The Correlator of Topological Charge Densities
at low $Q^2$ in QCD.

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Abstract

The correlator of topological charge densities $\chi(Q^2)$ in QCD is calculated in the domain of low $Q^2$. Basing on the Ward identities the low energy theorems are proved. The value of the first derivative $\chi'(0)$ found recently by QCD sum rules is used. The contributions of pseudoscalar quasi-Goldstone bosons as intermediate states in the correlator are calculated with the account of $\eta - \pi$ mixing.

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I. INTRODUCTION

The existence of topological quantum number is a very specific feature of non-abelian quantum field theories and, particularly, QCD. Therefore, the study of properties of the topological charge density operator in QCD

$$Q_5(x) = \frac{\alpha_s}{8\pi} G_{\mu\nu}^a(x) \tilde{G}_{\mu\nu}^a(x)$$

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and of the corresponding vacuum correlator

\[ \chi(q^2) = i \int d^4 x e^{i q x} \langle 0 \mid T \{ Q_5(x), Q_5(0) \} \mid 0 \rangle \]  

is of a great theoretical interest. \( G^{\mu\nu}_G \) is gluonic field strength tensor, \( \tilde{G}_{\mu\nu} = (1/2)\varepsilon_{\mu\nu\lambda\sigma}G^{\lambda\sigma} \) is its dual, \( n \) are the colour indeces, \( n = 1, 2, \ldots N_c^2 - 1 \), \( N_c \) is the number of colours, \( N_c = 3 \) in QCD. The existence of topological quantum numbers in non-abelian field theories was first discovered by Belavin et al. \[2\], their connection with non-conservation of \( U(1) \) chirality was established by t’Hooft \[3\]. Crewther \[4\] derived Ward identities related to \( \chi(0) \), which allowed him to prove the theorem, that \( \chi(0) = 0 \) in any theory where it is at least one massless quark. An important step in the investigation of the properties of \( \chi(q^2) \) was achieved by Veneziano \[5\] and Di Vecchia and Veneziano \[6\]. These authors considered the limit \( N_c \to \infty \). Assuming that in the theory there are \( N_f \) light quarks with the masses \( m_i \ll M \), where \( M \) is the characteristic scale of strong interaction, Di Vecchia and Veneziano found that

\[ \chi(0) = \langle 0 \mid \bar{q}q \mid 0 \rangle \left( \sum_i \frac{1}{m_i} \right)^{-1}, \]  

where \( \langle 0 \mid \bar{q}q \mid 0 \rangle \) is the common value of quark condensate for all light quarks and the terms of the order \( m_i / M \) are neglected. \[4\] The concept of \( \theta \)-term in the Lagrangian was succesfully exploited in \[4\] in deriving of (3). Using the same concept and studying the properties of the Dirac operator Leutwyler and Smilga \[7\] succeeded in proving eq.3 at any \( N_c \) for the case of two light quarks, \( u \) and \( d \).

In this paper I calculate \( \chi(q^2) \), at low \( |q^2| \ll M^2 \). In ref.1 the value \( \chi'(0) \) (more precise, its nonperturbative part) was found. On the basis of QCD sum rules in the external fields the connection of \( \chi'(0) \) with the part of the proton spin, carried by \( u, d, s \) quarks, \( \Sigma \) was established. By the use of experimental data on \( \Sigma \), as well as from the requirement of selfconsistency of the sum rule, it was obtained:

\[ \chi'(0) = (2.3 \pm 0.6) \times 10^{-3} \text{ GeV}^2 \]  

\[1\]The definition of \( \chi(q^2) \) used above eq.2, differs by sign from the definition used in \[4\]- \[6\].
in the limit of massless $u, d$ and $s$ quarks. At low $q^2$ the terms, proportional to quark masses, are related to the contributions of light pseudoscalar mesons as intermediate states in the correlator (2). These contributions are calculated below. In such calculation for the case of three quarks the mixing of $\pi^0$ and $\eta$ is of importance and it is accounted.

The presentation of the material in the paper is the following. In Sec.II the low energy theorems related to $\chi(0)$ are rederived with the account of possible anomalous equal-time commutator terms. (In [4]-[6] it was implicitly assumed that these terms are zero). In Sec.III the case of one and two light quarks are considered. It is proved, that the mentioned above commutator terms are zero indeed and for the case of two quarks eq.3 is reproduced without using $N_c \to \infty$ limit and the concepts of $\theta$–terms. In Sec.IV the case of three $u, d, s$ light quarks is considered in the approximation $m_u, m_d \ll m_s$. The problem of mixing of $\pi^0$ and $\eta$ states [8,9] is formulated and corresponding formulae are presented. The account of $\pi - \eta$ mixing allows one to get eq.(3) from low energy theorems, formulated in Sec.II for the case of three light quarks. (At $m_u, m_d \ll m_s$ it coincides with the two quark case). In Sec.V the $q^2$–dependence of $\chi(q^2)$ was found at low $|q^2|$ in the leading nonvanishing order in $q^2/M^2$ as well as in $m_q/M$.

II. LOW ENERGY THEOREMS.

Consider QCD with $N_f$ light quarks, $m_i \ll M \sim 1$ GeV, $i = 1, \ldots N_f$. Define the singlet (in flavour) axial current by

$$j_{\mu 5}(x) = \sum_{i} \bar{q}_i(x) \gamma_{\mu} \gamma_5 q(x)$$

and the polarization operator

$$P_{\mu \nu}(q) = i \int d^4x e^{iqx} \langle 0 | T \{j_{\mu 5}(x), j_{\nu 5}(0) \} | 0 \rangle.$$  \hfill (6)

The general form of the polarization operator is:

$$P_{\mu \nu}(q) = -P_L(q^2)\delta_{\mu \nu} + P_T(q^2)(-\delta_{\mu \nu}q^2 + q_\mu q_\nu)$$  \hfill (7)

Because of anomaly the singlet axial current is nonconserving:

$$\partial_\mu j_{\mu 5}(x) = 2N_f Q_5(x) + D(x),$$  \hfill (8)
where $Q_5(x)$ is given by (1) and

$$D(x) = 2i \sum_i N_f m_i \bar{q}_i(x) \gamma_5 q_i(x)$$

(9)

It is well known, that even if some light quarks are massless, the corresponding Goldstone bosons, arising from spontaneous violation of chiral symmetry do not contribute to singlet axial channel (it is the solution of $U(1)$ problem), i.e. to polarization operator $P_{\mu\nu}(q)$. $P_L(q^2)$ also have no kinematical singularities at $q^2 = 0$. Therefore

$$P_{\mu\nu}(q)q_\mu q_\nu = -P_L(q^2)q^2$$

(10)

vanishes in the limit $q^2 \rightarrow 0$. Calculate the left-hand side (lhs) of (10) in the standard way – put $q_\mu q_\nu$ inside the integral in (6) and integrate by parts. (For this it is convenient to represent the polarization operator in the coordinate space as a function of two coordinates $x$ and $y$.) Going to the limit $q^2 \rightarrow 0$ we have

$$\lim_{q^2 \rightarrow 0} P_{\mu\nu}(q)q_\mu q_\nu = i \int d^4x \langle 0 | T\{2N_f Q_5(x), 2N_f Q_5(0) + 2N_f Q_5(x), D(0) + D(x), 2N_f Q_5(0) + D(x), D(0)\} | 0 \rangle$$

$$+ \int d^4x \langle 0 | [\bar{q}_i(0)q_i(0), 2N_f Q_5(0)] | 0 \rangle \delta(x_0) = 0$$

(11)

In the calculation of (11) the anomaly condition (8) was used. The terms, proportional to quark condensates arise from equal time commutator $[j_{05}(x), D(0)]_{x_0=0}$, calculated by standard commutation relations. Relation (11) up to the last term was first obtained by Crewther [4]. The last term, equal to zero according to standard commutation relations and omitted in [4]-[6], is kept. The reason is, that we deal with very subtle situation, related to anomaly, where nonstandard Schwinger terms in commutation relations may appear. (It can be shown, that, in general the only Schwinger term in this problem is given by the last term in the lhs of (11): no others can arise.) Consider also the correlator:

$$P_\mu(q) = i \int d^4x \ e^{i qx} \langle 0 | T\{j_{\mu 5}(x), Q_5(0)\} | 0 \rangle$$

(12)
and the product $P_\mu(q)q_\mu$ in the limit $q^2 \to 0$ (or $q^2$ of order of the $m_\pi^2$, where $m_\pi$ is the mass of Goldstone boson). The general form of $P_\mu(q)$ is $P_\mu(q) = Aq_\mu$. Therefore nonvanishing values of $P_\mu(q)$ in the limit $q^2 \to 0$ (or of order of quark mass $m$, if $|q^2| \sim m_\pi^2$ – this limit will be also interesting for us later) can arise only from Goldstone bosons intermediate states in (12). Let us us estimate the corresponding matrix elements

$$\langle 0 | j_{\mu 5} | \pi \rangle = F q_\mu \quad (13)$$

$$\langle 0 | Q_5 | \pi \rangle = F' \quad (14)$$

$F$ is of order of $m$, since in the limit of massless quarks Goldstone bosons are coupled only to nonsinglet axial current. $F'$ is of order of $m_\pi^2 f_\pi \sim m$, where $f_\pi$ is the pion decay constant (not considered to be small), since in massless quark limit, the Goldstone boson is decoupled from $Q_5$. These estimations give

$$P_\mu q_\mu \sim \frac{q^2}{q^2 - m_\pi^2} m^2 \quad (15)$$

and it is zero at $q^2 \to 0$, $m_\pi^2 \neq 0$ and of order of $m^2$ at $q^2 \sim m_\pi^2$. In what follows I will restrict myself by the terms linear in quark masses. So, I can put $P_\mu(q)q_\mu = 0$ at $q \to 0$.

The integration by parts, in the right-hand side (rhs) of (12) gives:

$$\lim_{q^2 \to 0} P_\mu(q)q_\mu = -\int d^4 x \langle 0 | T\{2N_fQ_5(x), Q_5(0) + D(x), Q_5(0)\} | 0 \rangle$$

$$-\int d^4 x \langle 0 | [ j_{05}(x), Q_5(0) ] | 0 \rangle \delta(x_0) = 0 \quad (16)$$

After the substitution of (16) in (11) arise the low energy theorem:

$$i \int d^4 x \langle 0 | T\{2N_fQ_5(x), 2N_fQ_5(0)\} | 0 \rangle$$

$$-i \int d^4 x \langle 0 | T\{D(x), D(0)\} | 0 \rangle - 4 \sum_i N_f m_i \langle 0 | \bar{q}_i(0)q_i(0) | 0 \rangle$$

$$+ i \int d^4 x \langle 0 | [ j_{05}^0(x), 2N_fQ_5(0) ] | 0 \rangle \delta(x_0) = 0 \quad (17)$$

The low energy theorem (17), with the last term in the lhs omitted, was found by Crewther.
III. ONE AND TWO LIGHT QUARKS.

Consider first the case of one massless quark, \( N_f = 1, m = 0 \). This case can easily be treated by introduction of \( \theta \)-term in the Lagrangian,

\[
\Delta L = \theta \frac{\alpha_s}{4\pi} G_{\mu\nu}^n \tilde{G}_{\mu\nu}^n
\]

(18)

The matrix element \( \langle 0 | Q_5 | n \rangle \) between any hadronic state \( | n \rangle \) and vacuum is proportional

\[
\int d^4x \langle 0 | Q_5(x) | n \rangle \sim \langle 0 | \frac{\partial}{\partial \theta} \ln Z | n \rangle_{\theta=0},
\]

(19)

where \( Z = e^{iL} \) and \( L \) is the Lagrangian. The gauge transformation of the quark field

\[
\psi' \rightarrow e^{i\alpha \gamma_5} \psi
\]

results to appearance of the term

\[
\delta L = \alpha \left[ \partial_\mu j_{\mu 5} - \frac{\alpha_s}{4\pi} G_{\mu\nu}^n \tilde{G}_{\mu\nu}^n \right]
\]

(20)

in the Lagrangian. By the choice \( \alpha = \theta \) the \( \theta \)-term (18) will be killed and \((\partial/\partial \theta) \ln Z = 0\). Therefore, \( \chi(0) = 0 \) (Crewther theorem). The first term in (17) vanishes, as well as the second and third, since \( m = 0 \). From (17) we have, that indeed the anomalous commutator vanishes

\[
\langle 0 | \left[ j_{05}(x), Q_5(0) \right] | x_0 = 0 \rangle = 0,
\]

(21)

supporting the assumptions done in [4]–[6].

Let us turn now to the case of two light quarks, \( u, d, N_f = 2 \). This is the case of real QCD, where the strange quark is considered as a heavy. Define the isovector axial current

\[
j_{\mu 5}^{(3)} = (\bar{u} \gamma_\mu \gamma_5 u - \bar{d} \gamma_\mu \gamma_5 d)/\sqrt{2}
\]

(22)

and its matrix element between the states of pion and vacuum

\[
\langle 0 | j_{\mu 5}^{(3)} | \pi \rangle = f_\pi q_\mu,
\]

(23)

where \( q_\mu \) is pion 4-momentum, \( f_\pi = 133 \text{ MeV} \). Multiply (23) by \( q_\mu \). Using Dirac equations for quark fields, we have
\[
\frac{2i}{\sqrt{2}} \langle 0 \mid m_u \bar{u} \gamma_5 u - m_d \bar{d} \gamma_5 d \mid \pi \rangle = \frac{i}{\sqrt{2}} \langle 0 \mid (m_u + m_d)(\bar{u} \gamma_5 u - \bar{d} \gamma_5 d) + (m_u + m_d)((\bar{u} \gamma_5 u - \bar{d} \gamma_5 d) \mid \pi \rangle = f_\pi m^2_\pi,
\]

where \( m_u, m_d \) are \( u \) and \( d \) quark masses. The ratio of the matrix elements in lhs of (24) is of order

\[
\frac{\langle 0 \mid \bar{u} \gamma_5 u + \bar{d} \gamma_5 d \mid \pi \rangle}{\langle 0 \mid \bar{u} \gamma_5 u - \bar{d} \gamma_5 d \mid \pi \rangle} \sim \frac{m_u - m_d}{M},
\]

since the matrix element in the numerator violates isospin and this violaton (in the absence of electromagnetism, which is assumed) can arise from the difference \( m_u - m_d \) only. Neglecting this matrix element we have from (24)

\[
\frac{i}{\sqrt{2}} \langle 0 \mid \bar{u} \gamma_5 u - \bar{d} \gamma_5 d \mid \pi \rangle = \frac{f_\pi m^2_\pi}{m_u + m_d}.
\]

Let us find \( \chi(0) \) from low energy sum rule (17) restricting ourself to the terms linear in quark masses. Since \( D(x) \sim m \), the only intermediate state contributing to the matrix element

\[
\int d^4x \langle 0 \mid T\{D(x), \ D(0)\} \mid 0 \rangle
\]

in (17) is the one-pion state. Define

\[
D_q = 2i(m_u \bar{u} \gamma_5 u + m_d \bar{d} \gamma_5 d).
\]

Then

\[
\langle 0 \mid D_q \mid \pi \rangle = i\langle 0 \mid (m_u + m_d)(\bar{u} \gamma_5 u + \bar{d} \gamma_5 d) + (m_u - m_d)(\bar{u} \gamma_5 u - \bar{d} \gamma_5 d) \mid \pi \rangle = \sqrt{2} \frac{m_u - m_d}{m_u + m_d} f_\pi m^2_\pi,
\]

where the matrix element of singlet axial current was neglected and (26) was used. The substitution of (29) into (27) gives
\[ i \int d^4 x e^{iqx} \langle 0 \mid T \{ D_q(x), D_q(0) \} \mid 0 \rangle_{q \to 0} = \lim_{q \to 0} \left\{ -\frac{1}{q^2 - m^2} \frac{2}{m_u + m_d} f^2 \pi m^2 \right\} \]

\[ = -4 \frac{(m_u - m_d)^2}{m_u + m_d} \langle 0 \mid \bar{q}q \mid 0 \rangle \]  

(30)

In the last equality in (30) Gell-Mann-Oakes-Renner relation [10]

\[ \langle 0 \mid \bar{q}q \mid 0 \rangle = -\frac{1}{2} \frac{f^2 \pi m^2}{m_u + m_d} \]  

(31)

was substituted as well the SU(2) equalities

\[ \langle 0 \mid \bar{q}q \mid 0 \rangle = \langle 0 \mid \bar{d}d \mid 0 \rangle \equiv \langle 0 \mid \bar{q}q \mid 0 \rangle. \]  

(32)

From (17) and (30) we finally get:

\[ \chi(0) = i \int d^4 x \langle 0 \mid T \{ Q_5(x), Q_5(0) \} \mid 0 \rangle = \frac{m_u m_d}{m_u + m_d} \langle 0 \mid \bar{q}q \mid 0 \rangle \]  

(33)

in concidence with eq.3.

In a similar way matrix element \( \langle 0 \mid Q_5 \mid \pi \rangle \) can be found. Consider

\[ \langle 0 \mid j_{\mu 5} \mid \pi \rangle = F q_{\mu} \]  

(34)

The estimation of \( F \) gives

\[ F \sim \frac{m_u - m_d}{M} f_\pi \]  

(35)

and after multiplying of (34) by \( q_{\mu} \) the rhs of (34) can be neglected. In the lhs we have

\[ \langle 0 \mid D_\pi \mid \pi \rangle + 2N_f \langle 0 \mid Q_5 \mid \pi \rangle = 0 \]  

(36)

The substitution of (29) into (36) results in

\[ \langle 0 \mid Q_5 \mid \pi \rangle = -\frac{1}{2\sqrt{2}} \frac{m_u - m_d}{m_u + m_d} f_\pi m^2 \]  

(37)

The relation of this type (with a wrong numerical coefficient) was found in [9], the correct formula was presented in [11]. From comparison of (33) and (37) it is clear, that it would be wrong to calculate \( \chi(0) \) by accounting only pions as intermediate states in the lhs of (33) – the constant terms, reflecting the necessity of subtraction terms in dispersion relation and represented by proportional to quark condensate terms in (17) are extremingly important.

The cancellation of Goldstone bosons pole terms and these constant terms results in the Crewther theorem – the vanishing of \( \chi(0) \), when one of the quark masses, e.g. \( m_u \) is going to zero.
IV. THREE LIGHT QUARKS.

Let us dwell on the real QCD case of three light quarks, $u, d$ and $s$. Since the ratios $m_u/m_s, m_d/m_s$ are small, less than $1/20$, account them only in the leading order. When the $u$ and $d$ quark masses $m_u$ and $m_d$ are not assumed to be equal, the quasi-Goldstone states $\pi^0$ and $\eta$ are no more states of pure isospin 1 and 0 correspondingly: in both of these states persist admixture of other isospin [8,9] proportional to $m_u - m_d$. (The violation of isospin by electromagnetic interaction is small in the problem under investigation [8] and can be neglected. The $\eta' - \eta$ mixing is also neglected.) In order to treat the problem it is convenient to introduce pure isospin 1 and 0 pseudoscalar meson fields $\varphi_3$ and $\varphi_8$ in $SU(3)$ octet and the corresponding states $| P_3 \rangle, | P_8 \rangle$. Then in the $SU(3)$ limit

$$\langle 0 | j_{\mu 5}^{(3)} | P_3 \rangle = f_\pi q_\mu$$

$$\langle 0 | j_{\mu 5}^{(8)} | P_8 \rangle = f_\pi q_\mu,$$

where

$$j_{\mu 5}^{(8)} = (\bar{u} \gamma_\mu \gamma_5 u + \bar{d} \gamma_\mu \gamma_5 d - 2\bar{s} \gamma_\mu \gamma_5 s) / \sqrt{6}$$

and $j_{\mu 5}^{(3)}$ is given by (22). The states $| P_3 \rangle, | P_8 \rangle$ are not eigenstates of the Hamiltonian. In the free Hamiltonian

$$H = \frac{1}{2} \tilde{m}_\pi^2 \varphi_3^2 + \frac{1}{2} \tilde{m}_\eta^2 \varphi_8^2 + \langle P_8 | P_3 \rangle \varphi_3 \varphi_8 + \text{kinetic terms}$$

the nondiagonal term $\sim \varphi_3 \varphi_8$ is present ( $\tilde{m}_\pi^2$ and $\tilde{m}_\eta^2$ in (41) coincides with $m_\pi^2$ and $m_\eta^2$ up to terms quadratic in $\langle P_8 | P_3 \rangle$). The nondiagonal term was calculated in [8] on the basis of PCAC and current algebra (see also [9])

$$\langle P_8 | P_3 \rangle = \frac{1}{\sqrt{3}} m_\pi^2 \frac{m_u - m_d}{m_u + m_d}$$

The physical $\pi$ and $\eta$ states arise after orthogonalization of the Hamiltonian (41)

$$| \pi \rangle = cos \theta | P_3 \rangle - sin \theta | P_8 \rangle$$

$$| \eta \rangle = sin \theta | P_3 \rangle + cos \theta | P_8 \rangle$$
where the mixing angle $\theta$ is given by (at small $\theta$) \[8,9\]:

$$\theta = \frac{\langle P_8 | P_3 \rangle}{m_\pi^2 - m_\eta^2} \approx \frac{\langle P_8 | P_3 \rangle}{m_\pi^2} = \frac{1}{\sqrt{3}} \frac{m_\pi^2}{m_u - m_d}$$

(44)

In terms of the fields $\varphi_3$ and $\varphi_8$ PCAC relations take the form \[8\]:

$$\partial_\mu j^{(3)}_{\mu 5} = f_\pi (m_\pi^2 \varphi_3 + \langle P_8 | P_3 \rangle \varphi_8)$$

(45)

$$\partial_\mu j^{(8)}_{\mu 5} = f_\pi (m_\pi^2 \varphi_8 + \langle P_8 | P_3 \rangle \varphi_3)$$

(46)

Our goal now is to calculate the contribution of pseudoscalar octet states to the second term in the lhs of (17). It is convenient to use the full set of orthogonal states $| P_3 \rangle$, $| P_8 \rangle$ as the basis. Use the notation

$$D = D_q + D_s \quad D_s = 2im_s \bar{s} \gamma_5 s,$$

(47)

where $D_q$ is given by (28). The matrix element

$$\langle 0 | D_q | P_3 \rangle = \sqrt{2} \frac{m_u - m_d}{m_u + m_d} f_\pi m_\pi^2$$

(48)

can be found by the same argumentation, as that was used in the derivation of (29). In order to find $\langle 0 | D_q | P_8 \rangle$ take the matrix element of eq.45 between vacuum and $| P_8 \rangle$

$$\langle 0 | \partial_\mu j^{(3)}_{\mu 5} | P_3 \rangle = f_\pi \langle P_8 | P_3 \rangle$$

(49)

The substitution in the lhs of (49) of the expression for $\partial_\mu j^{(3)}_{\mu 5}$ through quark fields gives

$$\langle 0 | \bar{u} \gamma_5 u - \bar{d} \gamma_5 d | P_8 \rangle = - \frac{m_u - m_d}{m_u + m_d} \langle 0 | \bar{u} \gamma_5 u + \bar{d} \gamma_5 d | P_8 \rangle -$$

$$- i \sqrt{\frac{2}{3}} f_\pi m_\pi^2 \frac{m_u - m_d}{(m_u + m_d)^2}$$

(50)

In a similar way take matrix element of eq. (46) between $\langle 0 |$ and $| P_8 \rangle$ and substitute into it (50). We get
\[
\frac{i}{\sqrt{6}} \langle 0 \mid m_u + m_d - \frac{(m_u - m_d)^2}{m_u + m_d} \left( \bar{u}\gamma_5 u + \bar{d}\gamma_5 d \right) - 4 m_s \bar{s}\gamma_5 s \mid P_8 \rangle = f_\pi^2 m_\eta^2 \left( \frac{m_u - m_d}{m_u + m_d} \right)^2
\]

(51)

As follows from SU(3) symmetry of strong interaction

\[
\langle 0 \mid \bar{u}\gamma_5 u + \bar{d}\gamma_5 d \mid P_8 \rangle = -\langle 0 \mid \bar{s}\gamma_5 s \mid P_8 \rangle
\]

(52)

up to terms of order \( m_q/M \), which are neglected. From (51),(52) we find:

\[
\langle 0 \mid D_s \mid P_8 \rangle = -\sqrt{\frac{3}{2}} f_\pi m_\eta^2 \left[ 1 - \frac{1}{4} \frac{(m_u - m_d)^2}{m_u(m_u + m_d)} \right] \left[ 1 + \frac{m_u m_d}{m_s(m_u + m_d)} \right]
\]

(53)

\[
\langle 0 \mid D_q \mid P_8 \rangle = 4 \sqrt{\frac{2}{3}} f_\pi m_\eta^2 \frac{m_u m_d}{(m_u + m_d)^2}
\]

(54)

in notation (28),(47). When deriving (53) the SU(3) relation

\[
m_\eta^2 = \frac{4}{3} m_\pi^2 \frac{m_s}{m_u + m_d} \left( 1 - \frac{1}{4} \frac{m_u + m_d}{m_s} \right)
\]

(55)

was used. In (53) the small terms \( \sim m_u/m_s, m_d/m_s \) are accounted, because they are multiplied by large factor \( m_\eta^2 \). In (54) small terms are disregarded. The matrix element \( \langle 0 \mid D_s \mid P_3 \rangle \) can be found from (46). We have

\[
\frac{1}{\sqrt{6}} \langle 0 \mid D_q - 2D_s \mid P_3 \rangle = f_\pi \langle P_8 \mid P_3 \rangle
\]

(56)

The substitution of (48) and (42) into (56) gives

\[
\langle 0 \mid D_s \mid P_3 \rangle = 0
\]

(57)

Equations (48),(53),(54) and (57) allow one to calculate the interesting for us correlator

\[
i \int d^4x \langle 0 \mid T \{ D(x), D(0) \} \mid 0 \rangle
\]

(58)

when the sets \( \mid P_3 \rangle \langle P_3 \mid \) and \( \mid P_8 \rangle \langle P_8 \mid \) are taken as intermediate states. But \( \mid P_3 \rangle, \mid P_8 \rangle \) are not the eigenstates of the Hamiltonian, they mix in accord with (41). Therefore the transitions \( \langle P_8 \mid P_3 \rangle \) arising from the mixing term in (41) must be also accounted. There
are two such terms. The one corresponds to the transition $\langle 0 \mid D_s \mid P_s \rangle \langle P_3 \mid D_q \mid 0 \rangle$ and its contribution to (58) is given by

$$\lim_{q^2 \to 0} \left\{ -2\langle 0 \mid D_s \mid P_s \rangle \frac{1}{q^2 - m^2_\eta} \langle P_s \mid P_3 \rangle \frac{1}{q^2 - m^2_\pi} \langle P_3 \mid D_q \mid 0 \rangle \right\} = \frac{1}{2} f^2 \pi m^2 \left( \frac{m_u - m_d}{m_u + m_d} \right)^2$$

(59)

The other corresponds to the transition between two $D_s$ operators, where $\langle P_3 \mid P_3 \rangle$ enter as intermediate state. This contribution is equal to:

$$\lim_{q^2 \to 0} \left\{ -\langle 0 \mid D_s \mid P_s \rangle \frac{1}{q^2 - m^2_\eta} \langle P_s \mid P_3 \rangle \frac{1}{q^2 - m^2_\pi} \langle P_3 \mid D_s \mid 0 \rangle \right\} = \frac{1}{2} f^2 \pi m^2 \left( \frac{m_u - m_d}{m_u + m_d} \right)^2$$

(60)

It is enough to account only matrix elements, with $D_s$ operators, since they are enhanced by large factor $m^2_\eta$. All others are small in the ratio $m^2_\pi/m^2_\eta$.

Collecting all together, we get:

$$i \int d^4x \langle 0 \mid T\{D(x), D(0)\} \mid 0 \rangle = f^2 \pi m^2 \Bigg\{ 2 \left( \frac{m_u - m_d}{m_u + m_d} \right)^2 - 8 \frac{m_u m_d}{(m_u + m_d)^2} 
+ 2 \frac{m_s}{m_u + m_d} \left( 1 + \frac{1}{4} \frac{m_u + m_d}{m_s} \right) \left[ 1 - \frac{1}{2} \frac{m_u - m_d}{m_s (m_u + m_d)} \right] 
+ \frac{1}{2} \left( \frac{m_u - m_d}{m_u + m_d} \right)^2 \Bigg\}$$

(61)

The first term in the figure bracket in (61) comes from $\langle 0 \mid D_q \mid P_3 \rangle^2$, the second – from $\langle 0 \mid D_q \mid P_s \rangle \times \langle 0 \mid P_s \mid D_s \rangle$, the third – from $\langle 0 \mid D_s \mid P_q \rangle^2$, the last two terms are from (59),(60). Adding to (61) the proportional to quark condensate term

$$4(m_u + m_d + m_s) \langle 0 \mid \bar{q}q \mid 0 \rangle$$

(62)

in (17), we finally get for 3 quarks at $m_u, m_d \ll m_s$

$$i \int d^4x \langle 0 \mid T\{2N_f Q_5(x), 2N_f Q_5(x)\} \mid 0 \rangle = 36 \frac{m_u m_d}{m_u + m_d} \langle 0 \mid \bar{q}q \mid 0 \rangle$$

(63)
\[ \chi(0) = \frac{m_u m_d}{m_u + m_d} \langle 0 | \bar{q} q | 0 \rangle, \]  
(64)

since in this case \(4N_c^2 = 36\). Eq. 64 coincides with eq. 3, obtained \[ in \ N_c \to \infty \ limit. \ This \ fact \ demonstrates, \ that \ N_c \to \infty \ limit \ is \ irrelevant \ for \ determination \ of \ \chi(0) \ (at \ least \ for \ the \ cases \ of \ two \ or \ three \ light \ quarks \ and \ at \ m_u, m_d \ll m_s). \ \chi(0) \ for \ three \ light \ quarks \ at \ m_u, m_d \ll m_s – \ eq.64 \ coincides \ with \ \chi(0) \ in \ the \ two \ light \ quark \ case, \ (see \ [7] \ and \ [33]) \ i.e \ in \ this \ problem, \ when \ m_u, m_d \ll m_s, \ there \ is \ no \ difference \ if \ s-quark \ is \ considered \ as \ a \ heavy \ or \ light – \ it \ softly \ appears \ in \ the \ theory. \]

Determine the matrix elements \( \langle 0 | j_{\mu 5} | \eta \rangle \) and \( \langle 0 | Q_5 | \pi \rangle \). Following \[ 12] \ consider
\[ \langle 0 | j_{\mu 5} | \eta \rangle = \tilde{F} q_{\mu} \]
(65)

\( \tilde{F} \) is of order of \( f_\pi (m_s/M) \) and can be put to zero in our approxamation. By taking the divergence from (65), we have
\[ \langle 0 | D_s + 6Q_5 | \eta \rangle = 0 \]
(66)

The use of (53) gives (the \( \pi – \eta \) mixing as well terms of order \( m_u/m_s, \ m_d/m_s \) may be neglected here):
\[ \langle 0 | Q_5 | \eta \rangle = \frac{1}{2} \sqrt{\frac{1}{6} m_\eta^2} \]
(67)

Relation (67) was found in [12]. By the same reasoning it is easy to prove that
\[ \langle 0 | D_q | P_3 \rangle + \langle 0 | D_s | P_3 \rangle + 6\langle 0 | Q_5 | P_3 \rangle = 0 \]
(68)

The first term in (68) is given by (48), the second one is zero according (57). For the last term we can write using (43)
\[ \langle 0 | Q_5 | P_3 \rangle = \langle 0 | Q_5 | \pi \rangle + \theta \langle 0 | Q_5 | P_8 \rangle \]
(69)

Eq.’s (68),(69) give
\[ \langle 0 | Q_5 | \pi \rangle = -\frac{1}{2\sqrt{2}} f_\pi m_\eta^2 \frac{m_u - m_d}{m_u + m_d} \]
(70)

– the same formula as in the case of two light quarks.

It is clear, that the presented above considerations can be generalized to the case, when \( u,d \) and \( s \)-quark masses are comparable. The calculation became more cumbersome, but nothing principally new arises on this case.
V. $Q^2$–DEPENDENCE OF $\chi(Q^2)$ AT LOW $Q^2$.

Let us dwell on the calculation of the $q^2$–dependence of $\chi(q^2)$ at low $|q^2|$ in QCD, restricting ourselves by the first order terms in the ratio $q^2/M^2$, where $M$ is the characteristic hadronic scale, $M^2 \sim 1$ GeV$^2$. By $\chi(q^2)$ I mean its nonperturbative part with perturbative contribution subtracted from the total $\chi(q^2)$ defined by (2). The reason is that the perturbative part is strongly divergent, its contribution would be strongly dependent on regularization procedure and, therefore, physically meaningless. In this domain of $q^2$ $\chi(q^2)$ can be represented as

$$\chi(q^2) = \chi(0) + \chi'(0)q^2 + R(q^2) - R(0)$$  \hspace{1cm} (71)

$\chi(0)$ for the QCD case–three light quarks with $m_u, m_d \ll m_s$ – was determined in Sec.IV. $\chi'(0)$ (its nonperturbative part) for massless quarks was found in [1] basing on connection of $\chi'(0)$ with the part of proton spin $\sum$ carried by $u, d, s$ quarks. Its numerical value is given by (4). What is left, is the contribution of light pseudoscalar quasi–Goldstone bosons $R(q^2)$, which has nontrivial $q^2$–dependence and must be accounted separately. $R(q^2)$ vanishes for massless quarks and did not contribute to $\chi'(0)$, calculated in [1]. $R(0)$ must be subtracted from $R(q^2)$ since it was already accounted in $\chi(0)$. $R(q^2)$ can be written as

$$R(q^2) = -\langle 0 | Q_5 | \pi \rangle^2 \frac{1}{q^2 - m^2_\pi} - \langle 0 | Q_5 | \eta \rangle^2 \frac{1}{q^2 - m^2_\eta}$$  \hspace{1cm} (72)

The problem of $\eta – \pi$ mixing is irrelevant in the difference $R(q^2) - R(0)$ in any domain $|q^2| \sim m^2_\pi$ and $|q^2| \sim m^2_\eta$. The matrix elements entering (72) are given by (67), (70). Taking the difference $R(q^2) - R(0)$ and using the Euclidean variable $Q^2 = -q^2$, we have

$$\chi(Q^2) = \chi(0) - \chi'(0)Q^2 - \frac{1}{8} f^2_\pi Q^2 \left[ \left( \frac{m_u - m_d}{m_u + m_d} \right)^2 \frac{m^2_\pi}{Q^2 + m^2_\pi} + \frac{1}{3} \frac{m^2_\eta}{Q^2 + m^2_\eta} \right]$$  \hspace{1cm} (73)

Eq.(73) is our final result, where $\chi(0)$ is given by (64) and $\chi'(0)$ by (4). The accuracy of (73) is given by the parameters $Q^2/M^2$, $m^2_\pi/M^2$, $m^2_\eta/M^2 \ll 1$, (two last characterize the accuracy of $SU(3) \times SU(3)$). At $Q^2 \approx m^2_\eta$ the last term comprise about 20% of the second (the first term is very small, $\chi(0) \approx -4 \cdot 10^{-5}$ GeV$^4$). Evidently, the last term is much bigger in the Minkovski domain, $Q^2 < 0$, since there are pion and $\eta$ poles. As was mentioned above, $\chi(0)$ found here concides with $\chi(0)$ obtained in [3] by considering large $N_c$ limit. However,
the $q^2$-dependence is completely different. Namely, for $\chi(q^2)$ in [3] was found the relation (eq.(A4') in [3])
\[
\chi(q^2) = -\frac{aF^2_\pi}{2N_c} \left[ 1 - \frac{a}{N_c} \sum_i \frac{1}{q^2 - \mu_i^2} \right]^{-1},
\] (74)
where the Goldstone boson masses $\mu_i^2$ are related to quark condensate by
\[
\mu_i^2 = -2m_i \frac{1}{F^2_\pi} \langle 0 | \bar{q}q | 0 \rangle
\] (75)
and $a$ is some constant of order of hadronic mass square. At $q^2 = 0$ follows eq.3 for $\chi(0)$ if the inequality $a/N_c\mu_i^2 >> 1$ is assumed. However, at $|q^2| \sim m_\eta^2, m_\pi^2$ (74) strongly differs from (73): in (74) there are zeros at the points $q^2 = m_\eta^2, m_\pi^2$, but not poles, as it should be and as it take place in (73). And also the most important at low $Q^2$ hadronic term $\chi'(0)Q^2$ is absent in (74).

VI. SUMMARY

The $q^2$-dependence of topological charge density correlator $\chi(q^2)$ (2) in QCD was considered in the domain of low $q^2$. For the cases of two and three light quarks the values of $\chi(0)$, obtained earlier [4]-[7] were rederived basing on the low energy theorems and accounting of quasi-Goldstone boson ($\pi, \eta$) contributions. No large $N_c$ limit was used and it was no appeal to the $\theta$-dependence of QCD Lagrangian (except of the proof of absence of anomalous commutator in the sum rule (17)). The only concept, which was used, was the absence of Goldstone boson contribution as intermediate state in the singlet (in flavour) axial current correlator in the limit of massless quarks. In the three light quark case – the case of real QCD – the mixing of $\pi$ and $\eta$ is of importance and was widely exploited. The $q^2$ dependence of $\chi(q^2)$ was found as arising from two sources:

- 1. The contribution of hadronic states (besides $\pi$ and $\eta$). This contribution was determined from the established in [3] (basing on QCD sum rule approach) connection of $\chi'(0)$ with the part of the proton spin, carried by quarks.

- 2. The contributions of $\pi$ and $\eta$ intermediate states. These contributions were calculated by using low energy theorems only. The final result is presented in eq.73.
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