A SURVEY OF THE NUMBER OF SUPERSOLVABLE SUBGROUPS OF FINITE GROUPS

Primitivo B. Acosta-Humánez
Instituto de Matemática & Escuela de Matemática
Universidad Autónoma de Santo Domingo
Dominican Republic
pacosta-humanez@uasd.edu.do

Orieta Liriano
Instituto de Matemática & Escuela de Matemática
Universidad Autónoma de Santo Domingo
Ciclo de Estudios Generales
Universidad Iberoamericana
Dominican Republic
oliriano25@uasd.edu.do

Francis Mora-Ferreras
Instituto de Matemática
Universidad Autónoma de Santo Domingo
Dominican Republic
Instituto de Matemáticas
Universidad de Talca, Chile
fmora@inst-mat.utalca.cl

ABSTRACT
In this paper we survey a new criteria for solvability of finite groups in terms of number of supersolvable (also known as polycyclic) and non-supersolvable subgroups. In particular, we present original examples of supersolvable groups such as non-abelian groups with two cyclic generators. Additionally, we present some examples and applications in GAP system providing some description about recent criteria for solvability and properties of finite groups based on the number of supersolvable and non-supersolvable subgroups of a finite group $G$.

Keywords Finite groups · Polycyclic groups · Solvable groups · Supersolvable groups

Contents

1 Introduction 2

2 Preliminaries about classical Group Theory 3
  2.1 About $p$-groups and $p$-subgroups 4
  2.2 Homomorphisms 4
  2.3 Group action 5
  2.4 Semidirect product 5

3 Solvable and nilpotent groups 6
  3.1 Nilpotent groups 6
  3.2 Solvable groups 7

4 Supersolvable Groups and Zarrin Criteria for Solvable Groups 8
1 Introduction

Since the creation of the theory of groups, it is known as one of the most abstract theories of mathematics. However, owing to the development of computational applications to deal with problems associated with group theory, it has had a great boom in recent decades. Some researchers called this combination of group theory and computational applications as Computational Group Theory. Combinatorial aspects of Group Theory, such as the computation of the number of subgroups of certain type, has been developed by some researchers in recent decades and they called it Combinatorial Group Theory. In this sense, this paper falls in the topics Combinatorial Group Theory and Computational Group Theory.

Using the free access tool [zentralblatt] we found 585 documents on Combinatorial Group Theory, among which there are 272 papers, 226 chapter of books and 84 books. Likewise, a search was made for the theory of Computational Groups and 320 documents were found, among which are 139 papers, 150 chapter of books and 31 books. The intersection of the two sets is 14 documents, 1 book, and 14 papers in special volumes. Thus, the theory of finite groups is an important topic in mathematics because from the theoretical point of view it lead to new developments in other in mathematics and also it has been applied successfully in other areas such as physics, chemistry among others.

On the other hand, since the seminal work of E. Galois, in where the concept of Solvable Group, which involves abelian groups, was introduced, see [1][2], the solvability of groups has been deeply studied for a plenty of researchers in algebra, see [3]. In the Century XX was introduced an intermediate category of groups between abelian and solvable groups, the so-called supersolvable (see [4]), supersoluble ([5]) or polycyclic groups (see [6]).

In 1924 Schmidt [7] studied non nilpotent groups whose proper subgroups are nilpotent and proved that these groups are solvable. Such groups are now known as Schmidt groups. Rédei in [8] and Ballester-Bolinches, Esteban-Romero and Robinson in [9] classify Schmidt groups. In [9] they are called minimal non-nilpotent groups.

Then, later in 2012, Zarrin [10] extends Schmidt’s result and characterizes the semi-simple groups $G$ with at most 65 non-nilpotent subgroups. For them Zarrin introduces the class $S^n$; which is the class of groups with exactly $n$ non-nilpotent subgroups, i.e. $G$ is an $S^n$-group if it has $n$ non-nilpotent subgroups or $G \in S^n$. Zarrin proves the following results in [10]. Given $G$ a finite group that is a $S^n$-group with $n \leq 22$. Then $G$ is solvable and $G$ is isomorphic to alternating group of degree 5. Also if $G$ is a alternating group of degree 5, symmetric group of degree 5, or the special linear group of degree 2 over a field $\mathbb{F}_7$ is the set of $2 \times 2$ matrices with determinant 1.

Based on Zarrin’s results on [10], he obtains that the only simple groups with exactly 22 non-nilpotent subgroups is alternating group of degree 5 and that the only simple group with exactly 65 non-nilpotent subgroups is the special linear group of degree 2 over a field $\mathbb{F}_7$. A corollary to this is that if a non-solvable group $G$ is an $S^n$-group with $n \leq 22$, then $G/\Phi(G)$ is isomorphic to alternating group of degree 5.

In this same paper Zarrin [10] conjectured that a finite non-solvable group that is a $S^{22}$-group is isomorphic to alternating group of degree 5 or the special linear group of degree 2 over a field $\mathbb{F}_7$.

This conjecture was proven by Ballester-Bolinches, Esteban-Romero and Lu in 2017 in [11] Theorem A. That is, they proved the following result : given $G$ be a solvable group. Then $G$ has exactly 22 non-nilpotent subgroups if and only if it is isomorphic to alternating group of degree 5 or the special linear group of degree 2 over a field $\mathbb{F}_7$. 

P. ACOSTA-HUMÁNEZ, J. COLLADO, O. LIRIANO & F. MORA-FERRERAS - SUPERSOLVABLE GROUPS
This paper is structured as follows. Section 2 contains some theoretical aspects concerning basic group theory, necessaries to understand the rest of the paper. Section 3 contains concepts supersolvable groups and some remarkable results which are well known in this theory. Section 4 contains some relevant aspects of supersolvable groups and Zarrin criteria to obtain solvable groups. Finally, in Section 5 we introduce new results concerning the supersolvability of non-abelian groups generated by two cyclic generators and we included some programs developed in GAP system.

2 Preliminaries about classical Group Theory

This section contains the basic theoretical background that every undergraduate student of algebra knows. We start introducing concepts such as groups, subgroups, normal subgroups, homomorphisms, solvable groups. We follows some classical books such as Our start point is the well known book written by Doerk K. and Hawkes T., see [12, pag. 1].

In this way, a group is a non-empty set $G$ equipped with a binary operation which is associative

$$g(hw) = (gh)w \quad \text{for all } g, h, w \in G$$

(which we will usually denote multiplicative by juxtaposition) with the property that $G$ has an identity element $1 = 1_G$ that satisfies

$$1g = g1 = g \quad \text{for all } g \in G$$

and for each $g$ of $G$ has right and left inverse $g^{-1}$ satisfying

$$g^{-1}g = gg^{-1} = 1$$

If $hg = gh$ for all $g, h \in G$, we say that $G$ is an abelian group. The order of $G$ is the cardinality of the set and is written $|G|$. If $|G|$ is finite, we say that $G$ is finite.

Throughout this paper, $G$ will always denote a finite group.

A subset $U$ of $G$ is called subgroup if it is a group with respect to the binary operation defined on $G$: for this we write $U \leq G$ and $U < G$ when $U \neq G$. If $U < G$, we call $U$ a proper subgroup of $G$. We will use the symbol 1 to denote the identity subgroup $\{1\}$ of a group. If $U$ is a proper subgroup with the property that $U = V$ whenever $U \leq V < G$, we call $U$ a maximal subgroup of $G$; therefore, the maximal subgroups are precisely the maximal elements of the set of partially ordered proper subgroups by inclusion of $G$.

If $U \leq G$ and $g \in G$, we write $Ug$ instead of $U\{g\}$ and call $Ug$ a right coset of $U$ in $G$, the left cosets are defined analogously. Calling two elements $g$ and $h$ of $G$ equivalent if and only if $hg^{-1} \in U$, we obtain an equivalence relation on $G$ whose equivalence classes are exactly the right cosets of $U$ in $G$ (The set of right cosets of $U$ in $G$ will be denoted by $G/U$). It follows that these right cosets form a partition of $G$ and, in particular, that there exists a subset $T$ of $G$ such that

$$G = \bigcup_{t \in T} Ut, \quad y \quad Us \cap Ut = \emptyset$$

provided that $s, t \in T$ and $s \neq t$. This partition is called the right lateral decomposition of $G$ by $U$. A set $T$ of the type just described is called a right transversal of $U$ into $G$; is a set that contains exactly one element from each right coset, so the number of right cosets is $\prod_{t \in T} |Ut|

Example 1. Consider

$$G = D_8 := \langle s, r \mid r^4 = s^2 = 1, srs^{-1} = r^{-1} \rangle = \{1, r, r^2, r^3, s, sr, sr^2, sr^3\}$$

the dihedral group of order 8 and consider $\langle r^2 \rangle = \{1, r^2\}$.

Therefore, we observe

$$D_8/ \langle r^2 \rangle = \{g \{1, r^2\} : g \in D_8\} = \{\{1, r^2\}, \{r, r^3\}, \{s, sr^2\}, \{sr, sr^3\}\}$$

That is, there are 4 left cosets. Moreover, such left cosets are explicitly given by

$$D_8 = \{1, r^2\} \cup \{r, r^3\} \cup \{s, sr^2\} \cup \{sr, sr^3\}$$
There is a corresponding partition of $G$ into the left cosets of $U$, and a complete set of representatives of the left cosets is called a left traverse of $U$ into $G$.

The mapping $Ug \to g^{-1}U$ is a bijection from the set $G/U$ of right cosets to the set of left cosets of $U$ in $G$; the common cardinality of these two sets is called the index of $U$ in $G$ and is written $|G : U|$.

If $g \in G$, the mapping $u \to ug$ is a mapping from $U$ to the coset $Ug$. Thus, all right cosets have the cardinality $|U|$, and we get the following famous Lagrange theorem, see [12, Theorem 1.4, pag. 3].

**Theorem 1** (Lagrange’s Theorem). If $U$ is a subgroup of a group $G$, then $|G| = |G : U||U|$. In particular, if $G$ is finite, $|U|$ is a divisor of $|G|$.

From the previous example we have $|D_8| = |D_8 : \langle r^2 \rangle||\langle r^2 \rangle| = |D_8/\langle r^2 \rangle| = 2 \cdot 2$.

If $g$ is an element of a finite group $G$, there exists a smallest natural number $n$ such that $g^n = 1$; this is called the order of $g$ and is written $o(g)$. It is easy to see that $o(g) = |\langle g \rangle|$, the order of the cyclic subgroup generated by $g$, and therefore $o(g)$ divides $|G|$. The least common multiple of the integers $\{o(g) : g \in G\}$ is called the exponent of $G$ and is written $\text{Exp}(G)$.

### 2.1 About $p$-groups and $p$-subgroups

If $p$ is a prime number, we say that $G$ is a $p$-group if $G$ is finite and $|G| = p^n$ for some $n \in \mathbb{N}$. If we have a finite group $G$ and a prime number $p$, a $p$-subgroup of $G$ is a subgroup $H$ of $G$ whose order is a power of $p$ ($|H| = p^m$ for some $m \in \mathbb{N}$). If $|G| = p^n m$ where $p$ does not divide $m$, then a subgroup $H$ of order $p^n$ is called a $p$-Sylow subgroup of $G$.

The set of Sylow $p$-subgroups is denoted by $\text{Syl}_p(G)$. An almost immediate result is that if $H$ is a subgroup of a finite group $G$. Then $H \subseteq \text{Syl}_p(G)$ if and only if $|H|$ is a power of $p$ and also $|G : H|$ is not divisible by $p$.

**Theorem 2** (Sylow. Existence [13]). Let $G$ be a finite group. Then $\text{Syl}_p(G)$ is not an empty set.

A corollary that follows from this theorem and is attributed to Cauchy is that if $G$ is a finite group and $p$ is a prime that divides $|G|$, then there exists $x \in G$ of order $p$. Also a result that is not very difficult to prove is that for $p$ prime and $p^n \mid |G|$, then there exists a subgroup $H$ of $G$ of order $p^n$, since if $P \in \text{Syl}_p(G)$ then $P^n \mid |P|$.

**Corollary 1** (Sylow. Dominance, [13]). Let $G$ be a finite group and let $p$ be a prime. If $H$ is a $p$-subgroup of $G$ it is contained in some $p$-Sylow of $G$.

The following corollary follows from the theorem that states that $G$ is a finite group and $p$ is a prime. If $H$ is a $p$-subgroup of $G$ and $P \in \text{Syl}_p(G)$, then there exists $g \in G$ such that $H$ is a subset $P^g$.

**Corollary 2** (Sylow. Conjugacy [13]). Consider $P, Q \in \text{Syl}_p(G)$. Then there exists $g \in G$ such that $P = Q^g$.

### 2.2 Homomorphisms

Let $G$ and $H$ be groups. A mapping $\alpha : G \to H$ is called a homomorphism if

$$\alpha(xy) = \alpha(x)\alpha(y)$$

for all $x, y \in G$.

We would not be observing any hard and fast rules about writing maps left or right, but will simply choose whichever side seems most appropriate in a given context. As usual, an injective homomorphism (respectively surjective, bijective) is called a monomorphism (respectively epimorphism, isomorphism). Occasionally, the notation $\alpha : G \to H$ will mean a monomorphism and $\alpha : G \to H$ an epimorphism. If there is an isomorphism $\alpha : G \to H$, we say that $G$ is isomorphic with $H$ (or that $G$ and $H$ have the same type of isomorphism) and we write $G \cong H$.

A homomorphism $\alpha$ of $G$ is likewise called an endomorphism and when $\alpha$ is bijective, an automorphism of $G$. The set of all automorphisms of $G$ forms a group under the binary operation composition of mappings; this is called the automorphism group of $G$, denoted by $\text{Aut}(G)$, and is obviously a subgroup of $\text{Sym}(G)$.
If \( g, h \in G \), we set \( h^g = g^{-1}hg \), and if \( X \) is a non-empty subset of \( G \), we define \( X^g \) to be the set \( \{x^g : x \in X\} \). In addition we use \( X^G \) to denote the set

\[
X^G = \{x^g : g \in G\}
\]

from conjugates from \( X \) into \( G \). The mapping \( \rho_g : G \to G \) defined by

\[
\rho_g(h) = h^g
\]

for all \( h \in G \) is easily seen as an automorphism of \( G \); it is called the \textit{inner automorphism} induced by \( g \). An automorphism \( \alpha \) of \( G \) is called \textit{inner} if \( \alpha = \rho_g \) for some \( g \in G \), and the set of all inner automorphisms they are denoted by \( \text{Inn}(G) \).

Obviously, a group is abelian if and only if \( \text{Inn}(G) = 1 \). So if \( G \) is abelian and \( g, h \in G \) then

\[
\rho_g(h) = h^g = g^{-1}hg = g^{-1}gh = h.
\]

An \( N \) subgroup of \( G \) that is invariant under all inner automorphisms (for which therefore \( N^g = N \) for all \( g \in G \)) is called a \textit{normal subgroup of} \( G \). If \( N \) is a normal subgroup of \( G \), we denote it symbolically by \( N \trianglelefteq G \) (and by \( N \triangleleft G \) when \( N \neq G \)).

We can observe that \( 1 \) and \( G \) are always normal subgroups of \( G \), and a group \( G \neq 1 \) with no other normal subgroups is called a \textit{simple}. Thus \( G \) is simple if and only if it has precisely two normal subgroups.

A subgroup \( U \) of a group \( G \) is considered subnormal in \( G \) if there exists a chain of subgroups \( U_0, U_1, \ldots, U_r \) of \( G \) such that

\[
U = U_0 \leq U_1 \leq \cdots \leq U_{r-1} \leq U_r = G
\]

This is called a subnormal string from \( U \) to \( G \). We will call it a composition series if each of its \textit{factors}

\[
U_i/U_{i-1} (i = 1, \ldots, r)
\]

is \textit{simple}, in which case the factors are called \textit{composition factors} of \( G \).

### 2.3 Group action

We say that given a group \( G \) and \( X \) a non-empty set, an action of \( G \) on \( X \) is a function

\[
G \times X \longrightarrow X
\]

that assigns each element \( \sigma \in G \) and \( x \in X \) a single element \( \sigma x \in X \). The action of \( G \) on \( X \) checks the following properties:

1. If \( e \) is the identity element of \( G \), then \( \forall x \in X \) then \( ex = x \).
2. If \( \sigma, \tau \in G \), then \( (\sigma \tau) x = \sigma (\tau x) \).

**Definition 1.** If \( X \) is a set satisfying a action of \( G \), then we say that \( X \) is a \textit{G-Set}.

Here are some examples of group actions:

**Example 2.** Consider any non-empty set \( X \), let \( G \times X \longrightarrow X \) be defined with \( \sigma x = x \). It is evident that this is an action of \( G \) on \( X \), in this case \( G \) is said to act trivially on \( X \).

**Example 3.** If \( X \) is any non-empty set, let \( G = S_X \) be the group of permutations of \( X \). Then \( G \) acts on \( X \) permuting its elements, this means that \( G \times X \longrightarrow X \) is given by \( \sigma x = \sigma (x) \), where \( \sigma (x) \) is the function that calculates \( x \).

**Example 4.** Let \( G \) be a group and \( H \leq G \). Then \( H \) acts on \( G \) through the product of \( G \), that is, \( H \times G \longrightarrow G \) is given by \( (h, \sigma) \mapsto h\sigma \) (The action \( \sigma \) gives us a translation of \( h \) by the left). In particular, if \( G = H \) then the group \( G \) acts on itself.

### 2.4 Semidirect product

For this section we follow the references \([14, 15, 16]\). We start considering \( N \triangleleft G \), i.e., \( N \) is a normal subgroup of \( G \). For each element \( g \) of \( G \) we can define an automorphism of \( N \), \( \varphi_g : N \rightarrow N \), such that \( n \mapsto gng^{-1} \). We can see that \( \varphi \) induces the following homomorphism:

\[
\theta : G \to \text{Aut}(N), \quad g \mapsto \varphi_g | N.
\]
On the other hand, if there exists a subgroup \( Q \) of \( G \) such that \( G \rightarrow G/N \) induces the isomorphism \( Q \rightarrow G/N \), then we can recover \( G \) from \( N, Q \), together with the restriction of \( \theta \) to \( Q \). Moreover, an element \( g \) of \( G \) can be written exclusively in the form
\[
g = nq, \quad n \in N, \quad q \in Q
\]
\( q \) should be the only one element of \( Q \) mapping to \( gN \in G/N \), and \( n \) must be \( gq^{-1} \). Thus, we have a one-to-one correspondence of sets
\[
G \rightarrow N \times Q, \quad N \times Q \rightarrow G
\]
If \( g = nq \) and \( g' = n'q' \), then
\[
qq' = (nq)(n'q') = n(qn'q^{-1})qq' = n \cdot \theta(q)(n') \cdot qq'
\]

**Definition 2.** Let \( N \triangleleft G \) and \( Q \leq G \). A group \( G \) is a semidirect product of its subgroups \( N \) and \( Q \) whether the homomorphism \( G \rightarrow G/N \) induces an isomorphism \( Q \rightarrow G/N \).

In an equivalent way, \( G \) is a semidirect product of the subgroups \( N \) and \( Q \) whether
\[
N \triangleleft G; \quad Nq = G; \quad N \cap Q = \{1\}
\]
We can notice that \( Q \) do not need to be a normal subgroup of \( G \). When \( G \) is the semidirect product of subgroups \( N \) and \( Q \), we write \( G = N \rtimes Q \). That is, \( N \rtimes Q \), where \( \theta : Q \rightarrow Aut(N) \) gives the action of \( Q \) on \( N \) by inner automorphisms.

## 3 Solvable and nilpotent groups

Solvable groups are very important in the abstract group theory as well in applications. In particular, solvability of groups give us some knowledge about mathematical physics concerning the phenomena, see [17, 18, 19, 20]. In this section we present some aspects related to solvable and nilpotent groups, including properties, definitions and some known results. The concept of Frattini subgroup is also introduced.

### 3.1 Nilpotent groups

Following [15] for any (finite or infinite) group \( G \) defines the following subgroups inductively:
\[
Z_0(G) = 1, \quad Z_1(G) = Z(G) := \{ g \in G : \forall h \in G, \ gh = hg \}
\]
and \( Z_{i+1}(G) \) is the subgroup of \( G \) containing \( Z_i(G) \) such that
\[
Z_{i+1}(G)/Z_i(G) = Z(G/Z_i(G))
\]
(ie \( Z_{i+1}(G) \) is the entire preimage in \( G \) of the center of \( G/Z_i(G) \) under the natural projection). The chain of subgroups
\[
Z_0(G) \leq Z_1(G) \leq Z_2(G) \leq \cdots
\]
is called the upper central series of \( G \). (The use of the term upper indicates that \( Z_i(G) \leq Z_{i+1}(G) \).)

**Definition 3 (Nilpotent Group).** A group \( G \) is called nilpotent if it has a finite chain (or series) of subgroups \( \{G_i\}_{i=1}^n \subset G \):
\[
\{1_G\} = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_n = G
\]
where for each \( i = 0, 1, \ldots, n - 1 \) it is true that:
- \( G_i \) is a normal subgroup on \( G_{i+1} \), denoted norms like \( G_i \triangleleft G_{i+1} \)
- \( G_{i+1}/G_i \leq Z(G/G_i) \) for \( i = 0, 1, \ldots, n - 1 \).

**Remark 1.** A group \( G \) is nilpotent if \( Z_c(G) = G \) for some \( c \in \mathbb{Z} \). The smallest of such \( c \) is called the nilpotence class of \( G \). If \( G \) is abelian, then \( G \) is nilpotent, even more so of class 1, provided that \( |G| > 1 \), since in this case \( G = Z(G) = Z_1(G) \).

The following are some properties of nilpotent groups
1. Let \( G \) be a nilpotent group of class \( r \) and \( H \leq G \) and \( N \triangleleft G \). Then \( H \) and \( G/H \) are nilpotent.
2. Every finite \( p \)-group is nilpotent.
3. If \( G \) and \( G_1 \) are nilpotent groups then \( G \times G_1 \) is nilpotent.

4. Let \( G \) be a finite group. Then the following are equivalent.
   (a) \( G \) is nilpotent.
   (b) If \( H < G \) then \( H \neq N_G(H) \).
   (c) All maximal subgroups of \( G \) are normal.
   (d) All Sylow subgroups of \( G \) are normal.
   (e) \( ab = ba \) for all \( a, b \in G \) whose orders are coprime.
   (f) \( G \) is the direct product of its Sylows subgroups.

In the study of maximal subgroups, G. Frattini in \[21\] introduces what is known today as the Frattini subgroup, which is the intersection of the maximal subgroups of the group \( G \). That is to say

\[
\Phi(G) = \bigcap\{M \leq G \mid M \text{ maximal in } G\}.
\]

An important result, which has a simple proof, is the following. It gives a good argument to work with the normality of maximal subgroups and today we know it as Frattini’s argument.

**Theorem 3.** If \( G \) is finite then \( \Phi(G) \) is nilpotent.

This property can be used to characterize the nilpotency of a group \( G \). If \( G \) is a finite group then the following are equivalent:

1. \( G \) is nilpotent.
2. \( G/\Phi(G) \) is nilpotent.
3. Every maximal subgroup of \( G \) is normal in \( G \).
4. \( G' \leq \Phi(G) \)

We recall that \( G' \) denotes the derived of \( G \), i.e., \( G' = [G, G] \).

### 3.2 Solvable groups

We start introducing the definition of Solvable group, also known as Soluble group, see \[14, 15, 22, 23\].

**Definition 4 (Solvable group).** A group \( G \) is said to be solvable if there is a finite chain of subgroups \( \{G_i\}_{i=1}^n \subset G \):

\[
\{1_G\} = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_n = G
\]

where for each \( i = 0, 1, \ldots, n-1 \) it follows that:

- \( G_i \) is a normal subgroup on \( G_{i+1} \).
- The quotient group \( G_{i+1}/G_i \) is abelian.

Nilpotent groups are solvable and abelian groups are nilpotent. Unlike a solvable group, where the quotients of the normal series are abelian.

The following are some properties of solvable groups.

1. Every subgroup and every quotient of a solvable group is solvable.
2. If \( N \triangleleft G \) is such that \( N \) is solvable and \( G/N \) is solvable, then \( G \) is solvable.
3. Every finite \( p \)-group is solvable.

Moreover, for all group \( G \) satisfying \( \Phi(G) = 1 \), all maximal subgroups of \( G \) are solvable groups. Let \( N \) be a nontrivial normal subgroup of \( G \). Then, there exists a maximal subgroup \( M \) of \( G \) satisfying \( G = NM \). Thus, if \( N \) and \( M \) are solvable, then \( G \) is solvable too. It is consequence of Lemma[2]
4 Supersolvable Groups and Zarrin Criteria for Solvable Groups

In this section we provide the concepts and known results about Supersolvable groups, also known as Supersubtle groups or Polycyclic groups. Following [24] and [25] we have the following supersolvable group definition. Moreover, we present some results of Zarrin about non-nilpotent groups and their relation with Simple groups.

**Definition 5** (Supersolvable group). A group $G$ is supersolvable if there exist normal subgroups $G_i$ with

$$\{1_G\} = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_n = G$$

and where each factor $G_i/G_{i-1}$ is cyclic for $1 \leq i \leq n$. It is clear that groups whose order is a power of a prime $p$ are supersolvable.

**Example 5.** The group $S_3$ is supersolvable. It is due to $1_G \leq A_3 \leq S_3$

**Example 6.** The group $A_4$ is not a supersolvable group, it is because $K_4$ is the unique normal subgroup of $A_4$.

Finitely generated abelian groups are supersolvable. It is because if $G$ es finitely generated (cyclic) we can consider

$$G = \langle g_1, g_2, \cdots, g_n \rangle$$

which is an abelian group. We can set $G_i = \langle g_i, g_{i+1}, \cdots, g_n \rangle$, and we can notice $G_i/G_{i-1} = \langle g_iG_{i-1} \rangle$ y $G_i \triangleleft G, \forall 1 \leq i \leq n$, we have a normal chain of subgroups

$$1 = G_1 \triangleleft G_2 \triangleleft \cdots \triangleleft G_n = G$$

where the factors are cyclics. For instance, $G$ is supersolvable.

**Remark 2.** We can observe that supersolvable groups are finitely generated.

It can be seen that supersolvable groups are solvable since the factors $G_i/G_{i-1}$, are abelian (cyclic $\Rightarrow$ abelian) but being solvable does not imply being supersolvable (abelian is not necessarily cyclic).

**Example 7.** We observe that $S_4$ is solvable because

$$1 \leq K_4 \leq A_4 \leq S_4$$

but $S_4$ has only the normal subgroups $A_4$ and $K_4$, therefore $S_4$ is not supersolvable

**Theorem 4** (See [12] [25]). The following statements hold.

(a) Every subgroup of a supersolvable group is supersolvable.

(b) The image of homomorphisms of a supersolvable group is supersolvable.

To illustrate the reader, we write the proof of the previous theorem for completeness. It is because it is classical result. For further details see [25].

**Proof.** We proceed according to each item.

(a) Let $G$ be a supersolvable group and let

$$\{1\} = G_0 \leq G_1 \leq G_2 \leq \cdots \leq G_r = G$$

be a principal serie of $G$. For the subgroup $S$ of $G$,

$$\{1\} = S \cap G_0 \leq S \cap G_1 \leq S \cap G_2 \leq \cdots \leq S \cap G_r = S$$

is a normal serie of $S$ in where the factor $(S \cap G_i) / (S \cap G_{i-1})$ is isomorphic to the subgroup $(S \cap G_i)(G_{i-1}/G_{i-1})$ de $G_i/G_{i-1}$.

(b) For all $i$, $G_i/G_{i-1}$ is of first order, such that $(S \cap G_i) / (S \cap G_{i-1})$ is either trivial or of first order. Thus, the previous normal serie for $S$ will produce, after the elimination process of redundant terms, a principal serie of $S$ with cyclic factors. Therefore, $S$ is supersolvable.

**Theorem 5** (See [12] [25]). Assume $H \leq G, N \leq G$, where $G$ is a supersolvable group. Then $H$ and $G/N$ are supersolvable groups.
Theorem 6 (See [12, 25]). The following statements hold.

1. A direct product of a finite number of supersolvable groups is supersolvable.

2. If \( H_1, \ldots, H_n \) are normal subgroups of \( G \) and the groups \( G/H_1, \ldots, G/H_n \) are supersolvable, then \( G/\bigcap_{i=1}^n H_i \) is supersolvable.

We can summarize previous results as follows:

\[
\text{cyclic} \subset \text{abelian} \subset \text{nilpotent} \subset \text{solvable} \subset \text{all groups}
\]

We recall that under special conditions some nilpotent groups are supersolvable groups.

Remark. Given a class of groups \( \mathcal{X} \), the group \( G \) is said to be a non-minimal group on \( \mathcal{X} \), or a critical group on \( \mathcal{X} \), if \( G/\mathcal{X} \) but all proper subgroups of \( G \) are in \( \mathcal{X} \). The detailed knowledge of the structure of non-minimal groups in \( \mathcal{X} \) can provide insight into what makes a group a member of \( \mathcal{X} \). All groups considered in this document are finite. See [12, 23].

We recall the importance work of Zarrin because he generalized the result of Schmidt for solvability of groups from its non-nilpotent subgroups. Such result motivates a new criteria for solvability of groups through the number of their non-supersolvable and supersolvable subgroups. Zarrin introduced the class \( S^n \), that is, a group \( G \) belongs to \( S^n \) if it contains exactly \( n \) non-nilpotent subgroups. Thus, Zarrin proved the following results in [10].

Theorem 7. Let \( G \) be a finite group that is an \( S^n \)-group with \( n \leq 22 \). Then

1. \( G \) is solvable,
2. \( A_5 \) is an \( S^{22} \)-group.

Theorem 8. Let \( G \) be a semi-simple non-abelian finite group such that it is an \( S^n \)-group with \( n \leq 65 \). Then \( G \) is isomorphic to \( A_5, \operatorname{Sym}(5) \), or \( \operatorname{SL}_2(7) \).

We must take into account the following properties that satisfy the \( S^n \)-groups introduced by Zarrin:

1. Consider \( G = H \times K \in S^m \) and \( K \in S^m \). Then \( G \in S^t \) with \( mn \leq r \).
2. Given a group \( G \), and \( K \) a non-nilpotent normal subgroup of \( G \) such that \( G/K \in S^n \) and \( K \in S^m \). Then \( G \in S^t \) for \( t \geq m + n \).
3. Let \( \operatorname{SL}_2(q) \in S^m \), \( m \in \mathbb{N} \) and \( q \geq 4 \) be a power of \( p \) prime. Then \( \operatorname{SL}_2(q) \) has non-nilpotent subgroups \((\leq m)\).
4. If \( \operatorname{Sz}(q) \) with \( q = 2^{m+1}, \ m > 0 \). Then has \( \frac{q^2+q+2}{2} \leq n \) non-nilpotent subgroups.

To see these results, note that the groups given in Thompson’s classification have the subgroups described by Dickson.

We use the standard notation followed in Doerk and Awkes [12] or Huppert [23]. We use \( \operatorname{SL}_m(q) \) and \( \operatorname{PSL}_m(q) \) to denote the special linear group and the special projective linear group, respectively, of dimension \( m \) over the field with \( q \) elements, where \( q \) is a power of a prime. It can be seen that

\[
|\operatorname{PSL}_2(p^f)| = \frac{p^{f}(p^f - 1)(p^f + 1)}{(2, p^f - 1)}
\]
The Suzuki group. The group $Sz(F_q) = Sz(q)$ is defined as the set of linear maps over the vector space $V$ which preserve the inner product and also the restricted outer product. It can be shown that $Sz(q)$ acts 2-transitively on the set of points $q^2 + 1$, and that the two-point stabilizer has order $q - 1$. Therefore

$$|Sz(q)| = (q^2 + 1)q^2(q - 1).$$

We recall that a minimal simple group is a simple group whose maximal subgroups are solvable. In this way, Thompson in [26] classified the minimal simple groups starting from the work with Walter Feit [3]. As Thompson explained, this classification is a translation from solvable groups to simple groups. This allows us to know in a deep way the properties of solvable groups in the study of structures and solvable subgroups of a simple group. Briefly, Thompson opted for a classification of minimal simple groups for small orders.

**Theorem 9** (Thompson [26]). A minimal simple groups are the following groups:

1. $PSL_2(p)$, where $p$ is a prime number with $p > 3$ and $p^2 - 1 = 0 (5)$.
2. $PSL_2(2^f)$, where $q$ is a prime number.
3. $PSL_2(3^f)$, where $q$ is an odd prime number.
4. $PSL_3(3)$.
5. The Suzuki groups $Sz(2^f)$, where $q$ is an odd prime number.

In [27] Dickson et al investigated exhaustively the classical finite groups through linear groups. In particular, in [27] §12 the authors listed the subgroups of $PSL_2(p^f)$. This work of Dickson et al. was based on the previous works of Moore and Wiman. For more details see [28].

**Theorem 10** (Dickson [23, 27, 28]). The group $PSL_2(p^f)$ has only the following subgroups

1. Elementary $p$-abelian groups
2. Cyclic groups of order $z$ with $z | \frac{p^f+1}{k}$, where $k = mcd (p^f - 1, 2)$.
3. Dihedral groups of order $2z$ with $z$ such as in (2).
4. Alternating Group $A_4$ for $p > 2$ or $p = 2$ and $f = 0 (2)$.
5. Symmetric Group $S_4$ for $p^2 - 1 = 0 (16)$.
6. Alternating Group $A_5$ for $p = 5$ or $p^2 - 1 = 0 (5)$.
7. Semidirect products of elementary abelian groups with order $p^m$ with cyclic groups of order $t$; where $t | p^m - 1$ and $t | p^f - 1$
8. $PSL_2(p^m)$ for $m | f$ and $PGL_2(p^m)$ for $2m | f$.

Based on the results of Zarrin in [10]. Zarrin himself obtained that the only simple groups with exactly 22 non-nilpotent subgroups is $A_5$ and that the only simple group with exactly 65 non-nilpotent subgroups is $SL_2(7)$. A corollary to these facts is that if a non-solvable group $G$ is an $S^n$-group with $n < 22$, then $G/\Phi(G)$ is isomorphic to $A_5$. Zarrin in [10] conjectured that a finite non-solvable group that is a $S^{22}$-group is isomorphic to $A_5$ or $SL_2(7)$. This conjecture was proven by Ballester-Bolinches, Esteban-Romero and Lu in 2017, see [11 Theorem A] for further details. The following theorem corresponds to the proof of the mentioned conjecture.

**Theorem 11** (See [11]). Let $G$ be an insoluble group. Then $G$ has exactly 22 non-nilpotent subgroups if and only if it is isomorphic to $A_5$ or $SL_2(5)$.

Earlier, Huppert [29] proved that the word nilpotent in Schmidt’s theorem can be replaced by supersolvable obtaining the same conclusion mutatis mutandis.

### 4.1 Groups with non-supersolvable subgroups

One natural question that arise is about the minimum number of non-supersolvable subgroups to guarantee the solvability of a group $G$. The results of the paper [11] were motivated by the paper [10] where they shown that the only minimal simple group with exactly 6 non-supersolvable subgroups is $A_5$ and that an unsolvable group $G$ has exactly 6 non-supersolvable subgroups if and only if it is isomorphic to $A_5$ or $SL_2(5)$. The following results can be found in [11].
**Lemma 1.** Let $G$ be a group. The number of non-supersolvable subgroups of $G/\Phi(G)$ is not bigger than the number of supersolvable subgroups of $G$.

In the following lemma it is related $G/\Phi(G)$ with minimal simple groups. Remember that $\Phi(G)$ is the Frattini subgroup of $G$.

**Lemma 2.** Let $G$ be an non solvable group whose maximal subgroups are solvable. Then $G/\Phi(G)$ is a minimal simple group.

A consequence of the simple finite groups classification, according to Thompson, is that all non-abelian simple group contains a minimal simple subgroup.

**Lemma 3.** The number of non-supersolvable subgroups of a minimal simple group is at least 6. The only minimal simple group with exactly 6 non-supersolvable subgroups is $A_5$.

The proof of this lemma is obtained through Thompson classification on simple minimal groups together with Dickson classification. We proceed by making a discrimination of the subgroups of each minimal simple group to which $G$ can be isomorphic of the Thompson classification. See [23 Kapitel II, Bemerkung 7.5] and [23 Kapitel II, Hilfssatz 6.2]. Detailed proofs of these lemmas can be found in [11]. In this way, with these lemmas Ballester-Bolinches, Esteban-Romero and Lu in [11] prove the following theorem.

**Theorem 12 (Theorems B and C in [11]).** The following statements hold.

1. A group with less than 6 non-supersolvable subgroups is solvable.
2. Let $G$ be an insolvable group. Then $G$ has exactly 6 non-supersolvable subgroups if and only if it is isomorphic to $A_5$ or $SL_2(5)$.

For this proof a minimal counterexample is constructed. See the proof in [11]. To prove (1) the authors used the method of reductio ad absurdum considering false the result and assuming a contradiction of Lemma 3 is obtained. To prove (2) the authors used a similar argument and also they used (1).

### 4.2 Groups with supersolvable subgroups

Motivated for [11] the authors Jiakuan Lu and Jingjing Wang in [30] proved the solvability of a finite group with fewer than 53 supersolvable subgroups. Furthermore, they showed that a finite non-solvable group has exactly 53 supersolvable subgroups if and only if it is isomorphic to $A_5$.

**A class of groups:** In [30] the class $S^n$ was introduced as; $S^n$ as the set of finite groups having exactly $n$ supersolvable subgroups, and a group $G$ is said to be a $S^n$-group if $G \in S^n$.

**Lemma 4.** Let $G$ be a $S^n$-group with the Frattini subgroup $\Phi(G)$, and let $G/\Phi(G)$ be a $S^m$-group. Then $m \leq n$.

We can observe that this lemma is equivalent to Lemma 1 in the reference [11], which was the motivation of the recent paper [30] and contains a similar proof.

**Lemma 5.** Let $G = \text{PSL}_2(q)$, and let $p$ be a prime number. Assume $q = p^f \geq 8$ for $p = 2$, or $q = p^f \geq 13$ for $p > 2$ is prime. Then $G$ is a $S^n$-group for $n \geq 2 (q^2 + 1)$.

To see this lemma it is important to review the conjugation classes of simple groups, i.e., observing the Thompson classification and the quantity of supersolvable groups of $G = \text{PSL}_2(q)$. In particular, observing the number of dihedral subgroups, we conclude that they are supersolvable and they contain two conjugation classes: type $D_{2(q-1)}/d$ and type $D_{2(q+1)/d}$ respectively, where $d = (2, q - 1)$.

**Lemma 6.** Let $G$ be a non-solvable group whose maximal subgroups are solvable. Then $G/\Phi(G)$ is a minimal simple group.

**Lemma 7.** A minimal simple group is a $S_n$-group with $n \geq 53$. The only minimal simple group in $S^{53}$ is $A_5$.

In a similar way as Lemma 3 these lemmas are obtained through Thompson classification on simple minimal groups together with Dickson classification. Further, we provide examples to compute the number of supersolvable and non-supersolvable subgroups of $A_5$ as well their implementation in GAP system.
Theorem 13. Let $G$ be a $S^n$-group.

1) If $n$ smaller than 53, then $G$ is solvable.

2) If $G$ is a non-solvable $S^n$-group, then $n = 53$ if and only if $G \cong A_5$.

The proof of previous theorem is similar to the proof of Theorem 12. For this proof a minimal counterexample is constructed. See the proof in [30]. To prove (1) the authors assumed false the result (reductio ad absurdum) and also they considered $G \in S^n$ with $n < 53$ as the nonsolvable group with smaller order. Using Lemma 2 and Lemma 5 they arrived to a contradiction of Lemma 7. To prove (2) the authors used similar statements and item (1).

5 New contributions

In this section we present some algorithms in GAP related to the previous results, in particular, we present some programming modules about supersolvable, non-supersolvable and non-nilpotent subgroups of the alternating group of five elements $A_5$. Finally, we provide some results concerning the relation between $(p,q)$-groups and dihedral groups with supersolvability.

5.1 Programs in GAP System

Now we observe how to compute the quantity of subgroups of certain type simple groups, providing by Thompson classification, using Dickson classification for subgroups. Due to $A_5$ is the group with smaller order in the classification of simple groups, see ATLAS [31] for description of subgroups of $A_5$, we focus on $A_5$ to illustrate these programs.

Recall that $PSL_2(4) \cong A_5 \cong PSL_2(5)$ with $|PSL_2(5)| = 60$ and we know the description of the subgroups of $PSL_2(p)$ because result of Dickson. Thus, $PSL_2(5)$ has the following subgroups:

Type (1) elementary abelian $p$-groups (2-Sylow, 3-Sylow y 5-Sylow),
Type (2) cyclic groups ($C_2, C_3$),
Type (3) dihedral groups $D_6 \cong S_3$ y $D_{10}$,
Type (4) $A_4$,
Type (6) $A_5$,
Type (7) Semidirect products of elementary abelian groupos of order $p^m$ with cyclic groups of order $t$; where $t \mid p^m - 1$ and $t\mid p^f - 1; (C_2 \times C_2)$

We recall that the number of conjugated subgroups is given by the index of the normalizer

$$[G : N_G(H)] = |\{H^g | g \in G\}|$$

and that if $P$ is a Sylow $p$-subgroup, then

$$[G : P] = [G : N_G(P)][N_G(P) : P] \quad \text{or} \quad m = n_p[N_G(P) : P]$$

where $m$ is the index of Sylow subgroup. Thus, $n_p \equiv 1 \mod p$.

Also we recall that $G = PSL_2(5)$ is simple (it has no non-trivial normal proper subgroups). Thus, for all subgroup $H$ it is satisfied $N_G(H) = H$, because the normalizer is a normal subgroup. In this way, we can compute the number of conjugated subgroups to $D_6$ and $D_{10}$ as

$$[G : D_{10}] = \frac{60}{10} = 6, \quad [G : D_6] = \frac{60}{6} = 10.$$ 

In a similar way, of Type $A_4$ there $[G : A_4] = 60/12 = 5$ conjugated subgroups to $A_4$.

Similarly, we can apply the same methodology to each type of and observing the number of Sylow subgroups of each Sylow subgroup of $G$. Thus, we can obtain that the number of subgroups of $A_5$ is 59. Thus, the non-supersolvable
subgroups of $A_5$ are $A_4$ and $A_5$, being 5 of type $A_4$ and 1 of type $A_5$. In consequence, $A_5$ has 6 supersolvable subgroups and 53 supersolvable subgroups. Thus, due to $A_5$ is minimal simple group with smaller order, we can give some conjectures about the quantity of certain kind of subgroups of a particular group $G$ in order that $G$ must be a solvable group. In the same way, we can obtain the number of non-nilpotent subgroups of $G$ using the Thompson classification and the Dickson classification of $A_5$.

In this way, we present the following algorithm, implemented by the authors in GAP.

**Algorithm 1**

**Computing the number of non-supersolvable subgroups of a group of $A_5$**

```gap
gap> G:=AlternatingGroup(5);
Alt( [ 1 .. 5 ] )
gap> L:=LatticeSubgroups(G);
<subgroup lattice of Alt( [ 1 .. 5 ] ), 9 classes, 59 subgroups>
gap> C:=ConjugacyClassesSubgroups(L);
[ Group( () )^G, Group( [ (2,3)(4,5) ] )^G, Group( [ (3,4,5) ] )^G,
  Group( [ (2,3)(4,5), (2,4)(3,5) ] )^G, Group( [ (1,2,3,4,5) ] )^G,
  Group( [ (1,2)(4,5), (3,4,5) ] )^G, Group( [ (1,4)(2,3), (1,3)(4,5) ] )^G,
  Group( [ (3,4,5), (2,4)(3,5) ] )^G, Group( [ (2,4)(3,5), (1,2,5) ] )^G ]
gap>FNS:=Filtered (C, t -> not IsSupersolvable(Representative(t)));
[ Group( [ (3,4,5), (2,4)(3,5) ] )^G, Group( [ (2,4)(3,5), (1,2,5) ] )^G ]
gap> S:=Sum(List(FNS, Size));
6
```

**Algorithm 2**

**Computing the number of supersolvable subgroups of a group of $A_5$**

```gap
gap> G:=AlternatingGroup(5);
Alt( [ 1 .. 5 ] )
gap> L:=LatticeSubgroups(G);
<subgroup lattice of Alt( [ 1 .. 5 ] ), 9 classes, 59 subgroups>
gap> C:=ConjugacyClassesSubgroups(L);
[ Group( () )^G, Group( [ (2,3)(4,5) ] )^G, Group( [ (3,4,5) ] )^G,
  Group( [ (2,3)(4,5), (2,4)(3,5) ] )^G, Group( [ (1,2,3,4,5) ] )^G,
  Group( [ (1,2)(4,5), (3,4,5) ] )^G, Group( [ (1,4)(2,3), (1,3)(4,5) ] )^G,
  Group( [ (3,4,5), (2,4)(3,5) ] )^G, Group( [ (2,4)(3,5), (1,2,5) ] )^G ]
gap>FS:=Filtered (C, t -> IsSupersolvable(Representative(t)));
[ Group( () )^G, Group( [ (2,3)(4,5) ] )^G, Group( [ (3,4,5) ] )^G,
  Group( [ (2,3)(4,5), (2,4)(3,5) ] )^G, Group( [ (1,2,3,4,5) ] )^G,
  Group( [ (1,2)(4,5), (3,4,5) ] )^G, Group( [ (1,4)(2,3), (1,3)(4,5) ] )^G ]
gap>SS:=Sum(List(FS, Size));
53
```

5.2 Supersolvability of $(n,m)$-groups and dihedral groups

The following elementary result illustrates the concept of supersolvability.

**Proposition 1.** The dihedral group $D_n = \langle r, s \mid r^n = 1, s^2 = 1, srs = r^{-1} \rangle$ is supersolvable.

**Proof.** Due to the dihedral group contains only one normal proper subgroup, which corresponds to $R_n = \langle r \rangle$ and $D_n$ is semidirect product of $C_2$ with $R_n$, that is $D_n \cong C_2 \rtimes R_n$, then the homomorphism $\phi : D_n \rightarrow D_n/R_n$ induces the isomorphism $\phi : C_2 \rightarrow D_n/R_n$. In this way, $1 \leq R_n \leq D_n$ is a chain of normal subgroups and for instance $D_n/R_n$ is isomorphic to $C_2$ that is cyclic. Thus, we conclude $D_n$ is supersolvable. \[\square\]

The following example corresponds to the group of symmetries of an square, which is isomorphic to $D_4$. 
Algorithm 3

Computing the number of non-nilpotent subgroups of a group of $A_5$

```gap
gap> G:=AlternatingGroup(5);
Alt( [ 1 .. 5 ] )
gap> L:=LatticeSubgroups(G);
<subgroup lattice of Alt( [ 1 .. 5 ] ), 9 classes, 59 subgroups>
gap> C:=ConjugacyClassesSubgroups(L);
[ Group( () )^G, Group( [ (2,3)(4,5) ] )^G, Group( [ (3,4,5) ] )^G, Group( [ (2,3)(4,5), (2,4)(3,5) ] )^G, Group( [ (1,2,3,4,5) ] )^G, Group( [ (1,2)(4,5), (3,4,5) ] )^G, Group( [ (1,4)(2,3), (1,3)(4,5) ] )^G, Group( [ (3,4,5), (2,4)(3,5) ] )^G, Group( [ (2,4)(3,5), (1,2,5) ] )^G ]
gap> FN:=Filtered (C, t -> not IsNilpotent(Representative(t)));
[ Group( [ (1,2)(4,5), (3,4,5) ] )^G, Group( [ (1,4)(2,3), (1,3)(4,5) ] )^G, Group( [ (3,4,5), (2,4)(3,5) ] )^G, Group( [ (2,4)(3,5), (1,2,5) ] )^G ]
gap> SN:=Sum(List(FN, Size));
22
```

Example 8. The group $G := \langle (1,2,3,4),(1,3) \rangle \cong D_4$ is supersolvable, where $D_4$ is the well known dihedral group or order 8. Considering $G_1$ and $G_2$ subgroups of $G$ such that $G_1 = \langle (1,3)(2,4) \rangle$ and $G_2 = \langle (1,2,3,4) \rangle$, then

$1_G = G_0 \leq G_1 \leq G_2 \leq G_3 = G$

is a normal chain with cyclic factors. Thus, we can conclude that $D_4$ is supersolvable.

Although this is a review paper, we present this original result that is elementary, but it is new as far as we know.

Now groups of type $(n, m)$ introduced in [22, 32, 33] are defined. A group of type $(n, m)$ is a group of the following form

$G := \langle s, r \mid s^n = r^m = 1, sr = r^t s \rangle$ with $n, m \in \mathbb{Z}_+, 1 < t < m$.

Furthermore, each element $x$ of the group $G$ can be written uniquely as

$x = s^i r^j \quad i, j \in \mathbb{Z}, \quad 0 \leq i < n, \quad 0 \leq j < m.$

It can be seen that $m \geq 3$ and $G$ is not abelian, so $n \geq 2$ and from the definition it follows that $|G| = nm$. We say that this group $G$ is of type $(n, m)$ and generated $(s, r)$. In addition, the following properties are deduced:

(i) $s^k r^l = r^{k+1} s^l$, $k = 0, 1, 2, \ldots$

(ii) $s^n r = r^k a^k$, $k = 0, 1, 2, \ldots$

(iii) $s^i b^j = b^k c^j a^k$, $n \leq 0, \quad i \in \mathbb{Z}$

Theorem 14. Let $G$ be a group of type $(n, m)$ and generated by $(s, r)$, and if $H = \langle s \rangle$ and $N = \langle r \rangle$. So $G = H \rtimes N$ with $N$ normal in $G$.

Theorem 15. Let $G$ be a group of type $(n, m)$. Then $G$ is a supersolvable group.

Proof. It is an immediate consequence of the theorem [14]. We have that the homomorphism $G \to G/N$ induces the isomorphism $H \to G/N$. Also, both subgroups are cyclic, so we have

$\{1\} \subseteq N \subseteq G$

with $G/N \cong H$. Thus, $G$ is supersolvable.

Final Remarks

In this paper, which falls in the intersection of abstract group theory, combinatorial group theory and computational group theory, we made a review about the solvable, supersolvable and nilpotent groups relative to the computation
of subgroups. We studied the more important concepts, properties and theorems that allowed us the classification of groups with non-solvable, non-supersolvable and non-nilpotent subgroups. Finally, we gave as contribution the supersolvability analysis of \((n,m)\)-groups and dihedral groups.

In the last years the structure of groups through their subgroups have deeply studied, that is, the quantity of subgroups or certain class to determine the group. In particular, Zarrin proved that a group with more than 26 subgroups with normalizers are solvable groups. In addition, a group with more than 22 derived subgroups is a solvable. In other way, the only one minimal simple group with exactly 23 derived subgroups is \(A_5\).

Further research can involve more general aspects such as follows:

Given two classes of groups \(X\) and \(Y\), provide a characterization of groups with \(n\) subgroups non-\(X\)-subgroups not being \(Y\)-groups. In previous cases, \(X\) means nilpotent and \(Y\) means solvable. The interested reader can change \(X\) and \(Y\) by another structures.

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