STRONGLY COMPACT STRONG TRAJECTORY ATTRACTORS FOR
EVOLUTIONARY SYSTEMS AND THEIR APPLICATIONS

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ABSTRACT. We show that for any fixed accuracy and time length \( T \), a finite number of \( T \)-time length pieces of the complete trajectories on the global attractor are capable of uniformly approximating all trajectories within the accuracy in the natural strong metric after sufficiently large time when the observed dissipative system is asymptotically compact. Moreover, we obtain the strong equicontinuity of all the complete trajectories on the global attractor. These results follow by proving the existence of a strongly compact strong trajectory attractor. The notion of a trajectory attractor was previously constructed for a family of auxiliary systems including the originally considered one without uniqueness. Recently, Cheskidov and the author developed a new framework called evolutionary system, with which a (weak) trajectory attractor can be actually defined for the original system. In this paper, the theory of trajectory attractors is further developed in the natural strong metric for our purpose. We then apply it to both the 2D and the 3D Navier-Stokes equations and a general nonautonomous reaction-diffusion system.

Keywords: trajectory attractor, global attractor, evolutionary system, Navier-Stokes equations, reaction-diffusion system

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1. INTRODUCTION

The global attractor is a natural mathematical object describing the long-time behavior of solutions of many dissipative partial differential equations (PDEs). All the solutions converge to the attractor as time goes to infinity. Its studies goes back to the seminal work of Foias and Prodi [FP67], who proved that the long-time behavior of certain weak solutions of the 2D Navier-Stokes equations (NSE) is determined by the long-time behavior of a finite number of numerical parameters. For further development, see e.g. [La72, FT77, CFT85, Hal88, T88, Ha91, La91, BV92, Ro01, CV02, SY02, CLR13].

1.1. Main results and preliminary comments. In this paper we will show that the global attractor possesses a finite strong uniform tracking property. More precisely, we prove that,

Main Theorem 1. (Conclusion 3 of Theorem 3.12) Let $\mathcal{E}$ be an asymptotically compact evolutionary system\(^1\) satisfying the fundamental assumption\(^2\) A1 and let $\bar{\mathcal{E}}$ be the closure of $\mathcal{E}$. Then the global attractor $\mathcal{A}_s$ for $\mathcal{E}$ possesses the finite strong uniform tracking property, i.e., for any fixed accuracy $\epsilon > 0$ and time length $T > 0$, there exist $t_0$ and a finite set $P_T^f$ consisting of $T$-time length pieces on $[0, T]$ of the complete trajectories of $\mathcal{E}$ on $\mathcal{A}_s$, such that for any $t^* > t_0$, every trajectory $u(t)$ of $\mathcal{E}$ satisfies

$$d_s(u(t), v(t - t^*)) < \epsilon, \quad \forall t \in [t^*, t^* + T],$$

for some $T$-time length piece $v \in P_T^f$. Here $d_s$ is the natural strong metric.

Remark 1.1. We first give some explanation on terminologies in the theorem.

\(^1\)See Definitions 2.1 and 2.4 in preliminary Section 2 below, where we will briefly recall the basic definitions of the theory of evolutionary systems developed in [CF06, C09, CL09, CL14].

\(^2\)We will discuss the assumption a little later in this subsection.
One feature of our framework called evolutionary system is that the phase space $X$ (typically being a bounded absorbing set of the dissipative system under consideration) is endowed with both a weak metric and a strong metric. In applications, the strong metric induces the natural strong topology we are concerned about. Notations with subscript $s$ or $w$ are related to the strong metric $d_s$ or the weak metric $d_w$, respectively.

2. All Leray-Hopf weak solutions of the 3D Navier-Stokes equations with a fixed time-dependent force staying in $X$, for instance, form an evolutionary system satisfying the fundamental assumption A1. In general, evolutionary systems defined by PDEs of mathematical physics satisfy A1 (cf. e.g. [T88, CV02]).

3. Recently, Cheskidov and the author developed the framework of an evolutionary system in [CL14] by introducing a “closure of the evolutionary system $E$”. The structure of the global attractor $A_s$ for $E$ is obtained via all the complete trajectories $\bar{E}(\langle -\infty, \infty \rangle)$ of its closure $\bar{E}$. In applications, for an autonomous system, the closure $\bar{E}$ of the associated evolutionary system $E$ is identical to $E$ itself.

Intuitively, the finite strong uniform tracking property means that for any fixed accuracy $\epsilon$ and time length $T$, a finite number of $T$-time length pieces of the complete trajectories on the global attractor $A_s$ are capable of uniformly approximating all trajectories within the accuracy $\epsilon$ in the strong metric after sufficiently large time. The uniform tracking property indicates how the dynamics on $A_s$ describe the asymptotic behavior of all solutions of PDEs by picking up approximating pieces on $A_s$ one by one with smaller and smaller accuracies and longer and longer time lengths (see Corollary 3.14). It was studied in [V92, LR99] for some special cases and in [C09, CL14] for general cases. The main novelty of this theorem is the finiteness of the number of candidate approximating pieces for every fixed accuracy and time length. It follows by proving the existence of a strongly compact strong trajectory attractor, which we will discuss in Main Theorem 2 below.

The notions of a weak global attractor and a (weak) trajectory attractor were introduced for the autonomous 3D NSE by Foias and Temam [FT85] and by Sell [Se96], respectively. As the fundamental model for the flow of fluid, the NSE are of great physical importance. However, the problem of uniqueness is still a highlighted difficulty in the theory of PDEs. Their methods attempt to bypass this obstacle. The weak global attractor captures the long-time behavior of all Leray-Hopf weak solutions with respect to (w.r.t.) the weak topology of the natural phase space. The trajectory attractor is a global attractor in the space of trajectories, which is endowed with a suitable topology usually related to the weak topology. The set of points on all the trajectories in the trajectory attractor coincides with the weak global attractor [FT85, Se96, C09, CL14]. Only few papers are concerned with the strong topology (see [VZ96, C09, CL09, VZC10, CVZ11, CL14]).

The trajectory attractor was further studied in [Se96, CV97, CV02, SY02] for the nonautonomous system by investigating a family of auxiliary systems containing the originally considered one, but not just the original one. Its trajectory attractor does not always have

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3 Very recently, the nonuniqueness of weak solutions to the 3D NSE has greatly progressed. See [BuVi18, ABC22, CLuo22] and the references therein. Especially, the nonuniqueness of Leray-Hopf weak solutions have been proved [ABC22] for the 3D NSE in the whole space with a non-zero force.
to be the one for the original system as suggested by open problems in [CL09, CL14]. Precisely, it might not satisfy the minimality property\(^4\) for the original system. Recently, in our paper [CL14], Cheskidov and the author developed the framework called evolutionary system and made it natural to construct a (weak) trajectory attractor for the original system rather than for a family of systems. Indeed, the aforementioned minimality property does hold\(^5\). The notion of an evolutionary system \(\mathcal{E}\) was initiated in [CF06] to study a weak global attractor and a trajectory attractor for the autonomous 3D NSE, and continued in [C09] where the strong convergence of trajectories to the trajectory attractor was studied. Our new results in the current paper are systematic investigations on trajectory attractors involved in the natural strong topology.

In fact, the previous paper [CL14] presented primarily an approach that deals directly with the notion of a uniform global attractor for original nonautonomous systems. This notion, introduced by Haraux [Ha91], naturally generalizes that of a global attractor to nonautonomous ones. It was proved [CL14] that the uniform global attractor possesses the uniform tracking property under the assumption A1.

In this paper, we further develop the theory of trajectory attractors in the natural strong metric for our purpose. We will show the existence of a strongly compact strong trajectory attractor when an evolutionary system is asymptotically compact. As a consequence, we obtain that a finite number of pieces of the complete trajectories on the global attractor are enough to ensure the uniform tracking property in the strong metric, which is exactly Main Theorem 1. The proof relies on two main ingredients. One is a new point of view that identifies a weak global attractor possessing the weak uniform tracking property with the existence of a (weak) trajectory attractor. The other one is a simultaneous use of the weak and strong metrics.

With the new perspective, part of results in [CL14] can be reformulated into an inspirational form Theorem 3.6, then we conveniently generalize the related notions to strong metric versions and prove the following utilizing the weak and strong metrics at the same time.

**Main Theorem 2.** (Conclusions 4-5 of Theorem 3.6 and 1-2 of Theorem 3.12) Let \(\mathcal{E}\) be an asymptotically compact evolutionary system satisfying A1 and let \(\overline{\mathcal{E}}\) be the closure of \(\mathcal{E}\). Then

1. The strongly compact strong trajectory attractor\(^6\) \(\mathcal{A}_s\) exists, and it is the restriction of all the complete trajectories of \(\overline{\mathcal{E}}\) on \([0, \infty)\):

\[
\mathcal{A}_s = \Pi_{\mathbb{R}} \overline{\mathcal{E}}((\infty, \infty)) := \{ u(\cdot) |_{(0, \infty)} : u \in \overline{\mathcal{E}}((\infty, \infty)) \}.
\]

2. \(\mathcal{A}_s\) is a section of \(\mathcal{A}_s\):

\[
\mathcal{A}_s = \mathcal{A}_s(t) := \{ u(t) : u \in \mathcal{A}_s \}, \quad \forall t \geq 0.
\]

**Remark 1.2.** We supplement some notes.

1. We can see that all the complete trajectories \(\overline{\mathcal{E}}((\infty, \infty))\) of \(\overline{\mathcal{E}}\) lie in \(\mathcal{A}_s\) and their restriction on \([0, \infty)\) is \(\mathcal{A}_s\). Then, the approximating pieces in Main Theorem 1 lie

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\(^4\)See Definition 2.10.

\(^5\)We will have more detailed comments on our framework and the previous ones in next subsection.

\(^6\)See Definition 3.8.
in $\mathcal{A}_s$. Hence, it is also convenient to call that $\mathcal{A}_s$ or $\mathcal{E}$ possesses a finite strong uniform tracking property.

2. In applications, the evolutionary system $\mathcal{E}$ is only constructed from the solutions of the originally considered PDEs rather than those from a family of auxiliary PDEs, and thereby the attractors $\mathcal{A}_s$ and $\mathcal{A}_s$ are indeed the ones for the original system. They faithfully satisfy the minimality property for the original system. See next subsection for more detailed comments.

By the Arzalà-Ascoli Theorem, Main Theorem 2 has Main Theorem 1 and the following as corollaries.

**Main Theorem 3.** (Conclusion 4 of Theorem 3.12) Let $\mathcal{E}$ be an asymptotically compact evolutionary system satisfying A1 and let $\mathcal{E}$ be the closure of $\mathcal{E}$. Then, the strongly compact strong trajectory attractor $\mathcal{A}_s = \Pi_+ \mathcal{E}((-\infty, \infty))$ is strongly equicontinuous on $[0, \infty)$, i.e.,
\[
d_s(v(t_1), v(t_2)) \leq \theta(|t_1 - t_2|), \quad \forall t_1, t_2 \geq 0, \forall v \in \mathcal{A}_s,
\]
where $\theta(l)$ is a positive function tending to 0 as $l \to 0^+$. It is easy to see that, equivalently, all the complete trajectories $\mathcal{E}((-\infty, \infty))$ of $\mathcal{E}$ is strongly equicontinuous on $(-\infty, \infty)$. Roughly speaking, Main Theorem 3 excludes the existence of a complete trajectory on $\mathcal{A}_s$ that oscillates more and more drastically.

Global attractors are ever anticipated to be very complicated objects (fractals), which obstruct their applications. We expect that our finite strong uniform tracking property and strong equicontinuity, which are now described by the existence of a strongly compact strong trajectory attractor, will do some good for their practical utilization, for instance for numerical simulations.

Now we make some comments on the assumption A1 that appears in our above results. It is the following:

A1 The set consisting of all trajectories of $\mathcal{E}$ on $[0, \infty)$ is a precompact set in the space $C([0, \infty); X_w)$.

It provides the existence of a (weak) trajectory attractor for an evolutionary system. It also implies that the weak global attractor consists of points on the trajectories in the trajectory attractor (see Theorem 3.6). A1 is satisfied by the associated evolutionary systems of the 2D and the 3D NSE and a general dissipative reaction-diffusion system (RDS) that we will investigate in this paper. In general, the associated evolutionary systems of PDEs that arise in mathematical physics satisfy A1 (cf. e.g. [T88, CV02]). We will study more details on its relationships with the canonical closedness condition and continuity condition for evolutionary systems with uniqueness, which contain the classical frameworks of a semi-group for autonomous systems and a family of processes for nonautonomous systems (see Lemmas 3.28-3.30).

Our abstract theory can be directly applied to both the 2D and the 3D nonautonomous NSE on a bounded domain $\Omega$ with space periodic or with non-slip boundary conditions. We have the following for the 3D NSE.

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7It is the space of all weakly continuous $X$-valued functions on $[0, \infty)$. See Remark 1.1 or more details in Subsection 2.4.
Main Theorem 4. (Theorem 4.9) Assume that the external force \( g_0(t) \) is normal \(^8\) in \( L^2_{\text{loc}}(\mathbb{R}; V') \) and every complete trajectory of the closure \( \tilde{\mathcal{E}} \) of the corresponding evolutionary system \( \mathcal{E} \) of the 3D NSE with the fixed force \( g_0(t) \) is strongly continuous. Then

1. The global attractor \( \mathcal{A}_g \) satisfies the finite strong uniform tracking property, i.e., for any fixed accuracy \( \epsilon > 0 \) and time length \( T > 0 \), there exist \( t_0 \) and a finite set \( P^f_T \) consisting of \( T \)-time length pieces on \( [0, T] \) of the complete trajectories of \( \tilde{\mathcal{E}} \) on \( \mathcal{A}_g \), such that for any \( t^* > t_0 \), every Leray-Hopf weak solution \( u(t) \) satisfies

\[
|u(t) - v(t - t^*)| < \epsilon, \quad \forall t \in [t^*, t^* + T],
\]

for some \( T \)-time length piece \( v \in P^f_T \). Here \( |\cdot| \) is the \( (L^2(\Omega))^3 \)-norm.

2. The strongly compact strong trajectory attractor \( \mathfrak{A}_s = \Pi_s \mathcal{E}(\mathbb{R}; \infty) \), which is the restriction of all the complete trajectories of \( \mathcal{E} \) on \( [0, \infty) \), is strongly equicontinuous on \( [0, \infty) \), i.e.,

\[
|v(t_1) - v(t_2)| \leq \theta \left( |t_1 - t_2| \right), \quad \forall t_1, t_2 \geq 0, \forall v \in \mathfrak{A}_s,
\]

where \( \theta(l) \) is a positive function tending to 0 as \( l \to 0^+ \).

Remark 1.3. We have the following remarks on the theorem.

1. In our application, the phase space \( X \) is a bounded absorbing subset in the space \( H \), the whole of which the previous frameworks are used to taking as the natural phase space. The strong metric \( \delta_s \) of \( X \) is defined by the \( (L^2(\Omega))^3 \)-norm \( |\cdot| \) and induces the natural strong topology the same as before. The weak metric \( \delta_w \) of \( X \) induces the weak topology that is the restriction on \( X \) of the weak topology of \( (L^2(\Omega))^3 \) as mentioned earlier.

2. The normality condition on the force was introduced in [LWZ05] and the author [Lu06]. Note that the class of normal functions in \( L^2_{\text{loc}}(\mathbb{R}; V') \) is bigger than the class of so-called translation compact functions in \( L^2_{\text{loc}}(\mathbb{R}; V') \) (see Definition 5.19), which is necessary by applying previous frameworks [Se96, CV94, CV97, CV02, SY02].

3. Every complete trajectory in \( \tilde{\mathcal{E}}(\mathbb{R}; \infty) \) is a weak solution of the 3D NSE with the force \( g_0(t) \) or a force related to \( g_0(t) \). See the proof of Lemma 5.4 in [CL14]. For the autonomous 3D NSE, that is, the force is time-independent, \( \tilde{\mathcal{E}}(\mathbb{R}; \infty) = \mathcal{E}(\mathbb{R}; \infty) \).

4. The attractors \( \mathcal{A}_g \) and \( \mathfrak{A}_s \) are exactly for the original 3D NSE with a fixed force \( g_0(t) \). With a more restricted force \( g_0(t) \) that is a translation compact function in \( L^2_{\text{loc}}(\mathbb{R}; V') \), the previous frameworks (see e.g. [CV02]) are able to construct attractors for a family of auxiliary 3D NSE with forces related to \( g_0(t) \). However, it is not know whether they are actual attractors for the 3D NSE with the fixed force \( g_0(t) \) we originally consider. See Open Problem 4.16, or next subsection for more details.

5. For the 2D NSE, concerned on both weak and strong solutions, and a general RDS (1) below, we obtain similar results (see Theorems 4.21, 4.28 and 5.6, respectively).

\(^8\)See Definition 4.6.
Note that, in these cases, the solutions have already been proved to be strongly continuous (see Theorems 4.17, 4.23 and 5.2, respectively).

6. For the 2D cases, and for the RDS with more regular interaction terms, \( \tilde{E}((\infty, \infty)) \) can be known in more details due to the uniqueness and better regularity of their solutions (see Theorems 4.20 and 4.21, 4.27 and 4.28, 5.17 and 5.18, respectively).

It is worth to mention that we obtain the finite strong uniform attracting property and the strong equicontinuity of all the complete trajectories for these systems without additional conditions (cf. [CL14, LWZ05, Lu06, Lu07, CL09]).

The dissipative RDS:

\[
\partial_t u - a \Delta u + f(u, t) = g(x, t),
\]

on a bounded domain \( \Omega \) with Dirichlet or with Neumann or with periodic boundary conditions is another fundamental model in the theory of infinite dimensional dynamical systems. It is quite general that covers many examples arising in physics, chemistry and biology etc. We just list a few: the RDS with polynomial nonlinearity, Ginzburg-Landau equation, Chafee-Infante equation, FitzHugh-Nagumo equations and Lotka-Volterra competition system. See e.g. [M87, T88, Ro01, CV02, SY02] for more. The main difficulty of RDS different from NSE in previous literature is the time-dependence of the reaction terms \( f(\cdot, t) \). Nevertheless, we obtain the much deeper results that are similar to Main Theorem 4 under even less restrictions on \( f(\cdot, t) \): We make no additional assumption other than the necessary conditions of continuity, dissipativeness and growth (see (29)-(31) in Section 5). Contrastively, some conditions are imposed to construct a so-called symbol space, which is necessary for previous works [CV94, CV97, CV02, Lu07, CL09]. Then, as a by product, we give an answer to an open problem in [Lu07, CL09], which concerns how to describe the structure of global attractors for the RDS with general interaction terms \( f(\cdot, t) \).

It is not yet known whether previous frameworks can also deal with such above cases as we indicate in Open Problems 4.16 and 5.16 below (see also [CL09, CL14]).

1.2. More detailed comments on our new results. In this subsection, we discuss in more detail on the novelty of the paper, from both the theoretical and the applied points of view.

- **Theoretical point of view.** The main object of the paper is a notion of strongly compact strong trajectory attractor. In our previous work [CL14], a weak trajectory attractor is indeed defined for the original nonautonomous system under consideration rather than for a family of auxiliary systems as done with previous frameworks. Now, it is possible to establish one in strong metric also for the original system. The proofs of the main theoretical results of Main Theorems 1-3 rely on a new point of view and a simultaneous use of the weak and strong metrics. We also discuss the special evolutionary systems that include the classical frameworks of a semigroup and a family of processes.

Our theory is based on the previous works of the framework of an evolutionary system. It is originally designed in [CF06, C09] for autonomous systems and developed in [CL09, CL14] especially for nonautonomous systems. Note that the phase space \( X \) (typically taking a bounded absorbing set of the observed dissipative system) of an evolutionary
system is endowed with both a weak metric and a strong metric. The work [CL14] mainly focused on the notion of a uniform global attractor for the originally considered nonautonomous system rather than for a family of auxiliary systems. The open problems in [CL09, CL14] (see Open Problem 4.16) indicate that the uniform global and (weak) trajectory attractors constructed by the previous frameworks (see [Se96, CV97, CV02, SY02]) might not satisfy the minimality property with respect to uniformly attracting for the original nonautonomous system. Cheskidov and the author [CL14] overcame the difficulty by taking a closure of the associated evolutionary system $E$. The properties of the uniform global attractor for $E$ are able to be investigated via that for its closure $\bar{E}$.

Note that in applications the evolutionary system $E$ only consists of the solutions of the original nonautonomous PDEs, thereby its attractors do faithfully satisfy above mentioned minimality property. In contrast, the previous frameworks investigate all the solutions of, for instance, a family of 3D NSE with forces in a suitable closure of the translation family of the original force. It results in that a strong translation compactness condition on the force is necessary and such constructed attractors are only for the family of 3D NSE, which do not always have to coincide with those obtained in [CL14] (see Open Problem 4.16).

In theoretical settings, the previous mostly used framework developed by Chepyzhov and Vishik [CV97, CV02] was based on the use of the so-called time symbol (e.g. the external force in 3D NSE) and the related symbol space, which on one hand require from the very beginning some necessary restrictions on the time symbols, on the other hand, only provide auxiliary attractors. Moreover, as we will study in Section 5, in some interesting cases it is not even clear how to choose a symbol space. Our method [CL14] is able to deal directly with the actual attractors for the original nonautonomous systems with less restrictions on the time symbols, since we avoid constructing such a symbol space. See [CL14] as well as Subsection 3.3.2 below for more discussions on our framework and the previous ones.

In the previous paper [CL14], it is also possible and natural to define a trajectory attractor for the original nonautonomous system. The notion of a trajectory attractor was originally introduced in [Se96] to bypass the difficulty of uniqueness issue of the 3D NSE (cf. footnote 3). Its theory (see [Se96, CV97, CV02, SY02]) is usually related to the weak topologies of the functional settings of the PDEs. In this paper, we further develop the theory of strongly compact strong trajectory attractors, which is principally Main Theorem 2. As a consequence, we derive Main Theorem 1 as well as Main Theorem 3. Our theory and its applications are new for both autonomous and nonautonomous dissipative systems of or lack of uniqueness.

It is worthwhile to remind again that, such a strongly compact strong trajectory attractor is certainly for the original system under consideration. Especially, the minimality property with respect to uniformly strongly attracting (cf. Definitions 2.11 and 3.8) the original nonautonomous system is valid. This is possible on account of the theory established in our previous paper [CL14] and the reformulation with a new perspective in the ongoing paper. We will realize it as Main Theorem 2. Moreover, we will see that the existence a trajectory attractor describes the global attractor possessing the uniform tracking property (see Lemma 3.4) and its strong compactness provides the finiteness of the number of candidate strongly approximating pieces (see Main Theorem 1) and the strong equicontinuity of all the complete trajectories on the global attractor (see Main Theorem 3). It is also
worth to mention that we obtain simultaneously the strongly compact global attractor and the strongly compact strong trajectory attractor.

The proofs of Main Theorems 1-3 are proceed as follows. First, we give a convenient definition of the weak uniform tracking property slightly different from that in [C09, CL09, CL14]. Second, we show that, a weak global attractor for an evolutionary system $\mathcal{E}$ satisfying the weak uniform tracking property is equivalent to the existence of a (weak) trajectory attractor for $\mathcal{E}$ under the assumption A1. This equivalence makes it possible to reformulate the related results in the previous work [CL14] in a more concise and inspirational form. Third, following this new form, we set out to prove a version in the strong metric. A strong trajectory attractor is naturally defined as a (weak) trajectory attractor which is also a strong trajectory attracting set (see Definition 2.11). The strong trajectory attracting property was investigated in [VZ96, C09, CL09, VZC10, CVZ11, CL14], which are few papers involving the natural strong topology we are really interested in. Fourth, by taking advantage of a simultaneous use of the weak and strong metrics, we derive the strong compactness of the strong trajectory attractor, namely Main Theorem 2, when the evolutionary system is asymptotically compact. Finally, thanks to the Arzàla-Ascoli Theorem, we obtain the finite strong uniform tracking property and the strong equicontinuity of all the complete trajectories on the global attractor, that is, Main Theorems 1 and 3, respectively. In particular, the systems that possess strongly compact global attractors are asymptotically compact. Hence, in these cases, the strongly compact strong trajectory attractors follow immediately once property A1 holds (see Corollary 3.18 and Remark 3.19).

We investigate more details on the structure of the attractors, especially for closed evolutionary systems and evolutionary systems with uniqueness (see Definitions 3.20 and 2.8). These evolutionary systems include the classical frameworks of a semigroup and a family of processes as shown in [CL14]. Thereout, we generalize the results in [CV02, LWZ05, Lu06, Lu07, CL14] in that, without additional assumptions, we obtain the existence of strongly compact strong trajectory attractors, and the sequent consequences of the finite strong uniform tracking property, the strong equicontinuity of all the complete trajectories on global attractors and other properties following from Corollaries 3.13, 3.14 and 3.15. In the case of these evolutionary systems, we also compare A1 with the familiar related conditions of closedness and continuity imposed on the previous framework of a family of processes (see Lemmas 3.28-3.30).

- **Applied point of view.** The applications of our new theory are first focused on both the 3D and the 2D Navier-Stokes equations (NSE). The NSE are probably the most fundamental example in the theory of infinite dimensional dynamics systems, most part of which has been established by investigating this example. We also study a general reaction-diffusion system (RDS), which is another fundamental model.

We apply our abstract theory to the 3D NSE, concerned on Leray-Hopf weak solutions, and the 2D NSE, concerned on both weak and strong solutions, with a fixed force as continuations of [CL14], [Lu06] and [LWZ05], respectively. No extra condition is assumed. The property A1 is verified by the well-known compactness lemmas (see e.g. Lemma 4.4) on the solutions. The new results are Main Theorem 4 for the 3D case and analogues for the
2D cases. Besides the finite strong uniform tracking property, one of the new observation
is that, for instance, for the 2D NSE with a fixed normal force in $L^2_{\text{loc}}(\mathbb{R}; V')$, it denies the
existence of the more and more wildly oscillating complete weak solutions. Note that, actually,
for the autonomous 2D NSE, the strong compactness of the strong trajectory attractors
could have been deduced by the classical estimates (see (26)) that imply the uniqueness of
the solutions and the continuity of the associated semigroups. Nevertheless, this is not valid
for the nonautonomous case (see Remark 4.22). Now we are able to cope with this case.

Another fundamental model we study is a RDS (see (1)) with a fixed pair of a time-
dependent nonlinearity and a driving force. It is treated along the same line as the NSE.
Although the nonlinearity depending on time is a main difficulty in previous studies (see e.g. [CV97, CV02, Lu07, CL09]),
we assume less that the nonlinearity only satisfies the basic conditions of continuity, dissipativeness and growth (see (29)-(31)). This is owing to
no need to construct a so-called symbol space with our method. The three conditions neither
guarantee the unique solvability of (1) nor provide a suitable symbol space. The assumption
on the force is a translation boundedness condition, which is the weakest condition that
ensures the existence of a bounded uniformly w.r.t. the initial time absorbing ball. We
take this ball as a phase space $X$. The weak and strong metrics are metrics that induce the
usual weak and strong topologies of the space $^9\left(L^2(\Omega)\right)^N$ restricted on $X$, respectively.
We verify that all the weak solutions of the RDS (1) form an evolutionary system satisfying
A1. Therefore, we obtain the existence and structure of the weak attractors $A_{w}$ and $A_{w}$.
In addition, if the force is normal in $L^2_{\text{loc}}(\mathbb{R}; V')$, where $V'$ is the dual of $\left(H^1_0(\Omega)\right)^N$, the
evolutionary system is asymptotically compact. Then, the two weak attractors $A_{w}$ and $A_{w}$
are strongly compact strong attractors $A_{s}$ and $A_{s}$. Consequently, the finite strong uniform
tracking property for the RDS (1) holds, i.e., for any fixed accuracy $\epsilon$ and time length $T$,
a finite number of $T$-time length pieces of the complete trajectories on $A_{s}$ are enough to
uniformly approximate all weak solutions within the accuracy $\epsilon$ in the $\left(L^2(\Omega)\right)^N$-norm after
sufficiently large time. Moreover, all the complete trajectories on $A_{s}$ is equicontinuous on
$(-\infty, \infty)$ in the $\left(L^2(\Omega)\right)^N$-norm. Note again that there is no additional assumption on
the time-dependent nonlinearity. Hence, part of these results answer an open problem in
[Lu07, CL09], where an appropriate symbol space is absent (see Remark 5.8 for more
details).

We also consider the RDS (1) with more regular nonlinearities $f(\cdot, t)$. One more assumption imposed on $f(\cdot, t)$ is a translation compact condition (see Section 5.1), which
is necessary for the previous works to obtain the structure of the global attractors (see
[CV94, CV95, CV97, CV02, Lu07, CL09]). Analogously for the 3D NSE, we boost the
results in [CL09]. Another more assumption (see (51)) on $f(\cdot, t)$ guarantees the uniqueness
of the weak solutions of (1). Hence, we can know more properties about the structure of
the attractors and obtain similar results to those of 2D NSE, which generalize the results in
[CV02, Lu07].

At the end of the paper, we construct several interesting examples of nonlinearities $f(\cdot, t)$
that do not satisfy the two assumptions just mentioned any more, but our theory is still
applicable for the RDS (1) with these nonlinearities. The pointwise limit function $f_\infty(\cdot)$

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$^9N$ is the number of components of the unknown vector function $u$ of (1).
of one example as $t \to +\infty$ is discontinuous and the others even have no pointwise limit functions. These facts hint that there might not exist a suitable symbol space which is very required in previous studies [CV97, CV02, Lu07, CL09]. It is not clear whether the results of the RDS (1) with such nonlinearities can also be obtained with previous frameworks (see Open Problem 5.16).

1.3. Paper outline. The rest of this paper is organized as follows.

- In Sect. 2, we briefly recall the basic definitions of the theory of evolutionary systems and a criterion of their asymptotical compactness, developed in [CF06, C09, CL09, CL14]. For our convenience, we formulate as a definition the strongly trajectory attracting property obtained in [C09, CL09, CL14].
- In Sect. 3, we devote to prove main theoretical theorems of the paper. In Subsect. 3.1, we concern about the weak trajectory attractors and in Subsect. 3.2, the strongly compact strong trajectory attractors and the sequent corollaries, such as the finite strong uniform tracking property. Comparisons with known results are also mentioned.
- In Sect. 4, we apply our new theory to both the 3D and the 2D NSE with a fixed force. Similarly, we also present comparisons with existing literature.
- In Sect. 5, we apply the theory to the RDS with a fixed pair of a time-dependent nonlinearity and a driving force along the same line as the NSE. In Subsect. 5.1, the RDS with more regular nonlinearities are considered. In Subsect. 5.2, we collect and study some properties on the nonlinearities and give several examples with which our theory is applicable for the RDS while previous frameworks do not work.

2. EVOLUTIONARY SYSTEM

Now we briefly recall the basic definitions on evolutionary systems. See [C09, CL09, CL14] for details. An important property obtained in these references is formulated as a definition (see Definition 2.1) for the purpose of this paper. We also recall a criterion of the asymptotical compactness for evolutionary systems developed in [CF06, C09, CL09, CL14].

2.1. Phase space endowed with two metrics. Assume that a set $X$ is endowed with two metrics $d_s(\cdot, \cdot)$ and $d_w(\cdot, \cdot)$ respectively, satisfying the following conditions:

1. $X$ is $d_w$-compact.
2. If $d_s(u_n, v_n) \to 0$ as $n \to \infty$ for some $u_n, v_n \in X$, then $d_w(u_n, v_n) \to 0$ as $n \to \infty$.

Hence, we will refer to $d_s$ as a strong metric and $d_w$ as a weak metric. Let $\overline{A}$ be the closure of a set $A \subset X$ in the topology generated by $d_\bullet$. Here (the same below) $\bullet = s$ or $w$. Note that any strongly compact ($d_s$-compact) set is weakly compact ($d_w$-compact), and any weakly closed set is strongly closed.

2.2. (Autonomous) evolutionary system. Let

$$\mathcal{T} := \{I : I = [\tau, \infty) \subset \mathbb{R}, \text{ or } I = (-\infty, \infty)\} ,$$
and for each $s \in \mathbb{R}$, let $I + s$ be $[\tau, \infty)$ if $I = [\tau + s, \infty)$ or $(-\infty, \infty)$ if $I = (-\infty, \infty)$.

For any $I \in T$, denote by $F(I)$ the set of all $X$-valued functions on $I$. Now we define an evolutionary system $E$ as follows.

**Definition 2.1.** A map $E$ that associates to each $I \in T$ a subset $E(I) \subset F(I)$ will be called an evolutionary system if the following conditions are satisfied:

1. $E([0, \infty)) \neq \emptyset$.
2. $E(I + s) = \{ u(\cdot) : u(\cdot + s) \in E(I) \}$ for all $s \in \mathbb{R}$.
3. $\{ u(\cdot)|_t : u(\cdot) \in E(I) \} \subset E(I)$ for all pairs $I_1, I_2 \in T$, such that $I_2 \subset I_1$.
4. $E((-\infty, \infty)) = \{ u(\cdot) : u(\cdot)_{|_{\tau, \infty}} \in E([\tau, \infty)), \forall \tau \in \mathbb{R} \}$.

We will refer to $E(I)$ as the set of all trajectories on the time interval $I$. The set $E((-\infty, \infty))$ is called the kernel of $E$ and the trajectories in it are called complete. Let $P(X)$ be the set of all subsets of $X$. For every $t \geq 0$, define a set-valued map

$$R(t) : P(X) \to P(X),$$

$$R(t)A := \{ u(t) : u(0) \in A, u \in E([0, \infty)) \}, \quad A \subset X.$$  

Note that the assumptions on $E$ imply that $R(t)$ enjoys the following property:

$$R(t + s)A \subset R(t)R(s)A, \quad \forall A \subset X, \quad t, s \geq 0.$$  

**Definition 2.2.** A set $A_\bullet \subset X$ is a $d_\bullet$-global attractor if $A_\bullet$ is a minimal set that is

1. $d_\bullet$-closed.
2. $d_\bullet$-attracting: for any $B \subset X$ and $\epsilon > 0$, there exists $t_0$, such that

$$R(t)B \subset B_\bullet(A_\bullet, \epsilon) := \{ y \in X : \inf_{x \in A_\bullet} d_\bullet(x, y) < \epsilon \}, \quad \forall t \geq t_0.$$  

**Definition 2.3.** The $\omega_\bullet$-limit of a set $A \subset X$ is

$$\omega_\bullet(A) := \bigcap_{\tau \geq 0} \bigcup_{t \geq \tau} R(t)A.$$  

**Definition 2.4.** An evolutionary system $E$ is asymptotically compact if for any $t_n \to +\infty$ and any $x_n \in R(t_n)X$, the sequence $\{x_n\}$ is relatively strongly compact.

**Definition 2.5.** Let $E$ be an evolutionary system. If a map $E^1$ that associates to each $I \in T$ a subset $E^1(I) \subset E(I)$ is also an evolutionary system, we will call it an evolutionary subsystem of $E$, and denote by $E^1 \subset E$.

Note that it is in fact sufficient that for each $I \in T \setminus \{(-\infty, \infty)\}$, $E^1(I) \subset E(I)$, since $E^1((-\infty, \infty)) \subset E((-\infty, \infty))$ follows immediately from the definition of an evolutionary system.

In order to extend the notion of invariance from a semiflow to an evolutionary system, we will need the following mapping:

$$\tilde{R}(t)A := \{ u(t) : u(0) \in A, u \in E([0, \infty)) \}, \quad A \subset X, \quad t \in \mathbb{R}.$$  

**Definition 2.6.** A set $A \subset X$ is positively invariant if

$$\tilde{R}(t)A \subset A, \quad \forall t \geq 0.$$
A is invariant if
\[ \tilde{R}(t)A = A, \quad \forall t \geq 0. \]
A is quasi-invariant if for every \( a \in A \) there exists a complete trajectory \( u \in \mathcal{E}((-\infty, \infty)) \) with \( u(0) = a \) and \( u(t) \in A \) for all \( t \in \mathbb{R} \).

2.3. Nonautonomous evolutionary system and reducing to autonomous system. Let \( \Sigma \) be a parameter set and \( \{T(s) : s \geq 0\} \) be a family of operators acting on \( \Sigma \) satisfying \( T(s)\Sigma = \Sigma, \forall s \geq 0 \). Any element \( \sigma \in \Sigma \) will be called (time) symbol and \( \Sigma \) will be called (time) symbol space. For instance, in many applications \( \Sigma = \mathcal{T} \) semigroup and \( \Sigma \) is the translation semigroup and \( \Sigma \) is the translation family of the time-dependent items of the considered system or its closure in some appropriate topological space (for more examples see [CV94, CV02, CL14], the appendix in [CLR13]).

**Definition 2.7.** A family of maps \( \mathcal{E}_\sigma, \sigma \in \Sigma \) that for every \( \sigma \in \Sigma \) associates to each \( I \in \mathcal{T} \) a subset \( \mathcal{E}_\sigma(I) \subset \mathcal{F}(I) \) will be called a nonautonomous evolutionary system if the following conditions are satisfied:

1. \( \mathcal{E}_\sigma([\tau, \infty)) \neq \emptyset, \forall \tau \in \mathbb{R} \).
2. \( \mathcal{E}_\sigma(I + s) = \{u(\cdot) : u(\cdot + s) \in \mathcal{E}_{T(s)\sigma}(I)\}, \forall s \geq 0 \).
3. \( \{u(\cdot)|_{I_2} : u(\cdot) \in \mathcal{E}_{\sigma}(I_1)\} \subset \mathcal{E}_{\sigma}(I_2), \forall I_1, I_2 \in \mathcal{T}, I_2 \subset I_1 \).
4. \( \mathcal{E}_\sigma((-, \infty)) = \{u(\cdot) : u(\cdot)|_{[\tau, \infty)} \in \mathcal{E}_{\sigma}([\tau, \infty)), \forall \tau \in \mathbb{R} \} \).

It is shown in [CL09, CL14] that any nonautonomous evolutionary system can be viewed as an (autonomous) evolutionary system. Let\(^{10}\)
\[ \mathcal{E}_\Sigma(I) := \bigcup_{\sigma \in \Sigma} \mathcal{E}_\sigma(I), \quad \forall I \in \mathcal{T} \setminus \{(-\infty, \infty)\}, \]
and
\[ \mathcal{E}_\Sigma((-, \infty)) := \{u(\cdot) : u(\cdot)|_{[\tau, \infty)} \in \mathcal{E}_{\Sigma}([\tau, \infty)), \forall \tau \in \mathbb{R} \}. \]

Now we define an (autonomous) evolutionary system \( \mathcal{E} \) in the following way:
\[ \mathcal{E}(I) := \mathcal{E}_\Sigma(I), \quad \forall I \in \mathcal{T}. \]
It can be checked that all the conditions in Definition 2.1 are satisfied. Consequently, the above notions of invariance, quasi-invariance, and a global attractor for \( \mathcal{E} \) can be extended to the nonautonomous evolutionary system \( \{\mathcal{E}_\sigma\}_{\sigma \in \Sigma} \). The global attractor in the nonautonomous case will be conventionally called a uniform global attractor (or simply a global attractor). However, for some evolutionary systems constructed from nonautonomous dynamical systems the associated symbol spaces are not known. See [CL14] and the following sections for more details. Thus, we will not distinguish between autonomous and nonautonomous evolutionary systems. If it is necessary, we denote an evolutionary system with a symbol space \( \Sigma \) by \( \mathcal{E}_\Sigma \) and its global attractor by \( \mathcal{A}_\Sigma \), and so on.

**Definition 2.8.** An evolutionary system \( \mathcal{E}_\Sigma \) is a system with uniqueness if for every \( u_0 \in X \) and \( \sigma \in \Sigma \), there is a unique trajectory \( u \in \mathcal{E}_\sigma([0, \infty)) \) such that \( u(0) = u_0 \).

\(^{10}\)Here we fix a minor flaw in Section 2.3 in [CL14]. See Lemma 3.22 in Section 3.3 below.
2.4. **Weak trajectory attracting set and weak trajectory attractor.** Denote by $C([a, b]; X_w)$ the space of $d_w$-continuous $X$-valued functions on $[a, b]$ endowed with the metric

$$d_{C([a, b]; X_w)}(u, v) := \sup_{t \in [a, b]} d_w(u(t), v(t)).$$

Let also $C([a, \infty); X_w)$ be the space of $d_w$-continuous $X$-valued functions on $[a, \infty)$ endowed with the metric

$$d_{C([a, \infty); X_w)}(u, v) := \sum_{t \in \mathbb{N}} \frac{1}{2^t} d_{C([a, a+\ell]; X_w)}(u, v).$$

Note that the convergence in $C([a, \infty); X_w)$ is equivalent to uniform convergence on compact sets.

Now we suppose that evolutionary systems $E$ satisfy the following assumption:

$$E([0, \infty)) \subset C([0, \infty); X_w).$$

Define the family of translation operators $\{T(s)\}_{s \geq 0}$.

(2) \[ (T(s)u)(\cdot) := u(\cdot + s)|_{[0, \infty)}, \quad u \in C([0, \infty); X_w). \]

Due to the property (3) of the evolutionary system (see Definitions 2.1 and 2.7), we have that,

$$T(s)E([0, \infty)) \subset E([0, \infty)), \quad \forall s \geq 0.$$ 

Note that $E([0, \infty))$ may not be closed in $C([0, \infty); X_w)$. We consider the dynamics of the translation semigroup $\{T(s)\}_{s \geq 0}$ acting on the phase space $C([0, \infty); X_w)$. A set $P \subset C([0, \infty); X_w)$ weakly uniformly attracts a set $Q \subset E([0, \infty))$ if for any $\epsilon > 0$, there exists $t_0$, such that

$$T(t)Q \subset \left\{ v \in C([0, \infty); X_w) : \inf_{u \in P} \sup_{v \in Q} d_{C([0, \infty); X_w)}(u, v) < \epsilon \right\}, \quad \forall t \geq t_0.$$

**Definition 2.9.** A set $P \subset C([0, \infty); X_w)$ is a weak trajectory attracting set for an evolutionary system $E$ if it weakly uniformly attracts $E([0, \infty))$.

**Definition 2.10.** A set $A_w \subset C([0, \infty); X_w)$ is a weak trajectory attractor for an evolutionary system $E$ if $A_w$ is a minimal weak trajectory attracting set that is

(1) Closed in $C([0, \infty); X_w)$.

(2) Invariant: $T(t)A_w = A_w, \forall t \geq 0$.

It is easy to see that if a weak trajectory attractor exists, it is unique. In previous literature (see e.g. [Se96, CV97, CV02, SY02, C09, CL14]), this kind of attractor is just called a trajectory attractor. We now use the current name for a distinction, since in the paper we develop the theory of trajectory attractors related to the strong topology.

2.5. **Strong trajectory attracting set.** In [C09, CL14], a property of uniformly attracting $E([0, \infty))$ in $L^\infty_{loc}([0, \infty); X_w)$ was obtained. Here, analogy to Definition 2.9, we retell it as following Definition 2.11. A set $P \subset C([0, \infty); X_w)$ strongly uniformly attracts a set $Q \subset E([0, \infty))$ if for any $\epsilon > 0$ and $T > 0$, there exists $t_0$, such that

$$T(t)Q \subset \left\{ v \in C([0, \infty); X_w) : \inf_{u \in P} \sup_{s \in [0, T]} d_s(u(s), v(s)) < \epsilon \right\}, \quad \forall t \geq t_0.$$
Definition 2.11. A set \( P \subset C([0, \infty); X_w) \) is a strong trajectory attracting set for an evolutionary system \( \mathcal{E} \) if it strongly uniformly attracts \( \mathcal{E}([0, \infty)) \).

The advantage of such a definition will be seen in the next section, where we will naturally define a strong trajectory attractor analogy to Definition 2.10, and obtain its strong compactness.

2.6. Fundamental assumption A1. We will investigate evolutionary systems \( \mathcal{E} \) satisfying the following property:

A1 \( \mathcal{E}([0, \infty)) \) is a precompact set in \( C([0, \infty); X_w) \).

In general, the evolutionary systems defined by PDEs of mathematical physics satisfy A1 (cf. e.g. [T88, CV02]).

The evolutionary systems satisfying A1 are closely related to the concept of the uniform w.r.t. the initial time global attractor for a nonautonomous dynamical system, initiated by Haraux [Ha91]. For instance, as shown in [CL14], the uniform global attractor for an evolutionary system \( \mathcal{E}_\Sigma \) defined by a process \( \{U_\sigma(t, \tau)\} \) is the uniform w.r.t. the initial time global attractor for \( \{U_\sigma(t, \tau)\} \) due to Haraux. A stronger version of A1 used in [CF06, C09, CL09] is the following:

\( \bar{\text{A1}} \) \( \mathcal{E}([0, \infty)) \) is a compact set in \( C([0, \infty); X_w) \).

However, instead of the property \( \bar{\text{A1}} \), the evolutionary system \( \mathcal{E}_\Sigma \) usually satisfies only A1. For more details see [CL09, CL14].

2.7. Closure of an evolutionary system. Let

\[
\bar{\mathcal{E}}([\tau, \infty)) := \overline{\mathcal{E}([\tau, \infty))}^{C([\tau, \infty); X_w)}, \quad \forall \tau \in \mathbb{R},
\]

and

\[
\bar{\mathcal{E}}((\tau, \infty)) := \{ u(\cdot) : u(\cdot)|_{[\tau, \infty)} \in \bar{\mathcal{E}}([\tau, \infty)), \forall \tau \in \mathbb{R}\}.
\]

It can be checked that \( \bar{\mathcal{E}} \) is also an evolutionary system. We call \( \bar{\mathcal{E}} \) the closure of the evolutionary system \( \mathcal{E} \), and add for \( \bar{\mathcal{E}} \) the top-script \( \bar{\cdot} \) to the corresponding notations for \( \mathcal{E} \). For example, we denote by \( \bar{\mathcal{A}} \) the uniform \( \bar{d}_{s} \) global attractor for \( \bar{\mathcal{E}} \).

Obviously, if \( \mathcal{E} \) satisfies A1, then \( \bar{\mathcal{E}} \) satisfies \( \bar{\text{A1}} \). Note that, a nonautonomous evolutionary system \( \mathcal{E} \) usually satisfies only A1 rather than \( \bar{\text{A1}} \). Moreover, for some nonautonomous evolutionary systems, as we will see in Sections 4 and 5, there may not exist suitable symbol spaces associated to their closures. However, Cheskidov and the author showed in [CL14] that it is possible to investigate the global attractor as well as the weak trajectory attractor for any \( \mathcal{E} \) via those for its closure \( \bar{\mathcal{E}} \), no matter whether it is lack of uniqueness or lack of a symbol space, since our approach avoids the necessity of constructing a symbol space.

2.8. A criterion of asymptotical compactness. We recall a method to verify the asymptotical compactness of evolutionary systems satisfying these additional properties (see [CF06, C09, CL09, CL14]):

A2 (Energy inequality) Assume that \( X \) is a set in some Banach space \( H \) satisfying the Radon-Riesz property (see below) with the norm denoted by \( | \cdot | \), such that \( d_\phi(x, y) = |x - y| \) for \( x, y \in X \) and \( d_w \) induces the weak topology on \( X \). Assume
also that for any $\epsilon > 0$, there exists $\delta > 0$, such that for every $u \in \mathcal{E}([0, \infty))$ and $t > 0$,
\[ |u(t)| \leq |u(t_0)| + \epsilon, \]
for $t_0$ a.e. in $(t - \delta, t)$.

A3 (Strong convergence a.e.) Let $u_n \in \mathcal{E}([0, \infty))$ be such that, $u_n$ is $d_{C([0,T];X_w)}$-Cauchy sequence in $C([0,T];X_w)$ for some $T > 0$. Then $u_n(t)$ is $d_s$-Cauchy sequence a.e. in $[0,T]$.

A Banach space $B$ is said to satisfy the Radon-Riesz property if for any sequence \( \{x_n\} \subset B \),
\[ x_n \to x \text{ strongly in } B \iff \frac{\|x_n\|_B}{\|x\|_B} \to 1, \text{ as } n \to \infty. \]

In many applications, $H$ in A2 is a uniformly convex separable Banach space and $X$ is a bounded closed set in $H$. Then the weak topology of $H$ is metrizable on $X$, and $X$ is compact w.r.t. such a metric $d_w$. Moreover, the Radon-Riesz property is automatically satisfied in this case.

We have the following criterion of asymptotical compactness that is sufficient for the applications in this paper. For more, see Remark 3.19.

**Theorem 2.12.** [CL14] Let $\mathcal{E}$ be an evolutionary system satisfying A1, A2, and A3, and assume that its closure $\bar{\mathcal{E}}$ satisfies $\bar{\mathcal{E}}((-,\infty)) \subset C((-,\infty);X_s)$. Then $\mathcal{E}$ is asymptotically compact.

### 3. Attractors for Evolutionary System

An important property of a global attractor called uniform tracking property has been studied in [C09, CL09, CL14]. This property indicates how the dynamics on the global attractor describes a long-time behavior of every trajectory of an evolutionary system (see e.g. [Ro01]). We now show that a weak global attractor possessing the weak uniform tracking property is equivalent to the existence of a weak trajectory attractor. Inspired by this new point of view, we further develop in this section a notion of a strongly compact strong trajectory attractor for an evolutionary system $\mathcal{E}$, which connotes deep results on the strong uniform tracking property. It is remarkable that we obtain its existence at the same time we get the strongly compact strong global attractor.

#### 3.1. Weak trajectory attractor: Revisit with a new point of view.

In this subsection, we first investigate some properties of a weak trajectory attractor. Then, we reformulate related results obtained in [CL14]. We start with introducing the following definition.

**Definition 3.1.** A set $P \subset C([0, \infty);X_w)$ satisfies the weak uniform tracking property for an evolutionary system $\mathcal{E}$ if for any $\epsilon > 0$, there exists $t_0$, such that for any $t^* > t_0$, every trajectory $u \in \mathcal{E}([0, \infty))$ satisfies
\[ d_C([t^*,\infty);X_w](u(\cdot), v(\cdot - t^*)) < \epsilon, \]
for some trajectory $v \in P$. 

Lemma 3.2. Let $P \subset C([0, \infty); X_w)$ satisfy $T(s)P = P, \forall s \geq 0$. Then $P$ satisfies the weak uniform tracking property for an evolutionary system $\mathcal{E}$ if and only if for any $\epsilon > 0$, there exists $t_0$, such that for any $t^* > t_0$, every trajectory $u \in \mathcal{E}([0, \infty))$ satisfies

$$d_{C([t^*, \infty); X_w)}(u, v) < \epsilon,$$

for some trajectory $v \in P$.

Proof. By the assumption, for any $t^* \geq 0$, $v \in P$ if and only if there exists $v^* \in P$ that $T(t^*)v^* = v$. Hence, for any $\epsilon > 0$ and $u \in \mathcal{E}([0, \infty))$,

$$d_{C([t^*, \infty); X_w)}(u(\cdot), v(\cdot - t^*)) < \epsilon,$$

for some trajectory $v \in P$ is equivalent to

$$d_{C([t^*, \infty); X_w)}(u(\cdot), v^*(\cdot)) < \epsilon,$$

for some trajectory $v^* \in P$. \hfill $\square$

The latter property in Lemma 3.2 is in fact called weak uniform tracking property in [C09, CL09, CL14]. Instead, with this lemma in hand, we now use Definition 3.1 for our later convenience.

Now, we have the following relationship between the weak uniform tracking property and the weak trajectory attracting property.

Lemma 3.3. A set $P \subset C([0, \infty); X_w)$ satisfies the weak uniform tracking property for an evolutionary system $\mathcal{E}$ if and only if it is a weak trajectory attracting set for $\mathcal{E}$.

Proof. Suppose that $P$ is a weak trajectory attracting set for an evolutionary system $\mathcal{E}$. For any $\epsilon > 0$, there exists $t_0$, such that

$$T(t)\mathcal{E}([0, \infty)) \subset \{ v \in C([0, \infty); X_w) : d_{C([0, \infty); X_w)}(P, v) < \epsilon/2 \}, \forall t \geq t_0.$$

Hence, for any $t^* \geq t_0$ and every trajectory $u \in \mathcal{E}([0, \infty))$, we know that

(3) $$d_{C([0, \infty); X_w)}(v(\cdot), (T(t^*)u)(\cdot)) < \epsilon,$$

for some $v \in P$. By (2) of the definition of the family of translation operators \{$T(s)$\}, (3) is

(4) $$d_{C([0, \infty); X_w)}(v(\cdot), u(\cdot + t^*)) < \epsilon.$$

By a change of the variable, (4) is equivalent to

(5) $$d_{C([t^*, \infty); X_w)}(v(\cdot - t^*), u(\cdot)) < \epsilon.$$

Therefore, $P$ satisfies the weak uniform tracking property.

Conversely, assume that for any $\epsilon > 0$, there exists $t_0$, such that for any $t^* > t_0$, every trajectory $u \in \mathcal{E}([0, \infty))$ satisfies the inequality (5) for some $v \in P$. Equivalently, (3) is valid, which implies that

$$T(t)\mathcal{E}([0, \infty)) \subset \{ v \in C([0, \infty); X_w) : d_{C([0, \infty); X_w)}(P, v) < \epsilon \}, \forall t \geq t_0 + 1.$$

Thus, $P$ is a weak trajectory attracting set. \hfill $\square$
Let $\mathcal{E}$ be an evolutionary system satisfying A1. Due to Theorems 3.5 and 4.3 in [CL14], the weak global attractor $\mathcal{A}_w$ and the weak trajectory attractor $\mathfrak{A}_w$ for $\mathcal{E}$ exist, and satisfy

$$\mathfrak{A}_w = \Pi_+ \bar{\mathcal{E}}((-\infty, \infty)) := \{ u(\cdot)|_{[0, \infty)} : u \in \bar{\mathcal{E}}((-\infty, \infty)) \},$$

and

$$\mathcal{A}_w = \mathfrak{A}_w(t) := \{ u(t) : u \in \mathfrak{A}_w \}, \quad \forall t \geq 0.$$ 

Here, $\bar{\mathcal{E}}$ is the closure of $\mathcal{E}$. It follows from Lemma 3.3 that $\mathfrak{A}_w$ satisfies the weak uniform tracking property for $\mathcal{E}$, that is, for any $\epsilon > 0$, there exists $t_0$, such that for any $t^* > t_0$, every trajectory $u \in \mathcal{E}([0, \infty))$ satisfies

$$d_{C([t^*, \infty); X_w)}(u(\cdot), v(\cdot - t^*)) < \epsilon,$$

for some trajectory $v \in \mathfrak{A}_w$. In view of (6), there is $\tilde{v}(\cdot) \in \bar{\mathcal{E}}((-\infty, \infty))$ such that,

$$\tilde{v}(\cdot)|_{[0, \infty)} = v.$$

Especially, we have

$$d_{C([t^*, \infty); X_w)}(u(\cdot), \tilde{v}(\cdot - t^*)) < \epsilon.$$

Hence, we may conveniently call that $\bar{\mathcal{E}}((-\infty, \infty))$ or $\mathcal{E}$ itself satisfies the weak uniform tracking property. By (6) and (7), we know that, for any $\tilde{v}(\cdot) \in \bar{\mathcal{E}}((-\infty, \infty))$,

$$\{ \tilde{v}(t) : t \in \mathbb{R} \} \subset \mathcal{A}_w.$$

Then, we may also call that the weak global attractor $\mathcal{A}_w$ satisfies the weak uniform tracking property.

Contrarily, suppose that, for any $\epsilon > 0$, there exists $t_0$, such that for any $t^* > t_0$, every trajectory $u \in \mathcal{E}([0, \infty))$ satisfies

$$d_{C([t^*, \infty); X_w)}(u(\cdot), \tilde{v}(\cdot - t^*)) < \epsilon,$$

for some complete trajectory $\tilde{v}$ on $\mathfrak{A}_w$. Then, by Definition 3.1, $\Pi_+ \bar{\mathcal{E}}((-\infty, \infty))$ satisfies the weak uniform tracking property. It is deduced from Lemma 3.3 that it is a weak trajectory attracting set for $\mathcal{E}$. Obviously, $\Pi_+ \bar{\mathcal{E}}((-\infty, \infty))$ is invariant and closed in $C([0, \infty); X_w)$. As shown in the second part of the proof of Theorem 4.3 in [CL14], it is also a minimal weak trajectory attracting set for $\mathcal{E}$. That is, $\Pi_+ \bar{\mathcal{E}}((-\infty, \infty))$ is a weak trajectory attractor for $\mathcal{E}$. Therefore, we have the following equivalence.

**Lemma 3.4.** Let $\mathcal{E}$ be an evolutionary system satisfying A1. The weak global attractor $\mathcal{A}_w$ for $\mathcal{E}$ satisfies the weak uniform tracking property if and only if $\mathcal{E}$ possesses a weak trajectory attractor $\mathfrak{A}_w$.

Since the evolutionary system $\mathcal{E}$ satisfies A1, its weak trajectory attractor is always weakly compact, that is, it is compact in $C([0, \infty); X_w)$. Then, we introduce the following definition.

**Definition 3.5.** A set $P \subset C([0, \infty); X_w)$ satisfies the finite weak uniform tracking property for an evolutionary system $\mathcal{E}$ if for any $\epsilon > 0$, there exist $t_0$ and a finite subset $P^f \subset P$, such that for any $t^* > t_0$, every trajectory $u \in \mathcal{E}([0, \infty))$ satisfies

$$d_{C([t^*, \infty); X_w)}(u(\cdot), v(\cdot - t^*)) < \epsilon,$$

for some trajectory $v \in P^f$. 
With the new perspective of above Lemma 3.4, Theorems 3.5 and 4.3 in [CL14] can be restated in the following more concise form.

**Theorem 3.6.** Let $\mathcal{E}$ be an evolutionary system. Then

1. The weak global attractor $A_w$ exists, and $A_w = \omega_w(X)$.

Furthermore, assume that $\mathcal{E}$ satisfies A1. Let $\bar{\mathcal{E}}$ be the closure of $\mathcal{E}$. Then

2. $A_w = \omega_w(X) = \bar{\omega}_w(X) = \bar{A}_w$.

3. $A_w$ is the maximal invariant and maximal quasi-invariant set w.r.t. $\bar{\mathcal{E}}$:

$$A_w = \{ u_0 \in X : u_0 = u(0) \text{ for some } u \in \bar{\mathcal{E}}((-\infty, \infty)) \}.$$

4. The weak trajectory attractor $\mathfrak{A}_w$ exists, it is weakly compact, and

$$\mathfrak{A}_w = \Pi_+ \bar{\mathcal{E}}((-\infty, \infty)) := \{ u(\cdot)|_{[0, \infty)} : u \in \bar{\mathcal{E}}((-\infty, \infty)) \}.$$

Hence, $\mathfrak{A}_w$ satisfies the finite weak uniform tracking property for $\mathcal{E}$ and is weakly equicontinuous on $[0, \infty)$.

5. $A_w$ is a section of $\mathfrak{A}_w$:

$$A_w = \mathfrak{A}_w(t) := \{ u(t) : u \in \mathfrak{A}_w \}, \quad \forall t \geq 0.$$

*Proof.* The conclusions 1-3 are the corresponding results of Theorem 3.5 in [CL14]. The first part of the conclusion 4 and the conclusion 5 are just Theorem 4.3 in [CL14]. The conclusion 4 in Theorem 3.5 in [CL14] is incorporated in the existence of $A_w$ due to Lemmas 3.2 and 3.4. In other words, $A_w$ satisfies the weak uniform tracking property for $\mathcal{E}$. Thus, for any $\epsilon > 0$, there exists $t_0$, such that for any $t^* > t_0$, every trajectory $u \in \mathcal{E}([0, \infty))$ satisfies

$$(8) \quad d_{C([t^*, \infty); X_w]}(u(\cdot), v^*(\cdot - t^*)) < \epsilon/2,$$

for some trajectory $v^* \in \mathfrak{A}_w$. Thanks to the assumption A1, $\mathfrak{A}_w$ is weakly compact. Hence, we can take a set $P^f$ consisting of a finite number of trajectories,

$$P^f := \{ u_1, u_2, \cdots, u_K \} \subset \mathfrak{A}_w,$$

such that, for any $v \in \mathfrak{A}_w$,

$$d_{C([0, \infty); X_w]}(v, u_i) < \epsilon/2,$$

for some $u_i \in P^f$. Then, there exists some $u_j(\cdot) \in P^f$ satisfying

$$d_{C([0, \infty); X_w]}(v^*, u_j) < \epsilon/2,$$

which deduces that

$$d_{C([t^*, \infty); X_w]}(v^*(\cdot - t^*), u_j(\cdot - t^*)) < \epsilon/2.$$

Combining with (8), it implies that

$$d_{C([t^*, \infty); X_w]}(u(\cdot), u_j(\cdot - t^*)) < \epsilon,$$

for some trajectory $u_j \in P^f$. This means that $\mathfrak{A}_w$ satisfies the finite weak uniform tracking property for $\mathcal{E}$.

Denote by

$$\mathfrak{A}_w|_{[\alpha, \beta]} := \{ u(\cdot)|_{[\alpha, \beta]} : u \in \mathfrak{A}_w \}, \quad \forall \beta > \alpha \geq 0.$$
Note that $\mathfrak{A}_w|_{[0,1]}$ is compact in $C([0,1];X_w)$. It is deduced from the Arzelà-Ascoli compactness criterion that $\mathfrak{A}_w|_{[0,1]}$ is weakly equicontinuous on $[0,1]$. By the invariance of $\mathfrak{A}_w$, we have
\[
\{v(\cdot + \alpha) : v(\cdot) \in \mathfrak{A}_w|_{[\alpha,\alpha+1]}\} = \mathfrak{A}_w|_{[0,1]}, \quad \forall \alpha \geq 0.
\]
Thus, $\mathfrak{A}_w$ is weakly equicontinuous on $[0,\infty)$. □

Accordingly, we can call that $A_w$, or $\bar{E}((-\infty,\infty))$, or $E$ satisfies the finite weak uniform tracking property.

3.2. **Strongly compact strong trajectory attractor.** Such a form of Theorem 3.6 is indeed inspirational. We will establish a version of it in the strong metric in this subsection. We begin with the following.

**Lemma 3.7.** A strong trajectory attracting set for an evolutionary system $E$ is a weak trajectory attracting set for $E$.

**Proof.** Let $P$ is a strong trajectory attracting set for an evolutionary system $E$. Suppose that it is not a weak trajectory attracting set for $E$. Then, there exist $\epsilon_0 > 0$, and sequences $t_n \rightarrow \infty$ as $n \rightarrow \infty$, $u_n \in E([0,\infty))$, such that
\[
d_{C([0,\infty);X_w)}(P,T(t_n)u_n) > 3\epsilon_0. \tag{9}
\]
Let $l_0 \in \mathbb{N}$ satisfy
\[
\sum_{l > l_0} \frac{1}{2^l} \leq \epsilon_0.
\]
By the definition of the metric $d_{C([0,\infty);X_w)}$, we obtain from (9) that
\[
d_{C([0,l_0];X_w)}(P,T(t_n)u_n) \sum_{l \leq l_0} \frac{1}{2^l} + \sum_{l > l_0} \frac{1}{2^l} > 3\epsilon_0, \quad \forall n \in \mathbb{N},
\]
which yields that
\[
d_{C([0,l_0];X_w)}(P,T(t_n)u_n) > 2\epsilon_0, \quad \forall n \in \mathbb{N}. \tag{10}
\]
On the other hand, since $P$ is a strong trajectory attracting set, we have that
\[
\lim_{n \rightarrow \infty} \inf_{v \in P} \sup_{s \in [0,l_0]} d_w(v(s), (T(t_n)u_n)(s)) = 0.
\]
Passing to a subsequence and dropping a subindex, we can assume that there exists a sequence $\{v_n\} \subset P$, such that
\[
\lim_{n \rightarrow \infty} \sup_{s \in [0,l_0]} d_w(v_n(s), (T(t_n)u_n)(s)) = 0. \tag{11}
\]
Thanks to (10), there exists a sequence $\{s_n\} \subset [0,l_0]$, such that
\[
d_w(v_n(s_n), (T(t_n)u_n)(s_n)) > \epsilon_0, \quad \forall n \in \mathbb{N}. \tag{12}
\]
However, it follows from (11) that
\[
\lim_{n \rightarrow \infty} d_w(v_n(s_n), (T(t_n)u_n)(s_n)) = 0,
\]
which implies that
\[
\lim_{n \rightarrow \infty} d_w(v_n(s_n), (T(t_n)u_n)(s_n)) = 0.
\]
This contradicts to (12). We complete the proof. □

According to Definitions 2.10, 2.11 and Lemma 3.7, we naturally define a strong trajectory attractor as well as a strongly compact one.

**Definition 3.8.** A set $\mathcal{A}_s \subset C([0, \infty); X_w)$ is a strong trajectory attractor for an evolutionary system $\mathcal{E}$ if $\mathcal{A}_s$ is a minimal strong trajectory attracting set that is

1. **Closed in $C([0, \infty); X_w)$**.
2. **Invariant**: $T(t)\mathcal{A}_s = \mathcal{A}_s$, $\forall t \geq 0$.

It is said that $\mathcal{A}_s$ is strongly compact if it is compact in $C([0, \infty); X_s)$.

Hence, if a strong trajectory attractor exists, it is unique. Moreover, such a definition means that, a strong trajectory attractor is a weak trajectory attractor whenever it is also a strong trajectory attracting set.

We establish the following definition and lemmas that can be viewed as versions of Definition 3.1 and Lemmas 3.2 and 3.3 in the strong metric, respectively.

**Definition 3.9.** A set $P \subset C([0, \infty); X_w)$ satisfies the strong uniform tracking property for an evolutionary system $\mathcal{E}$ if for any $\epsilon > 0$ and $T > 0$, there exists $t_0$, such that for any $t^* > t_0$, every trajectory $u \in \mathcal{E}([0, \infty))$ satisfies

$$d_s(u(t), v(t^*)) < \epsilon, \quad \forall t \in [t^*, t^* + T],$$

for some $T$-time length piece $v \in P_T$. Here $P_T := \{v(\cdot)|_{[0,T]} : v \in P\}$.

**Lemma 3.10.** Let $P \subset C([0, \infty); X_w)$ satisfy $T(s)P = P$, $\forall s \geq 0$. Then $P$ satisfies the strong uniform tracking property for an evolutionary system $\mathcal{E}$ if and only if for any $\epsilon > 0$ and $T > 0$, there exists $t_0$, such that for any $t^* > t_0$, every trajectory $u \in \mathcal{E}([0, \infty))$ satisfies

$$d_s(u(t), v(t)) < \epsilon, \quad \forall t \in [t^*, t^* + T],$$

for some trajectory $v \in P$.

**Proof.** The proof is analogous to that of Lemma 3.2. Note that, by definition, $v \in P_T$ if and only if there is $\tilde{v} \in P$ such that $\tilde{v}|_{[0,T]} = v$. Hence, due to the assumption, for any $t^* \geq 0$, $v \in P_T$ if and only if there exists $v^* \in P$ that $T(t^*)v^* = \tilde{v}$. Then, for any $\epsilon > 0$, $T > 0$ and $u \in \mathcal{E}([0, \infty))$,

$$d_s(u(t), v(t^*)) < \epsilon, \quad \forall t \in [t^*, t^* + T],$$

for some $T$-time length piece $v \in P_T$ is equivalent to

$$d_s(u(t), v^*(t)) < \epsilon, \quad \forall t \in [t^*, t^* + T],$$

for some trajectory $v^* \in P$. □

Similarly, the latter property in Lemma 3.10 is called strong uniform tracking property in [C09, CL09, CL14]. However, we will soon see that, it is convenient to substitute Definition 3.9.

**Lemma 3.11.** A set $P \subset C([0, \infty); X_w)$ satisfies the strong uniform tracking property for an evolutionary system $\mathcal{E}$ if and only if it is a strong trajectory attracting set for $\mathcal{E}$. 

Proof. Assume that \( P \) is a strong trajectory attracting set for an evolutionary system \( \mathcal{E} \). Then, for any \( \epsilon > 0 \) and \( T > 0 \), there exists \( t_0 \), such that
\[
T(t)\mathcal{E}([0, \infty)) \subset \left\{ v \in C([0, \infty); X_w) : \inf_{u \in P} \sup_{t \in [0, T]} d_a(u(s), v(s)) < \frac{\epsilon}{2} \right\}, \quad \forall t \geq t_0.
\]
Hence, for any \( t^* \geq t_0 \) and every trajectory \( u \in \mathcal{E}([0, \infty)) \), we have
\[
\sup_{s \in [0, T]} d_a(\tilde{v}(s), (T(t^*)u)(s)) < \epsilon,
\]
for some \( \tilde{v} \in P \). Thanks to (2) of the definition of the family of translation operators \( \{T(s)\}_{s \geq 0} \), (13) is equivalent to
\[
\sup_{s \in [0, T]} d_a(v(s), u(s + t^*)) < \epsilon,
\]
with \( v = \tilde{v}|_{[0, T]} \). By a change of the variable, (14) is
\[
\sup_{s \in [t^*, t^* + T]} d_a(v(s - t^*), u(s)) < \epsilon.
\]
Therefore, \( P \) satisfies the strong uniform tracking property.

On the contrary, suppose that for any \( \epsilon > 0 \) and \( T > 0 \), there exists \( t_0 \), such that for any \( t^* > t_0 \), every trajectory \( u \in \mathcal{E}([0, \infty)) \) satisfies the inequality (15) for some \( v \in P_T \). Equivalently, (13) holds for some \( \tilde{v} \in P \) that \( \tilde{v}|_{[0, T]} = v \). Hence, we have,
\[
T(t)\mathcal{E}([0, \infty)) \subset \left\{ v \in C([0, \infty); X_w) : \inf_{u \in P} \sup_{t \in [0, T]} d_a(u(s), v(s)) < \epsilon \right\}, \quad \forall t > t_0,
\]
which means that \( P \) is a strong trajectory attracting set. \( \square \)

Due to Lemma 3.11, a strong trajectory attractor is the minimal set that is invariant, closed in \( C([0, \infty); X_w) \) and satisfying the strong uniform tracking property.

Now we arrive at one of the main theoretical results of the paper, which improves Theorems 3.6 and 4.4 in [CL14] by obtaining the strong compactness of strong trajectory attractors and its corollaries without additional condition.

**Theorem 3.12.** Let \( \mathcal{E} \) be an asymptotically compact evolutionary system. Then

1. The strong global attractor \( \mathfrak{A}_g \) exists, it is strongly compact, and \( \mathfrak{A}_g = \mathfrak{A}_w \).

Furthermore, assume that \( \mathcal{E} \) satisfies A1. Let \( \bar{\mathcal{E}} \) be the closure of \( \mathcal{E} \). Then

2. The strong trajectory attractor \( \mathfrak{A}_s \) exists and \( \mathfrak{A}_s = \mathfrak{A}_w = \Pi_1 \bar{\mathcal{E}}((0, \infty)) \), it is strongly compact.

3. \( \mathfrak{A}_s \) satisfies the finite strong uniform tracking property for \( \mathcal{E} \), i.e., for any \( \epsilon > 0 \) and \( T > 0 \), there exist \( t_0 \) and a finite subset \( P_T^f \subset \mathfrak{A}_s|_{[0, T]} \), such that for any \( t^* > t_0 \), every trajectory \( u \in \mathcal{E}([0, \infty)) \) satisfies
\[
d_a(u(t), v(t - t^*)) < \epsilon, \quad \forall t \in [t^*, t^* + T],
\]
for some \( T \)-time length piece \( v \in P_T^f \).
The sequences \( \{u_n(t) : t \in [0, \infty)\} \) is strongly equicontinuous on \([0, \infty)\), i.e.,
\[
d_s(v(t_1), v(t_2)) \leq \theta(|t_1 - t_2|), \quad \forall t_1, t_2 \geq 0, \forall v \in \mathcal{A}_s,
\]
where \( \theta(l) \) is a positive function tending to 0 as \( l \to 0^+ \).

**Proof:** The conclusion 1 is that of Theorem 3.6 in [CL14].

Due to Theorem 4.4 in [CL14] and Definition 2.11, the weak trajectory attractor \( \mathcal{A}_w \) is a strong trajectory attracting set that is invariant and compact in \( C([0, \infty); X_w) \). Suppose that \( P \subset C([0, \infty); X_w) \) is any other strong trajectory attracting set being invariant and closed in \( C([0, \infty); X_w) \). We know from Lemma 3.7 that \( P \) is also a weak trajectory attracting set. Hence, \( \mathcal{A}_w \subset P \). This concludes that \( \mathcal{A}_w \) is indeed a strong trajectory attractor \( \mathcal{A}_s \) according to Definition 3.8.

Now we demonstrate the compactness of \( \mathcal{A}_w \) in \( C([0, \infty); X_s) \).

First, we have \( \mathcal{A}_w \subset C([0, \infty); X_s) \). In fact, thanks to Theorem 3.6, for every \( u \in \mathcal{A}_w \), the set
\[
\{u(t) : t \in [0, \infty)\} \subset \mathcal{A}_w
\]
is precompact in \( X_s \). Hence, for every \( \tilde{u} \in [0, \infty) \), any weakly convergent sequence \( \{u(t_n) : t_n \geq 0\} \) with the limit \( u(\tilde{u}) \) as \( t_n \to \tilde{u} \) does strongly converge to \( u(\tilde{u}) \), which means \( u \in C([0, \infty); X_s) \).

Note that \( \mathcal{A}_w \) is compact in \( C([0, \infty); X_w) \). Now take a sequence \( \{u_n(t)\} \subset \mathcal{A}_w \) and \( u(t) \in \mathcal{A}_w \) that \( u_n(t) \to u(t) \) in \( C([0, \infty); X_w) \) as \( n \to \infty \). We claim that the convergence is indeed in \( C([0, \infty); X_s) \). Otherwise, there exist \( \epsilon_0 > 0, T_0 > 0 \), and sequences \( \{n_j\}, n_j \to \infty \) as \( j \to \infty \) and \( \{t_{n_j}\} \subset [0, T_0] \), such that
\[
d_s(u_{n_j}(t_{n_j}), u(t_{n_j})) > \epsilon_0, \quad \forall n_j.
\]
The sequences
\[
\{u_{n_j}(t_{n_j})\}, \{u(t_{n_j})\} \subset \mathcal{A}_s,
\]
are relatively strongly compact due to the strong compactness of \( \mathcal{A}_s \). Passing to a subsequence and dropping a subindex, we may assume that \( \{u_{n_j}(t_{n_j})\} \) and \( \{u(t_{n_j})\} \) are strongly convergent with limits \( x \) and \( y \), respectively. We have that
\[
d_w(x, y) \leq d_w(u_{n_j}(t_{n_j}), x) + d_w(u_{n_j}(t_{n_j}), u(t_{n_j})) + d_w(u(t_{n_j}), y), \quad \forall n_j.
\]
By the assumption,
\[
\lim_{j \to \infty} \sup_{t \in [0, T_0]} d_w(u_{n_j}(t), u(t)) = 0.
\]
Together with
\[
\lim_{j \to \infty} d_w(u_{n_j}(t_{n_j}), x) = 0,
\]
and
\[
\lim_{j \to \infty} d_w(u(t_{n_j}), y) = 0,
\]
it follows from (17) that \( x = y \), which is a contradiction to (16).

By Lemma 3.11, \( \mathcal{A}_s \) possesses the strong uniform tracking property for \( \mathcal{E} \). Therefore, for any \( \epsilon > 0 \) and \( T > 0 \), there exists \( t_0 \), such that for any \( t^* > t_0 \), every trajectory \( u \in \mathcal{E}([0, \infty)) \) satisfies
\[
d_s(u(t), v^*(t - t^*)) < \epsilon/2, \quad \forall t \in [t^*, t^* + T],
\]
for some $T$-time length piece $v^* \in \mathfrak{A}_s[0,T]$. Now since $\mathfrak{A}_s[0,T]$ is compact in $C([0,T]; X_s)$, we can take a finite number set,

$$P_T := \{u_1(t), u_2(t), \cdots, u_K(t)\} \subset \mathfrak{A}_s[0,T],$$

such that, for any $v \in \mathfrak{A}_s[0,T]$,

$$d_s(u_j(t), v(t)) < \epsilon/2, \quad \forall t \in [0,T],$$

for some $u_i(t) \in P_T$. Hence, there exists some $u_j(t) \in P_T$ satisfying

$$d_s(u_j(t), v^*(t)) < \epsilon/2, \quad \forall t \in [0,T],$$

which is equivalent to

$$d_s(u_j(t - t^*), v^*(t - t^*)) < \epsilon/2, \quad \forall t \in [t^*, t^* + T].$$

Together with (18), it implies that

$$d_s(u(t), u_j(t - t^*)) < \epsilon, \quad \forall t \in [t^*, t^* + T],$$

for some $T$-time length piece $u_j(t) \in P_T$. That is, $\mathfrak{A}_s$ satisfies the finite strong uniform tracking property for $\mathcal{E}$.

Finally, we show that $\mathfrak{A}_s = \Pi_+ \bar{\mathcal{E}}((\infty, \infty))$ is strongly equicontinuous on $[0, \infty)$. Without loss of generality, we assume that $|t_1 - t_2| \leq 1$. Hence, $t_1$ and $t_2$ belong to some interval $[\gamma, \gamma + 2],$ $\gamma \geq 0$. Denote by

$$\Pi_{[\alpha, \beta]} \bar{\mathcal{E}}((\infty, \infty)) := \{u(\cdot)|_{[\alpha, \beta]} : u \in \bar{\mathcal{E}}((\infty, \infty))\},$$

Notice that

$$\{v(\cdot + \gamma) : v(\cdot) \in \Pi_{[\gamma, \gamma + 2]} \bar{\mathcal{E}}((\infty, \infty))\} = \Pi_{[0,2]} \bar{\mathcal{E}}((\infty, \infty)).$$

Thus, we need only to verify that $\Pi_{[0,2]} \bar{\mathcal{E}}((\infty, \infty))$ is strongly equicontinuous on $[0, 2]$. Thanks to the Arzelà-Ascoli compactness criterion, this is a consequence of the fact that $\Pi_{[0,2]} \bar{\mathcal{E}}((\infty, \infty))$ is compact in $C([0,2]; X_s)$.

The proof is complete. \qed

Similarly, according to the above theorem and Theorem 3.6, it is also convenient to call that the global attractor $\mathcal{A}_s$, or $\bar{\mathcal{E}}((\infty, \infty))$, or $\mathcal{E}$ possesses a finite strong uniform tracking property. That is, for any fixed accuracy $\epsilon$ and time length $T$, a finite number of $T$-time length pieces on $[0, T]$ of the complete trajectories on $\mathfrak{A}_s$ are capable of uniformly approximating all trajectories within the accuracy $\epsilon$ in the strong metric after sufficiently large time. By applying Theorem 3.12 repeatedly, we have the following two corollaries that indicate how the dynamics on the global attractor determine the long-time dynamics of all trajectories of an evolutionary system.

**Corollary 3.13.** Let $\mathcal{E}$ be an asymptotically compact evolutionary system satisfying A1 and let $\bar{\mathcal{E}}$ be the closure of $\mathcal{E}$. Then, for any $\epsilon > 0$ and $T > 0$, there exist $\epsilon_0 \in \mathbb{N}$ and a finite number of $T$-time length pieces of the complete trajectories $\bar{\mathcal{E}}((\infty, \infty))$ on the global attractor for $\bar{\mathcal{E}}$:

$$P_T := \{u_1(t), u_2(t), \cdots, u_K(t)\} \subset \mathfrak{A}_s[0,T],$$
such that, for every trajectory \( u \in \mathcal{E}([0, \infty)) \), there is a sequence \( \{i_j\}_{j \geq j_0} \) with \( i_j \in \{1, 2, \cdots, K\} \) that satisfies
\[
d_s(u(t), u_{i_j}(t - jT)) < \epsilon, \quad \forall t \in [jT, (j + 1)T],
\]
for \( j \geq j_0 \) and \( u_{i_j} \in P_T^j \).

**Proof.** By Theorem 3.12, for any \( \epsilon > 0 \) and \( T > 0 \), there exist \( t_0 \) and a finite subset \( P_T^j \subset \mathfrak{A}_s[[0,T]] \) consisting of \( K \) elements,
\[
P_T^j := \{u_1(t), u_2(t), \cdots, u_K(t)\},
\]
such that for any \( t^* > t_0 \), every trajectory \( u \in \mathcal{E}([0, \infty)) \) satisfies
\[
d_s(u(t), u_i(t - t^*)) < \epsilon, \quad \forall t \in [t^*, t^* + T],
\]
for some \( T \)-time length piece \( u_i \in P_T^j \). Then, we obtain the conclusion if we apply this result successively by taking \( t^* = j_0T > t_0 \) and \( t^* = jT, j > j_0 \). \( \square \)

It is interesting to note that the number \( K \) of the candidate approximating pieces \( P_T^j \) depends only on the accuracy \( \epsilon \) and the time length \( T \), and then, for any fixed accuracy and time length, every trajectory is assigned a sequence of elements in the finite set \( \{1, 2, \cdots, K\} \).

**Corollary 3.14.** Let \( \mathcal{E} \) be an asymptotically compact evolutionary system satisfying A1 and let \( \bar{\mathcal{E}} \) be the closure of \( \mathcal{E} \). Give two sequences \( \{\epsilon_n\} \) and \( \{T_n\} \) satisfying
\[
\epsilon_1 > \epsilon_2 > \cdots > \epsilon_n \to 0, \quad \text{as } n \to \infty,
\]
and
\[
0 < T_1 < T_2 < \cdots < T_n < \cdots \to \infty, \quad \text{as } n \to \infty.
\]
Then, there exist a time \( t_0 \), a sequence \( \{J_n\} \) and a series of finite sets \( P_T^j \) consisting of \( T_n \)-time length pieces of the complete trajectories \( \bar{\mathcal{E}}((-\infty, \infty)) \) on the global attractor for \( \mathcal{E} \):
\[
P_T^j := \{u_1^n(t), u_2^n(t), \cdots, u_K^n(t)\} \subset \mathfrak{A}_s[[0,T_n]],
\]
such that, for every trajectory \( u \in \mathcal{E}([0, \infty)) \), there is a sequence
\[
i_1^n, i_2^n, \cdots, i_{J_1}^n, i_1^{n+1}, i_2^{n+1}, \cdots, i_{J_2}^{n+1}, \cdots, i_1^n, i_2^n, \cdots, i_{J_n}^n, \cdots,
\]
with \( i_j^n \in \{1, 2, \cdots, K\} \), \( 1 \leq j \leq J_n \) that satisfies \( u_{i_j}^n \in P_{T_n}^j \) and
\[
d_s\left(u(t), u_{i_j}^n\left(t - \left(t_0 + \sum_{l=1}^{n-1} J_l T_l + (j - 1)T_n\right)\right)\right) < \epsilon_n,
\]
\[
\forall t \in \left[t_0 + \sum_{l=1}^{n-1} J_l T_l + (j - 1)T_n, t_0 + \sum_{l=1}^{n-1} J_l T_l + jT_n\right].
\]

**Proof.** Due to Theorem 3.12, for any \( \epsilon_n > 0 \), and \( T_n > 0 \), there exist \( t_n \) and a finite subset \( P_{T_n}^j \subset \mathfrak{A}_s[[0,T_n]] \) consisting of \( K_n \) elements,
\[
P_{T_n}^j := \{u_1^n(t), u_2^n(t), \cdots, u_K^n(t)\},
\]
such that, for any \( t^*_n > t_n \), every trajectory \( u \in \mathcal{E}([0, \infty)) \), satisfies
\[
d_s(u(t), v(t - t^*_n)) < \epsilon, \quad \forall t \in [t_n, t_n^* + T],
\]
for some $T_n$-time length piece $v \in P_{T_n}$. Without loss of generality, we take $\{t_n\}$ satisfying $t_2 - t_1 > 1$ and $t_{n+1} - t_n > T_{n-1}$, $n \geq 2$.

Now we are going to determine $t_0$ and the sequences $\{J_n\}$ and $\{i_j^n\}$. We process inductively.

Let

$$t_0 := t_1 + 1,$$

and let

$$J_1 := \left\lfloor \frac{t_2 - t_0}{T_1} \right\rfloor + 1.$$

Here $\lfloor \cdot \rfloor$ is the greatest integer function. Then, $t_0 + J_1 T_1 \in (t_2, t_3)$. We apply the previous result in the first paragraph $J_1$ times with $\epsilon_1$ and $T_1$, and gain that, for every trajectory $u \in \mathcal{E}([0, \infty))$,

$$d_u \left( u(t), u_{i_j}^1 \left( t - (t_0 + (j-1)T_1) \right) \right) < \epsilon_1,$$

$$\forall t \in [t_0 + (j-1)T_1, t_0 + jT_1],$$

for $i_j^1 \in \{1, 2, \cdots, K_1\}$, $1 \leq j \leq J_1$ and $u_{i_j}^1 \in P_{T_1}^I$. Next, let 

$$J_2 := \left\lfloor \frac{t_3 - (t_0 + J_1 T_1)}{T_2} \right\rfloor + 1.$$

We apply the previous result in the first paragraph $J_2$ times with $\epsilon_2$ and $T_2$, and have that,

$$d_u \left( u(t), u_{i_j}^2 \left( t - (t_0 + J_1 T_1 + (j-1)T_2) \right) \right) < \epsilon_2,$$

$$\forall t \in [t_0 + J_1 T_1 + (j-1)T_2, t_0 + J_1 T_1 + jT_2],$$

for $i_j^2 \in \{1, 2, \cdots, K_2\}$, $1 \leq j \leq J_2$ and $u_{i_j}^2 \in P_{T_2}^I$. Note that, $t_0 + \sum_{l=1}^2 J_l T_l \in (t_3, t_4)$.

Suppose we have obtained $\{J_l\}$ and $\{i_j^l\}$, $1 \leq l \leq n$, $1 \leq j \leq J_l$, that

$$J_l := \left\lfloor \frac{t_{l+1} - (t_0 + \sum_{m=1}^{l-1} J_m T_m)}{T_l} \right\rfloor + 1,$$

and

$$i_j^l \in \{1, 2, \cdots, K_l\}, \quad u_{i_j}^l \in P_{T_l}^I,$$

satisfying

$$d_u \left( u(t), u_{i_j}^l \left( t - \left( t_0 + \sum_{m=1}^{l-1} J_m T_m + (j-1)T_l \right) \right) \right) < \epsilon_l,$$

$$\forall t \in \left[ t_0 + \sum_{m=1}^{l-1} J_m T_m + (j-1)T_l, t_0 + \sum_{m=1}^{l-1} J_m T_m + jT_l \right].$$

We know that $t_0 + \sum_{l=1}^n J_l T_l \in (t_{n+1}, t_{n+2})$. Now take

$$J_{n+1} := \left\lfloor \frac{t_{n+2} - (t_0 + \sum_{l=1}^n J_l T_l)}{T_{n+1}} \right\rfloor + 1.$$
We can consecutively apply the previous conclusion in the first paragraph \(J_{n+1}\) times with \(\epsilon_{n+1}\) and \(T_{n+1}\), and obtain that,

\[
\|s(u(t), u^{n+1}_{i+1} \left( t - \left( t_0 + \sum_{l=1}^{n} J_l T_l + (j-1) T_{n+1} \right) \right)) \| < \epsilon_{n+1},
\]

\[
\forall t \in \left[ t_0 + \sum_{l=1}^{n} J_l T_l + (j-1) T_{n+1}, t_0 + \sum_{l=1}^{n} J_l T_l + j T_{n+1} \right],
\]

for \(i+1 \in \{1, 2, \cdots, K_{n+1}\}, 1 \leq j \leq J_{n+1}\) and \(u^{n+1}_{i+1} \in P^{f}_{T_{n+1}}\).

The proof is completed.

Now we give a property of the complete trajectories \(\tilde{E}((-\infty, \infty))\) on \(A_s\).

**Corollary 3.15.** Let \(E\) be an asymptotically compact evolutionary system satisfying A1 and let \(\tilde{E}\) be the closure of \(E\). Then, every complete trajectory \(u(t) \in \tilde{E}((-\infty, \infty))\) is translation compact in \(C((-\infty, \infty); X_s)\), i.e., the set

\[
\{u(\cdot + h) : h \in \mathbb{R}\}^{C((-\infty, \infty); X_s)}
\]

is compact in \(C((-\infty, \infty); X_s)\).

**Proof.** It follows from Theorem 3.12 that, the set \(\{u(t) : t \in (-\infty, \infty)\} \subset A_s\) is precompact in \(X_s\), and \(u(t)\) is uniformly strongly continuous on \((-\infty, \infty)\). Then the conclusion follows from Proposition V.2.2 in [CV02].

The complete trajectories being periodic, quasi-periodic, almost periodic, homoclinic and heteroclinic on \(A_s\) are translation compact in \(C((-\infty, \infty); X_s)\). See [CV02] for more details.

**Remark 3.16.** Such forms of Theorems 3.6 and 3.12 suggest the following comments.

1. **Theorem 3.12** indicates that the notion of a strongly compact strong trajectory attractor is an apt description of the strongly compact strong global attractor possessing the finite strong uniform tracking property and the strong equicontinuity of all the complete trajectories on it.
2. **Comparing with Theorem 3.6, Theorem 3.12** implies that both the strong compactness of the strong global attractor and the strong trajectory attractor follow simultaneously once we obtain the asymptotical compactness of an evolutionary system.
3. **Theorems 3.6 and 3.12** show that the global attractor is a section of the trajectory attractor and the trajectory attractor consists of the restriction of all the complete trajectories on the global attractor on time semiaxis \([0, \infty)\); the notion of a global attractor stresses the property of attracting trajectories staring from sets in phase space \(X\) while the notion of a trajectory attractor emphasizes the uniform tracking property.

In fact, the asymptotical compactness of \(E\) is also a necessary condition.

**Theorem 3.17.** An evolutionary system \(E\) is asymptotically compact if and only if its strongly compact strong global attractor \(A_s\) exists.
Remark 3.19. With this corollary in hand, we are able to apply our theory to the systems and a general dissipative reaction-diffusion system. This idea in the next sections where we apply our theory to the 2D Navier-Stokes equations.

After reformulate them by our framework of evolutionary systems, the existence of strongly compact strong trajectory attractors follow immediately, once A1 is verified. We will see that for any \( t \rightarrow +\infty \) as \( n \rightarrow \infty \) and \( x_n \in R(t_n)X \). Then, for any positive integer \( k \), by Definition 2.2, there exist \( x_{n_k} \) and \( y_k \) such that, \( y_k \in A_s \) and

\[
d_d(x_{n_k}, y_k) < \frac{1}{k}.
\]

Since \( A_s \) is strongly compact, passing to a subsequence and dropping a subindex, we may assume that the sequence \( \{ y_k \} \) strongly converges with a limit \( y \). It follows that

\[
d_d(x_{n_k}, y) \leq d_d(x_{n_k}, y_k) + d_d(y_k, y) \quad \forall k,
\]

which means that the subsequence \( \{ x_{n_k} \} \) also strongly converges to \( y \). Hence, \( \{ x_n \} \) is relatively compact, that is, \( E \) is asymptotically compact. \( \square \)

**Corollary 3.18.** Let \( E \) be an evolutionary system satisfying A1 and let \( \bar{E} \) be the closure of \( E \). If the strongly compact strong global attractor \( A_s \) for \( E \) exists, then the strongly compact strong trajectory attractor \( \mathcal{A}_s \) for \( E \) exists. Hence,

1. \( \mathcal{A}_s = \Pi_+ \bar{E}((-\infty, \infty)) \) satisfies the finite strong uniform tracking property for \( E \), i.e., for any \( \epsilon > 0 \) and \( T > 0 \), there exist \( t_0 \) and a finite subset \( P^f_T \subset \mathcal{A}_s([0,T]) \) such that for any \( t^* > t_0 \), every trajectory \( u \in E([0,\infty)) \) satisfies

\[
d_u(u(t), v(t - t^*)) < \epsilon, \quad \forall t \in [t^*, t^* + T],
\]

for some \( T \)-time length piece \( v \in P^f_T \).

2. \( \mathcal{A}_s = \Pi_+ \bar{E}((-\infty, \infty)) \) is strongly equicontinuous on \([0, \infty)\), i.e.,

\[
d_u(v(t_1), v(t_2)) \leq \theta(|t_1 - t_2|), \quad \forall t_1, t_2 \geq 0, \forall v \in \mathcal{A}_s,
\]

where \( \theta(l) \) is a positive function tending to 0 as \( l \to 0^+ \).

**Remark 3.19.** With this corollary in hand, we are able to apply our theory to the systems for which the existence of strongly compact strong global attractors have been proved. After reformulate them by our framework of evolutionary systems, the existence of strongly compact strong trajectory attractors follow immediately, once A1 is verified. We will see this idea in the next sections where we apply our theory to the 2D Navier-Stokes equations and a general dissipative reaction-diffusion system.

### 3.3. Kernel of evolutionary system

In this subsection, we investigate further the kernels of evolutionary systems, especially of closed evolutionary systems and evolutionary systems with uniqueness.

#### 3.3.1. Closed evolutionary system

We introduce a closed evolutionary system \( \mathcal{E}_\Sigma \) with symbol space \( \Sigma \) and study its properties.

**Definition 3.20.** An evolutionary system \( \mathcal{E}_\Sigma \) is (weakly) closed if for any \( \tau \in \mathbb{R} \), \( u_n \in \mathcal{E}_{\sigma_n}((\tau, \infty)), \) the convergences \( u_n \to u \) in \( C([\tau, \infty); X_w) \) and \( \sigma_n \to \sigma \) in some topological space \( \mathbb{X} \) as \( n \to \infty \) imply \( u \in \mathcal{E}_{\sigma}((\tau, \infty)). \)

**Lemma 3.21.** Let \( \mathcal{E}_\Sigma \) be a closed evolutionary system satisfying A1. Then, \( \mathcal{E}_{\sigma}((\tau, \infty)) \) is nonempty for any \( \sigma \in \Sigma \).
Proof. Fix \( \sigma \in \Sigma \). Since for any \( \tau \in \mathbb{R} \), \( \mathcal{E}_\sigma([\tau, \infty)) \neq \emptyset \), we are able to take a sequence \( u_n(t) \in \mathcal{E}_\sigma([t_n, \infty)) \), where \( t_n \) is decreasing and \( t_n \to -\infty \) as \( n \to \infty \). Thanks to A1 and the definition of \( \mathcal{E}_\Sigma \), we have that, for every fixed \( t_m \), \( \mathcal{E}_\Sigma([t_m, \infty)) \) is precompact in \( C([t_m, \infty); X_w) \). Thus, together with the condition that \( \mathcal{E}_\Sigma \) is closed, the sequence

\[
\{ u_n(t) |_{[t_1, \infty)} \}_{n \geq 1} \subset \mathcal{E}_\sigma([t_1, \infty)),
\]

passing to a subsequence and dropping a subindex, converges in \( C([t_1, \infty); X_w) \) with a limit \( u^1(t) \in \mathcal{E}_\sigma([t_1, \infty)) \). Again, passing to a subsequence and dropping a subindex, the sequence

\[
\{ u_n(t) |_{[t_2, \infty)} \}_{n \geq 2} \subset \mathcal{E}_\sigma([t_2, \infty)),
\]

converges in \( C([t_2, \infty); X_w) \) with a limit \( u^2(t) \in \mathcal{E}_\sigma([t_2, \infty)) \). It is easy to see that, \( u^1(t) = u^2(t) \) on \([t_1, \infty)\). By a standard diagonalization process, we obtain a subsequence of \( u_n(t) \), still denoted by \( u_n(t) \), such that, for every \( m \in \mathbb{N} \),

\[
u_n(t)|_{[t_m, \infty)} \to u^m(t) \text{ in } C([t_m, \infty); X_w), \text{ as } n \to \infty,
\]

with \( u^m(t) \in \mathcal{E}_\sigma([t_m, \infty)) \). Note that, for any \( m \), \( u^m(t) = u^{m+1}(t) \) on \([t_m, \infty)\). Hence, we can define \( u(t) \) on \((-\infty, \infty)\), such that

\[
u(t) := u^m(t), \quad \forall t \in [t_m, \infty), \forall m \in \mathbb{N}.
\]

Obviously, it yields

\[
u(t)|_{[\tau, \infty)} \in \mathcal{E}_\sigma([\tau, \infty)), \quad \forall \tau \in \mathbb{R},
\]

which means that \( u(t) \in \mathcal{E}_\sigma((-\infty, \infty)) \) by Definition 2.7.

\[\square\]

Lemma 3.22. Let \( \mathbb{S} \) be some topological space and \( \Sigma \subset \mathbb{S} \) be sequentially compact in itself. Let \( \mathcal{E}_\Sigma \) be a closed evolutionary system. Then,

\[
\mathcal{E}_\Sigma((-\infty, \infty)) = \bigcup_{\sigma \in \Sigma} \mathcal{E}_\sigma((-\infty, \infty)).
\]

Proof. Thanks to Definition 2.7 and the definition of an evolutionary system \( \mathcal{E}_\Sigma \), we have

\[
\mathcal{E}_\Sigma((-\infty, \infty)) \supset \bigcup_{\sigma \in \Sigma} \mathcal{E}_\sigma((-\infty, \infty)).
\]

Take \( u(t) \in \mathcal{E}_\Sigma((-\infty, \infty)) \). Denote by

\[
u_n(t) := u(t)|_{[t_n, \infty)} \in \mathcal{E}_{\sigma_n}([t_n, \infty)),
\]

where \( \sigma_n \in \Sigma \), \( t_n \) is decreasing and \( t_n \to -\infty \) as \( n \to \infty \). Since \( \Sigma \) is sequentially compact in itself, passing to a subsequence and dropping a subindex, there exists \( \sigma \in \Sigma \), such that \( \sigma_n \to \sigma \) in \( \mathbb{S} \) as \( n \to \infty \). Note that, for every fixed \( m \),

\[
u_n(t)|_{[t_m, \infty)} = u(t), \quad \forall n \geq m, t \in [t_m, \infty).
\]

Hence, by the closedness of \( \mathcal{E}_\Sigma \),

\[
u(t)|_{[t_m, \infty)} \in \mathcal{E}_\sigma([t_m, \infty)), \quad \forall m \in \mathbb{N}.
\]

This implies that \( u(t) \in \mathcal{E}_\sigma((-\infty, \infty)) \) due to Definition 2.7. Therefore, we have

\[
\mathcal{E}_\Sigma((-\infty, \infty)) \subset \bigcup_{\sigma \in \Sigma} \mathcal{E}_\sigma((-\infty, \infty)).
\]
The proof is completed. □

**Lemma 3.23.** Let \( \mathcal{S} \) be some topological space and \( \Sigma \subset \mathcal{S} \) be sequentially compact in itself. Let \( \mathcal{E}_\Sigma \) be a closed evolutionary system. Then, for any \( \tau \in \mathbb{R} \), the set
\[
\mathcal{E}_\Sigma([\tau, \infty)) = \bigcup_{\sigma \in \Sigma} \mathcal{E}_\sigma([\tau, \infty))
\]
is closed in \( C([\tau, \infty); X_w) \).

**Proof.** Take a sequence \( \{u_n(t)\} \) that \( u_n(t) \in \mathcal{E}_{\sigma_n}([\tau, \infty)) \), \( \sigma_n \in \Sigma \) and \( u_n(t) \to u(t) \) in \( C([\tau, \infty); X_w) \) as \( n \to \infty \). Due to the sequential compactness of \( \Sigma \), passing to a subsequence and dropping a subindex, there exists \( \sigma \in \Sigma \) that is the limit of \( \{\sigma_n\} \) in \( \mathcal{S} \). Since \( \mathcal{E}_\Sigma \) is closed, we have \( u(t) \in \mathcal{E}_\sigma([\tau, \infty)) \). Hence, \( \mathcal{E}_\Sigma([\tau, \infty)) \) is a closed set in \( C([\tau, \infty); X_w) \). □

3.3.2. **Evolutionary system with uniqueness.** Let \( \mathcal{E}_\Sigma \) be an evolutionary system with symbol space \( \Sigma \), and let \( \Sigma \subset \Sigma \) be such that \( T(h) \Sigma = \Sigma \) for all \( h \geq 0 \). It is easy to check that the subfamily of maps \( \{\mathcal{E}_\sigma\} \), \( \sigma \in \Sigma \) is also an evolutionary system, with \( \Sigma \) as its symbol space. Then, it is an evolutionary subsystem of \( \mathcal{E}_\Sigma \). Denote it by \( \mathcal{E}_{\Sigma} \), as in Section 2. We suppose in this subsection that \( \bar{\Sigma} \) is the sequential closure of \( \Sigma \) in some topological space \( \mathcal{S} \).

**Theorem 3.24.** Let \( \mathcal{E}_\Sigma \) be an evolutionary system with uniqueness and with symbol space \( \Sigma \) satisfying A1 and let \( \bar{\mathcal{E}_\Sigma} \) be the closure of \( \mathcal{E}_\Sigma \). Let \( \bar{\Sigma} \) be the sequential closure of \( \Sigma \) in some topological space \( \mathcal{S} \) and \( \mathcal{E}_{\bar{\Sigma}} \supset \mathcal{E}_\Sigma \) be a closed evolutionary system with uniqueness and with symbol space \( \bar{\Sigma} \). Then, \( \mathcal{E}_{\bar{\Sigma}} \subset \bar{\mathcal{E}_\Sigma} \). Hence,

1. The three weak uniform global attractors \( \mathcal{A}_w^\Sigma, \bar{\mathcal{A}}_w^\Sigma \) and \( \bar{\mathcal{A}}_w^\Sigma \) for evolutionary systems \( \mathcal{E}_\Sigma, \bar{\mathcal{E}_\Sigma} \) and \( \mathcal{E}_{\bar{\Sigma}} \), respectively, exist.
2. \( \mathcal{A}_w^\Sigma, \bar{\mathcal{A}}_w^\Sigma \) and \( \bar{\mathcal{A}}_w^\Sigma \) are the maximal invariant and maximal quasi-invariant set w.r.t. \( \mathcal{E}_\Sigma \) and satisfy the following:
   \[
   \mathcal{A}_w^\Sigma = \bar{\mathcal{A}}_w^\Sigma = \bar{\mathcal{A}}_w^\Sigma = \{u_0 : u_0 = u(0) \text{ for some } u \in \mathcal{E}_\Sigma((\sigma, \infty))\}.
   \]
3. The three weak trajectory attractors \( \mathcal{A}_w^\Sigma, \bar{\mathcal{A}}_w^\Sigma \) and \( \bar{\mathcal{A}}_w^\Sigma \) for \( \mathcal{E}_\Sigma, \mathcal{E}_{\bar{\Sigma}} \) and \( \mathcal{E}_{\bar{\Sigma}} \), respectively, exist and satisfy the following:
   \[
   \mathcal{A}_w^\Sigma = \bar{\mathcal{A}}_w^\Sigma = \bar{\mathcal{A}}_w^\Sigma = \Pi_{\bar{\Sigma}}(\mathcal{E}_\Sigma((\sigma, \infty))).
   \]
Hence, the three weak trajectory attractors satisfy the finite weak uniform tracking property for all the three evolutionary systems and are weakly equicontinuous on \([0, \infty)\).
4. \( \mathcal{A}_w^\Sigma, \bar{\mathcal{A}}_w^\Sigma \) and \( \bar{\mathcal{A}}_w^\Sigma \) are sections of \( \mathcal{A}_w^\Sigma, \bar{\mathcal{A}}_w^\Sigma \) and \( \bar{\mathcal{A}}_w^\Sigma \):
   \[
   \mathcal{A}_w^\Sigma = \mathcal{A}_w^\Sigma = \mathcal{A}_w^\Sigma(t) = \mathcal{A}_w^\Sigma(t) = \mathcal{A}_w^\Sigma(t), \quad \forall t \geq 0.
   \]
Furthermore, assume that \( \bar{\Sigma} \subset \mathcal{S} \) is sequentially compact in itself. Then, \( \mathcal{E}_{\bar{\Sigma}} = \bar{\mathcal{E}_\Sigma} \). Hence,

5. The following relationships on kernels hold:
   \[
   \mathcal{E}_{\bar{\Sigma}}((\sigma, \infty)) = \mathcal{E}_{\bar{\Sigma}}((\sigma, \infty)) = \bigcup_{\sigma \in \Sigma} \mathcal{E}_\sigma((\sigma, \infty)),
   \]
and \( E_\sigma((\infty, \infty)) \) is nonempty for any \( \sigma \in \bar{\Sigma} \).

**Proof.** For any \( \tau \in \mathbb{R} \) and \( u_\sigma(t) \in E_\sigma([\tau, \infty)) \) with \( \sigma \in \bar{\Sigma}\setminus\Sigma \), we can take a sequence \( \{\sigma_n\} \subset \Sigma \) such that \( \sigma_n \to \sigma \) in \( \mathfrak{S} \) as \( n \to \infty \) since \( \mathfrak{S} \) is the sequential closure of \( \Sigma \). Because of the uniqueness of \( E_\Sigma \), there is a sequence \( \{u_n(t)\} \subset E_{\sigma_n}([\tau, \infty)) \) with \( u_n(\tau) = u_\sigma(\tau) \).

Passing to a subsequence and dropping a subindex, \( \{u_n(t)\} \) converges to a limit \( u(t) \) in \( C([\tau, \infty); X_w) \) due to A1. It follows that the limit \( u(t) \in E_\sigma([\tau, \infty)) \) for \( \sigma_n \to \sigma \) in \( \mathfrak{S} \) and \( E_\Sigma \) is closed. Note that \( u(\tau) = u_\sigma(\tau) \). By the uniqueness of \( E_\Sigma \), we have \( u(t) = u_\sigma(t) \), \( t \geq \tau \). Therefore, \( E_\Sigma \subset E_\Sigma \).

The existence of \( A^E_w, \bar{A}^E_w, \) and \( A^E_w \) follow from Theorem 3.6.

Obviously, we have, \( A^E_w \subset A^E_w \subset \bar{A}^E_w \), for \( E_\Sigma \subset E_\Sigma \subset \bar{E}_\Sigma \). On the other hand, since \( E_\Sigma \) satisfies A1, \( A^E_w = \bar{A}^E_w \), which deduces that \( A^E_w = A^E_w \). The rest results of the conclusion 2 are also obtained by Theorem 3.6.

Note that \( E_\Sigma \) satisfies A1 implies that \( \bar{E}_\Sigma \) satisfies \( \bar{A}1 \) and then \( E_\Sigma \) satisfies A1. Thus, the conclusions 3 and 4 follow again from Theorem 3.6 in a similar way.

Now suppose that \( \Sigma \) is sequentially compact in \( \mathfrak{S} \). Thanks to Lemma 3.23, for any \( \tau \in \mathbb{R} \), the set \( E_\Sigma([\tau, \infty)) \) is closed in \( C([\tau, \infty); X_w) \), which implies that \( \bar{E}_\Sigma \subset E_\Sigma \). Hence, \( E_\Sigma = E_\Sigma \). The last equality in the conclusion 5 is deduced from Lemma 3.22. Finally, by Lemma 3.21, \( E_\sigma((\infty, \infty)) \) is nonempty for every \( \sigma \in \Sigma \).

**Theorem 3.25.** Under the conditions of Theorem 3.24, assume that one of the followings is valid:

i). \( \bar{E}_\Sigma \) is asymptotically compact.

ii). \( \bar{E}_\Sigma \) satisfies A1, A2, and A3, and \( \bar{E}((\infty, \infty)) \subset C((\infty, \infty); X_a) \).

iii). \( \bar{E}_\Sigma \) possesses a strongly compact strong global attractor.

Then the three weak uniform global attractors in Theorem 3.24 are strongly compact strong uniform global attractors and the three weak trajectory attractors are strongly compact strong trajectory attractors. Moreover, the three trajectory attractors satisfy the finite strong uniform tracking property for all the three evolutionary systems and are strongly equicontinuous on \([0, \infty)\).

**Proof.** Assume that i) is valid. Obviously, \( E_\Sigma \) and \( \bar{E}_\Sigma \) are also asymptotically compact. All the conclusions follow from Theorem 3.12.

Assume that ii) holds. Thanks to Lemmas 3.1 and 3.4 in [CL14], \( \bar{E}_\Sigma \) satisfies \( \bar{A}1, \bar{A}2, \) and \( \bar{A}3 \). Therefore, by Theorem 2.12, \( \bar{E}_\Sigma \) is asymptotically compact. All the conclusions follow from i) just proved.

Now suppose that \( \bar{E}_\Sigma \) possesses a strongly compact strong global attractor. Due to Theorem 3.17, \( \bar{E}_\Sigma \) is also asymptotically compact. Then, all the conclusions are obtained.

We complete the proof. \( \square \)

**Remark 3.26.** The assumption that \( \bar{\Sigma} \subset \mathfrak{S} \) is the sequential closure of \( \Sigma \) was actually used in the proof of Theorem 3.10 in [CL14]. Here is a remediation for more preciseness. In many applications, the topology of \( \mathfrak{S} \) is metrizable on \( \Sigma \). Then, in these cases, the notions of a closure and a sequential closure are equivalent, as well as a compact set and a sequential compact set.
Theorems 3.24 and 3.25 generalize Theorems 3.10-3.13 in [CL14], and the related results in [CV02, LWZ05, Lu06, Lu07] where we were concerned with the classical cases of a process and a family of processes.

Note that a process and a family of processes always define evolutionary systems with uniqueness, as shown in [CL14]. Let \( \{U_\sigma(t, \tau)\} \), \( \sigma \in \Sigma \) be a family of processes acting on a separable reflexive Banach space \( H \): For any \( \sigma \in \Sigma \), \( \tau \in \mathbb{R} \), the following conditions are satisfied:

\[
U_\sigma(t, \tau) : H \to H \text{ is single-valued, } \quad \forall \ t \geq \tau,
U_\sigma(t, s) \circ U_\sigma(s, \tau) = U_\sigma(t, \tau), \quad \forall \ t \geq s \geq \tau,
U_\sigma(\tau, \tau) = \text{Identity operator.}
\]

Assume that it satisfies the following translation identity arising naturally from applications:

\[
U_\sigma(t + h, \tau + h) = U_{T(h)\sigma}(t, \tau), \quad \forall \ \sigma \in \Sigma, \ \ t \geq \tau, \ \tau \in \mathbb{R}, \ h \geq 0,
\]

and \( T(s)\Sigma = \Sigma, \forall s \geq 0 \). Here \( \{T(s)\}_{s \geq 0} \) is the translation semigroup. Assume further that the family of processes is dissipative, i.e., there exists a uniformly (w.r.t. \( \sigma \in \Sigma \)) absorbing ball \( B \): For any \( \tau \in \mathbb{R} \) and bounded set \( A \subset H \), there exists \( t_0 = t_0(A) \geq \tau \), such that,

\[
\bigcup_{\sigma \in \Sigma} \bigcup_{t \geq t_0} U_\sigma(t + \tau, \tau)A \subset B.
\]

Now take \( X = B \). Note that since \( X \) is a bounded subset of a separable reflexive Banach space, both the strong and the weak topologies on \( X \) are metrizable. Define a family of maps \( \mathcal{E}_\sigma, \sigma \in \Sigma \) in the following way:

\[
\mathcal{E}_\sigma([\tau, \infty)) := \{ u(\cdot) : u(t) = U_\sigma(t, \tau)u_\tau, u_\tau \in X, t \geq \tau \}.
\]

Conditions 1-4 in Definition 2.7 are verified by the definition of the family of processes \( \{U_\sigma(t, \tau)\}, \sigma \in \Sigma \). Hence, the family of processes \( \{U_\sigma(t, \tau)\}, \sigma \in \Sigma \) defines an evolutionary system \( \mathcal{E}_\Sigma \) with uniqueness and with symbol space \( \Sigma \). In applications, \( \Sigma \) is typically a closure of \( \Sigma \) in some appropriate functional space \( \mathfrak{X} \), where \( \Sigma := \{\sigma_0(\cdot + h) | h \in \mathbb{R}\} \) is the translation family of a fixed symbol \( \sigma_0 \). It can be checked [CL14] that the family of processes \( \{U_\sigma(t, \tau)\}, \sigma \in \Sigma \), or equivalently, the process \( \{U_{\sigma_0}(t, \tau)\} \), also defines an evolutionary system \( \mathcal{E}_\Sigma \) with uniqueness and with symbol space \( \Sigma \). It is an evolutionary subsystem of the former system.

In [LWZ05, Lu06, Lu07], analogous to the closedness condition, a (weak) continuity condition on a family of processes was used. A family of processes \( \{U_\sigma(t, \tau)\}, \sigma \in \Sigma \) is \( (H_\mathfrak{X} \times \Sigma, H_\mathfrak{X}) \) continuous, if for any \( t \geq \tau, \tau \in \mathbb{R} \), the mapping \( (u, \sigma) \to U_\sigma(t, \tau)u \) is continuous from \( H_\mathfrak{X} \times \Sigma \) to \( H_\mathfrak{X} \). In words of the associated evolutionary system \( \mathcal{E}_\Sigma \), we have the following definition.

**Definition 3.27.** An evolutionary system \( \mathcal{E}_\Sigma \) with uniqueness and with symbol space \( \Sigma \) is \( (X_\mathfrak{X} \times \Sigma, X_\mathfrak{X}) \) continuous if for any \( \tau \in \mathbb{R}, \sigma_n, \sigma \in \Sigma, u_n(t) \in \mathcal{E}_{\sigma_n}([\tau, \infty)) \), the convergences \( u_n(\tau) \to u_\tau \) in \( X_\mathfrak{X} \) and \( \sigma_n \to \sigma \) in some topological space \( \mathfrak{S} \) as \( n \to \infty \) imply \( u_n(t) \to u(t) \) in \( X_\mathfrak{X} \) for all \( t \geq \tau \) with a pointwise limit \( u(t) \in \mathcal{E}_\sigma([\tau, \infty)) \) and \( u(\tau) = u_\tau \).
Lemma 3.28. Let $\mathcal{E}_\Sigma$ be an evolutionary system with uniqueness and with symbol space $\Sigma$. If $\mathcal{E}_\Sigma$ is $(X_w \times \Sigma, X_w)$ continuous, then $\mathcal{E}_\Sigma$ is closed.

Lemma 3.29. Let $\mathcal{E}_\Sigma$ be an evolutionary system with uniqueness and with symbol space $\Sigma$. If $\mathcal{E}_\Sigma$ is $(X_w \times \Sigma, X_w)$ continuous, then, for any $\tau \in \mathbb{R}$, $\sigma_n \in \Sigma$, $u_n(t) \in \mathcal{E}_{\sigma_n}([\tau, \infty))$, there exist subsequences $\{\sigma_{n_j}\}, \{u_{n_j}(t)\}$, $\sigma_n \in \Sigma$ and $u(t) \in \mathcal{E}_\sigma([\tau, \infty))$ such that $u_{n_j}(t) \to u(t)$ in $X_w$ for all $t \geq \tau$ and $\sigma_{n_j} \to \sigma$ in $\Sigma$ as $n_j \to \infty$.

Lemma 3.30. Let $\mathcal{E}_\Sigma$ be an evolutionary system with uniqueness and with symbol space $\Sigma$. If $\mathcal{E}_\Sigma$ is closed and satisfies A1, then, for any $\tau \in \mathbb{R}$, $\sigma_n \in \Sigma$, $u_n(t) \in \mathcal{E}_{\sigma_n}([\tau, \infty))$, such that, $u_n(\tau) \to u_\tau$ in $X_w$ and $\sigma_n \to \sigma$ in some topological space $\mathcal{E}$ as $n \to \infty$, the convergence $u_n(t) \to u(t)$ in $C([\tau, \infty); X_w)$ holds with a limit $u(t) \in \mathcal{E}_\sigma([\tau, \infty))$ and $u(\tau) = u_\tau$. In particular, $\mathcal{E}_\Sigma$ is $(X_w \times \Sigma, X_w)$ continuous.

Proof: For any $\tau \in \mathbb{R}$, take $\sigma_n \in \Sigma$, $u_n(t) \in \mathcal{E}_{\sigma_n}([\tau, \infty))$, such that, $u_n(\tau) \to u_\tau$ in $X_w$ and $\sigma_n \to \sigma$ in $\mathcal{E}$ as $n \to \infty$. Thanks to A1, there exists a subsequence $\{u_{n_j}(t)\}$ such that

$$u_{n_j}(t) \to u(t) \text{ in } C([\tau, \infty); X_w),$$

for some $u(t) \in C([\tau, \infty); X_w)$ with $u(\tau) = u_\tau$. Since $\mathcal{E}_\Sigma$ is closed, we have $u(t) \in \mathcal{E}_\sigma([\tau, \infty))$. Suppose that

$$u_n(t) \to u(t) \text{ in } C([\tau, \infty); X_w),$$

as $n \to \infty$.

Thanks to A1 again, there is a subsequence of $\{u_{n_j}(t)\}$, such that

$$u_{n_j}(t) \to u^*(t) \text{ in } C([\tau, \infty); X_w),$$

as $n_j \to \infty$,

for some $u^*(t) \in C([\tau, \infty); X_w)$ with $u^*(\cdot) \neq u(\cdot)$. Note that $u^*(t) \in \mathcal{E}_\sigma([\tau, \infty))$ and $u^*(\tau) = u_\tau$. By uniqueness of $\mathcal{E}_\Sigma$, $u^*(t) = u(t)$, for all $t \geq \tau$, which is a contradiction. Hence,

$$u_n(t) \to u(t) \text{ in } C([\tau, \infty); X_w),$$

as $n \to \infty$.

Especially, $u_n(t) \to u(t)$ in $X_w$ for all $t \geq \tau$, which means that $\mathcal{E}_\Sigma$ is $(X_w \times \Sigma, X_w)$ continuous.

Remark 3.31. The condition A1 generally holds for dissipative systems in mathematical physics (cf. e.g. [T88, CV02]). See applications in the next sections.

4. Attractors for 2D and 3D Navier-Stokes Equations

In this section, we apply the new theory established in the previous section to the 2D and the 3D Navier-Stokes equations. Main new results are the existence of strongly compact strong trajectory attractors and sequent properties without further assumptions (cf. [LWZ05, Lu06, CL14]).
More precisely, we consider the space periodic 2D and 3D incompressible Navier-Stokes equations (NSE)

\[
\begin{cases}
  \frac{d}{dt}u - \nu \Delta u + (u \cdot \nabla)u + \nabla p = f(t), \\
  \nabla \cdot u = 0,
\end{cases}
\]

where \( u \), the velocity, and \( p \), the pressure, are unknowns; \( f(t) \) is a given driving force, and \( \nu > 0 \) is the kinematic viscosity coefficient of the fluid. By a Galilean change of variables, we can assume that the space average of \( u \) is zero, i.e.,

\[
\int_{\Omega} u(x,t) \, dx = 0, \quad \forall t,
\]

where \( \Omega = [0,L]^n, n = 2,3 \), is a periodic box.\(^1\!

First, let us introduce some notations and functional setting. Denote by \((\cdot, \cdot)\) and \( | \cdot | \) the \((L^2(\Omega))^n\)-inner product and the corresponding \((L^2(\Omega))^n\)-norm. Let \( V \) be the space of all \( \mathbb{R}^n \) trigonometric polynomials of period \( L \) in each variable satisfying \( \nabla \cdot u = 0 \) and \( \int_{\Omega} u(x) \, dx = 0 \). Let \( H \) and \( V \) to be the closures of \( V \) in \((L^2(\Omega))^n\) and \((H^1(\Omega))^n\), respectively. Define the strong and weak distances by

\[
d_s(u,v) := |u - v|, \quad d_w(u,v) := \sum_{\kappa \in \mathbb{Z}^n} \frac{1}{2|\kappa|} \frac{|u_\kappa - v_\kappa|}{1 + |u_\kappa - v_\kappa|}, \quad u,v \in H,
\]

where \( u_\kappa \) and \( v_\kappa \) are Fourier coefficients of \( u \) and \( v \) respectively. Note that the weak metric \( d_w \) induces the weak topology in any ball in \((L^2(\Omega))^n\).

Let also \( P_\sigma : (L^2(\Omega))^n \to H \) be the \( L^2 \)-orthogonal projection, referred to as the Leray projector. Denote by \( A = -P_\sigma \Delta = -\Delta \) the Stokes operator with the domain \( D(A) = (H^2(\Omega))^n \cap V \). The Stokes operator is a self-adjoint positive operator with a compact inverse. Let

\[
\|u\| := |A^{1/2}u|,
\]

which is called the enstrophy norm. Note that \( \|u\| \) is equivalent to the \( H^1 \)-norm of \( u \) for \( u \in D(A^{1/2}) \). The corresponding inner product is denoted by

\[
((u,v)) := (A^{1/2}u,A^{1/2}v).
\]

Let \( V' \) be the dual of \( V \). Now denote \( B(u,v) := P_\sigma (u \cdot \nabla v) \in V' \) for all \( u,v \in V \). This bilinear form has the following property:

\[
\langle B(u,v), w \rangle = -\langle B(u,w), v \rangle, \quad u,v,w \in V,
\]

in particular, \( \langle B(u,v), v \rangle = 0 \) for all \( u,v \in V \).

Now we can rewrite (19) as the following differential equation in \( V' \):

\[
\frac{d}{dt}u + \nu Au + B(u,u) = g,
\]

where \( u \) is a \( V \)-valued function of time and \( g = P_\sigma f \).

---

\(^1\) The no-slip case can be considered in a similar way, only with some adaption on the functional setting.
Definition 4.1. A weak solution of (19) on $[\tau, \infty)$ (or $(-\infty, \infty)$, if $\tau = -\infty$) is an $H$-valued function $u(t)$ defined for $t \in [\tau, \infty)$, such that
\[
\frac{d}{dt} u \in L^1_{\text{loc}}([\tau, \infty); V'), \quad u(t) \in C([\tau, \infty); H_w) \cap L^2_{\text{loc}}([\tau, \infty); V),
\]
and
\[
(u(t) - u(t_0), v) = \int_{t_0}^t (-\nu((u, v)) - \langle B(u, u), v \rangle + \langle g, v \rangle) \, ds,
\]
for all $v \in V$ and $\tau \leq t \leq t_0$.

Theorem 4.2 (Leray-Hopf). For every $u_0 \in H$ and $g \in L^2_{\text{loc}}(\mathbb{R}; V')$, there exists a weak solution of (19) on $[\tau, \infty)$ with $u(\tau) = u_0$ satisfying the following energy inequality
\[
|u(t)|^2 + 2\nu \int_{t_0}^t \|u(s)\|^2 \, ds \leq |u(t_0)|^2 + 2 \int_{t_0}^t \langle g(s), u(s) \rangle \, ds,
\]
for all $t \geq t_0$, $t_0$ a.e. in $[\tau, \infty)$.

Definition 4.3. A Leray-Hopf solution of (19) on the interval $[\tau, \infty)$ satisfying the energy inequality (20) for all $\tau \leq t_0 \leq t$, $t_0$ a.e. in $[\tau, \infty)$. The set $Ex$ of measure 0 on which the energy inequality does not hold will be called the exceptional set.

Now fix an external force $g_0$ that is translation bounded in $L^2_{\text{loc}}(\mathbb{R}; V')$, i.e.,
\[
\|g_0\|_{L^2_{\text{loc}}(\mathbb{R}; V')} := \sup_{t \in \mathbb{R}} \int_{t}^{t+1} \|g_0(s)\|_{V'}^2 \, ds < \infty.
\]
Denote by $L^2_{\text{loc}}(\mathbb{R}; V')$ the space $L^2_{\text{loc}}(\mathbb{R}; V')$ endowed with the local weak convergence topology. Then $g_0$ is translation compact in $L^2_{\text{loc}}(\mathbb{R}; V')$, i.e., the translation family of $g_0$,
\[
\Sigma := \{g_0(\cdot + h) : h \in \mathbb{R}\},
\]
is precompact in $L^2_{\text{loc}}(\mathbb{R}; V')$ (see [CV02]). Note that,
\[
\|g\|_{L^2_{\text{loc}}(\mathbb{R}; V')}^2 \leq \|g_0\|_{L^2_{\text{loc}}(\mathbb{R}; V')}^2, \quad \forall g \in \Sigma.
\]
Due to the energy inequality (20), we have
\[
|u(t)|^2 + \nu \int_{t_0}^t \|u(s)\|^2 \, ds \leq |u(t_0)|^2 + \frac{1}{\nu} \int_{t_0}^t \|g(s)\|_{V'}^2 \, ds, \quad \forall g \in \Sigma,
\]
for all $t \geq t_0$, $t_0$ a.e. in $[\tau, \infty)$. Here $u(t)$ is a Leray-Hopf solution of (19) with the force $g$ on $[\tau, \infty)$. By Grönwall’s inequality, there exists a uniformly (w.r.t. $\tau \in \mathbb{R}$ and $g \in \Sigma$) absorbing ball $B_\delta(0, R) \subset H$, where the radius $R$ depends on $L, \nu$, and $\|g_0\|_{L^2_{\text{loc}}(\mathbb{R}; V')}$. Let $X$ be a closed uniformly absorbing ball
\[
X = \{u \in H : |u| \leq R\},
\]
which is also weakly compact in $H$. Then, for any bounded set $A \subset H$, there exists a time $\bar{t} \geq 0$ independent of the initial time $\tau$, such that
\[
u (u(t) \in X, \quad \forall t \geq \bar{t} := \tau + \bar{t},
\]
for every Leray-Hopf solution \( u(t) \) with the force \( g \in \Sigma \) and the initial data \( u(\tau) \in A \). For any sequence of Leray-Hopf solutions \( u_k \), the following result (see e.g. [T88, CF89, Ro01, CL14]) holds.

**Lemma 4.4.** Let \( u_k(t) \) be a sequence of Leray-Hopf solutions of (19) with forces \( g_k \in \Sigma \), such that \( u_k(t) \in X \) for all \( t \geq t_1 \). Then

\[
\begin{align*}
    u_k & \text{ is bounded in } L^2(t_1, t_2; V) \text{ and } L^\infty(t_1, t_2; H), \\
    \frac{d}{dt} u_k & \text{ is bounded in } L^p(t_1, t_2; V'),
\end{align*}
\]

for all \( t_2 > t_1 \), with \( p = 2 \) if \( n = 2 \) and \( p = 4/3 \) if \( n = 3 \). Moreover, there exists a subsequence \( u_{k_j} \) converges to some solution \( u(t) \) in \( C([t_1, t_2]; H_w) \), i.e.,

\[
(u_{k_j}, v) \to (u, v) \text{ uniformly on } [t_1, t_2],
\]
as \( k_j \to \infty \), for all \( v \in H \).

**Remark 4.5.** [CL14] In the nonautonomous case, i.e., \( f(t) \) is dependent on \( t \), we don’t know here whether the limit \( u(t) \) is a Leray-Hopf solution yet when \( n = 3 \).

Consider an evolutionary system for which a family of trajectories consists of all Leray-Hopf solutions of the 2D or the 3D NSE with a fixed force \( g_0 \) in \( X \). More precisely, define

\[
\mathcal{E}([\tau, \infty)) := \{ u(\cdot) : u(\cdot) \text{ is a Leray-Hopf solution on } [\tau, \infty) \\
\text{with the force } g \in \Sigma \text{ and } u(t) \in X, \forall t \in [\tau, \infty) \}, \\tau \in \mathbb{R},
\]

\[
\mathcal{E}((-\infty, \infty)) := \{ u(\cdot) : u(\cdot) \text{ is a Leray-Hopf solution on } (-\infty, \infty) \\
\text{with the force } g \in \Sigma \text{ and } u(t) \in X, \forall t \in (-\infty, \infty) \}.
\]

Clearly, the properties 1-4 of an evolutionary system hold for \( \mathcal{E} \), if we utilize the translation semigroup \( \{ T(s) \}_{s \geq 0} \). Therefore, thanks to Theorem 3.6, the weak uniform global attractor \( A_w \) for this evolutionary system exists.

Now we give the definition of a normal function which was introduced in [LWZ05, Lu06].

**Definition 4.6.** Let \( B \) be a Banach space. A function \( \varphi(s) \in L^2_{\text{loc}}(\mathbb{R}; B) \) is said to be normal in \( L^2_{\text{loc}}(\mathbb{R}; B) \) if for any \( \varepsilon > 0 \), there exists \( \delta > 0 \), such that

\[
\sup_{t \in \mathbb{R}} \int_t^{t+\delta} \| \varphi(s) \|^2_B \, ds \leq \varepsilon.
\]

Note that the class of normal functions is a proper closed subspace of the class of translation bounded functions (see [LWZ05, Lu06] for more details). Then, we have the following.

**Lemma 4.7.** The evolutionary system \( \mathcal{E} \) of the 2D or the 3D NSE with the force \( g_0 \) satisfies \( A1 \) and \( A3 \). Moreover, if \( g_0 \) is normal in \( L^2_{\text{loc}}(\mathbb{R}; V') \) then \( A2 \) holds.

**Proof.** For the 3D case, it is just Lemma 5.7 in [CL14]. For the 2D case, it is derived in exactly the same way. \( \square \)
4.1. 3D Navier-Stokes equations. Now applying the theory in Section 3, we have the followings.

**Theorem 4.8. [CL14]** Let $g_0$ be translation bounded in $L^2_{\text{loc}}(\mathbb{R}; V')$. Then the weak uniform global attractor $A_w$ and the weak trajectory attractor $A_w$ for the 3D NSE with the fixed force $g_0$ exist, $A_w$ is the maximal invariant and maximal quasi-invariant set w.r.t. the closure $\bar{E}$ of the corresponding evolutionary system $E$ and

$$A_w = \omega_w(X) = \omega_s(X) = \{u(0) : u \in \bar{E}((-\infty, \infty))\},$$
$$\bar{A}_w = \Pi_{+\bar{E}}((-\infty, \infty)) = \{u(\cdot)|_{[0, \infty)} : u \in \bar{E}((-\infty, \infty))\},$$
$$\bar{A}_w = \bar{A}_w(t) = \{u(t) : u \in \bar{A}_w\}, \quad \forall t \geq 0.$$

Moreover, $\bar{A}_w$ satisfies the finite weak uniform tracking property and is weakly equicontinuous on $[0, \infty)$.

**Proof.** This theorem is just Theorem 5.8 and the first part of Theorem 5.10 in [CL14]. We reformulate these results according to Theorem 3.6.

**Theorem 4.9.** If $g_0$ is normal in $L^2_{\text{loc}}(\mathbb{R}; V')$ and every complete trajectory of $\bar{E}$ is strongly continuous, then the weak uniform global attractor $A_w$ is a strongly compact strong global attractor $A_s$, and the weak trajectory attractor $A_w$ is a strongly compact strong trajectory attractor $A_s$. Moreover,

1. $A_s = \Pi_{+\bar{E}}((-\infty, \infty))$ satisfies the finite strong uniform tracking property, i.e., for any $\epsilon > 0$ and $T > 0$, there exist $t_0$ and a finite subset $P_T^f \subset A_s|[0,T]$. such that for any $t^* > t_0$, every trajectory $u \in \bar{E}([0, \infty))$ satisfies

$$|u(t) - v(t - t^*)| < \epsilon, \quad \forall t \in [t^*, t^* + T],$$

for some $T$-time length piece $v \in P_T^f$.

2. $A_s = \Pi_{+\bar{E}}((-\infty, \infty))$ is strongly equicontinuous on $[0, \infty)$, i.e.,

$$|v(t_1) - v(t_2)| \leq \theta(|t_1 - t_2|), \quad \forall t_1, t_2 \geq 0, \forall v \in A_s,$$

where $\theta(l)$ is a positive function tending to $0$ as $l \to 0^+$.

The strong compactness of $A_s$ and the strong trajectory attracting property of $A_w$ have been proved in [CL14]. Here novelties are the strong compactness of $A_s$ and sequent properties.

**Proof.** The theorem follows by applying Lemma 4.7, Theorems 2.12 and 3.12.

4.1.1. Open problem. We give some supplementaries to Section 6 in [CL14]. In this subsection, we further assume that $g_0$ is translation compact in $L^2_{\text{loc}}(\mathbb{R}; V')$, i.e., the closure of the translation family $\Sigma$ of $g_0$ in $L^2_{\text{loc}}(\mathbb{R}; V')$,

$$\Sigma := \{g_0(\cdot + h) : h \in \mathbb{R}\},$$

is compact in $L^2_{\text{loc}}(\mathbb{R}; V')$. It is known [CV02] that $L^2_{\text{loc}}(\mathbb{R}; V')$ is metrizable and the corresponding metric space is complete. Note that the class of translation compact functions is also a closed subspace of the class of translation bounded functions, but it is a proper subset of the class of normal functions (for more details, see [LWZ05, Lu06]). Note also
that the argument of (21)-(24) is valid for $\Sigma$ replaced by $\bar{\Sigma}$ and Lemma 4.4 can be improved as follows (see e.g. [T88, CF89, CV02, CL14]).

**Lemma 4.10.** Let $n = 3$ and let $u_k(t)$ be a sequence of Leray-Hopf solutions of (19) with forces $g_k \in \bar{\Sigma}$, such that $u_k(t) \in X$ for all $t \geq t_1$. Then

$$u_k \text{ is bounded in } L^2(t_1, t_2; V) \text{ and } L^\infty(t_1, t_2; H),$$

$$\frac{d}{dt}u_k \text{ is bounded in } L^{4/3}(t_1, t_2; V'),$$

for all $t_2 > t_1$. Moreover, there exists a subsequence $k_j$, such that $g_{k_j}$ converges in $L^2_{\text{loc}}(\mathbb{R}; V')$ to some $g \in \bar{\Sigma}$ and $u_{k_j}$ converges in $C([t_1, t_2]; H_w)$ to some Leray-Hopf solution $u(t)$ of (19) with the force $g$, i.e.,

$$(u_{k_j}, v) \to (u, v) \text{ uniformly on } [t_1, t_2],$$

as $k_j \to \infty$, for all $v \in H$.

Now consider another evolutionary system with $\bar{\Sigma}$ as a symbol space. The family of trajectories for this evolutionary system consists of all Leray-Hopf solutions of the family of 3D NSE with forces $g \in \bar{\Sigma}$ in $X$:

$$\mathcal{E}_\bar{\Sigma}([\tau, \infty)) := \{u(\cdot) : u(\cdot) \text{ is a Leray-Hopf solution on } [\tau, \infty)$$

with the force $g \in \bar{\Sigma}$ and $u(t) \in X, \forall t \in [\tau, \infty]\}, \quad \tau \in \mathbb{R},$$

$$\mathcal{E}_\bar{\Sigma}((\infty, \infty)) := \{u(\cdot) : u(\cdot) \text{ is a Leray-Hopf solution on } (\infty, \infty)$$

with the force $g \in \bar{\Sigma}$ and $u(t) \in X, \forall t \in (\infty, \infty)\}.$

We have the following lemmas.

**Lemma 4.11.** [CL14] The evolutionary system $\mathcal{E}_\bar{\Sigma}$ of the family of 3D NSE with forces in $\bar{\Sigma}$ satisfies $A1$, $A2$ and $A3$.

**Lemma 4.12.** The evolutionary system $\mathcal{E}_\bar{\Sigma}$ of the family of 3D NSE with forces in $\bar{\Sigma}$ is closed.

**Proof.** For any $\tau \in \mathbb{R}$, take $g_k \in \bar{\Sigma}, u_k \in \mathcal{E}_{g_k}([\tau, \infty))$, such that, $u_k \to u$ in $C([\tau, \infty); X_w)$ and $g_k \to g$ in $L^2_{\text{loc}}(\mathbb{R}; V')$. By Lemma 4.10 and a standard diagonalization process, we have $u \in \mathcal{E}_g([\tau, \infty))$. That is, $\mathcal{E}_{\bar{\Sigma}}$ is closed. □

**Lemma 4.13.** Let $\mathcal{E}_\bar{\Sigma}$ be the evolutionary system of the family of 3D NSE with forces in $\bar{\Sigma}$. Then $\mathcal{E}_\bar{\Sigma} = \mathcal{E}_{\bar{\Sigma}}$.

**Proof.** By the assumption, $\bar{\Sigma}$ is metrizable and compact in $L^2_{\text{loc}}(\mathbb{R}; V')$. Thanks to Lemma 4.12, the corresponding evolutionary system $\mathcal{E}_{\bar{\Sigma}}$ is closed. It follows from Lemma 3.23 that, for any $\tau \in \mathbb{R}$, the set $\mathcal{E}_\bar{\Sigma}([\tau, \infty))$ is closed in $C([\tau, \infty); X_w)$. Hence, $\mathcal{E}_\bar{\Sigma} = \mathcal{E}_{\bar{\Sigma}}$. □

We have the following (cf. [CV02, CL14]) that now knows more on the structure of the kernel.
Theorem 4.14. Let \( g_0 \) be translation compact in \( L^2_{\text{loc}}(\mathbb{R}; V') \). Then the weak uniform global attractor \( \mathcal{A}_{\Sigma}^w \) and the weak trajectory attractor \( \mathfrak{A}_{\Sigma}^w \) for the family of 3D NSE with forces \( g \in \Sigma \) exist, \( \mathcal{A}_{\Sigma}^w \) is the maximal invariant and maximal quasi-invariant set w.r.t. the corresponding evolutionary system \( \mathcal{E}_{\Sigma} \), and

\[
\mathcal{A}_{\Sigma}^w = \{ u(0) : u \in \mathcal{E}_{\Sigma}((\infty, \infty)) \} = \left\{ u(0) : u \in \bigcup_{g \in \Sigma} \mathcal{E}_g((\infty, \infty)) \right\},
\]

\[
\mathfrak{A}_{\Sigma}^w = \Pi_+ \bigcup_{g \in \Sigma} \mathcal{E}_g((\infty, \infty)),
\]

\[
\mathcal{A}_{\Sigma}^w = \mathfrak{A}_{\Sigma}^w(t) = \left\{ u(t) : u \in \mathfrak{A}_{\Sigma}^w \right\}, \quad \forall t \geq 0,
\]

where \( \mathcal{E}_g((\infty, \infty)) \) is nonempty for any \( g \in \Sigma \). Moreover, \( \mathfrak{A}_{\Sigma}^w \) satisfies the finite weak uniform tracking property and is weakly equicontinuous on \( [0, \infty) \).

Proof. By Lemma 4.13, \( \mathcal{E}_{\Sigma} \) equals to its closure \( \bar{\mathcal{E}}_{\Sigma} \). Especially, we have

\[
\mathcal{E}_{\Sigma}((\infty, \infty)) = \bar{\mathcal{E}}_{\Sigma}((\infty, \infty)).
\]

It follows from Lemma 3.22 that

\[
\mathcal{E}_{\Sigma}((\infty, \infty)) = \bigcup_{g \in \Sigma} \mathcal{E}_g((\infty, \infty)).
\]

We know from Lemma 4.11 that \( \mathcal{E}_{\Sigma} \) satisfies \( \tilde{A}1 \). Therefore, \( \mathcal{E}_g((\infty, \infty)) \) is nonempty for any \( g \in \Sigma \) due to Lemma 3.21. The rest part of the conclusions follow by applying Theorem 3.6, which is in fact Theorems 6.3 and 6.6 in [CL14]. \( \square \)

Theorem 2.12 and Lemma 4.11 give a criterion for the strong compactness of the attractors. Thereout, we would obtain that all the complete Leray-Hopf solutions of the family of 3D NSE with forces \( g \in \Sigma \) satisfy the finite strong uniform tracking property and are strongly equicontinuous on \( (-\infty, \infty) \).

Theorem 4.15. Furthermore, if every complete trajectory of the family of 3D NSE with forces \( g \in \Sigma \) is strongly continuous, then the weak uniform global attractor \( \mathcal{A}_{\Sigma}^w \) is a strongly compact strong global attractor \( \mathfrak{A}_{\Sigma}^w \), and the weak trajectory attractor \( \mathfrak{A}_{\Sigma}^w \) is a strongly compact strong trajectory attractor \( \mathfrak{A}_{\Sigma}^w \). Moreover, \( \mathfrak{A}_{\Sigma}^w \) satisfies the finite strong uniform tracking property and is strongly equicontinuous on \( [0, \infty) \).

Let \( \bar{\mathcal{E}} \) be the closure of the evolutionary system \( \mathcal{E} \). It follows from Lemma 4.13 that \( \mathcal{E} \subseteq \bar{\mathcal{E}} \subseteq \mathcal{E}_{\Sigma} \). Then, an interesting problem naturally arises:

Open Problem 4.16. [CL14] Are the attractors \( \mathcal{A}_{\star} \), \( \mathfrak{A}_{\star} \) and \( \mathcal{A}_{\Sigma}^w \), \( \mathfrak{A}_{\Sigma}^w \) in Theorems 4.8 and 4.15 identical?

Due to Theorems 3.24 and 3.25, the answer is positive if the solutions of the 3D NSE are unique (cf. footnote 3). However, the negative answer would imply that the Leray-Hopf weak solutions are not unique, and especially that, the attractors \( \mathcal{A}_{\Sigma}^w \) and \( \mathfrak{A}_{\Sigma}^w \) for the auxiliary family of 3D NSE with forces in \( \Sigma \) do not satisfy the minimality property.
w.r.t. $d_\star$-attracting and $d_{C([0,\infty);X_\star)}$-attracting, respectively, for the evolutionary system $\mathcal{E}$ corresponding to the original 3D NSE with the fixed force $g_0$. For more details, see [CL14].

4.2. 2D Navier-Stokes equations: Weak solutions. In the 2D case, there are better properties of weak solutions than those in Theorem 4.2. The weak solutions are unique and strongly continuous w.r.t. time $t$, and the equality in (20) holds for every weak solution (see e.g. [T88, Ro01, CV02]). We will show in this subsection that these properties provide better results.

**Theorem 4.17. (Weak solutions)** Let $n = 2$. For every $u_0 \in H$ and $g \in L^2_{loc}(\mathbb{R}; V')$, the weak solution of (19) on $[\tau, \infty)$ with $u(\tau) = u_0$ is unique and satisfies

$$u(t) \in C([\tau, \infty); H).$$

Denote again by

$$\bar{\Sigma} := \{g_0(\cdot + h) : h \in \mathbb{R}\} L^2_{loc}(\mathbb{R}; V').$$

It can be known [CV02] that $\bar{\Sigma}$ endowed with the topology of $L^2_{loc}(\mathbb{R}; V')$ is metrizable and the corresponding metric space is compact. Due to the better properties of the weak solutions, we have the following better version of Lemma 4.4 (see e.g. [T88, Ro01, CV02]).

**Lemma 4.18.** Let $n = 2$ and let $u_k(t)$ be a sequence of weak solutions of (19) with forces $g_k \in \bar{\Sigma}$, such that $u_k(t) \in X$ for all $t \geq t_1$. Then

$$u_k \text{ is bounded in } L^2(t_1, t_2; V) \text{ and } L^\infty(t_1, t_2; H),$$

$$\frac{d}{dt} u_k \text{ is bounded in } L^2(t_1, t_2; V'),$$

for all $t_2 > t_1$. Moreover, there exists a subsequence $k_j$, such that $g_{k_j}$ converges in $L^2_{loc}(\mathbb{R}; V')$ to some $g \in \bar{\Sigma}$ and $u_{k_j}$ converges in $C([t_1, t_2]; H_w)$ to some weak solution $u(t)$ of (19) with the force $g$, i.e.,

$$(u_{k_j}, v) \to (u, v) \text{ uniformly on } [t_1, t_2],$$

as $k_j \to \infty$, for all $v \in H$.

Similarly, together with $\mathcal{E}$, we can also consider another evolutionary system with $\bar{\Sigma}$ as a symbol space. The family of trajectories for this evolutionary system consists of all weak solutions of the family of 2D NSE with forces $g \in \bar{\Sigma}$ in $X$:

$$\mathcal{E}_{\Sigma}([\tau, \infty)) := \{u(\cdot) : u(\cdot) \text{ is a weak solution on } [\tau, \infty) \text{ with}
\text{ the force } g \in \bar{\Sigma} \text{ and } u(t) \in X, \forall t \in [\tau, \infty)\}, \tau \in \mathbb{R},$$

$$\mathcal{E}_{\Sigma}((-\infty, \infty)) := \{u(\cdot) : u(\cdot) \text{ is a weak solution on } (-\infty, \infty) \text{ with}
\text{ the force } g \in \bar{\Sigma} \text{ and } u(t) \in X, \forall t \in (-\infty, \infty)\}.$$ Analogously, we have the following lemma. The proof is exactly the same as that of above Lemma 4.12.

**Lemma 4.19.** The evolutionary system $\mathcal{E}_{\Sigma}$ of the family of 2D NSE with forces in $\bar{\Sigma}$ is closed.

The following theorems recover and generalize the related results in [CV02, Lu06].
Theorem 4.20. Let \( g_0 \) be translation bounded in \( L^2_{\text{loc}}(\mathbb{R}; V') \). The two weak uniform global attractors \( A_w, \mathcal{A}_w^\Sigma \subset H_w \) and the two weak trajectory attractors \( \mathfrak{A}_w, \mathfrak{A}_w^\Sigma \subset C([0, \infty); H_w) \) for the 2D NSE with the fixed force \( g_0 \) and for the family of 2D NSE with forces \( g \in \Sigma \), respectively, exist, \( A_w \) and \( A_w^\Sigma \) are the maximal invariant and maximal quasi-invariant set w.r.t. the closure \( \mathcal{E} = \mathcal{E}_\Sigma \) of the corresponding evolutionary system \( \mathcal{E} \) and

\[
A_w = A_w^\Sigma = \{ u(0) : u \in \mathcal{E}_\Sigma((−\infty, \infty)) \} = \left\{ u(0) : u \in \bigcup_{g \in \Sigma} \mathcal{E}_g((−\infty, \infty)) \right\},
\]

\[
\mathfrak{A}_w = \mathfrak{A}_w^\Sigma = \Pi_+ \bigcup g \in \Sigma \mathcal{E}_g((−\infty, \infty)),
\]

\[
A_w = A_w^\Sigma(t) = \left\{ u(t) : u \in \mathfrak{A}_w^\Sigma \right\}, \quad \forall t \geq 0,
\]

where \( \mathcal{E}_g((−\infty, \infty)) \) is nonempty for any \( g \in \Sigma \). Moreover, \( A_w = A_w^\Sigma \) satisfies the finite weak uniform tracking property for \( \mathcal{E}_\Sigma \) and is weakly equicontinuous on \([0, \infty)\).

Proof. Thanks to Theorem 4.17, the evolutionary systems \( \mathcal{E} \) and \( \mathcal{E}_\Sigma \) are unique. Lemmas 4.7 and 4.19 indicate that \( \mathcal{E} \) satisfies A1 and that \( \mathcal{E}_\Sigma \) is closed, respectively. Then, the theorem follows by applying Theorem 3.24. \( \square \)

Theorem 4.21. If \( g_0 \) is normal in \( L^2_{\text{loc}}(\mathbb{R}; V') \), then the two weak global attractors \( A_w \) and \( A_w^\Sigma \) are strongly compact strong global attractors \( A_s \) and \( A_s^\Sigma \) in \( H \), and the two weak trajectory attractors \( \mathfrak{A}_w \) and \( \mathfrak{A}_w^\Sigma \) are strongly compact strong trajectory attractors \( \mathfrak{A}_s \) and \( \mathfrak{A}_s^\Sigma \) in \( C([0, \infty); H) \), respectively. Moreover,

1. \( \mathfrak{A}_s = \mathfrak{A}_s^\Sigma = \Pi_+ \bigcup g \in \Sigma \mathcal{E}_g((−\infty, \infty)) \) satisfies the finite strong uniform tracking property, i.e., for any \( \varepsilon > 0 \) and \( T > 0 \), there exist \( t_0 \) and a finite subset \( P^f_T \subset \mathfrak{A}_s([0,T]) \), such that for any \( t^* > t_0 \), every trajectory \( u \in \mathcal{E}_\Sigma([0, \infty)) \) satisfies

\[
|u(t) − v(t − t^*)| < \varepsilon, \quad \forall t \in [t^*, t^* + T],
\]

for some \( T \)-time length piece \( v \in P^f_T \).

2. \( \mathfrak{A}_s = \mathfrak{A}_s^\Sigma = \Pi_+ \bigcup g \in \Sigma \mathcal{E}_g((−\infty, \infty)) \) is strongly equicontinuous on \([0, \infty)\), i.e.,

\[
|v(t_1) − v(t_2)| \leq \theta (|t_1 − t_2|), \quad \forall t_1, t_2 \geq 0, \forall v \in \mathfrak{A}_s,
\]

where \( \theta(l) \) is a positive function tending to 0 as \( l \to 0^+ \).

Part of this theorem was obtained in [Lu06]. Here novelties are the existence of the strongly compact strong trajectory attractor \( \mathfrak{A}_s \) in \( C([0, \infty); H) \) and its corollaries.

Proof. We follow the proof of Theorem 4.20. The only thing we need to do is to verify the asymptotical compactness of the evolutionary system \( \mathcal{E}_\Sigma \). Note that, according to Lemma 4.7, \( \mathcal{E} \) also satisfies A2 and A3 when \( g_0 \) is normal in \( L^2_{\text{loc}}(\mathbb{R}; V') \). Moreover, Theorem 4.17 ensures that all the weak solutions of the family of 2D NSE with forces \( g \) in \( \Sigma \) are strongly continuous w.r.t. time \( t \). Hence, we obtain the conclusions by applying Theorem 3.25. \( \square \)
Remark 4.22. Let $u_1$ and $u_2$ be the solutions of (19) with forces $g_1(t), g_2(t) \in \bar{\Sigma}$, respectively. By the standard estimates (see e.g. [T88, Ro01, CV02, Lu06]), we have

\begin{equation}
|u_1(t_2) - u_2(t_2)|^2 \leq \left( |u_1(t_1) - u_2(t_1)|^2 + \frac{1}{\nu} \int_{t_1}^{t_2} ||g_1 - g_2||_V^2, ds \right) e^{C_1}.
\end{equation}

Here $C_1$ only depends on $L, \nu, A$ and increasingly on $t_2 - t_1$, $|u_1(t_1)|^2$, and $||g_1||^2_{L^2(\mathbb{R}; V)}$. Then, the uniqueness of the solutions in Theorem 4.17 is deduced from (25), so is their continuous dependence on the initial data and the forces, which informs the continuity of the associated family of processes (see [CV02]). In the autonomous case, i.e., the force $g(t)$ is independent of time $t$, (25) yields the following classical estimates

\begin{equation}
|u_1(t) - u_2(t)| \leq |u_1(0) - u_2(0)| e^{C_2(t)},
\end{equation}

where $C_2(t)$ depends increasingly on $t$. Hence, we obtain, for any $T > 0$,

\begin{equation}
|u_1(t) - u_2(t)| \leq |u_1(0) - u_2(0)| e^{C_2(T)}, \quad \forall t \in [0, T].
\end{equation}

Now consider a family of solutions $A_{[0, \infty]} := \{u(\cdot) : u(0) \in A\}$ with initial data in a compact subset $A \subset H$. We can see from (27) that $A_{[0,T]} := \{u(\cdot)|_{[0,T]} : u \in A_{[0,\infty]}\}$ is compact in $C([0,T]; H)$. Hence, $A_{[0,\infty]}$ is compact in $C([0, \infty); H)$. Especially, the trajectory attractor $\bar{\Sigma}_h$ in the above theorem is strongly compact. Although simple, to our knowledge, both the deducing process and the compactness result were not noticed before.

On the other hand, however, for the nonautonomous case, we can not obtain the similar compactness from (25).

4.3. 2D Navier-Stokes equations: Strong solutions. Concerning the strong solutions of the 2D NSE, there are similar results obtained by the same method as we do in previous subsection. We will present the main steps and omit details.

Now in this subsection, fix a more regular force $g_0$ that is translation bounded in $L^2_{\text{loc}}(\mathbb{R}; H)$, i.e.,

$$
\|g_0\|_{L^2_{\text{loc}}(\mathbb{R}; H)}^2 := \sup_{t \in \mathbb{R}} \int_{t}^{t+1} |g_0(s)|^2 ds < \infty.
$$

Then $g_0$ is translation compact in $L^{2,w}_{\text{loc}}(\mathbb{R}; H)$, that is, the translation family of $g_0$,

$$
\Sigma := \{g_0(\cdot + h) : h \in \mathbb{R}\}
$$

is precompact in $L^{2,w}_{\text{loc}}(\mathbb{R}; H)$ (see [CV02]). Denote again by

$$
\bar{\Sigma} := \overline{\{g_0(\cdot + h) : h \in \mathbb{R}\}}^{L^{2,w}_{\text{loc}}(\mathbb{R}; H)}.
$$

It is known [CV02] that $\bar{\Sigma}$ endowed with the topology of $L^{2,w}_{\text{loc}}(\mathbb{R}; H)$ is metrizable and the corresponding metric space is compact.

We have more regular solutions (see e.g. [T88, Ro01, CV02]).

Theorem 4.23. (Strong solutions) Let $n = 2$. For every $u_0 \in V$ and $g \in L^2_{\text{loc}}(\mathbb{R}; H)$, the solution of (19) on $[\tau, \infty)$ with $u(\tau) = u_0$ is unique and satisfies

$$
u(t) \in C([\tau, \infty); H) \cap L^\infty_{\text{loc}}([\tau, \infty); V) \cap L^2_{\text{loc}}([\tau, \infty); D(A)).$$
By the classical estimates, there exists a uniformly (w.r.t. \( \tau \in \mathbb{R} \) and \( g \in \bar{\Sigma} \)) absorbing ball \( B_{u}(0, R) \subset V \), where the radius \( R \) depends on \( L, \nu \), and \( \|g_{0}\|_{L_{\infty}^{2}(\mathbb{R}; H)}^{2} \). Let \( X \) be a closed uniformly absorbing ball

\[
X = \{ u \in V : \|u\| \leq R \},
\]

which is also weakly compact in \( V \). Then for any bounded set \( A \subset V \), there exists a time \( \bar{t} \geq 0 \) independent of \( \tau \), such that

\[
u(t) \in X, \quad \forall t \geq t_{1} := \tau + \bar{t},
\]

for every solution \( u(t) \) with \( \tau \in A \) and the initial data \( u(\tau) \in A \).

Due to the better regularity of the solutions, we have the following better version of Lemma 4.4 (see e.g. [T88, Ro01]).

**Lemma 4.24.** Let \( n = 2 \) and let \( u_{k}(t) \) be a sequence of strong solutions of (19) with forces \( g_{k} \in \bar{\Sigma} \), such that \( u_{k}(t) \in X \) for all \( t \geq t_{1} \). Then

\[
u_{k} \text{ is bounded in } L^{2}(t_{1}, t_{2}; D(A)), \quad \text{and } L^{\infty}(t_{1}, t_{2}; V),
\]

\[
du_{k} \text{ is bounded in } L^{2}(t_{1}, t_{2}; H),
\]

for all \( t_{2} > t_{1} \). Moreover, there exists a subsequence \( k_{j} \), such that \( g_{k_{j}} \) converges in \( L_{\text{loc}}^{2, w}(\mathbb{R}; H) \) to some \( g \in \bar{\Sigma} \) and \( u_{k_{j}} \) converges in \( C([t_{1}, t_{2}]; V) \) to some strong solution \( u(t) \) of (19) with the force \( g \), i.e.,

\[
(u_{k_{j}}, v) \to (u, v) \quad \text{uniformly on } [t_{1}, t_{2}],
\]

as \( k_{j} \to \infty \), for all \( v \in D(A) \).

Now we consider two evolutionary systems. One for which a family of trajectories consists of all strong solutions of the 2D NSE with the fixed force \( g_{0} \) in \( X \). More precisely, define

\[
E([\tau, \infty)) := \{ u(\cdot) : u(\cdot) \text{ is a strong solution on } [\tau, \infty) \text{ with} \}
\]

\[
\text{the force } g \in \bar{\Sigma} \text{ and } u(t) \in X, \forall t \in [\tau, \infty), \quad \tau \in \mathbb{R},
\]

\[
E((\infty, \infty)) := \{ u(\cdot) : u(\cdot) \text{ is a strong solution on } (\infty, \infty) \text{ with} \}
\]

\[
\text{the force } g \in \bar{\Sigma} \text{ and } u(t) \in X, \forall t \in (\infty, \infty), \quad \tau \in \mathbb{R},
\]

Another one we consider is with \( \bar{\Sigma} \) as a symbol space. The family of trajectories for this evolutionary system consists of all strong solutions of the family of 2D NSE with forces \( g \in \bar{\Sigma} \) in \( X \):

\[
E_{\bar{\Sigma}}([\tau, \infty)) := \{ u(\cdot) : u(\cdot) \text{ is a strong solution on } [\tau, \infty) \text{ with} \}
\]

\[
\text{the force } g \in \bar{\Sigma} \text{ and } u(t) \in X, \forall t \in [\tau, \infty), \quad \tau \in \mathbb{R},
\]

\[
E_{\bar{\Sigma}}((\infty, \infty)) := \{ u(\cdot) : u(\cdot) \text{ is a strong solution on } (\infty, \infty) \text{ with} \}
\]

\[
\text{the force } g \in \bar{\Sigma} \text{ and } u(t) \in X, \forall t \in (\infty, \infty), \quad \tau \in \mathbb{R},
\]

By analogous arguments to those of Lemmas 4.11 and 4.12, we have the following lemmas.

**Lemma 4.25.** The evolutionary system \( E \) of the 2D NSE with the fixed force \( g_{0} \) satisfies A1.
Lemma 4.26. The evolutionary system $\mathcal{E}_\Sigma$ of the family of 2D NSE with forces in $\bar{\Sigma}$ is closed.

Note that the strongly compact strong global attractor $\mathcal{A}_w^\Sigma$ for the family of 2D NSE with forces in $\Sigma$ had been obtained in [LWZ05]. Now, we have more: The following theorems recover and generalize the related results in [CV02, LWZ05].

Theorem 4.27. Let $g_0$ be translation bounded in $L^2_{\text{loc}}(\mathbb{R}; H)$. The two weak uniform global attractors $\mathcal{A}_w^\Sigma \subset V_w$ and the two weak trajectory attractors $\mathfrak{A}_w^\Sigma \subset C([0, \infty); V_w)$ for the 2D NSE with the fixed force $g_0$ and for the family of 2D NSE with forces $g \in \Sigma$, respectively, exist, $\mathcal{A}_w$ and $\mathcal{A}_w^\Sigma$ are the maximal invariant and maximal quasi-invariant set w.r.t. the closure $\hat{\mathcal{E}} = \mathcal{E}_\Sigma$ of the corresponding evolutionary system $\mathcal{E}$ and

$$\mathcal{A}_w = \mathcal{A}_w^\Sigma = \{ u(0) : u \in E^\Sigma((-\infty, \infty)) \} = \left\{ u(0) : u \in \bigcup_{g \in \Sigma} E_g((-\infty, \infty)) \right\},$$

$$\mathfrak{A}_w = \mathfrak{A}_w^\Sigma = \Pi_+ \bigcup_{g \in \Sigma} E_g((-\infty, \infty)),$$

$$\mathcal{A}_w = \mathfrak{A}_w(t) = \left\{ u(t) : u \in \mathfrak{A}_w^\Sigma \right\}, \quad \forall t \geq 0,$$

where $E_g((-\infty, \infty))$ is nonempty for any $g \in \Sigma$. Moreover, $\mathfrak{A}_w = \mathfrak{A}_w^\Sigma$ satisfies the finite weak uniform tracking property for $E^\Sigma$ and is weakly equicontinuous on $[0, \infty)$.

Proof: The proof is similar to that of Theorem 4.20. Due to Theorem 4.23, evolutionary systems $\mathcal{E}$ and $\mathcal{E}_\Sigma$ are unique. The facts that $\mathcal{E}$ satisfies A1 and $\mathcal{E}_\Sigma$ is closed are obtained by Lemmas 4.25 and 4.26, respectively. Then, the theorem follows by applying Theorem 3.24 again. \qed

Theorem 4.28. If $g_0$ is normal in $L^2_{\text{loc}}(\mathbb{R}; H)$, then the two weak global attractors $\mathcal{A}_w$ and $\mathcal{A}_w^\Sigma$ are strongly compact strong global attractors $\mathcal{A}_s$ and $\mathcal{A}_s^\Sigma$ in $V$, and the two weak trajectory attractors $\mathfrak{A}_w$ and $\mathfrak{A}_w^\Sigma$ are strongly compact strong trajectory attractors $\mathfrak{A}_s$ and $\mathfrak{A}_s^\Sigma$ in $C([0, \infty); V)$, respectively. Moreover,\n
1. $\mathfrak{A}_s = \mathfrak{A}_s^\Sigma = \Pi_+ \bigcup_{g \in \Sigma} E_g((-\infty, \infty))$ satisfies the finite strong uniform tracking property, i.e., for any $\varepsilon > 0$ and $T > 0$, there exist $t_0$ and a finite subset $P^f_T \subset \mathfrak{A}_s|[0,T]$, such that for any $t^* > t_0$, every trajectory $u \in E^\Sigma([0, \infty))$ satisfies

$$\| u(t) - v(t - t^*) \| < \varepsilon, \quad \forall t \in [t^*, t^* + T],$$

for some $T$-time length piece $v \in P^f_T$.

2. $\mathfrak{A}_s = \mathfrak{A}_s^\Sigma = \Pi_+ \bigcup_{g \in \Sigma} E_g((-\infty, \infty))$ is strongly equicontinuous on $[0, \infty)$, i.e.,

$$\| v(t_1) - v(t_2) \| \leq \theta (|t_1 - t_2|), \quad \forall t_1, t_2 \geq 0, \forall v \in \mathfrak{A}_s,$$

where $\theta(l)$ is a positive function tending to 0 as $l \to 0^+$.

Part of this theorem recovers the corresponding results in [LWZ05]. Novelties here are the existence of the strongly compact strong trajectory attractor $\mathfrak{A}_s$ in $C([0, \infty); V)$ and sequent properties.
Proof: Analogous to the proof of Theorem 4.21, we follow the proof of Theorem 4.27. Note that, by Theorem 3.24, $\mathcal{E}_s = \mathcal{E}_s$. Theorem 3.2 in [LWZ05] provides the existence of the strongly compact strong global attractor $\mathcal{A}_s^\Sigma$ in $V$ for the family of 2D NSE with forces $g \in \Sigma$ when $g_0$ is normal in $L_{loc}^2(\mathbb{R}; H)$. In other words, the evolutionary system $\dot{\mathcal{E}}_s = \mathcal{E}_s$ possesses a strongly compact strong global attractor $\mathcal{A}_s^\Sigma$. Hence, we obtain the conclusions by applying Theorem 3.25 again.

5. ATTRACTORS FOR REACTION-DIFFUSION SYSTEM

In this section, we study the long-time behavior of solutions of the following nonautonomous reaction-diffusion system (RDS):

\begin{equation}
\partial_t u - \Delta u + f(u, t) = g(x, t), \quad \text{in } \Omega,
\end{equation}

\begin{equation}
\begin{array}{l}
u = 0, \\
u|_{t=\tau} = u_\tau, \quad \tau \in \mathbb{R}.
\end{array}
\end{equation}

Here $\Omega$ is a bounded domain in $\mathbb{R}^n$ with a boundary $\partial \Omega$ of sufficient smoothness; $a = \{a_{ij}\}_{i=1}^N$ is an $N \times N$ real matrix with positive symmetric part $\frac{1}{2}(a + a^\ast) \geq \beta I$, $\beta > 0$; $u = u(x, t) = (u^1, \cdots, u^N)$ is the unknown function; $g = (g^1, \cdots, g^N)$ is the driving force and $f = (f^1, \cdots, f^N)$ is the interaction function.

Denote the spaces by $H := (L^2(\Omega))^N$ and $V := (H^1_0(\Omega))^N$, and denote by $(\cdot, \cdot)$ and $| \cdot |$ the $H$-inner product and the corresponding $H$-norm. Let $V'$ be the dual of $V$. Assume that $g(s) = g(\cdot, s)$ is translation bounded in $L_{loc}^2(\mathbb{R}; V')$, i.e.,

$$\|g\|^2_{L_{loc}^2(\mathbb{R}; V')} := \sup_{t \in \mathbb{R}} \int_t^{t+1} \|g(s)\|^2_{V'}, \quad ds < \infty,$$

and $f(v, s)$ satisfies the following conditions of continuity, dissipativeness and growth:

\begin{equation}
f(v, s) \in C(\mathbb{R}^N \times \mathbb{R}; \mathbb{R}^N),
\end{equation}

\begin{equation}
\sum_{i=1}^N \gamma_i |v^i|^{p_i} - C \leq \sum_{i=1}^N f_i(v, s)v^i = f(v, s) \cdot v, \quad \forall v \in \mathbb{R}^N,
\end{equation}

$$\gamma_k > 0, \quad k = 1, \cdots, N,$$

\begin{equation}
\sum_{i=1}^N |f_i(v, s)|^{\frac{p_i}{p_i-1}} \leq C \left( \sum_{i=1}^N |v^i|^{p_i} + 1 \right), \quad \forall v \in \mathbb{R}^N,
\end{equation}

where the letter $C$ denotes a constant which may be different in each occasion throughout this section.$^{12}$

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$^{12}$RDS (28) with other boundary conditions such as Neumann or periodic boundary conditions can be handled in the same way, and all results hold for these boundary conditions. For the Dirichlet boundary conditions, instead of considering $p_k \geq 2, k = 1, \cdots, N$, for simplicity, we may assume that $p_k > 1$. See Remarks II.4.1 and II.4.2 in [CV02] for more details.
Let \( q_k := p_k/(p_k - 1) \), \( r_k := \max\{1, n(1/2 - 1/p_k)\} \), \( k = 1, \ldots, N \), and denote by 
\[ p := (p_1, \ldots, p_N), \quad q := (q_1, \ldots, q_N), \quad r := (r_1, \ldots, r_N) \]
and
\[ \begin{align*}
L^p(\Omega) &:= L^{p_1}(\Omega) \times L^{p_2}(\Omega) \times \cdots \times L^{p_N}(\Omega), \\
H^{-\tau}(\Omega) &:= H^{-\tau_1}(\Omega) \times H^{-\tau_2}(\Omega) \times \cdots \times H^{-\tau_N}(\Omega), \\
L^p(\tau, t; L^p(\Omega)) &:= L^{p_1}(\tau, t; L^{p_1}(\Omega)) \times \cdots \times L^{p_N}(\tau, t; L^{p_N}(\Omega)), \\
L^q(\tau, t; H^{-\tau}(\Omega)) &:= L^{q_1}(\tau, t; H^{-\tau_1}(\Omega)) \times \cdots \times L^{q_N}(\tau, t; H^{-\tau_N}(\Omega)).
\end{align*} \]

**Definition 5.1.** A weak solution of (28) on \([\tau, \infty)\) (or \((-\infty, \infty)\), if \(\tau = -\infty\)) is a function 
\( u(x, t) \in L^p_{\text{loc}}(\tau, \infty; L^p(\Omega)) \cap L^q_{\text{loc}}(\tau, \infty; V) \) that satisfies (28) in the distribution sense of the space \(D'(\tau, \infty; H^{-\tau}(\Omega))\).

We recall the results on the existence of weak solutions of (28) (see e.g. [CV02]). Note that conditions (30)-(31) do not ensure the uniqueness of the solutions.

**Theorem 5.2.** Let \( f \) satisfy (29)-(31) and \( g \in L^2_{\text{loc}}(\mathbb{R}; V') \). For every \( u_\tau \in H \), there exists a weak solution \( u(t) \) of (28) satisfying 
\[ u \in C([\tau, \infty); H) \cap L^2_{\text{loc}}(\tau, \infty; V) \cap L^p_{\text{loc}}(\tau, \infty; L^p(\Omega)). \]
Moreover, the function \( |u(t)|^2 \) is absolutely continuous on \([\tau, \infty)\) and
\[ \frac{1}{2} \frac{d}{dt} |u(t)|^2 + (a \nabla u(t), \nabla u(t)) + (f(u(t), t), u(t)) = \langle g(t), u(t) \rangle, \]
for a.e. \( t \in [\tau, \infty) \).

Now, we consider a fixed pair of an interaction function \( f_0 \) and a driving force \( g_0 \), such that, \( f_0(v, t) \) satisfies (29)-(31) and \( g_0(t) \in L^2_{\text{loc}}(\mathbb{R}; V') \). Let 
\[ \sigma_0 := (f_0, g_0), \]
and 
\[ \Sigma := \{\sigma_0(\cdot + h) : h \in \mathbb{R}\}. \]
Obviously, for every \( \sigma = (f, g) \in \Sigma \), \( f \) satisfies (29)-(31) with the same constants, and 
\[ \|g\|_{L^2_{\text{loc}}(\mathbb{R}; V')} \leq \|g_0\|_{L^2_{\text{loc}}(\mathbb{R}; V')}. \]
Let \( u(t), t \in [\tau, \infty) \), be a weak solution of (28) with \( \sigma = (f, g) \in \Sigma \) guaranteed by Theorem 5.2. Thanks to (30) and (33), we obtain from (32) that
\[ \frac{d}{dt} |u(t)|^2 + \lambda_1 \beta |u(t)|^2 \leq C + \beta^{-1} \|g_0\|_{V'}^2, \]
for a.e. \( t \in [\tau, \infty) \). Here \( \lambda_1 \) is the first eigenvalue of the Laplacian with Dirichlet boundary conditions. Due to the absolute continuity of \( |u(t)| \) and Grönwall’s inequality, (34) implies that
\[ |u(t)|^2 \leq |u(\tau)|^2 e^{-\lambda_1 \beta(t-\tau)} + C, \quad \forall \ t \in [\tau, \infty). \]
Therefore, there exists a uniformly (w.r.t. \( \tau \in \mathbb{R} \) and \( \sigma \in \Sigma \)) absorbing ball \( B_{\delta}(0, R) \subset H \), where the radius \( R \) depends on \( \lambda_1, \beta \), the constant in (30) and \( \|g_0\|_{L^2_{\text{loc}}(\mathbb{R}; V')} \). We denote by \( X \) a closed absorbing ball
\[ X = \{u \in H : |u| \leq R\}. \]
That is, for any bounded set $A \subset H$, there exists a time $\bar{t} \geq 0$ independent of the initial time $\tau$, such that
\begin{equation}
    u(t) \in X, \quad \forall t \geq t_1 := \tau + \bar{t},
\end{equation}
for every weak solution $u(t)$ with $\sigma \in \Sigma$ and the initial data $u(\tau) \in A$. It is known that $X$ is weakly compact in $H$ and metrizable with a metric $d_w$ deducing the weak topology on $X$.

Consider an evolutionary system for which a family of trajectories consists of all weak solutions of $(38)$ with the fixed $\sigma_0$ in $X$. More precisely, define
\begin{align*}
    \mathcal{E}([\tau, \infty)) &:= \{ u(\cdot) : u(\cdot) \text{ is a weak solution on } [\tau, \infty) \} \quad \text{with } \sigma \in \Sigma \text{ and } u(t) \in X, \forall t \in [\tau, \infty], \quad \tau \in \mathbb{R}, \quad \mathcal{E}((-\infty, \infty)) := \{ u(\cdot) : u(\cdot) \text{ is a weak solution on } (-\infty, \infty) \} \quad \text{with } \sigma \in \Sigma \text{ and } u(t) \in X, \forall t \in (-\infty, \infty) \}.
\end{align*}

Clearly, the properties 1-4 in Definition 2.7 hold for $\mathcal{E}$ if we utilize the fact that is formulated as the translation identity: A weak solution of $(38)$ with $\sigma \in \Sigma$ initiating at time $\tau + h$ is also a weak solution of $(38)$ with $\sigma(\cdot + h) \in \Sigma$ initiating at time $\tau$. Thanks to Theorem 3.6, the weak global attractor $\mathcal{A}_w$ for this evolutionary system exists.

**Lemma 5.3.** Let $u_k(t)$ be a sequence of weak solutions of $(38)$ with $\sigma_k \in \Sigma$, such that $u_k(t) \in X$ for all $t \geq t_1$. Then
\begin{align*}
    u_k &\text{ is bounded in } L^2(t_1,t_2;V), \\
    \partial_t u_k &\text{ is bounded in } L^q(t_1,t_2;H^{-r}(\Omega)),
\end{align*}
for all $t_2 > t_1$. Moreover, there exists a subsequence $u_{k_j}$ converges in $C([t_1,t_2];H_w)$ to some $\phi(t) \in C([t_1,t_2];H)$, i.e.,
\begin{equation}
    (u_{k_j},v) \rightarrow (\phi,v) \text{ uniformly on } [t_1,t_2],
\end{equation}
as $k_j \rightarrow \infty$, for all $v \in H$.

**Proof.** The proof is analogous with that of Lemma 3.2 in [Lu07] and that of Lemma 2.1 in [R98]. Standard estimates (see e.g. [CV02]) show that, for all $t_2 > t_1$,
\begin{align}
    \{ u_k \} \text{ is bounded in } &L^2(t_1,t_2;V) \cap L^\infty(t_1,t_2;H) \cap L^p(t_1,t_2;L^p(\Omega)), \\
\text{and} \quad \{ \partial_t u_k \} \text{ is bounded in } &L^q(t_1,t_2;H^{-r}(\Omega)), \\
\{ f_k(u_k(x,t),t) \} \text{ is bounded in } &L^q(t_1,t_2;L^q(\Omega)).
\end{align}
By the embedding theorem (cf. Theorem II.1.4 in [CV02], Theorem 8.1 in [Ro01]), we obtain that
\begin{equation}
    \{ u_k \} \text{ is precompact in } L^2(t_1,t_2;H).
\end{equation}
Passing to a subsequence and dropping a subindex, we know from (38)-(41) that,
\begin{align}
    u_k(t) \rightarrow &\phi(t) \quad \text{weak-star in } L^\infty(t_1,t_2;H), \\
\text{weakly in } &L^2(t_1,t_2;V) \cap L^p(t_1,t_2;L^p(\Omega)),
\end{align}
strongly in $L^2(t_1, t_2; H)$,

and

\[
\Delta u_k(t) \to \Delta \phi(t) \quad \text{weakly in } L^2(t_1, t_2; V'),
\]

\[
\partial_t u_k(t) \to \partial_t \phi(t) \quad \text{weakly in } L^q(t_1, t_2; H^{-r}(\Omega)),
\]

\[
f_k(u_k(x, t), t) \to \psi(t) \quad \text{weakly in } L^9(t_1, t_2; L^9(\Omega)),
\]

for some

\[
\phi(t) \in L^\infty(t_1, t_2; H) \cap L^2(t_1, t_2; V) \cap L^p(t_1, t_2; L^p(\Omega)),
\]

and some

\[
\psi(t) \in L^9(t_1, t_2; L^9(\Omega)).
\]

Note that $g_0$ is translation compact in $L^2_{w, v}(\mathbb{R}; V')$ (see [CV02]). Thus, passing to a subsequence and dropping a subindex again, we also have,

\[
g_k(t) \to g(t) \quad \text{weakly in } L^2(t_1, t_2; V'),
\]

with some $g(t) \in L^2(t_1, t_2; V')$. Passing the limits yields the following equality

\[
\partial_t \phi - a \Delta \phi + \psi = g
\]

in the distribution sense of the space $\mathcal{D}'(t_1, t_2; H^{-r}(\Omega))$. Thanks to a vector version of Theorem II.1.8 in [CV02], (43)-(47) indicate that $\phi(t) \in C([t_1, t_2]; H)$.

Now we prove that $u_k(t) \to \phi(t)$ in $C([t_1, t_2]; H_w)$.

Thanks to the strong convergence in (42), we know that (passing to a subsequence and dropping a subindex in above procedures if it is necessary)

\[
u_k(t) \to \phi(t) \quad \text{strongly in } H, \quad \text{a.e. } t \in [t_1, t_2].
\]

Thus, for any test function $v \in (C_0^\infty(\Omega))^N$,

\[
(u_k(t), v) \to (\phi(t), v), \quad \text{a.e. } t \geq t_1.
\]

It follows from (38) that $\{(u_k(t), v)\}$ is uniformly bounded on $[t_1, t_2]$. On the other hand, by (39), for every $v \in (C_0^\infty(\Omega))^N$, $0 < \delta < 1$ and $t_1 \leq t \leq t + \delta \leq t_2$,

\[
\|(u_k(t + \delta) - u_k(t), v)\| = \left| \int_t^{t+\delta} \langle \partial_t u_k(s), v \rangle \, ds \right|
\leq C \delta^{\frac{1}{q}} \|v\|_{H^r} \|\partial_t u_k\|_{L^q(t, t+\delta; H^{-r})}
\leq C \delta^{\frac{1}{q}} \|v\|_{H^r}.
\]

That is, the sequence $\{(u_k(t), v)\}$ is equicontinuous on $[t_1, t_2]$. Hence,

\[
(u_k(t), v) \to (\phi(t), v) \quad \text{uniformly on } [t_1, t_2], \quad \forall v \in (C_0^\infty(\Omega))^N.
\]

Note that $(C_0^\infty(\Omega))^N$ is dense in $H$. We have

\[
(u_k(t), v) \to (\phi(t), v) \quad \text{uniformly on } [t_1, t_2], \quad \forall v \in H.
\]

We complete the proof.

Then, we have the following.
Lemma 5.4. The evolutionary system $\mathcal{E}$ of (28) with the fixed $\sigma_0$ satisfies A1 and A3. Moreover, if $g_0$ is normal in $L^2_{\text{loc}}(\mathbb{R}; V')$ then A2 holds.

Proof: The proof is analogous to that of Lemma 3.4 in [CL09]. First, by Theorem 5.2, $\mathcal{E}([0, \infty)) \subset C([0, \infty); X_s)$. Now take a sequence $\{u_k\} \subset \mathcal{E}([0, \infty))$. Owing to Lemma 5.3, there exists a subsequence, still denoted by $\{u_k\}$, which converges in $C([0, 1]; X_w)$ to some $\phi^1 \in C([0, 1]; X_s)$ as $k \to \infty$. Passing to a subsequence and dropping a subindex once more, we have that $u_k \to \phi^2$ in $C([0, 2]; X_w)$ as $k \to \infty$ for some $\phi^2 \in C([0, 2]; X_s)$. Note that $\phi^1(t) = \phi^2(t)$ on $[0, 1]$. Continuing this diagonalization process, we obtain a subsequence $\{u_{k_j}\}$ of $\{u_k\}$ that converges in $C([0, \infty); X_w)$ to some $\phi \in C([0, \infty); X_s)$ as $k_j \to \infty$. Therefore, A1 holds.

Take a sequence $\{u_k\} \subset \mathcal{E}([0, \infty))$ be such that it is a $d_{C([0,T]; X_w)}$-Cauchy sequence in $C([0, T]; X_w)$ for some $T > 0$. Thanks to Lemma 5.3 again, the sequence $\{u_k\}$ is bounded in $L^2(0, T; V)$. Hence, there exists some $\phi(t) \in C([0,T]; X_w)$, such that

$$\int_0^T |u_k(s) - \phi(s)|^2 \, ds \to 0, \quad \text{as } k \to \infty.$$ 

In particular, $|u_k(t)| \to |\phi(t)|$ as $k \to \infty$ a.e. on $[0, T]$, which means that $u_k(t)$ is a $d_u$-Cauchy sequence a.e. on $[0, T]$. Thus, A3 is valid.

For any $u \in \mathcal{E}([0, \infty))$ and $t > 0$, it follows from (34) and the absolute continuity of $|u(\cdot)|^2$ that

$$|u(t)|^2 \leq |u(t_0)|^2 + C(t - t_0) + \frac{1}{\beta} \int_{t_0}^t \|g_0\|_{V'}^2 \, ds,$$

for all $0 \leq t_0 < t$. Here $C$ is independent of $u$. Suppose now that $g_0$ is normal in $L^2_{\text{loc}}(\mathbb{R}; V')$. Then given $\epsilon > 0$, there exists $0 < \delta < \epsilon/2C$, such that

$$\sup_{t \in \mathbb{R}} \int_{t-\delta}^t \|g_0(s)\|_{V'}^2 \, ds \leq \frac{\beta \epsilon}{2}.$$ 

It follows from (48) that

$$|u(t)|^2 \leq |u(t_0)|^2 + \epsilon, \quad \forall t_0 \in (t - \delta, t),$$

which concludes that A2 holds.

We have the followings.

Theorem 5.5. Let $f_0$ satisfy (29)-(31) and $g_0$ be translation bounded in $L^2_{\text{loc}}(\mathbb{R}; V')$. Then the weak uniform global attractor $A_w$ and the weak trajectory attractor $\mathcal{A}_w$ for (28) with the fixed $\sigma_0 = (f_0, g_0)$ exist, $A_w$ is the maximal invariant and maximal quasi-invariant set w.r.t. the closure $\bar{\mathcal{E}}$ of the corresponding evolutionary system $\mathcal{E}$, and

$$A_w = \omega_w(X) = \omega_s(X) = \{u(0) : u \in \bar{\mathcal{E}}((-\infty, \infty))\},$$

$$\mathcal{A}_w = \Pi_+ \bar{\mathcal{E}}((-\infty, \infty)) = \{u(\cdot)|_{(0, \infty)} : u \in \bar{\mathcal{E}}((-\infty, \infty))\},$$

$$A_w = \mathcal{A}_w(t) = \{u(t) : u \in \mathcal{A}_w\}, \quad \forall t \geq 0.$$ 

Moreover, $\mathcal{A}_w$ satisfies the finite weak uniform tracking property and is weakly equicontinuous on $[0, \infty)$. 

Proof. It is known from Lemma 5.4 that the associated evolutionary system \( \mathcal{E} \) satisfies A1. Then the conclusions follow from Theorems 3.6. \( \square \)

**Theorem 5.6.** Furthermore, if \( g_0 \) is normal in \( L^2_{\text{loc}}(\mathbb{R}; V') \), then the weak global attractor \( \mathcal{A}_w \) is a strongly compact strong global attractor \( \mathcal{A}_s \), and the weak trajectory attractor \( \mathcal{A}_w \) is a strongly compact strong trajectory attractor \( \mathcal{A}_s \). Moreover,

1. \( \mathcal{A}_s = \Pi_+ \mathcal{E}((\mathbb{R}; [0,\infty)) \) satisfies the finite strong uniform tracking property, i.e., for any \( \epsilon > 0 \) and \( T > 0 \), there exist \( t_0 \) and a finite subset \( P^f_T \subset \mathcal{A}_s \), such that for any \( t^* > t_0 \), every trajectory \( u \in \mathcal{E}([0,\infty)) \) satisfies
   \[ |u(t) - v(t - t^*)| < \epsilon, \quad \forall t \in [t^*, t^* + T], \]
   for some \( T \)-time length piece \( v \in P^f_T \).
2. \( \mathcal{A}_s = \Pi_+ \mathcal{E}((\mathbb{R}; [0,\infty)) \) is strongly equicontinuous on \( [0,\infty) \), i.e.,
   \[ |v(t_1) - v(t_2)| \leq \theta (|t_1 - t_2|), \quad \forall t_1, t_2 \geq 0, \quad \forall v \in \mathcal{A}_s, \]
   where \( \theta(l) \) is a positive function tending to 0 as \( l \to 0^+ \).

Proof. According to Theorem 2.12, Theorem 5.2 and Lemma 5.4 mean that the associated evolutionary system \( \mathcal{E} \) is asymptotically compact. Hence, we obtain the theorem by applying Theorem 3.12. \( \square \)

**Remark 5.7.** The existence of \( \mathcal{A}_s \) is obtained in [CL09]. Thus, instead of Theorem 2.12, we can also utilize Theorem 3.17 to ensure the asymptotical compactness of the corresponding evolutionary system \( \mathcal{E} \).

**Remark 5.8.** The equality in the conclusion 1 of Theorem 5.6 answers an open problem in [Lu07, CL09], which concerns how to describe the structure of \( \mathcal{A}_s \) for (28) with general interaction terms, satisfying no additional assumption other than conditions of (29)-(31). Note that, only with these three conditions on nonlinearities, it is not known how to construct a suitable symbol space that is necessary for applying previous framework [CV02]. See Subsection 5.2 for more.

### 5.1. RDS with more regular interaction terms

In this subsection, we study (28) with more regular interaction functions.

Denote by \( \mathcal{M} \) the space \( C(\mathbb{R}^N; \mathbb{R}^N) \) endowed with the local uniform convergence topology. Denote by \( C^{p,p}(\mathbb{R}; \mathcal{M}) \) the space \( C(\mathbb{R}; \mathcal{M}) \) endowed with the topology of the following convergence: \( \varphi_k(s) \to \varphi(s) \) in \( C^{p,p}(\mathbb{R}; \mathcal{M}) \) as \( k \to \infty \), if \( \varphi_k(v, s) \) is uniformly bounded on any ball in \( \mathbb{R}^N \times \mathbb{R} \) and for every \( (v, s) \in \mathbb{R}^N \times \mathbb{R} \),

\[ \|\varphi_k(v, s) - \varphi(v, s)\|_{\mathbb{R}^N} \to 0, \quad \text{as} \quad k \to \infty. \]

Note that \( C^{p,p}(\mathbb{R}; \mathcal{M}) \) is in fact the space \( C(\mathbb{R}^N \times \mathbb{R}; \mathbb{R}^N) \) endowed with the usual weak topology. Let \( C^{p,p}(\mathbb{R}; \mathcal{M}) \) denote the space \( C(\mathbb{R}; \mathcal{M}) \) endowed with another topology of the following convergence: \( \varphi_k(s) \to \varphi(s) \) in \( C^{p,p}(\mathbb{R}; \mathcal{M}) \) as \( k \to \infty \), if \( \varphi_k(v, s) \) is uniformly bounded on any ball in \( \mathbb{R}^N \times \mathbb{R} \) and for every \( s \in \mathbb{R}, \quad \forall \|v\|_{\mathbb{R}^N} \leq R \),

\[ \max_{\|v\|_{\mathbb{R}^N} \leq R} \|\varphi_k(v, s) - \varphi(v, s)\|_{\mathbb{R}^N} \to 0, \quad \text{as} \quad k \to \infty. \]
Now we assume in addition that $f_0(v,t)$ is translation compact in $C^{p,u}(\mathbb{R}; \mathcal{M})$, i.e., the following closure

$$
\bar{\Sigma}_1 := \{ f_0(\cdot + h) : h \in \mathbb{R} \}^{C^{p,u}(\mathbb{R}; \mathcal{M})}
$$

is compact in $C^{p,u}(\mathbb{R}; \mathcal{M})$. Note that $\bar{\Sigma}_1$ is metrizable in the space $C^{p,u}(\mathbb{R}; \mathcal{M})$ and is compact w.r.t. such a metric. For convenience, we gather the properties of this kind of translation compact functions in Subsection 5.2 below. Denote by

$$
\bar{\Sigma}_2 := \{ g_0(\cdot + h) : h \in \mathbb{R} \}^{L^{2,w}_{loc}(\mathbb{R};V')}.
$$

Then, $\bar{\Sigma}_2$ endowed with the topology of $L^{2,w}_{loc}(\mathbb{R};V')$ is metrizable and the corresponding metric space is compact (see [CV02]).

Let

$$
\bar{\Sigma} := \{ \sigma_0(\cdot + h) : h \in \mathbb{R} \}^{C^{p,u}(\mathbb{R};\mathcal{M}) \times L^{2,w}_{loc}(\mathbb{R};V')} = \bar{\Sigma}_1 \times \bar{\Sigma}_2.
$$

Then $\bar{\Sigma}$ is compact in the product space $C^{p,u}(\mathbb{R}; \mathcal{M}) \times L^{2,w}_{loc}(\mathbb{R}; V')$ and is metrizable in the weaker space $C^{p,u}(\mathbb{R}; \mathcal{M}) \times L^{2,w}_{loc}(\mathbb{R}; V')$. For every $\sigma = (f,g) \in \bar{\Sigma}$, $g$ satisfies (33) by Proposition V.4.2 in [CV02] and there exists a sequence $\{ f_0(\cdot + h_n) \}$ that converges to $f$ in $C^{p,u}(\mathbb{R}; \mathcal{M})$ as $n \to \infty$. Then, $\{ f_0(\cdot + h_n) \}$ also converges to $f$ in the weak topology of $C(\mathbb{R}^N \times \mathbb{R}; \mathbb{R}^N)$. Note that the weak convergence of $C(\mathbb{R}^N \times \mathbb{R}; \mathbb{R}^N)$ is equivalent to the local uniform boundedness and the pointwise convergence of a sequence of functions of $C(\mathbb{R}^N \times \mathbb{R}; \mathbb{R}^N)$ (cf. [Di84, Me98]). Thus, $f$ satisfies (29)-(31) with the same constants.

It can be seen that the argument of (34)-(37) is still valid for $\Sigma$ replaced by $\bar{\Sigma}$ and Lemma 5.3 can be improved as follows.

**Lemma 5.9.** [CL09] Let $u_k(t)$ be a sequence of weak solutions of (28) with $\sigma_k \in \bar{\Sigma}$, such that $u_k(t) \in X$ for all $t \geq t_1$. Then

$$
u_k \text{ is bounded in } L^2(t_1,t_2;V),$$

$$
\partial_t \nu_k \text{ is bounded in } L^4(t_1,t_2; H^{-r}(\Omega)),
$$

for all $t_2 > t_1$. Moreover, there exists a subsequence $k_j$, such that $\sigma_{k_j}$ converges in $C^{p,u}(\mathbb{R}; \mathcal{M}) \times L^{2,w}_{loc}(\mathbb{R};V')$ to some $\sigma \in \bar{\Sigma}$, and $u_{k_j}$ converges in $C([t_1,t_2]; H_w)$ to some weak solution $u(t)$ of (28) with $\sigma$, i.e.,

$$
(u_{k_j}, v) \to (u, v) \text{ uniformly on } [t_1, t_2],
$$

as $k_j \to \infty$, for all $v \in H$.

**Proof.** Note that Lemma 5.3 still holds for $\Sigma$ replaced by $\bar{\Sigma}$ and the proof needs no modification at all. Now we continue the proof. Since $\bar{\Sigma}_1$ is also a compact set in $C^{p,p}(\mathbb{R}; \mathcal{M})$, $\{ f_k \}$ in (43) can be taken as a convergent sequence in $C^{p,p}(\mathbb{R}; \mathcal{M})$ with a limit $f \in \bar{\Sigma}_1$.

---

13If $K$ is a (relatively) weakly compact set in a Banach space $B$ and the dual $B'$ of $B$ contains a countable total set, then the $K^{weak}$ is metrizable. Recall that a set $A \subset B'$ is called total if $a(x) = 0$ for every $a \in A$ implies $x = 0$ (see [Di84], p18). Note that $\Sigma_1$ is also a compact set in $C^{p,p}(\mathbb{R}; \mathcal{M})$, i.e., a weakly compact set in $C(\mathbb{R}^N \times \mathbb{R}; \mathbb{R}^N)$ (cf. Theorem VII.1 in [Di84]). For any closed bounded ball $B \subset \mathbb{R}^N \times \mathbb{R}$, $\Sigma_1|_B := \{ f_{u} = \text{restriction of } f \text{ on } B : f \in \Sigma_1 \}$ is weakly compact in $C(B; \mathbb{R}^N)$. $C(B; \mathbb{R}^N)'$ contains a total set of Dirac $\delta$-measures on rational points of $B$. Hence $\Sigma_1|_B$ endowed with the weak topology of $C(B; \mathbb{R}^N)$ is metrizable. It follows that $\Sigma_1$ is metrizable in $C^{p,p}(\mathbb{R}; \mathcal{M})$ by means of the so-called Fréchet metric.
We claim that \( f(\phi(x, t), t) = \psi(t) \), which will imply that \( \phi(t) \) is a weak solution of (28) with the interaction function \( f \) and the driving force \( g \) in (46). Obviously, \( g \in \bar{\Sigma}_2 \). Thanks to the strong convergence in (42), we know that, passing to a subsequence if necessary,
\[
u_k(x, t) \to \phi(x, t), \quad a.e. \ (x, t) \in \Omega \times [t_1, t_2].
\]
Note that
\[
\|f_k(u_k(x, t), t) - f(\phi(x, t), t)\|_{\mathbb{R}^N} \leq \|f_k(u_k(x, t), t) - f_k(\phi(x, t), t)\|_{\mathbb{R}^N} \\
+ \|f_k(\phi(x, t), t) - f(\phi(x, t), t)\|_{\mathbb{R}^N}.
\]
Due to Theorem 5.22 in Subsection 5.2 below, all \( f_k \) satisfy (52) with the same function \( \theta \). Hence we obtain that
\[
(49) \quad \|f_k(u_k(x, t), t) - f(\phi(x, t), t)\|_{\mathbb{R}^N} \to 0, \quad a.e. \ (x, t) \in \Omega \times [t_1, t_2].
\]
On the other hand, according to Lemma II.1.2 in [CV02], the uniform boundedness of (40) and the pointwise convergence of (49) yield that
\[
f_k(u_k(x, t), t) \to f(\phi(x, t), t) \quad \text{weakly in } L^q(t_1, t_2; L^q(\Omega)).
\]
Therefore, \( f(\phi(x, t), t) = \psi(t) \) in \( L^q(t_1, t_2; L^q(\Omega)) \) for all \( t_2 > t_1 \).

Finally, thanks to Theorem 5.23 below, indeed, \( f_k \to f \) in \( C^{0,1}(\mathbb{R}; \mathcal{M}) \). Together with (46), it deduces that
\[
(50) \quad \sigma_k = (f_k, g_k) \to (f, g) \quad \text{in } C^{0,1}(\mathbb{R}; \mathcal{M}) \times L^{2,\infty}_{\text{loc}}(\mathbb{R}; V'),
\]
with \( \sigma = (f, g) \in \bar{\Sigma} \). \( \square \)

Similarly, we can now also consider another evolutionary system with \( \bar{\Sigma} \) as a symbol space. The family of trajectories for this evolutionary system consists of all weak solutions of the family of (28) with \( \sigma \in \bar{\Sigma} \) in \( X \):
\[
\mathcal{E}_{\Sigma}([\tau, \infty)) := \{ u(\cdot) : u(\cdot) \text{ is a weak solution on } [\tau, \infty) \}
\]
\[
\text{with } \sigma \in \bar{\Sigma} \text{ and } u(t) \in X, \forall t \in [\tau, \infty), \ \tau \in \mathbb{R},
\]
\[
\mathcal{E}_{\Sigma}((\infty, \infty)) := \{ u(\cdot) : u(\cdot) \text{ is a weak solution on } (-\infty, \infty) \}
\]
\[
\text{with } \sigma \in \bar{\Sigma} \text{ and } u(t) \in X, \forall t \in (-\infty, \infty).\]

We have the following lemmas.

**Lemma 5.10.** [CL09] The evolutionary system \( \mathcal{E}_{\Sigma} \) of the family of (28) with \( \sigma \in \bar{\Sigma} \) satisfies \( \bar{A}1 \).

**Lemma 5.11.** The evolutionary system \( \mathcal{E}_{\Sigma} \) of the family of (28) with \( \sigma \in \bar{\Sigma} \) is closed.

**Proof.** With Lemma 5.9 in hand, the proof is just the same as that of Lemma 4.12. \( \square \)

**Lemma 5.12.** Let \( \mathcal{E}_{\Sigma} \) be the evolutionary system of the family of (28) with \( \sigma \in \bar{\Sigma} \). Then \( \mathcal{E}_{\Sigma} = \mathcal{E}_{\Sigma} \).

**Proof.** Using Lemma 5.11, the argument is the same as that of Lemma 4.13. \( \square \)
Remark 5.13. In the proofs of Lemmas 5.10, 5.11 and 5.12, whenever utilizing Lemma 5.9, instead of (50), the convergence of \( \sigma_k \) in \( C^{p}((\mathbb{R}; \mathcal{M}) \times L^2_{\text{loc}}(\mathbb{R}; V')) \) is enough. Note again that \( \Sigma \) is compact in \( C^{p, u}((\mathbb{R}; \mathcal{M}) \times L^2_{\text{loc}}(\mathbb{R}; V')) \) and is metrizable in \( C^{p}((\mathbb{R}; \mathcal{M}) \times L^2_{\text{loc}}(\mathbb{R}; V')) \). Hence, \( \Sigma \) is sequentially compact in \( C^{p, u}((\mathbb{R}; \mathcal{M}) \times L^2_{\text{loc}}(\mathbb{R}; V')) \). Though it is also sequentially compact in \( C^{p}((\mathbb{R}; \mathcal{M}) \times L^2_{\text{loc}}(\mathbb{R}; V')) \) due to Corollary 5.24 below, it is not necessary for our current procedures.

Theorem 5.14. Let \( f_0 \) satisfy (29)-(31) and be translation compact in \( C^{p, u}(\mathbb{R}; \mathcal{M}) \), and \( \sigma_0 \) be translation bounded in \( L^2_{\text{loc}}(\mathbb{R}; V') \). Then the weak uniform global attractor \( \mathcal{A}_{w}^{\Sigma} \) and the weak trajectory attractor \( \mathcal{A}_{w}^{\Sigma} \) for the family of (28) with \( \sigma = (f, g) \in \Sigma \) exist, \( \mathcal{A}_{w}^{\Sigma} \) is the maximal invariant and maximal quasi-invariant set w.r.t. the corresponding evolutionary system \( \mathcal{E}_{\Sigma} \), and

\[
\mathcal{A}_{w}^{\Sigma} = \{ u(0) : u \in \mathcal{E}_{\Sigma}((-\infty, \infty)) \} = \left\{ u(0) : u \in \bigcup_{\sigma \in \Sigma} \mathcal{E}_{\sigma}((-\infty, \infty)) \right\},
\]

\[
\mathcal{A}_{w}^{\Sigma} = \bigcap_{\sigma \in \Sigma} \mathcal{E}_{\sigma}((-\infty, \infty)),
\]

\[
\mathcal{A}_{w}^{\Sigma}(t) = \left\{ u(t) : u \in \mathcal{A}_{w}^{\Sigma} \right\}, \quad \forall t \geq 0,
\]

where \( \mathcal{E}_{\sigma}((-\infty, \infty)) \) is nonempty for any \( \sigma \in \Sigma \). Moreover, \( \mathcal{A}_{w}^{\Sigma} \) satisfies the finite weak uniform tracking property and is weakly equicontinuous on \([0, \infty)\).

Proof. Utilizing Lemmas 5.11, 5.12 and 5.10, the proof is analogous to that of Theorem 4.14. \( \square \)

The existence of \( \mathcal{A}_{w}^{\Sigma} \) as well as \( \mathcal{A}_{w}^{\Sigma} \) was proved in [CL09]. Hence, we are able to obtain strong compactness of \( \mathcal{A}_{w}^{\Sigma} \) from that of \( \mathcal{A}_{w}^{\Sigma} \) by applying Corollary 3.18.

Theorem 5.15. Furthermore, if \( \sigma_0 \) is normal in \( L^2_{\text{loc}}(\mathbb{R}; V') \), then the weak uniform global attractor \( \mathcal{A}_{w}^{\Sigma} \) is a strongly compact strong uniform global attractor \( \mathcal{A}_{w}^{\Sigma} \) and the weak trajectory attractor \( \mathcal{A}_{w}^{\Sigma} \) is a strongly compact strong trajectory attractor \( \mathcal{A}_{w}^{\Sigma} \). Moreover, \( \mathcal{A}_{w}^{\Sigma} \) satisfies the finite strong uniform tracking property and is strongly equicontinuous on \([0, \infty)\).

Let \( \bar{\mathcal{E}} \) be the closure of the evolutionary system \( \mathcal{E} \). It follows from Lemma 5.12 that \( \mathcal{E} \subset \bar{\mathcal{E}} \subset \mathcal{E}_{\Sigma} \). Then, an interesting problem arises:

Open Problem 5.16. [CL09] Are the attractors \( \mathcal{A}_{w}, \mathcal{A}_{w}, \mathcal{A}_{w}^{\Sigma}, \mathcal{A}_{w}^{\Sigma} \) in Theorems 5.5 and 5.15 identical?

As indicated in Theorems 3.24 and 3.25, when the solutions of (28) with \( \sigma \in \Sigma \) are unique, the answer is positive. Otherwise, the negative answer would imply that the solutions are not unique and, moreover, that the attractors \( \mathcal{A}_{w}^{\Sigma} \) and \( \mathcal{A}_{w}^{\Sigma} \) for the auxiliary family of (28) with \( \sigma \in \Sigma \) do not satisfy the minimality property w.r.t. \( \mathcal{d}_{w} \)-attracting and \( \mathcal{d}_{C([0,\infty); X_{\sigma}]} \)-attracting, respectively, for the evolutionary system \( \mathcal{E} \) corresponding to the original (28) with the fixed \( \sigma_0 \). For more details, see [CL09, CL14].
5.1.1. **RDS with uniqueness.** Now we suppose further the following condition on the non-linearity $f_0(v, s)$.

\[
(f_0(v_1, s) - f_0(v_2, s), v_1 - v_2) \geq -C \|v_1 - v_2\|_{\mathbb{R}^N}^2, \forall v_1, v_2 \in \mathbb{R}^N, \forall s \in \mathbb{R}.
\]

It is known that the weak solutions provided by Theorem 5.2 are now unique (see e.g. [CV02]).

**Theorem 5.17.** Let $f_0$ satisfy (29)-(31), (51) and be translation compact in $C^{p,\infty}(\mathbb{R}; \mathcal{M})$, and $g_0$ be translation bounded in $L^2_{\text{loc}}(\mathbb{R}; V')$. Then the two weak uniform global attractors $A_w, A_w^\Sigma$ and the two weak trajectory attractors $\mathfrak{A}_w, \mathfrak{A}_w^\Sigma$ for (28) with the fixed $\sigma_0 = (f_0, g_0)$ and for the family of (28) with $\sigma = (f, g) \in \Sigma$, respectively, exist, $A_w$ and $A_w^\Sigma$ are the maximal invariant set and maximal quasi-invariant set w.r.t. the closure $\mathcal{E} = \mathcal{E}_\Sigma$ of the corresponding evolutionary system $\mathcal{E}$ and

\[
A_w = A_w^\Sigma = \{u(0) : u \in \mathcal{E}_\Sigma((-\infty, \infty))\} = \left\{u(0) : u \in \bigcup_{\sigma \in \Sigma} \mathcal{E}_\sigma((-\infty, \infty))\right\},
\]

\[
\mathfrak{A}_w = \mathfrak{A}_w^\Sigma = \Pi_+ \bigcup_{\sigma \in \Sigma} \mathcal{E}_\sigma((-\infty, \infty)),
\]

\[
A_w = A_w^\Sigma = \mathfrak{A}_w^\Sigma(t) = \left\{u(t) : u \in \mathfrak{A}_w^\Sigma\right\}, \quad \forall t \geq 0,
\]

where $\mathcal{E}_\sigma((-\infty, \infty))$ is nonempty for any $\sigma \in \hat{\Sigma}$. Moreover, $\mathfrak{A}_w = \mathfrak{A}_w^\Sigma$ satisfies the finite weak uniform tracking property for $\mathcal{E}_\Sigma$ and is weakly equicontinuous on $[0, \infty)$.

**Proof.** The proof is analogous to that of Theorem 4.20. By the assumptions, the associated evolutionary systems $\mathcal{E}$ and $\mathcal{E}_\Sigma$ are unique. It follows from Lemmas 5.4 and 5.11 that $\mathcal{E}$ satisfies A1 and that $\mathcal{E}_\Sigma$ is closed, respectively. Then, we obtain the theorem by Theorem 3.24. \(\square\)

**Theorem 5.18.** Furthermore, if $g_0$ is normal in $L^2_{\text{loc}}(\mathbb{R}; V')$, then the two weak uniform global attractors $A_w$ and $A_w^\Sigma$ are strongly compact strong uniform global attractors $A_w$ and $A_w^\Sigma$, and the two weak trajectory attractors $\mathfrak{A}_w$ and $\mathfrak{A}_w^\Sigma$ are strongly compact strong trajectory attractors $\mathfrak{A}_w$ and $\mathfrak{A}_w^\Sigma$, respectively. Moreover,

1. $\mathfrak{A}_w = \mathfrak{A}_w^\Sigma = \Pi_+ \bigcup_{\sigma \in \Sigma} \mathcal{E}_\sigma((-\infty, \infty))$ satisfies the finite strong uniform tracking property, i.e., for any $\epsilon > 0$ and $T > 0$, there exist $t_0$ and a finite subset $P_T^l \subset \mathfrak{A}_w|_{[0, T)}$, such that for any $t^* > t_0$, every trajectory $u \in \mathcal{E}_\Sigma([0, \infty))$ satisfies

\[
|u(t) - v(t - t^*)| < \epsilon, \quad \forall t \in [t^*, t^* + T],
\]

for some $T$-time length piece $v \in P_T^l$.

2. $\mathfrak{A}_w = \mathfrak{A}_w^\Sigma = \Pi_+ \bigcup_{\sigma \in \Sigma} \mathcal{E}_\sigma((-\infty, \infty))$ is strongly equicontinuous on $[0, \infty)$, i.e.,

\[
|v(t_1) - v(t_2)| \leq \theta (|t_1 - t_2|), \quad \forall t_1, t_2 \geq 0, \forall v \in \mathfrak{A}_w,
\]

where $\theta(l)$ is a positive function tending to zero as $l \to 0^+$.

Part of this theorem recovers the corresponding results in [Lu07]. Here we obtain in addition the existence of the strongly compact strong trajectory attractor and its corollaries.
Proof. We continue the proof of Theorem 5.17. We only need to obtain the asymptotical compactness of the evolutionary system $\mathcal{E}_\Sigma$. By Theorem 5.15, the strongly compact strong uniform global attractor $\mathcal{A}_\Sigma^\infty$ for $\mathcal{E}_\Sigma$ exists when $g_0$ is normal in $L^2_{\text{loc}}(\mathbb{R}; V')$. Hence, all conclusions follow from Theorem 3.25. □

5.2. On nonlinearity. In this subsection, we first collect properties of some kinds of interaction functions, with which (28) are studied in some previous literature (see [CV02, Lu07, CL09]). Then, we construct several counter examples that do not satisfy part of restrictions on the nonlinearity in these literature, especially do not belong to these classes of interaction functions. However, our Theorems 5.5 and 5.6 are still applicable for (28) with interaction terms being such examples. As indicated in Open Problem 5.16, it is not yet known how to obtain the same results by previous frameworks.

Definition 5.19. [CV95, CV02] Let $\Xi$ be a topological space of functions defined on $\mathbb{R}$. A function $\phi(s) \in \Xi$ is said to be translation compact in $\Xi$ if the closure
\[
\{\phi(\cdot + h) : h \in \mathbb{R}\}^\Xi,
\]
is compact in $\Xi$.

Denote by $C^{p,u}_{\text{tr.c.}}(\mathbb{R}; \mathcal{M})$, $C^{p,p}_{\text{tr.c.}}(\mathbb{R}; \mathcal{M})$ and $C_{\text{tr.c.}}(\mathbb{R}; \mathcal{M})$ the classes of translation compact functions in $C^{p,u}(\mathbb{R}; \mathcal{M})$, $C^{p,p}(\mathbb{R}; \mathcal{M})$ and $C(\mathbb{R}; \mathcal{M})$, respectively. We have the following characterizations and relationships of these spaces.

Theorem 5.20. [CV02] $\varphi(s) \in C_{\text{tr.c.}}(\mathbb{R}; \mathcal{M})$ if and only if for any $R > 0$, $\varphi(v, s)$ is bounded in $Q(R) = \{(v, s) : \|v\|_{\mathbb{R}^N} \leq R, s \in \mathbb{R}\}$, and
\[
\|\varphi(v_1, s_1) - \varphi(v_2, s_2)\|_{\mathbb{R}^N} \leq \theta(\|v_1 - v_2\|_{\mathbb{R}^N} + |s_1 - s_2|, R),
\]
\[
\forall (v_1, s_1), (v_2, s_2) \in Q(R),
\]
where $\theta(l, R)$ is a positive function tending to 0 as $l \to 0^+$. 

Theorem 5.21. [Lu07] $\varphi(s) \in C^{p,u}_{\text{tr.c.}}(\mathbb{R}; \mathcal{M})$ if and only if $\varphi(s) \in C^{p,p}_{\text{tr.c.}}(\mathbb{R}; \mathcal{M})$ and one of the following holds.
\begin{enumerate}
\item $\{\varphi(s) : s \in \mathbb{R}\}$ is precompact in $\mathcal{M}$.
\item For any $R > 0$, $\varphi(v, s)$ is bounded in $Q(R) = \{(v, s) : \|v\|_{\mathbb{R}^N} \leq R, s \in \mathbb{R}\}$, and
\[
\|\varphi(v_1, s_1) - \varphi(v_2, s_2)\|_{\mathbb{R}^N} \leq \theta(\|v_1 - v_2\|_{\mathbb{R}^N}, R), \quad \forall (v_1, s_1), (v_2, s_2) \in Q(R),
\]
where $\theta(l, R)$ is a positive function tending to 0 as $l \to 0^+$. 
\end{enumerate}

By Arzelà-Ascoli compactness criterion, the property 1 or 2 implies that the family $\{\varphi(\cdot, s) : s \in \mathbb{R}\}$ is equicontinuous on any ball $\{v \in \mathbb{R}^N : \|v\|_{\mathbb{R}^N} \leq R\}$ with radius $R > 0$.

Theorem 5.22. [Lu07] Let $\varphi \in C^{p,u}_{\text{tr.c.}}(\mathbb{R}; \mathcal{M})$ and denote by $\mathcal{H}^{p,u}(\varphi)$ the closure of its translation family $\{\varphi(\cdot + h) : h \in \mathbb{R}\}$ in $C^{p,u}(\mathbb{R}; \mathcal{M})$. Then
\begin{enumerate}
\item $\mathcal{H}^{p,u}(\varphi) \subset C^{p,u}_{\text{tr.c.}}(\mathbb{R}; \mathcal{M})$, moreover, $\mathcal{H}^{p,u}(\varphi) \subset \mathcal{H}^{p,u}(\varphi)$, $\forall \varphi_1 \in \mathcal{H}^{p,u}(\varphi)$.
\item $\varphi_1 \in \mathcal{H}^{p,u}(\varphi)$ satisfies (52) for the same $\theta(l, R)$.
\item The translation group $\{T(t)\}$ is invariant and continuous on $\mathcal{H}^{p,u}(\varphi)$ in the topology of $C^{p,u}(\mathbb{R}; \mathcal{M})$.
\end{enumerate}
**Theorem 5.23.** Let $\varphi \in C^{p,u}_{tr,c}(\mathbb{R}; \mathcal{M})$ and denote by $\mathcal{H}^{p,u}(\varphi)$ the closure of its translation family $\{\varphi(\cdot + h) : h \in \mathbb{R}\}$ in $C^{p,u}(\mathbb{R}; \mathcal{M})$. Let $\{\varphi_n\} \subset \mathcal{H}^{p,u}(\varphi)$ be such that $\varphi_n \to \varphi_0$ in $C^{p,p}(\mathbb{R}; \mathcal{M})$. Then

1. $\varphi_0 \in \mathcal{H}^{p,u}(\varphi)$.
2. $\varphi_n \to \varphi_0$ in $C^{p,u}(\mathbb{R}; \mathcal{M})$.

**Proof.** Note that, the compactness of $\mathcal{H}^{p,u}(\varphi)$ in $C^{p,u}(\mathbb{R}; \mathcal{M})$ implies its compactness in $C^{p,p}(\mathbb{R}; \mathcal{M})$. As indicated in the footnote 13, $\mathcal{H}^{p,u}(\varphi)$ endowed with the topology of $C^{p,p}(\mathbb{R}; \mathcal{M})$ is metrizable, and then the corresponding metric space is complete. Hence, $\varphi_0 \in \mathcal{H}^{p,u}(\varphi)$.

Due to Theorem 5.21, $\varphi$ satisfies (52). Now fix $s \in \mathbb{R}$, $R > 0$, and denote by $B_R := \{v \in \mathbb{R}^N : \|v\|_{\mathbb{R}^N} \leq R\}$. For any $\varepsilon > 0$, there exists $\delta > 0$ such that

\[ \|\varphi_0(v_1, s) - \varphi_0(v_2, s)\|_{\mathbb{R}^N} \leq \varepsilon/3, \quad \forall v_1, v_2 \in B_R, \|v_1 - v_2\|_{\mathbb{R}^N} \leq \delta, \]

and the function $\theta(l, R)$ in (52) is smaller or equal to $\varepsilon/3$ for $l \leq \delta$. For such $\delta$, there exists a finite number of points $\{v_1, \ldots, v_n\} \subset B_R$ being a $\delta$-net of $B_R$, that is, for any $v \in B_R$, there exists some $v_j$ satisfying $\|v - v_j\|_{\mathbb{R}^N} \leq \delta$. We have that, for any $v \in B_R$,

\[ \|\varphi_n(v, s) - \varphi_0(v, s)\|_{\mathbb{R}^N} \leq \|\varphi_n(v, s) - \varphi_n(v_j, s)\|_{\mathbb{R}^N} + \|\varphi_n(v_j, s) - \varphi_0(v_j, s)\|_{\mathbb{R}^N} + \|\varphi_0(v_j, s) - \varphi_0(v, s)\|_{\mathbb{R}^N} \]

Thanks to the conclusion 2 of Theorem 5.22, it follows that

\[ \|\varphi_n(v, s) - \varphi_n(v_j, s)\|_{\mathbb{R}^N} \leq \varepsilon/3, \quad \forall n \in \mathbb{N}. \]

Combining (53), (54) and the fact that $\varphi_n \to \varphi_0$ in $C^{p,p}(\mathbb{R}; \mathcal{M})$, we obtain

\[ \|\varphi_n(v, s) - \varphi_0(v, s)\|_{\mathbb{R}^N} \leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon, \quad \forall v \in B_R, \]

for sufficiently large $n$. This means that, for every $s \in \mathbb{R}$, $R > 0$,

\[ \max_{\|v\|_{\mathbb{R}^N} \leq R} \|\varphi_n(v, s) - \varphi_0(v, s)\|_{\mathbb{R}^N} \to 0, \quad \text{as } n \to \infty, \]

which is $\varphi_n(v, s) \to \varphi_0(v, s)$ in $C^{p,u}(\mathbb{R}; \mathcal{M})$.

We complete the proof. \(\square\)

**Corollary 5.24.** Let $\varphi \in C^{p,u}_{tr,c}(\mathbb{R}; \mathcal{M})$ and denote by $\mathcal{H}^{p,u}(\varphi)$ the closure of its translation family $\{\varphi(\cdot + h) : h \in \mathbb{R}\}$ in $C^{p,u}(\mathbb{R}; \mathcal{M})$. Then, $\mathcal{H}^{p,u}(\varphi)$ is sequentially compact in $C^{p,u}(\mathbb{R}; \mathcal{M})$.

**Proof.** Take a sequence $\{\varphi_n\} \subset \mathcal{H}^{p,u}(\varphi)$. $\{\varphi_n\}$ is compact in the metrizable topological space $(\mathcal{H}^{p,u}(\varphi), C^{p,p}(\mathbb{R}; \mathcal{M}))$. Hence, there is a subsequence $\{\varphi_{n_j}\}$, such that $\varphi_{n_j} \to \varphi_0$ in $C^{p,p}(\mathbb{R}; \mathcal{M})$ with $\varphi_0 \in C^{p,p}(\mathbb{R}; \mathcal{M})$ as $n_j \to \infty$. By Theorem 5.23, $\varphi_0 \in \mathcal{H}^{p,u}(\varphi)$ and the convergence is actually valid in $C^{p,u}(\mathbb{R}; \mathcal{M})$. \(\square\)

Let $C_b(\mathbb{R}; \mathcal{M})$ be the space of bounded continuous functions with values in $\mathcal{M}$ and endowed with the uniform convergence topology on $\mathbb{R}$. We have the following relationships.

**Theorem 5.25.** \cite{Lu07} $C_{tr,c}(\mathbb{R}; \mathcal{M}) \subset C^{p,u}_{tr,c}(\mathbb{R}; \mathcal{M}) \subset C^{p,p}(\mathbb{R}; \mathcal{M}) \subset C_b(\mathbb{R}; \mathcal{M})$ with all inclusions being proper and the former three sets being closed in $C_b(\mathbb{R}; \mathcal{M})$. \(\square\)
Now, we construct several examples in $C(\mathbb{R} \times \mathbb{R}; \mathbb{R})$ that satisfy conditions (30)-(31), but are not translation compact in $C^{p,p}_{\text{tr.c.}}(\mathbb{R}; C(\mathbb{R}; \mathbb{R}))$ and do not satisfy (51) as well. Note again that, Theorems 5.5 and 5.6 are applicable for (28) with such interaction functions.

In the following examples, let $T := \max\{0, t\}, \quad t \in \mathbb{R},$

and let $p \geq 2$ and $\mathcal{M} = C(\mathbb{R}, \mathbb{R})$.

**Example I.**

$$f(v, t) = \begin{cases} |v|^{p-2}v, & \text{if } v \leq 0, \\ -(1 + T)v, & \text{if } 0 \leq v \leq \frac{1}{1+T}, \\ v - \frac{1}{1+T}|v|^{p-1} - 1, & \text{if } v > \frac{1}{1+T}. \end{cases}$$

Note that, the family $\{f(\cdot, t) : t \in \mathbb{R}\}$ is not equicontinuous on $[0, 1]$, which means that $f(v, t)$ does not satisfy (52). The fact that

$$\partial_v f \left( \frac{1}{2(1+n)}, n \right) = -(1 + n) \rightarrow -\infty, \quad \text{as } n \rightarrow \infty,$$

implies that condition (51) does not hold for $f(v, t)$. Moreover, the pointwise limit function of $f(\cdot, t)$, as $t \rightarrow +\infty$, is a discontinuous function,

$$f_{\infty}(v) = \begin{cases} |v|^{p-2}v, & \text{if } v \leq 0, \\ |v|^{p-1} - 1, & \text{if } v > 0. \end{cases}$$

Hence, $f(v, t)$ does not even belong to $C^{p,p}_{\text{tr.c.}}(\mathbb{R}; \mathcal{M})$. In fact, for any sequence $\{f(\cdot, t_n)\}$ with $t_n \rightarrow +\infty$, the pointwise limit is $f_{\infty}$.

**Example II.**

$$f(v, t) = \begin{cases} |v + 2\pi|^{p-2}(v + 2\pi), & \text{if } v \leq -2\pi, \\ \rho(v) \sin(1 + T)v, & \text{if } -2\pi < v < 2\pi, \\ |v - 2\pi|^{p-1}, & \text{if } v \geq 2\pi, \end{cases}$$

where $\rho(\cdot)$ is a continuous function supported on $[-2\pi, 2\pi]$. For instance, $\rho(\cdot)$ is an infinitely differentiable function supported on $(-2\pi, 2\pi)$ and equals to 1 on $[-\pi, \pi]$. Note again that, the family $\{f(\cdot, t) : t \in \mathbb{R}\}$ is not equicontinuous in $[-2\pi, 2\pi]$. Hence, $f(v, t)$ does not satisfy (52). Moreover, there is no a pointwise limit function of any sequence $\{f(\cdot, t_n)\}$, as $t_n \rightarrow +\infty$. Therefore, $f(v, t) \notin C^{p,p}_{\text{tr.c.}}(\mathbb{R}; \mathcal{M})$. The fact that $\partial_v f(\cdot, \cdot)$ has no a uniform lower bound in $[-\pi, \pi] \times \mathbb{R}$ implies that $f(v, t)$ does not satisfy (51) either.

**Example III.**

$$f(v, t) = \begin{cases} |v + 2|^{p-2}(v + 2), & \text{if } v \leq -2, \\ \rho(v, T) \sin T^2, & \text{if } -2 < v < 2, \\ |v - 2|^{p-1}, & \text{if } v \geq 2, \end{cases}$$
where $\rho(\cdot, \cdot)$ is an infinitely differentiable function that $\rho(\cdot, t)$ is supported on
\[
\left(-2 + \frac{1}{2(1+T)}, 2 - \frac{1}{2(1+T)}\right)
\]
for any $t \in \mathbb{R}$, and equals to 1 on
\[
\left(-2 + \frac{1}{1+T}, 2 - \frac{1}{1+T}\right).
\]
Similarly, there exists no a uniform lower bound to $\partial_v f(\cdot, \cdot)$ in $[-2, 2] \times \mathbb{R}$ and the family $\{f(\cdot, t) : t \in \mathbb{R}\}$ is not equicontinuous on $[-2, 2]$. Hence, $f(v, t)$ does not satisfy (51) and (52). Moreover, there is also no a pointwise limit function of any sequence $\{f(\cdot, t_n)\}$, as $t_n \to +\infty$. Therefore, $f(v, t) \notin C^{1,1,c}_{tr.c.}(\mathbb{R}; M)$.

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