Time inversion, Self-similar evolution and Issue of time

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Abstract

We investigate the question, ”how does time flow?” and show that time may change by inversions as well. We discuss its implications to a simple class of linear systems. Instead of introducing any unphysical behaviour, inversions can lead to a new multi-time scale evolutionary path for the linear system exhibiting late time stochastic fluctuations. We explain how stochastic behaviour is injected into the linear system as a combined effect of an uncertainty in the definition of inversion and the irrationality of the golden mean number. We also give an ansatz for the nonlinear stochastic behaviour of (fractal) time which facilitates us to estimate the late and short time limits of a two-time correlation function relevant for the stochastic fluctuations in linear systems. These fluctuations are shown to enjoy generic $1/f$ spectrum. The implicit functional definition of the fractal time is shown to satisfy the differential equation $dx = dt$. We also discuss the relevance of intrinsic time in the present formalism, study of which is motivated by the issue of time in quantum gravity.

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1 Introduction

Time is one of the most enigmatic concepts in physics. Two of the most puzzling issues in the present day theoretical physics: i) the issue of time in quantum gravity and ii) the ubiquitous presence of $1/f$ spectra in diverse natural phenomena, though seemingly unrelated, are actually related to time. As it is well known, the timelessness of Wheeler-Dewitt equation, the quantal evolution equation of a closed gravity-matter system [1]. One of the key problems here is to give a realization of an intrinsic (internal) time, which will allow a self-consistent description of the intrinsic relative changes of the interacting degrees of freedom, although the dynamics of the total system appears ‘frozen’ from the point of view of the external time [2]. The issue of time is thus a mystery related to the quantal aspect of a closed system in the small time limit. The origin of $1/f$ spectrum, on the other hand, relates to self-similar fluctuations in a nonlinear system over very long time scales [3]. Although, much work have been done in this area over the last two decades [4], a re-examination of the generic $1/f$ spectrum in dynamical systems throwing new light into the problem would always be welcome. Even as the two issues are treated separately in the literature, we show, in the following, that these problems might have a common origin in an extended dynamical framework incorporating time inversion as yet another physical mode of time increment. The study will not only provide a general explanation of the generic $1/f$ spectrum, but also lead to fundamentally new insights into the structure of time. To explain our basic premise (framework) we treat here only a simple class of evolutionary equation in classical mechanics. Applications to a genuine quantum gravity model would be considered separately. As will become clear, some new insights into the issue of intrinsic time would be gained even at this level of our presentation.

To motivate the extension of the ordinary dynamical framework, we begin by posing the following two closely related questions: i) How does time flow? ii) Can a linear system support self-similar structures (fluctuations)? Obviously, answer to the second question is ‘no’, in the ordinary dynamics. Time (for that matter, any physical quantity (real variable)) is supposed to change (apart from trivial rescalings) by pure translation only. In the present paper, we however, advocate that time may change even by inversions, leading, in particular, to self-similar evolutions (fluctuations) even for a linear system. The definition of time inversion has an inherent uncertainty, injecting stochastic behaviour in the late time evolution of the system. In the short time limit, on the other hand, this stochasticity could be exploited to derive a timeless Wheeler-Dewitt-like equation for the purely fluctuating component of the system variable. Although a self-consistent realization of intrinsic time is available naturally in the present simple model, its extension to a quantum gravity model may have to surpass nontrivial problems. However, we believe that the present analysis would surely be of much utility in future studies of quantization problems of the gravitational fields.

It is worth pointing out here the relevance of the present work with recent studies. The possible presence of self-similar fluctuations/ dynamics in a linear system has been reported [5-7] recently. Bramwell et al [5] showed that a linear spin-wave theory of a critical ferromagnet could describe the fluctuation statistics of a closed turbulence experiment. Subsequently, a non-gaussian probability density function is derived [6], which turns out to represent an ‘effective universality class’ for a large number of strongly correlated systems. A common feature of these dissimilar systems is the presence of self- similar fluctuations over many length scales leading to fractal properties. It is also noted that the explicit nonlinearity may not have an essential role in generating these non-gaussian fluctuations. In ref.[7], on the otherhand, we pointed out the
presente of self- similar fluctuations over all time scales in a time dependent quantal evolution. As a consequence, the late time behaviour of the evolving state would exhibit non-gaussian fractal features described by a fractal (uncertainty) exponent $\nu = (\sqrt{5} - 1)/2$, the golden mean. The analysis makes use of an intrinsic sense of time [2], which is introduced in the evolution exploiting the emergent nonadiabatic geometric phase of the evolving state. As remarked above, the intrinsic time is meaningful in a closed system when the sense of time is defined internally from the relative changes of the interacting degrees of freedom (observables) of the system [2], in contrast to the externally defined Newtonian time in ordinary dynamics (of open systems). It turns out that the intrinsic time, which tracks the evolution of the purely fluctuating state when the mean dynamical evolution of the total state is removed, relates inversely with the external Newtonian time $t$, endowing an SL(2,R) form-invariance to the corresponding Schrödinger equation. Consequently, a fluctuation that is small in comparison to the mean evolution of the state in the scale $t \sim 1$, can be mapped to a large fluctuation (comparable to the mean) in the scale of $t_1 = \nu t \sim 1$ and is described analogously by an approximately self- similar Schrödinger equation, when $t_1$ acts as the time parameter. Contributions from all scales then lead to a nonvanishing exponent in the limit $t \to \infty$.

Clearly, the present study is in continuation of Refs.[2,7]. To understand the origin of a possible time inversion and the associated self-similar evolutions in a time dependent linear evolution more clearly, we study here a linear differential dynamical system in one space dimension, which may be considered as the simplest classical analogue of the Schrödinger equation. We show that even in the absence of a geometric phase, time inversion can be injected into a linear (classical) system by using the golden mean partition of unity: $\nu + \nu^2 = 1$. The meaning of time inversion and its relevance to linear systems is explained in Sec.2, by pointing out an intrinsic uncertainty in the definition of inversion. In Sec.3, we demonstrate how time inversions, in the context of the ordinary time, can lead to a new class of self-similar solutions to the linear equation. The origin of intrinsic time in the classical context is also pointed out. In Sec.4, we interpret the new solution as representing the late time stochastic behaviour (fluctuations) (over multiple time scales) of the standard solution, when randomness is injected into the system via time inversions at randomly distributed ‘transition moments’, transferring the evolution from one scale to other. This random multiscale evolution of a linear system thus reveals a fractal-like aspect in time itself. We also discuss the relation of the inversion induced uncertainty to the irrationality of the golden mean, giving rise to a fundamental limitation on measurability. In Sec.5, we present an ansatz for this stochastic fractal time, and discuss how this ansatz could successfully retrieve the late time stochastic behaviour of the linear system in a concrete way. We also point out here the close resemblance of the present stochastic equations with quantal evolution. The derivation of $1/f$ spectrum and applications to certain natural processes are discussed in Sec.6. In Sec.7, we show that our nonlinear ansatz for the fractal time $T$ represents a new class of fractal solutions for the equation $dT/dt = 1$. The role of the measurement uncertainty in formulating the fractal time concept is further highlighted here. We sum up our presentation in Sec.8. It will be clear that the discussion of Secs.2-4 forms the background of the stochastic formalise of Secs.5-7.

2 Time Inversion

Let us consider the simplest linear dynamical system given by
\[
\frac{dx}{dt} = (1 + v(t))x
\]

where \(t\) denotes the dimensionless (Newtonian) time. (We scale \(t\) suitably to adjust the dominant scale of evolution to \(t \sim 1\). The above equation is form-invariant under time translations. In ordinary treatment, leading to the standard solution \(x_s \sim \exp(t + \int vdt)\), the variable \(t\) (spacelike or timelike) behaves only as a labeling parameter. The above equation, being linear and deterministic, can not generate self-similar fluctuations in the ordinary framework. We, however, solve eq(1) as a true evolutionary process, by following the evolution (unfolding) of the system \(x\) as time changes (flows) successively in steps of a given unit. We also look for clues to extend the time translation form-invariance of eq(1) to the SL(2,R) form-invariance. We note that SL(2,R) represents the minimal extension of the translation group incorporating inversions.

Let us assume that the system given by eq(1) evolves out from the initial time \(t = 0\) with \(x(0) = 1\). The (implicitly time dependent) function \(v(t)\) may be considered to represent the time dependent interactions of the environment on the system \(x\). (This representation, though useful for our later discussion on intrinsic time, is not essential.) Following the Born-Oppenheimer ansatz (and using the terminology of stochastic processes, to be justified a posteriori), the evolution of the system can now be decomposed into two parts \(x = xx_1\), where \(\bar{x} = e^t\) is the explicitly time dependent ‘mean’ evolution and \(x_1\) is the implicitly time dependent purely ‘fluctuating’ component satisfying the reduced equation

\[
\frac{dx_1}{dt} = v(t)x_1
\]

We note that ordinarily a change in time, in the vicinity of a given instant \(t_0\) is indicated by a pure translation \(t = t_0 + \bar{t} \equiv t_0 + (t - t_0)\). In fact, the last equality is an identity (valid for all \(t\)). In the neighbourhood of \(t = 1(t > 1)\) (the scale of explicit time dependence) (say) an increase of time is thus indicated by \(t = 1 + \bar{t}, \bar{t} = t - 1 \approx (>0)\). The standard solution, \(x_s\), is clearly the unique translation form-invariant solution of eq(1). By inversion, on the otherhand, one means \(t = (1 + \bar{t})^{-1} \approx 1 - \bar{t}\). However, contrary to pure translations, the inversion leads to an equation (in \(t\), fixing \(t = 1\), for \(\bar{t} > 0\). Of course, one can consider the inversion as the definition for the variable \(\bar{t}\) itself. In that case, the inversion turns out to be a trivial representation of the translation \(t = 1 - \bar{t}, \bar{t} = 1 - t\) near \(t = (>)1\). Consequently, the possibility of an inversion is ordinarily ruled out. In the following we, however, show that a nontrivial realization of inversion (at the level of an identity) is indeed possible in the vicinity of \(t = 1\) with interesting physical implications in the context of an evolutionary equation of the form eq(1).

To this end, let us note that one is indeed free to interpretre inversion as a two-time transformation. Let \(t_\pm\) denote times \(t > 1\) and \(t < 1\) respectively. Then close to \(t = 1\), the inversion \(t_- = 1/(1 + (t_+ - 1))\) leads to the constraint \(1 - t_- = t_+ - 1\). The parametric representation of inversely related times is obviously given by \(t_- = 1 - \bar{t}\) and \(t_+ = 1 + \bar{t}\), \(0 < \bar{t} << 1\) (so that the constraint reduces to an identity, valid close to \(t = 1\)). With this reinterpretation, time inversion in the vicinity of \(t = 1\) acquires the status of a pure translation. If translation is considered to be the most natural mode of time increment, then there is no compelling reason of ignoring inversion as yet another natural mode of doing this. Consequently, it seems reasonable to assume that time changes from \(t_-\) to \(t_+\) not only by ordinary translation over the period \(t_+ - t_- = 2\bar{t}\), but also instantaneously by inversion.

To clarify the point further, we note that the above definition gives a new nontrivial solution
$t_- = 1/t_+$ to the constant $t_- + t_+ = 2$, in the vicinity of $t = 1$ over the linear solution $t_- = 2 - t_+$, ordinarily thought to be the only possible solution. The reason for this belief obviously relates to the idea that a change in a variable $t$ (say) is accomplished only by pure translation. Consequently, one may argue that the parametric form of our two-time (point) solution being clearly linear close to $t = 1$, is a trivial representation of the standard (linear) solution and must be devoid of any new dynamical content. Our intention, however, is to emphasize the contrary; although apparently linear, the two-time parametric solution, being inversely related, does indeed hide a structure of nonlinearity in the form of SL(2,R) group action, giving rise to an alternative means of inducing a change in the variable $t$. In view of this nonlinear possibility, the change (flow) of time could be visualized as an SL(2,R) group action, when the nontrivial SL(2,R) action is realized only near $t = 1$. Accordingly, time flows from $t = 0$ to $t = t_-$ by translation, and then may switch over to $t_+$ by inversion $t_+ = 1/t_-$, for another period of linear flow etc. In Sec.3 we explain in more detail how this scenario gets a natural application in the context of eq(1) leading to new dynamical features. The restriction on the applicability of the nontrivial generator of SL(2,R) (viz., the inversion), to be close to $t = 1$ is vital. Once this is removed, the inversion solves the constraint only for $t_- = t_+ = 1$. Consequently, the new dynamical features would arise mostly in connection with the short time (viz., close to the moment $t = 1$) and/or late time (by inversion) behaviour of the evolving system. We note incidentally that the above definition of inversion has an inbuilt uncertainty. The exact moment when an inversion is materialised, is rather irrelevant; it can be at any instant close to $t = 1$. (To put it in another way, the nature of the ‘inversion constraint’ allows the parameter $\bar{t}$ to be a random variable (c.f., Sec.5)). As will be pointed out below, this uncertainty is actually related to the irrationality of the golden mean $\nu$ and is also responsible for a loss of late time predictability even for the simple system eq(1).

Before closing this section, it is worthwhile to compare the present definition of time inversion with the usual time reversal (inversion) symmetry of an equation of the form eq(1). The usual time reversal symmetry means that the system $x(t)$ evolves not only forward in time from $t$ to $t + h, h > 0$, but it can also evolve backward; i.e., the state $x(t)$ can be reconstructed from the state $x(t + h)$. The parameter $t$ in eq(1) is thus ‘non-directed’, (cf. the remark below eq(1)) giving rise to the problem of time asymmetry (note that all the fundamental equations in Physics are time (reversal) symmetric). This is to be contrasted with a (time asymmetric) diffusion equation, having a well-defined temporal sense. We show below that the present definition of (time) inversion introduces a well-defined time-sense in the system’s evolution following eq(1), analogous to a diffusive process. More precise derivation of this equivalence will be given elsewhere.

3 New Solution

To discuss the salient features of time inversion in the context of ordinary time $t$, it suffices to restrict the class of equation (1) to the one given by $v(t) = v_0 t$. The parameter $v_0$ may be a slowly varying function of $t$ so that $|\frac{dv_0}{dt}| << v_0$. For simplicity, we may thus fix it to a (small) constant. (The $t$ dependence of $v(t)$ thus gets explicit. However, the slow implicit $t$ dependence of $v_0$ will be kept in view in the later discussion of intrinsic time). One thus considers the ‘linear regime’ of the system evolution, nonlinear time dependence (interactions), if any, would be felt only after an elapse of time $O(v_0^{-1})$. Let us now note that a translation near $t = 1$, treated as a transformation between two independent variables $t$ and $\bar{t}$, can lead to a scale changing
transformation of the form

\[ t = \frac{1 + \nu t_1}{1 - t_1} \]

(3)

where \( t_1 = \nu(t - 1) \approx (>0). \) Indeed, for \( t \approx (>1) \) eq(3) reduces, up to linear terms, \( t \approx 1 + \nu(\nu + 1)(t - 1) \), so that, \( \nu(\nu + 1) = 1. \) Thus, for \( \nu = \) the golden mean, eq(3) gives a nontrivial SL(2,R) representation of the pure translation. We note that in the context of pure translations, one is unable to interpret (utilize) an equation of the form eq(3). To explore the implications of eq(3) in relation to an inversion, it is advantageous to rewrite \( \nu_0 \), without any loss of generality, as \( \nu_0 = \lambda^2 \nu (>0). \) Moreover, we rescale \( t \) by \( \tilde{t} = \lambda t \) in the reduced equation,

\[ \frac{dx_1}{d\tilde{t}} = \nu \tilde{t} x_1 \]

(4)

(We drop tilde henceforth and reintroduce it later to discuss the scaling properties of \( x \). The parameter \( \lambda \) would indicate the strength of the nonlinear influence of time on the system.) To recall, eq(4) is derived by removing the mean \( \tilde{x} \) of the total system after an elapse of time \( t = t_- \) (say, for definiteness). By the inversion constraint \( 1 - t_- = t_+ - 1 \) the fluctuation \( x_1 \) is transported instantaneously to \( t_+ \), since \( t_- \) and \( t_+ \) are identified, so to speak, by inversion \( t_+(t_-) = t_-^{\dagger} = 1 + \tilde{t}. \) Noting \( dt_- = -dt_+ = -d\tilde{t}, \) and using eq(3) we get

\[ \frac{dx_1}{dt_1} = (1 + \nu t_1) x_1 \]

(5)

where \( t_1 = \nu \tilde{t} \approx (>0). \) Thus as the total system evolves up to \( t_- \approx 1 \) (say), the SL(2,R) representation eq(3) in conjunction with an explicit time inversion, restores the initial linear evolution to the fluctuation \( x_1 \) in time \( t_1. \) To emphasize, in the absence of time inversion, eq(5) would have been impossible. Eq(3) is valid only in the vicinity of \( t = 1, \) triggering the transition of eq(1) to eq(5) via eq(4), when eq(5), being a self-similar replica of eq(1) is valid at least up to \( t_1 \approx 1. \) Consequently, as time flows from \( t_+ \) onward linearly by translation, the fluctuation \( x_1 \) now evolves under eq(5) till \( t_1 \approx 1. \) The (-) sign is a nontrivial signature of time inversion. To understand its origin, we rederive eq(5) when the subtraction of the \( \tilde{x} \) is accomplished at a time \( t_+ (>1). \) Following the above steps, eq(4) now gets transformed to

\[ \frac{dx_1}{dt_1} = (1 - \nu t_1) x_1 \]

(6)

when the inverted form of eq(3) is used to indicate the transfer of the fluctuating system from \( t_+ \) to \( t_- \). One then needs to replace \( t_1 \to -t_1(t_1 > 0) \) to put eq(6) in the form eq(5). The need for a sign change in \( t_1 \) is necessary to counter the backward time flow generated by the transition \( t_+ \to t_- \). The small scale intrinsic evolution of the fluctuation \( x_1 \) given by eq(5) is thus realized in the opposite direction of the usual external time evolution eq(4) [7]. In other words, the possibility of a backward flow in time is avoided by flipping the direction of the self-similar evolutions of the system at successive iterates. To justify the term intrinsic, we note that the self-similar evolution in time \( t_1 \) is generated by splitting, so to speak, the time \( t \) itself into the inverted time in the form of the factor \((1 - t_1)^{-1}\) and the time \( (t_1) \) dependent interaction in eq(3). The self-similar eq(5) is thus a signature (and consequence) of the self-interaction due to the nonlinear structure in time, endowed by inversions.
To clarify it further, let us rederive eq(5) by yet another route, making the relationship between time inversion, intrinsic sense of time [2] and self-similar evolution more transparent. We start again from eq(4) but assume that the time of removal of $\bar{x}$ is $t_- = 1 - \bar{t}_1 (\bar{t}_1 > 0)$, instead of $t_- = 1 - t$, as in the previous cases. (To keep our notations clear, we denote a linear variable $t$ near $t = 1$ by $t = 1 + \bar{t}$, where $\bar{t}$ represents $t$ near $t = 0$.) We note that the small scale time $t_1$ is defined intrinsically by the interaction, which, in turn, introduces a squeezing of the ordinary time increment near $t = 1$: $\bar{t}_1 = \nu(t) = \nu\bar{v}(\approx 0), 0 < \nu < 1$. The possibility of two choices of $t_-$ could be ascribed to the uncertainty in the exact moment injecting an inversion into the system near $t = 1$. The proximity of the present moment to $t = 1$ over the other is indicated by the scale factor $\nu$. Noting that $t = \bar{v}(t_1)\bar{v}(t_1)[\bar{v}(t_1)]^{-1}$, $\bar{v}(t_1) = 1 + \nu\bar{t}_1$, we rewrite eq(4) as

$$\frac{-dx_1}{dt_-} = \bar{v}(\nu t_-)\bar{v}^{-1}x_1$$

This equation now leads to eq(5) provided two moments $t_-$ near $t = 1$ and $t_+$ near $t \sim \nu^{-1}$ are identified by inversion $t_+ - \nu^{-1} = 1 - t_- = \bar{t}_1$, i.e., when $\nu t_+ = \bar{v} \approx 1 + \nu\bar{t}_1 (dt_+ = -dt_- = d\bar{t}_1)$. The factors $\bar{v}, \bar{v}^{-1}$ (being the small scale replica of the interaction term), are introduced self-consistently from the intrinsically available informations (observables)[2] in the evolving fluctuation, eq(4), eliminating the external time variable $t$ of eq(1). This also removes any arbitrariness in the choice of the factors. Stated otherwise, the self-similar eq(5) may be considered to be an outcome of the self-measurement of $x_1$ by itself. In fact, the variable $\bar{t}_1$ indicates the time recorded, as it were, by an ‘internal clock’ stationed in the total system $x$ itself, whose rate of variation is correlated with (and determined by) the variation of the interaction via $\nu$, an implicit function of time $t$ (recall the slow time dependence of $v_0$). (Thus, in relation to eq(5), the system $x$ has the role of the ‘universe’[2,7].) Note that eq(4) is an extrinsic equation, since the changes in $x_1$ there is measured by an external clock. The inversion now carries the moment $t_-$ to the smaller scale $t_1$ near $t_1 = (>1)1$ via $\nu(1-t_-) = \nu t_+ - 1 \equiv t_1 - 1$, where $t_1 = 1 + t_2, t_2 = \nu^2 \bar{t}_1$. Consequently, as the linear Newtonian time changes from $t = 1$ onwards as $t = 1 + \bar{t}$, a concomitant flow of an intrinsic sense of time also gets developed in the system, which changes by inversion $t_-(1 - \bar{t}_1) = (1 + \bar{t}_1)^{-1}$, near $t = 1$, incorporating an explicit change of scale, leading to self-similarity, in the subsequent evolution over the period $t_1 \approx 0$ to $t_1 \approx 1$, and hence relating smaller scales near $t = 1$, successively, to longer scales as $t \to \infty$. This indicates how an otherwise uni-scale evolution of $x_1$ acquires multiscale self-similarity under inversions.

To summarize, the time inversion as defined in Sec.2 along with the SL(2,R) representation, eq(3), can induce a self-similar evolution eq(5) to the fluctuation $x_1$ when the mean evolution $\bar{x}$ is removed near $t \approx 1$. Time in the self-similar eq(5) is recorded, as it were, by a ‘clock’ stationed intrinsically in the total system $x$, so that the (intrinsic) time is measured in the unit of $\nu$, against the external flow in the unit of $t \sim 1$. We remark that the removal of the mean $\bar{x}$ can be accomplished at any instant in a neighbourhood of $t = 1$. This uncertainty, however, is not apparent in the self-similar eq(5), as it is obtained via inversion at a well-defined instants, $t_-$ say, near $t = 1$. In any case, the possibility of an instant like $t_-$ (and/or $t_+$) to act as a random variable is sufficient to inject a randomness in the late time evolution of the system. This point will be elaborated further in later sections.

Returning to the main discussion, we note that eq(5) is valid upto $t_1 \approx 1$ ($t \approx 1 + \nu^{-1}$), when (the linear regime) of eq(1) is valid upto $t \approx \nu^{-1}$. Near $t_1 = 1$ the system $x_1$ again makes a transition to the 2nd order self-similar fluctuation $x_2 : x_1 = e^{-t_1} x_2$ satisfying the equation
and so on, to finer and finer scales as \( t \to \infty \). The rate of this successive transitions is very slow. Indeed, as time \( t \) flows linearly to \( \infty \), the inversions at the successive transition moments \( \tau_n = \sum_{0}^{n} \frac{1}{\nu} \) generate, so to speak, an (overall) intrinsic time of time: \( T = \frac{1}{1 + \tau_0}, t \approx 1, T = [1, 1; 1 + t_2] \) at \( t \approx 1 + \nu^{-1} \) so on, so that \( T \to \nu \) through the sequence of the golden mean approximants as \( t \to \infty \), the rate of whose convergence is the slowest possible. Note that after first iteration, \( t_1 \) in eq(5) flows from \( t_1 \approx 0 \) to \( r_1 \approx 1 \) linearly, thus making a room for the 2nd iteration at \( \tau_1 \) and etc. The new intrinsic sense of time denoted \( T \) thus resembles, as it were, a cascaded flow down the ladders of the golden mean continued fraction, transporting the evolution of the system \( x \) successively to scales \( t_{n+1} \) at the transition moments \( \tau_n \). As pointed out already, the transition moments are inherently random, thus inducing a stochastic nature in the intrinsic time \( T \). We recall that an iteration always generates a sense of time in the context of a discrete dynamical system (map). The present scheme of iteration via inversions, however, relates to two senses of temporal directions: the intrinsic time sense \( T \) which converges to \( \nu \) and the ordinary linear sense of \( t \to \infty \), as \( n \to \infty \). In any case, the system \( x \), however, evolves uniquely along the intrinsic time flow, thus traversing finer and finer scales \( t_n \), as \( n \to \infty \). The scales \( t_n \) can be treated as independent variables related to each other successively by scaling equations \( t_{n+1} = \nu(t_n - 1) \) near \( t_n = 1, t_{n+1} = 0 \). In the limit \( t \to \infty \), the (intrinsic ) solution of eq(1) incorporating equal contributions from all scales thus has the form

\[
x_i = e^{(t-t_1+t_2-\cdots)}
\]

where \( t > t_1 > t_2 > \ldots \). We note that the randomness in \( \tau_n \) makes the variables \( t_n \) a set of stochastic variables, with randomness concentrating near \( t_n = 0 \) and \( t_n = 1 \). The contributions from these infinite number of (random) scales would of course lead to a very complicated fractal -like structure for \( x \). A somewhat simplified (but, nevertheless, useful) view of this fractal behaviour emerges when we look for the late time (i) asymptotic form of eq(8) using the scaling relations \( t_n = \nu^n t \) (c.f., our actual derivation of eq(8) following system’s evolution from \( t = 0 \) to \( t = \infty \) via inversions at well-defined transition moments and stretching each of the scales \( t_n \to \infty \) via scaling relations). We get \( x_f \sim \eta^n \), where, \( t_1 = \ln \eta^{-1}, \eta \to 0 \), and \( x_f = x_i/\tilde{x} \) stands for the renormalized form of \( x_i \) when the initial mean evolution (of the zeroth iterate) is subtracted out. We note incidentally that, in the absence of time inversion, the standard solution of eq(1), with a linear \( v(t) \), has the form \( x_s = e^{t + \frac{1}{2} \nu t^2} \). To pave the discussions of the subsequent sections, we compare, in the following section, the intrinsic solution (8) with the standard solution \( x_s \), and give an heuristic interpretation of \( x_f \) as a correlation function.

### 4 Interpretation

To interpret the new solution \( x_i \), let us proceed in several steps. We begin by noting that under inversion (i.e., the extended SL\((2,R)\) symmetry) the points \( t = 0 \) and \( t = \infty \) are identified. In the present context this means that the system would enjoy identical (equivalent) dynamical properties ( though their interpretations might vary, see below) for very short and late time. As a check, one can easily verify that eq(1) with \( v(t) = \nu t, t = t_0(1 + \eta), t_0 \) large, gets mapped to eq(5), under inversion, when \( \tilde{t} = t_0 \eta = (1 + \tilde{t}), \eta \approx t_0^{-1} \approx 0 \) and \( \tilde{x} = e^{(t_1+\nu t_0)^k} \) (c.f., eq(4)). We note next that the quadratic term in the exponential of \( x_s \) due to the linear time dependent
term $v(t) \propto t$ in eq(1), is replaced by the infinitely many scale dependent linear terms in the new SL(2,R) invariant solution (8). The evolution thus remains ‘truly’ linear over every time scale, in the sense that even the solution $\ln x_i$ is linear, though in multiple (random) scales (variables). The reason of this, as pointed out already, is that $v(t)$ continues to replicate using the ‘golden mean splitting ansatz’ eq(3) down the scales, thus driving the nonlinear (quadratic) contribution out to infinitely distant time, by keeping it always insignificantly small. Consequently, the solution (8) could as well be considered as the one valid in the short time limit near $t = 1$. In fact, this follows from the set of self-similar equations

$$\frac{dx_n}{dt_n} = (-1)^n(1 + \nu t_n)x_n$$

where $t_0 \equiv t$, when the variables $t_n$’s are allowed to vary in the finite interval $(0,1)$ (say). As $t_n \to 0$ in the limit $n \to \infty$, the solution gets contributions from smaller and smaller scales, inheriting the self-similarity of the underlying equations (9). Consequently, the late time scaling $x_f \sim \eta^s$ could as well be derived starting from eq(9) for a sufficiently large $n$, where

$$t_n = (-1)^n \ln \eta.$$  

Note that there is an ambiguity of sign depending upon $n$ being odd or even. However, this only reveals the essential equivalence of (the scaling) behaviours in the system’s evolution both in the short, $\eta \to 0$ ($n$ odd), and long, $\eta \to \infty$ ($n$ even), time limit. We remark that, in case one restricts all the variables $t_n$ in $(0,1)$, as above, the variable $t_0$ behaves only as a labeling parameter, the flow of time is solely indicated by $n$, the order of iteration. The iterated eq.(9) then defines a (discrete) map in the space of continuous functions.

Before interpreting the scaling, let us make a comparison of our solution $x_i$ with $x_s$. Interestingly, eq(8) follows indeed from $x_s$ provided we inject time inversions via $t_n = \frac{1}{1 + t_{n+1}}$, $t_{n+1} \approx 0$ in the quadratic term in the exponential, collecting together the dominant linear terms successively. As noted above, the subdominant quadratic term finally drops out in the limit $n \to \infty$. The time inversions thus lead the system to evolve following the flow generated by the intrinsic time sense. This proves our contention that the solution (8) is an SL(2,R) extension of the standard solution. We note that the solution (8) gives essentially the late (short) time scaling of the standard solution $x_s$ under inversions. This means that the solution $x_s$, in a physical application, represents the behaviour of the system (given by eq(1)) for moderately large $t$ ($\sim O(n)$, for moderate values of $n$). However, for a sufficiently large $t$, the system’s behaviour would slowly deviate from $x_s$, mimicking more and more the new solution $x_i$.

Finally, to interpret the scaling law, we note that the essential linearity of $\ln x_i$ endows every self-similar replica of the evolving system, $\ln x_n$, the status of a ‘time keeper’ (clock) which can be used to record time $t_n$ after the $n$th iteration. The variable $t_n$ can be defined as the $n$th generation Newtonian time for the system (fluctuation) $x_n$. (Note that in every generation of linear evolution, over the period $(0,1)$, $t_n$ acts as an ordinary (nonrandom) variable). However, as discussed in Sec.3, $t_n$, having constructed intrinsically from the $(n - 1)$th generation interaction $v_{n-1}$, corresponds to the intrinsic time relative to the $(n - 1)$ generation fluctuation $x_{n-1}$ (c.f., the relation of $t$ and $t_1$). The plethora of time variables $t_n$, related to each other by inversions, as explained, are the linearised remnants of the nonlinear, stochastic time, denoted $T(t)$, where the zeroth generation time $t$ is the ordinary Newtonian time. We note that once inversion is raised to a physically allowed mode of time flow, time as such becomes nonlinear. Our description above, however, identifies two simplified patterns of this nonlinear time (flow): i) the stochastic intrinsic flow $T$ following the ladders of the golden mean continued fraction ii) the extrinsic (linear) Newtonian flow $t$. However, the extrinsic time $t$ itself carries, thanks to (the possibility of) inversion(s), the seed of nonlinearity close to the instant
$t = 1$ (for instance). This fundamentally nonlinear behaviour of time could thus be represented succinctly as $T = t(T)$. Although, $T$ here may be considered to denote the ‘fully nonlinear’ time, it is sufficient for our purpose to identify $T$ with the intrinsic time. Incidentally, we note that $T = t(T) \Rightarrow t = T(t)$, because of the interchangeability (relativity) of the extrinsic and intrinsic time at successive scales near $t = 1$. Writing $\ln x_f \equiv \ln(x/\tilde{x}) = -T$, eq(4), rewritten as

$$\frac{dT(t)}{dt} = v(T)$$

(10)
can now be considered as a renormalization group (RG) flow equation for the stochastic (nonlinear) time $T$. The function $v(t)$ of eq(1), representing intrinsic (self-) interaction of the system (equivalently, the nonlinear time) now assumes the form of the RG $\beta$-function and has the form $v(T) = vT = \nu(1 + \nu t(T))$ close to $t = (T =)1$, where $t(T) \approx 0$. (The factor $\nu$ could be considered as the measure of the strength of the time-time self-interaction. The free parameter $\lambda$ (c.f., eq(4)) then denotes the strength of system’s coupling with the nonlinear structure of time. Eq(10), being the defining equation of the nonlinear time, in particular, has $\lambda = 1$). We note that nonlinear (stochastic) feature of time $T$ is revealed only close to $T = 1$. Assuming that time changes by pure translations only even in the vicinity of $t = 1$, we get the standard linear view of time. In this case a scale change is not allowed, so that $v(T) = 1$. Utilizing the SL(2,R) freedom near $t = 1$ one gets a flow of time which allows an evolving system to traverse finer scales, as explained in Secs.(2-3). Clearly, eq(8) gives the unique nontrivial solution of the SL(2,R) RG equation (10). The uniqueness, to within our definition of time inversion, concerns only with the form of the solution. The randomness in the scale-changing transition moments is likely to lead to very different final states in long time scales for two systems with slightly different initial conditions. In the next section, we present an ansatz for the nonlinear stochastic time $T$, and develop a method to compute the two-time correlation function $c(t)$ of a stochastic process obtained by replacing $t$ by $T$ in eq(1). Interestingly, our ansatz turns out to be an exact solution of eq(10) (c.f., Sec.7). As it turns out, the late (short) time power law of the solution $x_i$ correctly represents the same for the said correlation function. Consequently, the random transition moments $\tau_n$ (equivalently, the periods $t_n$ of successive linear evolutions, randomness concentrating near $t_n = 0$ and $t_n = 1$) could be considered as distributed with a probability distribution of a random walk process. The correlation function $c(t)$ then gives the probability of not making a transition at $t$, when it is known that no transition is made at $t = 0$. The nontrivial scaling now tells us that there is indeed a finite probability that the system does make a transition to a new scale in the long time limit, contradicting the linear time expectation that the probability should be 1. The asymptotic power law of the correlation function suggests that the late time evolution would resemble a Levy-like process [8]. This random walk scenario would be made more precise in Secs.5-7.

We note also that the short time scaling $x_f \sim \eta^\nu$, on the other hand, could, be interpreted as the indicative of a fat fractal-like structure [7,10] in the nonlinear time, with uncertainty exponent $\nu$, the golden mean. Each point of the fattened time axis would thus be structured, equivalent to a Cantor set of dimension $\nu$. The non-zero uncertainty exponent now tells us that, contrary to the accepted notions, the precise determination of an instant is almost impossible (even in the framework of the classical mechanics). In fact, this could be inferred directly from eq(3), which reveals the fundamental role of golden mean in the definition of inversion. In the framework of nonlinear time, the problem of measuring a duration of unit length $t = 1$ is equivalent to measuring a duration $\eta = \nu^{-1}, t = \nu \eta$, because $t$ is only a representative...
of the family of variables $t_n$. However, the *irrationality* of the golden mean makes a precise measurement of the later impossible. Hence, one can at best conclude that $t \approx 1$, giving rise to a *fundamental limitation on measurability*. As an afterthought, this inference could even be reached in the ordinary (linear time) framework. The problem of measuring $t = 1$ can always be reduced to measuring a variable $\eta$ when $\eta = \nu^{-1}, t = \nu \eta$. The practical limitation of a precise determination of an irrational then translates to the above measurement limitation, giving rise to the width (uncertainty) necessary for inversion to materialize. This fact may then be considered as offering a (theoretical) justification to our assumption that time inversion is a physically viable mode of time flow. A relatively small error in measuring the instant $t_1 = 1$ i.e., $t = \nu^{-1}$ (say), which is present unavoidably, now, allows the system to explore the inversion-induced intrinsic evolutionary path. In the process the initial error gets magnified, leading to the loss of final state predictability. In this framework of fractal time, time inversion can thus be interpreted as a consequence of the intrinsic uncertainty in ascertaining if a moment near $t = 1$ (say) is actually less or more than $t = 1$. We note finally that the model dependence in the late time scaling exponent ($= \lambda \nu$) of the solution $x_i$ can be retrieved by reintroducing $\tilde{t} = \lambda t$ (cf. eq(4)).

5 Stochastic versus quantum

Here, we present a framework to deal with an equation of the form eq(1) when time acquires a stochastic nature. The stochastic interpretation of time inversion allows one also to derive a Wheeler-Dewitt like equation for the fluctuating component $x_f$ of eq(1). In the light of the discussion of Sec.4, we envisage an ansatz for the stochastic (fractal) time, accommodating both an explicit inversion and the associated randomness: $T(t) = (1 + \mu \lambda(t) \tilde{T}(t))t$, where $0 < \lambda(t) \ll 1$ is a slowly varying function of the ordinary (Newtonian) time $t$, $\mu = \pm 1$ is a (time independent) symmetric random variable with $< \mu > = 0$, $< \mu^2 >= 1$, $<, >$ being the statistical average, and $\tilde{T}(t) = T(t^{-1})$. A natural choice of $\lambda$ is, $\lambda(t) = \epsilon \nu^{\lambda t}, \nu$ being the golden mean number, $n > 0$ is a sufficiently large random integer, and $\epsilon = \epsilon(1/t) \sim O(1)$ could be an yet another slowly varying function which is constant over moderate scales, but may become ‘active’ (i.e., large) near $t = 0$. We disregard it in the following, but recall its presence in discussing short time scale structure of $T$ in Sec.7. We note that the arbitrariness in $n$ indicates the uncertainty in the actual moment injecting an inversion close to $t = 1$ (say) and is used to model the uncertainty related to irrationality of $\nu$, whereas $\mu$ takes care of the definition of the inversion. (Recall that inversion reflects an uncertainty in the neighbourhood of $t = 1$ (for instance). Thus, measuring an instant $t = 1 + \tau, \tau > (\approx)0$ could very well end up with the result $t = 1 - \tau$ and vice versa.) For the sake of simplicity, we, however, choose $n$ to be a fixed integer (inversion is then assumed to materialize at a well-defined instant). We note, in particular that, $< T > = (1 - \nu^{2n})^{-1}t$, (which encodes our discussion on the limitation of measurability: although measurement of the instant $t = 1$ (say) is exact in the context of the ordinary linear time, there is a uncertainty $O(\pm \nu^n)$ in the case of the fractal time $T$ and $T_1 = \nu^n T = (1 + \mu \nu^a t_1 \tilde{T}_1) t_1, \tilde{T}_1 = \nu^{-n} \tilde{T} \approx 1$, near $t = \nu^{-n}$. The implicit definition of $T$ is thus rescaling symmetric, exhibiting its scale-free nature. Further, $t \tilde{T} = T/t \sim O(1)$, by inversion symmetric nature of the ansatz (c.f., Sec.(7)). However, this behaviour might change in the limit $t \to 0$ or $\infty$, because of large fluctuations generated due to the activation of $\epsilon$ factors which are ‘mute’ otherwise. Interestingly, the implicit definition of $T$ turns out to satisfy the differential relation $dT = \frac{T}{t} dt$, which is valid for $t > 0$. In the next section, we interprete this
as a new class of stochastic nonlinear solution of an ordinary equation $t\frac{dT}{dt} = T$ under inversion, indicating its relation with the measurement problem.

It now follows that the intrinsic scale dependent random flow of $T$ traverses the finer scale $O(\nu^n)$ (c.f., Sec.3), as time $t$ flows to $O(\nu^{-n})$. In other words, contribution of the finer scale $T_1$ in a evolving system becomes significant $\sim O(1)$ at time $t \sim O(\nu^{-n})$. This random scale free flow of fractal time $T$ is responsible for approximate self-similarity of the system’s evolution. To justify this, we note that a time dependent equation of the form eq(1) can be written generically as the stochastic equation

$$ \frac{dx}{dt} = (1 + \mu \nu^n t T) h(T) x $$

(11)

where $dT = \frac{T}{t} dt$. Note that, close to an instant $t$, $\frac{T}{t} h(T) \approx (1 + \mu \nu^n t T) h(t) + \mu \nu^n T \frac{dh}{dt}] = h(t) + [h^n T h(t) + \nu^n (T T') \frac{dh}{dt}]$, so that writing $x = e^\int h(t) dt \tilde{x}_f$, $(x(0) = 1)$, eq(11) reduces to

$$ t \frac{\tilde{x}_f}{dt} = [\nu^{2n} T h_1(T_1) + \zeta(t)] \tilde{x}_f $$

(12)

where $h_1(T_1) = h_1(t_1) + \mu \nu^n T_1 \frac{dh}{dt}$, close to the instant $t_1 = \nu^n \ln t = 0$, and $\zeta(t) = \nu^{2n} T \{(\mu \nu^n h(t) - h_1(t_1)) + \mu (\nu^n T \frac{dh}{dt} - \nu^n T \frac{dh}{dt})\}$, acts as a ‘noise’ at the scale changing transition point. The source of this noise is the mismatch of the boundary values of the function $h(t)$, because of the $\mu$ factors, although $h(t) = h_1(t_1), \frac{dh}{dt} = \nu^n \frac{dh}{dt}$, for $t = 1 + t_1$ (say). It follows from the relations $\mu T \approx \nu^{2n} < T >, \mu T^2 \approx 2 \nu^{2n} < T > \approx 2 \nu^{2n} < T >$, which can be proved using the representation $T = (\Sigma \mu \nu^n t) (\text{obtained by repeated applications of the ansatz over itself}), that $\zeta(t) = 0$, up to order $O(\nu^{2n})$. We remark, in passing, that, in this formalism, the expectation of the integrated residual noise is expected to remain finite in longer time scales. Further elaboration of this point would be considered separately.

It now follows that eq(12) is self-similar to eq(11), but for the zero mean noise term. (Eq(11) is, in fact, form invariant under rescalings when the nonrandom mean evolution of the system is removed.) However, removing the noise term by writing $\ln \tilde{x}_f = \int \zeta(t) t^{-1} dt + \ln x_f$, we finally get the correct self-similar equation for the fluctuation $x_f$:

$$ - \frac{dx_f}{dt_1} = (1 + \mu \nu^n \frac{T_1}{t_1}) h_1(T_1) x_f $$

(13)

In the derivation of this equation, via eq(12), we make use of $t_1 T_1 = t T = T/t, T_1 = \nu^n T$ whenever necessary. The (-) sign in the left hand side, and other symmetric changes in the right hand side are consequences of inversion near $t = 1$. We note that the logarithmic derivative in eq(12) ensures the initial condition $x_f(0) = 1$ at $t_1 = 0$ for eq(13). To explain eq(13), we note, following Sec.3, that the system $x$ is evolved till $t \approx 1$, when the mean (nonrandom part of the) evolution is removed. An inversion near $t = 1$ then induces a scale changing process infusing the fluctuating system with a noise, generated due to ‘boundary effects’. Once the noise is filtered out, the fluctuating component $x_f$ is found to evolve following a reduced equation which is self-similar to the original equation till $t_1 \sim 1$ (our approximations break down beyond $t_1 = 1$). Near the epoch $t_1 = 1$, the system again gets ready to accommodate another scale changing process $t_1 \rightarrow t_2$, and so on for the subsequent evolution. This completes our derivation of self-similarity of eq(1) in the framework of the stochastic time.

We note that removal of the total mean evolution by the ansatz $x = e^\int \mu T(t) dt \tilde{x}_f$, instead, would have resulted the equation
\[ t \frac{d\bar{x}_f}{dt} = (Th(T) - <Th(T)>)\bar{x}_f \] 

so that \(< \frac{d\bar{x}_f}{dt} >= 0\). This is analogous to the Wheeler-DeWitt-like equation in quantal evolution, indicating the reparametrization invariance of geometric phase [7]. In the present classical context, this could be interpreted as the reparametrization invariance of the fluctuating evolution, due to time inversion. However, \(< Th(T) >= h(t) + O(\nu^{2n})\), so that the above equality is only approximate, in all practical purposes; whence the self-similar eq(13) of the fluctuation could be retrieved (c.f., Sec.4, also see Sec.7). The reparametrization invariance of eq(14) thus offers a justification in calling the variable \(t_1\) an intrinsic time. A deeper analysis of this fractal time approach in (quantum) general relativity needs separate investigations.

6 \(1/f\) Spectrum and Applications

We now show that the two-point correlation function of the fluctuating system has a late time power law form, resembling a Levy like process [8]. Let \(c(t) = < \bar{x}(t)\bar{x}(0) >, \bar{x} = e^{-\int h(t) dt}x\) denote the correlation function of the evolving system \(x\) in eq(11). Here, \(<.,.\>\) denotes the statistical average over an ensemble of fluctuating solutions (for all possible realizations of the scale changing transition moments, in the terminology of Sec.4) with a common initial condition. Because of self-similarity of eqs.(11) and (13), the fluctuation \(x_f\) and the original system \(x\) have identical structures. The late time asymptotic form of the correlation function \(c(t) = < \bar{x}(t) >\) (for the initial condition \(\bar{x}(0) = x(0) = 1\)) can be easily obtained from eq(11) by first estimating it near \(t = 1\) and then taking the limit \(\epsilon \to 0\) of a slowness parameter \(\epsilon\) (a suitable power of \(\nu\), and not to be confused with one in the previous section), which occurs naturally as a rescaling parameter of the time variable: \(t \to \epsilon t\) (c.f., Sec.4), in the present scenario. Clearly, in the limit \(t = 1\), eq(11) (via eq(12), neglecting higher order terms) reduces to

\[ t^d\frac{\bar{x}}{dt} = \mu\nu^nT\bar{h_1}\bar{x} \] 

so that \(d < \bar{x} >= \nu^{2n}h_1 < \bar{x} > t^{-1}dt, h(1) = h_1\), when we use \(< \mu\bar{T}(t) >= \nu^{n} < \bar{T} >= \nu^{n}t^{-1}, up to O(\nu^n)\). One thus gets an ordinary differential equation, analogous to eq(4), for the correlation function \(c(t)\). Following Sec.3, this could be solved near \(t = 1\), accommodating a time inversion: \(t^{-1}dt \to -t^{-1}dt\). We thus get \(c(t) \sim t^{-\gamma}, \gamma = \nu^{2n}h_1, near t = 1\). The late time asymptotic form \(c(t)\) is now obtained by taking \(\epsilon \to 0\) and renormalizing it by a vanishingly small factor: \(C(t) = O(\epsilon^{\nu})c(t) \sim t^{-\gamma}\). Necessity of renormalizing \(c(t)\) by a small factor does not jeopardize its physical relevance. In fact, the relevant equation of \(c(t)\), (after an inversion) near \(t = 1\), \(dc = -\gamma t^{-1}c dt\), being rescaling symmetric, is valid even for large \(t\). However, the possible presence of a small factor might be responsible for not having any inkling so far about the relevance and implications of time inversion in the context of eq(1). Interestingly, we note here that the asymptotic form of \(c(t)\) mimics the late time behaviour of the new solution \(x_1\) in eq(8). (The power law form \(\sim \eta^{-\nu}\), in the paragraph below eq(8), is written in the log-scale of \(t_1\), instead of \(t\), as in here. Expressed in the log-scale of \(t\) the exponent is \(\nu^n\) in the exponent of \(c(t)\) is explained in the next section.) Consequently, the solution \(x_1\) could be identified as the correlation function for the stochastic process, eq(11). Finally, the power spectrum \(S(f)\) of the inverse power law correlation function \(c(t)\), \(S(f) = 2 \int_0^\infty c(t) \cos(2\pi ft) dt\), is known to diverge [3,9] with a power law tail \(\sim 1/f^{1-\gamma}\), in low frequency limit.
In the same spirit, we can also make an estimate of the short time scaling of the fractal time \( T \). Recalling \( \epsilon = 1 + \frac{1}{2}\epsilon_0 t^{-1} \), which appears with \( \nu^n \) as a constant (\( \approx 1 \)) factor for \( \epsilon_0 \ll 1 \) (a factor of 1/2 is included for convenience), and can now play a role to generate small scale time variations, \( < T > \) can be expressed as \( < T' > = [(1 - \nu^{2n} - \nu^{2n} t^\lambda)]^{-1} \), where \( t = \epsilon_0 t^{-1} \ll 1 \), and \( T' = t^{-1} T \), so that neglecting \( \nu^{2n} \) compared to 1, we get \( < T' > = (1 - \nu^{2n} t^\lambda)^{-1} = \tilde{\nu}^{2n}, \tilde{t} = 1 + \tilde{t} \).

Let us note further that, as remarked in Sec.5, \( < T' > \) diverges at \( t = 0 \) because of \( \epsilon \). However, away from \( t = 0 \), there always exists an interval of small \( t : \epsilon_0 \ll t \ll 1 \), when the above estimate of \( < T' > \) is well defined. Using \( \epsilon_0 \) as a cut off, one can treat \( T'/(\epsilon_0) = 1 + O(1)\mu \nu^n \), as a finite time independent random variable. The relevant correlation function \( c(t) = < T'(t)T'(0) > \sim < T'(t) > \), thus is given by \( c(t) \sim \tilde{\nu}^{2n}, \tilde{t} \approx 1 \). Following above arguments, we conclude that \( c'(t) \sim t^{\nu^2n} \) for small \( t \). Again this power law mimics the power law obtained in the last paragraph of Sec.4. We remark that the equivalence of the scalings of \( c(t) \) and \( c'(t) \) is indicative of the fact that the ‘a priori’ nature of Newtonian time gets blurred in longer time scales; system’s evolution is more accurately described by a class of intrinsic time variables, generated by the scale- free fluctuations, as inherited by the system from the fractal nature of time.

From the above discussions, it appears very natural to believe that observations of \( 1/f \)-like spectra in many natural phenomena must arise, at least partially, from the generic principle of time inversion induced stochastic fluctuations. We note that \( 1/f \) spectra in time series records, for instance, of quasar light intensity fluctuations, river (ocean) water level fluctuations [3], temperature (voltage) fluctuations under a steady current [9], etc could in fact be understood as generic consequences of time inversion. One can always conceive \( x \) in eq(1), as representing the relevant fluctuating variable, the rate of change of whose variation is supposed to be proportional to the system variable itself, where the ‘proportionality constant’ is a slowly varying function \( h(t) \) of time. In general, \( h(t) = h(t, x) \) may be a nonlinear function of the system variable \( x \) and other external influences. Here, we disregard explicit modelling of external influences, and assume that the time variation of \( h(t) \) arises purely from the nonlinear influences of the fractal time \( T \). The relevant equation, eq(1), then assumes the form of the stochastic equation (11), giving rise to the generic divergence in the corresponding spectrum. Numerical fits to time series data should be useful in estimating the model dependent index \( n \) of the spectrum.

### 7 Fractal time and Measurement limitation

Here, we present some salient features of our ansatz for the fractal time in Sec.5. In fact, we show that the ansatz constitutes a new class of stochastic solutions to the simplest linear differential equation \( \frac{dx}{dt} = x \) (c.f., eq(10)). Let us rewrite the ansatz in the form \( T(t) = (1 + \lambda t \tilde{T}(t)) t \), where \( \tilde{T}(t) = T(1/t) \) and \( \lambda \) may be an almost constant slowly varying function of \( t \). To begin with we disregard any explicit randomness in \( \lambda \). By symmetry, both \( T/t \) and \( t\tilde{T} \) satisfy coupled equations of the form \( x = 1 + \lambda y \) and \( y = 1 + \lambda x \), hence \( T(t)/t = t\tilde{T} \) for all \( t \) (when time variation of \( \lambda \) is disregarded). We assume that \( T(t) \) is continuously differentiable except possibly at \( t = 0 \) and \( \infty \). Noting that \( \frac{dT}{dt} = -t^{-2} \frac{d\tilde{T}}{dt} \), we get \( \frac{dT}{dt} = (1 + \lambda t \tilde{T}) + t\lambda \tilde{T} - \lambda \frac{d\tilde{T}}{dt} \), so that \( t\left(\frac{dT}{dt} - \frac{T}{t}\right) + \lambda(\frac{d\tilde{T}}{dt} - t\tilde{T}) = 0 \). It thus follows, \( \lambda \) being an arbitrary scale factor, that

\[
\frac{t}{T} \frac{dT}{dt} = T \tag{16}
\]

as asserted. The nontrivial role of inversion in obtaining the result needs to be emphasised. The
above equation is not satisfied for an ansatz with $T$, replacing $T$, in the bracket. It also follows that the relevant class of functions $T(t)$ must satisfy the condition $\frac{dT}{dt} = \frac{d\tilde{T}}{dT}$. We note that the equation is exact for a constant $\lambda$. However, the algebraic constraint $x = 1 + \lambda x$ implies $x = (1 - \lambda)^{-1}$, which means $T = (1 - \lambda)^{-1}t$. Consequently, for rational values of $\lambda$, when an exact evaluation of the scale factor is permissible, our ansatz, being the trivial (standard) solution of the above equation, fails to yield any new solution. However, for an irrational value of $\lambda$, the standard linear solution is valid only approximately $T \approx (1 - \lambda)^{-1}t$, in any physical application, which is a direct consequence of the measurement uncertainty discussed in Sec.4.

As stated already, we (partially) model this uncertainty by introducing a random parameter, $\lambda \to \mu \lambda$. In this case, the algebraic constraint assumes the status of a stochastic equation $x = 1 + \mu \lambda x$, so that the standard solution is obtained only in the mean: $< x > = 1 + \lambda^2 < x >$, $< T > = (1 - \lambda^2)^{-1}T$. The nonlinear stochastic function $T(t) = (1 + \mu \lambda t \tilde{T}(t))t$, then represents a nontrivial solution of eq(16). Finally, to fix the scaling parameter $\lambda$, we conceive the ideal situation of perfect time measurement $< T > = 1$ at $t = \nu$ leading to the value $\lambda = \nu$. In the present framework, this ideal time measurement is obviously precluded in natural phenomena, making a way to fluctuations over many time scales and complex structures. We remark that the choice of $\nu$ here is motivated by the SL(2,R) representation, eq(3), and the fact that the convergence rate of the golden mean approximants is the slowest possible, leading to the slowest ever rate of the intrinsic flow (c.f., Sec.3). For a rational value of $\lambda$, however, the scope of a nontrivial inversion is eliminated. The inversion in the implicit definition of $T$ then corresponds to the ordinary time reversal symmetry only (c.f., last paragraph of Sec.2). Mathematically, $\lambda$ can of course be any irrational number in $0 < \lambda < 1$.

To explore the nontrivial role of inversion (as defined in Sec.2) in the above discussion further, let us recall that the inversely related moments $t_-$ and $t_+$, with intrinsic uncertainties, are defined by $t_-t_+ = 1 + \delta$, where, $\delta$ is an $O(\tilde{T})$ random variable. In the present fractal time framework, one can model this random, nonlinear behaviour by the definition $t_+ = T(t_-)^{-1} = (1 + \lambda t_-^{-1}T(t_-))^{-1}$, so that $t_-t_+ = [1 + \lambda t_-^{-1}T(t_-)]$. One can now verify easily that $T = t_-T(t_-)^{-1}$ represents a nontrivial solution of eq(16). Clearly, the small (nonrandom part of ) parameter $\lambda$ avoids any clash with standard observations at moderate scales. In fact, $T = t$, till $t \sim O(\lambda^{1/2})$, influences of random multiple scales would be felt only in the longer time scales. The explicit form of a generic nontrivial solution indicating all the scales (analogous to the Weierstrass function) is still missing.

To understand the role of the index $n$ in the previous two sections, we now show that the fractal time $T(t)$ in fact represents a more general multiplicative process. Denoting $T_n = (1 + \mu \nu^nt_n\tilde{T}_n)t_n$, let us define the general fractal time $T$ by the multiplicative process, $T/t = (\tilde{T}/t\tilde{T}) = (\tilde{T})^2$ (by inversion symmetry), where $\ln \chi(t) = \Sigma \nu^{n+1} \ln T_n(t_n)$. It is now easy to check that $T$ satisfies the equation $t dT = T dt$, when each of $T_n$ solves $t_n dT_n = T_n dt_n$. To interprete $T$, we may imagine that a grand complex process represented by $T$ is materialized when the sequence of sub-processes $T_n$ is successfully materialized with respective probability of success $\nu^{n+1}$. The process is materialized (survived) at longer time scales $t \sim O(\nu^{-n})$ only with lesser probability. The short time correlation function $C(t)$ of the grand $T$ is obtained from the defining relation by taking expectation values: $\ln < T(t)t^{-1} > = 2\Sigma \nu^{n+1} \ln < T_n(t_n)^{-1} >$, so that $C(t) \sim t^2$. The correlation function $c(t)$ obtained in Sec.6 thus corresponds to the $n$th level fractal time $T_n$. The exponent of $C(t)$ correctly recovers the same obtained in Sec.4. Our method of solving eq(1) in Sec.3, giving rise to the solution $x_i$ in eq(8) thus directly leads to the correlation function $C(t)$ of the grand fractal time $T$. We note that any natural process, being manifested only over a finite period of time, would fail to display the grand fractal time.
exponent $\nu^2$, and would at most equal to a finite sum of powers of $\nu^2$, that too for large $n$.

8 Conclusion

Our study on the structure of time reveals a number of surprises. Time may indeed flow by inversions, leading to multiscale stochastic evolution for a linear system. Inversions ascribe a stochastic fractal like structure to time itself. We discuss two methods in uncovering the late time stochastic feature of a linear system. Repeated applications of inversion close to well defined scale dependent moments of the form $t_n = 1$, give rise to a new solution, self-similar over multiple scales, which can be interpreted as the two time correlation function of the corresponding stochastic equation, when the ordinary time variable $t$ is replaced by a fractal time $T(t)$. We present an ansatz for the fractal time which turns out to represent a new class of fractal solutions to the simplest linear differential equation. As a consequence, an ordinary linear system attains the status of a stochastic process. Our ansatz for fractal time provides a framework to compute the correlation function and the corresponding power spectrum of the process. We show that such a process would generically enjoy multiscale self-similarity leading to $1/f$ spectrum. We discuss intricate relations of the definition of time inversion, fractal time, irrationality of the golden mean and the fundamental uncertainty in measurability of a duration. Finally, we have discussed the relevance of intrinsic time in the present formalism drawing interesting analogies with the Wheeler-Dewitt equation. Our analysis show that the distinction between time and the system variable tends to get obliterated in longer time scales. We close with the remark that applications of the present fractal time approach is likely to yield new understanding mostly in the short and long time scales of dynamical systems. The scope of its applicability in quantum field theories also need not be overemphasised.

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