A Language Hierarchy of Binary Relations

Tara Brough1,3,*, Alan Cain2,3

Centro de Matemática e Aplicações, Faculdade de Ciências e Tecnologia, Universidade Nova de Lisboa, 2829–516 Caparica, Portugal

Abstract

Motivated by the study of word problems of monoids, we explore two ways of viewing binary relations on $X^*$ as languages. We exhibit a hierarchy of classes of binary relations on $X^*$, according to the class of languages the relation belongs to and the chosen viewpoint. We give examples of word problems of monoids distinguishing the various classes. Aside from the algebraic interest, these examples demonstrate that the hierarchy still holds when restricted to equivalence relations.

Keywords: binary relations on words, word problems, transducer, one-counter automaton, context-free grammar, ET0L-system, indexed grammar, linear indexed grammar

1. Introduction

In several applications of language theory in algebra and combinatorics, the issue arises of representing a relation in a way that is recognizable by an automaton or that can be defined by a grammar. For instance, automatic structures, defined for groups by Epstein et al. [1] and generalized to semigroups by Campbell et al. [2], are a way of defining a group or semigroup using binary relations, recognizable by a synchronous two-tape finite automaton that describe how generators for the group multiply normal-form words; in such groups and semigroups fundamental questions like the word problem are solvable. Automatic presentations for relational structures [3, 4] similarly use synchronous multi-tape finite automata that recognize, in terms of some regular language of representatives, the relations in the signature of a structure.

Possibly the most well-known type of relation in the application of language theory to algebra is the word problem, which is a binary relation that relates pairs of words over a generating set for a semigroup that represent the same
element of that semigroup. In the literature, this binary relation has been studied in the context of language theory from two perspectives. In one viewpoint, which we call the two-tape viewpoint, an element \((u, v)\) of the binary relation is thought of as being read (synchronously or asynchronously) by a two-tape automaton [5, 6, 7]. In the other, which we call the unfolded viewpoint, the element \((u, v)\) is represented by a word \(u \# v^{rev}\), where \('#'\) is a new symbol and \(v^{rev}\) denotes the reverse of \(v\) [5, 8, 9]. The latter viewpoint, which can also be thought of in terms of reading the first tape forwards and then the second tape in reverse, is a very natural representation for a binary relation when a stack is involved. In particular, hyperbolic groups in the sense of Gromov [10] can be characterized using context-free languages [11] in a way that closely resembles this, and it is this linguistic characterization that has given rise to the theory of word-hyperbolic semigroups [12, 13], since the geometric definition of hyperbolicity is less natural for semigroups. This in turn led to the study of semigroups with context-free word problem [14].

These considerations motivate the present paper, which compares and contrasts the binary relations that can be defined in the two-tape and unfolded perspectives for a number of language classes: rational \((R)\), one-counter \((O)\), context-free \((CF)\), ETO\(L\) \((ET\(O\)L), EDT\(O\)\(L\) \((EDT\(O\)L), linear indexed \((LIN)\), and indexed \((I)\). These classes were chosen because their applications to semigroups or groups have previously been studied; see [7, 8, 14, 5, 15, 16, 17, 18, 19].

What emerges is the language hierarchy illustrated in Figure 1. There is the expected straightforward containment of classes of binary relations representable in the two-tape perspective, following the containment of language classes, and similarly for those representable in the unfolded viewpoint. The two perspectives coincide for context-sensitive languages \((CSL)\). For each other language class, the class of unfolded binary relations is contained in the corresponding class of two-tape binary relations, with the containments being proven to be proper in each case except for \(I\), where it remains an open question. Some of the two-
tape classes are contained in unfolded classes corresponding to larger language classes. For instance, the two-tape \( CF \) relations are contained in the unfolded \( L \) relations (but not the unfolded \( ET \) relations). In other cases, we have proven incomparability of some classes of relations.

For several of these classes (inside the grey outline in Figure 1), the subclasses of binary relations between words over a 1-symbol alphabet coincide. It remains open whether the analogous subclasses of other classes coincide like this. Note that all the witnesses we give for proper containment and incomparability are over alphabets of at most 2 symbols, so there is no question of coincidence of subclasses of relations over 2-symbol alphabets.

2. Preliminaries

Throughout the paper, \( X \) will denote a finite alphabet and \( \varepsilon \) the empty word.

There are two ways to describe a binary relation \( \rho \) on \( X^* \) (that is, a subset \( \rho \) of \( X^* \times X^* \)) using languages:

- one can specify a sublanguage \( L_\rho \) of \( X^* # X^* \) (where \( # \) is a symbol not in \( A \)) such that \( L_\rho = \{ u#v^{rev} : u, v \in A^*, u \rho v \} \);

- one can specify a language \( L \) over the alphabet \( X_2^2 := (X \cup \{\varepsilon\}) \times (X \cup \{\varepsilon\}) \) such that \( \rho = L\pi \), where the map \( \pi \) is defined by

\[
(x_1, y_1) \cdots (x_k, y_k) \mapsto (x_1 \cdots x_k, y_1 \cdots y_k) \quad \text{for} \ x_i, y_i \in X \cup \{\varepsilon\}.
\]

(Note that there may be many choices for the language \( L \), and multiple words within \( L \) mapping onto a given element of \( \rho \).)

For a class of languages \( \mathcal{C} \), let

\[
T(\mathcal{C}) = \{ \rho : \rho = L\pi \text{ for some } L \in \mathcal{C} \}
\]

and

\[
U(\mathcal{C}) = \{ \rho : L_\rho \in \mathcal{C} \}.
\]

Mnemonically, \( T(\cdot) \) signifies ‘(two-)tape relation’; \( U(\cdot) \) signifies ‘unfolded relation’. In the remainder of the paper, we will often omit mention of the map \( \pi \) and simply think of grammars over \( X_2^2 \) as defining binary relations.

**Example 1.** Let \( |X| \geq 2 \) and let \( \rho = \{(w, w^{rev}) : w \in X^*\} \) be the relation that relates each word to its reverse. Then \( \rho \) is in \( T(CF) \) but not in \( U(CF) \).

**Proof.** The following context-free grammar defines a language \( L \) over \( X_2^2 \) with \( L\pi = \rho \).

\[
S \rightarrow (x, \varepsilon)S(\varepsilon, x) \mid \varepsilon \quad (\forall x \in X).
\]

However, \( L_\rho = \{ w#w : w \in X^*\} \), which is well known not to be context-free (this can be proved by the pumping lemma). \( \square \)

It is clear that if \( \mathcal{C} \) and \( \mathcal{D} \) are classes of languages and \( \mathcal{C} \subseteq \mathcal{D} \), then \( U(\mathcal{C}) \subseteq U(\mathcal{D}) \) and \( T(\mathcal{C}) \subseteq T(\mathcal{D}) \). The following result implies that proper containment is also inherited from language classes by classes of two-tape and
unfolded relations. The technical condition that language classes considered are closed under left quotient and left concatenation by a single symbol is a very weak property that is satisfied by all language classes considered in this paper and many others.

**Proposition 2.** Let \( \mathcal{C} \) and \( \mathcal{D} \) be classes of languages closed under homomorphism and under left quotient and left concatenation by a single symbol. If \( \mathcal{C} \setminus \mathcal{D} \) is non-empty, then the following classes of relations are all non-empty: \( \mathcal{C} \setminus \mathcal{D} \), \( \mathcal{D} \setminus \mathcal{D} \), \( \mathcal{C} \setminus \mathcal{D} \) and \( \mathcal{D} \setminus \mathcal{D} \).

**Proof.** For any language \( K \subseteq X^* \), let \( \rho_K = \{ (\varepsilon, w) : w \in K \} \). Then \( L_{\rho_K} = \{ \#w : w \in K \} \) is in \( \mathcal{C} \) (and hence \( \rho_K \in U(\mathcal{C}) \)) if and only if \( K \in \mathcal{C} \). If \( \rho \in T(\mathcal{C}) \) then the projections of \( \rho \) onto each component are languages in \( \mathcal{C} \), so we also have \( \rho_K \in T(\mathcal{C}) \) if and only if \( K \in \mathcal{C} \). Hence if \( K \in \mathcal{C} \setminus \mathcal{D} \), then \( \rho_K \) is in \( U(\mathcal{C}) \setminus U(\mathcal{D}) \), \( T(\mathcal{C}) \setminus T(\mathcal{D}) \), \( U(\mathcal{C}) \setminus U(\mathcal{D}) \) and \( T(\mathcal{C}) \setminus T(\mathcal{D}) \). \( \square \)

Since even a linear bounded automaton is powerful enough that there is no effective difference between the input \( (u, v) \) and the input \( u \# v^{rev} \), the following proposition is straightforward.

**Proposition 3.** If \( \mathcal{C} \) is the class of context-sensitive, recursive or recursively enumerable languages, we have \( U(\mathcal{C}) = T(\mathcal{C}) \).

Note that there are two definitions of linear indexed grammars in the literature. The one we have in mind and denote by \( \mathcal{LIN} \) is that in which, at each derivations step, flags are only copied to one of the (possibly multiple) non-terminals on the right-hand side of each production. The linear indexed grammars in this sense are equivalent to tree-adjoining grammars \( [20, \text{p.72}] \). The second definition (see for example \( [21] \)) requires there to be at most one non-terminal on the right hand side of any production. All of our results stated for \( \mathcal{LIN} \) in fact hold equally for the second definition.

### 3. Some further examples

Before moving on to establishing the hierarchy illustrated in Figure 1, we give several examples of binary relations and the language classes they belong to. While these examples do illustrate various aspects of the hierarchy, the intention in this section is not to prove these points but to explore some of the richness of binary relations from a language-theory viewpoint. The hierarchy will be systematically established in sections 4 and 5 with an attempt to give simple examples wherever possible.

**Example 4.** Any function \( f : N_0 \to N_0 \) can be viewed as a binary relation over an alphabet \( |X| = \{ x \} \), by defining \( \rho_f = \{ (x^n, x^{f(n)}) : n \in N_0 \} \). Consider the following functions:

- For \( e : N_0 \to N_0 \) given by \( n \mapsto n \), we have \( \rho_e \in T(\mathcal{R}) \setminus U(\mathcal{R}) \).
- For \( f : N_0 \to N_0 \) given by \( n \mapsto n \mod p \) (some \( p \in N \)), we have \( \rho_f \in U(\mathcal{R}) \).
- For \( g : N_0 \to N_0 \) given by \( n \mapsto n^2 \), we have \( \rho_g \in U(\mathcal{ETOL}) \setminus T(\mathcal{LIN}) \).
- For \( h : N \to N \) given by \( n \mapsto n^n \), we have \( \rho_h \in U(\mathcal{CSL}) \setminus T(\mathcal{I}) \).
Proof. The relation of the identity function \( e \) is \( \rho_e = \{ (x^n, x^n) : n \in \mathbb{N} \} = (x, x)^* \), which is rational. However, \( L_{\rho_e} = \{ x^n \# x^n : n \in \mathbb{N} \} \), which is easily seen by the pumping lemma to be non-rational.

We have \( L_{\rho_p} = x^n \# x^n \mod p \). This is recognised by a finite automaton with states \( \{q_0, \ldots, q_{p-1}, r_0, \ldots, r_{p-1}\} \), with \( q_0 \) the initial state, \( r_0 \) the unique final state, and transitions

\[
q_i \xrightarrow{x} q_{i+1 \mod p}, \quad q_i \xrightarrow{\#} r_i, \quad r_j \xrightarrow{x} r_{j-1},
\]

for \( 0 \leq i \leq p-1 \) and \( 1 \leq j \leq p-1 \).

An ET0L-system for the language \( L_{\rho_p} \) can be obtained by considering the expression of \( n^2 \) as the sum of the first \( n \) odd numbers. The system has axiom \( A \neq B \) and tables

- Table 1: \( A \rightarrow xA, \quad B \rightarrow xBC, \quad C \rightarrow x^2C; \)
- Table 2: \( A, B, C \rightarrow \varepsilon. \)

At each iteration of Table 1, the number of occurrences of \( A \) and \( B \) remains at one each, while the number of occurrences of \( C \) is increased by 1. One \( x \) is generated to the left of \( \# \) by \( A \), while \( 2e + 1 \) symbols \( x \) are generated to the right of \( \# \), where \( e \) is the number of occurrences of \( C \) in the current sentential form. Hence after \( n \) iterations of Table 1 and one application of Table 2, we obtain \( x^n \# x^{s(n)} \), where \( s(n) = \sum_{i=0}^{n-1} 2i + 1 = n^2 \). Thus \( \rho_g \in U(\mathcal{ET}0\mathcal{L}) \). The Parikh image of a linear indexed language is a semilinear set [21, Theorem 5.1], hence no language \( L \) with \( L\pi = \rho_g \) can be linear indexed, so \( \rho_g \notin U(\mathcal{ET}0\mathcal{L}). \)

Suppose \( \rho_h \in T(I) \); then the language \( \{ x^n : n \in \mathbb{N} \} \) (obtained by projecting onto the second tape) is indexed. By Gilman’s shrinking lemma [22] (with \( m = 1 \)), there exists \( k \) such that if \( n^m > k \) then there exist \( r \in \{2, \ldots, k\} \) and \( l_i \in \mathbb{N}_0 \) such that we can write \( n^m = \sum_{i=1}^{r} l_i \), and for any \( j \in \{1, \ldots, r\} \) there is a proper subset \( I \) of \( \{1, \ldots, r\} \) with \( i \in I \) and \( s_j := \sum_{i \in I} l_i = t^r \) for some \( t \in \mathbb{N} \). We may choose \( j \) such that \( l_j \geq n^m/k \), since \( r < k \). But if \( n > k \), we then have \( l_j \geq n^{m-1} - (n - 1)^{(n-1)} \), so that it is not possible for the sum \( s_j \) to be of the form \( t^r \) for \( t \in \mathbb{N} \). Hence \( \rho \notin T(I) \). However, \( L_{\rho_h} \) is recognised by a linear bounded automaton that writes input of the form \( x^n \# x^m \) onto the tape (rejecting input not of this form) and then checks whether \( m = n^m \) by essentially performing successive multiplications by \( n \), using the initial \( x^n \) portion of the tape both to count to \( n \) for the multiplication, and to ensure it is performed \( n \) times. Hence \( \rho \in U(\mathcal{CS}L) \).

The method of proof for \( \rho_g \in U(\mathcal{ET}0\mathcal{L}) \) should be extensible to all functions \( n \rightarrow n^k \) for \( k \in \mathbb{N} \) (the authors have shown it for \( k = 3 \)), and we conjecture more strongly:

**Conjecture 5.** Let \( p : \mathbb{N} \rightarrow \mathbb{N} \) be a polynomial function. Then \( \rho_p \in U(\mathcal{ET}0\mathcal{L}). \)

We note that a fact in some sense ‘inverse’ to Conjecture 5 is known: If \( p : \mathbb{N} \rightarrow \mathbb{N} \) is a polynomial function, then there exists an D0L language (D0L is a subclass of ET0L) with growth function \( p \) [23].

**Example 6.** Let \( X = \{1, \ldots, n\} \) and let

\[
o = \{ (w, 1^{|w|}, \ldots, n^{|w|}) : w \in X^* \}.
\]
The relation $\rel$ is a function that sorts a word into order. The language complexity of $\rel$ appears to increase as $n$ increases: $\rel$ is in $T(CF)$ for $n \leq 2$, in $U(LIN)$ for $n \leq 3$, in $T(LIN)$ for $n \leq 4$, and in $U(\mathcal{I})$ for all $n$.

**Proof.** We need only provide grammars for the largest $n$ claimed in each case, since the relations for smaller $n$ are homomorphic images of those for larger $n$.

For $n = 2$, a context-free grammar for $\rel$ is given by

$$
S \rightarrow (1, 1)S \mid (2, \varepsilon)S(\varepsilon, 2) \mid \varepsilon.
$$

In indexed grammars we will be using the notation $A^f$ for a non-terminal with flag $f$. Thus in the following grammars the superscripts in $N$ denote flags rather than powers of the non-terminal.

For $n = 3$, a linear indexed grammar for $\rel$ is given by

$$
S \rightarrow 1S1 \mid 2S2 \mid 3S3 \mid \#T
$$

Thus $\rel \in U(LIN)$ for $n \leq 3$, since $LIN$ is closed under homomorphism.

For $n = 4$, a linear indexed grammar for $\rel$ is given by

$$
S \rightarrow (1, 1)S \mid (2, \varepsilon)S \mid (3, \varepsilon)S \mid (4, \varepsilon)S(\varepsilon, 4) \mid T
$$

Hence $\rel \in U(LIN)$ for $n \leq 4$.

An indexed grammar for $\rel$ is given by the following productions, where $x$ ranges over all elements of $X$:

$$
S \rightarrow S^x \mid T\#T_1T_2\ldots T_n \quad T^x \rightarrow xT
$$

$$
T_x^z \rightarrow T_x \quad T^y \rightarrow \varepsilon \quad (\forall y \in X \setminus \{x\})
$$

Hence $\rel \in U(\mathcal{I})$ for all $n$. □

**Proposition 7.** The relation $\rel$ in Example 6 is not in $T(CF)$ for $n > 2$.

**Proof.** Suppose $n \geq 3$ and let $L$ be the language obtained by intersecting $\rel$ with $(123)^* \times X^*$ and then projecting onto the second tape. If $\rel \in T(CF)$, then $L$ is context-free. But $L = \{1^k2^k3^k \mid k \in \mathbb{N}_0\}$, which is not context-free. □

**Conjecture 8.** The relation $\rel$ in Example 6 is not in $T(\mathcal{E}\mathcal{T}0\mathcal{L})$ or in $T(LIN)$ for sufficiently large $n$.

**Proposition 9.** The cyclic permutation relation $\kappa = \{(uv, vu) : u, v \in X^*\}$ is in $U(\mathcal{E}\mathcal{T}0\mathcal{L})$ and in $U(LIN)$, but not in $T(CF)$ for $|X| \geq 2$. 

6
Symmetrically, if \( q \subseteq P \) then \( v \subseteq w \).

We proceed by case analysis, depending on the size of \( w \).

- **Case 1** \( |w| > k \). To maintain membership of \( L \) upon pumping, the strings in \( P \) must be in the same set \( A \) or \( Z \). They also must come from different tapes (so have different subscripts). Suppose first that \( P = \{ q_1, q_2 \} \). If \( q_1, q_2 \in A \), then \( v_2 \) is a substring of \( p \), while \( v_1 \) is a substring of \( rs \). Thus \( e_u \) is empty. Symmetrically, if \( q_1, q_2 \in Z \), then \( e_v \) is empty; and if instead \( P = \{ s_1, s_2 \} \), then \( e_u = \varepsilon \) if \( P \subseteq Z \), and \( e_v = \varepsilon \) if \( P \subseteq A \).
Now suppose \( P = \{q_1, s_2\} \subseteq Z \). Then \( u_1 \) and \( u_2 \) are contained in \( t \) and \( p \) respectively, so \( e_u = \varepsilon \). Symmetrically, if \( P = \{q_2, s_1\} \subseteq A \) then \( e_v = \varepsilon \).

However, if \( P = \{q_1, s_2\} \subseteq A \), we do not obtain separation. In this case \( v_1 \) is contained in \( rst \), while \( v_2 \) is contained in \( pqr \). Thus these strings overlap only in \( r \). Thus \( |e_u| \leq |r| < k \). Symmetrically, in the final subcase \( P = \{q_2, s_1\} \subseteq Z \), we have \( |e_u| < k \).

Case 2 \(|P| = 3\). In this case, all the strings in \( P \) must still be in the same set \( A \) or \( Z \), since on one of the tapes only one string (a power of a single letter) is being pumped. Moreover, \( P \) must always contain one of the pairs \( \{q_1, q_2\} \) or \( \{s_1, s_2\} \), so that this is actually a more restrictive situation than Case 1 and always leads to either \( e_u \) or \( e_v \) being empty.

Case 3 \(|P| = 4\). If all elements of \( P \) are in the same set \( A \) or \( Z \), then this is even more restricted than Case 2 and we have \( e_u \) or \( e_v \) empty. However, it is possible for \( P \) to contain strings from both \( A \) and \( Z \). The only situation in which this occurs is \( q_1, s_2 \in A \) and \( q_2, s_1 \in Z \). In this case, \( u_1 \) and \( u_2 \) are both contained in \( pqr \), while \( u_2 \) and \( v_1 \) are both contained in \( rst \). Hence \( e_u \) and \( e_v \) are both substrings of \( r \) and have length less than \( k \).

Thus we have established that in all cases either \(|e_u| < k\) or \(|e_v| < k\). The argument was valid for all \( n > k \), and so at least one of the following holds: for infinitely many \( n \), we have \(|e_u| > k\); or for infinitely many \( n \), we have \(|e_v| > k\).

Suppose first that \(|e_u| > k\) for infinitely many \( n \). Let \( \theta \) be the homomorphism given by deleting all occurrences of symbols from \( \{c, x, y, z\} \) on either tape, and let \( L' = L \theta \). Then \( L' \) contains \( \alpha = (a, \varepsilon)^n(b, \varepsilon)^{(c, a)}(\varepsilon, b)^n \) for arbitrarily large \( n \), since any ‘intermingling’ between symbols from \( A \times \{\varepsilon\} \) and \( \{c\} \times A \) in words with \(|e_u| < k\) occurs only among the symbols \((\varepsilon, c)\) and \((a, \varepsilon)\). Taking \( n \) greater than the pumping constant of \( L' \) (which, as a homomorphic image of \( L \), is context-free), there exist two substrings of \( w_n \) that can be simultaneously pumped, and these substrings must be contained within one of the subwords \((a, \varepsilon)^n(b, \varepsilon)^n\), \((b, \varepsilon)^n(\varepsilon, a)^n\) or \((\varepsilon, a)^n(\varepsilon, b)^n\). But this implies that \( L' \) contains words that cannot be the image under \( \theta \) of words in \( L \), contrary to the definition of \( L' \). Similarly, taking if \(|e_v| > k\) for infinitely many \( n \), then taking a homomorphism \( \theta' \) that deletes all occurrences of \( a, b, c \) and \( x \) again results in a non-context-free language. Thus \( L \) itself cannot have been context-free and so \( \kappa \notin T(\mathcal{C}) \).

In general, for \(|X| \geq 2\) we can modify the above proof for \(|X| \geq 6\) as follows. Let \( 0, 1 \in X \) and define \( a = 1, b = 10, c = 100, x = 1000, y = 10000 \) and \( z = 100000 \). The entire argument given above holds, with the modification that we must replace the homomorphisms \( \theta \) and \( \theta' \) by rational transductions in order to delete \( c, x, y, z \) (resp. \( a, b, c, x \)). \( \square \)

4. Comparing \( U(\mathcal{C}) \) and \( T(\mathcal{C}) \)

Although the following result is a special case of Proposition 12 we give its proof separately as a useful ‘warm-up’ for the proof of the later proposition.

**Proposition 10.** \( U(\mathcal{R}) \) is properly contained in \( T(\mathcal{R}) \).
Proposition 11. \( U(O) \) is properly contained in \( T(O) \).

Proof. Let \( \rho \) be a binary relation on \( X^* \) that lies in \( U(O) \). Consider a one-counter automaton \( A \) that recognizes \( L_\rho \). View \( A \) as having an integer counter that starts at 0 and which it increments or decrements as it reads each symbol of a word \( u \#^rev \), accepting if the counter has value 0 at the end of the input.

Build a one-counter automaton \( B \) over \( X_2^\# \) recognizing \( \{ (u, \varepsilon)(\varepsilon, v) : (u, v) \in \rho \} \) that functions as follows. It first reads all input from its first tape, simulating \( A \) on this input followed by \# . It then reads the input from its second tape, nondeterministically simulates \( A \) in reverse on this tape (so that increments to the counter become decrements, and vice versa). Since increments and decrements to the counter commute, it is clear that \( B \) accepts \((u, v)\) if and only if \( A \) accepts \( u \#^rev \).

An example of a relation in \( T(O) \) but not in \( U(O) \) (since it is not in \( U(CF) \)) is given in Proposition 15.

Proposition 12. Let \( \mathcal{C} \) be the class of indexed, linear indexed, \( ET0L \), \( EDT0L \), context-free or rational languages. Then \( U(\mathcal{C}) \subseteq T(\mathcal{C}) \).

Proof. Let \( \rho \) be a binary relation on \( X^* \) that lies in \( U(I) \) and let \( \Gamma \) be an indexed grammar for \( L_\rho \). We may assume that the set of non-terminals \( N \) is partitioned into sets \( N_L, N_\#, N_R \) such that all productions are of one of the types in the table below, where we adopt the convention that a nonterminal with a subscript \( H \in \{ L, \#, R \} \) is in \( N_H \), and the convention that a Greek letter with a subscript \( H \) denotes a word over \( N_H \) (potentially with flags).

We construct a new indexed grammar \( \Gamma' \) with non-terminals \( N' = \{ A' \mid A \in N \} \) and start symbol \( S'_\# \) (where \( S \) is the start symbol of \( \Gamma \)) in which each production is replaced by the corresponding production shown below, and where \( \alpha' \) denotes the word \( \alpha \) with each non-terminal \( A \) replaced by \( A' \), and each terminal \( a \) replaced by \( (a, \varepsilon) \) if \( \alpha \in N_L^I \) and by \( (\varepsilon, a) \) if \( \alpha \in N_R^I \). A flag \( f \) in brackets denotes that the flag may or may not be present in the production.
Let $\rho$.

By Proposition 12, it suffices to prove that $L = L_1 \cup L_2$.

Proof. The containment of linear indexed, EDT0L-indexed, context-free or rational grammar. Hence the containment of $U(\mathcal{C}) \subseteq T(\mathcal{C})$ also holds for all the classes mentioned.

Note that the transformation from $\Gamma$ to $\Gamma'$ preserves the property of being a linear indexed, EDT0L-indexed, context-free or rational grammar. Hence the containment of $U(\mathcal{C}) \subseteq T(\mathcal{C})$ also holds for all the classes mentioned.

$\square$

Proposition 13. The containment of $U(\mathcal{C})$ in $T(\mathcal{C})$ is proper.

Proof. See Example 1.

$\square$

Proposition 14. The containment of $U(\mathcal{L}_{\mathcal{L}})$ in $T(\mathcal{L}_{\mathcal{L}})$ is proper.

Proof. Let $\rho = \{ (a_1^0 a_2^0 a_3^0 b_1^0 b_2^0 b_3^0) : n \in \mathbb{N}_0 \}$. Then $\rho$ is obtained from the language $L_3 = \{ a_1^0 a_2^0 a_3^0 : n \in \mathbb{N}_0 \}$ by replacing each $a_i$ by $(a_i, b_i)$. Since $L_3$ is linear indexed by [40, Problem 4.1] (recalling that the classes of languages defined by linear indexed and tree-adjoining grammars coincide), $\rho$ lies in $T(L)\bigcup\mathcal{L}_{\mathcal{L}}$. But $L_\rho = \{ a_1^0 a_2^0 a_3^0 b_1^0 b_2^0 b_3^0 : n \in \mathbb{N}_0 \}$ has as a homomorphic image the language $L_5 = \{ a^n b^n c^n d^n e^n : n \in \mathbb{N}_0 \}$, which is not linear indexed [40, Problem 4.5]. Hence $\rho \notin U(\mathcal{L}_{\mathcal{L}})$.

$\square$

Proposition 15. The containment of $U(\mathcal{E}_{\mathcal{T}}\mathcal{L})$ in $T(\mathcal{E}_{\mathcal{T}}\mathcal{L})$ is proper, and moreover there exists a relation in $T(\mathcal{C}) \setminus U(\mathcal{E}_{\mathcal{T}}\mathcal{L})$.

Proof. By Proposition 14, $U(\mathcal{E}_{\mathcal{T}}\mathcal{L}) \subseteq T(\mathcal{E}_{\mathcal{T}}\mathcal{L})$. Let $K \subseteq X^*$ be a context-free language that is not EDT0L (an example on a 2-letter alphabet exists [41]). The language $\{ w#w^\mathcal{C} : w \in K \}$, which is $L_\sigma$ for the relation $\sigma = \{ (w, w) : w \in K \}$, is not EDT0L. However, a context-free grammar for $\sigma$ can be obtained by replacing every output symbol $x$ in a context-free grammar for $K$ by $(x, x)$.

$\square$

Proposition 16. $U(\mathcal{E}_{\mathcal{T}}\mathcal{L}) = T(\mathcal{E}_{\mathcal{T}}\mathcal{L})$.

Proof. By Proposition 14 it suffices to prove that $T(\mathcal{E}_{\mathcal{T}}\mathcal{L}) \subseteq U(\mathcal{E}_{\mathcal{T}}\mathcal{L})$. Let $\rho$ be a binary relation on $X^*$ that lies in $T(\mathcal{E}_{\mathcal{T}}\mathcal{L})$ and let $H = (V, X_2^*, \Delta, I)$ be an EDT0L-system for $\rho$. Define an EDT0L-system $H' = (V_1 \cup V_2, X \cup \{\#\}, \Delta', I')$ with $V_1, V_2$ copies of $V$, $I' = I_1 \# I_2$ and $\Delta'$ consisting of a set
of tables in one-to-one correspondence with the tables in $\Delta$, with each production $A \to \alpha$, $\alpha \in (V \cup X^2)^*$ being replaced by two productions $A_1 \to \alpha_1$ and $A_2 \to \alpha_2^\text{rev}$, where $\alpha_i$ is the same as $\alpha$ except with each non-terminal in $V$ replaced by the corresponding non-terminal in $V_i$, and the terminal symbols replaced by their $i$-th component. Since each table in $\Delta$ contains exactly one production from $A$ for each $A \in V$, applying a given sequence of tables in $\Delta'$ to $I'$ produces the word $u\#v^{\text{rev}}$, where $(u, v)$ is the word produced by applying the corresponding sequence of tables in $\Delta$ to $I$. Note that $\Delta'$ also contains only one production from each non-terminal in $V_1 \cup V_2$. Hence $H'$ is an EDT0L-system for $L_{o'}$.

Conjecture 17. $U(I)$ is properly contained in $T(I)$.

Example 18. Let $|X| \geq 2$ and for $a \in X$ define a relation $\rho$ on $X^*$ by $u \rho v$ iff $u$ and $v$ have the same length and $|u|_a = |v|_a$. This relation is in $T(O)$, since an automaton simply reads pairs of symbols (thus ensuring that two accepted words have the same length) and tracks the difference in the number of symbols $a$ on its two input tapes. Thus, by Proposition 23 below,

$$L_{\rho} = \{ u\#v : |u| = |v|, |u|_a = |v|_a \}$$

is a linear indexed language. Since $\mathcal{LIN}$ is closed under homomorphisms, the relation $\sigma = \{ (uv, uv) : |u| = |v|, |u|_a = |v|_a \}$, obtained by applying the homomorphism defined by $x \mapsto (x, x)$ (for all $x \neq \#$) and $\# \mapsto \epsilon$, is also in $T(\mathcal{LIN})$ and so in $T(I)$. However, it is unlikely that $\sigma$ is in $U(I)$, since recognizing

$$L_{\sigma} = \{ uv\#v^{\text{rev}}u^{\text{rev}} : |u| = |v|, |u|_a = |v|_a \}$$

seems to require two blind counters operating independently, which in turn does not seem to be workable with the nested stack automaton model of indexed languages. (Two independent non-blind counters are of course equivalent to a Turing machine.) Thus $\sigma$ is a potential witness for the proper containment of $U(I)$ in $T(I)$.

5. Comparing $U(C)$ and $T(D)$ for $D$ a subclass of $C$

Throughout this section, we will make use of Proposition 4 – in particular the fact that if $C \setminus D \neq \emptyset$ then $U(C) \setminus T(D) \neq \emptyset$ – without further comment.

Proposition 19. $T(R)$ is a proper subclass of $U(CF)$.

Proof. Let $\rho$ be a binary relation on $X^*$ that lies in $T(R)$ and let $\Gamma = (N, X^2, P, S)$ be a left rational two-tape grammar for $\rho$. Define a context-free grammar $\Gamma' = (N', X, P', S')$ whose productions are derived from $P$ as follows (for $A, B \in N$ and $a \in X$):

| Production in $\Gamma$ | Corresponding Production in $\Gamma'$ |
|------------------------|--------------------------------------|
| $A \to (a, \varepsilon) B$ | $A' \to aB'$ |
| $A \to (\varepsilon, a) B$ | $A' \to B' a$ |
| $A \to (a, \varepsilon)$ | $A' \to a#$ |
| $A \to (\varepsilon, a)$ | $A' \to # a$ |
| $A \to \varepsilon$ | $A' \to \#$ |
It is easy to see that \( L(\Gamma') = \{ u \# v^{rev} : (u, v) \in \rho \} = L_\rho \).

An example of a relation that lies in \( U(\mathcal{O}) \setminus T(\mathcal{R}) \) and thus in \( U(\mathcal{CF}) \setminus T(\mathcal{R}) \) is given in Proposition 52.

**Proposition 20.** The classes \( T(\mathcal{R}) \) and \( U(\mathcal{O}) \) are incomparable.

**Proof.** This follows from Proposition 2 and Proposition 52 below.

**Proposition 21.** The classes \( T(\mathcal{O}) \) and \( U(\mathcal{CF}) \) are incomparable.

**Proof.** Let \( X = \{a, b\} \) and \( \rho = \{ (a^n b^n, b^n a^n) : n \in \mathbb{N} \} \). Then \( \rho \) is one-counter, but \( L_\rho = \{ a^n b^n \# a^n b^n : n \in \mathbb{N} \} \) is not context-free. An example of a relation in \( U(\mathcal{CF}) \setminus T(\mathcal{O}) \) is given in Proposition 53.

**Corollary 22.** The classes \( T(\mathcal{CF}) \) and \( U(\mathcal{ET0L}) \) are incomparable.

**Proof.** This follows from Propositions 2 and 15.

**Proposition 23.** \( T(\mathcal{CF}) \) is a proper subclass of \( U(\mathcal{LIN}) \).

**Proof.** Let \( \rho \) be a relation on \( X^* \) such that \( \rho = L_\pi \) for a context-free language \( L \). Let \( \Gamma = (N, X_2^*, P, S) \) be a context-free grammar in Chomsky normal form for \( L \). We construct a linear indexed grammar with non-terminals \( \{ I, I_A \mid A \in N \} \), start symbol \( I \), flags \( N \cup \{\$\} \) (written as superscripts on the non-terminals) and productions:

\[
\begin{align*}
I & \rightarrow I^3_S \\
I_A & \rightarrow aI & \text{for } A \rightarrow (a, \varepsilon) \text{ in } P \\
I_A & \rightarrow I_B & \text{for } A \rightarrow BC \text{ in } P \\
I_A & \rightarrow Ia & \text{for } A \rightarrow (\varepsilon, a) \text{ in } P \\
I^8 & \rightarrow \#.
\end{align*}
\]

The idea is that the grammar uses the flags to perform a leftmost derivation in \( \Gamma \), with the symbols from the first tape being output to the left and the symbols from the second tape to the right (so that they appear in reverse). Each sentential form \((w_1, w_2)B_1 \ldots B_k\) in a leftmost derivation in \( \Gamma \) corresponds to the sentential form \( w_1IB_1B_2 \ldots IB_k\#w_2^{rev} \). When the derivation is complete, we will have obtained the expression \( uI^8v^{rev} \) for some \((u, v) \in \rho \). Finally, the production \( I^8 \) places the symbol \# to give \( u \# v^{rev} \), as desired. Hence \( L_\rho \) is linear indexed and so \( T(\mathcal{CF}) \subseteq U(\mathcal{LIN}) \).

The incomparability of \( \mathcal{LIN} \) and \( \mathcal{ET0L} \) is a straightforward consequence of results in [21, 23, 20]; for details, see [15]. Proofs of Corollaries 4.2 and 4.3. Hence Proposition 2 implies that amongst the classes \( U(\mathcal{LIN}) \), \( U(\mathcal{ET0L}) \), \( T(\mathcal{LIN}) \) and \( T(\mathcal{ET0L}) \), the only containments are \( U(\mathcal{LIN}) \subseteq T(\mathcal{LIN}) \) and \( U(\mathcal{ET0L}) \subseteq T(\mathcal{ET0L}) \).

**Conjecture 24.** \( T(\mathcal{O}) \) and \( U(\mathcal{ET0L}) \) are incomparable.

Let \( \rho \) be the relation defined in Example 13; recall that \( \rho \) was proven to lie in \( T(\mathcal{O}) \). It appears unlikely that \( L_\rho \) is ET0L, since there seems to be no way of using ET0L tables to maintain both the same length of words on each side of \# and the same number of symbols \( a \), while allowing the occurrences of \( a \) to appear in any combination of positions.
Conjecture 25. \( T(\mathsf{LIN}) \) and \( U(\mathcal{I}) \) are incomparable.

The relation \( \sigma \) mentioned as a potential witness for Conjecture 17 would also serve as a witness that \( T(\mathsf{LIN}) \not\subseteq U(\mathcal{I}) \).

Proposition 26. \( T(\mathcal{I}) \) is properly contained in \( U(\mathsf{CSL}) = T(\mathsf{CSL}) \).

Proof. We have \( T(\mathcal{I}) \subseteq T(\mathsf{CSL}) \) by containment of language classes, while \( U(\mathcal{I}) \setminus T(\mathcal{I}) \) is non-empty by Proposition 2, and by Proposition 3, \( U(\mathsf{CSL}) = T(\mathsf{CSL}) \). \( \square \)

6. The one-symbol case

Let \( \mathcal{B} \) consist of all binary relations on sets of words over one-symbol alphabets. Let

\[
T_1(\mathcal{C}) = T(\mathcal{C}) \cap \mathcal{B} \quad \text{and} \quad U_1(\mathcal{C}) = U(\mathcal{C}) \cap \mathcal{B}.
\]

Proposition 27. \( T_1(\mathcal{R}) = T_1(\mathsf{LIN}) \).

Proof. Let \( \rho \in T_1(\mathsf{LIN}) \) and let \( \Gamma \) be a two-tape context-free grammar for \( \rho \). Without loss of generality, assume that the set of terminal symbols of \( \Gamma \) is \( A = \{ (a, \varepsilon), (\varepsilon, a) \} \). The Parikh image of the language over \( A \) generated by \( \Gamma \) is a semi-linear set \( S \subseteq \mathbb{N}_0 \times \mathbb{N}_0 \) \([21, \text{Theorem 5.1}]\). Therefore \( \rho = \{ (a^\alpha, a^\beta) : (\alpha, \beta) \in S \} \), and so \( \rho \) is a rational relation and hence \( \rho \in T_1(\mathcal{R}) \). \( \square \)

Proposition 28. \( T_1(\mathcal{R}) = U_1(\mathcal{O}) \)

Proof. Let \( \rho \in T_1(\mathcal{R}) \) and consider a transducer \( T \) that recognizes \( \rho \). Construct a one-counter automaton \( C \) that accepts a word \( u \# v^{\text{rev}} \) if and only if \( (u, v) \) is accepted by \( T \) as follows. The one-counter automaton \( C \) keeps in its state a simulated copy of the state of \( T \), beginning with its start state. At some point before it reaches \#, while its simulated state is \( q \), it nondeterministically selects a transition of \( T \) starting at \( q \). Suppose this transition has label \( (a^k, a^\ell) \) (for \( k, \ell \in \mathbb{N} \cup \{0\} \)) and leads to state \( p \). Then \( C \) reads \( k \) symbols \( a \) from its input, failing if it reads \#, and increments its counter by \( \ell \). When the simulated state is a final state of \( T \), the automaton \( C \) can read \#. After having read \#, the automaton \( C \) simply reads \( v \) symbol-by-symbol, decrementing its counter by 1 each time, accepting if the counter is 0 when the end of the input is reached. It is clear that \( u \# v^{\text{rev}} \) is accepted if and only if the numbers of symbols \( a \) making up \( u \) and \( v \) are the numbers of symbols \( a \) on the two sides of a path leading from a start to a final state of \( T \). \( \square \)

Thus we have established that all classes of relations inside the box in Figure 1 are equal when \( |X| = 1 \).

Corollary 29. If \( \mathcal{C} \) is any class of languages intermediate between \( \mathcal{R} \) and \( \mathsf{LIN} \) inclusive, then \( T_1(\mathcal{C}) = T_1(\mathcal{R}) \). If \( \mathcal{D} \) is any class of languages intermediate between \( \mathcal{O} \) and \( \mathsf{LIN} \) inclusive, then \( U_1(\mathcal{D}) = T_1(\mathcal{R}) \).

Proof. The first statement follows immediately from Proposition 27, while the second follows from Propositions 27, 28 and the \( \mathsf{LIN} \) case of Proposition 12. \( \square \)
However, this is as far as the equality extends (amongst language classes considered in this paper). On the one hand, the identity function \( e : \mathbb{N} \rightarrow \mathbb{N} \) in Example 4 shows that \( U_1(R) \) is properly contained in \( T_1(R) \). On the other hand, we have the following:

**Proposition 30.** \( T_1(R) \) is properly contained in \( U_1(\mathcal{ET}0\mathcal{L}) \).

**Proof.** By Proposition 19, \( T_1(R) \subseteq U_1(\mathcal{CF}) \); hence \( T_1(R) \subseteq U_1(\mathcal{ET}0\mathcal{L}) \).

Let \( K = \{ x^n : n \in \mathbb{N} \} \) and \( \rho_K = \{ (w, w) : w \in K \} \). Then \( L_{\rho_K} \) is generated by an ET0L-system with axiom \( S \# S \) and two tables consisting of the single productions \( S \rightarrow SS \) and \( S \rightarrow x \). But \( \rho \) does not lie in \( T_1(\mathcal{CF}) \) since the projections onto each tape would then also be context-free and \( K \) is not context-free. Hence \( U_1(\mathcal{ET}0\mathcal{L}) \neq T_1(R) \). \( \square \)

It remains open whether there are any equalities between \( U_1(\mathcal{ET}0\mathcal{L}), \mathcal{T}_1(\mathcal{ET}0\mathcal{L}), U_1(I) \) and \( T_1(I) \).

**7. Word problems of monoids**

For a monoid \( M \) with finite generating set \( X \), the **word problem relation** of \( M \) with respect to \( X \) is \( \iota(M, X) = \{ (u, v) : u, v \in X^*, u =_M v \} \). Note that this is by definition an equivalence relation. We say that \( M \) has word problem in \( T(\mathcal{C}) \) if \( \iota(M, X) \in T(\mathcal{C}) \), and that \( M \) has word problem in \( U(\mathcal{C}) \) if \( L_{\iota(M, X)} \in U(\mathcal{C}) \). Note that \( L_{\iota(M, X)} \) was the first language-theoretic version of monoid word problems to be studied, and is often denoted WP(\( M, X \)).

In this section we exhibit examples of monoid word problems distinguishing the relation classes under consideration. Besides demonstrating the algebraic relevance of the relation classes, these examples also establish that the hierarchy shown in Figure 1 still holds when we restrict our attention to equivalence relations. The separation of the various classes by word problems is summarized, using the notation of this section, in Figure 2. For reasons of space, proofs of more technical results are omitted from this section and given in an appendix.

Denote the free monoid, the free inverse monoid, and the free group of rank \( n \) by, respectively, \( FM_n \), \( FIM_n \), and \( FG_n \).
Proposition 31. The word problem of the free monoid of rank 1 is in $T_1(\mathcal{R})$ and in $U_1(\mathcal{O})$ but not in $U_1(\mathcal{R})$.

Proof. Let $F_1$ be generated by $x$ and let $\rho = \iota(F_1, \{x\})$. We have $\rho = \{ (x^n, x^n) : n \in \mathbb{N}_0 \}$, which is the relation of the identity function $\epsilon(n) = n$. We showed in Example 4 that this relation is in $T(\mathcal{R}) \setminus U(\mathcal{R})$. Moreover, $L_\rho = \{ x^n \# x^n : n \in \mathbb{N}_0 \}$ is one-counter.

Proposition 32. The free group of rank 1 has word problem in $U(\mathcal{O})$ but not in $T(\mathcal{R})$. The free monoid of rank greater than 1 has word problem in $T(\mathcal{R})$ but not in $U(\mathcal{O})$.

Proof. Let $F$ be the free group of rank 1, generated as a monoid by $X = \{x, x^{-1}\}$. Then $L_{\iota(F, X)} = \{ u \# v : u, v \in X^*, |u|_x - |u|_{x^{-1}} = |v|_x - |v|_{x^{-1}} \}$. This language is easily recognised by a one-counter automaton that uses the stack to calculate $|u|_x - |u|_{x^{-1}}$ and then checks this against $|v|_x - |v|_{x^{-1}}$, accepting by empty stack. We require four states to record whether we are currently reading $u$ or $v$ and whether we currently have an excess of $x$’s or of $x^{-1}$’s. The only groups $G$ with $\iota(G)$ rational are finite \cite{7} Theorem 8.7.9, so $\iota(F, X)$ is not rational.

Therefore, let $|X| \geq 2$ and let $\rho$ be the equality relation on $X^*$, which is the two-tape word problem of $X^*$. Then $\rho$ is rational, but $L_\rho = \{ w \# w^{-1} : w \in X^* \}$ is not one-counter by \cite{8} Proposition 4.1 since $X^*$ has exponential growth.

Proposition 33. The word problem of the free group of rank 2 is in $U(\mathcal{CF})$ but not in $T(\mathcal{O})$.

Proof. Let $F_2$ be the free group on $X = \{x, y\}$ and let $X^\pm = \{x, y, x^{-1}, y^{-1}\}$. The word problem of $F_2$ is well known to be context-free: $L_{\iota(F_2, X^\pm)}$ is accepted by a pushdown automaton that pushes each symbol read onto the stack, unless it is the inverse of the current top-of-stack symbol (in which case the stack is popped) or $\#$ (in which case the stack is unchanged). Let $W = \iota(F_2, X)$ and suppose $W$ is one-counter. Then the set $W_1$ of all pairs $(w, \epsilon)$ in $W$ is also one-counter (we can modify an automaton accepting $W$ to move to a failure state if any symbols are read from the second tape). But $W_1$ is equivalent to the group word problem of $F_2$ (the set of all words equal to the identity in $F_2$), and a group has one-counter word problem if and only if it is cyclic \cite{8}.

Proposition 34. \cite{8} Theorem 1 and Corollary 1] The free inverse monoid of rank 1 has word problem in $U(\mathcal{EFT}\mathcal{L})$ but not in $T(\mathcal{CF})$.

Proposition 35. Let $X = \{a, b, \ell, r\}$ and let $M_1$ be the monoid with presentation

$$(X | \ell a^n b^n r = \ell b^n a^n r \ (n \in \mathbb{N})).$$

Then the word problem of $M_1$ is in $T(\mathcal{O})$ but not in $U(\mathcal{CF})$.

Proof. Let $\rho = \iota(M_1, X)$, Define a one-counter automaton $A$ with initial and final state $q_0$ and further states $p_1, q_1, p_2, q_2$ as follows: In state $q_0$, $A$ reads $(x, x)$ for $x \in X$, or on input $(\ell, \ell)$, $A$ may move to state $p_1$ or $q_1$. In state $p_1$, $A$ reads $(a, b)$ and increments the counter, or moves to state $q_2$ on input $(b, a)$, decrementing the counter. In state $p_2$, $A$ reads $(b, a)$ and decrements
Proposition 37. The monoid \( M_\varphi X \) for start symbol \( I \) has \( w_\pi \) under transductions, then the sublanguage where \( Y \) on the other hand, \( (L, \ell, r) \) is in \( C \) and contains \( A \) corresponding positions may be interchanged. That is, \( A \) accepts \( \rho \), so \( \rho \in T(O) \).

Let \( \phi \) be the homomorphism on \( X \cup \{\#\} \) that maps \( \ell, r \) and \# to \( e \). Then

\[
L_1 = \phi (L_\rho \cap a^a b^b r^# r a^a \ell) = \{ a^a b^b a^a r : n \in \mathbb{N}_0 \}
\]

is not context-free, hence \( L_\rho \) is also not context-free, so \( \rho \notin U(CF) \). \( \square \)

Our remaining examples are based on two general construction techniques similar to that used in the previous proposition. For \( \rho \subseteq X^* \times X^* \), define a monoid \( M[\rho] = (X, \ell, r) \) for \( \rho \). For \( L \subseteq X^* \), define a monoid \( M(L) = (X, \ell, r) \) for \( \rho \) in \( U(CF) \).

Proposition 36. Let \( \mathfrak{C} \) and \( \mathfrak{D} \) be classes of languages.

1. If \( \rho \in T(\mathfrak{C}) \) and \( \mathfrak{C} \) is closed under union, concatenation and Kleene star and contains \( \mathfrak{R} \), then \( M[\rho] \) has word problem in \( T(\mathfrak{C}) \).

2. If \( \mathfrak{C} \in \{CF, ET0L, EDTO, \mathbb{L}N, T\} \) and \( L \in \mathfrak{C} \) and furthermore \( \mathfrak{D} \) is closed under transductions, then \( M(L) \) has word problem in \( U(\mathfrak{C}) \).

Proof. First, let \( \rho \subseteq X^* \times X^* \) be in \( T(\mathfrak{C}) \). Let \( K \subseteq (X^2)^* \) such that \( \rho = K\pi \), \( Y = X \cup \{\ell, r\} \), \( Y_2 = \{(y, y) : y \in Y\} \) and \( \sigma = \iota(M_\rho, Y) \). Then \( \sigma = K' \pi \) for \( K' = Y_2 \cup Y_2(\ell, r) \). Since \( Y_2(\ell, r) \) is rational, by the closure properties of \( \mathfrak{C} \) we have \( K' \in \mathfrak{C} \) and hence \( \sigma \in T(\mathfrak{C}) \).

Second, let \( L \subseteq X^* \) be in \( \mathfrak{C} \) and \( \mathfrak{D} \), with \( Y \) and \( Y_2 \) as before, and let \( \sigma_1 = \iota(M(L), Y) \). Suppose \( \sigma_1 \in T(\mathfrak{D}) \) with \( \sigma' = K_1 \pi \) for \( K_1 \subseteq D \). If \( \mathfrak{D} \) is closed under transductions, then the sublanguage \( K_1 \) of \( K_1 \pi \) consists of all \( \epsilon \) such that \( w\pi \) has \( \ell \pi \) on the second tape is also in \( \mathfrak{D} \). But then \( K_1 \pi = \{ (\ell, \ell) \} (w \pi : w \in L \} \) is in \( T(\mathfrak{D}) \), implying that \( \{ \ell, \ell \pi \} \) (the projection of \( K_1 \pi \) onto the first tape) is in \( \mathfrak{D} \). But applying another transduction to this implies that \( L \) itself is in \( \mathfrak{D} \), which is false. Hence \( \sigma_1 = \iota(M(L), Y) \) is not in \( T(\mathfrak{D}) \).

On the other hand, \( L_\sigma \) consists of all \( u \# v \) such that either \( u = v = a_0 \epsilon_1 r_1 a_1 \ldots \epsilon_n r_n \) and \( v = a_0 \epsilon_1 r_1 a_1 \ldots \epsilon_n r_n \) for \( a_1, \ldots, a_n \in Y \) and \( (u_i, v_i) \in (L \times \{\epsilon\}) \cup \{\epsilon \} \). Let \( \Gamma \) be an indexed grammar for \( L \). Then an indexed grammar for \( L_\sigma \), of the same ‘type’ as \( \Gamma \) (context-free, ET0L-indexed etc.) with start symbol \( I \) can be defined by setting the following productions from \( I \):

\[
I \rightarrow aIa | \ell SrIr \ell | \ell IrS' \ell | \# (a \in Y),
\]

where \( S \) is the start symbol of \( \Gamma \) and \( S' \) is the start symbol of the grammar \( \Gamma^{rev} \) for \( L^{rev} \). Hence \( \iota(M(L), Y) = \sigma_1 \) is in \( U(\mathfrak{C}) \). \( \square \)

Let \( X = \{x, y\} \) and let \( K \subseteq X^* \) be a context-free language which is not ET0L, as in the proof of Proposition 15. Let \( X' = \{x' \mid x \in X\} \) and define a homomorphism \( \phi : X^* \rightarrow (X')^* \) by \( x\phi = x' \). Let \( \rho_K = \{ (w, w\phi) : w \in K \} \) and \( M_2 = M[\rho_K] \).

Proposition 37. The monoid \( M_2 \) has word problem in \( T(CF) \) but not in \( U(ET0L) \).
Proof. Let \( X = \{x, y\} \) and let \( K \subset X^* \) be a context-free language which is not ET0L, as in the proof of Proposition \ref{prop:context-free}. Let \( X' = \{x' \mid x \in X\} \) and define a homomorphism \( \phi : X^* \to (X')^* \) by \( x\phi = x' \). Then the language \( \{w(w\phi)^{rev} : w \in K\} \) is not ET0L \ref{prop:ET0L}. Let \( Y = \{r, \ell\} \cup X \cup X' \) and let \( M_2 \) be the monoid with presentation

\[
\langle Y \mid \ell \omega r = \ell (w\phi) r (w \in K) \rangle.
\]

Then \( L_{\iota(M_2, Y)} \cap \ell X^* r \# r (X')^* \ell = \{\ell \omega r (w\phi)^{rev} \ell : w \in K\} \). This language has as a homomorphic image the non-ET0L language mentioned above, and since the ET0L languages are closed under homomorphism, \( L_{\iota(M_2, Y)} \) cannot be ET0L.

However, \( M_2 = M[\rho] \) for the relation \( \rho = \{(w, w\phi) : w \in K\} \), which is in \( T(CF) \). Hence \( M_2 \) has word problem in \( T(CF) \) by Proposition \ref{prop:monoid-word-problem} as \( CF \) satisfies the required closure properties. \( \square \)

**Proposition 38.** There exists a monoid \( M_3 \) with word problem in \( U(LIN) \) but not in \( T(E\!T\!O\!L) \), and a monoid \( M_4 \) with word problem in \( U(E\!T\!O\!L) \) but not in \( T(LIN) \).

**Proof.** This follows immediately from Proposition \ref{prop:context-free} and the incomparability of \( E\!T\!O\!L \) and \( LIN \). \( \square \)

Let \( \rho = \{(a_t^1 a_t^2 a_t^3 b_t^1 b_t^2 b_t^3) : n \in \mathbb{N}\} \) and let \( M_5 = M[\rho] \). (Recall that the relation \( \rho \) was used in the proof of Proposition \ref{prop:context-free}.)

**Proposition 39.** The monoid \( M_5 \) has word problem in \( T(LIN) \) but not in \( U(LIN) \).

**Proof.** Let \( X = \{a_1, a_2, a_3, b_1, b_2, b_3, \ell, r\} \) and

\[
M_5 = \langle X \mid \ell a_1^n a_2^m a_3^n r = \ell b_1^p b_2^q b_3^r \ell (n, p, q, r \in \mathbb{N}) \rangle.
\]

Then \( M_5 = M[\rho] \) for the relation \( \rho \) given in the proof of Proposition \ref{prop:context-free} which is in \( T(LIN) \) but not in \( U(LIN) \). Hence by Proposition \ref{prop:monoid-word-problem} \( \iota(M_5, X) \in T(LIN) \). Intersecting \( L_{\iota(M_5, X)} \) with \( \ell a_1^n a_2^m a_3^n r \# r b_1^p b_2^q b_3^r \ell \) and applying a homomorphism yields \( L_\rho \), hence \( \iota(M_5, X) \notin U(LIN) \). \( \square \)

8. Relations of larger arity

The two-tape definition of binary relations is straightforwardly extendable to relations of any arity. It is less clear how to define the ‘unfolded’ version for non-binary relations. Indeed, which definition makes sense is likely to depend on the semantic content of the relation in question. For example, Gilman \cite{gilman} defines the multiplication table of a group with respect to a regular language \( L \) of normal forms as \( \{u \# v \# w^{rev} : u, v, w \in L, w =_G w\} \) (and the same definition is extended to semigroups in \cite{gilman}). However, for other ternary relations, such as the various betweenness relations, it might make little or no sense to reverse any of the component words. Meanwhile, for quaternary relations, in some cases the unfolding \((u, v, w, x) \mapsto u \# v \# w \# x^{rev}\) will make sense, as in the case where the relation represents some ternary operation with \( u \cdot v \cdot w = x \); while in others a more natural unfolding could be \((u, v, w, x) \mapsto u \# v \# x^{rev} \# w^{rev}\) (not even preserving the order of the components), for example if the relation consisted of tuples such that \(uv = wx\) in some algebraic structure.
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