GENERALIZED DUALITY, HAMILTONIAN FORMALISM AND NEW BRACKETS

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Abstract. It is shown that any singular Lagrangian theory: 1) can be formulated without the use of constraints by introducing a Clairaut-type version of the Hamiltonian formalism; 2) leads to a special kind of nonabelian gauge theory; 3) coincides with the many-time classical dynamics. A generalization of the Legendre transform to the case when the Hessian is zero is done by using the mixed (envelope/general) solution of the multidimensional Clairaut equation. The corresponding system of equations of motion is equivalent to the Lagrange equations and has a linear algebraic subsystem for “unresolved” velocities. Then the equations of motion are written in the Hamiltonian form by introducing a new bracket. This is a “shortened” formalism, since initially it does not contain “nondynamical” (degenerate) momenta at all, and therefore there is no notion of constraint. It is shown that any classical degenerate Lagrangian theory in its Clairaut-type Hamiltonian form is equivalent to the many-time classical dynamics.

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1. Introduction

Nowadays, many fundamental physical models are based on gauge field theories [13, 58]. On the classical level, they are described by singular (degenerate) Lagrangians, which makes the passage to a Hamiltonian description, which is important for quantization, highly nontrivial and complicated (see, e.g., [45, 51]).

A common way to deal with such theories is the Dirac approach [15] based on extending the phase space and constraints. This treatment of constrained theories has been deeply reviewed, e.g., in lecture notes [59] and books [23, 30]. In spite of its general success, the Dirac approach has some problems [16, 43, 57] and is not directly applicable in some cases, e.g., for irregular constrained systems (with linearly dependent constraints) [6, 41].

Therefore, it is of considerable interest to reconsider basic ideas of the Hamiltonian formalism in general from another point of view. The approach presented here does not depend on the structure of constraints and hopefully will be applicable to all cases.

In the standard approach for nonsingular theories [4, 49], the transition from Lagrangian to Hamiltonian description is done by the Legendre transform, and then finding the Hamiltonian as an envelope solution of the corresponding Clairaut equation [32]. The main idea of our formulation is the following [20]: for singular theories, instead of the Lagrange multiplier procedure developed by Dirac [15], alternatively we construct and solve the corresponding multidimensional Clairaut equation [32]. In this way, we state that the ordinary duality can be generalized to the Clairaut duality.

In this paper we develop our previous work [19, 20] to construct a self-consistent analog of the canonical (Hamiltonian) formalism and present a general algorithm to describe any Lagrangian system (singular or not) as a set of first-order differential equations without introducing Lagrange multipliers. The connection of our approach with the Dirac constraints method is shown in Appendix B.

The advantage of this “shortened” formalism, when the “nondynamical” (degenerate or “non-physical”) momenta do not enter in initial consideration, is its clarity and simplicity. Similar formulations (without primary constraints) were treated in [14, 25, 38], while here we explain the reason and source of constraints appearance from another viewpoint.

The “shortened” approach can play an important role for quantization of such complicated constrained systems as gauge field theories [34] and gravity [44]. These questions will be considered in future papers. To illustrate the power and simplicity of our method, we consider such examples, as the Cawley Lagrangian [9] (which led to difficulties in the Dirac approach), and the relativistic particle. A novel Hamiltonian-like form of the equations of motion is achieved by defining a new (non-Lie) bracket which is not anticommutative and does not satisfy the Jacobi identity. Note that quantization of such brackets can be done by means of non-Lie algebra methods (see, e.g., [42, 52]). This will be a subject of a subsequent paper.

While analyzing the part of the equations of motion corresponding to “unresolved” velocities, we arrive effectively at a kind of nonabelian gauge theory in the “degenerate” coordinate subspace, related to these velocities. Here we have shown that the Clairaut-type formulation is equivalent to the many-time classical dynamics developed in [17, 36] if “nondynamical” (degenerate) coordinates are treated as additional “times”. Finally, in an Appendix we show that, after introduction
of “nondynamical” momenta, corresponding Lagrange multipliers and respective constraints, the Clairaut-type formulation presented here corresponds to the Dirac approach [15].

In general, the paper has also a clear methodological aspect: it shows that one can deal with well-known problems from a different than commonly used viewpoint. This can lead to better understanding and new insights into such an important subject as Hamiltonian formalism in itself.

To simplify matters, we use coordinates, but all the statements can be readily reformulated in a coordinate free setting [8, 53]. We consider systems with a finite number of degrees of freedom. The Clairaut-type formulation of field theories will be done elsewhere.

2. The Legendre-Fenchel and Legendre Transforms

We start with a brief description of the standard Legendre-Fenchel and Legendre transforms for the theory with nondegenerate Lagrangian [5, 47]. Let

$$L(q^A, v^A), \quad A = 1, \ldots, n,$$

be a Lagrangian given by a function of 2n variables (n generalized coordinates $q^A$ and n velocities $v^A = \dot{q}^A = dq^A/dt$) on the configuration space $TM$, where $M$ is a smooth manifold. We consider the time-independent case for simplicity and conciseness, which will not influence the main procedure.

By the convex approach definition (see e.g. [5, 46]), a Hamiltonian $H(q^A, p_A)$ is a dual function on the phase space $T^*M$ (or convex conjugate [47]) to the Lagrangian (in the second set of variables $p_A$) and is constructed by means of the Legendre-Fenchel transform $L \mapsto H_{Fen}$ defined by [21, 46]

$$H_{Fen}(q^A, p_A) = \sup_{v^A} G(q^A, v^A, p_A), \quad (2.1)$$

$$G(q^A, v^A, p_A) = \sum_{B=1}^{n} p_B v^B - L(q^A, v^A). \quad (2.2)$$

Note that this definition is very general, and it can be applied to nonconvex [2] and nondifferentiable [56] functions $L(q^A, v^A)$, which can lead to numerous extended versions of the Hamiltonian formalism (see, e.g., [12, 31, 48]). Also, a generalization of the convex conjugacy can be achieved by substituting in (2.2) the form $p_A v^A$ by any function $\Psi(p_A, v^A)$ satisfying special conditions [26].

In the standard mechanics [29], one usually restricts to convex, smooth and differentiable Lagrangians (see, e.g., [5, 50]). Then the coordinates $q^A(t)$ are treated as fixed (passive with respect to the Legendre transform) parameters, and the velocities $v^A(t)$ are assumed independent functions of time.

According to our assumptions the supremum (2.1) is attained by finding an extremum point $v^A = v^A_{extr}$ of the (“pre-Hamiltonian”) function $G(q^A, v^A, p_A)$, which leads to the supremum condition

$$p_B = \frac{\partial L(q^A, v^A)}{\partial v^B} \bigg|_{v^A = v^A_{extr}}. \quad (2.3)$$

It is commonly assumed (see, e.g., [5, 29, 50]) that the only way to get rid of dependence on the velocities $v^A$ in the r.h.s. of (2.1) is to resolve (2.3) with respect

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1 We use indices in arguments, because we need to distinguish different kinds of coordinates (similar to [52]). For the same reason, we use the summation signs with explicit ranges.
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to velocities and find its solution given by a set of functions

\[ v^B_{\text{extr}} = V^B(q^A, p_A) \]  

(2.4)

This can be done only in the class of nondegenerate Lagrangians \( L(q^A, v^A) = L^{\text{nondeg}}(q^A, v^A) \) (in the second set of variables \( v^A \)), which is equivalent to the Hessian being non-zero

\[ \det \left| \frac{\partial^2 L^{\text{nondeg}}(q^A, v^A)}{\partial v^B \partial v^C} \right| \neq 0. \]  

(2.5)

Then substitute \( v^A_{\text{extr}} \) to (2.1) and obtain the standard Hamiltonian (see, e.g., [5, 29])

\[ H(q^A, p_A) \overset{\text{def}}{=} G(q^A, v^A_{\text{extr}}, p_A) = \sum_{B=1}^n p_B V^B(q^A, p_A) - L^{\text{nondeg}}(q^A, V^A(q^A, p_A)). \]  

(2.6)

The passage from the nondegenerate Lagrangian \( L^{\text{nondeg}}(q^A, v^A) \) to the Hamiltonian \( H(q^A, p_A) \) is called the Legendre transform which will be denoted by \( L^{\text{nondeg}} \overset{\text{Leg}}{\rightarrow} H \).

From the geometric viewpoint [1, 39, 54], the Legendre transform being applied to \( L(q^A, v^A) \) is tantamount to the Legendre transformation from the configuration space to the phase space \( \text{Leg} : TM \rightarrow T^*M \) (or between submanifolds in the presence of constraints [7, 22, 40]).

3. THE LEGENDRE-CLAIRAUT TRANSFORM

An alternative way to deal with the supremum condition (2.3) is to consider the related multidimensional Clairaut equation [19]. The connection between the Legendre transform, convexity and the Clairaut equation has a long story [33, 49] (see also [4]).

We differentiate (2.6) by the momenta \( p_A \) and use the supremum condition (2.3) to get

\[ \frac{\partial H(q^A, p_A)}{\partial p_B} = V^B(q^A, p_A) \]

\[ + \sum_{C=1}^n \left( p_C - \frac{\partial L(q^A, v^A)}{\partial v^C} \bigg|_{v^C = V^C(q^A, p_A)} \right) \frac{\partial V^C(q^A, p_A)}{\partial p_B} = V^B(q^A, p_A), \]  

(3.1)

which can be called the dual supremum condition (indeed this gives the first set of the Hamilton equations, see below). The relations (2.3), (2.6) and (3.1) together represent a particular case of the Donkin theorem (see e.g. [29]).

Then we substitute (3.1) in (2.6) and obtain

\[ H(q^A, p_A) = \sum_{B=1}^n p_B \frac{\partial H(q^A, p_A)}{\partial p_B} - L^{\text{nondeg}} \left( q^A, \frac{\partial H(q^A, p_A)}{\partial p_C} \right), \]  

(3.2)

which contains no manifest dependence on velocities at all. It is important that, for nonsingular Lagrangians, the relation (3.2) is an identity, which follows from (2.3), (2.6) and (3.1) by our construction.
Now we make the main step: to treat the equation (3.2) by itself (without referring to (2.3), (2.6) and (3.1)) as a definition of a new transform being a solution of the following nonlinear partial differential equation (the multidimensional Clairaut equation) [19]

\[ H^{Cl}(q^A, \bar{p}_A) = \sum_{B=1}^{n} \bar{p}_B \frac{\partial H^{Cl}(q^A, \bar{p}_A)}{\partial \bar{p}_B} - L \left( q^A, \frac{\partial H^{Cl}(q^A, \bar{p}_A)}{\partial \bar{p}_A} \right), \quad (3.3) \]

where \( \bar{p}_A \) are parameters (initially not connected with \( p_A \) defined by (2.3)) and \( L(q^A, v_A) \) is any differentiable smooth function of \( 2n \) variables, and here we do not demand the nondegeneracy condition (2.5). We call the transform defined by (3.3) \( L^{eg^{Cl}} \) a Clairaut duality transform (or the Legendre-Clairaut transform [19]).

Note that (2.3) is commonly treated as a definition of all dynamical momenta \( p_A \), but we should distinguish them from the parameters of the Clairaut duality transform \( \bar{p}_A \). In our approach, before solving the Clairaut equation and applying the supremum condition (2.3), the parameters \( \bar{p}_A \) are not connected with the Lagrangian. Thus, the corresponding Legendre-Clairaut transformation is the map \( L^{eg^{Cl}} : TM \to \bar{T}^*_M \), where the space \( \bar{T}^*_M \) in local coordinates is just \( (q^A, \bar{p}_A) \), and we call \( \bar{T}^*_M \) a Clairaut (extended) phase space.

The difference between the Legendre-Clairaut transform and the Legendre transform is crucial for degenerate Lagrangian theories [19]. Specifically, the multidimensional Clairaut equation (3.3) has solutions even for degenerate Lagrangians \( L(q^A, v_A) = L^{deg}(q^A, v_A) \) when the Hessian is zero

\[ \det \left| \frac{\partial^2 L^{deg}(q^A, v_A)}{\partial v^B \partial v^C} \right| = 0. \quad (3.4) \]

In this way, the Legendre-Clairaut transform (3.3) \( L^{eg^{Cl}} \) is another (along with \( L^{eg^{Fen}} \)) counterpart to the ordinary Legendre transform (2.6) in the case of degenerate Lagrangians.

To make this manifest and to find solutions of the Clairaut equation (3.3), we differentiate it by \( \bar{p}_C \) to obtain

\[ \sum_{B=1}^{n} \left[ \bar{p}_B - \frac{\partial L(q^A, v^A)}{\partial v^B} \bigg|_{v^B = \frac{\partial H^{Cl}(q^A, \bar{p}_A)}{\partial \bar{p}_B}} \right] \frac{\partial^2 H^{Cl}(q^A, \bar{p}_A)}{\partial \bar{p}_B \partial \bar{p}_C} = 0. \quad (3.5) \]

Now we apply the ordinary method of solving the Clairaut equation (see Appendix A). There are two possible solutions of (3.5), one in which the square brackets vanish (envelope solution) and one in which the double derivative in velocity vanishes (general solution). The l.h.s. of (3.5) is a sum over \( B \) and it is quite conceivable that one may vanish for some \( B \) and the other vanish for other \( B \). The physical reason of choosing the particular solution is presented in Section 4. Thus, we have two solutions of the Clairaut equation:

1) The envelope solution defined by the first multiplier in (3.5) being zero

\[ \bar{p}_B = p_B = \frac{\partial L(q^A, v^A)}{\partial v^B}, \quad (3.6) \]
which coincides with the supremum condition (2.3), together with (3.1). In this way, we obtain the standard Hamiltonian (2.6)

$$H^{\text{Cl}}_{\text{env}}(q^A, \bar{p}_A) |_{\bar{p}_A = p_A} = H(q^A, p_A).$$  (3.7)

Thus, in the nondegenerate case, the “envelope” Legendre-Clairaut transform $\mathcal{L}_{q^A \text{env}}: L \rightarrow H^{\text{Cl}}_{\text{env}}$ coincides with the ordinary Legendre transform by our construction here.

2) A general solution is defined by

$$\frac{\partial^2 H^{\text{Cl}}(q^A, \bar{p}_A)}{\partial \bar{p}_B \partial \bar{p}_C} = 0$$  (3.8)

which gives

$$\frac{\partial H^{\text{Cl}}(q^A, \bar{p}_A)}{\partial \bar{p}_B} = c^B, \text{ here } c^B \text{ are arbitrary smooth functions of } q^A,$n and the latter are considered in the Clairaut equation (3.3) as parameters (passive variables). Then the general solution acquires the form

$$H^{\text{Cl}}_{\text{gen}}(q^A, \bar{p}_A, c^A) = \sum_{B=1}^n \bar{p}_B c^B - L(q^A, c^A),$$  (3.9)

which corresponds to a “general” Legendre-Clairaut transform $\mathcal{L}_{q^A \text{gen}}: L \rightarrow H^{\text{Cl}}_{\text{gen}}$. Note that the general solution $H^{\text{Cl}}_{\text{gen}}(q^A, \bar{p}_A, c^A)$ is always linear in the variables $\bar{p}_A$ and the latter are not actually the dynamical momenta $p_A$, because we do not have the envelope solution condition (3.6), and therefore now there is no supremum condition (2.3). The variables $c^A$ are in fact unresolved velocities $v^A$ in the case of the general solution.

Note that in the standard way, $\mathcal{L}_{q^A \text{env}}$ can be also obtained by finding the envelope of the general solution [4], i.e. differentiating (3.9) by $c^A$ as

$$\frac{\partial H_{\text{gen}}^{\text{Cl}}(q^A, \bar{p}_A, c^A)}{\partial c^B} = \bar{p}_B - \frac{\partial L(q^A, c^A)}{\partial c^B} = 0$$  (3.10)

which coincides with (3.6) and (2.3). This means that $H_{\text{gen}}^{\text{Cl}}(q^A, \bar{p}_A, c^A) |_{c^A = v^A}$ is in fact the “pre-Hamiltonian” (2.2), which was needed to find the supremum in (2.1).

Let us consider the classical example of one-dimensional oscillator.

**Example 3.1.** Let $L(x, v) = mv^2/2 - kx^2/2$ ($m$, $k$ are constants), then the corresponding Clairaut equation (3.3) for $H = H^{\text{Cl}}(x, \bar{p})$ is

$$H = \bar{p}H_p' - \frac{m}{2} \left( H_p' \right)^2 + \frac{kx^2}{2},$$  (3.11)

where prime denotes partial differentiation with respect to $\bar{p}$. The general solution is

$$H_{\text{gen}}^{\text{Cl}}(x, \bar{p}, c) = \bar{p}c - \frac{mc^2}{2} + \frac{kx^2}{2},$$  (3.12)

where $c$ is an arbitrary function (“unresolved velocity” $v$). The envelope solution (with $\bar{p} = p$) can be found from the condition

$$\frac{\partial H^{\text{Cl}}}{\partial c} = p - mc = 0 \implies c_{\text{extr}} = \frac{p}{m},$$

which gives

$$H_{\text{env}}^{\text{Cl}}(x, p) = \frac{p^2}{2m} + \frac{kx^2}{2}$$  (3.13)

in the standard way.
Example 3.2. Let \( L(x, v) = x \exp kv \), then the corresponding Clairaut equation for \( H = H^{\text{Cl}}(x, \bar{p}) \) is

\[
H = \bar{p}H'_\bar{p} - x \exp \left(kH'_\bar{p}\right).
\]

The general solution is

\[
H^{\text{Cl}}_{\text{gen}}(x, \bar{p}) = \bar{p}c - x \exp kc,
\]

where \( c \) is any smooth function of \( x \).

The envelope solution (with \( \bar{p} = p \)) can be found by differentiating the general solution (3.15)

\[
\frac{\partial H^{\text{Cl}}}{\partial c} = p - x \exp kc = 0 \implies c_{\text{extr}} = \frac{1}{k} \ln \frac{p}{x},
\]

which leads to

\[
H^{\text{Cl}}_{\text{env}}(x, p) = \frac{p}{k} \ln \frac{p}{x} - p.
\]

4. The mixed Legendre-Clairaut transform

Now consider a singular Lagrangian \( L(q^A, v^A) = L^{\text{deg}}(q^A, v^A) \) for which the Hessian is zero (3.4). This means that the rank of the Hessian matrix \( W_{AB} = \frac{\partial^2 L(q^A, v^A)}{\partial v^B \partial v^C} \) is \( r < n \), and we suppose that \( r \) is constant. We rearrange indices of \( W_{AB} \) in such a way that a nonsingular minor of rank \( r \) appears in the upper left corner \[24\]. Represent the index \( A \) as follows: if \( A = 1, \ldots, r \), we replace \( A \) with \( i \) (the “regular” index), and, if \( A = r + 1, \ldots, n \) we replace \( A \) with \( \alpha \) (the “degenerate” index). Obviously, \( \det W_{ij} \neq 0 \), and \( \text{rank} W_{ij} = r \). Thus any set of variables labelled by a single index splits as a disjoint union of two subsets. We call those subsets regular (having Latin indices) and degenerate (having Greek indices).

The standard Legendre transform \( \text{Leg} \) is not applicable in the degenerate case because the condition (2.5) is not valid \[8, 53\]. Therefore the supremum condition (2.3) cannot be resolved with respect to degenerate \( A \), but it can be resolved only for regular \( A \), because \( \det W_{ij} \neq 0 \). On the contrary, the Clairaut duality transform given by (3.3) is independent of the Hessian being zero or not \[19\]. Thus, we state the main idea of the formalism we present here: the ordinary duality can be generalized to the Clairaut duality. This can be rephrased by saying that the standard Legendre transform \( \text{Leg} \) (given by (2.6)) can be generalized to the singular Lagrangian theory using the Legendre-Clairaut transform \( \text{Leg}^{\text{Cl}} \) given by the multidimensional Clairaut equation (3.3).

To find its solutions, we differentiate (3.3) by \( \bar{p}_A \) and split the sum (3.5) in \( B \) as follows

\[
\sum_{i=1}^{r} \left[ \bar{p}_i - \frac{\partial L(q^A, v^A)}{\partial v^i} \right] \cdot \frac{\partial^2 H^{\text{Cl}}(q^A, \bar{p}_A)}{\partial \bar{p}_i \partial \bar{p}_C} + \sum_{\alpha=r+1}^{n} \left[ \bar{p}_\alpha - \frac{\partial L(q^A, v^A)}{\partial v^\alpha} \right] \cdot \frac{\partial^2 H^{\text{Cl}}(q^A, \bar{p}_A)}{\partial \bar{p}_\alpha \partial \bar{p}_C} = 0.
\]

As \( \det W_{ij} \neq 0 \), we suggest to replace (4.1) by the conditions

\[
\bar{p}_i = p_i = \frac{\partial L(q^A, v^A)}{\partial v^i}, \quad i = 1, \ldots, r,
\]

\[
\frac{\partial^2 H^{\text{Cl}}(q^A, \bar{p}_A)}{\partial \bar{p}_\alpha \partial \bar{p}_C} = 0, \quad \alpha = r + 1, \ldots n.
\]
In this way we obtain a mixed envelope/general solution of the Clairaut equation as follows \[19\]. After resolving (4.2) by regular velocities \(v^i = V^i(q^A, p_i, c^\alpha)\) and writing down a solution of (4.3) as
\[
\frac{\partial H_{\text{Cl}}}{\partial \bar{p}_\alpha} q^A, p_i, c^\alpha \right) = \frac{\partial H_{\text{Cl}}}{\partial \bar{p}_\alpha} \left( q^A, p_i, c^\alpha, v^\alpha \right), \tag{4.4}
\]
which is the desired “mixed” Legendre-Clairaut transform \(L^\text{Leg} \rightarrow H_{\text{Cl}} \rightarrow H_{\text{mix}}\) written in coordinates.

Note that (4.4) coincides with the “slow and careful Legendre transformation” of \[55\] and with the “generalized Legendre transformation” of \[10\], while we have obtained in a new way.

**Example 4.1.** Let \(L(x, y, v_x, v_y) = my^2/2 + kxy_y\), then the corresponding Clairaut equation for \(H = H_{\text{Cl}}(x, y, p_x, p_y)\) is
\[
H = \bar{p}_x H'_{\bar{p}_x} + \bar{p}_y H'_{\bar{p}_y} - \frac{my}{2} \left( H'_{\bar{p}_x} \right)^2 - kxH'_{\bar{p}_y}. \tag{4.5}
\]
The general solution of (4.5) is
\[
H_{\text{gen}}(x, y, \bar{p}_x, \bar{p}_y, c_x, c_y) = \bar{p}_x c_x + \bar{p}_y c_y - \frac{my^2}{2} - kxy_c,
\]
where \(c_x, c_y\) are arbitrary functions of the passive variables \(x, y\). Then we differentiate
\[
\frac{\partial H_{\text{gen}}}{\partial c_x} = p_x - myc_x = 0, \quad \Rightarrow c_x^\text{extr} = \frac{p_x}{my},
\]
\[
\frac{\partial H_{\text{gen}}}{\partial c_y} = \bar{p}_y - kx.
\]
Finally, resolve only the first equation and set \(c_y \mapsto v_y\) an “unresolved velocity” to obtain the mixed Legendre-Clairaut transform (cf. \[55\], Examples 5 and 17)
\[
H_{\text{mix}}(x, y, p_x, \bar{p}_y, v_y) = \frac{p_x^2}{2my} + v_y(\bar{p}_y - kx). \tag{4.6}
\]

5. **Hamiltonian formulation of singular Lagrangian systems**

Let us use the mixed Hamilton-Clairaut function \(H_{\text{mix}}(q^A, p_i, \bar{p}_\alpha, v^\alpha)\) (4.4) to describe a singular Lagrangian theory by a system of ordinary first-order differential equations. In our formulation we divide the set of standard Lagrange equations of motion
\[
\frac{d}{dt} \frac{\partial L(q^A, v^A)}{\partial v^B} = \frac{\partial L(q^A, v^A)}{\partial q^B} \tag{5.1}
\]
into two subsets, according to the index \(B\) being regular \((B = i = 1, \ldots, r)\) or degenerate \((B = \alpha = r + 1, \ldots, n)\). We use the designation of “physical” momenta
in the regular subset only, such that the Lagrange equations become
\[
\frac{dp_i}{dt} = \frac{\partial L}{\partial q^i}, \quad (5.2)
\]
\[
\frac{dB_\alpha (q^A, p_i)}{dt} \bigg|_{v^\alpha = V^i(q^A, p_i, v^\alpha)} = \frac{\partial L}{\partial v^\alpha}, \quad (5.3)
\]
where
\[
B_\alpha (q^A, p_i) \overset{\text{def}}{=} \frac{\partial L}{\partial v^\alpha} \bigg|_{v^\alpha = V^i(q^A, p_i, v^\alpha)} \quad (5.4)
\]
are given functions which determine dynamics of the singular Lagrangian system in the “degenerate” sector. The functions \(B_\alpha (q^A, p_i)\) are independent of the unresolved velocities \(v^\alpha\) since \(\text{rank} W_{AB} = r\). One should also take into account that now
\[
\frac{dq^i}{dt} = V^i (q^A, p_i, v^\alpha), \quad \frac{dq^\alpha}{dt} = v^\alpha. \quad (5.5)
\]
An application of (5.2) yields the system of equations which gives a Hamiltonian-Clairaut description of a singular Lagrangian system
\[
\frac{\partial H_{\text{mix}}^{\text{Cl}}}{\partial p_i} = V^i (q^A, p_i, v^\alpha), \quad (5.6)
\]
\[
\frac{\partial H_{\text{mix}}^{\text{Cl}}}{\partial p_\alpha} = v^\alpha, \quad (5.7)
\]
\[
\frac{\partial H_{\text{mix}}^{\text{Cl}}}{\partial q^i} = -\frac{dp_i}{dt} + \sum_{\beta=r+1}^{n} [\bar{p}_\beta - B_\beta (q^A, p_i)] \frac{\partial v^\beta}{\partial q^i}, \quad (5.8)
\]
\[
\frac{\partial H_{\text{mix}}^{\text{Cl}}}{\partial q^\alpha} = dB_\alpha (q^A, p_i) dt + \sum_{\beta=r+1}^{n} [\bar{p}_\beta - B_\beta (q^A, p_i)] \frac{\partial v^\beta}{\partial q^\alpha}. \quad (5.9)
\]
An application of (5.2) yields the system of equations which gives a Hamiltonian-Clairaut description of a singular Lagrangian system
we can get rid of these difficulties, if we reformulate (5.6)–(5.9) by introducing a “physical” Hamiltonian

\[ H_{\text{phys}} (q^A, p_i) = H_{\text{mix}}^{\text{Cl}} (q^A, p_i, \bar{p}_\alpha, p^\alpha) - \sum_{\beta = r+1}^{n} \left[ \bar{p}_\beta - B_\beta (q^A, p_i) \right] v^\beta, \]  

(5.10)

which does not depend on the degenerate variables \( \bar{p}_\alpha \). The “physical” Hamiltonian (5.10) can be rewritten in the form

\[ H_{\text{phys}} (q^A, p_i) = \sum_{i=1}^{r} p_i V^i (q^A, p_i, v^\alpha) + \sum_{\alpha = r+1}^{n} B_\alpha (q^A, p_i) v^\alpha - L (q^A, V^i (q^A, p_i, v^\alpha), v^\alpha), \]  

(5.11)

Then using (4.2), one can show that the r.h.s. of (5.10) indeed does not depend on “nondynamical” momenta \( \bar{p}_\alpha \) and degenerate velocities \( v^\alpha \) at all (which justifies the term “physical”). Hereafter we will use the shorthand \( B_\beta := B_\beta (q^A, p_i) \), but be sure to remember it depends on \( (q^A, p_i) \).

Now we introduce a “\( q^\alpha \)-long derivative"

\[ D_\alpha X = \frac{\partial X}{\partial q^\alpha} + \{ B_\alpha, X \}_{\text{phys}}, \]  

(5.12)

where \( X = X (q^A, p_i) \) is a smooth scalar function, and

\[ \{ X, Y \}_{\text{phys}} = \sum_{i=1}^{n-r} \left( \frac{\partial X}{\partial q^i} \frac{\partial Y}{\partial p_i} - \frac{\partial Y}{\partial q^i} \frac{\partial X}{\partial p_i} \right) \]  

(5.13)

is the physical Poisson bracket (in regular variables \( q^i, p_i \)).

Then we obtain from (5.6)–(5.9) the main result of our Clairaut-type formulation: the sought-for system of ordinary first-order differential equations (the Hamilton-Clairaut system of equations) which describes any singular Lagrangian classical system has the form

\[ \frac{dq^i}{dt} = \{ q^i, H_{\text{phys}} (q^A, p_i) \}_{\text{phys}} - \sum_{\beta = r+1}^{n} \{ q^i, B_\beta \}_{\text{phys}} \frac{dq^\beta}{dt}, \quad i = 1, \ldots, r \]  

(5.14)

\[ \frac{dp_i}{dt} = \{ p_i, H_{\text{phys}} (q^A, p_i) \}_{\text{phys}} - \sum_{\beta = r+1}^{n} \{ p_i, B_\beta \}_{\text{phys}} \frac{dq^\beta}{dt}, \quad i = 1, \ldots, r \]  

(5.15)

\[ \sum_{\beta = r+1}^{n} F_{\alpha\beta} (q^A, p_i) \frac{dq^\beta}{dt} = D_\alpha H_{\text{phys}} (q^A, p_i), \quad \alpha = r + 1, \ldots, n. \]  

(5.16)

We also introduce in (5.10) a “\( q^\alpha \)-field strength” of the “\( q^\alpha \)-gauge fields”

\[ F_{\alpha\beta} (q^A, p_i) = \frac{\partial B_\beta}{\partial q^\alpha} - \frac{\partial B_\alpha}{\partial q^\beta} + \{ B_\alpha, B_\beta \}_{\text{phys}}, \]  

(5.17)

which is nonabelian due to the presence of the Poisson bracket in the physical phase space.

The system (5.14)–(5.16) is equivalent to the Lagrange equations of motion (5.1) by construction. Thus, the Clairaut-type formulation (5.14)–(5.17) is valid for any Lagrangian theory, as opposite to other approaches. A nonsingular system contains no “degenerate” variables at all, because the rank of the Hessian \( r \) is full (\( r = n \).
The distinguishing property of any singular system \((r < n)\) is clear and simple: it contains the additional system of linear algebraic equations (5.16) for “unresolved” velocities \(v^\alpha\) which can be analyzed and solved by standard linear algebra methods.

Indeed the system (5.16) gives a full classification of all singular Lagrangian theories, which is done in the next section.

Example 5.1 (Cawley [9]). Let \(L = \dot{x}\dot{y} + zy^2/2\), then the equations of motion are

\[
\ddot{x} = yz, \quad \ddot{y} = 0, \quad y^2 = 0.
\]

Because the Hessian has rank 2 and the velocity \(\dot{z}\) does not enter into the Lagrangian, the only degenerate velocity is \(\dot{z} (\alpha = z)\), the regular momenta are \(p_x = \dot{y}, p_y = \dot{x} (i = x, y)\). Thus, we have

\[
H_{\text{phys}} = p_x p_y - \frac{1}{2} z y^2, \quad B_z = 0.
\]

The equations of motion (5.14)–(5.15) are

\[
\dot{p}_x = 0, \quad \dot{p}_y = yz,
\]

and the condition (5.16) gives

\[
D_z H_{\text{phys}} = \frac{\partial H_{\text{phys}}}{\partial z} = -\frac{1}{2} y^2 = 0.
\]

Observe that (5.19) and (5.20) coincide with the initial Lagrange equations of motion (5.18).

Example 5.2. The Classical relativistic particle is described by

\[
L = -mR, \quad R = \sqrt{\dot{x}_0^2 - \sum_{i=x,y,z} \dot{x}_i^2},
\]

where a dot denotes derivative with respect to the proper time. Because the rank of the Hessian is 3, we consider velocities \(\dot{x}_i\) as regular variables and the velocity \(\dot{x}_0\) as a degenerate variable. Then for the regular canonical momenta we have \(p_i = m\dot{x}_i/R\) which can be resolved with respect to the regular velocities as

\[
\dot{x}_i = \dot{x}_0 \frac{p_i}{E}, \quad E = \sqrt{m^2 + \sum_{i=x,y,z} p_i^2}.
\]

Using (5.4) and (5.11) we obtain for Hamiltonians

\[
H_{\text{phys}} = 0, \quad h_{x_0} = -m \frac{\dot{x}_0}{R} = -E,
\]

and so the “physical sense” of \(h_{x_0}\) is that it is indeed the energy (5.22). Equations of motion (5.14)–(5.15) are

\[
\dot{x}_i = \dot{x}_0 \frac{p_i}{E}, \quad \dot{p}_1 = \frac{\partial h_{x_0}}{\partial x_i} \dot{x}_0 = 0,
\]

which coincide with the Lagrange equations following from (5.21) directly, the velocity \(\dot{x}_0\) is arbitrary.
6. Nonabelian gauge theory interpretation

Let us consider “$q^\alpha$-long derivative” \ref{eq:5.12} and “$q^\alpha$-field strength” \ref{eq:5.17} in more detail. Note that the “$q^\alpha$-long derivative” satisfies the Leibniz rule

$$D_\alpha \{B_\beta, B_\gamma\}_\text{phys} = \{D_\alpha B_\beta, B_\gamma\}_\text{phys} + \{B_\beta, D_\alpha B_\gamma\}_\text{phys} \tag{6.1}$$

which is valid while acting on “$q^\alpha$-gauge fields” $B_\alpha$. The commutator of the “$q^\alpha$-long derivatives” is now equal to the Poisson bracket with the “$q^\alpha$-field strength”

$$(D_\alpha D_\beta - D_\beta D_\alpha) X = \{F_{\alpha\beta} (q^A, p_i), X\}_\text{phys}, \tag{6.2}$$

In the particular case, acting on $F_{\alpha\beta} (q^A, p_i)$, this gives

$$D_\alpha D_\beta F_{\alpha\beta} (q^A, p_i) = 0. \tag{6.3}$$

Let us introduce the $B_\alpha$-transformation

$$\delta_{B_\alpha} X = \{B_\alpha, X\}_\text{phys}, \tag{6.4}$$

which satisfies

$$\begin{align*}
\delta_{B_\alpha} (D_\beta B_\gamma - D_\gamma B_\beta) B_\alpha &= \delta_{B_\alpha, B_\beta; B_\gamma} B_\gamma, \tag{6.5} \\
\delta_{B_\alpha} F_{\beta\gamma} (q^A, p_i) &= (D_\gamma D_\beta - D_\beta D_\gamma) B_\alpha, \tag{6.6} \\
\delta_{B_\alpha} \{B_\beta, B_\gamma\}_\text{phys} &= \{\delta_{B_\alpha} B_\beta, B_\gamma\}_\text{phys} + \{B_\beta, \delta_{B_\alpha} B_\gamma\}_\text{phys}. \tag{6.7}
\end{align*}$$

This means that the “$q^\alpha$-long derivative” $D_\alpha \ref{eq:5.12}$ is in fact a “$q^\alpha$-covariant derivative” with respect to the $B_\alpha$-transformation \ref{eq:6.4}. Indeed, observe that “$D_\alpha$ transforms as fields” \ref{eq:6.4}, which proves that it is really covariant (note the cyclic permutations)

$$\begin{align*}
\delta_{B_\alpha} D_\beta B_\gamma + \delta_{B_\beta} D_\alpha B_\gamma + \delta_{B_\gamma} D_\beta B_\alpha &= \{B_\alpha, D_\beta B_\gamma\}_\text{phys} + \{B_\gamma, D_\alpha B_\beta\}_\text{phys} + \{B_\beta, D_\gamma B_\alpha\}_\text{phys}, \tag{6.8}
\end{align*}$$

The “$q^\alpha$-Maxwell” equations of motion for the “$q^\alpha$-field strength” are

$$\begin{align*}
D_\alpha F_{\alpha\beta} (q^A, p_i) &= J_\beta (q^A, p_i), \tag{6.9} \\
D_\alpha F_{\beta\gamma} (q^A, p_i) + D_\gamma F_{\alpha\beta} (q^A, p_i) + D_\beta F_{\gamma\alpha} (q^A, p_i) &= 0, \tag{6.10}
\end{align*}$$

where $J_\alpha (q^A, p_i)$ is a “$q^\alpha$-current” which is a function of the initial Lagrangian \ref{eq:2.2} and its derivatives up to third order. Due to \ref{eq:6.3} the “$q^\alpha$-current” is conserved

$$D_\alpha J_\alpha (q^A, p_i) = 0. \tag{6.11}$$

Thus, a singular Lagrangian system leads effectively to some special kind of the nonabelian gauge theory in the “degenerate” coordinate subspace $q^\alpha$, in which “nonabelianity” appears not due to a Lie algebra, as in the Yang-Mills theory, but “classically”, due to the Poisson bracket in the physical phase space $(q^i, p_i)$. The corresponding manifold can perhaps be interpreted locally as the degenerate Poisson manifold being a direct product of real space of dimension $(n - r)$ and symplectic manifold of dimension $(r, r)$, where $r$ is the rank of Hessian.
7. Classification, gauge freedom and new brackets

Next we can classify singular Lagrangian theories as follows:

1. **Gaugeless theory.** The rank of the skew-symmetric matrix $F_{\alpha\beta}(q^A, p_i)$ is “full”, i.e. rank $F_{\alpha\beta}(q^A, p_i) = n - r$ and constant, and so the matrix $F_{\alpha\beta}(q^A, p_i)$ is invertible, and all the (degenerate) velocities $v^\alpha$ can be found from the system of linear equations (5.16) in a purely algebraic way.

2. **Gauge theory.** The skew-symmetric matrix $F_{\alpha\beta}(q^A, p_i)$ is singular. If rank $F_{\alpha\beta}(q^A, p_i) = r_f < n - r$, then a singular Lagrangian theory has $n - r - r_f$ gauge degrees of freedom. We can take them arbitrary, which corresponds to the presence of some symmetries in the theory. Note that the rank $r_f$ is even due to skew-symmetry of $F_{\alpha\beta}(q^A, p_i)$.

In the first (gaugeless theory) case one can resolve (5.16) as follows

$$v^\beta = \sum_{\alpha = r+1}^n \bar{F}^{\beta\alpha}(q^A, p_i) D_\alpha H_{\text{phys}}(q^A, p_i),$$  \hspace{1cm} (7.1)

where $\bar{F}^{\alpha\beta}(q^A, p_i)$ is the inverse matrix to $F_{\alpha\beta}(q^A, p_i)$, i.e.

$$\sum_{\beta = r+1}^n F_{\alpha\beta}(q^A, p_i) \bar{F}^{\beta\gamma}(q^A, p_i) = \sum_{\beta = r+1}^n \bar{F}^{\gamma\beta}(q^A, p_i) F_{\beta\alpha}(q^A, p_i) = \delta^\gamma_\alpha. \hspace{1cm} (7.2)$$

Substitute (7.1) in (5.14)–(5.15) to present the system of equations for a gaugeless degenerate Lagrangian theory in the Hamiltonian-like form as follows

$$\frac{dq^i}{dt} = \{q^i, H_{\text{phys}}(q^A, p_i)\}_{\text{new}}, \hspace{1cm} (7.3)$$

$$\frac{dp_i}{dt} = \{p_i, H_{\text{phys}}(q^A, p_i)\}_{\text{new}}, \hspace{1cm} (7.4)$$

where the bracket is defined by

$$\{X, Y\}_{\text{new}} = \{X, Y\}_{\text{phys}} - \sum_{\alpha = r+1}^n \sum_{\beta = r+1}^n \{X, B_\alpha\}_{\text{phys}} \bar{F}^{\alpha\beta}(q^A, p_i) D_\beta Y. \hspace{1cm} (7.5)$$

Then the time evolution of any function of dynamical variables $X(q^A, p_i)$ is also determined by the bracket (7.3) as follows

$$\frac{dX}{dt} = \{X, H_{\text{phys}}(q^A, p_i)\}_{\text{new}}. \hspace{1cm} (7.6)$$

In the second (gauge theory) case, with the singular matrix $F_{\alpha\beta}(q^A, p_i)$ of rank $r_f$, we rearrange its rows and columns to obtain a nonsingular $r_f \times r_f$ submatrix in the left upper corner. In such a way, the first $r_f$ equations of the system of linear (under also rearranged $v^\beta$) equations (5.16) are independent. Then we express the indices $\alpha$ and $\beta$ as pairs $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$, where $\alpha_1$ and $\beta_1$ denote the first $r_f$ rows and columns, while $\alpha_2$ and $\beta_2$ denote the rest of $n - r - r_f$ rows and

---

2Here we do not analyze these symmetries, which will be the subject of a separate paper.
columns. Correspondingly, we decompose the system \((7.10)\) as

\[
\sum_{\beta_1=r+1}^{r+r_f} F_{\alpha_1\beta_1} (q^A, p_i) v^{\beta_1} + \sum_{\beta_2=r+r_f+1}^{n} F_{\alpha_1\beta_2} (q^A, p_i) v^{\beta_2} = D_{\alpha_1} H_{\text{phys}} (q^A, p_i),
\]

\(7.7\)

\[
\sum_{\beta_1=r+1}^{r+r_f} F_{\alpha_2\beta_1} (q^A, p_i) v^{\beta_1} + \sum_{\beta_2=r+r_f+1}^{n} F_{\alpha_2\beta_2} (q^A, p_i) v^{\beta_2} = D_{\alpha_2} H_{\text{phys}} (q^A, p_i).
\]

\(7.8\)

Because the matrix \(F_{\alpha_1\beta_1} (q^A, p_i)\) is nonsingular by construction, we can find the first \(r_f\) velocities

\[
v^{\beta_1} = \sum_{\alpha_1=r+1}^{r+r_f} \bar{F}^{\beta_1\alpha_1} (q^A, p_i) D_{\alpha_1} H_{\text{phys}} (q^A, p_i)
\]

\[
- \sum_{\alpha_1=r+1}^{r+r_f} \bar{F}^{\beta_1\alpha_1} (q^A, p_i) F_{\alpha_1\beta_2} (q^A, p_i) v^{\beta_2},
\]

\(7.9\)

where \(\bar{F}^{\beta_1\alpha_1} (q^A, p_i)\) is the inverse of the nonsingular \(r_f \times r_f\) submatrix \(F_{\alpha_1\beta_1} (q^A, p_i)\) satisfying \((7.2)\).

Then, since \(\text{rank} F_{\alpha\beta} (q^A, p_i) = r_f\), the last \(n - r - r_f\) equations \((7.8)\) are linear combinations of the first \(r_f\) independent ones \((7.7)\), which gives

\[
F_{\alpha_2\beta_1} (q^A, p_i) = \sum_{\alpha_1=r+1}^{r+r_f} \lambda_{\alpha_2}^{\alpha_1} (q^A, p_i) F_{\alpha_1\beta_1} (q^A, p_i),
\]

\(7.10\)

\[
F_{\alpha_2\beta_2} (q^A, p_i) = \sum_{\alpha_1=r+1}^{r+r_f} \lambda_{\alpha_2}^{\alpha_1} (q^A, p_i) F_{\alpha_1\beta_2} (q^A, p_i),
\]

\(7.11\)

\[
D_{\alpha_2} H_{\text{phys}} (q^A, p_i) = \sum_{\alpha_1=r+1}^{r+r_f} \lambda_{\alpha_2}^{\alpha_1} (q^A, p_i) D_{\alpha_1} H_{\text{phys}} (q^A, p_i),
\]

\(7.12\)

where \(\lambda_{\alpha_2}^{\alpha_1} (q^A, p_i)\) are some \(r_f \times (n - r - r_f)\) smooth functions. Using the relation \((7.10)\) and invertibility of \(F_{\alpha_1\beta_1} (q^A, p_i)\) we eliminate the functions \(\lambda_{\alpha_2}^{\alpha_1} (q^A, p_i)\) and obtain

\[
F_{\alpha_2\beta_2} (q^A, p_i) = \sum_{\alpha_1=r+1}^{r+r_f} \sum_{\beta_1=r+1}^{r+r_f} F_{\alpha_2\beta_1} (q^A, p_i) \bar{F}^{\beta_1\alpha_1} (q^A, p_i) F_{\alpha_1\beta_2} (q^A, p_i),
\]

\(7.13\)

\[
D_{\alpha_2} H_{\text{phys}} (q^A, p_i) = \sum_{\alpha_1=r+1}^{r+r_f} \sum_{\beta_1=r+1}^{r+r_f} F_{\alpha_2\beta_1} (q^A, p_i) \bar{F}^{\beta_1\alpha_1} (q^A, p_i) D_{\alpha_1} H_{\text{phys}} (q^A, p_i).
\]

\(7.14\)

This indicates that a gauge theory is fully determined by the first \(r_f\) rows of the (rearranged) matrix \(F_{\alpha\beta} (q^A, p_i)\) and the first \(r_f\) (rearranged) derivatives \(D_{\alpha_1} H_{\text{phys}} (q^A, p_i)\) only. Next, using further linear combinations of full rows, we
can set all elements \( F_{\alpha_1\beta_2} (q^A, p_i) = 0 \), and obtain

\[
D_{\alpha_2} H_{\text{phys}} (q^A, p_i) = 0, \quad \alpha_2 = r + r_f + 1, \ldots, n, \tag{7.15}
\]

which means that functions \( D_{\alpha_2} H_{\text{phys}} (q^A, p_i) \) manifestly do not depend on the \( q^{\alpha_2} \).

Thus, in addition (to the gaugeless case), now we also have \( n - r - r_f \) relations (which are, in fact, combinations of motion integrals for the system (5.14)–(5.15)). We can make the unresolved \( n - r - r_f \) velocities vanish

\[
v^{\beta_2} = 0 \tag{7.16}
\]

by some “gauge fixing” condition. Then (7.9) becomes

\[
v^{\beta_1} = \sum_{\alpha_1=r+1}^{r+r_f} \bar{F}^{\beta_1\alpha_1} (q^A, p_i) D_{\alpha_1} H_{\text{phys}} (q^A, p_i). \tag{7.17}
\]

By analogy with (7.3)–(7.4), we can also write the system of equations for a singular Lagrangian theory in the Hamiltonian form (in the gauge case). Now we introduce another new bracket

\[
\{X, Y\}_\text{gauge} = \{X, Y\}_\text{phys} - \sum_{\alpha_1=r+1}^{r+r_f} \sum_{\beta_1=r+1}^{r+r_f} \{X, B_{\alpha_1}\}_\text{phys} F^{\alpha_1\beta_1} (q^A, p_i) D_{\beta_1} Y, \tag{7.18}
\]

Then substituting (7.10)–(7.17) into (5.14)–(5.15) and using (7.18), we obtain

\[
\frac{dq^i}{dt} = \{q^i, H_{\text{phys}} (q^A, p_i)\}_\text{gauge}, \tag{7.19}
\]
\[
\frac{dp_i}{dt} = \{p_i, H_{\text{phys}} (q^A, p_i)\}_\text{gauge}. \tag{7.20}
\]

This new bracket governs the time evolution in the gauge case

\[
\frac{dX}{dt} = \{X, H_{\text{phys}} (q^A, p_i)\}_\text{gauge}. \tag{7.21}
\]

The brackets (7.5) and (7.18) are not anticommutative and do not satisfy the Jacobi identity. Therefore, the standard quantization scheme is not applicable here directly. We expect that some more intricate further assumptions should make it possible to quantize consistently degenerate Lagrangian systems within the suggested approach (see, e.g., [42, 52]).

It is worthwhile to consider the \textit{limit case}, when \( r_f = 0 \), i.e.

\[
F_{\alpha\beta} (q^A, p_i) = 0 \tag{7.22}
\]

identically, which can mean that \( B_{\alpha} (q^A, p_i) = 0 \), so the Lagrangian can be independent of the degenerate velocities \( v^\alpha \). It follows from (5.10) that

\[
D_{\alpha} H_{\text{phys}} (q^A, p_i) = \frac{\partial H_{\text{phys}} (q^A, p_i)}{\partial q^{\alpha}} = 0, \tag{7.23}
\]

which leads to the “\textit{independence}” \textit{statement}: the “physical” Hamiltonian \( H_{\text{phys}} (q^A, p_i) \) does not depend on the degenerate coordinates \( q^\alpha \) if the Lagrangian does not depend on the velocities \( v^\alpha \). In this case, the bracket (7.18) coincides with the Poisson bracket in the reduced (“physical” space) \( \{, , \}_\text{gauge} = \{, , \}_\text{phys} \).
Example 7.1 (Christ-Lee model [11]). The Lagrangian of $SU(2)$ Yang-Mills theory in 0 + 1 dimensions is (in our notation)

$\mathcal{L}(x_i, y_\alpha, v_\alpha) = \frac{1}{2} \sum_{i=1,2,3} \left( v_i - \sum_{j,\alpha=1,2,3} \varepsilon_{ij\alpha} x_j y_\alpha \right)^2 - U \left( x^2 \right), \quad (7.24)$

where $i, \alpha = 1, 2, 3$, $x^2 = \sum_i x_i^2$, $v_i = \dot{x}_i$ and $\varepsilon_{ijk}$ is the Levi-Civita symbol. Because (7.24) is independent of degenerate velocities $\dot{y}_\alpha$, all $B_\alpha(q^A, p_i)$ (5.10) are zero, and therefore $F_{\alpha\beta}(q^A, p_i)$ (5.14) is zero, we have the limit gauge case of the above classification. The corresponding Clairaut equation (3.3) for $H = H^{Cl}(x_i, y_\alpha, \bar{p}_i, \bar{p}_\alpha)$ has the form

$H = \sum_{i=1,2,3} \bar{p}_i H'_{\bar{p}_i} + \sum_{\alpha=1,2,3} \bar{p}_\alpha H'_{\bar{p}_\alpha} - \frac{1}{2} \sum_{i=1,2,3} \left( H'_p - \sum_{j,\alpha=1,2,3} \varepsilon_{ij\alpha} x_j y_\alpha \right)^2 + U \left( x^2 \right), \quad (7.25)$

We show manifestly, how to obtain the envelope solution for regular variables and general solution for degenerate variables. Its general solution is

$H_{gen} = \sum_{i=1,2,3} \bar{p}_i c_i + \sum_{\alpha=1,2,3} \bar{p}_\alpha c_\alpha - \frac{1}{2} \sum_{i=1,2,3} \left( c_i - \sum_{j,\alpha=1,2,3} \varepsilon_{ij\alpha} x_j y_\alpha \right)^2 + U \left( x^2 \right), \quad (7.26)$

where $c_i, c_\alpha$ are arbitrary functions of coordinates.$^3$ We differentiate (7.26) by $c_i, c_\alpha$

\[ \frac{\partial H_{gen}}{\partial c_i} = \bar{p}_i - \left( c_i - \sum_{j,\alpha=1,2,3} \varepsilon_{ij\alpha} x_j y_\alpha \right), \quad (7.27) \]

\[ \frac{\partial H_{gen}}{\partial c_\alpha} = \bar{p}_\alpha, \quad (7.28) \]

and observe that only the first relation (7.27) can be resolved with respect to $c_i$, and therefore can lead to the envelope solution, while other $c_\alpha$ cannot be resolved, and therefore we consider only general solution of the Clairaut equation. So we can exclude half of the constants using (7.27) (with the substitution $\bar{p}_i \rightarrow p_i$) and get the mixed solution (4.4) to the Clairaut equation (7.26) as

$H^{Cl}_{mix}(x_i, y_\alpha, p_i, \bar{p}_\alpha, c_\alpha) = \frac{1}{2} \sum_{i=1,2,3} p_i^2 + \sum_{i,j,\alpha=1,2,3} \varepsilon_{ij\alpha} p_i x_j y_\alpha + \sum_{\alpha=1,2,3} \bar{p}_\alpha c_\alpha + U \left( x^2 \right). \quad (7.29)$

Using (4.10), we obtain the “physical” Hamiltonian

$H_{phys}(x_i, y_\alpha, p_i) = \frac{1}{2} \sum_{i=1,2,3} p_i^2 + \sum_{i,j,\alpha=1,2,3} \varepsilon_{ij\alpha} p_i x_j y_\alpha + U \left( x^2 \right). \quad (7.30)$

From the other side, the Hessian of (7.24) has rank 3, and we choose $x_i, v_i$ and $y_\alpha$ to be regular and degenerate variables respectively. The degenerate velocities $v_\alpha = \dot{y}_\alpha$ cannot be defined from (5.16) at all, they are arbitrary, and the first

$^3$Recall that $q^A$ are passive variables under the Legendre transform.
(7.31) of the system (5.14)–(5.15) become (also in accordance to the independence statement)

\[
\frac{\partial H_{\text{phys}}(x_i, y_\alpha, p_i)}{\partial y_\alpha} = \sum_{i,j=1,2,3} \varepsilon_{ij\alpha} p_i x_j = 0.
\]

The preservation in time (7.21) of (7.31) is fulfilled identically due to the antisymmetry properties of the Levi-Civita symbols. It is clear that only 2 equations from 3 of (7.31) are independent, so we choose

\[p_1 x_2 = p_2 x_1, \quad p_1 x_3 = p_3 x_1\]

and insert in (7.30) to get

\[
\tilde{H}_{\text{phys}} = \frac{1}{2} p_1^2 x_2^2 + U(x_2). \tag{7.32}
\]

The transformation \(\tilde{p}_1 = p_1 \sqrt{x_2^2/x_1}, \tilde{x} = \sqrt{x_2^2}\) gives the well-known result [11, 27]

\[
\tilde{H}_{\text{phys}} = \frac{1}{2} \tilde{p}_1^2 + U(\tilde{x}). \tag{7.33}
\]

8. Singular Lagrangian systems and many-time dynamics

The \(n\)-time classical dynamics and its connection with constrained systems were studied in [35, 36] as a generalization of some relativistic two-particle models [18]. Under the Clairaut-type approach, we now treat the degenerate coordinates \(q_\alpha\) as parameters analogous to \(n-r\) time variables (with \(n-r\) corresponding “Hamiltonians” \(-B_\alpha (q^4, p_i), \) see (5.12)). Indeed, let us introduce \(n-r+1\) “times” \(t_\mu\) and the corresponding “many-time Hamiltonians” \(\mathcal{H}_\mu (t^\mu, q^i, p_i), \mu = 0, \ldots, n-r\) by

\[
t \mapsto 0, \quad H_{\text{phys}} (q^i, q^\alpha, p_i) \mapsto \mathcal{H}_0 (q^i, t^0, p_i), \tag{8.1}
\]

\[
q^\alpha \mapsto t^\alpha, \quad -B_\alpha (q^i, q^\alpha, p_i) \mapsto \mathcal{H}_\alpha (q^i, t^\alpha, p_i), \alpha = 1, \ldots, n-r. \tag{8.2}
\]

Then the equations (5.14)–(5.15) can be presented in the differential form

\[
dq^i = \sum_{\mu=0}^{n-r} \{q^i, \mathcal{H}_\mu\} dt^\mu, \tag{8.3}
\]

\[
dp_i = \sum_{\mu=0}^{n-r} \{p_i, \mathcal{H}_\mu\} dt^\mu. \tag{8.4}
\]

The system of equations for degenerate velocities (5.16) becomes

\[
\sum_{\mu=0}^{n-r} \mathcal{G}_{\mu\nu} dt^\nu = 0, \tag{8.5}
\]

where

\[
\mathcal{G}_{\mu\nu} = \frac{\partial \mathcal{H}_\mu}{\partial v^\nu} - \frac{\partial \mathcal{H}_\nu}{\partial v^\mu} + \{\mathcal{H}_\mu, \mathcal{H}_\nu\}. \tag{8.6}
\]

If we introduce a 1-form \(\omega = p_i dq^i - \mathcal{H}_\mu dt^\mu\), then it follows from (8.5) that

\[
d\omega = \frac{1}{2} \sum_{\mu=0}^{n-r} \sum_{\nu=0}^{n-r} \mathcal{G}_{\mu\nu} dt^\mu \wedge dt^\nu = 0, \tag{8.7}
\]
which agrees with the action principle in the form $S = \int \omega$. The corresponding set of the Hamilton-Jacobi equations for $S = S(t, q^A) \mapsto \mathcal{S}(t^\mu, q^I)$ is

$$\frac{\partial \mathcal{S}}{\partial t^\mu} + \mathcal{H}_\mu \left( q^i, t^\alpha, \frac{\partial \mathcal{S}}{\partial q^i} \right) = 0. \quad (8.8)$$

Therefore we come to the conclusion that any singular Lagrangian theory (in the Clairaut-type formulation [19]) is equivalent to the many-time classical dynamics [17, 36]: the equations of motion are (8.3)–(8.4) and the integrability condition is $\mathcal{G}_{\mu\nu} = 0$.

9. Conclusions

To conclude, we have described Hamiltonian evolution of singular Lagrangian systems using $n - r + 1$ functions $H_{\text{phys}}(q^A, p_i)$ and $B_\alpha(q^A, p_i)$ of dynamical variables. This is done by means of the generalized Legendre-Clairaut transform, that is by solving the corresponding multidimensional Clairaut equation without introducing Lagrange multipliers. All variables are set as regular or degenerate according to the rank of the Hessian matrix of Lagrangian. We consider the restricted “physical” phase space formed by the regular coordinates $q^i$ and momenta $p_i$ only, while degenerate coordinates $q^\alpha$ play a role of parameters. There are two reasons why degenerate momenta $\bar{p}_\alpha$ corresponding to $q^\alpha$ need not be considered in the Clairaut-type formulation:

1) the mathematical reason: there is no possibility to find the degenerate velocities $v^\alpha$, as can be done for the regular velocities $v^i$ in (4.2), and the “pre-Hamiltonian” (2.2) has no extremum in degenerate directions;

2) the physical reason: momentum is a “measure of movement”, but in “degenerate” directions there is no dynamics, hence — no reason to introduce the corresponding “physical” momenta at all.

Note that some possibilities to avoid constraints were considered in a different context in [14, 13] and for special forms of the Lagrangian [15, 25].

The Hamiltonian form of the equations of motion (7.3)–(7.4) is achieved by introducing new brackets (7.5) and (7.18) which are responsible for the time evolution. However, they are not anticommutative and do not satisfy the Jacobi identity. Therefore quantization of such brackets requires non-Lie algebra methods (see, e.g., [32, 52]).

In the “nonphysical” coordinate subspace, we can formulate some kind of a non-abelian gauge theory, such that nonabelianity appears due to the Poisson bracket in the physical phase space.

Finally, we show that, in general, a singular Lagrangian system in the Clairaut-type formulation [19, 20] is equivalent to the many-time classical dynamics.

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Appendix A. Multidimensional Clairaut equation

The multidimensional Clairaut equation for a function \( y = y(x_i) \) of \( n \) variables \( x_i \) is \( [3,32] \)

\[
y = \sum_{j=1}^{n} x_j y'_{x_j} - f(y'_{x_i}), \tag{A.1}
\]

where prime denotes a partial differentiation by subscript and \( f \) is a smooth function of \( n \) arguments. To find and classify solutions of (A.1), we need to find first derivatives \( y'_{x_i} \) in some way, and then substitute them back to (A.1). We differentiate the Clairaut equation (A.1) by \( x_j \) and obtain \( n \) equations

\[
\sum_{i=1}^{n} y''_{x_i x_j} (x_i - f(y'_{x_i})) = 0. \tag{A.2}
\]

The classification follows from the ways the factors in (A.2) can be set to zero. Here, for our physical applications, it is sufficient to suppose that ranks of Hessians of \( y \) and \( f \) are equal

\[
\text{rank} y''_{x_i x_j} = \text{rank} f''_{y'_{x_i} y'_{x_j}} = r. \tag{A.3}
\]

This means that in each equation from (A.2) either the first or the second multiplier is zero, but it is not necessary to vanish both of them. The first multiplier can be set to zero without any additional assumptions. So we have

1) The general solution. It is defined by the condition

\[
y''_{x_i x_j} = 0. \tag{A.4}
\]

After one integration we find \( y'_{x_i} = c_i \) and substitution them to (A.1) and obtain

\[
y_{gen} = \sum_{j=1}^{n} x_j c_j - f(c_i), \tag{A.5}
\]

where \( c_i \) are \( n \) constants.

All second multipliers in (A.2) can be zero for \( i = 1, \ldots, n \), but this will give a solution, if we can resolve them under \( y'_{x_i} \). It may be possible, if the rank of Hessians \( f \) is full, i.e. \( r = n \). In this case we obtain

2) The envelope solution. It is defined by

\[
x_i = f(y'_{x_i}). \tag{A.6}
\]

We resolve (A.6) under derivatives as \( y'_{x_i} = C_i(x_j) \) and get

\[
y_{env} = \sum_{i=1}^{n} x_i C_i(x_j) - f(C_i(x_j)), \tag{A.7}
\]

where \( C_i(x_j) \) are \( n \) smooth functions of \( n \) arguments.
In the intermediate case, we can use the envelope solution (A.7) for first $s$ variables, while the general solution (A.5) for other $n - s$ variables, and obtain

3) The $s$-mixed solution, as follows

$$y_{\text{mix}}^{(s)} = \sum_{j=1}^{s} x_j C_j (x_j) + \sum_{j=s+1}^{n} x_j c_j - f(C_1 (x_j), \ldots, C_s (x_j), c_{s+1}, \ldots, c_n). \quad (A.8)$$

If the rank $r$ of Hessians $f$ is not full and a nonsingular minor of the rank $r$ is in upper left corner, then we can resolve first $r$ relations (A.6) only, and so $s \leq r$.

In our physical applications we use the limited case $s = r$.

**Example A.1.** Let $f(z_i) = z_1^2 + z_2^2 + z_3$, then the Clairaut equation for $y = y(x_1, x_2, x_3)$ is

$$y = x_1 y_x^1 + x_2 y_x^2 + x_3 y_x^3 - (y_x^1)^2 - (y_x^2)^2 - y_x^3, \quad (A.9)$$

and we have $n = 3$ and $r = 2$. The general solution can be found from (A.4) by one integration and using (A.5)

$$y_{\text{gen}} = c_1 (x_1 - c_1) + c_2 (x_2 - c_2) + c_3 (x_3 - 1), \quad (A.10)$$

where $c_i$ are constants.

Since $r = 2$, we can resolve only 2 relations from (A.6) by $y_x^1 = \frac{x_1}{2}, y_x^2 = \frac{x_2}{2}$. So there is no envelope solution (for all variables), but we have several mixed solutions corresponding to $s = 1, 2$:

$$
\begin{align*}
  y_{\text{mix}}^{(1)} &= \begin{cases} 
    \frac{x_1^2}{4} + c_2 (x_2 - c_2) + c_3 (x_3 - 1), \\
    c_1 (x_1 - c_1) + \frac{x_2^2}{4} + c_3 (x_3 - 1), 
  \end{cases} \\
  y_{\text{mix}}^{(2)} &= \frac{x_1^2}{4} + \frac{x_2^2}{4} + c_3 (x_3 - 1). 
\end{align*} 
\quad (A.11, A.12)
$$

The case $f(z_i) = z_1^2 + z_2^2$ can be obtained from the above formulas by putting $x_3 = c_3 = 0$, while $y_{\text{mix}}^{(2)}$ becomes the envelope solution $y_{\text{env}} = \frac{x_1^2}{4} + \frac{x_2^2}{4}$.

**APPENDIX B. CORRESPONDENCE WITH THE DIRAC APPROACH**

The connection of the Clairaut-type formulation with the Dirac approach is made by interpretation of the parameters $\bar{p}_\alpha$ entering to the general solution of the Clairaut equation as the “physical” degenerate momenta $p_\alpha$ using for them the same expression through the Lagrangian as

$$\bar{p}_\alpha = p_\alpha = \frac{\partial L(q^A, v^A)}{\partial v^{\alpha}}. \quad (B.1)$$

Then we obtain the primary Dirac constraints (in the resolved form and our notation (5.4))

$$
\Phi_\alpha (q^A, p_A) = p_\alpha - B_\alpha (q^A, p_i) = 0, \quad \alpha = r + 1, \ldots, n, \quad (B.2)
$$

which are defined now on the full phase space $T^*M$. Using (5.10) and (5.11), we can arrive at the complete Hamiltonian of the first-order formulation [24] (corresponding
to the total Dirac Hamiltonian [15]

\[ H_T(q^A, p_A, v^\alpha) = H_{mix}^{Cl}(q^A, p_\alpha, \bar{p}_\alpha, v^{\alpha}) \big|_{\bar{p}_\alpha = p_\alpha} \]

\[ = H_{phys}(q^A, p_i) + \sum_{\alpha=r+1}^{n} v^\alpha \Phi_\alpha(q^A, p_A) , \]

(B.3)

which is equal to the mixed Hamilton-Clairaut function [4.4] with the substitution (B.1) and use of (B.2). Then the Hamilton-Clairaut system of equations (5.14)–(5.15) coincides with the Hamilton system in the first-order formulation [24]

\[ \dot{q}^A = \{q^A, H_T\}_{full} , \quad \dot{p}_A = \{p_A, H_T\}_{full} , \quad \Phi_\alpha = 0 , \]

(B.4)

and (5.16) gives the second stage equations of the Dirac approach

\[ \{\Phi_\alpha, H_T\}_{full} = \{\Phi_\alpha, H_{phys}\}_{full} + \sum_{\beta=r+1}^{n} \{\Phi_\alpha, \Phi_\beta\}_{full} v^\beta = 0 , \]

(B.5)

where

\[ \{X, Y\}_{full} = \sum_{A=1}^{n} \left( \frac{\partial X}{\partial q^A} \frac{\partial Y}{\partial p_A} - \frac{\partial Y}{\partial q^A} \frac{\partial X}{\partial p_A} \right) \]

(B.6)

is the (full) Poisson bracket on the whole phase space \( T^*M \). Note that

\[ F_{\alpha\beta}(q^A, p_i) = \{\Phi_\alpha, \Phi_\beta\}_{full} , \]

(B.7)

\[ D_\alpha H_{phys}(q^A, p_i) = \{\Phi_\alpha, H_{phys}\}_{full} . \]

(B.8)

Our cases 2) and 1) of Section 7 work as counterparts of the first and the second class constraints in the Dirac classification [15], respectively. The limit case with zero “\( q^\alpha \)-field strength” \( F_{\alpha\beta}(q^A, p_i) = 0 \) (7.22) (see (B.7)) corresponds to the Abelian constraints [28, 37].

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