Solutions to WDVV from generalized Drinfeld-Sokolov hierarchies

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Abstract

The dispersionless limit of generalized Drinfeld-Sokolov hierarchies associated to primitive regular conjugacy class of Weyl group $W(g)$ is discussed. The map from these generalized Drinfeld-Sokolov hierarchies to algebraic solutions to WDVV equations has been constructed. Example of $g = D_4$ and $[w] = D_4(a_1)$ is considered in details and corresponding Frobenius structure is found.

1 Introduction and discussion

Equations of Witten, Dijkgraaf, Verlinde and Verlinde [WDVV]

$$c_{ijk}(t)\eta^{kl}c_{lmi}(t) = c_{njk}(t)\eta^{kl}c_{lmi}(t), \quad \text{where} \quad c_{ijk}(t) = \partial_i\partial_j\partial_k F(t)$$

$$\eta^{-1}_{mn} = \partial_m\partial_n F(t) - \text{constant}, \quad \sum_{i=1}^{n} d_i t_i F(t) = (3 - d) F(t)$$

describe deformations of two dimensional topological conformal quantum field theory. They have appeared to be intimately related to integrable systems. This was recognized by B. Dubrovin [Du1], who showed that each solution to WDVV equations gives rise to dispersionless bi Hamiltonian integrable system, that is with Poisson structures of hydrodynamic type [DN] and Hamiltonians not depending on field derivatives. This result was extended by one loop correction [DuZ], where it was shown that so constructed bi Hamiltonian structure is always a $W$ algebra, i.e. it contains a Virasoro subalgebra.

On the other hand, due to B. Dubrovin [Du4], it is possible, under certain assumption, to recover solution to WDVV equation from bi Hamiltonian structure of hydrodynamic type

$$\{u^i(x), u^j(y)\} = g^{ij}(u(x)) \delta'(x - y) + \Gamma^{ij}_k(u(x)) u'_k(x) \delta(x - y).$$

If matrix $g$ does not degenerate identically, it defines flat metrics [DN] on the target space.

Whitham averaging method [DN] provides Poisson structures of this type. This method allows to describe slow modulated $m$-phase solutions of non linear Hamiltonian system of equations. If one needs local Poisson structures of hydrodynamic type we should apply zero phase averaging being simply dispersionless limit.

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The dispersionless limit of Drinfeld-Sokolov hierarchies [DS], associated to non-twisted affine Lie algebra \( \hat{g} \) was shown [Kr] to provide polynomial solutions to WDVV equations (1.1). These solutions, viewed as Frobenius structures on orbits of Coxeter groups \( G \) [Du2], correspond to Weyl group \( G = W(g) \) of underlying simple Lie algebra \( g \). They correspond [Du5] to the orbit of braid group on Stocks matrices \( S \) passing through “Coxeter” point, i.e. \( S + S' \) defines a Carter graph [C] of Coxeter conjugacy class in \( W(g) \). Orbits given by primitive conjugacy classes were conjectured [Du5] to correspond to algebraic solutions of WDVV equations (1.1).

In this article we address the dispersionless limit of generalized Drinfeld-Sokolov hierarchies [GHM, BGHM], associated with (i) non-twisted affine Lie algebra \( \hat{g} \), (ii) its Heisenberg subalgebra \( H \), corresponding [KP] to regular primitive conjugacy class \([w]\) in \( W(g) \), and (iii) regular element \( \Lambda \) from it. The naive limit is shown to exist only for standard Drinfeld-Sokolov hierarchies, due to failure to parameterize the whole phase space by densities \( N_i \) of annihilators of the first Poisson structure. As a consequence, some fields generically evolve fast. We combine Hamiltonian reduction with Whitham averaging [M], restricting Poisson structures to the subvariety parameterized by annihilators’ densities and then taking the limit \( \epsilon \to 0 \). So constructed dispersionless limit is proved to give rise to algebraic solution to WDVV equations (1.1). Monodromy group of the Frobenius manifold coincides with \( W(g) \).

The paper is organized as follows. In section 2 we gather, to make the paper self-content, some preliminary information about gradations of Kac-Moody algebra and about its Heisenberg subalgebras. Section 3 reviews basic facts about generalized Drinfeld-Sokolov hierarchies. The properties of regular primitive conjugacy classes, relevant for the paper, are recollected in section 4. Section 5 brings together facts about type I integrable hierarchies with primitive regular conjugacy class \([w]\) and regular element \( \Lambda \). Quite rich underlying finite dimensional geometry is analysed in section 6. Finite group \( R \), acting non-linearly on Miura variables, and a ring of \( R \)-invariant polynomials are studied. The dispersionless limit of the hierarchies in consideration is examined in section 7. The non degeneracy of obtained metrics is proved, their flat coordinates are found and it is shown that Frobenius structure can be always extracted from this averaged Poisson structure. Finally, section 8 treats the simplest integrable hierarchy with non-Coxeter regular primitive conjugacy class, i.e. \( g = D_4 \) and \([w] = [D_4(a_1)]\). Corresponding solution to WDVV equations is found and finite group \( R \) is analysed.

2 Preliminaries

Let \( g \) be a simple Lie algebra of rank \( r \), and \( \hat{g} = g \otimes \mathbb{C} [z, z^{-1}] \oplus \mathbb{C} d \) — its affine Lie algebra. This algebra is well-known to be graded, with gradations being in one-to-one correspondence [K] with finite order inner automorphisms of \( g \).

**Definition 2.1.** Let \( s = (s_0, s_1, \ldots, s_r) \) — a sequence of non negative relatively prime integers. Let \( N_s = \sum_{i=1}^{r} k_i s_i \), where \( k_i \) — are Kac labels (such that \( \alpha_{\text{max}} = \sum_{i=1}^{r} k_i \alpha_i \) and \( k_0 = 1 \)). We then define the finite order automorphisms \( \sigma \) of \( g \) in some Cartan-Weyl basis by

\[
\sigma (H) = H, \quad \sigma (E_\alpha) = e^{2\pi i s_\alpha} E_\alpha ,
\]

with

\[
\delta_s = \frac{1}{N_s} \sum_{k=1}^{r} \frac{2}{\alpha_i^\vee \cdot \omega_k} s_k \omega_k ; \quad \alpha_i^\vee \cdot \omega_j = \delta_{ij} .
\]
Proposition 2.1 ([K]). Such automorphisms exhaust the finite order inner automorphisms of \( \mathfrak{g} \) up to conjugacy.

An inner automorphisms of \( \mathfrak{g} \) can be used to define a new \( \mathbb{Z} \)-gradation of \( \hat{\mathfrak{g}} \).

Definition 2.2. A gradation of \( \mathfrak{s} \) is defined via derivation \( d_{s} \)

\[
d_{s} = N_{s} \left( \frac{d}{dz} + \text{ad} \, H_{s} \right),
\]

where \( H_{s} \in \mathfrak{h} \) such that \( [H_{s}, E_{\alpha}] = (\alpha_{k} \cdot \delta_{s}) \, E_{\alpha_{k}} = s_{k} E_{\alpha_{k}} / N_{s} \).

Let us denote \( \hat{\mathfrak{g}}(s) \) the eigenspace of derivation \( d_{s} \) with an eigenvalue \( k \):

\[
\hat{\mathfrak{g}} = \bigoplus_{k \in \mathbb{Z}} \hat{\mathfrak{g}}(s_{w}) = k \hat{\mathfrak{g}}(s_{w}).
\]

Homogeneous gradation corresponds then to \( s_{h} = (1, 0, \ldots, 0) \), while principal to \( s_{p} = (1, 1, \ldots, 1) \). One can introduce a partial ordering on the set of gradation.

Definition 2.3. We say that \( s \succ s' \) if \( s_{i} \neq 0 \) whenever \( s'_{i} \neq 0 \).

Proposition 2.2 ([GHM]). If \( s \succ s' \) then the following is true

(i) \( \mathfrak{g}_{0}(s) \subseteq \mathfrak{g}_{0}(s') \),

(ii) \( \mathfrak{g}_{\geq 0}(s) \subseteq \mathfrak{g}_{\geq 0}(s') \) and \( \mathfrak{g}_{< 0}(s) \subseteq \mathfrak{g}_{< 0}(s') \),

(iii) \( \mathfrak{g}_{\geq 0}(s') \subseteq \mathfrak{g}_{\geq 0}(s) \) and \( \mathfrak{g}_{< 0}(s') \subseteq \mathfrak{g}_{< 0}(s) \).

Theorem 2.1 ([K]). Given a Kac-Moody algebra \( \hat{\mathfrak{g}} \), its maximal nilpotent subalgebras are called Heisenberg subalgebras. Heisenberg subalgebras are in one-to-one correspondence with conjugacy classes of Weyl group \( W(\mathfrak{g}) \) of \( \mathfrak{g} \).

Hence, they will be denoted by \( \mathcal{H}_{[w]} \). Let us recall the essentials of their construction.

Let \( w \) be a representative of a given conjugacy class \([w]\) acting naturally on \( \mathfrak{h} \) in some Cartan-Weyl basis marked by a prime to distinguish it from another basis (2.1) in use. Its action on Cartan subalgebra \( \mathfrak{h} \) is known to be inner in \( \mathfrak{g} \). Then, let \( Y \in \mathfrak{g} \) be such an element that \( w = \exp(2\pi i \text{ad} \, Y)_{\mathfrak{h}} \). Let \( \hat{w} \) be an obvious extension of \( w \) on the whole algebra \( \mathfrak{g} \):

\[
\hat{w} \, H'_{\alpha} = H'_{w(\alpha)}, \quad \hat{w} \, E'_{\alpha} = \psi_{\alpha} E'_{w(\alpha)},
\]

where \( \psi_{\alpha} \) are structure constant compatible factors taking values in \( S^{1} \) and may be chosen to be real. It can be shown ([K]) that the order of this extension \( N_{\mathfrak{w}} \) may only be either the same as or twice as the order of \( w \). The following construction does not, however, depend on this ambiguity.

Let \( a \in \mathfrak{g} \) be a representative of some orbit of \( \hat{w} \). We write \( a = \sum_{\xi} a_{\xi} \) as a sum of \( \hat{w} \) eigenvectors \( a_{\xi} \) with distinct eigenvalues \( \xi = \exp \left[ -2\pi i \frac{\kappa}{N_{\mathfrak{w}}} \right] \) and put \( \hat{a}(k) \in \hat{\mathfrak{g}} \) for any \( k \in \mathbb{Z} \) to be

\[
\hat{a}(k) = \exp \left[ i \varphi \frac{k}{N_{\mathfrak{w}}} \right] \exp \left[ -i \varphi \text{ad} \, Y \right] a_{\xi} + \delta_{k,0} \left( a|Y \right) c,
\]

\( \varphi \) being any real number.\( \square \)
where $c$ is the central element of Kac-Moody algebra. It is obvious that $\tilde{a}(k)$ is well-defined on $S^1$ and hence depends on $\varphi$ through $z = \exp(\im \varphi)$ only. It verifies commutation relations of Kac-Moody algebra:

$$\left[ \tilde{a}(k), \tilde{b}(l) \right] = [a, b] (k + l) + k\delta_{k,-l} (a|b) c .$$

The image of $\mathfrak{h}$ under this map is a subalgebra of $\hat{\mathfrak{g}}$ isomorphic to an infinite dimensional Heisenberg algebra:

$$\left[ \tilde{h}_1(k), \tilde{h}_2(l) \right] = k\delta_{k,-l} (h_1|h_2) c .$$

Factorizing out the center $z = Cc$ of Kac-Moody algebra $\hat{\mathfrak{g}}$, the Heisenberg subalgebras become maximal commutative ones. A practical realization of this construction faces the difficulty to find $Y$ explicitly.

Associated to each Heisenberg subalgebra $H^\mathfrak{g}[w]$ there is a distinguished gradation $s_w$ with the property that $H^\mathfrak{g}[w]$ admits $s_w$ grading $\mathfrak{g}$:

$$H^\mathfrak{g}[w] = \bigoplus_{k \in E} H^\mathfrak{g}[w]_k (s_w) ,$$

with

$$E = \{ k + N \mathbb{Z} | k \in I(w) \} ,$$

$$I(w) = \{ k_n \in \mathbb{Z}_+, n = 1, \ldots, r | 0 \leq k_1 \leq k_2 \leq \cdots \leq k_r < N \hat{\mathfrak{g}} , e^{2\pi ik_n/N \hat{\mathfrak{g}}} \text{ - eigenvalue of } w \} .$$

It is clear, that $s_w$ is the gradation corresponding to automorphism $w$ used to construct $H^\mathfrak{g}[w]$. Indeed, let us choose $Y \in \mathfrak{g}$ so that spectrum of $N \hat{\mathfrak{g}} \text{ ad } Y$ on $\mathfrak{h}$ coincides with $I(w)$. Then

$$d_{s} = N \hat{\mathfrak{g}} \left( z \frac{d}{dz} + \text{ad } Y \right) , \quad d_{s} \left( \tilde{h}(k) \right) = k \tilde{h}(k) .$$

**Definition 2.4.** An element $\Lambda \in \hat{\mathfrak{g}}$ is called semisimple if $\hat{\mathfrak{g}} = \text{Ker } (\text{ad } \Lambda) \oplus \text{Im } (\text{ad } \Lambda)$ and $\text{Ker } (\text{ad } \Lambda) \cap \text{Im } (\text{ad } \Lambda) = \emptyset$.

**Definition 2.5.** A semisimple element $\Lambda \in H^\mathfrak{g}[w]$ is called regular if $\text{Ker } (\text{ad } \Lambda) = H^\mathfrak{g}[w]$.

### 3 Generalized integrable hierarchies

Let $[w]$ be some conjugacy class in $\mathcal{W}(\mathfrak{g})$, $s_w$ – corresponding gradation. Pick up any $s \succ s_w$ and some constant semisimple element $\Lambda \in H^\mathfrak{g}[w]$ of certain $s_w$ grade $i \in E_+$. Let us define the matrix Lax operator $\mathcal{L}$:

$$\mathcal{L} = \partial_x + \Lambda + q(x) ,$$

where $q(x)$ takes value in

$$Q(i) = \hat{\mathfrak{g}}_{\geq 0} (s) \cap \hat{\mathfrak{g}}_{< i} (s_w) .$$
Proposition 3.1 ([GHM, DS, W]). There exist a unique formal series
\[ T(q) \in \text{Im}(\text{ad} \Lambda) \cap \hat{g} < 0(s_w) \]
that
\[ L = \exp[-\text{ad}T] L = \partial_x + \Lambda + h(q), \quad h(q) = \sum_{k<i} h_k(q) \quad (3.3) \]
where \( h(q) \in \mathcal{H}^{[w]}(s_w) \). Both \( P_k^{(s_w)} T(q) \) and \( P_k^{(s_w)} h(q) \) are polynomials in \( q(x) \) and its \( x \)-derivatives for any \( k \in \mathbb{Z} \).

There is a gauge symmetry acting on (3.1) and mapping \( Q(i) \) to \( Q(i) \):
\[ L \mapsto \exp[-\text{ad}n(x)] L, \quad (3.4) \]
where \( n(x) \) takes values in
\[ P = \hat{g}_0(s) \cap \hat{g} < 0(s_w). \quad (3.5) \]

Proposition 3.2 ([DS, FGMS]). If \( \Lambda \) is chosen so that
\[ \text{Ker}(\text{ad} \Lambda) \cap P = \emptyset \quad \Rightarrow \quad \text{ad} \Lambda : P \mapsto Q, \quad (3.6) \]
then elements of \( Q^{\text{can}}(i)^* \), where
\[ Q(i) = Q^{\text{can}}(i) \oplus [\Lambda, P], \quad (3.7) \]
are generators of the ring of gauge invariant polynomials in \( q(x) \) and its derivatives; (3.7) holds as equality of vector spaces. Lax operator \( L \) in (3.3) is gauge invariant.

Definition 3.1. Denote by \( \mathcal{M} \) the phase space, spanned by the elements of \( Q^{\text{can}}(i)^* \) and by \( \mathcal{F}(\mathcal{M}) \) the space of functionals of the type
\[ \varphi[q] = \int_{S^1} dx f(x, q, q', \ldots). \]

Proposition 3.3 ([GHM]). Dimension of phase space \( \mathcal{M} \) is independent on auxiliary gradation \( s \) and equals to
\[ \dim \mathcal{M} = \dim Q^{\text{can}}(i)^* = \sum_{k=0}^{i-1} \dim \hat{g}_k(s_w) \quad (3.8) \]

Remark 3.1. \( \dim \mathcal{M} \geq \dim \hat{g}_0(s_w) = \dim \hat{g}_0(s_p) = r \).

Condition (3.6) is automatically satisfied [FGMS] if \( \Lambda \) is chosen to be regular.

Theorem 3.1 ([GHM]). Given \( b \in \mathcal{H}^{[w]}(s_w) \), let \( \mathcal{A}(b) = \exp[\text{ad}T] b \). One defines two sets of time flows
\[ \frac{\partial}{\partial t_b} \mathcal{L} = \left[ P_{\geq 0}^{(s)} \mathcal{A}(b), \mathcal{L} \right], \quad \frac{\partial}{\partial t'_b} \mathcal{L} = \left[ P_{\geq 0}^{(s)} \mathcal{A}(b), \mathcal{L} \right], \quad (3.9) \]
being commutative within each set
\[ \forall b_1, b_2 \in H \left[ w \right] \quad \left[ \frac{\partial}{\partial t_{b_1}}, \frac{\partial}{\partial t_{b_2}} \right] = 0, \quad \left[ \frac{\partial}{\partial t'_{b_1}}, \frac{\partial}{\partial t'_{b_2}} \right] = 0. \]

Both time flows (3.9) preserve the phase space of gauge invariants \( M \). They coincide there and retain their commutativity property.

**Definition 3.2.** Introduce the pairing on the space of functions in \( C^\infty \left( S^1, \hat{g} \right) \)
\[
\langle A, B \rangle = \int_{S^1} dx \sum_{k \in \mathbb{Z}} \eta \left( P_k^A(x), P_k^B(x) \right) \hat{g}_0(s) \tag{3.10}
\]
for some gradation \( s \) and Killing form \( \eta \) on \( \hat{g}_0(s) \).

If \( \eta_{\hat{g}_0(s)} \) is properly normalized, the pairing does not depend on the gradation \( s \) chosen. We will assume this normalization chosen further on.

**Proposition 3.4 ([DS, GHM]).** Gauge invariant functionals \( H_b[q] = \langle b, h(q) \rangle \) are integrals of flows (3.9).

**Remark 3.2.** \( H_b \equiv 0 \) if \( \deg_{s_w}(b) \leq -i \).

**Definition 3.3.** For any \( \varphi \in M \) define gradient \( d_q \varphi \in \hat{g}_{\leq 0}(s) / \hat{g}_{< -i}(s_w) \) by
\[
\frac{d}{d\varepsilon} \varphi[q + \varepsilon r] \bigg|_{\varepsilon = 0} = \langle r, d_q \varphi \rangle \quad \forall r \in C^\infty \left( S^1, Q(i) \right)
\]

**Theorem 3.2 ([BGHM]).** Let \( s = s_w \), then
(i) there is a one parameter family of Hamiltonian structures on the gauge equivalence classes of the generalized Drinfeld-Sokolov hierarchy given by
\[
\{ \varphi, \psi \}_\lambda = \left\{ \lambda + q, [d_q \varphi, d_q \psi]_{R(s)} \right\} - \langle d_q \varphi, (d_q \psi)' \rangle \tag{3.11}
\]
where \( R(s) = \left( P_0^s - P_{<0}^s \right) / 2 - \lambda / z \). Expanding in powers of \( \lambda \), \( \{ \cdot, \cdot \}_\lambda = \{ \cdot, \cdot \}_2 + \lambda \{ \cdot, \cdot \}_1 \), we obtain two coordinated Hamiltonian structures on \( M \):
\[
\{ \varphi, \psi \}_1 = -\langle d_q \varphi, z^{-1} [d_q \psi, L] \rangle,
\]
\[
\{ \varphi, \psi \}_2 = \left\{ \lambda + q, [d_q \varphi, d_q \psi]_{R} \right\} - \langle d_q \varphi, (d_q \psi)' \rangle,
\]
where \( R = \left( P_0^s - P_{<0}^s \right) / 2 \). Under the time evolution in the coordinate \( t_b \), the following recursion relation holds:
\[
\frac{\partial \varphi}{\partial t_b} = \{ \varphi, H_{zb} \}_1 = \{ \varphi, H_b \}_2. \tag{3.12}
\]

(ii) Hamiltonians \( H_b \) with \( -i < \deg_{s_w} b < 0 \) are Casimirs of (3.11). Hamiltonians \( H_b \) with \( \deg_{s_w} b = 0 \) are Casimirs of the first bracket only.

Hierarchies (3.12) with \( \Lambda \) regular were dubbed in ref. [GHM] the hierarchies of type I.
Remark 3.3. The hierarchies with grade one element $\Lambda$ are distinguished by the fact that pencil of Poisson structures has no local annihilators. This is true also for the first structure as long as conjugacy class $[w]$ is chosen to be non degenerate, that is fixing no vector in $\mathfrak{h}$ and consequently $0 \notin I(w)$.

Corollary 3.1. Hamiltonians $H_b$, with $b \in \mathcal{H}^{[w]}$ of positive $s_w$ degree belonging to $I([w])$, annihilate the first Poisson structure. 

There are $r$ independent annihilators of $\{\cdot, \cdot\}_1$, that yet generate nontrivial flows with respect to $\{\cdot, \cdot\}_2$.

Proof. Choose $b$ such that $0 \leq \deg s_w (zb) < N_w$. Then $- N_w \leq \deg s_w (b) < 0$ and $H_b$ annihilates $\{\cdot, \cdot\}_2$ or vanishes identically.

To prove independence, it is enough to notice that equation (3.3) determining $h(q)$, projected on eigenspaces of $d_{s_w}$ reads

$$h_k + [\Lambda, T_{k+1}] = q_k + \text{Polynomial} (h_{i<k}, T_{j<k}, q_{i<k}).$$

Thus, $h_k$ with $k \in I(w)$ are linear in Drinfeld-Sokolov variable of scaling degree $k+1$ with successive terms involving variables of lower scaling weights and this proves independence.

Notice, that $s = s_h$ corresponds to the largest gauge group, while $s = s_w$ – to no gauge freedom at all. The former choice leads to generalized Drinfeld-Sokolov hierarchies, while the latter – to modified generalized Drinfeld-Sokolov hierarchies.

Theorem 3.3 (BGHM). Let $\mathcal{M}_m$ be the phase space of modified hierarchy, and $\{\cdot, \cdot\}_m$ – its second Poisson bracket:

$$\{\varphi, \psi\}_m = \langle q_m + \Lambda, [d_{q_m} \varphi, d_{q_m} \psi]_{R_m} \rangle - \langle d_{q_m} \varphi, (d_{q_m} \psi)' \rangle,$$ (3.13)

The Miura mapping is a (non invertible) Hamiltonian mapping,

$$\mu: (\mathcal{M}_m, \{\cdot, \cdot\}_m) \to (\mathcal{M}, \{\cdot, \cdot\}_2),$$

such that it defines a reduction of the dynamical equations of the hierarchy to those of the modified hierarchy.

Drinfeld-Sokolov hierarchies [DS] are recovered picking up Coxeter conjugacy class $[w_c]$ with representative $w_c = \prod_{k=1}^r r_{\alpha_k}$, associated principal gradation $(s_w = s_p)$ and regular element $\Lambda = z E_{-\max} + \sum_{k=1}^r E_{\alpha_k}$ of grade one.

4 Regular primitive conjugacy classes of $\mathcal{W}(\mathfrak{g})$

Conjugacy classes in Weyl group $\mathcal{W}(\mathfrak{g})$ of simple Lie algebra $\mathfrak{g}$ were uniformly described by R. Carter [C] in terms of Carter graphs.

Definition 4.1. Conjugacy class $[w] \subset \mathcal{W}(\mathfrak{g})$ is called non degenerate if $\det(1-w) \neq 0$ and is called primitive if $\det(1-w) = \det(1-w_c) = \det(K)$, $K$ – being the Cartan matrix.

In [BGHM] it was proved for partially modified hierarchies, that is for any $s$ such that $s_h \succeq s \succ s_w$.  

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Primitive conjugacy classes are distinguished as they have no representative in any proper Weyl subgroup $W' \subset W(g)$. All primitive conjugacy classes were listed by R. Carter [3] together with their characteristic polynomials $\det(t - w)$.

**Definition 4.2 ([DF])**. Conjugacy class $[w] \subset W(g)$ is called regular if associated Heisenberg subalgebra $H^w$ admits regular element $\Lambda$.

Regular conjugacy classes were elegantly studied by T. Springer in ref. [Sp], though another definition of regularity was used.

**Definition 4.3 ([Sp])**. If $G$ is a finite reflection group in a finite dimensional vector space $V$, then $v \in V$ is called regular if no nonidentity element of $G$ fixes $v$.

An element $g \in G$ is regular if it has a regular eigenvector.

Two definitions were shown [DF] (see also sec. 9 of [Sp]) to be equivalent. The authors of the reference [DF] studied generalized hierarchies associated with classical Lie algebras $g$ and regular conjugacy classes. They synthesized available information on regular primitive conjugacy classes.

Theorem 4.1 ([Sp]). Let $[w] \in W(g)$ be a regular conjugacy class of order $N$, i.e. $\forall w \in [w], w^N = 1$. Then eigenvalues of $w$ are $\exp(2\pi i k/N)$, where $k_n \in I(w_c)$ are exponents. The elements of the root system $R$ of $g$ are permuted by $\forall w \in [w]$ in orbits of length $N$.

So $\{1, N - 1\} \subset I(w)$, that implies $s_{w0} = 1$.

Theorem 4.2 ([Sp]). Let $[w] \subset W(g)$ be regular primitive conjugacy class. Let $\Lambda = I_+ + zC_{-(N-1)} \in H^w_{1-w}(s_w)$ be grade one regular element. Then

(i) $\exists I_- \in g$ of $s_w$ grade $-1$, such that $\rho = N_w s_w$, $I_+$ and $I_-$ form $sl_2$ subalgebra

$$[\rho, I_\pm] = \pm I_\pm \quad [I_+, I_-] = 2\rho; \quad (4.1)$$

(ii) Eigenvalues of $ad\rho$ on $g$ are integers;

(iii)

$$\dim g(0) = \dim g(\pm 1) = \frac{2}{N} \sum_{k=1}^{r} p_i = \frac{rh}{N}, \quad p_i \in I(w_c) \quad (4.2)$$

**Corollary 4.1.** The extension $\hat{w}$ of $w$ to the whole algebra $g$ is of order $N$.

**Proof.** The order of extension does not depend on the basis chosen. In the basis of (2.1) $\hat{w} = \exp [2\pi i ad \rho/N]$. Then $\hat{w}^N = 1$ as a straightforward consequence of statement (ii) of the theorem. $\square$

Dimensions of $g(k)$ may be read off from the character of this (reducible) representation of $sl_2$ on $g$:

$$\chi_{[w]}(q) = \Tr (q^{ad\rho}) = r + \sum_{\alpha \in R} q^{(\alpha, \rho)} = \sum_{k=-(N-1)}^{N-1} \dim g(k)q^k. \quad (4.3)$$

To determine the $s$ type of regular primitive conjugacy class we compare $\chi_{[w]}$ in the basis of (2.1) and in that of (2.3):

$$\Tr w = r + 2 \sum_{\alpha \in R^+} \cos (2\pi s_{w} \cdot \alpha) = \chi_{[w]}(e^{2\pi i/N_w}) \quad (4.4)$$
The knowledge of the characteristic polynomials of $[w]$ due to \cite{3} allows to compute the order $N_w$ of $[w]$. Then one searches for $s_w$ with $N_w = N$ and $s_w0 = 1$, satisfying (4.4) and (4.2). This and some more (see sec. 5) information about regular primitive conjugacy classes relevant for construction of generalized hierarchies is collected in appendix A.

5 Type-I, $i = 1$ hierarchies

Here and further on we restrict ourselves to hierarchies with a regular primitive conjugacy class and a regular element $I_+ + zC_{-(N-1)} = \Lambda \in \mathcal{H}[w]$ of $s_w$ grade one. Notice, that whatever auxiliary gradation $s$ we choose, $Q(i) \subset \mathfrak{g}$ and we are working with finite dimensional Lie algebra. Adjoint action of $\rho$ induces gradation of $\mathfrak{g}$:

$$\mathfrak{g} = \bigoplus_{k=-N+1}^{N-1} \mathfrak{g}_k.$$  

Notice, that $\hat{\mathfrak{g}}_0 (s_w)$ coincides with $\mathfrak{g}_0$.

Consider first homogeneous auxiliary gradation $s = s_h$. As follows from definition 3.3 and proposition 2.2 gradients have no negative $s_w$ grade part: $P_{<0}^{(s_w)} d_q \varphi = 0 \mod \hat{\mathfrak{g}}_{<0}(s_w)$. So the bi Hamiltonian structure (3.11) simplifies to

$${\{\varphi, \psi\}}_1 = -\langle d_q \varphi, [d_q \psi, C_{-(N-1)}] \rangle,$$

$${\{\varphi, \psi\}}_2 = \langle d_q \varphi, [d_q \psi, \partial_x + I_+ + q] \rangle. \quad (5.1)$$

As we have no $R$ matrix in commutator now, we can recognize in the second bracket a Kirillov-Poisson bracket corresponding to untwisted affinization $\hat{\mathfrak{g}}_{x}$ in $x$ of $\mathfrak{g}$. Introducing $J = I_+ + q$ we obtain

$${\{\varphi, \psi\}}_{KM} = \langle d_J \varphi, [d_J \psi, \partial_x + J] \rangle. \quad (5.2)$$

This fact enabled authors of ref. \cite{BFRFW} to invent a practical algorithm to compute Hamiltonian structure of Drinfeld-Sokolov hierarchies and their construction can be repeated here. They showed that the second Poisson structure (5.1) may be obtained by Hamiltonian reduction of (5.2) (see also \cite{CFMF}). Let us embed $Q^{can} \subset \hat{\mathfrak{g}}_{x}$ and lift flows on $Q^{can}$ to flows on $\hat{\mathfrak{g}}_{x}$. One may choose functional $\Psi$ on $\hat{\mathfrak{g}}_{x}$ coinciding with $\psi$ on $Q^{can}$ such that $\Psi$ flows in $\hat{\mathfrak{g}}_{x}$ preserve $Q^{can}$. Then

$$\delta q = [d_J \Psi|_{Q^{can}}, \partial_x + I_+ + q] = [d_q \psi(x) + r, \partial_x + I_+ + q(x)] \in Q^{can}, \quad (5.3)$$

where $\eta (r, Q^{can}) = 0$. Condition (5.3) determines $r$ uniquely as a function of $q$ and its $x$-derivatives.

The grading of Kac-Moody algebra induces a grading on the Poisson structure, namely – the following theorem holds true.

**Theorem 5.1 (BGH).** The dynamical equation (5.5) of the hierarchy generated by the Hamiltonian $H_b$ ($b \in \mathcal{H}[w](s_w)$) with respect to the second Hamiltonian structure, is invariant under the following rescaling

$$x \mapsto \lambda x, \quad t_b \mapsto \lambda^2 t_b, \quad q_{-k} \mapsto \lambda^{-(k+1)} q_{-k}, \quad (5.4)$$

where $q_{-k}$ is the component of $q(x)$ with $s_w$ grade $-k$.  

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This is a consequence of existence of Virasoro algebra due to Sugawara construction.

**Theorem 5.2 ([FGMS], [BFRFW]).** The second Poisson bracket of \((5.1)\) is a \(W\) algebra associated to \(sl_2\) subalgebra \((4.1)\) with Virasoro generator

\[
w_2(x) = \frac{1}{2} \eta (I_+ + q, I_+ + q) + \eta (\rho, q'),
\]

satisfying Virasoro algebra

\[
\{w_2(x), w_2(y)\} = -\eta (\rho, \rho) \delta''' (x - y) + 2 w_2(x) \delta' (x - y) + w_2'(x) \delta (x - y).
\]

There exists a minimal weight Drinfeld-Sokolov gauge \([BFRFW, DF]\)

\[
ad I_- (Q^{can}) = 0
\]

of conformal primaries:

\[
\{w_{k,i}(x), w_2(y)\} = kw_{k,i}(x) \delta' (x - y) + w_{k,i}'(x) \delta (x - y),
\]

where

\[
Q^{can} \ni q^{can} = w_2 \frac{I_-}{\eta (I_+, I_-)} + \sum w_{k,i} F_{k-1,i}, \quad [\rho, F_{k,i}] = -k F_{k,i}.
\]

Let us denote by \(P_{\text{w}}\) a set of scaling weights of fields of the \(W\)-algebra in question. \(P_{\text{w}}\) is invariant with respect to scale preserving changes of coordinates, so in particular it does not depend on the Drinfeld-Sokolov gauge slice chosen.

**Proposition 5.1.** \(I(w) \subseteq P_{\text{w}}.\) Multiplicities of \(N_w - 1\) in \(I(w)\) and in \(P_{\text{w}}\) are equal.

**Proof.** Let us take densities of Hamiltonians \(H_b, \deg_{\text{w}} b \in I(w).\) Their scaling degree are correspondingly \(\deg_{\text{w}} b + 1.\) These densities are gauge invariant, as follows from proposition 3.2. Due to corollary to the theorem 1.2 they are independent. Thus they may be taken as a part of coordinates on \(\mathcal{M}.\) So \(I(w) \subseteq P_{\text{w}}.\) In particular this implies that the multiplicity of \(N_w - 1\) in \(I(w)\) is less or equal to the one in \(P_{\text{w}}.\)

However, as was shown in ref. [Sp], for every \(C \in g(-N_w + 1)\) there exist such \(I \in g(1)\) that \(I + C\) is regular in \(g.\) So we have an opposite inequality of multiplicities. This completes the proof. \(\Box\)

**Remark 5.1.** One can choose \(F_{N_w - 1,1} = C_{-(N_w - 1)}\). Then the first Poisson structure can be always read off the second by shifting \(w_{N_w,1}\) by \(-\lambda\) and taking the linear term.

The character \([4.3]\) enables us to determine \(ad \rho\) eigenspace decomposition of \(Q^{can}\), and thus \(P_{\text{w}}.\) Indeed,

\[
(1 - q) \chi[w] = -\sum_{k=-N+1}^{-1} \dim Q_k^{can} q^k + \mathcal{O}(q),
\]

as follows from injectivity \([3.6]\) of mapping \(ad I_+ : Q_k \rightarrow Q_{k+1}\) for negative \(k\) and definition of \(Q^{can}_k\) \((5.7).\) Then \(P_{\text{w}}\) is the set of positive numbers \(k\) such that \(\dim Q^{can}_{-k} \neq 0,\) every number occurring \(\dim Q^{can}_{-k}\) times.
Proposition 5.2. Let
\[ U(w) = \{ 1 \leq k_1 \leq \cdots \leq k_{2n} < N_w - 1 \mid k_i \in \text{Pr}_w \text{ but } k_i \notin I(w) \} , \] (5.10)
where \(2n = \dim \mathcal{M} - r\). Then \( k_i + k_{2n+1-i} = N_w - 1\).

Proof. Brute force checking of data presented in Appendix A. We lack more elegant proof so far.

Now, consider modified hierarchy, i.e. \( s = s_w \). Its second Poisson structure (3.13) simplifies as well
\[ \{ \varphi, \psi \}_m = \langle q_m, [d_{q_m} \varphi, d_{q_m} \psi] \rangle - \langle d_{q_m} \varphi, (d_{q_m} \psi)' \rangle . \]
The phase space (3.2) of modified hierarchies \((g, \Lambda, s_w)\) reads
\[ \mathcal{M} = (Q^m(1))^* = (\hat{\mathfrak{g}}_0(s_w))^* \simeq \hat{\mathfrak{g}}_0(s_w) . \] (5.11)
Let \( X_i \) be a basis of \( \hat{\mathfrak{g}}_0(s_w) \subset \mathfrak{g} \) with Gramm matrix \( \mathcal{K} \) being restricted Killing form and let \( q_m = \sum_i \nu^i X_i \), then
\[ \{ \nu^i(x), \nu^j(y) \}_m = (\mathcal{K}^{-1})^{ij} \delta^r (x-y) + f^{ij}_k \nu^k(x) \delta^r (x-y) , \] (5.12)
where \( f^{ij}_k \) are structure constants of \( \hat{\mathfrak{g}}_0(s_w) \). Notice, that \( h \subseteq \hat{\mathfrak{g}}_0(s_w) \) due to proposition 2.2, and thus matrix \( f^{ij}_k \nu^k \) is of corank \( r \).

Given \([w]\) one obtains \( \hat{\mathfrak{g}}_0(s_w) \) from extended Dynkin diagram removing nodes with \( s_w \) and padding with \( u(1) \) to maintain the rank as follows from (2.2).

We collect all pertinent information about generalized Drinfeld-Sokolov hierarchies with regular primitive conjugacy class \([w]\) and regular element \( \Lambda \in \mathcal{H}[w] \) of \( s_w \) grade one in appendix A.

6 Finite dimensional geometry behind the hierarchy

Let us introduce the dispersion parameter \( \epsilon \) by rescaling all derivatives \( \partial \to \epsilon \partial \). Then, the leading term of \( \epsilon \to 0 \) expansion of a Poisson bracket on loop space will verify Jacobi identity, thus furnishing another Poisson structure:
\[ \{ w^i(x), w^j(y) \}_\lambda = \frac{1}{\epsilon} A^{ij}(w) \delta (x-y) . \] (6.1)
We write upper indices to emphasize the covariant nature of objects. The leading bracket (6.1) obviously defines finite dimensional one and we thus end up with pencil of finite dimensional Poisson brackets corresponding to \( s = s_h \) integrable hierarchy:
\[ \{ w^a, w^b \}_\lambda = A^{ij}(w) - \lambda B^{ij}(w) . \] (6.2)
Notice, that densities of annihilators of the first Poisson bracket (5.1) \( N^a = N^a(w) \), where \( 1 \leq a \leq r \) are Casimirs of the first finite bracket:
\[ \{ f, N^a \}_1 = 0 \quad \forall f , \] (6.3)
and are in involution with respect to the second
\[ \{ N^a, N^b \}_2 = 0 . \] (6.4)
Eqs. (6.3) and (6.4) show that Poisson tensor $A - \lambda B$ is of corank $r$ for all $\lambda$ and generic point of $\mathcal{M}$.

Due to Corollary 3.1 one may choose $w^i = (N^a, u^A)$ as coordinates on $\mathcal{M}$, where $u^A$ are Drinfeld-Sokolov variables with scaling weights $k_A + 1$, $k_A \in U(w)$.

Assume that the coordinate $N^r$ is chosen to be linear in Drinfeld-Sokolov variable $w_{N^r}$ (cf. Remark 5.1), and so the pencil (6.1) can be obtained by means of shifting of $N^r$ by $-\lambda$.

Lemma 6.1. Matrix $B^{AB} (u, N)$ is non degenerate with constant determinant.

Proof. As follows from scaling weight grading of Poisson structures

$$\deg_{sc} (A^{AB} - \lambda B^{AB}) = \deg u^A + \deg u^B - 1.$$ 

Since $\deg_{sc} \lambda = N_w$

$$B^{AB} = \begin{cases} 0 & \text{if } \deg_{sc} u^A u^B < N_w \\ \text{const.} & \text{if } \deg_{sc} u^A u^B = N_w \\ B^{AB} & \text{if } \deg_{sc} u^A u^B > N_w \end{cases}$$

Recall that we have assumed ascending ordering of $\deg_{sc} u^A$, and so matrix $B$ is lower triangular with respect to anti diagonal $\deg_{sc} u^A u^B = N_w$. This proves that $\det B$ equals to product of those constants. Since, by the choice of coordinates, we know that $|B^{AB}|$ is nondegenerate we conclude, that all those constants are nonzero.

Remark 6.1. Notice, that $N^1 = w_2$ annihilates the pencil of finite dimensional brackets: $\{ f, N \}^1 = 0$ for any $f$.

Proposition 6.1. Let $G^{A,a} (u, N) = \{ u^A, N^a \}^2$. Consider a variety $\mathcal{M}_r$ defined by equations $G^{A,2} (u, N) = 0$ and suppose that matrix $\left| \frac{\partial G^{A,2}}{\partial u^B} \right|$ is not degenerate on $\mathcal{M}_r$. Then

$$G^{B,b} (u, N)|_{\mathcal{M}_r} = 0 \quad \forall B, b.$$ 

Proof. By the implicit function theorem one may resolve the system of polynomial equations $G^{A,2} (u, N) = 0$ locally with respect to variables $u$. Consider the Hamiltonian flows in involution generated by $N^a$:

$$\frac{\partial \varphi}{\partial t_a} = \{ \varphi, N^a \}^2.$$ 

Then one has

$$\frac{\partial}{\partial t_a} \frac{\partial}{\partial t_2} u^A = \frac{\partial}{\partial t_2} \frac{\partial}{\partial t_a} u^A \Rightarrow \frac{\partial G^{A,2}}{\partial u^B} \frac{\partial G^{B,a}}{\partial u^B} = \frac{\partial G^{A,a}}{\partial u^B} \frac{\partial G^{B,2}}{\partial u^B}.$$ 

Restricting on $\mathcal{M}_r$ and using the non degeneracy of $\frac{\partial G^{A,2}}{\partial u^B}$ the result follows.

Remark 6.2. So, the equation $G^{A,2} = 0$, subject to condition of Proposition 6.1 defines an algebraic variety $\mathcal{M}_r$ of stationary points of all Hamiltonians in involution with respect to Poisson bracket (6.2). $\mathcal{M}_r$ “corresponds” to the kernel of Poisson tensor of (6.2).
As an immediate consequence of eqs. (6.3) and (6.4) we have the following

**Proposition 6.2.** Polynomials \( G^{A,a} (u, N) \) do not depend on \( N \).

As follows from generalized Drinfeld-Sokolov construction, this finite dimensional bracket is but Kirillov-Kostant bracket

\[
\{ \varphi, \psi \}_\lambda = (\Lambda + q [d_q \varphi, d_q \psi])
\]

restricted on the space of \( \text{ad} \, P \) invariant functions \( \varphi (\tilde{q}) = \varphi (q) \):

\[
\Lambda + \tilde{q} = \exp [\text{ad} \, n] (\Lambda + q) \quad n \in P
\] (6.5)

**Lemma 6.2.** Drinfeld-Sokolov coordinates \( w^i (q) \), in dispersionless limit, generate the ring of \( \text{ad} \, P \) invariant polynomials in \( q \).

**Proof.** Consider a ring of polynomial functions in \( q \). It is obviously generated by elements of \( Q^* \). Using the adjoint action of nilpotent group with Lie algebra being \( P \) one may always reduce \( \tilde{q} \) to canonical form \( \tilde{q} \in Q^\text{can} \) (see eq. (3.7)). Indeed, projecting eq. (6.6) on \( \text{ad} \, \rho \) eigenspaces we get

\[
\tilde{q}_i = q_i + [n_{i+1}, I_+] + \text{polynomial} (n_i, \ldots, n_1; q_{i-1}, \ldots, q_1)
\] (6.7)

where \([\rho, q_i] + iq_i = 0\) and \( 0 \leq i < N_w \). This allows to resolve, starting from \( i = 0 \) and proceeding inductively, for \( \tilde{q} \) and \( n(q) \) as polynomials in \( q \). Recapitulating, \( (\tilde{q}, n) \) generate a ring of polynomials in \( q \). The subring of gauge invariant polynomials is, then, obviously generated by \( \tilde{q} \), i.e. by Drinfeld-Sokolov coordinates.

Now consider the case of modified hierarchy, \( s = s_w \). The leading term Poisson structure gives Kirillov-Kostant bracket on \( \hat{g}_0 (s_w)^* \):

\[
\{ \nu^i, \nu^j \}_m = f^{ij}_k \nu^k.
\] (6.8)

The Miura map (1) provides a map from modified hierarchy \((s = s_w)\) into \( s = s_h \) one. Thus we have polynomial expressions \( w^i (\nu) = w^i (q_m) \) for dispersionless Drinfeld-Sokolov variables. Notice, that modified hierarchies have their “gauge algebra” \( P \) empty, and yet Miura coordinates \( \nu \) do not generate the ring of gauge invariant polynomials. The reason is that, imposing \( \tilde{q} \in Q_m \), equations (6.7) do not have a unique solution. Following ideas of ref. [BFRFW] we have

**Proposition 6.3.** There is a finite subgroup \( R \subset \exp (\text{ad} \, P) \) acting on \( Q_m = g_0 \).

**Proof.** Let \( n \in P \) corresponds to group element fixing \( Q_m \), i.e.

\[
I_+ + q_m = \exp [\text{ad} \, n] (I_+ + q_m) = \sum_{k=0}^{N_w+1} \frac{1}{k!} \text{ad}^k n (I_+ + q_m).
\] (6.10)

Projecting on \( \text{ad} \, \rho \) eigenspaces we gain the system of equations (6.7). Since \( q_m \in g_0 \), we have

\[
\tilde{q}_m = q_m + [n_1, I_+].
\] (6.11)
The rest of equations (6.7) for \( i > 0 \) determines \( n \). Given positive \( i \) there are \( \dim \mathfrak{g}_{-i} \) scalar equations. Due to injectivity of map \( \text{ad} I_+ : \mathfrak{g}_k \to \mathfrak{g}_{k+1} \) for negative \( k \) we may unambiguously solve for \( \dim \mathfrak{g}_{-i} \) unknowns contained in \( n_{i+1} \) and remain with \( m_i = \dim \mathfrak{g}_{-i} - \dim \mathfrak{g}_{i+1} \) scalar equations for \( n_j \), where \( j < i \). Notice, that by definition, \( m_i \neq 0 \) iff \( i \in \text{Pr}_w \) with \( m_i \) being the multiplicity of \( i \) in \( \text{Pr}_w \). Starting with \( i = 1 \) case

\[
[n_2, I_+] + \left[ n_1, q_m + \frac{1}{2} [n_1, I_+] \right] = 0 ,
\]

we solve for \( n_2 = n_1(q_m) \). Proceeding further with excluding \( n_k \), \( k > 1 \), we end up with \( m_i \) equations of order \( i + 1 \), for each distinct \( i \in \text{Pr}_w \), to determine \( \dim \mathcal{M} \) unknowns contained in \( n_1 \). Notice, that the number of equations \( \text{ord} \text{Pr}_w \) equals to the number of unknowns \( \dim \mathfrak{g}_{-1} = \dim \mathcal{M} \) and by Bezout theorem we obtain no more than \( \prod_{k \in \text{Pr}_w} (k + 1) \) solutions \( n_1 = n_1(q_m) \), which determine that number of nonlinear transformations \( q_m \to \tilde{q}_m \) of \( Q_m \). If only those \( \dim \mathcal{M} \) equations for \( n_1 \) are independent we obtain an equality in eq. (6.9).

Since \( n_1 \in \mathfrak{g}_{-1} \) completely determines any transformation from \( R \), as follows from the proof of Proposition 6.3, it is tempting to make an anz¨ ats for the simplest transformations as \( n \in \mathfrak{g}_{-1} \). It proves to be consistent only for simply laced Lie algebras \( \mathfrak{g} \). We may then rewrite eq. (6.10) as follows

\[
\tilde{q}_m = q_m + [n, I_+] + \sum_{k=1}^{N_w} \frac{1}{k!} \text{ad}^k n \left( \frac{1}{k+1} [n, I_+] + q_m \right). 
\]

To make the sum, contributing to negative \( \text{ad} \rho \) eigenspaces, vanish we need that

\[
[n, [n, H]] = 0 , \quad \forall H \in \mathfrak{g}_0(s_w), \quad (6.13a) \]
\[
[n, q_m + \frac{1}{2} [n, I_+] = 0 . \quad (6.13b) \]

To solve equations (6.13) we make use of Weyl group \( W(\mathfrak{g}_0) \) of semisimple Lie subalgebra \( \mathfrak{g}_0 \subset \mathfrak{g} \).

**Lemma 6.3.** Subspaces \( \mathfrak{g}_k \subset \mathfrak{g} \), \( k \in \mathbb{Z} \), are stable under natural action of \( W(\mathfrak{g}_0) \subset W(\mathfrak{g}) \). Each orbit \( Z \subset \mathfrak{g}_{-1} \) of \( W(\mathfrak{g}_0) \) is a commutative subalgebra of \( \mathfrak{g} \), if \( \mathfrak{g} \) is simply laced.

**Proof.** The action of Weyl group \( W(\mathfrak{g}_0) \) are inner in \( \mathfrak{g}_0 \) and in \( \mathfrak{g} \). For any root \( \alpha \), such that \( E_{\alpha} \in \mathfrak{g}_0 \), one has a reflection \( r_{\alpha} \) in the hyperplane perpendicular to the root acting on \( \mathfrak{g} \) as follows:

\[
r_{\alpha} = \exp [\text{ad} E_{\alpha}] \exp \left[ -\frac{2}{\alpha^2} \text{ad} E_{-\alpha} \right] \exp [\text{ad} E_{\alpha}]. \quad (6.14) 
\]

These reflections act canonically on Cartan subalgebra \( \mathfrak{h} \subset \mathfrak{g}_0 \subset \mathfrak{g} \). Since all elements of \( W(\mathfrak{g}_0) \) can be expressed as a product of these, we conclude that \( W(\mathfrak{g}_0) \) stabilizes \( \mathfrak{g}_k \). Take a root \( \beta \), such that \( E_{\beta} \in \mathfrak{g}_{-1} \), and consider the \( W(\mathfrak{g}_0) \) orbit \( Z \) that passes through it. Fix \( w \in W(\mathfrak{g}_0) \) and assume that \( w \) does not fix \( \beta \). It suffices to prove that \( \beta + w(\beta) \) is never a root.
Since \( \mathcal{W}(g_0) \) stabilizes \( g_{-1} \), \( \gamma \defeq w(\beta) - \beta \) is a linear combination of simple roots of \( g_0 \). For simply laced Lie algebras \( g \), \( \gamma \) itself is a root such that \( E_\gamma \in g_0 \). Assume that \( \beta + w(\beta) = 2\beta + \gamma \) is a root, then

\[
\frac{2(\beta|\gamma)}{2(\beta|\beta)} \leq -2.
\] (6.15)

This is impossible for any two roots \( \beta \) and \( \gamma \) of simply laced simple Lie algebra \( g \). \( \square 

\textbf{Corollary 6.1.} For any orbit \( Z \), and any element \( E_\beta \in Z \), \( \text{ad} E_\beta \) maps \( g_0 \) on \( Z \).

\textbf{Proof.} Consider the set \( S \defeq \{ [X, Y] \mid X \in g_0, Y \in Z \} \). Since \( h \subseteq g_0 \) we have \( Z' \subseteq S \). The set \( S \) is stable under the homomorphism \( \mathcal{W}(g_0) \) because \( g_0 \) and \( Z \) are stable. So \( S = Z \) and, hence, \( \text{ad} E_\beta \) maps \( g_0 \) in \( Z \). The surjectivity is obvious. \( \square 

\textbf{Remark 6.3.} Having the simple roots chosen in \( h^* \), it is plain to see, that different orbits \( Z \) may be labeled by simple roots of \( s_w \), degree one, i.e. such \( \alpha \) that \( E_\alpha \in g_1 \).

Choose an orbit \( Z \subset g_{-1} \), and let \( \{ X_q \} \) denote the set of root vectors that form the basis in \( Z \). We then solve (6.13a) by letting \( n = \sum x_q X_q \in Z \). Due to Corollary 6.1 we have

\textbf{Lemma 6.4.} Condition (6.13b) gives \( \dim Z \) homogeneous quadratic equations for parameters \( x_i \), that possess non trivial solutions.

Let us speculate a bit on the structure of the group \( R \). Fix \( q_m \in g_0 \), some orbit \( Z \), and let \( n(q_m) \in Z \) be a solution of eq. (6.13b), that gives the map \( n: q_m \rightarrow \tilde{q}_m \). Apply to the result another transformation with \( \tilde{n} \in Z \). We get another map \( \tilde{n}: \tilde{q}_m \rightarrow \tilde{q}_m \). Notice that their composition would be another map given by \( n'(q_m) = n(q_m) + \tilde{n}(q_m(q_m)), n': q_m \rightarrow \tilde{q}_m \). Both \( n \) and \( n' \) will verify eqs. (6.13) and obviously \( n \neq n' \). It is clear, that further iterations will again bring another solution of (6.13b). We therefore arrive to

\textbf{Proposition 6.4.} For each \( \mathcal{W}(g_0) \) orbit \( Z \subset g_{-1} \) there is subgroup \( R_Z \) of \( R \).

Notice, that if \( \dim Z = 1 \), \( n' \) must vanish and our simplest transformation must be reflections, i.e. \( R_Z = \mathbb{Z}_2 \).

Any transformation \( n \in P \) from \( R \) is uniquely determined by its \( g_{-1} \) projection \( n_{-1} \). That \( n_{-1} \) may be uniquely split \( n_{-1} = \sum h_{-1}^{(k)} \) with \( h_{-1}^{(k)} \in Z_k \) in distinct \( \mathcal{W}(g_0) \) orbits. It suggests that

\[
\exp \left[ \text{ad} \left( n(q_m) \right) \right] (I_+ + q_m) = \prod_i \exp \left[ \text{ad} \left( n_{-1}^{(k)}(q_m) \right) \right] (I_+ + q_m).
\]

To ensure that \( R \) is generated by the simplest transformations one must prove that each factor preserves \( g_0 \). We, however, do not have the proof.

\textbf{Remark 6.4.} In the case of standard Drinfeld-Sokolov hierarchy we have \( g_0 = h \). Then eq. (6.13a) implies that \( n = x_k E_{-\alpha_k} \in g_{-1} \) and eq. (6.13b) requires

\[
x_k = \frac{2}{(\alpha_k|\alpha_k)} \sum_{i=1}^{\alpha_k} (\alpha_k|\alpha_i) \nu^i \Rightarrow r_k (\nu^i) = \nu^i - \frac{2\delta_{jk}}{(\alpha_k|\alpha_k)} \sum_{i=1}^{\alpha_k} (\alpha_k|\alpha_i) \nu^i.
\]

(6.16)
The same transformation of $\nu$ results from the Weyl reflection, acting on $g_0 = \mathfrak{h}$, corresponding to simple root $\alpha_k$. Hence, polynomials $w^i(\nu)$ result to be Coxeter polynomials \cite{BFRFW}.

**Corollary 6.2.** Polynomials $w^i(\nu)$ of degrees $k_i + 1$ with $k_i \in \text{Pr}_w$ are invariant with respect to discrete group $R$ and generate the ring of $R$ invariant polynomials on $Q_m$.

**Proof.** By Lemma 6.2 $w^i(q)$ generate the ring of gauge invariant polynomials in $q$. Restricting them on $Q_m$ and restricting adjoint group to $R$ the proof follows.

**Remark 6.5.** Polynomials $w^i(\nu)$ are not however $\mathcal{W}(g_0)$ invariant. This means that they are not, for non trivial $\mathcal{W}(g_0)$, a restriction to $g_0$ of $\text{Ad}$ invariant polynomials on $g$.

Notice, $R$ invariance of $w^i(\nu)$ implies that Kirillov-Kostant brackets is equivariant with respect to action of $R$, i.e.

$$\{\tilde{\nu}^i, \tilde{\nu}^j\}_m = f^{ij} k \tilde{\nu}^k = \frac{\partial \tilde{\nu}^i}{\partial \nu^m} f^{mn} \nu^j \frac{\partial \nu^j}{\partial \nu^m}.$$

The $R$ invariant variety $\mathcal{M}_r$ is defined in $Q_m$, much the same, as fixed point of all Hamiltonians in involution:

$$F^k(\nu) = \{\nu^k, N^2\}_m = f^{kl} n \nu^n \frac{\partial N^2}{\partial \nu^l}. \quad (6.17)$$

Indeed, (6.17) implies $G^{A,2} = 0$ and so defines $\mathcal{M}_r$. It means, that one can restrict action of $R$ group on $\mathcal{M}_r$.

To construct “good” coordinates on $\mathcal{M}_r$ we consider Casimirs of Kirillov-Kostant bracket (6.8). Recall, that $g_0$ is semisimple Lie algebra, i.e $g_0 = \oplus_i g^{(i)}$, where $g^{(i)}$ is either simple or abelian. Let $J_{i,k}(\nu)$ be algebraically independent $\text{ad} g_0$ invariant polynomials of degree $k$ corresponding to $g^{(i)}$, i.e. invariant also under $\text{ad} g^{(i)}$. They obviously annihilate (6.8).

Introduce $r$ “abelian” coordinates $\mu$ on $h^* = \oplus_i (h^{(i)})^*$ defined by equations

$$J_{i,k}(\mu)|_{h^*} = J_{i,k}(\nu). \quad (6.18)$$

Coordinates $\mu$ have scaling weight one, as well as Miura coordinates.

**Proposition 6.5.** Coordinates $\mu$ do not depend on the choice of $\text{ad} g_0$ invariant polynomials $J_{i,k}$.

**Proof.** Indeed, let us choose another set of algebraically independent polynomials

$$f_j(\nu) = f_j(J(\nu)).$$

They are known to be of the same degrees. Using them we define new coordinates $\mu'$:

$$f_j(\mu')|_{h^*} = f_j(\nu) \quad f_j(J(\mu'))|_{h^*} = f_j(J(\nu)) = f_j(J(\mu))|_{h^*}.$$

Starting from functions $f$ of the lowest degree and proceeding up we conclude that $\mu'$ may be chosen to coincide with $\mu$. 

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We may complete $\mu$ by those Miura coordinates $\nu$ that correspond to $X_i \not\in \mathfrak{h}$ in $q_m = \sum \nu^i X_i$, to have coordinates on the whole phase space $\mathcal{M}$. Let us denote them $\eta^i$ so that $\eta^i = \mu^i$ for $1 \leq i \leq r$.

**Lemma 6.5.** Group $R$ admits restriction on the “abelian” coordinates $\mu$.

**Proof.** Indeed, Casimirs $J_{i,k}$, being invariant with respect to adjoint action of $\mathfrak{g}_0$, are not left invariant by action of group $R$. Let $\tilde{J}_{i,k}(\nu) = J_{i,k}(\nu)$, Then

$$0 = f^l p_n \eta^m \frac{\partial J_{i,k}(\nu)}{\partial \nu^m} = \{\nu^l, J_{i,k}(\nu)\}_m \Rightarrow 0 = f^l p_n \eta^m \frac{\partial J_{i,k}(\nu)}{\partial \nu^m}.$$ 

This implies that $\tilde{J}_{i,k}(\nu)$ results ad $\mathfrak{g}_0$ invariant and hence depends on $\mu$ only.

In the next section we shall make use of the following

**Lemma 6.6.** Define the following matrix

$$\left(\mathbb{K}^{-1}\right)_{mn} = \frac{\partial \eta^m}{\partial \nu^i} \left(\mathbb{K}^{-1}\right)_{ij} \frac{\partial \eta^n}{\partial \nu^j}. \quad (6.19)$$

Then its submatrix $\left(\mathbb{K}^{-1}\right)_{kl}$ with $1 \leq k, l \leq r$ is constant non degenerate matrix.

**Proof.** Since $J_{i,k}$ are ad $\mathfrak{g}_0$ invariant polynomials and so is Killing metrics, we may make use of Chevalley theorem

$$\frac{\partial J_1(\nu)}{\partial \nu^i} \left(\mathbb{K}^{-1}\right)_{ij} \frac{\partial J_2(\nu)}{\partial \nu^j} = F_{1,2}(J(\nu)). \quad (6.20)$$

stating that $F$ is a polynomial in ad $\mathfrak{g}_0$ invariant polynomials $J_{i,k}(\nu)$. Now using definition of coordinates $\mu$ $[6.18]$ and changing variables to $\eta$ we obtain

$$F_{1,2}(J(\mu)) = \frac{\partial J_1(\eta)}{\partial \eta^i} \left(\mathbb{K}^{-1}\right)_{ij} \frac{\partial J_2(\eta)}{\partial \eta^j} = \sum_{m,n=1}^r \frac{\partial J_1(\mu)}{\partial \mu^m} \left(\mathbb{K}^{-1}\right)_{mn} \frac{\partial J_2(\mu)}{\partial \mu^n}. \quad (6.21)$$

By another Chevalley theorem $J(\mu)$ are invariant with respect to Weyl group of $\mathfrak{g}_0$. This means that $r \times r$ submatrix $\mathbb{K}^{-1}$ is inverse of Killing metrics of rank $r$ semi-simple Lie algebra $\mathfrak{g}_0$ restricted to Cartan subalgebra. The latter coincides with Killing form of algebra $\mathfrak{g}$. Thus it is non degenerate due to celebrated Cartan’s criterion of semisimplicity.

**Corollary 6.3.** $\left(\mathbb{K}^{-1}\right)_{ij} = \left(\mathbb{K}^{-1}\right)_{ij}$ for $1 \leq i, j \leq r$.

**Lemma 6.7.** Group $R$ acts linearly on coordinates $\mu$.
Proof. Consider \( \bar{J}(\mu) = J(\hat{\mu}(\mu)) \). Rewrite (6.20) as follows

\[
F_{1,2}(\bar{J}(\mu)) = F_{1,2}(J(\hat{\mu})) = \sum_{m,n=1}^{r} \frac{\partial J_1(\hat{\mu})}{\partial \mu^m} (K^{-1})^{mn} \frac{\partial J_2(\hat{\mu})}{\partial \mu^n} = \sum_{m,n,k,l=1}^{r} \frac{\partial J_1(\mu)}{\partial \mu^m} \frac{\partial \mu^n}{\partial \mu^l} (K^{-1})^{kl} \frac{\partial J_2(\mu)}{\partial \mu^l} \frac{\partial \mu^n}{\partial \mu^m}.
\]

Since functions \( F_{1,2}(J) \) are not altered we have the same pairing and due to Lemma 6.4, \( R \) action preserves constant non degenerate submatrix of the matrix \( K^{-1} \). It is only possible if \( R \) acts linearly.

Remark 6.6. Since the action of \( R \) on \( \mu \) is defined via action on polynomials \( J(\mu) \), it can be determined only up to \( \text{ad} g_0 \), i.e. only as

\[
\bar{J}(\mu) = J(A \cdot \mu)
\]

for some constant matrix \( A \), determined up to transformations

\[
A \rightarrow w_1 A w_2, \quad \text{where} \quad w_1, w_2 \in W(g_0).
\]

Lemma 6.8. Non trivial transformations from \( R_Z \) correspond to the same matrix \( A_Z \) modulo this equivalence.

Proof. Fix an \( W(g_0) \) orbit \( Z \) in \( g_{-1} \). It suffices to show that \( W(g_0) \) permutes solutions of eq. (6.13b).

Notice, that \( W(g_0) \) invariant polynomials \( J \), corresponding to abelian constituents of \( g_0 \) are linear and may be chosen to be corresponding Miura variables as follows from (6.16). Then, due to Lemma 6.3, \( R \) groups acts on them linearly in terms of \( \mu \). Solving eqs. (6.13) for \( \mu(\nu) \) we get just \( \text{ord} W(g_0) \) solutions as follows from Bezout theorem. They correspond to different solutions of (6.13b).

Corollary 6.4. Eqs. (6.13b) have non more than \( \text{ord} W(g_0) \) non trivial solutions.

Corollary 6.5. Matrix \( A_Z \) may be chosen to be a reflection, i.e. \( A_Z^2 = 1 \).

Proof. Since \( R_Z \) is a subgroup, there exist two transformations product of which is identity. Since they correspond to the same matrix \( A_Z \), the proof follows.

Theorem 6.1. Let \( W(g_0) \) be Weyl group of \( g_0 \). Assume \( g \) simply laced. Then \( R|_{\mathcal{M}_r} \simeq W(g_0) \simeq W(g) \).

Proof. Weyl group \( W(g_0) \) preserves metrics \( K^{-1} \), and is generated by reflections (6.16) corresponding to simple roots of zero \( s_w \) grade. Group \( R \), restricted to \( \mathcal{M}_r \), also preserves the metrics and is generated by reflections associated with simple roots of \( s_w \) grade 1, due to Lemmas 6.3 and 6.4. We conclude that \( R|_{\mathcal{M}_r} \simeq W(g_0) \) is a finite group generated by \( r \) transformations associated to simple roots, that preserve \( K^{-1} \). Since it is an inverse of the Killing form of algebra \( g \) and \( W(g_0) \) acts canonically (6.16), it follows that \( R|_{\mathcal{M}_r} \simeq W(g_0) \) is isomorphic to Weyl group \( W(g) \).

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7 Dispersionless limit

As was proved by I. Krichever [Kr], the dispersionless limit, or zero phase Whitham averaging, of the standard Drinfeld-Sokolov hierarchies provides solutions to WDVV equations.

Introducing dispersion parameter $\epsilon$ the Poisson structure of hierarchies reads

$$\{w^i(x), w^j(y)\} = \sum_{k \geq 0} \epsilon^{k-1} \{w^i(x), w^j(y)\}^{(k)}$$

$$= \frac{1}{\epsilon} A^{ij}(w) \delta(x - y) + g^{ij}(w) \delta'(x - y) + G^{ij}(w) (w^k(x))' \delta(x - y) + O(\epsilon).$$

(7.1)

Remark 7.1. For the standard Drinfeld-Sokolov hierarchies $A \equiv 0$ because $\text{dim} \ M = r$ and hence annihilators of the first Poisson structure may be chosen as coordinates on the whole $M$. Note that due to Corollary 3.1 we only have $r \leq \text{dim} \ M$ independent annihilators, thus appearance of $A$ term should be, generally, expected.

Let us pick up $w^i = (N^a, u^A)$ as coordinates on $(Q^{can})^*$ as in section 6. We assume that they are obtained by ultralocal change of variables from Drinfeld-Sokolov variables, i.e. contain no derivative terms.

Due to eq. (6.4) hierarchy time flows of $N^a$ have polynomial $\epsilon$ expansion, but dynamics of $u^A$ coordinates does not enjoy this property

$$\frac{\partial N^a}{\partial t_b} = \{N^a, H_b\}_2 = \partial_x A^a(b) + O(\epsilon),$$

$$\frac{\partial u^A}{\partial t_b} = \{u^A, H_b\}_2 = \frac{1}{\epsilon} G^{A,b}_a(u, N) + \partial_x A^b(b) + O(\epsilon).$$

(7.2)

Hence brackets $\{N^a, N^b\}$ admit dispersionless limit, while others do not. $G^{A,b}_a$'s come from $A$ term in (7.1) and thus are responsible for fast dynamics of $u$ coordinates. If $u$ coordinates evolved so as to vanish $G^{A,a}_a$ identically, we would obtain a well defined dispersionless limit of the hierarchy. As we have seen in section 6 it happens on the algebraic subvariety $M_r \subset M$.

We thus supplement Dubrovin-Novikov prescriptions [DN] for the restriction of the Poisson structure (7.1) on the slow - modulated zero phase solutions by requirement of additional restriction on $M_r$. Dirac bracket provides restriction of the Hamiltonian structure on $M_r$ to $M_r$. We review briefly, for reader’s convenience, the construction of Dirac bracket, referring to [MR] for details.

Given constraint equation $G^{N^2,2} = 0$ defining $M_r$ and local coordinates $N^a$ there, we introduce new ones

$$\tilde{N}^a(x) = N^a(x) + \int dy \tau^a_A(x, y) G^{A,2}(w(y)).$$

such that $\tilde{N}^a|_{M_r} = N^a|_{M_r}$ and with $\tau$ subject to condition

$$\{\tilde{N}^a(x), G^{A,2}(w(y))\}|_{M_r} = 0.$$
Looking for solution of \( \tau \) as formal \( \epsilon \) series \( \tau = \sum_{m \geq 0} \epsilon^m \tau^{(m)} \), the equation above for \( \tau \) amounts to the following

\[
\{ N^a(x), G^{A,2}(y) \}_{(k)}^{(M_r)} + \sum_{m=0}^{k} \int dz \tau^{(m)}_B (x,z) \{ G^{B,2}(z), G^{A,2}(y) \}_{(k-m)}^{(M_r)} = 0 .
\]

(7.3)

Due to Proposition 6.1 we obtain that \( \tau^{(0)} = 0 \) provided that the matrix

\[
\left\{ \begin{array}{ll}
N^a(x), G^{A,2}(y) \\
N^b(y)
\end{array} \right\}|_{M_r}
\]

is nondegenerate.

**Definition 7.1.** Dirac bracket on \( M_r \) is defined as

\[
\left\{ N^a(x), N^b(y) \right\}_D = \left\{ \hat{N}^a(x), \hat{N}^b(y) \right\}|_{M_r} = \left\{ \hat{N}^a(x), N^b(y) \right\}|_{M_r} - \int dz_1 dz_2 \tau^{(1)}_A (x,z_1) \left\{ G^{A,2}(x), G^{B,2}(z_2) \right\}|_{M_r} \tau^b_B (y,z_2)
\]

(7.4)

Dirac bracket verifies \([MR]\) Jacobi identity. As an immediate consequence of (7.4) we have the following

**Lemma 7.1.** If \( \tau^{(0)} = 0 \) then

\[
\left\{ \hat{N}^a(x), N^b(y) \right\}_D^{(k)} = \left\{ N^a(x), N^b(y) \right\}|_{M_r}^{(k)} \quad k = 0, 1 .
\]

(7.5)

Because of (6.4) the \( \epsilon \to 0 \) expansion of Dirac bracket (7.4) starts with \( k = 1 \) term, and thus the bracket admits dispersionless limit.

**Corollary 7.1.** Dispersionless limit of bi Hamiltonian structure is bi Hamiltonian.

We thus arrive to the following theorem

**Theorem 7.1.** Consider Hamiltonian dynamical system admitting constant solutions. Let some \( N^a \) be the densities of the local commuting integrals of the system, considered as the parameters of the full family of the constant solutions. Let \( u \) denote the rest of dynamical variables. Assume that the matrix \( \{ G^{A,2}, G^{B,2} \}|_{M_r} \) does not degenerate identically. Then the dispersionless limit of the Hamiltonian structure restricted to \( M_r \), given by Dubrovin-Novikov formula

\[
\left\{ N^a(x), N^b(y) \right\}^* = g^{ab} (N(x)) \delta'(x-y) + \Gamma^{ab}_{c} (N(x)) (N^c(x))^t \delta(x-y),
\]

(7.6)

satisfies the Jacobi identity and does not depend on the choice of \( G \).

This result is a particular case of theorem due to A. Maltsev [M] who proved, using Dirac reduction procedure, that Dubrovin-Novikov averaging procedure yields, under certain assumptions, a Poisson structure on the space of \( m \)-phased solutions of dynamical equations of original Hamiltonian system.
Lemma 7.2. Assumptions of Theorem 7.1 verify for the hierarchies \((g, [w], \Lambda)\) with regular primitive conjugacy class \([w]\) and grade one regular element \(\Lambda\).

Proof. Due to (5.6), \(H_\Lambda = \int dx w_2 (x)\) is the momentum for the hierarchies in question, so their dynamical equations admit constant solutions, because of theorem 3.1.

Now we address the non degeneracy statement.

\[
A^{AB} = \{G^{A,2}, G^{B,2}\} = \frac{\partial G^{A,2}}{\partial w^i} \{w^i, w^j\} \frac{\partial G^{B,2}}{\partial w^j}.
\]

Restricting on \(\mathcal{M}_r\) and using (6.4) and \(\{u^A, N^a\}_2|_{\mathcal{M}_r} = 0\) we conclude

\[
A^{AB} = \partial G^{A,2} \{u (N), N\}_2 \frac{\partial G^{B,2}}{\partial u^D} (u (N), N).
\]

Thus, due to assumption of Proposition 6.1, it suffices to prove the non degeneracy of \(\tilde{A}^{CD} = \{u^C, u^D\}_2|_{\mathcal{M}_r}\). The latter is obvious because

\[
\tilde{A}^{CD} = \tilde{A}^{CD} + N^r B^{CD}|_{\mathcal{M}_r},
\]

where \(\tilde{A}\) does not depend on \(N^r\). Restriction on \(\mathcal{M}_r\) does not alter the linear dependence of \(\tilde{A}\) on \(N^r\) because of Proposition 6.2. Due to Lemma 6.1 \(\det B \neq 0\) and is constant. It, thus, remains constant after restriction to \(\mathcal{M}_r\).

Proposition 7.1 ([DN, Du4]). Given Poisson structure of hydrodynamic type (7.6), \(g^{ab}\) is the flat covariant metrics as long as \(\det g \neq 0\) and \(\Gamma\) is its connection, related to Levi-Civita connection by the following relation

\[
\Gamma^{cd}_e = -g^{ab} \Gamma^{cd}_e.
\]

Non degeneracy of matrix \(g^{ab} (N)\) obtained by Dirac reduction on \(\mathcal{M}_r\) of Poisson structure of generalized integrable hierarchies in consideration is not an obvious fact and needs to be proved.

Miura map ([1]) provides us with polynomial expressions of Drinfeld-Sokolov coordinates in terms of Miura ones and its derivatives:

\[
w_i^c (\nu) = w^i (\nu) + \epsilon \sum_{j=1}^{\dim \mathcal{M}} \left( \nu_j \wedge \nu \right) \left( \nu_j \right) + O (\epsilon^2).
\]

This change of coordinates provides a map of Miura bracket (5.12) to the second Poisson structure (5.1). Under this map we obtain for the first two matrix in (7.1) the following expressions

\[
A^{ij} (w) = \frac{\partial w^i}{\partial \nu^k} f^{klm} \frac{\partial w^j}{\partial \nu^m},
\]

\[
g^{ij} (w) = \frac{\partial w^i}{\partial \nu^k} \left( \kappa^{-1} \right)^{kl} \frac{\partial w^j}{\partial \nu^l} + \left( \nu_j \wedge \nu \right) \left( \nu_j \right) f^{klm} \nu^m. \quad (7.7)
\]

As an immediate consequence of definition of \(\mathcal{M}_r\) we have

Proposition 7.2. On \(\mathcal{M}_r (7.7)\) simplifies to the following expression:

\[
g^{ab} (N) = \sum_{k,j=1}^{\dim \mathcal{M}} \frac{\partial N^a}{\partial \nu^k} \left( \kappa^{-1} \right)^{kl} \frac{\partial N^b}{\partial \nu^l}. \quad (7.8)
\]
Proposition 7.3. \{\cdot, \cdot\}^*_2 \text{ can be read off } \{\cdot, \cdot\}^*_1 \text{ by appropriate shift of } N_r.

Proof. Due to Proposition 6.2 and Lemma 7.1 dispersionless limit of the second bracket remains linear in $N_r$. The result follows.

Note, that $N_1 = w_2$ satisfies Fuchs algebra

$$\{N^1, N^1\}^*_2 = 2N^1 \delta' + (N^2)' \delta.$$  

Thus we have arrived to

Proposition 7.4. Zero phase Whitham averaging maps graded bi Hamiltonian structure (5.1) into graded bi Hamiltonian structure of hydrodynamic type.

Due to Lemma 7.1 $N_a$ will remain annihilators of the first bracket:

$$\{N^a(x), N^b(y)\}^*_1 = \eta^{ab}(x - y).$$  

Remark 7.2. Due to scaling weight grading, and chosen field ordering $\eta^{ab}$ is anti-diagonal matrix.

Indeed, $\eta^{ab}$ vanishes if $\deg_{sc}(N^a N^b) \neq N_w + 2$ and $\deg_{sc}(N^a N^{r+1-a}) = N_w + 2$ due to $\deg_{sc} N^a = k_a + 1$ and ascending ordering of $k_a \in I(w)$. Notice, that this is consistent with previously assigned $N^1$ being $w_2$ and $N^r$ linear in $w N_w, 1$.

Proposition 7.5. Dispersionless limit of Dirac restriction of Poisson structure of modified hierarchy reads

$$\{\mu^i, \mu^j\}^* = (K^{-1})^{ij} \delta'(x - y).$$  

Proof. $\mu$ are Casimirs of finite dimensional Kirillov-Kostant bracket (6.8) and thus due to Lemma 7.1 and Lemma 6.6 the result follows.

Theorem 7.2. Metrics $g^{ab}(N)$ is not identically degenerate. Coordinates $\mu$ are flat coordinates for this metrics.

Proof. Let us choose coordinates $\eta$ as follows. Take the first $r$ coordinates to be $\mu$ and choose the rest to be canonical for finite dimensional bracket. The advantage of this choice is the simplicity of constraint equations (6.17)

$$\left. \frac{\partial N^2}{\partial \eta^k} \right|_{\mathcal{M}_r} = 0 \quad \text{Proposition 6.1} \quad \left. \frac{\partial N^a}{\partial \eta^k} \right|_{\mathcal{M}_r} = 0 \quad \forall r + 1 \leq k \leq \dim \mathcal{M}. \quad (7.12)$$

Due to Lemma 7.1 and Proposition 7.2 we have

$$g^{ab} = \sum_{k,l=1}^{\dim \mathcal{M}} \left. \frac{\partial N^a}{\partial \mu^k} (K^{-1})^{kl} \frac{\partial N^b}{\partial \mu^l} \right|_{\mathcal{M}_r}$$

$$= \sum_{k,l=1}^{r} \left. \frac{\partial N^a}{\partial \mu^k} (K^{-1})^{kl} \frac{\partial N^b}{\partial \mu^l} \right|_{\mathcal{M}_r}.$$
Both set of coordinates \( \mathcal{N} \) and \( \mu \) are local coordinates on \( \mathcal{M}_r \), and so \( J(\mu) = \det \left| \frac{\partial N^a}{\partial \mu^k} \right| \) does not degenerate at generic point of \( \mathcal{M}_r \). This proves non degeneracy.

The following identity along with Lemma 6.6 prove that \( \mu \) are indeed flat coordinates of metrics \( g \):

\[
\frac{\partial N^a}{\partial \mu^k} \Big|_{\mathcal{M}_r} = \frac{\partial N^a(\mu, \eta(\mu))}{\partial \mu^k} = \frac{\partial N^a}{\partial \mu^k} \Big|_{\mathcal{M}_r} + \sum_{i=r+1}^{\dim \mathcal{M}} \frac{\partial N^a}{\partial \eta^i} \frac{\partial \eta^i}{\partial \mu^k} \Big|_{\mathcal{M}_r} = \frac{\partial N^a}{\partial \mu^k} \Big|_{\mathcal{M}_r}.
\]

As a byproduct we conclude that brackets (7.6) and (7.11) define the same geometry on \( \mathcal{M}_r \):

\[
\{N^a(x), N^b(y)\}_2^* = \frac{\partial N^a}{\partial \mu^m}(x) \{\mu^m(x), \mu^n(y)\}_2^* \frac{\partial N^a}{\partial \mu^n}(y). \quad \text{on } \mathcal{M}_r \quad (7.13)
\]

This conclusion may be drawn also noting that both coordinates set are densities of mutually commuting integrals of corresponding exact brackets, the property that survives averaging.

**Corollary 7.2.** \( N^a\big|_{\mathcal{M}_r}(\mu) \) are invariant with respect to linear action of \( R \) group on \( \mathcal{M}_r \).

**Proof.** Indeed, due to Lemma 6.6, \( R \) admits a restriction of \( \mathcal{M}_r \) and preserves the latter. Hence, \( N^a \) were \( R \) invariant in \( \mathcal{M} \) and so they remain restricted on \( \mathcal{M}_r \).

Following K. Saito \[ Sa \] and using Theorem 7.2 we obtain the following

**Proposition 7.6.** Metrics \( \eta^{ab} \) is non degenerate.

**Proof.** Consider the following polynomial in \( \lambda \)

\[
P(\lambda) = \det \left| g^{ab}(\mathcal{N}) - \lambda \eta^{ab} \right| = \det \left| g^{ab}(N^1, \ldots, N^{r-1}, N^r - \lambda) \right| = \det \left| \eta^{ab}(N^r - \lambda)^n + \sum_{n=0}^{r-1} c_n (N^1, \ldots, N^{r-1}) (N^r - \lambda)^n \right|. \quad (7.14)
\]

When all \( \mathcal{N} \) but \( N^r \) vanish it simplifies to \( P(\lambda) = \det \left| \eta^{ab} \right| (N^r - \lambda)^n \). At this point \( J(\mu) \neq 0 \) and thus \( g^{ab} \) is non degenerate. Indeed, let \( \mu \) be eigenvector of some representative, which always exists, of \([w] \) in \( R \) with eigenvalue \( \xi = \exp[2i\pi/N] \). Then, due to homogeneity of \( N^a(\mu) \) and from their \( R \) invariance, we conclude that only variables of degree \( N \) may differ from zero at this point of \( \mathcal{M}_r \). If there are more than one such variable, then eigenvalue \( \xi \) is degenerate and we can always choose \( \mu \) so that only \( N^r \) does not vanish. Due to regularity of \([w] \), vector \( \mu \) is not left fixed by any transformation from \( R \), and thus the results follows.

**Proposition 7.7.** So obtained pencil of Hamiltonian structures provides us with quasi-homogeneous \[ Du \] flat pencil of metrics.

**Proof.** Let \( g \) be the metrics of the second Poisson structure and let \( \eta - \) of the first. Introduce function \( \tau = \mathcal{N}^1/N_w \), and introduce the following vector fields

\[
E^a = g^{ab} \partial_b \tau \quad e^a = \eta^{ab} \partial_b \tau. \quad (7.15)
\]
Notice, that with this choice $e^a = \delta_{a,r}$ and $\mathcal{L}_e g = \eta$ and $\mathcal{L}_e \eta = 0$ as follows from considerations above, and where we have assumed $\eta$ to be chosen anti diagonal with all nonzero entries being $N_w$. Then $E^a = \deg_{sc} (N^a)/N_w N^a$, as follows from (7.10). One sees immediately that $[e, E] = e$. The second Poisson structure is scaling weight graded and $N_w E$ is scaling weight Euler vector field. Thus $g$ must be an eigenvector of $\mathcal{L}_E : \mathcal{L}_E g = (d - 1)g$. In ref. [Du4] such flat pencils were called quasihomogeneous of degree $d$.

Theorem 7.3. Given $\{\cdot, \cdot\}_\lambda \star \lambda$ one may associate to it a solution to WDVV.

Proof. Since obtained pencil of Hamiltonian structures satisfies Jacobi identity we have a flat pencil of metrics. It is quasihomogeneous as was shown in proposition 7.7. Thus, following ref. [Du4], it is enough to show that the degree of quasihomogeneity $d \neq 1$.

As was said in the proof of proposition 7.7 $E^a = \deg_{sc} N^a/N_w N^a$. Due to quasihomogeneity of flat pencil we have $\mathcal{L}_E \eta = (d - 2)\eta$. But

$$\mathcal{L}_E \eta^{ab} = E^c \partial_c \eta^{ab} - \eta^{ac} \partial_c E^b - \eta^{cb} \partial_c E^a = \left(-\frac{\deg_{sc} N^b}{N_w} + \frac{\deg_{sc} N^a}{N_w}\right) \eta^{ab} = \frac{N_w + 2}{N_w} \eta^{ab}.$$ 

In the last line we have used the fact that $\eta^{ab}$ vanishes unless $\deg_{sc} (N^a N^b) = N_w + 2$, as follows from scaling weight grading of Poisson structures. From this we obtain

$$d = 1 - \frac{2}{N_w}.$$ 

Since $N_w$ is finite we obtain that $d < 1$. Note that for the Coxeter conjugacy class $N_w = h -$ Coxeter number, and we recover the formula of B. Dubrovin, obtained while constructing polynomial solutions to WDVV equations on the orbits of Coxeter groups [Du2].

Practically, we can find Frobenius potential $F(N)$ from the following relations

$$g^{ab} (N) = (d - 1 - d_a - d_b) \eta^{ac} \eta^{bd} \partial_c \partial_d F (N),$$

$$\Gamma_c^{ab} = \left(\frac{3 - d}{2} - d_a\right) \eta^{ad} \eta^{bf} \partial_d \partial_f \partial_c F,$$  

(7.16)

where $d_a \delta_a^b = \partial_a E^b$.

Thus, $N^a$ are Saito coordinates [Sa] on Frobenius manifold [Du3] being flat coordinates for the metric $\eta$. However, flat coordinates $\mu$ of the intersection metrics $g$ are also very important. They clarify the geometric origin of the Frobenius structure. In general it is a challenging task, given a flat metric, to find its flat coordinates. But in the case in question we were lucky to use the theory of integrable systems.

8 Example: $[w] = D_4 (a_1)$

To illustrate the developed technique we consider the example served as the motivation of the present work. Let $g = D_4$ – the simplest classical Lie algebra where non
Coxeter primitive conjugacy class occurs (see appendix [A]). Luckily it enjoys regularity property.

Take \( \{w\} = D_4(a_1) \). We have readily that \( I(w) = (1,1,3,3) \) and the set of conformal weights \( P_{w} = (1,1,1,2,3,3) \). Miura coordinates form a Kac Moody algebra of \( \mathfrak{g}_0(s_w) = u(1) \otimes \mathfrak{su}(2) \), thus we shall have three exact Casimirs and one will be computed in dispersion parameter expansion.

Positive roots of \( D_4 \) read

\[
\mathcal{R}_+ = \{ \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_2 + \alpha_4, \alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4 \}.
\]

Let us denote Lie algebra elements \( E_\alpha, \alpha \in \mathcal{R}_+ \) by its decomposition on simple roots. So if \( \alpha = \alpha_{\max} \) we write \( X_{12234} \), and for \( E_{\alpha_1+\alpha_2} \) write \( X_{12} \). Similarly for negative roots, substituting \( X \) with \( Y \). We fix Cartan-Weyl basis (2.1). So

\[
\rho = \sum_{i,j=1}^{r} K_{ij}^{-1} s_j H_{\alpha_i} = 2H_{\alpha_1} + 3H_{\alpha_2} + 2H_{\alpha_3} + 2H_{\alpha_4}.
\]

The Heisenberg subalgebra \( \mathcal{H}^{[w]} \) is spanned by \( z^k \Lambda_{1,1}, z^k \Lambda_{1,2} \) for \( i \in \{1,3\} \) and \( k \in \mathbb{Z} \).

\[
\begin{align*}
\Lambda_{1,1} &= X_1 + X_3 + zY_{12234} + X_{12} + X_{23} + X_{24}, \\
\Lambda_{2,1} &= X_1 - X_3 + X_4 - zY_{1234} - X_{12} + X_{23}, \\
\Lambda_{3,1} &= X_{1234} - \frac{z}{2} (Y_1 - Y_3 + 2Y_4 - Y_{12} + Y_{23}), \\
\Lambda_{3,2} &= -X_{12234} - \frac{z}{2} (Y_1 + Y_3 + Y_{12} + Y_{23} + 2Y_{24}).
\end{align*}
\]

These basis was just guessed, verifying linear independence and regularity. It turned out easier than proceed as in (2.4).

Let us choose \( \Lambda = \Lambda_{1,1} \) for our integrable hierarchy. The \( sl_2 \) subalgebra constituents read \( \rho, I_+, I_- \), where \( I_- = 3Y_1 + 3Y_3 + Y_{12} + Y_{23} + 4Y_{24} \) and \( I_+ = P_0^{\nu} \Lambda \).

The hierarchy Let us fix the minimal weight gauge (8.7) as follows

\[
q^{can} = \left( w_2 - \frac{1}{10} u_2 - \frac{1}{2} v_2 \right) \frac{1}{12} I_- + u_2 \frac{1}{20} (3Y_1 - 3Y_3 + 12Y_4) +
+ v_2 \frac{1}{20} (3Y_1 + 3Y_3 + 4Y_{24}) + w_3 \frac{1}{6} (Y_{123} - Y_{234} - 4Y_{124}) +
+ Y_{1234} u_4 + Y_{12234} w_4.
\]

Hamiltonians, annihilators of the first Poisson structure, read

\[
\begin{align*}
H_{\Lambda_{1,1}} &= \int dx \, w_2, \\
H_{\Lambda_{1,2}} &= \int dx \, u_2, \\
H_{\Lambda_{3,1}} &= \int dx \left( -u_4 + \frac{37}{3} u_2^2 - \frac{1}{6} w_2 u_2 + \frac{1}{6} w_2^2 + \frac{1}{6} w_2 v_2 -
- \frac{7}{6} u_2 v_2 + \frac{7}{12} v_2^2 \right), \\
H_{\Lambda_{3,2}} &= \int dx \left( w_4 - \frac{7}{3} u_2^2 + \frac{7}{24} w_2 u_2 + \frac{7}{3} u_2 v_2 \right).
\end{align*}
\]
We shall denote the densities of this annihilators as $N_1, N_2, N_3$ and $N_4$ respectively.

As was explained in section 5, fixing the gauge somehow, we are able to compute both Poisson structures exactly, but the output is enormous to be presented here. On the other hand we need then to pass to $\mathcal{N}, w_3, v_2$ coordinates and eliminate auxiliary coordinates $w_3$ and $v_2$. To do so we need to know $\mathcal{W}$ algebra at least up to $O(\epsilon)$ after rescaling. But we choose to omit these intermediate steps due to space restrictions and present the answer.

The simplest Hamiltonian generating $G$ terms is $H_{A_{1,2}}$. Its flows with respect to the second Hamiltonian structure read

$$\frac{\partial N_1}{\partial t_{\Lambda_{1,2}}} = N_2^2, \quad \frac{\partial N_2}{\partial t_{\Lambda_{1,2}}} = \frac{1}{3} (v_2 - N_2 + 5N_1)' ,$$

$$\frac{\partial N_3}{\partial t_{\Lambda_{1,2}}} = \frac{1}{3} \left( N_1N_2 + 3N_4 - \frac{1}{4}N_2^2 + \frac{1}{4}v_2N_2 + \frac{2}{5}w_3' \right)' ,$$

$$\frac{\partial N_4}{\partial t_{\Lambda_{1,2}}} = \frac{1}{3} \left( N_3 + \frac{1}{4}N_2^2 + \frac{1}{12}N_1(N_1 - N_2 + v_2) \right)' ,$$

$$\frac{\partial v_2}{\partial t_{\Lambda_{1,2}}} = \frac{12}{\epsilon} w_3 + \frac{1}{3} (5N_1 + 22N_2 + 2v_2)' ,$$

$$\frac{\partial w_3}{\partial t_{\Lambda_{1,2}}} = \frac{1}{\epsilon} \left( -4N_3 + \frac{1}{2}v_2^2 + \frac{3}{4}N_1v_2 - \frac{3}{4}N_1N_2 - \frac{1}{6}N_2v_2 - \frac{1}{6}N_2^2 + \frac{1}{4}N_1^2 \right) +$$

$$+ \frac{3}{20} \left( N_2 - v_2 \right)'' .$$

(8.3)

Following the recipe, we take $1/\epsilon$ terms as constraints. Using them we obtain equation for the phase space subvariety $\mathcal{M}_r$ of slow motion

$$w_3 = 0, \quad v_2 = N_2 - 2N_1 \pm \Delta, \quad \Delta = \sqrt{N_1^2 + 3N_2^2 + 48N_3} .$$

(8.4)

This finally leads to the following restricted bi Hamiltonian structure

$$\{N_i(x), N_j(y)\}^*_1 = 4 \delta_{i+j,5} \delta'(x - y) ,$$

$$\{N_i(x), N_j(y)\}^*_2 = \left[ \gamma^{ij} (N(x)) + \gamma^{ji} (N(y)) \right] \delta'(x - y) ,$$

(8.5)

where matrix $\gamma(N)$ reads

$$\gamma^{1,1} = \{N_1, N_2, 3N_3, 3N_4\}, \quad \gamma^{1,1} = \{N_1, N_2, N_3, N_4\},$$

$$\gamma^{2,2} = \frac{1}{3} (N_1 + 2\Delta), \quad \gamma^{2,3} = N_4 + \frac{1}{6}N_1N_2 + \frac{1}{12}N_2\Delta, \quad \gamma^{3,2} = 3\gamma^{2,3},$$

$$\gamma^{2,4} = \frac{1}{3} \left( N_3 - \frac{1}{12}N_2^2 + \frac{1}{4}N_2^2 + \frac{1}{12}N_1\Delta \right), \quad \gamma^{4,2} = 3\gamma^{2,4},$$

$$\gamma^{3,3} = \frac{1}{2} \left( N_1N_3 + \frac{3}{32}N_1N_4^2 + \frac{7}{288}N_4^3 \right) + \frac{1}{288} (N_1^2 + 12N_2^2 + 48N_3) \Delta ,$$

$$\gamma^{4,3} = \frac{1}{2} \left( N_2N_3 + \frac{7}{96}N_2N_4^2 + \frac{1}{32}N_2^3 + \frac{1}{48}N_4N_2\Delta \right), \quad \gamma^{3,4} = \gamma^{4,3},$$

$$\gamma^{4,4} = \frac{1}{6} \left( -N_1N_3 + \frac{19}{288}N_1^3 + \frac{7}{32}N_1N_2^2 + \frac{1}{144} (4N_1^2 + 3N_2^2 + 48N_3) \Delta \right).$$

The metric $g = \gamma + \gamma^{tr}$ is invertible, can be checked to be flat and forms, obviously, together with $\eta$ a flat pencil.
Modified hierarchy  We consider the modified hierarchy to exemplify discrete group $R$. Choosing coordinates as in (6.13), but indexing them with subscript to facilitate reading of following formulae, we have

$$K^{-1} = \left( \begin{array}{cc} K_{4 	imes 4}^{-1} & 0 \\ 0 & 1 \\ 1 & 0 \end{array} \right), \quad f \cdot \nu = \left( \begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\nu_5 & \nu_6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \nu_5 & 0 & 0 & \omega \\ 0 & -\nu_6 & 0 & 0 & -\omega \end{array} \right),$$

where $\omega = (\nu_1 - 2\nu_2 + \nu_3 + \nu_4)$. Notice, that quadratic Casimir of $su(2)$ constituent of $\mathfrak{g}_0(\mathfrak{s}_w)$ read

$$J_2 = (\nu_1 - 2\nu_2 + \nu_3 + \nu_4)^2 + 4\nu_5\nu_6. \quad (8.6)$$

According to (6.18) we introduce “abelian” coordinates $\mu$:

$$\mu_i = \nu_i, \quad i = 1, 3, 4,$$

$$(\mu_1 - 2\mu_2 + \mu_3 + \mu_4)^2 = (\nu_1 - 2\nu_2 + \nu_3 + \nu_4)^2 + 4\nu_5\nu_6. \quad (8.7)$$

It is invariant with respect to $su(2)$ Weyl group, acting on $\mu$ according to (6.16):

$$\mu_i \rightarrow \mu_i, \quad i = 1, 3, 4, \quad \mu_2 \rightarrow \mu_1 + \mu_3 + \mu_4 - \mu_2. \quad (8.8)$$

Then group $R$ are generated by the following elementary reflections corresponding to three simple roots of $\mathfrak{s}_w$ degree one: $\alpha_1, \alpha_3$ and $\alpha_4$. Each corresponds gauge transformation (6.4) with $n = x_1 E_{-\alpha_k} + x_2 E_{-\alpha_k + \alpha_2}$, with $k = 1, 3, 4$. For each $x$ we find quadratic equation (6.13b) and thus we obtain, for instance, the following transformation for $\alpha_1$:

$$R_1^\pm : \nu_1 \rightarrow \frac{\nu_1 + \nu_3 + \nu_4 \pm \sqrt{J_2}}{2}, \quad \nu_3 \rightarrow \nu_3, \quad \nu_4 \rightarrow \nu_4,$$

$$R_1^\pm : \nu_2 \rightarrow \frac{-1}{2 (\omega + \nu_5 - \nu_6)} (-2\nu_2 \nu_5 - 2 (2\nu_2 - 2 (\nu_3 - \nu_4) + \nu_5) \nu_6 +$$

$$+ (\nu_2 - 2 (\nu_3 + \nu_4) - 3\nu_6) \omega + \omega^2 \pm \sqrt{J_2} (\nu_1 + \nu_2 - \nu_3 - \nu_4 + \nu_6)), \quad R_1^\pm : \nu_5 \rightarrow \frac{2 (\nu_1 - \nu_2 + \nu_5) (\omega + 2 \nu_5 \pm \sqrt{J_2})}{2 (\omega + \nu_5 - \nu_6)},$$

$$R_1^\pm : \nu_6 \rightarrow \frac{(\nu_1 + \nu_2 - \nu_3 - \nu_4 + \nu_6) (\omega - 2\nu_6 \pm \sqrt{J_2})}{2 (\omega + \nu_5 - \nu_6)}.$$

As illustration to Lemma 6.7 note that

$$J_2 (R_1^\pm (\nu)) = \left( \frac{3\mu_1 - \mu_3 - \mu_4 \pm \sqrt{J_2}}{2} \right)^2.$$

One can check that $R_1^\pm$ are not reflections, because, for example,

$$\left( R_1^+ \right)^2 = \begin{cases} \text{Id}_{Q_\omega} & \text{if } 3\nu_1 - \nu_3 - \nu_4 \pm \sqrt{J_2} > 0, \\
R_1^- & \text{otherwise.} \end{cases}$$
We can get rid of sign ±, by making use of Weyl group \( \mathcal{W}(g_0) \) of \( sl(2) \) action \((8.8)\). We define action of \( R_1 \) as \( R_1^* \) for \( \nu_2, \nu_5, \nu_6 \) variables, choosing the following solution of eq. \((8.7)\)

\[
\mu_2 = \frac{1}{2} \left\{ \sqrt{J_2 + \mu_1 + \mu_3 + \mu_4} \right\}.
\]

This choice yields for \( \mu \) coordinates

\[
R_1: \mu_2 \to \mu_2, \quad \mu_3 \to \mu_3, \quad \mu_4 \to \mu_4, \quad \mu_1 \to -\mu_1 + \mu_2.
\]

Then \( R_1^* = R_2 R_1 R_2 \), where \( R_2 \) acts on \( \mu \) variables by \((8.8)\) and acts trivially on \( \nu_{2,5,6} \). It can be explicitly checked that so defined operation \( R_1 \) is a reflection \( R_1^2 = Id \). It means that we have well defined reflections on Riemann surface \((8.7)\) over \( Q_m^* \).

The same way we define reflections \( R_3 \) and \( R_4 \). Recall that shortcut \( \omega \) stands for \( \omega = \mu_1 + \mu_3 + \mu_4 + 2\nu_2 \)

\[
R_3: \mu_1 \to \mu_1, \quad \mu_2 \to \mu_2, \quad \mu_4 \to \mu_4, \quad \mu_3 \to -\mu_3 + \mu_2, \quad 
R_3: \nu_2 \to \frac{1}{2 (\omega - \nu_5 + \nu_6)} \left[ 2\nu_5 \nu_6 + \nu_6 \omega + \omega^2 - 2\mu_3 (2\nu_6 + \omega) + \nu_2 (-2\nu_5 + 4\nu_6 + 3\omega) + \sqrt{J_2} (\mu_1 - \nu_2 - \mu_3 + \mu_4 + \nu_6) \right],
R_3: \nu_5 \to \frac{2\mu_3 - \nu_2 - \nu_5}{2 (\omega - \nu_5 + \nu_6)} \left( -\omega + 2\nu_5 + \sqrt{J_2} \right),
R_3: \nu_6 \to \frac{(\mu_1 - \nu_2 - \mu_3 + \mu_4 + \nu_6)}{2 (\omega - \nu_5 + \nu_6)} (\omega + 2\nu_5 + \sqrt{J_2}).
\]

\[
R_4: \mu_1 \to \mu_1, \quad \mu_2 \to \mu_2, \quad \mu_4 \to \mu_4, \quad \mu_3 \to -\mu_3 + \mu_2,
R_4: \nu_2 \to \frac{\mu_1 + 2\nu_2 + \mu_3 - 3\mu_4 + \sqrt{J_2}}{2}, \quad \nu_6 \to \nu_6,
R_4: \nu_5 \to \frac{(-\nu_2 + 2\mu_4)}{2\nu_6} \left( \sqrt{J_2} - \omega \right).
\]

Notice, that \( \mu_k = \nu_k \) for \( k = 1, 3, 4 \) and we obtain that \( R \) acts linearly on these \( \mu \). It is easy to check for these linear transformations, and it certainly needs symbolic computation program to verify that

\[
R_k^2 = 1, \quad (R_k R_4)^2 = (R_3 R_4)^2 = 1, \quad (R_2 R_1)^3 = (R_2 R_3)^3 = (R_2 R_4)^3 = 1.
\]

That is they generate Weyl group of \( D_4 \) Lie algebra.

**Solution to WDVV** By theorem \((7.3)\) we can extract a solution to WDVV from bi Hamiltonian structure \((8.5)\). Since scaling degree of fields \( N_1, \ldots, N_4 \) are \( 2, 2, 4, 4 \) respectively, we can find Euler vector field

\[
E = \left( \frac{1}{2} N_1, \frac{1}{2} N_2, N_3, N_3 \right),
\]

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and the grade \( d = 1/2 \). Following [Du4] Frobenius potential \( F(N) \) can be extracted from (7.16). It should be noted that \( \Gamma_{k_j} = \partial_{k_j} \gamma^{j,i} \), making two relations equivalent. We thus find the claimed free energy

\[
\frac{1}{4} F(N) = N_2 N_3 N_4 + \frac{1}{2} N_1 N_4^2 + \frac{\Delta_5^6}{2^5 \cdot 3^4 \cdot 5} + \frac{1}{6} N_1 N_3^2 - \frac{1}{108} N_3 N_1^3 + \frac{19}{2^7 \cdot 3^5} N_1^2 N_2^2 + \frac{1}{3 \cdot 2^8} N_1 N_2^2 \cdot \tag{8.9}
\]

It may be explicitly checked to verify WDVV equations (1.1).

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**A Regular primitive conjugacy classes and their properties**

Here we collect information about regular primitive conjugacy classes of Weyl group \( \mathcal{W}(g) \) for simple Lie algebras \( g \). Classes are labeled by the type of Coxeter diagram. Recall, that it coincides with the Dynkin diagram for the Coxeter conjugacy class.

\( A_n \)

\[
[w] = A_n, \quad N_w = n + 1 \quad s_w = s_p \\
I(w) = (1, 2, 3, \ldots, n - 1, n) = \Pr_w \quad \hat{g}_0(s_w) = u(1)^{\otimes n}
\]

\( B_n, C_n \)

\[
[w] = B_n, C_n, \quad N_w = 2n \quad s_w = s_p \\
I(w) = (1, 3, 5, \ldots, 2n - 1) = \Pr_w \quad \hat{g}_0(s_w) = u(1)^{\otimes n}
\]

\( D_n \)

We pick four last roots to form \( D_4 \) subalgebra.

\[
[w] = D_n, \quad N_w = 2n - 2 \quad s_w = s_p \\
I(w) = (1, 3, \ldots, 2n - 1; n - 1) = \Pr_w \quad \hat{g}_0(s_w) = u(1)^{\otimes n}
\]

\[
[w] = D_{2n}(a_{n-1}), D_{2n}(b_{n-1}) \quad N_w = 2n \\
s_w = (1, 1, 0, 1, 0, \ldots, 1, 0, 1, 1)_{2n-2\text{times}} \\
I(w) = (1, 1, 3, 3, \ldots, 2n - 3, 2n - 3, 2n - 1, 2n - 1)_{2n \text{ numbers}}
\]
\[
\Pr_w = \left(1,1,2,\ldots, \frac{2n-3}{2}, \frac{2n-3}{2}, \frac{2n-3}{2}, \frac{2n-2}{2}, \frac{2n-1}{2}, \frac{2n-1}{2}\right)
\] 
\text{groups in 4 elements}

\hat{g}_0(s_w) = u(1)^{\otimes n+1} \oplus su(2)^{\otimes n-1}

G_2

\begin{align*}
[w] &= G_2 \quad N_w = 6 \quad s_w = s_p \\
I(w) &= (1,5) = \Pr_w \quad \hat{g}_0(s_w) = u(1)^{\otimes 2}
\end{align*}

F_4 \quad \alpha_1^2 = \alpha_2^2 = 2, \alpha_3^2 = \alpha_4^2 = 4 \text{ with double bond between the second and the third roots.}

\begin{align*}
[w] &= F_4 \quad N_w = 12 \quad s_w = s_p \\
I(w) &= (1,5,7,11) = \Pr_w \quad \hat{g}_0(s_w) = u(1)^{\otimes 4}
\end{align*}

E_6 \quad \text{Roots } \alpha_6, \alpha_2, \alpha_3, \alpha_4 \text{ form } D_4 \text{ subalgebra.}

\begin{align*}
[w] &= E_6 \quad N_w = 12 \quad s_w = s_p \\
I(w) &= (1,4,5,7,8,11) = \Pr_w \quad \hat{g}_0(s_w) = u(1)^{\otimes 6}
\end{align*}

E_7

\begin{align*}
[w] &= E_7 \quad N_w = 18 \quad s_w = s_p \\
I(w) &= (1,5,7,9,11,13,17) = \Pr_w \quad \hat{g}_0(s_w) = u(1)^{\otimes 7}
\end{align*}
\[
[w] = E_7(a_1) \quad N_w = 14
\]
\[
s_w = (1, 1, 1, 0, 1, 1, 1)
\]
\[
I(w) = (1, 3, 5, 7, 9, 11, 13)
\]
\[
Pr_w = (1, 3, 5, 5, 7, 8, 9, 11, 13)
\]
\[
\hat{g}_0(s_w) = u(1)^{\otimes 6} \oplus su(2)
\]

\[
[w] = E_7(a_4) \quad N_w = 6
\]
\[
s_w = (1, 0, 0, 1, 0, 1, 0)
\]
\[
I(w) = (1, 1, 1, 3, 5, 5)
\]
\[
Pr_w = (1, 1, 1, 1, 1, 2, 2, 2, 3, 3, 3, 3, 4, 4, 4, 5, 5, 5)
\]
\[
\hat{g}_0(s_w) = u(1)^{\otimes 2} \oplus su(2) \oplus su(3)^{\otimes 2}
\]

\[
E_8
\]
\[
[w] = E_8 \quad N_w = 30 \quad s_w = s_p
\]
\[
I(w) = (1, 7, 11, 13, 17, 19, 23, 29) = Pr_w \quad \hat{g}_0(s_w) = u(1)^{\otimes 8}
\]

\[
[w] = E_8(a_1) \quad N_w = 24
\]
\[
s_w = (1, 1, 1, 0, 1, 1, 1, 1)
\]
\[
I(w) = (1, 5, 7, 11, 13, 17, 19, 23)
\]
\[
Pr_w = (1, 5, 7, 9, 11, 13, 14, 17, 19, 23)
\]
\[
\hat{g}_0(s_w) = u(1)^{\otimes 7} \oplus su(2)
\]

\[
[w] = E_8(a_2) \quad N_w = 20
\]
\[
s_w = (1, 1, 1, 0, 1, 0, 1, 1, 1)
\]
\[
I(w) = (1, 3, 7, 9, 11, 13, 17, 19)
\]
\[
Pr_w = (1, 3, 5, 7, 8, 9, 11, 11, 13, 14, 17, 19)
\]
\[
\hat{g}_0(s_w) = u(1)^{\otimes 6} \oplus su(2)^{\otimes 2}
\]

\[
[w] = E_8(a_3), E_8(b_3) \quad N_w = 12
\]
\[
s_w = (1, 1, 0, 1, 0, 0, 1, 0, 0)
\]
\[
I(w) = (1, 1, 5, 5, 7, 7, 11, 11)
\]
\[
Pr_w = (1, 1, 1, 2, 3, 4, 5, 5, 5, 6, 7, 7, 7, 8, 9, 10, 11, 11)
\]
\[
\hat{g}_0(s_w) = u(1)^{\otimes 3} \oplus su(2)^{\otimes 3} \oplus su(3)
\]

\[
[w] = E_8(a_5), E_8(b_5) \quad N_w = 15
\]
\[
s_w = (1, 1, 0, 1, 0, 0, 1, 0, 0)
\]
\[
I(w) = (1, 2, 4, 7, 8, 11, 13, 14)
\]
\[
Pr_w = (1, 2, 3, 4, 5, 5, 7, 7, 7, 8, 9, 9, 11, 11, 13, 14)
\]
\[
\hat{g}_0(s_w) = u(1)^{\otimes 4} \oplus su(2)^{\otimes 4}
\]
\[ [w] = E_8(\alpha_6) \quad N_w = 10 \]
\[ s_w = (1, 0, 0, 1, 0, 1, 0, 0) \]
\[ I(w) = \begin{pmatrix} 1 & 1 & 3 & 3 & 7 & 7 & 9 & 9 \end{pmatrix} \]
\[ Pt_w = \begin{pmatrix} (1, 1, 1, 2, 3, 3, 3, 4, 4, 4, 5, 5, 5, 6, 6, 7, 7, 8, 9, 9) \end{pmatrix} \]
\[ \hat{g}_0(s_w) = u(1) \otimes^2 su(2)^{\otimes 2} \oplus su(3)^{\otimes 2} \]

\[ [w] = E_8(\alpha_8) \quad N_w = 6 \]
\[ s_w = (1, 0, 0, 0, 1, 0, 0, 0) \]
\[ I(w) = \begin{pmatrix} 1 & 1 & 1 & 1 & 5 & 5 & 5 \end{pmatrix} \]
\[ Pt_w = (1, \ldots, 1, 2, \ldots, 2, 3, \ldots, 3, 4, \ldots, 4, 5, \ldots, 5) \]
\[ \hat{g}_0(s_w) = u(1) \oplus su(4) \oplus su(5) \]

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