Classification of irreducible modules of $\mathcal{W}_3$ algebra with $c = -2$

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Abstract

We construct irreducible modules $V_\alpha, \alpha \in \mathbb{C}$ over $\mathcal{W}_3$ algebra with $c = -2$ in terms of a free bosonic field. We prove that these modules exhaust all the irreducible modules of $\mathcal{W}_3$ algebra with $c = -2$. Highest weights of modules $V_\alpha, \alpha \in \mathbb{C}$ with respect to the full (two-dimensional) Cartan subalgebra of $\mathcal{W}_3$ algebra are $\left(\frac{1}{2}\alpha(\alpha - 1), \frac{1}{6}\alpha(\alpha - 1)(2\alpha - 1)\right)$. They are parametrized by points $(t, w)$ on a rational curve $w^2 - \frac{1}{5}t^2(8t + 1) = 0$. Irreducible modules of vertex algebra $\mathcal{W}_{1+\infty}$ with $c = -1$ are also classified.

0 Introduction

In the study of two-dimensional conformal field theories extensions of conformal symmetry play an important role. The algebraic structures underlying the extended conformal symmetry are usually known as $\mathcal{W}$-algebras in literatures (see [BS, FF] and references therein). Mathematically $\mathcal{W}$-algebras can be put into the general framework of the theory of vertex algebras formulated first by Borcherds, cf. e.g. [B1, FLM, DL, LZ, FKRW, K, B2].

In contrast to vertex algebras associated to the Virasoro algebra, $\mathcal{W}$-algebras such as $\mathcal{W}_N$ algebras, have the feature that non-linearity terms appear in the operator product expansion of two generating fields, namely the commutator of two generators contains non-linear terms expressed by these generators themselves. Mainly due to the non-linear nature of $\mathcal{W}$-algebras, the study of their representation theory has been difficult and very non-trivial. Even the understanding of
representation theory of the Zamolodchikov $\mathcal{W}_3$ algebra $[Z3]$, which is the simplest example of $\mathcal{W}$-algebras beyond the Virasoro algebra, is far from satisfactory (see however $[BMP]$).

Apart from the Virasoro generators $L_n$, $n \in \mathbb{Z}$, $\mathcal{W}_3$ algebra has an additional set of generators $W_n$, $n \in \mathbb{Z}$. Denote by $\mathcal{U}(\mathcal{W}_3)$ the corresponding universal enveloping algebra. Define two generating series $T(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$ and $W(z) = \sum_{n \in \mathbb{Z}} W_n z^{-n-3}$.

It is well known that the vacuum module $\mathcal{V}\mathcal{W}_3$,c with central charge $c$ carries a vertex algebra structure. For a generic central charge $c$, $\mathcal{V}\mathcal{W}_3$,c is an irreducible representation of $\mathcal{U}(\mathcal{W}_3)$. In this case, the representation theory of vertex algebra $\mathcal{V}\mathcal{W}_3$,c is the same as that of $\mathcal{U}(\mathcal{W}_3)$. For a non-generic central charge $c$, $\mathcal{V}\mathcal{W}_3$,c is reducible and admits a unique maximal proper $\mathcal{U}(\mathcal{W}_3)$-submodule $I$ and thus a unique irreducible quotient, which is denoted by $\mathcal{W}_3$,c. $\mathcal{W}_3$,c inherits a vertex algebra structure from $\mathcal{V}\mathcal{W}_3$,c. Representation theory of $\mathcal{W}_3$,c with non-generic central charge $c$ becomes highly non-trivial since a module $M$ of $\mathcal{U}(\mathcal{W}_3)$ can be regarded as a module of $\mathcal{W}_3$,c if and only if the Fourier components of any field corresponding to any vector in $I$ annihilates the whole $M$.

In $[W2]$, in studying the vertex algebra $\mathcal{W}_{1+\infty}$ with central charge $-1$ (denoted by $\mathcal{W}_{1+\infty,-1}$) we explicitly constructed a number of irreducible modules of $\mathcal{W}_{3,-2}$ parametrized by integers and obtained full character formulas for these modules. We showed that the vertex algebra $\mathcal{W}_{1+\infty,-1}$ is isomorphic to a tensor product of $\mathcal{W}_{3,-2}$ and a Heisenberg vertex algebra generated by a free bosonic field by using Friedan-Martinec-Shenker bosonization technique $[FMS]$.

In this paper, we will continue the study of representation theory of $\mathcal{W}_{3,-2}$ and $\mathcal{W}_{1+\infty,-1}$. Note that $-2$ is a non-generic central charge for $\mathcal{W}_3$ algebra. We will explicitly construct irreducible modules $V_{\alpha}, \alpha \in \mathbb{C}$ of $\mathcal{W}_{3,-2}$ in terms of a free bosonic field. Then by locating key singular vectors in $\mathcal{V}\mathcal{W}_{3,-2}$ and then applying Zhu’s machinery $[Z]$ to our case we are able to prove that $V_{\alpha}, \alpha \in \mathbb{C}$ exhaust all the irreducible modules of $\mathcal{W}_{3,-2}$. It turns out that the set of all irreducibles of $\mathcal{W}_{3,-2}$ has an elegant description: highest weights of these irreducible modules are parametrized by points of a rational curve defined by $w^2 - \frac{1}{9}t^2(8t+1) = 0$. Combining with our results in $[W2]$ we also construct and classify all the irreducible modules of $\mathcal{W}_{1+\infty,-1}$. This latter classification result disproves a conjecture of Kac and Radul $[KR2]$.

Let us explain in more detail. Given a pair of $bc$ fields $b(z) = \sum_{n \in \mathbb{Z}} b(n) z^{-n}$ and $c(z) = \sum_{n \in \mathbb{Z}} c(n) z^{-n-1}$, we construct a Fock space
\( \mathcal{F} \) generated by the vacuum vector \( |bc\rangle \), satisfying
\[
b(n + 1)|bc\rangle = 0, \quad c(n)|bc\rangle = 0, \quad n \geq 0.
\]

Then \( j(z) =: b(z)c(z) := \sum_{n \in \mathbb{Z}} j_n z^{-n-1} \) is a free bosonic field. Take a scalar field \( \psi(z) \) such that \( j(z) = \partial \psi(z) \). Denote by \( \mathcal{H}_\alpha \) the Fock space of the Heisenberg algebra \( \{j_n, n \in \mathbb{Z}\} \) with vacuum vector \( |\alpha\rangle \) satisfying
\[
j_n |\alpha\rangle = \alpha \delta_{n,0} |\alpha\rangle, \quad n \geq 0.
\]

It is observed in [BCMN, W2] that the fields
\[
T(z) =: \partial b(z)c(z), \quad W(z) = \frac{1}{\sqrt{6}} \left( : \partial^2 b(z)c(z) : - : \partial b(z) \partial c(z) : \right),
\]
satisfy the \( \mathcal{W}_3 \) operator product expansions with central charge \(-2\). We can rewrite the fields \( T(z) \) and \( W(z) \) in terms of \( j(z) \) by means of boson-fermion correspondence (Proposition 3.1). \( \mathcal{F}_0 \) is thus isomorphic to \( \mathcal{H}_0 \). It is shown [W2] that the simple vertex algebra \( \mathcal{W}_{3,-2} \) is a vertex subalgebra of \( \mathcal{F}_0 \) and can be identified explicitly inside \( \mathcal{F}_0 \). Denote by \( V_\alpha \) the irreducible quotient of the \( \mathcal{W}_{3,-2} \)-submodule of \( \mathcal{H}_\alpha \) generated by the highest weight vector \( |\alpha\rangle \) in \( \mathcal{H}_\alpha \). Let \( \widetilde{W}(z) \equiv \sum_{n \in \mathbb{Z}} \widetilde{W}_n z^{-n-3} = \frac{1}{2} \sqrt{6} W(z) \). We will show that the highest weight of \( V_\alpha \) with respect to the full Cartan subalgebra \( \{L_0, \widetilde{W}_0\} \) of \( \mathcal{W}_{3,-2} \) is
\[
\left( \frac{1}{2} \alpha (\alpha - 1), \frac{1}{6} \alpha (\alpha - 1)(2\alpha - 1) \right).
\]

To show that the above irreducible modules \( V_\alpha, \alpha \in \mathbb{C} \) exhaust all the irreducible modules of \( \mathcal{W}_{3,-2} \), we invoke a powerful machinery due to Zhu in the general theory of vertex algebras [Z]. Zhu constructed an associative algebra \( A(V) \) for any vertex algebra \( V \) such that irreducible modules of the vertex algebra \( V \) one-to-one correspond to irreducible modules of the associative algebra \( A(V) \). Zhu’s constructions were generalized to vertex superalgebras in [KW]. By construction, the Zhu associative algebra \( A(V) \) is a certain quotient of \( V \). We denote by \([a]\) the image in \( A(V) \) of \( a \in V \). By studying the associative algebra \( A(V) \), one can often obtain useful information on highest weights of modules over \( V \).

We show that the Zhu associative algebra \( A(\mathcal{WW}_{3,-2}) \) is isomorphic to a polynomial algebra \( \mathbb{C}[t, w] \), where \( t \) and \( w \) correspond to elements \([L_{-2}|0\rangle]\) and \([\widetilde{W}_{-3}|0\rangle]\) in \( A(\mathcal{WW}_{3,-2}) \) respectively. Using some explicit results on singular vectors in \( \mathcal{WW}_{3,-2} \), we further show that Zhu associative algebra \( A(\mathcal{W}_{3,-2}) \) is isomorphic to (some quotient of) the quotient
We observe that all the solutions to the equation above can be written as of the form (0.1). But we have already constructed irreducible modules \( V_\alpha \) of \( \mathcal{W}_{3, -2} \) with a highest weight of any such form. This shows that \( V_\alpha \) (\( \alpha \in \mathbb{C} \)) are all irreducible \( \mathcal{W}_{3, -2} \)-modules and their highest weights are parametrized by points on the rational curve defined by the equation (0.2). The equation (0.2) as a necessary constraint on the highest weights of irreducible \( \mathcal{W}_{3, -2} \)-modules was anticipated in [H, EFHN] by some other arguments. 

In the remaining part of this paper we classify all the irreducible modules of vertex algebra \( \mathcal{W}_{1+\infty, -1} \). Recall in our paper [W2], we have shown that the vertex algebra \( \mathcal{W}_{1+\infty, -1} \) is isomorphic to a tensor product of the vertex algebra \( \mathcal{W}_{3, -2} \) and a Heisenberg vertex algebra generated by a free bosonic field. Therefore the classification of irreducible modules \( \mathcal{W}_{1+\infty, -1} \) follows from our classification of irreducible modules of \( \mathcal{W}_{3, -2} \) and the well-known description of all irreducible modules of a Heisenberg vertex algebra.

This paper is organized as follows. In Section 1, we recall the definition of a vertex algebra and review Zhu’s associative algebra theory. In Section 2, we recall the \( \mathcal{W}_3 \) algebra and study the case with central charge \(-2\) in some detail. In Section 3 we construct irreducible modules \( V_\alpha \) (\( \alpha \in \mathbb{C} \)) of \( \mathcal{W}_{3, -2} \) and determine their highest weights. In section 4 we calculate Zhu algebra \( A(\mathcal{W}_{3, -2}) \) and show that the list of irreducible modules constructed in Section 3 is complete. In Section 5 we classify all irreducible modules of \( \mathcal{W}_{1+\infty, -1} \).

1 Vertex algebras and Zhu’s associative algebra theory

Our definition of vertex algebras basically follows [FKRW, K]. It is known that our definition is essentially equivalent to other formulations in [B1, FLM, LZ, B2, B3]. Though it is not essential to have a gradation

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1I thank A. Honecker for mentioning these references to me.
in the definition of vertex algebras, we choose to keep it in this paper in order to present Zhu’s associative algebra theory.

**Definition 1.1** A vertex algebra consists of the following data: a \( \mathbb{Z}_+ \)-graded vector space \( V = \bigoplus_{n \in \mathbb{Z}_+} V_n \); a vector \( |0\rangle \in V \) (called the vacuum vector); an operator \( L_0 \) (called the degree operator) and an operator \( T \in \text{End} V \) (called the translation operator); a linear map from \( V \) to the space of fields \( a \mapsto Y(a,z) = \sum_{n \in \mathbb{Z}} a(n) z^{-n-1} \in \text{End} V[[z, z^{-1}]] \) (called the state-field correspondence). These data satisfy the following axioms:

- \((V)\) \( Y(|0\rangle, z) = I_V, \quad Y(a(z)|0\rangle|z=0 = a; \)
- \((G)\) \( L_0|V_n = nI_{V_n}, \quad [L_0, Y(a(z)] = \partial_z Y(a,z) + Y(L_0a(z); \)
- \((T)\) \( [T, Y(a(z)] = \partial_z Y(a,z), \quad T|0\rangle = 0; \)
- \((L)\) \( (z-w)^N[Y(a(z), Y(b,w)] = 0 \quad \text{for} \quad N \gg 0. \)

For \( a \in V_n, n \) is called the weight of \( a \), denoted by \( \text{wt} a \). Denote by \( o(a) = a(\text{wt} a - 1) \) for homogeneous \( a \in V \) and extends by linearity to the whole \( V \). The results of the remaining part of this section are due to Zhu [Z]. We refer the readers to [Z] for more detail.

**Definition 1.2** Define two bilinear operations \( \ast \) and \( \circ \) on \( V \) as follows. For a homogeneous, let

\[
\begin{align*}
a \ast b &= \text{Res}_z \left( Y(a,z) \frac{(z+1)^{\text{wt} a}}{z} b \right), \\
a \circ b &= \text{Res}_z \left( Y(a,z) \frac{(z+1)^{\text{wt} a}}{z^2} b \right),
\end{align*}
\]

then extend to \( V \times V \) by bilinearity. Denote by \( O(V) \) the subspace of \( V \) spanned by elements \( a \circ b \), and by \( A(V) \) the quotient space \( V/O(V) \).

It is convenient to introduce an equivalence relation \( \sim \) as in [W1]. For \( a, b \in V \), \( a \sim b \) means \( a - b \equiv 0 \mod O(V) \). For \( f, g \in \text{End} V \), \( f \sim g \) means \( f \cdot c \sim g \cdot c \) for any \( c \in V \). Denote by \([a]\) the image of \( a \) in \( V \) under the projection of \( V \) onto \( A(V) \).

**Lemma 1.1** 1) \( T + L_0 \sim 0. \)
2) For every homogeneous element \( a \in V \), and \( m \geq n \geq 0 \), one has

\[
\text{Res}_z \left( Y(a, z) \frac{(z + 1)^{\text{wt} \ a + n}}{z^{2+m}} \right) \sim 0.
\]

3) For homogeneous elements \( a, b \in V \), one has

\[
a \ast b \sim \text{Res}_z \left( Y(b, z) \frac{(z + 1)^{\text{wt} \ b - 1}}{z^a} \right).
\]

**Theorem 1.1**

1) \( O(V) \) is a two-sided ideal of \( V \) under the multiplication \( \ast \). Moreover, the quotient algebra \( (A(V), \ast) \) is associative.

2) \([1]\) is the unit element of the algebra \( A(V) \).

In the case that the vertex algebra \( V \) contains a Virasoro element \( \omega \), i.e. the corresponding field \( Y(\omega, z) \) is an energy-momentum tensor field, we have

**Lemma 1.2** \([\omega]\) is in the center of the associative algebra \( A(V) \).

The following proposition follows from the definition of \( A(V) \).

**Proposition 1.1** Let \( I \) be an ideal of \( V \). Then the associative algebra \( A(V/I) \) is isomorphic to \( A(V)/[I] \), where \([I]\) is the image of \( I \) in \( A(V) \).
2 \( \mathcal{W}_3 \) algebra with central charge \(-2\)

Denote by \( \mathcal{U}(\mathcal{W}_3) \) the quotient of the free associative algebra generated by \( L_m, W_m, \ m \in \mathbb{Z}, \) by the ideal generated by the following commutation relations (cf. e.g. [BMP]):

\[
\begin{align*}
[L_m, L_n] & = (m - n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,-n}, \\
[L_m, W_n] & = (2m - n)W_{m+n}, \\
[W_m, W_n] & = (m - n)(\frac{1}{15}(m + n + 3)(m + n + 2) - \frac{1}{6}(m + 2)(n + 2))L_{m+n} \\
& \quad + \beta(m - n)\Lambda_{m+n} + \frac{c}{360}m(m^2 - 1)(m^2 - 4)\delta_{m,-n},
\end{align*}
\]

where \( c \in \mathbb{C} \) is the central charge, \( \beta = 16/(22 + 5c) \) and

\[
\Lambda_m = \sum_{n \leq -2} L_n L_{m-n} + \sum_{n > -2} L_{m-n} L_n - \frac{3}{10}(m + 2)(m + 3)L_m.
\]

Denote

\[
\mathcal{W}_{3,\pm} = \{L_n, W_n, \pm n \geq 0\}, \quad \mathcal{W}_{3,0} = \{L_0, W_0\}.
\]

A Verma module \( \mathcal{M}_c(t, w) \) (or \( \mathcal{M}(t, w) \) whenever there is no confusion of central charge) of \( \mathcal{U}(\mathcal{W}_3) \) is the induced module

\[
\mathcal{M}(t, w) = \mathcal{U}(\mathcal{W}_3) \otimes_{\mathcal{U}(\mathcal{W}_{3,+} \oplus \mathcal{W}_{3,0})} \mathcal{C}_{t,w}
\]

where \( \mathcal{C}_{t,w} \) is the 1-dimensional module of \( \mathcal{U}(\mathcal{W}_{3,+} \oplus \mathcal{W}_{3,0}) \) generated by a vector \( |t, w\rangle \) such that

\[
\mathcal{W}_{3,+} |t, w\rangle = 0, \quad L_0 |t, w\rangle = t |t, w\rangle, \quad W_0 |t, w\rangle = w |t, w\rangle.
\]

\( \mathcal{M}(t, w) \) has a unique irreducible quotient which is denoted by \( \mathcal{L}(t, w) \) (or \( \mathcal{L}_c(t, w) \) when it is necessary to specify the central charge). A singular vector in a \( \mathcal{U}(\mathcal{W}_3) \)-module means a vector killed by \( \mathcal{W}_{3,+} \). For simplicity, we denote the vacuum vector \( |0, 0\rangle \) by \( |0\rangle \) in the case \( t = w = 0 \). It is easy to see that \( L_{-1}|0\rangle, W_{-1}|0\rangle, \) and \( W_{-2}|0\rangle \) are singular vectors in \( \mathcal{M}(0, 0) \). We denote by \( \mathcal{V}\mathcal{W}_{3,c} \) the vacuum module which is by definition the quotient of Verma module \( \mathcal{M}(0, 0) \) by the \( \mathcal{U}(\mathcal{W}_3) \)-submodule generated by the singular vectors \( L_{-1}|0\rangle, W_{-1}|0\rangle, \) and \( W_{-2}|0\rangle \). We also call \( \mathcal{L}(0, 0) \) the irreducible vacuum module. Let
$I$ be the maximal proper submodule of the Verma vacuum module $\mathcal{V}W_{3,c}$. Clearly $L(0,0)$ is the irreducible quotient of $\mathcal{V}W_{3,c}$. It is easy to see that $\mathcal{V}W_{3,c}$ has a linear basis

$$L_{-i_1-2} \cdots L_{-i_m-2}W_{-j_1-3} \cdots W_{-j_n-3}|0\rangle,$$

$$0 \leq i_1 \leq \cdots \leq i_m, \quad 0 \leq j_1 \leq \cdots \leq j_n, \quad m, n \geq 0.$$

The action of $L_0$ on $\mathcal{V}W_{3,c}$ gives rise to a principal gradation on $\mathcal{V}W_{3,c}$: $\mathcal{V}W_{3,c} = \bigoplus_{n \in \mathbb{Z}} (\mathcal{V}W_{3,c})^n$. Introduce the following fields

$$T(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}, \quad W(z) = \sum_{n \in \mathbb{Z}} W_n z^{-n-3}.$$

It is well known that the vacuum module $\mathcal{V}W_{3,c}$ (resp. irreducible vacuum module $L(0,0)$) carries a vertex algebra structure with generating fields $T(z)$ and $W(z)$. The $\mathcal{W}_3$ algebra with central charge $-2$ we have been referring to is the vertex algebra $L_{-2}(0,0)$, which we denote by $\mathcal{W}_{3,-2}$ in this paper. Fields $T(z)$ and $W(z)$ correspond to the vectors $L_{-2}|0\rangle$ and $W_{-3}|0\rangle$ respectively. The field corresponding to the vector $L_{-i_1-2} \cdots L_{-i_m-2}W_{-j_1-3} \cdots W_{-j_n-3}|0\rangle$ is

$$\partial^{(i_1)} T(z) \cdots \partial^{(i_m)} T(z) \partial^{(j_1)} W(z) \cdots \partial^{(j_n)} W(z),$$

where $\partial^{(i)}$ denotes $\frac{1}{i!} \partial^i$.

From now on we concentrate on the case of $\mathcal{W}_3$ algebra with central charge $c = -2$. We can rewrite (2.3) as the following OPEs in our central charge $-2$ case:

$$T(z)T(w) \sim -\frac{1}{(z - w)^4} + \frac{2T(w)}{(z - w)^2} + \frac{\partial T(w)}{z - w},$$

$$T(z)W(w) \sim 3\frac{W(w)}{(z - w)^2} + \frac{\partial W(w)}{z - w},$$

$$W(z)W(w) \sim -\frac{2/3}{(z - w)^6} + \frac{2T(w)}{(z - w)^4} + \frac{\partial T(w)}{(z - w)^3}$$

$$+ \frac{1}{(z - w)^2} \left( \frac{8}{3} : T(w)T(w) : - \frac{1}{2} \partial^2 T(w) \right)$$

$$+ \frac{1}{z - w} \left( \frac{4}{3} \partial : T(w)T(w) : - \frac{1}{3} \partial^3 T(w) \right).$$

Representation theory of the vertex algebra $\mathcal{V}W_{3,c}$ is just the same as that of $\mathcal{U}(\mathcal{W}_3)$. We see from the following lemma that $\mathcal{V}W_{3,-2}$ is reducible so its maximal proper submodule $I$ is not zero. Representation theory of $\mathcal{W}_{3,-2}$ becomes highly non-trivial due to the following
constraints: a module $M$ of the vertex algebra $\mathcal{VW}_{3,-2}$ can be a module of $\mathcal{W}_{3,-2}$ if and only if $M$ is annihilated by all the Fourier components of all fields corresponding to vectors in $I \subset \mathcal{VW}_{3,-2}$.

So it is important to find information of (top) singular vectors in the vacuum module $\mathcal{VW}_{3,-2}$. The following lemma can be proved by a tedious however direct calculation:

**Lemma 2.1**

1) There is no singular vector in $(\mathcal{VW}_{3,-2})_n$, $n \leq 5$.

2) There are two independent singular vectors in $(\mathcal{VW}_{3,-2})_6$, denoted by $v_s$ and $v_s'$:

$$v_s \equiv \left( \frac{3}{2} W_3 - \frac{19}{36} L_3^2 - \frac{8}{9} L_3 - \frac{14}{9} L_2 L_4 + \frac{44}{9} L_6 \right) |0\rangle,$$

$$v_s' \equiv \left( \frac{9}{2} W_6 + 9 L_3 W_3 - 6 L_2 W_4 \right) |0\rangle.$$  

3) $v_s' = \frac{27}{58} W_0 (v_s)$, $v_s = \frac{1}{36} W_0 (v_s')$. Equivalently we have

4) $W_0 \left( 6v_s \pm \frac{99}{27} v_s' \right) = \pm 6 \left( 6v_s \pm \frac{99}{27} v_s' \right)$.

**Remark 2.1** Vectors $v_s, v_s'$ are not singular vectors in the Verma module $\mathcal{M}(0,0)$.

### 3 Irreducible modules $V_\alpha (\alpha \in \mathfrak{c})$ of $\mathcal{W}_{3,-2}$

We first recall how we realize the $\mathcal{W}_{3,-2}$ algebra in terms of a pair of fermionic $bc$ fields $[W2]$. Take a pair of $bc$ fields

$$b(z) = \sum_{n \in \mathbb{Z}} b(n) z^{-n}, \quad c(z) = \sum_{n \in \mathbb{Z}} c(n) z^{-n-1}$$

with OPEs

$$b(z)c(w) \sim \frac{1}{z - w}, \quad b(z)b(w) \sim 0, \quad c(z)c(w) \sim 0. \quad (3.8)$$

Equivalently, we have the following commutation relations:

$$[b(m), c(n)]_+ = \delta_{m,-n}, \quad [b(m), b(n)]_+ = 0, \quad [c(m), c(n)]_+ = 0.$$

We denote by $\mathcal{F}$ the Fock space of the $bc$ fields, generated by $|bc\rangle$, satisfying

$$b(n+1)|bc\rangle = 0, \quad c(n)|bc\rangle = 0, \quad n \geq 0.$$
Then

\[ j(z) =: b(z)c(z) := \sum_{n \in \mathbb{Z}} j_n z^{-n-1} \]

is a free boson of conformal weight 1 with commutation relations

\[ [j_m, j_n] = m \delta_{m,-n}, \quad m, n \in \mathbb{Z}. \]

We further have the following commutation relations:

\[ [j_m, b(n)] = b(m + n), \quad [j_m, c(n)] = -c(m + n), \quad m, n \in \mathbb{Z}. \]

Then we have the \( bc \)-charge decomposition of \( F \) according to the eigenvalues of \( j_0 \):

\[ F = \bigoplus_{l \in \mathbb{Z}} F^l. \]

We denote by \( \mathcal{H}_\alpha (\alpha \in \mathbb{C}) \) the Fock space of the Heisenberg algebra generated by \( j_n, n \in \mathbb{Z} \), with vacuum vector \( |\alpha\rangle \) satisfying

\[ j_n |\alpha\rangle = \alpha \delta_{n,0} |\alpha\rangle, \quad n \geq 0. \]

Denote by \( \psi(z) = q + j_0 \ln z - \sum_{n \neq 0} j_n z^{-n} \), where the operator \( q \) satisfies \([q, j_n] = \delta_{n,0}\). Clearly \( j(z) = \partial \psi(z) \). (Note that our \( j(z), j_n, \cdots \) are denoted in [W2] by \(-j^{bc}(z), -j^{bc}_n, \cdots\)).

By the well-known boson-fermion correspondence, we have an isomorphism between \( F^l \) and \( \mathcal{H}_l \) as representations over the Heisenberg algebra generated by \( j_n, n \in \mathbb{Z} \). On the other hand, we may regard \( b(z) \) and \( c(z) \) as

\[ b(z) =: e^{\psi(z)} :, \quad c(z) =: e^{-\psi(z)} :. \quad (3.9) \]

Furthermore we have the following OPEs

\[ b(z)c(w) = \frac{1}{z-w} :b(z)c(w):, \quad c(z)b(w) = \frac{1}{z-w} :c(z)b(w):. \quad (3.10) \]

In particular it is well known that \( F^0 \) (and so \( \mathcal{H}_0 \)) is a vertex algebra.

Denote by \( \mathcal{F}^0 \) the kernel of the screening operator \( c(0) \) from \( F^0 \) to \( F^{-1} \).

It has a structure of a vertex subalgebra of \( \mathcal{F}^0 \). Let

\[ T(z) \equiv \sum_{n \in \mathbb{Z}} L_n z^{-n-2} =: \partial b(z)c(z) :. \quad (3.11) \]

Easy to check that \( T(z) \) is a Virasoro field with central charge \(-2\). We also define another field of conformal weight 3:

\[ W(z) \equiv \sum_{n \in \mathbb{Z}} W_n z^{-n-3} = \frac{1}{\sqrt{6}} \left( :\partial^2 b(z)c(z) - \partial b(z)\partial c(z) : \right). \quad (3.12) \]
We introduce a rescaled field \( \tilde{W}(z) = \frac{1}{2} \sqrt{6} W(z) \) for convenience later on, namely

\[
\tilde{W}(z) \equiv \sum_{n \in \mathbb{Z}} \tilde{W}_n z^{-n-3} = \frac{1}{2} \left( : \partial^2 b(z) c(z) : - : \partial b(z) \partial c(z) : \right). \quad (3.13)
\]

The following theorem is proved in \([W2]\).

**Theorem 3.1** The vertex algebra \( \mathcal{F}^0 \) is isomorphic to the simple vertex algebra \( \mathcal{W}_{3,-2} \) with generating fields \( T(z) \) and \( W(z) \) defined as in (3.11) and (3.12).

By the boson-fermion correspondence \( \mathcal{F}^0 \) and \( \mathcal{H}_0 \) are isomorphic as vertex algebras so we may view \( \mathcal{W}_{3,-2} \) as a vertex subalgebra of \( \mathcal{H}_0 \) as well. \( \mathcal{H}_\alpha \) is a module over the vertex algebra \( \mathcal{H}_0 \) and so can be regarded as a module over \( \mathcal{W}_{3,-2} \). Denote by \( V_\alpha \) the irreducible subquotient of the \( \mathcal{W}_{3,-2} \)-submodule of \( \mathcal{H}_\alpha \) generated by the highest weight \( |\alpha\rangle \).

We first rewrite the fields \( T(z) \) and \( W(z) \) defined as in (3.11) and (3.12) in terms of the field \( j(z) \) and its derivative fields.

**Proposition 3.1** Under the boson-fermion correspondence, the fields \( T(z) \) and \( W(z) \) in (3.11) and (3.12) can be expressed in terms of \( j(z) \) as

\[
T(z) = \frac{1}{2} \left( : j(z)^2 : + \partial j(z) \right), \quad (3.14)
\]

\[
\tilde{W}(z) = \frac{1}{12} \left( 4 : j(z)^3 : + 6 : j(z) \partial j(z) : + \partial^2 j(z) \right). \quad (3.15)
\]

**Proof.** By (3.11), we have

\[
\begin{align*}
b(z)c(w) & \sim \frac{1}{z-w} \left\{ 1 + (z-w) j(w) + \frac{(z-w)^2}{2} \left( : j(w)^2 : + \partial j(w) \right) \\
& \quad + \frac{(z-w)^3}{6} \left( : j(w)^3 : + 3 : j(w) \partial j(w) : + \partial^2 j(w) \right) \right\} \\
& \sim \frac{1}{z-w} + j(w) + \frac{1}{2} (z-w) \left( : j(w)^2 : + \partial j(w) \right) \\
& \quad + \frac{1}{6} (z-w)^2 \left( : j(w)^3 : + 3 : j(w) \partial j(w) : + \partial^2 j(w) \right).
\end{align*}
\]

From this we see that

\[
: \partial^2 b(w)c(w) : = \frac{1}{3} \left( : j(w)^3 : + 3 : j(w) \partial j(w) : + \partial^2 j(w) \right). \quad (3.16)
\]
On the other hand, by (3.10) we have
\[
c(z)b(w) = \frac{1}{z - w} \left\{ 1 - (z - w)j(w) + \frac{(z - w)^2}{2} \left( j(w)^2 : -\partial j(w) \right) \right.
\]
\[
+ \frac{(z - w)^3}{6} \left( -j(w)^3 : +3 : j(w)\partial j(w) : -\partial^2 j(w) \right) \right\}
\] + higher terms.

This implies that
\[
: \partial c(z)b(w) : = -\frac{1}{(z - w)^2} + \frac{1}{2} \left( j(w)^2 : -\partial j(w) \right)
\]
\[
+ \frac{1}{3} (z - w) \left( -j(z)^3 : +3 : j(z)\partial j(z) : \right)
\]
\[
+ \text{higher terms}. \tag{3.17}
\]

Equivalently we have by switching \( z \) and \( w \) in (3.17) and reversing the order between \( \partial c \) and \( b \) (we get a minus sign since \( bc \) fields are fermionic)
\[
: b(z)\partial c(w) : \sim \frac{1}{(z - w)^2} - \frac{1}{2} \left( j(z)^2 : -\partial j(z) \right)
\]
\[
+ \frac{1}{3} (z - w) \left( -j(z)^3 : +3 : j(z)\partial j(z) : \right)
\]
\[
\sim \frac{1}{(z - w)^2} - \frac{1}{2} \left( j(w)^2 : -\partial j(w) \right)
\]
\[
- \frac{1}{2} (z - w) \left( 2 : j(w)\partial j(w) : -\partial^2 j(w) \right)
\]
\[
+ \frac{1}{3} \left( -j(w)^3 : +3 : j(w)\partial j(w) : \right)
\]
\[
\sim \frac{1}{(z - w)^2} - \frac{1}{2} \left( j(w)^2 : -\partial j(w) \right)
\]
\[
- \frac{1}{2} (z - w) \left( 2 : j(w)\partial j(w) : -\partial^2 j(w) \right)
\]
\[
+ (z - w) \left( -\frac{1}{3} : j(w)^3 : +\frac{1}{6} \partial^2 j(w) \right). \tag{3.18}
\]

It follows from (3.18) that
\[
: \partial b(w)\partial c(w) : = -\frac{1}{3} : j(w)^3 : +\frac{1}{6} \partial^2 j(w). \tag{3.19}
\]

So by (3.13), (3.16) and (3.19) we have
\[
\tilde{W}(w) = \frac{1}{12} \left( 4 : j(w)^3 : +6 : j(w)\partial j(w) + \partial^2 j(w) \right).
\]
The proof of the identity (3.14) is similar. \qed

**Proposition 3.2** The highest weight of the $\mathcal{W}_{3,-2}$-module $V_\alpha$ ($\alpha \in \mathbb{C}$) with respect to $(L_0, \tilde{W}_0)$ is \((\frac{1}{2}\alpha(\alpha-1), \frac{3}{2}\alpha(\alpha-1)(2\alpha-1))\).

**Proof.** $L_0$ and $\tilde{W}_0$ can be written as an infinite sum of monomials in terms of $j_{-n}$, $n>0$ by Proposition 3.1. Indeed we have

\[
L_0 = \frac{1}{2} \left( \sum_{n<0} j_n j_{-n} + \sum_{n \geq 0} j_{-n} j_n \right) - \frac{1}{2} j_0
\]

\[
= \frac{1}{2} (j_0^2 - j_0) + \sum_{n>0} j_{-n} j_n.
\]

Since $j_n|\alpha\rangle = 0$, $n>0$ and $j_0|\alpha\rangle = \alpha|\alpha\rangle$, we have

\[
L_0|\alpha\rangle = \frac{1}{2} (j_0^2 - j_0)|\alpha\rangle = \frac{1}{2} \alpha(\alpha-1)|\alpha\rangle.
\]

Similarly, a little calculation shows that the only terms in $\tilde{W}_0$ which do not annihilate the vacuum vector $|\alpha\rangle$ are $\frac{1}{12} (4j_0^3 + 6j_0(-j_0) + 2j_0)$. So we have

\[
\tilde{W}_0|\alpha\rangle = \frac{1}{12} (4\alpha^3 + 6\alpha(-\alpha) + 2\alpha) = \frac{1}{6} \alpha(\alpha-1)(2\alpha-1).
\]

\[\Box\]

**Remark 3.1** The irreducible module $V_\alpha$ is isomorphic to the module $F^{-\alpha}$ constructed in $[W2]$ by comparing their highest weights for $\alpha \in \mathbb{Z}$. $V_\alpha$ is a proper subspace of $\mathcal{H}_\alpha$ in this case and its full character formula is given in $[W2]$. For $\alpha \notin \frac{1}{2}\mathbb{Z}$, we know that $\mathcal{H}_\alpha$ is irreducible as a module over the Virasoro algebra given by the field $T(z)$ with central charge $-2$ $[F]$, $[K]$ and so is irreducible as a module over $\mathcal{W}_{3,-2}$. Full character formulas of these $V_\alpha$ with respect to $\{L_0, \tilde{W}_0\}$ can be also calculated.

4 Classification of irreducible representations of $\mathcal{W}_{3,-2}$ algebra

We will show that irreducible modules $V_\alpha$, $\alpha \in \mathbb{C}$ exhaust all the irreducible modules of vertex algebra $\mathcal{W}_{3,-2}$ by calculating Zhu algebra in this case. We break the proof into a sequence of simple lemmas.
Lemma 4.1 Zhu algebra $A(VW_{3,c})$ is isomorphic to a polynomial algebra $\mathbb{C}[t, w]$, where $t, w$ correspond to $[L_{-2}|0\rangle]$ and $[\tilde{W}_{-3}|0\rangle]$ in $A(VW_{3,c})$.

Note that $L_{-2}|0\rangle$ is the Virasoro element in $VW_{3,c}$ so the element $[L_{-2}|0\rangle]$ lies in the center of $A(VW_{3,c})$ by Lemma 1.2. Proof of the above lemma is quite standard. See Lemma 4.1 in [W1] for a proof of a similar result. One can easily modify that proof to give a proof of our present lemma. We will not write it down here since it is not very illuminating.

Now specify $c = -2$. Let us denote by $\sigma$ the isomorphism from $A(VW_{3,-2})$ to $\mathbb{C}[t, w]$.

Lemma 4.2 Keeping the conventions in Lemma 4.1, under the isomorphism $\sigma$ we have

$$\sigma([v_s]) = w^2 - \frac{1}{9} t^2 (8t + 1), \quad \sigma([v_s']) = 0.$$

Proof. We will continue using the equivalence convention denoted by $\sim$ in the sense of Section 1. It follows from lemma 1.1 that for any $a \in VW_{3,-2},$

$$a \ast (\tilde{W}_{-3}|0\rangle) \sim (\tilde{W}_{-3} + 2\tilde{W}_{-2} + \tilde{W}_{-1}) a$$
$$a \ast (L_{-2}|0\rangle) \sim (L_{-2} + L_{-1}) a. \quad (4.20)$$

Recall that the isomorphism $\sigma$ from $A(VW_{3,-2})$ to $\mathbb{C}[t, w]$ sends elements $[L_{-2}|0\rangle]$ and $[\tilde{W}_{-3}|0\rangle]$ in $A(VW_{3,c})$ to $t, w$ respectively. By applying (4.20) to the first two terms of the singular vector $v_s$ given in Lemma 2.1 and then rewriting it in terms of the PBW basis of the form (2.5) by using the commutation relations (2.3), we get

$$v_s \sim w^2 + \left\{ -\left(2\tilde{W}_{-2} + \tilde{W}_{-1}\right) \tilde{W}_{-3} \right. $$
$$\frac{19}{36} L_{-3}^2 - \frac{8}{9} L_{-2}^3 - \frac{14}{9} L_{-2} L_{-4} + \frac{44}{9} L_{-6} \} |0\rangle$$

$$= w^2 + \left\{ -3 \left( \frac{8}{3} L_{-2} L_{-3} - \frac{10}{3} L_{-5} \right) $$
$$- \frac{3}{2} \left( \frac{8}{3} L_{-2}^2 - L_{-4} \right) $$
$$- \frac{19}{36} L_{-3}^2 - \frac{8}{9} L_{-2}^3 - \frac{14}{9} L_{-2} L_{-4} + \frac{44}{9} L_{-6} \} |0\rangle$$
\[ W_3 \text{ algebra} \]

\[ = w^2 + \left\{ -8L_{-2}L_{-3} + 10L_{-5} - 4L_{-2}^2 + \frac{3}{2}L_{-4} \right. \]

\[ \left. - \frac{19}{36}L_{-3}^2 - \frac{8}{9}L_{-2}^3 - \frac{14}{9}L_{-2}L_{-4} + \frac{44}{9}L_{-6} \right\} |0\rangle. \quad (4.21) \]

It is easy to show by induction and applying Lemma 1.1 that

\[ L_{-n} \sim (-1)^n \left( (n-1)(L_{-2} + L_{-1}) + L_0 \right), \quad n \geq 1. \quad (4.22) \]

By the equation (4.20) and repeated uses of (4.22) on the right hand side of (4.21), we get

\[ v_s \sim w^2 + 16t(t+3) - 40t - 4t(t+2) + \frac{9}{2}t \]

\[ - \frac{19}{18}t(2t+3) - \frac{8}{9}t(t+2)(t+4) - \frac{14}{3}t(t+4) + \frac{220}{9}t \]

\[ = w^2 - \frac{1}{9}t^2(8t+1). \quad (4.23) \]

This completes the proof that \( \sigma([v_s]) = w^2 - \frac{1}{9}t^2(8t+1) \). Similarly we can prove that \( \sigma([v'_s]) = 0 \). □

Denote \( f(t,w) = w^2 - \frac{1}{3}t^2(8t+1) \). Now the following lemma follows from Proposition 1.1, Lemma 4.1 and Lemma 4.2 (see Corollary 4.1 for a more precise statement).

**Lemma 4.3** The Zhu algebra \( A(W_{3,-2}) \) is a certain quotient of the quotient algebra \( \mathbb{C}[t,w]/\langle f(t,w) \rangle \), where \( \langle f(t,w) \rangle \) denotes the ideal of \( \mathbb{C}[t,w] \) generated by \( f(t,w) \in \mathbb{C}[t,w] \).

We have the following observation.

**Lemma 4.4** Solutions to the equation (0.4) are parametrized as follows:

\[ (t(\alpha), w(\alpha)) \equiv \left( \frac{1}{2}\alpha(\alpha - 1), \frac{1}{6}\alpha(\alpha - 1)(2\alpha - 1) \right), \quad \alpha \in \mathbb{C}. \quad (4.24) \]

**Proof.** First it is clear that \( t(\alpha) \) can take any complex value when \( \alpha \) ranges over \( \mathbb{C} \). Then by substituting \( t(\alpha) \) in the equation (0.2) we see that \( w(\alpha)^2 = \left[ \frac{1}{2}\alpha(\alpha - 1)(2\alpha - 1) \right]^2 \). We don’t lose any generality by letting \( w(\alpha) = \frac{1}{2}\alpha(\alpha - 1)(2\alpha - 1) \). The reason is that \( t(1-\alpha) = t(\alpha) \) while \( w(1-\alpha) = -w(\alpha) \). □
Remark 4.1 For different $\alpha, \alpha' \in \mathbb{C}$, $(t(\alpha), w(\alpha)) = (t(\alpha'), w(\alpha'))$ if and only if $\alpha = 0$ (resp. $1$), $\alpha' = 1$ (resp. $0$). Namely $V_0$ is isomorphic to $V_1$ and this is the only isomorphism among $V_\alpha, \alpha \in \mathbb{C}$.

Now we are ready to prove our classification theorem on irreducible modules over the $\mathcal{W}_{3,-2}$ algebra.

Theorem 4.1 $V_\alpha, \alpha \in \mathbb{C}$ are all the irreducible modules over the simple $\mathcal{W}_3$ algebra with central charge $-2$. Highest weights of these modules $V_\alpha$ are given by $\left(\frac{1}{2}\alpha(\alpha - 1), \frac{1}{6}\alpha(\alpha - 1)(2\alpha - 1)\right), \alpha \in \mathbb{C}$. They are parametrized by points $(t, w)$ on the rational curve defined by $w^2 = \frac{1}{9}t^2(8t + 1)$.

Proof. By Lemma 4.3, we see that any irreducible module of the associative algebra $\mathcal{A}(\mathcal{W}_{3,-2})$ is one-dimensional since $\mathcal{A}(\mathcal{W}_{3,-2})$ is commutative. Given $t, w \in \mathbb{C}$, let $\mathcal{C}_{t,w}$ be the one-dimensional module of $\mathcal{A}(\mathcal{W}_{3,-2})$, with $[L_{-2}0]$ acting as the scalar $t$ and $[\tilde{W}_{-3}0]$ as the scalar $w$. Then $(t, w)$ has to satisfy $w^2 = \frac{1}{9}t^2(8t + 1)$. Note that $o(L_{-2}0) = L_0$ and $o(\tilde{W}_{-3}0) = \tilde{W}_0$ by the definition of $o(\cdot)$ in Section 1. So by Theorem 1.2, the highest weight $(t, w)$ of any irreducible module of the vertex algebra $\mathcal{W}_{3,-2}$ with respect to $(L_0, \tilde{W}_0)$ has to satisfy the equation $w^2 = \frac{1}{9}t^2(8t + 1)$. By Lemma 4.4, we see all solutions to the above equation can be written as of the form $\left(\frac{1}{2}\alpha(\alpha - 1), \frac{1}{6}\alpha(\alpha - 1)(2\alpha - 1)\right), \alpha \in \mathbb{C}$. On the other hand, we have already constructed a family of irreducible modules $V_\alpha$ ($\alpha \in \mathbb{C}$) with highest weight exactly equal to $\left(\frac{1}{2}\alpha(\alpha - 1), \frac{1}{6}\alpha(\alpha - 1)(2\alpha - 1)\right)$. This completes the proof of the theorem.

We think it is remarkable that the set of all irreducible modules of $\mathcal{W}_{3,-2}$ has such a simple and elegant description in terms of a rational curve. It indicates that the non-rational vertex algebras may have very rich representation theory.

We have an immediate corollary of Theorem 4.1 which strengthens Lemma 4.3.

Corollary 4.1 The Zhu algebra $\mathcal{A}(\mathcal{W}_{3,-2})$ is isomorphic to the commutative associative algebra $\mathbb{C}[t, w]/ < f(t, w)>$.

Remark 4.2 1) Basing on the results of Theorem 4.1 and Corollary 4.1, it is natural to conjecture that the singular vectors $v_s$ and $v'_s$ generate the maximal proper submodule of the vacuum module $\mathcal{V}\mathcal{W}_{3,-2}$.
2) A Virasoro vertex algebra with a certain central charge is rational if and only if the corresponding vacuum module is reducible \([W_1]\). As our results show, \(W_3\) algebra provides new possibility, namely the simple vertex algebra \(W_{3,-2}\) is not rational but the corresponding vacuum module \(\mathcal{V}W_{3,-2}\) is reducible.

We further comment on why central charge \(c = -2\) is particularly interesting from a different point of view. There is the so-called quantized Drinfeld-Sokolov reduction (cf. e.g. [BH, FKW]) which allows one to establish connections between \(W_n\) algebra with central charge \(c^{(k)}_n\) and the affine Kac-Moody Lie algebra \(\hat{sl}_n\) with central charge \(k\). Here
\[
c^{(k)}_n = 2n^3 - n - 1 - n(n^2 - 1) \left( \frac{1}{k + n} + k + n \right).
\]

In particular, for \(k = -n + p/q\) one can rewrite \(c^{(k)}_n\) as follows:
\[
c^{(k)}_n = (n - 1) \left( 1 - \frac{n(n + 1)(p - q)^2}{pq} \right).
\]

The so-called minimal series central charges of the \(W_n\) algebra are those \(c^{(k)}_n\) for \(k = -n + p/q\), where \(p, q\) are coprime integers satisfying \(p, q \geq n\). They correspond to the admissible central charges \(k = -n + p/q\) for the affine algebra \(\hat{sl}_n\), with the same conditions imposed on \(p, q\) as above. The admissible representations with admissible central charges were first studied by Kac-Wakimoto [Ka].

Thus by means of Drinfeld-Sokolov reduction the central charge \(-2\) for the \(W_3\) algebra corresponds to the central charge \(k = -\frac{3}{2} \) or \(-\frac{7}{2}\) of \(\hat{sl}_3\). Observe that \(k = -\frac{3}{2} = -3 + \frac{2}{3}\) or \(-\frac{7}{2} = -3 + \frac{3}{2}\) corresponds to the "boundry" of the admissible central charges of \(\hat{sl}_3\).

However more than this is true. Consider the "boundry" of the admissible central charges of \(\hat{sl}_n\), i.e. \(k = -n + \frac{n}{n-1}\) or \(-n + \frac{n}{n-1}\). The corresponding central charge of the \(W_n\) algebra \(c^{(k)}_n = -2\), which is independent of \(n\). In this sense \(-2\) is a universal central charge for any \(W_n\) algebra. We expect that the representations of the affine algebra \(\hat{sl}_n\) with central charge equal to the "boundry" of the admissible central charges are of independent interest.
5 Classification of irreducible modules of vertex algebra $\mathcal{W}_{1+\infty,-1}$

Let $\mathcal{D}$ be the Lie algebra of regular differential operators on the circle. The elements

$$J^l_k = -t^{l+k}(\partial_t)^l, \quad l \in \mathbb{Z}_+, k \in \mathbb{Z},$$

form a basis of $\mathcal{D}$. $\mathcal{D}$ has also another basis

$$L^l_k = -t^kD^l, \quad l \in \mathbb{Z}_+, k \in \mathbb{Z},$$

where $D = t\partial_t$. Denote by $\hat{\mathcal{D}}$ the central extension of $\mathcal{D}$ by a one-dimensional center with a generator $C$, with commutation relation (cf. [KRI])

$$\left[t^r f(D), t^s g(D)\right] = t^{r+s} (f(D + s)g(D) - f(D)g(D + r)) + \Psi (t^r f(D), t^s g(D)) C, \quad (5.25)$$

where

$$\Psi (t^r f(D), t^s g(D)) = \begin{cases} 
\sum_{-r \leq j \leq -1} f(j)g(j + r), & r = -s \geq 0 \\
0, & r + s \neq 0.
\end{cases} \quad (5.26)$$

Letting weight $J^l_k = k$ and weight $C = 0$ defines a principal gradation

$$\hat{\mathcal{D}} = \bigoplus_{j \in \mathbb{Z}} \hat{\mathcal{D}}_j. \quad (5.27)$$

Then we have the triangular decomposition

$$\hat{\mathcal{D}} = \hat{\mathcal{D}}_+ \bigoplus \hat{\mathcal{D}}_0 \bigoplus \hat{\mathcal{D}}_- \quad (5.28),$$

where

$$\hat{\mathcal{D}}_\pm = \bigoplus_{j \in \pm \mathbb{N}} \hat{\mathcal{D}}_j, \quad \hat{\mathcal{D}}_0 = \mathcal{D}_r \bigoplus \mathbb{C}C.$$

Let $\mathcal{P}$ be the distinguished parabolic subalgebra of $\mathcal{D}$, consisting of the differential operators that extends into the whole interior of the circle. $\mathcal{P}$ has a basis $\{J^l_k, l \geq 0, l + k \geq 0\}$. It is easy to check that the 2-cocycle $\Psi$ defining the central extension of $\hat{\mathcal{D}}$ vanishes when restricted to the parabolic subalgebra $\mathcal{P}$. So $\mathcal{P}$ is also a subalgebra of $\hat{\mathcal{D}}$. Denote $\hat{\mathcal{P}} = \mathcal{P} \oplus \mathbb{C}C.$
Fix $c \in \mathbb{C}$. Denote by $\mathbb{C}_c$ the 1–dimensional $\hat{D}$ module by letting $C$ acts as scalar $c$ and $P$ acts trivially. Fix a non-zero vector $v_0$ in $\mathbb{C}_c$. The induced $\hat{D}$–module

$$M_c(\hat{D}) = U(\hat{D}) \otimes \mathbb{C}_c$$

is called the vacuum $\hat{D}$–module with central charge $c$. Here we denote by $U(\mathfrak{g})$ the universal enveloping algebra of a Lie algebra $\mathfrak{g}$. $M_c(\hat{D})$ admits a unique irreducible quotient, denoted by $\mathcal{W}_{1+\infty,c}$. Denote the highest weight vector $1 \otimes v_0$ in $M_c(\hat{D})$ by $|0\rangle$.

It is shown in [FKRW] that $\mathcal{W}_{1+\infty,c}$ carries a canonical vertex algebra structure, with vacuum vector $|0\rangle$ and generating fields

$$J_l(z) = \sum_{k \in \mathbb{Z}} J^l_k z^{-k-l-1},$$

of conformal weight $l + 1, l = 0, 1, \cdots$. The fields $J_l(z)$ corresponds to the vector $J_{l-1}^l|0\rangle$ in $\mathcal{W}_{1+\infty,c}$. Below we will concentrate on the particular case $\mathcal{W}_{1+\infty,-1}$.

The relation between vertex algebras $\mathcal{W}_{1+\infty,-1}$ and $\mathcal{W}_{3,-2}$ is made clear by the following theorem [W2].

**Theorem 5.1** The vertex algebra $\mathcal{W}_{1+\infty,-1}$ is isomorphic to a tensor product of the $\mathcal{W}_{3,-2}$ algebra, and the Heisenberg vertex algebra $\mathcal{H}_0$ with $J^0(z)$ as a generating field.

Then the classification of irreducible modules over $\mathcal{W}_{1+\infty,-1}$ follows from classification of those over $\mathcal{W}_{3,-2}$ since the classification of irreducible modules over a Heisenberg vertex algebra is well known. Also see Remark 4.1.

**Theorem 5.2** There exists a two-parameter family of irreducible modules over $\mathcal{W}_{1+\infty,-1}$. Any irreducible $\mathcal{W}_{1+\infty,-1}$-module can be written uniquely as a tensor product of a module $L(t(\alpha), w(\alpha))$ over $\mathcal{W}_{3,-2}$ with a module $\mathcal{H}_s$ over $\mathcal{H}_0$ ($\alpha \in \mathbb{C} - \{1\}$, $s \in \mathbb{C}$), with $(t(\alpha), w(\alpha))$ as defined in (4.24).

**Remark 5.1** Theorem 5.2 disproves a conjecture of Kac and Radul [KR2]. The list of irreducible modules of $\mathcal{W}_{1+\infty,-1}$ which were conjectured to be complete in [KR2] consists of those with $\alpha = 0$ in Theorem 5.3. (i.e. modules $\mathcal{M}^0_s$ in [W2]).
There are several questions which the author does not know the answers at present but hope to have a better understanding in the near future.

1) What are the fusion rules of $\mathcal{W}_{3,-2}$ (and thus of $\mathcal{W}_{1+\infty,-1}$)? The existence of reducible however indecomposable modules of $\mathcal{W}_{3,-2}$ seems to be related to the fact that there is a node at $(0,0)$ on the rational curve $w^2 - \frac{1}{9}t^2(8t+1) = 0$. It is likely that we may need to regard some reducible however indecomposable modules as basic objects when studying the fusion rules.

2) Recall the Cartan subalgebra of $\mathcal{W}_{1+\infty}$ is infinite dimensional. In [KK1] the highest weight of an irreducible quasifinite module over $\mathcal{W}_{1+\infty}$ is characterized in terms of a certain generating function $\Delta(x)$. The question is how to identify highest weights of the two-parameter family of irreducible modules of $\mathcal{W}_{1+\infty}$ with central charge $-1$ in Theorem 5.2 in terms of $\Delta(x)$. It would be very interesting to see if these irreducible modules of $\mathcal{W}_{1+\infty,-1}$ we have constructed are the first realizations of $\mathcal{W}_{1+\infty}$-modules with $\Delta(x) = \frac{p(x)e^{x}}{e^{x} - 1} + \cdots$ with some non-constant polynomial $p(x)$ (cf. [KK1] for notations).

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