Lattice-like operations and isotone projection sets*

A. B. Németh  
Faculty of Mathematics and Computer Science  
Babeş Bolyai University, Str. Kogălniceanu nr. 1-3  
RO-400084 Cluj-Napoca, Romania  
email: nemab@math.ubbcluj.ro

S. Z. Németh  
School of Mathematics, The University of Birmingham  
The Watson Building, Edgbaston  
Birmingham B15 2TT, United Kingdom  
email: nemeths@for.mat.bham.ac.uk

Abstract

By using some lattice-like operations which constitute extensions of ones introduced by M. S. Gowda, R. Sznajder and J. Tao for self-dual cones, a new perspective is gained on the subject of isotonicity of the metric projection onto the closed convex sets. The results of this paper are wide range generalizations of some results of the authors obtained for self-dual cones. The aim of the subsequent investigations is to put into evidence some closed convex sets for which the metric projection is isotonic with respect the order relation which give rise to the above mentioned lattice-like operations. The topic is related to variational inequalities where the isotonicity of the metric projection is an important technical tool. For Euclidean sublattices this approach was considered by G. Isac and respectively by H. Nishimura and E. A. Ok.

1. Introduction

The idea of solving operatorial equations via iterative methods based on ordering goes back to the beginnings of the ordered vector space theory (see e. g. [7]). If an operator has some “good properties” with respect to the ordering of the space, then these can be exploited to derive iterative processes for solving equations which involve such an operator.

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The most important role in this context have monotone or isotone operators, i. e., the operators maintaining the order relation. Iterative methods are widely used for solving various types of equilibrium problems (such as variational inequalities, complementarity problems etc.) Recently the isotonicity gained more and more ground for handling such problems (see [13], [1] and the large number of references in [1] related to ordered vector spaces).

The pioneers of this approach for complementarity problems are G. Isac and A. B. Németh. The metric projection onto a closed convex cone is the basis of many investigations about the solvability and/or the approximation of solutions of nonlinear complementarity problems associated with the cone. However, the idea of relating the ordering to the metric projection onto the cone is initiated in the paper [4], where isotone projection cones (i.e., generating pointed closed convex cones admitting an isotone projection onto themselves) are characterised.

The papers [5], [6], [12] consider the problem of solving nonlinear complementarity problems defined on isotone projection cones. The solution methods require repeated projection onto the underlying cone. It turned out later that this procedure is efficient for isotone projection cones [9].

A similar approach can be considered for variational inequalities if the projection onto the closed convex set associated with the variational inequality is monotone with respect to an appropriate order relation. One of the simplest such order relation, the coordinatwise ordering, was considered by G. Isac [3], who proved that the sublatticiality of a closed convex set implies the isotonicity of the projection onto it.

Recently H. Nishimura and E. A. Ok followed the footsteps of G. Isac and showed that the sublatticiality is also necessary for the the projection to be isotone [13]. They used the derived equivalence for several applications concerning variational inequalities defined on closed convex sublattices and other related equilibrium problems. Thus, the question of characterizing the closed convex sublattices with nonempty interior of the coordinatwise ordered Euclidean space which admit an isotone projection onto themselves with respect to this order arises very naturally. A partial answer to this question can be found in the early papers of D. M. Topkis [16] and A. F. Veinott Jr. [17] and a complete one in the paper of M. Queyranne and F. Tardella [14].

The nonnegative orthant of a Cartesian system in the Euclidean space is a self-dual latticial cone which is the positive cone of the coordinatwise ordering and defines “well behaved” lattice operations. Although important and therefore widely investigated, at the same time they are also very restrictive. There are several attempts to extend these lattice operations. One such extension proposed by M. S. Gowda, R. Sznajder and J. Tao [2] and related to self-dual cones is particularly meaningful to us, because, apart from inheriting several properties of the lattice operations, their operations seem to be useful for handling the problem of the isotonicity of the metric projections too.

In our recent paper [11] we have characterized the closed convex sets which are invariant with respect to these operations, showing that the metric projection onto these sets is isotone with respect to the order generated by the self-dual cone giving rise to the respective operations.
In this paper we show similar results for a general cone, by using two kinds of extended lattice operations: ones defined with the aid of the cone, while the others with the aid of its dual. These operations are strongly related: one can be expressed in terms of the other. However, the parallel usage of them offers some technical facilities. Apart from extending the results of [11], we show the equivalence between the sets which are invariant with respect to these operations and the ones which admit an isotone projection onto themselves.

The structure of this paper is as follows. In Section 2 we fix the terminology regarding vectorial ordering. In Section 3 we define our main tools: the so called lattice-like operations, emphasizing the relations with the lattice operations engendered by the nonnegative orthant and the extended lattice operations defined by Gowda Sznajder and Tao as well. In the same section we gather and prove the main properties of these operations. The main results and their proofs are contained in Section 4 namely in Theorems 1 and 2, and Corollary 1. They not only largely extend, but also strengthen the results in [11]. As a consequence a full geometric characterization of the closed convex sets admitting isotone projection with respect to the ordering induced by the Lorentz cone is gained. Besides the Lorentz cone, a special interest is focused on the latticial or simplicial cones in Section 5.

Finally, we end our paper by making some comments and raising some open questions in Section 6.

2. Some terminology

Denote by $\mathbb{R}^m$ the $m$-dimensional Euclidean space endowed with the standard inner product $\langle \cdot, \cdot \rangle : \mathbb{R}^m \times \mathbb{R}^m \rightarrow \mathbb{R}$.

Throughout this note we shall use some standard terms and results from convex geometry (see e.g. [15]).

Let $K$ be a convex cone in $\mathbb{R}^m$, i.e., a nonempty set with (i) $K + K \subset K$ and (ii) $tK \subset K$, $\forall t \in \mathbb{R}_+ = [0, +\infty)$. All cones used in this paper are convex. The convex cone $K$ is called pointed, if $(-K) \cap K = \{0\}$.

The cone $K$ is generating if $K - K = \mathbb{R}^m$.

For any $x, y \in \mathbb{R}^m$, by the equivalence $x \leq_K y \Leftrightarrow y - x \in K$, the convex cone $K$ induces an order relation $\leq_K$ in $\mathbb{R}^m$, that is, a binary relation, which is reflexive and transitive. This order relation is translation invariant in the sense that $x \leq_K y$ implies $x + z \leq_K y + z$ for all $z \in \mathbb{R}^m$, and scale invariant in the sense that $x \leq_K y$ implies $tx \leq_K ty$ for any $t \in \mathbb{R}_+$. If $\leq$ is a translation invariant and scale invariant order relation on $\mathbb{R}^m$, then $\leq = \leq_K$ with $K = \{x \in \mathbb{R}^m : 0 \leq x\}$. The vector space $\mathbb{R}^m$ endowed with the relation $\leq_K$ is denoted by $(\mathbb{R}^m, K)$. If $K$ is pointed, then $\leq_K$ is antisymmetric too, that is $x \leq_K y$ and $y \leq_K x$ imply that $x = y$. The elements $x$ and $y$ are called comparable if $x \leq_K y$ or $y \leq_K x$.

We say that $\leq_K$ is a latticial order if for each pair of elements $x, y \in \mathbb{R}^m$ there exist the least upper bound $\sup\{x, y\}$ (denoted by $x \lor y$) and the greatest lower bound $\inf\{x, y\}$
(denoted by $x \land y$) of the set $\{x, y\}$ with respect to the order relation $\leq_K$. In this case $K$ is said to be a latticial or simplicial cone, and $\mathbb{R}^m$ equipped with a latticial order is called Euclidean vector lattice.

The dual of the convex cone $K$ is the set

$$K^* := \{y \in \mathbb{R}^n : \langle x, y \rangle \geq 0, \forall x \in K\}.$$  

The cone $K$ is called self-dual, if $K = K^*$. If $K$ is self-dual, then it is a generating, pointed, closed cone.

Suppose that $\mathbb{R}^m$ is endowed with a Cartesian system. Let $x, y \in \mathbb{R}^m$, $x = (x^1, \ldots, x^m)$, $y = (y^1, \ldots, y^m)$, where $x^i$, $y^i$ are the coordinates of $x$ and $y$, respectively with respect to the Cartesian system. Then, the scalar product of $x$ and $y$ is the sum $\langle x, y \rangle = \sum_{i=1}^{m} x^i y^i$.

The set $\mathbb{R}^m_+ = \{x = (x^1, \ldots, x^m) \in \mathbb{R}^m : x^i \geq 0, i = 1, \ldots, m\}$ is called the nonnegative orthant of the above introduced Cartesian system. A direct verification shows that $\mathbb{R}^m_+$ is a self-dual cone.

Taking a Cartesian system in $\mathbb{R}^m$ and using the above introduced notations, the coordinatewise order $\leq$ in $\mathbb{R}^m$ is defined by

$$x = (x^1, \ldots, x^m) \leq y = (y^1, \ldots, y^m) \iff x^i \leq y^i, \ i = 1, \ldots, m.$$  

By using the notion of the order relation induced by a cone, defined in the preceding section, it is easy to see that $\leq = \leq_{\mathbb{R}^m}.$  

With the above representation of $x$ and $y$, we define

$$x \land y = (\min\{x^1, y^1\}, \ldots, \min\{x^m, y^m\}), \text{ and } x \lor y = (\max\{x^1, y^1\}, \ldots, \max\{x^m, y^m\}).$$  

Then, $x \land y$ is the greatest lower bound and $x \lor y$ is the least upper bound of the set $\{x, y\}$ with respect to the coordinatwise order. Thus, $\leq$ is a lattice order in $\mathbb{R}^m.$ The operations $\land$ and $\lor$ are called lattice operations.

The subset $M \subset \mathbb{R}^m$ is called a sublattice of the coordinatewise ordered Euclidean space $\mathbb{R}^m$, if from $x, y \in M$ it follows that $x \land y, x \lor y \in M.$

A hyperplane (through the origin), is a set of form

$$H(u, 0) = \{x \in \mathbb{R}^m : \langle u, x \rangle = 0\}, \ u \neq 0.$$  

(1)

For simplicity the hyperplanes will also be denoted by $H$. The nonzero vector $u$ in the above formula is called the normal of the hyperplane.

A hyperplane (through $a \in \mathbb{R}^m$) is a set of form

$$H(u, a) = \{x \in \mathbb{R}^m : \langle u, x \rangle = \langle u, a \rangle\}, \ u \neq 0.$$  

(2)

A hyperplane $H(u, a)$ determines two closed halfspaces $H_-(a, u)$ and $H_+(a, u)$ of $\mathbb{R}^m$, defined by

$$H_-(a, u) = \{x \in \mathbb{R}^m : \langle u, x \rangle \leq \langle u, a \rangle\},$$  

and

$$H_+(a, u) = \{x \in \mathbb{R}^m : \langle u, x \rangle \geq \langle u, a \rangle\}.$$
3. Metric projection and lattice-like operations

Denote by $P_D$ the projection mapping onto a nonempty closed convex set $D \subset \mathbb{R}^m$, that is the mapping which associates to $x \in \mathbb{R}^m$ the unique nearest point of $x$ in $D$:

$$P_Dx \in D, \text{ and } \|x - P_Dx\| = \inf\{\|x - y\| : y \in D\}.$$  

The nearest point $P_Dx$ can be characterized by

$$P_Dx \in D, \text{ and } \langle P_Dx - x, P_Dx - y \rangle \leq 0, \forall y \in D. \quad (3)$$

From the definition of the projection or the characterization $(3)$ there follow immediately the relations:

$$P_{x+D}y = x + P_D(y - x) \quad (4)$$

for any $x, y \in \mathbb{R}^m$,

$$P_D(tx + (1 - t)P_Dx) = P_Dx, \quad (5)$$

for any $x, y \in \mathbb{R}^m$ and any $t \in [0, 1]$.

We shall frequently use in the sequel the following simplified form of the theorem of Moreau [8]:

**Lemma 1** Let $K$ be a closed convex cone in $\mathbb{R}^m$ and $K^*$ its dual. For any $x$ in $\mathbb{R}^m$ we have $x = P_Kx - P_{K^*}(-x)$ and $\langle P_Kx, P_{K^*}(-x) \rangle = 0$. The relation $P_Kx = 0$ holds if and only if $x \in -K^*$.

Define the following operations in $\mathbb{R}^m$:

$$x \sqcap y = P_{x-K}y, \quad x \sqcup y = P_{x+K}y, \quad x \cap_* y = P_{x-K^*}y, \quad \text{and} \quad x \cup_* y = P_{x+K^*}y$$

Assume that the operations $\sqcup, \sqcap, \cap_*$ and $\cup_*$ have precedence over the addition of vectors and multiplication of vectors by scalars.

If $K$ is self-dual, then $\sqcup = \sqcup_*$ and $\sqcap = \cap_*$ and we arrive to the generalized lattice operations defined by Gowda, Sznajder and Tao in [2], and used in our paper [10]. In the particular case of $K = \mathbb{R}^m_+$, one can easily check that $\sqcap = \cap_* = \wedge, \quad \sqcup = \cup_* = \vee$. That is $\sqcap, \sqcup, \cap_*$ and $\cup_*$ are some lattice-like operations.

Although derived from the metric projection, the generalized lattice operations due to Gowda, Sznajder and Tao as well as the above considered lattice-like operations introduce a wieldy formalism, which allows the recognition of new interrelations in the field.

Lemma [1] suggests strong connections between the lattice-like operations. These connections are exhibited by the following lemma:

**Lemma 2** The following relations hold for any $x, y \in \mathbb{R}^m$:
The following relations hold for any \(x, y, z, w \in \mathbb{R}^m\) and any real scalar \(\lambda > 0\).

(i) \(x \sqcap y = x - P_K(x - y) = y - P_{K^*}(y - x),\)
\(x \sqcup y = x + P_K(y - x) = y + P_{K^*}(x - y),\)
\(x \sqcap_\ast y = x - P_{K^*}(x - y) = y - P_K(y - x),\)
\(x \sqcup_\ast y = x + P_K(y - x) = y + P_{K^*}(x - y).\)

(ii) \(x \sqcap_\ast y = y \sqcap x\) and \(x \sqcup_\ast y = y \sqcup x\).

Proof.

(i) From equation (iv) and Lemma 1 we have
\[
x \sqcup y = P_{x+Ky} = x + P_K(y - x) = x + (P_K(y - x) - P_{K^*}(x - y)) + P_{K^*}(x - y)
= x + (y - x) + P_{K^*}(x - y) = y + P_{K^*}(x - y).
\]

The other relations can be shown similarly.

(ii) It follows easily from item (i).

\(\square\)

Denote by \(\leq\) and \(\leq_\ast\) the relations defined by \(K\) and \(K^*\), respectively.

Lemma 3  The following relations hold for any \(x, y, z, w \in \mathbb{R}^m\) and any real scalar \(\lambda > 0\).

(i) \(x \sqcap y \leq x, x \sqcap y \leq_\ast y, x \sqcap_\ast y \leq_\ast x\) and \(x \sqcap_\ast y \leq y\), and equalities hold if and only if
\(x \leq_\ast y, y \leq x, x \leq y\) and \(y \leq_\ast x\), respectively.

(ii) \(x \leq x \sqcup y, y \leq x \sqcup y, x \leq_\ast x \sqcup y\) and \(y \leq x \sqcup_\ast y\), and equalities hold if and only if
\(y \leq_\ast x, x \leq y, x \leq_\ast y\) and \(y \leq_\ast x\), respectively.

(iii) \(x \sqcap y + x \sqcup_\ast y = x \sqcap_\ast y + x \sqcup y = x \sqcap y + y \sqcup x = x \sqcap_\ast y + y \sqcup_\ast x = x + y\).

(iv) \((x+z) \sqcap (y+z) = x \sqcap y + z, (x+z) \sqcup (y+z) = x \sqcup y + z, (x+z) \sqcap_\ast (y+z) = x \sqcap_\ast y + z\)
\((x+z) \sqcup_\ast (y+z) = x \sqcup_\ast y + z\).

(v) \((\lambda x) \sqcap (\lambda y) = \lambda x \sqcap y, (\lambda x) \sqcup (\lambda y) = \lambda x \sqcup y\) and \((\lambda x) \sqcap_\ast (\lambda y) = \lambda x \sqcap_\ast y\).
\((\lambda x) \sqcup_\ast (\lambda y) = \lambda x \sqcup_\ast y\).

(vi) \(\langle x - x \sqcap y, x \sqcup_\ast y - x \rangle = 0\) and \(\langle x - x \sqcap_\ast y, x \sqcup y - x \rangle = 0\).

(vii) \((-x) \sqcup (-y) = -x \sqcap y\) and \((-x) \sqcap_\ast (-y) = -x \sqcap_\ast y\).

(viii) \(\|x \sqcap y - z \sqcap w\| \leq \frac{3}{2}(\|x - z\| + \|y - w\|)\) and \(\|x \sqcup y - z \sqcup w\| \leq \frac{3}{2}(\|x - z\| + \|y - w\|)\).
(ix) \[ x \cap y = z \cap w, \quad \forall \ z = \lambda x + (1 - \lambda) x \cap y, \quad w = \mu y + (1 - \mu) x \cap y, \quad \lambda, \mu \in [0,1], \]
\[ x \sqcup y = z \sqcup w, \quad \forall \ z = \lambda x + (1 - \lambda) x \sqcup y, \quad w = \mu y + (1 - \mu) x \sqcup y, \quad \lambda, \mu \in [0,1]. \]

**Proof.**

(i) It follows from the definitions of the operations, Lemma 2 and Lemma 1.

(ii) It can be shown similarly to item (i).

Items (iii) and (iv) follow immediately from item (i) and (ii) of Lemma 2.

Item (v) follows easily from the positive homogeneity of \( P_K \) and \( P_{K^*} \), and item (i) of Lemma 2.

(vi) By using item (i) of Lemma 2 and Lemma 1, we get
\[ \langle x - x \cap y, x \sqcup_* y - x \rangle = \langle P_K(x - y), P_{K^*}(y - x) \rangle = 0. \]

Item (vii) follows from item (i) of Lemma 2.

To verify item (viii) we use item (i) of Lemma 2 and the Lipschitz property of the metric projection (19) as follows:
\[ \| x \cap y - z \cap w \| = \| x - P_K(x - y) - z + P_K(z - w) \| \leq \| x - z \| + \| P_K(x - y) - P_K(z - w) \| \leq \| x - z \| + \| (x - y) - (z - w) \| \leq 2 \| x - z \| + \| y - w \|, \]
and by symmetry
\[ \| x \cap y - z \cap w \| \leq \| x - z \| + 2 \| y - w \|. \]

By adding the obtained two relations we conclude the first relation in item (viii). The second relation can be deduced similarly.

By using the definition of \( x \cap y \), we have, according to the formula (5), that
\[ x \cap y = P_{x-K}y = P_{x-K}(\mu y + (1 - \mu) x \cap y) = P_{x-K}w = x \cap w = w \cap_* x. \]

Now, according to this formula, the formula \( z = \lambda x + (1 - \lambda) w \cap_* x = \lambda x + (1 - \lambda) P_{w-K^*}x \), and by swapping the roles of \( K \) and \( K^* \) and using a similar argument as above, we obtain
\[ w \cap_* x = P_{w-K^*}x = P_{w-K^*}(\lambda x + (1 - \lambda) P_{w-K^*}x) = P_{w-K^*}z = w \cap_* z = z \cap w. \]

In conclusion,
\[ x \cap y = z \cap w. \]

This is the first formula in item (ix). A similar argument yields the second relation in this item.

\[ \square \]

**Remark 1** Lemma 2 and Lemma 3 show that the operations \( \cap_* \) and \( \sqcup_* \) can be expressed everywhere in what follows by \( \cap \) and \( \sqcup \). However, we shall occasionally use the first ones too in order to emphasize certain assertions and to simplify the arguments.
4. Closed convex sets invariant with respect to the lattice-like operations

The set $M \subset \mathbb{R}^m$ is said to be invariant with respect to the operation $\sqcap$ if from $x, y \in M$ it follows that $x \sqcap y \in M$. The invariance of $M$ with respect to any of the operations $\sqcup, \sqcap_s$, and $\sqcup_s$ can be defined similarly.

The following lemma follows easily from item (ii) of Lemma 2.

Lemma 4 Let $K \subset \mathbb{R}^m$ be a closed convex cone. If $C$ is invariant with respect to one of the operations $\sqcap, \sqcap_s$ and one of the operations $\sqcup, \sqcup_s$, then $C$ is invariant with respect to all operations $\sqcap, \sqcap_s, \sqcup$ and $\sqcup_s$.

Let $K$ be a nonzero closed convex cone. We should simply call a set $M$ which is invariant with respect to the operations $\sqcap, \sqcap_s, \sqcup$ and $\sqcup_s$ $K$-invariant. By Lemma 4 it is enough to suppose that $M$ is invariant with respect to $\sqcap$ and $\sqcup$, or $\sqcap_s$ and $\sqcup$, or $\sqcap$ and $\sqcup$, or $\sqcap_s$ and $\sqcup_s$.

Recall that $\leq$ and $\leq_s$ denote the relations defined by $K$ and $K^*$, respectively.

If $K$ is a nonzero closed convex cone, then the closed convex set $C \subset \mathbb{R}^m$ is called a $K$-isotone ($K^*$-isotone) projection set or simply $K$-isotone ($K^*$-isotone) if $x \leq y$ implies $P_C x \leq P_C y$ (and respectively $x \leq_s y$ implies $P_C x \leq_s P_C y$). In this case we use equivalently the term $P_C$ is $K$-isotone (respectively $P_C$ is $K^*$-isotone).

Theorem 1 Let $K \subset \mathbb{R}^m$ be a closed convex cone. Then, $C$ is $K$-invariant, if and only if $P_C$ is $K$-isotone.

Proof. Assume that the closed convex set $C$ is $K$-invariant. Let $x, y \in \mathbb{R}^m$ with $x \leq y$ and denote $u = P_C x \in C$, $v = P_C y \in C$.

Assume that $u \leq v$ is false. Then, from $u \sqcup v \in C$, the definition of the projection and item (ii) of Lemma 3, we have $\| y - v \| < \| y - u \sqcup v \|$. Hence, from

$$\| y - v \|^2 = \| y - u \sqcup v \|^2 + \| u \sqcup v - v \|^2 + 2 \langle y - u \sqcup v, u \sqcup v - v \rangle,$$

it follows that

$$\| u \sqcup v - v \|^2 < 2 \langle u \sqcup v - y, u \sqcup v - v \rangle.$$

On the other hand, since $u \sqcap v \in C$, we have $\| x - u \| \leq \| x - u \sqcap v \|$, and thus we have similarly that

$$\| u \sqcap v - u \|^2 \leq 2 \langle u \sqcap v - x, u \sqcap v - u \rangle.$$

By summing up the latter two inequalities and using item (iii) of Lemma 3, it follows that

$$\langle u \sqcup v - v, u \sqcup v - v \rangle = \| u \sqcup v - v \|^2 < \langle u \sqcup v - y, u \sqcup v - v \rangle + \langle x - u \sqcap v, u \sqcup v - v \rangle.$$

Thus,

$$\langle y - x - (v - u \sqcap v), u \sqcup v - v \rangle < 0.$$
By combining the latter inequality with item (vi) of Lemma 3, we obtain that
\[ \langle y - x, u \sqcup v - v \rangle < 0. \]

But this is a contradiction, because \( y - x \in K \) and \( u \sqcup v - v \in K^* \) (by item (ii) of Lemma 3).

The obtained contradiction shows that \( P_C \) must be \( K \)-isotone.

Let us see now that if \( P_C \) is \( K \)-isotone, then \( C \) is \( K \)-invariant.

Assume the contrary: \( P_C \) is \( K \)-isotone, but there exist \( x, y \in C \) such that either \( x \sqcap y \notin C \), or \( x \sqcup y \notin C \).

Assume, that \( x \sqcap y \notin C \).

Since \( x \sqcap y \leq x \) and \( P_C \) is \( K \)-isotone, it follows that \( P_C(x \sqcap y) \leq x \), that is, \( P_C(x \sqcap y) \in x - K \). By our working hypothesis \( x \sqcap y \neq P_C(x \sqcap y) \) and by the definition of \( x \sqcap y = P_{x-K}y \), we must have
\[ \|y - x \sqcap y\| < \|y - P_C(x \sqcap y)\|. \]

Since \( y \in C \), by the characterization (3) of the projection we have:
\[
\langle y - P_C(x \sqcap y), x \sqcap y - y \rangle + \|y - P_C(x \sqcap y)\|^2 \\
= \langle y - P_C(x \sqcap y), x \sqcap y - y + y - P_C(x \sqcap y) \rangle \\
= \langle y - P_C(x \sqcap y), x \sqcap y - P_C(x \sqcap y) \rangle \leq 0,
\]
which, by using the Cauchy inequality, implies
\[ \|y - P_C(x \sqcap y)\|^2 \leq \langle y - P_C(x \sqcap y), y - x \sqcap y \rangle \leq \|y - P_C(x \sqcap y)\| \|y - x \sqcap y\|, \] (7)

If \( y = P_C(x \sqcap y) \), then the inequality
\[ \|y - P_C(x \sqcap y)\| \leq \|y - x \sqcap y\|, \] (8)
holds trivially. If \( y \neq P_C(x \sqcap y) \), then (8) follows from dividing (7) by \( \|y - P_C(x \sqcap y)\| \).

However, (8) contradicts (6).

The case of \( x \sqcup y \notin C \) can be handled similarly.

The obtained contradictions show that \( C \) must be \( K \)-invariant. \( \square \)

**Example 1** The set
\[ L_{m+1} = \{(x, x^{m+1}) \in \mathbb{R}^{m+1} : x \in \mathbb{R}^m, x^{m+1} \in \mathbb{R} \text{ and } \|x\| \leq x^{m+1}\}, \]
is a self-dual cone called \( m + 1 \)-dimensional second order cone, or \( m + 1 \)-dimensional Lorentz cone, or \( m + 1 \)-dimensional ice-cream cone (3).

By using Theorem 1, we can strengthen the main result in [1] regarding the Lorentz cone \( L_{m+1} \) as follows:

Let \( M \) be a closed convex subset with nonempty interior in \( \mathbb{R}^{m+1} \) with \( m > 1 \). Then, the following assertions are equivalent:
(i) $M$ is invariant with respect to the operations $\cap$ and $\cup$ defined by the Lorentz cone $L_{m+1}$.

(ii) $M$ is an $L_{m+1}$-isotone projection set.

(iii) 

$$M = C \times \mathbb{R},$$

where $C$ is a closed convex set with nonempty interior in $\mathbb{R}^m$.

In [11] we proved that (iii)$\Leftrightarrow$(ii)$\Rightarrow$(i), but not the implication (i)$\Rightarrow$(ii).

**Lemma 5** Let $K \subset \mathbb{R}^m$ be a closed convex cone. If $M, M_i, \subset \mathbb{R}^m, i \in I$ are $K$-invariant sets, then

(i) $\cap_{i \in I} M_i$ is also $K$-invariant,

(ii) $\eta M + a$ is also $K$-invariant for any $a \in \mathbb{R}^m$ and $\eta \in \mathbb{R}$.

**Proof.** The first assertion is trivial and the second follows easily from items (iv), (v) and (vii) of Lemma 3. ☐

**Lemma 6** The halfspace $H_-$ is $K$-invariant if and only if the hyperplane $H$ has this property.

**Proof.** According to Lemma 5, it is enough to carry out the proof for the case of the invariance with respect to $\cap$ and $\cup$.

According to item (ii) of Lemma 5, we can assume that $0 \in H$.

Suppose that $H$ is invariant, but $H_-$ is not. Then, there exist some $x, y \in H_-$ such that $x \cup y = y \cup x \notin H_-$ or $x \cap y \notin H_-$. Assume that $x \cap y \notin H_-$. Then, $x \cap y \in \text{int} H_-$. The line segment $[x, x \cap y]$ meets $H$ in $z = \lambda x + (1 - \lambda) x \cap y$, $\lambda \in (0, 1]$, while the line segment $[y, x \cap y]$ meets $H$ at $w = \mu y + (1 - \mu) x \cap y$, $\mu \in (0, 1]$. According to item (ix) in Lemma 3, we have then

$$z \cap w = x \cap y \notin H,$$

which contradicts the invariance of $H$.

Suppose now that $H_-$ is invariant, but $H$ is not. Then, there exist some $x, y \in H$ such that $x \cup y = y \cup x \notin H$ or $x \cap y \notin H$. Since $H_-$ is invariant, we can assume that $x \cup y \in \text{int} H_-$. Let $u$ be the normal of $H$. Then, $\langle u, x \cup y \rangle < 0$. By using the relation in item (iii) of Lemma 3, we have then

$$0 = \langle u, x + y \rangle = \langle u, x \cup y \rangle + \langle u, y \cap x \rangle.$$

Whereby, by using the relation $\langle u, x \cup y \rangle < 0$, we conclude that

$$\langle u, y \cap x \rangle > 0,$$

that is, $y \cap x \in \text{int} H_+$, contradicting the invariance of $H_-$. Similarly $x \cap y \notin H$ leads to a contradiction. ☐
Lemma 7 Suppose that $C$ is a $K$-invariant closed convex set with nonempty interior, and $H$ is a hyperplane tangent to $C$ in some point of $\text{bdr} C$. Then, $H$ is $K$-invariant.

Proof. According to item (ii) of Lemma 5, we can assume that $0 \in \text{bdr} C$, that $H$ is tangent to $C$ at $0$, and that $C \subset H$.

We shall prove our claim by contradiction: we assume that $H$ is not invariant.

Since $H$ is not invariant, there exist some $z, w \in H$ such that $z \sqcap w$ or $w \sqcup z$ is not in $H$. Suppose that $u$ is the normal of $H$. From the relation in item (iii) of Lemma 3 we have then

$$0 = \langle u, z + w \rangle = \langle u, z \sqcup w \rangle + \langle u, w \sqcap z \rangle,$$

whereby it follows that $z \sqcup w$ and $w \sqcap z$ are in opposite open half-spaces determined by $H$.

Suppose that $w \sqcap z \in \text{int} H_+$. By taking $x = z - (z + w)/2$, we have $-x = w - (z + w)/2$. Then, by our working hypothesis that $0 \in H$, it follows that the line segment $[-x, x] \subset H$.

Denote by $B$ the unit ball in $\mathbb{R}^m$, then there exists some $\delta > 0$ such that $(-x) \cap x + \delta B \subset \text{int} H_+$. (9)

Next we project $[-x, x]$ in the direction of $u$ onto $\text{bdr} C$. All the above reasonings are valid when we change $x$ with its positive multiple, hence we can chose $x$ small enough, so that the above projection to make a sense.

Denote by $\gamma(t)$ the image of $tx$ in $\text{bdr} C$ by this projection. Since $H$ is a tangent hyperplane, the segment $[-x, x]$ will be tangent to $\gamma$ at $t = 0$, $\gamma(0) = 0$, $\gamma'(0)$ exists, and $\gamma'(0) = x$.

Since $\gamma$ is differentiable in $t = 0$, we have the following representations around 0:

$$\gamma(t) = tx + \eta(t), \ t > 0,$$
and

$$\gamma(-t) = -tx + \zeta(-t), \ t > 0,$$

where

$$\frac{\eta(t)}{t} \to 0 \text{ and } \frac{\zeta(-t)}{t} \to 0, \text{ as } t \to 0, \ t > 0.$$

Using item (viii) of Lemma 3 as well as the relations (10) and (11), we have then

$$\|(-tx) \cap (tx) - \gamma(-t) \cap \gamma(t)\| \leq \frac{3}{2}(\| -tx - \gamma(-t)\| + \|tx - \gamma(t)\|) = \frac{3}{2}(\|\zeta(-t)\| + \|\eta(t)\|).$$

Dividing the last relation by $t > 0$, and using the relation in item (v) of Lemma 3, we obtain that

$$\|(-x) \cap x - \frac{1}{t}\gamma(-t) \cap \gamma(t)\| \leq \frac{3}{2} \left(\left\|\frac{\zeta(-t)}{t}\right\| + \left\|\frac{\eta(t)}{t}\right\|\right).$$

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Take now $t > 0$ small enough in order to have by (12)

$$\frac{3}{2} \left( \left\| \frac{\langle -t \rangle}{t} \right\| + \left\| \frac{\eta(t)}{t} \right\| \right) < \delta.$$ 

For such a $t > 0$ we have, by using (13), that

$$\frac{1}{t}(\gamma(-t) \cap \gamma(t)) \in \text{int} H_+,$$

and thus

$$\gamma(-t) \cap \gamma(t) \in \text{int} H_+,$$

that is, $\gamma(-t), \gamma(t) \in C$, but

$$\gamma(-t) \cap \gamma(t) \notin C,$$

contradicting the invariance of $C$.

The obtained contradiction shows that $H$ must be invariant with respect to the operations $\sqcup$ and $\sqcap$, that is, by Lemma 4, $H$ must be $K$-invariant.

\[\blacksquare\]

**Theorem 2** The closed convex set $C \subset \mathbb{R}^m$ with nonempty interior is $K$-invariant if and only if it is of the form

$$C = \cap_{i \in \mathbb{N}} H_-(u_i, a_i),$$

where each hyperplane $H(u_i, a_i)$ is tangent to $C$ and is $K$-invariant.

**Proof.** It is known (see e.g. [15], Theorem 25.5) that if $C \subset \mathbb{R}^m$ is a closed convex set with nonempty interior, then $\text{bdr} C$ contains a dense subset of points where this surface is differentiable. Since the topology of $\text{bdr} C$ possesses a countable basis, we can select from this dense set a countable dense set $\{a_i : i \in \mathbb{N}\} \subset \text{bdr} C$ such that there exist the tangent hyperplanes $H(u_i, a_i)$ to $C$ and $C \subset H_-(u_i, a_i), i \in \mathbb{N}$. Since the set $\{a_i, i \in \mathbb{N}\}$ is dense in $\text{bdr} C$, a standard convex geometric reasoning shows that in fact

$$C = \cap_{i \in \mathbb{N}} H_-(u_i, a_i).$$

(14)

Now, if $C$ is $K$-invariant, then so is $H(u_i, a_i), i \in \mathbb{N}$ by Lemma 7. Hence, the necessity of the condition in the theorem is proved.

Conversely, if we have the representation (14) with the hyperplanes $H(u_i, a_i), i \in \mathbb{N}$ $K$-invariant, then, by Lemma 6, the halfspaces $H_-(u_i, a_i), i \in \mathbb{N}$ are also $K$-invariant. Then, by using item (i) of Lemma 5 and the representation (14), we see that $C$ is $K$-invariant, and the sufficiency of the theorem is proved.

\[\blacksquare\]

By gathering and comparing the results of Theorem 1 and Theorem 2, we obtain the following corollary:
Corollary 1  Let $C$ be a closed convex set with nonempty interior. Then, the following assertions are equivalent:

(i) $C$ is a $K$-invariant set, i.e., it is invariant with respect to the operations $\cap$, $\cup$, $\cap^*$, and $\cup^*$;

(ii) The projection $P_C$ is $K$-isotone and $K^*$-isotone;

(iii) The set $C$ can be represented by

$$C = \cap_{i \in \mathbb{N}} H_-(u_i, a_i),$$  \hfill (15)

where each hyperplane $H(u_i, a_i), i \in \mathbb{N}$ is tangent to $C$ and is $K$-invariant;

(iv) $C$ can be represented by (15), where each hyperplane $H(u_i, a_i), i \in \mathbb{N}$ is tangent to $C$ and is a $K$-isotone projection and a $K^*$-isotone projection set.

Proof.

From Theorem 2 we have the equivalence

(i) $\iff$ (iii).

From Theorem 1 we have the equivalence

(i) $\iff$ (ii).

We have by Theorem 1 that a hyperplane $H(u_i, a_i)(i \in \mathbb{N})$ is $K$-isotone if and only if it is invariant. Hence, we have the equivalence

(iii) $\iff$ (iv). \hfill $\square$

5. The case of the simplicial cone

If the closed convex cone $K \subset \mathbb{R}^m$ induces a latticial ordering $\leq$ in $\mathbb{R}^m$, then it is called simplicial. The origin of this term relies in Youdine’s theorem [18], which says that in this case

$$K = \text{cone}\{e_1, ..., e_m\} = \{t_1 e_1 + ... + t_m e_m, t_i \in \mathbb{R}^+, i = 1, ..., m\},$$  \hfill (16)

where the vectors $e_1, ..., e_m$ form a basis of $\mathbb{R}^m$.

If $K$ is a simplicial cone, the set $M \subset \mathbb{R}^m$ is called a sublattice of $(\mathbb{R}^m, K)$ if from $x, y \in M$ it follows that $x \lor y, x \land y \in M$.

The cone $K \subset \mathbb{R}^m$ is called subdual if $K \subset K^*$.

The following lemma does not use the representation (16) of a simplicial cone $K$, but only the latticiality of $\leq$. Again we put $\leq$ for $\leq_K$.  

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Lemma 8  If the closed convex set $C \subset \mathbb{R}^m$ admits a $K$-isotone projection $P_C$ with respect to a subdual simplicial cone $K \subset \mathbb{R}^m$, then $C$ is a sublattice of the lattice $(\mathbb{R}^m, K)$.

Proof. Suppose that $P_C$ is $K$-isotone and take $x, y \in C$. Let us see that $x \lor y \in C$.

From the characterization (3) of the projection we have

$$
\langle P_C(x \lor y) - x \lor y, P_C(x \lor y) - y \rangle \leq 0.
$$

(17)

Since $x \leq x \lor y$ and $P_C$ is $K$-isotone, it follows that $x = P_Cx \leq P_C(x \lor y)$. Similarly, $y \leq P_C(x \lor y)$ and hence $x \lor y \leq P_C(x \lor y)$. We have also

$$
0 \leq P_C(x \lor y) - x \lor y \leq P_C(x \lor y) - y.
$$

(18)

Thus, the two terms in the scalar product (17) are in $K$ and since $K$ is subdual, we must have the equality:

$$
\langle P_C(x \lor y) - x \lor y, P_C(x \lor y) - y \rangle = 0.
$$

(19)

By using again the subduality of $K$, the relation (18), as well as (19), it follows that

$$
0 \leq \langle P_C(x \lor y) - x \lor y, (P_C(x \lor y) - y) - (P_C(x \lor y) - x \lor y) \rangle = -\|P_C(x \lor y) - x \lor y\|^2,
$$

thus we must have

$$
P_C(x \lor y) = x \lor y,
$$

and since $C$ is closed, $x \lor y \in C$.

Similar reasonings show that $x \land y \in C$.

$\Box$

The next corollary follows from Theorem 1 and Lemma 8.

Corollary 2  Let $K \subset \mathbb{R}^m$ be a subdual simplicial cone. If the closed convex set $C \subset \mathbb{R}^m$ is $K$-invariant, then $C$ is a sublattice of the lattice $(\mathbb{R}^m, K)$.

Remark 2  Since the $K$-invariance of a set is equivalent to its $K^*$-invariance, by replacing $K$ with $K^*$, similar results to Lemma 8 and Corollary 2 hold for superdual cones too.

It is well known (see e. g. [9]) that if the simplicial cone $K$ is represented by (16), then we have for its dual the representation

$$
K^* = \text{cone}\{u_1, \ldots, u_m\} = \{t_1u_1 + \ldots + t_mu_m ; t_i \in \mathbb{R}_+, i = 1, \ldots, m\},
$$

(20)

where the vectors $u_1, \ldots, u_m$ are obtained from the relations

$$
\langle e_i, u_j \rangle = \delta_{ij}, \text{ for any } i, j \in \{1, \ldots, m\},
$$

(21)

where $\delta_{ij}$ is the Kronecker delta.
Lemma 9 Let \( K \subset \mathbb{R}^m \) be a simplicial cone given by (10) such that its dual \( K^* \) is of form (20) with the vectors \( u_1, \ldots, u_m \) satisfying (21). The hyperplane \( H \) through 0 with the unit normal vector \( a \) is \( K \)-invariant if and only if
\[
\langle a, e_i \rangle \langle a, u_j \rangle \leq \delta_{ij},
\] (22)
for any \( i, j \in \{1, \ldots, m\} \). If \( a = \alpha^1 e_1 + \cdots + \alpha^m e_m = \beta^1 u_1 + \cdots + \beta^m u_m \), then \( \alpha^i = \langle a, u_i \rangle \) and \( \beta^j = \langle a, e_j \rangle \), and hence the system of inequalities (22) is equivalent to
\[
\alpha^i \beta^j \leq \delta_{ij},
\] (23)
Proof. By Theorem 1 it is enough to prove that \( P_H \) is isotone if and only if the conditions of the lemma hold.

Since \( P_H \) is linear, in order to characterize the hyperplane \( H \) with the property that \( x \leq y \) implies \( P_H x \leq P_H y \), it is sufficient to give necessary and sufficient conditions on the unit vector \( a \) such that
\[
0 \leq P_H e_i, \quad i = 1, \ldots, m.
\] (24)
Since \( a \) is a unit vector, the conditions (24) can be written in the form:
\[
0 \leq P_H e_i = e_i - \langle a, e_i \rangle a \quad i = 1, \ldots, m.
\] (25)
These conditions are equivalent to
\[
\langle e_i - \langle a, e_i \rangle a, u_j \rangle \geq 0,
\] (26)
for all \( i, j \in \{1, \ldots, m\} \), which is equivalent to (22). The relations \( \alpha^i = \langle a, u_i \rangle \) and \( \beta^j = \langle a, e_j \rangle \) follow easily from formulas (21).

Corollary 3 Let \( e_1, \ldots, e_m \) be an orthonormal system of vectors in \( \mathbb{R}^m \) and consider the system engendered by it in \( \mathbb{R}^m \). Then, the hyperplane \( H \) through 0 with the unit normal \( a = (a^1, \ldots, a^m) \) is \( K = \mathbb{R}_{m}^+ \)-invariant if and only if and only if
\[
a^i a^j \leq 0 \text{ whenever } i \neq j.
\] (27)
Proof. Using the notation in Lemma 9 we have in this case \( u_i = e_i, \quad i = 1, \ldots, m \), and \( \alpha^i = \beta^j = a^i, \quad i = 1, \ldots, m \). From the condition \( \|a\| = 1 \) we have that \( |a^i| \leq 1, \quad i = 1, \ldots, m \) and hence the conditions which corresponds to (22) for \( i = j \), that is, \( a^i a^j \leq 1, \quad i = 1, \ldots, m \) are automatically satisfied. The remaining conditions are exactly those in (27).

This corollary is in fact nothing else as Lemma 14 in [11].
Corollary 4 Let $K \subset \mathbb{R}^m$ be a simplicial cone given by (10) such that its dual $K^*$ is of form (20) with the vectors $u_1, \ldots, u_m$ satisfying (24). The existence of a $K$-invariant hyperplane $H$ through 0 with unit normal vector $a$ is equivalent to one of the following situations:

(i) The vector $a$ belongs to cone$\{e_p, -e_q\} \cap$ cone$\{u_p, -u_q\}$, for some $p, q \in \{1, \ldots, m\}$, $p \neq q$.

(ii) The inequality $\langle e_p, e_i \rangle \leq 0$ holds for some $p \in \{1, \ldots, m\}$ and any $i \in \{1, \ldots, m\}$ with $i \neq p$, and $a = \pm e_p/\|e_p\|$.

(iii) The inequality $\langle u_p, u_j \rangle \leq 0$ holds for some $p \in \{1, \ldots, m\}$ and any $j \in \{1, \ldots, m\}$ with $j \neq p$, and $a = \pm u_p/\|u_p\|$.

Proof.

Let us see first that the conditions are sufficient.

By Lemma 9, $H$ is a $K$-invariant hyperplane through 0 with unit normal vector $a$ if and only if

$$\alpha^i \beta^j \leq \delta_{ij},$$

(28)

for any $i, j \in \{1, \ldots, m\}$, where

$$a = \alpha^1 e_1 + \cdots + \alpha^m e_m = \beta^1 u_1 + \cdots + \beta^m u_m.$$  

(29)

Assume that item (i) holds. Then, $a = \alpha^p e_p + \alpha^q e_q = \beta^p u_p + \beta^q u_q$, where $\alpha^p \geq 0$, $\alpha^q \leq 0$, $\beta^p \geq 0$ and $\beta^q \leq 0$. Hence, we have $\alpha^p \beta^q - \alpha^q \beta^p \leq 0$. On the other hand, $1 = \|a\|^2 = \langle \alpha^p e_p + \alpha^q e_q, \beta^p u_p + \beta^q u_q \rangle = \alpha^p \beta^p + \alpha^q \beta^q$, $\alpha^p \beta^p \geq 0$ and $\alpha^q \beta^q \geq 0$ imply $\alpha^p \beta^p \leq 1$ and $\alpha^q \beta^q \leq 1$. Hence, conditions (28) are satisfied.

Assume that item (ii) holds. Then, $a = \pm 1/\|e_p\|$ and all other $\alpha^i$s are 0. We also have $\beta^i = \langle a, e_i \rangle = \pm (1/\|e_p\|) \langle e_p, e_i \rangle$. Thus, $\alpha^i \beta^j = 0 \leq \delta_{ij}$ if $i \neq p$ and $\alpha^p \beta^j = (1/\|e_p\|)^2 \langle e_p, e_j \rangle \leq \delta_{jp}$. Hence, conditions (28) are satisfied.

Assume that item (iii) holds. An argument similar to the one in the previous paragraph shows that conditions (28) are satisfied.

To see that the conditions are necessary, first suppose that more than two $\beta^j$s are nonzero. Then, there exists two $\beta^j$s which have the same sign. Without loss of generality we can suppose that they are both positive. Then, from inequalities (28), it follows that all $\alpha^i$s are nonpositive. Thus, $a \in -K$. If all $\beta^j$s are nonnegative, then $a \in K^*$. Hence, $a \in K^* \cap (-K) = \{0\}$, which is a contradiction. Therefore, there exists a $p \in \{1, \ldots, m\}$ such that $\beta^p < 0$. But then, by inequalities (28), all $\alpha^i$s with $i \neq p$ are nonnegative. Hence, all $\alpha^i$s, with $i \neq p$ are zero. Hence, from equation (29) we get $a = \alpha^p e_p$ and therefore $a = \pm e_p/\|e_p\|$. The same equation implies

$$\alpha^p e_p = \beta^1 u_1 + \cdots + \beta^m u_m.$$  

(30)
Take any $i \in \{1, \ldots, m\}$ with $i \neq p$. Since $a = \alpha^p e_p$ and $\alpha^k \leq 0$ for all $k \in \{1, \ldots, m\}$, it follows that $\alpha^p < 0$. Hence, inequality (28) implies $\beta^i \geq 0$. Thus, by multiplying equation (30) scalarly by $e_i$ and by using relations (21), we obtain
\[
\alpha^p \langle e_p, e_i \rangle = \beta^i \geq 0.
\] (31)

Now, if we divide (31) by $\alpha^p < 0$, then we obtain $\langle e_p, e_i \rangle \leq 0$. The latter inequality together with $a = \pm e_p/\|e_p\|$ implies that item (ii) holds.

Next, suppose that at most two $\beta^i$'s are nonzero.

Case 1. Suppose that there exists $p, q \in \{1, \ldots, m\}$ with $p \neq q$ such that $\beta^p \beta^q > 0$. Then, without loss of generality we can suppose that $\beta^p > 0$ and $\beta^q > 0$. Since all $\beta^i$'s are nonnegative and there exists two $\beta^i$'s which are positive, as above, we can show that this would imply that $a = 0$ which is a contradiction. Thus, this case cannot hold.

Case 2. Suppose that there exists $p, q \in \{1, \ldots, m\}$ with $p \neq q$ such that $\beta^p > 0$ and $\beta^q < 0$. Then, by using inequalities (28), we must have $\alpha^i = 0$, for all $i \in \{1, \ldots, m\} \setminus \{p, q\}$. By using again inequalities (28), we obtain $\alpha^p \geq 0$ and $\alpha^q \leq 0$. It follows that $a = \alpha^p e_p + \alpha^q e_q = \beta^p u_p + \beta^q u_q$. Thus, cone$\{e_p, -e_q\} \cap$ cone$\{u_p, -u_q\} \neq \{0\}$ and $a \in$ cone$\{e_p, -e_q\} \cap$ cone$\{u_p, -u_q\}$. So, in this case item (i) holds.

Case 3. Suppose that only one of the $\beta^i$ is nonzero, that is $a = \beta^p u_p$. Since $\|a\| = 1$, it follows that $a = \pm u_p/\|u_p\|$. An argument similar to the one following relation (30) shows that $\langle u_p, u_j \rangle \leq 0$, for any $j \in \{1, \ldots, m\}$, $j \neq p$. So, in this case item (iii) holds.

\[\square\]

Now, combining the results in Lemma 5, Corollary 1 and Lemma 9, we have the following result:

**Corollary 5** Let $K \subset \mathbb{R}^m$ be a simplicial cone given by (10) such that its dual $K^*$ is of form (21) with the vectors $u_1, \ldots, u_m$ satisfying (27). Let $C$ be a closed convex set with nonempty interior. Then, the following assertions are equivalent:

(i) $C$ is a $K$-invariant set, i. e., it is invariant with respect to the operations $\cap$, $\cup$, $\cap_*$, and $\cup_*$;

(ii) The projection $P_C$ is $K$-isotone and $K^*$-isotone;

(iii) The set $C$ can be represented by
\[
C = \cap_{i \in \mathbb{N}} H_-(a_i, b_i),
\] (32)

where each hyperplane $H(a_i, b_i)$, $i \in \mathbb{N}$ is tangent to $C$ and is $K$-invariant and $a_i$ are unit normals with
\[
a_i = \alpha^1_i e_1 + \cdots + \alpha^m_i e_m = \beta^1_i u_1 + \cdots + \beta^m_i u_m, \quad \alpha^k_i \beta^l_i \leq \delta_{kl}, \quad i \in \mathbb{N}.
\] (33)
Corollary 6 takes the form:

In the particular case of $K = \mathbb{R}_+^m$, where $\mathbb{R}_+^m$ is the positive orthant of a Cartesian system, taking into account that $\cap = \cap_+ = \land$, $\cup = \cup_+ = \lor$, the Corollary 5 (via Corollary 3 in place of Corollary 4) takes the form:

**Corollary 6** Let $C$ be a closed convex set with nonempty interior of the coordinatewise ordered Euclidean space $\mathbb{R}^m$. Then, the following assertions are equivalent:

(i) The set $C$ is a sublattice;

(ii) The projection $P_C$ is isotone;

(iii) $C = \cap_{i \in \mathbb{N}} H_-(a_i, b_i)$,

where each hyperplane $H(a_i, b_i)$ is tangent to $C$ and the normals $a_i$ are nonzero vectors $a_i = (a_{i1}, \ldots, a_{im})$ with the properties $a_i^k a_i^l \leq 0$ whenever $k \neq l$, $i \in \mathbb{N}$.

This corollary is exactly Corollary 4 in [11].

**Example 2**

1. Let $\oplus$ denote orthogonal direct sum of subspaces, $m = 2k$ and $\mathbb{R}^m = V_1 \oplus \cdots \oplus V_k$, where $V_1, \ldots, V_k$ are pairwise orthogonal two-dimensional subspaces of $\mathbb{R}^m$. Let $\{e_{2i-1}, e_{2i}\}$ be a basis of the subspace $V_i$, for any $i \in \{1, \ldots, k\}$. Then, $K := \text{cone}\{e_1, \ldots, e_m\}$ is a simplicial cone and let $K^* = \{u_1, \ldots, u_m\}$ its dual cone. By using the biorthogonality of the vectors $e_i$ and $u_j$, $i, j \in \{1, \ldots, m\}$, we obtain that $\text{cone}\{e_{2i-1}, -e_{2i}\} \cap \text{cone}\{u_{2i-1}, -u_{2i}\} \neq \{0\}$ if $\langle e_{2i-1}, e_{2i} \rangle \geq 0$ and $\text{cone}\{e_{2i-1}, -e_{2i}\} \cap \text{cone}\{u_{2i-1}, -u_{2i}\} = \text{cone}\{e_{2i-1}, -e_{2i}\} = \{0\}$ if $\langle e_{2i-1}, e_{2i} \rangle \leq 0$. Hence, by using item (i) of Corollary 4, any hyperplane $H$ through 0 with normal unit vector in $\text{cone}\{e_{2i-1}, -e_{2i}\} \cap \text{cone}\{u_{2i-1}, -u_{2i}\} \neq \{0\}$ is $K$-invariant.

2. Let $K$ be an isotone projection cone. Then, by item (ii) of Corollary 4, the $n-1$ dimensional hyperfaces of $K$ are $K$-invariant.

**6. Comments and open questions**

Motivated by isotone type iterative methods for variational inequalities in this paper we considered the following very general question: Which are the closed convex sets which possess a projection onto them which is isotone with respect to an order relation defined by a given cone? We showed that they are exactly the sets which are invariant with respect to some extended lattice operations defined by the projection onto the cone and presented
some examples for self-dual and simplicial cones. A related at least as interesting and still open question is: Which are the closed convex sets for which there exist a cone such that the projection onto them are isotone with respect to order relation defined by the cone? Both of the above questions can also be formulated for a general pair of cones and their corresponding orderings. These extended questions are also open.

We remark that several of our results do not use explicitly that the ambient space is Euclidean, and they hold in Hilbert spaces too.

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