Where Infinitesimals Come From ...

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Dedicated to Marie-Louise Nykamp

Abstract

The presence of infinitesimals is traced back to some of the most general algebraic structures, namely, semigroups, and in fact, magmas, [1], in which none of the structures of linear order, field, or the Archimedean property need to be present. Such a clarification of the basic structures from where infinitesimals can in fact emerge may prove to have a special importance in Physics, as seen in [4-16]. The relevance of the deeper and simpler roots of infinitesimals, as they are given in Definitions 3.1 and 3.2, is shown by the close connection in Theorem 4.1 and Corollary 4.1 between the presence of infinitesimals and the non-Archimedean property, in the particular case of linearly ordered monoids, a case which, however, has a wide applicative interest.

1. Preliminaries

Abraham Robinson, [3], considered it to be one of the important aspects of Nonstandard Analysis the first time rigorous and comprehensive formulation of a theory of infinitesimals. As mentioned in the instructive historical survey at the end of [3], Leibniz appears to be
the first to introduce the idea of infinitesimals, and show their usefulness in Calculus. And for the next two centuries, until in the second part of the 1800s Weierstrass introduced modern rigour into the subject, Calculus had much been based on a variety of intuitive, rather than rigorous uses of infinitesimals popping up in all kind of places and under any number of forms. In fact, in engineering or physics courses of Calculus, such loose appeal to infinitesimals has gone on until more recently.

As it happens, at various times, a number of facts have not been understood quite clearly related to the status of infinitesimals.

One such fact is that the Archimedean structure of the field $\mathbb{R}$ of usual real numbers does not allow the presence of infinitesimals. This is the reason why after the reform introduced by Weierstrass there has no longer been a place in Calculus for infinitesimals. In this regard, Robinson’s field $^*\mathbb{R}$ of nonstandard reals happens to be non-Archimedean, and as such, proves to be able to accommodate infinitesimals.

However, in pursuing Nonstandard Analysis, Robinson had further goals in addition to obtaining a rigorous foundation for infinitesimals. Indeed, among such goals was that $^*\mathbb{R}$ is a linearly ordered field extension of $\mathbb{R}$. Furthermore, it was aimed that a good deal of the usual properties of $\mathbb{R}$ would automatically remain valid for $^*\mathbb{R}$ as well, under the so called transfer principle.

A consequence of the above has been the tacit association of infinitesimals with:

- linear orders,
- fields,
- the Archimedean property.

As shown in this paper, however, the presence of infinitesimals can be traced back to far more general algebraic structures, namely, semigroups, and in fact, magmas, [1], in which none of the above three
structures need to be present.

Such a clarification of the basic structures from where infinitesimals can in fact emerge may prove to have a special importance in Physics, as seen in [4-16].

The relevance of the deeper and simpler roots of infinitesimals, as they are given in Definitions 3.1 and 3.2, is shown by the close connection in Theorem 4.1 and Corollary 4.1 between the presence of infinitesimals and the non-Archimedean property in the particular case of linearly ordered monoids, a case which, however, has a wide applicative interest.

2. Rich and Complex Structure of the Set of Additive Subgroups

It is useful to start by recalling the seldom considered and surprisingly rich and complex structure of the set of additive subgroups in $\ast \mathbb{R}$.

First, we recall that $\mathbb{R}$ is an additive subgroup of $\ast \mathbb{R}$. Further, we can distinguish the following four subgroups in $\ast \mathbb{R}$, namely

$$
\begin{align*}
\{0\} \subsetneq \text{Mon}(0) \subsetneq \text{Fin}(0) \subsetneq \ast \mathbb{R}
\end{align*}
$$

where $\text{Mon}(0)$ denotes the set of infinitesimals, that is, the so called monad at $0 \in \ast \mathbb{R}$, while $\text{Fin}(0)$ denotes the set of finite elements $x \in \ast \mathbb{R}$, thus $\text{Fin}(0) = \mathbb{R} \bigoplus \text{Mon}(0)$.

Clearly, when seen from $\mathbb{R}$, the subgroups (2.1) collapse to only two trivial instances, namely, $\{0\}$ and $\mathbb{R}$ itself.

Now the important fact to note is that there are many more additive subgroups in $\ast \mathbb{R}$, than listed in (2.1). Indeed, let $\epsilon \in \text{Mon}(0)$, $\epsilon > 0$, that is, a positive infinitesimal. Then, associated with this infinitesimal $\epsilon$ we obtain the following infinitely many pair-wise disjoint additive subgroups in $\ast \mathbb{R}$, namely.
And in fact, there are uncountably many of that type of pair-wise disjoint additive subgroups associated with the given infinitesimal $\epsilon$, namely

\begin{align*}
(2.2) \quad & \mathbb{R}\epsilon, \; \mathbb{R}\epsilon^2, \; \mathbb{R}\epsilon^n, \ldots \subseteq \text{Mon}(0) \\
(2.3) \quad & \mathbb{R}\epsilon^a \subseteq \text{Mon}(0), \quad a \in \mathbb{R}, \; a \geq 1
\end{align*}

Similarly, if we take any $X \in \ast \mathbb{R} \setminus \text{Fin}(0)$, $X > 0$, then associated with this infinitely large $X$ we obtain the following infinitely many pair-wise disjoint additive subgroups in $\ast \mathbb{R}$, namely

\begin{align*}
(2.4) \quad & \mathbb{R}X^a \subseteq \ast \mathbb{R}, \quad a \in \mathbb{R}, \; a \geq 1
\end{align*}

Needless to say, the additive subgroups in $\ast \mathbb{R}$ are far from being exhausted by those in (2.1) - (2.4).

As for the complexity of the relationships between various additive subgroups in $\ast \mathbb{R}$, we can note the following.

Let $\epsilon, \eta \in \text{Mon}(0)$, $\epsilon, \eta > 0$, then

\begin{align*}
(2.5) \quad & \mathbb{R}\epsilon \cap \mathbb{R}\eta \neq \phi \iff \mathbb{R}\epsilon = \mathbb{R}\eta \iff \epsilon/\eta \in \mathbb{R}
\end{align*}

and as is well known, the relation in the right hand of (2.5) is highly atypical among infinitesimals $\epsilon, \eta \in \text{Mon}(0)$.

Similarly, let $X, Y \in \ast \mathbb{R} \setminus \text{Fin}(0)$, $X, Y > 0$, then

\begin{align*}
(2.6) \quad & \mathbb{R}X \cap \mathbb{R}Y \neq \phi \iff \mathbb{R}X = \mathbb{R}Y \iff X/Y \in \mathbb{R}
\end{align*}

where again, the relation in the right hand of (2.6) is highly atypical among infinitely large $X, Y \in \ast \mathbb{R} \setminus \text{Fin}(0)$.

Let us now compare the above with the situation of additive subgroups of $\mathbb{R}$.

Given $x \in \mathbb{R}$, $x > 0$, then
hence no trace of the rich complexity of additive subgroups such as in (2.1) - (2.7).

Let us consider another example with infinitesimals, one that is closely connected with the reduced power algebras, and in particular, with \( {}^*\mathbb{R} \), see [4-16], namely the algebra \( \mathbb{R}^N \).

First we recall that we have the group isomorphism

\[(2.8) \quad \mathbb{R} \ni x \mapsto u_x = (x, x, x, \ldots) \in \mathcal{U}_{\mathbb{R}^N} \subset \mathbb{R}^N\]

Further, in the algebra \( \mathbb{R}^N \) one can distinguish the following additive semigroups

\[(2.9) \quad \{0\} \subsetneq \mathcal{I}_{\mathbb{R}^N} \subsetneq \mathcal{A}_{\mathbb{R}^N} \subsetneq \mathcal{B}_{\mathbb{R}^N} \subsetneq \mathbb{R}^N\]

where \( \mathcal{I}_{\mathbb{R}^N} \), \( \mathcal{A}_{\mathbb{R}^N} \) and \( \mathcal{B}_{\mathbb{R}^N} \) are, respectively, the set of sequences \( x = (x_0, x_1, x_2, \ldots) \in \mathbb{R}^N \), which converge to \( 0 \in \mathbb{R} \), converge to some element in \( \mathbb{R} \), respectively, are bounded.

In (2.9), in view of [4-16], one can see \( \mathcal{I}_{\mathbb{R}^N} \) as the monad of \( 0 \in \mathbb{R}^N \), that is, the set of infinitesimals in \( \mathbb{R}^N \), while \( \mathbb{R}^N \setminus \mathcal{B}_{\mathbb{R}^N} \) can be seen as the set of infinitely large elements in \( \mathbb{R}^N \).

Clearly, and unlike with \( {}^*\mathbb{R} \), in the algebra \( \mathbb{R}^N \) it is \textit{not} the case that

\[\epsilon \text{ infinitesimal, } \epsilon \neq 0 \quad \Rightarrow \quad 1/\epsilon \text{ infinitely large}\]

since it may happen that \( 1/\epsilon \) is not even defined. Similarly, it is \textit{not} the case that

\[X \text{ infinitely large} \quad \Rightarrow \quad 1/X \text{ infinitesimal}\]

since it may happen that \( 1/X \) is not even defined.
Similar with the situation in (2.2) - (2.6), and in fact, with an increased richness and complexity, one obtains the structure of the set of additive subgroups in \( \mathbb{R}^N \).

3. Defining Infinitesimals

The sharp contrast seen in section 2 between the richness and complexity of additive semigroups in \( \ast \mathbb{R} \) and \( \mathbb{R}^N \), and on the other hand, in \( \mathbb{R} \), suggests the following definition

**Definition 3.1**

A semigroup \((E, \ast)\) is called a *hyperspace*, if and only if it contains a sub-semigroup \( F \subsetneq E \), together with an infinite set \( \mathcal{I} \) of pair-wise disjoint sub-semigroups \( I \subsetneq F \).

In such a case the elements of the set

\[
I_{\mathcal{I}} = \bigcup_{I \in \mathcal{I}} I
\]

are called *infinitesimals*, when the set \( \mathcal{I} \) is maximal with its respective property.

Here, and in the sequel, two sub-semigroups are called disjoint, if and only if their intersection is void, or it is a set of idempotent elements.

Clearly, the above definition can be extended to *magmas*, [1], namely

**Definition 3.2**

A magma \((E, \ast)\) is called a *hyperspace*, if and only if it contains a sub-magma \( F \subsetneq E \), together with an infinite set \( \mathcal{I} \) of pair-wise disjoint sub-magmas \( I \subsetneq F \).

In such a case the elements of the set
are called \textit{infinitesimals}, when the set \( \mathcal{I} \) is maximal with its respective property.

\[ I_{\mathcal{I}} = \bigcup_{I \in \mathcal{I}} I \]

Similar with above, here, and in the sequel, two sub-magmas are called disjoint, if and only if their intersection is void, or it is a set of idempotent elements.

\textbf{Remark 3.1}

The above two definitions do not make use of any partial, let alone, linear order. Equally, they do not make use of but one single algebraic operation, unlike in the case of algebras or fields, where at least two operations, namely, addition and multiplication are involved. Finally, they do not make use of any kind of Archimedean or non-Archimedean property.

As for the algebraic operation \(*\) involved in the above two definitions, it takes the place of addition, rather than multiplication, as suggested by the example in section 2.

\textbf{4 Examples}

Let us start with two familiar and somewhat different versions of the concept of non-Archimedean structure within the setting of partially ordered monoids. Namely, let \((E, +, \leq)\) be a partially ordered monoid, thus we have satisfied

\[ x, y \in E_+ \implies x + y \in E_+ \]

where \( E_+ = \{ x \in E \mid x \geq 0 \} \).

A first intuitive version of the Archimedean condition, suggested in case \( \leq \) is a \textit{linear} order on \( E \), is
(4.2) \( \exists \ u \in E_+ : \ \forall \ x \in E : \ \exists \ n \in \mathbb{N} : \ nu \geq x \)

Here however is an alternative condition used in the literature when \( \leq \) may be a partial order on \( E \)

\[
\forall \ x \in E_+ :
\]

(4.3) \[
x = 0 \iff \left( \exists \ y \in E_+ : \ \forall \ n \in \mathbb{N} : \ nx \leq y \right)
\]

where clearly the implication ”\( \Rightarrow \)” is trivial, and thus condition (4.3) is equivalent with

(4.4) \( \forall \ x \in E_+ : \ Nx \) is bounded above \( \implies x = 0 \)

**Lemma 4.1**

We have the implication (4.2) \( \implies \) (4.3).

**Proof**

Assume that (4.3), hence (4.4) does not hold, then

\( \exists \ x \in E_+ : \ N x \) is bounded above, and \( x \neq 0 \)

thus

\( \exists \ u \in E_+ , \ x \in E : \ \forall \ n \in \mathbb{N} : \ nu \leq x \)

and (4.1) is contradicted.

\[ \square \]

As for the converse implication (4.3) \( \implies \) (4.2), we have

**Lemma 4.2**

If \( (E, +, \leq) \) is a *linearly* ordered monoid, then (4.3) \( \implies \) (4.2).
Proof

Assume indeed that (4.2) does not hold, then

∀ x ∈ E⁺ : ∃ y ∈ E : ∀ n ∈ ℕ : nx ̸⪰ y

and since ≤ is a linear order on E, we have

∀ x ∈ E⁺ : ∃ y ∈ E : ∀ n ∈ ℕ : nx ≤ y

Obviously, we can assume that y ∈ E⁺, thus (4.4) is contradicted.

Theorem 4.1

A nontrivial linearly ordered monoid (E, +, ≤) which is non-Archimedean in the sense of (4.2), is a hyperspace.

Proof.

Assume that contrary to (4.2), we have

(4.5) ∀ u ∈ E⁺ : ∃ x ∈ E : ∀ n ∈ ℕ : nu ≤ x

and we note that x ≥ u, thus in particular x ∈ E⁺.

Let us take u₁ > 0. Then (4.5) gives x₁ ≥ u₁ > 0, such that

(4.6) I₁ = ℤu₁ ≤ x₁

We take now u₂ > x₁, and as above, we obtain x₂ ≥ u₂, such that

(4.7) I₂ = ℤu₂ ≤ x₂

Continuing the procedure, we obtain

(4.8) 0 < u₁ ≤ x₁ < u₂ ≤ x₂ < ...
(4.9) \[ I_1 = \mathbb{Z}u_1 \leq x_1, \quad I_2 = \mathbb{Z}u_2 \leq x_2, \ldots \]

We show now that

(4.10) \[ I_1 \cap I_2 = \{0\} \]

Indeed, let \( y \in I_1 \cap I_2 \), then

(4.11) \[ y = n_1 u_1 = n_2 u_2, \quad y \neq 0 \]

for some \( n_1, n_2 \in \mathbb{Z} \). But (4.8), (4.9) give

(4.12) \[ y = n_1 u_1 \leq x_1 < u_2 \]

hence \( n_2 u_2 = y < u_2 \) which means

(4.13) \[ n_2 < 0 \]

and thus (4.11) yields

(4.14) \[ n_1 < 0 \]

It follows that

(*) \[ -y = n_1 u_1 = n_2 u_2 \in I_1 \cap I_2 \]

hence, from the start, we can assume in (4.11) that \( n_1, n_2 > 0 \).

This, however, contradicts (4.13), (4.14).

**Corollary 4.1**

A nontrivial linearly ordered monoid \((E, +, \leq)\) which is not a hyperspace, is Archimedean in the sense of (4.2).
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