K-THEORY OF n-COHERENT RINGS

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Abstract. Let $R$ be a strong $n$-coherent ring such that each finitely $n$-presented $R$-module has finite projective dimension. We consider $\mathcal{FP}_n(R)$ the full subcategory of $R$-Mod of finitely $n$-presented modules. We prove that $K_i(R) = K_i(\mathcal{FP}_n(R))$ for every $i \geq 0$ and obtain an expression of $\text{Nil}_i(R)$.

1. Introduction

Let $R$ be an associative ring with unit. We consider the following full subcategories of the category of left $R$-modules denoted by $R$-Mod:

$\mathcal{FP}_0(R) = \{ M \in R\text{-Mod} : M \text{ is finitely generated} \}$

$\mathcal{FP}_1(R) = \{ M \in R\text{-Mod} : M \text{ is finitely presented} \}$

$\mathcal{FP}_n(R) = \{ M \in R\text{-Mod} : M \text{ is finitely } n\text{-presented} \}$

$\mathcal{FP}_\infty(R) = \{ M \in R\text{-Mod} : M \text{ has a resolution by finitely generated free modules} \}$

$\text{Proj}(R) = \{ M \in R\text{-Mod} : M \text{ is finitely generated and projective} \}$.

Observe there is a chain of inclusions

$$\text{Proj}(R) \subseteq \mathcal{FP}_\infty(R) \subseteq \mathcal{FP}_n(R) \subseteq \ldots \subseteq \mathcal{FP}_1(R) \subseteq \mathcal{FP}_0(R) \subseteq R\text{-Mod}.$$ 

If $R$ is Noetherian then $\mathcal{FP}_0(R) = \mathcal{FP}_\infty(R)$ is an abelian category. If $R$ is coherent then $\mathcal{FP}_1(R) = \mathcal{FP}_\infty(R)$ is an abelian category. In this work we consider rings $R$ such that $\mathcal{FP}_n(R) = \mathcal{FP}_\infty(R)$ which are called strong $n$-coherent rings.

We prove in Theorem 2.5 that $\mathcal{FP}_n(R)$ also coincides with the category of $n$-coherent modules. We say that $R$ is $n$-regular if the projective dimension of each finitely $n$-presented module is finite. We investigate which techniques used in [21] for regular and coherent rings can be applied to $n$-regular and strong $n$-coherent rings. In this case we prove in Theorem 3.2 such that

$$K_i(R) \simeq K_i(\mathcal{FP}_n(R)) \quad \forall i \geq 0.$$ 

The main tool for this result is the Quillen’s Resolution Theorem given in [18] applied to $\text{Proj}(R) \hookrightarrow \mathcal{FP}_n(R)$.

Theorem 1.1 (Resolution Theorem). [8 Th 2.1] Let $\mathcal{M}$ be an exact category and let $\mathcal{P}$ be a full subcategory closed under extension in $\mathcal{M}$. Suppose in addition that:

1. If $0 \to M \to P \to P' \to 0$ is exact in $\mathcal{M}$ with $P, P'$ in $\mathcal{P}$, then $M$ is in $\mathcal{P}$.
2. For every $M$ in $\mathcal{M}$ there is a finite $\mathcal{P}$-resolution of $M$

$$0 \to P_k \to \ldots \to P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0$$

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Then the inclusion $\mathcal{P} \hookrightarrow \mathcal{M}$ induces isomorphisms

$$K_m(\mathcal{P}) \cong K_m(\mathcal{M})$$

for all $m \geq 0$.

We can also apply this theorem to the inclusion $\text{Nil}((\text{Proj}(R)) \hookrightarrow \text{Nil}(\mathcal{F}\mathcal{P}_n(R))$ and we obtain the Proposition 4.2 which give us an expression of $\text{Nil}_i(R)$. If $n = 1$ Swan in [21] prove that this expression is cero, then

$$K_i(R[t]) = K_i(R) \quad \text{and} \quad K_i(R[t, t^{-1}]) = K_i(R) \oplus K_{i-1}(R) \quad \text{for all} \quad i \geq 1.$$

This can be done applying Devissage Theorem to $\mathcal{F}\mathcal{P}_1(R) \hookrightarrow \text{Nil}(\mathcal{F}\mathcal{P}_1(R))$ because $\mathcal{F}\mathcal{P}_1(R)$ is abelian when $R$ is coherent. Let us recall the Quillen’s Devissage Theorem:

**Theorem 1.2** (Devissage Theorem). [8, Th 2.1] If $\mathcal{A}$ is an abelian category and $\mathcal{B}$ is a non-empty full subcategory closed under subobjects, quotient objects, and finite products in $\mathcal{A}$ and if every object $A$ in $\mathcal{A}$ has a finite filtration

$$0 = A_0 \subset A_1 \subset \ldots \subset A_k = A$$

with $A_i/A_{i-1}$ in $\mathcal{B}$ for each $i$, then the inclusion $\mathcal{B} \hookrightarrow \mathcal{A}$ induces isomorphisms

$$K_m(\mathcal{B}) \cong K_m(\mathcal{A})$$

for all $m \geq 0$.

Can we apply Devissage Theorem if $n > 1$? We can do it if $\mathcal{F}\mathcal{P}_n(R)$ is abelian but in Proposition 5.1 we prove this only happen when $R$ is coherent. The property which arise in $\mathcal{F}\mathcal{P}_n(R)$ when $R$ is strong $n$-coherent is thickness, $R$ is strong $n$-coherent if and only if $\mathcal{F}\mathcal{P}_n(R)$ is a thick category.

In Section 2 we state the definition of strong $n$-coherent and $n$-regular ring. We compare this regularity with the $n$-Von Neumann regularity. We show that $\text{Proj}(R) = \mathcal{F}\mathcal{P}_n(R)$ when $R$ is $n$-Von Neumann regular but this do not happen necessarily when $R$ is strong $n$-coherent and $n$-regular. In Section 3 we prove that $\text{Proj}(R) \hookrightarrow \mathcal{F}\mathcal{P}_n(R)$ induces isomorphisms $K_i(R) \cong K_i(\mathcal{F}\mathcal{P}_n(R))$ for all $i \geq 0$ when $R$ is strong $n$-coherent and $n$-regular. In Section 4 we generalize [21] Lemma 6.3 in order to apply Resolution Theorem to $\text{Nil}(\text{Proj}(R)) \hookrightarrow \text{Nil}(\mathcal{F}\mathcal{P}_n(R))$. This shows that $\text{Nil}_i(R)$ coincides with $\text{Nil}_n^\ast(R)$ which is the cokernel of the map

$$K_i(\mathcal{F}\mathcal{P}_n(R)) \to K_i(\text{Nil}(\mathcal{F}\mathcal{P}_n(R)))$$

induced by inclusion. In Section 5 we prove that we can use Devissage Theorem to prove $\text{Nil}_n^\ast(R) = 0$ if and only if $R$ is coherent, because $\mathcal{F}\mathcal{P}_n(R)$ is abelian only when $R$ is coherent. In the last section we present an application of Theorem 6.2 in the particular cases when $R$ is an arithmetic or a valuation ring.

The projective dimension of an $R$-module $M$ is denoted by $pd(M)$, the weak dimension of $M$ is denoted by $wd(M)$ and $w.dim(R)$ is the weak dimension of $R$. The expression $R$-module means left $R$-module.

2. $n$-COHERENT MODULES AND RINGS

Let $n \geq 0$ be a integer. Recall that an $R$-module $M$ is finitely $n$-presented if there exists a sequence

$$F_n \to F_{n-1} \to F_{n-2} \to \ldots \to F_0 \to M \to 0$$
where $F_i$ is a finitely generated and free (or projective) module, for every $0 \leq i \leq n$. Following [4], we denote by $FP_n(R)$ to the full subcategory of finitely $n$-presented modules. Note that $FP_0(R)$ is the category of finitely generated modules and $FP_1(R)$ is the category of finitely presented modules. Consider the $\lambda$-dimension of $M$

$$\lambda_R(M) = \sup\{n \geq 0 : M \text{ is finitely } n\text{-presented}\}.$$ 

If $M$ is not finitely generated we set $\lambda_R(M) = -1$. The category formed by modules that posses a resolution by finitely generated free (or projective) modules is $FP_\infty(R)$. We say that $M$ is \textit{finitely $\infty$-presented} if $M \in FP_\infty(R)$. We immediately observe the following chain of inclusions:

$$FP_\infty(R) = \bigcap_{n \geq 0} FP_n(R) \subseteq \ldots \subseteq FP_{n+1}(R) \subseteq FP_n(R) \subseteq \ldots FP_1(R) \subseteq FP_0(R).$$

The finitely $\infty$-presented modules can be found in the literature as pseudo coherent modules, see [24] Example 7.1.4, Ch. 2, [7]. The concept of $\lambda$-dimension and the finitely $n$-presented modules are related as follows.

**Lemma 2.1.** [4] Remark 1.5] For every $n \geq 0$.

1. $M \in FP_n(R)$ if and only if $\lambda_R(M) \geq n$.
2. $M \in FP_n(R) \setminus FP_{n+1}(R)$ if and only if $\lambda_R(M) = n$.
3. $M \in FP_\infty(R)$ if and only if $\lambda_R(M) = \infty$.

The $\lambda$-dimension of a ring $R$, denoted by $\lambda(R)$, was formulated by Vasconcelos in [22] in order to study the power series and polynomial rings. The $\lambda$-dimension of $R$ is the greatest integer $n$ (or $\infty$ if none such exits) such that $\lambda_R(M) \geq n$ implies $\lambda_R(M) = \infty$. Recall that $R$ is Noetherian ring if and only if $\lambda(R) = 0$ and $R$ is coherent if and only if $\lambda(R) \leq 1$.

As in [7] we extend the concept of coherent module:

**Definition 2.2** (n-coherent module). Let $n \geq 0$ be an integer. An $R$-module $M$ is called $n$-coherent if:

- $M$ belongs to the category $FP_n(R)$.
- For each submodule $N \rightarrowtail M$ such that $N$ is $(n-1)$-presented then $N$ is also $n$-presented.

Denote by $n$-Coh($R$) to the collection of all $n$-coherent $R$-modules. Note that if $n=1$ the definition is the same as coherent module.

**Remark 2.3.** A ring $R$ is $n$-coherent if it is $n$-coherent as a $R$-module with the regular action, (i.e. if each $(n-1)$-presented ideal of $R$ is $n$-presented). We say that $R$ is \textit{strong} $n$-coherent if each $n$-presented $R$-module is $(n+1)$-presented. The idea of $n$-coherent rings has been stated in the literature by D. L. Costa [5] but this terminology is not the same as in [7]. A strong $n$-coherent ring is equivalent to a $n$-coherent ring for $n=1$, but it is an open question for $n \geq 2$. A coherent ring is a 1-coherent ring (1-strong coherent ring) and 0-strong coherent ring is a Noetherian ring.

In our definition, $R$ is a strong $n$-coherent ring if and only if $\lambda(R) \leq n$. Denote by $n$-SCoh the class of all strong $n$-coherent rings. We observe the following chain of inclusions:

$$0$-SCoh $\subset$ $1$-SCoh $\subset$ $2$-SCoh $\subset \ldots \subset n$-SCoh $\subset \ldots$$
Example 2.4. [4] Example 1.3] Let $k$ be a field. Consider the following ring
\[ R = \frac{k[x_1, x_2, x_3, \ldots]}{(x_i x_j)_{i,j \geq 1}}. \]

In [4] it is shown that $FP_2(R) = FP_\infty(R)$, $(x_1) \in FP_0(R) \setminus FP_1(R)$ and $R/(x_1) \in FP_1(R) \setminus FP_2(R)$. We obtain strict inclusions
\[ FP_\infty(R) = FP_2(R) \subset FP_1(R) \subset FP_0(R) \]
which show that $R$ is strong 2-coherent but it is not coherent.

It is well known that a ring is a coherent ring if and only if the category of coherent modules ($1$-$\text{Coh}(R)$) coincides with the category of finitely presented modules ($FP_1(R)$). Let $n \geq 1$ and consider $n$-$\text{Coh}(R)$ the full subcategory of $n$-coherent modules.

**Theorem 2.5.** Let $n \geq 1$. If $R$ is a strong $n$-coherent ring then
\[ FP_n(R) = n$-$\text{Coh}(R). \]

**Proof.** Suppose $M \in FP_n(R)$. Then from [4] Proposition 1.1 we have a short exact sequence with $K \in FP_{n-1}(R)$.
\[ 0 \to K \to R^k \to M \to 0 \]
From our hypothesis we get $R^k$ is a $n$-coherent module by [7] Remark 3.5 and therefore $M \in n$-$\text{Coh}(R)$ by [7] Theorem 2.3. \qed

**Corollary 2.6.** [21 Corollary 2.7] If $R$ is a coherent ring then $FP_1(R) = 1$-$\text{Coh}(R)$.

The category of $FP_\infty(R)$ is thick, see [3] Theorem 1.8. That means it is closed under taking direct summands and for every short exact sequence
\[ 0 \to A \to B \to C \to 0 \]
with two of the three terms $A$, $B$, $C$ in $FP_\infty(R)$ so is the third. Because of [4] Theorem 2.4, $R$ is strong $n$-coherent if and only if $FP_n(R) = FP_\infty(R)$.

**Proposition 2.7.** [4 Corollary 2.6] Let $n \geq 1$. $R$ is a strong $n$-coherent ring if, and only if, the category $FP_n(R)$ is closed under taking kernels of epimorphisms.

**Corollary 2.8.** Let $n \geq 1$. If $R$ is a strong $n$-coherent ring then the category of $n$-coherent modules in $R$-$\text{Mod}$ is closed under direct summands, extensions and taking cokernels of monomorphisms and kernels of epimorphisms.

A result of Gersten [8] Theorem 2.3] shows that if $R$ is a regular coherent ring then the inclusion of the finitely generated projective modules in the category of $FP_1(R)$ induces isomorphisms $K_i(FP_1(R)) \simeq K_i(R)$ for all $i \geq 0$. Recall that a ring is said to be regular if every finitely generated ideal of $R$ has finite projective dimension. If $R$ is coherent, Quentel in [17] shows that $R$ is regular if and only if every finitely presented module has finite projective dimension, see [9] Theorem 6.2.1. Motivated by this fact, we introduce the following definition:

**Definition 2.9 (n-regular ring).** Let $n \geq 1$ be an integer. A ring $R$ is called $n$-regular if each finitely $n$-presented $R$-module has finite projective dimension.
A 1-regular ring is a regular ring. In the next section we extend the mentioned result of Gersten to strong $n$-coherent rings which are $n$-regular, see Theorem 3.2. The idea of strong $n$-coherent and $n$-regularity has been studied before for commutative rings. Following [5], let $n, d \geq 0$ recall that a ring is said to be $(n, d)$-Ring if every $n$-presented module has projective dimension at most $d$. This rings are strong $\text{sup} \{n, d\}$-coherent. In [5, Example 6.5] there is an example of a non-coherent ring $R$ of weak dimension one such that is a $(2, 1)$-Ring. In the commutative case, a ring is called $n$-Von Neumann regular if it is an $(n, 0)$-Ring. Thus, 1-Von Neumann regular rings are the Von Neumann regular rings, see [5, Theorem 1.3]. This notion of regularity do not coincides with Definition 2.9. Both concept are related as follows.

**Proposition 2.10.** [16, Theorem 2.1] A commutative ring $R$ is $n$-Von Neumann regular if and only if $R$ is a strong $n$-coherent and $n$-regular ring such that every finitely generated proper ideal of $R$ has non-zero annihilator.

The rings we are interested in are those with the first two conditions of Proposition 2.10. A ring $R$ such that every finitely generated proper ideal of $R$ has a non-zero annihilator is called CH-Ring. Another equivalent definition is as follows: every finitely generated submodule of a projective $R$-module $P$ is a direct summand of $P$ [2, Theorem 5.4].

Let $n \geq 1$. By [24, Theorem 3.9], every $n$-presented $R$-module is flat if and only if $R$ is $n$-Von Neumann regular. In particular every module of $FP_n(R)$ is projective. Then if $R$ is $n$-Von Neumann regular we obtain

$$\text{Proj}(R) = FP_n(R).$$

Under the hypothesis of $n$-Von-Neumann regularity there is no sense to study the relation between the K-theory of $R$ and $FP_n(R)$.

Let $k$ be a field and $E$ be a $k$-vector space with infinite rank. Set $B = k \times E$ the trivial extension of $k$ by $E$. Let $A$ be a Noetherian ring of global dimension 1, and $R = A \times B$ the direct product of $A$ and $B$. By [16, Theorem 3.4] $R$ is a $(2, 1)$-ring and is not a 2-Von Neumann regular ring. We obtain an example of a strong 2-coherent and 2-regular ring where

$$\text{Proj}(R) \neq FP_2(R).$$

3. K-theory of $FP_n(R)$

In this section we show $FP_n(R)$ is an exact category in the sense of [19, Definition 3.1.1]. We also prove that if $R$ is a strong $n$-coherent and $n$-regular ring then

$$K_i(FP_n(R)) \simeq K_i(R) \quad \forall i \geq 0.$$ 

**Proposition 3.1.** $FP_n(R)$ is an exact category.

**Proof.** The category $FP_n(R)$ is a full additive subcategory of $R$-Mod which is an abelian category. We have to prove that $FP_n(R)$ is closed under extension and it has a small skeleton. The first assertion follows from [4, Proposition 1.7]. The category of finitely generated $R$-modules $FP_0(R)$ has a small skeleton $S_0$ the set of quotient modules of

$$\{R^k : k \in \mathbb{N}\}.$$ 

The set $S_n = FP_n(R) \cap S_0$ is an skeleton of $FP_n(R)$. \qed
\textbf{Theorem 3.2.} If $R$ is a strong $n$-coherent and $n$-regular ring then
$$K_i(R) \simeq K_i(\text{n-Coh}(R)) \simeq K_i(\mathcal{FP}_n(R)) \quad \forall i \geq 0.$$  

\textit{Proof.} We are going to use the Resolution Theorem with $\mathcal{M} = \mathcal{FP}_n(R)$ and $\mathcal{P} = \text{Proj}(R)$. Note that $\text{Proj}(R) \subseteq \mathcal{FP}_\infty(R) \subseteq \mathcal{FP}_n(R)$. Consider the following exact sequence
$$0 \to P_1 \to M \to P_2 \to 0 \quad \text{with } P_1, P_2 \in \text{Proj}(R) \text{ and } M \in \mathcal{FP}_n(R).$$

Because $P_2$ is projective we obtain $M \simeq P_1 \oplus P_2$ then $M$ belongs to $\text{Proj}(R)$. We conclude that $\text{Proj}(R)$ is closed under extension in $\mathcal{FP}_n(R)$.

Let $0 \to M \to P_1 \to P_2 \to 0 \quad \text{with } P_1, P_2 \in \text{Proj}(R) \text{ and } M \in \mathcal{FP}_n(R).$

we have $P_1 \simeq M \oplus P_2$ and $M$ is also projective because $P_1$ is projective. As $\mathcal{FP}_\infty(R)$ is thick then $M$ belongs to $\mathcal{FP}_\infty(R)$ and in particular is finitely generated. We conclude $M \in \text{Proj}(R)$.

Let $M \in \mathcal{FP}_n(R)$ then there exists a resolution of free modules
$$F_n \xrightarrow{u_n} F_{n-1} \to \ldots \to F_1 \to F_0 \to M \to 0$$

As $R$ is strong $n$-coherent we can take a free resolution as long as we want
$$\ldots \to F_{n+1} \to F_n \to F_{n-1} \to \ldots \to F_1 \to F_0 \to M \to 0$$

By hypothesis we know $M$ has a finite projective dimension, then there exists $d$
$$0 \to \ker u_d \to F_d \xrightarrow{u_d} \ldots \to F_1 \to F_0 \to M \to 0$$

with $\ker u_d$ projective. As
$$\ker u_d = \text{Img} u_{d+1} \simeq F_{d+1}/\ker u_{d+1}$$

we obtain $\ker u_d$ is finitely generated. \hfill \Box

\textbf{Corollary 3.3.} \cite{2} Theorem 2.3] If $R$ is a coherent and regular ring then
$$K_i(R) \simeq K_i(\mathcal{FP}_1(R)) \quad \forall i \geq 0.$$  

4. Nil($\mathcal{FP}_n$)

Let $\mathcal{C}$ be an exact category, $\text{Nil}(\mathcal{C})$ is the category whose objects are the pairs $(A, \alpha)$ where $A \in \mathcal{C}$ and $\alpha : A \to A$ is a nilpotent endomorphism. A morphism $(A, \alpha) \to (B, \beta)$ in $\text{Nil}(\mathcal{C})$ is a morphism $f : A \to B$ in $\mathcal{C}$ such that $f \circ \alpha = \beta \circ f$. The category $\text{Nil}(\mathcal{C})$ is an exact category and there exist exact functors
$$\mathcal{C} \to \text{Nil}(\mathcal{C}) \quad A \mapsto (A, 0) \qquad \text{Nil}(\mathcal{C}) \to \mathcal{C} \quad (A, \alpha) \mapsto A.$$  

Taking $\mathcal{C} = \text{Proj}(R)$ and $\mathcal{N} = \text{Nil}(\text{Proj}(R))$, note that $K_i(\mathcal{N})$ is a direct summand of $K_i(\mathcal{N})$ and define $\text{Nil}_i(R)$ the cokernel of $K_i(R) \to K_i(\mathcal{N})$. We obtain
$$K_i(\mathcal{N}) = K_i(R) \oplus \text{Nil}_i(R) \quad i \geq 0.$$  

We consider $\mathcal{C} = \mathcal{FP}_n(R)$. Let $\mathcal{N}_n^i = \text{Nil}(\mathcal{FP}_n(R))$ and $\text{Nil}_i^n(R)$ the cokernel of $K_i(\mathcal{FP}_n(R)) \to K_i(\mathcal{N}_n^i)$. We obtain
$$K_i(\mathcal{N}_n^i) = K_i(\mathcal{FP}_n(R)) \oplus \text{Nil}_i^n(R) \quad i \geq 0.$$
Lemma 4.1.  (1) [21] Lemma 6.3 For every $(M, \alpha) \in \mathcal{N}^*_1$ there exist a $(P, \beta) \in \mathcal{N}$ and a epimorphism $\varphi : P \to M$ such that the following diagram is commutative

\[
\begin{array}{ccc}
P & \xrightarrow{\varphi} & M \\
\downarrow{\beta} & & \downarrow{\alpha} \\
P & \xrightarrow{\varphi} & M \\
\end{array}
\]

(2) If $R$ is coherent and regular then every $(M, \alpha) \in \mathcal{N}^*_1$ has a finite $\mathcal{N}$-resolution.

(3) If $R$ is strong $n$-coherent and $n$-regular then every $(M, \alpha) \in \mathcal{N}^*_n$ has a finite $\mathcal{N}$-resolution.

Proof.  (1) Consider $n \in \mathbb{N}$ such that $\alpha^{n+1} = 0$. Let $Q$ be a projective and finitely generated module with $f : Q \to M$. Let $P = Q^{n+1}$ and define

$\beta : P \to P \quad \beta(x_0, x_1, \ldots, x_n) = (0, x_0, \ldots, x_{n-1})$

$\varphi : P \to M \quad \varphi(x_0, x_1, \ldots, x_n) = f(x_0) + \alpha(f(x_1)) + \alpha^2(f(x_2)) + \ldots + \alpha^n(f(x_n)).$

(2) This case is a particular case of (3) with $n = 1$.

(3) Given $(M, \alpha) \in \mathcal{N}^*_n \subseteq \mathcal{N}^*_1$, by (1) there exist $(P_0, \beta_0) \in \mathcal{N}$ and a epimorphism in $\mathcal{N}^*_n$ such that $\varphi_0 : (P_0, \beta_0) \to (M, \alpha)$. Because $R$ is strong $n$-coherent then $\mathcal{FP}_n(R)$ is closed by taking kernel of epimorphisms. Then $(\ker(\varphi_0), \beta_0|_{\ker(\varphi_0)})$ belongs to $\mathcal{N}^*_n \subseteq \mathcal{N}^*_1$ and we can apply (1) again. There exist $(P_1, \beta_1) \in \mathcal{N}$ and an epimorphism in $\mathcal{N}^*_n$ such that $\varphi_1 : (P_1, \beta_1) \to (\ker(\varphi_0), \beta_0|_{\ker(\varphi_0)})$. By this way we construct a resolution

$\ldots (P_1, \beta_1) \xrightarrow{\varphi_1} (P_{i-1}, \beta_{i-1}) \xrightarrow{\varphi_{i-1}} \ldots (P_0, \beta_0) \xrightarrow{\varphi_0} (M, \alpha) \to 0$

As $R$ is $n$-regular, every $M \in \mathcal{FP}_n(R)$ has finite projective dimension, then there exist $k \in \mathbb{N}$ such that $P_k = 0$. We obtain a finite $\mathcal{N}$-resolution of $(M, \alpha)$.

Proposition 4.2. If $R$ is a strong $n$-coherent and $n$-regular ring then

$\text{Nil}^n_n(R) \simeq \text{Nil}_n(R)$.

Proof. We can regard $\mathcal{N}^*_n$ as the full subcategory of $R[t]$-modules consisting of modules $M$ which are $n$-coherent over $R$ and are such that the action of $t$ is a nilpotent endomorphism

$t : M \to M \quad m \mapsto t \cdot m.$

We obtain $\mathcal{N} \subseteq \mathcal{N}^*_n \subseteq R[t]$-Mod. The category $\mathcal{N}^*_n$ is exact and $\mathcal{N}$ is a full subcategory of $\mathcal{N}^*_n$ which is closed under kernels of epimorphisms and extensions. As $R$ is strong $n$-coherent and $n$-regular we can use the Lemma [11] [9] to obtain that every $M \in \mathcal{N}^*_n$ has a finite $\mathcal{N}$-resolution. Because of the Resolution Theorem we obtain that the inclusion $\mathcal{N} \hookrightarrow \mathcal{N}^*_n$ induces isomorphisms

$K_i(\mathcal{N}) \simeq K_i(\mathcal{N}^*_n)$.

In Theorem 5.2 we prove that the inclusion $\text{Proj}(R) \hookrightarrow \mathcal{FP}_n(R)$ induces isomorphisms

$K_i(\text{Proj}(R)) \simeq K_i(\mathcal{FP}_n(R)).$
Then we have the following diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & \text{Nil}_i(R) & \longrightarrow & K_i(N) & \longrightarrow & K_i(\text{Proj}(R)) & \longrightarrow & 0 \\
& & \downarrow & \simeq & \downarrow \simeq & & & \\
0 & \longrightarrow & \text{Nil}_i^\varnothing(R) & \longrightarrow & K_i(N_i^\varnothing) & \longrightarrow & K_i(\mathcal{F}\mathcal{P}_n(R)) & \longrightarrow & 0
\end{array}
\]

Because of the 5-Lemma applied to the previous diagram we obtain

\[\text{Nil}_i^\varnothing(R) \simeq \text{Nil}_i(R).\]

\[\square\]

**Remark 4.3.** The previous result with \(n = 1\) is \([21\text{ Proposition 6.2}].\)

5. **Categorical properties of** \(\mathcal{F}\mathcal{P}_n(R)\)

The Fundamental Theorem in K-theory relates the K-theory of \(R[t]\) and \(R[t, t^{-1}]\) with the K-theory of a ring \(R\). Let \(R\) be a ring then if \(i \geq 1\)

\[K_i(R[t]) \simeq K_i(R) \oplus \text{Nil}_{i-1}(R)\]

\[K_i(R[t, t^{-1}]) \simeq K_i(R) \oplus K_{i-1}(R) \oplus \text{Nil}_{i-1}(R) \oplus \text{Nil}_{i-1}(R).

We want to know when \(\text{Nil}_i(R) = 0\) in order to obtain an expression which relates the K-groups of \(R[t, t^{-1}]\) or \(R[t]\) only with the K-groups of \(R\).

If \(R\) is coherent it is proved in \([21\text{ Lemma 6.4}].\) that \(\text{Nil}_1^\varnothing(R) = 0\) using Devissage Theorem\([1,2].\) We are going to check that \(\mathcal{F}\mathcal{P}_1(R) \hookrightarrow N_1^\varnothing\) satisfies the hypothesis. The category \(N_1^\varnothing\) is abelian because \(\mathcal{F}\mathcal{P}_1(R)\) is abelian when \(R\) is coherent. We also have \(\mathcal{F}\mathcal{P}_1(R)\) is closed under subobjects, quotient object and finite product in \(N_1^\varnothing\). For each object \([M, \alpha] \in N_1^\varnothing\) we can filter \([M, \alpha]\) by \([M_i, \alpha|_{M_i}],\) where \(M_i = \alpha^{-i}(M)\) and \(n\) the lowest natural number such that \(\alpha^n(M) = 0.\) The quotient \([M_i, \alpha_i]/[M_{i-1}, \alpha_i-1] = [M_i/M_{i-1}, \overline{\alpha_i}],\) As \(\alpha(M_i) = M_{i-1}\) then \(\overline{\alpha} = 0.\) For this reason \([M_i, \alpha_i]/[M_{i-1}, \alpha_i-1] \in \mathcal{F}\mathcal{P}_1(R).\) If \(R\) is a regular coherent ring, by Proposition\([1,2]\) we have \(\text{Nil}_i(R) = \text{Nil}_i^\varnothing(R),\) as we seen above \(\text{Nil}_1^\varnothing(R) = 0\) and then \(\text{Nil}_i(R) = 0.\) It means that \(K_i(R[t]) = K_i(R)\) and \(K_i(R[t, t^{-1}]) = K_i(R) \oplus K_{i-1}(R)\) for all \(i \geq 1.\)

If \(R\) is a strong \(n\)-coherent and \(n\)-regular ring then by Proposition\([1,2]\) we have \(\text{Nil}_i^\varnothing(R) \simeq \text{Nil}_i(R).\) We can not follow the steps of the case \(n = 1\) because \(\mathcal{F}\mathcal{P}_n(R)\) is not necessarily abelian. We prove in Proposition\([5,1]\) that this only happen when \(R\) is coherent. Although \(\mathcal{F}\mathcal{P}_n(R)\) is not abelian if \(R\) is a non-coherent and strong \(n\)-coherent ring with \(n \geq 2,\) we have that \(\mathcal{F}\mathcal{P}_n(R)\) is a thick category. We expect that \(\text{Nil}_i^\varnothing(R)\) are easier to handle than \(\text{Nil}_i(R).\)

Recall from \([12]\) that a full subcategory \(\mathcal{C}\) of \(R\)-Mod is \(\text{wide}\) if it is abelian and closed under extensions. Let \(\mathcal{C}_0\) the wide subcategory generated by \(R.\) Observe \(\mathcal{C}_0\) contains all finitely presented modules then \(\mathcal{F}\mathcal{P}_1(R) \subseteq \mathcal{C}_0.\) By \([12\text{ Lema 1.6}].\) a ring \(R\) is coherent if and only if \(\mathcal{C}_0 = \mathcal{F}\mathcal{P}_1(R).\)

**Proposition 5.1.** Let \(n \geq 1.\) The category \(\mathcal{F}\mathcal{P}_n(R)\) is abelian if, and only if \(R\) is a coherent ring.

**Proof.** If \(\mathcal{F}\mathcal{P}_n(R)\) is abelian then \(\mathcal{C}_0 \subseteq \mathcal{F}\mathcal{P}_n(R) \subseteq \mathcal{F}\mathcal{P}_1(R).\) We obtain \(\mathcal{C}_0 = \mathcal{F}\mathcal{P}_1(R)\) then \(R\) is coherent. Reciprocally, if \(R\) is coherent then \(\mathcal{F}\mathcal{P}_1(R)\) is an abelian category. By \([21\text{ Theorem 2.4}].\) we obtain that \(\mathcal{F}\mathcal{P}_1(R) = \mathcal{F}\mathcal{P}_\infty(R) = \mathcal{F}\mathcal{P}_n(R)\) therefore \(\mathcal{F}\mathcal{P}_n(R)\) is an abelian category. \(\square\)
6. Arithmetic and valuation rings.

In this section R is a commutative ring. For coherent rings, the regularity condition is related to the weak dimension of R-modules. Examples of regular coherent rings includes Von Neumann regular and semihereditary rings. Coherent rings with finite weak dimension are regular, however the converse is not necessarily true.

Let us discuss the properties of R depending only on its weak dimension. If \( w.\dim(R) = 0 \) then R is Von Neumann regular which also is coherent. If \( w.\dim(R) = 1 \) the coherence is not guaranteed, see [14] Example 2.3, but in this case R is coherent if and only if \( R \) is semihereditary (see [10] Proposition 2.2). Finally \( w.\dim(R) \leq 1 \) if and only if R is an arithmetic reduced ring.

Recall a ring R is arithmetic if the ideals of \( R_M \) are totally order by inclusion for all maximal ideals M of R. It is proved in [6] Theorem 2.1] that arithmetic rings are strong 3-coherent and that reduced arithmetic rings are strong 2-coherent.

Arithmetical condition is not equivalent to strong n-coherence. The Example 2.4 is strong 2-coherent but is not arithmetical neither coherent, see [1] Example 3.13.

**Proposition 6.1.** Let R be an arithmetic ring.

1. The ring R is coherent if and only if the annihilator of every element is finitely generated. In particular, if \( w.\dim(R) < \infty \) then R is a supercoherent regular reduced ring.
2. If R is 3-regular:
   \[
   K_i(R) \simeq K_i(\text{3-Coh}(R)) \simeq K_i(\text{FP}_3(R))
   \]
   \( \forall i \geq 0 \).
3. If R is a reduced ring then:
   \[
   K_i(R) \simeq K_i(\text{2-Coh}(R)) \simeq K_i(\text{FP}_2(R))
   \]
   \( \forall i \geq 0 \).
4. If R has a Krull dimension 0 and is 2-regular then:
   \[ [\text{Spec}(R), \mathbb{Z}] = K_0(R) \simeq K_0(\text{2-Coh}(R)) \simeq K_0(\text{FP}_2(R)). \]

**Proof.**

1. By [15] 1.4 Fact, Ch XII, §Arithmetic Rings] an arithmetic ring is coherent if and only if the annihilator of every element is finitely generated. An arithmetical coherent ring with \( w.\dim(R) < \infty \) is a semihereditary ring.
2. It follows from Theorem 5.2 and [6] Theorem 2.1.
3. If R is an arithmetic reduced ring then \( w.\dim(R) \leq 1 \) (see [11] Theorem 4.2, [13] and \( \text{pd}(M) \leq 1 \) for all \( M \in \text{FP}_2(R) \) [3], Lemma 8].
4. By Pierce’s Theorem, [23] Theorem 2.2.2, \([\text{Spec}(R), \mathbb{Z}] = K_0(R) \) for every R with Krull dimension 0. By [6] Corollary 2.7] R is a strong 2-coherent ring.

Recall R is arithmetic if R is locally a valuation ring. A ring R is a valuation ring if the set of ideals of R is totally order by inclusion.

**Proposition 6.2.** Let R be a valuation ring, M its maximal ideal and Z the subset of its zero divisors.

1. If \( Z = 0 \) then R is a coherent ring.
2. If \( Z \neq 0 \), \( Z \neq M \) and R is 2-regular then:
   \[
   K_i(R) \simeq K_i(\text{2-Coh}(R)) \simeq K_i(\text{FP}_2(R))
   \]
   \( \forall i \geq 0 \).

   \( \text{FP}_\infty(R) = \text{FP}_2(R) \subseteq \text{FP}_1(R) \).
3. If $Z \neq 0$, $Z = M$ and $R$ is $2$-regular then:

$$K_i(R) \cong K_i(2 \text{Coh}(R)) \cong K_i(FP_2(R))$$

$\forall i \geq 0$.

$$FP_\infty(R) = FP_2(R) \subseteq FP_1(R).$$

**Proof.** Straightforward from Theorem 3.2 and [6, Theorem 2.11]. \hfill \Box

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