Unitarity of the SoV transform for the Toda chain.

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Abstract

The quantum separation of variables method consists in mapping the original Hilbert space where a spectral problem is formulated onto one where the spectral problem takes a simpler "separated" form. In order to realise such a program, one should construct the map explicitly and then show that it is unitary. In the present paper, we develop a technique which allows one to prove the unitarity of this map in the case of the quantum Toda chain. Our proof solely builds on objects and relations naturally arising in the framework of the so-called quantum inverse scattering method. Hence, with minor modifications, it should be readily transposable to other quantum integrable models solvable by the quantum separation of variables method. As such, it provides an important alternative to the proof of the map’s unitarity based on the group theoretical interpretation of the quantum Toda chain, which is absent for more complex quantum integrable models.

Introduction

The quantum inverse scattering method is a powerful tool for solving a vast class of so-called quantum integrable models. The idea of the method consists in tailoring a particular quadratic algebra, the Yang-Baxter algebra, associated with the model of interest, this in such a way that the original Hamiltonian belongs to a specific, one-parameter $\lambda$, commutative subalgebra $\{\tau(\lambda)\}_{\lambda \in \mathbb{R}}$ thereof. The algebraic relations associated with the Yang–Baxter algebra are most conveniently expressed on the level of the so-called monodromy matrix $T(\lambda)$ which is a matrix on some auxiliary space whose entries are operators on the model’s Hilbert space. The family $\{\tau(\lambda)\}_{\lambda \in \mathbb{R}}$ provides one with a set of commuting self-adjoint Hamiltonians, for instance through an expansion of the map $\lambda \mapsto \tau(\lambda)$ around some point $\lambda_0$. Thus, the problem of obtaining the spectrum of the original Hamiltonian is mapped into a multi-variable and multi-parameter spectral problem associated with the family of commuting self-adjoint operators $\{\tau(\lambda)\}_{\lambda \in \mathbb{R}}$. The resolution of this spectral problem can be carried out in two ways. The first one, referred to as the algebraic Bethe Ansatz has been developed in 1979 by Faddeev, Sklyanin and Takhtadjan [7] and takes its roots in the 1931 seminal paper of Bethe [3] where the so-called coordinate Bethe Ansatz solution of the XXX spin 1/2 Heisenberg has been proposed. The second one has been developed in 1985 by Sklyanin [30] and can be thought...
of as the quantum version of the classical separation of variables method. Although both methods take their roots in the representation theory of quantum groups, the algebraic Bethe Ansatz and the quantum separation of variables are, from a technical point of view, quite different. They can also be thought of as complimentary since, apart from very exceptional cases\(^1\), only one of the methods is applicable for solving the model. This paper deals with certain technical aspects arising in the implementation of the quantum separation of variables method. The idea behind the latter method consists in mapping bijectively the multi-variable, multi-parameter spectral problem associated with \(\{\tau(\lambda)\}_{\lambda \in \mathbb{R}}\) onto an auxiliary multi-parameter spectral problem in one variable which takes the form of a scalar \(\tau - Q\) equation \([2, 8, 30]\). To achieve such a setting, one builds a unitary map \(\mathcal{U}\) whose purpose is to send the original Hilbert space \(\mathcal{H}\) of the model onto another Hilbert space \(\mathcal{H}_{\text{sep}}\) where the separation of variables, in the above sense, occurs. Thus, the implementation of the method involves solving three tasks. One should first find a convenient explicit representation for the map \(\mathcal{U}\), second one should prove its unitarity and, third, one should establish the equivalence of the original spectral problem on \(\mathcal{H}\) with the scalar \(\tau - Q\) equation on \(\mathcal{H}_{\text{sep}}\). This paper deals with the quantum inverse scattering method approach to the resolution of the second problem. We shall now be slightly more explicit about the model of interest and the quantum separation of variables method. This will allow us to formulate the main achievement of the paper.

The quantum Toda chain

The very ideas of the quantum separation of variables take, in fact, their roots in the work of Gutzwiller on the quantum Toda chain \([18, 19]\) which refers to a quantum mechanical \(N + 1\)-body Hamiltonian

\[
H_{\kappa} = \sum_{a=1}^{N+1} \frac{p_a^2}{2} + \kappa e^{x_{N+1} - x_1} + \sum_{a=1}^{N} e^{x_a - x_{a+1}} \quad \text{acting on} \quad \mathcal{H} = \bigotimes_{n=1}^{N+1} \mathcal{H}_n \cong L^2(\mathbb{R}^{N+1}, \text{d}^{N+1}x),
\]

where \(\mathcal{H}_n \cong L^2(\mathbb{R}, \text{d}x)\) are "local" quantum spaces attached to the \(n^{\text{th}}\) "particle". Furthermore, in (0.1), \(p_n\) and \(x_n\) are pairs of conjugated variables satisfying the canonical commutation relations \([x_n, p_{\ell}] = i\hbar \delta_{n\ell}\). In the following, we shall choose the realization \(p_n = -i\hbar \partial_{x_n}\). Also, the index \(n\) present in the operators refers to the quantum space \(\mathcal{H}_n\) where these operators act non-trivially. When \(\kappa = 1\), one deals with the so-called closed Toda chain whereas, at \(\kappa = 0\), the model is referred to as the open Toda chain.

In the early ’80’s, Gutzwiller \([18, 19]\) has been able to characterize the spectrum of \(H_{\kappa=1}\) in the case of a small number \(N+1\) of particles, namely for \(N = 0, \ldots, 3\). He expressed the map realizing the quantum separation of variables for the \(N + 1\) particle chain at \(N = 0, \ldots, 3\) in the form of an integral transform. His main observation was that the non-trivial part of the integral transform’s kernel was given by the generalized eigenfunctions of the \(N\)-particle open Toda chain \(H_{\kappa=0}\). However, the real deep connection which allowed for a systematic development of the method is definitely to be attributed to Sklyanin. In \([30]\), by using an analogy with the classical separation of variables, Sklyanin gave a quantum inverse scattering method-based interpretation of the aforementioned integral transform. In the case of a quantum integrable model with a six-vertex \(R\)-matrix -such as the quantum Toda chain-, the transform corresponds precisely to the map that intertwines the \(T_{12}(\lambda)\) operator entry of the model’s monodromy matrix \(T(\lambda)\) with a multiplication operator. In other words, the kernel of the integral transform is given by the eigenfunctions, understood in the generalized sense, of \(T_{12}(\lambda)\). This observation, along with the set of algebraic relations stemming from the Yang-Baxter algebra satisfied by the entries \(T_{ab}(\lambda)\) of \(T(\lambda)\), allowed Sklyanin to construct the so-called quantum separation of variables representation on the space \(\mathcal{H}_{\text{sep}} = L^2(\mathbb{R}^{N+1}, \text{d}v)\). In fact, due to the translation invariance of the close quantum Toda chain, the measure \(\text{d}v\) factorizes \(\text{d}v = \text{d}x \otimes \text{d}\mu\) into a "trivial" one-dimensional Lebesgue measure \(\text{d}x\) that takes into account the spectrum \(\varepsilon\) of the momentum operator and a non-trivial part \(\text{d}\mu\) which is absolutely continuous in respect to \(\text{d}^{N+1}y\). The aforementioned map allows one to

\(^1\)such as certain sectors of the \(\mathfrak{sl}(2, \mathbb{C})\) or \(\mathfrak{sl}(2, \mathbb{R})\) XXX spin chain \([5, 6]\) or the so-called \(\mathfrak{sl}(2)\)-Gaudin model \([11]\)
represent functions $\Phi \in \mathfrak{h}$ as $\Phi(x_{N+1}) = \mathbb{W}[\hat{\Phi}](x_{N+1})$ where, for sufficiently well-behaved functions $\hat{\Phi}$,

$$\mathbb{W}[\hat{\Phi}](x_{N+1}) = \int_{\mathbb{R}^{N+1}} \varphi_{x_N}(x_N) \cdot e^{\frac{i}{\hbar}(\varepsilon-\mathcal{F}_{N+1})x_N} \cdot \hat{\Phi}(y_N; \varepsilon) \cdot d\varepsilon \otimes \frac{d\mu(y_N)}{\sqrt{N!}}. \quad \text{Here } \mathcal{F}_N = \sum_{a=1}^{N} y_a$$  

(0.2)

and $\varphi_{x_N}(x_N)$ is an integral kernel which will be of central interest to our study. Also, above, we have adopted notation, namely the subscript $\varphi$, to which the action of a multiplication operator whereas $\mathcal{U}_N$ constitutes the non-trivial part of $\mathbb{W}$. For $F \in L^1_{\text{sym}}(\mathbb{R}^N, d\mu(y_N))$, it is given by

$$\mathcal{U}_N[F](x_N) = \frac{1}{\sqrt{N!}} \int_{\mathbb{R}^N} \varphi_{x_N}(x_N) \cdot F(y_N) \cdot d\mu(y_N).$$  

(0.3)

The subscript $\text{sym}$ occurring in $L^1_{\text{sym}}$ indicates that the function $F$ is a symmetric function of its variables. In the following, we will refer to the integral transform $\mathcal{U}_N$ as the separation of variables (SoV) transform. The main advantage of the SoV transform is that it provides one with a very simple form for the eigenfunctions of the family of transfer matrices $\tau(\lambda)$. When focusing on a sector with a fixed momentum $\varepsilon$, any eigenfunction $\mathcal{V}_t$ of $\tau(\lambda)$ associated with the eigenvalue $t(\lambda)$ admits a factorized representation in the space $b_{\text{sep}}$, in the sense that

$$\hat{\mathcal{V}}_t(y_N; \varepsilon) = \prod_{a=1}^{N} q_t(y_a).$$  

(0.4)

The function of one variable $q_t(y)$ appearing above is entire and solves the scalar form of the so-called $\tau - Q$ equation

$$t(\lambda) \cdot q_t(\lambda) = (-i)^{N+1} q_t(\lambda + i\hbar) + (i)^{N+1} q_t(\lambda - i\hbar)$$  

(0.5)

under the below condition on the asymptotic behaviour of the solutions $[8]$

$$q_t(\lambda) = O(e^{-\frac{|\lambda|}{\hbar} |\mathcal{H}_{\text{sep}}(2^{\mathcal{G}(\lambda)}-1)|^{\frac{3}{2}}}) \quad \text{uniformly in } |\mathcal{G}(\lambda)| \leq \frac{\hbar}{2},$$  

(0.6)

with $t(\lambda)$ being a monic polynomial of degree $N + 1$. We do stress that the $\tau - Q$ equation is a joint equation for the coefficients of the polynomial $t(\lambda)$ and the solution $q_t$. The asymptotic behaviour and the regularity conditions on $q_t$ can be met simultaneously only for well tuned polynomials $t(\lambda)$ corresponding to eigenvalues of $\tau(\lambda)$; this effect gives rise to so-called quantization conditions for the Toda chain.

To phrase things more precisely, within the framework of the quantum separation of variables, the resolution of the spectral problem for the quantum Toda chain amounts to

i) building and characterizing the kernel $\varphi_{x_N}(x_N)$ of the SoV transform;

ii) establishing the unitarity of $\mathcal{U}_N : L^2(\mathbb{R}^N, d^N x) \rightarrow L^2_{\text{sym}}(\mathbb{R}^N, d\mu(y_N))$;

iii) characterizing all of the solutions to (0.5) and (0.6) and proving the equivalence of this spectral problem to the original one formulated on $\mathfrak{h}$.

Point iii) has been first argued by Sklyanin [30] and the correspondence proved by An [1].
The $GL(N, \mathbb{R})$-Whittaker function interpretation of $\varphi_{y_s}(x_N)$

The resolution of point $i)$ takes its roots in the work of Kostant [25]. The author of [25] found a way to quantize the integrals of motion for the open classical Toda chain hence showing the existence of an abelian ring of operators containing the quantum open Toda chain Hamiltonian \( \mathcal{H}_{e=1} \). Furthermore, he was able [26] to identify the system of joint generalized eigenfunctions to this ring as Whittaker functions for $GL(N, \mathbb{R})$. Kostant’s approach has been continued and extended so as to include other Toda Hamiltonians such as the closed one, $\mathcal{H}_{e=1}$, by Goodman and Wallach in [14, 15, 16]. In particular they provided an explicit construction of a set of generators for the aforesaid ring of operators in involution. Recall also that the systematic study of Whittaker functions has been initiated by Jacquet [21] and that the theory has been further developed by Hashizume [20] and Schiffmann [27]. At the time, the Whittaker function were constructed by purely group theoretical handlings, what allowed to represent them by means of the so-called Jacquet’s multiple integral. In 1990, Stade [33] obtained another multiple integral representations for the $GL(N, \mathbb{R})$ Whittaker functions. The authors of [11] proposed yet another multiple integral representation for these function which was based on the so-called Gauss decomposition of group elements. However, for many technical reasons, all these representations, although explicit, were hard to deal with or extract from them the sought informations on the functions. Nonetheless, this state of the art was already enough in what concerned applications to the quantum Toda chain.

Indeed, recall that Gutzwiller [18, 19] constructed the eigenfunctions for the closed $N + 1$-particle quantum Toda chain, at small values of $N$, be means of an integral transform whose kernel to the eigenfunctions of the open $N$-particle quantum Toda chain. Building on this idea and implicitly conjecturing that the ring of operators found by Kostant actually coincides with the quantum inverse scattering method issued integrals of motion for the open $N$-particle Toda chain, Kharchev and Lebedev [22] wrote down a multiple integral representation for the eigenfunctions for the closed periodic Toda chain $\mathcal{H}_{e=1}$ in the form \( \mathcal{W}[\hat{\mathcal{V}}] \), cf (0.2) and (0.4). Their construction worked for any value of $N$. The main point of their conjecture is that it allowed them to use Kostant’s characterization of the eigenfunctions of the open Toda abelian ring of operators as Whittaker functions for $GL(N, \mathbb{R})$ so as to identify the kernel $\varphi_{y_s}(x_N)$ with such Whittaker functions. At the time, they used the so-called Gauss decomposition based multiple integral representations for these Whittaker functions [11]. Later in [23, 24], the two authors managed to connect their approach with Sklyanin’s quantum separation of variables [30] approach to the quantum Toda chain.

More precisely, Sklyanin’s method relies on the observation that the integral kernel $\varphi_{y_s}(x_N)$ of the SoV transform corresponds, up to some minor modifications, to the eigenfunction of the $T_{12}(\lambda)$ operator entry of the monodromy matrix for an $N + 1$-particle Toda chain:

$$T_{12}(\lambda) \cdot \varphi_{y_s}(x_N) = e^{-\lambda x_{N+1}} \prod_{a=1}^{N} (\lambda - y_a) \cdot \varphi_{y_s}(x_N). \quad (0.7)$$

In [32], Sklyanin proposed a inductive scheme based on the recursive construction of the monodromy matrix which allowed one to build the eigenfunctions $\varphi_{y_s}(x_N)$ inductively. Kharchev and Lebedev [23, 24] managed to implement this scheme on the example of the open Toda chain, hence obtaining a new multiple integral representation for the $GL(N, \mathbb{R})$ Whittaker functions which they called Mellin-Barnes representation. Finally, in [9], Gerasimov, Kharchev and Lebedev established a clear connection between the group theoretical and the quantum inverse scattering method-based approaches to the open Toda chain. In particular, that paper proved the previously used conjecture relative to the concurrency between Kostant’s ring of operators on the one hand and the quantum inverse scattering issued conserved charges on the other.

There exists one more multiple integral based representation for the generalized eigenfunctions of the open Toda chain due to Givental [13]. The group theoretic interpretation of this type of multiple integral representation
has been given in [10]. Since the corresponding proof built on a specific type of Gauss decomposition for the group elements of $GL(N, \mathbb{R})$, this multiple integral representation bears the name Gauss–Givental. Furthermore, paper [10] also contained some comments relative a connection between the Gauss–Givental representation and the model’s $Q$-operator constructed earlier by Gaudin and Pasquier [8]. In fact, this connection, within the setting of another quantum separation of variables solvable model, the so-called non-compact XXX magnet, has been established, on a much deeper level of understanding, a few years earlier by Derkachov, Korchemsky and Manashov [5]. These authors observed that one can build the eigenfunctions of the $Q$-operator, this allowed Derkachov, Korchemsky and Manashov to propose a “pyramidal” representation for the model’s group elements of $GL(N, \mathbb{R})$. This allowed Derkachov, Korchemsky and Manashov to propose a “pyramidal” representation for the quantum SoV’s map kernel for the non-compact XXX magnet. Later, in [29], Silantyev applied the DKM method so as to re-derive the Gauss–Givental representation for the $GL(N, \mathbb{R})$ Whittaker functions $\varphi_{\lambda_N}(x_N)$.

The main result of the paper

Until now, we have not yet discussed the completeness and orthonormality of the system of generalized eigenfunctions of the open Toda chain $(T_{12}(\lambda))_{\lambda \in \mathbb{R}}$ abelian ring of operators. These properties, in fact, boil down to proving the unitarity of $\mathcal{U}_N$, viz the point ii) mentioned earlier. The completeness and orthonormality have, in fact, been already established within the framework of the group theoretical based approach to the model. Semenov-Tian-Shansky proved [28] the orthonormality of the system $\{x_N \mapsto \varphi_{\lambda_N}(x_N)\}_{\lambda_N \in \mathbb{R}^N}$. The latter, written formally, takes the form

$$\int_{\mathbb{R}^N} \left( \varphi_{\lambda_N}(x_N)' \right)^\dagger \varphi_{\lambda_N}(x_N) \cdot d^N x = \left[ \mu(y_N) \right]^{-1} \sum_{\sigma \in \mathfrak{S}_N} \prod_{a=1}^N \delta(y_a - y'_\sigma(a)) \cdot (0.8)$$

Also, the completeness of the system $\{y_N \mapsto \varphi_{\lambda_N}(x_N)\}_{x_N \in \mathbb{R}^N}$, which written formally, takes the form

$$\int_{\mathbb{R}^N} \left( \varphi_{\lambda_N}(x_N)' \right)^\dagger \varphi_{\lambda_N}(x_N) \mu(y_N) \cdot d^N y = \prod_{a=1}^{N+1} \delta(x_a - x'_a) \cdot (0.9)$$

follows from the material that can be found in chapters 15.9.1-15.9.2 and 15.11 of Wallach’s book [34].

We do stress that the proofs [28, 34] are technically involved, rather long, and completely disconnected from the QISM description of the model. It is, in fact, the last fact that is the most problematic from the point of view of implementing the quantum separation of variables to more complex quantum integrable models. Even for a relatively simple model such as the lattice discretization of the Sinh-Gordon model [4], point ii) remains an open question. It is the quest towards obtaining a simple and systematic approach to the resolution of analogues of problems outlined in point ii) but for more complex models that led us to developments described in the present paper. Namely, we propose a new approach to proving the unitarity of the quantum separation of variables map, that is to say the

Theorem The integral transform $\mathcal{U}_N$ defined by (0.3) for functions $F \in \mathcal{C}_c(\mathbb{R}^N)$ extends to a unitary map $\mathcal{U}_N : L^2_{\text{sym}}(\mathbb{R}^N, d\mu(y_N)) \rightarrow L^2(\mathbb{R}^N, d^N x)$ where

$$d\mu(y_N) = \mu(y_N) \cdot d^N y \quad \text{with} \quad \mu(y_N) = \frac{1}{(2\pi \hbar)^N} \prod_{k \neq p} \Gamma^{-1}(\frac{y_k - y_p}{i \hbar}) \cdot (0.10)$$

and the functions $\varphi_{\lambda_N}(x_N)$ are defined by their Mellin-Barnes multiple integral representation (1.1). The method we develop for proving this theorem

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• is completely independent from the previous scheme of works that require a group theoretical interpretation of the model in the spirit of [14, 15, 16, 25];

• solely relies on structures and objects naturally arising within the quantum inverse scattering method;

• is quite simple - on a formal level of rigour, it is almost immediate to implement- and relatively short.

Since our method naturally fits into the quantum inverse scattering method approach, its main advantage consists in allowing one, in principle, to apply it for proving the unitarity of the quantum separation of variables transform in the case of many other quantum integrable models. We chose to develop our method on the example of the Toda chain due to that model’s simplicity; this setting allowed us to avoid the technical ponderousness that would arise in the course of the analysis of more complex models.

In order to be slightly more specific about our approach, we remind that the $Q$-operator based DKM approach [5] allows one to derive the Gauss-Givental multiple integral representation for $\varphi_{yN}(x_N)$. In fact, it also allows one to establish, on a formal level of rigour, the relation (0.8), this in fairly simple way. Such a formal proof of the orthogonality condition has been given by Silantev [29]. In the present paper we, first of all, bring various elements of rigour to Sylantev’s manipulations [29] leading to a completely independent in respect to Semenov-Tian-Shansky’s work and much simpler proof of the isometric nature of $U_N$. Further, we provide a proof of the completeness relation (0.9) which, also, is completely independent from any group theoretical handlings. In fact, the proof we propose is based on the existence of two natural quantum inverse scattering method issued multiple integral representations for $\varphi_{yN}(x_N)$: the Gauss-Givental one and the Mellin-Barnes one. Knowing that (0.8) holds and that $\varphi_{yN}(x_N)$ satisfies a Mellin-Barnes multiple integral representation issued recurrence relation allows us to deduce (0.9). In this respect, our proof highlights a sort of beautiful duality between the two types of multiple integral representations. To the best of the author’s knowledge, this way of proving the completeness is based on completely new ideas. Furthermore, we do stress again that, on the formal level of rigour, the steps for proving the completeness relation are extremely easy.

The paper is organized as follows. In section 1 we introduce the Mellin-Barnes and Gauss-Givental multiple integral representations for $\varphi_{yN}(x_N)$ and establish several basic properties of the latter. Then, in section 2 we provide a proof, strongly inspired by the formal handlings of [29], of the isometric nature of the $U_N$ transform. Then, in section 3, we provide a proof of the isometric nature of the formal adjoint of $U_N$. All the ideas behind this proof are brand new, at least to the best of the author’s knowledge. Several results of technical nature are gathered in the appendices. In appendix A we build on the Mellin-Barnes integral representation for $\varphi_{yN}(x_N)$ so as to derive uniform in $yN \in \mathbb{R}^N$, $x_N \to \infty$ asymptotics of this function. Finally, in appendix B we establish a direct connection between the Mellin-Barnes and Gauss-Givental multiple integral representations, hence proving that, indeed, they do define the very same function. The proof given in B builds on several ideas introduced in [12]. Still, the main difference between our proof and the one of [12] is that we provide new arguments that allow us to circumvent the use of relations (0.8)-(0.9) in the proof.

1 The kernel of the SoV transform for the Toda chain

As it has been mentioned in the introduction, there exists two quantum inverse scattering method issued multiple integral representations for the integral kernel $\varphi_{yN}(x_N)$ of the transform $U_N$.

In this section, we shall review the structure of these two representations, present some short proofs of several known facts about these representations as well as prove certain, yet unestablished, properties thereof. This preliminary analysis will allow us to introduce all the concepts and tools that will be necessary for establishing the unitarity of $U_N$. 6
1.1 The Mellin-Barnes representation

Let \((y_N, x_N) \in \mathbb{R}^N \times \mathbb{R}^N\). The functions \(\varphi_{y_N}(x_N)\) occurring in (0.3) are defined as the unique solution to the induction

\[
\varphi_{y_{N+1}}(x_{N+1}) = \int_{(\mathbb{R} - ia_N)^N} e^{\pi i \sum_{N+1}^{N+1} \varphi_{w_N}(x_N) \varpi(w_N) | y_{N+1}) \cdot \frac{d^N w}{N!(2\pi i)^N},
\]

in which \(a > 0\) is a free parameter

\[
\varpi(w_N | y_{N+1}) = \prod_{a=1}^N \prod_{b=1}^{N+1} \left\{ \frac{i^b}{i^b} \right\} \left( \frac{y_{b} - w_{a}}{i\hbar} \right) \cdot \prod_{a \neq b} \Gamma^{-1} \left( \frac{y_{b} - w_{a}}{i\hbar} \right),
\]

and the inductions is subject to the initiation condition

\[
\varphi_y(x) = e^{i \varpi y}.
\]

It is straightforward to convince oneself that \(\varphi_{y_N}(x_N)\) admits the explicit expression

\[
\varphi_{y_N}(x_N) = e^{\pi i \sum_{s=1}^{N} \int_{(\mathbb{R} - ia_N)^N} \frac{d^N x}{(N - s)! (2\pi i)^{N-s}} \prod_{s=1}^{N} \varpi(\varpi_N, (x_N - x_N - s)) \prod_{s=1}^{N} \varpi(w_N(s) | w_N(s-1))},
\]

where we do agree upon \(0 < a_1 < \cdots < a_{N-1}\)

\[
w_N(0) = y_N \quad \text{and recall that} \quad \mathbb{R}_k = \sum_{a=1}^k x_a \quad \text{for} \quad k - \text{dimensional vectors} \quad x_k \in \mathbb{R}^k.
\]

The iterated integral converges strongly (exponentially fast), see eg [9], and defines a smooth function

\[
(y_N, x_N) \mapsto \varphi_{y_N}(x_N) \in L^\infty(\mathbb{R}^N \times \mathbb{R}^N, d^N y \otimes d^N x).
\]

This ensures that the integral transform with \(d\mu(y_N)\) as defined in (0.10)

\[
\mathcal{U}_N[F](x_N) = \frac{1}{\sqrt{N!}} \int_{\mathbb{R}^N} \varphi_{y_N}(x_N) \cdot F(y_N) \cdot d\mu(y_N)
\]

is well defined for any \(F \in L^1(\mathbb{R}^N, d\mu(y_N))\).

In fact, as we shall establish in appendix [A] the function \(\varphi_{y_N}(x_N)\) can be recast in terms of a combination of oscillatory and exponentially decreasing terms in \(x_N \in \mathbb{R}^N\), this uniformly in \(y_N \in \mathbb{R}^N\). Such a representation allows one to carry out integration by parts in (1.7) which readily lead to the below proposition whose proof is postponed to appendix [A]

**Proposition 1.1** Given any \(F \in C_c^\infty(\mathbb{R}^N)\), the integral transform \(\mathcal{U}_N[F]\) is well defined and belongs to the Schwartz class \(S(\mathbb{R}^N)\). In particular, for such functions \(F\), one has that \(\mathcal{U}_N[F] \in L^2(\mathbb{R}^N, d^N x)\).
1.2 The DKM construction of the eigenfunctions of the open Toda chain

We now discuss the Gauss–Givental multiple integral representation for the integral kernel \( \varphi_{y_{1}}(x_{N}) \) which has been obtained in [10, 13, 29], this by means of various different reasonings. For our purpose, we shall follow the presentation of [29] which was an adaptation of the method developed by Derkachov, Korchemsky and Manashov [5]. More precisely, these authors have observed on the example of the so-called \( \text{sl}(2, \mathbb{C}) \) XXX chain, that one can extract the elementary building blocks for the kernel \( \varphi_{y_{1}}(x_{N}) \) out of the integral kernel \( Q_{1}(x_{N}, x'_{N}) \) of the model’s \( Q \)-operator. The integral kernel of the \( Q \)-operator for the Toda chain has been constructed by Gaudin and Pasquier [8]. It admits the below representation

\[
Q_{1}(x_{N}, x'_{N}) = \prod_{n=1}^{N} V_{\pm_{n}}(x_{n} - x'_{n}) V_{\pm_{n}}(x_{n} - x'_{n-1}) \, ,
\]

(1.8)

where

\[
V_{\pm_{n}}(x) = \exp\left\{ - \frac{1}{\hbar} e^{\pm X} + i \frac{\lambda_{x}}{2\hbar} \right\} \, .
\]

(1.9)

By sending \( x'_{N} \rightarrow +\infty \) in the expression (1.8) for \( Q_{1}(x_{N}, x'_{N}) \) we get

\[
Q_{1}(x_{N}, x'_{N}) = \Lambda_{y_{1}}^{(N)}(x_{N} | x'_{N-1}) e^{-i \frac{\lambda}{\hbar} (x_{N} - x_{N-1})} \exp\left\{ - \frac{1}{\hbar} e^{\lambda_{y_{1}}} \cdot (1 + o(1)) \right\} \, ,
\]

(1.10)

with

\[
\Lambda_{y_{1}}^{(N)}(x_{N} | x'_{N-1}) = e^{i \frac{\lambda}{\hbar} (x_{N} + x_{N-1})} \prod_{n=1}^{N-1} V_{\pm_{n}}(x_{n} - x'_{n}) \prod_{n=2}^{N} V_{\pm_{n}}(x_{n} - x'_{n-1}) \, .
\]

(1.11)

We do stress here that \( x'_{N-1} = (x'_{1}, \ldots, x'_{N-1}) \) is a \( N - 1 \) dimensional vector obtained from \( x'_{N} \) by dropping the last coordinate \( x'_{N} \). The above function defines an integral kernel for the mapping

\[
\Lambda_{y_{1}}^{(N)} : L^{\infty}(\mathbb{R}^{N-1}) \rightarrow L^{\infty}(\mathbb{R}^{N})
\]

\[
f \mapsto \int_{\mathbb{R}^{N-1}} \Lambda_{y_{1}}^{(N)}(x_{N} | z_{N-1}) f(z_{N-1}) \prod_{a=1}^{N-1} dz_{a} \, ,
\]

(1.12)

and plays a crucial role in the construction of the so-called Gauss-Givental representation for the function \( \varphi_{y_{1}}(x_{N}) \).

We likewise define the conjugated operator

\[
\Lambda_{y_{1}}^{(N)} : S(\mathbb{R}^{N}) \rightarrow L^{\infty}(\mathbb{R}^{N-1})
\]

\[
f \mapsto \int_{\mathbb{R}^{N}} \Lambda_{y_{1}}^{(N)}(z_{N-1} | x_{N}) f(x_{N}) \prod_{a=1}^{N} dx_{a} \, ,
\]

(1.14)

where \( S(\mathbb{R}^{N}) \) refers to Schwartz functions on \( \mathbb{R}^{N} \) whereas the integral kernel reads

\[
\Lambda_{y_{1}}^{(N)}(z_{N-1} | x_{N}) = e^{-i \frac{\lambda}{\hbar} (x_{N} + x_{N-1})} \prod_{n=1}^{N-1} V_{\pm_{n}}(x_{n} - z_{n}) \prod_{n=2}^{N} V_{\pm_{n}}(x_{n} - z_{n-1}) \, .
\]

(1.16)

The operator \( \Lambda_{y_{1}}^{(N)} \) also defines a right-handed integral transform which is well behaved on appropriate classes of Schwartz functions in that it does preserve the regularity properties of this class. More precisely, one has that
Lemma 1.1 Let \( G(z_{N-1}, y) \in \mathcal{S}(\mathbb{R}^{N-1} \times \mathbb{R}) \), the function of \( N \)-variables

\[
\hat{G}_L(x_{N-2}) = \int_{\mathbb{R}^N} G(z_{N-1}, y) \cdot \Lambda_y^{(N-1)}(z_{N-1} \mid x_{N-2}) \cdot d^{N-1}z \cdot dy
\]  

(1.17)

belongs to \( \mathcal{S}(\mathbb{R}^{N-2}) \).

Proof — Since \( G \) belongs to the Schwartz class, for any \( k \in \mathbb{N}^* \), there exists \( C_k > 0 \) such that

\[
|G(z_{N-1}, y)| \leq \frac{C_k}{(1 + |y|)^{k+1}} \cdot \prod_{a=1}^{N-1} \left( \frac{1}{1 + |z_a|} \right)^{k+1}.
\]  

(1.18)

Thus, agreeing upon \( \|x_N\|_{\infty} = \max_{1 \leq a \leq N} |x_a| \), and using \( |\Lambda_y^{(N-1)}(z_{N-1} \mid x_{N-2})| \leq 1 \),

\[
|\hat{G}_L(x_{N-2})| \leq \int_{\|z_{N-1}\|_{\infty} = \max_{1 \leq a \leq N} |z_a|} |G(z_{N-1}, y)| \cdot \prod_{a=1}^{N-2} \exp \left\{ -\frac{1}{\hbar} x_{a} \cdot z_a - \frac{1}{\hbar} e^{z_{a+1} - x_a} \right\} \cdot d^{N-1}z \cdot dy
\]

\[
+ \sum_{p=1}^{N-1} \int_{\|x_{N-2}\|_{\infty}} C_k \cdot \prod_{a=1}^{N-1} \left( \frac{1}{1 + |z_a|} \right)^{k+1} \cdot d^{N-1}z \cdot dy \leq \frac{C}{(1 + \|x_{N-2}\|_{\infty})^{k}}.
\]  

(1.19)

The first line can be bounded due to the extremely quick decay of the integrand when \( \|x_{N-2}\|_{\infty} \to +\infty \) and the \( L^1(\mathbb{R}^{N-1} \times \mathbb{R}) \) nature of \( G \) whereas the bound on the \( \text{rhs} \) of the second line follows from a direct integration.

The action of the operator \( \Lambda_y^{(N)} \) to the right produces a slightly less regular behaviour.

Lemma 1.2 For any \( \Phi \in L^\infty(\mathbb{R}^{N-1}, d^{N-1}x) \), the function \((\Lambda_y^{(N)} \cdot \Phi)(x_N)\) satisfies the bounds

\[
\left| (\Lambda_y^{(N)} \cdot \Phi)(x_N) \right| \leq C \cdot \left\| \Phi \right\|_{L^\infty(\mathbb{R}^{N-1}, d^{N-1}x)} \cdot \prod_{n=1}^{N-1} \exp \left\{ -\frac{2}{\hbar} e^{\frac{z_{n+1} - x_n}{2\hbar}} \cosh(\tau) \right\} \cdot d\tau.
\]  

(1.20)

for some \( \Phi \)-independent constant \( C \), uniformly in \( x_N \) belonging to the domain

\[
\left\{ x_N \in \mathbb{R}^N : x_1 < 0 \text{ or } x_N > 0 \right\}.
\]  

(1.21)

In particular, the bounds guarantee an exponentially fast decay at infinity across the domain \((1.21)\).

Proof — Direct bounds lead to

\[
\left| (\Lambda_y^{(N)} \cdot \Phi)(x_N) \right| \leq \left\| \Phi \right\|_{L^\infty(\mathbb{R}^{N-1}, d^{N-1}x)} \cdot \prod_{n=1}^{N-1} \int_{\mathbb{R}} \exp \left\{ -\frac{2}{\hbar} e^{\frac{z_{n+1} - x_n}{2\hbar}} \cosh(\tau) \right\} \cdot d\tau.
\]  

(1.22)

Then, the bound for \( a > 0 \)

\[
\int_{\mathbb{R}} e^{-a \cosh(\tau)} \cdot d\tau \leq e^{-a} \int_{\mathbb{R}} e^{-\frac{\tau^2}{2}} \cdot d\tau \leq e^{-a} \sqrt{2\pi a}
\]  

(1.23)

accompanied by straightforward estimates allows one to conclude.

The two operators \( \Lambda_y^{(N)} \) and \( \Lambda_y^{(N)} \) satisfy an important exchange relation
Lemma 1.3 For any $\epsilon_1, \epsilon_N > 0$ one has

$$\int_{\mathbb{R}^N} \Lambda_y^{(N)}(\tau_{N-1} | x_N) \cdot e^{\frac{\tau_{N-1}}{\hbar}} \cdot \Lambda_y^{(N)}(x_N | \tau_{N-1}) \cdot d^N x = \Gamma(\frac{y - y' - i\epsilon_N}{\hbar}) \cdot \Gamma(\frac{y' - y - i\epsilon_1}{\hbar})$$

$$\times \left(\frac{e^{\tau_1} + e^{\tau'_1}}{\hbar}\right)^{\frac{N}{2}} \cdot \left(\frac{e^{-\tau_{N-1}} + e^{-\tau'_{N-1}}}{\hbar}\right)^{\frac{N}{2}} \left(\Lambda_y^{(N-1)} \cdot \Lambda_{y'}^{(N-1)}\right)(\tau'_{N-1} | \tau_{N-1}).$$  \hspace{1cm} (1.24)

Note that the $\epsilon_1, \epsilon_N$ dependent factors in the integrand are there so as to ensure the convergence of the integral.

Proof — The proof goes through a direct calculation. Namely, denote by $I_N(\tau_{N-1} | \tau_{N-1})$ the lhs of (1.24). Then,

$$I_N(\tau'_{N-1} | \tau_{N-1}) = I_1^{(\epsilon_1)}(\tau'_1, \tau_1) \cdot I_{2...N-1}(\tau'_{N-1}, \tau_{N-1}) \cdot I_N^{(\epsilon_N)}(\tau'_{N-1}, \tau_{N-1}).$$  \hspace{1cm} (1.25)

in which:

$$I_1^{(\epsilon_1)}(\tau'_1, \tau_1) = e^{\frac{\tau'_1}{\hbar}} e^{-\frac{\tau_1}{\hbar}} \int_{\mathbb{R}} e^{\frac{\tau_1}{\hbar}(y' + i\epsilon_1)} \exp \left\{ - \frac{1}{\hbar} e^{-\tau_1} \cdot (e^{\tau_1} + e^{\tau'_1}) \right\} \cdot dx$$

$$= e^{-\frac{\tau_1}{\hbar}} e^{\frac{\tau'_1}{\hbar}} \cdot \Gamma\left(\frac{y' - y - i\epsilon_1}{\hbar}\right) \cdot \left(\frac{e^{\tau_1} + e^{\tau'_1}}{\hbar}\right)^{\frac{N}{2}} \cdot \left[\frac{2}{\hbar} \cosh\left(\frac{\tau_1 - \tau'_1}{2}\right)\right]^{\frac{N}{2}}.$$  \hspace{1cm} (1.26)

as follow from implementing the change of variables $x = -\ln(\hbar) + \ln(e^{\tau_1} + e^{\tau'_1})$. Very similarly, one obtains

$$I_N^{(\epsilon_N)}(\tau'_{N-1}, \tau_{N-1}) = e^{-\frac{\tau_{N-1}}{\hbar}} e^{\frac{\tau'_{N-1}}{\hbar}} \cdot \Gamma\left(\frac{y - y' - i\epsilon_N}{\hbar}\right) \cdot \left(\frac{e^{-\tau_{N-1}} + e^{-\tau'_{N-1}}}{\hbar}\right)^{\frac{N}{2}} \cdot \left[\frac{2}{\hbar} \cosh\left(\frac{\tau_{N-1} - \tau'_{N-1}}{2}\right)\right]^{\frac{N}{2}}.$$  \hspace{1cm} (1.27)

Finally, using the identity

$$\int_{\mathbb{R}} V_{x+}(x_{p+1} - y)V_{x-}(x_p - y)V_{y+}(y - x'_p)V_{y-}(y - x'_{p+1}) \cdot dy$$

$$= \left(\frac{\cosh\left[(x_p - x'_{p+1})/2\right]}{\cosh\left[(x_{p+1} - x'_{p+1})/2\right]}\right)^{\frac{N}{2}} \int_{\mathbb{R}} V_{y+}(x_{p+1} - y)V_{y-}(x_p - y)V_{x+}(y - x'_p)V_{x-}(y - x'_{p+1}) \cdot dy.$$  \hspace{1cm} (1.28)

which is obtained through the change of variables

$$y = y' - \ln\left(\frac{e^{-x_p} + e^{-x'_p}}{e^{x_{p+1}} + e^{x'_{p+1}}}\right),$$  \hspace{1cm} (1.29)

one gets

$$I_{2...N-1}(\tau'_{N-1}, \tau_{N-1}) = \prod_{p=2}^{N-1} \left\{ \int_{\mathbb{R}} V_{y+}(x_p - y)V_{y-}(x_{p-1} - y)V_{x+}(x - x'_p)V_{x-}(x - x'_{p+1}) \cdot dx \right\}$$

$$= \left(\frac{\cosh\left[(\tau_{N-1} - \tau'_{N-1})/2\right]}{\cosh\left[\tau_1 - \tau'_1)/2\right]}\right)^{\frac{N}{2}} e^{\frac{\tau_1}{\hbar}(\tau_1 + \tau_{N-1})} e^{-\frac{\tau'_1}{\hbar}(\tau'_1 + \tau_{N-1})} \left(\Lambda_{y}^{(N-1)} \cdot \Lambda_{y'}^{(N-1)}\right)(\tau'_{N-1}, \tau_{N-1}).$$  \hspace{1cm} (1.29)

It remains to put all the three results together.

It is also readily checked that the operators $\Lambda_{y}^{(N)}$ commute in the sense that
Proposition 1.2 The operators $\Lambda_i(\mu)$ satisfy to the commutation relations $\Lambda_i(\mu)\Lambda_j(\mu^{-1}) = \Lambda_j(\mu)\Lambda_i(\mu^{-1})$ which, in coordinates, reads

$$\int_{\mathbb{R}^{N-1}} \Lambda_{\alpha}^{(\mu)}(x_N | z_{N-1}) \cdot \Lambda_{\beta}^{(\mu^{-1})}(z_{N-1} | x_{N-2}) \cdot \prod_{a=1}^{N-1} |z_a| = \int_{\mathbb{R}^{N-1}} \Lambda_{\beta}^{(\mu)}(x_N | z_{N-1}) \cdot \Lambda_{\alpha}^{(\mu^{-1})}(z_{N-1} | x_{N-2}) \cdot \prod_{a=1}^{N-1} |z_a| \quad (1.30)$$

Just as for the previous proposition, the proof goes through a direct calculation. We leave the details to the interested reader.

A repetitive application of proposition [3.1] shows that the integral kernel $\varphi_{y_N}(x_N)$ admits the alternative representation given by a multiple "pyramidal" action of the $\Lambda$ operators

$$\varphi_{y_N}(x_N) = (\Lambda_{i_1}^{(N)} \cdots \Lambda_{i_1}^{(1)})(x_N), \quad (1.31)$$

what corresponds precisely to the type of representations that have been developed for the SoV transform's kernel in [5] [29]. The representation (1.31) reads, in coordinates,

$$\varphi_{y_N}(x_N) = \int_{\mathbb{R}^{\Lambda(N)-1}} \Lambda_{j_1}^{(N)}(x_N | z_{N-1}^{(i_1)}) \cdot \Lambda_{j_2}^{(N-1)}(z_{N-1}^{(i_1)} | z_{N-2}^{(i_2)}) \cdots \Lambda_{j_N}^{(1)}(z_N^{(i_1-1)}) \cdot \prod_{k=1}^{N-1} |z_k| \quad (1.32)$$

There, the $-$ in the argument of $\Lambda_{j_1}^{(N)}(z_1^{(i_1-1)}) | -$) stresses that there is no dependence on the second type variable in this case. Furthermore, in (1.32), the integrations ought to be considered, a priori, starting from $z_1^{(N-1)}$ and then gradually going up to the "exterior" vector $z_{N-1}^{(i)}$.

2 Isometric nature of the $\mathcal{U}_N$ transform

Following the approach of [5] [29], we prove that the SoV transform $\mathcal{U}_N$ raises to an isometric map and, as such, is invertible from the left, the inverse being given by its formal adjoint $\mathcal{U}_N^*$, viz $\mathcal{U}_N \cdot \mathcal{U}_N^* = \text{id}_{L^2_{\text{sym}}(\mathbb{R}^N, \mu(y_N))}$.

We do stress that, in the course of the proof, we tackle the subtle problem of the exchangeability of various symbols which has not been considered in the aforesaid papers. The corresponding result reads

Theorem 2.1 The map $\mathcal{U}_N$ defined for $(L^1 \cap L^2)_{\text{sym}}(\mathbb{R}^N, d\mu(y_N))$ functions by the integral transform (1.7) extends into an isometric linear map

$$\mathcal{U}_N : L^2_{\text{sym}}(\mathbb{R}^N, d\mu(y_N)) \rightarrow L^2(\mathbb{R}^N, d\mu x) \quad (2.1)$$

In other words, one has the equality

$$||\mathcal{U}_{N}[F]||_{L^2(\mathbb{R}^N, d\mu x)} = ||F||_{L^2_{\text{sym}}(\mathbb{R}^N, d\mu(y_N))} \quad (2.2)$$

Written formally, the isometric character of $\mathcal{U}_N$ translates itself into the orthogonality relations (0.8).

The idea of the proof consists in using the pyramidal structure (1.31) of the kernels $\varphi_{y_N}(x_N)$ and the algebraic relations satisfied by the kernels $\Lambda_{i_1}^{(N)}$ and $\Lambda_{i_1}^{(1)}$ as stated in lemma [1.3]. The subtle point here is that the algebraic exchange relations, in the $\epsilon_1, \epsilon_N \rightarrow 0^+$ limit, are only satisfied in a weak sense. Thus, one has to recourse to various regularizing steps so as to justify the formal argument. We remind that the formal method goes back to the works of Derkachov, Manashov and Korchemsky [5] and has been applied by Silantev [29] to the case of the Toda chain.
Proof —
Take $\epsilon > 0$ and let $\Psi, \Phi \in \mathcal{C}_c^{\infty}(\mathcal{D}_e)$ where

$$\mathcal{D}_e = \left\{ y_N \in \mathbb{R}^N : \min_{a \neq b} |y_a - y_b| \geq 7\epsilon \right\},$$

(2.3)

and $\mathcal{C}_c^{\infty}(\mathcal{D}_e)$ refers to smooth compactly supported functions $\Psi(y_N)$ on $\mathcal{D}_e$ that are, furthermore, symmetric in $y_N$. Proposition (1.1) ensures that $\mathcal{U}_N[\Phi]$ and $\mathcal{U}_N[\Psi]$ are both well-defined and belong to $S(\mathbb{R}^N)$. In particular, the scalar product

$$P = N! \cdot \int_{\mathbb{R}^N} \left( \mathcal{U}_N[\Psi](x_N) \right)^* \mathcal{U}_N[\Phi](x_N) \cdot d^N x$$

(2.4)

is well defined. Let $\rho_\epsilon \in \mathcal{C}_c^{\infty}(\mathbb{R})$ be such that

$$0 \leq \rho_\epsilon \leq 1 \quad \text{supp}(\rho_\epsilon) \subset [-2\epsilon; 2\epsilon[ \quad \text{and} \quad \rho_\epsilon(\epsilon) = 1. $$

(2.5)

Then introduce the function

$$n_\epsilon(y_N, y'_N) = \sum \prod_{\sigma \in \Sigma_N, a \neq b} \frac{1}{n_\epsilon(y_N, y'_N)} \cdot \left( 1 - \rho_\epsilon(y_a - y'_b) \right) \quad \text{with} \quad \sum_{\sigma \in \Sigma_N} \sigma_\epsilon(y_N, y'_N) = 1$$

(2.6)

It is readily seen that, for any $(y_N, y'_N) \in \mathcal{D}_e \times \mathcal{D}_e$, one has $n_\epsilon(y_N, y'_N) \geq 1$. Hence, the functions

$$\sigma_\epsilon(y_N, y'_N) = \frac{1}{n_\epsilon(y_N, y'_N)} \cdot \prod_{\sigma \in \Sigma_N, a \neq b} \left( 1 - \rho_\epsilon(y_a - y'_b) \right)$$

(2.7)

provide one with a partition of unity on $(y_N, y'_N) \in \mathcal{D}_e \times \mathcal{D}_e$. As a consequence, one can recast the scalar product as

$$P = \sum_{\sigma \in \Sigma_N} P_{\sigma} \quad \text{with} \quad P_{\sigma} = \int_{\mathbb{R}^N} I_{\sigma}(x_N) \cdot d^N x \quad \text{(2.8)}$$

and

$$I_{\sigma}(x_N) = \int_{\mathbb{R}^{N \times N}} \varphi_{\sigma}^*(y_N) \cdot \varphi_{\sigma}(w_N) \cdot \Phi(y_N) \cdot \sigma(x) \cdot \Phi(y_N) \cdot d\mu(y_N) \cdot d\mu(w_N)$$

(2.9)

in which we agree upon $w_N = (y_{N,1}', \ldots, y_{N,N})$. Note that each integral over $x_N$ does converges: the functions $I_{\sigma}(x_N)$ all belong to the Schwartz class as can be seen by repeating the arguments that led to proposition (1.1).

We henceforth focus on the analysis of $P_{\sigma}$. For this purpose, we introduce

$$\varphi_{N-1}[\Psi \cdot \sigma(x) \cdot \Phi](y_1, w_N | \tau_{N-1}, \tau'_{N-1})$$

$$= \int_{\mathbb{R}^{N-1}} \prod_{a=1}^{N-1} dy_a \int_{\mathbb{R}^{N-1}} \prod_{a=2}^{N} dy_a \cdot \varphi_{N-1}^*(\tau_{N-1}) \cdot \varphi_{w_{N-1}}(\tau'_{N-1}) \cdot \Phi(y_N) \cdot \sigma(y_N, w_N) \cdot \mu(y_N) \cdot \mu(w_N),$$

(2.10)
where we agree upon
\[ y^{(k)}_N = (y_k, \ldots, y_N) \in \mathbb{R}^{N+1-k}. \]  

(2.11)

By repeating the arguments that led to proposition [5] one gets that, uniformly in \( y_1, w_N \) belonging to any fixed compact in \( \mathbb{R}, \mathcal{D}_{\tau_N-1}[\Psi \sigma_{id}\Phi](y_1, w_N \mid \tau_{N-1}, \tau'_{N-1}) \) is of Schwartz class in \( (\tau_{N-1}, \tau'_{N-1}) \). A straightforward application of Fubbini’s theorem then yields

\[ I_\sigma(x_N) = \int_{\mathbb{R}^N \times \mathbb{R}^N} f_0(x_N, \tau_{N-1}, \tau'_{N-1}, y_1, w_N) \cdot d^{N-1} \tau \cdot d^{N-1} \tau' \cdot dy_1 \cdot dw_N \]  

(2.12)

where

\[ f_\nu(x_N, \tau_{N-1}, \tau'_{N-1}, y_1, w_N) = e^{\tilde{\nu}(x_N-y_1)} \cdot \overline{\Lambda}^{(N)}_{w_N}(\tau'_{N-1} | x_N) \cdot \Lambda^{(N)}_{\nu}(x_N | \tau_{N-1}) \cdot \mathcal{D}_{\tau_N-1}[\Psi \sigma_{id}\Phi](y_1, w_N \mid \tau_{N-1}, \tau'_{N-1}). \]

Since \( \mathcal{D}_{\tau_N-1}[\Psi \sigma_{id}\Phi] \) is smooth and compactly supported in the two variables \( (y_1, w_N) \) and bounded in the variables \( (\tau_{N-1}, \tau'_{N-1}) \), it follows from lemma [12] that the function

\[ x_N \mapsto I_\sigma(x_N) \]  

(2.13)

is bounded by a function decreasing faster than an exponential on the domain

\[ \mathcal{D} = \{ x_N \in \mathbb{R}^N : x_i \leq 0 \text{ or } x_N \geq 0 \} \]  

(2.14)

Furthermore, since \( I_\sigma \in \mathcal{S}(\mathbb{R}^N) \), one gets that on \( \mathbb{R}^N \setminus \mathcal{D} \), given \( \nu \geq 0 \),

\[ \left| e^{\tilde{\nu}(x_N-x_1)} I_\sigma(x_N) \right| \leq C \cdot \prod_{a=1}^N \left( \frac{1}{1 + |x_a|} \right)^2. \]  

(2.15)

By the sur-exponential bounds on \( \mathcal{D} \), (2.15) holds, in fact, on the whole of \( \mathbb{R}^N \). As a consequence, by dominated convergence,

\[ P_\sigma = \lim_{\nu \to 0} \int_{\mathbb{R}^N} d^{N \cdot \mathcal{X}} \cdot \left\{ \int f_\nu(x_N, \tau_{N-1}, \tau'_{N-1}, y_1, w_N) \cdot d^{N-1} \tau \cdot d^{N-1} \tau' \cdot dy_1 \cdot dw_N \right\}. \]  

(2.16)

The \( \nu \)-regularization allows one to be in position so as to apply Fubbini’s theorem and take the \( x_N \)-integration first. Indeed, it is readily seen that \( |f_\nu| \leq g_\nu \) where, for some constant \( C > 0 \),

\[ g_\nu = C \cdot \frac{1}{N-1} \cdot \prod_{a=1}^{N-1} (1 + \tau_a) \cdot (1 + |\tau'_a|)^3 \cdot \exp \left\{ -\frac{e^{-\tau_1}}{h} (e^{\tau_1} + e^{\tau'_1}) - \frac{e^{\tau_N}}{h} (e^{-\tau_N} + e^{-\tau'_N}) \right\} \]

\[ \times \prod_{n=2}^{N-1} \exp \left\{ -\frac{2}{h} \sqrt{(e^{-\tau_n} + e^{-\tau'_n}) \cdot (e^{\tau_n} + e^{\tau'_n})} \cdot \cosh [x_n - s_n(\tau_{N-1}, \tau'_{N-1})] \right\} \]  

(2.17)

in which

\[ s_n(\tau_{N-1}, \tau'_{N-1}) = -\frac{1}{2} \ln \left( \frac{e^{-\tau_{n-1}} + e^{-\tau'_{n-1}}}{e^{\tau_n} + e^{\tau'_n}} \right) \]  

(2.18)
and $K$ denotes a compact such that supp$(\Psi) \cup$ supp$(\Phi) \subset K$. The positive function $g_{\nu}$ is readily seen to fullfill

\[
\int_{\mathbb{R}^N} g_{\nu}(x_N, \tau_{N-1}, \tau'_{N-1}, y_1, w_N) \cdot d^N x \leq C \cdot \frac{1_K(y_1) \cdot 1_K(w_N)}{\prod_{a=1}^{N-1} \left( (1 + |\tau_a|)^2 \cdot (1 + |\tau'_a|)^2 \right)} \cdot \Gamma^2(v/h)
\]

\[
\times \prod_{n=2}^{N-1} \left\{ \exp \left\{ -\frac{2}{h} \sqrt{(e^{-\tau_{n-1}} + e^{-\tau'_{n-1}}) \cdot (e^{\nu_n} + e^{\nu'_n})} \right\} \cdot \left( 1 + |\ln (e^{-\tau_{n-1}} + e^{-\tau'_{n-1}}) \cdot (e^{\nu_n} + e^{\nu'_n})| \right) \right\}^{\tau_{n-1} \cdot (e^{\nu_n} + e^{\nu'_n})}
\]

\[
\left( (e^{\nu_1} + e^{\nu'_1})(e^{-\tau_{N-1}} + e^{-\tau'_{N-1}}) \right)^{\tau_{N-1} \cdot (e^{\nu_1} + e^{\nu'_1})} \cdot \prod_{a=1}^{N-1} \left( (1 + |\tau_a|) \cdot (1 + |\tau'_a|) \right)
\]

(2.19)

what follows from the bound

\[
\int_{\mathbb{R}} e^{-a \cosh(r)} \cdot dr \leq C'(1 + |\ln(a)|) \cdot e^{-a} \quad \text{for some} \quad C' > 0.
\]

(2.20)

It is readily seen that the second line of (2.19) is a bounded function of $(\tau_{N-1}, \tau'_{N-1}) \in \mathbb{R}^{N-1} \times \mathbb{R}^{N-1}$. Hence,

\[
\int_{\mathbb{R}^{2N}} d^{N-1} \tau \cdot d^{N-1} \tau' \cdot dy_1 \cdot dw_N \cdot \left\{ \int_{\mathbb{R}^N} g_{\nu}(x_N, \tau_{N-1}, \tau'_{N-1}, y_1, w_N) \cdot d^N x \right\} < +\infty.
\]

(2.21)

In virtue of Fubini’s theorem, $g_{\nu} \in L^1$ and thus $f_\nu$ as well. Therefore one can change the orders of integration in (2.16) and compute the $x_N$-integration first. The latter can then be taken by means of lemma 1.3 thus leading to

\[
P_{\sigma} = \lim_{v \to 0^+} \int_{\mathbb{R}^{2N}} \Gamma \left( \frac{y_1 - w_N - iv}{i h} \right) \Gamma \left( \frac{w_N - y_1 + iv}{i h} \right) \cdot \left( \Lambda^{(N-1)}_{\nu_1} \cdot \Lambda^{(N-1)}_{w_N} \right) \left( \tau_{N-1} \mid \tau_{N-1} \right) \cdot \left( e^{\nu_1} + e^{\nu'_1} \right) \cdot \left( e^{-\tau_{N-1}} + e^{-\tau'_{N-1}} \right) \cdot \Psi \cdot \text{dr}_{\nu_1} \cdot \Phi \cdot \left( y_1, w_N \mid \tau_{N-1}, \tau'_{N-1} \right) \cdot d^{N-1} \tau \cdot d^{N-1} \tau' \cdot dy_1 \cdot dw_N.
\]

(2.22)

One can take the limit under the integral sign by applying the dominated convergence theorem. Indeed, the function $\text{dr}_{\nu_1}$ ensures that the $\Gamma$-functions are uniformly bounded on the support of integrand whereas the remaining part of the integrand can be bounded analogously to (2.19).

After a straightforward exchange of the order of integration, one is led to the representation

\[
P_{\sigma} = \int_{\mathbb{R}} dw_N \int_{\mathbb{R}^{N-1}} d^{N-1} \tau \cdot \int_{\mathbb{R}^{N-2}} d^{N-2} \xi \cdot \text{\overline{A}}^{(N-1)}_{w_N} (\xi_{N-2} \mid \tau_{N-1}) \cdot \left\{ \int_{\mathbb{R}^{N-1}} d\mu (y^{(2)}_N) \cdot \phi_{y^{(2)}_N}(\tau_{N-1}) \cdot \Psi (w_N, y^{(2)}_N \mid \xi_{N-2}) \right\},
\]

(2.23)

where

\[
\Psi (w_N, y^{(2)}_N \mid \xi_{N-2}) = \int_{\mathbb{R}} \frac{dy_1}{2\pi h} \gamma (y_1 - w_N) \int_{\mathbb{R}^{N-1}} d^{N-1} \tau' \cdot \Lambda^{(N)}_{y_1} (\tau'_{N-1} \mid \xi_{N-2}) \cdot \Phi (y_N) \cdot \text{d} \omega (y_N, w_N) \cdot \Psi (w_N) \cdot \mu (w_N) \phi_{y^{(2)}_N} (\tau'_{N-1}) \cdot \prod_{a=2}^{N} \gamma (y_a - y_1).
\]

(2.24)
Above, we have introduced
\[ \gamma(x) = \Gamma \left( \frac{x}{i\hbar}, -\frac{x}{i\hbar} \right) \quad \text{with} \quad \Gamma(x,y) = \Gamma(x) \cdot \Gamma(y). \quad (2.25) \]

It follows from lemma [1.1] that \( \Upsilon^1(w_N, y_N^{(2)} \mid \xi_{N-2}) \) is smooth and compactly supported in respect to the first set of variables \( w_N, y_N^{(2)} \) and of Schwartz class in respect to the second argument \( \xi_{N-2} \), this uniformly in \( w_N, y_N^{(2)} \).

The remainder of the proof goes by induction. One defines a sequence of functions
\[ \Upsilon_k(w_N, y_N^{(k+1)} \mid \xi_{N-k-1}) = \int_{\mathbb{R}} \frac{\text{d}y_k}{2\pi \hbar} \gamma(y_k - w_N) \int_{\mathbb{R}^{N-k}} \frac{\Lambda_{N-k}^{(k)}(\tau_{N-k} \mid \xi_{N-k-1}) \cdot \Upsilon_{k-1}(w_N, y_N^{(k)} \mid \tau_{N-k})}{\prod_{a=k}^{N} \gamma(y_a - y_{k+1})} \cdot \text{d}^{N-k} \tau \cdot \mu(\tau_{N-k}). \quad (2.26) \]

Then, the induction hypothesis claims that \( P_{sr} \) can be recast as
\[ P_{sr} = \int_{\mathbb{R}} \text{d}w_N \int_{\mathbb{R}^{N-k}} \text{d}^{N-k} \tau \int_{\mathbb{R}^{N-k-1}} \text{d}^{N-k-1} \xi \cdot \Lambda_{w_N}^{(N-k)}(\xi_{N-k-1} \mid \tau_{N-k}) \times \int_{\mathbb{R}^{N-k-1}} \varphi_{w_N}(\tau_{N-k}) \cdot \mu(y_N^{(k+1)}) \cdot \Upsilon_k(w_N, y_N^{(k+1)} \mid \xi_{N-k-1}) \prod_{a=k+1}^{N} \text{d}y_a. \quad (2.27) \]

In which the functions \( \Upsilon_k(w_N, y_N^{(k+1)} \mid \xi_{N-k-1}) \) are smooth and compactly supported in respect to the first set of variables \( w_N, y_N^{(k+1)} \) and of Schwartz class in the second ones \( \xi_{N-k-1} \). The properties of the functions \( \Upsilon_k \) are a consequence of lemma [1.1]. The remaining contents of the induction can be established with the help of bounds quite similar to those used in the "initialisation" part of the proof. The details are left to the reader.

All in all, upon \( k = N - 1 \) successive iterations, one gets
\[ P_{sr} = \int_{\mathbb{R}} \text{d}w_N \int_{\mathbb{R}^{N-1}} \text{d}^{N-1} \tau \cdot \Lambda_{w_N}^{(N-1)}(\tau \mid -) \cdot \Upsilon_{N-1}(w_N, y_N \mid -) \cdot \frac{\text{d}y_N}{2\pi \hbar} = \int_{\mathbb{R}} \Upsilon_{N-1}(w_N, w_N \mid -) \cdot \text{d}w_N. \]

Then, by tracing backwards the chains of transformations, we are led to the representation
\[ P_{sr} = \int_{\mathbb{R}^{N}} \text{d}w_N \int_{\mathbb{R}^{N-1}} \text{d}y_{N-1} \tau \left\{ \int_{\mathbb{R}^{N-1}} \text{d}y_{N-1} \text{d}y_{N-2} \prod_{a=1}^{N-1} \{ \gamma(y_a - w_N) \} \cdot \sigma_{id}(y_{N-1}, w_N) \times \Phi((y_{N-1}, w_N)) \cdot \Psi(w_N) \cdot \varphi_{\gamma_{N-1}}(\tau_{N-1}) \cdot \varphi_{y_{N-1}}(\tau_{N-1}) \cdot \mu((y_{N-1}, w_N)) \cdot \mu(w_N) \right\}. \quad (2.28) \]

Having established (2.28), it takes a straightforward induction to get
\[ P_{sr} = \int_{\mathbb{R}^{N}} \sigma_{id}(y_{sr}, y_{sr}) \cdot \Phi(y_{sr}) \cdot \Psi(y_{sr}) \cdot \text{d}\mu(y_{sr}). \quad (2.29) \]

Above, we have recast the integrand in terms of the original variables \( y'_{sr(a)} = w_{sr} \). Note that in (2.29) one does end up with \( \sigma_{id} \) instead of \( \sigma_{sr} \) since the first and second of its argument have both been permuted.
The integration in (2.29) runs, in fact, through $D_e$. Since, by definition, given any

$$y_N' \in D_e \quad \text{one has} \quad y_a' - y_b' \notin \text{supp}(\rho_e) \quad \text{for any} \quad a \neq b,$$

it follows that $\sigma_{\text{id}}(y_N', y_N') = \eta_{\text{id}}^{-1}(y_N', y_N')$. Furthermore, going back to the definition of the function $\eta_e$, it follows that the sole term that survives correspond to the identity permutation $\sigma = \text{id}$, since, for any $\sigma \in \mathfrak{S}_N \setminus \{\text{id}\}$, there exists $a \neq b$ such that $\sigma(a) = b$, viz $\rho_e(y_{\sigma(a)}') - y_b' = 1$. As a consequence, $\sigma_{\text{id}}(y_N', y_N') = 1$ leading to

$$N! \cdot \int_{\mathbb{R}^N} (\mathcal{U}_N[\Psi^*](x_N))^* \cdot \mathcal{U}_N[\Phi](x_N) \cdot d^N x = P = N! \cdot \int_{\mathcal{D}_e} \Phi(y_N^*) \cdot \Psi(y_N^*) \cdot d\mu(y_N^*).$$

The result (2.31) can be extended to functions $\Phi, \Psi \in \mathcal{C}^\infty_c(\mathbb{R}^N)$. Indeed, let $\eta_e \in \mathcal{C}^\infty(\mathbb{R}^N)$ be such that

$$0 \leq \eta_e \leq 1, \quad \text{supp}(\eta_e) \subset D_e \quad \text{and} \quad \eta_e|_{\mathcal{D}_e} = 1.$$

Then, given a sequence $e_M \to 0$, the sequences of functions $\Psi_M = \eta_e \cdot \Psi, \Phi_M = \eta_e \cdot \Phi$ satisfy to the hypothesis previously used and are such that

$$\Psi_M(y_N) \to_{M \to +\infty} 1_{\mathbb{R}^N \setminus \mathcal{D}_0}(y_N) \cdot \Psi(y_N), \quad \Phi_M(y_N) \to_{M \to +\infty} 1_{\mathbb{R}^N \setminus \mathcal{D}_0}(y_N) \cdot \Phi(y_N).$$

By using that $\mathbb{R}^N \setminus \mathcal{D}_0$ has zero Lebesgue measure, it follows,

$$\int_{\mathbb{R}^N} \Phi_M(y_N^*) \cdot \Psi_M(y_N^*) \cdot d\mu(y_N^*), \to_{M \to +\infty} \int_{\mathbb{R}^N} \Phi(y_N^*) \cdot \Psi(y_N) \cdot d\mu(y_N).$$

This implies that the sequence $\mathcal{U}_N[\Psi_M^*], \text{ resp. } \mathcal{U}_N[\Phi_M^*]$, is a Cauchy sequence in $L^2(\mathbb{R}^N, d^N x)$, and thus converges to some function $\tilde{\psi} \in L^2(\mathbb{R}^N, d^N x)$, resp. $\tilde{\phi} \in L^2(\mathbb{R}^N, d^N x)$. However, since

$$\mathcal{U}_N[\Psi_M^*](x_N) \to_{M \to +\infty} \mathcal{U}_N[\Psi^*](x_N) \quad \text{resp.} \quad \mathcal{U}_N[\Phi_M^*](x_N) \to_{M \to +\infty} \mathcal{U}_N[\Phi^*](x_N).$$

uniformly in $x_N \in \mathbb{R}^N$, it follows that $\tilde{\psi} = \mathcal{U}_N[\Psi^*]$, resp. $\tilde{\phi} = \mathcal{U}_N[\Phi]$. It remains to conclude by invoking the density of $\mathcal{C}^\infty_c(\mathbb{R}^N)$ in $L^2_{\text{sym}}(\mathbb{R}^N, d\mu(y_N))$.

\section{Isometric nature of the adjoint transform}

It is readily seen that the map $\overline{\mathcal{U}}_N$ defined on $L^1(\mathbb{R}^N, d^N x)$ through

$$\overline{\mathcal{U}}_N[F](y_N) = \frac{1}{\sqrt{N!}} \int_{\mathbb{R}^N} (\varphi_{y_N}(x_N))^* \cdot F(x_N) \cdot d^N x$$

(3.1)

is a formal adjoint of $\mathcal{U}_N$. The purpose of the present section is to show that $\overline{\mathcal{U}}_N$ extends to an isometric operator $L^2(\mathbb{R}^N, d^N x) \to L^2_{\text{sym}}(\mathbb{R}^N, d^N \mu(y_N))$. This will thus establish that $\mathcal{U}_N^* = \overline{\mathcal{U}}_N$ and that $\mathcal{U}_N$ is a unitary map.

In fact, since $(\varphi_{y_N}(x_N))^* = \varphi_{-y_N}(x_N)$, it is just as good to establish the isometricity of the operator $\mathcal{V}_N : L^2(\mathbb{R}^N, d^N x) \to L^2_{\text{sym}}(\mathbb{R}^N, d^N \mu(y_N))$ whose action on $(L^1 \cap L^2)(\mathbb{R}^N, d^N x)$ is given by the integral transform

$$\mathcal{V}_N[F](y_N) = \frac{1}{\sqrt{N!}} \int_{\mathbb{R}^N} \varphi_{y_N}(x_N) F(x_N) \cdot d^N x.$$

(3.2)
We do stress that the method allowing us to prove the isometric nature of $\mathcal{U}_N$ has never been proposed, even on a formal level of rigour, previously. Thus the whole content of the present section is completely new. Furthermore, from the point of view of formal manipulations, our method is quite simple. Our proof builds on isometric properties of two auxiliary integral transforms $\mathcal{H}_N$ and $\mathcal{J}_N$. The isometricity of the latter transforms is deduced from theorem 2.1 adjointed to the Mellin-Barnes type recurrence relation satisfied by the functions $\varphi_{y_{\ell}}(x_N)$. In this respect, the proof highlights a sort of duality between the isometric character of the operator $\mathcal{U}_N$ and $\mathcal{V}_N$. As soon as one isometry is proven, the second one follows from it. The steps of our proof do not rely on any property specifically associated with the Toda chain. Thus, it can quite probably be applied for proving the unitarity of SoV transforms arising in the context of other quantum integrable models solvable by the quantum separation of variables method.

3.1 The $\mathcal{H}_N$ transform

**Definition 3.1** Let $\mathcal{H}_N$ be the operator defined, for $G \in \mathcal{C}_{c,\text{sym}}^\infty(\mathbb{R}^N)$, as

$$\mathcal{H}_N[G](w_N) = \frac{1}{N!} \lim_{\alpha \to 0} \int_{\mathbb{R}^N} \prod_{a=1}^{N} (\Gamma(\frac{y_b - w_a + i\alpha}{i\hbar}) \cdot \prod_{a \neq b}^{N} \Gamma^{-1}(\frac{y_b - y_a}{i\hbar})) \cdot G(y_N) \frac{d^N y}{(2\pi\hbar)^N}. \quad (3.3)$$

Prior to establishing the isometric character of the operator $\mathcal{H}_N$, we establish certain of its most basic properties.

**Lemma 3.1** Let $G \in \mathcal{C}_{c,\text{sym}}^\infty(\mathbb{R}^N)$. Then the map

$$w_N \mapsto \prod_{a < b}^{N} (w_b - w_a) \cdot \mathcal{H}_N[G](w_N) \quad (3.4)$$

is smooth and $\mathcal{H}_N[G] \in L^2_{\text{sym}}(\mathbb{R}^N, d\tilde{\mu}(w_N))$ with the measure $\tilde{\mu}$ being given by

$$d\tilde{\mu}(w_N) = \tilde{\mu}(w_N) \cdot d^N w \quad \text{where} \quad \tilde{\mu}(w_N) = e^{i\pi w_N} \cdot \mu(w_N). \quad (3.5)$$

**Proof** —

We begin by showing that $w_N \mapsto \prod_{a < b}^{N} (w_b - w_a) \cdot \mathcal{H}_N[G](w_N)$ is smooth. For this purpose, we first need to define the skeleton $\Gamma^{(k)}$ in $\mathbb{C}^k$:

$$\Gamma^{(k)} = \partial \mathcal{D}_{y_1 + i\alpha, \frac{\pi}{k}} \times \cdots \times \partial \mathcal{D}_{y_k + i\alpha, \frac{\pi}{k}}. \quad (3.6)$$

Then, for pairwise distinct real numbers $\{w_a\}_1^N$ and $\{y_a\}_1^k$, one has that the contour integral

$$\oint \prod_{a=1}^{k} \frac{1}{(z_a - y_a - i\alpha)} \cdot \prod_{a > b}^{k} (z_a - z_b) \cdot \prod_{a=1}^{k} \prod_{b=1}^{N} \frac{1}{(z_a - w_b)} \cdot \frac{d^k z}{(2i\alpha)^k} \quad (3.7)$$

can be estimated by taking the one dimensional residues in each variable either in respect to the simple poles located inside of the respective contour or outside thereof. Choosing to compute the integral either by means of taking all residues inside or all outside of the contour leads to the identity

$$\prod_{r > \ell}^{k} (y_r - y_{\ell}) \cdot \prod_{r=1}^{k} \prod_{l=1}^{N} \frac{1}{y_r - w_{\ell} + i\alpha} = \sum_{b_1 \neq \cdots \neq b_k} \prod_{r=1}^{k} \prod_{s=1}^{N} \frac{1}{w_{b_r} - w_{b_{\ell}}} \cdot \prod_{r < \ell}^{k} \frac{1}{w_{b_r} - w_{b_{\ell}}} \cdot \prod_{s=1}^{k} \frac{1}{y_s - w_{b_s} + i\alpha}. \quad (3.8)$$
Implementing it at \( k = N \) yields, upon invoking the symmetry of the integrand in \( y_N \),

\[
\prod_{r < \ell}^{N} (w_r - w_{\ell}) \cdot \mathcal{H}_N[G](w_N) = \frac{1}{N!} \lim_{a \to 0} \int_{\mathbb{R}^N} \left( \frac{N!(ih)^N G(y_N)}{\prod_{a=1}^{N} (y_a - w_a + i\alpha)} \cdot \prod_{r < \ell}^{N} (y_r - y_{\ell}) \cdot \frac{\prod_{a,b=1}^{N} \Gamma\left(\frac{y_b - w_a}{ih} + 1\right)}{\prod_{a+b}^{N} \Gamma\left(\frac{y_b - y_a}{ih} + 1\right)} \cdot \frac{d^N y}{(2\pi h)^N} \right). \tag{3.9}
\]

Thus \( \prod_{r < \ell}^{N} (w_r - w_{\ell}) \cdot \mathcal{H}_N[G](w_N) \) can be recast as + boundary values of \( N \) one-dimensional Cauchy transforms.

Since \( G \) is smooth, it is readily seen by repeating the arguments holding for the one-dimensional case that

\[
\prod_{a \neq b}^{N} (w_b - w_a) \cdot \mathcal{H}_N[G](w_N) \text{ is smooth.}
\]

We shall now establish the \( L^2_{sym}(\mathbb{R}^N, \, d\bar{\mu}(w_N)) \) character of \( \mathcal{H}_N[G](w_N) \). This amounts to bounding the function when part of the variables goes to infinity. Since this function is manifestly symmetric in \( w_N \), it is enough to bound it on the domain \( \{ w_N \in \mathbb{R}^N : w_1 < \ldots < w_N \} \). Let \( R > 0 \) be such that \( \text{supp}(G) \), the support of \( G \), verifies \( \text{supp}(G) \subset ] - R ; R [^N \). We then introduce

\[
\mathcal{D}_{k,\ell} = \left\{ w_N \in \mathbb{R}^N : w_1 < \cdots < w_k < -R < w_{k+1} < \cdots < w_{\ell} < R < w_{\ell+1} < \cdots < w_N \right\}. \tag{3.10}
\]

It is readily seen that it is enough to prove the \( L^2 \) character of \( \mathcal{H}_N[G](w_N) \) on each \( \mathcal{D}_{k,\ell} \) with \( 0 \leq k, \ell \leq N \). \( \prod_{a \neq b}^{N} (w_b - w_a) \cdot \mathcal{H}_N[G](w_N) \) being smooth, it is bounded on \( \mathcal{D}_{0,N} \) and, as such, \( \mathcal{H}_N[G] \in L^2(\mathcal{D}_{0,N}, \, d\bar{\mu}(w_N)) \).

It thus remains to consider the case \( (k, \ell) \neq (0, N) \). Then, since

\[
(\mathcal{H}_N[G](w_N))^* = \mathcal{H}_N[G^{(-)}](w_N) \quad \text{with} \quad G^{(-)}(y_N) = G(-y_N) \tag{3.11}
\]

we may just as well assume that \( \ell \leq N - 1 \). Having this in mind, we define

\[
I_{k,\ell} = \left\{ \| 1 ; k \| \cup \| \ell + 1 ; N \| : k \geq 1 \right\} \quad \text{and} \quad I_{k,\ell}^c = \left\{ k + 1 ; \ell \right\}. \tag{3.12}
\]

Then, a direct application of (3.8) in respect to the variables \( w_a, a \in I_{k,\ell}^c \) yields

\[
\mathcal{H}_N[G](w_N) = \frac{[I_{k,\ell}^c]!}{N!} \prod_{a \neq b}^{[I_{k,\ell}^c]} \left( \frac{1}{w_a - w_b} \right) \cdot \prod_{a \in I_{k,\ell}^c} \left( \frac{\Gamma\left(\frac{w_a}{ih}\right)}{\prod_{a \in I_{k,\ell}^c} (y_a - w_a + i\alpha)} \right) \cdot \lim_{a \to 0} \int_{\mathbb{R}^N} \prod_{a \in I_{k,\ell}^c} \frac{G_{I_{k,\ell}^c}(y_N, w_N)}{\prod_{a \in I_{k,\ell}^c} (y_a - w_a + i\alpha)} \frac{(i\bar{w}_N)^{\omega_N}}{\left(\frac{h}{2\pi}\right)^N} \cdot \frac{d^N y}{(2\pi h)^N}. \tag{3.13}
\]

There, we have set,

\[
G_{I_{k,\ell}^c}(y_N, w_N) = (ih)^{N[I_{k,\ell}^c]} \cdot \frac{\left(\frac{i\bar{w}_N}{h}\right)^{\omega_N}}{\prod_{a \in I_{k,\ell}^c} \prod_{b=1}^{N} \Gamma\left(\frac{y_b - w_a}{ih} + 1\right)} \cdot \prod_{a+b}^{N} \Gamma\left(\frac{y_b - w_a}{ih} + 1\right) \cdot \prod_{a+b}^{N} \Gamma\left(\frac{y_b - y_a}{ih} + 1\right) \cdot \prod_{a \neq b}^{N} \Gamma\left(\frac{y_b - y_a}{ih} + 1\right) \cdot G(y_N). \tag{3.14}
\]
Further, we integrate by parts $1 + |l_{k,l}^N|_t$ times in respect to $y_N$ the term $(iw_N/h)^{-\mu_1}$ in (3.13) and also integrate by parts once each of the singular factors in the variables $y_a$, with $a \in I_{k,l}^N$, leading to

$$\mathcal{H}_N[G](w_N) = -\frac{|l_{k,l}^N|}{N!} \cdot \prod_{a \neq b} \left( \frac{1}{i(h_w - w_b)} \right) \cdot \prod_{a \in I_{k,l}^N} \left( \Gamma \left( -\frac{w_a}{i\hbar} \right) \right)^N \cdot \left( \frac{i\hbar}{\ln[iw_N/h]} \right)^{1+|l_{k,l}^N|}$$

$$\times \int_{\text{supp}(G)} \left( \frac{iw_N}{h} \right)^{-\frac{\mu_1}{2}} \prod_{a \in I_{k,l}^N} \left( \ln |w_a - w_a| + i\pi 1_{\mathbb{R}^+} (w_a - y_a) \right) \frac{\partial^{1+|l_{k,l}^N|}}{\partial y_a} \cdot \left( G_{k,l}(y_N, w_N) \right) \cdot \frac{d^N y}{(2\pi\hbar)^N}. \tag{3.15}$$

It follows from the uniform differentiability and uniformness in $y_N \in \text{supp}(G)$ of the remainder in the large $w_a$, $a \in I_{k,l}^N$, expansion

$$\prod_{a \in I_{k,l}^N} \prod_{b=1}^N \left( \Gamma \left( -\frac{y_b - w_a}{i\hbar} \right) \cdot \Gamma^{-1} \left( -\frac{w_a}{i\hbar} \right) \right) = \prod_{a \in I_{k,l}^N} \left\{ \left( \frac{w_a}{i\hbar} \right)^{\mu_1} \right\} \cdot \left\{ 1 + \sum_{a \in I_{k,l}^N} O \left( \frac{1}{w_a} \right) \right\} \tag{3.16}$$

that there exists a constant $C > 0$ such that

$$\left| \frac{\partial^{1+|l_{k,l}^N|}}{\partial y_a} \cdot \left( G_{k,l}(y_N, w_N) \right) \right| \leq C \left( \ln(-w_1) + \ln(w_N) + 1 \right)^{2|l_{k,l}^N|} \tag{3.17}$$

uniformly in $w_N \in D_{k,l}$ and $y_N \in \text{supp}(G)$. Using that the integration in (3.15) runs through a compact, it is readily inferred that there exists some constant $C'$ such that

$$\bar{\mu}(w_N) \cdot \left| \mathcal{H}_N[G](w_N) \right|^2 \leq C' \cdot \left[ \ln(-w_1) + \ln(w_N) + 1 \right]^{2|l_{k,l}^N|} \cdot \left( \frac{1}{|w_1| \cdot (\ln w_N)^2 \cdot |w_N| \cdot (\ln|w_N|)^2} \right)^{|l_{k,l}^N|} \cdot \prod_{p=1}^{\left\lfloor \frac{|l_{k,l}^N|}{2} \right\rfloor} \frac{1}{e^{\sum \left\lfloor w_{a,p}(0h_{k,l}(\ell)\pm 0h_{k,l}(\ell)w_{a,p}) \right\rfloor}}. \tag{3.18}$$

The last product in (3.18) utilises the parametrisation $I_{k,l} = \{ a_1, \ldots, a_{|l_{k,l}^N|} \}$ with the additional assumption $a_1 < \cdots < a_{|l_{k,l}^N|}$. The rhs is clearly in $L^1(D_{k,l}, d^N w)$.

We are now in position to prove the isometric character of $\mathcal{H}_N$ when restricted to $C_{c,\text{sym}}(\mathbb{R}^N)$. This represents the hardest result obtained in this paper.

**Proposition 3.1** For any $G \in C_{c,\text{sym}}(\mathbb{R}^N)$ one has

$$\| \mathcal{H}_N[G] \|_{L^2_{\text{sym}}(\mathbb{R}^N, d\bar{\mu}(w_N))} = \| G \|_{L^2_{\text{sym}}(\mathbb{R}^N, d\bar{\mu}(y_N))}, \tag{3.19}$$

where the operator $\mathcal{H}_N$ is as defined by (3.3).

It thus follows that $\mathcal{H}_N$ can naturally be extended into an isometric operator

$$\mathcal{H}_N : L^2_{\text{sym}}(\mathbb{R}^N, d\bar{\mu}(w_N)) \longrightarrow L^2_{\text{sym}}(\mathbb{R}^N, d\bar{\mu}(y_N)). \tag{3.20}$$

**Proof** —

Given any $F \in C_{c,\text{sym}}(\mathbb{R}^{N+1})$, we define

$$\overline{F}(y_{N+1}) = \hbar^N \mathcal{H}_{N+1} F \left( y_{N+1} \right). \tag{3.21}$$
An application of the Mellin-Barnes recurrence \((1.1)\) satisfied by the integral kernel \(\varphi_{y_{N+1}}(x_{N+1})\) of the transform \(\mathcal{U}_{N+1}\), cf \((1.7)\), followed by an application of Fubbini’s theorem\(^1\) leads to the representation

\[
\mathcal{U}_{N+1}[F](x_{N+1}) = \int_{(\mathbb{R}-i\alpha)^N} d\mu(w_N) \varphi_{w_N}(x_N) e^{-\tilde{\varphi}_{N+1} x_{N+1}} \cdot \int_{\mathbb{R}^{N+1}} e^{\tilde{\varphi}_{N+1} x_{N+1}} \frac{\sigma(w_N | y_{N+1})}{\sqrt{(N+1)!} \cdot \mu(w_N)} \cdot \tilde{F}(y_{N+1}) \cdot \frac{d\mu(y_{N+1})}{(2\pi)^N \cdot N!} \\
= \mathcal{U}_N[\tilde{S}_N[F](\cdot,x_{N+1})](x_N) = \frac{\mathcal{U}_N[\tilde{S}_N[F](\cdot,x_{N+1})](x_N)}{(2\pi)^N \cdot N! \cdot \sqrt{N+1}}, \tag{3.22}
\]

where \(*\) refers to the group of variables \(w_N\) in the function

\[
\tilde{S}_N[F](w_N,x) = e^{-\frac{1}{\hbar} \tilde{\varphi}_{w_N} \cdot x} \cdot \tilde{F}^{(N+1)}(\varphi_{w_N} : S_N[F](w_N,x)) \tag{3.23}
\]

on which the transform \(\mathcal{U}_N\) acts. The integral transform \(S_N\) introduced in \((3.22)\) reads

\[
S_N[F](w_N, x) = \lim_{\alpha \to 0^+} \int_{\mathbb{R}^{N+1}} e^{\tilde{\varphi}_{N+1} x} \prod_{a=1}^{N} \prod_{b=1}^{N+1} \Gamma(\frac{y_b - w_a + i\alpha}{\hbar}) \cdot \prod_{a \neq b} \Gamma^{-1}(\frac{y_b - w_a}{\hbar}) \cdot F(y_{N+1}) \cdot \frac{d^{N+1} y}{2\pi \hbar}. \tag{3.24}
\]

In fact, in order to establish the second line of \((3.22)\), one should check that

\begin{itemize}
  \item \(S_N[F](w_N - i\alpha e_N, x) \in L^1(\mathbb{R}^N, d\mu(w_N))\) with \(e_N = (1, \ldots, 1) \in \mathbb{R}^N\) so that the action of \(\mathcal{U}_N\) on this function is well defined;
  \item that \(|S_N[F](w_N - i\alpha e_N, x)|\) can be bounded by an \(L^1(\mathbb{R}^N, d\mu(w_N))\) function, independently of \(\alpha\) small enough.
\end{itemize}

These properties allow then to move the \(\alpha \to 0^+\) limit past the action of \(\mathcal{U}_N\). These two properties can be readily inferred by following the reasoning outlined in the proof of lemma \((3.1)\) so we do not reproduce the arguments once again. In fact, this very reasoning - with the sole difference being that one should integrate by parts and in respect to \(y_{N+1}\) the oscillating exponent \(\exp[\tilde{\varphi}_{N+1} x/\hbar]\) so as to generate an explicit algebraic decay in \(x\) - allows one to establish that \(\tilde{S}_N[F] \in L^2_{sym}(\mathbb{R}^N \times \mathbb{R}, d\mu(w_N) \otimes dx)\), and hence \(\tilde{S}_N[F]\) as well. Note that the subscript \(sym \times -\) refers to functions that are symmetric in respect to the first \(N\) variables, in accordance with the Carthesian product decomposition of \(\mathbb{R}^N \times \mathbb{R}\).

The isometric character of \(\mathcal{U}_{N+1}\) and \(\mathcal{U}_N\) leads to the relation

\[
N[F] = \|S_N[F]\|_{L^2_{sym}(\mathbb{R}^{N+1}, d\mu(w_N) \otimes dx)} = \sqrt{N+1} \cdot N! (2\pi \hbar)^N \cdot \|F\|_{L^2_{sym}(\mathbb{R}^{N+1}, \mu)_{N+1}}. \tag{3.25}
\]

We now build on the above two possible representations for \(N[F]\) so as to deduce the relation \((3.19)\), this by using a specific choice for the function \(F\). More precisely, from now on, we shall take \(F = F_K \in \mathcal{C}^\infty_{c sym}(\mathbb{R}^{N+1})\) given by

\[
F_K(y_{N+1}) = \text{Sym} \left\{ G(y_N) u(\tilde{y}_{N+1} - K) e^{\tilde{\varphi}_{N+1} (1+N/2)} \Gamma^N(\frac{\tilde{y}_{N+1}}{\hbar} + 1) \cdot \left( -\frac{\tilde{y}_{N+1}}{\hbar} + 1 \right)^{-N} \cdot (\frac{\tilde{y}_{N+1} + i}{\hbar})^{\frac{N}{2}} \right\}, \tag{3.26}
\]

where \(G\) and \(u\) are such that

\[
G \in \mathcal{C}^\infty_{c sym}(\mathbb{R}^N) \quad \text{and} \quad u \in \mathcal{C}^\infty_c(\mathbb{R}) \quad \text{with} \quad \int \mu(y)^2 \cdot \frac{dy}{2\pi \hbar} = 1. \tag{3.27}
\]

\(^1\)what is licit since the integral converges strongly in \((w_N,y_{N+1})\) due to the compact support of \(\tilde{F}\) and the fast decay in \(w_N\) of \(\varphi_{w_N}(x_N) \cdot \sigma(w_N | y_{N+1})\), cf eg \([9]\).
Finally, in (3.26), Sym stands for the symmetrization operator in respect to the variables $y_{N+1}$, viz. for any function $J$ of the variables $y_{N+1}$, one has

$$
\text{Sym}[J](y_{N+1}) = \frac{1}{(N+1)!} \sum_{\sigma \in \mathbb{S}_{N+1}} J(y^\sigma_{N+1}) \quad \text{where} \quad y^\sigma_{N+1} = (y_{\sigma(1)}, \ldots, y_{\sigma(N+1)}) .
$$

(3.28)

We shall compute the $K \to +\infty$ limit of $\mathcal{N}[F_K]$ in two ways by using (3.25). Ultimately, this will yield us the sought isometricity of $\mathcal{H}_N$.

**The $K \to +\infty$ limit of $\|F_K\|_{L^2_{\mu_0}(\mathbb{R}^{N+1}, \mu(y_{N+1}))}$**

Since $G$ is symmetric in $y_N$, it is readily seen that the sum over the permutation group in (3.26) can be recast as

$$
F_K(y_{N+1}) = \frac{u(\overline{y}_{N+1} - K)}{N + 1} \Gamma(N) \overline{y}_{N+1} + 1 + 1 - N \sum_{k=1}^{N+1} G(y_{N}^{(k)}) e^{\frac{\pi i}{N} (2+1) \overline{y}_N^{(2+1)} (N+1)} \cdot \left( \frac{\overline{y}_{N+1} + 1}{\hbar} \right)^{\overline{y}_N (2+1)} .
$$

(3.29)

in which, from now on, $y_N^{(k)}$ stands for the $N$- dimensional vector $4$

$$
y_N^{(k)} = (y_1, \ldots, y_{k-1}, y_{k+1}, \ldots, y_{N+1}) .
$$

(3.30)

Assume that is $K$ large enough. Then, the $k^{th}$ term in the above sum does not vanish only if $y_N^{(k)} \in \text{supp}(G)$.

Thus, since both supp$(G)$ and supp$(u)$ are compact, the function $G(y_N^{(k)}) u(\overline{y}_{N+1} - K)$ will be non zero solely for $y_k$ belonging to some $K$-independent compact centered at $K$. As a consequence, for $K$ large, the only non-zero contributions to the product $F_K F_K$ will stem from the diagonal terms, viz.

$$
\left| F_K(y_{N+1}) \right|^2 = \left| \frac{u(\overline{y}_{N+1} - K)}{N + 1} \Gamma(N) \overline{y}_{N+1} + 1 + 1 - N \sum_{k=1}^{N+1} G(y_{N}^{(k)}) e^{\frac{\pi i}{N} (2+1) \overline{y}_N^{(2+1)} (N+1)} \cdot \left( \frac{\overline{y}_{N+1} + 1}{\hbar} \right)^{\overline{y}_N (2+1)} \right|^2 .
$$

Inserting this expression into the integral, making the most of the symmetry of the integrand and finally changing the variables from $y_{N+1}$ to $s = \overline{y}_{N+1} - K$ recasts the norm in the form

$$
\mathcal{N}^2[F_K] = (N!)^2 (2\pi \hbar)^{2N} \cdot \int_{\mathbb{R}^N \times \mathbb{R}} |u(s)|^2 \cdot |G(y_N)|^2 \mathcal{J}_K(s, y_N) : d\mu(y_N) \cdot \frac{ds}{2\pi \hbar} .
$$

(3.31)

where

$$
\mathcal{J}_K(s, y_N) = \frac{e^{\frac{\pi i}{N} (1+) \overline{y}_N^{(1+)}}}{(1 + \frac{K + i}{\hbar})^N} \cdot \left( \frac{K + s + i}{K + s - i} \right)^{\overline{y}_N} \prod_{a=1}^{N} \left\{ \frac{\Gamma(1 + \frac{K + s}{\hbar}) \cdot \Gamma(1 + \frac{K - s}{\hbar})}{\Gamma(\frac{s + K - y_a - \overline{y}_N}{\hbar}, \frac{y_a + \overline{y}_N - s - K}{\hbar})} \right\} .
$$

(3.32)

It is readily seen that

$$
\mathcal{J}_K(s, y_N) \rightarrow 1 \quad \text{uniformly in} \quad (s, y_N)
$$

(3.33)

belonging to compact subsets of $\mathbb{R} \times \mathbb{R}^N$. As a consequence, since the integration runs through the compact support of $u(s)G(y_N)$, it follows that

$$
\lim_{K \to +\infty} \mathcal{N}^2[F_K] = (N!)^2 (2\pi \hbar)^{2N} \cdot \|G\|^2_{L^2_{\mu_0}(\mathbb{R}^{N+1}, \mu(y_{N+1}))} .
$$

(3.34)

4 We urge the reader not to confuse this notation with the one introduced in (2.11)
The \( K \rightarrow +\infty \) limit of \( \| S_N[F_K] \|_{L^2_{\text{sym}}(\mathbb{R}^N \times \mathbb{R}^N, dq(w_N) \cdot dx)} \)

We first note that the kernel of \( S_N \) obviously contains partially antisymmetric functions. Hence, we can drop the symmetrization operator when inserting the function \( (3.26) \) into the multiple integral \( (3.24) \). Further, when evaluating the \( L^2 \) norm of \( S_N[F_K] \), the integrals in respect to \( x \) can be taken due to the isometric character of the Fourier transform which allows one to recast the second representation for the norm as

\[
N^2[F_K] = \lim_{\alpha \to 0^+} \int d\mu(w_N) \cdot \left\{ \int_{\mathbb{R}^N} \prod_{a=1}^N \prod_{b=1}^{N+1} \left( \frac{y_b - w_a + i\alpha}{ih} \right) \cdot \prod_{a \neq b}^{N+1} \left( \frac{y_b - y_a}{ih} \right) \cdot F_K(y_{N+1}) \cdot \frac{d^{N+1}y}{2\pi h} \right\}
\times \lim_{\alpha \to 0^+} \left\{ \int \prod_{a=1}^N \prod_{b=1}^{N+1} \left( \frac{w_a - y_b + i\alpha}{ih} \right) \cdot \prod_{a \neq b}^{N+1} \left( \frac{y_b - y_a}{ih} \right) \cdot F_K(y'_{N+1}) \cdot \frac{d^{N}y'}{2\pi h} \right\}.
\tag{3.35}
\]

Above, the symbol \( \overline{y}_{N+1} = \overline{y}'_{N+1} \) means that one should replace \( y'_{N+1} = \overline{y}_{N+1} = \overline{y}'_{N+1} \) in the part of the integrand depending on \( y'_{N+1} \).

Next, we implement the change of variables \( y_{N+1} = s + K - \overline{y}_N \) (which also imposes the relation \( y'_{N+1} = s + K - \overline{y}'_N \)), pull out the \( \alpha \to 0^+ \) limit outside of the integrals and exchange the orders of integrations. Again, the justification for being able to do so basically parallels the proof of lemma \( (3.1) \). Thus, we solely sketch the main steps. One should first take the \( \alpha \to 0^+ \) limit in each of the integrals, \( ie \) represent each integrand in the spirit of \( (3.9) \), integrate by parts in respect to the relevant variables so as to "regularize" the singular part of the integrand and explicitly ensure the convergence (in respect to the \( w_N \) variables) at \( \infty \) (cf \( 3.15 \)). At that stage it is already possible to invoke Fubini's theorem so as to exchange the orders of integration. Further, one is then able to apply the dominated convergence theorem so as to recast the integral as an \( \alpha \to 0^+ \) limit. However, this time, the \( \alpha \to 0^+ \) limit symbol is outside of all the integration symbols. It then solely remains to "undo" all of the integrations by parts in respect to the \( y_N \) and \( y'_N \) variables. This last step is licit in that, for any \( \alpha > 0 \), the integral over \( w_N \) converges strongly. All in all, one obtains:

\[
N^2[F_K] = \lim_{\alpha \to 0^+} \int d^N y \cdot G(y_N) \int d^N y' \cdot G'(y'_N) \int \frac{ds}{2\pi h} |u(s)|^2 \prod_{a \neq b}^{N} \left( \frac{y_b - y_a}{ih} \right) \cdot L^2_K(s, y_N, y'_N),
\tag{3.36}
\]

where \( L^2_K \) is given by the \( N \)-dimensional integral

\[
L^2_K(s, y_N, y'_N) = \prod_{a=1}^N \left( \frac{y_b - w_a + i\alpha}{ih} \cdot \frac{w_a - y'_b + i\alpha}{ih} \right) \cdot \mathcal{J}^2_K(s; w_N, y_N, y'_N) \cdot d\mu(w_N).
\tag{3.37}
\]

The measure \( d\mu(w_N) \) has been already introduced in \( (3.5) \) whereas

\[
\mathcal{J}^2_K(s; w_N, y_N, y'_N) = e^{-\frac{1}{2} (2s + \overline{y}_N + \overline{y}'_N)} e^{-\frac{1}{2} \overline{y}_N \cdot \Gamma^N \left( 1 + \frac{s + K}{ih}, 1 - \frac{s + K}{ih} \right)} \cdot \left( -\frac{s + K}{ih} - 1 \right)^{-N} \cdot \left( \frac{s + K}{ih} - 1 \right)^{-N}
\times \left( \frac{s + K + i}{\hbar} \right)^{\overline{y}_N} \left( \frac{s + K - i}{\hbar} \right)^{-\overline{y}_N} \cdot \prod_{a=1}^N \left( \frac{w_a - s - \overline{y}_N + i\alpha}{ih} \cdot \frac{s + K - \overline{y}_N - w_a + i\alpha}{ih} \right)
\times \prod_{a=1}^N \left( \frac{y'_a + \overline{y}_N - s - K}{ih}, \frac{s + K - y'_a - \overline{y}_N}{ih}, \frac{y_a + \overline{y}_N - s - K}{ih}, \frac{s + K - y_a - \overline{y}_N}{ih} \right).
\tag{3.38}
\]
Because of the zeroes of the measure’s density\footnote{Here, we agree upon the shorthand notation $y_{N+1} = s - y_N + k$}, doing so, one will cross, individually in each variable so that the only poles that are crossed correspond to those lying closest to the real axis. Hence, in the course of belonging to compact subsets of $\mathbb{R}^{3N+1}$.

This observation is however not enough so as to conclude. Indeed, the integrand is expressed in a too singular way so as to be in position of invoking the dominated convergence theorem. Having this in mind, we implement several regularizing steps, allowing one, in the very end, to apply the latter theorem.

We start by deforming the contours in (3.37) from $\mathbb{R}^N$ up to ($\mathbb{R} + i\eta)^N$ where $\eta > \alpha$ is taken small enough so that the only poles that are crossed correspond to those lying closest to the real axis. Hence, in the course of doing so, one will cross, individually in each variable $w_a$, the poles at $w_a = y_b + ia$, with $b = 1, \ldots, N + 1$. Because of the zeroes of the measure’s density $\mu(w_N)$, poles corresponding to two or more coinciding coordinates (ie such that $w_a = w_b$) have zero residue. The evaluation of the residues that result from such a handling can be further simplified by using that the integrand is symmetric in $w_N$. It is enough to choose the last $k$ coordinates, $k \in \{1, \ldots, N\}$, of $w_N$ as those which will correspond to residue evaluation, weight it with the combinatorial factor $C_N^k$ and then sum up over all possible choices $k = 0, \ldots, N$. All in all, this yields

$$L_K^{(\alpha)}(s, y_N, y'_N) = \sum_{k=0}^{N} C_N^k \int_{(\mathbb{R} + i\eta)^N-k} \left\{ \prod_{a=1}^{N-k} \prod_{b=1}^{N} \left\{ \Gamma \left( \frac{y_b - w_a + i\alpha}{i\hbar}, \frac{w_a - y'_b + i\alpha}{i\hbar} \right) \right\} \sum_{p_1 \neq \cdots \neq p_k} (2i\alpha)^k \right\} ,$$

$$\text{Res} \left( \mu(w_{N-k}, x_k) \int_{(\mathbb{R} + i\eta)^N-k} \left\{ \prod_{b=1}^{N-k} \prod_{\ell=1}^{N} \left\{ \Gamma \left( \frac{y_b - x_\ell + i\alpha}{i\hbar}, \frac{x_\ell - y'_b + i\alpha}{i\hbar} \right) \right\} \right\} J_K^{(\alpha)}(s; (w_{N-k}, x_k), y_N, y'_N) \cdot d^N x \cdot x_a = y_{pa} + i\alpha \right\} ,$$

The above sum can be further simplified thanks to the effective symmetry of the function $L_K^{(\alpha)}$. Indeed, we are eventually interested in integrating it versus a symmetric function of $y_N$ in (3.36). It thus means that for our purpose, one can simplify further the expression for $L_K^{(\alpha)}$ by using the permutation invariance in respect to the components of $y_N$.

Then $k = 0$ term in the above sum is given by a $N$-dimensional integral over $\mathbb{R} + i\eta$ and it can be left as such. However, when $k \neq 0$, one should prepare the corresponding expression; we partition the sums over the indices $p_1 \neq \cdots \neq p_k$ depending on whether the residue at $y_{N+1} + i\alpha$ has been computed or not.

- Suppose that for any $a = 1, \ldots, k$ one has $p_a \neq N + 1$. There are, in total, $N!/(N-k)!$ such possible choices of the different configurations of the pairwise distinct integers ($p_1, \ldots, p_k$). Due to the aforementioned freedom of permuting the coordinates of $y_N$, it is enough to weight by $N!/(N-k)!$ the contribution of the configuration $p_a = a$, with $a = 1, \ldots, k$. viz. the one corresponding to $x_a = y_a + i\alpha$, $a = 1, \ldots, k$.

- Suppose that there is an $a \in \{1, \ldots, k\}$ such that $p_a = N + 1$. Then, the other ones necessarily take values in $1, \ldots, N$. In virtue of the symmetry of the integrand in the $x$’s and, again of the permutation freedom of the coordinates of $y_N$, it is enough to take into account the contribution of the configuration $p_a = a$ for $a = 1, \ldots, k - 1$ and $p_k = N + 1$, this weighted by the factor $k \cdot N!/(N-k+1)!$.

Hence, one has

$$L_K^{(\alpha)}(s, y_N, y'_N) \approx L_K^{(\alpha)}(s, y_N, y'_N) + \sum_{k=1}^{N} \frac{N! C_N^k}{(N-k)!} \cdot \left[ L_K^{(\alpha)}(s, y_N, y'_N) + \frac{k}{N-k+1} L_{k}^{(\alpha)}(s, y_N, y'_N) \right] .$$

(3.41)
where \( \approx \) means that the \( \text{lhs} \) and \( \text{rhs} \) give the same result when integrated versus a symmetric function of \( y_N \). Also, in (3.41), we have introduced
\[
\mathcal{L}_{K,0}^{(a)}(s, y_N, y'_N) = \int_{(\mathbb{R}+i)^N} \mathcal{V}_0(w_N, y_N + i\alpha e_N, y'_N - i\alpha e_N) \cdot \mathcal{J}_{K}^{(a)}(s; w_N, y_N, y'_N) \cdot d\mu(w_N) \tag{3.42}
\]
in which we do remind the notation \( e_k = (1, \ldots, 1) \in \mathbb{R}^k \) and have set
\[
\mathcal{V}_\ell(w_p, y_N, y'_N) = e^{\pi \theta \ell} \cdot \prod_{a=1}^{p} \prod_{b=\ell+1}^{N} \left\{ \Gamma \left( \frac{y_b - w_a}{ih} \right) \right\} \times \prod_{a=1}^{p} \prod_{b=\ell+1}^{N} \left\{ \Gamma \left( \frac{w_a - y_b}{ih} \right) \right\} \tag{3.43}
\]
Further
\[
\mathcal{L}_{K,k}^{(a)}(s, y_N, y'_N) = \prod_{b=1}^{N} \prod_{l=1}^{k} \left\{ \Gamma \left( \frac{y_l - y'_b + 2i\alpha}{ih} \right) \right\} \times \prod_{b=k+1}^{N} \prod_{l=1}^{N} \left\{ \Gamma \left( \frac{y_b - y_l}{ih} \right) \right\}
\]
\[
\times \int_{(\mathbb{R}+i)^{N-k}} \mathcal{V}_k(w_{N-k}, y_N + i\alpha e_N, y'_N - i\alpha e_N) \cdot \mathcal{J}_{K}^{(a)}(s; w_{N-k}, y_k + i\alpha e_k, y_N, y'_N) \cdot d\mu(w_{N-k}) \tag{3.44}
\]
Note that, since \( s, y_N \) and \( y'_N \) all belong to compact sets (and are thus bounded), for \( K \) large enough, the \( \alpha \rightarrow 0^+ \) limit of \( \mathcal{J}_{K}^{(a)}(s; w_{N-k}, y_k + i\alpha e_k, y_N, y'_N) \) exists and defines a smooth function. Finally,
\[
\mathcal{L}_{K,k}^{(a)}(s, y_N, y'_N) = \Gamma \left( \frac{y_N - \bar{y}_N + 2i\alpha}{ih} \right) \cdot \prod_{l=1}^{k-1} \Gamma \left( \frac{y_l - y'_b + 2i\alpha}{ih} \right) \cdot \prod_{b=k+1}^{N} \Gamma \left( \frac{y_b - y_N}{ih} \right)
\]
\[
\times \int_{(\mathbb{R}+i)^{N-k}} \mathcal{V}_{k-1}(w_{N-k}, y_N + i\alpha e_N, y'_N - i\alpha e_N) \cdot \mathcal{J}_{K}^{(a)}(s; w_{N-k}, y_{k-1} + i\alpha e_{k-1}, y_N, y'_N) \cdot d\mu(w_{N-k}) \tag{3.45}
\]
where
\[
\mathcal{J}_{K,k}^{(a)}(s, w_{N-k}, y_N, y'_N) = \prod_{b=1}^{k-1} \left\{ \Gamma \left( \frac{s + K - y_b - \bar{y}_N}{ih} \right) \cdot \Gamma \left( \frac{s + K - y'_b - \bar{y}_N + 2i\alpha}{ih} \right) \right\}
\]
\[
\times \prod_{l=1}^{N-k} \left\{ \Gamma \left( \frac{y_l + \bar{y}_N - s - K}{ih} \right) \cdot \Gamma \left( \frac{s + y'_b - \bar{y}_N + 2i\alpha}{ih} \right) \right\}
\]
\[
\times \prod_{a=1}^{N-k} \Gamma \left( \frac{w_a - s - (K + \bar{y}_N - i\alpha)}{ih} \right) \cdot \Gamma \left( \frac{s + K - w_a - \bar{y}_N + i\alpha}{ih} \right) \cdot \mathcal{J}_{K}^{(a)}(s; (w_{N-k}, y_{k-1} + i\alpha e_{k-1}), y_N, y'_N) \tag{3.46}
\]
The decomposition (3.41), implies the associated decomposition for the norm:
\[
N^2[F_K] = N_{K,0}^2 + \sum_{p=1}^{N} \frac{N!C_p^N}{(N-p)!} \left\{ N_{K,p}^2 + \frac{p \cdot N^2}{N-p+1} \right\}. \tag{3.47}
\]
We now study the \( K \rightarrow +\infty \) of each of these terms separately.
• Convergence of $N_{K,0}$

After some algebra, straightforwardly justifiable exchanges of orders of integration and a two-fold integration by parts, $N_{K,0}$ can be recast as

$$N_{K,0}^2 = \frac{N!}{\mathcal{G}} \longleftarrow \mathcal{G} \left[ J_K^{(0)}(w_N) \right] \cdot \left( -i \hbar^{-1} \ln \left| \frac{i w_N}{\hbar} \cdot \frac{K}{K - w_N} \right| \right)^{-2} \cdot d\mu(w_N)$$

(3.48)

in which the operator $\mathcal{G}$ is given by the below integral representation

$$\mathcal{G} \left[ J_K^{(0)}(w_N) \right] = \int_{\mathbb{R}^N} d^N y \int_{\mathbb{R}^N} d^N y' \int_{\mathbb{R}} \frac{d\xi}{2\pi \hbar} |u(s)|^2 G' (y_N') \cdot \left( \frac{i w_N}{\hbar} \cdot \frac{K}{K - w_N} \right)^{-2} \cdot \prod_{a \neq b}^{N} \left[ \Gamma^{-1} \left( \frac{y_b' - y_a'}{i\hbar} \right) \right] \cdot \mathcal{G} \left[ J_K^{(0)}(s; w_N, y_N, y_N') \right].$$

It is readily seen that for $w_N \in (\mathbb{R} + i\eta)^N$ and $(s, y_N, y_N')$ belonging to a compact subset of $\mathbb{R}^{2N + 1}$, one has

$$J_K^{(0)}(s; w_N, y_N, y_N') = M_K(w_N) \cdot \prod_{a=1}^{N} \left\{ \left( \frac{K - w_a}{K} \right)^{y_{\eta} - y_a} \cdot \exp \left\{ \frac{2\pi}{\hbar} (s - y_N) \right\} \right\} \cdot \left( 1 + O(K^{-1} \max_a \left| w_a - K \right|^{-1}) \right)$$

(3.49)

with $M_K(w_N)$ being given by

$$M_K(w_N) = e^{-\frac{\pi}{\hbar} y_N} \cdot \prod_{a=1}^{N} \left\{ \left( \frac{K - w_a}{K} \right)^{y_{\eta} - y_a} \cdot \Gamma^{-1} \left( \frac{K}{\hbar} \right) \right\}.$$  

(3.50)

It is readily seen that

$$M_K(w_N) \xrightarrow{K \to +\infty} 1 \quad \text{pointwise in } w_N \in (\mathbb{R} + i\eta)^N.$$  

(3.51)

Thus, $J_K^{(0)}$ converges to 1, uniformly in $(s, y_N, y_N')$ and pointwise in $w_N \in (\mathbb{R} + i\eta)^N$ in the $K \to +\infty$ limit. As a consequence

$$\lim_{K \to +\infty} \mathcal{G} \left[ J_K^{(0)}(w_N) \right] = \mathcal{G} \left[ 1(w_N) \right]$$

(3.52)

pointwise in $w_N$.

One needs however sharper bounds so as to apply the dominated convergence theorem to integrals involving $\mathcal{G} \left[ J_K(w_N) \right]$. Recall that given $\epsilon > 0$ and $(x, y) \in \mathbb{R}^2$ such that $|x + iy| > \epsilon$ and $|x| < 1/\epsilon$, one has the uniform bound

$$|\Gamma(x + iy)| \leq C \cdot |x + iy|^{-\frac{1}{2}} e^{-\pi|y|}$$

(3.53)

for some constant $C > 0$ depending on the choice of $\epsilon$. Thus, since $\Im(w_a) = \eta$, it follows that

$$|M_K(w_N)| \leq C' \cdot \prod_{a=1}^{N} \frac{K}{|w_a - K|}$$

(3.54)
for some constant $C' > 0$.

This bound along with \((3.49)\) and the remainder’s differentiable uniformness in \((y_N, y'_N)\) implies that there exists a constant \(\widetilde{C} > 0\) such that uniformly in \(K\) large, \(w_N \in (R + i\eta)^N\) and \((s, y_N, y'_N)\) belonging to a compact of \(\mathbb{R}^{2N+1}\), one has

\[
\max_{\ell=0,1,2} \left| \frac{d^\ell}{ds^\ell} \left( \frac{K}{K - w_N} \right)^{\frac{1}{2}} \mathcal{I}_K^{(0)}(s, w_N, y_N, y'_N) \right| \leq \widetilde{C} \prod_{a=1}^N \left| \frac{K}{w_a - K} \right|.
\]  

(3.55)

Furthermore, one also has

\[
\prod_{a,b=1}^N \left\{ \Gamma \left( \frac{y_b - w_a}{ih}, \frac{w_a - y'_b}{ih} \right) \right\} = \prod_{a=1}^N \Gamma (N) \prod_{a=1}^N \frac{(iw_a)^{\frac{K}{2}}}{(ih)^{\frac{N}{2}}} \cdot \left( \frac{w_a}{ih} \right)^{\frac{N}{2}} \cdot \left( 1 + O(\max_{a} |w_a|^{-1}) \right),
\]  

(3.56)

with a uniformly differentiable remainder in respect to \((y_N, y'_N)\) belonging to compact subsets of \(\mathbb{R}^{2N}\).

As a consequence, taking into account that the integration runs through the compact support of \(u(s)G(y_N)G^*(y'_N)\), one gets that, for some other constant \(\widetilde{C}\), given that \(\mathcal{R}(w_1) < \cdots < \mathcal{R}(w_N)\)

\[
\left| \left( \ln \left( \frac{\Gamma (\frac{y_b - w_a}{ih}, \frac{w_a - y'_b}{ih})}{\Gamma (\frac{K}{h}, \frac{K - w_N}{ih})} \right) \right)^{-2} \cdot \mathcal{I}(\mathcal{K}^{(0)})(w_N) \cdot \mathcal{I}(\mathcal{K}^{(0)})(w_N) \right| \leq \widetilde{C} \prod_{a,b=1}^N \left( \frac{\mathcal{R}(w_a) - \mathcal{R}(w_b)}{|w_a w_b|} \right) \times \prod_{a=1}^{N-1} \exp \left( \frac{\pi}{2} \mathcal{R}(w_a)(2a - N) - \mathcal{R}(w_a) \right) \cdot \prod_{a=1}^N \left( K(1 + \ln^2 |w_a|) \right) \cdot \frac{K}{|w_a(K - w_a)| \ln^2 |w_a|} \leq g_K(w_N)
\]

where the sequence \(g_K\) reads

\[
g_K(w_N) = C'' \prod_{a=1}^{N-1} \left\{ e^{-\pi[i\mathcal{K}(w_a)]} \right\} \cdot \frac{K}{|w_N(K - w_N)| \ln^2 |w_N|}.
\]  

(3.57)

and \(C'' > 0\) is some constant.

It follows from lemma [C.1] that the sequence \(g_K\) satisfies

\[
\lim_{K \to +\infty} \int_{(R+i\eta)^N} g_K(w_N) \cdot d^N w = \int_{(R+i\eta)^N} \left( \lim_{K \to +\infty} g_K(w_N) \right) \cdot d^N w.
\]  

(3.58)

Thus, by the dominated convergence theorem,

\[
\lim_{K \to +\infty} \mathcal{N}_{K,0}^2 = \int_{(R+i\eta)^N} d^N \mathcal{K}(w_N) \int_{(R+i\eta)^N} d^N y \int_{(R+i\eta)^N} d^N y' G^*(y'_N)G(y_N) \cdot \prod_{a,b=1}^N \left\{ \Gamma \left( \frac{y_b - w_a}{ih}, \frac{w_a - y'_b}{ih} \right) \right\} \prod_{a \neq b}^N \left\{ \Gamma \left( \frac{y_b - w_a}{ih}, \frac{w_a - y'_b}{ih} \right) \right\}.
\]  

(3.59)

\textbf{Convergence of} \(\mathcal{N}_{K,k}\)

It follows by expanding the singular part of the prefactor in \(L^{(a)}_{K,k}\) and then integrating the singular term by parts, that \(\mathcal{N}_{K,k}\) can be recast as

\[
\mathcal{N}_{K,k}^2 = \frac{N^1}{N - k + 1} \int_{(R+i\eta)^N} G_k \mathcal{I}_K^{(0)}(w_N - k) \cdot d^N \mathcal{K}(w_N - k).
\]  

(3.60)
where

\[
\mathcal{G}_k[J^{(0)}_K](w_{N-k}) = \int_{\mathbb{R}^N} \frac{d^N y}{\mathbb{R}^2} \int_{\mathbb{R}^N} d^N y' \int_{\mathbb{R}} d s \cdot \prod_{a=1}^k \left\{ \ln \left| y_a - y'_a \right| + i \pi \mathbb{1}_{\mathbb{R}^{+}}(y'_a - y_a) \right\} \\
\times \prod_{a=1}^k \frac{\partial}{\partial y'_a} \left( \widetilde{V}_k(s, w_{N-k}, y_N, y'_N) \cdot J^{(0)}_K(s, (w_{N-k}, y_k), y_N, y'_N) \right)
\]

(3.61)

with the function \( \widetilde{V}_k \) being given by

\[
\widetilde{V}_k(s, w_{N-k}, y_N, y'_N) = \frac{|u(s)|^2}{2\pi\hbar} G(y_N)G^*(y'_N) \prod_{a=1}^k \prod_{b=k+1}^N \left\{ \Gamma \left( \frac{y_b - y_a}{i\hbar} \right) \right\} \frac{1}{y'_a - y'_b} \prod_{b=1}^k \left\{ (y_b - y_a) \cdot (y'_b - y'_a) \right\} \times \prod_{a \neq b}^N \left( \Gamma^{-1} \left( \frac{y_b - y'_a}{i\hbar}, \frac{y'_b - y'_a}{i\hbar} \right) \right) \cdot \prod_{b=1}^k \left\{ i\hbar \left( \frac{y_a - y'_b}{i\hbar} + 1 \right) \right\} .
\]

(3.62)

The analysis of the \( K \to +\infty \) limit slightly differs depending on whether \( k = N \) or \( 1 \leq k \leq N - 1 \).

Suppose that \( k = N \), then it is readily seen from the uniform differentiability of the \( K \to +\infty \) remainder to \( J^{(0)}_K(s, y_N, y_N, y'_N) \) that

\[
\max_{A \subset [1 : N]} \left| \prod_{a \in A} \frac{\partial}{\partial y_a} \left[ J^{(0)}_K(s, y_N, y_N, y'_N) - 1 \right] \right| = 0
\]

(3.63)

and thus,

\[
\mathcal{G}_N[J^{(0)}_K] \longrightarrow \mathcal{G}_N[1] \quad i.e. \quad \lim_{K \to +\infty} N_{K,N} = N! \cdot \mathcal{G}_N[1].
\]

(3.64)

Now assume that \( 1 \leq k \leq N - 1 \). Then, we change the variables \( y'_N = v' - \overline{y}_{N-1} \) and integrate by parts the oscillatory asymptotic behaviour in \( w_{N-k} \). Agreeing upon

\[
v'_N = (y'_{N-1}, v' - \overline{y}_{N-1})
\]

(3.65)

we recast \( \mathcal{G}_k \) as

\[
\mathcal{G}_k[J^{(0)}_K](w_{N-k}) = \left\{ \begin{array}{ll}
-\frac{i}{\hbar} \ln \left( \frac{i\hbar N_k \cdot K}{\hbar(K - w_{N-k})} \right) & \text{if} \quad \overline{y}_N \neq \overline{y}_{N-1} \\
\left( \frac{i\hbar N_k \cdot K}{\hbar(K - w_{N-k})} \right)^2 & \text{if} \quad \overline{y}_N = \overline{y}_{N-1}
\end{array} \right.
\]

\[
\times \left( \frac{i\hbar N_k \cdot K}{\hbar(K - w_{N-k})} \right) \prod_{a=1}^k \frac{\partial}{\partial y'_a} \left( \frac{i\hbar N_k \cdot K}{\hbar(K - w_{N-k})} \right) \left( \frac{\partial}{\partial y_a} \right) \cdot \left( \frac{i\hbar N_k \cdot K}{\hbar(K - w_{N-k})} \right) \cdot \widetilde{V}_k(s, w_{N-k}, y_N, y'_N) J^{(0)}_K(s, (w_{N-k}, y_k), y_N, y'_N) .
\]

It then follows from the large \( w_a \) representation

\[
\mathcal{V}_k(w_{N-k}, y_N, y'_N) = e^{\frac{1}{\hbar} \sum_{a=1}^{N-k} \left( \frac{w_a}{\hbar} \cdot \overline{y}_a + \frac{\overline{y}_a}{\hbar} \cdot w_a \right)} \prod_{a=1}^k \left( \frac{w_a}{\hbar} \cdot \overline{y}'_a - \frac{\overline{y}_a}{\hbar} \cdot w_a \right) \cdot \left( 1 + \mathcal{O}(\text{max}|w_a|^{-1}) \right)
\]

(3.66)
that
\[
\max_{A \subseteq 1:k} \max_{\ell \in \{0;2\}} \left| \frac{\partial^\ell}{\partial y_N^\ell} \prod_{a \in A} \frac{\partial}{\partial y_a} \cdot \left\{ \left( i \frac{w_{N-k}}{\hbar} \right)^{\frac{2k}{2}} \mathcal{V}_k(s, w_{N-k}, y_N, v_N') \right\} \right|
\leq C \prod_{a=1}^{N-k-1} \left( 1 + \left| \ln |w_a| \right|^{2+k} \right) \cdot \prod_{a=1}^{N-k} \left( \frac{-w_a}{\hbar} \cdot \frac{w_a}{i\hbar} \right)^{N-k} . \tag{3.67}
\]
Likewise, it follows from (3.49) specialised to the case where \( w_N = (w_{N-k}, y_k) \) that
\[
\max_{A \subseteq 1:k} \max_{\ell \in \{0;2\}} \left| \frac{\partial^\ell}{\partial y_N^\ell} \prod_{a \in A} \frac{\partial}{\partial y_a} \cdot \left\{ \left( \frac{K}{K - w_{N-k}} \right)^{\frac{2k}{2}} : \mathcal{J}^{(0)}_K(s, (w_{N-k}, y_k), y_N, v_N') \right\} \right| \leq \tilde{C} \cdot \prod_{a=1}^{N-k} \left\{ \frac{K}{|w_a - K|} \right\} .
\]

Thus, by repeating the previously discussed schemes of majorations, we arrive to
\[
\left| \mathcal{G}_k[\mathcal{J}^{(0)}_K](w_{N-k}) \cdot \tilde{\mu}(w_{N-k}) \right| \leq \tilde{C}' \cdot \prod_{a=1}^{N-k-1} \left\{ K \cdot \left( 1 + \left| \ln |w_a| \right|^{2+k} \right) \cdot \frac{e^{-2 \frac{|R(w_a)|}{\hbar}}}{|w_a(K - w_a)|} \cdot \frac{K \cdot \left( \left| \ln |w_{N-k}| \right| \right)^{2} \left| w_{N-k}(K - w_{N-k}) \right|}{|w_{N-k}(K - w_{N-k})|} \right\} . \tag{3.68}
\]

The rhs of the last inequality does already fulfil the hypothesis on the dominant function in Lebesgue’s dominated convergence theorem, so that
\[
\lim_{k \to +\infty} N_{K,k}^2 = \frac{N!}{(N-k+1)!} \int_{(\mathbb{R}^N)^{N-k}} \mathcal{G}_k[1](w_N) \cdot d\tilde{\mu}(w_{N-k}) . \tag{3.69}
\]

- **Convergence of \( \mathcal{M}_{K,k} \)**

Upon repeating the aforementioned manipulations and carrying out the change of variables
\[
y_N = v - \overline{\mathcal{F}}_{N-1} \quad \text{and} \quad y_N' = v' - \overline{\mathcal{F}}_{N-1} \tag{3.70}
\]
it is readily seen that \( \mathcal{M}_{K,k} \) can be recast as
\[
\mathcal{M}_{K,k}^2 = N! \int_{\mathbb{R}(w_1) \cdots \mathbb{R}(w_{N-k})} \mathcal{G}_k[\mathcal{J}^{(0)}_{K,k}](w_N) \cdot d\tilde{\mu}(w_{N-k}) \tag{3.71}
\]
where
\[
\mathcal{G}_k[\mathcal{J}^{(0)}_{K,k}](w_{N-k}) = i\hbar \int_{\mathbb{R}^{N-1} \times \mathbb{R}^{N-1}} d^{N-1}y \cdot d\nu \int_{\mathbb{R}^{N-1} \times \mathbb{R}^{N-1}} d^{N-1}y' \cdot d\nu' \int_{\mathbb{R}} d\nu \int_{\mathbb{R}} d\nu' \sum_{a=1}^{k-1} \left\{ \ln |y_a - y'_a| + i\pi 1_{\mathbb{R}^+} (y'_a - y_a) \right\} \times \left\{ \ln |v - v'| + i\pi 1_{\mathbb{R}^+} (v' - v) \right\} \cdot \frac{\partial}{\partial y'_a} \prod_{a=1}^{k-1} \frac{\partial}{\partial y'_a} \cdot \left\{ \Gamma \left( \frac{v - v'}{i\hbar} + 1 \right) \mathcal{V}_{k-1}(s, w_{N-k}, v_N, y_N') \cdot \mathcal{J}^{(0)}_{K,k}(s, w_{N-k}, v_N, v_N') \right\} .
\]

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On this occasion, we do remind that
\[ v_N = (y_{N-1}, v - \bar{y}_{N-1}) \quad \text{and} \quad v'_N = (y'_{N-1}, v' - \bar{y}_{N-1}). \] (3.72)

The function \( J_{K,\varepsilon}^{(0)} \) exhibits the large \( K \) behaviour
\[
J_{K,\varepsilon}^{(0)}(s, w_{N-k}, v_N, v'_N) = \exp \left[ \frac{\pi}{\hbar} (N - k + 1)(v - s) - \frac{\pi}{2\hbar} (v + v' - 2\bar{y}_{K-1}) \right] 
\times \left( \frac{K}{\hbar} \right)^{v'-v} \cdot \hat{\Gamma} \left( -\frac{K}{\hbar}, \frac{K}{\hbar} \right) \cdot e^{-\frac{\pi \nu}{\hbar}} \prod_{a=1}^{N-k} \left( \frac{v_k - w_a}{K} \right)^{v_k - v} \left( 1 + O(K^{-1} + \max_a |w_a - K|^{-1}) \right) \] (3.73)

Furthermore, the behaviour of \( \bar{V}_{K-1} \) is given by (3.66)-(3.67), with the sole difference that one should replace in these expressions the variable \( \bar{y}_N \) by \( v \).

These informations along with the uniform differentiability of the remainders lead to the bound
\[
\left| \hat{\mu}(w_{N-k}) \frac{\partial}{\partial v'} \prod_{a=1}^{k-1} \frac{\partial}{\partial v_{a'}} \bigg( \left( \hat{\Gamma} \left( v - v' \right) + 1 \right) \bar{V}_{K-1}(s, w_{N-k}, v_N, v'_N) \cdot J_{K,\varepsilon}^{(0)}(s, w_{N-k}, v_N, v'_N) \bigg) \right| 
\leq C \cdot \hat{\Gamma} \left( -\frac{K}{\hbar}, \frac{K}{\hbar} \right) \cdot \prod_{a=b}^{N-k} \left( \frac{w_a - w_b}{w_a w_b} \right) \cdot \prod_{a=1}^{N-k} \left( |w_a|^{-2} (1 + |\ln w_a|) \right) \cdot \prod_{a=1}^{N-k} \left( 1 + \ln((K - w_a)/|K|) \right) 
\times \sum_{a=1}^{N-k} \left( e^{-\frac{\pi}{\hbar} \left( (N-k+1)|\Re(w_a)| - (2a-N+k+1)|\Re(w_a)| \right)} \right) \leq C' \cdot \hat{\Gamma} \left( -\frac{K}{\hbar}, \frac{K}{\hbar} \right) \prod_{a=1}^{N-k} \frac{1}{|w_a|^2}, \] (3.74)

for some constants \( C, C' > 0 \).

Taking into account that the \( d^N v \cdot d^N v' \) integration runs through the compact support of \( u(s)G(y_{N})G^*(y'_N) \), the above bound allow us to assert, in virtue of the dominated convergence theorem, that, independently of \( k \in \ll 1 : N \rr \),
\[
\lim_{k \to +\infty} N_{K,\varepsilon} = 0. \] (3.75)

In order to conclude, it is enough to carry out backwards, once that the \( K \to +\infty \) limits of interest have been taken, the chain of contour deformation that originally led to (3.47). The reverse chain of transformations leads to a representation for \( \lim_{K \to +\infty} N^2[F_K] \) that has the desired form. Indeed, all of our manipulations simply amount having set, from the very beginning, \( J_K^{(0)} = 1 \) in (3.37). We do stress that the role of the function \( J_K^{(0)} \) in generating pole contributions in the process of deforming the contours was passive, apart from generating poles at \( w_a = s + K - \bar{y}_N \). The latter have been shown to yield vanishing contributions in the \( K \to +\infty \) limit. Hence, setting \( J_K^{(0)} = 1 \) is not an obstruction to taking the reverse chain of transformations. Then, as soon as one sets \( J_K^{(0)} = 1 \) in (3.37), one can readily permute the orders of integrations in (3.36) and reconstruct the product of two \( \mathcal{H} \) transforms leading to
\[
\lim_{K \to +\infty} N^2[F_K] = (2\pi\hbar)^{2N} \cdot (N!)^2 \cdot \left\| \mathcal{H} \right\|_{L^2_{\gamma} (\mathbb{R} \times \mathfrak{d}([w_N]))}^2 \cdot \left\| \mathcal{H} \right\|_{L^2_{\gamma} (\mathbb{R} \times \mathfrak{d}([w_N]))}^2 \cdot (3.76)
\]

A comparison of (3.76) and (3.34) yields the claim. \( \blacksquare \)
3.2 The integral transform $\mathcal{J}_N$

We now use the isometry of the $\mathcal{H}_N$-transform so as to establish the one of a transform $\mathcal{J}_N$. The latter is already directly of interest to the problem. The results of the present section imply the isometricity of $\mathcal{V}_N$.

Proposition 3.2 Let $R \in \mathcal{C}_c^{\infty}(\mathbb{R}^{N-1} \times \mathbb{R})$, then the integral transform

$$
\mathcal{J}_N[R](\omega_N) = \lim_{a \to 0^+} \int_{\mathbb{R}^{N-1}} \frac{d^{N-1}y}{(2\pi \hbar)^{N-1}} \int_{\mathbb{R}} \frac{\epsilon N \cdot R(y_{N-1}, x)}{\sqrt{N} \cdot (N-1)!} \prod_{a \neq b}^{N-1} \Gamma^{-1}(\frac{y_a - y_b}{\hbar}) \prod_{a=1}^{N-1} \prod_{b=1}^{N-1} \Gamma\left(\frac{y_b - w_a + i\alpha}{\hbar}\right) (3.77)
$$

belongs to $\mathcal{S}(\mathbb{R}^N) \cap L^2_{sym}(\mathbb{R}^N, d\mu(\omega_N))$ and satisfies the identity

$$
\left\| \mathcal{J}_N[R] \right\|_{L^2_{sym}(\mathbb{R}^N, d\mu(\omega_N))} = \| R \|_{L^2_{sym}(\mathbb{R}^{N-1} \times \mathbb{R}, d\mu(\omega_{N-1}) \otimes dx)} (3.78)
$$

The isometric identity (3.78) allows one to raise $\mathcal{J}_N$ into an isometric operator

$$
\mathcal{J}_N : L^2_{sym}(\mathbb{R}^{N-1} \times \mathbb{R}, d\mu(\omega_{N-1}) \otimes dx) \to L^2_{sym}(\mathbb{R}^N, d\mu(\omega_N)). (3.79)
$$

Proof —

Given an $R \in \mathcal{C}_c^{\infty}(\mathbb{R}^{N-1} \times \mathbb{R})$, the $\mathcal{S}(\mathbb{R}^{N-1} \times \mathbb{R}) \cap L^2_{sym}(\mathbb{R}^N, d\mu(\omega_N))$ nature of the transform is established with the help of technique discussed in the course of the analysis of the $\mathcal{H}_N$ transform. Accordingly, we shall not reproduce these arguments once again.

Let $R \in \mathcal{C}_c^{\infty}(\mathbb{R}^{N-1} \times \mathbb{R})$ and $G_K$ be defined as

$$
G_K(\omega_N) = \text{Sym}\left\{ R(\omega_{N-1}, \ln(-\omega_N/K)) \cdot \frac{1_{\mathbb{R}^+}(\omega_N)}{\omega_N} \cdot e^{-i\pi N^{-1} \cdot \Gamma^{-1}(\frac{-\omega_N}{\hbar})} \cdot \Gamma^{-1}(\frac{\omega_N}{\hbar}) \right\} (3.80)
$$

where $1_A$ stands for the indicator function of the set $A$. It is then readily seen that $G_K \in \mathcal{C}_c^{\infty}(\mathbb{R}^N)$. It follows from the proposition [3.1] that

$$
\mathcal{N}[G_K] = \| G_K \|_{L^2_{sym}(\mathbb{R}^N, d\mu(\omega_N))} = \| \mathcal{H}_N[G_K] \|_{L^2_{sym}(\mathbb{R}^N, d\mu(\omega_N))}. (3.81)
$$

Just as in the proof of that proposition, we use this equality so as to estimate the $K \to +\infty$ limit of $\mathcal{N}[G_K]$ in two different ways. These estimations will then allow us to extract the isometricity relation (3.78).

First, we focus on the $L^2$-norm of $G_K$. Then, in virtue of the symmetry of the function $R$, we have the decomposition

$$
G_K(\omega_N) = \frac{1}{N} \sum_{p=1}^{N} R(\omega_{N-1}, \ln(-\omega_p/K)) \cdot \frac{1_{\mathbb{R}^+}(\omega_p)}{\omega_p} \cdot e^{-i\pi N^{-1} \cdot \Gamma^{-1}(\frac{-\omega_p}{\hbar})} \cdot \Gamma^{-1}(\frac{\omega_p}{\hbar}) (3.82)
$$

where

$$
\omega_{N-1} = (y_1, \ldots, y_{p-1}, y_{p+1}, \ldots, y_N). (3.83)
$$

The "off-diagonal" products in $|G_K(\omega_N)|^2$ will involve $R(\omega_{N-1}, \ln(-\omega_p/K)) \cdot (R(\omega_{N-1}, \ln(-\omega_j/K)))^* \text{ with } p \neq j$. However, since $R$ is compactly supported, it follows that there exists compacts $L \subseteq \mathbb{R}^{N-1}$ and $J \subseteq \mathbb{R}$ such that $R(\omega_{N-1}, x) = 0$ if either $\omega_{N-1} \notin L$ or $x \notin J$. Yet, $\ln(-\omega_p/K) \in J$ imples that $\omega_p \in Ke^{-J}$. Observe that, for $K$ large
enough, since \(0 \not\in \mathcal{C}^{-j}\), should \(y_j, y_p \in Ke^{-j}\) with \(j \neq p\), then necessarily \(y^{(p)}_{N-1, x} \not\in L\). As a consequence, then, 
\[ R(y^{(p)}_{N-1, x}, \ln(-y_p/K)) = 0 \] 
and the corresponding term does not contribute to the norm. All in all, this means

\[
\tilde{N}^2[G_K] = \frac{1}{N} \int_{\mathbb{R}^{N-1} \times \mathbb{R}} |R(y_{N-1, x})|^2 \cdot T_K^{(1)}(y_{N-1, x}) \cdot d\mu(y_{N-1}) \otimes \frac{d\mu}{(2\pi \hbar)^2} 
\]  
(3.84)

in which

\[
T_K^{(1)}(y_{N-1, x}) = \frac{2\pi \hbar}{Ke^{x}} \cdot e^{-\frac{\pi}{\hbar}(Ke^{x} + \bar{y}_{N-1})} \cdot \prod_{a=1}^{N-1} \left\{ \frac{\Gamma(-Ke^{x}/i\hbar, Ke^{x}/i\hbar)}{\Gamma(-Ke^{x}/i\hbar, Ke^{x}/i\hbar)} \right\}.
\]  
(3.85)

It is readily seen that, uniformly in \((y_{N-1, x})\) belonging to compact subsets of \(\mathbb{R}\),

\[
\lim_{K \to +\infty} T_K^{(1)}(y_{N-1, x}) = 1.
\]  
(3.86)

Thus, by the dominated convergence theorem,

\[
\lim_{K \to +\infty} \tilde{N}[G_K] = \frac{1}{\sqrt{N}} \cdot \frac{||R||_{\mathcal{L}_{\text{sym}}(\mathbb{R}^{N-1} \times \mathbb{R}, d\mu(y_{N-1}) \otimes d\mu)} .
\]  
(3.87)

We now estimate the same limit while using the second representation for \(N[G_K]\). Straightforward calculations based on the previously introduced ideas then lead to

\[
\tilde{N}^2[G_K] = \lim_{a \to 0^+} \int_{\mathbb{R}^N} d\mu(\omega_N) \int_{\mathbb{R}^{N-1} \times \mathbb{R}} d^{N-1}y \otimes d\nu \frac{R(y_{N-1, x})e^{i\bar{y}_{N-1}}}{N!(2\pi \hbar)^N} \prod_{a=1}^{N-1} \left\{ \Gamma(\frac{y_{N-1, x}}{\hbar}, \frac{y_{N-1, x}}{\hbar}) \right\} \prod_{a=1}^{N-1} \left\{ \Gamma(\frac{y_{N-1, x} - w_a + i\alpha}{\hbar}) \right\}
\times \int_{\mathbb{R}^{N-1} \times \mathbb{R}} d^{N-1}y' \otimes d\nu' \frac{R(y_{N-1, x}, y') e^{i\bar{y}_{N-1}x}}{N!(2\pi \hbar)^N} \prod_{a=1}^{N-1} \left\{ \Gamma(\frac{y_{N-1, x} - w_a + i\alpha}{\hbar}) \right\} \prod_{a=1}^{N-1} \left\{ \Gamma(\frac{y_{N-1, x} - w_a + i\alpha}{\hbar}) \right\} \]  
(3.88)

where, this time, we have set

\[
T_{K,0}^{(2)}(x, x', y_{N-1}, y'_{N-1} | \omega_N) = e^{-\frac{\pi}{\hbar}(e^{x} + e^{x'} + \bar{y}_{N-1} - i\alpha)} \prod_{a=1}^{N} \left\{ \begin{array}{c} (iKe^{x} + w_a - i\alpha)/\hbar, \\
-i(Ke^{x} + w_a + i\alpha)/\hbar, \\
iKe^{x}/\hbar, \\
iKe^{x}/\hbar \end{array} \right\}
\times e^{-\frac{\pi}{\hbar}(e^{x} + e^{x'} + \bar{y}_{N-1})} \prod_{a=1}^{N-1} \left\{ \begin{array}{c} iKe^{x}/\hbar, \\
iKe^{x}/\hbar, \\
iKe^{x}/\hbar, \\
iKe^{x}/\hbar \end{array} \right\}.
\]  
(3.89)

A straightforward computation shows that uniformly in \((y_{N-1, x}), (y'_{N-1, x})\) and \(w_N\) belonging to compact subsets of \(\mathbb{R}^N\), one has

\[
\lim_{K \to +\infty} T_{K,0}^{(2)}(x, x', y_{N-1}, y'_{N-1} | \omega_N) = 1.
\]  
(3.90)

It then remains to repeat the handleings outlined in the course of the proof of proposition \(3.1\) so as to show that this type of convergence is, in fact, enough so as to take the \(K \to +\infty\) limit under the integral sign. Since these are basically the same, we do not reproduce them here again. One thus gets

\[
\lim_{K \to +\infty} \tilde{N}[G_K] = \frac{1}{\sqrt{N}} \cdot \frac{||\mathcal{J}[R]||_{\mathcal{L}_{\text{sym}}(\mathbb{R}^N, d\mu(\omega_N))}}.
\]  
(3.91)

Equations (3.87) and (3.91) put together lead to the claimed identity. \(\Box\)
3.3 Isometric character of the transform $\mathcal{V}_N$

**Theorem 3.1** The transform $\mathcal{V}_N$ defined through (3.2) is such that given any

$$F \in C^\infty_c(\mathbb{R}^N), \quad \mathcal{V}_N[F] \in \mathcal{S}(\mathbb{R}^N) \cap L^2_{\text{sym}}(\mathbb{R}^N, d\mu(y_N)).$$

(3.92)

Furthermore, one has

$$\left\| \mathcal{V}_N[F] \right\|_{L^2_{\text{sym}}(\mathbb{R}^N, d\mu(y_N))} = \left\| F \right\|_{L^2(\mathbb{R}^N, d^N\lambda)}.$$  

(3.93)

As a consequence, $\mathcal{V}_N$ extends to an isometric operator

$$\mathcal{V}_N : L^2(\mathbb{R}^N, d^N\lambda) \to L^2_{\text{sym}}(\mathbb{R}^N, d\mu(y_N)).$$

(3.94)

This property, written formally, amounts to the so-called completeness of the system $\{\varphi_{y_N}(x_N)\}$, cf. (0.9).

**Proof** —

It follows from the recurrence relation (1.1) satisfied by the functions $\varphi_{y_N}(x_N)$ that $\mathcal{V}_N[F]$ can be recast as

$$\mathcal{V}_N[F](y_N) = \frac{i}{\sqrt{2\pi}}(N-1)! \lim_{\alpha \to 0^+} \int_{\mathbb{R}^{N-1} \times \mathbb{R}} \frac{d\mu((w_{N-1}, x))}{\sqrt{N} \cdot (N-1)!} e^{i \sqrt{N} \cdot (N-1)!} \mathcal{V}_{N-1}[F](x, w_{N-1})$$

$$= \frac{i}{\sqrt{2\pi}}(N-1)! \lim_{\alpha \to 0^+} \frac{1}{\sqrt{N}} \mathcal{F}_N[\mathcal{V}_{N-1}[F](x, w_{N-1})](y_N),$$

(3.95)

where $*$ indicates the couple of variables of the function $F$ on which the transform $\mathcal{V}_{N-1}$ acts and the $\mathcal{F}_N$ transform is understood to act on the variables $(w_{N-1}, x)$. The latter is defined as

$$\mathcal{F}_N[R](w_N) = \lim_{\alpha \to 0^+} \int_{\mathbb{R}^{N-1}} \frac{d^{N-1}y}{(2\pi\hbar)^{N-1}} \int_{\mathbb{R}} dx \cdot e^{i \sqrt{N} \cdot (N-1)!} \mathcal{V}_{N-1}[R(y_{N-1}, x)]$$

(3.96)

$$\mathcal{F}_N[R](w_N) = \lim_{\alpha \to 0^+} \int_{\mathbb{R}^{N-1}} \frac{d^{N-1}y}{(2\pi\hbar)^{N-1}} \int_{\mathbb{R}} dx \cdot e^{i \sqrt{N} \cdot (N-1)!} \mathcal{V}_{N-1}[R(y_{N-1}, x)].$$

(3.97)

Finally, the $\mathcal{V}_{N-1}$-transform is expressed, given any $G \in C^\infty_c(\mathbb{R}^{N-1})$, as

$$\mathcal{V}_{N-1}[G](x, w_{N-1}) = \frac{i}{\sqrt{2\pi}}(N-1)! e^{-i \sqrt{N} \cdot (N-1)!} \cdot \mathcal{V}_{N-1}[G](w_{N-1}).$$

(3.98)

Since the isometric nature of the $\mathcal{F}_N$-transform is equivalent to the one of the $\mathcal{V}_N$-transform, proposition [3.2] ensures that

$$\left\| \mathcal{V}_N[F] \right\|_{L^2_{\text{sym}}(\mathbb{R}^N, d\mu(y_N))} = \left\| \mathcal{V}_{N-1}[F(\cdot, x)] \right\|_{L^2_{\text{sym}}(\mathbb{R}^{N-1} \times \mathbb{R}, d\mu(y_{N-1})) \otimes d\lambda}.$$  

(3.99)

As a consequence, a straightforward induction leads to the claim.

**Conclusion**

In this paper we have developed a technique allowing one to prove the unitarity of the SoV transform in the case of integrable models with an infinite dimensional representation attached to each of its sites. Although we have developed the method on the example of the Toda chain, there do not seem to appear any obstruction to applying it to more complex models such as the lattice discretization of the sinh-Gordon model [4]. The original contribution
of this paper to the right invertibility of the transform, \( \mathcal{U}_N^+ \cdot \mathcal{U}_N = \text{id}_{\mathcal{L}^2(\mathbb{R}^N, d\varphi(y))} \) consisted in bringing
in several elements of rigour to the scheme invented in [5] and applied to the case of the Toda chain in [29]. However, the part relative to the left invertibility of the map \( \mathcal{U}_N \cdot \mathcal{U}_N^+ = \text{id}_{\mathcal{L}^2(\mathbb{R}^N, d\varphi(y))} \) was entirely based on brand new ideas. In the case of the Toda chain, our approach provides an alternative in respect to a purely group theoretic handling of the issue [14, 25, 28, 34]. On the one hand, our approach is much simpler as solely based on a direct calculation. On the other hand, our proof does not rely, at any stage, on the group theoretical interpretation of the Toda chain but solely on objects naturally arising in the framework of the quantum inverse scattering method approach to the quantum separation of variables. Hence, it will most probably work as well for other quantum integrable models through the separation of variables where the interpretation of the transform’s kernel in terms of a suitable Whittaker function does not exist. In a forthcoming publication, we plan to study the implementation of our method for proving the unitarity of the SoV transform to other quantum integrable models solvable by the quantum separation of variables.

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A Proof of proposition 1.1

The Mellin-Barnes multiple integral representation for \( \varphi_{y_N}(x_N) \) can be recast as

\[
\varphi_{y_N}(x_N) = e^{\pi N x_N y} \prod_{s=1}^{N-1} \int_{\mathbb{R}^{-i \alpha_s}^N} d^{N-s} y^{(s)} \prod_{s=1}^{N-1} \left( e^{\pi N x_N y^{(s)}} (x_N - y^{(s-1)}) \right) \cdot \prod_{b > a}^N \left( y_a - y_b \right) \times \prod_{s=1}^{N-1} \left\{ \prod_{b \neq a}^{N-s} \left( w_a^{(s)} - w_b^{(s)} \right) \right\} \cdot \mathcal{W}_N(\left( w^{(s)}_{N-s, 0} \right)^{N-1}), \quad (A.1)
\]

where \( \alpha_{N-1} > \cdots > \alpha_1 > 0 \). We also remind that \( w^{(0)}_N = y_N \) and agree upon

\[
\mathcal{W}_N(\left( w^{(s)}_{N-s, 0} \right)^{N-1}) = \prod_{b > a}^N \left( \frac{1}{y_a - y_b} \right) \prod_{s=1}^{N-1} \left( \frac{\varphi(\left( w^{(s)}_{N-s, 0} \right) | \left( w^{(s-1)}_{N-s, 0} \right)} \prod_{b \neq a}^{N-s} \left( w_a^{(s-1)} - w_b^{(s-1)} \right) \prod_{a \neq b}^{N-s} \left( w_a^{(s)} - w_b^{(s)} \right) \right) \right)^{(N-s)/(2\pi \hbar)^{N-s}}, \quad (A.2)
\]
It is a direct consequence of the identity \((3.8)\) that
\[
\prod_{b > a}^{N} (y_a - y_b) \cdot \prod_{s=1}^{N-s} \frac{\prod_{s'=b}^{N-s} (w_a^{(s')} - w_b^{(s')})}{\prod_{s'=1}^{N-s} (w_a^{(s')} - w_b^{(s')})} = \sum_{\sigma_s \in \mathfrak{S}_{N+1-s}} (-1)^{\sigma_s} \cdot \prod_{s=1}^{N-1} \prod_{a=1}^{N-s} \frac{1}{(w_a^{(s)}) - w_{\sigma_s(a)}^{(s)}}. \tag{A.3}
\]
Define the sequence of permutations \(\tau_s \in \mathfrak{S}_{N+1-s}\) as
\[
\tau_{N-1} = \sigma_{N-1} \quad \text{and} \quad \begin{cases} 
\tau_s(N + 1 - s) = \sigma_s(N + 1 - s) \\
\tau_s(a) = \sigma_s \circ \tau_{s+1} \quad \text{for} \quad s = 1, \ldots, N - 2,
\end{cases}
\tag{A.4}
\]
so that
\[
\prod_{a=1}^{N-s} \frac{1}{(w_a^{(s)}) - w_{\sigma(a)}^{(s)}} = \prod_{a=1}^{N-s} \frac{1}{(w_{\tau(a)}^{(s)}) - w_{\tau(a)}^{(s)}}. \tag{A.5}
\]
Since, for \(s = 1, \ldots, N - 2\)
\[
(-1)^{\tau_s} = (-1)^{\tau_{s+1}} \quad \text{it follows that} \quad \prod_{s=1}^{N-1} (-1)^{\tau_s} = (-1)^{\tau_1}. \tag{A.6}
\]
Then, the symmetry in each vector \(w^{(s)}_{N-s}\), \(s = 1, \ldots, N - 1\) taken singly of \(W_N([w^{(s)}_{N-s}]_{1}^{N-1})\) and its anti-symmetry in \(w^{(0)}_N = y_N\) implies that
\[
\varphi_{y_N}(x_N) = \prod_{s=2}^{N} s! \cdot \sum_{\tau_1 \in \mathfrak{S}_N} J(y_{N;\tau_1}) \quad \text{with} \quad y_{N;\tau_1} = (y_{\tau_1(1)}, \ldots, y_{\tau_1(N)}) \tag{A.7}
\]
where
\[
J(w^{(0)}_N) = e^{\pi \sum_{x=1}^{N-1} \int_{(x, y)} d^{N-s} w^{(s)} \prod_{s=1}^{N-1} \left( e^{\pi \sum_{a \in \mathfrak{S}_N} \frac{w^{(s)}_a}{w^{(s-1)}_a}} \right) \cdot \frac{W_N([w^{(s)}_{N-s}]_{1}^{N-1})}{\prod_{s=1}^{N-1} \prod_{a=1}^{N-s} (w^{(s)}_a - w^{(s-1)}_a)}}. \tag{A.8}
\]
Above, we agree upon
\[
r_s = x_{N-s} - x_{N-s+1}. \tag{A.9}
\]
In order to obtain an explicit formula allowing one to bound the function \(\varphi_{y_N}(x_N)\), one should move the contours of integration for variables associated with \(r_s > 0\) from the lower half-plane to the upper half-plane. The matter is that, in doing so, one will cross poles which will generate new type of exponents, say containing the combinations \(r_a + \cdots + r_b\) with \(a > b\). This last factor can be positive or negative. In the latter case, one should then move the integration in respect to the associated variables also to the upper half-plane. In the former case, there is nothing else to do. In fact, the most optimal way of expressing the result of contour shifting is in terms of a sum over all sequences \(R_{as}\) with \(a = 1, \ldots, N - s\) and \(s = 1, \ldots, N - 1\) that can be built according to the below algorithm. This algorithm is well defined provided that all the partial sums do not vanish, \(ie\)
\[
r_a + \cdots + r_b \neq 0 \quad \text{for any} \quad a \geq b. \tag{A.10}
\]
The case when some of the partial sums vanish is readily obtain by taking appropriate limits.
The algorithm starts at \(N - 1\)
The quantities $R_{a,N-2}$, $a = 1, 2$ are built as

- If $\begin{cases} r_{N-1} \leq 0 & \text{then set } R_{1,N-1} = r_{N-1} \\ r_{N-1} > 0 & \text{then pick } R_{1,N-1} \in \{0, r_{N-1}\} \end{cases}$.

Assume having chosen $R_{1,N-1}$. Note that, necessarily, for contours of integrations. Indeed, set $\begin{cases} r_{N-1} + r_{N-2} \leq 0 & \text{then set } R_{1,N-2} = r_{N-1} + r_{N-2} \\ r_{N-1} + r_{N-2} > 0 & \text{then pick } R_{1,N-2} \in \{0, r_{N-1} + r_{N-2}\} \end{cases}$.

- If $R_{1,N-1} \neq 0$, should $\begin{cases} r_{N-2} \leq 0 & \text{then set } R_{1,N-2} = r_{N-2} \\ r_{N-2} > 0 & \text{then pick } R_{1,N-2} \in \{0, r_{N-2}\} \end{cases}$.

- finally, if $\begin{cases} \text{if } R_{N-2} \leq 0 \text{ then set } R_{2,N-2} = r_{N-2} \text{.} \\ \text{if } R_{N-2} > 0 \text{ then pick } R_{2,N-2} \in \{0, r_{N-2}\}. \end{cases}$

Assume having chosen $\{R_{a,s}\}$ with $1 \leq a \leq N - s'$ and $s + 1 \leq s' \leq N - 1$. Then define

$$k_{a,s} = \min\{k \geq s + 1 : R_{a,k} = 0\} \text{ if } R_{a,s+1} = 0$$

$$k_{a,s} = s \text{ otherwise} \quad (A.11).$$

Note that, necessarily, for $k_{a,s} \geq s + 1$ one has $r_{k_{a,s}} + \cdots + r_{s+1} > 0$.

- If $\begin{cases} r_{k_{a,s}} + \cdots + r_{s} \leq 0 & \text{then set } R_{a,s} = r_{k_{a,s}} + \cdots + r_{s} \\ r_{k_{a,s}} + \cdots + r_{s} > 0 & \text{then pick } R_{a,s} \in \{0, r_{k_{a,s}} + \cdots + r_{s}\} \end{cases}$.

One continues in this way up to $s = 0$ when there is a unique choice possible for the sequence $R_{a,0}$. For $a = 1, \ldots, N - 1$

- if $\begin{cases} k_{a,0} = 0 & \text{then } R_{a,0} = 0 \\ k_{a,0} \neq 0 & \text{then } R_{a,0} = r_{k_{a,0}} + \cdots + r_{1} \text{ and } R_{N,0} = 0. \end{cases}$

We denote by $\mathcal{R}$ the set of all possible sequences $R_{a,s}$ that can be obtained by application of the above algorithm,

$$\mathcal{R} = \{\{R_{a,s}\} \text{, with } a = 1, \ldots, N - s \text{ such that } R_{a,s} \text{ built by algorithm}\}. \quad (A.12)$$

A given sequence $\{R_{a,s}\}$ defines uniquely which residues have been computed in the course of moving the contours of integrations. Indeed, set

$$\Omega_{a} = \{s \geq 1 : R_{a,s} = 0\} \text{ and } \Omega_{a}^{\pm} = \{s \geq 1 : \pm R_{a,s} > 0\}. \quad (A.13)$$

The set $\Omega_{a}$ will have $\ell_{a}$ neighbouring components in the sense that

$$\Omega_{a} = \{b_{a,1}, b_{a,1} + 1, \ldots, c_{a,1}\} \cup \cdots \cup \{b_{a,\ell_{a}}, \ldots, c_{a,\ell_{a}}\} \quad (A.14)$$

where

$$1 \leq b_{a,1} \leq c_{a,1} \leq b_{a,2} - 2 \leq \cdots \leq b_{a,p} \leq c_{a,p} \leq b_{a,p+1} - 2 \leq \cdots \leq c_{a,\ell_{a}} \leq N - a. \quad (A.15)$$
Furthermore, since there is at least one integer in between each of the "connected parts", one has $\ell_a \leq [(N - a)/2]$. Then, the result of contour deformation is to integrate solely over the variables $w_a^{(s)}$ with $s \in \Omega^+_a$, $a = 1, \ldots, N - 1$. The variables belonging to $\Omega_a$ should be reduced according to

$$w_a^{(b-a_0)} = \ldots = w_a^{(c-a_0+1)} \quad \text{with} \quad p = 1, \ldots, \ell_a.$$  

(A.16)

It is the reduction (A.16) that corresponds to computing the poles. Thus $J(w^{(0)}_N)$ is recast as

$$J(w^{(0)}_N) = e^{i\pi w^{(0)}_N} \cdot \sum_{(R,\alpha) \in R} \prod_{a=1}^{N-1} \left\{ e^{iR_a w_a^{(0)}} \right\} \times \prod_{a=1}^{N-1} \left\{ \prod_{s \in \Omega^+_a} \int \frac{d\mu(s)}{2\pi i} e^{iR_a w_a^{(s)}} \cdot \prod_{s \in \Omega^-_a} \int \frac{d\mu(s)}{2\pi i} e^{iR_a w_a^{(s)}} \right\} \cdot \tilde{W}^{(\text{red})(N)}((R,\alpha)_a; \{w_a^{(s)}\}_{N-1}^{N-1}).$$  

(A.17)

where $\eta_1 > \cdots > \eta_{N-1} > 0$ and $\alpha_{N-1} > \cdots > \alpha_1 > 0$,

$$\tilde{W}^{(\text{red})(N)}((R,\alpha)_a; \{w_a^{(s)}\}_{N-1}^{N-1}) = \frac{\tilde{W}^{(\text{red})(N)}((R,\alpha)_a; \{w_a^{(s)}\}_{N-1}^{N-1})}{\prod_{a=1}^{N-1} \left\{ \prod_{s \in \Omega^+_a} (w_a^{(s)} - w_a^{(s-1)}) \cdot \prod_{p=1}^{\ell_a} (w_a^{(c-a_0+1)} - w_a^{(b-a_0-1)}) \right\}}.$$  

(A.18)

Note that, above, we agree upon

$$B_a = \{b_a,1, \ldots, b_a,\ell_a\},$$  

(A.19)

and $\tilde{W}^{\text{(red)}}_N$ is obtained from $W_N$ by implementing the reduction (A.16).

It follows from the previous handlings that the integral transform $\mathcal{U}_N[F](x_N)$ can be recast as

$$\mathcal{U}_N[F](x_N) = \frac{N!}{\sqrt{\pi}^N} \cdot \int d^N y_F(y_N) \cdot e^{i\pi \omega y_N} \cdot \sum_{(R,\alpha) \in R} \prod_{a=1}^{N-1} \left\{ e^{iR_a y_a} \right\} \times \prod_{a=1}^{N-1} \left\{ \prod_{s \in \Omega^+_a} \int \frac{d\mu(s)}{2\pi i} e^{iR_a w_a^{(s)}} \cdot \prod_{s \in \Omega^-_a} \int \frac{d\mu(s)}{2\pi i} e^{iR_a w_a^{(s)}} \right\} \mu(y_N) \cdot \tilde{W}^{(\text{red})(N)}((R,\alpha)_a; \{w_a^{(s)}\}_{N-1}^{N-1})_{w_a^{(0)} = y_N}.$$  

Above, we agree upon the identification $r_N \equiv x_N$. We do stress that the singularities of $\tilde{W}^{(\text{red})(N)}_{x_N}$ at $y_a = y_b$, $a \neq b$, are compensated by the zeroes of the measure’s density $\mu(y_N)$. The integrand is thus a smooth, compactly supported function of $y_N$.

We now build on the above representation so as to ensure the Schwartz class of $\mathcal{U}_N[F]$. By the very construction of the sequence $(R,\alpha)_a$, for every fixed $a$, the numbers $R_{a,s}$ with $0 \leq s \leq N - a$ cannot all be zero. Defining

$$s_a = \min \left\{ s : R_{a,s} \neq 0 \right\} \quad \text{one has} \quad R_{a,s} = \begin{cases} r_{N-a} + \cdots + r_{s_a} & \text{if } s_a \geq 1 \\ r_{N-a} + \cdots + r_1 & \text{if } s_a = 0 \end{cases}.$$  

(A.20)

One can then integrate by parts in respect to all the variables $y_a$ such that $R_{a,0} \neq 0$ and bound the exponentially decreasing factors by a power law for all the variables $y_a$ such that $s_a \geq 1$. This readily shows that there exists a $k,F$-dependent constant $C_{k,F}$ such that

$$\left| \mathcal{U}_N[F](x_N) \right| \leq C_{k,F} \cdot \sum_{|R| = 0}^{N-1} \prod_{a=1}^{N-1} \left( \frac{1}{1 + |r_{N-a} + \cdots + r_{s_a}|} \right)^k.$$  

(A.21)
This condition is readily translated into one in respect to the "position" variables $x_N$, thus ensuring the Schwartz class of the integral transform $\mathcal{U}_N[F](x_N)$.

\section*{B From Mellin-Barnes to Gauss-Givental}

\textbf{Lemma B.1} Let $\mathcal{L}^{(N-1)}_y : L^\infty(\mathbb{R}^{N-1}) \rightarrow L^\infty(\mathbb{R}^{N-1})$ be an integral operator with a kernel

$$
\mathcal{L}^{(N-1)}_y(\mathbf{x}_{N-1} | \mathbf{y}_{N-1}) = e^{-i \mathbf{y}_{N-1} \cdot \mathbf{r}_{N-1}} \prod_{n=1}^{N-1} I(y_n - x_n - r_n) \prod_{n=1}^{N-2} I(y_{n+1} - x_{n+1} - r_{n+1}).
$$

Then, for $\mathfrak{g}(y) > \mathfrak{g}(w)$ the function $(\mathcal{L}^{(N-1)}_y \cdot \Lambda^{(N-1)}_w)(\mathbf{x}_{N-1} | \mathbf{z}_{N-2})$ is well-defined and one has the relation

$$
\mathcal{L}^{(N-1)}_y \cdot \Lambda^{(N-1)}_w = \hbar \mathfrak{g}(w-y) \Gamma \left( \frac{y-w}{i\hbar} \right) \times \Lambda^{(N-1)}_w \cdot \mathcal{L}^{(N-2)}_y.
$$

The proof of this lemma goes similarly to the one of lemma 1.3, so we do not reproduce it here. The operators $\mathcal{L}_y$ along with the above lemma have been introduced for the first time in [12].

\textbf{Lemma B.2} Let $s \in \mathbb{R}$. Let $y_1, \ldots, y_n$ and $x_1, \ldots, x_{n-1}$ be two sets of generic variables in $\mathbb{C}$ and $\mathcal{C}$ a contour that circumvents all the points $y_a + i\hbar n_a$, with $n_a \in \mathbb{N}$, $a = 1, \ldots, n$ from below whereas it circumvents the points $x_a - i\hbar n_a$, with $n_a \in \mathbb{N}$, $a = 1, \ldots, n-1$ from above. Then one has the integral identity

$$
\int_{\mathcal{C}^*} \frac{n! e^{\sum_{a=1}^{n} \frac{x_a - y_a}{i\hbar}}}{(2\pi \hbar)^n} \prod_{a \neq b} \frac{1}{\Gamma(\frac{y_a - x_b}{i\hbar})} \prod_{a=1}^{n} \frac{1}{\Gamma(\frac{w_a - x_a}{i\hbar})} \cdot \frac{d^iw}{(2\pi \hbar)^n} = \prod_{a=1}^{n} \frac{1}{\Gamma(\frac{w_a - x_a}{i\hbar})}.
$$

\textbf{Proof —}

The starting point is given by the multi-dimensional integral computed by Gustafsen [17]. Given any two generic sets of points $y_1, \ldots, y_{n+1}$ and $x_1, \ldots, x_{n+1}$ and a contour $\mathcal{C}$ that circumvents all the points $y_a + i\hbar n_a$, with $n_a \in \mathbb{N}$, $a = 1, \ldots, n + 1$ from below whereas it circumvents the points $x_a - i\hbar n_a$, with $n_a \in \mathbb{N}$, $a = 1, \ldots, n + 1$ from above, one has

$$
\int_{\mathcal{C}^*} \frac{n! e^{\sum_{a=1}^{n+1} \frac{x_a - y_a}{i\hbar}}}{(2\pi \hbar)^n} \prod_{a \neq b} \frac{1}{\Gamma(\frac{y_a - x_b}{i\hbar})} \prod_{a=1}^{n+1} \frac{1}{\Gamma(\frac{w_a - x_a}{i\hbar})} \cdot \frac{d^iw}{(2\pi \hbar)^n} = \prod_{a=1}^{n+1} \frac{1}{\Gamma(\frac{w_a - x_a}{i\hbar})}.
$$

For the time being, we assume that the parameters $\{x_a\}_{a=1}^{n+1}$ and $\{y_a\}_{a=1}^{n+1}$ are such that one can choose $\mathcal{C}$ lying sufficiently close to $\mathbb{R}$, i.e. $n = \sup_{w \in \mathfrak{g}} |\mathbf{g}(w)|$ is small, and that $\mathcal{C}$ avoids 0. We then set $y_{n+1} = -Ke^{-s}$ with $(K, s) \in \mathbb{R}^+ \times \mathbb{R}$ and $x_{n+1} = K$ and divide both sides of (B.4) by

$$
\Gamma^a\left( -\frac{K}{i\hbar}, -\frac{Ke^{-s}}{i\hbar} \right),
$$

what leads to

$$
\int_{\mathcal{C}^*} I_Ke^a\left( \frac{w_a}{i\hbar}; \frac{y_a}{i\hbar}; \frac{x_a}{i\hbar} | s \right) \frac{d^iw}{(2\pi \hbar)^n} = n! \prod_{a,b=1}^{n} \frac{1}{\Gamma\left( \frac{y_a - x_b}{i\hbar} \right)} \cdot u_K\left( \frac{y_a}{i\hbar}; \frac{x_a}{i\hbar} | s \right),
$$

as desired.
in which

\[
\mathcal{I}_K([y_n]^n_1; [x_n]^n_1 | s) = \frac{\Gamma(-K + \frac{e^{-s}}{ih})}{\Gamma(-K + \frac{e^{-s}}{ih})} \prod_{k=1}^{n} \frac{\Gamma(-\frac{Ke^{-s} + x_k - K}{ih})}{\Gamma(-\frac{K - Ke^{-s} - \frac{e^{-s}}{ih}}{ih})}
\]

(B.7)

whereas the integrand reads

\[
\mathcal{I}_K([w_n]^n_1; [y_n]^n_1; [x_n]^n_1 | s) = \prod_{k=1}^{n} \frac{\Gamma(-\frac{Ke^{-s} + w_k - K}{ih})}{\Gamma(-\frac{K - Ke^{-s} - \frac{e^{-s}}{ih}}{ih})} \prod_{a=1}^{n} \left\{ \prod_{b=1}^{n} \frac{\Gamma(\frac{y_b - w_a}{ih})}{\Gamma(\frac{w_a - w_b}{ih})} \right\}
\]

(B.8)

It is readily seen that pointwise in \([y_n]^n_1; [x_n]^n_1\) and in \(s\)

\[
u_K([y_n]^n_1; [x_n]^n_1 | s) \rightarrow \frac{\pi n}{\pi n} \cdot (1 + e^{-s})^{-\frac{\epsilon}{n}}.
\]

(B.9)

It is likewise readily seen that, pointwise in \([w_n]^n_1; [y_n]^n_1; [x_n]^n_1\) and in \(s\),

\[
\mathcal{I}_K([w_n]^n_1; [y_n]^n_1; [x_n]^n_1 | s) \rightarrow \frac{\pi n}{\pi n} \cdot \prod_{a=1}^{n} \frac{\prod_{b=1}^{n} \Gamma(\frac{y_b - w_a}{ih})}{\Gamma(\frac{w_a - w_b}{ih})}
\]

(B.10)

and that, furthermore,

\[
|\mathcal{I}_K([w_n]^n_1; [y_n]^n_1; [x_n]^n_1 | s)| \leq \tilde{C} \cdot \prod_{a=1}^{n} \|\mathcal{I}(w_a)\| \cdot \prod_{a=1}^{n} f_K(w_a),
\]

(B.11)

with \(\epsilon = \max_{t \in \mathcal{S}} |\mathcal{S}(t)|, S = \{x_1, \ldots, x_n, y_1, \ldots, y_n\}, \eta = \max_{w \in \mathcal{C}} |\mathcal{S}(w)|\) and

\[
f_K(w) = \frac{K \cdot e^{-\frac{\pi}{2}|\mathcal{S}(w)|}}{|w| \sqrt{|K - w| \cdot |w + Ke^{-s}|}} \cdot \frac{2\pi n}{\pi} \cdot \exp\left\{ \frac{\pi}{2h} \left[ (1 + e^{-s}) - |\mathcal{S}(K-w)| - |\mathcal{S}(Ke^{-s}+w)| \right] \right\}
\]

(B.12)

Elementary analysis then shows that, uniformly in \(K, w \in \mathcal{C} \mapsto f_K(w)\) is bounded\(^6\). As a consequence, we are thus in position to apply the dominated convergence theorem, leading to

\[
\int_{\mathcal{C}} \frac{e^{\frac{\pi n}{\pi n} \cdot \prod_{a=1}^{n} \frac{\prod_{b=1}^{n} \Gamma(\frac{y_b - w_a}{ih})}{\Gamma(\frac{w_a - w_b}{ih})}}}{(2\pi h)^{n}} \cdot d^nW = n! \cdot \prod_{a,b=1}^{n} \Gamma(\frac{y_b - x_a}{ih}) \cdot e^{\frac{\pi n}{\pi n} \cdot (1 + e^{-s})^{-\frac{\epsilon}{n}}}.
\]

(B.13)

It then follows from an analytic continuation that, in fact, the formula holds for \(|\mathcal{S}(s)| < \pi\) and for all sets of points \([x_n]^n_1\) and \([y_n]^n_1\) that can be separated by a curve \(\mathcal{C}\) in the sense of the statement of the present lemma.

In the newly obtained identity, we substitute

\[
x_n = K \quad s = v - \ln\left(\frac{-K}{ih}\right) \quad \text{with} \quad v \in \mathbb{R}
\]

(B.14)

\(^6\)we recall that \(0 \notin \mathcal{C}\) and that \(\mathcal{C}\) also avoids \(K\) and \(-Ke^{-s}\)
and again consider sets of points \{x_n\}, \{y_n\} such that one can take the contour \(C\) lying sufficiently close to \(\mathbb{R}\) and avoiding 0. Then, we divide both sides of (B.13) by \(\Gamma^n\left(-iK/\hbar\right)\). This yields the representation

\[
\mathcal{I}_K([w_a^n]; [y_a^n]; [x_a^n]) \cdot \frac{d^n w}{(2\pi i)^n} = n! \cdot \prod_{b=1}^{n} \Gamma\left(\frac{y_b - x_b}{i\hbar}\right) \cdot \mathcal{A}_K([y_a^n]; [x_a^n]) \cdot (v),
\]

in which

\[
\mathcal{A}_K([y_a^n]; [x_a^n]) \cdot (v) = \left(1 - \frac{i\hbar}{K} e^\pi \cdot \sum_{j=1}^{n} \cdot e^\frac{\pi}{2j} \cdot \prod_{a=1}^{n} \left(\Gamma\left(\frac{y_a - x_a}{i\hbar}\right) \cdot \Gamma^{-1}\left(-\frac{K}{i\hbar}\right) \cdot \left(-\frac{K}{i\hbar}\right)^{-\frac{n}{2}}\right) \right).
\]

whereas the integrand reads

\[
\mathcal{I}_K([w_a^n]; [y_a^n]; [x_a^n]) \cdot (s) = e^\frac{\pi}{2j} \cdot \prod_{a=1}^{n} \left(\frac{K}{i\hbar} \cdot \Gamma\left(\frac{w_a - K}{i\hbar}\right) \cdot \Gamma^{-1}\left(-\frac{K}{i\hbar}\right) \cdot \left(-\frac{K}{i\hbar}\right)^{-\frac{n}{2}}\right).
\]

It is readily seen that pointwise in \{y_a^n\}, \{x_a^n\} and in \(v\)

\[
\mathcal{A}_K([y_a^n]; [x_a^n]) \cdot (v) \xrightarrow{K \to \infty} e^\frac{\pi}{2j} \cdot \exp\left\{-e^\pi\right\},
\]

It is likewise readily seen that, pointwise in \{w_a^n\}, \{y_a^n\}, \{x_a^n\} and in \(s\),

\[
\mathcal{I}_K([w_a^n]; [y_a^n]; [x_a^n]) \cdot (s) \xrightarrow{K \to \infty} e^\frac{\pi}{2j} \cdot \prod_{a=1}^{n} \left|\prod_{a=1}^{n} \left|\prod_{a=1}^{n} \left|f_K(w_a)\right|\right|\right.
\]

and that, furthermore,

\[
\mathcal{I}_K([w_a^n]; [y_a^n]; [x_a^n]) \cdot (s) \leq C \cdot \left|\prod_{a=1}^{n} \left|\prod_{a=1}^{n} \left|f_K(w_a)\right|\right|\right.
\]

with

\[
\bar{f}_K(w) = \frac{\sqrt{K}}{\sqrt{|K - w|} \cdot |w|} \cdot \left|\prod_{a=1}^{n} \left|\prod_{a=1}^{n} \left|f_K(w_a)\right|\right|\right.
\]

We are, again, in position to apply the dominated convergence theorem, thus leading to the claim in the case of \(s \in \mathbb{R}\) and \(x_o, y_o\) lying sufficiently far from \(\mathbb{R}\). The general result then follows by analytic continuation in \((x_{n-1}, y_n)\) since the result already holds on an open subset of \(\mathbb{C}^{n-1} \times \mathbb{C}^n\).

**Proposition B.1** The unique solution \(\varphi_{y_n}(x_N)\) \((1.4)\) to the Mellin-Barnes induction \((1.1)\) satisfies the induction

\[
\varphi_{y_n}(x_N) = \int_{\mathbb{R}^{N-1}} \Lambda^{(N)}_{y_n}(x_N | \tau_{N-1}) \varphi_{y_{N-1}}(\tau_{N-1}) \cdot d^{N-1} \tau.
\]

(22)
The multiple integral \( B.22 \) is well defined since \( \varphi_{y_1} \in L^\infty(\mathbb{R}^N) \) and, for fixed \( x_N \in \mathbb{R}^N \), the function
\[
\tau_{N-1} \mapsto \Lambda_{y_1}^{(N)}(x_N \mid \tau_{N-1}) \in L^1(\mathbb{R}^{N-1}).
\]

The recurrence relation \( B.22 \) provides a connection between the Mellin-Barnes and Gauss-Givental representation of \( \varphi_{y_{N+1}}(x_{N+1}) \) and, in fact, shows that the Gauss-Givental representation, seen as an encased integral, is well defined. Proposition \( B.1 \) has been first derived in \([12]\). However, the proof given in \([12]\) utilizes the completeness and orthogonality of the system of functions \( \varphi_{y_N} \). Here, we provide a different proof of this induction that does not build on the completeness and orthogonality of the system \( \varphi_{y_N}(x_N) \).

**Proof** —

The statement holds for \( N = 0 \) since
\[
\varphi_{y_1}(x_1) = \Lambda_{y_1}^{(1)}(x_1 \mid -).
\]

Now assume that it holds up to some \( N \). A straightforward induction based on the exchange relation \( B.2 \) shows that given \( y_{N+1} \in \mathbb{R} \) and \( w_N \in (\mathbb{R} - i\alpha)^N, \alpha > 0 \) one has
\[
L_{y_{N+1}}^{(N)} \cdot \varphi_{w_N}(x_N) = \prod_{a=1}^{N} \left\{ h^{w_a-y_{N+1}} \cdot \Gamma\left(\frac{y_{N+1} - w_a}{ih}\right) \right\} \cdot \varphi_{w_N}(x_N).
\]

Thus

- inserting this relation into the Mellin-Barnes recurrence relation \( B.1 \),
- exchanging the order of \( \tau_N \) and \( w_N \) integrations,
- applying the Mellin-Barnes induction a second time to \( \varphi_{w_N}(\tau_N) \),
- exchanging the order of \( \gamma_N-1 \) and \( w_N \) integrations,

leads us to the representation
\[
\varphi_{y_{N+1}}(x_{N+1}) = \int_{\mathbb{R}^N} d^N\tau \int_{(\mathbb{R} - 2i\alpha)^N} d^{N-1}\gamma \cdot \frac{L_{y_{N+1}}^{(N)}(x_N \mid \tau_N)}{(N-1)! (2\pi\hbar)^{N-1}} \cdot \frac{\Gamma(\frac{y_{N+1} - \gamma}{\hbar})}{\prod_{a\neq b} \Gamma\left(\frac{w_a - w_b}{ih}\right)} \cdot h^{N(N-1)} \cdot \varphi_{\gamma_{N-1}}(\tau_{N-1}).
\]

Note that we were able to exchange the orders of integration twice since, in each case, the integrand can be bounded by a strictly positive function that is readily seen to be integrable for at least one ordering of the integration variables. By Fububini’s theorem, this is already enough. The integral arising in the last line of \( B.26 \) can be computed thanks to the results of lemma \( B.2 \) leading to
\[
\varphi_{y_{N+1}}(x_{N+1}) = \int_{\mathbb{R}^N} d^N\tau \cdot \frac{L_{y_{N+1}}^{(N)}(x_N \mid \tau_N)}{(N-1)! (2\pi\hbar)^{N-1}} \cdot \exp\left\{-\frac{1}{\hbar} \left( \varphi_{y_{N+1}} - \tau_N \right) \right\}
\]
\[
\times \int_{(\mathbb{R} - 2i\alpha)^N} d^{N-1}\gamma \cdot \frac{\Gamma(\frac{y_{N+1} - \gamma}{\hbar})}{\prod_{a\neq b} \Gamma\left(\frac{w_a - w_b}{ih}\right)} \cdot \varphi_{\gamma_{N-1}}(\tau_{N-1}).
\]

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Hence, using that
\[ \Lambda_{N+1}(x_{N+1} | \tau_N) = \mathcal{L}_{N+1}^{(N)}(x_N | \tau_N) \exp \left\{ -\frac{1}{\hbar} e^{x_{N+1} - \tau_N} \right\} \cdot e^{\mathcal{H}_{N+1}} \]  
we obtain the claim.

\[ \text{(B.28)} \]

\section{An auxiliary lemma}

\textbf{Lemma C.1} Let \( \eta > 0 \). The sequence
\[ f_K(s) = \frac{K}{\sqrt{s^2 + \eta^2} \cdot \ln \left[ s^2 + \eta^2 \right] \cdot \sqrt{(s - K)^2 + \eta^2}} \]  
\text{(C.1)}

satisfies
\[ \lim_{K \to +\infty} \int_{\mathbb{R}} f_K(s) \cdot ds = \int_{\mathbb{R}} \left[ \lim_{K \to +\infty} f_K(s) \right] \cdot ds \]  
\text{(C.2)}

\textbf{Proof} —

One has
\[ \int_{\mathbb{R}} f_K(s) \cdot ds = \int_{\mathbb{R}\setminus[1-\epsilon)K:(1+\epsilon)K]} f_K(s) \cdot ds + \int_{(1-\epsilon)K}^{(1+\epsilon)K} f_K(s) \cdot ds . \]  
\text{(C.3)}

Since
\[ \left| f_K(s)1_{\mathbb{R}\setminus[1-\epsilon)K:(1+\epsilon)K]}(s) \right| \leq \frac{C_\epsilon}{\sqrt{s^2 + \eta^2} \cdot \ln \left[ s^2 + \eta^2 \right]} \in L^1(\mathbb{R}, ds) \]  
\text{(C.4)}

for some \( \epsilon \)-dependent constant \( C_\epsilon > 0 \), one can apply the dominated convergence theorem to the first integral leading to
\[ \lim_{K \to +\infty} \int_{\mathbb{R}\setminus[1-\epsilon)K:(1+\epsilon)K]} f_K(s) \cdot ds = \int_{\mathbb{R}} \frac{ds}{\sqrt{s^2 + \eta^2} \cdot \ln \left[ s^2 + \eta^2 \right]} . \]  
\text{(C.5)}

It thus remains to establish that the second integral goes to zero. It is readily seen that
\[ \int_{(1-\epsilon)K}^{(1+\epsilon)K} f_K(s) \cdot ds = \int_{0}^{\epsilon K} h_K(s) \cdot \left[ \frac{1}{\sqrt{s^2 + \eta^2} - \frac{1}{s + i\eta}} \right] \cdot ds + \int_{\epsilon K}^{\infty} \frac{h_K(s)}{s + i\eta} \cdot ds \]  
\text{(C.6)}

where
\[ h_K(s) = \sum_{\nu = \pm} \frac{K \cdot [(s + \nu K)^2 + \eta^2]^{-\frac{1}{2}} \cdot \ln^2 \left[ (s + \nu K)^2 + \eta^2 \right]} \cdot \ln^2 \left[ (s + \nu K)^2 + \eta^2 \right] . \]  
\text{(C.7)}

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Since, uniformly in $K$, $h_K$ is bounded $[0: \epsilon K]$, one can apply the dominated convergence theorem to the first integral so as to get that it goes to zero with $K$. Finally,

$$\int_0^{\epsilon K} \frac{h_K(s)}{s + i\eta} \cdot ds = h_K(\epsilon K) \ln(\epsilon K + i\eta) - h_K(0) \ln(i\eta) - \int_0^{\epsilon K} h'_K(s) \ln(s + i\eta) \cdot ds.$$  \hfill (C.8)

A straightforward calculation leads to

$$\left\| h'_K \right\|_{L^\infty([0,\epsilon K])} \leq \frac{C}{K \ln^2(K)},$$  \hfill (C.9)

hence allowing one to conclude.

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