TORIC FANO MANIFOLDS WITH NEF TANGENT BUNDLES

QILIN YANG

Abstract. In this note we prove that any toric Fano manifold with nef tangent bundle is a product of projective spaces. In particular, it implies that Campana-Peternell's conjecture holds for toric manifolds.

1. Notation and main result

We will use standard notation for polytopes and toric varieties, as can be found in [CLS],[Fu],[Od].

Let $N \cong \mathbb{Z}^d$ be a $d$-dimensional lattice and $M = \text{Hom}_\mathbb{Z}(N, \mathbb{Z}) \cong \mathbb{Z}^d$ the dual lattice with $\langle \cdot, \cdot \rangle$ the nondegenerate pairing. As usual, $N_\mathbb{Q} = N \otimes \mathbb{Q}$ $\cong \mathbb{Q}^d$ and $M_\mathbb{Q} = M \otimes \mathbb{Q}$ $\cong \mathbb{Q}^d$ (respectively $N_\mathbb{R}$ and $M_\mathbb{R}$) will denote the rational (respectively real) scalar extensions.

A subset $P \subseteq M_\mathbb{R}$ is called a polytope if it is the convex hull of finitely many points in $M_\mathbb{R}$. The face of $P$ is denoted by $F \preceq P$. The set of vertices and facets of $P$ are denoted by $V(P)$ and $F(P)$ respectively. If $V(P) \subseteq M_\mathbb{Q}$ (respectively $V(P) \subseteq M_\mathbb{R}$) then $P$ is called a rational polytope (respectively a lattice polytope).

If $P$ is a rational polytope with $0 \in \text{int}P$, the dual polytope of $P$ is defined by

$$P^* := \{y \in N_\mathbb{R}|(x, y) \geq -1, \forall x \in P\},$$

which is also a rational polytope with $0 \in \text{int}P^*$. The fan $\mathcal{N}_P := \{\text{pos}(F) : F \preceq P^*\}$ is called the normal fan of $P$. Here pos$(F)$ denotes the cone positively generated by the face $F$ (also called positive hull of $F$). It is well-known that a fan $\Sigma$ in $N_\mathbb{R}$ defines a toric variety $X_\Sigma := X(N, \Sigma)$, which automatically admits a torus action and has a Zariski open and dense orbit:

$$T_N \times X_\Sigma \to X_\Sigma,$$

where $T_N \cong \text{Hom}_\mathbb{Z}(M, \mathbb{C}^*)$. We denote $X_P := X_{\mathcal{N}_P}$ the toric variety associated with the normal fan $\mathcal{N}_P$ of the polytope $P$. It is known that $X_P$ is nonsingular if and only if the vertices of any facet of $P^*$ form a $\mathbb{Z}$-basis of the lattice $M$.

A $d$-dimensional polytope $P \subseteq M_\mathbb{R}$ with $0 \in \text{int}P$ is called reflexive polytope if both $P$ and $P^*$ are lattice polytopes. A complex variety $X$ is called a Gorenstein Fano variety if $X$ is projective, normal and its anticanonical divisor is an ample Cartier divisor. The following theorem (see [Ni1]) classifies Gorenstein toric Fano varieties by reflexive polytopes:

**Theorem 1.1.** Under the map $P \mapsto X_P$ reflexive polytopes correspond uniquely up to isomorphism to Gorenstein toric Fano varieties. There are only finitely many isomorphism types of $d$-dimensional reflexive polytopes.

A Cartier divisor $D$ on a nonsingular variety $X$ is called a nef divisor if the intersection number $D \cdot C \geq 0$ for any irreducible curve $C \subseteq X$. A line bundle $L$ is called a nef line bundle if the associated Cartier divisor (i.e., $L = \mathcal{O}_X(D)$) is a nef divisor. A vector bundle $E$ over $X$ is called a nef vector bundle if the tautological
line bundle $\mathcal{O}_{P(E^*)}(1)$ on the projective bundle $P(E^*)$ is a nef line bundle. In [CP], Campana and Peternell conjectured that any Fano manifold with nef tangent bundle is a rational homogeneous manifold. In this note we confirm this conjecture for toric Fano manifold, in fact we get more and obtain the following main theorem:

**Theorem 1.2.** Any toric Fano manifold with nef tangent bundle is a product of projective spaces.

2. Cartier divisors on complete toric varieties

A fan $\Sigma$ in $N_\mathbb{R}$ is complete iff its support $|\Sigma| := \cup_{\sigma \in \Sigma} \sigma$ is the whole space $N_\mathbb{R}$, which is also equivalent to that the associated toric variety $X(N, \Sigma)$ is compact in classical topology ([CLS, Theorem 3.1.9]).

Let $\Sigma(k)$ denote the set of $k$-dimensional cones of the complete fan $\Sigma$. The elements in $\Sigma(1)$ are called rays, and given $\tau \in \Sigma(1)$, let $u_\tau$ denote the unique minimal generator of $N \cap \tau$. By orbit-cone correspondence ([CLS, Theorem 3.2.6]), a ray $\tau \in \Sigma(1)$ gives a $T_N$-invariant Cartier divisor $D_\tau$. On a complete toric variety we may write any Cartier divisor as a linear combination of $T_N$-invariant Cartier divisors. Let $D = \sum_{\tau \in \Sigma(1)} a_\tau D_\tau$ be a Cartier divisor on a complete toric variety $X_\Sigma$, its support function $\phi_D : N_\mathbb{R} \to \mathbb{R}$ is determined by the following properties:

1. $\phi_D$ is linear on each cone $\sigma \in \Sigma$.
2. $\phi_D(u_\tau) = -a_\tau$.
3. For each cone $\sigma \in \Sigma$ there is a $m_\sigma \in M$ such that $\phi_D(u) = \langle m_\sigma, u \rangle$ for all $u \in \sigma$ and $\langle m_\sigma, u_\tau \rangle = -a_\tau$ for all $\tau \in \sigma(1)$.

**Proposition 2.1.** ([CLS, Theorem 6.1.10 and Theorem 6.2.12]) Let $D = \sum_{\tau \in \Sigma(1)} a_\tau D_\tau$ be a Cartier divisor on a complete toric variety $X_\Sigma$ and denote

$$P_D = \{m \in M_\mathbb{R} | \langle m, u_\tau \rangle \geq -a_\tau, \forall \tau \in \Sigma(1)\}.$$  

Then the following are equivalent:

1. $D$ is basepoint free.
2. $D$ is a nef divisor.
3. $\phi_D$ is a upper convex function.
4. $m_\sigma \in P_D$ for all $\sigma \in \Sigma(d)$.
5. $\phi_D(u) = \min_{m \in P_D} \langle m, u \rangle$ for all $u \in N_\mathbb{R}$.

The support function $\phi_D$ of a Cartier divisor $D$ on a complete toric variety $X_\Sigma$ is called strictly convex if it is upper convex and for each $\sigma \in \Sigma(d)$ satisfies

$$\langle m_\sigma, u \rangle = \phi_D(u) \iff u \in \sigma.$$  

**Proposition 2.2.** ([CLS, Theorem 6.1.15 and Corollary 6.1.16]) A Cartier divisor $D$ on a complete toric variety $X_\Sigma$ is ample if and only if its support function $\phi_D$ is strictly convex. If $D$ is ample then $P_D$ is a full dimensional lattice polytope whose normal fan is $\Sigma$.

3. Complete toric variety with reductive automorphism group

The automorphism group $\text{Aut}(X_\Sigma)$ of a nonsingular complete toric variety $X_\Sigma$ was firstly studied by Demazure in [De, Section 4]. Identifying the elements of the Lie algebra of $\text{Aut}(X_\Sigma)$ with the invariant differential operators on the coordinate ring of $X_\Sigma$, Demazure gave a very simple description of the structure of the Lie algebra of $\text{Aut}(X_\Sigma)$ using the Demazure root system named after him. The Demazure root system $\mathcal{R}$ of $\text{Aut}(X_\Sigma)$ has a very simple description:

$$\mathcal{R} = \{m \in M | 2 \tau \in \Sigma(1) : \langle u_\tau, m \rangle = -1, \langle u_{\tau'}, m \rangle \geq 0, \forall \tau' \in \Sigma(1) \setminus \{\tau\}\}.$$  

Note here we use notation of [Ni1, Ni2], which are different form those in [De] by a minus signature. The Demazure roots in $\mathcal{R} \cap -\mathcal{R} = \{m \in \mathcal{R} | -m \in \mathcal{R}\}$ are called
semisimple roots. \( \text{Aut}(X_\Sigma) \) is a reductive algebraic group iff all Demazure roots in \( \mathcal{R} \) are semisimple, i.e., \( \mathcal{R} = -\mathcal{R} := \{ -m | m \in \mathcal{R} \} \). The following proposition of Nill, Benjamin’s will be used in proving our main theorem in the next section.

**Proposition 3.1.** [Ni2, Proposition 3.18] A \( d \)-dimensional complete toric variety is isomorphic to a product of projective spaces iff there are \( d \)-linearly independent semisimple roots.

4. Toric Fano manifolds with nef tangent bundles

The projective space \( \mathbb{P}^n \) is a toric Fano manifold and the following exact sequence

\[
0 \rightarrow \mathcal{O}_{\mathbb{P}^n} \rightarrow \mathcal{O}_{\mathbb{P}^n}(1)^{d+1} \rightarrow T_{\mathbb{P}^n} \rightarrow 0
\]

is called the Euler sequence of \( \mathbb{P}^d \). The following theorem is a toric generalization of this result.

**Theorem 4.1.** ([CLS, Theorem 8.1.6]) Let \( X_\Sigma \) be a toric manifold associated with the complete fan \( \Sigma \), then we have the following generalized Euler sequence

\[
0 \rightarrow \mathcal{O}_{X_\Sigma}^\oplus \rightarrow \oplus_{\tau \in \Sigma(1)} \mathcal{O}_{X_\Sigma}(D_\tau) \rightarrow T_{X_\Sigma} \rightarrow 0,
\]

where \( \rho \) is the Picard number of \( X_\Sigma \).

In particular, the canonical divisor and anticanonical divisor of the toric manifold \( X_\Sigma \) are respectively given by

\[
K = -\sum_{\tau \in \Sigma(1)} D_\tau; \quad K^* = \sum_{\tau \in \Sigma(1)} D_\tau.
\]

**Proposition 4.2.** Assume \( X_\Sigma \) is a toric manifold with nef tangent bundle, then for any \( \tau \in \Sigma(1) \), the associated \( T_{N^\tau} \)-invariant Cartier divisor \( D_\tau \) is a nef divisor.

**Proof.** Since \( D_\tau \) is \( T_{N^\tau} \)-invariant, it is a smooth hypersurface locating inside \( X_\Sigma \), we have the following exact sequence

\[
0 \rightarrow T_{D_\tau} \rightarrow T_{X_\Sigma} \bigg|_{D_\tau} \rightarrow N_{D_\tau} \rightarrow 0,
\]

Note the normal sheaf of \( D_\tau \) in \( X_\Sigma \) could be identified with \( \mathcal{O}_{X_\Sigma}(D_\tau) \). Since the tangent bundle \( T_{X_\Sigma} \) is a nef vector bundle, so is the quotient bundle \( N_{D_\tau} \) by [DPS, Proposition 1.15]. Hence \( D_\tau \) is a nef divisor. \( \square \)

**Theorem 4.3.** The tangent bundle of a toric manifold with nef tangent bundle is Griffiths semipositive.

**Proof.** Since \( \mathcal{O}_{X_\Sigma}(D_\tau) \) is a nef line bundle, \( D_\tau \) is basepoint free by Proposition 2.1. Hence \( \mathcal{O}_{X_\Sigma}(D_\tau) \) is a semipositive line bundle. Hence the direct sum bundle \( \oplus_{\tau \in \Sigma(1)} \mathcal{O}_{X_\Sigma}(D_\tau) \) is a Griffiths semipositive. The Griffiths semipositivity of tangent bundle \( T_{X_\Sigma} \) follows by the generalized Euler sequence via using [Ya, Proposition 3.5]. \( \square \)

Note Theorem 4.3 already implies that Campana-Peternell conjecture holds for toric fano manifolds. In fact, from Theorem 4.3 we know a toric fano manifold \( X_\Sigma \) with nef tangent bundle has nonnegative holomorphic bisectional curvature and positive Ricci curvature. By Mok’s theorem [Mo], \( X_\Sigma \) is biholomorphic to the product of Hermitian symmetric manifolds. In particular, \( \text{Aut}(X_\Sigma) \) is a reductive algebraic group. In the rest part of this note we will give a more precise structure description of a toric fano manifold with nef tangent bundle.
Let $\phi_{D_\tau}$ be the support function associated with $T_N$-invariant cartier divisor $D_\tau$. Then on each open cone $\sigma \in \Sigma(d)$,
$$\phi_{D_\tau}(u) = \langle m_\sigma, u \rangle, \ \forall u \in \sigma,$$
for some $m_\sigma \in M$.

**Proposition 4.4.** If the cartier divisor $D_\tau$ is nef then $\{m_\sigma \mid \sigma \in \Sigma(d)\}$ are semisimple Demazure roots of $\text{Aut}(X_\Sigma)$.

**Proof.** By the definition of the data $m_\sigma$ for the Cartier divisor $D_\tau$, we have $\phi_{D_\tau}(u_\tau) = \langle m_\sigma, u_\tau \rangle = -1$. Now fix a $\sigma \in \Sigma(d)$. By (4) of Proposition 2.1, $m_\sigma \in P_{D_\tau}$ if $D_\tau$ is basepoint free. Note now
$$P_{D_\tau} = \{ m \in M \mid \langle m, u_\tau \rangle \geq -1 \ \text{and} \ \langle m, u_\tau' \rangle \geq 0, \ \forall \tau' \in \Sigma(1) \setminus \{\tau\} \},$$
hence we have
$$\langle m_\sigma, u_\tau \rangle \geq 0, \ \forall \tau' \in \Sigma(1) \setminus \{\tau\},$$
therefore $m_\sigma$ is a Demazure root of $\text{Aut}(X_\Sigma)$. Since $\text{Aut}(X_\Sigma)$ is reductive, it is also a semisimple root. \hfill \square

**Proposition 4.5.** If $X_\Sigma$ is a toric Fano manifold with nef tangent bundle then $\text{Aut}(X_\Sigma)$ has $d$ linearly independent semisimple roots.

**Proof.** Let $\tau_1, \ldots, \tau_m$ denote all of 1-dimensional cones of $\Sigma$ and $u_\tau_1, \ldots, u_\tau_m$ their primitive generating vectors, and $D_1, \ldots, D_m$ the corresponding $T_N$-invariant basepoint free Cartier divisors. The support function $\phi_{D_i}$ of $D_i$ satisfies that
$$\phi_{D_i}(x) = \langle m_\sigma, x \rangle, \ \forall x \in \sigma \in \Sigma(d),$$
where $\langle m_\sigma, u_\tau \rangle = -1$ and $\langle m_\sigma, u_\tau_j \rangle \geq 0$ for $j \neq i$. Let $\{m_\sigma^j\}$ be the set of semisimple Demazure roots associated with the Cartier divisor $D_i$. Since each $D_i$ is a basepoint free divisor, the associated polytope
$$P_{D_i} = \{ m \in M_R \mid \langle m, u_\tau \rangle \geq -1 \ \text{and} \ \langle m, u_\tau_j \rangle \geq 0 \ \forall j \neq i \}$$
is a convex polytope. Note for $i \neq j$,
$$P_{D_i} \cap P_{D_j} = \{ m \in M_R \mid \langle m, u_\tau \rangle \geq 0 \ \text{for} \ j = 1, \ldots, m \} = \text{pos}(\tau_1, \ldots, \tau_m)^\vee$$
is $\{0\}$, since $\Sigma$ is complete the convex cone $\text{pos}(\tau_1, \ldots, \tau_m) = N_R$. The anticanonical divisor of $X_\Sigma$ is given by $K^* = D_1 + \cdots + D_m$ and it is an ample divisor, the associated polytope
$$P_{K^*} = \{ m \in M_R \mid \langle m, u_\tau \rangle \geq -1 \ \text{for} \ i = 1, \ldots, m \}$$
is a full dimensional polytope by Proposition 2.2. Note that
$$P_{K^*} = P_{D_1} \cup \cdots \cup P_{D_m}.$$Since $0 \neq m_\sigma^i \in P_{D_i}$, we have for any $\sigma, \sigma' \in \Sigma(d)$ that $m_\sigma^i \neq m_\sigma^j$, if $i \neq j$.

Note $\{m_\sigma^i\}$ are vertices of $P_{D_i}$, however none of them are vertices of $P_{K^*}$ though $\{m_\sigma^i \mid \sigma \in \Sigma(d)\} \subseteq P_{K^*}$. In fact $m_\sigma^i$ can’t lie in the intersection of two facets of $P_{K^*}$, hence it is not inside the facets with codimension $\geq 2$ of $P_{K^*}$. But each $m_\sigma^i$ is in the codimensional one facet $H_i = \{ x \in M \mid \langle m, u_\tau \rangle = -1 \}$ of $P_{K^*}$, and for $i \neq j$, the points $\{m_\sigma^i \mid \sigma \in \Sigma(d)\}$ and $\{m_\sigma^j \mid \sigma \in \Sigma(d)\}$ locate in the different facets of $P_{K^*}$.

Now let $v$ be any vertex of $P_{K^*}$ which has at least $d$ codimensional one facets of $P_{K^*}$, assume $H_i, \ldots, H_k (k \geq d)$ are those facets passing through the vertex $v$ and $H_i \cap \cdots \cap H_k = \{v\}$. Now fix a cone $\sigma \in \Sigma(d)$, since $\{ m \in M \mid \langle m, u_\tau \rangle = -1, j = 1, \ldots, k \}$ is a $d$-dimensional cone $\text{Cone}(P_{K^*} \cap M_R - v)$, the vectors $m_\sigma^i - v, \ldots, m_\sigma^k - v$ form a basis of $N_R$. Since $m_\sigma^i, \ldots, m_\sigma^k$ are on the different facets of cone $\text{Cone}(P_{K^*} \cap M_R - v)$, without loss of generality we may assume $m_\sigma^i - v, \ldots, m_\sigma^k - v$ are linearly independent. Then after a translation, $m_\sigma^i, \ldots, m_\sigma^k$
are still linearly independent. By Proposition 4.4, $m_1^{i_1}, \ldots, m_d^{i_d}$ are semisimple Demazure roots of $\text{Aut}(X_\Sigma)$, hence it has $d$ linearly independent semisimple roots. □

Now our main result Theorem 1.2 follows from Proposition 3.1 and Proposition 4.5.

REFERENCES

[CP] F. Campana and T. Peternell, Projective manifolds whose tangent bundles are numerically effective. *Math. Ann.* **289** (1991), 169–187.

[CLS] David A. Cox, John B. Little and Henry K. Schenck, *Toric varieties*, Graduate Studies in Mathematics, Vol **124**, American Mathematical Society, Providence, RI, 2011.

[DPS] J.-P. Demailly, T. Peternell and M. Schneider, Compact complex manifolds with numerically effective tangent bundles. *J. Algebraic Geom.* **3** (1994), 295–345.

[De] M. Demazure, Sous-groupes algébriques de rang maximum du groupe de Cremona. (French) *Ann. Sci. École Norm. Sup.* **3** (1970) 507–588.

[Fu] W. Fulton, *Introduction to toric varieties*, Annals of Mathematics Studies, **131**, The William H. Roever Lectures in Geometry. Princeton University Press, Princeton, NJ, 1993.

[Mo] N. Mok, The uniformization theorem for compact Kähler manifolds of nonnegative holomorphic bisectional curvature. *J. Differential Geom.* **27** (1988), 179–214.

[Ni1] B. Nill, Gorenstein toric Fano varieties. *Manuscripta Math.* **116** (2005), 183–210.

[Ni2] B. Nill, Complete toric varieties with reductive automorphism group. *Math. Z.* **252** (2006), 767–786.

[Od] T. Oda, Convex bodies and algebraic geometry. An introduction to the theory of toric varieties. Translated from the Japanese. Ergebnisse der Mathematik und ihrer Grenzgebiete **15**, Springer-Verlag, Berlin, 1988.

[Ya] Q.-L. Yang, $(k, s)$-positivity and vanishing theorems for compact Kähler manifolds. *Internat. J. Math.* **22** (2011), 545–576.

DEPARTMENT OF MATHEMATICS, SUN YAT-SEN UNIVERSITY, 510275, GUANGZHOU, P. R. CHINA.

E-mail address: yqli@mail.sysu.edu.cn