Phase Matching in Quantum Searching

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Abstract

Each iteration in Grover’s original quantum search algorithm contains 4 steps: two Hadamard-Walsh transformations and two amplitudes inversions. When the inversion of the marked state is replaced by arbitrary phase rotation $\theta$ and the inversion for the prepared state $|\gamma\rangle$ is replaced by rotation through $\phi$, we found that these phase rotations must satisfy a matching condition $\theta = \phi$. Approximate formula for the amplitude of the marked state after an arbitrary number of iterations are also derived. When phase matching is obtained, we found that the generalized quantum search is still a rotation in 2-dimensional space, but with a small angle $\beta' \approx (2 \sin \frac{\theta}{2}) \beta$, where $\theta$ is the angle of phase rotation of the marked state and $2\beta$ is the angle of $U(2)$ rotation when the phase rotations are the inversions in the original version. We give also a simple explanation of the phase matching requirement.

Keywords: quantum searching, phase matching, quantum computing

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Shor’s prime factoring algorithm and Grover’s quantum search algorithm are two of the great quantum algorithms \[1,2\]. In Grover’s original version, the algorithm is composed by: 1) inversion of the amplitude of the marked state; 2) inversion about the average. This second step can be constructed by two Hadamard-Walsh transformations and the inversion of the amplitudes of all basis states different from \(|0\rangle\). There are many applications of the algorithm. For example, using the algorithm, one can search a large database using only a single query \[3\]. It can be used in deciphering the DES encryption scheme \[4\]. It can be used in quantum counting \[3\]. As Grover’s algorithm involves only simple operations, it is easy to implement in experiment. It has been successfully implemented in NMR techniques \[5,6\] together with the simple D-J problem \[8,9\].

Grover’s algorithm is optimal. According to Benett et al \[10\], no quantum algorithm can solve the search better than \(O(\sqrt{N})\). Boyer et al \[11\] have given analytical expressions for the amplitude of the states in Grover’s search algorithm and tight bounds on the algorithm. Zalka \[12\] has improved this tight bounds and showed that Grover’s algorithm is optimal. Zalka also proposed \[13\] an improvement on Grover’s algorithm. Biron et al \[14\] generalized Grover’s algorithm to an arbitrarily distributed initial state. Pati \[15\] recast the algorithm in geometric language and studied the bounds on the algorithm. Very recently, Ozhigov \[16\] showed that quantum search can be further speeded up by a factor of \(\sqrt{2}\) by parallelism. Gingrich et al \[17\] also generalized Grover’s algorithm with parallelism with improvement. Jozsa \[18\] gave a simple explanation of Grover’s quantum search algorithm in simple geometry.

To generalize Grover’s quantum algorithm, one can do: 1) replace the Hadamard-Walsh algorithm by an arbitrary unitary transformation. This has been done neatly by Grover \[19\]; 2) one can replace the inversions of the amplitudes by arbitrary phase rotations as suggested in the same Letter by Grover \[19\]. It is believed that such a replacement will lead to a quantum search algorithm, though not as fast as the original version. Some authors have used this generalization to the advantage to produce a small iteration in a quantum search algorithm such that the amplitude of the marked state will be exactly unity \[3,13\], thus
avoiding the “over-cooking” problem in quantum searching. For instance, Zalka proposed to rotate the phase of the marked state by a small angle rather than $\pi$ to produce a smaller rotation of the state vector of the quantum computer [13]. However, it is found recently that an arbitrary phase (except $(2i + 1)\pi$, where $i$ is an integer) rotation of the marked state together with the inversion about average can not do any quantum search at all [20], contrary to what expected.

Grover has shown that in general $I_\gamma = I - 2|\gamma\rangle\langle\gamma|$ and $I_\tau = I - 2|\tau\rangle\langle\tau|$ together with an arbitrary unitary operator $U$ can be used to construct a quantum search algorithm $Q = -I_\gamma U^{-1}I_\tau U$. When $U^{-1} = U = W$, the Hadamard-Walsh transformation (denoted by $W$ hereafter), and $|\gamma\rangle = |0\rangle$, one recovers the original Grover’s quantum algorithm. It is worth pointing out that in his first paper [3], the inversion about average is realized by $-W I_n W$, where $I_n$ is the operation that inverses the amplitudes of the basis vectors $|n\rangle$ for $n \neq 0$. In Ref. [19], it is realized by $-W I_0 W$, where $I_0$ is to inverse the amplitude of the basis state $|0\rangle$. It can be shown easily that the two differ only by a phase factor. In general, if one rotates the phases of the basis states $|n\rangle$ except $n \neq \gamma$ through an angle $\phi$, it is equivalent to rotating the phase of the basis state $|\gamma\rangle$ through $-\phi$ angle and leaving other basis untouched.

In this Letter, instead of using inversions which are phase rotations through angle $\pi$, we use arbitrary phase rotations separately for $|\tau\rangle$ and $|\gamma\rangle$. Without confusion in the notations, we also use $I_\tau$ and $I_\gamma$ and $Q$ for these operations:

$$I_\gamma = I - (-e^{i\theta} + 1)|\gamma\rangle\langle\gamma|$$

$$I_\tau = I - (-e^{i\phi} + 1)|\tau\rangle\langle\tau|$$

$$Q = -I_\gamma U^{-1}I_\tau U$$

When $\theta = \phi = \pi$, $U = U^{-1} = W$ and $|\gamma\rangle = |0\rangle$ we recover the original Grover’s result.

It is easy to calculate that operator $Q$ is represented by the following matrix in the space
span by $|\gamma\rangle$ and $U^{-1}|\tau\rangle$.

$$Q \begin{pmatrix} |\gamma\rangle \\ U^{-1}|\tau\rangle \end{pmatrix} = \begin{pmatrix} -e^{i\theta} - (-e^{i\theta} + 1)(-e^{i\phi} + 1)|U_{\tau\gamma}|^2 \left(-e^{i\phi} + 1\right) U_{\tau\gamma} & e^{i\phi} \\ (-e^{i\theta} + 1)e^{i\phi}U^{*}_{\tau\gamma} \end{pmatrix} \begin{pmatrix} |\gamma\rangle \\ U^{-1}|\tau\rangle \end{pmatrix} \tag{4}$$

The basis vectors are not orthogonal. We orthonormalize them in a new basis

$$|1\rangle = \left(|\gamma\rangle > -U_{\tau\gamma}U^{-1}|\tau\rangle\right)/\xi \tag{5}$$

$$|2\rangle = U^{-1}|\tau\rangle, \tag{6}$$

where $\xi = -\sqrt{1 - |U_{\tau\gamma}|^2}$. The new matrix for operator $Q$ is

$$\begin{pmatrix} -e^{i\theta} - |U_{\tau\gamma}|^2 \left(1 - e^{i\theta}\right) - \left(1 - e^{i\theta}\right) U_{\tau\gamma}\sqrt{1 - |U_{\tau\gamma}|^2} \\ -U^*_{\tau\gamma}e^{i\phi} \sqrt{1 - |U_{\tau\gamma}|^2} - e^{i\phi} \left(1 - \left(1 - e^{i\theta}\right)|U_{\tau\gamma}|^2\right) \end{pmatrix} \tag{7}$$

To study the effect of phase rotations, we have studied, by directly multiplying the matrix $Q$ a given number of times with specified $\theta$ and $\phi$ values. For simplicity we put $N = 100$, $|\gamma\rangle = |0\rangle$. The unitary transformation $U$ is taken as $W$, the Hadamard-Walsh transformation. We have found that there exists a phase matching condition in constructing a useful quantum search algorithm. This relation is $\theta = \phi$.

In Fig.1 and Fig.2, we took $\theta = \pi/2$ and plotted the norm of the marked state amplitude after 7 and 8 iterations respectively for various values of $\phi$. We see the highest peak in both figures is at $\phi = \theta$. This shows that with phase matching the probability is the largest. For other values of $\phi$ the probability are small. When the number of iterations changes, the shape of the curve changes. For instance, in $|B_7|$ there are 4 peaks between $\pi/2$ and $\pi$, whereas in $|B_8|$, the number of peaks is 5. This is not what to be expected for a working quantum search algorithm, where as $j$ increases the norm of the amplitude should increase monotonically for small $j$ values.

In Fig.3 and 4, we have plotted $|B_j|$ versus $j$ for $\theta = \phi = \pi/2$, $\pi/10$ respectively. We see that when phase matching is satisfied, one can still construct a quantum search algorithm. The larger the rotating angle, the faster the algorithm. For instance when $\theta = \phi = \pi/2$, it requires approximately 10 iterations to reach 1. For $\theta = \phi = \pi/10$, it requires nearly 50 iterations to reach 1.
In Fig. 5, we have plotted the norm $|B_j|$ versus $j$ for a phase mismatching case: $\theta = \frac{\pi}{2}$ and $\phi = \frac{\pi}{10}$. We see that the norm of the amplitude of the marked state is restricted to a narrow range between 0.09 and 0.15. Even after 200 iterations, the amplitude still cannot be enhanced. Thus it cannot be used for searching at all.

To see the effect of phase matching on quantum searching algorithms, we have also plotted the norm of the amplitude of the marked state as a function of $\phi$ and $\theta$ in Fig. 6 for $|B_8|$. We see from the 3D plot clearly the mountain peaks along $\theta = \phi$ when the phases are matched. For other cases, the values are far less than 1.

In Fig. 7., we have given a 3D plot for $|B_j|$ as a function of $\theta$ and $j$. Here $\phi$ is chosen $\pi/2$. We see the $|\sin|$ like behavior of $|B_j|$ as a function of $j$ when phase matching is obtained. In other areas, the values of $|B_j|$ are small. It is only with phase matching that we can construct a quantum search algorithm.

Next, we discuss analytically the situations. In general, the matrix elements of the unitary transformation can be written as

$$U_{\tau \gamma} = e^{i \zeta} \sin (\beta).$$

(8)

The matrix of $Q$ can be written in a simple form

$$Q = \begin{pmatrix} ie^{i \phi} & 0 \\ 0 & -ie^{i(\phi+\frac{\pi}{2})} \end{pmatrix} \begin{pmatrix} i \cos \left(\frac{\theta}{2}\right) & 1 & 0 \\ -i \sin \left(\frac{\theta}{2}\right) & 0 & -1 \end{pmatrix} \begin{pmatrix} \cos (2\beta) & -\sin (2\beta) e^{i\zeta} \\ \sin (2\beta) e^{-i\zeta} & \cos (2\beta) \end{pmatrix}$$

(9)

In general $\beta$ is small and in the order of $1/\sqrt{N}$ where $N$ is the dimension of the database. Approximately we can write

$$Q \approx \begin{pmatrix} ie^{i \phi} & 0 \\ 0 & -ie^{i(\phi+\frac{\pi}{2})} \end{pmatrix} \begin{pmatrix} ie^{i \phi} & 2\sin \left(\frac{\theta}{2}\right) e^{i\zeta} \\ 2\sin \left(\frac{\theta}{2}\right) e^{-i\zeta} & -ie^{i \phi} \end{pmatrix}$$

(10)

By ignoring an overall phase factor $-e^{i \frac{\theta+\phi}{2}}$, we can rewrite the matrix as

$$Q = \begin{pmatrix} e^{i \frac{\theta+\phi}{2}} & -2i \sin \left(\frac{\theta}{2}\right) \beta e^{-i(\frac{\phi}{2}-\zeta)} \\ -2i \sin \left(\frac{\theta}{2}\right) \beta e^{i(\frac{\phi}{2}-\zeta)} & e^{i \frac{\theta+\phi}{2}} \end{pmatrix}$$

(11)
Let $\beta' = 2 \sin \left(\frac{\theta}{2}\right) \beta$, then

$$Q = \begin{pmatrix} e^{i \frac{\theta}{2}} & -i \beta' e^{-i \left(\frac{\phi}{2} - \zeta\right)} \\ -i \beta' e^{i \left(\frac{\phi}{2} - \zeta\right)} & e^{i \frac{\phi}{2} - \alpha} \end{pmatrix}. \tag{12}$$

When the phase matching condition is satisfied, $\theta = \phi$, we have

$$Q_{\text{new}} = \begin{pmatrix} 1 & -i \beta' e^{-i \left(\frac{\phi}{2} - \zeta\right)} \\ -i \beta' e^{i \left(\frac{\phi}{2} - \zeta\right)} & 1 \end{pmatrix} = I + \beta G \approx e^{\beta G}. \tag{13}$$

where

$$G = \begin{pmatrix} 0 & -ie^{-i \left(\frac{\phi}{2} - \zeta\right)} \\ -ie^{i \left(\frac{\phi}{2} - \zeta\right)} & 0 \end{pmatrix} \tag{14}$$

Using the properties of $G$, that is,

$$G^2 = -I, G^3 = -G, G^4 = I, \tag{15}$$

we can obtain for small $\beta$ the following expression for the product of $Q$,

$$Q^j = \cos j \beta' I + \sin j \beta' G = \begin{pmatrix} \cos (j \beta') & ie^{-i \left(\frac{\phi}{2} - \zeta\right)} \sin (j \beta') \\ ie^{i \left(\frac{\phi}{2} - \zeta\right)} \sin (j \beta') & \cos (j \beta') \end{pmatrix}$$

$$= \begin{pmatrix} \cos(j \beta') - \sin(j \beta') \\ \sin(j \beta') \cos(j \beta') \end{pmatrix}, \tag{16}$$

where in the last step the phase factor $-ie^{-i \left(\frac{\phi}{2} - \zeta\right)}$ is absorbed in the definition of the basis vector $|1\rangle$ (without loss of generality, we put this phase factor to 1 henceforth). Equation (16) is simply a rotation in the space span by $|1\rangle$ and $|2\rangle$. Each operation of $Q$ rotates the state vector of the quantum computer an angle $\beta' = 2 \sin \left(\frac{\theta}{2}\right) \beta$ towards the state $U^{-1} |\tau\rangle$.

After $j_m$ number of operations, the state vector of the computer is closest to $U^{-1} |\tau\rangle$, then another operation of $U$ will put the state of the computer to $|\tau\rangle$, the marked state. Then a measurement of the state of the computer will yield with near certainty the marked state.

Let's look at the $j_m$, the optimal number of iteration steps. Suppose the initial state of the
The computer is \( A_0|1\rangle + B_0|2\rangle = \cos \beta_0|1\rangle + \sin \beta_0|2\rangle \). Then the amplitude of the state \( U^{-1}|\tau\rangle \) after \( j \) iterations is

\[
B_j = \sin(j\beta') \cos \beta_0 + \cos(j\beta') \sin \beta_0 = \sin(j\beta' + \beta_0)
\]  

(17)

When \( \sin(j_m\beta' + \beta_0) \approx 1 \), we have maximum probability for the marked state. In this case,

\[
j_m\beta' + \beta_0 \approx \frac{\pi}{2}
\]

(18)

or

\[
j_m \sin \frac{\theta}{2} = (\frac{\pi}{2} - \beta_0)/2 \approx \frac{\frac{\pi}{4} - \beta_0}{\sin \frac{\theta}{2}}
\]

(19)

We have plotted the “exact” numerical results of \( j_m \sin(\theta)/2 \) for different values of \( N \). It is seen that the product is nearly a constant, validating our treatment.

We see from the equation (17) that when state \( |\tau\rangle \), the marked state and the prepared state \( |\gamma\rangle \) are rotated the same angle, we can construct a quantum search algorithm, but now it requires more steps. However, there may be cases where this can be used in the advantage, such as those proposed in Ref. \[5,13\]. In addition, a small rotation angle usually means a shorter realizing time, for instance in NMR, the angle of rotation is proportional to the duration of the radio frequency signals, this may be useful in some problems.

When phase matching is not satisfied, \( \theta \neq \phi \), we can give an approximate formula for small \( \beta \),

\[
Q^j \approx \begin{pmatrix}
\cos(j\delta) & -e^{-i(j-1)\delta} \left( \frac{1 - e^{2ij\delta}}{1 - e^{2i\delta}} \right) \\
e^{-i(j-1)\delta} \left( \frac{1 - e^{2ij\delta}}{1 - e^{2i\delta}} \right) & e^{-ij\delta}
\end{pmatrix} \beta'
\]

(20)

where \( \delta = (\theta - \phi)/2 \). We have omitted the high order terms of \( \beta \). When acted on an initial state of the form \( A_0|1\rangle + B_0|2\rangle \), the amplitude of the state \( |2\rangle = U^{-1}|\tau\rangle \) becomes

\[
B_j = e^{-i(j-1)\delta} \left( \frac{1 - e^{2ij\delta}}{1 - e^{2i\delta}} \right) \beta'A_0 + e^{ij\delta} B_0.
\]

(21)

Usually \( B_0 \) is small (for instance, in Grover’s original algorithm, the initial state is \( |0\rangle = -\sqrt{\frac{N-1}{N}}|1\rangle + \sqrt{\frac{1}{N}}|2\rangle \), \( B_0 = \sqrt{\frac{1}{N}} \)). The effectiveness of the algorithm relies heavily on the
first term. However, the factor \( \frac{1-e^{2ij\delta}}{1-e^{i\delta}} \) plays a crucial role in the phase matching. When \( \delta \) approaches zero, this factor approaches \( j \). More iterations of the operation will increase the amplitude. The factor can be written as a sum \( \sum_{j=0}^{j} e^{j\delta} \). When \( \delta \) is not zero (\( \delta \gg \beta' \)), from complex analysis, one sees that there are serious cancellations in the sum, and this makes this factor in an order of 1. Thus the operation does not increase the amplitude, and the quantum search algorithm fails.

To summarize, we have generalized Grover’s quantum search algorithm to arbitrary phase rotations. We have found that phase matching between the phase rotation of the marked state and the prepared state \( |\gamma\rangle \) is crucial in constructing a quantum search algorithm. Approximate formulas are given for the amplitude of the marked state after arbitrary number of iterations, and reasons are given to the failure of the algorithm when phase matching is not satisfied. The phase matching requirement has set a stringent condition in the experimental realization of the algorithm, as an imperfect gate operation may lead to phase mismatching and thus adversely affect the efficiency of the algorithm.

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REFERENCES

[1] P. Shor, in Proceedings of the 35th Annual Symposium on Foundations of Computer Science, 1994 (IEEE Computer Society Press, Los Alamos, CA, 1994) P.124-34.

[2] L. K. Grover, Phys. Rev. Lett. 79 (1997) 325.

[3] L. K. Grover, Phys. Rev. Lett., 79 (1997) 4709.

[4] G. Brassard, Science, 275 (1997) 627.

[5] G. Brassard, P. Høyer and A. Tapp, Lanl-eprint/quant-ph/9805082.

[6] I.L. Chuang, N. Gershenfeld, M. Kubinec, Phys. Rev. Lett. 80 (1998) 3408.

[7] J.A. Jones, M. Mosca, R. H. Hansen, Nature, 393 (1998) 344.

[8] J. A. Jones, M. Mosca, J. Chem. Phys. 109 (1998) 1648.

[9] I. L. Chuang, L.M.K. Vandersypen, Xinlan Zhou, D. W. Leung and S. Lloyd, Nature, 393 (1998) 143.

[10] C. Bennett et al, Lanl-eprint/quant-ph/9701001, also in SIAM journal on Computing.

[11] M. Boyer, G. Brassard, P. Høyer, A. Tapp, Lanl-eprint/quant-ph9605034; also in Fortsch. Phys. 46 (1998) 493.

[12] C. Zalka, Lanl-eprint/quant-ph/9711070.

[13] C. Zalka, Lanl-eprint/quant-ph/9902049.

[14] D. Biron et al, Lanl-eprint/quant-ph/9801066.

[15] A. Kumar Pati, Lanl-eprint/quant-ph/9807067.

[16] Y. Ozhigov, Lanl-eprint/quant-ph/9904039.

[17] R. Gingrich, C. P. Williams and N. Cerf, Lanl-eprint/quant-ph/9904049.

[18] R. Josca, Lanl-eprint/quant-ph/9901021.
[19] L.K. Grover, Phys. Rev. Lett., 80 (1998) 4329.

[20] G.L. Long, W.L. Zhang, Y. S. Li and L. Niu, Lanl-eprint/quant-ph/9904077, to appear in Communications in Theoretical Physics.
FIGURES

\begin{figure}
\centering
\includegraphics[width=\textwidth]{fig1}
\caption{$|B_\gamma|$ versus $\phi$ for $\theta = \frac{\pi}{2}$.}
\end{figure}

\begin{figure}
\centering
\includegraphics[width=\textwidth]{fig2}
\caption{$|B_8|$ versus $\phi$ for $\theta = \frac{\pi}{2}$.}
\end{figure}
FIG. 3. $|B_j|$ versus $j$ for $\theta = \phi = \frac{\pi}{2}$.

FIG. 4. $|B_j|$ versus $j$ for $\theta = \phi = \frac{\pi}{10}$.
FIG. 5. $|B_j|$ versus $j$ for $\theta = \frac{\pi}{2}$, $\phi = \frac{\pi}{10}$.

FIG. 6. 3D plot for $|B_8|$ versus $\theta$ and $\phi$.

FIG. 7. 3D plot for $|B_j|$ versus $\theta$ and $j$. $\phi = \pi/2$ and $\theta$ is in unit of $\pi/10$. 