A Mermin–Wagner theorem for Gibbs states on Lorentzian Triangulations

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Abstract

We establish a Mermin–Wagner type theorem for Gibbs states on infinite random Lorentzian triangulations (LT) arising in models of quantum gravity. Such a triangulation is naturally related to the distribution $P$ of a critical Galton–Watson tree, conditional upon non-extinction. At the vertices of the triangles we place classical spins taking values in a torus $\mathbb{M}$ of dimension $d$, with a given group action of a torus $\mathbb{G}$ of dimension $d' \leq d$. In the main body of the paper we assume that the spins interact via a two-body nearest-neighbor potential $U(x, y)$ invariant under the action of $\mathbb{G}$. We analyze quenched Gibbs measures generated by $U$ and prove that, for $P$-almost all Lorentzian triangulations, every such Gibbs measure is $\mathbb{G}$-invariant, which means the absence of spontaneous continuous symmetry-breaking.

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1 Introduction

The goal of this paper is to establish a Mermin-Wagner type result (cf. [10]) for a spin system on a random graph generated by a causal dynamical Lorentzian triangulation (CDLT for short). The model of a CDLT was introduced in an attempt to define a gravitational path integral in a theory of quantum gravity. See [8] for a review of the relevant literature; for a rigorous mathematical background of the model, cf. [1]. More precisely, we analyze a spin system on a random 2D graph $T$ sampled from some natural “uniform” measure corresponding to a critical regime (see below). A Gibbs random field corresponding to a given interaction potential is considered on graph $T$, giving rise to a quenched semi-direct product measure.

We address the question of whether a continuous symmetry assumed for the interaction potential will be inherited by an infinite-volume Gibbsian state. It had been proved that the Hausdorff dimension of the critical CDLT equals 1 a.s. [2]. This means that the emerging graph is essentially an infinite 2D lattice. Consequently, one can expect the absence of spontaneous breaking of continuous symmetry. To prove this fact rigorously, we apply techniques developed in the papers [5], [11], [6].

2 Basic definitions. The main result

2.1 Critical Lorentzian triangulations

We work with so-called rooted infinite CDLTs in a cylinder $C = S \times [0, \infty)$, where $S$ stands for a unit circle. Physically speaking, this is a $(1 + 1)$-type system (one spatial and one temporal dimension). In a more realistic $(3 + 1)$-type case, a Mermin–Wagner type result would have been surprising.

Definition 1 Consider a connected graph $G$ with countably many vertices, $V(G)$, embedded in cylinder $C$. A face of $G$ is a connected component of $C \setminus G$. We say that $G$ determines a CDLT $T$ of $C$ if

(i) all vertices of $G$ lie in the circles $S \times \{j\}, j \in \mathbb{N} = \{0, 1, \ldots\}$;

(ii) each face of $G$ is a triangle;

(iii) each face lies in some strip $S \times [j, j + 1]$, $j \in \mathbb{N}$, and has one vertex on one of the circles $S \times \{j\}, S \times \{j + 1\}$ and two vertices (together with corresponding edge) lying on the other circle;

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(iv) the number of edges on circle \( S \times \{j\} \) is positive for any \( j \in \mathbb{N} \).

It has to be mentioned that some care is needed here when one defines a triangle, due to self-loops and multiple edges; cf. [7]. We will say that the vertices lying in the circle \( v \in S \times \{j\} \) belong to the \( j \)-th layer of the CDLT. Next, define the size of a face as the number of edges incident to it, with the convention that an edge incident to the same face on two sides counts for two. We then call a face of size 3 (or 3-sided face) a triangle.

Note that two vertices of a triangle on the same circle, say \( S \times \{j\} \), may coincide with each other (in this case the corresponding edge stretches over the whole circle \( S \times \{j\} \), i.e., forms a loop). Such a situation occurs, in particular, on the circle \( S \times \{0\} \). In this paper we consider only the case where the number of edges on the zero-level circle \( S \times \{0\} \) is equal to 1. This is a technical assumption made for simplifying the argument.

**Definition 2** The root in a CDLT \( T \) is a triangle \( t \) of \( T \), called the root face, with the anti-clockwise ordering of its vertices \((x, y, z)\), where \( x \) and \( y \) (and hence the edge \((x, y)\)) lie in \( S \times \{0\} \) (and \( x \) coincides with \( y \)) whereas \( z \) belongs to \( S \times \{1\} \). Vertex \( x \) is referred to as the root vertex. A CDLT with a root is called rooted.

**Definition 3** Two rooted CDLTs, \( T \) and \( T' \), are equivalent if (i) \( T \) and \( T' \) are embeddings \( i_T \), \( i_{T'} \) of the same graph \( G \), (ii) there exists a homeomorphism \( h : C \to C \) such that (i1) \( hi_T = i_{T'} \), (i2) \( h \) transforms each circle \( S \times \{j\}, j \in \mathbb{N} \) to itself and (i3) \( h \) takes the root of \( T \) to the root of \( T' \).

In what follows, rooted CDLTs are considered up to the above equivalence.

In the same way we also can define a CDLT of a cylinder \( C_N = S \times [0, N] \). Let \( LT_N \) and \( LT_\infty \) denote the set of CDLTs with support \( C_N \) and \( C \) correspondingly.

### 2.2 Tree parametrization of Lorentzian triangulations

Given a CDLT \( T \in LT_N \), define the subgraph \( \tau \subset T \) by removing, for each vertex \( v \in T \), the leftmost edge going from \( v \) upwards and discarding all horizontal edges, see Figure [1]. The graph \( \tau \) is formed by all the remaining edges of \( T \). The graph thus obtained is a spanning tree of \( T \). Moreover, if one associates with each vertex of \( \tau \) its height in \( T \) then \( T \) can be completely
reconstructed when we know $\tau$ (cf. [7] and references therein). We call $\tau$ the tree parametrization of $T$.

Figure 1: Tree parametrization

In a similar way we can show that there exists a one-to-one bijection $m$ between the set $LT_\infty$ and the set of infinite planar rooted trees $T_\infty$:

$$m : T_\infty \rightarrow LT_\infty.$$  

We will use the same symbol $m$ for the bijection $T_N \rightarrow LT_N$ where $T_N$ is the set of all planted rooted trees of height $N$.

Owing to the tree-parametrization, we are able to identify a measure $P^{lt}_N$ on CDLTs $LT_N$ as induced by a measure defined on trees $T_N$. Indeed, let $P^{tree}_N$ be a probability measure on $T_N$. Then the measure $P^{lt}_N$ on $LT_N$ is defined by

$$P^{lt}_N(m(\tau)) = P^{tree}_N(\tau), \text{ for any } \tau \in T_N.$$

Furthermore, if $P^{tree}$ is a probability measure on $T_\infty$ then the measure $P^{lt}$ on $LT_\infty$ is defined as

$$P^{lt}(m(A)) = P^{tree}(A), \text{ for any } A \subset T_\infty.$$  

For formal details of this construction, see [4]. In what follows we omit indices in the notation for measure $P$.

To construct a measure on $T_\infty$, we let $\nu = \{p_k\}$ be the offspring distribution on $k \in \mathbb{N} = \{0, 1, \ldots\}$, with mean 1. Next, let $(\xi_t)_{t \geq 1}$ be the corresponding critical Galton–Watson (GW) process. Considered under the condition that it never dies, process $(\xi_t)$ turns into the so-called size-biased
process \((k_t)\). In our setting, \(k_t\) stands for the number of vertices on the circle \(S \times \{t\}\). We refer the reader to [9] for the detailed description of such branching processes.

It is known that the size-biased process under consideration is supported on the subset \(S\) of \(\mathcal{T}_\infty\) formed by single-spine trees. A single-spine tree consists of a single infinite linear chain \(s_0, s_1, \ldots\), (here \(s_0\) is the root vertex of a tree) called the spine, to each vertex \(s_j\) of which there is attached a finite random tree with their root at \(s_j\). Next, the generating function for the branching number at each vertex \(s_j\) is \(f'(x)\) where \(f(x)\) is the generating function of the offspring distribution \(\nu\). Moreover, the individual branches are independently and identically distributed in accordance with the original critical GW process [9]. The distribution of the size-biased process \((k_t)\) generates the measure \(P\) as above.

Let \(\sigma^2\) stand for the variance of offspring distribution \(\nu\). We have

\[
E(k_t \mid k_{t-1}) = k_{t-1} + \sigma^2. \tag{2.1}
\]

Indeed, let \(\tilde{\nu} = \{\tilde{\nu}_k\}\) be the size biased offspring distribution: \(\tilde{\nu}_k = kp_k\). The distribution of \(k_t\) conditioned on the value \(k_{t-1}\) is constructed as follows (cf. [9]). We choose uniformly one particle from \(k_{t-1}\) particles and generate the number of its descendants according to the distribution \(\tilde{\nu}\) with mean \(\sigma^2 + 1\). The descendants of other \(k_{t-1} - 1\) particles are generated by the distribution \(\nu\). This explain the relation (2.1).

### 2.3 Gibbs measure on Lorentzian triangulations

Let \(M\) be a \(d\)-dimensional torus with a flat metric \(\rho\). Given a CDLT \(T \in \mathbb{LT}_\infty\), the configuration space of the spin system over \(T\) is \(X^{V(T)}\) where \(V(T) \subset C\) is a collection of vertices in \(T\). In the main body of the paper we assume that spins interact via a two-body nearest-neighbor potential \(U:\ (x, y) \in M \times M \mapsto \mathbb{R}\). Nearest-neighbor means that the spins are attached to endpoints of the same edge of the triangulation; see below. Let \(G\) be a torus of dimension \(d' \leq d\) and assume \(G\) acts as a group on \(M\) preserving metric \(\rho\). This action is extended to \(M^A\) where \(A \subseteq V(T)\):

\[
(g \ast \vec{x}_A)_v = g \ast x_v; \ v \in A, \ \vec{x}_A = \{x_v, v \in A\} \in M^A, \ g \in G. \tag{2.2}
\]

Here \(A \subseteq V(T)\) stands for a finite collection of vertices in \(T\). Next, let \(\mu\) be a \(G\)-invariant finite measure on \(M\).

We adopt the following assumptions upon \(U\) (cf. [3]).
A Invariance. For any $g \in G$ and any $x, y \in M$
\[ U(g \ast x, g \ast y) = U(x, y). \]

B Differentiability. For any $x, y \in M$, there exist continuous second derivatives $\nabla_x \nabla_y U(x, y)$.

Given configurations $\bar{x}_A = \{x_v, v \in A\} \in M^A$ and $\bar{x}_V(T) \setminus A = \{x_v, v \in V(T) \setminus A\} \in M^V(T) \setminus A$, the energy $H(\bar{x}_A|\bar{x}_{V(T) \setminus A})$ of $\bar{x}_A$ in the external potential field generated by $\bar{x}_{V(T) \setminus A}$ is defined by
\[ H(\bar{x}_A|\bar{x}_{V(T) \setminus A}) = \sum_{\langle v, v' \rangle \in A \times A} U(x_v, x_{v'}) + \sum_{\langle v, v' \rangle \in A \times (V(T) \setminus A)} U(x_v, x_{v'}). \]
where we use a standard notation $\langle v, v' \rangle$ for a nearest-neighbor pair. The conditional Gibbs probability density in volume $A$ with the boundary condition $x_{V(T) \setminus A}$ is defined by
\[ P(\bar{x}_A | \bar{x}_{V(T) \setminus A}) = \frac{\exp\{-H(\bar{x}_A | \bar{x}_{V(T) \setminus A})\}}{\int \exp\{-H(\bar{x}^*_A | \bar{x}_{V(T) \setminus A})\} \mu_A(d\bar{x}^*_A)}, \]
where $\mu_A$ is the product-measure on $M^A$. A Gibbs measure $\mathcal{P}$ on $M^V(T)$ with potential $U$ is determined by the property that $\forall$ finite $A \subset V(T)$, the conditional density on $M^A$ generated by $\mathcal{P}$ coincides with $P(\bar{x}_A | \bar{x}_{V(T) \setminus A})$ for $\mathcal{P}$-a.s. $\bar{x}_{V(T) \setminus A} \in M^V(T) \setminus A$ and $\mu_A$-a.s. $\bar{x}_A \in M^A$. Let $M^T = M^V(T)$ stands for the space of all configurations of spins equipped with the $\sigma$-algebra $\mathcal{A}_T$.

The main result of this note is

**Theorem 2.1** Assume that the offspring distribution $\nu$ has the mean 1 with finite second moment. Let $\mathcal{P}$ be the corresponding size-biased Galton-Watson tree distribution.

If the potential $U$ satisfies assumptions A – B, then every Gibbs state $\mathcal{P}$ with this potential is a $G$-invariant measure on the space $(M^T, \mathcal{A}_T)$ for $\mathcal{P}$-a.a. CLDT $T \in LT_\infty$.

3 The proof: the Fröhlich–Pfister argument

The proof is based on techniques developed in [5], [11], [6]. First, we establish the following upper bound for the number of vertices $k_t$ on circle $S \times \{t\}$ under
the measure \( P: \exists \) a constant \( C \) (depending on a realization \( T \)) such that

\[
k_t \leq C t \ln^{\frac{1}{2}+\varepsilon} t, \quad t = 1, 2, \ldots \quad P - a.s.
\]  

(3.1)

In order to verify (3.1), we note that, according to (2.1) the process \( \tilde{k}_t = k_t - t \sigma^2 \) is a martingale:

\[
E(\tilde{k}_t | \tilde{k}_{t-1}) = E(k_t | k_{t-1}) - t \sigma^2 = k_{t-1} - (t - 1) \sigma^2 = \tilde{k}_{t-1}.
\]

It implies, in particular, that the series

\[
B_n = \sum_{t=1}^{n} \frac{\tilde{k}_t - \tilde{k}_{t-1}}{a_t}
\]

is a martingale for any sequence of numbers \( \{a_t\}, t = 1, 2, \ldots \). Hence, we have:

\[
\frac{1}{a_n} \tilde{k}_n = \frac{1}{a_n} \sum_{t=1}^{n} a_k (B_t - B_{t-1}).
\]

Moreover, we can show that for some \( C_1, C_2 > 0 \)

\[
E[(k_t - k_{t-1} - \sigma^2)^2 | k_{t-1}] \leq C_1 + C_2 k_{t-1}.
\]

This follows from the representation

\[
k_t - k_{t-1} - \sigma^2 = \xi_0 - E\xi_0 + \sum_{i=1}^{k_{t-1}-1} (\xi_i - E\xi_i),
\]

where \( \xi_0 \) has the distribution \( \tilde{\nu} \) and IID RVs \( \xi_i, i = 1, \ldots, k_{t-1} - 1 \) are distributed according to \( \nu \). Now the martingale property yields

\[
EB_n^2 = \sum_{t=1}^{n} \frac{C_1 + C_2 E\xi_{t-1}}{a_t^2} < \sum_{t=1}^{\infty} \frac{C_1' + C_2' t}{a_t^2}.
\]

(3.2)

This series converges if we choose \( a_t = t(ln t)^{\frac{1}{2}+\varepsilon} \), for any \( \varepsilon > 0 \). Thus, by Kroneker’s lemma we have

\[
\frac{k_t}{t(ln t)^{\frac{1}{2}+\varepsilon}} \to 0, \quad P\text{-a.s.}
\]

(3.3)

That proves the bound (3.1).
Set \( A \subset V(T_r) \) where \( T_r \) denotes the union of the first \( r \) layers of \( T \). As before \( k_t \) stands for the number of vertices in \( T \) lying in the layer \( t \). Let us identify \( g \in G \) with the vector of angles \( \theta \) and for any \( n > r + 1 \) define the gauge action \( g_n(v) \equiv g, v \in T_{r+1}, g_n(v) \equiv e, v \in T \setminus T_n \). Here \( e \) denotes the unit element of the group \( G \). On the layer \( j, r + 1 < j < n \), we set

\[
g_n(v) = \theta \frac{1}{Q(n - r)} \sum_{t = j + 1 - r}^{n-j} \frac{1}{t \ln t}, \quad \text{where } Q(n - r) = \sum_{t = 2}^{n-r} \frac{1}{t \ln t}. \tag{3.4}
\]

Our aim is to check the identity \( P(A) = P(g \ast A) \). Using technique developed in [6] (following [5], [11]), our task is reduced to checking the convergence of the series as \( n \to \infty \)

\[
\phi = \sum_{(v,v')} (g_n(v) - g_n(v'))^2 \leq \frac{|\theta|^2}{\ln \ln (n - r)} \sum_{t = 2}^{n-r} \frac{E_{t,t+1}}{t^2 \ln^2 t}. \tag{3.5}
\]

where \( E_{t,t+1} \) is the number of edges that connect the vertices of \( T \) from two different levels: \( S \times \{t\} \) and \( S \times \{t + 1\} \). By the construction of the triangulation, \( E_{t,t+1} = k_t + k_{t+1} \). Thus,

\[
\sum_{t = 3}^{n-r} \frac{E_{t,t+1}}{t^2 \ln^2 t} = \sum_{t = 3}^{n-r} \frac{k_t + k_{t+1}}{t^2 \ln^2 t} < Const \sum_{t \geq 2} \frac{k_t}{t^2 \ln^2 t}
\]

By (3.3) the last series is

\[
\sum_{t \geq 2} \frac{k_t}{t^2 \ln^2 t} < \infty \quad P \text{ - a.s.} \tag{3.6}
\]

Hence, the sum \( \phi \) in (3.5) goes to 0 when \( n \to \infty \). This finishes the proof of the theorem. \( \square \)

4 The case of the long range interaction

The above model can be naturally extended to the case of long range interaction. Namely, we define the energy \( H(\bar{x}_{T_N} \mid \bar{x}_{T_N}^c) \) of a spin configuration \( \bar{x}_{T_N} \in M^{T_N} \) in the external potential field induced by a configuration \( \bar{x}_{T_N}^c \in M^{T_N^c} \) (where \( T_N^c = V(T) \setminus T_N \)) as follows:

\[
H(\bar{x}_{T_N} \mid \bar{x}_{T_N}^c) = \sum_{(v,v') \in T_N \times T_N} U_{v,v'}(x_v, x_{v'}) + \sum_{(v,v') \in T_N \times T_N^c} U_{v,v'}(x_v, x_{v'}). \tag{4.1}
\]
Assume the following uniform upper bound on the interaction potentials $U_{v,v'}(x_v, x_{v'})$:

$$|U_{v,v'}(x, x')| \leq J(d(v, v')), \quad x, x' \in M.$$ 

Here $d(v, v')$ stands for the distance on graph $T$. Suppose the majorizing function $J$ satisfies, $\mathbb{P}$-a.s., the following properties:

$$\sup_{v \in \Lambda} \left[ \sum_{v' \in \Lambda} J(d(v, v'))d^2(v, v') \right] < \infty$$

and

$$\sup_{v \in \Lambda} \left[ \sum_{v' \in \Lambda} J(d(v, v'))1(d^2(v, v') \geq L) \right] \to 0, \text{ when } L \to \infty.$$ 

Then, as has been shown in [6], the assertion of Theorem 1 remains true. Using the a.s. estimation (3.1), the above conditions are satisfied if, for example,

$$J(r) \leq \left( \frac{1}{r \ln r} \right)^3.$$ 

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