Multiparticle correlations in the Schwinger mechanism

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Abstract
We discuss the Schwinger mechanism in scalar QED and derive the multiplicity distribution of particles created under an external electric field using the LSZ reduction formula. Assuming that the electric field is spatially homogeneous, we find that the particles of different momenta are produced independently, and that the multiplicity distribution in one mode follows a Bose-Einstein distribution. We confirm the consistency of our results with an intuitive derivation by means of the Bogoliubov transformation on creation and annihilation operators. Finally we revisit a known solvable example of time-dependent electric fields to present exact and explicit expressions for demonstration.

1 Introduction
A classic example of a non-perturbative tunneling phenomenon in quantum field theory is the decay of an electric field due to pair creation. The quantum vacuum is full of virtual particle-antiparticle pairs (i.e. the Dirac sea), which can occasionally gain enough energy from the external field to become real. The decay (or persistence) rate of the QED (quantum electrodynamics) vacuum in the presence of an external electric field was first deduced from the imaginary part of the Heisenberg-Euler Lagrangian [1] and formulated in Schwinger’s classic paper [2]. The phenomenon is commonly referred to as the Schwinger
mechanism (see ref. [3] for a comprehensive review.) In the case of QED the coupling constant $e$ is very small, and it is difficult in practice to achieve large enough electric fields; the probability of producing an electron-positron pair is, up to a prefactor, $\sim \exp[-\pi m_e^2/(eE)]$, and thus a very strong electric field, $E \sim m_e^2/e \simeq 1.3 \times 10^{18}$ V/m, is necessary to observe the phenomenon. To the best of our knowledge, the Schwinger mechanism in QED remains to be unambiguously observed experimentally.

Pair creation from an electric field became a phenomenologically much more relevant subject with the realization that the strong nuclear force is described by a gauge theory called QCD (quantum chromo-dynamics). A popular phenomenological view of QCD with confinement is a description in terms of a chromoelectric flux tube connecting the color charges of the quarks. If these quarks are then pulled apart by their momenta, the string formed by the chromoelectric field can decay via the Schwinger mechanism leading to the decay of the system into $q\bar{q}$ or color neutral mesons as a result of hadronization. In this case, the decay probability is characterized by $\sim \exp[-\pi m_q^2/\sigma]$, where the QCD string tension $\sigma \simeq 1$ GeV/fm is an energy stored in the chromoelectric flux tube per unit length. Applications of the particle production by the Schwinger mechanism range from $e^+e^-$ annihilation [4, 5] to early models of relativistic heavy ion collisions [6, 7, 8, 9, 10, 11, 12]. Recent extensive studies on thermal hadron production as a possible manifestation of the Hawking-Unruh effect, that is an equivalent formulation to the Schwinger mechanism in curved space-time, is found in refs. [13, 14, 15].

The QCD coupling constant $g$, although asymptotically small, is not as small as the QED one at phenomenologically interesting energies. Even more important is that the nonlinear dynamics of the gauge fields naturally leads, in some circumstances, to gauge fields that are parametrically large in the coupling, $A_\mu \sim 1/g$. A prime example of such a situation is caused by the large occupation numbers of gluonic states in high energy scattering. The transverse gluon density $\sim Q_s^2$ provides a typical energy scale $Q_s$ in such a system. There the nonlinear interactions among bremsstrahlung gluons with small Bjorken’s $x$ lead to gluon saturation, which is most conveniently described as a coherent color field radiated by static (in light cone time) sources. This description is referred to as the Color Glass Condensate (CGC) (for reviews, see [16, 17, 18, 19]). The collision of two objects whose wavefunction is characterized by $Q_s$ in the CGC formalism achieves a field configuration of longitudinal chromoelectric and chromomagnetic fields whose strength is also given by $Q_s$. This transient state of matter containing strong longitudinal fields is known as the glasma [20, 21, 22]. In the case of QCD it is of course difficult to achieve the canonical model case of a constant electrical field. Generically, a WKB-type non-perturbative evaluation such as in refs. [13, 14] could be expected to be valid in a case where the field strength $gA_\mu \sim Q_s$ is much larger than the typical (inverse) time and spatial scales of the fields. A perturbative calculation, on the other hand, is also feasible in the case when $g$ is small enough. The particle (gluons and quarks) production associated with strong CGC fields has been formulated based on the Lehmann-Symanzik-Zimmermann (LSZ) reduction formula [23, 24, 25, 26, 27, 28, 29, 30, 31] as well
as on the canonical formalism [32]. One of the aims of this paper is to establish a link, by a concrete example, between the formalism of LSZ reduction formulas, which is usually associated with perturbation theory only, and non-perturbative tunneling phenomena of the Schwinger mechanism. More concretely, we want to show that the LSZ perturbative framework automatically includes the particles produced by the Schwinger mechanism, provided the external field is properly resummed. Thus, this contribution does not need to be added separately by hand.

For applying the mechanism of pair creation from a classical field to phenomenology one needs, in addition to the vacuum decay rate or spectrum of pairs, the whole probability distribution of multiparticle production. In many practical applications of the Schwinger mechanism, there has been a confusion of terminology, with both the formulas and the concepts of the vacuum decay rate (or persistence probability calculated by Schwinger) and the pair production rate. The difference between the two was recently nicely discussed in ref. [33], where it is interpreted as a result of temporal correlations between the produced pairs. In fact these two were clearly distinguished already in a classical paper by Nikishov [34]. In the case of the typical QED discussion, the pair production rate is extremely small, in which case the probability distribution of produced pairs cannot be distinguished from a Poisson distribution. This seems to have been the assumption used, without any further justification, also in many QCD phenomenological applications (see e.g. [8] where this is very explicit). As we shall show explicitly in the following, this assumption is not true when the pair production is not strongly suppressed, as can typically be the case in e.g. heavy-ion collisions. Instead, the probability distribution of the produced pairs turns out naturally to be the appropriate (Bose-Einstein or Fermi-Dirac) quantum one. With explicit expressions for the probabilities to produce one, two, etc. particles, the distinction between the vacuum decay rate (related to the probability to produce no pairs) and the pair production rate (the expectation value of the number of pairs produced), becomes obvious. The fact that there is a quantum statistical (BE or FD) correlation has long ago been realized by some authors as a requirement that should in principle be built into Monte Carlo event generators [37, 38, 39]. Also, the full computation of the vacuum decay rate should encompass all the multiparticle production processes,–because of unitarity–, including the quantum statistics. To our knowledge, however, an explicit derivation of how the BE or FD correlations arise from the Schwinger mechanism has been lacking. Besides, the multiparticle distribution has scarcely drawn attention in the context of the Schwinger mechanism, probably because of the absence of experimental access. In fact the spectrum of multiparticle production is quite informative and precise data of charged hadron multiplicity fluctuations are already available in p-\bar{p} [40] and heavy-ion [41] collision experiments, where a negative binomial distribution gives a beautiful fit. Of course, to account for the high-energy experimental data, a simple treatment of spatially

\footnote{Note that the "inversion of spin statistics" discussed in [35, 36] refers to a formal expression of the vacuum decay rate as an integral over the BE or FD distribution function and not the actual probability distribution of produced particles.}
homogeneous (i.e. constant in space) background fields is inadequate, and re-
cently, it has been shown that an inhomogeneous configuration forming a certain
number of the plasma flux tubes leads to the negative binomial distribution [42].
This is beyond the scope of our current paper.

In the following, we shall study the case of scalar QED in a time-dependent
but spatially homogeneous external gauge field. We shall first introduce the
model and derive the probability distribution of produced pairs using the LSZ
reduction formula in sec. 2. Then, in sec. 3, we shall rederive the same results
using canonical quantization and interpret the calculation of sec. 2 in terms
of a Bogoliubov transformation. The discussion in sec. 3 to a large degree
follows that of Tanji [43], but takes the additional mathematically simple step
of actually writing down the whole probability distribution (see also [44]). Then,
in sec. 4 we shall demonstrate how this procedure works in practice using an
exactly solvable example (see also [45]) of a time dependent external potential,
from which we can take both the constant field and short pulse limits. We note
that, in all our discussions, we will solve the problem for a given external field
without taking into account the interplay between the field and the produced
particles which screen the external field, leading to plasma oscillation behavior
in time [43, 46, 47].

2 LSZ derivation

We here calculate the Schwinger mechanism in terms of the LSZ reduction
formulas. To this aim we develop a slightly modified version of the Schwinger-
Keldysh formalism to compute the generating functional of the particle and
antiparticle spectra.

2.1 Model

To avoid encumbering the discussion with unessential details, we consider scalar
QED, i.e. a charged scalar field $\phi$ coupled to an external vector potential $A^\mu$.
Moreover, in order to simplify things even further, we neglect any kind of self-
interactions among the scalar fields, and the coupling to the external electro-
magnetic field enters only via the covariant derivatives, $D^\mu = \partial^\mu - ieA^\mu$. Thus,
the Lagrangian of this model is:

$$\mathcal{L} = (D_\mu \phi) (D^\mu \phi)^* - m^2 \phi \phi^*. \quad (1)$$

In most of the considerations of this paper, we need not specify the precise form
of the background potential $A^\mu$. Only in the final section, we work out the case
of an explicit example of background electric field that leads to exact analytical
results.

2.2 Reduction formulas

We assume that the initial state of the system does not contain any particles or
antiparticles. However, because of the background field, transitions to populated
states are possible. Let us consider the following transition amplitudes,

\[ M_{m,n}(\{p_i\}, \{q_i\}) \equiv \langle p_1 \cdots p_m q_1 \cdots q_n \text{ out} | 0_\text{in} \rangle, \tag{2} \]

from the vacuum to a populated state. The conservation of electrical charge implies that an equal number of particles and antiparticles must be produced, i.e. that this general amplitude is proportional to \( \delta_{mn} \). This transition amplitude can be obtained from the expectation value of time-ordered products of fields:

\[
M_{m,n}(\{p_i\}, \{q_i\}) = \int \prod_{i=1}^m d^4 x_i e^{ip_i \cdot x_i} (\Box_{x_i} + m^2) \prod_{j=1}^n d^4 y_j e^{iq_j \cdot y_j} (\Box_{y_j} + m^2) \\
\times \langle 0_\text{out} | T \phi(x_1) \cdots \phi(x_m) \phi^*(y_1) \cdots \phi^*(y_n) | 0_\text{in} \rangle. \tag{3} \]

Here the on-shell boundary condition is the vacuum one, i.e. \( p_0^i \to \sqrt{p_i^2 + m^2} \) and \( q_0^i \to \sqrt{q_i^2 + m^2} \) for particles and antiparticles, respectively. This is adequate only if one chooses a gauge\(^2\) in which the background field \( A_\mu \) vanishes when time goes to \( +\infty \). Because the conjugate \( \phi^*(y_i) \) already takes care of antiparticle nature, \( q_0^i \) should also be chosen to be positive. Note that, in principle, each field in this formula should be accompanied by a wave-function renormalization factor, \( Z^{-1/2} \). However, since we do not include any self-interactions among the fields, these factors are equal to unity here and we can safely ignore them.

\[ \text{2.3 Generating functional: definition} \]

All the physical quantities related to particle production in this model can be constructed from the squared amplitudes \( |M_{m,n}|^2 \). A very useful object that contains all this information in a compact form is the generating functional defined by [26, 28]

\[
\mathcal{F}[z, \bar{z}] = \sum_{m,n=0}^\infty \frac{1}{m!n!} \int \prod_{i=1}^m d^3 p_i z(p_i) \prod_{j=1}^n d^3 q_j \bar{z}(q_j) \left| M_{m,n}(\{p_i\}, \{q_i\}) \right|^2. \tag{4} \]

In this functional, \( z \) and \( \bar{z} \) are two functions defined over the 1-particle momentum space (unlike what the notation may suggest, they are independent and not complex conjugates of each other).

If one sets the functions \( z \) and \( \bar{z} \) to constants equal to unity, one gets,

\[
\mathcal{F}[1,1] = \sum_{m,n=0}^\infty P_{m,n}, \tag{5} \]

\(^2\)It is always possible to find such a gauge if the electrical field vanishes when time goes to infinity, a necessary condition to be able to unambiguously define what we mean by “measuring a particle”. If one insists on using a gauge in which \( A_\mu \) is not zero when \( x^0 \to +\infty \), one must replace the ordinary derivatives by covariant derivatives and the plane waves by gauge transformed plane waves in eq. (3). The mass-shell condition for \( p_0^i \) and \( q_0^i \) should also be altered by the non-zero background gauge field.
where $P_{m,n}$ is the total probability to have $m$ particles and $n$ antiparticles in the final state. From unitarity, the sum of all these probabilities must be equal to one, hence

$$\mathcal{F}[1,1] = 1.$$ (6)

This is an important constraint on the generating functional $\mathcal{F}[z, \bar{z}]$, that leads to significant simplification in the computation of inclusive observables.

Assuming that this generating functional is known, one can obtain the single inclusive particle spectrum as

$$\frac{dN^+}{d^3p} = \frac{\delta^2 \mathcal{F}[z, \bar{z}]}{\delta z(p)} \bigg|_{z=z=1},$$ (7)

the single inclusive antiparticle spectrum as

$$\frac{dN^-}{d^3q} = \frac{\delta^2 \mathcal{F}[z, \bar{z}]}{\delta \bar{z}(q)} \bigg|_{z=z=1},$$ (8)

and the double inclusive particle-particle spectrum as

$$\frac{dN^{++}}{d^3p_1d^3p_2} = \frac{\delta^2 \mathcal{F}[z, \bar{z}]}{\delta z(p_1) \delta z(p_2)} \bigg|_{z=z=1}.$$ (9)

Other combinations of inclusive 2-particle spectra are given by

$$\frac{dN^{--}}{d^3q_1d^3q_2} = \frac{\delta^2 \mathcal{F}[z, \bar{z}]}{\delta \bar{z}(q_1) \delta \bar{z}(q_2)} \bigg|_{z=z=1}, \quad \frac{dN^{+-}}{d^3p_1d^3q} = \frac{\delta^2 \mathcal{F}[z, \bar{z}]}{\delta z(p_1) \delta \bar{z}(q)} \bigg|_{z=z=1}.$$ (10)

Note that from their definitions, these two particle spectra are normalized so that their integrals over $p$ and $q$ are, respectively,

$$\int d^3p_1d^3p_2 \frac{dN^{++}}{d^3p_1d^3p_2} = \langle N^+(N^+ - 1) \rangle,$$

$$\int d^3q_1d^3q_2 \frac{dN^{--}}{d^3q_1d^3q_2} = \langle N^-(N^- - 1) \rangle,$$

$$\int d^3p_1d^3q \frac{dN^{+-}}{d^3p_1d^3q} = \langle N^+N^- \rangle,$$ (11)

where $N^\pm$ denote the number operator for particles and antiparticles in the final state respectively. In terms of the total probability introduced in eq. (5) we will see that $\langle N^+ \rangle = \sum_{m,n} mP_{mn}$, $\langle N^- \rangle = \sum_{m,n} nP_{mn}$, $\langle N^+(N^+ - 1) \rangle = \sum_{m,n} m(m-1)P_{mn}$, etc, for which one can find a justification in the appendix A. What these equations mean in the $++$ and $--$ cases is that our 2-particle spectra are defined by summing over all possible pairs of distinct particles in every event. When summed over all momenta in a given event, this leads to $N^\pm(N^\pm - 1)$ where $N^\pm$ is the multiplicity of particles (resp. antiparticles) in that event. Naturally, this requirement of taking distinct particles has no incidence on the $+-$ case – since charged particles are always distinct from the corresponding antiparticles –, which explains the last of eqs. (11).
2.4 Generating functional: computation

Let us now proceed to the actual computation of the generating functional $F[z, \bar{z}]$. There is usually no closed form answer for this object. Since we are neglecting the self-interactions of the fields $\phi$ in our model, however, this becomes a much simpler calculation. It has been shown before [26, 28] that the generating functional $F[z, \bar{z}]$ is the sum of all the vacuum-vacuum graphs in a slightly modified version of the Schwinger-Keldysh formalism [48], where the off-diagonal components $G^{0+\mp}$ and $G^{0-\pm}$ of the free propagator are altered by the functions $z$ or $\bar{z}$. Explicitly, the propagators read:

\[ G^{0+\pm}(p) = \frac{i}{p^2 - m^2 + i\epsilon}, \quad G^{0-\pm}(p) = \frac{-i}{p^2 - m^2 - i\epsilon}, \]

As one can see, the off-diagonal free propagators are simply multiplied by $z(p)$ and $\bar{z}(p)$ respectively. Given these propagators, the rules for calculating $F[z, \bar{z}]$ are straightforward:

i. Draw all the vacuum-vacuum diagrams at the desired order. There are simply connected and multiply connected graphs. However, one can always limit the calculation to the simply connected ones, and then exponentiate the result in order to obtain the full result that also includes the multiply connected ones.

ii. For a given graph, sum over all the possible ways to assign + or − signs to the vertices.

iii. A − vertex is the complex conjugate of a + vertex. Let us denote by $V(A)$ the value of the coupling of $\phi, \phi^*$ to the background field in a + vertex. Note that this 'potential' is not simply $A_\mu$ itself, since there are both a $e(\partial_\mu \phi)\phi^* A^\mu$ and a $e^2 \phi \phi^* A_\mu A^\mu$ couplings – however, we will not need its detailed expression in the following. The corresponding − vertex is $V^*(A) = -V(A)$ (this identity follows from the hermiticity of the Lagrangian).

iv. Connect these vertices with the propagators defined in eq. (12).

Step i is trivial: there is only one topology of simply connected vacuum-vacuum graph in our model. These are the graphs made of a single closed loop, embedded with the background field $A^\mu$, as illustrated in fig. 1. We must sum over the number of insertions of the background potential $A_\mu$ (from zero to infinite insertions), and for each of these insertions we must sum over the type + and − for the corresponding vertex. This double summation can be organized in blocks, as illustrated in figs. 2 and 3.

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A vacuum-vacuum graphs are diagrams that have no external legs with respect to $\phi$. 
Figure 1: Topology of the connected vacuum-vacuum diagrams that contribute to $\ln \mathcal{F}$. The solid line denotes the free propagator of the charged scalar field $\phi$ (the arrow indicates the direction of the flow of positive electric charge). The wavy line terminated by a circled cross denotes the background gauge potential $V(A)$. Note that, in scalar QED, the background 'potential' is not simply $A_\mu$ itself, since there are both a $e(\partial_\mu \phi)^* A^\mu$ and a $e^2 \phi \phi^* A_\mu A^\mu$ couplings. The wavy line represents the sum of these two contributions.

Figure 2: Building blocks for the summation of field insertions having a fixed Schwinger-Keldysh vertex assignment.

Figure 3: Block decomposition of the double summation over the number of background field insertions and the $\pm$ assignments at the vertices.
If we denote by $T_+$ the sum of graphs in the first line of fig. 2 and by $T_-$ the sum of graphs on the second line of the same figure, we can take the remaining steps ii, iii, and iv and the sum of the vacuum-vacuum diagrams contributing to $\ln F$ can be written as

$$\ln F[\bar{z}, \tilde{z}] = \text{constant} + \sum_{n=1}^{\infty} \frac{1}{n} \text{tr} \left[ T_+ G^0_+ - T_- G^0_- \right]^n,$$

(13)

where the trace symbol ($\text{tr} (\cdots)$) denotes an integration over the space-time coordinates (not represented explicitly in the formula) of all the vertices. The first term in this formula, that we simply denoted “constant” but did not write explicitly, is independent of $z$ and $\bar{z}$. It is made of all the graphs in which all the vertices are of type $+$ or all of type $-$ (and thus cannot contain $z$ nor $\bar{z}$ since these come with the $G^0_{\pm \mp}$ propagators). In the second term of this formula, the index $n$ represents the number of the block consisting of one $+-$ and one $-+$ transitions, and the factor $1/n$ is a symmetry factor since we can rotate the graph by one block without altering it. Note that the index $n$ gives the order in $z$ and $\bar{z}$ of the corresponding term.

It is trivial to perform explicitly the summation in eq. (13) to find,

$$\ln F[\bar{z}, \tilde{z}] = \text{constant} - \text{tr} \ln \left( 1 - T_+ G^0_+ - T_- G^0_- \right).$$

(14)

The constant term that we did not write explicitly can be determined without any calculation so that $\ln F[1, 1] = 0$, as required from unitarity (6). Therefore, we have

$$F[\bar{z}, \tilde{z}] = \frac{\exp \left( -\text{tr} \ln \left( 1 - T_+ G^0_+ - T_- G^0_- \right) \right)}{\exp \left( -\text{tr} \ln \left( 1 - T_+ G^0_+ - T_- G^0_- \right)_{\bar{z} = \tilde{z} = 1} \right)}.$$ 

(15)

Although fairly formal, this formula contains all we need to know about the particle production by an external electromagnetic field in scalar QED.

In order to simplify the subsequent discussion, let us restrict ourselves to background electric fields that do not depend on the position $x$. In this case it is always possible to choose a gauge in which the background vector potential $A_\mu(x)$ is also independent of $x$, and its Fourier transform is proportional to a delta function $\delta(k)$ as far as its dependence on the spatial components of the momentum is concerned. In this particular case, the 2-point function $T_+ G^0_+ - T_+ G^0_-$ has the same entering and outgoing momenta. In order to make more explicit the $z$ and $\bar{z}$ dependence, let us introduce the notation:

$$\left[ T_+ G^0_+ - T_- G^0_- \right]_p \equiv z(p) \bar{z}(-p) L_p.$$

(16)

The important points here are that $L$ does not contain $z$ and $\bar{z}$, and that the functions $z$ and $\bar{z}$ carry the same momentum up to a relative sign. With this

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4We see explicitly here that the order in $z$ of a given term is the same as its order in $\bar{z}$, which reflects the fact that particles and antiparticles can be created only in pairs.

5Physically, the building block in eq. (16) is the amplitude squared for producing a single particle-antiparticle pair. Since the background field is uniform in space, the total momentum of this pair must be zero. Hence the opposite sign for the momentum argument of $z$ and $\bar{z}$. 

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compact notation:

\[
\mathcal{F}[z, \bar{z}] = \frac{\exp\left(-\text{tr} \ln \left[1 - z\bar{z}L\right]\right)}{\exp\left(-\text{tr} \ln \left[1 - L\right]\right)}.
\]  

(17)

From now on, it is simpler to calculate the trace in momentum space. Indeed, when the background potential is space independent, a unique momentum \( p \) runs around the loop.

To close this subsection, let us mention a generic property of the trace that appears in eq. (17). Strictly speaking, when the background electric field is independent of the location in space, this trace exhibits a factor \((2\pi)^3 \delta(0)\) in momentum space. This factor should be interpreted as the volume \( V \) of the system\(^6\), and its presence is an indication that the particle spectra are proportional to the overall volume.

2.5 Relation with wave propagation in the field \( A_\mu \)

So far, we have not attempted to calculate the object \( L \) that appears in the generating functional. Since it is built from the \( T_{\pm} \), which are Feynman (time-ordered) propagators amputated of their external legs, it is clear that \( L \) is related to the propagation of small fluctuations over the background electric field. However, knowing that \( L \) is related to the Feynman propagator is inconvenient for practical calculations because this propagator obeys complicated boundary conditions. In practice, one should try to rewrite \( L \) in terms of propagators that obey simpler boundary conditions, like the retarded propagator.

Let us start from the equation that defines \( T_+ \) to rewrite it into the retarded quantities. The resummation that leads to \( T_+ \) can be summarized by the following Lippmann-Schwinger equation:

\[
T_+ = \mathcal{V} + \mathcal{V} G^0_{++} T_+, \quad (18)
\]

where \( \mathcal{V} \) is the sum of the two couplings to the background field (the derivative coupling to a single \( A_\mu \) and the non-derivative coupling to \( A_\mu A^\nu \)). We do not need to specify more what \( \mathcal{V} \) is. Note that we could have written the equation in a slightly different form:

\[
T_+ = \mathcal{V} + T_+ G^0_{++} \mathcal{V}. \quad (19)
\]

(This just amounts to starting the expansion from the other end-point of the propagator.) Concerning \( T_- \), it is sufficient to note that it is the complex conjugate of \( T_+ \).

\(^6\)In order to check this, one can make the background electric field slightly space dependent, so that it has a compact support in space. One sees now that all the integrals are finite, and that the single particle spectrum is proportional to the size of the region where the background field is non-zero.
The amputated retarded propagator $T_R$ is defined from the same equation, but the free Feynman propagator $G^0_{++}$ is replaced by the free retarded propagator:

$$T_R = V + VG^0_R T_R = V + T_R G^0_R V ,$$

(20)

where the free retarded propagator $G^0_R(p)$ is defined as

$$G^0_R(p) = \frac{i}{p^2 - m^2 + ip^0 \epsilon} .$$

(21)

In order to express $T_+$ in terms of $T_R$, the first step is to relate the Feynman and the retarded propagators. This is done via the following well-known relationship:

$$G^0_+ = G^0_R + \rho^-, \quad (22)$$

where $\rho^-$ is a 2-point function whose definition in momentum space is

$$\rho^\pm(p) = 2\pi\theta(\pm p^0) \delta(p^2 - m^2) .$$

(23)

(Note that $\rho^-$ is nothing but $G^0_{+-}$ with $z = 1$.) From the above equations, we arrive trivially at

$$(1 - VG^0_R - V\rho_-)T_+ = (1 - VG^0_R)T_R ,$$

(24)

and subsequently at

$$(1 - (1 - VG^0_R)^{-1}V\rho_-)T_+ = T_R .$$

(25)

Thus, we have

$$T_+ = (1 - T_R \rho_-)^{-1} T_R .$$

(26)

Similarly, one can prove,

$$T_+ = T_R (1 - \rho_- T_R)^{-1} .$$

(27)

In fact it is easy to confirm that eqs. (26) and (27) are equivalent by expanding the inverse quantity in terms of $T_R \rho_-$ in eq. (26) and $\rho_- T_R$ in eq. (27). Taking the complex conjugate of eq. (26), we get,

$$T_- = T_+^* = (1 - T_R^* \rho_-)^{-1} T_R^* .$$

(28)

($\rho_-$ is purely real.) Multiplying eq. (27) by $\rho_-$ on the right, we finally obtain,

$$T_+ \rho_- = T_R (1 - \rho_- T_R)^{-1} \rho_- = T_R \rho_- (1 - T_R \rho_-)^{-1} .$$

(29)

Combining everything, we obtain the following expression for $z\bar{z}L$:
Thus we have managed to replace all the Feynman propagators in $L$ by retarded ones. The price to pay for this transformation is that we have now an expression that is no longer bilinear in the propagators, but has terms at any order $\geq 2$. As we shall see now, this apparent complication actually disappears thanks to an identity reminiscent of the optical theorem.

The Lippmann-Schwinger equation for $T_R$ is given in eq. (20). For $T_R^*$, it reads:

$$T_R^* = -\mathcal{V} - \mathcal{V}G^0_R T_R^* = -\mathcal{V} - T_R^* G^0_R \mathcal{V}. \tag{31}$$

Here, we have used the fact that, in a unitary theory (like scalar QED with a real background potential), we have $\mathcal{V}^* = -\mathcal{V}$. Adding up the equations for $T_R$ and $T_R^*$, we first obtain

$$T_R + T_R^* = -T_R^* G^0_R \mathcal{V} + \mathcal{V} G^0_R T_R^*. \tag{32}$$

Finally, we can eliminate $\mathcal{V}$ in the right hand side of this equation, in favor of $T_R$ or $T_R^*$. This leads easily to:

$$T_R + T_R^* = -T_R^* \left[ G^0_R + G^0_R \right] T_R = -T_R^* \left[ \rho_+ - \rho_- \right] T_R. \tag{33}$$

Note that this relation is a variant of the optical theorem\footnote{This is why the relation $\mathcal{V}^* = -\mathcal{V}$, that is the manifestation of unitarity in this calculation, is crucial in order to obtain eq. (33).} applied to a 2-point function. The left hand side is equal to the discontinuity of the 2-point function across the real energy axis, and the right hand side gives the expression of this discontinuity in terms of cut graphs. Thanks to this relationship, it is now straightforward to check that

$$(1 - T_R^* \rho_-)(1 - T_R \rho_-) = 1 + T_R^* \rho_+ T_R \rho_- . \tag{34}$$

By combining eqs. (30) and (34), we easily arrive at the following simplification;

$$1 - z\tilde{z}L = 1 - T_R^* z\rho_- (1 + T_R^* \rho_+ T_R \rho_-)^{-1} T_R^* \tilde{z}\rho_+$$

$$= (1 + T_R^* \rho_+ T_R \rho_-)^{-1} \left[ 1 - (z\tilde{z} - 1) T_R \rho_- T_R^* \rho_+ \right], \tag{35}$$

which leads to the generating functional:

$$\mathcal{F}[z, \tilde{z}] = \exp \left( -\text{tr} \ln \left[ 1 - (z\tilde{z} - 1) T_R \rho_- T_R^* \rho_+ \right] \right). \tag{36}$$

Here again, thanks to the fact that the background field is uniform, all the factors inside the logarithm share a single spatial momentum $p$. The function $z$ has argument $p$ and the $\tilde{z}$ is evaluated at $-p$.

At this point, we see that all the properties of the distribution of produced particles are determined by a single quantity, namely the amputated retarded propagator $T_R$ of a scalar particle on top of the background field $A^\mu$. Before
going further, it may be useful to rewrite the combination \( z \bar{z} T_R \rho_- T_R^* \rho_+ \) with all the momentum dependence:

\[
\left[ z \bar{z} T_R \rho_- T_R^* \rho_+ \right]_{p,q} = \bar{z}(-q) \rho_+(q) \int \frac{d^4k}{(2\pi)^4} z(k) T_R(p,k) \rho_-(k) T_R^*(k,q). \tag{37}
\]

This formula is completely general. Note that \( T_R^* \) is the same quantity as \( T_R \), in which all the retarded propagators are replaced by advanced ones and \( V(A) \) is replaced by \( V^*(A) \).

In eq. (37), the momenta \( k \) and \( q \) are forced to be on the in-vacuum mass-shell, since they appear inside the distribution \( \rho_\pm \). However, in the case of \( k \), it turns out to be simpler to have a momentum variable (which we denote here by \( \tilde{k} \)) that obeys the mass-shell condition imposed by the non-zero gauge potential at \( x^0 \to -\infty \). Let us denote \( G^\infty_R \) the retarded propagator evaluated in the presence of the (constant) background field

\[
A^\infty_\mu \equiv \lim_{x^0 \to -\infty} A_\mu(x). \tag{38}
\]

Since there is no electrical field at \( x^0 \to -\infty \), the gauge field is a pure gauge in this limit:

\[
A^\infty_\mu = \partial_\mu \chi(x), \tag{39}
\]

and the propagator \( G^\infty_\mu \) is simply obtained by a gauge transformation from the vacuum propagator \( G^0_\mu \):

\[
G^\infty_\mu(x,y) = e^{ie\chi(x)} G^0_\mu(x,y) e^{-ie\chi(y)}. \tag{40}
\]

Let us now rewrite the combination \( T_R \rho_- T_R^* \) that appears in eq. (37) in terms of the corresponding expressions \( T^\infty_R \) and \( \rho^\infty_\pm \) which are naturally functions of the modified mass-shell momentum \( \tilde{k} \). This can be done by writing eq. (37) as

\[
T_R \rho_- T_R^* = \frac{T_R G^0_\mu(G^\infty_R)^{-1} G^\infty_\mu(G^0_\mu)^{-1} \rho_- ((G^0_\mu)^{-1} G^\infty_\mu)^* ((G^\infty_R)^{-1} G^0_\mu T_R^*)^*}{(T^\infty_R)^*}. \tag{41}
\]

Note that because

\[
(D^\infty_x + m^2)G^\infty_\mu(x,y) = \delta(x-y), \tag{42}
\]

where \( D^\infty_x \) is the covariant derivative constructed with the asymptotic field \( A^\infty_\mu \) at \( y^0 \to -\infty \), and \( \rho^\infty_\pm(x,y) \) is still translationally invariant, it now projects momenta to the mass shell in presence of the background field at \( x^0 \to -\infty \). Since this gauge field is a pure gauge, it is easy to write the corresponding mass-shell conditions imposed by \( \rho^\infty_+ \) and \( \rho^\infty_- \):

\[
(\tilde{k} \pm eA^\infty_x)^2 = m^2, \tag{43}
\]

where the signs \( \pm \) are to be chosen for the positive and negative energy solutions respectively. In a gauge where \( A^0 = 0 \), as we shall chose later on, these read:

\[
\tilde{k}_0 = E^\infty_k \quad \text{and} \quad \tilde{k}_0 = -E^\infty_{-k}, \quad \text{where} \quad E^\infty_k \equiv \sqrt{(k + eA^\infty)^2 + m^2}. \tag{44}
\]
Therefore, eq. (37) can be rewritten as
\[
\left[ z^2 T_R \rho - T_R^* \rho_+ \right]_{p,q} = z(-q) \rho_+ (q) \int \frac{d^4 k}{(2\pi)^4} z(k) T_R^\infty (p, \tilde{k}) \rho_+ (\tilde{k}) (T_R^\infty (\tilde{k}, q))^* = z(p) z(-q) \rho_+ (q) \int \frac{d^4 k}{(2\pi)^4} \left| T_R^\infty (p, -\tilde{k}) \right|^2 \rho_+^\infty (\tilde{k}) .
\] (45)

In the second line, we have changed \( \tilde{k} \rightarrow -\tilde{k} \), and we have exploited the fact that \( T_R^\infty (p, -\tilde{k}) \) is proportional to \( \delta(p + k) \) in a uniform background field. Note that thanks to the constraints provided by the \( \rho_+ (q) \) and \( \rho_+^\infty (\tilde{k}) \) factors, the energies \( \tilde{k}^0 \) and \( q^0 \) are both positive. However, they obey different mass-shell conditions. The outgoing particle\(^8\) energy \( q^0 \) follows the in-vacuum dispersion relation, while \( \tilde{k}^0 \) obeys the dispersion relation in the presence of the background field \( A^\infty \).

For practical calculations of \( T_R^\infty (p, -\tilde{k}) \), it is best to relate this quantity to the Fourier coefficients of a plane wave propagating on top of the background field. Since \( T_R^\infty \) is obtained by amputating the retarded propagator \( R_R \) with \( (G_R^0)^{-1} \) on the right and with \( (G_R^\infty)^{-1} \) on the left, we can immediately write:
\[
T_R^\infty (p, -\tilde{k}) = \int d^4 x \ e^{ip \cdot x} \left( \Box + m^2 \right) \int d^4 y \ e^{i\tilde{k} \cdot y} (D_p^{\Box 2} + m^2) G_R (x, y)
\]
\[
= \lim_{x^0 \rightarrow +\infty} \int d^3 x \ e^{ip \cdot x} (\partial_{x^0} - iE^\text{out}_p) \eta_k (x) .
\] (46)

In these formulas \( E^\text{out}_p \) is the vacuum on-shell energy \( E^\text{out}_p = \sqrt{p^2 + m^2} \). Since the propagator \( G_R (x, y) \) is a Green’s function of the operator \( D^2 + m^2 \) (now, the covariant derivative \( D^\mu \) is defined with the full background field, not just its asymptotic value in the past),
\[
[D^2 + m^2] G_R (x, y) = \delta^4 (x - y) ,
\] (47)

it is easy to check that \( \eta_k (x) \) obeys the following equation of motion:
\[
[D^2 + m^2] \eta_k (x) = 0 ,
\] (48)

provided that \( \tilde{k} \) obeys the negative energy mass-shell condition (43). In order to find the boundary condition when \( x^0 \rightarrow -\infty \) for \( \eta_k (x) \), we can replace the full propagator \( G_R (x, y) \) in eq. (46) by the propagator \( G_R^\infty (x, y) \) that resums only the asymptotic field \( A^\infty \),
\[
\eta_k (x) = \int d^4 y \ e^{i\tilde{k} \cdot y} (D_y^{\Box 2} + m^2) G_R^\infty (x, y) = e^{i\tilde{k} \cdot x} .
\] (49)

\(^8\)Recall that we will need the trace of \( z^2 T_R \rho - T_R^* \rho_+ \) and thus \( q \) will be equal to the momentum of the produced particle \( p \).
In order to obtain the final formula, we have used:

\[(D_y^{\infty^2} + m^2) G_\infty^\infty(x, y) = \delta(x - y)\].

(50)

Thus, we see that the initial condition for \(\eta_k(x)\) is a plane wave, with a momentum \(\tilde{k}\) that obeys the mass-shell condition of eq. (43).

In a uniform background field, we can simplify a bit the notations by writing

\[T_\infty^\infty(p, \tilde{k}) \equiv -2i E_{\text{out}} p (2\pi)^3 \delta(p + \tilde{k}) |\beta_p|^2\]

(51)

so that

\[\left[z \bar{z} T_\infty^\infty \rho_\infty^\infty (T_\infty^\infty)^* \rho_+\right]_{p,q} = 2E_{\text{out}} p (2\pi)^3 \delta(p - q) z(p) \bar{z}(-q) \rho_+ (q) |\beta_p|^2\].

(52)

The only quantity that we need to determine in order to fully solve the problem is the coefficient \(\beta_p\). This is obtained by solving the equation of motion (48), with a plane wave initial condition when \(x^0 \rightarrow -\infty\). We note that the initial plane wave is chosen as an antiparticle-like one in eq. (49) and projected into a particle-like one in eq. (46), the intuitive meaning of which will be clear in discussions in sec. 3.

### 2.6 Multiparticle spectra

Let us now use eqs. (37) and (52) in order to obtain results about the spectra of the produced particles. From now on, let us simply denote \(E_{\text{out}}^p\) as \(E_p\) in this section, for we have chosen the definition of \(\beta_p\) so that \(E_{\text{in}}^p\) will never appear in the expressions. We shall wait for the next section where the difference between \(E_{\text{in, out}}^p\) and the physical interpretation of \(\beta_p\) will be more explicit. The single inclusive particle spectrum is obtained as the first derivative of the generating functional with respect to \(z(p)\). We obtain

\[
\frac{dN}{d^3p} = \frac{\delta}{\delta z(p)} \text{tr}\left[z \bar{z} T_\infty^\infty \rho_\infty^\infty (T_\infty^\infty)^* \right]_{z, \bar{z} = 1} \left|\beta_p\right|^2.
\]

(53)

It should be mentioned that, in accord with the definition (4), the functional differentiations with respect to \(z(p)\) and \(\bar{z}(q)\) are not accompanied by \((2\pi)^3\). In the second, we have made the trace explicit. The final \(\delta(p - k)\) comes from the differentiation with respect to \(z(p)\). In the last line, we have performed the \(d^4k\) integration explicitly, and we have interpreted the (infinite) factor \((2\pi)^3\delta(0)\) as the volume \(V\) of the system. Naturally, the spectrum of antiparticles is identical. For later reference, it will be useful to introduce more compact notations as follows:

\[
n_p \equiv \frac{dN}{d^3p} = \frac{V}{(2\pi)^3} |\beta_p|^2,
\]

\[
f_p \equiv \frac{(2\pi)^3}{V} n_p = |\beta_p|^2.
\]

(54)
Note that \( f_p \) has the interpretation of the occupation number for the produced particles of momentum \( p \). This is clear if we integrate eq. (54) over the momentum and write it as
\[
\langle N^+ \rangle = \int \frac{d^3p}{(2\pi)^3} f_p,
\]
where the properly normalized phase space measure \( d^3p \, d^3x/(2\pi)^3 \), or restoring the Planck constant \( d^3p \, d^3x/h^3 \), is explicit.

Let us now turn to the 2-particle spectra. For two particles, we obtain
\[
\frac{dN_{2^+}^-}{d^3p_1 \, d^3p_2} - \frac{dN_{1^+}^-}{d^3p_1} \frac{dN_{1^+}^-}{d^3p_2} = \frac{\delta^2}{\delta z(p_1) \delta z(p_2)} \text{tr} \left[ z \bar{z} \rho_+ T_\infty^\beta \rho_\infty^\beta (T_\infty^\beta)^* z \bar{z} \rho_+ T_\infty^\beta \rho_\infty^\beta (T_\infty^\beta)^* \right]_{z=\bar{z}=1} = \delta(p_1 - p_2) n_p f_p_1.
\]

The uncorrelated part of the 2-particle spectrum shows up naturally in this calculation, and we have absorbed it in the left hand side. The right hand side represents the correlated component of the 2-particle spectrum. As one can see, particles are correlated only if they have identical momenta. By integrating the previous equation over \( p_1 \) and \( p_2 \), and by using the first of eqs. (11), we obtain:
\[
\langle N^+ (N^+ - 1) \rangle - \langle N^+ \rangle^2 = \int d^3p \, n_p f_p,
\]
or equivalently
\[
\langle N^+ N^+ \rangle - \langle N^+ \rangle^2 = \int d^3p \, n_p (1 + f_p) = \int \frac{d^3p \, d^3x}{(2\pi)^3} f_p (1 + f_p).
\]
The form with an explicit integral over \( x \) is the one that would be applied to a system with a phase space density that depends (slowly) on the coordinate. The first term in the right hand side (the 1 in \( 1 + f_p \)) is the answer one would obtain for a Poisson distribution. That is, if the probability distribution is given by
\[
P_{mn} = \delta_{mn} \frac{e^{-\langle N \rangle} \langle N \rangle^m}{m!},
\]
which defines the Poisson distribution, one would have \( \langle N^+ N^+ \rangle - \langle N^+ \rangle^2 = \langle N^+ \rangle \) (i.e. the variance and the mean are identical). Thus, the deviations from a Poisson distribution are contained in the term proportional to \( n_p f_p \).

Equation (58) indicates that these correlations are Bose-Einstein correlations, i.e. due to stimulated emission of particles in a single quantum state.

For one particle and one antiparticle, we get:
\[
\frac{dN_{2^+}^-}{d^3p \, d^3q} - \frac{dN_{1^+}^-}{d^3p} \frac{dN_{1^+}^-}{d^3q} = \frac{\delta^2}{\delta z(p) \, \delta \bar{z}(q)} \left[ \text{tr} \left[ z \bar{z} \rho_+ T_\infty^\beta \rho_\infty^\beta (T_\infty^\beta)^* \right] + \text{tr} \left[ z \bar{z} \rho_+ T_\infty^\beta \rho_\infty^\beta (T_\infty^\beta)^* z \bar{z} \rho_+ T_\infty^\beta \rho_\infty^\beta (T_\infty^\beta)^* \right] \right]_{z=\bar{z}=1} = \delta(p + q) n_p (1 + f_p).
\]
In this case, the correlation can only exist if the particle and antiparticle have opposite spatial momenta. Again, the integrated form of this equation reads

$$\langle N^+ N^- \rangle - \langle N^+ \rangle \langle N^- \rangle = \int d^3p \, n_p (1 + f_p). \tag{61}$$

One may have wondered why the particle and antiparticle have correlations. This can be understood by the fact that, as we noted repeatedly, the pair of a particle and an antiparticle is created at once so that the particle and antiparticle production preserves the momentum conservation as well as the charge conservation. Therefore, the correlation (60) reflects the particle-particle correlation (56) with $p_1 = p$ and $p_2 = -q$, which explains the delta function of spatial momenta in eq. (60).

### 2.7 Nature of the distribution

The results of the previous subsection suggest that the distribution of produced particles obeys the following properties:

i. Two particles are correlated only if they have identical momenta.

ii. A particle and an antiparticle are correlated only if they have opposite momenta.

iii. In a given momentum mode, the distribution of produced particles follows a Bose-Einstein distribution.

Let us now present a more general justification of these results. Because the background electric field is uniform, a unique momentum $p$ runs around the loop. By using results obtained in the previous subsections, we can rewrite the generating functional explicitly as follows:

$$\mathcal{F}[z, \bar{z}] = \exp \left( -V \int \frac{d^3k}{(2\pi)^3} \ln \left[ 1 - \left( z(k) \bar{z}(-k) - 1 \right) f_k \right] \right). \tag{62}$$

In writing eq. (62), as already discussed previously in this paper, we have assumed that the system is placed in a finite volume $V$. This is indeed necessary in order to have a finite particle production rate in a constant (in space) external field. A consistent quantization of the system in a finite volume requires one to specify boundary conditions\(^9\) at the edges of $V$, which leads to the momentum $k$ being a discrete variable. The continuum in $k$ is recovered in the limit $V \to \infty$. Switching now to a notation which makes this explicit, and remembering that $d^3k/(2\pi)^3 = \sum_k$ we can write the generating functional as

$$\mathcal{F}[z, \bar{z}] = \prod_k \frac{1}{1 + f_k - z_k \bar{z}_k f_k}, \tag{63}$$

\(^9\)Periodic ones being the most convenient.
where we use a compact notation; \( z_k \equiv z(\mathbf{k}) \) and \( \bar{z}_k \equiv \bar{z}(-\mathbf{k}) \). From this formula, one sees immediately that the distributions of produced particles in the various modes are totally uncorrelated, since the generating functional factorizes as a product of generating functions for single modes:

\[
\mathcal{F}[z, \bar{z}] = \prod_k \mathcal{F}_k(z_k, \bar{z}_k), \quad \mathcal{F}_k(z_k, \bar{z}_k) \equiv \frac{1}{1 + f_k - z_k \bar{z}_k f_k}. \tag{64}
\]

By Taylor expanding this formula around \( z_k, \bar{z}_k = 0 \), it is easy to obtain the probability of having \( m_i \) particles and \( n_i \) antiparticles in the mode \( i \),

\[
P(\{m_i\}, \{n_i\}) = \prod_k \delta_{m_k, n_k} \left( \frac{f_k}{1 + f_k} \right)^{m_k}. \tag{65}
\]

Given the occupation numbers \( f_k \) (obtained from \( \beta_k \) by solving the equation of motion of a plane wave over the background field), this formula completely specifies the distribution of produced particles and antiparticles. Distinct modes are not correlated. In each mode, there must be an equal number of particles and antiparticles. The distribution of the particle multiplicity in the mode \( k \) is a Bose-Einstein distribution of occupation number \( f_k \). A Bose-Einstein distribution is in sharp contrast to a Poisson distribution (which would be the result in a complete absence of correlations), since its decrease at large \( m_k \) is much slower because of the absence of \( m_k! \) in a Poisson distribution (see the denominator of eq. (59)). As a result, final states with many particles in the same momentum mode are more likely.

### 3 Bogoliubov transformation interpretation

We can interpret the results obtained in the LSZ derivation as a Bogoliubov transformation. To do this explicitly it is useful to switch to canonical quantization, which we shall review here shortly as the following manipulations are very standard ones.

From the Lagrange density of eq. (1) one obtains the Hamiltonian of the theory (apart from the gauge part),

\[
H = \int d^3x \left[ \Pi^\dagger \Pi + ieA_0 \Pi \Phi - ieA_0 \Pi^\dagger \Phi^\dagger + (m^2 - e^2A_0^2)\Phi\Phi^\dagger + (\vec{D}\Phi) \cdot (\vec{D}\Phi)^\dagger \right], \tag{66}
\]

where \( \Phi \) and \( \Pi \) are operators in the Heisenberg picture satisfying the equal-time commutation relation,

\[
[\Phi(x), \Pi(y)] = i\delta(x - y). \tag{67}
\]

It will be convenient for the following discussion to choose a gauge where \( A_0 = 0 \), because in this gauge it is possible to directly associate the time dependence
of the wave function with the physical energy of the particle\textsuperscript{10}. We shall thus work with the following Hamiltonian,

\begin{equation}
H = \int d^3x \left[ \Pi \Pi + m^2 \Phi \Phi + (\vec{D}\Phi) \cdot (\vec{D}\Phi)^\dagger \right].
\end{equation}

The whole dynamics of the matter fields is determined by the equations of motion

\begin{align}
\partial_0 \Phi &= i [H, \Phi] = \Pi^\dagger \\
\partial_0 \Pi &= i [H, \Pi] = (\vec{D}^2 - m^2) \Phi^\dagger.
\end{align}

These can then be expressed as an equation of motion for \( \Phi \) only, but involving second order time derivatives. The important thing to realize is that because we are looking at a theory without self-interactions and coupled to a classical background field\textsuperscript{11}, the equations of motion of the field operators are linear; they are in fact the same as the classical equations of motion for the fields. The solution to the retarded field equations therefore contains all the information about the relation between the field operators \( \Phi \) and \( \Pi \) at \( x^0 \to -\infty \) and \( x^0 \to \infty \). In the Heisenberg picture, knowing the relations between the field operators is equivalent to knowing the dynamics of the theory; in particular the whole probability distribution of the produced particles.

We can now introduce the familiar decomposition of the field operators in terms of creation and annihilation operators. One can perform this decomposition in different bases of operators; in particular the ones that correspond to particles at \( x^0 \to -\infty \) (the “in” states) or \( x^0 \to \infty \) (the “out” states). Since these are just decompositions of the same operator \( \Phi \) in different bases, one gets the equality,

\begin{equation}
\Phi(x) = \int \frac{d^3k}{(2\pi)^3} \left[ a_{\text{in}, k} \phi^{+}_{\text{in}, k}(x) + b_{\text{in}, k}^\dagger \phi^{-}_{\text{in}, -k}(x) \right] = \int \frac{d^3k}{(2\pi)^3} \left[ a_{\text{out}, k} \phi^{+}_{\text{out}, k}(x) + b_{\text{out}, k}^\dagger \phi^{-}_{\text{out}, -k}(x) \right].
\end{equation}

Here we have denoted the dispersion relations of particles and antiparticles in the in-state as \( E^\text{in}_k \) and that in the out-state as \( E^\text{out}_k \). We assume that the background electric field is turned off adiabatically \( x^0 \to \pm \infty \), which means that \( A_\mu \) approaches a constant value \( A^\text{in/out}_\mu \). The dispersion relation for particles is then \( E^\text{in/out, +}_k = \sqrt{m^2 + (k + eA^\text{in/out}_\mu)^2} \) and the one for antiparticles

\textsuperscript{10}Consider for example a particle at rest in the vacuum: in the gauge \( A_\mu = 0 \) its wavefunction is \( e^{-imx^0} \). Performing a time-dependent gauge transformation with the function \( Mx^0/e \) will generate a (constant) gauge potential \( A_0 = -M/e \) and change the wave-function to \( e^{-i(M-m)x^0} \), which for \( M > m \) will seemingly look like a negative energy one.

\textsuperscript{11}Coupling to a quantum field would induce effective self-interactions through loop corrections.
$E_{k}^{\text{in}, \text{out}, -} = \sqrt{m^2 + (k - eA_{\text{in}, \text{out}})^2}$. In writing eq. (70) we have used the symmetry $E_{k}^{\text{in}} = E_{-k}^{\text{out}}$ to write everything in terms of the particle dispersion relation $E_{k}^{\text{in}, \text{out}} \equiv E_{k}^{\text{in}, \text{out}, +}$. Note that the momentum label $k$ refers to the “canonical” momenta, which are the variables describing the oscillation of the wavefunction in space. The momentum that is actually measured in a detector is the “kinetic” one, which in this case is $k + eA_{\text{in}, \text{out}}$ for particles and $k - eA_{\text{in}, \text{out}}$ for antiparticles. We are keeping the notations rather general in this section. In the physical situation we are interested in, the only particles that are measured are the “out”-ones. A convenient gauge choice, and the one adopted in sec. 4 is then to take $A_{\text{out}} = 0$, so that one need not distinguish between the canonical and kinematical momenta for particles in the final state—it is enough to remember that $E_{k}^{\text{in}}$ is different from $E_{k}^{\text{out}}$.

The choice of basis in the decomposition eq. (70) of the field operator is determined by the boundary conditions for the functions $\phi_{\text{in}, k}(x)$ and $\phi_{\text{out}, k}(x)$. When we require that they approach plane waves at asymptotic times:

\[
\phi^{+}_{\text{in}, k}(x) = e^{-iE_{k}^{\text{in}} x^0 + ik \cdot x} \quad \text{for } x^0 \to -\infty \tag{71}
\]
\[
\phi^{-}_{\text{in}, k}(x) = e^{iE_{k}^{\text{in}} x^0 + ik \cdot x} \quad \text{for } x^0 \to -\infty \tag{72}
\]
\[
\phi^{+}_{\text{out}, k}(x) = e^{-iE_{k}^{\text{out}} x^0 + ik \cdot x} \quad \text{for } x^0 \to +\infty \tag{73}
\]
\[
\phi^{-}_{\text{out}, k}(x) = e^{iE_{k}^{\text{out}} x^0 + ik \cdot x} \quad \text{for } x^0 \to +\infty \tag{74}
\]

the corresponding operators $a_{\text{in}, \text{out}}, b_{\text{in}, \text{out}}$ annihilate the in-state and out-state particles and antiparticles, respectively. Note that for further convenience our notation has been chosen such that the coordinate dependence in both $\phi^{+}_{\text{in}, k}(x)$ and $\phi^{-}_{\text{in}, k}(x)$ is $e^{ik \cdot x}$ and thus the usual negative energy plane wave $e^{ik \cdot x}$ corresponds to $\phi^{-}_{\text{in}, k}(x)$. The canonical commutation relation for $\Phi$ and $\Pi$ is satisfied if:

\[
[a_{\text{in}, k}, a^{\dagger}_{\text{in}, p}] = [b_{\text{in}, k}, a^{\dagger}_{\text{in}, p}] = [a_{\text{out}, k}, a^{\dagger}_{\text{out}, p}] = [b_{\text{out}, k}, b^{\dagger}_{\text{out}, p}] = (2\pi)^3 \delta(k - p). \tag{75}
\]

All the space-time dependence of the field operator $\Phi$ is in the coefficient functions $\phi^{\pm}_{\text{in}, \text{out}, k}(x)$: the creation and annihilation operators are time-independent. Because the equation of motion (69) for $\Phi$ is linear, the coefficient functions $\phi^{\pm}_{\text{in}, \text{out}, k}(x)$ must each independently satisfy the same equation. In fact the solution to the equation of motion $\eta_{k}(x)$ introduced in eq. (48) is nothing but $\phi^{-}_{\text{in}, -k}(x)$.

The relation between the field operators at $x^0 \to -\infty$ and at $x^0 \to +\infty$ is encoded in the Bogoliubov coefficients. They are in the transformation matrix between the in- and out-basis functions. The solution for $\phi^{+}_{\text{in}, k}(x)$ is again a superposition of plane waves at $x^0 \to +\infty$. If the background field depends only on time, the modes of different $k$ do not mix and we can introduce the Bogoliubov coefficients as the coefficients of this plane wave decomposition by

\[\text{Note that in here these commutation relations do not include a factor } 2E_{k} \text{ as is conventional. The normalization used here is simpler in the case where } E_{k}^{\text{in}} \text{ differs from } E_{k}^{\text{out}}.\]
By noticing that \( \phi^+_{\text{in},k}(x^0,-x) \) satisfies both the same initial condition as \( \phi_{\text{in},k}(x) \) and the same equation of motion, one finds the solution at \( x^0 \to \infty \) that starts as a negative energy wave as

\[
\lim_{x^0 \to +\infty} \phi^+_{\text{in},k}(x) = \sqrt{\frac{E_{\text{in}}}{E_{\text{out}}}} \left( \alpha_k e^{-iE_{\text{out}}x^0 + ik \cdot x} + \beta_k e^{iE_{\text{out}}x^0 + ik \cdot x} \right) .
\]

This can also been seen using \( e^{-ik \cdot x} \phi^+_{\text{in},k}(x) = [e^{-ik \cdot x} \phi_{\text{in},k}(x)]^* \). We can now deduce the relation,

\[
\phi^+_{\text{in},k}(x) = \sqrt{\frac{E_{\text{in}}}{E_{\text{out}}}} \left( \alpha_k \phi^+_{\text{out},k}(x) + \beta_k \phi^-_{\text{out},k}(x) \right) ,
\]

\[
\phi^-_{\text{in},k}(x) = \sqrt{\frac{E_{\text{in}}}{E_{\text{out}}}} \left( \alpha_k \phi^-_{\text{out},k}(x) + \beta_k \phi^+_{\text{out},k}(x) \right) .
\]

In the general case of a space dependent background field the Bogoliubov coefficients are not diagonal in momentum space. However, one can diagonalize the transformation matrix from the in- to the out-states, and our following discussion will equally well apply to the eigenstates of the more general transformation instead of individual momentum modes. Inserting the decompositions (78) and (79) into eq. (70) one gets

\[
a_{\text{out},k} = \alpha_k a_{\text{in},k} + \beta_k b^\dagger_{\text{in},-k} ,
\]

\[
b^\dagger_{\text{out},k} = \alpha^*_k b^\dagger_{\text{in},k} + \beta^*_k a_{\text{in},-k} .
\]

Consistency with the commutation relations (75) gives the normalization condition,

\[
|\alpha_k|^2 - |\beta_k|^2 = 1 .
\]

This normalization condition is a consequence of the charge conservation symmetry of our Lagrangian\(^\text{13}\). The above relations can be inverted to give

\[
a_{\text{in},k} = \alpha_k^* a_{\text{out},k} - \beta_k b^\dagger_{\text{out},-k} ,
\]

\[
b_{\text{in},k} = \alpha^*_k b_{\text{out},k} - \beta_k a^\dagger_{\text{out},-k} .
\]

\(^\text{13}\)One can verify that eq. (81) implies \( Q_{\text{out}} = Q_{\text{in}} \), where \( Q \equiv e \int \frac{d^3k}{(2\pi)^3} (a^\dagger_k a_k - b^\dagger_k b_k) \).
The vacuum is defined as the state that is annihilated by all the destruction operators (we shall drop the in- and out-labels for a moment),

$$a_k |0\rangle = b_k |0\rangle = 0 .$$  \hfill (84)

A properly normalized state with $n$ particles of momentum $k$ can be constructed as

$$|n_k\rangle = \left( \frac{a_k^\dagger}{\sqrt{n!V}} \right)^n |0\rangle .$$  \hfill (85)

Particles of momentum $k$ are counted with the particle number operator,

$$\frac{d\hat{N}}{d^3k} = \frac{a_k^\dagger a_k}{(2\pi)^3} .$$  \hfill (86)

For example, the expectation value of the number operator on a state with one particle of momentum $p$ is

$$\langle 1_p | \frac{d\hat{N}}{d^3k} | 1_p \rangle = \langle 0 | \frac{a_p a_p^\dagger a_k b_k^\dagger}{\sqrt{V}} \frac{a_k^\dagger}{\sqrt{V}} | 0 \rangle = \delta(p - k) .$$  \hfill (87)

After setting up these conventions let us return to the problem of particle production. We consider the situation where there are no particles in at $x^0 \to -\infty$, then the system is in the incoming vacuum state defined by

$$a_{\text{in},p} |0_{\text{in}}\rangle = b_{\text{in},p} |0_{\text{in}}\rangle = 0 .$$  \hfill (88)

We are working in the Heisenberg picture where there is no time evolution in the states, so the system stays in the $|0_{\text{in}}\rangle$ state. But at late time $x^0 \to +\infty$ particles are described by the “out” annihilation operators which do not necessarily annihilate $|0_{\text{in}}\rangle$. In order to count the number of outgoing particles we must count the number of out-particles contained in the state $|0_{\text{in}}\rangle$. For this it is useful to derive an expression for $|0_{\text{in}}\rangle$ in terms of the “out” quantities $a_{\text{out},p}$, $b_{\text{out},p}$ and $|0_{\text{out}}\rangle$. That is, we have to solve,

$$a_{\text{in},p} |0_{\text{in}}\rangle = (\alpha^*_p a_{\text{out},p} - \beta_p b_{\text{out},-p}^\dagger) |0_{\text{in}}\rangle = 0 ,$$  \hfill (89)

This above equation can be easily solved using the following ansatz;

$$|0_{\text{in}}\rangle = C \prod_k \exp(\lambda_k a_{\text{out},k}^\dagger b_{\text{out},-k}^\dagger) |0_{\text{out}}\rangle ,$$  \hfill (90)

where $C$ is a normalization constant. Applying the condition (89) to the ansatz (90) we find the condition,

$$\left[ \sum_{n=1}^\infty \frac{\lambda^n}{n!} a_{\text{out},k}^n V(a_{\text{out},k}^\dagger)^{n-1}(b_{\text{out},-k}^\dagger)^n - \sum_{n=0}^\infty \frac{\lambda^n}{n!} \beta_k (a_{\text{out},k}^\dagger)^n(b_{\text{out},-k}^\dagger)^{n+1} \right] |0_{\text{out}}\rangle = 0 .$$  \hfill (91)
Shifting the summation variable by one in the first term gives
\[\sum_{n=0}^{\infty} \frac{\lambda_k}{n!} \left( \lambda_k \alpha_k^* V - \beta_k \right) \left( a_{\text{out},k}^\dagger \right)^n \left( b_{\text{out},-k}^\dagger \right)^{n+1} |0_{\text{out}}\rangle = 0, \tag{92}\]
which is satisfied by
\[\lambda_p = V^{-1} \frac{\beta_p}{\alpha_p^*}. \tag{93}\]
Now we can fix the normalization constant from
\[\langle 0_{\text{out}} | \exp \left\{ \lambda_p a_{\text{out},p} b_{\text{out},-p}^\dagger \right\} \exp \left\{ \lambda_p a_{\text{out},p}^\dagger b_{\text{out},-p} \right\} |0_{\text{out}}\rangle = \frac{1}{1 - V^2 |\lambda_p|^2} = 1 + |\beta_p|^2, \tag{94}\]
where we used the normalization condition (81). We then have the expression for the initial state in the following form:
\[|0_{\text{in}}\rangle = \prod_k \left( 1 + |\beta_k|^2 \right)^{-1/2} \exp \left[ V^{-1} \frac{\beta_k}{\alpha_k^*} a_{\text{out},k}^\dagger b_{\text{out},-k}^\dagger \right] |0_{\text{out}}\rangle. \tag{95}\]
This has a clear physical interpretation that the initial vacuum is a superposition of states with out-state pairs of particles with \(k\) and antiparticles with \(-k\).

Armed with the explicit expression (95) it is now straightforward to calculate, for example, single and double inclusive spectra by taking expectation values of the number operator. The spectrum of particles is
\[\frac{dN^+}{d^3p} = \langle 0_{\text{in}} | a_{\text{out},p} a_{\text{out},p}^\dagger |0_{\text{in}}\rangle = \frac{V}{(2\pi)^3} (1 + |\beta_p|^2)^{-1} \sum_{n=1}^\infty \left( \frac{|\beta_p|^2}{\alpha_p^2} \right)^n = \frac{V}{(2\pi)^3} |\beta_p|^2. \tag{96}\]
This expression exactly coincides with eq. (53). The two particle spectrum (equal sign) is likewise
\[\frac{dN^{++}}{d^3p_1 d^3p_2} = \langle 0_{\text{in}} | a_{\text{out},p_1} a_{\text{out},p_2} a_{\text{out},p_2}^\dagger a_{\text{out},p_1}^\dagger - \delta(p_1 - p_2) \frac{a_{\text{out},p_1} a_{\text{out},p_1}^\dagger}{(2\pi)^3} \rangle |0_{\text{in}}\rangle
\]
\[= \frac{1}{(2\pi)^6} \langle 0_{\text{in}} | a_{\text{out},p_1}^\dagger a_{\text{out},p_2}^\dagger a_{\text{out},p_2} a_{\text{out},p_1} |0_{\text{in}}\rangle
\]
\[= \frac{dN^+_1}{d^3p_1} \frac{dN^+_2}{d^3p_2} + \delta(p_1 - p_2) \frac{V}{(2\pi)^3} |\beta_p|^4. \tag{97}\]
Note that we are explicitly subtracting the delta function contribution to agree with the definition (11). The variance is
\[\langle N^+ N^+ \rangle - \langle N^+ \rangle \langle N^+ \rangle = \int d^3p \ n_p (1 + f_p), \tag{98}\]
\[23\]
which should remind the reader of the particle number fluctuations in a Bose-Einstein system. It is actually easy to see that this is indeed what we have by looking directly at the probability distribution.

In order to see more clearly the structure of the probability distribution we note that we can decompose the Fock space into a direct product of the Fock spaces for different momenta \( k \). We then write an arbitrary state \(|\Psi\rangle\) as a tensor product

\[
|\Psi\rangle = \bigotimes_k |\Psi\rangle_k .
\]  

(99)

It is convenient to group together particles of momentum \( k \) and antiparticles of momentum \(-k\) under the same label \( k \). Thus we define the vacuum states \(|0,0\rangle_k\) of the subspaces \( k \) as

\[
a_{\text{out},k}|0,0\rangle_k = b_{\text{out},-k}|0,0\rangle_k = 0 .
\]  

(100)

We can now write \(|0_{\text{out}}\rangle\) as a tensor product of the vacuum states of the different modes,

\[
|0_{\text{out}}\rangle = \bigotimes_k |0,0\rangle_k .
\]  

(101)

A state with \( m \) particles of momentum \( k \) and \( n \) antiparticles of momentum \(-k\) at \( x^0 \to \infty \) is then denoted by

\[
|m,n\rangle_k = \left( \frac{a_{\text{out},k}}{\sqrt{m!V}} \right)^m \left( \frac{b_{\text{out},-k}}{\sqrt{n!V}} \right)^n |0,0\rangle_k .
\]  

(102)

By taking tensor products of the states \(|m,n\rangle_k\) for different \( k \) one can construct a complete basis for the Fock space.

Applying this decomposition to eq. (95) we write

\[
|0_{\text{in}}\rangle = \bigotimes_k \left\{ (1 + |\beta_k|^2)^{-1/2} \exp \left[ V^{-1} \frac{\beta_k}{\alpha_k} a_{\text{out},k}^\dagger b_{\text{out},-k}^\dagger \right] |0,0\rangle_k \right\} ,
\]  

(103)

and expanding the exponential we get

\[
|0_{\text{in}}\rangle = \bigotimes_k \left\{ (1 + |\beta_k|^2)^{-1/2} \sum_{m_k=0}^\infty \frac{1}{m_k!} \left( V^{-1} \frac{\beta_k}{\alpha_k} a_{\text{out},k}^\dagger b_{\text{out},-k}^\dagger \right)^{m_k} |0,0\rangle_k \right\}
\]  

(104)

\[
= \bigotimes_k \left\{ (1 + |\beta_k|^2)^{-1/2} \sum_{m_k=0}^\infty \left( \frac{\beta_k}{\alpha_k} \right)^{m_k} |m_k,k\rangle \right\} .
\]  

(105)

Thus we see that explicitly the “in” vacuum for a momentum mode \( k \) is a superposition of outgoing particle-antiparticle pair states. The amplitude for being in a state with \( m_k \) pairs is

\[
\mathcal{M}_k = \langle m_k,k|P_k|0_{\text{in}}\rangle = \frac{1}{\sqrt{1 + |\beta_k|^2}} \left( \frac{\beta_k}{\alpha_k} \right)^{m_k} ,
\]  

(106)
where we have to introduce $P_k$, the projection operator to the subspace $k$, to take the inner product\footnote{The states $|0_{in}\rangle$ and $|m_k, m_k\rangle_k$ live in different Hilbert spaces; one in the whole Fock space and the other one in its subspace $k$. In order to take an inner product one therefore has to project out the $k$ component of the state $|0_{in}\rangle$. Physically this projection means that we are not measuring the other momentum modes of the state $|0_{in}\rangle$ than $k$. Thus eq. (106) gives the amplitude to have $m_k$ pairs in the mode $k$ and any number of particles in the other momentum states.}. The corresponding probability to have $m_k$ pairs in the mode $k$ and any number of particles in the other momentum modes is

$$P(m_k) = \left| \langle m_k, m_k | P_k | 0_{in} \rangle \right|^2 = \frac{1}{1 + |\beta_k|^2} \left( \frac{|\beta_k|^2}{1 + |\beta_k|^2} \right)^{m_k} = \frac{1}{1 + f_k} \left( \frac{f_k}{1 + f_k} \right)^{m_k}.$$  

(107)

This law for the probabilities characterizes the Bose-Einstein, or geometrical, distribution. Since the momentum modes are independent, we can then write the combined probability distribution as

$$P(\{m_k\}) = \prod_k P(m_k) = \prod_k \frac{1}{1 + f_k} \left( \frac{f_k}{1 + f_k} \right)^{m_k},$$  

(108)

which is exactly the form at which we already arrived in eq. (65).

4 Exactly solvable example

So far our discussions and formulas are given in a rather general way. In what follows, we consider an example which is exactly solvable and demonstrate how the formulas in the preceding sections lead to the concrete evaluation of spectrum of produced particle under an external electric field.

The LSZ reduction method and the interpretation as a Bogoliubov transformation both require that the asymptotic states at $x^0 \to \pm\infty$ are well defined. This means that the external fields should be adiabatically vanishing so that we can define $|0_{in}\rangle$ and $|0_{out}\rangle$ without ambiguity. Thus, the imposed electric fields must be time-dependent, beginning with zero at $x^0 = -\infty$, growing finite with increasing $x^0$, and diminishing to zero again as $x^0 \to +\infty$.

4.1 Choice of the gauge potential

It is known that the Klein-Gordon or Dirac equation under the following time-dependent electric field $E = (0, 0, E(x^0))$ is exactly solvable [3, 34, 47];

$$E(x^0) = \frac{E}{[\cosh(\omega x^0)]^2}.$$  

(109)

This electric field exponentially goes to zero for $|x^0| \gg \omega^{-1}$. In the limit of $\omega \to 0$ the electric field becomes homogeneous in time. We can choose a gauge
in which \( A_0 = 0 \) and the vector potential associated with the electric field \((109)\) is \( A = -\nabla A^0 - \partial_0 A \). We can immediately recover eq. (109) from \( E = -\nabla A^0 - \partial_0 A \). After the integration we find the Sauter-type gauge potential,

\[
A_3(x) = \int_{x^0}^{x^\infty} dy^0 E(y^0) = \frac{E}{\omega} \left[ \tanh(\omega x^0) - 1 \right],
\]

where we have chosen the integration constant so as to make \( A_3(x) \to 0 \) at \( x^0 \to +\infty \). As we have discussed in sec. 3 this is a very natural choice in the present case. A constant \( A_3 \) amounts to a shift in the third component of the momentum \( p_3 \), which is interpreted as a different frame choice. It is most natural to sit in the frame in which the particles and antiparticles measured at \( x^0 = +\infty \) are at rest if their \( p_3 \) is zero. The gauge potential and the electric field are sketched in fig. 4.

\[
\begin{align*}
\text{Figure 4: Sketch of the chosen gauge potential and the associated electric field as a function of time } x^0. & \text{ The electric field has a peak at } x^0 = 0 \text{ whose height is } E \text{ and width is specified by } 1/\omega. & \text{The gauge potential has an offset from zero by } -2E/\omega \text{ in the in-vacuum at } x^0 = -\infty, \text{ meaning that the origin of } p_3 \text{ is shifted by this offset.}
\end{align*}
\]

### 4.2 Solving the equation of motion

An explicit solution for the Sauter-type gauge potential is already known, and so we will simply explain the necessary notation and then jump into the known expression of the solution. To make this paper as self-contained as possible we supplement the derivation in appendix B in more detail. Introducing the following notation,

\[
\lambda = \frac{eE}{\omega^2},
\]

the equation of motion in scalar QED is given by the gauged Klein-Gordon equation as

\[
\left[ \partial_0^2 - (\partial_3 - i\lambda \omega [\tanh(\omega x^0) - 1])^2 - \partial^2_\perp + m^2 \right] \phi(x) = 0,
\]

\[\text{We work with the (+, −, −, −) metric convention.}\]
which is immediately read from the Lagrangian density (1). Because the equation of motion does not have explicit dependence on $x$, we can factorize the wave function $\phi(x)$ into the time-dependent part and the spatial plane-wave part, i.e.

$$\phi(x) = \psi_k(x^0) e^{ik \cdot x}.$$  

The time dependence is governed by the differential equation,

$$\left[\partial_0^2 + (k_3 + \lambda \omega [\tanh(\omega x^0) - 1])^2 + k_\perp^2 + m^2\right] \psi_k(x^0) = 0.$$

We here change the variable $x^0$ into $\xi$, defined by

$$\xi \equiv \frac{1}{2} [\tanh(\omega x^0) + 1].$$

With this variable we can eliminate the hyperbolic function from the equation and express it only in terms of meromorphic functions. Moreover, $\xi$ has a convenient asymptotic behavior. We will later make use of

$$\xi^{ia} = \left(\frac{e^{2i\omega x^0}}{1 + e^{2i\omega x^0}}\right)^{ia} \rightarrow \begin{cases} 1 & \text{for } x^0 \to +\infty \\ e^{2i\omega x^0} & \text{for } x^0 \to -\infty \end{cases} \quad (116)$$

and also

$$(1 - \xi)^{ia} = \left(\frac{e^{-2i\omega x^0}}{1 + e^{-2i\omega x^0}}\right)^{ia} \rightarrow \begin{cases} e^{-2i\omega x^0} & \text{for } x^0 \to +\infty \\ 1 & \text{for } x^0 \to -\infty \end{cases} \quad (117)$$

to infer the plane-wave boundary conditions.

Also, for concise notation we introduce the dimensionless energies in the same way as in Dunne’s review [3];

$$\mu \equiv \frac{E_{k}^{\text{in}}}{2\omega}, \quad \nu \equiv \frac{E_{k}^{\text{out}}}{2\omega}, \quad (118)$$

where, with the time-dependent gauge potential, the energies in the in- and out-states are, respectively,

$$(E_{k}^{\text{in}})^2 = (k_3 - 2\lambda \omega)^2 + k_\perp^2 + m^2, \quad (E_{k}^{\text{out}})^2 = k_3^2 + k_\perp^2 + m^2. \quad (119)$$

We note that the energy of antiparticles is given by $E_{-k}^{\text{in}}$ in the in-vacuum and $E_{-k}^{\text{out}}$ in the out-vacuum, respectively. Also we should explain our convention of the electric charge $e$. Our choice is as follows; particles are negatively charged and antiparticles are positively charged for $e > 0$. These are reminiscent of electrons and positrons in real QED. Therefore, in the above, noticing that $k^3 = -k_3$ we see that the particle dispersion relation $E_{k}^{\text{in}}$ starts with a larger longitudinal momentum than $E_{k}^{\text{out}}$, which is understood as the deceleration by the Lorentz force in the direction anti-parallel to the external electric field.
Now we are ready to express the solution of the differential equation. One can express two linearly independent solutions in terms of the hypergeometric functions as

\[
\psi_k^{(1)}(\xi) = \xi^{-i\mu}(1-\xi)^{-i\nu} 2F_1\left(\frac{1}{2} - i(\lambda' + \mu + \nu), \frac{1}{2} + i(\lambda' - \mu - \nu); 1 - 2i\mu; \xi\right), \\
\psi_k^{(2)}(\xi) = \xi^{i\mu}(1-\xi)^{i\nu} 2F_1\left(\frac{1}{2} - i(\lambda' - \mu + \nu), \frac{1}{2} + i(\lambda' + \mu - \nu); 1 + 2i\mu; \xi\right),
\]

(120)

where \(\lambda' = \sqrt{\lambda^2 - 1/4}\) with \(\lambda\) defined in eq. (111).

### 4.3 Asymptotic behavior of the solution

Now we need to take an appropriate linear combination of the two solutions written in eq. (120), so that the boundary condition like eq. (48) can be fulfilled. In fact, it will turn out that these solutions already satisfy a simple plane-wave boundary condition. To make this explicit we should use eqs. (116) and (117) together with the general property of the hypergeometric function;

\[2F_1(a, b; c; \xi \to 0) \to 1.\]

Therefore, in the limit of \(x^0 \to -\infty\) (i.e. \(\xi \to 0\)), we see,

\[
\psi_k^{(1)}(\xi \to 0) \to e^{-2i\mu x^0} = e^{-i\xi k^0 x^0}, \quad \psi_k^{(2)}(\xi \to 0) \to e^{2i\mu x^0} = e^{i\xi k^0 x^0}.
\]

(121)

In accord with the identification defined by eqs. (71) and (72) we have

\[
\phi_{in, k}^+ = \psi_k^{(1)} e^{ikx}, \quad \phi_{in, k}^- = \psi_k^{(2)} e^{ikx}.
\]

(122)

The quantities necessary to compute the particle and antiparticle production are obtained as the coefficient of these solutions in the \(x^0 \to +\infty\) limit—when decomposed in terms of the \(\phi_{out, k}^\pm\). To this end, it is necessary to know the limit of \(2F_1(a, b; c; \xi \to 1)\). One must be, however, very careful when taking this limit because \(a, b,\) and \(c\) are complex numbers in this case. Thus, it is convenient to make a transformation in the argument from \(\xi\) to \(1 - \xi\), which is possible by means of the following mathematical identity;

\[
2F_1(a, b; c; x) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} 2F_1(a, b; 1 - c + a + b; 1 - x) \\
+ \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} (1-x)^{c-a-b} 2F_1(c-a, c-b; 1 + c - a - b; 1 - x),
\]

(123)

which leads to an alternative expression of the solution,

\[
\psi_k^{(1)}(\xi) = \xi^{-i\mu}(1-\xi)^{-i\nu} A_k 2F_1\left(\frac{1}{2} - i(\lambda' + \mu + \nu), \frac{1}{2} + i(\lambda' - \mu - \nu); 1 - 2i\nu; 1 - \xi\right) \\
+ \xi^{-i\mu}(1-\xi)^{i\nu} B_k 2F_1\left(\frac{1}{2} + i(\lambda' - \mu + \nu), \frac{1}{2} - i(\lambda' + \mu - \nu); 1 + 2i\nu; 1 - \xi\right),
\]

(124)

and

\[
\psi_k^{(2)}(\xi) = \xi^{i\mu}(1-\xi)^{-i\nu} A_k 2F_1\left(\frac{1}{2} - i(\lambda' - \mu + \nu), \frac{1}{2} + i(\lambda' + \mu - \nu); 1 - 2i\nu; 1 - \xi\right) \\
+ \xi^{i\mu}(1-\xi)^{i\nu} B_k 2F_1\left(\frac{1}{2} + i(\lambda' + \mu + \nu), \frac{1}{2} - i(\lambda' - \mu - \nu); 1 + 2i\nu; 1 - \xi\right),
\]

(125)
where we defined,
\[
A_k \equiv \frac{\Gamma(1-2i\mu)\Gamma(-2i\nu)}{\Gamma(\frac{1}{2} - i(\lambda' + \mu + \nu))\Gamma(\frac{1}{2} + i(\lambda' - \mu - \nu))},
\]
\[
B_k^* \equiv \frac{\Gamma(1-2i\mu)\Gamma(2i\nu)}{\Gamma(\frac{1}{2} + i(\lambda' - \mu + \nu))\Gamma(\frac{1}{2} - i(\lambda' + \mu + \nu))},
\]
(126)

Here we note that, if we take a naive limit of \(x \to 1\) using a textbook formula on \(\, _2F_1(a, b; c; 1)\), it would miss the second term in the identity (123) because of the factor \((1-x)^{c-a-b}\) which is zero if \(a, b, c\) are real but is an oscillatory finite function if \(a, b, c\) are complex. At this point it is straightforward to deduce the asymptotic plane-wave forms at \(x^0 \to \infty\), that is,
\[
\psi^{(1)}_k(\xi \to 1) \to A_k e^{-2i\omega_k x^0} + B_k^* e^{2i\omega_k x^0} = A_k e^{-iE_{\text{out}} k x^0} + B_k^* e^{iE_{\text{out}} k x^0},
\]
\[
\psi^{(2)}_k(\xi \to 1) \to A_k^* e^{2i\omega_k x^0} + B_k e^{-2i\omega_k x^0} = A_k^* e^{iE_{\text{out}} k x^0} + B_k e^{-iE_{\text{out}} k x^0}.
\]
(127)

Comparing the above behavior with the Bogoliubov transformations (76) and (77), we obtain the Bogoliubov coefficients as
\[
\alpha_k = \sqrt{\frac{E_{\text{out}} k}{E_{\text{in}} k}} A_k, \quad \beta_k = \sqrt{\frac{E_{\text{out}} k}{E_{\text{in}} k}} B_k,
\]
(128)

4.4 Particle spectrum

Before addressing the concrete expressions for the particle spectrum, we list all the necessary formulas to proceed with the calculations. The Gamma function generally satisfies,
\[
\Gamma(1+z) = z\Gamma(z), \quad \Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)},
\]
(129)
from which we can derive the following useful relations,
\[
|\Gamma(\alpha)|^2 = \frac{\pi}{\alpha \sinh(\pi \alpha)}, \quad |\Gamma(1+i\alpha)|^2 = \frac{\pi \alpha}{\sinh(\pi \alpha)}, \quad |\Gamma(\frac{1}{2} + i\alpha)|^2 = \frac{\pi}{\cosh(\pi \alpha)}.
\]
(130)

Then, after some algebra, we reach,
\[
|\alpha_k|^2 = \frac{\cosh[\pi(\lambda' + \mu + \nu)] \cosh[\pi(\lambda' - \mu - \nu)]}{\sinh(2\pi \mu) \sinh(2\pi \nu)}.
\]
(131)

and
\[
|\beta_k|^2 = \frac{\cosh[\pi(\lambda' - \mu + \nu)] \cosh[\pi(\lambda' + \mu - \nu)]}{\sinh(2\pi \mu) \sinh(2\pi \nu)}.
\]
(132)

We note here that, using \(\cosh(a + b) = \cosh(a) \cosh(b) + \sinh(a) \sinh(b)\) twice, we can easily check that
\[
|\alpha_k|^2 - |\beta_k|^2 = 1,
\]
(133)
which is consistent with the condition (81). Now that we have $|\beta_k|^2$ explicitly, we can get the general probability distribution which is characterized only in terms of $|\beta_k|^2$. The single inclusive spectrum, for example, is

$$\frac{dN^+}{d^3p} = \frac{V}{(2\pi)^3} \frac{\cosh[\pi(\lambda' - \mu_p + \nu_p)] \cosh[\pi(\lambda' + \mu_p - \nu_p)]}{\sinh(2\pi\mu_p) \sinh(2\pi\nu_p)}.$$  \hspace{1cm} (134)

From this expression we can get the occupation number $f_p$ which is obtained by removing the volume factor $V/(2\pi)^3$ of the single particle spectrum. Once $f_p$ is given, the whole probability distribution is known as discussed in the previous sections. We plot $f_p$ as a function of $p^3$ in the unit of $m_\perp \equiv \sqrt{p_\perp^2 + m^2}$ in fig. 5. In drawing fig. 5 we fixed $m_\perp$, and set the electric field to the value $E = \pi m_\perp^2/e$– which is sufficiently strong to create particles in view of the standard expression of the Schwinger mechanism—, and then we vary the time scale $\omega$.

![Figure 5: Time scale dependence of the produced particle distribution as a function of $p^3$.](image)

As $\omega \to 0$ the distribution approaches eq. (135) and $f_p$ extends between $p^3 \simeq -2eE/\omega$ and $p^3 \simeq 0$ as seen from the curve for $\omega = m_T$ in the figure. In contrast, with increasing $\omega$, the result approaches eq. (136) which spreads wider than the small-$\omega$ case with the distribution center located at $p^3 \simeq -eE/\omega$.

Now let us consider two extreme cases. First, we take the constant field limit ($\omega \to 0$), which make the above expression as simple as follows;

$$\frac{dN^+}{d^3p} \to \frac{V}{(2\pi)^3} \exp \left[ -\frac{\pi(p^2 + m^2)}{4eE} \left( \frac{1}{1 + \rho} - \frac{1}{\rho} \right) \right] \quad (\omega \to 0), \hspace{1cm} (135)$$

where $\rho \equiv \omega p^3/(2eE)$ taking a value in the range of $-2eE/\omega < p^3 < 0$ (i.e. $-1 < \rho < 1$). In the outside region, $p^3 > 0$ or $p^3 < -2eE/\omega$, the result is zero in the $\omega \to 0$ limit.
This distribution of particles in the $p^3$ direction corresponds to the momentum distribution as a result of the deceleration by the electric field while it is imposed. If $\omega$ is small, the electric field lives long, and the produced particles are pushed down by the Lorentz force along the $p^3$ direction for longer time. Note that this range bounded from $-2eE/\omega$ to zero coincides with the range of $A_3$ changing between $x^0 = \pm \infty$. One might expect that the total number of produced particles diverges in the case of a constant (in time) electric field. This is indeed true; the result of the $p^3$ integration is nearly proportional to the integration range $2eE/\omega$ when $\omega$ is small enough, which is divergent as $1/\omega$—i.e. as the time during which the external electric field is non-zero.

Next, we shall take a look at the opposite limit, i.e. a short-pulse limit, $\omega \to \infty$. In this limit eq. (134) is reduced to,

$$\frac{dN^+}{dp^3} \to \frac{V}{(2\pi)^3} \frac{E^\text{in}}{p^3} \frac{E^\text{out}}{p^3} \left( \frac{1}{E^\text{in}} - \frac{1}{E^\text{out}} \right)^2 \quad (\omega \to \infty). \quad (136)$$

To have non-zero value the electric field $eE$ should be larger than $\omega E^\text{out}$. Even though there is no exponential suppression\textsuperscript{16}, as compared to the small-$\omega$ case, the resulting $f_p$ is significantly suppressed by large $\omega$ for a fixed maximal strength of the electric field, as is apparent in fig. 5.

Let us finish this subsection with a comment on a related work by Cooper and Nayak [49]. In this paper, the authors study the Schwinger mechanism for the pair production of charged scalars in the presence of an arbitrary time-dependent background electric field. In particular, they calculate the pair production rate (i.e. the number of pairs created per unit of time)\textsuperscript{17}, and conclude that “the result has the same functional dependence on $E$ as the constant electric field $E$ result with the replacement: $E \to E(t)$”. However, the comparison of the two limits in eqs. (135) and (136) indicates that the time-dependent case is unlikely to be given by a mere replacement $E \to E(t)$ in the time-independent result.

5 Conclusions

We have calculated the multiplicity of produced particles under a spatially constant but time-dependent electric field in scalar QED using a formalism based on the LSZ reduction formula. We defined and computed the generating functional of the particle and antiparticle distribution, from which we determined the whole distribution of the production probability. We found that particle

\textsuperscript{16}When the electric field has a fast time dependence, the perturbative process $\gamma \to \phi\phi^*$ can produce particles.

\textsuperscript{17}One may question whether this is a well defined concept, since the presence of a time dependent external field makes the definition of proper particle states ambiguous. One may argue that the only quantities that can be defined unambiguously are those where measurements are done only after the external field has died out. From the point of view of the Bogoliubov transformation, the transformation coefficients are uniquely determined from the asymptotic plane-wave time-dependence. Of course one may use some working definition of the Bogoliubov coefficients at arbitrary intermediate time, but such a treatment implicitly assumes a quasi-static approximation.
production in one momentum mode follows a Bose-Einstein law, reflecting the statistics of scalar particles. We have also derived the same results by means of a Bogoliubov transformation on creation and annihilation operators.

A natural question may arise as follows: what is the distribution not in spinor (ordinary) QED instead of scalar QED? Our discussions in this paper were quite simple because we focused only on spin-zero scalar particles. In the case of spinor particles we need to deal with the spin structure, which brings unessential complication in, though the generalization is straightforward. We shall here just mention that the distribution of fermionic particles in the same momentum and spin mode follows a Fermi-Dirac distribution. We can confirm this immediately by replacing the commutation relation by the anticommutation one in the derivation based on the Bogoliubov transformation. The transformation (80) is unchanged since this originates from the asymptotic behavior of the wavefunctions. The normalization condition (81) should be $|\alpha_k|^2 + |\beta_k|^2 = 1$ then to preserve the anticommutation relation of the transformed operators. It is easy to show that $|\beta_p|^2$ gives the occupation number $f_p$, and using the above normalization condition we can arrive at the probability distribution,

$$ P(\{m_{s,k}\}) = \prod_{s,k} (1 - f_{s,k}) \left( \frac{f_{s,k}}{1 - f_{s,k}} \right)^{m_{s,k}}, \quad (137) $$

where $s$ refers to the spin and $m_{s,k}$ takes the values 0 or 1. Equation (65) (or equivalently (108)) and the above (137) are our central results. We see that, if we drop higher orders than the quadratic terms in $f_k$ for $f_k \ll 1$, all of the Bose-Einstein, Fermi-Dirac, and Poisson distributions are trivially reduced to identical answer; $1 - f_k$ for no-particle production and $f_k$ for one-particle production. The difference emerges at the quadratic order, and it is notable that the difference remains no matter how small $f_k$ is. For instance, the two-particle production probability in the same mode is $f_k^2/2$ if the distribution is a Poisson one, $f_k^2$ if a Bose-Einstein one, and zero if a Fermi-Dirac one. In other words one must properly take account of the quantum statistical nature if the multiparticle correlations are concerned.

In the final section of this paper, we have revisited a known exactly solvable example of time-dependent electric fields. The time-dependence of the Sauter-type potential is actually ideal to think of the Schwinger mechanism; since we can unambiguously define the asymptotic states in the infinite past and future. This property of adiabatically vanishing external fields is necessary to make the discussion of particle production meaningful.

Using the exact solution we took two extreme limits of constant ($\omega \to 0$) and short-pulse ($\omega \to \infty$) electric fields. In the constant case we found that $f_p$ distributes almost uniformly over the range $-2eE/\omega < p^3 < 0$. Therefore the total (integrated) number of produced particles diverges as $1/\omega$ that is interpreted as the time duration for which the external electric field is imposed. In the short-pulse case, on the other hand, $f_p$ is a double-peak structure and the minimum in-between is located at $p^3 = -eE/\omega$. In practice the latter would be useful because it is more difficult to sustain larger $eE/\omega$ in the laboratory.
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A Multiplicity distribution

In some cases, one is interested only in the overall number of particles and antiparticles in the final state, but not in their distribution in momentum space. For such observables, one can define a simpler generating function that does not contain any information relative to the momentum of the produced particles:

\[ G[u, \bar{u}] = \sum_{m,n=0}^{\infty} \left( \frac{u^m \bar{u}^n}{m! n!} \right) \left( \int \prod_{i=1}^{m} d^3 p_i \prod_{j=1}^{n} d^3 q_j \left| \mathcal{M}_{m,n} \{ \{ p_i \}, \{ q_j \} \} \right|^2 \right). \] (138)

Note that this is also equal to:

\[ G[u, \bar{u}] = \sum_{m,n=0}^{\infty} u^m \bar{u}^n P_{m,n}, \] (139)

where \( P_{m,n} \) is the probability to have exactly \( m \) particles and \( n \) antiparticles in the final state. Obviously, this new generating function can be obtained from the generating functional \( \mathcal{F}[z, \bar{z}] \) by setting the functions \( z(p) \) and \( \bar{z}(p) \) to constants respectively equal to \( u \) and \( \bar{u} \):

\[ G[u, \bar{u}] = \mathcal{F}[z(p) = u, \bar{z}(p) = \bar{u}]. \] (140)

Thanks to this relationship, one can obtain quantities such as those defined in eqs. (11) as ordinary derivatives of \( G[u, \bar{u}] \). For instance,

\[ \int d^3 p_1 d^3 p_2 \frac{dN_{++}}{d^3 p_1 d^3 p_2} = \left. \frac{\partial G[u, \bar{u}]}{\partial u^2} \right|_{u, \bar{u} = 1} = \sum_{m,n=0}^{\infty} m(m - 1) P_{m,n}. \] (141)

The last equality is obtained from eq. (139), and is the justification for the right hand side in the first of eqs. (11).

B Detailed derivation of the solution

With the variable change from \( x^0 \) to \( \xi \), simple algebraic procedures lead to \( \partial_0 = 2\omega \xi (1 - \xi) \partial_\xi \) and \( \partial_0^2 = 4\omega^2 \xi (1 - \xi) \partial_\xi^2 + (1 - 2\xi) \partial_\xi \), from which
we can rewrite the equation of motion (114) in the following form:

$$\left[ \xi (1 - \xi) \partial^2_\xi + (1 - 2\xi) \partial_\xi + \mu^2 \xi^{-1} + \nu^2 (1 - \xi)^{-1} - \lambda^2 \right] \psi_k(\xi) = 0 \, .$$

(142)

In what follows we will explain how to solve this differential equation. Before finding the analytical solution, from this form of the equation we can already confirm that the asymptotic behavior of the particle solution is $e^{\pm iE_k^0 x^0}$ at $x^0 \to -\infty$ and $e^{\pm iE_k^0 x^0}$ at $x^0 \to +\infty$ as it should be. We shall pick up the most singular terms out of the differential equation (142), which gives

$$\left[ \xi \partial^2_\xi + \partial_\xi + \mu^2 \xi^{-1} \right] \psi^\text{in}_k(\xi) = 0 \, ,$$

(143)

around $\xi = 0$ (i.e. $x^0 \to -\infty$). It is easy to find the solution of this equation as

$$\psi^\text{in}_k(\xi) = \xi^{\pm i\mu} \simeq e^{\pm iE_k^0 x^0} \, .$$

In the same way we can extract the behavior around $\xi \to 1$ (i.e. $x^0 \to +\infty$) from the singular terms;

$$\left[ (1 - \xi) \partial^2_\xi - \partial_\xi + \nu^2 (1 - \xi)^{-1} \right] \psi^\text{out}_k(\xi) = 0 \, ,$$

(144)

leading to $\psi^\text{out}_k(\xi) = (1 - \xi)^{\pm i\nu} \simeq e^{\mp iE_k^0 x^0}$.

Now let us return to solving eq. (142). Because we have seen the boundary condition, it is convenient to factorize the plane-wave pieces as follows;

$$\psi_k(\xi) = \xi^{-i\nu} (1 - \xi)^{-i\nu} \varphi_k(\xi) \, ,$$

(145)

then we can find the equation that $\varphi_k(\xi)$ should satisfy as

$$\bigg\{ \xi (1 - \xi) \partial^2_\xi + \left[ 1 - 2i\mu - (2i\mu - 2i\nu + 2)\xi \right] \partial_\xi \\
- \left( -i\mu - i\nu - i\lambda' + 1/2 \right) \left( -i\mu + i\nu + i\lambda' + 1/2 \right) \bigg\} \varphi_k(\xi) = 0 \, .$$

(146)

Here we recall that the hypergeometric differential equation,

$$\bigg\{ x(1 - x) \partial^2_x + \left[ c - (a + b + 1)x \right] \partial_x - ab \bigg\} f(x) = 0 \, ,$$

(147)

has two independent solutions given by

$$f^{(1)}(x) = _2F_1(a, b; c; x) , \quad f^{(2)}(x) = x^{1-c} _2F_1(a+1-c, b+1-c; 2-c; x) \, .$$

(148)

Therefore, by the identification of

$$a = \frac{1}{2} - i(\lambda' + \mu + \nu) , \quad b = \frac{1}{2} + i(\lambda' - \mu - \nu) , \quad c = 1 - 2i\mu \, ,$$

(149)

we finally arrive at the solution (120).
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