First order transition for the branching random walk at the critical parameter

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Abstract. Consider a branching random walk on the real line in the boundary case. The associated additive martingales can be viewed as the partition function of a directed polymers on a disordered tree. By studying the law of the trajectory of a particle chosen under the polymer measure, we establish a first order transition for the partition function at the critical parameter. This result is strongly related to the paper of Aïdékon and Shi in which they solved the problem of the normalisation of the partition function in the critical regime.

1 Introduction

We consider a real-valued branching random walk: Initially, a single particle sits at the origin. Its children together with their displacements, form a point process Θ on $\mathbb{R}$ and the first generation of the branching random walk. These children have children of their own which form the second generation, and behave relatively to their respective positions at birth like independent copies of the same point process $\Theta$. And so on.

Let $T$ be the genealogical tree of the particles in the branching random walk. Plainly, $T$ is a Galton-Watson tree. We write $|z| = n$ if a particle $z$ is in the $n$-th generation, and denote its position by $V(z)$. The collection of positions $(V(z), z \in T)$ is our branching random walk.

Following [1], assume throughout the paper the following conditions

\begin{align}
E\left(\sum_{|x|=1} e^{-V(x)}\right) &= 1, \quad E\left(\sum_{|x|=1} 1\right) > 1, \quad \text{and} \\
E\left(\sum_{|x|=1} V(x)e^{-V(x)}\right) &= 0, \quad \sigma^2 := E\left(\sum_{|x|=1} V(x)^2 e^{-V(x)}\right) < \infty.
\end{align}
The branching random walk is then said to be in the boundary case (Biggins and Ky- prianou [21]). We refer to [17] for detailed discussions on the nature of the assumption (1.1) and (1.2).

Let \( \Phi(t) := \log \mathbb{E} \left( \sum_{|x|=1} e^{-tV(x)} \right) \in (-\infty, +\infty] \), \( t \in \mathbb{R} \) and let

\[
W_{\beta,n} := \sum_{|x|=n} e^{-\beta V(x) - \Phi(\beta)n}, \quad \beta \in \mathbb{R},
\]

which can be viewed as the normalized partition function of a directed polymer on trees, see the forthcoming (1.12). In the literature \( W_{1,n} \) is called the critical additive martingale associated with the branching random walk. For notational simplification, we write \( W_{1,n} = W_n \) for any \( n \geq 0 \) \( [W_0 := 1] \).

Under (1.1), \( T \) is infinite with positive probability. Moreover the results stated here make only a trivial sense if the system dies out, so it is convenient to introduce:

\[
P^*(\cdot) := P(\cdot | T \text{ is infinite}).
\]

By Biggins [5], it is known that under some integrability conditions (for example under the forthcoming (1.5), we refer to Lyons [19] for the optimal conditions), we have

for \( \beta < 1 \), \( \lim_{n \to \infty} W_{\beta,n} = W_\beta > 0 \), \( P^* \text{a.s.} \)

for \( \beta \geq 1 \), \( \lim_{n \to \infty} W_{\beta,n} = 0 \), \( P^* \text{a.s.} \)

According to the terminology in the study of polymers (see e.g [13]), we say that the region \( \beta > 1 \) is the strong disorder regime, \( \beta \leq 1 \) the weak disorder regime and \( \beta = 1 \) the critical case.

We are interested here in the regularity of \( \beta \to W_\beta \) at \( \beta = 1 \). Biggins [17] proved that the martingale \( (W_{n,\beta})_{\beta \in \mathbb{C}} \) converges uniformly on any compact subset of a set \( \Lambda^* \subset \mathbb{C} \) almost surely and in mean. As a by-product, he obtained the analyticity of \( W_\beta \) on \((0,1)\). We shall show that there is a first order transition at \( \beta = 1 \). In order to state our main result, we need to assume that there exist \( \frac{1}{4} > \epsilon_0 > 0 \) and \( \frac{\epsilon_0}{2} > \delta_- > 0 \) such that

\[
\mathbb{E} \left( \left( \sum_{|x|=1} e^{-(1-2\delta_-)V(x)} \right)^{1+2\epsilon_0} \right) < \infty.
\]

Note that this condition (1.5), stronger than the Aïdékon-Shi [1]'s conditions (see (1.7)), implies

\[
\sup_{\beta \in [1-\delta_- , 1]} \mathbb{E} \left( \left( \sum_{|x|=1} e^{-\beta V(x)} \right)^{1+\epsilon_0} \right) < \infty.
\]

Let us also introduce the so-called derivative martingale defined by

\[
D_n := \sum_{|u|=n} V(u)e^{-V(u)}, \quad n \geq 1.
\]
Defining $X := \sum_{|x|=1} e^{-V(x)}$ and $\tilde{X} := \sum_{|x|=1} \max\{0, V(x)\} e^{-V(x)}$, Biggins and Kyprianou, [8], have shown that under the condition

$$E(X(\max(0, \log X))^2) < \infty, \quad E(\tilde{X} \max(0, \log \tilde{X})) < \infty,$$

there exists a random variable $D_\infty$ positive on the set of non-extinction such that

$$\lim_{n \to \infty} D_n = D_\infty, \quad P^\ast\text{a.s.}$$

Our first result in this paper is the following theorem:

**Theorem 1.1** Assume (1.1), (1.2) and (1.5). We have:

$$\lim_{\beta \uparrow 1} \frac{W_\beta}{1 - \beta} = 2D_\infty,$$

where the convergence holds in $P^\ast$ probability.

Theorem 1.1 relies on a study of the polymer measure at the critical point which will be our second result in this paper. Following Derrida and Spohn [14], we associate each vertex $x \in \mathbb{T}$ to $[\emptyset, x]$ the unique shortest path relating $x$ to the root $\emptyset$ and $x_i$ (for $0 \leq i \leq |x|$) the vertex on $[\emptyset, x]$ such that $|x_i| = i$. The trajectory of $x \in \mathbb{T}$ corresponds to the ancestor’s positions of $x$, i.e the vector $(V(x_1), ..., V(x_{|x|}))$, whereas $(V_s(x))_{s \in [0,1]}$ designates the linear interpolation of the trajectory of $x \in \mathbb{T}$ and is defined by

$$V_t(x) := \frac{1}{\sqrt{|x|}} V(x_{\lfloor |x|t \rfloor}) + (|x|t - \lfloor |x|t \rfloor) \frac{1}{\sqrt{|x|}} (V(x_{\lfloor |x|t \rfloor + 1}) - V(x_{\lfloor |x|t \rfloor})), \quad 0 \leq t \leq 1.$$

Then for each parameter $\beta > 0$ and $n \in \mathbb{N}$, we define the polymer measure $\mu_n^{(\beta)}$ on the disordered tree $\mathbb{T}_n$ by

$$\nu_n^{(\beta)}(x) := \frac{1}{W_{\beta,n}} e^{-\beta V(x) - \Phi(\beta)n} \delta_x, \quad x \in \mathbb{T}_n.$$

We will study the law of the trajectory of a particle chosen under the polymer measure. So let $(\mathcal{C}, ||\cdot||_\infty)$ be the set of continuous function on $[0,1]$ endowed with the sup-norm $||\cdot||_\infty$ and for any $A \in \mathcal{B}$ (the $\sigma$-algebra generated by the open sets of $(\mathcal{C}, ||\cdot||_\infty)$) define

$$\mu_n^{(\beta)}(A) := \frac{1}{W_{\beta,n}} \sum_{|x|=n} e^{-\beta V(x) - \Phi(\beta)n} \mathbb{1}_{\{(V_s(x))_{s \in [0,1]} \in A\}}.$$

The model of directed polymer on a disordered tree corresponds in some sense to a mean field limit when the dimension goes to infinity, see [14]. This approximation is can be understood in large lattice dimension : indeed, as $d$ increases, two independent paths $V_1$ and $V_2$ on the lattice have smaller probability to ever meet in the future. The models on the $d$-dimensional
lattice and on tree with branching number $b$ are asymptotically alike when $b = 2d \to \infty$. Many authors have already worked on this subject introduced in 1988 by Derrida and Spohn [14]. Recently Mörters and Ortgiese [21] studied the phase transition arising from the presence of a random disorder. Hu and Shi in [15] showed that the derivative martingale appears naturally in the rate convergence of $W_n \to 0$. Furthermore, Aidékon and Shi [1] proved the following theorem

Theorem A (Aidékon and Shi [1]) Assume (1.1), (1.2) and (1.7) we have

\begin{equation}
\lim_{n \to \infty} \sqrt{n} W_n \to \left( \frac{2}{\pi \sigma^2} \right)^{1/2}, \quad \text{in } P^* \text{ probability.}
\end{equation}

This result, used repeatedly in our paper, solved the problem of the normalisation of the partition function in the critical regime. Moreover we use the powerful method developed in [1] and establish our second result :

Theorem 1.2 Assume (1.1), (1.2) and (1.7). For any $F \in C_b(C, \mathbb{R}^+)$ we have

\begin{equation}
\mu_n^{(1)}(F) := \frac{1}{W_n} \sum_{|u| = n} e^{-V(u)} F \left( (V_s(u))_{s \in [0,1]} \right) \to E \left( F(\sigma(R_s))_{s \in [0,1]} \right), \quad \text{in } P^* \text{ probability},
\end{equation}

where $(R_s)_{s \in [0,1]}$ is a Brownian meander.

This convergence represents an important step in the proof of Theorem 1.1 but it may also have an independent interest. For example we mention an interesting paper by Alberts and Ortgiese [4] who also study a phase transition at the critical case. Theorem 1.2 would yield their Theorem 1.2.

Theorem 1.2 gives also an interesting consequence on the “overlap” of the branching random walk which is introduced in [14]. For $|u|, |v| = n$ we define the overlap by

$$Q_{u,v} = \sup \{ k \leq n, u_j = v_j \forall j \leq k \}.$$ 

 Similarly for $|u| = |v| = n$ we can introduce the fraction of time in which the two paths $(V(u_1), ..., V(u_n))$ and $(V(v_1), ..., V(v_n))$ are identical, i.e

$$\tilde{Q}_{u,v} = \sup \{ k \leq n, V(u_j) = V(v_j) \forall j \leq k \}.$$ 

Clearly, $0 < Q_{u,v} \leq \tilde{Q}_{u,v} \leq 1$.

Corollary 1.3 Assume (1.1), (1.2) and (1.7). For any $\delta > 0$, the following convergence is true

\begin{equation}
\frac{1}{W_n^2} \sum_{|u| = n, |v| = n} e^{-V(u)} e^{-V(v)} 1_{\{Q_{u,v} \geq \delta\}} \to 0, \quad \text{in } P^* \text{ probability.}
\end{equation}
Finally we stress that Theorem 1.2 is not true $\mathbb{P}^*$ almost surely. Indeed let us introduce
$$F_\epsilon \in C_b(\mathbb{C}, \mathbb{R}^+)$$
defined by
$$F_\epsilon(w) := \begin{cases} 0 & \text{if } w(1) \notin [-\epsilon, 2\epsilon], \\ \frac{\epsilon + x}{\epsilon} & \text{if } w(1) \in [-\epsilon, 0], \\ \frac{1}{\epsilon} & \text{if } w(1) \in [0, \epsilon], \\ \frac{2\epsilon - x}{\epsilon} & \text{if } w(1) \in [\epsilon, 2\epsilon], \end{cases}$$
\(\epsilon > 0, w \in \mathbb{C},\)
and
$$A_n := \{\exists \epsilon > 0, w \in \mathbb{C}, \epsilon \geq n \leq |x| \leq 2n, \frac{1}{2}\log n \leq V(x) \leq \frac{1}{2}\log n + C; \max_{n \leq k \leq 2n} \sqrt{nW_k} \leq C\}, \ n \in \mathbb{N}, C > 0.$$
According to Lemma 6.3 in [1] and Theorem 1.5 in [15], there exist $C, N > 0$ large and $c > 0$ small, such that for any $n > N$, $\mathbb{P}^*(A_n) \geq c$. Clearly for any $\epsilon > 0$, $n \geq \epsilon^2$, on the set $A_n$, we have $\mu_n(1)(F_\epsilon) \geq C^{-1}e^{-C}$. So
$$\limsup_{\epsilon \to 0} \limsup_{n \to \infty} \mathbb{E}^*(\mu_n(1)(F_\epsilon)) \geq cC^{-1}e^{-C} > 0 = \limsup_{\epsilon \to 0} \mathbb{E}\left(F_\epsilon\left[\sigma R_s s \in [0,1]\right]\right),$$
which implies that (1.14) can not hold $P^*$ almost surely.

Similarly, (1.15) can not be strengthened in $P^*$ almost sure convergence.

The rest of the paper is organized as follows. In Section 2 we present some preliminaries on branching random walks. Section 3 is devoted to the proof of the Theorem 1.2 and Corollary 1.3. Finally, in Section 4 we prove the Theorem 1.1.

## 2 Preliminaries

This section collects some preliminary results on the branching random walk (change of probabilities, an associated one-dimensional random walk), and it entirely comes from Aïdékon and Shi [1].

### 2.1 The many-to-one Lemma

Let $(V(x))$ be a branching random walk satisfying (1.1) and (1.2). Let $(S_n)_{n \geq 0}$ a random walk such that the law of $(S_1)$ is given by

$$E(f(S_1)) := E\left(\sum_{|z|=1} f(V(z))e^{-V(z)}\right), \quad \forall f : \mathbb{R} \to [0, \infty), \text{ measurable.}$$

The condition (1.1) and (1.2) implies that $(S_n)$ is a mean zero random walk and $E(S_1^2) = \sigma^2 < \infty$. From a simple induction it stems that for any $n \geq 0$ and $g : \mathbb{R}^n \to \mathbb{R}^+$ measurable
we have:
\[ E \left( \sum_{|x| = n} g(V(x_1), ..., V(x_n)) \right) = E \left( e^{S_n} g(S_1, ..., S_n) \right). \]

Equality (2.2) forms the so-called many-to-one Lemma which plays a fundamental role in many computations of expectations. The presence of the random walk \((S_i)\) is explained in Lyons, Pemantle, Peres [20], Lyons [19] and Biggins, Kyprianou [8].

2.2 The renewal function associated with a one-dimensional random walk

Associated to \((S_n)\), which is a centered random walk real-valued with \(\sigma^2 = E[S_1^2] \in (0, \infty)\), let \(h_0\) be its renewal function defined by
\[ h_0(u) := \sum_{j \geq 0} P \left( \min_{i \leq j-1} S_i > S_j \geq -u \right), \quad u \geq 0. \]

In the following this function will play an important role, so we collect here some facts about \(h_0\). For any \(u \geq 0\), \(h_0\) satisfies
\[ h_0(u) = E \left( h_0(S_1 + u) \mathbb{1}_{\{S_1 \geq -u\}} \right). \]

If we write
\[ S_n := \min_{j \leq n} S_j, \quad n \geq 0, \]

it is known that there exists \(c_0 > 0\) and \(\theta > 0\) such that
\[ c_0 := \lim_{u \to \infty} \frac{h_0(u)}{u}, \quad P \left( S_n \geq -u \right) \sim \frac{\theta h_0(u)}{n^{\frac{1}{2}}}, \quad \forall u \geq 0. \]

As a consequence there exists constants \(c, C > 0\) such that
\[ c_1 (1 + u) \leq h_0(u) \leq C_1 (1 + u), \quad u \geq 0. \]

As in [1], we will need the following uniform version of (2.6) as \(n \to \infty\):
\[ P \left( S_n \geq -u \right) = \frac{\theta h_0(u) + o(1)}{n^{\frac{1}{2}}}, \]

uniformly in \(u \in [0, (\log n)^{30}]\).

Finally we mention the inequality due to [1]: there exists \(c > 0\) such that for \(u > 0, a \geq 0, b \geq 0\) and \(n \geq 1,\)
\[ P \left( S_n \geq -a, b - a \leq S_n \leq b - a + u \right) \leq c \frac{(u + 1)(a + 1)(b + u + 1)}{n^{\frac{1}{2}}}. \]
2.3 A spine conditioned to stay positive

Let \((V(x))\) be a branching random walk satisfying (1.1) and (1.2), let \((\mathcal{F}_n)\) be the sigma-algebra generated by the branching random walk in the first \(n\) generations. Since Lyons [19], the spinal decomposition is a widespread technique to study the branching random walk. Usually we introduce the martingale \(W_n := \sum |z|=n e^{-V(z)}\) to define the probability \(Q\) satisfying for any \(n \in \mathbb{N}\), \(Q_{|\mathcal{F}_n} := W_n P_{|\mathcal{F}_n}\). Then we obtain a description with a spine of the branching random walk under \(Q\), moreover this spine behaves like a centered random walk. Here we will need a slightly different decomposition, we will work with a spine whose the law is as a random walk conditioned to stay positive.

First let us introduce some notations. For any vertex \(x \in \mathbb{T}\), let \(V(x) := \min_{y \in [0,x]} V(y)\). Then for \(\alpha \geq 0\) and \(u \geq -\alpha\) let \(h_\alpha(u) := h_0(u + \alpha)\). Finally we define the processes

\[
W^{(\alpha)}_n := \sum_{|x|=n} e^{-V(x)} 1_{\{V(x) \geq -\alpha\}}, \quad D^{(\alpha)}_n := \sum_{|x|=n} h_\alpha(V(x)) e^{-V(x)} 1_{\{V(x) \geq -\alpha\}}.
\]

From (2.4) and the branching property stem that for any \(\alpha \geq 0\), \((D^{(\alpha)}_n, n \geq 0)\) is a non-negative martingale with respect to \(\mathcal{F}_n\) (see Biggins Kyprianou [8] or [1] for a proof). So associated with \(D^{(\alpha)}_n\) we introduce the new probability measure \(Q^{(\alpha)}\) which satisfies for any \(n\),

\[
Q^{(\alpha)}_{|\mathcal{F}_n} := \frac{D^{(\alpha)}_n}{h_\alpha(0)} P_{|\mathcal{F}_n}.
\]

Now we will give a representation with spine of the branching random walk under \(Q^{(\alpha)}\). A justification of this representation can be founded in [1]. Recall that the point process which governs the law at the first generation of \((V(x), |x| = 1)\) is distributed under \(P\) as the point process \(\Theta\). For any \(u \geq -\alpha\), the new probability \(Q^{(\alpha)}\) makes appear the point process \(\Theta^{(\alpha)}_u\) whose distribution is the law of \((u + V(w), |x| = 1)\) under \(Q^{(\alpha+\alpha)}\). Then the branching random walk under \(Q^{(\alpha)}\) is governed by the followings rules:

- \(w_0^{(\alpha)} = \emptyset\) gives birth to particles distributed according to \(\Theta^{(\alpha)}_0\).
- Choose \(w_1^{(\alpha)}\) among children of \(w_0^{(\alpha)}\) with probability proportional to \(e^{-V(x)} 1_{\{V(x) \geq -\alpha\}} h_\alpha(V(x))\).
- \(\forall n \geq 1, w_n^{(\alpha)}\) gives birth to particles distributed according to \(\Theta^{(\alpha)}_u\) \((u = V(w_n^{(\alpha)})\).
- Choose \(w_{n+1}^{(\alpha)}\) among the children of \(w_n^{(\alpha)}\) with probability proportional to \(e^{-V(y)} 1_{\{V(y) \geq -\alpha\}} h_\alpha(V(y))\).
- Subtrees rooted at all other brother particles are independent branching random walks under \(P\).

See below three facts which we will use continuously:

(i) \(Q^{(\alpha)}(\text{non \textendash} \text{extinction}) = 1\) and \(\forall n \in \mathbb{N}, Q^{(\alpha)}(D^{(\alpha)}_n > 0) = 1\).

(ii) For any \(n\) and any vertex \(x\) with \(|x| = n\), we have

\[
Q^{(\alpha)}(w_n^{(\alpha)} = x|\mathcal{F}_n) = \frac{h_\alpha(V(x)) e^{-V(x)} 1_{\{V(x) \geq -\alpha\}}}{D^{(\alpha)}_n}.
\]
(iii) The spine process \((V(w_n^{(\alpha)}), n \geq 0)\) under \(Q^{(\alpha)}\), is distributed as a Markov chain with transition probabilities given by

\[
P^{(\alpha)}(u, dv) := 1_{\{v \geq -\alpha\}} \frac{h_\alpha(v)}{h_\alpha(u)} p(u, dv), \quad u \geq -\alpha,
\]

where \(p(u, dv) := \mathbb{P}(S_1 + u \in dv)\) is the transition probability of \((S_n)\). In the sense of Doob’s \(h\)-transform, \((V(w_n^{(\alpha)}))_{n \in \mathbb{N}}\) has the law of the random walk \((S_n)_{n \geq 0}\) conditioned to stay in \([-\alpha, \infty]\). A convey way to represent this processes is the following id entity: for any \(n \geq 1\) and any measurable function \(g: \mathbb{R}^{n+1} \to [0, \infty)\),

\[
E_{Q^{(\alpha)}} \left( g(V(w_i^{(\alpha)}), 0 \leq i \leq n) \right) = \frac{1}{h_\alpha(0)} E \left( g(S_i, 0 \leq i \leq n) h_\alpha(S_n) 1_{\{S_n \geq -\alpha\}} \right).
\]

Convention: Throughout the paper, \(c, c', c''\) denote generic constants which may change from paragraph to paragraph, but are independent of \(n\).

### 3 Proof of Theorem 1.2 and Corollary 1.3

Let us introduce \((R_s)_{s \in [0,1]}\) and \((R'_s)_{s \in [0,1]}\) two independent Brownian meander under \(P\). For any \(d \in \mathbb{N}^*, t = (t_1, ..., t_d) \in [0,1]^d\) and any process \((f_s)_{s \in [0,1]}\) we will denote the vector \((f_{t_1}, ..., f_{t_d})\) by \(f_t\).

This section is divided in two steps:

- A) We show Theorem 1.2 and Corollary 1.3 assuming the following assertion: Under the integrability conditions (1.1), (1.2) and (1.7), for any \(d \in \mathbb{N}^*, t = (t_1, ..., t_d) \in [0,1]^d, F \in \mathcal{C}_b(\mathbb{R}^d),\)

\[
\lim_{n \to \infty} \frac{1}{W_n} \sum_{|u| = n} e^{-V(u)} F(V_t(u)) = E(F(\sigma R_t)), \quad \text{in } \mathcal{P}^* \text{ probability.}
\]

- B) We prove assertion (3.1).

#### 3.1 Step A)

For any metric spaces \(E\) and \(F\) we denote \(\mathcal{C}_b(E, F) := \{f: E \to F, \text{ continuous and bounded}\}\). Let \(\mathcal{W}^+\) the law of the Brownian meander, and \((E^*(\mu_n \otimes \mu_n(\cdot)))_{n \geq 0}\) the sequence of probability measure on \(C^2 := \{f: [0,1] \to \mathbb{R}^2, \text{ continuous}\}\), defined by

\[
E^*(\mu_n \otimes \mu_n(F)) := E^* \left( \frac{1}{W_n^2} \sum_{|u| = |v| = n} e^{-V(u)} e^{-V(v)} F \left( [V_t(u), V_t(v)]_{t \in [0,1]} \right) \right), \quad F \in \mathcal{C}_b(C^2, \mathbb{R}).
\]
First we shall prove that \((3.1)\) implies :

\[
E^*(\mu_n \otimes \mu_n(\cdot)) \xrightarrow{weakly} \mathbb{W}^+ \otimes \mathbb{W}^+.
\]

In order to obtain \((3.3)\), recall from \([10]\) that for any continuous process the convergence of the finite-dimensional laws and the relative compactness imply the weak convergence. We will obtain the relative compactness via the following criteria (see \([23]\)):

A sequence \((P_n)\) of probability measures on \(C^2\) is weakly relatively compact if and only if the following two conditions hold :

i) for every \(\epsilon > 0\), there exist a number \(A\) and an integer \(n_0\) such that

\[P_n[|w(0)| > A] \leq \epsilon, \quad \text{for every } n \geq n_0.\]

ii) for every \(\eta, \epsilon > 0\), there exists a number \(\delta\) and an integer \(n_0\)

\[P_n[K(\cdot, \delta) > \eta] \leq \epsilon, \quad \text{for every } n \geq n_0.
\]

with

\[K(w, \delta) := \sup \{|w_1(t) - w_1(t')| + |w_2(t) - w_2(t')|; |t - t'| \leq \delta\}, \quad \forall \delta > 0, w = (w_1, w_2) \in C^2.
\]

Proof of the relative compactness of \((E^*(\mu_n \otimes \mu_n(\cdot)))\). The first condition is trivially satisfied. For the second we need to control \(E^*(\frac{1}{W_n^2} \sum_{|u|=n, |v|=n} e^{-V(V(u), V(v))} I_{\{K(V(u), V(v)) \geq \eta\}})\).

It is known that \(\limsup_{A \to \infty} \sup_{u \in \mathbb{N}} P^n(\frac{1}{W_n} \geq A \sqrt{n}) = \lim_{x \to -\infty} P^n(\min V(x) \leq -\alpha) = 0\), see for instance \((1.13)\) and (1.4) in \([3]\). Thus for any \(\epsilon > 0\) there exist \(A, \alpha > 0\) large enough such that for any \(n \in \mathbb{N}^*\), we have:

\[
E^*_n(\frac{1}{W_n^2} \sum_{|u|=n, |v|=n} e^{-V(u)-V(v)} I_{\{K(V(u), V(v)) \geq \eta\}}) \\
\leq E^*(\frac{1}{W_n} \sum_{|u|=n} e^{-V(u)} I_{\{\sup \{|V_t(u) - V_{t'}(u)|; |t-t'| \leq \delta\} \geq \frac{\alpha}{2}\}}) \\
\leq \epsilon + cA \sqrt{n} P \left( \sum_{|u|=n} e^{-V(u)} I_{\{V(u) \geq -\alpha\}} I_{\{|V_t(u) - V_{t'}(u)|; |t-t'| \leq \delta\} \geq \frac{\alpha}{2}\}} \right).
\]

Now using the Many-to-one Lemma we have

\[
E^*_n(\frac{1}{W_n^2} \sum_{|u|=n, |v|=n} e^{-V(u)-V(v)} I_{\{K(V(u), V(v)) \geq \eta\}}) \\
\leq \epsilon + cA \sqrt{n} P \left( \sup \{|S(n, t) - S(n, t')|; |t-t'| \leq \delta\} \geq \frac{\eta}{2} ; S_n \geq -\alpha \right),
\]

with \(S(n, t) := \frac{1}{\sqrt{n}} S_{[nt]} + (nt - [nt])(S_{[nt]+1} - S_{[nt]}), t \in [0, 1]\). Finally as

\[
\lim_{\delta \to 0} \limsup_{n \to \infty} \sqrt{n} P \left( \sup \{|S(n, t) - S(n, t')|; |t-t'| \leq \delta\} \geq \frac{\eta}{2} ; S_n \geq -\alpha \right) = 0, \quad (cf \([16]\) pp 615),
\]
we deduce that for any \( \eta, \epsilon > 0 \) there exist \( \delta > 0 \), \( N \in \mathbb{N} \) such that for any \( n \geq N \), 
\[
E^* [ \epsilon_n^2 ] \leq 3\epsilon.
\]
It ends the proof of the relative compactness of \( (\epsilon_n(\sigma_n, \cdot)) \).

\[\square\]

Now we have to prove that the finite dimensional distributions of \( E^*(\mu_n \otimes \mu_n(\cdot)) \) converge to those of \( \mathbb{W}^+ \otimes \mathbb{W}^+ \), i.e: for any \( d \in \mathbb{N}^* \), \( t = (t_1, ..., t_d) \in [0, 1]^d \) and \( F \in C_b(\mathbb{R}^2, \mathbb{R}) \),
\[
\lim_{n \to \infty} E^* \left( \frac{1}{W^2_n} \sum_{|u|=|v|=n} e^{-V(u)-V(v)} F(V_t(u), V_t(v)) \right) = E \left( F(\sigma_{R_t}, \sigma_{R'_t}) \right).
\]

According to (3.1), for any \( F, G \in (C_b(\mathbb{R}^d, \mathbb{R}))^2 \)
\[
\lim_{n \to \infty} \frac{1}{W^2_n} \sum_{|u|=n} e^{-V(u)} F(V_t(u)) = E(F(\sigma_{R_t})) \text{, in } P^* \text{ probability},
\]
\[
\lim_{n \to \infty} \frac{1}{W^2_n} \sum_{|v|=n} e^{-V(v)} G(V_t(v)) = E(G(\sigma_{R_t})) \text{, in } P^* \text{ probability}.
\]
The left hand terms of (3.6) and (3.7) are bounded, then by taking the expectation of the product of (3.6) and (3.7) we get
\[
\lim_{n \to \infty} E^* \left( \frac{1}{W^2_n} \sum_{|u|=|v|=n} e^{-V(u)-V(v)} F(V(u,t))G(V(v,t)) \right) = E \left( F(\sigma_{R_t}) \right) E \left( G(\sigma_{R'_t}) \right).
\]
This equality is sufficient to affirm that the law of \( (V(u,t), (V(v,t)) \) under \( E^*(\mu_n \otimes \mu_n(\cdot)) \) converges to this one \( (R_t, R'_t) \) under \( P \), which implies (3.5).

\[\square\]

To conclude step A) it remains to show that (3.3) implies Theorem 1.2 and Corollary 1.3. For any \( F \in C_b(C, \mathbb{R}) \) let \( F_{2*} \in C_b(C^2, \mathbb{R}) \) be the function defined by:
\[
F_{2*}((w_1(t), w_2(t))_{t \in [0,1]} := \left[ F((w_1(t))_{t \in [0,1]} \right) - E(F((R_s)_{s \in [0,1]})) \right] \times \left[ F((w_2(t))_{t \in [0,1]} \right) - E(F((R_s)_{s \in [0,1]})) \right], \quad w_1, w_2 \in C.
\]
Then according to (3.3), for any \( F \in C_b(C, \mathbb{R}) \) we have
\[
E^* \left( (\mu_n(F) - E[F(R)])^2 \right) = E^*(\mu_n \otimes \mu_n(F_{2*})) \to \mathbb{W}^+ \otimes \mathbb{W}^+(F_{2*}) = 0,
\]
which implies Theorem 1.2.

Concerning Corollary 1.3, observing that \( \{ Q_{u,v} \geq \delta \} \subset \{ V(u,s) = V(v,s), \forall s \leq \delta \} \), we deduce that (3.3) implies
\[
\lim_{n \to \infty} \sup_{t} E^* \left( \frac{1}{W^2_n} \sum_{|u|=n, |v|=n} e^{-V(u)-V(v)} 1_{\{Q_{u,v} \geq \delta \}} \right) \leq P(R = R'_s, \forall s \leq \delta/2) = 0,
\]
which gives Corollary 1.3.

\[\square\]

So we can turn now to the proof of (3.1).
3.2 Step B) : proof of (3.1)

Fix \(d \in \mathbb{N}^*\). Recall that for any \((t_1, \ldots, t_d) \in [0, 1]^d\) we denote

\[
t := (t_1, \ldots, t_d), \quad R_t := (R_{t_1}, \ldots, R_{t_d}), \quad \text{and} \quad V_t(u) := (V_{t_1}(u), \ldots, V_{t_d}(u)).
\]

Let us also introduce for any \(\alpha > 0\), \(F \in C_b(\mathbb{R}^d, \mathbb{R})\), \(t \in [0, 1]^d\), \(n \in \mathbb{N}\) and \(y \in \mathbb{R}^d\),

\[
(3.8) \quad F_t(y) := F(y) - \mathbb{E}(F'(\sigma R_t)), \quad \text{and} \quad W_n(\alpha, F_t) := \sum_{|x|=n} e^{-V(x)}1_{\{|x| \geq -\alpha\}} F_t(V_t(x)).
\]

Following [1], to prove (3.1), we firstly show the following result on \(Q^{(\alpha)}\) : for any \(\alpha \geq 0\), \(t \in \mathbb{R}^d_+\) and \(F \in C_b(\mathbb{R}^d, \mathbb{R})\),

\[
(3.9) \quad \lim_{n \to \infty} \mathbb{E}_Q^{(\alpha)} \left( \frac{1}{(W_n(\alpha))} \left( \sum_{|u|=n} e^{-V(u)}1_{\{|u| \geq -\alpha\}} F_t(V_t(u)) \right)^2 \right) = 0.
\]

Equality (3.9) represents the exact analogue of (3.1) under \(Q^{(\alpha)}\). Because of the relation (2.10) we will see at the end of this section how to obtain (3.1) from (3.9) by letting \(\alpha\) go to infinity. Working under \(Q^{(\alpha)}\) presents the following advantage : under \(Q^{(\alpha)}\) the spine remains above a barrier positioned at \(-\alpha\), then the random variables \((W_n^{(\alpha)}, D_n^{(\alpha)})\) are much more concentrated around their mean than \(W_n\) and \(D_n\) under \(P\).

Let us start the Proof of (3.9). As in [1] we need to rewrite \(W_n^{(\alpha)}, D_n^{(\alpha)}\) and \(W_n^{(\alpha), F_t}\) according to the position of spine \(V(w_n^{(\alpha)})\). For each vertex \(x\) with \(|x| = n\) and \(x \neq w_n^{(\alpha)}\), there is a unique \(i\) with \(0 \leq i < n\) such that \(w_i^{(\alpha)} \leq x\) and that \(w_i^{(\alpha)} \neq x\). For any \(i \geq 1\), let

\[
R_i^{(\alpha)} := \left\{ |x| = i : x > w_{i-1}^{(\alpha)}, x \neq w_i^{(\alpha)} \right\},
\]

(in words, \(R_i^{(\alpha)}\) stands for the set of “brothers” of \(w_i^{(\alpha)}\)). Accordingly,

\[
W_n^{(\alpha), F_t} = e^{-V(w_n^{(\alpha)})} F_t(V_t(w_n^{(\alpha)})) + \sum_{i=0}^{n-1} \sum_{y \in R_i^{(\alpha)}} \sum_{|x|=n, x \geq y} e^{-V(x)}1_{\{|x| \geq -\alpha\}} F_t(V_t(x)).
\]

Let \(k_n < n\) be an integer such that \(k_n \to \infty\) \((n \to \infty)\). We write

\[
W_n^{(\alpha), F_t, [0, k_n]} := \sum_{i=0}^{k_n-1} \sum_{y \in R_{i+1}^{(\alpha)}} \sum_{|x|=n, x \geq y} e^{-V(x)}1_{\{|x| \geq -\alpha\}} F_t(V_t(x)),
\]

\[
W_n^{(\alpha), F_t, [k_n, n]} := e^{-V(w_n^{(\alpha)})} F_t(V_t(w_n^{(\alpha)})) + \sum_{i=k_n}^{n-1} \sum_{y \in R_i^{(\alpha)}} \sum_{|x|=n, x \geq y} e^{-V(x)}1_{\{|x| \geq -\alpha\}} F_t(V_t(x)),
\]

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so that
\[(3.10) \quad W_n^{(a),F} = W_n^{(a),F}_{[0,k_n]} + W_n^{(a),F}_{[k_n,n]}, \quad \text{and similarly we can write,}
\[(3.11) \quad W_n^{(a)} = W_n^{(a),[0,k_n]} + W_n^{(a),[k_n,n]}, \quad D_n^{(a)} = D_n^{(a),[0,k_n]} + D_n^{(a),[k_n,n]}.
\]

Let also
\[(3.12) \quad E_{n,1} := \{ k_n^{1/3} \leq V(w_k^{(a)}) \leq k_n \} \cap \bigcap_{i=k_n}^n \{ V(w_i^{(a)}) \geq k_n \}, \]
\[(3.13) \quad E_{n,2} := \bigcap_{i=k_n}^{n-1} \left\{ \sum_{y \in \mathcal{R}_{i+1}} \left[ 1 + (V(y) - V(w_i^{(a)}))^+ \right] e^{-[V(y)-V(w_n^{(a)})]} \leq e^{V(w_i^{(a)})/2} \right\}, \]
\[(3.14) \quad E_{n,3} := \left\{ D_n^{(a),[k_n,n]} \leq \frac{1}{n^2} \right\}, \quad \text{and} \quad E_n := E_{n,1} \cap E_{n,2} \cap E_{n,3}.
\]

Under $Q^{(a)}$, $(V(w_n^{(a)}))_{n \geq 0}$ has the law of a centered random walk conditioned to stay positive. Moreover it is well known that a such process “tends to infinity” when $n$ goes to infinity. Then keeping this fact in minds we are inclined to affirm that $W_n^{(a),F}_{[k_n,n]}$ have a negligible contribution in (3.10) and (3.11). The following Lemma makes rigorous this affirmation,

**Lemma 3.1 (Aïdékon and Shi [1])** Let $\alpha \geq 0$. Let $k_n$ be such that $\frac{k_n}{\log n} \to \infty$ and that $\frac{k_n}{n^2} \to 0$, $n \to \infty$. Let $E_n$ be as in (3.14). Then

\[(3.15) \quad \lim_{n \to \infty} Q^{(a)}(E_n) = 1, \quad \lim_{n \to \infty} \inf_{u \in [k_n^{1/3},k_n]} Q^{(a)}(E_n | V(w_{k_n}^{(a)}) = u) = 1.
\]

Let $(k_n)_{n \in \mathbb{N}^*} := (\lfloor (\log n)^{30} \rfloor)_{n \in \mathbb{N}^*}$. Fix $\epsilon, \alpha > 0$, $t \in [0,1]^d$ and $F \in C_b(\mathbb{R}^d, \mathbb{R})$. As $||F|| < \infty$, for any event $A$, the expectation in (3.9) (:= $E_{Q^{(a)}}^{(3.9)}$) satisfies

\[(3.16) \quad E_{Q^{(a)}}^{(3.9)} \leq ||F||Q^{(a)}(A) + E_{Q^{(a)}}\left( \frac{1}{(W_n^{(a)})^2} \left( \sum_{|u|=n} e^{-V(u)} I_{\{V(u) \geq -\alpha\}} \bar{F}_t(V_t(u)) \right)^2 I_A \right).
\]

For any $n \in \mathbb{N}$ we set $A_n := \left\{ \frac{D_n^{(a)}}{(W_n^{(a)})^2} - \frac{\sqrt{n}}{\alpha} \right\} < \sqrt{n}$. According to Proposition 4.1 of [1] we have $\lim_{n \to \infty} Q^{(a)}(A_n) = 1$. Moreover on $A_n$, $\frac{1}{(W_n^{(a)})^2} \leq n \frac{(1+1/\beta)^2}{(D_n^{(a)})^2}$, thus the expectation in the right hand term of (3.16) is smaller than:

\[(3.17) \quad cnE_{Q^{(a)}}\left( \frac{W_n^{(a),t}F_t}{D_n^{(a)}} I_{A_n} \sum_{|u|=n} e^{-V(u)} I_{\{V(u) \geq -\alpha\}} \frac{h_\alpha(V(u)) \bar{F}_t(V_t(u))}{h_\alpha(V(u))} \right)
\]

\[= cnE_{Q^{(a)}}\left( I_{A_n} \frac{\bar{F}_t(V_t(w_n^{(a)})) W_n^{(a),t}}{h_\alpha(V(w_n^{(a)})) D_n^{(a)}} \right),
\]

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where we have recognize in (3.17) the expression of \( Q^{(\alpha)}(u_n^{(\alpha)} = u|F_n) \) as described in (2.11). Finally there exists \( n_0 > 0 \) such that for any \( n \geq n_0, \)

\[
E_{Q^{(\alpha)}}^{3.9} \leq \epsilon + cnE_{Q^{(\alpha)}}\left(1_{A_n} \frac{\mathcal{F}_t(V_t(u_n^{(\alpha)}))}{h_{\alpha}(V(u_n^{(\alpha)}))} \frac{W_n^{(\alpha)}F_t}{D_n^{(\alpha)}} \right).
\]

Now by using Lemma 3.1 we shall prove that we can replace \( \frac{W_n^{(\alpha)}F_t}{D_n^{(\alpha)}} \) by \( \frac{W_n^{(\alpha)}F_t}{D_n^{(\alpha)}}_{[0,k_n]} \) in (3.18). We define

\[
I_n := nE_{Q^{(\alpha)}}\left(\frac{W_n^{(\alpha)}F_t}{D_n^{(\alpha)}}_{[0,k_n]} \frac{1}{h_{\alpha}(V(u_n^{(\alpha)}))} \right), \quad \text{and}
\]

\[
L_n := nE_{Q^{(\alpha)}}\left(\frac{W_n^{(\alpha)}F_t}{D_n^{(\alpha)}}_{[0,k_n]} \frac{1}{h_{\alpha}(V(u_n^{(\alpha)}))} \right),
\]

In the following we shall prove that \( |I_n - L_n| \to 0 \). To achieve this goal we also introduce

\[
J_n := nE_{Q^{(\alpha)}}\left(\frac{W_n^{(\alpha)}F_t}{D_n^{(\alpha)}} \frac{1}{E_n} \frac{1}{h_{\alpha}(V(u_n^{(\alpha)}))} \right), \quad \text{and}
\]

\[
K_n := nE_{Q^{(\alpha)}}\left(\frac{W_n^{(\alpha)}F_t}{D_n^{(\alpha)}}_{[0,k_n]} \frac{1}{h_{\alpha}(V(u_n^{(\alpha)}))} \right),
\]

in order to prove that :

\[
|I_n - L_n| \leq |I_n - J_n| + |J_n - K_n| + |K_n - L_n| \to 0.
\]

(i) **Proof of** \( \lim_{n \to \infty} |I_n - J_n| = 0 \) : As \( \mathcal{F}_t \) is a bounded function, there exists \( c > 0 \) such that

\[
\frac{W_n^{(\alpha)}F_t}{D_n^{(\alpha)}} \leq c \frac{W_n^{(\alpha)}F_t}{D_n^{(\alpha)}}_{[0,k_n]},
\]

then

\[
|I_n - J_n| \leq c\|F\|E_{Q^{(\alpha)}}\left(\frac{1}{h_{\alpha}(V(u_n^{(\alpha)}))} \frac{W_n^{(\alpha)}}{D_n^{(\alpha)}} \right)
\]

\[
\leq c'E_{Q^{(\alpha)}}\left(\left(\frac{1}{h_{\alpha}(V(u_n^{(\alpha)}))} \right) \frac{W_n^{(\alpha)}}{D_n^{(\alpha)}} \right)^{\frac{1}{2}},
\]

From Lemma 4.4 \[1\] (pp15) (recall that \( Q^{(\alpha)}(A_n^*) + Q^{(\alpha)}(E_n^c) \) converges to zero) and Lemma 4.3 in \[1\] (pp14) we can affirm that

\[
E_{Q^{(\alpha)}}\left(\left(\frac{1}{h_{\alpha}(V(u_n^{(\alpha)}))} \right) \frac{W_n^{(\alpha)}}{D_n^{(\alpha)}} \right)^{\frac{1}{2}} = O\left(\frac{1}{\sqrt{n}}\right), \quad E_{Q^{(\alpha)}}\left(\left(\frac{W_n^{(\alpha)}}{D_n^{(\alpha)}} \right)^{2} \right)^{\frac{1}{2}} = O\left(\frac{1}{\sqrt{n}}\right).
\]
Combining (3.23) with (3.24) we get (i).

(ii) **Proof of** \( \lim_{n \to \infty} |J_n - K_n| = 0 \): By the triangular inequality observe that \( |J_n - K_n| \) is smaller

\[
nE_{Q_\alpha} \left( \frac{1_{E_n}}{h_\alpha(V(w_n^{(\alpha)}))} \left| \frac{W_n^{(\alpha), F_k,[k,n]} - W_n^{(\alpha), F_k,[0,k_n]} |}{D_n^{(\alpha)}} \right| \right) + nE_{Q_\alpha} \left( \frac{1_{E_n}}{h_\alpha(V(w_n^{(\alpha)}))} \left| \frac{W_n^{(\alpha), F_k,[k,n]} - W_n^{(\alpha), F_k,[0,k_n]} |}{D_n^{(\alpha)}} \right| \right)
\]

Recalling (3.14), on \( E_n \) we have \( |W_n^{(\alpha), F_k,[k,n]} | \leq cD_n^{(\alpha), [k,n]} \leq \frac{c}{n^2} \), then we deduce that

\[
\frac{|W_n^{(\alpha), F_k,[k,n]} |}{D_n^{(\alpha)}} \leq \frac{c}{n^2}, \quad \text{and} \quad \frac{|W_n^{(\alpha), F_k,[k,n]} |}{D_n^{(\alpha), [0,k_n]}} \leq \frac{c^2}{D_n^{(\alpha), [k,n]} n^2}, \quad \text{on} \ E_n
\]

Finally it stems that \( |J_n - K_n| \) is smaller

\[
\frac{1}{n} E_{Q_\alpha} \left( \frac{1_{E_n}}{h_\alpha(V(w_n^{(\alpha)}))} \right) \leq c \frac{1}{n k_n^2}
\]

which concludes the proof of (ii).

(iii) **Proof of** \( \lim_{n \to \infty} |K_n - L_n| = 0 \): Recall (3.20) and (3.22), first observe that

\[
1_{\{V(w_n^{(\alpha)}) \in [k_n, k_n] \}} - 1_{E_n} = 1_{E_n^c} 1_{\{V(w_n^{(\alpha)}) \in [k_n, k_n] \}}
\]

Then let us introduce the \( \sigma \)-field

\[
G_p := \sigma \left( (V(w_k^{(\alpha)}))_{k \leq p}, V(u) \text{ for } u \in \mathbb{T} \text{ such that } \exists z \in \bigcup_{k \in [1,p]} R_k^{(\alpha)} \text{ and } u \geq z \right), \quad p \in \mathbb{N}
\]

Clearly \( W_n^{(\alpha), [0,k_n]} \) and \( D_n^{(\alpha), [0,k_n]} \) are measurable with respect to \( G_{k_n} \), thus

\[
|K_n - L_n| \leq cnE_{Q_\alpha} \left( \frac{W_n^{(\alpha), [0,k_n]} 1_{\{V(w_k^{(\alpha)}) \in [k_n, k_n] \}} 1_{E_n^c}}{h_\alpha(V(w_n^{(\alpha)}))} \right)
\]

\[
= cnE_{Q_\alpha} \left( \frac{W_n^{(\alpha), [0,k_n]} 1_{\{V(w_k^{(\alpha)}) \in [k_n, k_n] \}}}{h_\alpha(V(w_n^{(\alpha)}))} \right) E_{Q_\alpha} \left[ \frac{1_{E_n^c}}{h_\alpha(V(w_n^{(\alpha)}))} |G_{k_n}| \right]
\]

Moreover, by the branching property, conditionally at \( V(w_k^{(\alpha)}) \), \( \frac{1_{E_n^c}}{h_\alpha(V(w_n^{(\alpha)}))} \) is independent of \( G_{k_n} \), so the term in (3.27) is equal to

\[
\leq cnE_{Q_\alpha} \left( \frac{W_n^{(\alpha), [0,k_n]} 1_{\{V(w_k^{(\alpha)}) \in [k_n, k_n] \}}}{h_\alpha(V(w_n^{(\alpha)}))} \right) \sup_{u \in [k_n, k_n]} E_{Q_\alpha} \left[ \frac{1_{E_n^c}}{h_\alpha(V(w_n^{(\alpha)}))} |G_{k_n}| \right].
\]

\[
\leq \frac{c}{n k_n^2}
\]

\[
\text{which concludes the proof of (iii).}
\]
According to [1] (see (4.9) p21) we know that

\[(3.28) \quad \limsup_{n \to \infty} \sqrt{n}E_{Q}^{(\alpha)}\left(\frac{W_{n}^{(\alpha)\cdot [0,k_{n}]}\mathbb{1}_{\{V(w_{k_{n}}^{(\alpha)}) \in [k_{n}^{\frac{1}{m}},k_{n}]\}}} {D_{n}^{(\alpha) \cdot [0,k_{n}]}\mathbb{1}_{\{V(w_{k_{n}}^{(\alpha)}) \in [k_{n}^{\frac{1}{m}},k_{n}]\}}}\right) \leq \theta.\]

Furthermore by the Cauchy-Schwartz inequality, for any $u \in [k_{n}^{\frac{1}{m}},k_{n}]$,

\[E_{Q}^{(\alpha)}\left(\frac{1}{h_{\alpha}(V(w_{k_{n}}^{(\alpha)}))}|V(w_{k_{n}}^{(\alpha)}) = u|\right) \leq E_{Q}^{(\alpha)}\left(\frac{1}{[h_{\alpha}(V(w_{k_{n}}^{(\alpha)}))]^{2}}|V(w_{k_{n}}^{(\alpha)}) = u|\right)^{\frac{1}{2}} Q^{(\alpha)}(E_{n}^{c}|V(w_{k_{n}}^{(\alpha)}) = u)^{\frac{1}{2}}.\]

From (3.15), we have $\sup_{u \in [k_{n}^{\frac{1}{m}},k_{n}]} Q^{(\alpha)}(E_{n}^{c}|V(w_{k_{n}}^{(\alpha)}) = u) \to 0$ when $n$ goes to infinity. Concerning the first term, according to (2.13), for any $u \in [k_{n}^{\frac{1}{m}},k_{n}]$ we have:

\[E_{Q}^{(\alpha)}\left(\frac{1}{h_{\alpha}(V(w_{k_{n}}^{(\alpha)}))^{2}}|V(w_{k_{n}}^{(\alpha)}) = u|\right) = E_{Q}^{(\alpha+u)}\left(\frac{1}{[h_{\alpha+u}(V(w_{k_{n}-k}^{(\alpha+u)})])^{2}}\right)\]

\[= \frac{1}{h_{\alpha+u}(0)}E\left(\frac{\mathbb{1}_{\{S_{n-k} \geq -(\alpha+u)\}}}{h_{\alpha+u}(S_{n-k})}\right) \leq \sum_{k=0}^{\sqrt{n}} c(k+1)^{-1}P_{\alpha+u}\left(S_{n-k} \geq 0, S_{n-k} \in [k,k+1]\right) + \frac{cn^{-\frac{1}{2}}}{h_{\alpha+u}(0)}P_{\alpha+u}\left(S_{n-k} \geq 0\right).

Recalling that $k_{n} = o(n^{\frac{1}{m}})$ and using (2.6) and (2.9), we get that for any $n \in \mathbb{N}$ large enough and any $u \in [k_{n}^{\frac{1}{m}},k_{n}]$,

\[(3.29) \quad E_{Q}^{(\alpha)}\left(\frac{1}{[h_{\alpha}(V(w_{k_{n}}^{(\alpha)}))]^{2}}|V(w_{k_{n}}^{(\alpha)}) = u|\right) \leq \frac{c}{h_{\alpha}(u)}\left(n^{-\frac{3}{2}}\sum_{k=0}^{\sqrt{n}} 1 + n^{-1}\right) \leq c' \frac{1}{n}.

Finally by combining (3.29) and (3.28) we obtain that $\lim_{n \to \infty} |K_{n} - L_{n}| = 0. \square$

It remains to prove that $L_{n} \to 0$ when $n$ goes to infinity. By using the Markov property (assuming $n$ large enough such that $k_{n} \leq \min t, n$), we get that

\begin{align*}
L_{n} &= nE_{Q}^{(\alpha)}\left(\frac{W_{n}^{(\alpha)}(\mathbb{T}_{t}\cdot [0,k_{n}])}{D_{n}^{(\alpha)\cdot [0,k_{n}]}\mathbb{1}_{\{V(w_{k_{n}}^{(\alpha)}) \in [k_{n}^{\frac{1}{m}},k_{n}]\}}}E_{Q}^{(\alpha)}\left[\frac{\mathbb{T}_{t}(V_{t}(w_{k_{n}}^{(\alpha)}))}{h_{\alpha}(V_{t}(w_{k_{n}}^{(\alpha)}))}G_{k_{n}}\right]\right) \\
&= nE_{Q}^{(\alpha)}\left(\frac{W_{n}^{(\alpha)}(\mathbb{T}_{t}\cdot [0,k_{n}])}{D_{n}^{(\alpha)\cdot [0,k_{n}]}\mathbb{1}_{\{V(w_{k_{n}}^{(\alpha)}) \in [k_{n}^{\frac{1}{m}},k_{n}]\}}}E_{Q}^{(\alpha)}\left[\frac{\mathbb{T}_{t}(V_{t}(w_{k_{n}}^{(\alpha)}))}{h_{\alpha}(V_{t}(w_{k_{n}}^{(\alpha)}))}V(w_{k_{n}}^{(\alpha)})\right]\right) \\
&\leq cnE_{Q}^{(\alpha)}\left(\frac{W_{n}^{(\alpha)}(\mathbb{T}_{t}\cdot [0,k_{n}])}{D_{n}^{(\alpha)\cdot [0,k_{n}]}\mathbb{1}_{\{V(w_{k_{n}}^{(\alpha)}) \in [k_{n}^{\frac{1}{m}},k_{n}]\}}}E_{Q}^{(\alpha)}\left[\frac{\mathbb{T}_{t}(V_{t}(w_{k_{n}}^{(\alpha)}))}{h_{\alpha}(V_{t}(w_{k_{n}}^{(\alpha)}))}V(w_{k_{n}}^{(\alpha)})\right]\right) \\
&\leq c'\sqrt{n}\sup_{u \in [k_{n}^{\frac{1}{m}},k_{n}]}E_{Q}^{(\alpha)}\left[\frac{\mathbb{T}_{t}(V_{t}(w_{k_{n}}^{(\alpha)}))}{h_{\alpha}(V_{t}(w_{k_{n}}^{(\alpha)}))}V(w_{k_{n}}^{(\alpha)}) = u\right].
\end{align*}

(3.30)
where we have used (3.28) in the last inequality. Recalling the definition of \( V_t \) in (1.10) using the Markov property at time \( k_n \), then (2.13) we can affirm that for any \( u \in [k_n^2, k_n] \), the expectation in (3.30) is equal to

\[
E_{\mathbf{Q}^{u+\alpha}} \left( \frac{\mathcal{T}_t \left( \frac{1}{\sqrt{n}} [V(w^{(u+\alpha)}_{nt_i-k_n}) + (nt - [nt]) (V(u^{(u+\alpha)}_{nt_i-k_n+1}) - V(w^{(u+\alpha)}_{nt_i-k_n}))]_{i \in \mathcal{I}[d]} \right)}{h_{\alpha+u}(V(w^{(u+\alpha)}_{nt_i-k_n}))} \right)
\]

\[
= \frac{1}{h_{\alpha+u}(0)} E \left( \frac{1}{\sqrt{n}} S_{[nt_i-k_n]} + (nt - [nt]) (S_{[nt_i-k_n+1]} - S_{[nt_i-k_n]}) \right)_{i \in \mathcal{I}[d]},
\]

which we rewrite (according to (2.6)),

\[
(3.31) \quad \frac{1}{h_{\alpha+u}(0)} E \left( F \left( \Delta^{(n)} S_t + S_{t(u+\alpha)}^{(n-k_n)} \right) I_{[nt_i-k_n \geq -u \alpha]} \right) - \frac{\theta}{\sqrt{n}} E(F(\sigma R_t)) + o \left( \frac{1}{\sqrt{n}} \right),
\]

with

\[
(3.32) S_{t(u+\alpha)}^{(n-k_n)} := \frac{(S_{[nt_i-k_n]} + (nt - [nt]) (S_{[nt_i-k_n+1]} - S_{[nt_i-k_n]}))_{i \in \mathcal{I}[d]}}{\sqrt{n - k_n}}, \quad \text{and}
\]

\[
(3.33) \Delta^{(n)} S_t := \frac{1}{\sqrt{n}} [S_{[nt_i-k_n]} + (nt - [nt]) (S_{[nt_i-k_n+1]} - S_{[nt_i-k_n]})]_{i \in \mathcal{I}[d]} - S_{t(u+\alpha)}^{(n-k_n)}.
\]

As a straight-forward consequence of the Lemma 5.1 in the Appendix, we have that: uniformly in \( u \in [k_n^2, k_n], \) as \( n \to \infty, \)

\[
(3.34) \quad \frac{1}{h_{\alpha+u}(0)} E \left( F \left( \Delta^{(n)} S_t + S_{t(u+\alpha)}^{(n-k_n)} \right) I_{[nt_i-k_n \geq -u \alpha]} \right) = \frac{\theta}{\sqrt{n}} E(F(\sigma R_t)) + o \left( \frac{1}{\sqrt{n}} \right).
\]

Then by combining (3.34), (3.30) and assertion (i), (ii), (iii) we obtain (3.9) \( \square \)

We turn now to the

**Proof of (3.7).** Let \( \epsilon > 0 \). From Theorem 1.1 [3], we know that

\[
(3.35) \quad \inf_{|x|=n} V(x) \to \infty, \quad \mathbf{P}^* \text{ a.s.}
\]

Then let \( k = k(\epsilon) > 0 \) such that \( \mathbf{P}^*(\Omega_k) \geq 1 - \epsilon \) with \( \Omega_k := \{ \inf_{|x| \geq 0} V(x) \geq -k \} \).

From (2.6) there exists \( \bar{M} = \bar{M}(\epsilon) > 0 \) such that

\[
(3.36) \quad c_0(1 - \epsilon)u \leq h_0(u) \leq c_0(1 + \epsilon)u, \quad \forall u \geq \bar{M}.
\]

Now we fix \( \alpha = \alpha(\epsilon) := k + \bar{M} \). Since \( h_{\alpha}(u) = h_0(u + \alpha) \), we have for all vertices \( x \)

\[
0 < c_0(1 - \epsilon)(V(x) + \alpha) \leq h_{\alpha}(V(x)) \leq c_0(1 + \epsilon)(V(x) + \alpha), \quad \text{on } \Omega_k.
\]

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We deduce that on \( \Omega_k \), for any \( n \in \mathbb{N}^* \),

\[
W_n^{(\alpha)} = W_n, \quad W_n^{(\alpha), \mathbb{F}_t} = \sum_{|u|=n} e^{-V(u)} F_t(V_t(u)) \quad \text{and} \quad 0 < c_0(1-\epsilon)(D_n + \alpha W_n) \leq D_n^{(\alpha)} \leq c_0(1+\epsilon)(D_n + \alpha W_n).
\]

Furthermore \( P^* \) a.s, \( D_n \to_{n \to \infty} D_\infty > 0 \), then let \( \eta(\epsilon) \), \( N(\epsilon) \) such that for any \( n \geq N \),

\[
P^*(O_n) \geq 1 - \epsilon \quad \text{with} \quad O_n := \{D_n \geq \eta\}.
\]

Gathering all these facts we finally deduce that for any \( n \geq N(\epsilon) \),

\[
E_{P^*}\left(\frac{1}{W_n^2} \left( \sum_{|z|=n} e^{-V(u)} F_t(V_t(z)) \right)^2 \right) \leq cE\left(\frac{1}{W_n^{(\alpha)^2}} \left( \sum_{|z|=n} 1_{\{V(u) \geq -\alpha\}} e^{-V(u)} F_t(V_t(z)) \right)^2 1_{\{\Omega_k \cap O_n\}} \right) + P^*(O_n^c) + P^*(\Omega_k^c)
\]

\[
\leq c' \alpha E_{Q^{(\alpha)}}\left(\frac{1}{D_n^{(\alpha)^2}} \left( \sum_{|z|=n} e^{-V(u)} 1_{\{V(u) \geq -\alpha\}} F_t(V_t(z)) \right)^2 1_{\{O_n\}} \right) + 2\epsilon
\]

\[
\leq \frac{c''}{\eta c_0(1-\epsilon)} E_{Q^{(\alpha)}}\left(\frac{1}{W_n^{(\alpha)^2}} \left( \sum_{|z|=n} e^{-V(u)} 1_{\{V(u) \geq -\alpha\}} F_t(V_t(z)) \right)^2 \right) + 2\epsilon,
\]

which is smaller that \( 3\epsilon \) for \( n \) large enough according to \( (3.9) \). This last inequality ends the proof of \( (3.1) \). \( \square \)

### 3.3 An extension of Theorem 1.2

As we will see in the next section, to prove Theorem 1.1, we will need a slightly extension of Theorem 1.2. Formally, it corresponds to the case where \( F_C(f) = e^{Cf(1)} \), \( f \in \mathcal{C}, C > 0 \):

**Proposition 3.2** Under \( (1.1), (1.2) \) and \( (1.5) \), for any \( C > 0 \), the following equality is true in \( P^* \) probability,

\[
\lim_{n \to \infty} \frac{1}{W_n} \sum_{|z|=n} e^{-V(z)} e^{C V(z)/\sqrt{n}} = E(e^{\sigma C R_1}),
\]

where \( R_1 \) denotes a Brownian meander at time 1.

**Proof of Proposition 3.2** From Theorem 1.2 we can affirm that

\[
\lim_{p \to \infty} \lim_{n \to \infty} \frac{1}{W_n} \sum_{|u|=n} e^{-V(u)} e^{C V(u)/\sqrt{p}} 1_{\{V(u) \leq p\}} = E\left(e^{\sigma C R_1}\right), \quad \text{in } P^* \text{ probability}.
\]
So in order to prove \((3.39)\) it remains to show that: for any \(\epsilon > 0\) as \(n \to \infty\) then \(p \to \infty\),
\[
\mathbf{P}^{\star\text{3.41}}(n, p, \epsilon) := \mathbf{P}^\star\left( \frac{1}{W_n} \sum_{|u|=n} e^{-V(u)} e^{\frac{V(u)}{\sqrt{n}}} \mathbf{1}_{\{V(u) \geq p\}} \geq \epsilon \right) \to 0.
\]

Let \(\epsilon > 0\). We choose \(k = k(\epsilon), \alpha = \alpha(\epsilon)\) as in the proof of \((3.1)\). From \((1.13)\) we recall that
\[
\lim_{A \to \infty} \sup_{n \in \mathbb{N}} \mathbf{P}^\star(\mathcal{U}_n^c, A) = 0 \text{ with } \mathcal{U}_n^c := \left\{ \frac{1}{W_n} \leq A \right\}.
\]
Recalling also the definition of \(\Omega_k\) we deduce that for \(A\) and \(n\) large enough, we have
\[
\mathbf{P}^{\star\text{3.41}}(n, p, \epsilon) \leq c \left( A \frac{1}{W_n} \sum_{|u|=n} e^{-V(u)} e^{\frac{V(u)}{\sqrt{n}}} \mathbf{1}_{\{V(u) \geq -\alpha, \frac{V(u)}{\sqrt{n}} \geq p\}} \geq \Omega_k, \mathcal{U}_n^c \right) + \mathbf{P}^\star(\Omega_k) + \mathbf{P}^\star(\mathcal{U}_n^c)
\]
\[
\leq \frac{cA}{\epsilon} \mathbf{E} \left( \sum_{|u|=n} e^{-V(u)} e^{\frac{V(u)}{\sqrt{n}}} \mathbf{1}_{\{V(u) \geq -\alpha, \frac{V(u)}{\sqrt{n}} \geq p\}} \right) + 2\epsilon
\]
\[
= \frac{cA}{\epsilon} \mathbf{E} \left( e^{\frac{C \sqrt{n}}{\epsilon}} S_n \geq -\alpha, S_n \geq p \sqrt{n} \right) + 2\epsilon,
\]
where in the last inequality we have used the identity \((2.2)\). By Lemma \(5.2\) (Appendix) this is smaller than \(\frac{cA}{\epsilon} e^{-\frac{p^4}{\epsilon^4}} + 2\epsilon\). Finally we conclude that
\[
\lim_{p \to \infty} \lim_{n \to \infty} \mathbf{P}^\star\left( \frac{1}{W_n} \sum_{|u|=n} e^{-V(u)} e^{\frac{V(u)}{\sqrt{n}}} \mathbf{1}_{\{V(u) \geq p\}} \geq \epsilon \right) = 0,
\]
which ends the proof of Proposition \(3.2\). \(\square\)

4 Proof of Theorem \(1.1\)

The proof of Theorem \(1.1\) is a straightforward consequence of the followings two results:

**Assume \((1.1)\), \((1.2)\) and \((1.5)\). We have:**
\[
\lim_{C \to \infty} \lim_{\beta \to 1, \beta < 1} \mathbf{P}^\star \left( \left| \frac{1}{\alpha} W_{\beta/\alpha} - 2D_\infty \right| > \epsilon \right) = 0, \quad \forall \epsilon > 0.
\]
and

**Lemma 4.1 Assume \((1.1)\), \((1.2)\) and \((1.5)\). We have \((\beta = 1 - \alpha)\)**
\[
\lim_{C \to \infty} \limsup_{\alpha \to 0} \mathbf{P}^\star \left( \frac{1}{\alpha} \left| W_{\beta} - W_{\beta/\alpha} \right| \geq \epsilon \right) = 0, \quad \forall \epsilon > 0.
\]

We first prove \((4.1)\).
Proof of (4.1). For any \( \alpha := 1 - \beta > 0 \) (small), \( C > 0 \) (large), let \( n = n(\alpha, C) := \lfloor C/\alpha^2 \rfloor \).

Assume (1.5), for small \( \alpha \), \( \Phi(\beta) = \Phi''(1)/2 + o(\alpha) = \frac{\sigma^2}{2} + o(\alpha^2) \), then

\[
W_{\beta,n} = \sum_{|u|=n} e^{-\sqrt{C}V(u)} e^{-\Phi(\beta)n} = e^{-\frac{C}{2} \sigma^2 + o(\alpha)} \sum_{|u|=n} e^{-V(u)} e^{\sqrt{C}V(u)/n} = e^{-\frac{C}{2} \sigma^2 + o(\alpha)} W_n \sum_{|z|=n} e^{-V(z)} e^{\sqrt{C}V(z)/n}.
\]

For any \( C > 0 \), \( \alpha \to 0 \) implies \( n \to \infty \), thus by Proposition 3.2 it stems that for any \( C > 0 \),

\[
\lim_{\alpha \to 0} \frac{1}{W_n} \sum_{|z|=n} e^{-\sqrt{C}V(z)} e^{\sqrt{C}V(z)/n} \to E(e^{\sigma \sqrt{C}R_1}), \quad \text{in } P^* \text{ probability.}
\]

On the other hand, by Aïdékton and Shi [1]

\[
\sqrt{n}W_n \to \sqrt{\frac{2}{\pi \sigma^2}} D_{\infty}, \quad \text{in } P^* \text{ probability.}
\]

Combining (4.3) and (4.4) we get for any \( C > 0 \),

\[
\lim_{\beta \to 1} W_{\beta, n} \to D_{\infty} e^{-\frac{C}{2} \sigma^2} \sqrt{\frac{2}{\pi \sigma^2}} E\left(e^{\sigma \sqrt{C}R_1}\right), \quad \text{in } P^* \text{ probability.}
\]

Since \( E(e^{\sigma \sqrt{C}R_1}) \sim e^{\sigma^2 \frac{C}{2} \sqrt{C \sigma^2 / 2\pi}} \) as \( C \to \infty \), we get

\[
\lim_{C \to \infty} e^{-\frac{C}{2} \sigma^2} \sqrt{\frac{2}{\pi \sigma^2}} E\left(e^{\sigma \sqrt{C}R_1}\right) = 2.
\]

Finally combining (4.5) and (4.6) we obtain (4.1). \( \square \)

In order to prove the Lemma 4.1, a first step consists to show the following assertion :

(♦) There exists \( c_\bullet > 0 \) and \( \alpha_0 < 1 \) such that :

\[
E\left(W_{\beta, 1+\frac{p}{2}}\right) \leq c_\bullet \quad \text{for any } 0 < \alpha < \alpha_0.
\]

Proof of (♦). We recall that under the condition (1.6) there exists \( \epsilon_0, \delta_- > 0 \) such that

\[
\sup_{\beta \in [1-\delta_-, 1]} E\left(W_{\beta, 1+\epsilon_0}^{1+\epsilon_0}\right). \quad \text{For any } \beta \in [1-\delta_- , 1], \text{ let } \alpha = 1 - \beta, \quad p = 1 + \frac{\alpha}{2}. \quad \text{The proof of (♦) is}
\]

similar to this one of Lemma 3 in [6]. Let us introduce the probability measure \( Q_\beta \) defined by

\[
Q_\beta := W_{\beta} \cdot P.
\]

We refer to [19], for the proof of the existence of this probability and the so called "spine decomposition" of \( Q_\beta \). We shall prove that there exists \( \alpha_0 < 1, \quad c_\bullet > 0 \) such that

\[
\sup_{\alpha \in (0, \alpha_0]} E(W_{\beta}^p) = \sup_{\alpha \in (0, \alpha_0]} E_{Q_\beta}(W_{\beta}^{p-1}) \leq c_\bullet.
\]
Under $Q_{\beta}$, we can decompose $W_\beta$ with respect to the "spine" $(w_n)_{n \in \mathbb{N}} \subset \mathbb{T}$, it leads to

\begin{equation}
W_\beta = \sum_{|u|=1} e^{-\beta V(u)-\Phi(\beta)} W_\beta(u) = e^{-\beta V(w_1)-\Phi(\beta)} \tilde{W}_\beta^{(1)}(1) + \sum_{v \neq w_1, |v|=1} e^{-\beta V(v)-\Phi(\beta)} W_\beta(v),
\end{equation}

with

\begin{equation}
W_\beta(u) := \lim_{k \to \infty} \sum_{|x|=n+k, x > u} e^{-\beta [V(x)-V(u)]}, \quad \tilde{W}_\beta^{(1)} := \lim_{k \to \infty} \sum_{|x|=n+k, x > w_1} e^{-\beta [V(x)-V(w_1)]}.
\end{equation}

By the branching property, the random variables $\tilde{W}_\beta^{(1)}$ and $(W_\beta(u))_{|u|=1, w \neq w_1}$ are independent, moreover $\tilde{W}_\beta^{(1)}$ is distributed as $W_\beta$ under $Q_{\beta}$ whereas for any $u$, $|u| = 1, w_1 \neq u$ $W_\beta(u)$ is distributed as $W_\beta$ under $P$. Introducing $B_i := \sum_{v \neq w_i} e^{-\beta [V(v)-V(w_i)]}-\Phi(\beta) W_\beta(v), i \in \mathbb{N}^*$, and iterating (4.9) $N$ times we get

\begin{equation}
W_\beta = \sum_{k=0}^{N-1} e^{-\beta V(w_k)-\Phi(\beta)k} B_{k+1} + e^{-\beta V(w_N)-\Phi(\beta)N} \tilde{W}_\beta^{(n)}.
\end{equation}

By convexity and observing that $\tilde{W}_\beta^{(n)}$ and $e^{-\beta V(w_N)-\Phi(\beta)N}$ are independent we deduce that

\begin{equation}
E_{Q_{\beta}}(W_{\beta}^{p-1}) \leq E_{Q_{\beta}}\left(\left[\sum_{k=0}^{N} e^{-\beta V(w_k)-\Phi(\beta)k} B_{k+1}\right]^{p-1}\right) + e^{-(p-1)\Phi(\beta)N} E_{Q_{\beta}}(e^{-\beta(p-1)V(w_N)}) E_{Q_{\beta}}(W_{\beta}^{p-1}).
\end{equation}

Furthermore some calculations provide $e^{-(p-1)\Phi(\beta)N} E_{Q_{\beta}}(e^{-\beta(p-1)V(w_N)}) = e^{\Phi(\beta)p-\Phi(\beta)N}$. As in addition $\Phi(\beta)p-\Phi(\beta) = -\frac{3}{8} \sigma^2 \alpha^2 + o_\alpha(\alpha^2)$, by choosing $N = \lfloor \frac{1}{\alpha^2} \rfloor$ (and $\alpha_0$ small enough) we obtain for any $\alpha \leq \alpha_0$,

\begin{equation}
E_{Q_{\beta}}(W_{\beta}^{p-1}) \leq e^{-(p-1)\Phi(\beta)N} E_{Q_{\beta}}(e^{-\beta(p-1)V(w_N)}) \leq \frac{1}{2}.
\end{equation}

Combining (4.12) and (4.13) lead to

\begin{equation}
E_{Q_{\beta}}(W_{\beta}^{p-1}) \leq 2 E_{Q_{\beta}}\left(\left[\sum_{k=0}^{N} e^{-\beta V(w_k)-\Phi(\beta)k} B_{k+1}\right]^{p-1}\right) \leq c N^{p-1} \times E_{Q_{\beta}}\left(\max_{k \leq N} e^{(-\beta V(w_k)-\Phi(\beta)k)(p-1)\max_{k \leq N} B_{k+1}^{p-1}}\right).
\end{equation}

Recalling $p - 1 = \frac{\alpha}{2}$, we have $N^{p-1} \leq c$, moreover using the Cauchy-Schwartz inequality, we get that

\begin{equation}
E_{Q_{\beta}}(W_{\beta}^{p-1}) \leq c E_{Q_{\beta}}\left(\max_{k \leq N} e^{(-\beta V(w_k)-\Phi(\beta)k)\alpha}\right)^{\frac{1}{2}} E_{Q_{\beta}}\left(\max_{k \leq N} B_{k+1}^{\alpha}\right)^{\frac{1}{2}}.
\end{equation}
We shall bound the two terms of the product, let us start by the first. We define the random walk 
\( \eta_k := \alpha(-\beta V(w_k) - \Phi(\beta) k) \). For any \( \alpha < 1 \) let \( t_0(\alpha) > 0 \) such that

\[
1 = \mathbb{E}_{\mathbf{Q}_\beta}(e^{t_0(\alpha)\eta_1}) = \mathbb{E}_{\mathbf{Q}_\beta}(e^{t_0(\alpha)\eta_k - t_0(\alpha)\Phi(\beta)}) = e^{\Phi(\beta(1+t_0(\alpha)) - \Phi(\beta)(1+t_0(\alpha)) - \frac{\alpha^2}{2}(t_0(\alpha)^2 - 1) + o_\alpha(\alpha^2)}.
\]

(4.15)

Then according to (4.15) we can choose \( \alpha_0 \) small enough such that \( \forall \alpha \leq \alpha_0, \ t_0(\alpha) > \frac{3}{2} \).

By definition of \( t_0(\alpha) \) the process \((e^{t_0(\alpha)\eta_k})_{k \in \mathbb{N}}\) is a martingale with mean 1, so by the Doob inequality we deduce that

\[
P\left(\max_{k \leq N} \eta_k > x\right) = P\left(\max_{k \leq N} e^{t_0(\alpha)\eta_k} \geq e^{t_0(\alpha)x}\right) \leq e^{-t_0x}, \quad x \geq 0,
\]

and thus

(4.16) \[ \mathbb{E}\left(\max_{k \leq N} e^{\eta_k}\right) \leq 1 + \int_0^{\infty} e^u P\left(\max_{k \leq N} \eta_k \geq u\right) du < c < \infty. \]

Now we need to bound \( \mathbb{E}_{\mathbf{Q}_\beta}\left(\max_{k \in [1,N]} B_{k+1}^\alpha\right) \). Let \( \kappa > \frac{1}{\epsilon_0} \). Noting that \((B_k)_{k \in \mathbb{N}}\) is a sequence of independent random variables identically distributed, we deduce that:

\[
\mathbb{E}_{\mathbf{Q}_\beta}\left(\max_{k \in [1,N]} B_{k+1}^\alpha\right) = \int_0^\infty \mathbb{Q}_\beta\left(\max_{k \in [1,N]} B_{k+1}^\alpha \geq t\right) dt = 2 + \int_2^\infty \mathbb{Q}_\beta\left(\max_{k \in [1,N]} B_{k+1}^\alpha \geq t\right) dt \leq 2 + N \int_2^\infty \mathbb{Q}_\beta(B_1^{\frac{1}{\alpha}} \geq t^{\frac{1}{\alpha}}) dt.
\]

Then by a trivial change of variable, with \( \alpha_0 \) small enough, it stems that for any \( \alpha \in [0, \alpha_0) \),

(4.17) \[ \mathbb{E}_{\mathbf{Q}_\beta}\left(\max_{k \in [1,N]} B_{k+1}^\alpha\right) \leq 2 + N\alpha \int_2^\infty \mathbb{Q}_\beta(B_1 \geq u) \frac{u^{\kappa\alpha}}{u} du \leq 2 + \frac{2N\kappa\alpha}{2^{1/(\kappa\alpha)}} \int_2^{-1/(\kappa\alpha)} \mathbb{Q}_\beta(B_1^{\frac{1}{\alpha}} \geq u) du \leq 2 + c \mathbb{E}_{\mathbf{Q}_\beta}(B_1^{\frac{1}{\alpha}}).
\]

Furthermore by convexity then the branching property we have

\[
\mathbb{E}_{\mathbf{Q}_\beta}(B_1^{\frac{1}{\alpha}}) = \mathbb{E}_{\mathbf{Q}_\beta}\left(\left(\sum_{v \neq u_1} e^{-\beta V(u) - \Phi(\beta) W_\beta^{(u)}}\right)^{\frac{1}{\alpha}}\right) \leq \mathbb{E}_{\mathbf{Q}_\beta}\left(\left(\sum_{v \neq u_1} e^{-\beta V(u) - \Phi(\beta) W_\beta^{(u)}}\right)^{\frac{1}{2}}\right) \leq \mathbb{E}_{\mathbf{Q}_\beta}\left(\left(\sum_{v \neq u_1} e^{-\beta V(u) - \Phi(\beta) W_\beta^{(u)}}\right)\right)^{\frac{1}{2}}
\]

As \( \mathbb{E}(W_\beta^{(u)}) = \mathbb{E}(W_\beta) = 1 \) and \( \kappa > \frac{1}{\epsilon_0} \), we deduce that

(4.18) \[ \mathbb{E}_{\mathbf{Q}_\beta}(B_1^{\frac{1}{\alpha}}) \leq \mathbb{E}_{\mathbf{Q}_\beta}(W_{1,\beta}^{\frac{1}{\alpha}}) = \mathbb{E}\left(W_{1,\beta}^{1+\frac{1}{\alpha}}\right) \leq \sup_{\beta \in [1-\delta,1]} \mathbb{E}(W_{b,1}^{1+\epsilon_0}).
\]
Combining (4.17) and (4.18) we conclude that there exists $c > 0$ such that for any $\alpha$ small enough,

$$
(4.19) \quad E_{Q_\beta} \left( \max_{k \in [1,N]} B_{k+1}^\alpha \right) \leq c.
$$

Finally assertion (♦) follows from (4.14), (4.16) and (4.19).

Now we can turn to the Proof of Lemma 4.1. In the following $n := \lfloor C_\alpha^2 \rfloor$ and $p := 1 + \frac{\alpha}{2}$. In order to avoid cumbersome notation, we will assume that $C_\alpha^2 \in \mathbb{N}$. The modifications needed to handle general case are minimal and straightforward, and therefore left to the reader. For any $u \in T$ such that $|u| = n$, let $W_{\beta}^{(u)} := \lim_{k \to \infty} \sum_{|x| = n+k, x > u} e^{-[V(x) - V(u)]}$ law $W_{\beta}$. Note that

$$
(4.20) \quad \frac{1}{\alpha} |W_{\beta} - W_{\beta,n}| = \frac{1}{\alpha} \sum_{|u| = n} e^{-\beta V(u) - \Phi(\beta)n} |(W_{\beta}^{(u)} - 1)| := \xi_n.
$$

Fix $\epsilon > 0$ and set $\tilde{\xi}_n := E(\xi^p | F_n)$. By the Markov inequality, we have

$$
(4.21) \quad P^*(\xi_n \geq \epsilon) \leq cP \left( \tilde{\xi}_n \geq \epsilon^{1+p} \right) + cE \left( \sum_{|u| = n} e^{-\Phi(\beta)n} |(W_{\beta}^{(u)} - 1)| \right).
$$

We shall prove that $\lim_{C \to \infty} \limsup_{\beta \to 1} P \left( \xi_n \geq \epsilon^{1+p} \right) = 0$. As $(W_{\beta}^{(u)} - 1)_{|u| = n}$ form a sequence of independent random variables with 0 mean, so by Petrov [22] ex 2.6.20 then (♦) we have

$$
(4.22) \quad E(\xi^p | F_n) \leq \frac{2}{\alpha^p} \sum_{|u| = n} e^{-p\beta V(u) - p\Phi(\beta)n} E_p \left( |W_{\beta} - 1|^p \right).
$$

Now let us recall the three following facts:

- For $\beta \to 1$, $\Phi(\beta) = \frac{\sigma^2}{2} \alpha^2 + o_\alpha(\alpha^2)$ and $p\beta = 1 - \frac{\alpha}{2} + o_\alpha(\alpha)$, thus

$$
\Phi(p\beta) - p\Phi(\beta) = \frac{\sigma^2}{2} \left[ (1-p\beta)^2 - p\alpha^2 \right] + o_\alpha(\alpha^2) = \frac{\sigma^2}{2} \left[ \alpha^2 / 4 - \alpha^2 \right] + o_\alpha(\alpha^2)
$$

$$
(4.23) \quad = -\frac{3}{8} \alpha^2 \sigma^2 + o_\alpha(\alpha^2).
$$

- For $\beta \to 1$, $\alpha^p = \alpha^{\alpha^{2/3}} \sim \alpha$.
- Let \( t(\alpha) := \frac{1-p\beta}{\alpha} (\to \frac{1}{2}) \). Observing that \( \frac{1}{\alpha} W_{p\beta,n} = \frac{t(\alpha)}{1-p\beta} W_{p\beta,\frac{1}{(1-p\beta)^2}} \), by combining (4.5) and (4.6) we can affirm that

\[
\lim_{C \to \infty} \lim_{n \to \infty} \frac{1}{\alpha} W_{p\beta,n} = D_\infty, \quad \text{in } \mathbb{P}^* \text{ probability.}
\]

Thus by combining (4.22) and (♦) to this three assertions we get that in \( \mathbb{P} \) probability,

\[
\lim_{C \to \infty} \lim_{\beta \to 1} E_{\mathbb{P}}(\xi_n^p|\mathcal{F}_n) \leq c D_\infty \lim_{C \to \infty} e^{-\frac{1}{2} \sigma^2 C} = 0, \quad \text{in } \mathbb{P} \text{ probability.}
\]

Therefore for any \( \epsilon > 0 \),

\[
\lim_{C \to \infty} \lim_{\beta \to 1} \mathbb{P}(\tilde{\xi}_n \geq \epsilon) = 0.
\]

With (4.26) and (4.21) we obtain Lemma 4.1. \( \Box \).

5 Appendix

Recall that \((S_n)_{n \geq 0}\) is a centred random walk with \( \mathbb{E}(S_1^2) := \sigma^2 < \infty \). For any \( t \in [0,1] \), let \( S_{tn} := S_{\lfloor tn \rfloor} \). Recall also the definition of \( \Delta S_t^{(n)} \) in (3.33). Let \((R_s)_{s \in [0,1]}\) a Brownian meander. The following Lemma and its proof are very similar to Lemma 2.2 in [18].

Lemma 5.1 Let \((k_n)_{n \in \mathbb{N}} := (\lfloor \log n \rfloor)_{n \in \mathbb{N}}\). For any \( t = (t_1, ..., t_d) \in (0,1]^d \) and any bounded continuous function \( F : [0, \infty)^d \to \mathbb{R} \), we have, as \( n \to \infty \),

\[
\mathbb{P} \left( \sqrt{n} \Delta S_t^{(n)} \notin B(0, k_n), S_n \geq -x \right) = o \left( \frac{h_0(x)}{\sqrt{n}} \right),
\]

and

\[
\mathbb{E}^x \left( F \left( \frac{y_1 + S_{t_1n}}{(n\sigma^2)^{\frac{1}{2}}} \ldots \frac{y_d + S_{t_dn}}{(n\sigma^2)^{\frac{1}{2}}} \right) 1_{\{S_n \geq 0\}} \right) = \frac{\theta h_0(x)}{n^{\frac{1}{2}}} \left( \mathbb{E}(F(R_t)) + o(1) \right),
\]

uniformly in \((x, y) \in [0, k_n] \times B(0, k_n)\).

Proof of (5.1). To prove (5.1) we can suppose without lost of generality that \( d = 1 \). Let \( t \in (0,1] \), recalling (3.33) observe there exists \( c > 0 \) such that for any \( n \in \mathbb{N} \) large enough we have

\[
|\Delta^{(n)} S_t| \leq c \frac{1}{\sqrt{n}} |S_{(n-k_n)t} - S_{n-t-k_n}| + \frac{k_n}{n^{\frac{1}{2}}} |S_{(n-k_n)t}|.
\]
It is clear that
\[(5.3) \quad P\left( |S_{(n-k_n)t}| \geq n, S_n \geq -x \right) \leq P\left( |S_{(n-k_n)t}| \geq n \right) = o\left( \frac{1}{\sqrt{n}} \right).
\]
Thus we only need to prove that uniformly in \( x \in [0, k_n] \),
\[(5.4) \quad P\left( |S_{(n-k_n)t} - S_{nt-k_n}| \geq k_n, S_n \geq -x \right) = o\left( \frac{h_0(x)}{\sqrt{n}} \right).
\]
According to the Markov property at time \( nt - k_n \), we have
\[
P\left( |S_{(n-k_n)t} - S_{nt-k_n}| \geq k_n, S_n \geq -x \right) \leq P\left( S_n \geq -x \right) P\left( \max\left( |S_{k_n(1-t)}|, |S_{k_n(1-t_j)}| \right) \geq k_n \right)
\]
\[
\leq \frac{h_0(x)}{\sqrt{n}} (k_n)^{-2} E\left( S_{k_n(1-t)}^2 \right) = o\left( \frac{h_0(x)}{\sqrt{n}} \right),
\]
which gives \( (5.4) \).

**Proof of (5.2).** Let \( t \in (0, 1]^d \) and \( F : [0, \infty)^d \to \mathbb{R} \) a continuous function bounded by \( M > 0 \). For any \((x, y) \in [0, b_n] \times B(0, b_n) \) we denote \( E_{x, y}^z \) the expectation in \( (5.1) \). According to [2] (p11) we have: for any \( i \in [1, d], \epsilon > 0 \), there exists \( A(\epsilon) \) large enough such that
\[
(5.5) \quad \sup_{x \in [0, b_n]} E_x^z \left( \frac{S_{t_n}}{\sigma \sqrt{n}} > A(\epsilon) \right) \leq \epsilon, \quad \sup_{x \in [0, b_n]} E_x^z \left( \frac{S_{t_n}}{\sigma \sqrt{n}} > A(\epsilon), S_n \geq 0 \right) \leq h_0(x) \frac{\epsilon}{\sqrt{n}}.
\]

Thus we can suppose that \( F \) is a continuous function with compact support. By approximation, we can also assume that \( F \) is Lipschitz. Let \((m_n)_{n \geq 0}\) be a sequence of integers such that \( \frac{n}{m_n} \) and \( \frac{m_n}{k_n} \) go to infinity. Decomposing \( E_{x, y}^z \) according to the time \( j \) such that \( S_j = S_{(n-k_n)t} \) gives:
\[
\left| E_{x, y}^z \left( S_j = S_{ij} \right) - \sum_{j=0}^{m_n} E_x \left( a_n(S_j + y_i, i \in [0, d], k), S_j = S_{ij} \right) \right| \leq ME \left( \sum_{j=0}^{m_n} 1\{S_j = S_{ij} \geq -x, \min_{i \in [0, d]} (S_i - S_j) \geq 0 \} \right)
\]
\[
\leq c(n - j + 1)^{-1/2} \sum_{j=0}^{m_n} P \left( S_j = S_j \geq -x \right),
\]
with
\[
(5.7) \quad a_n(z, j) := E \left( F \left( \frac{z_1 + S_{ij}}{(n\sigma^2)^{1/2}}, \ldots, \frac{z_d + S_{ij}}{(n\sigma^2)^{1/2}} \right), S_{ij} \geq 0 \right), \quad z \in \mathbb{R}^d.
\]

The amount in \( (5.6) \) is negligible, indeed according to \( (2.9) \) we have
\[
c \sum_{j=m_n+1}^{n} (n - j + 1)^{-1/2} P \left( S_j = S_j \geq 0 \right) \leq \sum_{j=m_n+1}^{n} (n - j + 1)^{-1/2} \times c \left( \frac{(x + 1)^2}{j^2} \right)
\]
\[
\leq c \frac{h_0(x)}{\sqrt{n}} \frac{k_n}{\sqrt{m_n}},
\]
\[
(5.8)
\]
Similarly, recalling that \( c_1(1 + x) \leq h_0(x) \leq C_1(1 + x) \) and \( x \leq k_n \), observe that

\[
\sum_{j=m_n+1}^{n} P^x (S_j = S_j \geq 0) \leq c(1 + x)^2 \sum_{j=m_n}^{n} (j + 1)^{-\frac{k}{2}}
\]

(5.9)

\[
\leq ch_0(x) \frac{k_n}{\sqrt{m_n}}.
\]

Going back to (5.8) let us study \( \mathbb{E}\left( F\left( \frac{z_1 + S_{t_n-j}}{\sigma \sqrt{n-j}}, \ldots, \frac{z_d + S_{t_n-j}}{\sigma \sqrt{n-j}} \right), S_{n-j} \geq 0 \right), j \leq m_n. \)

Recalling (5.7), and the definition of \( A(\epsilon) \) in (5.5), as \( F \) is Lipshitz, we have for any \( z \in B(0, k_n) \),

\[
|a_n(z, j) - \mathbb{E}\left( F\left( \frac{S_{t_1(n-j)}}{\sigma \sqrt{n-j}}, \ldots, \frac{S_{t_d(n-j)}}{\sigma \sqrt{n-j}} \right), S_{n-j} \geq 0 \right) | \leq (1) + (2),
\]

with

\[
(1) := \frac{c8A(\epsilon)\sqrt{m_n}}{\sqrt{n}} P\left( S_{n-j} \geq 0 \right),
\]

\[
(2) := \sum_{i=1}^{d} P\left( |z| + |S_{t_i(n-j)} - S_{t_i(n-j)}| + \frac{j}{n} S_{t_i(n-j)} \right) \geq 8A(\epsilon)\sqrt{m_n}, S_{n-j} \geq 0 \right).
\]

According to (2.6), for \( n \) large enough \( (m_n = o(n)) \) we have \( (1) \leq c'\frac{A(\epsilon)\sqrt{m_n}}{\sqrt{n}} \). Term (2) is quite similar to the expectation in (5.1). By using the Markov property and (5.5), we deduce that for any \( z \in B(0, k_n), j \leq m_n \),

\[
(2) \leq \frac{c}{\sqrt{n}} \sum_{i=1}^{d} \left[ P\left( S_{t_i(n-j)} \geq 0 \right) P\left( \max(|S_{j(1-t_i)}|, |S_{j(1-t_i)}|) \geq A(\epsilon)\sqrt{m_n} \right) + P\left( S_{t_i(n-j)} \geq A(\epsilon)\sqrt{n}, S_{n-j} \geq 0 \right) \right] \leq \frac{c\epsilon}{\sqrt{n}}.
\]

Finally we deduce that for \( n \) large enough we have

\[
(5.11) \sup_{z \in B(0, m_n), j \leq m_n} \left| a_n(z, j) - \mathbb{E}\left( F\left( \frac{S_{t_1(n-j)}}{\sigma \sqrt{n-j}}, \ldots, \frac{S_{t_d(n-j)}}{\sigma \sqrt{n-j}} \right), S_{n-j} \geq 0 \right) \right| \leq \frac{c\epsilon}{\sqrt{n}}.
\]

Furthermore we know that \( \left( \frac{S_{t_i(n-j)}}{\sigma \sqrt{n-j}} \right)_{i \in [0,1]} \) conditionally to \( S_n \geq 0 \) converges under \( P \) to the Brownian meander \([11]\). It implies that there exists \( (\eta_j)_{j \geq 0} \) tending to zero such that

\[
(5.12) \left| \mathbb{E}\left( F\left( \frac{S_{t_1(n-j)}}{\sigma \sqrt{n-j}}, \ldots, \frac{S_{t_d(n-j)}}{\sigma \sqrt{n-j}} \right), S_{n-j} \geq 0 \right) - P(\mathbb{S}_{n-j} \geq 0)\mathbb{E}(F(R_{t})) \right| \leq \eta_{n-j} P(\mathbb{S}_{n-j} \geq 0).
\]
Let $\epsilon > 0$. For $n$ large enough and $k \leq m_n$, we have from (2.9), $|P(S_{n-j} \geq 0)) - \frac{e}{\sqrt{n}}| \leq \frac{c}{\sqrt{n}}$. Combined with (5.11) and (5.12), for $n$ large enough and any $z \in B(0,k_n)$, $j \leq m_n$, this gives

$$|a_n(z,j) - P(S_{n-j} \geq 0)E(F(R_t))| \leq c\frac{\epsilon}{\sqrt{n}}$$

for $n$ greater than some $n_1$, $j \leq m_n$, $z \in B(0,k_n)$. We use this inequality for every $k = 0,\ldots,m_n$ and we obtain

$$\sum_{j=0}^{m_n} \left| E^x(a_n((S_j + y_i)_{i \in [0,d]}, j) - \frac{\theta}{\sqrt{n}}E(F(R_t)), S_j = S_j \geq 0) \leq c\frac{\epsilon}{\sqrt{n}} \sum_{j=0}^{m_n} P^x(S_j = S_j \geq 0) \leq c\frac{\epsilon}{\sqrt{n}} h_0(x).$$

Together with (5.8) and (5.9), it yields that there exists $n_0 \geq 0$ such that for any $n \geq n_0$, $x \in [0,k_n]$, $y \in B(0,k_n)$ we have

$$\left| E^x(F(\frac{y_1 + S_{t_{1n}}}{(n\sigma^2)\frac{1}{2}}, \ldots, \frac{y_d + S_{t_{dn}}}{(n\sigma^2)\frac{1}{2}}) 1_{\{S_n \geq 0\}} - \frac{\theta h_0(x)}{n^2} E(f(R_a)) \right| \leq c\frac{h_0(x)}{\sqrt{n}} \frac{k_n}{\sqrt{m_n}} + c\frac{\epsilon}{\sqrt{n}} h_0(x) \leq c\frac{\epsilon h_0(x)}{\sqrt{n}},$$

which yields (5.2). \hfill $\square$

The following lemma is a consequence of (12) (pp 8).

Lemma 5.2 Assume (1.1), (1.2) and (1.3). Let $(S_n)_{n \geq 0}$ be the centered random walk defined in (2.1). For any $C, \alpha > 0$, there exist $c(\alpha, C), n_0 > 0$ such that for any $p, n \geq n_0$,

$$(5.13) \quad E\left(e^{C S_n}; S_n \geq -\alpha, S_n \geq p\sqrt{n}\right) \leq \frac{c}{\sqrt{n}} e^{-p/2}.$$

Proof of Lemma 5.2 Fix $C, \alpha > 0$. According to (1.3) we have $\Phi(1 - \theta) = E(e^{\theta S_1}) = 1 + \sigma^2 \frac{\theta^2}{2} + o(\theta^2)$. We deduce that there exists $n_0 = n_0(C)$ such that for any $n, p \geq n_0$

$$(5.14) \quad E\left(e^{C S_n}; S_n \geq -\alpha, S_n \geq p\sqrt{n}\right) \leq \frac{c}{\sqrt{n}}.$$

Decomposing the expectation in (5.13) := $E(5.13)$ according to the time $k$ such that $S_k = S_n$ yields

$$E(5.13) = \sum_{k=0}^{n} E\left(1_{\{S_k = S_n \geq -\alpha\}} E\left(e^{C(x S_k - k) \sqrt{n}}; S_n \geq 0, S_{n-k} \geq p\sqrt{n-x}\right)\right)_{x=S_k} \leq \sum_{k=0}^{n} P(S_k = S_n \geq -\alpha) E\left(e^{C S_{n-k}}; S_n \geq 0, S_{n-k} \geq p\sqrt{n}\right) \leq (5.15)_{1} + (5.15)_{2}.$$
Following Caravenna [12] (pp 5), we define (5.15) for any $n \in \mathbb{N}$,

$$\sum_{k=0}^{n-1} P(S_k = S_k \geq -\alpha) P(S_{n-k} \geq p\sqrt{n}) \leq \sum_{k=0}^{n} \frac{c(1 + \alpha^2)}{n^2} \exp(-p) \leq c' \frac{(1 + \alpha)}{\sqrt{n}} \exp(-p),$$

(5.16)

(where we have used (5.14), the time reversal for $(S_j)_{j \leq k}$ and (2.9)) and

$$\sum_{k=0}^{n-1} P(S_k = S_k \geq -\alpha) E \left( e^{\sqrt{n-k}} ; S_{n-k} \geq 0, S_{n-k} \geq p\sqrt{n-k} \right).$$

(5.15)2

Now let us study for any $n \in \mathbb{N}$,

$$E \left( e^{\sqrt{n}} ; S_n \geq 0, S_n \geq p\sqrt{n} \right).$$

Following Caravenna [12] (pp 5), we define $(T_k, H_k)$ the strict ascending ladder variables process associated to the random walk $(S_n)_{n \in \mathbb{N}}$. Then according to (3.1) in [12] we have:

$$P \left( \frac{S_n}{\sqrt{n}} \in dx ; S_n \geq 0 \right) = \frac{1}{n} \sum_{m=0}^{n-1} \int_{0,\sqrt{m}} P \left( T_k = m, H_k \in dz \right) P \left( S_{n-m} \in \sqrt{m} dx - z \right)$$

$$= \frac{1}{n} \int_{0,1} \int_{0,1} \nu_n(\alpha, \beta) P \left( \frac{S_{n(1-\alpha)}}{\sqrt{n}} \in dx - \beta \right),$$

where $\nu_n$ is the finite measure on $[0,1] \times [0,\infty)$ defined by $\nu_n(A) := \frac{1}{\sqrt{n}} \sum_{k=0}^{\infty} P \left( (\frac{T_k}{n}, \frac{H_k}{\sqrt{n}}) \in A \right).$

Applied to our case it gives,

$$E \left( e^{\sqrt{n}} I_{\{S_n \geq p\sqrt{n} \}} ; S_n \geq 0 \right) = \frac{\sqrt{n}}{n} \int_{[p,\infty)} \int_{[0,1] \times [0,\infty)} \nu_n(\alpha, \beta) e^{c \sqrt{n} \alpha} P \left( \frac{S_{n(1-\alpha)}}{\sqrt{n}} \in dx - \beta \right)$$

$$= n^{-\frac{1}{2}} \int_{[0,1] \times [0,\infty)} \int_{[p,\infty)} e^{c \sqrt{n} \alpha} P \left( S_{n(1-\alpha)} \frac{S_{n(1-\alpha)}}{\sqrt{n}} \geq \max(0,p) \beta \right) I_{\{\beta \leq x\}} d\nu_n(\alpha, \beta)$$

$$= n^{-\frac{1}{2}} \int_{[0,1] \times [0,\infty)} E \left( e^{c \sqrt{n} \alpha} I_{\{S_n \geq p\sqrt{n} \}} \frac{S_{n(1-\alpha)}}{\sqrt{n}} \geq \max(0,p) \beta \right) d\nu_n(\alpha, \beta).$$

Furthermore as for any $p, n \geq n_0$,

$$E \left( e^{c \sqrt{n} \alpha} I_{\{S_n \geq p\sqrt{n} \}} \frac{S_{n(1-\alpha)}}{\sqrt{n}} \geq \max(0,p) \beta \right) \leq$$

$$= \begin{cases} cE \left( e^{c \sqrt{n} \alpha} \right) \leq c', & \text{if } (\alpha, \beta) \in [0,1] \times [\frac{p}{2}, \infty), \\ E \left( e^{c \sqrt{n} \alpha} I_{\{\frac{S_{n(1-\alpha)}}{\sqrt{n}} \geq \frac{p}{2} \}} \right) \leq c'' e^{-p/2}, & \text{if } (\alpha, \beta) \in [0,1] \times [0,p/2), \end{cases}$$

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we deduce that
\[
\mathbb{E} \left( e^{\frac{c}{\sqrt{n}} S_n \mathbb{1}_{(S_n \geq p \sqrt{n})}; S_n \geq 0} \right) \leq c n^{-\frac{1}{2}} \left( e^{-p/2} \nu_n([0, 1] \times [0, \infty)) + \nu_n ([0, 1] \times [p/2, \infty)) \right).
\]

Let \( \tau(x) := \inf\{k \geq 0, S_k \geq x\} \). Via some usual computations, we get that
\[
\nu_n ([0, 1] \times [p/2, \infty)) = \frac{1}{\sqrt{n}} \sum_{k=0}^{\infty} \mathbb{P} \left( T_k \leq n, H_k \geq \frac{p}{2} \sqrt{n} \right)
\]
\[
= \frac{1}{\sqrt{n}} \mathbb{E} \left( \mathbb{1}_{\{\tau(p/2 \sqrt{n}) \leq n\}} \sum_{k=0}^{\infty} \mathbb{1}_{\{T_k \leq n, H_k \geq \frac{p}{2} \sqrt{n}\}} \right)
= \frac{1}{\sqrt{n}} \mathbb{E} \left( \mathbb{1}_{\{\tau(p/2 \sqrt{n}) \leq n\}} \mathbb{E} \left( \sum_{k=0}^{\infty} \mathbb{1}_{\{T_k \leq n - \tau(p/2 \sqrt{n})\}} \right) \right)
\leq \mathbb{P} \left( \tau(p/2 \sqrt{n}) \leq n \right) \frac{1}{\sqrt{n}} \mathbb{E} \left( \sum_{k=0}^{\infty} \mathbb{1}_{\{T_k \leq n\}} \right)
\leq ce^{-\frac{p}{4} \nu_n([0, 1] \times [0, \infty))}.
\]

Finally as there exists \( c_1 > 0 \) such that \( \mathbb{P}(T_1 \geq n) \geq \frac{c_1}{\sqrt{n}} \), \( \forall n \geq 0 \), we deduce that \( \nu_n ([0, 1] \times [0, \infty)) \leq c \) for any \( n \geq 0 \) and thus
\[
(5.18) \quad (5.15) \leq \sum_{k=0}^{\frac{n}{2}} \mathbb{P} \left( S_k = S_k \geq -\alpha \right) \frac{ce^{-\frac{p}{4}}}{\sqrt{n-k}} \leq \frac{c(\alpha)}{\sqrt{n}} e^{-\frac{p}{2}}.
\]

Finally going back to (5.15), combining (5.16) and (5.18) we obtain
\[
(5.19) \quad \mathbb{E}[5.13] \leq \frac{c}{\sqrt{n}} e^{-\frac{p}{4}},
\]
and Lemma 5.2 follows. \( \square \)

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