EXISTENCE AND CHAOS OF POINTS THAT ARE RECURRENT BUT NOT BANACH RECURRENT

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Abstract. Recurrence is a classical and basic concept in the study of dynamical systems. According to the ‘recurrent frequency’ (i.e., the probability of finding the orbit of an initial point entering in a neighborhood), many different levels of recurrent points are found such as periodic points, almost periodic points, weakly almost periodic points, quasi-weakly almost periodic points and Banach recurrent points [19, 38]. In this paper we are mainly to search the existence on six new levels of recurrent points between Banach recurrence and general recurrence. Moreover, from the perspective of distributional chaos, we will show that five levels of the new recurrent points contain uncountable DC1-scrambled subsets.

1. Introduction

In classical study of dynamical systems, one fundamental problem in the study of dynamical systems is to consider the asymptotic behavior of the orbits. One of such important concepts is recurrence. Recurrent points such as periodic points, minimal points are typical objects to be studied. It is known that the set of recurrent points has full measure for any invariant measure under

\[ \omega(x) := \{ n \geq 0 | f^n(x) \in U \} \]

where \( U \) is non-wandering if

\[ N(U, V) := \{ n \geq 1 | U \cap f^{-n}(V) \neq \emptyset \} \]

and \( x \) is recurrent if

\[ N(B_\varepsilon(x), B_\varepsilon(x)) \neq \emptyset \text{ for any } \varepsilon > 0 \]

and

\[ N(x, B_\varepsilon(x)) \neq \emptyset \text{ for any } \varepsilon > 0 \]

where \( B_\varepsilon(x) \) denotes the open ball with radius \( \varepsilon \) centered at \( x \). It is known from the probabilistic perspective these two interesting sets with same asymptotic behavior have totally full measure (i.e., carrying full measure for any invariant measure). There is also a set with totally full measure [25] Theorem 2.5 on Page 120 such that for every point \( x \) in this set (called regular),

\[ \frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i(x)} \text{ converge to an ergodic measure } \mu_x \text{ and } \mu_x(U) > 0 \text{ for every open set } U \text{ containing } x. \]

2010 Mathematics Subject Classification. 37B10, 37B20.

Key words and phrases. Recurrence; Anosov system; Symbolic System.

Y. Jiang is supported by Fudan project FDUROP (Fudan’s Undergraduate Research Opportunities Program), and National University Student Innovation Program. X. Tian is the corresponding author and supported by National Natural Science Foundation of China (grant no. 11671093).
In particular, such $x$ is recurrent and moreover the probability of the orbit of $x$ entering any neighborhood of $x$ is positive (called weakly almost periodic in $[33]$), that is,

\begin{equation}
\liminf_{n \to +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi_{B_i(x)}(f^i(x)) > 0 \text{ (since large than } \mu_x(B(x)).
\end{equation}

This implies

\begin{equation}
\limsup_{n \to +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi_{B_i(x)}(f^i(x)) > 0 \text{ (called quasi-weakly almost periodic in $[33]$) and}
\end{equation}

\begin{equation}
\limsup_{|I| \to +\infty} \frac{1}{|I|} \sum_{i \in I} \chi_{B_i(x)}(f^i(x)) > 0 \text{ (called Banach recurrent in $[21]$),}
\end{equation}

here $I \subseteq \mathbb{N}$ is taken from finite continuous integer intervals and $|Y|$ denotes the cardinality of the set $Y$. It is obvious the order of above six asymptotic behavior from strongest to weakest is $(1.3)$, $(1.4)$, $(1.5)$, $(1.6)$, $(1.2)$, $(1.1)$ so that the sets of points with above six asymptotic behavior all have totally full measure. The sets of points with above six asymptotic behavior $(1.3)$, $(1.4)$, $(1.5)$, $(1.6)$, $(1.2)$, $(1.1)$ are denoted by $R(f), W(f), QW(f), BR(f), Rec(f), \Omega(f)$ respectively. From the topological perspective it is known that any set with totally full measure characterizes the dynamical complexity of the whole space, e.g. carrying full topological entropy (i.e., having entropy as same as the whole space) by $[8]$ so that the above six sets all have full topological entropy. A natural question is how about the complexity or chaotic degree in the gap-sets of $(Rec \setminus BR)$. Let $\chi_{B_i(x)}$ be the characteristic function of $B_i(x)$, $\chi_{B_i(x)}$ is still an open problem, since the technique described in $[33, 11]$ to characterize refined recurrence of recurrent points. Motivated by these ideas using density to describe the probability of one orbit entering one set, several new concepts were introduced in $[10]$ to describe the place where one general orbit can go and then give many refined characterization of general orbits. Let $S \subseteq \mathbb{N}$, define

\[ \bar{d}(S) := \limsup_{n \to +\infty} \frac{|S \cap \{0, 1, \ldots, n-1\}|}{n}, \quad d(S) := \liminf_{n \to +\infty} \frac{|S \cap \{0, 1, \ldots, n-1\}|}{n}, \]

and define $B^*(S) := \limsup_{|I| \to +\infty} \frac{|S \cap I|}{|I|}, \quad B_*(S) := \liminf_{|I| \to +\infty} \frac{|S \cap I|}{|I|}.$

These four concepts are called upper density and lower density of $S$, Banach upper density and Banach lower density of $S$, respectively, which are basic and have played important roles in the field of dynamical systems, ergodic theory and number theory, etc. A set $S \subseteq \mathbb{N}$ is called syndetic, if there is $N \in \mathbb{N}$ such that for any $n \in \mathbb{N}, S \cap \{n, n+1, \ldots, n+N\} \neq \emptyset$.

**Definition 1.1.** (Statistical $\omega$–limit sets) For $x \in X$ and $\xi = \{\bar{d}, d, B^*, B_*\}$, a point $y \in X$ is called $x - \xi$–accessible, if for any $\varepsilon > 0$, $N(x, V_{\varepsilon}(y))$ has positive density w. r. t. $\xi$, where $B_*(x)$ denotes the ball centered at $x$ with radius $\varepsilon$. Let

\[ \omega_{\xi}(x) := \{y \in X \mid y \text{ is } x - \xi - \text{accessible}\}. \]

For convenience, it is called $\xi - \omega$–limit set of $x$ or $\xi$–center of $x$. We also call $\omega_{B_2}(x)$ to be syndetic center of $x$.

With these definitions, one can immediately note that

\begin{equation}
\omega_{B_2}(x) \subseteq \omega_{B_1}(x) \subseteq \omega_{B_0}(x) \subseteq \omega_{B^*}(x) \subseteq \omega(f, x).
\end{equation}

For any $x \in X$, if $\omega_{B_2}(x) = \emptyset$, then we know that $x$ satisfies one and only one of following twelve cases:

**Case (1)** : $\omega_{B_2}(x) \subseteq \omega_{B_1}(x) = \omega_{B_0}(x) = \omega_{B^*}(x) = \omega(f, x);$

**Case (1')** : $\omega_{B_1}(x) \subseteq \omega_{B_0}(x) = \omega_{B^*}(x) = \omega_{B_2}(x) \subseteq \omega(f, x);$
Case (2) : $\omega_B(x) \subseteq \omega_d(x) = \omega(x) \subseteq \omega_B(x) = \omega(f, x)$;

Case (2') : $\omega_B(x) \subseteq \omega_d(x) \subseteq \omega(x) \subseteq \omega_B(x) \subseteq \omega(f, x)$;

Case (3) : $\omega_B(x) = \omega_d(x) \subseteq \omega(x) = \omega(f, x)$;

Case (3') : $\omega_B(x) = \omega_d(x) \subseteq \omega(x) = \omega_B(x) \subseteq \omega(f, x)$;

Case (4) : $\omega_B(x) \subseteq \omega_d(x) \subseteq \omega(x) = \omega_B(x) = \omega(f, x)$;

Case (4') : $\omega_B(x) \subseteq \omega_d(x) \subseteq \omega(x) = \omega_B(x) \subseteq \omega(f, x)$;

Case (5) : $\omega_B(x) = \omega_d(x) \subseteq \omega(x) \subseteq \omega_B(x) = \omega(f, x)$;

Case (5') : $\omega_B(x) = \omega_d(x) \subseteq \omega(x) \subseteq \omega_B(x) \subseteq \omega(f, x)$;

Case (6) : $\omega_B(x) \subseteq \omega_d(x) \subseteq \omega(x) \subseteq \omega_B(x) = \omega(f, x)$;

Case (6') : $\omega_B(x) \subseteq \omega_d(x) \subseteq \omega(x) \subseteq \omega_B(x) \subseteq \omega(f, x)$.

There are twelve cases rather than sixteen because $\omega(x)$ must be a nonempty set naturally (see [16]). Note that the recurrent points satisfying one of Cases (1)-(6) should have [16] so that the set of $(Rec \setminus BR)$ cannot contain Cases (1)-(6). Now we give a more refined result of Theorem A.

**Theorem B.** Let $f : M \to M$ be a transitive Anosov diffeomorphism on a compact Riemannian manifold. Then the set of $(Rec \setminus BR)$ contains points satisfying Cases (1')-(6'). Moreover, it contains five uncountable DC-1 scrambled subsets $S_i (i=2,3,4,5,6')$ such that for any $x \in S_i$, $x$ satisfies Case (i).

Note that this result is equivalent to it is stated for some $f^N$. It is well known for any transitive Anosov system, there exists subsystem of $f^N$ for some large $N$ such that it is topologically conjugated to a full shift with arbitrary many symbols so that we only need to give the proof for full shifts.

**Theorem C.** Given $k \geq 3$, let $(S_{k+1}, f)$ be the k-th symbolic dynamical system equipped with shift operator. Then the set of $(Rec \setminus BR)$ contains points satisfying Cases (1')-(6'). Moreover, it contains five uncountable DC-1 scrambled subsets $S_i (i=2,3,4,5,6')$ such that for any $x \in S_i$, $x$ satisfies Case (i).

## 2. Notations and Definitions

By $P(X, f)$ we denote the set of periodic points of the system $(X, f)$. By $C(X, f)$ we denote the measure center of the system $(X, f)$, which is the smallest invariant closed subset of $X$ with full measure under any borel measure. By $M(X)$ and $M(X, f)$ we denote the set of Borel measures and the set of invariant Borel measure on the measurable space $X$ respectively. By $S_\mu$ we denote the support of the measure $\mu$. By $V_f(x)$ we denote all the limit points of the set $\left\{ \frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i(x)} \right\}$.

We review the definition of distributional chaos [31][29]. Given a compact dynamical system $(X, f)$, for any positive integer $n$, points $x, y \in X$ and $t \in \mathbb{R}$ define

$$\Phi^{(n)}_{xy}(t) := \frac{1}{n} \left| \left\{ i \mid d(f^i(x), f^i(y)) < t, 0 \leq i < n \right\} \right|,$$

where $|A|$ denotes the cardinality of the set $A$. Let us denote by $\Phi_{xy}$ and $\Phi^*_{xy}$ the following functions:

$$\Phi_{xy}(t) := \liminf_{n \to \infty} \Phi^{(n)}_{xy}(t), \quad \Phi^*_{xy}(t) := \limsup_{n \to \infty} \Phi^{(n)}_{xy}(t).$$

Both functions $\Phi_{xy}$ and $\Phi^*_{xy}$ are nondecreasing. $\Phi_{xy}(t) = \Phi^*_{xy}(t) = 0$ for $t < 0$ and $\Phi_{xy}(t) = \Phi^*_{xy}(t) = 1$ for $t > \text{diam} \ X$. A pair of points $(x, y) \in X \times X$ is called distributionally chaotic of type 1 if $\Phi_{xy}(s) = 0$ for some $s > 0$ and $\Phi^*_{xy}(t) = 1$ for all $t > 0$. A set containing at least two points is called a distributionally chaos scrambled set of type 1 (DC1-scrambled, for short) if any pair of its distinct points is distributionally chaotic of type 1.

Given a positive integer $k$, by $S$ we denote $S = \{0, 1, \ldots, k-1\}$. Let $S_k = \prod_{n=1}^{k} S$ and define the metric on $S_k$ as

$$d(x, y) := \sum_{n=1}^{\infty} \frac{\delta(x_n, y_n)}{2^n}, \text{ for any } x = (x_1, x_2, \ldots), \ y = (y_1, y_2, \ldots) \in S_k.$$

By $(S_k, f)$ we denote the k-th symbolic dynamical system and by $f$ we denote the shift operator. $A$ is called a finite word in $S_k$ if $A = (a_1a_2 \cdots a_n)$ where $0 \leq a_i \leq k - 1$ and by $|A|$ we denote the length of $A$. For any finite word $A$, by $\#_k(A)$ we denote the number of $k$ in the word $A$. For any two finite words $A = (a_1a_2 \cdots a_n)$ and $B = (b_1b_2 \cdots b_m)$, by $AB$ we denote $AB = (a_1a_2 \cdots a_nb_1b_2 \cdots b_m)$. For any two finite words $A = (a_1a_2 \cdots a_n)$ and $B = (b_1b_2 \cdots b_m)$, $A$ is contained in $B$ if there exists a $t$ such that $a_i = b_{i+t}$ for any $1 \leq i \leq n$ and denote it by $A \subseteq B$. 
3. Proof

In this section, we will prove Theorem 3 through a stronger statement. We firstly review the irregular set \([27, 39, 13]\) and the regular set \([12, 17]\) of a dynamical system \((X, f)\) respect to a continuous function \(\varphi\) on \(X\). For a continuous function \(\varphi\) on \(X\), define the \(\varphi - \text{irregular set}\) as

\[
I_{\varphi}(f) := \left\{ x \in X \mid \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(x)) \text{ diverges} \right\}.
\]

Denote

\[
L_{\varphi} := \left[ \inf_{\mu \in M_f(X)} \int \varphi d\mu, \sup_{\mu \in M_f(X)} \int \varphi d\mu \right]
\]

and

\[
\text{Int}(L_{\varphi}) := \left( \inf_{\mu \in M_f(X)} \int \varphi d\mu, \sup_{\mu \in M_f(X)} \int \varphi d\mu \right).
\]

For any \(a \in L_{\varphi}\), define the level set as

\[
R_{\varphi}(a) := \left\{ x \in X \mid \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(x)) = a \right\}.
\]

Denote \(R_{\varphi} := \bigcup_{a \in L_{\varphi}} R_{\varphi}(a)\), called the regular points for \(\varphi\).

From the perspective of multifractal analysis, we state a stronger version of Theorem 3.

**Theorem D.** Given \(k \geq 3\), let \((S_{k+1}, f)\) be the \(k\)-th symbolic dynamical system equipped with shift operator. Suppose that \(\varphi\) is a continuous function on \(S_k\) and \(I_{\varphi}(f) \neq \emptyset\), then for any \(a \in \text{Int}(L_{\varphi})\), the recurrent level set of \(\{x \in \text{Rec} \setminus BR \mid x \text{ satisfies Case (i')}\}\) contains an uncountable DC1-scrambled subset in \(I_{\varphi}(f)\), \(R_{\varphi}(a)\) and \(R_{\varphi}\), respectively, \(i = 2, 3, 4, 5, 6\).

We will prove Theorem D by constructing an injective map \(T : S_k \to S_{k+1}\), which maps any point satisfying Case (i) to a point satisfying Case (i') \((i = 1, 2, 3, 4, 5, 6)\). Moreover, \(T\) has some invariant properties in the view of invariant measure, Birkhoff ergodic average and distributional chaos.

**Lemma 3.1.** [16] Suppose that \((X, f)\) is a compact dynamical system and \(x \in X\), then \(\omega(f, x) = \bigcap_{n \in N} S_{\mu_n}\) and \(\omega(f, x) = \bigcup_{n \in N} S_{\mu_n}\). Furthermore, let \(\Lambda = \omega(f, x)\), we have \(\omega_{B_\varepsilon}(x) = \bigcap_{n \in M(\Lambda, f)} S_{\mu_n}\) and \(\omega_{B_\varepsilon}(x) = \bigcup_{n \in M(\Lambda, f)} S_{\mu_n} = C(\Lambda, f)\).

**Lemma 3.2.** Suppose that \((X, f)\) is a compact dynamical system and \(\Lambda = \omega(f, x)\), then \(C(\Lambda, f)\) equals to the closure of the set \(\{x \in X \mid d(x, B_{\varepsilon}(x)) > 0, \text{ for any } \varepsilon > 0\}\).

**Proof.** Suppose that \(x \in X \mid d(x, B_{\varepsilon}(x)) > 0, \text{ for any } \varepsilon > 0\), which implies \(x \in \Lambda\). There exists an increasing sequence \(\{n_k\}\) such that

\[
\lim_{k \to \infty} \frac{1}{n_k} \sum_{i=0}^{n_k-1} \delta f^i(x) = \mu \in M(\Lambda, f).
\]

Then, we have for any \(\delta > 0\)

\[
\mu(B_{\delta}(x)) = \lim_{k \to \infty} \frac{1}{n_k} \sum_{i=0}^{n_k-1} \delta f^i(x)(B_{\delta}(x)) \geq d(x, B_{\delta}(x)) > 0,
\]

which implies \(x \in S_{\mu} \subseteq C(\Lambda, f)\).

On the other hand, by [14, 17] we have \(\mu(\{x \in X \mid d(x, B_{\varepsilon}(x)) > 0, \text{ for any } \varepsilon > 0\}) = 1\) for any \(x \in \Lambda\). Moreover, \(\mu(\{x \in X \mid d(x, B_{\varepsilon}(x)) > 0, \text{ for any } \varepsilon > 0\}) = 1\), which implies, for any \(x \in \Lambda\), \(S_{\mu_n}\) is contained in the closure of \(\{x \in X \mid d(x, B_{\varepsilon}(x)) > 0, \text{ for any } \varepsilon > 0\}\). Hence, by Lemma 3.1 we have \(C(\Lambda, f)\) is contained in the closure of \(\{x \in X \mid d(x, B_{\varepsilon}(x)) > 0, \text{ for any } \varepsilon > 0\}\). \(\square\)

**Lemma 3.3.** [11] Theorem E Suppose that \((S_k, f)\) is the \(k\)-th symbolic dynamical system. Notice that \((S_k, f)\) satisfies specification property. Then \(x \in \text{Rec} \mid x \text{ satisfies Case (i)}\), \(i = 2, 3, 4, 5, 6\) contains an uncountable DC1-scrambled subset. Further, if \(\varphi\) is a continuous function on \(X\) and \(I_{\varphi}(f) \neq \emptyset\) then for any \(a \in \text{Int}(L_{\varphi})\), the recurrent level set of \(\{x \in \text{Rec} \mid x \text{ satisfies Case (i)}\}\) contains an uncountable DC1-scrambled subset in \(I_{\varphi}(f)\), \(R_{\varphi}(a)\) and \(R_{\varphi}\), respectively, \(i = 2, 3, 4, 5, 6\).
For any \( y \in S_k \) we use \( Y_n \) to denote the first \( n \) letters of \( y \). The set of all finite words in \( S_k \) is countable, so it can be denoted as \( \{C_1, C_2, \ldots \} \). Let \( B_n = C_n Y_n \cdots Y_1 \), where \( Y_n \) repeats \( |A_n| \) times. Notice the fact that \( |B_n| \geq n|A_n|^2 \) which will be used repeatedly in the proof. Define an injective map \( T : S_k \to S_{k+1} \) as \( T(y) := \lim_{n \to \infty} A_n \), where \( A_n+1 = A_n B_n A_n \) and \( A_1 = (k) \). Meanwhile, we require \( |C_n| = o(|A_n|) \) as \( n \to \infty \). It is obvious that \( T \) is injective. This construction is mainly inspired by the constructions in \([28, 20]\).

**Proposition 3.1.** For any \( y \in S_k \), we have \( \omega_B(y) = S_k \sqsubseteq \omega(f, T(y)) \sqsubseteq S_{k+1} \).

**Proof.** Let \( x = T(y) \) and \( \Lambda = \omega(f, x) \). Firstly, notice the fact that \( x \) contains all the finite words in \( S_k \), so we have \( S_k \subseteq \Lambda \). Hence, by Lemma 3.1 and 3.2 we have \( S_k = \{x \in S_k \mid x \text{ is periodic}\} \subseteq \{x \in \Lambda \mid x \text{ is periodic}\} \subseteq \{x \in \Lambda \mid -d(x, B_k(x)) > 0, \text{ for any } \varepsilon > 0\} = C(\Lambda, f) = \omega_B(x) \), which implies \( S_k \subseteq \omega_B(T(y)) \).

Secondly, we prove \( \omega_B(x) \subseteq S_k \) by contradiction. Assume \( x \in \omega_B(x) \setminus S_k \neq \emptyset \). Further, by Lemma 3.2 we may assume \( d(z, B_k(z)) > 0 \) for any \( \varepsilon > 0 \). Without loss of generality, we denote \( z \) by \( z = (kz_2z_3 \cdots) \) where \( 0 \leq z_i \leq k \) and \( z_1 = k \). Let \( \{n_i\}_{i \geq 1} \) be all the indices that \( z_{n_i} = k \) and assume \( n_i < n_{i+1} \). Therefore, we have

\[
\liminf_{i \to \infty} \frac{i}{n_i} = \liminf_{i \to \infty} \frac{|N(z, B_1(z)) \cap \{0, 1, \ldots, n_i - 1\}|}{n_i} \geq \liminf_{n \to \infty} \frac{\# \{n_i \leq n\}}{n} = d(N(z, B_1(z)) > 0.
\]

On the other hand, we notice that \( y \in \Lambda = \omega(f, x) \) which implies for any \( i \) there exists an integer \( n_i \) such that \( n_i^{m}(x) = (z_1 z_2 \cdots z_n) \). By the construction of \( T \), \( x = (A_1 B_1 A_2 B_2 A_3 \cdots) \), so \( z_n \) only appears in \( A_n \). Assume \( (z_1 z_2 \cdots z_n) \) is contained in \( A_{p_1} B_{p_1+1} \cdots A_{q_1} \), \( z_1 = k \) appears in \( A_{p_1} \), and \( z_n \), appears in \( A_{q_1} \). Moreover, we assume possible \( i \) is the smallest possible index. (It is obvious that \( p_1 < q_1 \), otherwise \( q_1 \) will not be the smallest index.) Since \( p_1 < q_1 \), we have \( |A_{q_1}|^2 \leq |B_{q_1}| \leq n_i \). Moreover, notice that \( (z_1 z_2 \cdots z_n) \subset (A_{p_1} B_{p_1+1} \cdots A_{q_1}) \subset (A_{q_1} B_{q_1} A_{q_1}) \), which implies

\[
\frac{i}{n_i} = \frac{\#k(z_1 z_2 \cdots z_n)}{n_i} \leq \frac{\#(A_{q_1} B_{q_1} A_{q_1})}{n_i} \leq 2\frac{\#k(A_{q_1})}{n_i} \leq \frac{2|A_{q_1}|}{|B_{q_1}|} \leq \frac{2}{|A_{q_1}|},
\]

Hence, we have

\[
\liminf_{i \to \infty} \frac{i}{n_i} \leq \liminf_{i \to \infty} \frac{2}{|A_{q_1}|} = 0,
\]

which is contrary to (3.1).

Finally, \( S_k \subseteq \omega(f, x) \subseteq S_{k+1} \) which follows directly from \( x \in \omega(f, x) \setminus S_k \) and \( (kkk \cdots) \in S_{k+1} \setminus \omega(f, x) \). \( \square \)

Then, we consider the generated measure of \( y \) and \( T(y) \) where \( y \in S_k \).

**Proposition 3.2.** For any \( y \in S_k \), we have \( V(f)(T(y)) = V(f)(y) \).

**Proof.** By \( C(S_{k+1}) \) we denote the set of all complex value continuous functions on \( S_{k+1} \). Then \( (C(S_{k+1}), \|\cdot\|) \) is a Banach space equipped with uniform norm. Assume \( \{F_n\} \) is a dense countable subset of \( C(S_{k+1}) \), then there exists a metric on \( M(S_{k+1}) \) defined as

\[
d(\mu, \nu) := \sum_{l=1}^{\infty} \left| \int_{S_{k+1}} F_l d\mu - \int_{S_{k+1}} F_l d\nu \right| / 2^l \|F_l\|.
\]

Without loss of generality, we assume \( \|F_n\| = 1 \). The continuous module of \( F \) is defined as

\[
s(F, t) := \sup_{|x-y| \leq t} |F(x) - F(y)|.
\]

Let \( x = T(y) \), \( \mu_n = \frac{1}{n} \sum_{i=0}^{n-1} \delta \nu(x) \) and \( \nu_n = \frac{1}{n} \sum_{i=0}^{n-1} \delta \mu(y) \).

Firstly, we want to show \( V(f)(y) \subseteq V(f)(x) \). It suffices to prove for any \( \nu_n \in V(f)(y) \) exists \( \mu_m \in V(f)(x) \) such that \( d(\nu_n, \mu_m) = o(1) \) as \( n \to \infty \). Set \( m = |A_n| + |B_n| \) and let \( F \in C(S_{k+1}) \), \( \|F\| = 1 \). We have

\[
\left| \int_{S_{k+1}} F d\mu_m - \int_{S_{k+1}} F d\nu_n \right| = \frac{1}{m} \sum_{i=0}^{m-1} \left| F(f^i(x)) - \frac{1}{n} \sum_{i=0}^{n-1} F(f^i(y)) \right| \\
\leq \frac{1}{m} \sum_{i=0}^{m-1} F(f^i(x)) \quad \text{(A)} \\
+ \frac{1}{m} \sum_{i=0}^{m-1} F(f^i(x)) - \frac{1}{n} \sum_{i=0}^{n-1} F(f^i(y)) \quad \text{(B)}
\]

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For A we have
\[ A \leq \frac{|A_n| + |C_n|}{m} \leq \frac{2}{|A_n|}. \]

Let \( j + |A_n| + |C_n| \equiv i \mod n \) and \( 0 \leq j \leq n - 1 \), for B we have
\[ B = \frac{1}{m} \sum_{i=[A_n]+|C_n|}^{m-1} F(f^i(x)) - F(f^i(y)) + \frac{|A_n|^2}{m} - \frac{1}{n} \sum_{i=0}^{n-1} F(f^i(y)) \]
\[ \leq \frac{1}{m} \sum_{i=[A_n]+|C_n|}^{m-1} F(f^i(x)) - F(f^i(y)) + \frac{|A_n|^2}{m} - \frac{1}{n} \sum_{i=0}^{n-1} F(f^i(y)) \]
\[ \leq \frac{|A_n|^2}{m} \sum_{i=0}^{n-1} s(F, 2^{-i}) + \left| \frac{n |A_n|^2}{m} - 1 \right| \quad \text{(since } |f^i(x) - f^i(y)| \leq 2^{-n+1+j}) \]
\[ \leq 2 \frac{|A_n|}{m} + \frac{1}{n} \sum_{i=0}^{n-1} s(F, 2^{-i}). \]

Hence, for any \( F \in C(S_{k+1}) \) and \( \|F\| = 1 \), we have
\[ (3.2) \quad \left| \int_{S_{n+1}} F d\mu_n - \int_{S_{n+1}} F d\nu_n \right| \leq 4 \frac{|A_n|}{m} + \frac{1}{n} \sum_{i=0}^{n-1} s(F, 2^{-i}). \]

Therefore, for any \( N \geq 0 \) we have
\[ \limsup_{n \to \infty} d(\nu_n, \mu_m) = \limsup_{n \to \infty} 2^{-l} \left| \int_{S_{n+1}} F_i d\mu_n - \int_{S_{n+1}} F_i d\nu_n \right| \]
\[ \leq 2^{-N+1}. \]

Let \( N \to \infty \), we have \( d(\nu_n, \mu_m) = o(1) \) as \( n \to \infty \).

Secondly, we want to show \( V_f(x) \subseteq V_f(y) \). It suffices to prove for any \( \mu_m \) exists \( \nu_n \) such that \( d(\nu_n, \mu_m) = o(1) \) as \( m \to \infty \). We can choose \( n \) such that \( |A_n| \leq m < |A_{n+1}| \). Set \( \lambda = \frac{|A_n|}{m} \), then \( \mu_m = \lambda \mu + (1 - \lambda)\overline{\mu} \).

where \( \mu = \mu_{|A_n|} \) and \( \overline{\mu} = \frac{1}{m - |A_n|} \sum_{i=|A_n|+1}^{m-1} \delta f(x) \). As we do in the first part, we can obtain a similar estimate \( d(\mu, \nu_{n-1}) = o(1) \) as \( n \to \infty \) which implies \( d(\mu_n, \nu_n) = o(1) \) as \( m \to \infty \). Combined with the fact that \( d(\nu_{n-1}, \nu_n) = o(1) \) as \( m \to \infty \), finally we have \( d(\nu_n, \mu) = o(1) \) as \( m \to \infty \). Then, we estimate \( d(\nu_n, \overline{\mu}) \). Let \( j + |A_n| + |C_n| \equiv i \mod n \) and \( 0 \leq j \leq n - 1 \). We have
\[ \left| \int_{S_{n+1}} F d\overline{\mu} - \int_{S_{n+1}} F d\nu_n \right| \]
\[ = \left| \frac{1}{m - |A_n|} \sum_{i=|A_n|+1}^{m-1} F(f^i(x)) - \sum_{i=|A_n|+1}^{m-1} \frac{F(f^i(y)) - F(f^i(x))}{m - |A_n|} \right| \]
\[ \leq \frac{|C_n|}{m - |A_n|} + \sum_{i=|A_n|+1}^{m-1} \frac{F(f^i(x)) - F(f^i(y))}{m - |A_n|} \quad \text{(A)} \]
\[ \quad + \sum_{i=|A_n|+1}^{m-1} \frac{F(f^i(y))}{m - |A_n|} - \frac{1}{n} \sum_{i=0}^{n-1} s(F, 2^{-i}) \quad \text{(B)} \]

where A and B are equal to 0 if \( m \leq |A_n| + |C_n| \). For A we have
\[ A \leq \sum_{i=|A_n|+1}^{m-1} s(F, 2^{-i}) \]
\[ \leq 2 \frac{n-1}{n} \sum_{i=0}^{n-1} s(F, 2^{-i}). \]

For B we have
\[ B \leq \frac{1}{n} \sum_{i=0}^{n-1} \left| \frac{m - |A_n| - |C_n|}{n(m - |A_n|)} - \frac{1}{n} \right| s(F^i(y)) \]
\[ \leq \frac{|C_n|}{m - |A_n|}. \]
Hence, for any $F \in C(S_{k+1})$ and $\|F\| = 1$, we have

$$\left| \int_{S_{k+1}} Fd\nu - \int_{S_{k+1}} Fd\mu \right| \leq \frac{2|C_n|}{m - |A_n|} + \frac{2}{n} \sum_{i=0}^{n-1} s(F, 2^{-i}).$$

Therefore, for any $N \geq 0$ we have

$$\limsup_{m \to 0} (1 - \lambda)d(\nu_n, \mu_n) \leq \limsup_{m \to 0} (1 - |A_n|) \sum_{i=1}^{\infty} 2^{-l} \left| \int_{S_{k+1}} Fd\nu_n - \int_{S_{k+1}} Fd\mu \right| \leq \limsup_{m \to 0} (1 - |A_n|) \sum_{i=1}^{\infty} 2^{-l} \left[ \frac{2|C_n|}{m - |A_n|} + \frac{2}{n} \sum_{i=0}^{n-1} s(F, 2^{-i}) \right] + \sum_{l=N+1}^{\infty} 2^{-l+1} \leq 2^{-N+1}.$$ 

Let $N \to \infty$, $(1 - \lambda)d(\nu_n, \mu_n) = o(1)$ as $m \to \infty$, which implies $d(\nu_n, \mu_m) \to \lambda d(\nu_n, \mu) + (1 - \lambda)d(\nu_n, \mu)$ as $m \to \infty$. 

From the perspective of multifractal analysis, the regular set and the irregular set are invariant under $T$.

**Proposition 3.3.** For any continuous function $\varphi$ on $S_k$ and $a \in I_{\varphi}$, we have $T$ maps $I_{\varphi}(f)$ and $R_{\varphi}(a)$ into itself respectively.

**Proof.** We use the same notations as Proposition 3.2. Firstly, assume $y \in I_{\varphi}(f)$ and $x = T(y)$. By Proposition 3.2 for any increasing sequence $\{n_k\}$, we have an increasing sequence $\{m_k\}$ satisfies $d(\nu_{n_k}, \mu_{m_k}) \to 0$ as $k \to \infty$. Therefore,

$$\lim_{k \to \infty} \left| \frac{1}{n_k} \sum_{i=0}^{n_k-1} \varphi(f^i(y)) - \frac{1}{m_k} \sum_{i=0}^{m_k-1} \varphi(f^i(x)) \right| = \lim_{k \to \infty} \left| \int_{S_{k+1}} \varphi d\nu_{n_k} - \int_{S_{k+1}} \varphi d\mu_{m_k} \right| = 0.$$

Let $n_k = k$, then we have

$$\limsup_{k \to \infty} \left| \frac{1}{k} \sum_{i=0}^{k-1} \varphi(f^i(y)) \right| = \limsup_{k \to \infty} \left| \frac{1}{m_k} \sum_{i=0}^{m_k-1} \varphi(f^i(x)) \right| \leq \limsup_{k \to \infty} \left| \frac{1}{k} \sum_{i=0}^{k-1} \varphi(f^i(y)) - \frac{1}{m_k} \sum_{i=0}^{m_k-1} \varphi(f^i(x)) \right| \leq \liminf_{k \to \infty} \left| \frac{1}{k} \sum_{i=0}^{k-1} \varphi(f^i(x)) \right| = \liminf_{k \to \infty} \left| \frac{1}{m_k} \sum_{i=0}^{m_k-1} \varphi(f^i(x)) \right| = \limsup_{k \to \infty} \left| \frac{1}{m_k} \sum_{i=0}^{m_k-1} \varphi(f^i(x)) \right|.$$

This implies $x = T(y) \in I_{\varphi}(f)$.

On the other hand, assume $y \in R_{\varphi}(a)$ and $x = T(y)$. By Proposition 3.2 for any increasing sequence $\{m_k\}$, we have an increasing sequence $\{n_k\}$ satisfies $d(\mu_{n_k}, \nu_{m_k}) \to 0$ as $k \to \infty$. Therefore,

$$\lim_{k \to \infty} \left| \frac{1}{m_k} \sum_{i=0}^{m_k-1} \varphi(f^i(x)) - \frac{1}{n_k} \sum_{i=0}^{n_k-1} \varphi(f^i(y)) \right| = \lim_{k \to \infty} \left| \int_{S_{k+1}} \varphi d\mu_{n_k} - \int_{S_{k+1}} \varphi d\nu_{m_k} \right| = 0.$$

Hence, we have

$$\limsup_{k \to \infty} \left| \frac{1}{m_k} \sum_{i=0}^{m_k-1} \varphi(f^i(x)) \right| = \limsup_{k \to \infty} \left| \frac{1}{n_k} \sum_{i=0}^{n_k-1} \varphi(f^i(y)) \right| + \lim_{k \to \infty} \left| \frac{1}{m_k} \sum_{i=0}^{m_k-1} \varphi(f^i(x)) - \frac{1}{n_k} \sum_{i=0}^{n_k-1} \varphi(f^i(y)) \right| = \limsup_{k \to \infty} \left| \frac{1}{n_k} \sum_{i=0}^{n_k-1} \varphi(f^i(y)) \right| = a.$$
\[ \liminf_{k \to \infty} \frac{1}{m_k} \sum_{i=0}^{m_k-1} \varphi(f^i(x)) = \liminf_{k \to \infty} \frac{1}{m_k} \sum_{i=0}^{m_k-1} \varphi(f^i(y)) + \lim_{k \to \infty} \left( \frac{1}{m_k} \sum_{i=0}^{m_k-1} \varphi(f^i(x)) - \frac{1}{m_k} \sum_{i=0}^{m_k-1} \varphi(f^i(y)) \right) \]

This implies \( \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(x)) = a \), which means \( x = T(y) \in R_\varphi(a) \).

From the perspective of DC1, \( T \) maps a DC1 pair to a DC1 pair.

**Proposition 3.4.** Suppose \( y \) and \( \tilde{y} \) are DC1-scrambled, then \( T(y) \) and \( T(\tilde{y}) \) are DC1-scrambled.

**Proof.** Let \( x \) and \( \tilde{x} \) equal to \( T(y) \) and \( T(\tilde{y}) \) respectively. It suffices to prove that for any positive integer \( t \),

\[ \left| \Phi_{x,\tilde{x}}^{[n+1]}(2^{-t}) - \Phi_{y,\tilde{y}}^{[n]}(2^{-t}) \right| = o(1) \text{ as } n \to \infty. \]

Let \( j + |A_n| + |C_n| \equiv i \mod n \) and \( 0 \leq j \leq n - t \), then we have

\[ |f^i(x) - f^i(\tilde{x})| < 2^{-t} \iff |f^i(y) - f^i(\tilde{y})| < 2^{-t} \text{ for any } |A_n| + |C_n| \leq i < |A_n| + |B_n|. \]

Set \( N = \# \{ i : |f^i(y) - f^i(\tilde{y})| < 2^{-t}, 0 \leq i < n \}. \) Then we have

\[ |A_n|^2 (N - t) \leq \# \{ i : |f^i(x) - f^i(\tilde{x})| < 2^{-t}, |A_n| + |C_n| \leq i < |A_n| + |B_n| \} \leq |A_n|^2 (N + t), \]

which implies

\[ |A_n|^2 (N - t) \leq \# \{ i : |f^i(x) - f^i(\tilde{x})| < 2^{-t}, 0 \leq i < |A_{n+1}| \} \leq 3 |A_n| + |B_n|^2 (N + t). \]

Hence, we have

\[ \left| \Phi_{x,\tilde{x}}^{[n+1]}(2^{-t}) - \Phi_{y,\tilde{y}}^{[n]}(2^{-t}) \right| = O(\frac{1}{n}) = o(1) \text{ as } n \to \infty. \]

**Proof of Theorem 7** By Lemma 3.3 we have five uncountable DC1-scrambled sets \( D_i \) contained in the set \( \{ x \in Rec \mid x \text{ satisfies Case(i)} \} \) \( (i=2,3,4,5,6) \) respectively. For any \( x_i \in D_i \), by Proposition 3.4, we have

\[ \omega_{\varphi}(T(x_i)) = S_k \subseteq \omega(f, T(x_i)) \not\supseteq S_{k+1}; \]

by Proposition 3.2 and Lemma 3.1 we have

\[ \omega_{\varphi}(T(x_i)) = \omega_{\varphi}(x_i) \text{ and } \omega_{\varphi}(T(x_i)) = \omega_{\varphi}(x_i). \]

Hence, \( T(D_i) \) contained in the set \( \{ x \in Rec \mid x \text{ satisfies Case(i')} \} \) \( (i=2,3,4,5,6) \) respectively. Furthermore, by Proposition 3.1 we have \( T(D_i) \) is uncountable and DC1-scrambled; by Proposition 3.3 \( T(D_i) \) is contained in \( I_{\varphi}(f), R_\varphi(a) \) or \( R_\varphi \) if \( D_i \) is contained in \( I_{\varphi}(f), R_\varphi(a) \) or \( R_\varphi \) respectively \( (i = 2, 3, 4, 5, 6) \).

**Proof of Theorem 6** By Theorem 8 we have five uncountable DC1-scrambled sets contained in the set \( \{ x \in Rec \mid x \text{ satisfies Case(i')} \} \) \( (i = 2, 3, 4, 5, 6) \) respectively. For the existence of Case(1'), we need a new lemma.

**Lemma 3.4.** Assume \( K \) is a connected compact convex subset of \( M(S_k, f) \), then there exists \( y \in S_k \) such that \( V_f(y) = K \).

Take \( \mu \in M(S_k, f) \) with \( S_\mu = S_k \). Hence, by Lemma 3.4 we have a point \( y \in S_k \) such that \( V_f(y) = \{ \mu \} \). Let \( x = T(y) \), then by Proposition 3.2 we have \( V_f(x) = \{ \mu \} \). By Lemma 3.1 we have \( \omega_{\varphi}(x) = \omega_{\varphi}(x) = S_k \). Moreover, by Proposition 3.4 \( x \) satisfies Case(1').

**Acknowledgements** The authors would like to acknowledge Prof. Xiaoyi Wang for providing some references to us.
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