Abstract
The induced vertex stress of any vertex of a graph \( G \) measures the contribution of that vertex in the total stress of the graph \( G \). This paper investigates the total vertex stress in some merged graphs, cartesian product graphs and the join of two graphs. The induced vertex stress of geodetic graphs and its relation with Wiener index and betweenness centrality is also studied.

Keywords
vertex stress, induced vertex stress.

AMS Subject Classification
05C12, 05C07.

1. Introduction
The concept of stress is a shortest path dependent vertex parameter and was introduced by Shimbel(1953).

Let \( G(V(G),E(G)) \) be a simple connected undirected graph with vertex set \( V(G) \) and edge set \( E(G) \). Let \( n \) and \( m \) denote the number of its vertices and edges respectively. For any two vertices \( u, v \in V(G) \), the distance \( d(u, v) \) between \( u \) and \( v \) is the length of a shortest path between \( u \) and \( v \) in \( G \).

Definition 1.1. [6] For a simple connected undirected graph \( G(V,E) \), the stress of a vertex \( x \in V \) is defined as

\[
S_G(x) = \sum_{u \neq v \neq x} \sigma_{uv}(x)
\]

where \( \sigma_{uv}(x) \) is the number of shortest paths with vertices \( u \) and \( v \) as their end vertices and include the vertex \( x \).

The total vertex stress of \( G \) is given by, \( S(G) = \sum_{v \in V(G)} S_G(v) \).

The average vertex stress of \( G \) of order \( n \geq 1 \) and denoted by, \( \overline{S}(G) \) follows naturally as, \( \overline{S}(G) = \frac{1}{n} \sum_{v \in V(G)} S_G(v) \).

Definition 1.2. [5] Let \( V(G) = \{v_i : 1 \leq i \leq n\} \) and for the ordered vertex pair \((v_i, v_j)\) let there be \( k_G(i, j) \) distinct shortest paths of length \( l_G(i, j) \) from \( v_i \) to \( v_j \). Then, \( s_G(v_i) = \sum_{j=1, j \neq i}^{n} k_G(i, j)(l_G(i, j) - 1) \) and the total induced vertex stress of \( G \) is \( s(G) = \sum_{i=1}^{n} s_G(v_i) \).

2. Total vertex stress of some merged graphs
Merging two cycles:
Consider two cycles \( C_n \) and \( C_m \) having vertices \( v_1, v_2, \ldots, v_n \) and \( u_1, u_2, \ldots, u_m \) respectively. Merging any two vertices \( v_i \) and \( u_j \) we get two cycles with a common vertex.

Case 1. Merging one odd cycle \( C_{2t+1} \) and one even cycle \( C_{2s} \) denoted by \( C^o_{\triangle □}(t,s) \).

Let \( n = s + t \) then

Theorem 2.1. The total stress of \( C^o_{\triangle □}(t,s) \) is given by

\[
S(C^o_{\triangle □}(t,s)) = \frac{(2n-3)12-(2n^2-4n+1)t+2n^3-2n^2}{2}.
\]

Proof. The total stress of \( C^o_{\triangle □}(s,t) \) consists of the total stress of \( C_{2t+1} \), the total stress of \( C_{2(n-t)} \), the stress induced by \( 2(n-t)-1 \) vertices on the cycle \( C_{2t+1} \) and the stress induced...
The total stress of

$T$\( (a) \) The total stress of

$C_2(n-t)$ . Hence $S(C_2^{(a)}(t,s)) = \frac{2(n-t)(n-t+1)}{2} + 2t(n-1)(n-t+1)$.

Hence $S(C_2^{(a)}(t,s)) = \frac{(2n-3)t^2 - (2n^2-4n+1)t + 2n^3 - 2n^2}{2}$.

**Case 2.** Merging two even cycles $C_{2r}$ and $C_{2s}$ denoted by $C_{2r,2s}(t,s)$. Let $n = t + s$ then

**Proposition 2.2.** The total vertex stress of $C_{2r,2s}(t,s)$ is given by

$S(C_{2r,2s}(t,s)) = (n-2)t^2 - (n^2 - 2n)t + n^3 - n^2$

**Proof.** The total stress of $C_{2r,2s}(t,s)$ consists of

(a) The total stress of $C_{2r}$

(b) The total stress of $C_{2s}$

(c) The stress induced by $2r - 1$ vertices on the cycle $C_{2r}$

(d) The stress induced by $2(n-t) - 1$ vertices on the cycle $C_{2s}$

(a)+(b)+(c)+(d) gives the result

Case 3. Merging two odd cycles $C_{2r+1}$ and $C_{2s+1}$ denoted by $C_{2r,2s+1}(t,s)$. Let $n = t + s + 1$, then

**Proposition 2.3.** The total vertex stress of $C_{2r,2s+1}(t,s)$ is

$S(C_{2r,2s+1}(t,s)) = \frac{(2n-4)t^2 - (2n^2-6n+4)t + 2n^3 - 7n^2 + 7n - 2}{2}$

**Proof.** Similar to the proof of Proposition 2.1 and Proposition 2.2

Merging two paths:

Consider two paths $P_i$ having vertices $v_1, v_2, ..., v_i$ and $P_j$ having vertices $u_1, u_2, ..., u_j$. Merging any two vertices $v_i$ and $u_j$ gives a tree, denoted by $P^{(i,j)}_t(t,s)$.

Case 1. If two pendant vertices are merged, we get the path $P_{t+s-1}$. Hence $S(P^{(i,j)}_t(t,s)) = S(P_{t+s-1})$, for $i = 1, t$ and $j = 1, s$.

Merging a path with a cycle:

Consider the path graph $P_i$ and the cycle graph $C_n$. By merging the pendant vertex of $P_i$ with any vertex of $C_n$ we get a tadpole graph $T(n,t)$.

**Theorem 2.4.** The total vertex stress of $T(2r+1,t)$ is

$S(T(2r+1,t)) = S(P(t)) + S(C_{2r+1}) + r(t-1)(t+r-1)$.

**Proof.** The total vertex stress of $T(2r+1,t)$ consists of the total stress of the cycle $C_{2r+1}$, the total stress of the path $P(t)$, the stress induced by the $2r + 1$ vertices on the path $P_{t-1}$ and the stress induced by the $t - 1$ vertices on the cycle $C_{2r+1}$. Thus total stress of $T(2r+1,t)$ is $S(T(2r+1,t)) = S(P(t)) + S(C_{2r+1}) + \frac{2r(t-1)}{2} + \frac{2(1-t)(r-1)}{2}$.

**Corollary 2.5.** The total vertex stress of $T(2r,t)$ is given by

$S(T(2r,t)) = S(P(t)) + S(C_{2r}) + r(t-1)(t+r-1)$.
Consider \( u_i \in V(G + H) \). The total induced vertex stress of \( u_i \) is,
\[
\sigma_{G+H}(u_i) = \sum_{j=1, j \neq i}^{n+m} k_{G+H}(t,j)(\ell_{G+H}(t,j) - 1).
\]
Since \( u_i v_j \in E(G + H) \), for induced vertex stress of \( u_i \) we consider only \((u_i,u_j), j = 1,2, ... ,n\). Hence
\[
\sigma_{G+H}(u_i) = \sum_{j=1, j \neq i}^{n} k_{G+H}(t,j)(\ell_{G+H}(t,j) - 1).
\]

If \( \ell_{H}(t,j) = 2 \), then \( \ell_{G+H}(t,j) = 2 \) and \( k_{G+H}(t,j) = k_{G}(t,j) + m \). If \( \ell_{G}(t,j) > 2 \), then \( \ell_{G+H}(t,j) = 2 \) and \( k_{G+H}(t,j) = m \). Therefore
\[
\sigma_{G+H}(u_i) = \sum_{j=1, j \neq i}^{n} (k_{G}(t,j) + m) + \sum_{j=1, j \neq i}^{n} m.
\]
By a similar argument,
\[
\sigma_{G+H}(v_s) = \sum_{j=1, j \neq s}^{n} (k_{H}(s,j) + n) + \sum_{j=1, j \neq s}^{n} n.
\]
Hence the result.

5. Induced vertex stress of geodetic graphs

A geodetic graph is an undirected graph such that there exists a unique shortest path between each two vertices. The Wiener index or Wiener number \( W(G) \) of \( G \) is defined as
\[
W(G) = \frac{1}{2} \sum_{u \in V(G)} \sum_{v \in V(G)} d_G(u,v).
\]

**Theorem 5.1.** If \( G \) is a geodetic graph of order \( n \), then \( s(G) = 2W(G) - n(n-1) \).

**Proof.** Consider \( u_i \in V(G) \). Then induced vertex stress of \( u_i \) is given by
\[
\sigma_G(u_i) = \sum_{i=1, i \neq j}^{n} (\ell_G(t,i) - 1)\]
Hence
\[
s(G) = \sum_{i=1}^{n} \sum_{i=1, i \neq j}^{n} (\ell_G(t,i) - 1) = 2W(G) - n(n-1)
\]
Hence the total vertex stress of a geodetic graph \( G \) is given by \( S(G) = W(G) - \frac{n(n-1)}{2} \). Since the total betweenness centrality \( \sum_{v \in V(G)} B(v) = W(G) - \frac{n(n-1)}{2} \), the total betweenness centrality is same as total vertex stress in a geodetic graph.

6. Conclusion

In this paper, total vertex stress of some graph classes and some graph operations are computed and calculate the induced vertex stress of join of two graphs. Also establish a relation between the total induced vertex stress and Wiener index in geodetic graphs and proved that in geodetic graphs total betweenness centrality and total vertex stress are the same.

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