Bounded cohomology via quasi-trees

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Abstract

We show that for the standard “rank 1” groups $G$ (with no nontrivial finite normal subgroups) and arbitrary unitary representation $\rho$ of $G$ in a (nonzero) uniformly convex Banach space the vector space $H^2_b(G; \rho)$ is infinite dimensional. This is new even for representations $\rho : G \to O(n)$. The proof uses our earlier construction in [2] of group actions on quasi-trees.

1 Introduction

Let $G$ be a group and $E$ a normed vector space (usually complete, either over $\mathbb{R}$ or over $\mathbb{C}$). We denote by $O(E)$ the group of norm-preserving isomorphisms $E \to E$. Let $\rho : G \to O(E)$ be a linear representation of $G$; we will refer to it as a unitary representation. We sometimes write $\rho(g)x$ as $g(x)$ or $gx$.

Recall that $F : G \to E$ is a cocycle (with respect to $\rho$) if

$$F(gg') = F(g) + gF(g')$$

for all $g, g' \in G$. Equivalently, $g \mapsto (x \mapsto \rho(g)x + F(g))$ is an action of $G$ on $E$ by isometries with linear part $\rho$. Note that $F(g^{-1}) = -g^{-1}F(g)$.

Likewise, $F : G \to E$ is a quasi-cocycle if

$$\Delta(F) := \sup_{g, g' \in G} |F(gg') - F(g) - gF(g')| < \infty$$

A quasi-cocycle gives a quasi-action of $G$ on $E$ by isometries. We are interested in quasi-actions that cannot be boundedly perturbed to actions,

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i.e. in quasi-cocycles that are not a bounded distance away from a cocycle. Such quasi-cocycles are called essential. The space of all quasi-cocycles \( F : G \to E \) on \( G \) (with \( \rho \) fixed) is a vector space \( QC(G; \rho) \). The quotient \( QC(G; \rho) \) of this space by the subspace generated by cocycles and by bounded functions \( G \to E \) can naturally be identified with the kernel of the map

\[
H^2_b(G; \rho) \to H^2(G; \rho)
\]

from second bounded cohomology to second cohomology of \( G \). Thus a quasi-cocycle is essential if it induces a nonzero element of \( \tilde{QC}(G; \rho) \).

When the representation \( \rho : G \to O(\mathbb{R}) \) is trivial (denoted also \( \mathbb{R} \)), quasi-cocycles \( G \to \mathbb{R} \) are called quasi-morphisms and \( \tilde{QC}(G; \mathbb{R}) \) is usually denoted \( \tilde{QH}(G) \).

Brooks [8] gave a combinatorial argument to show that when \( G = F_k \) is a nonabelian free group, \( \tilde{QC}(G; \mathbb{R}) \) is infinite dimensional. The subsequent work of Epstein-Fujiwara [13] extended this construction to nonelementary hyperbolic groups \( G \), or more generally, to groups that act properly on \( \delta \)-hyperbolic spaces [14], and the work of Bestvina-Fujiwara [5] extended it further to allow non-proper actions, so it applies for example to the action of mapping class groups on curve complexes.

One can use the infinite dimensionality of \( H^2_b(H; \mathbb{R}) \) for (nonamenable) subgroups \( H \) of \( G \) to prove rigidity statements, e.g. that irreducible lattices in higher rank symmetric spaces do not embed in \( G \). This is based on the work of Burger-Monod [9] that for such lattices \( G \) the map \( H^2_b(G; \mathbb{R}) \to H^2(G; \mathbb{R}) \) is injective.

The work of Monod-Shalom [21, 22, 23] showed the importance of bounded cohomology with coefficients in \( \ell^p(G) \) in the study of rigidity of \( G \). For a very nice instance of this philosophy, see [10], where a bounded cohomology class is naturally associated to a group acting on a CAT(0) cube complex. Hamenstädt [16] constructs many quasi-cocycles on \( G \) with coefficients in \( \ell^p(G) \) by examining the boundary of \( G \), or the boundary of the space on which \( G \) acts.

In this paper we carry out a Brooks-like constructions of quasi-cocycles. The normed space \( E \) in this paper will be a uniformly convex Banach space. This means that for every \( \epsilon > 0 \) there is \( \delta > 0 \) such that \( x, y \in E, \ |x|, |y| \leq 1, \ |x - y| \geq \epsilon \) implies \( |x + y| \leq 1 - \delta \). For \( 1 < p < \infty \) spaces \( \ell^p \) are uniformly convex [11].

The original Brooks construction for free groups amounts to counting translates of a fixed segment in the Cayley graph between the identity and a group element. There is a relatively straightforward generalization to
arbitrary unitary representations. This is presented in Section 2. On the other hand, the proof that the constructed quasi-cocycles are nontrivial is much more involved than in the case of the trivial representation (the special case of $E = \ell^p(G)$ is considerably easier). This proof occupies Section 3 and uses in several places the assumption that the Banach space $E$ is uniformly convex. In some arguments the weaker assumption that $E$ is reflexive is sufficient (e.g. for finding fixed points of isometric actions, see Lemmas 3.3 and 3.7), but our proof of the key Lemma 3.4 that detects that our quasi-cocycles are unbounded uses the full strength of uniform convexity.

In Section 5 we generalize the construction of quasi-cocycles to the situation when a group $G$ acts on a quasi-tree with suitable conditions to guarantee the finiteness of the number of relevant translates of a given segment. The idea is to fix a quasi-isometry with a tree and transfer all counting to it. The nontriviality of the constructed quasi-cocycles is then derived from the free group case after making the assumption that $G$ contains a suitable free group as a subgroup.

Finally, when $G$ acts on a geodesic metric space $X$ with “strongly contracting” geodesics we use our earlier work [2] to replace this action with an action on a quasi-tree, and then apply the construction for quasi-trees. The strongly contracting property is satisfied for example by a geodesic in a Gromov-hyperbolic space, a rank-1 geodesic axis in a CAT(0) space, and the geodesic axis of a pseudo-Anosov element in the Teichmüller space (see Theorem 5.9). The bulk of the work is in Section 4, where we verify that axioms from [2] hold under suitable assumptions on the action of $G$ on $X$. This provides an alternative proof of the main theorems in [13, 5, 6] where a more involved counting scheme had to be used when the underlying space is not a tree. We also emphasize that there is no analog of this counting scheme for arbitrary unitary representations, and in fact an attempt to rectify this was our initial motivation for this work. It is somewhat surprising that no general conditions on a geodesic metric space $X$ are required for this construction.

In [17] Hull and Osin give a similar construction of quasi-cocycles, but from the point of view of hyperbolically embedded subgroups (for an introduction to this theory see [12]). The recent preprint by Osin [24] clarifies the relationship between the classes of groups where these constructions can be performed.

Here is a sample result (see Theorem 5.1 for the general statement). The first four examples were settled by Hamenstädt [16] in the case $E = \ell^p(G)$. Note that even in the case of a free group $G = F_2$ the result that
For an arbitrary representation \( \rho : G \to O(n) \) is new.

**Theorem.** In each of the following cases the space \( \mathcal{Q}\mathcal{C}(G;\rho) \) is infinite dimensional for any uniformly convex Banach space \( E \) and any unitary representation \( \rho : G \to O(E) \). In all examples assume \( G \) has no nontrivial finite normal subgroups, or more generally, that \( G \) has a maximal finite normal subgroup \( N \) and \( \rho(N) \) fixes a nonzero vector in \( E \).

- \( G \) is non-elementary word hyperbolic,
- \( G \) admits a non-elementary isometric action on a connected \( \delta \)-hyperbolic space such that at least one element is hyperbolic and WPD,
- \( G = \text{Mod}(S) \), the mapping class group of a compact surface which is not virtually abelian,
- \( G = \text{Out}(F_n) \) for \( n \geq 2 \),
- \( G \) admits a non-elementary isometric action on a \( \text{CAT}(0) \) space and at least one element is WPD and acts as a rank 1 isometry. ([6] for \( \rho = \mathbb{R} \) and ([15] for \( \rho = L^p(G) \), both under the assumption that the \( \text{CAT}(0) \) space \( X \) and the action of \( G \) are proper and \( G \) contains a rank-1 element).

We review the definition of *weak proper discontinuity* (WPD) from [5] (also called *weak acylindricity* in [10]). Suppose a group \( G \) acts on a geodesic metric space \( X \) by isometries. Assume that \( f \in G \) is a hyperbolic isometry, i.e. \( f \) acts on \( X \) with positive translation length. We say \( f \) is a *WPD element* if for any \( x \in X \) and \( D > 0 \), there exists \( M \) such that for \( Y \in \alpha \)

\[
\{ g \in G | d(x, g(x)) \leq D, d(f^M(x), g(f^M(x))) \leq D \}
\]

if finite. We say the action by \( G \) is WPD if every hyperbolic element \( f \in G \) is a WPD element.

A hyperbolic isometry of CAT(0) space is *rank-1* if its axis is (strongly) contracting (cf. [6]).

## 2 Quasi-cocycles from trees

Fix \( F_2 = \langle a, b \rangle \) and choose a word \( w \in F_2 \). For simplicity we will assume that \( w \) is cyclically reduced. Let \( E \) be a normed vector space and \( \rho : G \to O(E) \) a linear representation. Also choose a nonzero \( e \in E \). We now set up some notation that will be convenient for what will do later.
Let \([g, h]\) be an oriented segment in the Cayley graph for \(F_2\) with generators \(a\) and \(b\). Then we write \([g, h] \circ \subset [g', h']\) if \([g, h]\) is a subsegment of \([g', h']\) and the orientations of the two segments agree. We then define

\[
\begin{align*}
w_+(g) &= \{ h \in G | [h, hw] \circ \subset [1, g] \} \\
w_-(g) &= \{ h \in H | [h, hw] \circ \subset [g, 1] \}.
\end{align*}
\]

Now define a function \(H = H_{w,e} : F_2 \to E\) by

\[
H(g) = \sum_{h \in w_+(g)} h(e) - \sum_{h \in w_-(g)} h(e)
\]

In other words, to a translate \(h \cdot w\) we assign \(h(e)\) when traversed in the positive direction, and \(-h(e)\) when traversed in negative direction. For example, suppose there is a positive translate \(h \cdot w\) in \([1, g]\) and no other translates in \([1, g]\) (positive or negative). Then \(H(g) = h(e)\). In \([1, g^{-1}]\) there is also only one translate of \(w\), namely \(g^{-1}h \cdot w\) (and it is traversed negatively), so \(H(g^{-1}) = -g^{-1}h(e)\).

**Proposition 2.1.** The function \(H\) constructed above is a quasi-cocycle.

**Proof.** This is the standard Brooks argument. Consider the tripod spanned by \(1, g, gf\). Call the central point \(p\). We will see that contributions of copies of \(w\) in the tripod that do not cross \(p\) cancel out. We will partition the sets

\[
X = w_\pm(g), w_\pm(f), w_\pm(gf)
\]

into subsets \(X^-, X^0\) and \(X^+\) with \(|X^0| \leq |w|\) and

\[
\begin{align*}
w^-_+(g) &= w^-_+(gf), \\
w^-_-(g) &= w^-_-(gf), \\
w^+_+(g) &= gw^-_+(f), \\
w^+_-(g) &= gw^+_-(f), \\
w^+_+(gf) &= gw^-_+(f), \text{ and} \\
w^+_-(gf) &= gw^+_-(f).
\end{align*}
\]
If this can be done then
\[ H(gf) - H(g) - gH(f) = \sum_{h \in w_+^0(gf)} h(e) - \sum_{h \in w_+^0(gf)} h(e) - \sum_{h \in w_+^0(g)} h(e) + \sum_{h \in w_+^0(g)} h(e) - \sum_{h \in w_+^0(f)} gh(e) + \sum_{h \in w_+^0(f)} gh(e) \]
and therefore \( \Delta(H) \leq 6|w||e| \).

We are left to partition the sets. For \( w_+(g) \) we define
\[ w_+(g) = \{ h \in G | [h, hw]^{\circ} \subset [1, p] \} \]
and
\[ w_+(g) = \{ h \in G | [h, hw]^{\circ} \subset [p, g] \} \]
and
\[ w_+(g) = \{ h \in G | h \in w_+(g) \text{ and } p \in (h, hw) \}. \]
We partition \( w_+(gf) \) in the same way. For \( w_+(f) \) we replace \( p \) with \( g^{-1}(p) \) in the partitioning of the set. The partitioning of \( w_-(g) \), \( w_-(f) \) and \( w_-(gf) \) is similar. For example
\[ w_-(g) = \{ h \in G | [h, hw]^{\circ} \subset [g, p] \}. \]

It is straightforward to check that these partitions satisfy the required properties.

Example 2.2. Suppose \( w = abab \). Then \( H(a^n) = H(b^n) = 0 \), while \( H((ab)^n) = (1 + ab + (ab)^2 + \cdots + (ab)^{n-2})e \in E \). In the next section we will show that \( H \) is an essential quasi-cocycle, at least when \( \rho = \ell^p(G) \), \( 1 < p < \infty \) (see Lemma 3.4 and Lemma 3.7).

3 Nontriviality of quasi-cocycles

In Brooks’ original construction of quasi-morphisms it is easy to see that the quasi-morphisms are nontrivial. In particular \( H(w^n) \) will be unbounded while for the generators \( H(a^n) \) and \( H(b^n) \) will be zero. By this last fact if \( G \) is a homomorphism that is boundedly close to \( H \) then \( G \) must be bounded on powers of \( a \) and \( b \) and therefore \( G(a) = G(b) = 0 \). Since any homomorphism
is determined by its behavior on the generators we have \( G \equiv 0 \) and the nontriviality of \( H \) follows.

When the Brooks construction is extended to quasi-cocycles it is no longer clear that the quasi-cocycle is nontrivial. In particular if \( H = H_{w,e} \) it may the \( H(w^n) \) is bounded. In fact if 1 is not in the spectrum of \( \rho(w) \) then \( H(w^n) \) will be bounded for all choices of vectors \( e \). Even if 1 is in the spectrum, when \( e \) is chosen arbitrarily \( H(w^n) \) may be bounded. To show that the Brooks quasi-cocycles are unbounded we will need to restrict to the class of uniformly convex Banach spaces and to look at a wider class of words than powers of \( w \).

We will also have to work harder to show that a cocycle \( G \) that is bounded on powers of the generators is bounded everywhere. In fact we cannot do this in general but instead will show that in a reflexive Banach space (which includes uniformly convex Banach spaces) either the cocycle is bounded or the original representation, when restricted to a non-abelian subgroup, has an eigenvector. In this latter case it is easy to construct many nontrivial quasi-cocycles.

The following concept was introduced by Clarkson [11].

**Definition 3.1.** A Banach space \( E \) is uniformly convex if for every \( \epsilon > 0 \) there is \( \delta > 0 \) such that \( x, y \in E, |x| \leq 1, |y| \leq 1, |x - y| \geq \epsilon \) implies 
\[
\frac{|x + y|}{2} \leq 1 - \delta.
\]

The original definition in [11] replaces \( |x|, |y| \leq 1 \) above with equalities, but it is not hard to see that the two are equivalent.

**Proposition 3.2.** (i) \( \ell^p \) spaces are uniformly convex for \( 1 < p < \infty \) ([11]).

(ii) A uniformly convex Banach space is reflexive (the Milman-Pettis theorem).

(iii) If \( E \) is uniformly convex, then for any \( R > 0 \) there is \( \epsilon > 0 \) so that the following holds. If \( |v| \leq R \) and \( f : E \to \mathbb{R} \) is a functional of norm 1 with \( f(v) = |v| \) and if \( e \) is a vector of norm \( \geq 1/2 \) with \( f(e) \geq 0 \) then 
\[
|v + e| \geq |v| + \epsilon.
\]

**Proof.** We only prove (iii). Choose \( \delta \in (0, 1) \) so that \( |x|, |y| \leq 1, |x - y| \geq \frac{\delta}{2(\delta + 1)} \) implies \( |\frac{x + y}{2}| \leq 1 - \delta \). Let \( \epsilon = \delta/8 < 1 \). Suppose \( f, v, e \) are as above and \( |v + e| < R + \epsilon \). If \( |v| \leq 1/8 \) then \( |v + e| \geq |e| - |v| \geq 1/4 \geq |v| + 1/8 \) and we are done. So assume that \( |v| > 1/8 \). Then for \( x = \frac{v}{|v| + \epsilon}, y = \frac{v + e}{|v| + \epsilon} \), we have \( |x|, |y| \leq 1 \) and \( |x - y| \geq \frac{1}{2(|v| + 1)} > \frac{1}{2(R + 1)} \), so we must have \( |\frac{x + y}{2}| \leq 1 - \delta \).
Thus
\[ 1 - \delta \geq \frac{x + y}{2} = \frac{v + e/2}{|v| + e} \geq \frac{|v|}{|v| + \epsilon} \]
since \( f(v + e/2) = |v| + f(e)/2 \geq |v| \) and \( |f| = 1 \). From \( 1 - \delta \geq \frac{|v|}{|v| + \epsilon} \geq \frac{1}{8 + \epsilon} \), it follows that \( \epsilon > \delta/8 \), contradiction.

**Lemma 3.3.** Let \( \rho \) be a unitary representation of a group \( F \) on a reflexive Banach space \( E \). If there is a linear functional \( f \) and a vector \( e \in E \) such that the \( F \)-orbit of \( e \) lies in the half space \( \{ f \geq \delta \} \) with \( \delta > 0 \) then there is an \( F \)-invariant vector \( e' \neq 0 \in E \) and a \( F \)-invariant functional \( \phi \) with \( \phi(e') \geq \delta \). If \( e \) is \( F \)-invariant, then we can take \( e' = e \).

**Proof.** Let \( \Lambda \) be the convex hull of the \( F \)-orbit of \( e \) in the weak topology on \( E \). Since \( E \) is reflexive, \( \Lambda \) is weakly compact. The convex hull \( \Lambda \) is also \( F \)-invariant so by the Ryll-Nardzewski fixed point theorem it will contain an \( F \)-invariant vector \( e' \). Since \( e' \in \Lambda \), \( f(e') \geq \delta \) and therefore \( e' \neq 0 \).

Since \( e' \) is a functional on the reflexive Banach space \( E^* \) and the \( F \)-orbit of \( f \) will be contained in the half space \( \{ e' \geq \delta \} \) we similarly get a \( F \)-invariant vector \( \phi \in E^* \) with \( e'(\phi) = \phi(e') \geq \delta \).

Note that if \( E \) contains a nonzero vector that is \( F \)-invariant, then the Hahn-Banach theorem supplies a functional that satisfies the conditions of the lemma and so there is also a nonzero \( F \)-invariant functional.

**Lemma 3.4.** Let \( \rho \) be any unitary representation of \( F_2 = \langle a, b \rangle \) into a uniformly convex Banach space \( E \). Then one of the following holds:

(i) for every \( e \neq 0 \) and any \( 1 \neq w \in F_2 \) not of the form \( a^m b^n \) nor \( b^m a^n \) the quasi-cocycle \( H = H_{w,e} \) is unbounded on \( F_2 \), or

(ii) there is a free subgroup \( F_2 \cong F \subset F_2 \), a linear functional \( g \), a vector \( e \) and a \( \delta > 0 \) such that the \( F \)-orbit of \( e \) is contained in the half-space \( \{ g \leq -\delta \} \).

**Proof.** We first make some observations about words in \( F_2 \). Given a word \( w \) as in (i) we can find buffer words \( B \) and \( B' \) of the form \( a^m b^n \) or \( b^m a^n \) and a subgroup \( F = \langle a^m, b^m \rangle \) with \( m >> n \) such that if \( w' = BwB' \) and \( y_1, y_2, \ldots, y_n \in F \) then in the reduced word for the element \( x = y_1w'y_2w'\cdots w'y_n \) there is exactly one copy of \( w \) for each \( w' \) and no other copies of either \( w \) or \( w^{-1} \). Note that the word \( y_1w'y_2w'\cdots w'y_n \) may not be reduced and in its reduced version there may be cancellations in the \( w' \). However, the buffer words will prevent these cancellations from reaching \( w \).
The restrictions on \( w \) ensure that \( w \) does not appear as a subword of some \( y_t \). In particular, \( |H(w')| = |e| \) and \( H(xyw') = H(xy) + xyH(w') \) for any \( y \in F \).

For simplicity, normalize so that \( |e| = 1 \). Assume that (ii) doesn’t hold, and that \( H \) is bounded. Let \( F_w \) be the set of words of the form \( w'y_1w'y_2w' \cdots y_nw' \) and let \( R = \text{sup}_{x \in F_w} |H(x)| \). Choose an \( x \in F_w \) such that

\[
|H(x)| > R - \epsilon/3\text{ where }\epsilon\text{ is the constant from (iii) of Proposition 3.2. We will find a }y \in F\text{ such that }|H(xyw')| > R\text{ to obtain a contradiction.}
\]

Let \( \phi \) be a linear functional of norm 1 such that \( \phi(H(x)) = |H(x)| \). Let \( \psi = \phi \circ x \). Since (ii) doesn’t hold, there exists a \( y \in F \) with \( \psi(yH(w')) > -\delta \) where \( 0 < \delta < \min\{1/2, \epsilon/3\} \). If \( \psi(yH(w')) = \phi(xyH(w')) \geq 0 \) then by (iii) of Lemma 3.2 \( |H(xyw')| = |H(xy) + xyH(w')| \geq |H(x)| + \epsilon > R \), contradiction.

If \( \psi(yH(w')) = \phi(xyH(w')) < 0 \) let \( z_0 = \phi(xyH(w'))H(x)/|H(x)| \) and \( z_1 = xyH(w') - z_0 \) so that \( \phi(z_1) = 0 \), \( |z_0| < \delta \) and \( |z_1| \geq 1 - \delta > 1/2 \). By (iii) of Lemma 3.2 \( |H(x) + z_1| \geq |H(w')| + \epsilon \) so \( |H(xyw')| = |H(xy) + xyH(w')| = |H(x) + z_1 + z_0| \geq |H(w')| + \epsilon - \delta > R \), again a contradiction. \( \square \)

**Example 3.5.** Choose an embedding \( F_2 \subset U(2) \) so that every nontrivial element is conjugate to a matrix of the form

\[
\begin{pmatrix}
  e^{2\pi it} & 0 \\
  0 & e^{2\pi is}
\end{pmatrix}
\]

with \( t, s, \frac{t-s}{2}, \frac{k-1}{k} \) all irrational. (Such representations can be constructed by noting that they form the complement of countably many proper subvarieties in \( \text{Hom}(F_2, U(2)) \).) Then any \( H = H_{w,e} \) is bounded on any cyclic subgroup, but many are globally unbounded (since (ii) of Lemma 3.4 fails). The proof of the second statement follows by noting that the orbit of any unit vector under a nontrivial cyclic subgroup is dense in a torus \( S^1 \times S^1 \subset \mathbb{C}^2 \). For the first statement, observe that for a fixed \( g \in F \) the values \( H(g^n) \) can be computed, up to a bounded error, by adding sums of the form

\[
U_n = u(e) + gu(e) + \cdots + g^{n-1}u(e)
\]

one for every \( g \)-orbit of occurrences of \( w \) or \( w^{-1} \) along the axis of \( g \). Applying \( g \) we have

\[
g(U_n) = gu(e) + \cdots + g^nu(e)
\]

and so \( |g(U_n) - U_n| \leq 2|e| \), which implies that \( |U_n| \) is bounded, since \( g : \mathbb{C}^2 \to \mathbb{C}^2 \) moves every unit vector a definite amount. This gives an isometric
quasi-action of $F_2$ on $\mathbb{C}^2$ or $\mathbb{R}^4$ with unbounded orbits, but with every cyclic subgroup having bounded orbits.

**Example 3.6.** To see the importance of uniform convexity take the example of $\ell_*^0(F_2)$, the space of bounded functions on $F_2$ that decay to zero, i.e. for all $\epsilon > 0$ for all but finitely many elements in $F_2$ the function has absolute value $< \epsilon$. For the left regular representation of $F_2$ on $\ell_*^0(F_2)$ there are no non-trivial subgroups that satisfy condition (ii) but for many choices of the vector $e$, the quasi-cocycle $H_{w,e}$ will be bounded. For example if $e \in \ell_*^0(F_2)$ is defined by

$$e(x) = \begin{cases} 1 & x = 1 \\ 0 & x \neq 1 \end{cases}$$

then $|H_{w,e}(x)| = 1$ for all choices of $w$ and $x$ in $F_2$. On the other hand if we define $f \in \ell_*^0(F_2)$ by

$$f(x) = \begin{cases} 1/n & x = w^{-n}, n > 0 \\ 0 & \text{otherwise} \end{cases}$$

then $|H_{w,f}(w^n)| = \sum_{i=1}^n 1/i$ so $H_{w,f}$ is unbounded. This example emphasizes an inherent difficulty in extending our results to general Banach spaces.

Note that $\overline{QC}(G; \ell_*^\infty(G)) = 0$ for any group $G$ [20], so some assumption on the Banach space is necessary.

**Lemma 3.7.** Let $\rho$ be a unitary representation on a reflexive Banach space $E$ and $G$ a cocycle that is bounded on $\langle a^2, b \rangle$ and $\langle a^3, b \rangle$. Then one of the following holds.

(i) $G$ is bounded on $F_2$, or

(ii) There is a free subgroup $F_2 \cong F \subset F_2$ such that $\rho|_F$ fixes a nonzero vector in $E$.

**Proof.** The cocycle $G$ induces an action of $F_2$ on $E$ by affine isometries and the image of $G$ is the orbit of 0 under this action. If the restriction of this action to $\langle a^2, b \rangle$ is bounded (with respect to the norm topology) then the convex hull of the orbit (in the weak topology) will be $\langle a^2, b \rangle$-invariant and compact since $E$ is reflexive so by the Ryll-Nardzewski fixed point theorem $\langle a^2, b \rangle$ will have a fixed point. Thus $Fix(a^2) \cap Fix(b) \neq \emptyset$. If this intersection is not a single point then (ii) holds since $\rho$ restricted to $F = \langle a^2, b \rangle$ fixes the difference of any two vectors in the intersection. Similarly, (ii) holds if $Fix(a^3) \cap Fix(b) \neq \emptyset$ is not a single point. If the two intersections coincide then the intersection point is fixed by both $a = a^3(a^2)^{-1}$ and $b$, so $G$ is bounded. If the intersections are distinct then $F = \langle a^6, b \rangle$ fixes two distinct points, so (ii) holds.
Proposition 3.8. Let $\rho$ be a unitary representation of $F_2$ in a reflexive Banach space $E$ and let $F$ be a rank two free subgroup such that $\rho|_F$ has an invariant vector $e \neq 0$. Let $w'$ be a cyclically reduced word that is conjugate into $F$. Then quasi-cocycles of the form $H_{w,e}$ where $w$ is a reduced word that contains $w'$ as a subword generate an infinite dimensional subspace of $\tilde{QC}(F_2;\rho)$.

Proof. After possibly conjugating $F$ we can assume that $w'$ is contained in $F$. Since $w'$ is cyclically reduced its axis contains the identity in the Cayley graph for $F_2$. This implies that the minimal $F$-tree also contains the identity and allows us to find cyclically reduced words $\alpha$ and $\beta$ in $F$ such that the concatenation $w_k = w'\alpha_k\beta_k\alpha_k\beta_k$ is cyclically reduced. Furthermore we can assume that $\alpha$ and $\beta$ generate $F$. Let $H_k = H_{w_k,e}$. By Lemma 3.3 there exists an $F$-invariant linear functional $\phi$ with $\phi(e) \geq \delta > 0$. Then the restriction of the composition $\phi \circ H_k$ to $F$ is a quasi-morphism and similarly the restriction to $F$ of the composition of $\phi$ with any co-cycle $G$ is a homomorphism. We will show that the sequence $H_{k_0}, H_{k_0+1}, \ldots$ represents linearly independent elements in $\tilde{Q}(F_2;\rho)$ for $k_0$ so large that all powers of $\alpha$ and $\beta$ appearing in $w'$ are $<< k_0$ in absolute value. We will show that the restrictions of the $H_i$ to $F$ represent linearly independent quasi-morphisms on $F$. Indeed, if $H = H_k - c_{k_0}H_{k_0} - \cdots - c_{k-1}H_{k-1}$ for any constants $c_i$ then the quasi-morphism $\phi \circ H$ is 0 on the powers of $\alpha$ and $\beta$, so if a co-cycle $G$ is boundedly close $H$, then $\phi \circ G$ must be bounded, and therefore zero, on powers of $\alpha$ and $\beta$. Therefore $\phi \circ G$ is trivial when restricted to $F$. On the other hand a straightforward calculation shows that $\phi \circ H(w^n_k) \geq n\delta$ so $\phi \circ H$ is unbounded on $F$ and $H$ and $G$ cannot be boundedly close. 

Proposition 3.9. Let $\rho$ be a unitary representation of $F_2 = \langle a, b \rangle$ on a uniformly convex Banach space and assume that no nonabelian subgroup of $F_2$ fixes a nonzero vector. Then for any fixed $e \neq 0$ and a cyclically reduced word $w'$ the quasi-cocycles of the form $H_{w,e}$ span an infinite dimensional subspace of $\tilde{Q}(F_2;\rho)$, where $w$ range over cyclically reduced words that contain $w'$ as a subword.

Proof. Without loss of generality we can assume that $w'$ does not start with $b^{-1}$ or end with $a^{-1}$. Let $w_m = w'a^mb^mab^ma^mb^m$, $m \geq 1$, and $\gcd(m,6) = 1$. By Lemma 3.4 $H_m = H_{w_m,e}$ is unbounded. Furthermore $H_m$ is 0 on the subgroups $\langle a^2, b \rangle$ and $\langle a^3, b \rangle$ listed in Lemma 3.7. Let $H = H_m - \sum_{i<m} c_i H_i$.
for constants $c_i$. Then $H$ is also unbounded, since the $H_i$ for $i < m$ are visibly 0 on all words in $F_{w_m}$ from Lemma 3.4, but $H_m$ is unbounded on $F_{w_m}$.

Suppose $H$ differs from a cocycle $G$ by a bounded function. Then $G$ is also bounded on these subgroups and is unbounded on $F_2$, contradicting Lemma 3.7.

The main theorem of this section now follows immediately from the previous two propositions.

**Theorem 3.10.** Let $\rho$ be a unitary representation of $F_2$ on a uniformly convex Banach space $E \neq 0$. Then $\dim \tilde{QC}(F_2; \rho) = \infty$.

We remark that Pascal Rolli has a new construction, different from the Brooks construction, that he showed in [25] produces nontrivial quasi-cocycles on $F_2$ (and some other groups) when the Banach space $E$ is an $\ell^p$-space (or finite dimensional).

### 4 Strongly contracting geodesics

In this section we review the construction of quasi-trees from [2]. The construction is axiomatic and previously it has been applied in the setting of $\delta$-hyperbolic and CAT(0) spaces where the axioms are well-known to hold, and to the setting of Outer space where the axioms are verified by Algom-Kfir [1]. Here we verify that the axioms apply in arbitrary geodesic metric spaces with (some) strongly contracting geodesics. Many of our arguments are an adaptation of [1].

#### 4.1 Strongly contracting geodesics, elements and orbits

Let $\alpha$ be a geodesic in a geodesic metric space $X$ and $\pi = \pi_\alpha$ the nearest point projection from $X$ to $\alpha$. Note that in general $\pi(p)$ will be a subset of $\alpha$ rather than a single point. The geodesic $\alpha$ is $B$-strongly contracting if for every metric ball $V$ that is disjoint from $\alpha$, $\text{diam}(\pi(V)) \leq B$. A well known characterization of $\delta$-hyperbolic spaces is that every geodesic is $B$-strongly contracting for some fixed $B$. However, there are many examples of metric spaces where some but not all geodesics are strongly contracting.

Now assume that a group $G$ acts on $X$. The translation length of an element $f$ is

$$\tau(f) = \lim_{n \to \infty} \frac{d(x, f^n(x))}{n}.$$
The element $f$ is hyperbolic, with respect to this action, if $\tau(f) > 0$ or equivalently the $f$-orbit of $x$ is a quasi-geodesic. Then $f$ is a WPD element if $f$ acts hyperbolically on $X$ and for all $D > 0$ and $x \in X$ there exists an integer $M > 0$ such that

$$\{g \in G | d(x, g(x)) \leq D, d(f^M(x), g(f^M(x))) \leq D\}$$

is finite.

A geodesic $\alpha$ is an axis for a hyperbolic element $f \in G$ if $\alpha$ is $f$-invariant. We say that $f$ is axial if $f$ has an axis $\alpha$. Not all hyperbolic elements are axial.

We also say that the orbit $O_x = \{f^i(x) | i \in \mathbb{Z}\}$ of a hyperbolic element $f$ is $B$-strongly contracting if for every metric ball $V$ that is disjoint from $O_x$, $\text{diam}(\pi(V)) \leq B$, where $\pi: X \rightarrow O_x$ is the nearest point projection.

We now state a simplified version of the main result of this section.

**Theorem 4.1.** Let $G$ act on a geodesic metric space $X$ and let $f \in G$ be a WPD element with respect to this action with a strongly contracting orbit. Then $G$ acts on a quasi-tree $Q$ with a WPD element.

We will actually prove a stronger result, Theorem 4.25, that will give us some extra technical conditions that are necessary in the construction of quasi-cocycles. We also remark that with more work one can construct a $G$-quasi-tree $\tilde{Q}$ in which $f$ is a WPD element. See Theorem 4.26.

The following proposition is the main technical result that we’ll need.

**Proposition 4.2.** Let $\alpha$ be a $B$-strongly contracting geodesic with $\pi: X \rightarrow \alpha$ the nearest point projection. Let $p, q \in X$ with $x \in \pi(p)$ and $y \in \pi(q)$. If $d(x, y) \geq 5B$ then

$$d(p, q) \leq d(p, x) + d(x, y) + d(y, q) \leq d(p, q) + 10B.$$  

Exactly the same statement holds if $\alpha$ is replaced with a $B$-strongly contracting orbit of a hyperbolic isometry.

**Proof.** We give a proof for $\alpha$; the proof for $O_x$ is exactly the same.

The first inequality is just the triangle inequality.

Let $\beta$ be a geodesic from $p$ to $q$. We cover $\beta$ with a collection of balls disjoint from $\alpha$. Let $V_p = B(p, d(p, x))$ and $V_q = B(q, d(q, y))$ be a our first two balls. We inductively define the remaining balls. Fix $p_1 \in \beta$ with $d(p, p_1) = d(p, x)$. Let $R_1$ be the distance from $p_1$ to $\alpha$ and let $V_1 = B(p_1, R_1)$. To define $V_i$ we let $p_i$ be a point on $\beta$ such that $d(p_{i-1}, p_i) = R_{i-1}$
and \( d(p, p_{i-1}) < d(p, p_i) \). Let \( R_i \) be the distance from \( p_i \) to \( \alpha \) and then \( V_i = B(p_i, R_i) \). Let \( n \) be chosen to be the first integer where \( V_n \cap V_q \neq \emptyset \) (if such an \( n \) exists).

Note that \( n \geq 3 \) since \( d(x, y) \geq 5B \) and the projection of each ball to \( \alpha \) has diameter \( \leq B \).

We claim that \( R_i \leq 4B \) for some \( i = 1, \ldots, n - 1 \). Note that if all the \( V_i \) are disjoint from \( V_q \) then \( R_i \to 0 \) and the statement holds.

Now assume that \( n \) is finite and assume the claim fails. Then
\[
    d(p, q) \geq d(p, x) + (n - 1)4B + d(y, q)
\]
but
\[
    d(p, x) + (n + 2)B + d(y, q) \geq d(p, x) + d(x, y) + d(y, q) \\
    \geq d(p, q)
\]
which implies that
\[
    (n + 2)B \geq (n - 1)4B
\]
and \( n \leq 2 \), contradiction. Therefore \( R_k \leq 4B \) for some \( k = 1, \ldots, n - 1 \).

In fact we claim that the smallest such \( k \) is 1 or 2. Let \( x_k \) be a point in \( \pi(p_k) \). Then
\[
    d(p, p_k) \geq d(p, x) + (k - 1)4B
\]
and
\[
    d(p, p_k) \leq d(p, x) + d(x, x_k) + d(x_k, p_k) \\
    \leq d(p, x) + kB + 4B
\]
which implies that
\[
    (k + 4)B \geq 4(k - 1)B
\]
and therefore \( k \leq 8/3 \). Since \( k \) is an integer \( k \leq 2 \), as claimed.

We reverse the roles of \( p \) and \( q \) to define points \( q_i \). We let \( q_j \) be the first \( q_i \) such that the distance between \( q_i \) and \( \alpha \) is \( \leq 4B \) and as above \( j = 1 \) or 2. Let \( y_j \) be a point in \( \pi(q_j) \).

We can now finish the proof. We first note that
\[
    d(p_k, q_j) \leq d(p, q) - (d(p, x) + d(y, q)).
\]
Applying this we see that
\[
    d(x_k, y_j) \leq d(p, q) - (d(p, x) + d(y, q)) + 8B.
\]
We also have
\[ d(x, x_k) \leq 2B \]
with a similar inequality for \( b \) and \( y_j \). Combining inequalities we have
\[ d(x, y) \geq d(p, q) - (d(p, x) + d(y, q)) - 12B. \]
Rearranging this last inequality gives us the second inequality in the proposition with \( 10B \) replaced by \( 12B \). Note that there is better bound on \( d(p_k, q_j) \) when either or both of \( k \) and \( j \) are 2. There are similarly better bounds on \( d(x, x_k) \) and \( d(y, y_j) \) if \( k = 1 \) or \( j = 1 \), respectively. Using this one can get the slightly better bound of the proposition. As it is not essential for what follows, we leave the details to the reader.

We discussed the case such that \( p, q \) are not on \( \alpha \). If both of them are on \( \alpha \), then the claim is trivial, and if only one of them is on \( \alpha \), we can argue similarly (in fact, \( 10B \) can be replaced by \( 2B \)).

We have the following consequences:

**Corollary 4.3.** Let \( \alpha \) be a \( B \)-strongly contracting geodesic with \( \pi : X \to \alpha \) the nearest point projection. Let \( p, q \in X \) with \( x \in \pi(p) \) and \( y \in \pi(q) \). If \( d(x, y) \geq 5B \) then the path \( [p, x] \cup [x, y] \cup [y, q] \) is a \( (1, 10B) \)-quasi-geodesic.

**Corollary 4.4.** Let \( f \) be a hyperbolic isometry of \( X \) with a \( B \)-strongly contracting orbit \( \mathcal{O}_x \). Then the projection of any (not necessarily disjoint) metric ball of radius \( R \) to \( \mathcal{O}_x \) has diameter bounded by \( 2R + 10B \).

The next Proposition lets us talk about hyperbolic isometries with strongly contracting orbits.

**Proposition 4.5.** Let \( f \) be a hyperbolic isometry of \( X \). If the orbit \( \mathcal{O}_x \) is strongly contracting, then any other orbit \( \mathcal{O}_y \) is also strongly contracting.

**Proof.** Assume \( \mathcal{O}_x \) is \( B \) strongly contracting. Let \( D = d(x, y) \). First we argue that for any \( p \in X \) the projection \( \pi_y(p) \subset \mathcal{O}_y \) is contained in the \((3D + 10B)\)-neighborhood of \( \pi_x(p) \subset \mathcal{O}_x \). Without loss of generality, we may assume that \( x \in \pi_x(p) \) and let \( L = d(p, x) \). Thus \( d(p, y) \leq L + D \). Now let \( y' \in \pi_y(p) \), so in particular \( d(p, y') \leq d(p, y) \leq L + D \). Let \( x' \) be the corresponding point in \( \mathcal{O}_x \) so that \( d(x', y') = D \). Then \( d(p, x') \leq L + 2D \). From Proposition 4.2 (notice that \( \pi_x(x') = x' \) and we may assume that \( d(x', x) \geq 5B \)) we have
\[ L + 2D \geq d(p, x') \geq d(p, x) + d(x, x') - 10B = L + d(x, x') - 10B \]
and hence \( d(x, x') \leq 2D + 10B \), which implies \( d(x, y') \leq 3D + 10B \).

Now let \( V \) be an \( R \)-ball disjoint from \( \mathcal{O}_y \). If \( R \leq D \) then the projection of \( V \) to \( \mathcal{O}_x \) has bounded diameter by Corollary 4.3 and the same holds for \( \mathcal{O}_y \) by the first paragraph. If \( R > D \) and let \( V' \) be the concentric ball of radius \( R - D \). Thus \( V' \) is disjoint from \( \mathcal{O}_x \) and its projection to \( \mathcal{O}_x \) has diameter \( \leq B \), so by the first paragraph, its projection to \( \mathcal{O}_y \) is also bounded.

To finish the proof, note that every point in \( V - V' \) is within \( D \) of a point in \( V' \), so its projection to \( \mathcal{O}_x \), and hence \( \mathcal{O}_y \), is within a bounded distance of \( \pi_y(V') \). \( \square \)

The following is the well known Morse Lemma.

**Lemma 4.6.** Given \( A, C \) and \( B \) there exists a constant \( d > 0 \) such that the following holds. Let \( \alpha \) be a \( B \)-strongly contracting geodesic and \( \beta \) a \((A, C)\)-quasi-geodesic with endpoints on \( \alpha \). Then \( \beta \subset \mathcal{N}_d(\alpha) \). The same holds when \( \alpha \) is a \( B \)-strongly contracting orbit.

**Proof.** We will prove the statement for \( \alpha \) a geodesic and under the additional assumption that \( \beta \) is rectifiable and the length of any long subpath is bounded by \( A' \) times the distance between the endpoints. The general statement is then proved by the standard connect-the-dots approximation arguments.

Choose \( r >> A'B \). We will argue that the length of each subpath of \( \beta \) outside the \( r \)-neighborhood of \( \alpha \) is bounded by a function of \( A' \) times the distance between the endpoints. Hence taking \( d \) to be this bound plus \( r \) finishes the proof.

So assume \( \beta' \) is a segment of \( \beta \) with endpoints \( x, y \) at distance \( r \) from \( \alpha \) and with every point of \( \beta' \) at distance \( \geq r \) from \( \alpha \). Let \( L \) be the length of \( \beta' \). Cover \( \beta' \) with \( \leq L/r + 1 \) balls of radius \( r \) centered at points of \( \beta' \).

Each ball projects to a set in \( \alpha \) of diameter \( \leq B \), so we conclude that the projection of \( \beta' \) to \( \alpha \) has diameter \( \leq B(L/r + 1) \). Since the distance from \( \alpha \) to \( x \) and \( y \) is \( r \),

\[
d(x, y) \leq 2r + B(L/r + 1)
\]

and from our assumption we have

\[
L \leq A'd(x, y).
\]

Combining the two inequalities yields

\[
L \leq A'(2r + B(L/r + 1)),
\]

which gives the desired upper bound on \( L \). \( \square \)
4.2 Elementary closure

Note in particular that if $O_x$ is a $B$-strongly contracting orbit of some WPD element $f \in G$ and $g \in G$ maps $O_x$ into a Hausdorff neighborhood of $O_x$, then $g(O_x) \subset N_{d_0}(O_x)$ for $d_0$ independent of $g$. Define $EC^X(f)$, the elementary closure of $f$, to be the subgroup of $G$ consisting of $g \in G$ so that $g(O_x)$ is contained in a Hausdorff neighborhood of $O_x$. The definition is independent of $x$. Let $EC_X^0(f)$ be the subgroup that coarsely fixes every point in $O_x$. Namely, $g \in EC_X^0(f)$ if there exists a $D > 0$ such that the $g$-orbit of every $y \in O_x$ has diameter bounded by $D$. In most cases, when the space $X$ is understood, we will suppress the $X$ and simply write $EC(f)$ and $EC_0(f)$.

**Proposition 4.7.** Let $G$ act on a geodesic metric space and $f \in G$ a WPD element with strongly contracting orbits. Then $EC(f)$ is the unique maximal virtually cyclic subgroup that contains $f$, $EC_0(f)$ is finite and there is a short exact sequence

$$1 \to EC_0(f) \to EC(f) \to \mathbb{Z} \to 1$$

where $\mathbb{Z} \cong \mathbb{Z}$ or the infinite dihedral group $\mathbb{D}_\infty$. Furthermore if $h \notin EC(f)$ then $EC(f) \cap EC(hfh^{-1})$ is finite.

**Proof.** The set $Y = \bigcup_{g \in EC(f)} g(O_x)$ is contained in a Hausdorff neighborhood of $O_x$ and is therefore quasi-isometric to $\mathbb{R}$. If an element of $EC(f)$ has zero translation length it either interchanges the ends of $Y$ or it lies in $EC_0(f)$. If there are elements that interchange the ends, pass to a subgroup $H$ of index 2 consisting of elements that preserve the ends. On $H$ translation length then defines a homomorphism from this subgroup to $\mathbb{R}$ whose kernel is $EC_0(f)$. Suppose $h_i(f)$ are distinct cosets of $\langle f \rangle$ in $H$. By precomposing each $h_i$ with a power of $f$ we may assume that $d(x, h_i(x))$ is uniformly bounded. Then for any $k > 0$ either $d(f^k(x), h_i f^k(x))$ or $d(f^{-k}(x), h_i f^k(x))$ is uniformly bounded, but the latter possibility is ruled out for large $k$ by the assumption that $h_i$ preserve the ends of $Y$. Now the WPD condition implies that the collection of $h_i$’s is finite and therefore $EC_0(f)$ is finite and the image of the translation length homomorphism is a copy of $\mathbb{Z} \subset \mathbb{R}$. If $H$ is a proper subgroup of $EC(f)$ then we can extend the homomorphism from $H$ onto $\mathbb{Z}$ to a homomorphism from $EC(f)$ onto $\mathbb{D}_\infty$, the isometry group of $\mathbb{Z}$. To define this extension pick any $g \in EC(f) \setminus H$ and send it to an isometry of $\mathbb{Z}$ that switches the ends and this will uniquely determine the homomorphism. The kernel of the extended homomorphism will still be $EC_0(f)$. 

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We now prove the second statement. Let $h \not\in EC(f)$, so $h(O_x)$ is a quasi-line not in a Hausdorff neighborhood of $O_x$. By the Morse Lemma $\pi(h(O_x))$ is either bounded or it is a quasi-ray, where $\pi : X \to O_x$ is the nearest point projection. Suppose first it is bounded and choose some $\phi \in EC(f)$ with positive translation length. Then $d(x, x_n) \sim n \tau_{\phi}$ for any $x \in \pi(h(O_x))$ and $x_n \in \pi(\phi^n h(O_x))$. Therefore for large $n$, $\phi^n$ does not coarsely fix $h(O_x)$. This implies that $\phi^n$, and therefore $\phi$, is not in $EC(hf^{-1})$. So, each element $\phi \in EC(f) \cap EC(hfh^{-1})$ has zero translation length, therefore by WPD the intersection is finite.

The case when the projection is a quasi-ray, we use WPD and show it does not happen. If $h$ preserves an end of $O_x$, for any $n > 0$ the product $h\phi^n h^{-1} \phi^{-n}$ coarsely fixes a subray, so this list must contain only finitely many elements. It follows that $h$ commutes with a nontrivial power of $\phi$, so must coarsely preserve $O_x$. If $h$ does not preserve an end of $O_x$, then for $n > 0$ we have that $h\phi^n h^{-1} \phi^n$ coarsely fixes a subray and it similarly follows that $h$ conjugates a power of $\phi$ to its inverse. Again $h$ must coarsely preserve $O_x$, namely, $h \in EC(f)$, impossible.

**Lemma 4.8.** Every finite normal subgroup of $G$ is contained in $EC_0(f)$ and therefore $G$ contains a maximal finite normal subgroup.

**Proof.** Let $N$ be a normal subgroup of $G$. Then the $N$-orbit of two points in the same $G$-orbit will be isometric. In particular, if $N$ is finite then the diameter of the $N$-orbit of every point in $O_x$ will be uniformly bounded and hence $N \subset EC_0(f)$.

It is much more convenient to work with axial isometries. In the next lemma we improve $(X, f)$ with $f : X \to X$ hyperbolic so that $f$ becomes axial.

**Lemma 4.9.** Let $G$ act on a geodesic metric space $X$ and $f \in G$ a WPD element with strongly contracting orbits. Then $X$ is quasi-isometric to a geodesic metric space $Y$ and $G$ acts on $Y$ with $f$ an axial WPD element and with a strongly contracting axis $\alpha$. Moreover, we can arrange that $EC(f)$ preserves $\alpha$.

**Proof.** By Lemma 4.7 there is a short exact sequence

$$1 \to EC_0(f) \to EC(f) \to Z \to 1$$

where $Z \cong \mathbb{Z}$ or $\mathbb{D}_\infty$ and $EC_0(f)$ is finite. We may assume that $f$ maps to a generator of a maximal copy of $Z \subseteq Z$ after replacing it by a root.
Fix a point $p \in X$ and for each $g \in G$ add an edge $E_g$ of length $L$ to the point $g(p)$ and label the terminal endpoint $E_g$ by $\bar{g}$. Note that if $g(p) = h(p)$ but $g \neq h$ then $E_g \neq E_h$. We then identify all points that are in the same left coset of $EC_0(f)$ and attach an edge of length $\tau(f)/2$ between the cosets $hEC_0(f)$ and $hfEC_0(f)$. Note that if $g \in EC_0(f)$ then $hEC_0(f)$ and $hgEC_0(f)$ are the same coset but then so are $hfEC_0(f)$ and $hgEC_0(f)$ since $EC_0(f)$ is normal in $EC(f)$. In particular the edge does not depend on the choice of coset representative. This gives a new metric space $Y$ with a natural inclusion of $X$ in $Y$ and it is clear that the $G$ action on $X$ extends to a $G$ action on $Y$ since $EC_0(f)$ is normal in $EC(f)$.

To see that $Y$ is quasi-isometric to $X$ it is useful to define a metric space $Y'$ where the underlying space is the same and the only change is that the length of the added edges is $\tau(f)$ instead of $\tau(f)/2$. By choosing $L$ sufficiently large the inclusion of $X$ into $Y'$ will be an isometry. Since $X$ is coarsely dense in $Y'$ the two spaces are quasi-isometric. As $Y$ and $Y'$ are clearly quasi-isometric the inclusion of $X$ in $Y$ will also quasi-isometric via the inclusion.

Define paths $\alpha_h$ to be the union of the edges between $hf^iEC_0(f)$ and $hf^{i+1}EC_0(f)$ with $i \in \mathbb{Z}$. We claim that $\alpha_h$ is an axis for $hf^{-1}$. Clearly it is $hf^{-1}$-invariant and in fact by the identification, $EC(hf^{-1})$-invariant. To show that it is a geodesic in $Y$, let $h_0$ and $h_1$ be elements in $G$ such that $h_0$ and $h_1$ are vertices on $\alpha_h$ that are distance $n\tau(f)$ apart on $\alpha_h$. Then any path from $h_0$ to $h_1$ whose interior is disjoint from $\alpha_h$ will be of the form $E_{h_0} \cup \gamma \cup E_{h_1}'$ where $h_i$ and $h_i'$ are in the same coset. The cosets all have the same diameter so if we choose $L$ to be greater than this diameter then the length of $\gamma$ will be at least $n\tau(f) - 2L$ so the length of $E_{h_0} \cup \gamma \cup E_{h_1}'$ will be greater than $n\tau(f)$ and hence $\alpha_h$ is a geodesic.

To see that $\alpha = \alpha_1$ is strongly contracting we first argue that the $f$-orbits $O_x$ in $Y$ are strongly contracting. For this, observe that for every metric ball $V \subset Y$ disjoint from $O_x$ there is a (concentric) metric ball $V' \subset X$ disjoint from $O_x$ so that every point of $V$ is within a bounded distance of a point of $V'$. Then the proof that $O_x$ is strongly contracting follows as in the proof of Proposition 4.51 (where we argued that an orbit $O_y$ is strongly contracting).

Applying Proposition 4.51 we see that an orbit along $\alpha$ is strongly contracting, and then it is immediate that $\alpha$ is strongly contracting. (For the last claim, fix $x \in \alpha$. It suffices to show that the projection to $O_x$ and to $\alpha$ of any point $p$ is uniformly bounded. Indeed, set $y = \pi_x(p) \in \alpha$. Clearly, $y = \pi_y(p)$. By Proposition 4.51 $d(\pi_y(p), \pi_x(p))$ is uniformly bounded.) It is clear that $f$ is WPD on $Y$. 

4.3 \textit{L}-disjoint geodesics

\textbf{Lemma 4.10.} Let $\alpha$ and $\beta$ be $B$-strongly contracting geodesics. Assume that for some $L \geq 5B$ if $x, y \in \alpha \cap N_{d_0}(\beta)$ then $d(x, y) \leq L$, where $d_0$ is the constant from Lemma 4.6 for $(1, 10B)$ and $B$. Then $\text{diam} \, \pi_\alpha(\beta) \leq L$.

\textbf{Proof.} Let $p, q \in \beta$ with $x \in \pi_\alpha(p)$ and $y \in \pi_\alpha(q)$. If $d(x, y) \geq 5B$ then $[p, x] \cup [x, y] \cup [y, q]$ is a $(1, 10B)$-quasigeodesic and therefore is contained in $N_{d_0}(\beta)$ by Lemma 4.6. It follows that $d(x, y) \leq L$. \hfill \Box

Motivated by this lemma we say that two $B$-strongly contracting geodesics are \textit{L-disjoint} if whenever $x, y \in N_{d_0}(\alpha) \cap \beta$ then $d(x, y) \leq L$ with the same implication holding if $\alpha$ and $\beta$ are reversed. Note that we will always assume that $L \geq 5B$.

\textbf{Lemma 4.11.} Let $\alpha$ and $\beta$ be $B$-strongly contracting geodesics. If $x, y \in \alpha \cap N_D(\beta)$ for some $D > 0$ and $z \in \alpha$ is between $x$ and $y$ with $\min\{d(x, z), d(y, z)\} \geq 2D$ then $z \in N_{d_0}(\beta)$ where $d_0$ is the constant from Lemma 4.6 for $(1, 10B)$ and $B$. In particular if $\alpha \subset N_D(\beta)$ then $\alpha \subset N_{d_0}(\beta)$.

\textbf{Proof.} Let $p \in \pi_\beta(x)$ and $q \in \pi_\beta(y)$ and let $x' \in \pi_\alpha(p)$ and $y' \in \pi_\alpha(q)$. Then $d(x, x') \leq 2D$ and $d(y, y') \leq 2D$ so $z \in [x', y']$. As in the proof of Lemma 4.10 $[p, x'] \cup [x', y'] \cup [q, y']$ is contained in $N_{d_0}(\beta)$ so $z \in N_{d_0}(\beta)$. \hfill \Box

Let $\alpha$, $\beta$ and $\gamma$ be subsets of $X$ and define

$$d_\alpha^\pi(\beta, \gamma) = \text{diam}(\pi_\alpha(\beta), \pi_\alpha(\gamma))$$

where $\pi_\alpha$ is the nearest point projection to $\alpha$. Note that $d_\alpha^\pi$ is symmetric and satisfies the triangle inequality. When we use this notation $\alpha$ will be a strongly contracting geodesic and $\beta$ and $\gamma$ may be points, geodesic segments or complete, infinite geodesics.

The following two statements correspond to the Behrstock inequality (see axiom 3 after Proposition 4.17).

\textbf{Lemma 4.12.} Let $\alpha$ and $\beta$ be $B$-strongly contracting and $L$-disjoint geodesics with $L \geq 5B$. Then

$$\min\{d_\alpha^\pi(\beta, z), d_\beta^\pi(\alpha, z)\} \leq 2L + 10B$$

for all $z \in X$. 

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Proof. Fix \( p \in \pi_\alpha(z) \) and \( x \in \pi_\beta(z) \). Then fix \( q \in \pi_\alpha(x) \) and \( y \in \pi_\beta(p) \). Assume that the above minimum is \( M > 2L + 10B \). Then both \( d(p, q) \) and \( d(x, y) \) are \( \geq M - 2L > 10B \). By Proposition \([4,2]\) we have

\[
    d(z, x) \geq d(z, p) + M - 2L + d(q, x) - 10B
\]

and

\[
    d(z, p) \geq d(z, x) + M - 2L + d(y, p) - 10B
\]

which implies that \( M \leq 2L + 10B \). \( \blacksquare \)

Corollary 4.13. Let \( \alpha, \beta \) and \( \gamma \) be \( B \)-strongly contracting and mutually \( L \)-disjoint geodesics with \( L \geq 5B \). Then

\[
    \min \{ d^\pi_\alpha(\beta, \gamma), d^\pi_\beta(\alpha, \gamma) \} \leq 3L + 10B.
\]

Proof. Let \( z \) be any point in \( \gamma \). If \( d^\pi_\alpha(\beta, z) \leq M \) for \( z \in \gamma \) then \( d^\pi_\alpha(\beta, \gamma) \leq M + L \), with a similar bound holding for \( d^\pi_\beta(\alpha, \gamma) \). The corollary then follows from Lemma \([4,12]\). \( \blacksquare \)

Here is a useful way to think about WPD: the set of \( G \)-translates of the axis is “discrete”.

Lemma 4.14. Let \( G \) act on a geodesic metric space \( X \) and \( f \in G \) a WPD element with respect to this action that has a \( B \)-strongly contracting axis \( \alpha \). There exists \( L > 0 \) such that for all \( g \notin EC(f) \) the axes \( \alpha \) and \( g(\alpha) \) are \( L \)-disjoint.

Proof. Let \( D \) be the translation length of \( f \). Orient \( \alpha \) such that if \( x \in \alpha \) then \( x \) occurs before \( f(x) \). Fix \( z \in \alpha \). Let \( g \in G \setminus EC(f) \) such that \( \alpha \) and \( g(\alpha) \) are not \( L \)-disjoint. Then there exists \( \bar{x}, \bar{y} \in g(\alpha) \cap N_{d_0}(\alpha) \) with \( d(\bar{x}, \bar{y}) > L \) and \( \bar{x} \) occurring before \( \bar{y} \) in the orientation on \( g(\alpha) \) induced by \( \alpha \). Let \( x, y \in \alpha \) with \( d(x, y) = d(\bar{x}, \bar{y}) \), \( x \) occurring before \( y \) and \( z \) the midpoint between them. Let \( x' \in \pi_\alpha(\bar{x}) \) and \( y' \in \pi_\alpha(\bar{y}) \). \( x' \) may occur before or after \( y' \). Since \( d(x', \bar{x}), d(y', \bar{y}) \leq d_0 \) and \( d(x, y) = d(\bar{x}, \bar{y}) \) we have that \( |d(x', y') - d(x, y)| \leq 2d_0 \). For an appropriate choice of \( n, m \in \mathbb{Z} \) we have either:

1. \( d(gf^n x, \bar{x}), d(gf^n y, \bar{y}) \leq D/2 \) and \( d(x, f^m x'), d(y, f^m y') \leq D/2 + d_0 \) or
2. \( d(gf^n x, \bar{y}), d(gf^n y, \bar{x}) \leq D/2 \) and \( d(x, f^m y'), d(y, f^m x') \leq D/2 + d_0 \).
Replace $g$ with $f^m g f^n$, $\bar{x}$ with $f^m \bar{x}$ and $\bar{y} f^m \bar{y}$. Then (1) becomes

$$d(gx, \bar{x}), d(gy, \bar{y}) \leq D/2$$

and the triangle inequality then implies that $d(x, gx), d(y, gy) \leq D + 2d_0$. If instead (2) holds we get that $d(x, gy), d(y, gx) \leq D + 2d_0$ so $d(x, g^2x), d(y, g^2y) \leq D + 2d_0$. Now replace, $\bar{x}$ and $\bar{y}$ with points on $[g^2x, g^2y] \subset g^2(\alpha)$ that are more than $4D + d_0$ from the endpoints. By Lemma [4.11] we have $\bar{x}, \bar{y} \in \mathcal{N}_{d_0}(\alpha)$. Next replace $g$ with $g^2$. After suitably replacing $x, y$ and $x', y'$ we are now back in the situation of (1) except that $d(\bar{x}, \bar{y}) > L - 8(D + d_0)$.

From this discussion we see that if there is no $L > 0$ such that $\alpha$ and $g(\alpha)$ are $L$-disjoint for all $g \in G \setminus EC(f)$ then we can find a sequence $g_i \in G \setminus EC(f)$ and $x_i, y_i \in \alpha$ with $d(x_i, y_i) \to \infty$, $x_i$ occurring before $y_i$ and $z$ the midpoint between them such that $d(x_i, g_i x_i), d(y_i, g_i y_i) \leq 2D + d_0$.

Let $\alpha(t)$ be a unit speed parameterization of $\alpha$ that respects the orientation with $\alpha(0) = z$. By Lemma [4.11] if $|t| \leq d(x_i, y_i)/2 - (4D + 2d_0)$ then $g_i(\alpha(t)) \in \mathcal{N}_{d_0}(\alpha)$. Furthermore, for $t'$ with $\alpha(t') \in \pi(\alpha(\alpha(t)))$ we have $|t - t'| \leq 2D + 2d_0$ so, as above, via the triangle inequality, we have $d(\alpha(t), g_i(\alpha(t))) \leq 2D + 3d_0$. Let $M$ be the WPD constant for $f$, $z$ and $2D + 3d_0$. Then for $i$ sufficiently large we have $d(z, g_i z) \leq 2D + 3d_0$ and $d(f^M(z), g_i f^M(z)) \leq 2D + 3d_0$. Since $d(x_i, y_i) \to \infty$, infinitely many of the $g_i$ must be distinct, contradicting WPD.

To construct quasi-trees we need to know that for any two $G$-translates of the axis $\alpha$ there are only finitely many other translates where the first two translates have a large projection. This is the content of Proposition [4.17]. Before proving it we need two lemmas. In a hyperbolic space the verification is easy using the fact that the shortest geodesic between two axes is strongly contracting, but in general we have to work harder.

**Lemma 4.15.** Let $X$ be a geodesic metric space and $\alpha$, $\beta$ and $\gamma_i$, $i = 1, 2, 3$ be $B$-strongly contracting and mutually $L$-disjoint geodesics with $L > 5B$. Let $\xi = 3L + 10B$. For any $K > 2\xi$ if

$$d_{\gamma_i}(\alpha, \beta) \geq K$$

for $i = 1, 2, 3$ then

$$d_{\gamma_i}(\gamma_j, \gamma_k) \geq K - 2\xi$$

for some permutation of the $i, j, k$.

**Proof.** For convenience we set $\alpha^i = d_{\gamma_i}(\alpha, \gamma_j)$ and $\beta^i = d_{\gamma_i}(\gamma_j, \beta)$. We have the following three sets of inequalities:
1. \( \min\{\alpha_i^j, \alpha_j^i\} \leq \xi \) and \( \min\{\beta_i^j, \beta_j^i\} \leq \xi \);

2. \( \alpha_i^j + \beta_j^i \geq d_{\gamma_i}^\pi (\alpha, \beta) \geq K > 2\xi \);

3. \( \alpha_i^j + d_{\gamma_j}^\pi (\gamma_j, \gamma_k) + \beta_k^i \geq d_{\gamma_i}^\pi (\alpha, \beta) \geq K \).

The first inequality comes from Corollary 4.13 while the second and third are just the triangle inequality. Each inequality has six permutations (or for \([1]\) there are two sets of three) with each \(\alpha_i^j\) and \(\beta_j^i\) appearing in exactly one. By \([1]\), at least half of the twelve \(\alpha_i^j\) and \(\beta_j^i\) will be at most \(\xi\) while by \([2]\) at least half will be greater than \(\xi\). This implies that exactly half will be greater than \(\xi\) with one such term in each of the inequalities of type \([1]\) and \([2]\). Similarly if in each permutation of \([3]\) there is not exactly one of the \(\alpha_i^j\) or \(\beta_k^i\) that is greater than \(\xi\) there must be one permutation of \([3]\) where \(\alpha_i^j \leq \xi\) and \(\beta_k^i \leq \xi\) and therefore \(d_{\gamma_j}^\pi (\gamma_j, \gamma_k) \geq K - 2\xi\) which is what we want to prove. Therefore to finish the proof we assume that exactly one of the \(\alpha_i^j\) or \(\beta_k^i\) terms in each permutation is greater than \(\xi\) and obtain a contradiction.

Let \(G\) be the bipartite graph with vertices the \(\alpha_i^j\) and \(\beta_j^i\) and an edge between any two vertices if one term is greater than \(\xi\) and the other is at most \(\xi\). By the above paragraph each of the inequalities \([1]\), \([2]\) and \([3]\) will determine an edge in \(G\) and if we follow an edge of each type we have a path \(\alpha_i^j \rightarrow \alpha_j^i \rightarrow \beta_j^i \rightarrow \alpha_k^i\) of length three from \(\alpha_i^j\) to \(\alpha_k^i\) and therefore there is a cycle of length 9 from \(\alpha_i^j\) to itself. As a bipartite graph cannot have a cycle of odd length we have obtained our desired contradiction. \(\Box\)

**Lemma 4.16.** Let \(\alpha\) and \(\beta\) be \(B\)-strongly contracting and \(L\)-disjoint geodesics with \(L > 5B\). Let \(\phi : [a, b] \rightarrow X\) be a unit-speed geodesic and \(\xi = 2L + 10B\). If

\[
\min\{d_{\alpha}^\pi (\phi(a), \phi(b)), d_{\beta}^\pi (\phi(a), \phi(b))\} > 2\xi + L
\]

then the set

\[
E(\alpha, \beta; \phi) = \{t \in [a, b] | \max\{d_{\alpha}^\pi (\phi(t), \beta), d_{\beta}^\pi (\phi(t), \alpha)\} \leq \xi + 10B\}
\]

is non-empty.

**Proof.** Let \(\pi_\alpha\) be the nearest point projection to \(\alpha\). Given \(p, q \in X\) let \(x \in \pi_\alpha(p)\) and \(y \in \pi_\alpha(q)\). By Proposition 4.2

\[
d(x, y) \leq \max\{d(p, q) + 10B, 5B\} \leq d(p, q) + 10B.
\]
Let \( E_\alpha = \{ t \in [a, b] | d_\alpha^n(\phi(t), \beta) \leq \xi \} \). If \( t \in E_\alpha \) there exists \( t_i \in E_\alpha \) such that \( t_i \to t \). Therefore

\[
d_\alpha^n(\phi(t), \beta) \leq d_\alpha^n(\phi(t), \phi(t_i)) + d_\alpha^n(\phi(t_i), \beta) \leq |t - t_i| + 10B + \xi
\]

for all \( i \) so for all \( t \in E_\alpha \), \( d_\alpha^n(\phi(t), \beta) \leq \xi + 10B \). We similarly define \( E_\beta \) and conclude that for all \( t \in E_\beta \), \( d_\beta^n(\phi(t), \alpha) \leq \xi + 10B \). Also note that \( [a, b] = E_\alpha \cup E_\beta \) by Lemma 4.12.

Since \( d_\beta^n(\phi(a), \phi(b)) > 2\xi + L \) at least one of \( d_\beta^n(\phi(a), \alpha) \) and \( d_\beta^n(\phi(b), \alpha) \) must be greater than \( \xi \) and therefore by Lemma 4.12 either \( a \) or \( b \) is in \( E_\alpha \). In particular \( E_\alpha \neq \emptyset \) and similarly \( E_\beta \neq \emptyset \). A connected interval cannot be a disjoint union of non-empty closed intervals, so \( E_\alpha \cap E_\beta \neq \emptyset \) and \( E \supset E_\alpha \cap E_\beta \) and therefore \( E \neq \emptyset \). \( \square \)

**Proposition 4.17.** Let \( Y \) be a collection of \( L \)-disjoint, \( B \)-strongly contracting geodesics in a geodesic metric space \( X \). Then there exists \( K = K(B, L) \) such that for all \( \alpha, \beta \in Y \) we have

\[
\#\{ \gamma \in Y | d_\gamma^n(\alpha, \beta) > K \} < \infty.
\]

**Proof.** Let \( \xi = 3L + 10B \) and assume that \( K > 5\xi + 20B + 2L \). Let \( \phi : [a, b] \to X \) be a unit-speed, shortest geodesic connecting \( \alpha \) to \( \beta \) and let \( \gamma_i, i = 1, \ldots, n \), be a finite set of elements in \( Y \) with \( d_\gamma^n(\alpha, \beta) > K \). Then, \( d_\gamma^n(\phi(a), \phi(b)) > K - 2L \geq 5\xi + 20B \).

For each \( \gamma_i, \gamma_j \) let \( E_{ij} \) be the non-empty sets \( E(\gamma_i, \gamma_j; \phi) \) given by Lemma 4.10. For distinct \( i, j, k \) by Lemma 4.15 there is a permutation such that

\[
d_\gamma^n(\gamma_j, \gamma_k) \geq K - 2\xi > 3\xi + 20B + 2L.
\]

For \( t_j \in E_{ij} \) and \( t_k \in E_{ik} \) we then have

\[
3\xi + 20B < d_\gamma^n(\gamma_j, \gamma_k) \leq d_\gamma^n(\gamma_j, t_j) + d_\gamma^n(t_j, t_k) + d_\gamma^n(t_k, \gamma_k)
\]

and since \( d_\gamma^n(\gamma_j, t_j) \) and \( d_\gamma^n(t_k, \gamma_k) \) are both at most \( \xi + 10B \) this implies that \( d_\gamma^n(t_j, t_k) > \xi \).

In particular for any distinct \( i, j, k \) and \( p \in E_{ij}, q \in E_{ik} \) and \( r \in E_{jk} \), \( \text{diam}\{p, q, r\} \geq \xi \). The pigeonhole principle then implies that \( n \leq 2[(b - a)/\xi] + 1 \) and the proposition follows. \( \square \)
4.4 Constructing quasi-trees

We now apply this to a construction from [2]. We begin with some axioms. Let \( Y \) be a set and for each \( Y \in Y \) let \( d^\pi_Y : (Y\setminus Y)^2 \to \mathbb{R}^+ \) be a function that satisfies the following axioms:

1. \( d^\pi_Y(X, Z) = d^\pi_Y(Z, X) \);
2. \( d^\pi_Y(X, Z) + d^\pi_Y(Z, W) \geq d^\pi_Y(X, W) \);
3. There exists a \( \xi > 0 \) such that \( \min\{d^\pi_Y(X, Z), d^\pi_Y(Z, X)\} < \xi \);
4. There exists a \( K > 0 \) such that \( \#\{Y|d^\pi_Y(X, Z) \geq K\} \) is finite for all \( X, Z \in Y \).

Following [2] given a set \( Y \) and functions \( d^\pi_Y \) satisfying the axioms we define modified distance functions \( d_Y \). Define \( H(X, Z) \) to be the set of pairs \( (X', Z') \in Y \times Y \) such that one of the following holds:

- \( d^\pi_X(X', Z'), d^\pi_Z(X', Z') > 2\xi \);
- \( X = X' \) and \( d^\pi_Z(X, Z') > 2\xi \);
- \( Z = Z' \) and \( d^\pi_X(X', Z) > 2\xi \);
- \( (X', Z') = (X, Z) \).

We then define
\[
d_Y(X, Z) = \inf_{(X', Z') \in H(X, Z)} d^\pi_Y(X', Z').
\]

If \( X' = Y \) or \( Z' = Y \), then \( d^\pi_Y(X', Z') \) is defined to be 0.

We now define the projection complex \( P_K(Y) \). Let \( K > 0 \) be chosen such that Axiom (1)-(4) holds. The vertices of \( P_K(Y) \) is the set \( Y \). We connect two vertices \( X \) and \( Y \) with an edge if \( d_Z(X, Y) < K \) for all \( Z \in Y \setminus \{X, Y\} \).

We then have the following theorem:

**Theorem 4.18** ([2]). For \( K \) sufficiently large \( P_K(Y) \) is a quasi-tree.

To avoid specifying unimportant constants, we write \( A \sim B \) if the two constants differ at most a constant which depends only on the projection complex (\( \xi \) and \( K \)). We write \( A \succeq B \) if \( A - B > 0 \) or \( A \sim B \).

One could define the projection complex using the original functions \( d^\pi_Y \) and perhaps one would expect that this would give a quasi-isometric
complex. However, this is not the case and it is an open question if a projection complex defined directly from $d^\pi_Y$ is a quasi-tree.

We want to apply this theorem to our setting. Let $G$ act on a geodesic metric space $X$ with a WPD element $f$ that has a strongly contracting axis $\alpha$ and assume that $f$ is axial. Moreover, we may assume that $EC(f)$ preserves $\alpha$ by Lemma 4.9. Let $Y(G, X, f)$ be the set of $G$-translates of $\alpha$. We identify parallel translates. $G$ acts on $Y(G, X, f)$ and the stabilizer of the point $\alpha$ is $EC(f)$. Recapping our work above we have the following.

**Proposition 4.19.** The set $Y(G, X, f)$ and the functions $d^\pi_\alpha$ satisfy axioms (1)-(4).

We now add a group action to our axiomatic setup. If $G$ acts on $Y$ such that the functions $d^\pi_Y$ are $G$-invariant then $G$ will act isometrically on $P_K(Y)$ producing an action of $G$ on a quasi-tree. Conditions that guarantee that an element of $G$ has a WPD action on $P_K(Y)$ are given in Proposition 2.18 of [2], and a condition for and element to have an axis is given by Lemma 2.14 of [2]. We use two constants $K'' \geq K'$ from [2].

**Proposition 4.20.** There exists a $K' > 0$ such that the following holds. Let $g \in G$ such that:

(i) There exists a $Y \in Y$ and an $N > 0$ such that $d_Y(g^{-N}(Y), g^N(Y)) > K'$.

(ii) There exists an $m > 0$ such that the stabilizer of the vertices $Y, g^1(Y), \ldots, g^m(Y)$ is finite.

Then the action of $g$ on $P_K(Y)$ is (hyperbolic and) WPD.

Also, there exists a constant $K'' \geq K'$ such that if $d_Y(g^{-N}(Y), g^N(Y)) \geq K''$ then $g$ has an axis that contains all $g$-translates of $Y$ (for this we do not need (ii)).

We can also use $P_K(Y)$ to produce free groups in $G$. The following is Proposition 2.15 in [2] (set $L = K''$). In (iii), that $h$ has an axis in $S$ follows from the last part of Proposition 4.20.

**Proposition 4.21.** Let $Y, Z \in Y$ and $g_1, g_2 \in G$ such that

- $g_1(Z) = g_2(Z) = Y$;
- if $Z_0 = Z, Z_1 = g_1(Y)$ and $Z_2 = g_2(Y)$ then $d_Y(Z_i, Z_j) > K''$ if $i \neq j$.

Then
(i) \( F = \langle g_1^2, g_2^2 \rangle \) is a non-abelian free group.

(ii) There is an \( F \)-invariant trivalent tree \( S \subset \mathcal{P}_K(Y) \) where the \( F \)-action is proper and minimal.

(iii) For all non-trivial \( h \in F \) there exists a vertex \( W \in S \) such that
\[
d_W(h^{-1}(W), h(W)) > K''.
\]
h has an axis that is contained in \( S \).

Remark 4.22. By Proposition 4.7, the stabilizer of two distinct vertices in \( Y = Y(G, X, f) \) must be finite. Therefore for the \( G \)-action on \( \mathcal{P}_K(Y) \), Proposition 4.20 implies that the free group given by Proposition 4.21 is WPD, and each non-trivial element is axial and an axis is contained in \( S \). We will further see in the proof of Theorem 4.25 that the two conditions of Proposition 4.21 are also easily satisfied for the action on \( Y \).

By Proposition 4.7, \( EC(f) \) is virtually \( \mathbb{Z} \) and therefore two-ended. Some elements of \( EC(f) \) fix the ends while others will permute them. We say that \( EC(f) \) is end preserving if every element of \( EC(f) \) preserves the ends. We need the following lemma from [5] to find an \( f \) such that \( EC(f) \) is end preserving. Note that while in [5] it is assumed that every hyperbolic element is WPD it is only used that there is a free group of WPD elements.

Lemma 4.23. Let \( G \) act on a hyperbolic space \( X \) and \( F \subset G \) a non-abelian free subgroup of WPD elements. Then there exists a \( f \in F \) such that every \( g \in EC(f) \) preserves the ends of the quasi-axis of \( f \).

The following lemma gives a condition for the action of an element on \( \mathcal{P}_K(Y) \) to have an elementary closure that is end preserving.

Lemma 4.24. Let \( G \) act on a hyperbolic space \( X \) with \( f \) an axial WPD element such that \( EC^X(f) \) is end preserving. Given \( Y, Z \in Y = Y(G, X, f) \), there is a constant \( J \) such that if \( g(Z) = Y \) and \( d_Y^X(Z, g^2(Z)) \geq J \) then \( EC^{X_0}(g) \) is end preserving, where \( X_0 = \mathcal{P}_K(Y) \).

Proof. We first recall a definition from [2]. If \( h \) acts on \( X_0 \) with an orientated axis \( \alpha \) define the combinatorial axis by
\[
Y(h) = \{ W \in Y | W \in \beta \text{ for all geodesics } \beta \text{ parallel to } \alpha \}.
\]
Clearly \( Y(h) \) does not depend on the choice of the axis \( \alpha \), is contained in \( \alpha \) and possibly empty. We also record some facts about \( Y(h) \) that we will use [2 Lemma 2.14 and Proposition 2.18] :

(a) If \( Y(h) \) is nonempty, \( Y(h) \) has an order that is isomorphic to \( \mathbb{Z} \) coming from the order of the vertices on \( \alpha \).
(b) $\mathbf{Y}(h)$ is $EC^{X_0}(h)$ invariant and the induced action on $Z$ is an isometry.

(c) If $d_W(h^{-1}(W), h(W)) \geq K'$ then $h^n(W) \in \mathbf{Y}(h)$ for all $n \in \mathbb{Z}$. (In particular, $\mathbf{Y}(h)$ is nonempty.)

(d) If $W_0, W_1, W_2, W'_0, W'_2 \in \mathbf{Y}(h)$ and $W_1$ is between the pair $W_0$ and $W_2$ and the pair $W'_0$ and $W'_2$ then $d_{W_1}(W_0, W_2) \sim d_{Y_1}(W'_0, W'_2)$.

We can now prove the lemma. By the Axiom the number of large projections between $Y$ and $Z$ is finite so there exists a $K_0$ such that $d_{W'}(Y, Z) < K_0$ for all $W \in \mathbf{Y}\setminus \{Y, Z\}$. Choose $J >> K_0 + K'$, such that $K_0 + K' \geq J$ does not occur. Now assume that

$$d_Y(g^{-1}(Y), g(Y)) = d_Y(Z, g^2(Z)) \geq J.$$  

Then by (c) every $g$-translate of $Y$ is in $\mathbf{Y}(g)$. On the other hand, if $W \in \mathbf{Y}(g)$ is not a $g$-translate of $Y$ then by (a), $W$ lies between a $g$-translate of $Y$ and $Z$, say, $g^n(Y)$ and $g^n(Z)$. By (d)

$$d_W(g^{-1}(W), g(W)) \sim d_W(g^n(Y), g^n(Z)) = d_{g^{-n}(W)}(Y, Z) < K_0$$

and therefore $d_W(g^{-1}(W), g(W)) < J$.

Let $\phi \in EC^{X_0}(g)$ and assume that $\phi$ reverses the ends. By (b), $\phi$ and $g$ are isometries of the $Z$-order on $\mathbf{Y}(g)$ with $\phi$ reversing and $g$ preserving the orientation. Therefore $\phi(g(W)) = g^{-1}(\phi(W))$ for all $W \in \mathbf{Y}(g)$ and

$$d_{\phi(Y)}(g^{-1}(\phi(Y)), g(\phi(Y)) = d_{\phi(Y)}(\phi(g(Y)), \phi(g^{-1}(Y))) = d_Y(g(Y), g^{-1}(Y)) \geq J.$$  

Hence, by the above paragraph we must have $\phi(Y) = g^n(Y)$ for some $n \in \mathbb{Z}$. Let $\psi = g^{-n}\phi$. Then $\psi$ will also reverse the ends of $EC^{X_0}(g)$ and $\psi(Y) = Y$. We can assume that $Y$ corresponds to the axis of $f$ in $X$ and therefore $\psi \in EC^X(f)$. Let $\alpha_0$ be the axis for $f$ and $\alpha_{\pm 1}$ be the axes for $g^{\pm 1} fg^{\mp 1}$. The axes $\alpha_{\pm 1}$ correspond to the vertices $g^{\pm 1}(Y)$. Let $\pi$ be nearest point projection in $X$ to $\alpha_0$. Since $\psi$ fixes $Y$ and reverses ends in $EC^{X_0}(g)$ we have that $\psi(g^{\pm 1}(Y)) = g^{\mp 1}(Y)$ and therefore $\pi(\psi(\alpha_{\pm 1})) = \pi(\alpha_{\pm 1})$ which is only possible if $\psi$ reverses the ends of $\alpha_0$, and hence of $EC^X(f)$, a contradiction. 

We now prove the main theorem of this section.
Theorem 4.25. Let $G$ act on a geodesic metric space $X$ with $f \in G$ hyperbolic WPD element with a strongly contracting orbit. Then $G$ acts on a quasi-tree $Q$ and the largest finite normal subgroup $N \subset G$ acts trivially. Furthermore, if $G$ is not virtually cyclic, given any $\epsilon, D > 0$ there exists elements $a, b \in G$, an isometrically embedded tree $S \subset Q$ and a point $x_0$ such that

(i) $F = \langle a, b \rangle$ is a non-abelian free group of WPD elements;

(ii) $S$ is $F$-invariant and the $F$-action is proper and minimal;

(iii) $S$ is trivalent and all edges have length $> D$;

(iv) the basepoint $x_0$ is on the axis for $aba$ in $S$ and if $W = [x_0, aba x_0]$ and $\chi \in G$ with $\chi(W) \subset N_\epsilon(S)$ then $\chi = \psi h$ with $\psi \in F$ and $h \in N$.

Proof. If $G$ is virtually cyclic, then take the trivial action of $G$ on one point.

So, we assume that $G$ is not virtually cyclic. We will successively replace $X$ with projection complexes to obtain the desired action. By Lemma 4.9 we can assume that $f$ is axial and then let $X_0 = P_K(Y)$ where $Y = Y(G, X, f)$.

Each vertex in $Y$ is a translate of the axis of $f$ (we identify parallel translates) and its stabilizer is the elementary closure of the corresponding conjugate of $f$. Since $G$ is not virtually cyclic, $EC(X'_0)$ is a proper subgroup and $Y$ contains more than one point. By Lemma 4.8, $N$ is contained in every elementary closure so $N$ fixes every vertex and hence all of $P_K(Y)$.

Assume that $Y$ corresponds to the axis of $f$ in $X$ and let $Z \in Y$ with $Z \neq Y$. Since $G$ acts transitively on $Y$ there exists $\phi \in G$ such that $\phi(Z) = Y$. Since $d_Y(Z, f^k(\phi(Y))) \sim |k|_{T_f}$ by choosing $k >> l >> 0$ and letting $g_1 = f^k \phi$ and $g_2 = f^l \phi$ we have a $Y, Z$ and $g_1, g_2$ that satisfies the conditions of Proposition 4.21 and therefore $g_1^2$ and $g_2^2$ generate a free group and, as noted in Remark 4.22, this group is WPD. By Lemma 4.23 this free group contains an $f_0$ such that $EC^{X'_0}(f_0)$ is end preserving.

We now claim that we can find a sequence $X_i$ and $f_i \in G$ such that

- $G$ acts on $X_i$ with $f_i$ WPD and axis $\alpha_i$;
- $N$ acts trivially on $X_i$;
- $EC^{X'_i}(f_i)$ preserves its ends;
- either $EC^{X'_i}(f_i)$ is normal in $G$ or $|EC^{X'_i}(f_i)| > |EC^{X'_{i+1}}(f_{i+1})|$.
If $EC_{X_i}^{X_i}(f_i)$ is normal and hence equal to $N$ (by Lemma 4.8) we let $X_{i+1} = X_i$ and $f_{i+1} = f_i$ and we are done. If not there is a $\phi_i \in G$ such that $EC_{X_i}^{X_i}(f_i) \neq \phi_i EC_{X_i}^{X_i}(f_i) \phi_i^{-1}$. Let $Y_{i+1}$ and $Z_{i+1}$ be the vertices in $Y_i = Y(G, X_i, f_i)$ corresponding to the axes $\alpha_i$ and $\phi_i(\alpha_i)$. As above by letting $f_{i+1} = f_i^k \phi_i$ for $k_i$ large, Lemma 4.24 implies that the action of $f_{i+1}$ on $X_{i+1}$ is WPD and axial and $EC_{X_{i+1}}^{X_{i+1}}(f_{i+1})$ is end preserving, where $X_{i+1} = \mathcal{P}_K(Y_i)$.

If $\psi \in EC_{X_{i+1}}^{X_{i+1}}(f_{i+1})$ then $\psi(Y_{i+1}) = Y_{i+1}$ and $\psi(Z_{i+1}) = Z_{i+1}$ and therefore $\psi \in EC_{X_i}^{X_i}(f_i) \cap EC_{X_i}^{X_i}(\phi_i f_i \phi_i^{-1})$. By Proposition 4.7, $\psi$ has translation length 0 and since $EC_{X_i}^{X_i}(f_i)$ is end preserving we must have $\psi \in EC_{X_i}^{X_i}(f_i) \cap EC_{X_i}^{X_i}(\phi_i f_i \phi_i^{-1})$ and since $EC_{X_i}^{X_i}(f_i) \neq EC_{X_i}^{X_i}(\phi_i f_i \phi_i^{-1})$ we have $|EC_{X_i}^{X_i}(f_i)| > |EC_{X_{i+1}}^{X_{i+1}}(f_{i+1})|.$

Since $|EC_{X_i}^{X_i}(f_i)|$ is finite we must eventually have an $i$ such that $EC_{X_i}^{X_i}(f_i) = N$. Let $Q = X_i$. We now have our quasi-tree that is fixed by $N$ pointwise and we are left to prove (i)-(iv).

**Proof of (i), (ii), (iii).** As in the second paragraph, if we choose $l >> k_{i-1}$ then by Proposition 4.21 $(f_i)^2$ and $(f_{i-1}^l \phi_{i-1})^2$ generate a non-abelian free group $H \subset G$ of axial WPD elements. There is also an $H$-invariant tree $R \subset Q$ where the $H$-action is proper and minimal. We then let $a = f_i^{2n}$ and $b$ an $H$-conjugate of $f_i^{2m}$ where the conjugate is chosen such that the distance between the axes of $a$ and $b$ is $> D$. We also assume that $n >> m >> 0$, with specific constants to be determined later, so that the translation lengths of $a$ and $b$ are $> D$. Then $F = \langle a, b \rangle$ is a non-abelian free group and there will be a minimal $F$-invariant subtree $S \subset R$ that is trivalent with all edge lengths $> D$. Since the action of $H$ on $R$ is proper, the action of $F$ on $S$ will also be proper.

**Proof of (iv).** The elements $a$ and $b$ are conjugates in $G$ of the power of a single element so the axes of the $F$-conjugates of $a$ and $b$ are $L$-disjoint for some $L > 0$. Further, given $\epsilon > 0$ there is an $L' > 0$ such that if $a$ and $\beta$ are geodesics and $\mathcal{N}_\epsilon(\alpha) \cap \beta$ contains a segment of length $> L'$ then this segment contains a subsegment of length $> L$ in $\mathcal{N}_{d_0}(\alpha)$, where $d_0 = 2\delta$ with $\delta$ the hyperbolicity constant for the quasi-tree $Q$.

The axes of the $F$-conjugates of $a$ and $b$ will be a disjoint collection of geodesics in $S$ (and $Q$) which we label the $a$-lines and $b$-lines. The components of the complement of the $a$ and $b$-lines in $S$ will be a collection of connecting segments that have a single $F$-orbit. The edges of the trivalent tree $S$ are subsegments of $a$-lines, subsegments of $b$-lines and the connecting segments. The quotient $S/F$ is a “dumbbell graph”, the disjoint union of two circles connected by an arc.
The axes of $a$ and $b$ in $S$ are connected by a connecting segment. Place a basepoint $x_0$ where this segment meets the axis for $a$ and let $W = [x_0, aba(x_0)]$. Then $W$ can be partitioned into $W = A_1C_1BC_2A_2$ where the $A_i$ are $a$-edges from distinct $a$-lines, the $C_i$ are distinct connecting segments and $B$ is a $b$-edge. By choosing $m$ large we can assume that the $\epsilon$-neighborhood of two connecting segments that are adjacent along a $b$-line are at least $L'$ apart and by choosing $n$ even larger such that $|A_i| > |B|$. These lower bounds are chosen so that any geodesic segment in $N_r(c)(S)$ of length at least $|C_i| + 2\epsilon + 2L'$ will contain a subsegment of length $L'$ that is contained in the $\epsilon$-neighborhood of a single $a$ or $b$-line.

Fix a $\gamma \in G$ such that $\chi(W) \subset N_r(c)(S)$. Then by the above paragraph the geodesic segments $\chi(A_1), \chi(A_2)$ and $\chi(B)$ must each contain a subsegment of length $L'$ that is within the $\epsilon$-neighborhood of a single $a$-line or $b$-line. We label these lines $\gamma_1, \gamma_2$ and $\gamma_3$. Note that the $\gamma_i$ are distinct since the entire $a$-line containing $A_1$ will be mapped to $\gamma_1$ and the $a$-line and $b$-line containing $A_2$ and $B$ will be mapped to $\gamma_2$ and $\gamma_3$, respectively. Since the lines containing the $A_i$ and $B$ are distinct so must be the $\gamma_i$. We also note that if $\gamma_3$ is an $a$-line then the distance between $\chi(A_1)$ and $\chi(A_2)$ is strictly greater than $|A_i| + 2|C_i|$. However, this distance is equal to the distance between $A_1$ and $A_2$ which is $|B| + 2|C_i|$. Since $|B| < |A_i|$ we have a contradiction and therefore $\gamma_3$ is a $b$-line.

If $\gamma_3$ is a $b$-line then $\gamma_1$ and $\gamma_2$ must be $a$-lines. We also note that the distance between $\gamma_1$ and $\gamma_3$ is exactly $|C_1|$ and therefore $\chi(C)$ must be the connecting segment between $\gamma_1$ and $\gamma_3$. Thus there exists a $\psi \in F$ such that $\psi(C) = \chi(C)$. Since $\psi$ takes $a$-lines to $a$-lines we must have that $\psi^{-1}\chi$ fixes the $a$-line that contains $A_1$ which is the axis of $g$. In particular $\psi^{-1}\chi \in EC^Q(a) = EC^Q(f_i)$. In fact since $C$ is also fixed we $\psi^{-1}\chi$ has zero translation length and since $EC^Q(f_i)$ is end preserving we have that $h = \psi^{-1}\chi \in EC_0(f)$ and (iv) follows.

The following is a slight improvement of Theorem 4.11.

**Theorem 4.26.** Let $G$ act on a geodesic metric space $X$ and $f \in G$ is a WPD element with respect to this action with a strongly contracting orbit. Then $G$ acts on a quasi-tree $Q$ so that $f$ is a WPD element.

**Proof.** Since this improvement is not used in the rest of the paper, we only give a sketch. In [2] to the collection $Y = Y(G, X, f)$ we associated the complex $C(Y)$, by taking the disjoint union of lines, one for every coset of $EC(f)$, and if $d(X, Z) = 1$ in $\mathcal{P}_K(Y)$ we attach an edge of large length $L$ from every point in $\pi_X(Z)$ to every point in $\pi_Z(X)$. We then showed that

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\( \mathcal{C}(Y) \) is a quasi-tree. It remains to argue that \( f \) is WPD. The elementary closure \( EC(f) \) preserves the line \( \alpha \) in \( \mathcal{C}(Y) \) coming from the axis of \( f \). If \( h \not\in EC(f) \) then \( h(\alpha) \) has uniformly bounded projection to \( \alpha \) and the statement follows. See [2, Proposition 3.18].

5 Quasi-cocycles from quasi-trees

We will prove the following theorem.

**Theorem 5.1.** Let \( G \) act on a geodesic metric space \( X \) and assume that \( G \) contains a WPD element with a strongly contracting axis. Let \( \rho \) be a unitary representation of \( G \) in uniformly convex Banach space \( E \) and assume that for all finite normal subgroups \( N \) of \( G \) the restriction \( \rho|_N \) fixes a non-trivial subspace of \( E \). If \( G \) is not virtually cyclic then \( QC(G; \rho) \) is infinite dimensional.

**Remark 5.2.** The assumption that \( \rho|_N \) fixes a non-trivial subspace is necessary since if there exists a quasi-cocycle that is unbounded there must exists a non-zero vector that is fixed by \( \rho|_N \), as we now argue. Observe that if a finite group acts on \( E \) by isometries then the center of mass of any orbit will be fixed. If the action has two fixed points then it will fix a line and its rotational part will have a fixed nonzero vector. Now assume that \( G \) acts on \( E \) by isometries and \( N \subset G \) is finite and normal. Then for any two points \( x \) and \( y \) in the same \( G \)-orbit the \( N \) orbits of \( x \) and \( y \) will be isometric. If the translational part (the cocycle) of the \( G \)-action is unbounded then we can find \( x \) and \( y \) in the same \( G \)-orbit that are much further apart than the diameter of their \( N \)-orbits so the center of the mass of the two orbits will be distinct. In particular if the cocycle is unbounded then the rotational part, restricted to \( N \), will have a nonzero fixed vector. If one takes some care with the constants then the same argument shows that if \( G \) has an unbounded quasi-cocycle then for every finite normal subgroup \( N \), \( \rho|_N \) fixes a nonzero vector.

Here is an elementary example. Let \( G = \mathbb{Z}/2 \times F_2 \). \( G \) acts on the Cayley tree of the free group \( F_2 \) properly. Take \( \mathbb{R} \) as \( E \) and define \( \rho(a)(v) = -v \) for the non-trivial element \( a \in \mathbb{Z}/2 \), and \( \rho|_{F_2} \) is trivial. Then one can easily show that any quasi-cocycle is bounded.

5.1 Maps from quasi-trees to trees

To construct quasi-cocycles we will need to map the quasi-tree \( Q \) to a tree via a quasi-isometry. To show that the quasi-cocycles are non-trivial we
need to make this particularly nice. We do this next.

Let \( x, y \) and \( z \) be three points in a tree \( T \). These three points form a (possibly degenerate) tripod and we let \( m(x, y, z) \) be its central point. We note that if \( \phi \) is a quasi-isometry between trees then the distance between \( \phi(m(x, y, z)) \) and \( m(\phi(x), \phi(y), \phi(z)) \) is bounded uniformly by a constant that depends only on the QI-constants for \( \phi \).

The following lemma follows easily from this fact. We will use this lemma in the proof of Proposition 5.8. The proof of this lemma is a straightforward exercise which we leave to the reader.

**Lemma 5.3.** If \( p, q \in Q \) are sufficiently far apart then for any \( x, y, z \in Q \) and any isometry \( h : Q \to Q \), if

\[
[\phi(p), \phi(q)] \subset [\phi(x), m(\phi(x), \phi(y), \phi(z))]
\]

then

\[
[\phi(h(p)), \phi(h(q))] \not\subset [m(\phi(h(x)), \phi(h(y)), \phi(h(z))), \phi(h(y))].
\]

The lower bound for the distance between \( p \) and \( q \) depends only on the quasi-isometry constants for \( \phi : Q \to T \).

We state another lemma.

**Lemma 5.4.** For every \((K', L')\) there are \( K, L \) and \( D \) such that the following holds. Let \( \phi : S \to R \) be a \((K', L')\)-quasi-isometric embedding between simplicial trees with \( S \) trivalent and with edges of length \( > D \). Then there is a \((K, L)\)-quasi-isometric embedding \( \tilde{\phi} : S \to R \) such that \( \tilde{\phi} \) is a homeomorphism onto its image and \( \phi \) and \( \tilde{\phi} \) are uniformly close.

**Proof.** Let \( x \) be vertex in \( S \) and \( x_0, x_1, x_2 \) the three vertices adjacent to \( x \) and define \( \tilde{\phi}(x) = m(\phi(x_0), \phi(x_1), \phi(x_2)) \). We then extend \( \tilde{\phi} \) to be a linear map on the edges of \( S \). As noted before, for any three points \( p, q, r \in S \) the distance between \( \phi(m(p, q, r)) \) and \( m(\phi(p), \phi(q), \phi(r)) \) is uniformly bounded (with constants only depending on the qc-constants for \( \phi \)). It follows that \( \tilde{\phi} \) is uniformly close to \( \phi \) and therefore a quasi-isometric embedding (again with constants only depending on the ac-constants for \( \phi \)).

A map between trees is a global homeomorphism onto its image if and only if it is a local homeomorphism. Therefore to show that \( \tilde{\phi} \) is a homeomorphism we need to show that \( \phi(x_0), \phi(x_1) \) and \( \phi(x_2) \) are the vertices of a non-degenerate tripod. As both \( \phi(x) \) and \( \phi(x) \) and \( \phi(x_1) \) and \( \tilde{\phi}(x_1) \) are uniformly close, if the edge lengths for \( S \) are sufficiently long then the \( \phi(x_i) \) and
\( \phi(x_i) \) are closer to each other than they are to \( \phi(x) = m(\phi(x_0), \phi(x_1), \phi(x_2)) \) and it follows that \( m(\phi(x_0), \phi(x_1), \phi(x_2)) = m(\phi(x_0), \phi(x_1), \phi(x_2)) \) which in turn implies that the distance between \( \phi(x_i) \) and \( m(\phi(x_0), \phi(x_1), \phi(x_2)) \) is positive. Therefore the tripod is non-degenerate, completing the proof. □

5.2 Constructing quasi-cocycles

Let the triple \( \Phi = (T, \phi, x_0) \) consist of a quasi-isometry \( \phi \) between our quasi-tree \( Q \) and a tree \( T \) with \( x_0 \in Q \) a basepoint. Let \([x, y]\) and \([p, q]\) be two oriented segments in \( T \). If \([x, y]\) is a subsegment of \([p, q]\) and on the intersection the two orientations coincide, then we will write
\[
[x, y] \circ \subset [p, q].
\]

In \( Q \) there will not be a unique geodesic segment between any two points so for points \( x, y \in Q \), \([x, y]\) is the collection of all oriented geodesics between \( a \) and \( b \). If \( x, y, p \) and \( q \) are in \( Q \) then we write \([x, y] \subset N_\epsilon([p, q])\) if the set \([x, y]\) is contained in the \( \epsilon \)-Hausdorff neighborhood of \([p, q]\), the intersection of \([\phi(x), \phi(y)] \cap [\phi(p), \phi(q)]\) is non-empty and on the intersection the orientations of the two segments agree.

Given \( W = [x, y] \) we define sets that mimic the definition of \( w_\pm(g) \) in Section 2
\[
W_+(g) = \{ h \in G | \exists t \in G, [\phi(thx), \phi(thy)] \circ \subset [\phi(tx_0), \phi(tgx_0)] \}
\]

and
\[
W_-(g) = \{ h \in G | \exists t \in G, [\phi(thx), \phi(thy)] \circ \subset [\phi(tgx_0), \phi(tx_0)] \}.
\]

Note that in the definition of \( w_\pm(g) \) we did not look at translates by \( t \in G \). This is necessary here because the map \( \phi \) may not be \( G\)-equivariant. The sets \( W_\pm(g) \) depend on the choice of the triple \( \Phi \) but we suppress that in our notation.

The following lemma is elementary.

**Lemma 5.5.** There exists an \( \epsilon > 0 \), depending only on the quasi-isometry constants for \( \phi \), such that if \( h \in W_+(g) \) then \([hx, hy] \subset N_\epsilon([x_0, gx_0])\).

We then define sets
\[
\tilde{W}_+(g) = \{ h \in G | [hx, hy] \subset N_\epsilon([x_0, gx_0]) \}
\]

and
\[
\tilde{W}_-(g) = \{ h \in G | [hx, hy] \subset N_\epsilon([gx_0, x_0]) \}
\]
where the $\epsilon$ comes from Lemma 5.5. We then record the following corollary of the lemma:

**Corollary 5.6.**

$$W_\pm(g) \subset \tilde{W}_\pm(g).$$

Our next lemma is a restatement of the WPD property.

**Lemma 5.7.** Let $f \in G$ be an axial WPD element for the $G$-action on a quasi-tree $Q$ and fix a basepoint $x_0 \in Q$ on the axis of $f$. Then there exists an $M > 0$ such that if $w = f^M$, $W' = [x_0, wx_0]$ and $W = [x, y]$ with $W' \subset W$ then

- the sets $\tilde{W}_\pm(g)$ are finite;
- for all $\eta > 0$ there exists $N > 0$ such that for all $y \in Q$ there are at most $N$ elements $h \in \tilde{W}_\pm(g)$ with $[hx, hy] \cap N_\eta(y) \neq \emptyset$.

**Proof.** We can assume that $W = W'$ for as the segment gets larger the sets $\tilde{W}_\pm$ will get smaller.

Let $\epsilon$ be the constant from Lemma 5.5. Recall that $[x_0, g(x_0)]$ is the set of geodesics between $x_0$ and $g(x_0)$. Since $Q$ is a quasi-tree each geodesic in $[x_0, g(x_0)]$ is contained in a uniform neighborhood of any other. In particular there is a $\delta$, which does not depend on $g$, such that $N_\epsilon([x_0, g(x_0)])$ is covered by finitely many $\delta$-balls. Let $M$ be the WPD constant for $x = x_0$ and $D = 2\delta$. In particular, if $w = f^M$, $W = [x_0, wx_0]$ and $V_0, V_1$ are $\delta$-balls then there are at most $M'$ translates of $W$ that have initial and terminal endpoints in $V_0$ and $V_1$, respectively. Now, if $\tilde{W}_\pm(g)$ is infinite then by the pigeon hole principle there exists two $\delta$-balls in the cover such that infinitely many distinct translates of $W$ have their initial point in the first ball and there terminal point in the second ball, contradicting WPD. The same argument shows that $\tilde{W}_-(g)$ is finite.

Note that the bound on $|\tilde{W}_\pm(g)|$ depends on $M'$ and the number of $\delta$-balls that are needed to cover $N_\epsilon([x_0, g(x_0)])$. If we assume that the cover uses the minimal number of balls and that $h(W) \cap N_\eta(y) \neq \emptyset$ then there is a uniform bound on the number of $\delta$-balls in the cover that $h(W)$ can intersect. By the argument in the previous paragraph this gives a uniform bound of the possible $h \in \tilde{W}_\pm(g)$ with $h(W) \cap N_\eta(y) \neq \emptyset$. 

**Proposition 5.8.** Let $G$ act on a quasi-tree $Q$ with a WPD axial element $f \in G$ and let $\Phi = (T, \phi, x_0)$ be a triple consisting of a quasi-isometry $\phi : Q \to T$ to a tree and a basepoint $x_0$. Let $\rho : G \to E$ be a representation
to a normed vector space \( E \). Then there exists a subsegment \( W' \) of the axis for \( f \) such that if \( W = [x, y] \) is a geodesic segment in \( Q \) that contains \( W' \) and \( e \in E \) then

\[
H_{W, \phi, e}(g) = H(g) = \sum_{h \in W_+(g)} h(e) - \sum_{h \in W_-(g)} h(e)
\]

is a quasi-cocycle.

**Proof.** By Corollary \ref{cor:5.6}, \( W_\pm(g) \subset W_\pm(g) \). By Lemma \ref{lem:5.7} if \( W' \) is sufficiently long then \( W_\pm(g) \) is finite and hence \( W_\pm(g) \) is finite. Therefore \( H \) is defined.

We now mimic the proof of Proposition \ref{prop:2.1}, the Brooks construction for free groups. In particular for any \( g, h \in G \) we partition each of the sets \( X = W_\pm(g), W_\pm(h), W_\pm(gh) \) into 3 subsets \( X^-, X^0, X^+ \) with \( |X^0| \leq K \) for some \( K > 0 \), where \( K \) does not depend on \( g, h \), so that

- \( W^-_+(g) = W^-_+(gh) \),
- \( W^-_-(g) = W^-_-(gh) \),
- \( W^+_+(g) = gW^-_-(h) \),
- \( W^+_-(g) = gW^-_+(h) \),
- \( W^+_+(gh) = gW^-_+(h) \),
- \( W^+_-(gh) = gW^-_+(h) \).

Then it follows that \( H \) is a quasi-cocycle just like in the case of a free group.

We are left to define the partitions. Given \( t \in G \) let \( m_t = m(\phi(tx_0), \phi(tgx_0), \phi(tgx_0)). \) Again we follow the proof in the free group case and set

\[
W^-_+(g) = \{ h \in | \exists t \in G, [\phi(t x_0), \phi(t y)] \subset [\phi(t(x_0)), \phi(t m_t)] \}
\]

and

\[
W^+_+(g) = \{ h \in | \exists t \in G, [\phi(t x_0), \phi(t y)] \subset [\phi(t m_t), \phi(t gx_0)] \}
\]

and \( W^0_+(g) \) is the complement of \( W^-_+(g) \) and \( W^+_+(g) \) in \( W_+(g) \). We need to see that \( W^-_+(g) \) and \( W^+_+(g) \) are disjoint and that the cardinality of \( W^0_+(g) \) is uniformly bounded. After possibly lengthening \( w' \), the former follows from Lemma \ref{lem:5.3}. For the latter we note that if \( h \in W^0_+(g) \) then for some \( t \in G \), \( [\phi(t x_0), \phi(t y)] \subset [\phi(t x_0), \phi(t gx_0)] \); but for any \( t \in G \), \( [\phi(t x_0), \phi(t y)] \not\subset [\phi(t x_0), m_t] \) and \([\phi(t x_0), \phi(t y)] \not\subset [m_t, \phi(t gx_0)] \). Therefore \( m_t \in [\phi(t x_0), \phi(t y)] \) for some \( h \in W_+(g) \). As was noted before, for
\[ m(x, gx, ghx) = p \in Q, \phi(tp) \text{ is uniformly close to } m_t. \] It follows that there is an \( \eta > 0 \) such that \([hx, hy] \cap \mathcal{N}_\eta(p) \neq \emptyset\). Then Lemma 5.7 uniformly bounds the number of possible \( h \in W_+(g) \).

The partitions of \( W_-(g) \), \( W_+(h) \) and \( W_+(gh) \) are similarly defined and it is straightforward to check that the partitions satisfy the required properties.

**Proof of Theorem 5.1.** We will find elements \( a, b \in G \) that generate a rank two free subgroup \( F = \langle a, b \rangle \) of \( G \) such that for any cyclically reduced word \( w \) in \( a \) and \( b \) that contains \((aba)^n\) (for some fixed power \( n \)) there exists \( H \in QC(G; \rho) \) and an \( e \in E \) such that \( H|_F = H_{w,e} \) in \( QC(F; \rho|_F) \) where \( H_{w,e} \) are the quasi-cocycles from Section \( 2 \). Furthermore if \( \rho|_F \) doesn’t fix \( e \) then no non-abelian subgroup will have a \( \rho \)-invariant vector. The result will then follow from Propositions \( 3.8 \) and \( 3.9 \).

Let \( N \) be the maximal finite normal subgroup of \( G \). Let \( Q \) be the quasi-tree given by Proposition \( 4.25 \). Then there is a \((K', L')\)-quasi-isometry from \( Q \) to a tree \( T \). Let \( K, L \) and \( D \) be the constants given by Lemma \( 5.4 \) and \( \epsilon \) be given by Lemma \( 5.5 \) for a \((K, L)\)-quasi-isometry. Proposition \( 4.25 \) then gives \( a, b \in G \) such that

- \( F = \langle a, b \rangle \) is a free group of WPD elements;
- there is an \( F \)-invariant trivalent tree \( S \subset Q \) with the \( F \)-action proper and minimal and all edge lengths are \( > D; \)
- there is a basepoint \( x_0 \in S \) on the axis for \( aba \in F \) such that if \( \chi([x_0, abax]) \subset \mathcal{N}_\epsilon(S) \) then \( \chi = \psi h \) with \( \psi \in F \) and \( h \in N \).

By Lemma \( 5.4 \) we can modify the quasi-isometry from \( Q \) to \( T \) to obtain a \((K, L)\)-quasi-isometry \( \phi : Q \to T \) that restricted to \( S \) is a homeomorphism onto its image. Let \( \Phi = (T, \phi, x_0) \).

By Proposition \( 5.8 \) there is an \( n \) such that for any cyclically reduced word \( w \) that contains \((aba)^n\) and any \( e \in E \) if \( W = [x_0, (aba)^nx_0] \) then \( H = H_{W; \phi, e} \) is a quasi-cocycle. If \( e \) is fixed by \( \rho|_N \) we’ll show that \( H|_F = H_{w,e} \) in \( QC(F; \rho) \). We first observe that for all \( g \in F \) we have \( w_+(g) \subset W_+(g) \).

In fact for every \( h \in N \) and \( f \in w_+(g) \), since \( f(S) = S \), we also have \( fh \in W_+(g) \). Namely, \( w_+(g)N \subset W_+(g) \).

We claim that each of \(|W_+(g) \setminus w_+(g)N| \leq |N|K \). Indeed, by Corollary \( 5.9 \) \( W_+(g) \subset \tilde{W}_+(g) \), and for an arbitrary \( \chi \in W_+(g) \) we have \( \chi(W) \subset \mathcal{N}_\epsilon(S) \) by definition, and therefore \( \chi = fh \) with \( h \in N \) and \( f \in F \) by Proposition \( 4.25 \). Since \( h \) fixes \( S \) and \( F \) acts properly and minimally on \( S \)

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we must have either $f \in w_{\pm}(g)$ or $f(W) \cap N_{\varepsilon}(\{x_0, gx_0\}) \neq \emptyset$. By Lemma 5.7 the number of such $f$ (in the latter case) is bounded by some constant $K$.

Notice that for $f \in w_{\pm}(g)$ and $h \in N$ the contribution of $fh$ to $H$ is the same as the contribution of $f$ since $h$ fixes $e$. It follows that for all $g \in F$ we have

$$
\|H(g) - H_{w,e}(g)\| \leq \frac{2}{|N|}|N|K = 2K
$$

and therefore $H|F = H_{w,e}$ in $\widetilde{QC}(F; \rho)$. □

5.3 Standard examples

**Theorem 5.9.** Let $G$ be one of the following groups:

1. $G$ is a nonelementary hyperbolic group.

2. $G$ is the mapping class group of a closed surface (possibly with punctures) and $G$ is not virtually abelian.

3. $G = \text{Out}(F_n)$, $n \geq 3$.

4. $G$ is not virtually cyclic, it acts by isometries on a hyperbolic graph, and contains an element $g$ which is hyperbolic and WPD.

5. $G$ is a nonelementary (not virtually cyclic) group that acts by isometries on a CAT(0) space that contains an element $g$ which is rank-1 and WPD.

Let $\rho : G \to E$ be a representation into a uniformly convex Banach space such that if $N \subseteq G$ is a finite, normal subgroup then $\rho|_N$ has a nonzero fixed vector. Then $\widetilde{QC}(G; \rho)$ is infinite dimensional.

**Proof.** By Theorem 5.1 we need to show that $G$ acts on a geodesic metric space $X$ with a WPD element $g$ that has a strongly contracting axis. The simplest case is when $G$ acts properly discontinuously on a Gromov hyperbolic space as every geodesic will be strongly contracting and any hyperbolic element will obviously have WPD. In particular any element with positive translation distance will work. For groups that don’t have such nice actions we need to either find a non-properly discontinuous action on a hyperbolic space and work harder to show the WPD property or find a non-hyperbolic space where the action is properly discontinuous but still has strongly contracting axes. Fortunately in each case this is an established result and we only need to provide a reference.

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1. In this case $X$ is the Cayley graph of $G$ and any infinite order $g$ has a strongly contracting axis. Since $G$ acts properly on $X$ any infinite order $g$ will have WPD.

2. Here there are a number of choices. For example $G$ acts properly discontinuously on Teichmüller space and if $g \in G$ is pseudo-Anosov it will have a strongly contracting axis by [19]. Alternatively, one can use the action on the complex of curves [18]. That pseudo-Anosov mapping classes satisfy WPD was verified in [5], and the stronger property of *acylindricity* was proved in [7].

3. Again we have choices. For $Out(F_n)$ the space analogous to the Teichmüller space is Outer Space where $Out(F_n)$ acts properly discontinuously and any fully reducible element has a strongly contracting axis by [1]. Strictly speaking since the metric on Outer space is asymmetric our result doesn’t apply. The necessary technicalities are resolved in [1] but one could also use the space constructed in [4], or the complex of free factors, see [3].

4. In a hyperbolic space any geodesic is strongly contracting.

5. A rank-1 geodesic in a $CAT(0)$ space is strongly contracting, see [6].

□

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