Properties of the Bellman function related to the Carleson Imbedding theorem for the dyadic maximal operator

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Abstract

We provide a description for the Bellman function related to the Carleson Imbedding theorem, first mentioned in [4], with the use of the Hardy operator.

1 Introduction

The dyadic maximal operator on $\mathbb{R}^n$ is a useful tool in analysis and is defined by:

$$M_d\varphi(x) = \sup \left\{ \frac{1}{|Q|} \int_Q |\varphi(u)| \, du : x \in Q, Q \subseteq \mathbb{R}^n \text{ is a dyadic cube} \right\}, \quad (1.1)$$

for every $\varphi \in L^1_{\text{loc}}(\mathbb{R}^n)$ where the dyadic cubes are those formed by the grids $2^{-N}\mathbb{Z}^n$ for $N = 0, 1, \ldots$. As it is well known it satisfies the following weak type $(1,1)$ inequality

$$|\{x \in \mathbb{R}^n : M_d\varphi(x) > \lambda\}| \leq \frac{1}{\lambda} \int_{\{M_d\varphi > \lambda\}} |\varphi(u)| \, du, \quad (1.2)$$

for every $\varphi \in L^1(\mathbb{R}^n)$ and every $\lambda > 0$ from which it is easy to get the following $L^p$ inequality:

$$\|M_d\varphi\|_p \leq \frac{p}{p-1} \|\varphi\|_p, \quad (1.3)$$

for every $p > 1$ and every $\varphi \in L^p(\mathbb{R}^n)$. It is easy to see that the weak type inequality (1.2) is best possible. It has also been proved that (1.3) is best possible (see [1] and [2] for the general martingales and [7] for the dyadic ones).

In studying dyadic maximal operators as well as more general variants it would be convenient to work with functions supported in the unit cube $[0,1]^n$ and for this reason we replace $M_d$ by:

$$M'_d\varphi(x) = \sup \left\{ \frac{1}{|Q|} \int_Q |\varphi(u)| \, du : x \in Q \subseteq [0,1]^n \text{ is a dyadic cube} \right\} \quad (1.4)$$
and hence work completely in the measure space $[0,1]^n$. A standard definition and approximation argument allows one to pass to the operator $M_d$.

An approach for studying such maximal operators is the introduction of the so called Bellman functions (see [3]) related to them. Our interest is in the following Bellman type function:

$$B_p(f,F) = \sup \left\{ \frac{1}{|Q|} \int_Q (M_d \varphi)^p : \text{Av}_Q(\varphi^p) = F, \text{Av}_Q(\varphi) = f \right\},$$  \hspace{1cm} (1.5)

where $Q$ is a fixed dyadic cube, $\varphi \in L^p(Q)$ is nonnegative and $f, F$ satisfy $0 < f^p \leq F$.

The function (1.5) has been precisely computed in [4] and [5]. In fact the approach for the study of (1.5) has been given in a more general setting. Hence we will let $(X, \mu)$ be a nonatomic probability space and let $T$ be a family of measurable subsets of $X$ that have a tree-like structure similar to the one of the dyadic case (the precise definition will be given in the next section). Then we define the maximal operator associated to $T$ as follows:

$$M_T \varphi(x) = \sup \left\{ \frac{1}{\mu(I)} \int_I |\varphi| \, d\mu : x \in I \in T \right\},$$  \hspace{1cm} (1.6)

for every $\varphi \in L^1(X, \mu)$. Then the corresponding to (1.5) Bellman function is

$$B_T^p(f,F) = \sup \left\{ \int_X (M_T \varphi)^p \, d\mu : \varphi \geq 0, \varphi \in L^p(X, \mu), \int_X \varphi^p \, d\mu = F, \int_X \varphi \, d\mu = f \right\}. \hspace{1cm} (1.7)$$

In [4] and [5] the precise value of (1.7) has been given. More precisely it is proved that $B_T^p(f,F) = F \omega_p \left( \frac{f^p}{F} \right)^p$ for every pair $(f,F)$ such that $0 < f^p \leq F$, where $\omega_p : [0,1] \to \left[ 1, \frac{p}{p-1} \right]$ denotes the inverse of the function $H_p : \left[ 1, \frac{p}{p-1} \right] \to [0,1]$, which is given by $H_p(z) = -(p-1) z^p + p z^{p-1}$.

More general functions arise by adding variables on them, and the difficulty of their evaluation gets even harder. One of them is the following:

$$B_T^p(f,F,k) = \sup \left\{ \int_K (M_T \varphi)^p \, d\mu : \varphi \geq 0, \varphi \in L^p(X, \mu), \int_X \varphi^p \, d\mu = F, \int_X \varphi \, d\mu = f, K \subseteq X \text{ is } \mu\text{-measurable with } \mu(K) = k \right\}. \hspace{1cm} (1.8)$$

Here $k \in (0,1]$ and $0 < f^p \leq F$. Of course $B_T^p(f,F,1) = B_p(f,F)$. In [4] a linearization technique was introduced for the evaluation of (1.7) and (1.8). Additionally one can find in [4] the connection of the function (1.8) with the Carleson Imbedding theorem. In [5] and [6] it is used another technique (via a
symmetrization principle for $\mathcal{M}_T$) which enabled the authors to provide evaluation of them. More precisely it can be proved that

$$\left( \int_0^k \left( \frac{1}{t} \int_0^t g \right)^p dt \right) : \text{where } g : (0, 1] \to \mathbb{R}^+ \text{ is nonincreasing with } \int_0^1 g = f, \int_0^1 g^p = F$$

In this article we find a precise $g_k : (0, 1] \to \mathbb{R}$ for which this supremum is attained.

## 2 Preliminaries

Let $(X, \mu)$ be a nonatomic probability space (i.e. $\mu(X) = 1$). Then we give the following

**Definition 1.** A set $T$ of measurable subsets of $X$ will be called a tree if the following conditions are satisfied:

1. $X \in T$ and for every $I \in T$ we have $\mu(I) > 0$
2. For every $I \in T$ there corresponds a finite or countable subset $C(I)$ of $T$ containing at least two elements such that:
   a. the elements of $C(I)$ are pairwise disjoint subsets of $I$
   b. $I = \cup C(I)$
3. $T = \cup_{m \geq 0} T(m)$, where $T(0) = \{X\}$ and $T(m+1) = \cup_{I \in T(m)} C(I)$.
4. We have that $\lim_{m \to \infty} \sup_{I \in T(m)} \mu(I) = 0$.

Now we state some facts that appear in [4]. Fix $k \in (0, 1)$ and consider the functions

$$h_k(B) = \frac{(f - B)^p}{(1 - k)^{p-1}} + \frac{B^p}{k^{p-1}}, \quad (2.1)$$

defined for $0 \leq B \leq f$ and

$$\mathcal{R}_k(B) = \left( F - \frac{(f - B)^p}{(1 - k)^{p-1}} \right) \omega_p \left( \frac{B^p}{k^{p-1} \left( F - \frac{(f - B)^p}{(1 - k)^{p-1}} \right)} \right)^p, \quad (2.2)$$

defined for all $B \in [0, f]$ such that $h_k(B) \leq F$. Then as one can see in [4], the domain of $\mathcal{R}_k$ is an interval $[p_0(f, F, k), p_1(f, F, k)]$. We state the following from [4]:

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Lemma 1. i) For every $U \in [0, 1]$ the equation
\[
\sigma(z) = -(p - 1)z^p + (p - 1 + k)z^{p-1} - U \left[ 1 + (1 - k) \left( \frac{p - 1}{z} - p \right) \right] = 0 \quad (2.3)
\]
has a unique solution in the interval $\left[ 1, 1 + \frac{k}{p-1} \right]$ which is denoted by $\omega_{p,k}(U)$.

ii) The function $R_k$ defined on $[p_0(f, F, k), p_1(f, F, k)]$ assumes its absolute maximum at the unique interior point $B_0 \in \left( kf, \min \left( \frac{pk}{p - 1 + k}, p_1(f, F, k) \right) \right)$ such that
\[
\frac{f(1 - k)}{f - B_0} = \omega_{p,k}\left( \frac{fp}{F} \right).
\]
Moreover
\[
R_k(B_0) = \left[ F \omega_{p,k}\left( \frac{fp}{p} \right) - (1 - k)f^p \right] \cdot \left[ 1 - (1 - k) \left( \omega_{p,k}\left( \frac{fp}{p} \right) \right)^{-1} \right]^p \quad (2.4)
\]

iii) the value of $B_T^p(f, F, k)$ is given by (2.4).

In the next section we construct for any $k \in (0, 1]$ a nonincreasing $g_k : (0, 1] \rightarrow \mathbb{R}^+$ with $\int_0^1 g_k = f$, $\int_0^1 g_k^p = F$ for which $B_T^p(f, F, k) = \int_0^k \left( 1 \int_0^t g_k \right)^p dt$. The details are given in the next section.

3 Construction of the function $g_k$

We are going to prove the following:

**Theorem 1.** There exists a function $g : (0, 1] \rightarrow \mathbb{R}$ nonincreasing with $\int_0^1 g = f$ and $\int_0^1 g^p = F$ for which $B_T^p(f, F, k) = \int_0^k \left( 1 \int_0^t g \right)^p dt$.

More precisely an explicit function $g_k$ is given.

**Proof.** As it has been proven in [4] or [6]
\[
B_T^p(f, F, k) = \sup \{ R_k(B) : 0 \leq B \leq f, \text{ and } h_k(B) \leq F \} \quad (3.1)
\]
where $h_k(B)$ and $R_k(B)$ are given by (2.1) and (2.2) respectively. Note that $R_k$ is defined for those $B \in [0, f]$ for which
\[
h_k(B) \leq F \iff \frac{(f - B)^p}{(1 - k)p-1} + \frac{B^p}{kp-1} \leq F \iff 0 \leq \frac{B^p}{k^p-1 \left[ F - \frac{(f-B)^p}{(1-k)^p-1} \right]} \leq 1
\]
so that (2.2) makes sense in view of the definition of $\omega_p$.

By the proof of Lemma [1] as is given in [4], we see that the value $B_0$ satisfy the following:
\[
\omega_p(Z_0) = \frac{B_0}{k} \frac{1 - k}{f - B_0} \iff Z_0 = H_p \left( \frac{B_0}{k} \frac{1 - k}{f - B_0} \right) \quad (3.2)
\]
where $Z_0$ is given by:

$$Z_0 = \frac{B_0^p}{k^{p-1} \left( F - \frac{(f - B_0)^p}{(1-k)^{p-1}} \right)}.$$  \tag{3.3}

Then if we set $z = \frac{f(1-k)}{f - B_0}$, \textcolor{red}{
524 is equivalent to the equation $\sigma(z) = 0 \iff z = \omega_{p,k}(U)$, for $U = \frac{f^p}{F}$ of equivalently $f(1-k) = \omega_{p,k}(U)$.

Then

$$B_T^p (f, F, k) = R_k(B_0) = \left[ F - \frac{(f - B_0)^p}{(1-k)^{p-1}} \right] \left( \frac{B_0^p}{k^{p-1} \left( F - \frac{(f - B_0)^p}{(1-k)^{p-1}} \right)} \right)^p.$$

We search for a function $g_k : (0, 1] \to \mathbb{R}$ of the following form

$$g_k(t) = \begin{cases} A_1 t^{-1+\frac{1}{p}}, & t \in (0, k] \\ c, & t \in [k, 1] \end{cases}$$

with the property

$$B_T^p (f, F, k) = \int_0^k \left( \frac{1}{t} \int_0^t g_k \right)^p dt.$$  \tag{3.5}

We shall prove that such a function is continuous in $(0, 1]$ and constant on $[k, 1]$. That is we search for suitable $A_1$, $a$, $c$ that depend of $(f, F, k)$ for which it is satisfied

$$\int_0^1 g_k = f, \quad \int_0^1 g_k^p = F.$$  \tag{3.6}

We first work with the $L^1$-norm of $g_k$. We have that $\int_0^1 g = f \iff \int_0^k g + \int_k^1 g = f \iff \int_0^k g + c(1-k) = f.$  \tag{3.7}

We set now $c = \frac{f - B_0}{1-k}$. Thus we need to ensure that

$$\int_0^k g_k = B_0.$$

Secondly we work with

$$\int_0^1 g^p = F \iff \int_0^k g^p = F - \frac{(f - B_0)^p}{(1-k)^{p-1}}.$$  \tag{3.9}

Then \textcolor{red}{[3.8] is equivalent to

$$\int_0^k A_1 t^{-1+\frac{1}{p}} dt = B_0 \iff A_1 = \frac{B_0 k^{-1/a}}{a},$$

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so that we found $A_1$ as a function of $a$. We search now for $a$ such that (3.9) is satisfied, or equivalently

$$A_1^p \int_0^t t^{-\frac{p}{p+1}} dt = F - \frac{(f - B_0)^p}{(1-k)^{p-1}} \iff$$

$$B_0^p \frac{k^{-\frac{p}{a}}}{a^p} \frac{1}{1 + \frac{p}{1-p}} k^{1-p+\frac{p}{a}} = F - \frac{(f - B_0)^p}{(1-k)^{p-1}} \iff$$

$$B_0^p \frac{1}{k^{p-1} pa^{p-1} - (p-1)a^p} = F - \frac{(f - B_0)^p}{(1-k)^{p-1}} \iff$$

$$B_0^p \frac{1}{k^{p-1} H_p(a)} = F - \frac{(f - B_0)^p}{(1-k)^{p-1}} \iff H_p(a) = Z_0 \iff a = \omega_p(Z_0) \in \left[1, \frac{p}{p-1}\right] \quad (3.11)$$

Thus $a$ is given by (3.11) and $A_1$ by (3.10). Note that for every $t \in (0, k]$ we have that

$$\int_0^k g_k(u) du = t a g_k(t) \iff \int_0^t g_k = a g_k(t), \; \forall t \in (0, k].$$

Thus

$$\int_0^k \left( \frac{1}{t} \int_0^t g_k \right)^p dt = \int_0^k [a g(t)]^p dt = a^p \int_0^k g^p = \left[ F - \frac{(f - B_0)^p}{(1-k)^{p-1}} \right] \omega_p \left( \frac{B_0^p}{k^{p-1} \left( F - \frac{(f - B_0)^p}{(1-k)^{p-1}} \right)} \right)^p. \quad (3.12)$$

This last quantity that appears in (3.12), equals $B_0^p(f, F, k)$. We need only to prove that $g_k$ is continuous on $t_0 = k$. For this it is enough to prove that

$$\frac{f - B_0}{1-k} = A_1 k^{-\frac{1}{k+1}} \iff A_1 k^{-\frac{1}{k} + \frac{1}{2}} = \left( \frac{1-k}{f - B_0} \right)^{-1} \iff$$

$$A_1 k^{-\frac{1}{k} + \frac{1}{2}} = \left( \frac{B_0}{k} \frac{1-k}{f - B_0} \right)^{-1} \frac{B_0}{k} = \quad (3.13)$$

But on the other hand $a = \omega_p(Z_0) = \frac{B_0}{k} \frac{1-k}{f - B_0}$, (see section 2). Then (3.13) is equivalent to

$$A_1 k^{-\frac{1}{k} + \frac{1}{2}} = a^{-1} B_0 \iff A_1 k^{\frac{1}{2}} = \frac{B_0}{a} \iff$$

$$A_1 = \frac{B_0 k^{-\frac{1}{2}}}{a}, \text{ which is true in view of (3.10).}$$

Thus Theorem 1 is proved.
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