SOME RELATION BETWEEN SPECTRAL DIMENSION AND AHLFORS REGULAR CONFORMAL DIMENSION ON INFINITE GRAPHS

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Abstract

The spectral dimension $d_s$ of a weighted graph is an exponent associated with the asymptotic behavior of the random walk on the graph. The Ahlfors regular conformal dimension $\dim_{\text{ARC}}$ of the graph distance is a quasisymmetric invariant, where quasisymmetry is a well-studied property of homeomorphisms between metric spaces. In this paper, we give a typical example of a fractal-like graph with $d_s < \dim_{\text{ARC}} < 2$ and prove a sufficient condition for $\dim_{\text{ARC}} \leq d_s < 2$.

1 Introduction

This paper aims to evaluate a dimension of an infinite graph as a metric space, defined through quasisymmetric transformations. Let $(X, d)$ be a metric space and $\mu$ be a Borel measure on it. We first recall the definition of quasisymmetry.

**Definition 1.1** (Quasisymmetry, [17]). Let $X$ be a set and $d, \delta$ be metrics on $X$. We say $d$ is *quasisymmetric* to $\delta$, and write $d \sim_{\text{QS}} \delta$, if there exists a homeomorphism $\theta : [0, \infty) \to [0, \infty)$ such that for any $x, y, z \in X$ with $x \neq z$,

$$\delta(x, y)/\delta(x, z) \leq \theta(d(x, y)/d(x, z)).$$

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For example, \( d \sim_{QS} d^\alpha \) for any \( \alpha \in (0,1) \) and a metric space \( (X,d) \). The idea of this definition is that any annulus in \( (X,d) \) is comparable to one in \( (X,\delta) \). Beurling and Ahlfors [7] implicitly introduced quasisymmetry as a property of a map from \( \mathbb{R} \) to \( \mathbb{R} \), and characterized it as the boundary value of a quasiconformal map from the upper half-plane to itself. Kelingos [14] named it quasisymmetry. Tukia and Väisälä [29] generalized this notion for embedding maps from one metric space to another. Quasisymmetry has been studied in various fields, such as Gromov hyperbolic spaces ([8, 9, 19, 23] for example) and metric measure spaces ([11, 27] for example). Quasisymmetry is also used in studies of heat kernel estimates (see [4, 6, 13, 16, 21] for example).

Ahlfors regularity and the Ahlfors regular conformal dimension are defined as follows.

**Definition 1.2 (Ahlfors regularity).** For \( \alpha > 0 \), we say \( (X,d) \) is \( \alpha \)-Ahlfors regular if there exists \( C > 0 \) and a Borel measure \( \mu \) such that \( C^{-1}r^\alpha \leq \mu(B_d(x,r)) \leq Cr^\alpha \) for any \( x \in X \) and \( r \) with \( \inf_{y \in X \setminus \{x\}} d(x,y) \leq r \leq \operatorname{diam}(X,d) \) where \( B_d(x,r) = \{ y \in X \mid d(x,y) < r \} \) and \( \operatorname{diam}(X,d) = \sup_{x,y \in X} d(x,y) \).

**Definition 1.3 (Ahlfors regular conformal dimension).** The *Ahlfors regular conformal dimension* \( \dim_{ARC} \) of \( (X,d) \) is defined by

\[
\dim_{ARC}(X,d) = \inf \left\{ \alpha \in (0,\infty) \mid \text{there exists an } \alpha \text{-Ahlfors regular metric } \delta \text{ on } X \text{ with } d \sim_{QS} \delta \right\},
\]

where \( \inf \emptyset = \infty \).

Note that if \( \inf_{x,y} d(x,y) = 0 \) or \( \operatorname{diam}(X,d) = \infty \), and \( (X,d) \) is \( \alpha \)-Ahlfors regular then \( (X,d) \) is not \( \beta \)-Ahlfors regular for \( \beta \neq \alpha \). In particular, \( \alpha \) coincides with the Hausdorff dimension of \( (X,d) \) if \( (X,d) \) has no isolated points. Usually, we only consider the Ahlfors regularity and \( \dim_{ARC} \) in the latter case. However, in the present paper, we consider the case of \( \inf_{y} d(x,y) \neq 0 \) for some \( x \), to treat graph distances.

\( \dim_{ARC} \) was implicitly introduced for continuous metric spaces by Bourdon and Pajot [9] and named by Bonk and Kleiner [8]. In [8], this exponent is related to Cannon’s conjecture, which claims that for any hyperbolic group \( G \) whose boundary is homeomorphic to the 2-dimensional sphere, there exists a discrete, cocompact, and isometric action of \( G \) on the hyperbolic 3-space \( \mathbb{H}^3 \). Carrasco Piaggio [10] characterized \( \dim_{ARC}(X,d) \) as a critical value related to the *combinatorial p-modulus* \( \operatorname{Mod}_p \) of a family of curves \( \Gamma \) in a graph \( (V,E) \) (approximating \( (X,d) \)), where

\[
\operatorname{Mod}_p(\Gamma) := \inf \left\{ \sum_{v \in V} f(v)^p \mid f : V \to [0,\infty), \sum_{v \in \gamma} f(v) \geq 1 \text{ for any } \gamma \in \Gamma \right\}.
\]

In recent studies, this characterization of \( \dim_{ARC} \) has been also used to construct \( p \)-Sobolev spaces on fractals (see [18, 28], cf. [22]).
In \cite{17}, Kigami introduced the notion of a partition satisfying the basic framework (BF-partition for short) and used it to evaluate the Ahlfors regular conformal dimension of compact metric spaces. The author considered this notion on infinite graphs and extended Kigami’s results in \cite{24}. Roughly speaking, a BF-partition of an infinite graph is a successive unification of vertices of the given graph with some good conditions. The simplest example is that \((V, E) = (\mathbb{Z}_{\geq 0}, \{(n, m) \mid |n - m| = 1\})\) and a BF-partition is defined as (a technical extension of) a map \(K : \{(n, k) \mid n, k \geq 0\} \rightarrow 2^C\) by \(K(n, k) = \{m \in \mathbb{Z} \mid 2^nk \leq m \leq 2^nk + 1\}\). For another non-trivial example, see the first part of Section 3. For a given partition \(K : T \rightarrow V\), we consider the corresponding graph structures in the same way as \cite{17} and \cite{24}. Roughly speaking, we hierarchically divide \(T\) into \(\{T_n\}_n\) and for each \(n\) consider graph structures on \(T_n\) such that there is an edge between \(w, v \in T_n\) if \(K(w) \cap K(v) \neq \emptyset\) and \(w \neq v\). Then some potential theoretic exponents \(\overline{d}_p(K), \underline{d}_p(K)\) of this family of graphs, called the upper and lower \(p\)-spectral dimensions, for \(p > 0\) are defined. See Definitions 3.4 and 3.5 for the precise definitions of a BF-partition and the \(p\)-spectral dimensions. For these exponents, the following theorem that is essentially induced by the results in \cite{17} holds.

**Theorem 1.4** (\cite{24} Theorem 4.14, cf. \cite{17} Theorem 4.7.9). Let \(d\) be a metric on \(V\), satisfying some properties with respect to \(K\).

1. If \(p > \text{dim}_{ARC}(V, d)\) then \(p > \overline{d}_p(K) \geq \underline{d}_p(K) \geq \text{dim}_{ARC}(V, d)\).

2. If \(p \leq \text{dim}_{ARC}(V, d)\) then \(p \leq \underline{d}_p(K) \leq \overline{d}_p(K) \leq \text{dim}_{ARC}(V, d)\).

See Theorem 3.6 for the precise statement. We emphasize that the \(p\)-spectral dimensions are combinatorial exponents defined with a partition, and do not have any stochastic characterization. However, as pointed out in \cite{17} for the case of compact counterparts, if \((V, E)\) is the graphical Sierpiński gasket (see Figure 1) or a graphical generalized Sierpiński carpet (see Figure 2), \(K\) is a canonical partition and \(d\) is the graph distance on \((V, E)\), then \(\overline{d}_2(K)\) and \(\underline{d}_2(K)\) coincide with the spectral dimension \(d_s(V, E)\), defined as follows.

**Definition 1.5** (Spectral dimension). Let \((V, E)\) be a locally finite connected graph and \(\mu : E \rightarrow (0, \infty)\) be a symmetric weight function (i.e. \(\mu(x, y) = \mu(y, x)\) for any \((x, y) \in E\)). We also inductively let \(\mu(x) = \sum_{y(x, y) \in E} \mu(x, y), p(0, x, y) = \chi_{\{x\}}(y)/\mu(y)\) and \(p(n, x, y) = (\sum_{z(x, z) \in E} \mu(x, z)p(n - 1, z, y))/\mu(x)\). The spectral dimension of the weighted graph \((V, E, \mu)\) is defined by

\[
d_s(V, E, \mu) = -2 \lim_{n \to \infty} (\log p(2n, x, x)/\log n),
\]

if the limit exists for some (or equivalently, for any) \(x\). We simply write \(d_s(V, E)\) when \(\mu\) is the simple weight i.e. \(\mu \equiv 1\) on \(E\).
We remark that the value of the limit is independent of the choice of $x$ because $(V, E)$ is connected. Since $p(n, x, y)\mu(y)$ coincides with the transition probability of the weighted random walk on $(V, E, \mu)$ starting from $x$, the spectral dimension indicates the asymptotic behavior of the return probability of the random walk.

In this paper, we consider when same as the cases of the graphical Sierpiński gasket or graphical generalized Sierpiński carpets, the inequalities between the “geometric dimension” and the “stochastic exponent” $\dim_{\text{ARC}}(V, d) \leq d_s(V, E, \mu) < 2$ or $\dim_{\text{ARC}}(V, d) \geq d_s(V, E, \mu) \geq 2$ hold for a weighted graph $(V, E, \mu)$ and the graph distance $d$. (Recall that in the given examples, one of the inequality holds by $d_s^2 = d_s(V, E)$ and Theorem [1.4].)

We first construct a graph $(V, E)$ embedded in $\mathbb{C}$. This graph has spatial homogeneity in the following sense: let $S_{n,a,b} = \{z \in \mathbb{C} \mid 3^n a \leq \text{Re}(z) \leq 3^n (a + 1), 3^n b \leq \text{Im}(z) \leq 3^n (b + 1)\}$, then if $\text{int}(S_{n,a_j,b_j}) \cap V \neq \emptyset$ for some $n, a_j, b_j$ ($j = 1, 2$), then the restriction of $(V, E)$ to $S_{n,a_1,b_1}$ and to $S_{n,a_2,b_2}$ are isometric (see Figure 3). The first main result of this paper is to show that neither of the considered inequalities holds for this $(V, E)$ with the graph distance $d$ and the simple weight $\mu$. See Theorem 2.1 for the precise statement.

The proof of Theorem 2.1 implies that some type of symmetry for scaling is sufficient to show $d^2_s = d_s(V, E)$ and one of the considered inequalities. To justify this idea, we introduce the resistance on a weighted graph.

**Definition 1.6 (Resistance).** Let $(V', E', \mu)$ be a weighted graph, then for $A, B \subset V'$ with $A \cap B \neq \emptyset$ the *resistance* between $A$ and $B$ is defined by

$$R_{\mu}(A, B) = \left( \inf \left\{ \frac{1}{2} \sum_{(x,y) \in E'} (f(x) - f(y))^2 \mu(x, y) \mid f : \mathbb{R} \to \mathbb{R}, f|_A \equiv 1, f|_B \equiv 0 \right\} \right)^{-1}$$

It is known that $(R_{\mu}(A, B))^{-1}$ attains the minimum and $R_{\mu}(\{x\}, \{y\})$ is the distance on $V'$ (see [15] for example). Our second main theorem is the following.
Theorem 1.7. Let \((V', E')\) be an infinite, connected, locally finite graph, \(d'\) be the graph distance of \((V', E')\) and \(\mu\) be a weight on \(E\) such that

\[(p_0) \quad \text{for some } p_0 > 0, \quad \mu(x, y)/\mu(x, z) \geq p_0 \text{ for any } x, y, z \in V' \text{ with } (x, y), (x, z) \in E.\]

If there exist \(\alpha, \beta, C > 0\) such that \(\alpha + \beta > 2\), \(C^{-1}d'(x, y)\alpha \leq R_\mu(\{x\}, \{y\}) \leq Cd'(x, y)\alpha\) and \(C^{-1}n^\beta \leq \mu(B_d(x, n)) \leq Cn^\beta\) for any \(x, y \in V'\) and \(n \geq 0\), then the limit \(d_s(V', E', \mu)\) exists and

\[
\dim_{\text{ARC}}(V', d') \leq d_s(V', E', \mu) < 2.
\]

Here \(d_s(V', E', \mu) < 2\) follows from the assumption for \(R_\mu(\{x\}, \{y\})\), so we emphasize that in this theorem we only treat the case that the associated random walk is recurrent.

We remark that in the forthcoming paper [26] the author defines a variation \(d_s^r\) of \(d_s\) and proves \(\dim_{\text{ARC}}(X, \delta) \leq d^r_s < 2\) when \((X, \delta)\) is a (continuous) low dimensional fractal, even if it is not symmetric for scaling. In [26], the counterpart of Theorem 2.1 is also proved using resistance estimates in the present paper.

The structure of this paper is as follows. We define our targeting graph \((V, E)\), state the first main result, and evaluate resistances on \((V, E)\) in Section 2. Section 3 is devoted to introducing the notion of a BF-partition and related results, and then we evaluate \(\dim_{\text{ARC}}(V, d)\) using these results in Section 4. Finally, we prove Theorem 1.7 in Section 5.
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Notation

- For a set $X$, $\#X$ denotes the cardinality of $X$.

- $a \lor b$ (resp. $a \land b$) denotes $\max\{a, b\}$ (resp. $\min\{a, b\}$).

- Let $f, g$ be functions on a set $X$ and $A \subset X$. We say $f \lesssim g$ (resp. $f \gtrsim g$) for any $x \in A$ if there exists $C > 0$ such that $f(x) \leq Cg(x)$ (resp. $f(x) \geq Cg(x)$) for any $x \in A$. We also write $f \asymp g$ (for any $x \in A$) if $f \lesssim g$ and $f \gtrsim g$.

- Let $X$ be a set and $f : X \to X$ be a map, then we write $f^k$ instead of $\underbrace{f \circ \cdots \circ f}_k$. Moreover, $f^{-k}$ denotes $(f^{-1})^k$ for $k > 0$.

- For $A \subset \mathbb{C}$ and $\alpha, \beta \in \mathbb{C}$, $\alpha A + \beta$ denotes the set $\{\alpha z + \beta \mid z \in A\}$.

- $\sqcup_{\lambda \in \Lambda} A_\lambda$ denotes the disjoint union, that is, $\sqcup_{\lambda \in \Lambda} A_\lambda$ with $A_\lambda \cap A_\tau = \emptyset$ for any $\lambda, \tau \in \Lambda$ with $\lambda \neq \tau$.

- Let $\Theta$ be a variable defined by the minimum or maximum of some functions. We say $f$ is the optimal function for $\Theta$ if $f$ attains the minimum or maximum. For example, we say $f$ is the optimal function for $R_\mu(A, B)$ if $(R_\mu(A, B))^{-1} = (1/2) \sum_{(x,y) \in E} (f(x) - f(y))^2 \mu(x,y)$, $f|_A \equiv 1$ and $f|_B \equiv 0$.

- Let $(V, E, \mu)$ be a weighted graph. We treat $\mu$ as a discrete measure on $V$ i.e. $\mu(A) := \sum_{x \in A} \mu(x) = \sum_{x \in A} \sum_{y : (x,y) \in E} \mu(x,y)$ for $A \subset V$.

- $\mu_{(V,E)}$ denotes the simple weight on $(V, E)$. For simplicity, we write $R_{(V,E)}$ instead of $R_{\mu_{(V,E)}}$, same as the case of $d_s$.

- We abuse the notation $x$ instead of $\{x\}$ if no confusion may occur. For instance, we write $R(x, y)$ instead of $R(\{x\}, \{y\})$. 
2 Resistance estimate

In this section, we construct a fractal-like infinite graph \((V,E)\) appearing in the first main result, evaluate resistances on \((V,E)\) using combinatorial arguments, and calculate \(d_s(V,E)\) with the evaluation.

Let \(S = \{z \mid \text{Re}(z) \lor \text{Im}(z) \leq 1/2\} \subset \mathbb{C}\),

\[
p_j = \begin{cases} 
0 & \text{if } j = 0, \\
\exp(j\pi\sqrt{-1/4})/\sqrt{2} & \text{if } j = 1, 3, 5, 7, \\
\exp(j\pi\sqrt{-1/4})/2 & \text{if } j = 2, 4, 6, 8,
\end{cases}
\]

\(\varphi_j(z) = (z - p_j)/3 + p_j\), \(\Phi_0(A) = \bigcup_{j=0,1,3,5,7} \varphi_j(A)\), \(\Phi_1(A) = \bigcup_{j=1}^8 \varphi_j(A)\) for \(A \subset \mathbb{C}\) (see Figures 4, 5, 6). We define \(V_n \ (n \geq 0)\) by

\[
V_0 = \{p_0, p_1, p_3, p_5, p_7\}, \quad V_n = \Phi_F(n) \circ \cdots \circ \Phi_F(1) V_0 \ (n \geq 1),
\]

where

\[
F(n) = \begin{cases} 
1 & \text{if } k^2(k-1) < n \leq k^3 \text{ for some } k \in \mathbb{N}, \\
0 & \text{otherwise},
\end{cases}
\]

and a graph \((V,E)\) by

\[
V = \bigcup_{n \geq 0} 3^n \left( V_n + \frac{1 + \sqrt{-1}}{2} \right), \quad E = \{(x,y) \in V \times V \mid |x-y| = 2^{-1/2}\}.
\]

(see Figure 3). For the rest of this paper, \((V,E)\) denotes the graph defined above and \(d\) denotes the graph distance of \((V,E)\). Then our first main result is stated as follows.

**Theorem 2.1.**

\[
d_s(V, E) = 2 \frac{\log 5}{\log 3 + \log 5} < \dim_{\text{ARC}}(V, d) = \dim_{\text{ARC}}(SC, | \cdot |) < 2,
\]

where \(SC\) is the unique nonempty compact subset of \(\mathbb{C}\) with \(\Phi_1(SC) = SC\), called the (standard) Sierpiński carpet (see Figure 7).
Spectral dimension and ARC dimension on graphs

Figure 7: (Standard) Sierpiński carpet

Figure 8: Vicsek set

Remark. $2 \log 5 / (\log 3 + \log 5) = \dim_{\text{ARC}}(\text{Vic}, |\cdot|_C)$ where Vic is the unique nonempty compact subset of $\mathbb{C}$ with $\Phi_0(\text{SC}) = \text{SC}$, called the Vicsek set or the Vicsek tree (see Figure 8). On the other hand, the value of $\dim_{\text{ARC}}(\text{SC}, |\cdot|_C)$ is not known.

Now we proceed to evaluate resistances on $(V,E)$ and other associated graphs. Let

$$E_n = \{(x,y) \in V_n \times V_n \mid |x-y| = 3^{-n}2^{-1/2}\}$$

$$S_T = \{z \in S \mid \text{Im}(z) = 1/2\}, \quad S_B = \{z \in S \mid \text{Im}(z) = -1/2\},$$

$$R_{n,TB} = R_{(V_n,E_n)}(V_n \cap S_T, V_n \cap S_B), \quad R_{n,Pt} = R_{(V_n,E_n)}(p_1, p_5).$$

Theorem 2.2.

1. $R_{n,TB} \asymp R_{n,Pt}$ for any $n \geq 0$.

2. Let $m_1(n) = \#\{k \mid k \leq n, F(k) = 1\}$ and $m_2(n) = \#\{k \mid k \leq n, F(k) = 1, F(k-1) = 0\}$ for $n \geq 1$. Then there exist $C_1, C_2 > 0$ and $\rho > 1$ such that

$$\rho^{m_1(n)}3^{-m_1(n)}C_1^{m_2(n)} \leq R_{n,Pt} \leq 2\rho^{m_1(n)}3^{-m_1(n)}C_2^{m_2(n)}$$

for any $n \geq 0$.

3. There exists $M > 0$ such that $R_{n,Pt} \leq R_{n+M,Pt}/2$ for any $n \geq 0$.

Remark. Theorem 2.2.3 does not follows from Theorem 2.2.2. Indeed, we do not know whether $C_1$ equal to $C_2$ for Theorem 2.2.2.

In order to prove the above theorem, we introduce known results about resistance estimate for the graphical Sierpiński carpet and the idea of a flow on a graph. Let $V_1^\text{SC} = \{p_1, p_3, p_5, p_7\}$ and $V_n^\text{SC} = (\Phi_1)^n(V_0^\text{SC})$. Similar to the case of $(V,E)$, we set

$$E_n^\text{SC} = \{(x,y) \in V_n^\text{SC} \times V_n^\text{SC} \mid |x-y| = 3^{-n}\},$$

$$R_{n,TB}^\text{SC} = R_{(V_n^\text{SC},E_n^\text{SC})}(V_n^\text{SC} \cap S_T, V_n^\text{SC} \cap S_B),$$

$$R_{n,Pt}^\text{SC} = R_{(V_n^\text{SC},E_n^\text{SC})}(p_1, p_5).$$

We use the following fact.

Proposition 2.3. For some $\rho > 1$, $\rho^n \asymp R_{n,TB}^\text{SC} \asymp R_{n,Pt}^\text{SC}$ for any $n \geq 0$. 
Remark. 1. $\rho^n \simeq R_{n,\text{TB}}^{\text{SC}}$ follows in the same way as \cite{2}. On the other hand, we cannot apply this method to show $\rho^n \simeq R_{n,\text{Pt}}^{\text{SC}}$ because this method, especially a part of a potential theoretic argument in \cite[Section 4]{2}, uses the fact that the optimal function $g$ for $R_{n,\text{TB}}^{\text{SC}}$ satisfies $g \equiv 1$ on $V_n^{\text{sc}} \cap S$: we cannot obtain $R_{n+k,\text{Pt}}^{\text{SC}} \geq R_{n,\text{Pt}}^{\text{SC}} R_{k,\text{Pt}}^{\text{SC}}$ but only $R_{n+k,\text{Pt}}^{\text{SC}} \geq R_{n,\text{Pt}}^{\text{SC}} R_{k,\text{TB}}^{\text{SC}}$ for any $n, k \geq 0$ in this way. We will prove $R_{n,\text{TB}}^{\text{SC}} \geq R_{n,\text{Pt}}^{\text{SC}}$ in Appendix A.

2. The heat kernel estimates of the simple random walks on graphical Sierpiński carpets have been studied in \cite{3}, including transient cases without resistance estimates. We can also obtain $\rho^n \simeq R_{n,\text{Pt}}^{\text{SC}}$ by \cite[Theorem 1.5]{3} and \cite[Theorem 1.3]{5}.

Definition 2.4 (Unit flow). Let $(V', E')$ be a connected, locally finite graph. For $A, B \subset V'$ with $A \cap B = \emptyset$, $f : E' \to \mathbb{R}$ is called a unit flow from $A$ to $B$ if $f$ satisfies

- $f(x, y) = -f(y, x)$ for any $(x, y) \in E'$,
- $\sum_{y : (x, y) \in E'} f(x, y) = 0$ for any $x \notin A \cup B$,
- $\sum_{x \in A} \sum_{y : (x, y) \in E'} f(x, y) = 1$ and $\sum_{x \in B} \sum_{y : (x, y) \in E'} f(x, y) = -1$.

Let $\mu$ be a weight on $E'$, then it is known that

$$R_{\mu}(A, B) = \min \left\{ \frac{1}{2} \sum_{(x, y) \in E'} \frac{f(x, y)^2}{\mu(xy)} \left| \begin{array}{c} f \text{ is a unit flow from } A \text{ to } B \end{array} \right. \right\}. \quad (3)$$

We say $f$ is an optimal flow for $R_{\mu}(A, B)$, or an optimal flow from $A$ to $B$ if $f$ is an optimal function for the right hand side of (3).

Before the proof of Theorem 2.2 we show some technical lemmas.

Lemma 2.5. 1. $(1/4) R_{n,\text{Pt}} \leq R_{(G_n, E_n)}(p_1, p_3) \leq 4 R_{n,\text{Pt}}$ for any $n \geq 0$.

2. $R_{n+1,\text{Pt}} \geq R_{n,\text{Pt}}$ for any $n \geq 0$. In particular, if $F(n+1) = 0$ then $R_{n+1,\text{Pt}} = 3 R_{n,\text{Pt}}$.

Proof. Fix any $n \geq 0$. Let $g_{j,k}$ be the optimal functions for $R_{(V_n, E_n)}(p_j, p_k)$ ($1 \leq j, k \leq 8$).

1. By symmetry, $g_{13}(z) = 1/2$ for any $z \in V_n$ with Re($z$) = 0. This and minimality of the potential assure $g_{13}(p_7) \geq 1/2$ and $g_{13}(p_5) \leq 1/2$, so

$$f_1(z) := (g_{13}(z) + g_{75}(z) - g_{13}(p_5)) \wedge 1$$

satisfies $f_1(p_1) = f_1(p_7) = 1$ and $f_1(p_3) = f_1(p_5) = 0$. Therefore

$$(R_{n,\text{Pt}})^{-1} \leq R_{(V_n, E_n)}(\{p_1, p_7\}, \{p_3, p_5\})^{-1} \leq 4 R_{(V_n, E_n)}(p_1, p_3)^{-1}.$$
On the other hand, since \( g_{15}(z) = 1/2 \) for any \( z \in V_n \) with \( \text{Im}(z) = -\text{Re}(z) \),

\[
R_{(V_n,E_n)}(p_1, p_3) \leq R_{(V_n,E_n)}(\{p_1, p_7\}, \{p_3, p_5\}) \leq 4(R_{n,Pt})^{-1}
\]
similarly follows.

2. Let \( f_2 : V_{n+1} \to \mathbb{R} \) such that

\[
f_2(z) = \begin{cases}
g_{15}(\varphi_1^{-1}(z)) & \text{if } \text{Re}(z) + \text{Im}(z) \geq \frac{2}{3}, \\
g_{15}(\varphi_5^{-1}(z)) & \text{if } \text{Re}(z) + \text{Im}(z) \leq -\frac{2}{3}, \\
\frac{1}{2} & \text{otherwise}.
\end{cases}
\]

Then \( R_{n+1, Pt} \geq 2(\sum_{(x,y) \in E_{n+1}} (f_2(x) - f_2(y))^2)^{-1} = R_{n, Pt} \) follows immediately. If \( F(n + 1) = 0 \), then it is easy to check that \( R_{n, Pt} = R_{(V_{n+1}, E_{n+1})}(p_1, \varphi_1(p_5)) = R_{(V_{n+1}, E_{n+1})}((\varphi_1(p_5), \varphi_5(p_1))) = R_{(V_{n+1}, E_{n+1})}((\varphi_1(p_1), p_5)) \), so \( R_{n+1, Pt} \leq 3R_{n, Pt} \) by the triangle inequality. On the other hand, let \( f_3 : V_{n+1} \to \mathbb{R} \) defined by

\[
f_3(z) = \begin{cases}
\frac{1}{3}g_{15}(\varphi_1^{-1}(z)) + \frac{2}{3} & \text{if } z \in \varphi_1(S), \\
\frac{1}{3}g_{15}(\varphi_0^{-1}(z)) + \frac{1}{3} & \text{if } z \in \varphi_0(S), \\
\frac{1}{3}g_{15}(\varphi_5^{-1}(z)) & \text{if } z \in \varphi_5(S), \\
\frac{1}{2} & \text{otherwise}.
\end{cases}
\]

Then \( R_{n+1, Pt} \geq 2(\sum_{(x,y) \in E_{n+1}} (f_3(x) - f_3(y))^2)^{-1} = 3R_{n, Pt} \) follows.

\[\square\]

**Lemma 2.6.** Let \( l = l(n) := \max\{m \mid m \leq n, F(m) = 0\} \), then \( R_{n-l, Pt}^{SC}R_{l, Pt} \geq R_{n, Pt} \) for any \( n \).

**Proof.** For any \( x, y \in V_n \) with \( (x, y) \in E_{n-l}^{SC} \), there exists a unit flow \( f_{xy} \) from \( x \) to \( y \) on \( E_n \) such that \( f_{xy}(z,w) = 0 \) for any \( (z,w) \in E_n \) with \( |x - z| > 2^{1/2}3^{-(n-l)} \), and

\[
(1/2) \sum_{(z,w) \in E_n} f_{xy}(z,w)^2 \leq 4R_{l, Pt}
\]

by Lemma 2.5.1. Let \( g \) be the optimal flow for \( R_{n-l, Pt}^{SC} \). We define \( f : E_n \to \mathbb{R} \) by

\[
f(z,w) = \frac{1}{2} \sum_{(x,y) \in E_{n-l}^{SC}} g(x,y)f_{xy}(z,w),
\]

then we can see that \( f \) is a unit flow from \( p_1 \) to \( p_5 \). We also let

\[
L := \sup_{n \geq 0, z \in V_n} \#\{(x,y) \in E_{n-l}^{SC} \mid |x - z| \leq 2^{1/2}3^{-(n-l)}\},
\]
then
\[ R_{n,\text{Pt}} \leq \frac{1}{2} \sum_{(z,w) \in E_n} f(z,w)^2 \leq \frac{L}{8} \sum_{(x,y) \in E^\text{SC}_{n-l}} g(x,y)^2 \sum_{(z,w) \in E_n} f_{xy}(z,w)^2 \]
\[ \leq \frac{L}{2} R_{n-l,\text{Pt}} R_{l,\text{Pt}}. \]
Since \( L < \infty \), the claim follows. \( \square \)

**Proof of Theorem 2.2**

1. It is sufficient to show \( R_{n,\text{TB}} \geq R_{n,\text{Pt}} \) for \( n \geq 1 \).

- The case of \( F(n) = 0 \): let \( g_{n-1} \) be the optimal function for \( R_{n-1,\text{Pt}} \). Here we define \( f_n : V_n \to \mathbb{R} \) by
\[
f_n(z) = \begin{cases} 
1 & \text{if } z \in \varphi_1(S) \cup \varphi_3(S), \\
g_{n-1}(3z) + g_{n-1}(-3\sqrt{-1}z) - \frac{1}{2} & \text{if } z \in \varphi_0(S), \\
0 & \text{if } z \in \varphi_5(S) \cup \varphi_7(S).
\end{cases}
\]
Then \( (R_{n,\text{TB}})^{-1} \leq (1/2) \sum_{(x,y) \in E_n} (f_n(x) - f_n(y))^2 \leq 4(R_{n-1,\text{Pt}})^{-1} = 12(R_{n,\text{Pt}})^{-1} \) by Lemma 2.3,2. This completes the case.

- The case of \( F(n) = 1 \): let \( l(n) \) be the same as in Lemma 2.6. Using the former case, Proposition 2.3 and Lemma 2.6, we obtain \( R_{n-l,\text{TB}} R_{l,\text{TB}} \geq R_{n-l,\text{Pt}} R_{l,\text{Pt}} \geq R_{n,\text{Pt}} \) for any \( n \) with \( F(n) = 1 \). On the other hand, we can get \( R_{n,\text{TB}} \geq R_{n-l,\text{TB}} R_{l,\text{TB}} \) for any \( n \) by the same potential theoretic argument as in [2, Theorem 4.3], which completes the proof.

2. We inductively prove the upper bound. By Theorem 2.2.1, Proposition 2.3 and Lemma 2.6 there exist \( C_3, C_4 > 0 \) such that \( C_3 R_{n-l,\text{Pt}}^\text{SC} R_{l,\text{Pt}} \geq R_{n,\text{Pt}} \) for any \( n \in F^{-1}(1) \) and \( R_{n,\text{Pt}}^\text{SC} \leq C_4 \rho^k \) for any \( k \geq 0 \). We write \( C_2 = C_3 C_4 \). If \( F(n) = 0 \), the claim for \( n \) obviously follows from the claim for \( n-1 \) with Lemma 2.3.2. Otherwise, \( R_{n,\text{Pt}} \leq C_3 R_{n-l,\text{Pt}}^\text{SC} R_{l,\text{Pt}} \leq 2 \rho^{m_1(l)} 3^{l-m_1(l)} C_2^{m_2(l)} C_2 \rho^{n-l} = 2 \rho^{m_1(n)} 3^{n-m_1(n)} C_2^{m_2(n)} \) by the claim for \( l(n) \), which concludes the proof of the upper bound. The lower bound also follows in a similar way.

3. We first prove that there exists \( m \geq 1 \) such that if \( n \geq 0 \) satisfies \( F(n+1) = \cdots = F(n+m) \) then \( R_{n,\text{Pt}} \leq R_{n+m,\text{Pt}}/2 \). Indeed, if \( F(n+1) = 0 \) then \( R_{n+m,\text{Pt}} \geq R_{n+1,\text{Pt}} = 3R_{n,\text{Pt}} \) by Lemma 2.3.2. Otherwise, let \( l = l(n+m) = l(n) \) and \( L \) be the same as in Lemma 2.6, then
\[
R_{n,\text{Pt}} \leq \frac{L}{2} R_{n-l,\text{Pt}} R_{l,\text{Pt}} \leq \frac{L}{2} R_{l,\text{TB}} R_{n+l,\text{Pt}} R_{n-l,\text{TB}} R_{l,\text{Pt}} R_{n-l,\text{Pt}} R_{l,\text{Pt}} R_{n-l,\text{Pt}} R_{l,\text{Pt}} \frac{\rho^{n+m-l}}{R_{n-l,\text{TB}}^\text{SC}} \frac{\rho^m R_{n+m,\text{Pt}}}{R_{n-l,\text{TB}}^\text{SC}}
\]
by Lemma 2.6. Therefore Theorem 2.2, Proposition 2.3 and the potential theoretic argument shows that $R_{n, Pt} \leq R_{n+m, Pt}/2$ if $m$ is sufficiently large. Fix such an $m \geq 1$, then by the definition of $F$, for any $n \geq 0$ there exists $k_n$ such that $n + m^3 \leq k_n < n + m^3 + m$ and $f(k_n + 1) = \cdots = f(k_n + m)$. Therefore $R_{n, Pt} \leq R_{k_n, Pt} \leq R_{k_n+m, Pt}/2 \leq R_{n+m^3+2m, Pt}/2$ for any $n \geq 0$.

\[ \square \]

Next, we evaluate resistance metrics between points and check some properties associated with the metric.

**Proposition 2.7.** For $x_1, x_2 \in V$, we denote by $n(x_1, x_2)$ the minimal integer such that there exist $a_j, b_j \in \mathbb{Z}$ ($j = 1, 2$) with the property that

\[ x_j \in S_{n(x_1, x_2), a_j, b_j} \cup \{ 0 \}, \quad S_{n(x_1, x_2), a_1, b_1} \cap S_{n(x_1, x_2), a_2, b_2} \neq \emptyset \]

for $j = 1, 2$, where $S_{n,a,b}$ is defined by (1). Then $R_{(V,E)}(x_1, x_2) \asymp R_{n(x_1, x_2), TB}$ for any $x, y \in V$.

**Proof.** We first prove $R_{(V,E)}(x_1, x_2) \lesssim R_{n(x_1, x_2), TB}$. By (4), for $j = 1, 2$ we can inductively choose $x_{j,k}$ for $0 \leq k \leq n = n(x_1, x_2)$ such that $x_{j,k} \in V \cap 3^k \mathbb{Z}^2$ and

\[
\begin{align*}
|x_j - x_{j,0}| &\leq 2^{-1/2} & \text{if } k = 0, \\
\{ x_{j,k-1}, x_{j,k} \} &\subset 3^{k-1}(V_{k-1} + \frac{1}{2} + a + (\frac{1}{2} + b)\sqrt{-1}) \subset V & \text{otherwise.}
\end{align*}
\]

for some $a, b \in \mathbb{Z}$.

It is easy to check that $R_{(V,E)}(x_{j,k}, x_{j,k-1}) \leq (R_{k-1, Pt} \vee R_{(V,E)}(p_{k-1}, (p_1, p_3))) \leq 4R_{k-1, Pt}$, therefore we obtain $R_{(V,E)}(x_1, x_2) \leq 2 + 8 \sum_{k=0}^{n} R_{k, Pt} \leq 16MR_{n, Pt}$ by Theorem 2.2,3, so $R_{(V,E)}(x_1, x_2) \lesssim R_{n(x_1, x_2), TB}$ follows.

We next show $R_{(V,E)}(x_1, x_2) \gtrsim R_{n(x_1, x_2), TB}$. By the definition of $n = n(x_1, x_2)$, there exist $a, b \in \mathbb{Z}$ and an affine map $\psi$ such that $x_1 \in S_{n-1,a,b}$, $\psi(S_{n-1,a,b}) = \varphi_0(S)$ and $\psi(x_2) \notin S$ (see Figure 3). Let $f_1$ be a function on $\cup_{j=0}^{8} \varphi_j(V_{n-1})$ defined by

\[
f_1(z) = \begin{cases} 
g(\varphi_j^{-1}(z)) & \text{if } z \in \varphi_j(V_{n-1}) \text{ for } j = 1, 3, 5, 7, \\ 1 & \text{otherwise} \end{cases}
\]

where $g$ be the optimal function for $R_{n-1, TB}$. We also define $f_2 : V \to \mathbb{R}$ by

\[
f_2(z) = \begin{cases} 
\min\{ f_1(\sqrt{-1}^k \psi(z)) | k = 0, 1, 2, 3 \} & \text{if } \psi(z) \in S, \\ 0 & \text{otherwise}. \end{cases}
\]

Then $f_2(x_1) = 1, f_2(x_2) = 0$ and $(1/2) \sum_{(x,y) \in E} (f_2(x) - f_2(y))^2 \leq 12(R_{n-1, TB})^{-1}$. This concludes the proof.

\[ \square \]
Spectral dimension and ARC dimension on graphs

Figure 9: $\psi^{-1}(\varphi_0(S))$ and $\mathbb{C}\backslash\psi^{-1}(S)$ with values of $f_2(z)$

**Definition 2.8 (Volume doubling measure).** A Borel measure $\nu$ on a metric space $(X, \delta)$ is called a **volume doubling** measure with respect to $\delta$, if there exists $C > 0$ such that $0 < \nu(B_\delta(x, 2r)) \leq C\nu(B_\delta(x, r)) < \infty$ for any $x \in X$ and $r > 0$.

**Lemma 2.9.** Let $V_n = 8^{m_1(n)}5^{n-m_2(n)}$, where $m_1, m_2$ be the same as in Theorem 2.2.2. Then

1. $\mu_{(V,E)}(\{y \mid n(x,y) \leq m\}) \asymp V_m$ for any $x \in V$ and $m \geq 0$.
2. $\mu_{(V,E)}(B_{R_{(V,E)}}(x, R_{(V,E)}(x,y))) \asymp V_{n(x,y)}$ for any $x, y \in V$.
3. $\mu_{(V,E)}$ satisfies the volume doubling condition with respect to $R_{(V,E)}$.

The proof of Lemma 2.9 is straightforward and we omit it. Now, we are ready to prove our aim of this section.

**Theorem 2.10.** $d_s(V, E) = 2 \log 5/(\log 3 + \log 5)$.

**Proof.** By Lemma 2.9 and [24, Theorem 4.27], there exist a distance $\delta$ on $V$ and $\gamma > 1$ such that $\delta \sim_{QS} V, \delta(x,y)^\gamma \asymp \mu_{(V,E)}(B(x, R_{(V,E)}(x,y)))R_{(V,E)}(x,y)$ for any $x, y \in V$, and $p(2n, x, x) \asymp 1/\mu_{(V,E)}(B_\delta(x, n^{1/\gamma}))$ for any $x \in V$ and $n \geq 0$. It is easy to see that there exists $C > 0$ such that if $\delta(x,y)^\gamma \leq CR_{N,P_t}V_N$ then $n(x,y) \leq N$. Moreover, there exists $C' > 0$ such that for any $n \geq 0$ there exists $N = N(n)$ satisfying $C'R_{N,P_t} < n \leq CR_{N,P_t}$.
where \( C \) is same as above. Therefore
\[
\limsup_{k \to \infty} \frac{-\log p(2k, x, x)}{\log k} = \limsup_{k \to \infty} \frac{\log \mu(V,E)(B_\delta(x, k^{1/\gamma}))}{\log k} \\
\leq \limsup_{N \to \infty} \frac{\log \mu(V,E)(\{y \mid \delta(x,y) \leq (CR_{N,pt}N)^{1/\gamma}\})}{\log(R_{N,pt}N) + \log C''} \\
\leq \limsup_{N \to \infty} \frac{\log \mu(V,E)(\{y \mid n(x,y) \leq N\})}{\log(R_{N,pt}N) + \log C''} \\
\leq \limsup_{N \to \infty} \frac{\frac{m_1(N)}{N} \log 8 + (1 - \frac{m_1(N)}{N}) \log 5}{\frac{m_2(N)}{N} (\log 8 + \log \rho) + (1 - \frac{m_1(N)}{N})(\log 5 + \log 3) + \frac{m_2(N)}{N} \log C_1} \\
= 2 \frac{\log 5}{\log 3 + \log 5}
\]
because \( \lim_{N \to \infty} (m_1(N)/N) = \lim_{N \to \infty} (m_2(N)/N) = 0 \). Similarly to the above, we obtain \( \liminf_{k \to \infty} -2 \log p_{2k}(x, x)/\log k \geq 2 \log 5/(\log 3 + \log 5) \), which proves the theorem. \( \Box \)

3 Partition satisfying basic framework

In this section, we first introduce the notion of a BF-partition, which was introduced by [17] for the case of compact metric spaces and extended to the case of infinite graphs in [24]. Then, we also introduce results in [17, 24], which are necessary to evaluate the Ahlfors regular conformal dimension. See these papers for details.

Since the definition of a BF-partition is too complicated to directly understand, we begin by describing the idea of this notion in the case of \((V,E)\). For the convenience of the reader who jumps here from the introduction, we recall that \((V,E)\) is defined by [2]. Let \( T_* = \{S_{n,a,b} \mid n \geq 0, \text{int}(S_{n,a,b}) \cap V \neq \emptyset\} \), then the essential part of a partition \( K \) is defined as \( K : T_* \to 2^V \) by \( K(S_{n,a,b}) = V \cap S_{n,a,b} \). We remark that for any \( S_{n,a,b} \in T_* \), there exists a unique \( S_{n+1,a,b} \in T_* \) with \( K(S_{n,a,b}) \subset K(S_{n+1,a,b}) \). We will consider a graph on \( \{S_{n,a,b} \in T_*\} \) for each \( n \) such that there exists an edge between \( S_{n,a_1,b_1} \) and \( S_{n,a_2,b_2} \) if \( S_{n,a_1,b_1} \neq S_{n,a_2,b_2} \) and \( K(S_{n,a_1,b_1}) \cap K(S_{n,a_2,b_2}) \neq \emptyset \).

Here we return to introduce notions for the definition of a BF-partition.

**Definition 3.1 (Tree with a reference point).** Let \( T \) be a countable set and \( \pi : T \to T \) be a map such that the following conditions hold.
- \( \pi^n(w) \neq w \) for any \( n \geq 1 \) and \( w \in T \).
- For any \( w, v \in T \), there exist \( n, m \geq 0 \) such that \( \pi^n(w) = \pi^m(v) \).

Fix any \( \phi \in T \), and we call the triplet \((T, \pi, \phi)\) a bi-infinite tree with a reference point.

As the name shows, \((T, \pi, \phi)\) has a corresponding tree structure on \( T \) defined as follows.
Lemma 3.2 ([25, Lemma 3.2]).

1. Let \( b(w, v) = \min\{n \geq 0 | \pi^n(w) = \pi^m(v) \text{ for some } m \geq 0 \} \) for \( w, v \in T \), then \( \pi^{b(w, v)}(w) = \pi^{b(v, w)}(v) \).

2. Let \( A = \{ (w, v) \mid \pi(w) = v \text{ or } \pi(v) = w \} \) then \((T, A)\) is a tree.

For the rest of this section, we assume \((T, \pi, \phi)\) to be a bi-infinite tree with a reference point. We define \([w] = b(w, \phi) - b(\phi, w)\), \(T_n = \{ w \in T \mid [w] = n \}\) and \(T^w = \cup_{k \geq 0} \pi^{-k}(w)\) for any \( w \in T \) and \( n \in \mathbb{Z} \).

Definition 3.3 (Partition). Let \((V', E')\) be an infinite graph. A map \( K : T \to \{ A \subseteq V' \mid \#(A) < \infty \}\) is called a partition of \((V', E')\) parametrized by \((T, \pi, \phi)\) if the following conditions hold.

\[
(*) \quad \bigcup_{v \in \pi^{-1}(w)} K(v) = K(w) \text{ for any } w \in T.
\]

\[
(*) \quad \text{If } (w_k)_{k \in \mathbb{Z}} \subseteq T \text{ satisfies } w_k \in T_k \text{ and } \pi(w_{k+1}) = w_k \text{ for any } k, \text{ then there exist } n_0 \text{ and } (x, y) \in E' \text{ such that } K(w_n) = \{ x, y \} \text{ for any } n \geq n_0.
\]

\[(5) \quad \text{For any } (x, y) \in E', \text{ there exists } w \in T \text{ with } K(w) = \{ x, y \}.\]

Hereafter, we write \( K_w \) instead of \( K(w) \) for simplicity. Let

\[
\Lambda_e := \{ w \in T \mid \#(K_w) = 2 \text{ and } \#(K_{\pi(w)}) > 2 \}
\]
\[
T_e := \{ w \in T \mid T^w \cap \Lambda_e \neq \emptyset \} = \{ w \in T \mid \#(K_{\pi(w)}) > 2 \}.
\]

Definition 3.4 (Basic framework). Suppose \( \sup_{w \in T_e \setminus \Lambda_e} \#(\pi^{-1}(w)) < \infty \), \( K \) be a partition of a graph \((V', E')\) parametrized by \((T, \pi, \phi)\) and \( \delta \) be a metric on \( V' \). Let

\[
E'_n = \{ (w, v) \in (T_n \cap T_e) \times (T_n \cap T_e) \mid K_w \cap K_v \not= \emptyset \text{ and } w \not= v \}
\]

and let \( d'_n \) denote the graph distance of \((T_n, E'_n)\) allowing \( d'_n(w, v) = \infty \). We say \( K \) satisfies the basic framework with respect to \( \delta \) if the following conditions hold.

\[
(6) \quad K_w \neq K_v \text{ for any } w, v \in \Lambda_e \text{ with } w \neq v.
\]

\[
(7) \quad \text{There exists } \zeta \in (0, 1) \text{ such that } \text{diam}_\delta(K_w) \asymp \zeta^{[w]} \text{ for any } w \in T_e.
\]

\[
(8) \quad \text{There exists } \xi > 0 \text{ such that for each } w \in T_e, \text{ there exists } x_w \in K_w \text{ such that if } (x, y) \in E' \text{ and } \{ x, y \} \in B_\delta(x_w, \zeta^{[w]}) \text{ then } \{ x, y \} = K_v \text{ for some } v \in T^w.
\]

\[
(9) \quad \text{Let } \Delta_m(x, y) := \sup \{ n \mid x \in K_w, y \in K_v \text{ and } d'_n(w, v) \leq m \text{ for some } w, v \in T_n \}
\]

then there exists \( M_\ast \in \mathbb{N} \) with \( \delta(x, y) \asymp \zeta^{\Delta_m(x, y)} \) for any \( x, y \in V' \).

\[
(10) \quad L_\ast := \sup_{w \in T_e} \#(\{ v \mid (w, v) \in E'_{[w]} \}) < \infty
\]
Remark. To simplify notation, the formulation of the basic framework differs from the one in [24]. We can check the equivalence between these formulations, but skip here.

To the end of this section, we assume $K$ to be a BF-partition with respect to $\delta$. To use results in [24], we have to modify $T$ and $K$. Roughly speaking, we consider the modified version of $K$ as a partition (of a $\sigma$-compact metric space) of the corresponding cable system to $(V', E')$. Suppose $K$ to be a partition of $(V', E')$ satisfying the basic framework parametrized by $(T, \pi, \phi)$. Let

$$\tau_\zeta := \{(x, y, k, m) \mid (x, y) \in E', k \geq 1 \text{ and } 1 \leq m \leq 2^n(k)\}/\sim$$

where $n(k) \geq 0$ with $2^{-n(k)} \leq \zeta^k < 2^{1-n(k)}$, and $(x_1, y_1, k_1, m_1) \sim (x_2, y_2, k_2, m_2)$ if $x_1 = y_2$, $x_2 = y_1$, $k_1 = k_2$ and $m_1 + m_2 = 2^{k_1} + 1$. Then we define $T_\zeta = T_e \cup \tau_\zeta$, and $\pi': T_\zeta \to T_\zeta$ by

$$\pi'(w) = \begin{cases} 
\pi(w) & \text{if } w \in T_e, \\
v & \text{if } w = (x, y, 1, m) \in \tau_\zeta \text{ and } K_v = \{x, y\}, \\
(x, y, k - 1, j) & \text{if } w = (x, y, k, m) \in \tau_\zeta \text{ with } k \geq 1 \text{ and } j - 1 < m / (2^n(k) - n(k - 1)) \leq j.
\end{cases}$$

(Note that $\pi'$ is well-defined by (5) and (6).) Then it is easy to show that we can choose $\phi' \in T_\zeta$ with $[\cdot]'|_{T_\zeta} \equiv [\cdot]|_{T_e}$ where $[\cdot]'$ is the height defined with $(T_\zeta, \pi', \phi')$. Thus we consider $[\cdot]'$ as an extension of $[\cdot]$ and simply write $[\cdot]$ instead of $[\cdot]'$. We formally define $K_*: T_\zeta \to \{A \subset V'\}$ by

$$K_w^* = \begin{cases} 
K_w & \text{if } w \in T_e, \\
\{x\} & \text{if } w = (x, y, k, 1)(= (y, x, k, 2^n(k))) \in \tau_\zeta \text{ for some } x, y \in V' \text{ and } k \geq 1, \\
\emptyset & \text{otherwise}.
\end{cases}$$

Moreover, we also define $(T_\zeta)_n = \{w \in T_\zeta \mid [w] = n\}$ and $E^*_n \subset (T_\zeta)_n \times (T_\zeta)_n$ by

$$E^*_n = \left\{(v, w) \mid K_w^* \cap K_v^* \neq \emptyset, \text{ or } v = (x, y, k, m_1), w = (x, y, k, m_2) \text{ for some } x, y \in V', k, m_1, m_2 \geq 1 \text{ with } |m_1 - m_2| = 1 \right\}.$$
for any $n, f : (T_\zeta)_n \to \mathbb{R}, w \in T$ and $p > 0$, where $d^*_n$ is the graph distance of $((T_\zeta)_n, E^*_n)$ allowing $d^*_n(w, v) = \infty$. We define the upper $p$-spectral dimensions (parametrized by $M$) of the partition $K$ for $p > 0$ by

$$\overline{d}^*_p(K, M) = p \left(1 - \limsup_{k \to \infty} \frac{1}{k} \left(\sup_{w \in T_\zeta} \log \mathcal{E}^M_{p,k,w} \right)^{-1}\right)$$

and the lower $p$-spectral dimensions $\underline{d}^*_p(K, M)$ for $p > 0$ by (11) but replacing $\limsup$ by $\liminf$.

Now we are ready to state Theorem 1.4 precisely.

**Theorem 3.6** ([24], Theorem 4.14). If $M \geq M_*$ is sufficiently large (where $M_*$ is the integer in (9)), then the following holds.

1. $\inf \{p \mid \liminf_{k \to \infty} (\sup_{w \in T_\zeta} \mathcal{E}^M_{p,k,w}) = 0\} = \inf \{p \mid \limsup_{k \to \infty} (\sup_{w \in T_\zeta} \mathcal{E}^M_{p,k,w}) = 0\} = \dim_{\text{ARC}}(V', \delta)$.

2. If $p > \dim_{\text{ARC}}(V', \delta)$ then $p > \overline{d}^*_p(K, M) \geq \underline{d}^*_p(K, M) \geq \dim_{\text{ARC}}(V', \delta)$.

3. If $p \leq \dim_{\text{ARC}}(V', \delta)$ then $p \leq \underline{d}^*_p(K, M) \leq \overline{d}^*_p(K, M) \leq \dim_{\text{ARC}}(V', \delta)$.

**Remark.** This theorem is the discrete version of [17, Theorem 4.7.9]. The assumptions in Theorem 3.6.2 and 3 are slightly different from that in [24], Theorem 4.14 (and also [17, Theorem 4.7.9]), but these are justified by [17, Theorem 4.7.6].

### 4 Evaluation of Ahlfors regular conformal dimension

In this section, we prove $\dim_{\text{ARC}}(V, d) = \dim_{\text{ARC}}(SC, | \cdot |_C)$ and Theorem 2.1 using a BF-partition. Recall that $T_* = \{S_{n,a,b} \mid n \geq 0, \text{int} S_{n,a,b} \cap V \neq \emptyset\}$ and let

$$T = T_* \cup \{(n, S_{0,a,b}/2) \mid n \geq 1, a, b \in \mathbb{Z} \text{ with } S_{0,a,b}/2 \subset S_{0,a,b_*} \text{ for some } S_{0,a,b_*} \in T_*\}.$$

We define $\pi : T \to T$ by

$$\pi(w) = \begin{cases} S_{n+1,a,b_*} & \text{if } w = S_{n,a,b} \text{ and } S_{n,a,b} \subset S_{n+1,a,b_*}, \\ S_{0,a,b_*} & \text{if } w = (1, S_{0,a,b}/2) \text{ and } S_{0,a,b}/2 \subset S_{0,a,b_*}, \\ (n-1, S_{0,a,b}/2) & \text{if } w = (n, S_{0,a,b}/2) \text{ for } n \geq 2. \end{cases}$$

Then $(T, \pi, S_{0,0})$ is a bi-infinite tree with a reference point. Let $K(S_{n,a,b}) := S_{n,a,b} \cap V'$ for $S_{n,a,b} \in T_*$ and otherwise $K(n, S_{0,a,b}/2) := (S_{0,a,b}/2) \cap V'$, then we can see that $K$ is a partition of $V'$ satisfying the basic framework with respect to $| \cdot |_C$ as implied in the beginning of Section 3. We also note that $T_{-n} = \{S_{n,a,b} \in T\}$ for $n \geq 0$. 

**Proposition 4.1.** \( \dim_{\text{ARC}}(V, \cdot | c) = \dim_{\text{ARC}}(SC, \cdot | c) \).

**Proof.** Let

\[
X_k = \{ z \in S \mid \text{Im}(z) = \pm \text{Re}(z), \ \{\text{Re}(z), \text{Im}(z)\} \subset 2^{-1}3^{-k}\mathbb{Z} \},
\]

\[
V_k^a = \begin{cases} 
\Phi_F(a) \circ \cdots \circ \Phi_F(a-(k-1))(V_0) & \text{if } k \leq a, \\
\Phi_F(a) \circ \cdots \circ \Phi_F(1)(X_{k-a}) & \text{if } k > a,
\end{cases}
\]

\[
E_k^a = \{(x, y) \in V_k^a \times V_k^a \mid |x - y| = 3^{-a}/\sqrt{2}\}
\]

for \( a, k \geq 0 \). For \( p > 0 \) and any graph \((V', E')\) with \( V' \subset \mathbb{C} \), we also let

\[
\mathcal{E}_{p, TB}(V', E') = \inf \left\{ \frac{1}{2} \sum_{(x, y) \in E'} |f(x) - f(y)|^p \mid f : V' \to \mathbb{R}, f|_{V' \cap S_T} = 1, f|_{V' \cap S_B} = 0 \right\}.
\]

Then Theorem 3.6 and the same symmetry of \((V_k^a, E_k^a)\) as the unit square show that

\[
\dim_{\text{ARC}}(V, \cdot | c) = \inf \{ p \mid \limsup_{k \to \infty} \mathcal{E}_{p, TB}(V_k^a, E_k^a) = 0 \}.
\]

Since \( \dim_{\text{ARC}}(SC, \cdot | c) = \inf \{ p \mid \limsup_{k \to \infty} \mathcal{E}_{p, TB}(V_{kSC}^a, E_{kSC}^a) = 0 \} \) by [17, Example 4.6.7 and Theorem 4.6.10] and symmetry of the Sierpiński carpet, it is easy to see that \( \dim_{\text{ARC}}(V, \cdot | c) \geq \dim_{\text{ARC}}(SC, \cdot | c) \geq 1 \). To prove \( \dim_{\text{ARC}}(V, \cdot | c) \leq \dim_{\text{ARC}}(SC) \), we fix any \( p > \dim_{\text{ARC}}(SC, \cdot | c) \). Then there exists \( C > 0 \) such that if \( a, b, k \geq 0 \) with \( a \wedge k > b \) satisfy \( F(a) = \cdots = F(a - (b - 1)) = 1 \) then \( \mathcal{E}_{p, TB}(V_k^a, E_k^a) \leq C\mathcal{E}_{p, TB}(V_{k-b}^{a-b}, E_{k-b}^{a-b}) \). Indeed, this claim follows from in the same way as the potential theoretic argument in [20, Theorem 5.8] (which is based on [2, Theorem 4.3]) for resistances, because only symmetry of the square, convexity of \( f(x) = x^2 \) and the fact that the optimal functions are constant on \( S_T \) and \( S_B \) are used in the argument. Similarly, we may assume that if \( a, b, k \geq 0 \) with \( a \wedge k > b \) satisfy \( F(a) = \cdots = F(a - (b - 1)) = 0 \) then \( \mathcal{E}_{p, TB}(V_k^a, E_k^a) \leq C\mathcal{E}_{p, TB}(V_{k-b}^{a-b}, E_{k-b}^{a-b})^{3(1-p)b} \) without loss of generality. By [17, Proposition 4.7.5], there exists \( \gamma < 1 \) such that \( \mathcal{E}_{p, TB}(V_{kSC}^a, E_{kSC}^a) \leq \gamma^k \) for any \( k \geq 0 \), so there exist \( C' > 0 \) such that

\[
\mathcal{E}_{p, TB}(V_k^a, E_k^a) \leq C' \gamma^{m_2^a(k)} 3^{(1-p)(k - m_2^a(k))} C^{m_2^a(k)} \leq C'(3^{1-p} \vee \gamma)^k \left( \sup_{a \geq 0} C^{m_2^a(k)} \right)
\]

for any \( a, k \geq 0 \), where

\[
m_2^a(k) := \# \{ b \mid (a - k) \lor b < a, F(b) = 1 \},
\]

\[
m_2^a(k) := \# \{ b \mid (a - k) \lor b < a, F(b) \neq F(b - 1) \}.
\]

By definition of \( F \), \( \lim_{k \to \infty} \sup_a m_2^a(k)/k = 0 \) so \( \lim_{k \to \infty} \sup_a \mathcal{E}_{p, TB}(V_k^a, E_k^a) = 0 \). Since \( p > \dim_{\text{ARC}}(SC, \cdot | c) \) is arbitrary, this means \( \dim_{\text{ARC}}(G^*, \cdot | c) \leq \dim_{\text{ARC}}(SC, d_2) \), which concludes the proof. \( \qed \)
from 1 and

\[ b \]

above,

Proposition 2.7. On the other hand, by definition and reflection on edges similar to the Spectral dimension and ARC dimension on graphs.

\[ \text{ARC} \]

sition 4.1.

\[ \text{ARC} \]

can see that \( a \) (see Figure 10), which shows \( d \).

\[ \text{ARC} \]

Proof of Theorem 2.1.

Additionally, let \( l = n(n) \) be same as in Lemma 2.6 then by the reflection of edges, we can see that

\[ b_n \geq 3^{n-1}a_{l-1} \geq \frac{1}{2}((3^{n-1} - 2)e_{l-1} + 2c_{l-1}) \geq \frac{1}{2}a_n, \]

(see Figure 10), which shows \( a_n \leq b_n \leq c_n \leq e_n \) for any \( n \geq 0 \).

Since \( b_{n+1} \geq 3b_n \), we also obtain \( d(x, y) \leq c_{n(x,y)} \) for any \( x, y \in V \) in the same way as Proposition 2.7. On the other hand, by definition and reflection on edges similar to the above, \( d(x, y) \geq b_{n(x,y)} \) holds. This and \( c_{n+1} \leq 5c_n \) shows 1. 2 and 3 immediately follow from 1 and \( b_{n+1} \geq 3b_n, c_{n+1} \leq 5c_n \).

Remark. \( \text{diam}(V, d_n) \neq 3^n \) for \( n \geq 0 \). Indeed, \( b_{n+1} \geq b_n \) for any \( n \) and if \( F(n + 2) = F(n + 1) = 0 \) and \( F(n) = 1 \) then \( b_{n+2} \geq 30b_{n+1} \) because \( b_{n+2} \geq 3a_{n+1} \) and \( a_{n+1} \geq 10b_{n+2} \) (see Figure 11).

Proof of Theorem 2.7. By Lemma 4.2 and \( |x - y|_C \approx 3^{n(x,y)} \), it is easy to see that \( d \sim_{qs} \cdot |C| \). Therefore \( \text{dim}_{\text{ARC}}(V, d) = \text{dim}_{\text{ARC}}(V, \cdot |C|) \) by definition. Since it is known that \( \text{dim}_{\text{ARC}}(\text{SC}, \cdot |C|) \geq 1 + (\log 3/\log 2) \) (see [30] and [31] for example) and \( 1 + (\log 3/\log 2) > 1.5 > 2 \log 3/(\log 3 + \log 5) \), we obtain the desired inequality by Theorem 2.10 and Proposition 4.1.

Remark. By Theorem 2.2 and Proposition 2.7, we can also see \( R_{(V, E)} \sim_{qs} \cdot |C| \) and so \( \text{dim}_{\text{ARC}}(V, R_{(V, E)}) = \text{dim}_{\text{ARC}}(V, \cdot |C|) = \text{dim}_{\text{ARC}}(V, d) \).
5 Cases symmetric for scaling

In this section we first prove the following lemma:

Lemma 5.1. Under the same assumption as Theorem 1.7, there exists a BF-partition $K$ with respect to $d'$. 

For this partition $K$, we later prove $d_2(N, M) = d_s(V', E', \mu) < 2$ for sufficiently large $M$ and Theorem 1.7.

Before proving Lemma 5.1, we introduce a property of a metric space.

Definition 5.2. A metric space $(X, \delta)$ is called metric doubling if there exists $N > 0$ such that for any $x \in X$ and $r > 0$, there exist $\{x_j\}_{j=1}^{N}$ with $B_{\delta}(x, 2r) \subset \bigcup_{j=1}^{N} B_{\delta}(x_j, r)$.

Proof of Lemma 5.1. For $(x, y), (z, w) \in E'$, $H((x, y), (z, w))$ denotes the Hausdorff distance between them, that is,

$$
(d'(x, z) \wedge d'(x, w)) \vee (d'(x, z) \wedge d'(y, z)) \vee (d'(x, w) \wedge d'(y, w)).
$$

Then $H$ is the distance on $E'/\sim$ where $(x, y) \sim (z, w)$ if $\{x, y\} = \{z, w\}$. Since $d'$ is metric doubling (which follows from Ahlfors regularity) and $(V', E')$ is bounded degree, $H$ is also metric doubling. Fix some $w_s \in E'/\sim$ and applying [12, Theorem 2.2], we obtain $C_1, C_2 > 0$, $\zeta \in (0, 1)$, $T_k = \{w_n^k\}_{n \in \mathbb{N}} \subset E'/\sim$ $(k \in \mathbb{Z})$ and $Q : \sqcup_{k \in \mathbb{Z}} (T)_k \rightarrow \{A \subset E'/\sim\}$ with the following properties: for any $k, l \in \mathbb{Z}$ with $k \leq l$ and any $n, m \in \mathbb{N}$, $w_n^k = w_s$, $E'/\sim = \bigsqcup_{n \in \mathbb{N}} Q_{w_n^k}$, $B_H(w_n^k, C_1\zeta^k) \subset Q_{w_n^k} \subset B_H(w_n^k, C_2\zeta^k)$, and either $Q_{w_n^k} \subset Q_{w_m^k}$ or $Q_{w_n^k} \cap Q_{w_m^k} = \emptyset$ holds.

Let $T = \sqcup_{k \in \mathbb{Z}} (T)_k$. For any $w_n^k \in T$, let $\pi(w_n^k)$ be the unique vertex in $(T)_{k-1}$ and $K_{w_n^k} := \{x \mid (x, y) \in Q_{w_n^k} \text{ for some } y \in V'\}$, then $(T, \pi)$ is a bi-infinite tree and $K$ is a partition of $(V', E')$. Moreover, by definition of $H$, there exist $C_3, C_4$ and $x_n^k \in V'$ such that

$$
B_{\delta}(x_n^k, C_3\zeta^k) \subset K_{w_n^k} \subset B_{\delta}(x_n^k, C_4\zeta^k)
$$

for any $k \leq 0$ and $n \in \mathbb{N}$. Here we check that $K$ satisfies the basic framework.

Fix any $x, y \in V'$ with $d'(x, y) < \zeta^k$, then there exist $\{x_j\}_{j=1}^{N} \subset B_{\delta}(x, \zeta^k)$ for some $N$ such that $x_1 = x, x_N = y$ and $(x_j, x_{j+1}) \in E$ for any $j < N$. Let

$$
M_* := \sup_{x \in G} \sup_{k} \#\{w_n^k \in (T)_k \mid B(x, r_k) \cap K_{w_n^k} \neq \emptyset\} - 1,
$$

then $M_* < \infty$ because $(V', d')$ is metric doubling. This shows $\Delta_{M_*}(x, y) \geq k$, and the other properties also follow from metric doubling condition and [12].

Note that we can choose $\zeta > 0$ such that $T_e = \sqcup_{k \leq 0} (T)_k$. \hfill \Box
Proposition 5.3. Let $M_*$ be defined by (13). For any fixed $M \geq M_*$, $\mathcal{E}_{2,k,w}^M \gtrsim \zeta^{\kappa a}$ for any $k$ and $w = w_n^i \in \sqcup_{l \leq -k_l} T_l$.

Proof. In this proof, we use the idea of modulus of curves. Let $(V_*, E_*, \mu_*)$ be a weighted graph, $A, B \subset V_*$ with $A_* \cap B_* \neq \emptyset$. We also let

$$
P(V_*, E_*, \mu_*, A, B) = \left\{ \{x_j\}_{j=0}^m \mid m \in \mathbb{N}, x_0 \in A, x_m \in B \text{ and } (x_j, x_{j+1}) \in E \text{ for any } 0 \leq j < m \right\},$$

$$\mathcal{F}(V_*, E_*, \mu_*, A, B) = \left\{ f : G \to \mathbb{R} \mid \sum_{j=1}^m f(x_j) \geq 1 \text{ for any } \{x_j\}_{j=0}^m \in \mathcal{P}(V_*, E_*, \mu_*, A, B) \right\},$$

$$\mathcal{M}(V_*, E_*, \mu_*, A, B) = \min\{\sum_{x \in V_*} f(x)^2 \mu(x) \mid f \in \mathcal{F}(V_*, E_*, \mu_*, A, B)\},$$

which exists.

For the rest of the proof, we write $Q_{w,k}^M$ instead of the quintuplet

$$T_{|w|+k}, E_{|w|+k}, \mu((T_{|w|+k}, E_{|w|+k})), \pi^{-k}(w), C_{w,k}^M$$

where $C_{w,k}^M$ is defined in Definition 3.3 for simplicity. Since $\sup_{x \in V_*} \# \{(x, y) \in E'\} < \infty$ by (10), we know that $R_\mu(A, B)^{-1} \approx \mathcal{M}(V', E', \mu, A, B)$ for any $A, B \subset V'$. Similarly, $\mathcal{E}_{2,k,w}^M \approx \mathcal{M}(Q_{w,k}^M)$ for any $k \geq 0$ and $w \in \sqcup_{l \leq -k_l} T_l$ by (10).

Fix any $k \geq 0$, $w \in \sqcup_{l \leq -k_l} T_l$, and $f \in \mathcal{F}(Q_{w,k}^M)$.) We next construct a function $g \in \mathcal{F}(V', E', \mu, K_w, A_{w}^M)$ where $A_{w}^M := V' \setminus \cup \{K_v \mid v \in (T_{\zeta})_{|w|}, d_{|w|}(w, v) > M\}$, and prove

$$C^{-1} \zeta^{-[w]\alpha} \leq \sum_{x \in V'} g(x)^2 \mu(x) \leq C \zeta^{-([w]+k)\alpha} \sum_{x \in T_{|w|+k}} f(x)^2$$

for some $C > 0$ that is independent of $k, w$, and $f$, which suffices to show this proposition. Let $g_v$ be the optimal function for $\mathcal{M}(V', E', \mu, K_v, A_v^M)$. We define $g : V' \to \mathbb{R}$ by

$$g(x) = \max\{\tilde{f}(v)g_v(x) \mid v \in T_{|w|+k} \text{ such that } x \in \cup \{K_u \mid d_{|w|+k}(u, v) \leq M\}\}$$

where $\tilde{f}(v) := 2M \max\{f(u) \mid u \in T_{|w|+k}, d_{|w|+k}(u, v) \leq 2M\}$.

Claim. $g \in \mathcal{F}(V', E', \mu, K_w, A_w^M)$.

Proof of the claim. Fix any $\{x_j\}_{j=0}^m \in \mathcal{P}(V', E', \mu, K_w, A_w^M)$. Let $j_1 = \min\{j \mid x_j \in \bigcap \{A_v^M \mid \pi^k(v) = w\}\}$, then $j_1$ is well-defined because $A_v^M \subset \bigcap \{A_v^M \mid \pi^k(v) = w\}$. We inductively define $v_a \in T_{|w|+k}$ as the vertex satisfying $\{x_{j_a-1}, x_{j_a}\} \subset K_{v_a}$ and $j_a = \min\{j > j_{a-1} \mid x_j \in A_{v_{a-1}}^M\}$. Then we have some $a^* \geq 1$ such that $x_{m} \in V' \setminus A_{v_{a^*}}^M$ and

$$\sum_{j=1}^m g(x_j) \geq \sum_{a=1}^{a^*} \sum_{j=j_{a-1}+1}^{j_a} \tilde{f}(v_{a-1})g_{v_{a-1}}(x_j) \geq \sum_{a=1}^{a^*} \tilde{f}(v_{a-1}).$$
where \(j_0 = 0\) and \(v_0 \in \pi^{-k}(w)\) with \(x_0 \in K_0\). Moreover, there exists \(\{v_i^{*}\}_{l=0}^{(a^*+1)M} \in \mathcal{P}(\mathcal{M}_{w,k})\) with \(v_{aM} = v_a\) for \(0 \leq a \leq a^*\), so
\[
\sum_{a=1}^{a^*} \tilde{f}(v_{a-1}) \geq \sum_{a=1}^{a^*} \sum_{l=(a-1)M+1}^{aM} f(v_i^{*}) + \sum_{l=a^*M+1}^{(a^*+1)M} f(v_i^{*}) \geq 1.
\]

By \cite{24} Lemma 6.6 of the arXiv version and metric doubling property of \(d'\), it follows that for fixed \(c > 1\), \(R(B_{d'}(x, \phi), B_{d'}(x, c\phi)) \asymp \phi^\alpha\) for any \(\phi \geq 1\) and \(x \in V'\). This and \cite{7,8} and \cite{9} show that \(R(K_w, A_w^M) \asymp \zeta^{[w]^{[w]}}\) for any \(w \in T_c\). Therefore
\[
C^{-1}\zeta^{-[w]^{[w]}} \leq \mathcal{M}(V', E', \mu, K_w, A_w^M)
\leq \sum_{x \in V'} g(x)^2 \mu(x) \leq (L^* + 1)^{M+1} \sum_{v \in \Gamma[|w|+k]} \tilde{f}(v)^2 \sum_{x \in V'} g(v)(x^2)\mu(x)
\leq 4M^2(L^* + 1)^{3M+2}C\zeta^{-([w]^{[w]})^{[w]}} \sum_{v \in \Gamma[|w|+k]} f(v)^2
\]
for some \(C > 0\), where \(L^*\) is defined by (10). This concludes the proof.

PROPOSITION 5.4. For any fixed \(M \geq M_s\), \(\mathcal{E}_{2,k,w}^M \lesssim \zeta^{k\alpha}\) for any \(k\) and \(w = w_n^l \in \mathcal{T}_{l \leq -k}T_l\).

PROOF. We use the argument of flow. By \cite{5} Lemma 2.5], there exist \(C, C' > 1\) such that for any \(k \leq 0\) and \(u, v \in T_k\) with \(K_u \cap K_v \neq \emptyset\), there exists a unit flow \(f_{u,v}\) from \(u = u^k_a\) to \(v\) (as points of \(V'\)) satisfying \(f_{u,v}(x, y) = 0\) whenever \(\{x, y\} \not\subset B_d(x^k_a, C\zeta^k)\), and
\[
2^{-1}\sum_{(x,y) \in E'} f_{u,v}(x, y)^2 \mu(x, y) \leq C'\zeta^{k\alpha}.
\]
Additionally, since \((V', d')\) is metric doubling,
\[
\sup\{\#\{(u^k_a, u^k_b) \in J_k \mid x \in B_{d'}(x^k_a, C\zeta^k)\} \mid x \in V', k \leq 0\} < \infty
\]
same as Lemma 5.1. Similarly to Lemma 2.6 and Proposition 5.3, this shows \(C^{-1}\zeta^{[w]^{[w]}} \leq 2^{-1}\sum_{(x,y) \in E'} f(x, y)^2 \mu(x, y) \leq C\zeta^{([w]^{[w]})^{[w]}}\) for some \(C > 0\) and flow \(f\) from \(K_w\) to \(A_w^M\). This concludes the proof.

Proof of Theorem 1.7. Let \(M\) be sufficiently large. By Proposition 5.4, it is easy to see that \(\mathcal{E}_{2,k,w}^M \lesssim \zeta^{([w]^{[w]})^{[w]}}\) for \(w \in T_r \setminus \mathcal{T}_{l \leq -k}T_l\). Considering the resistance restricted on a path, we obtain that \(\alpha \leq 1\). Therefore \(\lim_{k \to \infty} \sup_{w \in T_r} (\mathcal{E}_{2,k,w}^M)^{1/k} = \zeta^{[w]^{[w]}}\) by Propositions 5.3 and 5.4. On the other hand, \(\nabla_s = \zeta^{-\beta}\) because \(n^\beta \asymp \mu(B_d(x, n))\), (10) and (12) hold. Therefore \(\mathcal{d}_2(K, M) = 2\beta/(\alpha + \beta) = d_s(V', E', \mu) < 2\), where the third equation follows from \cite{5} Theorem 1.3]. This with Theorem 3.6 suffices to prove the statement.

\[\square\]
Here we fix $S$pectral dimension and ARC dimension on graphs $R$ later follows from $\text{Prop. 2.3.}$

As we mentioned before, it suffices to show

**Proof of Proposition 2.3.**

We exist $\rho$ such that any $\text{Prop. A.1 (Harnack’s inequality for the graphical Sierpiński carpet). There exists $C > 0$ such that any $n \geq 0$ and any nonnegative harmonic function $f$ on $\{z \in D_n \mid \text{Re}(z) \lor \text{Im}(z) \neq \frac{1}{2}\}$ satisfy $c f(x) \geq f(y)$ for any $x, y \in D_n \cap \varphi_1(S)$.}

**Proof of Proposition 2.3.** As we mentioned before, it suffices to show $R_{n,TB}^{\text{SC}} \gtrsim R_{n,Pt}^{\text{SC}}$. Let $\tilde{R}_{n,TB}$ and $\tilde{R}_{n,\Delta}$ denote $\tilde{R}_{n,TB} = R_{(D_n,B_n)}(D_n \cap S_T, D_n \cap S_B)$, $\tilde{R}_{n,\Delta} = R_{(D_n,B_n)}(\{p_1 - \sqrt{-1} \cdot 3^n, p_1 - \frac{1}{2} \cdot 3^n\}, D_n \cap \{z \mid \text{Re}(z) + \text{Im}(z) = 0\})$ respectively (see Fig. 12). It is easy to see that $\tilde{R}_{n,TB} \asymp R_{n,TB}^{\text{SC}}$ and $\tilde{R}_{n,\Delta} \asymp R_{n,Pt}^{\text{SC}}$: the latter follows from $f_n^{\text{SC}}(z) - 1/2 = -f_n^{\text{SC}}(-z) + 1/2$, where $f_n^{\text{SC}}$ is the optimal function for $R_{n,Pt}^{\text{SC}}$.

Let $g_n$ be the optimal function for $\tilde{R}_{n,\Delta}$, then $g_n$ is harmonic on $D_n \cap \{z \mid \text{Re}(z) + \text{Im}(z) > 0\} \setminus \{p_1 - \sqrt{-1} \cdot 3^n, p_1 - \frac{1}{2} \cdot 3^n\}$.

Here we fix $k \geq 0$ such that $\tilde{R}_{n+k,\Delta} \geq 2\tilde{R}_{n,\Delta}$ for any $n \geq 0$ ($R_{n+k,Pt}^{\text{SC}} \gtrsim R_{n,Pt}^{\text{SC}} R_{k,TB}^{\text{SC}}$ and $\rho^n \asymp R_{n,TB}^{\text{SC}}$ assure the existence of such $k$), and define the function $h_n$ on $D_{n+k+2}$ by

\[
 h_n(x) = \begin{cases} 
 \tilde{R}_{n+k+2,\Delta} \cdot g_{n+k+2}(x) - \tilde{R}_{n+2,\Delta} \cdot g_{n+2}(\varphi_1^{-k}(x)) & \text{if } x \in \varphi_1^k(S), \\
 \tilde{R}_{n+k+2,\Delta} \cdot g_{n+k+2}(x) & \text{otherwise}.
\end{cases}
\]
Since \( g_n(p_1-2^{-1}3^{-n})-g_n(p_1-2^{-1}3^{-n}(1+\sqrt{-1})) = (\tilde{R}_{n,\triangle})^{-1}/2, \) \( h_n \) is harmonic on \( D_{n+k+2} \cap \varphi^k_1(\{z \mid \text{Re}(z) + \text{Im}(z) > 0\}) \), especially on \( D_{n+k+2} \cap \varphi^k_1(S) \). Moreover, since \( h_n(x) = \tilde{R}_{n+k+2,\triangle} \cdot g_{n+k+2}(x) \geq 0 \) on \( D_{n+k+2} \cap \varphi^k_1(\{\text{Im}(z) = -\text{Re}(z)\}) \), \( h_n \) is nonnegative. Therefore there exists \( C > 0 \) such that \( C h_n(x) \geq h_n(p_1-2^{-1}3^{-n}) = \tilde{R}_{n+k+2,\triangle} - \tilde{R}_{n+2,\triangle} \geq \tilde{R}_{n+2,\triangle} \) for any \( x \in D_{n+k+2} \cap \varphi^k_1(S) \) because of Proposition A.1. This shows

\[
R_{(D_{n+k+2},B_{n+k+2})}(\varphi^{k+2}_1(D_n), \varphi^{k+2}_5(D_n)) \\
\geq C^{-2}(\tilde{R}_{n+2,\triangle})^2(2((\tilde{R}_{n+k+2,\triangle})^{-2} + (\tilde{R}_{n+2,\triangle})^{-2}))^{-1} \\
\geq C^{-2}\frac{1}{4}(\tilde{R}_{n+2,\triangle})^2(\tilde{R}_{n+k+2,\triangle})^{-1}.
\]

On the other hand, by the potential theoretic argument as in [2, Theorem 4.3],

\[
R_{(D_{n+k+2},B_{n+k+2})}(\varphi^{k+2}_1(D_n), \varphi^{k+2}_5(D_n)) \\
\lesssim R_{(D_{n+k+2},B_{n+k+2})}(\varphi^{k+2}_1(D_0), \varphi^{k+2}_5(D_0))\tilde{R}_n
\]

for any \( n \geq 0 \). We also obtain that \((2 \cdot 3^k - 1)R_{n,\triangle}^{SC} \geq R_{n,k,\triangle}^{SC}\) by the triangle inequality of the resistance metric (see Fig. [3]). Therefore we obtain

\[
R_{n,TB}^{SC} \succ R_{n-2,TB}^{SC} \succ R_{n-2,TB}^{\triangle} \succ (\tilde{R}_{n,\triangle})^2/\tilde{R}_{n,k,\triangle} \succ (R_{n,\triangle}^{SC})^2/R_{n,k,\triangle}^{SC} \succ R_{n,\triangle}^{SC}
\]

for any \( n \geq 2 \).

\[
\square
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