Fourier Analysis of Advection-dominated Accretion Flows

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Abstract

We implement a new semi-analytical approach to investigate radially self-similar solutions for the steady-state advection-dominated accretion flows (ADAFs). We employ the usual $\alpha$-prescription for the viscosity, and all components of the energy–momentum tensor are considered. In this case, in the spherical coordinate, the problem reduces to a set of eighth-order, nonlinear differential equations with respect to the latitudinal angle $\theta$. Using the Fourier expansions for all the flow quantities, we convert the governing differential equations to a large set of nonlinear algebraic equations for the Fourier coefficients. We solve the algebraic equations via the Newton–Raphson method, and investigate the ADAF properties over a wide range of model parameters. We also show that the implemented series are truly convergent. The main advantage of our numerical method is that it does not suffer from the usual technical restrictions that may arise for solving ADAF differential equations near the polar axis. In order to check the reliability of our approach, we recover some widely studied solutions. Further, we introduce a new varying $\alpha$ viscosity model. New outflow and inflow solutions for ADAFs are also presented, using Fourier expansion series.

Key words: accretion, accretion disks – hydrodynamics

1. Introduction

Over recent decades, observations of the spectra of energetic astronomical objects have provided a good motivation for many astronomers to investigate the phenomena responsible for the enormous emitted energy of these objects, e.g., Fabbiano (1989), Makishima et al. (2000), Swartz et al. (2004). Theoretical models consistent with observational evidence have revealed that the accretion process around massive stars or black holes is a plausible mechanism for producing such energetic radiations. The standard theory of the accretion disks (see Shakura & Sunyaev 1973; Novikov & Thorne (1973)) is remarkably successful with regard to understanding some of the observational features of quasars, X-ray binaries, and AGNs; for more details, see Lynden-Bell & Pringle (1974), Watarai et al. (2001), Frank et al. (2002), Kato et al. (2008), and Abramowicz & Fragile (2013). In the standard Shakura & Sunyaev disk model (SSD), the energy released by the turbulent viscosity is radiated locally and the accretion flow is efficiently cooled. The SSD model is appropriate for accreting systems where the accretion rate is smaller than the Eddington rate, $\dot{M}_\text{Edd}$. At the same time, however, observational evidence puts some limitations on the validity of the SSD model: e.g., the disk is too cool to emit the hard X- and gamma-rays observed in black hole objects. From a mathematical point of view, these limitations are related to simplified assumptions made while deriving the energy equation, in which they neglected the heat transport by advective motions in the accretion flow. This mechanism is essential in the innermost disk structure, especially in black hole accretion. A detailed analysis of the inner region of the disks is indispensable because most of the disks’ radiation originates from there. Its importance is all the greater due to the fact that it gives rise to disk models that are distinct from the standard one.

An alternative model, which is applicable to structures where the accretion rate is much smaller than $\dot{M}_\text{Edd}$, is known as advection-dominated accretion flow (ADAF); this model was introduced by Narayan & Yi (1994, 1995) (hereafter NY94, NY95, respectively) and Ichimaru (1997). In this model, the heat generated by turbulence is stored as entropy and can be transported with the flow toward the central part, rather than being radiated away from the system immediately after generation. The ADAF model is of great interest because of its widespread applications in describing low-luminosity AGNs (Ludwig et al. 2012), as well as the quiescent and hard states of black hole binaries (Ho 2008; Narayan & McClintock 2008; Yuan 2007, 2011); for the latest review, see Yuan & Narayan (2014). This model has been also applied to the supermassive black hole at our Galactic Center, Sagittarius A* (Sgr A*); see Yuan et al. (2003).

Many improvements have been proposed to improve our understanding of the physics of hot accretion flows, including their multidimensional dynamics, disc-jet connection, radiation mechanisms, and various astrophysical applications. For a review, see Yuan & Narayan (2014). One interesting phenomenon associated with an accretion system, such as an ADAF, is launching of the winds or outflows. On the other hand, the structure of hot accretion flow is also remarkably affected by outflows, which carry huge amounts of mass, momentum, and energy from the disk (Bu et al. 2009; Kawabata & Mineshige 2009; Yuan et al. 2012a, 2012b; Bu et al. 2013; Samadi & Abbassi 2016). There is some direct and indirect observational evidence to confirm the existence of outflows in different accreting systems, such as low-mass X-ray binaries (Fender et al. 2004; Migliari & Fender 2006) and AGNs (Terashima & Wilson 2001; Pounds & Reeves 2009).

One of the main properties of NY94 solutions is that the flow possesses a positive value for the Bernoulli parameter at the regions near the pole, which implies that the flow is susceptible to produced wind. Recent hydrodynamic (HD) and magnetohydrodynamic (MHD) simulations have also confirmed emerging outflow in ADAFs (Igumenshchev & Abramowicz 1999; Stone et al. 1999; Igumenschev et al. 2000; Machida et al. 2001;
Stone & Pringle 2001; Yuan et al. 2012a, 2012b). Extensive theoretical efforts have been made to understand the outflow launching mechanism in ADAFs and its potential effects on its underlying accretion flows. For instance, see Xu & Chen (1997) (hereafter XC97), Blandford & Begelman (1999, 2004), Xue & Wang (2005), Tanaka & Menou (2006), Khajenabi & Shadmehri (2013), Degenaar et al. (2017), Samadi et al. (2014), Samadi & Abbassi (2016), Samadi et al. (2017), and Khajenabi et al. (2014). However, there are some observations that imply that outflow does not play an important role in the dynamics of some astronomical systems, e.g., cool-core clusters and several massive elliptical galaxies (Allen et al. 2006; Hlavacek-Larrondo & Fabian 2011). Therefore, the study of hot accretion flow systems without outflow can still be important. There are several analytical and numerical solutions in the relevant literature studying ADAFs structure in one or two dimensions; for example, see (e.g., cool-core clusters and several massive elliptical galaxies important role in the dynamics of some astronomical systems, observations that imply that out

Khajenabi & Shadmehri 2004, Xue & Wang 2005, Shadmehri (2014), Zeraatgari & Abbassi (2015), and Habibi et al. (2017). Among these solutions, NY95 is of particular importance. In that paper, the authors present numerical axisymmetric self-similar solutions for steady-state ADAFs structure in spherical coordinates \((r, \theta, \phi)\). In their model, self-similar scaling in the radial direction has been adopted while vertical structures have been found using two kinds of boundary conditions: numerical and relaxation method. They adopted the standard model for viscosity in which the viscosity coefficient \(\alpha\) is assumed to be constant. Furthermore, they assume that the radial self-similar solutions for all the quantities, we then find an eighth-order system of differential equations for latitudinal part of the functions. We next introduce proper physical boundary conditions for this system of differential equations. In Section 3, we briefly discuss the Fourier expansion method and the numerical procedure for finding a unique solution for the large nonlinear set of algebraic equations. In Section 4, we apply this approach to an ADAF system with zero \(\nu_0\) and find four type of solutions. In fact, it turns out that there are only two free parameters in the system: the viscosity parameter \(\tilde{\alpha}\), and the thermodynamic parameter \(\tilde{\epsilon}\). Without loss of generality, we fix \(\tilde{\alpha}\) and vary \(\tilde{\epsilon}\) to classify the solutions. We discuss the physics of the solutions and compare them to the solutions available in the literature. In Section 5, we check the dynamical stability of the solutions and find regions that are convectively unstable. Furthermore, we discuss whether this instability can change the global aspects of the solutions and produce outflow in the system. In Section 6, we apply this method to an ADAF system with nonzero \(\nu_0\). We show that, in this case, this method fails to find unique and convergent solutions. Finally, we summarize the results in Section 7. As a test for the reliability of the approach, we apply it to well-known solutions presented in NY95 for ADAFs with zero \(\nu_0\) and exactly recover all of the solutions. We show the results for this case in Appendix B.

2. General Formulation

In this section, we introduce the governing differential equations and discuss the appropriate boundary conditions. More specifically, we present the basic equations in Section 2.1. In Section 2.2, we introduce a new model for viscosity in which \(\alpha\) is not a constant. The relevant radial self-similar forms are presented in Section 2.3. We discuss the boundary conditions in Section 2.4.

2.1. Basic Equations

Let us begin with the basic equations. We use the spherical polar coordinates \((r, \theta, \phi)\), where \(r\) is the radial distance, and \(\theta\) and \(\phi\) are the polar and azimuthal angles, respectively. A black hole with mass \(M\) is at the origin and its gravitational potential is \(\psi(r) = -GM/r\). Furthermore, we assume that the system is axisymmetric and in a steady-state configuration.

It is important to note that the fluid is assumed to be viscous and we adopt the \(\alpha\) viscosity description (Igumenshchev et al. 1996; Igumenshchev & Abramowicz 1999; Stone et al. 1999; Igumenshchev et al. 2000; Fragile & Anninos 2005; Shadmehri 2014). Furthermore, we keep all the components of the energy–momentum tensor, \(T_{\mu\nu}\), throughout this paper.
The governing equations are those pertaining to continuity, momentum, and energy. These equations can be found in the HD textbooks (e.g., Mihalas & Mihalas 1984). For the sake of completeness, however, we rewrite them here. Therefore, the continuity equation becomes

$$\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \rho \nu_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\rho \sin \theta \nu_\theta) = 0,$$

and the three components of the momentum equations are written as

$$v_r \frac{\partial \nu_r}{\partial r} + v_\theta \left( \frac{\partial \nu_r}{\partial \theta} - \nu_\theta \right) - \nu_\phi \cot \theta = \frac{GM}{r^2} - \frac{1}{\rho} \frac{\partial p}{\partial r} + \frac{2\Psi}{\rho},$$

$$v_r \frac{\partial \nu_\theta}{\partial r} + v_\theta \left( \frac{\partial \nu_\theta}{\partial \theta} + \nu_r \right) - \frac{\nu_\phi \cot \theta}{r} = \frac{1}{\rho r} \frac{\partial p}{\partial \theta} + \frac{2\Phi}{\rho},$$

$$v_r \frac{\partial \nu_\phi}{\partial r} + v_\theta \frac{\partial \nu_\phi}{\partial \theta} + v_\phi (\nu_r + \nu_\theta \cot \theta) = \frac{2\Gamma}{\rho},$$

and the energy conservation equation is given by

$$\rho \left( v_r \frac{\partial e}{\partial r} + v_\theta \frac{\partial e}{\partial \theta} \right) + \frac{1}{\rho r} \frac{\partial p}{\partial r} + \frac{v_\theta \partial p}{\partial \theta} - \frac{2f\Lambda}{\rho} = 0.$$  

Equations (1)–(5) are the main equations; with an equation of state, they form a complete set for finding the unknown functions. In the above equations, the parameters $\Psi, \Phi, \Gamma,$ and $\Lambda$ are defined as

$$\Psi = \frac{1}{r \sin \theta \left[ \frac{\partial}{\partial r} (r^2 \sin \theta T_r) + \frac{\partial}{\partial \theta} (r \sin \theta T_\theta) \right]} - T_\theta + T_\phi \phi,$$

$$\Phi = \frac{1}{r \sin \theta \left[ \frac{\partial}{\partial r} (r \sin \theta T_\phi) + \frac{\partial}{\partial \theta} (\sin \theta T_\theta) \right]} - T_\phi \cot \theta - 2T_\theta,$$

$$\Gamma = \frac{1}{r} \left[ \frac{\partial}{\partial r} (r T_\phi) + \frac{\partial T_\phi}{\partial \theta} \right] + \frac{2}{r} (T_\phi + T_\phi \cot \theta),$$

and

$$\Lambda = \frac{\partial \nu_r}{\partial r} T_r + \frac{\partial \nu_\theta}{\partial r} T_\theta + \frac{\partial \nu_\phi}{\partial r} T_\phi + \left( \frac{\nu_r}{r} - v_\theta \right) T_\theta + \frac{1}{r} \left( \frac{\partial \nu_\theta}{\partial \theta} + v_\theta \right) T_\theta - \frac{\nu_\phi \cot \theta}{r} (T_\theta + T_\theta \cot \theta) + \frac{T_\phi r}{r} (v_r + v_\phi \cot \theta).$$

Here, $\rho$ is the gas density, $p$ is the gas pressure, and, $v_r, v_\theta,$ and $v_\phi$ are the radial, azimuthal, and toroidal components of the gas velocity. We also introduce $f$ as a fraction of the advected energy. Furthermore, $e$ denotes the specific internal energy of the fluid and can be written as $e = p/\rho(\gamma - 1)$, where $\gamma \equiv c_p/c_v$ is the ratio of specific heats. The independent components of the energy–momentum tensor in the polar spherical coordinate are given by

$$T_{rr} = \mu \frac{\partial \nu_r}{\partial r} - \frac{\mu}{3} \left[ \frac{1}{r^2} \frac{\partial (r^2 \nu_r)}{\partial r} + \frac{\partial}{\partial \theta} \left( \frac{\sin \theta \nu_\theta}{r \sin \theta} \right) \right],$$

$$T_{\theta\theta} = \mu \frac{\partial \nu_\theta}{\partial \theta} + \frac{\mu}{3} \left[ \frac{1}{r^2} \frac{\partial (r^2 \nu_\theta)}{\partial r} + \frac{\partial}{\partial \theta} \left( \frac{\sin \theta \nu_\theta}{r \sin \theta} \right) \right],$$

$$T_{\phi\phi} = \frac{\mu}{3} \left[ \frac{1}{r^2} \frac{\partial (r^2 \nu_\phi)}{\partial r} + \frac{\partial}{\partial \theta} \left( \frac{\sin \theta \nu_\phi}{r \sin \theta} \right) \right].$$

Hereafter, we define the isothermal sound speed as $c_\theta^2 = p/\rho$, and the net mass accretion rate is given by

$$\dot{M} = - \int 2\pi r^2 \sin \theta \rho(r, \theta) v_r(r, \theta) d\theta.$$  

### 2.2. A Latitudinally Varying $\alpha$ Viscosity Model

Our goal is to solve Equations (1)–(5) using our numerical method, to illustrate its ability and possible weaknesses. In doing so, we consider two cases with different viscosity prescriptions. In case (i), we first investigate an accretion flow with the commonly used $\alpha$-prescription, where the kinematic coefficient of viscosity is $\nu = \alpha c_\theta^2 (r, \theta) / \Omega_K(r) = \alpha r^2 \Omega_K(r) c_\theta^2 (\theta)$ and we have used the similarity solutions introduced in the next subsection; see Equation (20). Here, the Keplerian angular velocity is $\Omega_K(r) = \sqrt{\frac{GM}{r^3}}$. Note that viscosity coefficient $\alpha$ is the given model parameter and the viscosity is dependent on both the radial distance and the latitudinal angle. In case (ii), as a second illustrative configuration, we assume that viscosity is independent of the latitudinal angle. In this particular case, radially self-similar solutions are found if the kinematic viscosity coefficient is written as

$$\nu(r) = \alpha \nu^2 \Omega_K(r),$$

where $\alpha$ is a model parameter. In both cases (i) and (ii), the bulk viscosity, i.e., $\mu(r, \theta) = \rho(r, \theta) \nu$, depends upon $r$ and $\theta$. In case (i), bulk viscosity $\mu$ is proportional to pressure, i.e., $\mu(r, \theta) = \alpha p(r, \theta) / \Omega_K(r)$, whereas in case (ii), bulk viscosity $\mu$ is dependent on the density, i.e., $\mu(r, \theta) = \alpha p(r, \theta) r^2 \Omega_K(r)$.

The functional form of the introduced viscosity in case (ii) is actually a variant of the widely used $\alpha$-prescription, if the viscosity coefficient $\alpha$ is assumed to be dependent on the latitudinal angle as follows:

$$\alpha(\theta) = \frac{\tilde{\alpha}}{c_\theta^2(\theta)}.$$  

Although this prescription seems to be restrictive, there are hot accretion flow simulations that suggest viscosity coefficient $\alpha$ is not a constant and depends on the latitudinal angle.
this dependence, however, needs to be explored further using hot accretion flow simulations. However, it motivated some authors to explore properties of ADAFs with the viscosity parameter as a given function of the latitudinal angle. Habibi et al. (2017), for instance, found fully analytical no-wind solutions for the ADAFs with a viscosity parameter in proportion to \( \sin \theta \). In their analysis, however, the radial-azimuthal component of the stress tensor, \( T_{\rho \phi} \), is assumed to be dominant. This assumption is relaxed in our analysis by fully implementing all the stress components. The solutions that include the viscosity prescription introduced in case (ii) are direct generalizations from the analysis of Habibi et al. (2017).

In the next subsections, we present similarity solutions for ADAFs that have the viscosities given in cases (i) and (ii).

### 2.3. Self-Similar Solutions

By assuming radial self-similar solutions, one may convert the two-dimensional differential Equations (1)–(5) to a set of one-dimensional differential equations. To do so, we follow NY95 by using the following radial dependencies for the quantities

\[
\rho(r, \theta) = r^{-\alpha}\rho(\theta),
\]

\[
c_s(r, \theta) = \frac{MG}{r} c_s(\theta),
\]

\[
v_r(r, \theta) = \frac{GM}{r} v_r(\theta),
\]

\[
v_\theta(r, \theta) = \frac{GM}{r} \sin \theta \Omega(\theta).
\]

Furthermore, we define the function \( q(\theta) \) to include both models A and B. To do so, we express \( \mu \) as \( \mu(r, \theta) = \rho(r, \theta)^{2/3} \bar{c}_g q(\theta) \). The function \( q(\theta) \) is equal to \( \alpha c_s^2(\theta) \) for model A, and is equal to \( \bar{c}_g \) in model B.

Upon substituting the above solutions into Equations (1)–(5), the following ordinary differential equations are obtained:

\[
(3 - 2n) \rho v_r + 2v_\theta(\rho' + \rho \cot \theta) + 2\rho v_\theta' = 0, \tag{24}
\]

\[
\rho(6(n + 1)c_s^2 + q(12(n - 2)v_r + (4n - 17)v_\theta') + (4n - 17)\cot \theta v_\theta + 6\cot \theta v'_\theta + 6v''(\theta)) + 3(2\Omega(\theta)^2 \sin^2(\theta) - 2v'_\theta(v_\theta - q'\theta) - 3q'v_\theta + v_\theta^2 + 2v_\phi^2 - 2) + 3q'\rho(2v'_\theta - 3v_\theta) = 0, \tag{25}
\]

\[
\rho(6(n+1)c_s^2 \sin^2(\theta) \cos \theta + 3v_r(3v_\theta - q' \theta) - 2q' \theta) + v_\theta(3v_\theta - 4q' \theta) + 4\cot \theta q(\theta) v_\theta + 6c_s^2 \rho' - q(\rho(-6(n - 3)v'_r + v_r(-8\cos^2(\theta) + 9n - 6) + 8(\cot \theta v'_\theta + v'_\theta)) + 2\rho(3v_r + 4v_\theta' - 2\cot \theta v_\theta)) = 0, \tag{26}
\]

\[
q(-2\sin \theta \rho \Omega' - \rho(6 \cos \theta \Omega' + \sin \theta (2\Omega'' + 3(n - 2)\Omega))) + \rho(2 \sin \theta \Omega' (v_\theta - q') + \Omega \sin \theta v_r + 4 \Omega \cos \theta v_\theta) = 0, \tag{27}
\]

\[
36 \epsilon c_s^2(\rho((\gamma - 1)n - 1)v_r - (\gamma - 1)\rho' v_\theta) + \rho\left(36 \epsilon v_\theta c_s^2 + \frac{1}{2}(3\gamma - 5)q \cos^2(\theta) \times (54\Omega^2 \sin^2(\theta) + 8 \sin^2(\theta)(3 \sin^2(\theta) \Omega^2 + 3(\theta' v_\theta)^2 + 4(\theta'_r)^2 - 8 \sin \theta v_\theta(9 \sin \theta v'_\theta + 4 \cos \theta v'_\theta) + 48 \sin \theta v_r(\sin \theta v'_\theta + \cos \theta v'_\theta) + 72 \sin^2 \theta v_{\phi}^2 + (43 - 11 \cos 2\theta v_\phi)^2) \right) = 0, \tag{28}
\]

where prime stands for the derivative with respect to \( \theta \) and the new parameter \( \epsilon \) is defined as \( \epsilon = (5/3 - \gamma)/(\gamma - 1) \).

Equations (24)–(28) are the main non-linear differential equations that we shall solve in the subsequent sections for five unknowns: \( \rho, v_r, v_\theta, \Omega, \) and \( c_s \). It is necessary to emphasize that, in case (i) where \( q(\theta) = c_s^2(\theta) \), our equations are slightly different from Equations (B1)–(B5) presented in Xue & Wang (2005), as well as the equations implemented in XC97. We found some minor mistakes in Xue & Wang (2005). If we set \( v_\phi = 0 \) in Xue & Wang (2005) equations, the NY95 equations are not recovered.

### 2.4. Boundary Conditions

We now need a set of proper boundary conditions for solving the equations. Equations (24)–(28) constitute an eighth-order system of ordinary differential equations. Therefore, we need eight boundary conditions to solve them numerically. Definition of the net mass accretion rate provides the first boundary condition. If we do so, the integral (16) sets the normalization of \( \rho(\theta) \), i.e., one should fix the magnitude of \( \bar{m} = M/(2\pi \sqrt{G M}) \). More specifically, in the case of \( n = 3/2 \), this quantity does not depend on \( r \) and provides a suitable boundary condition. We will use this boundary condition for all the solutions presented in this paper for \( n = 3/2 \). Instead of this condition, one may use \( \rho(\pi/2) = 1 \) because we can simply scale \( \rho(\theta) \) by the accretion rate. We will use this condition for the cases where \( n \neq 3/2 \). One should note that \( \bar{m} \) can not be nonzero for these solutions. In other words, mass conservation implies that the net mass accretion rate is not dependent on \( r \).

On the other hand, the self-similar solutions yield \( \bar{m} = r^{3/2-n} \), where \( I \) is an integral given by

\[
I = -\int_0^\pi \rho(\theta) v_r(\theta) \sin \theta d\theta. \tag{29}
\]

The remaining seven boundary conditions are distributed between the equatorial plane and the rotation axis \( \theta = 0 \). At the equatorial plane, the boundary conditions can be written as

\[
\theta = \frac{\pi}{2}: \frac{d\rho}{d\theta} = \frac{dv_r}{d\theta} = \frac{d\Omega}{d\theta} = \frac{dc_s}{d\theta} = v_\phi = 0. \tag{30}
\]

The other two conditions can be obtained by fixing the magnitude of \( v_\phi(\pi/2) \) and \( c_s(\pi/2) \), as has been done in Xue & Wang (2005). In this case, we have an initial value problem. However, by fixing these conditions at \( \theta = 0 \), the system will be a boundary value problem. On the other hand, at \( \theta = 0 \), we expect that the solutions be well-behaved and nonsingular. In this case, we have

\[
\theta = 0: \frac{d\rho}{d\theta} = \frac{dv_r}{d\theta} = \frac{d\Omega}{d\theta} = \frac{dc_s}{d\theta} = 0. \tag{31}
\]
By imposing these conditions into Equations (24)–(28) at \( \theta = 0 \), one can easily verify that
\[ v_r = 0 \text{ or } v_r = -\frac{c_s^2}{2q}. \] (32)

Obviously, the number of conditions exceeds eight. However one should note that not all of them are independent. Technically, we chose a convenient eight-component subset of the conditions in order to solve the equations.

3. Semi-analytic Approach: Fourier Expansion

In this section, we follow the method introduced in XC97. However, we use the correct set of equations as well as a different set of boundary conditions. We also find new solutions when the latitudinal angle dependence of the viscosity parameter \( \alpha \) is permitted, as in case (ii). The idea is that we can always construct proper Fourier series for the physical quantities that satisfy the boundary conditions (30) and (31) by default. More specifically, in the simplest case, we can express the quantities as the following series
\[
\rho(\theta) = \sum_{i=0}^{N} a_i \cos(2i\theta), \quad c_i^2(\theta) = \sum_{i=0}^{N} w_i \cos(2i\theta)
\]
\[
v_r(\theta) = \sum_{i=0}^{N} b_i \cos(2i\theta), \quad \Omega(\theta) = \sum_{i=0}^{N} d_i \cos(2i\theta)
\]
\[
v_\theta(\theta) = \sin(2\theta) \sum_{i=0}^{N} h_i \cos(2i\theta). \] (33)

Substituting these series into Equations (24)–(28) and using an appropriate subset of boundary conditions, we obtain a \( 5(N+1) \) nonlinear algebraic equations for \( 5(N+1) \) coefficients. In other words, the problem reduces to solving algebraic equations rather than differential ones. We solve these equations via the Newton–Raphson method. In this method, one needs appropriate primary guesses for the solutions. We find them by using random number generators. In order to show how this procedure works, we have written the details for a toy model, in which \( \Omega = 0 \), in Appendix A. In practice, our equations are too long and we cannot write the main calculations in the paper.

The practical power of this approach is that the boundary conditions have been already included in the series; consequently, by finding the coefficients, one may analytically analyze the quantities in the whole interval \([0, \pi]\). In other words, these solutions cover the whole space and allow the dynamics to be investigated in a complete manner. Therefore, this approach may help improve our understanding of the ADAF structure.

However, the technical limitation is that we have to truncate the expansion in a specific \( N \). In fact, when \( N > 9 \), the parameter space of solutions gets extremely large and it becomes impossible to find a unique solution in practice. We start with \( N = 3 \) and increase \( N \) by checking the convergence of the solutions. More specifically, when the first and dominant coefficients in the series remain approximately constant by increasing \( N \), we decide that the numerical procedure is convergent. More specifically, in each step, we measure the fractional difference between the coefficients in \( N \) and \( N+1 \) cases. As a caveat, it should be noted that increasing \( N \) does not necessarily yield better solutions. In fact, for large \( N \), i.e., \( N > 10 \), the numeric errors dominate the calculation and the solution diverges and takes a highly oscillatory and unnatural form. In some cases, we use \( N = 9 \) for the number of coefficients. Fortunately, we find an acceptable convergence for \( 5 \leq N \leq 9 \) in all cases.

4. ADAF with \( v_\theta(\theta) = 0 \)

Our study begins with a simplified configuration that has \( v_\theta(\theta) = 0 \). The existence of similarity solutions implies that we have \( n = 3/2 \) in both cases (i) and (ii).

4.1. Model A: Solutions with \( \alpha \)-prescription

Accretion flow with the viscosity prescription introduced in case (i) is actually equivalent to the model studied in NY95. If we set \( q(\theta) = \alpha c_s^2 \) in Equations (24)–(28), these equations reduce to Equations (2.16)–(2.19) from NY95. To illustrate that the Fourier analysis method is an efficient tool for instigating an ADAF structure, we first retrieve NY95 results in Appendix B, along with verification that the implemented series are truly convergent. Although we use a different numerical method, our obtained solutions are consistent with NY95’s solutions to reasonable accuracy. This successful test problem is a good motivation to implement the Fourier analysis method for exploring ADAF structures with varying viscosity coefficients.

4.2. Model B: Solutions with the Latitudinal Angle Dependence of \( \alpha \)

Upon substituting \( q(\theta) = \tilde{\alpha} \) into the main Equations (24)–(28), we obtain
\[
\frac{v_r^2}{2} + \Omega^2 \sin^2 \theta = 1 - \frac{5c_s^2}{2} - \tilde{\alpha}(\cot \theta \nu'_r + \frac{(\rho \nu'_r)}{\rho} - v_r), \] (34)
\[
-\Omega^2 \sin \theta \cos \theta = -\frac{(\rho c_s^2)}{\rho} + \tilde{\alpha} \rho v_r + \frac{3 \tilde{\alpha} \nu'_r}{2}, \] (35)
\[
\frac{1}{2} \Omega v_r = \tilde{\alpha} \left( \frac{\rho \nu'_r}{\rho} + \Omega^2 + 3 \cot \theta \nu'_r - \frac{3}{4} \Omega \right), \] (36)
\[
-\frac{3c_s^2 v_r}{2\tilde{\alpha}} = 3v_r^3 + \frac{9}{4} \Omega^2 \sin^2 \theta + \sin^2 \theta (\Omega')^2 + (\nu'_r)^2. \] (37)

We are now in a position to solve the above equations using the Fourier analysis method. Before doing so, however, it is prudent to inspect these equations for general trends in the solutions. Equation (37), for instance, shows that its right hand side is always positive for \( \gamma < 5/3 \). This then leads to a negative radial velocity, irrespective of the angle \( \theta \).

A thorough consideration and classification of the numeric solutions revealed that there are two different branches of solutions: rotating inflow, \( \Omega = 0 \), and non-rotating inflow solutions, \( \Omega = 0 \). On the other hand, each branch can be divided into two main types: solutions that satisfy \( v_r(0) = 0 \), and those that satisfy \( v_r(0) = -\epsilon c_s^2 / 2\tilde{\alpha} \); see Equation (32). Consequently, we can categorize our solutions into four different cases.
therefore, the system deviates from a spherical symmetry and plane is about 35%. The pro

correspond to nearly spherical dynamic parameter \( \alpha \). However, the maximum values increases with the thermo-
density parameter \( \epsilon \). These solutions are representative of either ADAFs with \( f = 1 \) different values of \( \gamma \) or ADAFs with a constant adiabatic index \( \gamma \) and different values of \( f \).

For \( \epsilon = 0.2 \), the density varies by only 13% from the pole to the equatorial plane. Solutions with small \( \epsilon \), therefore, correspond to nearly spherical flows. For \( \epsilon = 0.5 \), on the other hand, the density contrast between the pole and the equatorial plane is about 35%. The profile of the angular velocity (bottom right) shows that it is more or less independent of the latitudinal angle, but its value increases with \( \epsilon \). For large values of \( \epsilon \), therefore, the system deviates from a spherical symmetry and tends to a rotationally flattened configuration. However, we find that, for \( \epsilon > 1 \), the density profile exhibits some physically implausible oscillations. For this reason, we do not report these solutions.

4.2.2. Rotating Inflow with \( v_r(0) = 0 \)

In this particular case, physical quantities are shown in Figure 2. We note that NY95 found no rotating solution for \( v_r(0) \neq 0 \), whereas our analysis for the \( \alpha \) varying model shows both rotating and non-rotating solutions when the radial velocity does not vanish at the poles. The behavior of the flow in this case is somehow opposite to the case where \( v_r(0) = 0 \).

4.2.3. Non-rotating Inflow with \( v_r(0) = 0 \)

The top row of Figure 3 shows solutions with \( v_r(0) = 0 \). We find that there is no solution when \( \epsilon \geq 0.5 \). Although the inflow does not rotate, its geometrical shape is not purely spherical. This trend is not surprising, in the sense that the \( \alpha \) viscosity profile is a function of \( \theta \). Furthermore, by increasing the thermodynamic parameter \( \epsilon \), the systematic deviation from a spherical configuration becomes more significant. Density contrast between the pole and the equatorial plane reaches 17% for \( \epsilon = 0.2 \). It should be noted that, in NY95, there is no nonrotating solution with \( v_r(0) = 0 \). In other words, the nonrotating solutions presented in NY95 possess a negative radial velocity at the poles.

4.2.4. Non-rotating Inflow with \( v_r(0) = -\epsilon c_s^2/2\alpha \)

The bottom row of Figure 3 displays solutions with \( v_r(0) = -\epsilon c_s^2/2\alpha \). These profiles exhibit different maximum and minimum points that are unlikely to be representative of any physical system. For the sake of completeness, however,
we report these solutions. It can be compared to the non-rotating solution presented in 
\textit{NY95}; see Appendix B. Their solution is purely spherical and none of the functions depend on \(\theta\). Therefore, their solution is reminiscent of the Bondi accretion in the presence of viscosity. However, our solution is not spherical. Furthermore, similar to other solutions presented in this paper, increasing the thermodynamic parameter \(\tilde{\alpha}\) causes more deviation from a spherical configuration. It can be seen from the left panel in the bottom row of Figure 3 that the density contrast can be more than 45%.

However, this deviation from a spherical configuration does not mean that there is no spherical non-rotating solution in our varying \(\alpha\) model. Let us briefly discuss a simple analytic non-rotating inflow spherical solution. When latitudinal angle dependence of the variables is neglected (i.e., spherical symmetry), we can find a non-rotating solution using Equations (34)--(37). Thus,

\[ v = \frac{\alpha(\epsilon + 5) - \sqrt{\alpha^2(\epsilon + 5)^2 + 2\epsilon^2}}{\epsilon} \]

\[ c_s^2 = \frac{2\alpha}{\epsilon^2} (\sqrt{\alpha^2(\epsilon + 5)^2 + 2\epsilon^2} - \alpha(\epsilon + 5)). \]

This solution corresponds to an inflow configuration, irrespective of \(\epsilon\). In other words, unlike in \textit{NY95}, it is not possible to produce wind by choosing a negative thermodynamics parameter \(\epsilon\).

4.2.5. Convergence of the Solutions

We now verify that the presented solutions are convergent. We increase the number of Fourier terms until the solution converges. In doing so, we define the fractional difference \(\Delta Q\) between solutions with \(N\) and \(N + 1\) Fourier coefficients, i.e.,

\[ \Delta Q = \frac{Q_{N+1} - Q_N}{Q_N} \times 100, \]

where \(Q\) stands for \(\rho, v_r, c_s^2\), or \(\Omega\). Obviously, the solutions are convergent if the fractional difference \(\Delta Q\) tends to zero with increasing \(N\). Figure 4 shows profiles of \(\Delta Q\) for the solutions presented in Figure 1 with \(\tilde{\alpha} = 0.1\) and \(\epsilon = 0.1\). Different colors belong to different \(N\). It is evident that, with increasing \(N\), the fractional difference decreases and becomes small for \(N \geq 5\). We generally find that the solutions correspond to fractional differences smaller than \(10^{-4}\%\), so long as the adopted terms in the Fourier expansions are larger than six. In other words, the obtained solutions are not modified by increasing the number of terms in the Fourier expansion to more than six. This implies that the solutions are convergent.

4.3. The Bernoulli Parameter

In order to study the occurrence of outflow, it is important to find the Bernoulli function of the flow. This parameter determines the whole energy per unit mass of the flow. As it has been pointed out in \textit{NY95}, whenever the Bernoulli parameter reaches a positive value, one may expect the existence

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**Figure 3.** Top row: Non-rotating self-similar solutions for \(\tilde{\alpha} = 0.1, v_\theta = 0, n = 3/2,\) and \(v_r(0) = 0\). Bottom row: Non-rotating self-similar solution with the same model parameters as in the top row, but with \(v_r(0) \neq 0\). Note that here the density and the sound speed are both normalized by their representative values at the pole.
of an outflow in the system. On the other hand, a positive value for the Bernoulli parameter means that the flow can escape to infinity as an outflow, due to it having sufficient energy to overcome gravity. Some researchers have pointed out that the positive value of the Bernoulli parameter is a consequence of self-similar solutions (Yuan 1999a; Abramowicz et al. 2000). Aside from that, in numerical HD and MHD simulations performed by Yuan et al. (2012a, 2012b), the Bernoulli function is positive in most regions. From this perspective, the Bernoulli parameter is useful to check the aforementioned possibility. As usual, let us define the dimensionless parameter \( b \) as follows:

\[
b = \frac{\text{Be}}{\Omega_p^2 r^2} = \frac{\frac{1}{2}(v_r^2 + v_\theta^2 + \sin^2 \theta \Omega^2)}{} - 1 + \frac{\gamma}{\gamma - 1} \frac{c_s^2}{}.\]

Figure 5 shows profiles of this parameter for the solutions presented thus far. The top left panel in Figure 5 belongs to a rotating solution presented in Figure 1. We find that the Bernoulli parameter is always positive within the allowed range of \( \epsilon \). Parameter \( b \) becomes larger when the thermodynamic parameter \( \epsilon \) is increased. This behavior is completely opposite to the corresponding solution explored in NY95 (i.e., model A) where increasing \( \epsilon \) leads to \( b < 0 \) for certain latitudinal angles. This figure suggests that the solutions in Figure 1 are able to produce outflows.

The top right panel in Figure 5 corresponds to the solution presented in Figure 2. As we have discussed, this solution possesses a nonzero radial velocity at the poles. Figure 5 shows that, for small \( \epsilon \), the parameter \( b \) is always positive. For larger values of \( \epsilon \), however, the Bernoulli parameter becomes negative in the interval near the poles. We find that this interval grow wider with increasing \( \epsilon \). The bottom left panel in Figure 5 shows the Bernoulli parameter for the non-rotating solution presented in the top row of Figure 3. In the case of small \( \epsilon \), parameter \( b \) becomes negative very close to the equatorial plane.

Finally, the bottom right panel of Figure 5 belongs to a non-rotating solution displayed in the bottom row of Figure 3. The Bernoulli parameter is negative within an interval of \( \theta \). However, it is positive near the poles and equatorial plane alike. There are, therefore, regions where the parameter \( b \) is positive and an outflow can exist. However, in order to make sure that an outflow occurs in the system, it is necessary to investigate the convective instability of the solutions. In the next section, we study this issue in more detail.

5. Dynamical Stability of the Solutions

The stability of the solutions is important in any physical system. For example, it is well-established that the ADAFs with zero \( v_\theta \) are intrinsically unstable to convective instabilities; see NY95 for more details. As we showed in Appendix B, when \( v_\theta = 0 \), there is no outflow in the system. However, the convective instability in some regions near the pole can play effectively and the resultant convection outflow can dominate the advection inflow. In other words, this instability can, in principle, change the global behavior of the solutions and induce outflow near the pole. From this perspective, and to be sure about the mathematical and physical viability of our solutions, it is necessary to study the response of the system to some pertinent instabilities.

The classic Solberg–Heiland criteria for stability of the rotating flow against local axisymmetric, adiabatic perturbations in the cylindrical coordinate system \((R, \phi, z)\) are given by (Tassoul 1978):

\[
-\frac{1}{\gamma \rho} \nabla P \cdot \nabla \ln P R^{\gamma-1} + \frac{1}{R^3} \frac{\partial R^2 \Omega^2}{\partial R} \geq 0,
\]

(41)

\[
-\frac{\partial P}{\partial z} \left( \frac{\partial R^2 \Omega^2}{\partial R} \frac{\partial \ln P}{\partial R} - \frac{\partial R^2 \Omega^2}{\partial z} \frac{\partial \ln P}{\partial R} \right) \geq 0.
\]

(42)

Condition (41) has been widely used to study the convective instability of astrophysical systems. It is convenient to write it as \( N^2 + \kappa^2 > 0 \), where \( N \) is the usual Brunt–Väisälä frequency and \( \kappa \) is the epicyclic frequency. Here, \( N^2 \) and \( \kappa^2 \) are given by the first and second terms on the left-hand side of (41). For a non-rotating flow, the epicyclic frequency is zero and (41) implies the existence of an inward increase of entropy, which is the well-known Schwarzschild criterion. For a rotating flow, the inward increase of entropy is a necessary condition for convective instability. In other words, the entropy decreases inward in a convectively stable flow. In an ADAF without radiation, the numerical simulations and analytical descriptions confirm that entropy increases inward. In other words, these systems are convectively unstable; e.g., NY94 and NY95.

It should be noted that, in the regions where the second condition (42) is violated, the local axisymmetric perturbations can, in principle, grow. However, these perturbations are local and can not change the global behavior of the solutions. Our focus, therefore, is on the regions where the first condition (41) is violated. These regions are convectively unstable and can affect the global properties of the flow, as explored in NY95 for their model A.
The left-hand side of (41) in the polar spherical coordinate system can be written as $r^{-3/2}H(\theta)$. Therefore, the sign of $H(\theta)$ determines the unstable regions. For $H(\theta) < 0$, the latitudinal direction $\theta$ is prone to convective instability. In Figure 6, we display the stability function $H(\theta)$ for the rotating solutions presented in Figure 5. It should be noted that $H(\theta)$ is negative within the interval $0 \leq \theta \leq \pi$ for the non-rotating solutions. Therefore, the non-rotating solutions are convectively unstable.

We can now explore stability of the rotating self-similar solutions. The left panel of Figure 6 shows that $\epsilon$ has a stabilizing effect on the system, in the sense that increasing $\epsilon$ leads to a wider stable interval. However, the stability function $H(\theta)$ is negative near the equatorial plane.

Thermodynamic parameter $\epsilon$ has a stabilizing effect on the rotating solutions with $v(0) \neq 0$. The right panel in Figure 6 shows that some parts of the system are stabilized by increasing parameter $\epsilon$. However, in this case, it does not necessarily extend the stable interval. Similar to the first type of rotating solutions, we find stability near the poles but instability can occur around the equatorial plane.

We have shown that all the solutions have regions where $b > 0$ and $H(\theta) < 0$. This trend, however, does not imply that convective instability is able to reverse direction of the flow and produce outflows. In other words, as in other hydrodynamic instabilities, the timescale for the growth of the instability should be sufficiently small, compared to other characteristic timescales in the system. In order to estimate the significance of the convective effects, compared to the advection, we follow and generalize the method presented in NY95. To do so, we assume that $f = 1$ and rewrite the energy Equation (37) in terms of convective energy flux $F_c$ and physical functions $c^2 (r, \theta)$, $v_c (r, \theta)$, $\rho(r, \theta)$ and $\Omega(r, \theta)$, as follows:

$$-rac{3\epsilon c^2 v_c \rho}{2r} = -\nabla \cdot F_c + \frac{\alpha \rho}{\Omega \kappa} \left( \frac{3v^2}{r^2} + \frac{9}{4} \Omega^2 \sin^2 \theta + \sin^2 \theta (\Omega')^2 + \frac{(v')^2}{r^2} \right).$$  

Three terms in this equation, i.e., one term on the left-hand side and two on the right-hand side, are representatives of the
advection, the convection, and the viscosity, respectively (NY95). The system is advection-dominated; as a consequence, we need to compare the convection and advection terms. As in NY95, we assume that the convective flux is proportional to the entropy gradient:

\[ F_c \simeq -K_c \rho(r, \theta)T(r, \theta) \frac{\partial s}{\partial r}, \]  

(44)

where \( K_c \) is a proportional constant and can be considered as an effective diffusion constant. Furthermore, the specific entropy \( s \) and temperature \( T \) are given by

\[ s = \frac{k_B}{(\gamma - 1)m} \ln \frac{\rho(r, \theta)}{\rho(r, \theta)^{\gamma}}, \quad T = \frac{m}{k_B \rho(r, \theta)}, \]  

(45)

where \( k_B \) is the Boltzmann constant and \( m \) is the mass of a single molecule, respectively. In order to complete this estimation, let us assume that \( K_c \) follows a profile similar to that of the viscosity, i.e., \( K_c = \tilde{\alpha}_c \Omega K \), where \( \tilde{\alpha}_c \) is different from \( \tilde{\alpha} \). In fact, in the standard case, this parameter can be larger than \( \alpha/2 \) (NY95). Here, we also assume that \( \tilde{\alpha}_c = 0.5\alpha \). Thus, it is easy to find the ratio of convective term to the advection terms:

\[ G(\theta) = \frac{2\pi \nabla \cdot F}{3e c_s^2 v_r \rho} = -\frac{\tilde{\alpha}_c}{v(\theta)}. \]  

(46)

The outflow regions can now be specified as regions where we have \( b > 0, H(\theta) < 0 \) and \( \ln G(\theta) > 0 \). Corresponding to our rotating solutions, however, there is not a region where all three conditions are satisfied. In other words, although the solutions are convectively unstable, the convective instability cannot revert the direction of the flow and cause outflow launching. We note that this is not the case in model A, and outflow can arise in the rotating solution.

On the other hand, we find that outflow can occur in the non-rotating solutions. We consider the non-rotating solution with \( v_\theta(0) = 0 \) (top panel in Figure 3). In this case, the Bernoulli parameter is positive everywhere except very close to the equatorial plane. Furthermore, we have \( H(\theta) < 0 \) for \( 0 \leq \theta \leq \pi \). Therefore, we need to check the third condition, i.e., \( \ln G(\theta) > 0 \). This condition is satisfied within the interval \( \theta \leq \theta_{\text{crit}} \). The critical angle \( \theta_{\text{crit}} \) depends on \( \epsilon \). For \( \epsilon = 0.08 \), 0.1, and 0.2, the critical angle is found as \( \theta_{\text{crit}} = 0.6, 0.53, \) and 0.4 rad, respectively. Our analysis shows that flows with a larger \( \epsilon \) are subject to a more collimated outflow.

We also find that outflow can occur in the second type of the non-rotating solutions presented in the bottom panel of Figure 3. For \( \epsilon = 0.08 \), outflow exists around the poles and the equatorial plane, i.e., within the ranges \( \theta < 0.29 \) and \( \theta > 1.3 \) rad. On the other hand, for \( \epsilon = 0.1 \), the outflow is limited to the regions with \( \theta > 1.36 \) and \( \theta < 0.25 \). Note that we have written these intervals for \( 0 \leq \theta \leq \pi/2 \), and one can simply generalize them to the whole space. For the larger values of \( \epsilon \), however, outflow exists only near the poles. When we have \( \epsilon = 0.2 \), for instance, the outflow region is restricted to \( \theta < 0.2 \). It is interesting that, although there is no rotation in the system, there is relatively collimated outflow around the poles.

6. ADAFs with \( v_\theta(\theta) = 0 \)

As we have already mentioned, ADAF solutions with a nonzero \( v_\theta(\theta) \) have been investigated by XC97 using the same numerical method implemented by us as well. In this case, the similarity exponent is not necessarily equal to \( n = 3/2 \). We explored a wide parameter space for \( n, \epsilon, \) and \( \alpha \), but our attempts to find solutions with convergent Fourier series were not successful. All obtained solutions that satisfy the main equations fail to fulfill our convergence criterion. We then tried to find convergent solutions by including sine functions in the Fourier series (33). In fact, this seems necessary in the sense that, without sine functions, the cosine functions cannot form a complete orthogonal system. This attempt, however, did not resolve the convergence problem. It is necessary to mention that the existence of sine terms does not alter our presented solutions for \( v_\theta = 0 \), because we found that the sine term coefficients are very small in this case.

The origin of this complexity is probably the intrinsic nonlinear nature of the governing equations when the nonzero \( v_\theta \) is included. More specifically, when \( v_\theta \) is zero, the Fourier...
approach leads to third-order algebraic equations. However, for nonzero $v_\theta$, the energy equation leads to fourth-order algebraic equations. Furthermore, the existence of $v_\theta$ substantially enlarges the number of terms in each algebraic equation. Consequently, that approach does not work for this case. Therefore, we do not confirm the outflow solutions presented in XC97. Furthermore, we think that XC97’s solutions are based on equations that have some mistakes.

7. Summary

In this paper, we have used the Fourier expansion to find semi-analytic solutions for ADAFs. More specifically, we assumed that the system is stationary and axisymmetric, and possesses radially self-similar structure. In this case, we have a one-dimensional system and the governing differential equations reduce to an eighth-order system. After setting an appropriate set of boundary conditions, we expand all the physical quantities using the Fourier expansion. In practice, one must truncate the expansions and keep a finite number of terms. In this paper, we keep five to nine terms in the expansions. Finally, instead of solving an eighth-order system of differential equations, we have solved a set of $5N$ non-linear algebraic equations to find $5N$ Fourier coefficients. This means that we find a semi-analytic function for all the physical quantities. The main practical benefit of this approach, compared to numerical integration of the differential equations, is that one obtains all the functions in the whole space. This makes it easy to study the properties of the system and straightforwardly interpret the results. For example, the stability issues of the flow can be easily checked. We remind the reader that one of the restrictions of the numerical integration of the governing equations is that one cannot start from the equatorial plane and reach the pole. In brief, one may say that the Fourier expansion analysis leads to analytical solutions, and analytical solutions are always helpful to simplify the analysis of the given system.

Using this approach for a new viscosity model, in which $\alpha$ varies with $\theta$, we have found four categories of the solutions in the absence of the latitudinal component of the velocity. The first rotating solution presented in Figure 1 corresponds to an inflow with $\tilde{\alpha} = 0.1$ and $\tilde{v}_\theta(0) = 0$. Although this solution is convectively unstable, the convection cannot reverse the direction of the flow. The second rotating and inflow solution is illustrated in Figure 2 with $\tilde{\alpha} = 0.1$ and $\tilde{v}_\theta(0) = 0$. Convection cannot reverse the direction of the flow in this solution either.

The third solution presented in the top row of Figure 3 is a non-rotating inflow with $\tilde{\alpha} = 0.1$ and $\tilde{v}_\theta(0) = 0$. Although the flow does not rotate, its geometrical shape is aspherical and it tends to a flattened configuration with increasing $\epsilon$. Furthermore, we showed that convection is dynamically important and it may contribute to launching of the outflows. More importantly, outflow can exist around the poles. Our last solution, which corresponds to a non-rotating inflow with $\tilde{\alpha} = 0.1$ and $\tilde{v}_\theta(0) = 0$, has been shown in the bottom row of Figure 3. We showed that convection in the system can produce outflows. For small $\epsilon$, outflow exists near the poles and also near the equatorial plane. For large values of $\epsilon$, however, outflow happens only around the poles.

Finally, we studied a case with $\tilde{v}_\theta \neq 0$. In this case, we could not find any convergent and unique solutions, due to the highly non-linear nature of the equations. We made attempts to generalize the method in various directions, but we could not find convergent solutions. Consequently, we do not confirm the outflow solutions previously reported in XC97.

As a final remark, we would like to mention that this paper has shown that the Fourier expansion method can help to study ADAF systems. Naturally, more careful and physically oriented investigations can be accomplished by taking into account more physics in the system. For example, one may add magnetic fields, the effect of thermal conduction, or the existence of radiation cooling, and use this method to derive the properties of the system. It is also possible, in practice, to use this expansion even in the radial direction. In that case, one may use radial eigenfunctions of the Laplace operator. In other words, one may study solutions that are not necessarily self-similar in the radial direction. It is even possible to study self-gravitating systems in which the central mass potential deviates from the standard Newtonian potential. For example, one may use the pseudo-Newtonian potential in order to include the relativistic effects. Therefore, more investigation is required to check the effectiveness of this approach.

Given these facts, the treatments in this paper are sufficiently general to describe many disk-wind substructures, such as inflow-outflow regions, coronae, disk jets, and collimated jets, which have appeared in simulations and are generally supposed to play important roles in the power spectrum of a system. The numerical approach presented by NY95 for vertical structure of disk is unable take into account these sub-structures because of the complexity of the numerical techniques. However, using Fourier analysis, we will be able to investigate the vertical structure uniquely by adding proper physics.

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Appendix A

In this appendix, we show the details of the Fourier approach for a toy model. In fact, we set $\Omega = 0$ in the main Equations (34)–(37) and keep only two terms in the expansions, for simplicity. In this case, Equation (36) is automatically satisfied and we deal only with differential Equations (34), (35), and (37). Furthermore, the physical variables are given by

$$\rho(\theta) = a_0 + a_1 \cos 2\theta,$$

$$c_s^2(\theta) = w_0 + w_1 \cos 2\theta,$$

$$v(\theta) = b_0 + b_1 \cos 2\theta.$$ (47)
Substituting these functions into Equations (34), (35), and (37), respectively, we find the following equations:

\[
2\alpha_1(2b_0 \cos 2\theta(b_1 \cos 2\theta - \alpha) - \alpha b_1(4 \cos 2\theta + 11 \cos 4\theta + 3) + b_1^2 \cos 32\theta + b_0^2 \cos 2\theta + \cos 2\theta \\
\times (5w_1 \cos 2\theta + 5w_0 - 2) + \alpha_0(-4b_0(\alpha - b_1 \cos 2\theta) - 4\alpha b_1(7 \cos 2\theta + 2) + b_1^2(\cos 4\theta + 1) + 2b_0^2 \\
+ 2(5w_1 \cos 2\theta + 5w_0 - 2)) = 0,
\]

\[\sin \theta \cos^3 \theta(60\alpha a_1 b_1 - 48a_1 w_1) + \cos \theta(\sin^2 \theta \\
\times (48a_1 w_1 - 60\alpha a_1 b_1) + \sin \theta \\
\times (24\alpha a_1 b_0 + 36\alpha a_0 b_1 - 24a_1 w_0 - 24a_0 w_1)) = 0,
\]

\[
12a_0 b_0^2 + 14a_1 b_1^2 + \sin^2 \theta(3b_1 w_1 - 2b_0^2) + \sin^2 \theta \\
\times (-24a_0 b_0 - 6b_1 w_0 - 6b_0 w_1) + \cos^2 \theta(3b_1 w_1 \\
- 2a_0 b_0^2) + \cos^2 \theta(24a_0 b_0 + 2a_0 b_1 + \sin^2 \theta(12a_1 b_1 - 18b_1 w_1) \\
+ 6b_1 w_0 + 6b_0 w_1) + 6b_1 w_0 + 6b_0 w_1 + 3b_1 w_1 = 0.
\]

Now we rewrite the products and powers of sine and cosine functions in terms of trigonometric functions with combined arguments. Therefore, Equations (48)–(50) take the following form:

\[
A_1 + A_2 \cos 2\theta + A_3 \cos 4\theta + a_1 b_1^2 \cos 6\theta = 0 \\
B_1 \sin 2\theta + B_2 \sin 4\theta = 0 \\
C_1 + C_2 \cos 2\theta + C_3 \cos 4\theta = 0,
\]

where the coefficients are defined as

\[
A_1 = -8a_0 a_1 b_0 - 16\alpha a_1 b_1 - 12\alpha a_1 b_1 + 4a_0 b_0^2 \\
+ 4a_1 b_1 b_0 + 2a_0 b_1^2 + 20a_0 w_0 + 10a_1 w_1 - 8a_0 \\
A_2 = -8a_1 b_0 - 8a_0 a_1 b_1 - 16\alpha a_1 b_1 + 4a_0 b_1^2 \\
+ 8a_0 b_1 b_0 + 3a_1 b_1^2 + 20a_0 w_0 + 20a_1 w_1 - 8a_1 \\
A_3 = -44a_0 a_1 b_0 + 2a_0 b_1^2 + 4a_1 b_1 b_0 + 10a_1 w_1 \\
B_1 = 4a_0 b_1 b_0 + 6a_0 a_0 b_1 - 4a_1 b_0 - 4a_0 w_1 \\
B_2 = 5a_0 a_1 b_1 - 4a_1 b_1 \\
C_1 = 12a_0 b_0^2 + 14a_1 b_1^2 + 6b_0 w_0 \epsilon + 3b_1 w_1 \epsilon \\
C_2 = 24a_0 b_0 b_1 + 6b_1 w_0 \epsilon + 6b_0 w_1 \epsilon \\
C_3 = 3b_1 w_1 \epsilon - 2a_0 b_0^2.
\]

Considering that we have kept only two terms in the expansions, now the coefficients of \(\cos m\theta\) and \(\sin m\theta\) for \(m = 0\) and \(m = 2\) must be set to zero. Therefore, for a given \(\alpha\) and \(\epsilon\), we have the following five algebraic equations for six unknowns \(a_0, a_1, b_0, b_1, w_0,\) and \(w_1:\)

\[
A_1 = 0, \quad A_2 = 0, \quad B_1 = 0, \quad C_1 = 0, \quad C_2 = 0.
\]

On the other hand, as discussed in Section 2.4, we have one more equation from the boundary condition \(\hat{m} = 0.23\). This constraint is given by

\[-2a_0 b_0 + \frac{2a_0 b_0}{3} + \frac{2a_0 b_1}{3} - \frac{14a_1 b_1}{15} = 0.23.\]

Finally, Equations (53) and (54) are six algebraic equations for six unknowns, and can be solved via the usual numeric procedure. From the solutions found, we then choose the physical ones. Consequently, although the coefficients are obtained using numerical methods, we have semi-analytic solutions given by (47). For a larger number of terms in the expansions, we use the same procedure to solve the main differential equations.

**Appendix B**

**Reproducing NY95’s Results**

In order to check the validity and correctness of Fourier analysis of ADAFs, we try to reproduce NY95’s results. We set \(n = 3/2\), and consequently \(\nu_r(r, \theta) = 0\). In this case, it is straightforward to show that the differential equations are sixth-order and we need six boundary conditions. As we mentioned, in this case, we use the net mass accretion rate to obtain one boundary condition \(\hat{m} = 0.23\), as was done in NY95—albeit, the magnitude of \(\hat{m}\) is not explicitly reported in NY95. Here, we chose \(\hat{m} = 0.23\) in order to find a lost relation to those of NY95. The boundary conditions (30) and (31) are automatically satisfied. We investigate both conditions on \(\nu_r(0)\) and compare the results to those presented in NY95, which have been obtained by a numerical relaxation technique. It is necessary to mention that there are two free parameters that control the physics of the system: \(\alpha\) and \(\epsilon\). In the following, we briefly report our results, and the physical interpretations of the solutions can be found in the comprehensive paper NY95.

Let us start with \(\nu_r(0) = 0\). It turns out that solutions are convergent for \(N \geq 6\). We illustrate this fact in Figure 7. More specifically, to show the convergence of the solutions, we have plotted the fractional difference between solutions obtained while retaining \(N\) and \(N + 1\) terms in the expansions. It is clear that increasing the number of terms \(N\) causes the fractional differences in all the physical quantities to shrink. In fact, for \(N \geq 6\), the fractional differences are almost zero.

We present the results for \(N = 9\). In other words, we keep 10 terms in the Fourier expansions. As mentioned in NY95, it turns out that this boundary condition belongs to rotating, i.e., \(\Omega \neq 0\), and fully advective solutions. The isodensity contours in the meridional plane have been illustrated in Figures 8. This figure can be compared to Figure 2 of NY95. For \(\epsilon = 0.1\), we have plotted the quantities for different values of \(\epsilon\) in Figure 9. This figure should be compared to Figure 1 of NY95.

As reported in NY95, there are some higher-order solutions for small \(\alpha\). In those solutions, the angular velocity reverses sign one or more times as a function of \(\theta\). Such solutions are unlikely to describe a real system. Only in NY95 has the isodensity contour of such a solution been reported. For completeness, we have also found a higher-order solution for \(\alpha = 0.01\) and \(\epsilon = 0.5\); the relevant quantities are shown in Figure 10. The corresponding isodensity contour is shown in the bottom right panel of Figure 8. As Figure 10 makes clear, \(\Omega\) changes sign six times in the interval \([0, \pi]\), while \(\nu_r\) is oscillatory but always negative.

Thus far, there is clearly an excellent agreement between our results and those presented in NY95. Finally, the only class of solutions we have not yet compared against NY95 is that for which \(\nu_r(0) = -\epsilon/2\alpha\). Applying the Fourier series approach to this case, we found that there is no rotating solution. For example, for \(\alpha = 0.1\), \(\epsilon = 0.1\), and \(\hat{m} = 0.23\), we found that \(\rho_r\), \(\nu_r\), and \(C_{\phi}^2\) are constant and are given by 0.23, 0.05, and 0.34, respectively. On the other hand, \(\Omega(\theta)\) oscillates in the narrow
interval $-2.4 \times 10^{-8} < \Omega < 3.2 \times 10^{-8}$, which is negligible compared to other velocity components. Our conclusion regarding the non-existence of rotation solutions for this boundary condition is completely in agreement with the analytical description presented in the appendix of NY95.

With this test, we have checked the reliability of the Fourier series approach. However, let us introduce another, more direct test to show that this method leads to true solutions for the main equations. To do so, we rewrite Equations (34)–(37), respectively, as follows:

$$ F_1(\theta) = 0, \quad F_2(\theta) = 0, \quad F_3(\theta) = 0, \quad F_4(\theta) = 0. \quad (55) $$

One may note that this method leads to semi-analytic solutions. For example, for our second type of solutions with $\alpha = 0.1$ and $\epsilon = 0.1$, when $N = 4$, the function $\rho(\theta)$ is given by

$$ \rho(\theta) \approx 1 + (4.7 \cos 2\theta + 9.2 \cos 4\theta + 2.1 \cos 6\theta) \times 10^{-4}. $$

We have similar expansions for other physical variables. Therefore, if we substitute them into Equations (55), we expect that the right-hand side of all of the equations in (55) should vanish. For different values of $N$ and for the aforementioned solution, we have plotted functions $F_i(\theta)$ in Figure 11. As this figure makes clear, those functions grow smaller with increasing $N$. When $N = 7$, all functions are smaller than $10^{-7}$ in the whole space, i.e., $|F_i(\theta)| < 10^{-7}$ in $0 < \theta < \pi$. These functions get even smaller for larger choices of $N$. This behavior is consistent with the convergence of the solutions presented in Figure 4. Therefore, one can conclude that solutions obtained with this Fourier approach are true for the main differential equations.
Figure 9. Self-similar solutions corresponding to $n = 3/2$, $\alpha = 0.1$, $\epsilon = 0.1$, 1, 10, and $n = 0.23$. Top left: density $\rho(\theta)$. Top right: the isothermal sound speed, $c_s^2$, with respect to $\theta$. Bottom left: radial velocity $v_r(\theta)$. Bottom right: the angular velocity $\Omega(\theta)$. 

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Figure 10. Physical quantities for a higher-order solution for which \( n = 3/2, \alpha = 0.01, \epsilon = 0.5, \) and \( m = 0.23. \) Top left: density \( \rho(\theta). \) Top right: the isothermal sound speed \( c_s^2 \) with respect to \( \theta. \) Bottom left: radial velocity \( v_r(\theta). \) Bottom right: the angular velocity \( \Omega(\theta). \)

Figure 11. Functions \( F_i \) in terms of \( \theta \) for different values of \( N \) when \( \epsilon = \alpha = 0.1 \) in our second rotating solution. The red, blue, green, and black curves belong to \( N = 4, N = 5, N = 6, \) and \( N = 7, \) respectively.

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