SPACEABILITY OF SETS OF NOWHERE $L^p$ FUNCTIONS

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Abstract. We say that a function $f : [0, 1] \to \mathbb{R}$ is nowhere $L^q$ if, for each nonvoid open subset $U$ of $[0, 1]$, the restriction $f|_U$ is not in $L^q(U)$. For a fixed $1 \leq p < \infty$, we will show that the set

$$S_p = \{ f \in L^p([0, 1]) : f \text{ is nowhere } L^q, \text{ for each } p < q \leq \infty \},$$

united with $\{0\}$, contains an isometric and complemented copy of $\ell_p$. In particular, this improves a result from [2] (which in turn is an improvement of a result from [4]), since $S_p$ turns out to be spaceable. In addition, our result is a generalization of one of the main results from [3].

1. Introduction

This note is a contribution to the study of large linear structures within essentially non-linear sets of functions which satisfy certain special properties. Given a topological vector space $X$, it is by now standard to say that a subset $S \subset X$ is lineable if $S \cup \{0\}$ contains an infinite dimensional subspace of $X$, and that $S$ is spaceable if $S \cup \{0\}$ contains a closed infinite dimensional subspace of $X$.

Our main object of study will be the $L^p[0, 1]$ spaces. For clearness, we will write $L^p$ instead of $L^p[0, 1]$. Two results motivate the present work. Muñoz-Fernández, Palmberg, Puglisi and Seoane-Sepúlveda showed the following:

Theorem 1.1 (from [4]). Let $1 \leq p < q$. Then $L^p \setminus L^q$ is $\mathfrak{c}$-lineable.

Here, $\mathfrak{c}$ indicates that we can find a vector space with a Hamel basis of cardinality $\mathfrak{c}$ contained in $(L^p \setminus L^q) \cap \{0\}$. The other result is from Botelho, Diniz, Fávaro and Pellegrino:

Theorem 1.2 (from [1]). For every $p > 0$, $\ell_p \setminus \cup_{0 < q < p} \ell_q$ is spaceable.

In view of Theorems 1.1 and 1.2 it is natural to ask whether we can obtain a result for $L^p$ spaces analogue to Theorem 1.2. Botelho, Fávaro, Pellegrino and Seoane-Sepúlveda have obtained a positive answer to that question:

Theorem 1.3 (from [2]). For each $p > 0$, $L^p \cup_{q \geq p} L^q$ is spaceable.

In this work we will present an improvement of Theorem 1.3; one could say, to the local level. We say that $f : [0, 1] \to \mathbb{R}$ is nowhere $L^q$, for some $1 \leq q \leq \infty$, if for each nonvoid open subset $U$ of $[0, 1]$, $f|_U$ is not in $L^q(U)$. For each $1 \leq p < \infty$, we define

$$S_p = \{ f \in L^p([0, 1]) : f \text{ is nowhere } L^q, \text{ for each } p < q \leq \infty \};$$

then $S_p$ satisfies the following property, which is our main result:

Theorem 1.4 (Main). For each $1 \leq p < \infty$, $(S_p \cup \{0\}) \subset L^p$ contains an isometric and complemented copy of $\ell_p$. In particular, $S_p$ is spaceable.

Section 2 will be dedicated to the proof of Theorem 1.4. Note that it only improves Theorem 1.3 for $p \geq 1$. We decided to restrict ourselves to this case for the sake of uniformity of arguments; the careful reader can verify that the same construction can be used to show the spaceability of $S_p$ for the quasi-Banach $p < 1$ case. Note that the elements from $S_p$ are nowhere essentially bounded (that is, nowhere $L^\infty$). It is worth mentioning that Theorem 1.4 generalizes one of the main theorems of [4], which states the following:

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Theorem 1.5 (from [3]). The set $G$ of $L^1$ functions which are nowhere essentially bounded is spaceable in $L^1$.

In the same paper, it was shown that $G$ is strongly $c$-algebraable, that is, $G \cup \{0\}$ contains a $c$-generated free algebra (with the pointwise multiplication). Let us mention that there are no nontrivial algebraic structures, not even 1-dimensional ones, within $C_p$, and even within $L^p \setminus L^q$, for $1 \leq p < q < \infty$. In effect, if $f \in L^p \setminus L^q$ for such $p$ and $q$, then $f^n$ is not in $L^p$ for large enough $n$.

2. Proof of the main result

We start by discriminating three objects that will be needed in order to construct a nice basic sequence of $L^p$ contained in $S_p$.

**First: an almost disjoint family of Cantor-built sets $(A_n)_n$.** First, we recall the construction of the Cantor set of (Lebesgue) measure one half, which we will denote by $C$. We start with the closed unit interval (denote it by $C_1$), and remove the center open interval of length 1/4, obtaining $C_2$, a disjoint union of two closed intervals. From each of them, we remove the center open interval of length 1/4², obtaining $C_3$, the disjoint union of four closed intervals. Repeating the process and taking the intersection of all $C_n$, we obtain $C$. For our convenience, we will call a hole of $C$ each open interval that was removed in each step of the construction of $C$. Let $t_n$ be the right endpoint of the interval which is the one more to the left, among the closed intervals that constitute $C_n$. The explicit formula for $t_n$ is $2^{2n-2} - 1$, and for our purposes what is important about the sequence $t_n$ is the following:

1. $t_n < 2^{3-2n}$ and
2. $m(C \cap [0, t_n]) = 2^n$.

This information will be needed later on. We now proceed introducing an specific notation for sets which are essentially unions of sets similar to $C$.

**Definition 2.1.**

1. For each subinterval $I$ from $[0, 1]$ and each subset $A \subset [0, 1]$, denote $A_I \triangleq (b - a)A + a$;

2. a nonvoid subset $A \subset [0, 1]$ of the form $\bigcup \{A_j : j \in \Gamma\}$ is said to be $C$-built (with $C$-components $A_j$) if there is an almost disjoint family $\{I_j : j \in \Gamma\}$ of subintervals of $[0, 1]$ such that $A_j = C I_j$, where $C$ is the Cantor set of measure 1/2.

According to the notation from [3], C-built sets are in particular Cantor-built, which means, roughly speaking, that they are essentially unions of Cantor sets. The holes of a set of the form $C_I$ are defined by the natural way: the set of all holes of $C_I$ is given by $\{J : J$ is a hole of $C\}$. With the notion of C-built sets in hands we can start partitioning $[0, 1]$. First, put $A_1 \triangleq C$. $A_2$ is defined as follows:

$$A_2 \triangleq \bigcup \{C_I : J$ is a hole of $A_1\}.$$

Note that $A_1$ and $A_2$ are almost disjoint, and that $m(A_2) = 1/4$. Assuming that we have defined $A_n$, for some $n \geq 2$, then $A_{n+1}$ is defined inductively:

$$A_{n+1} \triangleq \bigcup \{C_I : J$ is a hole of a Cantor component of $A_n\}.$$

This way, we obtain a sequence $(A_n)_n$ of almost disjoint C-built sets satisfying

1. $m(A_n) = 2^{-n}$, and
2. for any given nonempty open subset $U$ of $[0, 1]$, there exists an $n_0 \in \mathbb{N}$ such that $A_n$ has a C-component contained in $U$, for each $n \geq n_0$.

**Second: a convenient element $h$ of $L^p \setminus \cup_{p < q \leq \infty} L^q$.** Let $(r_j)_j$ be a strictly decreasing sequence of real numbers converging to $p$. For each $j$, the function $h_j : x \mapsto x^{-1/r_j}$ is in $L^p$, but not in $L^q$ for any $r_j \leq q \leq \infty$. Define $\tilde{h}$ by

$$\tilde{h} \triangleq \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{h_j}{\|h_j\|_p}.$$

**References.**
\( \tilde{h} \) is well-defined since the series converges absolutely and \( L^p \) is complete. Then \( \tilde{h} \in L^p \setminus \cup_{p < q \leq \infty} L^q \), and moreover:

**Lemma 2.2.** \( \tilde{h} = \chi_C \tilde{h} \in L^p \setminus \cup_{p < q \leq \infty} L^q \).

**Proof.** It is easily seen that \( h \in L^p \). To show that, for a fixed \( p < q < \infty \) we have \( h \notin L^q \), consider \( j_0 \) such that \( r_{j_0} < q \). Then

\[
\int_C |h|^q = \int_C |\tilde{h}|^q \geq \int_C \frac{h^q_{j_0}}{2^{k_0 q} \|h_{j_0}\|^p} \geq \frac{1}{2^{k_0 q} \|h_{j_0}\|^p} \int_C x^{-q/r_{j_0}}.
\]

We will show that the integral to the right converges to \(+\infty\). In effect, denoting \( s = q/r_{j_0} > 1 \) and considering the sequence \( t_n \) defined previously, we have that

\[
\int_C x^{-q/r_{j_0}} = \int_C x^{-s} \geq \int_{C \cap [0, t_n]} x^{-s} \geq m(C \cap [0, t_n]) \inf \{x^{-s} : x \in C \cap [0, t_n]\}
\]

\[
\geq m(C \cap [0, t_n]) t_n^{-s} \geq m(C \cap [0, t_n]) 2^{-n} 2^{s(2n-3)} > 2^{(2s-1)n}.
\]

Since the right hand side tends to infinity along with \( n \), it follows that \( h \notin L^q \). The remaining case, \( q = \infty \), is automatically covered, since \( L^\infty \subset L^q \), for each \( q < \infty \). ♠

**Third:** a disjoint infinite family of strictly increasing sequences of positive integers, \( \{n_j^k \} : k \in \mathbb{N} \}. It is a simple task for the reader to convince herself/himself that such family exists.

Before proceeding with the proof of the main result, let us establish the following notation: if \( I = [a, b] \) is a subinterval of \([0, 1]\) and \( f : [0, 1] \to \mathbb{R} \), we define \( f_I : [0, 1] \to \mathbb{R} \) by

\[
f_I(x) = \begin{cases} f \left( \frac{x-a}{b-a} \right), & \text{if } x \in I; \\ 0, & \text{if } x \notin I. \end{cases}
\]

Note that this notation is coherent with Definition 2.1 (1), since for any subset \( B \subset [0, 1] \), \( (\chi_B)_I = \chi_{B_I} \). It is easily seen that, for any subinterval \( I \) from \([0, 1]\) and any \( 1 \leq p \leq \infty \), \( f \) is in \( L^p \) if and only if \( f_I \) is in \( L^p \). In particular, for each subinterval \( I \) from \([0, 1]\), \( h_I \) is in \( L^p \setminus \cup_{p < q \leq \infty} L^q \).

**Proof (of Theorem 2.4).** Consider \( (A_j)_j \), \( h \) and \( \{n_j^k \} : k \in \mathbb{N} \} \) as defined above. For a fixed \( j \geq 2 \), let \( (B_j)_j \) be a sequence of all \( C \)-components of \( A_j \), and consider their respective convex hulls \( I_j \). Note that the \( I_j \) are disjoint. Define

\[
f_j = \sum_{l=1}^{\infty} \frac{1}{2^l} \frac{h_{I_j}}{\|h_{I_j}\|^p}.
\]

For \( j = 1 \) just put \( f_1 = \frac{h}{\|h\|^p} \). Then each \( f_j \) is in \( L^p \) with \( \|f_j\|^p = 1 \), and \( f_j \) is not in \( L^q(U) \), for any \( p < q \leq \infty \) and any open set \( U \) containing a \( C \)-component of \( A_j \). Note also, by the construction of \( h \), that \( f_j \) is zero outside of \( A_j \). For each \( k \), consider

\[
g_k = \sum_{j=1}^{\infty} \frac{1}{2^j} f_{n_j^k}.
\]

Then \( g_k \in S_p \) and \( \|g_k\|_p = 1 \), and since the functions \( g_k \) have almost disjoint supports, it follows that \( (g_k)_k \) is a complemented basic sequence in \( L^p \), isometrically equivalent to the canonical basis of \( \ell_p \). In particular, \( \text{span}(\{g_k : k \in \mathbb{N}\}) \) is complemented in \( L^p \) and isometrically isomorphic to \( \ell_p \).

To complete our proof, it remains to show that \( \text{span}(\{g_k : k \in \mathbb{N}\}) \subset S_p \cup \{0\} \). Any given nonzero element of \( \text{span}(\{g_k : k \in \mathbb{N}\}) \) is of the form

\[
g = \sum_{k=1}^{\infty} a_k g_k,
\]

where some \( a_{k_0} \) is assumed to be nonzero. Since the functions \( g_k \) have almost disjoint supports and \( g_{k_0} \) is nowhere \( L^q \) for any given \( p < q \leq \infty \), it follows that \( g \in S_p \). ♠
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