Two flags in a semimodular lattice generate an antimatroid

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Abstract

A basic property in a modular lattice is that any two flags generate a distributive sublattice. It is shown (Abels 1991, Herscovic 1998) that two flags in a semimodular lattice no longer generate such a good sublattice, whereas shortest galleries connecting them form a relatively good join-sublattice. In this note, we sharpen this investigation to establish an analogue of the two-flag generation theorem for a semimodular lattice. We consider the notion of a modular convex subset, which is a subset closed under the join and meet only for modular pairs, and show that the modular convex hull of two flags in a semimodular lattice of rank $n$ is isomorphic to a union-closed family on $[n]$. This family uniquely determines an antimatroid, which coincides with the join-sublattice of shortest galleries of the two flags.

Keywords: semimodular lattice, antimatroid, Jordan-Hölder permutation, modular pair

1 Introduction

One of the basic and classical results in lattice theory is that any two maximal chains (flags) in a modular lattice generates a distributive sublattice:

Theorem 1.1 (4). Let $L$ be a modular lattice with a finite rank, and let $C, D$ be flags of $L$. Then the sublattice generated by $C \cup D$ is a distributive lattice.

The present note addresses an extension of this theorem for a semimodular lattice. The distributive sublattice in the above theorem can be interpreted in terms of the graph of all flags, which is obtained by joining two flags differing on exactly one element. A path in this graph is usually referred to as a gallery. Abels [1, 2] studied the gallery-distance of two flags in a semimodular lattice, and showed that for the case of a modular lattice, the union of all shortest galleries between two flags coincides with the distributive sublattice generated by them. He also showed that in a semimodular lattice, the elements in shortest galleries between two flags form a join-closed subset of

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the original lattice. This direction was further addressed by Herscovic [10] in terms of reduced decompositions of the Jordan-Hölder permutation. He showed that the sublattice generated by two flags in a semimodular lattice no longer has good properties; it needs not to be ranked and is not known to be finite. He claims that in a semimodular lattice the join-closed subset of shortest galleries of two flags is a natural correspondent for the sublattice of two flags in a modular lattice.

In this note, we sharpen these investigations to establish an analogue of Theorem 1.1 for a semimodular lattice. The above "generation by two flags" is meant as repeated additions of join and meet of pairs from the elements in the flags. We here consider a restricted generation by join and meet only for modular pairs—pairs of elements satisfying the modularity equality for their rank. We introduce a modular convex subset as a subset that is closed under join and meet for modular pairs. The main result in this note is an extension of Theorem 1.1 in terms of the modular convex hull, which clarifies the above join-closed subset of two flags and establishes a link to antimatroids. Specifically, we show that in a semimodular lattice with rank $n$ the modular convex hull of two flags is isomorphic, as a poset, to a union-closed family $\mathcal{A} \subseteq 2^{[n]}$ having a (maximal) chain of length $n$. Such a union-closed family is referred here to as a pre-antimatroid, since it uniquely determines an antimatroid [6, 7, 11] as the union of all maximal chains. Its importance was recently recognized by [5, 14] in the literature of Knowledge Space Theory (KST) [8]. We show that this antimatroid coincides with the above join-closed subset of shortest galleries between the two flags.

Various classes of antimatroids and semimodular lattices are known. It would be an interesting future research to study how they are related via the modular convex hull of two flags. Our proof is constructive, and will be useful for such study.

2 Preliminaries

A lattice is a partially ordered set $L$ in which every pair $p, q$ of elements has join $p \lor q$ (the minimum common upper bound) and meet $p \land q$ (the maximum common lower bound). The partial order is denoted by $\leq$. By $p < q$ we mean $p \leq q$ and $p \neq q$. For a pair $p, q$ with $p \leq q$, the interval $[p, q]$ is defined as the set of elements $z$ with $p \leq z \leq q$. We say that $q$ covers $p$ if $p \neq q$ and $[p, q] = \{p, q\}$. A totally ordered subset is called a chain, which will be denoted simply as $p_1 < p_2 < \cdots$. The length of a chain is its cardinality minus one. A subset $S$ of a lattice is said to be join-closed if $p, q \in S \Rightarrow p \lor q \in S$. (Such a subset is called a subsemilattice in [2] and join-sublattice in [10].) Two posets $P, Q$ are said to be isomorphic if there is an order-preserving bijection from $P$ to $Q$. An order-preserving injection from $P$ to $Q$ is called an embedding.

A lattice $L$ considered in this paper has the maximum element and minimum element, which are denoted by 1 and 0. The rank of an element $p$ is defined as the maximum length of a chain from 0 to $p$, and is denoted by $r(p)$. The rank of the lattice $L$ is defined as the rank of 1. We only consider lattices with a finite rank.

For an positive integer $n$, let $[n] := \{1, 2, \ldots, n\}$. The family $2^{[n]}$ of all subsets of $[n]$ is viewed as a (Boolean) lattice by the inclusion order. A subfamily $\mathcal{F} \subseteq 2^{[n]}$ is regarded as a poset with respect to this order.

An (upper-)semimodular lattice is a lattice $L$ satisfying:

- if $a$ covers $a \land b$, then $a \lor b$ covers $b$. 

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It is known [9, Theorem 375] that a semimodular lattice is precisely a lattice whose rank function \( r \) satisfies the semimodularity inequality:

\[
  r(p) + r(q) \geq r(p \land q) + r(p \lor q) \quad (p, q \in L).
\] (2.1)

A pair \((p, q)\) of elements is said to be modular if it holds \((x \lor p) \land q = x \lor (p \land q)\) for every element \(x \in L\) with \(x \leq q\). In a semimodular lattice, modular pairs \((p, q)\) are precisely those pairs which satisfies (2.1) in equality; see [9, Theorem 381]. In particular, a pair \((p, q)\) is modular if and only if \((q, p)\) is modular (which is called M-symmetry).

An antimatroid on a finite set \(E\) is a family \(A \subseteq 2^E\) of subsets satisfying the following properties:

(A1) \(\emptyset, E \in A\).

(A2) For \(X, Y \in A\), it holds \(X \cup Y \in A\).

(A3) For a nonempty set \(X \in A\), there is \(e \in X\) such that \(X \setminus \{e\} \in A\).

See e.g., [3,11] for antimatroids. An antimatroid is a semimodular lattice with respect to inclusion order \(\subseteq\). Indeed, \(X \lor Y\) equals \(X \cup Y\), and \(X \land Y\) equals the union of all members contained in \(X \cap Y\). The rank is given by the cardinality \(X \mapsto |X|\), from which the semimodularity of the rank follows. It is known [6] that an antimatroid is precisely a realization of a special semimodular lattice, called a join-distributive lattice, in \(2^E\).

We introduce a weaker notion of an antimatroid that determines an antimatroid uniquely. A pre-antimatroid on a finite set \(E\) is a family \(K \subseteq 2^E\) of subsets satisfying (A1), (A2), and

(A3’) There is a (maximal) chain of length \(|E|\).

For a pre-antimatroid \(K\), let \(K^* \subseteq K\) be defined as the union of all chains of length \(|E|\).

Lemma 2.1 ([5]; see [14]). Let \(K\) be a pre-antimatroid. Then \(K^*\) is an antimatroid.

Proof. It suffices to verify that \(K^*\) satisfies the union-closedness (A2). Indeed, if \(X, Y\) are chains of length \(|E|\), \(X \in X\), and \(Y \in Y\), then \(\{X' \cup Y' \mid X' \in X, Y' \in Y\} \subseteq K^*\) contains a chain of length \(|E|\) having \(X \cup Y\). \qed

Consider the family of all flags in a semimodular lattice of rank \(n\), and endow it with a graph structure as: Two flags \(C, D\) are adjacent if and only if \(C \cap D = n - 1\). A path in this graph is called a gallery. The flag-distance \(\text{dist}(C, D)\) of two flags \(C, D\) is the minimum length of a shortest gallery between \(C\) and \(D\). It is known [14] that the flag-distance can be computed from Jordan-Hölder permutation; see Theorem 3.5 below.

3 Result

Let \(L\) be a semimodular lattice with rank \(n\). A subset \(S \subseteq L\) is said to be modular convex if for every modular pair of elements \(p, q \in S\) it holds \(p \lor q, p \land q \in S\). The modular convex hull of \(S\), denoted by \(\text{mConv} S\), is defined as the minimal modular convex set containing \(S\). Restricting the partial order of \(L\), the modular convex hull
mConv \( S \) is regarded as a subposet of \( L \). It is clear that in a modular lattice \( \text{mConv} \ S \) is equal to the sublattice generated by \( S \).

The main result of this note clarifies the relationship among the modular convex hull of two flags, their flag-distance interval, and an antimatroidal structure.

**Theorem 3.1.** Let \( C, D \) be flags of \( L \).

1. \( \text{mConv}(C \cup D) \) is isomorphic to a pre-antimatroid \( K \) on \([n]\).

2. A flag \( F \) belongs to \( \text{mConv}(C \cup D) \) if and only if \( F \) belongs to a shortest gallery between \( C \) and \( D \). In particular, the union of all flags of all shortest galleries between \( C \) and \( D \) is isomorphic to the antimatroid \( K^\ast \).

We start the proof. Suppose that \( C \) and \( D \) are given by \( 0 = c_0 < c_1 < \cdots < c_n = 1 \) and \( 0 = d_0 < d_1 < \cdots < d_n = 1 \), respectively. Consider \( L' := \{ p \in L \mid p \geq d_1 \} \), which is a semimodular sublattice with rank \( n - 1 \). Define flags \( C', D' \) of \( L' \) by \( C' := C \cup d_1 = \{ c_i \cup d_1 \}_{i=0,1,\ldots,n} \) and \( D' := D \cup d_1 = \{ d_i \}_{i=1,\ldots,n} \). By semimodularity, \( C' \) is actually a flag of \( L' \).

Let \( k \) be the maximum index for which \( c_k \not\geq d_1 \). Then \( C' \) is written as

\[
d_1 = c_0 \lor d_1 < c_1 \lor d_1 < \cdots < c_k \lor d_1 = c_{k+1} < \cdots < c_n = 1.
\]

Define \( \Delta \subseteq [0, c_k] \) by

\[
\Delta := \{ c_k \land q \mid q \in [d_1, c_{k+1}] \cap \text{mConv}(C' \cup D') : (c_k, q) \text{ is a modular pair} \}.
\] (3.1)

**Lemma 3.2.** \( \text{mConv}(C \cup D) \) equals \( \text{mConv}(C' \cup D') \cup \Delta \), and is join-closed.

**Proof.** Since \( (d_1, c_i) \) is a modular pair, it holds \( C' \subseteq \text{mConv}(C \cup D) \) and \( \text{mConv}(C' \cup D') \subseteq \text{mConv}(C \cup D) \). By definition (3.1), it holds \( \text{mConv}(C \cup D) \supseteq \text{mConv}(C' \cup D') \cup \Delta \).

Also \( (c_k, c_i \lor d_1) (0 \leq i \leq k) \) is a modular pair with \( c_k \land (c_i \lor d_1) = c_i \). This implies that \( \text{mConv}(C' \cup D') \cup \Delta \) contains both \( C \) and \( D \). Therefore we have

\[
\text{mConv}(C \cup D) \supseteq \text{mConv}(C' \cup D') \cup \Delta \supseteq C \cup D.
\] (3.2)

We show:

**Claim.** If \( \text{mConv}(C' \cup D') \) is join-closed and modular convex, then so is \( \text{mConv}(C' \cup D') \cup \Delta \).

Then, by (3.2) and induction, we have the statement. Take any \( u, v \in \text{mConv}(C' \cup D') \cup \Delta \). Our goal is to show \( u \lor v \in \text{mConv}(C' \cup D') \cup \Delta \), and \( u \land v \in \text{mConv}(C' \cup D') \cup \Delta \) if \( (u, v) \) is modular.

Case 1: \( u \in \text{mConv}(C' \cup D'), v \in \Delta \). Write \( v \) by \( v = q \land c_k \) as in (3.1). Since \( q \) covers \( v \) (by modularity), it must hold \( q = d_1 \lor v \). Then \( u \lor v = u \lor d_1 \lor v = u \lor q \). Therefore, if \( \text{mConv}(C' \cup D') \) is join-closed, then \( u \lor v \in \text{mConv}(C' \cup D') \). Suppose that \( (u, v) \) is a modular pair. Then \( (u, q) \) is also a modular pair. Indeed, \( u \land q \lor u \lor v \) or equals \( u \land v \lor u \lor v \); the latter is impossible by \( u \land q \geq d_1 \not\leq u \lor v \). Therefore \( u \lor q = u \lor v \in \text{mConv}(C' \cup D') \), and \( u \land q \in [d_1, c_{k+1}] \cap \text{mConv}(C' \cup D') \). Also \( (u \land q, c_k) \) is a modular pair with \( u \land q \land c_k = u \land v \). Thus \( u \land v \in \Delta \).

Case 2: \( u, v \in \Delta \). Note that \( z \mapsto d_1 \lor z \) is an embedding from \( \Delta \) to \([d_1, c_{k+1}] \cap \text{mConv}(C' \cup D') \) such that \( c_k \land (d_1 \lor z) = z \); see [9, Lemma 384]. Note that \( u \lor d_1, v \lor d_1 \in [d_1, c_{k+1}] \cap \text{mConv}(C' \cup D') \) with \( u = (u \lor d_1) \land c_k \) and \( v = (v \lor d_1) \land c_k \). Suppose
that $\text{mConv}(C' \cup D')$ is join-closed. Then $(d_1 \lor u) \lor (d_1 \lor v) = (d_1 \lor (u \lor v))$ exists in $[d_1, c_k] \cap \text{mConv}(C' \cup D')$, and necessarily covers $u \lor v$. Thus $((d_1 \lor u) \lor (d_1 \lor v)) \wedge c_k = u \lor v \in \Delta$. Suppose in addition that $(u, v)$ is a modular pair. Then $(u \lor d_1, v \lor d_1)$ is also modular. From $u \lor d_1, v \lor d_1 \in \text{mConv}(C' \cup D')$, it holds $(u \lor d_1) \wedge (v \lor d_1)$ belongs to $[d_1, c_{k-1}] \cap \text{mConv}(C' \cup D')$, and covers $u \lor v$. This means that $((u \lor d_1) \lor (v \lor d_1), c_k)$ is a modular pair with $(u \lor d_1) \wedge (v \lor d_1) \land c_k = u \lor v$, implying $u \lor v \in \Delta$. \hfill \Box

By using this lemma, we obtain a simple recursive algorithm to construct $\text{mConv}(C \cup D)$. This recursive algorithm can be naturally described by using the notion of the Jordan-Hölder permutation. For $i \in [n]$, let $\sigma(i) = \sigma_{C,D}(i)$ denote the smallest index $j$ such that $d_i \leq d_{i-1} \lor c_j$. The map $i \mapsto \sigma(i)$ is in fact a bijection on $[n]$, and is called the Jordan-Hölder permutation with respect to $C, D$ \cite{12, 13}. Observe that the above index $k$ is nothing but $\sigma(1) - 1$.

For $i = 1, 2, \ldots, n$, let $z_i, z'_i$ be elements defined by

$$z'_i := c_{\sigma(i)} \lor d_i (= c_{\sigma(i)} \lor d_{i-1}), \quad z_i := c_{\sigma(i)-1} \lor d_{i-1}. \quad (3.3)$$

Now $z_i$ is the maximal element of $C \lor d_{i-1}$ not greater than $d_i$. Therefore, the argument of the above lemma is applicable to two flags $C \lor d_{i-1}, D \lor d_{i-1}$ in $[d_{i-1}, 1]$, and obtains $\text{mConv}((C \lor d_{i-1}) \cup (D \lor d_{i-1}))$ from $\text{mConv}((C \lor d_i) \cup (D \lor d_i))$. Define $M_i \subseteq L (i = n, n - 1, \ldots, 1)$ by

$$M_n := \{d_{n-1}, d_n\}, \quad (3.4)$$

$$M_i := M_{i+1} \cup \{q \land z_i | q \in [d_i, z'_i] \cap M_{i+1} : (q, z_i) \text{ is a modular pair}\}. \quad (3.5)$$

**Corollary 3.3.** $\text{mConv}(C \cup D)$ is equal to $M_1$ and is join-closed.

Define $\varphi : \text{mConv}(C \cup D) \rightarrow 2^n$ by

$$\varphi(u) := \{i \in [n] | z_i \geq u\}. \quad (3.6)$$

**Lemma 3.4.** The map $\varphi$ is an embedding from $\text{mConv}(C \cup D)$ to $2^n$ such that for $u, v \in \text{mConv}(C \cup D)$,

1. $r(u) = n - |\varphi(u)|$, and
2. $\varphi(u \lor v) = \varphi(u) \cap \varphi(v)$.

**Proof.** Let $u \in \text{mConv}(C \cup D)$. Since $M_1 = \text{mConv}(C \cup D)$, $u$ is the meet of a subset of $\{z_i\}$. Therefore $u = \bigwedge_{i \in \varphi(u)} z_i$, and $\varphi$ is injective. Suppose that $u \in M_i \setminus M_{i+1}$. Then $u = q \land z_i$ for some $q \in [d_i, z'_i] \cap M_{i+1}$. It is clear that $r(u) = r(q) - 1$ and $\varphi(u) = \varphi(q) \cup \{z_i\}$ with $z_i \not\in \varphi(q)$. By induction on $i$ from $n$, we have (1), while (2) is obvious from the definition of $\varphi$. \hfill \Box

**Proof of Theorem 3.2 (1).** Let $\bar{\varphi} : \text{mConv}(C \cup D) \rightarrow 2^n$ by $u \mapsto [n] \setminus \varphi(u)$. Then $\text{mConv}(C \cup D)$ is isomorphic to the image of $\bar{\varphi}$, which is union-closed (A2) and contains $\emptyset, [n]$ (A1). By Lemma 3.3 (1), the image of $C$ (or $D$) is a flag required by (A3'). Therefore, the injective image is a pre-antimatroid. \hfill \Box
Next we show Theorem 3.1 (2). We note that the flag-distance \( \text{dist}(C, D) \) is computed from Jordan-Hölder permutation \( \sigma_{C,D} \). For a permutation \( \sigma \) on \([n]\), the inversion number of \( \sigma \) is defined as the number of pairs \( i, j \in [n] \) with \( i < j \) and \( \sigma(i) > \sigma(j) \).

**Theorem 3.5** ([1]; see [10]). For flags \( C, D \) in a semimodular lattice, the flag-distance \( \text{dist}(C, D) \) is equal to the inversion number of \( \sigma_{C,D} \).

**Sketch of Proof.** It is clear that the inversion number gives an upper bound of the flagdistance. In the setting of the proof of Lemma 3.2, the inversion number of a gallery of length \( k \) which consists of \( k \) flags

\[
0 = c_0 < c_1 < \cdots < c_i < c_i \vee d_1 < \cdots < c_k \vee d_1 = c_{k+1} < \cdots < c_n = 1
\]

for \( i = 0, 1, \ldots, k \).

**Proof of Theorem 3.1 (2).** We first show that any flag in any shortest gallery belongs to \( \text{mConv}(C \cup D) \). Choose any flag \( D' \) in a shortest gallery such that it is adjacent to \( D \). It suffices to show that \( D' \subseteq \text{mConv}(C \cup D) \). Here \( D' \) is given by

\[
0 = d_0 < d_1 < \cdots < d_i < d_i < d_{i+1} < \cdots < d_n = 1
\]

for \( d_i \neq d_i \). By Theorem 3.5 it holds \( j := \sigma_{C,D}(i) > \sigma_{C,D}(i + 1) =: j' \), and \( j = \sigma_{C,D'}(i + 1) \). Then \( d_{i+1} \leq d_i \vee c_j' \leq d_i \vee c_j \), and \( d_i \vee c_j = d_{i+1} \vee c_j \). Both \( d_{i-1} \vee c_j \) and \( d_i \vee c_j \) are covered by \( d_i \vee c_j = d_i \vee c_j \). Necessarily, they are equal. Now \( d_{i+1} \) and \( d_{i-1} \vee c_j \) form a modular pair such that their meet is \( d_i \). Also \( \text{mConv}(C, D) \) contains \( d_i \vee c_j \) since it is join-closed (Corollary 3.3). This concludes \( d_i \in \text{mConv}(C, D) \).

Next we show the conversion direction. Regard \( \text{mConv}(C \cup D) \) as a pre-antimatroid on \([n]\). Then the cardinality \( |X| \) for \( X \in \text{mConv}(C \cup D) \) equals the rank \( r(X) \) in \( L \). By this fact and the join-closedness, if \( X, Y \in \text{mConv}(C \cup D) \) are modular in \( L \), then \( X \cup Y, X \cap Y \in \text{mConv}(C \cup D) \). Therefore, \( \text{mConv}(C \cup D) \) is a subset of the sublattice generated by \( C, D \) in the Boolean lattice \( 2^n \). Abels [2] Proposition 1.8 showed that a flag in this sublattice is precisely a flag in a shortest gallery between \( C \) and \( D \) in \( 2^n \). On the other hand, the antimatroid \( A := (\text{mConv}(C \cup D))^\ast \subseteq 2^n \) is a semimodular lattice with respect to the inclusion order, where the join and the rank are the same as in \( L \) and as in \( 2^n \). Consequently, the Jordan-Hölder permutation \( \sigma_{C,D} \) in \( A \) is the same as in \( L \) and as in \( 2^n \). Together with Theorem 3.5 we conclude that any flag in \( \text{mConv}(C \cup D) \) belongs to a shortest gallery in \( L \). \[ \square \]

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