Solving Pure Yang-Mills in 2 + 1 Dimensions

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We analytically compute the spectrum of the spin zero glueballs in the planar limit of pure Yang-Mills theory in 2+1 dimensions. The new ingredient is provided by our computation of a new non-trivial form of the ground state wave-functional. The mass spectrum of the theory is determined by the zeroes of Bessel functions, and the agreement with large N lattice data is excellent.

The understanding of the non-perturbative dynamics of Yang-Mills theory is one of the grand problems of theoretical physics. In this letter we announce new analytical results pertaining to the spectrum of the spin zero glueballs of 2+1 dimensional Yang-Mills theory \[\text{YM}_2\text{+1}\]. This theory is expected on many grounds to share the essential features of its 3+1 dimensional cousin, such as asymptotic freedom and confinement, yet is distinguished by the existence of a dimensionful coupling constant. Here, we determine the ground state wave-functional in the plaquette. The new ingredient is our computation of a new theoretical results pertaining to the spectrum of the spin zero Yang-Mills theory in 2+1 dimensions. The new ingredient is provided by our computation of a new non-trivial form of the ground state wave-functional. The mass spectrum of the theory is determined by the zeroes of Bessel functions, and the agreement with large N lattice data is excellent.

The definition of $M$ implies a holomorphic invariance
\[
M(z, \bar{z}) \rightarrow M(z, \bar{z})h(\bar{z})
\]
\[
M'(z, \bar{z}) \rightarrow h(z)M'(z, \bar{z})
\]

where $h(z)$ is an arbitrary unimodular complex matrix whose matrix elements are independent of $\bar{z}$. This is distinct from the original gauge transformation, since it acts as right multiplication rather than left and is holomorphic. Under the holomorphic transformation, the gauge invariant variable $H$ transforms homogeneously
\[
H(z, \bar{z}) \rightarrow h(z)H(z, \bar{z})h^\dagger(\bar{z}).
\]

The theory written in terms of the gauge invariant $H$ fields will have its own local (holomorphic) invariance. The gauge fields, and the Wilson loop variables, know nothing about this extra invariance. We will deal with this by requiring that the physical state wave functionals be holomorphically invariant.

One of the most extraordinary properties of this parameterization is that the Jacobian relating the measures on the space of connections $C$ and on the space of gauge invariant variables $H$ can be explicitly computed
\[
d\mu[C] = \sigma d\mu[H]e^{2cA SWZW[H]}
\]

where $c_A$ is the quadratic Casimir in the adjoint representation of $SU(N)$ ($c_A = N$), $\sigma$ is a constant determinant factor and
\[
S_{WZW}(H) = -\frac{1}{12} \int d^3 x \ W^{\rho\sigma\lambda} W_\rho H_{\sigma} H_\lambda^{-1} \partial H_{\rho} \partial H_{\sigma} \partial H_{\lambda}
\]
is the level-$c_A$ $SU(N)$ Wess-Zumino-Witten action, which is both gauge and holomorphic invariant. Thus the inner product may be written as an overlap integral of gauge invariant wave functionals with non-trivial measure
\[
\langle 1|2 \rangle = \int d\mu[H]e^{2cA SWZW[H]} \Phi^*_1 \Psi_2.
\]
holomorphic-covariant derivative is given by

\[ J = \frac{c_A}{\pi} \partial_z HH^{-1} \]  

which transforms as a holomorphic connection

\[ J \mapsto hJh^{-1} + \frac{c_A}{\pi} \partial_z hh^{-1}. \]

Note that \( \bar{\partial} J \) transforms homogeneously, and a holomorphic-covariant derivative is given by

\[ D^{ab} = \delta^{ab} + i \frac{\pi}{c_A} f^{abc} J^c. \]

The standard \( YM_{2+1} \) Hamiltonian

\[ \mathcal{H}_{YM} = \int Tr \left( g_{YM}^2 E_i^2 + \frac{1}{g_{YM}^2} B^2 \right) \]

can be also explicitly rewritten in terms of gauge invariant variables. The collective field form \( \bar{\Psi} \) of this Hamiltonian (which we will refer to as the Karabali-Nair Hamiltonian) can be easily appreciated from its explicit form in terms of the currents as follows

\[ \mathcal{H}_{KN}[J] = T + V = m \left( \int_x J^a(x) \frac{\delta}{\delta J^a(x)} + \int_{x,y} \Omega_{ab}(x,y) \frac{\delta}{\delta J^a(x)} \frac{\delta}{\delta J^b(y)} \right) + \frac{\pi}{mc_A} \int_x \bar{\partial} J^a \bar{\partial} J^a \]

where

\[ m = \frac{g_{YM}^2 c_A}{2\pi}, \quad \Omega_{ab}(x,y) = \frac{c_A}{\pi^2} \frac{\delta_{ab}}{(x-y)^2} - \frac{i f^{abc} J^c(x)}{\pi (x-y)}. \]

Interpreted as a collective field theory, one can expect to compute, at large \( N \), correlators of gauge invariant operators. Note that the magnetic field is

\[ B = -2M^{1-1} \bar{\partial}(\partial HH^{-1}) M^1 = -\frac{2\pi}{c_A} M^{1-1} \bar{\partial} J M^1. \]

The derivation of this Hamiltonian involves carefully regulating certain divergent expressions in a gauge invariant manner \( \bar{\Psi}, \bar{\Psi} \). We note that the scale \( m \) is essentially the \( \hbar \) coupling.

The purpose of this letter is to determine masses of some of the lowest lying glueball states. To do so, we wish to determine the form of the vacuum wave functional and make use of the planar limit. Accordingly, we take the following ansatz for the vacuum wave functional

\[ \Psi_0 = \exp \left( -\frac{\pi}{2c_A m^2} \int \bar{\partial} J K(L) \bar{\partial} J + \ldots \right). \]

This form of the wavefunctional is explicitly gauge and holomorphic invariant. The kernel \( K \) is a formal Taylor expansion of \( L = (D \bar{\partial} + \bar{\partial} D)/2m^2 \), while the ellipsis contains terms higher order in \( \bar{\partial} J \) (or \( B \)). This wavefunctional has the form of a “generalized coherent state” appropriate to large \( N \) \( \bar{\Psi} \), but its form is not completely dictated by large \( N \) counting. The form of the ansatz, as we shall see, is sufficient to capture the mass spectrum of gauge invariant states, which we will probe using local operators. The large \( N \) limit ensures that these states are non-interacting, but we are also neglecting the size of the states by using local probes. (For further details on these points, see Ref. \( \bar{\Psi} \)).

In order to be physically sensible, \( K \) should have certain properties at long and short distances. We derive these properties below. In particular, the low momentum (large \( \hbar \) coupling) limit, \( p^2 \ll m^2 \), of the vacuum wave functional is easily determined to be of the form

\[ \Psi_0 = \exp \left( -\frac{1}{2g_{YM}^2 m} \int Tr B^2 \right). \]

(Equivalently, at low momentum, we should have \( K \to 1 \).) This wavefunctional provides a probability measure \( \bar{\Psi}_0 \) equivalent to the partition function of the Euclidean two-dimensional Yang-Mills theory with an effective Yang-Mills coupling \( g_{YM}^2 \equiv m g_{YM}^2 \). Using the results from \( \bar{\Psi} \), Karabali, Kim and Nair deduced the area law for the expectation value of the Wilson loop operator

\[ \langle \Phi \rangle \sim \exp(-\sigma A) \]

with the string tension following from the results of \( \bar{\Psi} \)

\[ \sigma = g_{YM}^4 \frac{N^2 - 1}{8\pi}. \]

This formula agrees nicely with extensive lattice simulations \( \bar{\Psi} \), and is consistent with the appearance of a mass gap as well as the large \( N \) ’t Hooft scaling.

Coming back to the derivation of the vacuum wave functional, we argue in Ref. \( \bar{\Psi} \) that operators \( \bar{O}_n \equiv \int \bar{\partial} JL^n \bar{\partial} J \), which would appear in a series expansion of \( K(L) \), satisfy

\[ \bar{T} \bar{O}_n = (2 + n)m \bar{O}_n + \ldots \]

where

\[ m = \frac{g_{YM}^2 c_A}{2\pi}, \quad \Omega_{ab}(x,y) = \frac{c_A}{\pi^2} \frac{\delta_{ab}}{(x-y)^2} - \frac{i f^{abc} J^c(x)}{\pi (x-y)}. \]
In Ref. [3] (see Sec. 3 and App. A), we have presented a series of calculations supporting this important result. Further evidence is provided by lattice considerations. Given this, we can formally write

$$TK(L) \rightarrow \frac{1}{L} \frac{d}{dL} [L^2 K(L)].$$

The full vacuum Schrödinger equation, combining all contributions self-consistently to quadratic order in $\partial J$

$$\mathcal{H}_{KN} \psi_0 = E_0 \psi_0 = \left[ \ldots + \int tr \partial J R \partial J + \ldots \right] \psi_0,$$

with suitable subtractions, then formally leads to the following differential equation for $K$

$$\frac{c_4 m}{\pi} R = -K - \frac{L}{2} \frac{d}{dL} [K(L)] + LR^2 + 1 = 0.$$  (21)

In this equation, the final term is the contribution of the potential $B^2$ term of the KN Hamiltonian, while the second to last term arises from the $\Omega$-term in the kinetic energy. Eq. 21 comes by consistently keeping all terms quadratic in $\partial J$ in the Schrödinger equation.

Although this equation is non-linear, it is easily solved by substituting $K = -y/2g$; the resulting equation may be recast as a Bessel equation. The only normalizable solution has the correct physical asymptotics for large and small $L$ and is given by

$$K(L) = \frac{1}{\sqrt{L}} \frac{J_2(4\sqrt{L})}{J_1(4\sqrt{L})}$$

(22)

where $J_n$ denotes the Bessel function of the first kind. This remarkable formula encodes information on the spectrum of the theory, as we show below. We note that this kernel has the following asymptotics (where $L \sim -p^2/4m^2$)

$$p \to 0, \ K \to 1; \quad p \to \infty, \ K \to 2m/p$$

(23)

consistent with confinement and asymptotic freedom, respectively.

In order to determine the spectrum, we factorize suitable correlation functions at large distances. The operators appearing in the correlation functions will have definite $J^{PC}$ quantum numbers, which will be inherited by the single particle poles contributing to the correlation function.

As a first example, we consider the $0^{++}$ states which may be probed by the operator $Tr \partial J \partial J$. We have

$$\langle Tr (\partial J \partial J)_x Tr (\partial J \partial J)_y \rangle \sim (K^{-1}(|x-y|))^2.$$  (24)

Here, we have computed the correlation function in the planar limit given our knowledge of the vacuum wavefunctional.

**TABLE I**: $0^{++}$ glueball masses in YM$_3$+1. All masses are in units of $\sqrt{\sigma}$. Ads/CFT computations[10] are also given for comparison. The percent difference between our prediction and lattice data is listed in the last column.

| State    | Lattice, $N \to \infty$ | Sugra | Our prediction | Diff, % |
|----------|-------------------------|-------|---------------|---------|
| $0^{++}$ | $4.065 \pm 0.055$       | $4.07 (input)$ | $4.10$       | $0.8$   |
| $0^{+++}$| $6.18 \pm 0.13$         | $7.02$ | $5.41$        | $12.5$  |
| $0^{++++}$| $7.99 \pm 0.22$        | $9.92$ | $6.72$        | $16$    |
| $0^{+}\ldots$| $9.44 \pm 0.38^a$     | $12.80$| $7.99$        | $15$    |

$^a$Mass of $0^{+++}$ state was computed on the lattice for SU(2) only [10]. The number quoted here was obtained by a simple rescaling of SU(2) result.

To proceed further, we note the identity

$$\frac{J_{\nu-1}(z)}{J_{\nu}(z)} = \frac{2\nu}{z} + 2z \sum_{n=1}^{\infty} \frac{1}{z^2 - j_{\nu,n}^2}$$

(25)

where $j_{\nu,n}$ are ordered zeros of the Bessel functions. For example, the first few zeros [8] of $J_2(z)$ are $j_{2,1} = 5.14$, $j_{2,2} = 8.42$, $j_{2,3} = 11.62$, $j_{2,4} = 14.80$, etc. Apart from additive constants, we then deduce

$$K^{-1}(k) = \frac{1}{2} \sum_{n=1}^{\infty} \frac{M_n^2}{M_n^2 + k^2}$$

(26)

where $M_n = j_{2,n}m/2$. The Fourier transform at large $|x - y|$ is

$$K^{-1}(|x - y|) = -\frac{1}{4\sqrt{2\pi}|x - y|} \sum_{n=1}^{\infty} (M_n)^{3/2} e^{-M_n|x - y|}.$$  (27)

In particular the $0^{++}$ correlator mentioned above is

$$\approx \frac{1}{32\pi|x - y|} \sum_{n, m=1}^{\infty} (M_n M_m)^{3/2} e^{-(M_n + M_m)|x - y|}.$$  (28)

Note that each term here has the correct $|\vec{x} - \vec{y}|$ dependence for a single particle pole of mass $M_n + M_m$ in 2 + 1 dimensions. The $0^{++}$ glueball masses are:

$$M_{0^{++}} = M_1 + M_1 = 5.14m$$
$$M_{0^{+++}} = M_1 + M_2 = 6.78m$$
$$M_{0^{++++}} = M_2 + M_2 = 8.42m$$
$$M_{0^{++}+++} = M_1 + M_3 = 8.35m$$
$$M_{0^{++++}+} = M_2 + M_3 = 10.02m.$$  (29)

Since $m$ is not a physical scale, we should re-write these results in terms of the string tension. Given equations presented above, at large $N$ we have $\sqrt{\sigma} \approx \sqrt{2\pi}m$. Our results are given in Table 1. Several comments are now in order. First, note that we have been able to predict masses of the $0^{++}$ resonances, as well as the lowest lying member, in contrast to the original results of Karabali
TABLE II: $0^{-}−$ glueball masses in YM$_{2+1}$. Columns are as in Table I.

| State | Lattice, $N \to \infty$ | Suqgra | Our prediction | Diff, % |
|-------|------------------------|--------|----------------|--------|
| $0^{-}$ | 5.91 ± 0.25 | 6.10 | 6.15 | 4 |
| $0^{-*}$ | 7.63 ± 0.37 | 9.34 | 7.46 | 2.3 |
| $0^{-*}$ | 8.96 ± 0.65 | 12.37 | 8.77 | 2.2 |

and Nair (which differ significantly numerically). The supergravity results listed in the table are a result of calculations \(10\) using the AdS/CFT correspondence \(11\). In that case, the overall normalization was not predicted but was determined by fitting to the lattice data, for example, to the mass of the lowest $0^{++}$ glueball. Our results for the excited state masses differ at the 10-15% level from the lattice results. We note that precisely these masses are more difficult to compute on the lattice \(12\), and thus the apparent 10 – 15% discrepancy may be illusory. \(14\)

Finally, we note that there is some interesting approximate degeneracies in the spectrum.

Let us move on to a discussion of the $0^{-}−$ glueball resonances. In this case, our predicted masses are much closer to the lattice data, which we believe to be more reliable in this case. We may probe these states with the operator $\text{Tr} \, \bar{J} \partial J \bar{J} \partial J$. We are thus interested in the correlation function \(12\)

$$
\langle \text{Tr} (\bar{J} \partial J \bar{J} \partial J) \rangle_x \text{ Tr} (\bar{J} \partial J \bar{J} \partial J) \rangle_y \sim (K^{-1}(|x−y|))^3.
$$

Using the results given above, we obtain glueball masses which are the sum of three $M_n$’s.

$$
M_{0−} = M_1 + M_1 + M_1 = 7.70 m \\
M_{0−*} = M_1 + M_2 + M_1 = 9.34 m \\
M_{0−*} = M_1 + M_2 + M_2 = 10.99 m. (31)
$$

These results are compared to lattice and supergravity data in Table II. We see that the resulting masses are within a few percent of the lattice data, and are much better than the supergravity predictions.

Our results suggest that there exist hidden constituent states as well as integrable structures in 2+1-dimensional Yang-Mills theory. Note that the full integrability of 2 + 1-dimensional pure Yang-Mills has been suspected for some time \(12\). In a longer paper \(3\), we will more carefully explain our techniques and results and will investigate other $J^{PC}$ glueball states and the corresponding Regge trajectories. It has not escaped our attention that a similar parameterization may be used in 3+1 dimensions in a variational sense and preliminary numerical results are encouraging.

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normalized poles. See Ref. [3] for a complete discussion.