Prescribing the curvature of Riemannian manifolds with boundary

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Abstract
Let $M$ be a compact connected surface with boundary. We prove that the signal condition given by the Gauss–Bonnet theorem is necessary and sufficient for a given smooth function $f$ on $\partial M$ (resp. on $M$) to be geodesic curvature of the boundary (resp. the Gauss curvature) of some flat metric on $M$ (resp. metric on $M$ with geodesic boundary). In order to provide analogous results for this problem with $n \geq 3$, we prove some topological restrictions which imply, among other things, that any function that is negative somewhere on $\partial M$ (resp. on $M$) is a mean curvature of a scalar flat metric on $M$ (resp. scalar curvature of a metric on $M$ and minimal boundary with respect to this metric). As an application of our results, we obtain a classification theorem for manifolds with boundary.

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1 Introduction
A natural problem in differential geometry is to find metrics with prescribed curvature, i.e, construct a Riemannian metric on a given smooth manifold $M$ whose curvature is equal to a given function $f$ on $M$.

On closed manifolds, the prescribed Gaussian (resp. scalar) curvature problem has been completely solved by Kazdan and Warner [19,21,22]. Here we address this problem for manifolds with boundary. For instance, let $M$ be a surface with boundary $\partial M$, given a smooth function $h$ defined on the boundary (or $f$ in the interior), is there a Riemannian metric $g$ such that the geodesic curvature $\kappa_{\partial M} = h$ (or Gaussian curvature $K_g = f$)? In fact such a problem is equivalent to solve a quasilinear system of partial differential equation with boundary conditions. We point out that, as a particular case, it is possible to solve this problem by conformal deformation of the metric, which consists in picking some metric $g_0$ on $M$ and seeking a conformally related metric to $g_0$, say $g = e^{2u}g_0$, for some positive
function $u$ to be found in order to satisfy

$$\begin{cases}
-\Delta_{g_0} u + 2K_{g_0} = 2fe^u & \text{in } M \\
\frac{\partial u}{\partial v} + 2\kappa_{g_0} = 2he^u & \text{on } \partial M,
\end{cases}$$

where $\Delta_{g_0}$, $K_{g_0}$ and $\kappa_{g_0}$ are the Laplacian, the Gauss curvature and the geodesic curvature of the boundary of $g_0$, respectively. Here $v$ is the outward unit normal on $\partial M$. On this subject, the literature is extensive and many results are known, see for instance [14,15,30], that includes the higher dimensional case as well as the recent works [9,24,25].

Suppose $M$ is a compact two-dimensional Riemannian manifold with boundary $\partial M$. The Gauss–Bonnet theorem states that

$$\int_M K dv + \int_{\partial M} \kappa d\sigma = 2\pi \chi(M),$$

where $K$ denotes the Gaussian curvature, $\kappa$ is the geodesic curvature of the boundary, $\chi(M)$ is the Euler characteristic, $dv$ is the element of volume and $d\sigma$ is the element of area. Besides establishing a link between the topology (Euler characteristic) and geometry of a surface, it also gives a necessary signal condition on the Gaussian curvature of a surface or geodesic curvature on the boundary in terms of its Euler characteristic.

Consider the following natural consequence given by the Gauss–Bonnet theorem when $M$ is a bounded domain $\Omega$ in $\mathbb{R}^2$ with smooth boundary (resp. compact connected 2-manifold with geodesic boundary):

- If $\chi(M) > 0$, then $\kappa$ (resp. $K$) must be positive somewhere.
- If $\chi(M) = 0$, then $\kappa$ (resp. $K$) must change sign unless it is $\kappa \equiv 0$.
- If $\chi(M) < 0$, then $\kappa$ (resp. $K$) must be negative somewhere. (1.1)

In the first result we prove that the obvious signal condition (1.1) is also sufficient to the problem of prescribing curvature. More precisely, we state the following theorem:

**Theorem 1.1** Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary. A function $\kappa \in C^\infty(\partial\Omega)$ is the geodesic curvature of a flat metric on $\Omega$ if only if $\kappa$ satisfies the signal condition (1.1).

We also prove the following result for manifolds with geodesic boundary.

**Theorem 1.2** Let $M^2$ be a compact connected surface with smooth geodesic boundary. A function $K \in C^\infty(M)$ is the Gaussian curvature of a metric on $M$ with geodesic boundary if only if $K$ satisfies the signal condition (1.1).

One of the key ingredients in the proof of Theorems 1.1 and 1.2 is the celebrated Osgood, Phillips, and Sarnak uniformization theorem for surfaces with boundary [29] (see also Brendle [4,5]). Namely, if the surface has boundary, in each conformal class of Riemannian metrics, there is a unique uniform metric of type I, i.e., a constant curvature metric with zero geodesic curvature, and a unique uniform metric of type II, i.e. the resulting Riemannian manifold $M$ is flat, i.e. the sectional curvature is zero and $\partial M$ has constant geodesic curvature on the boundary.

In order to generalize Theorems 1.1 and 1.2 we make use of a version of the uniformization theorem in higher dimensions. In this respect, we have the Yamabe problem for manifolds with boundary that consists in finding a conformal metric to the background one having constant scalar curvature and minimal boundary or having zero scalar curvature and constant...
mean curvature on \( \partial M \). Such a problem has inspiration in the closed case and it was solved in almost every case by Escobar \([11,12]\). We refer the interested reader to Marques \([26,27]\), Almaraz \([1]\), Brendle and Chen \([6]\) and Mayer and Ndiaye \([28]\) that studied many of the remaining cases.

Using results of existence of metrics with constant scalar curvature and minimal boundary or with zero scalar curvature whose boundary has constant mean curvature, we have the following theorems.

**Theorem 1.3** Let \( M^n, n \geq 3 \), be a compact connected manifold with boundary.

(i) Any function on \( \partial M \) that is negative somewhere is a mean curvature of a scalar flat metric on \( M \).

(ii) Every smooth function on \( \partial M \) is a mean curvature of a scalar flat metric if and only if \( M \) admits a scalar flat metric with positive constant mean curvature on the boundary.

**Theorem 1.4** Let \( M^n, n \geq 3 \), be a compact connected manifold with smooth boundary.

(i) Any function on \( M \) that is negative somewhere is a scalar curvature of a metric with minimal boundary.

(ii) Every smooth function on \( M \) is a scalar curvature of a metric with minimal boundary with respect to this metric if and only if \( M \) admits a metric with positive constant scalar curvature and minimal boundary.

Taking account some topological restrictions given in Sect. 6, we separate the compact manifolds with boundary into three groups:

**Theorem 1.5** Compact manifolds with boundary and dimension \( n \geq 3 \) can be divided into three classes:

(a) Any smooth function on \( \partial M \) (resp. \( M \)) is mean curvature of some scalar flat metric (resp. scalar curvature of a metric on \( M \) with minimal the boundary with respect to this metric); 

(b) A smooth function on \( \partial M \) (resp. \( M \)) is mean curvature of some scalar flat metric on \( M \) (resp. scalar curvature of a metric with minimal boundary with respect to this metric) if and only if it is either identically equal to zero or strictly negative somewhere; furthermore, any scalar flat metric having zero mean curvature is totally geodesic (resp. Ricci-flat).

(c) A smooth function on \( \partial M \) (resp. \( M \)) is mean curvature of some scalar flat metric (resp. scalar curvature of a metric with minimal the boundary with respect to this metric) if and only if it is strictly negative somewhere.

In short, every compact manifold with boundary of dimension \( n \geq 3 \) admits a scalar flat metric on \( M \) with constant negative mean curvature on \( \partial M \) (resp. a metric with constant negative scalar curvature and minimal boundary). Those in item (a) or (b) are scalar flat on \( M \) and have vanishing mean curvature on the boundary, and those in item (a) are scalar flat on \( M \) and have constant positive mean curvature on the boundary (resp. constant positive scalar curvature and minimal boundary).

This paper is organized as follows. In Sect. 2, we gather some important preliminary tools, discuss notations and formally present the second order linear operator we shall study. In Sect. 3, we prove the remarkable property that the map \( g \mapsto (R_g, 2H_{\partial M}) \) is almost always a surjection, which, together an approximation lemma contained in Sect. 4, allow us to prove in Sects. 5 and 6 results concerning what functions can be realized as scalar...
curvature or mean curvature of the boundary for dimension \( n \geq 2 \). To be more precise, we prove Theorems 1.1 and 1.2 in Sect. 5 and, discussing some topological obstructions results, we prove Theorems 1.3, 1.4 and 1.5 in Sect. 6.

2 Preliminaries

Let \( M^n \) be \( n \)-dimensional compact connected Riemannian manifold with boundary. Let \( S^2_{2, p} = \mathcal{W}^{2, p}(\text{Sym}^2(T^*M)) \) denote the section of class \( \mathcal{W}^{2, p} \) of symmetric \((0,2)\)-tensors. For \( p > n \), consider the operator

\[
\Psi(\cdot) := (R(\cdot), 2H(\cdot)) : \mathcal{M}^{2, p} \rightarrow L^p(M) \oplus W^{1, p}(\partial M),
\]

where \( \mathcal{M}^{2, p} \) denotes the open subset of \( S^2_{2, p} \) of the Riemannian metrics on \( M \). Since \( R_g \) and \( H_{g \# M} \) involve derivatives of \( g \) up to second order, by the Sobolev Embedding Theorem, for \( p > n \), we have that \( \Psi \) is a \( C^\infty \) map. Given an infinitesimal variation \( h \). We introduce

\[
\delta R_g h := \left. \frac{\partial}{\partial t} R_{g + th} \right|_{t=0}
\]

and

\[
\delta H_g h := \left. \frac{\partial}{\partial t} H_{g + th} \right|_{t=0},
\]

the variation of the scalar curvature \( R \) and of the mean curvature \( H \) in the direction of \( h \), respectively. Here \( \gamma = g_{\partial M} \). A classical computation, that can be found in [2], shows that

\[
\begin{cases}
\delta R_g h = -\Delta_g (\tr_g h) + \div_g \div_g h - \langle h, \Ric_g \rangle \\
2 \delta H_g h = [d(\tr_g h) - \div_g h](\nu) - \div_g X - \langle \Pi_g, h \rangle
\end{cases}
\]

(2.1)

where \( \nu \) is the outward unit normal to \( \partial M \), \( \Pi_g \) is the second fundamental form of \( \partial M \), \( X \) is the vector field dual to the one-form \( \omega(\cdot) = h(\cdot, \nu) \), \( \tr_g = g^{ij}h_{ij} \) is the trace of \( h \) and our convention for the Laplacian is \( \Delta_g f = \tr_g (\Hess_g f) \). The linearization of \( \Psi \) will be denoted by

\[
S_g(h) = D\Psi_g \cdot h = (\delta R_g h, 2\delta H_g h).
\]

Before proceeding, we need of the following lemma.

**Lemma 2.1** Let \( h \) be a symmetric \((0,2)\)-tensor and \( f \) be a smooth function on \( M \). Then

\[
\int_M f \div_g \div_g h \, dv = \int_M \langle \Hess_g f, h \rangle \, dv + \int_{\partial M} f \langle \div_g h, \nu \rangle - h(\nabla f, \nu) \, d\sigma \tag{2.2}
\]

The proof of Lemma 2.1 is just to apply the divergence theorem to the field \( X = f \div_g h - h(\nabla f, \cdot) \).

A direct calculation using the previous lemma gives that

\[
\int_M f \delta R_g h + 2 \int_{\partial M} f \delta H_g h = \int_M (-\Delta_g f (\tr_g h) + \langle \Hess_g f, h \rangle - f \langle h, \Ric_g \rangle)
\]

\[
+ \int_{\partial M} \tr_g h \frac{\partial f}{\partial v} - f \langle \Pi_g, h \rangle - f \div_g X - \omega(\nabla f)
\]

\[
= \int_M (-\Delta_g f (\tr_g h) + \langle \Hess_g f, h \rangle - f \langle h, \Ric_g \rangle)
\]

\[
+ \int_{\partial M} \tr_g h \frac{\partial f}{\partial v} - f \langle \Pi_g, h \rangle.
\]
where we have omitted the volume forms and used the fact that

\[ -f \, \text{div}_\gamma X = \omega(\nabla f) - h(v, v) \frac{\partial f}{\partial v}. \]

We first observe that the previous calculations clearly shows that

\[ (\delta R_g h, f)_{L^2(M)} - (A^* f, h)_{L^2(M)} = (B^* f, h)_{L^2(\partial M)} - 2(\delta H, h, f)_{L^2(\partial M)} \]

for all \( h \in \mathcal{M}^{2,p} \) and \( f \in W^{2,p}(M) \), where

\[ (u, v)_{L^2(M)} = \int_M u v \, dv \]

and

\[
\begin{align*}
A^*_g f &= - (\Delta_g f) g + \text{Hess}_g f - f \text{Ric}_g \quad \text{in } M \\
B^*_\gamma f &= \frac{\partial f}{\partial \nu} \gamma - f \Pi_\gamma \quad \text{on } \partial M.
\end{align*}
\]

Therefore, we introduce the formal \( L^2 \)-adjoint of \( S_g(h) \) to be the operator \( S^*_g : W^{2,p}(M) \to S^{0,p}_2(M) \oplus S^{1,p}_2(\partial M) \) given by

\[ S^*_g(f) = (A^*_g f, B^*_\gamma f), \]  

(2.3)

We claim that \( S^*_g(f) \) is an underdetermined elliptic operator, which in other words means that it has injective symbol (see [18] for definition). Indeed, the principal symbol of \( S^* \) is given by

\[
\begin{align*}
\sigma(A^*_g(\epsilon)) f &= (\|\epsilon\|^2 g - \epsilon \otimes \epsilon) f, \\
\sigma(B^*_\gamma(\epsilon)) f &= \langle \epsilon, \nu \rangle f \gamma,
\end{align*}
\]

for all \( \epsilon \in T^{*}_x M \) (cotangent space at \( x \)) and \( \epsilon \) normal to \( \partial M \) at \( x \). Note that the principal symbol of \( A^*_g \) is injective for \( \epsilon \neq 0 \). To see this, if we assume that \( \sigma(A^*_g(\epsilon)) f = 0 \), taking its trace, we see that \( (n - 1)\|\epsilon\|^2 f = 0 \). Moreover, for every linearly independent couple of vectors \( \epsilon \) and \( \eta \) belonging to \( \mathbb{R}^n \), the polynomial in the complex variable \( \tau \)

\[ \sigma(A^*_g)(\epsilon + \tau \eta) = g(\|\epsilon\|^2 + \tau^2 \|\eta\|^2) - [(\epsilon + \tau \eta) \otimes (\epsilon + \tau \eta)] \]

has exactly two roots, one with positive and one with negative imaginary part (Indeed, take the trace and obtain \( 0 = (n - 1)(\|\eta\|^2 \tau^2 + \|\epsilon\|^2) \)). Thus, \( A^*_g \) is a second order (overdetermined) elliptic operator. One condition has to be satisfied by \( S^*_g(f) \) to be an elliptic boundary problem,\(^1\) that is \( A^*_g \) to be elliptic on \( M \), and properly elliptic, and \( B^*_\gamma \) to satisfy the Shapiro–Lopatinskij condition at any point of the boundary, for precise definition see Section 20.1 of [18], Section 2.18 of [10] or chapter 2 of [23]. Since \( \nu \) is not tangent to \( \partial M \), it is possible to verify that the boundary problem satisfies the Shapiro–Lopatinskij condition (see for example (Ell2) in [10, page 108], as well all discussion in pp. 107–108 and Theorem 2.50 of [10] that relates elliptic boundary properties such as Fredholm operator, regularity of solutions and an a priori estimate).

\(^1\) Sometimes called of oblique derivative problem.


3 Local surjectivity

In this section, we study the surjectivity of the operator \( S^*_g \) which, in fact, relies on study the injectivity \( S^*_g \), i.e., we have to analyse the linear partial differential equation \( S^*_g f = 0 \). Consider \( f \in \text{Ker} S^*_g \). Taking the trace we get that

\[
\begin{cases}
\Delta f + \frac{R_g}{n-1} f = 0 & \text{in } M \\
\frac{\partial f}{\partial \nu} - \frac{H_g}{n-1} f = 0 & \text{on } \partial M.
\end{cases}
\]  

(3.1)

Therefore, \( S^*_g f = 0 \) can be rewritten as

\[
\text{Hess}_g f = \left( \text{Ric}_g - \frac{R_g}{n-1} g \right) f \text{ in } M \text{ and } f \left( \Pi_g - \frac{H_g}{n-1} \gamma \right) = 0 \text{ on } \partial M. \tag{3.2}
\]

In order to prescribe the curvature we have to show that the following \( g \mapsto (R_g, 2H_g) \) map is locally surjective. In this section our methods are based on those of Fischer and Marsden [16] (see also [7, 17]). We remark that the same conclusions in this section hold under the condition of metrics close in \( W^{s,p} \) norm for \( s > \frac{2}{p} + 1 \).

**Proposition 3.1** Given \( g \in \mathcal{M}^{2,p} \), \( p > \dim M \). Suppose that the scalar curvature (resp. Gauss curvature, if \( \dim M = 2 \)) in the interior vanishes with respect to \( g \) on \( M \). Then \( S^*_g : \mathcal{S}^{2,p} \to L^p(M) \oplus W^{2,p}(\partial M) \) is a surjection if one of the following holds:

(i) \( H_g \) (resp. \( \kappa_g \), if \( \dim M = 2 \)) is not a positive constant;

(ii) \( H_g = 0 \), but \( \Pi_g \) is not identically zero.

**Proof** Since \( S^*_g \) has injective symbol, it suffices to show that \( S^*_g \) is injective. We first prove part (ii), let \( f \in \text{Ker} S^*_g = \text{Ker}(A^*_g, B^*_g) \). It is immediate to see that if \( H_g = 0 \) in (3.1), then \( f \) is a constant function which together with \( f \Pi_g = 0 \) on \( \partial M \) implies that \( f \) is constant equal to zero.

For part (i), assume that \( \dim M > 2 \) and \( f \) is not identically zero on \( \partial M \). It follows from (3.2) that \( \Pi_g = \frac{H_g}{n-1} \gamma \). Recall that if \( \{e_i\}_{i=1}^{n-1} \in \text{span} T(\partial M) \), where \( e_n = \nu \), then we get by Codazzi equation that

\[
\nabla_i \nabla^M H_g = \nabla_i \nabla^M \Pi^j = \nabla_j \nabla^M \Pi^j_i + R_{jiv} = \nabla_j \nabla^M \Pi^j_i + R_{jiv}.
\]

where \( R_{ijkl} \) is the curvature tensor of \( (M, g) \). Thus,

\[
\nabla_i \nabla^M H_g = \nabla_i \nabla^M \left( \frac{H_g}{n-1} \gamma^j_i \right) + R_{jiv} = \frac{1}{n-1} \nabla_i \nabla^M H_g + R_{jiv}.
\]

However, from \( B^*_g f = 0 \) we have that

\[
0 = \nabla_i \left( \frac{\partial f}{\partial \nu} \gamma^j_i - f \Pi^j_i \right) = \nabla_i \nabla_v f - \frac{f}{n-1} \nabla_i \nabla^M H_g. \tag{3.3}
\]

Since \( R_g = 0 \) (and so \( \Delta_g f = 0 \) in \( M \)) and \( A^*_g f = 0 \), we have that \( \text{Hess}_g f = f \text{Ric}_g \), which together with all the above facts implies that \( \nabla^M H_g = 0 \), so \( H_g \) is constant. Notice that (3.1) implies that \( H_g \) is an eigenvalue of the Steklov problem of second order and hence \( H_g > 0 \), contradicting (i).
If \( \dim M = 2 \) and \( f \) is not identically zero on \( \partial M \), then (3.3) is automatically equivalent to \( V_{i} \partial_{M} \kappa_{\gamma} = 0 \) provided \( \text{Hess}_{g} f = f K_{g} g = 0 \). Arguing similarly, we conclude that \( f \) has to be zero and \( S^{*}_{g} \) is injective.

We claim that if there exists a point \( x_{0} \in \partial M \) so that \( f(x_{0}) = 0 \), then we must have \( \nabla^{\partial M} f(x_{0}) \neq 0 \). Reasoning by contradiction, assume that \( \nabla^{\partial M} f(x_{0}) = 0 \). We define \( h(t):= f(\alpha(t)) \), where \( \gamma \) is any geodesic on the boundary of \( M \) starting from \( x_{0} \). A direct calculation gives that \( h(t) \) satisfies the following linear second-order ODE:

\[
\frac{d^{2} h}{dt^{2}}(t) = \left( \text{Hess}_{g} f|_{\partial M} \right)_{\alpha(t)} \cdot (\alpha'(t), \alpha'(t))
\]

\[
= \left( \text{Hess}_{g} f \right)_{\alpha(t)} \cdot (\alpha'(t), \alpha'(t)) + \left\langle \nabla(f \circ \alpha), \Pi_{\gamma}(\alpha'(t), \alpha'(t)) \right\rangle
\]

\[
= \left( \text{Ric}_{g}(\alpha'(t), \alpha'(t)) - \frac{R_{g}}{n-1} \| \alpha'(t) \|^{2}_{g} + \frac{H_{\gamma}}{n-1} \Pi_{\gamma}(\alpha'(t), \alpha'(t)) \right) f \circ \alpha,
\]

where we have used (3.2). Because \( h(0) = 0 \) and \( h'(0) = 0 \), we have that \( h(t) \equiv 0 \) and, thus, \( f = 0 \) on \( \partial M \).

As consequence, 0 is a regular value of \( f|_{\partial M} \) which implies that \( \nabla^{\partial M} f \neq 0 \) on \( f^{-1}(0) \) and \( \nabla^{\partial M} H_{\gamma} = 0 \) on an open dense set and hence everywhere. As before, we obtain another contradiction. Thus, \( f \) is zero on \( \partial M \) and so \( S^{*}_{g} \) is injective. \( \square \)

Remark 3.2 If \( \dim M = 2 \), then item (ii) of Proposition 3.1 (or the second item of the next proposition) cannot hold.

Proposition 3.3 Given \( g \in \mathcal{M}^{2, p}, p > \dim M \). Suppose that the mean (resp. geodesic, if \( \dim M = 2 \)) curvature of \( \partial M \) vanishes with respect to the metric \( g \). Then \( S_{g}: S^{2-p} \to L^{p}(M) \oplus W^{2-p}(\partial M) \) is a surjection if one of the following holds:

(i) \( R_{g} \) (resp. \( K_{g} \), if \( \dim M = 2 \)) is not a positive constant;

(ii) \( R_{g} = 0 \), but \( \text{Ric}_{g} \) is not identically zero.

Proof Due to the similarity of the arguments, we merely sketch the proof. The condition in item (ii) implies that \( f \) is constant, in fact, equal to zero provided \( f \text{Ric}_{g} = 0 \).

We now sketch item (i). Since

\[
0 = \text{div}_{g} A^{*} f = f \nabla R_{g} \text{ on } M \text{ and } 0 = \text{tr}_{g} B^{*} f = \frac{\partial f}{\partial \nu} \text{ on } \partial M
\]

then \( R_{g} \) is constant if \( f \) is not identically zero. However, note that \( R_{g} \) is constant equal to zero because \( \frac{R_{g}}{n-1} \) is not a positive eigenvalue of the Neumann problem. We claim that \( \nabla f(x_{0}) \neq 0 \) whenever \( f \) is zero at some point. Indeed, consider the geodesic \( \alpha \) starting at \( x_{0} \). Defining \( h(t) = f \circ \alpha(t) \), a similar computation shows that \( h \), and thus \( f \), is identically equal to zero. Hence we can conclude that \( f \in \text{Ker} S_{g}^{*} \) is constant equal to zero. \( \square \)

Remark 3.4 For \( n = 2 \), the map \( \Psi \) cannot be onto a neighborhood of \( \Psi(g) = 0 \). For instance, the Gauss–Bonnet theorem for manifold with boundary shows that a hemisphere does not admit metric with Gauss curvature strictly positive or strictly negative and minimal boundary as well as an annulus with two boundaries components does not admit metric whose Gauss curvature vanishes and the boundary components have geodesic curvature strictly positive or strictly negative.

Now, for locally solve \( \Psi(g) = (f_{1}, f_{2}) \) in an appropriate topology, we use the implicit function theorem to prove that \( \Psi \) is a locally surjective map whenever \( S^{*} \) is injective. Moreover, we apply standard elliptic theory to \((A_{g} A^{*}_{g} f, B_{\gamma} B^{*}_{\gamma} f)\). Indeed, if \( A^{*} \) is an operator of
order 2 with injective symbol, then $A$ has also injective symbol and, thus, $A_g^* A_g^*$ is elliptic of order 4, provided $\sigma(A_g^* A_g^*) = \sigma(A_g^*) \sigma(A_g^*) = \sigma(A_g^*) \sigma(A_g^*)$ and $\sigma(A_g^*)$ injective implies $\sigma(A_g^*) \sigma(A_g^*)$ is an isomorphism. Moreover, since $B'_y$ and $B'_y$ satisfy the Shapiro–Lopatinskij condition, $B'_y B'_y$ also satisfies it.

**Theorem 3.5** Let $f = (f_1, f_2) \in L^p(M) \oplus W^{1,p}(\partial M)$, $p > n$. Assume that $S^*_{g_0}$ is injective, then there is an $\eta > 0$ such that if

$$\|f_1 - R_{g_0}\|_{L^p(M)} + \|f_2 - H_{g_0}\|_{W^{1/2,p}(\partial M)} < \eta,$$

then there is a metric $g_1 \in \mathcal{M}^{2,p}$ such that $\Psi(g_1) = f$. Moreover, $g$ is smooth in any open set where $f$ is smooth.

**Proof** In order to apply the implicit function theorem we consider the following operator $S: U \subset W^{4,p}(M) \to L^p(M) \oplus W^{1,p}(\partial M)$ defined by

$$S(u) = \left( R_{g_0} + A_{g_0}^* A_{g_0}^* u, H_{g_0} + B_{g_0}^* u \right),$$

where $U$ is sufficiently small neighborhood of zero in $W^{p,4}$. Indeed, this is an oblique boundary value problem for a second order quasilinear elliptic differential equation and by the Sobolev Embedding Theorem, with $n > p$, $S$ is a $C^1$ map from $W^{4,p}(M)$ onto $L^p(M) \oplus W^{1/2,p}(\partial M)$. We claim that $S'(0)$ is an isomorphism when restricted a small neighborhood of $W^{4,p}$ norm. In fact, $S(0) = \Psi(g_0)$ and

$$S'(0) v = (A_{g_0} A_{g_0}^* v, B_{g_0} B_{g_0}^* v) = S_{g_0} S_{g_0}^* v.$$

Hence $\text{Ker } S'(0) = \text{Ker } S_{g_0} S_{g_0}^* = 0$, provided $\text{Ker } S_{g_0} S_{g_0}^* = \text{Ker } S_{g_0}^*$. It follows from the implicit function theorem that $S$ maps a neighborhood of zero in $W^{p,4}$ onto a neighborhood of $S(0) = \Psi(g_0)$ in $L^p(M) \times W^{1/2,p}(\partial M)$. Considering $L^p(M) \oplus W^{1/2,p}(\partial M)$ with the norm

$$\|(h, w)\|_{L^p(M) \oplus W^{1/2,p}(\partial M)} = \|h\|_{L^p(M)} + \|w\|_{W^{1/2,p}(\partial M)},$$

there is an $\eta > 0$ such that if

$$\|f_1 - R_{g_0}\|_{L^p(M)} + \|f_2 - H_{g_0}\|_{W^{1/2,p}(\partial M)} < \eta,$$

then there exist a solution $g_1 = g_0 + S^* u$ of $\Psi(g_1) = f$. Using elliptic regularity and a bootstrap argument we have that if $f$ smooth, then $u$ is smooth.

Let $M$ be a manifold with boundary and $\rho: M \to \mathbb{R}$ is a smooth function. For $p > n$, we set

$$\mathcal{M}_{\rho, p}^{2,p} = \left\{ g \in \mathcal{M}^{2,p}; \ R_g = \rho \text{ and } H_{g|\partial M} = 0 \right\}$$

and

$$\mathcal{M}_{\tilde{\rho}, p}^{2,p} = \left\{ g \in \mathcal{M}^{2,p}; \ R_g = 0 \text{ and } H_{g|\partial M} = \tilde{\rho} \right\},$$

where $\tilde{\rho}: \partial M \to \mathbb{R}$ is a smooth function. The sets $\mathcal{M}_{\rho, p}^{2,p}$ and $\mathcal{M}_{\tilde{\rho}, p}^{2,p}$ are the set of metrics of prescribed scalar curvature and prescribed mean curvature of the boundary, respectively. It follows from Propositions 3.1, 3.3 and Theorem 3.5 the following result:

**Corollary 3.6** If $\rho$ and $\tilde{\rho}$ are not identical zero nor positive constants, then $\mathcal{M}_{\rho, p}^{2,p}$ and $\mathcal{M}_{\tilde{\rho}, p}^{2,p}$ are smooth submanifolds of $\mathcal{M}^{2,p}$.
In the next proposition we state a result about nonsurjectivity.

**Proposition 3.7** Let \((M, \overline{g})\) be a Riemannian manifold with boundary. The following assumptions imply that \(S_{\overline{g}}^n\) is not surjective:

(a) \(M\) is scalar flat with totally geodesic boundary \(\partial M\) or \(M\) is Ricci-flat with minimal boundary;

(b) \(M = \overline{\mathbb{R}^{n+1}}\) is the Euclidean ball of radius \(r_0\) in \(\mathbb{R}^{n+1}\) or \(M = S^n_+\) is the standard hemisphere of radius \(r_0\) in \(\mathbb{R}^{n+1}\).

**Proof** We easily see that in the conditions of item (a), \(\text{Ker} S_{\overline{g}}^n\) is composed of constant functions and \(S_{\overline{g}}^n\) is not surjective.

If \(M\) is a standard Euclidean unit ball \(B\), the Steklov eigenfunctions \(f\) with first nonzero eigenvalue \(n-1\) also satisfies

\[
\text{Hess}_{\overline{g}} f = 0 \quad \text{in} \quad M \quad \text{and} \quad \frac{\partial f}{\partial v} \frac{\overline{g}}{r_0^2} = \frac{f}{r_0^2} \frac{\overline{g}}{r_0^2} \quad \text{on} \quad \partial M.
\]

Therefore, \(S_{\overline{g}}^n\) is not surjective. Here, functions satisfying \(\Delta f = 0\) in \(M\) and \(\frac{\partial f}{\partial v} = \frac{f}{r_0^2}\) on \(\partial M\) belong to \(\text{Ker} S_{\overline{g}}^n\).

In contrast, when \(M = S^n_+\) with the metric \(\overline{g}\), the Robin eigenfunctions \(f\) with first nonzero eigenvalue \(n\) also satisfies

\[
\text{Hess}_{\overline{g}} f = -\frac{f}{r_0^2} \frac{\overline{g}}{r_0^2} \quad \text{in} \quad M \quad \text{and} \quad \frac{\partial f}{\partial v} \frac{\overline{g}}{r_0^2} = 0 \quad \text{on} \quad \partial M.
\]

Analogously, \(S_{\overline{g}}^n\) is not surjective and all function \(f \in W^{2,s}(M)\) satisfying \(\Delta f = -(n/r_0^2)f\) in \(M\) and \(\frac{\partial f}{\partial v} = 0\) on \(\partial M\) belongs to \(\text{Ker} S_{\overline{g}}^n\).

\(\square\)

### 4 Approximation lemma

In this section, inspired by Theorem 2.1 in [20], we show how to approximate a function arbitrarily closely in \(L^p(M)\) and \(W^{1,p}(\partial M)\) in order to apply Theorem 3.5. We proof the following lemma.

**Lemma 4.1** (Approximation lemma) Let \(M\) be a Riemannian manifold with boundary of dimension \(n \geq 2\).

(a) Let \(f, g \in C^\infty(\partial M)\). If the range of \(g\) is in the range of \(f\), that is, \(\min f \leq g(x) \leq \max f\) on \(\partial M\), then given any positive \(\varepsilon\) there is a diffeomorphism \(\varphi\) of \(M\) such that, for \(p > 2n\), we have that

\[
\|f \circ \varphi - g\|_{W^{1,p}(\partial M)} < \varepsilon.
\]

(b) Let \(f, g \in C^\infty(M)\). If the range of \(g\) is in the range of \(f\), that is, \(\min f \leq g(x) \leq \max f\) on \(M\), then given any positive \(\varepsilon\) there is a diffeomorphism \(\varphi\) of \(M\) such that, for \(p > n\), we have that

\[
\|f \circ \varphi - g\|_{L^p(M)} < \varepsilon.
\]
Proof For part (a), let \( \{ \Delta_i \} \) be a locally finite triangulation of \( \partial M \) so fine that \( g \) is nearly constant in each simplex. Then we can assume that for each \( i \) we have
\[
\max_{x,y \in \Delta_i} |g(x) - g(y)| < \delta
\]  \( (4.1) \)
where \( \delta = \varepsilon/4(4\tau \text{vol}(\partial M))^{1/p} \), where \( \tau \) is a constant chosen later. Let \( b_i \in \text{int}(\Delta_i) \). By continuity there exist disjoint open sets \( V_i \subset \partial M \), such that
\[
|f(x) - g(b_i)| < \delta
\]  \( (4.2) \)
for each \( i \) and \( x \in V_i \).

Choose a neighborhood \( Q \) of the \((n-1)\)-skeleton \( \partial M \), disjoint from \( b_i \), so small that
\[
\left( \max_{\partial M} |f| + \max_{\partial M} |g| \right)^p \cdot \text{vol} Q < \frac{\varepsilon^p}{2^p+3 \cdot \text{vol}(\partial M)}
\]  \( (4.3) \)
and
\[
\left( \max_{\partial M} |\nabla f| + \max_{\partial M} |\nabla g| \right)^p \cdot \text{vol} Q < \frac{\varepsilon^p}{2^p+3 \cdot \text{vol}(\partial M)}.
\]  \( (4.4) \)

Consider for each \( b_i \) a neighborhood \( U_i \) disjoint from \( Q \), and choose open sets \( O_1 \) and \( O_2 \), such that
\[
\partial M - Q \subset O_1 \subset \overline{O_1} \subset O_2 \subset \overline{O_2} \subset \partial M - \text{skeleton}.
\]

We will find a diffeomorphism \( \varphi \) of \( M \) so that \( \varphi(O_1 \cap \Delta_i) \subset V_i \). Firstly, there is a diffeomorphism \( \varphi_1 \) of \( M \) such that \( \varphi_1(U_i) \subset V_i \), a diffeomorphism \( \varphi_2 \) of \( M \) satisfying \( \varphi_2(O_1 \cap \Delta_i) \subset U_i \) and \( \varphi_2|\partial M - O_2 = id \) for each \( i \). This allow us to define \( \varphi = \varphi_1 \circ \varphi_2 \). Note that we are not interested in the behavior of the diffeomorphism in the interior of \( M \).

Recall that
\[
\|u\|_{W^{1,p}_0(\partial M)}^P := \|u\|_{L^p(\partial M)}^P + [u]_{W^{1,p}_0(\partial M)},
\]
where \([u]_{W^{1,p}_0(\partial M)} := \int_{\partial M} \int_{\partial M} \frac{|u(x) - u(y)|^p}{|x - y|^{n+\frac{p}{2}}} \) is the Glagliardo norm of \( u \).

We use \((4.1)-(4.3)\) to infer
\[
\|f \circ \varphi - g\|_{L^p}^P = \left( \int_Q + \int_{\partial M - Q} \right) |f \circ \varphi - g|^p
\]
\[
< \frac{\varepsilon^p}{2^{p+3}} + \sum_i \int_{O_1 \cap \Delta_i} |f \circ \varphi(y) - g(b_i) + g(b_i) - g(y)|^p
\]
\[
< \frac{\varepsilon^p}{2^{p+3}} + \sum_i 2^p \delta^p \text{vol}(\Delta_i) = \left( \frac{\varepsilon^p}{2^p+3} + \frac{\varepsilon^p}{2^{p+2}\tau} \right),
\]
where we have omitted the volume forms. The next step is to study the following identity.
\[
[f \circ \varphi - g]_{W^{1,p}_0(\partial M)} = \left( \int_Q + \int_{\partial M - Q} \right) \int_{\partial M} \frac{|(f \circ \varphi - g)(x) - (f \circ \varphi - g)(y)|^p}{|x - y|^{n+\frac{p}{2}}}.
\]
For $p > 2n$, setting $z = x - y$ we obtain that
\[
\int_{\partial Q} \int_{\partial M} \frac{|(f \circ \varphi - g)(x) - (f \circ \varphi - g)(y)|^p}{|x - y|^{n + \frac{p}{2}}}
\]
\[
\leq \int_{\partial M \setminus Q} \int_{\partial M \cap |z| \geq r_0} \frac{2(\max_{\partial M} |f \circ \varphi| + \max_{\partial M} |g|)^p}{|x - y|^{n + \frac{p}{2}}}
\]
\[
+ \int_{\partial M \setminus Q} \int_{\partial M \cap |z| < r_0} \frac{\max_{\partial M} |\nabla (f \circ \varphi)| |x - y| + \max_{\partial M} |\nabla g||x - y|^p}{|x - y|^{n + \frac{p}{2}}}
\]
\[
\leq \int_{\partial M \setminus Q} 2^p \left( \max_{\partial M} |f \circ \varphi| + \max_{\partial M} |g| \right)^p \int_{\partial M \cap |z| \geq r_0} \frac{1}{|z|^{n + \frac{p}{2}}}
\]
\[
+ \int_{\partial M \setminus Q} \left( \max_{\partial M} |\nabla (f \circ \varphi)| + \max_{\partial M} |\nabla g| \right)^p \int_{\partial M \cap |z| < r_0} |z|^\frac{p}{2} - n
\]
\[
< \frac{\varepsilon^p}{8} + \frac{\varepsilon^p}{2^{p+3}},
\]
where we have used the mean value theorem in the third line, (4.3) in the fourth line and (4.4) in the fifth line. Note that in the last line we have used the integrability of the kernel $\frac{1}{|z|^{n + \frac{p}{2}}}$ for $n + \frac{p}{2} > n$.

On the other hand,
\[
\int_{\partial M \setminus Q} \int_{\partial M} \frac{|(f \circ \varphi - g)(x) - (f \circ \varphi - g)(y)|^p}{|x - y|^{n + \frac{p}{2}}}
\]
\[
\leq \int_{\partial M \setminus Q} \int_{|z| \geq r_0} \frac{|f \circ \varphi(x) - g(b_i) - g(x) + g(b_i) - f \circ \varphi(y) + g(b_i) - g(y) - g(b_i)|^p}{|x - y|^{n + \frac{p}{2}}}
\]
\[
+ \int_{\partial M \setminus Q} \int_{|z| < r_0} \frac{\max_{\partial M} |\nabla (f \circ \varphi)||x - y| + \max_{\partial M} |\nabla g||x - y|^p}{|x - y|^{n + \frac{p}{2}}}
\]
\[
\leq \sum_i 4^p \delta^p \vol(\Delta_i) \int_{\partial M \cap |z| \geq r_0} \frac{1}{|z|^{n + \frac{p}{2}}}
\]
\[
+ \int_{\partial M \setminus Q} \left( \max_{\partial M} |\nabla (f \circ \varphi)| + \max_{\partial M} |\nabla g| \right)^p \int_{\partial M \cap |z| < r_0} |z|^\frac{p}{2} - n
\]
\[
< \frac{\varepsilon^p}{4} \frac{\vol(\partial M)}{r_0^{n + \frac{p}{2}}} + \frac{\varepsilon^p}{16},
\]
where we have used in the fourth line (4.1) and (4.2) and the mean value theorem in the third line. Here, the fifth line is less than $\frac{\varepsilon^p}{16}$ provided a constant $r_0$ can be chosen sufficiently small such that
\[
r_0 \cdot \vol(\partial M \cap (|z| < r_0)) \leq \frac{2^{-4}\varepsilon^p}{(\max_{\partial M} |\nabla (f \circ \varphi)| + \max_{\partial M} |\nabla g|)^p \vol(\partial M)}.
\]

Therefore, we see after a few calculation that if $\tau > 4 \frac{\vol(\partial M)}{r_0^{n + \frac{p}{2}}}$, then we have that $\|f \circ \varphi - g\|_{W^{1,p}(\partial M)} < \varepsilon$.

The remaining item follows from an easy modification in the argument of Theorem 2.1 of Kazdan and Warner [20].
Remark 4.2 It would be interesting to know if Lemma 4.1 can be generalized to include both curvatures at the same time in order to obtain results as in the spirit of [9,24], where it was considered the problem of prescribing the Gaussian and geodesic curvature of compact surfaces with boundary via conformal change of the metric.

5 Prescribing curvature: Proof of Theorem 1.1 and 1.2

In this section, using approximation Lemma 4.1 and Theorem 3.5, we prescribe the scalar (resp. Gaussian, if dim $M = 2$) curvature and mean (resp. geodesic, if dim $M = 2$) curvature of the boundary of a certain class of manifolds with boundary. More precisely, we show the following result.

Proposition 5.1 Assume $n \geq 2$. Let $(M^n, g_0)$ be a compact Riemannian manifold with boundary.

(a) Let $M$ be a scalar flat manifold with mean curvature on the boundary equal to $H_0$, and let $H$ be a smooth function on $\partial M$. If there is a constant $c > 0$ satisfying

$$\min cH \leq H_0 \leq \max cH,$$

Then $H$ is the mean curvature of the boundary of some scalar flat metric on $M$.

(b) Let $M$ be a manifold with constant scalar curvature $R_0$ and minimal boundary, and let $R$ be a smooth function on $M$. If there is a constant $c > 0$ satisfying

$$\min cR \leq R_0 \leq \max cR,$$

Then $R$ is scalar curvature of some metric with minimal boundary.

An immediate consequence is the following.

Corollary 5.2 Assume $n \geq 2$. Let $(M^n, g_0)$ be a compact Riemannian manifold with boundary.

(a) If $M$ is scalar flat manifold and its boundary has constant mean curvature $H_{g_0} = H_0$. Then any function $H$ on $\partial M$ having the same sign of $H_0$ somewhere is mean curvature of the boundary of some scalar flat metric on $M$, while if $H_0 \equiv 0$, then any function $H$ that changes sign is the mean curvature of some scalar flat metric.

(b) If $M$ is a manifold with constant scalar curvature $R_{g_0} = R_0$ and minimal boundary. Then any function $R$ on $M$ having the same sign of $R_0$ somewhere is scalar curvature of some metric with minimal boundary, while if $R_0 \equiv 0$, then any function $R$ that changes sign is the scalar curvature of some metric with minimal boundary with respect to this metric.

Proof of Proposition 5.1 We will prove item (a) since the other case is entirely analogous. If $Ker S_{g_0}^\omega = 0$ (for example, by Proposition 3.1, this may occur if $H_0$ is negative), then by Lemma 4.1 there is a diffeomorphism $\varphi$ of $M$ such that for $p > 2n$ we have

$$\|0 - R_{g_0}\|_{L^p(M)} + \|c(H \circ \varphi) - H_0\|_{W^{1/2,p}(\partial M)} = \|c(H \circ \varphi) - H_0\|_{W^{1/2,p}(\partial M)} < \eta.$$ 

In view of Theorem 3.5 there is a metric $g_1$ satisfying

$$\Psi(g_1) = (0, c(H \circ \varphi)).$$

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Throughout this section we will denote by $g = (\varphi^{-1})^*cg_1$ at each point in $M$. However, if $\text{Ker} S^n g_0 \neq 0$ (which by Proposition 3.1, says that $H_0$ is constant), one may perturb $g_0$ slightly in order to have non constant mean curvature $H_{g_0}$ and scalar flat metric in $M$ still satisfying $\min cH \leq H_0 \leq \max cH$. In order to obtain such a perturbation we may consider, for example, the following change of metric $\tilde{g} = \tilde{u}^{\frac{4}{n-2}}g_0$, where $\tilde{u}$ is a harmonic extension of a function $u$ defined on the boundary that is nearly equal to 1. Hence, we can repeat the previous argument.

We have now all ingredients to prove Theorems 1.1 and 1.2.

**Proof of Theorems 1.1 and 1.2** The proof goes along the lines of Proposition 5.1. Here the Osgood, Phillips and Sarnak uniformization theorem [29] (see also Brendle [4,5]) plays a fundamental role in our proof, because there is a unique uniform flat metric with constant geodesic curvature boundary and a unique uniform metric of constant curvature metric with geodesic boundary.

---

### 6 Existence of metrics with constant curvature

Let $(M^n, g)$, $n \geq 3$ be a complete, $n$-dimensional Riemannian manifold with boundary $\partial M$. Throughout this section $R_g$ will denote the scalar curvature with respect to the Riemannian metric $g$, while $H_g$ will be the mean curvature on the boundary. Denote by $\tilde{g} = u^{\frac{4}{n-2}}g$ a metric conformally related to $g$, where $u$ is a smooth positive function. It is well known that the transformation law for the scalar curvature and mean curvatures are given by:

$$
R_{\tilde{g}} = -\frac{4(n-1)}{(n-2)} \frac{\mathcal{L} u}{u^{n+2/(n-2)}} \quad \text{in } M
$$

$$
H_{\tilde{g}} = \frac{2}{(n-2)} \frac{B_g u}{u^{n/(n-2)}} \quad \text{on } \partial M,
$$

where $R_{\tilde{g}}$ and $H_{\tilde{g}}$ are the scalar curvature of and the mean curvature with respect to $\tilde{g}$, $\mathcal{L} = \Delta - \frac{n-2}{4(n-1)} R_g$ is the so-called conformal Laplacian and $B = \frac{\partial}{\partial v} - \frac{n-2}{2} H_g$ is an associated boundary operator.

The quadratic form associated with the operator $(\mathcal{L}, B)$ is

$$
E_g(u) = \int_M \left( \frac{4(n-1)}{n-2} |\nabla u|^2_{\tilde{g}} + R_g u^2 \right) dv + 2 \int_{\partial M} H_g u^2 d\sigma,
$$

where $dv$ and $d\sigma$ are the Riemannian measure on $M$ and the induced Riemannian measure on $\partial M$, respectively, with respect to the metric $g$.

For $a, b > 0$ let us define the following functional

$$
Q_{g}^{a,b}(u) = \frac{E_g(u)}{a(\int_M u^{2a/(n-2)} dv)^{n-2}/n + \int_{\partial M} u^{2b/(n-2)} d\sigma)^{n-2}/n};
$$

(6.1)

The Yamabe invariant is defined by

$$
\text{Yam}(M, \partial M) = \inf \left\{ Q_{g}^{a,b}(u); u > 0 \text{ in } C^\infty(M) \right\}
$$

(6.2)

---

2 In order to obtain a metric with non constant scalar curvature $R_{g_0}$ and minimal boundary still satisfying $\min cR \leq R_0 \leq \max cR$, we may consider a sufficiently small perturbation of the form $\tilde{g} = g_0 + h$ where $h(X, Y) = 0$ for all $X, Y \in T(\partial M)$.
which is invariant under conformal change of the metric $g$ for $(a, b) \in \{(0, 1), (1, 0)\}$ (see [11, 12]). It is not difficult to show that $Q_{g}^{1,0}(M, \partial M) \leq Q^{1,0}(\mathbb{S}_{+}^{n}, \partial \mathbb{S}_{+}^{n})$ (resp. $Q_{g}^{0,1}(M, \partial M) \leq Q^{0,1}(\mathbb{B}, \partial \mathbb{B})$). In [12], Escobar proved that if $Q_{g}^{1,0}(M, \partial M) \leq Q^{1,0}(\mathbb{S}_{+}^{n}, \partial \mathbb{S}_{+}^{n})$ (resp. $Q_{g}^{0,1}(M, \partial M) \leq Q^{0,1}(\mathbb{B}, \partial \math{B})$), then there exists a smooth metric $u^{\frac{4}{n-2}}g$, $u > 0$, of constant scalar curvature and zero mean curvature on the boundary (resp. zero scalar curvature with constant mean curvature on $\partial M$).

Moreover we have other invariants with respect to conformal geometry that are the eigenvalues of the boundary problem $(\mathcal{L}, \mathcal{B})$:

\[
\begin{cases}
\mathcal{L}\varphi = \lambda_{1}(\mathcal{L})\varphi & \text{in } M \\
\mathcal{B}\varphi = 0 & \text{on } \partial M
\end{cases}
\] (6.3)

and

\[
\begin{cases}
\mathcal{L}\varphi = 0 & \text{in } M \\
\mathcal{B}\varphi = \lambda_{1}(\mathcal{B})\varphi & \text{on } \partial M.
\end{cases}
\] (6.4)

An immediate consequence from the variation characterization of the first eigenvalue of problems (6.3) and (6.4) is the following.

**Proposition 6.1** (Escobar [12,13]) The first eigenfunction for problem (6.3) or (6.4) is strictly positive (or negative). Moreover, $\lambda_1(\mathcal{L})$ is positive (negative, zero) if and only if $\lambda_{1}(\mathcal{B})$ is positive (negative, zero).

We also have the following fundamental result due to Escobar [12,13] saying that there are three possibilities which are distinguished by the sign of the first eigenvalues $\lambda_{1}(\mathcal{B})$ and $\lambda_{1}(\mathcal{L})$ (In fact, there is an analogy with (1.1)).

**Proposition 6.2** (Escobar [12,13]) Let $(M^{n}, g)$ be a compact Riemannian manifold with boundary $n \geq 3$. There exists a metric conformally related to $g$ whose scalar curvature is zero and the mean curvature of the boundary does not change sign. The sign is uniquely determined by the conformal structure. Hence there are three mutually exclusive possibilities: $M$ admits a conformally related metric, which is scalar flat and of (i) positive, (ii) negative, or (iii) identically zero mean curvature of the boundary.

It is clear that holds an analogue result if there exists a metric conformally related of minimal boundary and whose scalar curvature does not change sign.

The Gauss–Bonnet theorem for surfaces with boundary gives obvious topological obstructions to prescribing the Gauss curvature and geodesic curvature. In fact, to best of our knowledge, there is no similar obstruction result for $n \geq 3$. Taking this into account we study the existence of metrics with constant scalar curvature or constant mean curvature. The strategy is first obtain metrics with $\lambda_{1}(\mathcal{B}) < 0$ and $\lambda_{1}(\mathcal{L}) < 0$. The key step is to construct a metric with negative finite total curvature

\[
\int_{M} R_{g} dv + \int_{\partial M} H_{g} d\sigma < 0
\]

for certain manifolds with boundary. Basically, we use the idea of Bérard–Bergery [3] to deform a metric in a small disk. We have the following result.

**Proposition 6.3** Assume $n \geq 3$. If $(M, g)$ is a manifold with smooth boundary, then either there exists a metric $g$ with $\lambda_{1}(\mathcal{B}) < 0$ or $\lambda_{1}(\mathcal{L}) < 0$. 

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Proof We can see from Proposition 6.1 that it is sufficient to prove that \( \lambda_1(\mathcal{L}) < 0 \). Pick an open disk \( D^n \) in \( M \) and let \( \overline{D}^d \times S^p \subset D^n \), where \( p + q = n \) with \( p \geq 1 \) and \( q \geq 2 \). Let \( f \) be a function \( f \) on \( \overline{D} \) depending only on the distance to origin and so that \( f \equiv 1 \) nearly \( \partial \overline{D} \).

On \( \overline{D} \times S^p \) we put the warped product metric 
\[
g_0 = f^{-\frac{2p}{n-1}}(g_d + f^2 g_s),
\]
where \( g_s \) and \( g_d \) are the standard metric on \( S^p \) and \( \overline{D} \), respectively. We next consider a metric \( g \) on \( M \) that coincides with \( g_0 \) on \( \overline{D} \times S^p \).

The integral of scalar curvature can be written as follows:
\[
\int_M R_g dv = \int_{M \setminus (\overline{D} \times S^p)} R_g dv + \int_{(\overline{D} \times S^p)} R_g dv.
\]
However, the second integral on the right hand side is equal to
\[
\int_{\overline{D} \times S^p} R_g dv_0 = \text{Vol}(S^p, g_s) \int_D f^{\frac{p}{n-1}-2} \left( R_{g_s} - \frac{p(n-1-p)}{n-1} |\nabla f|^2 \right) dv_{g_d},
\]
since \( p \leq n-2 \), the first integral in (6.5) and \( \int_{\partial M} H_g d\sigma \) do not depend on \( f \) we can choose \( f \) such that \( \int_{M} |\nabla f|^2 dv_{g_d} \), becomes sufficiently large in order to get \( \int_{M} R_g dv \) negative as we want. Thus by variational characterization of \( \lambda_1(\mathcal{L}) \), we have that \( \lambda_1(\mathcal{L}) < 0 \) as desired.

Remark 6.4 We can still prove that if \( \lambda_1(\mathcal{B}) < 0 \) or \( \lambda_1(\mathcal{L}) < 0 \), then there is a conformal metric, \( \tilde{g} \), which is scalar flat with mean curvature \( H_{\tilde{g}} = -1 \) on the boundary or has scalar curvature \( R_{\tilde{g}} = -1 \) and minimal boundary, respectively. Indeed, assume that \( \lambda_1(\mathcal{B}) < 0 \), so we have to solve the following problem
\[
\begin{cases}
\Delta_{\tilde{g}} u = 0 & \text{in } M \\
\frac{2}{n-2} \frac{\partial u}{\partial v} - H_{\tilde{g}} u = -u^\beta & \text{on } \partial M,
\end{cases}
\]
where \( \beta \) is the critical exponent \( n/(n-2) \). Let \( \varphi \) be the corresponding associated eigenfunction of (6.4). Now, choose constants \( 0 < c_- < c_+ \) such that \( 0 < -\lambda_1(\mathcal{B}) < (c_+ \varphi)^{\beta-1} \) and \( -\lambda_1(\mathcal{B}) > (c_- \varphi)^{\beta-1} \). Thus if \( u_\pm = c_\pm \varphi \), we have \( \lambda_1(\mathcal{B}) = B_\pm u_\pm \geq -u_\pm^\beta \) and \( \lambda_1(\mathcal{B}) < 0 \). By the sub- and super-solutions methods (cf. [30, Theorem 3.3]) the result is clear. The other case are left to the reader.

The following proposition implies the existence of metrics, depending on the case, having zero scalar curvature and minimal boundary.

Proposition 6.5 Assume \( n \geq 3 \). Let \( M^n \) be a compact manifold with smooth boundary. If \( M \) admits a scalar flat metric of positive mean curvature (resp. positive scalar curvature and minimal boundary), then it admits a scalar flat metric with zero mean curvature.

Proof Assume that \( M \) admits a scalar flat metric of positive mean curvature on the boundary. By supposition, \( \lambda_1(\mathcal{B}, g_+) > 0 \). On the other hand, it follows from Proposition 6.3 that there exists \( g_- \) such that \( g_- \) is scalar flat and has constant negative mean curvature. So we have that \( Q_{g_-}^{0.1}(M, \partial M) < 0 \) and \( Q_{g_+}^{0.1}(M, \partial M) > 0 \). Setting \( g_t = tg_- + (1-t)g_+ \), there exists \( t_0 \in (0, 1) \) such that \( 0 < Q_{g_{t_0}}^{0.1}(M, \partial M) < Q_{g_+}^{0.1}(\overline{M}, \partial \overline{M}) \) provided \( Q_{g}^{0.1}(M, \partial M) \) depends continuously\(^3\) on \( g \). Therefore, the result follows from Proposition 2.1 of [12].

\(^3\) The proof is similar to the proof that the Steklov eigenvalues \( \lambda(g) \) depend continuously on \( g \).
Combining Propositions 6.3 and 6.5 we arrive at the following proposition.

**Proposition 6.6** Let $M^n$, $n \geq 3$, be a manifold with smooth boundary.

(a) $M$ carries a scalar flat metric with constant negative mean curvature (resp. constant negative scalar curvature with minimal boundary).

(b) If $M$ carries a scalar flat metric $g$ whose boundary has mean curvature $H_\gamma \geq 0$ and $H_\gamma \neq 0$ (resp. $R_g \geq 0$ and $R_g \neq 0$ with minimal boundary), then there exists on $M$ a scalar flat metric with mean curvature $H_\gamma \equiv 1$ on $\partial M$ (resp. a metric that has scalar curvature $R_g \equiv 1$ and minimal boundary) and a scalar flat metric with zero mean curvature on $\partial M$.

Finally, combining Propositions 5.1, 6.6 and 6.1, we can draw the following conclusion:

**Corollary 6.7** (Theorem 1.3 and Theorem 1.4) Let $M^n$, $n > 2$, be a manifold with smooth boundary.

(i) Any function that is negative somewhere on $\partial M$ (resp. on $M$) is a mean curvature of a scalar flat metric (resp. a scalar curvature of a metric whose boundary has mean curvature zero).

(ii) Every smooth function on $\partial M$ (resp. on $\partial M$) is a mean curvature of a scalar flat metric (resp. a scalar curvature of a metric, where the mean curvature of the boundary is zero) if and only if $M$ admits a scalar flat metric with positive constant mean curvature on the boundary (resp. positive constant scalar curvature and minimal boundary).

7 Proof of Theorem 1.5

Because Theorems 1.3 and 1.4, we can be divide the class of the compact manifolds with boundary as follows:

- $M$ carries a scalar flat metric $g$ whose boundary has mean curvature $H_\gamma \geq 0$ and $H_\gamma \neq 0$ (resp. a metric with scalar curvature $R_g \geq 0$ and minimal boundary).
- $M$ carries no scalar flat metric $g$ with positive mean curvature (no metric with positive scalar curvature and minimal boundary), but do have one with $H_\gamma \equiv 0$ and $R_g \equiv 0$.
- $M$ carries a scalar flat metric whose mean curvature on $\partial M$ is negative somewhere (metric with negative scalar curvature and whose boundary is minimal).

Next we will prove Theorem 1.5 which follows from the topological restrictions discussed in Sect. 6.

**Proof of Theorem 1.5** The result is an immediate consequence of Propositions 5.1 and 6.6. However, it remains to show that if $M$ does not admit a scalar flat metric with positive mean curvature on the boundary, then any scalar curvature metric with zero mean curvature on the boundary has totally geodesic boundary (Similarly, we can prove the analogous if $M$ does not admit a metric with positive scalar curvature on the boundary and minimal boundary).

Assume that $\lambda_1(B) = 0$. Let $g(t)$ be a smooth family of metrics with $g(0) = g$ and infinitesimal variation $\frac{d}{dt} g(t)|_{t=0} = -\Pi_g$. If $\Pi_g \neq 0$, then it follows from Proposition 8.1 in “Appendix A” that

$$\frac{d}{dt} \lambda_1(B)|_{t=0} = \int_M |\Pi_g|^2 d\sigma > 0,$$

(7.1)
where we have used that \( g \) is scalar flat and has constant mean curvature on the boundary which implies that \( \psi = 1 \) and \( \Pi_g = \Pi_{\hat{g}} \). So \( \lambda_1(g(t)) > 0 \) for all \( t > 0 \) sufficiently small and we conclude from Proposition 6.1 that there is a metric with positive mean curvature, which is a contradiction unless \( \Pi_g \equiv 0 \) holds.

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### 8 Appendix A

Let \((M^n, g)\), \(n \geq 3\), be a \(n\)-dimensional Riemannian manifold with boundary \(\partial M \neq \emptyset\). Recall the following conformal boundary operator \((L, B)\), where \(L = \Delta - \frac{n-2}{4(n-1)} R_g\) in \(M\) and \(B = \frac{\partial}{\partial v} - \frac{n-2}{2} H_g\) on \(\partial M\). Recall the associated eigenvalue problems:

\[
\begin{cases}
L \phi = \lambda_1(L) \phi & \text{in } M \\
B \phi = 0 & \text{on } \partial M
\end{cases}
\quad (8.1)
\]

and

\[
\begin{cases}
L \phi = 0 & \text{in } M \\
B \phi = \lambda_1(B) \phi & \text{on } \partial M.
\end{cases}
\quad (8.2)
\]

Our next result concerns the variation of the first eigenvalues of the boundary problems (8.1) and (8.2).

**Proposition 8.1** Let \(\phi\) and \(\psi\) be normalized first eigenfunction of \((L, B)\) with respect to the interior and the boundary condition, respectively. Then

\[
\begin{align*}
(a) \quad & \frac{d}{dt} \lambda_1(L) \bigg|_{t=0} = - \int_M \phi^2(h, Ric_{\hat{g}}) dv; \\
(b) \quad & \frac{d}{dt} \lambda_1(B) \bigg|_{t=0} = - \int_M \psi^2(h, \Pi_{\hat{g}}) d\sigma;
\end{align*}
\]

where \(\hat{g} = \phi^{4/(n-2)} g\) and \(\tilde{g} = \psi^{4/(n-2)} g\).

**Proof** Let \(\varphi(t)\) and \(\psi(t)\) denote the first eigenfunctions associated to the first eigenvalues \(\lambda_1(L_t, g(t))\) and \(\lambda_1(B_t, g(t))\), respectively. Taking a derivative of \(L_t \varphi(t) = \lambda_1(L_t, g(t)) \varphi(t)\) and \(B_t \psi(t) = \lambda_1(B_t, g(t)) \psi(t)\) we get that

\[
\begin{align*}
L_t' \varphi(t) + L_t \varphi'(t) &= \lambda_1'(L_t, g(t)) \varphi(t) + \lambda_1(L_t, g(t)) \varphi'(t) \\
B_t' \psi(t) + B_t \psi'(t) &= \lambda_1'(B_t, g(t)) \psi(t) + \lambda_1(B_t, g(t)) \psi'(t)
\end{align*}
\]

where the prime denotes derivatives with respect to \(t\).

Then setting \(t = 0\) and using the divergence theorem we obtain that

\[
\frac{d}{dt} \lambda_1(L) \bigg|_{t=0} = \int_M \langle \varphi(0), L_0' \varphi(0) \rangle dv
\]

and

\[
\frac{d}{dt} \lambda_1(B) \bigg|_{t=0} = \int_{\partial M} \langle \psi(0), B'_0 \psi(0) \rangle d\sigma.
\]
The results now follow by a straightforward computation using (2.1) and the following identities:

\[
\left( \frac{d}{dt} \right|_{t=0} f = - \langle h, Hess_g f \rangle - \frac{1}{2} \langle \nabla (\text{tr} h), \nabla f \rangle + \omega(\nabla f),
\]

(8.5)

\[
\text{Ric}_{\hat{g}} = \text{Ric}_g - (n - 2)\text{Hess}_g f + (n - 2)|\nabla f|^2 - (\Delta f + (n - 2)|\nabla f|^2)g
\]

and

\[
\Pi_{\hat{g}} = e^f \Pi_g + \frac{\partial}{\partial v}(e^f)g,
\]

(8.7)

where \( \hat{g} = e^{2f}g \). We remark that these conformal change formulae (8.6) and (8.7) can be found at [11].

\[\square\]

References

1. Almaraz, S.: An existence theorem of conformal scalar flat metrics on manifolds with boundary. Pac. J. Math. 248(1), 1–22 (2010)
2. Araujo, H.: Critical points of the total scalar curvature plus total mean curvature functional. Indiana Univ. Math. J. 52(1), 85–107 (2003)
3. Berard-Bergery, L.: La courbure scalaire de variétés riemanniennes. Séminaire Bourbaki, Springer Lecture Notes, vol. 842, pp. 225–245 (1980)
4. Brendle, S.: Curvature flows on surfaces with boundary. Math. Ann. 324(3), 491–519 (2002)
5. Brendle, S.: A family of curvature flows on surfaces with boundary. Math. Z. 241(4), 829–869 (2002)
6. Brendle, S., Chen, S.: An existence theorem for the Yamabe problem on manifolds with boundary. J. Eur. Math. Soc. 16(5), 991–1016 (2014)
7. Chodosh, O.: Notes on linearisation stability. https://web.math.princeton.edu/~ochodosh/LinStabNOTES.pdf (2013)
8. Cox, G.: Scalar Curvature Rigidity Theorems for the Upper Hemisphere. Dissertation, Duke University (2011)
9. Cruz-Blázquez, S., Ruiz, D.: Prescribing Gaussian and geodesic curvature on disks. Adv. Nonlinear Stud. 18(3), 453–468 (2018)
10. Egorov, Y.V., Shubin, M.A.: Linear partial differential equations. Foundations of the classical theory. In: Partial Differential Equations-1, Itoiki Nauki i Tekhniki. Ser. Sovrem. Probl. Mat. Fund. Napr., vol. 30, pp. 5–255. VINITI, Moscow (1988)
11. Escobar, J.F.: The Yamabe problem on manifolds with boundary. J. Diff. Geom. 35, 21–84 (1992)
12. Escobar, J.F.: Conformal deformation of a Riemannian metric to a scalar flat metric with constant mean curvature on the boundary. Ann. Math. 136, 1–50 (1992)
13. Escobar, J.F.: Conformal deformation of a Riemannian metric to a constant scalar curvature metric with constant mean curvature on the boundary. Indiana Univ. Math. J. 45, 917–943 (1996)
14. Escobar, J.F.: Conformal metrics with prescribed mean curvature on the boundary. Calc. Var. Partial Differ. Equ. 4, 559–592 (1996)
15. Escobar, J.F., Garcia, G.: Conformal metrics on the ball with zero scalar curvature and prescribed mean curvature on the boundary. J. Funct. Anal. 211(1), 71–152 (2004)
16. Fischer, A., Marsden, J.: Deformations of the scalar curvature. Duke Math. J. 42(3), 519–547 (1975)
17. Fischer, A., Marsden, J.: Linearization stability of nonlinear partial differential equations. Proc. Symp. Pure Math. 2(2), 219–263 (1975)
18. Hörmander, L.: The Analysis of Linear Partial Differential Operators III: Pseudo-Differential Operators. Springer, Berlin, 2007 (1985). ISBN 978-3-540-49937-4
19. Kazdan, J., Warner, F.: Curvature functions for compact 2-manifold. Ann. Math. 99, 14–47 (1974)
20. Kazdan, J., Warner, F.: Existence and conformal deformation of metrics with prescribed Gaussian and scalar curvature. Ann. Math. 101(2), 317–331 (1975)
21. Kazdan, J., Warner, F.: Scalar curvature and conformal deformation of Riemannian structure. J. Differ. Geom. 10, 113–134 (1975)
22. Kazdan, J., Warner, F.: A direct approach to the determination of Gaussian and scalar curvature functions. Invent. Math. 28, 227–230 (1975)
23. Lions, J.L., Magenes, E.: Non-Homogeneous Boundary Value Problems and Applications I. Springer, Berlin (1972)
24. López-Soriano, R., Malchiodi, A., Ruiz, D.: Conformal metrics with prescribed Gaussian and geodesic curvatures. arXiv:1806.11533
25. López-Soriano, R., Ruiz, D.: Prescribing the Gaussian curvature in a subdomain of $S^2$ with Neumann boundary condition. J. Geom. Anal. 26(1), 630–644 (2016)
26. Marques, F.: Existence results for the Yamabe problem on manifolds with boundary. Indiana Univ. Math. J. 54, 1599–1620 (2005)
27. Marques, F.C.: Conformal deformations to scalar-flat metrics with constant mean curvature on the boundary. Commun. Anal. Geom. 15(2), 381–405 (2007)
28. Mayer, M., Ndiaye, C.B.: Barycenter technique and the Riemann mapping problem of Cherrier–Escobar. J. Differ. Geom. 107(3), 519–560 (2017)
29. Osgood, B., Phillips, R., Sarnak, P.: Extremals of determinants of Laplacians. J. Funct. Anal. 80, 148–211 (1988)
30. Schwartz, F.: The zero scalar curvature Yamabe problem on noncompact manifolds with boundary. Indiana Univ. Math. J. 55, 1449–1459 (2006)

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