POINCARÉ THEORY FOR COMPACT ABELIAN ONE–DIMENSIONAL SOLENOIDAL GROUPS

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Abstract. The notion of Poincaré rotation number for homeomorphisms of the unit circle is generalized to the case of homeomorphisms of a general compact abelian one–dimensional solenoidal group, which is also a one–dimensional foliated space; specifically, the theory is developed for the algebraic universal covering space of the circle. Poincaré’s dynamical classification theorem is also generalized to homeomorphisms of solenoids whose rotation element is an irrational element (i.e., monothetic generator) of the given group.

1. Introduction

In 1881, H. Poincaré (see [Poi]) introduced an invariant of topological conjugation for homeomorphisms of the unit circle

$$\rho : \text{Homeo}_+(S^1) \longrightarrow S^1, \quad f \mapsto \rho(f),$$

called the rotation number of $f$. He then proved a remarkable topological classification theorem for the dynamics of any orientation–preserving homeomorphism $f \in \text{Homeo}_+(S^1)$: $f$ has a periodic orbit if and only $\rho(f)$ is rational. If the rotation number $\rho(f)$ is irrational, then $f$ is semiconjugate to an irrational rotation $R_{\rho(f)}$. The semiconjugacy is actually a conjugacy if the orbits of $f$ are dense.

This study has been one of the most fruitful subjects in the theory of dynamical systems as witnessed by the works of A.N. Kolmogorov, V.I. Arnold, J. Moser, M.R. Herman, A.D. Brjuno, J.C. Yoccoz, among others (see [Kol], [Arn], [Mos], [Brj1, Brj2], [Yoc]; see also [Ghys], [Her], [Nav]).

In this paper we continue with this line of ideas and generalize the Poincaré rotation number to any compact abelian one–dimensional solenoidal group, which is also a compact abelian topological group obtained as the continuous homomorphic image of the algebraic universal covering space of the circle $S := \varprojlim \mathbb{R}/n\mathbb{Z}$ or, by Pontryagin duality, as a compact abelian topological group whose group of characters is an additive subgroup of the rational numbers with the discrete topology. In the case of the one-dimensional universal solenoidal group the group of characters is the whole discrete group $\mathbb{Q}$.

The theory is developed for this algebraic universal covering space of the circle since this is the paradigmatic example and all the ideas are already present there. It is considered only the case of homeomorphisms of solenoids which are isotopic to the identity. At the end of the paper it will be indicated the necessary modifications for the general case.

The algebraic universal covering space of the circle can be thought of as a generalized circle and can be realized as the orbit space of the locally trivial $\mathbb{Q}$–bundle structure $\mathbb{Q} \hookrightarrow A \longrightarrow A/\mathbb{Q}$, where $A$ is the adèle group of the rational numbers and $\mathbb{Q} \hookrightarrow A$ is a cocompact discrete subgroup of $A$. Since

$$S = \varprojlim \mathbb{R}/n\mathbb{Z} \cong A/\mathbb{Q},$$
it follows that $S$ is a compact abelian topological group with a locally trivial $\hat{\mathbb{Z}}$–bundle structure $\hat{\mathbb{Z}} \hookrightarrow S \rightarrow \mathbb{S}^1$ and also a one–dimensional foliated space whose leaves have a canonical affine structure isomorphic to the real one–dimensional affine space $\mathbb{A}^1$.

Using the notion of asymptotic cycle of Schwartzman (see [Sch]) the generalized Poincaré rotation element

$$\rho : \text{Homeo}_+(S) \rightarrow S, \quad f \mapsto \rho(f),$$

can be defined as follows. Let $f : S \rightarrow S$ be any homeomorphism isotopic to the identity which can be written as $f = \text{id} + \varphi$, where $\varphi : S \rightarrow S$ is the displacement function along the one–dimensional leaves of $S$ with respect to the affine structure. The suspension space of $f$ is defined as:

$$\Sigma_f(S) := S \times [0, 1]/(z, 1) \sim (f(z), 0).$$

Since $f$ is isotopic to the identity, it follows that $\Sigma_f(S) \cong S \times \mathbb{S}^1$ is a compact abelian topological group whose character group is given by

$$\text{Char}(\Sigma_f(S)) \cong \text{Char}(S) \times \text{Char}(\mathbb{S}^1) \cong \mathbb{Q} \times \mathbb{Z}.$$

The associated suspension flow $\phi_t : \Sigma_f(S) \rightarrow \Sigma_f(S)$ is given by:

$$\phi_t(z, s) := (f^m(z), t + s - m), \quad (m \leq t + s < m + 1).$$

Now, for any given character $\chi_{q,n} \in \text{Char}(\Sigma_f(S))$, there exists a unique 1–cocycle $C_{\chi_{q,n}} : \mathbb{R} \times \Sigma_f(S) \rightarrow \mathbb{R}$ associated to $\chi_{q,n}$ (see section 3 for complete information) such that

$$\chi_{q,n}(\phi_t(z, s)) = \exp(2\pi i C_{\chi_{q,n}}(t, (z, s))) \cdot \chi_{q,n}(z, s),$$

for every $(z, s) \in \Sigma_f(S)$ and $t \in \mathbb{R}$. From here it is obtained an explicit expression for the 1–cocycle $C_{\chi_{q,n}}(t, (z, s))$ and, by Birkhoff’s ergodic theorem, there is a well–defined homomorphism

$$H_f : \text{Char}(\Sigma_f(S)) \rightarrow \mathbb{R}$$

given by

$$H_f(\chi_{q,n}) := \int_{\Sigma_f(S)} C_{\chi_{q,n}}(1, (z, s)) d\nu,$$

where $\nu$ is a $\phi_t$–invariant Borel probability measure on $\Sigma_f(S)$. Finally, the well–defined continuous homomorphism

$$\rho(f) : \text{Char}(\Sigma_f(S)) \rightarrow \mathbb{S}^1$$

given by

$$\rho(f)(\chi_{q,n}) := \exp(2\pi i H_f(\chi_{q,n}))$$

determines an element in $\text{Char}(\text{Char}(\Sigma_f(S))) \cong S \times \mathbb{S}^1$ which does not depend on the second component. By Pontryagin’s duality theorem, it determines an element $\rho(f) \in S$ called the rotation element associated to $f$, which is the generalized Poincaré rotation number.

As expected, $\rho(f)$ is an element in the solenoid itself and it measures, in some sense, the average displacement of points under iteration of $f$ along the one–dimensional leaves with the Euclidean metric.

Since $S$ is torsion–free, it follows that there does not exist a notion of “rational” and so we only have to give a suitable definition of what “irrational” is. We proceed as follows:

**Definition:** An element $\alpha \in S$ is called irrational if the (additive) subgroup generated by $\alpha$ in $S$ is dense.

**Definition:** A homeomorphism $f : S \rightarrow S$ is said to have bounded mean variation if there exists $C > 0$ such that the sequence

$$\{F^n(z) - z - n\tau(F)\}_{n \geq 1}$$
is uniformly bounded by $C$. Here, $F$ is any lift of $f$, $\tau(F)$ is a lifting of $\rho(f)$ to $\mathbb{R} \times \hat{\mathbb{Z}}$ and $z \in \mathbb{R} \times \hat{\mathbb{Z}}$. (See section 4 for details.)

The generalized Poincaré theorem can be stated as follows:

**Theorem:** Suppose that $f : S \to S$ is any homeomorphism isotopic to the identity with irrational rotation element $\rho(f) \in S$. Then, $f$ is semiconjugated to the irrational rotation $R_{\rho(f)}$ if and only if $f$ has bounded mean variation. The semiconjugacy is actually a conjugacy if the orbits of $f$ are dense.

It should be pointed out that similar studies of the Poincaré theory have been developed very recently by several authors. In the paper [Jag], T. Jäger proved that a minimal homeomorphism of the $d$–dimensional torus is semiconjugated to an irrational rotation if and only if it is a pseudo–irrational rotation with bounded mean motion. J. Kwapisz (see [Kwa]) gave a definition of a rotation element for homeomorphisms of the real line with almost periodic displacement; when the displacement is limit periodic, the corresponding convex hull is a compact abelian one–dimensional solenoidal group. So, in this sense, our study of the rotation element is strongly related to that of Kwapisz. However, we started by considering $S$, a compact abelian group, as being a “generalized circle” and developing the theory from this perspective. We give two different (equivalent) definitions of the rotation element for homeomorphism isotopic to the identity, and also present a generalized notion when the homeomorphism is isotopic to a rotation by an element not in the base leaf, which produces a slightly different situation.

The paper is organized as follows: In section 2 are defined the algebraic universal covering space of the circle, its character group, the suspension of a homeomorphism isotopic to the identity and its corresponding character group. Section 3 introduces the notion of 1–cocycle and gives the definition of the generalized rotation element. In order to define this generalized rotation element $\rho(f)$, it is necessary to use the following ingredients: Pontryagin’s duality theory for compact abelian groups, the Bruschlinsky–Eilenberg homology theory and Schwartzman theory of asymptotic cycles as well as the notion of 1–cocycle and ergodic theory. The generalized Poincaré theorem is proved in section 4 and section 5 is dedicated to the study of minimal sets. Finally, section 6 indicates a second (equivalent) definition of the rotation element and also indicates a slightly general definition for the case of homeomorphisms isotopic to rotations whose rotation element is not in the base leaf. The study of the latter class as well as the developing of a Poincaré–Denjoy theory will be the subject of a forthcoming paper.

2. The algebraic universal covering space of the circle

2.1. The universal one–dimensional solenoid.

**Basic definitions.** It is well–known, by covering space theory, that for any integer $n \geq 1$, it is defined the unbranched covering space of degree $n$, $p_n : S^1 \to S^1$ given by $z \mapsto z^n$. If $n, m \in \mathbb{Z}^+$ and $n$ divides $m$, then there exists a unique covering map $p_{nm} : S^1 \to S^1$ such that $p_n \circ p_{nm} = p_m$. This determines a projective system of covering spaces $\{S^1, p_n\}_{n \geq 1}$ whose projective limit is the universal one–dimensional solenoid

$$S := \lim_{\leftarrow} S^1,$$

with canonical projection $S \to S^1$, determined by projection onto the first coordinate, which determines a locally trivial $\hat{\mathbb{Z}}$–bundle structure $\hat{\mathbb{Z}} \to S \to S^1$. $\hat{\mathbb{Z}} := \lim_{\leftarrow} \mathbb{Z}/m\mathbb{Z}$ is the profinite completion of $\mathbb{Z}$, which is a compact, perfect and totally disconnected abelian topological group homeomorphic to the Cantor set. Being $\hat{\mathbb{Z}}$ the profinite completion of $\mathbb{Z}$, it admits a canonical inclusion of $\mathbb{Z}$ whose image is dense.
S can also be realized as the orbit space of the $\mathbb{Q}$–bundle structure $\mathbb{Q} \to A \to A/\mathbb{Q}$, where $A$ is the adèle group of the rational numbers which is a locally compact abelian group, $\mathbb{Q}$ is a discrete subgroup of $A$ and $A/\mathbb{Q} \cong S$ is a compact abelian group (see [RV]). From this perspective, $A/\mathbb{Q}$ can be seen as a projective limit whose $n$–th component corresponds to the unique covering of degree $n \geq 1$ of $S^1$. $S$ is also called the algebraic universal covering space of the circle $S^1$. The Galois group of the covering is $\hat{\mathbb{Z}}$, the algebraic fundamental group of $S^1$.

By considering the properly discontinuously free action of $\mathbb{Z}$ on $\mathbb{R} \times \hat{\mathbb{Z}}$ given by

$$\gamma \cdot (x,t) := (x + \gamma, t - \gamma), \quad (\gamma \in \mathbb{Z}),$$

$S$ is identified with the orbit space $\mathbb{R} \times_{\mathbb{Z}} \hat{\mathbb{Z}}$. Here, $\mathbb{Z}$ is acting on $\mathbb{R}$ by covering transformations and on $\hat{\mathbb{Z}}$ by translations. The path–connected component of the identity element $0 \in S$ is called the base leaf and will be denoted by $L_0$. Clearly, $L_0$ is the image of $\mathbb{R} \times \{0\}$ under the canonical projection $\mathbb{R} \times \hat{\mathbb{Z}} \to S$ and it is homeomorphic to $\mathbb{R}$.

In summary, $S$ is a compact, connected, abelian topological group and also a one–dimensional lamination where each “leaf” is a simply connected one–dimensional manifold, homeomorphic to the universal covering space $\mathbb{R}$ of $S^1$, and a typical “transversal” is isomorphic to the Cantor group $\hat{\mathbb{Z}}$. $S$ also has a leafwise $C^\infty$ Riemannian metric (i.e., $C^\infty$ along the leaves) which renders each leaf isometric to the real line with its standard metric. So, it makes sense to speak of a rigid translation along the leaves. The leaves also have a natural order equivalent to the order of the real line.

Characters. Denote by $\text{Char}(S) := \text{Hom}(S, S^1)$ the topological group which consists of all continuous homomorphisms from $S$ into the multiplicative group $S^1$ endowed with the discrete topology. This group is called the Pontryagin dual of $S$ or, the Character group of $S$. From what has been said before, we know that $S \cong A/\mathbb{Q}$ and, since $A$ is self–dual (i.e., $A \cong \text{Char}(A)$), it follows that $\text{Char}(S) \cong \mathbb{Q}$. If $\hat{H}^1(S, \mathbb{Z})$ denotes the first Čech cohomology group of $S$ with coefficients in $\mathbb{Z}$, then $\hat{H}^1(S, \mathbb{Z}) \cong \text{Char}(S)$.

If $\chi : S \to S^1$ is any character, then $\chi$ is completely determined by its values when restricted to the dense one–parameter subgroup $L_0$. Since $L_0$ is canonically isomorphic to the additive group $(\mathbb{R}, +)$, the restriction of $\chi$ to $L_0$ is of the form $t \mapsto \exp(2\pi i t s)$. It is shown (see e.g., [RV]) that $s$ must be rational. Now, given any $z \in S$, there exists an integer $n \in \hat{\mathbb{Z}} \subset S$ such that $z + n \in L_0$. The value of the character $\chi$ at $z$ is given by

$$\chi(z) = \exp(2\pi i q(z + n)) = \exp(2\pi i qz)$$

for any $q \in \mathbb{Q}$. We will write $\chi(z) = \text{Exp}(2\pi i qz)$.

Homeomorphisms. We only consider the group which consists of all homeomorphisms of $S$ which are isotopic to the identity and can be written as $f = \text{id} + \varphi$, where $\varphi : S \to S$ is given by $\varphi(z) = f(z) - z$ and describes the displacement of points $z \in S$ along the leaf containing it. The symbol “–” refers to the additive group operation in the solenoid. Denote the set of all such functions $\varphi$ by $C_+(S)$.

Since $f$ is a homeomorphism which preserves the order in the leaves, it follows that there is a one to one correspondence between $C_+(S)$ and the set of real–valued continuous functions with the property that if $x$ and $y$ are in the same one–dimensional leaf and if $x < y$, then $x + \varphi(x) < y + \varphi(y)$. Therefore, $C_+(S)$ can be identified with the Banach space of real–valued continuous functions $C(S, \mathbb{R})$.

As mentioned in the introduction, the solenoid has a leafwise $C^\infty$ Riemannian metric (i.e., $C^\infty$ along the leaves) which renders each leaf isometric to the real line with its standard metric. Hence, the displacement function $\varphi$ can be thought of as a continuous real–valued function which we denote with the same symbol $\varphi$. In fact, since every leaf $L \subset S$ is dense, the restriction of this function to $L$, denoted by $\varphi_L$, completely determines the function. Furthermore, $\varphi_L$ is an almost periodic function whose convex hull is the solenoid and thus, $\varphi_L$ is a limit periodic function (see [Pon]).
Remark 2.1. Denote by $\text{Homeo}_+(S)$ the group of all homeomorphisms $f : S \longrightarrow S$ which are isotopic to the identity and can be written as $f = \text{id} + \varphi$, with $\varphi \in C_+(S)$; i.e.,

$$\text{Homeo}_+(S) := \{ f \in \text{Homeo}(S) : f = \text{id} + \varphi, \ \varphi \in C_+(S) \}.$$  

2.2. The suspension of a homeomorphism. Let $f : S \longrightarrow S$ be any homeomorphism isotopic to the identity. In $S \times [0,1]$ consider the equivalence relation

$$(z, 1) \sim (f(z), 0) \quad (z \in S).$$

The suspension of $f$ is the compact space

$$\Sigma_f(S) := S \times [0,1]/(z,1) \sim (f(z),0).$$

Since $f$ is isotopic to the identity, it follows that $\Sigma_f(S) \cong S \times S^1$ is a compact abelian topological group. In $\Sigma_f(S)$ there is a well–defined flow $\phi : \mathbb{R} \times \Sigma_f(S) \longrightarrow \Sigma_f(S)$, called the suspension flow of $f$, which is given by

$$\phi(t, (z,s)) := (f^m(z), t + s - m),$$

if $m \leq t + s < m + 1$. The canonical projection $\pi : S \times [0,1] \longrightarrow \Sigma_f(S)$ sends $S \times \{0\}$ homeomorphically onto its image $\pi(S \times \{0\}) \equiv S$ and every orbit of the suspension flow intersects $S$. The orbit of any $(z,0) \in \Sigma_f(S)$ must coincide with the orbit $\phi_t(z,0)$ at time $0 \leq t \leq T$ for $T$ an integer.

Characters of the suspension. Denote by $C(\Sigma_f(S), S^1)$ the topological space which consists of all continuous functions defined on $\Sigma_f(S)$ with values in the unit circle $S^1$ with the topology of uniform convergence on compact sets (i.e., the compact–open topology). Clearly, this is an abelian topological group under pointwise multiplication. The subset $R(\Sigma_f(S), S^1) \subset C(\Sigma_f(S), S^1)$ which consists of continuous functions $h : \Sigma_f(S) \longrightarrow S^1$ that can be written as $h(z,s) := \exp(2\pi i \psi(z,s))$ with $\psi : \Sigma_f(S) \longrightarrow \mathbb{R}$ a continuous function, is a closed subgroup. Hence, the quotient group $C(\Sigma_f(S), S^1)/R(\Sigma_f(S), S^1)$ is a topological group. By Bruschlinsky–Eilenberg’s theory (see [Sch]), it is known that

$$\hat{H}^1(\Sigma_f(S), \mathbb{Z}) \cong C(\Sigma_f(S), S^1)/R(\Sigma_f(S), S^1).$$

Since

$$\hat{H}^1(\Sigma_f(S), \mathbb{Z}) \cong \text{Char}(\Sigma_f(S)),$$

we conclude that

$$\text{Char}(\Sigma_f(S)) \cong C(\Sigma_f(S), S^1)/R(\Sigma_f(S), S^1).$$

On the other hand, $\Sigma_f(S) \cong S \times S^1$ implies that its character group is given by

$$\text{Char}(\Sigma_f(S)) \cong \text{Char}(S) \times \text{Char}(S^1) \cong \mathbb{Q} \times \mathbb{Z}.$$  

According with the definition of $\text{Exp}$ in [22], given any element $(q,n) \in \mathbb{Q} \times \mathbb{Z}$, the corresponding character $\chi_{q,n} \in \text{Char}(\Sigma_f(S))$ can be written as

$$\chi_{q,n}(z,s) = \text{Exp}(2\pi i qz) \cdot \text{Exp}(2\pi i ns)$$

$$= \text{Exp}(2\pi i (qz + ns)),$$

for any $(z,s) \in \Sigma_f(S)$.

Measures. Given any $f$–invariant Borel probability measure $\mu$ on $S$ and $\lambda$ the usual Lebesgue measure on $[0,1]$, the product measure $\mu \times \lambda$ leads to define a $\phi_t$–invariant Borel probability measure on $\Sigma_f(S)$. Reciprocally, given any $\phi_t$–invariant Borel probability measure $\nu$ on $\Sigma_f(S)$, it can be defined, by disintegration with respect to the fibers, an $f$–invariant Borel probability measure $\mu$ on $S$. Denote by $\mathcal{P}_f(S)$ the weak* compact convex space of $f$–invariant Borel probability measures defined on $S$. 

3. The Rotation element

3.1. 1–cocycles. A 1–cocycle associated to the suspension flow $\phi_t$ is a continuous function

$$C : \mathbb{R} \times \Sigma_f(S) \rightarrow \mathbb{R}$$

such that

$$C(t + u, (z, s)) = C(t, (z, s)),$$

for every $t, u \in \mathbb{R}$ and $(z, s) \in \Sigma_f(S)$. The set which consists of all 1–cocycles associated to $\phi_t$ is an abelian group denoted by $C^1(\phi)$. A 1–coboundary is the 1–cocycle determined by a continuous function $\psi : \Sigma_f(S) \rightarrow \mathbb{R}$ such that

$$C(t, (z, s)) := \psi(z, s) - \psi(\phi_t(z, s)).$$

The set of 1–coboundaries $\Gamma^1(\phi)$ is a subgroup of $C^1(\phi)$ and the quotient group

$$H^1(\phi) := C^1(\phi)/\Gamma^1(\phi),$$

is called the 1–cohomology group associated to $\phi_t$. The proof of the next proposition (for an arbitrary compact metric space) can be seen in [Ath].

**Proposition 3.1.** For every continuous function $h : \Sigma_f(S) \rightarrow S^1$ there exists a unique 1–cocycle $C_h : \mathbb{R} \times \Sigma_f(S) \rightarrow \mathbb{R}$ associated to $h$ such that

$$h(\phi_t(z, s)) = \exp(2\pi i C_h(t, (z, s))) \cdot h(z, s),$$

for every $(z, s) \in \Sigma_f(S)$ and $t \in \mathbb{R}$.

This proposition implies that there is a well–defined homomorphism

$$\text{Char}(\Sigma_f(S)) \cong H^1(\Sigma_f(S), \mathbb{Z}) \rightarrow H^1(\phi)$$

by sending any character $\chi_{q,n} \in \text{Char}(\Sigma_f(S))$ to the cohomology class $[C_{\chi_{q,n}}]$, where $C_{\chi_{q,n}}$ is the unique 1–cocycle associated to $\chi_{q,n}$.

Applying the above proposition to any character $\chi_{q,n} \in \text{Char}(\Sigma_f(S))$ we obtain

$$\chi_{q,n}(\phi_t(z, s)) = \exp(2\pi i C_{\chi_{q,n}}(t, (z, s))) \cdot \chi_{q,n}(z, s).$$

Using the explicit expressions for the characters on both sides we get

$$\chi_{q,n}(\phi_t(z, s)) = \chi_{q,n}(f^m(z), t + s - m) = \exp(2\pi i (qf^m(z) + nt + ns))$$

and

$$\chi_{q,n}(z, s) = \exp(2\pi i (qz + ns)).$$

Comparing the two expressions we obtain

$$C_{\chi_{q,n}}(t, (z, s)) = q(f^m(z) - z) + nt. \quad (3.1)$$

Now recall that $f : S \rightarrow S$ is a homeomorphism isotopic to the identity of the form $f = \text{id} + \varphi$, where $\varphi : S \rightarrow S$ is the displacement function, where, as described before, $\varphi$ can also be considered as a real–valued function on the solenoid. If $t = 1$, then $m = 1$ and the 1–cocycle at time $t = 1$ is

$$C_{\chi_{q,n}}(1, (z, s)) = q\varphi(z) + n. \quad (3.2)$$
3.2. The rotation element. If $\nu$ is any $\phi$–invariant Borel probability measure on $\Sigma_f(S)$, by Birkhoff’s ergodic theorem there is a well–defined homomorphism $H^1(\phi) \to \mathbb{R}$ given by

$$[C_\chi] \mapsto \int_{\Sigma_f(S)} C_\chi(1,(z,s))d\nu.$$ 

Now, composing the two homomorphisms

$$\text{Char}(\Sigma_f(S)) \to H^1(\phi) \to \mathbb{R}$$

it is obtained a well–defined homomorphism $H_{f,\nu} : \text{Char}(\Sigma_f(S)) \to \mathbb{R}$ given by

$$H_{f,\nu}(\chi_{q,n}) := \int_{\Sigma_f(S)} C_{\chi_{q,n}}(1,(z,s))d\nu.$$ 

Denote by $\mu$ the $f$–invariant Borel probability measure on $S$ obtained by disintegration of $\nu$ with respect to the fibers. Evaluating the above integral using equation 3.2 gives

$$H_{f,\nu}(\chi_{q,n}) = \int_{\Sigma_f(S)} (q\varphi + n)d\nu = q \int_S \varphi d\mu + n.$$ 

Hence, $H_{f,\nu}$ determines an element in $\text{Hom}(\text{Char}(\Sigma_f(S)), \mathbb{R})$ for each measure $\nu$ in $\Sigma_f(S)$, and therefore, for each measure $\mu \in \mathcal{P}_f(S)$. Hence, one gets a well–defined function

$$H_f : \mathcal{P}_f(S) \to \text{Hom}(\text{Char}(\Sigma_f(S)), \mathbb{R})$$

given by $\mu \mapsto H_{f,\mu}$, where $H_{f,\mu}$ is given by

$$H_{f,\mu}(\chi_{q,n}) = q \int_S \varphi d\mu + n.$$ 

By post–composing $H_f$ with the continuous homomorphism

$$\text{Hom}(\text{Char}(\Sigma_f(S)), \mathbb{R}) \to \text{Char}(\text{Char}(\Sigma_f(S)))$$

given by

$$H_{f,\mu} \mapsto \pi \circ H_{f,\mu},$$

where $\pi : \mathbb{R} \to S^1$ is the universal covering projection, we obtain a well–defined continuous function $\rho : \mathcal{P}_f(S) \to \text{Char}(\text{Char}(\Sigma_f(S)))$ given by

$$\mu \mapsto \rho_{\mu} := \pi \circ H_{f,\mu}.$$ 

That is, for each $\mu \in \mathcal{P}_f(S)$, there exists a well–defined continuous homomorphism

$$\rho_{\mu} : \text{Char}(\Sigma_f(S)) \to S^1$$

given by

$$\rho_{\mu}(\chi_{q,n}) := \exp(2\pi i H_{f,\mu}(\chi_{q,n})) = \exp \left(2\pi iq \int_S \varphi d\mu \right).$$ 

By Pontryagin’s duality theorem,

$$\text{Char}(\text{Char}(\Sigma_f(S))) \cong \Sigma_f(S)$$

and therefore $\rho_{\mu} \in \Sigma_f(S)$. Since $\Sigma_f(S) \cong S \times S^1$ and $\rho_{\mu}(\chi_{q,n}) = \rho_{\mu}(\chi_{q,0})$, it follows that $\rho_{\mu}$ does not depend on the second component and so, the identification $\rho_{\mu} = (\rho_{\mu}, 1) \in S \times S^1$ can be made. More precisely, it is well known that every character of $\text{Char}(S) \cong \mathbb{Q}$ is of the form $\chi_a$ for some $a \in \mathbb{A}$.
and the map $A \to \text{Char}(\mathbb{Q})$ given by $a \mapsto \chi_a$ induces an isomorphism $\text{Char}(\mathbb{Q}) \cong A/\mathbb{Q} \cong S$. This produces a genuine element $\rho_\mu \in S$.

**Definition 3.2.** The element $\rho_\mu(f) := \rho_\mu \in S$ defined above is the rotation element associated to $f$ with respect to the measure $\mu$.

**Remark 3.3.** By definition, $\rho_\mu(f)$ can be identified with the element $\int_S \varphi d\mu$ in the solenoid $S$ determined by the character of $\mathbb{Q}$ given by

$$q \mapsto \exp \left( 2\pi i q \int_S \varphi d\mu \right).$$

That is, $\rho_\mu(f)$ is solenoid–valued.

If $\mathcal{R} : \mathcal{P}_f(S) \to S$ is the map given by $\mu \mapsto \rho_\mu(f)$, then $\mathcal{R}$ is continuous from $\mathcal{P}_f(S)$ to $S$. Since $\mathcal{P}_f(S)$ is compact and convex, and $f$ is isotopic to the identity, the image $\mathcal{R}(\mathcal{P}_f(S))$ is a compact interval $I_f$ in the one–parameter subgroup $L_0$. This interval is called the rotation interval of $f$. Since $L_0$ is canonically isomorphic to $\mathbb{R}$, it is possible to identify $I_f$ with an interval in the real line. If $f$ is isotopic to a rotation $R_\alpha$ with $\alpha \notin L_0$, then $I_f \subset L_0 + \alpha$.

**Definition 3.4.** We say that $f$ is a pseudo–irrational rotation if $I_f$ consists of a single point $I_f = \{\alpha\}$ and $\alpha$ is an irrational element in $S$ (see section [4]).

In particular, if $f$ is uniquely ergodic, then the interval $I_f$ reduces to a point and the rotation element is a unique element of $S$.

The proof of the next proposition is clear from the definitions:

**Proposition 3.5.** If $\mu_1$ and $\mu_2$ are any two elements in $\mathcal{P}_f(S)$ which belong to the same measure class, then $\rho_{\mu_1}(f) = \rho_{\mu_2}(f)$.

**Remark 3.6.** The rotation element of $f$ can be interpreted as the exponential of an asymptotic cycle, in the sense of Schwartzman, of the suspension flow $\{\phi_t\}_{t \in \mathbb{R}}$ of $f$ (see [Sch]; see also [AK], [Pol]). If $A_\nu \in \text{Hom}(H^1(\Sigma_f(S), \mathbb{Z}), \mathbb{R}) = \text{Hom}(\text{Char}(\Sigma_f(S)), \mathbb{R})$ denotes the asymptotic cycle associated to the $\{\phi_t\}_{t \in \mathbb{R}}$–invariant measure $\nu$, then $\rho_\nu(f) = \exp(2\pi i A_\nu)$.

**Remark 3.7.** From Birkhoff’s ergodic theorem, for any ergodic $f$–invariant measure $\mu$,

$$\int_S \varphi d\mu = \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n} \varphi(f^j(z)),$$

for $\mu$–almost every point $z \in S$. We could have used this to define the rotation element with respect to an (ergodic) measure. Since we wanted to make explicit the role of the measure, we used the theory of asymptotic cycles in the sense of Schwartzman. (Compare [Kwa], Theorem 3.)

### 3.3. Basic example and properties.

**Basic example: Rotations.** Let $\alpha$ be any element in $L_0 \subset S$ and consider the rotation $R_\alpha : S \to S$ given by $z \mapsto z + \alpha$. The suspension flow $\phi_t : S \times S^1 \to S \times S^1$ is given by

$$\phi_t(z, s) := (z + m\alpha, t + s - m),$$

if $m \leq t + s < m + 1$. If $\chi_{q,n} \in \text{Char}(\Sigma_f(S))$ is any character, then

$$H_{R_\alpha,\mu}(\chi_{q,n}) = q \int_S \alpha d\mu + n = q\alpha + n.$$

This implies that

$$\rho_\mu(R_\alpha)(\chi_{q,n}) = \exp(2\pi i (q\alpha + n)) = \exp(2\pi i q\alpha)$$

and $\rho_\mu(R_\alpha) = \alpha$. 

Properties.

(1) (Invariance under conjugation) Let $f$ and $g$ be any two homeomorphisms isotopic to the identity and $h = \text{id} + \psi$. If $h \circ f = g \circ h$, then $\rho_\mu(f) = \rho_\mu(g)$. In particular, if $f$ is conjugated to a rotation $R_\alpha$, then $\rho_\mu(f) = \alpha$.

Proof. Observe first that $h \circ f = g \circ h$ implies that $h \circ f^m = g^m \circ h$ and $f^m + \psi \circ f^m = g^m \circ h - h + h$. That is

$$f^m - \text{id} = (g^m - \text{id}) \circ h + \psi - \psi \circ f^m.$$ 

Therefore, the 1–cocycle associated to any character $\chi_{q,n}$ at time $t = 1$ has the form

$$C_{\chi_{q,n}}(1, (z, s)) = q(f(z) - z) + n = q[(g(h(z)) - h(z)) + \psi(z) - \psi \circ f(z)] + n.$$ 

Since $\mu$ is both $f$ and $g$ invariant, we get

$$H_{f,\mu}(\chi_{q,n}) = q \int_{\mathbb{S}} (f(z) - z) \, d\mu + n = q \int_{\mathbb{S}} (g(h(z)) - h(z)) \, d\mu + n = H_{g,\mu}(\chi_{q,n}).$$

Hence, $\rho_\mu(f) = \rho_\mu(g)$. \hfill $\square$

(2) (Continuity) The function $\rho_\mu : \text{Homeo}_+ (\mathbb{S}) \to \mathbb{S}$ given by

$$f = \text{id} + \varphi \mapsto \int_{\mathbb{S}} \varphi \, d\mu$$

is continuous with respect to the uniform topology in $\text{Homeo}_+ (\mathbb{S})$.

(3) (The rotation element is equal to zero if and only if $f$ has a fixed point) Indeed, if $f$ has a fixed point $x$, then, $\varphi(x) = 0$; if $\mu = \delta_x$ is the Dirac mass at $x$, then, $\int_{\mathbb{S}} \varphi \, d\mu = 0$ and therefore $\rho_\mu(f) = 0$. On the other hand, if $\rho_\mu(f) = 0$, then $\int_{\mathbb{S}} \varphi \, d\mu = 0$ and $\varphi$ must vanish at some point $x$ which must be a fixed point of $f$.

4. POINCARÉ THEORY FOR COMPACT ABELIAN ONE–DIMENSIONAL SOLENOIDAL GROUPS

4.1. Irrational rotations. Since $\mathbb{S}$ is torsion–free, it follows that a non trivial rotation has no periodic points. This means the dichotomy rational–irrational does not appear in this context and we only have to define what “irrational” means. The following seems to be an appropriate definition:

Definition 4.1. We say that $\alpha \in \mathbb{S}$ is irrational if $\{n\alpha : n \in \mathbb{Z}\}$ is dense in $\mathbb{S}$. In classical terminology, $\mathbb{S}$ is said to be monothetic with generator $\alpha$.

Since $\mathbb{S}$ is a compact abelian topological group, the next theorem is classical (see e.g., [Gra] for the general statements).

Theorem 4.2. If $\alpha \in \mathbb{S}$, then the following propositions are equivalent:

a) The rotation $R_\alpha : \mathbb{S} \to \mathbb{S}$ given by $z \mapsto z + \alpha$ is ergodic with respect to the Haar measure on $\mathbb{S}$.

b) $\chi(\alpha) \neq 1$, for every non–trivial character $\chi \in \text{Char}(\mathbb{S})$.

c) $\mathbb{S}$ is a monothetic group with generator $\alpha$.

Remark 4.3. (a) Any non–trivial character $\chi \in \text{Char}(\mathbb{S})$ describes the solenoid $\mathbb{S}$ as a locally trivial fiber bundle over the circle $\mathbb{S}^1$ with typical fiber a Cantor group which is a closed subgroup of $\mathbb{Z}$.

In fact, there is such a fibration for each $q \in \mathbb{Q} \setminus \{1\}$. 

(b) For every $\alpha \in S$, $\chi \circ R_\alpha = R_{\chi(\alpha)} \circ \chi$.
(c) If $\alpha \in S$ is irrational, then $\chi(\alpha) \in S^1$ is irrational, for every non-trivial character $\chi \in \text{Char}(S)$.

4.2. Generalized Poincaré’s theorem. From now on, we fix a measure $\mu \in \mathcal{P}_f(S)$. This determines a rotation element $\rho_\mu(f)$ of $f$, which will be simply denoted by $\rho(f)$ when the measure is understood.

Recall $S$ is the orbit space of $\mathbb{R} \times \widehat{\mathbb{Z}}$ under the $\mathbb{Z}$-action
$$\gamma \cdot (x, t) = (x + \gamma, t - \gamma) \quad (\gamma \in \mathbb{Z}).$$

Denote by $p : \mathbb{R} \times \widehat{\mathbb{Z}} \to S$ the canonical projection. It is clear that $p$ is an infinite cyclic covering.

Let $F : \mathbb{R} \times \widehat{\mathbb{Z}} \to \mathbb{R} \times \widehat{\mathbb{Z}}$ be a lifting of $f$ to $\mathbb{R} \times \widehat{\mathbb{Z}}$. Then, $F$ has the form
$$F(x, t) = (F_t(x), R_\alpha(t)),$$
where $\widehat{\mathbb{Z}} \to \text{Homeo}(\mathbb{R})$ is a continuous function given by $t \mapsto F_t$, $F_t : \mathbb{R} \to \mathbb{R}$ is a homeomorphism with limit periodic displacement $\Phi_t(x)$ (i.e., $\Phi$ is a uniform limit of periodic functions) and $\alpha \in \widehat{\mathbb{Z}}$ is a monothetic generator.

The condition of $F$ being equivariant with respect to the $\mathbb{Z}$-action is:
$$F_{t-\gamma}(x + \gamma) = F_t(x) + \gamma,$$
for any $\gamma \in \mathbb{Z}$. That is, $F$ must commute with the integral translation $T_\gamma : \mathbb{R} \times \widehat{\mathbb{Z}} \to \mathbb{R} \times \widehat{\mathbb{Z}}$ given by $(x, t) \mapsto (x + \gamma, t)$ and also must be invariant under the $\mathbb{Z}$-action in $C(\widehat{\mathbb{Z}}, \text{Homeo}(\mathbb{R}))$.

Remark 4.4. It is very important to emphasize at this point that a lifting $F$ of $f$ exists and it is a homeomorphism of $\mathbb{R} \times \widehat{\mathbb{Z}}$ due to the fact that $f$ is isotopic to the identity, which implies that $f$ leaves invariant the one-dimensional leaves of the solenoid. As a consequence of this fact, $F$ leaves invariant the one-dimensional leaves of $\mathbb{R} \times \widehat{\mathbb{Z}}$. Since each leaf is canonically identified with $\mathbb{R}$, the displacement function along the leaves can be defined in an obvious way.

Definition 4.5. We say that $f$ has bounded mean variation if there exists $C > 0$ such that the sequence $\{F^n(z) - z - n\tau(F)\}_{n \geq 1}$ is uniformly bounded by $C$. Here, $F$ is any lift of $f$, $\tau(F)$ is a lifting of $\rho(f)$ to $\mathbb{R} \times \widehat{\mathbb{Z}}$ and $z \in \mathbb{R} \times \widehat{\mathbb{Z}}$.

We can now state and prove the generalized version of the Poincaré theorem: The first part of the proof follows closely the classical proof (see [Ghys], [Nav]).

Theorem 4.6. Let $f \in \text{Homeo}_+(S)$ with irrational rotation element $\rho(f)$. Then, $f$ is semiconjugated to the irrational rotation $R_{\rho(f)}$ if and only if $f$ has bounded mean variation. Furthermore, under the same hypothesis, if $f$ is minimal, then $f$ is conjugated to the rotation $R_{\rho(f)}$.

Proof. The function $H : \mathbb{R} \times \widehat{\mathbb{Z}} \to \mathbb{R} \times \widehat{\mathbb{Z}}$ given by
$$z \mapsto \sup_n \{F^n(z) - n\tau(F)\}$$
satisfies the following properties:

1. $H$ is surjective and continuous on the left.
2. $H \circ T_1 = T_1 \circ H$
3. $H \circ F = T_{\tau(F)} \circ H$.

Condition (2) implies that $H$ descends to a map $h : S \to S$. Condition (3) implies that $h \circ f = R_{\rho(f)} \circ h$. Following almost verbatim the arguments in [Ghys], [Nav], it follows that $h$ is continuous and semiconjugates $f$ to $R_{\rho(f)}$.

For the second part of the proof, suppose that $f$ is minimal and $h : S \to S$ is a semiconjugacy: $h \circ f = R_{\rho(f)} \circ h$. Let $\chi : S \to S^1$ be any continuous character and let $F := \{F_\theta = \chi^{-1}(\theta)\}_{\theta \in S^1}$ be the
collection of fibers. The translation $R_{\rho(f)}$ permutes the fibers in the following way: $F_\theta \mapsto F_{\chi(\rho)+\theta}$. Let $\mathcal{G} := \{G_\theta := h^{-1}(F_\theta)\}_{\theta \in \mathbb{S}}$. Then, $f(G_\theta) = G_{\chi(\rho)+\theta}$ since $h(f(G_\theta)) = R_{\rho(f)}(F_\theta) = F_{\chi(\rho)+\theta}$. Since $\rho(f)$ is irrational, $\chi(\rho) \in \mathbb{S}$ is also irrational. This implies that if $\theta_1, \theta_2 \in \mathbb{S}$ are such that $\theta_1 \neq \theta_2$, then $h(G_{\theta_1}) \neq h(G_{\theta_2})$. Since this is true for any character and characters separates points of $S$ we conclude that $h$ must be injective. Therefore, $h$ is a homeomorphism. \hfill \square

5. Minimal sets

**Proposition 5.1.** Suppose that $f \in \text{Homeo}_+(S)$ has irrational rotation element $\rho(f) \in S$ and satisfies the bounded mean variation condition. Then, there exists a compact $f$–invariant subset $K \subset S$ with the following properties:

a. $K = \omega(z) = \alpha(z)$ and $K$ is minimal.

b. Either, $K = S$ or, $K \subset S$ is a Cantor set.

c. $\text{supp}(\mu) = K$ for every $f$–invariant Borel probability measure.

**Proof.** By theorem 4.6, $f$ is semiconjugate to the rotation $R_{\rho(f)}$. In particular, we can assume that $f$ has no fixed points. This is equivalent to the condition that $\varphi$ does not vanish. Let $K$ be a minimal set for $f$ and let $\mathcal{L}$ be a leaf of the solenoid. We know that there exists an isometric embedding $\tau : \mathbb{R} \hookrightarrow \mathcal{L}$. Let $M := \mathcal{L} \cap K$ and $N := \tau^{-1}(M) \subset \mathbb{R}$. Then, $N$ is a closed and invariant subset of $\mathbb{R}$ under the homeomorphism $f_N := \tau^{-1} \circ f \circ \tau$.

We have one, and only one, of the following possibilities:

1. $N$ is a closed, infinite and discrete subset of $\mathbb{R}$ and $N = \tau^{-1}(\mathcal{O}_f(z))$ for some point $z \in \mathcal{L}$.

2. $N = \mathbb{R}$ and $K = S$.

The set of accumulation points of $N$, denoted by $N'$, is closed and invariant under $f_N$. Therefore, $\tau(N') \subset K$ is closed and invariant under $f$. By minimality of $K$, if $N'$ is non–empty, then $K = \tau(N')$ and therefore $N = N'$; i.e., $N$ is a perfect set.

On the other hand, if $N' = \emptyset$ we are precisely in case (1).

Suppose we are in case (1). Let $\psi_t : S \rightarrow S$, $z \mapsto z + \sigma(t)$ be the flow corresponding to the one–parameter subgroup $\sigma : \mathbb{R} \hookrightarrow S$ whose orbits are the leaves of the solenoid. We claim that $K = \overline{\mathcal{O}_f(z)}$ is a global cross–section of the flow $\{\psi_t\}_{t \in \mathbb{R}}$. Indeed, $K$ intersects every orbit of the flow and the intersection of $K$ with an orbit (or leaf) is a discrete infinite subset of the orbit. Let $\varphi_K$ be the restriction of $\varphi$ to $K$. Then there exists a re–parametrization of the flow such that for each $z \in K$, $\psi_1(z) = z + \varphi_K(z) = f(z)$. Then, $\psi_1(K) = K$ and $\psi_t(K) \neq K$ if $t \in (0, 1)$. This describes $S$ as a fibre bundle $\pi : S \rightarrow \mathbb{S}^1$ over the circle $\mathbb{S}^1$ and, as we explained before, this fibration is equivalent to the one given by a nontrivial character $\chi$. Therefore $K$ can be deformed isotopically to the kernel of $\chi$.

Suppose now that $N$ has nonempty interior. Let $I$ be a nontrivial interval contained in $N$. Then, there exists a constant $C > 0$ such that the diameter $\text{diam}(f^n_N(I)) > C$. This follows from the fact that $\inf_{z \in K} |\varphi(z)| > 0$. Let

$$E := \tau \left( \bigcup f^n_N(I) \right).$$

Then, $E \subset K$ is a nonempty closed set which is invariant under $f$. Therefore, $E = K$. Since $\text{diam}(f^n_N(I)) > C$, we have that every point $z \in K$ is a left extreme point of a nontrivial interval $I_z$ such that $\text{diam}(I_z) > C$ and which is contained in $K$ and in the leaf through $z$. The right extreme point of $I_z$ is also in $K$. From this it follows that $K = S$ and therefore, $N = \mathbb{R}$. \hfill \square
The following is the generalized version of Furstenberg’s theorem (see [Fur]):

**Theorem 5.2.** If \( \rho(f) \) is irrational and \( f \) has bounded mean variation, then \( f \) is uniquely ergodic. If, moreover, \( f \) is minimal, then \( f \) is strictly ergodic.

**Proof.** Let \( \mu \in \mathcal{P}_f(S) \) and \( K = \text{supp}(\mu) \). By theorem 4.6 there exists a continuous surjective map \( h : S \to S \) such that \( h \circ f = R_{\rho(f)} \circ h \). Then, for any continuous function \( \psi \in C(S) \) we have

\[
\int_S (\psi \circ R_{\rho(f)}) dh_\ast \mu = \int_S (\psi \circ h) df_\ast \mu = \int_S \psi dh_\ast \mu.
\]

This implies that \( h_\ast \mu \) is invariant under the translation \( R_{\rho(f)} \). By theorem 4.2, \( R_{\rho(f)} \) is ergodic with respect to the (normalized) Haar measure, so it follows that, \( h_\ast \mu \) is precisely the (normalized) Haar measure.

If \( K = S \), then \( h \) is a homeomorphism and \( \mu \) is unique. If \( K \neq S \) and considering any two measures \( \mu, \nu \in \mathcal{P}_f(S) \), standard arguments imply that \( \mu = \nu \) (compare [Her]). \( \square \)

### 6. General definitions and remarks

**6.1. The Rotation element à la de Rham.** If \( d\lambda \) denotes the usual Lebesgue measure on \( S^1 \), then, given any character \( \chi \in \text{Char}(\Sigma_f(S)) \) there is a well–defined closed differential one–form on \( \Sigma_f(S) \) given by

\[
\omega_\chi := \chi^* d\lambda.
\]

Let \( X \) be the vector field tangent to the flow \( \phi_t \) and let \( \nu \) be any \( \phi_t \)–invariant Borel probability measure on \( \Sigma_f(S) \). Define

\[
H_{f,\nu} : \text{Char}(\Sigma_f(S)) \to \mathbb{R}
\]

by

\[
H_{f,\nu}(\chi_{q,n}) := \int_{\Sigma_f(S)} \omega_{\chi_{q,n}}(X) d\nu
\]

and observe that this definition only depends on the cohomology class of \( \omega_{\chi_{q,n}} \) and the measure class of \( \nu \). Hence, we have a well–defined continuous homomorphism \( \rho(f) : \text{Char}(\Sigma_f(S)) \to S^1 \) given by

\[
\rho(f)(\chi_{q,n}) := \exp(2\pi i H_{f,\nu}(\chi_{q,n})).
\]

Thus, as before,

\[
\rho(f) \in \text{Char}(\text{Char}(\Sigma_f(S))) \cong \Sigma_f(S).
\]

**Proposition 6.1.** \( \rho(f) \) is the rotation element associated to \( f \) corresponding to \( \nu \).

**Example 6.2.** Let \( \alpha \) be any element in \( S \) and consider the rotation \( R_\alpha : S \to S \) given by \( z \mapsto z + \alpha \). The suspension flow \( \phi_t : S \times S^1 \to S \times S^1 \) is given by

\[
\phi_t(z,s) := (z + m\alpha, t + s - m) \quad (m \leq t + s < m + 1).
\]

Given any character \( \chi_{q,n} \in \text{Char}(\Sigma_f(S)) \) we have that

\[
\omega_{\chi_{q,n}} = qd\theta + nd\lambda
\]

and the vector field \( X \) associated to \( \phi_t \) is constant. In this case, \( H_{R_\alpha,\nu}(\chi_{q,n}) = \alpha q + n \) and therefore

\[
\rho(R_\alpha)(\chi_{q,n}) = \exp(2\pi i \alpha n).
\]

That is, \( \rho(R_\alpha) = \alpha \) which clearly coincides with the calculation made before.
6.2. The suspension of any homeomorphism isotopic to a translation is a compact abelian group. Let $G$ be a metrizable compact abelian group. Consider a minimal translation $T: z \mapsto \alpha z$. Let $\Sigma_T(G)$ be the suspension of $T$.

**Theorem 6.3.** $\Sigma_T(G)$ is a compact abelian group which contains $G$ as a closed subgroup and

$$\Sigma_T(G)/G \cong \mathbb{S}^1.$$

**Proof.** Let $d$ be any invariant distance on $G$; i.e.,

$$d(hg_1, hg_2) = d(g_1, g_2) \quad (h \in G).$$

(Such a distance always exists by applying the well-known averaging method using the Haar measure.)

Consider in $G \times \mathbb{R}$ the distance $\hat{d}$ given by

$$\hat{d}((g_1, t_1), (g_2, t_2)) := d(g_1, g_2) + |t_1 - t_2|.$$

Define $F: G \times \mathbb{R} \to G \times \mathbb{R}$ by $F(g, t) = (\alpha g, t + 1)$. Then, the distance $\hat{d}$ is invariant under $F$ and therefore induces a distance on $\Sigma_T(G)$. The canonical projection $p: G \times \mathbb{R} \to \Sigma_T(G)$ is a local isometry. With respect to the induced distance, the suspension flow $\{F_s\}_{s \in \mathbb{R}}$ acts by isometries on $\Sigma_T(G)$.

It is a well-known fact that the group of isometries of $\Sigma_T(G)$ is a compact metric space with respect to the compact-open topology. It follows that $\{F_s\}_{s \in \mathbb{R}} \subset \text{Isom}(\Sigma_T(G))$. Let $\Gamma = \{F_s\}_{s \in \mathbb{R}}$ and let $\bar{\Gamma}$ be the closure of $\Gamma$ in $\text{Isom}(\Sigma_T(G))$. Then, $\bar{\Gamma}$ is a compact abelian group.

Let $x \in \Sigma_T(G)$ and let $\Gamma(x) \subset \Sigma_T(G)$ be the orbit of $x$. Since $T$ is a minimal translation, $\Gamma(x)$ is dense in $\Sigma_T(G)$. Let $k: \Gamma(x) \to \Gamma$ be defined by $F_{s}(x) \mapsto F_{s}$. Clearly, $k$ is continuous and injective. Let us show that $k$ can be extended to a homeomorphism $\tilde{k}: \Sigma_T(G) \to \bar{\Gamma}$. Let $y \in \Sigma_T(G)$. Then, there exists a sequence of real numbers $\{t_n\}_{n \in \mathbb{N}}$ such that $F_{t_n}(x) \to y$ when $n \to \infty$.

By compactness of $\bar{\Gamma}$, there exists a subsequence $\{t_{n_i}\}_{i \in \mathbb{N}}$ such that $F_{t_{n_i}}$ converges to an isometry which we denote by $H_y$. This limiting isometry satisfies $H_y(x) = y$. Furthermore, if $L$ is any positive real number, define the segment of orbit of length $2L$

$$I(x, L) := \{F_s(x) | s \in [-L, L]\}.$$

Then, $H_y$ sends $I(x, L)$ isometrically onto

$$I(y, L) := \{F_s(y) | s \in [-L, L]\}.$$

Since $L$ can be taken arbitrarily large, we see that $H_y$ is independent of the sequence and only depends on $y$.

Let us define $\tilde{k}(y) = H_y$. Since $y$ was arbitrary this defines an extension of $k$ to all of $\Sigma_T(G)$. One can easily verify that $\tilde{k}$ is continuous and, since $\tilde{k}$ is injective on a dense subset of $\Sigma_T(G)$, it follows that it is injective. Since $\Sigma_T(G)$ is compact, $\tilde{k}$ is a homeomorphism.

Therefore, $\bar{\Gamma}$ is homeomorphic to $\Sigma_T(G)$ and, via this homeomorphism, we define the abelian group structure on $\Sigma_T(G)$.

Finally, there is a natural continuous group epimorphism (namely, a character) $\Sigma_T(G) \to \mathbb{S}^1$ whose kernel is a closed subgroup of $\Sigma_T(G)$ isomorphic to $G$, and hence

$$\Sigma_T(G)/G \cong \mathbb{S}^1.$$

$\square$

**Corollary 6.4.** Since the suspension only depends on the isotopy class of the homeomorphism, the suspension of any homeomorphism $f: G \to G$ isotopic to a minimal translation, is a compact abelian group.
Remark 6.5. The canonical examples of the above theorem are: the 2–torus, which is a suspension of an irrational rotation on the circle; and, the universal solenoid, which is a suspension of a minimal translation on \( \hat{\mathbb{Z}} \).

For the particular case of the universal solenoid \( S \), it is not true that any homeomorphism is isotopic to a translation. In fact, in [Odd] it is proved the following result:

Theorem 6.6. If \( \text{Homeo}_L(S) \) is the subgroup of \( \text{Homeo}(S) \) consisting of homeomorphisms of \( S \) that preserves the base leaf, then

\[
\text{Homeo}(S) \cong \text{Homeo}_L(S) \times \mathbb{Z} \hat{\mathbb{Z}}.
\]

For instance, by Pontryagin duality, the group of automorphisms of \( S \) is isomorphic to the group of automorphisms of \( \mathbb{Q} \), which is \( \mathbb{Q}^* \), since any automorphism is determined by its value at 1. Hence, any automorphism of \( S \) is never isotopic to a translation.

Remark 6.7. The elements in the same one–dimensional leaf \( L \) of \( S \), determines isotopic translations. If an element \( f \in \text{Homeo}(S) \) is isotopic to a translation, then \( f \) is isotopic to a translation of the form \( t + \gamma \), where \( \gamma \in L \cap \hat{\mathbb{Z}} \).

Concluding Remarks. As indicated in the introduction (see section 1), the theory developed in this paper can be rewritten verbatim for any compact abelian one–dimensional solenoidal group, since, by Pontryagin’s duality theory, any such group corresponds to a nontrivial additive subgroup \( G \subset \mathbb{Q} \), where \( \mathbb{Q} \) has the discrete topology. In a forthcoming paper we will continue the study of the generalized version of the classical Poincaré theory in its various aspects.

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