A NOTE ON TIME ANALYTICITY FOR ANCIENT SOLUTIONS OF
THE HEAT EQUATION

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Abstract. It is well known that generic solutions of the heat equation are not analytic
in time in general. Here it is proven that ancient solutions with exponential growth are
analytic in time in $M \times (-\infty, 0]$. Here $M = \mathbb{R}^n$ or is a manifold with Ricci curvature
bounded from below. Consequently a necessary and sufficient condition is found on the
solvability of backward heat equation in the class of functions with exponential growth.

1. Statement of result and proof

The goal of the note is to prove time analyticity of certain generic ancient solutions
to the heat equation on $M \times (-\infty, 0]$ where $M = \mathbb{R}^n$ or some noncompact Riemannian
manifolds. This is somewhat unexpected since it is well known that generic solutions to
the forward heat equation are not analytic in time. Even if $M = \mathbb{R}^1$ and $u$ is analytic in
$x$ when $t > 0$, time analyticity requires at least the initial value is analytic in $x$, c.f. [Wi]
Corollary 3.1 b. Ancient solutions of evolution equations are important since they not only
represent structures of solutions near a high value but also can be regarded as solutions
of the backward equation. Backward heat equations play important roles in stochastic
analysis and Ricci flows e.g. It is known that the Cauchy problem to the backward heat
equation is not solvable in general. One application of the main result is a necessary
and sufficient condition for solvability if the solutions grow no faster than exponential
functions. It is expected that the phenomenon observed here can be extended to many
other evolution equations.

Let us recall some relevant results for ancient solutions of the heat equation. Let
$M = \mathbb{R}^n$ or complete noncompact Riemannian manifold with nonnegative curvature. It
is proven in [SZ] that sublinear ancient solutions are constants. In [LZ] it is proven that
the space of ancient solutions of polynomial growth has finite dimension and the solutions
are polynomials in time. In the paper [CM1], a sharp dimension estimate of the space is
given. See the papers [Ca1] and [CM2] for applications to the study of mean curvature
flows, and also [Ha] in the graph case. Earlier, backward heat equations have been studied
by many authors, see [Mi], [Yo] e.g, and treated in many text books. A necessary and
sufficient solvability criteria seems lacking, except when $M$ is a bounded domain for which
semigroup theory gives an abstract criteria [Ca] Theorem 9.

In order to state the result, let us first introduce a bit of notations. We use $M$
to denote a $n$ dimensional, complete, noncompact Riemannian manifold, $\text{Ric}$ to denote the
Ricci curvature and 0 a reference point on $M$, $d(x, y)$ is the geodesic distance for $x, y \in M$. 

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Theorem 1.1. Let $M$ be a complete, $n$ dimensional, noncompact Riemannian manifold such that the Ricci curvature satisfies $\text{Ric} \geq -(n-1)K_0$ for a nonnegative constant $K_0$.

Let $u$ be a smooth, ancient solution of the heat equation $\Delta u - \partial_t u = 0$ of exponential growth, namely
\begin{equation}
|u(x,t)| \leq A_1 e^{A_2(d(x,0)+|t|)}, \quad \forall (x,t) \in M \times (-\infty,0],
\end{equation}
where $A_1, A_2$ are positive constants and $0$ is a reference point on $M$. Then $u = u(x,t)$ is analytic in $t$ with radius $\infty$, moreover,
\begin{equation}
u(x,t) = \sum_{j=0}^{\infty} a_j(x) \frac{t^j}{j!},
\end{equation}
with $\Delta a_j(x) = a_{j+1}(x)$. In addition,
\begin{equation}
|a_j(x)| \leq A_3 e^{A_4(j+\delta(x,0))}, \quad j = 0, 1, 2, \ldots;
\end{equation}
where $A_3, A_4$ are positive constants depending on $A_1, A_2, K_0, n$, and $A_3$ also depends on $|B(0,1)|^{-1}$.

Proof.

Let us start with a well known parabolic mean value inequality which can be found in Theorem 14.7 of $[L]$ e.g. Suppose $u$ is a positive subsolution to the heat equation on $M \times [0,T]$. Let $T_1, T_2 \in [0,T]$ with $T_1 < T_2$, $R > 0$, $p > 0$, $\delta, \eta \in (0,1)$. Then there exist positive constants $C_1$ and $C_2$, depending only on $p, n$ such that
\begin{equation}
\sup_{B(0,(1-\delta)R) \times [T_1, T_2]} u^p \leq C_1 \frac{\tilde{V}(R)}{|B(0,R)|} (R^{\sqrt{K_0}} + 1) \exp(C_2 \sqrt{K_0} (T_2 - T_1))
\end{equation}
\begin{equation}
\times \left( \frac{1}{\delta R} + \frac{1}{\eta T_1} \right)^{n+2} \int_{(1-\eta)T_1}^{T_2} \int_{B(0,R)} u^p(y,s) dy ds;
\end{equation}
Here $\tilde{V}(R)$ is the volume of geodesic balls of radius $R$ in the simply connected space form with constant sectional curvature $-K_0$, $|B(0,R)|$ is the volume of the geodesic ball $B(0,R)$ with center $0$ and radius $R$.

Let $u$ be an ancient solution to the heat equation. Then $u^2$ is a subsolution. Given a positive integer $j$, by shifting the time suitably, we can apply the mean value inequality to deduce
\begin{equation}
\sup_{B(0,j) \times [-j,0]} u^2 \leq C_1 e^{C_2 \sqrt{K_0} j} \int_{-j+1}^{0} \int_{B(0,j+1)} u^2(y,s) dy ds;
\end{equation}
where we have used the Bishop-Gromov volume comparison theorem. Note that the constants $C_1$, $C_2$ may have changed and $C_1$ now also depends on the lower bound of $|B(0,1)|$, i.e. the volume noncollapsing constant. For each $j$, observe that $\partial_t^j u$ is also a solution of the heat equation. Applying (1.5) with $u$ replaced by $\partial_t^j u$, we deduce
\begin{equation}
\sup_{Q(0,j)} (\partial_t^j u)^2 \leq C_1 e^{C_2 \sqrt{K_0} j} \int_{Q(0,j+1)} (\partial_t^j u)^2(y,s) dy ds,
\end{equation}
where we have used the notation $Q(0,j) \equiv B(0,j) \times [-j,0]$ to denote the space time cube of size $j$ with vertex $(0,0)$. Note that this is not the standard parabolic cube since spatial and time scale is the same.
Denote by \( \psi \) a standard Lipschitz cut off function supported in \( Q(0, j + 0.5) \) such that \( \psi = 1 \) in \( Q(0, j) \) and \( |\nabla \phi|^2 + |\partial_t \psi| \leq C \). Since \( u \) is a smooth solution to the heat equation, we compute

\[
\int_{Q(0, j+0.5)} (\Delta u)^2 \psi^2 \, dx \, dt = \int_{Q(0, j+0.5)} u_t \Delta u \psi^2 \, dx \, dt
\]

\[
= - \int_{Q(0, j+0.5)} ((\nabla u)_t \nabla u) \psi^2 \, dx \, dt - \int_{Q(0, j+0.5)} u_t \nabla u \nabla \psi^2 \, dx \, dt
\]

\[
= - \frac{1}{2} \int_{Q(0, j+0.5)} (|\nabla u|_t^2 \psi^2 \, dx \, dt - 2 \int_{Q(0, j+0.5)} u_t \psi \nabla u \nabla \psi \, dx \, dt
\]

\[
\leq \frac{1}{2} \int_{Q(0, j+0.5)} |\nabla u|^2 (\psi^2) \, dx \, dt + \frac{1}{2} \int_{Q(0, j+0.5)} (\psi)^2 \psi^2 \, dx \, dt
\]

\[
+ 2 \int_{Q(0, j+0.5)} |\nabla u|^2 |\nabla \psi|^2 \, dx \, dt.
\]

Therefore

\[
\int_{Q(0, j+0.5)} (\Delta u)^2 \psi^2 \, dx \, dt \leq C \int_{Q(0, j+0.5)} |\nabla u|^2 \, dx \, dt.
\]

This and the standard Cacciopoli inequality (energy estimate) between the cubes \( Q(0, j + 0.5) \) and \( Q(0, j + 1) \) show that

\[
(1.7) \int_{Q(0, j)} (\Delta u)^2 \, dx \, dt \leq C_0 \int_{Q(0, j+1)} u^2 \, dx \, dt.
\]

Here \( C_0 \) is a universal constant.

Since \( \partial_i^j u \) is also an ancient solution, we can replace \( u \) in (1.7) by \( \partial_i^j u \) to deduce

\[
\int_{Q(0, j+1)} (\partial_i^j u)^2 \, dx \, dt = \int_{Q(0, j+1)} (\Delta \partial_i^{j-1} u)^2 \, dx \, dt \leq C_0 \int_{Q(0, j+2)} (\partial_i^{j-1} u)^2 \, dx \, dt
\]

By induction, we deduce

\[
(1.8) \int_{Q(0, j+1)} (\partial_i^j u)^2 \, dx \, dt \leq C_0^j \int_{Q(0, 2j+1)} u^2 \, dx \, dt.
\]

Substituting (1.8) to (1.6), we find that

\[
\sup_{Q(0,j)} (\partial_i^j u)^2 \leq C_1 e^{C_2 \sqrt{\kappa_0 j}} C_0^j \int_{Q(0, 2j+1)} u^2 \, dx \, dt,
\]

for all \( i = 0, 1, 2, \ldots j \). This implies, by the exponential growth condition (1.1) and the Bishop-Gromov volume comparison theorem, that

\[
(1.9) \sup_{Q(0,j)} |\partial_i^j u| \leq C_1 C_3^j
\]

for all \( i = 0, 1, 2, \ldots j \). Here \( C_3 \) is a positive constant.

Fixing a number \( R \geq 1 \), for \( x \in B(0, R) \), choose an integer \( j \geq R \) and \( t \in [-j^2, 0] \). Taylor’s theorem implies that

\[
(1.10) u(x, t) - \sum_{i=0}^{j-1} \partial_i^j u(x, 0) \frac{t^i}{i!} = \frac{t^j}{j!} \partial_i^j u(x, s),
\]

where \( s \) is a point between \( x \) and \( x + t \partial_i^j u(x, 0) \).
where \( s = s(x, t, j) \in [t, 0] \). By (1.9), the right hand side of (1.10) converges to 0 uniformly on \( Q(0, R) \) as \( j \to \infty \). Hence

\[
(1.11) \quad u(x, t) = \sum_{i=0}^{\infty} \partial^i_t u(x, 0) \frac{t^i}{i!}
\]

i.e. \( u \) is analytic in \( t \) with radius \( \infty \). Writing \( a_j = a_j(x) = \partial^j_t u(x, 0) \). By (1.9) again, we have

\[
(1.12) \quad \partial_t u(x, t) = \sum_{j=0}^{\infty} a_{j+1}(x) \frac{t^j}{j!},
\]

\[
(1.13) \quad \Delta u(x, t) = \sum_{j=0}^{\infty} \Delta a_j(x) \frac{t^j}{j!},
\]

where both series converge uniformly on \( Q(0, R) \) for any fixed \( R > 0 \). Since \( u \) is a solution of the heat equation, this implies

\[
\Delta a_j(x) = a_{j+1}(x)
\]

with

\[
|a_j(x)| \leq A_3 e^{A_4(j + d(x, 0))}.
\]

Here \( A_3 \) and \( A_4 \) are positive constants with \( A_3 \) depending on \( |B(0, 1)|^{-1} \). This completes the proof of the theorem.

An immediate application is the following:

**Corollary 1.** Let \( M \) be as in the theorem. Then the Cauchy problem for the backward heat equation

\[
(1.14) \quad \begin{cases} 
\Delta u + \partial_t u = 0, \quad M \times [0, \infty); \\
u(x, 0) = a(x)
\end{cases}
\]

has a smooth solution of exponential growth if and only if

\[
(1.15) \quad |\Delta^j a(x)| \leq A_3 e^{A_4(j + d(x, 0))}, \quad j = 0, 1, 2, \ldots;
\]

where \( A_3, A_4 \) are positive constants depending on \( A_1, A_2, K_0, n, \) and \( A_3 \) also depends on \( |B(0, 1)|^{-1} \).

**Proof.**

Suppose (1.14) has a smooth solution of exponential growth, say \( u = u(x, t) \). Then \( u(x, -t) \) is an ancient solution of the heat equation with exponential growth. By the theorem

\[
u(x, -t) = \sum_{j=0}^{\infty} \Delta^j a(x) \frac{(-t)^j}{j!}
\]

Then (1.15) follows from the theorem since \( \Delta^j a(x) = a_j(x) \) in the theorem.

On the other hand, suppose (1.15) holds. Then it is easy to check that

\[
u(x, t) = \sum_{j=0}^{\infty} \Delta^j a(x) \frac{t^j}{j!}
\]
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is a smooth ancient solution of the heat equation. Indeed, the bounds (1.15) guaranty that the above series and the series

\[ \sum_{j=0}^{\infty} \Delta^{j+1} a(x) \frac{t^j}{j!}, \quad \sum_{j=0}^{\infty} \Delta^j a(x) \frac{\partial_t t^j}{j!}, \]

all converge absolutely and uniformly in \( Q(0, R) \) for any fixed \( R > 0 \). Hence \( \Delta u - \partial_t u = 0 \).

Moreover \( u \) has exponential growth since

\[ |u(x,t)| \leq \sum_{j=0}^{\infty} |\Delta^j a(x)| \frac{|t|^j}{j!} \leq A_3 e^{A_4 d(x,0)} \sum_{j=0}^{\infty} \left( e^{A_4 |t|} \right)^j = A_3 e^{A_4 d(x,0)+A_4 |t|}. \]

Thus \( u(x,-t) \) is a solution to the Cauchy problem of the backward heat equation (1.14) of exponential growth.

**Remark 1.1.** For the conclusion of the theorem to hold, some growth condition for the solution is necessary. Tychonov’s non-uniqueness example can be modified as follows. Let \( v = v(x,t) \) be Tychonov’s solution of the heat equation in \( \mathbb{R}^n \times (-\infty, \infty) \), which is 0 when \( t \leq 0 \) but nontrivial for \( t > 0 \). Then \( u \equiv v(x,t+1) \) is a nontrivial ancient solution. It is clearly not analytic in time. Note that \( |u(x,t)| \) grows faster than \( e^{c|x|^2} \) for some \( x \) and \( t \).

We are not sure if this bound is sufficient for time analyticity.

**Remark 1.2.** If \( M = \mathbb{R}^n \), then the ancient solution in the theorem is also analytic in space variables. For general manifolds, space analyticity requires certain bounds on curvature and its derivatives.

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