COHOMOLOGICAL CORRESPONDENCE CATEGORIES

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Abstract. We prove that homotopy invariance and cancellation properties are satisfied by any category of correspondences that is defined, via Calmès and Fasel’s construction, by an underlying cohomology theory. In particular, this includes any category of correspondences arising from a cohomology theory defined by a special linearly oriented ring spectrum.

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1. Introduction

Originally envisioned by Grothendieck, the theory of motives was set in new light by Beilinson’s conjecture on the existence of certain motivic complexes, from which one should derive a satisfactory motivic cohomology theory. This point of view ultimately led to Suslin and Voevodsky’s construction of the derived category of motives DM(k) over any field k [Voe00b]. The basic ingredient of this construction is the category Cor_A^[k] of finite correspondences over k. Finite correspondences define an additive category, and presheaves on this category—baptized presheaves...
with transfers—are exceptionally well behaved. Indeed, presheaves with transfers carry a very rich theory, satisfying fundamental properties such as preservation of homotopy invariance under sheafification \[\text{Voe00a}\], and a cancellation property with respect to smashing with \(G_m\) \[\text{Voe10}\]. These results are crucial in order to obtain a good category of motivic complexes.

Shortly after Suslin and Voevodsky’s introduction of motivic complexes, a “nonlinear” version of \(\text{DM}(k)\) was defined by Morel and Voevodsky \[\text{MV99}\] in the context of motivic homotopy theory. In this more general setting, the motivic stable homotopy category \(\text{SH}(k)\) was constructed, most notably via the \(A^1\)-localization and the \(\mathbf{P}^1\)-stabilization process. The category \(\text{SH}(k)\) is equipped with an adjunction

\[
\gamma^* : \text{SH}(k) \rightleftharpoons \text{DM}(k) : \gamma_* \tag{1.1}
\]

such that the image of the unit for the symmetric monoidal structure on \(\text{DM}(k)\) is mapped to the motivic Eilenberg–Mac Lane spectrum \(H\mathbb{Z}\) in \(\text{SH}(k)\) under \(\gamma_*\). In fact, this adjunction exhibits \(\text{DM}(k)\) as the category of modules over the ring spectrum \(H\mathbb{Z}\) (at least after inverting the exponential characteristic of \(k\)) \[\text{RØ08}\]. Furthermore, the restriction of \(\gamma_*\) to the heart of the homotopy t-structure on \(\text{DM}(k)\) is fully faithful. In fact, with rational coefficients, the category \(\text{SH}(k)_{\mathbb{Q}}\) splits into a plus part and a minus part, where the plus part is equivalent to \(\text{DM}(k, \mathbb{Q})\) \[\text{CD12}\]. Informally we can think of \(\text{DM}(k, \mathbb{Q})\) as consisting of the oriented part of \(\text{SH}(k)_{\mathbb{Q}}\).

Several alternate and refined versions of the category of correspondences have been introduced in the wake of Suslin and Voevodsky’s pioneering work, many of which attempt to provide a better approximation to the motivic stable homotopy category than \(\text{DM}(k)\). In particular, it is desirable to construct correspondences that capture also the unoriented information contained in \(\text{SH}(k)\). Examples include

- the category \(\mathbb{Z}F_k\), of linear framed correspondences, introduced by Voevodsky and further developed by Garkusha and Panin \[\text{GP18a}\];
- \(K^0_0\)- and \(K_0\)-correspondences, studied by Suslin and Walker in \[\text{Sus03; Wal96}\];
- the category \(\widetilde{\text{Cor}}_k\) of finite Milnor–Witt correspondences, introduced by Calmès–Déglise–Fasel; and
- the category \(\text{GWCor}_k\) of finite Grothendieck–Witt correspondences defined by the first author in \[\text{Dru18}\].

To exemplify to what extent the above categories succeed in providing better approximations to \(\text{SH}(k)\), let us mention that framed correspondences classify infinite \(\mathbf{P}^1\)-loop spaces \[\text{EHK}^{+18a}\], and the heart of the category \(\text{DM}(k)\) associated to \(\text{Cor}_k\) is equivalent to the heart of \(\text{SH}(k)\) (with respect to the homotopy t-structure) \[\text{AN18}\].

Along with the introduction of each new category of correspondences follows the need to prove the fundamental properties like strict homotopy invariance and cancellation in order to produce a satisfactory associated derived category of motives. For the above examples, this is achieved in \[\text{AGP18; GP18d; Sus03; FO17; DF17d; Dru18c; Dru18a}\]. The aim of this note is to establish these properties simultaneously for a certain class of correspondence categories, namely those that are defined by an underlying cohomology theory (see Definition 3.1 for the precise meaning). This includes Voevodsky’s finite correspondences—which can be defined using the cohomology theory \(\text{CH}^*\) of Chow groups—as well as finite MW-correspondences \(\text{Cor}_k\), which are defined using Chow–Witt groups \(\widetilde{\text{CH}}^*\). More generally, any special linearly oriented ring spectrum \(E \in \text{SH}(k)\) (in the sense of Panin–Walter \[\text{PW18}\]) gives rise to a cohomological correspondence category.

1.1. Outline. In Section 2 we introduce the axioms for a cohomology theory \(A^*\) needed to build the associated category \(\text{Cor}_k^A\) of finite \(A\)-correspondences. The definition the category \(\text{Cor}_k^A\) is given in Section 3. In addition, we give in Section 3 a number of constructions in the category \(\text{Cor}_k^A\). Most notably, Construction 3.13 ensures that a function on a smooth relative curve...
Section 4 is a brief comparison between our construction of A-correspondences and framed correspondences. This is done by constructing a functor from the category of framed correspondences $\text{Fr}_c(k)$ to $\text{Cor}^A_k$. Sections 5, 6, 7 and 8 are devoted to the proof of the strict homotopy invariance property of homotopy invariant presheaves on $\text{Cor}^A_k$. The proof breaks down into several excision results as well as a moving lemma, each of which is treated in its own section.

In Section 9 we show the cancellation theorem for finite A-correspondences, following the technique of Voevodsky’s original proof \[Voe10\]. Finally, in Section 10 we use the previous results to establish a well behaved category of motivic complexes $\text{DM}^A(k)$ associated to the category $\text{Cor}^A_k$, and we show several properties expected of this category. In particular, we define A-motivic cohomology in this category, and show that $\text{DM}^A(k)$ comes equipped with an adjunction to $\text{SH}(k)$ parallelly \[1.1\].

1.2. Relationship to other works. In the independent project \[EHK+18b\], the construction of the category $\text{Cor}^E_k$ of Section 3.1.1 is generalized to arbitrary ring spectra in $\text{SH}(S)$ over a base scheme $S$. Let us also mention that functors from the category of framed correspondences to other correspondence categories have been considered by several authors. The original construction of a functor $\text{Fr}_c \to \text{Cor}_k$ from framed correspondences to finite MW-correspondences was given by Déglise and Fasel in \[DF17a\]. In \[EHK+18\] \[§4.2\], the functor of Déglise and Fasel was refined to an hSpc-enriched functor $\Phi^E: \text{hCorr}^E(Sch_S) \to \text{hCorr}^E(Sch_S)$ from the homotopy category of the \(\infty\)-category of framed correspondences to finite $E$-correspondences.

1.3. Conventions and notation. Throughout, the symbol $k$ will denote a field. In certain sections we will also need to put some restrictions on the field $k$; this will be stated in the beginning of the relevant section.

We will mostly work with schemes that are smooth, separated and of finite type over $k$; the category of which will be denoted by $\text{Sm}_k$. If $X, Y \in \text{Sm}_k$, we write $X \times Y := X \times_k Y$. If $f: X \to Y$ is a morphism in $\text{Sm}_k$, we let $\omega_f := \omega_{X/k} \otimes f^*\omega_{Y/k}$ denote the relative canonical sheaf. Moreover, we abbreviate $\omega_{X \times Y/X}$ to $\omega_Y$. Throughout, we will let $i_0$ and $i_1$ denote the zero-, respectively the unit section $i_0: \text{Spec} k \to \mathbb{A}^1_k$. If we for example need to emphasize that $\mathbb{A}^2$ has coordinates $(x, y)$, we may for brevity denote this by $\mathbb{A}^2_{(x, y)}$. This notation will be used in the tables in Sections 5, 6, 7 and 8.

We denote by $\text{Map}_c(X, Y)$ the mapping spaces of an \(\infty\)-category $\mathcal{C}$, and write $[X, Y]_c := \pi_0 \text{Map}_c(X, Y)$. If $\mathcal{C}$ is any category, we let $\text{PSh}(\mathcal{C}) := \text{Fun}(\mathcal{C}^{\text{op}}, \text{Spc})$ denote the \(\infty\)-category of presheaves on $\mathcal{C}$. Moreover, we let $\text{PSh}_c(\mathcal{C})$ denote the full subcategory of $\text{PSh}(\mathcal{C})$ spanned by presheaves that carry finite coproducts to sums \[Lur09\] \[§5.5.8\].

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2. Twisted cohomology theories with support

Let \( S \) be a base scheme. We denote by \( \text{SmOp}_S^L \) the category of triples \((X, U, \mathcal{L})\), where \( X \in \text{Sm}_S \) is separated, smooth and of finite type over \( S \), \( U \) is an open subscheme of \( X \) and \( \mathcal{L} \) is a line bundle on \( X \). A morphism \((X, U, \mathcal{L}) \to (Y, V, \mathcal{M})\) in \( \text{SmOp}_S^L \) consists of a pair \((f, \alpha)\) of a morphism of \( S \)-schemes \( f: X \to Y \) such that \( f(U) \subseteq V \), and an isomorphism \( \alpha: \mathcal{L} \xrightarrow{\sim} f^* \mathcal{M} \). Note that there is an embedding \( \text{Sm}_S \to \text{SmOp}_S^L \) given by \( X \mapsto (X, \emptyset, \mathcal{O}_X) \). For any \((X, U, \mathcal{L}) \in \text{SmOp}_S^L\), we will write \( i_U \) for the inclusion \( i_U: U \to X \) and \( j_U \) for the inclusion \((X, \emptyset, \mathcal{L}) \to (X, U, \mathcal{L})\).

**Definition 2.1.** A twisted pre-cohomology theory is a graded functor
\[
A^*: (\text{SmOp}_S^L)^{\text{op}} \to \text{Ab}^Z
\]
which satisfies the following properties:

(a) (Localization) There is a natural transformation \( \partial: A^*(X, U, \mathcal{L}) \to A^{*+1}(U, i_U^* \mathcal{L}) \) of degree 1 which fits into an exact sequence
\[
A^*(X, \mathcal{L}) \xrightarrow{i_U^*} A^*(U, i_U^* \mathcal{L}) \xrightarrow{\partial} A^{*+1}(X, U, \mathcal{L}) \xrightarrow{j_U^*} A^{*+1}(X, \mathcal{L}).
\]

(b) (Étale excision) Suppose that \( f: X \to Y \) is an étale morphism of smooth \( S \)-schemes. Assume moreover that \( Z \subseteq Y \) is a closed subset such that \( f|_{f^{-1}(Z)}: f^{-1}(Z) \to Z \) is an isomorphism. Then the pullback homomorphism
\[
f^*: A^n(Y \setminus Z, \mathcal{L}) \to A^n(X \setminus f^{-1}(Z), f^* \mathcal{L})
\]
is an isomorphism for any line bundle \( \mathcal{L} \) on \( Y \) and any \( n \in Z \). If \((X, U, \mathcal{L}) \in \text{SmOp}_S^L\), let \( Z := X \setminus U \) be the closed complement of \( U \). We then write \( A^*_Z(X, \mathcal{L}) := A^*(X, U, \mathcal{L}) \). The map \( j_U^*: A^*_Z(X, \mathcal{L}) \to A^*(X, \mathcal{L}) \) is called the extension of support-homomorphism.

**Remark 2.2.** Definition 2.1 is nothing but a twisted version Panin and Smirnov’s definition of a cohomology theory considered for example in Panin09, except that for our purposes we need not assume the axiom of homotopy invariance. In the case of oriented homotopy invariant theories, our definition coincides with Panin and Smirnov’s definition.

**Remark 2.3.** The axiom of étale excision in Definition 2.1 implies that there is a canonical isomorphism \( A^*_Z(X, \mathcal{L}) \cong A^*_Z(X, \mathcal{L}) \oplus A^*_Z(X, \mathcal{L}) \).

In fact, Zariski excision is enough to prove most of the results below. The only places where we need étale excision is in the construction of the functor from framed correspondences to \( A \)-correspondences in Section 4, and in the proof that \( A \)-transfers are preserved under Nisnevich sheafification (Theorem 10.2). Moreover, the latter case only requires étale excision on local schemes. In Corollary 5.6 we show that a homotopy invariant cohomology theory satisfying Zariski excision will automatically satisfy étale excision on local schemes.

**Definition 2.4.** Let \( A^* \) be a twisted pre-cohomology theory. We say that \( A^* \) is a good cohomology theory if \( A^* \) satisfies the following properties:

(a) (Pushforward) Suppose that \( f: X \to Y \) is a morphism of smooth equidimensional \( S \)-schemes of constant relative dimension \( d \). Suppose moreover that \( Z \subseteq X \) is a closed
subset such that \( f|_Z \) is finite. Then, for any \( n \geq 0 \) and any line bundle \( \mathcal{L} \) on \( Y \), there is a pushforward homomorphism

\[
f_* : A^n_f(X, \omega_f \otimes f^* \mathcal{L}) \to A^{n-d}_{f|Z}(Y, \mathcal{L})
\]
satisfying \( \text{id}_* = \text{id} \).

(2) **(External product)** The cohomology theory is a ring cohomology theory, i.e., there is an associative product structure

\[
\times : A^n_{T_1}(X, \mathcal{L}) \otimes A^m_{T_2}(Y, \mathcal{M}) \to A^{n+m}_{T_1 \times T_2}(X \times_S Y, \mathcal{L} \boxtimes \mathcal{M})
\]

and a unit \( 1 \in A^0(S) \) turning \( A^*_{T_i}(X, \mathcal{L}) \) into an associative ring. Moreover, the external product \( \times \) commutes with pullbacks and pushforwards.

(3) **(Base change)** For any transversal square (defined in Definition 2.6) that is equipped with a set of compatible line bundles (defined in Definition 2.7) the diagram

\[
\begin{array}{ccc}
A^n_{(\phi^{-1})^*(T)}(Y', \mathcal{M}') & \xrightarrow{i_*'} & A^{n-d}_{(\phi^{-1})^*(T)}(X', \mathcal{L}') \\
\phi_Y^* & & \phi_X^* \\
A^n_{\phi^{-1}(T)}(Y, \mathcal{M}) & \xrightarrow{i_*} & A^n_{(T)}(X, \mathcal{L})
\end{array}
\]

is commutative.

(4) **(Projection formula)** Suppose that \( f : X \to Y \) and \( Z \subseteq X \) satisfy the hypotheses of (1), and let \( W \subseteq Y \) be a closed subset. Let moreover \( \mathcal{L} \) and \( \mathcal{M} \) be two line bundles on \( Y \). Given any two cohomology classes \( \alpha \in A^n_{\phi^{-1}(T)}(X, \mathcal{L}) \) and \( \beta \in A^m_{\phi^{-1}(T)}(Y, \mathcal{M}) \), we then have

\[
f_*(\alpha) \bowtie f_*(\beta) = f_*(\alpha \bowtie f^* \beta).
\]

(5) **(Graded commutativity)** For any \( (X, U, \mathcal{L}) \in \text{SmOp}^L_S \), the ring \( A^*(X, U, \mathcal{L}) \) is \((-1)\)-graded commutative, i.e., for any \( \alpha \in A^i(X, U, \mathcal{L}) \) and \( \beta \in A^j(X, U, \mathcal{L}) \), we have

\[
\alpha \bowtie \beta = (-1)^{ij}(\beta \bowtie \alpha).
\]

Here \((-1) \in A^0(S)\) is given as the pushforward \((-1) := (\text{id}_S, -1)_*(1)\); see Definition 3.19.

**Remark 2.5.** The existence of an external product \( \times \) as in Definition 2.4 (1) is equivalent to the existence of a cup product \( \smile : A^n_{T_1}(X, \mathcal{L}) \otimes A^m_{T_2}(X, \mathcal{M}) \to A^{n+m}_{T_1 \times T_2}(X, \mathcal{L} \otimes \mathcal{M}) \).

**Definition 2.6.** Let

\[
\begin{array}{ccc}
Y' & \xrightarrow{i'} & X' \\
\phi_Y & & \phi_X \\
Y & \xrightarrow{i} & X
\end{array}
\]

be a Cartesian square of smooth \( S \)-schemes, with \( i \) and \( i' \) of constant relative dimension \( d \) respectively \( d' \). The square (2.1) is called transversal if one of the following two conditions are satisfied:

(a) The morphisms \( i, i' \) are closed embeddings, \( \phi_X, \phi_Y \) are arbitrary, and the induced map \( d\phi_Y : \phi_Y^*(N_{Y/X}) \xrightarrow{\cong} N_{Y'/X'} \) is an isomorphism.

(b) The morphisms \( \phi_X, \phi_Y \) are smooth, \( i, i' \) are arbitrary, and the induced map \( di' : (i')^*\Omega^1_{\phi_X} \xrightarrow{\cong} \Omega^1_{\phi_X} \) is an isomorphism.

Note that for any transversal square, the isomorphism \( d\phi_Y \) induces an isomorphism \( d\phi_Y : \phi_Y^* \omega_i \xrightarrow{\cong} \omega_{i'} \).
Definition 2.7. Suppose that the square (2.1) is transversal. Then a compatible set of line bundles on the square (2.1) consists of the following data:
- Line bundles $\mathcal{L}, \mathcal{L}', \mathcal{M}, \mathcal{M}'$ on respectively $X, X', Y$ and $Y'$;
- isomorphisms of line bundles
  $\alpha: \phi_\ast^X \mathcal{L} \cong \mathcal{L}'; \quad \gamma: \mathcal{M} \cong i^\ast \mathcal{L} \otimes \omega_i$;
  $\beta: \phi_\ast^Y \mathcal{M} \cong \mathcal{M}'; \quad \delta: \mathcal{M}' \cong (i')^\ast \mathcal{L}' \otimes \omega_{i'}$.
We furthermore require that $\beta \circ \phi_\ast^Y(\gamma)$ corresponds to $\delta \circ (i')^\ast (\alpha \otimes \text{id}_{\omega_t})$ under the isomorphism $\text{Hom}_{\mathcal{O}_t}(\phi_\ast^Y i^\ast \mathcal{L} \otimes \phi_\ast^Y \omega_i, \mathcal{M}') \cong \text{Hom}_{\mathcal{O}_t}(\phi_\ast^X \mathcal{L} \otimes \omega_t, \mathcal{M}')$ induced by the canonical isomorphism $\phi_\ast^Y \omega_i \cong \omega_{i'}$ for the transversal square.

A transversal square equipped with a compatible set of line bundles may be denoted as follows:

$$
\begin{array}{ccc}
(Y', \mathcal{M}') & \xrightarrow{(i', \delta)} & (X', \mathcal{L}') \\
\downarrow (\phi_Y, \beta) & & \downarrow (\phi_X, \alpha)
\end{array}
$$

2.0.1. For later reference let us also recall the notion of an étale neighborhood:

Definition 2.8. Let $X \in \text{Sm}_S$ and let $Z \subseteq X$ be a closed subscheme. If $\pi: X' \to X$ is an étale morphism and $Z' \subseteq X'$ is a closed subscheme such that $\pi$ induces an isomorphism $Z' \cong \pi(Z)$, then we say that $\pi: (X', Z') \to (X, Z)$ is an étale neighborhood of $Z$ in $X$.

3. Cohomological correspondences

Let $k$ be a field, and suppose that $\Lambda^*$ is a good cohomology theory. We can then extend Calmès and Fasel’s definition of finite MW-correspondences [CF17] to our setting:

Definition 3.1. Let $X$ and $Y$ be smooth connected $k$-schemes of dimension $d_X$ respectively $d_Y$. We define the group of finite $\Lambda$-correspondences from $X$ to $Y$ as

$$
\text{Cor}_X^\Lambda(X, Y) := \lim_{T \in \mathcal{A}(X \times Y)} \Lambda^d_y(X \times Y, \omega_Y).
$$

If $X$ or $Y$ are not equidimensional, we sum over the equidimensional components.

3.0.1. In Definition 3.1 the symbol $\mathcal{A}(X, Y)$ denotes the partially ordered set of admissible subsets of $X \times Y$, i.e., closed subsets $T \subseteq X \times Y$ such that each irreducible component of $T_{\text{red}}$ is finite and surjective over $X$. See [CF17] §4.1 for more details.

3.1. The category $\text{Cor}_X^\Lambda$. We can define composition of finite $\Lambda$-correspondences in an identical manner as [CF17] §4.2: If $\alpha \in \text{Cor}_X^\Lambda(X, Y)$ and $\beta \in \text{Cor}_X^\Lambda(Y, Z)$, we define

$$
\beta \circ \alpha := (p_{XZ})_\ast (p_{XY}^\ast \alpha \sim p_{YZ}^\ast \beta). \quad (3.1)
$$

Here we write $p_{XY}$ for the projection $p_{XY}: X \times Y \times Z \to X \times Y$, and similarly for the other two maps. An identical proof as [CF17] Lemma 4.13 then shows that the groups $\text{Cor}_X^\Lambda(X, Y)$ form the mapping sets of a (discrete) category $\text{Cor}_X^\Lambda$ whose objects are the same as those of $\text{Sm}_k$.

Example 3.2. If $\Lambda^* = \text{CH}^*$ or $\text{CH}^!$, then the definition of $\text{Cor}_X^\Lambda$ recovers respectively Voevodsky’s category $\text{Cor}_X$ of finite correspondences, or Calmès–Dégilde–Fasel’s category $\text{Cor}_k$ of finite MW-correspondences. On the other hand we can define a good cohomology theory $\Lambda^*$ by letting $\Lambda^*_X(X, Z) := H^2_T(X, \Gamma^\ast, \mathcal{L}^\ast)$, where $\Gamma^\ast$ is the Nisnevich sheaf of powers of the fundamental ideal. Then $\text{Cor}_X^\Lambda$ is the category $\text{WCor}_k$ of finite Witt-correspondences considered in [CF17].
Remark 5.16. On the other hand, note that $\text{WCor}_k$ thus defined differs from the category of Witt correspondences defined in [Drul6]. However, arguing similarly as in [DF13] one can show that the associated derived categories of motives are equivalent after inverting the exponential characteristic of the ground field.

3.1.1. Special linearly oriented theories. More generally, we claim that any ring spectrum $E \in \text{SH}(k)$ that is a module over $\text{MSL}$, i.e., a special linearly oriented ring spectrum [PW18], defines a cohomological correspondence category. Let $X \in \text{Sm}_k$, and suppose that $i : Z \subseteq X$ is a closed subscheme. Let moreover $p : X \to \text{Spec} k$ be the structure map. Thus we have adjunctions

$$p^* : \text{SH}(k) \rightleftharpoons \text{SH}(X) : p_* \quad \text{and} \quad i^* : \text{SH}(Z) \rightleftharpoons \text{SH}(X) : i^!.$$

If $q : \mathcal{E} \to X$ is a vector bundle on $X$, let $s : X \to \mathcal{E}$ denote the zero section. Recall from [Hov17 §5.2] that this defines Thom transformations

$$\Sigma^\mathcal{E} := q# s_* : \text{SH}(X) \rightleftharpoons \text{SH}(X) : s^!q^* =: \Sigma^{-\mathcal{E}}.$$

In fact, these functors are defined for any class $\xi \in K(X)$ [BH18 §16.2].

Definition 3.3 ([DF17a; EHK⁺18]). Let $E \in \text{SH}(k)$ be a spectrum and let $X, Z$ be as above. Let furthermore $\xi \in K(Z)$. The $\xi$-twisted cohomology of $X$ with support on $Z$ and coefficients in $E$ is defined as

$$E_Z(X, \xi) := \text{Map}_{\text{SH}(k)}(1_k, p_* i_! i^* \Sigma^\xi p^* E).$$

The associated bigraded twisted cohomology groups with support is then given as

$$E^n_{p,q}(X, \xi) := [1_k, \Sigma^n p_* i_! i^* \Sigma^\xi p^* E]_{\text{SH}(k)}.$$

Proposition 3.4. Suppose that $E \in \text{SH}(k)$ is a special linearly oriented ring spectrum. Let $X \in \text{Sm}_k$, and suppose that $i : Z \subseteq X$ is a closed subscheme. For any line bundle $\mathcal{L}$ on $X$, set

$$A^*_{\mathcal{L}}(X, \mathcal{L}) := E^n_{2,0}(X, i^* \mathcal{L}).$$

Then $A^*_{\mathcal{L}}(X, \mathcal{L})$ defines a good cohomology theory and hence a cohomological correspondence category $\text{Cor}^E_k$.

Proof. The proposition follows from the six operations on $\text{SH}(k)$, as explained in [DF17a; DJK18] or [EHK⁺18]. Indeed, for the contravariant functoriality we refer to [DF17a §2.2], and for the definition of the cup product, see [DF17a §2.3.1]. The pushforward is given by the Gysin map

$$f_! : E_Z(X, f^* \xi + L_f) \to E_{f(X)}(Y, \xi),$$

where $L_f \in K(X)$ is the cotangent complex of $f$; see [DJK18; EHK⁺18]. In particular, for $\text{MSL}$ we have the Thom isomorphism $\Sigma^2 \text{MSL} \cong \Sigma^2 \text{det} \xi \otimes \Sigma^\xi \text{det} \xi \otimes \text{MSL}$ [BH18 Example 16.29]. When $\xi$ is a line bundle, this gives the pushforward

$$f_* : A^n_{\mathcal{L}}(X, \omega_f \otimes f^* \mathcal{L}) \to A^{n-d}_{f(X)}(Y, \mathcal{L}).$$

For the base change and projection formulas, see [DF17a Proposition 2.2.5] and [DF17a Remark 2.3.2].

3.1.2. Graph functors. We define a graph functor $\gamma_A : \text{Sm}_k \to \text{Cor}^A_k$ similarly as in [CF17, §4.3]: The functor $\gamma_A$ is the identity on objects, and if $f : X \to Y$ is a morphism in $\text{Sm}_k$, we let $\gamma_A(f) := i_*(1)$. Here $i : \Gamma_f \to \text{Spec } k$ is the embedding of the graph of $f$, and

$$i_* : A^0(\Gamma_f, \mathcal{O}_{\Gamma_f}) \to A^\dim Y(\text{Spec } \text{Spec } k, \omega_Y)$$

is the induced pushforward. We will frequently abuse notation and write simply $f$ instead of $\gamma_A(f)$. 

3.1.3. Defining $X \oplus Y := X \amalg Y$ turns $\text{Cor}^A_k$ into an additive category with zero-object the empty scheme. Moreover, $\text{Cor}^A_k$ is symmetric monoidal, with tensor product $\otimes$ defined by $X \otimes Y := X \times Y$ on objects, and given by the external product on morphisms.

3.1.4. Presheaves on $\text{Cor}^A_k$. Our basic object of study is the $\infty$-category $\text{PSh}_\Sigma(\text{Cor}^A_k)$ of presheaves on $\text{Cor}^A_k$ that take finite coproducts to sums. More generally we may of course also consider, for any coefficient ring $R$, the $\infty$-category $\text{PSh}_\Sigma(\text{Cor}^A_k, R)$ of presheaves of $R$-modules. For notational simplicity we will however mostly work with $R = \mathbb{Z}$.

3.1.5. The $\infty$-category $\text{PSh}_\Sigma(\text{Cor}^A_k)$ inherits a symmetric monoidal structure from that on $\text{Cor}^A_k$ via Day convolution. Moreover, the graph functor $\gamma_A : \text{Sm}_k \to \text{Cor}^A_k$ defines a “forgetful” functor

$$\gamma^A : \text{PSh}_\Sigma(\text{Cor}^A_k) \to \text{PSh}_\Sigma(\text{Sm}_k)$$

given by $\gamma^A(\mathcal{F}) := \mathcal{F} \circ \gamma_A$. Similarly as in [DF17a, §1.2], we deduce that the functor $\gamma^A$ admits a left adjoint $\gamma_A^*$ that is symmetric monoidal.

Lemma 3.5. The category $\text{Cor}^A_k$ is a (discrete) correspondence category in the sense of [EK18] (see also [Gar17, §2]).

Proof. This follows from the fact that $\text{Cor}^A_k$ and the graph functor $\gamma_A$ is symmetric monoidal [EK18].

3.1.6. Sheaves on $\text{Cor}^A_k$. For any Grothendieck topology $\tau$, we define the $\infty$-category $\text{Shv}_\tau(\text{Cor}^A_k)$ consisting of those presheaves $\mathcal{F} \in \text{PSh}_\Sigma(\text{Cor}^A_k)$ such that $\gamma^A(\mathcal{F})$ is a $\tau$-sheaf on $\text{Sm}_k$. The adjunction $(\gamma_A^*, \gamma^A_*)$ above then defines an adjunction

$$\gamma_A^* : \text{Shv}_\tau(\text{Sm}_k) \rightleftarrows \text{Shv}_\tau(\text{Cor}^A_k) : \gamma^A_*,$$

and the symmetric monoidal structure on $\text{PSh}(\text{Cor}^A_k)$ restricts to a symmetric monoidal structure on $\text{Shv}_\tau(\text{Cor}^A_k)$.

3.1.7. In this text, we will almost exclusively work with the case when $\tau = \text{Nis}$ is the Nisnevich topology. We show below (see Theorem 10.2) that the full inclusion $i : \text{Shv}_{\text{Nis}}(\text{Cor}^A_k) \to \text{PSh}_\Sigma(\text{Cor}^A_k)$ admits a left adjoint $a_{\text{Nis}} : \text{PSh}_\Sigma(\text{Cor}^A_k) \to \text{Shv}_{\text{Nis}}(\text{Cor}^A_k)$.

Definition 3.6. Let $X \in \text{Sm}_k$ be a smooth $k$-scheme. Following the notation of [CF17], we let $c_A(X) \in \text{PSh}(\text{Cor}^A_k)$ denote the representable presheaf on $\text{Cor}^A_k$ given by $U \mapsto \text{Cor}^A_k(U, X)$. Moreover, we let $Z_A(X) := a_{\text{Nis}}(c_A(X)) \in \text{Shv}_{\text{Nis}}(\text{Cor}^A_k)$ denote the Nisnevich sheaf associated to the presheaf $c_A(X)$.

3.2. Relative $A$-correspondences. We will need to consider also a relative version of the above construction of $\text{Cor}^A_k$.

Definition 3.7. Let $S$ be a base scheme, and suppose that $p : X \to S$ is a smooth map. Denote by $\mathcal{A}_0(X/S)$ the set of admissible subsets of $X$ relative to $S$—that is, closed subsets $T \subseteq X$ of $X$ such that each irreducible component of $T$ is finite and surjective over a component of $S$ via the map $p$. The set $\mathcal{A}_0(X/S)$ is partially ordered by inclusions. As the empty set has no irreducible components, it is admissible.
3.2.1. If \( S \) is equidimensional and \( p: X \to S \) is a smooth morphism of constant relative dimension \( d \), we define a covariant functor of abelian groups
\[
A_0(X/S) \to \text{Ab}
\]
by \( T \mapsto A^d_T(X, \omega_{X/S}) \). Indeed, if \( T' \subseteq T \), we have an extension of support-homomorphism \( A^d_{T'}(X, \omega_{X/S}) \to A^d_T(X, \omega_{X/S}) \). We will consider the colimit of this functor.

**Definition 3.8.** Suppose that \( S \) is equidimensional, and that \( p: X \to S \) is a smooth map of constant relative dimension \( d \). Assume first that \( X \) is also equidimensional. We then let
\[
C^A_0(X/S) := \lim_{T \in A_0(X/S)} A^d_T(X, \omega_{X/S})
\]
denote the group of finite relative \( A \)-cycles.

If \( X \) is not equidimensional, we may write \( X = \bigsqcup_j X_j \) where the \( X_j \)'s are the equidimensional components of \( X \). We then set
\[
C^A_0(X/S) := \prod_j C^A_0(X_j/S).
\]

3.2.2. The groups \( C^A_0(X/S) \) allow us to define a category of finite relative \( A \)-correspondences:

**Definition 3.9.** For \( S \in \text{Sm}_k \) a base scheme, let \( \text{Cor}^A_{S} \) denote the category whose objects are the same as the objects of \( \text{Sm}_S \), i.e., smooth separated schemes of finite type over \( S \), and morphisms defined as follows. Let \( X, Y \in \text{Sm}_S \), and suppose first that \( X \) and \( Y \) are equidimensional. We define the group of finite relative \( A \)-correspondences from \( X \) to \( Y \) as
\[
\text{Cor}^A_{S}(X, Y) := C^A_0(X \times_S Y/X).
\]
If \( X \) or \( Y \) are not equidimensional, let \( X = \bigsqcup_i X_i \) and \( Y = \bigsqcup_j Y_j \) be the equidimensional decomposition of \( X \) and \( Y \). Then we put \( \text{Cor}^A_{S}(X, Y) := \prod_{i,j} \text{Cor}^A_{S}(X_i, Y_j) \). We define composition of finite relative \( A \)-correspondences similarly as for finite \( A \)-correspondences.

As in Definition 3.1.2 we deduce that the category \( \text{Cor}^A_{S} \) is additive and symmetric monoidal. We refer to this category as the category of finite relative \( A \)-correspondences. Finally, we define a graph functor \( \gamma_{A,S}: \text{Sm}_S \to \text{Cor}^A_{S} \) identically as in the absolute case 3.1.2 above.

**Example 3.10.** We have \( \text{Cor}^A_{S}(X, S) = \Lambda^0(X) \) for any \( X \in \text{Sm}_S \). Note also that if \( S = \text{Spec} \, k \), then \( A_0(X \times_S Y/X) = A(X, Y) \) and \( \text{Cor}^A_{S}(X, Y) = \text{Cor}^A_{k}(X, Y) \).

3.2.3. For \( S \) a smooth \( k \)-scheme there is a functor \( \text{ext}_{S}: \text{Cor}^A_{k} \to \text{Cor}^A_{S} \) defined as follows. For any \( X \in \text{Sm}_k \), let \( X_S := X \times_S S \). Let \( X, Y \in \text{Sm}_k \); by working with one connected component at a time, we may assume that \( X \) and \( Y \) are connected. By the universal property of fiber products we have a morphism \( f: X_S \times_S Y_S \to X \times Y \), which induces a pullback morphism
\[
f^*: A^\dim_Y(X \times Y, \omega_Y) \to A^\dim_Y(T)(X_S \times_S Y_S, f^*\omega_Y)
\]
for any \( T \in A_0(X \times_k Y/X) = A(X, Y) \). As finiteness and surjectivity are preserved under base change we have \( f^{-1}(T) \in A_0(X_S \times_S Y_S/X_S) \). Now, recall that \( X \) and \( Y \) are assumed to be smooth of relative dimension \( \dim X \) respectively \( \dim Y \) over \( \text{Spec} \, k \). By base change for smooth morphisms, \( X_S \to S \) is smooth of relative dimension \( \dim X \), and similarly \( Y_S \to S \) is smooth of relative dimension \( \dim Y \). It then follows from [Har74, III Proposition 10.1] that \( X_S \times_S Y_S \) is smooth over \( X_S \) of relative dimension \( \dim Y \).

Moreover, the canonical sheaf \( \omega_{X/S} \) pulls back over \( X_S \) to \( \omega_{X_S/S} \), and similarly for \( \omega_{Y/k} \). Hence \( f^*\omega_{X_S \times_S Y_S} \cong \omega_{X_S \times_S Y_S/X_S} \). Since pullbacks commute with extension of support, we get an induced map on the colimit
\[
\text{Cor}^A_{k}(X, Y) \to \text{Cor}^A_{S}(X_S \times_S Y_S/X_S) = \text{Cor}^A_{S}(X_S, Y_S).
\]
All in all, we obtain a functor $\text{ext}_S: \text{Cor}_A^S \to \text{Cor}_A^S$.

3.2.4. In the opposite direction there is a “forgetful” functor $\text{res}_S: \text{Cor}_A^S \to \text{Cor}_A^S$ induced by pushforwards. Indeed, let $X, Y \in \text{Sm}_S$. Then there is a pullback diagram

$$X \times_S Y \xrightarrow{i_{X,Y}} X \times_k Y \xrightarrow{\Delta_S} S \times_k S,$$

where $\Delta_S \subseteq S \times S$ denotes diagonal. Moreover, we have isomorphisms

$$\omega_{X \times_S Y} \otimes i^* \omega_{X \times Y}^{-1} = \omega_{i_{X,Y}} \cong \omega_i = \omega_S^{-1}.$$

Thus there is, for any $T \in \mathcal{A}_S(X \times_S Y/X)$, a pushforward homomorphism

$$(i_{X,Y})_*: A^\dim Y(X \times_S Y, \omega_{Y/S}) \to A^\dim Y(T)(X \times Y, \omega_Y).$$

Passing to the colimit, we thus obtain a map $\text{Cor}_S^S(X, Y) \to \text{Cor}_A^S(X, Y)$. This defines a functor $\text{res}_S: \text{Cor}_A^S \to \text{Cor}_A^S$.

3.2.5. For any $X \in \text{Sm}_S, Y \in \text{Sm}_k$ and any admissible subset $T$ of $X \times Y$ we have a natural isomorphism $A^\dim X(Y, X \times Y, \omega) \cong A^\dim Y_S(X \times_S Y, \omega_{X \times_S Y_S}).$ These isomorphisms define a natural isomorphism $\text{Cor}_A^S(X, Y) \cong \text{Cor}_S^S(X, Y_S)$. Similarly as in [CF17, §6.2] we thus deduce that the functors $\text{res}_S$ and $\text{ext}_S$ form an adjunction

$$\text{res}_S: \text{Cor}_A^S \rightleftharpoons \text{Cor}_S^S: \text{ext}_S.$$

3.3. Correspondences of pairs. Most of our results will be formulated in terms of correspondences of pairs $(X, U) \in \text{SmOp}_S$. By abuse of notation, we will therefore also let $\text{Cor}_A^S$ denote the category of finite $A$-correspondences of pairs:

**Definition 3.11.** Let $\text{Cor}_A^S$ denote the category whose objects are those of $\text{SmOp}_S$ and whose morphisms are defined as follows. For $(X, U), (Y, V) \in \text{SmOp}_S$, with open immersions $j_X: U \to X$ and $j_Y: V \to Y$, let

$$\text{Cor}_A^S((X, U), (Y, V)) := \ker\left(\text{Cor}_A^S(X, Y) \oplus \text{Cor}_A^S(U, V) \xrightarrow{j_X^* - j_Y^*} \text{Cor}_S^S(U, Y)\right).$$

The composition in $\text{Cor}_A^S$ is defined by $(\alpha, \beta) \circ (\gamma, \delta) := (\alpha \circ \gamma, \beta \circ \delta)$. If $\beta$ is implicitly understood, we might write simply $\alpha$ instead of $(\alpha, \beta) \in \text{Cor}_A^S((X, Y), (Y, V))$.

**Definition 3.12.** Define the homotopy category $\text{Cor}_A^S$ of $\text{Cor}_A^S$ as follows. The objects of $\text{Cor}_A^S$ are the same as those of $\text{Cor}_S^S$, and the morphisms are given by

$$\text{Cor}_A^S((X, U), (Y, V)) := \text{Cor}_S^S((X, U), (Y, V))/\sim_A,$$

where $\sim_A$ is the equivalence relation induced by

$$A^1 \times (X, U) \xrightarrow{i_{X,U}} \text{Cor}_A^S((X, U), (Y, V))\xrightarrow{[\alpha, \beta]} \text{Cor}_S^S((X, U), (Y, V)).$$

Here $A^1 \times (X, U)$ is shorthand for $(A^1 \times X, A^1 \times U)$. If $(\alpha, \beta) \in \text{Cor}_S^S((X, U), (Y, V))$ is a finite relative $A$-correspondence of pairs, we write $[[\alpha, \beta]]$, or simply $[\alpha]$, for the image of $(\alpha, \beta)$ in $\text{Cor}_A^S((X, U), (Y, V))$.

3.4. Constructing correspondences from functions and trivializations. From now on we will assume that the base scheme $S$ is the spectrum of a field $k$. Later on we will also have to put more restrictions on $k$ (e.g., infinite or perfect); the appropriate assumptions will be stated in the beginning of each section were they are needed.
3.4.1. We will now describe how to construct a finite $A$-correspondence from the data of a regular function on a relative curve together with a trivialization of the relative canonical class. This construction can be thought of as an analogous statement to the defining axiom of a pretheory in the sense of Voevodsky [Voe02a], and will be used throughout.

Construction 3.13. Suppose that there is a diagram

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{f} & \mathbb{A}^1 \\
p & & \downarrow \\
U & & \\
\end{array}
\]

in $\text{Sm}_k$ satisfying the following properties:

1. $p: \mathcal{C} \to U$ is a smooth relative curve.
2. $Z(f) = Z \amalg Z'$, with $Z$ finite over $U$.
3. There is an isomorphism $\mu: \mathcal{O}_C \to \omega_{\mathcal{C}/U}$.

We can then define a finite relative $A$-correspondence $\langle f \rangle_Z^\mu \in \text{Cor}_U^A(U, \mathcal{C})$ as follows:

Let $\Gamma_f$ denote the graph of the morphism $f$, with embedding $\iota_f: \Gamma_f \hookrightarrow \mathcal{C} \times \mathbb{A}^1$. Consider the pushforward homomorphism $(\iota_f)_*: A^0(\Gamma_f, \mathcal{O}_{\Gamma_f} \otimes \omega_{\iota_f}) \to A^1_f(\mathcal{C} \times \mathbb{A}^1, \mathcal{O}_{\mathcal{C} \times \mathbb{A}^1})$, and let $dT: \mathcal{O}_{\mathbb{A}^1} \cong \omega_{\mathbb{A}^1}$ be the trivialization defined by the coordinate function $\mathbb{A}^1 \to \mathbb{A}^1$. Using the trivializations $-dT$ and $\mu$ we then obtain a homomorphism

\[ \iota_*: A^0(\Gamma_f, \mathcal{O}_{\Gamma_f}) \to A^1_f(\mathcal{C} \times \mathbb{A}^1, \omega_{\mathcal{C} \times \mathbb{A}^1}/U). \]

Let $\iota_*(1) \in A^1_f(\mathcal{C} \times \mathbb{A}^1, \omega_{\mathcal{C} \times \mathbb{A}^1}/U)$ denote the image of $1 \in A^0(\Gamma_f, \mathcal{O}_{\Gamma_f})$ under the map $\iota_*$ above.

Next, consider base change along the zero section $\iota_0: U \times 0 \to U \times \mathbb{A}^1$, and the pullback homomorphism $\iota^*_0: A^1_f(\mathcal{C} \times \mathbb{A}^1, \omega_{\mathcal{C} \times \mathbb{A}^1}/U) \to A^1_{Z(f)}(\mathcal{C}, \omega_{\mathcal{C}/U})$. Since $Z(f) = Z \amalg Z'$ we have

\[ A^1_{Z(f)}(\mathcal{C}, \omega_{\mathcal{C}/U}) = A^1_Z(\mathcal{C}, \omega_{\mathcal{C}/U}) \oplus A^1_Z(\mathcal{C}, \omega_{\mathcal{C}/U}) \]

by Remark 2.3.

Definition 3.14. In the above setting, define the finite relative $A$-correspondence $\langle f \rangle_Z^\mu \in \text{Cor}_U^A(U, \mathcal{C})$ as the image of $\iota_*(1) \in A^1_f(\mathcal{C} \times \mathbb{A}^1, \omega_{\mathcal{C} \times \mathbb{A}^1}/U)$ under the composite homomorphism

\[ A^1_f(\mathcal{C} \times \mathbb{A}^1, \omega_{\mathcal{C}/U}) \xrightarrow{\iota^*_0} A^1_{Z(f)}(\mathcal{C}, \omega_{\mathcal{C}/U}) \xrightarrow{\iota^*_Z} \text{Cor}_U^A(U, \mathcal{C}), \]

where the second map is the projection to the first coordinate, and the last map is the canonical homomorphism to the colimit.

3.4.2. We readily obtain a nonrelative $A$-correspondence by applying the functor $\text{res}_U: \text{Cor}_U^A \to \text{Cor}_k^A$.

Definition 3.15. Let $\mathcal{C}$, $Z$, $p$ and $f$ be as in Construction 3.13 and suppose that $g: \mathcal{C} \to X$ is a morphism in $\text{Sm}_k$. We then let

\[ \langle f \rangle_Z^{\mu, g} := g \circ \text{res}_U(\langle f \rangle_Z) \in \text{Cor}_k^A(U, X). \]

If it is clear from the context, we might drop the trivialization $\mu$ or the map $g$ from the notation.

Moreover, if $Z = Z(f)$ and $Z$ is finite over $U$, we may also abbreviate $\langle f \rangle_{Z(f)}$ to $\langle f \rangle$.

Lemma 3.16. Let $\mathcal{C}$, $Z$, $p$ and $f$ be as in Construction 3.13 and suppose that $Z = Z_1 \amalg Z_2$ with both $Z_1$ and $Z_2$ finite over $U$. Then $\langle f \rangle_{Z}^{\mu, g} = \langle f \rangle_{Z_1}^{\mu, g} \amalg \langle f \rangle_{Z_2}^{\mu, g}$.

Proof. The claim follows from the definition and Remark 2.3. \qed
**Definition 3.17.** Let $\mathcal{C}, U, \mu, Z, X, p, f$ and $g$ be as above and suppose that $U' \subseteq U$ and $X' \subseteq X$ are open subschemes such that $Z \times_U U' \subseteq g^{-1}(X')$. Write $f' := f|_{\mathcal{C} \times_U U'}$ and $g' := g|_{\mathcal{C} \times_U U'}$. This defines a correspondence of pairs

$$((f)_Z^\mu g, (f')_Z^\mu g') \in \text{Cor}_k^A((U, U'), (X, X')).$$

Similarly, define $(f)_Z^\mu g \in \text{Cor}_k^A((U, U'), (X, X'))$ for an étale neighborhood $\pi : (\mathcal{C}', Z') \to (\mathcal{C}, Z)$ satisfying $Z' \times_U U' \subseteq \pi^{-1}(X')$.

**Lemma 3.18.** Let $\mathcal{C}, U, \mu, Z, X, p, f$ and $g$ be as above and suppose that $U' \subseteq U$ and $X' \subseteq X$ are open subschemes. If $Z \cap g^{-1}(X \setminus X') = \emptyset$, then $(f)_Z^\mu g = 0 \in \text{Cor}_k^A((U, U'), (X, X'))$.

**Proof.** The correspondence $(f)_Z^\mu g \in \text{Cor}_k^A(U, X')$ defines the diagonal in the diagram

$$
\begin{array}{ccc}
X' & \rightarrow & X \\
\uparrow & & \uparrow \\
U' & \rightarrow & U.
\end{array}
$$

Moreover, the vertical arrows in the above diagram define the correspondence of pairs $(f)_Z^\mu g \in \text{Cor}_k^A((U, U'), (X, X'))$; it follows that $(f)_Z^\mu g$ factors through $(X', X')$, and is therefore zero. □

**Definition 3.19.** Let $U \in \text{Sm}_k$ and suppose that $\lambda$ is an invertible regular function on $U$. We can then consider the morphism $(\text{id}, \lambda) : (U \times U, \omega_U) \to (U \times U, \omega_U)$ in $\text{SmOp}_k^L$. We denote by

$$\langle \lambda \rangle \in \text{Cor}_k^A(U, U)$$

the image of $\text{id}_U \in \text{Cor}_k^A(U, U)$ under the corresponding pushforward map $(\text{id}, \lambda)_*$. In particular, if $\lambda = -1$, we will write $\epsilon$ for the finite $A$-correspondence

$$\epsilon := -(-1) \in \text{Cor}_k^A(U, U).$$

**Example 3.20.** Suppose that $A^* = \text{CH}^*$, so that $\text{Cor}_k^A$ is the category of finite MW-correspondences. Then $\langle \lambda \rangle \in \text{Cor}_k^A(U, U)$ coincides with the MW-correspondence $\lambda \cdot \text{id}_U \in \text{Cor}_k(U, U)$ given by multiplication with the quadratic form $\langle \lambda \rangle \in K_0^{\text{MW}}(U)$. In particular, the finite $A$-correspondence $\epsilon$ coincides with the usual $\epsilon$ defined in Milnor–Witt $K$-theory.

**Lemma 3.21.** Let $U, \mathcal{C}, p$ and $f$ be as in Construction 3.13 and let $g : \mathcal{C} \to X$ be a morphism in $\text{Sm}_k$. Suppose also that $p$ induces an isomorphism $Z(f) \cong U$, so that $Z(f)$ defines a section $s : U \to \mathcal{C}$ of $p$. Then the following hold:

(a) There is an invertible regular function $\lambda$ on $U$ such that $(f)_Z^\mu g = g \circ s \lambda$ in $\text{Cor}_k^A(U, X)$.

(b) If moreover $\mu_1 Z(f) = df$, where $df$ denotes the trivialization of the normal bundle $N_{Z(f)/\mathcal{C}}$ defined by $f$, then $(f)_Z^\mu g = g \circ s$.

**Proof.** (a) Let $j : Z(f) \to \Gamma_f, j_f : Z(f) \to \mathcal{C}$ and $i_f : \Gamma_f \to \mathcal{C} \times A^1$ denote the closed embeddings. Consider the following diagram consisting of two squares of varieties equipped with compatible sets of line bundles:

$$
\begin{array}{ccc}
(Z(f), \mathcal{O}_{Z(f)}) & \xrightarrow{(\text{id}, \mu)} & (Z(f), \omega_{Z(f)/U}) \\
\downarrow j & & \downarrow j \\
(\Gamma_f, \mathcal{O}_{\Gamma_f}) & \xrightarrow{(\text{id}, \mu)} & (\Gamma_f, \omega_{\Gamma_f/U}) \\
\end{array}
\begin{array}{ccc}
\rightarrow (\mathcal{C}, \omega_{\mathcal{C}/U}) \\
\downarrow i_f & & \downarrow i_f \\
(\mathcal{C} \times A^1, \omega_{\mathcal{C} \times A^1/U}).
\end{array}
$$
The first square is evidently transversal. To prove that the second one is transversal, it is enough to note that the homomorphism \( k[C][T] = k[C \times A^1] \to k[C] \) given by \( T \mapsto 0 \) takes the function \( f - T \) to \( f \) and induces an isomorphism

\[
N_{\Gamma_f/C \times A^1} \otimes k[C \times 0] = (f - T)/(f - T)^2 \otimes k[C][T]/(T) \cong (f)/f^2 = N_{Z(f)/C}.
\]

Hence the base change axiom gives us the following commutative diagram:

\[
\begin{array}{ccc}
A^0(\Gamma_f, \mathcal{O}_{\Gamma_f}) & \xrightarrow{\mu} & A^0(\Gamma_f, \omega_{\Gamma_f/U}) \\
\downarrow j^* & & \downarrow j^* \\
A^0(Z(f), \mathcal{O}_{Z(f)}) & \xrightarrow{\mu_{Z(f)}} & A^0(Z(f), \omega_{Z(f)/U} \otimes \omega_j) \\
& & \downarrow (j, \nu)_* \\
& & A^1_{Z(f)}(\mathcal{C}, \omega_{Z(f)/U} \otimes \omega_i) \\
\end{array}
\]

Here \( j^* \) and \( i_0^* \) are defined via the canonical isomorphisms \( j^*(\omega_{\Gamma_f/U}) \cong \omega_{Z(f)/U} \otimes \omega_j \) and \( i_0^*(\omega_{C \times A^1/u}) \cong \omega_{C/U} \otimes \omega_i \) induced by the short exact sequences of vector bundles

\[
0 \to T_{Z(f)} \to j^*(T_{\Gamma_f}) \to N_{Z(f)/\Gamma_f} \to 0
\]

and

\[
0 \to T_{C \times 0} \to i_0^*(T_{C \times A^1}) \to N_{C \times 0/C \times A^1} \to 0.
\]

Moreover, the homomorphism \( \mu_{Z(f)} \) is defined by the composition of \( \mu_{Z(f)} \) and the isomorphism \( j^*\omega_{\Gamma_f/U} \cong \omega_{Z(f)/U} \otimes \omega_j \); the homomorphism \( (j_f)_* \) is defined via the isomorphism \( j_f^*(\omega_i) \cong \omega_j \) induced by the canonical isomorphism \( \Gamma_f \cong C \); and the diagonal homomorphism \( (j_f, \nu)_* \) is induced by some trivialization \( \nu : \mathcal{O}_{Z(f)} \cong \omega_{Z(f)/U} \).

It follows from the construction that \( (f)_{Z(f)} = -dT(i_0^*(i_f)_*\mu(1)) \). Since the diagram is commutative we thus obtain \( (f)_{Z(f)} = (j_f, \nu)_* j^*(1) = s \circ \langle \lambda \rangle \), where \( \lambda \) is given as the fraction of \( \nu \) and the canonical isomorphism \( \omega_{Z(f)/U} \cong \mathcal{O}_U \) induced by the isomorphism \( p : Z(f) \cong U \).

(b) A straightforward computation with isomorphisms of line bundles shows that \( (j_f)_* \) is defined by the product of the canonical isomorphism \( \mathcal{O}_{Z(f)} \cong \omega_{Z(f)/U} \) with the invertible function

\[
\mu_{Z(f)}(\omega_i) \otimes df^{-1},
\]

where \( df : \mathcal{O}_{Z(f)} \cong \omega_{Z(f)/U} \) denotes the trivialization induced by the choice of the generator \( -f \) of the ideal \( (f) = I(Z(f)) \). So \( \lambda = 1 \), and the claim follows.

### 3.5. Some homotopies.

**Lemma 3.22.** Let \( U \) be an essentially smooth local scheme and let \( \lambda \in \Gamma(U, \mathcal{O}_U^*) \). Suppose that \( \lambda = w^2 \) for some invertible section \( w \) on \( U \). Then \( \langle \lambda \rangle \sim_{A^1} \text{id}_U \in \text{Cor}_k^A(U, U) \).

**Proof.** Let \([t_1 : t_2]\) be the coordinates of \( \mathbb{P}^1 \). For any matrix

\[
A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \text{GL}_2(\Gamma(U, \mathcal{O}_U))
\]

we will define a rational function \( r = s_1/s_2 \) on \( \mathbb{P}^1 \times U \) which is regular on \( (\mathbb{P}^1 \times U) \setminus Z(s_2) \). To do this, we let \( s_1 := a_{11}t_1 + a_{12}t_2 \) and \( s_2 := a_{21}t_1 + a_{22}t_2 \). The rational function \( r \) then defines an isomorphism

\[
e := r \circ \text{id}_U : (\mathbb{P}^1 \times U) \setminus Z(s_2) \cong A^1 \times U.
\]

Let \( t \) denote the coordinate function on \( A^1 \), let \( dt \) denote the standard trivialization of the canonical class of \( A^1 \), and let \( \langle t \rangle_{A^1} \in \text{Cor}_k^A(U, A^1 \times U) \) be the correspondence given by Construction 3.13. Define \( \langle A \rangle := \langle t \rangle_{A^1} \in \text{Cor}_k^A(U, \mathbb{P}^1 \times U) \), where \( i(A) : A^1 \times U \to \mathbb{P}^1 \times U \) denotes the composition of \( e^{-1} \) with the canonical embedding.
Now consider the following matrices:

\[
A_1 := \begin{pmatrix} 1 & 0 \\ 1 & w \end{pmatrix}, \quad A_2 := \begin{pmatrix} w & 0 \\ 1 & 1 \end{pmatrix}.
\]

Since \(\det(A_1) = \det(A_2) = w\), and since \(U\) is local, there is a sequence of elementary transformations connecting \(A_1\) and \(A_2\). Applying the construction above we get the sequence of \(A^1\)-homotopies connecting \(\langle A_1 \rangle\) and \(\langle A_2 \rangle\).

Note that \(\langle A_1 \rangle = \langle w \rangle\) and \(\langle A_2 \rangle = \langle w^{-1} \rangle\). Composing with \(\langle w \rangle\) and applying the functor \(\text{res}_U : \text{Cor}^A_k \rightarrow \text{Cor}^A\), we then get the required homotopy connecting \(\langle \lambda \rangle \in \text{Cor}^A_k(U, U)\) and \(\text{id}_U\).

Finally, since the homotopy is natural in \(U\) it defines homotopy of correspondences of pairs as well. \(\square\)

4. Connection to framed correspondences

Using similar techniques as in Construction 5.13 we can define a functor \(\Upsilon : \text{Fr}_*(k) \rightarrow \text{Cor}^A\) from the category of framed correspondences \([GP18a]\) to the category \(\text{Cor}^A_k\). See also \([EHK^{+18b}]\) for an alternative approach.

**Construction 4.1.** Let \(\Phi = (Z, \nu, \phi; g) \in \text{Fr}_n(X, Y)\) be an explicit framed correspondence. Thus \(Z\) is a closed subset in \(A^n \times X = A^n_X\); \((\nu, Z, \wedge) \rightarrow (A^n_X, Z)\) is an étale neighborhood; \(\phi = (\phi_i)\), where the \(\phi_i\)'s are regular functions on \(\nu\) such that \(Z = Z(\phi)\); and \(g\) is a morphism \(g : \nu \rightarrow Y\). For any \(\lambda \in k^\times\) we define a finite \(A\)-correspondence \(\Upsilon_\lambda(\Phi) \in \text{Cor}^A_k(X, Y)\) in the following way.

Let \(dt : \omega_{A^1} \cong \mathcal{O}_{A^1}\) denote the standard trivialization of the canonical class, and consider further two trivializations \(\mu_1, \mu_2 : \omega_{A^n} \cong \mathcal{O}_{A^n}\) given by \(\mu_1 = (dt)^{\wedge n}\) and \(\mu_2 = \lambda \mu_1\). Let \(\Gamma\) denote the graph \(\Gamma \subseteq A^n_X \times X \nu = A^n_X \times \nu\) of the relative morphism \(\nu \rightarrow A^n_X\) over \(X\). There is then a canonical projection \(\Gamma \rightarrow A^n_X\). Denote by \(i_X : X \rightarrow A^n_X\) and \(i_\nu : \nu \rightarrow A^n\) the embeddings given by the zero sections. Let \(g' : \nu \rightarrow X \times Y\) denote the product of the projection to \(X\) and \(g\).

The following diagram summarizes the situation:

\[
\begin{array}{ccc}
A^n \times Y & \xrightarrow{\Gamma} & A^n_X \\
\downarrow \quad \quad \downarrow & & \downarrow \\
A^n \times X & \xleftarrow{i_X} & X.
\end{array}
\]

We then define

\[
\Upsilon_\lambda(\Phi) := g'(i_\nu(\Gamma_* (1))),
\]

where we use the trivialization \(\mu_1\) of the canonical class \(\omega_{A^n}\), and the trivialization of \(\omega_{\nu/X}\) defined by the pullback of \(\mu_2\) along the étale morphism \(\nu \rightarrow A^n_X\).

In other words, the finite \(A\)-correspondence \(\Upsilon_\lambda(\Phi)\) is obtained as the image of \(i_\nu(\Gamma_* (1)) \in A^n_2(\nu, \omega_{\nu/X})\) under the composition

\[
A^n_2(\nu, \omega_{\nu/X}) \rightarrow \text{Cor}^A(X, \nu) \xrightarrow{\text{res}_X} \text{Cor}^A(X, \nu) \xrightarrow{g} \text{Cor}^A(X, Y),
\]

in which the last map is given by composition with \(g\).
4.0.1. The following alternative construction of the functor \( \text{Fr}_n(k) \to \text{Cor}_k^A \) using Thom classes was suggested to the authors by I. Panin. A similar approach is used in [EHK+ 18]; see [EHK+ 18, Lemma 4.3.24].

Construction 4.2. Let \( \Phi = (Z, V, \phi; g) \in \text{Fr}_n(X, Y) \) be an explicit framed correspondence. Consider the diagram

\[
\begin{array}{ccc}
Z & \xrightarrow{g} & V \\
\downarrow & & \downarrow \\
\text{Spec} k & \xrightarrow{i_0} & A^n, \\
\end{array}
\]

(4.2)

where \( i_0 : \text{Spec} k \to A^n \) denotes the zero section. Moreover, as in Construction 4.1 we let \( g' : V \to X \times Y \) be the product of the projection to \( X \) and \( g \), and use the trivialization of the canonical class \( \omega_{A^n} \) defined by coordinates. Then \( \Phi \) gives rise to an element \( \Upsilon'_\lambda(\Phi) \in \text{Cor}_k^A(X, Y) \) by letting

\[
\Upsilon'_\lambda(\Phi) := g'_*(\phi^*(i_0)_*(1)).
\]

Proposition 4.3. For any \( \Phi \in \text{Fr}_n(X, Y) \) we have \( \Upsilon_\lambda(\Phi) = \Upsilon'_\lambda(\Phi) \in \text{Cor}_k^A(X, Y) \).

Proof. Let \( j : Z \to V \) denote the closed immersion. Then the base change axioms applied to the diagrams

\[
\begin{array}{ccc}
Z & \xrightarrow{j} & \text{Spec} k \\
\downarrow & & \downarrow \\
V & \xrightarrow{\phi} & A^n, \\
\end{array}
\quad
\begin{array}{ccc}
Z & \xrightarrow{\phi} & V \\
\downarrow & & \downarrow \\
\text{Spec} k & \xrightarrow{j} & A^n \times X, \\
\end{array}
\]

implies that \( \Upsilon_\lambda(\Phi) = g'_*(j^*(1)) = \Upsilon'_\lambda(\Phi) \). \( \square \)

Theorem 4.4. For each unit \( \lambda \in k^* \), Construction 4.1 defines a functor

\[
\Upsilon_\lambda : \text{Fr}_n(k) \to \text{Cor}_k^A
\]

that carries the framed correspondence \( \sigma = (0, A^1, t, \text{pr} : A^1 \to \text{pt}) \in \text{Fr}_1(\text{pt}, \text{pt}) \) to \( (\lambda) \in \text{Cor}_k^A(\text{pt}, \text{pt}) \). Moreover, \( \Upsilon_\lambda \) factors through the category \( \mathbf{ZF}_n \) of linear framed correspondences.

Proof. The two (equivalent) constructions above give rise to a map \( \Upsilon_\lambda \) depending on the fraction \( \lambda \in k^* \) of the two trivializations of the canonical classes. We need to show that \( \Upsilon_\lambda \) is in fact a functor. For simplicity, write \( \Upsilon_\lambda \) for \( \Upsilon \). Thus, in order to show the claim we need to check the following:

(1) Equivalent explicit framed correspondences give rise to the same finite \( A \)-correspondence.
(2) For any \( \Phi_1 \in \text{Fr}_{n_1}(X_1, X_2), \Phi_2 \in \text{Fr}_{n_2}(X_2, X_3) \) we have \( \Upsilon(\Phi_2 \circ \Phi_1) = \Upsilon(\Phi_1) \circ \Upsilon(\Phi_2) \).
(3) For any \( \Phi = (Z, V, \phi; g) \in \text{Fr}_n(X_1, X_2) \) such that \( Z = Z_1 \amalg Z_2 \), we have \( \Upsilon(\Phi) = \Upsilon(Z_1, V, \phi; g) + \Upsilon(Z_2, V, \phi; g) \).

All three points are straightforward, so we omit the full proof and give here only a list of the properties of the cohomology theory \( A^* \) which are essential for proving the respective points above, along with a detailed proof only for the second point.

(1) The first point follows from the base change and étale excision axioms.
(2) The second point follows from the base change, ring structure, and projection formula axioms.
(3) The third point follows from the Zariski excision axiom for the cohomology theory \( A^* \).
To prove the second point, let $\Gamma_1: \mathcal{V}_1 \to \mathcal{V}_1 \times A^{n_1}$ and $\Gamma_2: \mathcal{V}_2 \to \mathcal{V}_2 \times A^{n_2}$ be the relative graphs, and let $g_1': \mathcal{V}_1 \to X_1 \times X_2$ and $g_2': \mathcal{V}_2 \to X_2 \times X_3$ be the morphisms defined as in Construction 4.1. We then have

$$\Upsilon(\Phi_1) = (g_1')_*(i_{\mathcal{V}_1}^*(\Gamma_1)_*(1)),$$

$$\Upsilon(\Phi_2) = (g_2')_*(i_{\mathcal{V}_2}^*(\Gamma_2)_*(1)).$$

On other hand, let $\mathcal{V} := \mathcal{V}_1 \times X_2 \mathcal{V}_1$, and let $\text{pr}_{\mathcal{V}_1}: \mathcal{V} \to \mathcal{V}_1$ and $\text{pr}_{\mathcal{V}_2}: \mathcal{V} \to \mathcal{V}_2$ denote the projections. Then the composition $\Phi := \Phi_2 \circ \Phi_1$ is given as $(Z, \mathcal{V}, \phi; g)$, where $Z := Z_1 \times X_2 Z_2$; $\phi := (\text{pr}_{\mathcal{V}_1}(\phi_1), \text{pr}_{\mathcal{V}_2}(\phi_2))$; and $g := g_2 \circ \text{pr}_{\mathcal{V}_2}$. Let $\text{pr}_1$ denote the projection

$$\text{pr}_1: X_1 \times X_2 \times X_3 \to X_2 \times X_3,$$

and similarly for $\text{pr}_2, \text{pr}_3$. Finally, as in Construction 4.1 we define maps $g': \mathcal{V} \to X_1 \times X_3$ and $g'': \mathcal{V} \to X_1 \times X_2 \times X_3$. Now, using base change, ring structure, and projection formula axioms for $A^*$ we find

$$\Upsilon(\Phi_2) \circ \Upsilon(\Phi_1) = (\text{pr}_2)_* (\text{pr}_1^*(g_2')_*(i_{\mathcal{V}_2}^*(\Gamma_2)_*(1)))) \sim \text{pr}_2^*(g_1')_*(i_{\mathcal{V}_1}^*(\Gamma_1)_*(1))).$$

Note that in the penultimate equality in the sequence above we essentially use that the trivializations $\mu_1$ and $\mu_2$ of the canonical class of $\omega_{A^{n_1}, n_2}$ appearing in Construction 4.1 are the products of the corresponding trivializations of $\omega_{A^{n_1}}$ and $\omega_{A^{n_2}}$. \hfill $\Box$

**Remark 4.5.** Note that Theorem 10.1 on strict homotopy invariance of sheaves on $\text{Cor}_A^k$ follows from the existence of a functor from framed correspondences to $\text{Cor}_A^k$ along with the fact that this theorem holds for framed correspondences by work of Garkusha–Panin [GP18]. Below we will however give an explicit proof not relying on framed correspondences.

5. **Injectivity on the relative affine line**

In this section we prove the following theorem, which is the first in a series of ingredients necessary to establish strict homotopy invariance (Theorem 10.1).

**Theorem 5.1.** Let $U$ be an affine smooth $k$-scheme, and suppose that $V_1 \subseteq V_2 \subseteq A^1_U$ are two open subschemes such that $A^1_U \setminus V_2$ and $V_2 \setminus V_1$ are finite over $U$. Let $i: V_1 \subseteq V_2$ denote the inclusion. Then, for any homotopy invariant presheaf with $A$-transfers $\mathcal{F}$, the restriction homomorphism $i^*: \mathcal{F}(V_2) \to \mathcal{F}(V_1)$ is injective.

5.0.1. We deduce Theorem 5.1 from the following result, which ensures the existence of a left inverse to $i^*$:

**Lemma 5.2.** Suppose that $V_1 \subseteq V_2 \subseteq A^1_U$ are open subschemes as in Theorem 5.1. Then there is a finite $A$-correspondence $\Phi \in \text{Cor}_A^k(V_2, V_1)$ such that $[i \circ \Phi] = [i_{V_2}] \in \text{Cor}_A^k(V_2, V_2)$.

**Proof.** To prove the claim we must construct a finite $A$-correspondence

$$\Phi \in \text{Cor}_A^k(V_2, V_1)$$

along with a homotopy

$$\Theta \in \text{Cor}_A^k(A^1 \times V_2, V_2)$$

satisfying $i \circ \Phi = \Theta \circ i_0$ and $\Theta \circ i_1 = id_{V_2}$. To do this, we will make use of the following functions:
Then, for any homotopy invariant presheaf with

Theorem 6.1.

Therefore, if we let

where \( pr^1_2 \): \( V_1 \times V_2 \to V_1 \) and \( pr^2_2 \): \( A^1 \times V_2 \times A^1 \to V_2 \) are the projections. Let moreover \( pr^2_2 \): \( V_2 \times V_2 \to V_2 \) denote the projection onto the second coordinate. Then the properties of \( f \) and \( h \) above that imply that

\[
\Theta' \circ i_0 = i \circ \Phi' ; \quad \Theta' \circ i_1 = ((y-x)g)^{dp,pr^2_2}_{Z(y-x)} + ((y-x)g)^{dp,pr^2_2}_{Z(y)}.
\]

Now, according to Lemma 5.21, the first summand in the last equality is equal to \( \langle \lambda \rangle \in \text{Cor}_k A(V_2, V_2) \) for some invertible function \( \lambda \). Moreover, the second summand is zero by Lemma 5.18. Therefore, if we let

\[
\Phi^+ := \Phi' \circ \langle \lambda^{-1} \rangle;
\]

\[
\Phi^- := \langle (y-x)g \rangle^{dp,pr^2_2}_{Z(y-x)} \circ \langle \lambda^{-1} \rangle;
\]

\[
\Phi := \Phi^+ - \Phi^-,
\]

it follows that

\[
[1dV]\mathbf{Z} \langle (y-x)g \rangle^{dp,pr^2_2}_{Z(y-x)} \circ \langle \lambda^{-1} \rangle = [i \circ \Phi] \in \text{Cor}_k A(V_2, V_2),
\]

as desired.

5.0.2. We will need the following two particular cases of Theorem 5.1:

Corollary 5.3. Suppose that \( \mathcal{F} \) is a homotopy invariant presheaf with A-transfers over a field \( k \). Then for any pair of open subschemes \( V_1 \subseteq V_2 \subseteq A^1_k \), the restriction homomorphism \( \mathcal{F}(V_2) \to \mathcal{F}(V_1) \) is injective.

Corollary 5.4. Suppose that \( \mathcal{F} \) is a homotopy invariant presheaf with A-transfers over a field \( k \). Then the restriction homomorphism \( \mathcal{F}(G_m^2) \to \mathcal{F}(G_m^2 \setminus \Delta_{G^2_{m^2}}) \) is injective, where \( \Delta_{G^2_{m^2}} \subseteq G_{m^2} \) denotes diagonal.

6. Excision on the relative affine line

The aim of this section is the prove the following excision result for open subsets of a relative affine line:

Theorem 6.1. Suppose that \( U \in \text{Sm}_k \) is an affine scheme or a local scheme, and let \( V_1 \subseteq V_2 \subseteq A^1_2 \) be a pair of open subschemes such that \( 0_U \in V_1 \). Let \( i : V_1 \subseteq V_2 \) denote the inclusion. Then, for any homotopy invariant presheaf with A-transfers \( \mathcal{F} \), the restriction homomorphism \( i^* \) induces an isomorphism

\[
i^* : \mathcal{F}(V_2 \setminus 0_U)/\mathcal{F}(V_2) \cong \mathcal{F}(V_1 \setminus 0_U)/\mathcal{F}(V_1).
\]
6.0.1. To prove the above theorem, we will show that \( i^* \) is injective and surjective, which amounts to constructing appropriate correspondences of pairs up to homotopy. Let us first show that \( i^* \) is injective:

**Lemma 6.2.** Suppose that \( V \subseteq A_U^1 \) is an open subscheme with \( 0_U \in V \). Then there is a finite \( A \)-correspondence of pairs

\[
\Phi \in \text{Cor}_k^A((A_U^1, A_U^1 \setminus 0_U), (V, V \setminus 0_U))
\]
such that \([i \circ \Phi] = [\text{id}_{(A_U^1, A_U^1 \setminus 0_U)}] \in \text{Cor}_k^A((A_U^1, A_U^1 \setminus 0_U), (A_U^1, A_U^1 \setminus 0_U))\).

**Proof.** We need to construct a finite \( A \)-correspondence

\[
\Phi \in \text{Cor}_k^A((A_U^1, A_U^1 \setminus 0_U), (V, V \setminus 0_U))
\]
along with a homotopy

\[
\Theta \in \text{Cor}_k^A(A^1 \times (A_U^1, A_U^1 \setminus 0_U), (A_U^1, A_U^1 \setminus 0_U))
\]
such that \( \Theta \circ i_0 = i \circ \Phi \) and \( \Theta \circ i_1 = \text{id}_{(A_U^1, A_U^1 \setminus 0_U)} \). To do this, we will make use of the following sections:

| \( s \in \Gamma \left( [t_0, t_\infty]_{U \times A^1}, O(n) \right) \) | \( \tilde{s} \in \Gamma \left( [t_0, t_\infty]_{U \times A^1 \times A^1}, O(n) \right) \) | \( s' \in \Gamma \left( [t_0, t_\infty]_{U \times A^1}, O(n-1) \right) \) |
|---|---|---|
| \( \tilde{s} \mid_{(P^1 \times U) \setminus V \times A^1} = t_0 \) | \( \tilde{s} \mid_{0 \times U \times A^1 \times A^1} = t_0 - t_{x_{\infty}} \) | \( \tilde{s} \mid_{0 \times U \times A^1} = t_0 - t_{x_{\infty}} \) |
| \( \tilde{s} \mid_{0 \times U \times 0} = s \) | \( \tilde{s} \mid_{\infty \times U \times A^1 \times A^1} = t_0 \) | \( \tilde{s} \mid_{\infty \times U \times A^1} = t_0 - t_{x_{\infty}} \) |
| \( \tilde{s} \mid_{0 \times U \times A^1} = t_0 - t_{x_{\infty}} \) | \( \tilde{s} \mid_{0 \times U \times A^1} = t_0 - t_{x_{\infty}} \) | \( \tilde{s} \mid_{Z(t_0 - t_{x_{\infty}}) \times U} = t_0 - t_{x_{\infty}} \) |

Since \( U \) is affine or local, it follows that \( O(1) \) is ample on \( P^1 \times U \times A^1 \) and \( P^1 \times U \times A^1 \times A^1 \). Hence, for \( n \) big enough, the Serre theorem ensures the existence of the sections \( s \) and \( s' \) as above. Having \( s \) and \( s' \), we then put \( \tilde{s} := (1 - \lambda)s + \lambda(t_0 - t_{x_{\infty}})s' \).

Now we define

\[
\Phi' := \langle s/t_{\infty} \rangle_{dy} \in \text{Cor}_k^A((A_U^1, A_U^1 \setminus 0_U), (V, V \setminus 0_U)),
\]

\[
\Theta' := \langle \tilde{s}/t_{\infty} \rangle_{dy} \in \text{Cor}_k^A(A_U^1 \times (A_U^1, A_U^1 \setminus 0_U), (A_U^1, A_U^1 \setminus 0_U)),
\]

where \( dy \) denotes the trivialization of the canonical class of the affine line \( A_U^1 \subseteq P^1 \) defined by the coordinate \( y := t_0/t_{\infty} \). Then it follows from the \( \tilde{s} \) and the properties of \( s \) and \( s' \) above that

\[
\Theta' \circ i_0 = i \circ \Phi' \quad \Theta' \circ i_1 = \langle (y - x)g \rangle_{Z(y-x)} + \langle (y - x)g \rangle_{Z(y)},
\]

where \( g := s'/t_{\infty} \in k[A^1 \times A^1 \times U] \). By Lemma \ref{3.21}, the first summand in the last equality is equal to \( \langle \lambda \rangle \) for some \( \lambda \in k[A_U^1]^x \). The second summand, \( \langle (y - x)g \rangle_{Z(y)} \), is zero by Lemma \ref{3.18} since \( Z(g) \cap (0 \times A^1 \times U) = \varnothing \). Now we define \( \Phi := \Phi' \circ \langle \lambda^{-1} \rangle \) and \( \Theta := \Theta' \circ \langle (\lambda^{-1}) \times \text{id}_{A_U^1} \rangle \). Then \( \Theta' \circ i_1 = \text{id}_{(A_U^1, A_U^1 \setminus 0_U)} \), and the claim follows.

6.0.2. The next step is to show surjectivity of \( i^* \):

**Lemma 6.3.** Suppose that \( V \subseteq A_U^1 \) is an open subscheme with \( 0_U \in V \). Then there is a finite \( A \)-correspondence of pairs

\[
\Psi \in \text{Cor}_k^A((A_U^1, A_U^1 \setminus 0_U), (V, V \setminus 0_U))
\]
such that \([\Psi \circ i] = [\text{id}_{(V, V \setminus 0_U)}] \in \text{Cor}_k^A((V, V \setminus 0_U), (V, V \setminus 0_U))\).
Proof. To prove the claim we need to construct a finite $A$-correspondence

$$\Psi \in \text{Cor}^n_A((A^1_v, A^1_V \setminus 0_U), (V, V \setminus 0_U))$$

along with a homotopy

$$\Theta \in \text{Cor}^0_A(A^1 \times (V, V \setminus 0_U), (V, V \setminus 0_U))$$

such that $\Theta \circ i_0 = \Psi \circ i$ and $\Theta \circ i_1 = \text{id}_{(V, V \setminus 0_U)}$. We do this via the following sections:

| $s$ | $s'$ |
|-----|------|
| $\tilde{s} |_{(P^1 \times U) \setminus V \times A^{1}_v} = \frac{t_{0}^n}{s}$ | $\tilde{s} |_{0 \times U \times A^{1}_v} = t_{0} - xt_{\infty}$ |
| $\tilde{s} |_{P^1 \times V \times 0} = f$ | $\tilde{s} |_{(A^1 \times V) \times V \times A^{1}_v} = \frac{t_{0}^n}{s}$ |
| $\tilde{s} |_{0 \times V \times A^{1}_v} = t_{0} - xt_{\infty}$ | $g |_{(A^{1}_v \times V) \times A^{1}_v} = \frac{t_{0}^n}{s}$ |
| $\tilde{s} |_{Z(t_{0} - xt_{\infty}) \times U} = f^n - 1$ | $s' |_{0 \times V} = f^n$ |

Here $g := s'/t_{\infty}^{n-1} \in k[A^1 \times V]$. Since $U$ is affine or local, it follows that $O(1)$ is ample on $P^1 \times A^1 \times U$ and $P^1 \times A^1 \times U \times A^1$. Hence the Serre theorem ensures the existence of the sections $s$ and $s'$ as above, provided $n$ is big enough. Having $s$ and $s'$, we then put $\tilde{s} := (1 - \lambda)s + \lambda(t_0 - xt_\infty)s'$. Next, define

$$\Psi' := (s/t_{\infty}^n)_{dy} \in \text{Cor}^n_A((A^1_v, A^1_V \setminus 0_U), (V, V \setminus 0_U)),$$

$$\Theta' := (\tilde{s}/t_{\infty}^n)_{dy} \in \text{Cor}^0_A(A^1 \times (A^1_v, A^1_V \setminus 0_U), (A^1_v, A^1_V \setminus 0_U)),$$

where $dy$ denotes the trivialization of the canonical class of the affine line $A^1 \subseteq P^1$ defined by the coordinate $y := t_0/t_{\infty}$. Then the properties of $s$ and $s'$ above imply that

$$\Theta' \circ i_0 = i \circ \Psi'; \quad \Theta' \circ i_1 = ((y - x)g)z(y - x) + ((y - x)g)z(y).$$

By Lemma 6.21 the first summand in the last equality is equal to

$$(\lambda) \in \text{Cor}^0_A((V, V \setminus 0_U), (V, V \setminus 0_U))$$

for some $\lambda \in k[V]^\times$. The second summand is zero by Lemma 6.18 since $Z(g) \cap (0 \times V) = \emptyset$. Hence the $A$-correspondences $\Psi := (\lambda^{-1}) \circ \Psi'$ and $\Theta := (\lambda^{-1}) \circ \Theta'$ have the desired properties. □

Proof of Theorem 6.1. Lemma 6.2 and Lemma 6.3 immediately imply the claim for the case of $V_2 = A^1_v$. In general, it follows that we have natural isomorphisms

$$\mathcal{F}(V_2 \setminus 0_U)/\mathcal{F}(V_2) \cong \mathcal{F}(A^1_v \setminus 0_U)/\mathcal{F}(A^1_v) \cong \mathcal{F}(V_1 \setminus 0_U)/\mathcal{F}(V_1),$$

which shows the claim. □

6.0.3. Arguing similarly as in the proof of Theorem 6.1 we obtain also an excision result for a nonrelativized affine line:

**Theorem 6.4.** Let $U$ be as in Theorem 6.1, and write $K := k(U)$. Let $z$ be a closed point in $A^1_K$, and let $i : V_1 \subseteq V_2$ be an inclusion of two open subschemes of $A^1_K$ such that $z \in V_1$. Then, for any homotopy invariant presheaf with $A$-transfers $\mathcal{F}$, the restriction homomorphism $i^*$ induces an isomorphism

$$i^* : \mathcal{F}(V_2 \setminus z)/\mathcal{F}(V_2) \xrightarrow{\cong} \mathcal{F}(V_1 \setminus z)/\mathcal{F}(V_1).$$

Proof. The proof is parallel to the proof of Theorem 6.1. All we need to do is to replace the line bundle $O(1)$ by $O(d)$, where $d := \text{deg}_{K_1} k(z)$; the section $t_0 \in \Gamma(P^1_{A^1_K}, O(1))$ by a section $\nu \in \Gamma(P^1_{A^1_K}, O(d))$ such that $Z(\nu) = z \times A^1_K$; and the section $t_\infty$ by $t_\infty^d$. □
7. Injectivity for semi-local schemes

In this section we will assume that the base field \( k \) is infinite.

**Theorem 7.1.** Let \( X \) be a smooth \( k \)-scheme and let \( x, \ldots, x_r \in X \) be finitely many closed points. Let \( U := \text{Spec} \, \mathcal{O}_{X,x_1, \ldots, x_r} \) and write \( j: U \to X \) for the canonical inclusion. Let \( Z \hookrightarrow X \) be a closed subscheme with \( x_1, \ldots, x_r \in Z \), and let \( i: X \setminus Z \to X \) be the open complement. Then, for any homotopy invariant presheaf with transfers \( F \), the homomorphism \( i^*: F(X) \to F(X \setminus Z) \) is injective.

7.0.1. Theorem 7.1 is an immediate consequence of the following moving lemma:

**Lemma 7.2.** Assume the hypotheses of Theorem 7.1. Then there exists a finite \( A \)-correspondence \( \Phi \in \text{Cor}_A^k(U, X \setminus Z) \) such that the diagram

\[
\begin{array}{ccc}
X \setminus Z & \xrightarrow{\Phi} & U \\
\downarrow & & \downarrow \\
U & \xrightarrow{j} & X
\end{array}
\]

commutes up to homotopy.

7.0.2. We prove Lemma 7.2 by constructing an appropriate relative curve \( C \) over \( U \) along with a good compactification \( \overline{C} \) of \( C \). The desired finite \( A \)-correspondence will then be defined by using certain sections on \( \overline{C} \).

**Lemma 7.3.** Assume the hypotheses of Theorem 7.1. Then there exists a diagram

\[
\begin{array}{ccc}
X & \xleftarrow{v} & C \\
\downarrow & & \downarrow \\
& \xrightarrow{\Delta} & U
\end{array}
\]

in \( \text{Sm}_k \), satisfying the following properties:

1. \( p: \overline{C} \to U \) is a relative projective curve, \( j: C \to \overline{C} \) is an open immersion, and the composition \( p \circ j \) is smooth.
2. The map \( p \circ j \) admits a section \( \Delta: U \to C \). By abuse of notation, we write \( \Delta \) also for the image of the morphism \( \Delta \).
3. If \( \mathcal{Z} := v^{-1}(Z) \subseteq C \), then \( \mathcal{Z} \) is finite over \( U \).
4. \( D := \overline{C} \setminus C \) is finite over \( U \).
5. The relative curve \( \overline{C} \) has an ample line bundle \( \mathcal{O}_{\overline{C}}(1) \).
6. There is a trivialization \( \mu: \mathcal{O}_C \xrightarrow{\cong} \omega_{\overline{C}/U} \).

**Proof.** We apply Lemma 8.2 with \( \pi = \text{id} : X \to X \). \( \square \)

**Proof of Lemma 7.2.** First of all we apply Lemma 7.3. Then it follows from Serre’s theorem that there is an integer \( l \gg 0 \) and a section \( d \in \Gamma(\overline{C}, \mathcal{O}(l)) \) such that \( D \subseteq \mathcal{Z}(d) \) and \( \mathcal{Z}(d) \) is finite over \( U \). For notational simplicity, let us redenote \( \mathcal{O}(l) \) by \( \mathcal{O}(1) \), and redenote \( D := \mathcal{Z}(d) \). Now our aim is to construct the following sections:

| \( s \in \Gamma(\overline{C}, \mathcal{O}(n)) \) | \( \tilde{s} \in \Gamma(\overline{C} \times A^1, \mathcal{O}(n)) \) | \( s' \in \Gamma(\overline{C}, \mathcal{O}(n) \otimes \mathcal{L}(\Delta)^{-1}) \) | \( \delta \in \Gamma(\overline{C}, \mathcal{L}(\Delta)) \) |
|---------------------------------|---------------------------------|---------------------------------|---------------------------------|
| \( Z(s|_{\mathcal{Z} \cup D}) = \emptyset \) | \( \tilde{s}|_{\mathcal{C} \times 0} = s \) | \( \tilde{s}'|_{\mathcal{C} \times 1} = s', \delta \) | \( \tilde{s}|_{D \times A^1} = s \) |
| \( Z(\delta|_{\mathcal{Z} \cup D \cup \Delta}) = \emptyset \) | \( Z(s'|_{\mathcal{Z} \cup D \cup \Delta}) = \emptyset \) | \( Z(\delta) = \Delta \) |
To do this, let δ be the zero section of \( \mathcal{L}(\Delta) \) and choose an integer \( n \gg 0 \) such that the restriction maps

\[
\begin{align*}
\Gamma(\mathcal{C}, \mathcal{O}(n) \otimes \mathcal{L}(\Delta)^{-1}) & \to \Gamma(\mathcal{Z} \amalg D \amalg \Delta, \mathcal{O}(n) \otimes \mathcal{L}(\Delta)^{-1}), \\
\Gamma(\mathcal{C}, \mathcal{O}(n)) & \to \Gamma(\mathcal{Z} \amalg D, \mathcal{O}(n))
\end{align*}
\]

are surjective. We can then find a global section \( s' \) of \( \mathcal{O}(n) \otimes \mathcal{L}(\Delta)^{-1} \) such that \( s'|_{\mathcal{Z} \amalg D \amalg \Delta} \) is invertible. Let \( s \) be a lift of \( s'|_{\mathcal{Z} \amalg D} \in \Gamma(\mathcal{Z} \amalg D, \mathcal{O}(n)) \), and define \( \delta := (1 - \lambda)s + \lambda s'\delta \). We can then put

\[
\Phi' := (s/d^n)|_{\mathcal{Z}(s)}; \quad \Theta' = (\delta/d^n)|_{\mathcal{Z}(\delta)}.
\]

Then the properties of the sections above imply that \( \Theta' \circ i_0 = i \circ \Phi' \), and Lemma 3.21 implies that \( \Theta' \circ i_1 = j \circ (\lambda) \) for some \( \lambda \in k[U]^\times \). Let now \( \Phi := \Phi' \circ (\lambda^{-1}) \). Then \( \Theta := \Theta' \circ (\lambda^{-1}) \) gives the required homotopy, satisfying \( \Theta \circ i_0 = i \circ \Phi \) and \( j = \Theta \circ i_1 \).

\[\square\]

### 8. Étale excision

In this section we assume that the base field is infinite. The main result of the section is the following étale excision result for homotopy invariant presheaves with A-transfers:

**Theorem 8.1.** Let \( X \in \text{Sm}_k \) and suppose that \( \pi: (X', Z') \to (X, Z) \) is an étale neighborhood of \( Z \) in \( X \). Assume also that \( z \in Z \) and \( z' \in Z' \) are two closed points such that \( \pi(z') = z \). Write \( U := X_z = \text{Spec} \mathcal{O}_{X_z} \) for the corresponding local scheme, and similarly \( U' := X'_{z'} \). Then, for any homotopy invariant presheaf with A-transfers \( \mathcal{F} \), the map \( \pi^* \) induces an isomorphism

\[
\pi^*: \mathcal{F}(X_z \setminus Z_z)/\mathcal{F}(X_z) \xrightarrow{\cong} \mathcal{F}(X'_{z'} \setminus Z'_{z'})/\mathcal{F}(X'_{z'}).
\]

8.0.1. Before proving Theorem 8.1 we need some preparations. In particular, we need to construct appropriate relative curves over \( X \) and \( X' \).

**Lemma 8.2.** Let \( X \) be a smooth \( k \)-scheme and let \( Z \subseteq X \) be a closed subscheme. Suppose that \( \pi: (X', Z') \to (X, Z) \) is an étale neighborhood of \( Z \) in \( X \). Let moreover \( z \in Z \) and \( z' \in Z' \) be closed points such that \( \pi(z') = z \), and write \( U := X_z \) and \( U' := X'_{z'} \) for the corresponding local schemes. Then there is a commutative diagram

\[
\begin{array}{cccccc}
U' & \xleftarrow{p''} & \mathcal{C}'' & \xrightarrow{j''} & X' \\
\downarrow & & \downarrow & & \downarrow \\
U & \xleftarrow{p'} & \mathcal{C}' & \xrightarrow{j'} & X' \\
\downarrow & & \downarrow & & \downarrow \\
U & \xleftarrow{p} & \mathcal{C} & \xrightarrow{j} & X
\end{array}
\]

in \( \text{Sm}_k \), such that the following properties hold:

1. \( p, p', p'' \) are relative projective curves; \( j, j', j'' \) are open immersions; \( \pi, \pi' \) are étale; \( \mathcal{C}, \mathcal{C}' \) are finite; and \( p \circ j, p' \circ j' \) are smooth. Moreover, \( \mathcal{C}'' = \mathcal{C}' \times_U U' \); and there are trivializations of the relative canonical classes \( \mu: \mathcal{O}_C \cong \omega_{C/U} \) and \( \mu': \mathcal{O}_{C'} \cong \omega_{C'/U} \).

2. The schemes \( Z := v^{-1}(Z) \), \( Z' := v'^{-1}(Z') \) and \( Z'' := v''^{-1}(Z') \) are finite over \( U \) and \( U' \), respectively.
(3) There are closed subschemes \( \Delta_{Z} \subseteq Z \), \( \Delta'_{Z} \subseteq Z' \) and \( \Delta''_{Z} \subseteq Z'' \) such that \( p, p' \) and \( p'' \) induce isomorphisms \( w: \Delta_{Z} \cong Z \times X U, w': \Delta'_{Z} \cong Z \times X U \) and \( w'': \Delta''_{Z} \cong Z' \times X' U' \). Moreover, \( v|_{Z} \circ w = \text{pr}_{Z}^{Z \times X U}, v'|_{Z} \circ w' = \pi_{Z} \circ \text{pr}_{Z}^{Z \times X U} \), and \( v''|_{Z} \circ w'' = \text{pr}_{Z}^{Z' \times X' U'} \).

(4) There are closed subschemes \( \Delta \subseteq C \) and \( \Delta' \subseteq C' \) such that \( p \) and \( p'' \) induce isomorphisms \( p|_{\Delta}: \Delta \cong U \) and \( p|_{\Delta'}: \Delta' \cong U' \). Moreover, the compositions \( v \circ p|_{\Delta} \) and \( v \circ p|_{\Delta'} \) are equal to the canonical morphisms \( U \to X \) and \( U' \to X' \), respectively.

(5) The schemes \( D := C \setminus C, D' := C' \setminus C' \) and \( D'' := C'' \setminus C'' \) are finite over \( U \) and \( U' \) respectively. Furthermore, \( D'' \cong \overline{\pi^{-1}(D)} \), and \( D' \cong \overline{\pi^{-1}(D)} \).

(6) There is an ample line bundle \( O(1) \) on \( \overline{\mathcal{C}} \) and a section \( d \in \Gamma(\overline{\mathcal{C}}, O(1)) \) such that \( Z(d) = D \).

Proof. We use \([Dru18c, \text{Lemma 3.7}]\) to deduce the claim. Define relative curves \( C := \mathcal{X} \times S U, C' := \mathcal{X}' \times S U \) and \( C'' := \mathcal{X}'' \times S U' \), where \( \mathcal{X}, \mathcal{X}', \mathcal{X}'' \) and \( S \) are given by \([Dru18c, \text{Lemma 3.7}]\). By applying base change along \( U' \to S \) and \( U' \to S \) we can define the data of (1), (2), (5), (6).

The closed subschemes of (3) and (4) are then defined as the graphs of the morphisms \( U \to X, U' \to X' \), \( Z \times X U \to X, Z' \times X' U' \to X' \) and the composition \( Z \times X U \cong Z' \times X' U' \to X' \). Then all properties except (4) and (5) are satisfied.

Nevertheless we have a line bundle \( O(1) \) on \( \overline{\mathcal{C}} \) which is the pullback of the ample bundle \( O(1) \) on \( \overline{\mathcal{C}} \). As \( U \to S \) is a morphism of local schemes, it follows that \( O(1) \) is ample. Serre’s theorem then tells us that, for some \( n > 0 \), we can find a section \( d_{\text{new}} \in \Gamma(\overline{\mathcal{C}}, O_{\overline{\mathcal{C}}}(n)) \) satisfying \( Z(d_{\text{new}}) \cap Z = \emptyset \) and \( Z(d_{\text{new}}) \supset D \). Now define \( D_{\text{new}} := Z(d_{\text{new}}), D'_{\text{new}} := \overline{\pi^{-1}(D_{\text{new}})} \cup D' \) and \( D''_{\text{new}} := \overline{\pi^{-1}(D'_{\text{new}})} \). Replacing \( d, D, D' \) and \( D'' \) with \( d_{\text{new}}, D_{\text{new}}, D'_{\text{new}} \) and \( D''_{\text{new}} \), it follows that all conditions are satisfied.

\( \square \)

8.0.2. The proof of Theorem 8.1 now relies on the following lemmas.

**Lemma 8.3.** Under the assumptions of Theorem 8.1 there is a finite \( \Lambda \)-correspondence \( \Phi \in \text{Cor}_{\Lambda}(U, X') \) satisfying \( \pi \circ \Phi \sim_{\Lambda, i} \), where \( i: U \to X \) denotes canonical embedding.

**Proof.** Applying Lemma 8.2 we obtain relative curves \( C \) and \( C' \) over \( U \) and \( U' \), respectively. Our first aim is to prove that there is an integer \( N \) such that for all \( n \geq N \), there exists the following set of data: a line bundle \( \mathcal{L} \) on \( \overline{\mathcal{C}} \); a section \( d \in \Gamma(\overline{\mathcal{C}}, \mathcal{L} \otimes O(n)) \); and moreover sections satisfying the following conditions:

\[
\begin{align*}
\text{for } s &\in \Gamma(\overline{\mathcal{C}}, O(n)) \quad \delta s &\in \Gamma(\overline{\mathcal{C}} \times \mathcal{A}^{1}, O(n)) \quad s' &\in \Gamma(\overline{\mathcal{C}}, O(n) \otimes \mathcal{L}^{-1}(\Delta))
\end{align*}
\]

\[
\begin{align*}
\text{for } s |_{\overline{\mathcal{C}}} &\neq \emptyset \\
\text{for } s |_{\overline{\mathcal{C}}} &\neq \emptyset \\
\text{for } s |_{\overline{\mathcal{C}}} &\neq \emptyset \\
\text{for } s |_{\overline{\mathcal{C}}} &\neq \emptyset
\end{align*}
\]

In addition, we will require that \( Z(s) = Z_0 \cup Z_0' \) and that there exists a regular map \( l: Z_0 \to C' \) satisfying \( \varphi \circ l = \text{id}_{Z_0} \).

To do this, first apply \([Dru18c, \text{Lemma 4.1}]\) to construct, for some \( m \in \mathbf{Z} \), a section \( \xi \in \Gamma(C', O_{\mathcal{C}'}(m)) \) such that \( Z(\xi) \to \mathcal{C} \) is a closed embedding and \( Z(\xi|_{\overline{\mathcal{A}}^{-1}(\Delta)}) = \Delta_{Z} \). Define \( Z_0 := \overline{\pi^{-1}(Z(\xi))} \subseteq C \subseteq \overline{\mathcal{C}} \) and put \( \mathcal{L} := \mathcal{L}(Z) \). Let \( \zeta \in \Gamma(\overline{\mathcal{C}}, \mathcal{L}) \) be a section with \( Z(\zeta) = Z \). Then \( Z(\zeta|_{\overline{\mathcal{C}}}) = \Delta_{Z} \).
Choose \( N \in \mathbb{Z} \) such that for all \( n \geq N \), the restriction homomorphisms
\[
\Gamma(\mathcal{C}, \mathcal{O}(n) \otimes \mathcal{L}^{-1}) \to \Gamma(Z \amalg D, \mathcal{O}(n) \otimes \mathcal{L}^{-1})
\]
\[
\Gamma(\mathcal{C}, \mathcal{O}(n)) \to \Gamma((Z \cup \Delta) \amalg D, \mathcal{O}(n))
\]
are surjective. Then, since \( Z \amalg D \) is semi-local, there is a section \( \zeta' \in \Gamma(\mathcal{C}, \mathcal{O}(n) \otimes \mathcal{L}^{-1}) \) such that \( \zeta'|_{Z \amalg D} \) is invertible. Define \( s := \zeta' \in \Gamma(\mathcal{C}, \mathcal{O}(n)) \).

Now choose a section \( s_1 \in \Gamma(\mathcal{C}, \mathcal{O}(n)) \) such that \( s_1|_{\Delta} = 0 \) and \( s_1|_{Z} = s \). We then put \( \delta := (1 - \lambda)s + \lambda s_1 \). Since \( s_1|_{\Delta} = 0 \), there is a section \( s' \in \Gamma(\mathcal{C}, \mathcal{O}(n) \otimes \mathcal{L}(\Delta)^{-1}) \) such that \( s_1 = \delta s' \), where \( \delta \in \Gamma(\mathcal{C}, \mathcal{L}(\Delta)) \) satisfies \( Z(\delta) = \Delta \). Moreover, since by construction \( Z(s_1|_{Z}) = \Delta_Z = Z(\delta|_{Z}) \), it follows that \( s'|_{Z} \) is invertible and so \( Z(s'|_{Z}) = \emptyset \). Hence the desired sections \( s \), \( s \), and \( s' \) are constructed.

By construction, the morphism \( \varpi \) induces an isomorphism between the closed subschemes \( l(Z_0) \subseteq C' \) and \( Z_0 \). Since \( \varpi \) is étale, it follows that \( \varpi^{-1}(Z_0) = l(Z_0) \amalg \bar{Z}_0 \). Hence we can define an étale neighborhood \( \varpi^+: (C' \setminus \bar{Z}_0, l(Z_0)) \to (C, Z_0) \) such that \( \varpi^+(Z_0) = l(Z_0) \).

Applying Construction 3.13 we define
\[
\Phi' := (\varpi^*(s/d^m))|_{Z_0} = \varphi|_{(U, U \setminus Z \times X U)}, (X', X' \setminus Z')
\]
and
\[
\Theta' := \langle \delta/d \rangle^{\text{oprt}} = \varphi|_{(U, U \setminus Z \times X U)}, (X, X \setminus Z),
\]
where \( \text{pr}: C \times A^1 \to C \) is the projection. It follows from the list of properties above, Lemma 8.21 and Lemma 8.18 that \( \Theta' \circ i_1 = i \circ \langle \lambda \rangle \) for some invertible function \( \lambda \in k[U]^{\times} \). If we let \( \Phi := \Phi' \circ \langle \lambda^{-1} \rangle \) and \( \Theta := \Theta' \circ \langle \lambda^{-1} \rangle \), it follows that \( \Theta \circ i_1 = i \). So to prove the lemma it is enough to show that \( \Theta \circ i_0 = \pi \circ \Phi \).

Since \( \varpi \) is finite, it is affine. Hence for some Zariski neighborhood \( V' \) of \( l(Z_0) \) in \( C' \setminus \bar{Z}_0 \), the restriction \( \varpi|_{V'} \) is affine. Then, for some Zariski neighborhood \( V \) of \( Z_0 \) in \( C \), there is a closed embedding \( c: V'' \subseteq A^r \times V \), where \( V'' := V' \cap \varpi^{-1}(V) \), and such that \( c(l(Z_0)) = 0 \times Z_0 \). Let \( f_1, \ldots, f_r \in k[A^r \times V] \) be functions satisfying \( f_i|_{c(V')} = 0 \) and \( f_i|_{A^r \times Z_0} = x_i \), where the \( x_i \)'s denote the coordinate functions on \( A^r \). For \( i = 1, \ldots, r \), let \( \tilde{f}_i := (1 - \lambda)f_i + \lambda x_i \) and consider the closed subscheme \( Z(\tilde{f}_1, \ldots, \tilde{f}_r) \subseteq A^r \times V \times A^1 \). Then the projection \( \text{pr}: Z(\tilde{f}_1, \ldots, \tilde{f}_r) \to V \times A^1 \) is étale over \( Z_0 \times A^1 \). Let \( W \subseteq Z(\tilde{f}_1, \ldots, \tilde{f}_r) \) be a Zariski neighborhood of \( 0 \times Z_0 \times A^1 \) such that the restriction of the projection \( \text{pr}_W: W \to V \times A^1 \) is étale. Furthermore, let \( \iota \) be the pullback of \( s/d^m \) from \( V \) to \( W \), and let \( i_0: V \to C \) denote the open embedding. Then
\[
\langle t \rangle^{\text{pr}_W}_{|_{0 \times Z_0 \times A^1}} \in \varphi|_{(U, U \setminus Z \times X U)}, (X, X \setminus Z)
\]
is the homotopy connecting \( \iota \circ \Phi' = (\varpi^*(s/d^m))|_{Z \times X U}, (X, X \setminus Z) \)
and \( \Theta \circ i_0 = (s/d^m)^{\text{oprt}} \).

**Lemma 8.4.** Suppose that \( \text{char} k \neq 2 \), and let \( X \in \text{Sm}_k \). Let \( Z \subseteq X \) be a closed subscheme and \( z \in X \) a closed point. Write \( U \) for the essentially smooth local scheme \( U := X^\circ_{\nu} = \text{Spec} \mathcal{O}^\nu_{X, z} \), and let \( \lambda \in k[U]^{\times} \) be an invertible regular function satisfying \( \lambda|_{Z \times X U} = 1 \). Then
\[
i \circ \langle \lambda \rangle \sim_A; i \in \varphi|_{(U, U \setminus Z \times X U)}, (X, X \setminus Z),
\]
where \( i \) denotes the canonical morphism \( i: U \to X \).

**Proof.** Lift \( \lambda \) to an invertible section on some affine Zariski neighborhood \( V \subseteq X \) of the point \( z \in X \). Then \( \lambda|_{Z \times X V} = 1 \) for some other Zariski neighborhood \( V' \subseteq V \) of \( z \); shrinking \( X \) to \( V' \) we may assume that \( \lambda \in k[X]^{\times} \) with \( \lambda|_{Z} = 1 \).
Consider the étale covering \( \pi : X' \to X \), where \( X' = \text{Spec } k[X]/(w^2 - \lambda) \). Let \( Z' \) be the closed subscheme of \( X' \) given by \( Z' := \text{Spec } k[Z]/(w - 1) \), so that \( Z' \cong Z \). Then \( (X', Z') \to (X, Z) \) is an étale neighborhood. By Lemma 5.3 there exists a correspondence of pairs

\[
\Phi \in \text{Cor}_k^A((U, U \setminus Z \times X U), (X', X' \setminus Z'))
\]

such that \( \pi \circ \Phi \sim_{\mathbf{A}^1} i \) in \( \text{Cor}_k^A((U, U \setminus Z \times X U), (X, X \setminus Z)) \). On other hand, Lemma 5.22 implies that

\[
\langle \lambda \circ \pi = \pi \circ (w^2) \rangle = \pi \circ (w^2) \sim_{\mathbf{A}^1} \pi \in \text{Cor}_k^A((X, X \setminus Z), (X, X \setminus Z)).
\]

Hence \( i \circ (\pi') = \langle \lambda \circ i \sim_{\mathbf{A}^1} \lambda \circ \pi \sim_{\mathbf{A}^1} \pi \circ \Phi \sim_{\mathbf{A}^1} i \). \( \square \)

**Lemma 8.5.** Let \( i' : U' = X'_{\nu} \to X' \) denote the canonical embedding. Then under the assumptions of Theorem 8.1 there exists \( \Phi \in \text{Cor}_k^A((U, X') \to (X', X')) \) such that \( \Phi \circ \pi \sim_{\mathbf{A}^1} i' \).

**Proof.** Using Lemma 8.2 we construct relative projective curves \( p' : \mathcal{C}' \to U, p'' : \mathcal{C}'' \to U' \), along with the other data related to the first two rows of the diagram 8.1.

Since \( U' \) is essentially smooth, \( \Delta'' \cong U' \). Moreover, since \( p' : \mathcal{C}' \to U' \) is a smooth morphism with fibers of dimension one, it follows that \( \Delta'' \) is a smooth divisor on \( \mathcal{C}'' \). Hence it is a smooth divisor on \( \mathcal{C}' \) as well and there is an invertible bundle \( \mathcal{L}(\Delta'') \) on \( \mathcal{C}' \) and a section \( \delta \in \Gamma(\mathcal{C}', \mathcal{L}(\Delta'')) \) such that \( Z(\delta') = \Delta'' \).

Since \( Z' \) is finite over the local scheme \( U', Z' \) is semi-local. Let \( \delta' \in k[Z'] \) be a regular function such that \( \delta'|_{\Delta'_Z} = 0 \), and such that \( \delta' \) is invertible on the closed points of \( Z' \) outside \( \Delta_Z \). Then the closed fibers of \( Z(\delta') \) and \( \Delta_Z \) coincide. Now \( Z(\delta') \) is finite over \( U \) since it is a closed subset in \( Z' \). Moreover, \( \Delta_Z \) is finite over \( U \) since \( \Delta_Z \) is isomorphic to the closed subscheme \( U \setminus X \times X Z \) in \( U \). Hence \( Z(\delta') = \Delta_Z \) by Nakayama’s lemma.

Define \( \mathcal{O}_{\mathcal{C}'}(1) := \mathcal{O}_{\mathcal{C}'}(\mathcal{O}(1)) \) and \( \mathcal{O}_{\mathcal{C}''}(1) := \mathcal{O}_{\mathcal{C}''}(\mathcal{O}(1)) \) (see Lemma 8.2). Then, since \( \mathcal{O}(1) \) is ample and \( \mathcal{O}_{\mathcal{C}'}(1) \) are finite, it follows that \( \mathcal{O}_{\mathcal{C}'}(1) \) and \( \mathcal{O}_{\mathcal{C}''}(1) \) are ample. The Serre theorem then tells us that there is an integer \( n \in \mathbf{Z} \) such that the restriction homomorphisms

\[
\begin{align*}
\Gamma(\mathcal{C}', \mathcal{O}(n)) &\to \Gamma(\mathcal{Z}'' \amalg D'', \mathcal{O}(n) \otimes \mathcal{L}(\Delta'')) ; \quad (8.2) \quad \\
\Gamma(\mathcal{C}'', \mathcal{O}(n) \otimes \mathcal{L}(\Delta'')) &\to \Gamma(\mathcal{Z}'' \amalg D'', \mathcal{O}(n) \otimes \mathcal{L}(\Delta'')) ; \quad (8.3)
\end{align*}
\]

are surjective. As mentioned above, \( Z \) and \( D \) are finite over \( U \), so it follows that \( \mathcal{C}' \) and \( \mathcal{C}'' \) are semi-local, and moreover that there are trivializations \( \xi'_Z : \mathcal{O}_{\mathcal{C}'} \cong \mathcal{O}_{\mathcal{C}''}(1)|_{Z'} \) and \( \xi_D : \mathcal{O}_{\mathcal{C}''} \cong \mathcal{O}_{\mathcal{C}''}(1)|_{D'} \). Now using surjectivity of the map 8.2 we find a section

\[
s \in \Gamma(\mathcal{C}', \mathcal{O}(n)), \quad s|_{Z'} = \delta \otimes \xi^\otimes_{Z'}, \quad s|_{D'} = \xi^\otimes_{D'}.
\]

By the same reason as above there is some trivialization \( \xi'_Z : \mathcal{O}_{\mathcal{C}''} \cong \mathcal{L}(\Delta'')|_{Z''} \). Then \( b_1 = \mathcal{O}_{\mathcal{C}'}(\delta') \) and \( b_2 = \delta \otimes \xi^{-1}_{Z''} \) are two regular functions on \( Z'' \) such that \( Z(b_1) = Z(b_2) = \Delta_Z'' \).

Hence there is an invertible function \( \nu \in k[Z'']^* \) such that

\[
\mathcal{O}(\nu) = \delta \otimes \xi^{-1}_{Z''}.
\]

Indeed, \( \nu \) is uniquely defined by the equality \( b_1 \nu = b_2 \) on the closed subscheme \( Z(I) \subseteq Z'' \).

Here \( I := \ker(m^{b_1}) \), where \( m^{b_1} \in \text{End}(k[Z'']) \) is defined by multiplication by \( b_1 \). Moreover, the equality \( b_1 \nu = b_2 \) implies that \( \nu \) is invertible on \( Z(I) \), and any lift of \( \nu \) to a regular function on \( Z'' \) satisfies the equality \( b_1 \nu = b_2 \) as well. So it is enough to choose a lift such that \( \nu \) is nonzero at the closed points of \( Z'' \setminus Z(I) \).

Using surjectivity of the second map 8.3, we find a section

\[
s' \in \Gamma(\mathcal{C}'', \mathcal{O}(n) \otimes \mathcal{L}(\Delta'')^{-1}), \quad s'|_{Z''} = \mathcal{O}_{\mathcal{C}''}(\nu), \quad s'|_{D''} = \mathcal{O}_{\mathcal{C}''}(\nu) \otimes \delta|_{D''}^{-1}.
\]
Note that the section \( \delta|_{D''} \) is well defined since \( \Delta'' \cap D'' = \emptyset \).

Now define \( \tilde{s} := (1 - \lambda)s + \lambda s' \). Then we have:

| \( s \in \Gamma(C, O(n)) \) | \( \tilde{s} \in \Gamma(C \times A^I, O(n)) \) | \( s' \in \Gamma(C, O(n)) \otimes \mathcal{L}(\Delta')^{-1} \) |
|---|---|---|
| \( Z(s|_{D''}) = \emptyset \) | \( \tilde{s}|_{C \times 0} = \varphi^\prime(s) \) | \( \tilde{s}|_{C \times 1} = \delta s' \) |
| \( Z(\tilde{s}|_{D''}) = \varphi^\prime(\varphi^\prime(s)) \) | \( s'|_{Z \times Z} = \delta' s' \) | \( \tilde{s}|_{Z \times A^I} = \delta s' \) |

Applying Construction 3.13 we let

\[
\Phi^\prime := (s/d^n)^{\pi^\prime(n), v'} \in \text{Cor}^\Lambda_k(U, X'),
\]

\[
\Theta^\prime := (\tilde{s}/d^n)^{\pi^\prime(n), v''} \in \text{Cor}^\Lambda_k(U' \times A^I, X').
\]

Then, by construction,

\[
\Theta^\prime \circ i_0 = \Phi^\prime \circ \pi; \quad \Theta^\prime \circ i_1 = (s/d^n)^{\pi^\prime(n), v''} + (\tilde{s}/d^n)^{\pi^\prime(n), v''}.
\]

By Lemma 3.13 we have

\[
(s/d^n)^{\pi^\prime(n), v''} = 0 \in \text{Cor}^\Lambda_k((U', U'' \setminus Z' \times X, U'), (X', X'' \setminus Z')).
\]

Furthermore, Lemma 3.21 tells us that \( (s/d^n)^{\pi^\prime(n), v''} = i' \circ (\lambda') \) for some \( \lambda' \in k[U']^\times \). Let

\[
\nu \in k[U]_+^{\times} \text{ be an invertible function on } U \text{ satisfying } \pi^*(\nu)(z) = \lambda'(z)^{-1}.
\]

Define \( \Phi := \Phi^\prime \circ \nu \) and \( \Theta := \Theta^\prime \circ (\pi^*(\nu)) \). Then \( \Theta \circ i_1 = i' \circ (\lambda' \cdot \pi^*(\nu)) \) and so Lemma 3.13 yields the claim.

**Proof of Theorem 8.1.** Lemmas 8.3 and 8.5 establish respectively injectivity and surjectivity of the map \( \pi^* \). \( \square \)

**Corollary 8.6.** Suppose that \( \Lambda^* \) is a graded presheaf of abelian groups that satisfies all properties of a good cohomology theory except the \( \acute{e}tale \) excision axiom. Instead, assume that \( \Lambda^* \) satisfies Zariski excision and homotopy invariance. In other words, for any \( X \in \text{Sm}_k \), any line bundle \( \mathcal{L} \) on \( X \), and for any open subscheme \( j: U \subseteq X \) and any closed subscheme \( Z \subseteq X \) such that \( Z \subseteq U \), the maps

\[
\text{pr}^*: \Lambda^a(X, \mathcal{L}) \xrightarrow{\simeq} \Lambda^a(X \times A^I, \text{pr}^* \mathcal{L}),
\]

\[
j^*: \Lambda^a(X, X \setminus Z, \mathcal{L}) \xrightarrow{\simeq} \Lambda^a(U, U \setminus Z, j^* \mathcal{L})
\]

are isomorphisms.

Then \( \Lambda^* \) satisfies the \( \acute{e}tale \) excision axiom on local schemes, i.e., for any \( X \in \text{Sm}_k \), \( Z \subseteq X \), \( \pi: (X', Z') \to (X, Z) \), \( z \in Z \) and \( z' \in Z' \) as in Theorem 8.1, \( \pi^* \) induces an isomorphism

\[
\pi^*: \Lambda^*(X_z, Z_z \setminus Z) \xrightarrow{\simeq} \Lambda^*(X'_z, Z'_z \setminus Z_z).
\]

**Proof.** Consider the category \( \text{Cor}^\Lambda_k \) of correspondences built from \( \Lambda^* \) in the sense of Definition 3.1. First of all we see that the proof of Lemmas 8.3 and 8.5 and Construction 3.13 do not use the \( \acute{e}tale \) excision axiom for \( \Lambda^* \). Thus we have morphisms \( \Phi_i, \Phi_r \in \text{Cor}^\Lambda_k(U, X') \) such that \( \pi \circ \Phi_r = i \), and \( \Phi_r \circ \pi = i' \). Then \( \Phi_r \) induces the right inverse homomorphism \( \Lambda^*(X'_{z'}, X'_z \setminus Z'_z) \to \Lambda^*(X_z, X'_{z'} \setminus Z_z) \) to \( \pi^* \) and \( \Phi_r \) induces the left inverse. \( \square \)
9. The cancellation theorem

In this section we show the cancellation theorem for A-correspondences by suitably adapting Voevodsky’s proof for the case of COR. For the sake of brevity we will omit the steps that are identical to Voevodsky’s original proof, and rather focus on the details that are specific to our situation. We refer the interested reader to \[\text{Voe10}\] for the remaining formal aspects of the proof.

Definition 9.1. Recall that the Karoubi envelope of COR is the preadditive category whose objects are pairs \((X, p)\) with \(X \in \text{Sm}_k\) and \(p \in \text{Cor}_k^A(X, X)\) an idempotent. The morphisms are given by

\[
\text{Cor}_k^A((X, p), (X', p')) = \text{im}(\text{Cor}_k^A(X, X') \xrightarrow{p'\circ(-)\circ p} \text{Cor}_k^A(X, X')).
\]

Any object \(X \in \text{Sm}_k\) can be considered as an object in the Karoubi envelope of COR by \(X \mapsto (X, \text{id}_X)\). By abuse of notation, we will write \(\text{Cor}_k^A\) also for the Karoubi envelope of COR.

Define \(X \land G_m := \ker(\text{pr}_1 : X \times G_m \to X)\) as an object in the Karoubi envelope of COR, and let \(\text{pr}^\land : G_m^{\land 2} \to G_m^{\land 2}\) denote the canonical projection. The external product on A-correspondences defines a functor

\[
(-) \otimes G_m^{\land 1} : \text{Cor}_k^A \to \text{Cor}_k^A
\]
given by \(X \to X \land G_m, \alpha \mapsto \alpha \times \text{id}_{G_m^{\land 1}}\). Furthermore, for any \(X \in \text{Sm}_k\) we let \(c_\alpha(X) \land G_m\) denote the presheaf \(U \mapsto \text{Cor}_k^A(U \land G_m, X \land G_m)\).

Lemma 9.2. Let \(\tau^\land : G_m^{\times 2} \to G_m^{\times 2}\) denote the twist automorphism, and let

\[
\tau^\land : G_m^{\land 2} \to G_m^{\land 2}
\]
be the morphism induced by \(\text{pr}^\land \circ \tau^\land\). Then \(\tau^\land\) is an A-homotopy to \(-1 \in \text{Cor}_k^A(G_m^{\land 2}, G_m^{\land 2})\).

Proof. Let us first show that \(\tau^\land \circ \iota \sim_{\text{A}} \iota \circ \epsilon\), where \(\iota : G_m^{\times 2} \setminus \Delta \to G_m^{\land 2}\) is the composition of the open immersion \(G_m^{\times 2} \setminus \Delta \to G_m^{\land 2}\) and the projection \(\text{pr}^\land : G_m^{\times 2} \to G_m^{\land 2}\). The homotopy is given by

\[
(\tau^\land + (-1)) \circ \iota = ((t-x_1)(t-x_2))(t-x_2) + ((t-x_1)(t-x_2))(t-x_1) \circ \iota \circ ((x_2-x_1)^{-1})
\]

\[
= ((t-x_1)(t-x_2)) \circ \iota \circ ((x_2-x_1)^{-1})
\]

\[
\sim_{\text{A}} ((t-x_1)(t-x_2)) \circ \iota \circ ((x_2-x_1)^{-1})
\]

\[
= ((t-x_1)(t-x_2)) \circ \iota \circ ((x_2-x_1)^{-1})
\]

\[
= (\nu_1 + \nu_2) \circ \iota \circ ((1-x_1)(x_2-x_1)^{-1}) \in \text{Cor}_k^A(G_m^{\times 2} \setminus \Delta, G_m^{\times 2}).
\]

Here \(\nu_1 : G_m^{\times 2} \to G_m^{\times 2}\) is the morphism \((x_1, x_2) \mapsto (x_1, x_2, 1)\), while \(\nu_2 : G_m^{\times 2} \to G_m^{\times 2}\) is defined by \((x_1, x_2) \mapsto (1, x_1, x_2)\). Since \(\text{pr}^\land \circ \nu_1 = 0\) and \(\text{pr}^\land \circ \nu_2 = 0\) in \(\text{Cor}_k^A(G_m^{\times 2} \setminus \Delta, G_m^{\times 2})\), it follows that

\[
(\tau^\land + (-1)) \circ \iota = 0 \in \text{Cor}_k^A(G_m^{\times 2} \setminus \Delta, G_m^{\times 2}).
\]

Now Corollary 5.3 yields the claim, since \(\text{Cor}_k^A(-, G_m^{\land 2})\) is a homotopy invariant presheaf with A-transfers. \(\square\)

Definition 9.3. Let \(G_m \times G_m\) have coordinates \((t_1, t_2)\). For any \(n \geq 1\), define the functions \(g_n^+ : g_n^- \in k[G_m \times G_m]\) by

\[
g_n^+ := t_1^n + 1, \quad g_n^- := t_1^n + t_2.
\]

Moreover, let \(Z_n^\pm\) denote the support of the principal divisor \(Z(g_n^\pm)\) on \(G_m \times G_m\) defined by \(g_n^\pm\).
Remark 9.4. The functions $g^+_n/g^-_n$ differ by a sign from Voevodsky’s functions $g_n$ defined in \cite{MR2062607} §4. However, the same proof as that of \cite{MR2062607} Lemma 4.1 goes through to show that for any closed subset $T$ of $G_m \times X \times G_m \times Y$ finite and surjective over $G_m \times X$, there is an integer $N$ such that for all $n \geq N$, the divisor of $g^+_n/g^-_n$ intersects $T$ properly over $X$, and the associated cycle is finite over $X$. The only reason for our choice of functions is to make the finite $A$-correspondence in Lemma 9.0 homotopic to $(1)$, and not $(-1)$. Of course, in the situation of \cite{MR2062607} this choice does not matter, as Voevodsky’s correspondences are oriented.

Definition 9.5. For any $Y \in \text{Sm}_k$ and any $n \geq 1$, we define maps of presheaves

$$c_A(Y) \xrightarrow{\rho_n} c_A(Y) \wedge G_m$$

as follows. Let $X \in \text{Sm}_k$, and let $T$ be any admissible subset of $X \times Y$. Then the homomorphism

$$\theta: \mathcal{A}^{\dim Y}(X \times Y, \omega_Y) \rightarrow \mathcal{A}^{\dim Y+1}(X \times G_m \times Y \times G_m, \omega_{Y \times G_m})$$

is given by

$$\theta := (-) \times \text{id}_{G_m} = (-) \times \Delta(1),$$

where $\Delta: G_m \rightarrow G_m \times G_m$ is the diagonal. On the other hand, the map

$$\rho_n: \mathcal{A}^{\dim Y+1}(X \times G_m \times Y \times G_m, \omega_{Y \times G_m}) \rightarrow \mathcal{A}^{\dim Y+1}(X \times Y, \omega_Y)$$

is defined by the formula

$$\rho_n := p_*(-) \circ q^*(\langle g^+_n \rangle - \langle g^-_n \rangle),$$

where $p$ and $q$ are the projections

$$p: X \times G_m \times Y \times G_m \rightarrow X \times Y, \quad q: X \times G_m \times Y \times G_m \rightarrow G_m \times G_m.$$ 

By naturality, these homomorphisms extend to maps of presheaves.

9.0.1. The maps $\rho_n$ form an “exhausting sequence of partially defined homomorphisms” in the sense that for any finite subset $F \subseteq \text{Cor}^A(X \times G_m, Y \times G_m)$, there is an integer $N(F)$ such that for all $n \geq N(F)$, $\rho_n(\alpha)$ is defined for all $\alpha \in F$. Indeed, this condition is satisfied by Remark 9.4.

Lemma 9.6. Let $q': G_m \times G_m \rightarrow \text{Spec } k$ denote the projection, and let $\Delta: G_m \rightarrow G_m \times G_m$ be the diagonal. Then there is an $A^1$-homotopy

$$q'_*(\Delta_*(\langle g^+_n \rangle) - \langle g^-_n \rangle)) \sim_{A^1} (1) \in \mathcal{A}^{0}(\text{Spec } k, \mathcal{O}_{\text{Spec } k}).$$

Proof. We deduce the claim from the following computation:

$$q'_*(\Delta_*(\langle g^+_n \rangle) - \langle g^-_n \rangle)) = \langle \Delta^*(g^+_n) \rangle^\text{pr}_{G_m} - \langle \Delta^*(g^-_n) \rangle^\text{pr}_{G_m}$$

(9.1)

$$= \langle \Delta^*(g^+_n) \rangle^\text{pr}_{\text{pt}A^1} - \langle \Delta^*(g^-_n) \rangle^\text{pr}_{\text{pt}A^1}_{Z(\langle g^+_n \rangle)_{|G_m}}$$

(9.2)

$$= \langle t^n + 1 \rangle^\text{pr}_{\text{pt}A^1} - \langle t^n + 1 \rangle^\text{pr}_{\text{pt}A^1}_{Z(t^n+1)}$$

(9.3)

$$\sim_{A^1} \langle t^n + 1 \rangle^\text{pr}_{\text{pt}A^1}_{Z(t^n+1)} = \langle 1 \rangle.$$

(9.4)

Here the homotopy is given by $t^n + \lambda t + (1 - \lambda) \in k[A^1 \times A^1]$. \qed
9.0.2. We are now ready to prove the cancellation theorem for A-correspondences.

**Theorem 9.7.** For any \( X, Y \in \Sm_k \), the map \( \theta = (-) \boxtimes G_m^{\Lambda_1} \) induces a quasi-isomorphism of complexes of presheaves with A-transfers

\[
C_* (\theta) : \Cor_k (\Delta^* \times X, Y) \simeq \Cor_k ( (\Delta^* \times X) \land G_m, Y \land G_m).
\]

**Proof.** The proof follows the same approach as Voevodsky’s cancellation theorem for the category \( \Cor_k \) \[\text{[Voe10]}\]. Thus many aspects of the proof will be the same as those of Voevodsky’s proof, and we will therefore focus on the details that are specific to our context.

To prove that \( C_* (- \boxtimes G_m^{\Lambda_1}) \) is a quasi-isomorphism it is enough to show that the maps \( \rho_n \) and \( \theta \) are inverse to each other up to natural \( A^1 \)-homotopy.

First, note that the functions \( f_i^+ \) and \( g_i^- \) enjoy the following properties:

1. \( g_i^n |_{\Delta} = t^n + a_1 t^{n-1} + \cdots + a_n - 1 t + 1 \), and \( g_i^n |_{\Delta} = t^n + b_1 t^{n-1} + \cdots + b_n - 1 t^2 + t \) (in fact, \( g_i^n |_{\Delta} = t^n + 1 \) and \( g_i^n |_{\Delta} = t^n + t \));
2. \( g_i^n |_{G_m \times 1} = g_i^n |_{G_m \times 1} \neq 0 \).

Let \( p \) and \( q \) be the projections

\[
p : X \times G_m \times Y \times G_m \to X \times Y, \quad q : X \times G_m \times Y \times G_m \to G_m \times G_m.
\]

Let moreover \( p' : X \times Y \to \Spec k \) and \( q' : G_m \times G_m \to \Spec k \) denote the projections, so that we have a pullback square

\[
\begin{array}{ccc}
X \times G_m \times Y \times G_m & \xrightarrow{q} & G_m \times G_m \\
p \downarrow & & \downarrow q' \\
X \times Y & \xrightarrow{p'} & \Spec k
\end{array}
\]

Property (1) then implies that the composition \( \rho_n \circ \theta \) is \( A^1 \)-homotopic to the identity, by the following computation:

\[
\begin{align*}
& p_* ( (\alpha \times \Delta, (1) ) \sim q^* ( (g_i^n - (g_i^n)) ) & \quad (9.6) \\
& = p_* (p^* (\alpha) \sim q^* ( (\Delta, (1) ) \sim (g_i^n - (g_i^n)) ) ) & \quad (9.7) \\
& = \alpha \sim p_* (q^* (\Delta, (1) ) \sim (g_i^n - (g_i^n)) ) & \quad (9.8) \\
& = \alpha \sim (p')^* (q'_* (\Delta, (1) ) \sim (g_i^n - (g_i^n)) ) & \quad (9.9) \\
& = \alpha \sim (p')^* (q'_* (\Delta, (\Delta^* (g_i^n)) - (\Delta^* (g_i^n)) ) ) & \quad (9.10) \\
& \sim_{A^1} \alpha \sim (p')^* (1) ) & \quad (9.11) \\
& = \alpha. & \quad (9.12)
\end{align*}
\]

Here the equality (9.3) follows from the projection formula, (9.9) follows from base change applied to the diagram above, and the homotopy (9.11) follows from Lemma 9.6.

Similarly, property (2) implies that for any \( a \in \Cor_k (X, Y) \), the classes \( \rho_n ((\alpha \boxtimes \id G_m^{\Lambda_1}) \circ i_X) \), \( \rho_n (i_Y \circ (\alpha \boxtimes \id G_m^{\Lambda_1}) \circ i_X) \) and \( \rho_n (i_Y \circ (\alpha \boxtimes \id G_m^{\Lambda_1})) \) are equal to \( 0 \) up to natural homotopy. Next, in a standard way Lemma 9.2 and Voevodsky’s trick implies that \( \rho_n \) is also right inverse up to \( A^1 \)-homotopy. \( \square \)

### 10. The category of A-motives

In this section we assume that the base field \( k \) is infinite, perfect and of characteristic different from 2.
10.1. Strict homotopy invariance.

**Theorem 10.1.** Let $\mathcal{F}$ be a homotopy invariant presheaf with $A$-transfers. Then the associated Nisnevich sheaf $\mathcal{F}_{\text{Nis}}$ is strictly homotopy invariant, i.e., the projection $p: X \times A^1 \to X$ induces an isomorphism
\[ p^*: H^n_{\text{Nis}}(X, \mathcal{F}_{\text{Nis}}) \cong H^n_{\text{Nis}}(X \times A^1, \mathcal{F}_{\text{Nis}}) \]
for all $X \in \text{Sm}_k$ and all $n \geq 0$.

**Proof.** The theorem is a consequence of the injectivity and excision theorems proved in Sections 5, 6, 7 and 8. The deduction of strict homotopy invariance from these results is standard; see for example [GP18] or [Dru18c]. □

10.2. Nisnevich localization.

**Theorem 10.2.** The category of Nisnevich sheaves with $A$-transfers is abelian. The Nisnevich sheafification $\mathcal{F}_{\text{Nis}}$ of any presheaf with $A$-transfers $\mathcal{F}$ is equipped with $A$-transfers in a unique and natural way, and there is a natural isomorphism
\[ \text{Ext}^i_{\text{Shv Nis}}(\text{Cor}_A^k(U), \mathcal{F}_{\text{Nis}}) \cong H^i_{\text{Nis}}(X, \mathcal{F}_{\text{Nis}}). \]

**Proof.** By [Dru18b, Theorem 3.1] it is enough to prove that for any local henselian essentially smooth scheme $U$ and any $X \in \text{Sm}_k$, the canonical homomorphisms $X \to X$ induce an isomorphism
\[ \bigoplus_{x \in X} \text{Cor}_A^k(U, X^h_x) \cong \text{Cor}_A^k(U, X). \]
Here $\text{Cor}_A^k(U, X^h_x) := \varprojlim \text{Cor}_A^k(U, X')$, where the limit ranges over all étale neighborhoods $v: (X', x) \to (X, x)$ of $x$ in $X$. It follows from the étale excision axiom that
\[ \text{Cor}_A^k(U, X^h_x) = \varprojlim_{T \in A(U, X^h_x), x \in T} A^{d^x_T}(U \times X^h_x, \omega_{X^h_x}), \]
Here $A^{d^x_T}(U \times X^h_x, \omega_{X^h_x}) := \varprojlim_{v \in V} A^{d^x_{v,T}}(U \times X', \omega_{X'})$, where the limit ranges over the set of all étale neighborhoods $v$ of $x$ in $X$. For any étale neighborhood $v$, $T' \subseteq U \times X'$ maps isomorphically onto its image $T$ in $U \times X^h$ as $T$ is finite over the henselian local scheme $U$. Furthermore, we have $T = \prod_{x \in T} T_x$. Hence, by Zariski excision we obtain
\[ A^{d^x_T}(U \times X, \omega_X) = \varprojlim_{x \in X} A^{d^x_{T_x}}(U \times X, \omega_X). \]
Now the claim follows, since injective limits commute with direct sums. □

**Remark 10.3.** The category of finite $A$-correspondences $\text{Cor}_A^k$ is a strict $V$-category in the sense of [Gar17, Definition 2.3], and a $V$-ringoid in sense of [GP14, Definition 2.4]. So, alternatively, Theorem 10.2 can be proved by using the technique of [GP14].

Note also that the proof of Theorem 10.2 could be obtained by following the original approach of the proof of the similar theorem for Suslin–Voevodsky’s finite correspondences in [Voe00], i.e., showing that the cone of the morphism $c_A(U^*) \to c_A(U)$ is acyclic, where $U^*$ is the Čech complex associated to a Nisnevich covering $U \to U$ of a smooth $k$-scheme $U$. 
10.3. Effective A-motives.

**Definition 10.4.** The ∞-category $DM^\text{eff}_A(k)$ of effective A-motives is the localization of the derived category $D^-(\text{Shv}_{Nis}(\text{Cor}^A_k))$ with respect to the morphisms of the form $X \times A^1 \to X$. Let $M^\text{eff}_A : \text{Sm}_k \to DM^\text{eff}_A(k)$ be the functor defined as the composition of the localization $D^-(\text{Shv}_{Nis}(\text{Cor}^A_k)) \to DM^\text{eff}_A(k)$ with the functor $\text{Sm}_k \to D^-(\text{Shv}_{Nis}(\text{Cor}^A_k))$ given by $X \mapsto Z_A(X)[0]$. For any $X \in \text{Sm}_k$, we refer to $M^\text{eff}_A(X)$ as the effective A-motive of $X$. If $X = \text{Spec } k$, we abbreviate $M^\text{eff}_A(\text{Spec } k)$ to $Z_A$. Finally, we define the Tate object $Z_A(1)$ as

$$Z_A(1) := \text{cofib}(Z_A \to M^\text{eff}_A(G_m))[-1],$$

where $Z_A \to M^\text{eff}_A(G_m)$ is the map induced by the rational point $1 : \text{Spec } k \to G_m$.

10.3.1. Note that there is a symmetric monoidal structure on $DM^\text{eff}_A(k)$ inherited from that on $\text{Shv}_{Nis}(\text{Cor}^A_k)$, satisfying $M^\text{eff}_A(X) \otimes M^\text{eff}_A(Y) \cong M^\text{eff}_A(X \times Y)$. The motive of a point, $Z_A$, is then the unit for this monoidal structure. For any $n \geq 1$, we can use this monoidal structure to define $Z_A(n) := Z_A(1)^{\otimes n}$.

**Theorem 10.5.** The ∞-category $DM^\text{eff}_A(k)$ of effective A-motives is equivalent to the full subcategory of $D^-(\text{Shv}_{Nis}(\text{Cor}^A_k))$ spanned by motivic complexes, i.e., complexes whose cohomology sheaves are strictly homotopy invariant.

**Theorem 10.6.** For any $X \in \text{Sm}_k$ and any motivic complex $\mathcal{F}^\bullet \in D^-(\text{Shv}_{Nis}(\text{Cor}^A_k))$ there is a natural isomorphism

$$[M^\text{eff}_A(X), \mathcal{F}^\bullet]_{D^-(\text{Shv}_{Nis}(\text{Cor}^A_k))} \cong H^i_{\text{Nis}}(X, \mathcal{F}^\bullet).$$

4. The category of A-motives. As in the classical case, we obtain the category $DM_A(k)$ of A-motives via a stabilization process with respect to tensoring with the Tate object.

**Definition 10.7.** The ∞-category $DM_A(k)$ of A-motives is obtained from $DM^\text{eff}_A(k)$ by ⊗-inverting $Z_A(1)$. There is then a canonical functor $\Sigma^\infty : DM^\text{eff}_A(k) \to DM_A(k)$. Define the functor $M_A : \text{Sm}_k \to DM_A(k)$ as the composition of $M^\text{eff}_A$ and $\Sigma^\infty$.

10.4.1. It follows similarly as in [DF17a] that $DM_A(k)$ is a presentably symmetric monoidal stable ∞-category equipped with an adjunction

$$\Sigma^\infty : DM^\text{eff}_A(k) \rightleftarrows DM_A(k) : \Omega^\infty.$$

10.4.2. The following theorem is a consequence of the cancellation theorem for A-correspondences:

**Theorem 10.8.** The canonical functor $\Sigma^\infty : DM^\text{eff}_A(k) \to DM_A(k)$ is fully faithful, and for any $X \in \text{Sm}_k$ and any motivic complex $\mathcal{F}^\bullet \in D^-(\text{Shv}_{Nis}(\text{Cor}^A_k))$, there is a natural isomorphism

$$[M_A(X), \Sigma^\infty \mathcal{F}^\bullet]_{DM_A(k)} \cong H^i_{\text{Nis}}(X, \mathcal{F}^\bullet).$$

**Definition 10.9.** Let $X \in \text{Sm}_k$. For any pair of integers $p, q \in \mathbb{Z}$, we define the integral A-motivic cohomology of $X$ in bidegree $(p, q)$ as

$$H^p_{\text{A}}(X, \mathbb{Z}) := [M_A(X), Z_A(q)[p]]_{DM_A(k)}.$$

10.4.3. The adjunction $\gamma^A_A : \text{PSh}_E(\text{Sm}_k) \rightleftarrows \text{PSh}_E(\text{Cor}^A_k) : \gamma^A_A$ descends to an adjunction

$$\gamma^A_A : \text{SH}(k) \rightleftarrows DM_A(k) : \gamma^A_A \quad (10.1)$$

of stable ∞-categories, which allows us to compare $DM_A(k)$ with the motivic stable homotopy category $\text{SH}(k)$.

**Definition 10.10.** In the adjunction $(10.1)$, let $H_A \in \text{SH}(k)$ denote the Eilenberg–Mac Lane spectrum $\gamma^A_A \gamma^A_A(1) = \gamma^A_A(Z_A)$. 
10.4.4. The cancellation theorem for A-correspondences implies that $H_Z^A$ is an $\Omega P_1$-spectrum in $\text{SH}(k)$ which represents A-motivic cohomology. More precisely, for any $X \in \text{Sm}_k$ and any pair of integers $p, q$, there is a natural isomorphism

$$[\Sigma^\infty_+ X, \Sigma^{p,q} H_Z^A]|_{\text{SH}(k)} \cong H_A^{p,q}(X, \mathbb{Z}).$$

10.4.5. The combination of Lemma 3.5 and [EK18] shows moreover that in the above adjunction (10.1), the right adjoint is monadic:

**Theorem 10.11.** Let $e$ denote the exponential characteristic of $k$. Then there is an equivalence of presentably symmetric monoidal stable $\infty$-categories

$$\text{Mod}_{H_Z^A[1/e]}(\text{SH}(k)) \simeq \text{DM}_A(k, \mathbb{Z}[1/e]),$$

where $\text{Mod}_{H_Z^A[1/e]}(\text{SH}(k))$ denotes motivic spectra equipped with an action from $H_Z^A[1/e]$.

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