Weak convergence of the Stratonovich integral with respect to a class of Gaussian processes

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Abstract

For a Gaussian process $X$ and smooth function $f$, we consider a Stratonovich integral of $f(X)$, defined as the weak limit, if it exists, of a sequence of Riemann sums. We give covariance conditions on $X$ such that the sequence converges in law. This gives a change-of-variable formula in law with a correction term which is an Itô integral of $f'''$ with respect to a Gaussian martingale independent of $X$. The proof uses Malliavin calculus and a central limit theorem from [9]. This formula was known for fBm with $H = 1/6$ [10]. We extend this to a larger class of Gaussian processes.

1 Introduction

Let $X = \{X_t, t \geq 0\}$ be a centered Gaussian process, and let $f: \mathbb{R} \to \mathbb{R}$ be a $C^\infty$ function such that $f$ and its derivatives have at most polynomial growth. We define the Stratonovich integral of $f'$ with respect to $X$, denoted,

\[ \int_0^t f'(X_s) d^o X_s, \]

as the limit in probability of the trapezoidal Riemann sum,

\[ \frac{1}{2} \sum_{i=0}^{\lfloor nt \rfloor - 1} \left[ f'(X_{\frac{i+1}{n}}) + f'(X_{\frac{i}{n}}) \right] \left( X_{\frac{i+1}{n}} - X_{\frac{i}{n}} \right) \]  

(1)

when that limit exists. In a 2005 paper, Gradinaru et al. [5] studied this integral, and identified conditions on $X$ under which the Riemann sum converges. In particular, the sum converges for any fractional Brownian motion (fBm) $\{B_t, t \geq 0\}$ with Hurst parameter $H > 1/6$, in which case the following change-of-variable formula holds:

\[ f(B_t) = f(0) + \int_0^t f'(B_s) d^o B_s. \]  

(2)

Subsequently, [10] examined the end point case $H = 1/6$. Here, it was proved that (1) converges weakly to an Itô-like expansion formula, consisting of a stochastic integral and a correction term in the form of an Itô integral of $f'''$. In this case, we have the weak change-of-variable formula,

\[ f(B_t) \overset{L}{=} f(0) + \int_0^t f'(B_s) d^c B_s - \frac{\sqrt{6}}{12} \int_0^t f^{(3)}(X_s) dW_s, \]  

(3)

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where \( W \) is a Brownian motion, independent of \( B \), with variance given by (8).

In this paper, we consider the behavior of (1) for a more general class of Gaussian processes, which are characterized by conditions on the covariance. Convergence follows from a central limit theorem first proved by Nourdin and Nualart in [9]. This theorem is based on Malliavin calculus, and applies to a sequence of multiple Skorohod integrals. In this paper, we give a set of six covariance conditions on the process \( X \), which lead to weak convergence of the form (3). These conditions are satisfied by fBM with \( H = 1/6 \). As an application, we have found that the conditions are met for three fBM-derived processes, including

- Bifractional Brownian motion (bBm) with parameters \( HK = 1/6 \);
- ‘extended’ bBm with \( K \in (1, 2) \) and \( HK = 1/6 \); and
- sub-fractional Brownian motion with parameter \( h = 1/3 \).

In the prequel to this paper ([6]), we applied the same central limit theorem to a ‘midpoint’ Riemann sum of the form

\[
\sum_{j=1}^{\lfloor nt/2 \rfloor} f'(X_{2j-1})\left( X_{2j} - X_{2j-2} \right).
\]

For this sum, we found a slightly different weak change-of-variable formula for a process that acts similar to fBM with \( H = 1/4 \). In that case, the sum converges weakly to a stochastic integral plus a correction term in the form of an Itô integral of \( f'' \), namely

\[
f(X_t) \overset{\mathcal{L}}{=} f(X_0) + \int_0^t f'(X_s) \, dX_s + \frac{1}{2} \int_0^t f''(X_s) \, dB_s,
\]

where \( B \) is a scaled Brownian motion, independent of \( X \), with a given variance. As is suggested in [9], there are other forms of Riemann sums that can be tried, including one for any fBM with \( H > 1/10 \). We expect that our theoretical tools could be applied to the \( H = 1/10 \) case as well, but this is not pursued in the present paper.

A brief outline of the paper is as follows. In Section 2, we give some background on Malliavin calculus, and the main analytical tools that will be used in this paper. We also recall the central limit theorem (proved in [6]) that provides the main theoretical basis for our result. In Section 3, we identify the covariance conditions on the process \( X \) for the CLT to hold, and prove (3). The proof essentially consists of restating the Riemann sum (1) as a sequence of terms dominated in probability by 3-fold Skorohod integrals; then verifying that the CLT conditions are met. Section 4 discusses the three examples listed above, and demonstrates the procedure for verifying the covariance conditions. In Section 5 we give proofs for three of the longer lemmas from Section 3.

The main inspirations for this paper were [9] and [10]. Most of the notation follows that in [9] and [6].

2 Preliminaries and notation

Let \( X = \{X(t), t \geq 0\} \) be a centered Gaussian process defined on a probability space \((\Omega, \mathcal{F}, P)\) with continuous covariance function

\[
\mathbb{E}[X(t)X(s)] = R(t, s), \quad s, t \geq 0.
\]

We will always assume that \( \mathcal{F} \) is the \( \sigma \)-algebra generated by \( X \). Let \( \mathcal{E} \) denote the set of step functions on \([0, T]\) for \( T > 0 \); and let \( \mathcal{H} \) be the Hilbert space defined as the closure of \( \mathcal{E} \) with respect to the scalar product

\[
\langle 1_{[0,t]}, 1_{[0,s]} \rangle_{\mathcal{H}} = R(t, s), \quad s, t \geq 0.
\]
The mapping $I_{[0, t]} \mapsto X(t)$ can be extended to a linear isometry between $\mathcal{H}$ and the Gaussian space spanned by $W$. We denote this isometry by $h \mapsto X(h)$. In this way, $\{X(h), h \in \mathcal{H}\}$ is an isonormal Gaussian process. For integers $q \geq 1$, let $\mathcal{H}^{\otimes q}$ denote the $q^{th}$ tensor product of $\mathcal{H}$. We use $\mathcal{H}^{\otimes q}$ to denote the symmetric tensor product.

For integers $q \geq 1$, let $\mathcal{H}_q$ be the $q^{th}$ Wiener chaos of $X$, that is, the closed linear subspace of $L^2(\Omega)$ generated by the random variables $\{H_q(X(h)), h \in \mathcal{H}, \|h\|_{\mathcal{H}} = 1\}$, where $H_q(x)$ is the $q^{th}$ Hermite polynomial, defined as

$$H_q(x) = (-1)^q e^{\frac{x^2}{2}} \frac{d^q}{dx^q} e^{-\frac{x^2}{2}}.$$  

In particular, $H_1(x) = x$, $H_2(x) = x^2 - 1$, and $H_3(x) = x^3 - 3x$. For $q \geq 1$, it is known that the map

$$I_q(h^{\otimes q}) = H_q(X(h))$$

provides an isometry between the symmetric product space $\mathcal{H}^{\otimes q}$ (equipped with the modified norm $\frac{1}{\sqrt{q}} \cdot \|\cdot\|_{\mathcal{H}^{\otimes q}}$) and $\mathcal{H}_q$. By convention, $\mathcal{H}_0 = \mathbb{R}$ and $I_0(x) = x$.

### 2.1 Elements of Malliavin Calculus

Following is a brief description of some identities that will be used in the paper. The reader may refer to [9] for a brief survey, or to [11] for detailed coverage of this topic. Let $\mathcal{S}$ be the set of all smooth and cylindrical random variables of the form $F = g(X(\phi_1), \ldots, X(\phi_n))$, where $n \geq 1$; $g : \mathbb{R}^n \to \mathbb{R}$ is an infinitely differentiable function with compact support, and $\phi_i \in \mathcal{H}$. The Malliavin derivative of $F$ with respect to $X$ is the element of $L^2(\Omega, \mathcal{H})$ defined as

$$DF = \sum_{i=1}^n \frac{\partial g}{\partial x_i}(X(\phi_1), \ldots, X(\phi_n))\phi_i.$$  

In particular, $DX(h) = h$. By iteration, for any integer $q \geq 1$ we can define the $q^{th}$ derivative $D^q F$, which is an element of $L^2(\Omega, \mathcal{H}^{\otimes q})$.

For any integer $q \geq 1$ and real number $p \geq 1$, let $\mathbb{D}^{q,p}$ denote the closure of $\mathcal{S}$ with respect to the norm $\|\cdot\|_{\mathbb{D}^{q,p}}$ defined as

$$\|F\|_{\mathbb{D}^{q,p}}^p = \mathbb{E}[|F|^p] + \sum_{i=1}^q \mathbb{E}[\|D^i F\|_{\mathcal{H}^{\otimes i}}^p].$$

We denote by $\delta$ the Skorohod integral, which is defined as the adjoint of the operator $D$. This operator is also referred to as the divergence operator in [11]. A random element $u \in L^2(\Omega, \mathcal{H})$ belongs to the domain of $\delta$, $\text{Dom} \, \delta$, if and only if,

$$\mathbb{E}\left[\langle DF, u \rangle_{\mathcal{H}}\right] \leq c_u \sqrt{\mathbb{E}|F|^2}$$

for any $F \in \mathbb{D}^{1,2}$, where $c_u$ is a constant which depends only on $u$. If $u \in \text{Dom} \, \delta$, then the random variable $\delta(u) \in L^2(\Omega)$ is defined for all $F \in \mathbb{D}^{1,2}$ by the duality relationship,

$$\mathbb{E}[F\delta(u)] = \mathbb{E}\left[\langle DF, u \rangle_{\mathcal{H}}\right].$$

This is sometimes called the Malliavin integration by parts formula. We iteratively define the multiple Skorohod integral for $q \geq 1$ as $\delta(\delta^{q-1}(u))$, with $\delta^0(u) = u$. For this definition we have,

$$\mathbb{E}[F\delta^q(u)] = \mathbb{E}\left[\langle D^q F, u \rangle_{\mathcal{H}^{\otimes q}}\right],$$

where $u \in \text{Dom} \, \delta^q$ and $F \in \mathbb{D}^{q,2}$. Moreover, if $h \in \mathcal{H}^{\otimes q}$, then we have $\delta^q(h) = I_q(h)$. 


For \( f \in \mathcal{S}^{\otimes p} \) and \( g \in \mathcal{S}^{\otimes q} \), the following integral multiplication formula holds:

\[
\delta^p(f)\delta^q(g) = \sum_{r=0}^{p} r! \binom{p}{r} \binom{q}{r} \delta^{p+q-2r}(f \otimes_r g),
\]

where \( \otimes_r \) is the contraction operator (see, e.g., [11], Sec. 1.1).

We will use the Meyer inequality for the Skorohod integral, (see, for example Prop. 1.5.7 of [11]). Let \( \mathbb{D}^{k,p}(\mathcal{S}^{\otimes k}) \) denote the corresponding Sobolev space of \( \mathcal{S}^{\otimes k} \)-valued random variables. Then for \( p \geq 1 \) and integers \( k \geq q \geq 1 \), we have,

\[
\|\delta^q(u)\|_{\mathbb{D}^{k-q,p}} \leq c_{k,p}\|u\|_{\mathbb{D}^{k,p}(\mathcal{S}^{\otimes q})}
\]

for all \( u \in \mathbb{D}^{k,p}(\mathcal{S}^{\otimes k}) \) and some constant \( c_{k,p} \).

The following three results are well known, and will be used extensively in this paper. The reader may refer to [9] and [11] for details.

**Lemma 2.1.** Let \( q \geq 1 \) be an integer.

(a) Assume \( F \in \mathbb{D}^{0,2} \), \( u \) is a symmetric element of \( \text{Dom} \ \delta^q \), and \( \langle D^r F, \delta^j(u) \rangle_{\mathcal{S}^{\otimes r}} \) \( \in L^2(\Omega, \mathcal{S}^{\otimes q-r-j}) \) for all \( 0 \leq r + j \leq q \). Then \( \langle D^r F, u \rangle_{\mathcal{S}^{\otimes r}} \in \text{Dom} \ \delta^r \) and

\[
F \delta^q(u) = \sum_{r=0}^{q} \binom{q}{r} \delta^{q-r} \langle (D^r F, u)_{\mathcal{S}^{\otimes r}} \rangle.
\]

(b) Suppose that \( u \) is a symmetric element of \( \mathbb{D}^{j+k,2}(\mathcal{S}^{\otimes j}) \). Then we have,

\[
D^k \delta^j(u) = \sum_{i=0}^{j\wedge k} \binom{k}{i} \binom{j}{i} i! \delta^{j-i}(D^{k-i}u).
\]

(c) Let \( u, v \) be symmetric functions in \( \mathbb{D}^{2q,2}(\mathcal{S}^{\otimes q}) \). Then

\[
\mathbb{E}[\delta^q(u)\delta^q(v)] = \sum_{i=0}^{q} \binom{q}{i}^2 \mathbb{E} \left[ \langle D^{q-i}u, D^{q-i}v \rangle_{\mathcal{S}^{\otimes (2q-i)}} \right].
\]

In particular,

\[
\|\delta^q(u)\|_{L^2(\Omega)}^2 = \mathbb{E} \left[ \delta^q(u)^2 \right] \leq \sum_{i=0}^{q} \binom{q}{i}^2 \mathbb{E} \left[ \|D^{q-i}u\|_{\mathcal{S}^{\otimes (2q-i)}}^2 \right].
\]

### 2.2 Criteria for convergence in the Skorohod space \( \mathbb{D}(0, \infty) \)

**Definition 2.2.** Assume \( F_n \) is a sequence of \( d \)-dimensional random variables defined on a probability space \( (\Omega, \mathcal{F}, P) \), and \( F \) is a \( d \)-dimensional random variable defined on \( (\Omega, \mathcal{G}, P) \), where \( \mathcal{F} \subset \mathcal{G} \). We say that \( F_n \) converges stably to \( F \) as \( n \to \infty \), if, for any continuous and bounded function \( f : \mathbb{R}^d \to \mathbb{R} \) and \( \mathbb{R} \)-valued, \( \mathcal{F} \)-measurable random variable \( Z \), we have

\[
\lim_{n \to \infty} \mathbb{E}(f(F_n)Z) = \mathbb{E}(f(F)Z).
\]

The following central limit theorem first appeared in [9], and the present multi-dimensional version was proved in [5]. This will be the main theoretical tool of the paper.
Theorem 2.3. Let $q \geq 1$ be an integer, and suppose that $F_n$ is a sequence of random variables in $\mathbb{R}^d$ of the form $F_n = \delta^q(u_n) = (\delta^q(u_n^1), \ldots, \delta^q(u_n^d))$, for a sequence of $\mathbb{R}^d$-valued symmetric functions $u_n$ in $\mathbb{D}^{2q,2q}(\mathcal{S}^\otimes q)$. Suppose that the sequence $F_n$ is bounded in $L^1(\Omega, \mathcal{F})$ and that:

(a) $\langle u_n^i, D_t^m F_n^j \otimes h, \mathcal{F}^q \rangle$ converges to zero in $L^1(\Omega)$ for all integers $1 \leq j, j_0 \leq d$, all integers $1 \leq a_1, \ldots, a_m, r \leq q - 1$ such that $a_1 + \cdots + a_m + r = q$; and all $h \in \mathcal{F}^q \otimes r$.

(b) For each $1 \leq i, j \leq d$, $\langle u_n^i, D^q F_n^j \otimes h, \mathcal{F}^q \rangle$ converges in $L^1(\Omega, \mathcal{F})$ to a random variable $s_{ij}$, such that the matrix $\Sigma := (s_{ij})_{d \times d}$ is nonnegative definite (that is, $\lambda^T \Sigma \lambda \geq 0$ for all nonzero $\lambda \in \mathbb{R}^d$).

Then $F_n$ converges stably to a random variable in $\mathbb{R}^d$ with conditional Gaussian law $\mathcal{N}(0, \Sigma)$ given $X$.

Remark 2.4. Conditions (a) and (b) mean that for $q \geq 1$, some combinations of lower-order derivative products are negligible. In particular, for $q = 3$, then the following scalar products will converge to zero in $L^1(\Omega, \mathcal{F})$:

- $\langle u_n^i, h \rangle_{\mathcal{F}^3}$ for all $h \in \mathcal{F}^3$.
- $\langle u_n^i, D^q F_n^j \otimes h \rangle_{\mathcal{F}^3}$ for all $h \in \mathcal{F}^2$ and all $j$ (including $i = j$).
- $\langle u_n^i, D^q F_n^j \otimes h \rangle_{\mathcal{F}^3}$ for all $h \in \mathcal{F}$ and all $j$.
- $\langle u_n^i, D^q F_n^j \otimes D^q F_n^k \otimes D^q F_n^\ell \rangle_{\mathcal{F}^3}$ for all $1 \leq k, j, \ell \leq d$.

Only the $3^{rd}$-order derivative products $\langle u_n^i, D^3 F_n^j \rangle_{\mathcal{F}^3}$ converge to a nontrivial random variable.

Remark 2.5. It suffices to impose condition (a) for $h \in \mathcal{S}_0$, where $\mathcal{S}_0$ is a total subset of $\mathcal{F}^3$.

For the main result of this paper, we will also require that the sequence $F_n$ satisfies a relative compactness condition in order to establish convergence in the Skorohod space $\mathbb{D}[0, \infty)$.

Corollary 2.6. Suppose $\{G_n(t), t \geq 0\}$ is a sequence of $\mathbb{R}$-valued processes of the form $G_n(t) = \delta^q(u_n(t))$, where $u_n(t)$ is a sequence of symmetric functions in $\mathbb{D}^{2q,2q}(\mathcal{S}^\otimes q)$. Assume that for any finite set of times $\{0 = t_0 < t_1 < \cdots < t_d\}$, the sequence

$$ (G_n(t_1) - G_n(t_0), \ldots, G_n(t_d) - G_n(t_{d-1})) $$

satisfies Theorem 2.3; where the $d \times d$ matrix $\Sigma$ is diagonal with entries $s^2(t_i) - s^2(t_{i-1})$. Suppose further that there exist real numbers $C > 0$, $\gamma > 0$, and $\beta > 1$ such that for each $n$ and for any $0 \leq t_1 < t < t_2$, we have

$$ E \left[ |G_n(t) - G_n(t_1)|^\gamma |G_n(t_2) - G_n(t)|^\gamma \right] \leq C \left( \frac{|nt_2| - |nt_1|}{n} \right)^\beta. $$

Then the family of stochastic processes $\{G_n, n \geq 1\}$ converges as $n \to \infty$ to the process $G = \{G_t, t \geq 0\}$, where $G(t)$ is a Gaussian random variable with mean zero and variance $s^2(t)$. Equivalently, we can say that $G_n(t) \overset{L}{\to} \sqrt{s^2(t)} Z$ as $n \to \infty$, where $Z \sim \mathcal{N}(0, 1)$.

This convergence criteria in $\mathbb{D}$ is well known (see, e.g., [2], Thm. 13.5).
3 Convergence of the Stratonovich integral

3.1 Covariance conditions

Consider a Gaussian stochastic process $X := \{X_t, t \geq 0\}$ with covariance function $E[X_sX_t] = R(s,t)$. Assume $R(s,t)$ satisfies the following bounds: for any $T > 0$, $0 < s \leq 1$, and $s \leq r, t \leq T$:

(i) $E[(X_t - X_{t-s})^2] \leq C_1 s^\frac{3}{2}$, for a positive constant $C_1$.

(ii) If $t > s$,

$$|E[X_t^2 - X_{t-s}^2]| \leq C_2 s^{\frac{3}{2} + \theta} (t-s)^{-\theta}$$

for some $C_2$ and $1/2 < \theta < 1$.

(iii) For $t \geq 4s$,

$$|E[(X_t - X_{t-s})^2 - (X_{t-s} - X_{t-2s})^2]| \leq C_3 s^{\frac{3}{2} + \nu} (t-2s)^{-\nu}$$

for some constants $C_3$ and $\nu > 1$.

(iv) There is a constant $C_4$ and a real number $\lambda \in \left(\frac{1}{3}, \frac{1}{2}\right]$ such that

$$|E[X_r(X_t - X_{t-s})]| \leq \begin{cases} C_4 s^\lambda (t-s)^{\lambda - 1} + |t-r|^{\lambda - 1} & \text{if } |t-r| \geq 2s \text{ and } t \geq 2s \\ C_4 s^\lambda & \text{otherwise} \end{cases}$$

(v) There is a constant $C_5$ and a real number $\gamma > 1$ such that for $t \wedge r \geq 2s$ and $|t-r| \geq 2s$,

$$|E[(X_t - X_{t-s})(X_r - X_{r-s})]| \leq C_5 s^{\frac{3}{2} + \gamma} |t-r|^{-\gamma}.$$

(vi) For integers $n > 0$ and integers $0 \leq j, k \leq nT$, define $\beta_n(j, k) := E\left[(X_{\frac{j+1}{n}} - X_{\frac{j}{n}})(X_{\frac{k+1}{n}} - X_{\frac{k}{n}})\right]$.

Then for each real number $0 \leq t \leq T$,

$$\lim_{n \to \infty} \sum_{j,k=0}^{\lfloor nt \rfloor - 1} \beta_n(j, k)^3 = \eta(t), \quad (8)$$

where $\eta(t)$ is a continuous and nondecreasing function with $\eta(0) = 0$. As we will see, $\eta(t)$ is comparable to the ‘cubic variation’ $[X, X, X]$, discussed in [5] and [10]. As described in [10], these terms are related by Theorem 10 of [12].

In particular, it can be shown that the above conditions are satisfied by fBm with Hurst parameter $H = 1/6$. In Section 4 we will show additional examples.

In addition to conditions (i) - (vi) on $X$, we will also assume the following condition (0) on the test function $f$:

(0) Let $f : \mathbb{R} \to \mathbb{R}$ be a $C^\infty$ function, such that $f$ and all its derivatives have at most polynomial growth.

Consider a uniform partition of $[0, \infty)$ with increment length $1/n$. The Stratonovich integral of $f'(W)$ will be defined as the limit in probability of the sequence (see [10]):

$$\Phi_n^X(t) := \frac{1}{2} \sum_{j=0}^{\lfloor nt \rfloor - 1} \left[f'(X_{\frac{j}{n}}) + f'(X_{\frac{j+1}{n}})\right] \Delta X_{\frac{j}{n}} \quad (9)$$

where $\Delta X_{\frac{j}{n}} = X_{\frac{j+1}{n}} - X_{\frac{j}{n}}$.

The following is the major result of this section.
Theorem 3.1. Let $f$ be a real function satisfying condition (0), and let $X = \{X_t, t \geq 0\}$ be a Gaussian process satisfying conditions (i) through (vi). Then:

$$\left( X_t, \Phi^X_n(t) \right) \xrightarrow{\mathcal{L}} \left( X_t, f(X_t) - f(X_0) + \frac{\sqrt{6}}{12} \int_0^t f^{(3)}(X_s) \, dB_s \right)$$

as $n \to \infty$ in the Skorohod space $\mathbf{D}[0, \infty)$, where $B = \{B_t, t \geq 0\}$ is a scaled Brownian motion, independent of $X$, and with variance $\mathbb{E} \left[ B^2_t \right] = \eta(t)$ for the function $\eta$ defined in condition (vi).

The proof follows from Theorem 2.3 and Corollary 2.6, and is given in a series of lemmas. Following is an outline of the proof. After a preliminary technical lemma, we use a Taylor expansion to decompose

$$\Phi^X_n(t) = f(X_t) - f(X_0) + \frac{1}{12} F_n(t) + \Delta_n(t).$$

We first show that $\Delta_n(t) \xrightarrow{P} 0$ as $n \to \infty$; then we show that $F_n(t)$ satisfies Theorem 2.3. Next we show that $F_n(t) + \Delta_n(t)$ is relatively compact in the sense of Corollary 2.6, and the result follows.

Introduce the following notation, which is similar to that of [6] and [9]. Let $\varepsilon_t = 1_{[0,t]}; \partial_{\varepsilon_t} = 1_{[\frac{1}{n^2}, \frac{1}{n^2+1}]}$; let $\hat{X}_{\varepsilon} = \frac{1}{\varepsilon} \left( X_{\varepsilon} + X_{\varepsilon+1} \right)$; and $\hat{\varepsilon}_t = \frac{1}{\varepsilon} (\varepsilon_t + \varepsilon_{t+1})$. In the following, the term $C$ represents a generic positive constant, which may change from line to line. The constant $C$ may depend on $T$ and the constants in conditions (0) and (i) - (vi) listed above. By the isometry between the space generated by $X$ and the Hilbert space $\mathcal{H}$, we will use the terms $\mathbb{E}[X_s X_t] = \langle \varepsilon_s, \varepsilon_t \rangle_{\mathcal{H}}$ interchangeably.

We begin with the following technical results, which follow from conditions (i) through (v).

**Lemma 3.2.** Assume $\{X_t, 0 \leq t \leq T\}$ satisfies conditions (i), (ii), (iv) and (v). For integers $n \geq 1$, $r \geq 1$ and integers $0 \leq a < b < c \leq \lfloor nT \rfloor$, there exists a constant $C > 0$, which does not depend on $a, b, c$ or $r$, such that:

(a) $$\sup_{0 \leq j, k \leq \lfloor nT \rfloor} \left| \left\langle \partial_{\varepsilon_{j}}, \partial_{\varepsilon_{k}} \right\rangle_{\mathcal{H}} \right| \leq C n^{-\frac{3}{4}}; \quad \text{and} \quad \sup_{0 \leq u \leq T} \sup_{0 \leq j \leq \lfloor nT \rfloor} \left| \left\langle \varepsilon_u, \partial_{\varepsilon_{j}} \right\rangle_{\mathcal{H}} \right| \leq C n^{-\lambda}.$$

(b) $$\sum_{j=a}^{b} \left| \left\langle \hat{\varepsilon}_t, \partial_{\varepsilon_{j}} \right\rangle_{\mathcal{H}} \right| \leq C n^{-\frac{3}{4}} (b - a + 1)^{1-\theta}; \quad \text{and} \quad \sum_{j=a}^{b} \left| \left\langle \hat{\varepsilon}_t, \partial_{\varepsilon_{j}} \right\rangle_{\mathcal{H}} \right|^r \leq C n^{-\frac{3}{4}} \quad \text{for } r > 1.$$

(c) For $0 \leq u, v \leq T$,

$$\sum_{j=a}^{b} \left| \left\langle \varepsilon_u, \partial_{\varepsilon_{j}} \right\rangle_{\mathcal{H}} \right| \leq C; \quad \text{and} \quad \sum_{j=a}^{b} \left| \left\langle \varepsilon_v, \partial_{\varepsilon_{j}} \right\rangle_{\mathcal{H}} \right| \left| \left\langle \varepsilon_v, \partial_{\varepsilon_{j}} \right\rangle_{\mathcal{H}} \right| \leq C n^{-2\lambda}.$$

(d) $$\sum_{j, k=a}^{b} \left| \left\langle \partial_{\varepsilon_{j}}, \partial_{\varepsilon_{k}} \right\rangle_{\mathcal{H}} \right|^r \leq C (b - a + 1) n^{-\frac{5}{4}}.$$
\( \sum_{k=b+1}^{c} \sum_{j=a}^{b} \left| \left\langle \partial_{\pi_j}, \partial_{\pi_k} \right\rangle_{g_j} \right|^r \leq C(c-b)^r n^{-\frac{\epsilon}{r}} \)  

(15)

where \( \epsilon = \max\{1, \theta, 2 - \gamma\} \).

**Proof.** We may assume \( a = 0 \). For part (a), the first inequality follows immediately from condition (i) and Cauchy-Schwarz; and the second inequality is just a restatement of condition (iv). For (b), applying condition (i) for \( j = 0 \) and condition (ii) for \( j \geq 1 \), we have:

\[
\sum_{j=0}^{b} \left| \mathbb{E} \left[ X_{j+1}^2 - X_j^2 \right] \right| \leq Cn^{-\frac{1}{4}} \sum_{j=1}^{b} j^{-\theta} + Cn^{-\frac{1}{2}} \leq Cn^{-\frac{1}{4}} \int_{0}^{b} u^{-\theta} du + Cn^{-\frac{1}{2}} \leq Cn^{-\frac{1}{4}}(b+1)^{1-\theta}. 
\]

Then if \( r \geq 2 \),

\[
\sum_{j=0}^{b} \left| \mathbb{E} \left[ X_{j+1}^2 - X_j^2 \right] \right|^r \leq Cn^{-\frac{1}{4}} \sum_{j=1}^{b} j^{-r\theta} + Cn^{-\frac{1}{2}} \leq Cn^{-\frac{1}{4}}
\]

because \( \theta > 1/2 \) implies \( j^{-r\theta} \) is summable.

For (c), define the set \( J_c = \{ j : 0 \leq j \leq b, j = 0 \text{ or } |j - nu| < 2 \text{ or } |j - nv| < 2 \} \), and note that \( |J_c| \leq 7 \). Then we have by (a) and condition (iv),

\[
\sum_{j=0}^{b} \left| \left\langle \varepsilon_u, \partial_{\pi_j} \right\rangle_{g_j} \right| \leq \sum_{j \in J_c} Cn^{-\lambda} + Cn^{-\lambda} \sum_{j \not\in J_c} (j^{\lambda-1} + |j - nu|^{\lambda-1}) \leq Cn^{-\lambda} + Cn^{-\lambda}(b+1)^\lambda \leq C,
\]

and

\[
\sum_{j=0}^{b} \left| \left\langle \varepsilon_u, \partial_{\pi_j} \right\rangle_{g_j} \left\langle \varepsilon_v, \partial_{\pi_j} \right\rangle_{g_j} \right| \leq \sum_{j \in J_c} Cn^{-2\lambda} + Cn^{-2\lambda} \sum_{j \not\in J_c} (j^{2\lambda-2} + |j - nu|^{2\lambda-2} + |j - nv|^{2\lambda-2}) \leq Cn^{-2\lambda} + Cn^{-2\lambda} \sum_{p=1}^{\infty} p^{2\lambda-2} \leq Cn^{-2\lambda}
\]

because \( \lambda \leq 1/3 \).

For (d), define the set: \( J_d = \{ j, k : j \land k < 1 \text{ or } |j - k| < 2 \} \), and note that \( |J_d| \leq 6(b+1) \). Then we have by (a) and condition (v)

\[
\sum_{j,k=0}^{b} \left| \left\langle \partial_{\pi_j}, \partial_{\pi_k} \right\rangle_{g_j} \right|^r \leq \sup_{j,k} \left| \left\langle \partial_{\pi_j}, \partial_{\pi_k} \right\rangle \right|^r \sum_{j,k=0}^{b} \left| \left\langle \partial_{\pi_j}, \partial_{\pi_k} \right\rangle_{g_j} \right| \leq Cn^{-\frac{r-1}{r}} \sum_{j,k=0}^{b} \left| \left\langle \partial_{\pi_j}, \partial_{\pi_k} \right\rangle_{g_j} \right| \leq Cn^{-\frac{r-1}{r}} \left( \sum_{(j,k) \in J_d} n^{-\frac{1}{2}} + n^{-\frac{1}{4}} \sum_{(j,k) \not\in J_d} |j - k|^{-\gamma} \right) \leq C(b+1)n^{-\frac{r}{r}}.
\]
In particular, if \( r = 3 \) and \( b = \lfloor nt \rfloor - 1 \) (as in condition (vi)), the sum converges absolutely, and the sum vanishes if \( r > 3 \).

For (e), we consider the maximal case, which occurs when \( a = 0 \):

\[
\sum_{k=b+1}^{c} \sum_{j=0}^{b} \left| \langle \partial_{x_k}, \partial_{x_j} \rangle_{S_j} \right|^r = \sum_{k=b+1}^{c} \left| \langle \partial_{x_k} \rangle_{S_j} \right|^r + \sum_{k=b+1}^{c} \sum_{j=1}^{b-1} \left( \langle \partial_{x_k}, \partial_{x_j} \rangle_{S_j} \right)^r + \sum_{k=b+1}^{c} \left| \langle \partial_{x_k} \rangle_{S_j} \right|^r.
\]

Note that \( \partial_{x_k} = \varepsilon_{\lambda_k} \). By part (c) and condition (v), respectively, this is

\[
\leq \sum_{k=b+1}^{c} \left| \langle \varepsilon_{\lambda_k} \rangle_{S_j} \right|^r + Cn^{-\frac{5}{6}} \sum_{k=b+1}^{c} \sum_{j=1}^{b-1} (k-j)^{-\gamma} + Cn^{-\frac{5}{6}} \sum_{k=b+1}^{c} (k-b)^{-\gamma}
\]

\[
\leq Cn^{-\frac{5}{6}} (c-b)^{1-\theta} + Cn^{-\frac{5}{6}} (c-b)^{2-\gamma} + Cn^{-\frac{5}{6}}
\]

\[
\leq Cn^{-\frac{5}{6}} (c-b)^{\gamma},
\]

where \( \epsilon = \max\{1-\theta, 2-\gamma\} < 1 \). \( \square \)

### 3.2 Taylor expansion of \( \Phi_n^X(t) \)

The details of this expansion were mainly inspired by Lemma 5.2 of [10]. We begin with the telescoping series,

\[
f(X_t) = f(0) + f(X_t) - f(X_{\lfloor nt \rfloor}) + \sum_{j=0}^{\lfloor nt \rfloor - 1} \left[ f(X_{\lfloor nt + 1 \rfloor}) - f(X_{\lfloor nt \rfloor}) \right].
\]

By continuity of \( f \) and \( X \), we know that for large \( n \), \( f(X_t) - f(X_{\lfloor nt \rfloor}) \to 0 \) uniformly on compacts in probability (ucp), so this term may be neglected. For each \( j \), we use a Taylor expansion of order 6 with residual term. Let \( h_j := \frac{1}{4} \left[ X_{\lfloor nt + 1 \rfloor} - X_{\lfloor nt \rfloor} \right] \). Then:

\[
f(X_{\lfloor nt \rfloor}) - f(X_{\lfloor nt \rfloor}) = \left( f(X_{\lfloor nt \rfloor} + h_j) - f(X_{\lfloor nt \rfloor}) \right) - \left( f(X_{\lfloor nt \rfloor} - h_j) - f(X_{\lfloor nt \rfloor}) \right)
\]

\[
= \frac{6}{2} f'(X_{\lfloor nt \rfloor}) h_j + R_n^+(j) - \left( \frac{6}{2} f'(X_{\lfloor nt \rfloor}) h_j + R_n^-(j) \right)
\]

\[
f'(X_{\lfloor nt \rfloor}) \Delta X_{\lfloor nt \rfloor} = \frac{1}{24} f^{(3)}(X_{\lfloor nt \rfloor}) \Delta X_{\lfloor nt \rfloor}^3 + \frac{1}{2451} f^{(5)}(X_{\lfloor nt \rfloor}) \Delta X_{\lfloor nt \rfloor}^5 + R_n^+(j) - R_n^-(j)
\]

where \( \Delta X_{\lfloor nt \rfloor} = 2h_j \); and \( R_n^+(j), R_n^-(j) \) are Taylor series remainder terms of order 7. Next we compute:

\[
\frac{f'(X_{\lfloor nt \rfloor}) + f'(X_{\lfloor nt \rfloor})}{2} - f'(X_{\lfloor nt \rfloor}) = \frac{1}{2} \left( f'(X_{\lfloor nt \rfloor} + h_j) - f'(X_{\lfloor nt \rfloor}) \right) + \frac{1}{2} \left( f'(X_{\lfloor nt \rfloor} - h_j) - f'(X_{\lfloor nt \rfloor}) \right)
\]

\[
= \frac{1}{2} \sum_{k=1}^{5} f^{(1+k)}(X_{\lfloor nt \rfloor}) h_j^k + K_n^+(j) + \frac{1}{2} \sum_{k=1}^{5} (-1)^k f^{(1+k)}(X_{\lfloor nt \rfloor}) h_j^k + K_n^-(j)
\]

\[
= \frac{1}{8} f^{(3)}(X_{\lfloor nt \rfloor}) \Delta X_{\lfloor nt \rfloor}^3 + \frac{1}{2451} f^{(5)}(X_{\lfloor nt \rfloor}) \Delta X_{\lfloor nt \rfloor}^5 + \frac{1}{2} (K_n^+ + K_n^-),
\]

where \( K_n^+, K_n^- \) are remainder terms of order 6. Combining the two equations, we obtain

\[
f(X_{\lfloor nt \rfloor}) - f(X_{\lfloor nt \rfloor}) = \frac{f'(X_{\lfloor nt \rfloor}) + f'(X_{\lfloor nt \rfloor})}{2} \Delta X_{\lfloor nt \rfloor} - \frac{1}{12} f^{(3)}(X_{\lfloor nt \rfloor}) \Delta X_{\lfloor nt \rfloor}^3 - \frac{4}{2451} f^{(5)}(X_{\lfloor nt \rfloor}) \Delta X_{\lfloor nt \rfloor}^5
\]

\[
+ R_n^+(j) - R_n^-(j) - \frac{1}{4} [K_n^+(j) + K_n^-(j)] \Delta X_{\lfloor nt \rfloor}.
\]

Our first task is to show that the \( f^{(5)} \) terms and the remainder term vanish in probability.
Lemma 3.3. For each integer $n \geq 1$ and real numbers $0 \leq t_1 < t_2 \leq T$,

$$
E \left[ \left( \sum_{j=\lfloor nt_1 \rfloor}^{\lfloor nt_2 \rfloor - 1} f^{(5)}(\hat{X}_{\frac{j}{n}}) \Delta X_{\frac{j}{n}}^5 \right)^2 \right] \leq Cn^{-\frac{3}{2}} (\lfloor nt_2 \rfloor - \lfloor nt_1 \rfloor). \tag{16}
$$

The proof of this lemma is technical, and is deferred to Section 5.

Lemma 3.4. For integers $n \geq 1$, let

$$
Z_n(t) = \sum_{j=0}^{\lfloor nt \rfloor - 1} \left[ R_n^+(j) - R_n^-(j) + \frac{1}{4} (K_n^+(j) + K_n^-(j)) \Delta X_{\frac{j}{n}} \right].
$$

Then for real numbers $0 \leq t_1 < t_2 \leq T$, we have

$$
E \left[ (Z_n(t_2) - Z_n(t_1))^2 \right] \leq Cn^{-\frac{3}{2}} (\lfloor nt_2 \rfloor - \lfloor nt_1 \rfloor)^2. \tag{17}
$$

Proof. We may assume $t_1 = 0$. Observe that each term in the sum $Z_n(t)$ has the form

$$
f^{(7)}(\xi_j) \Delta X_{\frac{j}{n}}^7,
$$

where $\xi_j$ is an intermediate value between $X_{\frac{j}{n}}$ and $X_{\frac{j+1}{n}}$. Using the Hölder inequality, for each $0 \leq j, k < \lfloor nt_2 \rfloor$ we have

$$
E \left[ f^{(7)}(\xi_j) f^{(7)}(\xi_k) \Delta X_{\frac{j}{n}}^7 \Delta X_{\frac{k}{n}}^7 \right] \leq \left( \sup_{0 \leq u < 1} E \left[ f^{(7)}(uX_{\frac{j}{n}} + (1-u)X_{\frac{j+1}{n}})^4 \right] \right)^{\frac{1}{4}} \leq C n^{-\frac{3}{4}}.
$$

By condition (0), the first two terms are bounded. By condition (i), $E \left[ \Delta X_{\frac{j}{n}}^2 \right] \leq C_1 n^{-\frac{1}{2}}$; and we have by the Gaussian moments formula that

$$
E \left[ \Delta X_{\frac{j}{n}}^{28} \right] \leq 27!! \left( C_1 n^{-\frac{1}{2}} \right)^{14},
$$

hence it follows that

$$
C \sum_{j,k=0}^{\lfloor nt_2 \rfloor - 1} E \left[ f^{(7)}(\xi_j) f^{(7)}(\xi_k) \Delta X_{\frac{j}{n}}^7 \Delta X_{\frac{k}{n}}^7 \right] \leq C \sum_{j,k=0}^{\lfloor nt_2 \rfloor - 1} n^{-\frac{3}{2}} \leq C |nt_2|^2 n^{-\frac{7}{2}}.
$$

\hfill \square

3.3 Malliavin calculus representation of $3^{rd}$ order term

From Lemmas 3.3 and 3.4, it follows that as $n$ becomes large, then $f(X_t)$ behaves asymptotically as

$$
f(X_0) + \frac{1}{2} \sum_{j=0}^{\lfloor nt \rfloor - 1} \left[ f'(X_{\frac{j}{n}}) + f'(X_{\frac{j+1}{n}}) \right] \Delta X_{\frac{j}{n}} - \frac{1}{12} \sum_{j=0}^{\lfloor nt \rfloor - 1} f^{(3)}(\hat{X}_{\frac{j}{n}}) \Delta X_{\frac{j}{n}}^3.
$$

We now turn to the $f^{(3)}(\hat{X}_{\frac{j}{n}})$ term in the Taylor expansion.
Remark 3.5. It may happen that the upper bound of condition (v) is such that
\[ \eta(t) \leq \lim_{n \to \infty} \sum_{j,k=0}^{\lfloor nt \rfloor - 1} |\beta_n(j,k)^3| = 0 \]
for all \( t \), which implies
\[ \lim_{n} \mathbb{E} \left[ \left( \sum_{j=0}^{\lfloor nt \rfloor - 1} f^{(3)}(\hat{X}_n^j) \Delta X_n^3 \right)^2 \right] = 0 \]
for any function \( f \) satisfying condition (0). In this case, \( \Phi_n^X(t) \) converges in probability and we have the change-of-variable formula (2). Indeed, this corresponds to the case of zero cubic variation discussed in [5]. In the rest of this section, we will assume that \( \eta(t) \) is non-trivial.

Consider the 3\textsuperscript{rd} Hermite polynomial \( H_3(y) = y^3 - 3y \). For \( y = \frac{\Delta X_n^3}{\Delta X_n^3_{\text{Lt}}} \), it follows that
\[ \sum_{j=0}^{\lfloor nt \rfloor - 1} f^{(3)}(\hat{X}_n^j) \Delta X_n^3 = \sum_{j=0}^{\lfloor nt \rfloor - 1} \|\Delta X_n^3\|_{L^2} f^{(3)}(\hat{X}_n^j) H_3 \left( \frac{\Delta X_n^3}{\|\Delta X_n^3\|_{L^2}} \right) \]
\[ + 3 \sum_{j=0}^{\lfloor nt \rfloor - 1} \|\Delta X_n^3\|_{L^2} f^{(3)}(\hat{X}_n^j) \Delta X_n^3 \]
The second term is dealt with in the next lemma. The proof is technical, and is deferred to Section 5.

**Lemma 3.6.** For integers \( n \geq 1 \) and integers \( 0 \leq a < b \leq nT \),
\[ \mathbb{E} \left[ \sum_{j=a}^{b-1} \|\Delta X_n^3\|_{L^2} f^{(3)}(\hat{X}_n^j) \Delta X_n^3 \right] \leq C n^{-\frac{1}{2}}. \]

Next, we consider the \( H_3 \) term. By (4) and Lemma 2.1.a we have
\[ \sum_{j=0}^{\lfloor nt \rfloor - 1} \|\Delta X_n^3\|_{L^2} f^{(3)}(\hat{X}_n^j) H_3 \left( \frac{\Delta X_n^3}{\|\Delta X_n^3\|_{L^2}} \right) = \sum_{j=0}^{\lfloor nt \rfloor - 1} f^{(3)}(\hat{X}_n^j) \delta^3 (\hat{\partial}^{(3)} \hat{X}_n^j) \]
\[ = \sum_{j=0}^{\lfloor nt \rfloor - 1} \delta^3 \left( f^{(3)}(\hat{X}_n^j) \hat{\partial}^{(3)} \hat{X}_n^j \right) + 3 \delta \left( \left\langle D f^{(3)}(\hat{X}_n^j), \hat{\partial}^{(3)} \hat{X}_n^j \right\rangle_{B_3} \right) \]
\[ + 3 \delta \left( \left\langle D^2 f^{(3)}(\hat{X}_n^j), \hat{\partial}^{(3)} \hat{X}_n^j \right\rangle_{B_3} \right) + \left\langle D^3 f^{(3)}(\hat{X}_n^j), \hat{\partial}^{(3)} \hat{X}_n^j \right\rangle_{B_3^{(3)}} \]
\[ = \sum_{j=0}^{\lfloor nt \rfloor - 1} \delta^3 \left( f^{(3)}(\hat{X}_n^j) \hat{\partial}^{(3)} \hat{X}_n^j \right) + P_n(t). \]

As \( n \to \infty \), we show that the term \( P_n(t) \) vanishes in probability.

**Lemma 3.7.** For integers \( n \geq 1 \) and real numbers \( 0 \leq t_1 < t_2 \leq T \),
\[ \mathbb{E} \left[ P_n(t)^2 \right] \leq C (\lfloor nt_2 \rfloor - \lfloor nt_1 \rfloor) n^{-\frac{1}{2}}. \]
Proof. We may assume $t_1 = 0$. We want to show

$$\mathbb{E}\left[\left(\delta^2 \left( \sum_{j=0}^{[nt_2]-1} \left\langle D^f(3)(\hat{X}_n^j), \hat{\partial^{(3)}_n} \right\rangle \right) \right)^2\right] \leq C [nt_2] n^{-\frac{4}{5}}; \quad (18)$$

$$\mathbb{E}\left[\left(\delta \left( \sum_{j=0}^{[nt_2]-1} \left\langle D^2f(3)(\hat{X}_n^j), \hat{\partial^{(3)}_n} \right\rangle \right) \right)^2\right] \leq C [nt_2] n^{-\frac{1}{5}}; \quad (19)$$

and

$$\mathbb{E}\left[\left( \sum_{j=0}^{[nt_2]-1} \left\langle D^3f(3)(\hat{X}_n^j), \hat{\partial^{(3)}_n} \right\rangle \right)^2\right] \leq C [nt_2] n^{-2}. \quad (20)$$

Proof of (18). By (6) we have

$$\mathbb{E}\left[\left(\delta^2 \left( \sum_{j=0}^{[nt_2]-1} \left\langle D^f(3)(\hat{X}_n^j), \hat{\partial^{(3)}_n} \right\rangle \right) \right)^2\right] \leq \mathbb{E}\left[ \left( \sum_{j=0}^{[nt_2]-1} \left\langle D^f(3)(\hat{X}_n^j), \hat{\partial^{(3)}_n} \right\rangle \right)^2 \right]$$

$$+ 2\mathbb{E}\left[ \left( \sum_{j=0}^{[nt_2]-1} \left\langle D^f(3)(\hat{X}_n^j), \hat{\partial^{(3)}_n} \right\rangle \right)^2 \right] \leq \sup_{0 \leq j < [nt_2]} \mathbb{E}\left[ f^4(\hat{X}_n^j)^2 \right] \sum_{j=0}^{[nt_2]-1} \left\langle \hat{\partial}_n^j, \hat{\partial}_n^j \right\rangle \left\langle \hat{\partial}_n^j, \hat{\partial}_n^j \right\rangle$$

$$+ \sup_{0 \leq j < [nt_2]} \mathbb{E}\left[ f^5(\hat{X}_n^j)^2 \right] \sum_{j=0}^{[nt_2]-1} \left\langle \hat{\partial}_n^j, \hat{\partial}_n^j \right\rangle \left\langle \hat{\partial}_n^j, \hat{\partial}_n^j \right\rangle$$

By condition (0) and Lemma 3.2.a and 3.2.d,

$$\sup_{0 \leq j < [nt_2]} \mathbb{E}\left[ f^4(\hat{X}_n^j)^2 \right] \leq C n^{-\frac{4}{5}} [nt_2]; \quad \text{and}$$

$$\sup_{0 \leq j < [nt_2]} \mathbb{E}\left[ f^5(\hat{X}_n^j)^2 \right] \leq C n^{-\frac{2}{5}} [nt_2].$$
Then by condition (0), Lemma 3.2.a and 3.2.b,

\[
\sup_{0 \leq j \leq [nt_2]} E \left[ f^{(6)}(\hat{X}_n) \right]^2 \sum_{j,k=0}^{[nt_2]-1} \left| \langle \hat{\varepsilon}_n, \varpartial \hat{\pi} \rangle_{\mathcal{H}} \right|^3 \left| \langle \hat{\varepsilon}_n, \varpartial \hat{\pi} \rangle_{\mathcal{H}} \right|^3 \leq C \sup_{0 \leq j \leq [nt_2]} \left( \frac{[nt_2]-1}{2} \sum_{j=0}^{[nt_2]-1} \left( \langle \hat{\varepsilon}_n, \varpartial \hat{\pi} \rangle_{\mathcal{H}} \right)^2 \right)^2 \leq C n^{-2}.
\]

**Proof of (19) and (20):** The same estimates apply for the other terms, since

\[
E \left[ \left( \delta \sum_{j=0}^{[nt_2]-1} \langle D^2 f^{(3)}(\hat{X}_n), \varpartial \varpartial_{\varpartial} H \rangle_{\mathcal{H}} \right) \right]^2 \leq E \left[ \sum_{j,k=0}^{[nt_2]-1} \left| \langle D^2 f^{(3)}(\hat{X}_n), \varpartial \varpartial_{\varpartial} H \rangle_{\mathcal{H}} \right|^2 \right] \]

and (20) is bounded in the above computation as well. □

### 3.4 Weak convergence of non-trivial part of 3rd order term

We are now ready to apply Theorem 2.3 to the term

\[
\sum_{j=0}^{[nt_1]-1} \delta^3 \left( f^{(3)}(\hat{X}_n) \varpartial \varpartial_{\varpartial} H \right).
\]

Let \(0 = t_0 < t_1 < \cdots < t_d \leq T\) be a finite set of real numbers, and for \(i = 1, \ldots, d\) define

\[
u_i^n = \sum_{j=0}^{[nt_1]-1} f^{(3)}(\hat{X}_n) \varpartial \varpartial_{\varpartial} H.
\]

and define the \(d\)-dimensional vector

\[
F_n = (F_1^n, \ldots, F_d^n),
\]

where each \(F_i^n = \delta^3(u_i^n)\).

To satisfy the CLT conditions, we must deal with terms of the following forms (see Remark 2.4):

1. \(\langle u_i^n, h \rangle_{\mathcal{H}} \) for \(h \in \mathcal{H} \otimes \mathcal{H} \).
2. \(\langle u_i^n, D F_i^n \otimes h \rangle_{\mathcal{H}} \) for \(h \in \mathcal{H} \otimes \mathcal{H} \).
3. \(\langle u_i^n, D^2 F_i^n \otimes h_1 \rangle_{\mathcal{H}} \) for \(h_1, h_2 \in \mathcal{H} \), and
4. \(\langle u_i^n, D^3 F_i^n \rangle_{\mathcal{H}} \) for \(i = 1, \ldots, d\).

We must show that all terms converge to zero except for the terms \(\langle u_i^n, D^3 F_i^n \rangle_{\mathcal{H}} \), \(i = 1, \ldots, d\), which will converge stably to a Gaussian random vector (Lemma 3.11).
Lemma 3.8. For each \(i, j, k, \ell = 1, \ldots, d\), the following terms vanish in \(L^1(\Omega)\) as \(n \to \infty\):
(a) \(\langle u_n^i, h \rangle_{\mathcal{F}_{\Omega}}\) for each \(h \in \mathcal{F}^\otimes 3\).
(b) \(\langle u_n^i, DF^j_n \otimes h \rangle_{\mathcal{F}_{\Omega}}\) for each \(h \in \mathcal{F}^\otimes 2\).
(c) \(\langle u_n^i, D^2 F^j_n \otimes h \rangle_{\mathcal{F}_{\Omega}} + \langle u_n^i, DF^j_n \otimes DF^k_n \otimes h \rangle_{\mathcal{F}_{\Omega}}\) for \(h \in \mathcal{F}\).

Proof. We begin with two estimates that will be needed. For each \(1 \leq i \leq d\),
\[ \mathbb{E} \left\| DF^i_n \right\|_{\mathcal{F}}^2 < C; \quad \text{and} \]
\[ \mathbb{E} \left\| D^2 F^i_n \right\|_{\mathcal{F}}^2 < C. \] (21)
(22)

Proof of (21). Let \(a_i = \lfloor nt_i-1 \rfloor\) and \(b_i = \lfloor nt_i \rfloor\). By Lemma 2.1.b,
\[ DF^i_n = \delta^3(Du_n) + 3\delta^2(u_n^i). \]
Hence, using Lemma 2.1.c,
\[ \mathbb{E} \left\| DF^i_n \right\|_{\mathcal{F}}^2 \leq 2 \sum_{j,k=a_i}^{b_i-1} \mathbb{E} \left[ \delta^3 \left( f^{(4)}(\hat{X}_n^j) \hat{\partial}_{\hat{\pi}}^3 \right) \delta^3 \left( f^{(4)}(\hat{X}_n^k) \hat{\partial}_{\hat{\pi}}^3 \right) \right] \langle \hat{\pi}^3, \hat{\pi}^3 \rangle_{\mathcal{F}} \]
\[ + 18 \sum_{j,k=a_i}^{b_i-1} \mathbb{E} \left[ \delta^2 \left( f^{(3)}(\hat{X}_n^j) \hat{\partial}_{\hat{\pi}}^2 \right) \delta^2 \left( f^{(3)}(\hat{X}_n^k) \hat{\partial}_{\hat{\pi}}^2 \right) \right] \langle \hat{\pi}^2, \hat{\pi}^2 \rangle_{\mathcal{F}} \]
\[ = 2 \sum_{j,k=a_i}^{b_i-1} \sum_{\ell=0}^{3} \left( \frac{3}{\ell} \right)^2 \mathbb{E} \left[ f^{(7-\ell)}(\hat{X}_n^j) f^{(7-\ell)}(\hat{X}_n^k) \right] \langle \hat{\pi}^2, \hat{\pi}^2 \rangle_{\mathcal{F}}^3 \langle \hat{\pi}^3, \hat{\pi}^3 \rangle_{\mathcal{F}} \]
\[ + 18 \sum_{j,k=a_i}^{b_i-1} \sum_{\ell=0}^{2} \left( \frac{2}{\ell} \right)^2 \mathbb{E} \left[ f^{(5-\ell)}(\hat{X}_n^j) f^{(5-\ell)}(\hat{X}_n^k) \right] \langle \hat{\pi}^2, \hat{\pi}^2 \rangle_{\mathcal{F}}^3 \langle \hat{\pi}^3, \hat{\pi}^3 \rangle_{\mathcal{F}} \]
\[ \leq C \]
by condition (0) and Lemma 3.2.d. The proof of (22) follows the same lines, using Lemma 2.1.b to obtain
\[ D^2 F^i_n = \delta^3(D^2 u_n) + 6\delta^2(Du_n) + 6\delta(u_n^i). \]

Now for the main proof. Without loss of generality, we may assume that each \(h \in \mathcal{F}\) is of the form \(\varepsilon_{\tau}\) for some \(0 \leq \tau \leq T\) (see Remark 2.5). Then for (a) we have:
\[ \mathbb{E} \left| \langle u_n^i, h \rangle_{\mathcal{F}_{\Omega}} \right| = \sup_{a_i \leq m \leq b_i} \left\| \mathbb{E} \left[ f^{(3)}(\hat{X}_m^j) \hat{\partial}_{\hat{\pi}}^3 \right] \right\|_{\mathcal{F}_{\Omega}} \sup_{\tau, m} \left\| \langle \varepsilon_{\tau}, \hat{\partial}_{\hat{\pi}}^3 \rangle_{\mathcal{F}} \right\| \sum_{m=a_i}^{b_i-1} \left\| \langle \varepsilon_{\tau}, \hat{\partial}_{\hat{\pi}}^3 \rangle_{\mathcal{F}} \right\| \]
\[ \leq C n^{-2\alpha}, \]
where we used Lemma 3.2.a and Lemma 3.2.c. For (b),
\[ \mathbb{E} \left| \langle u_n^i, DF^j_n \otimes \varepsilon_{\tau} \otimes \varepsilon_u \rangle_{\mathcal{F}_{\Omega}} \right| \leq \sqrt{\sup_{a_i \leq m \leq b_i} \left\| f^{(3)}(\hat{X}_m^j) \right\|_{\mathcal{F}}^2 \sum_{m=a_i}^{b_i-1} \left\| \mathbb{E} \left[ \hat{\partial}_{\hat{\pi}}^3, DF^j_n \otimes \varepsilon_{\tau} \otimes \varepsilon_u \right] \right\|_{\mathcal{F}_{\Omega}}^2} \]
\[ \leq C \sup_m \left\| \hat{\partial}_{\hat{\pi}}^3 \right\|_{\mathcal{F}} \sqrt{\mathbb{E} \left\| DF^j_n \right\|_{\mathcal{F}}^2 \sum_{m=a_i}^{b_i-1} \left\| \langle \hat{\partial}_{\hat{\pi}}^3, \varepsilon_{\tau} \rangle_{\mathcal{F}} \right\|_{\mathcal{F}} \left\| \langle \hat{\partial}_{\hat{\pi}}^3, \varepsilon_u \rangle_{\mathcal{F}} \right\|_{\mathcal{F}}}. \]
By condition (i), \( \| \partial \|_B \leq Cn^{-\frac{1}{4}} \), and so by (21) and Lemma 3.2.c we have an upper bound of \( Cn^{-\frac{1}{4} - 2\lambda} \).

For (c), by similar reasoning along with Lemma 3.2.c and (22),

\[
E \left| \langle u_{n,i}^i, D^2F_{n,j}^j \otimes \varepsilon_T \rangle_{B^\otimes 3} \right| \leq \sqrt{\sup_{m \leq m = b_i} E \left| f^{(3)} (\hat{X}_n^m) \right|^2} \sum_{m = a}^{b_i - 1} \left( E \left| \partial^{(3)}_n, D^2F_{n,j}^j \otimes \varepsilon_T \right|^2 \right)^{\frac{1}{2}} \leq C \sup_m \| \partial \|_B \sqrt{E \| D^2F_{n,j}^j \|_{B^\otimes 2}^2} \sum_{m = a}^{b_i - 1} \left| \langle \partial \|_B, \varepsilon_T \rangle \right| \leq Cn^{-\frac{1}{4}}.
\]

The estimate is similar for the term \( E \left| \langle u_{n,i}^i, D^2F_{n,j}^j \otimes DF_{n,k}^k \otimes \varepsilon_T \rangle_{B^\otimes 3} \right| \). \( \square \)

Now we focus on the terms \( \langle u_{n,i}^i, D^3F_{n,j}^j \rangle_{B^\otimes 3} \). By Lemma 2.1.b,

\[
D^3F_{n,i}^j = \delta^3(D^3u_{n,i}^i) + 9\delta^2(D^2u_{n,i}^i) + 18\delta(Du_{n,i}^i) + 6u_{n,i}^i,
\]

so that \( \langle u_{n,i}^i, D^3F_{n,j}^j \rangle_{B^\otimes 3} \) can be written as,

\[
\sum_{\ell,m} f^{(3)}(\hat{X}_n^m)\delta^3 \left( f^{(6)}(\hat{X}_n^m)\partial^{(3)}_n \right) \left| \langle \partial \|_B, \hat{\varepsilon}_m^\ell \rangle \right|^3_{B^\otimes 3} + 9 \sum_{\ell,m} f^{(3)}(\hat{X}_n^m)\delta^2 \left( f^{(5)}(\hat{X}_n^m)\partial^{(3)}_n \right) \left| \langle \partial \|_B, \hat{\varepsilon}_m^\ell \rangle \right|^2_{B^\otimes 3} + 18 \sum_{\ell,m} f^{(3)}(\hat{X}_n^m)\delta \left( f^{(4)}(\hat{X}_n^m)\partial^{(3)}_n \right) \left| \langle \partial \|_B, \hat{\varepsilon}_m^\ell \rangle \right|^2_{B^\otimes 3} + 6 \langle u_{n,i}^i, u_{n,j}^j \rangle_{B^\otimes 3},
\]

where \( m, \ell \) are the indices for \( u_{n,i}^i, D^3F_{n,j}^j \), respectively, with \( |nt_{i-1}| \leq m \leq |nt_i| \), and \( |nt_{j-1}| \leq \ell \leq |nt_j| \).

**Lemma 3.9.** For each \( 1 \leq i,j \leq d \) we have

\[
\left| \langle u_{n,i}^i, D^3F_{n,j}^j \rangle_{B^\otimes 3} - 6 \delta_{ij} \langle u_{n,i}^i, u_{n,j}^j \rangle_{B^\otimes 3} \right| \rightarrow 0
\]

as \( n \rightarrow \infty \), where \( \delta_{ij} \) is the Kronecker delta.

**Proof.** We will show that for each \( 1 \leq i,j \leq d \)

\[
\lim_{n \rightarrow \infty} E |A_n(i,j)| = \lim_{n \rightarrow \infty} E |B_n(i,j)| = \lim_{n \rightarrow \infty} E |C_n(i,j)| = 0.
\]

and moreover, if \( i \neq j \) then \( \lim_{n \rightarrow \infty} E \left| \langle u_{n,i}^i, u_{n,j}^j \rangle_{B^\otimes 3} \right| = 0 \).

To begin with, observe that if \( g(x) \) is a function satisfying condition (0), then it follows from condition (i) and Lemma 2.1.c that for that for \( q = 1, 2, 3, \)

\[
\sup_j \left\| \partial^q \left( g(\hat{X}_n^m)\partial^{(3)}_n \right) \right\|_{L^2} \leq C \sup_j \left\| \partial^q \right\|_{B^\otimes 3} \leq \frac{Cn^{-\frac{7}{4}}}{}. \tag{23}
\]

For the terms \( A_n(i,j), B_n(i,j), C_n(i,j) \) we include the case \( i = j \). We have

\[
E |A_n(i,j)| \leq \sup_m \left\| f^{(3)}(\hat{X}_n^m) \right\|_{L^2} \sup_{\ell,m} \left\| \partial^3 \left( f^{(6)}(\hat{X}_n^m)\partial^{(3)}_n \right) \right\|_{L^2} \sup_{\ell,m} \left| \langle \partial \|_B, \hat{\varepsilon}_m^\ell \rangle \right| \sum_{\ell,m} \left| \langle \partial \|_B, \hat{\varepsilon}_m^\ell \rangle \right|^2_{B^\otimes 3}.
\]
Using condition (0), \(23\), and Lemma 3.2.a, respectively, we have
\[
\mathbb{E} |A_n(i, j)| \leq C n^{-\frac{1}{6} - \lambda} \sum_{\ell, m} \left\langle \partial_{\mathcal{W}, \ell} \Phi, \varepsilon_{\mathcal{Z}} \right\rangle_{\mathcal{B}_1},
\]
so that Lemma 3.2.c gives \(\mathbb{E} |A_n(i, j)| \leq C n^{-\frac{1}{6} - 3\lambda}\). Next, using condition (0), \(23\) and Lemma 3.2.a,
\[
\mathbb{E} |B_n(i, j)| \leq 9 \sup_m \left\| f^{(3)}(\hat{X}_{\mathcal{W}}) \right\|_{L^2} \sup_\ell \left\| \delta^2 \left( f^{(5)}(\hat{X}_{\mathcal{Z}}) \partial_{\mathcal{W}} \right) \right\|_{L^2} \sup_{\ell, m} \left\langle \partial_{\mathcal{W}, \ell} \Phi, \varepsilon_{\mathcal{Z}} \right\rangle_{\mathcal{B}_1} \leq C n^{-\frac{1}{6} - 2\lambda} \sum_{\ell, m} \left\langle \partial_{\mathcal{W}, \ell} \Phi, \varepsilon_{\mathcal{Z}} \right\rangle_{\mathcal{B}_1};
\]
and so by Lemma 3.2.d,
\[
\mathbb{E} |B_n(i, j)| \leq C n^{-\frac{1}{6} - 2\lambda} \max\{|nt_i|, |nt_j|\},
\]
which converges to zero since \(2\lambda > 1/3\). Similarly for \(C_n(i, j)\) using Lemma 3.2.d,
\[
\mathbb{E} |C_n(i, j)| \leq 18 \sup_m \left\| f^{(3)}(\hat{X}_{\mathcal{W}}) \right\|_{L^2} \sup_\ell \left\| \delta \left( f^{(4)}(\hat{X}_{\mathcal{Z}}) \partial_{\mathcal{W}} \right) \right\|_{L^2} \sup_{\ell, m} \left\langle \partial_{\mathcal{W}, \ell} \Phi, \varepsilon_{\mathcal{Z}} \right\rangle_{\mathcal{B}_1} \sum_{\ell, m} \left\langle \partial_{\mathcal{W}, \ell} \Phi, \varepsilon_{\mathcal{Z}} \right\rangle_{\mathcal{B}_1} \leq C n^{-\frac{1}{6} - \lambda} \sum_{\ell, m} \left\langle \partial_{\mathcal{W}, \ell} \Phi, \varepsilon_{\mathcal{Z}} \right\rangle_{\mathcal{B}_1} \leq C n^{-\lambda + \frac{1}{6}}.
\]

For the second part, we may assume \(i < j\). Using Lemma 3.2.e,
\[
\mathbb{E} \left| \langle u_n^i, u_n^j \rangle_{\mathcal{B}_2} \right|^2 \leq \sup_m \left\| f^{(3)}(\hat{X}_{\mathcal{W}}) \right\|_{L^2} \sum_{\ell, m} \left\langle \partial_{\mathcal{W}, \ell} \Phi, \varepsilon_{\mathcal{Z}} \right\rangle_{\mathcal{B}_1} \sum_{\ell, m} \left\langle \partial_{\mathcal{W}, \ell} \Phi, \varepsilon_{\mathcal{Z}} \right\rangle_{\mathcal{B}_1} \leq C n^{-1} |nt_j|^\epsilon,
\]
which converges to zero because \(\epsilon < 1\). \(\square\)

**Lemma 3.10.** Using the above notation, for each \(1 \leq i, j, k, l \leq d\) we have
\[
\lim_{n \to \infty} \mathbb{E} \left[ \langle u_n^i, D^2 F_n^j \otimes D F_n^k \rangle_{\mathcal{B}_3} \right] = 0, \text{ and}
\]
\[
\lim_{n \to \infty} \mathbb{E} \left[ \langle u_n^i, D F_n^j \otimes D F_n^k \otimes D F_n^l \rangle_{\mathcal{B}_3} \right] = 0.
\]

The proof of this lemma is deferred to Section 5.

Lemmas 3.8, 3.9 and 3.10 show that condition (a) of Theorem 2.3 is satisfied, and moreover that the only non-trivial terms are of the form \(6 \langle u_n^i, u_n^j \rangle_{\mathcal{B}_3} \). It remains to establish the convergence of these terms to a non-negative random variable \(6s_i^2\). With this result, it follows from Theorem 2.3 that the couple \((X, F_n)\) converges stably to \((X, \zeta)\), where \(\zeta = (\zeta_1, \ldots, \zeta_d)\) is a vector whose components are conditionally independent Gaussian random variables with mean zero and variance \(6s_i^2\).

**Lemma 3.11.** For each \(1 \leq i \leq d\), conditioned on \(X\),
\[
\lim_{n \to \infty} \langle u_n^i, u_n^i \rangle_{\mathcal{B}_3} = s_i^2,
\]
where each \(s_i^2\) has the form
\[
s_i^2 = s(t_i)^2 - (t_i)^2 = \int_{t_{i-1}}^{t_i} f^{(3)}(X_s)^2 \eta(ds).
\]
It follows that on the subinterval \([t_{i-1}, t_i]\) we have the conditional result
\[
6 \langle u^i_n, u^i_n \rangle_{\mathcal{F}^3} \rightarrow 6 \int_{t_{i-1}}^{t_i} f^{(3)}(X_s)^2 \eta(ds),
\]
almost surely as \(n \to \infty\), which implies
\[
F^i_n \xrightarrow{L^2} \sqrt{6} \int_{t_{i-1}}^{t_i} f^{(3)}(X_s) dB_s,
\]
where \(\{B_t, t \geq 0\}\) is a Brownian motion, independent of \(X\), with variance \(\eta(t)\).

**Proof.** Let \(a = \lfloor nt_{i-1} \rfloor\) and \(b = \lfloor nt_i \rfloor\), and recall \(\beta_n(j, k) = \langle \partial^j \hat{X}, \partial^k \hat{X} \rangle_{\mathcal{F}^3}\), from condition (vi). We have
\[
\langle u^i_n, u^i_n \rangle_{\mathcal{F}^3} = \sum_{j,k=a}^{b-1} f^{(3)}(\hat{X}_n^j) f^{(3)}(\hat{X}_n^k) \beta_n(j, k)^3.
\]
For each \(n\), define a discrete measure on \(\{1, 2, \ldots\}^2\) by
\[
\mu_n := \sum_{j,k=0}^{\infty} \beta_n(j, k)^3 \delta_{jk},
\]
where \(\delta_{jk}\) denotes the Kronecker delta. It follows from condition (vi) that for each \(t > 0\),
\[
\mu([0, t]^2) := \lim_{n \to \infty} \mu_n([nt], [nt]) = \lim_n \sum_{j,k=0}^{\lfloor nt \rfloor - 1} \beta_n(j, k)^3 = \eta(t).
\]
Moreover, if \(0 < s < t\) then
\[
\mu_n([ns], [nt]) = \mu_n([ns], [ns]) + \sum_{j=0}^{\lfloor nt \rfloor - 1} \sum_{k=\lfloor ns \rfloor}^{\lfloor nt \rfloor - 1} \beta_n(j, k)^3,
\]
which converges to zero because the disjoint sum vanishes by Lemma 3.2.e. Hence we can conclude that \(\mu_n\) converges weakly to the measure given by \(\mu([0, s] \times [0, t]) = \eta(s \land t)\). It follows by continuity of \(f^{(3)}(X_t)\) and Portmanteau theorem that
\[
\sum_{j,k=0}^{\lfloor nt \rfloor - 1} f^{(3)}(\hat{X}_n^j) f^{(3)}(\hat{X}_n^k) \beta_n(j, k)^3 = \int_{\mathbb{R}^2} f^{(3)}(X_s) f^{(3)}(X_u) 1_{s \leq t} 1_{u \leq t} \mu_n(ds, du)
\]
converges in \(L^1(\Omega, \mathcal{F})\) to
\[
\int_0^t f^{(3)}(X_s)^2 \eta(ds).
\]
It follows that on the subinterval \([t_{i-1}, t_i]\) we have the result
\[
\langle u^i_n, u^i_n \rangle_{\mathcal{F}^3} \rightarrow \int_{t_{i-1}}^{t_i} f^{(3)}(X_s)^2 \eta(ds)
\]
in \(L^1(\Omega, \mathcal{F})\) as \(n \to \infty\). Using the Itô isometry for the above integral, we conclude \((26)\). \(\square\)
3.5 Relative compactness of the sequence $F_n(t) + \Delta_n(t)$

Let

$$G_n(t) := \frac{1}{2} \sum_{j=0}^{[nt]-1} \left( f'(X_{\frac{j+1}{n}}) + f'(X_{\frac{j}{n}}) \right) \Delta X_{\frac{j}{n}}.$$

To establish convergence of $G_n(t)$ in $D[0, \infty)$, we need to show that $\{G_n(t)\}$ is relatively compact. For this, it is enough to show that there exist real numbers $\alpha > 0$, $\beta > 1$ such that for each $T > 0$ and any $0 \leq t_1 < t < t_2 \leq T$ we have,

$$\mathbb{E} \left[ |G_n(t) - G_n(t_1)|^\alpha |G_n(t_2) - G_n(t)|^\beta \right] \leq C \left( \frac{|nt_2| - |nt_1|}{n} \right)^\beta.$$

We will do this in several parts. From the preceding section we have,

$$G_n(t) = f(X_{\frac{nt}{n}}) - f(X_0) + \frac{1}{12} \sum_{j=0}^{[nt]-1} \delta^3 \left( f^{(3)}(\hat{X}_{\frac{j}{n}}) \mathcal{J}_{n}^3 \right) + \frac{4}{25!} \sum_{j=0}^{[nt]-1} f^{(5)}(\hat{X}_{\frac{j}{n}}) \Delta X_{\frac{j}{n}}^5 \delta \partial_{\frac{j}{n}} + Z_n(t) + \frac{3}{12} \sum_{j=0}^{[nt]-1} \| \Delta X_{\frac{j}{n}} \|_{L^2} \left( f^{(3)}(\hat{X}_{\frac{j}{n}}) \delta \partial_{\frac{j}{n}} + P_n(t) \right)$$

$$= f(X_{\frac{nt}{n}}) - f(X_0) + \frac{1}{12} \sum_{j=0}^{[nt]-1} \delta^3 \left( f^{(3)}(\hat{X}_{\frac{j}{n}}) \mathcal{J}_{n}^3 \right) + \Delta_n(t).$$

By Lemmas 3.3, 3.4, and 3.7 we have

$$\mathbb{E} \left[ \left( \sum_{j=0}^{[nt]-1} f^{(5)}(\hat{X}_{\frac{j}{n}}) \Delta X_{\frac{j}{n}}^5 \delta \partial_{\frac{j}{n}} \right)^2 \right] \leq C n^{-\frac{9}{4}} \left( [nt_2] - [nt_1] \right);$$

$$\mathbb{E} \left[ (Z_n(t_2) - Z_n(t_1))^2 \right] \leq C n^{-\frac{9}{4}} \left( [nt_2] - [nt_1] \right); \text{ and}$$

$$\mathbb{E} \left[ P_n(t)^2 \right] \leq C n^{-\frac{9}{4}} \left( [nt_2] - [nt_1] \right).$$

Each of these estimates has the form

$$\mathbb{E} \left[ (U_n(t_2) - U_n(t_1))^2 \right] \leq C n^{-\beta} \left( [nt_2] - [nt_1] \right)^\zeta \leq C \left( \frac{|nt_2| - |nt_1|}{n} \right)^\beta,$$

where $\zeta < \beta$ and $\beta > 1$, hence it follows by Cauchy-Schwarz that for $t_1 < t < t_2$ we have

$$\mathbb{E} \left[ |U_n(t) - U_n(t_1)| \ |U_n(t_2) - U_n(t)| \right] \leq C \left( \frac{|nt_2| - |nt_1|}{n} \right)^\beta,$$

so each of these individual sequences is relatively compact. For the term,

$$Y_n(t) = \sum_{j=0}^{[nt]-1} \| \Delta X_{\frac{j}{n}} \|_{L^2} \left( f^{(3)}(\hat{X}_{\frac{j}{n}}) \delta \partial_{\frac{j}{n}} \right),$$

we have by Lemma 3.6 that $Y_n(t)$ vanishes in probability. However, to show relative compactness we need a different estimate.
Lemma 3.12. For \( 0 \leq t_1 < t_2 \leq T \) such that \(|nt_2| - |nt_1| \geq 1\), we have
\[
E \left[ (Y_n(t_2) - Y_n(t_1))^4 \right] \leq Cn^{-4} (|nt_2| - |nt_1|)^2 + Cn^{-\frac{4}{3} - 4\lambda} (|nt_2| - |nt_1|)^{\frac{4}{3} + 4\lambda}.
\]
It follows that the sequence \( \{Y_n(t)\} \) is relatively compact.

Proof. Let \( \Phi := \Phi_n(j_1, j_2, j_3, j_4) = \prod_{i=1}^{4} f^{(3)}(\hat{X}_{\hat{u}_i}) \), and let \( a = |nt_1|, b = |nt_2| \). We have
\[
E \left[ \left( \sum_{j=a}^{b-1} \| \Delta X_j \|_L^2 \right)^8 f^{(3)}(\hat{X}_{\hat{u}_i}) \delta(\partial_{\hat{u}_i}) \right]^4 \leq Cn^{-\frac{4}{3}} \sum_{j_1, j_2, j_3, j_4} \left| E \left[ D \left[ \Phi_n \delta(\partial_{\hat{u}_i}) \right] \right] \right| \leq Cn^{-\frac{4}{3}} \sum_{j_1, j_2, j_3, j_4} \left| E \left[ D \left[ \Phi_n^{(r)} \delta(\partial_{\hat{u}_i}) \right] \right] \right|
\]
\[
= Cn^{-\frac{4}{3}} \sum_{j_1, j_2, j_3, j_4} \left| E \left[ D \left[ \Phi_n^{(r)} \delta(\partial_{\hat{u}_i}) \right] \right] \right| + 3Cn^{-\frac{4}{3}} \sum_{j_1, j_2, j_3, j_4} \left| E \left[ D \left[ \Phi_n^{(r)} \delta(\partial_{\hat{u}_i}) \right] \right] \right|
\]
where
\[
\Phi_n^{(r)} = f^{(4)}(\hat{X}_{\hat{u}_i}) \prod_{i=1}^{4} f^{(3)}(\hat{X}_{\hat{u}_i}).
\]
Continuing this process, we obtain terms of the form:
\[
Cn^{-\frac{4}{3}} \sum_{j_1, j_2, j_3, j_4} \left| E \left[ D \left[ \Phi_n^{(r)} \delta(\partial_{\hat{u}_i}) \right] \right] \right|,
\]
\[
Cn^{-\frac{4}{3}} \sum_{j_1, j_2, j_3, j_4} \left| E \left[ D^2 \Phi_n^{(r)} \delta(\partial_{\hat{u}_i}) \right] \right|,
\]
where \( \partial^k \Phi \) represents the appropriate \( k \)-th derivative of \( \Phi \). By Lemma 3.2.c and 3.2.d, the sums of each type have, respectively, upper bounds of the form
\[
Cn^{-2}(b-a)^{-2} + Cn^{-\frac{4}{3}}(b-a) + Cn^{-\frac{4}{3} - 4\lambda}(b-a)^{4\lambda},
\]
hence we conclude that
\[
E \left[ (Y_n(t_2) - Y_n(t_1))^4 \right] \leq C \left( n^{-2} (|nt_2| - |nt_1|)^2 + n^{-\frac{4}{3} - 4\lambda} (|nt_2| - |nt_1|)^{\frac{4}{3} + 4\lambda} \right),
\]
As for above terms, it follows by Cauchy-Schwarz that
\[
E \left[ (Y_n(t_2) - Y_n(t_1))^2 \right] \leq C \left( n^{-2} (|nt_2| - |nt_1|)^2 + n^{-\frac{4}{3} - 4\lambda} (|nt_2| - |nt_1|)^{\frac{4}{3} + 4\lambda} \right),
\]
and thus \( \{Y_n(t)\} \) is relatively compact. \( \square \)
**Tightness of $F_n$.**

To conclude the proof of Theorem 3.1, we want to show that the sequence $\{F_n(t)\}$ satisfies the relative compactness condition.

**Lemma 3.13.** For $0 \leq t_1 < t_2 \leq T$, write

$$F_n(t_2) - F_n(t_1) = \sum_{j=[nt_1]+1}^{[nt_2]-1} \delta^3\left(f^{(3)}(X_u^\varepsilon)\vartheta^{33}\right)$$

Then given $0 \leq t_1 < t < t_2 \leq T$, there exists a positive constant $C$ such that

$$\mathbb{E}\left[|F_n(t) - F_n(t_1)|^2|F_n(t_2) - F_n(t)|^2\right] \leq C\left(\frac{|nt_2| - |nt_1|}{n}\right)^2. \quad (27)$$

**Proof.** We begin with a general claim about the norm of $DF_n$. Suppose $a, b$ are nonnegative integers. Let

$$g_a = \sum_{j=[nt_1]}^{[nt_2]-1} f^{(a)}(\hat{X}_u^\varepsilon)\vartheta^{33}. \quad (28)$$

Then we have

$$\mathbb{E}\left[\|D^bg_a\|_{L^2(\mathbb{P})}^2 \right] \leq C\left(\frac{|nt_2| - |nt_1|}{n}\right)^2. \quad (27)$$

**Proof of (28).** For each $b$ we can write

$$\mathbb{E}\left[\|D^bg_a\|_{L^2(\mathbb{P})}^2 \right] = \mathbb{E}\left[\left(\sum_{j,k=[nt_1]}^{[nt_2]-1} f^{(a+b)}(\hat{X}_u^\varepsilon)f^{(a+b)}(\hat{X}_u^\varepsilon)\left\langle \hat{\vartheta}^{33} \right|_{\mathbb{P}} , \hat{\vartheta}^{33} \right|_{\mathbb{P}} \right)^2 \right]$$

$$\leq \sup_{|nt_1| \leq j < |nt_2|} \left(\mathbb{E}\left|f^{(a+b)}(\hat{X}_u^\varepsilon)\right|^4 \right)^{\frac{1}{2}} \left(\sup_{j,k=[nt_1]} \left\langle \hat{\vartheta}^{33} , \hat{\vartheta}^{33} \right|_{\mathbb{P}} \right)^{2b} \left(\sum_{j,k=[nt_1]}^{[nt_2]-1} \left\langle \vartheta^{33} , \vartheta^{33} \right|_{\mathbb{P}} \right)^3 \right)^2$$

$$\leq Cn^{-2} (|nt_2| - |nt_1|)^2,$$

by Lemma 3.2.d.

**Proof of (27).** By the Meyer inequality (7) there exists a constant $c_{2,4}$ such that

$$\mathbb{E}\left(|\delta^3(u_n)|^4\right) \leq c_{3,4}\|u_n\|_{L^3(\mathbb{P})}^3.$$

where in this case,

$$u_n = \sum_{j=[nt_1]}^{[nt_2]-1} f^{(3)}(\hat{X}_u^\varepsilon)\vartheta^{33}$$

and

$$\|u_n\|_{L^3(\mathbb{P})}^3 = \mathbb{E}\|u_n\|_{L^3(\mathbb{P})}^3 + \mathbb{E}\|Du_n\|_{L^3(\mathbb{P})}^3 + \mathbb{E}\|D^2u_n\|_{L^3(\mathbb{P})}^3 + \mathbb{E}\|D^3u_n\|_{L^3(\mathbb{P})}^3.$$

From (28) we have $\mathbb{E}\|D^b u_n\|_{L^2(\mathbb{P})}^2 \leq Cn^{-2} (|nt_2| - |nt_1|)^2$ for $b = 0, 1, 2, 3$. From this result, given $0 \leq t_1 < t < t_2$, it follows from the Hölder inequality that

$$\mathbb{E}\left[|F_n(t) - F_n(t_1)|^2|F_n(t_2) - F_n(t)|^2\right] \leq C\left(\frac{|nt_2| - |nt_1|}{n}\right)^2.$$
4 Examples of suitable processes

4.1 Bifractional Brownian motion

The bifractional Brownian motion is a generalization of fractional Brownian motion, first introduced by Houdré and Villa [7]. It is defined as a centered Gaussian process $B^{H,K} = \{B^{H,K}_t, t \geq 0\}$, with covariance given by,

$$E[B^{H,K}_t B^{H,K}_s] = \frac{1}{2K} \left( t^{2H} + s^{2H} \right)^K - \frac{1}{2K} |t-s|^{2HK},$$

where $H \in (0, 1)$, $K \in (0, 1)$ (Note that the case $K = 1$ corresponds to fractional Brownian motion with Hurst parameter $H$). The reader may refer to [13] and [8] for further discussion of properties.

In this section, we show that the results of Section 3 are valid for bifractional Brownian motion with parameter values $H,K$ such that $HK \geq 1/6$.

**Proposition 4.1.** Let $B_t = \{B^{H,K}_t, t \geq 0\}$ be a bifractional Brownian motion with parameters $H,K$ satisfying $HK = 1/6$. Then conditions (i) - (v) are satisfied, with $\theta = 2/3; \lambda = 1/3$;

$$\nu = \begin{cases} 5/3 & \text{if } H < 1/2 \\ 4H - \frac{1}{3} & \text{if } H \geq 1/2 \end{cases}, \quad \text{and} \quad \gamma = \begin{cases} 2/3 + 2H & \text{if } H \leq 1/2 \text{ and } K < 1 \\ 5/3 & \text{otherwise} \end{cases}.$$

Proof. Condition (i). From Prop. 3.1 of [7] we have

$$E \left[ (B_t - B_{t-s})^2 \right] \leq Cs^{2HK} = Cs^{\frac{2}{3}}.$$

Condition (ii). By Fundamental Theorem of Calculus,

$$|E [B^2_t - B^2_{t-s}]| = t^{2HK} - (t-s)^{2HK} = \int_{t-s}^0 2HK(t+\xi)^{2HK-1}d\xi \leq Cs(t-s)^{-\frac{2}{3}}.$$

Condition (iii).

$$E \left[ (B_t - B_{t-s})^2 - (B_{t-s} - B_{t-2s})^2 \right] = E \left[ (B_t - B_{t-s})(B_t - 2B_{t-s} + B_{t-2s}) \right]$$

$$= t^{2HK} - \frac{2}{2K} \left[ t^{2H} + (t-s)^{2H} \right]^K + \frac{1}{2K} \left[ t^{2H} + (t-2s)^{2H} \right]^K$$

$$- \frac{1}{2K} \left[ (t-s)^{2H} + (t-2s)^{2H} \right]^K + \frac{2}{2K} \left[ (t-s)^{2H} + (t-2s)^{2H} \right]^K.$$

In absolute value, this is bounded by

$$\frac{1}{2K} \left| t^{2H} + t^{2H} \right|^K - 2 \left| t^{2H} + (t-s)^{2H} \right|^K - 2 \left| (t-s)^{2H} + (t-2s)^{2H} \right|^K.$$

Both terms have the form

$$2^{-K} |g(t) - g(t-s) + g(t-2s)| \leq C s^2 \sup_{x \in [t-2s,t]} |g''(x)|,$$

for $g(x) = (c + x^{2H})^K$. We show an upper bound for the first term $g''(x)$, with the other one similar.

We have

$$\sup_{x \in [t-2s,t]} |g''(x)| \leq 4H^2 K (1 - K) \left[ t^{2H} + x^{2H} \right]^{K-2} \left( x^{4H-2} + 2HK \right)^2 (2H - 1) \left[ t^{2H} + x^{2H} \right]^{K-1} x^{2H-2}.$$
For the above values of $H, K$ we have
\[
\sup_{x \in [t-2s,t]} 2HK \left| 2H - 1 \right| \left[ r^{2H} + x^{2H} \right]^{K-1} x^{2H-2} \leq C(t-2s)^{2HK-2}.
\]
For the first term, if $H < 1/2$ then
\[
\sup_{x \in [t-2s,t]} 4H^2K(1 - K) \left[ t^{2H} + x^{2H} \right]^{K-2} x^{4H-2} \leq C(t-2s)^{2HK-4H+4H-2} = C(t-2s)^{2HK-2}.
\]
On the other hand, if $H \geq 1/2$, then $t \geq 4s$ implies $t \geq 2(t-2s)$, hence
\[
\sup_{x \in [t-2s,t]} 4H^2K(1 - K) \left[ t^{2H} + x^{2H} \right]^{K-2} x^{4H-2} \leq 4H^2K(1 - K)3^{K-2}(t-2s)^{2HK-2}x^{4H-2} \leq C(t-2s)^{2HK-4H} = C(t-2s)^{2\tilde{H}} - 4H.
\]
Condition (iv). First, for the case $|t-r| < 2s$ or $t < 2s$, we have
\[
\mathbb{E} \left[ B_r(B_t - B_{t-s}) \right] = \mathbb{E} \left[ B_rB_t - B_rB_{t-s} \right]
\leq \frac{1}{2K} \left( r^{2H} + t^{2H} \right)^K - \left[ r^{2H} + (t-s)^{2H} \right]^K + \frac{1}{2K} \left| r - t + s \right|^{2HK} - \left| r - t \right|^{2HK}
\leq C_s^{2HK} = C_s^{2\tilde{H}}
\]
using the inequality $a^r - b^r \leq (a-b)^r$ for $0 < r < 1$. For $|t-r| \geq 2s$, $t \geq 2s$, we consider two cases. First, assume $r \geq t + 2s$.
\[
\mathbb{E} \left[ B_r(B_t - B_{t-s}) \right] = \mathbb{E} \left[ B_rB_t - B_rB_{t-s} \right]
\leq \frac{1}{2K} \left( r^{2H} + t^{2H} \right)^K - \left[ r^{2H} + (t-s)^{2H} \right]^K + \frac{1}{2K} \left| r - t + s \right|^{2HK} - \left| r - t \right|^{2HK}
\leq 2^{1-K} HK s(t-s)^{-\tilde{H}} + 2^{1-K} HK s(r-t)^{-\tilde{H}},
\]
where we used the fact that $r-t \geq 2s$ implies $r-t \geq 2(r-t-s)$. On the other hand, if $r \leq t - 2s$, then the estimate for
\[
\frac{1}{2K} \left( r^{2H} + t^{2H} \right)^K - \left[ r^{2H} + (t-s)^{2H} \right]^K
\]
is the same, and for the other term we have,
\[
\frac{1}{2K} \left| r - t + s \right|^{2HK} - \left| r - t \right|^{2HK} \leq \frac{1}{2K} \left| t - r \right|^{2HK-1} d\xi \leq 2^{1-K} HK s(t-r-s)^{2HK-1} \leq 2^{1-K} HK s(t-r)^{-\tilde{H}},
\]
hence for either case we have an upper bound of $C_s \left( (t-s)^{\lambda-1} + |t-r|^\lambda \right)$ for $\lambda = \frac{1}{2}$.
Condition (v). Assume $t \land r \geq 2s$ and $|t-r| \geq 2s$. We have
\[
\mathbb{E} \left[ (B_r - B_{t-s})(B_r - B_{t-r-s}) \right] = \frac{1}{2K} \left( t^{2H} + r^{2H} \right)^K - \left[ t^{2H} + (r-s)^{2H} \right]^K - \left[ (t-s)^{2H} + r^{2H} \right]^K + \left[ (t-s)^{2H} + (r-s)^{2H} \right]^K
\]
\[
+ \frac{1}{2K} \left( |t-r| + s \right)^{2HK} - 2\left| t - r \right|^{2HK} + |t - r - s|^{2HK}.
\]
This can be interpreted as the sum of a position term, \( \frac{1}{2r} \varphi(t, r, s) \), and a distance term, \( \frac{1}{2r} \psi(t - r, s) \), where

\[
\varphi(t, r, s) = \left[ t^{2H} + r^{2H} \right]^K \left[ t^{2H} + (r - s)^{2H} \right]^K - \left[ (t - s)^{2H} + r^{2H} \right]^K - \left[ (t - s)^{2H} + (r - s)^{2H} \right]^K;
\]

and

\[
\psi(t - r, s) = |t - r + s|^{2HK} - 2|t - r|^{2HK} + |t - r - s|^{2HK}.
\]

We begin with the position term. Note that if \( K = 1 \), then \( \varphi(t, r, s) = 0 \), so we may assume \( K < 1 \) and \( H > \frac{1}{6} \). Without loss of generality, assume \( 0 < 2s \leq r \leq t \). We can write \( \varphi(t, r, s) \) as

\[
2HK \int_0^s \left( \left[ t^{2H} + (r - \xi)^{2H} \right]^{K-1} (r - \xi)^{2H-1} - \left[ (t - s)^{2H} + (r - \xi)^{2H} \right]^{K-1} (r - \xi)^{2H-1} \right) d\xi
\]

\[
= \int_0^s \int_0^\xi 4H^2K(1 - K) \left[ (t - \eta)^{2H} + (r - \xi)^{2H} \right]^{K-2} (t - \eta)^{2H-1} (r - \xi)^{2H-1} d\xi d\eta,
\]

so that

\[
|\varphi(t, r, s)| \leq 4H^2K(1 - K)s^2 \left[ (t - s)^{2H} + (r - s)^{2H} \right]^{K-2} (t - s)^{2H-1} (r - s)^{2H-1}.
\]

Using (29), there are 3 cases to consider:

- **If** \( H < 1/2 \), then for \( 2s \leq r \leq t - 2s \), we have \( t - r < t - s \) and

  \[
  Cs^2 \left[ (t - s)^{2H} + (r - s)^{2H} \right]^{K-2} (t - s)^{2H-1} (r - s)^{2H-1} \leq Cs^2(t - r)^{2HK-2H-1} (r - s)^{2H-1}
  \]

  \[
  = C \left( \frac{s}{r - s} \right)^{1-2H} s^{\frac{3}{2} + 2H} (t - r)^{-\frac{5}{2} - 2H}
  \]

  \[
  \leq Cs^{\frac{3}{2} + \gamma}|t - r|^{-\gamma},
  \]

  where \( \gamma = \frac{3}{2} + 2H > 1 \).

- **If** \( H = 1/2 \), then \( K = 1/3 \) and for \( 2s \leq r \leq t - 2s \)

  \[
  s^2 \left[ (t - s)^{2H} + (r - s)^{2H} \right]^{K-2} (t - s)^{2H-1} (r - s)^{2H-1} \leq s^2|t - r|^{-\frac{2}{3}}.
  \]

- **If** \( H > 1/2 \), then note that for \( 2s \leq r \leq t - 2s \)

  \[
  s^2 \left[ (t - s)^{2H} + (r - s)^{2H} \right]^{K-2} (t - s)^{2H-1} (r - s)^{2H-1} \leq s^2(t - s)^{2HK-2} \leq s^2|t - r|^{-\frac{2}{3}}.
  \]

Next, consider the distance term \( \psi(t - r, s) \). Without loss of generality, assume \( 2s \leq r \leq t - 2s \). We have

\[
|\psi(t - r, s)| = \left| |t - r + s|^{2HK} - 2|t - r|^{2HK} + |t - r - s|^{2HK} \right|
\]

\[
= \int_0^s \int_{-\xi}^{\xi} 2HK(2HK - 1)|t - r + \eta|^{2HK-2} d\eta d\xi
\]

\[
\leq Cs^2(t - r - s)^{2HK-2} \leq Cs^2|t - r|^{-\frac{2}{3}},
\]

since \( |t - r| \geq 2s \) implies \( (t - r - s)^{-\frac{2}{3}} \leq 2s^\gamma|t - r|^{-\frac{2}{3}} \). Note that when \( K < 1 \), then \( H < 1/2 \) implies \( \gamma \leq 5/3 \), so the upper bound is controlled by \( \varphi(t, r, s) \) in this \( K = 1 \) case. \( \square \)
Proposition 4.2. Let \( \{B_t^{H,K}, t \geq 0\} \) be a bifractional Brownian motion with parameters \( H \leq 1/2 \) and \( HK = 1/6 \). Then Condition (vi) of Section 4 holds, with the function \( \eta(t) = C_K t \), where

\[
C_K = \frac{1}{8^K} \left( 8 + 2 \sum_{m=1}^{\infty} \left( (m+1)^{\frac{3}{2}} - 2m^{\frac{3}{2}} + (m-1)^{\frac{3}{2}} \right) \right).
\]

Proof. First of all, we write

\[
\sum_{j,k=0}^{[nt]-1} \beta_n(j, k)^3 = 2 \sum_{j=0}^{[nt]-1} \beta_n(j, 0)^3 + \sum_{j,k=1}^{[nt]-1} \beta_n(j, k)^3.
\]

When \( j \geq 2 \), we have

\[
|\beta_n(j, 0)| = \left| \mathbb{E} \left[ X_{\frac{j+1}{n}} (X_{\frac{j+1}{n}} - X_{\frac{j}{n}}) \right] \right| \\
\leq \frac{1}{2^{K_n^{\frac{3}{2}}}} \left( [1 + (j+1)^{2H}]^K - [1 + j^{2H}]^K \right) + \frac{1}{2^{K_n^{\frac{3}{2}}}} |(j-1)^{2HK} - j^{2HK}| \\
\leq \frac{1}{2^{K_n^{\frac{3}{2}}}} \int_0^1 2HK [1 + (j+x)^{2H}]^{K-1} (j+x)^{2H-1} dx + \frac{1}{2^{K_n^{\frac{3}{2}}}} \int_0^1 2HK (j-1+y)^{2HK-1} dy \\
\leq C n^{-\frac{3}{2}} (j-1)^{-\frac{3}{2}}.
\]

Therefore, using Lemma 3.2.a for \( \beta_n(0, 0) \) and \( \beta_n(1, 0) \),

\[
\sum_{j=0}^{[nt]-1} |\beta_n(j, 0)^3| \leq 2Cn^{-1} + \sum_{j=2}^{[nt]-1} Cn^{-1} (j-1)^{-2} \leq Cn^{-1},
\]

and in the rest of the proof we will always assume \( j, k \geq 1 \).

As in Prop. 4.1, we use the decomposition,

\[
\beta_n(j, k) = \frac{1}{2^{K_n}} \varphi \left( \frac{j+1}{n}, \frac{k+1}{n}, \frac{1}{n} \right) + \frac{1}{2^{K_n}} \psi \left( \frac{j-k}{n}, \frac{1}{n} \right) = 2^{-K_n \frac{3}{2}} \varphi(j+1, k+1, 1) + 2^{-K_n \frac{3}{2}} \psi(j-k, 1),
\]

which gives

\[
\beta_n(j, k)^3 = \frac{1}{8^{K_n}} \left( \varphi^3 + 3\varphi^2 \psi + 3\varphi \psi^2 + \psi^3 \right).
\]

To begin, we want to show that

\[
\lim_{n \to \infty} \sum_{j,k=1}^{[nt]-1} n^{-1} \left| \varphi(j+1, k+1, 1) \right| = 0.
\] (30)

Proof of (30). Note that \( \varphi = 0 \) if \( K = 1 \), so we may assume \( K < 1 \) and \( H > 1/6 \). From (29), when \( t \wedge r \geq 2s \) and \( |t-r| \geq 2s \) we have

\[
|\varphi(t, r, s)| \leq 4H^2 K(1-K)s^2 \left[ (t-s)^{2H} + (r-s)^{2H} \right] K^{-2} (t-s)^{2H-1} (r-s)^{2H-1} \leq Cs^2 (t-s)^{HK-1} (r-s)^{HK-1},
\]

so that

\[
|\varphi(j+1, k+1, 1)| \leq Cn^{-2HK} j^{HK-1} k^{HK-1}.
\]
Recalling the notation \( J_d \) from Lemma 3.2.d, we have
\[
n^{-1} \sum_{j,k=0}^{\lfloor nt \rfloor-1} |\varphi(j+1,k+1,1)| = n^{-1} \sum_{(j,k) \in J_d} |\varphi(j+1,k+1,1)| + n^{-1} \sum_{(j,k) \not\in J_d} |\varphi(j+1,k+1,1)|
\]
\[
\leq C|nt|n^{-\frac{s}{p}} + Cn^{-\frac{s}{p}} \left( \sum_{j=2}^{\lfloor nt \rfloor-1} j^{HK-1} \right)^{2}
\]
\[
\leq C|nt|n^{-\frac{s}{p}} + C|nt|^{2HK} n^{-\frac{s}{p}} \leq Cn^{-\frac{s}{p}},
\]
where we used the fact (which follows from Lemma 3.2.a and the definition of \( \eta \)) that \(|\varphi(j+1,k+1,1)|\) is bounded.

Hence, (30) is proved. It follows from (30) that
\[
\frac{1}{8^Kn} \sum_{j,k=0}^{\lfloor nt \rfloor-1} |\varphi^3 + 3\varphi^2\psi + 3\varphi\psi^2| \to_n 0,
\]
(31)
since \( \varphi \) and \( \psi \) are both bounded. Hence, it is enough to consider
\[
\eta(t) = \lim_{n \to \infty} \frac{1}{8^Kn} \sum_{j,k=0}^{\lfloor nt \rfloor-1} \psi(j-k,1)^3.
\]
To evaluate (32), we have
\[
\frac{1}{8^Kn} \sum_{j,k=0}^{\lfloor nt \rfloor-1} \psi(j-k,1)^3
\]
\[
= \frac{1}{8^Kn} \sum_{j,k=0}^{\lfloor nt \rfloor-1} \left( |j-k+1|^\frac{s}{2} - 2|j-k|^\frac{s}{2} + |j-k-1|^\frac{s}{2} \right)^3
\]
\[
= \frac{1}{8^Kn} \sum_{j=0}^{\lfloor nt \rfloor} 2^3 + \frac{2}{8^Kn} \sum_{j=0}^{\lfloor nt \rfloor-1} \sum_{k=0}^{\lfloor nt \rfloor-1} \left( (j-k+1)^\frac{s}{2} - 2(j-k)^\frac{s}{2} + (j-k-1)^\frac{s}{2} \right)^3
\]
\[
= \frac{8|nt|}{8^Kn} + \frac{2}{8^Kn} \sum_{j=1}^{\lfloor nt \rfloor} \sum_{m=1}^{\lfloor nt \rfloor} \left( (m+1)^\frac{s}{2} - 2m^\frac{s}{2} + (m-1)^\frac{s}{2} \right)^3
\]
\[
= \frac{8|nt|}{8^Kn} + \frac{2}{8^Kn} \sum_{j=1}^{\lfloor nt \rfloor} \sum_{m=1}^{\lfloor nt \rfloor} \sum_{m=1}^{\infty} \left( (m+1)^\frac{s}{2} - 2m^\frac{s}{2} + (m-1)^\frac{s}{2} \right)^3
\]
\[
- \frac{2}{8^Kn} \sum_{j=1}^{\lfloor nt \rfloor} \sum_{m=j}^{\infty} \left( (m+1)^\frac{s}{2} - 2m^\frac{s}{2} + (m-1)^\frac{s}{2} \right)^3,
\]
where the last term tends to zero since
\[
Cn^{-1} \sum_{j=1}^{\lfloor nt \rfloor} \sum_{m=j}^{\infty} \left( (m+1)^\frac{s}{2} - 2m^\frac{s}{2} + (m-1)^\frac{s}{2} \right)^3 \leq Cn^{-1} \sum_{j=1}^{\infty} j^{-4} \to_n 0.
\]

We therefore conclude that \( \eta(t) = C_K t \), where
\[
C_K = \frac{1}{8^K} \left( 8 + 2 \sum_{m=1}^{\infty} \left( (m+1)^\frac{s}{2} - 2m^\frac{s}{2} + (m-1)^\frac{s}{2} \right)^3 \right).
\]
This number is approximately $\frac{7.188}{26}$.

As an immediate consequence of our proof of Theorem 3.1, we have an alternate proof and extension of previous results in Gradinaru et al. In [5], it was proved that (2) holds for any fractional Brownian motion with $H > 1/6$, that is, the correction term vanishes. Following Remark 3.5, we may conclude the following:

**Corollary 4.3.** Let $B_t = \{B_t^{H,K}, t \geq 0\}$ be a bifractional Brownian motion with parameters $1/6 < HK < 1$. Then on a fixed interval $[0, T]$ and for $0 < s \leq 1$, $B$ satisfies (2).

**Proof.** Notice that $s \leq 1$ implies $s^{2HK} \leq s^{\frac{1}{2}}$. With small modifications to the proof of Prop. 4.1, it is easy to verify that conditions (i) - (v) are satisfied when $HK > 1/6$. In accordance with Remark 3.5, we want to show that

$$\lim_{n \to \infty} \sum_{j,k=0}^{[nt]-1} |\beta_n(j,k)|^3 = 0.$$ 

We may assume $K < 1$. From Prop. 3.1 of [7], we have that

$$\mathbb{E}[(B_t - B_{t-s})^2] \leq C s^{2HK}.$$ 

Recalling the notation $J_d$ from Lemma 3.2.d, Cauchy-Schwarz implies for $(j, k) \in J_d$, we have $|\beta_n(j, k)| \leq C n^{-2HK}$. For $(j, k) \notin J_d$, by (29) we have

$$|\phi(j + 1, k + 1, 1)| \leq 4H^2 K (1 - K) \left[j^{2H} + k^{2H}\right]^{K-2} j^{2H-1} k^{2H-1} \leq C |j - k|^{-1-2H(1-K)},$$

and similar to Prop. 4.1, we have

$$|\psi(j - k, 1)| \leq C |j - k|^{2HK-2};$$

hence for $(j, k) \notin J_d$ we have $|\beta_n(j, k)| \leq C n^{-2HK} |j - k|^{-\gamma}$ for $\gamma = \min\{1 + 2H(1-K), 2HK - 2\}$. It follows that

$$\left|\sum_{j,k=0}^{[nt]-1} \beta_n(j,k)^3\right| \leq \sum_{(j,k)\in J_d} |\beta_n(j,k)^3| + \sum_{(j,k)\notin J_d} |\beta_n(j,k)^3| \leq \sum_{(j,k)\in J_d} C n^{-2HK} + C n^{-6HK} \sum_{(j,k)\notin J_d} |j - k|^{-\gamma} \leq C n^{-6HK} [nt]$$

so $|\eta(t)| = 0$ because $HK > 1/6$.

### 4.2 Extended bifractional Brownian motion

This process is discussed in a recent paper by Bardina and Es-Sebaiy [11]. The covariance has the same formula as standard bBm, but it is ‘extended’ in the sense that $1 < K < 2$, with $H$ restricted to satisfy $0 < HK < 1$. Within the context of this paper, this allows us to consider values of $1/12 < H < 1/6$. As in section 4.1, we show computations only for the case $HK = 1/6$. A result similar to Cor. 4.3 can also be shown by modification to the proposition below.

**Proposition 4.4.** Let $Y = \{Y_t^{1/6}, t \geq 0\}$ be an extended bifractional Brownian motion with parameters $1 < K < 2$, $HK = 1/6$. Then $Y$ satisfies conditions (i) - (vi), with $\theta = 2/3$, $\lambda = (2H) \wedge \frac{1}{3}$, and with $\nu$, $\gamma$ and $\eta(t)$ as given in Prop. 4.1.
Proof. Conditions (ii) and (v) are the same as for standard bBm, as shown in Prop. 4.1. In particular, the decomposition into $\phi(t,r,s)$ and $\psi(t-r,s)$ for condition (v) is the same, so it follows that $\eta(t)$ of condition (vi) has the same form. The proofs for conditions (i), (iii) and (iv) require some modifications to accept the case $K > 1$.

Condition (i). From Prop. 3 of [1] we have

$$\mathbb{E} \left[ (Y_t - Y_{t-s})^2 \right] \leq s^{2HK} = s^\frac{4}{3}.$$  

Condition (iii). First, we have

$$\mathbb{E} \left[ (Y_t - Y_{t-s})^2 - (Y_{t-s} - Y_{t-2s})^2 \right]$$

$$= t^{2HK} - \frac{2}{2K} \left[ t^{2H} + (t-s)^{2H} \right]^K + \frac{2}{2K} \left[ (t-s)^{2H} + (t-2s)^{2H} \right]^K - \frac{2}{2K} (t-2s)^{2HK}$$

$$\leq 2t^{2HK} - \frac{2}{2K} \left[ t^{2H} + (t-s)^{2H} \right]^K - 2(t-2s)^{2HK} + \frac{2}{2K} \left[ (t-s)^{2H} + (t-2s)^{2H} \right]^K$$

$$= 4HK \int_{t-s}^t (t+\xi)^{2HK-1} - (t-s+\xi)^{2HK-1} \, d\xi$$

$$\leq Cs^2(t-2s)^{2HK-2}.$$  

On the other hand,

$$t^{2HK} - \frac{2}{2K} \left[ t^{2H} + (t-s)^{2H} \right]^K + \frac{2}{2K} \left[ (t-s)^{2H} + (t-2s)^{2H} \right]^K - \frac{2}{2K} (t-2s)^{2HK}$$

$$\geq 2(t-s)^{2HK} - \frac{2}{2K} \left[ t^{2H} + t^{2H} \right]^K + \frac{2}{2K} \left[ (t-s)^{2H} + (t-2s)^{2H} \right]^K - 2(t-s)^{2HK}$$

$$= -4HK \int_0^s (t-s+\eta)^{2HK-1} - (t-2s+\eta)^{2HK-1} \, d\eta$$

$$\geq -Cs^2(t-2s)^{2HK-2},$$

hence the term is bounded in absolute value as required, with $\nu = 2 - 2HK = 5/3$.

Condition (iv).

$$|\mathbb{E} [Y_t (Y_t - Y_{t-s})]| = \frac{1}{2K} \left( \left[ t^{2H} + t^{2H} \right]^K - \left[ t^{2H} + (t-s)^{2H} \right]^K \right) + \frac{1}{2K} \left( |r-t+s|^{2HK} - |r-t|^{2HK} \right)$$

$$\leq \frac{1}{2K} \left( \left[ t^{2H} + t^{2H} \right]^K - \left[ t^{2H} + (t-s)^{2H} \right]^K \right) + \frac{1}{2K} \left( |r-t+s|^{2HK} - |r-t|^{2HK} \right).$$

We consider two cases for the first term. If $t < 2s$, then by Fundamental Theorem of Calculus,

$$\frac{1}{2K} \left( \left[ t^{2H} + t^{2H} \right]^K - \left[ t^{2H} + (t-s)^{2H} \right]^K \right) \leq \frac{1}{2K} \left( \left[ t^{2H} + (2s)^{2H} \right]^K - r^{2HK} \right)$$

$$= \frac{K}{2K} \int_0^t 2H \left[ t^{2H} + u \right]^{K-1} \, du \leq Cs^{2H}.$$  

If $t \geq 2s$, then

$$\frac{1}{2K} \left( \left[ t^{2H} + t^{2H} \right]^K - \left[ t^{2H} + (t-s)^{2H} \right]^K \right) = 2HK \int_{t-s}^t \left[ t^{2H} + (t+u)^{2H} \right]^{K-1} (t+u)^{2H-1} \, du$$

$$\leq Cs^{2H(K-1)}(t-s)^{2H-1} \leq Cs(t-s)^{2H-1}.$$
In particular, if $|r-t| < 2s$ then this is bounded by
\[
C_s^{2H} \left( \frac{s}{t-s} \right) \leq C_s^{2H}.
\]

For the second term, if $|r-t| < 2s$, then it easily follows that
\[
\frac{1}{2K} \left( (|r-t| + s)^{2HK} - |r-t|^{2HK} \right) \leq C_s^{2HK} \leq C_s^{2H};
\]
and if $|r-t| \geq 2s$, then by Mean Value
\[
\frac{1}{2K} \left( (|r-t| + s)^{2HK} - |r-t|^{2HK} \right) \leq C_s|r-t|^{2HK-1} \leq C_s T^{2H(K-1)} |r-t|^{2H-1} \leq C_s|r-t|^{2H-1}.
\]
In particular, if $t < 2s$ then
\[
C_s|r-t|^{2H-1} = C_s^{2H} \left( \frac{s}{|r-t|} \right) \leq C_s^{2H}.
\]
Hence, we have shown that
\[
|E[Y_r(Y_t - Y_{t-s})]| \leq \begin{cases} 
C_s \left[ (t-s)^{2H-1} + |r-t|^{2H-1} \right] & \text{if } T \geq 2s \text{ and } |r-t| \geq 2s \\
C_s^{2H} & \text{otherwise}
\end{cases}
\]
and so condition (iv) is satisfied by taking $\lambda = \min\{2H, \frac{1}{3}\}$, where $K \in (1, 2)$ implies $\lambda > 1/6$.

4.3 Sub-fractional Brownian motion

Another variant on fBm is the process known as sub-fractional Brownian motion (sfBm). This is a centered Gaussian process $\{Z_t, t \geq 0\}$, with covariance defined by:
\[
R_h(s, t) = s^h + t^h - \frac{1}{2} \left[ (s + t)^h + |s - t|^h \right],
\] (33)
with real parameter $h \in (0, 2)$. Some properties of sfBm are given in [3] and [4]. Note that $h = 1$ is a standard Brownian motion, and also note the similarity of $R_h(t, s)$ to the covariance of fBm with $H = h/2$. Indeed, in [4] it is shown that sfBm may be decomposed into an fBm with $H = h/2$ and another centered Gaussian process.

Similar to Section 4.1, we discuss only the case $h = 1/3$. For $h > 1/3$, it can be shown that conditions (i)-(vi) are satisfied with $\eta(t) = 0$, hence (2) holds.

**Proposition 4.5.** Let $Z = \{Z_t, t \geq 0\}$ be a sub-fractional Brownian motion with covariance and parameter $h = 1/3$. Then $Z$ satisfies conditions (i) - (vi) of Section 3; hence Theorem 3.1 holds. For condition (vi) we have $\eta(t) = C_h t$, where
\[
C_h = 1 + \frac{1}{4} \sum_{m=1}^{\infty} \left( (m+1)^{\frac{4}{3}} - 2m^{\frac{4}{3}} + (m-1)^{\frac{4}{3}} \right)^3.
\]

**Proof.** Condition (i). We have
\[
E \left[ (Z_t - Z_{t-s})^2 \right] = R_h(t, t) + R_h(t-s, t-s) - 2R_h(t, t-s)
= \frac{9h}{2} + \frac{1}{2} (2t-s)^h - \frac{9h}{2} (t-s)^h + \frac{1}{2} (2t-s)^h + s^h
= \frac{1}{2} \left[ (2t)^h - (2t-s)^h \right] - \frac{1}{2} \left[ (2t-2s)^h - (2t-s)^h \right] + s^h.
\]
This is bounded in absolute value by $C s^h$, using the inequality $a^h - b^h \leq (a - b)^h$.

**Condition (ii).**

$$|E [Z_t^2 - Z_{t-s}^2]| = \left| 2t^h - \frac{2h}{2} t^h - 2(t-s)^h + \frac{2h}{2} (t-s)^h \right| = \frac{|4 - 2h|}{2} [t^h - (t-s)^h].$$

By Mean Value this is bounded by

$$C s(t-s)^{h-1} = C s(t-s)^{-\frac{2}{3}},$$

which implies (ii) with $\theta = 2/3$.

**Condition (iii).**

$$E [(Z_t - Z_{t-s})^2 - (Z_{t-s} - Z_{t-2s})^2] = R_h(t, t) - 2R_h(t, t - s) + 2R_h(t - s, t - 2s) - R_h(t - 2s, t - 2s)$$

$$= -\frac{(2t)^h}{2} + (2t - s)^h - (2t - 3s)^h + \frac{1}{2} (2t - 4s)^h$$

$$= -\frac{1}{2} [(2t)^h - 2(2t - s)^h + (2t - 2s)^h] + \frac{1}{2} [(2t - 2s)^h - 2(2t - 3s)^h + (2t - 4s)^h].$$

By Mean Value, these terms are bounded in absolute value by

$$C s^2(2t - 4s)^{h-2} \leq C s^{\frac{5}{3} + \nu} (t - s)^{-\nu}$$

for $\nu = 5/3$.

**Condition (iv).**

$$|E [Z_t(Z_t - Z_{t-s})]| = |R_h(r, t) - R_h(r, t - s)|$$

$$= |t^h - (t-s)^h - \frac{1}{2} [(r + t)^h - (r + t - s)^h] + \frac{1}{2} (|r - t + s|^h - |r - t|^h)|$$

Note that the above expression is always bounded by $C s^h$ by the inequality $a^h - b^h \leq (a - b)^h$. Hence, the bound is satisfied for the cases $t < 2s$ or $|t - r| < 2s$. Assuming $t \geq 2s$, $|r - t| \geq 2s$, we have

$$|t^h - (t-s)^h - \frac{1}{2} [(r + t)^h - (r + t - s)^h] + \frac{1}{2} (|r - t + s|^h - |r - t|^h)|$$

$$\leq h \int_{-s}^{0} (t+u)^{h-1} du + \frac{h}{2} \int_{-s}^{0} (r + t + u)^{h-1} du + \frac{h}{2} \int_{0}^{s} (|r - t| + u)^{h-1} du$$

$$\leq C s(t-s)^{-\frac{2}{3}} + C s(|r - t| - s)^{-\frac{2}{3}}.$$
Assuming that $|t - r| \geq 2s$, by Mean Value this is bounded in absolute value by
\[
Cs^2|t - r - s|^{h-2} \leq Cs^2|t - r|^{h-2}
\]
since $|t - r| \geq 2s$ implies $|t - r - s| \geq \frac{1}{2}|t - r|$. If $h < 1$, then we take $\gamma = 2 - h = 5/3$, and we have an upper bound of
\[
Cs^{h+2-h}|t - r|^{h-2} = Cs^{h+\gamma}|t - r|^{-\gamma}.
\]

Condition (vi). First assume $h = 1/3$. Referring to condition (v) above, we can decompose $\beta_n(j, k)$ as
\[
\beta_n(j, k) = \frac{1}{2n^h} \omega(j, k, 1) + \frac{1}{2n^h} \psi(j - k, 1),
\]
where $\omega(j, k, 1) = -(j + k + 2)^h + 2(j + k + 1)^h - (j + k)^h$ and $\psi(j - k, 1) = |j - k + 1|^h - 2|j - k|^h + |j - k - 1|^h$.

Note that $\psi(j - k, 1)$ is identical to the $\psi$ used in Prop. 4.1, where in this case $h = 2HK$. Following the proof of Prop. 4.2, it is enough to show
\[
\lim_{n \to \infty} n^{-1} \sum_{j,k=0}^{[nt]-1} |\omega(j, k, 1)| = 0,
\]
(34)
so that, similar to (32) in the proof of Prop. 4.2, we have
\[
\eta(t) = \lim_{n \to \infty} \frac{1}{8H^3K} \sum_{j,k=0}^{[nt]-1} \psi(j - k, 1)^3 = C_h t,
\]
where
\[
C_h = 1 + \frac{1}{4} \sum_{m=1}^{\infty} \left( (m + 1)^{\frac{1}{3}} - 2m^{\frac{1}{3}} + (m - 1)^{\frac{1}{3}} \right)^3.
\]
That is, $C_h$ corresponds to the constant $C_K$ from Prop. 4.2 with $K = 1$.

Proof of (34). By Mean Value and the above computation for condition (v), $|\omega(j, k, 1)| \leq C(j + k)^{-\gamma}$ for some $\gamma > 1$. Hence, for each $j \geq 2$,
\[
\sum_{k=0}^{[nt]-1} |\omega(j, k, 1)| \leq C \sum_{k=0}^{[nt]-1} (j + k)^{-\gamma} \\
\leq C \int_{j-1}^{\infty} u^{\gamma} du \leq C(j - 1)^{1-\gamma}.
\]
It follows that we have
\[
n^{-1} \sum_{j,k=0}^{[nt]-1} |\omega(j, k, 1)| = n^{-1} \sum_{k=0}^{[nt]-1} (|\omega(0, k, 1)| + |\omega(1, k, 1)|) + n^{-1} \sum_{j=2}^{[nt]-1} \sum_{k=0}^{[nt]-1} |\omega(j, k, 1)|
\]
\[
= n^{-1} \sum_{k=0}^{[nt]-1} \left( [(k + 2)^h - 2(k + 1)^h + k^h] + [(k + 3)^h - 2(k + 2)^h + (k + 1)^h] \right)
\]
\[
+ Cn^{-1} \sum_{j=2}^{[nt]-1} (j - 1)^{1-\gamma}
\]
\[
\leq Cn^{-1} + Cn^{-1}[nt]^{2-\gamma}
\]
which converges to 0 since $\gamma > 1$. \qed
5 Proof of Some Technical Lemmas

5.1 Proof of Lemma 3.3

We may assume \( t_1 = 0 \). For this proof we use Malliavin calculus to represent \( \Delta X^p_n \) as a Skorohod integral. Consider the Hermite polynomial identity \( x^5 = H_5(x) + 10H_3(x) + 15H_1(x) \). Using the isometry \( H_p(X(h)) = \delta^p(h^\otimes p) \) (when \( \|h\|_B = 1 \)) we obtain for each \( 0 \leq j \leq \lfloor nt_2 \rfloor - 1 \),

\[
\Delta X^5_n = \delta^5(\partial^\otimes 5) + 10\|\Delta X^1_n\|^2_{L^2} \delta^3(\partial^\otimes 3) + 15\|\Delta X^1_n\|^4_{L^2} \delta^1(\partial^\otimes 1). \tag{35}
\]

With this representation, we can expand

\[
\sum_{j,k=0}^{\lfloor nt_2 \rfloor - 1} \mathbb{E} \left[ f^{(5)}(\hat{X}_n^j) f^{(5)}(\hat{X}_n^k) \Delta X^5_n \Delta X^5_n \right]
\]

into 9 sums of the form

\[
C \sum_{j,k=0}^{\lfloor nt_2 \rfloor - 1} \|\Delta X^5_n\|_{L^2}^p \|\Delta X^5_n\|_{L^2}^q \mathbb{E} \left[ f^{(5)}(\hat{X}_n^j) f^{(5)}(\hat{X}_n^k) \delta^p(\partial^\otimes p) \delta^q(\partial^\otimes q) \right] \tag{36}
\]

where \( p, q \) take values 1, 3, or 5. By the integral multiplication formula \([6]\); and using the Malliavin duality \([5]\), each term of the form \([36]\) can be further expanded into terms of the form

\[
C \sum_{j,k=0}^{\lfloor nt_2 \rfloor - 1} \left\langle \partial_n^j, \partial_n^k \right\rangle_B \|\Delta X^5_n\|_{L^2}^p \|\Delta X^5_n\|_{L^2}^q \mathbb{E} \left[ f^{(5)}(\hat{X}_n^j) f^{(5)}(\hat{X}_n^k) \delta^p(\partial^\otimes p) \delta^q(\partial^\otimes q) \right] \tag{36a}
\]

where \( 0 \leq r \leq p \wedge q \) and \( p, q \in \{1, 3, 5\} \). For \( 0 \leq m = p + q - 2r \leq 10 \), we have

\[
D^m \left[ f^{(5)}(\hat{X}_n^j) f^{(5)}(\hat{X}_n^k) \right] = \sum_{a+b=m} D^a \left( f^{(5)}(\hat{X}_n^j) \right) D^b \left( f^{(5)}(\hat{X}_n^k) \right) = \sum_{a+b=m} f^{(5+a)}(\hat{X}_n^j) f^{(5+b)}(\hat{X}_n^k) \hat{\varepsilon}^\otimes a_n \otimes \hat{\varepsilon}^\otimes b_n.
\]

Hence, we expand \([36a]\) again into terms of the form:

\[
C \sum_{j,k=0}^{\lfloor nt_2 \rfloor - 1} \left\langle \partial_n^j, \partial_n^k \right\rangle_B \|\Delta X^5_n\|_{L^2}^p \|\Delta X^5_n\|_{L^2}^q \mathbb{E} \left[ f^{(5+a)}(\hat{X}_n^j) f^{(5+b)}(\hat{X}_n^k) \right] 
\times \left\langle \hat{\varepsilon}^\otimes a_n \otimes \hat{\varepsilon}^\otimes b_n, \delta^p(\partial^\otimes p) \delta^q(\partial^\otimes q) \right\rangle_B \tag{36b}
\]

where \( a + b = p + q - 2r \). With this representation, we are now ready to develop estimates for each term. By condition (0),

\[
\left| \mathbb{E} \left[ f^{(5+a)}(\hat{X}_n^j) f^{(5+b)}(\hat{X}_n^k) \right] \right| \leq \left( \sup_{0 \leq j < \lfloor nt_2 \rfloor} \mathbb{E} \left[ f^{(5+a)}(\hat{X}_n^j) \right] \right)^{\frac{1}{2}} \left( \sup_{0 \leq k < \lfloor nt_2 \rfloor} \mathbb{E} \left[ f^{(5+b)}(\hat{X}_n^k) \right] \right)^{\frac{1}{2}} \leq C;
\]
and by condition (i),
\[
\sup_{0 \leq j < [nt_2]} \| \Delta X_j \|_{L_2^p} \leq Cn^{-\frac{10-(p+q)}{6}}.
\]

If \( a \geq 1 \) with \( a + b = p + q - 2r \), by condition (iv)
\[
\left| \left< \partial_{\alpha} \partial_{\beta} \partial_{\gamma} X, \partial_{\alpha} \partial_{\beta} \partial_{\gamma} X \right>_{B^a_{r+b+t}} \right| \leq Cn^{-(a+b-1)\lambda} \left| \partial_{\alpha} \partial_{\beta} \partial_{\gamma} X \right|_{B^a_{r+b+t}},
\]
with a similar term in \( k \) if \( a = 0 \) and \( b \geq 1 \). Hence, assuming \( a \geq 1 \), each term in the expansion of \( \hat{c} \) has an upper bound of
\[
C \sum_{j,k=0}^{[nt_2]-1} \left| \left< \partial_{\alpha} \partial_{\beta} \partial_{\gamma} X, \partial_{\alpha} \partial_{\beta} \partial_{\gamma} X \right>_{B^a_{r+b+t}} \right|^r n^{-\frac{10-(p+q)}{6}} n^{-(a+b-1)\lambda} \left| \partial_{\alpha} \partial_{\beta} \partial_{\gamma} X \right|_{B^a_{r+b+t}}.
\]
To show each term has the desired upper bound, first assume \( r \geq 1 \). Then \( \left| \left< \partial_{\alpha} \partial_{\beta} \partial_{\gamma} X, \partial_{\alpha} \partial_{\beta} \partial_{\gamma} X \right>_{B^a_{r+b+t}} \right| \leq Cn^{-\lambda} \), and by Lemma 3.2.d we have an upper bound of
\[
Cn^{-\frac{10-(p+q)}{6}-(a+b)\lambda} \sum_{j,k=0}^{[nt_2]-1} \left| \partial_{\alpha} \partial_{\beta} \partial_{\gamma} X \right|_{B^a_{r+b+t}} \leq C[nt_2] n^{-\frac{10-(p+q)}{6}-(a+b)\lambda} = C[nt_2] n^{-\frac{10-(p+q)}{6}-(a+b)\lambda},
\]
which is less than or equal to \( C[nt_2] n^{-\frac{5}{6}} \) because \( \lambda > 1/6 \). For cases with \( r = 0 \), then either \( a \geq 1 \) or \( b \geq 1 \), so without loss of generality assume \( a \geq 1 \). For this case with Lemma 3.2.c we have an upper bound of
\[
C \sum_{j,k=0}^{[nt_2]-1} n^{-\frac{10-(p+q)}{6}-(a+b-1)\lambda} \left| \partial_{\alpha} \partial_{\beta} \partial_{\gamma} X \right|_{B^a_{r+b+t}} \leq C[nt_2] n^{-\frac{10-(p+q)}{6}-(a+b)\lambda} \lambda,
\]
which is less than \( C[nt_2] n^{-\frac{4}{6}} \) since \( \lambda \leq 1/3 \).

### 5.2 Proof of Lemma 3.6

Without loss of generality, assume \( a = 0 \). First we want to show that for each integer \( 0 \leq k \leq b - 1 \),
\[
E \left| \sum_{j=0}^{k} f^{(3)}(X_{\frac{n}{2^k}}) \Delta X_{\frac{n}{2^k}} \right| \leq C.
\]  
(37)

Using the Taylor expansion similar to Section 3.2,
\[
f''(X_{\frac{n}{2^{k+1}}}) - f''(X_{\frac{n}{2^k}}) = \left( f''(X_{\frac{n}{2^{k+1}}}) - f''(X_{\frac{n}{2^k}}) \right) = \left( f''(X_{\frac{n}{2^{k+1}}}) - f''(X_{\frac{n}{2^k}}) \right)
\]
\[
= f^{(3)}(X_{\frac{n}{2^k}}) \Delta X_{\frac{n}{2^k}} + \frac{1}{24} f^{(5)}(X_{\frac{n}{2^k}}) \Delta X_{\frac{n}{2^k}}^3 + \frac{1}{255!} f^{(7)}(X_{\frac{n}{2^k}}) \Delta X_{\frac{n}{2^k}}^5 + B^+(n) - B^-(n)
\]

where \( B^+(n) \), \( B^-(n) \) have the form \( C f^{(3)}(\xi_j) \Delta X_{\frac{n}{2^k}} \). Hence we can write,
\[
E \left| \sum_{j=0}^{k} f^{(3)}(X_{\frac{n}{2^k}}) \Delta X_{\frac{n}{2^k}} \right| \leq \left| \sum_{j=0}^{k} f''(X_{\frac{n}{2^k}}) \Delta X_{\frac{n}{2^k}} \right| + \frac{1}{24} \left| \sum_{j=0}^{k} f^{(5)}(X_{\frac{n}{2^k}}) \Delta X_{\frac{n}{2^k}}^3 \right|
\]
\[
+ \frac{1}{255!} \left| \sum_{j=0}^{k} f^{(7)}(X_{\frac{n}{2^k}}) \Delta X_{\frac{n}{2^k}}^5 \right| + E \left| B^+(n) \right| - \left| B^-(n) \right|.
\]
We have the following estimates: By condition (0),

\[ \mathbb{E} \left| \sum_{j=0}^{k} \left( f''(X_{\frac{j+1}{n}}) - f''(X_{\frac{j}{n}}) \right) \right| \leq \mathbb{E} \left| f''(X_{\frac{k+1}{n}}) - f''(X_0) \right| \leq C; \]

by Lemma 3.3,

\[ \mathbb{E} \left| \sum_{j=0}^{k} f^{(7)}(\hat{X}_{\frac{j}{n}})\Delta X^5_X \right| \leq Cn^{-\frac{2}{3}}(k+1)^{\frac{2}{3}}; \]

and by Lemma 3.4,

\[ \mathbb{E} \left| \sum_{j=0}^{k} B^+_{\frac{j}{n}}(j) \right| + \mathbb{E} \left| B^-_{\frac{j}{n}}(j) \right| \leq Cn^{-\frac{2}{3}}(k+1)^{2}. \]

This leaves the \( \Delta X^3 \) term. Using the Hermite polynomial identity \( y^3 = H_3(y) + 3H_1(y) \), we can write

\[ \mathbb{E} \left| \sum_{j=0}^{k} f^{(5)}(\hat{X}_{\frac{j}{n}})\Delta X^3_X \right| \leq \mathbb{E} \left| \sum_{j=0}^{k} f^{(5)}(\hat{X}_{\frac{j}{n}})\delta^3(\delta^{\otimes 3}) \right| + \mathbb{E} \left| \sum_{j=0}^{k} \left| \Delta X_{\frac{j}{n}} \right|_{L_2}^2 f^{(5)}(\hat{X}_{\frac{j}{n}})\delta(\delta^{\otimes 3}). \right| \]

For the first term we have

\[ \mathbb{E} \left| \sum_{j=0}^{k} f^{(5)}(\hat{X}_{\frac{j}{n}})\delta^3(\delta^{\otimes 3}) \right|^2 \]

\[ = \sum_{j,\ell=0}^{k} \mathbb{E} \left[ f^{(5)}(\hat{X}_{\frac{j}{n}}) f^{(5)}(\hat{X}_{\frac{\ell}{n}}) \delta^3(\delta^{\otimes 3}) \delta^3(\delta^{\otimes 3}) \right] \]

\[ = \sum_{j,\ell=0}^{k} \sum_{r=0}^{3} \binom{r}{3}^2 \mathbb{E} \left[ f^{(5)}(\hat{X}_{\frac{j}{n}}) f^{(5)}(\hat{X}_{\frac{\ell}{n}}) \delta^{6-2r}(\delta^{\otimes 3-r} \otimes \delta^{\otimes 3-r}) \delta^{6-2r}(\delta^{\otimes 3-r} \otimes \delta^{\otimes 3-r}) \right] \]

\[ \leq \sum_{j,\ell=0}^{k} \sum_{r=0}^{3} \sum_{a+b=4}^{\infty} \left| \langle \hat{\varepsilon}^{a,b}_{\frac{j}{n}} \otimes \hat{\varepsilon}^{a,b}_{\frac{\ell}{n}}, \delta^{\otimes 3-r} \otimes \delta^{\otimes 3-r} \rangle \right|_{S^3_{6-2r}} \left| \langle \delta^{\otimes 3-r} \otimes \delta^{\otimes 3-r} \rangle \right|_{S^3_{6-2r}}. \]

For this sum, if \( r = 0 \) we use Lemma 3.2.a and 3.2.b for each pair \((a,b)\) to obtain terms of the form

\[ \leq C n^{-1-3\lambda}(k+1), \]

where we use the fact that \( r = 0 \) implies \( a \geq 3 \) or \( b \geq 3 \). If \( r \geq 1 \), we use Lemma 3.2.a and 3.2.d to obtain terms of the form

\[ C n^{-(6-2r)\lambda} \sum_{j,\ell=0}^{k} \left| \langle \delta^{\otimes 3}, \delta^{\otimes 3} \rangle \right|_{S^3_{6-2r}} \leq C n^{-(6-2r)\lambda-\frac{2}{3}}(k+1), \]

noting that \((6-2r)\lambda + \frac{2}{3} > 1\).
For the other term, we have by Lemma 3.2.b,

\[
E \left( \sum_{j=0}^{k} \left\| \Delta X_j \right\|^2_{L^2} f^{(5)}(\tilde{X}_\alpha) \delta(\partial_\alpha) \right)^2 \\
\leq \sup_{0 \leq j \leq k} \left\| \Delta X_j \right\|^4_{L^2} \sum_{j, \ell=0}^{k} \left| E \left[ f^{(5)}(\tilde{X}_\alpha)f^{(5)}(\tilde{X}_\alpha) \left( \delta^2 (\partial_\alpha \otimes \partial_\alpha) + \left< \partial_\alpha, \partial_\alpha \right>_{\delta_2} \right) \right] \right|
\]
\[
\leq Cn^{-\frac{2}{3}} \sum_{j, \ell=0}^{k} \left| E \left[ D^2 \left[ f^{(5)}(\tilde{X}_\alpha)f^{(5)}(\tilde{X}_\alpha) \right] \delta^{2} (\partial_\alpha \otimes \partial_\alpha) + \left< \partial_\alpha, \partial_\alpha \right>_{\delta_2} \right] \right|
\leq Cn^{-\frac{2}{3}} \sum_{j, \ell=0}^{k} \left| \left< \tilde{e}_\alpha, \partial_\alpha \right>_{\delta_2} \right|
\]
\[
\leq Cn^{-\frac{2}{3}} \left( \sum_{j=0}^{k} \left| \left< \tilde{e}_\alpha, \partial_\alpha \right>_{\delta} \right|^2 \right) \left( \sum_{j=0}^{k} \left| \left< \tilde{e}_\alpha, \partial_\alpha \right>_{\delta} \right|^2 \right) + Cn^{-\frac{2}{3}} \sum_{j=0}^{k} \left| \left< \tilde{e}_\alpha, \partial_\alpha \right>_{\delta} \right|^2
\]
\[
\leq Cn^{-\frac{2}{3}} \left( \sum_{j=0}^{k} \left| \left< \tilde{e}_\alpha, \partial_\alpha \right>_{\delta} \right|^2 \right) + Cn^{-\frac{2}{3}} \sum_{j=0}^{k} \left| \left< \tilde{e}_\alpha, \partial_\alpha \right>_{\delta} \right|^2
\]
\[
\leq Cn^{-\frac{2}{3}} \left( \sum_{j=0}^{k} \left| \left< \tilde{e}_\alpha, \partial_\alpha \right>_{\delta} \right|^2 \right)
\]

where the estimates follow from Lemma 3.2.c and 3.2.d. Hence, by Cauchy-Schwarz

\[
E \left( \sum_{j=0}^{k} \left\| \Delta X_j \right\|^2_{L^2} f^{(5)}(\tilde{X}_\alpha) \delta(\partial_\alpha) \right)^2 \leq C,
\]

which proves (37). Now we define

\[
G_n(j) = \sum_{k=0}^{j} f^{(3)}(\tilde{X}_\alpha) \Delta X_k,
\]

and by Abel’s formula and condition (iii) we have

\[
E \left( \sum_{j=0}^{b-1} \left\| \Delta X_j \right\|^2_{L^2} f^{(3)}(\tilde{X}_\alpha) \Delta X_j \right) \leq \left\| \Delta X_b \right\|^2_{L^2} E |G_n(b-1)| + \sum_{j=0}^{b-1} E |G_n(j)| \left( \left\| \Delta X_j \right\|^2_{L^2} - \left\| \Delta X_b \right\|^2_{L^2} \right)
\]
\[
\leq Cn^{-\frac{2}{3}} + Cn^{-\frac{2}{3}} \sum_{j=0}^{b-1} (j-1)^{-\nu}
\]
\[
\leq Cn^{-\frac{4}{3}}.
\]

5.3 Proof of Lemma 3.10

Proof of (24). Let \( a_j = \lfloor nt_{j-1} \rfloor \) and \( b_j = \lfloor nt_j \rfloor \). By Lemma 2.1.b,

\[
D^2 F_n^j = \sum_{k=a_j}^{b_j-1} D^2 \delta^3 \left( f^{(3)}(\tilde{X}_\alpha) \partial_\alpha^{\otimes 3} \right) + \sum_{k=a_j}^{b_j-1} \left\{ \delta^3 \left( f^{(5)}(\tilde{X}_\alpha) \partial_\alpha^{\otimes 3} \right) \tilde{e}_\alpha^{\otimes 2} + 6 \delta^2 \left( f^{(4)}(\tilde{X}_\alpha) \partial_\alpha^{\otimes 2} \right) \partial_\alpha \otimes \tilde{e}_\alpha + 6 \delta \left( f^{(3)}(\tilde{X}_\alpha) \partial_\alpha \right) \partial_\alpha^{\otimes 2} \right\}
\]
and
\[
DF^j_n = \sum_{m=a_k}^{b_k-1} \left\{ \delta^3 \left( f^{(4)}(X_m') \partial^4 \right) \frac{\partial}{X_m'} + 3 \delta^2 \left( f^{(3)}(X_m') \partial^3 \right) \frac{\partial}{X_m'} \right\}.
\]

(38)

With these two expansions, it follows that the expectation
\[
E \left[ \left( u^j_n, D^2 F^j_n \otimes D F^k_n \right)^2 \right]
\]
consists of terms of the form
\[
\sum_{p,p'=a_k}^{b_k-1} \sum_{q,q'=a_k}^{b_k-1} \sum_{m,m'=a_k}^{b_k-1} \sum_{n,n'=a_k}^{b_k-1} \sum_{p,p'=a_k}^{b_k-1} E \left[ G(p,p') \delta^r (g_1(X_m) \partial^r) (g_2(X_m') \partial^r) (g_4(X_m') \partial^r) \right] \delta^r (g_3(X_m') \partial^r)
\]
\[
\times \left( \langle \frac{\partial^m}{X_m} \partial^m \rangle \langle \frac{\partial^m}{X_m} \partial^m \rangle \right)^{r_1-1} \left( \langle \frac{\partial^m}{X_m} \partial^m \rangle \langle \frac{\partial^m}{X_m} \partial^m \rangle \right)^{r_2-1} \left( \langle \frac{\partial^m}{X_m} \partial^m \rangle \langle \frac{\partial^m}{X_m} \partial^m \rangle \right)^{r_3-1} \left( \langle \frac{\partial^m}{X_m} \partial^m \rangle \langle \frac{\partial^m}{X_m} \partial^m \rangle \right)^{r_4-1}
\]
\[
\leq C n^{-\Lambda^2}
\]

where the exponent \( \Lambda \) is determined by \( \{r_1, \ldots, r_4\} \) as follows: First, suppose \( r_1 = 3 \). Then by Lemma 3.2.a and 3.2.c,
\[
\sum_{p=0}^{\lfloor nt \rfloor-1} \left( \langle \frac{\partial^m}{X_m} \partial^m \rangle \langle \frac{\partial^m}{X_m} \partial^m \rangle \right)^{r_2-2} \left( \langle \frac{\partial^m}{X_m} \partial^m \rangle \langle \frac{\partial^m}{X_m} \partial^m \rangle \right)^{r_3-2} \left( \langle \frac{\partial^m}{X_m} \partial^m \rangle \langle \frac{\partial^m}{X_m} \partial^m \rangle \right)^{r_4-2}
\]
\[
\leq C n^{-\lambda (r_3-2) \lambda (3-r_3)}
\]

On the other hand, if \( r_1 = 1 \) or 2 then by Lemma 3.2.a and 3.2.d,
\[
\sum_{p=0}^{\lfloor nt \rfloor-1} \left( \langle \frac{\partial^m}{X_m} \partial^m \rangle \langle \frac{\partial^m}{X_m} \partial^m \rangle \right)^{r_2-2} \left( \langle \frac{\partial^m}{X_m} \partial^m \rangle \langle \frac{\partial^m}{X_m} \partial^m \rangle \right)^{r_3-2} \left( \langle \frac{\partial^m}{X_m} \partial^m \rangle \langle \frac{\partial^m}{X_m} \partial^m \rangle \right)^{r_4-2}
\]
\[
\leq C n^{-\lambda (r_3-2) \lambda (3-r_3)}
\]

\[
= C n^{-\lambda (r_3-2) \lambda (3-r_3)}
\]
Combining this with a similar computation for the sum over \( p' \), we obtain

\[
\Lambda = \lambda(R - 6) + \frac{1}{3} (12 - R) = 2 - \left( \frac{1}{3} - \lambda \right)(R - 6).
\]

In particular, \( \Lambda = 2 \) if \( R = 6 \) and \( \Lambda = \frac{5}{7} + \lambda \) for \( R = 7 \). It follows that we want to find bounds for terms of the form

\[
C^{n - \Lambda} \sup_{p, p'} \sum_{j_1, j_2, j_3, j_4} \left| \mathbb{E} \left[ G(p, p') \prod_{i=1}^{4} \delta^{r_i} \left( g_i(\hat{X}_{\tilde{\psi}}) \partial^{\otimes r_i} \right) \right] \right|.
\]

(40)

By repeated use of (6), we can expand each product of the form

\[
\prod_{i=1}^{4} \delta^{r_i} \left( g_i(\hat{X}_{\tilde{\psi}}) \partial^{\otimes r_i} \right)
\]

into a sum of terms of the form

\[
C_M \delta^M \left( \Psi_n \partial^{b_1} \otimes \partial^{b_2} \otimes \partial^{b_3} \otimes \partial^{b_4} \right) \langle \partial^{\alpha_1}, \partial^{\alpha_2}, \partial^{\alpha_3}, \partial^{\alpha_4} \rangle_{\delta^M}
\]

where \( C_M \) is a combinatorial constant from (6), \( \Psi_n = \prod_{i=1}^{4} g_i(\hat{X}_{\tilde{\psi}}) \); each \( \alpha_i \in \{0, 1, 2\} \), such that \( A := \sum_{i=1}^{6} \alpha_i \leq R/2 \); each nonnegative integer \( b_i \) satisfies \( b_i \leq r_i \); and the exponent \( M \) satisfies:

\[
M = b_1 + b_2 + b_3 + b_4 = R - 2A.
\]

With this representation, and using the Malliavin duality (5), we want to bound terms of the form

\[
C^{n - \Lambda} \sup_{p, p'} \sum_{j_1, j_2, j_3, j_4} \left| \mathbb{E} \left[ \left( D^M G(p, p'), \Psi_n \partial^{b_1} \otimes \cdots \otimes \partial^{b_4} \right) \right] \langle \partial^{\alpha_1}, \partial^{\alpha_2}, \partial^{\alpha_3}, \partial^{\alpha_4} \rangle_{\delta^M} \right|.
\]

(41)

Consider first the case \( A = 0 \). Then \( M = R \geq 6 \), and each \( b_i = r_i \geq 1 \). Hence

\[
C^{n - \Lambda} \sup_{p, p'} \sum_{j_1, j_2, j_3, j_4} \left| \mathbb{E} \left[ \left( D^M G(p, p'), \Psi_n \partial^{b_1} \otimes \cdots \otimes \partial^{b_4} \right) \right] \langle \partial^{\alpha_1}, \partial^{\alpha_2}, \partial^{\alpha_3}, \partial^{\alpha_4} \rangle_{\delta^M} \right|
\]

\[
\leq C^{n - \Lambda} \sup_{p} \left| \langle \hat{\xi}_{\tilde{\psi}}, \partial^{\frac{4}{n}} \rangle_{\delta^M} \right|^4 \left( \prod_{j=0}^{R-4} \left| \langle \hat{\xi}_{\tilde{\psi}}, \partial^{\frac{4}{n}} \rangle_{\delta^M} \right| \right)
\]

By Lemma 3.2.a and 3.2.c, this is bounded by \( C n^{1-2\Lambda} \), since \( \Lambda \geq 1 \) for all \( R \) and \( R \geq 6 \).

If \( A \geq 1 \), by permutation of indices we may assume that \( \alpha_1 \geq 1 \), so (41) may be bounded using Lemma 3.2.c and 3.2.d:

\[
C^{n - \Lambda} \sup_{p, p'} \sum_{j_1, j_2, j_3, j_4} \left| \mathbb{E} \left[ \left( D^M G(p, p'), \Psi_n \partial^{b_1} \otimes \cdots \otimes \partial^{b_4} \right) \right] \langle \partial^{\alpha_1}, \partial^{\alpha_2}, \partial^{\alpha_3}, \partial^{\alpha_4} \rangle_{\delta^M} \right|
\]

\[
\leq C |n|^{3} n^{-\Theta},
\]
where
\[ \Theta = 4 + (R - 6 + M)\lambda - \frac{R - A}{3} = 4 + (R - A)(2\lambda - \frac{1}{3}) - 6\lambda. \]

Since \( A \leq R/2, R \geq 6, \) and \( \lambda > 1/6, \) we have \( \Theta > 3 \) for all cases except when \( R = 6, A = 3. \) This case has the form,
\[ C_n^{-2} \sup_{j_1,j_2,j_3,j_4} \left| \mathbb{E} \left[ G(p,p') \Psi_n \left( \partial_{\mu_1}^{\alpha_1} \partial_{\mu_2}^{\alpha_2} \cdots \partial_{\mu_6}^{\alpha_6} \right) \right] \right| \]
\[ \leq C_n^{-2} \sup_{j_1,j_2,j_3,j_4} \left| \partial_{\mu_1}^{\alpha_1} \partial_{\mu_2}^{\alpha_2} \cdots \partial_{\mu_6}^{\alpha_6} \right| \sup_{j_3,j_4} \left| \partial_{\mu_3}^{\alpha_3} \partial_{\mu_4}^{\alpha_4} \right| \sup_{j_5,j_6} \left| \partial_{\mu_5}^{\alpha_5} \partial_{\mu_6}^{\alpha_6} \right| \]
\[ \leq C |nt| n^{-2} \left[ \left( \sup_k \sum_{j=0}^{n|t|-1} \left| \partial_{\mu_1}^{\alpha_1} \partial_{\mu_2}^{\alpha_2} \right| \right)^2 + \sup_{j,k} \left| \partial_{\mu_3}^{\alpha_3} \partial_{\mu_4}^{\alpha_4} \right| \right] \]
\[ \leq C |nt|^2 n^{-3}, \]
by Lemma 3.2.d.

Proof of (25). For this term, we see that
\[ \mathbb{E} \left[ \left( \hat{u}_n^{(i)} \right)^2 \right] \]
consists of terms with the form
\[ \sum_{p,p'=a_i} \sum_{j_1,j_2=a_i} \sum_{j_3,j_4=a_k} \sum_{j_5,j_6=a_k} \mathbb{E} \left[ G(p,p') \delta^{r_1} \left( g_1(\hat{X}_{\mu_1}) \partial_{\mu_1}^{\otimes r_1} \right) \cdots \delta^{r_6} \left( g_6(\hat{X}_{\mu_6}) \partial_{\mu_6}^{\otimes r_6} \right) \right] \]
\[ \times \left( \hat{\varepsilon}_{\mu_1} \partial_{\mu_1}^{r_1} \right) \left( \hat{\varepsilon}_{\mu_2} \partial_{\mu_2}^{r_2} \right) \cdots \left( \hat{\varepsilon}_{\mu_6} \partial_{\mu_6}^{r_6} \right) \]
where each \( r_i \in \{2,3\} \) and \( G(p,p'), g_i(x) \) are as defined above. As with (24) above, we assume that all components are defined over the time interval \([0,t]\) for some \( t \leq T. \) As above, let \( R = \sum_{i=1}^{n|t|} r_i, \) and note that for this case \( 12 \leq R \leq 18. \) Similar to the above case, we obtain
\[ \sum_{p,p'} \left| \left( \hat{\varepsilon}_{\mu_1} \partial_{\mu_1}^{r_1} \right) \left( \hat{\varepsilon}_{\mu_2} \partial_{\mu_2}^{r_2} \right) \cdots \left( \hat{\varepsilon}_{\mu_6} \partial_{\mu_6}^{r_6} \right) \right| \leq C n^{-\Lambda}, \]
where \( \Lambda = 2 - (\frac{1}{2} - \lambda)(R - 12). \) It follows that, similar to (10), we want to obtain bounds for terms of the form
\[ C_n^{-\Lambda} \sup_{p,p'} \left| \mathbb{E} \left[ G(p,p') \prod_{i=1}^{n|t|} \delta^{r_i} \left( g_i(\hat{X}_{\mu_1}) \partial_{\mu_1}^{\otimes r_1} \right) \right] \right|. \]
Using (8) and the Malliavin duality as before, we obtain terms of the form
\[ C_n^{-\Lambda} \sup_{p,p'} \left| \mathbb{E} \left[ \left( \partial_{\mu_1}^{\alpha_1} \partial_{\mu_2}^{\alpha_2} \cdots \partial_{\mu_6}^{\alpha_6} \right) \right] \right| \]
(42)
where \( \hat{\Psi}_n = \prod_{i=1}^{n|t|} g_i(\hat{X}_{\mu_1}), \) each \( \alpha_i \) and each \( b_i \) take values from \( \{0,1,2,3\}; \) and the product includes all 15 possible pairs from the set \( \{j_1, \ldots, j_6\} \) such that \( A := \sum_{i=1}^{n|t|} \alpha_i \leq R/2. \) As in the above case, for each \( R \) we have \( M \) and \( A \) satisfying \( M = \sum_{i=1}^{n|t|} b_i \) and \( M = R - 2A. \)
In the product
\[ \left\langle D^M G(p, p), \delta^M \left( \bar{\psi}_n \partial^{\oplus b_1} \otimes \cdots \otimes \partial^{\oplus b_6} \right) \right\rangle_{\mathcal{F}^{\oplus M}} \prod_{\{i_1, \ldots, i_6\}} \left\langle \partial^{\oplus b_6} \right\rangle_{\mathcal{F}_{i_j}} \]
(43)
each of the indices \( \{j_1, \ldots, j_6\} \) must appear at least once. Note that by Lemma 3.2.a we have (possibly up to a fixed constant)
\[ \sup_{0 \leq j, k \leq \lfloor nt \rfloor} \left| \left\langle \partial^{\oplus b_6} \right\rangle_{\mathcal{F}_{i_6}} \right| \leq \sup_{0 \leq j, p \leq \lfloor nt \rfloor} \left| \left\langle \hat{\epsilon}_p^{\oplus b_6} \right\rangle_{\mathcal{F}_{i_6}} \right|, \]
and by Lemma 3.2.c and 3.2.d we have
\[ \sup_{0 \leq k \leq \lfloor nt \rfloor} \sum_{j=0}^{\lfloor nt \rfloor - 1} \left| \left\langle \partial^{\oplus b_6} \right\rangle_{\mathcal{F}_{i_6}} \right| \leq \sup_{p, j, k} \left| \left\langle \hat{\epsilon}_p^{\oplus b_6} \right\rangle_{\mathcal{F}_{i_6}} \right|, \]
Hence, we may conclude that (43) contains terms less than or equal to
\[ \left\langle \hat{\epsilon}_p^{\oplus b_6} \right\rangle_{\mathcal{F}_{i_6}} \]
and, by Lemma 3.2.c, (42) is bounded in absolute value by
\[ C n^{-3} \sum_{j_4, j_5, j_6 = 0}^{\lfloor nt \rfloor - 1} \left| \left\langle \partial^{\oplus b_6} \right\rangle_{\mathcal{F}_{i_6}} \right|, \]
where, using the fact that \( R = M + 2A, \)
\[ \Theta' = 2 - \left( \frac{1}{3} - \lambda \right)(R - 12) + (M - 3)\lambda + \frac{A}{3} = 6 + (R - A)(2\lambda - \frac{1}{3}) - 15\lambda, \]
Observe that \( \Theta' > 3 \) whenever \( R - A > 6. \) The case \( R - A = 6 \) occurs only when \( R = 12, A = 6, \) and \( M = 0; \) so in this case we have an upper bound of
\[ C n^{-2} \sum_{j_4, j_5, j_6 = 0}^{\lfloor nt \rfloor - 1} \left| \left\langle \partial^{\oplus b_6} \right\rangle_{\mathcal{F}_{i_6}} \right|^3 \left( \sup_{p, j, k} \sum_{j=0}^{\lfloor nt \rfloor - 1} \left| \left\langle \partial^{\oplus b_6} \right\rangle_{\mathcal{F}_{i_6}} \right| \right)^3 \]
\[ \leq C n^{-2} \Theta'^{\frac{6}{5}} \leq C n^{-1}. \]

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References
[1] X. Bardina and K. Es-Sebaiy (2011), “An extension of bifractional Brownian motion”, Comm. on Stochastic Analysis, 5(2): 333-340.
[2] P. Billingsley, Convergence of Probability Measures, 2nd Ed., Wiley, 1999.
[3] T. Bojdecki, L. Gorostiza and A. Talarczyk (2004), “Sub-fractional Brownian motion and its relation to occupation times”, Statist. Probab. Letters, 69: 405-419.
[4] J. Ruiz de Chavez and C. Tudor (2009), “A decomposition of sub-fractional Brownian motion”, *Math. Reports*, 11(61): 67-74.

[5] M. Gradinaru, I. Nourdin, F. Russo and P. Vallois (2005), “m-order integrals and generalized Itô’s formula: the case of a fractional Brownian motion with any Hurst index,” *Ann. Inst. Henri Poincaré Probab. Stat.*, 41(4):781-806.

[6] D. Harnett and D. Nualart (2011), “Central limit theorem for a Stratonovich integral with Malliavin calculus,” *Preprint* (arXiv: 1105.4841).

[7] C. Houdré and J. Villa (2003), “An example of infinite dimensional quasi-helix”, *Contemp. Math.* 336: 3-39.

[8] P. Lei and D. Nualart (2009), “A decomposition of the bifractional Brownian motion and some applications”, *Statist. Probab. Letters*, 79: 619-624.

[9] I. Nourdin and D. Nualart (2010), “Central limit theorems for multiple Skorokhod integrals”, *J. Theor. Probab.*, 23: 39-64.

[10] I. Nourdin, A. Réveillac and J. Swanson (2010), “The weak Stratonovich integral with respect to fractional Brownian motion with Hurst parameter 1/6,” *Electron. J. Probab.*, 15: 2087-2116.

[11] D. Nualart, *The Malliavin Calculus and Related Topics*, 2nd Ed., Springer, 2006.

[12] D. Nualart and S. Ortiz-Latorre (2008), “Central limit theorems for multiple stochastic integrals and Malliavin calculus,” *Stochastic Processes and Appl.*, 118: 614-628.

[13] F. Russo and C.A. Tudor (2006), “On bifractional Brownian motion,” *Stochastic Processes and Appl.*, 116 (5): 830-856.