HOPF ALGEBRA OF MULTI-DECORATED ROOTED FORESTS, FREE MATCHING ROTA-BAXTER ALGEBRAS AND GRÖBNER-SHIRSHOV BASES

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Abstract. Recent advances in stochastic PDEs, Hopf algebras of typed trees and integral equations have inspired the study of algebraic structures with replicating operations. To understand their algebraic and combinatorial nature, we first use rooted forests with multiple decoration sets to construct free Hopf algebras with multiple Hochschild 1-cocycle conditions. Applying the universal property of the underlying operated algebras and the method of Gröbner-Shirshov bases, we then construct free objects in the category of matching Rota-Baxter algebras which is a generalization of Rota-Baxter algebras to allow multiple Rota-Baxter operators. Finally the free matching Rota-Baxter algebras are equipped with a cocycle Hopf algebra structure.

1. Introduction

This paper applies rooted forests to obtain free objects for generalized cocycle Hopf algebras and then for matching Rota-Baxter algebras. To pass from the former to the latter, we interpret rooted forests as bracketed words and utilize the method of Gröbner-Shirshov bases.
1.1. **Rooted tree Hopf algebras and Rota-Baxter algebras.** The Hopf algebra of rooted forests arose from the study of Connes and Kreimer \[1\] on renormalization of quantum field theory where the rooted forests serves as a baby model of Feynman diagrams. The importance of this Hopf algebra and its noncommutative analog, the Foissy-Holtkamp Hopf algebra \([11, 23]\), has been investigated from various points of view. From the algebraic viewpoint, this importance is revealed by its universal property in the category of cocycle Hopf algebras \([11, 25, 29]\), in terms of a Hopf algebra with a grafting operator tied together by the Hochschild 1-cocycle condition. It is also characterized as the free operated algebra \([18, 38]\), namely an algebra equipped with a linear operator, and thus are interpreted in terms of bracketed words and Motzkin paths. Recently such a universal property was generalized to braided Hopf algebras of rooted forests à la Connes-Kreimer \([12, 27]\).

Another algebraic structure playing a key role in the work of Connes and Kreimer, and in the context of operated algebras, is the Rota-Baxter algebra which has its origin in the work of G. Baxter \([4]\) in fluctuation theory in probability. Connections of Rota-Baxter algebras have been established with broad areas in mathematics and mathematical physics, including quasi-symmetric functions, operads, integrable system and renormalization methods. See for example \([1, 2, 21, 36]\).

Free Rota-Baxter algebras can also be realized on rooted forests \([10, 38]\). In fact, the realization can be obtained from certain generalization of the noncommutative Connes-Kreimer Hopf algebra. This connection of free Rota-Baxter algebras with generalized Connes-Kreimer Hopf algebras not only highlights the combinatorial nature of Rota-Baxter algebras which attracted the attentions of outstanding combinatorists such as Cartier and Rota \([7, 33]\), but also provides a natural Hopf algebra structure on free Rota-Baxter algebras from that on the rooted forests.

Further various Hochschild 1-cocycle conditions have been applied to establish or characterize other Hopf like algebraic structures, such as Hopf algebras on free commutative modified Rota-Baxter algebras \([42]\), left counital Hopf algebras on free (commutative) Nijenhuis algebras \([17, 24]\) and free Rota-Baxter systems \([31, 32]\), as well as the Loday-Ronco Hopf algebra of binary rooted trees \([40]\) and infinitesimal bialgebras of rooted forests \([39]\).

1.2. **Matching Rota-Baxter algebras and outline of the paper.** As a multi-operator generalization of the Rota-Baxter algebra, the recent notion of a matching Rota-Baxter algebra \([41]\) has its motivation from the study of multiple pre-Lie algebras \([13]\) originated in the important work of Bruned, Hairer and Zambotti \([6]\) on algebraic renormalization of regularity structures and further motivated by the studies of associative Yang-Baxter equations, Volterra integral equations and linear structure of Rota-Baxter operators \([6, 20]\). See Section 3.1 for a summary of the broad connections of matching Rota-Baxter algebras.

Our purpose of this paper is to construct free matching Rota-Baxter algebras from rooted forests by generalizing the construction of free Rota-Baxter algebras mentioned above from the cocycle Hopf algebra of rooted forests, from one operator to multiple operators. Thus we first introduce a class of decorated rooted forests which will serve as the carrier of the free Hopf algebra with multiple Hochschild 1-cocycle conditions. Free matching Rota-Baxter algebras will be a quotient of this free cocycle Hopf algebra modulo the operated ideal generated by the matching Rota-Baxter algebra relations. We next display a basis of this quotient for which we apply the method of Gröbner-Shirshov bases. As this method works better with the algebraic notion of bracketed words, we utilize the dictionary between rooted forests and bracketed words provided in \([18]\) which allows us to take advantage of both the combinatorial structure of rooted forests
for their ease description of the coproduct and the algebraic structure of bracketed words for their amenability for detailed computations.

In the process, we also take advantage of the two gradings and filtrations on rooted forests (and the corresponding bracketed words), one by the number of vertices and one by the depths of the forests.

To provide more details, our first step is to construct free multiple cocycle Hopf algebras from rooted forests Section 2. Thus we will need to work with a set $X$ of generators and a set $\Omega$ of operators that satisfy the cocycle condition. For this purpose, we introduce rooted forests with two decoration sets $X$ and $\Omega$, with $X$ only allowed to decorate the leaf vertices. We then show that the resulting space $H_{RT}(X, \Omega)$ meets our needs for the free $\Omega$-cocycle Hopf algebra on $X$ (Theorem 2.10). Then the free matching Rota-Baxter algebra on $X$ is the quotient of $H_{RT}(X, \Omega)$ modulo the matching Rota-Baxter algebra relation. In order to apply the method of Gröbner-Shirshov bases to give an explicit construction of the free matching Rota-Baxter algebra in the next section, we utilize the one-one correspondence between rooted forests and bracketed words [12] to rephrase Theorem 2.10 in terms of (multiple) bracketed words (Corollary 2.14).

Our goal in Section 3 is to give an explicit construction of free matching Rota-Baxter algebras by applying the method of Gröbner-Shirshov bases to obtain a canonical linear basis of the quotient of the free $\Omega$-cocycle Hopf algebra $H_{RT}(X, \Omega)$ (reinterpreted in terms of bracketed words) modulo the matching Rota-Baxter algebra relations. First the notion of matching Rota-Baxter algebras is recalled together with a list of their properties. Then with a suitable monomial order, it is established in Theorem 3.3 that the matching Rota-Baxter algebra relations form a Gröbner-Shirshov basis and thus give rise to a linear basis of the free matching Rota-Baxter algebra as the aforementioned quotient. The operations of the matching Rota-Baxter algebra are given in terms of this linear basis. In view of establishing a cocycle Hopf algebra structure on the free matching Rota-Baxter algebra in the next section, we apply the one-one correspondence between rooted forests and bracketed words again and rephrase the free matching Rota-Baxter algebra in terms of decorated rooted forests.

In Section 4, we equip the free matching Rota-Baxter algebra with an $\Omega$-cocycle Hopf algebra structure descending from the one on $H_{RT}(X, \Omega)$, by first establishing an $\Omega$-cocycle bialgebra structure and then verifying the needed connected condition in order to obtain the Hopf algebra structure in Theorem 4.7.

**Notations.** Throughout this paper, we fix a unitary commutative ring $k$ which will be the base ring of all modules, algebras, coalgebras, bialgebras, tensor products, as well as linear maps. By an algebra, we mean a unitary associative algebra unless otherwise specified.

2. Multi-operated Hopf algebras of decorated rooted forests

In this section, we construct free operated algebras with multiple operators by decorated rooted forests, as well as by bracketed words.

The rooted forests we consider have different decorations on their leafs and (internal) vertices, but can still be obtained as a suitable subset of the classical noncommutative Connes-Kreimer Hopf algebra $H_{RT}(\Omega)$, that is, Foissy-Holtkamp Hopf algebra [11, 23], of rooted forests with all their vertices (leafs and internal vertices) decorated by the same set. Thus we first recall the notions for $H_{RT}(\Omega)$ for later applications.
2.1. Noncommutative Connes-Kreimer Hopf algebras. Various Hopf algebras of decorated planar rooted trees and forests are commonly studied in combinatorics, algebra and other fields. We recall the needed notions and results to be applied in our constructions in this paper.

A rooted tree is a connected and simply connected set of vertices and oriented edges such that there is precisely one distinguished vertex, called the root. A planar rooted tree is a rooted tree with a fixed embedding into the plane.

Let $\mathcal{T}$ denote the set of planar rooted trees and $\mathcal{F}$ the set of planar rooted forests, expressed algebraically as the free monoid $\mathcal{F} := M(\mathcal{T})$ generated by $\mathcal{T}$ with the concatenation product $m_{RT}$ which is usually suppressed for brevity. The empty tree elements of $\Omega$ trees (resp. forests) whose vertices, including the leaves and internal vertices, are decorated by $\Delta$. See \([\ref{19}]\). A rooted forest $F$ is denoted by $1$, the unit of $\mathcal{F}(\Omega)$ (resp. $\mathcal{F}(\Omega) := M(\mathcal{T}(\Omega))$) denote the set of planar rooted trees (resp. forests) whose vertices, including the leaves and internal vertices, are decorated by elements of $\Omega$. Define the free $k$-module spanned by the set $\mathcal{F}(\Omega)$:

$$H_{RT}(\Omega) := k\mathcal{F}(\Omega) = kM(\mathcal{T}(\Omega)) = k\langle \mathcal{T}(\Omega) \rangle,$$

which is also the noncommutative polynomial algebra on the set $\mathcal{T}(\Omega)$ with the concatenation.

The noncommutative Connes-Kreimer Hopf algebra $H_{RT}(\Omega)$ introduced by Foissy \([\ref{1}]\) and Holtkamp \([\ref{2}]\) is the above algebra equipped with a coproduct which can be defined in several ways, by subforests, by admissible cuts and by a 1-cocycle condition. A subforest of a planar rooted forest $F \in \mathcal{F}(\Omega)$ is the forest consisting of a set of vertices of $F$ together with their descendants and edges connecting all these vertices. Let $\mathcal{F}_F$ be the set of subforests of $F$, including the empty tree $1$ and the full subforest $F$. Define

$$\Delta_{RT}(F) := \sum_{G \in \mathcal{F}_F} G \otimes (F/G),$$

where $F/G$ is obtained by removing the subforest $G$ and edges connecting $G$ to the rest of the tree \([\ref{19}]\). Here we use the convention that $F/G = 1$ when $F = G$, and $F/G = F$ when $G = 1$. See \([\ref{19}]\) for a description of the coproduct by admissible cuts.

The 1-cocycle condition that characterizes the coproduct $\Delta_{RT}$ is given by the grafting operators. For $\omega \in \Omega$, define

$$B_\omega^+: H_{RT}(\Omega) \to H_{RT}(\Omega)$$

to be the linear grafting operation by sending a rooted forest in $\mathcal{F}(\Omega)$ to its grafting with the new root decorated by $\omega$ and sending $1$ to $\bullet_\omega$.

Then a recursive description of $\Delta_{RT}$ is the 1-cocycle condition for $T \in \mathcal{T}$

$$\Delta_{RT} B_\omega^+(T) := B_\omega^+(T) \otimes 1 + (id \otimes B_\omega^+) \Delta_{RT}(T)$$

with the convention $\Delta_{RT}(1) = 1 \otimes 1$. In particular, we have

$$\Delta_{RT}(\bullet_\omega) = \bullet_\omega \otimes 1 + 1 \otimes \bullet_\omega, \ \omega \in \Omega.$$

For a decorated rooted forest $F = T_1 \cdots T_m \in \mathcal{F}(\Omega)$ with $m \geq 2$, we have

$$\Delta_{RT}(F) = \Delta_{RT}(T_1) \cdots \Delta_{RT}(T_m).$$

Also define $\epsilon_{RT} : k\mathcal{F}(\Omega) \to k$ by taking $\epsilon_{RT}(F) = 0$ for $F \in \mathcal{F}(\Omega)$, and $\epsilon_{RT}(1) = 1$. Let $u_{RT} : k \to H_{RT}(\Omega)$ be the linear map given by $1_k \mapsto 1$. 

Recall [15, 19, 28] that a bialgebra \((H, m, u, \Delta, \varepsilon)\) is called \textbf{graded} if there are \(k\)-submodules 
\[ H^{(n)} \], \( n \geq 0 \), of \( H \) such that 
\[ H = \bigoplus_{n \geq 0} H^{(n)}, \quad H^{(p)}H^{(q)} \subseteq H^{(p+q)}, \quad \Delta(H^{(n)}) \subseteq \bigoplus_{p+q=n} H^{(p)} \otimes H^{(q)}, \quad n, p, q \geq 0. \]

The bialgebra \( H \) is called \textbf{connected graded} if in addition \( H^{(0)} = imu (= k) \) and \( \ker \varepsilon = \bigoplus_{n \geq 1} H^{(n)}. \) It is well known that a connected graded bialgebra is a Hopf algebra.

\textbf{Theorem 2.1.} [11, 23] With the degree of a rooted forest defined by its number of vertices, the quintuple \((H_{RT}(\Omega), m_{RT}, u_{RT}, \Delta_{RT}, \varepsilon_{RT})\) is a connected graded bialgebra and hence a Hopf algebra.

\section{2.2. Multi-decorated planar rooted trees and forests.} We now give a generalization of the Hopf algebra \( H_{RT}(\Omega) \) of decorated rooted forests by allowing the leaf vertices and internal vertices decorated by different sets. We then show that it gives a realization of the free object in the category of algebras with multiple operators, thus automatically equipping the free object with a Hopf algebra structure given by a cocycle condition.

Let \( X \) be a set and let \( \Omega \) be a nonempty set disjoint from \( X \). Replacing \( \Omega \) by \( X \sqcup \Omega \) in \( H_{RT}(\Omega) \), we obtain the Hopf algebra \( H_{RT}(X \sqcup \Omega) = k(\mathcal{F}(X \sqcup \Omega)) \) as in Theorem 2.1.

Let \( \mathcal{T}(X, \Omega) \) (resp. \( \mathcal{T}(X, \Omega) \)) denote the subset of \( \mathcal{T}(X \sqcup \Omega) \) (resp. \( \mathcal{T}(X \sqcup \Omega) \)) consisting of vertex decorated planar rooted trees (resp. forests) with the property that elements of only \( \Omega \) can decorate the internal vertices, namely vertices which are not leafs. The unique vertex of the tree \( \bullet \) is regarded as a leaf vertex. In other words, elements of \( X \) can only be used to decorate the leaf vertices. Of course, some of the leaf vertices can also be decorated by elements from \( \Omega \). For example,

\[ 1, \ldots, x, 1^\beta_x, 1^\alpha_x, \gamma^y_{\alpha_x}, \gamma^x_{\alpha_x}, \beta^y_{\alpha_x}, \beta^x_{\alpha_x}, \gamma^y_{\beta_x}, \gamma^x_{\beta_x}, x, y \in X, \alpha, \beta, \gamma \in \Omega, \]

are in \( \mathcal{T}(X, \Omega) \) whereas, the following are not in \( \mathcal{T}(X, \Omega) \):

\[ 1^y_x, 1^\gamma_x, \alpha^y_{\gamma_x}, \beta^x_{\gamma_x}, x, y \in X, \alpha, \beta, \gamma \in \Omega. \]

\textbf{Remark 2.2.} Now we give some special cases of our decorated planar rooted forests.

(a) If \( X = \emptyset \), then \( \mathcal{T}(X, \Omega) = \mathcal{T}(\Omega) \) is the linear basis in the decorated Foissy-Holtkamp Hopf algebra \( H_{RT}(\Omega) \) [11].

(b) If \( \Omega \) is a singleton, then \( \mathcal{T}(X, \Omega) \) was introduced and studied in [33] to construct a cocycle Hopf algebra on decorated planar rooted forests.

(c) The subset of \( \mathcal{T}(X, \Omega) \) consisting of rooted forests whose (all) leaf vertices are decorated by elements of \( X \) and whose internal vertices are decorated by elements of \( \Omega \), are introduced in [13] to construct free operated \textit{nonunitary} semigroups and free operated nonunitary algebras.

Define

\[ H_{RT}(X, \Omega) := k\mathcal{T}(X, \Omega) = kM(\mathcal{T}(X, \Omega)) \]

to be the free \( k \)-module spanned by \( \mathcal{T}(X, \Omega) \).

We define the degree \( \text{deg}(F) \) of \( F \in \mathcal{T}(X, \Omega) \) to be its number of vertices. For \( n \geq 0 \), let \( \mathcal{T}^{(n)} \) denote the set of \( F \in \mathcal{T}(X, \Omega) \) with degree \( n \) and let \( H_{RT}(X, \Omega)^{(n)} := H_{RT}^{(n)} := k\mathcal{T}^{(n)}. \) For
\[ F = T_1 \cdots T_k \in \mathcal{F}(X, \Omega) \] with \( k \geq 0 \) and \( T_1, \cdots, T_k \in \mathcal{F}(X, \Omega) \), define \( \text{bre}(F) := k \) to be the breadth of \( F \) with the convention that \( \text{bre}(1) = 0 \) when \( k = 0 \).

**Theorem 2.3.** The quintuple \((H_{RT}(X, \Omega), m_{RT}, 1, \Delta_{RT}, \varepsilon_{RT})\) is a connected graded subsbialgebra of \( H_{RT}(X \sqcup \Omega) \) with the grading \( H_{RT}(X, \Omega) = \oplus_{n \geq 0} H^n_{RT} \) and hence is a Hopf algebra.

**Proof.** Note that \( H_{RT}(X, \Omega) \) is a subspace of \( H_{RT}(X \sqcup \Omega) \). It is sufficient to show that \( H_{RT}(X, \Omega) \) is closed under the multiplication \( m_{RT} \) and the coproduct \( \Delta_{RT} \). Recall that forests in \( H_{RT}(X, \Omega) \) are characterized by the condition that their internal vertices are decorated by elements from \( \Omega \) only. If two forests have this condition, then their concatenation also has this condition, since an internal vertex of the concatenated forest is an internal vertex of one of the two forests. For the same reason, for any forest \( F \) with this condition, any of its subforest \( G \) and the quotient forest \( F/G \) still have this condition. Thus the subspace \( H_{RT}(X, \Omega) \) is closed under the concatenation product and the coproduct defined in Eq. (1). \( \square \)

Combining the degree and the grafting operators, we can equip \( \mathcal{F}(X, \Omega) \) with other structures.

**Definition 2.4.** An \( \Omega \)-operated algebra \((R, P_\Omega)\) with a grading \( R = \oplus_{n \geq 0} R^n \) (resp. an increasing filtration \( \{R_n\}_{n \geq 0} \)) is called an \( \Omega \)-operated graded algebra (resp. \( \Omega \)-operated filtered algebra) if \((R, \oplus_{n \geq 0} R^n)\) is a graded algebra (resp. \((R, \{R_n\}_{n \geq 0})\) is a filtered algebra) and

\[
P_\omega(R^0) \subseteq R^{\omega+1} \quad (\text{resp. } P_\omega(R_n) \subseteq R_{n+1}).
\]

As is well-known, a graded algebra \((R, \oplus_{n \geq 0} R^n)\) is a filtered algebra with the filtration \( R_n := \oplus_{k \leq n} R^k, n \geq 0 \). Further by definition, the grafting operator \( B^\omega \) increases the degree of a rooted forest by one. Thus we have

**Lemma 2.5.** The \( \Omega \)-operated algebra \( H_{RT}(X, \Omega) \) with its grading \( \oplus_{n \geq 0} H^n_{RT} \) and the associated filtration \( H_{RT,(n)} := \oplus_{k \leq n} H^k_{RT} \), is an \( \Omega \)-operated graded algebra and an \( \Omega \)-operated filtered algebra.

For our later applications to matching Rota-Baxter algebras, we introduce another increasing filtration on \( \mathcal{F}(X, \Omega) \) which leads to the notion of the depth of a decorated rooted forest. For distinction, we use \( \mathcal{F}_n \) and \( H_{RT,n} \) for the previous filtration defined by degree and \( \mathcal{F}_n \) and \( H_{RT,n} \) for the new filtration defined by depth.

Denote \( \bullet_X := \{\bullet_x \mid x \in X\} \) and set

\[
\mathcal{F}_0 := M(\bullet_X) = S(\bullet_X) \sqcup \{1\},
\]

where \( M(\bullet_X) \) (resp. \( S(\bullet_X) \)) is the submonoid (resp. subsemigroup) of \( \mathcal{F}(X, \Omega) \) generated by \( \bullet_X \).

Here the using of the notations \( M \) and \( S \) are justified since \( M(\bullet_X) \) (resp. \( S(\bullet_X) \)) is indeed isomorphic to the free monoid (resp. semigroup) generated by \( \bullet_X \). Suppose that \( \mathcal{F}_n \) has been defined for an \( n \geq 0 \). Then define

\[
\mathcal{F}_{n+1} := M(\bullet_X) \sqcup (\sqcup_{\omega \in \Omega} B^\omega(\mathcal{F}_n))).
\]

Thus we obtain \( \mathcal{F}_n \subseteq \mathcal{F}_{n+1} \) and

\[
\mathcal{F}(X, \Omega) = \lim_{\rightarrow} \mathcal{F}_n = \bigcup_{n=0}^{\infty} \mathcal{F}_n.
\]

Elements \( F \in \mathcal{F}_n \setminus \mathcal{F}_{n-1} \) are said to have depth \( n \), denoted by \( \text{dep}(F) = n \). Here are some examples:

\[
\text{dep}(1) = \text{dep}(\bullet_x) = 0, \quad \text{dep}(\bullet_\omega) = \text{dep}(B^\omega(1)) = 1, \quad \text{dep}(1^\omega_1) = \text{dep}(B^\omega_1(B^\omega_1(1))) = 2,
\]
where \( \alpha, \omega \in \Omega \) and \( x, y \in X \).

Note the subtle difference of this depth from the usual notion of depth of a rooted tree, defined to be the length of the longest path from a root of \( F \) to its leafs. The advantage of this new depth is that it is consistent with the natural depth of bracketed words to be introduced in Section 2.4. This difference is most easily seen in \( \text{dep}(\bullet, x) = 0 \) while \( \text{dep}(\bullet, y) = 1 \). In general our notion \( \text{dep}(F) \) of depth for \( F \in \mathcal{T} \) is the same as the usual depth when all the longest paths from a root of \( F \) to the leafs end in vertices with decorations from \( X \); while \( \text{dep}(F) \) is the usual depth of \( F \) plus one if one of these longest paths ends in a vertex with decoration from \( \Omega \).

2.3. Free \( \Omega \)-cyclo Hopf algebras of decorated planar rooted forests. In this subsection, we will combine the notions of \( \Omega \)-operated algebras and Hopf algebras to define an \( \Omega \)-(operated) Hopf algebra and \( \Omega \)-(operated) cocycle Hopf algebra. We then show that \( H_{RT}(X, \Omega) \) is a free \( \Omega \)-cyclo Hopf algebra on the set \( X \). For this purpose, we recall the following concepts.

**Definition 2.6.** [9] Let \( \Omega \) be a nonempty set.

(a) An \( \Omega \)-operated algebra is an algebra \( A \) together with a family of linear operators \( P_\omega : A \to A, \omega \in \Omega \).

(b) Let \((A, (P_\omega)_{\omega \in \Omega})\) and \((A', (P'_\omega)_{\omega \in \Omega})\) be \( \Omega \)-operated algebras. A linear map \( \phi : A \to A' \) is called an \( \Omega \)-operated algebra homomorphism if \( \phi \) is an algebra homomorphism such that \( \phi P_\omega = P'_\omega \phi \) for \( \omega \in \Omega \).

(c) A free \( \Omega \)-operated algebra on a set \( X \) is an \( \Omega \)-operated algebra \((A, (P_\omega)_{\omega \in \Omega})\) together with a set map \( j_X : X \to A \) with the property that, for any \( \Omega \)-operated algebra \((A', (P'_\omega)_{\omega \in \Omega})\) and any set map \( f : X \to A' \), there is a unique homomorphism \( f : A \to A' \) of \( \Omega \)-operated algebras such that \( \tilde{f} j_X = f \).

Now we enrich these notions by adding the bialgebra structures.

**Definition 2.7.**

(a) An \( \Omega \)-operated bialgebra is a bialgebra \((H, m, 1_H, \Delta, \varepsilon)\) which is also an \( \Omega \)-operated algebra \((H, (P_\omega)_{\omega \in \Omega})\).

(b) Let \((H, (P_\omega)_{\omega \in \Omega})\) and \((H', (P'_\omega)_{\omega \in \Omega})\) be \( \Omega \)-operated bialgebras. A linear map \( \phi : H \to H' \) is called an \( \Omega \)-operated bialgebra homomorphism if \( \phi \) is a bialgebra homomorphism such that \( \phi P_\omega = P'_\omega \phi \) for \( \omega \in \Omega \).

(c) An \( \Omega \)-cyclo bialgebra is an \( \Omega \)-operated bialgebra \((H, m, 1_H, \Delta, \varepsilon, (P_\omega)_{\omega \in \Omega})\) which satisfies the cyclo condition:

\[
\Delta P_\omega = P_\omega \otimes 1_H + (\id \otimes P_\omega) \Delta \quad \text{for } \omega \in \Omega.
\]

If the bialgebra in an \( \Omega \)-cyclo bialgebra is a Hopf algebra, then it is called an \( \Omega \)-cyclo Hopf algebra.

(d) The free \( \Omega \)-cyclo bialgebra on a set \( X \) is an \( \Omega \)-cyclo bialgebra \((H_X, m_X, 1_X, \Delta_X, \varepsilon_X, (P_\omega)_{\omega \in \Omega})\) together with a set map \( j_X : X \to H_X \) with the property that, for any cyclo bialgebra \((H, m, 1_H, \Delta, \varepsilon, (P'_\omega)_{\omega \in \Omega})\) and set map \( f : X \to H \) whose images are primitive (that is, \( \Delta(f(x)) = f(x) \otimes 1_H + 1_H \otimes f(x) \)), there is a unique homomorphism \( \tilde{f} : H_X \to H \) of \( \Omega \)-operated bialgebras such that \( \tilde{f} j_X = f \). The concept of a free \( \Omega \)-cyclo Hopf algebra is defined in the same way.

We are indebted to Foissy for the following result.
Lemma 2.8. Let \((H, m, 1_H, \Delta, \varepsilon, (P_\omega)_{\omega \in \Omega})\) be an \(\Omega\)-cyclic bialgebra. Then for each \(\omega \in \Omega\),

\[
P_\omega(H) := \{P_\omega(h) \mid h \in H\}
\]

is a coideal of \(H\).

Proof. Let \(\omega \in \Omega\). We first show \(P_\omega(h) \subseteq \ker \varepsilon\). Let \(h' := P_\omega(h) \in H\) be arbitrary with \(h \in H\). Using the Sweedler notation, we can write

\[
\varepsilon(h') = \varepsilon\left(\sum_{(h')} h'_1 \varepsilon(h'_2)\right) = \sum_{(h')} \varepsilon(h'_1) \varepsilon(h'_2) = (\varepsilon \otimes \varepsilon)\Delta(h').
\]

Thus

\[
\varepsilon(h') = \varepsilon P_\omega(h) = (\varepsilon \otimes \varepsilon)\Delta(h') = (\varepsilon \otimes \varepsilon)\Delta(P_\omega(h))
\]

\[
= (\varepsilon \otimes \varepsilon)(P_\omega(h) \otimes 1_H + (\text{id} \otimes P_\omega)\Delta(h)) \quad \text{(by Eq. (\ref{eq:delta}))}
\]

\[
= (\varepsilon \otimes \varepsilon)(P_\omega(h) \otimes 1_H) + (\varepsilon \otimes \varepsilon P_\omega)\Delta(h)
\]

\[
= \varepsilon P_\omega(h) + \sum_{(h)} \varepsilon(h_{(1)}) \varepsilon P_\omega(h_{(2)})
\]

\[
= \varepsilon P_\omega(h) + \varepsilon P_\omega\left(\sum_{(h)} \varepsilon(h_{(1)})(h_{(2)})\right)
\]

\[
= \varepsilon P_\omega(h) + \varepsilon P_\omega(h),
\]

which implies that \(\varepsilon P_\omega(h) = 0\).

Secondly, for any \(h \in H\),

\[
\Delta(P_\omega(h)) = P_\omega(h) \otimes 1_H + (\text{id} \otimes P_\omega)\Delta(h) \quad \text{(by Eq. (\ref{eq:delta}))}
\]

\[
= P_\omega(h) \otimes 1_H + (\text{id} \otimes P_\omega)(\sum_{(h)} (h_{(1)} \otimes h_{(2)}))
\]

\[
= P_\omega(h) \otimes 1_H + \sum_{(h)} h_{(1)} \otimes P_\omega(h_{(2)}) \in P_\omega(H) \otimes H + H \otimes P_\omega(H).
\]

Thus \(P_\omega(H)\) is a coideal. \(\square\)

The following result generalizes the universal properties of several related structures studied in \([8, 18, 23, 38]\). See \([8, \text{Theorem 2.3}]\) for the commutative case.

Lemma 2.9. \([39]\) Let \(j_X : X \hookrightarrow H_{RT}(X, \Omega)\), \(x \mapsto \bullet_x\) be the nature embedding and \(m_{RT}\) be the concatenation product. The quadruple \((H_{RT}(X, \Omega), m_{RT}, 1, (B^+_\omega)_{\omega \in \Omega})\) together with \(j_X\) is the free \(\Omega\)-operated algebra on \(X\).

We next strengthen Lemma 2.9 to include the bialgebra structure.

Theorem 2.10. Let \(j_X : X \hookrightarrow H_{RT}(X, \Omega)\), \(x \mapsto \bullet_x\) be the nature embedding and \(m_{RT}\) be the concatenation product.

(a) The sextuple \((H_{RT}(X, \Omega), m_{RT}, 1, \Delta_{RT}, \varepsilon_{RT}, (B^+_\omega)_{\omega \in \Omega})\) together with \(j_X\) is the free \(\Omega\)-cocycle bialgebra on \(X\).

(b) The Hopf algebra given by the connected graded bialgebra \((H_{RT}(X, \Omega), m_{RT}, 1, \Delta_{RT}, \varepsilon_{RT}, (B^+_\omega)_{\omega \in \Omega})\) together with \(j_X\) is the free \(\Omega\)-cocycle Hopf algebra on \(X\).
Proof. By Theorem 2.3, the quintuple \((H_{RT}(X, \Omega), m_{RT}, 1, \Delta_{RT}, \varepsilon_{RT})\) is a bialgebra. Furthermore, by Eq. (2), the sextuple \((H_{RT}(X, \Omega), m_{RT}, 1, \Delta_{RT}, \varepsilon_{RT}, (B^+_{\omega})_{\omega \in \Omega})\) is an \(\Omega\)-cosemibialgebra.

To verify the freeness, let \((H, m, 1_H, \Delta, \varepsilon, (P_{\omega})_{\omega \in \Omega})\) be an \(\Omega\)-cosemibialgebra and \(f : X \to H\) a set map such that
\[
\Delta(f(x)) = f(x) \otimes 1_H + 1_H \otimes f(x) \quad \text{for all } x \in X.
\]
In particular, \((H, m, 1_H, (P_{\omega})_{\omega \in \Omega})\) is an \(\Omega\)-bimodule. It follows from Lemma 2.9 that there exists a unique \(\Omega\)-bimodule homomorphism \(\tilde{f} : H_{RT}(X, \Omega) \to H\) such that \(\tilde{f}j_X = f\).

It remains to check the following two compatibilities between the coproducts \(\Delta\) and \(\Delta_{RT}\), and between the counit \(\varepsilon\) and \(\varepsilon_{RT}\).

\[
\Delta(\tilde{f}(x)) = \Delta(f(x)) = f(x) \otimes 1_H + 1_H \otimes f(x) \quad \text{for all } x \in X.
\]

Similarly, to verify Eq. (10), we just need to show that \(\Delta(\tilde{f}(x)) = \Delta(f(x))\) for all \(x \in X\).

To verify Eq. (11), we consider the set
\[
\mathcal{A} := \{F \in H_{RT}(X, \Omega) | \Delta(\tilde{f}(F)) = (\tilde{f} \otimes \tilde{f})\Delta_{RT}(F)\}.
\]

By Lemma 2.9, \(H_{RT}(X, \Omega)\) is generated by \(X\) as an \(\Omega\)-bimodule. Thus to verify Eq. (11), we just need to show that \(\mathcal{A}\) is a subalgebra of \(H_{RT}(X, \Omega)\) that contains \(X\).

Since \(\tilde{f}\) is an \(\Omega\)-bimodule homomorphism, and \(\Delta_{RT}\) and \(\Delta\) are algebra homomorphisms from \(H_{RT}(X, \Omega)\) and \(H\), respectively, we get \(1 \in \mathcal{A}\) and \(\mathcal{A}\) is a subalgebra of \(H_{RT}(X, \Omega)\). For any \(x \in X\), we have
\[
\Delta(\tilde{f}(x)) = \Delta(f(x)) = f(x) \otimes 1_H + 1_H \otimes f(x) \quad \text{for all } x \in X.
\]

Thus \(\mathcal{A}\) is stable under \(B^+_{\omega}\) for any \(\omega \in \Omega\) and so \(\mathcal{A} = H_{RT}(X, \Omega)\).

Similarly, to verify Eq. (11), we just need to show that the subset
\[
\mathcal{B} := \{F \in H_{RT}(X, \Omega) | \varepsilon(\tilde{f}(F)) = \varepsilon_{RT}(F)\} \subseteq H_{RT}(X, \Omega).
\]

is an \(\Omega\)-bimodule subalgebra of \(H_{RT}(X, \Omega)\).
Since $\tilde{f}$ is an $\Omega$-operated algebra homomorphism, $\varepsilon_{RF}$ and $\varepsilon$ are algebra homomorphisms from $H_{RT}(X, \Omega)$ and $H$, respectively. So we get $1 \in \mathcal{B}$ and $\mathcal{B}$ is a subalgebra of $H_{RT}(X, \Omega)$. For any $x \in X$, by Eq. (1) and the left counicity, we obtain
\[(\varepsilon \otimes \text{id})\Delta(f(x)) = \varepsilon(f(x))1_H + 1_{k} \otimes f(x) = \beta_{\ell}(f(x)),\]
which implies that $\varepsilon(f(x)) = 0$. Then
\[\varepsilon(\tilde{f}(\bullet_{x})) = \varepsilon(f(x)) = 0 = \varepsilon_{RT}(\bullet_{x}),\]
showing $\bullet_{x} \in \mathcal{B}$. For $F \in \mathcal{B}$ and $\omega \in \Omega$, we have
\[\varepsilon(\tilde{f}(B_{\omega}^{+}(F))) = \varepsilon(P_{\omega}(\tilde{f}(F))) \text{ (by } \tilde{f} \text{ being an } \Omega\text{-operated algebra homomorphism)}
\[= \varepsilon P_{\omega}(\tilde{f}(F)) = 0 \quad \text{(by Lemma 2.8)}
\[= \varepsilon_{RT}(B_{\omega}^{+}(F)).\]
Hence $\mathcal{B}$ is stable under $B_{\omega}^{+}$ for any $\omega \in \Omega$ and so $\mathcal{B} = H_{RT}(X, \Omega)$. This completes the proof. 

(\[\square\] The proof follows from Item (\[\square\]) and the well-known fact that any bialgebra homomorphism between two Hopf algebras is compatible with the antipodes \[\square\] Lemma 4.04.

If $X = \emptyset$, we obtain the freeness of $H_{RT}(\emptyset, \Omega) = H_{RT}(\Omega)$, which is the decorated noncommutative Connes-Kreimer Hopf algebra by Remark 2.2 (\[\square\]).

**Corollary 2.11.** The sextuple $(H_{RT}(\Omega), m_{RT}, 1, \Delta_{RT}, \varepsilon_{RT}, (B_{\omega}^{+})_{\omega \in \Omega})$ is free $\Omega$-cocy cle Hopf algebra on the empty set, that is, the initial object in the category of $\Omega$-cyclic Hopf algebras.

Further taking $\Omega = \{\omega\}$ to be a singleton in Corollary 2.11, all decorated planar rooted forests have the same decoration and hence can be rendered undecorated as in the Foissy-Holtkamp Hopf algebra \[\square\] 2.3. Thus we have, similar to \[\square\], \[\square\], \[\square\].

**Corollary 2.12.** Let $\mathcal{F}$ be the set of planar rooted forests without decorations. Then sextuple $(k\mathcal{F}, m_{RT}, 1, \Delta_{RT}, \varepsilon_{RT}, B_{\omega}^{+})$ is the free cocycle Hopf algebra on the empty set, that is, the initial object in the category of cocycle Hopf algebras.

### 2.4. Free $\Omega$-operated monoids and algebras.

Our next goal is to construct free matching Rota-Baxter algebras by applying the method of Gröbner-Shirshov bases which works better in the context of bracketed words. Thus in this subsection, we recall the construction of a free $\Omega$-operated monoid and $\Omega$-operated algebra in terms of bracketed words on a set $X$ and identify them with the free $\Omega$-operated algebra $H_{RT}(X, \Omega) = k\mathcal{F}(X, \Omega)$. See \[\square\] for more details of these bracketed words.

Given an $\omega \in \Omega$ and a set $Y$, let $[Y]_{\omega}$ denote the set $\{[y]_{\omega} \mid y \in Y\}$, so it is indexed by $Y$ but disjoint with $Y$. We also assume that the sets $[Y]_{\omega}$ to be disjoint with each other as $\omega$ varies in $\Omega$.

We now define the free $\Omega$-operated monoid over the set $X$ as the limit of a direct system
\[\{i_{n,n+1} : \mathcal{M}_{n} \rightarrow \mathcal{M}_{n+1}\}_{n=0}^{\infty}\]
of inductively defined free monoids $\mathcal{M}_{\omega}$, where the transition homomorphisms $i_{n+1,n}$ are natural embeddings. For the initial step of $n = 0$, we define $\mathcal{M}_{0} := M(X)$ and then define
\[\mathcal{M}_{1} := M(X \sqcup (\sqcup_{\omega \in \Omega}[\mathcal{M}_{0}]_{\omega}))\]
with the natural embedding
\[i_{0,1} : \mathcal{M}_{0} = M(X) \hookrightarrow \mathcal{M}_{1} = M(X \sqcup (\sqcup_{\omega \in \Omega}[\mathcal{M}_{0}]_{\omega})).\]
Note that \([\mathcal{M}_0, \omega] \subseteq \mathcal{M}_1\) for each \(\omega \in \Omega\). In particular, \(1 \in \mathcal{M}_0\) is sent to \(1 \in \mathcal{M}_1\). Inductively assume that, for any given \(n \geq 1\), \(\mathcal{M}_k, k \geq n\) with the natural embedding
\[
i_{n-1,n} : \mathcal{M}_{n-1} \hookrightarrow \mathcal{M}_n
\]
have been defined. We then define
\[
\mathcal{M}_{n+1} := M (X \sqcup (\sqcup_{\omega \in \Omega} [\mathcal{M}_{n-1}, \omega])).
\]
The natural embedding in Eq. (12) induces the natural embedding
\[
[\mathcal{M}_{n-1}, \omega] \hookrightarrow [\mathcal{M}_{n}, \omega],
\]
yielding a monomorphism of free monoids
\[
i_{n,n+1} : \mathcal{M}_n = M (X \sqcup (\sqcup_{\omega \in \Omega} [\mathcal{M}_{n-1}, \omega])) \hookrightarrow M (X \sqcup (\sqcup_{\omega \in \Omega} [\mathcal{M}_n, \omega])) = \mathcal{M}_{n+1}.
\]
This completes the inductive construction of the direct system. Finally we define the direct limit
\[
\mathfrak{M}(X, \Omega) := \varinjlim M_n = \bigcup_{n \geq 0} \mathfrak{M}_n
\]
with identity 1. Elements in \(\mathfrak{M}(X, \Omega)\) are called \(\Omega\)-bracketed words in \(X\) and elements of \(\mathfrak{M}_n \setminus \mathfrak{M}_{n-1}\) are said to have depth \(n\), denoted by \(\text{dep}_{\mathfrak{M}}(w) = n\). Define
\[
P_\omega : \mathfrak{M}(X, \Omega) \to \mathfrak{M}(X, \Omega), \ u \mapsto [u], \omega \in \Omega,
\]
and extend it by linearity to a linear operator on \(k\mathfrak{M}(X, \Omega)\), still denoted by \(P_\omega\). Then the pair \((\mathfrak{M}(X, \Omega), (P_\omega)_{\omega \in \Omega})\) is an \(\Omega\)-operated monoid and its linear span \((k\mathfrak{M}(X, \Omega), (P_\omega)_{\omega \in \Omega})\) is an \(\Omega\)-operated algebra.

Let \(X\) be a set and \(\Omega\) a nonempty set disjoint with \(X\). Taking direct limit in Eq. (13) we obtain
\[
\mathfrak{M}(X, \Omega) = M (X \sqcup (\sqcup_{\omega \in \Omega} [\mathfrak{M}(X, \Omega)]))
\]
Thus any \(1 \neq u \in \mathfrak{M}(\Omega, X)\) has a unique factorization
\[
u = w_1 \cdots w_k, \quad w_i \in X \cup \mathfrak{M}(\Omega, X), \ 1 \leq i \leq k, \ k \geq 1.
\]
We call \(k\) the breadth of \(u\) and denote it by \(|u|\). For \(u = 1 \in \mathfrak{M}(\Omega, X)\), we define \(|u| := 0\).

**Proposition 2.13.** \([\mathfrak{M}]\) Corollary 3.6] Let \(X\) be a set and \(\Omega\) a nonempty set. Let \(j_x : X \to k\mathfrak{M}(X, \Omega)\) be the natural embedding and let \(\cdot\) be the concatenation product. Then the triple \((k\mathfrak{M}(X, \Omega), \cdot, (P_\omega)_{\omega \in \Omega})\) together with \(j_x\) is the free \(\Omega\)-operated algebra on \(X\).

Lemma 2.9 and the uniqueness of the free objects in the category of \(\Omega\)-operated algebras then yield the isomorphism of \(\Omega\)-operated algebras
\[
\theta : (k\mathfrak{M}(X, \Omega), \cdot, (P_\omega)_{\omega \in \Omega}) \cong (k\mathfrak{F}(X, \Omega), \cdot, (B^+_{\omega})_{\omega \in \Omega}),
\]
sending \(x \in X\) to \(\theta(x) := \cdot_x\). Comparing Eqs. (3) and (13), we see that \(\theta\) preserves the filtrations of bracketed words in \(\mathfrak{M}(X, \Omega)\) and forests in \(\mathfrak{F}(X, \Omega)\) given by depths:
\[
\theta([\mathcal{M}_n]) = \mathfrak{F}_n, \quad n \geq 0.
\]

Further, for \(w \in \mathfrak{M}(X, \Omega)\), let \(\text{deg}_{\mathfrak{M}}(w)\), called the total degree of \(w\), denote the total number (counting multiplicities) of the appearances of elements of \(X\) and brackets \([\cdot]_{\omega}\), \(\omega \in \Omega\), in \(w\). So \([xy]_{\omega}\) has \(\text{deg}_{\mathfrak{M}}(w) = 6\) since the letters appear four time and the operators appear twice.

For \(n \geq 0\), let \(S_n\) denote the subset of \(\mathfrak{M}(X, \Omega)\) with total degree \(n\) and let \(S_n\) denote the union \(\bigcup_{k \leq n} S_{k}^{(k)}\). Then we have a grading and a filtration
\[
k\mathfrak{M}(X, \Omega) = \bigoplus_{n \geq 0} k\mathfrak{M}^{(n)}, \quad k\mathfrak{M}_{(n)} \subseteq k\mathfrak{M}_{(n+1)}, \quad n \geq 0.
\]
Since the map $\theta$ sends $x \in X$ to $\bullet$, and $[w]_\omega$ to $B^\omega_\omega(\theta(w))$, it preserves the degrees: $\deg_m(w) = \deg(\theta(w))$, and the resulting gradings and filtrations. Thus as a consequence of Lemma 2.9, we have

**Corollary 2.14.** With the grading and its associated filtration on $k\mathfrak{M}(X, \Omega)$ defined by the total degree $\deg_t$ in Eq. (13), the free $\Omega$-operated algebra $k\mathfrak{M}(X, \Omega)$ is an $\Omega$-operated graded algebra and an $\Omega$-operated filtered algebra, isomorphic to the ones for $k\mathfrak{T}(X, \Omega)$ in Lemma 2.9.

### 3. Gröbner-Shirshov bases and free matching Rota-Baxter algebras

In this section we construct free matching Rota-Baxter algebras from bracketed words and decorated rooted forests by the method of Gröbner-Shirshov bases. We begin with a brief review of matching Rota-Baxter algebras emphasizing their many connections. We then recall the Composition-Diamond (CD) Lemma for the Gröbner-Shirshov bases of operated algebras. With these preparations, the Gröbner-Shirshov bases for matching Rota-Baxter algebras is then obtained. This gives the desired construction of free matching Rota-Baxter algebras in terms of bracketed words. We finally apply the isomorphism in Eq. (16) to give a construction of free matching Rota-Baxter algebras in terms of decorated rooted forests.

#### 3.1. Matching Rota-Baxter algebras

In this subsection, we recall the concept of matching Rota-Baxter algebras, which generalizes that of Rota-Baxter algebras.

**Definition 3.1.** [11] Let $\Omega$ be a nonempty set and let $\lambda_\Omega := (\lambda_\omega)_{\omega \in \Omega} \subseteq k$ be a parameterized family of scalars with index set $\Omega$. More precisely, $\lambda_\Omega$ is a map $\Omega \to k$.

(a) A **matching (multiple) Rota-Baxter algebra** of weight $\lambda_\Omega$ is a pair $(R, P_\Omega)$ consisting of an algebra $R$ and a family $P_\Omega := (P_\omega)_{\omega \in \Omega}$ of linear operators $P_\omega : R \to R$, $\omega \in \Omega$, that satisfy the **matching Rota-Baxter equation**

$$P_\alpha(x)P_\beta(y) = P_\alpha(xP_\beta(y)) + P_\beta(P_\alpha(x)y) + \lambda_\beta P_\alpha(xy) \quad \text{for all } x, y \in R, \alpha, \beta \in \Omega.$$  

When $\lambda_\Omega = \{\lambda\}$ in $(R, \lambda_\Omega)$, that is, when $\lambda_\Omega : \Omega \to k$ is constant, we also call the matching Rota-Baxter algebra to have weight $\lambda$.

(b) Let $(R, P_\Omega)$ and $(R', P'_\Omega)$ be matching Rota-Baxter algebras of the same weight $\lambda_\Omega$. A linear map $\phi : R \to R'$ is called a **matching Rota-Baxter algebra homomorphism** if $\phi$ is an algebra homomorphism such that $\phi P_\omega = P'_\omega \circ \phi$ for all $\omega \in \Omega$.

To motivate our study of free matching Rota-Baxter algebras, we list some properties of matching Rota-Baxter algebras and refer the reader to [11], [14], [20] for further details.

(a) Any Rota-Baxter algebra of weight $\lambda$ can be viewed as a matching Rota-Baxter algebra of weight $\lambda$ by taking $\Omega$ to be a singleton.

(b) When $\lambda = 0$, the matching Rota-Baxter equation is Lie compatible in the sense that

$$[P_\alpha(x), P_\beta(y)] = P_\alpha([x, P_\beta(y)]) + P_\beta([P_\alpha(x), y]).$$

Here the Lie bracket is taking as the commutator. In the case when $|\Omega| = 2$, this has been studied in [13].

(c) [11], Proposition 2.5] Matching Rota-Baxter algebras provide a solution to the linearity of the set of Rota-Baxter operators on an algebra as follows. Let $(R, (P_\omega)_{\omega \in \Omega})$ be a matching Rota-Baxter algebra of weight $\lambda$. Then any finite linear combination

$$P := \sum_{\omega \in \Omega} k_\omega P_\omega, \quad k_\omega \in k,$$
with $k_\omega \in k$ is a Rota-Baxter operator of weight $\lambda \sum k_\omega$. In particular, if $\sum k_\omega = 1$, then $P$ is a Rota-Baxter algebra of weight $\lambda$. Thus any element in the linear span $\sum k_\omega P_\omega$ of $P_\Omega$ is a Rota-Baxter operator of certain weight.

(d) [11, Corollary 4.5] Matching Rota-Baxter algebras have a close connection with matching (multiple) pre-Lie algebras introduced by Foissy [13]. Let $(R, (P_\omega)_{\omega \in \Omega})$ be a matching Rota-Baxter algebra of weight $\lambda_\Omega$. Define

$$x \ast_\omega y := P_\omega(x)y - y P_\omega(x) - \lambda_\omega xy \text{ for } x, y, z \in R, \omega \in \Omega.$$

Then the pair $(R, (\ast_\omega)_{\omega \in \Omega})$ is a matching (multiple) pre-Lie algebra.

(e) [11, Theorem 3.4] A matching Rota-Baxter algebra $(R, (P_\omega)_{\omega \in \Omega})$ of weight $\lambda_\Omega$ induces a matching dendriform algebra $(R, (<_\omega)_{\omega \in \Omega}, (>_\omega)_{\omega \in \Omega})$, where

$$x <_\omega y := x P_\omega(y) + \lambda_\omega xy, \quad x >_\omega y := P_\omega(x)y \text{ for } x, y \in R, \omega \in \Omega.$$

(f) [11, Example 2.3], [13] Consider the $\mathbb{R}$-algebra $R := \text{Cont}(\mathbb{R})$ of continuous functions on $\mathbb{R}$. Let $K_\omega(x, t)$ be a parameterized family of kernels of continuous functions on $\mathbb{R}^2$ and let

$$(I_\omega : R \longrightarrow R, \quad f(x) \mapsto \int_0^x K_\omega(x, t)f(t)\, dt, \quad \omega \in \Omega),$$

be the corresponding family of Volterra integral operators [17]. Then when $K_\omega(x, t)$ is independent of $x$, the pair $(R, (I_\omega)_{\omega \in \Omega})$ is a matching Rota-Baxter algebra of weight zero.

(g) [11, § 2.2] For $r, s \in R \otimes R$, let

$$r_{13}s_{12} - r_{12}s_{23} + r_{23}s_{13} = -\lambda s_{13}$$

be the polarized associative Yang-Baxter equation of weight $\lambda$. Then a solution of this equation gives a matching Rota-Baxter operator of weight $\lambda$.

The purpose of this section is to construct free matching Rota-Baxter algebras. Since by Proposition 2.13, $\mathfrak{k}\mathfrak{m}(X, \Omega)$ is the free $\Omega$-operated algebra on a set $X$, the free matching Rota-Baxter algebra on $X$ is obtained by taking the quotient of $\mathfrak{k}\mathfrak{m}(X, \Omega)$ modulo the operated ideal generated by the matching Rota-Baxter algebra relations. More precisely, let $\text{Id}(S)$ be the operated ideal of $\mathfrak{k}\mathfrak{m}(X, \Omega)$ generated by the set

$$S := \{ [x_\alpha | y_\beta] - [x_\alpha | y_\beta]_\alpha - [x_\alpha | y_\beta]_\beta - \lambda_\beta [x_\alpha y_\beta]_\alpha | x, y \in \mathfrak{m}(\Omega, X), \alpha, \beta \in \Omega \}. $$

Then the free matching Rota-Baxter algebra $\mathfrak{m}^{\text{NC}}(X, \Omega)$ is given by the quotient $\mathfrak{k}\mathfrak{m}(X, \Omega)/\text{Id}(S)$. We will identify a canonical subset of $\mathfrak{m}(X, \Omega)$ which gives a linear basis of this quotient and express the operation of the matching Rota-Baxter algebra in terms of this basis.

Of course the free matching Rota-Baxter algebra is also given by taking the quotient of the other realization $\mathfrak{k}\mathcal{F}(X, \Omega)$ of the free $\Omega$-operated algebra on the set $X$ modulo the matching Rota-Baxter algebra relations: $\mathfrak{k}\mathcal{F}(X, \Omega)/\text{Id}(\Xi)$, where $\text{Id}(\Xi)$ is the $\Omega$-operated ideal of $\mathfrak{k}\mathcal{F}(X, \Omega)$ generated by

$$\Xi := \left\{ B_\alpha^\ast(F_1) \circ_\ell B_\beta^\ast(F_2) - B_\alpha^\ast(F_1 \circ_\ell B_\beta^\ast(F_2)) - B_\alpha^\ast(B_\beta^\ast(F_1) \circ_\ell F_2) - \lambda_\beta B_\alpha^\ast(F_1 \circ_\ell F_2) | \alpha, \beta \in \Omega \right\}.$$  

We choose to work with $\mathfrak{k}\mathfrak{m}(X, \Omega)$ since it provides a simpler context to apply the method of Gröbner-Shirshov bases, as we will carry out next.

3.2. Gröbner-Shirshov bases of free matching Rota-Baxter algebras. In this subsection, we recall the Composition-Diamond Lemma for the $\Omega$-operated (unitary) algebra $\mathfrak{k}\mathfrak{m}(X, \Omega)$ and apply it to construct a linear basis of the free matching Rota-Baxter algebra on a set.
3.2.1. Composition-Diamond Lemma for free $\Omega$-operated algebras. For further details on the notations and background, we refer the reader to [1, 2, 22].

Let $X$ be a set and $\Omega$ a nonempty set. $\star \not\in X$, and $X^* := X \sqcup \{\star\}$. By a $\star$-bracketed word on $X$, we mean any bracketed word in $\mathcal{M}(\Omega, X) := \mathcal{M}(\Omega, X^*)$ with exactly one occurrence of $\star$, counting multiplicities. For $q \in \mathcal{M}(\Omega, X)^*$ and $u \in \mathcal{M}(\Omega, X)$, we define

$$q_{|u} := q|_{\star \mapsto u}$$

to be the bracketed word on $X$ obtained by replacing the unique occurrence of $\star$ in $q$ by $u$. For $q \in \mathcal{M}(\Omega, X)^*$ and $s = \Sigma_i c_i q_{|u_i} \in k\mathcal{M}(\Omega, X)$, where $c_i \in k$ and $u_i \in \mathcal{M}(\Omega, X)$, we define

$$q_{|s} := \Sigma_i c_i q_{|u_i},$$

and extend this notation to any $q \in k\mathcal{M}^*(\Omega, X)$ by linearity. Note that the element $q_{|u}$ is usually not a bracketed word but a bracketed polynomial.

A monomial order on $\mathcal{M}(\Omega, X)$ is a well order $\leq$ on $\mathcal{M}(\Omega, X)$ such that

$$u \leq v \Rightarrow q_{|u} < q_{|v}, \text{ for all } u, v \in \mathcal{M}(\Omega, X) \text{ and all } q \in \mathcal{M}^*(\Omega, X).$$

Here, as usual, we denote $u < v$ if $u \leq v$ but $u \neq v$. Since $\leq$ is a well order, it follows from Eq. (23) that $1 \leq u$ and $u < |u|_{\text{le}}$ for all $u \in \mathcal{M}(\Omega, X)$ and $\omega \in \Omega$.

Let $\leq$ be a monomial order on $\mathcal{M}(\Omega, X)$ and let $f \in k\mathcal{M}(\Omega, X)$.

(a) If $f \not\in k$, the unique largest monomial $\bar{f}$ appearing in $f$ is called the leading bracketed word (monomial) of $f$.

(b) The coefficient of $\bar{f}$ in $f$ is called the leading coefficient of $f$, which is denoted by $c(\bar{f})$.

(c) If $f \not\in k$ and $c(f) = 1$, then $f$ is monic with respect to the monomial order $\leq$ and a subset $S \subset k\mathcal{M}(\Omega, X)$ is monic with respect to $\leq$ if every $s \in S$ is monic with respect to $\leq$.

The notion of a Gröbner-Shirshov basis is given in terms of compositions and triviality of compositions, encoding the notion of critical pairs in a rewriting system [3].

**Definition 3.2.** Let $f, g \in k\mathcal{M}(\Omega, X)$ be $\Omega$-bracketed polynomials monic with respect to $\leq$. Let $\bar{f}$ be the leading monomial of $f$, and let $|\bar{f}|$ denote its breadth.

(a) If there exist $u, v, w \in \mathcal{M}(\Omega, X)$ such that $w = \bar{f}u = v \bar{g}$ with max $\{|\bar{f}|, |\bar{g}|\} < w < |\bar{f}| + |\bar{g}|$, then the $\Omega$-bracketed polynomial

$$(f, g)_w := (f, g)_{\text{int}(u, v, w)} := f u - v g$$

is called the **intersection composition of $f$ and $g$ with respect to $(u, v)$**.

(b) If there exist $q \in \mathcal{M}^*(\Omega, X)$ and $w \in \mathcal{M}(\Omega, X)$ such that $w = \bar{f} := (f, g)_q := q|_g$, then the $\Omega$-bracketed polynomial

$$(f, g)_w := f - q|_g$$

is called the **including composition of $f$ and $g$ with respect to $q$**.

In both cases, the bracketed word $w$ is called the **ambiguity** for the compositions.

Now we arrive at the key notion of a Gröbner-Shirshov basis in which the confluency of critical pairs is captured by a triviality condition.

**Definition 3.3.** Let $S \subseteq k\mathcal{M}(\Omega, X)$ be a set of $\Omega$-bracketed polynomials that is monic with respect to a monomial order $\leq$, and let $w \in \mathcal{M}(\Omega, X)$.

(a) An element $u \in k\mathcal{M}(X, \Omega)$ is called **trivial modulo** $(S, w)$ if $u$ can be written as a linear combination

$$\sum_i c_i q_i|_{s_i},$$

with $0 \neq c_i \in k, q_i \in \mathcal{M}(\Omega, X)^*, s_i \in S$ and $q_i|_{s_i} < w$. Then we denote

$$u \equiv 0 \text{ mod } (S, W).$$

(b) The set $S$ is called a **Gröbner-Shirshov bases** (with respect to $\leq$), if for each pair $f, g \in S$ with $f \neq g$, every including composition and including composition $(f, g)_w$ of $f$ and $g$ is trivial modulo $(S, w)$.
For $u, v \in k\mathcal{M}(\Omega, X)$, we say $u$ and $v$ are congruent modulo $(S, w)$ and denote by $u \equiv v \mod (S, w)$ if $u - v$ is trivial modulo $(S, w)$.

The following theorem is the Composition-Diamond Lemma for $\Omega$-(unitary) algebras, adapting from the case for $\Omega$-nonunitary algebras in [5]. See also [32].

**Theorem 3.4.** [4, Theorem 3.13] Let $X$ be a set and $\Omega$ a nonempty set, and let $\preceq$ be a monomial order on $\mathcal{M}(\Omega, X)$. Let $S$ be a set of $\Omega$-bracketed polynomials in $k\mathcal{M}(\Omega, X)$ which are monic with respect to $\preceq$ and let $\text{Id}(S)$ be the $\Omega$-operated ideal of $k\mathcal{M}(\Omega, X)$ generated by $S$. Then the following statements are equivalent:

(a) $S$ is a Gröbner-Shirshov basis in $k\mathcal{M}(\Omega, X)$.

(b) For every non-zero $f \in \text{Id}(S)$, then $\bar{f} = q|\bar{s}$ for some $q \in \mathcal{M}(\Omega, X)^*$ and $s \in S$.

(c) Let

$$\text{Irr}(S) := \{w \in \mathcal{M}(\Omega, X) | w \neq q|\bar{s}, q \in \mathcal{M}(\Omega, X)^*, s \in S\} = \mathcal{M}(X, \Omega) \setminus \{q|\bar{s} | q \in \mathcal{M}(\Omega, X)^*, s \in S\}.$$

Then there is a linear decomposition $k\mathcal{M}(X, \Omega) = \text{Id}(S) \oplus \text{Irr}(S)$. Thus $\text{Irr}(S)$ modulo $\text{Id}(S)$ is a $k$-linear basis of $k\mathcal{M}(X, \Omega)/\text{Id}(S)$.

3.2.2. Gröbner-Shirshov bases for free matching Rota-Baxter algebras. We now show that the matching Rota-Baxter relations form a Gröbner-Shirshov basis of the free $\Omega$-operated algebras $k\mathcal{M}(\Omega, X)$, and hence gives rise to a linear basis of the free matching Rota-Baxter algebra thanks to the Composition-Diamond Lemma in Theorem 3.4.

Let $X$ and $\emptyset \neq \Omega$ be well-ordered sets. For notational convenience, we also denote $P_\omega(u) = [u]_\omega$. So one appearance of $P_\omega$ in a bracketed work $w \in \mathcal{M}(X, \Omega)$ means one appearance of a bracket $[u]_\omega$ in $w$. For $u = u_1 \cdots u_r \in M(X)$ with $u_1, \ldots, u_r \in X$, define $\deg_X(u) = r$ if $u \neq 1$ and $\deg_X(1) = 0$. Extend the well order $\preceq$ on $X$ to the **degree lexicographical order** $\preceq$ on $M(X)$ by taking, for any $u = u_1 \cdots u_r, v = v_1 \cdots v_s \in M(X) \setminus \{1\}$, where $u_1, \ldots, u_r, v_1, \ldots, v_s \in X$,

\[ u \prec v \iff \begin{cases} \deg_X(u) < \deg_X(v), \\ \text{or } \deg_X(u) = \deg_X(v) = r \text{ and } (u_1, \ldots, u_r) \preceq (v_1, \ldots, v_s) \text{ lexicographically,} \end{cases} \]

(24)

Here we use the convention that the empty word $1 \leq u$ for all $u \in M(X)$. Then $\preceq$ is a well order on $M(X) [5].$

Further we extend $\preceq$ to $\mathcal{M}(X, \Omega)$. Applying Eq. (13) and grouping adjacent letters in $X$ together, we find that every $u \in \mathcal{M}(X)$ may be uniquely written as a product in the form

\[ u = u_0 P_{\alpha_1}(u^*_1)^{u_1} P_{\alpha_2}(u^*_2)^{u_2} \cdots P_{\alpha_r}(u^*_r)^{u_r}, \]

where

\[ u_0, \ldots, u_r \in M(X), u^*_1, \ldots, u^*_r \in \mathcal{M}_{n-1}(X) \text{ and } \alpha_1, \ldots, \alpha_r \in \Omega. \]

Denote by $\deg_P(u)$ the number of occurrence of $P_\omega = [u]_\omega, \omega \in \Omega$, and define the $P$-breadth $\breve{p}(u)$ of $u$ to be $r$. For example, we have

\[ u := x_0 P_{\alpha_1}(x_1)x_2 P_{\alpha_2}(x_3P_{\alpha_3}(x_4))x_5x_6 = u_0 P_{\alpha_1}(u^*_1)^{u_1} P_{\alpha_2}(u^*_2)^{u_2} u_2, \quad x_0, \ldots, x_6 \in X, \quad \alpha_1, \alpha_2, \alpha_3 \in \Omega, \]

where $u_0 = x_0, u_1 = x_2, u_2 = x_5x_6, u^*_1 = x_1, u^*_2 = x_3P_{\alpha_3}(x_4), \deg_P(u) = 3$ and $\breve{p}(u) = 2$.

Let $u, v \in \mathcal{M}(X)$ and write them uniquely in the form of Eq. (25):

\[ u = u_0 P_{\alpha_1}(u^*_1)^{u_1} P_{\alpha_2}(u^*_2)^{u_2} \cdots P_{\alpha_r}(u^*_r)^{u_r} u_r \quad \text{and} \quad v = v_0 P_{\beta_1}(v^*_1)^{v_1} P_{\beta_2}(v^*_2)^{v_2} \cdots P_{\beta_s}(v^*_s)^{v_s}. \]

We define $u \leq_P v$ by induction on $\deg(u) + \deg(v) \geq 0$. For the initial step of $\deg(u) + \deg(v) = 0$, we have $u, v \in M(X)$ and use the degree lexicographical order given in Eq. (24). For the induction step of
dep(u) + dep(v) ≥ 1, we define
\[ u \leq \text{db} v \Longleftrightarrow \begin{cases} \deg_P(u) < \deg_P(v), \\
\text{or } \deg_P(u) = \deg_P(v) \text{ and } \bre_P(u) < \bre_P(v), \\
\text{or } \deg_P(u) = \deg_P(v) \text{ and } \bre_P(u) = \bre_P(v)(= r) \text{ and} \\
(P_{\alpha_1}, u_1', \ldots, P_{\alpha_t}, u_t, 0, \ldots, 0_t) \leq (P_{\beta_1}, v_1', \ldots, P_{\beta_t}, v_t, 0, \ldots, 0_t) \text{ lexicographically.} \end{cases} \]

Here \( P_{\alpha_i} \leq P_{\beta_i} \) is compared by the order on \( \Omega \) and \( u'_t \leq \text{db} v'_t \) and \( u_t \leq \text{db} v_t \) are compared by the induction hypothesis. With a similar argument to the case of \( \text{rs}_\text{db} \) on \( \mathfrak{W}(X, \Omega) \), the above defined \( \leq \text{db} \) is a monomial order on \( \mathfrak{W}(X, \Omega) \). In fact when \( \Omega \) is a singleton, the above defined \( \leq \text{db} \) is exactly the one given in [13] on \( \mathfrak{W}(X) \). See also [22].

**Theorem 3.5.** With the order \( \leq \text{db} \) on \( \mathfrak{W}(\Omega, X) \), the set
\[ S = \{ [x]_\alpha [y]_\beta - [x]_\alpha [y]_\beta - \lambda [x,y]_\alpha \mid x, y \in \mathfrak{W}(\Omega, X), \alpha, \beta \in \Omega \} \]
is a Gröbner-Shirshov basis in \( k\mathfrak{W}(\Omega, X) \).

**Proof.** With the leading terms from \( S \) in the form of \( [x]_\alpha [y]_\beta \), all the possible ambiguities for compositions of \( \Omega \)-bracketed polynomials in \( S \) are of the following three forms.
\[ w_1 := [x]_\alpha [y]_\beta [z]_\gamma, \quad w_2 := u[y]_\delta [y]_\beta [z]_\gamma, \quad w_3 := [z]_\delta [u][y]_\beta [z]_\gamma. \]
where \( x, y, z \in \mathfrak{W}(\Omega, X), \alpha, \beta, \gamma, \delta \in \Omega, \ u \in \mathfrak{W}(\Omega, X)^* \). We now check that all these compositions are trivial.

**Case 1.** \( w_1 = [x]_\alpha [y]_\beta [z]_\gamma \).
In this case, we may write
\[ f := f_{\alpha, \beta}(x, y) = [x]_\alpha [y]_\beta - [x]_\alpha [y]_\beta - \lambda [x,y]_\alpha, \]
\[ g := g_{\beta, \gamma}(y, z) = [y]_\beta [z]_\gamma - [y]_\beta [z]_\gamma - \lambda [y,z]_\beta. \]
Then we have
\[ \tilde{f} = [x]_\alpha [y]_\beta \quad \text{and} \quad \tilde{g} = [y]_\beta [z]_\gamma. \]
Thus
\[
(f, g)_{w_1} = f[l]_\gamma - [l]_\alpha g
\]
\[ = [x]_\alpha [y]_\beta [z]_\gamma - [x]_\alpha [y]_\beta [z]_\gamma - [x]_\alpha [y]_\beta [z]_\gamma - \lambda [x,y]_\alpha [z]_\gamma
\]
\[ - [x]_\alpha [y]_\beta [z]_\gamma - [x]_\alpha [y]_\beta [z]_\gamma - [x]_\alpha [y]_\beta [z]_\gamma + \lambda [x,y]_\alpha [z]_\gamma
\]
\[ = - [x]_\alpha [y]_\beta [z]_\gamma - [x]_\alpha [y]_\beta [z]_\gamma - [x]_\alpha [y]_\beta [z]_\gamma + \lambda [x,y]_\alpha [z]_\gamma
\]
\[ = - f_{\alpha, \beta}(x, y, z) - g_{\beta, \gamma}(x, y, z) - f_{\alpha, \beta}(x, y, z) - g_{\beta, \gamma}(x, y, z) - \lambda [x,y]_\alpha [z]_\gamma
\]
}\]
which is trivial modulo \((S, w_1)\) since

\[
\begin{align*}
[x \star \frac{f_g, y(x, y), z}{g_{x, y}(x, y)}]_{\alpha} &= [x \star \frac{y_{x, y}, z}{y_{x, y}, z}]_{\alpha} = [x \star y_{x, y}, z]_{\alpha} <_{\text{bd}} [x \star y_{x, y}, z]_{\alpha} = w_1, \\
\star [f_{g, y}(x, y), z] &= \star [y_{x, y}, z] = [y_{x, y}, z] <_{\text{bd}} [y_{x, y}, z] = w_1, \\
\star [g_{x, y}(x, y), z] &= \star [x, z] = [x, z] <_{\text{bd}} [x, z] = w_1, \\
\star [f_{g, y}(x, y), z] &= \star [y_{x, y}, z] = [y_{x, y}, z] <_{\text{bd}} [y_{x, y}, z] = w_1, \\
\star [f_{g, y}(x, y), z] &= \star [y_{x, y}, z] = [y_{x, y}, z] <_{\text{bd}} [y_{x, y}, z] = w_1, \\
\star [f_{g, y}(x, y), z] &= \star [y_{x, y}, z] = [y_{x, y}, z] <_{\text{bd}} [y_{x, y}, z] = w_1.
\end{align*}
\]

\textbf{Case 2.} \(w_2 = [u_{x, y}, z]_{\alpha} \). In this case, we may write

\[
q := [u_{x, y}, z]_{\alpha}, \ g := g_{x, y}(x, y) = [x_{y}, y]_{x} - [x_{y}, y]_{x} - [x_{y}, y] - \lambda y, \ x_{y},
\]

\[
f := f_{a, \alpha}(u_{x, y}, z) = [u_{x, y}, z]_{\alpha} - [u_{x, y}, z]_{\alpha} - [u_{x, y}, z]_{\alpha} - \lambda y, a, u_{x, y}, z.
\]

Then we have

\[
\tilde{f} = [u_{x, y}, z]_{\alpha}, \ \tilde{g} = [x_{y}, y]_{x}, \ \text{and} \ \tilde{f} = q \tilde{g}.
\]

Thus

\[
(f, g)_{w_2} = f - q g = [u_{x, y}, z]_{\alpha} - [u_{x, y}, z]_{\alpha} - [u_{x, y}, z]_{\alpha} - \lambda y, a, u_{x, y}, z,
\]

which is trivial modulo \((S, w_2)\) since

\[
\begin{align*}
\star [u_{g, x}(y, x), z] &= \star [u_{g, x}(y, x), z] = [u_{g, x}(y, x), z]_{\alpha} <_{\text{bd}} [u_{g, x}(y, x), z]_{\alpha} = w_2, \\
\star [u_{g, x}(y, x), z] &= \star [u_{g, x}(y, x), z] = [u_{g, x}(y, x), z]_{\alpha} <_{\text{bd}} [u_{g, x}(y, x), z]_{\alpha} = w_2,
\end{align*}
\]

\[
\begin{align*}
\star [f_{a, \alpha}(u_{x, y}, z)] &= \star [u_{x, y}, z] = [u_{x, y}, z]_{\alpha} <_{\text{bd}} [u_{x, y}, z]_{\alpha} = w_2, \\
\star [f_{a, \alpha}(u_{x, y}, z)] &= \star [u_{x, y}, z] = [u_{x, y}, z]_{\alpha} <_{\text{bd}} [u_{x, y}, z]_{\alpha} = w_2,
\end{align*}
\]
Thus Section 3.4 and 3.5 will give a linear basis of the free matching Rota-Baxter algebras on the set $X$. By Theorems 3.4 and 3.5, the set consists of bracketed words in $\mathfrak{M}(X, \Omega)$ that do not contain subwords of the form $[u]_\beta [v]_\alpha$ for any $u, v \in \mathfrak{M}(X, \Omega)$ and $\alpha, \beta \in \Omega$. As in the case of one operator \([19]\), we give a description of $\text{Irr}(S)$ by inclusion conditions, rather than the above exclusive conditions. For subsets $U, V \subseteq \mathfrak{M}(X, \Omega)$ and $r \geq 1$, we use the abbreviations

\[
UV := \{uv \mid u \in U, v \in V\}, \quad U^r := \{u_1 \cdots u_r \mid u_i \in U, 1 \leq i \leq r\}, \quad [U]_\Omega := \{[u]_\omega \mid u \in U, \omega \in \Omega\}.
\]

**Definition 3.6.** Let $Y, Z$ be subsets of $\mathfrak{M}(\Omega, X)$. Define the **alternating products** of $Y$ and $Z$ by

\[
\Lambda(Y, Z) := \left( \bigcup_{r \geq 1} (Y[Z]_\Omega)^r \right) \bigcup \left( \bigcup_{r \geq 0} (Y[Z]_\Omega)^r \cdot Y \right) \bigcup \left( \bigcup_{r \geq 1} ([Z]_\Omega)^r \cdot Y \right) \bigcup \left( \bigcup_{r \geq 0} ([Z]_\Omega)^r \cdot [Z]_\Omega \right) \bigcup \{1\},
\]

where $1$ is the identity in $\mathfrak{M}(\Omega, X)$.

We observe that $\Lambda(Y, Z) \subseteq \mathfrak{M}(\Omega, X)$. Then we recursively define

\[
X_0 := M(X) = S(X) \cup \{1\} \quad \text{and} \quad X_n := \Lambda(S(X), X_{n-1}), \quad n \geq 1.
\]

Thus $X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n \subseteq \cdots$. Finally we define

\[
X_\infty := \lim_{\rightarrow} X_n = \bigcup_{n \geq 0} X_n.
\]

Elements in $X_\infty$ are called **matching Rota-Baxter words** (MRBWs). For a MRBW $w \in X_\infty$, we call $\text{dep}(w) := \min\{n \mid w \in X_n\}$ the **depth** of $w$, which agrees with the depth of $w$ as an element of $\mathfrak{M}(X, \Omega)$ in Section 2.4.

The following properties of MRBWs are easy to verify as in the case of one operator \([19]\).

**Lemma 3.7.** Every MRBW $w \neq 1$ has a unique alternating decomposition: $w = w_1 \cdots w_m$, where $w_i \in X \cup [X_\infty]_\Omega$, $1 \leq i \leq m$, $m \geq 1$, and no consecutive elements in the sequence $w_1, \cdots, w_m$ are in $[X_\infty]_\Omega$.

Recall that $k\mathfrak{M}(\Omega, X)/\text{Id}(S)$ is a free matching Rota-Baxter algebra on the set $X$, and has a linear basis $\text{Irr}(S)$ by Theorems 3.4 and 3.5. Thus with the evident identification of $\text{Irr}(S)$ with $X_\infty$ thanks to Lemma 3.7, we obtain

**Theorem 3.8.** The set $\text{Irr}(S) = X_\infty$ modulo $\text{Id}(S)$ is a linear basis of the free matching Rota-Baxter algebra $k\mathfrak{M}(\Omega, X)/\text{Id}(S)$ of weight $\lambda_\Omega$ on $X$.

Thus the quotient map $\varphi : k\mathfrak{M}(\Omega, X) \to k\mathfrak{M}(\Omega, X)/\text{Id}(S)$ of $\Omega$-operated algebras becomes the projection

\[
(26) \quad \varphi : k\mathfrak{M}(\Omega, X) = \text{Id}(S) \oplus kX_\infty \longrightarrow kX_\infty.
\]

Let $\mathfrak{M}^N(\Omega, X) := kX_\infty$ be the resulting free matching Rota-Baxter algebra on $X$ with its matching Rota-Baxter operators $P_\omega, \omega \in \Omega$ and multiplication $\varphi_\omega$. We first note the inclusions $[X_\infty]_\alpha \subseteq X_\infty, \alpha \in \Omega$. Thus the matching Rota-Baxter operator $P_\omega$ on $kX_\infty$ is simply

\[
P_\omega(w) = [w]_\omega.
\]
For the product \( \circ_w \) in \( \Pi_{\Omega}^{NC}(X, \Omega) \), we have the following algorithm in analog to those for the product of two Rota-Baxter words (resp. Rota-Baxter system words) in the free Rota-Baxter algebra \([19]\) (resp. free Rota-Baxter system \([32]\)).

Algorithm 3.9. Let \( w, w' \in X_\infty \). We define the product \( w \circ_w w' \) inductively on the sum of depths \( n := \text{dep}(w) + \text{dep}(w') \geq 0 \).

(a) If \( n = 0 \), then \( w, w' \in X_0 = M(X) \) and define \( w \circ_w w' := ww' \).

(b) Let \( k \geq 0 \). Suppose that \( w \circ_w w' \) have been defined for \( 0 \leq n \leq k \) and consider the case of \( n = k + 1 \). We need to consider the following two cases.

Case 1. \( \text{bre}(w), \text{bre}(w') \leq 1 \). We define

\[
(28) \quad w \circ_w w' = \left\{ \begin{array}{ll}
[w \circ_w w']_{\alpha} + [w \circ_w w']_{\beta} + \lambda_{\beta}[w \circ_w w']_{\alpha}, & \text{if } w = [w]_{\alpha} \text{ and } w' = [w']_{\beta}, \\
0, & \text{otherwise},
\end{array} \right.
\]

where \( \alpha, \beta \in \Omega \).

Case 2. \( \text{bre}(w) \geq 2 \) or \( \text{bre}(w') \geq 2 \). Let \( w = w_1 \cdots w_m \) and \( w' = w'_1 \cdots w'_m \) be the alternating decompositions of \( w \) and \( w' \), respectively. Define

\[
(29) \quad w \circ_w w' := w_1 \cdots w_m - (w_m \circ_w w'_1)w'_2 \cdots w'_m,
\]

where \( w_m \circ_w w'_1 \) is defined by Eq. (28) and the rest of the products are given by the concatenation.

Different from the filtration defined by the depth, the grading \( \mathcal{M}(n) \) and filtration \( \mathcal{M}(n) \) on \( \mathcal{M}(X, \Omega) \) defined by the total degree \( \text{deg}_{\mathcal{E}_{\mathcal{M}}} \) in Eq. (19) restrict those on \( X_\infty \):

\[
(30) \quad X^{(n)} := \mathcal{M}(n) \cap X_\infty, \quad X^{(n)} := \mathcal{M}(n) \cap X_\infty, \quad n \geq 0.
\]

The resulting grading \( \Pi_{\Omega}^{NC}(X, \Omega) = \bigoplus_{n \geq 0} kX^{(n)} \) holds only linearly since the multiplication does not preserve the grading. However, we have the following compatibility for the filtered structures.

**Proposition 3.10.** The free matching Rota-Baxter algebra \( \Pi_{\Omega}^{NC}(X, \Omega) \), with the filtration

\[
(31) \quad \Pi_{\Omega}^{NC}(X, \Omega)(n) := kX^{(n)}, \quad n \geq 0,
\]

is an \( \Omega \)-operated filtered algebra as defined in Definition [24].

Moreover, the homomorphism \( \varphi : k\mathcal{M}(X, \Omega) \rightarrow \Pi_{\Omega}^{NC}(X, \Omega) \) of \( \Omega \)-operated algebras preserves the filtrations:

\[
(32) \quad \varphi(k\mathcal{M}(X, \Omega)(n)) \subseteq k\mathcal{M}(X, \Omega)(n), \quad n \geq 0.
\]

**Proof.** The first inclusion in Eq. (31) follows from the definition of the operators \( P_{\alpha} \) in Eq. (27).

For the second inclusion in Eq. (31), by linearity we just need to prove

\[
(33) \quad \mathcal{X}_{(p)} \circ_w \mathcal{X}_{(q)} \subseteq k\mathcal{X}_{(p+q)}, \quad p, q \geq 0,
\]

for which we apply induction on \( p + q \geq 0 \), with the general remark that \( \text{deg}_{\mathcal{E}_{\mathcal{M}}} \) is additive with respect to the concatenation product. Thus for \( w \in \mathcal{X}_{(p)} \) and \( w' \in \mathcal{X}_{(q)} \), we have \( w \circ_w w' \in \mathcal{X}_{(p+q)} \) as long as \( w \circ_w w' = ww' \) is the concatenation.

When \( p + q = 0 \), we have \( p = q = 0 \). Since \( \mathcal{X}_{(0)} = \mathcal{X}^{(0)} = M(X) = \mathcal{M}_0 \) on which the product \( \circ_w \) is the concatenation, the inclusion holds by the above general remark. Let \( k \geq 0 \). Assume that the inclusion holds for \( p + q < k \) and consider the case when \( p + q = k + 1 \). If either \( p \) or \( q \) is zero, then \( \mathcal{X}_p \circ_w \mathcal{X}_q \) is the concatenation and the desired inclusion again follows. If none of \( p \) or \( q \) is zero and consider \( w = w_1 \cdots w_m \in \mathcal{X}_p \) and \( w' = w'_1 \cdots w'_m \in \mathcal{X}_q \) with their alternating decompositions. Then \( w \circ_w w' \) is again the concatenation and \( w \circ_w w' \) is in \( \mathcal{X}_{(p+q)} \) except when \( w_m = [\mathcal{W}]_{\alpha} \) and \( w'_1 = [\mathcal{W}'_1]_{\beta} \). In which case, Eq. (28) gives

\[
(34) \quad w \circ_w w' = w_1 \cdots w_{m-1} [\mathcal{W}_1 \circ_w \mathcal{W}'_1]_{\alpha} w'_2 \cdots w'_{m'} + w_1 \cdots w_{m-1} [w \circ_w w']_{\beta} w'_2 \cdots w'_{m'}
\]

\[
+ \lambda_{\beta} w_1 \cdots w_{m-1} [\mathcal{W} \circ_w \mathcal{W}']_{\alpha} w'_2 \cdots w'_{m'}.
\]
Since all the products are the concatenation except the ones in the brackets, by the general remark again, we just need to show that each of the brackets is in $X_{\text{deg}(w_0)+\text{deg}(w_1)}$. But by the induction hypothesis, the $\circ_w$-products inside the three brackets are in $kX_{\text{deg}(w_0)+\text{deg}(w_1)+1}$. Hence the three brackets are in $kX_{\text{deg}(w_0)+\text{deg}(w_1)}$ by the first inclusion in Eq. (51). This completes the induction.

We finally prove Eq. (52) by induction on $n \geq 0$. The initial case of $n = 0$ holds since $\mathcal{M}_0 = M(X)$ equals $X_0$ on which $\varphi$ is the identity. For a given $k \geq 0$, assume that $\varphi(k\mathcal{M}_n) \subseteq kX_n$ for $n \geq k$ and consider $1 \neq w \in \mathcal{M}_{k+1}$. If the width of $w$ is one, then $w$ is either in $X$ or is of the form $[\varpi]_w$, $\varpi \in \mathcal{M}_w$, $\omega \in \Omega$. The former case is already proved. For the latter case, we have $\varphi(w) = P_w(\varphi(\varpi))$ is in $kX_{k+1}$ by the first inclusion in Eq. (51). If the width of $w$ is greater than one, then $w = w_1w_2$ with $w_1, w_2 \in \mathcal{M}_k$. Thus by the induction hypothesis, $\varphi(w) \in kX_{\text{deg}(w)}$, $i = 1, 2$. Since $\varphi$ is an algebra homomorphism, by the second inclusion in Eq. (51), we have

$$\varphi(w) = \varphi(w_1) \circ_w \varphi(w_2) \in kX_{\text{deg}(w_1)} \circ_w X_{\text{deg}(w_2)} \subseteq kX_{\text{deg}(w)}.$$ This completes the induction. \hfill \Box

3.4. Free matching Rota-Baxter algebras on decorated rooted forests. In view of the convenience of working with rooted forests for Hopf algebra structures in the next section, we apply the isomorphism $\theta : k\mathcal{M}(X, \Omega) \to k\mathcal{F}(X, \Omega)$ in Eq. (15) of $\Omega$-operated algebras to reformulate the main results in this section in terms of rooted forests. We first write

$$\mathcal{U}_n := \theta(X_n), \quad n \geq 0 \quad \text{and} \quad \mathcal{U}_\infty := \lim_{n \to \infty} \mathcal{U}_n = \lim_{n \to \infty} \theta(X_n) = \theta(X_\infty),$$

giving rise to linear isomorphisms

$$\Pi^{\text{NC}}_{\Omega}(X, \Omega) = k\mathcal{U}_\infty \xrightarrow{\theta} \Pi^{\text{NC}}_{\text{RT}}(X, \Omega) := k\mathcal{U}_\infty, \quad \text{Id}(S) \xrightarrow{\theta} \text{Id}([\Xi])$$

for the operated ideals generated by $S$ and $[\Xi]$ defined in Eqs. (17) and (22) respectively. Similar to matching Rota-Baxter bracketed words, elements in $\mathcal{U}_\infty$ are called matching Rota-Baxter forests (MRBFs).

Further we obtain a homomorphism of $\Omega$-operated algebras

$$\psi := \theta \circ \theta^{-1} : (k\mathcal{M}(X, \Omega), \cdot, (B^{+}_\omega)_{\omega \in \Omega}) \to (\Pi^{\text{NC}}_{\text{RT}}(X, \Omega), \circ_L, (B^{+}_\omega)_{\omega \in \Omega}),$$

yielding the commutative diagram

$$\begin{array}{ccc}
\mathcal{M}(X, \Omega) = \text{Id}(S) \oplus \Pi^{\text{NC}}_{\Omega}(X, \Omega) & \xrightarrow{\psi} & \Pi^{\text{NC}}_{\text{RT}}(X, \Omega) \\
downarrow{\theta} & & \downarrow{\theta} \\
k\mathcal{F}(X, \Omega) = \text{Id}([\Xi]) \oplus \Pi^{\text{NC}}_{\text{RT}}(X, \Omega) & \xrightarrow{\psi} & \Pi^{\text{NC}}_{\text{RT}}(X, \Omega)
\end{array}$$

The following result shows an elementary property of $\psi$.

**Lemma 3.11.** Let $i : \Pi^{\text{NC}}_{\text{RT}}(X, \Omega) = \mathcal{U}_\infty \to k\mathcal{F}(X, \Omega)$ be the natural inclusion. Then $\psi i = \text{id}_{\Pi^{\text{NC}}_{\text{RT}}(X, \Omega)}$.

Consequently, $\psi$ is idempotent.

With this transporting of structures, the free matching Rota-Baxter algebra structure on $\Pi^{\text{NC}}_{\Omega}(X, \Omega)$ gives rise to a free matching Rota-Baxter algebra structure on $\Pi^{\text{NC}}_{\text{RT}}(X, \Omega)$. More precisely, define a product

$$\circ_L : \Pi^{\text{NC}}_{\text{RT}}(X, \Omega) \otimes \Pi^{\text{NC}}_{\text{RT}}(X, \Omega) \to \Pi^{\text{NC}}_{\text{RT}}(X, \Omega)$$
by taking

$$F \circ_L F' := \theta(\theta^{-1}(F) \circ_w \theta^{-1}(F')) \quad \text{for} \quad F, F' \in \Pi^{\text{NC}}_{\text{RT}}(X, \Omega).$$

Also define a linear operator on $\Pi^{\text{NC}}_{\text{RT}}(X, \Omega)$ by $\theta P \theta^{-1}$ which turns out to be just the grafting operator $B^{+}_\omega$. 
Moreover, the degree deg on \( \mathcal{T}(X, \Omega) \) and its derived grading \( \mathcal{T}^{(n)} \) and filtration \( \mathcal{F}_{(n)} \) restrict to a grading and filtration on \( k\mathcal{X}_\infty \). By Eq. (37), they are compatible with the ones on \( \mathcal{X}_\infty \). More precisely,
\[
\mathcal{L}^{(n)} := \mathcal{T}^{(n)} \cap \mathcal{X}_\infty = \theta(\mathcal{X}^{(n)}), \quad \mathcal{L}^{(n)} := \mathcal{F}_{(n)} \cap \mathcal{X}_\infty = \theta(\mathcal{X}_{(n)}), \quad n \geq 0.
\]

Therefore by Proposition 3.11, we have

**Proposition 3.12.** Let \( j_X : X \hookrightarrow \Pi^{NC}_{RT}(X, \Omega), \ x \mapsto \bullet, x \in X, \) be the natural embedding. Then the triple \( (\Pi^{NC}_{RT}(X, \Omega), \circ, (B^\omega)_{\omega \in \Omega}) \) together with \( j_X \) is the free matching Rota-Baxter algebra of weight \( \lambda_X \) on \( X \).

Further, \( \Pi^{NC}_{RT}(X, \Omega) \) with \( k\mathcal{X}_{(n)}, n \geq 0, \) is an \( \Omega \)-operated filtered algebra and \( \psi \) is a homomorphism of \( \Omega \)-operated filtered algebras.

4. \( \Omega \)-cocycle Hopf algebras and free matching Rota-Baxter algebras

In this section, we first derive an \( \Omega \)-cocycle bialgebraic structure on the free matching Rota-Baxter algebra \( \Pi^{NC}_{RT}(X, \Omega) \), via a construction of a suitable coproduct. We then show that this \( \Omega \)-cocycle bialgebra is connected cofiltered and so a Hopf algebra.

Let \( u_\ell : k \to \Pi^{NC}_{RT}(X, \Omega) \) be the linear map given by \( 1_k \mapsto 1 \). By Proposition 3.12, the triple \( (\Pi^{NC}_{RT}(X, \Omega), \circ, u_\ell) \) is an algebra. We now define a linear map \( \Delta_\ell : \Pi^{NC}_{RT}(X, \Omega) \to \Pi^{NC}_{RT}(X, \Omega) \otimes \Pi^{NC}_{RT}(X, \Omega) \) by setting
\[
\Delta_\ell(F) := (\psi \otimes \psi)\Delta_{RT}i(F) \quad \text{for all } F \in \Pi^{NC}_{RT}(X, \Omega),
\]
where \( \psi : \Pi^{NC}_{RT}(X, \Omega) \to k\mathcal{T}(X, \Omega) \) is the natural inclusion. In other words, \( \Delta_\ell \) is defined so that the diagram
\[
\begin{array}{ccc}
\Pi^{NC}_{RT}(X, \Omega) & \xrightarrow{\Delta_\ell} & \Pi^{NC}_{RT}(X, \Omega) \otimes \Pi^{NC}_{RT}(X, \Omega) \\
\downarrow & & \downarrow \psi \otimes \psi \\
k\mathcal{T}(X, \Omega) & \xrightarrow{\Delta_{RT}i} & k\mathcal{T}(X, \Omega) \otimes k\mathcal{T}(X, \Omega)
\end{array}
\]
commutes.

Define \( \epsilon_\ell : \Pi^{NC}_{RT}(X, \Omega) \to k \) by setting
\[
\epsilon_\ell(F) = \begin{cases} 
0, & \text{if } F \neq 1, \\
1, & \text{if } F = 1.
\end{cases}
\]

We first verify that the coproduct \( \Delta_\ell \) on \( \Pi^{NC}_{RT}(X, \Omega) \) satisfies the Hochschild 1-cocycle condition.

**Lemma 4.1.** Let \( F = B^+_\omega(\overline{F}) \) be in \( \Pi^{NC}_{RT}(X, \Omega) \). Then
\[
\Delta_\ell(B^+_\omega(\overline{F})) = B^+_\omega(\overline{F}) \otimes 1 + (1 \otimes B^+_\omega)\Delta_{\ell}(\overline{F}).
\]

**Proof.** By the linearity, we just need to verify Eq. (40) for \( F \in \mathcal{X}_\infty \). Then
\[
\Delta_{\ell}(B^+_\omega(\overline{F})) = (\psi \otimes \psi)\Delta_{RT}i(B^+_\omega(\overline{F})) \quad (\text{By Eq. (38)})
\]
\[
= (\psi \otimes \psi)\Delta_{RT}(B^+_\omega(\overline{F})) \quad (\text{by } i \text{ being an inclusion map})
\]
\[
= (\psi \otimes \psi)(F \otimes 1 + (1 \otimes B^+_\omega)\Delta_{RT}(\overline{F})) \quad (\text{by Eq. (34)})
\]
\[
= \psi(F) \otimes \psi(1) + (\psi \otimes \psi B^+_\omega)\Delta_{RT}(\overline{F})
\]
\[
= \psi i(F) \otimes \psi(1) + (\psi \otimes \psi B^+_\omega)\Delta_{RT}i(\overline{F}) \quad (\text{by } i \text{ being an inclusion map})
\]
\[
= F \otimes 1 + (\psi \otimes \psi B^+_\omega)\Delta_{RT}i(\overline{F}) \quad (\text{by Lemma 3.13})
\]
\[
= F \otimes 1 + (\psi \otimes \psi B^+_\omega(\psi \otimes \psi)\Delta_{RT}i(\overline{F})
\]
\[
(\text{by } \psi \text{ being an operated algebra homomorphism in Eq. (34)})
\]
\[
= F \otimes 1 + (\text{id} \otimes B^+_\omega(\psi \otimes \psi)\Delta_{RT}i(\overline{F})
\]
\[
= F \otimes 1 + (\text{id} \otimes B^+_\omega(\psi \otimes \psi)\Delta_{RT}i(\overline{F})
\]
\[ = F \otimes 1 + (\text{id} \otimes B_\alpha^+) \Delta_\ell(\mathcal{F}) \] (by Eq. \((38)\)).

This completes the proof. □

Next we verify the compatibility of \(\Delta_\ell\) with \(\circ_\ell\), starting with a special case.

**Lemma 4.2.** Let \(F, F' \in \mathcal{L}_\infty\) with \(F \circ_\ell F' = FF'\). Then

\[ \Delta_\ell(F \circ_\ell F') = \Delta_\ell(F) \circ_\ell \Delta_\ell(F'). \]

**Proof.** We have

\[
\Delta_\ell(F \circ_\ell F') = (\psi \otimes \psi) \Delta_{RT} i(FF')
\]

(by \(i\) being an inclusion map)

\[
= (\psi \otimes \psi) \left( \Delta_{RT}(F) \Delta_{RT}(F') \right) \quad \text{(by \(\Delta_{RT}\) being an algebra homomorphism)}
\]

\[
= \left( \left( \psi \otimes \psi \right) \Delta_{RT}(F) \right) \circ_\ell \left( \left( \psi \otimes \psi \right) \Delta_{RT}(F') \right) \quad \text{(by \(\psi\) being an algebra homomorphism)}
\]

\[
= \left( \psi \otimes \psi \right) \Delta_{RT}(F) \circ_\ell \left( \left( \psi \otimes \psi \right) \Delta_{RT}(F') \right) \quad \text{(by \(i\) being an inclusion map)}
\]

\[
= \Delta_\ell(F) \circ_\ell \Delta_\ell(F') \quad \text{(by Eq. \((38)\)).}
\]

This completes the proof. □

In general, we have

**Lemma 4.3.** Let \(F, F' \in \mathcal{L}_{RT}^{NC}(X, \Omega)\). Then

\[ \Delta_\ell(F \circ_\ell F') = \Delta_\ell(F) \circ_\ell \Delta_\ell(F') \quad \text{and} \quad \epsilon_\ell(F \circ_\ell F') = \epsilon_\ell(F) \epsilon_\ell(F'). \]

**Proof.** The second equation follows from the definition of \(\epsilon_\ell\) in Eq. \((39)\).

For the first equation, by the linearity, we just need to consider the case when \(s = \text{dep}(F) + \text{dep}(F') \geq 0\). For the initial step of \(s = 0\), we have \(\text{dep}(F) = \text{dep}(F') = 0\) and so \(F \circ_\ell F' = FF'\). Then Eq. \((41)\) follows from Lemma \(4.2\).

Let \(t \geq 0\). Assume that Eq. \((41)\) holds for \(s = t\) and consider the case of \(s = t + 1\). In this case, we first consider the case when \(\text{bre}(F) = \text{bre}(F') = 1\). If \(F \circ_\ell F' = FF'\), then Eq. \((41)\) follows from Lemma \(4.2\). If \(F \circ_\ell F' \neq FF'\), then we have \(F = B_\alpha^+ (\mathcal{F})\) and \(F' = B_\beta^+ (\mathcal{F})\) for some \(\alpha, \beta \in \Omega\) and \(\mathcal{F}, \mathcal{F}' \in \mathcal{L}_\infty\). Write

\[ \Delta_\ell(\mathcal{F}) := \sum_{(F)} \mathcal{F}(1) \otimes \mathcal{F}(2) \quad \text{and} \quad \Delta_\ell(\mathcal{F}') := \sum_{(F)} \mathcal{F}'(1) \otimes \mathcal{F}'(2). \]

Then

\[
\Delta_\ell(F \circ_\ell F')
\]

\[
= \Delta_\ell(B_\alpha^+(\mathcal{F}) \circ_\ell B_\beta^+(\mathcal{F}'))
\]

\[
= \Delta_\ell(L_\alpha^+(\mathcal{F}) \circ L_\beta^+(\mathcal{F}')) + \lambda_\beta B_\alpha^+(\mathcal{F}) \circ_\ell (1 + B_\beta^+(\mathcal{F}'))
\]

\[
= \Delta_\ell(L_\alpha^+(\mathcal{F}) \circ L_\beta^+(\mathcal{F}')) + \lambda_\beta \Delta_\ell(B_\alpha^+(\mathcal{F} \circ_\ell \mathcal{F}'))
\]

\[
= \Delta_\ell(B_\alpha^+(\mathcal{F} \circ_\ell \mathcal{F}')) + \lambda_\beta \Delta_\ell(B_\alpha^+(\mathcal{F} \circ_\ell \mathcal{F}'))
\]

\[
= \Delta_\ell(F \circ_\ell F') \quad \text{and} \quad \epsilon_\ell(F \circ_\ell F') = \epsilon_\ell(F) \epsilon_\ell(F').
\]
\[ + \lambda_\beta (\text{id} \otimes B_{\omega}^-(F)) \left( \Delta_\ell(2) \otimes \Delta_\ell(F) \right) \] (by the induction hypothesis on \( s \))
\[ = (F \otimes F') \otimes 1 + \sum_{(F)} (\Delta_\ell(2) \otimes \Delta_\ell(F')) + \sum_{(F)} (F \otimes F') \otimes B_{\beta}^+(F) \] (by Eqs. (4.10) and (4.12))
\[ = \left( F \otimes 1 + \sum_{(F)} (\Delta_\ell(2) \otimes \Delta_\ell(F')) \right) \otimes F' \] (by the case when \( \text{bre}(F) = \text{bre}(F') = 1 \))
\[ = \Delta_\ell(F) \otimes \Delta_\ell(F'). \]

This completes the proof. \( \square \)

**Theorem 4.4.** The sextuple \( (\mathbb{I}^{\text{NC}}_{\text{RT}}(X, \Omega), \otimes_\ell, u_\ell, \Delta_\ell, \epsilon_\ell, (B_{\omega}^\pm)_{\omega \in \Omega}) \) is an \( \Omega \)-cocylic bialgebra.

**Proof.** By Lemmas 4.1 and 4.3, we only need to verify the coassociativity of \( \Delta_\ell \) and the counicity of \( \epsilon_\ell \).

For the coassociativity of \( \Delta_\ell \), following the idea of the proof of Theorem 2.10, we just need to show that the set
\[ \mathcal{C} := \{ F \in \mathbb{I}^{\text{NC}}_{\text{RT}}(X, \Omega) \mid (\Delta_\ell \otimes \text{id})\Delta_\ell(F) = (\text{id} \otimes \Delta_\ell)\Delta_\ell(F) \} \]
is an \( \Omega \)-operated subalgebra of \( \mathbb{I}^{\text{NC}}_{\text{RT}}(X, \Omega) \).

Note that \( 1 \) is in \( \mathcal{C} \). By Lemma 4.3, \( \Delta_\ell \) is an algebra homomorphism. Then \( (\Delta_\ell \otimes \text{id})\Delta_\ell \) and \( (\text{id} \otimes \Delta_\ell)\Delta_\ell \) are also algebra homomorphisms, implying that \( \mathcal{C} \) is a subalgebra of \( \mathbb{I}^{\text{NC}}_{\text{RT}}(X, \Omega) \). For any \( x \in X \), we have
\[ (\Delta_\ell \otimes \text{id})\Delta_\ell(\bullet_x) = (\Delta_\ell \otimes \text{id})(1 \otimes x + x \otimes 1) \] (by Eq. (4.13))
\[ = 1 \otimes 1 \otimes x + x \otimes 1 + x \otimes 1 \otimes 1 = (\text{id} \otimes \Delta_\ell)(\bullet_x). \]

Thus \( \bullet_x \in \mathcal{C} \). For any \( F \in \mathcal{C} \) and \( \omega \in \Omega \), we have
\[ (\Delta_\ell \otimes \text{id})\Delta_\ell(B_{\omega}^-(F)) \]
\[ = (\Delta_\ell \otimes \text{id})(B_{\omega}^-(F) \otimes 1 + (\text{id} \otimes B_{\omega}^+(F)) \Delta_\ell(F)) \] (by Lemma 4.1)
\[ = \Delta_\ell(B_{\omega}^-(F)) \otimes 1 + (\Delta_\ell \otimes B_{\omega}^+(F)) \Delta_\ell(F) \]
\[ = B_{\omega}^-(F) \otimes 1 \otimes 1 + (\text{id} \otimes B_{\omega}^+(F)) \Delta_\ell(F) \otimes 1 + (\text{id} \otimes \text{id} \otimes B_{\omega}^+(F)) (\Delta_\ell \otimes \text{id})\Delta_\ell(F) \] (by Lemma 4.3)
Thus $\mathcal{C}$ is a subalgebra of $D$ which means that

\[ (\text{id} \otimes \Delta)(F) \triangleq \Delta(F) \]

(Definition 4.5). For any $x \in X$, by Eq. (4.1), we have

\[ (\text{id} \otimes \Delta)(\bullet_x) = \Delta(\bullet_x) \]

and

\[ (\text{id} \otimes \epsilon)(\Delta)(\bullet_x) = (\bullet_x \otimes 1 + 1 \otimes \bullet_x) \]

Hence $\bullet_x \in \mathcal{D}$. For any $F \in \mathcal{D}$ and $\omega \in \Omega$, we have

\[ (\epsilon \otimes \text{id})(\Delta)(B^+_{\omega}(F)) \]

\[ = (\epsilon \otimes \text{id})(B^+_{\omega}(F) \otimes 1 + (\epsilon \otimes B^+_{\omega})\Delta(F)) \]

(Definition 4.1))

\[ = \epsilon(\Delta(\epsilon)B^+_{\omega}(F)) \]

\[ \otimes 1 + (\epsilon \otimes \text{id})(\epsilon \otimes B^+_{\omega})\Delta(F) \]

\[ = \epsilon(B^+_{\omega}(F)) \]

\[ \otimes 1 + (\epsilon \otimes B^+_{\omega})(\epsilon \otimes \text{id})\Delta(F) \]

\[ = 0 + (\epsilon \otimes B^+_{\omega})\beta(\epsilon) \]

(Definition 4.1) and $F \in \mathcal{D}$)

\[ = 1 \otimes B^+_{\omega}(F) = \beta(\epsilon)(B^+_{\omega}(F)) \]

\[ (\text{id} \otimes \epsilon)(\Delta)(B^+_{\omega}(F)) \]

\[ = (\text{id} \otimes \epsilon)(B^+_{\omega}(F) \otimes 1 + (\text{id} \otimes B^+_{\omega})\Delta(F)) \]

(Definition 4.1))

\[ = B^+_{\omega}(F) \]

\[ \otimes 1 + (\text{id} \otimes \epsilon)(\text{id} \otimes B^+_{\omega})\Delta(F) \]

\[ = B^+_{\omega}(F) \]

\[ \otimes 1 + \sum_{F(1)} F(1) \otimes \epsilon(\text{id} \otimes B^+_{\omega}(F(2))) \]

\[ = B^+_{\omega}(F) \]

\[ \otimes 1 + 0 \]

(Definition 4.1) and $F \in \mathcal{D}$

\[ = \beta(\epsilon)(B^+_{\omega}(F)). \]

Thus $B^+_{\omega}(F) \in \mathcal{D}$ and $\mathcal{D}$ is stable under $B^+_{\omega}$ for any $\omega \in \Omega$. Consequently, $\mathcal{D} = \mathcal{III}^{\mathcal{NC}}_{\mathcal{RT}}(X, \Omega)$ and $\epsilon$ is a counit. This completes the proof. \qed

We now introduce the connectedness condition on coalgebras [3].

**Definition 4.5.** (a) A coalgebra $(C, \Delta, \epsilon)$ is called **coaugmented** if there is a linear map $u : k \to C$, called the **coaugmentation**, such that $\epsilon u = \text{id}_k$. (b) A coaugmented coalgebra $(C, u, \Delta, \epsilon)$ is called **cofiltered** if there is an exhaustive increasing filtration $\{C(n)\}_{n \geq 0}$ of $H$ such that

\[ \text{im } u \subseteq C(n), \quad \Delta(C(n)) \subseteq \sum_{p+q=n} C(p) \otimes C(q), \quad n \geq 0, p, q \geq 0 \]
Elements in \( C(n) \ \backslash \ C(n-1) \) are said to have degree \( n \). \( C \) is called connected (filtered) if in addition \( C^0 = \text{im} \ u (= \ k) \).

By the coaugmented condition, we have \( C = \text{im} \ u \oplus \ker \varepsilon \). Then from \( \text{im} \ u \subseteq C(n) \) and modularity, we have \( C(n) = \text{im} \ u \oplus (C(n) \cap \ker \varepsilon) \), as originally stated in [5].

We quote the following condition [5, Theorem 3.4] for Hopf algebras. See also [2, 28, 30].

**Lemma 4.6.** Let \( H = (H, m, \mu, \Delta, \varepsilon) \) be a bialgebra such that \((H, \Delta, \varepsilon)\) is a connected cofiltered coaugmented coalgebra. Then \( H \) is a Hopf algebra.

Finally, we arrive at our main result on Hopf algebraic structure on free matching Rota-Baxter algebras.

**Theorem 4.7.** The sextuple \((\Pi_{RT}^{NC}(X, \Omega), \diamondsuit, \mu, \Delta, \varepsilon, (B^+_\omega)_{\omega \in \Omega})\) is an \( \Omega \)-cocycle Hopf algebra.

**Proof.** By Theorem 4.3. The quintuple \((\Pi_{RT}^{NC}(X, \Omega), \diamondsuit, \mu, \Delta, \varepsilon, (B^+_\omega)_{\omega \in \Omega})\) is an \( \Omega \)-cocycle bialgebra. In particular, \((\Pi_{RT}^{NC}(X, \Omega), \diamondsuit, \mu, \Delta, \varepsilon, (B^+_\omega)_{\omega \in \Omega})\) is a coaugmented coalgebra. To apply Lemma 4.6, we just need to check that \((\Pi_{RT}^{NC}(X, \Omega), \diamondsuit, \mu, \Delta, \varepsilon)\) is connected and cofiltered, with respect to the filtration \( k\mathcal{F}(n), n \geq 0 \). The connectedness is clear since \( \mathcal{F}(0) = 1 \). Further by Proposition 3.12 and Eq. (38), we have

\[
\Delta_{\ell}(\psi_n) = (\psi \otimes \psi)(\Delta_{\ell}(\psi_n)) \leq (\psi \otimes \psi)\left( \sum_{p+q=n} (k\mathcal{F}(p) \otimes (k\mathcal{F}(q)) \right) \leq \sum_{p+q=n} (k\omega(p) \otimes (k\omega(q))
\]

Now the conclusion follows from Lemma 4.6. \(\square\)

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**References**

[1] M. Aguiar, On the associative analog of Lie bialgebras, *J. Algebra* **244** (2001), 492-532.
[2] C. Bai, L. Guo and X. Ni, Nonabelian generalized Lax pairs, the classical Yang-Baxter equation and PostLie algebras, *Comm. Math. Phys.* **297** (2010), 553-596.
[3] F. Baader and T. Nipkow, 1998. Term Rewriting and All That. Cambridge U. P., Cambridge.
[4] G. Baxter, An analytic problem whose solution follows from a simple algebraic identity, *Pacific J. Math.* **10** (1960), 731-742.
[5] L. A. Bokut, Y. Chen and J. Qiu, Gröbner-Shirshov bases for associative algebras with multiple operators and free Rota-Baxter algebras, *J. Pure Appl. Algebra* **214** (2010), 89-100.
[6] Y. Bruned, M. Hairer and L. Zambotti, Algebraic renormalisation of regularity structures, *Invent. math.* **215** (2019), 1039-1156.
[7] P. Cartier, On the structure of free Baxter algebras, *Adv. in Math.* **9** (1972), 253-265.
[8] P. Clavier, L. Guo, S. Paycha, B. Zhang, Locality and renormalisation: universal properties and integrals on trees, *J. Math. Phys.* **61** (2020), 022301.
[9] A. Connes and D. Kreimer, Hopf algebras, renormalization and non-commutative geometry, *Comm. Math. Phys.* **199** (1) (1998), 203-242.
[10] K. Ebrahimi-Fard and L. Guo, Rota-Baxter algebras and dendriform algebras, *J. Pure Appl. Algebra* **212** (2008) 320-339.
[11] L. Foissy, Les algebres de Hopf des arbres enracines décorés. I & II *Bull. Sci. Math.* **126** (2002), 193-239, 249-288.
[12] L. Foissy, Quantization of the Hopf algebras of decorated planar rooted trees, preprint, 2008.
[13] L. Foissy, Algebras structures on typed decorated rooted trees, arXiv:1811.07572.
[14] X. Gao and L. Guo, Rota’s Classification Problem, rewriting systems and Gröbner-Shirshov bases, *J. Algebra* **470** (2017), 219-253.
[15] X. Gao and L. Guo, A note on connected cofiltered coalgebras, conilpotent coalgebras and Hopf algebras, *Southeast Asian Bull. Math.* **43** (2019), 313-321.
[16] X. Gao, L. Guo and Y. Zhang, Commutative matching Rota-Baxter operators, shuffle products with decorations and matching Zinbiel algebras, *J. Algebra*, accepted, arXiv:2007.12095v2.
[17] X. Gao, P. Lei and T. Zhang, Left counital Hopf algebras on free Nijenhuis algebras, *Comm. Algebra* **46** (2018), 4868-4883.

[18] L. Guo, Operated semigroups, Motzkin paths and rooted trees, *J. Algebraic Combin.* **29** (2009), 35-62.

[19] L. Guo, An Introduction to Rota-Baxter Algebra, International Press, 2012.

[20] L. Guo, R. Gustavsson and Y. Li, An algebraic study of Volterra integral equations and their operator linearity, arXiv:2008.06756.

[21] L. Guo, S. Paycha and B. Zhang, Renormalization of conical zeta values and the Euler-Maclaurin formula, *Duke Math. Jour.* **166** (2017) 537-571.

[22] L. Guo, W. Sit and R. Zhang, Differential type operators and Gröbner-Shirshov bases, *J. Symb. Comput.* **52** (2013), 97-123.

[23] R. Holtkamp, Comparison of Hopf algebras on trees. *Arch. Math. (Basel)* **80** (2003), 368-383.

[24] D. Kreimer, On the Hopf algebra structure of perturbative quantum field theories, *Adv. Theor. Math. Phys.* **2** (1998), 303-334.

[25] D. Kreimer and E. Panzer, Renormalization and Mellin transforms, Computer Algebra in Quantum Field Theory, *Texts Monographs in Symbolic Computation* XII (2013)195-223.

[26] J.-L. Loday and M. O. Ronco, On the structure of cofree Hopf algebras, *J. Reine Angew. Math.* **592** (2006), 123-155.

[27] Y. Li and L. Guo, Braided dendriform and tridendriform algebras, and braided Hopf algebras of rooted trees, *J. Algebraic Combin.* (2020), https://doi.org/10.1007/s10801-020-00957-0.

[28] D. Manchon, Hopf algebras, from basics to applications to renormalization, Comptes-rendus des Rencontres mathematiques de Glaon, 2001.

[29] I. Moerdijk, On the Connes-Kreimer construction of Hopf algebras, *Contemp. Math.* **271** (2001), 311-321.

[30] S. Montgomery, Hopf algebras and their actions on rings, *Regional Conference Series in Mathematics* Providence, RI: American Mathematical Society, 1993.

[31] X. S. Peng, Y. Zhang, X. Gao and Y. F. Luo, Left counital Hopf algebras on bi-decorated planar rooted forests and Rota-Baxter systems, *Bull. Belg. Math. Soc. Simon Stevin* **27** (2020), no. 2, 219-243.

[32] J. Qiu and Y. Chen, Free Rota-Baxter systems and a Hopf algebra structure, *Comm. Algebra* **46** (2018), 3913-3925.

[33] G.-C. Rota, Baxter algebras and combinatorial identities I, II, *Bull. Amer. Math. Soc.* **75** (1969), 325-329, 330-334.

[34] H. Strohmayer, Operads of compatible structures and weighted partitions, *J. Pure Appl Algebra* **212** (2008), 2522-2534.

[35] M. E. Sweedler, Hopf Algebras, Mathematics Lecture Note Series W. A. Benjamin, Inc., New York, 1969.

[36] H. Yu, L. Guo and J.-Y. Thibon, Weak quasi-symmetric functions, Rota-Baxter algebras and Hopf algebras, *Adv. Math.* **344** (2019), 1-34.

[37] S. Zemyan, The Classical Theory of Integral Equations, Birkhäuser, New York, 2011.

[38] T. Zhang, X. Gao and L. Guo, Hopf algebras of rooted forests, cocycles, and free Rota-Baxter algebras, *J. Math. Phys.* **57** (2016), 101701.

[39] Y. Zhang, D. Chen, X. Gao and Y. F. Luo, Weighted infinitesimal unitary bialgebras on rooted forests and weighted cocycles, *Pacific J. Math.* **302** (2019), 741-766.

[40] Y. Zhang and X. Gao, Hopf algebras of planar binary trees: an operated algebra approach, *J. Algebraic Combin.* **51** (2020), 567-588.

[41] Y. Zhang, X. Gao and L. Guo, Matching Rota-Baxter algebras, matching dendriform algebras and matching pre-Lie algebras, *J. Algebra* **552** (2020), 134-170.

[42] X. Zhang, X. Gao and L. Guo, Modified Rota-Baxter algebras, shuffle products and Hopf algebras, *Bull. Malays. Math. Sci. Soc.* **42** (2019), 3047-3072.

[43] S. Zheng, X. Gao, L. Guo and W. Sit, Rota-Baxter type operators, rewriting systems and Gröbner-Shirshov bases, *J. Symbolic Comput.* accepted, arXiv:1412.8055.

[44] S. Zheng and L. Guo, Left counital Hopf algebra structures on free commutative Nijenhuis algebras, *Scientia Cinica Mathematica* (2019), https://doi.org/10.1360/SCM-2017-0662, arXiv:1711.04823.
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