DUALITIES OF DIFFERENTIAL GEOMETRIC INVARIANTS ON CUSPIDAL EDGES ON FLAT FRONTS IN THE HYPERBOLIC SPACE AND THE DE SITTER SPACE

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Abstract. We compute the differential geometric invariants of cuspidal edges on flat surfaces in hyperbolic 3-space and in de Sitter space. Several dualities of invariants are pointed out.

1. Introduction

In [6], one Weierstrass type representation formula for flat surfaces in the hyperbolic 3-space $H^3$ is obtained, and two representations are given in [21]. In [21], a notion flat fronts for flat surfaces which admit certain singularities is defined and the Weierstrass type representation formula is extended to the representation formula for flat fronts. This formula represents flat fronts via two holomorphic functions (called the Weierstrass data). In [20], criteria for cuspidal edges and swallowtails are given in terms of the Weierstrass data, and it is shown that these singularities are generic singularities of flat fronts in $H^3$. For a flat fronts in $H^3$, it is known that its unit normal vector taking the value in the de Sitter 3-space $S^3_1$ (called the de Sitter Gauss map) is also a flat front in $S^3_1$, and it is constructed from the same data. It is known that the sets of singular points of these two flat fronts coincide. Flat fronts are special cases of linear Weingarten fronts. In [22], a global Weierstrass type representation of linear Weingarten fronts is given, and global properties of them are studied and correspondence between dual surfaces of them are given (see [22] for detail, see also [7,17]).

One can regard that these two flat surfaces are in a dual relation. It is introduced in [11], that a formulation considering this duality as a double Legendrian fibration in contact geometry. See [13,14] for studies of linear Weingarten surfaces from this viewpoint. On the other hand, a front is a surface in a 3-space with well-defined unit normal vector even on the set of singular points. Since there is a unit normal vector, fronts can be studied from the differential geometric viewpoint. Many geometric invariants of cuspidal edges and other singular points are introduced and geometric properties of them are investigated. (see [2,5,9,10,15,25,29,33,35] for example). Since flat fronts are determined by the Weierstrass data, it is natural to consider a relation between the data and the invariants of singular points. This relation can be regarded as geometric meanings of the data on singular points.

In this paper, we give explicit expressions for invariants of cuspidal edges in terms of the Weierstrass data $(\alpha, \beta)$ (Theorem 3.4). We can observe several dualities of
geometric invariants and singularities of cuspidal edges on flat fronts in $H^3$ and $S^3_1$. Using such expressions, we characterize the condition that the set of singular points which consists of cuspidal edges is a line of curvature (Proposition 4.1). Furthermore, we show a relation between lines of curvature and cone-like singular points (Corollary 4.3). For geometric meanings of cone-like singularities of flat fronts in $H^3$, see [8, 20, 21].

2. Preliminaries

2.1. Fronts in $H^3$ and $S^3_1$. Let $R^4_1$ be the Lorentz-Minkowski 4-space with the inner product $\langle \cdot, \cdot \rangle = (-+++)$. Let

$$H^3 = \{ x = (x_0, x_1, x_2, x_3) \in R^4_1 \mid (x, x) = -1, x_0 > 0 \},$$

$$S^3_1 = \{ x \in R^4_1 \mid (x, x) = 1 \}$$

be the hyperbolic and the de Sitter 3-spaces. Although dual relations of surfaces in these two pseudo-spheres has been known (see [13] (2.3), [19] (1.9) for example), we use the formulation in [11] to working on Legendrian dualities. Following [11], we set $\Delta_1 \subset H^3 \times S^3_1$ by

$$\Delta_1 = \{ (x, y) \in H^3 \times S^3_1 \mid \langle x, y \rangle = 0 \},$$

and set $\pi_1 : \Delta_1 \to H^3$, $\pi_2 : \Delta_1 \to S^3_1$ by

$$\pi_1(x, y) = x, \quad \pi_2(x, y) = y.$$

Moreover, we set two 1-forms

$$\theta_1 = -x_0 dy_0 + \sum_{i=1}^{3} x_i dy_i \bigg|_{\Delta_1} , \quad \theta_2 = -y_0 dx_0 + \sum_{i=1}^{3} y_i dx_i \bigg|_{\Delta_1}.$$ 

Then $\theta_1^{-1}(0)$ and $\theta_2^{-1}(0)$ determine the same tangent plane field over $\Delta_1$ which is denoted by $K$. It is well-known that the pair $(\Delta_1, K)$ is a contact manifold and $\pi_1$ and $\pi_2$ are Legendrian fibrations ([11, Theorem 2.2]). This formulation is introduced for investigating surfaces in the lightcones. See [11] for other dualities among pseudo-spheres. It should be remarked that dualities of hypersurfaces in pseudo-spheres in the Lorentz-Minkowski space are also found independently (see [8, 23, 24]).

Let $(N, ds^2)$ be a Riemannian or a semi-Riemannian 3-manifold. Let $U \subset R^2$ be a domain. A map $k : U \to N$ is a frontal if there exists a map $L : U \to T_1N$ such that $\pi \circ L = k$ and $ds^2(dk(X), L(p)) = 0$, where $\pi : T_1N \to N$ is the unit tangent bundle. The lift $L$ is called an isotropic lift of $k$. A frontal $k$ is a front if an isotropic lift can be taken as an immersion. Since the unit tangent bundle of $H^3$ can be identified with $H^3 \times S^3_1$, we can rewrite this setting by using $\Delta_1$. A map $L = (f, g) : U \to \Delta_1$ is isotropic if $L^*\theta_1 = 0$. A map $f : U \to H^3$ (respectively, $g : U \to S^3_1$) is a frontal if there exists a map $g : U \to S^3_1$ (respectively, $f : U \to H^3$) such that the pair $(f, g) : U \to \Delta_1$ is isotropic. A frontal $f : U \to H^3$ (respectively, $g : U \to S^3_1$) is a front if the map $L = (f, g)$ can be taken as an immersion. If $(f, g) : U \to \Delta_1$ is isotropic, then we say that $f$ and $g$ are $\Delta_1$-dual each other, $g$ is a $\Delta_1$-dual of $f$, and $f$ is a $\Delta_1$-dual of $g$. 


2.2. Matrix representation of $H^3$ and $S^3_1$. Let $\text{Herm}(2)$ be the set of $2 \times 2$ Hermitian matrices. We take elements $e_0, e_1, e_2, e_3 \in \text{Herm}(2)$ as

$$e_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

where $i = \sqrt{-1}$. Then we have an identification $\iota : \mathbb{R}^4 \to \text{Herm}(2)$

$$\iota(x) = \sum_{k=0}^{3} x_k e_k = \begin{pmatrix} x_0 + x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_0 - x_3 \end{pmatrix},$$

where $x = (x_0, x_1, x_2, x_3)$ with the metric

$$\langle X, Y \rangle = \frac{1}{2} \text{trace} \left( X e_2 Y e_2 \right),$$

$X, Y \in \text{Herm}(2)$. In particular,

$$\langle X, X \rangle = -\det X \quad (X \in \text{Herm}(2)).$$

By this identification, $H^3$ and $S^3_1$ are rewritten as

$$H^3 = \{ X \in \text{Herm}(2) \mid \det X = 1, \text{trace } X > 0 \} = \{ AA^\ast \mid A \in SL(2, \mathbb{C}) \},$$

$$S^3_1 = \{ X \in \text{Herm}(2) \mid \det X = -1 \} = \{ Ae_3 A^\ast \mid A \in SL(2, \mathbb{C}) \}.$$  

Furthermore, the exterior product in $T_pH^3$ and $T_pS^3_1$ are rewritten as

$$X \times Y = \frac{i}{2}(X p^{-1} Y - Y p^{-1} X)$$

for $X, Y \in T_pH^3$, or $X, Y \in T_pS^3_1$.

2.3. Singularities of fronts and their differential geometric invariants. Let $(N, ds^2)$ be a Riemannian or a semi-Riemannian 3-manifold. Let $U \subset \mathbb{R}^2$ be a domain, and $k : U \to N$ a front. For $p \in U$, considering $L(p) = (p, \nu(p)) \in T^1_pN$, we consider the signed area density function

$$\Omega(k_u(u, v), k_v(u, v), \nu(u, v))$$

for a coordinate system $(u, v)$, and $(u)_u = \partial/\partial u$, $(v)_v = \partial/\partial v$, where $\Omega$ is the volume form. A function $\lambda : U \to \mathbb{R}$ is called an identifier of singularity if it is a non-zero functional multiplication of $\Omega(k_u(u, v), k_v(u, v), \nu(u, v))$. A singular point $p$ of $k$ is called non-degenerate if $d\lambda(p) \neq 0$, where $\lambda$ is an identifier of singularity.

Let $p \in U$ be a non-degenerate singular point of a front $k : U \to N$. Then by the non-degeneracy of $p$, the set of singular points $\Sigma(k)$ is a regular curve near $p$, and hence we take a parameterization $\gamma(t)$ ($\gamma(0) = p$) of it. We call $\gamma$ a singular curve. We set $\dot{\gamma} = k \circ \gamma$. One can show that there exists a vector field $\eta$ on a sufficient small neighborhood of $p$ such that if $p \in \Sigma(k)$, then $\ker dk_p = \langle \eta_p \rangle$. We call $\eta$ the null vector field. The restriction of $\eta$ on $\Sigma(k)$ can be parameterized by the parameter $t$ of $\gamma$. We denote by $\eta(t)$ the null vector field along $\gamma$.

**Definition 2.1.** Let $\mathbb{R}^3$ be the Euclidean 3-space and $k : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ a $C^\infty$ map-germ. The map-germ $k$ at 0 is a cuspidal edge (respectively, a swallowtail) if it is $\mathcal{A}$-equivalent to the map-germ $(u, v) \mapsto (u, v^2, v^3)$ (respectively, $(u, v) \mapsto (u, 4v^3 + 2uv, 3u^4 + uv^2)$) at the origin. Here, two map-germs $g_1, g_2 : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ are $\mathcal{A}$-equivalent if there exist diffeomorphism-germs $\Xi_s : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ and $\Xi_t : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ satisfying $g_2 \circ \Xi_s = \Xi_t \circ g_1$. 


If $k: (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ is a cuspidal edge or a swallow, then it is a front and 0 is a non-degenerate singular point. There are useful criteria for them. Let $k: U \to \mathbb{R}^3$ be a front and $p \in U$ a non-degenerate singular point. Let $\gamma(t)$ be a parametrization $(\gamma(0) = p)$ of the singular curve, and $\eta(t)$ a null vector field. We set $\delta(t) = \det(\gamma'(t), \eta(t))$. For a non-degenerate singular point $p$ of $k$, the map-germ $k$ at $p$ is cuspidal edge (respectively, swallowtail) if and only if $\delta(0) = 0$ (respectively, $\delta'(0) = 0$). See [20, Proposition 1.3].

Let $M$ be $H^3$ or $S^3_1$. Let $f: U \to M$ be a front and let $f$ at $p$ be a cuspidal edge, and $g$ a $\Delta_1$-dual of $f$. A pair of vector fields $(\xi, \eta)$ is called adapted if $\xi$ is tangent to the singular set and $\eta$ is a null vector field of $f$. By the criterion for cuspidal edge, $\eta \neq 0$ holds. Thus $\xi f$ and $\nabla_{\eta f}$ are linearly independent, in particular, $\xi f \neq 0$ and $\xi f \times \nabla_{\eta f} \neq 0$ hold. We define

$$\kappa_s(t) = \sgn(\lambda) \frac{\Omega(\xi f, \nabla_\xi(\xi f), g)}{|\xi f|^3} \bigg|_{(u,v) = \gamma(t)}, \quad \kappa_n(t) = \frac{\langle \nabla_\xi(\xi f), g \rangle}{|\xi f|^2} \bigg|_{(u,v) = \gamma(t)},$$

$$\kappa_t(t) = \frac{\Omega(\xi f, \nabla_\eta(\eta f), \nabla_\xi(\eta f))}{|\xi f \times \nabla_\eta(\eta f)|^2} - \frac{\Omega(\xi f, \nabla_\eta(\eta f), \nabla_\xi(\xi f)) \langle \xi f, \nabla_\eta(\eta f) \rangle}{|\xi f|^2 |\xi f \times \nabla_\eta(\eta f)|^2} \bigg|_{(u,v) = \gamma(t)},$$

$$\kappa_c(t) = \frac{|\xi f|^3/2 \Omega(\xi f, \nabla_\eta(\eta f), \nabla_\xi(\eta f))}{|\xi f \times \nabla_\eta(\eta f)|^{3/2}} \bigg|_{(u,v) = \gamma(t)},$$

where $\gamma(t)$ is a parametrization of $\Sigma(f)$, $\lambda$ is an identifier of singularity, and $\Omega$ is the volume form and under the identification $T_pR^4_1 = R^4_1$, it can be calculated by $\Omega(X, Y, Z) = (X \times Y, Z)$ for $X, Y, Z \in T_{f(p)}H^3$ or for $X, Y, Z \in T_{f(p)}S^3_1$. Here, $\nabla$ is the metric connection defined for a vector $\zeta \in T_{f(p)}H^3$,

$$\nabla_\zeta k = \zeta k + \langle \zeta, k \rangle f \in T_{f(p)}H^3,$$

and for a vector $\zeta \in T_{f(p)}S^3_1$,

$$\nabla_\zeta k = \zeta k - \langle \zeta, g \rangle g \in T_{f(p)}S^3_1.$$

The above functions $\kappa_s$, $\kappa_n$, $\kappa_t$ and $\kappa_c$ do not depend on the choices of the parameters of $\Sigma(f)$, $\xi(\gamma)$ nor on the choice of $(\xi, \eta)$, and they are called singular curvature, limiting normal curvature, cuspidal-directional torsion (cuspidal torsion) and cuspidal curvature respectively. See [29] (for $\kappa_s$ and $\kappa_n$), [26] (for $\kappa_t$) and [27] (for $\kappa_c$) for detail.

3. Flat fronts in $H^3$ and $S^3_1$

Let $U \subset C$ be a domain, and $\alpha, \beta : U \to C \setminus \{0\}$ holomorphic functions. We consider a solution $A : U \to SL(2, C)$ of the differential equation

$$A'(z) = A(z)D(z) \quad \left( D(z) = \begin{pmatrix} 0 & \alpha(z) \\ \beta(z) & 0 \end{pmatrix} \right), \quad = \frac{d}{dz}.$$  

Then one can see that $\det A$ is a constant, we take a solution satisfying $\det A = 1$. We define maps $f: U \to H^3$ and $g: U \to S^3_1$ by

$$f(z) = A(z)A^*(z), \quad g(z) = A(z)e_3A^*(z).$$
Then it is known that $f$ is (zero intrinsic curvature) and $g$ is spacelike flat on their regular point sets [21, Proposition 2.5]. Furthermore, $f$ and $g$ are $\Delta_1$-dual each other.

By a calculation, we have

$$f' = ADA^*, \quad f_\tau = AD^+A^*, \quad g' = ADe_3A^*, \quad g_\tau = Ae_3D^+A^*.$$  

Thus by

$$f' \times f_\tau = \frac{i}{2} \lambda Ae_3A^*, \quad g' \times g_\tau = \frac{-i}{2} \lambda AA^*,$$

the singular sets $\Sigma(f)$ and $\Sigma(g)$ are coincide, where $\lambda$ is defined by

$$\lambda = \alpha\overline{\alpha} - \beta\overline{\beta}.$$  

We see that $\lambda$ is an identifier of singularity for each $f$ and $g$. It is also known that the set of singular points of flat front in the 3-sphere $S^3$ and that of its dual coincide. In [16], a characterization of the flat torus in $S^3$ is obtained.

3.1. Criteria for singularities of $f, g$. We give conditions for cuspidal edges and swallowtails appearing on the flat fronts in $H^3$ and in $S^3_1$ by the Weierstrass data $(\alpha, \beta)$. The case of the flat fronts in $H^3$, it is obtained in [20, Theorem 1.1], and the case of the flat fronts in $S^3_1$, the similar conditions are obtained in [4, Proposition 6]. We shall state here conditions for cuspidal edges and swallowtails on flat fronts in $S^3_1$ in terms of the Weierstrass data $(\alpha, \beta)$.

Let $f : U \to H^3$ and $g : U \to S^3_1$ be flat fronts constructed by the pair $(\alpha, \beta)$ of holomorphic functions as in (3.2). Let $p$ be a non-degenerate singular point of $f$ or $g$, and $\gamma(t)$ a parametrization of singular curve near $p$. The tangent vector $\dot{\gamma}(t) = (d/dt)\gamma(t)$ to $\gamma$ can be expressed by

$$\xi(t) = -i\lambda(z(\gamma(t))) = -i(\alpha\overline{\alpha} - \beta\overline{\beta})(\gamma(t))\partial_z + i(\alpha'\overline{\alpha} - \beta'\overline{\beta})(\gamma(t))\partial_{\overline{z}}$$

(see [20]) under the identification $T_pU$ with $R^2$ and $C$ via the correspondence

$$\zeta = a + bi \in C \leftrightarrow (a, b) \in R^2 \leftrightarrow a\partial_u + b\partial_v = \zeta\partial_z + \overline{\zeta}\partial_{\overline{z}},$$

where $z = u + iv$. On the other hand, following [20, p322], one can take null vector fields as follows:

**Lemma 3.1.** The vector field $\eta_h$ gives a null vector field of $f$ and $\eta_g$ gives a null vector field of $g$, where

$$\eta_h = \frac{i}{\sqrt{\alpha\beta}} = \frac{i}{\sqrt{\alpha\beta}}\partial_z - \frac{i}{\sqrt{\alpha\beta}}\partial_{\overline{z}}, \quad \eta_d = \frac{1}{\sqrt{\alpha\beta}} = \frac{1}{\sqrt{\alpha\beta}}\partial_z + \frac{1}{\sqrt{\alpha\beta}}\partial_{\overline{z}}.$$

**Proof.** See [20, p322] for $\eta_h$. The directional derivative $\eta_d g$ of $g$ in the direction of $\eta_d$ can be calculated as

$$\eta_d g = A \begin{pmatrix} 0 & -\frac{\alpha}{\sqrt{\alpha\beta}} + \overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} A^*.$$  

Since $\alpha\overline{\alpha} - \beta\overline{\beta} = 0$ on $\Sigma(g)$, it follows that $\eta_d g = 0$ on $\Sigma(g)$.
We set
\[ C_h = \text{Re} \left( \frac{i\lambda'}{\sqrt{\alpha\beta}} \right) = \frac{\text{Re}(i\sqrt{\alpha\beta}\lambda')}{|\alpha|^2} \]
(3.7)
\[ = -\text{Im} \left( \frac{\lambda'}{\sqrt{\alpha\beta}} \right) = \text{Im} \left( \frac{\lambda'}{\sqrt{\alpha\beta}} \right) = \frac{\text{Im}(\sqrt{\alpha\beta}\lambda')}{|\alpha|^2}, \]
\[ C_d = \text{Re} \left( \frac{\lambda'}{\sqrt{\alpha\beta}} \right) = \frac{\text{Re}(\sqrt{\alpha\beta}\lambda)}{|\alpha|^2} \]
(3.8)
\[ = \text{Im} \left( \frac{i\lambda'}{\sqrt{\alpha\beta}} \right) = \frac{\text{Im}(i\sqrt{\alpha\beta}\lambda)}{|\alpha|^2}. \]

Then we have the following proposition.

**Proposition 3.2.** (I) ([20, Theorem 1.1]) Let \( f : U \rightarrow H^3 \) be a flat front constructed by the data \((\alpha, \beta)\) as in (3.2). Let \( p \) be a non-degenerate singular point of \( f \). Then
1. \( f \) at \( p \) is a cuspidal edge if and only if \( C_h \neq 0 \) at \( p \).
2. \( f \) at \( p \) is a swallowtail if and only if \( C_h = 0 \) and
\[ \text{Re} \left( \frac{S(\alpha) - S(\beta)}{\alpha\beta} \right) \neq 0 \]
at \( p \), where \( S(\alpha) \) is the Schwarzian derivative of the primitive function of \( \alpha \) with respect to \( z \):
\[ S(\alpha) = \left( \frac{\alpha'}{\alpha} \right)' - \frac{1}{2} \left( \frac{\alpha'}{\alpha} \right)^2. \]

(II) Let \( g : U \rightarrow S_1^3 \) be a flat front constructed by the data \((\alpha, \beta)\) as in (3.2). Let \( p \) be a non-degenerate singular point of \( g \). Then
1. \( g \) at \( p \) is a cuspidal edge if and only if \( C_d \neq 0 \) at \( p \).
2. \( g \) at \( p \) is a swallowtail if and only if \( C_d = 0 \) and
\[ \text{Re} \left( \frac{S(\alpha) - S(\beta)}{\alpha\beta} \right) \neq 0 \]
at \( p \).

**Proof.** See [20, Theorem 1.1] for the proof of (I). Since
\[ \{ z | (\eta_h f, \eta_d f) = (0, 0) \} = \{ z | (\eta_h g, \eta_d g) = (0, 0) \} = \emptyset, \]
and \( f \) and \( g \) are dual each other, \( f \) and \( g \) are fronts. By [20, Proposition 1.3], it is enough to show that the condition (II), (I) in the proposition is equivalent to \( \delta_d(0) \neq 0 \), where \( \delta_d(t) = \det(\xi(t), \eta_d(t)) \).

\[ \delta_d(t) = -i \left( \frac{\lambda'}{\sqrt{\alpha\beta}} + \frac{\lambda'}{\sqrt{\alpha\beta}} \right) (\gamma(t)) = -2i \text{Re} \left( \frac{\lambda'}{\alpha\beta} \right) (\gamma(t)) \]
(3.10)
\[ = -2i \text{Re} \left( \sqrt{\alpha\beta}\lambda \frac{\gamma(t)}{\alpha(\gamma(t))} \right) \]
and the relation \( \alpha(\gamma) = |\alpha|^2/(\alpha\beta) \) holds along \( \gamma \), the first assertion holds.

Next we show the second assertion of (II). We note that
\[ \lambda' = |\alpha|^2 \left( \frac{\alpha'}{\alpha} - \frac{\beta'}{\beta} \right) \]
on \( \gamma \). Thus by (3.10), \( \delta_d \) is proportional to
\[ \tilde{\delta}_d = \text{Re} \left( \frac{1}{\sqrt{\alpha\beta}} \left( \frac{\alpha'}{\alpha} - \frac{\beta'}{\beta} \right) \right). \]
We assume that \( \delta_d(0) = 0 \), namely, \( \tilde{\delta}_d(0) = 0 \). Then since
\[
\frac{d}{dt} \left( \frac{1}{\sqrt{\alpha \beta}} \left( \frac{\alpha'}{\alpha} - \frac{\beta'}{\beta} \right) \right)(\gamma(t)) \nabla \left\{ \left( \frac{\alpha'}{\alpha} - \frac{\beta'}{\beta} \right) - \frac{1}{\sqrt{\alpha \beta}} \left( \frac{\alpha'}{\alpha} - \frac{\beta'}{\beta} \right) \right\} \gamma = \frac{\gamma}{\sqrt{\alpha \beta}}(S(\alpha) - S(\beta)).
\]
Thus we have
\[
\frac{d}{dt} \left( \delta_d(\gamma(t)) \right) = \frac{1}{2} \left( \frac{\gamma}{\sqrt{\alpha \beta}}(S(\alpha) - S(\beta)) + \frac{\gamma}{\sqrt{\alpha \beta}}(S(\alpha) - S(\beta)) \right).
\]
Here it holds that
\[
\frac{\gamma}{\sqrt{\alpha \beta}} = -i|\alpha|^2 \left( \frac{\alpha'}{\alpha} - \frac{\beta'}{\beta} \right) = i|\alpha|^2 \sqrt{\alpha \beta} \left( \frac{\alpha'}{\alpha} - \frac{\beta'}{\beta} \right),
\]
and
\[
\frac{\gamma}{\sqrt{\alpha \beta}} = i|\alpha|^2 \left( \frac{\alpha'}{\alpha} - \frac{\beta'}{\beta} \right),
\]
at \( p = \gamma(0) \) because (3.35) and \( \delta_d(0) = 0 \). Hence we have
\[
\frac{d}{dt} \left( \delta_d \right)_{t=0} = \frac{1}{2} \left( \frac{\gamma}{\sqrt{\alpha \beta}}(S(\alpha) - S(\beta)) + \frac{\gamma}{\sqrt{\alpha \beta}}(S(\alpha) - S(\beta)) \right)(p)
\]
\[
= i|\alpha|^2 \sqrt{\alpha \beta} \left( \frac{\alpha'}{\alpha} - \frac{\beta'}{\beta} \right) \left( \frac{S(\alpha) - S(\beta)}{\alpha \beta} \right)(p)
\]
\[
= i|\alpha|^2 \sqrt{\alpha \beta} \left( \frac{\alpha'}{\alpha} - \frac{\beta'}{\beta} \right) \text{Re} \left( \frac{S(\alpha) - S(\beta)}{\alpha \beta} \right)(p).
\]
We note that \( (d/dt)\delta_d \neq 0 \) is equivalent to \( (d/dt)\tilde{\delta}_d \neq 0 \) under the condition \( \delta_d(0) = 0 \). Thus by [20, Proposition 1.3] again, we have the conclusion. \( \square \)

Remark 3.3. Assuming that \( u(z) \) is a solution of the differential equation
\[
(3.11) \quad u''(z) - \alpha(z)u(z) = 0,
\]
satisfying \( \det A = 1 \), where \( A = (u(z), u'(z)) \), then the matrix \( A \) satisfies (3.1). This equation (3.11) is called the SL-form of the hypergeometric equation. The flat fronts \( AA^* \) and \( Ae_3 A^* \) in \( H^3 \) and \( S^3_1 \) constructed from the data (\( a, 1 \)) can be regarded generalizations of the Schwarz map of the hypergeometric equation. They are called hyperbolic and de Sitter Schwarz map (6.8, see also [11,32]). Thus setting \( \beta = 1 \), then all invariants given in Section 8 can be regarded as invariants of these Schwarz map.

3.2. Geometric invariants of cuspidal edges. For flat fronts \( f : U \to H^3 \) and \( g : U \to S^3_1 \) given by (3.2). Let \( p \in \bar{U} \) be a singular point, and let \( f \) at \( p \) be a cuspidal edge. Let \( \gamma(t) \) be a parametrization of \( \Sigma(f) \) or \( \Sigma(g) \). Let \( \kappa^h, \kappa^t, \kappa^c \) (respectively, \( \kappa^h, \kappa^t, \kappa^c \)) be the singular curvature, cuspidal torsion and the cuspidal curvature of \( f \) on \( \Sigma(f) \) (respectively, of \( g \) on \( \Sigma(g) \)). Then we have the following theorem.
Theorem 3.4. It holds that

\begin{align}
\kappa^h_s(t) &= - \frac{|\lambda|^2}{4|\alpha|^2 \left| \text{Im} \left( \sqrt{\alpha \beta} \lambda \right) \right|} \bigg|_{(u,v)=\gamma(t)} = - \frac{|\lambda|^2}{4|\alpha|^2 C_h} \bigg|_{(u,v)=\gamma(t)} \\
\kappa^h_t(t) &= \frac{\text{Re} \left( \sqrt{\alpha \beta} \lambda \right)}{\text{Im} \left( \sqrt{\alpha \beta} \lambda \right)} \bigg|_{(u,v)=\gamma(t)} = \frac{C_d}{C_h} \bigg|_{(u,v)=\gamma(t)} \\
\kappa^h_c(t) &= \frac{4|\alpha|^2 \left| \text{Im} \left( \sqrt{\alpha \beta} \lambda \right) \right|^{3/2}}{|\text{Re} \left( \sqrt{\alpha \beta} \lambda \right)|} \bigg|_{(u,v)=\gamma(t)} = \frac{4|\alpha| C_d}{|C_h|^{3/2}} \bigg|_{(u,v)=\gamma(t)} \tag{3.14}
\end{align}

and

\begin{align}
\kappa^d_s(t) &= - \frac{|\lambda|^2}{4|\alpha|^2 \left| \text{Re} \left( \sqrt{\alpha \beta} \lambda \right) \right|} \bigg|_{(u,v)=\gamma(t)} = - \frac{|\lambda|^2}{4|\alpha|^4 C_d} \bigg|_{(u,v)=\gamma(t)} \\
\kappa^d_t(t) &= - \frac{\text{Im} \left( \sqrt{\alpha \beta} \lambda \right)}{\text{Re} \left( \sqrt{\alpha \beta} \lambda \right)} \bigg|_{(u,v)=\gamma(t)} = - \frac{C_h}{C_d} \bigg|_{(u,v)=\gamma(t)} \\
\kappa^d_c(t) &= - \frac{4|\alpha|^2 \left| \text{Re} \left( \sqrt{\alpha \beta} \lambda \right) \right|^{3/2}}{|\text{Re} \left( \sqrt{\alpha \beta} \lambda \right)|} \bigg|_{(u,v)=\gamma(t)} = - \frac{4|\alpha| C_d}{|C_d|^{3/2}} \bigg|_{(u,v)=\gamma(t)} \tag{3.17}
\end{align}

In particular, both \(\kappa^h_s\) and \(\kappa^d_s\) are strictly negative if \(f\) and \(g\) have only cuspidal edges.

We can observe dual relations between \(\kappa^h_t\) and \(\kappa^d_t\) and between \(\kappa^h_c\) and \(C_d\) (respectively, \(\kappa^d_c\) and \(C_h\)). We will see this duality in Section 3.3. We remark that since the Gaussian curvature is bounded, \(\kappa_v\) vanishes identically for \(f\) and \(g\).

Proof. We first calculate the invariant \(\kappa_s\) for \(f\) and \(g\). By a direct calculation, we have

\[
\xi f = iA \begin{pmatrix} 0 & -\lambda \alpha + \lambda' \beta \\ -\lambda' \beta + \lambda \alpha & 0 \end{pmatrix} A^*,
\]

\[
\xi g = iA \begin{pmatrix} 0 & \lambda \alpha + \lambda' \beta \\ -\lambda' \alpha + \lambda \beta & 0 \end{pmatrix} A^*,
\]

and on the singular set,

\[
\xi f = i\frac{-\lambda \alpha + \lambda' \beta}{A} ADA^* = 2\beta \sqrt{\frac{\beta}{A}} C_h ADA^*,
\]

\[
\xi g = i\frac{\lambda \beta + \lambda' \alpha}{\beta} AD e_3 A^* = 2i \sqrt{\frac{\beta}{A}} C_d AD e_3 A^*. \tag{3.18}
\]

Thus on the singular set, we have

\[
\nabla \xi f \equiv -2i\beta \frac{\beta}{\alpha} C_h AD h A^* \mod \langle f \rangle \xi(\nu),
\]

\[
\nabla \xi g \equiv -2i\alpha \frac{\beta}{\beta} C_d AD d A^* \mod \langle g \rangle \xi(\nu),
\]

where

\[
\hat{D}_h = -\lambda \psi D^2 - \lambda \psi D' + \lambda' DD^*, \quad \hat{D}_d = -\lambda \psi D^2 e_3 - \lambda \psi D' e_3 + \lambda' D e_3 D^*. 
\]
Here $\mathcal{E}(U) = \{ h : U \to \mathbb{R} \}$ is the ring consists of function-germs on $U$, and $\langle k_1, \ldots, k_r \rangle_{\mathcal{E}(U)} = \{ a_1 k_1 + \cdots + a_r k_r \mid a_1, \ldots, a_r \in \mathcal{E}(U) \}$. The formula $f_1 \equiv f_2 \mod A$ stands for $f_1 - f_2 \in A$. Hence, we have

\[ \xi f \times \nabla_\xi f \equiv 4\beta^2 C_\beta^2 \lambda_\beta (\alpha' \beta - \alpha \beta') \beta g \equiv -4\beta^2 C_\beta^2 \lambda_\beta |\lambda|^2 g \mod \langle \xi f \times f \rangle_{\mathcal{E}(U)} \]

and

\[ \xi g \times \nabla_\xi g \equiv -4\alpha^2 C_{\gamma, \beta}^2 \lambda_\beta (\alpha' \beta - \alpha \beta') \beta f \equiv 4\alpha^2 C_{\gamma, \beta}^2 |\lambda|^2 f \mod \langle \xi g \times g \rangle_{\mathcal{E}(U)} \]

on the singular set, where we see $\lambda_\beta (\alpha' \beta - \alpha \beta') = -\lambda_\beta (\alpha / \beta)(\alpha \alpha' - \beta \beta')$ on the singular set. On the other hand, the signed area density function for $f$ and $g$ are $A = \Omega(\xi f, \eta f, g)$ and $A_d = \Omega(\xi g, \eta dg, f)$, respectively. Then we see that $\eta_h A_h = -|\lambda_h|^2$, and $\eta_d A_d = |\lambda_d|^2$. Thus we have \(3.19\) and \(3.20\).

Now we proceed to calculate the cuspidal torsion and the cuspidal curvature. To show this, we prove several formulas of differentials of $f$ and $g$ needed later.

**Lemma 3.5.** On the singular set, it holds that

\[ f' \times f'' = f' \times f'' = f \times f'' = g' \times g'' = 0 \]

and

\[ f' \times f'' = \frac{i}{2}(\alpha' \beta - \alpha \beta')g, \quad f' \times f'' = \frac{i}{2} \lambda \gamma g, \]

\[ f' \times f'' = -\frac{i}{2} \lambda g, \quad f' \times f'' = -\frac{i}{2} \lambda \gamma f, \]

\[ f' \times f'' = \frac{i}{2} \lambda g, \quad f' \times f'' = \frac{i}{2} \lambda \gamma f. \]

In particular, all vectors in \(3.20\) are parallel to $g$, and that in \(3.21\) are parallel to $f$.

**Proof.** By a direct calculation, we have

\[ f'' = A \left( \begin{array}{ll} \alpha & \alpha' \\ \beta' & \alpha \beta \end{array} \right) A^*, \quad f'' = A \left( \begin{array}{ll} |\alpha|^2 & 0 \\ 0 & |\beta|^2 \end{array} \right) A^*, \quad f'' = A \left( \begin{array}{ll} \alpha / \gamma & \beta / \gamma \\ \gamma / \alpha & \gamma / \beta \end{array} \right) A^* \]

and

\[ g'' = A \left( \begin{array}{ll} \alpha & -\alpha' \\ \beta' & -\alpha \beta \end{array} \right) A^*, \quad g'' = A \left( \begin{array}{ll} -|\alpha|^2 & 0 \\ 0 & |\beta|^2 \end{array} \right) A^*, \quad g'' = A \left( \begin{array}{ll} \alpha / \gamma & \beta / \gamma \\ \gamma / \alpha & \gamma / \beta \end{array} \right) A^*. \]

This shows the assertion.

**Lemma 3.6.** On the singular set, it holds that

\[ \xi f \times \nabla_\xi f \equiv \frac{2}{|\alpha|^2} \Im \left( \sqrt{\alpha \beta} \lambda \gamma \right) \equiv g, \]

\[ \xi f \times \nabla_\xi f \equiv \frac{-2}{|\alpha|^2} \Re \left( \sqrt{\alpha \beta} \lambda \gamma \right) \equiv f. \]

**Proof.** By definition, we have

\[ \nabla_\xi f = \eta_\xi f + (\eta_\xi f, f) f, \quad \nabla_\xi f = \eta_\xi f - (\eta_\xi f, g) g. \]
Since \( \langle \eta_h f, f \rangle = \langle \eta_d g, g \rangle = 0 \), it holds that \( \langle \eta_h \eta_h f, f \rangle + \langle \eta_h f, \eta_h f \rangle = 0 \) and \( \langle \eta_d \eta_d g, g \rangle + \langle \eta_d g, \eta_d g \rangle = 0 \). Moreover, on the singular set, \( \eta_h f = \eta_d g = 0 \), and hence \( \langle \eta_h \eta_h f, f \rangle = \langle \eta_d \eta_d g, g \rangle = 0 \) on \( \Sigma(f) = \Sigma(g) \). In particular, it holds that \( \nabla_{\eta_h} \eta_h f = \eta_h \eta_h f \) and \( \nabla_{\eta_d} \eta_d g = \eta_d \eta_d g \), and we have \( \xi f \times \nabla_{\eta_h} \eta_h f = \xi f \times \eta_h \eta_h f \) and \( \xi g \times \nabla_{\eta_d} \eta_d g = \xi g \times \eta_d \eta_d g \) on the singular set.

On the singular set, it follows that

\[
(3.24) \quad \eta_h \eta_h f = \frac{-f''}{\alpha \beta} + \frac{2f'g'}{|g|^2} - \frac{f''}{\alpha \beta} - \frac{1}{2\alpha \beta} \left( \frac{\alpha' + \beta'}{\alpha + \beta} \right) f' + \frac{1}{2\alpha \beta} \left( \frac{\alpha' + \beta'}{\alpha + \beta} \right) f,
\]

\[
(3.25) \quad \eta_d \eta_d g = \frac{g''}{\alpha \beta} + \frac{2g'f'}{|f|^2} + \frac{g''}{\alpha \beta} - \frac{1}{2\alpha \beta} \left( \frac{\alpha' + \beta'}{\alpha + \beta} \right) g' - \frac{1}{2\alpha \beta} \left( \frac{\alpha' + \beta'}{\alpha + \beta} \right) g.
\]

By (3.19), (3.20), (3.21), (3.24), (3.25) and the relation \( |\alpha|^2(\alpha' \beta - \alpha \beta') = \alpha \beta \lambda' \) which holds on \( \Sigma(f) = \Sigma(g) \), we have the assertion.

By Lemma 3.6 for any \( X \in T_{f(p)} H^3 \) and \( Y \in T_{g(p)} S^1 \), we have

\[
(3.26) \quad \Omega(\xi f, \nabla_{\eta_h} \eta_h f, X) = \frac{\lambda'}{2|\alpha|^4} \left( \text{Im} \left( \sqrt{\alpha \beta \lambda'} \right) \right)^2 \langle g, X \rangle,
\]

\[
\Omega(\xi g, \nabla_{\eta_d} \eta_d g, Y) = -\frac{\lambda'}{2|\alpha|^2} \left( \text{Re} \left( \sqrt{\alpha \beta \lambda} \right) \right)^2 \langle f, Y \rangle.
\]

**Lemma 3.7.** On the singular set, \( \langle \nabla_{\eta_h} \eta_h f, \xi f \rangle = \langle \nabla_{\eta_d} \eta_d g, \xi g \rangle = 0 \) holds.

**Proof.** By the above arguments, it follows that \( \langle \nabla_{\eta_h} \eta_h f, \xi f \rangle = \langle \eta_h \eta_h f, \xi f \rangle \) and \( \langle \nabla_{\eta_d} \eta_d g, \xi g \rangle = \langle \eta_d \eta_d g, \xi g \rangle \) on \( \Sigma(f) = \Sigma(g) \). By (3.22) and (3.23),

\[
\langle f', f' \rangle = \alpha \beta, \quad \langle f', f \rangle = |\alpha|^2, \quad \langle f', f_\alpha \rangle = \alpha \beta f,
\]

\[
\langle f', f'' \rangle = \frac{1}{2} (\alpha' \beta + \alpha \beta'), \quad \langle f', f_{\alpha' \beta'} \rangle = 0, \quad \langle f', f_{\alpha \beta} \rangle = \frac{1}{2} (\alpha' \beta + \alpha \beta'),
\]

\[
\langle f_{\alpha' \beta'}, f'' \rangle = 0, \quad \langle f_{\alpha' \beta'}, f_{\alpha \beta} \rangle = \frac{1}{2} (\alpha' \beta + \alpha \beta'), \quad \langle f_{\alpha' \beta'}, f_{\alpha' \beta'} \rangle = 0,
\]

\[
\langle g', g' \rangle = -\alpha \beta, \quad \langle g', g \rangle = |\alpha|^2, \quad \langle g', g_\alpha \rangle = -\alpha \beta,
\]

\[
\langle g', g'' \rangle = \frac{1}{2} (\alpha' \beta + \alpha \beta'), \quad \langle g', g_{\alpha' \beta'} \rangle = 0, \quad \langle g', g_{\alpha \beta} \rangle = 0, \quad \langle g', g_{\alpha' \beta'} \rangle = \frac{1}{2} (\alpha' \beta + \alpha \beta'),
\]

\[
\langle g_{\alpha' \beta'}, g'' \rangle = 0, \quad \langle g_{\alpha' \beta'}, g_\alpha \rangle = 0, \quad \langle g_{\alpha' \beta'}, g_{\alpha' \beta'} \rangle = \frac{1}{2} (\alpha' \beta + \alpha \beta'),
\]

hold on the singular set. Moreover, we have \( \alpha' \alpha + \beta' \beta = |\alpha|^2 (\alpha' / \alpha + \beta' / \beta) \), \( \alpha' \beta + \alpha \beta' = \alpha \beta (\alpha' / \alpha + \beta' / \beta) \) on the singular set. Thus we have \( \langle \eta_h \eta_h f, \xi f \rangle = \langle \eta_d \eta_d g, \xi g \rangle = 0 \) on the singular set.

We turn to calculate \( \kappa^h \) and \( \kappa^d \). By Lemma 3.4, the second terms of \( \kappa^h \) and \( \kappa^d \) in (2.1) vanish. Thus for the calculations of them, only the first terms of \( \kappa^h \) and \( \kappa^d \) in (2.1) are needed. Since \( \langle f'', g \rangle = \langle f', g \rangle = \langle f_{\alpha' \beta'}, g \rangle = 0 \) and \( \langle g'', f \rangle = \langle g_{\alpha' \beta'}, f \rangle = 0 \),
\[ \langle g_{\infty}, f \rangle = 0 \] hold on \( \Sigma(f) = \Sigma(g) \), and by (3.24) and (3.25), it holds that
\[
\nabla_\xi \nabla_{\eta h} f = -i\lambda \left( \frac{-1}{\alpha \beta} f''' + \frac{2}{|\alpha|^2} f'' - \frac{1}{\alpha \beta} f'' \right) \\
+ i\lambda' \left( \frac{-1}{\alpha \beta} f'' + \frac{2}{|\alpha|^2} f' - \frac{1}{\alpha \beta} f' \right) \quad \text{mod} \; F,
\]
\[
\nabla_\xi \nabla_{\eta d} g = -i\lambda \left( \frac{1}{\alpha \beta} g''' + \frac{2}{|\alpha|^2} g'' + \frac{1}{\alpha \beta} g' \right) \\
+ i\lambda' \left( \frac{1}{\alpha \beta} g'' + \frac{2}{|\alpha|^2} g' + \frac{1}{\alpha \beta} g' \right) \quad \text{mod} \; G,
\]
where \( F = \langle f, f', f'', f'_\infty, f_{\infty} \rangle_{E(U)}, \; G = \langle g, g', g'', g'_\infty, g_{\infty} \rangle_{E(U)} \). We show the following lemma:

**Lemma 3.8.** On the singular set, it holds that
\[
\langle f'''', g \rangle = -\frac{\alpha \beta}{2|\alpha|^2} \lambda', \quad \langle f''', g \rangle = \frac{\lambda'}{2}, \quad \langle f_{\infty}, g \rangle = \frac{\lambda_{\infty}}{2}, \quad \langle f_{\infty}, g \rangle = -\frac{\lambda_{\infty}}{2|\alpha|^2} \lambda_{\infty},
\]
(3.27)
\[
\langle g''', f \rangle = \frac{\alpha \beta}{2|\alpha|^2} \lambda', \quad \langle g''', f \rangle = \frac{\lambda'}{2}, \quad \langle g_{\infty}, f \rangle = \frac{\lambda_{\infty}}{2}, \quad \langle g_{\infty}, f \rangle = -\frac{\lambda_{\infty}}{2|\alpha|^2} \lambda_{\infty}.
\]
(3.28)

**Proof.** By a direct calculation, we have
\[
f''' = A \left( \frac{\alpha \beta}{\beta''} + \frac{\alpha^2 \beta}{2\alpha' \beta + \alpha' \beta'} \right) A^*, \quad f'' = A \left( \frac{\alpha \beta}{\beta'} + \frac{\alpha^2 \beta}{2\alpha' \beta + \alpha' \beta'} \right) A^*,
\]
\[
f_{\infty} = A \left( \frac{\alpha \beta}{\beta''} + \frac{\alpha^2 \beta}{2\alpha' \beta + \alpha' \beta'} \right) A^*, \quad f_{\infty} = A \left( \frac{\alpha \beta}{\beta'} + \frac{\alpha^2 \beta}{2\alpha' \beta + \alpha' \beta'} \right) A^*,
\]
and
\[
g''' = A \left( \frac{-\alpha \beta}{\beta''} + \frac{\alpha^2 \beta}{2\alpha' \beta + \alpha' \beta'} \right) A^*, \quad g_{\infty} = A \left( \frac{-\alpha \beta}{\beta'} + \frac{\alpha^2 \beta}{2\alpha' \beta + \alpha' \beta'} \right) A^*.
\]
(3.29)

This shows the assertion. \( \square \)

Let us continue the calculations for \( \kappa_i^\alpha \) and \( \kappa_i^\beta \). By (3.24) and (3.25),
\[
\Omega(\xi, f, \nabla_{\eta h} f, \nabla_\xi \nabla_{\eta h} f) = \frac{i}{|\alpha|^8} \left( \frac{\lambda(\lambda_{\infty})^2}{\alpha \beta} \lambda_{\infty}^2 - \alpha \beta \lambda_{\infty}^2 \right)^2 \\
= 4 \text{Re} \left( \sqrt{\alpha \beta} \lambda_{\infty} \right) \left( \frac{\lambda_{\infty}^2}{|\alpha|^8} \lambda_{\infty}^2 \right)^2,
\]
\[
\Omega(\xi, g, \nabla_{\eta d} g, \nabla_\xi \nabla_{\eta d} g) = i \left( \frac{\lambda_{\infty}^2}{|\alpha|^8} \lambda_{\infty}^2 \right)^2 \\
= 4 \text{Im} \left( \sqrt{\alpha \beta} \lambda_{\infty} \right) \left( \frac{\lambda_{\infty}^2}{|\alpha|^8} \lambda_{\infty}^2 \right)^3.
\]
hold on the singular set. Therefore by (2.1), the formulas (3.13) and (3.16) are proven. Finally, we consider the cuspidal curvatures \( \kappa^h_c \) and \( \kappa^d_c \). By (3.15),

\[
|\xi f| = 2|\alpha|^2|C_h| = 2\left| \text{Im}\left( \sqrt{\alpha\beta\lambda z} \right) \right|, \quad |\xi g| = 2|\alpha|^2|C_d| = 2\left| \text{Re}\left( \sqrt{\alpha\beta\lambda z} \right) \right|
\]

hold on the singular set. Moreover, we have

\[
\nabla_{\eta h} \nabla_{\eta h} \eta h f = \frac{i}{\sqrt{\alpha\beta}} \left( \frac{-1}{\alpha\beta} f''' + \frac{2}{|\alpha|^2} f'' - \frac{1}{\alpha\beta} f' \right)
\]

\[
- \frac{i}{\sqrt{\alpha\beta}} \left( \frac{-1}{\alpha\beta} f''' + \frac{2}{|\alpha|^2} f'' - \frac{1}{\alpha\beta} f' \right) \mod \mathcal{F},
\]

\[
\nabla_{\eta g} \nabla_{\eta g} \eta d g = \frac{1}{\sqrt{\alpha\beta}} \left( \frac{1}{\alpha\beta} g''' + \frac{2}{|\alpha|^2} g'' + \frac{1}{\alpha\beta} g' \right)
\]

\[
+ \frac{1}{\sqrt{\alpha\beta}} \left( \frac{1}{\alpha\beta} g''' + \frac{2}{|\alpha|^2} g'' + \frac{1}{\alpha\beta} g' \right) \mod \mathcal{G}
\]

on the singular set. Thus we have

\[
\langle g, \nabla_{\eta h} \nabla_{\eta h} \eta h f \rangle = \frac{2i}{|\alpha|^2} \left( \frac{\lambda'}{\sqrt{\alpha\beta}} - \frac{\lambda'}{\sqrt{\alpha\beta}} \right) = \frac{4 \text{Im}\left( \sqrt{\alpha\beta\lambda} \right)}{|\alpha|^4},
\]

\[
\langle f, \nabla_{\eta d} \nabla_{\eta d} \eta d g \rangle = \frac{2}{|\alpha|^2} \left( \frac{\lambda'}{\sqrt{\alpha\beta}} + \frac{\lambda'}{\sqrt{\alpha\beta}} \right) = \frac{4 \text{Re}\left( \sqrt{\alpha\beta\lambda} \right)}{|\alpha|^4}
\]

on the singular set by (3.27). Hence we obtain (3.14) and (3.17). \(\square\)

We note that \( \kappa^h_c < 0 \) follows from the general theory (29 Theorem 3.1) since the extrinsic Gaussian curvature of a flat front \( f \) is \( 1 > 0 \). However, \( \kappa^d_c < 0 \) does not follow by the general theory since the extrinsic Gaussian curvature of a flat front \( g \) is \( -1 < 0 \).

3.3. Relationships between curvatures and singularities of the dual surfaces. For a non-degenerate singular point \( p \) of \( f \) (resp. \( g \)), it is known that \( f \) at \( p \) (resp. \( g \) at \( p \)) is not a cuspidal edge if and only if \( C_h(p) = 0 \) (resp. \( C_d(p) = 0 \)), where \( C_h \) (resp. \( C_d \)) is as in (3.7) (resp. (3.8)) (cf. 29). By Theorem 3.3 it follows that a flat front \( f \) (resp. \( g \)) in \( H^3 \) (resp. \( S^3_1 \)) is not a cuspidal edge at a non-degenerate singular point \( p \) if and only if \( \kappa^h \) (resp. \( \kappa^d \)) vanishes at \( p \). Therefore we consider relation between the type of singularity which is not a cuspidal edge of \( f \) or \( g \) and behavior of \( \kappa^d \) or \( \kappa^h \).

Let \( U \subset C \) be a simply-connected domain and \( \mathcal{O}(U) \) the set of holomorphic functions on \( U \). Then, for \( h \in \mathcal{O}(U) \), we can construct flat fronts

\[
f = f_h : U \rightarrow H^3, \quad g = g_h : U \rightarrow S^3_1
\]

which are represented by a pair of holomorphic functions \( (\alpha, \beta) = (e^h, 1) \). Converse is also true, namely, for a flat front \( f : U \rightarrow H^3 \) without umbilical point, then we can choose a suitable complex coordinate \( z \) such that \( \alpha \, dz = e^h \, dz \) and \( \beta \, dz = dz \). (see 20 p 323).

Let \( f : U \rightarrow H^3 \) and \( g : U \rightarrow S^3_1 \) be flat fronts defined by (3.2) by the data \( (\alpha, \beta) = (e^h, 1) \). Then the sets of singular points of \( f \) and \( g \) are

\[
\Sigma(f) = \Sigma(g) = \{ z \in U \mid h(z) + h(z) = 0 \}.
\]

Moreover, \( \lambda = h + \overline{h} \) is an identifier of singularity. By Proposition 3.2 it holds that:
We have the following theorem.

**Theorem 3.9.** Let \( f : U \to H^3 \) and \( g : U \to S^3 \) be flat fronts and \( p \) be a non-degenerate singular point of both \( f \) and \( g \). Let \( \gamma(t) \) be a singular curve through \( p = \gamma(0) \). Assume that \( f \) at \( p \) is a cuspidal edge and \( \kappa_1^e(p) = 0 \) (resp. \( g \) at \( p \) is a cuspidal edge and \( \kappa_1^e(p) = 0 \)). Then \( g \) (resp. \( f \)) is a swallowtail at \( p \) if and only if \( \frac{\partial}{\partial t} \kappa_1^e(p) \neq 0 \) (resp. \( \frac{\partial}{\partial t} \kappa_1^e(p) \neq 0 \)).

**Proof.** Since this is a local situation, we may assume that both \( f \) and \( g \) are constructed by the data \((\alpha, \beta) = (e^h, 1)\), for \( h \in \mathcal{O}(U) \). Let \( \gamma(t) \) be a parametrization of \( \Sigma(f) = \Sigma(g) \). By Theorem 3.4 \( \kappa_0^e, \kappa_c^e, \kappa_t^e \) and \( \kappa_t^d \) are given by

\[
\begin{align*}
\kappa_0^e(t) &= \frac{4 \text{Im}(e^{-\frac{\sqrt{2}}{2}h'})}{|\text{Im}(e^{-\frac{\sqrt{2}}{2}h'})|^{3/2}}(\gamma(t)), \\
\kappa_c^e(t) &= \frac{-4 \Re(e^{-\frac{\sqrt{2}}{2}h'})}{|\text{Re}(e^{-\frac{\sqrt{2}}{2}h'})|^{3/2}}(\gamma(t)), \\
\kappa_t^e(t) &= \frac{\text{Re}(e^{-\frac{\sqrt{2}}{2}h'})}{\text{Im}(e^{-\frac{\sqrt{2}}{2}h'})}(\gamma(t)), \\
\kappa_t^d(t) &= \frac{\text{Im}(e^{-\frac{\sqrt{2}}{2}h'})}{\text{Re}(e^{-\frac{\sqrt{2}}{2}h'})}(\gamma(t)).
\end{align*}
\]  

(3.30)

First, we assume that \( \kappa_t^d = 0 \) and \( \kappa_t^d \neq 0 \) at \( p \). This is equivalent to the condition that \( \text{Im}(e^{-\frac{\sqrt{2}}{2}h'}) = 0 \) and \( \text{Re}(e^{-\frac{\sqrt{2}}{2}h'}) \neq 0 \) at \( p \). Thus \( (d/dt)\kappa_t^d(0) \neq 0 \) if and only if \( (d/dt)\text{Im}(e^{-\frac{\sqrt{2}}{2}h'})(0) \neq 0 \). By a direct calculation, we see that

\[
\frac{d}{dt} \left( \text{Im}(e^{-\frac{\sqrt{2}}{2}h'})(\gamma(t)) \right) = \frac{1}{2t} \left( e^{-\frac{\sqrt{2}}{2}} \left( h'' - \frac{1}{2}(h')^2 \right) \gamma + e^{-\frac{\sqrt{2}}{2}} \left( \overline{h''} - \frac{1}{2}(\overline{h'})^2 \right) \overline{\gamma} \right)(t) = \frac{1}{2} \left( -e^{-\frac{\sqrt{2}}{2}} \left( h'' - \frac{1}{2}(h')^2 \right) h' + e^{-\frac{\sqrt{2}}{2}} \left( \overline{h''} - \frac{1}{2}(\overline{h'})^2 \right) \overline{h'} \right)(t).
\]

Since we assume that \( h + \overline{h} = 0 \) and \( e^{-\frac{\sqrt{2}}{2}h'} - e^{-\frac{\sqrt{2}}{2}h'} = 0 \) at \( p \), we have \( h' = e^{-h}h' \) and \( h' = e^{-\overline{h}}h' \). Hence it holds that

\[
\frac{d}{dt} \left( \text{Im}(e^{-\frac{\sqrt{2}}{2}h'})(\gamma(t)) \right) \bigg|_{t=0} = \frac{1}{2} \left( -e^{-h} \left( h'' - \frac{1}{2}(h')^2 \right) e^{-\frac{\sqrt{2}}{2}}h' + e^{-\overline{h}} \left( \overline{h''} - \frac{1}{2}(\overline{h'})^2 \right) e^{-\frac{\sqrt{2}}{2}h'} \right)(p) = -\left( e^{-\frac{\sqrt{2}}{2}h'}(p) \text{Re} \left( \left( e^{-h} \left( h'' - \frac{1}{2}(h')^2 \right) \right)(p) \right) \right),
\]

where we used the relation \( e^{-\frac{\sqrt{2}}{2}h'} - e^{-\frac{\sqrt{2}}{2}h'} = 0 \) at \( p \) again to have the second equality. Thus we have the assertion for \( f \).

The case for \( g \) can be proven by the same computation using (3.30).
In [22, Proposition 3.8], an equivalent statement of Theorem 3.9 is obtained from a different viewpoint. In fact, \( \kappa \) coincides with \( \Delta \) in [22, p. 1910] when \( \varepsilon = 0 \) (a linear Weingarten front is a flat front if \( \varepsilon = 0 \)) in their notation. Thus Theorem 3.9 clarifies the geometric meaning of \( \Delta \) in [22] for a flat front in \( H^3 \). In [22], it is shown that the co-orientabilities and orientabilities of the original front and its dual have a connection. Furthermore, zig-zag numbers of them are studied.

4. Lines of curvature and cone-like singular points

We consider the condition that the singular curve \( \gamma \) of \( f \) and \( g \) are a line of curvature. Let \( k : U \to N \) be a frontal into a Riemannian or semi-Riemannian 3-manifold \( N \). Let \( I \) be an interval. A curve \( \gamma : I \to U \) is a line of curvature if \( I(k \circ \gamma') \) is parallel to \( II(k \circ \gamma') \), where \( I \) and \( II \) are the first and the second fundamental matrices, and they are regarded as linear maps \( T_p \nu^\perp \to T_p \nu^\perp \subset T_p N \) and \( k \circ \gamma'(t) \in T_p \nu^\perp (p = \gamma(t)) \).

**Proposition 4.1.** Let \( f : U \to H^3 \) and \( g : U \to S^3_1 \) be flat fronts constructed by (3.2). Let \( p \in \Sigma(f) = \Sigma(g) \) be a cuspidal edge of \( f \) (resp. \( g \)) and \( \gamma \) a singular curve through \( \gamma(0) = p \). Then \( \gamma \) is a line of curvature of \( f \) (resp. \( g \)) if and only if \( \kappa_h \) (resp. \( \kappa_d \)) vanishes identically on \( \gamma \).

Although this is shown for the case of fronts in Euclidean 3-space (see [34, Proposition 3.3], [15, p. 95]), and the same proof works, we give a proof here using the representation formula.

**Proof.** The curve \( \gamma \) is a line of curvature of \( f \) (resp. \( g \)) if and only if the following equation holds:

\[
\Omega(\xi f, g, \xi g)(\gamma(t)) = 0 \quad (\text{resp. } \Omega(\xi g, f, \xi f)(\gamma(t)) = 0).
\]

By a direct calculation, we see that

\[
\Omega(\xi f, g, \xi g)(\gamma(t)) = \langle \xi f \times g, \xi g \rangle(\gamma(t)) = 4 \Re(\sqrt{\alpha \beta \lambda}z(\gamma(t))) \Im(\sqrt{\alpha \beta \lambda}z(\gamma(t)));
\]

\[
(\text{resp. } \Omega(\xi g, f, \xi f)(\gamma(t)) = \langle \xi g \times f, \xi f \rangle(\gamma(t)) = -4 \Re(\sqrt{\alpha \beta \lambda}z(\gamma(t))) \Im(\sqrt{\alpha \beta \lambda}z(\gamma(t)))).
\]

Thus we have the assertion by Theorem 3.4 if \( f \) (resp. \( g \)) has cuspidal edges along \( \gamma \). \( \square \)

We define another kind of singular point so called cone-like singularity ([20, p. 305]).

**Definition 4.2.** Let \( h : U \to M \) be a frontal from a region into a 3-dimensional manifold. Let \( p \in U \) be a non-degenerate singular point. Then \( p \) is a cone-like singularity if there exists a neighborhood \( V \) of \( p \) such that for any \( q \in \Sigma(h) \), a null vector field \( \eta_q \) is tangent to \( \Sigma(h) \).

Assume that the singular curve \( \gamma \) of \( f \) (resp. \( g \)) consists of cuspidal edges. By Proposition 4.1, if \( \gamma \) is a line of curvature of \( f \) (resp. \( g \)), then \( \xi \) and \( \eta_d \) (resp. \( \xi \) and \( \eta_h \)) are parallel along \( \gamma \). Thus we have the following corollary.
Corollary 4.3. Let $f : U \to H^3$ and $g : U \to S^3_1$ be flat fronts and $p \in \Sigma(f) = \Sigma(g)$ a non-degenerate singular point. If the singular curve $\gamma$ passing through $p$ consists of cuspidal edges of $f$ (resp. $g$) and is a line of curvature of $f$ (resp. $g$), then $g$ (resp. $f$) has a cone-like singularity at $p$.

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