Finite-temperature effective potential for gauge models in de Sitter space

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Abstract. The 1-loop effective potential for gauge models in static de Sitter space at finite temperatures is computed by means of the $\zeta$-function method. We found a simple relation which links the effective potentials of gauge and scalar fields at all temperatures.

In the de Sitter invariant and zero-temperature states the potential for the scalar electrodynamics is explicitly obtained, and its properties in these two vacua are compared. In this theory the two states are shown to behave similarly in the regimes of very large and very small radii $a$ of the background space. For the gauge symmetry broken in the flat limit ($a \to \infty$) there is a critical value of $a$ for which the symmetry is restored in both quantum states.

Moreover, the phase transitions which occur at large or small $a$ are of the first or second order, respectively, regardless of the vacuum considered. The analytical and numerical analysis of the critical parameters of the above theory is performed. We also established a class of models for which the kind of phase transition occurring depends on the choice of the vacuum.

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1. Introduction

The aim of this paper is to investigate the properties of the finite-temperature (FT) effective potential of the gauge models in the de Sitter space. The temperature $\beta^{-1}$ is introduced as the temperature of quantum fields which are in thermal equilibrium in the static de Sitter coordinates. Thus, the corresponding quantum state is determined by the Green’s function which is periodical along the time coordinate of the static frame with period $i\beta$. This enables one to use the Euclidean formulation of the theory. In particular, one can define the partition function $Z_{\beta}$ for arbitrary $\beta$ as a functional integral, where the field configurations belong to the spherical domain $S^d_{\beta}$ with conical singularities on the Euclidean horizon.

There are several motivations for this work.

The first one is to obtain the 1-loop effective potential on such non-trivial spaces such as $S^d_{\beta}$ in an analytical form. This is a continuation of the pioneering studies by Shore [1]
and Allen [2, 3] which investigated the effective actions on hyperspheres \( S^4 \). Furthermore, the paper extends the results of [4] concerning the scalar fields, to more interesting gauge models with spontaneous symmetry breaking. A second motivation is related to the possible application of these results to cosmological problems since the exponentially expanding phase of the early universe [5] is described by the de Sitter geometry. Our analysis has become possible after the spectrum of vector Laplacians on singular \( d \)-spheres has been found explicitly in [6]. Finally, it is worth pointing out that quantum field theory on manifolds with conical singularities have drawn considerable attention in connection with the off-shell formulation of black-hole thermodynamics (see [7] for a review). However, as far as we know, the only computation of 1-loop effective action in four-dimensional spaces with conical singularities is contained in [4]. Thus, we think that another example of such a kind of computation, such as the one performed in the present analysis, would certainly be interesting.

In this paper special attention is given to the calculation and comparison of the effective potentials for two physically relevant quantum states: the \( dSI \) invariant (\( dSI \)) and the zero-temperature (\( ZT \)) states. They can both be considered as possible vacua in quantum-field theory in the de Sitter space.

The \( dSI \) state resembles the Poincaré invariant vacuum of the Minkowski spacetime since it preserves the symmetry of the de Sitter space. The corresponding temperature is equal to \((\beta_H)^{-1} = (2\pi a)^{-1}\), where \( a \) denotes the de Sitter radius [8] and the Euclidean section of the de Sitter space is the hypersphere \( S^4 \). In analogy with black-hole physics this temperature can be called the Hawking temperature. In the Gibbons–Hawking path integral [9] the Euclidean gravitational action with a positive cosmological constant has the extremum on a 4-sphere \( S^4 \). Thus, the choice of the \( dSI \) state has a natural explanation in the semiclassical treatment of the quantum cosmology. For this reason the 1-loop effective action for quantum fields on \( S^4 \) has been studied in a number of works [2–4, 10, 11]. In particular, some authors [3, 10, 11] investigated the phase structure of the GUT models with respect to the value of the de Sitter radius \( a \).

The \( dSI \) vacuum is analogous to the Hartle–Hawking vacuum [12] introduced for quantum fields around an eternal black hole. From this point of view the \( ZT \) state \((\beta \to \infty)\) in the de Sitter space is analogous to the Boulware vacuum [13]. The properties of this state are much less investigated, and here we use the explicit expressions of the effective potential to see how the choice of the vacuum affects the phase structure of the gauge theories with spontaneous symmetry breaking.

The paper is organized as follows. In section 2, we briefly discuss why the Euclidean functional integral on the spherical domains \( S^4_\beta \) can be interpreted as a partition function in the static de Sitter space. The FT effective potential for gauge models is computed with the \( \zeta \)-function method in section 3. In section 4 we compare the symmetry-breaking mechanisms at different values of \( a \) in the two vacua for models of scalar electrodynamics. A summary of the results is given in section 5. Technical details concerning the derivation of the \( \zeta \)-function are given in the appendix.

2. Partition function

In the static coordinates the line element of the de Sitter space with radius \( a \) can be written in the form

\[
\text{d}s^2 = \cos^2 \chi \text{d}r^2 - a^2 (d\chi^2 + \sin^2 \chi \text{d}\theta_1^2 + \sin^2 \chi \sin^2 \theta_1 \text{d}\theta_2^2), \quad (2.1)
\]
where $-\infty < t < +\infty$, $|\chi| \leq \pi/2$, and $0 \leq \theta_1, \theta_2 \leq \pi$. The above coordinates (2.1) only cover a part of the space. One can consider a Killing vector field generating a one-parameter group of isometries, subgroup of $SO(1, 4)$. Then coordinates (2.1) can be related to the timelike part of the Killing field associated with the translations along time $t$. This region is restricted by the Killing horizon where the Killing field is null. It also coincides with the event horizon for the observers with coordinates $\chi = 0$.

Let $G(x, x')$ be a scalar Green function in the de Sitter spacetime for the dSI state (some explicit expressions can be found in [14–16]). This function is periodic when the time coordinates of points $x$ or $x'$ are independently increased by $i/\beta_H$ where $\beta_H = 2\pi a$. Thus, the Euclidean Green function

$$G^E(\tau - \tau', u, u') = iG(i(\tau - \tau'), u, u'),$$

which is obtained from $G$ with the help of a Wick rotation, is defined on a 4-sphere $\mathbb{S}^4$ with the line element

$$ds^2 = \cos^2 \chi d\tau^2 + a^2 d\chi^2 + \sin^2 \chi (d\theta_1^2 + \sin^2 \chi d\theta_2^2),$$

where $0 \leq \tau \leq \beta_H$. Note that because of the periodicity property in imaginary time, the function $G(x, x')$ can be interpreted as a Green function for a canonical ensemble at temperature $1/\beta_H$. The physical meaning behind this interpretation is that any freely moving observer experiences the dSI state as a thermal bath at temperature $1/\beta_H$, see [8].

A natural generalization of $G(x, x')$ is a Green function $G^E_\beta(x, x')$ in the de Sitter space which is periodic in imaginary time with an arbitrary period $\beta$. Functions of this kind can be constructed from $G(x, x')$ with the help of a reperiodization formula suggested by Dowker [17, 18]. One can interpret $G^E_\beta(x, x')$ as a Green function of a canonical ensemble of particles in the static part (2.1) of de Sitter space at temperature $1/\beta_H$. It is also possible to introduce the Euclidean function $G^E_\beta(x, x')$ by using a definition analogous to equation (2.2). The space that $G^E_\beta(x, x')$ is set on is a spherical domain $S^4_\beta$, which is described by the metric (2.3) of the 4-sphere, but now with period $\beta$ along $\tau$.

The main property of $S^4_\beta$ is that it has conical singularities at the points $\chi = \pm \pi/2$, near which the space looks like $S^2 \times C$, where $C$ is a cone with deficit angle $2\pi(1 - \beta/\beta_H)$.

Let us now consider the functional integral

$$Z_\beta = \int [D\phi] e^{-iL_E[\phi, g_{\mu\nu}]}\),$$

where $L_E[\phi, g_{\mu\nu}]$ is the classical Euclidean action for the fields $\phi$ on $S^4_\beta$, $g_{\mu\nu}$ is the metric tensor defined by equation (2.3) and $[D\phi]$ is a covariant integration measure. The function $G^E_\beta$ can be obtained from this integral in the standard way. For instance, for a real scalar field $\phi$ it reads

$$G^E_\beta(x, x') = Z_\beta^{-1} \int [D\phi] \phi(x)\phi(x') e^{-iL_E[\phi, g_{\mu\nu}]}\).$$

It follows from (2.4) and (2.5) that integral $Z_\beta$ is analogous to a statistical-mechanical partition function $\text{Tr} e^{-\beta H}$ of a canonical ensemble of particles at temperature $1/\beta_H$. Thus, we will call $Z_\beta$ the partition function and use it for the definition of the effective potential in the theory. The corresponding quantum states will be called FT states. Some justification of this method is that at $\beta = \beta_H$ the integral $Z_\beta$ defines the quantum theory and effective potential in the dSI state. It should be noted, however, that in the case of the Killing horizons $Z_\beta$ and statistical-mechanical partition function $\text{Tr} e^{-\beta H}$ are not completely equivalent.$^\dagger$

$^\dagger$ Note that this disagreement is also present at $\beta = \beta_H$. 

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Their relation was investigated in the case of black-hole geometries with the aim of establishing the statistical explanation of the black-hole entropy, see for instance [19–22]. We will not dwell on this issue further since it is not relevant for our purposes.

It must also be mentioned that in the presence of the Killing horizon the function \( G_\beta^F (x, x') \) does not have the Hadamard form [23] when \( \beta \neq \beta_H \). In this case the stress–energy tensor of a quantum field has a non-integrable divergence on the horizon of the chosen static coordinate system [17]. As was discussed in [24], this property alone may be not sufficient to exclude FT states as unphysical, since the computation of the stress tensor neglects the backreaction effects. Such effects are very strong near the horizon and a reliable computation requires a non-perturbative approach which is still not developed. However, a non-Hadamard form of \( G_\beta^F (x, x') \) is not an obstacle for the definition of the partition function (2.4). The conical singularities of \( S^4_\beta \) result only in a number of additional ultraviolet divergent terms in the effective action \( -\ln Z_\beta \) and one can use the standard renormalization procedure to give a meaning to the integral (2.4). We will discuss this later.

3. Effective potential

3.1. The model

We now focus on the definition and computation of the FT effective potential for the scalar electrodynamics in the de Sitter space. A similar computation has been done before by Shore [1] and Allen [2] for the dS1 state. We restrict our analysis to the Abelian theories because the generalization to the non-Abelian case, as was shown in [3], is almost straightforward.

Let us consider the model of a complex self-interacting scalar field \( \phi \), which also interacts with an Abelian gauge field \( A_\mu \). The classical Lorentzian action reads

\[
I[A_\mu, \phi] = \int \sqrt{-g} \, dx^4 \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} (D_\mu \phi)(D^\mu \phi)^* - V(\phi) \right],
\]  

(3.1)

where

\[
F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu,
\]  

(3.2)

\[
D_\mu \phi = \partial_\mu \phi - ie A_\mu \phi,
\]  

(3.3)

\[
V(\phi) = \frac{1}{2} (m^2 + \xi R) \phi^* \phi + \frac{\lambda}{4!} (\phi^* \phi)^2.
\]  

(3.4)

The coupling \( e \) stands for the electric charge, \( \lambda \) is the constant of the self-interaction, which is assumed to be positive, and \( \xi \) denotes the coupling to the scalar curvature \( R = 12/a^2 \). According to equation (2.4), the partition function \( Z_\beta \) for this model at an arbitrary temperature \( \beta^{-1} \) is the Euclidean functional integral

\[
Z_\beta = \int [D\phi] [DA_\mu] [D\bar{c}] [Dc] \exp \left( -I_E[A_\mu, \phi, \bar{c}, c] \right),
\]  

(3.5)

\[
I_E[A_\mu, \phi, \bar{c}, c] = I_E[A_\mu, \phi] + \Delta I[A_\mu, \phi] + I_{gh}[\bar{c}, c, \phi],
\]  

(3.6)

where all the fields are given on the spherical domain \( S^4_\beta \), equation (2.3). The functional \( I_E \) denotes the Euclidean form of action (3.1). The term \( \Delta I \) is a gauge fixing term, \( \bar{c}, c \) are the corresponding ghost fields and \( I_{gh}[\bar{c}, c, \phi] \) is their action. The explicit form of these quantities will be fixed below. The proof of the gauge independence of \( Z_\beta \) on \( S^4_\beta \) repeats the analogous proof given in [2] for \( S^4_\beta \).

We will use the effective potential method to study the phase transitions in the theory (3.1). The symmetry breaking in our model is characterized by the average value
\( \langle \hat{\phi} \rangle_{\beta} \) of the scalar field in the given FT state. If this average does not depend on time and the spatial coordinates \( \chi, \theta_1, \theta_2 \), the effective potential \( V_{\text{eff}}(\phi, \beta) \) can be introduced by following the method of [4]. To this aim one can separate the field \( \phi \) into a constant part \( \phi \) (which can be chosen real) and the excitations \( \phi' = \phi_1 + i\phi_2 \)

\[
\phi = \phi + \phi' = \phi + \phi_1 + i\phi_2.
\] (3.7)

The effective potential is determined by the relation

\[
Z_{\beta} = \int d\phi \exp(-\beta V_{\text{eff}}(\phi, \beta)),
\] (3.8)

\[
\exp(-\beta V_{\text{eff}}(\phi, \beta)) = \int [D\phi][DA_\mu][D\tilde{c}][Dc]\exp(-I_E[A_\mu, \phi', \tilde{c}, c]),
\] (3.9)

where \( \beta = \frac{1}{4} \pi \alpha' \beta \) is the volume of \( S^4_\beta \). Note that the integral (3.8) is the usual integral.

As was shown in [4], a point \( \phi_0 \) where \( V_{\text{eff}}(\phi, \beta) \) has a minimum, coincides with the 1-loop value of the average \( \langle \hat{\phi} \rangle_{\beta} \). The real part of \( V_{\text{eff}}(\phi, \beta) \) is a sum of the classical potential energy \( V(\phi) \) and a quantum correction. If the field configuration \( \phi_0 \) is unstable, then \( V_{\text{eff}}(\phi_0, \beta) \) has a non-vanishing imaginary part which determines the decay probability of this configuration.

To compute \( V_{\text{eff}} \) with the help of equation (3.9) it is suitable to use the t’Hooft gauge with the gauge fixing term

\[
\Delta I[A_\mu, \phi] = \frac{1}{2} \alpha \int_{S^4_\beta} \sqrt{g} \, dx^4 \left[ \nabla_\mu A^\mu - \alpha^{-1} e \varphi \phi_2 \right]^2,
\] (3.10)

where \( \alpha \) is an arbitrary parameter. Then, by expanding the classical potential to second order in \( \phi_1 \) and \( \phi_2 \), one obtains

\[
I_E[A_\mu, \phi] + \Delta I[A_\mu, \phi] = \frac{1}{2} \int_{S^4_\beta} \left[ A^\mu_T(\Delta^{(T)}_{\mu\nu} + e^2 \varphi^2 g_{\mu\nu})A^\nu_T + \alpha \left( -\nabla_\mu \nabla_\nu + \alpha^{-1} e^2 \varphi^2 g_{\mu\nu} \right) A^\nu_L + \phi_1 \left( -\nabla^2 + V''(\varphi) \right) \phi_2 + \phi_2 \left( -\nabla^2 + \alpha^{-1} e^2 \varphi^2 + \varphi^{-1} V'(\varphi) \right) \phi_2 \right] \sqrt{g} \, dx^4
\] (3.11)

where \( A^\mu_T \) and \( A^\mu_L = \nabla_\mu \chi \) are the transverse and longitudinal components of the vector field, respectively, and \( \Delta^{(T)}_{\mu\nu} = -g_{\mu\nu} \nabla^\alpha \nabla_\alpha + R_{\mu\nu} \) is the transverse Hodge–deRham operator.

Hereafter, we will be interested in the situation where the 1-loop quantum effects can change the form of the classical potential significantly. As was pointed out by Coleman and Weinberg [25], who investigated the model (3.1) in the flat space, one has to assume to this aim that the gauge coupling \( e^4 \) is of the order of \( \lambda \). In this case the quantum corrections due to the gauge fields will be comparable with the classical potential. On the other hand, to provide the convergence of the perturbation expansions one has to assume that \( \lambda \ll 1 \). In this case one can safely neglect the contributions to the effective potential of the scalar loops which will be proportional to \( \lambda^2 \ll e^4 \).

### 3.2. Representation for the potential

In order to simplify the computations it is convenient to choose the Landau gauge for which \( \alpha \to \infty \). Then the contribution of the ghost fields does not depend on the average value of the field \( \varphi \), and since it does not affect the symmetry breaking in the model, it can be neglected. As was explained above, the contribution of the scalar excitations can be neglected as well. After that one arrives to the obvious result

\[
V_{\text{eff}}(\phi, \beta) = V(\varphi) + \frac{3}{8 \pi \alpha' \beta} \log \det[\mu^{-2}(\Delta^{(T)}_{\mu\nu} + e^2 \varphi^2 g_{\mu\nu})].
\] (3.12)
Further we use the ζ-function regularization method [26, 27] which defines the determinant of an operator \( A \) with eigenvalues \( \lambda_n \)

\[
\log \det[\mu^{-2}A] = -\zeta_A'(0) - \zeta_A(0) \log \mu^2
\]  

(3.13)

in terms of the generalized \( \zeta_A \)-function

\[
\zeta_A(z) = \sum_{n, (\lambda_n \neq 0)} \lambda_n^{-z}
\]  

(3.14)

and its derivative \( \zeta_A'(z) = d\zeta_A(z)/dz \). An arbitrary mass parameter \( \mu \) is introduced in equation (3.13) to keep the right dimensionality.

Let us note that the ζ-function method automatically gives a finite expression for the quantum determinants \(^{†}\). The last term on the right-hand side of equation (3.13) corresponds to the finite counterterms which are always present after renormalization. The conical singularities are known to introduce additional ultraviolet divergences, see for instance [29, 30]. The renormalization on manifolds with conical singularities has recently been discussed in a number of papers [29, 31–36, 6].

In the case of the Hodge-deRham operator all 1-loop divergences which are linear in the conical deficit angle, as shown in [6], are removed under standard renormalization of the gravitational couplings [37] in the bare gravitational action. The renormalization of the remaining divergences of order \((\beta - \beta_H)^2 \) in the deficit angle or higher is more involved, and requires additional counterterms in the effective action. These terms have the form of integral invariants defined on the singular surface [33]. The values of the additional couplings cannot be predicted in the theory, and so to simplify the analysis we will assume that these couplings in \( V_{\text{eff}} \) are absent.

The definition and representation of the ζ-function \( \zeta(T)(z) \) for the transverse Hodge-deRham operator is given in the appendix. It is convenient to parametrize this function of \( z \), in terms of the parameters \( \beta \) and \( \sigma \equiv \frac{1}{4} - e^2a^2\phi^2 \). Thus, equation (3.12) reads

\[
V_{\text{eff}}(\phi, \beta) = V(\phi) - \frac{3}{8\pi a^3 \beta} \left[ d \frac{d \zeta(T)(0, \beta, \sigma)}{dz} + \zeta(T)(0, \beta, \sigma) \log a^2 \right].
\]  

(3.15)

Our key result, whose derivation is given in the appendix, is that on \( S^4_\beta \) the ζ-function \( \zeta(T)(z, \beta, \sigma) \) of the vector operator \( a^2 \Delta_T^{\mu \nu} + g_{\mu \nu}(\frac{1}{4} - \sigma) \) and the ζ-function \( \zeta^{(0)}(z, \beta, \sigma) \) of the scalar operator \(-a^2\Delta^2 + \frac{9}{4} - \sigma \) are related in a simple way, see equation (A.8). In particular, one can show with the help of equations (A.9), (A.10) and (A.12) that for arbitrary \( \beta \)

\[
\frac{d}{dz} \zeta^{(T)}(0, \beta, \sigma) = 3 \frac{d}{dz} \zeta^{(0)}(0, \beta, \sigma) - 2 \left( \int_{\frac{1}{2} + \sqrt{\sigma}}^{\frac{3}{2} + \sqrt{\sigma}} + \int_{\frac{3}{2} - \sqrt{\sigma}}^{\frac{1}{2} - \sqrt{\sigma}} \right) (\frac{3}{2} - u) \psi(u) du
\]

(3.16)

\[
\zeta^{(T)}(0, \beta, \sigma) = 3 \zeta^{(0)}(0, \beta, \sigma) + \frac{11}{12} - \sigma,
\]  

(3.17)

where \( \psi(u) \equiv \Gamma'(u)/\Gamma(u) \) is the digamma function [38]. Thus, the quantum correction \( V_{\text{eff}}^g \) to the potential due to gauge fields and the correction \( V_{\text{eff}}^s \) from the scalars are related in the universal way

\[
V_{\text{eff}}^g(\sigma, \beta) = 3V_{\text{eff}}^s(\sigma, \beta) + \beta^{-1} \Omega(\sigma),
\]  

(3.18)

\(^{†}\) For massless fields on an infinite cone, the definition of the ζ-function meets an additional difficulty which was discussed in [28]. However, in the considered case the ζ-function is well defined due to the compactness of space \( S^4_\beta \), and its analytical properties at \( \beta = 2\pi \) and \( \beta \neq 2\pi \) are the same.
where the function $\Omega(\sigma)$ is temperature independent
\[
\Omega(\sigma) = \frac{3}{4\pi a^3} \left[ \left( \int_{\frac{1}{2}}^{\frac{1}{2}+\sqrt{\sigma}} + \int_{\frac{1}{2}}^{\frac{1}{2}-\sqrt{\sigma}} \right) \left( \frac{3}{2} - u \right) \psi(u) \, du \right.
- 4\zeta'(1, -1, \frac{3}{2}) + \left. \frac{1}{2} (\sigma - \frac{11}{12}) \log \mu^2 a^2 \right].
\] (3.19)

Hence, the study of the effective potential (3.15) is reduced to the investigation of the scalar functional $V_{\text{eff}}(\sigma, \beta)$ which was done in [4]. Let us note, however, that even the structure of $V_{\text{eff}}(\sigma, \beta)$ is rather complicated and in general one may rely only on numerical calculations.

What is interesting is that $V_{\text{eff}}(\sigma, \beta)$ can be found in an analytical form in the two most interesting limits: in the dSI state and in the ZT state. We use this fact to consider the phase transitions in the gauge model at different values of the de Sitter radius and compare the phase structures of the theory in these two cases.

### 3.3. de Sitter-invariant state

We assume that the renormalized mass of the field $\phi$ is zero and so the classical potential in equation (3.15) is
\[
V(\phi) = \frac{1}{2} R \phi^2 + \frac{\lambda}{4!} \phi^4.
\] (3.20)
with $R = 12/a^2$. Then the expression for the potential, which follows from equation (A.13), is
\[
V_{\text{eff}}(\phi, \beta_H) = \frac{1}{2} R \phi^2 + \frac{\lambda}{4!} \phi^4 - \frac{3}{(4\pi)^2 a^4} \left[ \frac{1}{4} e^4 a^2 \phi^2 + \frac{1}{4} e^4 a^4 \phi^4 - \frac{19}{192} + 2\zeta'(12, -3, \frac{1}{2})
- \frac{9}{2} \zeta'(-1, \frac{3}{2}) - \left( \int_{\frac{1}{2}}^{\frac{1}{2}+\sqrt{\sigma}} + \int_{\frac{1}{2}}^{\frac{1}{2}-\sqrt{\sigma}} \right) u(u - \frac{1}{2})(u - 3) \psi(u) \, du \right.
+ \left. \left( \frac{1}{4} e^4 a^4 \phi^4 + e^2 a^2 \phi^2 + \frac{19}{50} \right) \log \mu^2 a^2 \right].
\] (3.21)

One can also define the energy $E$ and entropy $S$ of the quantum fields at the Hawking temperature. Thus, by using (A.13), we find
\[
E = 4\pi a^3 \frac{\partial}{\partial \beta} [\beta V_{\text{eff}}]_{\beta=\beta_H} = 4\pi a^3 \frac{3}{3} \left( \frac{1}{2} R \phi^2 + \frac{\lambda}{4!} \phi^4 \right) + \frac{3}{4\pi a} \left[ \frac{1}{12} e^4 a^4 \phi^4 - \frac{7}{5} e^2 a^2 \phi^2 - \frac{19}{180} + \frac{1}{12} (e^4 a^4 \phi^4 + 2 e^2 a^2 \phi^2) (\psi(\frac{1}{2} + \sqrt{\sigma}) + \psi(\frac{1}{2} - \sqrt{\sigma}) - \log \mu^2 a^2) \right].
\] (3.22)
The entropy $S$ of the quantum field in the dSI state is defined as $S = 2\pi a (E - V_{\text{eff}})$ and its expression follows from equations (3.21) and (3.22).

### 3.4. Zero-temperature state

As follows from equation (A.17), the effective potential in this case reads
\[
V_{\text{eff}}(\phi, \beta = \infty) = \frac{1}{2} R \phi^2 + \frac{\lambda}{4!} \phi^4 - \frac{3}{(4\pi)^2 a^4} \left[ \frac{1}{12} e^4 a^4 \phi^4 - \frac{11}{12} e^2 a^2 \phi^2 + \frac{317}{192} \right.
- 6\zeta'(12, -3, \frac{1}{2}) + \left. \frac{3}{2} \zeta'(-1, \frac{3}{2}) \right].
\]
It is interesting to note that the quantum correction to the potential due to gauge fields at ZT has the same form of correction due to the scalar fields multiplied by a factor of 3, see equation (3.18).

Note that formulae (3.21) and (3.23) for the potential include integrals \( \int_{\frac{1}{2}}^{b} u \psi(u) \, du \) which can be evaluated by using the results by Dowker and Kirsten [39]. The final expressions are represented in terms of \( \Gamma \)-functions and the derivatives of \( \zeta_R \) with the arguments depending on \( \sqrt{\sigma} \). However, these expressions are more lengthy than equations (3.21) and (3.23) and we do not give them here.

### 3.5. Flat limit

By taking into account that for \( z \gg 1 \)

\[
\text{Re} \psi(\frac{1}{2} + iz) = \log z + \frac{1}{52} z^{-2} - \frac{127}{960} z^{-4} + O(z^{-8}),
\]

\[
\text{Re} \psi(\frac{1}{2} + iz) = \log z - \frac{1}{32} z^{-2} - \frac{7}{960} z^{-4} + O(z^{-8}),
\]

one can check that the energy density \( (4\pi a^3/3)^{-1} E \) and the effective potentials \( V_{\text{eff}}(\psi, \beta_H) \) and \( V_{\text{eff}}(\psi, \beta = \infty) \) coincide in the limit \( a \rightarrow \infty \) with the effective potential in the Minkowski vacuum \( V_{\text{Mink}} \)

\[
V_{\text{Mink}}(\psi) = \frac{\lambda}{4!} \psi^4 + \frac{3e^4}{64\pi^2} \psi^4 \left[ \log \left( \frac{e^2 \psi^2}{\mu^2} \right) - \frac{3}{2} \right].
\]

The last expression enables one to fix the unknown constant \( \mu \) in terms of the physical parameters.

The effective potential (3.26) has a minimum at a non-zero value \( \psi = \phi_0 \) where

\[
\phi_0^2 = \frac{\mu^2}{e^2} \exp \left( 1 - \frac{8\pi^2\lambda}{9e^4} \right).
\]

Thus, the quantum state is characterized by a field configuration with the non-zero average \( \langle \phi \rangle \simeq \phi_0 \), where the gauge symmetry is spontaneously broken and the gauge field becomes a vector boson with the mass \( M = e|\phi_0| \). Potential (3.26), written in terms of the parameter \( M \), reads

\[
V_{\text{Mink}}(\psi) = \frac{3e^4}{64\pi^2} \psi^4 \left[ \log \frac{e^2 \phi^2}{M^2} - \frac{1}{2} \right].
\]

It coincides with the Coleman–Weinberg expression [25]. The result that in the flat limit dSI and ZT states coincide with the Minkowski vacuum is not surprising. In this limit the de Sitter group converts into the Poincaré one and the Hawking temperature vanishes.
3.6. Limit of large curvatures

It is also interesting to compare the form of potentials (3.21) and (3.23) in the opposite limit, when the curvature of the spacetime is very large, namely $e^2 a^2 \varphi^2 \ll 1$. Since we are interested in the phase transitions, terms in the potential which do not depend on $\varphi$ are irrelevant. Hence, from now on it is more convenient to deal with the difference $V_{\text{eff}}(\varphi, \beta) - V_{\text{eff}}(0, \beta)$ which vanishes for $\varphi = 0$.

By taking into account equation (3.21) one can easily prove that the dominant contribution for the large curvature at $\beta = \beta_H$ is

$$V_{\text{eff}}(\varphi, \beta) - V_{\text{eff}}(0, \beta_H) \simeq \frac{3 e^2 \varphi^2}{16 \pi^2 a^2} \left[ P + \frac{19}{6} - 2 \gamma - \log a^2 M^2 \right]$$

$$+ \frac{3 e^2 \varphi^2}{64 \pi^2} \left[ 3 - 2 \gamma - \log a^2 M^2 \right],$$

(3.29)

where

$$P = \frac{32 \pi^2 \xi}{e^2} - \frac{8 \pi^2 \lambda}{9 e^4} - \frac{3}{2},$$

(3.30)

and $\gamma = 0.5772 \ldots$ is the Euler constant. As far as the ZT state is concerned, one can prove the analogous formula

$$V_{\text{eff}}(\varphi, \beta = \infty) - V_{\text{eff}}(0, \beta = \infty) \simeq \frac{3 e^2 \varphi^2}{64 \pi^2 a^2} \left[ Q + K - \log a^2 M^2 \right]$$

$$+ \frac{3 e^4 \varphi^4}{64 \pi^2} \left[ K + 20 - \log a^2 M^2 \right],$$

(3.31)

where

$$Q = \frac{128 \pi^2 \xi}{e^2} - \frac{8 \pi^2 \lambda}{9 e^4} + \frac{14}{3},$$

(3.32)

$$K = 12 \int_0^{1/2} (\frac{1}{4} - z^2) \left[ \psi(\frac{1}{2} + z) + \psi(\frac{1}{2} - z) \right] dz = -5.97011 \ldots.$$  

(3.33)

Equations (3.29) and (3.31) show that in the limit when the de Sitter curvature is large the potentials are different but have similar structures.

4. Symmetry breaking

In order to compare phase transitions in the scalar electrodynamics (3.1) in dSI and ZT states we rewrite expressions (3.21) and (3.23) in the following dimensionless form†

$$A_1(x, y) = \frac{64 \pi^2}{3 M^4} [V_{\text{eff}}(\varphi, \beta_H) - V_{\text{eff}}(0, \beta_H)] = \frac{4 \xi}{y} (P + \frac{13}{6} - \log y)$$

$$- x^2 \log y + \frac{4}{y^2} \left( \int^{\frac{1}{2} + \sqrt{\sigma}}_{\frac{1}{2}} + \int^{\frac{1}{2} - \sqrt{\sigma}}_{\frac{1}{2}} \right) u(u - \frac{1}{2})(u - 3) \psi(u) \, du,$$

(4.1)

$$A_2(x, y) = \frac{64 \pi^2}{3 M^4} [V_{\text{eff}}(\varphi, \beta = \infty) - V_{\text{eff}}(0, \beta = \infty)] = \frac{x}{y} (Q - \log y) + x^7 \left[ 2 - \log y \right]$$

$$- \frac{12}{y^2} \left( \int^{\frac{1}{2} + \sqrt{\sigma}}_{\frac{1}{2}} + \int^{\frac{1}{2} - \sqrt{\sigma}}_{\frac{1}{2}} \right) u(u - \frac{1}{2})(u - 1) \psi(u) \, du.$$

† For dSI, state representation (4.1) was suggested in [2].
\[ + \frac{12}{y^2} \sqrt{\frac{1}{4} - xy} \left( \int_{\frac{1}{2}}^{\frac{1}{2} + \sqrt{\sigma}} - \int_{\frac{1}{2}}^{\frac{1}{2} - \sqrt{\sigma}} \right) u(u - 1) \psi(u) \, du. \] (4.2)

Here
\[ x = \frac{e^2 \phi^2}{M^2}, \quad y = a^2 M^2, \quad \sigma = \frac{1}{4} - xy, \] (4.3)
and the renormalization parameter \( \mu \) has been expressed, according to equation (3.27), in terms of the vector boson mass as \( \mu^2 = M^2 \exp \left( \frac{8 \pi^2 \lambda}{9 e^4} - 1 \right) \). Interestingly, all the information about the constants \( e^2, \lambda \) and \( \xi \) of the considered model is contained in the parameters \( P \) and \( Q \) defined in equations (3.30) and (3.32), respectively.

For arbitrary values of constants \( e^2, \lambda \) and \( \xi \) the parameters \( P \) and \( Q \) are independent.

We now consider the phase structure of the theory in the limit of large and small curvatures of the de Sitter space, where it can be investigated analytically.

### 4.1. Phase transitions at small curvatures

In this limit \( e^2 a^2 \phi^2 \gg 1 \) and one has
\[ A_1(x, y) = x^2 (\log x - \frac{1}{2}) + 4 \frac{x}{y} (\log x + P + \frac{1}{2}), \] (4.4)
\[ A_2(x, y) = x^2 (\log x - \frac{1}{2}) + \frac{x}{y} (\log x + Q - \frac{1}{2}). \] (4.5)

Equation (4.4) coincides with the result of [2]. As was shown in [2] the symmetry breaking in this limit in dSI state is characterized by first-order phase transitions. Under such transitions, when the curvature radius \( a \) reduces to a critical value \( a_c \), the mass of the vector boson changes discontinuously from a value \( M_c = e \phi_c \) to zero. By comparing equations (4.4) and (4.5) one can see that the effective potential at ZT has a similar structure, so in this case one may expect an analogous behaviour.

Let \( x_c \) and \( y_c \) be critical values of parameters \( x \) and \( y \) in the point of the phase transition. They are related to the critical radius \( a_c \) and mass \( M_c \) as follows
\[ x_c = \frac{M_c^2}{M^2}, \quad y_c = a_c^2 M^2 \] (4.6)
where, as before, \( M \) is the mass of the vector boson in the Minkowski spacetime. In the given approximation the values of \( x_c \) and \( y_c \), with \( x_c > 0 \), can be found explicitly. They must satisfy the following conditions
\[ \frac{d}{dx} A_i(x, y)|_{x=x_c, y=y_c} = 0, \] (4.7)
\[ A_i(x_c, y_c) = 0, \] (4.8)
which can be resolved. For the dSI state the results coincide with those of [2]
\[ M_c^2 = M^2 \exp \left( -\frac{5}{2} - P \right), \] (4.9)
\[ a_c^2 = 2 M_c^{-2} [P + \sqrt{P^2 - 4}]. \] (4.10)

For the ZT we find the following critical mass \( \tilde{M}_c \) and radius \( \tilde{a}_c \)
\[ \tilde{M}_c^2 = M^2 \exp \left[ -\frac{11}{3} - Q \right], \] (4.11)
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Figure 1. The above curves, from top to bottom, represent the function $A_1(x, y)$ for $P = 10$ and $\sqrt{\tau} = 7.4, 7.8, 8.49, 9$ and $9.5$, respectively (see [2]). The critical value of the de Sitter radius ($\sqrt{\tau} \approx 8.49$) corresponds to the first-order phase transition.

Figure 2. The curves, from top to bottom, illustrate $A_2(x, y)$ for $Q = 16$ and $\sqrt{\tau} = 3.5, 3.8, 4.2, 4.7$ and $5$, respectively. The first-order phase transition takes place at $\sqrt{\tau} \approx 4.2$.

\[ a_c^2 = \frac{1}{2} \bar{M}_c^{-2} \left[ Q - \frac{37}{6} + \sqrt{(Q - \frac{37}{6})^2 - 4} \right]. \] (4.12)

Equations (4.9) and (4.10) are valid for $P \geq 2$, and equations (4.11) and (4.12) are valid for $Q \geq \frac{29}{6}$. Thus, we can conclude that in the limit of small curvatures and for the given values of $P$ and $Q$ the dSI and ZT states have qualitatively similar properties. In both cases one has first-order phase transitions, but with different values of critical radii and masses. These properties are shown in figures 1 and 2, where the functions $A_1(x, y)$ and $A_2(x, y)$ are evaluated for $P = 10$ and $Q = 16$, respectively. The numerical computations of the critical parameters are in agreement with expressions (4.9)–(4.12).

4.2. Phase transitions at large curvatures

In this regime we can use asymptotics (3.29) and (3.31). As one can see from these equations, the terms proportional to $\varphi^2$ change the sign from positive to negative when $a$ becomes larger than a critical value $a_c$. In this case, to have the symmetry breaking in the theory at $a > a_c$, the terms $\varphi^4$ in (3.29) and (3.31) have to be positive. When $a$ becomes smaller than $a_c$, the mass of the vector boson gradually vanishes. This situation corresponds to second-order phase transitions.
The critical radius $a_c$ in the dSI state is [2]

$$a_c^2 = M^{-2} \exp \left[ P + \frac{19}{6} - 2\gamma \right].$$ (4.13)

In order to have a second-order phase transition one must ensure that the coefficient of the $\varphi^4$ term in (3.29) is positive. By substituting (4.13) into equation (3.29) we find that the above condition is satisfied [2] for

$$3 - 2\gamma - \log a_c^2 M^2 = -P - \frac{1}{6} > 0 \implies P < -\frac{1}{6}.$$ (4.14)

The value $P = P_{cr} = -\frac{1}{6}$ actually represents a critical boundary which separates the models of the scalar electrodynamics with different kinds of phase transition. For models with $P$ above $P_{cr}$ the system undergoes first-order phase transitions, whereas for models with $P$ below $P_{cr}$ one has second-order phase transitions. The value of $P_{cr}$ is also confirmed by the numerical analysis.

A similar expression for the critical radius $\tilde{a}_c$ at ZT can be found from equation (3.31)

$$\tilde{a}_c^2 = M^{-2} \exp(Q + K).$$ (4.15)

The difference between the values of the critical radii can be estimated by their ratio

$$\frac{\tilde{a}_c^2}{a_c^2} = \exp \left( \frac{96\pi^2 \xi}{e^2} + K + 3 + 2\gamma + \frac{16\pi^2}{e^2} \right).$$ (4.16)

It depends only on $\xi$ and $e^2$, and for $\xi \gg e^2$ one can conclude that $\tilde{a}_c \gg a_c$. For the minimal coupling ($\xi = 0$) $\tilde{a}_c^2/a_c^2 \simeq 1.63$.

In the case of the ZT state, the critical value $Q_{cr}$ can also be determined. By substituting equation (4.15) into the coefficient of the quartic term of equation (3.31) one obtains

$$K + \frac{20}{3} - \log \tilde{a}_c^2 M^2 = \frac{20}{3} - Q > 0 \implies Q < \frac{20}{3}.$$ (4.17)

Thus, $Q_{cr} = \frac{20}{3}$ and it is confirmed by the numerical computations.

The typical behaviour of the effective potentials in the case of second-order phase transitions is illustrated by figures 3 and 4, which are obtained by using formulae (4.1) and (4.2). Figure 3 depicts the function $A_1(x, y)$ for $P = -\frac{1}{2}$ and $\sqrt{\gamma} = 1.95, 2.13, 2.3, 2.5$ and 2.8, respectively (see [2]). The critical value $\sqrt{\gamma} \approx 2.13$ corresponds to the second-order transition.
4.3. Classification of models

As we have shown, the first-order phase transitions always take place in the models of scalar electrodynamics (3.1) when constants $P$ and $Q$ are large and positive. On the other hand, if these constants are negative with large absolute values, the symmetry breaking corresponds to second-order phase transitions.

An analytical study of the region of parameters $\xi$, $\lambda$, and $\epsilon^2$ (or $P_{cr}$ and $Q_{cr}$), where the kind of the phase transition changes from first to second order, is difficult because the effective potentials have quite involved forms (4.1), (4.2). Thus, in principle, one should use here numerical methods. It is interesting, however, that the analytical estimates $P_{cr} = -\frac{1}{6}$ and $Q_{cr} = \frac{20}{3}$ of section 4.2 are in very good agreement with the numerical results. For the dSI state this was first pointed out by Allen [2].

As we already mentioned, the symmetry-breaking mechanism in the considered models is completely determined by the parameters $P$ and $Q$. By using their definitions (3.30), (3.32) one finds that

$$Q = 4P + \frac{8\pi^2\lambda}{3\epsilon^2} + \frac{32}{3}.$$  (4.18)

Note that the parameter $\lambda$ must be chosen to be positive in order to have a scalar potential bounded from below. We also assumed that $\lambda \simeq \epsilon^4$ in order to have a considerable quantum correction to the classical potential coming from the gauge fields. For values $\lambda \gg \epsilon^4$ our perturbative approach is not reliable, so we have to restrict ourselves to the interval of parameters $0 \leq \lambda \epsilon^{-2} \leq \epsilon^2$. Thus, by taking into account that $\epsilon^2/4\pi \simeq 10^{-2}$, we find with the help of equation (4.18) the allowed region in the $P$--$Q$ plane $0 \leq Q - 4P - \frac{32}{3} \leq 1$. There we can make definite predictions about the properties of the considered models. This region is shown in figure 5. It consists of three subregions (I), (II) and (III), bounded by broken lines. Each subregion indicates a class of models with the same properties.

Region (I) is determined by inequalities $P < -\frac{1}{6}$ and $Q \leq \frac{20}{3}$. If the coupling constants $\xi, e, \lambda$ satisfy these restrictions, the corresponding models undergo second-order phase transitions in both dSI and ZT states.

In region (II) one has $P \geq -\frac{1}{6}$ and $Q > \frac{20}{3}$ and first-order phase transitions in both dSI and ZT vacua.

Region (III) is the most interesting, since in this case $P < -\frac{1}{6}$ but $Q > \frac{20}{3}$, thus the system undergoes a first-order phase transition in the ZT case, whereas it has a second-order phase transition in the dSI case.
Figure 5. The regions (I), (II) and (III), bounded by broken lines, represent the allowed areas in the $P-Q$ plane. If the parameters $P$ and $Q$ are chosen in regions (I) or (II), one a second- or first-order phase transition occurs, respectively, in both dSI and ZT states. In region (III) first-order phase transitions take place in the ZT state, whilst the dSI state shows second-order phase transitions.

In the case of minimal coupling ($\xi = 0$), according to equations (3.30) and (3.32), $P \leq -\frac{3}{2} < -\frac{1}{6}$ and $Q \leq \frac{14}{3} < \frac{32}{3}$, and so only second-order phase transitions are possible in both states.

5. Summary

We investigated the effective potential for the gauge models on the singular spherical domains $S^{4}_\beta$. The spectrum of the corresponding wave operators on these backgrounds can be found exactly and it enables one to calculate the potential with the help of the $\zeta$-function. We have shown that for arbitrary values of $\beta$ there is a simple relation (3.18) between the 1-loop corrections to the potential from gauge and scalar fields. Thus, in studying the gauge models in the de Sitter space one can use the computations of the scalar potential of [4]. The effective action on $S^{4}_\beta$ can be interpreted as the free energy of fields in static de Sitter space, where the parameter $\beta$ is the inverse temperature of the system. Thus, by changing $\beta$ one can learn how the presence of temperature affects the properties of such a quantum theory. The aim of the paper was to compare the phase structures of the gauge models in the de Sitter invariant vacuum, which is usually used in the quantum cosmology, with the structure of the normal vacuum which is the state at ZT.

It is worth pointing out that our results, obtained in the simplest case of scalar electrodynamics, allow a straightforward generalization for non-Abelian gauge theories. In this case, as outlined in [3, 11], the 1-loop effective potential for scalar fields takes contributions from all gauge bosons becoming massive through the scalar coupling, Higgs mechanism (see for example equations (3.3) and (3.6) of [11]). Thus, the main element to obtain the 1-loop effective action for non-Abelian theories is the exact knowledge of the gauge bosons mass spectrum. This study, that in principle is not conceptually difficult, for general gauge theories can be a computational hard task.

According to our analysis both vacua have very similar properties in the extreme regimes of very small and very large curvatures. If the restoration of the gauge symmetry happens at large radius $a$ (small curvature) of the de Sitter space, then the corresponding phase transition is of first order regardless what vacuum is considered. Analogously, if the symmetry is restored at small values of $a$ (large curvature), one has second-order phase transitions. The critical masses and radii are completely determined by the constants $P$ and $Q$, which are the functions of the parameters $\xi$, $e^2$ and $\lambda$. Expressions (4.11), (4.12) and (4.15) for the
critical parameters in the ZT state represent the new result.

We have also found the critical value \( Q_{cr} = \frac{20}{3} \) for the parameter \( Q \), which separates models with second- \( (Q < \frac{20}{3}) \) and first- \( (Q > \frac{20}{3}) \) order phase transitions in the ZT state. Finally, we proved the existence of a class of the models \( (P < -\frac{1}{6} \) and \( Q > \frac{20}{3} \) ) where the kinds of phase transition in the two vacua are different.

The detailed analysis of the effective potential for all \( \beta \)'s is not given here, but it can be carried out in principle by using the properties of the \( \zeta \)-function. In the most interesting cases, however, it is sufficient to restrict the study to the expansions near the zero and Hawking temperatures, as was done in [4].

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Appendix. The \( \zeta \)-function

Here we consider the \( \zeta \)-function

\[
\zeta(T)(z) = \sum_{n=0}^{\infty} \sum_{l=1}^{\infty} D^T_n(\Lambda^T_{n,l})^{-z} + \sum_{n=1}^{\infty} \bar{D}^T_n(\Lambda^T_{n,0})^{-z} \tag{A.1}
\]

for the transverse operator in equation (3.12) which is defined for convenience in the dimensionless form as \( a^2(\Delta^T_{\mu\nu} + g_{\mu\nu}e^2\phi^2) \).

According to [6] the degeneracies in four dimensions read

\[
D^T_n = 3(n+1)(n+2), \quad \bar{D}^T_n = \frac{n}{2}(3n+5), \tag{A.2}
\]

while the eigenvalues \( \Lambda^T_{n,l} \) are

\[
\Lambda^T_{n,l} = (n + \gamma l)(n + \gamma l + 3) + 2 + e^2a^2\phi^2, \tag{A.3}
\]

where \( \gamma = \beta H/\beta \). As in section 3, it is suitable to parametrize the function (A.1) by the parameters \( \beta \) and \( \sigma = \frac{1}{4} - e^2a^2\phi^2 \). Thus, one can write

\[
\zeta(T)(z, \beta, \sigma) = \sum_{n=0}^{\infty} \sum_{l=1}^{\infty} D^{(0)}_n(\Lambda^{(0)}_{n,l}(\sigma))^{-z} + \sum_{n=1}^{\infty} \bar{D}^{(0)}_n(\Lambda^{(0)}_{n,0}(\sigma))^{-z}, \tag{A.4}
\]

where

\[
\Lambda^{(0)}_{n,l}(\sigma) = (n + \gamma l + \frac{3}{2})^2 - \sigma. \tag{A.5}
\]

Function (A.4) has an interesting relation to the \( \zeta \)-function of the scalar Laplace operator \(-a^2\nabla^\mu \nabla_\mu + \frac{9}{4} - \sigma \) on the same spherical domain \( S^4_\beta \). These operators were studied in [4], where they were shown to possess the eigenvalues coinciding with (A.5). So the scalar \( \zeta \)-function is represented in the form

\[
\zeta^{(0)}(z, \beta, \sigma) = \sum_{n=0}^{\infty} \sum_{l=1}^{\infty} D^{(0)}_n(\Lambda^{(0)}_{n,l}(\sigma))^{-z} + \sum_{n=1}^{\infty} \bar{D}^{(0)}_n(\Lambda^{(0)}_{n,0}(\sigma))^{-z}, \tag{A.6}
\]

where the degeneracies

\[
D^{(0)}_n = (n+1)(n+2), \quad \bar{D}^{(0)}_n = \frac{1}{2}(n+1)(n+2) \tag{A.7}
\]
This case was investigated by Allen in [2] and our aim here is to rederive his result for \( \zeta(T) \). Temperatures close to the Hawking temperature can be obtained in the simple analytical form. However, the analysis of the derivative of the generalized \( \zeta \)-function has been studied in detail in [2, 4], and this makes the analysis of \( \zeta(T) \) much more easy. In particular, by taking into account the result of [4] and equation (A.8) one obtains

\[
\zeta(T)(0, \beta, \sigma) = 3\zeta(0)(0, \beta, \sigma) + \frac{11}{12} - \sigma
\]

(A.9)

However, the analysis of the derivative of the generalized \( \zeta \)-function is more involved and we consider only two of the most interesting cases where the result can be obtained in the simple analytical form.

### A.1. Temperatures close to the Hawking temperature

This case was investigated by Allen in [2] and our aim here is to rederive his result for \( \zeta(T)(0, \beta_H, \sigma) \) in a different way with the help of equation (A.8). For the scalar \( \zeta \)-function one can find the following representation [2]

\[
\frac{d}{dz} \zeta(0, \beta_H, \sigma) = -\frac{1}{2} \left( \int_{\frac{1}{2}}^{1+\sqrt{\sigma}} + \int_{\frac{1}{2}}^{1-\sqrt{\sigma}} \right) u(u - \frac{1}{2})(u - 1)\psi(u) du + \frac{1}{12}\sigma^2 + \frac{1}{2} \sigma
\]

\( \frac{d}{dz} \zeta(0, \beta_H, \sigma) = -\frac{1}{2} \left( \int_{\frac{1}{2}}^{1+\sqrt{\sigma}} + \int_{\frac{1}{2}}^{1-\sqrt{\sigma}} \right) u(u - \frac{1}{2})(u - 1)\psi(u) du + \frac{1}{12}\sigma^2 + \frac{1}{2} \sigma
\]

(A.11)

where \( \psi(u) \equiv \Gamma(u)/\Gamma(u) \). By using relation \( \zeta_R(n + 1, z) = (-1)^{n+1} \frac{\sigma^n}{n!} \psi(z)/n! \) it is possible to show that

\[
-\sigma \psi(\frac{3}{2}) + \sum_{k=1}^{\infty} \frac{\sigma^{k+1}}{k+1} \zeta_R(2k + 1, \frac{3}{2}) = \left( \int_{\frac{1}{2}}^{1+\sqrt{\sigma}} + \int_{\frac{1}{2}}^{1-\sqrt{\sigma}} \right) u(u - \frac{1}{2})\psi(u) du.
\]

(A.12)

Thus, by substituting equations (A.11) and (A.12) into (A.8) and using the properties of the \( \psi \)-function one obtains

\[
\frac{d}{dz} \zeta(T)(0, \beta, \sigma) = -\frac{1}{2} \left( \int_{\frac{1}{2}}^{1+\sqrt{\sigma}} + \int_{\frac{1}{2}}^{1-\sqrt{\sigma}} \right) u(u - \frac{1}{2})(u - 3)\psi(u) du + \frac{1}{2}\sigma^2 - \frac{11}{24} \sigma
\]

\[
+2\zeta_R(-3, \frac{3}{2}) - \frac{9}{2} \zeta_R(-1, \frac{3}{2}) - 3 \left( \frac{\beta}{\beta_H - 1} \right) \left[ \frac{441}{144}\sigma^2 - \frac{973}{5760} \right] + \frac{1}{192} (16\sigma^2 - 40\sigma + 9)(\psi(\frac{3}{2} + \sqrt{\sigma}) + \psi(\frac{3}{2} - \sqrt{\sigma})) + O((\beta - \beta_H)^2).
\]

(A.13)

As one can see, this expression coincides at \( \beta = \beta_H \) with the result reported in [2].
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A.2. Vanishing temperature

To derive $d\zeta(T)/dz$ in the ZT limit one can use the method suggested in [4]. Unfortunately, the final formula given in [4] has a wrong form because of a misprint, so here we use the opportunity to represent the correct answer. According to relation (A.8) and equation (B2) of [4], one has

$$\beta_H \lim_{\beta \to \infty} \left( \beta^{-1} \frac{d}{dz} \zeta(T)(0, \beta, \sigma) \right) = 3 \beta_H \lim_{\beta \to \infty} \left( \beta^{-1} \frac{d}{dz} \zeta(0)(0, \beta, \sigma) \right)$$

$$= 3 \sum_{k=0}^{\infty} C_k(z) \left( \frac{\sqrt{\sigma}}{2} \right)^{2k} \left( 2z + 2k - 1 \right) \left[ \zeta_R(2z + 2k - 3, \frac{3}{2}) \right] - \frac{1}{2} \zeta_R(2z + 2k - 1, \frac{1}{2}) \equiv 3 f(z, \sigma). \tag{A.14}$$

The function $f(z, \sigma)$ was introduced in [4], where it was shown to be related to $\zeta(0)(z, \beta_H, \sigma)$, see (A.11), by the differential equation

$$\frac{d}{d\sqrt{\sigma}} [f(z, \sigma)(\sqrt{\sigma})^{2z-1}] = 3\sigma^{-1} \zeta(0)(0, \beta_H, \sigma). \tag{A.15}$$

Thus, one can write

$$\frac{d}{dz} f(0, \sigma) = \sqrt{\sigma} \int_0^{\sqrt{\sigma}} \frac{dy}{y^2} \left[ 3 \left( \frac{d}{dz} \zeta(0)(0, \beta_H, y^2) \right) - \frac{d}{dz} \zeta(0)(0, \beta_H, 0) \right]$$

$$- 2(f(0, y^2) - f(0, 0)) \right] \left( 3 \frac{d}{dz} \zeta(0)(0, \beta_H, 0) - 2f(0, 0) \right). \tag{A.16}$$

By using (A.11) in (A.16) one finally finds

$$\beta_H \lim_{\beta \to \infty} \left( \beta^{-1} \frac{d}{dz} \zeta(T)(0, \beta, \sigma) \right) = 3 \left( \int_{\frac{1}{2}}^{\frac{1}{2} + \sqrt{\sigma}} + \int_{\frac{1}{2}}^{\frac{1}{2} - \sqrt{\sigma}} \right) u(u - \frac{1}{2})(u - 1) \psi(u) du$$

$$- 3\sqrt{\sigma} \left( \int_{\frac{1}{2}}^{\frac{1}{2} + \sqrt{\sigma}} - \int_{\frac{1}{2}}^{\frac{1}{2} - \sqrt{\sigma}} \right) u(u - 1) \psi(u) du$$

$$+ \frac{1}{12} \sigma^2 + \frac{1}{6} \sigma + \frac{17}{160} - 6 \zeta_R'(3, \frac{3}{2}) + \frac{1}{2} \zeta_R'(1, \frac{1}{2}). \tag{A.17}$$

References

[1] Shore G M 1980 Ann. Phys. 128 376
[2] Allen B 1983 Nucl. Phys. B 226 228
[3] Allen B 1985 Ann. Phys. 161 152
[4] Fursaev D V and Miele G 1994 Phys. Rev. 49 987
[5] Linde A D 1990 Particle Physics and Inflationary Cosmology (New York: Harwood Academic)
[6] De Nardo L, Fursaev D V and Miele G 1997 Class. Quantum Grav. 14 1059
[7] Frolov V P, Fursaev D V and Zelnikov A I 1996 Phys. Rev. D 54 2711
[8] Gibbons G W and Hawking S W 1977 Phys. Rev. D 15 2738
[9] Gibbons G W and Hawking S W 1977 Phys. Rev. D 15 2752
[10] Buccella F, Esposito G and Miele G 1992 Class. Quantum Grav. 9 1499
[11] Esposito G, Miele G and Rosa L 1994 Class. Quantum Grav. 11 2031
[12] Hartle J B and Hawking S W 1976 Phys. Rev. D 13 2188
[13] Boulware D G 1975 Phys. Rev. 11 1401
[14] Tagirov E A 1973 Ann. Phys. NY 76 561
[15] Candelas P and Raine D 1975 Phys. Rev. D 12 965
[16] Dowker J S and Critchley R 1976 Phys. Rev. D 13 224
[17] Dowker J S 1978 Phys. Rev. 18 1856
[18] Dowker J S 1977 J. Phys. A: Math. Gen. 10 115
[19] deAlwis S P and Ohta N 1995 Phys. Rev. D 52 3529
[20] Demers J-G, Lafrance R and Myers R C 1995 Phys. Rev. D 52 2245
[21] Solodukhin S N 1996 Phys. Rev. D 53 824
[22] Frolov V P, Fursaev D V and Zelnikov A I 1997 Nucl. Phys. B 486 339
[23] Kay B S and Wald R M 1991 Phys. Rep. 207 49
[24] Fursaev D V and Miele G 1995 Class. Quantum Grav. 12 393
[25] Coleman S and Weinberg E 1973 Phys. Rev. D 7 1888
[26] Critchley R and Dowker J S 1976 Phys. Rev. D 13 3224
[27] Hawking S W 1977 Commun. Math. Phys. 55 133
[28] Zerbini S, Cognola G, and Vanzo L 1996 Phys. Rev. D 54 2699
[29] Cognola G, Kirsten K and Vanzo L 1994 Phys. Rev. D 49 1029
[30] Fursaev D V 1994 Phys. Lett. B 334 53
[31] Susskind L and Uglum J 1994 Phys. Rev. D 50 2700
[32] Solodukhin S N 1995 Phys. Rev. D 51 609
[33] Fursaev D V 1995 Mod. Phys. Lett. A 10 649
[34] Fursaev D V and Solodukhin S N 1996 Phys. Lett. B 365 51
[35] Kabat D 1995 Nucl. Phys. B 453 281
[36] Fursaev D V and Miele G 1997 Nucl. Phys. B 484 697
[37] Birrell N D and Davies P C W 1982 Quantum Fields in Curved Space (Cambridge: Cambridge University Press)
[38] Abramowitz M and Stegun I A 1972 Handbook of Mathematical Functions (New York: Dover)
[39] Dowker J S and Kirsten K Spinors and forms on generalized cones Preprint MUTP/96/23, hep-th/9608189