On the list color function threshold

Hemanshu Kaul | Akash Kumar | Jeffrey A. Mudrock
Patrick Rewers | Paul Shin | Khue To

1Department of Applied Mathematics, Illinois Institute of Technology, Chicago, Illinois, USA
2Department of Mathematics, College of Lake County, Grayslake, Illinois, USA
3Department of Mathematics and Statistics, University of South Alabama, Mobile, Alabama, USA

Correspondence
Hemanshu Kaul, Department of Applied Mathematics, Illinois Institute of Technology, Chicago, IL 60616, USA.
Email: kaul@iit.edu

Abstract
The chromatic polynomial of a graph $G$, denoted $P(G, m)$, is equal to the number of proper $m$-colorings of $G$. The list color function of graph $G$, denoted $P_{\ell}(G, m)$, is a list analogue of the chromatic polynomial that has been studied since the early 1990s, primarily through comparisons with the corresponding chromatic polynomial. It is known that for any graph $G$ there is a $k \in \mathbb{N}$ such that $P_{\ell}(G, m) = P(G, m)$ whenever $m \geq k$. The list color function threshold of $G$, denoted $\tau(G)$, is the smallest $k \geq \chi(G)$ such that $P_{\ell}(G, m) = P(G, m)$ whenever $m \geq k$. In 2009, Thomassen asked whether there is a universal constant $\alpha$ such that for any graph $G$, $\tau(G) \leq \chi_{\ell}(G) + \alpha$, where $\chi_{\ell}(G)$ is the list chromatic number of $G$. We show that the answer to this question is no by proving that there exists a positive constant $C$ such that $\tau(K_{2, l}) - \chi_{\ell}(K_{2, l}) \geq C \sqrt{l}$ for $l \geq 16$.

KEYWORDS
chromatic polynomial, list color function, list coloring

MATHEMATICAL SUBJECT CLASSIFICATION
05C15, 05C30

1 | INTRODUCTION

In this paper, all graphs are nonempty, finite, simple graphs unless otherwise noted. Generally speaking, we follow West [17] for terminology and notation. The set of natural numbers is $\mathbb{N} = \{1, 2, 3, \ldots\}$. For $m \in \mathbb{N}$, we write $[m]$ for the set $\{1, \ldots, m\}$. If $G$ is a graph and...
$S \subseteq V(G)$, we use $G[S]$ for the subgraph of $G$ induced by $S$. If $u$ and $v$ are adjacent in $G$, $uv$ or $vu$ refers to the edge between $u$ and $v$. We write $K_{n,l}$ for complete bipartite graphs with partite sets of size $n$ and $l$. If $G$ and $H$ are vertex disjoint graphs, we write $G \lor H$ for the join of $G$ and $H$.

### 1.1 List coloring and counting list colorings

In classical vertex coloring one wishes to color the vertices of a graph $G$ with up to $m$ colors from $[m]$ so that adjacent vertices in $G$ receive different colors, a so-called proper $m$-coloring. The chromatic number of a graph, denoted $\chi(G)$, is the smallest $m$ such that $G$ has a proper $m$-coloring. List coloring is a generalization of classical vertex coloring introduced independently by Vizing [15] and Erdős et al. [8] in the 1970s. In list coloring, we associate a list assignment $L$ with a graph $G$ so that each vertex $v \in V(G)$ is assigned a list of available colors $L(v)$ (we say $L$ is a list assignment for $G$). We say $G$ is $L$-colorable if there is a proper coloring $f$ of $G$ such that $f(v) \in L(v)$ for each $v \in V(G)$ (we refer to $f$ as a proper $L$-coloring of $G$). A list assignment $L$ is called a $k$-assignment for $G$ if $|L(v)| = k$ for each $v \in V(G)$. We say $G$ is $k$-choosable if $G$ is $L$-colorable whenever $L$ is a $k$-assignment for $G$. The list chromatic number of a graph $G$, denoted $\chi^e(G)$, is the smallest $k$ such that $G$ is $k$-choosable. It is immediately obvious that for any graph $G$, $\chi(G) \leq \chi^e(G)$. Moreover, it is well-known that the gap between the chromatic number and list chromatic number of a graph can be arbitrarily large as the following result illustrates (see e.g., [13] for further details).

**Theorem 1** (Folklore). For $n \in \mathbb{N}$, $\chi^e(K_{n,l}) = n + 1$ if and only if $l \geq n^2$.

In 1912 Birkhoff [3] introduced the notion of the chromatic polynomial with the hope of using it to make progress on the four-color problem. For $m \in \mathbb{N}$, the chromatic polynomial of a graph $G$, $P(G,m)$, is the number of proper $m$-colorings of $G$. It is well-known that $P(G,m)$ is a polynomial in $m$ of degree $|V(G)|$ (e.g., see [5]). For example, $P(K_n,m) = \prod_{i=0}^{n-1} (m-i)$, $P(C_n,m) = (m-1)^n + (-1)^n (m-1)$, $P(T,m) = m(m-1)^{n-1}$ whenever $T$ is a tree on $n$ vertices, and $P(K_{2,l},m) = m(m-1)^l + m(m-1)(m-2)^l$ (see [2] and [17]).

The notion of chromatic polynomial was extended to list coloring in the early 1990s by Kostochka and Sidorenko [12]. If $L$ is a list assignment for $G$, we use $P(G,L)$ to denote the number of proper $L$-colorings of $G$. The list color function $P_L(G,m)$ is the minimum value of $P(G,L)$ where the minimum is taken over all possible $m$-assignments $L$ for $G$. Since an $m$-assignment could assign the same $m$ colors to every vertex in a graph, it is clear that $P_L(G,m) \leq P(G,m)$ for each $m \in \mathbb{N}$. In general, the list color function can differ significantly from the chromatic polynomial for small values of $m$. One reason for this is that a graph can have a list chromatic number that is much higher than its chromatic number. On the other hand, in 1992, Donner [6] showed that for any graph $G$ there is a $k \in \mathbb{N}$ such that $P_L(G,m) = P(G,m)$ whenever $m \geq k$.

It is also known that $P_L(G,m) = P(G,m)$ for all $m \in \mathbb{N}$ when $G$ is a cycle or chordal (see [12] and [11]). Moreover, if $P_L(G,m) = P(G,m)$ for all $m \in \mathbb{N}$, then $P_L(G \lor K_n,m) = P(G \lor K_n,m)$ for each $n, m \in \mathbb{N}$ (see [9]).
1.2 The list color function threshold

We now introduce a notion that has received some attention (under different names) in the literature.* Given any graph $G$, the list color function number of $G$, denoted $\nu(G)$, is the smallest $t \geq \chi(G)$ such that $P_t(G) = P(G, t)$. The list color function threshold of $G$, denoted $\tau(G)$, is the smallest $k \geq \chi(G)$ such that $P_k(G, m) = P(G, m)$ whenever $m \geq k$. By Donner's 1992 result, we know that both $\nu(G)$ and $\tau(G)$ are well-defined for any graph $G$. Furthermore, $\chi(G) \leq \chi_k(G) \leq \nu(G) \leq \tau(G)$.

In 2009, Thomassen [14] showed that for any graph $G$, $\tau(G) \leq |V(G)|^{10} + 1$. Then, in 2017, Wang et al. [16] showed that for any graph $G$, $\tau(G) \leq (|E(G)| - 1)/\ln(1 + \sqrt{2}) + 1$. Two well-known open questions on the list color function can be stated using the list color function number and list color function threshold.

**Question 2** (Kirov and Naimi [11]). For every graph $G$, is it the case that $\nu(G) = \tau(G)$?

**Question 3** (Thomassen [14]). Is there a universal constant $\alpha$ such that for any graph $G$, $\tau(G) - \chi_k(G) \leq \alpha$?

Question 2, which is asking whether the list color function of a graph and the corresponding chromatic polynomial stay the same after the first point at which they are both nonzero and equal, remains open. However, the DP-coloring analogue of Question 2 was answered in the negative in [4], where it was studied under the notion of chromatic adherence (see [10] for an introduction to the DP color function, the DP-coloring analogue of list color function).

In [14], it was shown that in Question 3, $\alpha$ cannot be zero. In this paper, we show that the answer to Question 3 is no in a fairly strong sense. Specifically, we prove the following.

**Theorem 4.** Suppose $G = K_{2,l}$ and $l \geq 16$. Let $q = \lfloor l/4 \rfloor$. Then,

$$\tau(G) > \left(\frac{q}{\ln(16/7)}\right)^{1/2} + 1.$$

Consequently, there is a constant $C > 0$ such that for each $l \geq 16$, $\tau(K_{2,l}) - \chi_k(K_{2,l}) = \tau(K_{2,l}) - 3 \geq C\sqrt{l}$.

We have made no attempt to optimize the leading constant above. However, we believe that this lower bound captures the behavior of $\tau(K_{2,l})$.

**Conjecture 5.** $\tau(K_{2,l}) = \Theta(\sqrt{l})$ as $l \to \infty$.

In light of the bound of Wang, Qian, and Yan, Thomassen’s Question 3, and Theorem 4, it is natural to study the asymptotic behavior of the list color function threshold as the size of the graphs

---

*It is worth mentioning that a DP-coloring (see [7]) analogue of the list color function threshold was recently introduced and studied in [1].
we consider tends toward infinity. We define the extremal functions \( \delta_{\text{max}}(t) = \max \{ \tau(G) - \chi_{\ell}(G) : G \text{ is a graph with at most } t \text{ edges} \} \) and \( \tau_{\text{max}}(t) = \max \{ \tau(G) : G \text{ is a graph with at most } t \text{ edges} \} \). By Theorem 4 and the bound of Wang, Qian, and Yan, we know that there exist positive constants \( C_1, C_2 \) such that \( C_1 \sqrt{t} \leq \delta_{\text{max}}(t) \leq C_2 t \) for large enough \( t \). The same asymptotic bounds hold for \( \tau_{\text{max}}(t) \) as well.

**Question 6.** What is the asymptotic behavior of \( \delta_{\text{max}}(t) \)?

Since \( \chi_{\ell}(G) = O(\sqrt{|E(G)|}) \) as \( |E(G)| \to \infty \), if \( \tau_{\text{max}}(t) = \omega(\sqrt{t}) \) as \( t \to \infty \), then \( \delta_{\text{max}}(t) \sim \tau_{\text{max}}(t) \) as \( t \to \infty \).

**Question 7.** What is the asymptotic behavior of \( \tau_{\text{max}}(t) \)? In particular, is \( \tau_{\text{max}}(t) = \omega(\sqrt{t}) \)?

Understanding \( \tau(K_{n,t}) \) would be the first natural candidate toward answering Questions 6 and 7.

## 2 Proof of Theorem 4

To prove a lower bound on \( \tau(G) \), we need an upper bound on \( P(\ell, G, m) \) that is smaller than \( P(\ell, G, m) \) for some \( m \). Our first step is to give an enumerative generalization† of the “if” direction of Theorem 1. We generalize the folklore “bad” list assignment from Theorem 1 and count the number of such list colorings to get an upper bound on \( P(\ell, K_{n,n^t}, m) \).

**Lemma 8.** Let \( n, m, t \in \mathbb{N} \) with \( n \geq 2 \) and \( m \geq n + 1 \), and let \( G = K_{n,n^t} \) with bipartition \( X = \{x_1, \ldots, x_n\}, Y = \{y_1, \ldots, y_{n^t}\} \). Let \( S_k = \{m + n(k - 2) + \ell : \ell \in [n] \} \) for each \( k \in [n] \), and let \( A = \{s_1, \ldots, s_n : s_k \in S_k \text{ for each } k \in [n] \} \). Suppose \( A = \{A_0, \ldots, A_{n-1}\} \). Let \( L \) be the \( m \)-assignment for \( G \) defined by \( L(x_k) = [m - n] \cup S_k \) for each \( k \in [n] \) and \( L(y_k) = [m - n] \cup A_{(k-1)/t} \) for each \( k \in [n^t] \). Then‡

\[
P(G, L) = n^n \prod_{i=0}^n (m - i)^i \left( \binom{n}{i} \right)^{(n-1)n-i} + \sum_{N=1}^n \sum_{S=0}^{n-N} \left[ n^S \binom{n}{S} \binom{m-n}{N} \sum_{i=0}^{N-1} (-1)^i \binom{N}{i} (N-i)^{n-S} \right] \cdot \prod_{i=0}^S (m - N - i)^i \left( \binom{S}{i} \right)^{(n-1)^{S-n^S}}.
\]

**Proof:** Let \( C \) be the set of all proper \( L \)-colorings of \( G \). Let \( T = \{(0, n)\} \cup \{(N, S) \in \mathbb{Z} \times \mathbb{Z} : 1 \leq N \leq n \text{ and } 0 \leq S \leq n - N \} \). For each \( (N, S) \in T \), let \( T_{(N, S)} \) be the set of proper \( L \)-colorings \( f \) of \( G \) such that \( |f(X) \cap [m - n]| = N \) and \( |f(X) \cap \bigcup_{k=1}^n S_k| = S \). Notice that \( P(G, L) = |C| = \sum_{(N, S) \in T} |T_{(N, S)}| \).

†While we really only need the generalization when \( n = 2 \), we prove the result for general \( n \) for completeness.
‡If \( a < b \), then we interpret \( \binom{a}{b} \) as being equal to zero.
Let $L'$ be the restriction of $L$ to $X$. We compute $|T_{(0,n)}|$ in two steps: first, we count the number of proper $L'$-colorings $h$ of $G[X]$ such that $h(X) \subseteq \bigcup_{k=1}^{n} S_k$. Then, given a proper $L'$-coloring $h$ of $G[X]$ such that $h(X) \subseteq \bigcup_{k=1}^{n} S_k$, we count the number of proper $L$-colorings $f$ of $G$ such that $f(v) = h(v)$ for each $v \in X$. We will find that the number obtained in the second step does not depend on $h$, so that $|T_{(0,n)}|$ equals the number obtained in the first step times the number obtained in the second step.

For step one, notice that $|S_k| = n$ for each $k \in [n]$. So, the number of proper $L'$-colorings $h$ of $G[X]$ such that $h(X) \subseteq \bigcup_{k=1}^{n} S_k$ is $\prod_{k=1}^{n} |S_k| = n^n$. For step two, suppose $h$ is a proper $L'$-coloring of $G[X]$ such that $h(X) \subseteq \bigcup_{k=1}^{n} S_k$. For each $k \in [n]$, suppose $h(v_k) = s_k$. For each integer $i$ with $0 \leq i \leq n$, let $R_i = \{j \in [n^{ni}] : |L(y_j) \cap \{s_k : k \in [n]\}| = i\}$. Notice that $R_0, ..., R_n$ form a partition of $[n^{ni}]$, and that $|R_i| = t \cdot |\{j \in [n^{n-1}] \cup \{0\} : |A_j \cap \{s_k : k \in [n]\}| = i\}|$ for each integer $i$ with $0 \leq i \leq n$. By the definition of $A$, it is easy to see $|R_i| = t \binom{n}{i} (n-1)^{n-i}$. Then, the number of proper $L'$-colorings $f$ of $G$ such that $f(v) = h(v)$ for each $v \in X$ is given by

$$\prod_{i=0}^{n} (m-i)^{|R_i|} = \prod_{i=0}^{n} (m-i)^{t \binom{n}{i} (n-1)^{n-i}}$$

from which we conclude

$$|T_{(0,n)}| = n^n \prod_{i=0}^{n} (m-i)^{t \binom{n}{i} (n-1)^{n-i}}.$$

We now compute $|T_{(N,S)}|$ for arbitrary $(N, S) \in T \setminus \{(0, n)\}$. To do so, we again employ a two-step process: first, we count the number of proper $L'$-colorings $h$ of $G[X]$ such that $|h(X) \cap [m-n]| = N$ and $h(X) \cap \bigcup_{k=1}^{n} S_k! = S$. Then, given a proper $L'$-coloring $h$ of $G[X]$ such that $|h(X) \cap [m-n]| = N$ and $|h(X) \cap \bigcup_{k=1}^{n} S_k!| = S$, we count the number of proper $L$-colorings $f$ of $G$ such that $f(v) = h(v)$ for each $v \in X$. Again, we will find that the number obtained in the second step does not depend on $h$, so that $|T_{(N,S)}|$ equals the number obtained in the first step times the number obtained in the second step.

For step one, we can generate all such proper $L'$-colorings $h$ of $G[X]$ via the following four-part process: first, choose a subset $P_k$ of $[n]$ of size $S$, and let $P_0 = [n] - P$. Second, choose a subset $O$ of $[m-n]$ of size $N$. Third, color the vertices in $\{x_k : k \in P_0\}$ with the colors in $O$ such that each color in $O$ is used at least once. Lastly, for each $k \in P$, color $x_k$ with a color in $S_k$. The first part can be done in $\binom{n}{S}$ ways. The second part can be done in $\binom{m-n}{N}$ ways. By some simple counting and the Inclusion–Exclusion Principle, the third part can be done in

$$\sum_{i=0}^{N} (-1)^i \binom{N}{i} (N-i)^{N-S} = \sum_{i=0}^{N-1} (-1)^i \binom{N}{i} (N-i)^{N-S}$$

ways. Finally, for each $k \in P$, there are $|S_k| = n$ ways to color $x_k$ with a color in $S_k$. Thus, the final part can be done in $n^n$ ways. Hence, the number of proper $L'$-colorings $h$ of $G[X]$ such that $|h(X) \cap [m-n]| = N$ and $h(X) \cap \bigcup_{k=1}^{n} S_k! = S$ is
\[
\binom{n}{S} \binom{m-n}{N} \left( \sum_{i=0}^{N-1} (-1)^i \binom{N}{i} (N-i)^{n-S} \right)n^S.
\]

For step two, suppose \( h \) is a proper \( L' \)-coloring of \( G[X] \) such that \( |h(X) \cap [m-n]| = N \) and \( |h(X) \cap \bigcup_{k=1}^n S_k| = S \). Let \( P \) = \( \{k \in [n] : h(x_k) \in [m-n]\} \) and \( P = \{k \in [n] : h(x_k) \in S_k\} \). Notice that \( P \) and \( P \) form a partition of \([n]\). Suppose \( h(X) \cap [m-n] = \{o_1, ..., o_n\}\). For each \( k \in P \), suppose \( h(x_k) = s_k \). For each \( i \in Z \) with \( 0 \leq i \leq |P| = S \), let \( R_i = \{j \in [n^P] : L(y_j) \cap \{s_k : k \in P_i\} = i\} \). Notice that \( R_0, ..., R_S \) form a partition of \([n^P]\), and that \( |R_i| = t \cdot |\{j \in [n^P - 1] \cup \{0\} : |A_j \cap \{s_k : k \in P_i\}| = i\} | \) for each \( i \in Z \) with \( 0 \leq i \leq S \). By the definition of \( A \), it is easy to see \( |R_i| = t(S)n^{S-i}n^{n-S} \). Then, the number of proper \( L \)-colorings \( f \) of \( G \) such that \( f(v) = h(v) \) for each \( v \in X \) is given by

\[
\prod_{i=0}^S (m - N - i)^{|R_i|} = \prod_{i=0}^S (m - N - i)^t(S)n^{S-i}n^{n-S},
\]

from which we conclude

\[
|\mathcal{T}(N,S)| = n^S \binom{n}{S} \binom{m-n}{N} \left( \sum_{i=0}^{N-1} (-1)^i \binom{N}{i} (N-i)^{n-S} \right)
\times \prod_{i=0}^S (m - N - i)^t(S)n^{S-i}n^{n-S}.
\]

The result follows. 

We can use Lemma 8 to find appropriate \( m, n, \) and \( t \) such that \( P(G, L) < P(G, m) \), which would imply \( \tau(G) > m \). Since the focus of Theorem 4 is \( n = 2 \), we will now slightly generalize the list assignment constructed in the statement of Lemma 8 in the case \( n = 2 \). The notion of “balanced” list assignment given below captures the essence of what makes this list assignment “bad” as well as nice to work with.

Suppose \( G = K_{2,l} \), the bipartition of \( G \) is \( \{x_1, x_2\}, \{y_1, ..., y_l\} \), and \( L \) is an \( m \)-assignment for \( G \) such that \( L(x_1) = \{m\} \) and \( L(x_2) = \{m-2\} \cup \{m+1, m+2\} \). Let \( z_i = \{|j \in [l] : L(y_j) = \{m-2\} \cup \{m-1, m+1\}\} \), \( z_2 = \{|j \in [l] : L(y_j) = \{m-2\} \cup \{m-1, m+1\}\} \), \( z_3 = \{|j \in [l] : L(y_j) = \{m-2\} \cup \{m+1, m+2\}\} \), and \( z_4 = \{|j \in [l] : L(y_j) = \{m-2\} \cup \{m+1, m+2\}\} \). Then, we say the list assignment \( L \) is balanced if \( \sum_{i=1}^4 z_i = l \) and \( |z_j - z_i| \leq 1 \) whenever \( i, j \in [4] \).

We will now use the formula and list assignment for \( G = K_{2,4l} \) in Lemma 8, to determine how large \( t \) must be to ensure the existence of a balanced \( m \)-assignment \( L \) for \( G \) that demonstrates \( P_t(G, m) < P(G, m) \).

**Lemma 9.** Suppose \( G = K_{2,4l} \) and \( m \geq 3 \). If

\[
t > \max \left\{ \frac{\ln(\varepsilon/(4(m-2)))}{2\ln((m-2)/(m-1))}, \frac{\ln((2-\varepsilon)/4)}{\ln(1 - 1/(m-1)^2)} \right\}
\]

then

\[
\binom{n}{S} \binom{m-n}{N} \left( \sum_{i=0}^{N-1} (-1)^i \binom{N}{i} (N-i)^{n-S} \right)n^S.
\]
for some real number $\epsilon$ with $0 < \epsilon < 2$, then there is a balanced $m$-assignment $L$ for $G$ such that $P(G, L) < P(G, m)$.

**Proof.** Suppose $G = K_{2,4t}$ and the bipartition of $G$ is $\{x_1, x_2\}$, $\{y_1, ..., y_{4t}\}$. Clearly, $P(G, m) = m(m - 1)^{4t} + m(m - 1)(m - 2)^{4t}$. If $L$ is the $m$-assignment for $G$ described in the statement of Lemma 8, then $L$ is a balanced $m$-assignment and

$$P(G, L) = (m - 2)(m - 1)^{4t} + (m - 3)(m - 2)^{4t+1} + 4(m - 2)^{2t+1}(m - 1)^{2t} + 4(m - 2)^t(m - 1)^{2t}m'$$

by Lemma 8. We will show $P(G, L) < P(G, m)$. Let $b = 1 + 1/(m - 1)$ and $s = 1 - 1/(m - 1)$. Notice

$$t > \frac{\ln(\epsilon/(4(m - 2)))}{2\ln((m - 2)/(m - 1))}$$

which implies $4(m - 2)s^{2t} < \epsilon$, as $2\ln((m - 2)/(m - 1)) < 0$.

Also

$$t > \frac{\ln((2 - \epsilon)/4)}{\ln(1 - 1/(m - 1)^2)}$$

which implies $4s^tb^t < 2 - \epsilon$.

Combining these inequalities yields

$$4(m - 2)s^{2t} + 4s^tb^t < 2$$

which implies $4(m - 2)s^{2t} + 4s^tb^t < 2 + (4m - 6)s^{4t}$.

Then,

$$4(m - 1)^{2t}(m - 2)^{2t+1} + 4m^t(m - 2)^t(m - 1)^{2t}$$

$$< 2(m - 1)^{4t} + m(m - 1)(m - 2)^{4t} - (m - 2)^{4t+1}(m - 3).$$

From which we obtain

$$(m - 2)(m - 1)^{4t} + (m - 2)^{4t+1}(m - 3) + 4(m - 1)^{2t}(m - 2)^{2t+1}$$

$$+ 4m^t(m - 2)^t(m - 1)^{2t}$$

$$< m(m - 1)^{4t} + m(m - 1)(m - 2)^{4t}$$

as desired. □

We now establish some notation that will be used for the remainder of the paper. Suppose $G = K_{2,l}$, the bipartition of $G$ is $\{x_1, x_2\}$, $\{y_1, ..., y_l\}$, and $L$ is an $m$-assignment for $G$. For each $(a_1, a_2) \in L(x_1) \times L(x_2)$, let $C_{(a_1, a_2)}$ be the set of proper $L$-colorings of $G$ in which $x_i$ is colored with $a_i$ for each $i \in [2]$. Notice $P(G, L) = \sum_{(a_1, a_2) \in L(x_1) \times L(x_2)} |C_{(a_1, a_2)}|$. 

Generally speaking, our strategy for proving Theorem 4 is inductive. We wish to show that if there is a balanced $m$-assignment $L$ for $G = K_{2,4t}$ that demonstrates $P{\ell}(G, m) < P(G, m)$ (Lemma 8 will be the key to proving such an $L$ exists) and $t$ is sufficiently large, then for any $l \geq 4t$ there is a balanced $m$-assignment $L'$ for $G' = K_{2,l}$ that demonstrates $P{\ell}(G', m) < P(G', m)$. The next two lemmas make the inductive idea precise.

**Lemma 10.** Suppose $G = K_{2,l}$ and $G' = K_{2,l+1}$. If $L$ is a balanced $m$-assignment for $G$ with $m \geq 3$, $P(G, L) < P(G, m)$, $\varepsilon \in (0, 2)$, and $l$ satisfies

$$\left| \frac{l}{4} \right| > \max \left\{ \frac{\ln(\varepsilon/(2(m-2)))}{2\ln((m-2)/(m-1))}, \frac{\ln((2-\varepsilon)/4)}{\ln(1-1/(m-1)^2)} \right\},$$

then there is a balanced $m$-assignment $L'$ for $G'$ such that $P(G', L') < P(G', m)$.

**Proof.** For simplicity, suppose the bipartitions of $G$ and $G'$ are $\{x_1, x_2\}, \{y_1, \ldots, y_l\}$ and $\{x_1, x_2\}, \{y_1, \ldots, y_{l+1}\}$, respectively. We know that $P(G, L) < P(G, m) = m(m-1)^{l} + m(m-1)(m-2)^{l}$. We also know that $P(G', m) = m(m-2)^{l+1} + m(m-1)(m-2)^{l+1}$. As such

$$P(G', m) - P(G, m) = m(m-1)^{l}(m-2) + m(m-1)(m-2)^{l}(m-3).$$

Let $c_{(i,j)} = |C_{(i,j)}|$, with regard to $G$. We know $P(G, L) = \sum_{(i,j) \in E(X,Y)} c_{(i,j)}$. Without loss of generality, assume $z_i \leq z_j$ for each $j \in [2, 3, 4]$. Let $L'$ be the $m$-assignment for $G'$ given by $L'(v) = L(v)$ if $v \in V(G)$ and $L'(y_{l+1}) = [m-2] \cup [m-1, m+1]$. Clearly, $L'$ is a balanced $m$-assignment for $G'$.

With some simple counting, we see that:

$$P(G', L') = (m - 1) \sum_{i=1}^{m-2} c_{(i,i)} + (m - 2) \sum_{(i,j) \in E \setminus E_{m-2}} c_{(i,j)} + (m - 2) \sum_{i=1}^{m-2} c_{(m-1,i)}$$

$$+ (m - 1) \sum_{i=1}^{m-2} c_{(m,i)} + (m - 2) \sum_{i=1}^{m-2} c_{(m,i+1)} + (m - 1) \sum_{i=1}^{m-2} c_{(i,m+2)}$$

$$+ (m - 2)c_{(m-1,m+1)} + (m - 1)[c_{(m-1,m+2)} + c_{(m,m+1)}] + mc_{(m,m+2)}$$

$$= (m - 2)P(G, L) + 2c_{(m,m+2)} + c_{(m-1,m+2)} + c_{(m,m+1)} + \sum_{i=1}^{m-2} c_{(i,m+2)}$$

$$+ \sum_{i=1}^{m-2} c_{(m,i)} + \sum_{i=1}^{m-2} c_{(i,i)}.$$

This implies $P(G', L') - P(G, L) = (m - 3)P(G, L) + J \leq (m - 3)[m(m-1)^{l} + m(m-1)(m-2)^{l}] + J$, where $J = P(G', L') - (m - 2)P(G, L)$. We will show if

$$\left| \frac{l}{4} \right| > \max \left\{ \frac{\ln(\varepsilon/(2(m-2)))}{2\ln((m-2)/(m-1))}, \frac{\ln((2-\varepsilon)/4)}{\ln(1-1/(m-1)^2)} \right\},$$

then there is a balanced $m$-assignment $L'$ for $G'$ such that $P(G', L') < P(G', m)$. The proof is complete.
then \( J < m(m-1)^l \). Notice, if \( J < m(m-1)^l \) we have \( P(G', L') - P(G, L) < P(G', m) - P(G, m) \) which implies \( P(G', L') < P(G', m) \). Since \( z_j \leq z_i \) where \( j \in [4] \), \( z_i = \lfloor l/4 \rfloor \). So,

\[
\frac{\ln(\varepsilon/(2(m - 2)))}{\ln((m - 2)/(m - 1))} \quad \text{which implies} \quad 2(m - 2)\left(\frac{m - 2}{m - 1}\right)^{z_l} < \varepsilon.
\]

Since \( 2z_i \leq z_2 + z_4, 2z_l \leq z_3 + z_4, \) and \( (m - 2)/(m - 1) < 1 \), we have

\[
(m - 2)\left(\frac{m - 2}{m - 1}\right)^{z_2 + z_4} + (m - 2)\left(\frac{m - 2}{m - 1}\right)^{z_3 + z_4} < \varepsilon. \tag{1}
\]

Similarly,

\[
\frac{\ln((2 - \varepsilon)/4)}{\ln(1 - 1/(m - 1)^2)} \quad \text{which implies} \quad 4\left(\frac{m - 2}{m - 1}\right)^{z_l} < 2 - \varepsilon.
\]

Now, let \( b = 1 + 1/(m - 1) \) and \( s = 1 - 1/(m - 1) \). The most recent inequality becomes \( 4(bs)^{z_l} < 2 - \varepsilon \), and since \( s < 1 \), \( 2(bs)^{z_l} \geq 2b^{z_2}s^{z_4} \). We will show that \( b^{z_2}s^{z_4} + b^{z_4}s^{z_2} \leq 2(bs)^{z_l} \). Notice that \( z_2 = z_3 \) or \( \max\{z_2, z_3\} = z_4 + 1 \) and \( \min\{z_2, z_3\} = z_4 \).

Assume \( z_2 = z_3 \). Since \( bs < 1 \), \( b^{z_2}s^{z_2} + b^{z_4}s^{z_4} = 2(bs)^{z_l} \leq 2(bs)^{z_l} \). Now, without loss of generality, assume \( z_2 = z_4 \) and \( z_3 = z_4 + 1 \). Then, \( b^{z_2}s^{z_2} + b^{z_4}s^{z_4} = (bs)^{z_l}(b + s) = 2(bs)^{z_l} \).

So, we have \( b^{z_2}s^{z_2} + b^{z_4}s^{z_4} \leq 2(bs)^{z_l} \). As a result, \( 2b^{z_2}s^{z_2} + b^{z_4}s^{z_4} + b^{z_4}s^{z_2} < 2 - \varepsilon \). This along with (1) implies

\[
2b^{z_2}s^{z_4} + b^{z_4}s^{z_2} + b^{z_4}s^{z_4} + (m - 2)s^{z_2 + z_4} + (m - 2)s^{z_3 + z_4} < 2.
\]

This implies

\[
2m^{z_l}(m - 1)^{z_2 + z_4}(m - 2)^{z_4} + m^{z_l}(m - 1)^{z_2 + z_4}(m - 2)^{z_4} + m^{z_l}(m - 1)^{z_2 + z_4}(m - 2)^{z_4}
\]
\(+(m - 2)(m - 1)^{z_2 + z_4}(m - 2)^{z_4} + (m - 2)(m - 1)^{z_2 + z_4}(m - 2)^{z_4} < 2(m - 1)^l \)

which implies

\[
2m^{z_l}(m - 1)^{z_2 + z_4}(m - 2)^{z_4} + m^{z_l}(m - 1)^{z_2 + z_4}(m - 2)^{z_4}
\]
\(+m^{z_l}(m - 1)^{z_2 + z_4}(m - 2)^{z_4} + (m - 2)(m - 1)^{z_2 + z_4}(m - 2)^{z_4}
\]
\(+(m - 2)(m - 1)^{z_2 + z_4}(m - 2)^{z_4} + (m - 2)(m - 1)^l < m(m - 1)^l. \)

Recall \( J = 2c_{(m,m+2)} + c_{(m-1,m+2)} + c_{(m,m+1)} + \sum_{i=1}^{m-2} c_{(i,m+2)} + \sum_{i=1}^{m-2} c_{(i,m)} + \sum_{i=1}^{m-2} c_{(i,i)}. \) Thus, \( J < m(m - 1)^l. \) \( \square \)

**Lemma 11.** Suppose \( G = K_{2,r} \), and there is a balanced \( m \)-assignment \( L \) for \( G \) with \( m \geq 3 \) such that \( P(G, L) < P(G, m) \). Suppose there is an \( \varepsilon \in (0, 2) \) such that \( r \) satisfies
Then, for each \( l \geq r \), if \( G' = K_{2,l} \), there is a balanced \( m \)-assignment \( L' \) for \( G' \) such that \( P(G', L') < P(G', m) \). Consequently, \( P_e(K_{2,l}, m) < P(K_{2,l}, m) \) (i.e., \( \tau(K_{2,l}) > m \)) whenever \( l \geq r \).

**Proof.** The proof is by induction on \( l \). When \( l = r \) the desired statement is true since \( G' = K_{2,r} \). Suppose \( l > r \) and the desired statement holds for all natural numbers greater than \( r - 1 \) and less than \( l \). Since \( l - 1 \geq r \), there is a balanced \( m \)-assignment, \( L \), for \( H = K_{2,l-1} \) such that \( P(H, L) < P(H, m) \). Since there is an \( \epsilon \in (0, 2) \) such that

\[
\begin{align*}
\left| \frac{l - 1}{4} \right| &> \frac{r}{4} > \max \left\{ \frac{\ln(\epsilon/(2(m - 2)))}{2 \ln((m - 2)/(m - 1))}, \frac{\ln((2 - \epsilon)/4)}{\ln(1 - 1/(m - 1)^2)} \right\},
\end{align*}
\]

Lemma 10 implies there is a balanced \( m \)-assignment \( L' \) for \( G' = K_{2,l} \) such that \( P(G', L') < P(G', m) \).

The next lemma follows immediately from Lemmas 11 and 9.

**Lemma 12.** If \( t \in \mathbb{N}, m \geq 3 \), and

\[
\begin{align*}
t &> \max \left\{ \frac{\ln(\epsilon/(4(m - 2)))}{2 \ln((m - 2)/(m - 1))}, \frac{\ln((2 - \epsilon)/4)}{\ln(1 - 1/(m - 1)^2)} \right\},
\end{align*}
\]

for some \( \epsilon \in (0, 2) \), then \( \tau(K_{2,l}) > m \) whenever \( l \geq 4t \).

Now, we are ready to prove Theorem 4.

**Proof.** With the intent of using Lemma 12, we will show

\[
q > \max \left\{ \frac{\ln(\epsilon/(4(m - 2)))}{2 \ln((m - 2)/(m - 1))}, \frac{\ln((2 - \epsilon)/4)}{\ln(1 - 1/(m - 1)^2)} \right\}
\]

when \( \epsilon = 1/4 \), and \( m = \lfloor (q/\ln(16/7))^{1/2} + 1 \rfloor \). Notice \( l \geq 16 \) implies \( q \geq 4 \) and \( m \geq 3 \). Clearly, \( q \geq (m - 1)^2 \ln(16/7) \).

Let \( f : (2, \infty) \to \mathbb{R} \) and \( g : (2, \infty) \to \mathbb{R} \) be given by \( f(x) = (x - 1)\ln(16/7) \) and \( g(x) = (1/2)\ln(16(x - 2)) \). Suppose \( h : (2, \infty) \to \mathbb{R} \) is given by \( h(x) = f(x) - g(x) \), we have \( h'(x) = \ln(16/7) - 1/(2(x - 2)) \). Notice \( h'(x) > 0 \) when \( x \geq 3 \), and \( h(3) = 2\ln(16/7) - \ln(16)/2 > 0 \). So, \( f(x) > g(x) \) when \( x \geq 3 \).

Now, since \( q \geq (m - 1)f(m) \), \( q \geq (m - 1)^2 \ln(16/7) > ((m - 1)/2)\ln(16(m - 2)) \). Let \( \epsilon = 1/4 \). Using the fact that \( \ln(1 + x) < x \) when \( x \neq 0 \) and \( x > -1 \), we have

\[
(m - 1)^2 \ln(16/7) = \frac{\ln(16/7)}{-1/(m - 1)^2} > \frac{\ln((2 - \epsilon)/4)}{\ln(1 - 1/(m - 1)^2)}
\]

and
\[
(m - 1) \frac{\ln(16(m - 2))}{2} = \frac{\ln(1/(16(m - 2)))}{-2/(m - 1)} > \frac{\ln(\epsilon/(4(m - 2)))}{2\ln((m - 2)/(m - 1))}.
\]

Since \( q = \lfloor l/4 \rfloor, l \geq 4q \). So Lemma 12 implies

\[
\tau(K_{2,1}) > \left[ \frac{q}{\ln(16/7)} \right]^{1/2} + 1.
\]

\\

ACKNOWLEDGMENTS

This paper is based on a research project conducted with undergraduate students Akash Kumar, Patrick Rewers, Paul Shin, and Khue To at the College of Lake County during the summer and fall of 2021. The support of the College of Lake County is gratefully acknowledged. The authors also thank Dan Cranston and Seth Thomason for helpful conversations.

DATA AVAILABILITY STATEMENT

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

ORCID

Hemanshu Kaul https://orcid.org/0000-0002-6691-0176

REFERENCES

1. J. Becker, J. Hewitt, H. Kaul, M. Maxfield, J. Mudrock, D. Spivey, S. Thomason, T. Wagstrom, The DP color function of joins and vertex-gluing of graphs, Discrete Math. 345 (2022), Article 113093.
2. N. Biggs, Algebraic graph theory, Cambridge University Press, New York, NY, 1994.
3. G. D. Birkhoff, A determinant formula for the number of ways of coloring a map, Annals Math. 14 (1912), 42–46.
4. M. Bui, H. Kaul, M. Maxfield, J. Mudrock, P. Shin, S. Thomason, Non-chromatic-adherence of the DP color function via generalized theta graphs, Graphs Combin. 39 (2023), article 42.
5. F. Dong, K. M. Koh, and K. L. Teo, Chromatic polynomials and chromaticity of graphs, World Scientific, Singapore, 2005.
6. Q. Donner, On the number of list-colorings, J. Graph Theory. 16 (1992), 239–245.
7. Z. Dvořák and L. Postle, Correspondence coloring and its application to list-coloring planar graphs without cycles of lengths 4 to 8, J. Combin. Theory Series B. 129 (2018), 38–54.
8. P. Erdős, A. L. Rubin, and H. Taylor, Choosability in graphs, Congressus Num. 26 (1979), 125–127.
9. H. Kaul and J. Mudrock, Criticality, the list color function, and list coloring the Cartesian product of graphs, J. Combin. 12 (2021), no. 3, 479–514.
10. H. Kaul and J. Mudrock, On the chromatic polynomial and counting DP-colorings of graphs, Adv. Appl. Math. 123 (2021), Article 103121.
11. R. Kirov and R. Naimi, List coloring and \( n \)-monophilic graphs, Ars Combin. 124 (2016), 329–340.
12. A. V. Kostochka, and A. Sidorenko, Problem session of the prachatice conference on graph theory, Fourth Czechoslovak Symposium on Combinatorics, Graphs and Complexity, Ann. Discrete Math. 51 (1992), 380.
13. J. Mudrock, On the list coloring problem and its equitable variants, Ph.D. Thesis, Illinois Institute of Technology, 2018.
14. C. Thomassen, The chromatic polynomial and list colorings, J. Combin. Theory Series B. 99 (2009), 474–479.
15. V. G. Vizing, *Coloring the vertices of a graph in prescribed colors*, Diskret. Analiz. Metody Diskret. Anal. v Teorii Kodovi Skhem. 101 (1976), no. 29, 3–10.

16. W. Wang, J. Qian, and Z. Yan, *When does the list-coloring function of a graph equal its chromatic polynomial*, J. Combin. Theory Series B. 122 (2017), 543–549.

17. D. B. West, *Introduction to Graph Theory*, Prentice Hall, Upper Saddle River, NJ, 2001.

How to cite this article: H. Kaul, A. Kumar, J. A. Mudrock, P. Rewers, P. Shin and K. To, *On the list color function threshold*, J. Graph Theory. 2024;105:386–397.  
https://doi.org/10.1002/jgt.23045