CFT duals of Kerr-Taub-NUT and beyond.

Malcolm J. Perry, a,b,c Maria J. Rodriguez d,e

aDepartment of Physics and Astronomy, Queen Mary University of London, Mile End Road, London E1 4NS, UK.
bDAMTP, Centre for Mathematical Sciences, Wilberforce Road, Cambridge, CB3 0WA, UK.
cTrinity College, Cambridge, CB2 1TQ, UK.
dDepartment of Physics, Utah State University, 4415 Old Main Hill Road, UT 84322, USA.
eInstituto de Fisica Teorica UAM/CSIC, Universidad Autonoma de Madrid, 13-15 Calle Nicolas Cabrera, 28049 Madrid, Spain.

E-mail: malcolm@maths.cam.ac.uk, maria.rodriguez@gmail.com

Abstract: The duality relating the four-dimensional Kerr-Taub-NUT black hole to a thermal two-dimensional CFT with central charges $c_L = c_R = 12J_0$ is analyzed in detail, generalizing an argument given recently for Kerr within the soft-hair approach. The hidden conformal symmetry is realized in the form of $Vir_L \times Vir_R$ diffeomorphisms which act non-trivially on the black hole horizon. Semiclassical formulae are derived for the temperature and central charges of the dual CFT. Assuming the applicability of the Cardy formula, these CFT quantities precisely reproduce the macroscopic Bekenstein-Hawking area law. Various further generalizations including the complete family of black holes in four dimensions are discussed.
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1 Introduction

Many black holes in 4-spacetime dimensions, including Kerr black holes with spin $J$, are conjectured to have a dual two-dimensional conformal field theory (2D CFT) description with central charges $c_L = c_R = 12J$. If one assumes the applicability of the Cardy formula, the central charges reproduce the macroscopic area-entropy law. The synthesis between 4D-gravity and 2D CFT can be obtained in part from the fact that we can relate the description of the four-dimensional black hole horizon as a quotient of a deformed AdS$_3$ to the conformal field theory description $^1$. This turns out to be enough information to show that the extremal 4D-black holes are dual to two-dimensional CFT $^[1, 2]$. 

Another fascinating result stemming from the symmetry analysis of the black hole Klein-Gordon equation (rather than an analysis of the spacetime) led to the observation that generic non-extreme black holes are dual to finite 2D CFT at finite temperatures $^[3]$. The 'hidden conformal symmetry' which act on solutions of the scalar wave equation at low frequencies seems to persist for generic non-extreme values of the black hole spin $J$.

A more refined version of the duality between gravity in 4D and a CFT comes from the so-called soft hair approach. It was recently shown in $^[4]$ that the hidden conformal symmetry in the form of $Vir_L \times Vir_R$ diffeomorphisms act non-trivially on the black

$^1$More precisely, the black hole near bifurcation surface metrics are quotients of the deformed AdS$_3$ geometries and a ‘θ-leave’ over the bundle.
hole horizon generating soft hair. This soft hair is realized as finite covariant right(left)-moving Iyer-WaldVirasoro charges and the central terms. To guarantee the integrability and associativity of the charges a Wald-Zoupas counterterm was proposed. Despite these developments there has been relatively little effort to prove uniqueness of the counterterm. Assuming the validity of the Cardy formula, it is now clear that with the Wald-Zoupas counterterm originally proposed in [4] only reproduces the area law for the Kerr(Newsman) black hole [4, 5]. Already in [6], the authors found that for AdS-Kerr black holes an alternative proposal to the Wald-Zoupas counterterm which was consistent with all previous works [2, 7]. Further, this generalized term produced the dual 2D CFT at temperatures $(T_L, T_R)$ and central charges consistent with the thermodynamics of these black holes.

Previous potentially related attempts to validate the Wald-Zoupas counterterm for 4D black hole solutions include [8, 9].

In this paper we take steps towards a more global understanding of the counterterm uniqueness by studying a generalized version of the counterterm valid for the complete family of black holes in 4D. The linearized charge contribution from the proposed counterterm to the charges is defined by

$$
\delta Q_{ct} = \frac{1}{16\pi} \int_{\partial \Sigma} F_{(ct)} \, d\Sigma^{ab}
$$

where $N$ is the volume two-form on the normal bundle to the $\Sigma_{bif}$.

$$
F_{(ct)} = -4N d^c \nabla_{\xi} (\xi_a h^d b) ,
$$

where $\xi$ is associated to a diffeomorphisms acting on the horizon, and $h$ an on-shell linearized fluctuation around a fixed black hole background.

Our most interesting results concerns the soft hair charges of the Kerr-Taub-NUT black hole spacetimes with spin $J_0$. We find that the counterterm (1.1) satisfies the known consistency requirements and yields central charges $c_L = c_R = 12 J_0$. One would like to go beyond this and explicitly identify the 2D CFT description within the complete family of Plebanski-Demianski black hole classes. In all cases, employing the generalized counterterm (1.1), within the soft hair and monodromy approach we are able to determine the central charges $c_L = c_R$ and dual temperatures. We then apply the thermodynamic Cardy formula relating the microscopic entropy of a unitary CFT to its temperature and central charge. The resulting entropy agrees exactly with the area-entropy law, providing corroboration for our proposal that generic black holes in 4D-gravity are dual to a two-dimensional CFT.

We can list the proposed central charges $c_L = c_R$ and dual temperatures in 4D-gravity/2D CFT be extreme or not. Table 1 summarizes the results obtained using Kerr/CFT, soft hair approach, monodromy technique during the past years. For the details on this published work we refer the reader to the papers cited 2.

This paper is organized as follows. In section 2, we will review the Kerr-Taub-NUT black hole metric. We will also present the Smarr law and first law for the solution. In section 3, we will briefly discuss the so-called monodromy technique [10] that also serves to

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2The temperatures and central charges for Kerr-Taub-NUT are derived in the present paper.
identify the conjugate variables, and dual 2D CFT quantities. The conformal coordinates in which the Virasoro action takes the simple form are presented in section 4. The linearized covariant charges are also found in this section and the central terms computed. The identifications we previously derived for the \((T_L,T_R)\) temperatures are also shown to arise from imposing the first law of thermodynamics in section 5. In Section 6 we take the rotating Taub-NUT black hole solution with an overall distribution of the Misner string, and show that similar identifications yield a 2D CFT dual. Further generalizations including the computation of a well-defined charge for the most general 4D black hole solution of General Relativity are discussed in Section 7. In the discussion section we present some final remarks. Further technical points are relegated to the appendices.

Throughout this paper we use units such that \(c = k = G = \hbar = 1\).
2 Kerr-Taub-NUT

In this section we review the Kerr-Taub-NUT black hole and fix our notation. This will provide the setup for studying the covariant charges and computing the central terms in Section 4.

We begin by considering the four-dimensional Kerr-Taub-NUT black hole solution which has also been recently the center of many other works \cite{14, 15}. The Kerr-Taub-NUT metric in Boyer-Lindquist co-ordinates $(t, r, \theta, \phi)$ is

$$ds^2 = \rho^2 \left( \frac{dr^2}{\Delta} + d\theta^2 \right) + \frac{\sin^2 \theta}{\rho^2}(-a \, dt + \Sigma \, d\phi)^2 - \frac{\Delta}{\rho^2}(dt - \Xi \, d\phi)^2 \quad \text{(2.1)}$$

where

$$\rho^2 = r^2 + (N + a \cos \theta)^2, \quad \Delta = r^2 - 2mr - N^2 + a^2, \quad \Xi = a \sin^2 \theta - 2N \cos \theta, \quad \Sigma = r^2 + N^2 + a^2 \quad \text{(2.2)}$$

This solution to Einstein’s vacuum equations $R_{\mu\nu} - (1/2) R g_{\mu\nu} = 0$ carries three parameters: $m, a, N$ which represent respectively the mass, rotation and nut charge. The location of the outer/inner horizons defined via $\Delta = 0$ are $r_{\pm} = M \pm \sqrt{M^2 - a^2 + N^2}$.

With these geometric quantities in hand, let us now turn towards thermodynamics. The corresponding thermodynamic quantities can be found in \cite{16} \footnote{The Kerr-Taub-NUT solution considered in our paper corresponds to $s = 0$ solutions in \cite{16} containing two Misner strings that are ‘symmetrically distributed’ making both axes equally singular.}. The Kerr-Taub-NUT black hole exhibits a mass, angular momentum and angular velocity defined by

$$M = m, \quad J = \frac{a\Sigma_+}{2r_+} = a \left( M + \frac{N^2}{r_+} \right), \quad \Omega_\pm = \frac{a}{\Sigma_\pm} \quad \text{(2.4)}$$

The entropy and temperature can be assigned as

$$S_\pm = \pi \Sigma_\pm, \quad T_\pm = \frac{\kappa_\pm}{2\pi} = \frac{\Delta'(r_\pm)}{4\pi \Sigma_\pm}, \quad \text{(2.5)}$$

and NUT charge and Misner potential by

$$N_\pm = -\frac{2\pi N^2 (N \mp a)}{r_+}, \quad \psi_\pm = \frac{1}{8\pi N}, \quad \text{(2.6)}$$

where $\Sigma_\pm = \Sigma(r_\pm) = r_\pm^2 + a^2 + N^2$ and $\Delta'(r_\pm) = \pm (r_+ - r_-)$. These thermodynamic quantities obey a generalized first law:

$$\delta M = T \delta S + \Omega \delta J + \psi_+ \delta N_+ + \psi_- \delta N_-, \quad \text{(2.7)}$$

together with the corresponding Smarr relation

$$M = 2(T S + \Omega J + \Psi_+ N_+ + \Psi_- N_-). \quad \text{(2.8)}$$

Let us also note the following two interesting facts about the total angular momentum $J$. The total angular momentum $J$ actually differs from the asymptotic charge $J_0 = aM$ by the Misner string contribution $J_s = J - J_0 = aN^2/r_+$. Both $J_s$ and $J_0$ are also in agreement with the values directly computed via the Komar integrals. As we will now show the asymptotic charge $J_0$ will become the relevant quantity for the CFT description.
3 Monodromies and 2D CFT

The monodromy technique developed in [10] relies on the monodromy data of the Klein-Gordon equation in a curved background to identify the \((T_L,T_R)\) temperatures of the dual 2D CFT. As we now describe, using the monodromy technique we are able to both successfully apply the method to a larger body of data and to derive the conjugate 2D CFT values for the Kerr-Taub-NUT black hole solution. In this section we also describe, assuming the Cardy formula, the corresponding central charge for this solution.

The focus in this section is the Klein-Gordon equation for a scalar massless field

\[
\frac{1}{\sqrt{-g}} \partial_{\mu}(\sqrt{-g} g^{\mu\nu} \partial_{\nu} \Psi) = 0. \tag{3.1}
\]

One can easily verify the equation is separable in the Kerr-Taub-NUT background (2.1). The radial equation for \(\Psi(t, \phi, r, \theta) = e^{-i\omega t + im\phi} R(r) S(\theta)\) reduces to

\[
\partial_r [\Delta \partial_r R(r)] + \left[ K_l + \frac{(\Sigma \omega - am)^2}{\Delta} \right] R(r) = 0, \tag{3.2}
\]

Here \(K_l\) is the separation constant and the functions \(\Delta, \Sigma\) are defined in (2.2). This equation contains three singularities: two regular singular points located at the outer/inner horizons \(r = r_{\pm}\) and an irregular singular point at \(r = \infty\). Solutions to (3.2) will have branch cuts at the regular singular points. One can immediately see that the series expansion of the two linearly independent solutions around \(r = r_{\pm}\) points can be expanded respectively as

\[
R(r) = (r - r_{\pm})^{\pm \alpha_+} [1 + O(r - r_{\pm})] \quad \text{and} \quad R(r) = (r - r_{\pm})^{\pm \alpha_-} [1 + O(r - r_{\pm})]. \tag{3.3}
\]

The monodromies at each of these regular singular points are:

\[
\alpha_{\pm} = \frac{(\omega - \Omega_{\pm} m)}{2(\pm \kappa_{\pm})}. \tag{3.4}
\]

This monodromy data possesses highly intricate but universal properties which follow from geometry alone. Moreover, recent analyses indicate that the monodromy data encodes information about black hole thermodynamics, hidden conformal symmetry and evidence for a 2D CFT description of the thermal properties of black hole microstate [17–19].

Following [10] we consider instead the linear combination of the monodromies \(^4\) for the energy eigenvalues \(\omega_{L,R}\) of \((i\partial_{t_R}, i\partial_{t_L})\)

\[
\omega_L = \alpha_+ - \alpha_-, \quad \omega_R = \alpha_+ + \alpha_. \tag{3.5}
\]

with \(t_{L,R}\) precisely the coordinates in the monodromy basis. Thus, we can write

\[
\Psi(t, \phi, r, \theta) = e^{-i\omega t + im\phi} R(r) S(\theta) = e^{i\omega_L t_L - i\omega_R t_R} R(r) S(\theta) \tag{3.6}
\]

using the explicit form of the monodromies (3.4) we find the conjugate variables \(^5\)

\[
t_L = 2\pi T_L \phi - \frac{1}{2M} t, \quad t_R = 2\pi T_R \phi, \tag{3.7}
\]

\(^4\)Note that the monodromy definitions here \(\alpha_{\pm}\) are related to the definitions in [20] via \(\pm i2\alpha_{\pm} = \theta_{\pm}\).

\(^5\)In references [3] the conjugate variable are defined at \(t_{\pm}\) such that the relation \(e^{-i\omega_L t_L - i\omega_R t_R} = e^{-\omega_L t_R - \omega_R t_L}\). These are in agreement with our notation in that \(t_- = -t_L\) and \(t_+ = +t_R\).
with \((t, \phi)\) the Boyer-Lindquist coordinates and right/left temperatures
\[
T_R = \frac{r_+ - r_-}{4\pi a}, \quad T_L = \frac{M^2 + N^2}{2\pi a M} = \frac{(r_+ + r_-)^2 + 4N^2}{4\pi a(r_+ + r_-)}.
\]

For vanishing NUT charge \(N = 0\) we recover the Kerr black hole results [4]. Another interesting case is the extreme limit (see Table 1) for which \(T_R \to 0\) as \(r_+ = r_-\).

Having found the conjugate CFT temperatures for the Kerr-Taub-NUT black hole, we would like to understand whether or not there is some way for us to recover the Bekenstein-Hawking formula for black hole entropy. Employing Cardy formula for a chiral CFT, we find the entropy obeys
\[
S_\pm = S_{\text{Cardy}} = \frac{\pi^2}{3} c (T_L \pm T_R), \quad (3.9)
\]
provided the central charge \(c = c_L = c_R\) for the black hole geometry obeys
\[
c = 12aM = 12J_0. \quad (3.10)
\]
The CFTs have many universal properties that one would like to map. So far in our explicit matching between gravity and the CFT, we have found agreement between the Cardy and the Bekenstein-Hawking entropy formulas. Rather than trying to prove that the Cardy formula is applicable, we take the precise entropy agreement resulting from our monodromy analysis as the first piece of evidence of this match. We bring this numerological observation a few steps closer to an actual explanation of the entropy. Within the soft-hair approach [4] we now turn to the vector fields which generate (local) symmetries of the near-bifurcation surface Kerr-Taub-NUT geometry and compute the non-vanishing boundary charges. More evidence will appear below in section 5.

4 Conformal coordinates and Covariant charges

In this section we derive the relation \(c_R = 12J_0\) for the central charge of Kerr-NUT black holes. To this end, following the construction of linearized covariant charges associated to the diffeomorphisms acting on the horizon [4] we evaluate the central term in the Virasoro charge algebra.

First, we must define conformal coordinates that are well-adapted to an analysis of the 4D black hole mirroring that of the 3D BTZ black holes [21]. The richness follows in part from the fact that 4D black hole geometries contain locally on the horizon a three-dimensional metric that is a quotient of a deformed AdS3. This resonates the duality between conformal field theory and quantum gravity on AdS3, as well as the central charge of the conformal field theory, derived some time ago in [22]. To make this manifest for the Kerr-Taub-NUT black holes (2.1) we can now conjecture the conformal coordinates \((w^\pm, y)\) which seem to most clearly exhibit the conformal structure as follows
\[
w^+ = R(r) e^{t_R}, \quad w^- = R(r) e^{t_L}, \quad y = Q(r) e^{(t_L + t_R)/2} \quad (4.1)
\]
with \((t_L, t_R)\) defined in (3.7). Applying the coordinate transformation (4.1) with \(R^2+Q^2 = 1\) and the functions \(R^2(r) = (r-r_+)/r-r_-\) we find that the black hole metric around the bifurcation surface \(w^\pm = 0\) (to leading and sub-leading order) becomes

\[
\begin{align*}
    ds^2 &= 4\rho_+^2 dw^+ dw^- + \frac{16M^2 a^2 \sin^2 \theta}{y^2 \rho_+^2} dy^2 + \rho_+^2 d\theta^2 \\
    &- \frac{8M w^+}{y^2 \rho_+^2} (r_+ - r_-) (\Sigma_+ + 2Na \cos \theta) dw^- dy \\
    &+ \frac{64M^2 a^2 \sin^2 \theta}{y^3} w^- \left[ \frac{(r_+ + M)}{4M \rho_+^2} - \frac{(4\pi M T_L - \Xi)(4\pi M (T_L + T_R) - \Xi)}{8M^2 \rho_+^2 \sin^2 \theta} \right] dw^+ dy \\
    &+ \ldots,
\end{align*}
\]

(4.2)

where

\[
\rho_+^2 \equiv \rho(r_+)^2 = r_+^2 + (N + a \cos \theta)^2, \quad \Sigma_+ = r_+^2 + N^2 + a^2.
\]

Interestingly, this limit is finite only for the specific choice of (3.8). Moreover, this approach implies a very specific form of the geometry on the black hole bifurcation surface at leading order (4.2); for Kerr-Taub-NUT the ‘\(\theta\)-leaves’ of fixed polar angle take the form of a quotient of a deformed \(AdS_3\).

For future reference, a quantity that will be later employed in the computations of the surface integrals involves the volume element at leading order on the bifurcation surface

\[
\epsilon_{+-y\theta} = \frac{8aM \sin \theta \rho_+^2}{y^3} + \ldots.
\]

(4.4)

as well as the inverse metric

\[
\begin{align*}
    g^{yy} &\sim \frac{y^2 \rho_+^2}{16M^2 a^2 \sin^2 \theta}, \quad g^{\theta\theta} \sim \frac{1}{\rho_+^2}, \quad g^{+-} \sim \frac{y^2}{2\rho_+^2}, \\
    g^{+y} &\sim \frac{(r_+ - r_-)}{8M^2 \rho_+^2 \sin^2 \theta} (\Sigma_+ + 2Na \cos \theta) w^+ y, \\
    g^{-y} &\sim \left[ \frac{(r_+ + M)}{4M \rho_+^2} - \frac{(4\pi M T_L - \Xi)(4\pi M (T_L + T_R) - \Xi)}{8M^2 \rho_+^2 \sin^2 \theta} \right] w^- y.
\end{align*}
\]

(4.5)

**Covariant Charge**

In the previous subsection a set of conformal coordinates were proposed. These coordinates are well-adapted in that they make the deformed quotients of \(AdS_3\) on the black hole bifurcation surface geometry explicit for Kerr-Taub-NUT black holes. Similar results were reported for Kerr and Ker–Newman black holes [4, 5]. The main feature to notice is that at leading order on the bifurcation surface the metric (4.2) is invariant under the vector fields

\[
\begin{align*}
    \zeta_0 &= w^+ \partial_+ + \frac{1}{2} y \partial_y, \quad \bar{\zeta}_0 = w^- \partial_- + \frac{1}{2} y \partial_y \\
    \zeta_n &= \epsilon_n \partial_+ + \frac{1}{2} \partial_+ \epsilon_n y \partial_y, \quad \bar{\zeta}_n = \epsilon_n \partial_- + \frac{1}{2} \partial_- \epsilon_n y \partial_y.
\end{align*}
\]

(4.6)

We can consider more general conformal vector fields

\[
\begin{align*}
    \zeta_n &= \epsilon_n \partial_+ + \frac{1}{2} \partial_+ \epsilon_n y \partial_y, \quad \bar{\zeta}_n = \epsilon_n \partial_- + \frac{1}{2} \partial_- \epsilon_n y \partial_y.
\end{align*}
\]

(4.7)
and restrict the full set of functions \((\epsilon, \bar{\epsilon})\) so that \((\zeta, \bar{\zeta})\) are invariant under \(2\pi\) azimuthal rotations is
\[
\epsilon_n = 2\pi T_R (w^+) \exp^{\frac{2\pi n}{2\pi T_R}} , \quad \bar{\epsilon}_n = 2\pi T_L (w^-) \exp^{\frac{2\pi n}{2\pi T_L}} .
\]
(4.8)
Taking \(\zeta_n \equiv \zeta(\epsilon_n)\) and \(\bar{\zeta}_n = \bar{\zeta}(\epsilon_n)\), one can easily verify that the vector fields (4.7) obey the Lie bracket algebra
\[
[\zeta_m, \zeta_n] = i(n-m)\zeta_{m+n} , \quad [\bar{\zeta}_m, \bar{\zeta}_n] = i(n-m)\bar{\zeta}_{m+n} .
\]
(4.9)
and the two set commuting with another
\[
[\zeta_m, \bar{\zeta}_n] = 0 .
\]
(4.10)
The generator of a diffeomorphism \(\zeta\) is a conserved charge \(Q\). Under Dirac brackets, the charges obey the same algebra as the symmetries themselves, up to a possible central term \(K_{m,n}\). For well-defined integrable charges one has for the the Dirac bracket algebra
\[
\{Q_n, Q_m\} = (m-n)Q_{m+n} + K_{m,n} .
\]
(4.11)
where the central term is given by
\[
K_{m,n} = \delta_m Q(\zeta_n, \mathcal{L}_{\zeta_n} g, g) .
\]
(4.12)
The interpretation of \(\delta_m Q\) is the infinitesimal charge differences between neighboring geometries \(g_{\mu\nu}\) and \(g_{\mu\nu} + h_{\mu\nu}\) with the variation explicitly given by \(h_{\mu\nu} = \mathcal{L}_{\zeta_n} g_{\mu\nu}\) and \(h = h^{ab} g_{ab}\). Moreover, under certain conditions, it has been proven that the central term must be constant on the phase space and given by
\[
K_{m,n} = \frac{c_R m^3}{12} \delta_{m+n,0} .
\]
(4.13)
for some constant central charge \(c_R\). The last step in this section involves working out the central charge \(c_R\) by computing the central term in the Virasoro charge algebra \(\delta Q\).

The general form for the linearized charge associated to a diffeo \(\zeta\) on a surface \(\Sigma\) with boundary \(\partial \Sigma\) is
\[
\delta_m Q = \delta Q_{IW} + \delta Q_{ct} .
\]
(4.14)
This soft hair is realized as finite covariant right(left)- moving Iyer-Wald Virasoro charges \(\delta Q_{IW}\) and the central terms. To guarantee the integrability and associativity of the charges we consider the linearized charge \(\delta Q_{ct}\) contribution defined in (1.1) from a generalized version of the counterterm valid for the Kerr-Taub-NUT family of black holes in 4D. This term reduces exactly to the Wald-Zoupas counterterm for Kerr black holes in \([4, 5]\), and more importantly, the same counterterm proposed for AdS-Kerr black holes in \([6]\).

On the one hand, the Iyer-Wald linear charge contribution yields
\[
\delta Q_{IW} = \frac{1}{16\pi} \int_{\partial \Sigma} \ast F_{IW} = \frac{1}{16\pi} \int d\theta dw^+ ( -4\epsilon_y + \theta - h^y - \tilde{\zeta} y \Gamma_{y-} ) .
\]
(4.15)
And, on the other hand, we find a linear charge contribution from the counterterm

\[ \delta Q_{ct} = \frac{1}{16\pi} \int_{\partial \Sigma} F_{(ct)\,ab} d\Sigma^{ab} \]  
(4.16)

where \( N \) is the volume two-form on the normal bundle to the \( \Sigma_{bif} \).

\[ F_{(ct)\,ab} = -4N_a^c \nabla_c (\zeta_{[a} h^{d\,b]} \), \]
(4.17)

Note that the addition of this counterterm is justified to achieve integrability. The nonzero contributions to \( K_{n,m} \) come only from

\[ \delta Q = \frac{1}{16\pi} \int d\theta dw^+ \epsilon_{\theta^+ \gamma} (F_{(IW)}^{\gamma^+} + F_{(ct)}^{\gamma^+}) \]
(4.18)

We find that the integrand

\[ F_{(IW)}^{\gamma^+} = 4h^{\gamma^-} \zeta^y \Gamma^{\gamma^-}, \]
(4.19)

and considering \( N_{+} = 1 \), \( N_{-} = -1 \) then

\[ F_{(ct)}^{\gamma^+} = -2\nabla_+ (\zeta^- h^{\gamma^+}) + 2\nabla_+ (\zeta^y h^{\gamma^+}) + 2\nabla_- (\zeta^y h^{-\gamma}) - 2\nabla_- (\zeta^y h^{-\gamma}) \]
(4.20)

\[ = 2\zeta^y (\nabla_+ h^{\gamma^+}) + 2(\nabla_- \zeta^-) h^{-\gamma} - 2\zeta^y (\nabla_- h^{-\gamma}) \]
(4.21)

\[ = 2\zeta^y h^{-\gamma} (\Gamma^{\gamma^+} + \Gamma_{-} - 2\Gamma) \]
(4.22)

\[ = 2\zeta^y h^{-\gamma} (\Gamma^{\gamma^+} - \Gamma_{-}) \]
(4.23)

Note that

\[ h^{\gamma^+} = 0, \quad h^{-\gamma} = 0, \quad \nabla_+ \zeta^- = 0, \quad \nabla_- \zeta^- = \Gamma^- \zeta^y, \]
(4.24)

and also

\[ \nabla_+ h^{\gamma^+} = 0, \quad \nabla_+ h^{-\gamma} = \Gamma^{\gamma^+} h^{-\gamma}, \quad \nabla_- h^{-\gamma} = 0, \quad \nabla_- h^{-\gamma} = 2\Gamma h^{-\gamma}. \]
(4.25)

Adding (4.19) and (4.23) the terms together in (4.18) one finds

\[ \delta Q = \frac{1}{16\pi} \int d\theta dw^+ \epsilon_{\gamma+ \gamma} (4h^{\gamma^-} \zeta^y \Gamma^{\gamma^-} + 2\zeta^y h^{-\gamma} (\Gamma^{\gamma^+} + \Gamma_{-})) \]
(4.26)

\[ = \frac{1}{16\pi} \int d\theta dw^+ \frac{8Ma \sin \theta \rho^2}{y^3} 2h^{\gamma^-} \zeta^y (\Gamma_{\gamma^+} + \Gamma_{\gamma^-}) \]
(4.27)

By working at small \( w^+ \) and taking the \( w^+ \to 0 \) limit (which amounts to approaching \( \Sigma_{bif} \) along the future horizon) one finds

\[ h^{\gamma^-} = g^{\gamma^+} \partial_+ \zeta^y = \frac{y^3 g^{\gamma^+}}{4\rho^2} \quad \text{with} \quad \partial_+ = \partial_+ \]
(4.28)

\[ \int d\theta \sin \theta = 2, \quad \text{and} \quad \Gamma^{\gamma^+} + \Gamma_{\gamma^-} = -\frac{2}{y}. \]
(4.29)
Choosing \( \zeta \) to be \( \zeta_n \) and \( \tilde{\zeta} \) to be \( \zeta_m \), the variation becomes

\[
K_{m,n} = \delta Q = \frac{1}{16\pi} \int d\theta dw^+ \frac{8Ma \sin \theta \rho_+^2}{y^3} \left( \frac{y^3 \epsilon'_m \epsilon^n}{4\rho_+^2} \right) \left( \frac{1}{2} y \epsilon'_n \left( \frac{-2}{y} \right) \right)
\]

(4.30)

\[
= -\frac{1}{16\pi} \int d\theta dw^+ 4Ma \sin \theta \epsilon'_m \epsilon^n
\]

(4.31)

\[
= -\frac{1}{16\pi} \int dw^+ \frac{w^+ S_M a}{w^+} \epsilon'_m \epsilon^n
\]

(4.32)

\[
= -\frac{1}{16\pi} (4\pi^2 T_R) 8Ma \frac{im^3}{2\pi T_R} \delta_{m+n,0}
\]

(4.33)

\[
= -Ma im^3 \delta_{m+n,0}
\]

(4.34)

Here we have computed the Dirac bracket of two charges. Passing to the commutator rule of Dirac brackets to commutators \( \{.,.\} \rightarrow -i[.,.] \) as introduces a factor of \(-i\). The central charge of Kerr-Taub-NUT black holes can be easily identified via (4.13)

\[
c_R = 12aM = 12J_0.
\]

(4.35)

The central charge computed from soft hair arguments is in agreement with (3.10). Recall that the latter computation was totally independent to the one in this section, involving the monodromies of the scalar wave solutions of Klein-Gordon equation. We also note that (4.35) is a function only of the Kerr-Taub-NUT black hole angular momenta \( J_0 \). The two Misner strings considered here are attached to the horizon of the black hole, symmetrically distributed, and spinning with \( J_s \). See Section 2 for more details. Moreover, the central charge \( c_R \) will continue to be independent on the details of the Misner string distribution in these solutions as we show in Section 6.

5 First Law of Thermodynamics

As given in [4], another way of writing the relation between the left, right frequencies and uniquely fixing the identification of the temperatures \((T_L, T_R)\) arises from imposing the first law of thermodynamics

\[
\delta M = T \delta S + \Omega \delta J,
\]

(5.1)

with the identification

\[
\omega = \delta M, \quad m = \delta J.
\]

(5.2)

For the Kerr-Taub-NUT black holes we consider the first law (2.7) for fixed values of the Misner charges, that yields

\[
\delta M = \pm T_\pm \delta S_\pm + \Omega_\pm \delta J.
\]

(5.3)

and argue that the remarkable validity of the relation

\[
\delta S_\pm = \frac{\delta E_L}{T_L} \pm \frac{\delta E_R}{T_R},
\]

(5.4)
where
\[
\delta E_L = 2M(2\pi T_L) \delta M, \\
\delta E_R = 2M(2\pi T_R) \delta M - \delta J, 
\]
with the right and left temperatures \(T_{L,R}\) defined in (3.8). This gives further evidence to the choice of counterterm in the computation of the covariant charges for Kerr-NUT black holes. Any other identifications for the \(T_{L,R}\) temperatures will not satisfy the thermodynamic relations (5.4).

6 Kerr-Taub-NUT and general distribution of Misner string

The rotating Taub-NUT black hole solution can also include a dimension-full physical parameter which governs the overall distribution and strength of the Misner string. The parameter \(s\) parametrizing the Misner distribution can be formally added by performing in the black hole metric (2.1) the ‘large coordinate transformation’
\[
t \to t + 2s\phi. 
\]
The absence of closed timelike and null geodesics requires \(|s/N| \leq 1\). In particular, when \(s = +N\), there is only one Misner string and it is located on the north pole (\(\cos \theta = +1\)) axis, while the south pole (\(\cos \theta = -1\)) axis is completely regular. The choice \(s = -N\) corresponds to the opposite situation, while for \(s = 0\) the two Misner strings are ‘symmetrically distributed’ and both axes are ‘equally singular’. See [16] for details.

The solution takes the form (2.1) with the following definitions of the functions
\[
\rho^2 = r^2 + (N + a \cos \theta)^2, \quad \Delta = r^2 - 2Mr - N^2 + a^2, \\
\Xi = a\sin^2 \theta - 2N \cos \theta - 2s, \quad \Sigma = r^2 + N^2 + a^2 - 2as 
\]
It is important to note that the location of the outer/inner horizons remain invariant under the change of the Misner string distribution. These are defined via \(\Delta = 0\) are \(r_{\pm} = M \pm \sqrt{M^2 - a^2 + N^2}\).

The black hole horizon can be assigned the following entropy
\[
S_\pm = \pi (r_{\pm}^2 + a^2 + N^2 - 2as). 
\]
Note that this quantity explicitly depends on the parameter \(s\). This means that the total Bekenstein-Hawking entropy actually differs from the special case where the strings are symmetrically distributed with the \(s = 0\) case above. This suggests that the identification of the dual CFT quantities will incorporate the dependence of the Misner string distribution parameter \(s\).

With the explicit expression for the black hole entropy in hand (6.4), let us now turn towards the monodromies (3.4). Following the same steps as in our previous sections we identify the conjugate variables
\[
t_L = 2\pi T_L \phi - \frac{1}{2M} t, \quad t_R = 2\pi T_R \phi, 
\]
with \((t, \phi)\) the Boyer-Lindquist coordinates and

\[
T_R = \frac{r_+ - r_-}{4\pi a}, \quad T_L = \frac{M^2 + N^2 - as}{2\pi a M} = \frac{(r_+ + r_-)^2 + 4N^2 - 4as}{4\pi a(r_+ + r_-)}.
\]  

(6.6)

Employing Cardy formula

\[
S_\pm = S_{\text{Cardy}} = \frac{\pi^2}{3} c(T_L \pm T_R),
\]  

(6.7)

one finds that the central charge \(c = c_L = c_R\) is defined by

\[
c = 12aM.
\]  

(6.8)

The same central charge is found from the soft hair arguments. Note that the large diffeomorphism (6.1) in addition to the coordinate transformation (6.5) will lead to a metric containing a deformed quotient of \(AdS_3\) close to the bifurcation (4.2). While we do not present here the full derivation of the soft hair charges, we argue that the structure of the metric (4.2) close to the bifurcation surface suffices to straightforwardly carry out the steps in Section 4.

Having now applied the soft hair holds for rotating black holes with charges, we now turn to general classes of 4D black holes. We will argue that the generalized counterterm (1.1) leads to well defined soft hair linearized charges.

7 Soft hair charges on general classes of 4D black holes

The complete family of (non-accelerating) electric and magnetically charged black-hole like space-times in four space-time dimensions [23] can be written in Boyer-Lindquist coordinates \(^6\) as

\[
ds^2 = \frac{Q}{\rho^2} \left[ dt + (2N \cos \theta - a \sin^2 \theta) \frac{d\phi}{K} \right]^2 + \frac{\rho^2}{Q} dr^2 + \frac{P}{\rho^2} \left[ a dt - (r^2 + a^2 + N^2) \frac{d\phi}{K} \right]^2 + \frac{\rho^2}{P} \sin^2 \theta d\theta^2,
\]  

(7.1)

\[
(7.2)
\]

where

\[
\rho^2 = r^2 + (N + a \cos \theta)^2
\]  

(7.3)

\[
P = \sin^2 \theta \left( 1 + \frac{\Lambda}{3} 4aN \cos \theta + \frac{\Lambda}{3} a^2 \cos^2 \theta \right)
\]  

(7.4)

\[
Q = (\omega^2 k + e^2 + g^2) - 2mr + er^2 - \frac{\Lambda}{3} r^4
\]  

(7.5)

\[
K = 1 - a^2 / L^2,
\]  

(7.6)

\(^6\)Note that the metric here differs respect to [23] by a coordinate transformation \(\phi^{(here)} \to \phi/\Xi\) and \(t^{(here)} \to t + 2N \phi/K\)
with $\epsilon, n$ and $k$ are given by

$$
\epsilon = \frac{\omega^2 k}{a^2 - N^2} - \left( a^2 + 3N^2 \right) \frac{\Lambda}{3}, \quad (7.7)
$$

$$
n = \frac{\omega^2 k N}{a^2 - N^2} + \left( a^2 - N^2 \right) N \frac{\Lambda}{3}, \quad (7.8)
$$

$$
k = \left( 1 - N^2 \Lambda \right) \left( \frac{\omega^2}{a^2 - N^2} \right)^{-1}. \quad (7.9)
$$

satisfying Einstein’s equations

$$
R_{\mu\nu} - (1/2) R g_{\mu\nu} + \Lambda g_{\mu\nu} = 2 \left( F_{\mu\alpha} F^\alpha_{\nu} - (1/4) g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right), \quad (7.10)
$$

$$
\partial_\mu \left( \sqrt{-g} F^{\mu\nu} \right) = 0, \quad (7.11)
$$

where $\Lambda = -3/L^2$ is the cosmological constant, and the Faraday tensor $F_{\mu\nu} = \partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu$ is given in terms of the electromagnetic potential

$$
\tilde{A} = -\frac{e r}{\rho^2} \left[ dt - \Xi \frac{d\phi}{K} \right] + \frac{g}{\rho^2} \left[ p dt - (\rho^2 \cos \theta + p \Xi) \frac{d\phi}{K} \right]. \quad (7.12)
$$

where

$$
\Xi = a \sin^2 \theta - 2N \cos \theta, \quad p = N + a \cos \theta. \quad (7.13)
$$

This vector potential reduces to the gauge field expression in [24] when the NUT charge is set to zero $N = 0$.

This solution to Einstein’s equations contains seven parameters: $m, e, g, a, N, \Lambda$ and $\omega$ which represent respectively the mass, electric charge, magnetic charge, rotation, nut charge, cosmological constant and remaining scaling freedom. Of these, the first six can be varied independently, and $\omega$ can be set to any convenient value if $a$ or $N$ are not both zero.

The event horizon of the black hole is located at $r = r_+$ where $Q(r_+) = 0$. The metric (7.1) has also a Cauchy horizon $r = r_-$ when $Q(r_-) = 0$ and $0 < r_- < r_+$.

To elucidate the thermodynamics of the metric, we now consider the black hole event horizon area computed from (7.1) by integrating

$$
Area \equiv \int \sqrt{g_{\theta\theta}} \, d\theta \, d\phi = \frac{4\pi (r^2_+ + a^2 + N^2)}{K}. \quad (7.14)
$$

such that the entropy yields

$$
S_\pm \equiv \frac{Area}{4} = \frac{\pi (r^2_+ + a^2 + N^2)}{K}. \quad (7.15)
$$

The expressions for the Hawking temperature and angular velocity are similar to those for the uncharged AdS-Kerr-Taub-NUT black hole presented in [14]

$$
T_\pm = \frac{Q'_+}{4\pi (r^2_+ + a^2 + N^2)}, \quad \Omega_\pm = \frac{a K}{r^2_+ + a^2 + N^2}. \quad (7.16)
$$
However, in present case, the outer and inner horizons $r_{\pm}$ involve the electric and magnetic charges $e$ and $g$. One can see from these thermodynamic expressions that in the vanishing NUT charge ($N = 0$) limit we recover the quantities given in [24]. Further, these quantities are also consistent with [14] for $e = g = 0$.

As advocated in [4, 5, 25], the null-tetrad representation can be used in the derivation of the near black hole bifurcation surface region to simplify the computations. In particular, the corresponding null-tetrad representation for the space-time (7.1) is

$$ g^{ab} = -l^a n^b - n^a l^b + m^a \tilde{m}^b + \tilde{m}^a m^b \quad (7.17) $$

with $a, b = t, r, \theta, \phi$ and the null tetrad defined as

$$ l^a = \omega \left[ (r^2 + a^2 + N^2) \partial_t - Q \partial_r + a K \partial_\phi \right] \quad (7.18) $$

$$ n^a = \frac{\omega}{2Q(r^2 + p^2)} \left[ (r^2 + a^2 + N^2) \partial_t + Q \partial_r + a K \partial_\phi \right] \quad (7.19) $$

$$ m^a = -\frac{\omega}{\sqrt{2}P(r^2 + p^2)} \left[ \frac{p^2 - (a^2 + N^2)}{a} \partial_t + i \frac{P}{\sin \theta} \partial_\theta - K \partial_\phi \right] \quad (7.20) $$

and $p$ defined in (7.13).

One may now wonder if all 4D black holes (7.1) exhibit a conformal structure in the near bifurcation surface geometry. As we will see in the next section, it is in fact that adapted $(w_{\pm}, y, \theta)$ conformal coordinates which naturally arise from the study of the Kerr black holes give rise to this universal near horizon geometry feature in black holes. The conformal coordinates were useful in the context of extending the Kerr/CFT proposal to non-extremal black holes [3] by explicitly describing the conformal symmetry of the space on which the field propagates via e.g. the Klein-Gordon equation rather than the symmetry of the spacetime geometry. In the next section of this paper, however, we will give a different interpretation. In particular, we will connect the close to the bifurcation surface geometry to the soft hair analysis and CFT dual.

### 7.1 Near bifurcation surface metric for generic black holes

In order to better understand these local conformal symmetries in the metrics and the validity of the counterterms in soft-hair approach leading to the Hawking entropy we will explicitly compute the near bifurcation surface metrics for all generic black holes solutions of Einstein equations in 4D [26]. We will argue that stationary non-accelerating black hole metrics of the Plebański–Demiański class (7.1), to leading order at the bifurcation surface completely determine the soft hair charge and its Bekenstein-Hawking entropy.

We start by considering the perturbative expansion of a metric around the event horizon $r = r_{+}$. The event horizon radius $r_{+}$ sets the size of the black hole (7.1). We define the zoom in coordinates on the event horizon as

$$ r \to r_{+} + \frac{w_{+}w_{-}}{y^2} Q'(r_{+}) \quad (7.21) $$

$$ \phi \to (2\alpha)^{-1} \log \left( \frac{w_{+}y^2}{w_{-}} \right) \quad (7.22) $$

$$ t \to (2\delta)^{-1} \log \left( \frac{w_{-}y^2}{w_{+}} \right) - \frac{\gamma}{\delta} \phi \quad (7.23) $$
By applying an expansion for \( w^+ \ll r^+ \), the near bifurcation surface metric can be cast into an elegant simple geometry, defined by

\[
\begin{align*}
    ds^2 &\approx \frac{1}{y^2} \left( 4 \rho^2_+ dw_+ dw_- + \frac{4 a^2 P}{\delta^2 \rho^2_+} dy^2 \right) + \frac{\rho^2_+ \sin^2 \theta}{P} d\theta^2 
\end{align*}
\]

for the following choices of the parameters

\[
\begin{align*}
    \alpha &= \frac{Q'_+}{2 a K}, \quad \gamma = \frac{(\Omega_+ + \Omega_-)}{(\Omega_- - \Omega_-)} \alpha, \quad \delta = -\Omega_+ (\gamma + \alpha). \quad (7.25)
\end{align*}
\]

and where for any function \( X \) we have defined its evaluation on the horizon \( X_+ = X(r^+) \).

The universal expression for the near bifurcation surface metric of black holes that we find signals at an underlying connection to a locally defined \( SL(2,R)_L \times SL(2,R)_R \) symmetry of the near-horizon geometry and near-region scalar field equation. For more details on (7.24), we refer the interested reader to [14] where symmetries and its thermodynamics is studied.

### 7.2 Central charge

The immediate goal of this subsection is to identify the right and left-temperatures arising from the CFT for general classes of 4D black holes. The systematic approach presented in the previous section unveiled that 4D black holes mirror the geometry of the 3D BTZ black holes in the regions close to the bifurcation surface. The periodicities analysis under the azimuthal identification \( \phi \rightarrow \phi + 2\pi \) of the conformal variables yields

\[
\begin{align*}
    w^+ &\sim e^{2\pi \alpha}, \quad w^- \sim e^{2\pi \gamma}, \quad y \sim e^{\pi (\alpha + \gamma)} \quad (7.26)
\end{align*}
\]

This is the same as the identification employed in [21] that turns AdS\(_3\) in Poincare coordinates into BTZ with temperatures

\[
\begin{align*}
    \alpha &\rightarrow 2\pi T_R, \quad \gamma \rightarrow 2\pi T_L. \quad (7.27)
\end{align*}
\]

Under the holographic duality the dimensionless left and right temperatures are then

\[
\begin{align*}
    T_R = \frac{Q'}{4 \pi a K}, \quad T_L = \frac{(\Omega_+ + \Omega_-)}{(\Omega_- - \Omega_-)} \frac{Q'}{4 \pi a K}. \quad (7.28)
\end{align*}
\]

According to the Cardy formula the entropy for a unitary CFT obeys

\[
\begin{align*}
    S_\pm = S_{\text{Cardy}} = \frac{\pi^2}{3} c (T_L \pm T_R), \quad (7.29)
\end{align*}
\]

Using (7.28) we find the microscopic entropy for the dual black hole (7.1) regarded the central charge of the CFT \( c = c_L = c_R \) is defined by

\[
\begin{align*}
    c = - \frac{6a}{\delta}. \quad (7.30)
\end{align*}
\]

While this very general analysis gives the central charge of the dual CFT, we need to construct the linearized charge associated to a diffeo (4.14) acting on the horizon and see if they are finite. It can easily be verified that the general form of the near bifurcation surface metric (7.24) gives a finite linearized charge yielding (7.30).
8 Discussion

In this work we have constructed the 2D CFT identifications for the 4D Kerr-Taub-NUT and Plebianski-Demianski spacetimes. Extending three different approaches — monodromies, soft-hair and thermodynamics — for these 4D black hole geometries we have found explicit expressions for the dual temperatures \((T_L, T_R)\) and central charge. Remarkably, comparing the microscopic entropy with the macroscopic area law, we see an impressive and detailed match for general \((m, a, N, g, e, \Lambda)\) black holes. We expect this to have implications for the hypothesis that hidden conformal symmetry explains the leading microstate degeneracy for such 4D black hole backgrounds.

We conclude with two miscellaneous observations.

As we stated earlier, the choice of the boundary counterterm relevant for the linearized soft-hair charge contribution is not unique. Originally the Wald-Zoupas counterterm was proposed to reproduce the area law for the Kerr(Newman) black hole \([4, 5]\). In \([6]\), the authors found that for AdS-Kerr black holes a more general proposal \((1.1)\) was expected for constructing the 2D CFT that was consistent with all other results from the monodromy techniques, the thermodynamic analysis and all extremal black holes studies (Table 1). Extending the analysis of \([6]\) for Kerr-Taub-NUT and the complete family of 4D black holes we explicitly showed that the more general counterterm \((1.1)\) gives rise to well-defined charge and gives a central charge \(c_R = c_L\). Our argument is sharpened version of those previously made and is perhaps the most general in spirit given that it works for the complete family of 4D black holes.

We do not herein proved uniqueness of the counterterm, but instead presented extensive evidence of the applicability of a more general counterterm for the general classes of 4D black holes. This indicates that the applicability of the Wald-Zoupas is indeed limited. It is also worth emphasizing that the counterterm \((1.1)\) reduces to the Wald-Zoupas counterterm for Kerr-Newman black holes, and hence includes the results found in \([4, 5]\). The observation of the inconsistent results derived from the extension of the Wald-Zoupas counterterm to Kerr-Taub-NUT can be found in Appendix A. While we kept the identification of the dual temperatures \((3.8)\) in the computations, we found that the charge associated with the diffeomorphisms on the horizon supported with the Wald-Zoupas counterterm led to a central charge defined by \(c_L \neq c_R\). Moreover, the central charge compatible with the Cardy formula was further not quantized — in contrast to our results where \(c_L = c_R = 12J_0\) — and is inconsistent with previously know results from extensions of the Kerr/CFT conjecture for extreme black holes.

Our final comment in this concluding section will focus on the Kleinian aspects of the (Lorentzian) Kerr-Taub-NUT black holes and 2D CFTs. For the sake of simplicity, let us consider the Kerr-Taub-NUT metric, which is determined by three parameters \((M, N, a)\). Kleinian signature metric \([27]\) as shown in \([15, 28, 29]\) for black holes, can construct its Kleininan signature metric following the analytic continuation \(t \rightarrow it\) and \(\rightarrow i\theta\) supplemented by \(a \rightarrow ia\) and \(N \rightarrow iN\) required by the reality of the metric. When the mass and NUT charge are equal, \(M = \pm N\) the geometry is self-dual for any value of the Kerr rotation parameter \(a\). In fact, \(a\) can be absorbed into a shift of the \(r\)-coordinate. Via a coordinate
transformation involving both \((r, \theta)\) the geometry can be shown to be diffeomorphic to the self-dual Kleinian Taub-NUT metric \([15]\). Inspecting the 2D CFT quantities we derived for the Kerr-Taub-NUT metrics, we argue that the Kleinian Kerr-Taub-NUT metric will have (if any) a 2D CFT dual with temperatures

\[ T_L = 0, \quad T_R = \frac{1}{2\pi} \]

and central charge

\[ c = 12J. \]  

Remarkably, this is exactly the opposite representations of the one copy of the Virasoro algebra found for the extreme Kerr black holes within the Kerr/CFT correspondence.

Kleinian metrics have a natural interpretation in terms of scattering amplitudes. Perhaps interesting simplifications and structures occur in a 2D CFT description of these spacetimes, which can lead to further implications for constructing scattering amplitudes.

There are still plenty of open questions. Is there a holographic dual for the Kleinian Kerr-Taub-NUT? Does an explicit microscopic construction of 4D black holes exist beyond the semiclassical limit considered here? These appear to be interesting avenues for future pursuit.

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A Alternative counterterm

The Wald-Zoupas term is of the form

\[
\delta Q_{ct} = \frac{1}{16\pi} \int_{\partial\Sigma} \zeta (\ast X) \\
= \frac{1}{16\pi} \int d\theta dw^+ \epsilon_{y+\theta} \left( -2 h^y \zeta^y \left( n_+ \partial t^+ + n_+ \Gamma^+_y t^+ \right) \right) \\
= \frac{1}{16\pi} \int d\theta dw^+ \epsilon_{y+\theta} \left( -2 \right) h^y \zeta^y \left( \frac{2T_R}{y(T_L + T_R)} + \Gamma^+_y \right),
\]

(A.1)

In order to have null vectors normal to the bifurcation surface that are periodic we take \(l^\mu \sim -\frac{1}{\sqrt{2}} y^\mu T_R / (T_L + T_R) \partial_+\) and \(n^\mu \sim -\frac{1}{\sqrt{2}} y^\mu T_R / (T_L + T_R) \partial_-\) normalized such that \(l.n = -1\).

The charge associated to the diffeomorphism \(\zeta_n\) reduces to a boundary integral

\[
\delta Q = \frac{1}{16\pi} \int d\theta dw^+ \epsilon_{y+\theta} \left( -2 \right) h^y \zeta^y \left( 2 \Gamma^+_y + \frac{2T_R}{y(T_L + T_R)} + \Gamma^+_y \right)
\]

(A.2)
One finds after a simple integral the central charge gives

\[ c_R = \frac{12Ma}{((T_L + T_R)(T_L + T_R) - \Theta)} \left[(T_L + T_R)^2 - T_L \Theta \right] \tag{A.3} \]

where we labeled \( \Theta = \frac{N^2}{(2\pi aM)} \). In our case when the NUT parameter vanishes we recover previous results in [4] for the Kerr black hole.

**Christoffel symbols**

The Christoffel symbols relevant for this computation are found to be

\[
\Gamma^+_{+y} = -\frac{a^2 K(M - 4\pi M(T_L + T_R)) - M(r_+ - r_-)(\Sigma + 2aN\cos \Theta) - \rho^+ - 2aM \left( a\sin^2 \theta(M + r_+) - 8\pi^2 aM T_L(T_L + T_R) \right)}{y\rho^+},
\]

\[
\Gamma^-_{-y} = -\frac{a^2 K(M - 4\pi M(T_L + T_R)) - M(r_+ - r_-)(\Sigma + 2aN\cos \Theta) + \rho^+ - 2aM \left( a\sin^2 \theta(M + r_+) - 8\pi^2 aM T_L(T_L + T_R) \right)}{y\rho^+},
\]

where

\[
\Gamma^+_{+y} + \Gamma^-_{-y} = -\frac{2}{y}. \tag{A.4}
\]

And the integrals are

\[
\int_0^\pi d\theta \sin \theta \Gamma^+_{+y} = -\frac{2(r_+ + r_-)}{y r_+}, \quad \int_0^\pi d\theta \sin \theta \Gamma^-_{-y} = -\frac{2(r_+ - r_-)}{y r_+}. \tag{A.5}
\]

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