Higher cumulants of voltage fluctuations in current-biased diffusive contacts

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The third and fourth cumulants of voltage in a current-biased diffusive metal contact of resistance \( R \) are calculated for arbitrary temperatures and voltages using the semiclassical cascade approach. The third cumulant equals \( c^2 R^3 T^1/3 \) at high temperatures and \( 4e^2 R^3 T/15 \) at low temperatures, whereas the fourth cumulant equals \( 2e^2 R^3 T/3 \) at high temperatures and \( (34/105)e^3 R^3 I \) at low temperatures.

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Recently, higher cumulants of current in mesoscopic conductors received a significant attention of theorists.\(^1\) This work was pioneered by Levitov and Lesovik,\(^2\) who found that the charge transmitted through a single-channel quantum contact at zero temperature is distributed according to a binomial law. Subsequently, these calculations were extended to conductors with a large number of quantum channels such as diffusive wires\(^3\) and chaotic cavities.\(^4\)–\(^6\) More recently, the third cumulant of current was calculated for a tunnel contact with interacting quasiparticles.\(^7\) The third and fourth cumulants of current were also calculated for diffusive-metal contacts for arbitrary temperatures and voltages.\(^8\),\(^9\)

Common to all these papers was that they considered the fluctuations of current or charge transmitted through a contact at a constant voltage drop across it. This allowed the authors to treat independently the charge transmitted through different quantum channels and at different energies. The assumption of constant voltage is justified if the resistance of the external circuit is much smaller than that of the conductor. In actual experiments, the opposite relation is quite possible and in this case one can speak of fluctuations of the voltage drop across the conductor at a constant current. For a system with an Ohmic conduction the second cumulant of voltage is just the second cumulant of current in the voltage-biased regime times \( R^2 \), where \( R \) is the resistance of the conductor. One might think that higher cumulants of voltage and current are related in a similar way, but this is not the case. Very recently Kindermann, Nazarov, and Beenakker\(^10\) showed that higher cumulants of voltage and current in the current- and voltage-biased conductors are not related in such a simple way. In particular, they calculated the low-temperature third and fourth cumulants of charge transmitted through a multichannel conductor connected in series with a macroscopic resistor to a voltage source and found that these cumulants present nonlinear functions of current cumulants of the same conductor in the voltage-biased mode.

The purpose of this paper is to calculate the third and fourth cumulants of voltage in a current-biased diffusive-metal contact for arbitrary temperatures and to show how the recently proposed semiclassical cascade approach should be modified in that case. The key point for this approach is that the system is described by at least two distinct variables whose fluctuations are characterized by essentially different time scales. Fluctuations of the "slow" variable modulate the intensity of noise sources for the "fast" variable and result in additional higher-order correlations. Therefore the higher cumulants of the fast variable may be recursively expressed in terms of its lower-order cumulants. Originally, this method was proposed for diffusive metals.\(^9\) Later it was proved to be equivalent to the rigorous quantum-mechanical approach for chaotic cavities.\(^12\) More recently similar recursive relations were obtained as a saddle-point expansion of a stochastic path integral.\(^13\) The papers\(^9\),\(^12\) addressed the case of purely elastic scattering in the low-frequency limit at constant voltage bias where the fluctuations of the electric potential are inessential and the cascade expansions can be made with respect to only one parameter, the distribution function \( f(z) \). For a current-biased conductor this is not the case any more and the electric potential explicitly enters into the expressions. Here we show how the cascade expansion should be generalized to the case where the system is described by two different slow variables.

Consider a quasi-one-dimensional diffusive contact of length \( L \) and resistance \( R \) connected in series with a resistor of larger cross section that has yet much larger resistance \( R_S \gg R \) (see Fig. 1). Because of a strong energy relaxation in the resistor the local distribution of electrons is Fermian with a temperature equal to that of the bath, yet the local electric potential may fluctuate. Assume that the resistor is connected to the left end of the contact and the right end of the contact is grounded.

![FIG. 1. Mesoscopic conductor connected in series with a voltage source and an external load.](https://example.com/fig1.png)
A large constant voltage is applied to the left end of the series resistor. Therefore the noise of the resistor may be considered as Gaussian. The fluctuations of current in the circuit are determined by the larger resistance, hence the spectral density of the current noise $S_I = 4T/R_S$ is extremely small because of large $R_S$. Therefore the current through the contact may be considered as constant even at a nonzero temperature. We will be interested in fluctuations of the electric potential at the left end of the contact, which actually presents the voltage drop across it.

In what follows we consider only fluctuations in the zero-frequency limit and will neglect the pile-up of charge. It will be implied that all the subsequent equations contain only low-frequency Fourier transforms of the corresponding quantities. A fluctuation of current inside the contact is given by

$$\delta j = -\sigma \nabla \delta \phi + \delta j^{\text{ext}},$$  \hspace{1cm} (1)

where $\sigma$ is the conductivity of the metal, $\delta \phi$ is a fluctuation of the electric potential, and

$$\delta j^{\text{ext}}(r) = eN_F \int d\delta F^{\text{ext}}.$$  \hspace{1cm} (2)

The Fourier transform of the correlator of extraneous sources $\delta F^{\text{ext}}$ is expressed in terms of the average distribution function $f$ via a formula\textsuperscript{14}

$$\langle \delta F^{\text{ext}}(\varepsilon, r, \phi_{0}) \delta F^{\text{ext}}(\varepsilon', r') \rangle = \frac{D}{N_F} \delta (r - r') \times \delta (\varepsilon - \varepsilon') \delta (\phi_{0}) [1 - f(\varepsilon)].$$  \hspace{1cm} (3)

Integrating Eq. (1) over the contact volume, one obtains

$$\delta I = \frac{eN_F}{L} \int d^3r \int d\varepsilon \delta F^{\text{ext}} + \frac{1}{R} \delta \phi_0.$$  \hspace{1cm} (4)

where $\delta \phi_0$ is a fluctuation of electric potential at the left end of the contact and $R$ is the contact resistance. As the current fluctuations are negligibly small, one may set $\delta I = 0$, so that at low frequencies

$$\delta \phi_0 = -R \frac{eN_F}{L} \int d^3r \int d\varepsilon \delta F_x.$$  \hspace{1cm} (5)

Hence the second cumulant of voltage $\phi_0$ is related to the second cumulant of current in the voltage-biased contact in a trivial way

$$\langle \langle \phi_0^3 \rangle \rangle = R^2 \langle \langle I^2 \rangle \rangle |_{\phi_0 = \text{const}},$$

$$\langle \langle I^2 \rangle \rangle = \frac{2}{RL} \int_0^L dx \int d\varepsilon f(\varepsilon, x) [1 - f(\varepsilon, x)].$$  \hspace{1cm} (6)

The characteristic time scale for $\delta j^{\text{ext}}$ and $\delta F^{\text{ext}}$ is the elastic scattering time. As this time is much shorter than the $RC$ time and the time of diffusion across the contact that describe the evolution of fluctuations of voltage $\delta \phi_0$ and the distribution function $\delta f(\varepsilon, r)$. Hence one may perform a cascade expansion of higher cumulants with respect to these slow variables.

Consider first the third cumulant of voltage $\langle \langle \phi_0^3 \rangle \rangle$. Since the third cumulant of extraneous currents is vanishingly small in the diffusive limit,\textsuperscript{9} the bare third cumulant of voltage fluctuations obtained by a direct multiplication of three equations (5) is also small. Hence the third cumulant of voltage should be given by a cascade correction

$$\langle \langle \phi_0^3 \rangle \rangle = 3 \int_0^L dx \int d\varepsilon \frac{\delta \langle \langle \phi_0^2 \rangle \rangle}{\delta f(\varepsilon, x)} \langle \delta f(\varepsilon, x) \delta \phi_0 \rangle,$$  \hspace{1cm} (7)

where the functional derivative

$$\frac{\delta \langle \langle \phi_0^3 \rangle \rangle}{\delta f(\varepsilon, x)} = \frac{2R}{L} [1 - 2f(\varepsilon, x)].$$  \hspace{1cm} (8)

is easily calculated from Eq. (6). The key difference from the case of constant voltage is that the fluctuation $\delta f$ should be calculated now taking into account the feedback from the environment. The fluctuations of voltage $\phi_0$ caused by random scattering in the contact result in fluctuations of the distribution function at the right end of the resistor, which presents the boundary condition for the distribution function in the contact. Hence the fluctuation of a distribution function is a sum

$$\delta f(\varepsilon, x) = \delta f(\varepsilon, x) + \frac{\partial f(\varepsilon, x)}{\partial \phi_0} \delta \phi_0,$$  \hspace{1cm} (9)

where $\delta \phi_0$ is given by Eq. (5) and

$$\delta f(\varepsilon, x) = (D \nabla^2)^{-1} \nabla \delta F^{\text{ext}}$$

is the “intrinsic” part of fluctuation directly caused by random scattering. The last term in Eq. (9) mixes together fluctuations at different energies so that they are not independent any more. In the case of purely elastic scattering in the contact, the average distribution function $f(\varepsilon, x)$ in the contact is given by

$$f(\varepsilon, x) = \psi(x) f_0(\varepsilon - e\phi_0) + \bar{\psi}(x) f_0(\varepsilon)$$  \hspace{1cm} (10)

where $\phi_0 = IR$ is the voltage drop across the contact and $\psi(x) = 1 - x/L$ and $\bar{\psi}(x) = x/L$ are the characteristic potentials\textsuperscript{15} of the left and right electrodes. Hence the derivative in Eq. (9) is just

$$\frac{\partial f(\varepsilon, x)}{\partial \phi_0} = -e\psi(x) \frac{\partial f_0(\varepsilon - e\phi_0)}{\partial \varepsilon}.$$  \hspace{1cm} (11)

Multiplying Eqs. (9) and (5) and averaging the product with use of Eq. (3), one easily obtains that

$$\langle \delta f(\varepsilon, x) \delta \phi_0 \rangle = -2eRU(\varepsilon, x) + R^2 \frac{\partial f(\varepsilon, x)}{\partial \phi_0} \langle \langle I^2 \rangle \rangle,$$  \hspace{1cm} (12)
where

\[ U(\varepsilon, x) = \frac{1}{T}(\nabla^2)^{-1} \frac{\partial}{\partial x} f(1 - f). \]

A substitution of Eqs. (8) and (12) into Eq. (7) gives

\[ \langle\langle \phi_0^3 \rangle\rangle = \frac{1}{30} eIR^2 \left[ 8T^2 \sinh \left( \frac{eIR}{T} \right) - 2eIRT \right] + 4eIRT \cosh \left( \frac{eIR}{T} \right) - 5e^2 I^2 R^2 \coth \left( \frac{eIR}{2T} \right) \]

\[ \left[ T \sinh^2 \left( \frac{eIR}{T} \right) \right]. \tag{13} \]

This expression reduces to

\[ \langle\langle \phi_0^3 \rangle\rangle = \frac{1}{3} e^2 R^3 I \]

at low current or high temperature \( eIR \ll T \) and to

\[ \langle\langle \phi_0^3 \rangle\rangle = \frac{4}{15} e^2 R^3 I \]

at high current or low temperature \( eIR \gg T \). In the former case \( \langle\langle \phi_0^3 \rangle\rangle \) coincides with \(-R^3\langle\langle I^3 \rangle\rangle\), but in the latter case it is four times larger. On the whole, the temperature dependence of \( \langle\langle \phi_0^3 \rangle\rangle \) at a given current appears to be more flat than that of \( \langle\langle I^3 \rangle\rangle \) at a given voltage. Equation (13) is in an agreement with the temperature-dependent third cumulant of voltage obtained by Beenakker et al.\(^{11}\) One may also obtain its low-temperature limit from the formula for the third cumulant of current of Kindermann et al.\(^{10}\) using voltage-biased cumulants for a diffusive contact.\(^{8}\)

Unlike the third cumulant, the fourth cumulant cannot be expressed in terms of only functional derivatives with respect to \( \delta f \). The point is that the correlator (12) explicitly depends on the voltage drop \( \phi_0 \) through the derivative \( \partial f_0(\varepsilon - e\phi_0)/\partial \varepsilon \). Therefore one has to perform the cascade expansion with respect to \( \delta \tilde{f} \) and \( \delta \phi_0 \) considering them as different stochastic variables. The rules for constructing the diagrams remain basically the same as for the case of a voltage-biased contact,\(^{9}\) but now a variation of any fluctuating quantity should be taken twice, i.e. with respect to \( \delta \tilde{f}(\varepsilon, x) \) and \( \delta \phi_0 \). Hence the number of terms significantly increases:

\[ \langle\langle \phi_0^4 \rangle\rangle = 6S_1 + 12S_2 + 6S_3 + 12(S_4 + S_5 + S_6 + S_7) + 3S_8 + 6S_9 + 3S_{10}, \tag{14} \]

where

\[ S_1 = \int d\varepsilon_1 \int d\varepsilon_2 \int dx_1 \int dx_2 \frac{\delta^2 \langle\langle \phi_0^2 \rangle\rangle}{\delta f(\varepsilon_1, x_1) \delta f(\varepsilon_2, x_2)} \]

\[ S_2 = \int d\varepsilon \int dx \frac{\delta^2 \langle\langle \phi_0^2 \rangle\rangle}{\delta \tilde{f}(\varepsilon, x) \delta \phi_0} \langle\langle \delta \tilde{f}(\varepsilon, x) \delta \phi_0 \rangle\rangle \langle\langle \phi_0^2 \rangle\rangle, \tag{16} \]

\[ S_3 = \frac{\partial^2 \langle\langle \phi_0^2 \rangle\rangle}{\partial \phi_0^2} \langle\langle \phi_0^2 \rangle\rangle, \tag{17} \]

\[ S_4 = \int d\varepsilon_1 \int d\varepsilon_2 \int dx_1 \int dx_2 \frac{\delta \langle\langle \phi_0^2 \rangle\rangle}{\delta f(\varepsilon_1, x_1)} \]

\[ \times \left( \delta \tilde{f}(\varepsilon_1, x_1) \delta \phi_0 \right) \langle\langle \delta \tilde{f}(\varepsilon_2, x_2) \delta \phi_0 \rangle\rangle, \tag{18} \]

\[ S_5 = \int d\varepsilon \int dx \frac{\delta \langle\langle \phi_0^2 \rangle\rangle}{\delta \tilde{f}(\varepsilon, x)} \langle\langle \delta \tilde{f}(\varepsilon, x) \delta \phi_0 \rangle\rangle \langle\langle \phi_0^2 \rangle\rangle, \tag{19} \]

\[ S_6 = \frac{\partial \langle\langle \phi_0^2 \rangle\rangle}{\partial \phi_0} \int d\varepsilon \int dx \frac{\delta \langle\langle \phi_0^2 \rangle\rangle}{\delta \tilde{f}(\varepsilon, x)} \langle\langle \delta \tilde{f}(\varepsilon, x) \delta \phi_0 \rangle\rangle, \tag{20} \]

\[ S_7 = \left( \frac{\partial \langle\langle \phi_0^2 \rangle\rangle}{\partial \phi_0} \right)^2 \langle\langle \phi_0^2 \rangle\rangle, \tag{21} \]

\[ S_8 = \int d\varepsilon_1 \int d\varepsilon_2 \int dx_1 \int dx_2 \frac{\delta \langle\langle \phi_0^2 \rangle\rangle}{\delta f(\varepsilon_1, x_1)} \]

\[ \times \left( \delta \tilde{f}(\varepsilon_1, x_1) \delta \phi_0 \right) \langle\langle \delta \tilde{f}(\varepsilon_2, x_2) \delta \phi_0 \rangle\rangle, \tag{22} \]

\[ S_9 = \int d\varepsilon \int dx \frac{\delta \langle\langle \phi_0^2 \rangle\rangle}{\delta \tilde{f}(\varepsilon, x)} \langle\langle \delta \tilde{f}(\varepsilon, x) \delta \phi_0 \rangle\rangle \frac{\partial \langle\langle \phi_0^2 \rangle\rangle}{\partial \phi_0}, \tag{23} \]

and

\[ S_{10} = \left( \frac{\partial \langle\langle \phi_0^2 \rangle\rangle}{\partial \phi_0} \right)^2 \langle\langle \phi_0^2 \rangle\rangle. \tag{24} \]

The numerical prefactors 6, 12, 3 in Eq. (14) present the numbers of inequivalent permutations of \( \phi_0 \) in the corresponding expressions. The functional derivative with respect to \( \phi_0 \) is defined as

\[ \frac{\delta \langle\langle \phi_0 \rangle\rangle}{\delta \phi_0} = \frac{\partial \langle\langle \phi_0 \rangle\rangle}{\partial \phi_0} + \int d\varepsilon \int dx \frac{\delta \langle\langle \phi_0 \rangle\rangle}{\delta \tilde{f}(\varepsilon, x)} \frac{\partial \tilde{f}(\varepsilon, x)}{\partial \phi_0}. \]

The sum \( 6S_1 + 12S_4 + 3S_8 \) gives just \( R^4 \langle\langle I^4 \rangle\rangle \), where \( \langle\langle I^4 \rangle\rangle \) is the fourth cumulant of current for a current-biased contact with a voltage drop \( \langle\phi_0\rangle \). The sum \( 12S_2 + 12S_5 \) is easily brought to a form

\[ 12 \int d\varepsilon \int dx \frac{\partial}{\partial \phi_0} \left( \frac{\delta \langle\langle \phi_0^2 \rangle\rangle}{\delta \tilde{f}(\varepsilon, x)} \langle\langle \delta \tilde{f}(\varepsilon, x) \delta \phi_0 \rangle\rangle \right) \langle\langle \phi_0^2 \rangle\rangle \]
\[ \langle \langle \langle \tilde{I}^3 \rangle \rangle \rangle \partial_{\phi_0} (\langle \langle \tilde{I}^2 \rangle \rangle) = 4R^4 \frac{\partial (\langle \langle \tilde{I}^3 \rangle \rangle)}{\partial_{\phi_0}} (\langle \langle \tilde{I}^2 \rangle \rangle). \]

The integrals in \( S_6 \) and \( S_9 \) also present cumulants \( \langle \langle \tilde{I}^3 \rangle \rangle \) for a voltage-biased contact. The whole expression (14) assumes the form

\[ \langle \langle \phi_4^0 \rangle \rangle = R^4 \langle \langle \tilde{I}^4 \rangle \rangle + 6R^6 \frac{\partial^2 (\langle \langle \tilde{I}^2 \rangle \rangle)}{\partial_{\phi_0}^2} (\langle \langle \tilde{I}^2 \rangle \rangle)^2 \]

\[ + 15R^6 \left( \frac{\partial (\langle \langle \tilde{I}^2 \rangle \rangle)}{\partial_{\phi_0}} \right)^2 (\langle \langle \tilde{I}^2 \rangle \rangle) - 6R^5 \frac{\partial (\langle \langle \tilde{I}^3 \rangle \rangle)}{\partial_{\phi_0}} (\langle \langle \tilde{I}^2 \rangle \rangle). \]  \hspace{1cm} (25)

Substituting the cumulants of current for the voltage-biased contact, one easily obtains

\[ \langle \langle \phi_4^0 \rangle \rangle = \frac{1}{2520} e^2 R^3 \left\{ 51eIRT^2 \cosh \left( \frac{5eIR}{2T} \right) \right. \]

\[ + 72T^3 \sinh \left( \frac{5eIR}{2T} \right) \]

\[ - (456T^3 + 224e^2 I^2 R^2 T) \sinh \left( \frac{3eIR}{2T} \right) \]

\[ + (70e^3 T^3 R^3 - 399eIRT^2) \cosh \left( \frac{3eIR}{2T} \right) \]

\[ + 1008T^3 \sinh \left( \frac{eIR}{2T} \right) \]

\[ + (560e^3 T^3 R^3 + 348eIRT^2) \cosh \left( \frac{eIR}{2T} \right) \}

\[ / \left[ T^2 \sinh \left( \frac{eIR}{2T} \right) \right]^5. \]  \hspace{1cm} (26)

This expression reduces to

\[ \langle \langle \phi_4^0 \rangle \rangle = \frac{2}{3} e^2 R^3 T \]

in the high-temperature limit, which differs from the corresponding cumulant of current in a voltage-biased contact just by a factor \( R^4 \). In the high-current limit,

\[ \langle \langle \phi_4^0 \rangle \rangle = \frac{34}{105} e^3 R^4 I. \]

This value is in an agreement with the formula for the low-temperature fourth cumulant of transmitted charge of Kindermann et al.\(^\text{10}\) Unlike the fourth cumulant of current, the fourth cumulant of voltage is positive for all currents and temperatures and its numerical prefactor in the high-current limit is larger by more than an order of magnitude than that of the fourth cumulant of current at high voltages.

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