**Static-response theory and the roton-maxon spectrum of a flattened Bose-dipolar condensate**

R. N. Bisset, P. B. Blakie, and S. Stringari

1INO-CNR BEC Center and Dipartimento di Fisica, Università di Trento, Povo, Italy  
2Institut für Theoretische Physik, Leibniz Universität Hannover, Hannover, Germany  
3Department of Physics, Centre for Quantum Science, and Dodd-Walls Centre for Photonic and Quantum Technologies, University of Otago, Dunedin, New Zealand

Important information for the roton-maxon spectrum of a flattened dipolar Bose-Einstein condensate can be extracted by applying a static perturbation exhibiting a periodic in-plane modulation. By solving the Gross-Pitaevskii equation in the presence of this weak perturbation we evaluate the linear density response of the gas and use it, together with sum rules, to provide a Feynman-like upper-bound prediction for the excitation spectrum, finding excellent agreement with the numerical predictions of full Bogoliubov calculations. By suddenly removing the static perturbation, while still maintaining the trap, we find that the density modulations – as well as the weights of the perturbation-induced side peaks of the momentum distribution – undergo an oscillatory behavior with double the characteristic frequency of the excitation spectrum. The measurement of these oscillations could provide an easy determination of the roton-maxon dispersion, as well as the phonon dispersion for nondipolar condensates.

Bose-Einstein condensates (BECs) of highly-magnetic dipolar atoms are now achievable using chromium [1, 2], dysprosium [3, 4] and erbium [5]. While remaining in the weakly-interacting regime, dipolar condensates possess several phenomena reminiscent of strongly-interacting systems thanks to the long-ranged and anisotropic nature of dipole-dipole interactions [6–8]. Among these analogue phenomena is the prediction of a roton-maxon dispersion [9, 10], the possibility to create a supersolid [11], and the experimental demonstration of dilute self-bound droplets that are stabilized by quantum fluctuations [12, 13] (note that related droplets were also observed in binary condensates [14, 15]). While BECs offer a highly-controllable platform from an experimental perspective, their diluteness also allows theorists to probe such phenomena using a host of well-established tools.

First observed in liquid helium, rotons are elementary excitations that reside near a local minimum of the energy dispersion, as can be seen in the lowest band of Fig. 1(b1). Preceding these are the maxons, constituting a local maximum. While the rotons of liquid helium rely on strong correlations, it is remarkable that an analogous roton-maxon dispersion relation was predicted in 2003 to occur for weakly-interacting dipolar condensates [9, 10]. There has also been intense interest in rotons of other weakly-interacting systems. Roton-like dispersions have recently been observed in shaken optical lattices [17], condensates with synthetic spin-orbit coupling [18, 19], and in the presence of a cavity [20]. Dipolar rotons are fundamentally different, though, since they genuinely arise from interactions instead of external driving.

Among the key challenges for their experimental realization, weakly-interacting dipolar rotons require the condensate to be highly anisotropic, with a short axis along the direction of dipole polarization. The existence of rotons also creates a vulnerability to condensate collapse [21–23] (or droplet formation), that can even be triggered by thermal density fluctuations [24]. Nevertheless, recently there has been a significant number of proposals for experimental probes for rotons. One of these is based on using Bragg spectroscopy to extract the dynamic structure factor, as well as a roton peak of the static structure factor [25]. Another, is based on high-resolution imaging, utilizing that density fluctuations should be greatly enhanced at the length scale of the roton wavelength, especially at finite temperature [26, 27]. Other proposals are based on applying a one-dimensional (1D) lattice to either trigger a roton collapse of the condensate [28, 29], or to detect a peak of the momentum distribution for lattice wavelengths near the roton minimum [30]. Recently, a landmark experiment has produced the first evidence for the existence of dipolar rotons [31]. This was obtained by the observation of momentum side peaks, created by the macroscopic occupation of roton modes during the early stages of a condensate collapse following an interaction quench.

We propose an approach to extract the weakly-interacting roton-maxon spectrum based on the application of a static 1D lattice in the plane of a flattened dipolar BEC. The response of the density is highly sensitive to the lattice wavelength and, with the help of sum rules, can be used to provide an upper bound for the energy dispersion. We calculate this upper bound numerically using a three-dimensional (3D) Gross-Pitaevskii equation (GPE), and show that it provides an almost exact prediction for the roton-maxon dispersion obtained directly from Bogoliubov-de Gennes (BdG) calculations. Such close agreement is highly nontrivial given that the trap confinement in the tight direction introduces a band structure, while the roton-maxon spectrum is only present in the lowest band [see Fig. 1(b1)]. In principle,
higher bands also contribute to the density response so it is remarkable that the sum-rule method captures the roton-maxon spectrum so precisely. To compliment the possibility of extracting the density response in position space using in situ imaging, we demonstrate that the side peaks of the momentum distribution - of relevance to expansion experiments - can also be used to give the dispersion relation. Finally, we propose another set of experiments where the static lattice is suddenly removed while the trap remains on. The resulting oscillation frequencies in position space, as well as for the momentum side peaks, provide a means to extract the dispersion relation without having to calibrate the lattice strength or the magnitude of the density response. Interestingly, we find a phase inversion of the momentum side peak oscillations for rotons compared to maxons. This intriguing behavior can be quantitatively described by perturbation theory without using any fitting parameters.

Formalism: We consider a flattened 3D Bose-Einstein condensate that is harmonically trapped only along the z direction, characterized by frequency \( \omega_z \). Along the untrapped directions the components of the in-plane wavevector \( \mathbf{k}_\rho = (k_x, k_y) \) provide good quantum numbers. No assumptions are made about the density profile along the z direction and this must be solved numerically. With regard to this last point, it was demonstrated that accurate treatment of the tight direction can be crucial for providing quantitatively useful results [32].

The primary motivation for proposing experiments in the radially untrapped regime is that rotons are strongly ‘attracted’ to high density, tightly confining them to a small central region within a harmonically-trapped condensate [27-29]. This reduction of the rotonized portion of the system acts to reduce the roton signal in observed quantities [33]. Nevertheless, as a check, we have also performed calculations in the presence of harmonic trapping in all directions (not shown here), and observe qualitatively consistent results.

The generalized GPE takes the form [12, 34-37]

\[
\frac{i\hbar}{\partial t} \psi(x) = \left[ -\frac{\hbar^2 \nabla^2}{2m} + \frac{m\omega_z^2 z^2}{2} + \int d^2x'U(x-x')|\psi(x')|^2 + \gamma_{QF}|\psi|^3 \right] \psi(x),
\]

with the interaction potential being well-described by the pseudopotential \( U(r) = g_s \delta(r) + U_{dd}(r) \). The contact interaction strength is \( g_s = 4\pi a_s \hbar^2/m \), for s-wave scattering length \( a_s \) and mass \( m \). The dipoloes are polarized along z and the corresponding dipole-dipole interactions are described by \( U_{dd}(r) = (3g_{dd}/4\pi)(1 - 3\cos^2 \theta)/r^3 \), where \( \theta \) is the angle between \( \mathbf{r} \) and the z axis. Their strength is given by \( g_{dd} = \mu_0 \mu_r^2/3 \), for magnetic dipole moment \( \mu_r \). The dipolar Lee-Huang-Yang (LHY) correction is added in the local density sense, being proportional to \( \gamma_{QF} = (32g_s/2\sqrt{\alpha_s^2}/\pi(1 + 3\alpha_s^2/2) \) [35] [39], where the ratio \( \epsilon_{dd} = g_{dd}/g_s \) is useful since \( \epsilon_{dd} > 1 \) signals the dipole-dominated regime. It should be noted that the effect of the LHY term is not qualitatively important to the results presented here, but does shift the scattering length of the roton instability downwards by around 8%.

To benchmark our approach we obtain excitation energies and wavefunctions by solving the BdG equations. These can be obtained by linearizing about Eq. (1) in the absence of any perturbing lattice [40]. Solving these in the present regime cannot be done analytically, so we use the numerical techniques outlined in [41] but here we include the LHY term.

**Sum rules and the static density response:** We consider the condensate response to the 1D periodic lattice perturbation

\[
V_{pert} = 2V_L \cos(k_L \cdot x),
\]

where \( V_L \) is a constant and \( k_L = (k_L, 0, 0) \). To do this we solve for ground states of the time-independent GPE.
including $V_{\text{pert}}$. In the limit of small $V_L$, the spatial density oscillation arising from the perturbation furnishes the static density response function

$$\chi(k_L) = \lim_{V_L \to 0} \frac{\Delta n}{2V_L},$$

where the amplitude of the density perturbation is

$$\Delta n = \frac{\max\{n(x)\} - \min\{n(x)\}}{2n_0},$$

for the 2D density $n(x) = \int |\psi(x)|^2 dz$ and its unperturbed value $n_0 \ [42]$. A rigorous upper bound for the lowest-energy band can then be obtained by making use of the sum-rule result,

$$\epsilon(k) \leq \hbar \sqrt{\frac{m_1}{m-1}} \frac{\epsilon_0(k)}{\chi(k)/2},$$

where $\epsilon_0(k) = \hbar^2 k^2 / 2m$ is the noninteracting dispersion relation, and $m_p = \int d\omega \omega^2 S(k, \omega)$ are the $p$-moments of the dynamic structure factor. Actually, the upper bound [5] provides a better estimate than the Feynman upper bound $\epsilon_F(k) = h m n_1(k)/m_0(k) = \epsilon_0 / S(k)$, where $S(k) = m_0$ is the static structure factor.

**Roton-maxon dispersion:** As a realistic example we focus on a condensate of $^{164}$Dy atoms with a trapping frequency $\omega_0 = 2\pi \times 100$ Hz, density of $n_0 = 300 \text{ } \mu\text{m}^{-3}$, and a scattering length $a_s = 85.5a_0$, giving $\epsilon_{\text{ad}} \approx 1.5$. Three-body losses are expected to be minimal since the unperturbed peak 3D density is only $6.6 \times 10^{19}$ particles $\text{m}^{-3}$ and the scattering length is well within the range already realized in experiments [4] [43].

In Fig. 1 (a) we show the static density response function $\chi$ calculated by applying a static periodic perturbation with wave vector $k_L$ and using Eq. (5). For the dipolar condensate [Fig. 1 (a1)], a large response peak dominates, indicative of a rotonized dispersion relation, see also [39]. A similar sharp peak is known to characterize the static response of superfluid $^4$He as a consequence of the presence of the rotonic excitation [44]. In contrast, for the non-dipolar condensate [Fig. 1 (a2)], the response is almost two orders of magnitude lower and monotonically decreases.

Excitation energies calculated from BdG theory (solid lines) are displayed in Fig. 1 (b1) for the dipolar condensate, and in Fig. 1 (b2) for the non-dipolar one. For the dipolar case, a roton-maxon character is clearly visible in the lowest band. The upper bound [5], involving the static response $\chi$, provides a very accurate prediction for the lowest band of the dipolar gas, practically indistinguishable from the BdG solution. Such a result is highly nontrivial since our 3D calculations inherently include the contributions from higher bands [see Fig. 1 (b)]. In contrast, and to highlight the importance of this result, Fig. 1 (b2) shows that for the non-dipolar condensate the upper bound exhibits large deviations from the exact BdG energy of the lowest band for $k_{pL} \gtrsim 0.6$. The upper bound’s success in predicting the roton-maxon dispersion is partly thanks to the low roton energy, and partly due to the lowest band experiencing the most-attractive interactions at moderate to large $k_p$. As an interesting side point, note that, for the nondipolar condensate [Fig. 1 (b2)], the lowest bands tend to become degenerate in a pairwise fashion at large momentum. This behavior arises as the excitations become more surface-like [45] and the two planar surfaces essentially uncouple.

Dynamics after lattice removal: Another approach for extracting the roton-maxon spectrum is to suddenly remove the perturbing lattice, and then to follow the ensuing in-trap dynamics either with the position space observable $\Delta n(t)$ [1] or with momentum space observable $N_{kL}(t)$. A clear experimental advantage of directly measuring the oscillation frequency is that the dispersion relation can be extracted without the need for precise calibration of the lattice strength nor the density response amplitude. For reference, the roton minimum in Fig. 1 corresponds to a wavelength of 4.3$\mu$m, a value that should be well-resolved in the current generation of experiments with in situ imaging resolution of around 1$\mu$m (e.g. see [4]).

We simulate this starting with a ground state in the presence of the lattice and then evolving it according to the GPE [1] with the lattice suddenly removed (i.e. $V_L = 0$ for $t > 0$). Such GPE dynamics are shown as symbols in Fig. 2 (a) for a lattice near the roton wavelength ($k_{pL} l_z = 0.4$), and in Fig. 2 (b) for a roton ($k_{pL} l_z = 1$). Both $\Delta n$ and $N_{kL}$ are seen to exhibit oscillations at twice the frequency of the dispersion relation. From an analytic perspective we can also predict these quantities using linear response theory, where in the single mode approximation they are:

$$\Delta n(t) = \frac{4V_L \epsilon_0(k_L)}{\epsilon^2(k_L)} \left| \cos \left( \frac{\epsilon(k_L)t}{\hbar} \right) \right|,$$
FIG. 2. In-trap dynamics of the density contrast \( \Delta n \) [Eq. (4)] and momentum response \( N_{kL} \) after sudden lattice removal at \( t = 0 \). Lattices with (a) a maxon and (b) a roton wavelength are considered. The single-mode predictions (7) and (8) – with \( \epsilon \) extracted from Fig. 1 – are shown as solid blue lines, while the time-dependent GPE results appear as symbols.

\[
\frac{N_{kL}(t)}{N} = \frac{V_L^2}{\epsilon^2(k_L)} \left[ \frac{\epsilon^2(k_L)}{\epsilon^2(k_L)} \cos^2 \frac{\epsilon(k_L)t}{\hbar} + \sin^2 \frac{\epsilon(k_L)t}{\hbar} \right].
\]  

Equations (7) and (8) are included in Fig. 2 as solid blue lines, where their excellent agreement with the symbols confirms that the GPE oscillation frequencies are indeed representative of the lowest-band dispersion [Fig. 1(b1)].

While, as expected, Fig. 2 shows that \( \Delta n(t) \) always decreases immediately after the lattice is removed (at \( t = 0 \)), it is interesting to note that the behavior for \( N_{kL}(t) \) is qualitatively different. Although \( N_{kL}(t) \) initially decreases for the roton case (b), it instead sharply increases for the maxon case (a). From a detectability viewpoint, these large upward oscillations for maxons should more than compensate for their relatively weak static response \( t < 0 \). This behavior can be explained by considering the ratio of the non-interacting dispersion to the interacting one, in Eq. (8), and noting that \( \epsilon_0/\epsilon < 1 \) for maxons while \( \epsilon_0/\epsilon > 1 \) for rotons. An intuitive picture can be understood as follows. Maxons (as well as phonons) have \( \epsilon_0/\epsilon < 1 \) because of an effectively repulsive interaction at the relevant wavevector. At the moment that the lattice is removed the density perturbation is maximal and hence so too is the interaction energy. A quarter of an oscillation period later, the density is flat and the interaction energy is now minimal, with the difference being converted into kinetic energy which manifests as an increased \( N_{kL}(t) \). Rotons experience an effectively attractive interaction, which explains why the situation is reversed.

**Extent of the linear regime:** Larger perturbations will be easier to detect, but may deviate from the linear response regime. Additionally, large perturbations can trigger the rotonized condensate to collapse [28, 29]. In Fig. 3, we address these issues with the same two lattice wavelengths as in Fig. 2, i.e. (a) a maxon and (b) a roton. GPE results are shown as symbols and we see that \( \max\{\Delta n(t)\} \propto V_L \), while \( \max\{N_{kL}(t)\} \propto V_L^2 \), in good agreement with the predictions from Eqs. (7) and (8). We have checked that for all \( V_L \) considered, the in-trap oscillation frequencies (see Fig. 3) coincide with high precision (within 1%) to the BdG roton-maxon frequencies in Fig. 1. In fact, the excellent agreement in Fig. 2(b) is for one of the most nonlinear cases, having a density contrast of \( \max(2\Delta n(t)) \approx 0.6 \), as indicated by the dashed line in Fig. 3(b). This robustness of the linear response regime constitutes an important result. Similarly, the GPE energy predictions extracted from the static density response \( \Delta n(t = 0) \) [using (7)] show excellent agreement with the BdG energies (within 1%). Due to the full 3D character of our simulations, the energy predictions from \( N_{kL}(t = 0) \) [using (8)] do not perfectly map to those from \( \Delta n(t = 0) \), even in the linear limit, but this discrepancy never exceeds 0.025\( \hbar \omega_L \).

It should be noted that all results shown in Fig. 3 are within the stable regime. For larger \( V_L \), the stationary states become dynamically unstable and the remaining translational symmetry breaks, i.e. the high-density stripes break up to form quantum droplets [4, 12, 13]. Despite this, the stability window should be large enough since \( N_{kL} \) is sizeable and the density contrast is already quite large, i.e. \( 2\Delta n \sim 0.5 \).

**Conclusions:** We have outlined an approach for the quantitative extraction of the roton-maxon dispersion relation in a flattened dipolar condensate. By measuring the static density response in position space – or the corresponding side peaks of the momentum distribution – a sum-rule upper bound provides an almost exact prediction for the roton-maxon dispersion of the lowest band, a property not shared by non-dipolar condensates in the flattened geometry. By suddenly removing the lattice...
and observing the ensuing in-trap dynamics, we demonstrated that the responses oscillate in a stable manner at twice the characteristic frequency of the dispersion relation. Crucial for experimental observability, the oscillation frequency remains constant even for large perturbation amplitudes. Interestingly, the side peak weights of the momentum distribution oscillate in a qualitatively different way for rotons as compared to phonons and maxons. This should provide a clear signature for the effectively attractive interactions experienced by the roton excitations. We quantitatively explained this behavior using perturbation theory.

Note added. We have become aware of the recent preprint [47], in which Bragg spectroscopy is used to measure the dynamic structure factor of a rotonized dipolar BEC.

Acknowledgments: We acknowledge useful discussions with Lauriane Chomaz, Franco Dalfovo and Francesca Ferlaino. This work was supported by the QUIC grant of the Horizon 2020 FET program, the Provincia Autonoma di Trento, and the DFG/FWF (FOR 2247). RNB was supported by the European Union’s Horizon 2020 research and innovation programme under the Marie Sklodowska-Curie grant agreement No. 793504 (DDQF). PBB was supported by the Marsden Fund of New Zealand.

[1] A. Griesmaier, J. Werner, S. Hensler, J. Stuhler, and T. Pfau, Phys. Rev. Lett. 94, 160401 (2005).
[2] Q. Beaufils, R. Chicireanu, T. Zanon, B. Laburthe-Tolra, E. Maréchal, L. Vernac, J.-C. Keller, and O. Gorceix, Phys. Rev. A 77, 061601 (2008).
[3] M. Lu, N. Q. Burdick, S. H. Youn, and B. L. Lev, Phys. Rev. Lett. 107, 190401 (2011).
[4] H. Kadau, M. Schmitt, M. Wenzel, C. Wink, T. Maier, I. Ferrier-Barbut, and T. Pfau, Nature 530, 194 (2016).
[5] K. Aikawa, A. Frisch, M. Mark, S. Baier, A. Rietzler, R. Grimm, and F. Ferlaino, Phys. Rev. Lett. 108, 210401 (2012).
[6] T. Lahaye, C. Menotti, L. Santos, M. Lewenstein, and T. Pfau, Rep. Prog. Phys. 72, 126401 (2009).
[7] M. A. Baranov, M. Daimonte, G. Pupillo, and P. Zoller, Chemical Reviews 112, 5012 (2012) pMID: 22877362.
[8] L. Pitaevskii and S. Stringari, Bose-Einstein Condensation and Superfluidity (Oxford University Press, 2016).
[9] L. Santos, G. V. Shlyapnikov, and M. Lewenstein, Phys. Rev. Lett. 90, 250403 (2003).
[10] Giovanazzi, S. and O’Dell, D. H.J., Eur. Phys. J. D 31, 439 (2004).
[11] H. Saito, Y. Kawaguchi, and M. Ueda, Phys. Rev. Lett. 102, 230403 (2009).
[12] L. Chomaz, S. Baier, D. Petter, M. J. Mark, F. Wächter, L. Santos, and F. Ferlaino, Phys. Rev. X 6, 041039 (2016).
[13] M. Schmitt, M. Wenzel, F. Bittcher, I. Ferrier-Barbut, and T. Pfau, Nature 539, 259 (2016).
[14] D. S. Petrov, Phys. Rev. Lett. 115, 155302 (2015).
[15] C. R. Cabrera, L. Tanzi, J. Sanz, B. Naylor, P. Thomas, P. Cheiney, and L. Tarruell, Science 359, 301 (2018).
[16] G. Semeghini, G. Ferioli, L. Masi, C. Mazzinghi, L. Wolswijk, F. Minardi, M. Modugno, G. Modugno, M. Inguscio, and M. Fattori, Phys. Rev. Lett. 120, 235301 (2018).
[17] L.-C. Ha, L. W. Clark, C. V. Parker, B. M. Anderson, and C. Chin, Phys. Rev. Lett. 114, 055301 (2015).
[18] G. I. Martone, Y. Li, L. P. Pitaevskii, and S. Stringari, Phys. Rev. A 86, 063621 (2012).
[19] M. A. Khamehchi, Y. Zhang, C. Hammer, T. Busch, and P. Engels, Phys. Rev. A 90, 063624 (2014).
[20] S.-C. Ji, L. Zhang, X.-T. Xu, Z. Wu, Y. Deng, S. Chen, and J.-W. Pan, Phys. Rev. Lett. 114, 105301 (2015).
[21] J. Léonard, A. Morales, P. Zupancic, T. Esslinger, and T. Donner, Nature 543, 87 (2017).
[22] T. Lahaye, J. Metz, B. Fröhlich, T. Koch, M. Meister, A. Griesmaier, T. Pfau, H. Saito, Y. Kawaguchi, and M. Ueda, Phys. Rev. Lett. 101, 080401 (2008).
[23] R. M. Wilson, S. Ronen, and J. L. Bohn, Phys. Rev. A 80, 023614 (2009).
[24] E. B. Linscott and P. B. Blakie, Phys. Rev. A 90, 053605 (2014).
[25] P. B. Blakie, D. Baillie, and R. N. Bisset, Phys. Rev. A 86, 021604 (2012).
[26] M. Klawunn, A. Recati, L. P. Pitaevskii, and S. Stringari, Phys. Rev. A 84, 033612 (2011).
[27] R. N. Bisset and P. B. Blakie, Phys. Rev. Lett. 110, 265302 (2013).
[28] J. P. Corson, R. M. Wilson, and J. L. Bohn, Phys. Rev. A 87, 051605 (2013).
[29] J. P. Corson, R. M. Wilson, and J. L. Bohn, Phys. Rev. A 88, 013614 (2013).
[30] M. Jona-Lasinio, K. Lakomy, and L. Santos, Phys. Rev. A 88, 026603 (2013).
[31] L. Chomaz, R. M. W. van Bijnen, D. Petter, G. Faraoni, S. Baier, J. H. Becher, M. J. Mark, F. Wächter, L. Santos, and F. Ferlaino, Nat. Phys. 14, 442 (2018).
[32] R. M. Wilson and J. L. Bohn, Phys. Rev. A 83, 023623 (2011).
[33] M. Jona-Lasinio, K. Lakomy, and L. Santos, Phys. Rev. A 88, 013619 (2013).
[34] F. Wächter and L. Santos, Phys. Rev. A 93, 061603 (2016).
[35] A. R. P. Lima and A. Pelster, Phys. Rev. A 84, 041604 (2011).
[36] R. N. Bisset, R. M. Wilson, D. Baillie, and P. B. Blakie, Phys. Rev. A 94, 033619 (2016).
[37] T. Ferrier-Barbut, H. Kadau, M. Schmitt, M. Wenzel, and T. Pfau, Phys. Rev. Lett. 116, 215301 (2016).
[38] To prevent Fourier copies along the \( z \) direction from interacting we truncate the range of the dipole-dipole interaction [45].
[39] A. R. P. Lima and A. Pelster, Phys. Rev. A 86, 063609 (2012).
[40] D. Baillie, R. M. Wilson, and P. B. Blakie, Phys. Rev. Lett. 119, 255302 (2017).
[41] D. Baillie and P. B. Blakie, New Journal of Physics 17, 033028 (2015).
[42] We assume a homogenous density along the \( y \) direction.
except when testing for dynamic instability.

[43] I. Ferrier-Barbut, M. Wenzel, F. Böttcher, T. Langen, M. Isoard, S. Stringari, and T. Pfau, Phys. Rev. Lett. 120, 160402 (2018)

[44] F. Dalfovo and S. Stringari, Phys. Rev. B 46, 13991 (1992)

[45] F. Dalfovo, S. Giorgini, M. Guilleumas, L. Pitaevskii, and S. Stringari, Phys. Rev. A 56, 3840 (1997)

[46] S. Stringari, arXiv:1805.11325

[47] D. Petter, G. Natale, R. M. W. van Bijnen, A. Patscheider, M. J. Mark, L. Chomaz, and F. Ferlaino, arXiv:1811.12115

[48] S. Ronen, D. C. E. Bortolotti, and J. L. Bohn, Phys. Rev. A 74, 013623 (2006)