Off-diagonal mixed state phases in unitary evolution

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Off-diagonal mixed state phases based upon a concept of orthogonality adapted to unitary evolution and a proper normalisation condition are introduced. Some particular instances are analysed and parallel transport leading to the off-diagonal mixed state geometric phase is delineated. A complete experimental realisation of the off-diagonal mixed state geometric phases in the qubit case using polarisation-entangled two-photon interferometry is proposed.

1 Introduction

When two quantal states are orthogonal their relative phase is indeterminate as they do not interfere. Yet, there may be a unitary path connecting such states and along this path there accumulates phase information that in part reflects the curvature of the subjacent state space. Thus, it seems pertinent to ask: is there a way to retain this particular information about the curvature when the path connects orthogonal states?

This issue was settled in the pure state case only quite recently by Manini and Pistolesi\textsuperscript{1}. In essence, their idea may be understood by considering a unitarity $U$ and a complete set of orthonormalised pure states represented by the one dimensional projectors $\{P_k = |A_k\rangle\langle A_k|\}$ in terms of which one may define a family of off-diagonal phase factors

$$
\gamma^{(l)}_{j_1 j_2 \ldots j_l} = \Phi[\text{Tr}(U P_{j_1} U P_{j_2} \ldots U P_{j_l})], \ l = 1, \ldots, N.
$$

Here, $\Phi[z] = z/|z|$, $N$ is the dimensionality of the Hilbert space, and all $j_k$ in the set are different. These phase factors are manifestly independent of the choice of Hilbert space representatives $\{|A_k\rangle\}$, and hence measurable in principle. Furthermore, they are independent of cyclic permutations of the

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indexes $j_1, j_2, \ldots j_l$, they contain the standard Pancharatnam phase factor as $l = 1$, they reflect the curvature of the subjacent state space if $U$ parallel transports $\{|A_k\}\}$, and experimental test of the $l = 2$ case has been reported for spin polarised neutrons.

In this paper, we wish to elaborate on the concept of off-diagonal phases for mixed states proposed in. For parallel transporting unitarities this defines a family of off-diagonal mixed state geometric phases that reflects the geometry of the subjacent state space. These phases extend the concept of mixed state phase in to cases where the latter is undefined.

Unitary maps of a complete orthonormal set of states are characterised and the concomitant concept of quantum parallel transport is delineated in the next section. We propose an operationally natural notion of orthogonality adapted to unitarily connected density matrices in section 3. This is used for the off-diagonal mixed state geometric phase proposed in section 4. Section 5 contains some explicit examples and a complete experimental realisation of the off-diagonal mixed state geometric phases in the qubit case is proposed in section 6. The paper ends with the conclusions.

## 2 $U(N)$, $SU(N)$, and quantum parallel transport

Any complete orthonormal basis of a finite dimensional Hilbert space is mapped unitarily to another complete orthonormal basis. Here, we provide some general remarks on unitary maps of such bases that are pertinent in the context of off-diagonal mixed state phases.

Consider Hilbert space $\mathcal{H}$ of finite dimension $N$. Any unitary map acting on $\mathcal{H}$ can be decomposed as $U(N) = U(1) \times SU(N)$. As is clear from Eq. (1), the $U(1)$ part factors out and contributes a factor $[U(1)]^l$ to $\gamma^{(l)}$.

We may also consider parallel transporting unitarities. Indeed, parallel transport of a pure quantum state, probably first put forward in, plays an important role in the theory of geometric phases as in this case the dynamical contributions along the path are assured to vanish. In the context of off-diagonal phases, it proves useful to extend this and consider parallel transport of a set of orthonormal pure states.

Consider a complete orthonormal basis $|A_k\rangle$ of $\mathcal{H}$. A continuous one-parameter family of unitarities $U(s)$ is said to parallel transport the basis $|A_k\rangle$ if it fulfils

$$\langle A_k|U^\dagger \dot{U}|A_k\rangle = 0, \quad \forall k,$$

which is equivalent to having no local accumulation of phase along the unitary path for each $|A_k\rangle$. We may further notice that any unitarity $U(s)$ may be
written as

\[ U(s) = \mathcal{P} \exp \left( -i \int_0^s J(s') ds' \right) , \]

(3)

\(J(s)\) being Hermitian and \(\mathcal{P}\) is path ordering. Thus, equivalent to Eq. (2) is

\[ \langle A_k | J(s) | A_k \rangle = 0, \ \forall k, \]

(4)

which entails that \(J(s)\) has to be off-diagonal in the parallel transported basis and therefore traceless in any basis. Thus, \(U \in SU(N)\) is a necessary (but not sufficient) condition for \(U\) being parallel transporting a complete basis.

The conditions in (2) or equivalently in (4) define a nontrivial fibre bundle with structure group being isomorphic to the \(N\) torus. In the context of mixed states, this is the relevant bundle structure in \(^5\). Uhlmann \(^7\) has provided another concept of mixed state parallel transport that defines a \(U(N)\) bundle and differs both conceptually and physically \(^8\) from that of \(^5\). In this report, we focus on the extension of \(^5\) to the off-diagonal case.

3 Orthogonality

Generalisation of the off-diagonal phases to the mixed state case requires an appropriate notion of “orthogonality” between unitarily connected density matrices. In this section, we propose a simple definition of this based upon interference.

In order to develop this idea, let us first suppose \(|A\rangle\) and \(|B\rangle\) are Hilbert space representatives of two arbitrary pure quantal states \(A\) and \(B\), and assume further that \(|A\rangle\) is exposed to the variable \(U(1)\) shift \(e^{i\chi}\). The resulting interference pattern obtained in superposition is determined by the intensity profile

\[ I = \left| e^{i\chi} |A\rangle + |B\rangle \right|^2 = 2 + 2 |\langle A|B\rangle| \cos[\chi - \arg \langle A|B\rangle]. \]

(5)

The key point here is to note that \(A\) and \(B\) are orthogonal if and only if \(I\) is independent of \(\chi\).

To extend this idea to the mixed state case, consider a pair of unitarily connected density operators

\[ \rho_A = \sum_k \lambda_k |A_k\rangle \langle A_k| \rightarrow \rho_B = \sum_k \lambda_k |B_k\rangle \langle B_k|, \]

(6)

\(^d\)Less restrictive conditions may be put on \(U\) by considering parallel transport in a fixed \(K < N\) dimensional subspace of \(\mathcal{H}\). In such a case \(U(N) = SU(K) \times U(N - K)\), where \(SU(K)\) parallel transport some basis of this subspace. Such unitarities are useful when considering rank \(< N\) density operators.

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where each $|B_k\rangle = U|A_k\rangle$. Evidently, each such orthonormal pure state component of the density operator contributes to the interference according to Eq. (5). Thus, the total intensity profile becomes

$$I = \sum_k \lambda_k \left| e^{i\chi}|A_k\rangle + |B_k\rangle \right|^2 = 2 + 2 \sum_k \lambda_k |\langle A_k|B_k\rangle| \cos[\chi - \text{arg}\langle A_k|B_k\rangle],$$

where we have used that the $\lambda$'s sum up to unity. Following the above pure state case, we say that $\rho_A$ and $\rho_B$ are orthogonal if and only if $I$ is independent of $\chi$ for all Hilbert space representatives $\{|A_k\rangle\}$ and $\{|B_k\rangle\}$ of the eigenstates of $\rho_A$ and $\rho_B$, respectively. It follows that $\rho_A \perp \rho_B$ if and only if $\langle A_k|B_k\rangle = 0$, $\forall k$.

For an $N$ dimensional Hilbert space $\mathcal{H}$, we may generate a set of $N$ mutually orthogonal density operators as follows. Assume there is a unitary operator $U_g$ such that $|A_n\rangle = (U_g)^n - 1^n|A_1\rangle$, $n = 1, \ldots, N$, is a complete orthonormal basis of $\mathcal{H}$. Explicitly, we may write

$$U_g = |A_1\rangle\langle A_N| + |A_N\rangle\langle A_{N-1}| + \ldots + |A_2\rangle\langle A_1|.$$  

(8)

Now, if $\rho_1|A_k\rangle = \lambda_k|A_k\rangle$ then

$$\rho_n = (U_g)^{n-1}\rho_1(U_g)^{n-1}, \ n = 1, \ldots, N$$  

(9)

is a set of mutually orthogonal density operators. Explicitly, this entails that

$$\rho_1 = \lambda_1|A_1\rangle\langle A_1| + \lambda_2|A_2\rangle\langle A_2| + \ldots + \lambda_N|A_N\rangle\langle A_N|,$$

$$\rho_2 = \lambda_1|A_2\rangle\langle A_2| + \lambda_2|A_3\rangle\langle A_3| + \ldots + \lambda_N|A_1\rangle\langle A_1|,$$

$$\ldots,$$

$$\rho_N = \lambda_1|A_N\rangle\langle A_N| + \lambda_2|A_{N-1}\rangle\langle A_{N-1}| + \ldots + \lambda_N|A_1\rangle\langle A_1|. $$

(10)

### 4 Off-diagonal mixed states phases

In this section, we propose the off-diagonal phases for mixed states, based upon the concept of orthogonality described above. To do this, we only need to determine how the mutually orthogonal density operators should appear in the trace. This may be resolved by noting that:

- For $U = U_g^\dagger$ that permutes $|A_N\rangle \rightarrow |A_{N-1}\rangle \rightarrow \ldots \rightarrow |A_1\rangle \rightarrow |A_N\rangle$, we have

$$\text{Tr}(U_g^\dagger P_1 U_g^\dagger P_2 \ldots U_g^\dagger P_N) = 1, $$

(11)

due to normalisation $\langle A_k|A_k\rangle = 1$, $\forall k$. 

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The ansatz $\rho^{p/q}_k$ reduces to $P_k$ in the limit of pure states, if $p$ and $q$ are integers. Note here that $\rho^{p/q}_k$ is well-defined as $\rho_k \geq 0$.

Replace $P_k$ with $\rho^{p/q}_k$ in Eq. (1) and consider $l = N$ with $j_k = k$, $\forall k$. For $U = U_g^\dagger$ we have

$$\text{Tr}(U_g^{l/p/q} U_g^{l/p/q} \ldots U_g^{l/p/q}) = \text{Tr}(\rho_N^{N/p/q}),$$

where we have used that $(U_g \rho U_g^\dagger)^{p/q} = U_g \rho^{p/q} U_g^\dagger$ for any integers $p$ and $q$. Normalisation yields $p = 1$ and $q = N$. Now, for general $l \leq N$, assume that $p$ and $q$ are completely determined by $l$, we may write $p(l) = \sum_m a_m l^m$ and $q(l) = \sum_m b_m l^m$, with $l$–independent integer coefficients $\{a_m\}$ and $\{b_m\}$. From $p(N) = 1$ and $q(N) = N$ for any $N$, we obtain $a_m = \delta_{m,0}$ and $b_m = \delta_{m,1}$. Thus, $p = 1$ and $q = l$.

We are now ready to state our main result: the off-diagonal mixed state phase for an ordered set of $l \leq N$ mutually orthogonal density matrices $\rho_{j_k}$, $k = 1, \ldots, l$, transported by $U$ is naturally given by

$$\gamma_{l/p/q}^{(l)} \equiv \Phi[\text{Tr}(U \sqrt{\rho_{j_1}} U \sqrt{\rho_{j_2}} \ldots U \sqrt{\rho_{j_l}})].$$

This is manifestly gauge invariant and independent of cyclic permutations of the indexes $j_1, j_2, \ldots, j_l$. By construction it reduces to Eq. (1) in the limit of pure states. If $U$ is parallel transporting then Eq. (13) defines a family of off-diagonal mixed state geometric phases. Furthermore, just as in the pure state case, any $U(1)$ component of $U$ contributes here with a factor $[U(1)]^l$ to $\gamma^{(l)}$.

The mixed state phase

$$\gamma_{p_1}^{(1)} = \Phi[\text{Tr}(U \rho_{j_1})]$$

proposed in $^5$ may be seen as a natural consequence of this general framework if we put $l = 1$. In section $^6$ we propose an experimental realisation of the $l = 2$ case

$$\gamma_{p_1, p_2}^{(2)} = \Phi[\text{Tr}(U \sqrt{\rho_{j_1}} U \sqrt{\rho_{j_2}})]$$

in polarisation-entangled two-photon interferometry.

Note that if $\rho_1$ and $U_g$ commutes for some $n$ then $\rho_n = \rho_1$. This may happen for $n \neq 1$ if and only if $\rho_1 = I/N$, $I$ being the identity operator on $\mathcal{H}$, i.e. for the maximally mixed state. In such a case $\rho_1 = \rho_2 = \ldots = \rho_N$ and all $\gamma^{(l)}$ solely reflect properties of $U$. 


5 Examples

In the qubit case, consider an $SU(2)$ operator

$$U = U_{11} |A_1⟩⟨A_1| + U_{12} |A_1⟩⟨A_2| + U_{21} |A_2⟩⟨A_1| + U_{22} |A_2⟩⟨A_2|$$  \hspace{1cm} (16)

in the common eigenbasis of the mutually orthogonal $ρ_1$ and $ρ_2$. This yields

$$\text{Tr}(U ρ_1) = η(λ_1 e^{iα} + λ_2 e^{-iα})$$

$$\text{Tr}(U ρ_2) = η(λ_1 e^{-iα} + λ_2 e^{iα})$$

$$\text{Tr}(U \sqrt{ρ_1} U \sqrt{ρ_2}) = 2η^2 \sqrt{λ_1 λ_2} \cos 2α - 1 + η^2,$$  \hspace{1cm} (17)

where we have used $U_{11} = U_{22} = η e^{iα}$ and $U_{12} U_{21} = -\det U + U_{11} U_{22} = -1 + η^2$ for $SU(2)$. If $U$ is parallel transporting then $α$ is the geodesically closed solid angle enclosed by the Bloch vector.

In the nondegenerate case $λ_1 \neq λ_2$, the $l = 1$ phases are indeterminate only for $η = 0$, for which the $l = 2$ phase is well-defined since $\text{Tr}(U \sqrt{ρ_1} U \sqrt{ρ_2}) = -1$. In the degenerate case $λ_1 = λ_2 = \frac{1}{2}$, $\text{Tr}(U ρ_1)$ and $\text{Tr}(U ρ_2)$ have additional nodal points as discussed in $9,10$. These occur whenever $\cos α = 0$, at which angles $\cos 2α = -1$ and we again have $\text{Tr}(U \sqrt{ρ_1} U \sqrt{ρ_2}) = -1$. Thus, $γ^{(1)}$ and $γ^{(2)}$ never become indeterminate simultaneously and thus provide a complete phase characterisation of the qubit case.

The complexity of the analysis increases rapidly with $N$. For simplicity we therefore focus on two important special kinds of unitarities for $N \geq 2$:

(i) Diagonal unitarities

$$U_d = \sum_{k=1}^{N} U_{kk} |A_k⟩⟨A_k|.$$  \hspace{1cm} (18)

Here, all $|U_{kk}| = 1$ and for $SU(N)$ we have $U_{11} U_{22} \ldots U_{NN} = +1$. If $U_d$ is parallel transporting then $U_{kk} = γ^{(1)}_k$, $γ^{(1)}_k$ being the cyclic geometric phase factor of the pure state $A_k$.

(ii) Permuting unitarities, which may be written as

$$U_p = U_{12} |A_1⟩⟨A_2| + U_{23} |A_2⟩⟨A_3| + \ldots + U_{N1} |A_N⟩⟨A_1|.$$  \hspace{1cm} (19)

Here, all $|U_{12}| = |U_{23}| = \ldots = |U_{N1}| = 1$ and in the case of $SU(N)$ we have $U_{12} U_{23} \ldots U_{N1} = (-1)^{N-1}$.

These two cases can be considered as extremes in the sense that (i) corresponds to cyclic evolution of the common eigenstates of the $ρ$’s while (ii) is a particular instance where each of these eigenstates evolves into an orthogonal state. Combinations of these two extremes are discussed in $4$. 

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First, let us consider the diagonal case. We have

\[ \text{Tr}(U_d \sqrt{\rho_{j_1}} \ldots U_d \sqrt{\rho_{j_l}}) = \sum_{k=1}^{N} (U_{kk})^l \sqrt{\lambda_{k_1} \ldots \lambda_{k_l}}. \]  

(20)

As each term contains precisely \( l \) \( \lambda \)'s, it follows that all \( \text{Tr}(U_d \sqrt{\rho_{j_1}} \ldots U_d \sqrt{\rho_{j_l}}) \) must vanish if \( l > \text{rank of the } \rho \)'s.

In the permutation case we first notice that

\[ U_p \sqrt{\rho_k} = \sqrt{\lambda_1} U_{k-1,k} |A_{k-1}\rangle \langle A_k| + \ldots + \sqrt{\lambda_{N-k+1}} U_{N-1,N} |A_{N-1}\rangle \langle A_N| + \sqrt{\lambda_{N-k+2}} U_{N1} |A_N\rangle \langle A_1| + \ldots + \sqrt{\lambda_N} U_{k-2,k-1} |A_{k-2}\rangle \langle A_{k-1}| \]  

(21)

with the identifications \( U_{01} \equiv U_{N1} \) and \( |A_0\rangle \equiv |A_N\rangle \). Thus, multiplying \( l \) such factors results in a sum of operators of the form \( |A_N\rangle \langle A_N| \ldots |A_1\rangle \langle A_1| \). For \( SU(N) \), this implies that we may write

\[ \text{Tr}(U_p \sqrt{\rho_{j_1}} \ldots U_p \sqrt{\rho_{j_N}}) = (-1)^{N-1} f^{(N)}_{\rho_{j_1} \ldots \rho_{j_N}} (\lambda_1, \ldots, \lambda_N), \]  

(22)

where each \( f^{(N)} \) is determined by the sequence of \( \rho \)'s.

The \( f \)'s in Eq. (22) have some interesting properties. First, it can be seen that

\[ f^{(N)}_{\rho_{j_1} \ldots \rho_{j_N}} = 1, \forall N. \]  

(23)

Thus, there exist at least one well-defined off-diagonal mixed state phase for \( U_p \), independent of the rank of the \( \rho \)'s. Secondly, we have

\[ f^{(N)}_{\rho_{j_1} \ldots \rho_{j_N}} \geq 0, \forall j_1, \ldots, j_N. \]  

(24)

This implies that the off-diagonal mixed state phases for \( U_p \) are completely determined by the dimension of the Hilbert space \( \mathcal{H} \). Indeed, for sequences where \( f^{(N)} \neq 0 \) we have

\[ \gamma^{(N)} = -1, \text{ if dim}(\mathcal{H}) \text{ even}, \]

\[ \gamma^{(N)} = +1, \text{ if dim}(\mathcal{H}) \text{ odd}. \]  

(25)
6 Two-photon experiment

When considering the issue of experimental realisation of the off-diagonal mixed state phases we immediately encounter a problem: how do we experimentally implement the \( l \)th root of density operators? Fortunately, this may be resolved in the \( l = 2 \) case in the sense of purification, i.e. by adding an ancilla system in a certain way. Here, we demonstrate this in the qubit case in terms of an explicit experiment for polarisation-entangled photon pairs. The set up is sketched in Fig. 1.

Consider an ensemble of linearly polarised photons with polarisation degree \( r \). In the horizontal-vertical (\( h - v \)) basis, there are two possible unitarily equivalent and orthogonal representations of the ensemble in terms of the density operators
\[
\rho_1 = \frac{1 + r}{2} |h\rangle\langle h| + \frac{1 - r}{2} |v\rangle\langle v|,
\rho_2 = \frac{1 - r}{2} |h\rangle\langle h| + \frac{1 + r}{2} |v\rangle\langle v|.
\]
(26)

A purification of any of these density operators may be achieved by adding an ancilla photon in such a way that the photon pair is in a pure polarisation state whose partial trace over the ancilla is the density operator. The polarisation-entangled state
\[
|\Psi_1\rangle = \sqrt{\frac{1}{2}(1 + r)} |h\rangle \otimes |h\rangle + \sqrt{\frac{1}{2}(1 - r)} |v\rangle \otimes |v\rangle,
\]
(27)

which has been demonstrated in\(^{12}\), is an example of a purification of \( \rho_1 \).

For simplicity, we consider unitarities that rotate linear polarisation states along great circles an angle \( \beta \) on the Poincaré sphere. This amounts to
\[
U(\beta, \theta) = \exp \left( -i\beta \left[ \cos \theta (|h\rangle\langle v| + |v\rangle\langle h|) + \sin \theta (-i|h\rangle\langle v| + i|v\rangle\langle h|) \right] \right),
\]
(28)

which fulfils the parallel transport conditions in Eqs. (2) and (4) with respect to the \( h - v \) basis. \( U(\beta, \pi/2) \) takes linear polarisation into linear polarisation with plane of polarisation rotated an angle \( \beta \). An important special case is the polarisation flip \( F = U(\pi/2, \pi/2) \) that connects \( \rho_1 \) and \( \rho_2 \). Furthermore, for \( \theta = 0 \) and \( \beta = \pi/4 \), circular polarisation states are obtained.

We shall now demonstrate how purification may be used in the set up shown in Fig. 1 to test \( \gamma^{(1)} \) and \( \gamma^{(2)} \) for \( \rho_1 \) and \( \rho_2 \) in the case of \( U = U(\beta, \theta) \). With \( |\Psi_1\rangle \) as input, the intensity detected in coincidence is\(^{13}\)
\[
I = |U_s \otimes U_a |\Psi_1\rangle + V_s \otimes V_a |\Psi_1\rangle|^2
= 2 + 2\Re \left[ \langle\Psi_1| U_s^\dagger V_s \otimes U_a^\dagger V_a |\Psi_1\rangle \right],
\]
(29)
where we have used that simultaneous detection occurs only the photons both either took the shorter path or the longer path (assuming sufficiently short coincidence window). By appropriate choices of the unitarities shown in Fig. 1, we may obtain the $\gamma$’s as follows.

- $\gamma^{(1)}_{\rho_1}$: Choose $U_s = e^{i\chi}$, $V_s = U(\beta, \theta)$, and $U_a = V_a = I$ yielding
  \[
  \langle \Psi_1 | U_1^\dagger V_s \otimes U_1^\dagger V_a | \Psi_1 \rangle = e^{-i\chi} \text{Tr} [U(\beta, \theta) \otimes I | \Psi_1 \rangle \langle \Psi_1 |] \\
  = e^{-i\chi} \text{Tr}_s [U(\beta, \theta) \rho_1],
  \]
  where we have used that $\text{Tr}_a [\rho_1] = \rho_1$. Thus, $\arg \gamma^{(1)}_{\rho_1}$ is the shift obtained by variation of $\chi$. Explicit calculation for $U(\beta, \theta)$ in Eq. (28) entails that $\gamma^{(1)}_{\rho_1}$ is real-valued and changes sign at $\beta = (j + \frac{1}{2})\pi$, $j$ integer, corresponding to a sequence of phase jumps of $\pi$.

- $\gamma^{(1)}_{\rho_2}$: Choose $U_s = e^{i\chi} F$, $V_s = U(\beta, \theta) F$, and $U_a = V_a = I$ yielding
  \[
  \langle \Psi_1 | U_1^\dagger V_s \otimes U_1^\dagger V_a | \Psi_1 \rangle = e^{-i\chi} \text{Tr} [U(\beta, \theta) F \otimes I | \Psi_1 \rangle \langle \Psi_1 | F^\dagger] \\
  = e^{-i\chi} \text{Tr}_s [U(\beta, \theta) \rho_2],
  \]
  where we have used that $\text{Tr}_a [F | \Psi_1 \rangle \langle \Psi_1 | F^\dagger] = \rho_2$. Thus, $\arg \gamma^{(1)}_{\rho_2}$ is the shift obtained by variation of $\chi$. Also $\gamma^{(1)}_{\rho_2}$ is real-valued and changes sign at $\beta = (j + \frac{1}{2})\pi$, $j$ integer, for $U(\beta, \theta)$.

- $\gamma^{(2)}_{\rho_1, \rho_2}$: Choose $U_s = e^{i\chi} F$, $V_s = U(\beta, \theta)$, $U_a = F$, and $V_a = U(\beta, -\theta)$ yielding
  \[
  \langle \Psi_1 | U_1^\dagger V_s \otimes U_1^\dagger V_a | \Psi_1 \rangle = e^{-i\chi} \text{Tr} [U(\beta, \theta) \otimes U(\beta, -\theta) \\
  \times | \Psi_1 \rangle \langle \Psi_1 | F^\dagger \otimes F^\dagger].
  \]
  Explicit calculation yields
  \[
  \text{Tr} [U(\beta, \theta) \otimes U(\beta, -\theta) | \Psi_1 \rangle \langle \Psi_1 | F^\dagger \otimes F^\dagger] = \text{Tr}_s [U \sqrt{\rho_1} U \sqrt{\rho_2}].
  \]
  Thus, $\arg \gamma^{(2)}_{\rho_1, \rho_2}$ is the shift obtained by variation of $\chi$. Furthermore, we may compute the expected output as
  \[
  \text{Tr}_s [\sqrt{\rho_1} U \sqrt{\rho_2} U] = \sqrt{1 - r^2} \cos^2 \beta - \sin^2 \beta,
  \]
  which is independent of $\theta$ and can be positive and negative for $r \neq 1$ depending upon $\beta$. Such an experiment would test that the off-diagonal geometric phase is either $0$ or $\pi$ for mixed qubit states.
7 Conclusions

The concept of geometric phase has recently been extended to cases where the standard definition breaks down. Such cases occur if a unitarity connects orthogonal pure states\(^1\) or if a unitarity connects mixed states\(^5\). Here we have reported on a unification of these extensions: the off-diagonal mixed state phase that also covers situations where mixed states do not interfere in the sense of\(^5\).

Although the present off-diagonal mixed state phases are properties of the system (they are expressed solely in terms of a set of density operators pertaining to the system) experimental realisations thereof seem to require control and measurement of one or possibly several additional ancilla systems. We have proposed an explicit Franson interferometer set up for polarisation-entangled photon pairs as a complete experimental realisation of the off-diagonal mixed state phase in the qubit case. Such an experiment would in particular demonstrate a nontrivial sign change property of the off-diagonal qubit phase that is associated with the mixed state case. We hope that the ideas reported here would trigger new experimental tests as well as to further theoretical considerations of off-diagonal phases.

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References

1. N. Manini and F. Pistolesi, Phys. Rev. Lett. 85, 3067 (2000).
2. S. Pancharatnam, Proc. Indian Acad. Sci. A 44, 247 (1956).
3. Y. Hasegawa, R. Loidl, M. Baron, G. Badurek, and H. Rauch, Phys. Rev. Lett. 87, 070401 (2001).
4. S. Filipp and E. Sjöqvist, http://xxx.lanl.gov/abs/quant-ph/0209087.
5. E. Sjöqvist, A.K. Pati, A. Ekert, J.S. Anandan, M. Ericsson, D.K.L. Oi, and V. Vedral, Phys. Rev. Lett. 85, 2845 (2000).
6. B. Simon, Phys. Rev. Lett. 51, 2167 (1983).
7. A. Uhlmann, Rep. Math. Phys. 24, 229 (1986).
8. M. Ericsson, A.K. Pati, E. Sjöqvist, J. Brännlund, and D.K.L. Oi, http://xxx.lanl.gov/abs/quant-ph/0206063.
9. R. Bhandari, Phys. Rev. Lett. 89, 268901 (2002).
10. J.S. Anandan, E. Sjöqvist, A.K. Pati, A. Ekert, M. Ericsson, D.K.L. Oi, and V. Vedral, Phys. Rev. Lett. 89, 268902 (2002).
11. J.D. Franson, Phys. Rev. Lett. 62, 2205 (1989).
12. P.G. Kwiat, E. Waks, A.G. White, I. Appelbaum, and P.H. Eberhard, Phys. Rev. A 60, R773 (1999).
13. B. Hessmo and E. Sjöqvist, Phys. Rev. A 62, 062301 (2000).
Figure Captions

Fig.1. Franson set up for polarisation-entangled photon pairs. In the longer arms, the system and ancilla photons are exposed to the unitarities $U_s$ and $U_a$, respectively, and similarly $V_s$ and $V_a$ in the shorter arms.
