HYPERBOLIC GEOMETRY ON NONCOMMUTATIVE POLYBALLS

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Abstract. This paper is an introduction to the hyperbolic geometry of noncommutative polyballs $B_n(H)$ in $B(H)^{n_1+\cdots+n_k}$, where $n = (n_1, \ldots, n_k) \in \mathbb{N}^k$ and $B(H)$ is the algebra of all bounded linear operators on a Hilbert space $H$. We use the theory of free pluriharmonic functions on polyballs and noncommutative Poisson kernels on tensor products of full Fock spaces to define hyperbolic type metrics on $B_n(H)$, study their properties, and obtain hyperbolic versions of Schwarz-Pick lemma for free holomorphic functions on polyballs. As a consequence, the polyballs can be viewed as noncommutative hyperbolic spaces. When specialized to the regular polydisk $D^k(H)$ (which corresponds to the case $n_1 = \cdots = n_k = 1$), our hyperbolic metric $\delta_H$ is complete and invariant under the group $\text{Aut}(D^k)$ of all free holomorphic automorphisms of $D^k(H)$, and the $\delta_H$-topology induced on $D^k(H)$ is the usual operator norm topology. The restriction of $\delta_H$ to the scalar polydisk $D^k$ is equivalent to the Kobayashi distance on $D^k$. Most of the results of this paper are presented in the more general setting of Harnack (resp. Poisson) parts of the closed polyball $B_n(H)^-$. 

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Introduction

A theory of free holomorphic functions on noncommutative polydomains which admit universal operator models has been developed in [18], [19], [23], [24], [26], and [28]. These results played a crucial role in our work on the curvature invariant [25], the Euler characteristic [27], and the group of free holomorphic automorphisms on noncommutative regular polyballs [28]. The regular polyball $B_n$, $n = (n_1, \ldots, n_k) \in \mathbb{N}^k$, is a noncommutative analogue of the scalar polyball $(\mathbb{C}^{n_1})_1 \times \cdots \times (\mathbb{C}^{n_k})_1$ and has a universal model $S := \{S_{i,j}\}$ consisting of left creation operators acting on the tensor product $F^2(H_{n_1}) \otimes \cdots \otimes F^2(H_{n_k})$ of full Fock spaces. As a consequence, the theory of free holomorphic functions on noncommutative polydomains is related, via noncommutative Berezin transforms, to the study of the operator algebras generated by the universal models associated with the polydomains, as well as to the theory of functions in several complex variable([13], [30], [31]). We remark that, in general, one can view the free holomorphic functions on noncommutative polydomains as noncommutative functions in the sense of [10].

Recently [29], we obtained structure theorems characterizing the bounded (resp. positive) free $k$-pluriharmonic functions on regular polyballs. These results will play an important role in the present paper which is an introduction to the hyperbolic geometry of noncommutative polyballs. The main goal is to introduce hyperbolic type metrics on these polyballs, study their basic properties, and provide an...
analogue of Schwarz-Pick lemma in this setting. As a consequence, the regular polyballs can be viewed as noncommutative hyperbolic spaces.

Poincaré’s discovery of a conformally invariant metric on the open unit disc $\mathbb{D} := \{ z \in \mathbb{C} : |z| < 1 \}$ of the complex plane is at the heart of geometric function theory. The hyperbolic (Poincaré) distance is defined on $\mathbb{D}$ by
\[
\delta_p(z, w) := \frac{1}{2} \ln \frac{1 + |\varphi_z(w)|}{1 - |\varphi_z(w)|}, \quad z, w \in \mathbb{D},
\]
where $\varphi_z$ is the automorphism of $\mathbb{D}$ given by $\varphi_z(w) = \frac{w-z}{1-z\overline{w}}$, and it is invariant under the conformal automorphisms of $\mathbb{D}$, i.e.
\[
\delta_p(\varphi(z), \varphi(w)) = \delta_p(z, w), \quad z, w \in \mathbb{D},
\]
for all $\varphi \in \text{Aut}(\mathbb{D})$. Moreover, $(\mathbb{D}, \delta_p)$ is a complete metric space and the $\delta_p$-topology induced on the open disk is the usual planar topology. Schwarz-Pick lemma asserts that any analytic function $f : \mathbb{D} \to \mathbb{D}$ is distance-decreasing with respect to $\delta_p$, i.e.
\[
\delta_p(f(z), f(w)) \leq \delta_p(z, w), \quad z, w \in \mathbb{D}.
\]
This result has had profound implications in the development of geometric function theory. It has been generalized to higher dimensional complex spaces in various ways (see [11], [12]). Bergman (see [3]) introduced an analogue of the Poincaré distance for the open unit ball $B_m := \{ z \in \mathbb{C}^m : \|z\|_2 < 1 \}$, which has properties similar to those of $\delta_p$. There is a large literature concerning invariant metrics, hyperbolic manifolds, and the geometric viewpoint of complex function theory (see [11], [12], [9], [36], and [14] and the references there in). There are several extensions of the Poincaré-Bergman distance and related topics to more general domains. We mention the work of by L. Harris ([6], [7], [8]) in the setting of $\mathbb{D}$, and [14] and the references there in). Extending Poincaré’s result [13] that the open ball of $\mathbb{C}^m$ is not biholomorphic equivalent to the polydisk $\mathbb{D}^k$, $k \geq 2$, we proved in [23] that the noncommutative ball $[B(\mathbb{H})^m]_1$ is not free biholomorphic equivalent to the regular polyball $B_n(\mathbb{H})$, where $n = (n_1, \ldots, n_k)$ and...
k \geq 2. As in the classical case ([30], [31], [13]), one expects significant differences between the hyperbolic geometry of the ball \([B(\mathcal{H})]^m\] and that of the regular polyball \(B_n(\mathcal{H})\), and differences regarding the theory of free holomorphic (resp. pluriharmonic) functions on these noncommutative domains.

If \(A, B \in B_n(\mathcal{H})^\sim\), we say that \(A\) and \(B\) are Harnack equivalent (and denote \(A \overset{H}{\sim} B\)) if there exists \(\epsilon > 1\) such that
\[
\frac{1}{c^2} F(rB) \leq F(rA) \leq c^2 F(rB), \quad r \in [0, 1),
\]
for any positive free \(k\)-pluriharmonic function \(F : B_n(\mathcal{H}) \to B(\mathcal{E}) \otimes_{\min} B(\mathcal{H})\), where \(\mathcal{E}\) is a separable Hilbert space, in the sense of [29]. In this case, we write \(A \overset{H}{\sim} B\). The equivalence classes with respect to the equivalence relation \(\overset{H}{\sim}\) are called Harnack parts of \(B_n(\mathcal{H})^\sim\). In Section 1, we prove a Harnack type inequality for positive free \(k\)-pluriharmonic functions on regular polyballs and use it to show that the Harnack part containing the zero element coincides with the open polyball \(B_n(\mathcal{H})\).

Given a Harnack part \(\Delta\) of \(B_n(\mathcal{H})^\sim\), we define the map \(\delta_H : \Delta \times \Delta \to \mathbb{R}^+\) by setting
\[
\delta_H(A, B) := \ln \inf \left\{ c > 1 : A \overset{H}{\sim} B \right\}.
\]
In Section 2, we prove that \(\delta_H\) is a metric on \(\Delta\) and provide a Schwarz-Pick type result for free holomorphic functions on regular polyballs with respect to \(\delta_H\). We show that if \(\Phi = (\Phi_1, \ldots, \Phi_n) : B_n(\mathcal{H}) \to [B(\mathcal{H})]^m\] is a free holomorphic function on the regular polyball and \(X, Y \in B_n(\mathcal{H})\), then \(\Phi(X) \overset{H}{\sim} \Phi(Y)\) and
\[
\delta_H(\Phi(X), \Phi(Y)) \leq \delta_H(X, Y),
\]
where \(\delta_H\) is the hyperbolic metric defined on the Harnack parts of \([B(\mathcal{H})]^m\] and on the polyball \(B_n(\mathcal{H})\), respectively. Using the description of the group \(\text{Aut}(B_n)\) of all free holomorphic automorphisms of \(B_n\) (see [28]), we prove that
\[
\delta_H(A, B) = \delta_H(\Psi(A), \Psi(B)), \quad \Psi \in \text{Aut}(B_n),
\]
for any \(A, B \in B_n(\mathcal{H})^\sim\) such that \(A \overset{H}{\sim} B\). In particular, the hyperbolic distance \(\delta_H\) on the open polyball is invariant under the automorphism group \(\text{Aut}(B_n)\).

If \(A\) and \(B\) are in \(B_n(\mathcal{H})^\sim\), we say that they are Poisson equivalent (and denote \(A \overset{P}{\sim} B\)) if there exists \(c > 1\) such that
\[
\frac{1}{c^2} \mathcal{P}(rB) \leq \mathcal{P}(rA) \leq c^2 \mathcal{P}(rB), \quad r \in [0, 1),
\]
where \(X \mapsto \mathcal{P}(X, B)\) is the free pluriharmonic Poisson kernel on the regular polyball (see Section 1). In this case we write \(A \overset{P}{\sim} B\). The equivalence classes with respect to equivalence relation \(\overset{P}{\sim}\) are called Poisson parts of \(B_n(\mathcal{H})^\sim\). We prove in Section 1 that the Poisson part containing the zero element coincides with the open polyball \(B_n(\mathcal{H})\).

Given a Poisson part \(\Delta\) of \(B_n(\mathcal{H})^\sim\), we define the map \(\delta_P : \Delta \times \Delta \to \mathbb{R}^+\) by setting
\[
\delta_P(A, B) := \ln \inf \left\{ c > 1 : A \overset{P}{\sim} B \right\}.
\]
In Section 3, we prove that \(\delta_P\) is a metric on \(\Delta\) and obtain an explicit formula for it (see Theorem 3.3) in terms of certain noncommutative Cauchy kernels acting on tensor products of full Fock spaces. Moreover, we prove that \(\delta_P\) is a complete metric on \(B_n(\mathcal{H})\) and that the \(\delta_P\)-topology coincides with the operator norm topology on \(B_n(\mathcal{H})\).

In Section 4, we consider the regular polydisk \(D^k(\mathcal{H}) := B_{\{1, \ldots, k\}}(\mathcal{H})\), which consists of all tuples \(X = (X_1, \ldots, X_k)\) of commuting strict contractions such that \(\Delta_X(I) > 0\). We remark that in this case we have
\[
\Delta_X(I) = \sum_{p_1, \ldots, p_k \in \{0, 1\}} (-1)^{p_1 + \cdots + p_k} T_1^{p_1} \cdots T_k^{p_k} (T_k^* T_k)^{p_k} \cdots (T_1^* T_1)^{p_1}
\]
and \(D^k(\mathbb{C}) = \mathbb{D}^k\). Using a characterization of positive free \(k\)-pluriharmonic functions on regular polydisks (see Theorem 4.2) and the results of the previous sections, we prove that \(A \overset{H}{\sim} B\) if and only if \(A \overset{P}{\sim} B\),
for any $A, B \in \mathcal{D}^k(\mathcal{H})^{-}$. Consequently, the metrics $\delta_H$ and $\delta_P$ coincide on the Harnack (resp. Poisson) parts of $\mathcal{D}^k(\mathcal{H})^{-}$. We show that the hyperbolic metric $\delta_H$ is complete on the regular polydisk and the $\delta_H$-topology coincides with the operator norm topology on $\mathcal{D}^k(\mathcal{H})$. As a consequence, $(\mathcal{D}^k(\mathcal{H}), \delta_H)$ is a complete hyperbolic space. Moreover, $\delta_H$ is invariant under the automorphism group $Aut(\mathcal{D}^k)$ and

$$\delta_H(f(X), f(Y)) \leq \delta_H(X, Y), \quad X, Y \in \mathcal{D}^k(\mathcal{H})$$

for any free holomorphic function $f : \mathcal{D}^k(\mathcal{H}) \to [B(\mathcal{H})^m]_1$. Therefore, the hyperbolic metric $\delta_H$ on $\mathcal{D}^k(\mathcal{H})$ has similar properties to those of the Poincaré distance on the open unit disc $\mathbb{D}$. In addition, we prove that if $A$ and $B$ are in $\mathcal{D}^k(\mathcal{H})$, then

$$\delta_H(A, B) = \ln \max \left\{ \|C_A(R)C_B(R)^{-1}\|, \|C_B(R)C_A(R)^{-1}\| \right\},$$

where

$$C_X(R) := (I \otimes \Delta_X(I)^{1/2}) \prod_{i=1}^k (I - R_i \otimes X^*_i), \quad X = (X_1, \ldots, X_k) \in \mathcal{D}^k(\mathcal{H}),$$

and $R_1, \ldots, R_k$ are the shift operators on the Hardy space $H^2(\mathbb{D}^k)$. In particular, we show that $\delta_H|_{\mathbb{D}^k \times \mathbb{D}^k}$ is equivalent to the Kobayashi distance on the polydisk $\mathbb{D}^k$ (see [9]) and

$$\delta_H(z, w) = \frac{1}{2} \ln \frac{\prod_{i=1}^k (1 + |\psi_z(w_i)|)}{\prod_{i=1}^k (1 - |\psi_z(w_i)|)}$$

for any $z = (z_1, \ldots, z_k)$ and $w = (w_1, \ldots, w_k) \in \mathbb{D}^k$, where $\psi_z := (\psi_{z_1}, \ldots, \psi_{z_k})$ is the involutive automorphisms of $\mathbb{D}^k$ such that $\psi_z(0) = z$ and $\psi_z(z_i) = 0$.

We remark that, according to the results of the present paper and those from [21], the metrics $\delta_H$ and $\delta_P$ coincide for the regular polydisk $\mathcal{D}^k(\mathcal{H})$ and for the noncommutative ball $[B(\mathcal{H})^m]_1$. It remains an open problem whether the same result is true for any regular polyball. On the other hand, it will be interesting to see to what extent these results extend to more general polydomains in $B(\mathcal{H})^m$ such as those studied in [24] and [26].

1. HARNACK AND POISSON EQUIVALENCES ON THE CLOSED POLYBALL

In this section, we recall some basic facts concerning the noncommutative Berezin transforms on polyballs and introduce the Harnack and the Poisson equivalence relations on the closed polyball $B_n(\mathcal{H})^{-}$.

We provide a Harnack type inequality for positive free $k$-pluriharmonic functions and use it to show that the Harnack (resp. Poisson) equivalence class containing the zero element coincides with the open polyball $B_n(\mathcal{H})$.

Let $H_{n_1}$ be an $n_1$-dimensional complex Hilbert space with orthonormal basis $e_1^1, \ldots, e_{m_1}^1$. We consider the full Fock space of $H_{n_1}$ defined by $F^2(H_{n_1}) := \mathbb{C}1 \otimes \bigoplus_{p \geq 1} H_{n_1}^{\otimes p}$, where $H_{n_1}^{\otimes p}$ is the (Hilbert) tensor product of $p$ copies of $H_{n_1}$. Let $\mathbb{F}_{n_1}^+$ be the unitail free semigroup on $n_1$ generators $g_1^1, \ldots, g_{m_1}^1$ and the identity $g_0^1$. Set $e_{\alpha}^j := e_{j_1}^1 \otimes \cdots \otimes e_{j_p}^p$ if $\alpha = g^1_{j_1} \cdots g^p_{j_p} \in \mathbb{F}_{n_1}^+$ and $e_{g_0^1}^j := 1 \in \mathbb{C}$. The length of $\alpha \in \mathbb{F}_{n_1}^+$ is defined by $|\alpha| := 0$ if $\alpha = g_0^1$ and $|\alpha| := p$ if $\alpha = g^1_{j_1} \cdots g^p_{j_p}$, where $j_1, \ldots, j_p \in \{1, \ldots, n_1\}$. We define the left creation operator $S_{i,j}$ acting on the Fock space $F^2(H_{n_1})$ by setting $S_{i,j}e_{\alpha} := e_{j}^i \otimes e_{\alpha}$, $\alpha \in \mathbb{F}_{n_1}^+$, and the operator $S_{i,j}$ acting on the Hilbert tensor product $F^2(H_{n_1}) \otimes \cdots \otimes F^2(H_{n_k})$ by setting

$$S_{i,j} := I \otimes \cdots \otimes I \otimes S_{i,j} \otimes I \otimes \cdots \otimes I,$$

where $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, n_i\}$. We denote $S := (S_1, \ldots, S_k)$, where $S_i := (S_{i,1}, \ldots, S_{i,n_i})$, or $S := \{S_{i,j}\}$. The noncommutative Hardy algebra $F^\infty_n$ (resp. the polyball algebra $A_n$) is the weakly closed (resp. norm closed) non-selfadjoint algebra generated by $\{S_{i,j}\}$ and the identity. Similarly, we define the right creation operator $R_{i,j} : F^2(H_{n_1}) \to F^2(H_{n_1})$ by setting $R_{i,j}e_{\alpha} := e_{i}^j \otimes e_{\alpha}$ for $\alpha \in \mathbb{F}_{n_1}^+$, and the operator $R_{i,j}$ acting on the Hilbert tensor product $F^2(H_{n_1}) \otimes \cdots \otimes F^2(H_{n_k})$ by setting

$$R_{i,j} := I \otimes \cdots \otimes I \otimes R_{i,j} \otimes I \otimes \cdots \otimes I,$$

where $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, n_i\}$. We denote $S := (S_1, \ldots, S_k)$, where $S_i := (S_{i,1}, \ldots, S_{i,n_i})$, or $S := \{S_{i,j}\}$. The noncommutative Hardy algebra $F^\infty_n$ (resp. the polyball algebra $A_n$) is the weakly closed (resp. norm closed) non-selfadjoint algebra generated by $\{S_{i,j}\}$ and the identity. Similarly, we define the right creation operator $R_{i,j} : F^2(H_{n_1}) \to F^2(H_{n_1})$ by setting $R_{i,j}e_{\alpha} := e_{i}^j \otimes e_{\alpha}$ for $\alpha \in \mathbb{F}_{n_1}^+$, and the operator $R_{i,j}$ acting on the Hilbert tensor product $F^2(H_{n_1}) \otimes \cdots \otimes F^2(H_{n_k})$ by setting

$$R_{i,j} := I \otimes \cdots \otimes I \otimes R_{i,j} \otimes I \otimes \cdots \otimes I,$$
The polyball algebra $\mathcal{R}_n$ is the norm closed non-selfadjoint algebra generated by $\{R_{i,j}\}$ and the identity.

We recall (see [17, 29]) some basic properties for the noncommutative Berezin transforms associated with regular polyballs. Let $X = (X_1, \ldots, X_k) \in \mathcal{B}_n(H)$ with $X_i := (X_{i,1}, \ldots, X_{i,n})$. We use the notation $X_{i, \alpha, 1} := X_{i,j_1} \cdots X_{i,j_p}$ if $\alpha = g_{j_1} \cdots g_{j_p} \in F^+_n$, and $X_{i, b_0} := I$. The noncommutative Berezin kernel associated with any element $X$ in the noncommutative polyball $\mathcal{B}_n(H)$ is the operator

$$K_X : H \to F^2(H_{n_k}) \otimes \cdots \otimes F^2(H_{n_k}) \otimes \Delta_X(I)(H)$$

defined by

$$K_X h := \sum_{\beta_i \in F^+_n, i = 1, \ldots, k} e^{\frac{1}{2}}_{\beta_1} \cdots e^{\frac{1}{2}}_{\beta_k} \otimes \Delta_X(I)^{1/2} X_{1, \beta_1}^* \cdots X_{k, \beta_k}^* h, \quad h \in H,$$

where $\Delta_X(I)$ is the defect operator. A very important property of the Berezin kernel is that $K_X X^*_{i,j} = (S^*_{i,j} \otimes I)K_X$ for any $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, n_i\}$. The Berezin transform at $X \in \mathcal{B}_n(H)$ is the map $B_X : B(\otimes_{i=1}^k F^2(H_{n_k})) \to B(H)$ defined by

$$B_X[g] := K_X^* (g \otimes I_H)K_X, \quad g \in B(\otimes_{i=1}^k F^2(H_{n_k})).$$

If $g$ is in the $C^*$-algebra $C^*(S)$ generated by $S_{i,1}, \ldots, S_{i,n_i}$, where $i \in \{1, \ldots, k\}$, we define the Berezin transform at $X \in \mathcal{B}_n(H)$ by

$$B_X[g] := \lim_{r \to 1} K^*_{rX}(g \otimes I_H)K_{rX}, \quad g \in C^*(S),$$

where the limit is in the operator norm topology. In this case, the Berezin transform at $X$ is a unital completely positive linear map such that

$$B_X(S_{\alpha}S_{\beta}^*) = X_{\alpha}X_{\beta}^*, \quad \alpha, \beta \in F^+_n \times \cdots \times F^+_n,$$

where $S_{\alpha} := S_{1, \alpha_1} \cdots S_{k, \alpha_k}$ if $\alpha := (\alpha_1, \ldots, \alpha_k) \in F^+_n \times \cdots \times F^+_n$.

The Berezin transforms will play an important role in this paper. More properties concerning noncommutative Berezin transforms and multivariable operator theory on noncommutative balls and polydomains, can be found in [17] and [29]. For basic results on completely positive (resp. bounded) maps we refer the reader to [13] and [16].

For each $m \in \mathbb{Z}$, we set $m^+ := \max\{m, 0\}$ and $m^- := \max\{-m, 0\}$. A function $F$ with operator-valued coefficients in $B(E)$, where $E$ is separable Hilbert space, is called free $k$-pluriharmonic on the abstract polyball $\mathcal{B}_n$ if it has the form

$$F(X) = \sum_{m_1 \in \mathbb{Z}} \cdots \sum_{m_k \in \mathbb{Z}} \sum_{\alpha_1, \beta_1 \in F^+_n, i \in \{1, \ldots, k\}} A_{(\alpha_1, \ldots, \alpha_k; \beta_1, \ldots, \beta_k)} X_{1, \alpha_1} \cdots X_{k, \alpha_k} X_{1, \beta_1}^* \cdots X_{k, \beta_k}^*,$$

where the multi-series converge in the operator norm topology for any $X = (X_1, \ldots, X_k) \in \mathcal{B}_n(H)$, with $X_i := (X_{i,1}, \ldots, X_{i,n})$, and any Hilbert space $H$. Without loss of generality, we can assume throughout this paper that $H$ is a separable infinite dimensional Hilbert space. According to [29], the order of the series in the definition above is irrelevant. Note that any free holomorphic function on $\mathcal{B}_n$ is $k$-pluriharmonic. Indeed, according to [25], any free holomorphic function on the polyball $\mathcal{B}_n$ with coefficients in $B(E)$ has the form

$$f(X) = \sum_{m_1 \in \mathbb{N}} \cdots \sum_{m_k \in \mathbb{N}} \sum_{\alpha_1, \alpha_k \in F^+_n, i \in \{1, \ldots, k\}} C_{(\alpha_1, \ldots, \alpha_k)} X_{1, \alpha_1} \cdots X_{k, \alpha_k}, \quad X \in \mathcal{B}_n(H),$$

where the multi-series converge in the operator norm topology.

Now, we introduce a preorder relation $\preceq^H$ on the closed ball $\mathcal{B}_n(H)$. If $A$ and $B$ are in $\mathcal{B}_n(H)$, we say that $A$ is Harnack dominated by $B$, and denote $A \preceq^H B$, if there exists $c > 0$ such that

$$F(cA) \leq c^2 F(cB)$$
for any positive free $k$-pluriharmonic function $F$ with operator valued coefficients and any $r \in [0, 1)$. When we want to emphasize the constant $c$, we write $A^H_c < B$. Since $<^H$ is a preorder relation on $B_n(\mathcal{H})^\sim$, it induces an equivalence relation $\sim^H$ on $B_n(\mathcal{H})^\sim$, which we call Harnack equivalence. The equivalence classes with respect to $\sim^H$ are called Harnack parts of $B_n(\mathcal{H})^\sim$. Let $A$ and $B$ are in $B_n(\mathcal{H})^\sim$. It is easy to see that $A$ and $B$ are Harnack equivalent (we denote $A^H_c \sim B^H_c$) if and only if there exists $c \geq 1$ such that

\[(1.1) \quad \frac{1}{c^2} F(rB) \leq F(rA) \leq c^2 F(rB)\]

for any positive free $k$-pluriharmonic function $F$ with operator-valued coefficients and any $r \in [0, 1)$. We also use the notation $A^H_c \prec B$ if $A^H_c < B$ and $B^H_c \prec A$.

We denote by $C^*(S)$ the $C^*$-algebra generated by $S_{i,j}$, where $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, n_i\}$. A completely positive (c.p.) linear map $\mu_X : C^*(S) \to B(\mathcal{H})$ is called representing c.p. map for the point $X \in B_n(\mathcal{H})^\sim$ if

$$
\mu_X(S_{1,\alpha_1} \cdots S_{k,\alpha_k}^* X_{1,\beta_1} \cdots X_{k,\beta_k}) = X_{1,\alpha_1} \cdots X_{k,\alpha_k}^* X_{1,\beta_1} \cdots X_{k,\beta_k}
$$

for any $(\alpha_1, \ldots, \alpha_k)$ and $(\beta_1, \ldots, \beta_k)$ in $F^+_{n_1} \times \cdots \times F^+_{n_k}$ with $\alpha_i, \beta_i \in \mathbb{F}_{n_i}^+$, $|\alpha_i| = m_i^-, |\beta_i| = m_i^+$, and $m_i \in \mathbb{Z}$.

Next, we obtain characterizations for the Harnack equivalence on the closed regular polyball $B_n(\mathcal{H})^\sim$.

**Proposition 1.1.** Let $A$ and $B$ are in $B_n(\mathcal{H})^\sim$ and let $c > 1$. Then the following statements are equivalent:

(i) $A^H_c \prec B$;

(ii) $c^2 B_B - B_A$ and $c^2 B_A - B_B$ are completely positive linear map on the operator space span{$A^*_n A_n$}$^{-\| \|}$, where $B_X$ is the noncommutative Berezin transform at $X \in B_n(\mathcal{H})^\sim$;

(iii) there are representing c.p. maps $\mu_A$ and $\mu_B$ for $A$ and $B$, respectively, such that

$$
\frac{1}{c^2} \mu_B \leq \mu_A \leq c^2 \mu_B.
$$

**Proof.** Let $A$ and $B$ be elements in the regular closed polyball $B_n(\mathcal{H})^\sim$ and let $c > 1$. First we prove that $A^H_c \prec B$ if and only if $c^2 B_B - B_A$ is a completely positive linear map on the operator space span{$A^*_n A_n$}$^{-\| \|}$.

Assume that $A^H_c \prec B$ and let $g \in \mathcal{P} := \text{span} \{A^*_n A_n\}^{-\| \|}$ be a positive operator. According to Theorem 2.4 from [29], the map

$$
F(X) = B_X[g] := K_X[g \otimes I_{\mathcal{H}}] K_X, \quad X \in B_n(\mathcal{H}),
$$

is a positive free $k$-pluriharmonic function on $B_n(\mathcal{H})$ which has a continuous extension (in the operator norm topology) to the closed ball $B_n(\mathcal{H})^\sim$. Since $A^H_c \prec B$, we have $F(rA) \leq c^2 F(rB)$ for any $r \in [0, 1)$, which is equivalent to

$$
(c^2 B_B - B_A)[g] \geq 0, \quad r \in [0, 1).
$$

Since $B_B[g] := \lim_{r \to 1} B_B[g] = F(B)$ exists in the operator norm topology, and a similar result holds if we replace $B$ with $A$, we deduce that $c^2 B_B - B_A$ is a positive linear map on span{$A^*_n A_n$}$^{-\| \|}$. Similarly, passing to matrices one can prove that $c^2 B_B - B_A$ is completely positive. Hence, we deduce that $c^2 B_B - B_A$ is completely positive on the operator space span{$A^*_n A_n$}$^{-\| \|}$.

Conversely, assume that $c^2 B_B - B_A$ is completely positive on the operator space span{$A^*_n A_n$}$^{-\| \|}$. Then $c^2 B_B^{ext} - B_A^{ext}$ is positive on $\mathcal{P}_E := B(\mathcal{E}) \otimes_{\min} \text{span} \{A^*_n A_n\}^{-\| \|}$, where

$$
B_X^{ext}[g] := (I_{\mathcal{E}} \otimes K_X)[g \otimes I_{\mathcal{H}}] (I_{\mathcal{E}} \otimes K_X), \quad X \in B_n(\mathcal{H}).
$$
Let \( F : B_n(\mathcal{H}) \to B(\mathcal{E}) \otimes_{\min} B(\mathcal{H}) \) be a positive free \( k \)-pluriharmonic function on \( B_n(\mathcal{H}) \). Then, for each \( r \in [0, 1) \), we have
\[
F(rX) = B_X^{\mathcal{E}^k}[F(rS)] \geq 0
\]
and \( F(rS) \in \mathcal{P}_\mathcal{E} \). Consequently, we have
\[
c^2 F(rB) - F(rA) = (c^2B_X^{\mathcal{E}^k} - B_X^{\mathcal{E}^k})(F(rS)) \geq 0
\]
for any \( r \in [0, 1) \), which proves our assertion. Now, the equivalence of (i) and (ii) is clear.

It remains to prove that (ii) is equivalent to (iii). To this end, assume that item (ii) holds. According to Arveson’s extension theorem [2], there are completely positive maps \( \varphi, \psi : C^*(\mathcal{S}) \to B(\mathcal{H}) \) such that
\[
\varphi(g) = c^2B_A[g] - B_B[g] \quad \text{and} \quad \psi(g) = c^2B_B[g] - B_A[g]
\]
for any \( g \in \text{span}\{A_i^*A_n\}^{-\min}_\|\| \). Hence, we deduce that
\[
B_A[g] = \frac{c^2}{c^4 - 1}(c^2\varphi + \psi) \quad \text{and} \quad B_B[g] = \frac{c^2}{c^4 - 1}(c^2\psi + \varphi).
\]
Now, we define \( \mu_A : C^*(\mathcal{S}) \to B(\mathcal{H}) \) and \( \mu_B : C^*(\mathcal{S}) \to B(\mathcal{H}) \) by setting
\[
(1.2) \qquad \mu_A := \frac{c^2}{c^4 - 1}(c^2\varphi + \psi) \quad \text{and} \quad \mu_B := \frac{c^2}{c^4 - 1}(c^2\psi + \varphi)
\]
and note that \( B_A[g] = \mu_A(g) \) and \( B_B[g] = \mu_B(g) \) for any \( g \in \text{span}\{A_i^*A_n\}^{-\min}_\|\| \). Due to the properties of the noncommutative Berezin transform, it is clear that \( \mu_A \) and \( \mu_B \) are representing c.p. maps for \( A \) and \( B \), respectively. The inequalities \( \frac{1}{c}\mu_B \leq \mu_A \leq c\mu_B \) are simple consequences of relation (1.2) and the fact that \( c > 1 \). To complete the proof, it is enough to prove that (iii) \( \implies \) (i). To this end, assume that (iii) holds for some \( c > 1 \) and let \( F : B_n(\mathcal{H}) \to B(\mathcal{E}) \otimes_{\min} B(\mathcal{H}) \) be a positive free \( k \)-pluriharmonic function on \( B_n(\mathcal{H}) \). Then \( F(rS) \geq 0 \) for any \( r \in [0, 1) \). Since \( F(rS) \in B(\mathcal{E}) \otimes_{\min} \text{span}\{A_i^*A_n\}^{-\min}_\|\| \), we deduce that
\[
\frac{1}{c^2}(id \otimes \mu_B)[F(rS)] \leq (id \otimes \mu_A)[F(rS)] \leq c^2(id \otimes \mu_B)[F(rS)]
\]
for any \( r \in [0, 1) \), which implies \( A \overset{H}{\sim} B \) and completes the proof. \( \square \)

A bounded linear operator \( A \in B(\mathcal{E} \otimes \bigotimes_{i=1}^k F^2(H_{n_i})) \) is called \( k \)-multi-Toeplitz with respect to the universal model \( R := (R_1, \ldots, R_k) \), where \( R_i := (R_{i,1}, \ldots, R_{i,n_i}) \), if
\[
(I_{\mathcal{E}} \otimes R_{s,t})A(I_{\mathcal{E}} \otimes R_{s,t}) = \delta_{st}A, \quad s, t \in \{1, \ldots, n_i\},
\]
for every \( i \in \{1, \ldots, k\} \). Let \( \mathcal{T}_n \) be the set of all \( k \)-multi-Toeplitz operators on \( \mathcal{E} \otimes \bigotimes_{i=1}^k F^2(H_{n_i}) \). In [29], we proved that
\[
\mathcal{T}_n = \text{span}\{f^*g : f, g \in B(\mathcal{E}) \otimes_{\min} A_n\}^{-\text{SOT}} = \text{span}\{f^*g : f, g \in B(\mathcal{E}) \otimes_{\min} A_n\}^{-\text{WOT}},
\]
where \( A_n \) is the polyball algebra. In what follows, we provide a Harnack type inequality for positive free \( k \)-pluriharmonic function on the regular polyballs.

**Theorem 1.2.** Let \( F \) be a positive free \( k \)-pluriharmonic function on the regular polyball \( B_n \), with operator coefficients in \( B(\mathcal{E}) \) and let \( 0 \leq r < 1 \). Then
\[
F(0) \left( \frac{1 - r}{1 + r} \right)^k \leq F(X) \leq F(0) \left( \frac{1 + r}{1 - r} \right)^k
\]
for any \( X \in rB_n(\mathcal{H})^- \).
Proof. If $F$ is a positive free $k$-pluriharmonic function on the regular polyball $B_n$, with operator coefficients in $B(EE)$, then there exist coefficients $A(\alpha_1,\ldots,\alpha_k;\beta_1,\ldots,\beta_k) \in B(EE)$ with $\alpha_i, \beta_i \in \mathbb{F}_n^+$, $|\alpha_i| = m_{i-1}, |\beta_i| = m_i^+$ such that, for any $r \in [0,1)$,

$$F(rS) = \sum_{m_1 \in \mathbb{Z}} \cdots \sum_{m_k \in \mathbb{Z}} \sum_{\alpha_i, \beta_i \in \mathbb{F}_n^+, i \in \{1,\ldots,k\}} A(\alpha_1,\ldots,\alpha_k;\beta_1,\ldots,\beta_k) \otimes r^{\sum_{i=1}^k(|\alpha_i| + |\beta_i|)} S_{1,\alpha_1}S_{1,\beta_1} \otimes \cdots \otimes S_{k,\alpha_k}S_{k,\beta_k},$$

where the multi-series is convergent in the operator norm topology and the sum does not depend on the order of the series. In this case, $F(rS)$ is a positive $k$-multi-Toeplitz operator and

$$F(rS) = F(rS_1,\ldots,rS_k) = \sum_{m_k \in \mathbb{Z}} \sum_{|\alpha_i| = m_{i-1}, |\beta_i| = m_i^+} C(\alpha_k;\beta_k) \otimes r^{|\alpha_k| + |\beta_k|} S_{k,\alpha_k}S_{k,\beta_k}^*.$$

Note that $F(rS)$ is also a positive 1-multi-Toeplitz operator with respect to $R_k = (R_{k,1},\ldots,R_{k,n_k})$, with coefficients in $B(EE) \otimes_{\min} B(F^2(H_{n_k})) \otimes \cdots \otimes \min B(F^2(H_{n_k-1}))$. Applying the noncommutative Berezin transform at $X_k = (X_{k,1},\ldots,X_{k,n_k})$, we deduce that

$$G(X_k) := \sum_{m_k \in \mathbb{Z}} \sum_{|\alpha_i| = m_{i-1}, |\beta_i| = m_i^+} C(\alpha_k;\beta_k) \otimes r^{\max(|\alpha_k|,|\beta_k|)} X_{k,\alpha_k}X_{k,\beta_k}^*,$$

is a positive pluriharmonic function on the unit ball $[B(H)^{n_k}]_1$. Using the Harnack type inequality from [21], we deduce that

$$G(0) \frac{1-r}{1+r} \leq G(rS_k) \leq G(0) \frac{1+r}{1-r}.$$ 

Note that $G(rS_k) = F(rS_1,\ldots,rS_k)$ and $G(0) = F(rS_1,\ldots,rS_{k-1},0)$ is a positive $(k-1)$-multi-Toeplitz operator. As above, we obtain

$$F(rS_1,\ldots,rS_{k-1},0,0) \frac{1-r}{1+r} \leq F(rS_1,\ldots,rS_{k-1},0) \leq F(rS_1,\ldots,rS_{k-2},0,0) \frac{1+r}{1-r}.$$ 

Continuing this process, we obtain

$$F(0,\ldots,0) \frac{1-r}{1+r} \leq F(rS_1,0,\ldots,0) \leq F(0,\ldots,0) \frac{1+r}{1-r}.$$ 

Now, combining all these inequalities, we deduce that

$$F(0) \left( \frac{1-r}{1+r} \right)^k \leq F(rS_1,\ldots,rS_k) \leq F(0) \left( \frac{1+r}{1-r} \right)^k$$

for any $r \in [0,1)$. Applying the Berezin transform at $X \in rB_n(\mathcal{H})$ to the latter inequalities, we complete the proof.

We define the **free pluriharmonic Poisson kernel** on the regular polyball $B_n$ by setting

$$\mathcal{P}(R,X) := \sum_{m_1 \in \mathbb{Z}} \cdots \sum_{m_k \in \mathbb{Z}} \sum_{\alpha_i, \beta_i \in \mathbb{F}_n^+, i \in \{1,\ldots,k\}} R_{1,\alpha_1} \cdots R_{k,\beta_k} \otimes X_{1,\alpha_1} \cdots X_{k,\beta_k},$$

for any $X \in B_n(\mathcal{H})$, where the convergence is in the operator norm topology, and $\tilde{\alpha}_i = g_{\alpha_i} \cdots g_{\alpha_1}$ if $\alpha_i = g_{\alpha_i} \cdots g_{\alpha_1} \in \mathbb{F}_n^+$. According to [20], the map $X \mapsto \mathcal{P}(R,X)$ is a positive free $k$-pluriharmonic function on $B_n(\mathcal{H})$. In this case, Theorem 1.2 implies

$$\left( \frac{1-r}{1+r} \right)^k I \leq \mathcal{P}(R,X) \leq \left( \frac{1+r}{1-r} \right)^k I$$

for any $X \in rB_n(\mathcal{H})$.

Now, we introduce a preorder relation $\preceq$ on the closed ball $B_n(\mathcal{H})$. If $A$ and $B$ are in $B_n(\mathcal{H})$, we say that $A$ is **Poisson dominated** by $B$, and denote $A \preceq B$, if there exists $c > 0$ such that

$$\mathcal{P}(R,rA) \leq c^2 \mathcal{P}(R,rB)$$
for any \( r \in [0, 1) \). When we want to emphasize the constant \( c \), we write \( P < c B \). Since \( \prec \) is a preorder relation on \( B_n(\mathcal{H})^- \), it induces an equivalence relation \( P \) on \( B_n(\mathcal{H})^- \), which we call Poisson equivalence. The equivalence classes with respect to \( P \) are called Poisson parts of \( B_n(\mathcal{H})^- \). Let \( A \) and \( B \) are in \( B_n(\mathcal{H})^- \). It is easy to see that \( A \) and \( B \) are Poisson equivalent (we denote \( A P B \)) if and only if there exists \( c \geq 1 \) such that

\[
\frac{1}{c^2} \mathcal{P}(R, rB) \leq \mathcal{P}(R, rA) \leq c^2 \mathcal{P}(R, rB)
\]

for any \( r \in [0, 1) \). We also use the notation \( A P B \) if \( A P B \) and \( B P A \). We remark that in the particular case when \( k = 1 \), the Poisson equivalence coincides with the Harnack equivalence (see [21]).

We recall that the spectral radius of an \( n_i \)-tuple \( A_i := (A_{i,1}, \ldots, A_{i,n_i}) \) of operators is given by

\[
r(A_i) := \lim_{p \to \infty} \left\| \sum_{\beta_i \in \mathbb{F}_n^+_{i}, |eta_i| = p} A_{i,\beta_i} A_{i,\beta_i}^* \right\|^{1/2p}.
\]

When \( n_i = 1 \) we find again the usual spectral radius of an operator.

**Lemma 1.3.** Let \( A = (A_1, \ldots, A_k) \in B_n(\mathcal{H})^- \). Then \( A P 0 \) if and only if the joint spectral radius \( r(A_i) < 1 \) for any \( i \in \{1, \ldots, k\} \).

**Proof.** Assume that \( A P 0 \). Then there is \( c > 0 \) such that \( \mathcal{P}(R, rA) \leq c^2 I \) for any \( r \in [0, 1) \). Set

\[
w := \sum_{\alpha \in \mathbb{F}_n^+_{i}} e_{\alpha}^i \otimes h_{\alpha} \in F^2(H_{n_i}) \otimes \mathcal{H},
\]

where \( h_{\alpha} \in \mathcal{H} \) and \( \sum_{\alpha \in \mathbb{F}_n^+_{i}} \|h_{\alpha}\|^2 < \infty \), and let

\[
\tilde{w} := \sum_{\alpha \in \mathbb{F}_n^+_{i}} (1 \otimes \cdots \otimes 1 \otimes e_{\alpha}^i \otimes 1 \otimes \cdots \otimes 1) \otimes h_{\alpha}
\]

be in \( F^2(H_{n_i}) \otimes \cdots \otimes F^2(H_{n_i}) \). Note that

\[
\langle P(R_i, rA_i)w, w \rangle = \langle \mathcal{P}(R, rA)\tilde{w}, \tilde{w} \rangle \leq c^2 \|\tilde{w}\|^2 = c^2 \|w\|^2
\]

for any \( w \in F^2(H_{n_i}) \otimes \mathcal{H} \), where \( P(R_i, rA_i) \) is the Poisson kernel associate with the row contraction \( rA_i \).

Applying Theorem 1.2 from [21], we deduce that \( A \prec 0 \) and, consequently, \( r(A_i) < 1 \).

Conversely, let \( A = (A_1, \ldots, A_k) \in B_n(\mathcal{H})^- \) and assume that \( r(A_i) < 1 \) for any \( i \in \{1, \ldots, k\} \). Due to the fact that, for each \( i \in \{1, \ldots, k\} \), \( R_{i,1}, \ldots, R_{i,n_i} \) are isometries with orthogonal ranges, we have

\[
r(A_i) = r(R_{i,1} \otimes A_{i,1}^* + \cdots + R_{i,n_i} \otimes A_{i,n_i}^*).
\]

Setting \( A_i := R_{i,1}^* \otimes A_{i,1} + \cdots + R_{i,n_i}^* \otimes A_{i,n_i} \), we have \( r(A_i) = r(A_i) < 1 \) and deduce that the spectrum of \( A_i \) is included in \( \mathbb{D} := \{ \zeta \in \mathbb{C} : |\zeta| < 1 \} \). Since \( \lambda \mapsto (\lambda I - A_i)^{-1} \) is continuous on the resolvent \( \rho(A_i) \), one can easily see that \( M_i := \text{sup}_{r \in (0,1)} \| (I - rA_i)^{-1} \| < \infty \). According to [29], \( \mathcal{P}(R, rS) \) is equal to

\[
\prod_{i=1}^k (I - R_{i,1}^* \otimes S_{i,1} - \cdots - R_{i,n_i}^* \otimes S_{i,n_i})^{-1}(I \otimes \Delta r_S(I)) \prod_{i=1}^k (I - R_{i,1} \otimes S_{i,1}^* - \cdots - R_{i,n_i} \otimes S_{i,n_i})^{-1}
\]

for any \( r \in [0, 1) \). Since the noncommutative Berezin transform \( B_A \) is continuous in the operator norm and completely positive, so is \( id \otimes B_A \). Using the map \( id \otimes B_A \), we deduce that

\[
\mathcal{P}(R, rA) = (id \otimes B_A) [\mathcal{P}(R, rS)]
\]

\[
= \prod_{i=1}^k (I - rA_i)^{-1} (I \otimes \Delta r_A(I)) \prod_{i=1}^k (I - rA_i)^{-1}
\]

\[
\leq M_1^2 \cdots M_k^2 I
\]

for any \( r \in [0, 1) \). This shows that \( A P 0 \) and completes the proof. \( \square \)
Now, we show that the Harnack (resp. Poisson) equivalence class containing the zero element coincides with the open polyball $B_n(H)$.

**Theorem 1.4.** Let $A = (A_1, \ldots, A_k) \in B_n(H)^\sim$. Then the following statements are equivalent.

(i) $A^\sim H 0$;
(ii) $r(A_i) < 1$ for any $i \in \{1, \ldots, k\}$ and there exists $a > 0$ such that
\[ P(R, rA) \geq aI, \quad r \in [0, 1); \]
(iii) $A \in B_n(H)$;
(iv) $A^\sim P 0$.

**Proof.** Note that if $F : B_n(H) \to B(\mathcal{E}) \otimes_{\min} B(\mathcal{H})$ is a free $k$-pluriharmonic function then, for each $i \in \{1, \ldots, k\}$, the map
\[ X_i \mapsto F(0, \ldots, 0, X_i, 0, \ldots, 0), \quad X_i \in [B(\mathcal{H})^{n_i}]_1, \]
is a pluriharmonic function on $[B(\mathcal{H})^{n_i}]_1$. Conversely, if $F_i : [B(\mathcal{H})^{n_i}]_1 \to B(\mathcal{E}) \otimes_{\min} B(\mathcal{H})$ is a free pluriharmonic function on the open unit ball $[B(\mathcal{H})^{n_i}]_1$, then the map $X = (X_1, \ldots, X_k) \mapsto F_i(X_i)$ is a free $k$-pluriharmonic function on $B_n(H)$. Consequently, if $A = (A_1, \ldots, A_k)$ and $B = (B_1, \ldots, B_k)$ are in the closed regular polyball $B_n(H)^\sim$ and $A_i^\sim H B$, then $A_i \sim B_i$ for any $i \in \{1, \ldots, k\}$.

Now, assume that $A^\sim H 0$. Due to the remark above, we must have $A_i^\sim H 0$ for any $i \in \{1, \ldots, k\}$. Applying Theorem 1.6 from [21] to $A_i$, we deduce that $A_i \in [B(\mathcal{H})^{n_i}]_1$. In particular, the joint spectral radius $r(A_i) < 1$. Since $A^\sim H 0$ and the map $X \mapsto P(R, X)$ is a positive free $k$-pluriharmonic function on $B_n(H)$, there is $a > 0$ such that $P(R, rA) \geq aI$ for any $r \in [0, 1)$. Therefore, item (ii) holds.

Now, we prove that (ii) $\implies$ (iii). As in the proof of Lemma 1.3, we have
\[ P(R, rA) = \prod_{i=1}^k (I - rA_i)^{-1} (I \otimes \Delta_{rA}(I)) \prod_{i=1}^k (I - rA_i^*)^{-1}, \]
where $A_i := R_{i,1}^* \otimes A_{i,1} + \cdots + R_{i,n_i}^* \otimes A_{i,n_i}$. Moreover, the spectral radius of the operator $A_i$ coincides with $r(A_i) < 1$ and
\[ 0 < M := \sup_{r \in [0,1]} \| \prod_{i=1}^k (I - rA_i)^{-1} \| < \infty. \]

Hence and using the fact that $P(R, rA) \geq aI$ for any $r \in [0, 1)$, we obtain
\[ I \otimes \Delta_{rA}(I) \geq \frac{c}{M^2} I \otimes I, \quad r \in [0, 1). \]

Hence $\Delta_{A}(I) \geq \frac{c}{M^2} I$, which shows that $A \in B_n(H)$. Therefore, item (iii) holds. To prove the implication (iii) $\implies$ (i), assume that $A \in B_n(H)$. Define
\[ \gamma_A = m_{B_n}(A) := \inf \{ t > 0 : A \in tB_n(H) \}. \]
Due to Proposition 2.6 from [29], we have $m_{B_n}(A) < 1$ and, consequently, $rA \in \gamma_A B_n(H)^\sim$ for any $r \in [0, 1)$. Applying the Harnack type inequality of Theorem 1.2, we obtain the inequalities
\[ F(0) \left( \frac{1 - \gamma_A}{1 + \gamma_A} \right)^k \leq F(rA) \leq F(0) \left( \frac{1 + \gamma_A}{1 - \gamma_A} \right)^k \]
for any positive free $k$-pluriharmonic function $F$ on $B_n(H)$ and any $r \in [0, 1)$. Consequently, $A^\sim H 0$, which completes the proof of the implication (iii) $\implies$ (i). Note that if $A^\sim H 0$, then we automatically have $A^\sim P 0$. On the other hand, taking into account Lemma 1.3 one can easily see that (iv) $\implies$ (ii). This completes the proof. \qed
2. Hyperbolic metric on the Harnack parts of the closed polyball

In this section, we introduce a hyperbolic type metric on the Harnack parts of the closed regular polyball and show that it is invariant under the automorphism group of all free biholomorphic functions of the polyball. We provide a Schwarz-Pick type result for free holomorphic functions on regular polyballs with respect to the hyperbolic metric.

Given $A, B \in B_n(\mathcal{H})^-$ in the same Harnack part, i.e. $A \overset{\mathcal{H}}{\sim} B$, we introduce

$$\omega_H(A, B) := \inf \left\{ c > 1 : A \overset{\mathcal{H}}{\sim} cB \right\}.$$ 

**Lemma 2.1.** Let $\Delta$ be a Harnack part of $B_n(\mathcal{H})^-$ and let $A, B, C \in \Delta$. Then the following properties hold:

(i) $\omega_H(A, B) \geq 1$;
(ii) $\omega_H(A, B) = 1$ if and only if $A = B$;
(iii) $\omega_H(A, B) = \omega_H(B, A)$;
(iv) $\omega_H(A, C) \leq \omega_H(A, B) \omega_H(B, C)$.

**Proof.** First, note that (i) and (iii) are consequences of the definition of $\omega_H(A, B)$ and relation (1.1). If $\omega_H(A, B) = 1$, then there is a sequence $c_n > 1$ with $c_n \to 1$ such that

$$\frac{1}{c_n^2} F(rB) \leq F(rA) \leq c_n^2 F(rB)$$

for any positive free $k$-pluriharmonic function $F$ with operator-valued coefficients and any $r \in [0, 1)$. In particular, since the map $X \mapsto P(R, X)$ is a positive free $k$-pluriharmonic function on $B_n(\mathcal{H})$, we have

$$\frac{1}{c_n^2} P(R, rB) \leq P(R, rA) \leq c_n^2 P(R, rB)$$

for any $n \in \mathbb{N}$ and $r \in [0, 1)$. Since $c_n \to 1$ we deduce that $P(R, rA) = P(R, rA)$ for any $r \in [0, 1)$. For each $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, n_i\}$, let $x := 1 \otimes \cdots \otimes 1 \otimes e_j^i \otimes 1 \otimes \cdots \otimes 1$ and $y := 1 \otimes \cdots \otimes 1$ be in $F^2(H_{n_i}) \otimes \cdots \otimes F^2(H_{n_k})$. If $\alpha_i, \beta_i \in F_n^+$ with $|\alpha_i| = m_i^-$, $|\beta_i| = m_i^+$, we have

$$\langle R_{1, \bar{a}_1} \cdots R_{k, \bar{a}_k} R_{1, \bar{b}_1} \cdots R_{k, \bar{b}_k} x, y \rangle = 1$$

if $\beta_s = g_s^*$ for $s \in \{1, \ldots, k\}$, and $\alpha_s = g_s^*$ for $s \in \{1, \ldots, k\} \setminus \{i\}$ and $\alpha_i = g_j^*$, while

$$\langle R_{1, \bar{a}_1} \cdots R_{k, \bar{a}_k} R_{1, \bar{b}_1} \cdots R_{k, \bar{b}_k} x, y \rangle = 0$$

otherwise. Consequently, we have

$$\langle P(R, X)(x \otimes h), y \otimes \ell \rangle = \langle X_{i,j}h, \ell \rangle$$

for any $X := (X_1, \ldots, X_k) \in B_n(\mathcal{H})$ with $X_i = (X_{i,1}, \ldots, X_{i,n_i})$ and any $h, \ell \in \mathcal{H}$. Now, it is clear that relation (2.1) holds.

Due to the definition (1.1), for any positive free $k$-pluriharmonic function $F$ with operator-valued coefficients and any $r \in [0, 1)$, we have

$$\frac{1}{\omega_H(A, B)^2} F(rB) \leq F(rA) \leq \omega_H(A, B)^2 F(rB)$$

and

$$\frac{1}{\omega_H(B, C)^2} F(rC) \leq F(rB) \leq \omega_H(B, C)^2 F(rC).$$

Combining these inequalities, we deduce that

$$\frac{1}{\omega_H(A, B)^2 \omega_H(B, C)^2} F(rC) \leq F(rA) \leq \omega_H(A, B)^2 \omega_H(B, C)^2 F(rC).$$

Hence, we obtain that $\omega_H(A, C) \leq \omega_H(A, B) \omega_H(B, C)$, which completes the proof.

Now, we can introduce a hyperbolic type metric on the Harnack parts of $B_n(\mathcal{H})^-$. 

\[\square\]
Proposition 2.2. Let $\Delta$ be a Harnack part of $B_n(H)^-$ and define $\delta_H : \Delta \times \Delta \rightarrow \mathbb{R}^+$ by setting
$$\delta_H(A, B) := \ln \omega_H(A, B), \quad A, B \in \Delta.$$ Then $\delta_H$ is a metric on $\Delta$.

Proof. The result follows from Lemma 2.1.

Theorem 2.3. Let $F$ be a free $k$-pluriharmonic function on the regular polyball $B_n$ with operator coefficients in $B(E)$ and let $G = (G_1, \ldots, G_k) : B_n \rightarrow B_n$ be a free holomorphic function such that each $G_i : [B(H)^n]|_1 \rightarrow [B(H)^n]|_1$ is a free holomorphic function. Then $F \circ G$ is a free $k$-pluriharmonic function on $B_n$.

Proof. According to [29], $F$ is a free $k$-pluriharmonic function on the regular polyball $B_n$ with operator coefficients in $B(E)$ if and only if there exist coefficients $A_{(\alpha_1, \ldots, \alpha_k; \beta_1, \ldots, \beta_k)} \in B(E)$ with $\alpha_i, \beta_i \in F_{n_i}^+$, $|\alpha_i| = m^{-}_i, |\beta_i| = m^{+}_i$ such that, for any $r \in [0, 1)$,
$$F(rS) = \sum_{m_1 \in \mathbb{Z}} \cdots \sum_{m_k \in \mathbb{Z}} \sum_{\alpha_i, \beta_i \in F_{n_i}^+} A_{(\alpha_1, \ldots, \alpha_k; \beta_1, \ldots, \beta_k)} \otimes r^{\sum_{i=1}^k (|\alpha_i| + |\beta_i|)} S_{1, \alpha_1} \cdots S_{k, \alpha_k} S^*_{1, \beta_1} \cdots S^*_{k, \beta_k},$$
where the multi-series is convergent in the operator norm topology and its sum does not depend on the order of the series. In this case, $F(rS)$ is a $k$-multi-Toeplitz operator in $\mathcal{A}_n$,
$$\text{span}\{f^*g : f, g \in B(E) \otimes \text{min} \mathcal{A}_n\} - \|\cdot\|,$$
where $\mathcal{A}_n$ is the polyball algebra.

Let $G = (G_1, \ldots, G_k) : B_n \rightarrow B_n$ be a free holomorphic function with $G_i = (G_{i,1}, \ldots, G_{i,n_i})$, where each $G_i : [B(H)^n]|_1 \rightarrow [B(H)^n]|_1$ is a free holomorphic function. According to Proposition 2.2 from [28], range $G \in B_n(H)$ if and only if $G(rS) \in B_n(\otimes_{i=1}^k F^2(H_{n_i}))$ for any $r \in [0, 1)$, where $S = (S_{1,1}, \ldots, S_{n,1})$, $S_i = (S_{i,1}, \ldots, S_{i,n_i})$, is the universal model of the regular polyball $B_n$. Consequently, $F(G(rS))$ is equal to
$$\sum_{m_1 \in \mathbb{Z}} \cdots \sum_{m_k \in \mathbb{Z}} \sum_{\alpha_i, \beta_i \in F_{n_i}^+} A_{(\alpha_1, \ldots, \alpha_k; \beta_1, \ldots, \beta_k)} \otimes G_{1, \alpha_1}(rS_1) \cdots G_{k, \alpha_k}(rS_k) G_{1, \beta_1}(rS_1)^* \cdots G_{k, \beta_k}(rS_k)^*,$$
where the multi-series is convergent in the operator norm topology. Now, we prove that $F(G(rS))$ is a $k$-multi-Toeplitz operator with respect to the universal model $R$. It is enough to show that
$$T := G_{1, \alpha_1}(rS_1) \cdots G_{k, \alpha_k}(rS_k) G_{1, \beta_1}(rS_1)^* \cdots G_{k, \beta_k}(rS_k)^*$$
is $k$-multi-Toeplitz operator with respect to $R$, when $\alpha_i, \beta_i \in F_{n_i}^+$, $|\alpha_i| = m^{-}_i, |\beta_i| = m^{+}_i$. First, note that, for each $i \in \{1, \ldots, k\}$ and $s, t \in \{1, \ldots, n_i\}$
$$R^*_{i,s} G_{i, \alpha_i}(rS_i) G_{i, \beta_i}(rS_i)^* R_{i,t} = \delta_{s,t} G_{i, \alpha_i}(rS_i) G_{i, \beta_i}(rS_i)^*.$$
Hence, we deduce that
$$R^*_{i,s} T R_{i,t} = \left[ \prod_{p \in \{1, \ldots, k\}} G_{p, \alpha_p}(rS_p) \right] R^*_{i,s} G_{i, \alpha_i}(rS_i) G_{i, \beta_i}(rS_i)^* R_{i,t} \left[ \prod_{p \in \{1, \ldots, k\}} G_{p, \alpha_p}(rS_p) \right]^*$$
for each $i \in \{1, \ldots, k\}$, which proves our assertion.

Using Theorem 1.5 from [29] we deduce that, for each $r \in [0, 1)$, $F(G(rS))$ has a unique Fourier representation
$$\varphi(S) := \sum_{m_1 \in \mathbb{Z}} \cdots \sum_{m_k \in \mathbb{Z}} \sum_{\alpha_i, \beta_i \in F_{n_i}^+} B^{(r)}_{(\alpha_1, \ldots, \alpha_k; \sigma_1, \ldots, \sigma_k; \omega_1, \ldots, \omega_k)} \otimes r^{\sum_{i=1}^k (|\sigma_i| + |\omega_i|)} S_{1, \sigma_1} \cdots S_{k, \sigma_k} S^*_{1, \omega_1} \cdots S^*_{k, \omega_k}.$$
On the other hand, note that if

\[(2.2)\]

for every holomorphic function on

Theorem 2.4.

\[B_k\]

can see that each coefficient

where

is convergent in the operator norm topology for any \(t \in [0, 1]\). Moreover \(F(G(rS)) = \text{SOT-}\lim_{t \to 1} \varphi(tS)\) and

\[(2.1)\]

where \(x := x_1 \otimes \cdots \otimes x_k\), \(y = y_1 \otimes \cdots \otimes y_k\) with

\[\left\{ \begin{array}{ll}
  x_i = e^{i} & \text{if } m_i \geq 0 \\
  x_i = 1 & \text{if } m_i < 0
\end{array}\right.\]

for every \(i \in \{1, \ldots, k\}\). We need to show that each coefficient \(B_{(\sigma_1, \ldots, \sigma_k; \omega_1, \ldots, \omega_k)}^{(r)}\) does not depend on \(r \in [0, 1]\). Indeed, using the relations above, we deduce that

\[\langle F(G(rS)) \rangle(h \otimes x), \ell \otimes y)\]

On the other hand, note that if \(m_i \geq 0\), then \(x_i = e^{i}_{\omega_i}\) and \(y_i = 1\). Consequently, we have

\[\langle G_{i, \beta}(rS_1)^{\ast}x_i, G_{i, \alpha}(rS_1)^{\ast}y_i \rangle = \langle G_{i, \beta}(rS_1)^{\ast}x_i, G_{i, \alpha}(rS_1)^{\ast}y_i \rangle = G_{i, \alpha}(0) \langle e^{i}_{\omega_i}, G_{i, \beta}(rS_1)^{\ast}y_i \rangle = r^{\omega_i}M(\alpha_i, \beta_i, \omega_i),\]

where \(M(\alpha_i, \beta_i, \omega_i)\) is a constant which does not depend on \(r\). Similarly, if \(m_i < 0\), we deduce that

\[\langle G_{i, \beta}(rS_1)^{\ast}x_i, G_{i, \alpha}(rS_1)^{\ast}y_i \rangle = r^{\omega_i}M(\alpha_i, \beta_i, \sigma_i),\]

where \(M(\alpha_i, \beta_i, \sigma_i)\) is a constant which does not depend on \(r\). Now, using relations \((2.1)\) and \((2.2)\), one can see that each coefficient \(B_{(\sigma_1, \ldots, \sigma_k; \omega_1, \ldots, \omega_k)}^{(r)}\) does not depend on \(r \in [0, 1]\). Therefore we can write

\[B_{(\sigma_1, \ldots, \sigma_k; \omega_1, \ldots, \omega_k)}^{(r)} = B_{(\sigma_1, \ldots, \sigma_k; \omega_1, \ldots, \omega_k)}^{(0)}\].

Since, due to the considerations above, the multi-series

\[\sum_{m_1 \in \mathbb{Z}} \cdots \sum_{m_k \in \mathbb{Z}} \sum_{\sigma, \omega \in \mathbb{F}_{+}^{k}, i \in \{1, \ldots, k\}} B_{(\sigma_1, \ldots, \sigma_k; \omega_1, \ldots, \omega_k)} \otimes (rt)^{\sum_{i=1}^{k}(|\sigma_i| + |\omega_i|)}S_{1, \sigma_1} \cdots S_{k, \sigma_k}S_{1, \omega_1} \cdots S_{k, \omega_k},\]

is convergent in the operator norm topology for any \(t, r \in [0, 1]\), we conclude that \(F \circ G\) is a free \(k\)-pluriharmonic function. The proof is complete. \(\square\)

The following result is a Schwarz-Pick lemma for free holomorphic functions on the regular polyball \(B_n\) with operator-valued coefficients, with respect to the hyperbolic metric.

**Theorem 2.4.** Let \(G : B_n(\mathcal{H}) \to B_n(\mathcal{H})^\ast\) be free holomorphic functions such that

\[G(X) := (G_1(X_1), \ldots, G_k(X_k)) \in B_n(\mathcal{H})^\ast, \quad X := (X_1, \ldots, X_k) \in B_n(\mathcal{H}),\]

where \(G_i := (G_{i,1}, \ldots, G_{i,n_1}) : [B(\mathcal{H})^{n_1}] \to [B(\mathcal{H})^{n_1}]^\ast\) and each \(G_{i,j} : [B(\mathcal{H})^{n_1}] \to B(\mathcal{H})\) is a free holomorphic function on \([B(\mathcal{H})^{n_1}]\). If \(X, Y \in B_n(\mathcal{H})\), then \(G(X)^{\mathcal{H}}G(Y)\) and

\[\delta_\mathcal{H}(G(X), G(Y)) \leq \delta_\mathcal{H}(X, Y),\]

where \(\delta_\mathcal{H}\) is the hyperbolic metric defined on the Harnack parts of the closed polyball \(B_n(\mathcal{H})^\ast\).
Proof. Let $F : \mathcal{B}_n(\mathcal{H}) \to B(\mathcal{E}) \otimes_{\text{min}} B(\mathcal{H})$ be a positive free $k$-pluriharmonic function on $\mathcal{B}_n(\mathcal{H})$ with coefficients in $B(\mathcal{E})$. Due to Theorem 2.4, the map $F \circ \gamma \mathcal{G}$ is a positive free $k$-pluriharmonic function on $\mathcal{B}_n(\mathcal{H})$ for any $\gamma \in [0, 1)$. If $X, Y \in \mathcal{B}_n(\mathcal{H})$, Theorem 1.4 shows that $A \overset{H}{\sim} \mathcal{B}$ for some $c \geq 1$. Consequently,
\[ \frac{1}{c^2}(F \circ \gamma \mathcal{G})(rY) \leq (F \circ \gamma \mathcal{G})(rX) \leq c^2(F \circ \gamma \mathcal{G})(rY) \]
for any $\gamma, r \in [0, 1)$. Taking $r \to 1$ and using the continuity of $F \circ \gamma \mathcal{G}$ on $\mathcal{B}_n(\mathcal{H})$ in the operator norm topology, we deduce that
\[ \frac{1}{c^2}(F \circ \gamma \mathcal{G})(Y) \leq (F \circ \gamma \mathcal{G})(X) \leq c^2(F \circ \gamma \mathcal{G})(Y) \]
for any $\gamma \in [0, 1)$. Hence $\mathcal{G}(X) \overset{H}{\sim} \mathcal{G}(Y)$. Using the definition of the hyperbolic metric defined on the Harnack parts of the closed polyball $\mathcal{B}_n(\mathcal{H})^-$, one can complete the proof. □

The next result is a Schwarz-Pick lemma for free holomorphic functions from the regular polyball $\mathcal{B}_n(\mathcal{H})$ to the unit ball $[B(\mathcal{H})^m]_1$, with respect to the hyperbolic metric.

**Proposition 2.5.** Let $\Phi = (\Phi_1, \ldots, \Phi_m) : \mathcal{B}_n(\mathcal{H}) \to [B(\mathcal{H})^m]_1$ be a free holomorphic function on the regular polyball. If $X, Y \in \mathcal{B}_n(\mathcal{H})$, then $\Phi(X) \overset{H}{\sim} \Phi(Y)$ and
\[ \delta_{\mathcal{H}}(\Phi(X), \Phi(Y)) \leq \delta_{\mathcal{H}}(X, Y), \]
where $\delta_{\mathcal{H}}$ is the hyperbolic metric defined on the Harnack parts of $[B(\mathcal{H})^m]_1$ and on the polyball $\mathcal{B}_n(\mathcal{H})$, respectively.

**Proof.** Let $F : [B(\mathcal{H})^m]_1 \to B(\mathcal{E}) \otimes_{\text{min}} B(\mathcal{H})$ be a positive free pluriharmonic function. According to [19], there are some operators $C(\alpha) \in B(\mathcal{E})$, $\alpha \in \mathbb{F}_m$, such that
\[ F(Y_1, \ldots, Y_m) = \sum_{q=1}^{\infty} \sum_{\alpha \in \mathbb{F}_m, |\alpha|=q} C(\alpha) \otimes Y_\alpha^* + \sum_{q=0}^{\infty} \sum_{\alpha \in \mathbb{F}_m, |\alpha|=q} C(\alpha) \otimes Y_\alpha, \quad (Y_1, \ldots, Y_m) \in [B(\mathcal{H})^m]_1, \]
where the convergence is in the operator norm topology. If $G = (G_1, \ldots, G_m) : \mathcal{B}_n(\mathcal{H}) \to [B(\mathcal{H})^m]_1$ is a free holomorphic function on the regular polyball, i.e., each $G_i : \mathcal{B}_n(\mathcal{H}) \to B(\mathcal{H})$ is free holomorphic, then, using Theorem 2.4 from [28], we deduce that $F \circ G$ is a positive free $k$-pluriharmonic function on $\mathcal{B}_n(\mathcal{H})$. Applying this result when $G_i = r \Phi G_j$, $r \in [0, 1)$, and $j \in \{1, \ldots, m\}$, we have that $F \circ r \Phi$ is a positive free $k$-pluriharmonic function. Now, the rest of the proof is similar to that of Theorem 2.4. We leave it to the reader. □

Let $n = (n_1, \ldots, n_k) \in \mathbb{N}_k$ and let $\sigma$ be a permutation of the set $\{1, \ldots, k\}$ such that $n_{\sigma(i)} = n_i$. Then the map $p_\sigma : \mathcal{B}_n(\mathcal{H})^- \to \mathcal{B}_n(\mathcal{H})^-$, defined by
\[ p_\sigma(X) = (X_{\sigma(1)}, \ldots, X_{\sigma(k)}), \quad X := (X_1, \ldots, X_k) \in \mathcal{B}_n(\mathcal{H})^-, \]
is a homeomorphism of $\mathcal{B}_n(\mathcal{H})^-$ and $p_\sigma|_{\mathcal{B}_n(\mathcal{H})}$ a free holomorphic automorphism of $\mathcal{B}_n(\mathcal{H})$. If each $U_i \in \mathbb{C}^{n_i}$, $i \in \{1, \ldots, k\}$, is a unitary operator and $U \in B(\mathbb{C}^{n_1 + \cdots + n_k})$ is the direct sum $U = U_1 \oplus \cdots \oplus U_k$, then the map $\Phi_U : \mathcal{B}_n(\mathcal{H})^- \to \mathcal{B}_n(\mathcal{H})^-$ defined by $\Phi_U(X) := UX$ is also a free holomorphic automorphism of $\mathcal{B}_n(\mathcal{H})$ and homeomorphism of $\mathcal{B}_n(\mathcal{H})^-$. In [28], we obtained a complete description of the group $\text{Aut}(\mathcal{B}_n)$ of all free holomorphic automorphisms of the regular polyball $\mathcal{B}_n$. More precisely, we proved that if $\Psi \in \text{Aut}(\mathcal{B}_n)$ and $\lambda = (\lambda_1, \ldots, \lambda_k) := \Psi^{-1}(0)$, then there are unique unitary operators $U_i \in B(\mathbb{C}^{n_i})$, $i \in \{1, \ldots, k\}$, and a unique permutation $\sigma \in S_k$ with $n_{\sigma(i)} = n_i$ such that
\[ \Psi = p_\sigma \circ \Phi_U \circ \Psi_\lambda, \]
where $U := U_1 \oplus \cdots \oplus U_k$, $\Psi_\lambda := (\Psi_{\lambda_1}, \ldots, \Psi_{\lambda_k})$, and $\Psi_\lambda$ is the involutive free holomorphic automorphisms of the open unit ball $[B(\mathcal{H})^m]_1$ (see [28]). Moreover, we showed that if $\Psi = (\Psi_1, \ldots, \Psi_k)$ is the boundary function with respect to the universal model $\mathcal{S} = \{S_{i,j}\}$, i.e. $\Psi := \lim_{r \to 1} \Psi(r\mathcal{S})$, then the following statements hold:
such chain, we set \( \Psi \) is a free holomorphic function on the regular polyball \( \gamma B_n \) for some \( \gamma > 1 \).

(ii) \( \hat{\Psi} \) is a pure element in the polyball \( B_n(\otimes_{i=1}^{k} F^{2}(H_{\sigma}))^{-} \) and \( \Psi = \Psi(S) \). Each \( \hat{\Psi}_i = (\hat{\Psi}_{i,1}, \ldots, \hat{\Psi}_{i,n}) \) is an isometry with entries in the noncommutative disk algebra generated by \( S_{i,1}, \ldots, S_{i,n_i} \) and the identity.

(iii) \( \hat{\Psi} \) is a homeomorphism of \( B_n(H)^{-} \) onto \( B_n(H)^{-} \).

In what follows we show that the hyperbolic metric is invariant under the group \( \text{Aut}(B_n) \) of all free holomorphic automorphisms of \( B_n \).

**Theorem 2.6.** Let \( A \) and \( B \) be in \( B_n(H)^{-} \) such that \( A \sim B \). Then

\[
\delta_H(A, B) = \delta_H(\hat{\Psi}(A), \hat{\Psi}(B)), \quad \hat{\Psi} \in \text{Aut}(B_n).
\]

**Proof.** Let \( A \) and \( B \) be in \( B_n(H)^{-} \) and let \( \Psi \in \text{Aut}(B_n) \). First, we prove that if \( c \geq 1 \), then \( A \sim B \) if and only if \( \Psi(A) \sim_c \Psi(B) \). Assume that \( A \sim B \) and let \( F : B_n(H) \to B(\mathcal{E}) \otimes \min B(\mathcal{H}) \) be a positive free \( k \)-pluriharmonic function on \( B_n(H) \). Due to Theorem 2.3 and the remarks above, the map \( F \circ \gamma \Psi \) is a positive free \( k \)-pluriharmonic function on \( B_n(H) \) for any \( \gamma \in [0, 1] \). If \( A \sim B \), then

\[
(F \circ \gamma \Psi)(rA) \leq c^2(F \circ \gamma \Psi)(rB)
\]

for any \( \gamma, r \in [0, 1] \). Taking \( r \to 1 \) in the inequality above and using the fact that \( \gamma \Psi(rA) \to \gamma \Psi(A) \) and \( (F \circ \gamma \Psi)(rA) \to (F \circ \gamma \Psi)(A) \) in the operator norm topology, we deduce that

\[
F(\gamma \Psi(A)) \leq c^2 F(\gamma \Psi(B))
\]

for any \( \gamma \in [0, 1] \), which shows that \( \Psi(A) \sim_c \Psi(B) \). Conversely, if \( \Psi(A) \sim_c \Psi(B) \), then applying the direct implication to \( \Psi^{-1} \), we obtain that \( \Psi^{-1}(\Psi(A)) \sim \Psi^{-1}(\Psi(B)) \). Since \( \Psi^{-1} \circ \id = \id \) on \( B_n(H)^{-} \), we deduce that \( A \sim B \). Now, it is clear that \( A \sim B \) if and only if \( \Psi(A) \sim_c \Psi(B) \), which shows that \( \delta_H(A, B) = \delta_H(\Psi(A), \Psi(B)) \) for any \( \Psi \in \text{Aut}(B_n) \). The proof is complete. \( \square \)

Fix \( n = (n_1, \ldots, n_k) \in \mathbb{N}^k \) and let \( \sigma \) be a permutation of the set \( \{1, \ldots, k\} \) such that \( n_{\sigma(i)} = n_i \). Let \( \mathcal{G} \) be the set of all free holomorphic functions of the form \( p_{\sigma} \circ G : B_n(H) \to B(H)^{n_1+\cdots+n_k} \), where \( G \) has the form

\[
G(X) := (G_1(X_1), \ldots, G_k(X_k)) \in B_n(H), \quad X := (X_1, \ldots, X_k) \in B_n(H),
\]

with \( G_i := (G_{i,1}, \ldots, G_{i,n_i}) : [B(H)^{n_i}]_1 \to B(H)^{n_i} \) and each \( G_{i,j} : [B(H)^{n_i}]_1 \to B(H) \) is a free holomorphic function on \([B(H)^{n_i}]_1 \). Note that the group \( \text{Aut}(B_n) \) of all free holomorphic automorphisms of the regular polyball \( B_n \) is included in \( \mathcal{G} \).

In what follows we define a Kobayashi type pseudo-distance on domains \( M \subset B(H)^{n_1+\cdots+n_k} \) with respect to the hyperbolic metric \( \delta_H \) of the regular polyball \( B_n(H) \). Given two points \( X, Y \in M \), we consider a **chain of free holomorphic polyballs** from \( X \) to \( Y \). That is, a chain of elements

\[
X = X_0, X_1, \ldots, X_m = Y
\]

in \( M \), pairs \( (A^{(1)}, B^{(1)}), \ldots, (A^{(m)}, B^{(m)}) \) of elements in \( B_n(H) \), and free holomorphic functions \( F_1, \ldots, F_m \) in \( \mathcal{G} \) with values in \( M \) such that

\[
F_j(A^{(j)}) = X^{(i-1)} \quad \text{and} \quad F_j(B^{(j)}) = X^{(i)} \quad \text{for} \quad j = 1, \ldots, m.
\]

Denote this chain by \( \gamma \) and define its length by

\[
\ell(\gamma) := \delta_H(A^{(1)}, B^{(1)}) + \cdots + \delta(A^{(m)}, B^{(m)}),
\]

where \( \delta_H \) is the hyperbolic metric on \( B_n(H) \). We define the Kobayashi type pseudo-distance

\[
\delta^M_{B_n}(X, Y) := \inf \ell(\gamma),
\]

where the infimum is taken over all chains \( \gamma \) of free holomorphic polyballs from \( X \) to \( Y \). If there is no such chain, we set \( \delta^M_{B_n}(X, Y) = \infty \). In general, \( \delta^M_{B_n} \) is not a true distance on \( M \).
Proposition 2.7. If $M = B_n(\mathcal{H})$, then $\delta^M_{B_n}$ is a true distance and $\delta^M_{B_n} = \delta_H$.

Proof. Fix $X, Y \in B_n(\mathcal{H})$ and let $\gamma$ be a chain of free holomorphic polyballs from $X$ to $Y$, as described above. Since $\delta_H$ is a metric, Theorem 2.3 implies
$$\delta_H(X, Y) \leq \delta_H(X_0, X_1) + \cdots + \delta_H(X_{m-1}, X_m)$$
$$= \delta_H(F_1(A^{(1)}), F_1(B^{(1)})) + \cdots + \delta_H(F_k(A^{(m)}), F_k(B^{(m)}))$$
$$\leq \delta_H(A^{(1)}, B^{(1)}) + \cdots + \delta(A^{(m)}, B^{(m)}) = \ell(\gamma).$$

Taking the infimum over all chains $\gamma$ of free holomorphic polyballs from $X$ to $Y$, we deduce that $\delta_H(X, Y) \leq \delta^M_{B_n}(X, Y)$. Taking $F$ the identity on $B_n(\mathcal{H})$, we obtain $\delta_H(X, Y) = \delta^M_{B_n}(X, Y)$. The proof is complete. \hfill \Box

We remark that, in the particular case when $k = 1$ and $n_1 = 1$, Proposition 2.7 implies the well-known result that the Kobayashi distance on the open unit disc $D$ coincides with the Poincaré metric. On the other hand, if $k = 1$ and $n_1 \in \mathbb{N}$, we find again a result from [21].

3. A metric on the Poisson parts of the closed polyball

In this section, we introduce the Poisson metric $\delta_P$ on Poisson parts of the closed polyball and obtain an explicit formula for $\delta_P$ in terms of certain noncommutative Cauchy kernels acting on tensor products of full Fock spaces. We also prove that $\delta_P$ is a complete metric on $B_n(\mathcal{H})$ and that the $\delta_P$-topology coincides with the operator norm topology on $B_n(\mathcal{H})$.

Given $A, B \in B_n(\mathcal{H})^-$ in the same Poisson part, i.e. $A \overset{P}{\sim} B$, we introduce
$$\omega_P(A, B) := \inf \left\{ c > 1 : A \overset{P}{\sim} c B \right\}.$$

Lemma 3.1. Let $\Delta$ be a Poisson part of $B_n(\mathcal{H})^-$ and let $A, B, C \in \Delta$. Then the following properties hold:

(i) $\omega_P(A, B) \geq 1$;
(ii) $\omega_P(A, B) = 1$ if and only if $A = B$;
(iii) $\omega_P(A, B) = \omega_P(B, A)$;
(iv) $\omega_P(A, C) \leq \omega_P(A, B) \omega_P(B, C)$.

Proof. The proof is similar to Lemma 2.1. \hfill \Box

Now, we can introduce a metric on the Poisson parts of $B_n(\mathcal{H})^-$. Chang

Proposition 3.2. Let $\Delta$ be a Poisson part of $B_n(\mathcal{H})^-$ and define the function $\delta_P : \Delta \times \Delta \to \mathbb{R}^+$ by setting
$$\delta_P(A, B) := \ln \omega_P(A, B), \quad A, B \in \Delta.$$

Then $\delta_P$ is a metric on $\Delta$.

Proof. The result follows from Lemma 3.1. \hfill \Box

Theorem 3.3. If $A$ and $B$ are in the open ball $B_n(\mathcal{H})$, then
$$\delta_P(A, B) = \ln \max \left\{ \| C_A(R) C_B(R)^{-1} \|, \| C_B(R) C_A(R)^{-1} \| \right\},$$

where
$$C_X(R) := (I \otimes \Delta X(I))^{1/2} \prod_{i=1}^k (I - R_{n_i} \otimes X^*_{i, n_i} - \cdots - R_{i, n_i} \otimes X^*_{i, n_i})^{-1}$$
for any $X = (X_1, \ldots, X_k) \in B_n(\mathcal{H})$ with $X_i = (X_{i,1}, \ldots, X_{i,n_i})$. 

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Proof. Due to Theorem 1.4, the open polyball $B_n(H)$ is the Poisson part of $B_n(H)^-$ containing the zero element. Let $A, B \in B_n(H)$ and assume that $A \overset{p}{\sim} B$ for some $c \geq 1$. Then

$$
\frac{1}{c^2} \mathcal{P}(rA) \leq \mathcal{P}(rB) \leq c^2 \mathcal{P}(rA)
$$

for any $r \in [0, 1)$. According to Theorem 4.2 from [29], we have

$$
\mathcal{P}(rA) = C_X(r)^*C_X(r), \quad X \in B_n(H),
$$

where $C_X(r)$ is given in the theorem. Since the $X \mapsto \mathcal{P}(rA)$ is continuous on $B_n(H)$ in the operator norm topology, and taking $r \to 1$ in relation (3.1), we obtain that

$$
\frac{1}{c^2} C_B(r)^*C_B(r) \leq C_A(r)^*C_A(r) \leq c^2 C_B(r)^*C_B(r)
$$

which, due to the fact that $C_A(r)$ and $C_B(r)$ are invertible operators, implies

$$
(C_A(r)^{-1})^*C_B(r)^*C_B(r)C_A(r)^{-1} \leq c^2 I
$$

and

$$
(C_B(r)^{-1})^*C_A(r)^*C_B(r)C_A(r)^{-1} \leq c^2 I.
$$

Consequently,

$$
t := \max \left\{ \|C_A(r)C_B(r)^{-1}\|, \|C_B(r)C_A(r)^{-1}\| \right\} \leq c
$$

which implies $\ln t \leq \delta_P(A, B)$. To prove the reverse inequality, note that

$$
\|C_A(r)C_B(r)^{-1}\| \leq t \quad \text{and} \quad \|C_B(r)C_A(r)^{-1}\| \leq t.
$$

These inequalities imply

$$
\frac{1}{t^2} C_B(r)^*C_B(r) \leq C_A(r)^*C_A(r) \leq t^2 C_B(r)^*C_B(r)
$$

which is equivalent to

$$
\frac{1}{t^2} \mathcal{P}(rB) \leq \mathcal{P}(rA) \leq t^2 \mathcal{P}(rB)
$$

and shows that $t \geq 1$. Hence, $\frac{1}{t} \mathcal{P}(S, B) \leq \mathcal{P}(r, A) \leq t^2 \mathcal{P}(S, B)$. Applying the Berezin transform at $rX$, $r \in [0, 1)$, and using the fact that $B_{rR} \otimes id$ is a unital completely positive map, we deduce that $A \overset{t}{\sim} B$. Consequently, $\delta_P(A, B) \leq \ln t$, which completes the proof of the theorem. \qed

We remark that if $A$ and $B$ are in $B_n(H)^-$, then $A \overset{p}{\sim} B$ if and only if $rA \overset{p}{\sim} rB$ for any $r \in [0, 1)$ and $\sup_{r \in [0, 1)} \omega_P(rA, rB) < \infty$. In this case, Theorem 3.3 implies

$$
\delta_P(A, B) = \ln \max \left\{ \sup_{r \in [0, 1)} \|C_A(r)C_B(r)^{-1}\|, \sup_{r \in [0, 1)} \|C_B(r)C_A(r)^{-1}\| \right\}.
$$

According to Lemma 1.3, if $X = (X_1, \ldots, X_k) \in B_n(H)^-$, then $X \overset{p}{\sim} 0$ if and only if the joint spectral radius $r(X_i) < 1$ for any $i \in \{1, \ldots, k\}$. Set

$$
B_n(H)_0^- := \left\{ X \in B_n(H)^- : X \overset{p}{\sim} 0 \right\}
$$

and note that Theorem 4.4 implies that $B_n(H) \subset B_n(H)_0^-$. We define the map $d_P$ on $B_n(H)_0^- \times B_n(H)_0^-$ by setting

$$
d_P(A, B) := \sup_{r \in [0, 1)} \|\mathcal{P}(r, rA) - \mathcal{P}(r, rB)\|, \quad A, B \in B_n(H)_0^-.
$$

Note that if $X \in B_n(H)_0^-$, then $X \overset{p}{\sim} 0$, which shows that there is $c \geq 1$ such that $\mathcal{P}(r, rA) \leq c^2 I$ for any $r \in [0, 1)$. Consequently, $d_P(A, B) < \infty$ for any $A, B \in B_n(H)_0^-$. If $d_P(A, B) = 0$, then $d_P(A, B) = 0$ and, as in the proof of Lemma 2.1, we deduce that $A = B$. Now, it is clear that $d_P$ is a metric on $B_n(H)_0^-$.}

**Theorem 3.4.** The map $d_P$ has the following properties:
(i) $d_P$ is a complete metric on $B_n(H)_0$;
(ii) the $d_P$-topology is stronger than the norm topology on $B_n(H)_0$;
(iii) the $d_P$-topology coincides with the norm topology on $B_n(H)$.

Proof. Let $A = (A_1, \ldots, A_k)$ and $B = (B_1, \ldots, B_k)$ be in $B_n(H)_0$. Set $w := \sum_{\alpha \in F^+_n} e_\alpha \otimes h_\alpha \in F^2(H_n) \otimes H$, where $h_\alpha \in H$ and $\sum_{\alpha \in F^+_n} \|h_\alpha\|^2 < \infty$, and let

$$\tilde{w} := \sum_{\alpha \in F^+_n} (1 \otimes \cdots \otimes 1 \otimes e_\alpha \otimes 1 \otimes \cdots \otimes 1) \otimes h_\alpha$$

be in $F^2(H_n) \otimes \cdots \otimes F^2(H_n)$. Note that

$$\langle P(R_i, rA_i)w, w \rangle = \langle \mathcal{P}(R, rA)\tilde{w}, \tilde{w} \rangle$$

for any $w \in F^2(H_n) \otimes H$, where $P(R_i, rA_i)$ is the Poisson kernel associate with the row contraction $rA_i$ and where $R_i = (R_{i,1}, \ldots, R_{i,n_i})$ is the $n_i$-tuple of right creation operators acting on the full Fock space $F^2(H_n)$.

Setting $\Lambda_{A_q} := R^*_q A_{q,1} + \cdots + R^*_q A_{q,n_q}$, $q \in \{1, \ldots, k\}$, we deduce that

$$rA_{A_q} = \frac{1}{2\pi} \int_0^{2\pi} e^{-it} P(re^{it} R_q, A_q), \quad r \in [0, 1),$$

and, consequently,

$$\|rA_q - rB_q\| = \|rA_{A_q} - rA_{B_q}\|
\leq \frac{1}{2\pi} \int_0^{2\pi} \|P(re^{it} R_q, A_q) - P(re^{it} R_q, B_q)\| dt
\leq \sup_{r \in [0,1)} \|P(re^{it} R_q, A_q) - P(re^{it} R_q, B_q)\|$$

for any $r \in [0, 1)$. Hence and using relation (3.2), we deduce that

$$\|A_q - B_q\| \leq d_P(A, B), \quad A, B \in B_n(H)_0.$$

Let $A^{(m)} := (A^{(m)}_1, \ldots, A^{(m)}_k)$ be a $d_P$-Cauchy sequence in $B_n(H)_0$. Due to the latter inequality, for each $i \in \{1, \ldots, k\}$, $\{A^{(m)}_i\}_{m=1}^\infty$ is a Cauchy sequence in the norm topology of $[B(H)^{n_i}]_1 \otimes H$. Consequently, there exists $T_i \in [B(H)^{n_i}]_1$ such that $\|A^{(m)}_i - T_i\| \to 0$, as $m \to \infty$. Since $A^{(m)} \in B_n(H)_0$, so is $T = (T_1, \ldots, T_k)$. Since $A^{(m)}$ is a $d_P$-Cauchy sequence, there is $m_0 \in \mathbb{N}$ such that $d_P(A^{(m)}, A^{(m_0)}) \leq 1$ for any $m \geq m_0$. On the other hand, $A^{(m_0)} \in B_n(H)_0$ which shows that $A^{(m)} \overset{P}{\to} 0$ and, consequently, there is $c \geq 1$ such that $\mathcal{P}(R, rA^{(m_0)}) \leq c^2 I$ for any $r \in [0, 1)$. Hence, we deduce that

$$\mathcal{P}(R, rA^{(m)}) \leq (d_P(A^{(m)}, A^{(m_0)}) + \|\mathcal{P}(R, rA^{(m_0)})\|_I) I \leq (c^2 + 1)I$$

for any $m \geq m_0$. Using the continuity of the map $X \mapsto \mathcal{P}(R, X)$ on $B_n(H)$ and taking $m \to \infty$, we obtain $\mathcal{P}(R, rT) \leq (c^2 + 1)I$ for any $r \in [0, 1)$, which shows that $T \overset{P}{\to} 0$. Thus $T \in B_n(H)_0$.

Set $G_r := \mathcal{P}(S, rA^{(m)}) - \mathcal{P}(S, rT)$ and let $0 \leq r_1 < r_2 < 1$. Using the noncommutative Berezin transform we have $\left(\mathcal{B}_{2\pi} R \otimes \text{id}\right) [G_{r_2}] = G_{r_1}$. Hence, $\|G_{r_1}\| \leq \|G_{r_2}\|$. Using this result and the fact that

$$\mathcal{P}(R, X) = C_X(R)^* C_X(R), \quad X \in B_n(H),$$

where

$$C_X(R) := (I \otimes \Delta_X(I)^{1/2}) \prod_{i=1}^k (I - R_{i,1} \otimes X_{i,1}^* - \cdots - R_{i,m_i} \otimes X_{i,m_i}^*)^{-1}$$
for any $X = (X_1, \ldots, X_k) \in \mathcal{B}_n(\mathcal{H})$ with $X_i = (X_{i,1}, \ldots, X_{i,n_i})$, we obtain
\[
d_P(A^{(m)}, T) = \left\| \lim_{r \to 1} \left[ \mathcal{P}(\mathcal{R}, rA) - \mathcal{P}(\mathcal{R}, rB) \right] \right\|
= \left\| \lim_{r \to 1} \left[ C_{A^{(m)}}(\mathcal{R})^* C_{A^{(m)}}(\mathcal{R}) - C_{T}(\mathcal{R})^* C_{T}(\mathcal{R}) \right] \right\|
= \|C_{A^{(m)}}(\mathcal{R})^* C_{A^{(m)}}(\mathcal{R}) - C_{T}(\mathcal{R})^* C_{T}(\mathcal{R})\|.
\]

The latter equality holds due to the fact that the spectral radius is upper semicontinuous and, for each $q \in \{1, \ldots, k\}$, the spectral radii of $A_{q^{(m)}}$ and $A_{r^{(q)}}$ are strictly less than 1. Indeed, in this case we have $C_{rT}(\mathcal{R}) \to C_{A^{(m)}}(\mathcal{R})$ and $C_{T}(\mathcal{R}) \to C_{T}(\mathcal{R})$ in the norm topology, as $r \to 1$. On the other hand, since that map $Y \to Y^{-1}$ is norm continuous on the open set of invertible operators, $C_{A^{(m)}}(\mathcal{R}) \to C_{T}(\mathcal{R})$ in norm as $m \to \infty$. Consequently, $d_P(A^{(m)}, T) \to 0$ as $m \to \infty$, which completes the proof of part (i).

Due to relation (3.3), part (ii) is clear.

To prove part (iii), assume that $A, B \in \mathcal{B}_n(\mathcal{H})$. Note that, as above, we have
\[
d_P(A, B) = \left\| \lim_{r \to 1} \left[ \mathcal{P}(\mathcal{R}, rA) - \mathcal{P}(\mathcal{R}, rB) \right] \right\|
= \|C_{A}(\mathcal{R})^* C_{A}(\mathcal{R}) - C_{T}(\mathcal{R})^* C_{T}(\mathcal{R})\|.
\]

According to Proposition 4.1 from [29], the map $X \to C_X(\mathcal{R})$ is continuous in the operator norm topology. Now, one can easily complete the proof of part (iii). \hfill \square

**Lemma 3.5.** If $A, B \in \mathcal{B}_n(\mathcal{H})^-$ are such that $A \leq^P B$, then
\[
\delta_P(A, B) = \frac{1}{2} \sup \left\{ \ln \frac{\langle \mathcal{P}(\mathcal{R}, rA)x, x \rangle}{\langle \mathcal{P}(\mathcal{R}, rB)x, x \rangle} \right\},
\]
where the supremum is taken over all $x \in F^2(H_{n_1}) \otimes \cdots \otimes F^2(H_{n_k}) \otimes \mathcal{H}$, $x \neq 0$, and all $r \in [0, 1)$.

**Proof.** Let $A, B \in \mathcal{B}_n(\mathcal{H})^-$ be such that $A \leq^P B$ with $c \geq 1$. Then
\[
\frac{1}{c^2} \langle \mathcal{P}(\mathcal{R}, rB)x, x \rangle \leq \langle \mathcal{P}(\mathcal{R}, rA)x, x \rangle \leq c^2 \langle \mathcal{P}(\mathcal{R}, rB)x, x \rangle
\]
for any $r \in [0, 1)$ and any $x \in F^2(H_{n_1}) \otimes \cdots \otimes F^2(H_{n_k}) \otimes \mathcal{H}$ with $x \neq 0$. Since $\mathcal{P}(\mathcal{R}, rB)$ is an invertible operator, we deduce that
\[
-\ln c \leq \frac{1}{2} \ln \frac{\langle \mathcal{P}(\mathcal{R}, rA)x, x \rangle}{\langle \mathcal{P}(\mathcal{R}, rB)x, x \rangle} \leq \ln c
\]
which implies
\[
M := \frac{1}{2} \sup \left\{ \ln \frac{\langle \mathcal{P}(\mathcal{R}, rA)x, x \rangle}{\langle \mathcal{P}(\mathcal{R}, rB)x, x \rangle} \right\} \leq \delta_P(A, B).
\]

To prove the reverse inequality, note that
\[
\frac{1}{2} \left| \ln \frac{\langle \mathcal{P}(\mathcal{R}, rA)x, x \rangle}{\langle \mathcal{P}(\mathcal{R}, rB)x, x \rangle} \right| \leq M
\]
which is equivalent to
\[
e^{-2M} \langle \mathcal{P}(\mathcal{R}, rB)x, x \rangle \leq \langle \mathcal{P}(\mathcal{R}, rA)x, x \rangle \leq e^{2M} \langle \mathcal{P}(\mathcal{R}, rB)x, x \rangle
\]
for any $r \in [0, 1)$ and any $x \in F^2(H_{n_1}) \otimes \cdots \otimes F^2(H_{n_k}) \otimes \mathcal{H}$ with $x \neq 0$. Consequently, $\delta_P(A, B) \leq m$. The proof is complete. \hfill \square

**Theorem 3.6.** Let $\Delta$ be a Poisson part of $\mathcal{B}_n(\mathcal{H})^0$. Then the following properties hold:

(i) $\delta_P$ is a complete metric on $\Delta$.

(ii) the $\delta_P$-topology is stronger than the $d_P$-topology on $\Delta$.

(iii) the $\delta_P$-topology, the $d_P$-topology, and the operator norm topology coincide on the open polyball $\mathcal{B}_n(\mathcal{H})$.

(iv) the $\delta_H$-topology is stronger than the $\delta_P$-topology on $\mathcal{B}_n(\mathcal{H})$. 

Proof. Let $A, B \in \Delta$. Due to the definition of $\omega_p$, we have
\[ \mathcal{P}(R, rA) \leq \omega_p(A, B) \mathcal{P}(R, rB), \quad r \in [0, 1), \]
which implies
\[ \mathcal{P}(R, rA) - \mathcal{P}(R, rB) \leq [\omega_p(A, B)^2 - 1]M_B, \quad r \in [0, 1), \]
where $M_B := \sup_{r \in (0, 1)} \|\mathcal{P}(R, rB)\| < \infty$ due to the fact that $B \overset{p}{\to} 0$. Similarly, we can obtain the inequality
\[ \mathcal{P}(R, rB) - \mathcal{P}(R, rA) \leq [\omega_p(A, B)^2 - 1]M_A, \quad r \in [0, 1). \]
Consequently, since $\mathcal{P}(R, rB) - \mathcal{P}(R, rA)$ is a self-adjoint operator, we obtain
\[ \|\mathcal{P}(R, rB) - \mathcal{P}(R, rA)\| \leq \max\{M_A, M_B\}[\omega_p(A, B)^2 - 1], \quad r \in [0, 1). \]

Hence, we deduce that
\[ d_p(A, B) \leq \max\{M_A, M_B\} \left( \epsilon^{2\delta_p(A, B)} - 1 \right). \]

Now, we prove that $d_p$ is a complete metric on $\Delta$. Let $\{A^{(m)}\}_{m=1}^\infty$ be a $d_p$-Cauchy sequence in $\Delta$. For any $\epsilon > 0$, there is $m_0 \in \mathbb{N}$ such that
\[ \delta_p(A^{(m)}, A^{(p)}) < \epsilon, \quad m, p \geq m_0. \]
Since $A^{(m)} \overset{P}{\prec} A^{(m_0)}$ and $A^{(p)} \overset{P}{\prec} 0$, we have
\[ \mathcal{P}(R, rA^{(m)}) \leq \omega_p(A^{(m)}, A^{(m_0)})\mathcal{P}(R, rA^{(m_0)}) \leq \omega_p(A^{(m)}, A^{(m_0)}) \sup_{r \in [0, 1)} \|\mathcal{P}(R, rA^{(m_0)})\|I \]
for any $r \in [0, 1)$. Hence and using (3.5), we obtain
\[ \|\mathcal{P}(R, rA^{(m)})\| \leq \sup_{r \in [0, 1)} \|\mathcal{P}(R, rA^{(m_0)})\|\epsilon^{2\epsilon} \]
for any $m \geq m_0$ and $r \in [0, 1)$. Since $\sup_{r \in [0, 1)} \|\mathcal{P}(R, rA^{(m_0)})\| < \infty$, we deduce that the sequence $\left\{\sup_{r \in [0, 1)} \|\mathcal{P}(R, rA^{(m)})\|\right\}_{m=1}^\infty$ is bounded. The inequality (3.4), implies that $\{A^{(m)}\}_{m=1}^\infty$ is a $d_p$-Cauchy sequence. According to Theorem 3.4, there exists $A \in B_n(H)_0$ such that
\[ \lim_{m \to \infty} d_p(A^{(m)}, A) = 0. \]

Combining relations (3.5) and (3.6)
\[ \mathcal{P}(R, rA^{(m)}) \leq \omega_p(A^{(m)}, A^{(m_0)})\mathcal{P}(R, rA^{(m_0)}) \leq \epsilon^{2\epsilon} \sup_{r \in [0, 1)} \mathcal{P}(R, rA^{(m_0)}) \]
for any $m \geq m_0$ and $r \in [0, 1)$. Using relation (3.7) and passing to the limit as $m \to \infty$ in relation (3.8), we obtain
\[ \mathcal{P}(R, rA) \leq \epsilon^{2\epsilon}\mathcal{P}(R, rA^{(m_0)}) \]
for any $r \in [0, 1)$, which shows that $A \overset{P}{\prec} A^{(m_0)}$. On the other hand, since $A^{(m_0)} \overset{P}{\prec} A^{(m)}$ for any $m \geq m_0$, relation (3.5) implies
\[ \mathcal{P}(R, rA^{(m_0)}) \leq \omega_p(A^{(m_0)}, A^{(m)})\mathcal{P}(R, rA^{(m)}) \leq \epsilon^{2\epsilon}\mathcal{P}(R, rA^{(m)}) \]
for any $m \geq m_0$ and $r \in [0, 1)$. Due to Theorem 3.4, the $d_p$-topology is stronger than the norm topology on $B_n(H)_0$. Therefore, relation (3.7) implies $A^{(m)} \to A \in B_n(H)_0$ in the operator norm topology. Taking $m \to \infty$ in relation (3.10), we obtain
\[ \mathcal{P}(R, rA^{(m_0)}) \leq \epsilon^{2\epsilon}\mathcal{P}(R, rA) \]
for any $m \geq m_0$ and $r \in [0, 1)$.
for any \( r \in [0, 1) \). Consequently, \( A^{(m_0)} \overset{P}{\prec} A \) which together with relation (3.10) imply \( A^{(m_0)} \overset{P}{\prec} A \). Thus \( A \in \Delta \). Note that the inequalities (3.9) and (3.11) show that \( \omega_P(A^{(m_0)}, A) \leq e^{2r} \) and, therefore, \( \delta_P(A^{(m_0)}, A) \leq \epsilon \). Now, using relation (3.5), we obtain \( \delta_P(A^{(m)}, A) \leq 2\epsilon \) for any \( m \geq m_0 \). This shows that \( \delta_P(A^{(m)}, A) \to 0 \) as \( m \to \infty \) and completes the proof that \( \delta_P \) is a complete metric on \( \Delta \). Note that we have already proved part (ii) of the theorem.

Now, we prove part (iii). Assume that \( A, B \in B_n(H) \). Since \( \mathcal{P}(R, B) \) is a positive invertible operator, we have \( I \leq \|\mathcal{P}(R, B)^{-1}\|\mathcal{P}(R, B) \). Using the fact that the Berezin transform \( B_{R \otimes id} \) is a completely positive linear map, we deduce that \( I \leq \|\mathcal{P}(R, B)^{-1}\|\mathcal{P}(R, rB) \) for any \( r \in [0, 1) \). Consequently,

\[
\frac{\langle \mathcal{P}(R, rA)x, x \rangle}{\langle \mathcal{P}(R, rB)x, x \rangle} - 1 \leq \frac{\|\mathcal{P}(R, B)^{-1}\|}{\|x\|} \langle (\mathcal{P}(R, rA) - \mathcal{P}(R, rB))x, x \rangle \\
\leq \|\mathcal{P}(R, B)^{-1}\| d_P(A, B)
\]

for any \( x \in F^2(H_{n_1}) \otimes \cdots \otimes F^2(H_{n_k}) \otimes H \), \( x \neq 0 \), and all \( r \in [0, 1) \). Hence, we obtain

\[
\ln \frac{\langle \mathcal{P}(R, rA)x, x \rangle}{\langle \mathcal{P}(R, rB)x, x \rangle} \leq \ln (1 + \|\mathcal{P}(R, B)^{-1}\| d_P(A, B))
\]

Interchanging \( A \) with \( B \), we obtain a similar inequality. Putting the two inequalities together, we deduce that

\[
\left| \ln \frac{\langle \mathcal{P}(R, rA)x, x \rangle}{\langle \mathcal{P}(R, rB)x, x \rangle} \right| \leq \ln (1 + \max \{\|\mathcal{P}(R, A)^{-1}\|, \|\mathcal{P}(R, B)^{-1}\|\}) d_P(A, B)
\]

for any \( x \in F^2(H_{n_1}) \otimes \cdots \otimes F^2(H_{n_k}) \otimes H \), \( x \neq 0 \), and all \( r \in [0, 1) \). Using Lemma 3.15 we obtain

\[
(3.12) \quad \delta_P(A, B) \leq \frac{1}{2} \ln (1 + \max \{\|\mathcal{P}(R, A)^{-1}\|, \|\mathcal{P}(R, B)^{-1}\|\}) d_P(A, B)
\]

Let \( \{A^{(m)}\}_{m=1}^{\infty} \) be a sequence of elements in \( B_n(H) \) and \( A \in B_n(H) \) be such that \( d_P(A^{(m)}, A) \to 0 \), as \( m \to \infty \). This implies \( \mathcal{P}(R, A^{(m)}) \to \mathcal{P}(R, A) \) in the operator norm topology, as \( m \to \infty \). Since \( A^{(m)}, A \in B_n(H) \), the operators \( \mathcal{P}(R, A^{(m)}) \) and \( \mathcal{P}(R, A) \) are invertible and, consequently, \( \mathcal{P}(R, A^{(m)})^{-1} \to \mathcal{P}(R, A)^{-1} \) in the operator norm topology, as \( m \to \infty \). Now, it is clear that there is \( M > 0 \) such that \( \|\mathcal{P}(R, A^{(m)})^{-1}\| \leq M \) for any \( m \in \mathbb{N} \). Applying inequality (3.12), we obtain

\[
\delta_P(A^{(m)}, A) \leq \frac{1}{2} \ln \left[ 1 + Md_P(A^{(m)}, A) \right], \quad m \in \mathbb{N}.
\]

Since \( d_P(A^{(m)}, A) \to as m \to \infty, we deduce that \( \delta_P(A^{(m)}, A) \to 0 \) as well. This shows that the \( d_P \)-topology on \( B_n(H) \) is stronger than the \( \delta_P \)-topology. On the other hand, due to part (ii) of this theorem and the fact that \( B_n(H) \) is a Poisson part in \( B_n(H)_0 \), we conclude that the \( \delta_P \)-topology coincides with the \( d_P \)-topology on \( B_n(H) \). Now, using Theorem 3.4 part (iii), we complete the proof of item (iii).

According to Theorem 4.4 the open unit polyball \( B_n(H) \) is the Harnack (respectively, Poisson) part of \( B_n(H)^{-} \) which contains the origin. Since \( \delta_P(X, Y) \leq \delta_H(X, Y) \) for \( X, Y \in B_n(H) \), item (iv) follows.

The proof is complete.

**Corollary 3.7.** If \( \Delta \) is a Poisson part of \( B_n(H)_0 \), then

\[
\delta_P(A, B) \geq \frac{1}{2} \ln \left( 1 + \frac{d_P(A, B)}{\max \{\sup_{r \in [0, 1]} \|\mathcal{P}(R, A)\|, \sup_{r \in [0, 1]} \|\mathcal{P}(R, B)\|\} \} \right).
\]

If \( A, B \in B_n(H) \), then

\[
\delta_P(A, B) \leq \frac{1}{2} \ln \left( 1 + \max \{\|\mathcal{P}(R, A)^{-1}\|, \|\mathcal{P}(R, B)^{-1}\|\} \right) d_P(A, B).
\]
4. Hyperbolic metric on the regular polydisk

Using a characterization of positive free $k$-pluriharmonic functions on regular polydisks and the results of the previous sections, we prove that the Harnack parts and the Poisson parts on $D^k(H)$ coincide, and so are the metrics $\delta_H$ and $\delta_P$. We show that the hyperbolic metric $\delta_H$ on $D^k(H)$ has similar properties to the Poincaré distance on the open unit disk $D$.

Let $\Omega \subset F_n^+ \times F_n^+$ be the set of all pairs $(\alpha, \beta)$ where $\alpha = (\alpha_1, \ldots, \alpha_k), \beta = (\beta_1, \ldots, \beta_k)$ are in $F_n^+ := F_n^{+1} \otimes \cdots \otimes F_n^{+n}$ such that $\alpha_1, \beta_1 \in F_n^{+1}, |\alpha_i| = m_i^-$, and $|\beta_i| = m_i^+$ for some $m_i \in \mathbb{Z}$. In [29], we proved that a map $F : B_n(H) \to B(E) \otimes_{min} B(H)$, with $F(0) = I$, is a positive free $k$-pluriharmonic function on the regular polyball if and only if it has the form

$$F(X) = \sum_{(\alpha, \beta) \in \Omega} P_\varepsilon V_\alpha V_\beta \varepsilon \otimes X_\alpha X_\beta,$$

where $V = (V_1, \ldots, V_k)$ is a $k$-tuple of commuting row isometries on a space $K \supset E$ such that

$$\sum_{(\alpha, \beta) \in \Omega} P_\varepsilon V_\alpha V_\beta \varepsilon \otimes r^{(\alpha) + (\beta)} S_\alpha S_\beta \geq 0, \quad r \in [0, 1),$$

the series is convergent in the operator topology, and $\tilde{\alpha} = (\tilde{\alpha}_1, \ldots, \tilde{\alpha}_k)$ is the reverse of $\alpha = (\alpha_1, \ldots, \alpha_k)$, i.e. $\tilde{\alpha}_i = g_{i}^+ \cdots g_{k}^+$ if $\alpha_i = g_{i}^- \cdots g_{k}^- \in F_n^{+1}$. As a consequence of this result we have the following characterization of positive free $k$-pluriharmonic function on regular polydisks. We include a proof for completeness. In what follows we consider the regular polydisc $D^k(H) := B_{(1, \ldots, 1)}(H)$.

**Proposition 4.1.** Let $F : D^k(H) \to B(E) \otimes_{min} B(H)$ be a free $k$-pluriharmonic function with $F(0) = I$. Then $F$ is positive if and only if

$$F(X) = (P_\varepsilon \otimes I) \mathcal{P}(U, X)|_{\varepsilon \otimes H},$$

where $U = (U_1, \ldots, U_k)$ is a $k$-tuple of commuting unitaries on a Hilbert space $K \supset E$, and the free pluriharmonic Poisson kernel $\mathcal{P}(U, X)$ is equal to

$$\sum_{m_1, \ldots, m_k} \cdots \sum_{m_k} (U_1^{*m_1^-} \cdots (U_k^{*m_k^-}) V_1^{m_1^+} \cdots (U_k^{m_k^+}) \otimes X_1^{m_1^-} \cdots (X_k^{m_k^-})^{m_k^+}$$

for any $X = (X_1, \ldots, X_k) \in D^k(H)$, where the convergence of the multi-series is in the operator norm topology.

**Proof.** Assume that $F$ is a positive free $k$-pluriharmonic function with $F(0) = I$. According to the above-mentioned result, there is a $k$-tuple $V = (V_1, \ldots, V_k)$ of commuting isometries on a Hilbert space $G \supset H$ such that

$$F(X) = \sum_{m_1, \ldots, m_k} \cdots \sum_{m_k} (V_1^{*m_1^-} \cdots (V_k^{*m_k^-}) V_1^{m_1^+} \cdots (V_k^{m_k^+}) \otimes X_1^{m_1^-} \cdots (X_k^{m_k^-})^{m_k^+}$$

for any $X = (X_1, \ldots, X_k) \in D^k(H)$, where the convergence is in the operator norm topology. Due to Itô’s theorem (see [34]), there is a $k$-tuple $U = (U_1, \ldots, U_k)$ of commuting unitaries on a Hilbert space $K \supset G$ such that $U_i^{*} = V_i$ for any $i \in \{1, \ldots, k\}$. Due to Fuglede’s theorem (see [4]), the unitaries are doubly commuting, i.e. $U_i U_j^* = U_j U_i$ for any $i, j \in \{1, \ldots, k\}$. Consequently, we have $F(X) = (P_\varepsilon \otimes I) \mathcal{P}(U, X)|_{\varepsilon \otimes H}$. The converse of the theorem is due to Theorem 4.2 from [29]. The proof is complete. \qed

Given a completely bounded linear map $\mu : \text{span}\{R_n^+ R_n\} \to B(E)$, we introduce the noncommutative Poisson transform of $\mu$ to be the map $\mathcal{P}\mu : B_n(H) \to B(E) \otimes_{min} B(H)$ defined by

$$(\mathcal{P}\mu)(X) := \tilde{\mu}[\mathcal{P}(R, X)], \quad X \in B_n(H),$$

where the completely bounded linear map

$$\tilde{\mu} := \mu \otimes \text{id} : \text{span}\{R_n^+ R_n\}^\perp \otimes_{min} B(H) \to B(E) \otimes_{min} B(H)$$

is uniquely defined by $\tilde{\mu}(A \otimes Y) := \mu(A) \otimes Y$ for any $A \in \text{span}\{R_n^+ R_n\}$ and $Y \in B(H)$. 

Using Corollary 4.5 from [29] and Proposition 4.1 we obtain the following structure theorem for positive $k$-harmonic functions on the regular polydisk $D^k(\mathcal{H})$, which extends the corresponding classical result in scalar polydisks [30].

**Theorem 4.2.** Let $F : D^k(\mathcal{H}) \to B(\mathcal{E}) \otimes_{\min} B(\mathcal{H})$ be a free $k$-pluriharmonic function. Then the following statements are equivalent:

(i) $F$ is positive;
(ii) there exists a completely positive linear map $\mu : C^*(\mathbb{R}) \to B(\mathcal{E})$ such that $F = \mathcal{P}\mu$;
(iii) there exists a $k$-tuple $U = (U_1, \ldots, U_k)$ of commuting unitaries acting on a Hilbert space $\mathcal{K} \supset \mathcal{E}$ and a bounded operator $W : \mathcal{E} \to \mathcal{K}$ such that

$$F(X) = (W^* \otimes I) [C_X(U)^* C_X(U)] (W \otimes I),$$

where

$$C_X(U) := (I \otimes \Delta_X(I^{1/2})) \prod_{i=1}^k (I - U_i \otimes X_i^*), \quad X = (X_1, \ldots, X_k) \in D^k(\mathcal{H}).$$

**Proof.** In the particular case when $F(0) = I$, the implication (i) $\implies$ (ii) is due to Corollary 4.5 from [29]. Now, we consider the general case when $F : D^k(\mathcal{H}) \to B(\mathcal{E}) \otimes_{\min} B(\mathcal{H})$ is an arbitrary positive free $k$-pluriharmonic function of the form

$$\sum_{m_1 \in \mathbb{Z}} \cdots \sum_{m_k \in \mathbb{Z}} A_{m_1 \cdots m_k : m_1^+ \cdots m_k^+} \otimes X_1^{m_1^+} \cdots X_k^{m_k^+}$$

for any $X = (X_1, \ldots, X_k) \in D^k(\mathcal{H})$, where the convergence is in the operator norm topology. For each $\epsilon > 0$, set

$$G_{\epsilon} := \left[ (A + \epsilon I)^{-1/2} \otimes I \right] (F + \epsilon I) \left[ (A + \epsilon I)^{-1/2} \otimes I \right],$$

where $A \otimes I := F(0)$ with $A \geq 0$. Since $G_{\epsilon}$ is a positive free $k$-pluriharmonic function on $D^k(\mathcal{H})$ with $G_{\epsilon}(0) = I$, we can apply Corollary 4.5 from [29] and find a completely positive linear map $\mu_{\epsilon} : C^*(\mathbb{R}) \to B(\mathcal{E})$ such that

$$\mu_{\epsilon} \left( (R_1^*)^{m_1^+} \cdots (R_k^*)^{m_k^+} R_1^{m_1^-} \cdots R_k^{m_k^-} \right) = (A + \epsilon I)^{-1/2} A_{m_1 \cdots m_k : m_1^+ \cdots m_k^+} (A + \epsilon I)^{-1/2}.$$ 

Define the completely positive linear map $\nu_{\epsilon} : C^*(\mathbb{R}) \to B(\mathcal{E})$ by setting

$$\nu_{\epsilon}(g) := (A + \epsilon I)^{-1/2} \mu_{\epsilon}(g) (A + \epsilon I)^{1/2}, \quad g \in C^*(\mathbb{R}).$$

Note that $\nu_{\epsilon} \left( (R_1^*)^{m_1^+} \cdots (R_k^*)^{m_k^+} R_1^{m_1^-} \cdots R_k^{m_k^-} \right) = A_{m_1 \cdots m_k : m_1^+ \cdots m_k^+}$ if $(m_1, \ldots, m_k) \neq (0, \ldots, 0)$, and $\nu_{\epsilon}(I) = A + \epsilon I$. Define $\mu : C^*(\mathbb{R}) \to B(\mathcal{E})$ by setting

$$\mu \left( (R_1^*)^{m_1^+} \cdots (R_k^*)^{m_k^+} R_1^{m_1^-} \cdots R_k^{m_k^-} \right) = A_{m_1 \cdots m_k : m_1^+ \cdots m_k^+}$$

if $(m_1, \ldots, m_k) \neq (0, \ldots, 0)$, and $\mu(I) = A$. It is clear that $\nu_{\epsilon}(g) = \mu(g) + \epsilon \langle g(1), 1 \rangle I$ for any $g \in C^*(\mathbb{R})$, and $\nu_{\epsilon}(g) \to \mu(g)$ as $\epsilon \to 0$. Therefore, $\mu$ is a completely positive linear map and $F = \mathcal{P}\mu$. Using Corollary 4.5 from [29] and Proposition 4.1 one deduce the implications (ii) $\implies$ (iii) and (iii) $\implies$ (i). The proof is complete. \qed

We recall [9] that the Kobayashi distance for the polydisc $\mathbb{D}^k$ is given by

$$K_{\mathbb{D}^k}(z, w) = \frac{1}{2} \ln \frac{1 + \|\psi_z(w)\|_\infty}{1 - \|\psi_z(w)\|_\infty},$$

where $\psi_z$ is the involutive automorphisms of $\mathbb{D}^k$ given by

$$\psi_z = \left( \frac{w_1 - z_1}{1 - z_1 w_1}, \ldots, \frac{w_k - z_k}{1 - z_k w_k} \right)$$

for any $z = (z_1, \ldots, z_k)$ and $w = (w_1, \ldots, w_k)$ in $\mathbb{D}^k$.

**Theorem 4.3.** Let $D^k(\mathcal{H})$ be the regular polydisk. The following statements hold.
(i) If $A, B \in D^k(\mathcal{H})^-$, then $A \overset{H}{\sim} B$ if and only if $A \overset{P}{\sim} B$.
(ii) The metrics $\delta_H$ and $\delta_P$ coincide on the Harnack parts of $D^k(\mathcal{H})^-$. 
(iii) If $A$ and $B$ are in $D^k(\mathcal{H})^-$ and $A \overset{H}{\sim} B$, then
$$\delta_H(A, B) = \delta_H(\Psi(A), \Psi(B)), \quad \Psi \in \text{Aut}(D^k).$$
(iv) If $A$ and $B$ are in $D^k(\mathcal{H})$, then
$$\delta_H(A, B) = \ln \max \left\{ \| C_A(R)C_B(R)^{-1} \|, \| C_B(R)C_A(R)^{-1} \| \right\},$$
where
$$C_\mathcal{X}(R) := \left( I \otimes \Delta_{\mathcal{X}}(I)^{1/2} \right) \prod_{i=1}^k \left( I - R_i \otimes X_i^* \right), \quad \mathcal{X} = (X_1, \ldots, X_k) \in D^k(\mathcal{H}).$$
(v) $\delta_H|_{D^k \times D^k}$ is equivalent to the Kobayashi distance on the polydisk $D^k$ and
$$\delta_H(z, w) = \frac{1}{2} \ln \prod_{i=1}^k \left( 1 + |\psi_{z_i}(w_i)| \right) \prod_{i=1}^k \left( 1 - |\psi_{z_i}(w_i)| \right)$$
for any $z = (z_1, \ldots, z_k)$ and $w = (w_1, \ldots, w_k)$ in $D^k$, where $\psi_z := (\psi_{z_1}, \ldots, \psi_{z_n})$ is the involutive automorphisms of $D^k$ such that $\psi_z(0) = z$ and $\psi_z(z_i) = 0$.
(vi) The hyperbolic metric $\delta_H$ is complete on the Harnack parts of $D^k(\mathcal{H})^-$.
(vii) The hyperbolic metric $\delta_H$-topology coincides with the operator norm topology on the regular polydisk $D^k(\mathcal{H})$.

Proof. Let $A, B \in D^k(\mathcal{H})^-$ and recall that the map $\mathcal{X} \mapsto \mathcal{P}(R, \mathcal{X})$ is a positive free $k$-pluriharmonic function on $D^k(\mathcal{H})$. Consequently, if $A \overset{H}{\sim} B$, then $A \overset{P}{\sim} B$. To prove the converse, assume that $A \overset{P}{\sim} B$.

Then we have
$$(4.1) \quad \mathcal{P}(R, rA) \leq e^2 \mathcal{P}(R, rB)$$
for any $r \in [0, 1)$. Let $F : D^k(\mathcal{H}) \to B(\mathcal{E}) \otimes_{min} B(\mathcal{H})$ be an arbitrary positive free $k$-pluriharmonic function with coefficients in $B(\mathcal{E})$. According to Theorem 4.2, there exists a completely positive linear map $\mu : C^*(R) \to B(\mathcal{E})$ such that $F(\mathcal{X}) = (\mathcal{P}_\mu)(\mathcal{X}) = (\mu \otimes id)[\mathcal{P}(R, \mathcal{X})]$ for any $\mathcal{X} \in D^k(\mathcal{H})$. Consequently, using relation (4.1), we deduce that $F(rA) \leq e^2 F(rB)$ for any $r \in [0, 1)$. This shows that $A \overset{H}{\sim} B$ and completes the proof of item (i). As a consequence, we deduce that the Harnack parts of $D^k(\mathcal{H})^-$ coincide with the Poisson parts. Moreover, since $A \overset{H}{\sim} B$ if and only if $A \overset{P}{\sim} B$, part (ii) holds. Note that part (iii) is a particular case of Theorem 2.6 while part (iv) is a particular case of Theorem 3.3. On the other hand, the items (vi) and (vii) are due to part (i) and Theorem 3.9.

It remains to prove part (v). Due to part (iv), we have
$$(4.2) \quad \delta_H(z, 0) = \ln \max \left\{ \| C_z(R) \|, \| C_z(R)^{-1} \| \right\},$$
where
$$C_z(R) = \prod_{i=1}^k \left( 1 - |z_i|^2 \right)^{1/2} \prod_{i=1}^k \left( I - z_i R_i \right)^{-1}.$$ 
Since $\| (I - z_i R_i)^{-1} \| \leq \frac{1}{1 - |z_i|}$, we deduce that
$$(4.3) \quad \| C_z(R) \| \leq \prod_{i=1}^k \left( 1 + |z_i| \right)^{1/2} \left( \frac{1}{1 - |z_i|} \right)^{1/2}.$$ 
Now, we calculate $\| C_z(R)^{-1} \|$. First, note that
$$\left\| \prod_{i=1}^k (I - z_i R_i) \right\| \leq \prod_{i=1}^k (1 + |z_i|).$$
Due to Riesz representation theorem we have
\[
\sup_{(w_1, \ldots, w_k) \in \mathbb{D}^k} \left| \prod_{i=1}^k (1 + \bar{z}_i w_i) \right| = \prod_{i=1}^k (1 + |z_i|).
\]
The von Neumann inequality for regular polyballs (see [26]) implies
\[
\left| \prod_{i=1}^k (1 + z_i w_i) \leq \left| \prod_{i=1}^k (I - \bar{z}_i R_i) \right| .
\]
Now, combining these relations, we deduce that
\[
\left\| \prod_{i=1}^k (I - \bar{z}_i R_i) \right\| = \prod_{i=1}^k (1 + |z_i|),
\]
which implies
\[
\|C_z(R)^{-1}\| = \prod_{i=1}^k \left( \frac{1 + |z_i|}{1 - |z_i|} \right)^{1/2}.
\]
Hence and using relations (4.2) and (4.3), we obtain
\[
(4.4) \quad \delta_H(z, 0) = \frac{1}{2} \ln \prod_{i=1}^k \frac{1 + |z_i|}{1 - |z_i|}.
\]
Now, let \( \psi_z := (\psi_{z_1}, \ldots, \psi_{z_k}) \) be the involutive automorphism of \( \mathbb{D}^k \) such that \( \psi_z(0) = z_i \) and \( \psi_z(z_i) = 0 \). Using part (iii) and relation (4.4), we deduce that
\[
\delta_H(z, w) = \delta_H(\psi_z(z), \psi_z(w)) = \delta_H(0, \psi_z(w))
\]
\[
= \frac{1}{2} \ln \prod_{i=1}^k \frac{1 + |\psi_z(w_i)|}{1 - |\psi_z(w_i)|} = \sum_{i=1}^k \delta_\mathbb{D}(z_i, w_i),
\]
where \( \delta_\mathbb{D} \) is the Poincaré distance on the open disk \( \mathbb{D} \). Since the function \( t \mapsto \ln \frac{1 + t}{1 - t} \) is increasing on \( [0, 1) \), we have
\[
\max_{i=1, \ldots, k} \frac{1}{2} \ln \frac{1 + |\psi_z(w_i)|}{1 - |\psi_z(w_i)|} = \frac{1}{2} \ln \frac{1 + \|\psi_z(w)\|_\infty}{1 - \|\psi_z(w)\|_\infty},
\]
which is the Kobayashi distance for the polydisc (see (11)). Consequently, \( \delta_H|_{\mathbb{D}^k \times \mathbb{D}^k} \) is equivalent to the Kobayashi distance on the polydisk \( \mathbb{D}^k \). This completes the proof of part (v).

**Corollary 4.4.** Let \( f = (f_1, \ldots, f_m) : \mathbb{D}^k(\mathcal{H}) \to [B(\mathcal{H})^m]_1 \) be a free holomorphic function on the regular polydisk. If \( X, Y \in \mathbb{D}^k(\mathcal{H}) \), then
\[
\delta_H(f(X), f(Y)) \leq \delta_H(X, Y),
\]
where \( \delta_H \) is the hyperbolic metric. In particular, if \( f(0) = 0 \), then
\[
\frac{1 + \|f(z)\|_2}{1 - \|f(z)\|_2} \leq \prod_{i=1}^k \frac{1 + |z_i|}{1 - |z_i|}
\]
for any \( z = (z_1, \ldots, z_k) \) in \( \mathbb{D}^k \).

**Proof.** The first part of this corollary is due to Proposition 2.5. To prove the second part, we use Theorem 4.3 part (ii) and (v). Note that
\[
\delta_H(z, 0) = \frac{1}{2} \ln \prod_{i=1}^k \frac{1 + |z_i|}{1 - |z_i|} \quad \text{and} \quad \delta_H(f(z), 0) = \frac{1}{2} \ln \frac{1 + \|f(z)\|_2}{1 - \|f(z)\|_2}.
\]
Since \( \delta_H(f(z), 0) \leq \delta_H(z, 0) \), one can complete the proof. \( \square \)
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