POSITIVE TOPOLOGICAL ENTROPY AND $\ell_1$

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Abstract. We characterize positive topological entropy for quasi-state space homeomorphisms induced from $C^*$-algebra automorphisms in terms of dynamically generated subspaces isomorphic to $\ell_1$. This geometric condition is also used to give a description of the spaces of subspaces isomorphic to $\ell_1$. In particular we obtain a geometric characterization of positive entropy for topological dynamical systems, as well as an analogue for completely positive topological entropy of Glasner and Weiss’s combinatorial characterization of completely positive Kolmogorov-Sinai entropy.

1. Introduction

In [8] E. Glasner and B. Weiss showed that if a homeomorphism from a compact metric space $X$ to itself has zero topological entropy, then so does the induced homeomorphism on the space of probability measures on $X$ with the weak* topology. One of the two proofs they gave of this striking result established a remarkable connection between topological dynamics and the local theory of Banach spaces. The key geometric fact is the exponential dependence of $k$ on $n$ given an approximately isometric embedding of $\ell_1^n$ into $\ell_\infty^k$, which they deduced from the work of T. Figiel, J. Lindenstrauss, and V. D. Milman on almost Hilbertian sections of unit balls in Banach spaces [6].

The first author showed in [10] that Glasner and Weiss’s geometric approach can be conceptually simplified from a functional-analytic viewpoint using Voiculescu-Brown entropy (see Remark 3.10 in [10]) and also more generally applied to show that if an automorphism of a separable exact $C^*$-algebra has zero Voiculescu-Brown entropy then the induced homeomorphism on the quasi-state space has zero topological entropy. In this case the crucial Banach space fact is the exponential dependence of $k$ on $n$ given an approximately isometric embedding of $\ell_1^n$ into the matrix $C^*$-algebra $M_k$ [10, Lemma 3.1], which can be deduced from the work of N. Tomczak-Jaegermann on the Rademacher type 2 constants of Schatten $p$-classes [11]. Following a procedure similar to that of [8] but in a matrix setting, the key Banach space map in [10] is constructed by passing suitable elements in the $C^*$-algebra through a matrix algebra $M_k$ in a completely positive approximation and then evaluating on states which register the entropic growth, thus producing a map from the trace class $C_1^k$ to $\ell_\infty^k$ which, after composing with a suitable projection and taking the dual, yields the desired approximately isometric embedding of $\ell_1^n$ into $M_k$. 

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As observed in Remark 3.11 of [10], Lemma 3.1 of [10] can also be used directly at the $C^*$-algebra level with respect to the completely positive approximations that appear in the definition of Voiculescu-Brown entropy. Indeed if a finite set $\Omega$ of elements in the $C^*$-algebra forms a standard basis for an isomorphic copy of $\ell^1_1$ then a completely positive approximation to $\Omega$ of suitably fine tolerance will yield an isomorphic embedding of $\ell^1_1$ into the associated matrix algebra with a similar isomorphism constant (see Lemma 3.1 in Section 3). We may thus ask if the geometric phenomena in [8, 10] connected to positive topological entropy on the quasi-state space can be entirely captured within the Banach space structure of the $C^*$-algebra itself. The principal aim of this note is to answer this question in the affirmative. Specifically, we show that positive entropy on the quasi-state space can be characterized by the existence of a subspace of the $C^*$-algebra isomorphic to $\ell^1_1$ which is generated in a canonical way by a single element along a set of iterates of positive density. This forms the content of Section 2, the first of the two sections comprising the main body of the paper.

In Section 3 we apply arguments similar to those from Section 2 in conjunction with results from [10] to obtain a geometric characterization of the topological Pinsker algebra. This dynamical object is the $C^*$-algebraic manifestation of the maximal zero entropy factor of a topological dynamical system [3] and is an analogue of the Pinsker $\sigma$-algebra in ergodic theory. We show that the self-adjoint elements of the topological Pinsker algebra are precisely those which do not dynamically generate in a canonical way a subspace isomorphic to $\ell^1_1$ along a set of iterates of positive density. As a corollary we obtain a geometric characterization of positive entropy for topological dynamical systems which answers a question posed by V. Pestov after the first author’s talk at the Winter 2002 CMS Meeting in Ottawa, as well as an analogue for completely positive topological entropy of Glasner and Weiss’s combinatorial characterization of completely positive Kolmogorov-Sinai entropy [8].

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2. A geometric characterization of positive entropy on the quasi-state space

We begin by establishing notation. Let $A$ be a $C^*$-algebra and $\alpha$ an automorphism of $A$. Following the notation of [10], we denote by $S_\alpha$ the homeomorphism of the closed unit ball of $A^*$ (with the weak$^*$ topology) given by $S_\alpha(\sigma) = \sigma \circ \alpha$, and by $\tilde{T}_\alpha$ the restriction of $S_\alpha$ to the quasi-state space $Q(A)$, i.e., the weak$^*$ compact convex set of positive linear functionals on $A$ of norm at most one. Also, when $A$ is unital we denote by $T_\alpha$ the restriction of $\tilde{T}_\alpha$ to the state space $S(A)$, i.e., the weak$^*$ compact
convex set of positive unital linear functionals on $A$. The real linear subspaces of self-adjoint elements of $A$ and $A^*$ will be denoted by $A_{sa}$ and $(A^*)_{sa}$, respectively.

Let $X$ be a compact topological space and $T : X \to X$ a homeomorphism. Let $d$ be a pseudo-metric on $X$. A set $E \subseteq X$ is said to be $(n, \varepsilon)$-separated (with respect to $T$ and $d$) if for every $x, y \in E$ with $x \neq y$ there exists a $0 \leq k \leq n - 1$ such that $d(T^k x, T^k y) > \varepsilon$. We denote by $\text{sep}_n(T, \varepsilon)$ the largest cardinality of an $(n, \varepsilon)$-separated set, and we define

$$h_d(T) = \sup_{\varepsilon > 0} \limsup_{n \to \infty} \frac{1}{n} \log \text{sep}_n(T, \varepsilon).$$

If $d$ is a metric compatible with the topology on $X$ then $h_d(T)$ agrees with the topological entropy $h_{\text{top}}(T)$ defined as the supremum of $h_{\text{top}}(T, \mathcal{U})$ over all finite open covers $\mathcal{U}$ of $X$, where

$$h_{\text{top}}(T, \mathcal{U}) = \lim_{n \to \infty} \frac{1}{n} \log N(\mathcal{U} \vee T^{-1} \mathcal{U} \vee \cdots \vee T^{-(n-1)} \mathcal{U})$$

and $N(\cdot)$ denotes the smallest cardinality of a subcover. See [5, 13] for references on topological entropy.

We will have occasion to consider Banach spaces over both the real and complex numbers. If the scalar field is not clear from the context then it will be assumed to be the complex numbers unless the notation is tagged with an $\mathbb{R}$. In particular the Banach spaces $\ell_1$ over some index set which appear in the statements of our results are complex, although this is not essential as the analogous statements using real scalars are also valid.

**Definition 2.1.** Let $X$ be a complex Banach space and $\alpha : X \to X$ an isometric isomorphism. Let $x \in X$. We say that an infinite subset $I \subseteq \mathbb{Z}$ is an $\ell_1$ isomorphism set for $x$ if there is an isomorphism from $\ell_1^I$ to $\text{span}\{\alpha^i(x) : i \in I\}$ sending the standard basis element of $\ell_1^I$ associated with $i \in I$ to $\alpha^i(x)$.

Recall that the density of a set $J \subseteq \mathbb{Z}$ is defined as the limit

$$\lim_{n \to \infty} \frac{|J \cap \{-n, -n + 1, \ldots, n\}|}{2n + 1}$$

if it exists. If $X$ and $Y$ are Banach spaces and $\Gamma : X \to Y$ is an isomorphism then we say that $\Gamma$ is a $K$-isomorphism if $\|\Gamma\|\|\Gamma^{-1}\| \leq K$.

**Theorem 2.2.** Let $A$ be a separable $C^*$-algebra and $\alpha$ an automorphism of $A$. Then the following are equivalent:

1. $h_{\text{top}}(\tilde{T}_\alpha) = \infty$,
2. there exist an $a \in A$, constants $K, d > 0$, a sequence $\{n_k\}_{k \in \mathbb{N}}$ in $\mathbb{N}$ going to $\infty$, and sets $I_k \subseteq \{0, 1, \ldots, n_k - 1\}$ of cardinality at least $dn_k$ for each $k \in \mathbb{N}$ such that, for each $k \in \mathbb{N}$, the linear map from $\ell_1^{I_k}$ to $\text{span}\{\alpha^i(a) : i \in I_k\}$ which sends the standard basis element of $\ell_1^{I_k}$ associated with $i \in I_k$ to $\alpha^i(a)$ is a $K$-isomorphism,
3. there exists an $a \in A$ with an $\ell_1$ isomorphism set of positive density.
We may moreover take \( a \) in (2) and (3) to be in any given countable total subset \( W \) of \( A_{sa} \). When \( A \) is unital then the conditions (1)–(3) are also equivalent to

\[(4) \quad h_{top}(T_a) = \infty.\]

**Proof.** \((1) \Rightarrow (2)\). Let \( W = \{a_1, a_2, a_3, \ldots \} \) be a countable total subset of \( A_{sa} \). Without loss of generality we may assume that \( W \) is a subset of the unit ball \( B_1(A_{sa}) \). Define a metric on \( B_1(A^*) \) by

\[d(\sigma, \omega) = \sum_{j=1}^{\infty} 2^{-j} ||(\sigma - \omega)(a_j)||.\]

Then there exist \( \varepsilon > 0, \lambda > 0 \), a sequence \( \{n_k\}_{k\in \mathbb{N}} \) in \( \mathbb{N} \) going to \( \infty \), and an \( (n_k, 4\varepsilon) \)-separated subset \( E_k \) of \( Q(A) \) with cardinality at least \( e^{\lambda n_k} \) for each \( k \in \mathbb{N} \). Take an \( N > 0 \) with \( 2^{-N} < \varepsilon \) and define the pseudo-metric

\[d'(\sigma, \omega) = \sum_{j=1}^{N} 2^{-j} ||(\sigma - \omega)(a_j)||\]

on \( B_1(A^*) \). Then \( |d(\sigma, \omega) - d'(\sigma, \omega)| < \varepsilon \) for all \( \sigma, \omega \in Q(A) \), and so \( E_k \) is \( (n_k, 2\varepsilon) \)-separated with respect to \( d' \). In particular for distinct \( f, g \in E_k \) there exist \( 1 \leq j \leq N \) and \( 0 \leq i \leq n_k - 1 \) such that \( |(f - g)(\alpha^i(a_j))| > \varepsilon \).

Define next an \( \mathbb{R} \)-linear map \( \phi : (A^*)_{sa} \rightarrow (\ell_\infty^{N\times n_k})_{\mathbb{R}} \) by \( (\phi(f))_{ji} = f(\alpha^i(a_j)) \), where the standard basis of \( (\ell_\infty^{N\times n_k})_{\mathbb{R}} \) is indexed by \( \{1, \ldots, N\} \times \{0, \ldots, n_k - 1\} \). Then \( \phi(E_k) \) is \( \varepsilon \)-separated. As in the proof of Proposition 2.1 of \( \cite{[8]} \), there are constants \( d, \delta > 0 \) depending only on \( \varepsilon \) and \( \lambda \) such that for all sufficiently large \( k \) there is a set \( J_k \subseteq \{1, \ldots, N\} \times \{0, \ldots, n_k - 1\} \) with

\[(i) \quad |J_k| \geq dNn_k, \quad \text{and} \]

\[(ii) \quad \pi'((\phi(B_1((A^*)_{sa})))) \supseteq B_\delta((\ell_\infty^N)_{\mathbb{R}}), \quad \text{where} \quad \pi : (\ell_\infty^{N\times n_k})_{\mathbb{R}} \rightarrow (\ell_\infty^N)_{\mathbb{R}} \text{ is the canonical projection.} \]

Then for any such sufficiently large \( k \) there exist some \( 1 \leq j(k) \leq N \) and a set \( I_k \subseteq \{0, \ldots, n_k - 1\} \) such that \( |I_k| \geq dn_k \) and \( \{j(k)\} \times I_k \subseteq J_k \). Consequently \( \pi'((\phi(B_1((A^*)_{sa})))) \supseteq B_\delta((\ell_\infty^N)_{\mathbb{R}}) \), where \( \pi' : (\ell_\infty^{N\times n_k})_{\mathbb{R}} \rightarrow (\ell_\infty^N)_{\mathbb{R}} \) is the canonical projection. The dual \( (\pi' \circ \phi)^* \) is an injection of \( (\ell_\infty^N)^* = (\ell_1^N)_{\mathbb{R}} \) into \( ((A^*)^*)^* = (A_{sa})^* \) and the norm of the inverse of this injection is bounded above by \( \delta^{-1} \). Notice that \( A_{sa} \subseteq ((A^*)_{sa})^* \), and from our definition of \( \phi \) it is clear that \( (\pi' \circ \phi)^* \) sends the standard basis element of \( \ell_1^N \) associated with \( i \) in \( I_k \) to \( \alpha^j(a_{j(k)}) \).

Let \( \Gamma_k \) be the complexification of the map \( \pi' \circ \phi^* : (\ell_1^N)_{\mathbb{R}} \rightarrow span_{\mathbb{R}} \{\alpha^j(a_{j(k)}) : j \in I_k \} \subseteq A \). Since \( a_{j(k)} \) is self-adjoint, the inverse of \( \Gamma_k \) evidently has norm no bigger than \( K := 2\delta^{-1} \). Since \( 1 \leq j(k) \leq N \) there is a \( j_0 \) such that \( j(k) = j_0 \) for infinitely many \( k \). By taking a subsequence of \( \{n_k\}_{k\in \mathbb{N}} \) we may assume that \( j(k) = j_0 \) for all \( k \). Now set \( a = a_{j(0)} \).

\((2) \Rightarrow (1)\). Multiplying \( a \) by a scalar we may assume that \( ||a|| = 1 \). Take a dense sequence \( a_1 = a, a_2, a_3, \ldots \) in the unit ball \( B_1(A) \) and define a metric on \( B_1(A^*) \) in the same way as in the first part of the proof. Denote \( span\{\alpha^i(a) : i \in I_k\} \) by \( V_k \), and let \( \Gamma_k \) denote the linear map from \( \ell_1^N \) to \( V_k \) sending the standard basis element of \( \ell_1^N \).
associated with \( i \in I_k \) to \( \alpha^i(a) \). For each \( f \in (\ell^k)^* \) we have \((\Gamma^{-1}_k)^*(f) \in (V_k)^* \) and \( \| (\Gamma^{-1}_k)^*(f) \| \leq K \| f \| \). By the Hahn-Banach theorem we may extend \((\Gamma^{-1}_k)^*(f)\) to an element in \( A^* \), which we will still denote by \((\Gamma^{-1}_k)^*(f)\). Let \( 0 < \varepsilon < (2K)^{-1} \), and let \( M = \lfloor (2K\varepsilon)^{-1} \rfloor \) be the largest integer no greater than \((2K\varepsilon)^{-1}\). Let \( \{g_i : i \in I_k\} \) be the standard basis of \((\ell^k)^* = \ell^k_\alpha\). For each \( f \in \{1, \ldots, M\}^{I_k} \) set \( \tilde{f} = \sum_{i \in I_k} 2f(i)\varepsilon g_i \).

Then \( f' := (\Gamma^{-1}_k)^*(\tilde{f}) \) is in \( B_1(A^*) \).

We claim that the set \( \{ f' : f \in \{1, \ldots, M\}^{I_k} \} \) is \((n_k, \varepsilon)\)-separated. Suppose \( f, g \in \{1, \ldots, M\}^{I_k} \) and \( f(i) < g(i) \) for some \( i \in I_k \). Then

\[
\begin{align*}
    d(S^*_\alpha(f'), S^*_\alpha(g')) &\geq |(S^*_\alpha(f'))(a) - (S^*_\alpha(g'))(a)| \\
    &\geq |f'(\alpha^i(a)) - g'(\alpha^i(a))| \\
    &\geq 2|g(i) - f(i)|\varepsilon \\
    &> \varepsilon.
\end{align*}
\]

This establishes our claim. Thus \( \text{sep}_{n_k}(S_\alpha, \varepsilon) \geq M^{I_k} \geq M^{dn_k} \). It follows that \( h_{\text{top}}(S_\alpha) \geq d \log M \). Letting \( \varepsilon \to 0 \) we get \( h_{\text{top}}(S_\alpha) = \infty \). Thus \( h_{\text{top}}(\tilde{T}_\alpha) = \infty \) by Lemma 3.3 of [10].

(2) \(\Rightarrow\) (3). Let \( Y_\alpha \) be the collection of sets \( I \subseteq \mathbb{Z} \) such that the linear map from \( \ell_1^I \) to \( \text{span}\{\alpha^i(a) : i \in I\} \) which sends the standard basis element of \( \ell_1^I \) associated with \( i \in I \) to \( \alpha^i(a) \) is a \( K \)-isomorphism. If we identify the subsets of \( \mathbb{Z} \) with elements of \( \{0, 1\}^\mathbb{Z} \) via their characteristic functions then \( Y_\alpha \) is a closed shift-invariant subset of \( \{0, 1\}^\mathbb{Z} \). It follows from the second paragraph of the proof of Theorem 3.2 in [8] that \( Y_\alpha \) has an element \( J \) with density at least \( d \). Clearly \( J \) is an \( \ell_1 \) isomorphism set for \( \alpha \).

(3) \(\Rightarrow\) (2). This is immediate in view of the fact that \( \alpha \) is isometric.

Finally, we note that the equivalence (1) \(\Leftrightarrow\) (4) in the unital case follows from Lemmas 2.2 and 3.3 of [10]. \(\Box\)

**Remark 2.3.** (i) It is easily seen that for the countable subset \( W \subseteq A_{sa} \) we need in fact only assume that \( \bigcup_{j \in \mathbb{Z}} \alpha^j(W) \) is total in \( A \).

(ii) It is desirable to be able to relax the condition \( W \subseteq A_{sa} \) to \( W \subseteq A \). However, we have been unable to determine if this is possible.

(iii) The proof of Theorem 3.4 in [10] uses the fact that the linear maps in the definition of Voiculescu-Brown entropy are self-adjoint. Using Theorem 2.2 above and Lemma 3.1 of [10] (or more precisely Lemma 3.1 in Section 3 below) one can prove Theorem 3.4 of [10] using only the fact that these linear maps are contractions, and hence more in the spirit of a Banach space approach.

(iv) The implication (1) \(\Rightarrow\) (2) is useful for proving \( h_{\text{top}}(\tilde{T}_\alpha) = 0 \) for certain \( \alpha \). See Corollary 2.4 below for an example.

**Corollary 2.4.** Let \( \sigma \) be a permutation of \( \mathbb{Z} \). Let \( \sigma_* \) be the corresponding free permutation automorphism of the full free group \( C^*\)-algebra \( C^*(F_\mathbb{Z}) \) sending \( u_j \) to \( u_{\sigma(j)} \), where \( \{u_j\}_{j \in \mathbb{Z}} \) is the set of canonical unitaries of \( C^*(F_\mathbb{Z}) \). Then \( h_{\text{top}}(T_{\sigma_*}) = \infty \) if and only if \( \sigma \) has an infinite orbit.
Proof. The “if” part follows from the proof of Proposition 2.4 of \cite{10}. For the “only if” part, suppose that \( \sigma \) has no infinite orbits. For each \( g \in F_Z \) let \( u_g \) be the corresponding unitary in \( C^*(F_Z) \). Then by taking \( W = \{(u_g + u_g^*)/2, (u_g - u_g^*)/(2i) : g \in F_Z \} \) in Theorem \( \ref{2.2} \) we get \( h_{\text{top}}(T_{\sigma_g}) = 0 \), as desired. \( \square \)

3. A geometric characterization of the topological Pinsker algebra

In \cite{3} F. Blanchard and Y. Lacroix introduced the maximal zero entropy factor of a topological dynamical system. This was called the topological Pinsker factor in \cite{7} and is defined as follows. Let \( X \) be a compact metric space and \( T : X \to X \) a homeomorphism. A pair \( (x, y) \in X \times X \) with \( x \neq y \) is called an entropy pair if \( h_{\text{top}}(T, \mathcal{U}) > 0 \) for every two-element open cover \( \mathcal{U} = \{U, V\} \) with \( x \in \text{int}(X \setminus U) \) and \( y \in \text{int}(X \setminus V) \) \cite{11}. The topological Pinsker factor is the quotient system arising from the closed \( T \)-invariant equivalence relation on \( X \) generated by the collection of entropy pairs.

Let \( \alpha_T \) be the automorphism of \( C(X) \) given by \( \alpha_T(f) = f \circ T \) for all \( f \in C(X) \). The topological Pinsker factor corresponds at the \( C^* \)-algebra level to the \( \alpha_T \)-invariant \( C^* \)-subalgebra \( P_{X,T} \) of \( C(X) \) whose elements are those functions \( f \in C(X) \) which satisfy \( f(x) = f(y) \) for every entropy pair \( (x, y) \). We refer to \( P_{X,T} \) as the topological Pinsker algebra. It is an analogue of the Pinsker \( \sigma \)-algebra in ergodic theory.

We also refer to the real Banach algebra \( (P_{X,T})_{\text{sa}} \) of self-adjoint elements of \( P_{X,T} \) as the real topological Pinsker algebra. The goal of this section is to obtain a geometric description of the elements of \( (P_{X,T})_{\text{sa}} \) and \( P_{X,T} \).

We will employ the notation used in Section 2 of \cite{10} for the terms involved in the definition of Voiculescu-Brown entropy \cite{31}, in particular the completely positive rank \( \text{rcp}(\Omega, \delta) \) and the local Voiculescu-Brown entropy \( h_l(\alpha, \Omega) \). For other notation and terminology see the beginning of Section \ref{2}

The following lemma essentially appeared in Remark 3.11 of \cite{10}.

**Lemma 3.1.** Let \( A \) be an exact \( C^* \)-algebra. Let \( a_1, \ldots, a_n \in A \) and suppose that the linear map \( \Gamma : \ell^1 \to \text{span}\{a_1, \ldots, a_n\} \) sending the \( i \)th standard basis element of \( \ell^1 \) to \( a_i \) for each \( i = 1, \ldots, n \) is an isomorphism. Let \( \delta > 0 \) be such that \( \delta < \|\Gamma^{-1}\|^{-1} \). Then

\[
\log \text{rcp}(\{a_1, \ldots, a_n\}, \delta) \geq nc\|\Gamma\|^{-2}\|\Gamma^{-1}\|^{-1} - \delta^2
\]

where \( c > 0 \) is a universal constant.

**Proof.** Let \( \pi : A \to \mathcal{B}(\mathcal{H}) \) be a faithful \( * \)-representation, and suppose \( (\phi, \psi, B) \in \text{CPA}(\pi, \{a_1, \ldots, a_n\}, \delta) \). For any linear combination \( \sum c_ia_i \) of the \( a_i \)'s we have

\[
\left\| \sum c_ia_i \right\| \leq \left\| \pi \left( \sum c_ia_i \right) - (\psi \circ \phi) \left( \sum c_ia_i \right) \right\| + \left\| (\psi \circ \phi) \left( \sum c_ia_i \right) \right\|
\]

\[
\leq \delta \sum |c_i| + \left\| \phi \left( \sum c_ia_i \right) \right\|
\]

\[
\leq \delta \|\Gamma^{-1}\| \left\| \sum c_ia_i \right\| + \left\| \phi \left( \sum c_ia_i \right) \right\|
\]

and so \( \left\| \phi \left( \sum c_ia_i \right) \right\| \geq (1 - \delta \|\Gamma^{-1}\|) \left\| \sum c_ia_i \right\| \). Thus, since \( \phi \) is contractive, its restriction to \( \text{span}\{a_1, \ldots, a_m\} \) is a \((1 - \delta \|\Gamma^{-1}\|)^{-1}\)-isomorphism onto its image in
for some universal constant $c > 0$. For a function $f \in C(X)$ where $X$ is a compact Hausdorff space, we denote by $d_f$ the pseudo-metric on $X$ given by

$$d_f(x, y) = |f(x) - f(y)|$$

for all $x, y \in X$.

**Theorem 3.2.** Let $X$ be a compact metric space and $T : X \to X$ a homeomorphism. Then for any $f \in C(X)_{\text{sa}}$ the following are equivalent:

1. $ht(\alpha_T, \{f\}) > 0$,
2. there exists an entropy pair $(x, y) \in X \times X$ with $f(x) \neq f(y)$,
3. $h_d(T) > 0$,
4. there exist $K, d > 0$, a sequence $\{n_k\}_{k \in \mathbb{N}}$ in $\mathbb{N}$ going to $\infty$, and sets $I_k \subseteq \{0, 1, \ldots, n_k - 1\}$ of cardinality at least $dn_k$ for each $k \in \mathbb{N}$ such that, for each $k \in \mathbb{N}$, the linear map from $\ell^1_k$ to $\text{span}\{\alpha^i_T(f) : i \in I_k\}$ which sends the standard basis element of $\ell^1_k$ associated with $i \in I_k$ to $\alpha^i_T(f)$ is a $K$-isomorphism,
5. $f$ has an $\ell_1$ isomorphism set of positive density.

**Proof.** (1)⇒(2)⇒(3). These implications follow from the proofs of Theorem 4.3 and Lemma 4.2, respectively, in [10].

(3)⇒(4)⇒(5). Here we can apply the same arguments as in the proofs of the respective implications (1)⇒(2)⇒(3) in Theorem 2.2.

(5)⇒(1). By assumption there exists a set $I \subseteq \mathbb{Z}$ of density greater than some $d > 0$ and an isomorphism $\Gamma : \ell^1_1 \to \text{span}\{\alpha^i_T(f) : i \in I\}$ sending the standard basis element of $\ell^1_1$ associated with $i \in I$ to $\alpha^i_T(f)$. Let $0 < \delta < \|\Gamma^{-1}\|^{-1}$. By Lemma 3.1 for every $n \in \mathbb{N}$ we have

$$\text{log rcp}(\{\alpha^i_T(f) : i \in I \cap \{−n, −n + 1, \ldots, n\}\}, \delta) \geq |I \cap \{−n, −n + 1, \ldots, n\}| \cdot c \cdot \|\Gamma\|^{-2}(\|\Gamma^{-1}\|^{-1} - \delta)^2$$

for some universal constant $c > 0$. Since $I$ has density greater than $d$ and

$$\text{rcp}(\{f, \alpha_T(f), \ldots, \alpha^{2n}_T(f)\}, \delta) = \text{rcp}(\{\alpha^{-n}_T(f), \alpha^{-n+1}_T(f), \ldots, \alpha^n_T(f)\}, \delta),$$

we infer that

$$\text{log rcp}(\{f, \alpha_T(f), \ldots, \alpha^{2n}_T(f)\}, \delta) \geq \text{d}(2n + 1)c\|\Gamma\|^{-2}(\|\Gamma^{-1}\|^{-1} - \delta)^2$$

for all sufficiently large $n \in \mathbb{N}$. Hence $ht(\alpha_T, \{f\}) > 0$. □

**Remark 3.3.** (i) By restricting to the $\alpha_T$-invariant unital $C^\ast$-subalgebra of $C(X)$ generated by $f$ (which is separable and therefore has metrizable pure state space) we need only assume that $X$ is a compact Hausdorff space for the equivalences (1)⇔(3)⇔(4)⇔(5) in Theorem 3.2.

(ii) Since $h_{\text{top}}(T_{\alpha_T}) = \infty$ if and only if $h_{\text{top}}(T) > 0$ (see [8] and [10]) if and only if there is an entropy pair in $X \times X$ [1], Theorem 3.2 shows that whenever $A$ is unital
and commutative and $Y \subseteq A_{sa}$ generates $A$ as a unital $C^*$-algebra (or, equivalently, separates the pure states of $A$) we can require $a \in Y$ in conditions (2) and (3) in Theorem 2.2.

**Corollary 3.4.** Let $A$ be a unital $C^*$-algebra and $\alpha$ an automorphism of $A$. Let $\alpha'$ be the automorphism of $C(S(A))$ given by $\alpha'(f)(\sigma) = f(\sigma \circ \alpha)$ for all $f \in C(S(A))$ and $\sigma \in S(A)$. Let $a \mapsto \bar{a}$ be the order isomorphism from $A_{sa}$ to $C(S(A))$ (or, equivalently, separates the pure states of $A$). Let $a \in A_{sa}$. Then $\text{ht}(\alpha', \{\bar{a}\}) > 0$ implies $\text{ht}(\alpha, \{a\}) > 0$.

**Proof.** The map $a \mapsto \bar{a}$ is an isomorphism of Banach spaces which is isometric on $A_{sa}$ and conjugates $\alpha$ to $\alpha'|_{C(S(A))}$, and so we obtain the conclusion from the implication (1) $\Rightarrow$ (4) in Theorem 3.2 and an appeal to Lemma 3.1. \qed

**Theorem 3.5.** The real topological Pinsker algebra is equal to the set of all $f \in C(X)_{sa}$ which do not have an $\ell_1$ isomorphism set of positive density.

**Proof.** By Theorem 4.3 of [10] a function $f \in C(X)$ lies in the topological Pinsker algebra if and only if $\text{ht}(\alpha_T, \{f\}) > 0$, and so an appeal to Theorem 3.2 yields the result. \qed

Since $P_{X,T} = (P_{X,T})_{sa} + i(P_{X,T})_{sa}$ we obtain the following corollary.

**Corollary 3.6.** The topological Pinsker algebra is equal to the set of all $f \in C(X)$ whose real and imaginary parts both lack an $\ell_1$ isomorphism set of positive density.

Since positive topological entropy implies the existence of an entropy pair [1], the topological Pinsker algebra is equal to $C(X)$ precisely when $T$ has zero topological entropy and is equal to the scalars precisely when the system $(X, T)$ has completely positive entropy (which means that every non-trivial factor has positive topological entropy [2]). We thus also have the following two corollaries.

**Corollary 3.7.** The homeomorphism $T$ has positive topological entropy if and only if there is an $f \in C(X)$ with an $\ell_1$ isomorphism set of positive density.

**Corollary 3.8.** The system $(X, T)$ has completely positive entropy if and only if every non-constant $f \in C(X)_{sa}$ has an $\ell_1$ isomorphism set of positive density.

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