TORSORS OF ISOTROPIC LOOP REDUCTIVE GROUPS OVER LAURENT POLYNOMIALS

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Abstract. Let $k$ be a field of characteristic 0. Let $G$ be a reductive group over the ring of Laurent polynomials $R = k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. Assume that $G$ is loop reductive, that is, $G$ contains a maximal $R$-torus, and that every semisimple normal subgroup of $G$ contains $G_m, R$. We show that the natural map $H_1^\text{ét}(R, G) \to H_1^\text{ét}(k(x_1, \ldots, x_n), G)$ has trivial kernel. This settles in positive the conjecture of V. Chernousov, Ph. Gille, and A. Pianzola that $H_1^\text{Zar}(R, G)$ is trivial.

1. Introduction

Let $R$ be a commutative ring. Let $G$ be a reductive group scheme over $R$ in the sense of [SGA3]. We say that $G$ has isotropic rank $\geq n$ if every normal semisimple reductive $R$-subgroup of $G$ contains $(G_m, R)^n$.

V. Chernousov, Ph. Gille, and A. Pianzola proposed the following conjecture.

Conjecture. [ChGP17, Conjecture 5.4] Let $k$ be a field of characteristic 0. Let $G$ be a loop reductive group over the ring of Laurent polynomials $R = k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. Assume that $G$ has isotropic rank $\geq 1$. Then $H_1^\text{Zar}(R, G)$ is trivial.

We prove this conjecture. Previously, this statement was known to hold if $G$ is defined over $k$ [GP2], if $k$ is algebraically closed, $n = 2$ and $G$ is simply connected [SZ12]; for some twisted forms of $GL_n$ [Art95] and of orthogonal groups [Par83].

The proof relies on the “diagonal argument” trick for loop reductive groups [St16] and on the established cases of the Serre–Grothendieck conjecture [PSV15, FP15].

2. Preliminaries on loop reductive groups

Let $k$ be a field of characteristic 0. We fix once and for all an algebraic closure $\bar{k}$ of $k$ and a compatible set of primitive $m$-th roots of unity $\xi_m \in \bar{k}$, $m \geq 1$.

P. Gille and A. Pianzola [GP3, Ch. 2, 2.3] compute the étale (or algebraic) fundamental group of the $k$-scheme

$$X = \text{Spec } k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$$

at the natural geometric point $e : \text{Spec } \bar{k} \to X$ induced by the evaluation $x_1 = x_2 = \ldots = x_n = 1$. Namely, let $k_\lambda, \lambda \in \Lambda$ be the set of finite Galois extensions of $k$ contained in $\bar{k}$.

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Let $I$ be the subset of $\Lambda \times \mathbb{Z}_{>0}$ consisting of all pairs $(\lambda, m)$ such that $\xi_m \in k_\lambda$. The set $I$ is directed by the relation $(\lambda, m) \leq (\mu, k)$ if and only if $k_\lambda \subseteq k_\mu$ and $m|k$. Consider

$$X_{\lambda,m} = \text{Spec} \ k_\lambda[x_1^{\pm \frac{1}{m}}, \ldots, x_n^{\pm \frac{1}{m}}]$$

as a scheme over $X$ via the natural inclusion of rings. Then $X_{\lambda,m} \to X$ is a Galois cover with the Galois group

$$\Gamma_{\lambda,m} = (\mathbb{Z}/m\mathbb{Z})^n \rtimes \text{Gal}(k_\lambda/k),$$

where $\text{Gal}(k_\lambda/k)$ acts on $k_\lambda[x_1^{\pm \frac{1}{m}}, \ldots, x_n^{\pm \frac{1}{m}}]$ via its canonical action on $k_\lambda$, and each $(\bar{k}_1, \ldots, \bar{k}_n) \in (\mathbb{Z}/m\mathbb{Z})^n$ sends $x_i^{1/m}$ to $\xi_{\bar{k}_m}^k x_i^{1/m}$, $1 \leq i \leq n$. The semi-direct product structure on $\Gamma_{\lambda,m}$ is induced by the natural action of $\text{Gal}(k_\lambda/k)$ on $\mu_m(k_\lambda) \cong \mathbb{Z}/m\mathbb{Z}$. We have

$$(2.1) \quad \pi_1(X,e) = \varprojlim_{(\lambda,m) \in I} \Gamma_{\lambda,m} = \hat{\mathbb{Z}}(1)^n \rtimes \text{Gal}(k),$$

where $\hat{\mathbb{Z}}(1)$ denotes the profinite group $\varprojlim_m \mu_m(\bar{k})$ equipped with the natural action of the absolute Galois group $\text{Gal}(k) = \text{Gal}(\bar{k}/k)$.

For any reductive group scheme $G$ over $X$, we denote by $G_0$ the split, or Chevalley–Demazure reductive group in the sense of [SGA3] of the same type as $G$. The group $G$ is a twisted form of $G_0$, corresponding to a cocycle class $\xi$ in the étale cohomology set $H^1_{\text{ét}}(X, \text{Aut}(G_0))$.

**Definition 2.1.** [GP3] Definition 3.4] The group scheme $G$ is called loop reductive, if the cocycle $\xi$ is in the image of the natural map

$$H^1(\pi_1(X,e), \text{Aut}(G_0)(\bar{k})) \to H^1_{\text{ét}}(X, \text{Aut}(G_0)).$$

Here $H^1(\pi_1(X,e), \text{Aut}(G_0)(\bar{k}))$ stands for the non-abelian cohomology set in the sense of Serre [Se]. The group $\pi_1(X,e)$ acts continuously on $\text{Aut}(G_0)(\bar{k})$ via the natural homomorphism $\pi_1(X,e) \to \text{Gal}(\bar{k}/k)$.

This definition can be reformulated as follows.

**Theorem.** [GP3] Corollary 6.3] A reductive group scheme over $X$ is loop reductive if and only if $G$ has a maximal torus over $X$.

The definition of a maximal torus is as follows.

**Definition 2.2.** [SGA3] Exp. XII Déf. 3.1] Let $G$ be a group scheme of finite type over a scheme $S$, and let $T$ be a $S$-torus which is an $S$-subgroup scheme of $G$. Then $T$ is a maximal torus of $G$ over $S$, if $T_{k(s)}^{\gamma(s)}$ is a maximal torus of $G_{k(s)}^{\gamma(s)}$ for all $s \in S$.

Our main result is based on the following observation.

**Lemma 2.3 ("diagonal argument").** [St16] Lemma 4.1] Let $k$ be a field of characteristic 0. Let $G$ be a loop reductive group over $R = k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. For any integer $d > 0$, denote by $f_{z,d}$ (respectively, $f_{w,d}$) the composition of $k$-homomorphisms

$$R \to k[z_1^{\pm 1}, \ldots, z_n^{\pm 1}, w_1^{\pm 1}, \ldots, w_n^{\pm 1}] \to k[z_1^{\pm 1}, \ldots, z_n^{\pm 1}, (z_1 w_1^{-1})^{\pm \frac{1}{d}}, \ldots, (z_n w_n^{-1})^{\pm \frac{1}{d}}]$$

sending $x_i$ to $z_i$ (respectively, to $w_i$) for every $1 \leq i \leq n$. Then there is $d > 0$ such that

$$f_{z,d}(G) \cong f_{w,d}(G).$$
as group schemes over \( k[z_1^{\pm 1}, \ldots, z_n^{\pm 1}, (z_1w_i^{-1})^{\pm 1/d}, \ldots, (z_nw_i^{-1})^{\pm 1/d}] \).

We introduce additional notation that will be used every time when we apply Lemma 2.3 in proofs of other statements.

**Notation 2.4.** In the setting of the claim of Lemma 2.3, set
\[
t_i = (z_iw_i^{-1})^{1/d}, \quad 1 \leq i \leq n,
\]
where \( z_i, w_i, \) and \( d \) are as in that lemma. Note that this is equivalent to
\[
z_i = w_it_i^d, \quad 1 \leq i \leq n.
\]
We denote by \( G_z \) the group scheme over \( k[z_1^{\pm 1}, \ldots, z_n^{\pm 1}] \) which is the pull-back of \( G \) under the \( k \)-isomorphism
\[
k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \overset{\phi_i \otimes z_i}{\longrightarrow} k[z_1^{\pm 1}, \ldots, z_n^{\pm 1}].
\]
The group scheme \( G_w \) over \( k[w_1^{\pm 1}, \ldots, w_n^{\pm 1}] \) is defined analogously. Note that \( G_z \) and \( G_w \) are isomorphic after pull-back to
\[
k[z_1^{\pm 1}, \ldots, z_n^{\pm 1}, t_1^{\pm 1}, \ldots, t_n^{\pm 1}] = k[w_1^{\pm 1}, \ldots, w_n^{\pm 1}, t_1^{\pm 1}, \ldots, t_n^{\pm 1}].
\]

3. Proof of the conjecture

The following statement was obtained in [St19] as a joint corollary of the corresponding statement for simply connected semisimple reductive groups [PSV15 Theorem 1.6], and of the result of I. Panin and R. Fedorov on the Serre–Grothendieck conjecture [FP15].

**Theorem.** [St19] Theorem 4.2] Assume that \( R \) is a regular semilocal domain that contains an infinite field, and let \( K \) be its fraction field. Let \( G \) be a reductive group scheme over \( R \) of isotropic rank \( \geq 1 \). Then for any \( n \geq 1 \) the natural map
\[
H^1_{\text{et}}(R[x_1, \ldots, x_n], G) \to H^1_{\text{et}}(K[x_1, \ldots, x_n], G)
\]
has trivial kernel.

**Lemma 3.1.** Let \( G \) be a reductive group of isotropic rank \( \geq 1 \) over a regular local ring \( A \) containing an infinite field \( k \). Let \( f(x) \in A[x] \) be a non-zero polynomial. Then \( H^1_{\text{et}}(\mathbb{A}^1_A, G) \to H^1_{\text{et}}((\mathbb{A}^1_A)_f, G) \) has trivial kernel.

**Proof.** Let \( K \) be the fraction field of \( A \). By [St19] Theorem 4.2] the map \( H^1_{\text{et}}(A[x], G) \to H^1_{\text{et}}(K[x], G) \) has trivial kernel. By [CTO] Proposition 2.2] the map \( H^1_{\text{et}}(K[x], G) \to H^1_{\text{et}}(K(x), G) \) has trivial kernel. Hence the claim. \( \square \)

The following lemma is based on a classical trick of Quillen [Q].

**Lemma 3.2.** Let \( G \) be a reductive group of isotropic rank \( \geq 1 \) over a regular ring \( A \) containing an infinite field \( k \). Let \( f(x) \in A[x] \) be a monic polynomial. Then \( H^1_{\text{et}}(\mathbb{A}^1_A, G) \to H^1_{\text{et}}((\mathbb{A}^1_A)_f, G) \) has trivial kernel.

**Proof.** Let \( \xi \in H^1_{\text{et}}(\mathbb{A}^1_A, G) \) be in the kernel. Since \( f \) is monic, for any maximal ideal \( m \) of \( A \) the image of \( f \) in \( A_m[x] \) is non-zero. Then by Lemma 3.1 the \( G \)-bundle \( \xi|_{\mathbb{A}^1_A} \) is trivial. Since \( A \) is regular, \( G \) is \( A \)-linear by [Tho87] Corollary 3.2]. Then by [AHW18] Theorem 3.2.5] (see also [Mos] Korollar 3.5.2]) the fact that for any maximal ideal \( m \) of \( A \) the \( G \)-bundle \( \xi|_{\mathbb{A}^1_A} \) is trivial implies that \( \xi \) is extended from \( A \).
Set \( y = x^{-1} \) and choose \( g(y) \in A[y] \) so that \( x^{\deg(f)}g(y) = f(x) \). Then \( g(0) \in A^\times \) and \( A[x]_{xf} = A[y]_{gy} \). We have \( \mathbb{P}^1_A = \mathbb{A}^1_A \cup \text{Spec}(A[y]_g) \), and \( \mathbb{A}^1_A \cap \text{Spec}(A[y]_g) = (\mathbb{A}^1_A)_{xf} \). Hence we can extend \( \xi \) to a bundle \( \xi \) on \( \mathbb{P}^1_A \) by patching it with the trivial bundle on \( \text{Spec}(A[y]_g) \).

Let \( \eta = \xi|_{\text{Spec}(A[y])} \). By assumption, \( \eta \) is trivial on \( \text{Spec}(A[y]_g) \). Since \( g(0) \in A^\times \), by the same argument as above \( \eta \) is extended. However, \( g(0) \) is invertible and \( \eta \) is trivial at \( y = 0 \), hence \( \eta \) is trivial. Hence \( \xi \) is trivial at \( x = y = 1 \). Hence \( \xi \) is trivial. □

**Lemma 3.3.** Let \( G \) be a reductive group of isotropic rank \( \geq 1 \) over a regular ring \( A \) containing an infinite field \( k \). Let \( f(x) \in A[x] \) be a monic polynomial such that \( f(0) \in A^\times \). Then \( H^1_{\text{ét}}((\mathbb{A}^1_A)_x, G) \rightarrow H^1_{\text{ét}}((\mathbb{A}^1_A)_{xf}, G) \) has trivial kernel.

**Proof.** Since \( f(0) \in A^\times \), any \( G \)-bundle in the kernel can be extended to \( \mathbb{A}^1_A \) by patching with a trivial \( G \)-bundle on \( (\mathbb{A}^1_A)_f \). Then it is trivial by Lemma 3.2 applied to \( xf \). □

**Lemma 3.4.** Under the assumptions of Lemma 3.3 for any \( n \geq 0 \) the natural map

\[
H^1_{\text{ét}}(A[t_1^{\pm 1}, \ldots, t_n^{\pm 1}], G) \rightarrow H^1_{\text{ét}}(A \otimes_k k(t_1, \ldots, t_n), G)
\]

has trivial kernel.

**Proof.** We prove the claim by induction on \( n \); the case \( n = 0 \) is trivial. Set \( l = k(t_1, \ldots, t_{n-1}) \).

By the inductive hypothesis, the map

\[
H^1_{\text{ét}}(A[t_1^{\pm 1}, \ldots, t_n^{\pm 1}], G) \rightarrow H^1_{\text{ét}}(A[t_n^{\pm 1}] \otimes_k l, G) = H^1_{\text{ét}}(A \otimes_k l[t_n^{\pm 1}], G)
\]

has trivial kernel, so it remains to prove the triviality of the kernel for the map

\[
H^1_{\text{ét}}(A \otimes_k l[t_n^{\pm 1}], G) \rightarrow H^1_{\text{ét}}(A \otimes_k l(t_n), G).
\]

We have \( l(t_n) = \lim_{g \rightarrow l} |l(t_n)|_{t_n,g} \), where \( g \in l(t_n) \) runs over all monic polynomials with \( g(0) \in l^\times \).

Since \( H^1_{\text{ét}}(-, G) \) commutes with filtered direct limits, it remains to show that every map

\[
(1) \quad H^1_{\text{ét}}(A \otimes_k l[t_n^{\pm 1}], G) \rightarrow H^1_{\text{ét}}(A \otimes_k l[t_n]_{t_n,g}, G)
\]

has trivial kernel. This is the claim of Lemma 3.3. □

**Lemma 3.5.** Let \( k \) be an infinite field, \( A \) be a regular ring containing \( k \), and let \( G \) be a reductive group of isotropic rank \( \geq 1 \) over \( A[z_1^{\pm 1}, \ldots, z_n^{\pm 1}] \). For any set of integers \( d_i > 0 \), \( 1 \leq i \leq n \), the map

\[
\psi: H^1_{\text{ét}}(A[z_1^{\pm 1}, \ldots, z_n^{\pm 1}, t_1, \ldots, t_n], G) \xrightarrow{z_i \mapsto w_i t_1^{d_i}} H^1_{\text{ét}}(A \otimes_k k(w_1, \ldots, w_n)[t_1^{\pm 1}, \ldots, t_n^{\pm 1}], \psi^*(G))
\]

has trivial kernel.

**Proof.** We prove the claim by induction on \( n \geq 0 \). The case \( n = 0 \) is trivial. To prove the induction step for \( n \geq 1 \), it is enough to show that

\[
\phi: H^1_{\text{ét}}(A \otimes_k k[z_1^{\pm 1}, \ldots, z_n^{\pm 1}, t_1, \ldots, t_n], G) \xrightarrow{z_1 \mapsto w_1 t_1^{d_1}} H^1_{\text{ét}}(A \otimes_k k(w_1)[t_1^{\pm 1}][z_2^{\pm 1}, \ldots, z_n^{\pm 1}, t_2, \ldots, t_n], \phi^*(G))
\]

has trivial kernel. Indeed, after that we can apply the induction assumption with \( k \) substituted by \( k(w_1) \) and \( A \) substituted by \( A \otimes_k k(w_1)[t_1^{\pm 1}] \). Set

\[
B = A[z_2^{\pm 1}, \ldots, z_n^{\pm 1}, t_2, \ldots, t_n]
\]
and omit for simplicity the subscript 1. Then we need to show that the map
\[
\phi : H^1_{\text{ét}}(B[z^{\pm 1}, t], G) \xrightarrow{z \mapsto wt^d} H^1_{\text{ét}}(B \otimes_k k(w)[t^{\pm 1}], \phi^*(G))
\]
has trivial kernel. Here \( G \) is defined over \( B[z^{\pm 1}] \). We have
\[
B \otimes_k k(w)[t^{\pm 1}] = \lim_{g} B \otimes_k k[w^{\pm 1}][t^{\pm 1}] = \lim_{g} B \otimes_k k[w^{\pm 1}, t^{\pm 1}],
\]
where \( g = g(w) \) runs over all monic polynomials in \( k[w] \) with \( g(0) \neq 0 \). Let \( N = \deg(g) \geq 1 \).
Since \( \phi(z) = wt^d \), we have \( g(w) = g(\phi(z)t^{-d}) = t^{-Nd} f(t) \), where \( f(t) \) is a polynomial in \( t \) with coefficients in \( k[\phi(z)^{\pm 1}] \) such that its leading coefficient is in \( k \setminus 0 \), and \( f(0) = \phi(z)^N \). Then
\[
B \otimes_k k[w^{\pm 1}, t^{\pm 1}]_g = B \otimes_k k[\phi(z)^{\pm 1}, t]_{tf}.
\]
The group scheme \( \phi^*(G) \) is defined over \( B \otimes_k k[\phi(z)^{\pm 1}] \). Both terminal coefficients of \( tf(t) \) are invertible in \( k[\phi(z)^{\pm 1}] \), hence by Lemma 3.2 the map
\[
H^1_{\text{ét}}(B[z^{\pm 1}, t], G) \xrightarrow{z \mapsto wt^d} H^1_{\text{ét}}(B \otimes_k k[w^{\pm 1}, t^{\pm 1}]_g, \phi^*(G)) = H^1_{\text{ét}}(B \otimes_k k[\phi(z)^{\pm 1}, t]_{tf}, \phi^*(G))
\]
has trivial kernel.
Since \( H^1_{\text{ét}}(-, G) \) commutes with filtered direct limits, we conclude that \( \phi \) has trivial kernel.

**Theorem 3.6.** Let \( k \) be a field of characteristic 0, and let \( G \) be a loop reductive group of isotropic rank \( \geq 1 \) over \( R = k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \). For any regular ring \( A \) containing \( k \), the natural map
\[
H^1_{\text{ét}}(k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \otimes_k A, G) \to H^1_{\text{ét}}(k(x_1, \ldots, x_n) \otimes_k A, G)
\]
has trivial kernel.

**Proof.** We apply Lemma 2.3 to \( G \), and we use Notation 2.4. Consider the following commutative diagram. In this diagram, the horizontal maps \( j_1 \) and \( j_2 \) are the natural ones, and all maps always take variables \( t_i \) to \( t_i \), \( 1 \leq i \leq n \), and \( A \) to \( A \). The bijections \( g_1 \) and \( g_2 \) exist by Lemma 2.3

\[
\begin{array}{ccc}
H^1_{\text{ét}}(k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \otimes_k A, G) & \xrightarrow{j_1} & H^1_{\text{ét}}(k(x_1, \ldots, x_n) \otimes_k A, G) \\
\downarrow f_1: x_i \mapsto z_i & & \downarrow f_2: x_i \mapsto z_i \\
H^1_{\text{ét}}(k[z_1^{\pm 1}, \ldots, z_n^{\pm 1}, t_1, \ldots, t_n] \otimes_k A, G_z) & \xrightarrow{h: z_i \mapsto wt_i^d} & H^1_{\text{ét}}(k(z_1, \ldots, z_n, t_1, \ldots, t_n) \otimes_k A, G_z) \\
\downarrow g_1 & & \cong \downarrow g_2: z_i \mapsto wt_i^d \\
H^1_{\text{ét}}(k(w_1, \ldots, w_n)[t_1^{\pm 1}, \ldots, t_n^{\pm 1}] \otimes_k A, G_z) & \xrightarrow{j_2} & H^1_{\text{ét}}(k(w_1, \ldots, w_n, t_1, \ldots, t_n) \otimes_k A, G_w)
\end{array}
\]

In order to prove that \( j_1 \) has trivial kernel, it is enough to show that all maps \( j_2, g_1, h, f_1 \) have trivial kernels. The map \( j_2 \) has trivial kernel by Lemma 3.4. As explained above, \( g_1 \)
is bijective. The map $h$ is has trivial kernel by Lemma 3.5. Finally, the map $f_1$ has trivial kernel, since it has a retraction. Therefore, the map $j_1$ has trivial kernel. □

Corollary 3.7. Let $k$ be a field of characteristic 0, and let $G$ be a loop reductive group of isotropic rank $\geq 1$ over $R = k[\frac{x_1}{1}, \ldots, \frac{x_n}{1}]$. Then $H^1_{\text{Zar}}(R,G) = H^1_{\text{Nis}}(R,G)$ is trivial.

Proof. This is clear, since $H^1_{\text{Zar}}(k(x_1, \ldots, x_n), G) = H^1_{\text{Nis}}(k(x_1, \ldots, x_n), G)$ is trivial. □

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