ON THE VARIATION OF THE POISSON STRUCTURES
OF CERTAIN MODULI SPACES

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Abstract. Given a Lie group G whose Lie algebra is endowed with a nondegenerate invariant symmetric bilinear form, we construct a Poisson algebra of continuous functions on a certain open subspace \( \mathcal{R}(\pi, G) \) of the space of representations in G of the fundamental group \( \pi \) of a compact connected orientable topological surface with finitely many boundary circles; when G is compact and connected, \( \mathcal{R}(\pi, G) \) may be taken dense in the space of all representations. The space \( \mathcal{R}(\pi, G) \) contains spaces of representations where the values of those generators of the fundamental group which correspond to the boundary circles are constrained to lie in fixed conjugacy classes and, on these representation spaces, the Poisson algebra restricts to stratified symplectic Poisson algebras constructed elsewhere earlier. Hence the Poisson algebra on \( \mathcal{R}(\pi, G) \) gives a description of the variation of the stratified symplectic Poisson structures on the smaller representation spaces as the chosen conjugacy classes move.
1. Introduction

Given a Lie group $G$ whose Lie algebra is endowed with a nondegenerate invariant symmetric bilinear form, certain subspaces $\text{Rep}(\pi,G)_C$ (the subscript will be explained below) of the space $\text{Rep}(\pi,G)$ of representations in $G$ of the fundamental group $\pi$ of an orientable topological surface with finitely many boundary circles are known to inherit symplectic or more generally Poisson structures; see e. g. [7] and the literature there. These subspaces are obtained when the values of those generators of $\pi$ which correspond to the boundary circles are constrained to lie in fixed conjugacy classes of $G$; apart from trivial cases, these subspaces have strictly positive codimension in $\text{Rep}(\pi,G)$ and hence are genuinely smaller than $\text{Rep}(\pi,G)$. But what can be said about the variation of the Poisson structures when the conjugacy classes are allowed to move? It is the objective of the present paper to offer an answer to this question, phrased in the world of Poisson geometry. Our main result will give a Poisson structure on a suitable subspace $\mathcal{R}(\pi,G)$ of $\text{Rep}(\pi,G)$ which is a union of distinct spaces of the kind $\text{Rep}(\pi,G)_C$; this Poisson structure will induce the Poisson structures on the spaces $\text{Rep}(\pi,G)_C$. When $G$ is compact and connected, $\mathcal{R}(\pi,G)$ may in fact be taken to be dense in $\text{Rep}(\pi,G)$. We now explain this in some more detail.

Let $\Sigma$ be a compact connected orientable topological surface of genus $\ell$ with boundary $\partial \Sigma$ consisting of $n$ circles $S_1, \ldots, S_n$. To avoid trivial cases or inconsistencies, when the genus is zero, we suppose that $n \geq 3$. Consider the usual presentation

\[ P = \langle x_1, y_1, \ldots, x_\ell, y_\ell, z_1, \ldots, z_n; r \rangle, \quad r = \Pi_{j=1}^{\ell} [x_j, y_j] z_1 \ldots z_n, \]

of the fundamental group $\pi = \pi_1(\Sigma)$. Let $\cdot$ be a nondegenerate invariant symmetric bilinear form on the Lie algebra $\mathfrak{g}$ of $G$. Extending the approach in [7], from these data, we shall construct a Poisson structure on a certain open subspace $\mathcal{R}(\pi,G)$ of $\text{Rep}(\pi,G)$ which is a union of distinct spaces of the kind $\text{Rep}(\pi,G)_C$; this Poisson structure will induce the Poisson structures on the spaces $\text{Rep}(\pi,G)_C$. When $G$ is compact and connected, $\mathcal{R}(\pi,G)$ may in fact be taken to be dense in $\text{Rep}(\pi,G)$. We now explain this in some more detail.

Given an $n$-tuple $C = (C_1, \ldots, C_n)$ of conjugacy classes in $G$, denote by $\text{Hom}(\pi,G)_C$ the space of homomorphisms $\chi$ from $\pi$ to $G$ for which the value $\chi(z_k)$ of each generator $z_k$ lies in $C_k$, for $1 \leq k \leq n$, and denote by $\text{Rep}(\pi,G)_C$ the corresponding space of representations, that is, the orbit space for the action of $G$ on the space $\text{Hom}(\pi,G)_C$ by conjugation. From these data, in [7], we constructed the structure of a stratified symplectic space on $\text{Rep}(\pi,G)_C$, in particular, a Poisson algebra $(C^\infty(\text{Rep}(\pi,G)_C), \{\cdot, \cdot\})$ of continuous functions on $\text{Rep}(\pi,G)_C$; the latter endows a certain top stratum with the structure of a symplectic manifold. Henceforth we shall refer to a Poisson structure which is part of the structure of a stratified symplectic space as a stratified symplectic Poisson structure; thus a stratified symplectic Poisson algebra on a space with a single stratum is just a smooth symplectic Poisson algebra in the usual sense.

Let $O$ be an open connected $\text{Ad}$-invariant neighborhood of zero in $\mathfrak{g}$ which, under the exponential map $\exp$ from $\mathfrak{g}$ to $G$, is mapped diffeomorphically onto an open invariant neighborhood $B$ of the identity of $G$. When $G$ is compact, connected and simply connected, and hence semisimple, a maximal such subset $O$ is given by the largest connected neighborhood of the origin of $\mathfrak{g}$ where the exponential map is
be dense in $\text{Rep}(\mathbf{g})$. $\mathbf{g}$ is a momentum mapping for the action of $\mathbf{G}$ in the sense of Lie groups (they correspond to the stronger condition $0 < |\nu| < 1$) whence $B$ is certainly dense in $G$. For example, for $G = \text{SU}(2)$, we then have $B = G \setminus \{-\text{Id}\}$. What happens for $G = \text{SU}(n)$ when $n \geq 3$ will be briefly explained in (3.5) below. On the other hand, when $G$ is still compact, connected and semisimple but not simply connected, things get more complicated; for example, for $G = \text{SO}(3)$, an appropriate subset $B$ is the ball in $\text{SO}(3)$ which arises when the conjugacy class

\[
\begin{bmatrix}
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

is removed from $\text{SO}(3)$; this conjugacy class is a copy of the real projective plane, and its points are still regular values for the exponential map.

Returning to the general case, let $\mathcal{H}(\pi, G)$ be the space of homomorphisms $\chi$ from $\pi$ to $G$ for which the value $\chi(z_k)$ of each generator $z_k$ lies in $B$ and let $\mathcal{R}(\pi, G)$ be the corresponding space of $G$-orbits; we do not indicate the dependence on $B$. We shall say that a conjugacy class $C$ is $B$-regular if it is contained in $B$. It is manifest that $\mathcal{R}(\pi, G)$ is the union in $\text{Rep}(\pi, G)$ of the spaces $\text{Rep}(\pi, G)_C$ for those $n$-tuples $C = (C_1, \ldots, C_n)$ of conjugacy classes which are $B$-regular in the sense that every $C_k$ is $B$-regular, and $\mathcal{R}(\pi, G)$ is plainly open in $\text{Rep}(\pi, G)$.

**Theorem.** The data determine the structure of a Poisson algebra of continuous functions on $\mathcal{R}(\pi, G)$ which, on each moduli space $\text{Rep}(\pi, G)_C$ (that lies in $\mathcal{R}(\pi, G)$), restricts to the stratified symplectic Poisson algebra on $\text{Rep}(\pi, G)_C$ determined by the data. When $G$ is compact and connected, the subspace $\mathcal{R}(\pi, G)$ may be taken to be dense in $\text{Rep}(\pi, G)$.

In particular, the strata of the spaces $\text{Rep}(\pi, G)_C$ in $\mathcal{R}(\pi, G)$, which carry symplectic structures in view of what has been said earlier, appear as symplectic leaves (in a generalized sense) for the Poisson structure on $\mathcal{R}(\pi, G)$. The first statement of the theorem will be proved in (2.18) below, and the second statement, which says that, for $G$ compact and connected, the subspace $\mathcal{R}(\pi, G)$ may be taken to be dense in $\text{Rep}(\pi, G)$, will be justified in Section 3.

We now explain briefly how this theorem is proved. By purely finite dimensional methods, we shall construct a suitable extended moduli space, that is to say, a smooth symplectic manifold $\mathcal{M}$ and a hamiltonian action of the group $G_0 \times G_1 \times \cdots \times G_n$ on $\mathcal{M}$, each $G_j$ being a copy of $G$, $0 \leq j \leq n$, with momentum mapping $\mu_j: \mathcal{M} \to \mathfrak{g}_j^*$ for the action of the $j$'th copy $G_j$ of $G$, in such a way that

$$
\mu_0 \times \mu_1 \times \cdots \times \mu_n: \mathcal{M} \to \mathfrak{g}_0^* \times \mathfrak{g}_1^* \times \cdots \times \mathfrak{g}_n^*
$$

is a momentum mapping for the action of $G_0 \times G_1 \times \cdots \times G_n$ on $\mathcal{M}$ and that the following holds: Consider the reduced space $\mathcal{M}_0 = \mu_0^{-1}(0)/G$ for the hamiltonian action of the zero'th copy $G_0$ of $G$ on $\mathcal{M}$, with its structure $(C^\infty(\mathcal{M}_0), \{\cdots\})$ of a stratified symplectic space [20]. The action of $G_0 \times G_1 \times \cdots \times G_n$ on $\mathcal{M}$ induces a hamiltonian action (in the sense of stratified symplectic spaces) of $\Gamma = G_1 \times \cdots \times G_n$ on $\mathcal{M}_0$, and there is a canonical map from $\mathcal{M}_0$ to $\mathcal{R}(\pi, G)$ which

\[
\begin{bmatrix}
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]
induces a homeomorphism between the space of orbits $M_0/\Gamma$ and $R(\pi,G)$. The subalgebra $(C^\infty(M_0))^\Gamma$ of $\Gamma$-invariant functions in $C^\infty(M_0)$ then yields a Poisson algebra $(C^\infty(R(\pi,G)),\{\cdot,\cdot\})$ of continuous functions on $R(\pi,G)$ having the asserted properties.

The philosophy is the same as that in [7]: The concept of fundamental group is too weak to handle peripheral structures; in that paper, the fundamental group has been replaced by two more general concepts which enabled us to overcome the difficulties with the peripheral structure: by that of a group system, to handle the global structure, and by that of a fundamental groupoid, to handle the infinitesimal structure. In the present paper, we shall study the global structure by means of the fundamental groupoid. This leads to the extended moduli space $M$ mentioned above; this space is somewhat “larger” than the extended moduli space in [7] in the sense that the extended moduli space in [7] which arises from a choice of conjugacy classes that are $B$-regular is obtained by symplectic reduction from the extended moduli space $M$ to be explored in the present paper, see (2.14) below; on the other hand, those conjugacy classes which are not $B$-regular, whatever choice of $B$, cannot be incorporated in the approach in terms of this larger extended moduli space, though. We shall explain the obstacles in Remark 2.19 below.

Write $\Sigma^\bullet$ for the corresponding punctured surface, which contains $\Sigma$ and, for each boundary component of $\Sigma$, has a single puncture in such a way that, for $1 \leq k \leq n$, the $k$’th boundary circle $S_k$ surrounds the $k$’th puncture and no other puncture. For $G = U(r)$, the unitary group, after a choice of complex structure on $\Sigma^\bullet$ has been made, a space of the kind $\text{Rep}(\pi,G)_{\mathbb{C}}$ admits an interpretation as moduli space of semistable rank $r$ parabolic bundles of parabolic degree zero, with flags and weights given by $\mathbb{C}$. In fact, the search for a sensible moduli space of rank $r$ holomorphic vector bundles on $\Sigma^\bullet$ in the category of complex varieties led to the introduction of the additional structure of flags and weights at the punctures of $\Sigma^\bullet$ [17]. Now, whether or not $G$ is the unitary group, in our representation space picture, a complex structure of $\Sigma^\bullet$ does not come into play and the complex structure on the moduli space is not visible but, from the results of [7], $\text{Rep}(\pi,G)_{\mathbb{C}}$ comes with a structure of a stratified symplectic space. The main result of the present paper, the theorem spelled out above, gives a Poisson structure on a certain ambient space $R(\pi,G)$ which, among other things, yields a description of the variation of the stratified symplectic structures across the pieces $\text{Rep}(\pi,G)_{\mathbb{C}}$ of $R(\pi,G)$ as $\mathbb{C}$ moves. For $G$ the unitary group, the variation of the complex structure of the parabolic moduli spaces as the flags and weights change has been studied in [3], but not by means of an ambient space.

We keep the notations of [7], and most unexplained concepts may be found in that paper. A leisurely introduction into Poisson geometry of certain moduli spaces (related to those studied in the present paper but not exactly the same ones) may be found in [11] where also more references are given. A space similar to our extended moduli space $M$ has been given in [13] and, by infinite dimensional methods from gauge theory, a symplectic structure on that space has been constructed. See Remark 2.20 below for details. I am much indebted to A. Weinstein for discussions at various stages; he was presumably the first to notice the significance of groupoids for moduli spaces. Thanks are also due to S. Helgason and R. Steinberg for having pointed out to me the relevant literature where the simple connectedness of conjugacy classes
of compact, connected, and simply connected Lie groups has been established.

2. Details and Proofs

The system $(\pi; \pi_1, \ldots, \pi_n)$, with $\pi = \pi_1(\Sigma)$ and $\pi_k = \pi_1(S_k) \cong \mathbb{Z}$, for $1 \leq k \leq n$, is what is called a group system [7, 21]. When the boundary $\partial \Sigma$ of $\Sigma$ is non-empty the group $\pi$ is free; yet it is convenient to use the presentation (1.1).

Next we recall the appropriate fundamental groupoid: Pick a base point $p_0$ not on the boundary and, moreover, for each boundary component $S_k$ of $\Sigma$, pick a base point $p_k$. This determines the fundamental groupoid $\tilde{\pi} = \Pi(\Sigma; p_0, p_1, \ldots, p_n)$. To obtain a presentation of it we decompose $\Sigma$ into cells as follows where we do not distinguish in notation between the chosen edge paths and their homotopy classes relative to their end points: Let $x_1, y_1, \ldots, x_\ell, y_\ell$ be closed paths which (i) do not meet the boundary, (ii) have $p_0$ as starting point, and (iii) yield the generators respectively $x_1, y_1, \ldots, x_\ell, y_\ell$ of the fundamental group $\pi = \pi_1(\Sigma, p_0)$; for $j = 1, \ldots, n$, let $a_k$ be the boundary path of the $k$'th boundary circle, having $p_k$ as starting point, and let $\gamma_k$ be a path from $p_0$ to $p_k$. When we cut $\Sigma$ along these 1-cells we obtain a disk whose boundary yields the defining relation of $\tilde{\pi} = \Pi(\Sigma; p_0, p_1, \ldots, p_n)$. The resulting presentation of $\tilde{\pi}$ looks like

\begin{equation}
\tilde{\pi} = \langle x_1, y_1, \ldots, x_\ell, y_\ell, a_1, \ldots, a_n, \gamma_1, \ldots, \gamma_n; \tilde{\tau} \rangle, \quad \tilde{\tau} = \Pi_\ell \prod_{k=1}^n \gamma_k a_k \gamma_k^{-1}.
\end{equation}

Let $\tilde{F}$ denote the groupoid which is free on the generators of (2.1). To have a neutral notation, whenever necessary, we shall write $\tilde{\Pi}$ for either $\tilde{F}$ or $\tilde{\pi}$; accordingly we write $\Pi$ for either $F$ or $\pi$. This is consistent with the presentation (1.1) of the fundamental group if we identify, for $1 \leq k \leq n$, the generator $z_k$ with $\gamma_k a_k \gamma_k^{-1}$. More precisely, the assignments

\begin{equation}
\begin{aligned}
i(e) &= p_0, & i(x_j) &= x_j, & i(y_j) &= y_j, & i(z_k) &= \gamma_k a_k \gamma_k^{-1}, \\
\rho(p_j) &= e, & \rho(x_j) &= x_j, & \rho(y_j) &= y_j, & \rho(a_k) &= z_k, & \rho(\gamma_k) &= \text{Id},
\end{aligned}
\end{equation}

where $1 \leq j \leq \ell$ and $1 \leq k \leq n$, yield morphisms of presentations $i: \mathcal{P} \to \tilde{\mathcal{P}}$ and $\rho: \tilde{\mathcal{P}} \to \mathcal{P}$ and hence functors $i: \Pi \to \tilde{\Pi}$ and $\rho: \tilde{\Pi} \to \Pi$ (where the notation $i$ and $\rho$ is abused) inducing a deformation retraction of $\tilde{\Pi}$ onto $\Pi$; cf. e. g. [6 (6.5.13)] for the latter notion.

As usual, view the group $G$ as a groupoid with a single object $e$, identified with the neutral element of $G$. Denote by $\tilde{\Pi}_0$ the set $\{p_0, \ldots, p_n\}$ of objects of $\tilde{\Pi}$, and write $\text{Hom}(\tilde{\Pi}, G)$ for the space of groupoid homomorphisms from $\tilde{\Pi}$ to $G$. The obvious action of $G$ on $\text{Hom}(\Pi, G)$ by conjugation extends to an action of the group $G^{\tilde{\Pi}_0} \cong G \times \cdots \times G$ ($n + 1$ copies of $G$) on $\text{Hom}(\tilde{\Pi}, G)$ in the following way: We denote by $s$ and $t$ the source and target mappings from $\tilde{\Pi}$ to the object set $\tilde{\Pi}_0$. Given a homomorphism $\alpha$ from $\tilde{\Pi}$ to $G$ and $\vartheta \in G^{\tilde{\Pi}_0}$, the homomorphism $\vartheta \alpha$ is defined by

$$\vartheta \alpha(w) = \vartheta(t(w)) \alpha(w)(\vartheta(s(w)))^{-1}.$$ 

The orbit space for the $G^{\tilde{\Pi}_0}$-action on $\text{Hom}(\tilde{\Pi}, G)$ will be denoted by $\text{Rep}(\tilde{\Pi}, G)$. The structure of the space $\text{Rep}(\Pi, G)$ of representations of the group $\Pi$ can now
be studied by looking at $\text{Rep}(\tilde{\Pi}, G)$ instead. More precisely: The functors $i$ and $\rho$ induce maps

$$i^* : \text{Hom}(\tilde{\Pi}, G) \to \text{Hom}(\Pi, G), \quad \rho^* : \text{Hom}(\Pi, G) \to \text{Hom}(\tilde{\Pi}, G)$$

which, for $\Pi = F$ and $\tilde{\Pi} = \tilde{F}$, are manifestly smooth. We shall occasionally refer to $i^*$ and $\rho^*$ as restriction and corestriction, respectively. The following is obvious.

**Proposition 2.3.** The restriction mapping induces a homeomorphism

$$i^* : \text{Rep}(\tilde{\Pi}, G) \to \text{Rep}(\Pi, G).$$

We apply a variant of the construction in [9] to the presentation $\tilde{\mathcal{P}}$: Let $O_0, O_1, \ldots, O_n$ be open $G$-invariant subsets of the Lie algebra $\mathfrak{g}$ of $G$ where the exponential map is regular; at this stage, we just take for each of $O_0, O_1, \ldots, O_n$ the same maximal open connected $G$-invariant neighborhood of the origin of $\mathfrak{g}$ where the exponential map is regular; the subscripts then constitute merely a notational device. In (2.18) below we shall take $O_1 = \cdots = O_n = O$, where $O$ refers to the chosen invariant connected neighborhood of zero where the exponential map is a diffeomorphism. With the present choice of $O_0, O_1, \ldots, O_n$, define the space $\mathcal{H}(\tilde{\mathcal{P}}, G)$ by means of the pull back square

$$\begin{array}{ccc}
\mathcal{H}(\tilde{\mathcal{P}}, G) & \xrightarrow{\tilde{\eta}} & O_0 \times O_1 \times \cdots \times O_n \\
\downarrow & & \downarrow \text{exp} \times \cdots \times \text{exp} \\
\text{Hom}(\tilde{F}, G) & \xrightarrow{\tilde{r}, a_1, \ldots, a_n} & G_0 \times G_1 \times \cdots \times G_n,
\end{array}$$

(2.4)

where $\tilde{r}$ and $\tilde{a}_1, \ldots, \tilde{a}_n$ denote the smooth maps induced by, respectively, $r$ and $a_1, \ldots, a_n$, and where we have written $G_0 = G, G_1 = G, \ldots, G_n = G$; here $\tilde{\eta}$ is just a name for the corresponding map which results from the pull back construction. Since $\text{Hom}(\tilde{F}, G)$ is a smooth manifold, and since the exponential map, restricted to any of the $O_j$’s, is regular, $1 \leq j \leq n$, the space $\mathcal{H}(\tilde{\mathcal{P}}, G)$ is a smooth manifold, too.

Keeping the notation used in [9], for a group or groupoid $\Pi$, we denote by $(C_*(\Pi, R), \partial)$ the chain complex of its inhomogeneous reduced normalized bar resolution over $R$. Let $c$ be an absolute 2-chain of $F$ which represents a 2-cycle for the group system $(\pi_1, \pi_2, \ldots, \pi_n)$. Its image in the 2-chains of the fundamental group $\tilde{\pi}$ of the closed (!) surface $\tilde{\Sigma}$ is then closed. Write $\tilde{\kappa} \in H_2(\tilde{\pi})$ for its class. When the genus $\ell$ is different from zero, $\tilde{\pi}$ is non-trivial and the canonical map from the second homology group $H_2(\pi_1, \{\pi_k\})$ of the group system (cf. [7]) to $H_2(\tilde{\pi})$ is an isomorphism identifying the fundamental classes. Furthermore, when the relators $z_1, \ldots, z_n$ are added to (1.1) we obtain the presentation

$$\tilde{\mathcal{P}} = \langle x_1, y_1, \ldots, x_\ell, y_\ell, z_1, \ldots, z_n; r, z_1, \ldots, z_n \rangle$$

(2.5)

of the fundamental group $\tilde{\pi} = \pi_1(\tilde{\Sigma})$ of the closed surface $\tilde{\Sigma}$ resulting from capping of the $n$ boundaries.
We now apply a variant of the construction in Theorem 1 of [9]: Write $\tilde{\pi}_\partial$ for the free subgroupoid of $\tilde{F}$ having $p_1, \ldots, p_n$ as objects and $a_1, \ldots, a_n$ as morphisms; this groupoid may also be viewed as a subgroupoid of the fundamental groupoid $\tilde{\pi}$ of $\Sigma$, and we do no distinguish in notation between the two subgroupoids. Abstractly, $\tilde{\pi}_\partial$ amounts of course to a disjoint union of the $n$ free cyclic groups $\pi_1, \ldots, \pi_n$. We pick $c$ in such a way that

$$\partial c = [r] - [z_1] - \cdots - [z_n]$$

in the reduced normalized inhomogeneous bar resolution of $F$. This can always be done, cf. what is said in Section 2 of [7]. View $c$ as a 2-chain of $\tilde{F}$ by the embedding $\iota$ of $F$ into $\tilde{F}$ (cf. (2.2)) and let

$$\tilde{c} = c + \sum_j \left( [\gamma_j^{-1}|\gamma_j a_j] - [\gamma_j a_j|\gamma_j^{-1}] \right),$$

cf. [7 (8.7)]. Then

$$\partial \tilde{c} = [\tilde{r}] - [a_1] - \cdots - [a_n]$$

whence, in particular, $\tilde{c}$ is manifestly a relative cycle for $(\tilde{\pi}, \tilde{\pi}_\partial)$, cf. step 3 in the proof of the key lemma (8.4) in [7]. Notice that $c$ itself is not a relative cycle for $(\tilde{\pi}, \tilde{\pi}_\partial)$.

Let $\cdot$ be an invariant symmetric bilinear form on $\mathfrak{g}$, at this stage not necessarily nondegenerate, and let $\Omega = \omega_1 \cdot \omega_2 \in \Omega^2(G^2)$ be the indicated 2-form arising from the Maurer-Cartan form $\omega$ on $G$ with values in $\mathfrak{g}$, cf. [7, 9, 23]. Write

$$E: \tilde{F}^2 \times \text{Hom}(\tilde{F}, G) \to G^2$$

for the evaluation map, and let

$$\omega_{\tilde{c}} = \langle \tilde{c}, E^*\Omega \rangle,$$

the result of pairing $\tilde{c}$ with the induced form, cf. [7 (5.3)] and [9 (13)]. This is a $G$-invariant 2-form on $\text{Hom}(\tilde{F}, G)$. In view of [9 (15)] we have

$$d\omega_{\tilde{c}} = \langle \partial \tilde{c}, E^*\lambda \rangle.$$

Here $\lambda$ is the fundamental 3-form on $G$, cf. [7, 9, 23]. Let $h$ be the standard homotopy operator on forms on $\mathfrak{g}$ arising from integration along straight line segments, let $\beta = h(\exp^*\lambda)$, and define the 2-form $\omega_{\tilde{c}, \tilde{P}}$ on $\mathcal{H}(\tilde{P}, G)$ by

$$\omega_{\tilde{c}, \tilde{P}} = \tilde{\eta}^* \omega_{\tilde{c}} - \tilde{r}^* \beta + \tilde{a}_1^* \beta + \cdots + \tilde{a}_n^* \beta.$$

In view of (2.8) and (2.10), this 2-form is closed; cf. the reasoning in Section 7 of [7].
Let $\psi : g \to g^*$ be the adjoint of the given invariant symmetric bilinear form $\cdot$ on $g$, and write

$$\mu : \mathcal{H} (\tilde{P}, G) \to g_0^* \times \cdots \times g_n^*$$

for the composite

$$\mathcal{H} (\tilde{P}, G) \xrightarrow{\tilde{r}, a_1, \ldots, a_n} O_0 \times \cdots \times O_n \to g_0 \times \cdots \times g_n \xrightarrow{\psi \times (n+1)} g_0^* \times \cdots \times g_n^*.$$

Further, for $0 \leq j \leq n$, we write

$$\mu_j : \mathcal{H} (\tilde{P}, G) \to g_j^*$$

for the composite of $\mu$ with the projection onto the $j$'th copy $g_j^*$ of $g^*$ in $g_0^* \times \cdots \times g_n^*$.

The same reasoning as in Section 7 of [7] shows that

$$-\omega_{\tilde{c}, \tilde{P}} (X_H, \cdot) = d(X \circ \mu),$$

that is to say, the formal momentum mapping property is satisfied where we have written $H = \mathcal{H} (\tilde{P}, G)$ for short. Here $X_H$ denotes the vector field on $H$ coming from $X \in g_0 \times \cdots \times g_n$ via the action of $G_0 \times \cdots \times G_n$ on $H$. Likewise, for $0 \leq j \leq n$, $\mu_j$ is formally a momentum mapping for the corresponding action of the $j$'th copy $G_j$ of $G$ on $\mathcal{H} (\tilde{P}, G)$, that is,

$$-\omega_{\tilde{c}, \tilde{P}} (X_H, \cdot) = d(X \circ \mu_j),$$

where now $X_H$ refers to the vector field on $H$ coming from $X \in g_j$ via the $G_j$-action on $H$.

We now suppose that the given invariant symmetric bilinear form $\cdot$ on $g$ is nondegenerate; this allows us to identify $g$ tacitly with its dual $g^*$. In particular, this will always be meant below when we refer to the “preimage of the center with respect to the momentum mapping” and when we identify adjoint orbits with coadjoint ones.

**Theorem 2.13.** Near the zero locus of the map $\tilde{r} : \mathcal{H} (\tilde{P}, G) \to O_0$ brought into play in (2.4) or, more generally, near the preimage with respect to $\tilde{r}$ of the center of $g$, the $2$-form $\omega_{\tilde{c}, \tilde{P}}$ on $\mathcal{H} (\tilde{P}, G)$, cf. (2.11.1), is nondegenerate, that is, a symplectic structure.

We note that, when $O_1 = \cdots = O_n = O$, the chosen invariant connected neighborhood of zero, $O$, where the exponential map is a diffeomorphism, the smooth map $\tilde{c}$ (cf. (2.4)) is injective and the zero locus of $\tilde{r}$ may then be identified with the space $\text{Hom}(\tilde{r}, G)$.

To prepare for the proof, which will be given in (2.17) below, and to gain additional insight, we proceed as follows: Let $O = (O_1, \ldots, O_n)$ be an $n$-tuple of (co)adjoint orbits, each $O_k$ being in $O_k$, and let $C = (C_1, \ldots, C_n)$ be the corresponding $n$-tuple of conjugacy classes, where $C_k = \exp (O_k)$, for $1 \leq k \leq n$. We may in fact identify $O$ with the (co)adjoint orbit $O_1 \times \cdots \times O_n$ for $\Gamma = G_1 \times \cdots \times G_n$ and $C$ with the
conjugacy class $C_1 \times \cdots \times C_n$ of $\Gamma$. Since each $O_k$ is in $O_k$, each projection map $O_k \to C_k$ is a covering projection. Consider the mapping

$$
\mu_1 \times \mu_2 \times \cdots \times \mu_n : H(\tilde{P}, G) \to \mathfrak{g}_1^* \times \cdots \times \mathfrak{g}_n^* ;
$$

in view of (2.12.1) and (2.12.2), this is a momentum mapping for the action of $\Gamma$ on $H(\tilde{P}, G)$, with reference to the closed 2-form $\omega_{c, \tilde{P}}$. Whether or not the latter is nondegenerate, we can form the reduced space

$$
H(\tilde{P}, G)_{c, \tilde{P}}, \omega
$$

with reference to the $n$-tuple

$$
\omega = (\omega_1, \ldots, \omega_n)
$$

of the Kirillov forms $\omega_k$ on the $O_k$'s, the reduced space $H(\tilde{P}, G)_{c, \tilde{P}}$ inherits a closed 2-form

(2.14.1)

$$
\omega_{c, \tilde{P}, \omega},
$$

the corresponding reduced form, in the standard way. As usual, the notation $\mathcal{O}_1$, $\mathcal{O}_2$ etc. here indicates the (co)adjoint orbit $O_1$, $O_2$ etc., endowed with the negative of the symplectic (i.e. Kirillov) structure of $O_1$, $O_2$ etc. We may in fact view $\omega$ as the Kirillov form of the (co)adjoint orbit $O_1 \times \cdots \times O_n$ for $\Gamma = G_1 \times \cdots \times G_n$. The hamiltonian action of $G_0 \times \cdots \times G_n$ on $H(\tilde{P}, G)$ induces a hamiltonian action of $G_0$ on $H(\tilde{P}, G)_{c, \tilde{P}}$, preserving $\omega_{c, \tilde{P}, \omega}$ and having momentum mapping

(2.14.2)

$$
\mu_0 : H(\tilde{P}, G)_{c, \tilde{P}} \to \mathfrak{g}_0^*
$$

induced by $\mu_0$ or, what amounts to the same, induced by the map $\hat{r} : H(\tilde{P}, G) \to O_0$ given in (2.4). As a space,

$$
H(\tilde{P}, G)_{c, \tilde{P}} = (\mu_1 \times \cdots \times \mu_n)^{-1} (O_1 \times \cdots \times O_n) / (G_1 \times \cdots \times G_n).
$$

On the other hand, define the smooth manifold $H(\mathcal{P}, G)_{\Sigma}$ by means of the pull back square

(2.14.3)

$$
\begin{array}{c}
H(\tilde{P}, G)_{c, \tilde{P}} \\
\downarrow \eta^* \\
\Hom(F, G)_C \\
\downarrow \exp \times \exp \times \cdots \times \exp \\
O_0 \times O_1 \times \cdots \times O_n \to G_0 \times C_1 \times \cdots \times C_n,
\end{array}
$$

where the notation $\hat{r}, \tilde{z}_1, \ldots, \tilde{z}_n$ etc. is the obvious one. When the action of $\Gamma$ is divided out, the groupoid generators $\gamma_k$ are no longer relevant in the sense that the functor $i$ from $F$ to $\tilde{F}$ spelled out in (2.2) induces a diffeomorphism from $H(\tilde{P}, G)_{c, \tilde{P}}$ onto $H(\mathcal{P}, G)_{\Sigma}$, cf. also (2.3), and we identify henceforth the two spaces; notice in particular that at this stage the distinction in notation between the $a_k$'s and the $z_k$'s is no longer necessary.
In [7 (5.2)], for arbitrary conjugacy classes \(C_1, \ldots, C_n\), (i.e. not necessarily consisting of regular points for the exponential map), the space \(\mathcal{H}(\mathcal{P}, G)_{C}\), from which the extended moduli in that paper is obtained, is defined by means of the pull back square

\[
\begin{array}{ccc}
\mathcal{H}(\mathcal{P}, G)_{C} & \xrightarrow{(\hat{r}, \mathbf{z}_1, \ldots, \mathbf{z}_n)} & O_0 \times C_1 \times \cdots \times C_n \\
\downarrow \eta & & \downarrow \exp \times \text{Id} \times \cdots \times \text{Id} \\
\text{Hom}(F, G)_{C} & \xrightarrow{(r, \mathbf{z}_1, \ldots, \mathbf{z}_n)} & G_0 \times C_1 \times \cdots \times C_n,
\end{array}
\]

where \(\hat{r}\) and \(\mathbf{z}_1, \ldots, \mathbf{z}_n\) denote the smooth maps induced by, respectively, \(r\) and \(z_1, \ldots, z_n\); again \(\eta\) is just a name for the corresponding map which results from the pull back construction. Hence the obvious morphism of squares from (2.14.3) to (2.14.4) and the functor \(i\) from \(F\) to \(\hat{F}\) plainly induce a smooth mapping

\[
i^\sharp: \mathcal{H}(\tilde{\mathcal{P}}, G)_{\mathbf{\eta}} \to \mathcal{H}(\mathcal{P}, G)_{C}
\]

of \(G\)-spaces, \(G\) being, roughly speaking, the residual copy \(G_0\) of \(G\) in \(G_0 \times G_1 \times \cdots \times G_n\).

Recall that the corresponding momentum mapping

\[
\mu: \mathcal{H}(\mathcal{P}, G)_{C} \to g^\ast
\]

is induced by \(\hat{r}: \mathcal{H}(\mathcal{P}, G)_{C} \to O_0\); in [7 (Section 7)] this momentum mapping is written \(\mu: \mathcal{H}(\mathcal{P}, G)_{C} \to \mathfrak{g}^\ast\). It is obvious that \(i^\sharp\) is compatible with the momentum mappings (2.14.2) and (2.14.6) since both come from the relator \(r\). We remind the reader that, at this stage, for each \(j\), the exponential map, restricted to \(O_j\), is a covering projection onto \(C_j\), for \(0 \leq j \leq n\).

**Theorem 2.14.** Near the zero locus of the momentum mapping \(\mu_{O}\) brought into play in (2.14.2) or, more generally, near the preimage with respect to \(\mu_{O}\) of the center of \(g\), the reduced 2-form \(\omega_{\mathcal{P}, \mathcal{P}, \omega}\) on \(\mathcal{H}(\mathcal{P}, G)_{\mathbf{\eta}}\), cf. (2.14.1), is symplectic, and the map \(i^\sharp\) from \(\mathcal{H}(\tilde{\mathcal{P}}, G)_{\mathbf{\eta}}\) to \(\mathcal{H}(\mathcal{P}, G)_{C}\) is a covering projection of symplectic manifolds with hamiltonian \(G\)-action.

The proof will be postponed until (2.17) below; some prerequisites for it will be given in (2.15) and (2.16).

(2.15) **Presymplectic Reduction.** Let \(M\) be a smooth manifold endowed with a presymplectic (i.e. closed) 2-form \(\omega\) and an action of a Lie group \(K\) which is hamiltonian in the sense that there is a smooth \(K\)-equivariant map \(\mu\) from \(M\) to the dual \(\mathfrak{k}^\ast\) of the Lie algebra \(\mathfrak{k}\) which satisfies the formal momentum property

\[-\omega(X_M, \cdot) = d(X \circ \mu)(= X \circ d\mu),\]

where \(X_M\) refers to the vector field on \(M\) coming from \(X \in \mathfrak{k}\) via the \(K\)-action. Let \(p\) be a point of \(M\), and let

\[Z_p = \ker(d\mu_p) \subseteq T_p M, \quad B_p = T_p(Gp) \subseteq T_p M.\]

As usual, we write \(B_p^\omega\) etc. for the annihilator of \(B_p\) in \(T_p M\), with reference to \(\omega\).
2.15.1. The annihilator of $B_p$ coincides with $Z_p$, that is, $B^\omega_p = Z_p$.

Next we suppose that $\mu(p) = 0$, that is to say, that $p$ lies in the zero locus of the momentum mapping $\mu$.

2.15.2. The vector space $B_p$ is isotropic, that is, $B_p \subseteq B^\omega_p = Z_p$.

In fact, the restriction of $\mu$ to the orbit $Gp$ is constant, having constant value zero. Hence the differential $d\mu_p$, restricted to $B_p$ is zero whence, for $X, Y \in \mathfrak{k}$,

$$-\omega(X_M, Y_M)_p = X d\mu_p(Y_M) = 0.$$

2.15.3. The vector space $Z_p$ is coisotropic, that is, $Z^\omega_p \subseteq Z_p$.

In fact, since $B_p \subseteq Z_p$, we have $Z^\omega_p \subseteq B^\omega_p = Z_p$.—Let $H_p = Z_p/B_p$.

**Lemma 2.15.4.** The 2-form $\omega$ is nondegenerate near the point $p$ in the zero locus of $\mu$ if and only if the following holds: The alternating bilinear form $\omega_p$ on $H_p$ induced by $\omega$ is nondegenerate and the image $d\mu_p(T_pM)$ in $\mathfrak{k}^*$ consists precisely of the linear forms on $\mathfrak{k}$ which annihilate the Lie algebra $\mathfrak{k}_p$ of the stabilizer $K_p$ of the point $p$ of $M$.

**Proof.** It is a standard fact that the condition is necessary. Sufficiency is readily verified using a variant of the argument given in Section 5 of [9]: When $\omega_p$ is symplectic, the alternating bilinear form $\omega$ on $T_pM$, restricted to $Z_p$, has degeneracy space equal to the subspace $B_p$ of $Z_p$. The momentum mapping property, combined with the condition involving the stabilizer $\mathfrak{k}_p$ at $p$, then implies that on $T_pM$, the form $\omega$ is nondegenerate: Let $X \in T_pM$, and suppose that

$$\omega(X, Y) = 0, \text{ for every } Y \in T_pM.$$

Taking $Y = U_M$ for $U \in \mathfrak{k}$ and using the formal momentum mapping property, we see that $Ud\mu_p(X) = 0$ for every $U \in \mathfrak{k}$ whence $d\mu_p(X) = 0$ and hence $X \in Z_p$. The nondegeneracy of $\omega_p$ on $H_p$ then implies that $X \in B_p$, that is, $X = V_M$ for some $V \in \mathfrak{k}$. The formal momentum mapping property then yields $Vd\mu_p(Y) = 0$ for every $Y \in T_pM$, that is, $V$ annihilates the image $d\mu_p(T_pM)$ in $\mathfrak{k}^*$. Hence $V$ lies in $\mathfrak{k}_p$ whence $X = V_M$ vanishes at the point $p$. □

(2.16) PARABOLIC COHOMOLOGY. Let $\phi \in \text{Hom}(F, G)$, and suppose that $\phi(r)$ lies in the center of $G$. Then the composite of $\phi$ with the adjoint action of $G$ induces the structure of a left $\pi$-module on the Lie algebra $\mathfrak{g}$, and we write $\mathfrak{g}_\phi$ for $\mathfrak{g}$, viewed as a $\pi$-module in this way. The Reidemeister-Fox calculus, applied to the presentation (1.1), yields free resolution $\mathbf{R}(\mathcal{P})$, cf. [7 (2.2)], and application of the functor $\text{Hom}_{\mathbb{R}^\pi}(\cdot, \mathfrak{g}_\phi)$ to this free resolution yields the chain complex

$$C(\mathcal{P}, \mathfrak{g}_\phi) : C^0(\mathcal{P}, \mathfrak{g}_\phi) \xrightarrow{\delta^0_\phi} C^1(\mathcal{P}, \mathfrak{g}_\phi) \xrightarrow{\delta^1_\phi} C^2(\mathcal{P}, \mathfrak{g}_\phi),$$

cf. [9 (4.1)] and [7 (4.1)]; this chain complex computes the group cohomology $H^\ast(\pi, \mathfrak{g}_\phi)$. We note that there are canonical isomorphisms

$$C^0(\mathcal{P}, \mathfrak{g}_\phi) \cong \mathfrak{g}, \quad C^1(\mathcal{P}, \mathfrak{g}_\phi) \cong \mathfrak{g}^{2\ell+n}, \quad C^2(\mathcal{P}, \mathfrak{g}_\phi) \cong \mathfrak{g}.$$
To recall the geometric significance of this chain complex, denote by $\alpha_\phi$ the smooth map from $G$ to $\operatorname{Hom}(F,G)$ which assigns $x\phi x^{-1}$ to $x \in G$, maintain the notation $r$ for the smooth map from $\operatorname{Hom}(F,G)$ to $G$ induced by the relator $r$ so that the pre-image of the neutral element $e$ of $G$ equals the space $\operatorname{Hom}(\pi,G)$, and write $R_\phi: g^{2\ell+n} \to T_\phi \operatorname{Hom}(F,G)$ and $R_{r\phi}: g \to T_{r(\phi)}G$ for the corresponding operations of right translation. The tangent maps $T_e\alpha_\phi$ and $T_\phi r$ make commutative the diagram

\[
\begin{array}{ccc}
T_eG & \xrightarrow{T_e\alpha_\phi} & T_\phi \operatorname{Hom}(F,G) \\
\uparrow\text{Id} & & \uparrow R_\phi \\
g & \xrightarrow{\delta_0^\phi} & g^{2\ell+n} & \xrightarrow{\delta_1^\phi} & g.
\end{array}
\]

(cf. [7 (4.2)] and [9 (4.2)]. We now suppose that our chosen $\phi \in \operatorname{Hom}(F,G)$ lies in $\operatorname{Hom}(F,G)_C$, viewed as a subspace of $\operatorname{Hom}(F,G)$. For $k = 1, \ldots, n$, denote by $h_k$ the image in $g$ of the linear endomorphism given by $\operatorname{Ad}(\phi(z_k)) - \text{Id}$, so that there results the exact sequence

\[
(2.16.3) \quad 0 \to s_k \to g \to h_k \to 0
\]

of vector spaces, where $s_k$ denotes the Lie algebra of the stabilizer of $\phi(z_k)$; notice that $h_k$ amounts to the tangent space of the conjugacy class $C_k$. By Proposition 4.4 of [7], the values of the operator $\delta_0^\phi$ in (2.16.2) lie in $g^{2\ell} \times h_1 \times \cdots \times h_n$, viewed as a subspace of $C^1(\mathcal{P},g_\phi) \cong g^{2\ell} \times g^n$, and the first cohomology group of the resulting chain complex

\[
(2.16.4) \quad C_{\partial}(\mathcal{P},g_\phi): g \xrightarrow{\delta_0^\phi} g^{2\ell} \times h_1 \times \cdots \times h_n \xrightarrow{\delta_1^\phi} g
\]

equals the first parabolic cohomology group $H^1_{\partial}(\pi,\{\pi_j\};g_\phi)$ in the sense of [22]; the latter may in fact be defined to be the image of the relative cohomology group $H^1(\pi,\{\pi_j\};g_\phi)$ in the absolute cohomology group $H^1(\pi,g_\phi)$ under the canonical map. We now extend this notion by defining the parabolic cohomology $H^*_{\partial}(\pi,\{\pi_j\};g_\phi)$ of $(\pi,\{\pi_j\})$ with values in $g_\phi$ to be the cohomology of (2.16.4); in view of the cited Proposition 4.4 in [7], this yields the correct notion in degree 1. Further, it is straightforward to generalize this definition of parabolic cohomology to an arbitrary $\pi$-module $V$ but we shall not need this here.

We now determine $H^0_{\partial}(\pi,\{\pi_j\};g_\phi)$ and $H^2_{\partial}(\pi,\{\pi_j\};g_\phi)$. Let

\[
\Phi = (\operatorname{Ad}(z_1) - \text{Id}, \ldots, \operatorname{Ad}(z_n) - \text{Id}): g^n \to g^n.
\]

Application of the functor $\operatorname{Hom}_{R_{\pi}}(-,g_\phi)$ to the quotient complex $R(\tilde{\mathcal{P}},\{\pi_j\})$ constructed in [7 (2.8)] yields the cochain complex

\[
C(\tilde{\mathcal{P}},\{\pi_j\};g_\phi): g \xrightarrow{\delta_0^\phi} g^{2\ell} \times g^n \xrightarrow{\delta_1^\phi} g,
\]

and application of this functor to the comparison map [7 (2.12)] yields the cochain map

\[
(2.16.5) \quad (\text{Id},(\text{Id},\Phi),\text{Id}): C(\tilde{\mathcal{P}},\{\pi_j\};g_\phi) \to C(\mathcal{P},g_\phi)
\]
which induces the canonical map

\[ H^*(\pi, \{\pi_j\}; g_\phi) \to H^*(\pi, g_\phi). \]

By construction, (2.16.5) may be factored through (2.16.4); when we display this, we obtain the commutative diagram

\[ C(P, g_\phi): \quad g \xrightarrow{\delta_0_\phi} g^{2\ell} \times g^n \xrightarrow{\delta^1_\phi} g \]

\[ \begin{array}{c}
\text{Id} \\
\downarrow \text{(Id, incl)} \\
\downarrow \text{Id}
\end{array} \]

(2.16.6)

\[ C_{\text{par}}(P, g_\phi): \quad g \xrightarrow{\delta_0_\phi} g^{2\ell} \times h_1 \times \cdots \times h_n \xrightarrow{\delta^1_\phi} g \]

\[ \begin{array}{c}
\text{Id} \\
\downarrow \text{(Id, proj)} \\
\downarrow \text{Id}
\end{array} \]

\[ C(\tilde{P}, \{\pi_j\}; g_\phi): \quad g \xrightarrow{\delta_0_\phi} g^{2\ell} \times g^n \xrightarrow{\delta^1_\phi} g \]

where the notation \( \delta_0_\phi \) and \( \delta^1_\phi \) is slightly abused; here “proj” and “incl” refer to the obvious projection and inclusion mappings, and the composite incl \( \circ \) proj equals \( \Phi \). From the commutativity of (2.16.6) we deduce at once the following canonical isomorphisms

\[ H^0_{\text{par}}(\pi, \{\pi_j\}; g_\phi) \cong H^0(\pi, g_\phi); \quad H^2_{\text{par}}(\pi, \{\pi_j\}; g_\phi) \cong H^2(\pi, \{\pi_j\}; g_\phi). \]

Poincaré duality for the surface group system \((\pi; \pi_1, \ldots, \pi_n)\) yields the duality isomorphism

\[ \cap \kappa: H^2(\pi, \{\pi_j\}; g_\phi) \to H_0(\pi, g_\phi) \]

where \( \kappa \) refers to the fundamental class in \( H_2(\pi, \{\pi_j\}; \mathbb{R}) \), cf. Section 3 of [7]. Consequently Poincaré duality yields a canonical duality isomorphism

\[ \cap \kappa: H^2_{\text{par}}(\pi, \{\pi_j\}; g_\phi) \to H_0(\pi, g_\phi). \]

Since the chosen invariant symmetric nondegenerate bilinear from \( \cdot \) on \( g \) induces a nondegenerate bilinear pairing

\[ H^0(\pi, g_\phi) \otimes H_0(\pi, g_\phi) \to \mathbb{R} \]

as usual, from (2.16.8), we obtain the nondegenerate bilinear pairing

\[ H^0_{\text{par}}(\pi, \{\pi_j\}; g_\phi) \otimes H^2_{\text{par}}(\pi, \{\pi_j\}; g_\phi) \to \mathbb{R}. \]

It has been shown in Section 3 of [7] that \( \cdot \) induces a symplectic structure

\[ H^1_{\text{par}}(\pi, \{\pi_j\}; g_\phi) \otimes H^1_{\text{par}}(\pi, \{\pi_j\}; g_\phi) \to \mathbb{R} \]

in degree 1.
These observations yield the infinitesimal structure of the spaces $\text{Hom}(F,G)_C$ and $\mathcal{H}(P,G)_C$ by means of the commutative diagram

\[
\begin{array}{ccc}
T_eG & \xrightarrow{T_e\alpha} & T_\phi(\text{Hom}(F,G)_C) \\
\downarrow\text{Id} & & \downarrow R_\phi \\
C_{\text{par}}(P, g_\phi): g & \xrightarrow{\delta^0} & g^{2\ell} \times \mathfrak{h}_1 \times \cdots \times \mathfrak{h}_n \\
\downarrow R_{\tau(\phi)} & & \downarrow \delta^1 \\
& & g,
\end{array}
\]

(2.16.11)

having its vertical arrows isomorphisms of vector spaces. This diagram arises from the diagram (2.16.2) by restriction.

(2.17) **Proofs of Theorems 2.14 and 2.13.** The hypothesis that the given invariant symmetric bilinear form $\cdot \cdot$ on $g$ be nondegenerate is still in force. We now spell out explicitly an argument already used in Section 5 of [9], in Section 2 of [10], and in Section 8 of [7].

For $1 \leq k \leq n$, let $\tau_k$ be the 2-form on $C_k$ constructed in [7 (6.2)] and satisfying

\[(2.17.1) \quad \exp^* \tau_k = \beta - \omega_k\]
on $O_k$. We recall that the equivariant 2-form $\omega_{c, P, C}$ on $\mathcal{H}(P, G)_C$ given in [7 (7.1.1)] is defined by

\[(2.17.2) \quad \omega_{c, P, C} = \eta^* \omega_c - \tau_1^* \beta + \tau_2^* \tau_1 + \cdots + \tau_n^* \tau_n.\]

**Lemma 2.17.3.** *Near the zero locus of the momentum mapping $\mu_C: \mathcal{H}(P, G)_C \to g_0^*$ brought into play in (2.14.6), that is, near the space $\text{Hom}(\pi, G)_C$, viewed as a subspace of $\mathcal{H}(P, G)_C$ or, more generally, near the preimage with respect to $\mu_C$ of the center of $g$, the 2-form $\omega_{c, P, C}$ on $\mathcal{H}(P, G)_C$ is nondegenerate.*

**Proof.** Let $K = G$ and $M = \mathcal{H}(P, G)_C$, let $p$ be a point of $\mathcal{H}(P, G)_C$ having the property that the group (!) homomorphism $\phi = \eta(p)$ from $F$ to $G$, cf. (2.14.4) for the notation $\eta$, sends $r$ to a central element of $G$, and apply Lemma 2.15.4, with $\omega_{c, P, C}$ playing the role of $\omega$. To see that we are in the situation of that Lemma, we proceed as follows. Inspection of (2.16.11) shows that right translation identifies the vector space $H_p$ in (2.15.4) with the parabolic cohomology group $H^1_{\text{par}}(\pi, \{\pi_j\}; g_\phi)$ and the form $\omega_p$ with the induced 2-form (2.16.10). Moreover, the stabilizer Lie algebra $\mathfrak{k}_p$ and the cokernel of $d\mu_p: T_p M \to g^*$ amount to the zero'th and second cohomology group, respectively, of the chain complex (2.16.4) defining parabolic cohomology whence

$$
\mathfrak{k}_p \cong H^0_{\text{par}}(\pi, \{\pi_j\}; g_\phi), \quad \text{coker}(d\mu_p) \cong H^2_{\text{par}}(\pi, \{\pi_j\}; g_\phi).
$$

The nondegeneracy of the resulting pairing (2.16.9) implies that the image $d\mu_p(T_p M)$ in $g^*$ consists precisely of the linear forms on $g$ which annihilate the stabilizer Lie algebra $\mathfrak{k}_p$. Furthermore, when $p$ does not lie in the zero locus of the momentum mapping, adding a suitable constant, we may change the momentum mapping so that $p$ lies in the zero locus without changing the remaining geometrical data.
Hence the hypotheses of Lemma 2.15.4 are satisfied, whence the form $\omega_{c,p,c}$ is nondegenerate near the point $p$ as asserted. □

Proof of 2.14. The mapping $i^\sharp$, cf. (2.14.5), fits into the pull back square

$$
\begin{array}{c}
\mathcal{H}(\tilde{P},G)_\Sigma \\
\downarrow i^\sharp \\
\mathcal{H}(\mathcal{P},G)_C
\end{array}
\xrightarrow{(z_1,\ldots,z_n)}
\begin{array}{c}
O_1 \times \cdots \times O_n \\
\exp \times \cdots \times \exp \\
C_1 \times \cdots \times C_n
\end{array}
$$

(2.14.7)

whose vertical maps are the corresponding induced maps; since the exponential maps, restricted to the $O_k$'s, are covering projections, so is the map $i^\sharp$. Plainly, then,

$$
(i^\sharp)^* \omega_{c,p,c} = (i^\sharp)^* \eta^* \omega_c - (i^\sharp)^* \tau^* \beta + \tilde{z}_1^*(\beta - \omega_1) + \cdots + \tilde{z}_n^*(\beta - \omega_n).
$$

(2.14.8)

Comparison with (2.11) shows that this is the 2-form $\tilde{\omega}_{c,\tilde{P}}$ on $\mathcal{H}(\tilde{P},G)_\Sigma$ to which the closed 2-form $\omega_{c,\tilde{P}}$ on $\mathcal{H}(\mathcal{P},G)_C$ descends by reduction from $\mathcal{H}(\mathcal{P},G)$ to $\mathcal{H}(\mathcal{P},G)_\Sigma$. More precisely, the constituents, respectively,

$$
\tilde{\eta}^* \omega_c, \tilde{\tau}^* \beta, \tilde{a}_1^* \beta, \ldots, \tilde{a}_n^* \beta
$$

and

$$
(i^\sharp)^* \eta^* \omega_c, (i^\sharp)^* \tau^* \beta, \tilde{z}_1^* \beta, \ldots, \tilde{z}_n^* \beta
$$

correspond to each other, and the additional terms $-\tilde{z}_1^* \omega_1, \ldots, -\tilde{z}_n^* \omega_n$ arise since reduction is applied to $\mathcal{H}(\tilde{P},G)$ at the (co)adjoint orbit $O_1 \times \cdots \times O_n$ for $\Gamma = G_1 \times \cdots \times G_n$. The distinction between the groupoid chain $\tilde{c}$ and the group chain $c$ disappears when the action of $\Gamma$ is divided out, since then the groupoid generators $\gamma_k$ are no longer relevant. By Lemma 2.17.3, the 2-form $\omega_{c,\tilde{P}}$ on $\mathcal{H}(\mathcal{P},G)_C$ is nondegenerate near the zero locus. Consequently the 2-form $\omega_{c,\tilde{P}}$ on $\mathcal{H}(\mathcal{P},G)_\Sigma$ is nondegenerate. We have already observed that the momentum mappings (2.14.2) and (2.14.6) correspond as well. This proves Theorem 2.14. □

Proof of 2.13. Let $q \in \mathcal{H}(\mathcal{P},G)$. The image of $q$ under the momentum mapping

$$
\mu = \mu_1 \times \cdots \times \mu_n : \mathcal{H}(\mathcal{P},G) \to g_1^* \times \cdots \times g_n^*
$$

lies in precisely one (co)adjoint orbit $O_1 \times \cdots \times O_n$ for $\Gamma = G_1 \times \cdots \times G_n$ and, by construction, each $O_k$ consists of regular points for the exponential map. The reduced space $\mathcal{H}(\mathcal{P},G)_\Sigma$ arises from reduction at zero, applied to the diagonal action of $\Gamma = G_1 \times \cdots \times G_n$ on

$$
\mathcal{H}(\mathcal{P},G) \times \mathcal{O}_1 \times \cdots \times \mathcal{O}_n
$$

which is hamiltonian, with momentum mapping

$$
\mu - \nu : \mathcal{H}(\mathcal{P},G) \times \mathcal{O}_1 \times \cdots \times \mathcal{O}_n \to g_1^* \times \cdots \times g_n^*,
$$
where \( \iota \) denotes the inclusion of \( \mathcal{O}_1 \times \cdots \times \mathcal{O}_n \) into \( \mathfrak{g}_1^* \times \cdots \times \mathfrak{g}_n^* \). The resulting presymplectic structure on \( \mathcal{H}(\tilde{P}, G) \times \mathcal{O}_1 \times \cdots \times \mathcal{O}_n \) is given by

\[
(\omega_{\tilde{c},\tilde{p}}, -\omega_1, \ldots, -\omega_n);
\]

here \( \omega_{\tilde{c},\tilde{p}} \) is the 2-form (2.11.1). Let

\[
p = (q, \mu(q)) \in M = \mathcal{H}(\tilde{P}, G) \times \mathcal{O}_1 \times \cdots \times \mathcal{O}_n,
\]

and write \([p] \in \mathcal{H}(\tilde{P}, G)_{\mathcal{T}}\) for the point represented by \( p \). Then, with reference to the notation introduced in (2.15), \( \omega_{c,p,c} \) now playing the role of \( \omega \), the canonical map from \( H_p \) to \( T_p \mathcal{H}(\tilde{P}, G)_{\mathcal{T}} \) is an isomorphism identifying the form \( \omega_p \) with the reduced form \( \omega_{\tilde{c},\tilde{p},\omega} \) (cf. (2.14.1)) at \([p]\). A dimension count shows that, in this case, the stabilizer Lie algebra \( \mathfrak{t}_p \) is trivial and that the point \( p \) is regular for the momentum mapping \( \mu - \iota \). By (2.14), the 2-form \( \omega_{\tilde{c},\tilde{p},\omega} \) is nondegenerate at \([p]\). Hence the hypotheses of (2.15.4) are satisfied whence the 2-form (2.13.1) on \( \mathcal{H}(\tilde{P}, G) \times \mathcal{O}_1 \times \cdots \times \mathcal{O}_n \) is nondegenerate near \( p \) and hence the 2-form \( \omega_{\tilde{c},\tilde{p}} \) on \( \mathcal{H}(\tilde{P}, G) \) is nondegenerate near \( q \). This completes the proof of (2.13).

(2.18) The Proof of Main Result. We now prove the theorem spelled out in the introduction.

Consider the smooth manifold \( \mathcal{H}(\tilde{P}, G) \), with 2-form \( \omega_{\tilde{c},\tilde{p}} \), cf. (2.11.1); by (2.13), near the zero locus of \( \tilde{\pi}: \mathcal{H}(\tilde{P}, G) \to O_0 \) or, more generally, near the preimage of the center of \( \mathfrak{g} \), this 2-form is nondegenerate, that is, a symplectic structure. Let \( \mathcal{M} \subseteq \mathcal{H}(\tilde{P}, G) \) be an open \((G_0 \times \cdots \times G_n)\)-invariant submanifold where \( \omega_{\tilde{c},\tilde{p}} \) is nondegenerate and which contains the preimage of the center of \( \mathfrak{g} \). Abusing the notation introduced in (2.11), we shall denote the restrictions of \( \omega_{\tilde{c},\tilde{p}}, \mu, \mu_0, \mu_1 \), etc. to \( \mathcal{M} \) by, respectively, \( \omega_{\tilde{c},\tilde{p}} \), \( \mu \), \( \mu_0 \), \( \mu_1 \), etc. as well. The induced action of \( G_0 \times \cdots \times G_n \) on \( \mathcal{M} \) is hamiltonian, with momentum mapping

\[
\mu = \mu_0 \times \mu_1 \times \cdots \times \mu_n: \mathcal{M} \to \mathfrak{g}_0^* \times \mathfrak{g}_1^* \times \cdots \times \mathfrak{g}_n^*.
\]

The data \( (\mathcal{M}, \omega_{\tilde{c},\tilde{p}}, \mu) \) constitute the extended moduli space which we are looking for.

By the main result of [20], the reduced space \( \mathcal{M}_0 = \mu_0^{-1}(0)/G_0 \), for the action of the zero’th copy \( G_0 \) of \( G \) on \( \mathcal{M} \), inherits a stratified symplectic space structure. The action of \( G_0 \times G_1 \times \cdots \times G_n \) on \( \mathcal{M} \) manifestly induces a hamiltonian action (in the sense of stratified symplectic spaces) of \( \Gamma = G_1 \times \cdots \times G_n \) on \( \mathcal{M}_0 \). The subalgebra \( (C^\infty(\mathcal{M}_0))^\Gamma \) of \( \Gamma \)-invariant functions in \( C^\infty(\mathcal{M}_0) \) then yields a Poisson algebra \( (C^\infty(\mathcal{M}_0/\Gamma), \{\cdot, \cdot\}) \) of continuous functions on the quotient space \( \mathcal{M}_0/\Gamma \).

Recall that the smooth map \( \tilde{\eta}: \mathcal{H}(\tilde{P}, G) \to \text{Hom}(\tilde{F}, G) \) has been brought into play in (2.4), and write \( \mathcal{H}(\tilde{\pi}, G) = \tilde{\varphi}(\mu_0^{-1}(0)) \). This is the \( \tilde{\varphi} \)-image in \( \text{Hom}(\tilde{F}, G) \) of the zero locus \( \mu_0^{-1}(0) \) of the map \( \mu_0 \) from \( \mathcal{H}(\tilde{P}, G) \) to \( O_0 \); the space \( \mathcal{H}(\tilde{\pi}, G) \) clearly lies in \( \mathcal{M} \) as the zero locus \( \mu_0^{-1}(0) \) of \( \mu_0 \), now viewed as a map from \( \mathcal{M} \) to \( O_0 \). Restriction from the groupoid \( \tilde{\pi} \) to the group \( \pi \), cf. (2.3), induces a homeomorphism

\[
i^*: \text{Rep}(\tilde{\pi}, G) = \text{Hom}(\tilde{\pi}, G)/G^{n+1} \to \text{Rep}(\pi, G)
\]
which passes to a homeomorphism

\[ i^\natural: \mathcal{H}(\pi, G)/G^{n+1} \to \mathcal{R}(\pi, G). \]

Hence \( \tilde{\eta} \) induces a projection map from \( \mathcal{M}_0 = \mu_0^{-1}(0)/G \) onto \( \mathcal{H}(\pi, G)/G^{n+1} \) and hence a projection map

\[ q: \mathcal{M}_0/\Gamma \to \mathcal{R}(\pi, G). \]

We now suppose that the \( O_j \)'s, \( 1 \leq j \leq n \), are all equal to the chosen invariant connected neighborhood of zero, \( O \), where the exponential map from \( \mathfrak{g} \) to \( G \) is a diffeomorphism. Then the map \( \tilde{\eta} \) is injective and \( q \) is a homeomorphism. Thus the Poisson algebra \( (C^\infty(\mathcal{M}_0/\Gamma), \{ \cdot, \cdot \}) \) then yields a Poisson algebra \( (C^\infty(\mathcal{R}(\pi, G), \{ \cdot, \cdot \}) \) of continuous functions on \( \mathcal{R}(\pi, G) \); this is the Poisson algebra which we are looking for. Comparison of the construction of this Poisson structure with that of \( (C^\infty(\text{Rep}(\pi, G)_C), \{ \cdot, \cdot \}) \) in [7] shows that, on each moduli space \( \text{Rep}(\pi, G)_C \) (that lies in \( \mathcal{R}(\pi, G) \)), the Poisson structure on \( \mathcal{R}(\pi, G) \) restricts to the stratified symplectic Poisson algebra on \( \text{Rep}(\pi, G)_C \). The proof of the theorem spelled out in the introduction is now complete save that the additional statement saying that, for \( G \) compact, the subspace \( \mathcal{R}(\pi, G) \) may be taken dense in \( \text{Rep}(\pi, G) \), has not been justified yet. This will be achieved in the next section.

Remark 2.19. The chosen neighborhood \( O \) of the origin of the Lie algebra \( \mathfrak{g} \), and hence its image \( B \) in \( G \), inherits a Poisson structure from \( \mathfrak{g} \) (or rather from \( \mathfrak{g}^* \)), and this structure cannot be extended to \( G \). It would be interesting to know whether the obstruction to extending the Poisson structure on \( \mathcal{R}(\pi, G) \) to one on \( \text{Rep}(\pi, G) \) is formally of the same kind. The simplest case occurs perhaps for \( \ell = 0 \) and \( n = 3 \). (When \( \ell = 0 \) and \( n = 1 \) or \( n = 2 \) the space \( \text{Rep}(\pi, G)_C \) is either empty or consists of a single point.) Then \( \text{Rep}(\pi, G)_C \) is the space of \( G \)-orbits of triples \( (a, b, c) \in C_1 \times C_2 \times C_3 \) satisfying \( abc = e \) while \( \text{Rep}(\pi, G) \) is the space of \( G \)-orbits of triples \( (a, b, c) \in G \times G \times G \) satisfying \( abc = e \). Even though the induced Poisson structure on \( B \subseteq G \) cannot be extended to a Poisson structure on \( G \) it may still be possible to extend the Poisson structure on \( \mathcal{R}(\pi, G) \) to one on \( \text{Rep}(\pi, G) \). Another special case worth looking at arises with \( G = \text{SU}(2) \) and a surface \( \Sigma \) of genus \( \ell > 0 \) (say) with a single boundary circle. In this case, when \( O \) is taken to be the open ball in \( \mathfrak{g} = \text{su}(2) \) consisting of all \( X \) such that \( -\psi(X, X) < 8\pi^2 \) where \( \psi \) is the Killing form, the image \( B \) of \( O \) under the exponential map is the open ball \( \text{SU}(2) \setminus \{ \text{Id} \} \) in \( \text{SU}(2) \), and \( \text{Rep}(\pi, G) \) is the disjoint union of \( \mathcal{R}(\pi, G) \) and \( \text{Rep}_{-1}(\pi, G) \); here \( \text{Rep}_{-1}(\pi, G) \) refers to the space of twisted representations \( \phi \) of the fundamental group \( \pi \) of the corresponding closed surface \( \Sigma \) having the property that, after a choice \( x_1, y_1, \ldots, x_\ell, y_\ell \) of generators for \( \pi \) has been made,

\[
[\phi x_1, \phi y_1] \cdots [\phi x_\ell, \phi y_\ell] = -\text{Id} \in G = \text{SU}(2).
\]

The construction in the present paper endows \( \mathcal{R}(\pi, G) \) with a Poisson algebra of continuous functions, and the space \( \text{Rep}_{-1}(\pi, G) \) has been known for a while to carry a structure of a smooth symplectic manifold and hence in particular is endowed with the corresponding symplectic Poisson algebra structure [2, 7, 9, 12, 14, 18]. In the vector bundle picture, after a choice of (compatible) complex structure on \( \Sigma \) has been made, \( \text{Rep}_{-1}(\pi, G) \) may be identified with the space of stable holomorphic
vector bundles on $\Sigma$ having rank 2, degree 1, and trivial determinant. (There is no difference between semistable and stable vector bundles in this case.) But is there a Poisson algebra of continuous functions on $\text{Rep}(\pi, G)$ which, on $\mathcal{R}(\pi, G)$ and $\text{Rep}_{-1}(\pi, G)$, restricts to the corresponding Poisson algebras?

For general $\ell$ and $n$ and a general group $G$, the difference $\text{Rep}(\pi, G) \setminus \mathcal{R}(\pi, G)$ is the union in $\text{Rep}(\pi, G)$ of all spaces $\text{Rep}(\pi, G)_C$ for those $n$-tuples $C = (C_1, \ldots, C_n)$ of conjugacy classes which are not $B$-regular in the sense that at least one of the $C_k$ is not $B$-regular. However, in view of the main result of [7], the spaces $\text{Rep}(\pi, G)_C$ also arise by symplectic reduction, applied to suitable extended moduli spaces and hence inherit structures of stratified symplectic spaces. Among the obstacles to extending the Poisson structure from $\mathcal{R}(\pi, G)$ to $\text{Rep}(\pi, G)$ is the following one: The space $\mathcal{H}(\tilde{P}, G)$ coming into play in (2.4) is contained in a larger space $H(\tilde{P}, G)$ to be defined by means of a pull back square

$$
\begin{array}{ccc}
H(\tilde{P}, G) & \overset{(\tilde{r}, \tilde{a}_1, \ldots, \tilde{a}_n)}{\longrightarrow} & g_0 \times g_1 \times \cdots \times g_n \\
\downarrow \tilde{n} & & \downarrow \exp \times \cdots \times \exp \\
\text{Hom}(\tilde{F}, G) & \overset{(\tilde{r}, a_1, \ldots, a_n)}{\longrightarrow} & G_0 \times G_1 \times \cdots \times G_n.
\end{array}
$$

However, $H(\tilde{P}, G)$ cannot naively serve as an extended moduli space since it is not a smooth manifold: its singularities arise from the non-regular points of the exponential map. Moreover the relationship between the zero locus of $\tilde{r}$ and $\text{Rep}(\pi, G)$ is rather intricate and does not just come down to a homeomorphism after the action of $\Gamma$ on the zero locus has been divided out.

Remark 2.20. In [13 (5.2)], a certain space denoted $\mathcal{N}^{g,N}$ has been constructed and, by gauge theory methods, a symplectic structure on a smooth open part of $\mathcal{N}^{g,N}$ together with a hamiltonian action of a product $G^N$ of finitely many copies of $G$ on this space and a momentum mapping into $(g^*)^N$ have been obtained. An appropriate smooth open part of the space $\mathcal{N}^{g,N}$, where $N = n$, arises from $\mathcal{H}(\tilde{P}, G)$ by reduction with respect to the momentum mapping $\mu_0$. We do not know whether the symplectic structure on $\mathcal{N}^{g,N}$ constructed in [13 (5.2)] coincides with the symplectic structure obtained above by a purely finite dimensional construction via reduction from (2.11.1). An investigation of the difference between adjoint orbits and conjugacy classes, crucial for the construction of the Poisson structure on the ambient space, has not been undertaken in [13], though.
3. The case when $G$ is compact and connected

Throughout this section, $G$ will be a compact and connected Lie group. Our aim is to show that, roughly speaking, there is a “nice” choice for the open neighborhood $O$ of the origin in the Lie algebra $\mathfrak{g}$ of $G$ having dense image $B$ in $G$ under the exponential map.

Recall that a point of $G$ is called singular if it lies in two distinct maximal tori and regular otherwise, cf. e. g. [5 p. 168]. The set of regular points is denoted by $G_r$. The set $G \setminus G_r$ of singular points has codimension $\geq 3$ [5 (V.2.6)] and hence $G_r$ is, in particular, dense in $G$. Thus, in view of the well known structure of $G_r$ for $G$ compact, connected, and simply connected, (and hence semisimple) [5 (V.7.11)], we could in fact choose $O$ in $\mathfrak{g}$ in such a way that the image $B = \exp(O)$ coincides with $G_r$. With this choice, we would immediately get $\mathcal{R}(\pi,G)$ dense in $\mathrm{Rep}(\pi,G)$.

This choice of $O$ (and hence $B$) will not be “nice”, though, since $B$ would not then contain e. g. the neutral element of $G$. We now show how $G_r$ may be enlarged to a suitable subset $B$ of $G$ having better properties for our purposes.

Choose a root of the equation $z^2 + 1 = 0$ and denote it by $i$ as usual. Let $O$ be the neighborhood of the origin in $\mathfrak{g}$ consisting of those $X$ in $\mathfrak{g}$ which have the property that the endomorphism $\mathrm{ad}(X)$ of $\mathfrak{g}$ has only eigenvalues $\lambda = 2\pi i \nu$ with $|\nu| < 1$. The following is presumably known; we did not find it in the literature.

**Lemma 3.1.** When $G$ is and simply connected (and hence semisimple), the exponential map, restricted to $O$, is a diffeomorphism onto its image $B$ in $G$.

We now prepare for the proof of this Lemma. To fix notation, we recall that, after a choice of maximal torus $T$ in $G$ has been made, the global weights $\vartheta:T \to S^1$ of the adjoint representation are called global roots, the corresponding linear forms $\Theta:T \to i\mathbb{R}$ the infinitesimal roots, and the real linear forms $\alpha:T \to \mathbb{R}$ such that $\Theta = 2\pi i \alpha$ are said to be the real roots of $G$. Given a point $X \in \mathfrak{g}$ which is singular for the exponential map, $\exp(X)$ is a singular point of $G$ but the converse is not necessarily true:

**Proposition 3.2.** A point $X$ of $\mathfrak{g}$ is singular for the exponential map if and only if $\exp(X)$ is a singular point of $G$ (in the sense of Lie groups) in such a way that, for some real root $\alpha$ and some integer $k \neq 0$, $\alpha(X) = k$.

**Proof.** Let $X \in \mathfrak{g}$; it lies in a Cartan subalgebra of $\mathfrak{g}$, and the point $\exp(X)$ lies in the corresponding maximal torus $T$. With respect to a basis of weight vectors, the complexification of $\mathrm{ad}(X)$ is then in diagonal form, with entries $\Psi(X)$ and 0 where $\Psi$ runs through the infinitesimal roots. Now $\exp(X)$ is a singular point of $G$ if and only if, for some real root $\alpha$ and some integer $k$ (which may be zero), $\alpha(X) = k$. On the other hand, the exponential map is non-regular at a point $X$ of the Lie algebra $\mathfrak{g}$ if and only if the endomorphism $\mathrm{ad}(X)$ of $\mathfrak{g}$ has an eigenvalue of the kind $2\pi ik, k \neq 0$, $k$ being a nonzero integer, cf. Corollary 1 on p. 106 of [16]. This implies the assertion. □

We now spell out two immediate consequences of (3.2).

**Corollary 3.3.** The subset $O$ is the (uniquely determined) largest connected neighborhood $O$ of the origin of $\mathfrak{g}$ where the exponential map is regular. □
Write $O_r$ for the subset of $O$ consisting of those $X$ in $\mathfrak{g}$ which have the property that $\text{ad}(X)$ has only eigenvalues $\lambda = 2\pi i \nu$ with $0 < |\nu| < 1$.

**Corollary 3.4.** When $G$ is simply connected, the subset $O_r$ is mapped diffeomorphically onto the set $G_r$ of regular points of $G$. \(\square\)

We choose a maximal torus $T$ in $G$, with Lie algebra $\mathfrak{t}$; let $r = \dim T (= \text{rank}(G))$. The *diagram* is the inverse image in $\mathfrak{t}$ of the set of singular points (in the sense of Lie groups) in $T$, with respect to the exponential map [5 (V.7)]. Alternatively, in view of (3.2), the diagram consists of the hyperplanes $L_{\alpha n} = \alpha^{-1}(n)$ in $\mathfrak{t}$ where $\alpha$ runs through the real roots of $G$ and $n$ through the integers. Further, a choice of fundamental Weyl chamber corresponds to a choice of a system $S$ of simple roots and vice versa; the *walls* of the Weyl chamber are then the hyperplanes $L_{\alpha} = L_{\alpha 0}$ where $\alpha$ runs through the corresponding simple roots.

We now choose a fundamental Weyl chamber; in terms of the corresponding system $S$ of simple (real) roots, this Weyl chamber consists of those $X$ in $\mathfrak{t}$ which satisfy the condition $\alpha(X) > 0$ for every $\alpha \in S$. The choice of fundamental Weyl chamber uniquely determines an alcove (in $\mathfrak{t}$), that is, a connected component $P$ of the complement of the diagram, in such a way that the origin of $\mathfrak{t}$ is a vertex of the closure $\overline{P}$ and that, for each wall of the fundamental Weyl chamber, a certain (uniquely determined) convex subset thereof containing the origin of $\mathfrak{t}$ constitutes a wall of $P$ (but $P$ will have other walls). The map from $(G/T) \times P$ to $G$ which assigns $\exp(\Lambda) x^{-1}$ to $(x T , \Lambda) \in (G/T) \times P$ is known to be a universal covering with a connected total space onto the subset $G_r$ of regular elements of $G$, in fact, a diffeomorphism when $G$ is simply connected, see e. g. [5 (V.7.11)].

Let $\tilde{P}$ be the subspace of the closed alcove $\overline{P}$ which is obtained when those walls of $\overline{P}$ are removed which do not lie in any of the walls of the chosen Weyl chamber. In other words,

$$\tilde{P} = \{ X \in \mathfrak{t}; \; \alpha(X) \geq 0 \text{ for } \alpha \in S, \; \alpha(X) < 1 \text{ for } \alpha \not\in S \}.$$

Since, for any $X \in \mathfrak{g}$, with respect to a basis of weight vectors, the complexification of $\text{ad}X$ is in diagonal form, with entries $\pm 2\pi i \alpha(X)$ and 0 where $\alpha$ runs through the positive roots of $G$, the subspace $\tilde{P}$ of $\overline{P}$ consists precisely of those $X$ in $\mathfrak{t}$ which have the property that the endomorphism $\text{ad}(X)$ of $\mathfrak{g}$ has only eigenvalues $\lambda = 2\pi i \nu$ with $|\nu| < 1$. Consequently the image of the map from $(G/T) \times \tilde{P}$ to $\mathfrak{g}$ which assigns $\text{Ad}(x) \Lambda$ to $(x T , \Lambda) \in (G/T) \times \tilde{P}$ is precisely the neighborhood $O$ of the origin in $\mathfrak{g}$. In view of (3.2), every point of this neighborhood $O$ is regular for the exponential map.

**Proof of Lemma 3.1.** Suppose that $G$ is simply connected. We claim that then the restriction of the exponential map to $O$ is injective. It is clearly injective when restricted to the subset $O_r$ since the map from $G/T \times P$ to $G$ which sends $(x T , \Lambda) \in (G/T) \times P$ to $\exp(\Lambda) x^{-1}$ is a diffeomorphism onto the subset $G_r$ of regular elements of $G$. Furthermore, $G$ being simply connected, the map from $\overline{P}$ to $G$ which maps $\Lambda$ to $\exp(\Lambda)$ yields a bijective correspondence between the points of $\overline{P}$ and the conjugacy classes of $G$ and hence between adjoint orbits and conjugacy classes of $G$. We now claim that this implies the injectivity of the restriction of
the exponential map to $O$. In fact, let $X, Y$ be two points of $O$ having the same image under the exponential map. If one of them is in $O_r$, the other one must be in $O_r$, too, since $O_r$ is the preimage of the regular points of $G$, and hence $X$ and $Y$ must coincide, since the corresponding map from $(G/T) \times O_r$ to $G_r$ is a diffeomorphism. If the points $X, Y$ of $O$ are both not in $O_r$, they necessarily lie in the same adjoint orbit $O \subseteq O$ since the map from $\mathcal{P}$ to $G$ sending $\Lambda \in \mathcal{P}$ to $\exp \Lambda$ yields a bijective correspondence between adjoint orbits and conjugacy classes of $G$. Now, the exponential map, still being regular on $O$, is just a covering projection from $O$ onto its image, which is a conjugacy class (say) $C$. However, $G$ being simply connected, in view of an unpublished result of Bott’s, cf. [4] (Theorem 3.4) and [19], the conjugacy class $C$ is simply connected or, equivalently, the centralizer $Z_x$ of any $x \in G$ is connected; see also what is said on p. 351 of [8]. Hence the covering projection from $O$ onto $C$ is a diffeomorphism. Consequently the exponential map, restricted to $O$, is injective. □

Next we consider a compact, connected, semisimple Lie group $G$ which is not necessarily simply connected. Then the fundamental group $\pi_1 G$ acts on the ball $\hat{B}$ of regular values of the exponential map from the Lie algebra $\mathfrak{g}$ to the universal cover $\tilde{G}$, and we may choose for $B$ the image in $G$ under the covering projection of a suitable fundamental domain for the $\pi_1 G$-action on $\hat{B}$. We now describe this somewhat more explicitly: The closed alcove $\mathcal{P}$ is a regular polyhedron, (referred to sometimes as fundamental polyhedron in the literature,) and the fundamental group $\pi_1 G$ may be realized as a finite group of automorphisms thereof, cf. [5] (V.7.17) Ex. 5]. With reference to this group of automorphisms, we then choose a fundamental domain $D$ in $P$ whose closure $\overline{D}$ contains the origin and meets each wall of the Weyl chamber in a convex subset of dimension $r - 1$. Let $\overline{D}$ be the subspace of $\mathcal{P}$ which is obtained when those walls of $\overline{D}$ are removed which do not lie in any of the walls of the chosen Weyl chamber, and let $O$ be the image of $\mathcal{P}$ in $\mathfrak{g}$ under the canonical map from $G/T \times \overline{D}$ to $\mathfrak{g}$; this set $O$ is a neighborhood of the origin in $\mathfrak{g}$. Further, the exponential map, restricted to $O$, is a diffeomorphism from $O$ onto its image $B$ in $G$, and $B$ is a neighborhood of the neutral element of $G$ which contains $G_r$ by construction and hence is, in particular, dense in $G$. For $G = \text{SO}(3)$, the closed alcove $\mathcal{P}$ is a line segment having the origin $o$ of $\mathfrak{t} \cong \mathbb{R}$ as a vertex, and $\overline{D} = oQ$, $Q$ being the midpoint of $P$, is half of $\mathcal{P}$ which still contains the origin $o$ as a vertex; then $\overline{D} = \overline{D} \setminus \{Q\}$, and the point $Q$ corresponds to the conjugacy class of

$$
\begin{bmatrix}
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{bmatrix};
$$

this explains the remark made about $\text{SO}(3)$ in the introduction.

Finally, let $G$ be an arbitrary compact connected Lie group. Then a suitable covering $\tilde{G}$ of $G$ may be written as a product $\tilde{G} = T \times K$ where $T$ is a torus and $K$ is compact and simply connected. We may then apply to $K$ what has been said already in order to find an open neighborhood $O_K$ of the origin of the Lie algebra $\mathfrak{t}$ of $K$ and, furthermore, we may choose an open subset $O_T$ of the origin of the Lie algebra $\mathfrak{t}$ of $T$, in such a way that the exponential maps are diffeomorphisms onto their images $B_T$ and $B_K$ in $T$ and $K$, respectively, and that these images are dense in the corresponding groups. A suitable choice of fundamental domain in $O_T \times O_K$ for the action of the group of deck transformations of the covering from $T \times K$ onto $G$ will then yield an open neighborhood $O$ of the origin of the Lie
algebra $\mathfrak{g}$ of $G$ such that the exponential map, restricted to $O$, is a diffeomorphism onto its image $B$ in $G$, and so that $B$ is, furthermore, dense in $G$.

These considerations show that, for any compact and connected Lie group $G$, the subspace $R(\pi, G)$ may be taken dense in $\text{Rep}(\pi, G)$ in a certain "nice way" made precise above. The proof of the theorem in the introduction is now complete.

**Example 3.5.** We give a brief description of adjoint orbits and conjugacy classes and their mutual relationships, for the group $SU(n)$; since the other compact semisimple linear groups appear as closed subgroups of the special unitary group this gives certain information about other groups for free.

The group $SU(n)$ consists of complex $(n \times n)$-matrices $A$ of determinant 1 such that $\overline{A^tA} = E$, the standard maximal torus being the subgroup of diagonal matrices $\text{diag}(z_1, \ldots, z_n)$ with $z_1 \cdots z_n = 1$ where $z_j = \exp(2\pi i \nu_j)$, for $1 \leq j \leq n$. Its Lie algebra $\mathfrak{su}(n)$ is the space of skew-hermitian matrices of trace zero. We write such a matrix as $M = 2\pi i N$; the adjoint orbit through such an $M$ is determined by its eigenvalues $\lambda_1 = 2\pi i \nu_1, \ldots, \lambda_n = 2\pi i \nu_n$. The Lie algebra $\mathfrak{t}$ of the maximal torus is the space of matrices $M = 2\pi i N$ where $N = \text{diag}(\nu_1, \ldots, \nu_n)$ with $\nu_1 + \cdots + \nu_n = 0$.

The simple real roots $\alpha_j$ are given by $\alpha_j = \nu_j - \nu_{j+1}$, for $1 \leq j < n$, and the fundamental Weyl chamber consists of the matrices $2\pi i N$ where $N = \text{diag}(\nu_1, \ldots, \nu_n)$ with $\nu_1 \geq \cdots \geq \nu_n$. The subspace $\overline{P}$ of the closed alcove $\overline{P}$ introduced above is then given by the matrices $2\pi i \text{diag}(\nu_1, \ldots, \nu_n)$ in the fundamental Weyl chamber which satisfy the additional condition $\nu_1 - \nu_n < 1$. (The root $\nu_1 - \nu_n$ is the highest weight of the adjoint representation.)

The stabilizer $Z_M$ in $SU(n)$ of $M = 2\pi i N$ in $\mathfrak{t}$ depends on how many coincidences there are amongst the eigenvalues. Thus, if the first $n_1$ are equal, the next $n_2$ are equal, and so on,

$$Z_M = (U(n_1) \times U(n_2) \times \cdots \times U(n_s)) \cap SU(n).$$

Then, if $M$ lies in the fundamental Weyl chamber, it lies in $n_1 + n_2 + \cdots + n_s - s$ walls thereof, with the convention that this number to be zero means that $M$ lies in the interior (which only happens when all the $n_j$ equal 1 and $s = n$). The adjoint orbit $\mathcal{O}_M$ through $M$ is then the homogeneous space $SU(n)/Z_M$. When $M$ lies in $\overline{P}$, the exponential map identifies $\mathcal{O}_M$ with the conjugacy class $C_m$ through $m = \exp(M) \in SU(n)$. However, when $M$ lies in $\overline{P} \setminus \overline{P}$, the stabilizer $Z_m$ of $m = \exp(M) \in SU(n)$ is strictly larger than the stabilizer $Z_M$ of $M$ and the resulting fibre bundle projection from the adjoint orbit $\mathcal{O}_M$ onto the conjugacy class $C_m$ through $m$ has as fibre the homogeneous space $Z_m/Z_M$ which has strictly positive dimension.

For example, when $n = 3$, the fundamental Weyl chamber is given by the triples $(\nu_1, \nu_2, \nu_3)$ with $\nu_1 \geq \nu_2 \geq \nu_3$ and $\nu_1 + \nu_2 + \nu_3 = 0$; its two walls consist of the points $2\pi i \text{diag}(\nu, \nu, -2\nu)$ and $2\pi i \text{diag}(2\nu, -\nu, -\nu)$, where $\nu \geq 0$. The alcove $P$ is determined by the additional condition $\nu_1 - \nu_3 < 1$. The conjugacy class through a regular point of $SU(3)$ is the six dimensional flag manifold $SU(3)/(S^1 \times S^1)(\cong U(3)/(S^1)^3)$. The points of $\overline{P}$ which lie in any of the two walls of the Weyl chamber look like $M = 2\pi i \text{diag}(\nu, \nu, -2\nu)$ or $M = 2\pi i \text{diag}(2\nu, -\nu, -\nu)$ with $0 \leq \nu < \frac{1}{3}$; for $\nu > 0$, the adjoint orbit $\mathcal{O}_M$ through such a point $M$ is a 2-dimensional complex
projective space $P_2\mathbb{C}$, and the exponential map induces a diffeomorphism onto the corresponding conjugacy class. However, the points of $\overline{P} \setminus \tilde{P}$ are precisely of the kind $M = 2\pi i \text{diag}(\nu, 1 - 2\nu, \nu - 1)$ with $\frac{1}{3} \leq \nu \leq \frac{2}{3}$. In this case, for $\frac{1}{3} < \nu < \frac{2}{3}$, the conjugacy class $\mathcal{O}_M$ through such a point $M$ is still a six dimensional flag manifold of the kind $U(3)/(S^1)^3$ while the conjugacy class $C_m$ through $m = \exp M$ is a 2-dimensional complex projective space, the projection mapping from $\mathcal{O}_M$ to $C_m$ being the canonical one from the flag manifold to $P_2\mathbb{C}$ having as fibre a 1-dimensional complex projective space. In the extreme cases $\nu = \frac{1}{3}$ and $\nu = \frac{2}{3}$, we get here the two points $M_1 = 2\pi i \text{diag}(\frac{1}{3}, \frac{1}{3}, -\frac{2}{3})$ and $M_2 = 2\pi i \text{diag}(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3})$, respectively, each of which lies in a wall of the fundamental Weyl chamber as a vertex of $P$ but, beware, $M_1$ and $M_2$ do not lie in $\tilde{P}$; the adjoint orbits $\mathcal{O}_{M_1}$ and $\mathcal{O}_{M_2}$ of these points are 2-dimensional complex projective spaces but their images $m_1$ and $m_2$, respectively, in $G$ under the exponential map are central elements of $G$ and hence the conjugacy classes of these images consist each of a single point.

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