Limiting eigenvectors of outliers for Spiked Information-Plus-Noise type matrices

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Abstract

We consider an Information-Plus-Noise type matrix where the Information matrix is a spiked matrix. When some eigenvalues of the random matrix separate from the bulk, we study how the corresponding eigenvectors project onto those of the spikes. Note that, in an Appendix, we present alternative versions of the earlier results of [3] ("no eigenvalue outside the support of the deterministic equivalent measure") and [11] ("exact separation phenomenon") where we remove some technical assumptions that were difficult to handle.

1 Introduction

In this paper, we consider the so-called Information-Plus-Noise type model

\[ M_N = \Sigma_N \Sigma_N^* \text{ where } \Sigma_N = \sigma \frac{X_N}{\sqrt{N}} + A_N, \]

defined as follows.

- \( n = n(N), n \leq N, c_N = n/N \rightarrow_{N \rightarrow +\infty} c \in [0;1]. \)
- \( \sigma \in ]0; +\infty[. \)
- \( X_N = [X_{ij}]_{1 \leq i \leq n; 1 \leq j \leq N} \) is an infinite set of complex random variables such that \( \{\Re(X_{ij}), \Im(X_{ij}), i \in \mathbb{N}, j \in \mathbb{N}\} \) are independent centered random variables with variance \(1/2\) and satisfy

\[ 1. \text{ There exists } K > 0 \text{ and a random variable } Z \text{ with finite fourth moment for which there exists } x_0 > 0 \text{ and an integer number } n_0 > 0 \text{ such that, for any } x > x_0 \text{ and any integer numbers } n_1, n_2 > n_0, \text{ we have} \]

\[ \frac{1}{n_1 n_2} \sum_{i \leq n_1, j \leq n_2} P(|X_{ij}| > x) \leq KP(|Z| > x). \]  

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2. 
\[ \sup_{(i,j) \in \mathbb{N}^2} \mathbb{E}(|X_{ij}|^3) < +\infty. \]  

- Let \( \nu \) be a compactly supported probability measure on \( \mathbb{R} \) whose support has a finite number of connected components. Let \( \Theta = \{ \theta_1; \ldots; \theta_J \} \) where \( \theta_1 > \ldots > \theta_J \geq 0 \) are \( J \) fixed real numbers independent of \( N \) which are outside the support of \( \nu \). Let \( k_1, \ldots, k_J \) be fixed integer numbers independent of \( N \) and \( r = \sum_{j=1}^J k_j \). Let \( \beta_j(N) \geq 0 \), \( r + 1 \leq j \leq n \), be such that \( \frac{1}{n} \sum_{j=r+1}^{n} \delta_{\beta_j(N)} \) weakly converges to \( \nu \) and

\[ \max_{r+1 \leq j \leq n} \text{dist} (\beta_j(N), \text{supp}(\nu)) \xrightarrow{N \to \infty} 0 \]  

where \( \text{supp}(\nu) \) denotes the support of \( \nu \). Let \( \alpha_j(N), j = 1, \ldots, J \), be real nonnegative numbers such that

\[ \lim_{N \to +\infty} \alpha_j(N) = \theta_j. \]

Let \( A_N \) be a \( n \times N \) deterministic matrix such that, for each \( j = 1, \ldots, J \), \( \alpha_j(N) \) is an eigenvalue of \( A_N A_N^* \) with multiplicity \( k_j \), and the other eigenvalues of \( A_N A_N^* \) are the \( \beta_j(N) \), \( r + 1 \leq j \leq n \). Note that the empirical spectral measure of \( A_N A_N^* \) weakly converges to \( \nu \).

**Remark 1.1.** Note that assumption such as (i) appears in [14]. It obviously holds if the \( X_{ij} \)'s are identically distributed with finite fourth moment.

For any Hermitian \( n \times n \) matrix \( Y \), denote by \( \text{spect}(Y) \) its spectrum, by

\[ \lambda_1(Y) \geq \ldots \geq \lambda_n(Y) \]

the ordered eigenvalues of \( Y \) and by \( \mu_Y \) the empirical spectral measure of \( Y \):

\[ \mu_Y := \frac{1}{n} \sum_{i=1}^{n} \delta_{\lambda_i(Y)}. \]

For a probability measure \( \tau \) on \( \mathbb{R} \), denote by \( g_\tau \) its Stieltjes transform defined for \( z \in \mathbb{C} \setminus \mathbb{R} \) by

\[ g_\tau(z) = \int_{\mathbb{R}} \frac{d\tau(x)}{z - x}. \]

When the \( X_{ij} \)'s are identically distributed, Dozier and Silverstein established in [15] that almost surely the empirical spectral measure \( \mu_{M_N} \) of \( M_N \) converges weakly towards a nonrandom distribution \( \mu_{\sigma,\nu,c} \) which is characterized in terms of its Stieltjes transform which satisfies the following equation: for any \( z \in \mathbb{C}^+ \),

\[ g_{\mu_{\sigma,\nu,c}}(z) = \frac{1}{(1 - \sigma^2 g_{\mu_{\sigma,\nu,c}}(z))z - \frac{1}{1 - \sigma^2 g_{\mu_{\sigma,\nu,c}}(z)} - \sigma^2(1 - c)} d\nu(t). \]  

This result of convergence was extended to independent but non identically distributed random variables by Xie in [32]. (Note that, in [19], the authors in-
vestigated the case where \(\sigma\) is replaced by a bounded sequence of real numbers.) In [11], the author carries on with the study of the support of the limiting spectral measure previously investigated in [16] and later in [29, 25] and obtains that there is a one-to-one relationship between the complement of the limiting support and some subset in the complement of the support of \(\nu\) which is defined in [16] below.

**Proposition 1.2.** Define differentiable functions \(\omega_{\sigma, \nu, c}\) and \(\Phi_{\sigma, \nu, c}\) on respectively \(\mathbb{R} \setminus \text{supp}(\mu_{\sigma, \nu, c})\) and \(\mathbb{R} \setminus \text{supp}(\nu)\) by setting

\[
\omega_{\sigma, \nu, c} : \mathbb{R} \setminus \text{supp}(\mu_{\sigma, \nu, c}) \rightarrow \mathbb{R}
\]

\[
x \mapsto x(1 - \sigma^2 c g_{\mu_{\sigma, \nu, c}}(x) - \sigma^2(1 - c)(1 - \sigma^2 c g_{\mu_{\sigma, \nu, c}}(x)) (5)
\]

and

\[
\Phi_{\sigma, \nu, c} : \mathbb{R} \setminus \text{supp}(\nu) \rightarrow \mathbb{R}
\]

\[
x \mapsto x(1 + \sigma^2 g_{\nu}(x) + \sigma^2(1 - c) + \sigma^2 g_{\nu}(x)).
\]

Set

\[
E_{\sigma, \nu, c} := \left\{ x \in \mathbb{R} \setminus \text{supp}(\nu), \Phi'_{\sigma, \nu, c}(x) > 0, g_{\nu}(x) > -\frac{1}{\sigma^2 c} \right\}.
\]

\(\omega_{\sigma, \nu, c}\) is an increasing analytic diffeomorphism with positive derivative from \(\mathbb{R} \setminus \text{supp}(\mu_{\sigma, \nu, c})\) to \(E_{\sigma, \nu, c}\), with inverse \(\Phi_{\sigma, \nu, c}\).

Moreover, extending previous results in [25] and [8] involving the Gaussian case and finite rank perturbations, [11] establishes a one-to-one correspondence between the \(\theta_i\)'s that belong to the set \(E_{\sigma, \nu, c}\) (counting multiplicity) and the outliers in the spectrum of \(M_N\). More precisely, setting

\[
\Theta_{\sigma, \nu, c} = \left\{ \theta \in \Theta, \Phi'_{\sigma, \nu, c}(\theta) > 0, g_{\nu}(\theta) > -\frac{1}{\sigma^2 c} \right\},
\]

and

\[S = \text{supp}(\mu_{\sigma, \nu, c}) \cup \{\Phi_{\sigma, \nu, c}(\theta), \theta \in \Theta_{\sigma, \nu, c}\},\]

we have the following results.

**Theorem 1.3.** [11] For any \(\epsilon > 0\),

\[\mathbb{P}[\text{for all large } N, \text{spect}(M_N) \subset \{x \in \mathbb{R}, \text{dist}(x, S) \leq \epsilon\}] = 1.\]

**Theorem 1.4.** [11] Let \(\theta_j\) be in \(\Theta_{\sigma, \nu, c}\) and denote by \(n_{j-1} + 1, \ldots, n_{j-1} + k_j\) the descending ranks of \(\alpha_j(N)\) among the eigenvalues of \(A_N A_N^*\). Then the \(k_j\) eigenvalues \((\lambda_{n_{j-1}+i}(M_N), 1 \leq i \leq k_j)\) converge almost surely outside the support of \(\mu_{\sigma, \nu, c}\) towards \(\rho_{\theta_j} := \Phi_{\sigma, \nu, c}(\theta_j)\). Moreover, these eigenvalues asymptotically separate from the rest of the spectrum since (with the conventions that \(\lambda_0(M_N) = +\infty\) and \(\lambda_{N+1}(M_N) = -\infty\)) there exists \(\delta_0 > 0\) such that almost surely for all large \(N\),

\[
\lambda_{n_{j-1}}(M_N) > \rho_{\theta_j} + \delta_0 \text{ and } \lambda_{n_{j-1}+k_j+1}(M_N) < \rho_{\theta_j} - \delta_0.
\]
Remark 1.5. Note that Theorems 1.3 and 1.4 were established in [11] for $A_N$ as in (14) below and with $S \cup \{0\}$ instead of $S$ but they hold true as stated above and in the more general framework of this paper. Indeed, these extensions can be obtained sticking to the proof of the corresponding results in [11] but using the new versions of [3] and of the exact separation phenomenon of [11] which are presented in the Appendix A of the present paper.

The aim of this paper is to study how the eigenvectors corresponding to the outliers of $M_N$ project onto those corresponding to the spikes $\theta_i$’s. Note that there are some pioneering results investigating the eigenvectors corresponding to the outliers of finite rank perturbations of classical random matricial models: [28] in the real Gaussian sample covariance matrix setting, and [7, 8] dealing with finite rank additive or multiplicative perturbations of unitarily invariant matrices. For a general perturbation, dealing with sample covariance matrices, S. Péché and O. Ledoit [23] introduced a tool to study the average behaviour of the eigenvectors but it seems that this did not allow them to focus on the eigenvectors associated with the eigenvalues that separate from the bulk. It turns out that further studies [10, 5] point out that the angle between the eigenvectors of the outliers of the deformed model and the eigenvectors associated to the corresponding original spikes is determined by Biane-Voiculescu’s subordination function. For the model investigated in this paper, such a free interpretation holds but we choose not to develop this free probabilistic point of view in this paper and we refer the reader to the paper [13]. Here is the main result of the paper.

**Theorem 1.6.** Let $\theta_j$ be in $\Theta_{\sigma,\nu,c}$ (defined in (7)) and denote by $n_j-1+1, \ldots, n_j-1+k_j$ the descending ranks of $\alpha_j(N)$ among the eigenvalues of $A_N A_N^\ast$. Let $\xi(j)$ be a normalized eigenvector of $M_N$ relative to one of the eigenvalues $(\lambda_{n_j-1+q}(M_N), 1 \leq q \leq k_j)$. Denote by $\| \cdot \|_2$ the Euclidean norm on $\mathbb{C}^n$. Then, almost surely

1. $\lim_{N \to +\infty} \| P_{Ker(\alpha_j(N)I_N-A_N A_N^\ast)} \xi(j) \|_2^2 = \tau(\theta_j)$ where

$$\tau(\theta_j) = \frac{1 - \sigma^2c_\nu(\rho_{\theta_j})}{\omega_{\sigma,\nu,c}(\rho_{\theta_j})} = \frac{\Phi_{\sigma,\nu,c}(\theta_j)}{1 + \sigma^2c_\nu(\theta_j)} \quad (10)$$

2. For any $\theta_i$ in $\Theta_{\sigma,\nu,c} \setminus \{\theta_j\}$,

$$\lim_{N \to +\infty} \| P_{Ker(\alpha_i(N)I_N-A_N A_N^\ast)} \xi(j) \|_2 = 0.$$

The sketch of the proof of Theorem 1.6 follows the analysis of [10] as explained in Section 2. In Section 3, we prove a universal result allowing to reduce the study to estimating expectations of Gaussian resolvent entries carried on Section 4. In Section 5, we explain how to deduce Theorem 1.6 from the previous Sections. In an Appendix A, we present alternative versions on the one
hand of the result in [3] about the lack of eigenvalues outside the support of the
deterministic equivalent measure, and, on the other hand, of the result in [11]
about the exact separation phenomenon. These new versions deal with random
variables whose imaginary and real parts are independent but remove the tech-
nical assumptions ((1.10) and “\(b_1 > 0\)” in Theorem 1.1 in [3] and “\(\omega_{\sigma,\nu,c}(b) > 0\)” in Theorem 1.2 in [11]). This allows us to claim that Theorem 1.4 holds in our
context (see Remark 1.5). Finally, we present, in an Appendix B, some technical
lemmas that are used throughout the paper.

2 Sketch of the proof

Throughout the paper, for any \(m \times p\) matrix \(B\), \((m, p) \in \mathbb{N}^2\), we will denote by \(\|B\|\) the largest singular value of \(B\), and by \(\|B\|_2 = (\text{Tr}(BB^*))^{1/2}\) its Hilbert-
Schmidt norm.

The proof of Theorem 1.6 follows the analysis in two steps of [10].

**Step A.** First, we shall prove that, for any orthonormal system \((\xi_1, \cdots, \xi_k)\) of eigenvectors associated to the \(k_j\) eigenvalues \(\lambda_{n_j-1+q}(M_N), 1 \leq q \leq k_j\), the following convergence holds almost surely:

\[
\forall l = 1, \ldots, J,
\sum_{p=1}^{k_j} \left\| P_{\ker(\alpha_l(N))} I_N - A_N A_N^* \xi_p \right\|_2^2 \to_{N \to +\infty} k_j \delta_{jl} \frac{1 - \sigma^2 c g_{\nu,\nu,c}(\rho_{\theta_j})}{\omega_{\nu,\nu,c}(\rho_{\theta_j})}. \tag{11}
\]

Note that for any smooth functions \(h\) and \(f\) on \(\mathbb{R}\), if \(v_1, \ldots, v_n\) are eigenvectors associated to \(\lambda_1(M_N), \ldots, \lambda_n(M_N)\) and \(w_1, \ldots, w_n\) are eigenvectors associated to \(\lambda_1(A_N A_N^*), \ldots, \lambda_n(A_N A_N^*)\), one can easily check that

\[
\text{Tr} [h(M_N) f(A_N A_N^*)] = \sum_{m, p=1}^n h(\lambda_p(M_N)) f(\lambda_m(A_N A_N^*)) \langle v_m, w_p \rangle^2. \tag{12}
\]

Thus, since \(\alpha_l(N)\) on one hand and the \(k_j\) eigenvalues of \(M_N\) in \((\rho_{\theta_j} - \varepsilon, \rho_{\theta_j} + \varepsilon)\) (for \(\varepsilon\) small enough) on the other hand, asymptotically separate from the rest of the spectrum of respectively \(A_N A_N^*\) and \(M_N\), a fit choice of \(h\) and \(f\) will allow the study of the restrictive sum \(\sum_{p=1}^{k_j} \left\| P_{\ker(\alpha_l(N))} I_N - A_N A_N^* \xi_p \right\|_2^2\). Therefore proving \(11\) is reduced to the study of the asymptotic behaviour of \(\text{Tr} [h(M_N) f(A_N A_N^*)]\) for some functions \(f\) and \(h\) respectively concentrated on a neighborhood of \(\theta_l\) and \(\rho_{\theta_j}\).

**Step B:** In the second, and final, step, we shall use a perturbation argument identical to the one used in [10] to reduce the problem to the case of a spike with multiplicity one, case that follows trivially from Step A.

Step B closely follows the lines of [10] whereas Step A requires substantial
work. We first reduce the investigations to the mean Gaussian case by proving the following.
Proposition 2.1. Let $X_N$ as defined in Section 1. Let $G_N = [G_{ij}]_{1 \leq i \leq n, 1 \leq j \leq N}$ be a $n \times N$ random matrix with i.i.d. standard complex normal entries. Let $h$ be a function in $C^\infty(\mathbb{R}, \mathbb{R})$ with compact support, and $\Gamma_N$ be a $n \times n$ Hermitian matrix such that

$$\sup_{n,N} \| \Gamma_N \| < \infty \text{ and } \sup_{n,N} \text{rank}(\Gamma_N) < \infty. \quad (13)$$

Then almost surely,

$$\text{Tr} \left( h \left( \left( \frac{X_i}{\sqrt{N}} + A_N \right) \left( \frac{X_i}{\sqrt{N}} + A_N \right)^* \right) \Gamma_N \right)$$

$$- \mathbb{E} \left( \text{Tr} \left[ h \left( \left( \frac{G_{ij}}{\sqrt{N}} + A_N \right) \left( \frac{G_{ij}}{\sqrt{N}} + A_N \right)^* \right) \Gamma_N \right] \right) \to_{N \to \infty} 0.$$

The asymptotic behaviour of $\mathbb{E} \left( \text{Tr} \left[ h \left( \left( \frac{G_{ij}}{\sqrt{N}} + A_N \right) \left( \frac{G_{ij}}{\sqrt{N}} + A_N \right)^* \right) f(A_N A_N^*) \right] \right)$ can be deduced, by using the bi-unitarily invariance of the distribution of $G_N$, from the following Proposition 2.2 and Lemma 5.1.

Proposition 2.2. Let $G_N = [G_{ij}]_{1 \leq i \leq n, 1 \leq j \leq N}$ be a $n \times N$ random matrix with i.i.d. complex standard normal entries. Assume that $A_N$ is such that

$$A_N = \begin{pmatrix} d_1(N) & 0 \\ \vdots & \ddots \\ 0 & d_n(N) \end{pmatrix} \quad (14)$$

where $n = n(N)$, $n \leq N$, $c_N = n/N \to_{N \to \infty} c \in [0; 1]$, for $i = 1, \ldots, n$, $d_i(N) \in \mathbb{C}$, $\sup_N \max_{i=1, \ldots, n} |d_i(N)| < +\infty$ and $\frac{1}{n} \sum_{i=1}^n |d_i(N)|^2$ weakly converges to a compactly supported probability measure $\nu$ on $\mathbb{R}$ when $N$ goes to infinity. Define for all $z \in \mathbb{C} \setminus \mathbb{R}$,

$$G^q_N(z) = \left( zI - \left( \frac{G_N}{\sqrt{N}} + A_N \right) \left( \frac{G_N}{\sqrt{N}} + A_N \right)^* \right)^{-1}.$$

Define for any $q = 1, \ldots, n$,

$$\gamma_q(N) = (A_N A_N^*)_{qq} = |d_q(N)|^2. \quad (15)$$

There is a polynomial $P$ with nonnegative coefficients, a sequence $(u_N)$ of nonnegative real numbers converging to zero when $N$ goes to infinity and some nonnegative real number $l$, such that for any $(p, q)$ in $\{1, \ldots, n\}^2$, for all $z \in \mathbb{C} \setminus \mathbb{R}$,

$$\mathbb{E} \left( \left( G^q_N(z) \right)_{pq} \right) = \frac{1 - \sigma^2 \omega_{\sigma, \nu, c}(z)}{\omega_{\sigma, \nu, c}(z) - \gamma_q(N)} \delta_{pq} + \Delta_{p,q,N}(z), \quad (16)$$

with

$$|\Delta_{p,q,N}(z)| \leq (1 + |z|)^l P(|3z|^{-1}) u_N.$$
3 Proof of Proposition 2.1

In the following, we will denote by \( o_C(1) \) any deterministic sequence of positive real numbers depending on the parameter \( C \) and converging for each fixed \( C \) to zero when \( N \) goes to infinity. The aim of this section is to prove Proposition 2.1.

Define for any \( C > 0 \),
\[
Y_{ij}^C = \Re X_{ij} 1_{|\Re X_{ij}| \leq C} - \E (\Re X_{ij} 1_{|\Re X_{ij}| \leq C}) + \sqrt{-1} \{ \Im X_{ij} 1_{|\Re X_{ij}| \leq C} - \E (\Im X_{ij} 1_{|\Im X_{ij}| \leq C}) \}.
\] (17)

Set
\[
\theta^* = \sup_{(i,j) \in \mathbb{N}^2} \E(|X_{ij}|^3) < +\infty.
\]

We have
\[
\E (|X_{ij} - Y_{ij}^C|^2) = \E (|\Re X_{ij}|^2 1_{|\Re X_{ij}| > C}) + \E (|\Im X_{ij}|^2 1_{|\Im X_{ij}| > C}) - \E (\Re X_{ij} 1_{|\Re X_{ij}| > C})^2 - \E (\Im X_{ij} 1_{|\Im X_{ij}| > C})^2
\]
\[
\leq \frac{\E (|\Re X_{ij}|^3) + \E (|\Im X_{ij}|^3)}{C}
\]
so that
\[
\sup_{i \geq 1, j \geq 1} \E (|X_{ij} - Y_{ij}^C|^2) \leq \frac{2\theta^*}{C}.
\]

Note that
\[
1 - 2\E (|\Re Y_{ij}^C|^2) = 1 - 2\E \left\{ (|\Re X_{ij}|^2 1_{|\Re X_{ij}| \leq C} - \E (|\Re X_{ij}|^2 1_{|\Re X_{ij}| \leq C}))^2 \right\}
\]
\[
= 2 \left[ \frac{1}{2} - \E (|\Re X_{ij}|^2 1_{|\Re X_{ij}| \leq C}) \right] + 2 \E (|\Re X_{ij}|^2 1_{|\Re X_{ij}| > C})^2
\]
\[
= 2\E (|\Re X_{ij}|^2 1_{|\Re X_{ij}| > C}) + 2 \E (|\Re X_{ij}|^2 1_{|\Re X_{ij}| > C})^2,
\]
so that
\[
\sup_{i \geq 1, j \geq 1} |1 - 2\E (|\Re Y_{ij}^C|^2)| \leq \frac{4\theta^*}{C}.
\]

Similarly
\[
\sup_{i \geq 1, j \geq 1} |1 - 2\E (|\Im Y_{ij}^C|^2)| \leq \frac{4\theta^*}{C}.
\]

Let us assume that \( C > 8\theta^* \). Then, we have
\[
\E (|\Re Y_{ij}^C|^2) > \frac{1}{4} \quad \text{and} \quad \E (|\Im Y_{ij}^C|^2) > \frac{1}{4}.
\]

Define for any \( C > 8\theta^* \), \( X^C = (X_{ij}^C)_{1 \leq i \leq n; 1 \leq j \leq N} \), where for any \( 1 \leq i \leq n, 1 \leq j \leq N \),
\[
X_{ij}^C = \frac{\Re Y_{ij}^C}{\sqrt{2\E (|\Re Y_{ij}^C|^2)}} + \sqrt{-1} \frac{\Im Y_{ij}^C}{\sqrt{2\E (|\Im Y_{ij}^C|^2)}}.
\] (18)
Lemma 3.1. Establish the following approximation result.

Denote by $U$ and $\cdot$ almost surely, for all large $N$, $X = \frac{X^C + \alpha U}{\sqrt{1 + \alpha^2}}$.

Now, for any $n \times N$ matrix $B$, let us introduce the $(N + n) \times (N + n)$ matrix

$$M_{N+n}(B) = \begin{pmatrix} 0_{n \times n} & B & A_N \\ B^* & A_N^* & 0_{N \times N} \end{pmatrix}.$$

Define for any $z \in \mathbb{C} \setminus \mathbb{R}$,

$$\tilde{G}(z) = \left( zI_{N+n} - M_{N+n} \left( \frac{X_N}{\sqrt{N}} \right) \right)^{-1},$$

and

$$\tilde{G}^{\alpha,C}(z) = \left( zI_{N+n} - M_{N+n} \left( \frac{X^{\alpha,C}}{\sqrt{N}} \right) \right)^{-1}.$$

Denote by $\mathcal{U}(n + N)$ the set of unitary $(n + N) \times (n + N)$ matrices. We first establish the following approximation result.

**Lemma 3.1.** There exist some positive deterministic functions $u$ and $v$ on $[0, +\infty)$ such that $\lim_{C \to +\infty} u(C) = 0$ and $\lim_{\alpha \to 0} v(\alpha) = 0$, and a polynomial $P$ with nonnegative coefficients such that for any $\alpha$ and $C > 8\theta^*$, we have that

- almost surely, for all large $N$,

$$\sup_{U \in \mathcal{U}(n + N)} \sup_{(i,j) \in \{1, \ldots, n + N\}^2} \sup_{z \in \mathbb{C} \setminus \mathbb{R}} |\Im z|^2 \left| (U^* \tilde{G}^{\alpha,C}(z)U)_{ij} - (U^* \tilde{G}(z)U)_{ij} \right| 
\leq u(C) + v(\alpha), \quad (19)$$

- for all large $N$,

$$\sup_{U \in \mathcal{U}(n + N)} \sup_{(i,j) \in \{1, \ldots, n + N\}^2} \sup_{z \in \mathbb{C} \setminus \mathbb{R}} \frac{1}{P(|\Im z|^{-1})} \left| \mathbb{E} \left( (U^* \tilde{G}^{\alpha,C}(z)U)_{ij} - (U^* \tilde{G}(z)U)_{ij} \right) \right| 
\leq u(C) + v(\alpha) + o_C(1). \quad (20)$$

**Proof.** Note that

$$X_{ij}^C - Y_{ij}^C = RX_{ij}^C \left( 1 - \sqrt{2} \mathbb{E} (|R_{ij}^C|^2)^{1/2} \right) + \sqrt{-1} \Im X_{ij}^C \left( 1 - \sqrt{2} \mathbb{E} (|\Im X_{ij}^C|^2)^{1/2} \right)$$

$$= RX_{ij}^C \frac{1 - 2 \mathbb{E} (|R_{ij}^C|^2)}{1 + \sqrt{2} \mathbb{E} (|R_{ij}^C|^2)^{1/2}} + \sqrt{-1} \Im X_{ij}^C \frac{1 - 2 \mathbb{E} (|\Im Y_{ij}^C|^2)}{1 + \sqrt{2} \mathbb{E} (|\Im Y_{ij}^C|^2)^{1/2}}.$$

Then,

$$\left\{ \sup_{(i,j) \in \mathbb{N}^2} \mathbb{E} (|X_{ij}^C - Y_{ij}^C|^2) \right\}^{1/2} \leq \frac{4\theta^*}{C}, \text{ and } \sup_{(i,j) \in \mathbb{N}^2} \mathbb{E} (|X_{ij}^C - Y_{ij}^C|^3) < \infty.$$
It is straightforward to see, using Lemma 5.8, that for any unitary \((n + N) \times (n + N)\) matrix \(U\),
\[
\left| (U^* \tilde{G}^{\alpha,C}(z) U)_{ij} - (U^* \tilde{G}(z) U)_{ij} \right|
\leq \frac{\sigma}{|z|^2} \left\| \frac{X_N - X^{\alpha,C}}{\sqrt{N}} \right\|
\leq \frac{\sigma}{|z|^2} \left\{ \left\| \frac{X_N - Y^C}{\sqrt{N}} \right\| + \left\| \frac{X^C - Y^C}{\sqrt{N}} \right\|
+ \left( 1 - \frac{1}{\sqrt{1 + \alpha^2}} \right) \left\| \frac{X^C}{\sqrt{N}} \right\| + \alpha \left\| \frac{G}{\sqrt{N}} \right\| \right\}.
\] (21)

From Bai-Yin’s theorem (Theorem 5.8 in [2]), we have
\[
\left\| \frac{G}{\sqrt{N}} \right\| = 2 + o(1).
\]

Applying Remark 5.4 to the \((n + N) \times (n + N)\) matrix \(\tilde{B} = \begin{pmatrix} 0_{n \times n} & B \\ B^* & 0_{N \times N} \end{pmatrix}\) for \(B \in \{ X_N - Y^C, X^C - Y^C, X^C \}\) (see also Appendix B of [14]), we have that almost surely
\[
\limsup_{N \to +\infty} \left\| \frac{X^C}{\sqrt{N}} \right\| \leq 2 \sqrt{2}, \quad \limsup_{N \to +\infty} \left\| \frac{X^C - Y^C}{\sqrt{N}} \right\| \leq \frac{8 \sqrt{2\theta^*}}{C},
\]
and
\[
\limsup_{N \to +\infty} \left\| \frac{X_N - Y^C}{\sqrt{N}} \right\| \leq \frac{\sqrt{\theta^*}}{C}.
\]

Then, (19) readily follows.

Let us introduce
\[
\Omega_{N,C} = \left\{ \left\| \frac{G}{\sqrt{N}} \right\| \leq 4, \left\| \frac{X^C}{\sqrt{N}} \right\| \leq 4, \left\| \frac{X_N - Y^C}{\sqrt{N}} \right\| \leq \frac{8 \sqrt{\theta^*}}{C}, \left\| \frac{X^C - Y^C}{\sqrt{N}} \right\| \leq \frac{16 \theta^*}{C} \right\}.
\]

Using (21), we have
\[
\left| \mathbb{E} \left( (U^* \tilde{G}^{\alpha,C}(z) U)_{ij} - (U^* \tilde{G}(z) U)_{ij} \right) \right|
\leq \frac{4 \sigma}{|z|^2} \left[ 2 \sqrt{\frac{\theta^*}{C}} + \frac{4 \theta^*}{C} + \alpha + \left( 1 - \frac{1}{\sqrt{1 + \alpha^2}} \right) \right]
+ \frac{2}{|z|^2} P(\Omega_{N,C}^c).
\]

Thus (20) follows.
Now, Lemma 5.9 Lemma 3.1 and Lemma 5.10 readily yields the following approximation lemma.

**Lemma 3.2.** Let $h$ be in $C^\infty(\mathbb{R}, \mathbb{R})$ with compact support and $\tilde{\Gamma}_N$ be a $(n + N) \times (n + N)$ Hermitian matrix such that such that
\[
\sup_{n,N} \|\tilde{\Gamma}_N\| < \infty \quad \text{and} \quad \sup_{n,N} \text{rank}(\tilde{\Gamma}_N) < \infty.
\]
(22)

Then, there exist some deterministic functions $u$ and $v$ on $[0, +\infty[$ such that $\lim_{C \to +\infty} u(C) = 0$ and $\lim_{\alpha \to 0} v(\alpha) = 0$, such that for all $C > 0$, $\alpha > 0$, we have almost surely for all large $N$,
\[
\left| \text{Tr} \left[ h \left( \mathcal{M}_{N+n} \left( \frac{X^{\alpha,C}}{\sqrt{N}} \right) \right) \tilde{\Gamma}_N \right] - \text{Tr} \left[ h \left( \mathcal{M}_{N+n} \left( \frac{X_N}{\sqrt{N}} \right) \right) \tilde{\Gamma}_N \right] \right| \leq a_{C,\alpha}^{(1)},
\]
(23)
and for all large $N$,
\[
\left| \mathbb{E} \text{Tr} \left[ h \left( \mathcal{M}_{N+n} \left( \frac{X^{\alpha,C}}{\sqrt{N}} \right) \right) \tilde{\Gamma}_N \right] - \mathbb{E} \text{Tr} \left[ h \left( \mathcal{M}_{N+n} \left( \frac{X_N}{\sqrt{N}} \right) \right) \tilde{\Gamma}_N \right] \right| \leq a_{C,\alpha,N}^{(2)},
\]
(24)
where
\[
a_{C,\alpha}^{(1)} = u(C) + v(\alpha), \quad a_{C,\alpha,N}^{(2)} = u(C) + v(\alpha) + o_C(1).
\]

Note that the distributions of the independent random variables $\mathcal{R}(X^{\alpha,C}_{ij})$, $\mathcal{I}(X^{\alpha,C}_{ij})$ are all a convolution of a centred Gaussian distribution with some variance $\nu_{\alpha}$, with some law with bounded support in a ball of some radius $R_{C,\alpha}$; thus, according to Lemma 5.11 they satisfy a Poincaré inequality with some common constant $C_{P1}(C, \alpha)$ and therefore so does their product (see the Appendix B). An important consequence of the Poincaré inequality is the following concentration result.

**Lemma 3.3.** Lemma 4.4.3 and Exercise 4.4.5 in [1] or Chapter 3 in [24]. There exists $K_1 > 0$ and $K_2 > 0$ such that for any probability measure $\mathbb{P}$ on $\mathbb{R}^M$ which satisfies a Poincaré inequality with constant $C_{P1}$, and for any Lipschitz function $F$ on $\mathbb{R}^M$ with Lipschitz constant $|F|_{Lip}$, we have
\[
\forall \epsilon > 0, \quad \mathbb{P} \left( |F - \mathbb{E}_\mathbb{P}(F)| > \epsilon \right) \leq K_1 \exp \left( - \frac{\epsilon}{K_2 \sqrt{C_{P1}} |F|_{Lip}} \right).
\]

In order to apply Lemma 3.3 we need the following preliminary lemmas.

**Lemma 3.4.** (see Lemma 8.2 [10]) Let $f$ be a real $C_L$-Lipschitz function on $\mathbb{R}$. Then its extension on the $N \times N$ Hermitian matrices is $C_L$-Lipschitz with respect to the Hilbert-Schmidt norm.

**Lemma 3.5.** Let $\tilde{\Gamma}_N$ be a $(n + N) \times (n + N)$ matrix and $h$ be a real Lipschitz function on $\mathbb{R}$. For any $n \times N$ matrix $B$,
\[
\left\{ (\mathcal{R}B(i,j), \mathcal{I}B(i,j))_{1 \leq i \leq n, 1 \leq j \leq N} \right\} \mapsto \text{Tr} \left[ h \left( \mathcal{M}_{N+n}(B) \right) \tilde{\Gamma}_N \right]
\]
is Lipschitz with constant bounded by $\sqrt{2} \|\tilde{\Gamma}_N\|_2 \|h\|_{Lip}$. 

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Proof. \[ \left| \text{Tr} \left[ h(\mathcal{M}_{N+p}(B))\hat{\Gamma}_N \right] - \text{Tr} \left[ h(\mathcal{M}_{N+p}(B'))\hat{\Gamma}_N \right] \right| \]
\[ \leq \left\| \hat{\Gamma}_N \right\|_2 \left\| h(\mathcal{M}_{N+p}(B)) - h(\mathcal{M}_{N+p}(B')) \right\|_2 \]
\[ \leq \left\| \hat{\Gamma}_N \right\|_2 \| h \|_{\text{Lip}} \left\| \mathcal{M}_{N+p}(B) - \mathcal{M}_{N+p}(B') \right\|_2 \] \hspace{1cm} (25)
where we used Lemma 3.4 in the last line. Now,
\[ \left\| \mathcal{M}_{N+p}(B) - \mathcal{M}_{N+p}(B') \right\|_2 \leq \| B - B' \|_2 \] \hspace{1cm} (26)
Lemma 3.5 readily follows from (25) and (26).

Lemma 3.6. Let \( \hat{\Gamma}_N \) be a \((n+N)\times(n+N)\) matrix such that \( \sup_{N,n} \left\| \hat{\Gamma}_N \right\|_2 \leq K \). Let \( h \) be a real Lipschitz function on \( \mathbb{R} \). The random variable \( F_N = \text{Tr} \left[ h \left( \mathcal{M}_{N+p} \left( \frac{X^{\alpha,C}}{\sqrt{N}} \right) \right) \hat{\Gamma}_N \right] \) satisfies the following concentration inequality
\[ \forall \epsilon > 0, \mathbb{P}(\left| F_N - \mathbb{E}(F_N) \right| > \epsilon) \leq K_1 \exp \left( -\frac{\epsilon \sqrt{N}}{K_2(\alpha,C)K\| h \|_{\text{Lip}}} \right), \]
for some positive real numbers \( K_1 \) and \( K_2(\alpha,C) \).

Proof. Lemma 3.6 follows from Lemmas 3.5 and 3.3 and basic facts on Poincaré inequality recalled at the end of the Appendix B.

By Borel-Cantelli’s Lemma, we readily deduce from the above Lemma the following

Lemma 3.7. Let \( \hat{\Gamma}_N \) be a \((n+N)\times(n+N)\) matrix such that \( \sup_{N,n} \left\| \hat{\Gamma}_N \right\|_2 \leq K \). Let \( h \) be a real \( C^1 \) function with compact support on \( \mathbb{R} \).
\[ \text{Tr} \left[ h \left( \mathcal{M}_{N+p} \left( \frac{X^{\alpha,C}}{\sqrt{N}} \right) \right) \hat{\Gamma}_N \right] \xrightarrow{a.s.} 0 \rightarrow +\infty \] \hspace{1cm} (27)

Now, we will establish a comparison result with the Gaussian case for the mean values by using the following lemma (which is an extension of Lemma 4.1 below to the non-Gaussian case) as initiated by [22] in Random Matrix Theory.

Lemma 3.8. Let \( \xi \) be a real-valued random variable such that \( \mathbb{E}(|\xi|^{p+2}) < \infty \). Let \( \phi \) be a function from \( \mathbb{R} \) to \( \mathbb{C} \) such that the first \( p+1 \) derivatives are continuous and bounded. Then,
\[ \mathbb{E}(\xi \phi(\xi)) = \sum_{a=0}^{p} \frac{\kappa_{a+1}}{a!} \mathbb{E}(\phi^{(a)}(\xi)) + \epsilon, \]
where \( \kappa_a \) are the cumulants of \( \xi \), \( |\epsilon| \leq K \sup_t |\phi^{(p+1)}(t)| \mathbb{E}(|\xi|^{p+2}), K \) only depends on \( p \).
Lemma 3.9. Let $G_N = [G_{ij}]_{1 \leq i \leq n, 1 \leq j \leq N}$ be a $n \times N$ random matrix with i.i.d. complex $N(0,1)$ Gaussian entries. Define

$$\tilde{G}^\phi(z) = \left(zI_{N+n} - M_{N+n} \left(\sigma \frac{G_N}{\sqrt{N}}\right)\right)^{-1}$$

for any $z \in \mathbb{C} \setminus \mathbb{R}$. There exists a polynomial $P$ with nonnegative coefficients such that for all large $N$, for any $(i, j) \in \{1, \ldots, n + N\}^2$, for any $z \in \mathbb{C} \setminus \mathbb{R}$, for any unitary $(n + N) \times (n + N)$ matrix $U$,

$$\left|\mathbb{E} \left[(U^* \tilde{G}^\phi(z)U)_{ij}\right] - \mathbb{E} \left[(U^* \tilde{G}(z)U)_{ij}\right]\right| \leq \frac{1}{\sqrt{N}} P(|\mathbb{R}z|^{-1}). \quad (29)$$

Moreover, for any $(N+n) \times (N+n)$ matrix $\Gamma_N$ such that

$$\sup_{n,N} \|\Gamma_N\| < \infty \text{ and } \sup_{n,N} \text{rank}(\Gamma_N) < \infty,$$  \quad (30)

and any function $h$ in $C^\infty(\mathbb{R}, \mathbb{R})$ with compact support, there exists some constant $K > 0$ such that, for any large $N$,

$$\left|\mathbb{E} \left[\text{Tr} \left(h \left(M_{N+n} \left(\sigma \frac{\sqrt{N}}{X_N}\right)\right) \Gamma_N\right)\right]\right| - \mathbb{E} \left[\text{Tr} \left(h \left(M_{N+n} \left(\sigma \frac{\sqrt{N}}{X_N}\right)\right) \Gamma_N\right)\right| \leq \frac{K}{\sqrt{N}}. \quad (31)$$

Proof. We follow the approach of [27] chapters 18 and 19 consisting in introducing an interpolation matrix $X_N(\alpha) = \cos \alpha X_N + \sin \alpha G_N$ for any $\alpha \in [0; \frac{\pi}{2}]$ and the corresponding resolvent matrix $\tilde{G}(\alpha, z) = \left(zI_{N+n} - M_{N+n} \left(\sigma \frac{X_N(\alpha)}{\sqrt{N}}\right)\right)^{-1}$ for any $z \in \mathbb{C} \setminus \mathbb{R}$. We have, for any $(s, t) \in \{1, \ldots, n + N\}^2$, 

$$\mathbb{E} \tilde{G}_{st}^\phi(z) - \mathbb{E} \tilde{G}_{st}(z) = \int_0^{\pi} \mathbb{E} \left(\frac{\partial}{\partial \alpha} \tilde{G}_{st}(\alpha, z)\right) d\alpha$$

with

$$\frac{\partial}{\partial \alpha} \tilde{G}_{st}(\alpha, z) = \frac{\sigma}{2\sqrt{N}} \sum_{l=1}^{n} \sum_{k=n+1}^{n+N} \left\{ \tilde{G}_{st}(\alpha, z)\tilde{G}_{kt}(\alpha, z) + \tilde{G}_{sk}(\alpha, z)\tilde{G}_{lt}(\alpha, z) \right\}$$

$$\times \left[ -\sin \alpha \Re X_{l(k-n)} + \cos \alpha \Im G_{l(k-n)} \right]$$

$$+ i \left[ \tilde{G}_{st}(\alpha, z)\tilde{G}_{kt}(\alpha, z) - \tilde{G}_{sk}(\alpha, z)\tilde{G}_{lt}(\alpha, z) \right]$$

$$\times \left[ -\sin \alpha \Im X_{l(k-n)} + \cos \alpha \Re G_{l(k-n)} \right].$$

Now, for any $l = 1, \ldots, n$ and $k = n+1, \ldots, n+N$, using Lemma [3.8] for $p = 1$ and for each random variable $\xi$ in the set $\{\Re X_{l(k-n)}, \Re G_{l(k-n)}, \Im X_{l(k-n)}, \Im G_{l(k-n)}\}$, and for each $\phi$ in the set

$$\left\{(U^* \tilde{G}(\alpha, z)U)_{ij}, (p, q) = (l, k) \text{ or } (k, l), (i, j) \in \{1, \ldots, n + N\}^2\right\},$$
one can easily see that there exists some constant $K > 0$ such that

$$\left| E(U^* \tilde{G}(z)U)_{ij} - E(U^* \tilde{G}(z)U)_{ij} \right| \leq \frac{K}{N^{3/2}} \sup_{Y \in H_{n+N}(\mathbb{C})} \sup_{V \in U(n+N)} S_V(Y)$$

where $H_{n+N}(\mathbb{C})$ denotes the set of $(n + N) \times (n + N)$ Hermitian matrices and $S_V(Y)$ is a sum of a finite number independent of $N$ and of terms of the form

$$\sum_{i=1}^{n} \sum_{k=n+1}^{n+N} \left| (U^* R(Y))_{i k} (R(Y))_{p k} (R(Y))_{p k} (R(Y)U)_{p k} \right|$$

with $R(Y) = (zI_{n+N} - Y)^{-1}$ and $\{p_1, \ldots, p_6\}$ contains exactly three $k$ and three $l$.

When $(p_1, p_6) = (k, l)$ or $(k, l)$, then, using Lemma 5.8,

$$\sum_{i=1}^{n} \sum_{k=n+1}^{n+N} \left| (U^* R(Y))_{i p_1} (R(Y))_{p_2 p_3} (R(Y))_{p_4 p_5} (R(Y)U)_{p_6} \right|$$

$$\leq \frac{1}{|3z|^2} \sum_{k, l=1}^{n+N} \left| (U^* R(Y))_{i l} (R(Y)U)_{k j} \right|$$

$$\leq \frac{(N + n)}{|3z|^2} \left( \sum_{l=1}^{n+N} \left| (U^* R(Y))_{i l} \right|^2 \right)^{1/2} \left( \sum_{k=1}^{n+N} \left| (R(Y)U)_{k j} \right|^2 \right)^{1/2}$$

$$= \frac{(N + n)}{|3z|^2} \left( \left| (U^* R(Y)R(Y)^*U)_{i l} \right| \right)^{1/2} \left( \left| (U^* R(Y)^*R(Y)U)_{j j} \right| \right)^{1/2}$$

When $p_1 = p_6 = k$ or $l$, then, using Lemma 5.8,

$$\sum_{i=1}^{n} \sum_{k=n+1}^{n+N} \left| (U^* R(Y))_{i p_1} (R(Y))_{p_2 p_3} (R(Y))_{p_4 p_5} (R(Y)U)_{p_6} \right|$$

$$\leq \frac{N + n}{|3z|^2} \sum_{l=1}^{n+N} \left| (U^* R(Y))_{i l} (R(Y)U)_{j j} \right|$$

$$\leq \frac{(N + n)}{|3z|^2} \left( \sum_{l=1}^{n+N} \left| (U^* R(Y))_{i l} \right|^2 \right)^{1/2} \left( \sum_{l=1}^{n+N} \left| (R(Y)U)_{j j} \right|^2 \right)^{1/2}$$

$$= \frac{(N + n)}{|3z|^2} \left( \left| (U^* R(Y)R(Y)^*U)_{i l} \right| \right)^{1/2} \left( \left| (U^* R(Y)^*R(Y)U)_{j j} \right| \right)^{1/2}$$

[20] readily follows.
Then by Lemma 5.10, there exists some constant $K > 0$ such that, for any $N$ and $n$, for any $(i, j) \in \{1, \ldots, n + N\}^2$, any unitary $(n + N) \times (n + N)$ matrix $U$,

$$
\limsup_{y \to 0^+} \left| \int \left[ \mathbb{E}(U^* \tilde{G}(t + iy)U)_{ij} - \mathbb{E}(U^* \tilde{G}^2(t + iy)U)_{ij} \right] h(t) \, dt \right| \leq \frac{K}{\sqrt{N}}.
$$

(33)

Thus, using (97) and (30), we can deduce (31) from (33).

The above comparison lemmas allow us to establish the following convergence result.

**Proposition 3.10.** Let $h$ be a function in $C^\infty(\mathbb{R}, \mathbb{R})$ with compact support and let $\tilde{\Gamma}_N$ be a $(n + N) \times (n + N)$ matrix such that $\sup_{n,N} \text{rank}(\tilde{\Gamma}_N) < \infty$ and $\sup_{n,N} \|\tilde{\Gamma}_N\| < \infty$. Then we have that almost surely

$$
\text{Tr} \left[ h \left( (M_{N+n} \left( \frac{X_N}{\sqrt{N}} \right) ) \tilde{\Gamma}_N \right) - \mathbb{E} \left[ \text{Tr} \left[ h \left( (M_{N+n} \left( \frac{X_N}{\sqrt{N}} \right) ) \tilde{\Gamma}_N \right) \right] \right] \right] 
\longrightarrow_{N \to +\infty} 0.
$$

(34)

**Proof.** Lemmas 3.2, 3.7 and 3.9 readily yield that there exist some positive deterministic functions $u$ and $v$ on $[0, +\infty]$ with $\lim_{C \to +\infty} u(C) = 0$ and $\lim_{\alpha \to 0} v(\alpha) = 0$, such that for any $C > 0$ and any $\alpha > 0$, almost surely

$$
\limsup_{N \to +\infty} \left| \text{Tr} \left[ h \left( (M_{N+n} \left( \frac{X_N}{\sqrt{N}} \right) ) \tilde{\Gamma}_N \right) - \mathbb{E} \left[ \text{Tr} \left[ h \left( (M_{N+n} \left( \frac{X_N}{\sqrt{N}} \right) ) \tilde{\Gamma}_N \right) \right] \right] \right] \right| 
\leq u(C) + v(\alpha).
$$

The result follows by letting $\alpha$ go to zero and $C$ go to infinity.

Now, note that, for any $N \times n$ matrix $B$, for any continuous real function function $h$ on $\mathbb{R}$, and any $n \times n$ Hermitian matrix $\Gamma_N$, we have

$$
\text{Tr} \left( h \left( (B + A_N)(B + A_N)^* \right) \Gamma_N \right) = \text{Tr} \left[ \tilde{h} \left( (M_{N+n} (B)) \tilde{\Gamma}_N \right) \right]
$$

where $\tilde{h}(x) = h(x^2)$ and $\tilde{\Gamma}_N = \begin{pmatrix} \Gamma_N & 0 \\ 0 & (0) \end{pmatrix}$. Thus, Proposition 3.10 readily yields Proposition 2.1.

### 4 Proof of Proposition 2.2

The aim of this section is to prove Proposition 2.2 which deals with Gaussian random variables. Therefore we assume here that $A_N$ is as (14) and set $\gamma_q(N) = (A_N A_N^*)_{qq}$. In this section, we let $X$ stand for $\tilde{G}_N$, $A$ stands for $A_N$, $G$ denotes the resolvent of $M_N = \Sigma \Sigma^*$ where $\Sigma = \sigma \frac{X_N}{\sqrt{N}} + A_N$ and $g_N$ denotes the mean of the Stieltjes transform of the spectral measure of $M_N$, that is

$$
g_N(z) = \mathbb{E} \left( \frac{1}{n} \text{Tr} G(z) \right), \quad z \in \mathbb{C} \setminus \mathbb{R}.
$$
4.1 Matricial master equation

To obtain the equation (35) below, we will use many ideas from [17]. The following Gaussian integration by part formula is the key tool in our approach.

**Lemma 4.1.** [Lemma 2.4.5 [1]] Let \( \xi \) be a real centered Gaussian random variable with variance \( 1 \). Let \( \Phi \) be a differentiable function with polynomial growth of \( \Phi \) and \( \Phi' \). Then,

\[
E(\xi \Phi(\xi)) = E(\Phi'(\xi)).
\]

**Proposition 4.2.** Let \( z \) be in \( \mathbb{C} \setminus \mathbb{R} \). We have for any \((p, q)\) in \( \{1, \ldots, n\}^2 \),

\[
E(G_{pq}(z)) \left\{ z(1 - \sigma^2 c_N g_N(z)) - \frac{\gamma(N)}{1 - \sigma^2 c_N g_N(z)} - \sigma^2(1 - c_N) + \frac{\sigma^2}{N} \sum_{p=1}^{n} \nabla_{pp}(z) \right\} = \delta_{pq} + \nabla_{pq}(z),
\]

where

\[
\nabla_{pq} = \frac{1}{1 - \sigma^2 c_N g_N} \left\{ \frac{\sigma^2}{N} \frac{E(G_{pq})}{1 - \sigma^2 c_N g_N} \Delta_3 + \Delta_2(p, q) + \Delta_1(p, q) \right\},
\]

\[
\Delta_1(p, q) = \sigma^2 E \left\{ \left[ \frac{1}{N} TrG - E \left( \frac{1}{N} TrG \right) \right] (G \Sigma \Sigma^*)_{pq} \right\},
\]

\[
\Delta_2(p, q) = \sigma^2 E \left\{ Tr(GA \Sigma^*) [G_{pq} - E(G_{pq})] \right\},
\]

\[
\Delta_3 = \sigma^2 E \left\{ \left[ \frac{1}{N} TrG - E \left( \frac{1}{N} TrG \right) \right] Tr(\Sigma^* GA) \right\}.
\]

**Proof.** Using Lemma 4.1 with \( \xi = \Re X_{ij} \) or \( \xi = \Im X_{ij} \) and \( \Phi = G_{pi} \Sigma_{qj} \), we obtain that for any \( j, q, p \),

\[
E \left( \left[ G \frac{\sigma X}{\sqrt{N}} \right]_{pj} \Sigma_{qj} \right) = \sum_{i=1}^{n} E \left( G_{pi} \frac{\sigma X_{ij}}{\sqrt{N}} \Sigma_{qj} \right) \left[ \frac{\sigma^2}{N} \sum_{i=1}^{n} E \left[ (G \Sigma)_{ij} G_{ji} \Sigma_{qj} \right] + \frac{\sigma^2}{N} E(G_{pq}) \right] \left[ Tr(G \Sigma) (G \Sigma)_{ij} \Sigma_{qj} \right] + \frac{\sigma^2}{N} E(G_{pq}) \left[ \frac{\sigma^2}{N} \sum_{i=1}^{n} E \left[ TrG (G \Sigma)_{ij} \Sigma_{qj} \right] + \frac{\sigma^2}{N} E(G_{pq}) \right]
\]

On the other hand, we have

\[
E \left( (GA)_{pj} \Sigma_{qj} \right) = E \left( (GA)_{pj} \Sigma_{qj} \right) + \sum_{i=1}^{n} E \left( G_{pi} A_{ij} \frac{\sigma X_{ij}}{\sqrt{N}} \Sigma_{qj} \right) \left[ \frac{\sigma^2}{N} \sum_{i=1}^{n} E \left[ TrG (G \Sigma)_{ij} \Sigma_{qj} \right] + \frac{\sigma^2}{N} E(G_{pq}) \right]
\]

\[
= E \left( (GA)_{pj} \Sigma_{qj} \right) + \frac{\sigma^2}{N} E \left( G_{pq} (\Sigma^* GA)_{ij} \right) \left[ \frac{\sigma^2}{N} \sum_{i=1}^{n} E \left[ TrG (G \Sigma)_{ij} \Sigma_{qj} \right] + \frac{\sigma^2}{N} E(G_{pq}) \right]
\]
where we applied Lemma 4.1 with $\xi = \Re X_{qj}$ or $\xi = \Im X_{qj}$ and $\Psi = G_{pi} A_{ij}$.

Summing (42) and (44) yields

$$E \left[ (G \Sigma)_{pj} \Sigma_{qj} \right] = \frac{\sigma^2}{N} E (G_{pq}) + \frac{\sigma^2}{N} E \left[ TrG \left( G \Sigma \right)_{pj} \Sigma_{qj} \right]$$

(45)

$$+ \frac{\sigma^2}{N} E \left[ G_{pq} (\Sigma^* GA)_{jj} \right] + E \left[ (GA)_{pj} A_{qj} \right].$$

(46)

Define

$$\Delta_1(j) = \frac{\sigma^2}{N} E \left[ TrG \left( G \Sigma \right)_{pj} \Sigma_{qj} \right] - \frac{\sigma^2}{N} E [TrG] E \left[ (G \Sigma)_{pj} \Sigma_{qj} \right].$$

From (46), we can deduce that

$$E \left[ (G \Sigma)_{pj} \Sigma_{qj} \right] = \frac{1}{1 - \sigma^2 c_{NgN}} \left\{ \frac{\sigma^2}{N} E (G_{pq}) + \frac{\sigma^2}{N} E \left[ G_{pq} (\Sigma^* GA)_{jj} \right] 
+ E \left[ (GA)_{pj} A_{qj} \right] + \Delta_1(j) \right\}.$$ 

(47)

Then, summing over $j$, we obtain that

$$E \left[ (G \Sigma^*)_{pq} \right] = \frac{1}{1 - \sigma^2 c_{NgN}} \left\{ \frac{\sigma^2}{N} E (G_{pq}) + \frac{\sigma^2}{N} E \left[ G_{pq} Tr (\Sigma^* GA) \right] 
+ E \left[ (GAA^*)_{pq} \right] + \Delta_1(p, q) \right\},$$

where $\Delta_1(p, q)$ is defined by (37). Applying Lemma 4.1 with $\xi = \Re X_{ij}$ or $\Im X_{ij}$ and $\Psi = (GA)_{ij}$, we obtain that

$$E \left[ Tr \left( \frac{\sigma \sqrt{N}}{\sqrt{N}} GA \right) \right] = \frac{\sigma^2}{N} E [TrG Tr (\Sigma^* GA)].$$

Thus,

$$E [Tr (\Sigma^* GA)] = E [Tr (A^* GA)] + \sigma^2 c_{NgN} E [Tr (\Sigma^* GA)] + \Delta_3,$$

where $\Delta_3$ is defined by (39) and then

$$E [Tr (\Sigma^* GA)] = \frac{1}{1 - \sigma^2 c_{NgN}} \left\{ E [Tr (GAA^*)] + \Delta_3 \right\}. \quad (48)$$

(48) and (38) imply that

$$\frac{\sigma^2}{N} E \left[ G_{pq} Tr (\Sigma^* GA) \right] = \frac{\sigma^2}{N} \frac{E (G_{pq})}{1 - \sigma^2 c_{NgN}} \left\{ E [Tr (GAA^*)] + \Delta_3 \right\} + \Delta_2(p, q),$$

(49)

where $\Delta_2(p, q)$ is defined by (38). We can deduce from (47) and (49) that

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\[
\begin{align*}
E \left[ (G \Sigma \Sigma^*)_{pq} \right] &= \frac{1}{1 - \sigma^2 c_N g N} \left\{ \sigma^2 E(G_{pq}) + E \left[ (GAA^*)_{pq} \right] + \frac{\sigma^2}{N} \frac{E[G_{pq}]}{1 - \sigma^2 c_N g N} E[Tr(GAA^*)] \right. \\
&\quad + \frac{\sigma^2}{N} \frac{E(G_{pq})}{1 - \sigma^2 c_N g N} \Delta_3 + \Delta_1(p,q) + \Delta_2(p,q) \left. \right\}. \tag{50}
\end{align*}
\]

Using the resolvent identity and (50), we obtain that
\[
E(G_{pq}) = \frac{1}{1 - \sigma^2 c_N g N} \left\{ \sigma^2 E(G_{pq}) + E \left[ (GAA^*)_{pq} \right] \right. \\
&\quad + \frac{\sigma^2}{N} \frac{E[G_{pq}]}{1 - \sigma^2 c_N g N} E[Tr(GAA^*)] \left. \right\} + \delta_{pq} + \nabla_{pq}. \tag{51}
\]

where \( \nabla_{pq} \) is defined by (36). Taking \( p = q \) in (51), summing over \( p \) and dividing by \( n \), we obtain that
\[
\begin{align*}
z g N &= \frac{\sigma^2 g N}{1 - \sigma^2 c_N g N} + \frac{Tr[E(G)AA^*]}{n(1 - \sigma^2 c_N g N)} \\
&\quad + \frac{\sigma^2 g N Tr[E(G)AA^*]}{N(1 - \sigma^2 c_N g N)^2} + 1 + \frac{1}{n} \sum_{p=1}^{n} \nabla_{pp}. \tag{52}
\end{align*}
\]

It readily follows that
\[
\begin{align*}
\frac{Tr[E(G)AA^*]}{n(1 - \sigma^2 c_N g N)} \left( \frac{\sigma^2 g N}{1 - \sigma^2 c_N g N} + 1 \right) &= \left( z - \frac{\sigma^2}{1 - \sigma^2 c_N g N} \right) g N - \frac{1}{n} \sum_{p=1}^{n} \nabla_{pp}. \\
\text{Therefore} \\
\frac{Tr[E(G)AA^*]}{n(1 - \sigma^2 c_N g N)} &= z g N (1 - \sigma^2 c_N g N) - \frac{\gamma q}{1 - \sigma^2 c_N g N} - \sigma^2 (1 - c N) + \frac{\sigma^2}{N} \sum_{p=1}^{n} \nabla_{pp}. \tag{54}
\end{align*}
\]

(4.1) and (51) yield
\[
\begin{align*}
E(G_{pq}) \times \left\{ z(1 - \sigma^2 c_N g N) - \frac{\gamma q}{1 - \sigma^2 c_N g N} - \sigma^2 (1 - c N) + \frac{\sigma^2}{N} \sum_{p=1}^{n} \nabla_{pp} \right\} = \delta_{pq} + \nabla_{pq}.
\end{align*}
\]

Proposition 4.2 follows. \( \square \)

4.2 Variance estimates

In this section, when we state that some quantity \( \Delta_N(z) \), \( z \in \mathbb{C} \setminus \mathbb{R} \), is equal to \( O(\frac{1}{N^p}) \), this means precisely that there exist some polynomial \( P \) with nonnegative coefficients and some positive real number \( l \) which are all independent of \( N \) such that for any \( z \in \mathbb{C} \setminus \mathbb{R} \),
\[
|\Delta_N(z)| \leq \frac{(|z| + 1)^d P(|3z|^{-1})}{N^p}.
\]
We present now the different estimates on the variance. They rely on the following Gaussian Poincaré inequality (see the Appendix B). Let \( Z_1, \ldots, Z_q \) be \( q \) real independent centered Gaussian variables with variance \( \sigma^2 \). For any \( C^1 \) function \( f : \mathbb{R}^q \to \mathbb{C} \) such that \( f \) and \( \text{grad} f \) are in \( L^2(N(0, \sigma^2 I_q)) \), we have
\[
\mathbb{V} \{ f(Z_1, \ldots, Z_q) \} \leq \sigma^2 \mathbb{E} \left( \| \text{grad} f(Z_1, \ldots, Z_q) \|_2^2 \right), \tag{55}
\]
denoting for any random variable \( a \) by \( \mathbb{V}(a) \) its variance \( \mathbb{E}(|a - \mathbb{E}(a)|^2) \). Thus, \( (Z_1, \ldots, Z_q) \) satisfies a Poincaré inequality with constant \( C_{PI} = \sigma^2 \).

The following preliminary result will be useful to these estimates.

**Lemma 4.3.** There exists \( K > 0 \) such for all \( N \),
\[
\mathbb{E} \left( \lambda_1 \left( \frac{XX^*}{N} \right) \right) \leq K.
\]

**Proof.** According to Lemma 7.2 in [20], we have for any \( t \in [0; N/2] \),
\[
\mathbb{E} \left[ \text{Tr} \left( \exp t \frac{XX^*}{N} \right) \right] \leq n \exp \left( (\sqrt{c_N} + 1)^2 t + \frac{1}{N} (c_N + 1)^2 t \right). \tag{56}
\]

By the Chebychev’s inequality, we have
\[
\exp \left( t \mathbb{E} \left( \lambda_1 \left( \frac{XX^*}{N} \right) \right) \right) \leq \mathbb{E} \left( \exp t \lambda_1 \left( \frac{XX^*}{N} \right) \right) \leq \mathbb{E} \left[ \text{Tr} \left( \exp t \frac{XX^*}{N} \right) \right] \leq n \exp \left( (\sqrt{c_N} + 1)^2 t + \frac{1}{N} (c_N + 1)^2 t \right).
\]

It follows that
\[
\mathbb{E} \left( \lambda_1 \left( \frac{XX^*}{N} \right) \right) \leq \frac{1}{t} \log n + (\sqrt{c_N} + 1)^2 + \frac{1}{N} (c_N + 1)t.
\]

The result follows by optimizing in \( t \). \( \square \)

**Lemma 4.4.** There exists \( C > 0 \) such that for all large \( N \), for all \( z \in \mathbb{C} \setminus \mathbb{R} \),
\[
\mathbb{E} \left( \left| \frac{1}{n} \text{Tr} G - \mathbb{E} \left( \frac{1}{n} \text{Tr} G \right) \right|^2 \right) \leq \frac{C}{N^2 |\Im z|^4}, \tag{56}
\]
\[
\forall (p, q) \in \{1, \ldots, n\}^2, \mathbb{E} \left( |G_{pq} - \mathbb{E}(G_{pq})|^2 \right) \leq \frac{C}{N |\Im z|^4}, \tag{57}
\]
\[
\mathbb{E} \left( |\text{Tr} \Sigma^* G A - \mathbb{E}(\text{Tr} \Sigma^* G A)|^2 \right) \leq \frac{C(1 + |z|)^2}{|\Im z|^4}. \tag{58}
\]
Proof. Let us define $\Psi : \mathbb{R}^{2(n \times N)} \to M_{n \times N}(\mathbb{C})$ by

$$\Psi : \{x_{ij}, y_{ij}, i = 1, \ldots, n, j = 1, \ldots, N\} \to \sum_{i=1}^{n} \sum_{j=1}^{N} (x_{ij} + \sqrt{-1}y_{ij}) e_{ij},$$

where $e_{ij}$ stands for the $n \times N$ matrix such that for any $(p, q) \in \{1, \ldots, n\} \times \{1, \ldots, N\}$, $(e_{ij})_{pq} = \delta_{ip}\delta_{jq}$. Let $F$ be a smooth complex function on $M_{n \times N}(\mathbb{C})$ and define the complex function $f$ on $\mathbb{R}^{2(n \times N)}$ by setting $f = F \circ \Psi$. Then,

$$\|\text{grad} f(u)\|_2 = \sup_{V \in M_{n \times N}(\mathbb{C}), \text{Tr}VV^* = 1} \left| \frac{d}{dt} F(\Psi(u) + tV)_{|t=0} \right|.$$

Now, $X = \Psi(\Re(X_{ij}), \Im(X_{ij}), 1 \leq i \leq n, 1 \leq j \leq N)$ where the distribution of $\{\Re(X_{ij}), \Im(X_{ij}), 1 \leq i \leq n, 1 \leq j \leq N\}$ is $N(0, \frac{1}{2}I_{2nN})$.

Hence consider $F : H \to \frac{1}{n} \text{Tr} \left( zI_n - \left( \frac{\sigma}{\sqrt{n}} + A \right) \left( \frac{\sigma}{\sqrt{n}} + A \right)^* \right)^{-1}$.

Let $V \in M_{n \times N}(\mathbb{C})$ such that $\text{Tr}VV^* = 1$.

$$\frac{d}{dt} F(X + tV)_{|t=0}$$

$$= \frac{1}{n} \left\{ \text{Tr} \left( G \frac{\sigma}{\sqrt{n}} \left( \sigma \frac{X}{\sqrt{n}} + A \right)^* G \right) + \text{Tr} \left( G \left( \sigma \frac{X}{\sqrt{n}} + A \right) \sigma \frac{V^*}{\sqrt{n}} G \right) \right\}.$$

Moreover using Cauchy-Schwartz’s inequality and Lemma [68] we have

$$\left| \frac{1}{n} \text{Tr} \left( G \frac{\sigma}{\sqrt{n}} \left( \sigma \frac{X}{\sqrt{n}} + A \right)^* G \right) \right|$$

$$\leq \frac{\sigma}{n} (\text{Tr}VV^*)^{\frac{1}{2}} \left[ \frac{1}{N} \text{Tr} \left( \left( \sigma \frac{X}{\sqrt{n}} + A \right) \left( \sigma \frac{X}{\sqrt{n}} + A \right)^* \right) G^2 (G^*)^2 \right]^{\frac{1}{2}}$$

$$\leq \frac{\sigma}{\sqrt{Nn|\Im z|^2}} \left[ \lambda_1 \left( \left( \sigma \frac{X}{\sqrt{n}} + A \right) \left( \sigma \frac{X}{\sqrt{n}} + A \right)^* \right) \right]^{\frac{1}{2}}.$$

We get obviously the same bound for $|\frac{1}{n} \text{Tr} \left( G \left( \sigma \frac{X}{\sqrt{n}} + A \right) \sigma \frac{V^*}{\sqrt{n}} G \right)|$. Thus

$$\mathbb{E} \left( \|\text{grad} f(\Re(X_{ij}), \Im(X_{ij}), 1 \leq i \leq n, 1 \leq j \leq N)\|^2 \right)$$

$$\leq \frac{4\sigma^2}{|\Im z|^4 Nn} \lambda_1 \left( \left( \sigma \frac{X}{\sqrt{n}} + A \right) \left( \sigma \frac{X}{\sqrt{n}} + A \right)^* \right). \quad (59)$$

\[59\] readily follows from \[67], \[69], Theorem A.8 in [2], Lemma A.3, and the fact that $\|A_N\|$ is uniformly bounded. Similarly, considering

$$F : H \to \text{Tr} \left[ \left( zI_n - \left( \frac{\sigma}{\sqrt{n}} + A \right) \left( \frac{\sigma}{\sqrt{n}} + A \right)^* \right)^{-1} E_{qp} \right],$$

where $E_{qp}$ is the $n \times n$ matrix such that $(E_{qp})_{ij} = \delta_{qi}\delta_{pj}$, we can obtain that, for any $V \in M_{n \times N}(\mathbb{C})$ such that $\text{Tr}VV^* = 1$,
\[
\left| \frac{d}{dt} F(X + tV)_{t=0} \right| \leq \frac{\sigma}{\sqrt{N}} \left\{ \left[ (G^*G)_{pp} (G^*\Sigma^*G)_{qq} \right]^{1/2} + \left[ (G^*G)_{qq} (G\Sigma^*G^*)_{pp} \right]^{1/2} \right\}.
\]

Thus, one can get (57) in the same way. Finally, considering

\[
F : H \to \text{Tr} \left[ \left( \sigma \frac{H}{\sqrt{N}} + A \right)^* \left( zI_N - \left( \sigma \frac{H}{\sqrt{N}} + A \right) \left( \sigma \frac{H}{\sqrt{N}} + A \right)^* \right)^{-1} A \right],
\]

we can obtain that, for any \( V \in M_{n \times N}(\mathbb{C}) \) such that \( \text{Tr}VV^* = 1 \),

\[
\left| \frac{d}{dt} F(X + tV)_{t=0} \right| \leq \sigma \left\{ \left( \frac{1}{N} \text{Tr} \Sigma^*GA^*GG^* \Sigma A^* \Sigma \right)^{1/2} + \left( \frac{1}{N} \text{Tr} G\Sigma^*G^* \Sigma A^* \Sigma \right)^{1/2} \right\} + \left( \frac{1}{N} \text{Tr} G^*A^*G^* \right)^{1/2}.
\]

Using Lemma 5.8 (i), Theorem A.8 in [2], Lemma 4.3, the identity \( \Sigma^*G = G\Sigma^* = -I + zG \), and the fact that \( \|A_N\| \) is uniformly bounded, the same analysis allows to prove (58).

**Corollary 4.5.** Let \( \Delta_1(p, q), \Delta_2(p, q), (p, q) \in \{1, \ldots, n\}^2 \), and \( \Delta_3 \) be as defined in Proposition 4.2. Then there exist a polynomial \( P \) with nonnegative coefficients and a nonnegative real number \( l \) such that, for all large \( N \), for any \( z \in \mathbb{C} \setminus \mathbb{R} \),

\[
\Delta_3(z) \leq \frac{P(|z|^{-1}(1 + |z|)^l)}{N}, \tag{60}
\]

and for all \( (p, q) \in \{1, \ldots, n\}^2 \),

\[
\Delta_1(p, q)(z) \leq \frac{P(|z|^{-1}(1 + |z|)^l)}{N}, \tag{61}
\]

\[
\Delta_2(p, q)(z) \leq \frac{P(|z|^{-1}(1 + |z|)^l)}{N\sqrt{N}}. \tag{62}
\]

**Proof.** Using the identity

\[
GM_N = -I + zG,
\]

(61) readily follows from Cauchy-Schwartz inequality, Lemma 5.8 and (60). (62) and (60) readily follows from Cauchy-Schwartz inequality and Lemma 4.3.
4.3 Estimates of Resolvent entries

In order to deduce Proposition 2.2 from Proposition 4.2 and Corollary 4.5, we need the two following Lemma 4.6 and Lemma 4.7.

**Lemma 4.6.** For all \( z \in \mathbb{C} \setminus \mathbb{R} \),

\[
\frac{1}{|1 - \sigma^2 c_N g_N(z)|} \leq \frac{|z|}{|3z|}, \tag{63}
\]

\[
\frac{1}{|1 - \sigma^2 c g_{\mu,\nu,c}(z)|} \leq \frac{|z|}{|3z|}. \tag{64}
\]

**Proof.** Since \( \mu_{M_N} \) is supported by \([0, +\infty[\), (63) readily follows from

\[
\left| \frac{1}{1 - \sigma^2 c_N g_N(z)} \right| = \frac{|z|}{|z - \sigma^2 c_N g_N(z)|} \leq \frac{|z|}{|3z - \sigma^2 c_N g_N(z)|} \leq \frac{|z|}{|3z|} \left( 1 + \sigma^2 c_N \mathbb{E} \int |z - t|^2 d\mu_{M_N}(t) \right).
\]

(64) may be proved similarly.

Corollary 4.5 and Lemma 4.6 yields that, there is a polynomial \( Q \) with nonnegative coefficients, a sequence \( b_N \) of nonnegative real numbers converging to zero when \( N \) goes to infinity and some nonnegative integer number \( l \), such that for any \( p, q \) in \( \{1, \ldots, n\} \), for all \( z \in \mathbb{C} \setminus \mathbb{R} \),

\[
\nabla_{pq} \leq (1 + |z|)l Q(|\Im z|^{-1}) b_N, \tag{65}
\]

where \( \nabla_{pq} \) was defined by (36).

**Lemma 4.7.** There is a sequence \( v_N \) of nonnegative real numbers converging to zero when \( N \) goes to infinity such that for all \( z \in \mathbb{C} \setminus \mathbb{R} \),

\[
\left| g_N(z) - g_{\mu,\nu,c}(z) \right| \leq \left\{ \frac{|z|^2 + 2}{|3z|^2} + \frac{1}{|3z|} \right\} v_N. \tag{66}
\]

**Proof.** First note that it is sufficient to prove (66) for \( z \in \mathbb{C}^+ := \{ z \in \mathbb{C}; \Im z > 0 \} \) since \( g_N(\bar{z}) - g_{\mu,\nu,c}(\bar{z}) = g_N(z) - g_{\mu,\nu,c}(z) \). Fix \( \epsilon > 0 \). According to Theorem A.8 and Theorem 5.11 in [2], and the assumption on \( A_N \), we can choose \( K > \max \{ 2/\epsilon; x, x \in \text{supp}(\mu_{\nu,c}) \} \) large enough such that \( \mathbb{P}(\|M_N\| > K) \) goes to zero as \( N \) goes to infinity. Let us write

\[
g_N(z) = \mathbb{E} \left( \frac{1}{n} \text{Tr} G_N(z) 1_{\|M_N\| \leq K} \right) + \mathbb{E} \left( \frac{1}{n} \text{Tr} G_N(z) 1_{\|M_N\| > K} \right). \tag{67}
\]
For any $z \in \mathbb{C}^+$ such that $|z| > 2K$, we have

$$\left| E \left( \frac{1}{n} \text{Tr} G_N(z) 1_{\|M_N\| \leq K} \right) \right| \leq \frac{1}{K} \leq \frac{\epsilon}{2} \quad \text{and} \quad |g_{\mu,\nu,c}(z)| \leq \frac{1}{K} \leq \frac{\epsilon}{2}.$$  

Thus, $\forall z \in \mathbb{C}^+$, such that $|z| > 2K$, we can deduce that

$$\left| E \left( \frac{1}{n} \text{Tr} G_N(z) 1_{\|M_N\| \leq K} \right) - g_{\mu,\nu,c}(z) \right| \leq \frac{2}{3z},$$

and

$$\left| E \left( \frac{1}{n} \text{Tr} G_N(z) 1_{\|M_N\| \leq K} \right) - g_{\mu,\nu,c}(z) \right| \leq \frac{2}{3z}.$$  

Now, it is clear that $E \left( \frac{1}{n} \text{Tr} G_N(z) 1_{\|M_N\| \leq K} \right)$ is a sequence of locally bounded holomorphic functions on $\mathbb{C}^+$ which converges towards $g_{\mu,\nu,c}$. Hence, by Vitali’s Theorem, $E \left( \frac{1}{n} \text{Tr} G_N(z) 1_{\|M_N\| \leq K} \right)$ converges uniformly towards $g_{\mu,\nu,c}$ on each compact subset of $\mathbb{C}^+$. Thus, there exists $N(\epsilon) > 0$, such that for any $N \geq N(\epsilon)$, for any $z \in \mathbb{C}^+$, such that $|z| \leq 2K$ and $\exists z \geq \epsilon$,

$$\left| E \left( \frac{1}{n} \text{Tr} G_N(z) 1_{\|M_N\| \leq K} \right) - g_{\mu,\nu,c}(z) \right| \leq \frac{2}{3z}.$$  

Finally, for any $z \in \mathbb{C}^+$, such that $\exists z \in ]0; \epsilon[$, we have

$$\left| E \left( \frac{1}{n} \text{Tr} G_N(z) 1_{\|M_N\| \leq K} \right) - g_{\mu,\nu,c}(z) \right| \leq \frac{2}{3z}.$$  

It readily follows from (68), (69) and (70) that for $N \geq N(\epsilon)$,

$$\sup_{z \in \mathbb{C}^+} \left\{ E \left( \frac{1}{n} \text{Tr} G_N(z) 1_{\|M_N\| \leq K} \right) - g_{\mu,\nu,c}(z) \right\} \leq \frac{2}{3z}.$$  

Moreover, for $N \geq N'(\epsilon) \geq N(\epsilon)$, $\mathbb{P}(\|M_N\| > K) \leq \epsilon$. Therefore, for $N \geq N'(\epsilon)$, we have for any $z \in \mathbb{C}^+$,

$$|g_N(z) - g_{\mu,\nu,c}(z)| \leq \frac{|z|^2 + 2}{3z} \leq \frac{1}{3z}.$$  

(71)
Thus, the proof is complete by setting

\[ v_N = \sup_{z \in \mathbb{C}} \left\{ \left| g_N(z) - g_{\mu_{\sigma,\nu,c}}(z) \right| \left( \frac{|z|^2 + 2}{|3z|^2} + \frac{1}{3z} \right)^{-1} \right\}. \]

Now set

\[ \tau_N = (1 - \sigma^2 c_N g_N(z))z - \frac{\gamma_q(N)}{1 - \sigma^2 c_N g_N(z)} - \sigma^2 (1 - c_N) \]

and

\[ \tilde{\tau}_N = (1 - \sigma^2 c g_{\mu_{\sigma,\nu,c}}(z))z - \frac{\gamma_q(N)}{1 - \sigma^2 c g_{\mu_{\sigma,\nu,c}}(z)} - \sigma^2 (1 - c). \quad (72) \]

Lemmas 4.6 and 4.7 yield that there is a polynomial \( R \) with nonnegative coefficients, a sequence \( w_N \) of nonnegative real numbers converging to zero when \( N \) goes to infinity and some nonnegative real number \( l \), such that for all \( z \in \mathbb{C} \setminus \mathbb{R}, \)

\[ |\tau_N - \tilde{\tau}_N| \leq (1 + |z|) R(|3z|^{-1}) w_N. \quad (73) \]

Now, one can easily see that,

\[ \left| (1 - \sigma^2 c g_{\mu_{\sigma,\nu,c}}(z))z - \frac{\gamma_q(N)}{1 - \sigma^2 c g_{\mu_{\sigma,\nu,c}}(z)} - \sigma^2 (1 - c) \right| \geq |3z|, \quad (74) \]

so that

\[ \left| \frac{1}{\tilde{\tau}_N} \right| \leq \frac{1}{|3z|}. \quad (75) \]

Note that

\[ \frac{1}{\tilde{\tau}_N} = \frac{(1 - \sigma^2 c g_{\mu_{\sigma,\nu,c}}(z))}{\omega_{\sigma,\nu,c}(z) - \gamma_q(N)}. \quad (76) \]

Then, (10) readily follows from Proposition 12 (65), (73), (75), (76), and (ii) Lemma 5.8. The proof of Proposition 2.2 is complete.

## 5 Proof of Theorem 1.6

We follow the two steps presented in Section 2.

**Step A.** We first prove \( \ll \).

Let \( \eta > 0 \) small enough and \( N \) large enough such that for any \( l = 1, \ldots, J, \)

\( \omega_l(N) \in [\theta_l - \eta, \theta_l + \eta] \) and \( [\theta_l - 2\eta, \theta_l + 2\eta] \) contains no other element of the spectrum of \( A_N^* A_N \) than \( \omega_l(N) \). For any \( l = 1, \ldots, J, \) choose \( f_{\eta,l} \) in \( C^\infty(\mathbb{R}, \mathbb{R}) \) with support in \([\theta_l - 2\eta, \theta_l + 2\eta]\) such that \( f_{\eta,l}(x) = 1 \) for any \( x \in [\theta_l - \eta, \theta_l + \eta] \) and \( 0 \leq f_{\eta,l} \leq 1 \). Let \( 0 < \epsilon < \delta_0 \) where \( \delta_0 \) is introduced in Theorem 1.3.

Choose \( h_{\epsilon,j} \) in \( C^\infty(\mathbb{R}, \mathbb{R}) \) with support in \([\rho_{\theta_j} - \epsilon, \rho_{\theta_j} + \epsilon]\) such that \( h_{\epsilon,j} \equiv 1 \) on \([\rho_{\theta_j} - \epsilon/2, \rho_{\theta_j} + \epsilon/2]\) and \( 0 \leq h_{\epsilon,j} \leq 1 \).
Almost surely for all large $N$, $M_N$ has $k_j$ eigenvalues in $[\rho_{\theta_j} - \varepsilon/2, \rho_{\theta_j} + \varepsilon/2]$. According to Theorem 1.4, denoting by $(\xi_1, \ldots, \xi_{k_j})$ an orthonormal system of eigenvectors associated to the $k_j$ eigenvalues of $M_N$ in $(\rho_{\theta_j} - \varepsilon/2, \rho_{\theta_j} + \varepsilon/2)$, it readily follows from (12) that almost surely for all large $N$,

$$\sum_{n=1}^{k_j} \left\| P_{\ker(\alpha_l(N)I_n - A_N A_N^*)} \xi_n \right\|^2 = \text{Tr} [h_{\varepsilon,j}(M_N) f_{\eta,l}(A_N A_N^*)].$$

Applying Proposition 2.1 with $\Gamma_N = f_{\eta,l}(A_N A_N^*)$ and $K = k_l$, the problem of establishing (11) is reduced to prove that

$$E \left( \text{Tr} \left[ h_{\varepsilon,j} \left( \left( \sigma G_N^{\sqrt{N}} + A_N \right) \left( \sigma G_N^{\sqrt{N}} + A_N \right)^* \right) f_{\eta,l}(A_N A_N^*) \right] \right) \rightarrow_{N \to +\infty} \frac{k_j \delta_{jl} \left( 1 - \sigma^2 c g_{\mu,\nu,c}(\rho_{\theta_j}) \right)}{\omega'_{\sigma,\nu,c}(\rho_{\theta_j})}. \quad (77)$$

Using a Singular Value Decomposition of $A_N$ and the biunitarily invariance of the distribution of $G_N$, we can assume that $A_N$ is as (14) and such that for any $j = 1, \ldots, J$,

$$(A_N A_N^*)_{ii} = \alpha_j(N) \quad \text{for} \ i = k_1 + \ldots + k_{j-1} + l, \ l = 1, \ldots, k_j.$$ 

Now, according to Lemma 5.9,

$$E \left( \text{Tr} \left[ h_{\varepsilon,j} \left( \left( \sigma G_N^{\sqrt{N}} + A_N \right) \left( \sigma G_N^{\sqrt{N}} + A_N \right)^* \right) f_{\eta,l}(A_N A_N^*) \right] \right) = - \lim_{y \to 0^+} \frac{1}{\pi} \int \Im \text{Tr} \left[ G_N^0(t + iy) f_{\eta,l}(A_N A_N^*) \right] h_{\varepsilon,j}(t) dt,$$

with, for all large $N$,

$$E \text{Tr} \left[ G_N^0(t + iy) f_{\eta,l}(A_N A_N^*) \right] = \sum_{k_1 + \ldots + k_{l-1} + 1}^{k_1 + \ldots + k_l} f_{\eta,l}(\alpha_l(N)) E[G_N^0(t + iy)]_{kk}$$

$$= \sum_{k=1}^{k_1 + \ldots + k_{l-1} + 1} E[G_N^0(t + iy)]_{kk}.$$ 

Now, by considering

$$\tau' = (1 - \sigma^2 c g_{\mu,\nu,c}(z))z - \frac{\theta_l}{1 - \sigma^2 c g_{\mu,\nu,c}(z)} - \sigma^2 (1 - c)$$

instead of dealing with $\hat{\tau}_N$ defined in (72) at the end of the proof of Proposition 2.2, one can prove that there is a polynomial $P$ with nonnegative coefficients, a sequence $(u_N)_N$ of nonnegative real numbers converging to zero when $N$


The second (middle) term is simply the integral and noting that the integrals along the vertical lines tend to zero as $iy$

Now, according to Lemma 5.10, we have

$$E \left( \left( G_N^G(z) \right)_{kk} \right) = \frac{1 - \sigma^2 c g_{\mu,\nu,c}(z)}{\omega_{\sigma,\nu,c}(z) - \theta_l} + \Delta_{k,N}(z),$$

with

$$|\Delta_{k,N}(z)| \leq (1 + |z|)^s P(|\Re z|^{-1})u_N.$$

Thus,

$$E \text{Tr} \left[ G_N^G(t + iy)^i A_N A_N^* \right] = k_l \frac{1 - \sigma^2 c g_{\mu,\nu}(t + iy)}{\omega_{\sigma,\nu}(z) - \theta_l} + \Delta_N(t + iy),$$

where for all $z \in \mathbb{C} \setminus \mathbb{R}$, $\Delta_N(z) = \sum_{k = k_1 + k_1 + k_2 + 1}^{k_1 + \cdots + k_l} \Delta_{k,N}(z)$, and $|\Delta_N(z)| \leq k_l(1 + |z|)^s P(|\Re z|^{-1})u_N$.

First let us compute

$$\lim_{y \to 0} \frac{k_l}{\pi} \int_{\rho_{\theta_j} - \varepsilon}^{\rho_{\theta_j} + \varepsilon} \Re \frac{h_{\varepsilon,j}(t)(1 - \sigma^2 c g_{\mu,\nu,c}(t + iy))}{\theta_l - \omega_{\sigma,\nu,c}(t + iy)} dt.$$

The function $\omega_{\sigma,\nu,c} (\Re z) = \omega_{\sigma,\nu,c}(z)$ and $g_{\mu,\nu,c}(\Re z) = g_{\mu,\nu,c}(z)$, so that $\Re \frac{1 - \sigma^2 c g_{\mu,\nu,c}(t + iy)}{\theta_l - \omega_{\sigma,\nu,c}(t + iy)} = \frac{1}{2\pi i} \frac{1}{\theta_l - \omega_{\sigma,\nu,c}(t + iy)} dt$. As in [10], the above integral is split into three pieces, namely $\int_{\rho_{\theta_j} - \varepsilon}^{\rho_{\theta_j} + \varepsilon} / \int_{\rho_{\theta_j} - \varepsilon/2}^{\rho_{\theta_j} + \varepsilon/2} + \int_{\rho_{\theta_j} + \varepsilon/2}^{\rho_{\theta_j} + \varepsilon}$.

Each of the first and third integrals are easily seen to go to zero when $y \to 0$ by a direct application of the definition of the functions involved and of the (Riemann) integral. As $h_{\varepsilon,j}$ is constantly equal to one on $[\rho_{\theta_j} - \varepsilon/2; \rho_{\theta_j} + \varepsilon/2]$, the second (middle) term is simply the integral

$$\frac{k_l}{2\pi i} \int_{\rho_{\theta_j} - \varepsilon/2}^{\rho_{\theta_j} + \varepsilon/2} \frac{1 - \sigma^2 c g_{\mu,\nu,c}(t + iy)}{\theta_l - \omega_{\sigma,\nu,c}(t + iy)} dt.$$

Completing this to a contour integral on the rectangular with corners $\rho_{\theta_j} \pm \varepsilon/2 \pm iy$ and noting that the integrals along the vertical lines tend to zero as $y \to 0$ allows a direct application of the residue theorem for the final result, if $l = j$, 

$$\frac{k_l (1 - \sigma^2 c g_{\mu,\nu,c}(\rho_{\theta_j}))}{\omega'_{\sigma,\nu,c}(\rho_{\theta_j})}.$$

If we consider $\theta_l$ for some $l \neq j$, then $z \mapsto (1 - \sigma^2 c g_{\mu,\nu,c}(z)) (\theta_l - \omega_{\sigma,\nu,c}(z))^{-1}$ is analytic around $\rho_{\theta_j}$, so its residue at $\rho_{\theta_j}$ is zero, and the above argument provides zero as answer.

Now, according to Lemma 5.10, we have

$$\lim_{y \to 0^+} \left( u_N^{-1} \right) \left| \int h_{\varepsilon,j}(t) \Delta_N(t + iy) dt \right| < +\infty.$$
so that
\[
\lim_{N \to +\infty} \limsup_{y \to 0^+} \left| \int h_{\varepsilon,j}(t) \Delta_N(t + iy) dt \right| = 0. \tag{79}
\]
This concludes the proof of (11).

Step B: In the second, and final, step, we shall use a perturbation argument identical to the one used in [10] to reduce the problem to the case of a spike with multiplicity one, case that follows trivially from Step A. A further property of eigenvectors of Hermitian matrices which are close to each other in the norm will be important in the analysis of the behaviour of the eigenvectors of our matrix models. Given a Hermitian matrix \( M \in M_N(\mathbb{C}) \) and a Borel set \( S \subseteq \mathbb{R} \), we denote by \( E_M(S) \) the spectral projection of \( M \) associated to \( S \). In other words, the range of \( E_M(S) \) is the vector space generated by the eigenvectors of \( M \) corresponding to eigenvalues in \( S \). The following lemma can be found in [3].

Lemma 5.1. Let \( M \) and \( M_0 \) be \( N \times N \) Hermitian matrices. Assume that \( \alpha, \beta, \delta \in \mathbb{R} \) are such that \( \alpha < \beta, \delta > 0 \), \( M \) and \( M_0 \) has no eigenvalues in \( [\alpha - \delta, \alpha] \cup [\beta, \beta + \delta] \). Then,
\[
\|E_M((\alpha, \beta)) - E_{M_0}((\alpha, \beta))\| < \frac{4(\beta - \alpha + 2\delta)}{\pi \delta^2} \|M - M_0\|.
\]
In particular, for any unit vector \( \xi \in E_{M_0}((\alpha, \beta))(\mathbb{C}^N) \),
\[
\| (I_N - E_M((\alpha, \beta))) \xi \|_2 < \frac{4(\beta - \alpha + 2\delta)}{\pi \delta^2} \|M - M_0\|.
\]

Assume that \( \theta_i \) is in \( \Theta_{\sigma,\nu,c} \) defined in (7) and \( k_i \neq 1 \). Let us denote by \( V_1(i), \ldots, V_k(i) \), an orthonormal system of eigenvectors of \( A_N A_N^* \) associated with \( \alpha_i(N) \). Consider a Singular Value Decomposition \( A_N = U_N D_N V_N \) where \( V_N \) is a \( N \times N \) unitary matrix, \( U_N \) is a \( n \times n \) unitary matrix whose \( k_i \) first columns are \( V_1(i), \ldots, V_{k_i}(i) \) and \( D_N \) is as (12) with the first \( k_i \) diagonal elements equal to \( \sqrt{\alpha_i(N)} \).

Let \( \delta_0 \) be as in Theorem 1.4. Almost surely, for all \( N \) large enough, there are \( k_i \) eigenvalues of \( M_N \) in \((\rho_{\theta_i} - \frac{\delta_0}{4}, \rho_{\theta_i} + \frac{\delta_0}{4})\), namely \( \lambda_{n_i - 1 + q} + \lambda_{n_i + k_i} \), \( q = 1, \ldots, k_i \) (where \( n_i - 1, \ldots, n_i + k_i \) are the descending ranks of \( \alpha_i(N) \) among the eigenvalues of \( A_N A_N^* \)), which are moreover the only eigenvalues of \( M_N \) in \((\rho_{\theta_i} - \delta_0, \rho_{\theta_i} + \delta_0)\). Thus, the spectrum of \( M_N \) is split into three pieces:
\[
\{\lambda_1(M_N), \ldots, \lambda_{n_i - 1}(M_N)\} \subset (\rho_{\theta_i}, \rho_{\theta_i} + \infty],
\]
\[
\{\lambda_{n_i - 1 + 1}(M_N), \ldots, \lambda_{n_i - 1 + k_i}(M_N)\} \subset (\rho_{\theta_i} - \delta_0, \rho_{\theta_i} + \delta_0),
\]
\[
\{\lambda_{n_i - 1 + k_i + 1}(M_N), \ldots, \lambda_N(M_N)\} \subset [0, \rho_{\theta_i} - \delta_0).
\]
The distance between any of these components is equal to \( 3\delta_0/4 \). Let us fix \( \epsilon_0 \) such that \( 0 \leq \theta_i (2\epsilon_0 N + c_0^2 k_i^4) < \text{dist}(\theta_i, \text{supp} \nu \cup \nu_i \neq \theta_i) \) and such that
by the eigenvectors associated to \( \sigma_1, \nu, \alpha \) defined by (84). For any \( 0 < \epsilon < \epsilon_0 \), define the matrix \( A_N(\epsilon) \) as \( A_N(\epsilon) = U_N D_N(\epsilon) V_N \) where

\[
(D_N(\epsilon))_{m,m} = \sqrt{\alpha_i(N)(1 + \epsilon(k_i - m + 1))}, \quad \text{for } m \in \{1, \ldots, k_i\},
\]
and \( (D_N(\epsilon))_{pq} = (D_N)_{pq} \) for any \( (p, q) \notin \{(m, m) : m \in \{1, \ldots, k_i\}\} \).

Set

\[
M_N(\epsilon) = \left( \sigma \frac{X_N}{\sqrt{N}} + A_N(\epsilon) \right) \left( \sigma \frac{X_N}{\sqrt{N}} + A_N(\epsilon) \right)^*.
\]

For \( N \) large enough, for each \( m \in \{1, \ldots, k_i\} \), \( \alpha_i(N)(1 + \epsilon(k_i - m + 1))^2 \) is an eigenvalue of \( A_N A_N(\epsilon) \) with multiplicity one. Note that, since \( \sup_N \|A_N\| < +\infty \), it is easy to see that there exist some constant \( C \) such that for any \( N \) and for any \( 0 < \epsilon < \epsilon_0 \),

\[
\|M_N(\epsilon) - M_N\| \leq C \epsilon \left( \frac{\|X_N\|}{\sqrt{N}} + 1 \right).
\]

Applying Remark 5.4 to the \((n + N) \times (n + N)\) matrix \( \tilde{X}_N = \left( \begin{array}{cc} 0_{n \times n} & X_N \\ X_N^* & 0_{N \times N} \end{array} \right) \) (see also Appendix B of [14]), it readily follows that there exists some constant \( C' \) such that a.s for all large \( N \), for any \( 0 < \epsilon < \epsilon_0 \),

\[
\|M_N(\epsilon) - M_N\| \leq C' \epsilon. \tag{80}
\]

Therefore, for \( \epsilon \) sufficiently small such that \( C' \epsilon < \delta_0/4 \), by Theorem A.46 [2], there are precisely \( n_{i-1} \) eigenvalues of \( M_N(\epsilon) \) in \([0, \rho_\theta - 3\delta_0/4] \), precisely \( k_i \) in \((\rho_\theta - \delta_0/2, \rho_\theta + \delta_0/2) \) and precisely \( N - (n_{i-1} + k_i) \) in \((\rho_\theta + 3\delta_0/4, +\infty) \). All these intervals are again at strictly positive distance from each other, in this case \( \delta_0/4 \).

Let \( \xi \) be a normalized eigenvector of \( M_N \) relative to \( \lambda_{n_{i-1}+q}(M_N) \) for some \( q \in \{1, \ldots, k_i\} \). As proved in Lemma 5.1 if \( E(\epsilon) \) denotes the subspace spanned by the eigenvectors associated to \( \{\lambda_{n_{i-1}+1}(M_N(\epsilon)), \ldots, \lambda_{n_{i-1}+k_i}(M_N(\epsilon))\} \) in \( \mathbb{C}^N \), then there exists some constant \( C \) (which depends on \( \epsilon_0 \)) such that for \( \epsilon \) small enough, almost surely for large \( N \),

\[
\|P_{E(\epsilon)} \cdot \xi\|_2 \leq C \epsilon. \tag{81}
\]

According to Theorem 1.4 for \( j \in \{1, \ldots, k_i\} \), for large enough \( N \), \( \lambda_{n_{i-1}+j}(M_N(\epsilon)) \) separates from the rest of the spectrum and belongs to a neighborhood of \( \Phi_{\sigma,\nu,c}(\theta_i^{(j)}(\epsilon)) \) where

\[
\theta_i^{(j)}(\epsilon) = \theta_i(1 + \epsilon(k_i - j + 1))^2.
\]

If \( \xi_j(\epsilon, i) \) denotes a normalized eigenvector associated to \( \lambda_{n_{i-1}+j}(M_N(\epsilon)) \), Step A above implies that almost surely for any \( p \in \{1, \ldots, k_i\} \), for any \( \gamma > 0 \), for all large \( N \),

\[
\left| \langle V_p(i), \xi_j(\epsilon, i) \rangle \right|^2 - \frac{\delta_{pj}}{\omega_{\sigma,\nu,c}(\Phi_{\sigma,\nu,c}(\theta_i^{(j)}(\epsilon)))} \left( 1 - \sigma^2 c_{\mu^{\sigma,\nu,c}}(\Phi_{\sigma,\nu,c}(\theta_i^{(j)}(\epsilon))) \right) < \gamma. \tag{82}
\]
The eigenvector $\xi$ decomposes uniquely in the orthonormal basis of eigenvectors of $M_N(\epsilon)$ as $\xi = \sum_{j=1}^{k_i} c_j(\epsilon) \xi_j(\epsilon, i) + \xi(\epsilon)^\perp$, where $c_j(\epsilon) = \langle \xi_j(\epsilon, i) \rangle$ and $\xi(\epsilon)^\perp = P_{E(\epsilon)^\perp} \xi$; necessarily $\sum_{j=1}^{k_i} |c_j(\epsilon)|^2 + \|\xi(\epsilon)^\perp\|^2_2 = 1$. Moreover, as indicated in relation (81), $\|\xi(\epsilon)^\perp\|_2 \leq C\epsilon$. We have

$$P_{\ker(\alpha_i(N)I_N - A_N A_N^{\perp})} \xi = \sum_{j=1}^{k_i} c_j(\epsilon) P_{\ker(\alpha_i(N)I_N - A_N A_N^{\perp})} \xi_j(\epsilon, i) + P_{\ker(\alpha_i(N)I_N - A_N A_N^{\perp})} \xi(\epsilon)^\perp$$

$$= \sum_{j=1}^{k_i} c_j(\epsilon) \sum_{l=1}^{k_i} \langle \xi_j(\epsilon, i) | V_l(i) \rangle V_l(i) + P_{\ker(\alpha_i(N)I_N - A_N A_N^{\perp})} \xi(\epsilon)^\perp.$$

Take in the above the scalar product with $\xi = \sum_{j=1}^{k_i} c_j(\epsilon) \xi_j(\epsilon, i) + \xi(\epsilon)^\perp$ to get

$$\langle P_{\ker(\alpha_i(N)I_N - A_N A_N^{\perp})} \xi | \xi \rangle = \sum_{j=1}^{k_i} c_j(\epsilon) \langle \xi_j(\epsilon, i) | V_l(i) \rangle \overline{c_s(\epsilon)} \langle V_l(i) | \xi_s(\epsilon, i) \rangle$$

$$+ \sum_{j=1}^{k_i} c_j(\epsilon) \sum_{l=1}^{k_i} \langle \xi_j(\epsilon, i) | V_l(i) \rangle \langle V_l(i) | \xi(\epsilon)^\perp \rangle$$

$$+ \langle P_{\ker(\alpha_i(N)I_N - A_N A_N^{\perp})} \xi(\epsilon)^\perp | \xi \rangle.$$

Relation (82) indicates that

$$\sum_{j,l,s=1}^{k_i} c_j(\epsilon) \langle \xi_j(\epsilon, i) | V_l(i) \rangle \overline{c_s(\epsilon)} \langle V_l(i) | \xi_s(\epsilon, i) \rangle$$

$$= \sum_{j=1}^{k_i} |c_j(\epsilon)|^2 |\langle V_j(i) | \xi_j(\epsilon, i) \rangle|^2 + \Delta_1$$

$$= \sum_{j=1}^{k_i} |c_j(\epsilon)|^2 \frac{1 - \sigma^2 c_{g_{u,\mu,c}}(\Phi_{\sigma,\mu,c}(\theta_{l_j}(\epsilon)))}{\omega_{\sigma,\mu,c} \Phi_{\sigma,\mu,c}(\theta_{l_j}(\epsilon))} + \Delta_1 + \Delta_2,$$

where for all large $N$, $|\Delta_1| \leq \sqrt{\gamma} k_i^3$ and $|\Delta_2| \leq \gamma$. Since $\|\xi(\epsilon)^\perp\|_2 \leq C\epsilon$,

$$\left| \sum_{j=1}^{k_i} c_j(\epsilon) \sum_{l=1}^{k_i} \langle \xi_j(\epsilon, i) | V_l(i) \rangle \langle V_l(i) | \xi(\epsilon)^\perp \rangle \right.$$  

$$+ \langle P_{\ker(\alpha_i(N)I_N - A_N A_N^{\perp})} \xi(\epsilon)^\perp | \xi \rangle \left| \right| \leq (k_i^2 + 1) C\epsilon.$$  

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Thus, we conclude that almost surely for any $\gamma > 0$, for all large $N$,

$$\left| \langle P_{\ker(\alpha_i(N)I_N - A_N A_N^*)}\xi, \xi \rangle - \sum_{j=1}^{k_i} |c_j(\epsilon)|^2 \left( 1 - \sigma^2 cg_{\mu_{\sigma,\nu,c}}(\Phi_{\sigma,\nu,c}(\theta_i^{(j)}(\epsilon))) \right) \right| \leq (k_i^2 + 1)C\epsilon + \sqrt{\gamma}k_i^3 + \gamma. \quad (83)$$

Since we have the identity

$$\langle P_{\ker(\alpha_i(N)I_N - A_N A_N^*)}\xi, \xi \rangle = \|P_{\ker(\alpha_i(N)I_N - A_N A_N^*)}\xi\|^2$$

and the three obvious convergences $\lim_{\epsilon \to 0} \omega'_{\sigma,\nu,c}(\Phi_{\sigma,\nu,c}(\theta_i^{(j)}(\epsilon))) = \omega'_{\sigma,\nu,c}(\rho_{\theta_i})$, $\lim_{\epsilon \to 0} g_{\mu_{\sigma,\nu,c}}(\Phi_{\sigma,\nu,c}(\theta_i^{(j)}(\epsilon))) = g_{\mu_{\sigma,\nu,c}}(\rho_{\theta_i})$ and $\lim_{\epsilon \to 0} \sum_{j=1}^{k_i} |c_j(\epsilon)|^2 = 1$, relation (83) concludes Step B and the proof of Theorem 1.6. (Note that we use (2.9) of [11] which is true for any $x \in \mathbb{C} \setminus \mathbb{R}$ to deduce that $1 - \sigma^2 cg_{\mu_{\sigma,\nu,c}}(\Phi_{\sigma,\nu,c}(\theta_i)) = \frac{1}{1 + \sigma^2 cg_{\nu}(\theta_i)}$ by letting $x$ go to $\Phi_{\sigma,\nu,c}(\theta_i)$).

**Appendix A**

We present alternative versions on the one hand of the result in [3] about the lack of eigenvalues outside the support of the deterministic equivalent measure, and on the other hand of the result in [11] about the exact separation phenomenon. These new versions (Theorems 5.3 and 5.6 below) deal with random variables whose imaginary and real parts are independent, but remove the technical assumptions ((1.10) and "$b_1 > 0$" in Theorem 1.1 in [3] and "$\omega_{\sigma,\nu,c}(b) > 0$" in Theorem 1.2 in [11]). The proof of Theorem 5.3 is based on the results of [6]. The arguments of the proof of Theorem 1.2 in [11] and Theorem 5.3 lead to the proof of Theorem 5.6.

**Theorem 5.2.** Consider

$$M_N = (\sigma \frac{X_N}{\sqrt{N}} + A_N)(\sigma \frac{X_N}{\sqrt{N}} + A_N)^*, \quad (84)$$

and assume that

1. $X_N = [X_{ij}]_{1 \leq i \leq n, 1 \leq j \leq N}$ is a $n \times N$ random matrix such that $[X_{ij}]_{i \geq 1, j \geq 1}$ is an infinite array of random variables which satisfy (1) and (2) and such that $\Re(X_{ij}), \Im(X_{ij}), (i, j) \in \mathbb{N}^2$, are independent, centered with variance $1/2$.
2. $A_N$ is an $n \times N$ nonrandom matrix such that $\|A_N\|$ is uniformly bounded.
3. $n \leq N$ and, as $N$ tends to infinity, $c_N = n/N \to c \in [0, 1]$. 

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4. $[x, y]$, $x < y$, is such that there exists $\delta > 0$ such that for all large $N$, $|x - \delta; y + \delta| \subset \mathbb{R} \setminus \text{supp}(\mu_{\sigma, \mu_{AN} A_N^*, c_N})$ where $\mu_{\sigma, \mu_{AN} A_N^*, c_N}$ is the nonrandom distribution which is characterized in terms of its Stieltjes transform which satisfies the equation \[ \frac{1}{\lambda} \] where we replace $c$ by $c_N$ and $\nu$ by $\mu_{AN} A_N^*$.

Then, we have

$$\mathbb{P}[\text{for all large } N, \text{spec}(M_N) \subset \mathbb{R} \setminus [x, y]] = 1.$$  

Since, in the proof of Theorem 5.2, we will use tools from free probability theory, for the reader’s convenience, we recall the following basic definitions from free probability theory. For a thorough introduction to free probability theory, we refer to \[30\].

- A $C^*$-probability space is a pair $(A, \tau)$ consisting of a unital $C^*$-algebra $A$ and a state $\tau$ on $A$ i.e a linear map $\tau : A \to \mathbb{C}$ such that $\tau(1_A) = 1$ and $\tau(aa^*) \geq 0$ for all $a \in A$. $\tau$ is a trace if it satisfies $\tau(ab) = \tau(ba)$ for every $(a, b) \in A^2$. A trace is said to be faithful if $\tau(aa^*) > 0$ whenever $a \neq 0$. An element of $A$ is called a noncommutative random variable.

- The noncommutative $\ast$-distribution of a family $a = (a_1, \ldots, a_k)$ of noncommutative random variables in a $C^*$-probability space $(A, \tau)$ is defined as the linear functional $\mu_a : P \mapsto \tau(P(a, a^*))$ defined on the set of polynomials in $2k$ noncommutative indeterminates, where $(a, a^*)$ denotes the $2k$-uple $(a_1, \ldots, a_k, a_1^*, \ldots, a_k^*)$. For any selfadjoint element $a_1$ in $A$, there exists a probability measure $\nu_{a_1}$ on $\mathbb{R}$ such that, for every polynomial $P$, we have

$$\mu_{a_1}(P) = \int P(t) d\nu_{a_1}(t).$$

Then we identify $\mu_{a_1}$ and $\nu_{a_1}$. If $\tau$ is faithful then the support of $\nu_{a_1}$ is the spectrum of $a_1$ and thus $\|a_1\| = \sup\{|z|, z \in \text{support}(\nu_{a_1})\}$.

- A family of elements $(a_i)_{i \in I}$ in a $C^*$-probability space $(A, \tau)$ is free if for all $k \in \mathbb{N}$ and all polynomials $p_1, \ldots, p_k$ in two noncommutative indeterminates, one has

$$\tau(p_1(a_{i_1}, a_{i_1}^*) \cdots p_k(a_{i_k}, a_{i_k}^*)) = 0$$

whenever $i_1 \neq i_2, i_2 \neq i_3, \ldots, i_{k-1} \neq i_k, (i_1, \ldots, i_k) \in I^k$, and $\tau(p_l(a_{i_1}, a_{i_1}^*)) = 0$ for $l = 1, \ldots, k$.

- A noncommutative random variable $x$ in a $C^*$-probability space $(A, \tau)$ is a standard semicircular random variable if $x = x^*$ and for any $k \in \mathbb{N}$,

$$\tau(x^k) = \int t^k d\mu_{sc}(t)$$

where $d\mu_{sc}(t) = \frac{1}{2\pi} \sqrt{4 - t^2} I_{[-2, 2]}(t) dt$ is the semicircular standard distribution.
Let $k$ be a nonnull integer number. Denote by $\mathcal{P}$ the set of polynomials in $2k$ noncommutative indeterminates. A sequence of families of variables $(a_n)_{n\geq 1} = (a_1(n), \ldots, a_k(n))_{n\geq 1}$ in $C^*$-probability spaces $(\mathcal{A}_n, \tau_n)$ converges in $\ast$-distribution, when $n$ goes to infinity, to some $k$-tuple of noncommutative random variables $a = (a_1, \ldots, a_k)$ in a $C^*$-probability space $(\mathcal{A}, \tau)$ if the map $P \in \mathcal{P} \mapsto \tau_n(P(a_{n}, a_{n}^*))$ converges pointwise towards $P \in \mathcal{P} \mapsto \tau(P(a, a^*))$.

$k$ noncommutative random variables $a_1(n), \ldots, a_k(n)$, in $C^*$-probability spaces $(\mathcal{A}_n, \tau_n)$, $n \geq 1$, are said asymptotically free if $(a_1(n), \ldots, a_k(n))$ converges in $\ast$-distribution, as $n$ goes to infinity, to some noncommutative random variables $(a_1, \ldots, a_k)$ in a $C^*$-probability space $(\mathcal{A}, \tau)$ where $a_1, \ldots, a_k$ are free.

We will also use the following well known result on asymptotic freeness of random matrices. Let $\mathcal{A}_n$ be the algebra of $n \times n$ matrices with complex entries and endow this algebra with the normalized trace defined for any $M \in \mathcal{A}_n$ by $\tau_n(M) = \frac{1}{n} \text{Tr}(M)$. Let us consider a $n \times n$ so-called standard G.U.E matrix, i.e a random Hermitian matrix $\mathcal{G}_n = [\mathcal{G}_{jk}]_{j,k=1}^n$, where $\mathcal{G}_{ii}, \sqrt{2\text{Re}(\mathcal{G}_{ij})}$, $\sqrt{2\text{Im}(\mathcal{G}_{ij})}$, $i < j$ are independent centered Gaussian random variables with variance 1. For a fixed real number $t$ independent from $n$, let $H_{n}^{(1)}, \ldots, H_{n}^{(t)}$ be deterministic $n \times n$ Hermitian matrices such that $\max_{1 \leq l \leq t} \sup_{n} \|H_{n}^{(l)}\| \leq +\infty$ and $(H_{n}^{(1)}, \ldots, H_{n}^{(t)})$, as a t-tuple of noncommutative random variables in $(\mathcal{A}_n, \tau_n)$, converges in distribution when $n$ goes to infinity. Then, according to Theorem 5.4.5 in [H], $\frac{\mathcal{G}_n}{\sqrt{n}}$ and $(H_{n}^{(1)}, \ldots, H_{n}^{(t)})$ are almost surely asymptotically free i.e almost surely, for any polynomial $P$ in $t+1$ noncommutative indeterminates,

$$\tau_n \left\{ P \left( H_{n}^{(1)}, \ldots, H_{n}^{(t)}, \frac{\mathcal{G}_n}{\sqrt{n}} \right) \right\} \to_{n \to +\infty} \tau \left( P(h_1, \ldots, h_t, s) \right) \quad (86)$$

where $h_1, \ldots, h_t$ and $s$ are noncommutative random variables in some $C^*$-probability space $(\mathcal{A}, \tau)$ such that $(h_1, \ldots, h_t)$ and $s$ are free, $s$ is a standard semi-circular noncommutative random variable and the distribution of $(h_1, \ldots, h_t)$ is the limiting distribution of $(H_{n}^{(1)}, \ldots, H_{n}^{(t)})$.

Finally, the proof of Theorem 5.2 is based on the following result which can be established by following the proof of Theorem 1.1 in [G]. First, note that the algebra of polynomials in non-commuting indeterminates $X_1, \ldots, X_k$, becomes a $\ast$-algebra by anti-linear extension of $(X_{i_1}X_{i_2} \ldots X_{i_m})^{*} = X_{i_m} \ldots X_{i_2}X_{i_1}$.

**Theorem 5.3.** Let us consider three independent infinite arrays of random variables, $[W_{ij}^{(1)}]_{i \geq 1, j \geq 1}$, $[W_{ij}^{(2)}]_{i \geq 1, j \geq 1}$ and $[X_{ij}]_{i \geq 1, j \geq 1}$ where

- for $l = 1, 2$, $W_{ii}^{(l)}$, $\sqrt{2}\text{Re}(W_{ij}^{(l)})$, $\sqrt{2}\text{Im}(W_{ij}^{(l)})$, $i < j$, are i.i.d centered and bounded random variables with variance 1 and $W_{ij}^{(l)} = W_{ij}^{(l)}$. 

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\begin{itemize}
  \item \{\Re(X_{ij}), \Im(X_{ij}), i \in \mathbb{N}, j \in \mathbb{N}\} are independent centered random variables with variance 1/2 and satisfy (1) and (2).
\end{itemize}

For any \((N, n) \in \mathbb{N}^2\), define the \((n + N) \times (n + N)\) matrix:

\[
W_{n+N} = \begin{pmatrix}
  W^{(1)}_n & X_N \\
  X_N^* & W^{(2)}_N
\end{pmatrix}
\]  \hspace{1cm} (87)

where \(X_N = [X_{ij}]_{1 \leq i, j \leq n}\), \(W^{(1)}_n = [W^{(1)}_{ij}]_{1 \leq i, j \leq n}\), \(W^{(2)}_N = [W^{(2)}_{ij}]_{1 \leq i, j \leq N}\).

Assume that \(n = n(N)\) and \(\lim_{N \to +\infty} \frac{n}{N} = c \in [0, 1]\).

Let \(t\) be a fixed integer number and \(P\) be a selfadjoint polynomial in \(t+1\) noncommutative indeterminates.

For any \(N \in \mathbb{N}^2\), let \((P^{(1)}_{n+N}, \ldots, B^{(t)}_{n+N})\) be a \(t\)-tuple of \((n+N) \times (n+N)\) deterministic Hermitian matrices such that for any \(u = 1, \ldots, t\), \(\sup_N \|P^{(u)}_{n+N}\| < \infty\).

Let \((A, \tau)\) be a \(C^*\)-probability space equipped with a faithful tracial state and \(s\) be a standard semi-circular noncommutative random variable in \((A, \tau)\). Let \(b_{n+N} = (b^{(1)}_{n+N}, \ldots, b^{(t)}_{n+N})\) be a \(t\)-tuple of noncommutative selfadjoint random variables which is free from \(s\) in \((A, \tau)\) and such that the distribution of \(b_{n+N}\) in \((A, \tau)\) coincides with the distribution of \((B^{(1)}_{n+N}, \ldots, B^{(t)}_{n+N})\) in \((M_{n+N}(\mathbb{C}), \frac{1}{n+N} \text{Tr})\).

Let \([x, y]\) be a real interval such that there exists \(\delta > 0\) such that, for any large \(N\), \([x-\delta, y+\delta]\) lies outside the support of the distribution of the noncommutative random variable \(P\left(s, b^{(1)}_{n+N}, \ldots, b^{(t)}_{n+N}\right)\) in \((A, \tau)\). Then, almost surely, for all large \(N\),

\[
\text{spectP}\left(W^{(1)}_{n+N} \sqrt{\frac{A}{n+N}}, B^{(1)}_{n+N}, \ldots, B^{(t)}_{n+N}\right) \subset \mathbb{R} \setminus [x, y].
\]

Proof. We start by checking that a truncation and Gaussian convolution procedure as in Section 2 of [6] can be handled for such a matrix as defined by (87), to reduce the problem to a fit framework where,

\[(H)\] for any \(N\), \((W^{(1)}_{n+N})_{ij}, \sqrt{2} \Re((W^{(1)}_{n+N})_{ij}), \sqrt{2} \Im((W^{(1)}_{n+N})_{ij}), i < j, i \leq n + N, j \leq n + N\), are independent, centered random variables with variance 1, which satisfy a Poincaré inequality with common fixed constant \(C_{PI}\).

Note that, according to Corollary 3.2 in [24], \((H)\) implies that for any \(p \in \mathbb{N},\)

\[
\sup_{N \geq 1} \sup_{1 \leq i, j \leq n+N} \mathbb{E}(\|(W^{(1)}_{n+N})_{ij}\|^p) < +\infty.
\]  \hspace{1cm} (88)

Remark 5.4. Following the proof of Lemma 2.1 in [6], one can establish that, if \((V_{ij})_{i \geq 1, j \geq 1}\) is an infinite array of random variables such that \(\{\Re(V_{ij}), \Im(V_{ij}), i \in \mathbb{N}, j \in \mathbb{N}\}\) are independent centered random variables which satisfy (1) and (2), then almost surely we have

\[
\limsup_{N \to +\infty} \left\| Z^{(n+N)} \sqrt{\frac{1}{N+n}} \right\| \leq 2\sigma^*.
\]
where
\[
Z_{n+N} = \begin{pmatrix} (0) & V_N \\ V_N^* & (0) \end{pmatrix} \text{ with } V_N = [V_{ij}]_{1 \leq i \leq n, 1 \leq j \leq N} \text{ and } \sigma^* = \left\{ \sup_{(i,j) \in \mathbb{N}^2} \mathbb{E}(|V_{ij}|^2) \right\}^{1/2}.
\]

Then, following the rest of the proof of Section 2 in [8], one can prove that for any polynomial \( P \) in \( 1 + t \) noncommutative variables, there exists some constant \( L > 0 \) such that the following holds. Set \( \theta^* = \sup_{i,j} \mathbb{E} \left( |X_{ij}|^3 \right). \) For any \( 0 < \epsilon < 1 \), there exist \( C_\epsilon > 8\sigma^* \) (such that \( C_\epsilon > \max_{l=1,2} |W_{11}^{(l)}| \text{ a.s.} \)) and \( \delta_\epsilon > 0 \) such that almost surely for all large \( N \),
\[
\left\| P \left( \frac{W_{n+N}}{\sqrt{n+N}}, B_{n+N}^{(1)}, \ldots, B_{n+N}^{(t)} \right) - P \left( \frac{\tilde{W}_{n+N}^{C,\delta}}{\sqrt{n+N}}, B_{n+N}^{(1)}, \ldots, B_{n+N}^{(t)} \right) \right\| \leq L \epsilon,
\]
where, for any \( C > 8\sigma^* \) such that \( C > \max_{l=1,2} |W_{11}^{(l)}| \text{ a.s.} \), and for any \( \delta > 0 \), \( \tilde{W}_{n+N}^{C,\delta} \) is a \((n+N) \times (n+N)\) matrix which is defined as follows. Let \( \{G_{ij} \}_{i \geq 1, j \geq 1} \) be an infinite array which is independent of \( \{X_{ij}, W_{ij}^{(1)}, W_{ij}^{(2)} (i,j) \in \mathbb{N}^2 \} \) and such that \( \sqrt{2} \text{Re} G_{ij}, \sqrt{2} \text{Im} G_{ij}, i < j, G_{ii}, \) are independent centred standard real gaussian variables and \( G_{ij} = G_{ji} \). Set \( G_{n+N} = [G_{ij}]_{1 \leq i,j \leq n+N} \) and define \( X_N^C = [X_{ij}^C]_{1 \leq i,j \leq n} \) as in [18]. Set
\[
\tilde{W}_{n+N}^{C,\delta} = \left( \begin{array}{c} W_{n+N}^{(1)} \\ \left( X_N^C \right)^* \\ W_{n+N}^{(2)} \end{array} \right) \text{ and } \tilde{W}_{n+N}^{C,\delta} = \frac{\tilde{W}_{n+N}^{C,\delta} + \delta G_{n+N} \sqrt{1 + \delta^2}}{\sqrt{1 + \delta^2}}.
\]
\( \tilde{W}_{n+N}^{C,\delta} \) satisfies (H) (see the end of Section 2 in [8]). (89) readily yields that it is sufficient to prove Theorem 5.3 for \( \tilde{W}_{n+N}^{C,\delta} \). Therefore, assume now that \( W_{n+N}^{C,\delta} \) satisfies (H). As explained in Section 6.2 in [6], to establish Theorem 5.3, it is sufficient to prove that for all \( m \in \mathbb{N} \), all self-adjoint matrices \( \gamma, \alpha, \beta_1, \ldots, \beta_t \) of size \( m \times m \) and all \( \epsilon > 0 \), almost surely, for all large \( N \), we have
\[
spect(\gamma \otimes I_{n+N} + \alpha \otimes \frac{W_{n+N}}{\sqrt{n+N}} + \sum_{u=1}^t \beta_u \otimes B_{n+N}^{(u)}) \subset spect(\gamma \otimes 1_A + \alpha \otimes s + \sum_{u=1}^t \beta_u \otimes b_{n+N}^{(u)}) + \epsilon, \epsilon]. \quad (90)
\]
(90) is the analog of Lemma 1.3 for \( r = 1 \) in [6]. Finally, one can prove (90) by following Section 5 in [6].

We will need the following lemma in the proof of Theorem 5.2.

**Lemma 5.5.** Let \( A_N \) and \( c_N \) be defined as in Theorem 5.2. Define the following \((n+N) \times (n+N)\) matrices: \( P = \begin{pmatrix} I_N & (0) \\ (0) & (0) \end{pmatrix} \) and \( Q = \begin{pmatrix} (0) & (0) \\ (0) & I_N \end{pmatrix} \)

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and $A = \begin{pmatrix} 0 & A_N \\ 0 & 0 \end{pmatrix}$. Let $s, p_N, q_N, a_N$ be noncommutative random variables in some $C^*$-probability space $(\mathcal{A}, \tau)$ such that $s$ is a standard semi-circular variable which is free with $(p_N, q_N, a_N)$ and the $*$-distribution of $(A, P, Q)$ in 

\[
\left( M_{N+n}(\mathbb{C}), \frac{1}{N+n} \text{Tr} \right)
\]

coincides with the $*$-distribution of $(a_N, p_N, q_N)$ in $(\mathcal{A}, \tau)$. Then, for any $\epsilon \geq 0$, the distribution of $(\sqrt{1 + c_N sp_N s q_N + \sqrt{1 + c_N sp_N s q_N}} + a_N + a_N^*)^2 + c p_N$ is

\[
\frac{n}{N+n} T \ast \mu_{\sigma, \mu_{A_N A_N^*} c_N} + \frac{n}{N+n} \delta_0
\]

where

\[
T \ast \mu_{\sigma, \mu_{A_N A_N^*} c_N}
\]

is the pushforward of $\mu_{\sigma, \mu_{A_N A_N^*} c_N}$ by the map $z \mapsto z + \epsilon$.

**Proof.** Here $N$ and $n$ are fixed. Let $k \geq 1$ and $C_k$ be the $k \times k$ matrix defined by

\[
C_k = \begin{pmatrix} 0 & 1 \\ \vdots & \ddots \\ 1 & 0 \end{pmatrix}.
\]

Define the $k(n+N) \times k(n+N)$ matrices

\[
\hat{A}_k = C_k \otimes A, \quad \hat{P}_k = I_k \otimes P, \quad \hat{Q}_k = I_k \otimes Q.
\]

For any $k \geq 1$, the $*$-distributions of $(\hat{A}_k, \hat{P}_k, \hat{Q}_k)$ in $(M_{k(N+n)}(\mathbb{C}), \frac{1}{k(n+n)} \text{Tr})$ and $(A, P, Q)$ in $(M_{(N+n)}(\mathbb{C}), \frac{1}{(N+n)} \text{Tr})$ respectively, coincide. Indeed, let $\mathcal{K}$ be a noncommutative monomial in $\mathbb{C}(X_1, X_2, X_3, X_4)$ and denote by $q$ the total number of occurrences of $X_3$ and $X_4$ in $\mathcal{K}$. We have

\[
\mathcal{K}(\hat{P}_k, \hat{Q}_k, \hat{A}_k) = C_k^q \otimes \mathcal{K}(P, Q, A, A^*),
\]

so that

\[
\frac{1}{k(n+N)} \text{Tr} \left[ \mathcal{K}(\hat{P}_k, \hat{Q}_k, \hat{A}_k, \hat{A}_k^*) \right] = \frac{1}{k} \text{Tr}(C_k^q) \frac{1}{(n+N)} \text{Tr} \left[ \mathcal{K}(P, Q, A, A^*) \right].
\]

Note that if $q$ is even then $C_k^q = I_k$ so that

\[
\frac{1}{k(n+N)} \text{Tr} \left[ \mathcal{K}(\hat{P}_k, \hat{Q}_k, \hat{A}_k, \hat{A}_k^*) \right] = \frac{1}{(n+N)} \text{Tr} \left[ \mathcal{K}(P, Q, A, A^*) \right]. \tag{91}
\]

Now, assume that $q$ is odd. Note that $PQ = QP = 0$, $AQ = A$, $QA = 0$, $AP = 0$ and $PA = A$ (and then $QA^* = A^*$, $A^*Q = 0$, $PA^* = 0$ and $A^*P = A^*$). Therefore, if at least one of the terms $X_1X_2, X_2X_1, X_2X_3, X_3X_1, X_4X_2$ or $X_3X_4$ appears in the noncommutative product in $\mathcal{K}$, then $\mathcal{K}(P, Q, A, A^*) = 0$, so that (91) still holds. Now, if none of the terms $X_1X_2, X_2X_1, X_2X_3, X_3X_1, X_4X_2$ or $X_3X_4$ appears in the noncommutative product in $\mathcal{K}$, then we have $\mathcal{K}(P, Q, A, A^*) = \tilde{\mathcal{K}}(A, A^*)$ for some noncommutative monomial $\tilde{\mathcal{K}} \in \mathbb{C}(X, Y)$ with degree $q$. Either the noncommutative product in $\tilde{\mathcal{K}}$ contains a term such as $X^p$ or $Y^p$ for some $p \geq 2$ and then, since $A^2 = (A^*)^2 = 0$, we have $\tilde{\mathcal{K}}(A, A^*) = 0$, or $\tilde{\mathcal{K}}(X, Y)$ is one of the monomials $(XY)^2 \xrightarrow{\mu} X$ or $Y(XY)^2 \xrightarrow{\mu} Y$. In both cases,
we have $\text{Tr}\tilde{K}(A, A^*) = 0$ and (91) still holds. Now, define the $k(N + n) \times k(N + n)$ matrices

$$
\hat{P}_k = \begin{pmatrix} I_{kn} & 0 \\ 0 & 0 \end{pmatrix}, \quad \hat{Q}_k = \begin{pmatrix} 0 & 0 \\ 0 & I_{kN} \end{pmatrix}, \quad \tilde{A}_k = \begin{pmatrix} 0 & A_N \\ \vdots & \vdots \\ A_N & 0 \end{pmatrix}
$$

where $\tilde{A}$ is the $kn \times kN$ matrix defined by

$$
\tilde{A} = \begin{pmatrix} 0 \\ & \ddots \\ & & 0 \end{pmatrix}.
$$

It is clear that there exists a real orthogonal $k(N + n) \times k(N + n)$ matrix $O$ such that $\hat{P}_k = O\hat{P}_kO^*$, $\hat{Q}_k = O\hat{Q}_kO^*$ and $\tilde{A}_k = O\tilde{A}_kO^*$. This readily yields that the noncommutative $*$-distributions of $(\hat{A}_k, \hat{P}_k, \hat{Q}_k)$ and $(\tilde{A}_k, \hat{P}_k, \hat{Q}_k)$ in $(M_{k(N+n)}(\mathbb{C}), \frac{1}{k(N+n)}\text{Tr})$ coincide. Hence, for any $k \geq 1$, the distribution of $(\hat{A}_k, \hat{P}_k, \hat{Q}_k)$ in $(M_{k(N+n)}(\mathbb{C}), \frac{1}{k(N+n)}\text{Tr})$ coincides with the distribution of $(a_N, p_N, q_N)$ in $(\mathcal{A}, \tau)$. By Theorem 5.4.5 in [1], it readily follows that the distribution of $(\frac{\sqrt{kN}c_N\sigma}{\sqrt{k(N+n)}\tilde{Q}_k} + \sqrt{\frac{c_N}{kN} + \frac{c_n}{kN}})\tilde{P}_k$ goes to infinity, of $(\frac{\sqrt{kN}c_N\sigma}{\sqrt{k(N+n)}\tilde{Q}_k} + \sqrt{\frac{c_N}{kN} + \frac{c_n}{kN}})\tilde{P}_k$ in $(M_{k(N+n)}(\mathbb{C}), \frac{1}{k(N+n)}\text{Tr})$, where $\mathcal{G}$ is a $k(N + n) \times k(N + n)$ GUE matrix with entries with variance 1. Now, note that

$$
\left[\sqrt{1 + c_N}\sigma\left(\frac{\mathcal{G}}{\sqrt{k(N+n)}} + \tilde{Q}_k\frac{\mathcal{G}}{\sqrt{k(N+n)}}\tilde{P}_k\right) + \tilde{A}_k + \tilde{A}_k^*\right]^2 + \epsilon\tilde{P}_k
$$

$$
= \left(\sigma\mathcal{G}_{kn \times kn} + \tilde{A}\right)(\sigma\mathcal{G}_{kn \times kn} + \tilde{A})^* + \epsilon I_{kn}
$$

where $\mathcal{G}_{kn \times kn}$ is the upper right $kn \times kN$ corner of $\mathcal{G}$. Thus, noticing that $\mu_{\tilde{A}\tilde{A}} = \mu_{A_N A_N}$, the lemma follows from [15].

**Proof of Theorem 5.2** Let $W$ be a $(n + N) \times (n + N)$ matrix as defined by (87) in Theorem 5.3. Note that, with the notations of Lemma 5.5, for any $\epsilon \geq 0$,

$$
\left(\sigma\frac{X_N}{N} + A_N\right)(\sigma\frac{X_N}{N} + A_N)^* + \epsilon I_n
$$

$$
= \left(\sigma\frac{X_N}{N} + A_N\right)^2 + \epsilon P
$$

$$
= \left(\sqrt{1 + c_N P}\frac{\sigma W}{\sqrt{N+n}} + \sqrt{1 + c_N Q}\frac{\sigma W}{\sqrt{N+n}} P + A + A^*\right)^2 + \epsilon P.
$$
Thus, for any $\epsilon \geq 0$,
\[
\text{spect} \left\{ \left( \sigma \frac{X}{\sqrt{N}} + A \right) \left( \sigma \frac{X}{\sqrt{N}} + A \right)^* + \epsilon I_n \right\} 
\subset \text{spect} \left\{ \left( \sqrt{1 + c_N} P \frac{\sigma W}{\sqrt{N + n}} Q + \sqrt{1 + c_N} Q \frac{\sigma W}{\sqrt{N + n}} P + A + A^* \right)^2 + \epsilon P \right\}.
\]

Let $[x, y]$ be such that there exists $\delta > 0$ such that for all large $N$, $|x - \delta; y + \delta| \subset \mathbb{R} \setminus \text{supp}(\mu_{\sigma, \mu_{AN}, \delta, c_N})$.

(i) Assume $x > 0$. Then, according to Lemma 5.5 with $\epsilon = 0$, there exists $\delta' > 0$ such that for all large $n$, $|x - \delta'; y + \delta'|$ is outside the support of the distribution of $(\sqrt{1 + c_N} \sigma p_N s q_N + \sqrt{1 + c_N} \sigma q_N s p_N + a_N + a_N^*)^2$. We readily deduce that almost surely for all large $N$, according to Theorem 5.3, there is no eigenvalue of $(\sqrt{1 + c_N} P \frac{\sigma W}{\sqrt{N + n}} Q + \sqrt{1 + c_N} Q \frac{\sigma W}{\sqrt{N + n}} P + A + A^*)^2 [x, y]$. Hence, by (92) with $\epsilon = 0$, almost surely for all large $N$, there is no eigenvalue of $M_N$ in $[x, y]$.

(ii) Assume $x = 0$ and $y > 0$. There exists $0 < \delta' < y$ such that $[0, 3\delta']$ is for all large $N$ outside the support of $\mu_{\sigma, \mu_{AN}, \delta, c_N}$. Hence, according to Lemma 5.5, $[\delta'/2, 3\delta']$ is outside the support of the distribution of $(\sqrt{1 + c_N} \sigma p_N s q_N + \sqrt{1 + c_N} \sigma q_N s p_N + a_N + a_N^*)^2 + \delta p_n$. Then, almost surely for all large $N$, according to Theorem 5.3, there is no eigenvalue of $(\sqrt{1 + c_N} P \frac{\sigma W}{\sqrt{N + n}} Q + \sqrt{1 + c_N} Q \frac{\sigma W}{\sqrt{N + n}} P + A + A^*)^2 + \delta' P$ in $[\delta', 2\delta']$ and thus, by (92), no eigenvalue of $(\sigma \frac{X}{\sqrt{N}} + A_N)(\sigma \frac{X}{\sqrt{N}} + A_N)^* + \delta' I_n$ in $[\delta', 2\delta']$. It readily follows that, almost surely for all large $N$, there is no eigenvalue of $(\sigma \frac{X}{\sqrt{N}} + A_N)(\sigma \frac{X}{\sqrt{N}} + A_N)^*$ in $[0, \delta']$. Since moreover, according to (i), almost surely for all large $N$, there is no eigenvalue of $(\sigma \frac{X}{\sqrt{N}} + A_N)(\sigma \frac{X}{\sqrt{N}} + A_N)^*$ in $[\delta', y]$, we can conclude that there is no eigenvalue of $M_N$ in $[x, y]$.

The proof of Theorem 5.2 is now complete.

We are now in a position to establish the following exact separation phenomenon.

**Theorem 5.6.** Let $M_n$ as in (53) with assumptions [1-4] of Theorem 5.2. Assume moreover that the empirical spectral measure $\mu_{AN, \delta}$ of $A_N A_N^*$ converges weakly to some probability measure $\nu$. Then for $N$ large enough,
\[
\omega_{\sigma, \nu, c}(x, y) = [\omega_{\sigma, \nu, c}(x); \omega_{\sigma, \nu, c}(y)] \subset \mathbb{R} \setminus \text{supp}(\mu_{AN, \delta}),
\]

where $\omega_{\sigma, \nu, c}$ is defined in (1). With the convention that $\lambda_0(M_N) = \lambda_0(A_N A_N^*) = +\infty$ and $\lambda_{n+1}(M_N) = \lambda_{n+1}(A_N A_N^*) = -\infty$, for $N$ large enough, let $i_N \in \{0, \ldots, n\}$ be such that
\[
\lambda_{i_N+1}(A_N A_N^*) < \omega_{\sigma, \nu, c}(x) \quad \text{and} \quad \lambda_{i_N}(A_N A_N^*) > \omega_{\sigma, \nu, c}(y).
\]
Then
\[ P[ \text{for all large } N, \lambda_{iN+1}(M_N) < x \text{ and } \lambda_{iN}(M_N) > y ] = 1. \] (95)

**Remark 5.7.** Since \( \mu_{\sigma, \mu AN A^*N, cN} \) converges weakly towards \( \mu_{\sigma, \nu, c} \), assumption 4. implies that \( 0 < \tau < \delta, [x - \tau; y + \tau] \subset \mathbb{R} \setminus \text{supp } \mu_{\sigma, \nu, c} \).

**Proof.** (93) is proved in Lemma 3.1 in [11].

- If \( \omega_{\sigma, \nu, c}(x) < 0 \), then \( i_N = n \) in (94) and moreover we have, for all large \( N \), \( \omega_{\sigma, \mu AN A^*N, cN}(x) < 0 \). According to Lemma 2.7 in [11], we can deduce that, for all large \( N \), \( [x, y] \) is on the left hand side of the support of \( \mu_{\sigma, \mu AN A^*N, cN} \), so that \( -\infty; y + \delta \) is on the left hand side of the support of \( \mu_{\sigma, \mu AN A^*N, cN} \).

- If \( \omega_{\sigma, \nu, c}(x) \geq 0 \), we first explain why it is sufficient to prove (95) for \( x \) such that \( \omega_{\sigma, \nu, c}(x) > 0 \). Indeed, assume for a while that (95) is true whenever \( \omega_{\sigma, \nu, c}(x) > 0 \). Let us consider any interval \( [x, y] \) satisfying condition 4. of Theorem 5.2 and such that \( \omega_{\sigma, \nu, c}(x) = 0 \); then \( i_N = n \) in (94). According to Proposition 1.2, \( \omega_{\sigma, \nu, c}(x + y) > 0 \) and then almost surely for all large \( N \), \( \lambda_n(M_N) > y \). Finally, sticking to the proof of Theorem 1.2 in [11] leads to (95) for \( x \) such that \( \omega_{\sigma, \nu, c}(x) > 0 \).

\( \square \)

**Appendix B**

We first recall some basic properties of the resolvent (see [22], [12]).

**Lemma 5.8.** For a \( N \times N \) Hermitian matrix \( M \), for any \( z \in \mathbb{C} \setminus \text{spect}(M) \), we denote by \( G(z) := (zI_N - M)^{-1} \) the resolvent of \( M \).

Let \( z \in \mathbb{C} \setminus \mathbb{R} \),

(i) \( \|G(z)\| \leq |3z|^{-1} \).

(ii) \( |G(z)_{ij}| \leq |3z|^{-1} \) for all \( i, j = 1, \ldots, N \).

(iii) \( G(z)M = MG(z) = -I_N + zG(z) \).

Moreover, for any \( N \times N \) Hermitian matrices \( M_1 \) and \( M_2 \),

\( (zI_N - M_1)^{-1} - (zI_N - M_2)^{-1} = (zI_N - M_1)^{-1}(M_1 - M_2)(zI_N - M_2)^{-1} \).

The following technical lemmas are fundamental in the approach of the present paper.
Lemma 5.9. [Lemma 4.4 in [5]] Let \( h : \mathbb{R} \to \mathbb{R} \) be a continuous function with compact support. Let \( B_N \) be a \( N \times N \) Hermitian matrix and \( C_N \) be a \( N \times N \) matrix. Then

\[
\text{Tr} [h(B_N)C_N] = -\lim_{y \to 0^+} \frac{1}{\pi} \int \Im \text{Tr} [(t + iy - B_N)^{-1}C_N] h(t) dt.
\]

(96)

Moreover, if \( B_N \) is random, we also have

\[
\mathbb{E} \text{Tr} [h(B_N)C_N] = -\lim_{y \to 0^+} \frac{1}{\pi} \int \Im \mathbb{E} \text{Tr} [(t + iy - B_N)^{-1}C_N] h(t) dt.
\]

(97)

Lemma 5.10. Let \( f \) be an analytic function on \( \mathbb{C} \setminus \mathbb{R} \) such that there exist some polynomial \( P \) with nonnegative coefficients, and some positive real number \( \alpha \) such that

\[
\forall z \in \mathbb{C} \setminus \mathbb{R}, \quad |f(z)| \leq (|z| + 1)^\alpha P(|\Im z|^{-1}).
\]

Then, for any \( h \) in \( \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}) \) with compact support, there exists some constant \( \tau \) depending only on \( h, \alpha \) and \( P \) such that

\[
\limsup_{y \to 0^+} \left| \int \mathbb{R} h(x) f(x + iy) dx \right| < \tau.
\]

We refer the reader to the Appendix of [12] where it is proved using the ideas of [21].

Finally, we recall some facts on Poincaré inequality. A probability measure \( \mu \) on \( \mathbb{R} \) is said to satisfy the Poincaré inequality with constant \( C_{PI} \) if for any \( \mathcal{C}^1 \) function \( f : \mathbb{R} \to \mathbb{C} \) such that \( f \) and its gradient \( \text{grad} f \) are in \( L^2(\mu) \),

\[
\mathbb{V}(f) \leq C_{PI} \int |f|^2 d\mu,
\]

with \( \mathbb{V}(f) = \int |f - \int f d\mu|^2 d\mu \).

We refer the reader to [9] for a characterization of the measures on \( \mathbb{R} \) which satisfy a Poincaré inequality.

If the law of a random variable \( X \) satisfies the Poincaré inequality with constant \( C_{PI} \) then, for any fixed \( \alpha \neq 0 \), the law of \( \alpha X \) satisfies the Poincaré inequality with constant \( \alpha^2 C_{PI} \).

Assume that probability measures \( \mu_1, \ldots, \mu_M \) on \( \mathbb{R} \) satisfy the Poincaré inequality with constant \( C_{PI} \) respectively. Then the product measure \( \mu_1 \otimes \cdots \otimes \mu_M \) on \( \mathbb{R}^M \) satisfies the Poincaré inequality with constant

\[
\max_{i \in \{1, \ldots, M\}} C_{PI}(i)
\]

in the sense that for any differentiable function \( f \) such that \( f \) and its gradient \( \text{grad} f \) are in \( L^2(\mu_1 \otimes \cdots \otimes \mu_M) \),

\[
\mathbb{V}(f) \leq C_{PI}^* \int \|	ext{grad} f\|^2 d\mu_1 \otimes \cdots \otimes d\mu_M
\]

with \( \mathbb{V}(f) = \int |f - \int f d\mu_1 \otimes \cdots \otimes d\mu_M|^2 d\mu_1 \otimes \cdots \otimes d\mu_M \) (see Theorem 2.5 in [18]).
Lemma 5.11. [Theorem 1.2 in [4]] Assume that the distribution of a random variable $X$ is supported in $[-C; C]$ for some constant $C > 0$. Let $g$ be an independent standard real Gaussian random variable. Then $X + \delta g$ satisfies a Poincaré inequality with constant $C_{PI} \leq \delta^2 \exp(4C^2/\delta^2)$.

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