A Note on Multiple-Processor Multitask Scheduling

Wenxin Li
Department of ECE
The Ohio State University
wenxinliwx.1@gmail.com
li.7328@osu.edu

Ness Shroff
Department of ECE and CSE
The Ohio State University
shroff.11@osu.edu

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Abstract

In this paper we study the multiple-processor multitask scheduling problem in the deterministic and stochastic models. We consider and analyze M-SRPT, a simple modification of the shortest remaining processing time algorithm, which always schedules jobs according to SRPT whenever possible, while processes tasks in an arbitrary order. The modified SRPT algorithm is shown to achieve an competitive ratio of $\Theta(\log \alpha + \beta)$ for minimizing flow time, where $\alpha$ denotes the ratio of maximum job workload and minimum job workload, $\beta$ represents the ratio between maximum non-preemptive task workload and minimum job workload. The algorithm is shown to be optimal (up to a constant factor) when there are constant number of machines. We further consider the problem under poisson arrival and general workload distribution, M-SRPT is proved to be asymptotic optimal when the traffic intensity $\rho$ approaches 1, if the task size is upper bound by the derived upper bound $\eta$.

1 Introduction

With widespread applications in various manufacturing industries, scheduling jobs to minimize the total flow time (also known as response time, sojourn time and delay) is one of the most classic and fundamental problem in operation research and has been extensively studied. As an important metric measuring the quality of a scheduler, flow time, is formally defined as the difference between job completion time and releasing date, and characterizes the amount of time that the job spends in the system.

Optimizing the objective of flow time has been considered both in offline and online scenarios. If preemption is allowed, shortest remaining processing time (SRPT) discipline is shown to be optimal in single machine environment. Many generalizations of this basic formulation become NP-hard, for example, non-preemptive single machine model and preemptive model with two machines [5]. When jobs arrive online, no information about jobs is known to the algorithm in advance, several algorithms with logarithmic competitive ratio are proposed in various settings [5, 1]. On the other hand, while SRPT minimizes the mean response time sample-path wise, it requires the knowledge of remaining job service time. Gittins proved that the Gittins index policy minimizes the mean delay in an M/G/1 queue, which only requires the access to the information about job size distribution.

Though much progresses have been made in single-task job scheduling, there is a lack of theoretical understanding on multiple-processor multitask scheduling (MPMS). Jobs with multiple tasks
are common and relevant in practice, as jobs and tasks can take many different forms in modern computing environment. For example, for the objective of computing matrix vector product, we can divide matrix elements and vector elements into groups of columns and rows respectively, then the tasks correspond to the block-wise multiplication operations. Moreover, tasks can also be map, shuffle and reduce procedures in MapReduce framework. To this end, in this paper, we investigate how to minimize the total flow time of multitask jobs in a multiserver system, where a job is considered to be completed until all the tasks within the job are finished.

With the tremendous increasing in data size and job complexity, we cannot emphasize too much the importance of multiple-processor multitask scheduling in modern era. More specifically, distributed computing has indeed become a useful tool to tackle many large-scale computational challenges, since parallel algorithms can be advantageous over their sequential counterparts, by dividing computational expensive jobs over machines as multiple tasks, to utilize the combination computational power of processors. For example, there are two basic perspectives to design distributed scalable machine learning methods [4], data-parallel and model-parallel. In the first perspective, the data set is partitioned and dispersed into different machines, each machine has a local copy of the whole model, while model-parallel framework partitions and distributes the model parameters on different workers and update a subset of parameters on each worker. A natural question arising is, how to design efficient scheduling algorithms to minimize the total amount time that the multitask jobs spend in the system.

On the other hand, the ability to preempt jobs is important for desirable performance in flow time minimization [3, p.506]. When preemption is not available, the approach of checkpoint based preemption is suggested [3, p.506]. Checkpointing is a tolerate failure technique to avoid applications with large processing time being forced to restart from the very beginning. Similarly we can take extra time to checkpoint jobs and restart again from the last checkpoint, to provide more flexibility for scheduling jobs. The number of checkpoints can be varied, it is important to understand the effects of checkpoints on the system performance. If there are only a few checkpoints, the performance is close to that under non-preemptive disciplines, otherwise we need to pay a large amount of extra time for saving job states and restarting jobs, when the number of checkpoints is large. It is natural to ask, how to choose the number of checkpoints to ensure good performance.

**Related Work.** For the MapReduce framework, Wang et al. [12] studied the problem of scheduling map tasks with data locality, and proposed a map task scheduling algorithm consisting of the Join the Shortest Queue policy the MaxWeight policy. The algorithm asymptotically minimizes the number of backlogged tasks (which is directly related to the delay performance based on Little’s law), when the arrival rate vector approaches the capacity region boundary. Zheng et al. [14] proposed an online scheduler called available shortest remaining processing time (ASRPT), which is shown to achieve an efficiency ratio no more than two.

However, little is known about multitask scheduling. Scully et. al [9] presented the first theoretical analysis of single-processor multitask scheduling problem, and gave an optimal policy that is easy to compute for batch arrival, together with the assumption that the processing time of tasks satisfies the aged Pareto distributions. To model the scenario when the scheduler has incomplete information about job size, Scully et. al [10] introduced the multistage job model and proposed an optimal scheduling algorithm for multistage job scheduling in M/G/1 queue. In addition, the closed-form expression of mean response time is given for the optimal scheduler. Sun et al. [11] studied the multitask scheduling problem when all the tasks are of unit size, and proved that
among causal and non-preemptive policies, fewest unassigned tasks first (FUT) policy, earliest due date first (EDD) policy, and first come first serve (FCFS) are near delay-optimal in distribution (stochastic ordering) for minimizing the metric of average delay, maximum lateness and maximum delay respectively.

Contributions. In this paper we answer the aforementioned questions and our contributions are summarized as follows. The analysis in this paper follows from and is closely related to that in [2, 6].

• We present Algorithm 1 in Section 3, which is a simple modification of SRPT and achieves a competitive ratio of $O(\log \alpha + \beta)$, where $\alpha$ is the maximum-to-minimum job workload ratio, $\beta$ represents the ratio between maximum non-preemptive task workload and minimum job workload. In addition, it can be shown that no $o(\log \alpha + \beta)$-competitive algorithm exists when the number of machines is constant. For the class of work-conserving algorithms, $O(\alpha + \beta^{1-\epsilon})$ is the best possible competitive ratio.

• Under certain probabilistic structure on the problem instances, we further reveal the following conclusion about the algorithm in Section 4, by utilizing our aforementioned result in the adversarial setting. Assuming that jobs arrive according to a poisson process, we prove that Algorithm 1 is optimal when load $\rho \to 1$, as long as the workload of non-preemptive tasks are upper bounded by threshold $\eta$ specified in equation (10).

2 Model and preliminaries

Deterministic Model. We are given a set $J = \{J_1, J_2, \ldots, J_n\}$ of $n$ jobs arriving online over time, together with a set of $N$ identical machines. Job $i$ consists of $n_i$ tasks and its workload $p_i$ is equal to the total summation of the processing time of tasks, i.e., $p_i = \sum_{\ell \in n_i} p_i^{(\ell)}$, where $p_i^{(\ell)}$ represents the processing time of task $\ell$. Tasks can be either preemptive or non-preemptive. A task is non-preemptive if it is not allowed to interrupt the task once it starts service, i.e., the task is run to completion. All the information of job $i$ is unknown to the algorithm until its releasing date $r_i$. Under any given scheduling algorithm, the completion time of job $j$ under the algorithm, denoted by $C_j$, is equal to the maximum completion time of individual tasks within the job. Formally, let $C_j^{(\ell)}$ be the completion time of task $\ell$ in job $j$, then $C_j = \max_{\ell \in [n]} C_j^{(\ell)}$. The flow time of job $j$ is defined as $F_j = C_j - r_j$, our objective is to minimize the total flow time $\sum_{j \in [n]} F_j$. Note that different tasks within the same job may or may not be allowed to be processed in parallel, our analysis holds for both scenarios.

Throughout the paper we use $\alpha = \max_{i \in [n]} p_i / \min_{i \in [n]} p_i$ to denote the ratio of maximum and minimum job workload. Let $\eta = \max\{p_i^{(\ell)} | \text{task } \ell \text{ of job } i \text{ is non-preemptive}\}$ be the maximum processing time of a non-preemptive task, $\beta = \eta / \min_{i \in [n]} p_i$ be the ratio between $\eta$ and minimum job workload. In some sense, parameters $\beta$ and $\eta$ represent the degree of non-preemptivity and exhibits a trade-off between the preemptive and non-preemptive setting, since the problem degenerates to the preemptive case when $\eta = 1$, and the problem approaches the non-preemptive case when $\eta$ increases to $\max_{i \in [n]} p_i$.

The definitions of work-conserving algorithms and competitive are formally given as following, notations of this paper are summarized in Table 1.
Definition 1 (Work-conserving scheduling algorithm [3]). A scheduling algorithm $\pi$ is called work-conserving if it never idles machines when there exists at least one feasible job or task awaiting the execution in the system. Here a job or task is called feasible, if it satisfies all the given constraints of the system (e.g., precedence constraint, preemptive and non-preemptive constraint, etc).

Definition 2 (Competitive ratio). The competitive ratio of online algorithm $A$ refers to the worst ratio of the cost incurred by $A$ and that of optimal offline algorithm $A^*$ over all input instances $\omega$ in $\Omega$, i.e.,

$$CR_A = \max_{\omega \in \Omega} \frac{\text{Cost}_A(\omega)}{\text{Cost}_{A^*}(\omega)}.$$ 

In the multiple-processor multitask scheduling problem, the cost is the total flow time under instance $\omega = \{(r_i, p_i^{(t)})\}_{t\in[n],i\in[n]}$.

**Stochastic Model.** In the stochastic setting, we assume that jobs arrive into the system according to a Poisson process with rate $\lambda$. Job processing time are i.i.d distributed with probability density function $f(\cdot)$. The analysis relies on the concept of busy period, which is defined as following.

Definition 3 (Busy Period [3]). Busy period is defined to be the longest time interval in which no machines are idle.

We use $B(w)$ to denote the length of a busy period with started by a workload of $w$. It can be seen that the $B(\cdot)$ is an additive function [3 p.460], i.e., for $\forall w_1, w_2$,

$$B(w_1 + w_2) = B(w_1) + B(w_2),$$

as a busy period with initial workload of $w_1 + w_2$ can be regarded as a busy period started by initial workload $w_2$, following a busy period started by initial workload $w_1$. Moreover, the length of a busy period with initial workload of $w$ and load $\rho$ is shown to be equal to

$$B(w) = \frac{\mathbb{E}[w]}{1 - \rho}. \quad (1)$$

| Notation | Description |
|----------|-------------|
| $N$      | number of machines |
| $n$      | number of jobs |
| $r_i$    | arrival time of job $i$ |
| $p_i$    | total workload of job $i$ |
| $\rho \leq y$ | load composed of jobs with size 0 to $y$: $\rho_y = \lambda \cdot \int_0^y tf(t)dt$ |
| $\alpha$ | job size ratio: $\alpha = \max_{i\in[n]} \frac{p_i}{\min_{i\in[n]} p_i}$ |
| $\eta$  | maximum processing time of a single task |
| $\beta$ | $\frac{\eta}{\min_{i\in[n]} p_i}$ |
| $C_i$   | completion time of job $i$ |

Table 1: Notation Table
3 Competitive Ratio Analysis

The main idea of Algorithm 1 is similar as SPRT, i.e., we utilize as many resources as possible on the job with smallest remaining workload, to reduce the number of alive jobs in a greedy manner, while satisfying all the given constraints.

Algorithm 1: Modified SRPT (M-SRPT)

1. At each time slot $t$, maintain the following quantities:
   - For each job $i \in [n]$, maintain
     - $W_i(t)$ // remaining workload
     - $w_i(t)$ // remaining workload of the shortest single task being processed (if exists) or alive
   - $J_1(t) \leftarrow \{i \in [n] | w_i(t) = 0\}$ // Jobs with tasks that are finished at time $t$
   - $d(t) \leftarrow |J_1(t)|$ // Number of machines to be reallocated and assign jobs alive to the $d(t)$ machines, where jobs with lower remaining workload have a higher priority. When parallelism is not allowed, at most one machine is allocated to a single job.

3.1 Performance Analysis

Our main result is stated in the following theorem.

Theorem 4. Algorithm 1 achieves a competitive ratio that is no more than

$$CR_{M-SRPT} \leq 4 \log \frac{p_{\max}}{p_{\min}} + 2 \frac{\eta}{p_{\min}} + 8.$$  

To show the competitive ratio above, we divide the jobs into different classes and compare the remaining number of jobs under Algorithm 1 with that under optimal algorithm $\pi^*$. For any algorithm $\pi$, at time slot $t$, we divide the unfinished jobs into $\Theta(\log \alpha)$ classes $\{C_k(\pi,t)\}_{k \in [\log \alpha + 1]}$, based on their remaining workload. Jobs with remaining workload that is no more than $2^k$ and larger than $2^{k-1}$ are assigned to the $k$-th class. Formally,

$$C_k(\pi,t) = \left\{ i \in [n] \mid W_i(\pi,t) \in (2^{k-1}, 2^k) \right\},$$

where $W_i(\pi,t)$ represents the unfinished workload of job $i$ at time $t$. In the following analysis, we use $C^{[k]}(\pi,t) = \bigcup_{i=1}^{k} C_i(\pi,t)$ to denote the collection of jobs in the first $k$ classes, and let $W^{[k]}_\pi(t) = \sum_{i=1}^{k} W^{(i)}_\pi(t)$ represent the total remaining workload of jobs in the first $k$ classes, where $W^{(k)}_\pi(\pi,t)$ denotes the amount of remaining workload of jobs in class $C_k(\pi,t)$. $W^{[k]}_{\pi^*}(t)$ and $W^{[k]}_{\pi^*}(t)$ are defined in a similar way for the optimal scheduling algorithm $\pi^*$.

We first prove the following lemma, which relates the remaining workload in M-SRPT with that under optimal algorithm $\pi^*$.

Lemma 5. For $\forall t \geq 0$, the unfinished workload under Algorithm 1 can be upper bounded as

$$W_{M-SRPT}^{[k]}(t) \leq W_{\pi^*}^{[k]}(t) + N \cdot (2^{k+1} + \eta + 1).$$  

(2)
Proof: In the proof we always divide jobs into different classes according to the remaining workload under M-SRPT, we suppress reference to M-SRPT in the notation of $C_k$. Without loss of generality we can assume that $W_{M-SRPT}^{[k]}(t) > W_{\pi^*}^{[k]}(t)$, otherwise Lemma 2 already holds. Since the remaining workload under M-SRPT is strictly larger than that under the optimal algorithm, we claim that there must exist time slots in $(0, t)$, at which either

- Idle machines exist under M-SRPT;
- Jobs with remaining workload (under M-SRPT) larger than $2^k$ are processed.

Otherwise, all the machines will be processing jobs belonging to set $C_{[k]}(t)$ before time $t$, while no jobs in higher classes, *i.e.*, $\bigcup_{i>k} C_i(t)$, will be switched into class $C_{[k]}(t)$. Combining with the fact that the initial workload under Algorithm 1 and optimal algorithm are identical, *i.e.*, $W_{M-SRPT}^{[k]}(0) = W_{\pi^*}^{[k]}(0)$, we can see that $W_{M-SRPT}^{[k]}(t)$ should be no more than $W_{\pi^*}^{[k]}(t)$ and the contradiction appears. Now consider the following two collection of time slots before $t$:

$$
T_k^{(1)} = \{ \bar{t} \in [0, t] \} \text{ At time } \bar{t}, \text{ at least one machine is idle under Algorithm 1}, \\
T_k^{(2)} = \{ \bar{t} \in [0, t] \} \text{ At time } \bar{t}, \text{ there exists } i > k \text{ such that at least one machine is} \\
\text{processing jobs in } C_i \text{ under Algorithm 1}.
$$

Let $\tilde{t}_k^{(i)} = \max\{ t \mid t \in T_k^{(i)} \} (i \in \{1, 2\})$ be the last time slot in $T_k^{(i)}$, based on which we divide our proof into the following two cases.

Case 1: $\tilde{t}_k^{(1)} \geq \tilde{t}_k^{(2)}$. From the definition of $\tilde{t}_k^{(1)}$, it can be seen that during $(\tilde{t}_k^{(1)}, t]$, no machines are idle or process jobs with remaining workload larger than $2^k$ under Algorithm 1 while the increment in remaining workload incurred by newly arriving jobs are identical for Algorithm 1 and $\pi^*$. In addition, it is important to point out that $([n] \setminus C_{[k]}(\tilde{t}_k^{(1)})) \cap C_{[k]}(\bar{t}) = \emptyset$ for $\forall \bar{t} \in (\tilde{t}_k^{(1)}, t]$, *i.e.*, no job will switch from a higher class to $C_{[k]}$ during $(\tilde{t}_k^{(1)}, t]$. Hence

$$W_{M-SRPT}^{[k]}(t) - W_{\pi^*}^{[k]}(t) \leq W_{M-SRPT}^{[k]}(\tilde{t}_k^{(1)}) - W_{\pi^*}^{[k]}(\tilde{t}_k^{(1)}).$$

It suffices to prove the workload difference inequality (2) for $t = \tilde{t}_k^{(1)}$, *i.e.,*

$$W_{M-SRPT}^{[k]}(\tilde{t}_k^{(1)}) \leq W_{\pi^*}^{[k]}(\tilde{t}_k^{(1)}) + N \cdot (2^k + 1 + \eta + 1). \tag{3}$$

Note that there exists some idle machines at time $t = \tilde{t}_k^{(1)}$, which implies that under Algorithm 1 the number of jobs alive must be less than $N$. Hence $W_{M-SRPT}^{[k]}(\tilde{t}_k^{(1)}) \leq (N - 1) \cdot 2^k$ and (3) holds.

Case 2: $\tilde{t}_k^{(1)} < \tilde{t}_k^{(2)}$. According to the definition of $\tilde{t}_k^{(2)}$, there exist jobs with remaining workload larger than $2^k$ being processed at $\tilde{t}_k^{(2)}$, we use $\mathcal{J}(\tilde{t}_k^{(2)}) = [n] \setminus C_{[k]}(\tilde{t}_k^{(2)})$ to denote the collection of such jobs.

When all the tasks are processed preemptively, we can obtain (2) easily, as we are able to conclude that there are at most $N - 1$ jobs in $C_{[k]}(\tilde{t}_k^{(2)})$. This is because that tasks are allowed
to be preempted, while Algorithm 1 selects a job with remaining workload larger than $2^k$ at the beginning of time $\bar{t}_k^{(2)}$. Consequently $W^{[k]}_{M-SRPT}(\bar{t}_k^{(2)}) \leq n_{M-SRPT}^{[k]}(\bar{t}_k^{(2)}) \cdot 2^k$ and for $\forall t > \bar{t}_k^{(2)}$,

$$W^{[k]}_{M-SRPT}(t) - W^{[k]}_{\pi^*}(t) \leq W^{[k]}_{M-SRPT}(\bar{t}_k^{(2)}) - W^{[k]}_{\pi^*}(\bar{t}_k^{(2)}) + [N - n_{M-SRPT}^{[k]}(\bar{t}_k^{(2)})] \cdot 2^k \leq N \cdot 2^k,$$

where the first inequality follows from the fact that no more than $N - n_{M-SRPT}^{[k]}(\bar{t}_k^{(2)})$ jobs switches from higher classes to $C^{[k]}(t)$, as there are at most $N - n_{M-SRPT}^{[k]}(\bar{t}_k^{(2)})$ jobs with remaining workload larger than $2^k$ are being processed at time $\bar{t}_k^{(2)}$. Hence Lemma 5 holds.

Now for the case when there exist non-preemptive tasks, arguments above does not work since machines may be processing tasks with remaining workload larger than $2^k$ and $n_{M-SRPT}^{[k]}(\bar{t}_k^{(2)})$ may be larger than $N$. Let $r \in [N]$ be the number of tasks that are being processed at time $\bar{t}_k^{(2)}$ and belongs to $[n] \setminus C^{[k]}(\bar{t}_k^{(2)})$, and $t_s \leq \bar{t}_k^{(2)}$ be the latest starting processing time of these tasks. We divide our analysis into the following two subcases:

- **Case 2.1:** No jobs switch from set $[n] \setminus C^{[k]}(t_s)$ to $C^{[k]}(\bar{t}_k^{(2)})$ under Algorithm 1. We use $\Delta_k$ to represent the increment of $W^{[k]}_{\pi^*}$ incurred by the newly arriving jobs during time period $[t_s, \bar{t}_k^{(2)}]$. Then we have:

$$W^{[k]}_{M-SRPT}(\bar{t}_k^{(2)}) - W^{[k]}_{M-SRPT}(t_s) = -(N - r)(\bar{t}_k^{(2)} - t_s) + \Delta_k. \quad (4)$$

On the other hand, $W^{[k]}_{\pi^*}$, the remaining workload of jobs in class $C^{[k]}$ under optimal algorithm, decreases at a speed that is no more than $N$ units of workload per time slot, hence

$$W^{[k]}_{\pi^*}(\bar{t}_k^{(2)}) - W^{[k]}_{\pi^*}(t_s) \geq -N \cdot (\bar{t}_k^{(2)} - t_s) + \Delta_k. \quad (5)$$

According to the definition of $\bar{t}_k^{(2)}$, no jobs with remaining workload larger than $2^k$ are processed in $[\bar{t}_k^{(2)}, t]$. Compared with time $\bar{t}_k^{(2)}$, there are at most $r$ jobs switch from $[n] \setminus C^{[k]}(\bar{t}_k^{(2)})$ to set $C^{[k]}(\bar{t}_k^{(2)} + 1)$. Therefore

$$W^{[k]}_{M-SRPT}(\bar{t}_k^{(2)} + 1) - W^{[k]}_{\pi^*}(\bar{t}_k^{(2)} + 1) \leq W^{[k]}_{M-SRPT}(\bar{t}_k^{(2)}) - W^{[k]}_{\pi^*}(\bar{t}_k^{(2)}) + r \cdot 2^k. \quad (6)$$

Combining inequalities (4) - (6), we can obtain

$$W^{[k]}_{M-SRPT}(t) - W^{[k]}_{\pi^*}(t) \leq W^{[k]}_{M-SRPT}(\bar{t}_k^{(2)} + 1) - W^{[k]}_{\pi^*}(\bar{t}_k^{(2)} + 1) \leq W^{[k]}_{M-SRPT}(t_s) - W^{[k]}_{\pi^*}(t_s) + r \cdot [2^k + (\bar{t}_k^{(2)} - t_s)] \leq (N - 1) \cdot 2^k + r \cdot (\bar{t}_k^{(2)} - t_s) \leq N \cdot (2^k + \eta).$$

The third inequality above holds since at time $t_s$, Algorithm 1 is required to do job selection and a job with remaining workload larger than $2^k$ is selected. The last inequality follows from the fact that $\bar{t}_k^{(2)} - t_s \leq \eta$, as $t_s$ is the starting time of a non-preemptive task that is still alive at time $\bar{t}_k^{(2)}$.  

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• **Case 2.2:** There exist jobs switching from set \([n] \setminus \mathcal{C}_{[k]}(t_s)\) to \(\mathcal{C}_{[k]}(t_k^{(2)})\) under Algorithm [1]. We use \(\mathcal{J}_s\) to denote the collection of such switching jobs. It is essential to bound the number of switching jobs, which will incur an increment of \(|\mathcal{J}_s| \cdot 2^k\) in the remaining workload of class \(\mathcal{C}_{[k]}\). A straightforward bound is \(|\mathcal{J}_s| \leq N \cdot (\bar{t}_k^{(2)} - t_s) \leq N \cdot \eta\), since at most \(N\) jobs receive service at each time slot, and hence the number of switching jobs is no more than \(N\). However, this bound is indeed loose and we argue that

\[
|\mathcal{J}_s| \leq N - r. \tag{7}
\]

Notice that after a job switches to class \(\mathcal{C}_{[k]}\) during \([t_s, \bar{t}_k^{(2)}]\), it will only be preempted by jobs that are also in class \(\mathcal{C}_{[k]}\), which is due to the SRPT rule. According to the precondition of this case, there are \(r\) jobs in set \([n] \setminus \mathcal{C}_{[k]}\) that are continuously being processed during \([t_s, \bar{t}_k^{(2)}]\), hence at most \(N - r\) units of resources per time slot are available for the remaining jobs. Note that resources that are allocated to jobs in \(\mathcal{C}_{[k]}\) will not be utilized for switching a job from a higher class to \(\mathcal{C}_{[k]}\). In addition, finished jobs will have no contribution to the total remaining workload \(W^{[k]}_{M-SRPT}(t)\). Hence \(|\mathcal{J}_s|\) is no more than \(N - r\).

Furthermore, we can derive the following conclusion:

\[
W^{[k]}_{M-SRPT}(t) - W^{[k]}_{\pi^*}(t)
\leq W^{[k]}_{M-SRPT}(\bar{t}_k^{(2)} + 1) - W^{[k]}_{\pi^*}(\bar{t}_k^{(2)} + 1)
\leq W^{[k]}_{M-SRPT}(\bar{t}_k^{(2)}) - W^{[k]}_{\pi^*}(\bar{t}_k^{(2)}) + r \cdot 2^k + N \quad \text{(job switching at } t^{(2)})
\leq [W^{[k]}_{M-SRPT}(t_s) - W^{[k]}_{\pi^*}(t_s)] + (N - r) \cdot 2^k + N \cdot (\bar{t}_k^{(2)} - t_s)] + r \cdot 2^k + N \quad \text{(job switching during } [t_s, \bar{t}_k^{(2)}])
\leq N \cdot (2^{k+1} + \eta + 1).
\]

The proof is complete. □

We are ready to prove the competitive ratio of Algorithm [1].

**Proof of Theorem 4.** Let \(n_{M-SRPT}(t)\) and \(n_{\pi^*}(t)\) represent the number of jobs alive at time \(t\) under Algorithm [1] and optimal scheduler respectively. For \(\forall t \geq 0\),

\[
n_{\pi^*}(t) \geq \sum_{k=\log p_{\min}}^{\log p_{\max}+1} \frac{W^{[k]}_{\pi^*}(t)}{2^k} = \sum_{k=\log p_{\min}}^{\log p_{\max}+1} \frac{W^{[k]}_{\pi^*}(t) - W^{[k-1]}_{\pi^*}(t)}{2^k} \quad \text{(definition of } W^{[k]}_{\pi^*}(t))
\]

\[
= \frac{W^{[k]}_{M-SRPT}(t)}{2^{\log p_{\max}+1}} + \sum_{k=\log p_{\min}}^{\log p_{\max}+1} \frac{W^{[k]}_{M-SRPT}(t)}{2^{k+1}} \geq \sum_{k=\log p_{\min}}^{\log p_{\max}+1} \frac{W^{[k]}_{\pi^*}(t)}{2^{k+1}}. \tag{8}
\]

On the other hand, the number of jobs alive under Algorithm [1] can be upper bounded in a similar
conserving algorithms have a competitive ratio of $\Omega(\log n)$. There exists no algorithm achieving a competitive ratio of $O(\log n)$ for the above problem.

The following lower bounds mainly follow from the observation that, multiple-processor multitask scheduling problem generalizes both the preemptive and non-preemptive settings. The proof is complete.

**Fact 6.** For multiple-processor multitask scheduling problem with constant number of machines, there exists no algorithm achieving a competitive ratio of $O(\log \alpha + \beta)$.

**Proof:** When $p_{\min} = \eta = 1$, the problem degenerates to preemptive setting and no algorithm can achieve a competitive ratio of $o(\log \alpha)$. When $\eta = p_{\max}$, the problem degenerates to the non-preemptive setting and $O(\beta)$ is the best possible competitive ratio if the number of machines is constant. The proof is complete.

**Fact 7.** For multiple-processor multitask scheduling problem, the competitive ratio of any work-conserving algorithms have an competitive ratio of $\Omega(\log \alpha + \beta^{1-\varepsilon})$ for $\forall \varepsilon > 0$.

**Proof:** The reasoning is similar as Fact 6 since work-conserving algorithms cannot achieve a competitive ratio of $o(\beta^{1-\varepsilon})$ in the non-preemptive setting.

### 3.2 Competitive Ratio Lower Bound

The following lower bounds mainly follow from the observation that, multiple-processor multitask scheduling problem generalizes both the preemptive and non-preemptive settings.

**Fact 6.** For multiple-processor multitask scheduling problem with constant number of machines, there exists no algorithm achieving a competitive ratio of $O(\log \alpha + \beta)$.

**Proof:** When $p_{\min} = \eta = 1$, the problem degenerates to preemptive setting and no algorithm can achieve a competitive ratio of $o(\log \alpha)$. When $\eta = p_{\max}$, the problem degenerates to the non-preemptive setting and $O(\beta)$ is the best possible competitive ratio if the number of machines is constant. The proof is complete.

**Fact 7.** For multiple-processor multitask scheduling problem, the competitive ratio of any work-conserving algorithms have an competitive ratio of $\Omega(\log \alpha + \beta^{1-\varepsilon})$ for $\forall \varepsilon > 0$.

**Proof:** The reasoning is similar as Fact 6 since work-conserving algorithms cannot achieve a competitive ratio of $o(\beta^{1-\varepsilon})$ in the non-preemptive setting.
4 Optimality with Poisson Arrival

In this section we show that under mild probabilistic assumptions, Algorithm 1 is asymptotic optimal for minimizing the total flow time in the heavy traffic region. The main result is stated as following.

**Theorem 8.** Let $F_{\rho}^{\text{M-SRPT}}$ and $F_{\rho}^{*}$ be the mean flow time incurred by Algorithm 1 and optimal algorithm respectively, when the traffic intensity is equal to $\rho$. In an $M/G/N$ with job size distribution satisfying either (1) bounded or (2) unbounded with tail function of upper Matuszewska index less than $-2$, Algorithm 1 is heavy traffic optimal, i.e.,

$$\lim_{\rho \to 1} \frac{\mathbb{E}[F_{\rho}^{\text{M-SRPT}}]}{\mathbb{E}[F_{\rho}^{*}]} = 1,$$  

(9)

as long as the size of a single task is no more than

$$\eta = \begin{cases} 
\Theta\left(\frac{1}{(1-\rho) \int_0^1 \frac{f(x)}{1-x} dx}\right) & \text{Case (1)} \\
\Theta\left(\frac{1}{(1-\rho) G^{-1}(\rho) \int_0^1 \frac{f(x)}{1-x} dx}\right) & \text{Case (2)} 
\end{cases}$$  

(10)

**Remark.** The probabilistic assumptions (1) and (2) here are all with respect to the distribution of job size, i.e., the total workload of tasks. For the processing time of a single task, the only assumption we have is the upper bound $\eta$. It can be seen that the optimality result in [2] corresponds to the special case when $\eta = 1$, while the bound derived in [10] could be extremely large when $\rho$ approaches 1. On the other hand, for the integral above, we have the following rough estimation,

$$\int_0^\infty \frac{f(x)}{1-\rho \leq x} dx \leq \int_0^\infty \frac{x f(x)}{1-\rho \leq x} dx + \int_0^1 \frac{f(x)}{1-\rho \leq x} dx \leq \log \left(\frac{1}{1-\rho}\right) + \frac{1}{1-\rho}.$$

**Lower bound on minimum flow time** $\mathbb{E}[F_{\rho}^{*}]$. To start with, we consider the benchmark system consisting of a single machine with speed $N$, where all the tasks can be allowed to be served in preemptive fashion, i.e., the concept of task is indeed unnecessary in this setting. The performance of SRPT for this single server system is summarized by the following fact.

**Fact 9 ([7]).** In an $M/G/1$ with service distribution satisfying either (1) bounded or (2) unbounded with tail function of upper Matuszewska index less than $-2$, then

$$\mathbb{E}[F_{\rho}^{\text{SRPT}^{-1}}] = \begin{cases} 
\Theta\left(\frac{1}{1-\rho}\right) & \text{Case (1)} \\
\Theta\left(\frac{1}{(1-\rho) G^{-1}(\rho)}\right) & \text{Case (2)} 
\end{cases}$$

where $G^{-1}(\cdot)$ denotes the inverse of $G(x) = \rho_{\leq x}/\rho$.

It is clear to see that the mean flow time under SRPT for this system can be performed as a valid lower bound for the multitask problem, i.e.,

$$\mathbb{E}[F_{\rho}^{*}] \geq \mathbb{E}[F_{\rho}^{\text{SRPT}^{-1}}].$$  

(11)
Proof of Theorem \[8\]: Our main goal is to derive an analytical upper bound on the quantity \( \mathbb{E}[F_{\rho}^M-SRPT] \). The proof mainly follows from techniques in \([2, 8]\), which relates the flow time of the tagged job with an appropriate busy period.

Consider a tagged job with remaining workload \( x \), arriving time \( r_x \) and completion time \( C_x \). The computing resources of \( N \) servers must be spent on the following types of job during time \([r_x, C_x]\):

1. The system may be dealing with jobs with remaining workload larger than \( x \), or some machines are idle, while the tagged job is in service, because the number of jobs alive is smaller than \( N \). We use \( W_{\text{waste}}(r_x) \) to represent the amount of such resources, then
   \[
   W_{\text{waste}}(r_x) \leq (N - 1) \cdot x, \tag{12}
   \]
   which is indeed the same as Lemma 5.1 in \([2]\). The reason is straightforward—the tagged job must be in service according to Algorithm \([1]\) hence the number of such time slots should not exceed \( x \) and thus (12) holds.

2. The system may be dealing with jobs with remaining workload no more than \( x \) at time \( r_x \), the amount of resources spent on this class is no more than \( W_{M-SRPT}^\leq(r_x) \). Here we use \( W_{\leq}^M-SRPT(t) \) to denote the total workload of jobs with remaining workload no more than \( x \) at time \( t \).

3. The system may be dealing with jobs which have a remaining workload larger than \( x \) at time \( t = r_x \), while the tagged job is not in service. This is possible and happens only if the system may be processing tasks which belong to a job with total remaining workload larger than \( x \), the tasks are in service before time \( r_x \) and the non-preemptive rule allows the task to be served from time \( r_x \) onwards. Let \( W_{\text{non-pm}}(r_x) \) denote the total units of computing resources spent on this class of jobs during \([r_x, C_x]\). Our main argument for this class of jobs is
   \[
   W_{\text{non-pm}}(r_x) \leq (N^2 + N) \cdot \eta + N \cdot x, \tag{13}
   \]
To see the correctness of inequality (13), we consider time intervals \([r_x, r_x + \eta]\) and \((r_x + \eta, C_x]\) separately.

- Note that there are \( N \cdot \eta \) computing resources during time \([r_x, r_x + \eta]\) in total, hence it is obvious to see that the amount of resources spent on this collection of jobs during \([r_x, r_x + \eta]\) cannot exceed \( N \cdot \eta \).
- We next show that in time interval \((r_x + \eta, C_x]\), the total amount of computing resources spent on such jobs is no more than \( N^2 \cdot \eta + N \cdot x \). Consider the following two types of jobs:
  - Note that jobs of this class that have a remaining workload larger than \( x \) at time \( t = r_x + \eta \) will be processed after time \( t = r_x + \eta \) only if the tagged job is in service, hence the amount of resources spending on such jobs are already taken into account in the first class above, i.e., the quantity \( W_{\text{waste}}(r_x) \), and we can ignore this subclass.
  - For the collection of jobs with remaining workload no more than \( x \) at time \( t = r_x + \eta \), we first consider the setting when different tasks within the same job can be processed in parallel. It is clear to see that the remaining workload of such jobs at time \( t = r_x \)
must be no more than \(x + N \cdot \eta\). Since there are at most \(N\) such jobs in total, we can conclude that the remaining workload of jobs in this subclass must be no more than \(N \cdot (x + N \cdot \eta) = N \cdot x + N^2 \cdot \eta\), which implies that \(W_{\text{non-pm}}(r_x) \leq N \cdot x + N^2 \cdot \eta + N \eta\) and (13) holds.

4. Tagged job itself. The amount of resources is equal to \(x\), the size of the tagged job.

5. Newly arriving jobs during \([r_x, C_x]\) with size no more than \(x\).

Hence \(f_{M-SRPT}^x\), the flow time of the tagged job, is no more than the length of a busy period with arrival rate \(\rho \leq x\) and initial workload of \(W_{\text{waste}}(r_x) + W_{\text{non-pm}}(r_x) + W_{\text{M-SRPT}}^x(r_x) + x\). Hence we have

\[
\begin{align*}
    f_{M-SRPT}^x \leq & \quad B^{(\rho \leq x)} \left( W_{\text{waste}}(r_x) + W_{\text{non-pm}}(r_x) + W_{\text{M-SRPT}}^x(r_x) + x \right) \\
    (a) \quad = & \quad B^{(\rho \leq x)} \left( W_{\text{waste}}(r_x) + W_{\text{non-pm}}(r_x) + x \right) + B^{(\rho \leq x)} \left( W_{\text{M-SRPT}}^x(r_x) \right) \\
    (b) \quad \leq & \quad B^{(\rho \leq x)} \left( N^2 \cdot \eta + N \cdot (2x + \eta) \right) + B^{(\rho \leq x)} \left( W_{\text{M-SRPT}}^x(r_x) \right) \\
    (c) \quad \leq & \quad B^{(\rho \leq x)} \left( 3N^2 \cdot (\eta + x) \right) + B^{(\rho \leq x)} \left( W_{\text{SRPT}-1}^x(r_x) \right),
\end{align*}
\]

where (a) follows from the additivity of busy period; In (b) we utilize the upper bounds established in (12) and (13) and (c) follows from Lemma 10.

Note that the average flow time under SRPT in a single server system is lower bounded as

\[
\mathbb{E}[F_{\text{SRPT}-1}] \geq \mathbb{E}[B^{(\rho \leq x)}(W_{\text{SRPT}-1}^x(t))] = \mathbb{E}_{x,r_x}[B^{(\rho \leq x)}(W_{\text{SRPT}-1}^x(r_x))] = \mathbb{E}_{x,r_x}[\Sigma_2],
\]

where the first equality holds due to the Poisson Arrivals See Time Average (PASTA) property \([13]\).

Note that

\[
\mathbb{E}[\Sigma_1] = O \left( \mathbb{E} \left[ B^{(\rho \leq x)}(\eta + x) \right] \right) = O \left( \mathbb{E} \left[ \frac{\eta + x}{1 - \rho \leq x} \right] \right) \\
= O \left( \log \frac{1}{1 - \rho} \right) + \eta \cdot O \left( \int_0^\infty \frac{f(x)}{1 - \rho \leq x} \, dx \right).
\]

To achieve heavy traffic optimality, it suffices to show that the difference between average flow time under Algorithm 1 and optimal algorithm is a lower order term, \(i.e.,\)

\[
\lim_{\rho \to 1} \frac{\mathbb{E}[F_{\text{M-SRPT}}^x] - \mathbb{E}[F_{\text{SRPT}-1}]}{\mathbb{E}[F_{\text{SRPT}-1}]} = 0.
\]

Note that

\[
\mathbb{E}[F_{\text{M-SRPT}}^x] = \mathbb{E}_{x,r_x}[f_{M-SRPT}^x] = \mathbb{E}_{x,r_x}[\Sigma_1] + \mathbb{E}_{x,r_x}[\Sigma_2],
\]

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can be simplified to
\[
\lim_{\rho \to 1} \frac{\eta \cdot O \left( \int_0^\infty \frac{f(x)}{1-x} dx \right)}{\mathbb{E}[F^{\text{SRPT}}_{\rho}]} = 0,
\]
since \(\log(1/(1 - \rho))\) is always a lower order term, compared with the optimal flow time \(\mathbb{E}[F^{\text{SRPT}}_{\rho}]\).

Hence
\[
\eta = \begin{cases} 
\mathcal{o}\left(\frac{1}{(1-\rho)\int_0^\infty \frac{f(x)}{1-x} dx}\right) & \text{Case (1)} \\
\mathcal{o}\left(\frac{1}{(1-\rho)G^{-1}(\rho)\int_0^\infty \frac{f(x)}{1-x} dx}\right) & \text{Case (2)}
\end{cases}
\]

Lemma 10. The difference of under Algorithm 7 and optimal algorithm is upper bounded by
\[
W^{M-\text{SRPT}}_{\leq y}(t) - W^{\text{SRPT}}_{\leq y}(t) \leq N \cdot (y + \eta + 1), \forall y, t \geq 0.
\]

Proof: The proof is similar as that of Lemma 5.

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