_x}{v - w^y_y} \right)^2 + \left( \frac{w - w^z_z}{v - w^y_y} \right)^2 \right]^{-1/2} \\
\frac{w - w^z_z}{S_d} &= \left[ 1 + \left( \frac{u - w^s_x}{w - w^z_z} \right)^2 + \left( \frac{v - w^y_y}{w - w^z_z} \right)^2 \right]^{-1/2} \\
\end{align*}$$

The first equation of the system Eq. 19 can be recast after introducing the decomposition with the static and dynamic components as:

$$S_{d,0} + S_d = \left( u_0 + u' - w^s_{x,0} - w^s_{z,0} \right)^2$$

$$+ \left( v_0 + v' - w^y_{y,0} - w^y_{z,0} \right)^2$$

$$+ \left( w_0 + w' - w^z_{z,0} - w^z_{z,0} \right)^2$$

This last expression links the flame surface, the flow and flame displacement static and dynamic speeds. This is a general expression where fluctuation can be from turbulence, broadband noise or acoustics/vortical modulation. When considering only static components (dynamic fluctuations are zero) this expression is equal to Eq. 14. This expression states that both the static and dynamic components of the flame speed, flame displacement speed and flow velocities are balanced.

**Expression for turbulent stabilized flames.** Now, the dynamic fluctuation is assumed turbulent only or acoustic, or mixed. In the next section, the dynamic fluctuations will be assumed to be dominated by harmonic flow modulation. By elevating Eq. 20 to the power two, assuming the static flame surface $w_s = 0$, and identifying terms on left and right hand sides by the order of fluctuations, one obtains:

$$\begin{align*}
S_{d,0}^2 &= u_0^2 + v_0^2 + w_0^2 \\
S_d^2 &= u_0(u' - w^s_x)^2 + v_0(v' - w^y_y)^2 + w_0(w' - w^z_z)^2
\end{align*}$$

The first equation of the system Eqs. 21 has been discussed in the previous section. The second equation reflects the coupling of the static and dynamic flow, flame and flame surface speeds components. This coupling could be responsible for transition phenomena from laminar to turbulent combustion or transition to blowout and flashback as coupling between static and dynamic component occurs. Indeed, disturbances (including flame surface elements) are propagated (convected) by the static flow and that equation reflects that coupling. Assuming a form of disturbances such as of harmonic form with $\exp(-i\omega t)$ will directly link the flow speed $u$ to the wavevectors $k$ because $k = \omega/\omega$ where $\omega = 2\pi f$. The last expression is on the dynamic components only and shows that for a stabilized flame, the propagation of the flame surface is solely due to turbulence and local dynamic flame speed. This equation provides the balance between disturbances amplitudes and phases. The impact of those results are multiples. First, the flame stabilizes where the static component of flow and flame speed satisfy the first condition. Secondly, at that location, the flame stabilizes (the dynamic flame surface component is nearly oscillating around zero) in the turbulent flow at the local dynamic flame speed. This local dynamic flame speed is close to the laminar flame speed as the only physical values that can be taken by the fluctuating local dynamic flame speed should be of the order of the laminar flame speed. Finally, at that location the flame surface propagates through turbulence fluctuations by alternating positive and negative fluctuations. The turbulent flame propagates (the flame surface changes position due to convection and wrinkles) through this mechanism with the turbulent fluctuations below or above the flame speed $S_d$ values. These fluctuations change the flame position around the static component. In other words, the propagation depends on the turbulence level with respect to $S_d$. This could further lead to flashback of blowout triggering for very low or high values of the turbulent fluctuation. Amplitudes and frequencies of the fluctuations are then important to the flame stabilization process. Indeed, while the flame propagates for values of the turbulent fluctuations below or above the laminar flame speed order of magnitude, the stabilization is consequently ensured for alternating periodic positive and negative turbulent/acoustic fluctuations around the static field.

**Expression for modulated stabilized flames.** In this section, the dynamic component (the fluctuation field) is made of an harmonic modulation and turbulent fluctuations and the amplitude of the harmonic modulation is dominating over the turbulent fluctuation. We consider the specific case of Eq. 20 when the static component of the flame surface speed is zero so that Eq. 14 is satisfied. This
leads, after expanding the power square terms and neglecting the product of fluctuations, to:

\[
\frac{S'_d}{S_{d,0}} = \frac{u' - w'^r}{u_0} \left( \frac{u_0}{S_{d,0}} \right)^2 + \frac{v' - w'^\theta}{v_0} \left( \frac{v_0}{S_{d,0}} \right)^2 + \frac{w' - w'^r}{w_0} \left( \frac{w_0}{S_{d,0}} \right)^2
\]  

(22)

This last equation can be reformulated in cylindrical coordinates. The starting point consists of the expressions of the components of velocities \( v \) and \( w \) in cylindrical coordinates:

\[
\begin{align*}
\{v \, e_r\} &= u_r \cos \theta \, e_r - u_\theta \sin \theta \, e_\theta \\
\{w \, e_\theta\} &= u_r \sin \theta \, e_r + u_\theta \cos \theta \, e_\theta
\end{align*}
\]  

(23)

These relationships can also be expressed with their static and dynamic components. For the \( v \) term, one obtains assuming small angle fluctuations:

\[
[\{v(0) + v'\} e_r]^2 = [\{u_{r,0} + u'_r\} \cos (\theta_0) e_r - (u_{\theta,0} + u'_\theta) \sin (\theta_0) e_\theta]^2
\]  

(24)

That last equation can be expanded as:

\[
\begin{align*}
(v_0 + v')^2 e_r \cdot e_r &= (u_{r,0} + u'_r)^2 \cos^2 (\theta_0) e_r \cdot e_r \\
- 2(u_{r,0} + u'_r) \cos (\theta_0)(u_{\theta,0} + u'_\theta) \sin (\theta_0) e_r \cdot e_\theta \\
+ (u_{\theta,0} + u'_\theta)^2 \sin^2 (\theta_0) e_\theta \cdot e_\theta
\end{align*}
\]  

(25)

which leads to:

\[
\begin{align*}
v'_0 + 2v_0 v' + v'^2 &= [(u_{r,0} + u'_r) \cos \theta_0]^2 + [(u_{\theta,0} + u'_\theta) \sin \theta_0]^2
\end{align*}
\]  

(26)

The identification of the static and remaining components leads by neglecting the second order term, to:

\[
\begin{align*}
v'_0 &= u_{r,0}^2 \cos^2 \theta_0 + u_{\theta,0}^2 \sin^2 \theta_0 \\
2v_0 v' &= 2u_{r,0} u'_r \cos^2 \theta_0 + 2u_{\theta,0} u'_\theta \sin^2 \theta_0
\end{align*}
\]  

(27)

The ratio of these two last quantities and a similar calculation for the term \( w \) give the expressions below that relates velocities in cartesian and cylindrical coordinates:

\[
\begin{align*}
\frac{v'}{v_0} &= \frac{u'_r \cos^2 \theta_0 + u'_\theta \sin^2 \theta_0}{u_{r,0}^2 \cos^2 \theta_0 + u_{\theta,0}^2 \sin^2 \theta_0} \\
\frac{w'}{w_0} &= \frac{u'_r \sin^2 \theta_0 + u'_\theta \cos^2 \theta_0}{u_{r,0}^2 \sin^2 \theta_0 + u_{\theta,0}^2 \cos^2 \theta_0}
\end{align*}
\]  

(28)

The expressions of the fluctuating ratios are now evaluated at \( \theta = \pi/2 \) to simplify the calculations. This is justified for axisymmetric static flow and axisymmetric flow perturbations. It leads to the following ratios:

\[
\begin{align*}
v'/v_0 &= u'_r/u_{r,0} \quad \text{and} \quad w'/w_0 = u'_r/u_{r,0}
\end{align*}
\]  

Similar expressions can be obtained within the same assumptions for the flame surface speed. Substituting those expressions into Eq. 22 leads to:

\[
\begin{align*}
\frac{S'_d}{S_{d,0}} &= \frac{u' - w'^r}{u_0} \left( \frac{u_0}{S_{d,0}} \right)^2 + \frac{u'_r - w'^r}{u_{r,0}} \left( \frac{u_{r,0}}{S_{d,0}} \right)^2 + \frac{w'_r - w'^r}{w_{r,0}} \left( \frac{w_{r,0}}{S_{d,0}} \right)^2
\end{align*}
\]  

(29)

Next, the equation is developed so that, one obtains:

\[
\begin{align*}
\frac{S'_d}{S_{d,0}} &= \left[ \frac{u'_0 - w'_r}{u_{r,0}} \left( \frac{u_{r,0}}{S_{d,0}} \right)^2 + \frac{u'_r - w'_r}{u_{r,0}} \left( \frac{u_{r,0}}{S_{d,0}} \right)^2 \right] \frac{u'_r}{u_{r,0}}
\end{align*}
\]  

(30)

It is then possible to show that the ratios \( u'_{r,0} / u_{r,0} \) and \( u'_{\theta,0} / u_{\theta,0} \) are equivalent if dominated by the harmonic modulation, see the derivation in Ref. 32 reported next. Indeed, expressions for the azimuthal and radial fluctuating amplitudes and their wavevectors can be obtained. The starting point consists of the linearized Euler momentum equations in cylindrical coordinates along the radial and azimuthal directions with the axisymmetric assumption for the flow. These equations are the governing equations that can describe the non-reacting unsteady flow downstream of a swirler in linear regime with low amplitude of modulation. These results provide the mode of propagation of azimuthal and radial perturbations due to mode conversion occurring at a swirler. The fluctuating quantities are assumed to depend only on the longitudinal direction \( x \). These equations are:

\[
\begin{align*}
\rho \frac{\partial u'_r}{\partial t} - \rho \frac{\partial u'_0}{r} + \frac{\partial}{\partial x} \left( \rho u'_0 u'_r \right) &= 0
\end{align*}
\]  

(31)

\[
\begin{align*}
\rho \frac{\partial u'_\theta}{\partial t} + \frac{\partial}{\partial x} \left( \rho u'_\theta u'_r \right) &= 0
\end{align*}
\]  

(32)

Where \( \rho \) is the fluid density and \( (u_r, u_\theta, u_z) \) the velocity components.

The fluctuating density has an upstream and downstream propagating component. The associated wavevectors components \( k'_r \) and \( k'_\theta \) and the amplitudes are \( \rho^- \) and \( \rho^+\):

\[
\rho' = \left( \rho^- \exp (i k'_r x) + \rho^+ \exp (-i k'_r x) \right) \times \exp (-i2\pi ft)
\]  

(33)

To reflect the convective nature (downstream propagation) of the fluctuating velocities \( u'_r \) and \( u'_\theta \) along the axial direction, the following Fourier decompositions are used (where the associated wavevector components are \( k_{r,0} \) and \( k_{\theta,0} \) and the associated amplitudes are \( \tilde{u}'_r \) and \( \tilde{u}'_\theta \)):

\[
\begin{align*}
u'_r &= \tilde{u}'_r \exp (i k_{r,0} x) \exp (-i2\pi ft)
\end{align*}
\]  

(34)
\[ \hat{u}' - \hat{u}_0 \exp (ikx) \exp (-i2\pi ft) \]  

Inserting Eq. (33), Eq. (34) and Eq. (35) into Eq. (31) and Eq. (32) leads to two equations which are functions of the fluctuating quantities \( \hat{u}' \) and \( \hat{u}_0' \) and their respective wavevector components \( k_u \) and \( k_w \). Adding the first obtained equation to the second one multiplied by the ratio \( (\hat{u}_0/\hat{u}_0') \) leads to:

\[ \hat{u}' \left[ -\hat{p}2\pi f + \hat{p}\hat{u}_0k_u + \hat{p}\hat{u}_0^2 \right] \exp (ikx) + \hat{u}_0' \left[ -2\hat{p}\hat{u}_0 \hat{u}_0' \hat{r} + \hat{p}\hat{u}_0^2 \right] \exp (ikx) = 0 \]  

The next step consists in expressing the real and imaginary parts of Eq. 36 and expressed them for \( x = 0 \), which corresponds to the swirler outlet. It leads for the real part to:

\[ \hat{u}' \left[ \hat{p}\hat{u}_0^2 \hat{r} + \hat{u}_0' \left[ -2\hat{p}\hat{u}_0 \hat{u}_0' \hat{r} + \hat{p}\hat{u}_0^2 \right] \right] = 0 \]  

And for the imaginary part it leads to:

\[ \hat{u}' \left[ -\hat{p}2\pi f + \hat{p}\hat{u}_0k_u \right] + \hat{u}_0' \left[ -2\hat{p}\hat{u}_0 \hat{u}_0' \hat{r} + \hat{p}\hat{u}_0^2 \right] \exp (ikx) = 0 \]  

As a consequence, the ratio of radial and azimuthal convective wave amplitudes is obtained from the real part Eq. 37:

\[ \frac{\hat{u}'_r}{\hat{u}_0'} = \frac{\hat{u}_r}{\hat{u}_0} \]  

Making use of \( k_u = 2\pi f / \nu_r \) and noting \( \nu_r \) the convective velocity of the radial fluctuations, the azimuthal component of the wavevector is obtained from the imaginary part Eq. 38:

\[ k_u = 2\pi f / \nu_r \times \left[ 1 + \left( \frac{\hat{u}_r}{\hat{u}_0} \right)^2 \times \left( 1 - \frac{\hat{u}_r}{\nu_r} \right) \right] \]  

It has been shown in previous work\(^{33}\) that the azimuthal fluctuations propagate at the convective axial velocity of the flow implying that the term in brackets of Eq. 40 is unity. It implies that \( \nu_r = \hat{u}_r \). This demonstrates that the radial fluctuations also propagate at the axial velocity of the flow. To conclude this analytical development, the wavevector and amplitude calculation leads to the equality \( u'_r/u_{0,r} = u'_0/u_{0,0} \). Taking into account that equality and rearranging the terms, the following expression is finally determined for Eq. 30:

\[ \frac{S'_d}{S_{d,0}} = \zeta \frac{u'_r}{u_{0,r}} + \chi_e \frac{u'_0}{u_{0,0}} \]  

where:

\[ \zeta = \left[ \frac{u^2_0}{S^2_{d,0}} - \frac{u^2_{0,r}}{u^2_{0,r}} \right] \]  

and:

\[ \chi_e = \left[ \frac{u^2_0}{S^2_{d,0}} \left( 1 - \frac{w^2_{x,r}}{u^2_{0,r}} \right) \right] \]  

In Eq. 41, one can see that the expression is similar to previous work\(^{32}\) except that terms here are evaluated at the flame front. In previous work\(^{34}\) the flow velocity ratios were expressed upstream the flame front. To retrieve that expression, a phase shift should be included into Eq. 41. This result confirms that the turbulent burning speed is a function of the axial and azimuthal components and leads to an exact expression for the previously reported result. The terms ratios of flame surface speed to velocity fluctuation along each direction in that last equation are expected to be of the order of unity as the flow speed will dominate the overall budget between flow, flame and flame surface speeds.

### Derivation of the fluctuating flame speed based on thermal energy equation

The second derivation now undertaken is based on an analogy between the temperature governing equation and the G-Equation.\(^{13}\) In that analogy, the flame speed can be written as:

\[ S_d = \frac{\nabla \cdot \left( \frac{\lambda}{c_p} \rho \nabla T \right) + \dot{\rho} T}{c_p} \]  

This expression of the flame speed is compared with the flow velocity for a laminar methane-air atmospheric 1D flame at equivalence ratio of 0.7 in Fig. 2.

The numerical simulation of that steady flame was conducted in Cantera with a complex chemistry mechanism on a 5 µm spacing mesh. In this figure, the red curve represents the flow speed throughout the flame front while the blue curve represents the flame speed obtained with Eq. 44 by post-processing the temperature and reacting terms. It is worth to point out that the two curves have a nearly perfect matching and the unburnt gas unstretched laminar flame speed is retrieved as measured on the upstream flame side.

In expression Eq. 44 for the flame speed, there are two main terms: a thermal diffusion term and a reacting term. It is now possible to split this last expression by considering each variable as the sum of a static and dynamic component. This splitting leads to:

\[ (S_{d,0} + S'_{d,0})(\rho_0 + \rho')(\nabla(T_0 + T'_0)) = \nabla \cdot \left( \frac{\lambda}{c_p} \nabla(T_0 + T'_0) + (\dot{\rho} T + \dot{T}' \nabla T) \right) \]  

where:

\[ \zeta = \left[ \frac{u^2_0}{S^2_{d,0}} - \frac{u^2_{0,r}}{u^2_{0,r}} \right] \]  

and:

\[ \chi_e = \left[ \frac{u^2_0}{S^2_{d,0}} \left( 1 - \frac{w^2_{x,r}}{u^2_{0,r}} \right) \right] \]
Next, the thermal gradient modulus is expanded as follow by considering only first order terms and by considering that the thermal fluctuation are along the thermal gradient direction and thus \( \cos(VT_0, VT_0') \) is unity, i.e. that the thermal fluctuation are perpendicular to the thermal flame front:

\[
|VT| = \left( \left( \frac{\partial T_0}{\partial x} \right)^2 + \left( \frac{\partial T_0}{\partial y} \right)^2 + \left( \frac{\partial T_0}{\partial z} \right)^2 \right)^{1/2}
\]

\[
= \left( \left( \frac{\partial T_0}{\partial x} + \frac{\partial T_0'}{\partial x} \right)^2 + \left( \frac{\partial T_0}{\partial y} + \frac{\partial T_0'}{\partial y} \right)^2 \right)^{1/2}
\]

\[
+ \left( \frac{\partial T_0}{\partial z} + \frac{\partial T_0'}{\partial z} \right)^2 \right)^{1/2}
\]

\[
+ \left( \frac{\partial T_0}{\partial x} + \frac{\partial T_0'}{\partial x} \right)^2 + \left( \frac{\partial T_0}{\partial y} + \frac{\partial T_0'}{\partial y} \right)^2 + 2 \frac{\partial T_0}{\partial x} \frac{\partial T_0'}{\partial y}
\]

\[
+ \left( \frac{\partial T_0}{\partial z} + \frac{\partial T_0'}{\partial z} \right)^2 + 2 \frac{\partial T_0}{\partial x} \frac{\partial T_0'}{\partial z} \right)^{1/2}
\]

\[
= |VT_0 + VT_0'| + \left( |VT_0|^2 + |VT_0'|^2 + 2VT_0 \cdot VT_0' \right)^{(1/2)}
\]

\[
= |VT_0| \left[ 1 + \frac{|VT_0'|^2}{|VT_0|^2} + 2 \frac{VT_0 \cdot VT_0'}{|VT_0|^2} \right]^{1/2}
\]

\[
= |VT_0| \left[ 1 + \frac{|VT_0'|^2}{|VT_0|^2} + 2 \frac{|VT_0||VT_0'| \cos(VT_0, VT_0)}{|VT_0|^2} \right]^{1/2}
\]

\[
= |VT_0| \left[ 1 + 2 \frac{|VT_0'|}{|VT_0|} \right]^{1/2}
\]

\[
= |VT_0| \left[ 1 + |VT_0'| \right]
\]

\[
= |VT_0| + |VT_0'|
\]

By developing Eq. (45) and using the previous expression derived, one can find:

\[
\left[ S_{d,0} \rho_0 + S_{d,0} \rho_0' + S_{d,0}' \rho_0 + S_{d,0}' \rho_0' \right] \left[ |VT_0| + |VT_0'| \right]
\]

\[
\frac{\lambda}{c_p} \Delta T_0 + |VT_0| \frac{\lambda}{c_p} \Delta T_0 + |VT_0'| \frac{\lambda}{c_p} \Delta T_0 + \frac{\lambda}{c_p} \frac{\partial T_0}{c_p} + \frac{\lambda}{c_p} \frac{\partial T_0'}{c_p}
\]

(47)

By further developing this expression, subtracting the static component of the flame speed and performing order of fluctuation equivalency, one obtains for the fluctuating flame speed:

\[
S'_{d,0} = \frac{\nabla \cdot \left[ \frac{\lambda}{c_p} VT_0' \right] + \frac{\lambda}{c_p} VT_0 + \rho_0 |VT_0'| + \rho_0' |VT_0|}{\rho_0 |VT_0|}
\]

(48)

By then taking the ratio of the fluctuating displacement flame speed to its static value, one obtains:

\[
\frac{S'_{d,0}}{S_{d,0}} = \frac{\nabla \cdot \left[ \frac{\lambda}{c_p} VT_0' \right] + \frac{\lambda}{c_p} VT_0 + \rho_0 |VT_0'| + \rho_0' |VT_0|}{\rho_0 |VT_0|}
\]

(49)

One can furthermore consider that the second right hand term is negligible compared to the first term so that the expression reduced to:

\[
\frac{S'_{d,0}}{S_{d,0}} = \frac{\nabla \cdot \left[ \frac{\lambda}{c_p} VT_0' \right] + \frac{\lambda}{c_p} VT_0}{\rho_0 |VT_0|}
\]

(50)

The neglected term includes two sub-terms: (1) a non-dimensional thermal gradient fluctuation term, and (2) a non-dimensional density fluctuation term. Physically, these non-dimensional fluctuations are negligible compared to heat release fluctuation and thermal diffusion fluctuations due to amplitude of heat release fluctuation and length scale at work, respectively.

**Derivation of the flame transfer function coefficients**

The procedure to determine the swirling flame transfer function coefficients is as follow. The two expressions Eq. 41 and Eq. 50 are equalized to determine the coefficients \( \xi \) and \( \chi \) of the swirling flame response. To do so, the terms of Eq. 50 are first recast making use of the temperature governing equation. The starting point is the following equation deduced from Eq. (44) by taking the static component:

\[
S_{d,0} \rho_0 |VT_0| = \nabla \cdot \left[ \frac{\lambda}{c_p} VT_0 \right] + \frac{\lambda}{c_p} \frac{\partial T_0}{c_p}
\]

(51)
Making use of Eq. 51 with Eq. 50, one gets:

$$\nabla \cdot \left[ \frac{\lambda}{c_p} \nabla T_0 \right] + \frac{\omega_T}{c_p} = S_{d,0} \rho_0 |\nabla T_0|$$  \hspace{1cm} (52)

The thermal energy equation for the temperature is now written in cylindrical coordinates as:

$$\rho \left[ \frac{\partial T}{\partial t} + u_r \frac{\partial T}{\partial r} + \frac{u_0}{r} \frac{\partial T}{\partial \theta} + u_z \frac{\partial T}{\partial z} \right]$$

$$= \nabla \cdot \left[ \frac{\lambda}{c_p} \nabla T \right] + \frac{\omega_T}{c_p}$$  \hspace{1cm} (53)

The static equation writes:

$$\rho_0 \left[ \frac{\partial T_0}{\partial t} + u_r,0 \frac{\partial T_0}{\partial r} + \frac{u_{0,0}}{r} \frac{\partial T_0}{\partial \theta} + u_z,0 \frac{\partial T_0}{\partial z} \right]$$

$$= \nabla \cdot \left[ \frac{\lambda}{c_p} \nabla T_0 \right] + \frac{\omega_T,0}{c_p}$$  \hspace{1cm} (54)

Subtracting Eq. 54 from the expanded form of Eq. 53 results in the following expression by retaining only first order term and subtracting static component:

$$\nabla \cdot \left[ \frac{\lambda}{c_p} \nabla T_0 \right] + \frac{\omega_T,0}{c_p}$$

$$= \rho_0 \left[ \frac{\partial T_0}{\partial t} + u_r,0 \frac{\partial T_0}{\partial r} + \frac{u_{0,0}}{r} \frac{\partial T_0}{\partial \theta} + u_z,0 \frac{\partial T_0}{\partial z} \right]$$

$$+ u_z,0 \frac{\partial T_0}{\partial z} + u_0 \frac{\partial T_0}{\partial r}$$  \hspace{1cm} (55)

It is next assumed that in the linear regime, the fluctuation of the temperature gradients components are of lower amplitude than their static values. It is also assumed that unsteadiness of the temperature fluctuation are negligible with respect to the products of velocity fluctuation terms and temperature gradient components. In addition, the static flow is assumed to be axisymmetric. These assumptions lead to the following equation for the relative ratio of flame displacement speed:

$$\frac{S_d}{S_{d,0}} = \left[ \frac{u_0,0}{S_{d,0} |\nabla T_0|} \right] \frac{u'}{u_0} + \left[ \frac{u_{r,0}}{S_{d,0} |\nabla T_0|} \frac{\partial T_0}{\partial r} \right] \frac{u'_{\theta}}{u_{0,0}}$$  \hspace{1cm} (56)

Next, one approximate the thermal gradient magnitude as: $|\nabla T_0| = (T_b - T_a) / \delta_{\theta}$ and introduce trigonometry with the flame angle to estimate both $\partial T_0 / \partial r$ and $\partial T_0 / \partial \theta$. This equation can be thus simplified further and the terms identified to Eq. 41 leading to the following estimates for the coefficients $\xi_{\theta}$ and $\chi_{\theta}$ of the premixed swirling flame transfer function:

$$\begin{align*}
\xi_{\theta} &= \frac{u_0,0}{S_{d,0}} \sin \alpha \\
\chi_{\theta} &= -\frac{u_0,0}{S_{d,0}} \cos \alpha
\end{align*}$$  \hspace{1cm} (57)

where $\alpha$ is the swirling flame angle. These coefficients are local values and are the premixed swirling flame transfer function coefficients.

As pointed in Ref.24, these coefficients are also linked by the relationship: $\xi_{\theta} = -\chi_{\theta}$ to satisfy the unity flame response gain value at zero frequency. Thus, equating both relationships leads to $\tan \alpha = u_{r,0} / u_{z,0}$ which is of interest for practical application.

**Swirling flame transfer function coefficients**

Experimental work documented17 in literature has shown that these coefficients have the following values: $\zeta = 0.4$ and $\chi = -0.4$. Recent estimated values from numerical simulations led to slightly different values than those.24 It is thus possible to evaluate the expressions of Eq. 57 to describe the evolution of the velocity ratios $u_{r,0} / S_{d,0}$ and $u_{z,0} / S_{d,0}$ as a function of the flame angle $\alpha$.

This process leads to the results depicted in Fig. 3. On this figure are given the evolution of the ratios of axial $u_{z,0} / S_{d,0}$ and radial $u_{r,0} / S_{d,0}$ velocity to local flame speed $S_{d,0}$ as a function of the flame angle $\alpha$. The blue curve corresponds to a constant $\zeta = 0.4$ coefficient while the red curve corresponds to a constant $\chi = -0.4$ coefficient. The horizontal line corresponds to an experimental flame angle $\alpha = 38^\circ$. The horizontal lines corresponds to the values of the velocity ratios for this flame angle. The obtained ratios of velocities for this flame angle are extracted from this figure and documented in Tab. 2.

![Figure 3. Ratios of axial $u_{z,0} / S_{d,0}$ and radial $u_{r,0} / S_{d,0}$ velocity to local flame speed $S_{d,0}$ as a function of the flame angle $\alpha$.](image-url)
Table 2. List of variables for local displacement flame speed assumptions 1 to 3.

| Variable                        | Assumption 1          | Assumption 2          | Assumption 3          |
|--------------------------------|-----------------------|-----------------------|-----------------------|
| Local flame speed $S_{d,0}$    | 0.2 ($S_d = \frac{S_0}{2}$) | 1.0 ($S_d \approx S_1$) | 1.64 ($S_d \approx S_T$) |
| Ratio $u_{r,0}/S_{d,0}$        | 0.5                   | 0.5                   | 0.5                   |
| Ratio $u_{\theta,0}/S_{d,0}$   | 0.33                  | 0.33                  | 0.33                  |
| Ratio $u_x/S_{d,0}$            | 0.65                  | 0.65                  | 0.65                  |
| Local radial velocity $u_{r,0}$| 0.1                   | 0.5                   | 0.82                  |
| Local azimuthal velocity $u_{\theta,0}$ | 0.066                | 0.33                  | 0.54                  |
| Local axial velocity $u_x$     | 0.13                  | 0.65                  | 1.07                  |
| Axial bulk velocity $U_b$      | 2.67                  | 2.67                  | 2.67                  |
| Flame angle $\alpha$          | 38°                   | 38°                   | 38°                   |
| Global flame speed $S_{T,0}$   | 1.64                  | 1.64                  | 1.64                  |
| Coefficient $\chi_e$          | -0.4                  | -0.4                  | -0.4                  |
| Coefficient $\zeta_e$         | 0.4                   | 0.4                   | 0.4                   |

Figure 4. Premixed swirling flame transfer function coefficients $\zeta$ and $\chi$ plotted as function of the flame displacement speed $S_{d,0}$ (y)-axis and the velocity ratios (x)-axis.
The obtained coefficients $\zeta$ and $\chi$ are now represented with spatial maps as functions of the displacement speed $S_{d,0}$ and the local velocity ratios $u_{r,0}/S_{d,0}$ and $u_{s,0}/S_{d,0}$. The results are depicted in Fig. 4. On each figure, the vertical lines correspond to the velocity ratio extracted from Fig. 3. These maps are obtained for a flame angle $\alpha = 38^\circ$. These maps can be used to determine the coefficients $\chi_e$ and $\zeta_e$ knowing the local flame displacement speed $S_{d,0}$ and the local velocity ratios $u_{r,0}/S_{d,0}$ and $u_{s,0}/S_{d,0}$. The white spaces on these maps are due to undefined/unphysical data. These spaces are the consequences of the static flowfield assumptions and the equations obtained. These maps are obtained with the assumption that $u_{\theta,0} = 0.6 u_{s,0}$.

**Premixed swirling flame transfer function**

While the previous sections have established links between flame speed definitions and the determination of the premixed swirling flame transfer coefficients, the present section focuses on investigating the effect of the flame angle $\alpha$ on the flame response. Because the focus of the paper is on the coefficients of the swirling flame transfer function, it is logical to include here an example of premixed swirling flame response with the values of those coefficients. In addition, theoretical estimates for the flame response trend with respect to the flame angle evolution have not been reported previously. These FTF results provide a path for future validation and verification that the assumption of constant flame response coefficients holds.

The flame transfer function of such flames$^{27}$ which is a function of the flame response expression of “V” flames$^{36}$ is first written for completeness:

$$F(\omega, \alpha) = \left( \frac{2}{\omega_e^2 (1 - \cos^2 \alpha)} \times \left[ \exp(i \omega_e) - 1 - \frac{( \exp(i \omega_e \cos^2 \alpha) - 1)}{\cos^2 \alpha} \right] \right.$$  
$$+ \frac{2i}{\omega_e (1 - \cos^2 \alpha)} \times \left[ ( \exp(i \omega_e \cos^2 \alpha) - \exp(i \omega_e)) \right]$$  
$$\times [1.0 - (\zeta + \chi \exp(i \phi))]$$

where $\omega_e = \omega R/S_{T,0} \cos \alpha$, the radius of the injector $R$ is 0.011 m, and the phase measured experimentally is $\phi = \omega \times 0.012 - 1.0$. While it has been shown in the previous section that the swirling premixed flame transfer function coefficients are dependent on the flame angle, it is assumed that the coefficients $\chi$ and $\zeta$ are constant for all swirl number (i.e. all flame angle) whereas the coefficients depend upon that flame angle. This assumption should be further investigated in future works on other swirled-stabilized

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**Figure 5.** Premixed swirling flame transfer function gain and phase.

Tab. 2 presents a list of variables of interest and estimated for three assumptions referred 1 to 3. Each assumption corresponds to a local flame speed value which support one of its possible order of magnitude. In assumption 1, the displacement flame speed $S_d$ is assumed to be equivalent to the local unstretched laminar flame speed $S_d^0$. In assumption 2, the flame speed $S_d$ is assumed to be of the order of the local stretched laminar flame speed. Finally, in assumption 3, the local flame speed is assumed to be of the order of the turbulent flame speed $S_T$. The objective of these three assumptions is to evaluate the resulting local flow velocities based on the velocity ratio determined from Fig. 3 and from the following expression used for the local azimuthal to flame speed ratio:

$$\left( \frac{u_{r,0}}{S_{d,0}} \right)^2 + \left( \frac{u_{\theta,0}}{S_{d,0}} \right)^2 + \left( \frac{u_{s,0}}{S_{d,0}} \right)^2 = 1 \quad (58)$$

The local velocity ratios and local speeds are also listed in Tab. 2. The unstretched flame speed was obtained for a methane/air premixture at atmospheric condition and equivalence ratio of 0.7. The stretched flame speed was estimated from computations data of swirled flame$^{10}$ and the turbulent speed using correlation in Ref.$^{35}$.

It is shown that whatever the flame speed value, the estimates of local velocity components are of similar amplitude reflecting the fact that the static flame stabilizes in regions of low local flow speeds.$^{14}$
configurations as for each swirl number these two quantities may change. It implicitly assumes that the velocity ratios obey to those given in Fig. 3. The results of the flame transfer function are given in Fig. 5 for the gain (top figure) and the phase (bottom figure) for varying flame angle from 31° to 59° and the baseline flame angle is 38°.

One first describes the gain curves of Fig. 5(a). In the range 0 to 65 Hz, the amplitude gain curves are nearly superimposed and the gain variation is limited. Particularly, the location of the minimum flame transfer function gain is recorded at the same frequency and same amplitude throughout all flame angles. Beyond 65 Hz, the gain curves show a clear amplitude reduction trend as the flame angle increases. This reflects that the flame transfer function gain decreases as the swirl number increases. Regarding the phases curves of Fig. 5(b), the curves present undulations of the phase at the minimum of the gain and are similar for all flame angle up to 100 Hz where strong differences begin to occurs beyond that point. These results are consistent with observations from numerical and experimental works documenting the effect of swirl number on the flame response or on combustion instability amplitude.

Conclusions

In this article, an attempt to link turbulent combustion and combustion instability was undertaken by uniting definitions of the flame speed. It is pointed out that the flame displacement speed is a key quantity that can be used to link flame stabilization and dynamic stability. A theoretical derivation was conducted to identify and determined premixed swirling flame transfer functions parameters. Results indicate that existing experimental values of these coefficients compare well to the theoretical predictions for various conditions. In this article, the effect of flame angle (id est implicitly of the swirl number) was investigated. The theoretical predictions indicate a gain drop as the flame angle is increased (id est the swirl number is increased) which is supported by numerical and experimental observations in the literature.

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