Coexistence of zero Lyapunov exponent and positive Lyapunov exponent for new quasi-periodic Schrödinger operator

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Abstract In this paper we solve a problem about the Schrödinger operator with potential \( v(\theta) = 2\lambda \cos^2(\pi \theta) / (1 - \alpha \cos 2\pi \theta) \), (|\alpha| < 1) in physics. With the help of the formula of Lyapunov exponent in the spectrum, the coexistence of zero Lyapunov exponent and positive Lyapunov exponent for some parameters is first proved, and there exists a curve that separates them. The spectrum in the region of positive Lyapunov exponent is purely pure point spectrum with exponentially decaying eigenfunctions for almost every frequency and almost every phase. From the research, we realize that the infinite potential \( v(\theta) = 2\lambda \tan^2(\pi \theta) \) has zero Lyapunov exponent for some energies if 0 < |\lambda| < 1.

Keywords quasi-periodic, Schrödinger operators, Lyapunov exponent, spectrum, pure point

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1 Introduction and main results

Since the 1970’s, Schrödinger operators have been popular in solid-state physics. Schrödinger operators with random or quasi-periodic potentials can describe the influence of an external magnetic field on the electrons of a crystal, and model Hamiltonians of quantum mechanical systems. Anderson model and almost Mathieu operator are two widely studied examples. The spectral type including absolutely continuous spectrum, singular continuous spectrum and pure point spectrum is one of the major researches.

In physics, Anderson localization [1] describes insulating behavior in the sense that quantum states are localized in a bounded region all the time. The quantum state satisfying Anderson localization is called localized state, otherwise is called extended state. In mathematics, Anderson localization means pure point spectrum with exponentially decaying eigenfunctions.

The almost Mathieu operator \( (v(\theta) = 2\lambda \cos 2\pi \theta) \) has been throughly studied and has purely spectral types in three different cases [11], [2]: for almost every pair (\( \theta, b \)) the Schrödinger operator has purely absolutely continuous spectrum if |\lambda| < 1, purely pure point spectrum with exponentially decaying eigenfunctions if |\lambda| > 1 and purely singular continuous spectrum if |\lambda| = 1. Therefore, phase transition occurs when \( \lambda \) goes from |\lambda| > 1 to |\lambda| < 1. However, we don’t know what is happening in the transition

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region ($\lambda \approx 1$). In fact, we have a deep understanding of the nature of Schrödinger operators with "large" ($|\lambda| \gg 1$) and "small" ($|\lambda| \ll 1$) analytic potentials: for "small" $|\lambda|$ the Schrödinger operator has purely absolutely continuous spectrum and has zero Lyapunov exponent in the spectrum [8]; for "large" $|\lambda|$ the Schrödinger operator has pure point spectrum (for almost every $\theta$) with exponentially decaying eigenfunctions and positive Lyapunov exponent for all energies in $\mathbb{R}$ [7]. Nonetheless, the phase transition between absolutely continuous and pure point spectrum has been considerably harder to understand. Hence it is essential to study mixed spectral types for Schrödinger operators though such examples for one-frequency discrete case have been considered difficult to construct explicitly. Bourgain [6] constructed a quasi-periodic Schrödinger operator with two frequencies which has both absolutely continuous and pure point spectrum. Bjerklöv [5] gave examples which have positive Lyapunov exponent on certain regions of the spectrum and zero on other regions. In his examples, operators with arbitrarily large potentials may have zero Lyapunov exponent for certain energies. However, he failed to prove the coexistence of more than two spectral types. Zhang [15] proved these examples have coexistence of absolutely continuous and pure point spectrum for some parameters as well as coexistence of absolutely continuous and singular continuous spectrum for some other parameters. Avila [2] showed that perturbations of the critical almost Mathieu operator (with potential $v(\theta) = 2\cos 2\pi \theta$) may have arbitrarily many alternances between subcritical and supercritical regimes.

We consider the following one-dimensional quasi-periodic Schrödinger operator on $l^2(\mathbb{Z})$:

$$ (H_{\alpha, \theta, b} u)_n = u_{n+1} + u_{n-1} + v(nb + \theta)u_n, \ n \in \mathbb{Z}, $$

(1.1)

where

$$ v(\theta) = 2\lambda \frac{\cos(2\pi \theta)}{1 - \alpha \cos(2\pi \theta)}, \ \alpha \in (-1, 1), $$

(1.2)

$v$ is the potential, $\alpha$ is the parameter, $\theta \in T = \mathbb{R}/\mathbb{Z}$ is the phase, $\lambda \in \mathbb{R}$ is the coupling, the frequency $b \in \mathbb{R}$ is irrational. It is trivial when $\lambda = 0$.

This model was first introduced by Ganeshan S, Pixley J H and Sarma S D in the top journals (Physical Review Letters) [10]. They did some calculations based on self-dual condition and did some realistic experiments about atomic optical lattices and photonic waveguides to numerically verify the following problem:

**Problem 1.1.** For the model 1.1, there exists a one-dimensional mobility edge

$$ \alpha E = 2\text{sgn}(\lambda)(1 - |\lambda|), $$

(1.3)

that separates localized states from extended states if they coexist.

See Figure 1 for illustration. The x-coordinate represents the parameter $\alpha$, and the y-coordinate represents the corresponding energy in the spectrum. The localization properties of a quantum state can
be numerically quantified by IPR and TDOS (see the definitions in [10]). Pure cyan denotes $IPR = 0$ for the extended state and pure black denotes $IPR = 1$ for the localized state. In addition, pure cyan denotes maximum TDOS values between 1 and 10 for the extended state and pure black denotes $TDOS = 0$ for the localized state. It is clear that the red curve (1.3) separates localized states from extended states.

In this paper, we give the first strict proof of the Problem 1.1. In addition, we show that zero Lyapunov exponent and positive Lyapunov exponent coexist under some conditions. Moreover, we give an example that has infinite potential and zero Lyapunov exponent for some energies.

Our main results of this paper are the following theorems:

**Theorem 1.2.** For the model (1.1), we have the formula of Lyapunov exponent

$$ L(b, A) = \max \{ \log \frac{\alpha E + 2\lambda \pm \sqrt{(\alpha E + 2\lambda)^2 - 4\alpha^2}}{2(1 + \sqrt{1 - \alpha^2})}, 0 \} $$

for every $E$ in the spectrum.

**Theorem 1.3.** In the model (1.1), for every $E$ in the spectrum,

$$ L(b, A) > 0 \iff \begin{cases} \alpha E > 2\text{sgn}(\lambda)(1 - |\lambda|) & \lambda > 0, \\ \alpha E < 2\text{sgn}(\lambda)(1 - |\lambda|) & \lambda < 0, \end{cases} $$

and

$$ L(b, A) = 0 \iff \begin{cases} \alpha E \leq 2\text{sgn}(\lambda)(1 - |\lambda|) & \lambda > 0, \\ \alpha E \geq 2\text{sgn}(\lambda)(1 - |\lambda|) & \lambda < 0, \\ \forall E \in \sum_{b,v} \lambda = 0. \end{cases} $$

In particular, when $\lambda \neq 0$ the curve

$$ \alpha E = 2\text{sgn}(\lambda)(1 - |\lambda|) $$

(1.6)

separates the spectrum with positive Lyapunov exponent from the spectrum with zero Lyapunov exponent if they coexist.

**Theorem 1.4.** In the model (1.1), there are both positive Lyapunov exponent and zero Lyapunov exponent in the spectrum if $1 - |\alpha| < |\lambda| < 1 + |\alpha|$; there is only positive Lyapunov exponent in the spectrum if $|\lambda| > (1 + |\alpha|)^2$; there is only zero Lyapunov exponent in the spectrum if $|\lambda| \leq (1 - |\alpha|)^2$.

**Corollary 1.5.** Consider the potential $v(\theta) = 2\lambda\tan^2\pi\theta$, there are both zero Lyapunov exponent and positive Lyapunov exponent in the spectrum if $0 < |\lambda| < 1$.

This potential is infinite and has a singular point in $\mathbb{T}$.

**Theorem 1.6.** Suppose $b \in \mathbb{T}$ is Diophantine and $\theta \in \mathbb{T}$ is non-resonant. For the model (1.1), spectrum in the region of positive Lyapunov exponent is purely pure point spectrum with exponentially decaying eigenfunctions (Anderson localization).

It is known that Lebesgue almost every $b \in \mathbb{T}$ is Diophantine and Lebesgue almost every $\theta \in \mathbb{T}$ is non-resonant. Thus the theorem holds for almost every frequency and almost every phase.

## 2 The formula of Lyapunov exponent

In this section, we use Herman’s subharmonicity methods and Avila’s global theroy to obtain the formula of Lyapunov exponent in the spectrum.

Given a bounded map $v : \mathbb{Z} \to \mathbb{R}$, the solutions of Schrödinger equation

$$ (H_{\alpha, \theta, b}u)_n = u_{n+1} + u_{n-1} + v(nb + \theta)u_n = zu_n, $$

(2.1)

which, for $n \geq 1$, can be expressed as

$$ \begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix} = A_n(\theta, z) \begin{pmatrix} u_1 \\ u_0 \end{pmatrix}, $$

(2.2)
\[ A_n(\theta, z) = A(\theta + (n-1)b) \ldots A(\theta), \quad A_{-n}(\theta, z) = A_n(\theta - nb)^{-1}, \quad n > 0. \]

\[ A(\theta, z) = A^{(z-v)}(\theta) = \begin{pmatrix} z - v(\theta) & -1 \\ 1 & 0 \end{pmatrix}, \quad \theta \in \mathbb{T}, \quad z \in \mathbb{Z}. \quad (2.3) \]

The spectrum \( \sigma(H_{\alpha, \theta, b}) \) of the operator \( H_{\alpha, \theta, b} \) is a nonempty, compact subset of \( \mathbb{R} \). When \( b \) is irrational it is the same for almost every \( \theta \in \mathbb{T} \), we denote it by \( \sum_{b,v} \). The Lyapunov exponent as usual is defined by

\[ L(z) = L(b, A^{(z-v)}) = \lim_{n \to \infty} \frac{1}{n} \int_{\mathbb{T}} \log \| A_n(\theta, z) \| \, d\theta. \quad (2.4) \]

The Lyapunov exponent defined above is non-negative since \( \det(A_n) = 1 \) for every \( n \in \mathbb{Z} \).

Now, we use Herman’s subharmonicity argument and extensions to estimate a lower bound of the Lyapunov exponent of the model (1.1).

**Theorem 2.1.** For the model (1.1), the Lyapunov exponent has a lower bound at all energies \( z \in \mathbb{C} \):

\[ L(z) \geq \max \{ \log \left| \frac{\alpha z + 2\lambda \pm \sqrt{(\alpha z + 2\lambda)^2 - 4\alpha^2}}{2(1 + \sqrt{1 - \alpha^2})} \right|, 0 \}. \]

**Proof.** Note that

\[ A(\theta, z) = \begin{pmatrix} z - 2\lambda \frac{\cos(2\pi \theta)}{1 - \alpha \cos(2\pi \theta)} & -1 \\ 1 & 0 \end{pmatrix} = \frac{1}{2 - \alpha e^{2\pi i} - \alpha e^{-2\pi i}} \begin{pmatrix} 2z - (\alpha z + 2\lambda)(e^{2\pi i} + e^{-2\pi i}) & -2 + \alpha e^{2\pi i} + \alpha e^{-2\pi i} \\ 2 - \alpha e^{2\pi i} - \alpha e^{-2\pi i} & 0 \end{pmatrix}. \]

Denote

\[ B(\theta, z) = (2 - \alpha e^{2\pi i} - \alpha e^{-2\pi i})A(\theta, z) \]

\[ = \begin{pmatrix} 2z - (\alpha z + 2\lambda)(e^{2\pi i} + e^{-2\pi i}) & -2 + \alpha e^{2\pi i} + \alpha e^{-2\pi i} \\ 2 - \alpha e^{2\pi i} - \alpha e^{-2\pi i} & 0 \end{pmatrix}. \]

Similarly, we can define \( B_n(\theta, z) = B(\theta + (n-1)b) \ldots B(\theta) \).

According to Birkhoff theorem and Jensen theorem, we have

\[ L(z) = \lim_{n \to \infty} \frac{1}{n} \int_{\mathbb{T}} \log \| B_n(\theta, z) \| \, d\theta - \int_{\mathbb{T}} \log |2 - \alpha e^{2\pi i} - \alpha e^{-2\pi i}| \, d\theta, \]

and

\[ \int_{\mathbb{T}} \log |2 - \alpha e^{2\pi i} - \alpha e^{-2\pi i}| \, d\theta = \log |1 + \sqrt{1 - \alpha^2}|. \]

Setting \( \omega = e^{2\pi i} \), we find that

\[ \omega B(\theta, z) = \begin{pmatrix} 2z\omega - (\alpha z + 2\lambda)(\omega^2 + 1) & -2\omega + \alpha \omega^2 + \alpha \\ 2\omega - \alpha \omega^2 - \alpha & 0 \end{pmatrix}. \]

We define a new function

\[ N_n(\omega) = \omega^n B_n(\theta, z), \]
initially on $|\omega|=1$, then $N_n$ extends to an entire function and hence $\omega \mapsto \log \| N_n(\omega) \|$ is subharmonic. Hence it holds that
\[
\int_T \frac{1}{n} \log \| B_n(\theta, z) \| \, d\theta = \int_T \frac{1}{n} \log \| \omega^n B_n(\theta, z) \| \, d\theta = \int_T \log \| N_n(\omega) \| \, d\theta \geq \log \| N_n(0) \| = \log \left( \begin{pmatrix} -\alpha z - 2\lambda & \alpha \\ -\alpha & 0 \end{pmatrix}^n \right).
\]

Therefor,
\[
L(z) = \lim_{n \to \infty} \int_T \frac{1}{n} \log \| B_n(\theta, z) \| \, d\theta - \log|1 + \sqrt{1 - \alpha^2}|
= \lim_{n \to \infty} \int_T \frac{1}{n} \log \| N_n(\omega) \| \, d\theta - \log|1 + \sqrt{1 - \alpha^2}|
\geq \lim_{n \to \infty} \log \left( \begin{pmatrix} -\alpha z - 2\lambda & \alpha \\ -\alpha & 0 \end{pmatrix}^n \right) \|-\log|1 + \sqrt{1 - \alpha^2}|
= \max \{ \log \frac{\alpha z + 2\lambda + \sqrt{(\alpha z + 2\lambda)^2 - 4\alpha^2}}{2(1 + \sqrt{1 - \alpha^2})} \}.
\]

Due to the non-negativity of the Lyapunov exponent, we have
\[
L(z) \geq \max \{ \log \frac{\alpha z + 2\lambda + \sqrt{(\alpha z + 2\lambda)^2 - 4\alpha^2}}{2(1 + \sqrt{1 - \alpha^2})} , 0 \}.
\]

Denote
\[
A_\varepsilon(\theta, z) = A(\theta + \varepsilon i, z), \; \theta \in T, \; z \in \mathbb{C}, \; \varepsilon \in \mathbb{R},
\]

it is easy to know that $2 - \alpha e^{2\pi(\theta + \varepsilon)i} - \alpha e^{-2\pi(\theta + \varepsilon)i}$ has at most two real roots about $\varepsilon$ and they respectively satisfy
\[
e^{2\pi i} = e^{2\pi} \frac{1 + \sqrt{1 - \alpha^2}}{\alpha} \text{ or } e^{2\pi i} = e^{2\pi} \frac{1 - \sqrt{1 - \alpha^2}}{\alpha}.
\]

Take modulus
\[
1 = e^{2\pi} \frac{1 + \sqrt{1 - \alpha^2}}{\alpha} \text{ or } 1 = e^{2\pi} \frac{1 - \sqrt{1 - \alpha^2}}{\alpha}.
\]

When $|\varepsilon| < \frac{1}{2\pi} \log \left| \frac{1 + \sqrt{1 - \alpha^2}}{\alpha} \right|$, we have
\[
1 < e^{2\pi} \frac{1 + \sqrt{1 - \alpha^2}}{\alpha} \text{ and } 1 > e^{2\pi} \frac{1 - \sqrt{1 - \alpha^2}}{\alpha},
\]

it means that $2 - \alpha e^{2\pi(\theta + \varepsilon)i} - \alpha e^{-2\pi(\theta + \varepsilon)i}$ has no real roots about $\varepsilon$ under this condition. Thus, $A(\cdot, z) \in C^\omega(\mathbb{R}/\mathbb{Z}, SL(2, \mathbb{C}))$ admits a holomorphic extension to $|Im \cdot| < \frac{1}{2\pi} \log \left| \frac{1 + \sqrt{1 - \alpha^2}}{\alpha} \right|$ and $A_\varepsilon(\cdot, z) \in C^\omega(\mathbb{R}/\mathbb{Z}, SL(2, \mathbb{C}))$ is well-defined. Denote $\delta = \frac{1}{2\pi} \log \left| \frac{1 + \sqrt{1 - \alpha^2}}{\alpha} \right|$, by Jensen theorem, we have
\[
\int_T \log |2 - \alpha e^{2\pi(\theta + \varepsilon)i} - \alpha e^{-2\pi(\theta + \varepsilon)i}| d\theta = \log|1 + \sqrt{1 - \alpha^2}| \quad (if \ |\varepsilon| < \delta).
\]

When $|\varepsilon| < \delta$, it holds that
\[
L(b, A_\varepsilon) = L(b, B_\varepsilon) - \int_T \log |2 - \alpha e^{2\pi(\theta + \varepsilon)i} - \alpha e^{-2\pi(\theta + \varepsilon)i}| d\theta
= L(b, B_\varepsilon) - \log|1 + \sqrt{1 - \alpha^2}|.
\]
We consider the class of 1-periodic functions on \( \mathbb{R} \) which have analytic extension to some strip \(|\Im z| < \eta\) and take values in complex \(2 \times 2\) matrices. We denote them by \(C^\omega(T,M_2(\mathbb{C}))\).

Cocycles \((b, D)\) with \(D \in C^\omega(T,M_2(\mathbb{C}))\) are called analytic. For analytic cocycles, we have a similar definition of Lyapunov exponent \(L(b, D) : T \times C^\omega(T,M_2(\mathbb{C})) \to (-\infty, \infty)\) and it is jointly continuous at every \((b, D)\) with \(b \in \mathbb{R} \setminus \mathbb{Q}\).\(^\text{[2] [12] [13]}\) Given any analytic cocycle \((b, D)\), we consider its holomorphic extension \((b, D_\varepsilon)\) with \(|\varepsilon| \leq \eta\).

The Lyapunov exponent \(L(b, D_\varepsilon)\) is easily seen to be a convex function of \(\varepsilon\). Thus we can introduce the acceleration of \((b, D_\varepsilon)\).

\[
\omega(b, D_\varepsilon) = \frac{1}{2\pi} \lim_{\varepsilon \to 0^+} \frac{L(b, D_{\varepsilon+h}) - L(b, D_\varepsilon)}{h}. \tag{2.6}
\]

It follows from convexity and continuity of the Lyapunov exponent that the acceleration is an upper semicontinuous function in parameter \(\varepsilon\).

**Definition 2.2.** \((b, A) \in (\mathbb{R} \setminus \mathbb{Q}) \times C^\omega(\mathbb{R}/\mathbb{Z}, SL(2, \mathbb{C}))\) is regular if \(L(b, A_\varepsilon)\) is affine for \(\varepsilon\) in a neighborhood of 0.

**Remark 2.3.** If \(A\) takes values in \(SL(2, \mathbb{R})\), then \(\varepsilon \mapsto L(b, A_\varepsilon)\) is an even function. By convexity, \(\omega(b, A) \geq 0\). And if \(b \in \mathbb{R} \setminus \mathbb{Q}\), then \((b, A)\) is regular if and only if \(\omega(b, A) = 0\).

The acceleration was first introduced and the above results were proved in \(\text{[2]}\) for analytic \(SL(2, \mathbb{C}) -\) cocyles. It was extended to the general case \(M_2(\mathbb{C})\) in \(\text{[13]}\).

The acceleration satisfies the following theorem:

**Theorem 2.4** (Quantization of acceleration \(\text{[2] [3] [14]}\)). Consider a cocycle \((b, D)\) with \(\det D(x)\) bounded away from 0 on the strip \(\mathbb{T}_\varepsilon = \{z : |\Im z| < \varepsilon\}\), then \(\omega(b, D_\varepsilon) \in \frac{1}{2} \mathbb{Z}\). Moreover, \(\omega(b, D_\varepsilon) \in \mathbb{Z}\) for \(SL(2, \mathbb{C}) -\) cocyles.

**Lemma 2.5.** \(\text{[2]}\) \(E \notin \sum_{b,v}\) if and only if \((b, A)\) is uniformly hyperbolic.

**Lemma 2.6.** \(\text{[2]}\) If \(L(b, A) > 0\), then \((b, A)\) is regular if and only if \((b, A)\) is uniformly hyperbolic.

**Lemma 2.7.** \(\text{[2]}\) If \((b, A) \in (\mathbb{R} \setminus \mathbb{Q}) \times C^\omega(\mathbb{R}/\mathbb{Z}, SL(2, \mathbb{C}))\), then \((b, A)\) is regular if and only if \(\omega(b, A) = 0\).

**Remark 2.8.** The above lemmas suggest that the acceleration \(\omega(b, A)\) is positive for every energy \(E\) in the spectrum if the corresponding Lyapunov exponent \(L(b, A)\) is positive.

Next, we use Avila’s global theory to obtain the formula of Lyapunov exponent in the spectrum. The key point is to prove that acceleration \(\omega(b, E) = 1\) for \(\varepsilon \geq 0\) in the spectrum.

**PROOF OF THEOREM**\(^\text{[1.3]}\) Similarly, denote

\[
B_\varepsilon(\theta, E) = (2 - \alpha e^{2\pi(\theta+\varepsilon)i} - \alpha e^{-2\pi(\theta+\varepsilon)i})A_\varepsilon(\theta, E), \quad E \in \mathbb{R}.
\]

Recalling the definition of acceleration \((\text{2.6)}\) and equation \((\text{2.5)}\), we have

\[
\omega(b, A_\varepsilon) = \omega(b, B_\varepsilon), \quad |\varepsilon| < \frac{1}{2\pi} \log \left|\frac{1 + \sqrt{1 - \alpha^2}}{\alpha}\right|. \tag{2.7}
\]

If \(\varepsilon\) is sufficiently large, it holds that

\[
B_\varepsilon(\theta, E) = \varepsilon e^{2\pi i - 2\pi \theta i} \begin{pmatrix} -\alpha E - 2\lambda & \alpha \\ -\alpha & 0 \end{pmatrix} + o(1).
\]

Putting the above into the definition of the Lyapunov exponent yields

\[
L(b, B_\varepsilon) = \max \{\log |\alpha E + 2\lambda \pm \sqrt{(\alpha E + 2\lambda)^2 - 4\alpha^2}|\} + 2\pi \varepsilon + o(1),
\]

if \(\varepsilon\) is sufficiently large.
Applying Theorem 2.4, one obtains that

\[ \omega(b, B_{\varepsilon}) = 1, \]

and

\[ L(b, B_{\varepsilon}) = \max\{\log\left|\frac{\alpha E + 2\lambda \pm \sqrt{(\alpha E + 2\lambda)^2 - 4\alpha^2}}{2}\right|\} + 2\pi\varepsilon, \]  \hspace{1cm} (2.8)\]

if \( \varepsilon \) is sufficiently large.

Since \( L(b, B_{\varepsilon}) \) is a convex function of \( \varepsilon \), we combine Remark 2.3 and equation (2.8) to see that

\[ \omega(b, A_{\varepsilon}) = \omega(b, B_{\varepsilon}) = 0 \text{ or } 1, \hspace{1cm} 0 \leq \varepsilon < \frac{1}{2\pi} \log \left| \frac{1 + \sqrt{1 - \alpha^2}}{\alpha} \right|. \]  \hspace{1cm} (2.9)\]

If \( L(b, A) > 0 \) and \( E \in \sum_{b,v} \), we employ Remark 2.8 and equation (2.9) to obtain

\[ \omega(b, A_{\varepsilon}) = \omega(b, B_{\varepsilon}) = 1, \hspace{1cm} 0 \leq \varepsilon < \frac{1}{2\pi} \log \left| \frac{1 + \sqrt{1 - \alpha^2}}{\alpha} \right|. \]

Thanks to equation (2.8) and the continuity of the Lyapunov exponent, one has

\[ \omega(b, B_{\varepsilon}) = 1, \hspace{1cm} \varepsilon \geq 0. \]

Obviously, there holds

\[ L(b, B_{\varepsilon}) = \max\{\log\left|\frac{\alpha E + 2\lambda \pm \sqrt{(\alpha E + 2\lambda)^2 - 4\alpha^2}}{2}\right|\} + 2\pi\varepsilon, \hspace{1cm} \varepsilon \geq 0. \]

According to equation (2.5) and the non-negativity of the Lyapunov exponent, we have

\[ L(b, A) = L(b, B) - \log|1 + \sqrt{1 + \alpha^2}| \]
\[ = \max\{\log\left|\frac{\alpha E + 2\lambda \pm \sqrt{(\alpha E + 2\lambda)^2 - 4\alpha^2}}{2(1 + \sqrt{1 - \alpha^2})}\right|, 0\}. \]  \hspace{1cm} (2.10)\]

If \( L(b, A) = 0 \) and \( E \in \sum_{b,v} \), it also satisfies the above equation (2.10) by Theorem 2.1.  \hspace{1cm} \( \square \)

**Corollary 2.9.** Let \( \alpha = 0 \), then the model (1.1) becomes the almost Mathieu operator, and

\[ L(E) = \max\{\log|\lambda|, 0\} \] for every \( E \in \sum_{b,v} \).

**Corollary 2.10.** Let \( v(\theta) = 2\lambda \frac{1 - \cos(2\pi\theta)}{1 - \alpha \cos(2\pi\theta)} \), then

\[ L(E) = \max\{\log\left|\frac{E + 2\lambda \pm \sqrt{(E + 2\lambda)^2 - 4\alpha^2}}{2}\right|, 0\}, \forall E \in \sum_{b,v}. \]

If \( \alpha = 1 \), then it becomes the periodic model and the Lyapunov exponent is \( L(E) = 0 \) for every \( E \in \sum_{b,v} \); if \( \alpha = -1 \), then the potential is \( v(\theta) = 2\lambda \tan^2(\pi\theta) \) and the Lyapunov exponent is

\[ L(E) = \max\{\log\left|\frac{E + 2\lambda \pm \sqrt{(E + 2\lambda)^2 - 4\alpha^2}}{2}\right|\} \]

for every \( E \in \sum_{b,v} \).

**Proof.** If \( \alpha = 1 \), then \( v(\theta) = 2\lambda \), the spectrum is contained in \([-2 + 2\lambda, 2 + 2\lambda]\) and \(|E - 2\lambda| \leq 2\). Obviously, there holds \( L(E) = 0 \) for every \( E \in \sum_{b,v} \).

Note that

\[ 2\lambda \frac{1 - \cos(2\pi\theta)}{1 - \alpha \cos(2\pi\theta)} = 2\lambda + 2\lambda(\alpha - 1) \frac{\cos 2\pi\theta}{1 - \alpha \cos 2\pi\theta}, \]

so the new coupling is \( \lambda(\alpha - 1) \), the new energy is \( E - 2\lambda \). The corollary follows by Theorem 1.2.  \hspace{1cm} \( \square \)
3 Coexistence of zero Lyapunov exponent and positive Lyapunov exponent

In this section, we use the formula of the Lyapunov exponent in Theorem 1.2 to study the coexistence of zero Lyapunov exponent and positive Lyapunov exponent.

By operator theory, it is easy to see that \( \sum_{b,v} \subseteq [-2 + \min(v), 2 + \max(v)] \), and we have the following lemma.

**Lemma 3.1.** If \( \lambda > 0 \), then \( \sum_{b,v} \subseteq [-2 - \frac{2\lambda}{1+\alpha}, 2 + \frac{2\lambda}{1-\alpha}] \).

If \( \lambda < 0 \), then \( \sum_{b,v} \subseteq [-2 + \frac{2\lambda}{1-\alpha}, 2 - \frac{2\lambda}{1+\alpha}] \).

If \( \lambda = 0 \), then \( \sum_{b,v} = [-2, 2] \).

**Proof.** Just consider the fact that \( \sum_{b,v} \subseteq [-2 + \min(v), 2 + \max(v)] \). When \( \lambda > 0 \), \( \min(v) = -\frac{2\lambda}{1+\alpha} \), \( \max(v) = \frac{2\lambda}{1-\alpha} \); when \( \lambda < 0 \), \( \min(v) = \frac{2\lambda}{1-\alpha} \), \( \max(v) = -\frac{2\lambda}{1+\alpha} \); when \( \lambda = 0 \), it is a periodic model and \( \sum_{b,v} = [-2, 2] \).

Then we use the formula of Lyapunov exponent and Lemma 3.1 to prove the Theorem 1.3.

**Proof of Theorem 1.3.** For every \( E \in \sum_{b,v} \), according to the formula of the Lyapunov exponent 1.2, we have

\[
L(E) = \max\{\log \frac{|\alpha|}{1 + \sqrt{1 - \alpha^2}}, 0\} = 0, \text{ if } |\alpha E + 2\lambda| < 2|\alpha|.
\]

Hence it holds that

\[
L(E) > 0 \iff \begin{cases} 
\alpha E + 2\lambda + \sqrt{(\alpha E + 2\lambda)^2 - 4\alpha^2} > 2(1 + \sqrt{1 - \alpha^2}) \quad \alpha E + 2\lambda > 2|\alpha|, \\
\alpha E + 2\lambda - \sqrt{(\alpha E + 2\lambda)^2 - 4\alpha^2} < -2(1 + \sqrt{1 - \alpha^2}) \quad \alpha E + 2\lambda < -2|\alpha|, \\
\end{cases}
\]

\[
\iff |\alpha E + 2\lambda| > 2.
\]

Assume \( \alpha E + 2\lambda < -2 \) when \( \lambda > 0 \), we have

\[
\sum_{b,v} \subseteq [-2 - \frac{2\lambda}{1+\alpha}, 2 + \frac{2\lambda}{1-\alpha}],
\]

and

\[
\begin{cases} 
E < \frac{-2 - 2\lambda}{\alpha} < -2 - \frac{2\lambda}{1+\alpha} \quad \alpha > 0, \\
E > \frac{-2 + 2\lambda}{\alpha} > 2 + \frac{2\lambda}{1-\alpha} \quad \alpha < 0.
\end{cases}
\]

This contradiction means that \( \alpha E + 2\lambda < -2 \) when \( \lambda > 0 \).

Assume \( \alpha E + 2\lambda > 2 \) when \( \lambda < 0 \), we have

\[
\sum_{b,v} \subseteq [-2 + \frac{2\lambda}{1-\alpha}, 2 - \frac{2\lambda}{1+\alpha}],
\]

and

\[
\begin{cases} 
E > \frac{2 + 2\lambda}{1-\alpha} > 2 + \frac{2\lambda}{1-\alpha} \quad \alpha > 0, \\
E < \frac{-2 - 2\lambda}{\alpha} < -2 + \frac{2\lambda}{1+\alpha} \quad \alpha < 0.
\end{cases}
\]

This contradiction means that \( \alpha E + 2\lambda > 2 \) when \( \lambda < 0 \).

It is known that \( L(E) = 0 \) for every \( E \in \sum_{b,v} \) when \( \lambda = 0 \).

In conclusion, for every \( E \in \sum_{b,v} \) the Lyapunov exponent \( L(E) \) is positive if and only if

\[
\begin{cases} 
\alpha E > 2\text{sgn}(\lambda)(1 - |\lambda|) \quad \lambda > 0, \\
\alpha E < 2\text{sgn}(\lambda)(1 - |\lambda|) \quad \lambda < 0,
\end{cases}
\]

for every \( E \in \sum_{b,v} \), the Lyapunov exponent \( L(E) \) is zero if and only if

\[
\begin{cases} 
\alpha E \leq 2\text{sgn}(\lambda)(1 - |\lambda|) \quad \lambda > 0, \\
\alpha E \geq 2\text{sgn}(\lambda)(1 - |\lambda|) \quad \lambda < 0, \\
\forall E \in \sum_{b,v} \quad \lambda = 0.
\end{cases}
\]
The boundary is
\[ \alpha E = 2\text{sgn}(\lambda)(1 - |\lambda|). \]

Furthermore, it is easy to prove Theorem 1.4 by Theorem 1.3 and Lemma 3.1.

**PROOF OF THEOREM 1.4.** For simplicity, we only prove the theorem in the case of \( \lambda > 0 \) and \( \alpha > 0 \), the proof is the same in the other cases.

Assume \( \{E : L(E) = 0, E \in \sum_{b,v}\} = \emptyset \) when \( \lambda > 0, \alpha > 0, \lambda < 1 + \alpha \). By Theorem 1.3, \( E \geq \frac{2(1 - \lambda)}{\alpha} \) for every \( E \in \sum_{b,v} \). Select \( n \) such that \( \cos(2\pi (nb + \theta)) < \frac{1 - \lambda}{\alpha} \) (\( n \) exists because of \( \frac{1 - \lambda}{\alpha} > -1 \)), we have
\[ \frac{2(1 - \lambda)}{\alpha} > \langle \delta_n, H_{\alpha,\theta,\delta}\delta_n \rangle = \frac{2\lambda\cos(2\pi (nb + \theta))}{1 - \alpha\cos(2\pi (nb + \theta))} = \int_{\mathbb{R}} Ed\mu_{\delta_n} > \frac{2(1 - \lambda)}{\alpha}, \]
\( \mu_{\delta_n} \) is the spectral measure on \( \sigma(H_{\alpha,\theta,\delta}) \), see the definitions (4.2). This contradiction means \( \{E : L(E) = 0, E \in \sum_{b,v}\} \neq \emptyset \) when \( \lambda > 0, \alpha > 0, \lambda > 1 - \alpha \).

Assume \( \{E : L(E) > 0, E \in \sum_{b,v}\} = \emptyset \) when \( \lambda > 0, \alpha > 0, \lambda > 1 - \alpha \). By Theorem 1.3, \( E \leq \frac{2(1 - \lambda)}{\alpha} \) for every \( E \in \sum_{b,v} \). Select \( n \) such that \( \cos(2\pi (nb + \theta)) > \frac{1 - \lambda}{\alpha} \) (\( n \) exists because of \( \frac{1 - \lambda}{\alpha} < 1 \)), we have
\[ \frac{2(1 - \lambda)}{\alpha} < \langle \delta_n, H_{\alpha,\theta,\delta}\delta_n \rangle = \frac{2\lambda\cos(2\pi (nb + \theta))}{1 - \alpha\cos(2\pi (nb + \theta))} = \int_{\mathbb{R}} Ed\mu_{\delta_n} \leq \frac{2(1 - \lambda)}{\alpha}. \]

This contradiction means \( \{E : L(E) > 0, E \in \sum_{b,v}\} \neq \emptyset \) when \( \lambda > 0, \alpha > 0, \lambda < 1 + \alpha \). The above discussions imply that there are both zero Lyapunov exponent and positive Lyapunov exponent in the spectrum when \( \lambda > 0, \alpha > 0, 1 - \alpha < \lambda < 1 + \alpha \).

According to Lemma 3.1, the spectrum \( \sum_{b,v} \) is contained in \([-2 - \frac{2\lambda}{1 + \alpha}, 2 + \frac{2\lambda}{1 - \alpha}] \). By Theorem 1.3, there is no positive Lyapunov exponent in the spectrum if \( \frac{2(1 - \lambda)}{\alpha} \geq 2 + \frac{2\lambda}{1 - \alpha} \), i.e., \( \lambda \leq (1 - \alpha)^2 \). There is no zero Lyapunov exponent in the spectrum if \( \frac{2(1 - \lambda)}{\alpha} < -2 - \frac{2\lambda}{1 + \alpha} \), i.e., \( \lambda > (1 + \alpha)^2 \).

Repetition of the same arguments leads to
\[
\begin{align*}
\exists E_1, E_2 \in \sum_{b,v}, L(E_1) > 0, L(E_2) = 0 & \quad \text{if } 1 - |\alpha| < |\lambda| < 1 + |\alpha|, \\
\forall E \in \sum_{b,v}, L(E) > 0 & \quad \text{if } |\lambda| > (1 + |\alpha|)^2, \\
\forall E \in \sum_{b,v}, L(E) = 0 & \quad \text{if } |\lambda| \leq (1 - |\alpha|)^2.
\end{align*}
\]

**PROOF OF COROLLARY 1.5.** Notice that
\[ v(\theta) = 2\lambda\tan^2 \pi \theta = 2\lambda \frac{1 - \cos 2\pi \theta}{1 + \cos 2\pi \theta} = 2\lambda - 4\lambda \frac{\cos 2\pi \theta}{1 + \cos 2\pi \theta}, \]
so the new coupling is \(-2\lambda\), the new energy is \( E - 2\lambda \), the parameter is \(-1\). By Theorem 1.4, there are both zero Lyapunov exponent and positive Lyapunov exponent in the spectrum if \( 0 < |\lambda| < 1 \).

**Remark 3.2.** The potential is infinite and non-analytic in \( \mathbb{T} \), and the Schrödinger equation has zero Lyapunov exponent in some energies. It is very different from the Maryland model \( v(\theta) = 2\lambda\tan(\pi \theta) \) which has positive Lyapunov exponent if \( |\alpha| \neq 0 \).

4 Anderson localization and proof of Problem 1.1.

In this section, we use similar methods in [11] to prove that the spectrum in the region of positive Lyapunov exponent is purely pure point spectrum with exponentially decaying eigenfunctions (Anderson localization).
To study the spectral properties of the operator $H_{α,γ,b}$, we have to introduce its universal spectral measure $μ_{α,γ,b}$ on $σ(H_{α,γ,b})$ defined as follows:

$$μ_{α,γ,b} = \frac{1}{2}(μ_{α,γ,b,δ_0} + μ_{α,γ,b,δ_1}) \quad δ_i(n) = δ_{i,n}, \quad i = 0, 1$$

(4.1)

where the spectral measures $μ_{δ,γ,b,δ_i}, \quad i = 0, 1$ are uniquely defined by

$$⟨δ_i, (H_{δ,γ,b} - zI)^{-1}δ_i⟩ = \int_{σ(H_{α,γ,b})} \frac{dμ_{α,γ,b,δ_i}(t)}{t - z}, \quad ∀z ∈ C, \; ∃z > 0.$$  

They are probability measures.

The universal spectral measure $μ_{α,γ,b}$ has a unique Lebesgue decomposition:

$$μ_{α,γ,b} = μ_{α,γ,b,pp} + μ_{α,γ,b,sc} + μ_{α,γ,b,ac},$$

where $μ_{α,γ,b,pp}$ is a pure point measure, $μ_{α,γ,b,sc}$ is a singular continuous measure, and $μ_{α,γ,b,ac}$ is an absolutely continuous measure. This means that $μ_{α,γ,b,pp}(R \setminus C) = 0$ for some countable set $C$, $μ_{α,γ,b,sc}(E) = 0$ for every $E ∈ R$, $μ_{α,γ,b,sc}(R \setminus N) = 0$ for some set $N$ of zero Lebesgue measure, and $μ_{α,γ,b,ac}(N) = 0$ for every set $N$ of zero Lebesgue measure. We also denote $μ_{α,γ,b,c} = μ_{α,γ,b,sc} + μ_{α,γ,b,ac}$ for the continuous part and $μ_{α,γ,b,s} = μ_{α,γ,b,pp} + μ_{α,γ,b,sc}$ for the singular part of $μ_{α,γ,b}$.

Given this measure decomposition, we can define the following subsets of $l^2(Z)$:

$$l^2(Z)_{ac} = \{ψ ∈ l^2(Z) : μ_{α,γ,b} = μ_{α,γ,b,ac}\},$$
$$l^2(Z)_{sc} = \{ψ ∈ l^2(Z) : μ_{α,γ,b} = μ_{α,γ,b,sc}\},$$
$$l^2(Z)_{pp} = \{ψ ∈ l^2(Z) : μ_{α,γ,b} = μ_{α,γ,b,pp}\}.$$

Each of these subsets turns out to be a closed subspace, and

$$l^2(Z) = l^2(Z)_{ac} ⊕ l^2(Z)_{sc} ⊕ l^2(Z)_{pp}.$$

We also considers the continuous subspace:

$$l^2(Z)_{c} = l^2(Z)_{ac} ⊕ l^2(Z)_{sc}.$$

The spectrum of the restriction of $H_{α,γ,b}$ to $l^2(Z)_{ac}$ is denoted by $σ_{ac}(H_{α,γ,b})$ and called the absolutely continuous spectrum of $H_{α,γ,b}$. The sets $σ_{sc}(H_{α,γ,b})$, $σ_{pp}(H_{α,γ,b})$, $σ_{c}(H_{α,γ,b})$ are defined similarly and called singular continuous, pure point and continuous spectrum of $H_{α,γ,b}$, respectively. We have $σ(H_{α,γ,b}) = supp \ μ_{α,γ,b}$, $σ_{ac} = supp \ μ_{α,γ,b,ac}$, $σ(H_{α,γ,b,sc}) = supp \ μ_{α,γ,b,sc}$ and $σ(H_{α,γ,b,pp}) = supp \ μ_{α,γ,b,pp}$.

For $[n_1, n_2] = n ∈ Z : n_1 ≤ n ≤ n_2$, denote by $H_{[n_1, n_2]}$ the restriction of $H$ to this interval with zero boundary conditions at $n_1 - 1$ and $n_2 + 1$: that is, $H_{[n_1, n_2]} = P_{[n_1, n_2]}[H]P_{[n_1, n_2]}^*$ where $P_{[n_1, n_2]} : l^2(Z) → l^2([n_1, n_2])$ is the canonical projection, and $P_{[n_1, n_2]}^* : l^2([n_1, n_2]) → l^2(Z)$ is the canonical embedding.

Moreover, for $E ∉ σ(H_{[n_1, n_2]})$ and $n, m ∈ [n_1, n_2]$, let

$$G_{[n_1, n_2]}(n, m; E) = ⟨δ_n, (H_{[n_1, n_2]} - E)^{-1}δ_m⟩,$$

it is called Green function.

We say that $E ∈ R$ is a generalized eigenvalue if equation (2.1) has a non-trival solution $u_E$, called the corresponding generalized eigenfunction, satisfying

$$|u_E| ≤ C(1 + |n|)^δ$$

for suitable finite contents $C$ and $δ$, and every $n ∈ Z$. 

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Definition 4.1.  An irrational number \( b \in \mathbb{T} \) is called Diophantine if there are constants \( c = c(b) > 0 \) and \( r = r(b) > 1 \) such that
\[
|\sin(2\pi nb)| > \frac{c}{|n|^r} \quad \text{for every} \quad n \in \mathbb{Z} \setminus \{0\},
\]
and \( \theta \in \mathbb{T} \) is called resonant respect to above number \( b \) if the relation
\[
|\sin(2\pi(\theta + \frac{n}{2} b))| < e^{-|n|^{\frac{1}{r}}}
\]
holds for infinitely many \( n \in \mathbb{Z} \); otherwise \( \theta \) is called non-resonant.

Remark 4.2.  Lebesgue almost every \( b \in \mathbb{T} \) is Diophantine, and the set of resonant \( \theta \)'s is a dense \( G_\delta \) set (as can be seen directly from the definition) of zero Lebesgue measure (by Borel-Cantelli) so almost every \( \theta \in \mathbb{T} \) is non-resonant respect to \( b \).

Definition 4.3.  Let
\[
P_k(\theta, E) = \det[[H_{\alpha, \theta, b} - E]|_{[0,k-1]}]
\]
and
\[
K = \{k \in \mathbb{Z}_+ : \exists \theta \in \mathbb{T} \text{ with } |P_k(\theta, E)| \geq \frac{1}{\sqrt{2}} e^{kL(E)}\}.
\]

In fact,
\[
det[[H_{\alpha, \theta, b} - E]|_{[0,k-1]}] = \det\begin{pmatrix}
v_\theta(0) - E & 1 \\ 1 & v_\theta(1) - E & 1 \\ & \ddots & \ddots & \ddots \\ & & 1 & v_\theta(k-1) - E
\end{pmatrix},
\]

where \( v_\theta(j) = v(jb + \theta), \; j = 0, \ldots, k-1 \).

Lemma 4.4.  There are coefficients \( b_j, 0 \leq j \leq k \), such that
\[
P_k(\theta, E) = \frac{\sum_{j=0}^{k} b_j(\cos(2\pi(\theta + \frac{k-1}{2} b)))^j}{\prod_{j=0}^{k-1} (1 - \cos(2\pi(\theta + j b)))}.
\]

Proof.  Since \( \cos \) is an even function, denote \( U \) the change of basis \( \delta_j \mapsto \delta_{k-1-j} \), then
\[
U^{-1}H_{\alpha, \theta - \frac{k-1}{2} b, b}|_{[0,k-1]}U = H_{\alpha, \theta - \frac{k-1}{2} b, b}|_{[0,k-1]}.
\]
Thus,
\[
P_k(\theta - \frac{k-1}{2} b, E) = P_k(-\theta - \frac{k-1}{2} b, E).
\]
Denote
\[
Q_k(\theta, E) = P_k(\theta, E) \prod_{j=0}^{k-1} (1 - \cos(2\pi(\theta + j b)))
\]
Due to
\[
\prod_{j=0}^{k-1} (1 - \cos(2\pi(\theta - \frac{k-1}{2} b + j b))) = \prod_{j=0}^{k-1} (1 - \cos(2\pi(-\theta - \frac{k-1}{2} b + j b))).
\]
we obtain
\[
Q_k(\theta - \frac{k-1}{2} b, E) = Q_k(-\theta - \frac{k-1}{2} b, E). \quad (4.3)
\]
Thus, the Fourier expansion of \( \theta \mapsto Q_k(\theta - \frac{k-1}{2} b, E) \) reads

\[
Q_k(\theta, E) = \sum_{j=0}^{k} a_j \cos(2\pi j(\theta + \frac{k-1}{2} b))
\]

since all the sin terms are absent due to (4.3) and the degree obviously does not exceed \( k \). The lemma follows since the linear span of \( \{1, \cos(2\pi x), \cos(2\pi 2x), \ldots, \cos(2\pi kx)\} \) is equal to that of \( \{1, \cos(2\pi x), \cos^2(2\pi x), \ldots, \cos^k(2\pi x)\} \).

**Theorem 4.5** (Kingman 1973). Suppose \((\Omega, \mu, T)\) is ergodic. If \( f_n : \Omega \to \mathbb{R} \) are measurable, obey \( \| f_n \|_\infty \leq n \) and the subadditivity condition

\[
f_{n+m}(\theta) \leq f_n(\theta) + f_m(T^n \theta),
\]

then

\[
\lim_{n \to \infty} \frac{1}{n} f_n(\theta) = \inf_{n \geq 1} \frac{1}{n} \mathbb{E}(f_n)
\]

for \( \mu \)-almost every \( \theta \in \Omega \).

**Lemma 4.6.** For every \( k \in \mathbb{Z}_+ \), at least one of \( k, k+1, k+2 \) belongs to \( K \).

**Proof.** Recall that the transfer matrix \( M_E(k, \theta) \) may be written as

\[
A_n(\theta, E) = \begin{pmatrix}
P_k(\theta, E) & -P_{k-1}(\theta + b, E) \\
P_{k-1}(\theta, E) & -P_{k-2}(\theta + b, E)
\end{pmatrix}.
\]  

(4.4)

Therefore, the statement of lemma follows from Kingman’s subadditive ergodic Theorem 4.5.

When the Lyapunov exponent is positive, on average the transfer matrices have exponentially large norm, and hence some of the entries must be exponentially large. These entries in turn appear in a description of the Green’s function of the operator restricted to a finite interval. Namely, by Cramer’s Rule, we have for \( n_1, n_2 = n_1 + k - 1 \), and \( n \in [n_1, n_2] \),

\[
|G_{[n_1,n_2]}(n_1, n; E)| = \left| \frac{P_{n_2-n_1}(\theta + (n+1)b, E)}{P_k(\theta + n_1 b, E)} \right|,
\]  

(4.5)

\[
|G_{[n_1,n_2]}(n_1, n_2; E)| = \left| \frac{P_{n-n_1}(\theta + n_1 b, E)}{P_k(\theta + n_1 b, E)} \right|.
\]  

(4.6)

**Theorem 4.7** (Furman 1997). Suppose \((\Omega, T)\) is uniquely ergodic. If \( f_n : \Omega \to \mathbb{R} \) are continuous and obey the subadditivity condition \( f_{n+m}(\theta) \leq f_n(\theta) + f_m(T^n \theta) \), then

\[
\lim sup_{n \to \infty} \frac{1}{n} f_n(\theta) \leq \inf_{n \geq 1} \frac{1}{n} \mathbb{E}(f_n)
\]

for every \( \theta \in \Omega \) and uniformly on \( \Omega \).

**Lemma 4.8.** For every \( E \in \mathbb{R} \) and \( \varepsilon > 0 \), there exists \( k(E, \varepsilon) \) such that

\[
|P_k(\theta, E)| < e^{(L(E)+\varepsilon)k}
\]

for every \( k > k(E, \varepsilon) \) and every \( \theta \in \mathbb{T} \).

**Proof.** It is a consequence of equation (4.4) and Theorem 1.7.
**Lemma 4.10.** Fix $E \in \mathbb{R}$ and $\gamma \in \mathbb{R}$. A point $n \in \mathbb{Z}$ will be called $(\gamma, k)$-regular if there exists an interval $[n_1, n_2]$, containing $n$ such that

1. $n_2 = n_1 + k - 1$,
2. $n \in [n_1, n_2]$,
3. $|n - n_i| > \frac{k}{5}$,
4. $|G_{[n_1, n_2]}(n, n_i; E)| < e^{-\gamma|n-n_i|}$.

Otherwise, $n$ is called $(\gamma, k)$-singular.

**Lemma 4.11.** Suppose $n$ is $(L(E) - \varepsilon, k) -$ singular for some $0 < \varepsilon < \frac{L(E)}{4}$ and $k > 4k(E; \frac{\theta}{E}) + 1$. Then, for every $j$ with

$$n - \frac{3}{4}k \leq j \leq n - \frac{3}{4}k + \frac{k + 1}{2},$$

we have that

$$|P_k(\theta + jb, E)| \leq e^{k(L(E) - \frac{\theta}{E})}.$$ 

**Proof.** Since $n$ is $(L(E) - \varepsilon, k) -$ singular, it follows that for every interval $[n_1, n_2]$ of length $k$ containing $n$ with $|n - n_i| > \frac{k}{5}$, we have that

$$|G_{[n_1, n_2]}(n, n_i; E)| \geq e^{-(L(E) - \varepsilon)|n-n_i|}.$$ 

By equation (4.5), this means that

$$|P_{n_2-n}(\theta + (n + 1)b, E)| \geq e^{-(L(E) - \varepsilon)|n-n_i|}.$$ 

We can choose $n_1$ to be equal to the $j$ in question and set $n_2 = j + k - 1$. Then we find, using Lemma 4.8,

$$|P_k(\theta + jb, E)| \leq |P_{j+k-1-n}(\theta + (n + 1)b, E)|e^{(L(E) - \varepsilon)|n-j|}$$

$$\leq e^{(L(E) + \frac{\theta}{E})j + k - 1 - n}e^{(L(E) - \varepsilon)(n-j)}$$

$$= e^{(k-1)L(E) + \varepsilon\left(\frac{j + k - 1 - n}{6} + j - n\right)}$$

$$\leq e^{(k-1)L(E) + \varepsilon\left(\frac{j}{2} - \frac{j}{2}k\right)}$$

$$< e^{k(L(E) - \varepsilon) + \frac{\varepsilon}{6}L(E) - L(E)}$$

$$< e^{k(L(E) - \varepsilon)}.$$ 

**Lemma 4.12.** Suppose $n \in [n_1, n_2] \subseteq \mathbb{Z}$ and $u$ is a solution of the equation $Hu = Eu$. Then,

$$u(n) = -G_{[n_1, n_2]}(n, n_1; E)u(n_1 - 1) - G_{[n_1, n_2]}(n, n_2; E)u(n_2 + 1).$$

In particular, if $u_E$ is a generalized eigenfunction, then every point $n \in \mathbb{Z}$ with $u_E(n) \neq 0$ is $(\gamma, k)$-singular for $k > k_1 = k_1(E, \gamma, \theta, n)$.

**Lemma 4.12.** For every $n \in \mathbb{Z}$, $\varepsilon > 0$, $\tau < 2$, there exists $k_2 = k_2(\theta, b, n, \varepsilon, \tau, E)$ such that for every $k \in K$ with $k > k_2$, we have that

$$m, n \text{ are both } (L(E) - \varepsilon, k) - \text{ singular and } |m - n| > \frac{k+1}{2} \Rightarrow |m - n| > k^{\tau}.$$ 

**Proof.** Assume that $m_1$ and $m_2$ are both $(L(E) - \varepsilon, k) -$ singular with

$$d = m_2 - m_1 > \frac{k}{2}.$$
Let
\[ n_i = m_i - \left\lfloor \frac{3}{4} k \right\rfloor, \quad i = 1, 2. \]

By Lemma 4.4, there is a polynomial \( R_k \) of degree \( k \) such that
\[
P_k(\theta, E) \prod_{j=0}^{k-1} (1 - \cos(2\pi(\theta + jb))) = R_k(\cos(2\pi(\theta + {k-1 \over 2} b))).
\]

Let
\[
\theta_j = \begin{cases} 
\theta + (n_1 + {k-1 \over 2} + j)b, & j = 0, 1, \ldots, \lfloor {k+1 \over 2} \rfloor - 1, \\
\theta + (n_2 + {k-1 \over 2} + j - \lfloor {k+1 \over 2} \rfloor)b, & j = \lfloor {k+1 \over 2} \rfloor, \lfloor {k+1 \over 2} \rfloor + 1, \ldots, k.
\end{cases}
\]

The points \( \theta_0, \theta_1, \ldots, \theta_k \) are distinct. Lagrange interpolation then shows
\[
|R_k(z)| = \left| \sum_{j=0}^k R_k(\cos(2\pi \theta_j)) \prod_{l \neq j} (z - \cos(2\pi \theta_l)) \prod_{l \neq j} \cos(2\pi \theta_j) - \cos(2\pi \theta_l)) \right|.
\]

Due to
\[
\frac{1}{k} \sum_{j=0}^{k-1} \log(1 - \alpha \cos(2\pi(\theta + jb))) \to \log \frac{1 + \sqrt{1 - \alpha^2}}{2}, \quad \text{as} \quad k \to \infty.
\]

there exists \( k_3 \) such that for \( k > k_3 \)
\[
eq e^{k\log \theta 16 \sqrt{1 - \alpha^2} \pi} < | \prod_{j=0}^{k-1} (1 - \alpha \cos(2\pi(\theta + jb))) | < e^{k\log \theta 16 \sqrt{1 - \alpha^2} \pi}.
\]

and by Lemma 4.8 there exists \( k_4 \) such that for \( k > k_4 \)
\[
|P_k(\cos(2\pi \theta_j))| < e^{k(L(E) - {\pi} \over 2)}, \quad j = 0, 1, \ldots, k.
\]

By Lemma 7 in [11], we know that if \( d < k^\tau \) for some \( \tau < 2 \), there exists \( k_5 \) so that for \( k > k_5 \), we have
\[
\frac{| \prod_{l \neq j} (z - \cos(2\pi \theta_l)) |}{| \prod_{l \neq j} (\cos(2\pi \theta_j) - \cos(2\pi \theta_l)) |} \leq e^{\frac{k_5}{\pi}} \text{ for } z \in [-1, 1], \quad 0 \leq j \leq k.
\]

(4.8)

Given \( \tau < 2 \), consider \( k \in \mathcal{K} \) with \( k > \max\{k_3, k_4, k_5\} \) and \( \tilde{\theta} \) with
\[
|P_k(\tilde{\theta})| \geq \frac{1}{\sqrt{2}} e^{kL(E)}.
\]

But assuming \( d < k^\tau \), we also have the following upper bound,
\[
|P_k(\tilde{\theta})| \leq e^{-k\log \theta 16 \sqrt{1 - \alpha^2} \pi} (k + 1)e^{kL(E) - {\pi} \over 2} e^{k\log \theta 16 \sqrt{1 - \alpha^2} \pi} e^{kL(E) - {\pi} \over 2}.
\]

This contradiction shows that \( d < k^\tau \) is impossible.

Then, we provide the detailed proof of Theorem 1.6.

**PROOF OF THEOREM 1.6.** Let \( E(\theta) \) be a generalized eigenvalue of \( H_{\alpha, \theta, b, b} \), and denote the corresponding generalized eigenfunction by \( u_E \). Assume without loss of generality \( u_E(0) \neq 0 \) (otherwise replace 0 by 1). By Lemma 4.11 and Lemma 4.12 if
\[
|n| > \max\{k_1(E, L(E) - \varepsilon, \theta, 0), k_2(\theta, b, 0, \varepsilon, 1.5, E)\} + 1,
\]
the point \( n \) is \((L(E) - \varepsilon, k)\)-regular for some \( k \in \{|n| - 1, |n|, |n| + 1\} \cap \mathcal{K} \neq \emptyset \), since 0 is \((L(E) - \varepsilon, k)\)-singular. Thus, there exists an interval \([n_1, n_2]\) of length \( k \) containing \( n \) such that

\[
\frac{1}{5}(|n| - 1) \leq |n - n_i| \leq \frac{4}{5}(|n| + 1),
\]

and

\[
|G_{[n_1, n_2]}(n, n_i)| < e^{-(L(E) - \varepsilon)|n - n_i|}.
\]

By above inequation and equation (4.7), we obtain that

\[
|u_E(n)| \leq 2C(2|n| + 1)\delta e^{-(L(E) - \varepsilon)(|n| - 1)},
\]

where \( C \) is a constant. This implies exponential decay in the region of positive Lyapunov exponent if \( \varepsilon \) is chosen small enough.

We need a theorem to prove Problem 1.1.

**Theorem 4.13** (Ruelle 1979). Suppose \( A_n \in SL(2, \mathbb{C}) \) obey

\[
\lim_{n \to \infty} \frac{1}{n} ||A_n|| = 0
\]

and

\[
\lim_{n \to \infty} ||A_n \ldots A_1|| = \gamma > 0.
\]

Then there exists a one-dimensional subspace \( V \subseteq \mathbb{R}^2 \) such that

\[
\lim_{n \to \infty} ||A_n \ldots A_1|| = -\gamma \quad \text{for } v \in V \setminus \{0\}
\]

and

\[
\lim_{n \to \infty} ||A_n \ldots A_1|| = \gamma \quad \text{for } v \notin V.
\]

Then, it is easy to prove the problem 1.1.

**PROOF OF PROBLEM 1.1.** By Theorem 4.13, localized states only appear in the region of positive Lyapunov exponent. Therefore, Problem 1.1 is the consequence of Theorem 1.3 and Theorem 1.6.

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