NILPOTENT LEFT QUASIGROUPS

MARCO BONATTO

Abstract. In this paper we investigate central congruence of left quasigroups in the sense of Freese and McKenzie [FM87] and we extend some known results for quandles. In particular, we can extend the characterization of finite nilpotent latin quandles and the characterization of distributive varieties of quandles to the setting of idempotent left quasigroups.

1. Introduction

Left quasigroups are binary algebraic structures with a very combinatorial flavour. Many algebraic structures of interest have an underlying left quasigroup structure (e.g. racks and quandles [AG03, Joy82, Mat82], cycle sets [Rum07] etc) and therefore it is worth to study such structures as a common ground. In this paper we basically show that many results stated for quandles in [Bon20, Bon21b, BF21] can be extended to (idempotent) left quasigroups with a focus on nilpotent left quasigroups in the sense of the commutator theory developed in [H.76, FM87]. This theory generalized the well-known concepts of abeliannes, solvability and nilpotency in group theory to arbitrary algebraic structures and it has been specialized to the setting of racks and quandles in [BS21].

One of the goals of the paper is to remark the interplay between the lattice of congruences and the lattice of admissible subgroups defined in [Bon21a]. The two lattices are related by a Galois connection (see [Bon21a, Theorem 1.10]) that can be used to transfer information from one lattice to another.

For racks, the admissible subgroups encode many information, including also the property of abeliannes and centrality of congruences. In particular, such properties of congruences are reflected by the properties of the relative displacement groups [BS21, Lemma 5.1, Proposition 5.2]. The structure of the displacement group is also effected by the properties of congruences: for instance a rack is solvable (resp. nilpotent) if and only if its displacement group is solvable (resp. nilpotent) [BS21, Lemma 6.1, Lemma 6.2]. Moreover, we have a prime-power decomposition theorem for quandle [BS21, Section 6.2] and in particular connected racks of prime power order are nilpotent. For left quasigroups the relation between congruences and admissible subgroups is not so tight, nevertheless we show that some weaker results hold in this direction as Corollary 4.8 and Lemma 4.9.

Central congruences and semiregular admissible subgroups are closely related. In particular, some properties of central congruences are actually determined by the property of being semiregular for the action of the displacement group on every block of the congruence. Thus, we investigate congruences obtained from semiregular admissible subgroups through the Galois connection (under some further mild assumptions every central congruence arise in this way) and semiregular left quasigroups. These property are determined by a family of the equivalence relations already introduced for semimedial left quasigroups in [Bon21a]. We also obtain that semiregular quandles are the building blocks of idempotent left quasigroups (see Proposition 3.5). Moreover, abelian left quasigroups (in the sense of [FM87]) are semiregular and the idempotent ones are actually quandles (see Corollary 4.3(ii)).

The results collected in the first part of the paper can be used to investigate Malt’sev varieties of idempotent left quasigroups. In particular, Theorem 5.3, Proposition 5.4, Proposition 5.5 and Corollary 5.7 are direct generalization of known results for quandles [BF21].

The paper is organized as follows: in Section 2 we recall some basic facts about left quasigroups, congruences and admissible subgroups and the Cayley kernel. In Section 3 we define and study semiregular left quasigroups and semiregular admissible subgroups. In Section 4 we recall some basic facts about commutator theory and we explore central congruences and central extensions of left quasigroups. In the last Section we turn out attention to Malt’sev varieties of left quasigroups and we extend some results from [BF21].

1.1. Notation and terminology. An algebraic structure is given by a set $A$ with an arbitrary set of basic operations $F$. A term on $A$ is either a variable or an expression $f(t_1, \ldots, t_n)$, where $t_1, \ldots, t_n$ are
terms on \( A \) and \( f \) is a basic \( n \)-ary operation of \( A \). A term \( t \) is idempotent if \( t(x, \ldots, x) \equiv x \) holds. An algebra \( A \) is idempotent if every term on \( A \) is idempotent.

Let \( \alpha \) be an equivalence relation on a set \( A \). We denote the class of \( x \) with respect to \( \alpha \) by \([x]_{\alpha}\) (or simply by \([x]\) in some cases). An equivalence relation \( \alpha \) on \( A \) is a congruence if it is compatible with the algebraic structure. Namely, if \( f \) is an \( n \)-ary basic operation then

\[
f(x_1, x_2, \ldots, x_n) \equiv f(y_1, y_2, \ldots, y_n)
\]

provided \( x_i \equiv y_i \) for \( i = 1, \ldots, n \). The set of congruences of \( Q \) is a lattice denoted by \( \text{Con}(A) \) with top element \( 1_A = A \times A \) and bottom element \( 0_A = \{(x, x) : x \in A\} \). Congruences and morphisms are essentially the same thing. Indeed if \( h : A \to A' \) is a morphism then \( \ker(h) = \{(x, y) \in A : h(x) = h(y)\} \) is a congruence of \( A \). On the other hand if \( \alpha \) is a congruence, the factor set is endowed with a natural algebraic structure: indeed for every basic \( n \)-ary operation \( f \) on \( A \), the map given by

\[
f([x_1]_\alpha, \ldots, [x_n]_\alpha) = [f(x_1, \ldots, x_n)]_\alpha
\]

for every \( x_1, \ldots, x_n \in A \) is a well-defined \( n \)-ary operation on \( A/\alpha \). The canonical map

\[
A \to A/\alpha, \quad x \mapsto [x]_\alpha
\]

is a morphism with respect to such structure.

The congruences of \( A/\alpha \) are \( \{\beta/\alpha : \beta \leq \alpha \in \text{Con}(A)\} \) where \( \beta/\alpha \) is defined by setting

\[
[x]_\beta \equiv [y]_\alpha \iff x \beta y
\]

for every \( x, y \in A \). Note that if \( A \) is idempotent then the blocks of congruences of \( A \) are subalgebras.

A variety is a class of algebraic structures closed under subalgebras, homomorphic images and direct products. We denote by \( \mathcal{S}(A) \) the set of subalgebras of \( A \).

A Maltsev term is a ternary (idempotent) term \( m \) satisfying the identities

\[
m(x, y, y) = m(x, y, x) \equiv x.
\]

We say that an algebraic structure \( A \) has a Maltsev term if the variety generated by \( A \) has a Maltsev term.

Let \( \mathcal{K} \) be a class of idempotent algebraic structures. We say that \( \mathcal{K} \) is closed under extensions if \( A \in \mathcal{K} \) provided that \( A/\alpha \in \mathcal{K} \) and \([x]_\alpha \in \mathcal{K}\) for every \( x \in A \) for some \( \alpha \in \text{Con}(A) \).

Let us recall some group theoretical terminology. Let \( G \) be a group acting on a set \( Q \) and \( x \in Q \). We denote the pointwise stabilizer of \( x \) by \( G_x \) and the orbit of \( x \) under the action of \( G \) by \( x^G \). The group \( G \) is semiregular if the pointwise stabilizers of the action of \( G \) are trivial and transitive if \( Q = x^G \) for every \( x \in Q \). If \( G \) is semiregular and transitive on \( Q \), we say that \( G \) is regular on \( Q \).

2. Preliminary results

2.1. Left quasigroups. A left quasigroup is a binary algebraic structure \( (Q, \cdot, \setminus) \) that satisfies the following identities

\[
x \cdot (x \setminus y) = y \equiv x \setminus (x \cdot y).
\]

We define the left and right multiplication mapping as

\[
L_x : y \mapsto x \cdot y, \quad R_x : y \mapsto y \cdot x
\]

for every \( x \in Q \). The map \( L_x \) is a permutation for every \( x \in Q \) (note that \( x \setminus y = L_x^{-1}(y) \) for every \( x, y \in Q \)) and so we can introduce the left multiplication group as \( \text{LMlt}(Q) = \{L_x, x \in Q\} \). In the following we will denote the \( \cdot \) operation just by juxtaposition. We define the set of idempotent elements of \( Q \) as \( E(Q) = \{x \in Q : xx = x\} \). Note that if \([x]_\alpha \in E(Q/\alpha)\) then the block of \( x \) with respect to \( \alpha \) is a subalgebra of \( Q \). We say that \( Q \) is

(i) idempotent if \( Q = E(Q) \), i.e. the identity \( xx \equiv x \) holds.

(ii) Projection if the identity \( xy \equiv y \) holds. We denote by \( P_n \) the projection left quasigroups with \( n \) elements.

(iii) A rack if the identity \( x(yz) \equiv (xy)(xz) \) holds (or equivalently \( L_x \in \text{Aut}(Q) \) for every \( x \in Q \)). Idempotent racks are called quandles.

(iv) Latin if the right multiplications are bijective. In this case a binary operation can be defined as \( x/y = R_y^{-1}(x) \) for \( x, y \in Q \). If \( Q \) is infinite, the universal algebraic features (e.g. congruences, subalgebras etc) of the associated quasigroup structure \( (Q, \cdot, \setminus) \) might be different from the one of the left quasigroup \( (Q, \cdot, \setminus) \).
2.2. Congruences and subgroups. Let $Q$ be a left quasigroup and $\alpha$ be a congruence of $Q$. Then we have a canonical surjective group homomorphism, defined on generators as

$$\pi_\alpha : \text{LMlt}(Q) \rightarrow \text{LMlt}(Q/\alpha), \quad L_x \mapsto L_{[x/\alpha]}.$$

In particular, we have that $\pi_\alpha(h)([x/\alpha]) = [h(x)]/\alpha$ for every $x \in Q$ and every $h \in \text{LMlt}(Q)$. The kernel of the map $\pi_\alpha$ is denoted by $\text{LMlt}^\alpha$. We also define the displacement group relative to $\alpha$ as

$$\text{Dis}_\alpha = \{hL_xL_{y^{-1}}h^{-1}, x, y, h \in \text{LMlt}(Q)\},$$

namely the normal closure of the set $\{L_xL_{y^{-1}} : x, y \in Q\}$ in $\text{LMlt}(Q)$. In particular we denote by $\text{Dis}(Q)$ the displacement group relative to $1_Q$ and by $\text{Dis}^\alpha = \text{Dis}(Q) \cap \text{LMlt}^\alpha$. We also define the blockwise stabilizer of $x \in Q$ as $\text{Dis}(Q)[x]_\alpha = \pi_\alpha^{-1}(\text{Dis}(Q/\alpha)[x]) = \{h \in \text{Dis}(Q) : h(x) = \alpha x\}$. In particular note that $\text{Dis}(Q)_x \leq \text{Dis}(Q)[x]_\alpha$ for every $x \in Q$ and

$$\text{Dis}_\alpha \leq \text{Dis}^\alpha = \bigcap_{[x/\alpha] \oplus \alpha} \text{Dis}(Q)[x]_\alpha = \{h \in \text{Dis}(Q) : h(x) = \alpha x \text{ for every } x \in Q\}.$$

The subgroups $\text{Dis}_\alpha$ and $\text{Dis}^\alpha$ can be defined in the very same way as in (2) and as in (3) for every binary relation $\alpha$ on $Q$. Therefore we have two operators, $\text{Dis}$ and $\text{Dis}^\alpha$ from the equivalence relations on $Q$ to the subgroups of $\text{Dis}(Q)$ (we can also define the operator $\text{LMlt}^\alpha : \alpha \mapsto \text{LMlt}^\alpha \leq \text{LMlt}(Q)$).

**Lemma 2.1.** [BS21] Proposition 3.2] Let $Q$ be a left quasigroup and $\alpha, \beta \in \Con(Q)$ such that $\alpha \leq \beta$. Then:

$$\text{Dis}_{\beta/\alpha} = \pi_\alpha(\text{Dis}_\beta), \quad \text{Dis}^\beta/\alpha = \pi_\alpha(\text{Dis}^\beta), \quad \text{LMlt}^{\beta/\alpha} = \pi_\alpha(\text{LMlt}^\beta).$$

In particular, the map $(\mathbb{1})$ restricts and corestricts to $\text{Dis}(Q)$ and $\text{Dis}(Q/\alpha)$ and the kernel of the restricted map is $\text{Dis}^\alpha$.

**Proposition 2.2.** Let $Q$ be a left quasigroup and $\{\alpha_i : i \in I\} \subseteq \Con(Q)$, $\beta = \bigwedge_{i \in I} \alpha_i$ and $\gamma = \bigvee_{i \in I} \alpha_i$. Then:

(i) $\text{Dis}(Q)[x]_\beta = \bigcap_{i \in I} \text{Dis}(Q)[x]_{\alpha_i}$ for every $x \in Q$.

(ii) $\text{Dis}^\beta = \bigcap_{i \in I} \text{Dis}^{\alpha_i}$.

(iii) $\text{Dis}_\gamma = \{\text{Dis}_{\alpha_i}, i \in I\}.$

**Proof.** (i) We have that

$$\text{Dis}(Q)[x]_\beta = \left\{h \in \text{Dis}(Q) : h(x) = \beta x\right\}$$

$$= \{h \in \text{Dis}(Q) : h(x) = \alpha_i x \text{ for all } i \in I\} = \bigcap_{i \in I} \{h \in \text{Dis}(Q) : h(x) = \alpha_i x\}$$

$$= \bigcap_{i \in I} \text{Dis}(Q)[x]_{\alpha_i}.$$

(ii) According to (3) and (1), we have $\text{Dis}^\beta = \bigcap_{i \in I} \text{Dis}^{\alpha_i}$.

(iii) It is easy to see that $\text{Dis}_{\alpha_i} \leq \text{Dis}_\gamma$, and thus $\{\text{Dis}_{\alpha_i}, i \in I\} \subseteq \text{Dis}_\gamma$. For the other inclusion, let $x \gamma y$, and take the witnesses $x = z_1, \ldots, z_n = y$ such that $z_k \alpha_j z_{k+1}$, for some $j_k \in I$ and every $k$. Then

$$L_xL_{y^{-1}} = L_{z_1}L_{z_1^{-1}} \cdots L_{z_n}L_{z_n^{-1}} \in \text{Dis}_{\alpha_1} \cap \text{Dis}_{\alpha_2} \cap \cdots \cap \text{Dis}_{\alpha_n},$$

and thus every generator $f L_xL_{y^{-1}}f^{-1}$ of $\text{Dis}_\gamma$ belongs to $\{\text{Dis}_{\alpha_i}, i \in I\}$. □

For a left quasigroup $Q$ and a subgroup $N \leq \text{LMlt}(Q)$ we can define two equivalence relations as

$$x \bigodot y \text{ if and only if } x = h(y) \text{ for some } h \in N,$n

$$x \con N y \text{ if and only if } L_xL_y^{-1} \in N.$$n

Hence we have two operators $\con_\bullet$ and $\con_\circ$ from the set of subgroups of $\text{LMlt}(Q)$ to the set of the equivalence relations on $Q$.

We say that $Q$ is connected by $N$ if $N$ is transitive on $Q$, i.e. $\con_\circ N = 1_Q$. If $\text{LMlt}(Q)$ is transitive we simply say that $Q$ is connected. Note that if $Q$ is a connected idempotent left quasigroups or a left quasigroup with a Mal’cev term then $Q$ is connected by $\text{Dis}(Q)$ [BF21] Proposition 3.6]. If all the subalgebras of $Q$ are connected we say that $Q$ is superconnected (in particular $Q$ is connected). The class of superconnected left quasigroups is closed under subalgebras and homomorphic images and the class of superconnected idempotent left quasigroups is also closed under extensions [Bon211B Corollary 1.12].
In particular, finite latin left quasigroups are superconnected but the converse is not true in general (see [Bon21b] Example 1.8(ii)) for an example of an infinite latin quandle that is not superconnected.

The admissible subgroups of $Q$, as defined in [Bon21a], are

$$\text{Norm}(Q) = \{N \leq \text{LMlt}(Q) : \mathcal{O}_N \leq \text{con}_N\} = \{N \leq \text{LMlt}(Q) : \text{Dis}_{\mathcal{O}_N} \leq N\}.$$  

The admissible subgroups form a sublattice of the lattice of the normal subgroups of $\text{LMlt}(Q)$. If $N \in \text{Norm}(Q)$ then $\mathcal{O}_N$ is a congruence of $Q$ and $\text{Dis}_\alpha, \text{Dis}^\alpha, \text{LMlt}^\alpha \in \text{Norm}(Q)$ [Bon21a Lemma 1.7, Corollary 1.9]. It is easy to verify that

$$\mathcal{O}_{\text{Dis}_\alpha} \leq \mathcal{O}_{\text{Dis}^\alpha} \leq \mathcal{O}_{\text{LMlt}^\alpha} \leq \alpha \leq \text{con}_{\text{Dis}_\alpha} \leq \text{con}_{\text{Dis}^\alpha}$$

for every congruence $\alpha$.

**Lemma 2.3.** Let $Q$ be an idempotent left quasigroup and $\alpha \in \text{Con}(Q)$. If the blocks of $\alpha$ are connected then $\alpha = \mathcal{O}_{\text{Dis}_\alpha} = \mathcal{O}_{\text{Dis}^\alpha}$.

**Proof.** In general we have $\mathcal{O}_{\text{Dis}_\alpha} \leq \mathcal{O}_{\text{Dis}^\alpha} \leq \alpha$. Let $x \in Q$ and consider the groups $H = \{L_y, y \in [x]\}$ and $D = \{hL_yL_z^{-1}h^{-1}, y, z \in [x], h \in H\} \leq \text{Dis}_\alpha$. The action of $D$ coincides with the action of $\text{Dis}([x])$ that is transitive on $[x]$. Therefore, $\alpha = \mathcal{O}_{\text{Dis}_\alpha}$.

The following Proposition shows that the correspondence theorem for normal subgroups restricts to the sublattices of admissible groups.

**Proposition 2.4.** Let $Q$ be a left quasigroup and $\alpha$ be a congruence of $Q$. Then the mappings

$$\{N \in \text{Norm}(Q) : \text{LMlt}^\alpha \leq N\} \leftrightarrow \text{Norm}(Q/\alpha)$$

$$N \mapsto \pi_\alpha(N)$$

$$\pi_\alpha^{-1}(K) \leftrightarrow K$$

provides an isomorphism of lattices.

**Proof.** The pair of maps above clearly define a bijective correspondence between the lattice of normal subgroups of $\text{LMlt}(Q)$ and the lattice of normal subgroups of $\text{LMlt}(Q/\alpha) \cong \text{LMlt}(Q)/\text{LMlt}^\alpha$.

We already proved that if $N \in \text{Norm}(Q)$ then $\pi_\alpha(N) \in \text{Norm}(Q/\alpha)$ in [Bon21a Lemma 1.7]. Assume that $K \in \text{Norm}(Q/\alpha)$ and let $H = \pi_\alpha^{-1}(K)$. Clearly $H$ is normal in $\text{LMlt}(Q)$. If $h \in H$ we have

$$\pi_\alpha(hL_xL_{x'}^{-1}) = L_{h(x)}L_{x'}^{-1} = L_{\pi_\alpha(h)(x)}L_{x'}^{-1} \in \text{Dis}_{\text{Norm}_\alpha} \leq \text{K}.$$  

Thus, $L_{h(x)}L_{x'}^{-1} \in H$ and so $\text{Dis}_{\text{Norm}_\alpha} \leq H$.  

Given a left quasigroup $Q$, note that the images of the operators $\text{Dis}^*$ and $\text{Dis}_*$ lie in the sublattice

$$\text{Norm}'(Q) = \{N \in \text{Norm}(Q) : N \leq \text{Dis}'(Q)\} = \{N \cap \text{Dis}(Q) : N \in \text{Norm}(Q)\}.$$  

The correspondence given in Proposition 2.4 can be restricted to this sublattice.

**Corollary 2.5.** Let $Q$ be a left quasigroup and $\alpha$ be a congruence of $Q$. Then the mappings

$$\{N \in \text{Norm}'(Q) : \text{Dis}^\alpha \leq N\} \leftrightarrow \text{Norm}'(Q/\alpha)$$

$$N \mapsto \pi_\alpha(N)$$

$$\pi_\alpha^{-1}(K) \leftrightarrow K$$

provides an isomorphism of lattices.

The lattice of congruences of a left quasigroups and the lattice of admissible subgroups are related by a monotone Galois connection. Since the image of the operator $\text{Dis}^*$ lie in the sublattice defined in 5 we can restate [Bon21a, Theorem 1.10] as follows.

**Theorem 2.6.** [Bon21a, Theorem 1.10] Let $Q$ be a left quasigroup. The pair of mappings $\mathcal{O}_\alpha$ and $\text{Dis}^*$ provides a monotone Galois connection between $\text{Con}(Q)$ and $\text{Norm}'(Q)$.  

4
2.3. The Cayley kernel. Let $Q$ be a left quasigroup, we introduce the Cayley kernel of $Q$ as the equivalence relation $\lambda_Q = \text{con}_1$. Namely, the Cayley kernel is defined by setting

$$x \lambda_Q y \text{ if and only if } L_x = L_y.$$  

The Cayley kernel is not a congruence in general. If $\lambda_Q$ is a congruence we say that $Q$ is a Cayley left quasigroup (e.g. racks are Cayley left quasigroups).

**Remark 2.7.** Let us point out two easy properties of the Cayley kernel of a left quasigroup $Q$. Let $\alpha \in \text{Con}(Q)$:

(i) $\alpha \leq \lambda_Q$ if and only if $\text{Dis}_\alpha = 1$.

(ii) If $x \lambda_Q y$ then $[x]_\alpha = [y]_\alpha$. Indeed: if $L_x = L_y$, then $L_{[x]_\alpha} = \pi_\alpha(L_x) = \pi_\alpha(L_y) = L_{[y]_\alpha}$.

(iii) According to [Bon21a Proposition 1.6], $\lambda_Q/\alpha = \text{con}_{\text{Dis}^\alpha/\alpha}$.

The Cayley kernel plays a role in the universal algebraic theory of left quasigroups. Indeed in [BS19] we established that strongly abelian congruences of left quasigroups in the sense of [HD88] are those below the Cayley kernel. Such congruences form a complete sublattice of the lattice of congruences.

**Proposition 2.8.** Let $Q$ be a left quasigroup. The set $\{\alpha \in \text{Con}(Q) : \alpha \leq \lambda_Q\}$ is a complete sublattice of $\text{Con}(Q)$.

**Proof.** Let $(\alpha_i : i \in I) \subseteq \text{Con}(Q)$ and assume that $\alpha_i \leq \lambda_Q$ for every $i \in I$, namely $\text{Dis}_{\alpha_i} = 1$ for every $i \in I$. Clearly $\wedge_{i \in I} \alpha_i \leq \lambda_Q$. Let $\beta = \vee_{i \in I} \alpha_i$. According to Proposition 2.7(ii) $\text{Dis}_\beta = 1$, namely $\beta \leq \lambda_Q$ by Remark 2.7(i).

The congruences defined by orbits are related to the Cayley kernel.

**Lemma 2.9.** Let $Q$ be a left quasigroup, $\alpha \in \text{Con}(Q)$, $\beta = \text{O}_{\text{Dis}^\alpha}$ and $\gamma = \text{O}_{\text{Dis}_\alpha}$. Then:

(i) $\text{Dis}^\beta = \text{Dis}^\gamma$ and $\alpha / \beta \leq \lambda_Q / \beta$.

(ii) $\alpha / \gamma \leq \lambda_Q / \gamma$.

**Proof.** (i) The pair $\text{O}_\alpha$ and $\text{Dis}^\alpha$ provides a Galois connection, therefore $\text{Dis}^\beta = \text{Dis}^{\text{O}_{\text{Dis}^\alpha}} = \text{Dis}^\alpha$. Moreover, $\text{Dis}_{\alpha} \leq \text{Dis}^\alpha = \text{Dis}^\beta$ and so $\text{Dis}_{\alpha/\beta} = \text{Di}_\beta(\text{Dis}_{\alpha}) = 1$. Thus, $\alpha / \beta \leq \lambda_Q / \beta$ by Remark 2.7(i).

(ii) Clearly $\gamma \leq \alpha$ and $\text{Dis}_\alpha \leq \text{Dis}^\gamma$. Therefore $\text{Dis}_{\alpha/\gamma} = \text{Di}_\gamma(\text{Dis}_{\alpha}) = 1$, i.e. $\alpha / \gamma \leq \lambda_Q / \gamma$ (see Remark 2.7(i)).

Let $Q$ be a left quasigroup. If $\lambda_Q = 0_Q$ we say that $Q$ is faithful. The class of faithful left quasigroups is closed under direct products and the class of idempotent faithful left quasigroups is closed under extensions [Bon21b Corollary 1.12].

**Lemma 2.10.** Let $Q$ be a left quasigroup, $\alpha \in \text{Con}(Q)$, $\{\alpha_i : i \in I\} \subseteq \text{Con}(Q)$ and $\beta = \wedge_{i \in I} \alpha_i$.

(i) If $Q / \alpha$ is faithful then $\lambda_Q \leq \alpha$.

(ii) If $Q / \alpha_i$ is faithful for every $i \in I$ then $Q / \beta$ is faithful.

**Proof.** (i) Let $L_x = L_y$. Then by Remark 2.7(ii) we have $L_{[x]_\alpha} = L_{[y]_\alpha}$ and so $[x]_\alpha = [y]_\alpha$ since $Q / \alpha$ is faithful.

(ii) According to Remark 2.7(iii), $\alpha_i = \text{con}_{\text{Dis}^\alpha}$, for every $i \in I$ and we need to prove that $\beta = \text{con}_{\text{Dis}^\beta}$. Using that $\text{Dis}^\beta = \bigcap_{i \in I} \text{Dis}^\alpha_i$, we have $\text{con}_{\text{Dis}^\beta} = \wedge_{i \in I} \text{con}_{\text{Dis}^\alpha_i} = \wedge_{i \in I} \alpha_i = \beta$. The converse of Lemma 2.10(ii) does not hold. Given a left quasigroup $Q$ and $\alpha = \text{O}_{\text{Dis}(Q)}$ then $\lambda_{Q/\alpha} = 1_Q$. [Bon21a Corollary 1.9]. It is easy to construct a left quasigroup such that $\lambda_Q \leq \alpha$ and that $Q / \alpha$ is not trivial (e.g. a faithful left quasigroup that is not connected by its displacement group).

Let $Q$ be a left quasigroup. If all the subalgebras of $Q$ are faithful, $Q$ is said to be superfaithful. The class of superfaithful left quasigroups is closed under direct subalgebras and the class of idempotent superfaithful left quasigroups is closed under extensions [Bon21b Corollary 1.12].

**Lemma 2.11.** Let $Q$ be a left quasigroup.

(i) If $Q$ has injective right multiplications then $Q$ is superfaithful.

(ii) If $Q$ is superfaithful then $\mathcal{P}_2 \notin \mathcal{S}(Q)$.

(iii) If $Q$ is idempotent and $\mathcal{P}_2 \notin \mathcal{S}(Q)$ then $Q$ is superfaithful.

**Proof.** (i) The statement follows by the same proof of [Bon21a Lemma 1.1].

(ii) The left quasigroup $\mathcal{P}_2$ is not faithful.

(iii) It follows by [Bon21b Lemma 1.9].
According to Lemma 2.11, latin left quasigroups are superfaithful and the class of superfaithful idempotent left quasigroups is the class of idempotent left quasigroups such that the following implication holds:

\[(6) \quad xy = y \text{ and } yx = x \Rightarrow x = y.\]

Let \(Q\) be an idempotent left quasigroup \(Q\). Then \(Fix(L_x) = \{x\}\) for every \(x \in Q\) if and only if the implication

\[(7) \quad xy = y \Rightarrow x = y\]

holds. The class of idempotent left quasigroups satisfying (6) (resp. (7)) is closed under subalgebras and direct products as it is a quasi-variety [BSS1] and extensions. Indeed assume that \(Q/\alpha\) and \([x]\) are is such class and \(xy = y\). Then \([x][y] = [y]\). So \([x] = [y]\) and thus \(x = y\).

**Corollary 2.12.** Let \(Q\) be an idempotent left quasigroup.

(i) If right multiplications are injective, then \(Fix(L_x) = \{x\}\) for every \(x \in Q\).

(ii) If \(Fix(L_x) = \{x\}\) for every \(x \in Q\) then \(Q\) is superfaithful.

**Proof.** (i) Assume that \(xy = y\). Then \(R_y(x) = xy = yyy = y\) and so \(x = y\).

(ii) Clear since (7) implies (6). \(\square\)

According to Lemma 2.11 (iii), superconnected idempotent left quasigroups are superfaithful. In particular we have the following.

**Lemma 2.13.** Let \(Q\) be a superconnected idempotent left quasigroup. Then \(O_{Dis_\alpha} = O_{Dis_\alpha^\alpha} = \alpha = \text{con}_{Dis_\alpha} = \text{con}_{Dis_\alpha^\alpha}\) for every \(\alpha \in \text{Con}(Q)\).

**Proof.** Let \(\alpha \in \text{Con}(Q)\). The blocks of \(\alpha\) are connected, so according to Lemma 2.3 we have \(\alpha = O_{Dis_\alpha} = O_{Dis_\alpha^\alpha}\). According to Remark 2.7 (iii) we have that \(\lambda_{Q/\alpha} = \text{con}_{Dis_\alpha^\alpha}/\alpha\). The left quasigroup \(Q\) and its factors are faithful, and so \(\alpha = \text{con}_{Dis_\alpha^\alpha}\). \(\square\)

3. **Semiregular left quasigroups**

3.1. **Semiregular left quasigroups.** We say that a left quasigroup \(Q\) is semiregular if \(\text{Dis}(Q)\) is semiregular on \(Q\). A class of semiregular left quasigroups is the class of principal quandles studied in [Bon20]. Some of the results that hold for semiregular quandles can be easily extended to left quasigroups.

It is easy to see that the class of semiregular left quasigroups is closed under subalgebras and direct products but it is not closed under homomorphic images. For instance, the quandle SmallQuandle(12, 1) in the RIG database of GAP is semiregular, but it has a factor that is not semiregular [Ven15]. The next lemma characterizes semiregular factors of a left quasigroup.

**Lemma 3.1.** Let \(Q\) be a left quasigroup and \(\alpha\) be a congruence of \(Q\).

(i) \(Q/\alpha\) is semiregular if and only if \(\text{Dis}_\alpha^\alpha = \text{Dis}(Q/\alpha)_{[x]}\) for every \(x \in Q\).

(ii) Let \(\beta = \bigwedge_{i \in I} \alpha_i \in \text{Con}(Q)\). If \(Q/\alpha_i\) is semiregular for every \(i \in I\), then \(Q/\beta\) is semiregular.

**Proof.** (i) Since \(\text{Dis}_\alpha^\alpha \leq \text{Dis}(Q/\alpha)_{[x]} = \pi_\alpha^{-1}(\text{Dis}(Q)_{[x]}),\) we have that \(\text{Dis}(Q/\alpha)_{[x]} = 1\) if and only if \(\text{Dis}(Q)_{[x]} = \text{Dis}_\alpha^\alpha\).

(ii) Let \(\beta = \bigwedge_{i \in I} \alpha_i\). Then \(Q/\beta\) embeds into the direct product of semiregular left quasigroups \(\prod_{i \in I} Q/\alpha_i\), and so it is semiregular as well. \(\square\)

For semiregular left quasigroups, the property of being faithful and superfaithful collapse.

**Lemma 3.2.** Let \(Q\) be a semiregular left quasigroup. The following are equivalent:

(i) \(Q\) is faithful.

(ii) The right multiplication mappings of \(Q\) are injective.

(iii) \(Q\) is superfaithful.

**Proof.** The implication (iii) \(\Rightarrow\) (i) is clear and (ii) \(\Rightarrow\) (iii) is Lemma 2.11 (i).

(i) \(\Rightarrow\) (ii) Let \(xz = yz\). Then \(L_y^{-1}L_x \in \text{Dis}(Q)_{[x]} = 1\) and so \(L_x = L_y\). Thus \(x = y\). \(\square\)

**Lemma 3.3.** Semiregular idempotent left quasigroups are quandles.
Proof. Let $Q$ be an idempotent semiregular left quasigroup and $x, y \in Q$. According to [Bon21a] Lemma 1.4, $L_x L_y L_x^{-1} L_{xy} = \text{Dis}(Q)$. Moreover

$$L_x L_y L_x^{-1} L_{xy}(xy) = xy$$

and so $L_x L_y L_x^{-1} L_{xy} \in \text{Dis}(Q)$. Therefore $L_{xy} = L_x L_y L_x^{-1}$, i.e. $Q$ is a quandle. \hfill \Box

3.2. The equivalence $\sigma_Q$. Let $Q$ be a left quasigroup and $N \not\subseteq \text{LMlt}(Q)$. We define an equivalence relation as

$$x \sigma_N y \text{ if and only if } N x = N y.$$

Lemma 3.4. Let $Q$ be a left quasigroup and $N \not\subseteq \text{LMlt}(Q)$. The blocks of $\sigma_N$ are blocks with respect to the action of $\text{LMlt}(Q)$. If $Q$ is idempotent then they are subalgebras of $Q$.

Proof. Let $x \sigma_N y$, i.e. $N x = N y$ and $h \in \text{LMlt}(Q)$. The group $N$ is normal in $\text{LMlt}(Q)$, so we have $N h(x) = h(N x) = h N x$. Hence $h(x) \sigma_N h(y)$ and the classes of $\sigma_N$ are blocks with respect to the action of $\text{LMlt}(Q)$.

If in addition $Q$ is idempotent, then

$$N L_x^+(x) = L_x^4 N x L_x^4 = L_x^4 N y L_x^4 = N L_y^+(x) = N x,$$

so $[x]_{\sigma_N}$ is a subalgebra of $Q$. \hfill \Box

In [BS21] we already introduced the equivalence relation $\sigma_{\text{Dis}(Q)}$ and we denoted it simply by $\sigma_Q$. Semiregularity of $Q$ is captured by the equivalence $\sigma_Q$. Indeed $Q$ is semiregular if and only if $\sigma_Q = 1_Q$.

Such a relation is related to central congruences of left quasigroups (see [BS21]) and its properties have been investigated for quandles in [Bon21a]. Some of its properties can be extended to idempotent left quasigroups.

Proposition 3.5. Let $Q$ be an idempotent left quasigroup. The classes of $\sigma_Q$ are semiregular subquandles of $Q$. In particular every idempotent left quasigroup is a disjoint union of semiregular quandles.

Proof. Let $H = \langle L_y, y \in [x]_{\sigma_Q} \rangle$. The action of the displacement group of $[x]_{\sigma_Q}$ is the action of the group $D = \langle h L_y L_x^{-1} h^{-1}, y \in [x]_{\sigma_Q}, h \in H \rangle$ restricted to $[x]_{\sigma_Q}$. So if $h \in \text{Dis}([x]_{\sigma_Q})$, then $h = g(x)_{\sigma_Q}$ for some $g \in D$. If $h(x) = x$ then $h(y) = g(y) = y$ for every $y \in [x]_{\sigma_Q}$, i.e. $h = 1$ and $[x]_{\sigma_Q}$ is semiregular. According to Lemma 3.3 the classes of $\sigma_Q$ are semiregular quandles. \hfill \Box

Let $Q$ be a left quasigroup, $x \in Q$ and let $\bar{x} = N \text{Dis}(Q)(N x)$. Then $[x]_{\text{Dis}(Q)} \cap [x]_{\sigma_Q} = x^{\bar{x}}$. Indeed $h(x) \sigma_N x$ if and only if $N h(x) = h N x = N x$, i.e. $h \in \bar{x}$. The equivalence $\sigma_N$ can be trivial and in this case $\bar{x} = N x$ and $Z(\text{Dis}(Q)) = 1$ ($Z(\text{Dis}(Q)) \leq \bar{x}$ for every $x \in Q$).

Corollary 3.6. Let $Q$ be a connected idempotent left quasigroup and $N \in \text{Norm}(Q)$. Then $[x]_{\sigma_Q} = x^{\bar{x}}$ for every $x \in Q$.

Proof. Since $Q$ is connected, $x^{\bar{x}} = [x]_{\sigma_Q} \cap x \text{Dis}(Q) = [x]_{\sigma_Q}$. \hfill \Box

3.3. Semiregular admissible subgroups. Let us consider admissible semiregular subgroups. Note that, if $Q$ is a left quasigroup and $N$ is an admissible subgroup, then $N$ is semiregular if and only if $\sigma_N = 1_Q$. Indeed, if $N x = N y$ for every $x \in Q$, we have $\sigma_N = 1_Q$. On the other hand, if $\sigma_N = 1_Q$ and $h \in N x$, then $h(y) = y$ for every $y \in N$. Thus $h = 1$ and so $N$ is semiregular.

Lemma 3.7. Let $Q$ be a left quasigroup, $N \in \text{Norm}'(Q)$ and $\alpha = O_N$. If $N$ is semiregular, then $\text{Dis}(Q)[x] \cong N \times \text{Dis}(Q)[x]$ and $\text{Dis}^\alpha \cong N \times \text{Dis}^\alpha[x]$ for every $x \in Q$.

Proof. Let $x \in Q$. Recall that $\text{Dis}_N \leq N \leq \text{Dis}^\alpha \leq \text{Dis}(Q)[x]$ and $\text{Dis}(Q)_x \leq \text{Dis}(Q)[x]$. The subgroup $N$ is a normal subgroup of $\text{Dis}(Q)[x]$. The group $N$ is regular on $[x]$, so the elements of $N$ are representatives of the cosets of $N$ with respect to $\text{Dis}(Q)_x$ and so $\text{Dis}(Q)[x] = N \text{Dis}(Q)_x$. Moreover $N \cap \text{Dis}(Q)_x = 1$, therefore the block stabilizer splits as a semidirect product of the $N$ and $\text{Dis}(Q)_x$. The same argument shows that a similar decomposition holds for $\text{Dis}^\alpha$. \hfill \Box

Lemma 3.8. Let $Q$ be a finite faithful left quasigroup, $N \in \text{Norm}'(Q)$ and $\alpha = O_N$. If $N$ is semiregular, then:

(i) $\alpha = \text{con}_N$ and $N = \text{Dis}_N = \{L_y L_x^{-1}, y \in [x]\}$ for every $x \in Q$.

(ii) If $[x] \in E(Q/\alpha)$, then $[x]$ is a semiregular latin left quasigroup and $N \cong \text{Dis}([x])$. 

7
Proof. (i) Let \( x \in Q \) and let \( \varphi \) be the mapping:

\[
\varphi : N \to \text{Dis}_Q, \quad n \mapsto L_{n(x)}L_x^{-1}.
\]

If \( L_{n(x)}L_x^{-1} = \varphi(n) = \varphi(m) = L_{m(x)}L_x^{-1} \) then \( n(x) = m(x) \) since \( Q \) is faithful. The subgroup \( N \) is semiregular and so \( n = m \). Therefore \( \varphi \) is bijective and so \( N = \{ L_{n(x)}L_x^{-1} : n \in N \} \leq \text{Dis}_Q \). Since it is always the case that \( \text{Dis}_Q \leq N \), we can conclude that \( N = \text{Dis}_Q \). So, if \( y \in \text{con}_N x \) (i.e. \( L_yL_x^{-1} \in N \)) then \( y = n(x) \) for some \( n \in N \). Thus \( \mathcal{O}_N = \text{con}_N \).

(ii) If \( [x] \in E(Q/\alpha) \), the block \( [x] \) is a subalgebra. Let \( H = \{ L_y, y \in [x] \} \). The subgroup \( N \) is normal in \( \text{LMlt}(Q) \) and so by item (i) \( N = hL_yL_x^{-1} = x \in [y], h \in H = \{ L_yL_x^{-1} : y \in [x] \} \). So, we have

\[
\text{Dis}([x]) \equiv \{ hL_yL_x^{-1}h^{-1} : h \in H, y \in [x] \} = N/[x].
\]

Since the group \( N \) is semiregular, then \( K = \cap_{y \in [x]} N_y = 1 \) and so \( \text{Dis}([x]) \equiv N/[x] \equiv N/K = N \). In particular \([x]\) is semiregular.

Assume that \( yx = zx \) for \( y, z \in [x] \). Then \( L_y^{-1}L_z \in \text{Dis}_\alpha_x = 1 \) and using that \( Q \) is faithful we have \( y = z \). Therefore the subalgebra \([x]\) is latin. \( \square \)

**Corollary 3.9.** Let \( Q \) be a finite faithful idempotent left quasigroup and let \( N \in \text{Norm}'(Q) \) be semiregular. Then:

(i) the blocks of \( \mathcal{O}_N \) are semiregular latin idempotent subquandles of \( Q \) and \( N \) is solvable.

(ii) If \( Q/\alpha \) is superfaithful (resp. superconnected, resp. \( \text{Fix}(L_{[x]}) = \{ [x] \} \) for every \( [x] \in Q/\alpha \) then \( Q \) is superfaithful (resp. superconnected, resp. \( \text{Fix}(L_{[x]}) = \{ [x] \} \) for every \( x \in Q \)).

**Proof.** (i) According to Proposition 3.8 the blocks are latin semiregular subalgebras and so they are quandles by Lemma 3.3. The subgroup \( N \) is isomorphic to the displacement group of the blocks, that is solvable [Ste01, Theorem 1.3].

(ii) Such properties are closed under extensions and the blocks of \( \alpha \) are finite latin quandles (see item (i)), so they have such properties. \( \square \)

## 4. Nilpotent Left Quasigroups

### 4.1. Commutator theory

The notion of commutator of congruences and the related concepts of center, solvability and nilpotency have been developed for arbitrary algebraic structure in [FM87].

Let \( A \) be an algebraic structure and \( \alpha, \beta, \delta \in \text{Con}(A) \). We say that \( \alpha \) centralizes \( \beta \) over \( \delta \), and write \( C(\alpha, \beta; \delta) \), if for every \((n+1)\)-ary term operation \( t \), every pair \( x, y \) and every \( z_1 \beta u_1, \ldots, z_n \beta u_n \) we have

\[
t(x, z_1, \ldots, z_n) \delta t(x, u_1, \ldots, u_n) \quad \text{implies} \quad t(y, z_1, \ldots, z_n) \delta t(y, u_1, \ldots, u_n).
\]

The **commutator** of \( \alpha, \beta \in \text{Con}(A) \), denoted by \([\alpha, \beta]\), is the smallest congruence \( \delta \) such that \( C(\alpha, \beta; \delta) \).

A congruence \( \alpha \) is called:

- **abelian** if \( C(\alpha, \alpha; 0_A) \), i.e., if \([\alpha, \alpha] = 0_A \),
- **central** if \( C(\alpha, 1_A; 0_A) \), i.e., if \([\alpha, 1_A] = 0_A \).

The **center** of \( A \), denoted by \( \zeta_A \), is the largest congruence of \( A \) such that \( C(\zeta_A, 1_A; 0_A) \). An algebraic structure \( A \) is called **abelian** if \( \zeta_A = 1_A \), or, equivalently, if the congruence \( 1_A \) is abelian. We can define a series of congruence of \( A \) as

\[
\zeta_1(A) = \zeta_A, \quad \zeta_n+1(A)/\zeta_n(A) = \zeta_{\zeta(A)/\zeta_n(A)}
\]

for every \( n \in \mathbb{N} \). The algebraic structure \( A \) is called **nilpotent** of length \( n \) if \( \zeta_n(A) = 1_A \).

In [BS21] we adapted the universal algebraic theory of commutators to the setting of rings and quandles. The main results of our work can be partially applied to the setting of left quasigroups.

**Lemma 4.1.** [BS21, Lemma 5.1] Let \( Q \) be a left quasigroup, \( \alpha, \beta \) its congruences. If \( C(\alpha, \beta; 0_Q) \) holds then \( \text{Dis}_\alpha, \text{Dis}_\beta \equiv 1 \) and \( \alpha \leq \sigma_{\text{Dis}_\beta} \).

**Corollary 4.2.** Let \( Q \) be a left quasigroup and \( \alpha \leq \zeta_Q \). Then \( \text{Dis}_\alpha \) is central in \( \text{Dis}(Q) \) and \( \alpha \leq \sigma_Q \).

According to Lemma 4.1 abelian left quasigroups are semiregular, indeed we have the following Corollary applying directly Lemma 3.2 and Lemma 3.3.

**Corollary 4.3.** Let \( Q \) be an abelian left quasigroup.

(i) If \( Q \) is idempotent then \( Q \) is a quandle.

(ii) If \( Q \) is finite and faithful then \( Q \) is latin.
4.2. Central Congruences. In this paper we exploit the results of the Section 3.3 to study central congruences. The key observation is the following: let Q be a left quasigroup connected by Dis(Q) and N ≤ Z(Dis(Q)), then N is semiregular. And so, in particular if α is a central congruence then Dis_α is semiregular (see Corollary 4.2). For this reason the results of this section are stated for left quasigroups congruences. The key observation is the following: let Q be a left quasigroup connected by Dis(Q) and N ≤ Z(Dis(Q)). According to Lemma 3.7 we have that Dis_α is semiregular and so, in particular if α is a central congruence then Dis_α is semiregular. Therefore Dis_α and Dis_β are connected by their displacement group (and so they cover connected idempotent left quasigroup and left quasigroups with a Malt’sev term).

Central congruences are reflected by the structure of the displacement group.

Lemma 4.4. Let Q be a left quasigroup connected by Dis(Q), α be a central congruence and β = O Dis_α. Then:

(i) Dis(Q)_α ≅ Dis_α × Dis(Q)_x and Dis_β ≅ Dis_α × Dis_β x for every x ∈ Q.

(ii) Dis_β embeds into Dis_α_β. In particular, Dis_β is abelian.

(iii) If Q is finite, [Dis_β] divides [x] || Q/β for every x ∈ Q.

Proof. (i) The congruence α is central and so the group Dis_α is central in Dis(Q) by virtue of Corollary 4.2. A central subgroup of a transitive group is semiregular, and then Dis_α is regular over the blocks of β. According to Lemma 3.7 we have that Dis(Q)_x ≅ Dis_α × Dis(Q)_x and Dis_β ≅ Dis_α × Dis_β x for every x ∈ Q. The subgroup Dis_α is central, therefore the semidirect product is actually a direct product.

(ii) The mapping

\[ \operatorname{Dis}_β \rightarrow \prod_{x \in Q/β} \operatorname{Sym}_x, \quad h \mapsto \{ h[x] : [x] \in Q/β \} \]

is an injective group homomorphism. The congruence β is central, so Dis_β x = Dis_β whenever x ∈ β. Then the action of Dis_β x on the block of x is trivial. Therefore Dis_β[x] = Dis_α[x] ≅ Dis_α and so the image of S is contained in Dis_β_α, which is abelian.

(iii) If Q is finite, [x] || Dis_α and so [Dis_β] divides [x] || Q/β.

□

Corollary 4.5. Let Q be a finite faithful left quasigroup connected by Dis(Q) and α be a central congruence. Then:

(i) O Dis_α = α = con Dis_α.

(ii) Dis(Q)_α ≅ Dis_α × Dis(Q)_x and Dis_α ≅ Dis_α × Dis_α x for every x ∈ Q.

(iii) Dis_α embeds into Dis([x]) || Q/α. In particular, Dis_α is abelian and [Dis_α] divides [x] || Q/α for every x ∈ Q.

Proof. Let x ∈ Q. The subgroup Dis_α is a central subgroup of the transitive group Dis(Q), see Corollary 4.2. Therefore Dis_α is semiregular. According to Lemma 3.3(ii), O Dis_α = con Dis_α. The inclusion O Dis_α ≤ α ≤ con Dis_α holds, then (i) holds.

For (ii) and (iii) we can apply Lemma 4.4 since α = O Dis_α.

□

According to Corollary 4.5, we can apply Corollary 3.3 to central congruences of finite connected faithful idempotent left quasigroup.

Corollary 4.6. Let Q be a finite connected faithful idempotent left quasigroup and α be a central congruence. Then:

(i) the blocks of α are affine latin quandles.

(ii) If Q/α is superfaithful (resp. superconnected, resp. Fix(L_{[x]}) = [x]) for every x ∈ Q/α then Q is superfaithful (resp. superconnected, resp. Fix(L_{[x]}) = [x] for every x ∈ Q).

Let us show a converse of Corollary 4.4(ii).

Lemma 4.7. Let Q be a finite connected idempotent left quasigroup and α be a central congruence. If Fix(L_α) = {x} for every x ∈ Q and Fix(L_{[x]}) = [x] for every x ∈ Q/α.

Proof. Note that Q is faithful. Assume that [x][y] = [y]. Then L_x L_y^{-1} ∈ Dis(Q)_y. According to Corollary 4.1 the block stabilizer is the direct product of Dis_α = {L_z L_y^{-1} : z ∈ [y]} and the stabilizer of y in Dis(Q). Thus, there exists u α y and h ∈ Dis(Q)_y = Dis(Q)_u such that L_x L_y^{-1} = h L_u L_y^{-1}. Then L_x = h L_u ∈ LMin(Q)_u and accordingly xy = u. Therefore, x = u α y and so Fix(L_{[x]}) = [x].

□

A rack Q is nilpotent if and only if Dis(Q) is nilpotent and if Q is a connected quandle and |Q| = p^n for some prime p, then Q is nilpotent and Dis(Q) is a p-group. Moreover, the displacement group of finite latin quandles is solvable [Sto01, BS21]. In general, the relationship between nilpotency of left
Corollary 4.8. Let \( Q \) be a superconnected idempotent left quasigroup. If \( Q \) is nilpotent of length \( n \) then \( \text{Dis}(Q) \) is solvable of length at most \( n \).

Proof. If \( Q \) is abelian, then \( \text{Dis}(Q) \) is abelian. Assume that \( Q \) is nilpotent of length \( n \). According to Lemma 2.13 \( \zeta_Q = \mathcal{O}_{\text{Dis}_Q} \) and so applying Lemma 4.4 we have that \( \text{Dis}^{\zeta_Q} \) is abelian. The factor \( Q/\zeta_Q \) is nilpotent of length \( n - 1 \) and by induction \( \text{Dis}(Q/\zeta_Q) \) is solvable of length at most \( n - 1 \). Thus, \( \text{Dis}(Q) \) is solvable of length at most \( n \). \( \square \)

Let \( Q \) be a finite left quasigroup connected by \( N \). Then we have that \(|N| = |x^N||N_x| = |Q||N_x|\) for every \( x \in Q \). If \( p \) divides \(|Q|\) then clearly \( p \) divides \(|N|\). Under further assumptions the converse holds.

Proposition 4.9. Let \( Q \) be a finite idempotent superconnected left quasigroup and let \( p \) be a prime. If \( Q \) is nilpotent, then \( p \) divides \(|Q|\) if and only if \( p \) divides \(|\text{Dis}(Q)|\). In particular, if \(|Q| = p^n \) then \( \text{Dis}(Q) \) is a \( p \)-group.

Proof. Let assume that \( p \) divides \(|\text{Dis}(Q)|\) then, by induction \( p \) divides \(|Q/\zeta_Q|\) and so it divides also \( |Q| = |Q/\zeta_Q|\rceil_{x_{\zeta_Q}} \). If \( p \) divides \(|\text{Dis}^{\zeta_Q}|\) then, according to Lemma 4.4 \( p \) divides \(|x|\rceil_{\zeta_Q} \) since \( \zeta_Q = \mathcal{O}_{\text{Dis}_Q} \). Hence, \( p \) divides \(|Q| = |Q/\zeta_Q|\rceil_{x_{\zeta_Q}} \). \( \square \)

4.3. Central extension. Let \( Q \) be a left quasigroup, \( A \) an abelian group, \( f \in \text{Aut}(A) \), \( g \in \text{End}(A) \) and \( \theta : Q \times Q \rightarrow A \) be a map. The algebraic structure \( E = (Q \times A, \cdot) \) where

\[(x, a) \cdot (y, b) = (x \cdot y, g(a) + f(b) + \theta(x, y))\]

is a left quasigroup and it is called a central extension of \( Q \) by \( A \). The map \( p_1 : E \rightarrow Q \), \( (x, a) \mapsto x \)

is a morphism of left quasigroups. We denote the left quasigroup \( E \) by \( \text{Aff}(Q, A, g, f, \theta) \). If \(|Q| = 1 \), we can identify \( Q \times A \) with \( A \) and (9) reads

\[a \cdot b = g(a) + f(b) + c\]

for some \( c \in A \). For this special case we use the notation \( E = \text{Aff}(A, g, f, c) \) and we say that \( E \) is an affine left quasigroup over \( A \). Affine left quasigroups are abelian (they are reducts of modules).

Remark 4.10. Note that:

(i) \( E \) is idempotent if and only if \( Q \) is idempotent, \( g = 1 - f \) and \( \theta_{x, x} = 0 \).

(ii) \( E \) is latin if and only if \( Q \) is latin and \( g \in \text{Aut}(A, +) \). In particular, if \( E \) is idempotent then \( E \) is latin if and only if \( Q \) and \( [x]_{\text{ker}(p)} = \text{Aff}(A, 1 - f, f, 0) \) are latin.

Central extensions are define in the framework of Malt’sev algebraic structures. In particular, every nilpotent algebraic structure with a Malt’sev term can be constructed by a chain of central extensions (in the case of quandles this lead to a similar result [BS21] Section 7).

5. Malt’sev left quasigroups

Malt’sev classes of left quasigroups have been investigated in [BF21]. In particular Malt’sev left quasigroups are superconnected. In this section we extend some results on Malt’sev quandles and Malt’sev varieties of quandles to the setting of idempotent left quasigroups.

Lemma 5.1. Let \( Q \) be a Malt’sev left quasigroup. Then \( \mathcal{O}_{\text{Dis}_x} = \mathcal{O}_{\text{Dis}^{\alpha}} = \alpha \) for every \( \alpha \in \text{Con}(Q) \).

Proof. Let \( \alpha \in \text{Con}(Q) \) and \( \beta = \mathcal{O}_{\text{Dis}_x} \leq \mathcal{O}_{\text{Dis}^{\alpha}} \leq \alpha \). The left quasigroup \( Q \) is Malt’sev and so it has no non-trivial strongly abelian congruence, so \( \text{Dis}_x \neq 1 \). According to Lemma 2.10 (iii) we have \( \alpha/\beta \leq \lambda_{Q/\beta} \), and so \( \alpha = \beta \) since also \( Q/\beta \) is Malt’sev. Hence, \( \alpha = \mathcal{O}_{\text{Dis}_x} = \mathcal{O}_{\text{Dis}^{\alpha}} \). \( \square \)

Using Lemma 5.1 if \( Q \) is a finite nilpotent Malt’sev left quasigroup then \( \zeta_Q = \mathcal{O}_{\text{Dis}_Q} \) and so Lemma 4.4 applies to \( \zeta_Q \). Hence we can employ the same argument of Proposition 4.4 to prove the following.
Proposition 5.2. Let $Q$ be a finite nilpotent Mal’tsev left quasigroup and let $p$ be a prime. Then $p$ divides $|Q|$ if and only if $p$ divides $\text{Dis}(Q)$. In particular, if $|Q| = p^n$ then $\text{Dis}(Q)$ is a $p$-group.

Using central extensions, we can extend [Bon21b, Theorem 2.15].

Theorem 5.3. Idempotent Mal’tsev nilpotent left quasigroups are latin. In particular, idempotent Mal’tsev abelian left quasigroups are affine latin quandles.

Proof. According to Corollary 4.3(ii), if $Q$ is an idempotent abelian left quasigroup, then $Q$ is a quandle. Hence $Q$ is latin according to [Bon21b, Theorem 2.15] and so affine (connected abelian quandles are affine, see HSV16, Section 7).

Assume that $Q$ is nilpotent of length $n$, then $Q$ is a central extension of $Q/\zeta Q$. Then $Q/\zeta Q$ is latin by induction on the nilpotency length and the blocks of $\zeta Q$ are latin, since they are abelian. Thus, we can conclude that $Q$ is latin by Remark 4.10(ii).

Let us show a characterization of finite nilpotent idempotent left quasigroups with a Mal’tsev term.

Proposition 5.4. Let $Q$ be a finite nilpotent idempotent left quasigroup. The following are equivalent:

(i) $Q$ is connected and $\text{Fix}(L_x) = \{x\}$ for every $x \in Q$.
(ii) $Q$ is superconnected.
(iii) $Q$ is Mal’tsev.
(iv) $Q$ is latin.

Proof. The implication (iv) $\Rightarrow$ (i) is true in general (latin left quasigroup are connected and Corollary 2.12(i) applies).

(i) $\Rightarrow$ (ii) Note that $Q$ is faithful. Let us proceed by induction on the nilpotency length. If $Q$ is abelian then $Q$ is latin by Corollary 4.3(ii) and therefore $Q$ is superconnected. Let $Q$ be nilpotent of length $n$.

By Lemma 4.7, $Q/\zeta Q$ satisfies the hypothesis and then by induction $Q/\zeta Q$ is superconnected. Hence, we can conclude by Corollary 4.6(ii).

(ii) $\Rightarrow$ (iii) According to [BF21, Corollary 3.8], $Q$ has a Malcev term.

(iii) $\Rightarrow$ (iv) We can apply Theorem 5.3.

The key fact in the sequel of the section is the following: idempotent semiregular left quasigroups are quandles (see Lemma 3.3), and so in particular abelian idempotent left quasigroups are quandles (see Corollary 4.3(i)).

According to [BF21, Theorem 4.21] the class of semiregular quandles of a Mal’tsev variety of quandles is a subvariety. A direct consequence of such a theorem and the fact above is the following.

Corollary 5.5. Let $V$ be a Mal’tsev variety of idempotent left quasigroup. The class of semiregular left quasigroups of $V$ is a subvariety of quandles of $V$.

A variety $V$ is said to be congruence distributive if the congruence lattice of all the algebraic structures in $V$ is distributive. We can extend the characterization of distributive varieties of quandles to varieties of idempotent left quasigroups.

Proposition 5.6. Let $V$ be a variety of idempotent left quasigroup. The following are equivalent:

(i) $V$ contains an abelian left quasigroup.
(ii) $V$ contains an abelian quandle.
(iii) $V$ contains a finite quandle.

Proof. (i) $\Rightarrow$ (ii) Idempotent abelian left quasigroups are quandles, see Corollary 4.3(ii).

(ii) $\Rightarrow$ (iii) Let $Q \in V$ be an abelian quandle. According to [BF21, Theorem 4.22] the variety generated by $Q$ contains a finite quandle.

(iii) $\Rightarrow$ (i) Let $Q \in V$ be a finite quandle. Then the variety generated by $Q$ contains an abelian quandle according to [BF21, Theorem 4.22].

Corollary 5.7. Let $V$ be a variety of idempotent left quasigroups. The following are equivalent:

(i) $V$ is congruence distributive.
(ii) $V$ does not contain a finite quandle.
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(M. Bonatto) Dipartimento di matematica e informatica - UNIFE

Email address: marco.bonatto.87@gmail.com

12