HILBERT SPECIALIZATION OF PARAMETRIZED VARIETIES

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Abstract. Hilbert specialization is an important tool in Field Arithmetic and Arithmetic Geometry, which has usually been intended for polynomials, hence hypersurfaces, and at scalar values. In this article, first, we extend this tool to prime ideals, hence affine varieties, and offer an application to the study of the irreducibility of the intersection of varieties. Then, encouraged by recent results, we consider the more general situation in which the specialization is done at polynomial values, instead of scalar values.

1. Introduction

Hilbert Irreducibility Theorem has been a core result in Field Arithmetic for many decades. A simple form, see for example [FJ08, Page 218], says that, given an irreducible polynomial $P(T, Y)$ in $\mathbb{Q}(T)[Y]$, one can find infinitely many $t \in \mathbb{Q}$ such that the so-called specialized polynomial $P(t, Y)$ is irreducible in $\mathbb{Q}[Y]$.

This result has been generalized under many aspects. First of all, the notion of Hilbertian field has been introduced to identify all those fields $K$ for which the previous statement is verified in the case of polynomials, which are separable in $Y$, if we replace $\mathbb{Q}$ with $K$, see for example [FJ08, Page 218]. Moreover, if $K$ is of characteristic 0 or imperfect, the same result holds if we replace a separable irreducible polynomial in two variables $P(T, Y)$ in $K(T)[Y]$ with several irreducible polynomials $P_1(T, Y), \ldots, P_n(T, Y)$ in $K(T)[Y]$ in two arrays of variables, $\underline{T} = (T_1, \ldots, T_r)$ and $\underline{Y} = (Y_1, \ldots, Y_s)$ for $r, s$ positive integers: we can find a Zariski-dense subset $H \subset \mathbb{A}_K^r$ such that the polynomial $P_i(t, \underline{Y})$ is irreducible in $K[Y]$ for every $t \in H$ and $i = 1, \ldots, n$, see for example [FJ08, Section 12.1].

If we look at this statement from a geometric point of view, another potential generalization arises naturally. Giving an irreducible polynomial $P(\underline{T}, \underline{Y})$ in $K(\underline{T})[\underline{Y}]$ is equivalent to giving an irreducible $K(\underline{T})$-hypersurface $V_{K(\underline{T})}(P(\underline{T}, \underline{Y}))$ in the $s$-dimensional affine space $\mathbb{A}_K^s$ over $K(\underline{T})$. In these terms, for $K$ Hilbertian and of characteristic 0 or imperfect, Hilbert Irreducibility says that for a Zariski-dense set of choices $\underline{t}$ of the variables $\underline{T}$ in $K^r$, the specialized algebraic set $V_K(P(\underline{t}, \underline{Y})) \subset \mathbb{A}_K^s$ is an irreducible $K$-hypersurface. It is then natural to ask if an analogous result

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The precise definition will be given in the following section.
holds in the case of a \( K(\mathcal{T}) \)-variety of codimension bigger than 1. In algebraic terms this is equivalent to finding values \( \mathcal{t} \) in \( K^r \) of \( \mathcal{T} \) such that a nonzero prime ideal \( p_\mathcal{T} \) in \( K(\mathcal{T})[Y] \) remains a nonzero prime ideal of \( K[Y] \) when specializing \( \mathcal{T} \) at \( \mathcal{t} \). This is indeed one of the problems we are addressing in this article.

We want to make another step further. Until now the Hilbert specialization has typically been intended for scalar values in \( K^r \). However, there is a recent result, see [BDN20], in which a polynomial version of the Schinzel hypothesis is proved by specializing at polynomial values instead of scalar values. Given irreducible polynomials \( P_i(\mathcal{T}, Y) \), for \( i = 1, \ldots, n \), in a polynomial ring \( R[\mathcal{T}, Y] \) for \( R \) an integral domain, the variables \( \mathcal{T} \) are replaced by polynomials \( Q(Y) = (Q_1(Y), \ldots, Q_i(Y)) \), in the other variables and, under appropriate assumptions, all the specialized polynomials \( P_i(\mathcal{T}, Y) \) are shown to remain irreducible in \( R[Y] \). This recent development encourages to pursue the study of the specialization at polynomials. Indeed, another goal of this article will be to apply the results of the first part to obtain a more general version of them where the specialization of the variables \( \mathcal{T} \) occurs at polynomials in \( (K[Y])^r \) instead of scalars in \( K^r \).

1.1. Notation and main results. In this paper, given a field \( F \), a set of variables \( \mathbf{X} = \langle X_1, \ldots, X_m \rangle \) and polynomials \( f_1, \ldots, f_n \) in \( F[\mathbf{X}] \), we denote by \( V_F(f_1, \ldots, f_n) \) the affine subvariety of \( \mathbb{A}^m_F \) of equations \( \{ f_i(x_1, \ldots, x_m) = 0, i = 1, \ldots, n \} \). More formally, following [Liu02] Definition 3.4.7, it is the affine scheme associated to the finitely generated \( F \)-algebra \( F[\mathbf{X}]/(f_1, \ldots, f_n) \). As, by extension of scalars, the same set of elements gives rise to different varieties over different fields, to avoid any ambiguity, we use the word \( F \)-variety to specify the base field. If the ideal \( \langle f_1, \ldots, f_n \rangle \) is prime in \( F[\mathbf{X}] \), we say that the \( F \)-variety is irreducible. Moreover, if the \( F \)-variety has codimension \( 1 \), we call it an \( F \)-hypersurface. Finally we say that an irreducible \( F \)-variety is separable if \( \text{Frac}(F[\mathbf{X}]/(f_1, \ldots, f_n)) \) is a separable extension of \( F(\mathbf{X}) \), in the general sense as in [FJ08] Lemma 2.6.1.

We state the first result of the article.

**Theorem 1.1.** Let \( K \) be a Hilbertian field, \( \mathcal{P}(\mathcal{T}, Y) = \{ P_1(\mathcal{T}, Y), \ldots, P_r(\mathcal{T}, Y) \} \) a set of polynomials in \( K(\mathcal{T}, Y) \) such that \( V_T = V_{K(\mathcal{T})}(\mathcal{P}(\mathcal{T}, Y)) \) is a separable irreducible \( K(\mathcal{T}) \)-variety. Then for every \( \mathcal{t} = (t_1, \ldots, t_r) \) in \( K^r \) in some Zariski-dense subset of \( \mathbb{A}^r_K \), the \( K \)-variety \( V_\mathcal{t} = V_K(\mathcal{P}(\mathcal{t}, Y)) \supseteq \mathbb{A}^m_K \), where \( \mathcal{P}(\mathcal{t}, Y) \) is the set made of the specialized polynomials at \( \mathcal{t} \) is irreducible and its dimension \( \dim_K V_\mathcal{t} \) is equal to \( \dim_{K(\mathcal{T})} V_T \), the dimension of \( V_T \) as a \( K(\mathcal{T}) \)-variety.

**Remarks 1.2.** (a) Even if the polynomials in \( \mathcal{P} \) are irreducible, it is not enough that the corresponding specialized polynomials \( P_i(\mathcal{t}, Y) \) are irreducible to conclude that the variety \( V_K(\mathcal{P}(\mathcal{t}, Y)) \) is irreducible. Generally speaking, it may be that an ideal

\footnote{We define the dimension of a variety as the Krull dimension of the ring \( F[\mathbf{X}]/(f_1, \ldots, f_n) \).}
is generated by irreducible polynomials in $K[Y]$ but is not a prime ideal. Take for example the ideal $\langle Y - X, Y - X^2 \rangle$, which contains the product $X(X - 1)$.

(b) Hilbert specialization in the case of polynomials can be also performed over rings instead of fields, as in [BDKN20] Theorem 1.6 and Remark 4.4. It is then natural to ask if we can extend Theorem 1.1 to a ring $R$ as well, at least when $R$ is a Unique Factorization Domain. As our proof and [BDKN20] suggest, it may be possible that such a version holds if a nonzero element $\varphi \in R$ is inverted, i.e. if $R$ is replaced by $R[\varphi^{-1}]$ (a restriction that cannot be avoided in general). This, however, remains unclear at the moment.

(c) Theorem 1.1 can be seen as a Hilbertian version of Bertini’s Theorem. If we look at the statement of Bertini’s Theorem given in [FJ08, Corollary 10.4.3], we can see the similarity between the two statements. However, the two statements go on parallel routes. Bertini’s Theorem demands an algebraically closed base field $K$. In particular the case $s = 1$ is excluded in the Bertini context while it is a significant situation in the Hilbert context of Theorem 1.1.

A recurrent tool in the article will be the generic polynomial. Given an integer $D \geq 0$ we define the generic polynomial of degree $D$:

$$Q_D(\Lambda, Y) = \sum_{i=1}^{N_D} \Lambda_i Q_i(Y)$$

where $Q_i(Y)$ varies over all the power products $Y_1^{\beta_1} \cdots Y_s^{\beta_s}$, $\beta_i \geq 0$, in the variables $Y$ of degree smaller or equal than $D$ and $N_D$ is the number of such power products and $\Lambda = (\Lambda_1, \ldots, \Lambda_{N_D})$ is a set of auxiliary variables, which correspond to the “generic” coefficients.

The application of Theorem 1.1, with these variables $\Lambda$ playing the role of the variables $T$ in the statement of Theorem 1.1 is the main tool of the proof of the next two statements.

The first one is about the intersection between an irreducible $K$-variety and a “generic” $K(T)$-hypersurface.

**Theorem 1.3:** Let $K$ be a Hilbertian field of characteristic 0. For $l \geq 1$, let $P_1(Y), \ldots, P_l(Y)$ be polynomials in $K[Y]$ such that $V = V_K(P_1, \ldots, P_l)$ is an irreducible $K$-variety of positive dimension $d$. Then, for every $\Delta \in K^{N_D}$ in a Zariski-dense subset of $K^{N_D}$, the $K$-variety $V \cap V_K(Q_D(\Delta, Y))$ is irreducible and $\dim_K V \cap V_K(Q_D(\Delta, Y)) = d - 1$.

We will see that a more general result, in fact, holds. If we replace the “generic” hypersurface by an intersection of $\rho$ “generic” hypersurfaces, with $\rho \leq d$, the intersection with the original variety $V$ will “often” be an irreducible $K$-variety of dimension $d - \rho$, see Corollary 3.6.
The generic polynomial will be central also in the proof of the last main result of this paper, where the goal is to generalize Theorem 1.1 to the situation in which the variables are specialized at polynomials.

We note that the set $K[\mathbb{Y}]_D$ of all the polynomials of degree smaller or equal than $D$ can be endowed with a Zariski topology through the natural isomorphism with $A_K^{ND}$ which associates to a polynomial $P(\mathbb{Y})$ the point in $A_K^{ND}$ having the coefficients of $P$ as coordinates.

**Theorem 1.4:** Let $K$ be a Hilbertian field of characteristic 0. For $l \geq 1$, let $P_1(T, \mathbb{Y}), \ldots, P_l(T, \mathbb{Y})$ be polynomials in $K[T, \mathbb{Y}]$ such that $V_T = V_{K[T]}(P_1, \ldots, P_l)$ is an irreducible $K[T]$-variety of dimension $d$. Fix non-negative integers $D_1, \ldots, D_r$.

Then for every $U = (U_1(\mathbb{Y}), \ldots, U_r(\mathbb{Y}))$ in a Zariski-dense subset of $\prod_{i=1}^r K[\mathbb{Y}]_{D_i}$, the $K$-variety $V_U = V_{K}(P_1(U, \mathbb{Y}), \ldots, P_l(U, \mathbb{Y}))$ is an irreducible $K$-variety of dimension $\dim_K V_U = d$.

**Remarks 1.5.** (a) The case $l = r = 1$ (i.e. one polynomial and one variable $T$) yields the Schinzel hypothesis for the polynomial ring $K[\mathbb{Y}]$, as stated in [BDN20, Section 1.1]: given an irreducible polynomial $P(T, \mathbb{Y})$ in $K[T, \mathbb{Y}]$, for every $U(\mathbb{Y})$ in some Zariski-dense subset of $K[\mathbb{Y}]_D$, the polynomial $P(U(\mathbb{Y}), \mathbb{Y})$ is irreducible in $K[\mathbb{Y}]$.

(b) By taking $D_i = 0$, for every $i$, Theorem 1.4 implies Theorem 1.1 in characteristic 0 (see Remark 3.9).

**1.2. Hilbert sets.** In this introduction, we have restricted the choice of the base field to a Hilbertian field. However, these statements can be generalized to every field thanks to the notion of Hilbert sets. We are giving only a quick review on this topic; refer to [FJ08, Sections 12,13] for more details.

Given a set of irreducible polynomials $P_1(T, \mathbb{Y}), \ldots, P_l(T, \mathbb{Y})$ in $K(T)[\mathbb{Y}]$, we define the following set:

$$H_K(P_1, \ldots, P_l) = \{ t \in K \mid f_i(t, \mathbb{Y}) \text{ is irreducible in } K[\mathbb{Y}] \text{ for each } i = 1, \ldots, l \}.$$ 

Such sets and their intersections with non-empty open Zariski-subsets are called Hilbert subsets of $K^r$. Alternatively, if we do not specify the dimension $r$, we say that $H_K(P_1, \ldots, P_l)$ is a Hilbert set of $K$ to say that it is a Hilbert subset of $K^r$ for some $r$. This definition does not require any hypothesis on the field.

These sets can be empty: take for example $K = \mathbb{C}$ and $P(T, Y) \in \mathbb{C}(T)[Y]$, an irreducible complex monic polynomial, such that $\deg_Y P > 1$.

Therefore we define the Hilbertian fields, which we have already mentioned, as the fields for which all these sets are Zariski-dense in $K^r$. Of course Hilbertian fields are the most convenient setting because the sets of elements for which our results hold are generally as big as possible.

By definition, Hilbert sets are stable under finite intersection. We will show, in the next sections, that the Zariski-dense subsets involved in Theorems 1.1, 1.3 and
are, in fact, Hilbert sets, with no assumption on the field $K$. Consequently, our main results hold, in fact, for a finite number of varieties/ideals.

In particular, the original Hilbert irreducibility property in its full form, i.e. for several polynomials $P_1, \ldots, P_l$, follows from Theorem 1.1: just apply it to each of the prime ideals $\langle P_i \rangle$ and then take the intersection of the Hilbert sets.

The paper is organized as follows. In Section 2 we will focus on Theorem 1.1: we will see the two steps of its proof and some further remarks. Some more preliminary tools will be added when convenient. In Section 3 we will first give some results about generic polynomials and then, finally, we will use them to obtain the proofs of Theorems 1.3 and 1.4.

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2. Proof of Theorem 1.1

The cornerstone of the article is the proof of Theorem 1.1, which will be essential to prove Theorems 1.3 and 1.4. The statement we are actually going to prove hereinafter is a more general version of Theorem 1.1.

**Theorem 2.1:** Let $K$ be a field. Assume that $p_T = \langle P_1(T, Y), \ldots, P_l(T, Y) \rangle$ is a prime ideal in $K[T, Y]$ such that $p_T \cap K[T] = \{0\}$ and Frac $\left( \frac{K[T, Y]}{p_T} \right)$ is separable over $K(T)$. Denote by $d$ the dimension of $\frac{K[T, Y]}{p_T}$ as a $K[T]$-algebra. Then for every $t$ in a Hilbert subset of $K^r$, the following equivalent statements hold:

(i) The quotient $\frac{K[Y]}{p_t}$ is an integral algebra of dimension $d$ over $K$, where $p_t = \langle P_1(t, Y), \ldots, P_l(t, Y) \rangle$ is the specialized ideal.

(ii) The ideal $p_t$ is prime and its height $\text{ht} p_t$ is equal to the height $\text{ht} p_T$ of $p_T$.

(iii) The $K$-variety $V_t = V(P(t, Y))$, where $P(t, Y)$ is the set made of the specialized polynomials at $t$, is irreducible and its dimension $\dim_K V_t$ is equal to $\dim_{K(T)} V_T$, the dimension of $V_T$ as a $K(T)$-variety.

**Remark 2.2.** Consider the extreme case for which the ideal $p_T$ in Theorem 2.1 is maximal as an ideal in $K(T)[Y]$. Then the quotient $\frac{K(T)[Y]}{p_T}$ is an algebraic separable field extension of $K(T)$ of finite degree. In this case, Theorem 2.1 implies the well-known fact that the degree of the extension is preserved under specialization at every $t$ in a Hilbert set of $K$. Moreover, if the extension is Galois, then also the Galois group of the extension is preserved. The study of the specialization of Galois extensions is central in Inverse Galois Theory, see for example [Vö96, FJ08].

The equivalence between the three statements is easy. The proof is given right after the following lemma, which we will frequently use in the paper.
Lemma 2.3: Let $K$ be a field. Then, given a prime ideal 
\[ p = \langle P_1(T, Y), \ldots, P_l(T, Y) \rangle \subset K[T, Y] \]
such that $p \cap K[T] = \{0\}$, the ideal 
\[ \tilde{p} = \langle P_1(T, Y), \ldots, P_l(T, Y) \rangle \subset K(T)[Y] \]
is prime and its height $\text{ht} \tilde{p}$ is equal to the height $\text{ht} p$ of $p$.

Proof. Denoting $S = K[T] \setminus \{0\}$, we remark that $S$ is a multiplicative subset and $S^{-1}K[T, Y] = K(T)[Y]$. The natural morphism sending an element $a$ in $K[T, Y]$ to $\frac{a}{1}$ in $S^{-1}K[T, Y]$ induces a bijective correspondence between prime ideals in $K[T, Y]$ having empty intersection with $S$ and prime ideals in $K(T)[Y]$ [AM69 Proposition 3.11(iv)]. So we can consider the prime ideal $\tilde{p}$ associated to $p$ by this correspondence, which is the ideal generated by the image of $p$ under the aforementioned morphism: the ideal 
\[ \tilde{p} = \langle P_1(T, Y), \ldots, P_l(T, Y) \rangle \subset K(T)[Y] \]
is prime.

Now we want to check that the height is preserved by this extension. Consider a maximal chain of primes in $p$ in $K[T, Y]$,
\[ p_1 \subset p_2 \subset \ldots \subset p. \]

As $p_i \subset p$ for each $i$, we have $p_i \cap S = \emptyset$, so $\tilde{p}_i$ is prime in $K(T)[Y]$. Since the inclusions are conserved for $\tilde{p}_i$, we have $\text{ht} p_i \leq \text{ht} \tilde{p}$. Vice versa, assume that a chain of primes $\tilde{p}_i$ inside $\tilde{p}$ is longer than $\text{ht} p$: by the previous correspondence we can build a chain of ideals inside $p$ longer than $\text{ht} p$, which is a contradiction. So $\text{ht} p = \text{ht} \tilde{p}$. \hfill \square

We can now give the proof of the equivalence between the three statements of Theorem 2.1.

Proof. (i)$\iff$(ii) It easily follows from the fact that the height of the ideal is equal to the codimension of the quotient algebra by the ideal.

(ii)$\iff$(iii) By Lemma 2.3 $\text{ht} p_T = \text{ht} \tilde{p}_T$. By statement (ii), $\text{ht} p_T = \text{ht} p_T$. So $\text{ht} p_T = \text{ht} \tilde{p}_T$. As the height of a prime ideal is the codimension of the associated variety and the rings $K(T)[Y]$ and $K[Y]$ have the same Krull dimension, statement (iii) follows. The converse is easily shown in the same manner. \hfill \square

There are two requirements for statement (i) of Theorem 2.1: we want $K[T, Y]/p_T$ to be integral and of the correct dimension. We are going to prove the two parts separately: first we show that each integral component of $K[Y]/p_t$ is of dimension $d$, then that there is only one such component.
2.1. **First part.** This part is mostly geometric. We are quickly recalling some tools that we are going to use. The statements are taken from [FGI+05].

**Lemma 2.4 (Local freeness, Lemma 5.11):** Let $A$ be a Noetherian domain and $B$ a finite-type $A$-algebra. Let $M$ be a finite $B$-module. Then there exists $c \in A$, $c \neq 0$ such that the localisation $M[c^{-1}]$ is a free module over $A[c^{-1}]$.

This result, due to Grothendieck, has a deep consequence, the so-called **Generic flatness**.

**Theorem 2.5 (Generic flatness, Theorem 5.12):** Let $S$ be a Noetherian and integral scheme. Let $p : X \to S$ be a finite type morphism and let $F$ be a coherent sheaf of $\mathcal{O}_X$-modules. Then there exists a non-empty open subscheme $U \subset S$ such that the restriction of $F$ to $X_U = p^{-1}(U)$ is flat over $\mathcal{O}_U$.

We do not want to go into the details of this statement, as it falls outside of the aim of this paper. The only things we need to know is, first, that if a ring $A$ is Noetherian, then the structural sheaf of Spec $A$ is coherent as a sheaf of modules over itself [Har77, 5.2.1]. Moreover, if $S$ is an affine integral scheme, i.e. $S = \text{Spec } A$ for some domain $A$, then the open subscheme $U$ in Theorem 2.5 is, indeed, Spec $A[c^{-1}]$ for some $c$ coming from local freeness.

Now we can begin the actual proof of Theorem 2.1.

Here is a diagram including all the maps involved, so to give also the necessary notation:

$$
\begin{array}{c}
K[T, Y] / p \downarrow \quad sp_t \downarrow \\
\downarrow i_t \quad \downarrow i_t \\
K[T] \quad sp_t \quad K
\end{array}
$$

Here $sp_t$ is the specialization map at the fixed point $t \in K^r$. We will show that both maps $i_t$ are injective.

As, by assumption, $p_T \cap K[T] = \{0\}$, we have that $i_T$ is an injection.

For $i_t$ to be well defined and injective, we show that $p_t \cap K = \{0\}$, which is equivalent to showing that $p_t \neq K[Y]$. Consider the ideal $\tilde{p}_T$, which, by Lemma 2.3, satisfies $\tilde{p}_T \subsetneq K(T)[Y]$. By Weak Nullstellensatz [EJ08 Proposition 9.4.1], if $1 \notin \tilde{p}_T \subset K(T)[Y]$, then there exists

$$x(T) = (x_1(T), \ldots, x_s(T)) \in K(T)
$$

such that

$$P_i(T, x(T)) = 0 \quad \forall i = 1, \ldots, l.
$$

For every $t$ outside of a proper Zariski-closed set $C$ of values, we can extend the morphism of specialization $sp_t$ to the $x_i$’s (e.g. [Deb09 Lemma 1.7.3]). Then, denoting $x(t) = (sp_t(x_1(T)), \ldots, sp_t(x_s(T))) \in K^n$, we have that

$$P_i(t, x(t)) = 0 \quad \forall i = 1, \ldots, l
$$

(1)
which implies that \(1 \notin p_t\), so \(p_t \neq K[Y]\).

The above diagram of ring morphisms induces a diagram of scheme morphisms on the spectra of the rings

\[
\begin{array}{ccc}
\text{Spec} \left( K[T,Y]/p_T \right) & \xrightarrow{\text{sp}_{i_T}} & \text{Spec} \left( K[Y]/p_t \right) \\
\downarrow i_T & & \downarrow i_t \\
\text{Spec} K[T] & \xrightarrow{\text{sp}_{i_T}} & \text{Spec} K
\end{array}
\]

We look at the map \(i_T^*\):

- As \(K[T]\) is a Noetherian domain, \(\text{Spec} K[T]\) is a Noetherian and integral scheme;
- As \(K[T,Y]/p_T\) is an algebra of finite type over \(K[T]\), \(i_T^*\) is a morphism of finite type;
- Let \(\mathcal{F}\) be the structural sheaf of \(\text{Spec} K[T]\). Then \(\mathcal{F}\) is coherent on itself.

Then we can apply Generic Flatness (Theorem 2.5): there exists \(c(T) \in K[T]\) such that the following restriction of \(i_T^*\)

\[
i_T^* : \text{Spec} \left( K[T,Y]/p_T \right) \xrightarrow{\text{sp}_{i_T}} \text{Spec} \left( K[Y]/p_t \right)
\]

is flat. This implies, by \cite{Har77} Proposition 9.5, Corollary 9.6, that every irreducible component of \(\text{Spec} \left( K[T,Y]/p_T \right) \left[ c(T)^{-1} \right]\) has dimension \(d\).

This yields the following restriction of the initial diagram for every \(t \in K^r\) such that \(c(t) \neq 0\) and \(t \notin C\):

\[
\begin{array}{ccc}
\text{Spec} \left( K[T,Y]/p_T \left[ c(T)^{-1} \right] \right) & \xrightarrow{\text{sp}_{i_T}} & \text{Spec} \left( K[Y]/p_t \right) \\
\downarrow i_T & & \downarrow i_t \\
\text{Spec} \left( K[T] \left[ c(T)^{-1} \right] \right) & \xleftarrow{\text{sp}_{i_T}} & \text{Spec} K
\end{array}
\]

As the dimension of the fiber at a point is preserved by base change, we can conclude that every irreducible component of \(\text{Spec} K[Y]/p_t\) has dimension \(d\) for every value of \(t \in K^r\) such that \(c(t) \neq 0\) and \(t \notin C\), i.e. for every value of \(t \in K^r\) outside of two proper Zariski-closed sets, whose union is still a Zariski-closed set. Denote this set by \(C_1\).

2.2. Second part. The second stage of the proof is to find \(t\) in \(K^r \setminus C_1\) such that the specialized quotient \(K[Y]/p_t\) is integral. This part has a more algebraic approach and relies on the Noether Normalization Lemma. We are stating below a complete version of this result, coming from the merge of the statements in \cite{Hoc10} and \cite{Eis95} Corollary 13.18. It is readily checked that the two proofs can also be merged to yield the following statement.
LEMMA 2.6 (Noether Normalization Lemma): Let $A$ be an algebra of finite type of dimension $d$ over a domain $R$. Then there exist a nonzero element $c \in R$ and elements $z_1, \ldots, z_d$ in $A \langle c^{-1} \rangle$, algebraically independent over $R \langle c^{-1} \rangle$, such that $A \langle c^{-1} \rangle$ is a module of finite type over its subring $R \langle c^{-1} \rangle \{ z \} := R \langle c^{-1} \rangle \{ z_1, \ldots, z_d \}$. Moreover, set $F = \text{Frac} \ R$ and $L = \text{Frac} \ A$. If $L$ is separable over $F$, then $z$ can be chosen so to be a separating transcendence basis of the extension.

An interesting remark is that the element $c$ satisfying Lemma 2.4 and Theorem 2.5 can also be chosen to satisfy Lemma 2.6. This is clear by looking at the proofs of these results.

Therefore, going back to the proof of Theorem 2.5, we can apply Lemma 2.6 to the situation $A = K[T, Y]/\mathfrak{p}_T$ and $R = K[T]$. We get that

$$K[T, Y]/\mathfrak{p}_T [c(T)^{-1}] = K[T][c(T)^{-1}][z(T)]/\mathfrak{p}_T$$

for $c(T) \in K[T]$ the same as in Section 2.3 $z(T) = (z_1(T), \ldots, z_d(T))$ a separating transcendence basis in $K[T, Y]/\mathfrak{p}_T$ and $\theta(T) = (\theta_1(T), \ldots, \theta_m(T))$ the elements generating $K[T, Y]/\mathfrak{p}_T[c(T)^{-1}]$ as a $K[T][c(T)^{-1}][z(T)]$-module. Moreover, $z(T)$ is also separating, i.e. the field $\text{Frac} (K[T, Y]/\mathfrak{p}_T[c(T)^{-1}])$ is algebraically separable over $K(T, z(T))$.

Set $R_T := K[T][c(T)^{-1}]$ and $A_T := K[T, Y]/\mathfrak{p}_T[c(T)^{-1}]$. We apply the Primitive Element Theorem as in [Mil20, Theorem 5.1]: there exists an element $\alpha(T) \in \text{Frac}(A_T)$ such that

$$\text{Frac}(A_T) = K(T, z(T))((\theta(T)) = K(T, z(T))(\alpha(T)).$$

(2)

Moreover, by [Mil20, Remark 5.2], $\alpha(T)$ can be written as a linear combination

$$\alpha(T) = \sum_{i=1}^{m} \alpha_i(T) \theta_i(T)$$

(3)

with $\alpha_i(T) \in K[T, z(T)]$ and chosen to be integral over $R_T[z(T)]$ (up to multiplying the $\alpha_i(T)$ by some element of $K(T, z(T))$).

For $i = 1, \ldots, m$, let $\delta_i \in R_T[z(T)]$ such that $\delta_i \theta_i(T)$ is integral over $R_T[z(T)]$.

Let $d(T) \in R_T[z(T)]$ be the product of $\delta_1 \cdots \delta_m$ with the discriminant of the $K(T, z(T))$-basis

$$1, \alpha(T), \ldots, \alpha(T)^{m-1}$$

of the $m$-dimensional $K(T, z(T))$-vector space $K(T, z(T))(\alpha(T))$.

As $R_T[z(T)]$ is integrally closed, it is classical (e.g. [Deb09, Théorème 1.3.15(a)]) that

$$d(T) \theta_i(T) \in R_T[z(T)][\alpha(T)] \quad \forall i = 1, \ldots, m.$$

(4)

Moreover, our choice of $\alpha(T)$ implies that its minimal polynomial $p(T, z(T), Y)$ over $K(T, z(T))$ is in $R_T[z(T), Y]$.
The field $K(\mathcal{T}, z(\mathcal{T}), Y)$ is isomorphic to the field $K(\mathcal{T}, W, Y)$ where $W$ is a new set of variables independent of $\mathcal{T}$. Consider the polynomial $p(\mathcal{T}, W, Y)$ image of $p(\mathcal{T}, z(\mathcal{T}), Y)$ via this isomorphism and let

$$H = \{t \in K^r \mid p(t, W, Y) \text{ is irreducible in } K[W, Y]\}$$

be the Hilbert set of $p$.

For every $t \in H \subseteq K^r$, the polynomial $p(t, W, Y)$ is irreducible in $K[W, Y]$.

It is important to remark that, for every $t \in K^r \setminus (C_1 \cup C_2)$, where $C_2$ is the closed set defined by $c(t) = 0$, a specialization morphism can be defined that maps $A_T$ to $A_t[c(t)^{-1}]$ where $A_t = K[Y]_{p_t} / [c(t)^{-1}]$. We denote the images of $z(\mathcal{T})$ and $\mathbf{h}(\mathcal{T})$ via this morphism by $z(t)$ and $\mathbf{h}(t)$ respectively.

Furthermore, after specialization in $T = t \in K^r \setminus (C_1 \cup C_2)$, the elements $z_i(t)$ are still algebraically independent as Section 2.3 implies that the transcendence degree is preserved through specialization at $t$, i.e.

$$d = \text{trdeg}_K \text{Frac}(A_T) = \text{trdeg}_K \text{Frac}(A_t) = \text{trdeg}_K K(z_1(t), \ldots, z_d(t))$$

Therefore, for $t$ outside of $(C_1 \cup C_2)$, $K[z(t)]$ is still a polynomial ring of dimension $d$, hence isomorphic to $K[W]$. As a result, denoting by $\alpha(t)$ the specialization of $\alpha(\mathcal{T})$ given by (3), the polynomial $p(t, z(t), Y) \in K[z(t), Y]$ must also be irreducible for $t \in H \setminus (C_1 \cup C_2)$ so

$$K(z(t))[\alpha(t)] \cong K(z(t))[Y] / (p(t, z(t), Y))$$

is a field.

Specializing $\mathcal{T}$ in $t \in K^r$ outside of the Zariski-closed set $C_3$ defined by $d(t) = 0$, conclusion (4) implies that $\theta_i(t) \in K(z(t))[\alpha(t)]$ for every $i$.

Finally, for $t \in H \setminus (C_1 \cup C_2 \cup C_3)$, which is a Hilbert set, $\theta_i(t) \in K(z(t))[\alpha(t)]$ for $i = 1, \ldots, m$ so $K[z(t)][\theta(t)]$ is a subring of $K(z(t))[\alpha(t)]$, which is a field, so

$$K[z(t)][\theta(t)] \cong K[Y]_{p_t}$$

must be integral. This proves statement (i) of Theorem 2.1.

3. Theorems 1.3 and 1.4

Before discussing the other two main results, we want to focus on an important tool for their proofs: quasi-generic polynomials.

3.1. Quasi-generic polynomials. In the Introduction, we have briefly talked about generic polynomials. In fact, we want to define a larger class of polynomials, the quasi-generic polynomials, of which the generic polynomial is the principal example.
Definition 3.1: Let $K$ be a field, $K[Y]$ the ring of polynomials with coefficients in $K$ and variables $Y$. Given an integer $D \geq 0$, a set

$$S = \{Q_1(Y), \ldots, Q_{|S|}(Y)\} \subseteq \{Y_1^{\beta_1} \cdots Y_s^{\beta_s}, \beta_i \geq 0 \text{ and } \sum_{i=1}^s \beta_i \leq D\}$$

of power products of degree at most $D$, which always contains $Q_1(Y) = 1$ and a polynomial $R(Y) \in K[Y]$, we define the quasi-generic polynomial of base $S, R$:

$$Q_{S,R}(\underline{\lambda}, Y) = \sum_{i=1}^{|S|} \lambda_i Q_i(Y) + R(Y)$$

where $\underline{\lambda} = (\lambda_1, \ldots, \lambda_{|S|})$ is a new set of variables called the set of parameters.

We note that, by taking all the power products for $i = 1, \ldots, |S|$ and $R(Y) = 0$, we obtain the generic polynomial of degree $D$.

The importance of such polynomials is shown in the following lemma.

Lemma 3.2: Let $K$ be a field. Let $p$ be a prime ideal in $K[Y]$ of height $ht \ p$ and $Q_{S,R}(\underline{\lambda}, Y)$ a quasi-generic polynomial. Assume that $S, p$ and $R(Y)$ satisfy hypothesis (H) stated below. Denote by $\mathfrak{P}$ the ideal $(p, Q_{S,R}) \subseteq K[\underline{\lambda}, Y]$ and by $\mathfrak{Q}$ the ideal $(p, Q_{S,R}) \subseteq K(\underline{\lambda})[Y]$. Then $\mathfrak{P}$ is a prime ideal of height $ht \mathfrak{P} = ht \ p + 1$.

To state hypothesis (H), consider the set $E$ of elements in $B := K[Y]/p$ which are algebraic over $K$. Clearly $E$ is a field containing $K$. Let then

$$\varphi_{S,R} : K^{[S]-1} \to B/E$$

be the map sending an $(|S| - 1)$-uple $(a_2, \ldots, a_{|S|})$ to the coset modulo $E$ of the element $\sum_{i=2}^{|S|} a_i Q_i(Y) + R(Y)$.

Definition 3.3: The triple $(p, S, R(Y))$ satisfies hypothesis (H) if

(H) \quad \varphi_{S,R} \text{ is not identically zero.}

Lemma 3.4: (i) If the triple $(p, S, R(Y))$ satisfies hypothesis (H), then $p$ is a non-maximal ideal of $K[Y]$.

(ii) If $p$ is a non-maximal of $K[Y]$ and $\{Y_1, \ldots, Y_s\} \subset S \cup \{R(Y)\}$, then the triple $(p, S, R(Y))$ satisfies hypothesis (H).

Proof. (i) By contradiction, assume that $p$ is maximal. Then $B$ is a $K$-algebra of finite type and a field, so by [AM69 Corollary 5.24] $B$ is an algebraic extension of $K$, hence $B = E$ and $\varphi_{S,R}$ is identically zero.

(ii) By contradiction, assume that $\varphi_{S,R}$ is identically zero. Then, $\varphi_{S,R}(0, \ldots, 0) = 0$ and $\varphi_{S,R}(e_i) = 0$ for every $i$, where $\{e_i, i = 1, \ldots, |S| - 1\}$ is the canonical base of $K^{[S]-1}$ as a $K$-vector space. Then $Y_i \in E$ for every $i$, but $\{Y_i, i = 1, \ldots, s\}$ generates $B$ over $K$, hence $B = E$, i.e. $B$ is a field, which is a contradiction with $p$ being non-maximal. \qed
Proof of Lemma 7.2 First step. We show that $\mathfrak{P}$ is a prime ideal of $K[\Lambda, Y]$.

Using a similar strategy as in [BDN20] Lemma 2.1(a), consider the ring automorphism

$$f : K[\Lambda, Y] \to K[\Lambda, Y]$$

which is the identity on $K[\Lambda_2, \ldots, \Lambda_{|S|}, Y]$ and sends $\Lambda_1$ to $\Lambda_1 - \sum_{i=2}^{|S|} \Lambda_i Q_i(Y) - R(Y)$. The ideal $(p, \mathcal{Q}_{S,R})$ is then sent to the ideal $(p, \Lambda_1)$.

Now consider the specialization morphism, $f_0 : K[\Lambda, Y] \to K[\Lambda_2, \ldots, \Lambda_{|S|}, Y]$ sending $\Lambda_1$ to 0. The ideal $(p)$ in $K[\Lambda_2, \ldots, \Lambda_{|S|}, Y]$ is prime as the following isomorphism shows

$$K[\Lambda_2, \ldots, \Lambda_{|S|}, Y]/(p) \cong K[Y]/p[\Lambda_2, \ldots, \Lambda_{|S|}].$$

So its preimage under $f_0$, i.e. the ideal $p + \ker f_0 = (p, \Lambda_1)$ is also prime.

As a result, the ideal $\mathfrak{P} = (p, \mathcal{Q}_{S,R})$ is prime in $K[\Lambda, Y]$, being sent to a prime ideal by $f$.

Second step. We show that $\mathcal{Q}_{S,R}$ is not invertible in the ring $B_\Lambda := K(\Lambda)[Y]/\hat{p}_\Lambda$, where $\hat{p}_\Lambda$ is the extension of $p$ to $K(\Lambda)[Y]$.

We note that the quotient $B_\Lambda$ is integral and non-trivial. Indeed, the ideal $\hat{p}_\Lambda$ is prime in $K(\Lambda)[Y]$: the ideal $p_\Lambda = (p) \subset K[\Lambda, Y]$ is prime (proceed similarly as in (H)) and $p_\Lambda \cap K[\Lambda] = \{0\}$ because, otherwise, if there was some nonzero $P(\Lambda)$ in $p_\Lambda$, then for every $\Lambda \in K[|S|]$ such that $P(\Lambda) \neq 0$, $P(\Lambda) \in p$, which is a contradiction because $p \neq K[Y]$.

Now, by contradiction, assume that $\mathcal{Q}_{S,R}$ is invertible in $B_\Lambda$.

Then, there exists $\alpha \in B_\Lambda$ such that $\alpha \mathcal{Q}_{S,R} = 1$. As $B_\Lambda = \mathcal{S}^{-1}B[\Lambda]$ with $\mathcal{S} = K[\Lambda]$, we can write $\alpha = \frac{N(\Lambda)}{\mathcal{S}(\Lambda)}$ for $N \in B[\Lambda]$ and $P \in K[\Lambda]$, $P \neq 0$.

As, by hypothesis (H), $\varphi_{S,R}$ is not identically 0, the linear subvariety $V = \varphi_{S,R}(0)$ is of dimension strictly smaller that $|S| - 1$.

Define the set

$$Z := \{ \underline{a} = (a_2, \ldots, a_r) \in K^{|S|-1} : P(\Lambda_1, a_2, \ldots, a_{|S|}) = 0 \}.$$  

If we write $P(\Lambda) = \sum_{i=1}^k p_i(\Lambda_2, \ldots, \Lambda_{|S|}) \Lambda_i^i$, then we see that $Z = \bigcap_{i=0}^k V(p_i)$, where $V(p_i)$ is the zero locus of $p_i$ in $K^{|S|-1}$. We distinguish two cases.

First case. Assume that $K$ is infinite.

The polynomial $P(\Lambda)$ is nonzero, so, in particular, there exists $i$ such that $p_i \neq 0$. As $Z \subset V(p_i)$, then $Z \cup V \subset V(p_i) \cup V$. The set $V(p_i) \cup V$ is a proper closed set because union of two proper closed sets. Thus, if $K$ is infinite, $V(p_i) \cup V \neq K^{|S|-1}$, hence $V \cup Z \neq K^{|S|-1}$.

Take then $\underline{a} \in K^{|S|-1} \setminus (V \cup Z)$. Recall that $N(\Lambda)\mathcal{Q}_{S,R}(\Lambda, Y) = P(\Lambda)$. So, as $\mathcal{Q}_{S,R}(\Lambda, Y) \divides P(\Lambda)$ in $B[\Lambda]$, it follows that $m(\Lambda_1) := \mathcal{Q}_{S,R}(\Lambda_1, \underline{a}, Y)$ divides $p(\Lambda_1) := P(\Lambda_1, \underline{a})$ in $B[\Lambda_1]$.

By construction of $\underline{a}$, we have $p(\Lambda_1) \neq 0$ and $m(\Lambda_1) = \Lambda_1 + Q(Y)$ with $Q(Y) = \sum_{i=2}^{|S|} a_i Q_i(Y) + R(Y)$. Then $\Lambda_1 = -Q(Y)$ is a root of $p(\Lambda_1) = 0$ which, by
construction, has coefficients in $K$ so its roots are algebraic over $K$. But $Q(Y)$ is transcendental over $K$: the coset modulo $E$ of $Q(Y)$ is $\varphi_{S,R}(\underline{a}) \neq 0$ because $\underline{a} \notin V$, so $Q(Y) \notin E$. This is a contradiction.

Second case. Assume that $K$ is finite. Let $K'$ be an algebraic closure of $K$. By [AM69, Theorem 5.10], there exists a prime ideal $p'$ in $K'Y$ such that $p' \cap K[Y] = p$. Moreover, by the Going-up Theorem [AM69, Theorem 5.11] and the incomparability property [AM69, Corollary 5.9] we have $\text{ht } p' = \text{ht } p$.

Replacing $K$ and $p$ by $K'$ and $p'$ we can get back to the first case. Indeed, define $B' := K'[Y]/p'$ and $B'_A := K(\underline{A}) \otimes_{K[\underline{A}]} B'[\underline{A}]$ and apply the first case to the image of $Q_{S,R}$ under the induced homomorphism $B_A \to B'_A$. The image of the polynomial $Q_{S,R}$ is then not invertible in $B'_A$, which implies that $Q_{S,R}$ is not invertible in $B_A$, for otherwise the previous homomorphism yields an invertible element in $B'_A$.

Third step. The fact that $Q_{S,R}$ is not invertible in the ring $B_A$ implies that $\mathfrak{P} \cap K[\underline{A}] = \{0\}$. If this was not the case, we would have $\mathfrak{P} = K(\underline{A})/Y$. But, then, we could find $A(\underline{A}, Y), B(\underline{A}, Y) \in K(\underline{A})/Y$ and $P(Y) \in p$ such that

$$A(\underline{A}, Y)P(Y) + B(\underline{A}, Y)Q_{S,R}(\underline{A}, Y) = 1.$$ 

Reducing this equality modulo $\bar{p}_A$, we would obtain that $Q_{S,R}$ is invertible in $B_A$, which is a contradiction.

Saying that that $\mathfrak{P} \cap K[\underline{A}] = \{0\}$ is also equivalent to saying that $\mathfrak{P}$ is a prime ideal of $K(\underline{A})/Y$, by bijective correspondence [AM69, Proposition 3.11(iv)].

Fourth step. The polynomial $Q_{S,R}$ is not contained in $\bar{p}_A$, i.e. $\bar{p}_A \not\subseteq \mathfrak{P}$. Otherwise, we could write the following relation

$$Q_{S,R}(\underline{A}, Y) = \sum_{i=1}^n A_i(\underline{A}, Y)P_i(Y)$$

for $A_i \in K(\underline{A})/Y$ and $P_i(Y) \in p$. Specializing this equality in $\underline{A} = \underline{0}$ and $\underline{A} = (1, 0, \ldots, 0)$, we would find that $R$ and $1 + R$, respectively, belong to $p$, so $1 \in p$, which is a contradiction.

Fifth step. It follows from $\mathfrak{P}$ being a prime ideal and $\bar{p}_A \not\subseteq \mathfrak{P}$ that the quotient $\mathfrak{P}/\bar{p}_A$ is a nonzero prime ideal of $B_A$.

The ring $B_A$ is integral and Noetherian by construction and the element $Q_{S,R}$ mod $\bar{p}_A$ is nonzero and is not invertible in $B_A$. By Krull’s Height Theorem [Hart77, Theorem 1.11A], the ideal $\mathfrak{P}/\bar{p}_A \subseteq B_A$ has height 1, so the ideal $\mathfrak{P}$ has height $\text{ht } p_A + 1 = \text{ht } p + 1$ in $K(\underline{A})/Y$. \hspace{1cm} \Box

Now, consider the ideal $\mathfrak{P} = \langle p, Q_{S,R} \rangle \subseteq K[\underline{A}, Y]$. If we assume that $K$ has characteristic 0, we have just showed that $\mathfrak{P}$ satisfies all the hypotheses of Theorem 2.1. Its conclusion already proves Theorem 1.3 as it is stated in the Introduction. As promised we will establish a more general version using several quasi-generic polynomials.
In the following sections we present two recursive generalizations of Lemma 3.2 and see how they imply Theorem 3.3 (generalized) and Theorem 3.4.

3.2. Intersection of varieties. Fix $\rho > 0$. For $i = 1, \ldots, \rho$, fix a non-negative integer $D_i$ and then consider the quasi-generic polynomial $Q_{S_i, R_i}(\Lambda, Y)$ of basis of all the power products in the variables $Y$ of degree $\leq D_i$ and $R_i = 0$; the additional variables $\Lambda_i$ form the “set of parameters” of Definition 3.1. In fact, given this choice of $S_i$ and $R_i$, the polynomial $Q_{S_i, R_i}(\Lambda, Y)$ is the generic polynomial of degree $D_i$, so, in this section, we will denote it by $Q_{D_i}(\Lambda, Y)$.

Set $K[\Lambda, Y] = K[\Lambda_1, \ldots, \Lambda_{\rho}, Y]$, where $\Lambda = (\Lambda_1, \ldots, \Lambda_{\rho})$.

The following statement generalizes Lemma 3.2 for this set of data.

**Theorem 3.5:** Let $K$ be a field. Let $p$ be a non-maximal prime ideal of $K[\Lambda, Y]$ such that $\dim_K(K[\Lambda, Y]/p) = d > 0$. Let $Q_{D_1}(\Lambda_1, Y), \ldots, Q_{D_{\rho}}(\Lambda_{\rho}, Y)$ be the generic polynomials defined above for $0 < \rho \leq d$. Then the ideal $P_\rho = (p, Q_{D_1}, \ldots, Q_{D_{\rho}})$ is a prime ideal of $K[\Lambda, Y]$ such that $P_\rho \cap K[\Lambda] = \{0\}$ and

$$\dim_K(K[\Lambda]/P_\rho) = d - \rho,$$

where $\tilde{P}_\rho$ is the extension of $P_\rho$ to $K[\Lambda]/\Lambda$.

**Proof.** We proceed by recursion on $\rho$.

The case $\rho = 1$ is exactly Lemma 3.2 where $P_1$ is the ideal $P$ in the statement of the lemma and, consequently, $\tilde{P}_1$ is the ideal $P$. As previously remarked, the fact that $\text{ht } P_1 = \text{ht } p + 1$ is equivalent to saying that

$$\dim_K(K[\Lambda]/P_1) = \dim_K(K[\Lambda]/p) - 1 = d - 1.$$

For simplicity in the notation, we only explain the case $\rho = 2$. It will then be clear how to prove the case for an arbitrary $\rho \leq d$.

Let $P_1 = (p, Q_{D_1}) \subset K[\Lambda_1, Y]$ be the ideal obtained as in the case $\rho = 1$. As $\dim K[\Lambda_1, Y] \geq \dim K[\Lambda, Y]$ and $\text{ht } P_1 = \text{ht } p + 1$, the ideal $P_1$ is not maximal. Moreover, by Lemma 3.2(ii), the triple $(P_1, S_2, 0)$ satisfies hypothesis (I) because $Q_{D_2}$ is the generic polynomial of degree $D_2$. Therefore, we can apply Lemma 3.2 to $P_1$ and $Q_{D_2}$ and obtain that $P_2 = (p, Q_{D_1}, Q_{D_2})$ is prime in $K[\Lambda_1, \Lambda_2, Y]$ and has height $\text{ht } P_2 = \text{ht } p + 2$, i.e.

$$\dim_K(K[\Lambda]/P_2) = \dim_K(K[\Lambda]/p) - 2 = d - 2.$$

Denote by $V_{p, \Lambda}$ the variety defined by $Q_{D_1}(\Lambda_1, Y), \ldots, Q_{D_{\rho}}(\Lambda_{\rho}, Y)$. If $0 < \rho \leq s$, as it is the case if $0 < \rho \leq d$ as above, a recursive application of Lemma 3.2 starting with $p = (Q_{D_1}(\Lambda_1, Y))$, easily shows that $V_{p, \Lambda}$ is, in fact, an irreducible $K[\Lambda]$-variety of codimension $\rho$, i.e. $Q_{D_1}(\Lambda_1, Y), \ldots, Q_{D_{\rho}}(\Lambda_{\rho}, Y)$ is a prime ideal of height $\rho$ in $K[\Lambda, Y]$. We call $V_{p, \Lambda}$ the generic $K[\Lambda]$-subvariety of codimension $\rho$. 


Using this remark, a general version of Theorem 1.3 follows from conjoining Theorem 3.5 and Theorem 2.1.

**Corollary 3.6.** Let $K$ be a field of characteristic 0. Let $V = V_K(p)$ be an irreducible $K$-variety such that $\dim_K\left(K[\mathcal{Y}]_p\right) = d > 0$. Let $V_{\rho,\lambda}$ be the generic $K[\mathcal{A}]$-subvariety defined above. Then for $\lambda = (\lambda_1, \ldots, \lambda_p)$ in some Hilbert subset of $K^{N_{D_1} + \cdots + N_{D_\rho}}$, the intersection $V \cap V_{\rho,\lambda}$ of $V$ with the $K$-variety $V_{\rho,\lambda}$ obtained by specializing $\mathcal{A}$ at $\lambda$ is an irreducible $K$-variety of dimension $d - \rho$.

Theorem 1.3 is the special case for which $\rho = 1$ and $Q_{S,R}(\mathcal{A}, \mathcal{Y})$ is the generic polynomial of degree $D$.

**Proof.** By Theorem 3.5, the ideal $\mathfrak{P}_p = (p, Q_{D_1}, \ldots, Q_{D_\rho})$ is prime in $K[\mathcal{A}, \mathcal{Y}]$ and $\mathfrak{P}_p \cap K[\mathcal{A}] = \{0\}$. Moreover, as $K$ has characteristic 0, $\text{Frac}\left(K[\mathcal{A}, \mathcal{Y}]_{\mathfrak{P}_p}\right)$ is separable over $K[\mathcal{A}, \mathcal{Y}]$. Then we can apply Theorem 2.1 to $\mathfrak{P}_p$, using statement (iii) of the theorem, for $\lambda = (\lambda_1, \ldots, \lambda_p)$ in a Hilbert subset of $K^{N_{D_1} + \cdots + N_{D_\rho}}$, the $K$-variety

$$V_K(p, Q_{D_1}(\lambda_1, \mathcal{Y}), \ldots, Q_{D_\rho}(\lambda_\rho, \mathcal{Y})) = V \cap V_{\rho,\lambda}$$

is an irreducible $K$-variety of dimension $d - \rho$. \qed

**Remark 3.7.** At the beginning of the section, we chose to take as $Q_{S,R}(\mathcal{A}, \mathcal{Y})$ the generic polynomial of degree $D_i$. However, if we fix the ideal $p$ at the beginning, Corollary 3.6 holds more generally if we take for $S$, a subset of all possible monomials such that the triple $(p, S, 0)$ satisfies hypothesis (H) and Theorem 3.5.

### 3.3. Specialization at polynomials.

In the previous sections the surrounding ring used to define the quasi-generic polynomials was $K[\mathcal{Y}]$, while in this section it will be $K[\mathcal{T}, \mathcal{Y}]$.

Fix $\rho > 0$. For $i = 1, \ldots, \rho$, fix a non-negative integer $D_i$ and then consider the quasi-generic polynomial $Q_{S_i,R_i}(\mathcal{A}, \mathcal{Y})$ of basis the set $S_i$ of all the power products in the variables $\mathcal{Y}$ of degree $\leq D_i$ and $R_i = -T_i$; the additional variables $\mathcal{A}_i$ form the “set of parameters” of Definition 3.1. Thus, we have

$$Q_{S_i,R_i}(\mathcal{A}_i, \mathcal{T}, \mathcal{Y}) = \sum_{j=1}^{N_{D_i}} A_{i,j} Q_j(\mathcal{Y}) - T_i = U_{D_i}(\mathcal{A}_i, \mathcal{Y}) - T_i.$$

Note that $U_{D_i}(\mathcal{A}_i, \mathcal{Y})$, as defined above, is the generic polynomial of degree $D_i$ in the variables $\mathcal{Y}$.

According to the definition of quasi-generic polynomial, the power products could be taken in the variables $\mathcal{T}$ and $\mathcal{Y}$, but we take them only in the variables $\mathcal{Y}$ for our purpose.

The following statement generalizes Lemma 3.2 for this set of data.

**Theorem 3.8.** Let $K$ be a field. Let $p$ be a prime ideal of $K[\mathcal{T}, \mathcal{Y}]$ such that $p \cap K[\mathcal{T}] = \{0\}$ and $\dim_K\left(K[\mathcal{T}]_{\mathfrak{p}}\right) = d > 0$, where $\mathfrak{p}$ is the extension of $p$
to \( K(T)[Y] \). Let \( Q_{S_1,R_1}(\Delta_1, T, Y), \ldots, Q_{S_p,R_p}(\Delta_p, T, Y) \) be the quasi-generic polynomials defined above for \( 0 < \rho \leq r \). Then the ideal \( \mathfrak{P}_S = \langle p, Q_{S_1,R_1}, \ldots, Q_{S_p,R_p} \rangle \) is a prime ideal of \( K[\Delta, T, Y] \) such that \( \mathfrak{P}_S \cap K[\Delta] = \{0\} \) and
\[
\dim_{K(\Delta)} \left( K(\Delta)[T, Y]/\mathfrak{P}_S \right) = d + r - \rho
\]
where \( \mathfrak{P}_S \) is the extension of \( \mathfrak{P}_S \) to \( K(\Delta)[T, Y] \).

**Proof.** We proceed by recursion on \( \rho \).

Assume \( \rho = 1 \). As \( p \cap K[T] = \{0\} \), in particular, \( p \cap K[T_2, \ldots, T_r] = \{0\} \), so the ideal \( \mathfrak{P}_T = \langle p \rangle \subset K(T_2, \ldots, T_r)[T_1, Y] \) is non-maximal by Lemma 2.3.

By construction, the set \( S_1 \) contains all the power products in the variables \( Y \) of degree \( \leq D_1 \), hence \( Y_j \), for all \( j = 1, \ldots, s \), and we have set \( R_1(T, Y) = -T_1 \). So, by Lemma 3.3(ii), the triple \( (\mathfrak{P}_T, S_1, -T_1) \) satisfies hypothesis \( H \) with the ring \( K[Y] \) in Lemma 3.4 replaced by \( K(T_2, \ldots, T_r)[T_1, Y] \).

Therefore, by Lemma 3.2, the ideal
\[
\mathfrak{P}_{T_2,S_1} = \langle p, Q_{S_1,R_1} \rangle \subset K(\Delta_1, T_2, \ldots, T_r)[T_1, Y]
\]
is a prime ideal and \( \dim_{K(\Delta_1)} \mathfrak{P}_{T_2,S_1} = \dim_{K(\Delta_1)} \mathfrak{P}_T + 1 \).

By the classical bijective correspondence between extended and contracted ideals in rings of fractions (e.g. [AM69, Proposition 3.11(iv)]), to the ideal \( \mathfrak{P}_{T_2,S_1} \) we associate the prime ideal
\[
\mathfrak{P}_{S_1} = \langle p, Q_{S_1,R_1} \rangle \subset K(\Delta_1)[T, Y].
\]
In the same manner, we associate the prime ideal
\[
\mathfrak{P}_{S_1} = \langle p, Q_{S_1,R_1} \rangle \subset K[\Delta_1, T, Y]
\]
and, in addition, we have \( \mathfrak{P}_{S_1} \cap K[\Delta_1] = \{0\} \).

Moreover, by Lemma 2.3, \( \dim_{K(\Delta_1)} \mathfrak{P}_{S_1} = \dim_{K(\Delta_1)} \mathfrak{P}_{T_2,S_1} = \dim_{K(\Delta_1)} \mathfrak{P}_T + 1 \), so
\[
\dim_{K(\Delta_1)} \left( K(\Delta_1)[T, Y]/\mathfrak{P}_{S_1} \right) = r + s - \left( \dim_{K(\Delta_1)} \mathfrak{P}_T + 1 \right) = d + r - 1.
\]

For simplicity in the notation, we explain only the case \( \rho = 2 \). The case of an arbitrary \( \rho \leq r \) can be easily deduced.

Consider the prime ideal
\[
\mathfrak{P}_{T_3,S_1} = \langle p, Q_{S_1,R_1} \rangle \subset K(\Delta_1, T_3, \ldots, T_r)[T_1, T_2, Y]
\]
deduced from \( \mathfrak{P}_{S_1} \) by applying the classical bijective correspondence. Denote by
\[
\mathfrak{P}_{T_2,T_3} \subset K(\Delta_1, T_3, \ldots, T_r)[T_2, Y]
\]
the ideal obtained by replacing \( T_1 \) with the generic polynomial, previously denoted by \( D_{T_1}(\Delta_1, Y) \), in \( \mathfrak{P}_{T_3,S_1} \). For \( \rho = 2 \), the ideal \( \mathfrak{P}_{T_2,T_3} \) will play the role played by \( \mathfrak{P}_T \) in the case \( \rho = 1 \).
The ideal $\Phi_{T^*_1,T_3}$ is formally constructed through the quotient morphism, which we denote by $\pi_1$, sending $\Phi_{T_3,S_1}$ to
\[
\Phi_{T_3,S_1}/(Q_{S_1, R_1}) \cong \Phi_{T^*_1,T_3}. \tag{3}
\]
It follows that $\Phi_{T^*_1,T_3}$ is prime. Moreover, by [Eis95, Proposition 9.2], we have $\text{ht} \Phi_{T^*_1,T_3} = \text{ht} \Phi_{T_3,S_1} - 1$. Now, by Lemma 2.3 and the case $\rho = 1$, we have $\text{ht} \Phi_{T_3,S_1} = \text{ht} \Phi_{S_1} = \text{ht} \rho + 1$. Therefore, we obtain:
\[
\text{ht} \Phi_{T^*_1,T_3} = \text{ht} \rho, \tag{7}
\]
so $\Phi_{T^*_1,T_3}$ is a non-maximal prime ideal of $K(\Lambda_1, T_3, \ldots, T_r)[T_2, Y]$.

Consider the polynomial $Q_{S_2, R_2}(\Lambda_2, T, Y)$. By construction, $S_2$ contains $Y_j$ for all $j = 1, \ldots, s$ and $R_2(T, Y) = -T_2$. By Lemma 3.3(ii), the triple $(\Phi_{T^*_1,T_3}, S_2, -T_2)$ satisfies hypothesis (H) with the ring $K[Y]$ in Lemma 3.3 replaced by the ring $K(\Lambda_1, T_3, \ldots, T_r)[T_2, Y]$.

From Lemma 3.2 applied to $\Phi_{T^*_1,T_3}$ and $Q_{S_2, R_2}$, we deduce that the ideal
\[
\Phi_{T^*_1,T_3,S_2} := (\Phi_{T^*_1,T_3}, Q_{S_2, R_2}) \subset K(\Lambda_1, \Lambda_2, T_3, \ldots, T_r)[T_2, Y]
\]
is prime and has height $\text{ht} \Phi_{T^*_1,T_3,S_2} = \text{ht} \rho + 1$.

Using the morphism $\pi_1$, we obtain that the ideal
\[
\Phi_{T^*_1,T_3,S_2} + \ker \pi_1 = (\Phi_{T_3,S_1}, Q_{S_2, R_2}) = (\rho, Q_{S_1, R_1}, Q_{S_2, R_2})
\]
is a prime ideal of $K(\Lambda_1, \Lambda_2, T_3, \ldots, T_r)[T_1, T_2, Y]$ and that its height is equal to
\[
\text{ht} \Phi_{T^*_1,T_3,S_2} + 1 = \text{ht} \rho + 2.
\]

Applying the classical bijective correspondence, the ideal
\[
\Phi_{S_1, S_2} := (\rho, Q_{S_1, R_1}, Q_{S_2, R_2}) \subset K(\Lambda_1, \Lambda_2, T, Y)
\]
is prime and such that $\Phi_{S_1, S_2} \cap K[\Lambda_1, \Lambda_2] = \{0\}$. Moreover, the ideal
\[
\Phi_{S_1, S_2} := (\rho, Q_{S_1, R_1}, Q_{S_2, R_2}) \subset K(\Lambda_1, \Lambda_2)[T, Y]
\]
is prime. By Lemma 2.3 both these ideals have height $\text{ht} \rho + 2$.

In terms of dimensions, this is equivalent to saying that
\[
\dim_{K(\Lambda_1, \Lambda_2)}\left(\frac{K(\Lambda_1, \Lambda_2)[T, Y]}{\Phi_{S_1, S_2}}\right) = d + r - 2.
\]

\[\square\]

Fix $\rho \equiv r$. Theorem 1.4 follows from Theorem 3.8 conjoined with Theorem 2.1. Differently from Theorem 1.1, we need to assume $K$ of characteristic 0 to guarantee the separability required in the statement of Theorem 1.1.

\[\text{3Recall that } Q_{S_1, R_1} = \mathcal{U}_{D_1}(\Lambda_1, Y) - T_1\]
Proof of Theorem \[1.4\] By assumption, \( V_T \) is a \( K(T) \)-variety so the ideal \( \mathfrak{p} := \langle P_1, \ldots, P_l \rangle \) is a prime ideal of \( K(T)[Y] \). Equivalently, \( \mathfrak{p} := \langle P_1, \ldots, P_l \rangle \) is a prime ideal of \( K[T] \) and \( \mathfrak{p} \cap K[T] = \{0\} \).

For \( i = 1, \ldots, r \), let \( Q_{S_i, R_i} \) be the quasi-generic polynomials defined at the beginning of the section, i.e.

\[
Q_{S_i, R_i}(\Lambda_i, T, Y) = \sum_{j=1}^{N_{D_i}} \Lambda_{i,j} Q_j(Y) - T_i = U_{D_i}(\Lambda_i, Y) - T_i.
\]

The polynomials \( Q_{S_i, R_i} \) and \( \mathfrak{p} \) satisfy the hypotheses of Theorem \[3.5\], so the ideal \( \mathfrak{p}_S := \langle p, Q_{S_i, R_i}, \ldots, Q_{S_i, R_i} \rangle \) is a prime ideal of \( K[\Lambda, T, Y] \) such that \( \mathfrak{p}_S \cap K[\Lambda] = \{0\} \) and \( \text{ht} \mathfrak{p}_S = \text{ht} \mathfrak{p} + r \).

Denote by \( \mathfrak{p}_S \) the extension of \( \mathfrak{p}_S \) to \( K(\Lambda)[T, Y] \). By the classical bijective correspondence the ideal \( \mathfrak{p}_S \) is prime.

For \( i = 1, \ldots, r \), denote by \( \pi_i \) the quotient morphism by the ideal \( (Q_{S_i, R_i}) \). Denote by \( \mathfrak{p}_\Lambda \) the ideal of \( K(\Lambda)[Y] \) obtained by replacing \( T_i \) with \( U_{D_i} \) for every \( i \).

Applying, recursively, all the morphisms \( \pi_i \) to the ideal \( \mathfrak{p}_S \) in the same manner as for \( \mathfrak{p}_S \), we obtain that \( \mathfrak{p}_\Lambda \) is a prime ideal of \( K(\Lambda)[Y] \) and

\[
\text{ht} \mathfrak{p}_\Lambda = \text{ht} \mathfrak{p}.
\]

By bijective correspondence, the ideal

\[
\mathfrak{p}_\Lambda = \langle P_1(U(\Lambda, Y), Y), \ldots, P_l(U(\Lambda, Y), Y) \rangle,
\]

where \( U(\Lambda, Y) = \langle U_{D_1}(\Lambda, Y), \ldots, U_{D_l}(\Lambda, Y) \rangle \), is a prime ideal of \( K[\Lambda, T, Y] \) such that \( \mathfrak{p}_\Lambda \cap K[\Lambda] = \{0\} \) and \( \text{ht} \mathfrak{p}_\Lambda = \text{ht} \mathfrak{p} \).

Finally, we can apply \( \text{Theorem 2.4} \) to \( \mathfrak{p}_\Lambda \). For \( \Lambda \) in \( K^{D_1+\ldots+D_r} \), consider the ideal

\[
\mathfrak{p}_\Lambda = \langle P_1(U(\Lambda, Y), Y), \ldots, P_l(U(\Lambda, Y), Y) \rangle = \langle P_1(U(Y), Y), \ldots, P_l(U(Y), Y) \rangle
\]

where \( U(\Lambda) = \langle U_1(\Lambda), \ldots, U_1(\Lambda) \rangle \) and \( U(Y) = \langle U_1(Y), \ldots, U_1(Y) \rangle \) with \( U_i(Y) = U_i(\Lambda, Y) \). By \( \text{Theorem 2.4} \) for every \( \Lambda \) in some Hilbert subset of \( K^{D_1+\ldots+D_r} \), the ideal \( \mathfrak{p}_\Lambda \) is prime and has height \( \text{ht} \mathfrak{p} \), i.e. \( V_U = V_K(\mathfrak{p}_\Lambda) \) is an irreducible \( K \)-variety and

\[
\dim_K V_U = \dim_K \left( K[Y]/\mathfrak{p}_\Lambda \right) = s - \text{ht} \mathfrak{p} = d.
\]

Recalling the isomorphism between \( K_{K^{D_1+\ldots+D_r}} \) and \( \prod_{i=1}^{r} K[\Lambda_i]_{D_i} \) that we mentioned in the Introduction, taking a Hilbert subset of \( K^{D_1+\ldots+D_r} \) is equivalent to taking a Hilbert subset of \( \prod_{i=1}^{r} K[\Lambda_i]_{D_i} \).

\( \square \)

Remark 3.9. As we mentioned in Remark \[3.5\] (b), taking \( D_i = 0 \), for every \( i \), implies \( \text{Theorem 3.4} \) in characteristic \( 0 \). Indeed, for every \( i = 1, \ldots, r \), take

\[
Q_{S_i, R_i} = \Lambda_{i,1} - T_i.
\]
The map $\varphi_{R_i,S_i}$ sends 0, the only point of $K^0$, to $-T_i$. The element $-T_i$ is clearly transcendental over $K$ and, by hypothesis, $-T_i$ is not in $p$, so $\varphi_{R_i,S_i}$ is not identically 0 for every $i$. Therefore, we apply Theorem 1.4 and Theorem 1.1 follows.

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