The truncated EM method for stochastic differential delay equations with variable delay

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Abstract
This paper mainly investigates the strong convergence and stability of the truncated Euler-Maruyama (EM) method for stochastic differential delay equations with variable delay whose coefficients can be growing super-linearly. By constructing appropriate truncated functions to control the super-linear growth of the original coefficients, we present a type of the truncated EM method for such SDDEs with variable delay, which is proposed to be approximated by the value taken at the nearest grid points on the left of the delayed argument. The strong convergence result (without order) of the method is established under the local Lipschitz plus generalized Khasminskii-type conditions and the optimal strong convergence order 1/2 can be obtained if the global monotonicity with $U$ function and polynomial growth conditions are added to the assumptions. Moreover, the partially truncated EM method is proved to preserve the mean-square and $H_\infty$ stabilities of the true solutions. Compared with the known results on the truncated EM method for SDDEs, a better order of strong convergence is obtained under more relaxing conditions on the coefficients, and more refined technical estimates are developed so as to overcome the challenges arising due to variable delay. Lastly, some numerical examples are utilized to confirm the effectiveness of the theoretical results.

Keywords: Stochastic differential delay equations, truncated EM method, variable delay, strong convergence, stability.

1. Introduction

Stochastic differential delay equations (SDDEs) play a significant part in many application fields, such as economy, finance, automatic control and population dynamics (see, e.g., \cite{1,2,3,4}). In general, SDDEs rarely have explicit solutions available, and therefore one has to construct appropriate numerical schemes so as to approximate their solutions. Moreover, many important SDDE models often possess super-linear growth coefficients in practice, for example, stochastic delay Lotka-Volterra model arising in population dynamics of the form (\textsuperscript{2})

\[
dX(t) = \text{diag}(X_1(t), \cdots, X_d(t)) \left[ \left( b + AX(t - \tau) \right) dt + \sigma X(t) dB(t) \right].
\]  

(1.1)
where $B(t)$ is a Brownian motion and $\sigma = (\sigma_{ij})_{d \times d}$ is a matrix representing the intensity of noise. If we apply the explicit Euler-Maruyama (EM) scheme to the model (1.1), it can be shown that this EM approximation may not converge to the true solution in the strong mean-square sense at a finite time point (see, e.g., [5]). As a result, this paper concerns the strong convergence analysis of numerical scheme for such SDDEs with super-linearly growing drift and diffusion coefficients.

When the delay term vanishes, the underlying SDDEs reduce to the usual stochastic differential equations (SDEs), numerical methods for which have been extensively investigated for the past decades under the global Lipschitz condition (e.g., [6], [7], [8]). In 2012, Hutzenthaler et al. [9] firstly introduced an explicit method called the tamed EM method to solve SDEs with super-linearly growing drift coefficient and linearly growing diffusion coefficient. Since then, several explicit schemes have been proposed for SDEs whose coefficients can be growing super-linearly under the local Lipschitz condition, e.g., balanced EM method [10], [11], truncated EM method [12], [13]. Recently, the attention of some researches was attracted to the strong convergence of explicit numerical methods for super-linear SDEs with delay, i.e., SDDEs. Motivated by the work of Mao [12], [13], Guo et al. [14] discussed the strong convergence of the truncated EM method for SDDEs under the local Lipschitz condition plus the generalized Khasminskii-type condition (2.3). In a subsequent paper, Gao et al. [15] took the delay and jumps into consideration, they extended convergence results from [14] to [16] to the case of SDDEs with Poisson jumps. By using a different estimate for the difference between the original and the truncated coefficients, Fei et al. [17] relaxed the restrictive condition on the step size which is required to extremely small and thus improved the convergence results of [14]. Moreover, Song et al. [18] achieved a better convergence order than [14] and [17] by adopting the truncation techniques from [19] for such SDDEs. The applications of the partially truncated and modified truncated EM methods for SDDEs can be found in [20], [21]. Other explicit numerical methods for super-linear SDDEs, say tamed EM, balanced EM, truncated Milstein, projected EM, are discussed in [22], [23], [24], [25], [26], [27].

In the preceding discussion, the focus of the interest is on the numerical method of SDEs with constant delay. In reality, delay can behave as a function of time, namely, variable delay. Mao and Sabanis [28] were the first authors to consider the strong convergence of numerical method for such SDDEs. In contrast to the constant delay, the main difficulty in the construction of the computational approach in case of SDDEs with variable delay is how to numerically approximate the values of the solution at the delayed instants. Mao and Sabanis [28] proposed to use the approximate values at the nearest grid points on the left of the delayed arguments to estimate the variable delay, and they derived an upper bound for the difference of this approximation. Influenced by [28], several explicit and implicit variants of the EM method have been developed to study the convergence of the numerical solutions to stochastic equations with variable delay. For instance, we refer to [29] for the convergence in probability of EM method for highly nonlinear neutral stochastic differential equations with variable delay, and to [30] for the strong error analysis of EM method for SDEs with variable and distributed delays. Further, in [31], strong convergence rates are derived for the split-step theta method applied to stochastic age-dependent population equations with Markovian switching and variable delay.

Based on the above discussion, the objective of this work is to study the strong convergence (including the stability) of explicit numerical method of super-linear SDDEs with variable delay. By constructing appropriate truncated functions to control the super-linear growth of the original coefficients, we introduce a type of the truncated EM method for such SDDEs with variable delay, which can be approximated by the value taken at the left endpoint of interval containing the delayed argument. We then show that the method is convergent in the sense of
mean-square according to the properties (3.5) and (3.6) of the truncated functions: preservation of Khasminskii-type condition and linear growth for a fixed step size. In addition, we discuss the mean-square and $H_\infty$ stabilities of the method.

The main contribution of this paper is to improve the strong convergence results of the truncated EM method for SDDEs. We obtain a better strong convergence order than the existing results such as Cong et al. [32 Theorem 3.12], Guo et al. [14 Theorem 5.3], and Fei et al. [17 Theorem 3.6], under the more relaxing conditions: a generalized Khasminskii-type and global monotonicity conditions with $U$ function, see Remarks 4.15 and 4.16. It should be pointed out that the appearance of $U$ function in global monotonicity condition will make the choice of the coefficients for SDDEs more flexible, see e.g., Fei et al. [17 Example 6.2]. Our technical estimates are more refined than those of Guo et al. [14], Fei et al. [17] and Song et al. [18] in that we develop new techniques to overcome the challenges arising due to variable delay. Moreover, it is proved that the partially truncated EM method has the properties of the mean-square and $H_\infty$ stabilities.

The remainder of the paper is organised as follows. The second section introduces some basic notions and assumptions. The next section describes the truncated EM scheme for SDDEs with variable delay. Strong convergence results are established in the forth section. In the fifth section, we prove some stability theorems. Numerical simulations are provided in the sixed section. In the final section, we close the paper by our conclusion.

2. Preliminaries

Throughout this paper, let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a filtration $\{ \mathcal{F}_t \}_{t \geq 0}$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). Let $\tau > 0$ be a constant and denote by $C([\tau, 0]; \mathbb{R}^d)$ the space of all continuous functions from $[-\tau, 0]$ to $\mathbb{R}^d$ with the norm $\| \phi \| = \sup_{-\tau \leq \theta \leq 0} | \phi(\theta) |$. Let $B(t)$ be an $m$-dimensional Brownian motion. If $A$ is a vector or matrix, its transpose is denoted by $A^T$. If $X \in \mathbb{R}^d$, then $|X|$ is the Euclidean norm. If $A$ is a matrix, its trace norm is denoted by $|A| = \sqrt{\text{tr}(A^T A)}$. For two real numbers $a$ and $b$, $a \vee b := \max(a, b)$ and $a \wedge b := \min(a, b)$. For a set $G$, its indicator function is denoted by $\mathbb{1}_G$. The scalar product of two vectors $X, Y \in \mathbb{R}^d$ is denoted by $\langle X, Y \rangle$ or $X^T Y$. The largest integer which is less or equal to a real number $a$ is denoted by $\lfloor a \rfloor$. In addition, we use $C$ to denote the generic constant that may change from place to place.

Let $\delta : [0, +\infty) \to [0, \tau]$ be the delay function which is Borel measurable. Consider the following SDDE of the form

$$dX(t) = f(X(t), X(t - \delta(t)))dt + g(X(t), X(t - \delta(t)))dB(t), \ t \geq 0,$$

with the initial data

$$X(\theta) : [-\tau \leq \theta \leq 0] = \xi \in C([\tau, 0]; \mathbb{R}^d),$$

where $f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ and $g : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^{d \times m}$ are Borel-measurable functions. We introduce the following assumptions:

**Assumption 2.1 (Local Lipschitz Condition).** For any $R > 0$, there exists a constant $L_R$ depending on $R$ such that

$$|f(x_1, y_1) - f(x_2, y_2)|^2 + |g(x_1, y_1) - f(x_2, y_2)|^2 \leq L_R(|x_1 - x_2|^2 + |y_1 - y_2|^2),$$

for any $x_1, x_2, y_1, y_2 \in \mathbb{R}^d$ with $|x_1| \vee |x_2| \vee |y_1| \vee |y_2| \leq R$. 

...
Assumption 2.2 (Generalized Khasminskii-type Condition). There exist constants $K_1 > 0$, $K_2 \geq 0$, $K_3 \geq 0$ and $\beta > 2$ such that
\[
2(x, f(x, y)) + |g(x, y)|^2 \leq K_1(1 + |x|^2 + |y|^2) - K_2|x|^\beta + K_3|y|^{\beta}, \quad \forall x, y \in \mathbb{R}^d. \tag{2.3}
\]

Assumption 2.3. Let $\delta : [0, +\infty) \to [0, 1]$ be continuously differentiable and there is $\hat{\delta} \in [0, 1)$ such that $|d\delta(t)| \leq \hat{\delta}$, for any $t \geq 0$.

Lemma 2.4. Suppose that Assumptions 2.1, 2.2 and 2.3 hold with $K_2 > \frac{K_3}{(1 - \hat{\delta})} \geq 0$. Then for any given initial data \((\pi, \beta)\), there is a unique global solution $X(t)$ to (2.1) on $t \in [-\tau, +\infty)$. Moreover, the solution $X(t)$ has the property that
\[
\sup_{-\tau \leq t \leq T} \mathbb{E}|X(t)|^2 < \infty. \tag{2.4}
\]
The proof of this result can be found in Song et al. [33].

3. The truncated EM scheme for SDDEs with variable delay

We first choose a strictly increasing continuous functions $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\mu(R) \to \infty$ as $R \to \infty$ and
\[
\sup_{|x| \leq r} \frac{|f(x, y)|}{(1 + |x| + |y|)} \vee \frac{|g(x, y)|}{(1 + |x| + |y|)} \leq \mu(r), \quad \forall r \geq 1. \tag{3.1}
\]
Denote by $\mu^{-1}$ the inverse function of $\mu$ and we see that $\mu^{-1} : [\mu(1), \infty) \to \mathbb{R}_+$ is a strictly increasing continuous function. We then choose a constant $\hat{h} \geq 1 / \mu(1)$ and a strictly decreasing function $\varphi : (0, 1) \to [\mu(1), +\infty)$ such that
\[
\lim_{\Delta \to 0} \varphi(\Delta) = \infty \quad \text{and} \quad \Delta^{1/4}\varphi(\Delta) \leq \hat{h}, \quad \forall \Delta \in (0, 1]. \tag{3.2}
\]
For a given step size $\Delta \in (0, 1]$, let us define a truncation mapping $\pi_{\Delta} : \mathbb{R}^d \to \{x \in \mathbb{R}^d : |x| \leq \mu^{-1}(h(\Delta))\}$ by
\[
\pi_{\Delta}(x) = \left(|x| \wedge \mu^{-1}(\varphi(\Delta))\right) \frac{x}{|x|}, \quad \forall x \in \mathbb{R}^d, \tag{3.3}
\]
when $x = 0$, we set $x/|x| = 0$. That is, $\pi_{\Delta}$ maps to itself if $|x| \leq \mu^{-1}(\varphi(\Delta))$ and to $\mu^{-1}(\varphi(\Delta))x/|x|$ if $|x| > \mu^{-1}(\varphi(\Delta))$. Define the following truncated functions
\[
f_{\Delta}(x, y) = f(\pi_{\Delta}(x), \pi_{\Delta}(y)) \quad \text{and} \quad g_{\Delta}(x, y) = g(\pi_{\Delta}(x), \pi_{\Delta}(y)), \quad \forall x, y \in \mathbb{R}^d. \tag{3.4}
\]
From (3.1), (3.3) and (3.4), we have
\[
|f_{\Delta}(x, y)| \vee |g_{\Delta}(x, y)| \leq \varphi(\Delta)(1 + |\pi_{\Delta}(x)| + |\pi_{\Delta}(y)|) \leq \varphi(\Delta)(1 + |x| + |y|), \quad \forall x, y \in \mathbb{R}^d, \tag{3.5}
\]
which means that the truncated coefficients $f_{\Delta}$ and $g_{\Delta}$ grow at most linearly for a fixed step size $\Delta$, but $f$ and $g$ may not. The following lemma shows that these truncated coefficients can conserve the generalized Khasminskii-type condition for any $\Delta \in (0, 1]$, the proof of the lemma is similar to that of Fei et al. [17] Lemma 3.2] and so is omitted.
Lemma 3.1. Let Assumptions hold with $K_2 \geq K_1 \geq 0$. Then for any $\Delta \in (0, 1]$,
\begin{equation}
2(x, f_\Delta(x, y)) + |g_\Delta(x, y)|^2 \leq \hat{K}_1 (1 + |x|^2 + |y|^2) - K_2|\sigma_\Delta(x)|^2 + K_3|\sigma_\Delta(y)|^2, \; \forall x, y \in \mathbb{R}^d,
\end{equation}
where $\hat{K}_1 = 2K_1 \left(1 + \lfloor 1/\mu^{-1} (\varphi(1)) \rfloor \right)$.

Assume that step size $\Delta \in (0, 1]$ is a fraction of $\tau$. Take $\Delta = \tau/M$ for some sufficiently large integer $M$. Define $t_k = k\Delta$ and $\delta_k = \lfloor \delta(k\Delta)/\Delta \rfloor$, for any integer $k \geq 0$. Then the boundedness of $\delta$ gives
\begin{equation}
0 \leq \delta_k \leq \tau/\Delta = M.
\end{equation}
Define $s(t) := [t/\Delta]$, for any $t \geq -\tau$. The discrete-time truncated EM scheme is defined as follows:
\begin{equation}
y_{k+1} = y_k + f_\Delta(y_k, y_{k-\delta_k})\Delta + g_\Delta(y_k, y_{k-\delta_k})\Delta B_k, \; k \geq 0,
y_k = \xi(t_k), \; k = -M, -M + 1, \cdots, 0,
\end{equation}
where $\Delta B_k = B(t_{k+1}) - B(t_k)$. Define the continuous-time step approximations $Z_1(t)$ and $Z_2(t)$ by
\begin{equation}
Z_1(t) = \sum_{k=-M}^{\infty} y_k[I_{[t, t_{k+1})}(t)], \; \forall t \geq -\tau,
\end{equation}
\begin{equation}
Z_2(t) = \sum_{k=0}^{\infty} y_{k-\delta_k}[I_{[t, t_{k+1})}(t)], \; \forall t \geq 0.
\end{equation}
where $I$ is the indicator function. Define the continuous-time approximation $Y_\Delta(t)$ on $t \in [-\tau, \infty)$ by
\begin{equation}
Y_\Delta(t) = \xi(0) + \int_{-\tau}^{t} f_\Delta(Z_1(s), Z_2(s))ds + \int_{-\tau}^{t} g_\Delta(Z_1(s), Z_2(s))dB(s), \; t > 0,
Y_\Delta(t) = \xi(t), \; \tau \leq t \leq 0.
\end{equation}
Thus $Y_\Delta(t)$ is an Itô process on $t \geq 0$ with Itô differential
\begin{equation}
dY_\Delta(t) = f_\Delta(Z_1(t), Z_2(t))dt + g_\Delta(Z_1(t), Z_2(t))dB(t).
\end{equation}
It is useful to know that for any $t \in [t_k, t_{k+1})$ with $k \geq 0$,
\begin{equation}
Z_1(t) = Y_\Delta(t_k) = y_k \quad \text{and} \quad Z_2(t) = Y_\Delta(t_k - \lfloor \delta(t_k)/\Delta \rfloor \Delta) = y_{k-\delta_k},
\end{equation}
as well as
\begin{equation}
Y_\Delta(t) - Z_1(t) = \int_{t_k}^{t} f_\Delta(Z_1(s), Z_2(s))ds + \int_{t_k}^{t} g_\Delta(Z_1(s), Z_2(s))dB(s),
\end{equation}
which means that the $Y_\Delta(t)$ and $Z_1(t)$ coincide with the discrete solution at the grid points.

Remark 3.2. By (3.5), we see from (3.8) that for a given step size $\Delta \in (0, 1]$, any $p \geq 2$ and any $k \geq 1$,
\begin{equation}
E|y_k|^p \leq C_{\rho, \|\xi\|, \hat{K}_1},
\end{equation}
Moreover, this and (3.14) guarantee that for a given step size \( \Delta \in (0, 1] \) and any \( p \geq 2 \),
\[
\mathbb{E}[Y_A(t)]^p < \infty, \quad \forall t \geq 0.
\] (3.16)

However, we cannot conclude that this bound is independent of \( \Delta \). As a result of this observation, we need not apply stopping time arguments in the proof Lemma 4.3 due to the fact that for any \( \Delta \in (0, 1] \) and any \( p \geq 2 \),
\[
\mathbb{E} \int_0^T |g_A(Z_1(s), Z_2(s))|^p ds < \infty.
\] (3.17)

In fact, by (3.3), we have
\[
y_1 = y_0 + f_A(y_0, y_{-\delta})\Delta + g_A(y_0, y_{-\delta})\Delta R_0.
\] (3.18)

Thus, for a given step size \( \Delta \in (0, 1] \) and any \( p \geq 2 \), by (3.5), we have
\[
\mathbb{E}[y_1]^p \leq 3^{p-1}(E[\xi(0)]^p + \mathbb{E}[f_A(y_0, y_{-\delta})]^p \Delta^p + \mathbb{E}[g_A(y_0, y_{-\delta})]^p \Delta^0.5p)
\leq 3^p(h(\Delta))^p\|\xi\|^p \Delta^{0.5p} \leq 3^p\|\xi\|^p \Delta^{0.5p}.
\] (3.19)

Thus, by induction, we can show that (3.15) holds. Moreover, rewrite (3.14) as
\[
Y_A(t) = y_k + f_A(y_k, y_{k-\delta})(t - t_k) + g_A(y_k, y_{k-\delta})(B(t) - B(t_k)), \quad \forall t \in [t_k, t_{k+1}), \quad k \geq 0.
\] (3.20)

Then, for a given step size \( \Delta \in (0, 1] \) and any \( p \geq 2 \)
\[
\mathbb{E}[Y_A(t)]^p \leq c_p(\mathbb{E}[y_k]^p + \mathbb{E}[f_A(y_k, y_{k-\delta})]^p \Delta^p + \mathbb{E}[g_A(y_k, y_{k-\delta})]^p \Delta^0.5p)
\leq c_p(\varphi(\Delta))^p(\mathbb{E}[y_k]^p + \mathbb{E}[y_{k-\delta}]^p) \Delta^{0.5p}
\leq C_p\|\xi\|^p \|k\hat{\Delta}, \forall t \in [t_k, t_{k+1}), \quad k \geq 0,
\]
which gives (3.16).

4. Strong convergence

4.1. Strong convergence (without order) at time \( T \)

**Lemma 4.1.** For any \( \Delta \in (0, 1] \) and \( p > 0 \),
\[
\mathbb{E}[|Y_A(t) - Z_1(t)|^p |F_{k(t)}] \leq CA^{p/2}(\varphi(\Delta))^p(1 + |Z_1(t)|)^p + |Z_2(t)|)^p, \quad \forall t \geq 0,
\] (4.1)

where \( C \) is a positive constant independent of \( \Delta \).

**Proof.** For any \( p \geq 2 \), by (3.5), we see from (3.14) that for any \( t \geq 0 \)
\[
\mathbb{E}[|Y_A(t) - Z_1(t)|^p |F_{k(t)}] = \mathbb{E}[\int_{s(t)}^{t} f_A(Z_1(s), Z_2(s))ds + \int_{s(t)}^{t} g_A(Z_1(s), Z_2(s))dB(s) |F_{k(t)}]
\leq C\mathbb{E}[|f_A(Z_1(t), Z_2(t))(t - \kappa(t))| |F_{k(t)}] + C\mathbb{E}[|g_A(Z_1(t), Z_2(t))(B(t) - B(\kappa(t)))| |F_{k(t)}]
\leq CA^p(\varphi(\Delta))^p(1 + |Z_1(t)|^p + |Z_2(t)|^p) + CA^{p/2}(\varphi(\Delta))^p(1 + |Z_1(t)|^p + |Z_2(t)|^p)
\leq CA^{p/2}(\varphi(\Delta))^p(1 + |Z_1(t)|^p + |Z_2(t)|^p),
\]
this also holds for any \( p \in (0, 2) \) due to the Hölder inequality. □
Lemma 4.2. Let Assumption 2.3 hold. For any \( k \in \{0, 1, 2, \cdots \} \), let \( k - [\delta(k\Delta)/\Delta] = u \), where \( u \in \{-M, -M + 1, \cdots, 0, 1, \cdots, k\} \). Then

\[
\#\left\{ j \in \{0, 1, 2, \cdots \} : j - [\delta(j\Delta)/\Delta] = u \right\} \leq [(1 - \delta)^{-1}] + 1, \tag{4.2}
\]

where \( \#S \) denotes the number of elements of the set \( S \).

This lemma provides an upper bound for the maximal number of indices \( k \in \{1, 2, \cdots \} \) for which the expressions \( k - \delta_k \) are equal, the proof can be found in Milošević [44, Lemma 3].

Lemma 4.3. Let Assumptions 2.1, 2.2 and 2.3 hold with \( K_2 \geq ((1 - \delta)^{-1}) + 1 \) and \( K_3 \geq 0 \). Then for any \( \Delta \in (0, 1] \),

\[
\sup_{0 \leq t \leq T} \mathbb{E}[Z_t(t)]^2 \leq C \quad \text{or} \quad \sup_{0 \leq t \leq T} \mathbb{E}[y_t]^2 \leq C, \quad \forall T > 0. \tag{4.3}
\]

where \( C \) is a positive constant independent of \( \Delta \).

Proof. For any integer \( k \geq 0 \), we conclude from (3.8) that

\[
|y_{k+1}|^2 = |y_k|^2 + 2(y_k, f_k(y_k, y_{k-\delta}))\Delta + |g_\Delta(y_k, y_{k-\delta})|^2\Delta + |f_\Delta(y_k, y_{k-\delta})|^2\Delta^2 + J_k, \tag{4.4}
\]

where

\[
J_k = 2\langle y_k, g_\Delta(y_k, y_{k-\delta})\Delta B_k \rangle + 2\langle f_\Delta(y_k, y_{k-\delta}), g_\Delta(y_k, y_{k-\delta})\Delta B_k \rangle \Delta + |g_\Delta(y_k, y_{k-\delta})|^2(\|\Delta B_k\|^2 - \Delta).
\]

Obviously, \( EJ_k = 0 \). By (3.1), we have

\[
E[y_{k+1}]^2 \leq E[y_k]^2 + \hat{K}_1 E(1 + |y_k|^2 + |y_{k-\delta}|^2)\Delta + E[|f_\Delta(y_k, y_{k-\delta})|^2] \Delta^2 + E\left[ -K_2\|\Delta y_k\|^2 + K_3\|\Delta y_{k-\delta}\|^2 \right] \Delta, \quad \forall k \geq 0. \tag{4.5}
\]

Moreover, by (3.5), we have

\[
E|f_\Delta(y_k, y_{k-\delta})|^2 \Delta^2 \leq (\varphi(\Delta))^2 E(1 + |y_k|^2 + |y_{k-\delta}|^2) \Delta^2 \\
\leq \hat{h}^2 E(1 + |y_k|^2 + |y_{k-\delta}|^2) \Delta \leq \hat{h}^2 E(1 + |y_k|^2 + |y_{k-\delta}|^2) \Delta. \tag{4.6}
\]

Inserting (4.6) into (4.5) gives

\[
E[y_{k+1}]^2 \leq E[y_k]^2 + (\hat{K}_1 + \hat{h}^2) E(1 + |y_k|^2 + |y_{k-\delta}|^2) \Delta \\
+ E\left[ -K_2\|\Delta y_k\|^2 + K_3\|\Delta y_{k-\delta}\|^2 \right] \Delta, \quad \forall k \geq 0. \tag{4.7}
\]

Thus, we have

\[
E[y_k]^2 \leq \|\xi\|^2 + (\hat{K}_1 + \hat{h}^2) \sum_{j=0}^{k-1} E(1 + |y_j|^2 + |y_{j-\delta}|^2) \Delta \\
+ E\left[ \sum_{j=0}^{k-1} \left( -K_2\|\Delta y_j\|^2 + K_3\|\Delta y_{j-\delta}\|^2 \right) \right] \Delta, \quad \forall k \geq 1. \tag{4.8}
\]
By Lemma 4.2, we yields that

\[\sum_{j=0}^{k-1} |\pi_a(y_{i-j})|^\beta \Lambda \leq ((1 - \hat{\delta})^{-1} + 1) \sum_{j=-M}^{k-1} |\pi_a(y_j)|^\beta \Lambda\]

\[= ((1 - \hat{\delta})^{-1} + 1) \sum_{j=-M}^{-1} |\pi_a(y_j)|^\beta \Lambda + ((1 - \hat{\delta})^{-1} + 1) \sum_{j=0}^{k-1} |\pi_a(y_j)|^\beta \Lambda\]

\[\leq ((1 - \hat{\delta})^{-1} + 1) r||\xi||^\beta + ((1 - \hat{\delta})^{-1} + 1) \sum_{j=0}^{k-1} |\pi_a(y_j)|^\beta \Lambda, \forall k \geq 1, \quad (4.9)\]

where \(\delta_\epsilon = [\delta(k\Lambda)/\Delta].\) Thus,

\[\sum_{j=0}^{k-1} \left[- K_2 |\pi_a(y_j)|^\beta + K_3 |\pi_a(y_{i-j})|^\beta \right] \Lambda\]

\[\leq K_3 ((1 - \hat{\delta})^{-1} + 1) r||\xi||^\beta - (K_2 - K_3 ((1 - \hat{\delta})^{-1} + 1)) \sum_{j=0}^{k-1} |\pi_a(y_j)|^\beta \Lambda\]

\[\leq K_3 ((1 - \hat{\delta})^{-1} + 1) r||\xi||^\beta. \quad (4.10)\]

Inserting this into (4.8), we have

\[\mathbb{E}|y_k|^2 \leq (||\xi||^2 + K_3 ((1 - \hat{\delta})^{-1} + 1) r||\xi||^\beta) + (K_1 + \hat{h}^3) \sum_{j=0}^{k-1} \left[1 + \mathbb{E}|y_j|^2 + \mathbb{E}|y_{i-j}|^2 \right] \Lambda\]

\[\leq (||\xi||^2 + K_3 ((1 - \hat{\delta})^{-1} + 1) r||\xi||^\beta) + (K_1 + \hat{h}^3) \sum_{j=0}^{k-1} \left[1 + 2 \sup_{-M \leq i \leq j} \mathbb{E}|y_j|^2 \right] \Lambda, \forall k \geq 1. \quad (4.11)\]

As this holds for any integer \(k\) satisfying \(1 \leq k \leq [T/\Delta],\) while the sum of the right-hand-side (RHS) terms is non-decreasing in \(k,\) we then have

\[\sup_{1 \leq k \leq [T/\Delta]} \mathbb{E}|y_k|^2 \leq (||\xi||^2 + K_3 ((1 - \hat{\delta})^{-1} + 1) r||\xi||^\beta) + (K_1 + \hat{h}^3) \sum_{j=0}^{k-1} \left[1 + 2 \sup_{-M \leq i \leq j} \mathbb{E}|y_j|^2 \right] \Lambda. \quad (4.12)\]

which implies that

\[\sup_{-M \leq k \leq T} \mathbb{E}|y_k|^2 \leq (||\xi||^2 + K_3 ((1 - \hat{\delta})^{-1} + 1) r||\xi||^\beta) + (K_1 + \hat{h}^3) \sum_{j=0}^{k-1} \left[1 + 2 \sup_{-M \leq i \leq j} \mathbb{E}|y_j|^2 \right] \Lambda, \forall k = 1, 2, \cdots, [T/\Delta]. \quad (4.13)\]

By the discrete Gronwall inequality, we get the desired assertion (4.3). □

**Assumption 4.4.** There is a pair of constants \(K_4 > 0\) and \(g \in (0, 1]\) such that

\[|\xi(t) - \xi(s)| \leq K_4 |t - s|^g, \forall s, t \in [-\tau, 0]. \quad (4.14)\]
Lemma 4.5. Let Assumptions 2.1, 2.2, and 2.3 hold with $K_2 \geq ((1 - \delta)^{-1} + 1)K_3 \geq 0$. For any $R \geq \|\xi\|$, define $\tau_R = \inf\{t \geq 0 : |X(t)| \geq R\}$ and $\bar{\rho}_{A,R} = \inf\{t \geq 0 : |Z_t(t)| \geq R\}$. Then

$$P(\tau_R \leq T) \leq \frac{C}{R^2} \quad \text{and} \quad P(\bar{\rho}_{A,R} \leq T) \leq \frac{C}{R^2}, \quad \forall T > 0, \quad (4.15)$$

where $C$ is a positive constant independent of $\Delta$.

**Proof.** By the Itô formula and Assumption 2.2 we have that for any $t \in [0, T]$

$$E|X(t \wedge \tau_R)|^2 \leq \|\xi(0)\|^2 + K_1 E \int_0^{t \wedge \tau_R} \left(1 + |X(s)|^2 + |X(s - \delta(s))|^2\right)ds + \frac{K_1}{1 - \delta} E \int_0^{t \wedge \tau_R} |X(s)|^\beta ds + \frac{\tau||\xi||^\beta}{1 - \delta} \int_0^{t \wedge \tau_R} \|X(s)\| ds \leq C + 2K_1 \int_0^T \left(\sup_{0 \leq s \leq s} E|X(u \wedge \tau_R)|^2\right)ds, \quad (4.16)$$

where we have used the following estimates

$$\int_0^{t \wedge \tau_R} |X(s - \delta(s))|^\beta ds \leq \frac{1}{1 - \delta} \int_{-\delta(0)}^{0} |X(u)|^\beta du \leq \frac{1}{1 - \delta} \int_{-\tau(t)}^{0} |X(u)|^\beta du \leq \frac{\tau||\xi||^\beta}{1 - \delta} \int_0^{\tau(t)} \|X(s)\| ds, \quad (4.17)$$

and

$$\int_0^T E|X((s - \delta(s)) \wedge \tau_R)|^2 ds \leq \int_0^T \left(\sup_{0 \leq s \leq s} E|X(u \wedge \tau_R)|^2\right)ds \leq \frac{T||\xi||^2}{1 - \delta} + \frac{1}{1 - \delta} \int_0^T \|X(s)\| ds. \quad (4.18)$$

Consequently,

$$\sup_{0 \leq s \leq T} E|X(u \wedge \tau_R)|^2 \leq C + 4K_1 \int_0^T \left(\sup_{0 \leq s \leq s} E|X(u \wedge \tau_R)|^2\right)ds.$$

The Gronwall inequality gives

$$\sup_{0 \leq s \leq T} E|X(u \wedge \tau_R)|^2 \leq C.$$

Thus,

$$E|X(T \wedge \tau_R)|^2 \leq C.$$

Finally, using the Chebyshev inequality gives

$$P(\tau_R \leq T) \leq \frac{C}{R^2}.$$
Now, we begin to establish the second assertion in (4.15). The remaining proof of this lemma is similar to that of Li et al. [19, Lemma 3.2], but more refined techniques are needed to overcome the difficulty due to variable delay. We observe that \( \tilde{\theta}_{\Delta,R} = \theta_{\Delta,R} \Delta \), where

\[
\theta_{\Delta,R} := \inf\{k > 0 : |y_k| \geq R\}.
\]

Clearly, \( \tilde{\theta}_{\Delta,R} \) and \( \theta_{\Delta,R} \) are \( \mathcal{F}_t \) and \( \mathcal{F}_{\tilde{t}_k} \) stopping times, respectively. It is useful to know that

\[
y(k+1)_{\theta_{\Delta,R}} - y_{k_{\theta_{\Delta,R}}} = \mathbb{I}_{\{k<\theta_{\Delta,R}\}}(y_{k+1} - y_k), \ \forall k \geq 0,
\]

see Shiryaev [35, p. 477]. Thus, from (3.8) and (4.30), we have

\[
y(k+1)_{\theta_{\Delta,R}} = y_{k_{\theta_{\Delta,R}}} + \left[f_\Delta(y_k, y_{k-\tilde{\theta}})\Delta + g_\Delta(y_k, y_{k-\tilde{\theta}})\Delta B_k\right]|_{\theta_{\Delta,R}}
\]

Consequently,

\[
\mathbb{E}[y(k+1)_{\theta_{\Delta,R}}^2] = \mathbb{E}[y_{k_{\theta_{\Delta,R}}}^2 + 2 \int_0^{\Delta} \mathbb{E}[f_\Delta(y_{k_{\theta_{\Delta,R}}}, y_{k-\tilde{\theta}_{\Delta,R}})\Delta B_k]|_{\theta_{\Delta,R}}^2 + \mathbb{E}[g_\Delta(y_{k_{\theta_{\Delta,R}}}, y_{k-\tilde{\theta}_{\Delta,R}})\Delta B_k]|_{\theta_{\Delta,R}}^2 |_{\theta_{\Delta,R}}^2]
\]

where

\[
\hat{J}_k = 2 \int_0^{\Delta} \mathbb{E}[f_\Delta(y_{k_{\theta_{\Delta,R}}}, y_{k-\tilde{\theta}_{\Delta,R}})\Delta B_k]|_{\theta_{\Delta,R}}^2 + 2 \int_0^{\Delta} \mathbb{E}[g_\Delta(y_{k_{\theta_{\Delta,R}}}, y_{k-\tilde{\theta}_{\Delta,R}})\Delta B_k]|_{\theta_{\Delta,R}}^2.
\]

Note that

\[
\Delta B_k|_{\theta_{\Delta,R}} = B(t_{k+1}) - B(t_k).
\]

Since \( B(t) \) is a continuous martingale, by the Doob martingale stopping time theorem, we have that

\[
\mathbb{E}[\Delta B_k^2|_{\theta_{\Delta,R}} | \mathcal{F}_{t_{k_{\theta_{\Delta,R}}}}] = 0 \text{ and for any } A \in \mathbb{R}^{d\times N}
\]

\[
\mathbb{E}[|A\Delta B_k^2|_{\theta_{\Delta,R}} | \mathcal{F}_{t_{k_{\theta_{\Delta,R}}}}] = |A|^2 \mathbb{E}[|B(t_{k+1}) - B(t_k)| | \mathcal{F}_{t_{k_{\theta_{\Delta,R}}}}] = |A|^2 \mathbb{E}[|B(t_{k+1}) - B(t_k)| | \mathcal{F}_{t_{k_{\theta_{\Delta,R}}}}] \Delta,
\]

see Li et al. [19, p. 12]. Then

\[
\mathbb{E} J_k = 2 \mathbb{E}[y_{k_{\theta_{\Delta,R}}}^2] \mathbb{E}[\Delta B_k^2|_{\theta_{\Delta,R}} | \mathcal{F}_{t_{k_{\theta_{\Delta,R}}}}] + 2 \mathbb{E}[\hat{J}_k] = 0.
\]

Moreover, by (3.5) and Lemma 4.3, we have

\[
\mathbb{E} \left[ f_\Delta(y_k, y_{k-\tilde{\theta}})^2 \right] \Delta^2 \leq C(\varphi(\Delta))^2 \mathbb{E} \left[ 1 + |y_k|^2 + |y_{k-\tilde{\theta}}|^2 \right] \Delta^2 \leq C \Delta^{3/2} \leq C \Delta.
\]
Plugging (4.23), (4.24) into (4.20) and using Lemmas 3.1, 4.3, we have

\[
E[y_{(k+1)\theta,x}]^2 \leq E[y_{(k+1)\theta,x}^2] + C\Delta + E\left[2\gamma_k y_{(k+1)\theta,x}\right] + \|\delta_\Delta(y_{(k+1)\theta,x})y_{(k+1)\theta,x}\|^2 \Delta
\]

\[
= E[y_{(k+1)\theta,x}^2] + C\Delta + \hat{H}_k \leq E[y_{(k+1)\theta,x}^2] + C\Delta + \hat{H}_k \leq E[y_{(k+1)\theta,x}^2] + C\Delta + \hat{H}_k, \forall k \geq 0,
\]

(4.25)

where

\[
\hat{H}_k = E\left[(-K_3|\Delta(y)|)^2 + K_3|\Delta(y)|^2\right] \leq \Delta.
\]

Thus, for any integer \(k\) satisfying \(1 \leq k \leq \lfloor T/\Delta \rfloor\), we conclude from (4.25) that

\[
E[y_{(k+1)\theta,x}]^2 \leq \|\xi\|^2 + Ck\Delta + \sum_{j=0}^{T/\Delta - 1} E\left[(-K_3|\Delta(y)|)^2 + K_3|\Delta(y)|^2\right] \leq \|\xi\|^2 + C + \sum_{j=0}^{T/\Delta - 1} (-K_3|\Delta(y)|)^2 + K_3|\Delta(y)|^2\] \[
\]

\[
E[y_{(k+1)\theta,x}]^2 \leq \|\xi\|^2 + CT + E\left[\sum_{j=0}^{T/\Delta - 1} (-K_3|\Delta(y)|)^2 + K_3|\Delta(y)|^2\right] \leq \|\xi\|^2 + CT + E\left[\sum_{j=0}^{T/\Delta - 1} (-K_3|\Delta(y)|)^2 + K_3|\Delta(y)|^2\right]
\]

(4.26)

By Lemma 4.2, we get

\[
\sum_{j=0}^{T/\Delta - 1} |\Delta(y)|^2 \leq \left(\lfloor 1 - \tilde{\delta} \rfloor^{-1} + 1\right) \sum_{i=0}^{T/\Delta - 1} |\Delta(y)|^2
\]

\[
= \left(\lfloor 1 - \tilde{\delta} \rfloor^{-1} + 1\right) \sum_{i=0}^{T/\Delta - 1} |\Delta(y)|^2 \leq \left(\lfloor 1 - \tilde{\delta} \rfloor^{-1} + 1\right) \sum_{i=0}^{T/\Delta - 1} |\Delta(y)|^2
\]

(4.27)

Consequently,

\[
\sum_{j=0}^{T/\Delta - 1} \left[(-K_3|\Delta(y)|)^2 + K_3|\Delta(y)|^2\right] \Delta \leq K_3\left(\lfloor 1 - \tilde{\delta} \rfloor^{-1} + 1\right)(M\Delta)\|\xi\|^2 + (K_3\left(\lfloor 1 - \tilde{\delta} \rfloor^{-1} + 1\right) - K_2) \sum_{j=0}^{T/\Delta - 1} |\Delta(y)|^2 \Delta
\]

\[
\leq K_3\left(\lfloor 1 - \tilde{\delta} \rfloor^{-1} + 1\right)\|\xi\|^2, \forall k \geq 1.
\]

(4.28)
Inserting this into (4.26) gives
\[ \mathbb{E}[y_{k}\xi, \theta_{k,h}]^{2} \leq \langle \xi \rangle^{2} + c_{1}T + K_{3}((1 - \delta)^{-1} + 1)\tau \| \xi \|^{p}, \ \forall k = 1, 2, \cdots, [T/\Delta]. \]
In particular, we have
\[ \mathbb{E}[y_{T/\Delta}, \theta_{k,h}]^{2} \leq C, \]
or equivalently,
\[ \mathbb{E}[Z_{1}(T \wedge \hat{\theta}_{\Delta,k})]^{2} \leq C \]
which implies that
\[ R^{2}\mathbb{P}(\hat{\theta}_{\Delta,k} \leq T) \leq \mathbb{E}[\mathbb{I}_{[\theta_{\Delta,k} \leq T]}|Z_{1}(\hat{\theta}_{\Delta,k})^{2}] \leq \mathbb{E}[Z_{1}(T \wedge \hat{\theta}_{\Delta,k})]^{2} \leq C. \]
Thus, the proof is finished. □

**Remark 4.6.** It should be pointed out that if we use the usual continuous proof of Guo et al. [14] Lemma 3.3] to estimate the second assertion in (4.15), then there will be a term \( J^{*} \) with the following form we have to address.

\[ J^{*} := \int_{0}^{t} |\pi_{\Delta}(Z_{2}(s))|^{p} ds - \bar{\kappa} \int_{0}^{t} |\pi_{\Delta}(Z_{1}(s))|^{p} ds, \quad (4.29) \]

where \( \bar{\kappa} = [(1 - \delta)^{-1} + 1] \) and \( \theta \) is a stopping time. Of course, by the known conditions, we can show that
\[ J^{*} \leq \bar{\kappa} \tau \| \xi \|^{p} + \langle |\pi_{\Delta}(\gamma(\theta/\Delta - \delta|\theta_{\Delta}))|^{p} - \bar{\kappa} |\pi_{\Delta}(\gamma(\theta/\Delta))|^{p} \theta - |\theta/\Delta| \rangle. \]

But we see from (4.30) than this estimate remains a "tail", namely, the second term on the right hand side of (4.30), and we have on other method to address \( J^{*} \) properly. However, if we take \( \theta \) to be the grid point, then the "tail" vanishes. This motivates the above discrete proof in Lemma 4.5.

The following theorem establish the strong convergence (without order) results of the truncated EM method.

**Theorem 4.7.** Let Assumptions 2.1, 2.2, 2.3 and 4.4 hold with \( K_{2} \geq ((1 - \delta)^{-1} + 1)K_{3} \geq 0. \) Then for any \( q \in [1, 2), \)
\[ \lim_{\Delta \rightarrow 0} \mathbb{E}[X(T) - Z_{1}(T)]^{q} = 0, \ \forall T > 0. \quad (4.31) \]

**Proof.** Let \( \tau_{R} \) and \( \hat{\theta}_{R,k} \) be the same as before. Let \( q \in [1, 2]. \) Define \( \hat{\theta}_{R,k} = \tau_{R} \wedge \hat{\theta}_{R,k} \) and
\[ \hat{\theta}_{R}(T) = X(T) - Z_{1}(T), \] for any \( T > 0. \) By the Young inequality, for any \( \eta > 0, \) we have
\[ \mathbb{E}[\hat{\theta}_{R}(T)]^{q} = \mathbb{E}[|\hat{\theta}_{R}(T)|^{q}]_{[\theta_{k,h} > Tk+1]} + \mathbb{E}[|\hat{\theta}_{R}(T)|^{q}]_{[\theta_{k,h} \leq Tk+1]} \]
\[ \leq \mathbb{E}[|\hat{\theta}_{R}(T)|^{q}]_{[\theta_{k,h} > Tk+1]} + \frac{q\eta}{2} \mathbb{E}[\hat{\theta}_{R}(T)]^{2} + \frac{2 - q}{2q\eta(2-q)} \mathbb{E}[\hat{\theta}_{R,k} \leq Tk+1]. \]
In this theorem, \( C_{R} \) denotes a positive constant depending on \( R \) but independent of \( \Delta, \) its value may be different for different appearance. By Lemmas 2.4 and 4.3 we get that
\[ \mathbb{E}[\hat{\theta}_{R}(T)]^{q} \leq 2\mathbb{E}[X(T)]^{q} + 2\mathbb{E}[Z_{1}(T)]^{2} \leq C. \]
While from Lemma 4.5 we have

$$P(\theta_{A,R} \leq T + 1) \leq P(\tau_R \leq T + 1) + P(\rho_{A,R} \leq T + 1) \leq \frac{C}{R^2}. $$

Consequently, we have

$$\mathbb{E}[\hat{\varepsilon}_A(T)^q] \leq \frac{q\eta C}{2} + \frac{(2 - q)C}{2R^2 \eta^q(2 - q)} + \mathbb{E}[|\hat{\varepsilon}_A(T)^q|_{\|\rho_{A,R} > T + 1\|}].$$  \hspace{1cm} (4.32)

Let \( \hat{\epsilon} > 0 \) be arbitrary. Choose \( \eta > 0 \) sufficiently small for \( \frac{q\eta C}{2} \leq \hat{\epsilon} \) and then choose \( R \) sufficiently large for \( \frac{(2 - q)C}{2R^2 \eta^q(2 - q)} \leq \hat{\epsilon} \). Then for such chosen \( R \), we see from (4.32) that

$$\mathbb{E}[\hat{\varepsilon}_A(T)^q] \leq \mathbb{E}[|\hat{\varepsilon}_A(T)^q|_{\|\rho_{A,R} > T + 1\|}] + 2\hat{\epsilon}. $$

If we can show that

$$\lim_{\Delta \to 0} \mathbb{E}[|\hat{\varepsilon}_A(T)^q|_{\|\rho_{A,R} > T + 1\|}] = 0, \hspace{1cm} (4.33)$$

the desired assertion (4.31) follows. Define the truncated functions

$$F_R(x, y) = f\left((|x| \cap R) \frac{x}{|x|}, (|y| \cap R) \frac{y}{|y|}\right)$$

and

$$G_R(x, y) = g\left((|x| \cap R) \frac{x}{|x|}, (|y| \cap R) \frac{y}{|y|}\right),$$

for any \( x, y \in \mathbb{R}^d \). Without loss of any generality, we assume that \( \Delta^* \) is sufficiently small for \( \mu^{-1}(b(\Delta^*)) \geq R \). Then, for any \( \Delta \in (0, \Delta^*], \) we get that

$$f_\Delta(x, y) = F_R(x, y) \quad \text{and} \quad g_\Delta(x, y) = G_R(x, y)$$

for any \( x, y \in \mathbb{R}^d \) with \( |x| \cup |y| \leq R \). Consider the following SDE:

$$dz(t) = F_R(z(t), z(t - \delta(t)))dt + G_R(z(t), z(t - \delta(t)))dB(t), \quad t \geq 0, $$

(4.34)

with the initial data \( z(t) = \xi(t) \) on \( t \in [-\tau, 0] \). By Assumption 2.1, we observe that \( F_R(x, y) \) and \( G_R(x, y) \) are globally Lipschitz continuous with the Lipschitz constant \( L_R \). Hence, SDE (4.33) has a unique global solution \( z(t) \) on \( t \geq -\tau \) satisfying

$$P(z(t \wedge \tau_R) = X(t \wedge \tau_R) \text{ for any } t \in [0, T]) = 1. $$

(4.35)

On the other hand, for any \( \Delta \in (0, \Delta^*], \) we apply the (classical) EM method to the SDE (4.34) and we denote \( z_A(t) \) and \( z_A(t) \) by the continuous-time continuous-sample and the piecewise constant EM solutions, respectively. Then we see from Mao and Sabanis [28, Theorem 2.1] that continuous-time continuous-sample EM solution \( z_A(t) \) has the property

$$\mathbb{E}\left[\sup_{0 \leq t \leq T} |z_A(t) - z_A(t)|^q\right] \leq C_2 \Delta^p(0.5 \epsilon_0).$$

(4.36)
where \( c_2 \) is a positive constant dependent of \( L_R, T, \xi, q \) but independent of \( \Delta \). From this and the fact that
\[
\mathbb{E} [z_\Delta (T \wedge (\hat{\rho}_{\Delta R} - 1)) - \hat{z}_\Delta (T \wedge (\hat{\rho}_{\Delta R} - 1))] \leq C_R \Delta^{0.5(q/\sigma_0)}, \tag{4.37}
\]
see Mao and Sabanis [28, Corollary 3.4], we conclude that
\[
\mathbb{E} [z(T \wedge (\hat{\rho}_{\Delta R} - 1)) - z_\Delta (T \wedge (\hat{\rho}_{\Delta R} - 1))] \leq C_R \Delta^{0.5(q/\sigma_0)}. \tag{4.38}
\]
Moreover,
\[
\mathbb{P} \left( Z_\Delta (t \wedge (\hat{\rho}_{\Delta R} - 1)) = z(t \wedge (\hat{\rho}_{\Delta R} - 1)) \right) \text{ for any } t \in [0, T] = 1. \tag{4.39}
\]
Consequently,
\[
\mathbb{E}[|\hat{e}_\Delta (T)|^q \|_{\hat{\rho}_{\Delta R} > T+1}] = \mathbb{E}[|\hat{e}_\Delta (T \wedge (\hat{\rho}_{\Delta R} - 1))|^{q/\sigma_0} \|_{\hat{\rho}_{\Delta R} > T+1}]
\leq \mathbb{E}[|X(T \wedge (\hat{\rho}_{\Delta R} - 1)) - Z_\Delta (T \wedge (\hat{\rho}_{\Delta R} - 1))|^q]
= \mathbb{E}[|z(T \wedge (\hat{\rho}_{\Delta R} - 1)) - \hat{z}_\Delta (T \wedge (\hat{\rho}_{\Delta R} - 1))|^q]
\leq C_R \Delta^{q(0.5/\sigma_0)}, \tag{4.40}
\]
which establish (4.31). Thus, the proof is finished. \( \square \)

4.2. Strong convergence order at time \( T \)

In this section, we mainly discuss the strong convergence order of the truncated EM method for (2.1). We need a stronger condition than (2.3).

**Assumption 4.8.** There is a pair of constants \( p_0 > 2 \) and \( \bar{K}_1 > 0 \) such that
\[
\langle x, f(x, y) \rangle + \frac{p_0 - 1}{2} |g(x, y)|^2 \leq \bar{K}_1 (1 + |x|^2 + |y|^2), \ \forall x_1, x_2 \in \mathbb{R}^d.
\]
We cite a known result as a Lemma, see e.g., [36].

**Lemma 4.9.** Suppose that Assumption 2.1 and 4.8 hold. Then for any given initial data (2.2), there is a unique global solution \( X(t) \) to (2.1) on \( t \in [-\tau, +\infty) \). Moreover, the solution \( X(t) \) has the property that
\[
\sup_{-\tau \leq t \leq T} \mathbb{E}|X(t)|^{p_0} < \infty.
\]

**Lemma 4.10.** Let Assumption 4.8 hold. Then for any \( \Delta \in (0, 1) \),
\[
\langle x, f_\Delta (x, y) \rangle + \frac{p_0 - 1}{2} |g_\Delta (x, y)|^2 \leq 2 \bar{K}_1 \left( 1 \vee \frac{1}{\mu^{-1}(\varphi(1))} \right) (1 + |x|^2 + |y|^2), \ \forall x_1, x_2 \in \mathbb{R}^d.
\]
The proof can be found in Fei et al. [17, Lemma 3.2].

Using the similar techniques in the proofs of Lemma 4.3, we obtain the following lemma which provides an upper bound for the \( p_0 \)-th moment of the numerical solution \( Y_\Delta \).
Lemma 4.11. Let Assumptions 2.1, 4.8, 2.3 hold. Then
\[
\sup_{0 \leq \Delta \leq 1} \sup_{0 \leq t \leq T} \mathbb{E}|Y_\Delta(t)|^{p_0} \leq C,
\]
where C is a positive constant independent of \( \Delta \).

Let \( \mathcal{U} \) be the family of continuous functions \( U : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_+ \) such that for any \( b > 0 \), there exists a positive constant \( \kappa_b \) for which
\[
U(x_1, x_2) \leq \kappa_b |x_1 - x_2|^2, \quad \forall x_1, x_2 \in \mathbb{R}^d.
\]

Assumption 4.12 (Global Monotonicity with \( U \) Function and Polynomial Growth conditions).
There exist constants \( p_1 > 2, l \geq 0 \) and \( \bar{K}_2 > 0 \) as well as a function \( u \in \mathcal{U} \) such that
\[
\langle x_1 - x_2, f(x_1, y_1) - f(x_2, y_2) \rangle + \frac{p_1 - 1}{2} |g(x_1, y_1) - g(x_2, y_2)|^2 \leq \bar{K}_2 (|x_1 - x_2|^2 + |y_1 - y_2|^2) - \frac{1}{1 - \delta} U(x_1, x_2) + U(y_1, y_2), \quad \forall x_1, y_1, x_2, y_2 \in \mathbb{R}^d
\]
and
\[
|f(x_1, y_1) - f(x_2, y_2)|^2 + |g(x_1, y_1) - g(x_2, y_2)|^2 \leq \bar{K}_2 (1 + |x_1|^l + |x_2|^l + |y_1|^l + |y_2|^l) (|x_1 - x_2|^2 + |y_1 - y_2|^2), \quad \forall x_1, y_1, x_2, y_2 \in \mathbb{R}^d.
\]

Thus, from (4.42), we have the following growth condition
\[
|f(x, y)|^2 + |g(x, y)|^2 \leq \bar{K}_3 (1 + |x|^{2+l} + |y|^{2+l}), \quad \forall x, y \in \mathbb{R}^d,
\]
and
\[
|f_\Delta(x, y)|^2 = |f(\pi_\Delta(x), \pi_\Delta(y))|^2 \leq \bar{K}_3 (1 + |\pi_\Delta(x)|^{2+l} + |\pi_\Delta(y)|^{2+l}) \leq \bar{K}_3 (1 + |x|^{2+l} + |y|^{2+l}), \quad \forall x, y \in \mathbb{R}^d,
\]
as well as
\[
|g_\Delta(x, y)|^2 \leq \bar{K}_3 (1 + |x|^{2+l} + |y|^{2+l}), \quad \forall x, y \in \mathbb{R}^d,
\]
where \( \bar{K}_3 \) is a positive constant depending on \( \bar{K}_2 \).

Lemma 4.13. Let Assumptions 2.1, 4.8, 2.3, 4.4 hold with \( p_0 \geq 2 + l \). Then for any \( \Delta \in (0, 1] \) and any \( p \in \left[ 2, \frac{p_0}{1 + 1/2} \right] \)
\[
\mathbb{E}|Y_\Delta(t) - Z_\Delta(t)|^p \leq C \Delta^{0.5p}, \quad \forall t \geq 0,
\]
and
\[
\mathbb{E}|Y_\Delta(t - \delta(t)) - Z_\Delta(t)|^p \leq C \Delta^{(0.5p)}, \quad \forall t \geq 0,
\]
where C is a positive constant independent of \( \Delta \).
Proof. Let \( p \in \left[ 2, \frac{p_0}{1 + l/2} \right] \). For any \( t \in [t_k, t_{k+1}] \) with \( k \geq 0 \), we see from (3.14) that
\[
\mathbb{E}[Y(t) - Z(t)]^p = \mathbb{E}[Y(t) - Y(t_k)]^p \\
\leq C \left( \Delta^p \mathbb{E}[f_\delta(y_k, y_{k-\delta})]^p + \Delta^{0.5p} \mathbb{E}[g_\delta(y_k, y_{k-\delta})]^p \right).
\]
By (4.45), we have
\[
\mathbb{E}[f_\delta(y_k, y_{k-\delta})]^p \leq C \mathbb{E} \left( 1 + |y_k|^{2+\epsilon} + |y_{k-\delta}|^{2+\epsilon} \right)^{p/2} \\
\leq C \left( 1 + \mathbb{E}|y_k|^{1+\epsilon/2}p + \mathbb{E}|y_{k-\delta}|^{1+\epsilon/2}p \right) \leq C,
\]
where Lemma 4.9 has been used. Similarly, we can show that
\[
\mathbb{E}[g_\delta(y_k, y_{k-\delta})]^p \leq C.
\]
Thus,
\[
\mathbb{E}[Y(t) - Z(t)]^p \leq C \Delta^p + C \Delta^{0.5p} \leq C \Delta^{0.5p}.
\]
We now begin to establish assertion (4.48) and recall that \( \delta_k = \lfloor \delta(t_k)/\Delta \rfloor \) and
\[
Y(t - \delta(t)) - Z(t) = Y(t - \delta(t)) - Y_k((k - \delta_k)\Delta).
\]
By Assumption 2.3, we have the following useful estimate
\[
|t - \delta(t)) - (k - \delta_k)\Delta| (\lceil \delta \rceil + 4)\Delta,
\]
see Deng et al. [37] Lemma 4.6. From now on, we will use \( c_p \) to denote genetic positive constants dependent only on \( p \) and its values may change between occurrences. Now consider the following four possible cases.

Case 1: If \( t - \delta(t) \geq (k - \delta_k)\Delta \geq 0 \) or \( (k - \delta_k)\Delta \geq t - \delta(t) \geq 0 \), then it follows from (4.51) that
\[
\mathbb{E}[Y(t - \delta(t)) - Z(t)]^p \leq c_p |t - \delta(t) - (k - \delta_k)\Delta|^{p-1} \int_{(k-\delta_k)\Delta}^{(k-\delta_k)\Delta} \mathbb{E}[f_\delta(Z_1(s), Z_2(s))] ds \\
+ c_p |t - \delta(t) - (k - \delta_k)\Delta|^{0.5p-1} \int_{(k-\delta_k)\Delta}^{(k-\delta_k)\Delta} \mathbb{E}[g_\delta(Z_1(s), Z_2(s))] ds \\
\leq C (|t - \delta(t) - (k - \delta_k)\Delta|^{0.5p} \\
\leq C (\lceil \delta \rceil + 4)\Delta^{0.5p},
\]
where the Burkholder-Davis-Gundy inequality, (4.49), (4.50) and (4.52) have been used.

Case 2: If \( t - \delta(t) \leq (k - \delta_k)\Delta \leq 0 \) or \( (k - \delta_k)\Delta \leq t - \delta(t) \leq 0 \), by Assumption 4.4 and (4.52) we have
\[
\mathbb{E}[Y(t - \delta(t)) - Z(t)]^p = \mathbb{E}[\xi(t - \delta(t)) - \xi((k - \delta_k)\Delta)]^p \\
\leq (K^p \lceil \delta \rceil + 4)\Delta^{0p}.
\]
Case 3: If \( t - \delta(t) \geq (k - \delta_k)\Delta \), then

\[
t - \delta(t) \leq ((\hat{\delta}) + 4)\Delta \quad \text{and} \quad - (k - \delta_k)\Delta \leq ((\hat{\delta}) + 4)\Delta.
\]

Thus, we have

\[
\mathbb{E}[Y_A(t - \delta(t)) - Z_\Delta(t)]^p = \mathbb{E}[Y_A(t - \delta(t)) - Y_A((k - \delta_k)\Delta)]^p
\leq 2^{p-1}\mathbb{E}[Y_A(t - \delta(t)) - \xi(0)]^p + 2^{p-1}\mathbb{E}[\xi(0) - \xi((k - \delta_k)\Delta)]^p
\leq C(t - \delta(t))^{0.5p} + CK_4^p((- (k - \delta(k)))^p
\leq C((\hat{\delta}) + 4)^{0.5p} + ((\hat{\delta}) + 4)^{0.5p} \Delta^{0.5p}.
\]

Case 4: If \((k - \delta_k)\Delta \geq 0 \geq t - \delta(t)\), in a similar way as (4.55) was obtained, we also have

\[
\mathbb{E}[Y_A(t - \delta(t)) - Z_\Delta(t)]^p \leq C((\hat{\delta}) + 4)^{0.5p} + ((\hat{\delta}) + 4)^{0.5p} \Delta^{0.5p}.
\]

Combining these different cases together, we obtain the desired assertion (4.48).

We then can obtain the optimal rate of strong convergence for truncated EM approximation.

**Theorem 4.14.** Let Assumptions 4.8, 4.12, 2.3 and 4.4 hold with \( p_0 \geq 2 + 3l \). Then for any \( \Delta \in (0, 1] \),

\[
\mathbb{E}[X(T) - Y_\Delta(T)]^2 \leq C \big( \Delta^{2p} + [\mu^{-1}(\varphi(\Delta))]^{(p_0 - l - 2)} \big), \quad \forall T > 0,
\]

and

\[
\mathbb{E}[X(T) - Z_\Delta(T)]^2 \leq C \big( \Delta^{2p} + [\mu^{-1}(\varphi(\Delta))]^{(p_0 - l - 2)} \big), \quad \forall T > 0,
\]

where \( C \) is a positive constant independent of \( \Delta \). In particular, let

\[
\mu(r) = \hat{K}^{1/2} r^{l/2}, \quad \forall r \geq 1 \quad \text{and} \quad \varphi(\Delta) = \hat{h} \Delta^{-1/4}, \quad \forall \Delta \in (0, 1].
\]

Then for any \( \Delta \in (0, 1] \),

\[
\mathbb{E}[X(T) - Y_\Delta(T)]^2 \leq C\Delta^{-l/2} \quad \text{and} \quad \mathbb{E}[X(T) - Z_\Delta(T)]^2 \leq C\Delta^{-l/2}, \quad \forall T > 0.
\]

**Remark 4.15.** If constant delay is considered in (2.1), then Theorem 4.14 reduces to Fei et al. [17] Theorem 3.6. Comparing with the two theorems, we observe the following differences:

- Theorem 4.14 requires a slightly stronger condition on the parameters, namely, \( p_0 \geq 2 + 3l \), while Fei et al. [17] Theorem 3.6] requires \( p_0 > 2 + l \);
- Theorem 4.14 allows a slightly weaker control function \( \mu \) for truncation, namely, \( \mu(r) = C r^{l/2} \), while Fei et al. [17] Theorem 3.6] allows \( \mu(r) = C r^{l/2} \);
- In Theorem 4.14 the truncated EM approximation \( Y_\Delta(T) \) achieves a better order of \( L^2 \)-convergence which is \( 2\Delta \wedge 1 \), while in Fei et al. [17] Theorem 3.6] the corresponding convergence order is \( 2\Delta \wedge (1 - 2\varepsilon) \wedge 2\varepsilon(p_0 - l - 2)/(2 + l) \) for some \( \varepsilon \in (0, 1/4] \), which can be close to the rate of Theorem 4.14 but \( p_0 \) should be required to sufficiently large.

**Remark 4.16.** It is worth mentioning how our work compares with that of Cong et al. [32], who proved the strong convergence results of partially truncated EM method applied to the SDDEs with variable delay and Markovian switching. What differentiates our work from [32] are:
• We remove the restrictive condition (3.25) in [32, Theorem 3.12], i.e.,

\[
h(\Delta) \geq \mu((\Delta^{2^\nu} \lor \Delta^2)^{-(p-2)}),
\]

which could force the step size to be so small that the truncated EM method would be inapplicable;

• We relax the grown constraint on the Khasminskii-type condition (2.13) in [32, Assumption 2.3], i.e.,

\[
2(x, f(x, y)) + |g(x, y)| \leq K(1 + |x|^2 + |y|^2), \quad \forall x, y \in \mathbb{R}^d,
\]

by a generalized Khasminskii-type condition (2.3), and global monotonicity condition (3.15) in [32, Assumption 3.9], i.e.,

\[
\langle x_1 - x_2, f(x_1, y_1) - f(x_2, y_2) \rangle + \frac{p_1 - 1}{2}|g(x_1, y_1) - g(x_2, y_2)|^2 \\
\leq K(|x_1 - x_2|^2 + |y_1 - y_2|^2), \quad \forall x_1, y_1, x_2, y_2 \in \mathbb{R}^d
\]

by the global monotonicity condition with U function (4.41);

• We obtain the optimal order 1/2 of strong convergence which is higher than that of Cong et al. [32, Theorem 3.12], i.e.,

\[
\mathbb{E}|X(T) - Y_\Delta(T)|^2 \leq C(\Delta \lor \Delta^2(\Delta)),
\]

under a slightly stronger condition on the parameters.

**Proof of Theorem 4.1** Let \( p_0 \geq 2 + 3\nu \) and \( R \geq ||\xi|| \), define the stopping time

\[
\rho_R = \inf\{t \geq 0 : |X(t)| \lor |Y_\Delta(t)| \geq R\}.
\]

Set \( e_\Delta(t) = X(t) - Y_\Delta(t) \), for any \( t \in [-\tau, T] \), which means that \( e_\Delta(t) = 0 \), for any \( t \in [-\tau, 0] \). By the Itô formula, we have that for any \( t \in [0, T] \),

\[
\mathbb{E}|e_\Delta(t \wedge \rho_R)|^2 \leq \mathbb{E} \int_0^{t \wedge \rho_R} \left( (X(s) - Y_\Delta(s), f(X(s), X(s - \delta(s))) - f_\Delta(Z_1(s), Z_2(s))) + |g(X(s), X(s - \delta(s))) - g_\Delta(Z_1(s), Z_2(s))|^2 \right) ds \\
+ \mathbb{E} \int_0^{t \wedge \rho_R} \left( (Y_\Delta(s), f(X(s), X(s - \delta(s))) - f(Y_\Delta(s), Y_\Delta(s - \delta(s)))) + (p_1 - 1)|g(X(s), X(s - \delta(s))) - g(Y_\Delta(s), Y_\Delta(s - \delta(s)))|^2 \right) ds \\
+ \mathbb{E} \int_0^{t \wedge \rho_R} \left( 2(X(s) - Y_\Delta(s), f(Y_\Delta(s), Y_\Delta(s - \delta(s))) - f_\Delta(Z_1(s), Z_2(s))) + \frac{p_1 - 1}{p_1 - 2}|g(Y_\Delta(s), Y_\Delta(s - \delta(s))) - g_\Delta(Z_1(s), Z_2(s))|^2 \right) ds \\
\leq \Pi_1 + \Pi_2,
\]

(4.64)
where

\[
\Pi_1 := \mathbb{E} \int_0^{\mathcal{E}^{\eta, \rho}_T} \left( |e_\Delta(s)|^2 + 2K_2(|e_\Delta(s)|^2 + |X(s - \delta(s)) - Y_\Delta(s - \delta(s))|^2 \right)
- \frac{2}{1 - \delta} U(X(s), Y_\Delta(s)) + 2U(X(s - \delta(s)), Y_\Delta(s - \delta(s))) ds,
\]

\[
\Pi_2 := \mathbb{E} \int_0^{\mathcal{E}^{\eta, \rho}_T} \left( |f(Y_\Delta(s), Y_\Delta(s - \delta(s))) - f_\Delta(Z_1(s), Z_2(s))|^2 
+ \frac{p_1 - 1}{p_1 - 2} |g(Y_\Delta(s), Y_\Delta(s - \delta(s))) - g_\Delta(Z_1(s), Z_2(s))|^2 \right) ds,
\]

where Assumption \ref{4.12} has been used. Noticing that \(U(X(s), Y_\Delta(s)) = 0\) for any \(s \in [-\tau, 0]\), we then have

\[
\int_0^{\mathcal{E}^{\eta, \rho}_T} U(X(s - \delta(s)), Y_\Delta(s - \delta(s))) ds \leq \frac{1}{1 - \delta} \int_0^{\mathcal{E}^{\eta, \rho}_T} U(X(s), Y_\Delta(s)) ds
\]

and

\[
\int_0^{\mathcal{E}^{\eta, \rho}_T} |X(s - \delta(s)) - Y_\Delta(s - \delta(s))|^2 ds \leq \frac{1}{1 - \delta} \int_0^{\mathcal{E}^{\eta, \rho}_T} |X(s) - Y_\Delta(s)|^2 ds.
\]

Consequently,

\[
\Pi_1 \leq \left(1 + 2K_1 + \frac{2K_2}{1 - \delta}\right) \mathbb{E} \int_0^{\mathcal{E}^{\eta, \rho}_T} |e_\Delta(s)|^2 ds \leq \left(1 + 2K_1 + \frac{2K_2}{1 - \delta}\right) \mathbb{E} \int_0^{\mathcal{E}^{\eta, \rho}_T} |\varphi_\Delta(s, \rho_R)|^2 ds. \quad (4.65)
\]

In order to estimate \(\Pi_2\), we have the

\[
\Pi_2 \leq \Pi_{21} + \Pi_{22},
\]

where

\[
\Pi_{21} = 2 \int_0^T \mathbb{E}[f(Y_\Delta(s), Y_\Delta(s - \delta(s))) - f_\Delta(Y_\Delta(s), Y_\Delta(s - \delta(s)))|^2 ds 
+ \frac{2(p_1 - 1)}{p_1 - 2} \int_0^T \mathbb{E}[g(Y_\Delta(s), Y_\Delta(s - \delta(s))) - f_\Delta(Y_\Delta(s), Y_\Delta(s - \delta(s)))|^2 ds,
\]

and

\[
\Pi_{22} = 2 \int_0^T \mathbb{E}[f_\Delta(Y_\Delta(s), Y_\Delta(s - \delta(s))) - f_\Delta(Z_1(s), Z_2(s))|^2 ds 
+ \frac{2(p_1 - 1)}{p_1 - 2} \int_0^T \mathbb{E}[g_\Delta(Y_\Delta(s), Y_\Delta(s - \delta(s))) - g_\Delta(Z_1(s), Z_2(s))|^2 ds.
\]

Noting that \(p_0 \geq 2 + 3l > 2 + l\), in a similar way/fashion as Estimate (3.15) from \cite{17} Theorem 3.6 was proved, we also can show that

\[
\Pi_{21} \leq C [\mu^{-1}(\varphi(\Delta))]^{-(p_0 - l - 2)}. \quad (4.66)
\]
Recall [17, Lemma 3.3] that

\[ |f_\Delta(x_1, y_1) - f_\Delta(x_2, y_2)|^2 \leq K_2 (1 + |x_1|^2 + |x_2|^2 + |y_1|^2 + |y_2|^2)(|x_1 - x_2|^2 + |y_1 - y_2|^2), \]

for any \( x_1, y_1, x_2, y_2 \in \mathbb{R}^d \). By the condition that \( p_0 \geq 2 + 3l \) which implies \( \frac{2p_0}{p_0 - l} \leq \frac{p_0}{1 + l/2} \), \(4.42\) and the Hölder inequality as well as Lemma 4.13, we have that for \( s \in [0, T] \),

\[
\mathbb{E}|f_\Delta(Y_\Delta(s), Y_\Delta(s - \delta(s))) - f_\Delta(Z_1(s), Z_2(s))|^2 \\
\leq K_2 \mathbb{E} \left[ |Y_\Delta(s) - Z_1(s)|^2 + |Y_\Delta(s - \delta(s)) - Z_2(s)|^2 \right] \\
\times \left[ 1 + |Y_\Delta(s)|^2 + |Y_\Delta(s - \delta(s))|^2 + |Z_1(s)|^2 + |Z_2(s)|^2 \right] \\
\leq C \left( \mathbb{E}|Y_\Delta(s) - Z_1(s)|^{2p_0/(p_0 - l)} + \mathbb{E}|Y_\Delta(s - \delta(s)) - Z_2(s)|^{2p_0/(p_0 - l)} \right)^{(p_0 - l)/p_0} \\
\times (1 + \mathbb{E}|Y_\Delta(s)|^p + \mathbb{E}|Y_\Delta(s - \delta(s))|^p + \mathbb{E}|Z_1(s)|^p + \mathbb{E}|Z_2(s)|^p)^{l/p_0} \\
\leq C \left( \mathbb{E}|Y_\Delta(s) - Z_1(s)|^{2p_0/(p_0 - l)} \right)^{(p_0 - l)/p_0} + C \left( \mathbb{E}|Y_\Delta(s - \delta(s)) - Z_2(s)|^{2p_0/(p_0 - l)} \right)^{(p_0 - l)/p_0} \\
\leq C \Delta^{1/2e}. \tag{4.67}
\]

Similarly, we can show that

\[
\mathbb{E}|g_\Delta(Y_\Delta(s), Y_\Delta(s - \delta(s))) - g_\Delta(Z_1(s), Z_2(s))|^2 \leq C \Delta^{1/2e}. \tag{4.68}
\]

Combining \(4.64\) and \(4.68\), we get

\[
\mathbb{E}|e_\Delta(t \wedge \rho_R)|^2 \leq \mathbb{E} \int_0^{t \wedge \rho_R} |e_\Delta(s \wedge \rho_R)|^2 ds + C \left( \Delta^{2e/1} \vee |\mu^{-1}(\varphi(\Delta))|^{-(p_0 - l - 2)} \right). \tag{4.69}
\]

The Gronwall inequality gives

\[
\mathbb{E}|e_\Delta(t \wedge \rho_R)|^2 \leq C \left( \Delta^{2e/1} \vee |\mu^{-1}(\varphi(\Delta))|^{-(p_0 - l - 2)} \right). \tag{4.70}
\]

Letting \( R \to \infty \) gives assertion \(4.57\). \(4.58\) follows from \(4.57\) and Lemma 4.13. Finally, from \(4.44\) and \(3.1\), we may define \( \mu(R) \) and \( \varphi(\Delta) \) by \(4.59\), e.g.,

\[
\mu(r) = K_1^{1/2} r^{1/2}, \quad \forall r \geq 1 \quad \text{and} \quad \varphi(\Delta) = \int_0^{\Delta} \varphi(\Delta), \quad \forall \Delta \in (0, 1].
\]

Then

\[
|\mu^{-1}(\varphi(\Delta))|^{-(p_0 - l - 2)} = C \Delta^{0.5(p_0 - l - 2)/l} \leq C \Delta, \tag{4.71}
\]

due to \( p_0 \geq 2 + 3l \), which implies that \( 0.5(p_0 - l - 2)/l \geq 1 \). From \(4.71\) and \(4.57\) as well as \(4.58\), we obtain the assertion \(4.60\). Thus, the proof is complete. \( \square \)

5. Mean-square and \( H_\infty \) stabilities

In this section, we mainly discuss the mean-square and \( H_\infty \) stabilities of the truncated EM method for SDDE \(2.1\). Noting that the truncated functions \( f_\Delta \) and \( g_\Delta \) can preserve the Khasminskii-type condition \(3.6\), unfortunately they cannot preserve the stability condition. Thus, we can
only hope that the terms that work for the stability in the coefficients grows at most linearly, while for those that grow super-linearly, they have no stabilizing effect. In our truncated method for stability, we only use the truncation technique to the super-linear terms in the coefficients.

Lemma 5.4 show that those partially truncated functions defined by (5.10) have the property of keeping the stability condition.

As a result, we assume that \( f \) and \( g \) can be decomposed as \( f(x,y) = F_1(x,y) + F(x,y) \) and \( g(x,y) = G_1(x,y) + G(x,y) \), where \( F_1,F : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d \) and \( G_1,G : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times m} \). Moreover, \( F_1(0,0) = F(0,0) = G_1(0,0) = G(0,0) = 0 \), the coefficients \( F_1, F, G_1, G \) satisfy the following conditions.

**Assumption 5.1.** For any \( R > 0 \), there exists constants \( L \) and \( \bar{L}_R \) depending on \( R \) such that

\[
|F_1(x_1,y_1) - F_1(x_2,y_2)|^2 \leq L(|x_1 - x_2|^2 + |y_1 - y_2|^2),
\]  
for any \( x_1, x_2, y_1, y_2 \in \mathbb{R}^d \) and

\[
|F(x_1,y_1) - F(x_2,y_2)|^2 \leq \bar{L}_R(|x_1 - x_2|^2 + |y_1 - y_2|^2),
\]  
for any \( x_1, x_2, y_1, y_2 \in \mathbb{R}^d \) with \( |x_1| \leq |x_2| \leq |y_1| \leq |y_2| \leq R.\]

**Assumption 5.2.** There exist nonnegative constants \( \theta, \lambda_1, \lambda_2, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \) and \( \beta > 2 \) such that

\[
2(x,F_1(x,y)) + (1 + \theta)|G_1(x,y)|^2 \leq -\lambda_1|x|^2 + \lambda_2|y|^2, \quad \forall x,y \in \mathbb{R}^d, \tag{5.2}
\]

\[
2(x,F(x,y)) + (1 + \theta^{-1})|G(x,y)|^2 \leq \alpha_1|x|^2 + \alpha_2|y|^2 - \alpha_3|x|^{\beta} + \alpha_4|y|^{\beta}, \quad \forall x,y \in \mathbb{R}^d. \tag{5.3}
\]

When \( \theta = 0 \), we set \( \theta^{-1} |G(x,y)|^2 = 0 \), when \( \theta = \infty \), we set \( \theta |G(x,y)|^2 = 0 \). Clearly, Assumption 5.2 implies that

\[
2(x,f(x,y)) + |g(x,y)|^2 \leq -(\lambda_1 - \alpha_1)|x|^2 + (\lambda_2 + \alpha_2)|y|^2 - \alpha_3|x|^{\beta} + \alpha_4|y|^{\beta}, \quad \forall x,y \in \mathbb{R}^d. \tag{5.4}
\]

We conclude from Song et al. [33, Theorem 3.6] that the SDDE (2.1) is stable in mean square sense, which can be stated by the following lemma.

**Lemma 5.3.** Let Assumptions 5.1, 5.2, and 2.3 hold with

\[
\lambda_1 > \alpha_1 + \frac{1}{4} \lambda_2 + ((1 - \delta)^{-1} - 1)(\lambda_2 + \alpha_2) \quad \text{and} \quad \alpha_3 > ((1 - \delta)^{-1} - 1)\alpha_4 \geq 0, \tag{5.5}
\]

where \( ((1 - \delta)^{-1} - 1) = \lfloor 1/(1 - \delta) \rfloor + 1 \). Then for any given initial data \( (2.2) \), the unique global solution \( X(t) \) to (2.1) has the property that

\[
\limsup_{t \to \infty} \frac{\log \mathbb{E}[X(t)]^2}{t} \leq - \frac{\gamma^*}{\tau} \log \left( \frac{\alpha_3}{((1 - \delta)^{-1} - 1)\alpha_4} \right), \tag{5.6}
\]

and

\[
\int_0^\infty \mathbb{E}[X(t)]^2 dt < \infty, \tag{5.7}
\]

where \( \gamma^* > 0 \) is the unique root to the following equation

\[
\lambda_1 = \left( \alpha_1 + \frac{1}{4} \lambda_2 \right)^2 + ((1 - \delta)^{-1} - 1)(\lambda_2 + \alpha_2)e^{\gamma^*} + \gamma^*. \tag{5.8}
\]
Lemma 5.4. Let Assumptions 5.1, 5.2, and 5.3 hold with

\[ \lambda_1 > \alpha_1 + \frac{1}{4} \lambda_2 + (1 - \delta^{-1}) + (\alpha_2 + \alpha_3) \quad \text{and} \quad \alpha_3 > (1 - \delta^{-1}) + \alpha_4 \geq 0. \]  

For a given step size \( \Delta \in (0, 1] \), define

\[ f\alpha(x, y) = F_1(x, y) + F\alpha(x, y) \quad \text{and} \quad g\alpha(x, y) = G_1(x, y) + G\alpha(x, y), \quad \forall x, y \in \mathbb{R}^d, \]  

where \( F\alpha(x, y) = F(\pi\alpha(x), \pi\alpha(y)) \) and \( G\alpha(x, y) = G(\pi\alpha(x), \pi\alpha(y)) \). Then

\[
2(x, f\alpha(x, y)) + |g\alpha(x, y)|^2 \leq -(\lambda_1 - \alpha_1 - \frac{1}{4} \alpha_2) |x|^2 + (\lambda_2 + \alpha_2) |y|^2
- \alpha_3 |\pi\alpha(x)|^2 + \alpha_4 |\pi\alpha(y)|^2, \quad \forall x, y \in \mathbb{R}^d,
\]

and

\[
|f\alpha(x, y)|^2 \Delta \leq \epsilon\alpha(|x|^2 + |y|^2), \quad \forall x, y \in \mathbb{R}^d,
\]

where \( \epsilon\alpha = (4\bar{L} + 2\bar{L}_1)\Delta + 8(\varphi(\Delta))^2\Delta \).

Proof. For any \( x \in \mathbb{R}^d \) with \( |x| \leq \mu^{-1}(\varphi(\Delta)) \) and any \( y \in \mathbb{R}^d \), (5.11) follows from Assumption 5.2. While for any \( x \in \mathbb{R}^d \) with \( |x| > \mu^{-1}(\varphi(\Delta)) \) and any \( y \in \mathbb{R}^d \), in a similar way as Fei et al. [17]. Inequality (3.4) was obtained, we also derive from Assumption 5.2 that

\[
2(x, f\alpha(x, y)) + |g\alpha(x, y)|^2 \leq -\lambda_1 |x|^2 + \lambda_2 |y|^2 - \alpha_3 |\pi\alpha(x)|^2 + \alpha_4 |\pi\alpha(y)|^2
+ \frac{|x|}{\mu^{-1}(\varphi(\Delta))} (\alpha_1 |\pi\alpha(x)|^2 + \alpha_2 |\pi\alpha(y)|^2)
\leq -\lambda_1 |x|^2 + \lambda_2 |y|^2 - \alpha_3 |\pi\alpha(x)|^2 + \alpha_4 |\pi\alpha(y)|^2
+ \alpha_1 |x|^2 + \alpha_2 |y|^2.
\]

Now, using the inequality \( |x||y| \leq \frac{1}{4} |x|^2 + |y|^2 \) for any \( x, y \in \mathbb{R}^d \) and rearranging the RHS terms of the last inequality in (5.13), we obtain (5.11). Now let us establish (5.12). From Assumption 5.1 and the property of truncated function, we have that

\[
|f\alpha(x, y)|^2 = |F_1(x, y) + F\alpha(x, y)|^2
\leq 2|F_1(x, y)|^2 + 2|F\alpha(x, y)|^2 \leq 2(\bar{L} + \bar{L}_1)(|x|^2 + |y|^2), \quad \text{if} \quad |x| \leq 1, \ |y| \leq 1,
\]

and

\[
|f\alpha(x, y)|^2 \leq 2\bar{L}(|x|^2 + |y|^2) + 2(\varphi(\Delta))^2(1 + |x| + |y|^2)
\leq (2\bar{L} + 8(\varphi(\Delta))^2)(|x|^2 + |y|^2), \quad \text{if} \quad |x| \leq 1, \ |y| > 1.
\]

This also holds for \( |x| > 1, |y| \leq 1 \) or \( |x| \geq 1, |y| \geq 1 \). Thus, the proof is finished. □

The following theorem shows that the partially truncated EM solution can share the mean-square and \( H_{\infty} \) stabilities of the true solution.
Theorem 5.5. Let Assumptions 5.1, 5.2 and 2.3 hold with

\[ A_1 > \alpha_1 + \frac{1}{4} \lambda_2 + (\alpha_1 + \alpha_2) \quad \text{and} \quad \alpha_3 > (\alpha_1 + \alpha_2 + \epsilon_3) \geq 0. \]

Choose \( \Delta^* \in (0, 1] \) satisfying

\[ \epsilon_\Delta^* = (4 \hat{L} + 2 \hat{L}_1) \Delta^* + 8(h(\Delta^*))^2 \Delta^* = \frac{A_1 - \alpha_1 - \frac{1}{4} \lambda_2 - (A_1 - \lambda_1 + 1)(\alpha_2 + \alpha_3)}{1 + ((1 - \lambda_1 + 1) + 1).} \tag{5.16} \]

Then for any \( \Delta \in (0, \Delta^*) \) and any initial data \( (y_k)_{k \geq 0} \), the truncated EM approximation \( (y_k)_{k \geq 0} \) with the truncated coefficients \( f_\Delta \) and \( g_\Delta \) given by (5.10) has the property that

\[ \limsup_{k \to \infty} \frac{\log \mathbb{E}[y_k^2]}{t_k} \leq -\left( \gamma_\Delta^* \wedge \frac{1}{\tau} \log \frac{\alpha_3}{((1 - \lambda_1 + 1) + \alpha_4)} \right), \tag{5.17} \]

where \( \gamma_\Delta^* \) is the unique root to the following equation

\[ A_1 = (A_1 + \frac{1}{4} \lambda_2 + \epsilon_3) + ((1 - \lambda_1 + 1)(\alpha_2 + \epsilon_3) e^\gamma_\Delta + 1 - e^{-\gamma_\Delta} \Delta). \tag{5.18} \]

Moreover,

\[ \lim_{\Delta \to 0} \gamma_\Delta^* = \gamma^*, \tag{5.19} \]

and

\[ \int_0^\infty \mathbb{E}[Z_t(s)]^2 ds < \infty. \tag{5.20} \]

Proof. From (5.8), we have

\[ |y_{k+1}|^2 = |y_k|^2 + 2(y_k, f_\Delta(y_k, y_{k-\Delta})) + g_\Delta(y_k, y_{k-\Delta})^2 + |f_\Delta(y_k, y_{k-\Delta})|^2 \Delta^2 + J_k, \forall k \geq 0, \tag{5.21} \]

where

\[ J_k = 2(y_k, g_\Delta(y_k, y_{k-\Delta}) B_k) + 2(f_\Delta(y_k, y_{k-\Delta}), g_\Delta(y_k, y_{k-\Delta}) \Delta B_k) + g_\Delta(y_k, y_{k-\Delta})^2 (|DB_k|^2 - \Delta). \]

Obviously, \( \mathbb{E}[J_k] = 0 \). Using Lemma 5.4 yields

\[ 2(y_k, f_\Delta(y_k, y_{k-\Delta})) + |g_\Delta(y_k, y_{k-\Delta})|^2 \leq -(A_1 - \alpha_1 - \frac{1}{4} \lambda_2) |y_k|^2 + (A_1 + \alpha_2) |y_{k-\Delta}|^2 - \alpha_3 |\pi_\Delta(y_k)|^2 + \alpha_4 |\pi_\Delta(y_{k-\Delta})|^2. \tag{5.22} \]

Inserting (5.22) into (5.21) and using (5.12), we have

\[ |y_{k+1}|^2 \leq |y_k|^2 - (A_1 - \alpha_1 - \frac{1}{4} \lambda_2 - \epsilon_3) |y_k|^2 \Delta + (A_1 + \alpha_2 + \epsilon_3) |y_{k-\Delta}|^2 \Delta \\
- \alpha_3 |\pi_\Delta(y_k)|^2 \Delta + \alpha_4 |\pi_\Delta(y_{k-\Delta})|^2 \Delta + J_k, \forall k \geq 0. \tag{5.23} \]
For an arbitrary constant $r > 1$, we see from (5.23) that
\[
\begin{align*}
\mu r^{(k+1)\lambda} E[y_{k+1}^2] &- r^{k\lambda} E[y_k^2] \\
&\leq (\mu r^{(k+1)\lambda} - r^{k\lambda}) E[y_k^2] + (\lambda_1 - \alpha_1 - 1) r^{(k+1)\lambda} E[y_k^2] \Delta + (\lambda_2 + \alpha_2 + \epsilon_\Delta) r^{(k+1)\lambda} E[y_k] \Delta \\
&+ E \left[ -\alpha_3 r^{(k+1)\lambda} |\pi_{\Delta} (y_k)|^\beta \Delta + \alpha_4 r^{(k+1)\lambda} |\pi_{\Delta} (y_{k-\delta})|^\beta \right] \Delta, \quad \forall k \geq 0,.
\end{align*}
\]  

(5.24)

Consequently,
\[
\begin{align*}
\mu r^{(k+1)\lambda} E[y_{k+1}^2] &\leq |\xi(0)|^2 + \left( (\lambda_1 - \alpha_1 - 1) r^\Delta + 1 - r^{-\Delta} \right) \sum_{j=0}^k r^{(j+1)\lambda} E[y_j]^2 \\
&+ (\lambda_2 + \alpha_2 + \epsilon_\Delta) \sum_{j=0}^k r^{(j+1)\lambda} E[y_j] \Delta \\
&+ E \left[ -\alpha_3 \sum_{j=0}^k r^{(j+1)\lambda} |\pi_{\Delta} (y_j)|^\beta + \alpha_4 \sum_{j=0}^k r^{(j+1)\lambda} |\pi_{\Delta} (y_{j-\delta})|^\beta \right] \Delta, \quad \forall k \geq 0,.
\end{align*}
\]  

(5.25)

By Lemma 4.2 we get
\[
\begin{align*}
\sum_{j=0}^k r^{(j+1)\lambda} |y_{j-\delta}|^2 &\leq \left( ((1-\hat{\delta})^{-1} + 1) r^\Delta \sum_{j=-M}^k r^{(j+1)\lambda} |y_j|^2 \\
&= \left( ((1-\hat{\delta})^{-1} + 1) r^\Delta \right) \sum_{j=-M}^k r^{(j+1)\lambda} |y_j|^2 + \left( (1-\hat{\delta})^{-1} + 1 \right) r^\Delta \sum_{j=0}^k r^{(j+1)\lambda} |y_j|^2 \\
&\leq \left( \frac{(1-\hat{\delta})^{-1} + 1}{1 - r^{-\Delta}} \right) ||\xi||^2 + \kappa r^\Delta \sum_{j=0}^k r^{(j+1)\lambda} |y_j|^2, \quad \forall k \geq 0,
\end{align*}
\]  

(5.26)

and
\[
\begin{align*}
\sum_{j=0}^k r^{(j+1)\lambda} |\pi_{\Delta} (y_{j-\delta})|^\beta &\leq \left( \frac{(1-\hat{\delta})^{-1} + 1}{1 - r^{-\Delta}} \right) ||\xi||^\beta + \kappa r^\Delta \sum_{j=0}^k r^{(j+1)\lambda} |\pi_{\Delta} (y_j)|^\beta, \quad \forall k \geq 0.
\end{align*}
\]  

(5.27)

Inserting (5.26) and (5.27) into (5.25) gives that
\[
\begin{align*}
\mu r^{(k+1)\lambda} E[y_{k+1}^2] &\leq H_0(r, \Delta) - H_1(r, \Delta) \sum_{j=0}^k r^{(j+1)\lambda} E[y_j] \Delta \\
&- H_2(r) \sum_{j=0}^k r^{(j+1)\lambda} E[|\pi_{\Delta} (y_j)|^2 \Delta], \quad \forall k \geq 0,
\end{align*}
\]  

(5.28)

where
\[
\begin{align*}
H_0(r, \Delta) &= ||\xi||^2 + \left( (1-\hat{\delta})^{-1} + 1 \right) r^\Delta \left[ (\lambda_2 + \alpha_2 + \epsilon_\Delta) ||\xi||^2 + \alpha_4 ||\xi||^\beta \right] \frac{\Delta}{1 - r^{-\Delta}}, \\
H_1(r, \Delta) &= \left( \lambda_1 - \alpha_1 - \frac{1}{4} \alpha_2 - \epsilon_\Delta - (1-\hat{\delta})^{-1} + 1 \right) r^\Delta (\lambda_2 + \alpha_2 + \epsilon_\Delta) - \frac{1 - r^{-\Delta}}{\Delta}, \\
H_2(r) &= \alpha_3 - (1-\hat{\delta})^{-1} + 1 \right) r^\Delta \alpha_4.
\end{align*}
\]  

(5.29)
Choose \( \Delta^* \in (0, 1] \) such that (5.16) holds, i.e.,

\[
\epsilon_{\Delta^*} = \frac{\lambda_1 - \alpha_1 - \frac{1}{4}\lambda_2 - (1(1 - \delta)^{-1} j_1 + 1)(\alpha_2 + \lambda_2)}{1 + (1(1 - \delta)^{-1} j_1 + 1)}.
\]

Then for any \( \Delta \in (0, \Delta^*) \), we have

\[
H_1(1, \Delta) = \lambda_1 - \alpha_1 - \frac{1}{4}\lambda_2 - \epsilon_\Delta - (1(1 - \delta)^{-1} j_1 + 1)(\alpha_2 + \lambda_2 + \epsilon_\Delta) > 0 \quad (5.30)
\]

and

\[
H_1(\bar{r}, \Delta) = \frac{1 - e^{-\Delta}}{\Delta} < 0, \text{ with } \left( \frac{\lambda_1 - \alpha_1 - \frac{1}{4}\lambda_2 - \epsilon_\Delta}{((1(1 - \delta)^{-1} j_1 + 1)(\alpha_2 + \lambda_2 + \epsilon_\Delta))} \right)^{1/r} > 1 \quad (5.31)
\]

From (5.30), (5.31) and (5.32), there is a positive constant \( r^*_1 = r^*_1(\Delta) \in (1, \bar{r}) \) such that \( H_1(r^*_1, \Delta) = 0 \). Let \( r^* = r^*(\Delta) = r^*_1(\Delta) \wedge r^*_2 = \left( \frac{\alpha_3}{((1(1 - \delta)^{-1} j_1 + 1)\alpha_1)} \right)^{1/r} > 1 \), then for any \( 1 < r < r^*(\Delta) \), we have \( H_1(r, \Delta) > 0 \) and \( H_2(r) > 0 \), thus we conclude from (5.28) that

\[
j^{(k+1)\Delta}E[y_{k+1}]^2 \leq H_0(r, \Delta) < \infty, \quad \forall k \geq 0.
\]

Therefore,

\[
\limsup_{k \to \infty} \frac{\log E[y_k]^2}{t_k} \leq -\log r. \quad (5.33)
\]

Bearing in mind that \( r^*_2 = e^{\gamma_1} \) and setting \( r = e^{\gamma_1}, \ r^*_1 = e^{\gamma_1}, \) then (5.33) becomes (5.17). Finally, noticing that \( \epsilon_\Delta \to 0 \) and \( 1 - e^{-r^*_1 \Delta} \to r^*_1 \) as \( \Delta \to 0 \), comparing (5.8) with (5.18), we obtain the assertion (5.19).

Finally, we begin to establish (5.20). By (5.23), we have

\[
E[y_{k+1}]^2 \leq |\xi(0)|^2 - \left( \lambda_1 - \alpha_1 - \frac{1}{4}\alpha_2 - \epsilon_\Delta \right) \sum_{j=0}^{k} E[y_j]^2 \Delta + \left( \alpha_2 + \lambda_2 + \epsilon_\Delta \right) \sum_{j=0}^{k} E[y_{j-\delta}]^2 \Delta
\]

\[
+ \sum_{j=0}^{k} \left[ \alpha_3 \sum_{j=0}^{k} |\sigma_\Delta(y_j)|^2 + \alpha_4 \sum_{j=0}^{k} |\sigma_\Delta(y_{j-\delta})|^2 \right] \Delta, \quad \forall k \geq 0. \quad (5.34)
\]

While Lemma 4.2 gives that

\[
\sum_{j=0}^{k} |y_{j-\delta}|^2 \leq ((1 - \delta)^{-1} j_1 + 1) M \|\xi\|^2 + \sum_{j=0}^{k} |y_j|^2 \quad (5.35)
\]
\[ \sum_{j=0}^{k} |\sigma_{\lambda}(y_{j+1})|^\beta \leq ((1 - \hat{\delta})^{-1}j + 1)M|\xi|^\beta + \sum_{j=0}^{k} |\sigma_{\lambda}(y_j)|^\beta. \] (5.36)

Substituting (5.35) and (5.36) into (5.34), we obtain that for any \( k \geq 0 \),
\[ 0 \leq \mathbb{E} |y_{k+1}|^2 \leq |\xi|^2 + ((1 - \hat{\delta})^{-1}j + 1)\tau((\lambda_2 + \alpha_2 + \epsilon_\lambda)|\xi|^2 + \alpha_4|\xi|^\beta) - (\alpha_3 - ((1 - \hat{\delta})^{-1}j + 1)\alpha_4)\mathbb{E} \left[ \sum_{j=0}^{k} |\sigma_{\lambda}(y_j)|^\beta \right] \Delta \]
\[ - \left[ \lambda_1 - \alpha_1 - \frac{1}{4} \alpha_2 - \epsilon_\alpha - ((1 - \hat{\delta})^{-1}j + 1)\alpha_2 + \epsilon_\lambda \right] \sum_{j=0}^{k} \mathbb{E} |y_j|^2 \Delta, \forall k \geq 0, \] (5.37)
which means
\[ \sum_{j=0}^{k} \mathbb{E} |\sigma_{\lambda}(y_j)|^2 \Delta \leq \frac{|\xi|^2 + ((1 - \hat{\delta})^{-1}j + 1)\tau((\lambda_2 + \alpha_2 + \epsilon_\lambda)|\xi|^2 + \alpha_4|\xi|^\beta)}{\lambda_1 - \alpha_1 - \frac{1}{4} \alpha_2 - \epsilon_\alpha - ((1 - \hat{\delta})^{-1}j + 1)\alpha_2 + \epsilon_\lambda} < \infty, \] (5.38)
holds for any \( k \geq 0 \). Letting \( k \to \infty \) gives (5.20). Thus, the proof is finished. \( \square \)

6. Numerical examples

**Example 6.1.** Consider a highly nonlinear scalar SDDE (see [17])
\[ dX(t) = \left[ -9X^3(t) + |X(t - \delta(t))|^{3/2} \right] dt + X^2(t)dB(t), \quad t \geq 0, \] (6.1)
with initial data \( X(t) : -\tau \leq t \leq 0 = \xi \in C([-\tau, 0]; \mathbb{R}) \), where \( B(t) \) is a one-dimensional Brownian motion. Assume that \( \delta \) satisfies Assumption [2.7] Clearly, the coefficients
\[ f(x, y) = -9x^3 + |y|^{3/2} \quad \text{and} \quad g(x, y) = x^2, \forall x, y \in \mathbb{R}, \] (6.2)
are locally Lipschitz continuous. Moreover, if \( p_0 = 18.5 \), then
\[ xf(x, y) + \frac{p_0 - 1}{2} = -9x^4 + x|y|^{3/2} + 8.75x^4 \leq -9x^4 + 8.75x^4 + 0.25x^4 + 0.75y^2 = 0.75y^2, \] (6.3)
which means that Assumption [4.8] is satisfied. For any \( x_1, x_2, y_1, y_2, \in \mathbb{R} \), we have
\[ (x_1 - x_2)(f(x_1, y_1) - f(x_2, y_2)) \leq -4.5(x_1^2 + x_2^2)|x_1 - x_2|^2 + 0.5|x_1 - x_2|^2 + 2.25|y_1 - y_2|^2 + 2.25(y_1^2 + y_2^2)|y_1 - y_2|^2 \] (6.4)
and
\[ |g(x_1, y_1) - g(x_2, y_2)|^2 = |x_1^2 - x_2^2|^2 \leq 2(x_1^2 + x_2^2)|x_1 - x_2|^2, \] (6.5)
Thus according to \[17\, \text{Corollary 3.7}\], for any \( \tau \), we may estimate the exact solution in the sense that
\[
\Delta = \frac{\| x_1 - x_2 \|}{2} + 2.25\| y_1 - y_2 \|^2 - (5.5 - p_1)(x_1^2 + x_2^2)|x_1 - x_2|^2 + 1.125(y_1^2 + y_2^2)|y_1 - y_2|^2. \quad (6.6)
\]

If we set \( p_1 = 3.25, \delta = 0.5 \) and \( U(x_1, x_2) = 1.125(x_1^2 + x_2^2)|x_1 - x_2|^2 \), then \( (6.6) \) becomes
\[
(x_1 - x_2)(f(x_1, y_1) - f(x_2, y_2)) + \frac{p_1 - 1}{2}g(x_1, y_1) - g(x_2, y_2)\| \leq 0.5|x_1 - x_2|^2 + 2.25|y_1 - y_2|^2 - \frac{1}{1 - \delta}U(x_1, x_2) + U(y_1, y_2). \quad (6.7)
\]

Moreover, it is straightforward to show that \((4.12)\) is satisfied with \( l = 4\). Thus, we have verified Assumption \((4.12)\) with \( p_0 \geq 2 + 3l \). From \((6.2)\) and \((3.1)\), we may set
\[
\mu(R) = 10R^3, \quad \forall R \geq 1 \quad \text{and} \quad \varphi(\Delta) = 10\Delta^{-1/4}, \quad \forall \Delta \in (0, 1]. \quad (6.8)
\]

Then, by Theorem 4.14, for any \( \Delta \in (0, 1] \) the truncated EM solution \( Y_\Delta(t) \) will converge to the exact solution \( X(t) \) in the sense that
\[
E|X(T) - Y_\Delta(T)|^2 \leq C\Delta \quad \text{and} \quad E|X(T) - Z_\Delta(T)|^2 \leq C\Delta, \quad \forall T > 0. \quad (6.9)
\]

However, if constant delay is considered in SDDE \((6.1)\), we may also apply the truncated EM method from \([17]\) to \((6.1)\) by setting
\[
\mu(R) = 10R^3, \quad \forall R \geq 1 \quad \text{and} \quad \varphi(\Delta) = 10\Delta^{-1/5}, \quad \forall \Delta \in (0, 1], \quad (6.10)
\]
due to
\[
\sup_{|x|, |y| \leq R} |f(x, y) \vee g(x, y)| \leq 10R^3, \quad \forall R \geq 1. \quad (6.11)
\]

Thus according to \([17\, \text{Corollary 3.7}]\), for any \( \Delta \in (0, 1] \) the truncated EM solution will approximate the exact solution in the sense that
\[
E|X(T) - Y_\Delta(T)|^2 \leq C\Delta^{1/5} \quad \text{and} \quad E|X(T) - Z_\Delta(T)|^2 \leq C\Delta^{3/5}, \quad \forall T > 0, \quad (6.12)
\]

see \([17\, \text{p. 2086}]\). Comparing \((6.9)\) and \((6.12)\), we conclude that our scheme establishes a better rate of convergence result than that of Fei et al. \([17]\) under almost the same conditions. Set \( \tau = 1, \delta(t) = 0.5 - 0.5 \sin(t) \) and \( X(t) = 2 \) for any \( t \in [-\tau, 0] \). Truncated EM solution with step size \( \Delta = 2^{-14} \) is taken as the replacement of the true solution. The root of mean-square errors with different step sizes \( 2^{-7}, 2^{-8}, \ldots, 2^{-11} \) for 500 at time \( T = 10 \) simulations is illustrated in Fig. 1(a). A least square fit of errors yields the strong convergence order 0.5134 and thus is close to the theoretical value 0.5.

**Example 6.2.** Let us consider the stochastic delay power logistic model (see e.g., \([3\, \text{or} 2]\))
\[
dX(t) = X(t)[a + bX(t - \delta(t)) - X^2(t)]dt + cX(t)X(t - \delta(t))dB(t), \quad t \geq 0 \quad (6.13)
\]
with initial data \(X(t): -\tau \leq t \leq 0 = \xi \in C([-\tau, 0]; \mathbb{R})\), where \(B(t)\) is a one-dimensional Brownian motion and \(a, b, c\) are all constants. Assume that \(\xi\) satisfies Assumption \([4.4]\) with \(K_4 = 2, q = 0.5\) and \(\delta\) satisfies Assumption \([2.3]\). Set

\[
f(x, y) = F_1(x, y) + F(x, y) \quad \text{and} \quad g(x, y) = G_1(x, y) + G(x, y)
\]

where

\[
F_1(x, y) = ax, \quad G_1(x, y) = 0, \quad F(x, y) = bxy - x^2, \quad G(x, y) = cxy,
\]

for any \(x, y \in \mathbb{R}\). Clearly, Assumption \([5.1]\) holds. Put \(\theta = \infty\). Then

\[
2xF_1(x, y) + (1 + \theta)G_1(x, y) = 2ax^2,
\]

and the elementary inequality yields that

\[
2xF_1(x, y) + (1 + \theta^{-1})G(x, y)^2 = 2x(bxy - x^3) + (cxy^2)
\]

\[
\leq 0.5x^4 + 0.5(2bxy)^2 - 2x^4 + 0.5x^4 + 0.5(cxy^2)^2 = 2b^2y^2 - x^4 + 0.5c^2y^4.
\]

Thus,

\[
\lambda_1 = -2a, \quad \lambda_2 = 0, \quad \lambda_1 = 0, \quad \alpha_2 = 2b^2, \quad \alpha_3 = 1, \quad \alpha_4 = 0.5c^4, \quad \beta = 4.
\]

If we let

\[
0.5c^4((1 - \hat{\delta})^{-1} + 1) < 1 \quad \text{and} \quad -2a > 2b^2((1 - \hat{\delta})^{-1} + 1),
\]

which means that condition \((5.9)\) is satisfied, then by Lemma \([5.4]\) the exact solution \(X(t)\) to SDDE \((6.13)\) has the property that

\[
\limsup_{t \to \infty} \frac{\log \mathbb{E}[X(t)^2]}{t} \leq -\left(\gamma^* \wedge \frac{1}{r} \log \frac{1}{0.5c^4((1 - \hat{\delta})^{-1} + 1)}\right),
\]

and \(\int_0^\infty \mathbb{E}[X(t)^2] dt < \infty\), where \(\gamma^* > 0\) is the unique root to the following equation

\[
-2a = \gamma^* + 2b^2((1 - \hat{\delta})^{-1} + 1)e^{\gamma^*}.
\]

On the other hand, take

\[
\mu(R) = ((|b| + 1) \vee |c|)R^2, \quad \forall R \geq 1 \quad \text{and} \quad \varphi(\Delta) = ((|b| + 1) \vee |c|)\Delta^{-1/4}, \quad \forall \Delta \in (0, 1].
\]

We apply partially truncated EM scheme \((3.8)\) with coefficients \(f_\Delta\) and \(g_\Delta\) given by \((5.10)\) and we denote \(\{y_k\}_{k \geq 0}\) by the discrete truncated EM approximation. According to Theorem \([5.5]\) for any \(\Delta \in (0, \Delta^*)\) and any initial data, the truncated EM approximation \(\{y_k\}_{k \geq 0}\) has the property that

\[
\limsup_{k \to \infty} \frac{\log \mathbb{E}[|y_k|^2]}{\ell_k} \leq -\left(\gamma^*_\Delta \wedge \frac{1}{\ell} \log \frac{1}{0.5c^4((1 - \hat{\delta})^{-1} + 1)}\right),
\]
Table 1: $\epsilon_\Delta$ and $\gamma^\star_\Delta$ with different step sizes for solving (6.19)

| $\Delta$ | $10^{-4}$ | $10^{-5}$ | $10^{-6}$ | $10^{-7}$ | $10^{-8}$ | $10^{-9}$ |
|----------|-----------|-----------|-----------|-----------|-----------|-----------|
| $\epsilon_\Delta$ | 0.3220  | 0.1014  | 0.0320  | 0.0101  | 0.0032  | 0.0010  |
| $\gamma^\star_\Delta$ | 0.6982  | 1.1728  | 1.3272  | 1.3764  | 1.3920  | 1.3970  |

and $\sum_{k=0}^{\infty} \mathbb{E}|y_k|^2 \Delta < \infty$, where $\gamma^\star_\Delta > 0$ is the unique root to the following equation

$$-2a = \frac{1 - e^{-\gamma^\star_\Delta}}{\Delta} + \epsilon_\Delta + \epsilon_\Lambda + (1 - \hat{\delta})^{-1} (2b^2 + \epsilon_\Lambda) e^{\gamma^\star_\tau}.$$  

(6.19)

Numerically, if we set $\hat{\delta} = 0.05$, $\tau = 0.1$, $a = -3$, $b = 1$, $c = 0.5$, $\varphi(\Delta) = 2\Delta^{-1/4}$, then

$$(1 - \hat{\delta})^{-1} (1 + 1) = 2, \quad \gamma^* = 1.3992, \quad L = 5, \quad L_R = (3\sqrt{2} + 1), \quad \epsilon_\Delta = 20\Delta + 32\Delta^{1/2},$$

and solving (5.16) gives $\Delta^* = 4.2308 \times 10^{-4}$. Computational results for $\epsilon_\Delta$ and $\gamma^\star_\Delta$ with different step sizes $\Delta$ are shown in Table 1. Fig illustrates a simple path of the truncated EM solution $Y_\Delta(t)$ with step size $\Delta = 10^{-4}$ and $\hat{\delta}(t) = 0.05 - 0.05 \sin(t)$.

![Graph of root of mean-square errors for (6.1)](image1)

![Graph of a sample path of $Y_\Delta(t)$ for (6.13)](image2)

Fig. 1. Numerical simulations for (6.1) and (6.13)

7. Conclusion

In this paper, we mainly study the strong convergence and stability of truncated EM scheme for SDDEs with variable delay. The results show that our method has a better convergence order...
than those of the existing literature on the truncated EM for SDDEs under more relaxing conditions. Numerical simulations are provided to show the effectiveness of the theoretical results. In the future, we will consider the strong convergence of the numerical scheme for SDDEs driven by Lévy process where all the three coefficients can grow super-linearly.

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