THE TOPOLOGICAL STRUCTURE OF (HOMOGENEOUS) SPACES AND GROUPS WITH COUNTABLE cs*-CHARACTER

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Abstract. In this paper we introduce and study three new cardinal topological invariants called the cs*--, cs-, and sb-characters. The class of topological spaces with countable cs*-character is closed under many topological operations and contains all N-spaces and all spaces with point-countable cs*-network. Our principal result states that each non-metrizable sequential topological group with countable cs*-character has countable pseudo-character and contains an open kω-subgroup. This result is specific for topological groups: under Martin Axiom there exists a sequential topologically homogeneous kω-space X with N0 = csχ*(X) < ψ(X).

Introduction

In this paper we introduce and study three new local cardinal invariants of topological spaces called the sb-character, the cs-character and cs*-character, and describe the structure of sequential topological groups with countable cs*-character. All these characters are based on the notion of a network at a point x of a topological space X, under which we understand a collection N of subsets of X such that for any neighborhood U ⊂ X of x there is an element N ∈ N with x ∈ N ⊂ U, see [Lin].

A subset B of a topological space X is called a sequential barrier at a point x ∈ X if for any sequence (xn)n∈ω ⊂ X convergent to x, there is m ∈ ω such that xn ∈ B for all n ≥ m, see [Lin]. It is clear that each neighborhood of a point x ∈ X is a sequential barrier for x while the converse is true for Fréchet-Urysohn spaces.

Under a sb-network at a point x of a topological space X we shall understand a network at x consisting of sequential barriers at x. In other words, a collection N of subsets of a topological space X is a sb-network at x if for any neighborhood U of x there is an element N ∈ N such that for any sequence (xn) ⊂ X convergent to x the set N contains almost all elements of (xn). Changing two quantifiers in this definition by their places we get a definition of a cs-network at x.

Namely, we define a family N of subsets of a topological space X to be a cs-network (resp. a cs*-network) at a point x ∈ X if for any neighborhood U ⊂ X of x and any sequence (xn) ⊂ X convergent to x there is an element N ∈ N such that N ⊂ U and N contains almost all (resp. infinitely many) members of the sequence (xn). A family N of subsets of a topological space X is called a cs-network (resp. cs*-network) if it is a cs-network (resp. cs*-network) at each point x ∈ X, see [Na].

The smallest size |N| of an sb-network (resp. cs-network, cs*-network) N at a point x ∈ X is called the sb-character (resp. cs-character, cs*-character) of X at the point x and is denoted by sbχ(X, x) (resp. csχ(X, x), csχ*(X, x)). The cardinals sbχ(X) = supx∈X sbχ(X, x), csχ(X) = supx∈X csχ(X, x) and csχ*(X) = supx∈X csχ*(X, x) are called the sb-character, cs-character and cs*-character of the
topological space $X$, respectively. For the empty topological space $X = \emptyset$ we put $sb_\chi(X) = cs_\chi(X) = cs_\chi^*(X) = 1$.

In the sequel we shall say that a topological space $X$ has countable sb-character (resp. cs-, cs*-character) if $sb_\chi(X) \leq \aleph_0$ (resp. $cs_\chi(X) \leq \aleph_0$, $cs_\chi^*(X) \leq \aleph_0$). In should be mentioned that under different names topological spaces with countable sb- or cs-character have already occurred in topological literature. In particular, a topological space has countable cs-character if and only if it is csf-countable in the sense of [Lin]; a (sequential) space $X$ has countable sb-character if and only if it is universally csf-countable in the sense of [Lin] (if and only if it is weakly first-countable in the sense of [Ar1] if and only if it is 0-metrizable in the sense of Nedev [Ne]). From now on, all the topological spaces considered in the paper are $T_1$-spaces. At first we consider the interplay between the characters introduced above.

**Proposition 1.** Let $X$ be a topological space. Then

1. $cs_\chi^*(X) \leq cs_\chi(X) \leq sb_\chi(X) \leq \chi(X)$;
2. $\chi(X) = sb_\chi(X)$ if $X$ is Fréchet-Urysohn;
3. $cs_\chi^*(X) < \aleph_0$ iff $cs_\chi(X) < \aleph_0$ iff $sb_\chi(X) < \aleph_0$ iff $cs_\chi^*(X) = 1$ iff $cs_\chi(X) = 1$ iff each convergent sequence in $X$ is trivial;
4. $sb_\chi(X) \leq 2^{cs_\chi^*(X)}$;
5. $cs_\chi(X) \leq cs_\chi^*(X) \cdot \sup\{\|\kappa\|^{\omega} : \kappa < cs_\chi^*(X)\} \leq (cs_\chi^*(X))^{\aleph_0}$ where $\|\kappa\|^{\omega} = \{A \subseteq \kappa : |A| < \aleph_0\}$.

Here “iff” is an abbreviation for “if and only if”. The Arens’ space $S_2$ and the sequential fan $S_\omega$ give us simple examples distinguishing between some of the characters considered above. We recall that the Arens’ space $S_2$ is the set $\{(0,0), (\frac{1}{n},0), (\frac{1}{m}, \frac{1}{nm}) : n, m \in \mathbb{N}\} \subset \mathbb{R}^2$ carrying the strongest topology inducing the original planar topology on the convergent sequences $C_0 = \{(0,0), (\frac{1}{n},0) : n \in \mathbb{N}\}$ and $C_n = \{(\frac{1}{n},0), (\frac{1}{m}, \frac{1}{nm}) : m \in \mathbb{N}\}, n \in \mathbb{N}$. The quotient space $S_\omega = S_2/C_0$ obtained from the Arens’ space $S_2$ by identifying the points of the sequence $C_0$ is called the sequential fan, see [Lin]. The sequential fan $S_\omega$ is the simplest example of a non-metrizable Fréchet-Urysohn space while $S_2$ is the simplest example of a sequential space which is not Fréchet-Urysohn.

We recall that a topological space $X$ is sequential if a subset $A \subset X$ if closed if and only if $A$ is sequentially closed in the sense that $A$ contain the limit point of any sequence $(a_n) \subset A$, convergent in $X$. A topological space $X$ is Fréchet-Urysohn if for any cluster point $a \in X$ of a subset $A \subset X$ there is a sequence $(a_n) \subset A$, convergent to $a$.

Observe that $\aleph_0 = cs_\chi(S_2) = cs_\chi(S_\omega) = sb_\chi(S_2) = \chi(S_2) = \varnothing$ while $\aleph_0 = cs_\chi(S_\omega) = cs_\chi(S_\omega) = sb_\chi(S_\omega) = \chi(S_\omega) = \varnothing$. Here $\varnothing$ is the well-known in Set Theory small uncountable cardinal equal to the cofinality of the partially ordered set $\mathbb{N}^\omega$ endowed with the natural partial order: $(x_n) \leq (y_n)$ iff $x_n \leq y_n$ for all $n$, see [Va]. Besides $\varnothing$, we will need two other small cardinals: $b$ defined as the smallest size of a subset of uncountable cofinality in $(\mathbb{N}^\omega, \leq)$, and $p$ equal to the smallest size $|\mathcal{F}|$ of a family of infinite subsets of $\omega$ closed under finite intersections and having no infinite pseudo-intersection in the sense that there is no infinite subset $I \subset \omega$ such that the complement $I \setminus F$ is finite for any $F \in \mathcal{F}$, see [Va], [yD]. It is known that $\aleph_1 \leq p \leq b \leq \varnothing \leq \varnothing \leq \omega$ where $\omega$ stands for the size of continuum. Martin Axiom implies $p = b = \varnothing = \omega$, [MS]. On the other hand, for any uncountable regular
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...cardinals $\lambda \leq \kappa$ there is a model of ZFC with $p = b = \delta = \lambda$ and $c = \kappa$, see [vD, 5.1]. Unlike to the cardinal invariants $cs_\kappa$, $sb_\kappa$, and $\chi$ which can be distinguished on simple spaces, the difference between the cardinal invariants $cs_\chi$ and $cs^*_\chi$ is more subtle: they cannot be distinguished in some models of Set Theory!

**Proposition 2.** Let $X$ be a topological space. Then $cs^*_\chi(X) = cs_\chi(X)$ provided one of the following conditions is satisfied:

1. $cs^*_\chi(X) < p$;
2. $\kappa^{\aleph_0} \leq cs^*_\chi(X)$ for any cardinal $\kappa < cs^*_\chi(X)$;
3. $p = \omega$ and $\lambda^\omega \leq \kappa$ for any cardinals $\lambda < \kappa \geq \omega$;
4. $p = \omega$ (this is so under MA) and $X$ is countable;
5. the Generalized Continuum Hypothesis holds.

Unlike to the usual character, the $cs^*$-, $cs$-, and $sb$-characters behave nicely with respect to many countable topological operations.

Among such operation there are: the Tychonov product, the box-product, producing a sequentially homeomorphic copy, taking image under a sequentially open map, and forming inductive topologies.

As usual, under the box-product $\square_{i\in I}X_i$ of topological spaces $X_i$, $i \in I$, we understand the Cartesian product $\prod_{i\in I}X_i$ endowed with the box-product topology generated by the base consisting of products $\prod_{i\in I}U_i$ where each $U_i$ is open in $X_i$. In contrast, by $\prod_{i\in I}X_i$ we denote the usual Cartesian product of the spaces $X_i$, endowed with the Tychonov product topology.

We say that a topological space $X$ carries the inductive topology with respect to a cover $\mathcal{C}$ of $X$ if a subset $F \subset X$ is closed in $X$ if and only if the intersection $F \cap C$ is closed in $C$ for each element $C \in \mathcal{C}$. For a cover $\mathcal{C}$ of $X$ let $\text{ord}(\mathcal{C}) = \sup_{x \in X} \text{ord}(\mathcal{C}, x)$ where $\text{ord}(\mathcal{C}, x) = |\{C \in \mathcal{C} : x \in C\}|$. A topological space $X$ carrying the inductive topology with respect to a countable cover by closed metrizable (resp. compact, compact metrizable) subspaces is called an $\mathcal{M}_\omega$-space (resp. a $k_\omega$-space, $\mathcal{MK}_\omega$-space).

A function $f : X \to Y$ between topological spaces is called sequentially continuous if for any convergent sequence $(x_n)$ in $X$ the sequence $(f(x_n))$ is convergent in $Y$ to $f(\lim x_n)$; $f$ is called a sequential homeomorphism if $f$ is bijective and both $f$ and $f^{-1}$ are sequentially continuous. Topological spaces $X, Y$ are defined to be sequentially homeomorphic if there is a sequential homeomorphism $h : X \to Y$. Observe that two spaces are sequentially homeomorphic if and only if their sequential coreflexions are homeomorphic. Under the sequential coreflexion $\sigma X$ of a topological space $X$ we understand $X$ endowed with the topology consisting of all sequentially open subsets of $X$ (a subset $U$ of $X$ is sequentially open if its complement is sequentially closed in $X$; equivalently $U$ is a sequential barrier at each point $x \in U$). Note that the identity map $id : \sigma X \to X$ is continuous while its inverse is sequentially continuous, see [Lin].

A map $f : X \to Y$ is sequentially open if for any point $x_0 \in X$ and a sequence $S \subset Y$ convergent to $f(x_0)$ there is a sequence $T \subset X$ convergent to $x_0$ and such that $f(T) \subset S$. Observe that a bijective map $f$ is sequentially open if its inverse $f^{-1}$ is sequentially continuous.

The following technical Proposition is an easy consequence of the corresponding definitions.
Proposition 3. (1) If $X$ is a subspace of a topological space $Y$, then $cs^*_X(X) \leq cs^*_X(Y)$, $cs_X(Y) \leq cs_X(Y)$ and $sb_X(Y) \leq sb(Y)$.

(2) If $f : X \to Y$ is a surjective continuous sequentially open map between topological spaces, then $cs^*_X(Y) \leq cs^*_X(X)$ and $sb_X(Y) \leq sb_X(X)$.

(3) If $f : X \to Y$ is a surjective sequentially continuous sequentially open map between topological spaces, then $\min\{cs^*_X(Y), \aleph_1\} \leq \min\{cs^*_X(X), \aleph_1\}$, $\min\{cs_X(Y), \aleph_1\} \leq \min\{cs_X(X), \aleph_1\}$, and $\min\{sb_X(Y), \aleph_1\} \leq \min\{sb_X(X), \aleph_1\}$.

(4) If $X, Y$ are sequentially homeomorphic topological spaces, then $\min\{cs^*_X(X), \aleph_1\} = \min\{cs_X(Y), \aleph_1\} = \min\{cs^*_X(Y), \aleph_1\}$, and $\min\{sb_X(Y), \aleph_1\} = \min\{sb_X(X), \aleph_1\}$.

(5) $\min\{sb_X(X), \aleph_1\} = \min\{sb_X(\sigma X), \aleph_1\} \leq sb_X(\sigma X) \leq sb_X(X)$ and $cs_X(X) \leq cs_X(\sigma X) \geq \min\{cs_X(\sigma X), \aleph_1\} = \min\{cs_X(\sigma X), \aleph_1\} = \min\{cs_X(X), \aleph_1\} = \min\{cs_X(\sigma X), \aleph_1\} \leq cs^*_X(\sigma X) \geq cs^*_X(X)$ for any topological space $X$.

(6) If $X = \prod_{i \in I} X_i$ is the Tychonov product of topological spaces $X_i$, $i \in I$, then $cs^*_X(X) \leq \sum_{i \in I} cs^*_X(X_i)$, $cs_X(X) \leq \sum_{i \in I} cs_X(X_i)$ and $sb_X(X) \leq \sum_{i \in I} sb_X(X_i)$.

(7) If $X = \boxprod_{i \in I} X_i$ is the box-product of topological spaces $X_i$, $i \in I$, then $cs^*_X(X) \leq \sum_{i \in I} cs^*_X(X_i)$ and $cs_X(X) \leq \sum_{i \in I} cs_X(X_i)$.

(8) If a topological space $X$ carries the inductive topology with respect to a cover $C$ of $X$, then $cs^*_X(X) \leq \sup_{C \in C} cs^*_X(C)$.

(9) If a topological space $X$ carries the inductive topology with respect to a point-countable cover $C$ of $X$, then $cs_X(X) \leq \sup_{C \in C} cs_X(C)$.

(10) If a topological space $X$ carries the inductive topology with respect to a point-finite cover $C$ of $X$, then $sb_X(X) \leq \sup_{C \in C} sb_X(C)$.

Since each first-countable space has countable cs*-character, it is natural to consider the class of topological spaces with countable cs*-character as a class of generalized metric spaces. However, this class contains very non-metrizable spaces like $\beta \mathbb{N}$, the Stone-Čech compactification of the discrete space of positive integers. The reason is that $\beta \mathbb{N}$ contains no non-trivial convergent sequence. To avoid such pathologies we shall restrict ourselves by sequential spaces. Observe that a topological space is sequential if $X$ carries the inductive topology with respect to a cover by sequential subspaces. In particular, each $\mathcal{M}_\omega$-space is sequential and has countable cs*-character. Our principal result states that for topological groups the converse is also true. Under an $\mathcal{M}_\omega$-group (resp. $\mathcal{MK}_\omega$-group) we understand a topological group whose underlying topological space is an $\mathcal{M}_\omega$-space (resp. $\mathcal{MK}_\omega$-space).

Theorem 1. Each sequential topological group $G$ with countable cs*-character is an $\mathcal{M}_\omega$-group. More precisely, either $G$ is metrizable or else $G$ contains an open $\mathcal{MK}_\omega$-subgroup $H$ and is homeomorphic to the product $H \times D$ for some discrete space $D$.

For $\mathcal{M}_\omega$-groups the second part of this theorem was proven in [Ba1]. Theorem 1 has many interesting corollaries.

At first we show that for sequential topological groups with countable cs*-character many important cardinal invariants are countable, coincide or take some fixed values. Let us remind some definitions, see [En1]. For a topological space $X$ recall that
• the \textit{pseudocharacter} \(\psi(X)\) is the smallest cardinal \(\kappa\) such that each one
point set \(\{x\} \subset X\) can be written as the intersection \(\{x\} = \cap \mathcal{U}\) of some
family \(\mathcal{U}\) of open subsets of \(X\) with \(|\mathcal{U}| \leq \kappa\);
• the \textit{cellularity} \(c(X)\) is the smallest cardinal \(\kappa\) such that \(X\) contains no family
\(\mathcal{U}\) of size \(|\mathcal{U}| > \kappa\) consisting of non-empty pairwise disjoint open subsets;
• the \textit{Lindelöf number} \(l(X)\) is the smallest cardinal \(\kappa\) such that each open
cover of \(X\) contains a subcover of size \(\leq \kappa\);
• the \textit{density} \(d(X)\) is the smallest size of a dense subset of \(X\);
• the \textit{tightness} \(t(X)\) is the smallest cardinal \(\kappa\) such that for any subset \(A \subset X\)
and a point \(a \in A\) from its closure there is a subset \(B \subset A\) of size \(|B| \leq \kappa\)
with \(a \in B\);
• the \textit{extent} \(e(X)\) is the smallest cardinal \(\kappa\) such that \(X\) contains no closed
discrete subspace of size \(> \kappa\);
• the \textit{compact covering number} \(kc(X)\) is the smallest size of a cover of \(X\) by
compact subsets;
• the \textit{weight} \(w(X)\) is the smallest size of a base of the topology of \(X\);
• the \textit{network weight} \(nw(X)\) is the smallest size \(|\mathcal{N}|\) of a topological network
for \(X\) (a family \(\mathcal{N}\) of subsets of \(X\) is a \textit{topological network} if for any open
set \(U \subset X\) and any point \(x \in U\) there is \(N \in \mathcal{N}\) with \(x \in N \subset U\));
• the \textit{k-network weight} \(knw(X)\) is the smallest size \(|\mathcal{N}|\) of a \(k\)-network for
\(X\) (a family \(\mathcal{N}\) of subsets of \(X\) is a \(k\)-\textit{network} if for any open set \(U \subset X\)
and any compact subset \(K \subset U\) there is a finite subfamily \(\mathcal{M} \subset \mathcal{N}\) with
\(K \subset \cup \mathcal{M} \subset U\)).

For each topological space \(X\) these cardinal invariants relate as follows:

\[
\max\{e(X), l(X), e(X)\} \leq nw(X) \leq knw(X) \leq w(X).
\]

For metrizable spaces all of them are equal, see [En1, 4.1.15].

In the class of \(k\)-spaces there is another cardinal invariant, the \(k\)-ness introduced
by E. van Douwen, see [vD, §8]. We remind that a topological space \(X\) is called a
\(k\)-\textit{space} if it carries the inductive topology with respect to the cover of \(X\) by all
compact subsets. It is clear that each sequential space is a \(k\)-space. The \(k\)-\textit{ness}
\(k(X)\) of a \(k\)-space is the smallest size \(|\mathcal{K}|\) of a cover \(\mathcal{K}\) of \(X\) by compact subsets such
that \(X\) carries the inductive topology with respect to the cover \(\mathcal{K}\). It is interesting
to notice that \(k(\mathbb{N}) = 0\) while \(k(\mathbb{Q}) = \aleph_0\), see [vD]. Proposition 3(8) implies that
\(cs^\prec(X) \leq k(X) \cdot \psi(X) \geq kc(X)\) for each \(k\)-space \(X\). Observe also that a topological
space \(X\) is a \(k\)\(_n\)-space if and only if \(X\) is a \(k\)-space with \(k(X) \leq \aleph_0\).

Besides cardinal invariants we shall consider an ordinal invariant, called the
sequential order. Under the \textit{sequential closure} \(A^{(1)}\) of a subset \(A\) of a topological
space \(X\) we understand the set of all limit point of sequences \((a_n) \subset A\), convergent
in \(X\). Given an ordinal \(\alpha\) define the \(\alpha\)-th sequential closure \(A^{(\alpha)}\) of \(A\) by transfinite
induction: \(A^{(\alpha)} = \bigcup_{\beta < \alpha} (A^{(\beta)})^{(1)}\). Under the \textit{sequential order} \(so(X)\) of a topological
space \(X\) we understand the smallest ordinal \(\alpha\) such that \(A^{(\alpha+1)} = A^{(\alpha)}\) for any
subset \(A \subset X\). Observe that a topological space \(X\) is Fréchet-Urysohn if and only
if \(so(X) \leq 1\); \(X\) is sequential if and only if \(cl_X(A) = A^{(so(X))}\) for any subset \(A \subset X\).

Besides purely topological invariants we shall also consider a cardinal invariant,
specific for topological groups. For a topological group \(G\) let \(ib(G)\), the \textit{boundedness
index} of \(G\) be the smallest cardinal \(\kappa\) such that for any nonempty open set \(U \subset G\)
there is a subset \(F \subset G\) of size \(|F| \leq \kappa\) such that \(G = F \cdot U\). It is known that
\[ \text{ib}(G) \leq \min\{c(G), l(G), e(G)\} \text{ and } w(G) = \text{ib}(G) \cdot \chi(G) \text{ for each topological group, see } [Tk]. \]

**Theorem 2.** Each sequential topological group \( G \) with countable \( cs^{*} \)-character has the following properties: \( \psi(G) \leq \aleph_0, \text{sb}_\chi(G) = \chi(G) \in \{1, \aleph_0, \varnothing\}, \) \( \text{ib}(G) = e(G) = d(G) = l(G) = e(G) = nw(G) = knw(G), \) and \( \text{so}(G) \in \{1, \omega_1\}. \)

We shall derive from Theorems 1 and 2 an unexpected metrization theorem for topological groups. But first we need to remind the definitions of some of \( \alpha_{\gamma} \)-spaces, \( i = 1, \ldots, 6 \) introduced by A.V. Arkhangelski in \([Ar_2], [Ar_4]\). We also define a wider class of \( \alpha_{\gamma} \)-spaces.

A topological space \( X \) is called

- an \( \alpha_1 \)-space if for any sequences \( S_n \subset X, n \in \omega, \) convergent to a point \( x \in X \) there is a sequence \( S \subset X \) convergent to \( x \) and such that \( S_n \setminus S \) is finite for all \( n \);
- an \( \alpha_3 \)-space if for any sequences \( S_n \subset X, n \in \omega, \) convergent to a point \( x \in X \) there is a sequence \( S \subset X \) convergent to \( x \) and such that \( S_n \cap S \neq \emptyset \) for infinitely many sequences \( S_n \);
- an \( \alpha_7 \)-space if for any sequences \( S_n \subset X, n \in \omega, \) convergent to a point \( x \in X \) there is a sequence \( S \subset X \) convergent to some point \( y \in X \) and such that \( S_n \cap S \neq \emptyset \) for infinitely many sequences \( S_n \);

Under a sequence converging to a point \( x \) of a topological space \( X \) we understand any countable infinite subset \( S \) of \( X \) such that \( S \setminus U \) if finite for any neighborhood \( U \) of \( x \). Each \( \alpha_1 \)-space is \( \alpha_4 \) and each \( \alpha_4 \)-space is \( \alpha_7 \). Quite often \( \alpha_7 \)-spaces are \( \alpha_4 \), see Lemma 7. Observe also that each sequentially compact space is \( \alpha_7 \). It can be shown that a topological space \( X \) is an \( \alpha_7 \)-space if and only if it contains no closed copy of the sequential fan \( S_\omega \) in its sequential coreflexion \( \sigma X \). If \( X \) is an \( \alpha_4 \)-space, then \( \sigma X \) contains no topological copy of \( S_\omega \).

We remind that a topological group \( G \) is **Weil complete** if it is complete in its left (equivalently, right) uniformity. According to \([PZ, 4.1.6]\), each \( k_\omega \)-group is Weil complete. The following metrization theorem can be easily derived from Theorems 1, 2 and elementary properties of \( M\mathcal{K}_{k_\omega} \)-groups.

**Theorem 3.** A sequential topological group \( G \) with countable \( cs^{*} \)-character is metrizable if one of the following conditions is satisfied:

1. \( \text{so}(G) < \omega_1 \);
2. \( \text{sb}_\chi(G) < \varnothing \);
3. \( \text{ib}(G) < k(G) \);
4. \( G \) is Fréchet-Urysohn;
5. \( G \) is an \( \alpha_7 \)-space;
6. \( G \) contains no closed copy of \( S_\omega \) or \( S_2 \);
7. \( G \) is not Weil complete;
8. \( G \) is Baire;
9. \( \text{ib}(G) < |G| < 2^{\aleph_0} \).

According to Theorem 1, each sequential topological group with countable \( cs^{*} \)-character is an \( M_{\omega} \)-group. The first author has proved in \([Ba_3]\) that the topological structure of a non-metrizable punctiform \( M_{\omega} \)-group is completely determined by its density and the compact scatteredness rank.
Recall that a topological space $X$ is \textit{punctiform} if $X$ contains no compact connected subspace containing more than one point, see [En$_2$, 1.4.3]. In particular, each zero-dimensional space is punctiform.

Next, we remind the definition of the scatteredness height. Given a topological space $X$ let $X_{(1)} \subset X$ denote the set of all non-isolated points of $X$. For each ordinal $\alpha$ define the $\alpha$-th derived set $X_{(\alpha)}$ of $X$ by transfinite induction: $X_{(\alpha)} = \bigcap_{\beta < \alpha} (X_{(\beta)})_{(1)}$. Under the \textit{scatteredness height} $\text{sch}(X)$ of $X$ we understand the smallest ordinal $\alpha$ such that $X_{(\alpha+1)} = X_{(\alpha)}$. A topological space $X$ is \textit{scattered} if $X_{(\alpha)} = \emptyset$ for some ordinal $\alpha$. Under the \textit{compact scatteredness rank} of a topological space $X$ we understand the ordinal $\text{scr}(X) = \sup \{ \text{sch}(K) : K \text{ is a scattered compact subspace of } X \}$.

**Theorem 4.** Two non-metrizable sequential punctiform topological groups $G, H$ with countable $c^*$-character are homeomorphic if and only if $d(G) = d(H)$ and $\text{scr}(G) = \text{scr}(H)$.

This theorem follows from Theorem 1 and “Main Theorem” of [Ba$_3$] asserting that two non-metrizable punctiform $M_{\omega}$-groups $G, H$ are homeomorphic if and only if $d(G) = d(H)$ and $\text{scr}(G) = \text{scr}(H)$. For countable $k_{\omega}$-groups this fact was proven by E.Zelenyuk [Ze$_1$].

The topological classification of non-metrizable sequential locally convex spaces with countable $c^*$-character is even more simple. Any such a space is homeomorphic either to $\mathbb{R}^\infty$ or to $\mathbb{R}^\infty \times Q$ where $Q = [0, 1]^\omega$ is the Hilbert cube and $\mathbb{R}^\infty$ is a linear space of countable algebraic dimension, carrying the strongest locally convex topology. It is well-known that this topology is inductive with respect to the cover of $\mathbb{R}^\infty$ by finite-dimensional linear subspaces. The topological characterization of the spaces $\mathbb{R}^\infty$ and $\mathbb{R}^\infty \times Q$ was given in [Sa]. In [Ba$_2$] it was shown that each infinite-dimensional locally convex $MK_{\omega}$-space is homeomorphic to $\mathbb{R}^\infty$ or $\mathbb{R}^\infty \times Q$. This result together with Theorem 1 implies the following classification

**Corollary 1.** Each non-metrizable sequential locally convex space with countable $c^*$-character is homeomorphic to $\mathbb{R}^\infty$ or $\mathbb{R}^\infty \times Q$.

As we saw in Theorem 2, each sequential topological group with countable $c^*$-character has countable pseudocharacter. The proof of this result is based on the fact that compact subsets of sequential topological groups with countable $c^*$-character are first countable. This naturally leads to a conjecture that compact spaces with countable $c^*$-character are first countable. Surprisingly, but this conjecture is false: assuming the Continuum Hypothesis N. Yakovlev [Ya] has constructed a scattered sequential compactum which has countable sb-character but fails to be first countable. In [Ny$_2$] P.Nyikos pointed out that the Yakovlev construction still can be carried under the assumption $\mathfrak{b} = \mathfrak{c}$. More precisely, we have

**Proposition 4.** Under $\mathfrak{b} = \mathfrak{c}$ there is a regular locally compact locally countable space $Y$ whose one-point compactification $\alpha Y$ is sequential and satisfies $\aleph_0 = \text{sb}(\alpha Y) < \psi(\alpha Y) = \mathfrak{c}$.

We shall use the above proposition to construct examples of topologically homogeneous spaces with countable $c^*$-character and uncountable pseudocharacter. This shows that Theorem 2 is specific for topological groups and cannot be generalized to topologically homogeneous spaces. We remind that a topological space $X$
is **topologically homogeneous** if for any points \( x, y \in X \) there is a homeomorphism \( h : X \to X \) with \( h(x) = y \).

**Theorem 5.**

1. There is a topologically homogeneous countable regular \( k_\omega \)-space \( X_1 \) with \( \aleph_0 = \text{sb}(X_1) < \chi(X_1) = d \) and \( \text{so}(X_1) = \omega \);
2. Under \( b = c \) there is a sequential topologically homogeneous zero-dimensional \( k_\omega \)-space \( X_2 \) with \( \aleph_0 = \text{cs}(X_2) < \psi(X_2) = c \);
3. Under \( b = c \) there is a sequential topologically homogeneous totally disconnected space \( X_3 \) with \( \aleph_0 = \text{sb}(X_3) < \psi(X_3) = c \).

We remind that a space \( X \) is **totally disconnected** if for any distinct points \( x, y \in X \) there is a continuous function \( f : X \to \{0, 1\} \) such that \( f(x) \neq f(y) \), see [En2].

**Remark 1.** The space \( X_1 \) from Theorem 5(1) is the well-known Arkhangel’ski-Franklin example [AF] (see also [Co, 10.1]) of a countable topologically homogeneous \( k_\omega \)-space, homeomorphic to no topological group (this also follows from Theorem 2). On the other hand, according to [Ze2], each topologically homogeneous countable regular space (in particular, \( X_1 \)) is homeomorphic to a quasitopological group, that is a topological space endowed with a separately continuous group operation with continuous inversion. This shows that Theorem 2 cannot be generalized onto quasitopological groups (see however [Zd] for generalizations of Theorems 1 and 2 to some other topologo-algebraic structures).

Next, we find conditions under which a space with countable \( \text{cs}^* \) -character is first-countable or has countable \( \text{sb} \) -character. Following [Ar3] we define a topological space \( X \) to be **c-sequential** if for each closed subspace \( Y \subset X \) and each non-isolated point \( y \) of \( Y \) there is a sequence \( (y_n) \subset Y \setminus \{y\} \) convergent to \( y \). It is clear that each sequential space is c-sequential. A point \( x \) of a topological space \( X \) is called **regular \( G_\delta \)** if \( \{x\} = \bigcap B \) for some countable family \( B \) of closed neighborhood of \( x \) in \( X \), see [Lin].

First we characterize spaces with countable \( \text{sb} \) -character (the first three items of this characterization were proved by Lin [Lin, 3.13] in terms of (universally) \( \text{cs}^* \)-countable spaces).

**Proposition 5.** For a Hausdorff space \( X \) the following conditions are equivalent:

1. \( X \) has countable \( \text{sb} \) -character;
2. \( X \) is an \( \alpha_1 \) -space with countable \( \text{cs}^* \) -character;
3. \( X \) is an \( \alpha_4 \) -space with countable \( \text{cs}^* \) -character;
4. \( \text{cs}^*_x(X) \leq \aleph_0 \) and \( \text{sb}_x(X) < p \).

Moreover, if \( X \) is c-sequential and each point of \( X \) is regular \( G_\delta \), then the conditions (1)-(4) are equivalent to:

5. \( \text{cs}^*_x(X) \leq \aleph_0 \) and \( \text{sb}_x(X) < d \).

Next, we give a characterization of first-countable spaces in the same spirit (the equivalences (1) \( \iff \) (2) \( \iff \) (5) were proved by Lin [Lin, 2.8]).

**Proposition 6.** For a Hausdorff space \( X \) with countable \( \text{cs}^* \) -character the following conditions are equivalent:

1. \( X \) is first-countable;
2. \( X \) is Fréchet-Urysohn and has countable \( \text{sb} \) -character;
3. \( X \) is Fréchet-Urysohn \( \alpha_7 \) -space;
Moreover, if each point of $X$ is regular $G_{\delta}$, then the conditions (1)–(4) are equivalent to:

(5) $X$ is a sequential space containing no closed copy of $S_2$ or $S_{\omega}$;
(6) $X$ is a sequential space with $\chi(X) < \omega$.

For Fréchet-Urysohn (resp. dyadic) compacta the countability of the $cs^*$-character is equivalent to the first countability (resp. the metrizability). We remind that a compact Hausdorff space $X$ is called dyadic if $X$ is a continuous image of the Cantor discontinuum $\{0,1\}^\kappa$ for some cardinal $\kappa$.

**Proposition 7.**

(1) A Fréchet-Urysohn countably compact space is first-countable if and only if it has countable $cs^*$-character.

(2) A dyadic compactum is metrizable if and only if its has countable $cs^*$-character.

In light of Proposition 7(1) one can suggest that $cs^*_\chi(X) = \chi(X)$ for any compact Fréchet-Urysohn space $X$. However that is not true: under CH, $cs^*_\chi(\alpha D) \neq \chi(\alpha D)$ for the one-point compactification $\alpha D$ of a discrete space $D$ of size $|D| = \aleph_2$. Surprisingly, but the problem of calculating the $cs^*$- and $cs$-characters of the spaces $\alpha D$ is not trivial and the definitive answer is known only under the Generalized Continuum Hypothesis. First we note that the cardinals $cs^*_\chi(\alpha D)$ and $cs^*_\chi(\alpha D)$ admit an interesting interpretation which will be used for their calculation.

**Proposition 8.** Let $D$ be an infinite discrete space. Then

(1) $cs^*_\chi(\alpha D) = \min\{w(X) : X$ is a (regular zero-dimensional) topological space of size $|X| = |D|$ containing non no-trivial convergent sequence$\};$
(2) $cs^*_\chi(\alpha D) = \min\{w(X) : X$ is a (regular zero-dimensional) topological space of size $|X| = |D|$ containing no countable non-discrete subspace$\}.$

For a cardinal $\kappa$ we put $\log \kappa = \min\{\lambda : \kappa \leq 2^{\lambda}\}$ and $\text{cof}(\alpha|^{\leq \omega})$ be the smallest size of a collection $C \subset [\alpha|^{\leq \omega}$ such that each at most countable subset $S \subset C$ lies in some element $C \in C$. Observe that $\text{cof}(\alpha|^{\leq \omega}) \leq \kappa^{\omega}$ but sometimes the inequality can be strict: $1 = \text{cof}(\aleph_0|^{\leq \omega}) < \aleph_0$ and $\aleph_1 = \text{cof}(\aleph_1|^{\leq \omega}) < \aleph_1^{\aleph_0}$. In the following proposition we collect all the information on the cardinals $cs^*_\chi(\alpha D)$ and $cs^*_\chi(\alpha D)$ we know.

**Proposition 9.** Let $D$ be an uncountable discrete space. Then

(1) $\aleph_1 \cdot \log |D| \leq cs^*_\chi(\alpha D) \leq cs^*_\chi(\alpha D) \leq \min\{|D|, 2^{\aleph_0} \cdot \text{cof}(\log |D|)^{\leq \omega}\}$ while $sb^*_\chi(\alpha D) = \chi(\alpha D) = |D|$;
(2) $cs^*_\chi(\alpha D) = cs^*_\chi(\alpha D) = \aleph_1 \cdot \log |D|$ under GCH.

In spite of numerous efforts some annoying problems concerning $cs^*$- and $cs$-characters still rest open.

**Problem 1.** Is there a (necessarily consistent) example of a space $X$ with $cs^*_\chi(X) \neq cs^*_\chi(X)$? In particular, is $cs^*_\chi(\alpha \kappa) \neq cs^*_\chi(\alpha \kappa)$ in some model of ZFC?
In light of Proposition 8 it is natural to consider the following three cardinal characteristics of the continuum which seem to be new:

\[ w_1 = \min \{ w(X) : X \text{ is a topological space of size } |X| = \aleph_0 \text{ containing no non-trivial convergent sequence} \} ; \]

\[ w_2 = \min \{ w(X) : X \text{ is a topological space of size } |X| = \aleph_0 \text{ containing no non-discrete countable subspace} \} ; \]

\[ w_3 = \min \{ w(X) : X \text{ is a } P\text{-space of size } |X| = \aleph_0 \} . \]

As expected, a \( P\)-space is a \( T_1\)-space whose any \( G_\delta\)-subset is open. Observe that \( w_1 = c^* \chi(\alpha \cdot \aleph_0) \) while \( w_2 = c^* \chi(\alpha \cdot \aleph_0) \). It is clear that \( \aleph_1 \leq w_1 \leq w_2 \leq w_3 \leq \aleph_0 \) and hence the cardinals \( w_i, i = 1, 2, 3 \), fall into the category of small uncountable cardinals, see [Va].

**Problem 2.** Are the cardinals \( w_i, i = 1, 2, 3 \), equal to (or can be estimated via) some known small uncountable cardinals considered in Set Theory? Is \( w_1 < w_2 < w_3 \) in some model of ZFC?

Our next question concerns the assumption \( b = \aleph_1 \) in Theorem 5.

**Problem 3.** Is there a ZFC-example of a sequential space \( X \) with \( sb_\chi(X) < \psi(X) \) or at least \( cs^*_\chi(X) < \psi(X) \)?

Propositions 1 and 5 imply that \( sb_\chi(X) \in \{ 1, \aleph_0 \} \cup [\aleph_0, \aleph_1] \) for any \( c\)-sequential topological space \( X \) with countable \( cs^*\)-character. On the other hand, for a sequential topological group \( G \) with countable \( cs^*\)-character we have a more precise estimate \( sb_\chi(G) \in \{ 1, \aleph_0, \aleph_1 \} \).

**Problem 4.** Is \( sb_\chi(X) \in \{ 1, \aleph_0, \aleph_1 \} \) for any sequential space \( X \) with countable \( cs^*\)-character?

As we saw in Proposition 7, \( \chi(X) \leq \aleph_0 \) for any Fréchet-Urysohn compactum \( X \) with \( cs_\chi(X) \leq \aleph_0 \).

**Problem 5.** Is \( sb_\chi(X) \leq \aleph_0 \) for any sequential (scattered) compactum \( X \) with \( cs_\chi(X) \leq \aleph_0 \) ?

Now we pass to proofs of our results.

**ON SEQUENCE TREES IN TOPOLOGICAL GROUPS**

Our basic instrument in proofs of main results is the concept of a sequence tree. As usual, under a **tree** we understand a partially ordered subset \((T, \leq)\) such that for each \( t \in T \) the set \( \downarrow t = \{ \tau \in T : \sigma \leq t \} \) is well-ordered by the order \( \leq \). Given an element \( t \in T \) let \( \uparrow t = \{ \tau \in T : \tau \geq t \} \) and \( \text{succ}(t) = \min(\uparrow t \setminus \{t\}) \) be the set of successors of \( t \) in \( T \). A maximal linearly ordered subset of a tree \( T \) is called a **branch** of \( T \). By \( \text{max} T \) we denote the set of maximal elements of the tree \( T \).

**Definition 1.** Under a **sequence tree** in a topological space \( X \) we understand a tree \((T, \leq)\) such that

- \( T \subset X \);
- \( T \) has no infinite branch;
- for each \( t \notin \text{max} T \) the set \( \text{min}(\uparrow t \setminus \{t\}) \) of successors of \( t \) is countable and converges to \( t \).
Saying that a subset $S$ of a topological space $X$ converges to a point $t \in X$ we mean that for each neighborhood $U \subset X$ of $t$ the set $S \setminus U$ is finite.

The following lemma is well-known and can be easily proven by transfinite induction (on the ordinal $s(a, A) = \min\{\alpha : a \in A^{(\alpha)}\}$ for a subset $A$ of a sequential space and a point $a \in \bar{A}$ from its closure).

**Lemma 1.** A point $a \in X$ of a sequential topological space $X$ belongs to the closure of a subset $A \subset X$ if and only if there is a sequence tree $T \subset X$ with $\min T = \{a\}$ and $\max T \subset A$.

For subsets $A, B$ of a group $G$ let $A^{-1} = \{x^{-1} : x \in A\} \subset G$ be the inversion of $A$ in $G$ and $AB = \{xy : x \in A, y \in B\} \subset G$ be the product of $A, B$ in $G$. The following two lemmas will be used in the proof of Theorem 1.

**Lemma 2.** A sequential subspace $F \subset X$ of a topological group $G$ is first countable if the subspace $F^{-1}F \subset G$ has countable sb-character at the unit $e$ of the group $G$.

**Proof.** Our proof starts with the observation that it is sufficient to consider the case $e \in F$ and prove that $F$ has countable character at $e$.

Let $\{S_n : n \in \omega\}$ be a decreasing sb-network at $e$ in $F^{-1}F$. First we show that for every $n \in \omega$ there exists $m > n$ such that $S_m \cap (F^{-1}F) \subset S_n$. Otherwise, for every $m \in \omega$ there would exist $x_m, y_m \in S_m$ with $x_my_m \in (F^{-1}F) \setminus S_n$. Taking into account that $\lim_{m \to \infty} x_m = \lim_{m \to \infty} y_m = e$, we get $\lim_{m \to \infty} x_my_m = e$. Since $S_n$ is a sequential barrier at $e$, there is a number $m$ with $x_my_m \in S_n$, which contradicts to the choice of the points $x_m, y_m$.

Now let us show that for all $n \in \omega$ the set $S_n \cap F$ is a neighborhood of $e$ in $F$. Suppose, conversely, that $e \in \text{cl}_F(F \setminus S_n)$ for some $n_0 \in \omega$.

By Lemma 1 there exists a sequence tree $T \subset F$, $\min T = \{e\}$ and $\max T \subset F \setminus S_n$. To get a contradiction we shall construct an infinite branch of $T$. Put $x_0 = e$ and let $m_0$ be the smallest integer such that $S_{m_0} \cap F^{-1}F \subset S_n$.

By induction, for every $i \geq 1$ find a number $m_i > m_{i-1}$ with $S_{m_i} \cap F^{-1}F \subset S_{m_{i-1}}$ and a point $x_i \in \text{succ}(x_{i-1}) \cap (x_{i-1}S_{m_i})$. To show that such a choice is always possible, it suffices to verify that $x_{i-1} \notin \max T$. It follows from the inductive construction that $x_{i-1} \in F \cap (S_{m_0} \cdots S_{m_{i-1}}) \subset F \cap S_{m_0}^2 \subset S_n$ and thus $x_{i-1} \notin \max T$ because $\max T \subset F \setminus S_n$.

Therefore we have constructed an infinite branch $\{x_i : i \in \omega\}$ of the sequence tree $T$ which is not possible. This contradiction finishes the proof.

**Lemma 3.** A sequential $\omega_1$-subspace $F$ of a topological group $G$ has countable sb-character provided the subspace $F^{-1}F \subset G$ has countable cs-character at the unit $e$ of $G$.

**Proof.** Suppose that $F \subset G$ is a sequential $\omega_1$-space with $\text{cs}_\lambda(F^{-1}F, e) \leq \aleph_0$. We have to prove that $\text{sb}_\lambda(F, x) \leq \aleph_0$ for any point $x \in F$. Replacing $F$ by $Fx^{-1}$, if necessary, we can assume that $x = e$ is the unit of the group $G$. Fix a countable family $\mathcal{A}$ of subsets of $G$ closed under group products in $G$, finite unions and finite intersections, and such that $F^{-1}F \in \mathcal{A}$ and $\mathcal{A}|F^{-1}F = \{A \cap (F^{-1}F) : A \in \mathcal{A}\}$ is a cs-network at $e$ in $F^{-1}F$. We claim that the collection $\mathcal{A}|F = \{A \cap F : A \in \mathcal{A}\}$ is a sb-network at $e$ in $F$.

Assuming the converse, we would find an open neighborhood $U \subset G$ of $e$ such that for any element $A \in \mathcal{A}$ with $A \cap F \subset U$ the set $A \cap F$ fails to be a sequential barrier at $e$ in $F$. 
Let 

\[ \mathcal{A}' = \{ A \in \mathcal{A} : A \subset F \cap \Omega \} = \{ A_n : n \in \omega \} \text{ and } B_n = \bigcup_{k \leq n} A_k. \]

Let \( m_{n-1} = 0 \) and \( U_{n-1} \subset U \) be any closed neighborhood of \( e \) in \( G \). By induction, for every \( k \in \omega \) find a number \( m_{k+1} > m_k \), a closed neighborhood \( U_k \subset U_{k-1} \) of \( e \) in \( G \), and a sequence \((x_{k,i})_{i<\omega} \) convergent to \( e \) so that the following conditions are satisfied:

(i) \( \{x_{k,i} : i \in \omega\} \subset U_{k-1} \cap F \setminus B_{m_{k+1}}; \)

(ii) the set \( F_k = \{ x_{n,i} : n \leq k, i \in \omega \} \setminus B_{m_k} \) is finite;

(iii) \( U_k \cap (F_k \cup \{x_{i,j} : i,j \leq k\}) = \emptyset \) and \( U_k^2 \subset U_{k-1} \).

The last condition implies that \( U_0 U_1 \cdots U_k \subset U \) for every \( k \geq 0 \).

Consider the subspace \( X = \{x_{k,i} : k,i \in \omega\} \) of \( F \) and observe that it is discrete (in itself). Denote by \( \bar{X} \) the closure of \( X \) in \( F \) and observe that \( \bar{X} \setminus X \) is closed in \( F \). We claim that \( e \) is an isolated point of \( \bar{X} \setminus X \). Assuming the converse and applying Lemma 1 we would find a sequence tree \( T \subset X \) such that \( \text{min}T = \{e\} \), \( \text{max}T \subset X \), and \( \text{succ}(e) \subset \bar{X} \setminus X \).

By induction, construct a (finite) branch \((t_i)_{i<\omega+1}\) of the tree \( T \) and a sequence \( \{C_i : i \leq n\} \) of elements of the family \( \mathcal{A} \) such that \( t_0 = e \), \( |\text{succ}(t_i) \setminus t_i \cdot C_i| < \aleph_0 \) and \( C_i \subset t_i \cap (F \setminus F^{F-1}) \), \( t_i \setminus t_i \cdot C_i \), for each \( i \leq n \). Note that the infinite set \( \sigma = \text{succ}(t_n) \cap t_n \cdot C_n \subset X \) converges to the point \( t_n \neq e \).

On the other hand, \( \sigma \subset t_0 \cdot C_0 \subset t_0 \cdot C = t_0 \cdot C \subset \cdots \subset t_0 \cdot C_0 \cdots C_n \subset U_0 \cdots U_n \subset U \). It follows from our assumption on \( A \) that \( C_0 \cdots C_n \in \mathcal{A} \) and thus \( (C_0 \cdots C_n) \cap F \subset B_{m_k} \) for some \( k \). Consequently, \( \sigma \subset X \cap B_{m_k} \) and \( \sigma \subset \{x_{j,i} : j \leq k, i \in \omega\} \) by the item (i) of the construction of \( X \). Since \( e \) is a unique cluster point of the set \( \{x_{j,i} : j \leq k, i \in \omega\} \), the sequence \( \sigma \) cannot converge to \( t_n \neq e \), which is a contradiction.

Thus \( e \) is an isolated point of \( \bar{X} \setminus X \) and consequently, there is a closed neighborhood \( W \) of \( e \) in \( G \) such that the set \( V = \{e\} \cup X \) is closed in \( F \).

For every \( n \in \omega \) consider the sequence \( S_n = W \cap \{x_{n,i} : i \in \omega\} \) convergent to \( e \). Since \( F \) is an \( \alpha_\tau \)-space, there is a convergent sequence \( S \subset F \) such that \( S \cap S_n = \emptyset \) for infinitely many sequences \( S_n \). Taking into account that \( V \) is a closed subspace of \( F \) with \( |V \cap S| = \aleph_0 \), we conclude that the limit point \( \lim S \) of \( S \) belongs to the set \( V \). Moreover, we can assume that \( S \subset V \). Since the space \( X \) is discrete, \( \lim S \in V \setminus X = \{e\} \). Thus the sequence \( S \) converges to \( e \). Since \( \mathcal{A}' \) is a \( \alpha_\gamma \)-network at \( e \) in \( F \), there is a number \( n \in \omega \) such that \( A_n \) contains almost all members of the sequence \( S \). Since \( S_m \cap (S_k \cup A_n) = \emptyset \) for \( m > k \geq n \), the sequence \( S \) cannot meet infinitely many sequences \( S_m \). But this contradicts to the choice of \( S \). \( \square \)

Following [vD, §8] by \( \mathbb{L} \) we denote the countable subspace of the plane \( \mathbb{R}^2 \):

\[ \mathbb{L} = \{ (0,0), \left( \frac{1}{n}, \frac{1}{nm} \right) : n, m \in \mathbb{N} \} \subset \mathbb{R}^2. \]

The space \( \mathbb{L} \) is locally compact at each point except for \( (0,0) \). Moreover, according to Lemma 8.3 of [vD], a first countable space \( X \) contains a closed topological copy of the space \( \mathbb{L} \) if and only if \( X \) is not locally compact.

The following important lemma was proven in [Ba1] for normal sequential groups.

**Lemma 4.** If a sequential topological group \( G \) contains a closed copy of the space \( \mathbb{L} \), then \( G \) is an \( \alpha_\tau \)-space.

**Proof.** Let \( h : \mathbb{L} \rightarrow G \) be a closed embedding and let \( x_0 = h(0,0), x_{n,m} = h\left( \frac{1}{n}, \frac{1}{nm} \right) \) for \( n, m \in \mathbb{N} \). To show that \( G \) is an \( \alpha_\tau \)-space, for every \( n \in \mathbb{N} \) fix a sequence
operation on $G$ and using the continuity of the group operation, show that $x_0 \notin A$ is a cluster point of $A$ in $G$. Consequently, the set $A$ is not closed and by the sequentiality of $G$, there is a sequence $S \subset A$ convergent to a point $a \notin A$. Since every space $D_n$ is closed and discrete in $G$, we may replace $S$ by a subsequence, and assume that $|S \cap D_n| \leq 1$ for every $n \in \mathbb{N}$. Consequently, $S$ can be written as $S = \{x_{n_i, m_i} : i \in \omega\}$ for some number sequences $(n_i)$ and $(m_i)$ with $n_{i+1} > n_i$ for all $i$. It follows that the sequence $(x_{n_i, m_i})_{i \in \omega}$ converges to $x_0$ and consequently, the sequence $T = \{y_{n_i, m_i}\}_{i \in \omega}$ converges to $x_0^{-1} \cdot a$. Since $T \cap \{y_{n_i, m_i}\}_{m \in \mathbb{N}} \neq \emptyset$ for every $i$, we conclude that $G$ is an $\alpha_\gamma$-space. \hfill $\square$

Lemma 4 allows us to prove the following unexpected

**Lemma 5.** A non-metrizable sequential topological group $G$ with countable cs-character has a countable cs-network at the unit, consisting of closed countably compact subsets of $G$.

**Proof.** Given a non-metrizable sequential group $G$ with countable cs-character we can apply Lemmas 2–4 to conclude that $G$ contains no closed copy of the space $\mathbb{L}$. Fix a countable cs-network $\mathcal{N}$ at $e$, closed under finite intersections and consisting of closed subspaces of $G$. We claim that the collection $\mathcal{C} \subset \mathcal{N}$ of all countably compact subsets $N \in \mathcal{N}$ forms a cs-network at $e$ in $G$.

To show this, fix a neighborhood $U \subset G$ of $e$ and a sequence $(x_n) \subset G$ convergent to $e$. We must find a countably compact set $M \in \mathcal{N}$ with $M \subset U$, containing almost all points $x_n$. Let $A = \{A_k : k \in \omega\}$ be the collection of all elements $N \subset U$ of $\mathcal{N}$ containing almost all points $x_n$. Now it suffices to find a number $n \in \omega$ such that the intersection $M = \bigcap_{k \leq n} A_k$ is countably compact. Suppose to the contrary, that for every $n \in \omega$ the set $\bigcap_{k \leq n} A_k$ is not countably compact. Then there exists a countable closed discrete subspace $K_0 \subset A_0$ with $K_0 \neq e$. Fix a neighborhood $W_0$ of $e$ with $W_0 \cap K_0 = \emptyset$. Since $\mathcal{N}$ is a cs-network at $e$, there exists $k_1 \in \omega$ such that $A_{k_1} \subset W_0$.

It follows from our hypothesis that there is a countable closed discrete subspace $K_1 \subset \bigcap_{k \leq k_1} A_k$ with $K_1 \ni e$. Proceeding in this fashion we construct by induction an increasing number sequence $(k_n)_{n \in \omega} \subset \omega$, a sequence $(K_n)_{n \in \omega}$ of countable closed discrete subspaces of $G$, and a sequence $(W_n)_{n \in \omega}$ of open neighborhoods of $e$ such that $K_n \subset \bigcap_{k \leq k_n} A_k$, $W_n \cap K_n = \emptyset$, and $A_{k_n+1} \subset W_n$ for all $n \in \omega$.

It follows from the above construction that $\{e\} \cup \bigcup_{n \in \omega} K_n$ is a closed copy of the space $\mathbb{L}$ which is impossible. \hfill $\square$

**Proofs of Main Results**

**Proof of Proposition 1.** The first three items can be easily derived from the corresponding definitions. To prove the fourth item observe that for any cs*-network $\mathcal{N}$ at a point $x$ of a topological space $X$, the family $\mathcal{N}' = \{\bigcup \mathcal{F} : \mathcal{F} \subset \mathcal{N}\}$ is an sb-network at $x$. 

$$(y_{n,m})_{m \in \mathbb{N}} \subset G,$$ convergent to the unit $e$ of $G$. Denote by $*: G \times G \to G$ the group operation on $G$.
The proof of fifth item is more tricky. Fix any cs*-network $\mathcal{N}$ at a point $x \in X$ with $|\mathcal{N}| \leq \text{cs}^*_c(X)$. Let $\lambda = \text{cof}(|\mathcal{N}|)$ be the cofinality of the cardinal $|\mathcal{N}|$ and write $\mathcal{N} = \bigcup_{\alpha < \lambda} \mathcal{N}_\alpha$ where $\mathcal{N}_\alpha \subset \mathcal{N}_\beta$ and $|\mathcal{N}_\alpha| < |\mathcal{N}|$ for any ordinals $\alpha \leq \beta < \lambda$. Consider the family $\mathcal{M} = \{ \cup C : C \in [\mathcal{N}_\alpha]^{\leq \omega}, \alpha < \lambda \}$ and observe that $|\mathcal{M}| \leq \lambda \cdot \sup\{ |[\kappa]^{\leq \omega} : \kappa < |\mathcal{N}| \}$ where $[\kappa]^{\leq \omega} = \{ A \subset \kappa : |A| \leq \aleph_0 \}$. It rests to verify that $\mathcal{M}$ is a cs-network at $x$.

Fix a neighborhood $U \subset X$ of $x$ and a sequence $S \subset X$ convergent to $x$. For every $\alpha < \lambda$ choose a countable subset $C_\alpha \subset \mathcal{N}_\alpha$ such that $\cup C_\alpha \subset U$ and $S \cap (\cup C_\alpha) = S \cap (\cup \{ N \in \mathcal{N}_\alpha : N \subset U \})$. It follows that $\cup C_\alpha \subset \mathcal{M}$. Let $S_\alpha = S \cap (\cup C_\alpha)$ and observe that $S_\alpha \subset S_\beta$ for $\alpha \leq \beta < \lambda$. To finish the proof it suffices to show that $S \setminus S_\alpha$ is finite for some $\alpha < \lambda$. Then the element $\cup C_\alpha \subset U$ of $\mathcal{M}$ will contain almost all members of the sequence $S$.

Separately, we shall consider the cases of countable and uncountable $\lambda$. If $\lambda$ is uncountable, then it has uncountable cofinality and consequently, the transfinite sequence $(S_\alpha)_{\alpha < \lambda}$ eventually stabilizes, i.e., there is an ordinal $\alpha < \lambda$ such that $S_\beta = S_\alpha$ for all $\beta \geq \alpha$. We claim that the set $S \setminus S_\alpha$ is finite. Otherwise, $S \setminus S_\alpha$ would be a sequence convergent to $x$ and there would exist an element $N \in \mathcal{N}$ with $N \subset U$ and infinite intersection $N \cap (S \setminus S_\alpha)$. Find now an ordinal $\beta \geq \alpha$ with $N \in \mathcal{N}_\beta$ and observe that $S \cap N \subset S_\beta = S_\alpha$ which contradicts to the choice of $N$.

If $\lambda$ is countable and $S \setminus S_\alpha$ is infinite for any $\alpha < \lambda$, then we can find an infinite pseudo-intersection $T \subset S$ of the decreasing sequence $(S \setminus S_\alpha)_{\alpha < \lambda}$. Note that $T \cap S_\alpha$ is finite for every $\alpha < \lambda$. Since sequence $T$ converges to $x$, there is an element $N \in \mathcal{N}$ such that $N \subset U$ and $N \cap T$ is infinite. Find $\alpha < \lambda$ with $N \in \mathcal{N}_\alpha$ and observe that $N \cap S \subset S_\alpha$. Then $N \cap T \subset N \cap T \cap S_\alpha \subset T \cap S_\alpha$ is finite, which contradicts to the choice of $N$.

Proof of Proposition 2. Let $X$ be a topological space and fix a point $x \in X$.

(1) Suppose that $\text{cs}^*_c(X) < p$ and fix a cs*-network $\mathcal{N}$ at the point $x$ such that $|\mathcal{N}| < p$. Without loss of generality, we can assume that the family $\mathcal{N}$ is closed under finite unions. We claim that $\mathcal{N}$ is a cs-network at $x$. Assuming the converse we would find a neighborhood $U \subset X$ of $x$ and a sequence $S \subset X$ convergent to $x$ such that $S \setminus N$ is infinite for any element $N \in \mathcal{N}$ with $N \subset U$. Since $\mathcal{N}$ is closed under finite unions, the family $\mathcal{F} = \{ S \setminus N : N \in \mathcal{N}, N \subset U \}$ is closed under finite intersections. Since $|\mathcal{F}| \leq |\mathcal{N}| < p$, the family $\mathcal{F}$ has an infinite pseudo-intersection $T \subset S$. Consequently, $T \cap N$ is finite for any $N \in \mathcal{N}$ with $N \subset U$. But this contradicts to the facts that $T$ converges to $x$ and $\mathcal{N}$ is a cs*-network at $x$.

The items (2) and (3) follow from Propositions 1(5) and 2(1). The item (4) follows from (1,2) and the inequality $\chi(X) \leq \kappa$ holding for any countable topological space $X$.

Finally, to derive (5) from (3) use the well-known fact that under GCH, $\lambda^{\aleph_0} \leq \kappa$ for any infinite cardinals $\lambda < \kappa$, see [HJ, 9.3.8].

Proof of Theorem 1. Suppose that $G$ is a non-metrizable sequential group with countable cs*-character. By Proposition 2(1), $\text{cs}^*_c(G) = \text{cs}^*_c(G) \leq \aleph_0$.

First we show that each countably compact subspace $K$ of $G$ is first-countable. The space $K$, being countably compact in the sequential space $G$, is sequentially compact and so are the sets $K^{-1}K$ and $(K^{-1}K)^{-1}(K^{-1}K)$ in $G$. The sequential compactness of $K^{-1}K$ implies that it is an $\alpha_0$-space. Since $\text{cs}^*_c((K^{-1}K)^{-1}(K^{-1}K)) \leq$
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$\text{cs}_1(G) \leq \aleph_0$ we may apply Lemmas 3 and 2 to conclude that the space $K^{-1}K$ has countable sb-character and $K$ has countable character.

Next, we show that $G$ contains an open $\mathcal{MK}_\omega$-subgroup. By Lemma 5, $G$ has a countable $c$-network $\mathcal{K}$ consisting of countably compact subsets. Since the group product of two countably compact subspaces in $G$ is countably compact, we may assume that $\mathcal{K}$ is closed under finite group products in $G$. We can also assume that $\mathcal{K}$ is closed under the inversion, i.e. $K^{-1} \in \mathcal{K}$ for any $K \in \mathcal{K}$. Then $H = \cup \mathcal{K}$ is a subgroup of $G$. It follows that this subgroup is a sequential barrier at each of its points, and thus is open-and-closed in $G$. We claim that the topology on $H$ is inductive with respect to the cover $\mathcal{K}$. Indeed, consider some $U \subset H$ such that $U \cap K$ is open in $K$ for every $K \in \mathcal{K}$. Assuming that $U$ is not open in $H$ and using the sequentiality of $H$, we would find a point $x \in U$ and a sequence $(x_n)_{n \in \omega} \subset H \setminus U$ convergent to $x$. It follows that there are elements $K_1, K_2 \in \mathcal{K}$ such that $x \in K_1$ and $K_2$ contains almost all members of the sequence $(x^{-1}x_n)$.

Then the product $K = K_1 \cdot K_2$ contains almost all $x_n$ and the set $U \cap K$, being an open neighborhood of $x$ in $K$, contains almost all members of the sequence $(x_n)$, which is a contradiction.

As it was proved before each $K \in \mathcal{K}$ is first-countable, and consequently $H$ has countable pseudocharacter, being the countable union of first countable subspaces. Then $H$ admits a continuous metric. Since any continuous metric on a countably compact space generates its original topology, every $K \in \mathcal{K}$ is a metrizable compactum, and consequently $H$ is an $\mathcal{MK}_\omega$-subgroup of $G$.

Since $H$ is an open subgroup of $G$, $G$ is homeomorphic to $H \times D$ for some discrete space $D$.

**Proof of Theorem 2.** Suppose $G$ is a non-metrizable sequential topological group with countable $c^*$-character. By Theorem 1, $G$ contains an open $\mathcal{MK}_\omega$-subgroup $H$ and is homeomorphic to the product $H \times D$ for some discrete space $D$ . This implies that $G$ has point-countable $\kappa$-network. By a result of Shibakov [Shi], each sequential topological group with point-countable $k$-network and sequential order $< \omega_1$ is metrizable. Consequently, $\text{so}(G) = \omega_1$. It is clear that $\psi(G) = \psi(H) \leq \aleph_0$, $\chi(G) = \chi(H)$, $\text{sb}_\chi(G) = \text{sb}_\chi(H)$ and $\text{ib}(G) = \text{ib}(H)$, $\text{c}(G) = \text{c}(H)$, $\text{d}(G) = \text{d}(H)$, $\text{e}(G) = \text{e}(H)$, $\text{nw}(G) = \text{nw}(H) = |D| \cdot \aleph_0$.

To finish the proof it rests to show that $\text{sb}_\chi(H) = \chi(H) = \frak{d}$. It follows from Lemmas 2 and 3 that the group $H$, being non-metrizable, is not $\omega_1$ and thus contains a copy of the sequential fan $S_\omega$. Then $\frak{d} = \chi(S_\omega) = \text{sb}_\chi(S_\omega) \leq \text{sb}_\chi(H) \leq \chi(H)$.

To prove that $\chi(H) \leq \frak{d}$ we shall apply a result of K. Sakai [Sa] asserting that the space $\mathbb{R}_\infty \times Q$ contains a closed topological copy of each $\mathcal{MK}_\omega$-space and the well-known equality $\chi(\mathbb{R}_\infty \times Q) = \chi(\mathbb{R}_\infty) = \frak{d}$ (following from the fact that $\mathbb{R}_\infty$ carries the box-product topology, see [Sch, Ch.II, Ex.12]).

**Proof of Theorem 5.** First we describe two general constructions producing topologically homogeneous sequential spaces. For a locally compact space $Z$ let $\alpha Z = Z \cup \{\infty\}$ be the one-point extension of $Z$ endowed with the topology whose neighborhood base at $\infty$ consists of the sets $\alpha Z \setminus K$ where $K$ is a compact subset of $Z$. Thus for a non-compact locally compacts space $Z$ the space $\alpha Z$ is noting else but the one-point compactification of $Z$. Denote by $2^n = \{0,1\}^\omega$ the Cantor cube.
Consider the subsets

$$\Xi(Z) = \{(c, (z_i))_{i \in \omega} \in 2^\omega \times (\alpha Z)^\omega : z_i = \infty \text{ for all but finitely many indices } i \}$$

and

$$\Theta(Z) = \{(c, (z_i))_{i \in \omega} \in 2^\omega \times (\alpha Z)^\omega : \exists n \in \omega \text{ such that } z_i \neq \infty \text{ if and only if } i < n \}.$$ 

Observe that $\Theta(Z) \subseteq \Xi(Z)$.

Endow the set $\Xi(Z)$ (resp. $\Theta(Z)$) with the strongest topology generating the Tychonov product topology on each compact subset from the family $\mathcal{K}_Z$ (resp. $\mathcal{K}_\Theta$), where

$$\mathcal{K}_Z = \{2^\omega \times \prod_{i \in \omega} C_i : C_i \text{ are compact subsets of } \alpha Z \text{ and almost all } C_i = \{\infty\}\};$$

$$\mathcal{K}_\Theta = \{2^\omega \times \prod_{i \in \omega} C_i : \exists i_0 \in \omega \text{ such that } C_{i_0} = \alpha Z, C_i = \{\infty\} \text{ for all } i > i_0 \text{ and } C_i \text{ is a compact subsets of } Z \text{ for every } i < i_0\}.$$

**Lemma 6.** Suppose $Z$ is a zero-dimensional locally metrizable locally compact space. Then

1. the spaces $\Xi(Z)$ and $\Theta(Z)$ are topologically homogeneous;
2. $\Xi(Z)$ is a regular zero-dimensional $k_\omega$-space while $\Theta(Z)$ is a totally disconnected $k$-space;
3. if $Z$ is Lindelöf, then $\Xi(Z)$ and $\Theta(Z)$ are zero-dimensional $\mathcal{MK}_\omega$-spaces with
   $$\chi(\Xi(Z)) = \chi(\Theta(Z)) = 0;$$
4. $\Xi(Z)$ and $\Theta(Z)$ contain copies of the space $\alpha Z$ while $\Theta(Z)$ contains a closed copy of $Z$;
5. $\text{cs}_\chi(\Xi(Z)) = \text{cs}_\chi(\Theta(Z)) = \text{cs}_\chi(\alpha Z)$, $\text{cs}_\chi(\Xi(Z)) = \text{cs}_\chi(\Theta(Z)) = \text{cs}_\chi(\alpha Z)$,
   $\text{sb}_\chi(\Theta(Z)) = \text{sb}_\chi(\alpha Z)$, and $\psi(\Xi(Z)) = \psi(\Theta(Z)) = \psi(\alpha Z)$;
6. the spaces $\Xi(Z)$ and $\Theta(Z)$ are sequential if and only if $\alpha Z$ is sequential;
7. if $Z$ is not countably compact, then $\Xi(Z)$ contains a closed copies of $S_2$ and $S_\omega$ and $\Theta(Z)$ contains a closed copy of $S_2$.

**Proof.** (1) First we show that the space $\Xi(Z)$ is topologically homogeneous.

Given two points $(c, (z_i))_{i \in \omega}, (c', (z'_i))_{i \in \omega}$ of $\Xi(Z)$ we have to find a homeomorphism $h$ of $\Xi(Z)$ with $h((c, (z_i))_{i \in \omega}) = (c', (z'_i))_{i \in \omega}$. Since the Cantor cube $2^\omega$ is topologically homogeneous, we can assume that $c \neq c'$. Fix any disjoint closed-and-open neighborhoods $U, U'$ of the points $c, c'$ in $2^\omega$, respectively.

Consider the finite sets $I = \{i \in \omega : z_i \neq \infty\}$ and $I' = \{i \in \omega : z'_i \neq \infty\}$. Using the zero-dimensionality and the local metrizability of $Z$, for each $i \in I$ (resp. $i \in I'$) fix an open compact metrizable neighborhood $U_i$ (resp. $U'_i$) of the point $z_i$ (resp. $z'_i$) in $Z$. By the classical Brouwer Theorem [Ke, 7.4], the products $U \times \prod_{i \in I} U_i$ and $U' \times \prod_{i \in I'} U'_i$, being zero-dimensional compact metrizable spaces without isolated points, are homeomorphic to the Cantor cube $2^\omega$. Now the topological homogeneity of the Cantor cube implies the existence of a homeomorphism $f : U \times \prod_{i \in I} U_i \rightarrow U' \times \prod_{i \in I'} U'_i$ such that $f((c, (z_i))_{i \in I}) = (c', (z'_i))_{i \in I'}$.

Let

$$W = \{(x, (x_i))_{i \in \omega} \in \Xi(Z) : x \in U, x_i \in U_i \text{ for all } i \in I\}$$

and

$$W' = \{(x', (x'_i))_{i \in \omega} \in \Xi(Z) : x' \in U', x'_i \in U'_i \text{ for all } i \in I'\}.$$

It follows that $W, W'$ are disjoint open-and-closed subsets of $\Xi(Z)$. Let $\chi : \omega \setminus I \rightarrow \omega \setminus I$ be a unique monotone bijection.

Now consider the homeomorphism $\tilde{f} : W \rightarrow W'$ assigning to a sequence $(x, (x_i))_{i \in \omega}) \in W$ the sequence $(x', (x'_i))_{i \in \omega}) \in W'$ where $(x', (x'_i))_{i \in I'}) = f((x, (x_i))_{i \in I})$ and $x'_i =
$x_{\chi(i)}$ for $i \notin I$. Finally, define a homeomorphism $h$ of $\Xi(Z)$ letting
\[
h(x) = \begin{cases} 
x & \text{if } x \notin W \cup W'; \\
h(W) & \text{if } x \in W; \\
h^{-1}(x) & \text{if } x \in W'.
\end{cases}
\]
and observe that $h(c,(z_i)_{i\in\mathbb{N}}) = (c',(z'_i)_{i\in\mathbb{N}})$ which proves the topological homogeneity of the space $\Xi(Z)$.

Replacing $\Xi(Z)$ by $\Theta(Z)$ in the above proof, we shall get a proof of the topological homogeneity of $\Theta(Z)$.

The items (2–4) follow easily from the definitions of the spaces $\Xi(Z)$ and $\Theta(Z)$, the zero-dimensionality of $\alpha Z$, and known properties of $k_\omega$-spaces, see [FST] (to find a closed copy of $Z$ in $\Theta(Z)$ consider the closed embedding $e : Z \rightarrow \Theta(Z)$, $e : z \mapsto (z, z_0, z, \infty, \infty, \ldots)$, where $z_0$ is any fixed point of $Z$).

To prove (5) apply Proposition 3(6,8,9,10). (To calculate the $c^\ast$-, $c$-, and sb-characters of $\Theta(Z)$, observe that almost all members of any sequence $(a_n) \subset \Theta(Z)$ convergent to a point $a = (c, (z_i)) \in \Theta(Z)$ lie in the compactum $2^x \times \prod_{i\in\omega} C_i$, where $C_i$ is a clopen neighborhood of $z_i$ if $z_i \neq \infty$, $C_i = \alpha Z$ if $i = \min\{ j \in \omega : z_j = \infty \}$ and $C_i = \{ \infty \}$ otherwise. By Proposition 3(6), the $c^\ast$-, $c$-, and sb-characters of this compactum are equal to the corresponding characters of $\alpha Z$.)

(6) Since the spaces $\Xi(Z)$ and $\Theta(Z)$ contain a copy of $\alpha Z$, the sequentiality of $\Xi(Z)$ or $\Theta(Z)$ implies the sequentiality of $\alpha Z$. Now suppose conversely that the space $\alpha Z$ is sequential. Then each compactum $K \in \mathcal{K}_\Xi \cup \mathcal{K}_\Theta$ is sequential since a finite product of sequential compacta is sequential, see [En1, 3.10.I(b)]. Now the spaces $\Xi(Z)$ and $\Theta(Z)$ are sequential because they carry the inductive topologies with respect to the covers $\mathcal{K}_\Xi$, $\mathcal{K}_\Theta$ by sequential compacta.

(7) If $Z$ is not countably compact, then it contains a countable closed discrete subspace $S \subset Z$ which can be thought as a sequence convergent to $\infty$ in $\alpha Z$. It is easy to see that $\Xi(S)$ (resp. $\Theta(S)$) is a closed subset of $\Xi(Z)$ (resp. $\Theta(Z)$). Now it is quite easy to find closed copies of $S_2$ and $S_\omega$ in $\Xi(S)$ and a closed copy of $S_2$ in $\Theta(S)$. \hfill \Box

With Lemma 6 at our disposal, we are able to finish the proof of Theorem 5. To construct the examples satisfying the conditions of Theorem 5(2,3), assume $b = c$ and use Proposition 4 to find a locally compact locally countable space $Z$ whose one-point compactification $\alpha Z$ is sequential and satisfies $\kappa_0 = \text{sb}_{\chi}(\alpha Z) < \psi(\alpha Z) = c$. Applying Lemma 6 to this space $Z$, we conclude that the topologically homogeneous $k$-spaces $X_2 = \Xi(Z)$ and $X_3 = \Theta(Z)$ give us required examples.

The example of a countably topologically homogeneous $k_\omega$-space $X_1$ with $\text{sb}_{\chi}(X_1) < \chi(X_1)$ can be constructed by analogy with the space $\Theta(\mathbb{N})$ (with that difference that there is no necessity to involve the Cantor cube) and is known in topology as the Ankhangeski-Franklin space, see [AF]. We briefly remind its construction. Let $S_0 = \{0, \frac{1}{n} : n \in \mathbb{N} \}$ be a convergent sequence and consider the countable space $X_1 = \{(x_i)_{i\in\omega} \in S_0^\omega : \exists n \in \omega \text{ such that } x_i \neq 0 \text{ iff } i < n \}$ endowed with the strongest topology inducing the product topology on each compactum $\prod_{i\in\omega} C_i$ for which there is $n \in \omega$ such that $C_n = S_0$, $C_i = \{0\}$ if $i > n$, and $C_i = \{x_i\}$ for some $x_i \in S_0 \setminus \{0\}$ if $i < n$. By analogy with the proof of Lemma 6 it can be shown that $X_1$ is a topologically homogeneous $k_\omega$-space with $\kappa_0 = \text{sb}_{\chi}(X_1) < \chi(X_1) = 0$ and $so(X_1) = \omega$. 

\[\text{ON GROUPS WITH COUNTABLE } c^\ast\text{-CHARACTER } 17\]
Proof of Proposition 5. The equivalences (1) \(\iff\) (2) \(\iff\) (3) were proved by Lin [Lin, 3.13] in terms of (universally) csf-countable spaces. To prove the other equivalences apply

Lemma 7. A Hausdorff topological space \(X\) is an \(\alpha_1\)-space provided one of the following conditions is satisfied:

1. \(X\) is a Fréchet-Urysohn \(\alpha_7\)-space;
2. \(X\) is a Fréchet-Urysohn countably compact space;
3. \(\text{sb}(X) < p\);
4. \(\text{sb}(X) < \mathfrak{d}\), each point of \(X\) is regular \(G_\delta\), and \(X\) is \(c\)-sequential.

Proof. Fix any point \(x \in X\) and a countable family \(\{S_n\}_{n \in \omega}\) of sequences convergent to \(x\) in \(X\). We have to find a sequence \(S \subset X \setminus \{x\}\) convergent to \(x\) and meeting infinitely many sequences \(S_n\). Using the countability of the set \(\bigcup_{n \in \omega} S_n\) find a decreasing sequence \((U_n)_{n \in \omega}\) of closed neighborhoods of \(x\) in \(X\) such that \((\bigcap_{n \in \omega} U_n) \cap (\bigcup_{n \in \omega} S_n) = \{x\}\). Replacing each sequence \(S_n\) by its subsequence \(S_n \cap U_n\), if necessary, we can assume that \(S_n \subset U_n\).

(1) Assume that \(X\) is a Fréchet-Urysohn \(\alpha_7\)-space. Let \(A = \{a \in X : a\) is the limit of a convergent sequence \(S \subset X\) meeting infinitely many sequences \(S_n\}\). It follows from our assumption on \((S_n)\) and \((U_n)\) that \(A \subset \bigcap_{n \in \omega} U_n\).

It suffices to consider the non-trivial case when \(x \notin A\). In this case \(x\) is a cluster point of \(A\) (otherwise \(X\) would be not \(\alpha_7\)). Since \(X\) is Fréchet-Urysohn, there is a sequence \((a_n) \subset A\) convergent to \(x\). By the definition of \(A\), for every \(n \in \omega\) there is a sequence \(T_n \subset X\) convergent to \(a_n\) and meeting infinitely many sequences \(S_n\). Without loss of generality, we can assume that \(T_n \subset \bigcup_{i \geq n} S_i\) (because \(a \in A \setminus \{x\}\) and thus \(a \notin \bigcup_{n \in \omega} S_n\)). It is easy to see that \(x\) is a cluster point of the set \(\bigcup_{n \in \omega} T_n\).

Since \(X\) is Fréchet-Urysohn, there is a sequence \(T \subset \bigcup_{n \in \omega} T_n\) convergent to \(x\).

Now it rests to show that the set \(T\) meets infinitely many sequences \(S_n\). Assuming the converse we would find \(n \in \omega\) such that \(T \subset \bigcup_{i \leq n} S_n\). Then \(T \subset \bigcup_{i \leq n} T_n\), which is not possible since \(\bigcup_{i \leq n} T_i\) is a compact set failing to contain the point \(x\).

(2) If \(X\) is Fréchet-Urysohn and countably compact, then it is sequentially compact and hence \(\alpha_7\), which allows us to apply the previous item.

(3) Assume that \(\text{sb}(X) < p\) and let \(\mathcal{N}\) be a sb-network at \(x\) of size \(|\mathcal{N}| < p\). Without loss of generality, we can assume that the family \(\mathcal{N}\) is closed under finite intersections. Let \(S = \bigcup_{n \in \omega} S_n\) and \(F_{N,n} = N \cap (\bigcup_{i \geq n} S_i)\) for \(N \in \mathcal{N}\) and \(n \in \omega\).

It is easy to see that the family \(\mathcal{F} = \{F_{N,n} : N \in \mathcal{N}, n \in \omega\}\) consists of infinite subsets of \(S\), has size \(|\mathcal{F}| < p\), and is closed under finite intersection. Now the definition of the small cardinal \(p\) implies that this family \(\mathcal{F}\) has an infinite pseudo-intersection \(T \subset S\). Then \(T\) is a sequence convergent to \(x\) and intersecting infinitely many sequences \(S_n\). This shows that \(X\) is an \(\alpha_1\)-space.

(4) Assume that the space \(X\) is \(c\)-sequential, each point of \(X\) is regular \(G_\delta\), and \(\text{sb}(X) < \mathfrak{d}\). In this case we can choose the sequence \((U_n)\) to satisfy \(\bigcap_{n \in \omega} U_n = \{x\}\).

Fix an sb-network \(\mathcal{N}\) at \(x\) with \(|\mathcal{N}| < \mathfrak{d}\). For every \(n \in \omega\) write \(S_n = \{x_{n,i} : i \in \mathbb{N}\}\). For each sequential barrier \(N \in \mathcal{N}\) find a function \(f_N : \omega \to \mathbb{N}\) such that \(x_{n,i} \in N\) for every \(n \in \omega\) and \(i \geq f_N(n)\). The family of functions \(\{f_N : N \in \mathcal{N}\}\) has size < \(\mathfrak{d}\) and hence is not cofinal in \(\mathbb{N}^\omega\). Consequently, there is a function \(f : \omega \to \mathbb{N}\) such that \(f \not\subseteq f_N\) for each \(N \in \mathcal{N}\). Now consider the sequence \(S = \{x_{n,f(n)} : n \in \omega\}\). We claim that \(x\) is a cluster point of \(S\). Indeed, given any neighborhood \(U\) of \(x\),
find a sequential barrier $N \in \mathcal{N}$ with $N \subseteq U$. Since $f \not\leq f_N$, there is $n \in \omega$ with $f(n) > f_N(n)$. It follows from the choice of the function $f_N$ that $x_{n,f(n)} \in N \subseteq U$.

Since $S \setminus U_n$ is finite for every $n$, $\{x\} = \bigcap_{n \in \omega} U_n$ is a unique cluster point of $S$ and thus $\{x\} \cup S$ is a closed subset of $X$. Now the $c$-sequentiality of $X$ implies the existence of a sequence $T \subseteq S$ convergent to $x$. Since $T$ meets infinitely many sequences $S_n$, the space $X$ is $\alpha_4$. □

**Proof of Proposition 6.** Suppose a space $X$ has countable $c^*$-character. The implications $(1) \Rightarrow (2, 3, 4, 5)$ are trivial. The equivalence $(1) \iff (2)$ follows from Proposition 1(2). To show that $(3) \Rightarrow (2)$, apply Lemma 7 and Proposition 5(3 $\Rightarrow$ 1).

To prove that $(4) \Rightarrow (2)$ it suffices to apply Proposition 5(4 $\Rightarrow$ 1) and observe that $X$ is Fréchet-Urysohn provided $\chi(X) \leq p$ and $X$ has countable tightness. This can be seen as follows.

Given a subset $A \subseteq X$ and a point $a \in \overline{A}$ from its closure, use the countable tightness of $X$ to find a countable subset $N \subseteq A$ with $a \in \overline{N}$. Fix any neighborhood base $\mathcal{B}$ at $x$ of size $|\mathcal{B}| < p$. We can assume that $\mathcal{B}$ is closed under finite intersections. By the definition of the small cardinal $p$, the family $\{B \cap N : B \in \mathcal{B}\}$ has infinite pseudo-intersection $S \subseteq N$. It is clear that $S \subseteq A$ is a sequence convergent to $x$, which proves that $X$ is Fréchet-Urysohn.

$(5) \Rightarrow (2)$. Assume that $X$ is a sequential space containing no closed copies of $S_\omega$ and $S_2$ and such that each point of $X$ is regular $G_\delta$. Since $X$ is sequential and contains no closed copy of $S_2$, we may apply Lemma 2.5 [Lin] to conclude that $X$ is Fréchet-Urysohn. Next, Theorem 3.6 of [Lin] implies that $X$ is an $\alpha_4$-space. Finally apply Proposition 5 to conclude that $X$ has countable sb-character and, being Fréchet-Urysohn, is first countable.

The final implication $(6) \Rightarrow (2)$ follows from $(5) \Rightarrow (2)$ and the well-known equality $\chi(S_\omega) = \chi(S_2) = \emptyset$.

**Proof of Proposition 7.** The first item of this proposition follows from Proposition 6(3 $\Rightarrow$ 1) and the observation that each Fréchet-Urysohn countable compact space, being sequentially compact, is $\alpha_7$.

Now suppose that $X$ is a dyadic compact with $c^*_s(X) \leq \aleph_0$. If $X$ is not metrizable, then it contains a copy of the one-point compactification $\alpha D$ of an uncountable discrete space $D$, see [En1, 3.12.12(i)]. Then $c^*_s(\alpha D) \leq c^*_s(X) \leq \aleph_0$ and by the previous item, the space $\alpha D$, being Fréchet-Urysohn and compact, is first-countable, which is a contradiction.

**Proof of Proposition 8.** Let $D$ be a discrete space.

(1) Let $\kappa = c^*_s(\alpha D)$ and $\lambda_1$ ($\lambda_2$) is the smallest weight of a (regular zero-dimensional) space $X$ of size $|X| = |D|$, containing no non-trivial convergent sequence. To prove the first item of proposition 8 it suffices to verify that $\lambda_2 \leq \kappa \leq \lambda_1$. To show that $\lambda_2 \leq \kappa$, fix any $c^*_s$-network $\mathcal{N}$ at the unique non-isolated point $\infty$ of $\alpha D$ of size $|\mathcal{N}| \leq \kappa$. The algebra $\mathcal{A}$ of subsets of $D$ generated by the family $\{D \setminus N : N \in \mathcal{N}\}$ is a base of some zero-dimensional topology $\tau$ on $D$ with $w(D, \tau) \leq \kappa$. We claim that the space $D$ endowed with this topology contains no infinite convergent sequences. To get a contradiction, suppose that $S \subseteq D$ is an infinite sequence convergent to a point $a \in D \setminus S$. Then $S$ converges to $\infty$ in $\alpha D$ and hence, there is an element $N \in \mathcal{N}$ such that $N \subseteq \alpha D \setminus \{a\}$ and $N \cap S$ is infinite. Consequently, $U = D \setminus N$ is a neighborhood of $a$ in the topology $\tau$ such that $S \setminus U$
is infinite which contradicts to the fact that $S$ converges to $a$. Now consider the equivalence relation $\sim$ on $D$: $x \sim y$ provided for every $U \in \tau$ ($x \in U \Leftrightarrow (y \in U)$. Since the space $(D, \tau)$ has no infinite convergent sequences, each equivalence class $[x]_\sim \subset D$ is finite (because it carries the anti-discrete topology). Consequently, we can find a subset $X \subset D$ of size $|X| = |D|$ such that $x \not\sim y$ for any distinct points $x, y \in X$. Clearly that $\tau$ induces a zero-dimensional topology on $X$. It rests to verify that this topology is $T_1$. Given any two distinct point $x, y \in X$ use $x \not\sim y$ to find an open set $U \in A$ such that either $x \in U$ and $y \not\in U$ or $x \not\in U$ and $y \in U$. Since $D \setminus U \in A$, in both cases we find an open set $W \in A$ such that $x \in W$ but $y \not\in W$. It follows that $X$ is a $T_1$-space containing no non-trivial convergent sequence and thus $\lambda_2 \leq w(X) \leq |A| \leq |N| \leq \kappa$.

To show that $\kappa \leq \lambda_1$, fix any topology $\tau$ on $D$ such that $w(D, \tau) \leq \lambda_1$ and the space $(D, \tau)$ contains no non-trivial convergent sequences. Let $\mathcal{B}$ be a base of the topology $\tau$ with $|\mathcal{B}| \leq \lambda_1$, closed under finite unions. We claim that the collection $\mathcal{N} = \{\alpha D \setminus B : B \in \mathcal{B}\}$ is a $c^\omega$-network for $\alpha D$ at $\infty$. Fix any neighborhood $U \subset \alpha D$ of $\infty$ and any sequence $S \subset D$ convergent to $\infty$. Write $\{x_1, \ldots, x_n\} = \alpha D \setminus U$ and by finite induction, for every $i \leq n$ find a neighborhood $B_i \in \mathcal{B}$ of $x_i$ such that $S \setminus \bigcup_{j=1}^i B_j$ is infinite. Since $\mathcal{B}$ is closed under finite unions, the set $N = \alpha D \setminus (B_1 \cup \cdots \cup B_n)$ belongs to the family $\mathcal{N}$ and has the properties: $N \subset U$ and $N \cap S$ is infinite, i.e., $\mathcal{N}$ is a $c^\omega$-network at $\infty$ in $\alpha D$. Thus $\kappa \leq |\mathcal{N}| \leq |\mathcal{B}| \leq \lambda_1$. This finishes the proof of (1).

An obvious modification of the above argument gives also a proof of the item (2).

**Proof of Proposition 9.** Let $D$ be an uncountable discrete space.

1. The inequalities $n_1 \cdot \log |D| \leq c_{\Kappa}^\omega(\alpha D) \leq c_{\omega}^{\omega}(\alpha D)$ follows from Propositions 7(1) and 1(2,4) yielding $|D| = \chi(\alpha D) = sb\chi(\alpha D) \leq 2^{c_{\Kappa}^\omega(\alpha D)}$. The inequality $c_{\omega}^{\omega}(\alpha D) \leq \epsilon \cdot \text{cof}(|\log |D||^{\leq \omega})$ follows from proposition 8(2) and the observation that the product $\{0, 1\}^{\log |D|}$ endowed with the $\aleph_0$-box product topology has weight $\leq \epsilon \cdot \text{cof}(|\log |D||^{\leq \omega})$. Under the $\aleph_0$-box product topology on $\{0, 1\}^{\omega}$ we understand the topology generated by the base consisting of the sets $\{f \in \{0, 1\}^{\kappa} : f|C = g|C\}$ where $g \in \{0, 1\}^{\omega}$ and $C$ is a countable subset of $\kappa$.

The item (2) follows from (1) and the equality $n_1 \cdot \log \kappa = 2^{n_0} \cdot \min\{\kappa, (\log \kappa)^\omega\}$ holding under GCH for any infinite cardinal $\kappa$, see [HJ, 9.3.8]
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