Non-Fermi Liquid Effects in QCD at High Density

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Abstract

We study non-Fermi liquid effects due to the exchange of unscreened magnetic gluons in the normal phase of high density QCD by using an effective field theory. A one-loop calculation gives the well known result that magnetic gluons lead to a logarithmic enhancement in the fermion self energy near the Fermi surface. The self energy is of the form \( \Sigma(\omega) \sim \omega \gamma \log(\omega) \), where \( \omega \) is the energy of the fermion, \( \gamma = O(g^2) \), and \( g \) is the coupling constant. Using an analysis of the Dyson-Schwinger equations we show that, in the weak coupling limit, this result is not modified by higher order corrections even in the regime where the logarithm is large, \( \gamma \log(\omega) \sim 1 \). We also show that this result is consistent with the renormalization group equation in the high density effective field theory.
I. INTRODUCTION

It is theoretically well established that dense quark matter is not a Fermi liquid. Attractive interaction between pairs of quarks that are anti-symmetric in color cause an instability in the quark-quark scattering amplitude if the quark momenta lie on opposite sides of the Fermi surface \[1, 2, 3\]. This instability is resolved by the formation of a diquark condensate which breaks the color gauge symmetry. It is also well known that this is not the only non-Fermi liquid effect in dense quark matter. Unscreened magnetic gluon exchanges lead to a logarithmic singularity in the quark self-energy close to the Fermi surface \[4, 5, 6, 7, 8, 9, 10, 11\]. This logarithmic singularity may lead to a breakdown of perturbation theory in the normal phase of dense QCD at very low energies, \(\omega \sim \mu \exp(-c_{nft}/g^2)\), where \(\mu\) is the chemical potential, \(g\) is the coupling constant, and \(c_{nft} = 9\pi^2\).

This scale is exponentially small as compared to the scale of superconductivity, \(\omega \sim \mu \exp(-c_{bcs}/g)\), where \(c_{bcs} = 3\pi^2/\sqrt{2} \[12\]. Understanding non-Fermi liquid effects in the normal phase of quark matter is nevertheless important. First of all, understanding the normal phase is necessary in order to put calculations in the superfluid phase on a solid footing. Also, in order to establish the possible existence of a superconducting phase of quark matter from the observation of neutron stars we have to compute the properties of both the normal and the superconducting phase. And finally, non-Fermi liquid effects may play a role if the dominant superconducting phase is suppressed by non-zero quark masses, lepton chemical potentials, or magnetic fields.

If electric charge neutrality is taken into account, then a non-zero strange quark mass leads to approximately equal differences between the Fermi momenta of strange and up as well as up and down quarks \[13\]. This implies that if the strange quark mass exceeds a critical value, only pairing between quarks of the same flavor is possible. Pairing between equal flavors requires order parameters with non-zero spin, and the corresponding gaps are suppressed by roughly two orders of magnitude compared to the spin zero gap \[14\]. The gap can be further suppressed by a non-zero temperature or magnetic field. Finally, flavor symmetry breaking may lead to the appearance of gapless fermion modes even in the superconducting phase \[15, 16\]. Depending on whether there is magnetic screening in this phase the gapless modes will also lead to interesting non-Fermi liquid effects.

In this work we study non-Fermi liquid effects in QCD using the high density effective
The paper is organized as follows. In Sect. II we discuss power counting in the high density effective theory. In Sects. III and IV we study the Dyson-Schwinger equation for the quark self energy in the normal phase of dense quark matter. In Sect. V we consider the renormalization group equation for the quark propagator. We discuss some of the implications of our results in Sect. VI. Non-Fermi liquid effects due to unscreened gauge boson exchanges were first discussed by Holstein, Norton and Pincus in the case of a cold electron gas. Similar effects due to dynamical gauge fields in systems of strongly correlated electrons were studied by Polchinski, Nayak and Wilczek, and others.

II. HIGH DENSITY EFFECTIVE THEORY

At high baryon density the relevant degrees of freedom are particle and hole excitations which move with the Fermi velocity $v$. Since the momentum $p \sim v\mu$ is large, typical soft scatterings cannot change the momentum by very much and the velocity is approximately conserved. An effective field theory of particles and holes in QCD is given by

$$\mathcal{L} = \sum_v \psi_v^\dagger (iv \cdot D) \psi_v - \frac{1}{4} G^{a}_{\mu\nu} G^{a}_{\mu\nu} + \ldots,$$

where $v_{\mu} = (1, \vec{v})$. The field $\psi_v$ describes particles and holes with momenta $p = \mu \vec{v} + l$, where $l \ll \mu$. We will write $l = l_0 + l_{\parallel} + l_{\perp}$ with $l_{\parallel} = \vec{v}(\vec{l} \cdot \vec{v})$ and $l_{\perp} = \vec{l} - l_{\parallel}$. In order to take into account all low energy degrees of freedom we have to cover the Fermi surface with patches labeled by the local Fermi velocity.

Higher order terms in the effective lagrangian are suppressed by inverse powers of the chemical potential. There are two types of higher order corrections, operators that only involve fields in a given patch, and operators with four or more fermion fields that connect fields in different patches. In order to understand the importance of higher order corrections we have to develop a power counting scheme for the high density effective theory. We first discuss a “naive” attempt to count powers of the small scale $l$. In the naive power counting we assume that $v \cdot D$ scales as $l$, $\psi_v$ scales as $l^{3/2}$, $A_\mu$ scales as $l$, and every loop integral scales as $l^4$. We also assume that $\vec{D}_{\perp}, \vec{v} \cdot D \sim l$, where $\vec{v}_{\mu} = (1, -\vec{v})$. In this case it is easy to see that a general diagram with $V_k$ vertices of scaling dimension $k$ scales as $l^6$.
FIG. 1: Counting hard loops in the effective field theory. If all (soft) gluon lines are removed the remaining fermionic loops contain sums over the velocity index.

with

\[ \delta = 4 + \sum_k V_k (k - 4). \]  

(2)

A general vertex is of the form

\[ \psi^a (v \cdot D)^b (\bar{v} \cdot D)^c (D_\perp)^d (1/\mu)^e, \]  

(3)

and has mass dimension \(3a/2 + b + c + d - e = 4.\) Since \(k = 3a/2 + b + c + d\) and \(e \geq 0\) we have \(k - 4 \geq 0.\) This implies that the power counting is trivial: All diagrams constructed from the leading order lagrangian have the same scaling, all diagrams with higher order vertices are suppressed, and the degree of suppression is simply determined by the number and the scaling dimension of the vertices.

Complications arise because not all loop diagrams scale as \(l^4.\) In fermion loops sums over patches and integrals over transverse momenta can combine to give integrals that are proportional to the surface area of the Fermi sphere,

\[ \frac{1}{2\pi} \sum_{\psi} \int \frac{d^2l_\perp}{(2\pi)^2} = \frac{\mu^2}{2\pi^2} \int \frac{d\Omega}{4\pi}. \]  

(4)

These loop integrals scale as \(l^2,\) not \(l^4.\) In the following we will refer to loops that scale as \(l^2\) as “hard loops” and loops that scale as \(l^4\) as “soft loops”. In order to take this distinction into account we define \(V_k^S\) and \(V_k^H\) to be the number of soft and hard vertices of scaling dimension \(k.\) A vertex is called soft if it contains no fermion lines. In order to determine the \(l\) counting of a general diagram in the effective theory we remove all gluon lines from
the graph, see Fig. 1. We denote the number of connected pieces of the remaining graph by \( N_C \). Using Euler identities for both the initial and the reduced graph we find that the diagram scales as \( l^\delta \) with

\[
\delta = \sum_k \left[ (k - 4)V_k^S + (k - 2 - f_k)V_k^H \right] + E_Q + 4 - 2N_C. \tag{5}
\]

Here, \( f_k \) denotes the number of fermion fields in a hard vertex, and \( E_Q \) is the number of external quark lines. We observe that in general the scaling dimension \( \delta \) still increases with the number of higher order vertices, but now there are two important exceptions.

First we observe that the power counting for hard vertices is modified by a factor that counts the number of fermion lines in the vertex. It is easy to see that four-fermion operators without extra derivatives are leading order \((k - 2 - f_k = 0)\), but terms with more than four fermion fields, or extra derivatives, are suppressed. This result is familiar from the effective field theory analysis of theories with short range interactions \[27, 28\].

The second observation is that the number of fermion loops that become disconnected if soft gluons are removed, \( N_C \), reduces the power \( \delta \). Each disconnected loop contains at least one power of the coupling constant, \( g \), for every soft vertex. As a result, fermion loop insertions in gluon \( n \)-point functions spoil the power counting if the gluon momenta satisfy \( l \sim g\mu \). This implies that for \( l < g\mu \) the high density effective theory becomes non-perturbative and fermion loops in gluon \( n \)-point functions have to be resummed. This resummation corresponds to the familiar hard dense loop (HDL) resummation \[29, 30\]. Note, however, that in the high density effective theory we do not perform a hard dense loop resummation of Green functions with external fermion lines.

There is a simple generating functional for hard dense loops in gluon \( n \)-point functions which is given by \[30\]

\[
\mathcal{L}_{\text{HDL}} = -\frac{m^2}{2} \sum_v G^a_{\mu\alpha} \frac{v^\alpha v^\beta}{(v \cdot D)^2} G^b_{\mu\beta}, \tag{6}
\]

where \( m^2 = N_f g^2 \mu^2 / (4\pi^2) \) is the dynamical gluon mass and the sum over patches corresponds to an average over the direction of \( \vec{v} \). For momenta \( l < g\mu \) we have to add the HDL generating functional to the HDET effective action, \( \mathcal{L}_{\text{HDET}} \to \mathcal{L}_{\text{HDET}} + \mathcal{L}_{\text{HDL}} \). This means that we work with hard dense loop resummed gluon propagators and vertices. If we analyze the low energy behavior in the vicinity of a generic point on the Fermi surface then there is no double counting involved, because we no longer have to consider sums over patches. It is
FIG. 2: Fig. a) shows the dominant contribution to the fermion self energy in the high density effective theory. The solid square denotes an insertion of the gluon self energy, see Fig. b). The dot denotes the free quark-gluon vertex.

interesting to note that the velocity index on the quark field acts like a flavor label, and that the diagrams selected by the large $\mu$ limit are the diagrams of the large $N_f$ approximation.

The hard dense loop action describes static screening of electric fields and dynamic screening of magnetic modes. Since there is no screening of static magnetic fields low energy gluon exchanges are dominated by magnetic modes. The resummed transverse gauge boson propagator is given by

$$D_{ij}(k) = \frac{\delta_{ij} - \hat{k}_i\hat{k}_j}{k^2 - \hat{k}^2 + i\eta |k_0|/|\vec{k}|},$$

(7)

where $\eta = \frac{\pi}{2} m^2$ and we have assumed that $|k_0| < |\vec{k}|$. We observe that the gluon propagator becomes large in the regime $k \sim (\eta k_0)^{1/3} \gg k_0$. This implies that the power counting for very low energy gluons has to be modified. Landau damped gluons satisfy the scaling laws $k^0 \sim l$ and $|\vec{k}| \sim l^{1/3}$. As we shall see in the next section, this modified scaling relation has important consequences for the structure of the fermion self energy.

III. DYSON-SCHWINGER EQUATION

The leading logarithmic term in the one-loop fermion self energy in the high density effective theory is determined by the Feynman diagram shown in Fig. 2. We find

$$\Sigma(\omega, l) = \frac{g^2}{9\pi^2} \omega \log \left( \frac{\Lambda}{\omega} \right),$$

(8)

where $\omega = l_0$ and $\Lambda$ is a cutoff. This result implies that for $\omega \sim \Lambda \exp(-9\pi^2/g^2)$ the perturbative correction is of order 1, and higher order terms of the form $g^{2n} \log^n(\Lambda/\omega)$ may have to be included. In order to study this problem we consider the Dyson-Schwinger
equation for the fermion self energy

\[-i \Sigma(p) = - \int \frac{d^4k}{(2\pi)^4} \Gamma^a_{\mu} S(p + k) \Gamma^b_{\nu} D_{\mu\nu}^{ab}(k).\]  

(9)

Here, \(-i \Sigma(p) = S^{-1}(p) - S_0^{-1}(p)\) is the fermion self energy, \(D_{\mu\nu}^{ab}\) is the gluon propagator, and \(\Gamma^a_{\mu}\) is the quark-gluon vertex function. Following the arguments given in the previous section we consider the contribution from transverse gauge bosons only and use the HDL resummed transverse propagator. We will use the free quark-gluon vertex \(\Gamma_i^a = g v_i \lambda^a / 2\). We shall solve the Dyson-Schwinger equation for the quark propagator but we will not solve a self-consistency equation for the gluon propagator or the quark-gluon vertex. We will justify these assumptions below.

We note that the infrared divergence in the fermion self energy depends only on the energy and not on the momentum of the quark. Fermion momenta scale as \(l_p \sim l\) while gluon momenta scale as \(|\vec{k}| \sim l^{1/3}\). As a consequence we can neglect the dependence on the external fermion momentum and we will assume that the quark self energy is a function of the energy only. The Dyson-Schwinger equation is

\[-i \Sigma(p_0) = g^2 C_F \int \frac{d^4k}{(2\pi)^4} \frac{1 - (\vec{v} \cdot \vec{k})^2}{p_0 + k_0 - l_{p+k} + \Sigma(p_0 + k_0)} \frac{1}{k_0^2 - \vec{k}^2 + i\eta|k_0|/|\vec{k}|},\]  

(10)

where \(l_p = \vec{v} \cdot \vec{p} - \mu\) and \(C_F = (N_c^2 - 1)/(2N_c)\). After analytic continuation to euclidean space we have

\[\Sigma(p_4) = g^2 C_F \int \frac{dk_4}{2\pi} \int \frac{k^2 dk}{(2\pi)^2} \int_{-1}^{1} dx \frac{1 - x^2}{i(p_4 + k_4) - l_p - kx + i\Sigma(p_4 + k_4)} \frac{1}{k_4^2 + k^2 + \eta^2 k_4^2/k^2},\]  

(11)

where \(l_k = \vec{v} \cdot \vec{k} \equiv kx\) and we have defined \(\Sigma(p_4) \equiv \Sigma_E(p_4) \equiv -i\Sigma(p_0)\). Because \(l_p \ll |\vec{k}|\) we can neglect the dependence on \(l_p\). The angular integration can be carried out analytically, leaving

\[\Sigma(p_4) = \frac{g^2 C_F}{4\pi^3} \int dk_4 \int dk \left( \frac{(p_4 + k_4 + \Sigma(p_4 + k_4))^2 + k^2}{k^2} \arctan \left( \frac{k}{p_4 + k_4 + \Sigma(p_4 + k_4)} \right) \right) - \frac{p_4 + k_4 + \Sigma(p_4 + k_4)}{k} \frac{1}{k_4^2 + k^2 + \eta |k_4|^2/k},\]  

(12)

This expression will be analyzed numerically in the next section. In order to derive an approximate analytic expression we note that the gluon propagator becomes large in the regime \(k \sim (\eta k_4)^{1/3} \gg k_4\). Therefore, we can in first approximation neglect the \(k_4^2\) term in
the gluon propagator. In addition to that, we can approximate $1 - x^2 \simeq 1$ in the numerator of eq. (11). We get

$$\Sigma(p_4) = 2C_F g^2 \int \frac{dk_4}{2\pi} \int \frac{dk}{(2\pi)^2} \arctan \left( \frac{k}{p_4 + k + \Sigma(p_4 + k_4)} \right) \frac{1}{k^2 + \eta^4 k^4}. \quad (13)$$

The non-analytic contribution to the self-energy can be extracted from

$$\frac{d}{dp_4} \Sigma(p_4) = 2g^2 C_F \int \frac{dk_4}{2\pi} \int \frac{dk}{(2\pi)^2} \left\{ \pi \delta(k_4 + p_4 + \Sigma(p_4 + k_4)) \right. \\
- \frac{k}{(k_4 + p_4 + \Sigma(k_4 + p_4))^2 + k^2} \left. \right\} \frac{1 + \Sigma'(p_4 + k_4)}{k^2 + \eta^4 k^4}. \quad (14)$$

Only the first term in the curly brackets has a logarithmic singularity in the limit $p_4 \to 0$. We get

$$\Sigma(p_4) \simeq \frac{g^2 C_F p_4}{4\pi^2} \int dk \frac{k}{k^2 + \eta^4 k^4} \simeq \frac{g^2 C_F}{12\pi^2} p_4 \log \left( \frac{\Lambda}{p_4} \right), \quad (15)$$

which is equal to the result of the one-loop calculation. This implies that as long as $g$ is small the one-loop self energy is a solution of the Dyson-Schwinger equation even in the regime $g^2 \log(\Lambda/p_4) \sim 1$. It also means that there are no contributions of the form $g^{2n}[\log(\Lambda/p_4)]^n$ with $n > 1$. A similar conclusion was reached by Polchinski in his analysis of the spinon gauge theory in 2+1 dimension [25].

We finally return to the question of including vertex corrections in the Dyson-Schwinger equation. Brown et al. showed that vertex corrections are not logarithmically enhanced except in a small kinematic window where the collinear momentum transfer, $\Delta l_p$, is much smaller than the energy transfer, $\Delta \omega$. This conclusion is unchanged if self energy corrections to the propagator are included. The regime $\Delta l_p \ll \Delta \omega$ does not contribute to the Dyson-Schwinger equation at leading order in the coupling constant. As a consequence vertex corrections do not have to be included at leading order in the weak coupling limit. This result is analogous to Migdal’s theorem for the electron-phonon interaction [31]. We should emphasize, however, that the physical picture that underlies Migdal’s theorem is quite different from what happens in the QCD case.

We have also checked that fermion self energy insertions do not modify the gluon self energy, Fig. 2b, at leading order in the coupling constant. This result is related to the fact that the leading term in the self energy depends only on the energy, $\omega$, and not on the momentum, $l_p$, of the quark.
\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|}
\hline
$\mu$ [GeV] & $\alpha_s$ & $m$ [GeV] & $\omega_{bcs}$ [GeV] & $\omega_{nfl}$ [GeV] \\
\hline
0.5 & 1.1 & 0.50 & $2.2 \cdot 10^{-2}$ & $7 \cdot 10^{-4}$ \\
1 & 0.52 & 0.70 & $2.2 \cdot 10^{-2}$ & $8 \cdot 10^{-7}$ \\
100 & 0.12 & 33 & $1.1 \cdot 10^{-2}$ & $2 \cdot 10^{-25}$ \\
$10^{10}$ & 0.029 & $1.7 \cdot 10^9$ & $7.9 \cdot 10^{-1}$ & $1 \cdot 10^{-98}$ \\
\hline
\end{tabular}
\caption{Characteristic scales in dense quark matter. We compare the dynamical screening mass $m$, the BCS scale $\omega_{bcs}$, and the scale of non-Fermi liquid effects $\omega_{nfl}$. We also show the one-loop running coupling constant $\alpha_s$ evaluated at the chemical potential $\mu$.}
\end{table}

IV. NUMERICAL ANALYSIS

In the previous section we presented analytic arguments which suggest that the infrared enhancement in the fermion self energy is one-loop exact in the weak coupling limit. In this section we shall strengthen these arguments by performing a numerical study of the Dyson-Schwinger equation (12). This will also provide an estimate of the size of higher order corrections at non-asymptotic densities.

Let us first give numerical estimates for the relevant scales. We shall assume that the strong coupling constant $\alpha_s$ is given by the $N_f = 3$ one-loop running coupling constant evaluated at the scale $\mu$. In Table I we compare the dynamical screening scale $m$, the scale of superconductivity $\omega_{bcs} \sim b_0 \mu g^{-5} \exp(-c_{bcs}/g)$, and the scale of non-Fermi liquid effects $\omega_{nfl} \sim \frac{m}{\exp(-c_{nfl}/g^2)}$. We use $c_{bcs} = 3\pi^2/\sqrt{2}$ and $c_{nfl} = 9\pi^2$ given above as well as $b_0 = 512\pi^4 \exp(-\left(\pi^2 + 4\right)/8)$. Even if the chemical potential is very large the screening scale is close to the Fermi energy. The scale of superfluidity varies very little whereas the scale of non-Fermi liquid effects becomes extremely small if the chemical is large.

We have solved the Dyson-Schwinger equation for two different values of the chemical potential, a low value of $\mu = 0.5$ GeV relevant for the physics of neutron stars and an asymptotically large value of $\mu = 100$ GeV. At the high scale perturbation theory is expected to be applicable but at the lower scale the coupling is not small and the usefulness of perturbation theory is in doubt. It was shown, however, that the numerical solution of the gap equation in the superconducting phase is close to the asymptotic solution even if the coupling is not small, and that the gap on the Fermi surface is quite consistent with values
FIG. 3: Numerical solution of the Dyson-Schwinger equation for the fermion self energy. We show the euclidean self energy $\Sigma(p_4)$ as a function of $p_4$. The two sets of curves correspond to two different chemical potentials, $\mu = 0.5$ GeV and $\mu = 100$ GeV. The solid lines show the numerical solution, the dashed curves show the analytic result for the leading logarithm, and the dotted curves show a power-like self energy $\Sigma(p_4) \sim p_4(\Lambda/p_4)^\gamma$.

obtained from phenomenological models [32]. We have chosen the cutoff of the effective theory to be equal to the dynamical screening scale $m$.

Our results are shown in Fig. 3. The solid line shows the numerical solution and the dashed line shows the analytic result for the leading logarithm, equ. (8). For comparison we also show a self energy function that scales as a fractional power of energy, $\Sigma(p_4) = p_4(\Lambda/p_4)^\gamma$ with $\gamma = g^2/(9\pi^2)$. This behavior was proposed by Boyanovsky and de Vega on the basis of
FIG. 4: Comparison between the numerical solution of the Dyson-Schwinger equation and the analytic leading log result. The solid lines show the ratio of the numerical solution over the leading log term for two different chemical potentials. The dotted lines show the same ratio for the power-like self energy $\Sigma(p_4) \sim p_4(\Lambda/p_4)^\gamma$.

For very small values of the energy, $p_4 \ll \Lambda$, we find excellent agreement between the numerical results and the leading log expression. This is seen even more clearly in Fig. 4 where we show the ratio of the numerical solution over the leading logarithm. Significant deviations only occur for energies near the cutoff, but some higher order corrections are still present for energies one or two orders of magnitude below the cutoff. Our results show no evidence for higher order terms of the form $g^{2n} \log^n(p_4)$ with $n > 1$ and support the arguments given in the previous section. We conclude that in the limit of weak coupling and low energy the self energy is given by equ. (8) even in the regime where the logarithm is large.
Boyanovsky and Vega argued, on the basis of a renormalization group analysis, that the inverse quark propagator is of the form 

\[ S^{-1}(\omega, l) = \omega (\omega / \Lambda)^\gamma - l \]

where \( \gamma = g^2 / (9\pi^2) \) in the weak coupling limit. A similar result in the QED case was presented by Gan and Wong [23]. These arguments contradict the solution of the Dyson-Schwinger equation presented in the preceding two sections. In order to clarify the situation we consider the renormalization group equation for the two-point function in the high density effective theory. We shall consider the leading order lagrangian

\[
\mathcal{L} = \psi_\dagger (\omega - v_F l) \psi_v + g v_F \psi_\dagger \hat{v} \cdot \vec{A} \psi_v + \ldots ,
\]

where we have explicitly included the Fermi velocity \( v_F \). This is necessary because at finite density Lorentz invariance is broken and we need separate wave function renormalization factors for the energy and momentum dependent terms in the action. The one-loop fermion self energy is given by

\[
\Sigma(\omega, l) = \frac{g^2 v_F}{9\pi^2} \omega \log \left( \frac{\Lambda}{\omega} \right),
\]

where we have neglected terms that do not contain logarithms. Equ. (18) has a logarithmic divergence in the effective field theory. This divergence can be removed by adding a counter-term to the lagrangian. We define the bare lagrangian

\[
\mathcal{L}_0 = \mathcal{L} + \mathcal{L}_{ct}
\]

where

\[
\mathcal{L}_0 = \psi_\dagger (Z\omega - ZZ_F v_F l) \psi_v + Z_g g v_F \psi_\dagger \hat{v} \cdot \vec{A} \psi_v
\]

\[
= \psi_\dagger (\omega - v_{0,F} l) \psi_{0,v} + g_0 v_{0,F} \psi_\dagger \hat{v} \cdot \vec{A} \psi_{0,v},
\]

as well as the bare fields and coupling constants

\[
\psi_{0,v} = Z^{1/2} \psi_v, \quad v_{0,F} = Z_F v_F, \quad g_0 = \frac{Z_g}{ZZ_F} g.
\]

Equ. (18) implies that at one-loop order \( Z \sim \log(\Lambda) \) and that \( ZZ_F = 1 \). We showed in the previous section that, in the kinematic regime of interest, there is no logarithmic divergence in the quark-gluon vertex. As a consequence we can take \( Z_g = 1 \). Equ. (18) also suggests, however, that the effective coupling constant is

\[
\alpha = \frac{g^2 v_F}{4\pi},
\]
which acquires an anomalous dimension because of the scaling of the Fermi velocity. The bare and renormalized Green functions are related by

\[ G_0^{(n)}(\omega_i, v_{0,Fi}, \alpha_0) = Z^{n/2} G^{(n)}(\omega_i, v_{Fi}, \alpha), \]

where \( n \) denotes the number of external fermion fields, \((\omega_i, l_i)\) are the external energies and momenta. Differentiating this relation with respect to \( \Lambda \) gives the renormalization group equation

\[ \left\{ \Lambda \frac{\partial}{\partial \Lambda} + \beta(\alpha) \frac{\partial}{\partial \alpha} - \gamma_F(\alpha) l_i \frac{\partial}{\partial l_i} + \frac{n}{2} \gamma(\alpha) \right\} G^{(n)}(\omega_i, l_i, \alpha) = 0, \]

where we have defined the beta function and the anomalous dimensions

\[ \beta(\alpha) = \Lambda \frac{\partial \alpha}{\partial \Lambda}, \quad \gamma(\alpha) = \Lambda \frac{\partial \log Z}{\partial \Lambda}, \quad \gamma_F(\alpha) = \Lambda \frac{\partial \log Z_F}{\partial \Lambda}. \]

In deriving eq. (23) we have used the fact that \( G^{(n)} \) depends on the Fermi velocity only through \( \alpha \) and \( v_{Fi} \). At one-loop order we have

\[ \beta(\alpha) = -\gamma_F(\alpha) \alpha, \quad \gamma(\alpha) = -\gamma_F(\alpha) = \frac{4\alpha}{9\pi}. \]

We note that the beta function is positive and the effective theory is infrared free. This means that the perturbative analysis of the low energy behavior is reliable. The fact that the effective coupling is weak at low energy is related to the fact that the Fermi velocity vanishes as the quasi-particle energy goes to zero. We have checked that the one-loop anomalous dimensions do not depend on the gauge parameter in a generalized Coulomb gauge. In general, of course, there is no reason to expect the anomalous dimensions to be gauge invariant. Only physical properties of the solutions of the renormalization group equation, such as the quasi-particle properties, are gauge invariant.

Boyanovsky and de Vega solved the renormalization group equation under the assumption that the beta function vanishes \cite{9}. In this case we have

\[ \left\{ \Lambda \frac{\partial}{\partial \Lambda} + \gamma \left[ l \frac{\partial}{\partial l} + 1 \right] \right\} S(w, l, \alpha) = 0, \]

where we have used \( \gamma_F = -\gamma \). We observe that the inverse propagator satisfies the renormalization group equation \{\( \Lambda \partial/(\partial \Lambda) + \gamma[l \partial/(\partial l) - 1] \} S^{-1}(\omega, l, \alpha) = 0 \}. It is easy to see that this equation is solved by

\[ S^{-1}(\omega, l) = \omega \left( \frac{\Lambda}{\omega} \right)^\gamma - v_{Fl}, \]
where we have imposed the boundary condition $S^{-1}(\omega = \Lambda, l) = \omega - v_F l$. Equ. (27) is the result of Boyanovksy and Vega. We showed, however, that the beta function does not vanish. The complete renormalization group equation is

$$\left\{ \Lambda \frac{\partial}{\partial \Lambda} + \gamma \left[ \alpha \frac{\partial}{\partial \alpha} + i \frac{\partial}{\partial l} + 1 \right] \right\} S(w, l, \alpha) = 0. \quad (28)$$

The propagator with the one-loop self energy included

$$S^{-1}(\omega, l) = \omega \left( 1 + \gamma \log \left( \frac{\Lambda}{\omega} \right) \right) - v_F l \quad (29)$$

is a solution of the complete renormalization group equation. We can also study the possible presence of higher order terms of the form $S^{-1} \sim \alpha^n \log^n(\Lambda/\omega)$. Consider the ansatz

$$S^{-1}(\omega, l) = \sum_k a_k \alpha^k \omega \left[ \log \left( \frac{\Lambda}{\omega} \right) \right]^k - v_F l. \quad (30)$$

Inserting this ansatz into the renormalization group equation we obtain $a_{k+1} = a_k \gamma_1 (k - 1)/(k + 1)$, where we have used the one-loop anomalous dimension $\gamma(\alpha) = \gamma_1 \alpha$. This shows that $a_k = 0$ for $k > 2$ and terms of order $\alpha^2 \log^2(\Lambda/\omega)$ or higher are absent.

VI. SUMMARY AND DISCUSSION

We have studied non-Fermi liquid effects due to unscreened transverse gauge boson exchanges in the normal phase of high density QCD. We find that if the coupling is weak the fermion self energy is given by $\gamma \omega \log(\Lambda/\omega)$ with $\gamma = g^2/(9\pi^2)$. This result is reliable even if $\gamma \log(\Lambda/\omega) \sim 1$. We established this result using two different methods, the Dyson-Schwinger equation and the renormalization group. In the context of the Dyson-Schwinger equation the absence of higher order corrections is a consequence of the special kinematics of ungapped fermions interacting with Landau damped gluons. In the kinematic regime of interest the right hand side of the Dyson-Schwinger equation is independent of the fermion self energy. As a consequence, there is no difference between the one-loop result and the self-consistent solution. In the context of the renormalization group the absence of higher order terms follows from the relations $\gamma = -\gamma_F$ and $\beta = -\gamma_F \alpha$. The relation between the anomalous dimension of the fermion field and the Fermi velocity is again due to the special kinematics. The relation between the beta function and the anomalous dimension of the Fermi velocity is a consequence of gauge invariance.
The weak coupling result implies that the quark propagator has a cut rather than a pole, and that the naive quasi-particle description breaks down. The spectral density is given by
\[ \rho(\omega) = \frac{\gamma \omega}{[\omega(1 + \gamma \log(\Lambda/\omega)) - l]^2 + \pi^2 \gamma^2 \omega^2}. \] (31)
For non-zero momentum \( l \) this is approximately a Breit-Wigner distribution, but the wave function normalization and Fermi velocity vanish as \( l \to 0 \). An important consequence of the breakdown of Fermi liquid theory is an anomalous term in the specific heat. Ipp et al. showed that
\[ C_{\text{anom}} = \gamma C_{\text{free}} \log \left( \frac{\Lambda}{T} \right) = N_f (N_c^2 - 1) \frac{g^2 \mu^2 T}{72 \pi^2} \log \left( \frac{\Lambda}{T} \right), \] (32)
where \( C_{\text{free}} = N_c N_f \mu^2 T / 3 \). They also computed the argument of the logarithm as well as terms that include fractional powers \( T^{5/3} \) and \( T^{7/3} \). Our results suggest that equ. (32) is reliable even if \( g^2 \log(\Lambda/T) \gg 1 \). The anomalous term in the fermion self energy does not lead to an anomalous term in the thermodynamic potential \( \Omega \) at \( T = 0 \). The two-loop contribution to \( \Omega \) is infrared finite. Instead, this graph has an ultraviolet divergence in the effective field theory. This means that the thermodynamic potential has to be determined in the microscopic theory. The result is
\[ \Omega = -\frac{N_f N_c \mu^4}{12 \pi^2} \left\{ 1 - \frac{3(N_c^2 - 1)}{4N_c} \left( \frac{\alpha_s}{\pi} \right) + O(\alpha_s^2) \right\}. \] (33)
In the superconducting phase the infrared enhancement in the fermion self energy is cutoff for energies less than the gap. Because \( \Delta \sim \exp(-c_{bcs}/g) \) the correction to the self energy never exceeds \( \gamma \log(\Delta) \sim O(g) \). As a consequence non-Fermi liquid effects in the normal phase do not qualitatively alter the superconducting phase of QCD, but they give a correction to the gap which is enhanced by one power of \( 1/g \) relative to its naive order in the coupling constant. This correction reduces the gap by a factor \( \exp[-(\pi^2 + 4)(N_c - 1)/16] \sim 0.18 \), where we have set \( N_c = 3 \). \[ 20, 34, 35, 36 \]

In this paper we did not study Green functions with more than two external fermion lines. It is not clear whether more complicated \( n \)-point functions exhibit additional infrared divergences. It would be interesting, for example, to study the propagation of zero sound in the normal phase of dense quark matter. With regard to the physics of neutron stars it would also be interesting to study the thermal conductivity as well as the neutrino emissivity and opacity.
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