RECENT PROGRESS ON RICCI SOLITONS

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Abstract. In recent years, there has seen much interest and increased research activities in Ricci solitons. Ricci solitons are natural generalizations of Einstein metrics. They are also special solutions to Hamilton’s Ricci flow and play important roles in the singularity study of the Ricci flow. In this paper, we survey some of the recent progress on Ricci solitons.

The concept of Ricci solitons was introduced by Hamilton [64] in mid 80’s. They are natural generalizations of Einstein metrics. Ricci solitons also correspond to self-similar solutions of Hamilton’s Ricci flow [62] and often arise as limits of dilations of singularities in the Ricci flow [66, 10, 25, 91]. They can be viewed as fixed points of the Ricci flow, as a dynamical system, on the space of Riemannian metrics modulo diffeomorphisms and scalings. Ricci solitons are of interests to physicists as well and are called quasi-Einstein metrics in physics literature (see, e.g., [50]). In this paper, we survey some of the recent progress on Ricci solitons as well as the role they play in the singularity study of the Ricci flow. This paper can be regarded as an update of the article [13] written by the author a few years ago.

1. Ricci Solitons

1.1. Ricci Solitons. Recall that a Riemannian metric $g_{ij}$ is Einstein if its Ricci tensor $R_{ij} = \rho g_{ij}$ for some constant $\rho$. A smooth $n$-dimensional manifold $M^n$ with an Einstein metric $g$ is an Einstein manifold. Ricci solitons, introduced by Hamilton [64], are natural generalizations of Einstein metrics.

Definition 1.1. A complete Riemannian metric $g_{ij}$ on a smooth manifold $M^n$ is called a Ricci soliton if there exists a smooth vector field $V = (V^i)$ such that the Ricci tensor $R_{ij}$ of the metric $g_{ij}$ satisfies the equation

$$R_{ij} + \frac{1}{2} (\nabla_i V_j + \nabla_j V_i) = \rho g_{ij},$$

for some constant $\rho$. Moreover, if $V$ is a gradient vector field, then we have a gradient Ricci soliton, satisfying the equation

$$R_{ij} + \nabla_i \nabla_j f = \rho g_{ij},$$

for some smooth function $f$ on $M$. For $\rho = 0$ the Ricci soliton is steady, for $\rho > 0$ it is shrinking and for $\rho < 0$ expanding. The function $f$ is called a potential function of the Ricci soliton.

Since $\nabla_i V_j + \nabla_j V_i$ is the Lie derivative $L_V g_{ij}$ of the metric $g$ in the direction of $V$, we also write the Ricci soliton equations (1.1) and (1.2) as

$$Rc + \frac{1}{2} L_V g = \rho g \quad \text{and} \quad Rc + \nabla^2 f = \rho g$$

(1.3)

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respectively.

When the underlying manifold is a complex manifold, we have the corresponding notion of Kähler-Ricci solitons.

**Definition 1.2.** A complete Kähler metric $g_{\alpha \overline{\beta}}$ on a complex manifold $X^n$ of complex dimension $n$ is called a Kähler-Ricci soliton if there exists a holomorphic vector field $V = (V^\alpha)$ on $X$ such that the Ricci tensor $R_{\alpha \overline{\beta}}$ of the metric $g_{\alpha \overline{\beta}}$ satisfies the equation

$$R_{\alpha \overline{\beta}} + \frac{1}{2}(\nabla_{\overline{\beta}} V^\alpha + \nabla^\alpha V_{\overline{\beta}}) = \rho g_{\alpha \overline{\beta}}$$

for some (real) constant $\rho$. It is called a gradient Kähler-Ricci soliton if the holomorphic vector field $V$ comes from the gradient vector field of a real-valued function $f$ on $X^n$ so that

$$R_{\alpha \overline{\beta}} + \nabla_{\overline{\beta}} \nabla_{\alpha} f = \rho g_{\alpha \overline{\beta}}, \quad \nabla_{\alpha} \nabla_{\beta} f = 0.$$  

Again, for $\rho = 0$ the soliton is steady, for $\rho > 0$ it is shrinking and for $\rho < 0$ expanding.

Note that the case $V = 0$ (i.e., $f$ being a constant function) is an Einstein (or Kähler-Einstein) metric. Thus Ricci solitons are natural extensions of Einstein metrics. In fact, we will see below that there are no non-Einstein compact steady or expanding Ricci solitons. Also, by a suitable scale of the metric $g$, we can normalize $\rho = 0, +1/2, \text{ or } -1/2$.

**Lemma 1.1.** (Hamilton [67]) Let $g_{ij}$ be a complete gradient Ricci soliton with potential function $f$. Then we have

$$R + |\nabla f|^2 - 2\rho f = C$$

for some constant $C$. Here $R$ denotes the scalar curvature.

**Proof.** Let $g_{ij}$ be a complete gradient Ricci soliton on a manifold $M^n$ so that there exists a potential function $f$ such that the soliton equation (1.2) holds. Taking the covariant derivatives and using the commutating formula for covariant derivatives, we obtain

$$\nabla_i R_{jk} - \nabla_j R_{ik} + R_{ijkl} \nabla_l f = 0.$$  

Taking the trace on $j$ and $k$, and using the contracted second Bianchi identity

$$\nabla_j R_{ij} = \frac{1}{2} \nabla_i R,$$

we get

$$\nabla_i R = 2R_{ij}\nabla_j f.$$  

Thus

$$\nabla_i (R + |\nabla f|^2 - 2\rho f) = 2(R_{ij} + \nabla_i \nabla_j f - \rho g_{ij}) \nabla_j f = 0.$$  

Therefore

$$R + |\nabla f|^2 - 2\rho f = C$$

for some constant $C$. □

**Proposition 1.1.** (cf. Hamilton [67], Ivey [70]) On a compact manifold $M^n$, a gradient steady or expanding Ricci soliton is necessarily an Einstein metric.
Proof. Taking the trace in equation (1.2), we get

\[ R + \Delta f = n\rho. \]  

(1.8)

Taking the difference of (1.6) in Lemma 1.1 and (1.8), we get

\[ \Delta f - |\nabla f|^2 + 2\rho f = n\rho - C. \]

When \( M \) is compact and \( \rho \leq 0 \), it follows from the maximum principle that \( f \) must be a constant and hence \( g_{ij} \) is a Einstein metric. \( \square \)

More generally, we have

**Proposition 1.2.** Any compact steady or expanding Ricci soliton must be Einstein.

Proof. This follows from Proposition 1.1 and Perelman’s results that any compact Ricci soliton is necessarily a gradient soliton (see Propositions 2.1-2.4). \( \square \)

For compact shrinking Ricci solitons in low dimensions, we have

**Proposition 1.3.** (Hamilton [64] for \( n = 2 \), Ivey [70] for \( n = 3 \)) In dimension \( n \leq 3 \), there are no compact shrinking Ricci solitons other than those of constant positive curvature.

1.2. **Examples of Ricci Solitons.** When \( n \geq 4 \), there exist nontrivial compact gradient shrinking solitons. Also, there exist complete noncompact Ricci solitons (steady, shrinking and expanding) that are not Einstein. Below we list a number of such examples. It turns out most of the examples are rotationally symmetric and gradient, and all the known examples of nontrivial shrinking solitons so far are Kähler.

**Example 1.1.** (Compact gradient Kähler shrinkers) For real dimension 4, the first example of a compact shrinking soliton was constructed in early 90’s by Koiso [72] and the author [10] on compact complex surface \( \mathbb{CP}^2 \# (-\mathbb{CP}^2) \), where \( (-\mathbb{CP}^2) \) denotes the complex projective space with the opposite orientation. This is a gradient Kähler-Ricci soliton, has \( U(2) \) symmetry and positive Ricci curvature. More generally, they found \( U(n) \)-invariant Kähler-Ricci solitons on twisted projective line bundle over \( \mathbb{CP}^{n-1} \) for \( n \geq 2 \).

**Remark 1.1.** If a compact Kähler manifold \( M \) admits a non-trivial Kähler shrinker then \( M \) is Fano (i.e., the first Chern class \( c_1(M) \) of \( M \) is positive), and the Futaki-invariant [51] is nonzero.

**Example 1.2.** (Compact toric gradient Kähler shrinkers) In [96], Wang-Zhu found a gradient Kähler-Ricci soliton on \( \mathbb{CP}^2 \# 2(-\mathbb{CP}^2) \) which has \( U(1) \times U(1) \) symmetry. More generally, they proved the existence of gradient Kähler-Ricci solitons on all Fano toric varieties of complex dimension \( n \geq 2 \) with non-vanishing Futaki invariant.

**Example 1.3.** (Noncompact gradient Kähler shrinkers) Feldman-Ilmanen-Knopf [17] found the first complete noncompact \( U(n) \)-invariant shrinking gradient Kähler-Ricci solitons, which are cone-like at infinity. It has positive scalar curvature but the Ricci curvature does not have a fixed sign.

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1See alternative proofs in (Proposition 5.21, [31]) or (Proposition 5.1.10, [17]), and [30].

2See [17] for alternative proofs.

3The author’s construction was carried out in 1991 at Columbia University. When he told his construction to S. Bando that year in New York, he also learned the work of Koiso from Bando.
Example 1.4. (The cigar soliton) In dimension two, Hamilton [64] discovered the first example of a complete noncompact steady soliton on $\mathbb{R}^2$, called the **cigar soliton**, where the metric is given by
\[ ds^2 = \frac{dx^2 + dy^2}{1 + x^2 + y^2} \]
with potential function
\[ f = -\log(1 + x^2 + y^2). \]
The cigar has positive (Gaussian) curvature and linear volume growth, and is asymptotic to a cylinder of finite circumference at $\infty$.

Example 1.5. (The Bryant soliton) In the Riemannian case, higher dimensional examples of noncompact gradient steady solitons were found by Robert Bryant [5] on $\mathbb{R}^n$ ($n \geq 3$). They are rotationally symmetric and have positive sectional curvature. Furthermore, the geodesic sphere $S^{n-1}$ of radius $s$ has the diameter on the order $\sqrt{s}$. Thus the volume of geodesic balls $B_r(0)$ grow on the order of $r^{(n+1)/2}$.

Example 1.6. (Noncompact gradient steady Kähler solitons) In the Kähler case, the author [10] found two examples of complete rotationally noncompact gradient steady Kähler-Ricci solitons

(a) On $\mathbb{C}^n$ (for $n = 1$ it is just the cigar soliton). These examples are $U(n)$ invariant and have positive sectional curvature. It is interesting to point out that the geodesic sphere $S^{2n-1}$ of radius $s$ is an $S^1$-bundle over $\mathbb{C}P^{n-1}$ where the diameter of $S^1$ is on the order 1, while the diameter of $\mathbb{C}P^{n-1}$ is on the order $\sqrt{s}$. Thus the volume of geodesic balls $B_r(0)$ grow on the order of $r^n$, $n$ being the complex dimension. Also, the curvature $R(x)$ decays like $1/r$.

(b) On the blow-up of $\mathbb{C}^n/\mathbb{Z}_n$ at the origin. This is the same space on which Eguchi-Hansen [43] ($n = 2$) and Calabi [8] ($n \geq 2$) constructed examples of Hyper-Kähler metrics. For $n = 2$, the underlying space is the canonical line bundle over $\mathbb{C}P^1$.

Example 1.7. (Noncompact gradient expanding Kähler solitons) In [11], the author constructed a one-parameter family of complete noncompact expanding solitons on $\mathbb{C}^n$. These expanding Kähler-Ricci solitons all have $U(n)$ symmetry and positive sectional curvature, and are cone-like at infinity.

More examples of complete noncompact Kähler-Ricci expanding solitons were found by Feldman-Ilmanen-Knopf [47] on "blow-ups" of $\mathbb{C}^n/\mathbb{Z}_k$, $k = n+1, n+2, \ldots$. (See also Pedersen et al [83].)

Example 1.8. (Sol and Nil solitons) Non-gradient expanding Ricci solitons on Sol and Nil manifolds were constructed by J. Lauret [74] and Baird-Laurent [2].

Example 1.9. (Warped products) Using doubly warped product and multiple warped product constructions, Ivey [71] and Dancer-Wang [40] produced noncompact gradient steady solitons, which generalize the construction of Bryant’s soliton. Also, Gastel-Kronz [55] produced a two-parameter family (doubly warped product metrics) of gradient expanding solitons on $\mathbb{R}^{m+1} \times N$, where $N^n$ ($n \geq 2$) is an Einstein manifold with positive scalar curvature.

Example 1.10. Very recently, Dancer-Wang [39] produced new examples of gradient shrinking, steady and expanding Kähler solitons on bundles over the product
of Fano Kähler-Einstein manifolds, generalizing those in Examples 1.1, 1.3, 1.6, 1.7 and those by Pedersen et al [83].

We conclude our examples with Example 1.11. (Gaussian solitons) \((\mathbb{R}^n, g_0)\) with the flat Euclidean metric can be also equipped with both shrinking and expanding gradient Ricci solitons, called the Gaussian shrinker or expander.

(a) \((\mathbb{R}^n, g_0, |x|^2/4)\) is a gradient shrinker with potential function \(f = |x|^2/4:\)

\[
Rc + \nabla^2 f = \frac{1}{2} g_0.
\]

(b) \((\mathbb{R}^n, g_0, -|x|^2/4)\) is a gradient expander with potential function \(f = -|x|^2/4:\)

\[
Rc + \nabla^2 f = -\frac{1}{2} g_0.
\]

Remark 1.2. We’ll see later that the Gaussian shrinker is very special because it has the largest reduced volume \(\tilde{V} = 1\) (see Section 4.2).

2. Variational Structures

In this section we describe Perelman’s \(F\)-functional and \(W\)-functional and the associated \(\lambda\)-energy and \(\nu\)-energy respectively. The critical points of the \(\lambda\)-energy (respectively \(\nu\)-energy) are precisely given by compact gradient steady (respectively shrinking) solitons. We also consider the \(W\)-functional and the corresponding \(\nu\)-energy introduced by Feldman-Ilmanen-Ni [48] whose critical points are expanding solitons. Throughout this section we assume that \(M^n\) is a compact smooth manifold.

2.1. The \(F\)-functional and \(\lambda\)-energy. In [84] Perelman considered the functional

\[
\mathcal{F}(g_{ij}, f) = \int_M (R + |\nabla f|^2)e^{-f}dV
\]

defined on the space of Riemannian metrics and smooth functions on \(M\). Here \(R\) is the scalar curvature and \(f\) is a smooth function on \(M^n\). Note that when \(f = 0\), \(F\) is simply the total scalar curvature of \(g\), or the Einstein-Hilbert action on the space of Riemannian metrics on \(M\).

Lemma 2.1. (First Variation Formula of \(F\)-functional, Perelman [84]) If \(\delta g_{ij} = v_{ij}\) and \(\delta f = \phi\) are variations of \(g_{ij}\) and \(f\) respectively, then the first variation of \(\mathcal{F}\) is given by

\[
\delta \mathcal{F}(v_{ij}, \phi) = \int_M \left[ -v_{ij}(R_{ij} + \nabla_i \nabla_j f) + \frac{\nu}{2} - \phi \right] (2\Delta f - |\nabla f|^2 + R)e^{-f}dV
\]

where \(\nu = g^{ij}v_{ij}\).

Next we consider the associated energy

\[
\lambda(g_{ij}) = \inf \{ \mathcal{F}(g_{ij}, f) : f \in \mathcal{C}^{\infty}(M), \int_M e^{-f}dV = 1 \}.
\]

Clearly \(\lambda(g_{ij})\) is invariant under diffeomorphisms. If we set \(u = e^{-f/2}\), then the functional \(\mathcal{F}\) can be expressed as

\[
\mathcal{F} = \int_M (Ru^2 + 4|\nabla u|^2)dV.
\]
Thus

$$\lambda(g_{ij}) = \inf \left\{ \int_M (Ru^2 + 4|\nabla u|^2) dV : \int_M u^2 dV = 1 \right\},$$

the first eigenvalue of the operator $-4\Delta + R$. Let $u_0 > 0$ be a first eigenfunction of the operator $-4\Delta + R$ so that

$$-4\Delta u_0 + Ru_0 = \lambda(g_{ij})u_0.$$ 

Then $f_0 = -2\log u_0$ is a minimizer of $\lambda(g_{ij})$:

$$\lambda(g_{ij}) = F(g_{ij}, f_0).$$

Note that $f_0$ satisfies the equation

$$-2\Delta f_0 + |\nabla f_0|^2 - R = \lambda(g_{ij}).$$

For any symmetric 2-tensor $h = h_{ij}$, consider the variation $g_{ij}(s) = g_{ij} + sh_{ij}$. It is an easy consequence of Lemma 2.1 and Eq. (2.1) that the first variation $D g \lambda(h)$ of $\lambda(g_{ij})$ is given by

$$\left. \frac{d}{ds} \right|_{s=0} \lambda(g_{ij}(s)) = \int M -h_{ij}(R_{ij} + \nabla_i \nabla_j f)e^{-f} dV,$$

where $f$ is a minimizer of $\lambda(g_{ij})$. In particular, the critical points of $\lambda$ are precisely steady gradient Ricci solitons.

We remark that by considering the quantity

$$\bar{\lambda}(g_{ij}) = \lambda(g_{ij})(Vol(g_{ij}))^{\frac{2}{n}},$$

which is a scale invariant version of $\lambda(g_{ij})$, Perelman [34] also showed the following result.

**Proposition 2.2.** $\bar{\lambda}(g_{ij})$ is nondecreasing along the Ricci flow whenever it is non-positive; moreover, the monotonicity is strict unless we are on a gradient expanding soliton. In particular, any (compact) expanding Ricci soliton is necessarily a gradient soliton.

2.2. The $\mathcal{W}$-functional and $\nu$-energy. In order to study shrinking Ricci solitons, Perelman [34] introduced the $\mathcal{W}$-functional

$$\mathcal{W}(g_{ij}, f, \tau) = \int_M |\tau (R + |\nabla f|^2) + f - n|(4\pi \tau)^{-\frac{n}{2}} e^{-f} dV,$$

where $g_{ij}$ is a Riemannian metric, $f$ a smooth function on $M^n$, and $\tau$ a positive scale parameter. Clearly the functional $\mathcal{W}$ is invariant under simultaneous scaling of $\tau$ and $g_{ij}$ (or equivalently the parabolic scaling), and invariant under diffeomorphism. Namely, for any positive number $a$ and any diffeomorphism $\varphi$ we have

$$\mathcal{W}(a\varphi^* g_{ij}, \varphi^* f, a\tau) = \mathcal{W}(g_{ij}, f, \tau).$$
Lemma 2.2. (First Variation of $W$-functional, Perelman [84]) If $v_{ij} = \delta g_{ij}$, $\phi = \delta f$, and $\eta = \delta \tau$, then

$$
\delta W(v_{ij}, \phi, \eta) = \int_M -\tau v_{ij}(R_{ij} + \nabla_i f \nabla_j f - \frac{1}{4\pi} g_{ij})(4\pi\tau)^{-\frac{n}{2}} e^{-f} dV
+ \int_M \left(\frac{\phi}{2} - \phi - \frac{\eta}{2\tau}\right)[(\tau + 2\Delta f - |\nabla f|^2) + f - n - 1](4\pi\tau)^{-\frac{n}{2}} e^{-f} dV
+ \int_M \eta(R + |\nabla f|^2 - \frac{n}{4\tau})(4\pi\tau)^{-\frac{n}{2}} e^{-f} dV.
$$

Here $v = g^{ij}v_{ij}$ as before.

Similar to the $\lambda$-energy, we can consider

$$
\mu(g_{ij}, \tau) = \inf \{W(g_{ij}, f, \tau) : f \in C^\infty(M), (4\pi\tau)^{-\frac{n}{2}} \int_M e^{-f} dV = 1 \}.
$$

Note that if we let $u = e^{-f/2}$, then the functional $W$ can be expressed as

$$
W(g_{ij}, f, \tau) = \int_M [\tau(Ru^2 + 4|\nabla u|^2)] - u^2 \log u^2 - nu^2)(4\pi\tau)^{-\frac{n}{2}} dV,
$$

and the constraint $\int_M (4\pi\tau)^{-\frac{n}{2}} e^{-f} dV = 1$ becomes $\int_M u^2(4\pi\tau)^{-\frac{n}{2}} dV = 1$. Therefore $\mu(g_{ij}, \tau)$ corresponds to the best constant of a logarithmic Sobolev inequality.

Since the nonquadratic term is subcritical (in view of Sobolev exponent), it is rather straightforward to show that $\mu(g_{ij}, \tau)$ is achieved by some nonnegative function $u \in H^1(M)$ which satisfies the Euler-Lagrange equation

$$
\tau(-4\Delta u + Ru) - 2u \log u - nu = \mu(g_{ij}, \tau)u.
$$

One can further show that the minimizer $u$ is positive and smooth (see Rothaus [88]). This is equivalent to say that $\mu(g_{ij}, \tau)$ is achieved by some minimizer $f$ satisfying the nonlinear equation

$$
\tau(2\Delta f - |\nabla f|^2 + R) + f = n = \mu(g_{ij}, \tau).
$$

Proposition 2.3. (Perelman [84]) Suppose $g_{ij}(t), 0 \leq t < T$ is a solution to the Ricci flow on a compact manifold $M^n$. Then $\mu(g_{ij}(t), T-t)$ is nondecreasing in $t$; moreover, the monotonicity is strict unless we are on a shrinking gradient soliton. In particular, any (compact) shrinking Ricci soliton is necessarily a gradient soliton.

Remark 2.1. Recently, Naber [77] has shown that if $(M^n, g)$ is a complete noncompact shrinking Ricci soliton with bounded curvature $|Rm| < C$ with respect to some smooth vector field $V$, then there exists a smooth function $f$ on $M$ such that $(M^n, g)$ is a gradient soliton with $f$ as a potential function. This in particular means that $V = \nabla f + X$ for some Killing field $X$ on $M$.

The associated $\nu$-energy is defined by

$$
\nu(g_{ij}) = \inf \{W(g, f, \tau) : f \in C^\infty(M), \tau > 0, (4\pi\tau)^{-\frac{n}{2}} \int e^{-f} dV = 1 \}.
$$

One checks that $\nu(g_{ij})$ is realized by a pair $(f, \tau)$ that solve the equations

$$
\tau(-2\Delta f + |Df|^2 - R) - f + n + \nu = 0, \quad (4\pi\tau)^{-\frac{n}{2}} \int f e^{-f} = \frac{n}{2} + \nu.
$$
Consider variations $g_{ij}(s) = g_{ij} + sh_{ij}$ as before. Using Lemma 2.2 and (2.6), one calculates the first variation $D_g \nu(h)$ to be

$$\left. \frac{d}{ds} \right|_{s=0} \nu(g_{ij}(s)) = (4\pi \tau)^{-\frac{2}{3}} \int -h_{ij}[\tau(R_{ij} + \nabla_i \nabla_j f) - \frac{1}{2} \tau g_{ij}] e^{-f} dV.$$

A stationary point of $\nu$ thus satisfies

$$R_{ij} + \nabla_i \nabla_j f - \frac{1}{2} \tau g_{ij} = 0,$$

which says that $g_{ij}$ is a gradient shrinking Ricci soliton.

As before, $D_g \nu(h)$ vanishes on Lie derivatives. By scale invariance it also vanishes on multiplies of the metric. Inserting $h_{ij} = -2(R_{ij} + \nabla_i \nabla_j f - \frac{1}{2} \tau g_{ij})$, one recovers Perelman’s formula that finds that $\nu(g_{ij}(t))$ is monotone on a Ricci flow, and constant if and only if $g_{ij}(t)$ is a gradient shrinking Ricci soliton.

2.3. The $W_-$-functional and $\nu_-$-energy. In [48], Feldman-Ilmanen-Ni introduced the dual $W_-$-functional (corresponding to expanders)

$$W_-(g_{ij}, f, \sigma) = \int_M [\sigma(R + |\nabla f|^2) - (f - n)](4\pi \sigma)^{-\frac{2}{3}} e^{-f} dV,$$

the $\mu_-$-energy

$$\mu_-(g_{ij}, \sigma) = \inf \{W_-(g_{ij}, f, \tau) : f \in C^\infty(M), (4\pi \sigma)^{-\frac{2}{3}} \int_M e^{-f} dV = 1\},$$

and the corresponding $\nu_-$-entropy

$$\nu_-(g_{ij}) = \sup_{\sigma > 0} \{\mu_-(g_{ij}, \sigma)\}.$$ 

Here, $\sigma$ is a positive parameter. They proved that

**Proposition 2.4. (Feldman-Ilmanen-Ni [48])**

(a) $\mu_-(g_{ij}, \sigma)$ is achieved by a unique $f$; $\mu_-(g_{ij}(t), t-t_0)$ is nondecreasing under the Ricci flow; moreover, the monotonicity is strict unless we are on an expanding gradient soliton.

(b) If $\lambda(g) < 0$, then $\nu_-(g_{ij})$ is achieved by a unique $\sigma$; $\nu_-(g_{ij}(t))$ is nondecreasing under the Ricci flow, and is constant only on an expanding soliton.

Furthermore, if $\lambda(g) < 0$ then $\nu_-$ is achieved by a unique pair $(f, \sigma)$ that solve the equations

$$\sigma(-2\Delta f + |Df|^2 - R) + f - n + \nu_- = 0, \quad (4\pi \tau)^{-\frac{2}{3}} \int f e^{-f} = \frac{n}{2} - \nu_-.$$

3. Ricci Solitons and Ricci Flow

3.1. Ricci solitons as self-similar solutions of the Ricci flow. Let us first examine how Einstein metrics behave under Hamilton’s Ricci flow

$$\frac{\partial g_{ij}(t)}{\partial t} = -2R_{ij}(t).$$

If the initial metric is Ricci flat, so that $R_{ij} = 0$ at $t = 0$, then clearly the metric does not change under the Ricci flow: $g_{ij}(t) = g_{ij}(0)$. Hence any Ricci flat metric is a stationary solution. This happens, for example, on a flat torus or on any $K3$-surface with a Calabi-Yau metric.
If the initial metric \( g_{ij}(0) \) is Einstein with positive scalar curvature, then the metric will shrink under the Ricci flow by a time-dependent factor. Indeed, if at \( t = 0 \) we have
\[
R_{ij}(0) = \frac{1}{2} g_{ij}(0).
\]
Then
\[
g_{ij}(t) = (1 - t) g_{ij}(0),
\]
which shrinks homothetically to a point as \( t \to T = 1 \), while the scalar curvature \( R \to \infty \) like \( 1/(T - t) \) as \( t \to T \). Note that \( g(t) \) exists for \( t \in (-\infty, T) \), hence an *ancient solution*.

By contrast, if the initial metric is an Einstein metric of negative scalar curvature, the metric will expand homothetically for all times. Suppose
\[
R_{ij}(0) = -\frac{1}{2} g_{ij}(0)
\]
at \( t = 0 \). Then the solution to the Ricci flow is given by
\[
g_{ij}(t) = (1 + t) g_{ij}(0).
\]
Hence the evolving metric \( g_{ij}(t) \) exists and expands homothetically for all time, and the curvature will fall back to zero like \(-1/t\). Note that now the evolving metric \( g_{ij}(t) \) only goes back in time to \(-1\), when the metric explodes out of a single point in a "big bang".

Now suppose we have a one-parameter group of diffeomorphisms \( \psi_t, -\infty < t < \infty \), which is generated by some vector field \( V \) on \( M \), and suppose \( g_{ij}(t) = \psi_t^* \hat{g}_{ij} \) is a solution to the Ricci flow, called a *self-similar solution*, with initial metric \( \hat{g}_{ij} \). Then
\[
-2Rc = \mathcal{L}_V g
\]
for all \( t \). In particular, the initial metric \( g_{ij}(0) = \hat{g}_{ij} \) satisfies the steady Ricci soliton equation in (1.3).

Conversely, suppose we have a steady Ricci soliton \( \hat{g} = (\hat{g}_{ij}) \) on a smooth manifold \( M^n \) so that
\[
2 \hat{R}_c + \mathcal{L}_V \hat{g} = 0,
\]
for some smooth vector field \( V = (V^i) \). Assume the vector field \( V \) is complete (i.e., \( V \) generates a one-parameter group of diffeomorphisms \( \psi_t \) of \( M \)). Then clearly
\[
g_{ij}(t) = \psi_t^* \hat{g}_{ij} \quad -\infty < t < \infty,
\]
is a self-similar solution of the Ricci flow with \( \hat{g}_{ij} \) as the initial metric.

More generally, we can consider self-similar solutions to the Ricci flow which move by diffeomorphisms and also shrinks or expands by a (time-dependent) factor at the same time. Such self-similar solutions correspond to either shrinking or expanding Ricci solitons \( (M, \hat{g}, V) \) with the vector field \( V \) being complete. For example, a shrinking gradient Ricci soliton satisfying the equation
\[
\check{R}_{ij} + \nabla_i \nabla_j \check{f} - \frac{1}{2} \hat{g}_{ij} = 0,
\]
with \( V = \nabla \check{f} \) complete, corresponds to the self-similar Ricci flow solution \( g_{ij}(t) \) of the form
\[
g_{ij}(t) := (1 - t) \varphi_t^* (g_{ij}), \quad t < 1,
\]
where \( \varphi_t \) are the diffeomorphisms generated by \( V/(1-t) \). (Compare Eq. (3.2) with Eq. (3.1) for \( \rho = 1/2 \).)
Thus, we see a complete gradient Ricci soliton with respect to some complete vector field corresponds to the self-similar solution of the Ricci flow it generates. For this reason we often do not distinguish the two.

Remark 3.1. If \( M^n \) is compact, then \( V \) is always complete. But if \( M \) is noncompact then \( V \) may not be complete in general. Recently Z.-H. Zhang [101] has observed that for any complete gradient (steady, shrinking, or expanding) Ricci soliton \( g_{ij} \) with potential function \( f \), \( V = \nabla f \) is a complete vector field on \( M \).

In particular, a complete gradient Ricci soliton always corresponds to the self-similar solution of the Ricci flow it generates.

3.2. Ricci Solitons and Singularity Models of the Ricci Flow. Ricci solitons play an important role in the study of the Ricci flow. They are intimately related to the Li-Yau-Hamilton (also called differential Harnack) type estimates (cf. [67] and [17]) in such a way that the Li-Yau-Hamilton quantity vanishes on (expanding) Ricci solitons (see, e.g., Section 2.5 in [17]). More importantly, Ricci solitons often arise as the below-up limits of singularities in the Ricci flow which we now describe.

Consider a solution \( g_{ij}(t) \) to the Ricci flow on \( M^n \times [0, T) \), \( T \leq +\infty \), where either \( M^n \) is compact or at each time \( t \) the metric is complete and has bounded curvature. We say that \( g_{ij}(t) \) is a maximal solution of the Ricci flow if either \( T = +\infty \) or \( T < +\infty \) and the norm of its curvature tensor \( |Rm| \) is unbounded as \( t \to T \). In the latter case, we say \( g_{ij}(t) \) is a singular solution to the Ricci flow.

Clearly, a round sphere \( S^3 \) will shrink to a point under the Ricci flow in some finite time. Also, as Yau suggested to Hamilton in mid 80’s, if we take a dumbbell metric on \( S^3 \) with a neck like \( S^2 \times B^1 \), we expect the neck will shrink because the positive curvature in the \( S^2 \) direction will dominate the slightly negative curvature in the \( B^1 \) direction. In some finite time we expect the neck will pinch off. If we dilate in space and time at the maximal curvature point, then we expect the limit of dilations converge to the round infinite cylinder \( S^2 \times \mathbb{R} \). This intuitive picture was justified by Angenent-Knopf [1] on \( S^{n+1} \) with suitable rotationally symmetric metrics. These are examples of so called Type I singularities. Hamilton [67] also described an intuitive picture of a degenerate neck-pinchning: imagine the dumbbell is not symmetric and one side is bigger than the other. Then one could also pinch off a small sphere from a big one. If we choose the sizes of the little one and the large one to be just right, then we expect a degenerate singularity: the little sphere pinches off and there is nothing left on the other side. H.-L Gu and X.-P. Zhu [61] recently verified this picture by showing that such a degenerate neck-pinchning, a Type II singularity, can be formed on \( S^n \) with suitable rotationally symmetric metric for all \( n \geq 3 \).

As in the minimal surface theory and harmonic map theory, one usually tries to understand the structure of a singularity by rescaling the solution (or blow up) to obtain a sequence of solutions and study its limit. For the Ricci flow, the theory was first developed by Hamilton in [67].

Denote by

\[
K_{\text{max}}(t) = \sup_{x \in M} |Rm(x, t)|_{g_{ij}(t)}.
\]

According to Hamilton [67], one can classify maximal solutions into three types; every maximal solution is clearly of one and only one of the following three types:

**Type I:** \( T < +\infty \) and \( \sup(T - t)K_{\text{max}}(t) < +\infty \);
Type II(a): $T < +\infty$ but $\sup(T - t)K_{\text{max}}(t) = +\infty$;
Type II(b): $T = +\infty$ but $\sup tK_{\text{max}}(t) = +\infty$;
Type III: $T = +\infty$, $\sup tK_{\text{max}}(t) < +\infty$

For each type of maximal solution, Hamilton defines a corresponding type of limiting singularity model.

Definition 3.1. A solution $g_{ij}(x, t)$ to the Ricci flow on the manifold $M$, where either $M$ is compact or at each time $t$ the metric $g_{ij}(\cdot, t)$ is complete and has bounded curvature, is called a singularity model if it is not flat and of one of the following three types:

Type I: The solution exists for $t \in (-\infty, \Omega)$ for some constant $\Omega$ with $0 < \Omega < +\infty$ and

$$|Rm| \leq \Omega/(\Omega - t)$$

everywhere with equality somewhere at $t = 0$;

Type II: The solution exists for $t \in (-\infty, +\infty)$ and

$$|Rm| \leq 1$$

everywhere with equality somewhere at $t = 0$;

Type III: The solution exists for $t \in (-A, +\infty)$ for some constant $A$ with $0 < A < +\infty$ and

$$|Rm| \leq A/(A + t)$$

everywhere with equality somewhere at $t = 0$.

Definition 3.2. A solution of the Ricci flow is said to satisfy the injectivity radius condition if for every sequence of (almost) maximum points $\{(x_k, t_k)\}$, there exists a constant $c_2 > 0$ independent of $k$ such that

$$\text{inj}(M, x_k, g_{ij}(t_k)) \geq \frac{c_2}{\sqrt{K_{\text{max}}(t_k)}}, \quad \text{for all } k.$$  

Here, by a sequence of (almost) maximum points, we mean $\{(x_k, t_k) \in M \times [0, T)\}$, $k = 1, 2, \cdots$, has the following property: there exist positive constants $c_1$ and $\alpha \in (0, 1]$ such that

$$|Rm(x_k, t_k)| \geq c_1K_{\text{max}}(t), \quad t \in [t_k - \frac{\alpha}{K_{\text{max}}(t_k)}, t_k]$$

for all $k$.

In [84], Perelman proved an important no local collapsing theorem, which yields the following result conjectured by Hamilton in [67].

Theorem 3.1. (Little Loop Lemma) Let $g_{ij}(t)$, $0 \leq t < T < +\infty$, be a solution of the Ricci flow on a compact manifold $M^n$. Then there exists a constant $\delta > 0$ having the following property: if at a point $x_0 \in M$ and a time $t_0 \in [0, T)$,

$$|Rm(\cdot, t_0)| \leq r^{-2} \text{ on } B_{t_0}(x_0, r)$$

for some $r \leq \sqrt{T}$, then the injectivity radius of $M$ with respect to the metric $g_{ij}(t_0)$ at $x_0$ is bounded from below by

$$\text{inj}(M, x_0, g_{ij}(t_0)) \geq \delta r.$$
Clearly by the above Little Loop Lemma a maximal solution on a compact manifold with the maximal time $T < +\infty$ always satisfies the injectivity radius condition. Also, by the Gromoll-Meyer injectivity radius estimate [57], a solution on a complete noncompact manifold with positive sectional curvature also satisfies the injectivity radius condition. We refer the reader to [17] (Chapter 4) for more detailed discussions.

**Theorem 3.2. (Hamilton [67])** For any maximal solution to the Ricci flow which satisfies the injectivity radius condition and is of Type I, II, or III, there exists a sequence of dilations of the solution which converges in $C^\infty_{\text{loc}}$ topology to a singularity model of the corresponding Type.

In the case of manifolds with nonnegative curvature operator, or Kähler metrics with nonnegative holomorphic bisectional curvature, we can bound the Riemannian curvature $Rm$ by the scalar curvature $R$ up to a constant factor depending only on the dimension. Then we can slightly modify the statements in the previous theorem as follows

**Theorem 3.3. (Hamilton [67])** For any complete maximal solution to the Ricci flow with bounded and nonnegative curvature operator on a Riemannian manifold, or on a Kähler manifold with bounded and nonnegative holomorphic bisectional curvature, there exists a sequence of dilations which converges to a singular model. For Type I solutions: the limit model exists for $t \in (-\infty, \Omega)$ with $0 < \Omega < +\infty$ and has
\[
R \leq \frac{\Omega}{\Omega - t}
\]
everywhere with equality somewhere at $t = 0$;
For Type II solutions: the limit model exists for $t \in (-\infty, +\infty)$ and has
\[
R \leq 1
\]
everywhere with equality somewhere at $t = 0$;
For Type III solutions: the limit model exists for $t \in (-A, +\infty)$ with $0 < A < +\infty$ and has
\[
R \leq \frac{A}{A + t}
\]
everywhere with equality somewhere at $t = 0$.

For Type II or Type III singularity models with nonnegative curvature we have the following results.

**Theorem 3.4. (Hamilton [66])** Any Type II singularity model of the Ricci flow with nonnegative curvature operator and positive Ricci curvature must be a steady Ricci soliton.

**Theorem 3.5. (Cao [11])**
(i) Any Type II singularity model on a Kähler manifold with nonnegative holomorphic bisectional curvature and positive Ricci curvature must be a steady Kähler-Ricci soliton;
(ii) Any Type III singularity model on a Kähler manifold with nonnegative holomorphic bisectional curvature and positive Ricci curvature must be a shrinking Kähler-Ricci soliton.
Theorem 3.6. (Chen-Zhu [25]) Any Type III singularity model of the Ricci flow with nonnegative curvature operator and positive Ricci curvature must be a homothetically expanding Ricci soliton.

We remark that the basic idea in proving the above theorems is to apply the Li-Yau-Hamilton estimates for the Ricci flow [65] or the Kähler-Ricci flow [9], and the strong maximum principle type arguments.

On the other hand, by exploring Perelman’s $\mu$-entropy (defined by Eq. (2.3)), Sesum [91] studied compact Type I singularity model and obtained the following

Theorem 3.7. (Sesum [91]) Let $(M, g_{ij}(t))$ be a compact Type I singularity model obtained as a rescaling limit of a Type I maximal solution. Then $(M, g_{ij}(t))$ must be a gradient shrinking Ricci soliton.

Remark 3.2. Recently Naber [77] showed that a suitable rescaling limit of any Type I maximal solution is a gradient shrinking soliton. However, it is not guaranteed that such a limit soliton is non-flat. It remains an interesting question whether a noncompact Type I singularity model is a gradient shrinking soliton.

4. Reduced Distance, Reduced Volume and Gradient Shrinkers

In Section 2, we saw compact shrinkers are critical points of the $\mu$-energy, which is monotone under the Ricci flow. In [34], Perelman also introduced a space-time distance function $l(x, t)$, the reduced distance, which is analogous to the distance function introduced by Li-Yau [75]. Perelman used the reduced distance and its associated reduced volume to deal with complete noncompact solutions with bounded curvature. He established the comparison geometry to the reduced distance function, and proved that the reduced volume is monotone under the Ricci flow. This monotonicity formula is more useful for local considerations, and is used to prove the noncollapsing theorem for complete solutions to the Ricci flow with bounded curvature on noncompact manifolds.

4.1. The reduced distance. We will write the Ricci flow in the backward version

$$\frac{\partial}{\partial \tau} g_{ij} = 2R_{ij}$$

on a manifold $M$. In practice one often takes $\tau = t_0 - t$ for some fixed time $t_0$. Throughout the section we assume that $(M, g_{ij}(\tau))$ is complete with bounded curvature.

To each space curve $\gamma(\tau)$ in $M$, $0 \leq \tau_1 \leq \tau \leq \tau_2$, its $L$-length is defined as

$$L(\gamma) = \int_{\tau_1}^{\tau_2} \sqrt{\tau} \left[ R(\gamma(\tau), \tau) + |\dot{\gamma}(\tau)|^2_{g(\tau)} \right] d\tau.$$ 

A curve $\gamma(\tau)$ is called an $L$-geodesic if the tangent vector field $X = \dot{\gamma}(\tau)$ along $\gamma$ satisfies the $L$-geodesic equation

$$\nabla_X X - \frac{1}{2} \nabla R + \frac{1}{2\tau} X + 2Rc(X, \cdot) = 0.$$ 

Given any space time point $(p, \tau_1)$ ($\tau_1 \geq 0$) and a tangent vector $v \in T_p M$, there exists a unique $L$-geodesic $\gamma(\tau)$ starting at $p$ with $\lim_{\tau \to \tau_1} \sqrt{\tau} \dot{\gamma}(\tau) = v$. Also, for any

4For most parts of the discussion in this section, it suffices to assume Ricci curvature bounded from below; see Ye [99].
$q \in M$ and $\tau_2 > \tau_1$, there always exists a shortest $L$-geodesic $\gamma(\tau) : [\tau_1, \tau_2] \to M$ connecting $p$ and $q$.

Now we fix a point $p \in M$ and set $\tau_1 = 0$. The $L$-distance function on the space-time $M \times \mathbb{R}^+$ is denoted by $L(q, \bar{\tau})$ and defined to be the $L$-length of the $L$-shortest curve $\gamma(\tau)$ connecting $p = \gamma(0)$ and $q = \gamma(\bar{\tau})$.

The reduced distance $l(q, \bar{\tau})$, from the space-time origin $(p, 0)$ to $(q, \bar{\tau})$, is defined as

$$l(q, \bar{\tau}) \triangleq \frac{1}{2\sqrt{\bar{\tau}}} L(q, \bar{\tau}).$$

Remark 4.1. In the case that $(M^n, g)$ is a static solution to the Ricci flow (i.e., $g$ is Ricci flat), it is easy to see that $L(q, \bar{\tau}) = d^2(p, q) / 2\sqrt{\bar{\tau}}$, so

$$l(q, \bar{\tau}) = \frac{d^2(p, q)}{4\bar{\tau}}.$$ 

In particular, the reduced distance function $l(x, \tau)$ on $\mathbb{R}^n$ with respect to the origin is given by the Gaussian shrinker potential function (see Example 1.11(a))

$$l(x, \tau) = \frac{|x|^2}{4\tau} = \frac{|x|^2}{4(1-t)}.$$ 

Remark 4.2. One has $l(x, \tau) = n/2$ for positive Einstein manifold $(M^n, g(\tau))$, normalized with $R = n/2\tau$, $\tau = 1 - t$. Furthermore, if $(M^n, g_{ij}(x, t), f(x, t))$, $-\infty < t < 1$, is a complete gradient shrinker with bounded curvature, then the reduced distance function only differs from the potential function $f$ by a constant: $l(x, \tau) = f(x, 1 - \tau) + C$ with $\tau = 1 - t > 0$, see, e.g., Lemma 7.77 in [34].

In [84], Perelman computed the first and second variations of the $L$-length and obtained

**Lemma 4.1.** For the reduced distance $l(q, \bar{\tau})$ defined above, there hold

$$\frac{\partial l}{\partial \bar{\tau}} = -\frac{l}{\tau} + R + \frac{1}{2\tau^{3/2}} K$$

$$|\nabla l|^2 = -R + \frac{l}{\tau} - \frac{1}{\tau^{3/2}} K$$

$$\Delta l \leq -R + \frac{n}{2\tau} - \frac{1}{2\tau^{3/2}} K.$$ 

where

$$K = \int_0^{\bar{\tau}} \tau^{\frac{3}{2}} Q(X) d\tau,$$

and

$$Q(X) = -R\tau - \frac{R}{\tau} - 2 < \nabla R, X > + 2Rc(X, X)$$

is the trace Li-Yau-Hamilton quadratic. Moreover, the equality holds in Eq.(4.3) if and only if the solution along the $L$ minimal geodesic $\gamma$ satisfies the gradient soliton equation

$$R_{ij} + \nabla_i \nabla_j l - \frac{1}{2\tau} g_{ij} = 0.$$
4.2. The reduced volume. The reduced volume of \((M, g(\tau))\) is defined as

\[
\tilde{V}(\tau) = \int_M (4\pi\tau)^{-\frac{n}{2}} \exp(-l(q, \tau))dvol_\tau(q).
\]

**Remark 4.3.** For the Gaussian soliton on \(\mathbb{R}^n\), one finds

\[
\tilde{V}(\tau) = 1
\]

for all \(\tau > 0\). Also, for a Ricci flat manifold \((M^n, g)\) one has

\[
\tilde{V}(\tau) = \int_M (4\pi\tau)^{-\frac{n}{2}} \exp(-d^2(p, q)/4\tau).
\]

(4.4)

Note that it follows from Eqs. (4.1)-(4.3) in Lemma 4.1 that

\[
\frac{d}{d\tau} \tilde{V}(\tau) \leq 0,
\]

provided \(M\) is compact, and the equality holds if and only if we are on a gradient shrinking soliton.

More generally, Perelman [34] showed that the monotonicity of \(\tilde{V}(\tau)\) also holds on noncompact manifolds by establishing a Jacobian comparison for the \(\mathcal{L}\)-exponential map \(\exp(\tau) : T_p M \to M\) associated to the \(\mathcal{L}\)-length.

**Theorem 4.1. (Monotonicity of the reduced volume)** Let \((M^n, g_{ij}(\tau))\) be a complete solution to the backward Ricci flow \(\frac{\partial}{\partial \tau}g_{ij} = 2R_{ij}\) with bounded curvature. Fix a point \(p \in M\) and let \(l(q, \tau)\) be the reduced distance from \((p, 0)\).

(i) the reduced volume \(\tilde{V}(\tau)\) is nonincreasing in \(\tau\), and \(\tilde{V}(\tau) \leq 1\) for all \(\tau\);

(ii) the monotonicity is strict unless we are on a gradient shrinking soliton.

**Remark 4.4.** As pointed out by Yokota [100], \(\tilde{V}(\tau) = 1\) for some \(\tilde{\tau} > 0\) if and only if \((M^n, g_{ij}(\tau))\) is the Gaussian shrinker on \(\mathbb{R}^n\).

Using the monotonicity of the reduced volume, one can derive a version of no local collapsing theorem for noncompact solutions with bounded curvature.

**Definition 4.1.** Given any positive constants \(\kappa > 0\) and \(r > 0\), we say a solution to the Ricci flow is \(\kappa\)-noncollapsed at \((x_0, t_0)\) on the scale \(r\) if it satisfies the following property: if \(|Rm|(x, t) \leq r^{-2}\) for all \((x, t) \in B_{t_0}(x_0, r) \times [t_0 - r^2, t_0]\), then

\[
\text{vol}_{t_0}(B_{t_0}(x_0, r)) \geq \kappa r^n.
\]

**Theorem 4.2. (Perelman [34])** Let \((M^n, \hat{g}_{ij})\) be a complete Riemannian manifold with bounded curvature \(|Rm| \leq k_0\) and with injectivity radius bounded from below by \(\text{inj}(M, \hat{g}_{ij}) \geq i_0\), for some positive constants \(k_0\) and \(i_0\). Let \(g_{ij}(t), t \in [0, T]\) be a smooth solution to the Ricci flow with bounded curvature for each \(t \in [0, T]\) and \(g_{ij}(0) = \hat{g}_{ij}\). Then there is a constant \(\kappa = \kappa(k_0, i_0, T) > 0\) such that the solution is \(\kappa\)-noncollapsed on scales \(\leq \sqrt{T}\).

In the special case of gradient shrinking solitons, the above result has been improved recently by Naber [77] and further by Carrillo-Ni [19].

**Proposition 4.1. (Carrillo-Ni [19])** Let \((M^n, g, f)\) be a gradient shrinking soliton satisfying Eq. (1.2) for \(\rho = 1/2\) and with either bounded Ricci curvature \(|Rc| \leq C\) or nonnegative Ricci curvature \(Rc \geq 0\). Then there exists a positive constant \(\kappa > 0\) such that \(\text{vol}(B(x_0, 1)) \geq \kappa\) whenever \(|Rc| \leq 1\) on \(B(x_0, 1)\).
Remark 4.5. In [77], Naber showed a similar result but requires the bound on the curvature tensor $Rm$.

5. GEOMETRY OF GRADIENT RICCI SOLITONS

In this section, we examine the geometric structures of gradient steady and shrinking Ricci solitons, in particular the sort we get as Type I or Type II limits.

We start by examining ancient solutions with nonnegative curvature, and two geometric quantities, the asymptotic scalar curvature ratio and the asymptotic volume ratio.

5.1. Ancient solutions of the Ricci Flow. A complete solution $g_{ij}(t)$ to the Ricci flow is called ancient if it is defined for $-\infty < t < T$. By definition, Type I and Type II singularity models are ancient, and so are steady and shrinking Ricci solitons.

Now let us recall the definitions of the asymptotic scalar curvature ratio and the asymptotic volume ratio (cf. [67]).

Suppose $(M^n, g)$ is an $n$-dimensional complete noncompact Riemannian manifold with nonnegative Ricci curvature $Rc \geq 0$. Let $O$ be a fixed point on a Riemannian manifold $M^n$. We consider the geodesic ball $B(O, r)$ centered at $O$ of radius $r$. The well-known Bishop volume comparison theorem tells us that the ratio $Vol(B(O, r))/r^n$ is nonincreasing in $r \in [0, +\infty)$. Thus there exists a limit

$$\nu_M = \lim_{r \to +\infty} \frac{Vol(B(O, r))}{r^n},$$

which is called the asymptotic volume ratio of $(M^n, g)$ which is invariant under dilation and is independent of the choice of the origin.

Remark 5.1. By Lemma 8.10 in [34], if $(M^n, g)$ is a complete Ricci flat manifold, then the asymptotic volume ratio $\nu_M$ is equal to the asymptotic reduced volume:

$$\nu_M = \lim_{\tau \to \infty} \tilde{V}(\tau).$$

Next, let $s(x)$ denote the distance from $x \in M$ to the fixed point $O$, and $R$ the scalar curvature. The asymptotic scalar curvature ratio of $(M^n, g)$ is defined by

$$A = \limsup_{s \to +\infty} Rs^2,$$

which is also independent of the choice of the fixed point $O$ and invariant under dilation.

Remark 5.2. The concept of asymptotic scalar curvature ratio is particular useful on manifolds with positive sectional curvature. The first gap type theorem was obtained by Mok-Siu-Yau [76]. Yau (see [58]) suggested that this should be a general phenomenon, which confirmed by Greene-Wu [58], Eschenberg-Shrader-Strake [45] and Drees [42]. They showed that any complete noncompact $n$-dimensional (except $n = 4$ or 8) Riemannian manifold of positive sectional curvature must have positive asymptotic scalar curvature ratio $A > 0$. Similar results on complete noncompact Kähler manifolds of positive holomorphic bisectional curvature were obtained by Chen-Zhu [26] and Ni-Tam [81].
Proposition 5.1. (Hamilton [67]) For a complete ancient solution to the Ricci flow with bounded and nonnegative curvature operator (or Kähler with bounded and nonnegative bisectional curvature), the asymptotic scalar curvature ratio $A$ is constant (i.e., independent of time).

It is then natural to ask when is $A = \infty$, or $A < \infty$ for an ancient solution. Intuitively, we see a paraboloid has $A = \infty$, while a cone has $0 < A < \infty$.

Proposition 5.2. (Hamilton [67]) Suppose $g_{ij}(t)$ is a complete ancient solution to the Ricci flow with bounded and positive curvature operator. Assume $g_{ij}(t)$ is also Type-I like so that

$$\sup_{t \in (-\infty, T)} R(T - t) < +\infty$$

and assume the asymptotic scalar curvature ratio $A < \infty$. Then

(a) The asymptotic volume ratio $\nu_M$ is positive; and

(b) for any origin $O$ and any time $t$, there exists $\phi(O, t) > 0$ such that $R(x, t)s(x)^2 \geq \phi(O, t)$ at every point $x$, where $s(x)$ is the distance from $x$ to any origin $O$.

On the other hand, we have

Theorem 5.1. (Perelman [84]) Let $g_{ij}(t)$ be a complete non-flat ancient solution to the Ricci flow with bounded and nonnegative curvature operator on a noncompact manifold $M^n$. Then the asymptotic volume ratio $\nu_M(t) = 0$ for all $t$.

In the Kähler case, the same result as Theorem 5.1 (assuming nonnegative curvature operator) is obtained independently by Chen-Zhu [27]. Moreover, we have the following stronger result in which the assumption of nonnegativity of curvature operator is replaced by the weaker assumption of nonnegative holomorphic bisectional curvature.

Theorem 5.2. (Chen-Tang-Zhu [28] and Cao [12] for $n = 2$, Ni [78] for $n \geq 3$) Let $g_{\alpha\beta}(t)$ be a complete non-flat ancient solution to the Kähler-Ricci flow with bounded and nonnegative holomorphic bisectional curvature on a noncompact complex manifold $X^n$. Then the asymptotic volume ratio $\nu_X(t) = 0$ for all $t$.

Note that combining Theorem 5.1 with Proposition 5.2 we can deduce the following

Proposition 5.3. Any complete noncompact ancient Type I-like solution to the Ricci flow with bounded and positive curvature operator on an $n$-dimensional manifold must have infinite asymptotic scalar curvature ratio $A = \infty$.

Remark 5.3. Proposition 5.3 was first proved by Chow-Lu [32] for $n = 3$.

Of course Theorem 5.1 implies that any complete non-flat shrinking or steady soliton with bounded and nonnegative curvature operator has zero asymptotic volume ratio $\nu_M = 0$. On the other hand, for gradient shrinking or expanding Ricci solitons, Carrillo-Ni [19] recently showed the same result under weaker curvature conditions.

Proposition 5.4. (Carrillo-Ni [19])

(a) If $(M^n, g, f)$ is a gradient shrinking soliton with nonnegative Ricci curvature $Rc \geq 0$, then the asymptotic volume ratio $\nu_M = 0$.

(b) If $(M^n, g, f)$ is a gradient expanding soliton with nonnegative scalar curvature $R \geq 0$, then the asymptotic volume ratio $\nu_M > 0$. 
Remark 5.4. It was known to Hamilton that if \((M^n, g, f)\) is a gradient expanding soliton with bounded and nonnegative Ricci curvature \(0 \leq Rc \leq C\), then the asymptotic volume ratio \(\nu_M > 0\).

We like to point out that, due to the Hamilton-Ivey pinching theorem\(^{5}\), it turns out a 3-dimensional complete ancient solution \((M^3, g_{ij}(t))\) with bounded curvature \(|Rm| \leq C\) has nonnegative sectional curvature. Also, every complete ancient solution \((M^n, g_{ij}(t))\) with bounded curvature \(|Rm| \leq C\) has nonnegative scalar curvature. This follows from by applying the standard maximum principle to the evolution equation of the scalar curvature \(R\)

\[
\frac{\partial}{\partial t} R = \Delta R + 2|Rc|^2.
\]

Recently, B.-L. Chen\(^{24}\) was able to remove the curvature bound assumption in both results mentioned above.

**Theorem 5.3. (B.-L. Chen\(^{24}\))** A 3-dimensional complete ancient solution has nonnegative sectional curvature.

**Proposition 5.5. (B.-L. Chen\(^{24}\))** Let \(g_{ij}(t)\) be a complete ancient solution on a noncompact manifold \(M^n\). Then the scalar curvature \(R\) of \(g_{ij}(t)\) is nonnegative for all \(t\).

To illustrate the local arguments used in\(^{24}\), below we sketch the proof of Proposition 5.5 which is relatively simpler.

**Proof.** Suppose \(g_{ij}\) is defined for \(-\infty < t \leq T\) for some \(T > 0\). We can divide the arguments in\(^{24}\) into two steps:

Step 1: Consider any complete solution \(g_{ij}(t)\) defined on \([0, T]\). For any fixed point \(x_0 \in M\), pick \(r_0 > 0\) sufficiently small so that

\[
|Rc|(-, t) \leq (n - 1)r_0^{-2} \quad \text{on} \quad B_t(x_0, r_0)
\]

for all \(t \in [0, T]\). Then for any positive number \(A > 2\), pick \(K_A > 0\) such that \(R \geq -K_A\) on \(B_0(x_0, Ar_0)\) at \(t = 0\). We claim that there exists a universal constant \(C > 0\) (depending on the dimension \(n\)) such that

\[
R(-, t) \geq \min\{-\frac{n}{t + \frac{3A}{4}r_0}\}, \quad \text{on} \quad B_t(x_0, \frac{3A}{4}r_0)
\]

(5.1)

for each \(t \in [0, T]\).

Indeed, take a smooth nonnegative decreasing function \(\phi\) on \(\mathbb{R}\) such that \(\phi = 1\) on \((-\infty, 7/8]\), and \(\phi = 0\) on \([1, \infty)\). Consider the function

\[
u(x, t) = \phi\left(\frac{d_t(x_0, x)}{Ar_0}\right)R(x, t).
\]

Then we have

\[
(\frac{\partial}{\partial t} - \Delta)u = \frac{\phi'}{Ar_0}(\frac{\partial}{\partial t} - \Delta)d_t(x_0, x) - \frac{\phi'' R}{(Ar_0)^2} + 2\phi |Rc|^2 - 2\nabla \phi \cdot \nabla R
\]

at smooth points of the distance function \(d_t(x_0, \cdot)\).

\(^{5}\)See, e.g., Theorem 2.4.1 in\(^{17}\) or Theorem 6.44 in\(^{33}\)

\(^{6}\)Though Proposition 5.5 was not stated explicitly in\(^{24}\), the essential arguments are there (see Proposition 2.1(i) and Corollary 2.3(i) in\(^{24}\)).
Let $u_{\text{min}}(t) = \min_M u(\cdot, t)$. Whenever $u_{\text{min}}(t_0) \leq 0$, assume $u_{\text{min}}(t_0)$ is achieved at some point $\bar{x} \in B_t(x_0, r_0)$, then $\phi' R(\bar{x}, t_0) \geq 0$. On the other hand, by Lemma 8.3(a) of Perelman [84] (or Lemma 3.4.1(i), [17]), we know that

$$
\frac{\partial}{\partial t} - \Delta) d_t(x_0, x) \geq -\frac{5(n-1)}{3r_0}
$$

outside $B_t(x_0, r_0)$. Following (Section 3, Hamilton [63]), we define

$$
\frac{d}{dt} \bigg|_{t=t_0} u_{\text{min}} = \liminf_{h \to 0^+} \frac{u_{\text{min}}(t_0 + h) - u_{\text{min}}(t_0)}{h},
$$

the liminf of all forward difference quotients, then

$$
\frac{d}{dt} \bigg|_{t=t_0} u_{\text{min}} \geq -\frac{5(n-1)}{3Ar_0^2} \phi' R + \frac{2}{n} \phi R^2 + \frac{1}{(Ar_0)^2} (\frac{2\phi'^2}{\phi} - \phi'') R.
$$

Hence,

$$
\frac{d}{dt} \bigg|_{t=t_0} u_{\text{min}} \geq -\frac{n}{t + \frac{1}{K}} \frac{C}{Ar_0^2} - \frac{C^2}{(Ar_0)^2},
$$

provided $u_{\text{min}}(t_0) \leq 0$. Now integrating the above inequality, we get

$$
u_{\text{min}}(t) \geq \min\left\{ -\frac{n}{t + \frac{1}{K}}, -\frac{C}{Ar_0^2} \right\} \quad \text{on } B_t(x_0, 3A r_0),
$$

and the inequality (5.1) in our claim follows.

Step 2: Now if our solution $g_{ij}(t)$ is ancient, we can replace $t$ by $t - \alpha$ in (5.1) and get

$$
R(\cdot, t) \geq \min\left\{ -\frac{n}{t - \alpha + \frac{1}{K}}, -\frac{C}{Ar_0^2} \right\} \quad \text{on } B_t(x_0, 3A r_0).
$$

Letting $A \to \infty$ and then $\alpha \to -\infty$, we completed the proof of Theorem 5.3.

We also would like to mention Yokota [100] recently proved an interesting gap theorem for ancient solutions.

**Proposition 5.6.** There exists a (small) constant $\epsilon > 0$ depending only on the dimension $n$ which satisfies the following property: suppose $(M^n, g(t))$, $-\infty < t \leq T_0$ is a complete ancient solution to the Ricci flow with Ricci curvature bounded below such that

$$
\lim_{\tau \to -\infty} \tilde{V}(\tau) \geq 1 - \epsilon.
$$

Then $(M^n, g(t))$ is a Gaussian shrinker on $\mathbb{R}^n$. Here $\tilde{V}(\tau)$ is the reduced volume with respect to $(p, 0)$ for some base point $p \in M$, and $\tau = T_0 - t$.

We end this subsection by noting the following classification result of Hamilton [67] on 2-dimensional ancient $\kappa$-solutions.

**Theorem 5.4.** The only 2-dimensional non-flat ancient solutions to the Ricci flow with bounded and nonnegative curvature and $\kappa$-noncollapsed on all scales are the round sphere and the round real projective plane.
5.2. **Geometry of gradient steady and expanding Ricci solitons.** We now turn to gradient steady and expanding solitons.

**Proposition 5.7.** (Hamilton [67]) Suppose we have a complete noncompact gradient steady Ricci soliton \((M^n, g_{ij})\) so that

\[
R_{ij} = \nabla_i \nabla_j f
\]

for some potential function \(f\) on \(M\). Assume the Ricci curvature is positive \(R_c > 0\), and the scalar curvature \(R\) attains its maximum \(R_{\text{max}}\) at a point \(x_0 \in M^n\). Then

\[
|\nabla f|^2 + R = R_{\text{max}}
\]
everywhere on \(M^n\). Furthermore, the function \(f\) is convex and attains its minimum at \(x_0\).

**Remark 5.5.** It was observed by Hamilton and the author [15] that \(f\) is also a exhaustion function of linear growth. As a consequence, the underlying manifold of the gradient steady soliton is diffeomorphic to the Euclidean space \(\mathbb{R}^n\).

**Remark 5.6.** In case of a complete gradient expanding soliton with nonnegative Ricci curvature, the potential function \(f\) is a convex exhaustion function of quadratic growth. Hence we have

**Proposition 5.8.** Let \((M^n, g_{ij}, f)\) be a gradient expanding soliton with \(R_c \geq 0\). Then \(M^n\) is diffeomorphic to \(\mathbb{R}^n\).

In the Kähler setting we have the following strong uniformization type result.

**Proposition 5.9.** (Bryant [6] and Chau-Tam [22]) Suppose we have a complete noncompact gradient steady Kähler-Ricci soliton \((X^n, g_{i\bar{j}})\). Assume Ricci curvature is positive \(R_c > 0\), and the scalar curvature \(R\) attains its maximum \(R_{\text{max}}\) at a point \(x_0 \in X^n\). Then \(X^n\) is biholomorphic to the complex Euclidean space \(\mathbb{C}^n\).

**Remark 5.7.** Under the same assumption as in Proposition 5.9, it was observed earlier by Hamilton and the author [15] (see also [12]) that \(X^n\) is Stein and diffeomorphic to \(\mathbb{R}^{2n}\).

**Remark 5.8.** Chau-Tam [22] also showed

**Proposition 5.10.** Let \((X^n, g_{i\bar{j}})\) be a complete noncompact gradient expanding Kähler-Ricci soliton with nonnegative Ricci curvature, then \(X^n\) is biholomorphic to \(\mathbb{C}^n\).

**Proposition 5.11.** (Hamilton [67]) For a complete noncompact gradient steady Ricci soliton with bounded curvature and positive sectional curvature of dimension \(n \geq 3\) where the scalar curvature assume its maximum at a point \(O \in M\), the asymptotic scalar curvature ratio is infinite

\[
A = \limsup_{s \to +\infty} R s^2 = +\infty.
\]

**Remark 5.9.** In fact, by examining Hamilton’s proof in [67], one gets the stronger conclusion that

\[
A_{1+\varepsilon} = \limsup_{s \to +\infty} R s^{1+\varepsilon} = +\infty
\]
for arbitrarily small \(\varepsilon > 0\).
Remark 5.10. For $n = 3$, also see the work of Chu [36] for more geometric information.

One of the basic questions is to classify steady Ricci solitons with positive curvature. When dimension $n = 2$, we have the following important uniqueness result of Hamilton.

**Theorem 5.5. (Hamilton [64])** The only complete steady Ricci soliton on a two-dimensional manifold with bounded (scalar) curvature $R$ which assumes its maximum $R_{\text{max}} = 1$ at an origin is the cigar soliton on the plane $\mathbb{R}^2$ with the metric

$$ds^2 = \frac{dx^2 + dy^2}{1 + x^2 + y^2}.$$

Remark 5.11. For $n = 3$, Prelman ([84], 11.9) claimed that any complete non-compact $\kappa$-noncollapsed gradient steady soliton with bounded positive curvature must be the Bryant soliton. He also conjectured that any complete noncompact three-dimensional $\kappa$-noncollapsed ancient solution with bounded positive curvature is necessarily a Bryant soliton.

5.3. **Geometry of shrinking solitons.** In this subsection, we describe recent progress on complete shrinking Ricci solitons.

First of all, by the splitting theorem of Hamilton [63], a complete shrinking Ricci soliton with bounded and nonnegative curvature operator either has positive curvature operator everywhere or its universal cover splits as a product $N \times \mathbb{R}^k$, where $k \geq 1$ and $N$ is a shrinking soliton with positive curvature operator. On the other hand, we know that compact shrinking solitons with positive curvature operator are isometric to finite quotients of round spheres, thanks to the works of Hamilton [62, 63] (for $n = 3, 4$) and B"ohm-Wilking [4] (for $n \geq 5$).

For dimension $n = 3$, Perelman [84] proved the following

**Proposition 5.12. (Perelman [84])** There does not exist a three-dimensional complete noncompact $\kappa$-noncollapsed gradient shrinking soliton with bounded and positive sectional curvature.

In other words, a three-dimensional complete $\kappa$-noncollapsed gradient shrinking soliton with bounded and positive sectional curvature must be compact.

Based on the above proposition, Perelman [85] obtained the following important classification result (see also Lemma 6.4.1 in [17]), which is an improvement of a result of Hamilton (Theorem 26.5, [67]).

**Theorem 5.6. (Perelman [84])** Let $g_{ij}(t)$ be a nonflat gradient shrinking soliton to the Ricci flow on a three-manifold $M^3$. Suppose $g_{ij}(t)$ has bounded and nonnegative sectional curvature and is $\kappa$-noncollapsed on all scales for some $\kappa > 0$. Then $(M, g_{ij}(t))$ is one of the following:

(i) the round three-sphere $S^3$, or one of its metric quotients;
(ii) the round infinite cylinder $S^2 \times \mathbb{R}$, or its $\mathbb{Z}_2$ quotient.

Thus, the only three-dimensional complete noncompact $\kappa$-noncollapsed gradient shrinking soliton with bounded and nonnegative sectional curvature are either $\mathbb{R}^3$ or quotients of $S^2 \times \mathbb{R}$.

In the past a few years, there has been a lot of attempts to improve and generalize the above results of Perelman. Ni-Wallach [79] and Naber [77] dropped the assumption on $\kappa$-noncollapsing condition and replaced nonnegative sectional
curvature by nonnegative Ricci curvature. In addition, Ni-Wallach \[79\] allows the curvature $|Rm|$ to grow as fast as $e^{ar(x)}$, where $r(x)$ is the distance function to some origin and $a > 0$ is a suitable small positive constant. In particular, they proved

**Proposition 5.13. (Ni-Wallach \[79\])** Any 3-dimensional complete noncompact non-flat gradient shrinking soliton with nonnegative Ricci curvature $Rc \geq 0$ and with $|Rm|(x) \leq Ce^{ar(x)}$ must be a quotient of the round cylinder $S^2 \times \mathbb{R}$.

Recently, based on Theorem 5.3 and Proposition 5.13, B.-L. Chen, X.-P. Zhu and the author \[14\] observed that one can actually remove all the curvature bound assumptions.

**Theorem 5.7. (Cao-Chen-Zhu \[14\])** Let $(M^3, g_{ij})$ be a 3-dimensional complete noncompact non-flat shrinking gradient soliton. Then $(M^3, g_{ij})$ is a quotient of the round cylinder $S^2 \times \mathbb{R}$.

For $n = 4$, Ni and Wallach \[80\] showed that any 4-dimensional complete gradient shrinking soliton with nonnegative curvature operator and positive isotropic curvature, satisfying certain additional assumptions, is a quotient of either $S^4$ or $S^3 \times \mathbb{R}$. Partly based on this result, Naber \[77\] proved

**Proposition 5.14. (Naber \[77\])** Any 4-dimensional complete noncompact shrinking Ricci soliton with bounded and nonnegative curvature operator is isometric to either $\mathbb{R}^4$, or a finite quotient of $S^3 \times \mathbb{R}$ or $S^2 \times \mathbb{R}^2$.

For higher dimensions, H. Gu and X.-P. Zhu \[61\] first proved that any complete, rotationally symmetric, non-flat, $n$-dimensional ($n \geq 3$) shrinking Ricci soliton with $\kappa$-noncollapsing on all scales and with bounded and nonnegative sectional curvature must be the round sphere $S^n$ or the round cylinder $S^{n-1} \times \mathbb{R}$. Subsequently, Kotschwar \[73\] proved that the only complete shrinking Ricci solitons (without the curvature sign and bound assumptions) of rotationally symmetric metrics (on $S^n$, $\mathbb{R}^n$ and $R \times S^{n-1}$) are, respectively, the round, flat, and standard cylindrical metrics.

Recently, various authors proved classification results on gradient shrinking solitons with vanishing Weyl curvature tensor which include the rotationally symmetric ones as a special case.

**Proposition 5.15. (Ni-Wallach \[79\])** Let $(M^n, g)$ be a complete, locally conformally flat gradient shrinking soliton with nonnegative Ricci curvature. Assume that

$$|Rm|(x) \leq e^{a(r(x)+1)}$$

for some constant $a > 0$, where $r(x)$ is the distance function to some origin. Then its universal cover is $\mathbb{R}^n$, $S^n$ or $S^{n-1} \times \mathbb{R}$.

**Remark 5.12.** In the compact case, Eminenti-La Nave-Mantegazza \[44\] first showed that every compact shrinking Ricci soliton with vanishing Weyl tensor is a quotient of $S^n$.

In \[86\], Petersen-Wylie removed the assumption on nonnegative Ricci curvature and replaced the pointwise curvature growth assumption by certain integral bound on the norm of Ricci tensor.

\[ ^{7}\text{See also \[18\] for an alternative proof.} \]
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Proposition 5.16. (Petersen-Wylie [80]) Let \((M^n, g)\) be a complete gradient shrinking Ricci soliton with potential function \(f\). Assume the Weyl tensor \(W = 0\) and
\[
\int_M |Rc|^2 e^{-f} dvol < \infty,
\]
then \((M^n, g)\) is a finite quotient of \(\mathbb{R}^n\), \(S^n\) or \(S^{n-1} \times \mathbb{R}\).

On the other hand, Z.-H. Zhang [102] removed all curvature bound assumptions in Proposition 5.15 and Proposition 5.16, thus giving an extension of Theorem 5.7 to higher dimensions.

Theorem 5.8. (Z.-H. Zhang [102]) Any complete gradient shrinking soliton with vanishing Weyl tensor must be a finite quotients of \(\mathbb{R}^n\), \(S^n\) or \(S^{n-1} \times \mathbb{R}\).

Remark 5.13. In the Kähler case, we have a rather satisfying result in all dimensions by Ni [78].

Proposition 5.17. (Ni [78]) In any complex dimension, there is no complete noncompact gradient shrinking Kähler-Ricci soliton with positive holomorphic bi-sectional curvature.

Remark 5.14. Wylie [97] showed that a complete shrinking Ricci soliton has finite fundamental group. In the compact case, the result was first proved by Derdzinski [41] and Fernández-López and García-Río [49] (see also a different proof by Eminenti-La Nave-Mantegazza [44]). This can be viewed as an extension of the well-known Myers’ theorem. Moreover, Fang-Man-Zhang [46] proved that a complete gradient shrinking Ricci soliton with bounded scalar curvature has finite topological type.

6. Stability of Ricci Solitons

In this section we describe the second variation formulas for Perelman’s \(\lambda\)-energy and \(\nu\)-energy due to Hamilton, Ilmanen and the author [16].

6.1. Second Variation of \(\lambda\)-energy. Recall that the \(\lambda\)-energy is defined by
\[
\lambda(g_{ij}) = \inf \{ \mathcal{F}(g_{ij}, f) : f \in C^\infty(M), \int_M e^{-f} dV = 1 \}
\]
and its first variation is given by
\[
\frac{d}{ds} \bigg|_{s=0} \lambda(g(s)) = \int_M -h_{ij}(R_{ij} + \nabla_i \nabla_j f)e^{-f} dV,
\]
where \(f\) is a minimizer.

For any symmetric 2-tensor \(h = h_{ij}\) and 1-form \(\omega = \omega_i\), denote \(Rm(h, h) := R_{ijkl}h_{ik}h_{jl}\), \(\text{div } \omega := \nabla_i \omega_i\), \(\text{div } h := \nabla_j h_{ji}\), \((\text{div}^* \omega)_{ij} := -(\nabla_i \omega_j + \nabla_j \omega_i)/2 = -(1/2)L_{\omega^\#}g_{ij}\), where \(\omega^\#\) is the vector field dual to \(\omega\).

Theorem 6.1. (Cao-Hamilton-Ilmanen [16]) Let \((M^n, g)\) be a compact Ricci flat manifold and consider variations \(g(s) = g + sh\). Then the second variation \(D^2 g \lambda(h, h)\) of \(\lambda\) at \(g\) is given by
\[
\frac{d^2}{ds^2} \bigg|_{s=0} \lambda(g(s)) = \int_M <Lh, h> dV,
\]
where
\[
Lh := \frac{1}{2} \Delta h + \text{div}^* \text{div } h + \frac{1}{2} \nabla^2 v_h + Rm(h, \cdot),
\]
and \( v_h \) satisfies
\[
\Delta v_h = \text{div} \text{div} h.
\]

Note if we decompose \( C^\infty(\text{Sym}^2(T^*M)) \) as
\[
\ker \text{div} \oplus \text{im} \text{div}^*.
\]

One verifies that \( L \) vanishes on \( \text{im} \text{div}^* \), that is, on Lie derivatives. On \( \ker \text{div} \) one has
\[
L = \frac{1}{2} \Delta_L,
\]
where
\[
\Delta_L h := \Delta h + 2Rm(h, \cdot) - Rc \cdot h - h \cdot Rc
\]
is the Lichnerowicz Laplacian on symmetric 2-tensors. We call a critical point \( g \) of \( \lambda \) linearly stable if \( L \leq 0 \).

**Example 6.1.** A Calabi-Yau K3 surface has \( \Delta_L \leq 0 \) according to Guenther-Isenberg-Knopf [59]. More generally, Dai-Wang-Wei [37] showed that any manifold with a parallel spinor has \( \Delta_L \leq 0 \). So these manifolds are linearly stable in the sense presented above.

**Example 6.2.** Let \( g \) be compact and Ricci flat. Following [7, 59], we examine conformal variations. It is convenient to replace \( ug \) by
\[
h = Su := (\Delta u)g - D^2 u
\]
which differs from the conformal direction only by a Lie derivative and is divergence free. We have
\[
\Delta_L Su = (S \Delta u)g,
\]
so \( \Delta_L \) has the same eigenvalues as \( \Delta \). In particular, \( L \leq 0 \) in the conformal direction. This contrasts with the Einstein functional.

6.2. **Second Variation of \( \nu \)-energy.** Recall that the \( \nu \)-energy is defined by
\[
\nu(g_{ij}) = \inf \{ W(g, f, \tau) : f \in C^\infty(M), \tau > 0, \frac{1}{(4\pi\tau)^{n/2}} \int_M e^{-f} dV = 1 \}
\]
and its first variation is given by
\[
\frac{d}{ds} \bigg|_{s=0} \nu(g_{ij}(s)) = \frac{1}{(4\pi\tau)^{n/2}} \int_M -h_{ij} [\tau(R_{ij} + \nabla_i \nabla_j f) - g_{ij}/2] e^{-f} dV.
\]

We now describe the second variation of the \( \nu \)-energy in [16] for positive Einstein metrics and compact shrinking Ricci solitons.

First, for Einstein metrics normalized by \( Rc = g/2\tau \), we have

**Theorem 6.2.** (Cao-Hamilton-Ilmanen [16]) Let \( (M, g) \) be a Einstein manifold of positive scalar curvature and consider variations \( g(s) = g + sh \). Then the second variation \( D^2_{g} \nu(h, h) \) is given by
\[
\frac{d^2}{ds^2} \bigg|_{s=0} \nu(g(s)) = \frac{\tau}{\text{vol}(g)} \int_M < Nh, h >,
\]
where
\[
Nh := \frac{1}{2} \Delta h + \text{div}^* \text{div} h + \frac{1}{2} \nabla^2 v_h + Rm(h, \cdot) - \frac{g}{2n\tau \text{vol}(g)} \int_M \text{tr} g h.
\]
and $v_h$ is the unique solution of

$$\Delta v_h + \frac{v_h}{2\tau} = \text{div} \text{div} h, \quad \int_M v_h = 0.$$  

As in the previous case, $N$ is degenerate negative elliptic and vanishes on $\text{im} \text{div}^*$. Write

$$\ker \text{div} = (\ker \text{div})_0 \oplus \mathbb{R}g$$

where $(\ker \text{div})_0$ is defined by $\int \text{tr} g h = 0$. Then on $(\ker \text{div})_0$ we have

$$N = \frac{1}{2} \left( \Delta_L + \frac{1}{\tau} \right)$$

where $\Delta_L$ is the Lichnerowicz Laplacian. So the linear stability of a shrinker comes down to the (divergence free) eigenvalues of the Lichnerowicz Laplacian. Let us write $\mu_L$ for the maximum eigenvalue of $\Delta_L$ on symmetric 2-tensors and $\mu_N$ for the maximum eigenvalue of $N$ on $(\ker \text{div})_0$. We now quote several examples from [16]:

**Example 6.3.** The round sphere is linearly stable: $\mu_N = -2/(n-1)\tau < 0$. In fact, it is geometrically stable (i.e. nearby metrics are attracted to it up to scale and gauge) by the results of Hamilton [62, 63, 64] and Huisken [69].

**Example 6.4.** For complex projective space $\mathbb{CP}^m$, the maximum eigenvalue of $\Delta_L$ on $(\ker \text{div})_0$ is $\mu_L = -1/\tau$ by work of Goldschmidt [56], so $\mathbb{CP}^m$ is neutrally linearly stable, i.e. the maximum eigenvalue of $m$ on $(\ker \text{div})_0$ is $\mu_N = 0$.

**Example 6.5.** Any product of two Einstein manifolds $M = M_1^{n_1} \times M_2^{n_2}$ is linearly unstable, with $\mu_N = 1/2\tau$. The destabilizing direction $h = g_1/n_1 - g_2/n_2$ corresponds to a growing discrepancy in the size of the factors.

**Example 6.6.** Any compact Kähler-Einstein manifold $X^n$ of positive scalar curvature with $\dim H^{1,1}(X) \geq 2$ is linearly unstable. Indeed, we can compute $\mu_N$ as follows. Let $\sigma$ be a harmonic 2-form and $h$ be the corresponding metric perturbation; then $\Delta_L h = 0$, and if $\sigma$ is chosen perpendicular to the Kähler form, then as above we obtain $\mu_N = 1/2\tau$.

**Example 6.7.** Let $Q^m$ denote the complex hyperquadric in $\mathbb{CP}^{m+1}$ defined by

$$\sum_{i=0}^{m+1} z_i^2 = 0,$$

a Hermitian symmetric space of compact type, hence a Kähler-Einstein manifold of positive scalar curvature.

(a) $Q^2$ is isometric to $\mathbb{CP}^1 \times \mathbb{CP}^1$, the simplest example of the above instability phenomenon.

(b) $Q^3$ has $\dim H^{1,1}(Q^3) = 1$, so the above discussion does not apply. But the maximum eigenvalue of $\Delta_L$ on $(\ker \text{div})_0$ is $\mu_L = -2/3\tau$ by work of Gasqui and Goldschmidt [53] (or see [54]). The proximate cause is a representation that appears in the sections of the symmetric tensors but not in scalars or vectors. Therefore, $Q^3$ is linearly unstable with

$$\mu_N = \frac{1}{6\tau}.$$
(c) For $Q^4$, the maximum eigenvalue of $\Delta_L$ on symmetric tensors is $\mu_L = -1/\tau$ by work of Gasqui and Goldschmidt [52] (or see [54]). So $Q^4$ is neutrally linearly stable: $\mu_N = 0$.

For compact shrinkers, we have the following general second variation formula.

**Theorem 6.3. (Cao-Hamilton-Ilmanen [16])** Let $(M, g)$ be a shrinker with the potential function $F$. The second variation $D_2^2\nu(h, h)$ is given by

$$\left. \frac{d^2}{ds^2} \right|_{s=0} \nu(g(s)) = \frac{\tau}{(4\pi\tau)^{n/2}} \int_M e^{-F} < N h, h >,$$

where

$$N h := \frac{1}{2} \Delta_F h + \text{div}_F^* \text{div}_F h + \frac{1}{2} D_F^2 v_h + Rm(h, \cdot) - \frac{\text{Rc}}{n(4\pi\tau)^{n/2}} \int_M e^{-F} \text{tr}_g h,$$

and $v_h$ is the unique solution of

$$\Delta_F v_h + \frac{v_h}{2\tau} = \text{div}_F \text{div}_F h, \quad \int_M \text{tr}_g h = 0.$$

Here we define

$$\text{div}_F h = (e^{-F} \text{div} e^F) h = (\text{div} - D_j F) h = D_j h_{ij} - D_j F h_{ij},$$

and

$$\Delta_F = \Delta - D_p F D_p.$$
7. The Gaussian Density of Shrinking Ricci Solitons

In [16], the notion of Gaussian density (or central density) of a shrinking Ricci soliton is introduced. In case of a compact shrinking soliton \((M^n, g)\), the Gaussian density is simply given by

\[
\Theta(M) = \Theta(M, g) := e^{\nu(M, g)}
\]

where \(\nu(M, g) = \nu(g_{ij})\) is the \(\nu\)-energy of the shrinker \((M, g)\).

In the following discussion, we will normalize Einstein manifolds of positive scalar curvature by \(Rc = g/(n - 1)\) (i.e., \(\tau = \frac{1}{2(n-1)}\)), so that the round sphere \(S^n\) has radius 1. As shown in [16], we have the following facts.

1. \(\Theta(S^n) = \left(\frac{n-1}{2\pi e}\right)^{n/2} \text{vol}(S^n)\).

2. If \(M\) is a Einstein manifold of positive scalar curvature, then

\[
\Theta(M) = \left(\frac{1}{4\pi^2 e}\right)^{n/2} \text{vol}(M) \leq \Theta(S^n),
\]

with equality if and only if \(M = S^n\).

3. \(\Theta(\mathbb{C}P^n) = \left(\frac{m+1}{\pi e}\right)^N \frac{\text{vol}(S^{2m+1})}{2\pi}\).

4. The Kähler-Einstein manifold \(M = \mathbb{C}P^2 \# k(-\mathbb{C}P^2),\) \(k = 0, 3, \ldots, 8,\) has \(\Theta(M) = \frac{(9-k)}{2e^2}\).

5. \(\Theta(M_1 \times M_2) = \Theta(M_1)\Theta(M_2)\).

As in [16], we say that one shrinking soliton decays to another if there is a small perturbation of the first whose Ricci flow develops a singularity modelled on the second. Because the \(\nu\)-invariant is monotone during the flow, decay can only occur from a shrinking soliton of lower density to one of higher density. This creates a “decay lowerarchy”.

8. 4-D Einstein Manifolds and Shrinking Ricci Solitons

In this section we collect information about stability and Gaussian density values of all known (orientable) positive Einstein 4-manifolds and 4-dimensional compact shrinking Ricci solitons. Below is a list containing all the information we know so far. Note that on \(\mathbb{C}P^2 \# (-\mathbb{C}P^2)\) we have the Kähler-Ricci soliton metric of [72, 10] and the Page non-Kähler Einstein metric [82], while on \(\mathbb{C}P^2 \# 2(-\mathbb{C}P^2)\) we have the Kähler-Ricci soliton metric of [96] and the very recent Non-Kähler Einstein metric of [29]. Also, Headrick-Wiseman [68] found out \(\Theta = .455\) for the Wang-Zhu soliton through numerical computation\(^8\).

Remark 8.1. On any compact Einstein 4-manifold \(M^4\), the Hitchin-Thorpe inequality (see e.g., [3]) states that

\[
2\chi(M) \geq 3|\tau(M)|,
\]

where \(\chi(M)\) is the Euler characteristic and \(\tau(M)\) is the signature of \(M^4\). An interesting question is whether the Hitchin-Thorpe inequality holds for 4-dimensional

\(^8\)Added in Proof: Very recently Stuart Hall has found the value of \(\Theta = .4552\) for the Non-Kähler Einstein metric in [29]. His calculation also yielded the value of \(\Theta = .4549\) for Wang-Zhu soliton, in line with [68].
compact shrinkers. Note that for the Kähler-Ricci shrinkers $M = \mathbb{CP}^2 \# k(-\mathbb{CP}^2)$ ($k = 1, 2$), $\chi(M) = 3 + k$ and $|\tau(M)| = k - 1$, so $2\chi(M) \geq 3|\tau(M)|$ is valid.

| Shrinking Solitons | Type | $\Theta$ | $\Theta$ | Stability  |
|-------------------|------|---------|---------|-----------|
| $S^4$             | Einstein | $6/e^2$ | .812    | Stable    |
| $\mathbb{CP}^2$  | Einstein | $9/2e^2$ | .609    | Stable    |
| $S^2 \times S^2$ | Einstein product | $4/e^2$ | .541    | Unstable  |
| $\mathbb{CP}^2 \# (-\mathbb{CP}^2)$ | Kähler-Ricci soliton ($[72],[10]$) | $3.826/e^2$ | .518    | Unknown   |
| $\mathbb{CP}^2 \# (-\mathbb{CP}^2)$ | non-Kähler Einstein ($[82]$) | $3.821/e^2$ | .517    | Unknown   |
| $\mathbb{CP}^2 \# 2(-\mathbb{CP}^2)$ | Kähler-Ricci soliton ($[96]$) | $3.361/e^2$ | .4549  | Unknown   |
| $\mathbb{CP}^2 \# 2(-\mathbb{CP}^2)$ | non-Kähler Einstein ($[29]$) | $< 7/2e^2$ | .4552  | Unknown   |
| $\mathbb{CP}^2 \# 3(-\mathbb{CP}^2)$ | Kähler-Einstein | $3/e^2$   | .406    | Unstable  |
| $\mathbb{CP}^2 \# 4(-\mathbb{CP}^2)$ | Kähler-Einstein | $5/e^2$   | .338    | Unstable  |
| $\mathbb{CP}^2 \# 5(-\mathbb{CP}^2)$ | Kähler-Einstein | $2/e^2$   | .271    | Unstable  |
| $\mathbb{CP}^2 \# 6(-\mathbb{CP}^2)$ | Kähler-Einstein | $3/e^2$   | .203    | Unstable  |
| $\mathbb{CP}^2 \# 7(-\mathbb{CP}^2)$ | Kähler-Einstein | $1/e^2$   | .135    | Unstable  |
| $\mathbb{CP}^2 \# 8(-\mathbb{CP}^2)$ | Kähler-Einstein | $1/2e^2$  | .068    | Unstable  |

9. **Open Problems**

We conclude with several problems/questions:

(i) Find a non-product, non-Einstein, Riemannian (non-Kähler) shrinking Ricci soliton.

(ii) Show that any noncompact Type I singularity model obtained as a rescaling limit of a Type I maximal solution is a gradient shrinking soliton. Naturally, one should try to explore Perelman’s reduced volume approach.

(iii) Show that any 3-dimensional complete noncompact gradient steady soliton with bounded and positive curvature is the Bryant soliton. Are there non-rotationally symmetric steady soliton with positive curvature in dimensions $n \geq 4$?

(iv) For $n \geq 4$, are there any complete noncompact ($\kappa$-noncollapsed) gradient shrinking Ricci soliton with positive curvature operator or positive Ricci curvature?
So far there seems no known example of noncompact non-product shrinker with nonnegative Ricci curvature.

(v) Is it true that linearly stable compact 4-dimensional shrinking solitons are necessarily Einstein? Are the only linearly stable Einstein 4-manifolds either the round sphere $S^4$ or the complex projective space $\mathbb{CP}^2$ with the Fubini-Study metric?

(vi) Does the Hitchin-Thorpe inequality hold for compact 4-dimensional shrinking solitons?

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