FATOU-BIEBERBACH DOMAINS

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ABSTRACT. We show that for any $m \in \mathbb{N} \cup \{\infty\}$ there exist $m$ disjoint FB domains whose union is dense in $\mathbb{C}^k$. In fact we show that any point not in the union is a boundary point for all the domains. We construct FB domains that contains arbitrary countable collections of subvarieties of $\mathbb{C}^k$, and we construct FB domains that intersect elements of countable collections of affine subspaces of $\mathbb{C}^k$ in connected proper subsets. Moreover, we show that any Runge FB domain is the attracting basin for a sequence of automorphisms of $\mathbb{C}^k$, although not necessarily if you only allow iteration of one automorphism. We also show that an increasing sequence of Runge $\mathbb{C}^k$’s is a $\mathbb{C}^k$.

1. INTRODUCTION

This paper is inspired by the paper [10], and is organized as follows: We start by giving some definitions in Section 2. In Section 3 we will prove a generalization of Theorem 9.1 in [10] that, together with results on approximations of biholomorphisms by automorphisms of $\mathbb{C}^k$, due to Andersen, Lempert, Forstneric, and Rosay, will be a very effective tool for constructing various Fatou-Bieberbach Domains. In Section 4 we develop some results regarding polynomial convexity needed in Section 6. In Section 5 we prove the following theorem:

**Theorem 1.** For any $m \in \mathbb{N} \cup \{\infty\}$ there exists a union of Fatou-Bieberbach Domains $\Omega = \bigcup_{j=1}^m \Omega_j$ that satisfies the following:

(i) $\Omega_k \cap \Omega_l = \emptyset$ for all $k \neq l$,

(ii) For any $q \in \mathbb{C}^k \setminus \Omega$ we have that $q \in \partial \Omega_i$ for all $i$.

This answers a question posed by Rosay and Rudin in [10]. Rosay and Rudin also posed a couple of questions regarding intersections between Fatou-Bieberbach domains and complex lines in $\mathbb{C}^2$. One question is the following: Can the intersection between a Fatou-Bieberbach domain and a complex line in $\mathbb{C}^2$ be connected? Globevnik has given a positive answer to this question by constructing a Fatou-Bieberbach domain intersecting the $z$-plane in approximately a disc [8]. Another question is the following: How many complex lines can a Fatou-Bieberbach domain in $\mathbb{C}^2$ contain? Buzzard and Fornæss have shown that a Fatou-Bieberbach domain in $\mathbb{C}^2$ can contain any finite number of complex lines [2]. In Section 6 we prove the following two theorems:

**Theorem 2.** Let $\{L_j\}_{j \in \mathbb{N}}$ be a collection of affine subspaces of $\mathbb{C}^k$. Then there exists a Fatou-Bieberbach Domain $\Omega$ such that $\Omega \cap L_i$ is connected, and such that $L_i \setminus \Omega \neq \emptyset$ for all $i \in \mathbb{N}$.
Theorem 3. Let \( \{V_j\}_{j \in \mathbb{N}} \) be a collection of closed subvarieties of \( \mathbb{C}^k \). Then there exists a Fatou-Bieberbach Domain \( \Omega \) such that \( \bigcup_{j=1}^{\infty} V_j \subset \Omega \).

Notice that \( \{V_j\} \) could be dense in \( \mathbb{C}^k \). Theorem 3 generalizes a result in [10] stating that a Fatou-Bieberbach domain can contain any countable set of points.

In Section 7, we give an example of a Fatou Bieberbach domain that is not the basin of attraction of an automorphism of \( \mathbb{C}^k \), but on the other hand, we prove that any Runge Fatou-Bieberbach domain is the basin of attraction for a sequence of automorphism of \( \mathbb{C}^k \). Lastly, we show that an increasing union of Runge domains that are biholomorphic to \( \mathbb{C}^k \) is again a \( \mathbb{C}^k \). This last result gives a partial answer to a question posed in [7, p.4].

Apart from being interesting in their own right, constructions of Fatou-Bieberbach domains with special properties can have useful applications in other areas. See [11] for an application to proper holomorphic embeddings.

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2. Definitions and Notation

Throughout the article, we will use the following notation: If \( p \in \mathbb{C}^k \) and if \( \varepsilon > 0 \), we let \( B_\varepsilon(p) \) denote an open ball centered at \( p \) with radius \( \varepsilon \). If \( k = 1 \) this set is denoted \( \triangle_\varepsilon(p) \). If \( p \) is the origin, these sets will be denoted \( B_\varepsilon \) and \( \triangle_\varepsilon \), and if in addition \( \varepsilon = 1 \), these sets are denoted \( B \) and \( \triangle \) respectively. If nothing else is stated, \( k \) is always assumed to be larger than or equal to two.

Definition 1. Let \( \text{Aut}_p(\mathbb{C}^k) \) denote the group of holomorphic automorphisms of \( \mathbb{C}^k \) fixing the point \( p \in \mathbb{C}^k \). If all the eigenvalues \( \lambda_i \) of \( dF(p) \) satisfy \( |\lambda_i| < 1 \) we say \( F \) is attracting at \( p \).

3. Fatou-Bieberbach Domains As Sequence Attracting Basins

It was proven in [10] that \( \Omega_{\{F\}}^p \) is a Fatou-Bieberbach domain for all attracting \( F \in \text{Aut}_p(\mathbb{C}^k) \). A weaker result was also proven, one in which was put an additional condition on the eigenvalues of \( dF(p) \). To help us construct our domains, we will generalize this weaker result, and Theorem 4 should be compared with Theorem 9.1 in [10].

Lemma 1. Let \( \Gamma(s, r, \rho) \) be the family of biholomorphic maps \( F: B_\rho \to \mathbb{C}^k \) that fixes the origin and satisfies \( s\|z\| \leq \|F(z)\| \leq r\|z\| \) for all \( z \in B_\rho \), where \( s, r, \rho \in \mathbb{R}^+ \). Let \( A_F = dF(0) \). There exists a \( C > 0 \) such that we for all \( F \in \Gamma \) have that

\[
\|A_F^{-1}F(z) - z\| \leq C\|z\|^2
\]

for all \( z \in B_\rho \).
Proof. For any $F \in \Gamma(s,r,\rho)$, the map $G = A_F^{-1}F - I$ has no terms of order less than two. This means that there exists a $C$ for this particular map. But as we can choose such a $C$ depending only on the supremum of $G$ over $B_p$, it follows from the boundedness of $\Gamma(s,r,\rho)$ on $B_p$ that one single $C$ must work for all maps. \qed

**Theorem 4.** Let $0 < s < r < 1$ such that $r^2 < s$, let $\delta > 0$, and let $\{F_j\} \subset \text{Aut}_p(\mathbb{C}^k)$ such that $s\|z - p\| \leq \|F_j(z) - p\| \leq r\|z - p\|$ for all $z \in B_\delta(p)$ and all $j \in \mathbb{N}$. Then there exists a biholomorphic map

$$
\Phi: \Omega^p_{\{F_j\}} \to \Phi(\Omega^p_{\{F_j\}}) = \mathbb{C}^k.
$$

Proof. We may assume that $p = 0$ and that $\delta < 1$, and we write $\Omega = \Omega^0_{\{F_j\}}$. Let $A_j = dF_j(0)$. We will prove that the sequence of automorphisms $\Phi_j$ defined by

$$
\Phi_j = A(j)^{-1}F(j)
$$

converges to the desired map uniformly on compacts in $\Omega$. For all $z \in B_\delta$ and all $j \in \mathbb{N}$ we have that $\|F(j)(z)\| \leq r^j\|z\| < r^j$, so the sequence $\{F(j)\}$ is uniformly attracting on $B_\delta$. It follows that

$$
\Omega = \cup_{j=0}^\infty F(j)^{-1}(B_\delta),
$$

from which it follows that $\Omega$ is an open and connected subset of $\mathbb{C}^k$.

To prove convergence of the sequence $\Phi_j$ on compacts in $\Omega$ it is enough to prove convergence of $\Phi_j$ on $B_\delta$. For if $K \subset \Omega$ is a compact set it is clear that for a $k \in \mathbb{N}$ we have that $F(k)(K) \subset B_\delta$, and for any $z \in K$, the limit

$$
\Phi(z) = \lim_{j \to \infty} \Phi_j(z)
$$

can be written as

$$
\Phi(z) = \lim_{j \to \infty} A(k)^{-1}\Phi^k_j F(k)(z) = A(k)^{-1}\Phi^k F(k)(z),
$$

where $\Phi^k_j$ is the composition

$$
\Phi^k_j = A(k + 1,j)^{-1}F(k + 1,j),
$$

and $\Phi^k$ is the limit

$$
\Phi^k = \lim_{j \to \infty} \Phi^k_j.
$$

Remember that $\{F_j\}$ is an arbitrary sequence of maps in $\Gamma(s,r,\rho)$. Lemma 1 gives us the following estimate:

$$
\|\Phi_{j+1}(z) - \Phi_j(z)\| = \|A(j)^{-1}(A_{j+1}^{-1}F_{j+1}(F(j)(z)) - F(j)(z))\|
\leq s^{-j}C\|F(j)(z)\|^2 \leq C\delta^2(r^2/s)^j
$$

for all $z \in B_\delta, j \in \mathbb{N}$. Since $\sum_{j=0}^{\infty}(r^2/s)^j$ exists, this shows that the sequence is uniformly convergent, i.e. $\Phi$ is a holomorphic map from $\Omega$ into $\mathbb{C}^k$. Since we have that $d\Phi_j(0) = I$ for all $j \in \mathbb{N}$, we have that $d\Phi(0) = I$, and it follows from a standard result that the limit map has to be one to one onto its image. It remains to show that the image is the whole of $\mathbb{C}^k$.

Notice that there exists an $R \in \mathbb{R}^+$ such that

$$
(\text{a}) \ \Phi^k(B_\delta) \subset B_R,
$$

for all $k \in \mathbb{N}$. 
We claim that there exists an \( \epsilon > 0 \) such that for any \( k \in \mathbb{N} \) we have that
\[
(b) \ B_\epsilon \subset \Phi^k(B_\delta).
\]
For if not there is a sequence of positive numbers \( \epsilon_i \downarrow 0 \) with a corresponding sequence \( \{k_i\} \) such that
\[
(c) \ B_{\epsilon_i} \setminus \Phi^{k_i}(B_\delta) \neq \emptyset
\]
for all \( i \in \mathbb{N} \). By \((a)\) we have that \( \{\Phi^{k_i}\} \) is a normal family on \( B_\delta \) so we may assume that
\[
\lim_{i \to \infty} \Phi^{k_i} = \tilde{\Phi},
\]
where \( \tilde{\Phi} \) is a biholomorphic map on \( B_\delta \) fixing the origin. This leads to a contradiction as \( \tilde{\Phi}(B_\delta) \) clearly would have to contain a ball of some positive radius, contradicting \((c)\).

For an arbitrary \( M \in \mathbb{R}^+ \) there exists a \( k \in \mathbb{N} \) such that \( B_M \subset A(k)^{-1}(B_\epsilon) \). It follows from \((b)\) that
\[
B_M \subset A(k)^{-1}(B_\epsilon) \subset A(k)^{-1}(\Phi^k(B_\delta)) \subset A(k)^{-1}\Phi^kF(k)(\Omega) = \Phi(\Omega),
\]
which means that \( C^k = \Phi(\Omega) \); thus the proof is finished. \( \square \)

In Section 7 we will prove that even though quite restrictive, the above theorem is (theoretically) sufficient for constructing all Runge Fatou-Bieberbach domains. A non-Runge Fatou-Bieberbach domain would obviously not be a basin of attraction.

It is an open question whether the theorem would still hold if we drop the condition \( r^2 < s \). It is however clear that one in general needs some upper and lower bound for the family of automorphisms. Without an upper bound one could choose a sequence of linear maps approaching the identity so fast that the basin would simply be the origin. A more interesting example can be found in [6]. In this paper, Fornæss constructs sequences of automorphisms where the lower bound decreases fast to zero. The resulting basins are increasing unions of holomorphic balls, and the infinitesimal Kobayashi metric vanishes identically on the basins. Nevertheless they fail to be biholomorphic to \( \mathbb{C}^k \) due to the fact that they carry nonconstant bounded plurisubharmonic functions.

4. POLYNOMIAL CONVEXITY

For our constructions in connection with subvarieties of \( \mathbb{C}^k \), we will need some results concerning polynomial convexity. These results are however not needed in Section 5.

The holomorphically convex hull of a compact set \( K \subset U \), with respect to the set \( U \), is defined as
\[
\hat{K}_{\mathcal{O}(U)} = \{ z \in U; \|f(z)\| \leq \|f\|_K, \forall f \in \mathcal{O}(U) \}
\]
If \( U = \mathbb{C}^k \) we suppress the subscript, and write \( \hat{K} \) instead of \( \hat{K}_{\mathcal{O}(\mathbb{C}^k)} \). If for a compact set \( K \) we have that \( K = \hat{K} \), we say that \( K \) is polynomially convex.

**Lemma 2.** Let \( K \subset \mathbb{C}^k \) be polynomially convex, let \( V \subset \mathbb{C}^k \) be a closed subvariety, and let \( K' \subset V \) be compact such that \( K \cap V \subset K' \). Then we have that
\[
\hat{K} \cup \hat{K'} = K \cup \hat{K'}_{\mathcal{O}(V)} = K \cup \hat{K'}.
\]
**Proof.** Write $C = K \cup K'$. Let $p \in \mathbb{C}^k \setminus (C \cup V)$. There is an $f \in \mathcal{O}(\mathbb{C}^k)$ such that $\|f\|_K < 1$ and such that $\|f(p)\| > 1$, and there is an $h \in \mathcal{O}(\mathbb{C}^k)$ such that $h \mid V \equiv 0$ and such that $h(p) \neq 0$. So if we define $g_k(z) = h(z) \cdot f(z)^k$, we have that $\|g_k(p)\| > \|g\|_C$ for a large enough $k$. It follows from the Local Maximum Modulus Principle [9] that $\hat{C} = K \cup \hat{K}$. Now if $q \in V \setminus \hat{K}$, there exists a $\varphi \in \mathcal{O}(V)$ such that $\|\varphi(q)\| > \|\varphi\|_{\hat{K} \setminus \mathcal{O}(V)}$. Since we for arbitrarily large $R \in \mathbb{R}$ have that $V' = V \cap \overline{\mathcal{F}}_R$ is polynomially convex, $\varphi$ can be approximated uniformly on $V'$ by an entire function, and the result follows. \hfill \Box

**Lemma 3.** Let $K \subset \mathbb{C}$ be polynomially convex, let $p_1, p_2 \in K$, and let $Q = \{q_1, \ldots, q_m\} \subset \mathbb{C} \setminus K$. Then there exists a polynomially convex set $K'$ such that $K \subset K'$, such that $p_1$ and $p_2$ are in the same path-connected component of $K'$, and such that $Q \subset \mathbb{C} \setminus K'$.

**Proof.** Choose $R \in \mathbb{R}$ such that $K \subset \Delta_R$, and let $\gamma_i : [0, 1] \mapsto \mathbb{C} \setminus K$ be a path connecting $q_i$ and a point $p \in \mathbb{C} \setminus \Delta_R$ for $i = 1, \ldots, m$. Let $\Gamma : [0, 1] \mapsto \Delta_R$ be a smooth curve such that $\Gamma(0) = p_1$ and such that $\Gamma(1) = p_2$, and such that $\Gamma \cap Q = \emptyset$. We assume that $p_1$ and $p_2$ are in different path-components of $K$, or else the lemma is trivial. Let $\{(t_{1k}, t_{2k}), \ldots, t_{1k}, t_{2k}\}$ be an increasing sequence of numbers in the closed unit interval such that if we let $\Gamma = \gamma \setminus \{(t_{1k}, t_{2k})\}_k$, we have that the following is satisfied:

(i) $\gamma(t_{ik}) \in K$ for all $k$ and all $i$.
(ii) $\Gamma \cap \gamma_i = \emptyset$ for all $i = 1, \ldots, m$.
(iii) $\gamma((t_{ik}, t_{2k})) \cap K = \emptyset$ for all $k = 1, \ldots, i$.

This means that $Q$ is not in the polynomial hull of $K_0 = K' \cup \Gamma$. If the polynomial hull of $K_0 \cup \gamma((t_{1k}, t_{2k}))$ does not intersect $Q$ we define $K_1 = K_0 \cup \gamma((t_{1k}, t_{2k}))$. If not we do the following: The only possibility for $K_1$ to intersect $Q$ is for $\gamma(t_{1k})$ and $\gamma(t_{2k})$ to be in same connected component of $K_0$. Denote this component $C$. But this means that if we let $\mu$ be a path close enough to $C$ that connects $\gamma(t_{1k})$ and $\gamma(t_{2k})$, and define $K_1 = K_0 \cup \mu$, then the polynomial hull of $K_1$ will not intersect $Q$. Do the same thing for the rest of the intervals, and the set $K' = \hat{K}_1$ will satisfy the claims of the lemma. \hfill \Box

5. **Disjoint Fatou-Bieberbach Domains Whose Union is Dense in $\mathbb{C}^k$**

In this and in the next section we will let $A : \mathbb{C}^k \to \mathbb{C}^k$ denote the linear map defined by

$$A : (z_1, \ldots, z_k) \mapsto \left(\frac{z_1}{2}, \ldots, \frac{z_k}{2}\right).$$

Fix a $\rho > 0$. By Schwarz Lemma there exists a positive number $\delta(\rho)$, and two numbers $r, s \in \mathbb{R}^+$ with $r^2 < s$, such that for a biholomorphic map $F : \mathcal{F}_\rho \to \mathbb{C}^k$ fixing the origin we have that

$$\|F - A\|_{\mathcal{F}_\rho} < \delta(\rho) \Rightarrow F \in \Gamma(s, r, \rho'),$$

for some positive $\rho'$ smaller than $\rho$ (the family $\Gamma(s, r, \rho')$ is defined in Lemma 1). By Theorem 4 then, if $\{F_j\} \subset Aut_p(\mathbb{C}^k)$ is a sequence of automorphisms satisfying

$$\big(\ast\big) \|F_j(z) - A(z - p) - p\| < \delta(\rho), z \in B_\rho(p)$$
for all \( j \in \mathbb{N} \), then \( \Omega_1^{(F_j)} \) is biholomorphic to \( \mathbb{C}^k \). The notation \( \delta(\rho) \) will be used in the following proof.

\textit{Proof of Theorem 1:} We will prove the result in the case of \( m = \infty \), and we will indicate at the end of the proof what to do in the finite case. Let \( \epsilon_j > 0 \). To construct the domains, we will inductively construct a sequence of attracting automorphisms. At each step in the construction we will generate one more basin of attraction, and we will make sure that enough points gets pulled into all of the basins to ensure the claims of the theorem.

Let \( p_1 = q_1 \) be the origin and let \( \rho_1 = \frac{1}{2} \). We start our construction by letting \( F_1 \) be the linear map \( A \).

Having constructed \( j \) automorphisms, let the following be the situation \( S_j \): We have constructed automorphisms \( \{F_1, \ldots, F_j\} \), we have chosen two sets of distinct points \( \{p_1, \ldots, p_j\} \) and \( \{q_1, \ldots, q_j\} \), and a set of positive numbers \( \{\rho_1, \ldots, \rho_j\} \). For each \( \rho_i \) there is a corresponding \( \delta(\rho_i) \). The following are satisfied:

\begin{enumerate}[(a)]
    \item \( \overline{B}_{\rho_i}(q_i) \cap \overline{B}_{\rho_k}(q_k) = \emptyset \) for all \( i \neq k \),
    \item \( F(j)(p_1) = q_1 \) for \( i = 1, \ldots, j \),
    \item \( F_i(q_k) = q_k \) for \( k \leq i \) for all \( i = 1, \ldots, j \),
    \item \( \|F_i(z) - A(z - q_i) - q_i\| < \delta(\rho_k) \) for all \( z \in \overline{B}_{\rho_k}(q_k) \) and \( k \leq i \).
\end{enumerate}

We also assume that \( \bigcup_{i=1}^{j} \overline{B}_{\rho_i}(q_i) \) is polynomially convex. When these points, numbers and automorphisms are chosen at a certain step, they will stay with us throughout the construction. Notice that because of (c), (d) and Theorem 4; for any sequence of automorphisms constructed so as to satisfy these conditions at each step, we have that the basin of attraction of each point \( q_i \) is a Fatou-Bieberbach domain.

We will now demonstrate how to construct the automorphism \( F_{j+1} \). In addition to ensuring that the four stated claims are satisfied at the next step, we must make sure that we get enough points into the basins to satisfy the other claims of the theorem.

Let \( K_j = F(j)^{-1}(\cup_{i=1}^{j} \overline{B}_{\rho_i}(q_i)) \). Choose sets of points \( T_i = \{t_i^1, \ldots, t_i^{m_i} \} \subset B_{j+1} \setminus K_j \) for \( i = 1, \ldots, j + 1 \) such that the sets \( T_i \) are pairwise disjoint. Make sure that for any \( q \in B_{j+1} \setminus K_j \) and each \( i = 1, \ldots, j + 1 \) there is a \( t_k^i \) such that

\begin{enumerate}[(c)]
    \item \( \|t_k^i - q\| < \epsilon_j \).
\end{enumerate}

Let \( \tilde{t}_k^i = F(j)(t_k^i) \). Next choose a point \( p_{j+1} \in \mathbb{C}^k \setminus (\bigcup_{i=1}^{j+1} T_i \cup K_j) \), and write \( q_{j+1} = F(j)(p_{j+1}) \). We may now choose \( \rho_{j+1} > 0 \) such that the set \( \overline{B}_{\rho_{j+1}}(q_{j+1}) \) does not contain any of the \( \tilde{t}_k^i \)'s and does not intersect \( F(j)(K_j) \). Make sure that \( \rho_{j+1} \) is small enough so that \( F(j)(K_j) \cup \overline{B}_{\rho_{j+1}}(q_{j+1}) \) is polynomially convex.

For a \( \mu > 0 \) let \( B_{\mu}(q_i) \) denote a \( \mu \)-neighborhood of \( \overline{B}_{\rho_i}(q_i) \) for \( i = 1, \ldots, j + 1 \). For the construction of the automorphism we will invoke Theorem 2.3 in [5]. By this result, if \( \mu \) is small enough, for any \( \epsilon > 0 \) there exists an automorphism \( \varphi \in \text{Aut}(\mathbb{C}^k) \) such that the following is satisfied:

\begin{enumerate}[(f)]
    \item \( \|\varphi(z) - A(z - q_i) - q_i\| < \epsilon \) for \( z \in B_{\mu}(q_i) \) for \( i = 1, \ldots, j + 1 \).
\end{enumerate}

We may also assume that \( \varphi(q_i) = q_i \). By the same theorem there exists now an automorphism \( \varphi \in \text{Aut}(\mathbb{C}^k) \) such that

\begin{enumerate}[(g)]
    \item \( \|\varphi(z) - z\| < \epsilon \) for \( z \in \overline{B}_{\rho_i}(q_i) \) for \( i = 1, \ldots, j + 1 \),
    \item \( \varphi(\tilde{t}_k^i) \in \varphi^{-1}(B_{\rho_i}(q_i)) \) for all \( \tilde{t}_k^i \in T_i \) for \( i = 1, \ldots, j + 1 \).
\end{enumerate}
Again we may assume that \( \phi(q_i) = q_i \). If we choose \( \epsilon \) small enough, we see that (a),(b),(c) and (d) is satisfied at the new step \( S_{j+1} \) if we define

\[
F_{j+1} = \varphi \circ \phi.
\]

. Notice that if \( \epsilon \) is small enough, we also have

(1) \( F(j+1)(t_k^i) \subset B_{\rho_i}(q_i) \) for all \( t_k^i \in T_i \) for \( i = 1, ..., j + 1 \).

We have now inductively constructed an infinite sequence of automorphisms \( \{F_j\} \). As commented on earlier, all the basins \( \Omega_i = \Omega_{F_j}^{\epsilon_i} \) are biholomorphic to \( \mathbb{C}^k \).

Since the basins are clearly disjoint, we have ensured (i). Let \( \Omega \) denote the union of all the basins, let \( q \in B_j \cap (\mathbb{C}^k \setminus \Omega) \), and let \( \Omega_i \) be an arbitrary basin. By (e) and (i) there is a sequence of points \( \{t_j^1\}_{j=1}^{\infty} \subset \Omega_i \) such that \( \|t_j^1 - q\| < \epsilon_j \) for all \( j \in \mathbb{N} \).

This shows that \( q \in \partial \Omega_i \), and we have (ii).

In the case \( m \in \mathbb{N} \) we prove the theorem in the exact same manner except that we stop generating new basins at the appropriate step in the construction. \( \square \)

6. Intersections With Subvarieties

In this section we prove Theorem 2 and Theorem 3. For both proofs let \( A: \mathbb{C}^k \to \mathbb{C}^k \) be the linear map defined in the previous section.

**Proof of Theorem 2:** We will construct a sequence of automorphisms \( \{F_j\} \subset Aut_0(\mathbb{C}^k) \) such that \( \Omega^0_{\{F_j\}} \) is a Fatou-Bieberbach Domain containing connected subsets of the affine spaces. Let \( \{p_i^j\}_{i \in \mathbb{N}} \) be a dense set of points in \( L_j \) for all \( j \in \mathbb{N} \), and make the following induction hypothesis \( I_j \): We have automorphisms \( \{F_1, ..., F_j\} \subset Aut_0(\mathbb{C}^k) \), and a set of points \( \{q_1, ..., q_j\} \) such that \( q_i \in L_i \). For each \( k \leq j \) there are paths \( \gamma^k_{lm} \subset L_k \) connecting \( p_i^k \) and \( p_m^k \) for \( l, m \leq j - k + 1 \). The following are satisfied:

1. Each \( F_i \) is a composition of maps satisfying (\( * \)) (section 5),
2. \( F(j)(p_i^k) \subset B \) for \( i \leq j - k + 1, k = 1, ..., j \),
3. \( F(j)(\gamma^k_{lm}) \subset B \) for \( l, m \leq j - k + 1, k = 1, ..., j \),
4. \( F(j)(q_i) \subset \mathbb{C}^k \setminus \mathcal{B} \) for \( i = 1, ..., j \).

We may assume that \( I_1 \) is true with \( F_1 = A \). Assume now that \( I_j \) is true. We will construct \( F_{j+1} \) so as to ensure that we have \( I_{j+1} \).

We want to make sure that there is a path \( \gamma^1_{(j+1)i} \subset F(j)(L_1) \) connecting the images of \( p_i^{j+1} \) and \( p_i^1 \), while at the same time we have an automorphism tucking the path into the unit ball while keeping the \( F(j)(q_i) \)'s at a distance. Notice that this will ensure that this is also the case for paths \( \gamma^1_{(j+1)i} \) and points \( p_i^1 \) for \( i = 2, ..., j \).

Now \( K = F(j)^{-1}(\mathcal{B}) \) is a polynomially convex set. Let \( l_1 \) be the complex line containing \( p_i^{j+1} \) and \( p_i^1 \). If none of the \( q_i \)'s lie in \( l_1 \), it follows from Lemma 2 that we for any path \( \gamma \subset l_1 \) connecting \( p_j^{j+1} \) and \( p^1_1 \) have that

\[
\{q_1, ..., q_j\} \cap \overline{K \cup \gamma} = \emptyset.
\]

If \( Q = \{q_1, ..., q_i\} \subset l_1 \), Lemma 3 tells us that there is a polynomially convex compact set \( K'_i \subset l_1 \) containing \( K \cap l_1 \), such that \( Q \cap K'_i = \emptyset \), and such that \( K'_i \)
contains a path $\gamma$ connecting $p_{j+1}^1$ and $p_1^1$. Lemma 2 tells us that

$$K \cup K_1' = K \cup K_1' \Rightarrow \{q_1, ..., q_j\} \cap K \cup K_1' = \emptyset.$$ 

Let $K' = F(j)(K \cup K_1')$. Now choose $s \in \mathbb{N}$ such that $A^s(K') \subset B$. For any $\mu > 0$, by Theorem 2.1 in [5] there exists an automorphism $\sigma \in Aut_0(\mathbb{C}^k)$ such that $\|\sigma - id\|_{K'} < \mu$, and such that $\sigma(F(j)(q_i)) \in \mathbb{C}^k \setminus A^{-s}(B_2)$ for $i = 1, ..., j$. Define $\psi_1 = A^s \circ \sigma$. Make sure that $\mu$ is small enough for $\psi_1$ to be a composition of maps satisfying the condition $(*)$ in the beginning of the previous section.

Now repeat this procedure for the rest of the indices $i = 2, ..., j + 1$. That is: Construct automorphisms that tuck the $p_{j+2-i}$'s along with the paths into the unit ball, while keeping the $q_i$'s away. Call the automorphisms $\psi_i$. We will have $I_{j+1}$ if we define $F_{j+1} = \psi_{j+1} \circ ... \circ \psi_1$, and choose a $q_{j+1} \in L_{j+1}$ such that $F(j + 1)(q_{j+1}) \notin \overline{B}$.

We have inductively defined a sequence of automorphisms $\{F_j\}$ and we claim that $\Omega = \Omega^0_{\{F_j\}}$ is the Fatou-Bieberbach Domain that we are after. It follows from Theorem 4 and the choices of automorphisms that $\Omega$ is biholomorphic to $\mathbb{C}^k$. Let $U_1$ and $U_2$ be connected components of $\Omega \cap L_j$ for a $j \in \mathbb{N}$. Since the set $\{p_i^j\}_{i \in \mathbb{N}}$ is dense in $L_j$, there is a $p_i^j \in U_1$ and a $p_{j+n}^j \in U_2$ and we have a path $\gamma_{lm}^j \subset L_j$ connecting the two points while satisfying $\gamma_{lm}^j \subset \Omega$. Thus $U_1 = U_2$, and we must have that $\Omega \cap L_j$ is connected. Lastly we have that $q_j \subset L_j \setminus \Omega$ for all $j \in \mathbb{N}$. □

It should now be clear how to the prove Theorem 3. For each subvariety we choose a compact exhaustion, and we construct the sequence of automorphisms such that we for each new step tuck more and larger compact sets into the ball. Of course we have to make sure that the basin is not the whole of $\mathbb{C}^k$.

**Proof of Theorem 3:** We will construct a sequence of automorphisms $\{F_j\} \subset Aut_0(\mathbb{C}^k)$ such that $\Omega^0_{\{F_j\}}$ is a Fatou-Bieberbach Domain containing $V = \cup_{i=1}^\infty V_i$.

Let $K_1' = V_j \cap \overline{B}_1$, and let $p \in \mathbb{C}^k \setminus V \cup \overline{B}_2$.

We make the following induction hypothesis $I_j$: We have a collection of automorphisms $\{F_1, ..., F_j\}$ such that $F(j)(K_{j-k+1}) \subset B$ for $k \leq j$, and such that $F(j)(p) \subset \mathbb{C}^k \setminus \overline{B}$. By letting $V_1$ be a coordinate axis, $I_1$ is satisfied with $F_1 = A$.

Assume that we have $I_j$. $F(j)^{-1}(\overline{B})$ is a polynomially convex compact set, and by repeated use of Lemma 2, the polynomial hull of $K = F(j)^{-1}(\overline{B}) \cup K_1^{j+1} \cup K_2 \cup ... \cup K_1^{j+1}$ does not contain the point $p$. Let $s \in \mathbb{N}$ such that $A^s(F(j)(K)) \subset B$. For any $\mu > 0$, by Theorem 2.1 in [5] there is an automorphism $\phi \in Aut_0(\mathbb{C}^k)$ such that $\|\phi - id\|_{F(j)(K)} < \mu$, and such that $\phi(F(j)(p)) \in \mathbb{C}^k \setminus A^{-s}(B_2)$. Now let $F_{j+1} = A^s \circ \phi$. This gives us $I_{j+1}$. Make sure that $\mu$ is chosen such that $F_{j+1}$ is a composition of maps satisfying the condition $(*)$ in the beginning of the previous section.

We have inductively constructed a sequence of automorphisms $\{F_j\}$. It follows from Theorem 4 that $\Omega = \Omega^0_{\{F_j\}}$ is biholomorphic to $\mathbb{C}^k$. It is clear from the construction that all the $V_i$'s will be in the basin, and also that the point $p$ will not be. Thus $\Omega$ is the desired Fatou-Bieberbach Domain. □
7. Attracting Basins

A natural question is the following: Are all Fatou-Bieberbach domains attracting basins for sequences of automorphisms? We can not answer this in general, but in the case of Fatou-Bieberbach domains that are also Runge we have the following results:

**Proposition 1.** There exists a Fatou-Bieberbach domain $\Omega$ such that there is no $F \in \text{Aut}(\mathbb{C}^k)$ with $\Omega = \Omega(F^p)$ for a $p \in \mathbb{C}^k$.

**Proof.** For any polynomially convex compact set $K \subset \mathbb{C}^k$, there exists a Fatou-Bieberbach domain $\Omega$ lying dense in $\mathbb{C}^k \setminus K$ (See [10] for the strictly convex case).

Now, let $U \subset \mathbb{C}^k$ be an open set such that $(U) = U$ with $U$ polynomially convex, and let $\Omega$ be a Fatou-Bieberbach domain that lies dense in the complement of $U$. If $\Omega$ is to be the attracting basin for an automorphism of $\mathbb{C}^k$, it is clear that this automorphism will have to be an automorphism of $U$. So if we choose $U$ such that no automorphism of $U$ extends holomorphically to an automorphism that is attracting at a point outside of $U$, we know that the Fatou-Bieberbach domain in question cannot be the attracting basin of an automorphism of $\mathbb{C}^k$. And since no automorphism of the unit ball extends holomorphically to such an attracting fix-point automorphism, we can let $U$ be the unit ball. \qed

To prove that all Runge Fatou-Bieberbach domains are sequence-attracting basins we will need the following Lemma:

**Lemma 4.** Let $\Omega$ be a Runge Fatou-Bieberbach domain. For a compact set $K \subset \Omega$, a bounded open set $U$ such that $K \subset U$, and an $\epsilon > 0$, there exists a $\Phi \in \text{Aut}(\mathbb{C}^k)$ such that $\|\Phi(z) - z\|_K < \epsilon$, $\Phi(\mathbb{C}^k \setminus \Omega) \cap U = \emptyset$.

**Proof.** Let $\Psi: \mathbb{C}^k \to \Omega$ be a Fatou-Bieberbach map. By [1] there exists a sequence of automorphisms $\{F_j\}$ such that $F_j \to \Psi$ uniformly on compacts in $\mathbb{C}^k$. So $F_j \circ \Psi^{-1} \to \text{id}$ on $K$. This means that we can let $\tilde{\Psi}$ be a Fatou-Bieberbach map $\tilde{\Psi}: \Omega \to \mathbb{C}^k$ such that $\|\tilde{\Psi}(z) - z\| < 1$ for all $z \in K$. By [1] and Corollary 5.3 in [4, p.141] there exists a sequence of automorphisms $\{\Phi_j\}$ such that $\Phi_j \to \tilde{\Psi}$ uniformly on compacts in $\Omega$. So for a large enough $j$ we have the $\epsilon$-estimate. And by Corollary 5.3 in [4, p.141] we have that $\|\Phi_j\| \to \infty$ uniformly on $\mathbb{C}^k \setminus \Omega$, so the result follows with $\Phi = \Phi_m$ for a large enough $m$. \qed

**Proposition 2.** Let $\Omega$ be a Runge Fatou-Bieberbach domain. For any $p \in \Omega$ there exists a sequence $\{\varphi_j\} \subset \text{Aut}_p(\mathbb{C}^k)$ such that $\Omega = \Omega(\varphi_j)$. Moreover, in the terminology from Lemma 1, we may assume that we have $|\varphi_i(z + p) - p| \in \Gamma(s,r,\rho)$ with $r^2 < s$ for all $i \in \mathbb{N}$.

**Proof.** We may assume $p = 0$, and we let $r_j \downarrow 0$ such that $B_{r_j} \subset \Omega$. Choose a compact exhaustion $K_0 \subset K_1 \subset \cdots \subset K_j \subset \cdots$ of $\Omega$ where $K_0 = \overline{B}_{r_0}$. Let $F_0 = A$, where $A$ is the linear map defined in Section 5.
Now, make the following induction hypothesis $I_j$: We have automorphisms 
\{F_0, \ldots, F_j\} \subset \text{Aut}_0(\mathbb{C}^k)$ such that the following are satisfied
\begin{enumerate}[(a)]
\item $F(j)(K_j) \subset B_{r_j}$.
\item $F(j)(\mathbb{C}^k \setminus \Omega) \cap \overline{B}_{r_0} = \emptyset$.
\end{enumerate}
$I_0$ is obviously true if $r_0$ is chosen to be small enough.
Let $r \geq j + 1$ such that $B_{r_0} \subset F(j)(K_r)$. There exists an $s \in \mathbb{N}$ such that
\[ A^s(F(j)(K_r)) \subset B_{r_{j+1}}. \]

Let $U$ be a bounded open set such that $A^{-s}(B_{r_0}) \subset U$. Since we have that $F(j)(K_r) \subset F(j)(\Omega)$ which is a Fatou-Bieberbach domain, Lemma 4 gives us a $\phi_j \in \text{Aut}(\mathbb{C}^k)$ such that
\begin{enumerate}[(c)]
\item $\phi_j \approx \text{id}$ on $F(j)(K_r)$, \label{eq:conjugacy}
\item $\phi_j(\mathbb{C}^k \setminus (F(j)(\Omega)) \cap U = \emptyset$. \label{eq:conjugacy2}
\end{enumerate}

We may also assume that $\phi_j(0) = 0$. Then we can define $F_{j+1} = A^s \circ \phi_j$, and $I_{j+1}$ follows. It is now clear that $\lim_{j \to \infty} F(j)(z) \to p$ uniformly on compacts in $\Omega$, and it is clear that for any $z \in \mathbb{C}^k \setminus \Omega$, we do not have convergence of $F(j)(z)$. Lastly, since $A \circ \phi$ can be made arbitrary close to $A$ on $\overline{B}_{r_0}$ we may assume that $F_{j+1}$ is a composition of maps $\varphi_i$ all elements in $\Gamma(s, r, r_0)$. \hfill $\square$

**Proposition 3.** Let $\{\Omega_j\}$ be an increasing sequence of Fatou-Bieberbach domains in $\mathbb{C}^k$ that are all Runge. Then $\Omega = \bigcup_{j=1}^{\infty} \Omega_j$ is biholomorphically equivalent to $\mathbb{C}^k$.

**Proof.** Let $\{K_j\}$ be an increasing sequence of compact sets that exhausts $\Omega$ such that $K_j \subset \Omega_j$ for all $j \in \mathbb{N}$, and let $B_j$ denote the ball with radius $j$ in $\mathbb{C}^k$.

We will inductively construct a sequence of biholomorphisms that converges to a biholomorphism $\Phi: \Omega \to \mathbb{C}^k$. To start the induction let $\varphi_1$ be any biholomorphism $\varphi_1: \Omega_1 \to \mathbb{C}^k$, and assume that we have constructed maps $\varphi_j: \Omega_j \to \mathbb{C}^k$ for $j = 1, 2, \ldots, k$. What we want to do is to construct $\varphi_{k+1}$ such that it is very close to $\varphi_k$ on $K_k$, and such that their inverses are very close on $\overline{B}_k$. For any $\varepsilon$ we can, by the same argument as in the proof of Lemma 4, assume that we have a biholomorphism $\phi_{k+1}: \Omega_{k+1} \to \mathbb{C}^k$ such that
\begin{enumerate}[(a)]
\item $\|\phi_{k+1}(z) - z\| < \varepsilon$ for all $z \in K_k$,
\item $\|\phi_{k+1}^{-1}(z) - z\| < \varepsilon$ for all $z \in \varphi_k^{-1}(\overline{B}_k)$. \label{eq:close}
\end{enumerate}

Here $B_k(h)$ denotes an $h$-neighborhood of $B_k$. For any $\delta > 0$, by [1] we may assume that we have an $F_{k+1} \in \text{Aut}(\mathbb{C}^k)$ such that
\begin{enumerate}[(c)]
\item $\|F_{k+1}(z) - \varphi_k(z)\| < \delta$ for all $z \in K_k(h)$,
\item $\|F_{k+1}^{-1}(z) - \varphi_k^{-1}(z)\| < \delta$ for all $z \in \overline{B}_k$. \label{eq:convergence}
\end{enumerate}

Now we can define $\varphi_{k+1} = F_{k+1} \circ \phi_{k+1}$, and for any $\rho_{k+1} > 0$ we can make sure that
\begin{enumerate}[(e)]
\item $\|\varphi_{k+1}(z) - \varphi_k(z)\| < \rho_{k+1}$, \label{eq:convergence2}
\item $\|\varphi_{k+1}^{-1}(z) - \varphi_k^{-1}(z)\| < \rho_{k+1}$, \label{eq:convergence3}
\end{enumerate}

by letting $\varepsilon$ and $\delta$ be small enough. We may now assume that we have an infinite sequence of biholomorphisms $\varphi_j: \Omega_j \to \mathbb{C}^n$ that satisfies (e) and (f) where the sequence $\{\rho_j\}$ is chosen to be sumable. In the terminology from [4] we now have that $(\varphi_j, \Omega_j) \to (\Phi, \Omega)$. And by making sure that the $\rho_j$'s are small enough we can guaranty that $\Phi$ is not degenerate at every point, which tells us that it is 1-1 onto its image. By Theorem 5.2 in [4, p. 140] we also have that $(\varphi_j^{-1}, \mathbb{C}^k) \to (\Phi^{-1}, \Phi(\Omega))$, so by the convergence of $\{\varphi_j^{-1}\}$, we must have that $\Phi(\Omega) = \mathbb{C}^k$. \hfill $\square$
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