LAGRANGIAN THEORY OF GRAVITATIONAL INSTABILITY OF FRIEDMAN-LEMAITRE COSMOLOGIES - GENERIC THIRD ORDER MODEL FOR NON-LINEAR CLUSTERING

by

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Summary: The Lagrangian perturbation theory on Friedman-Lemaître cosmologies investigated and solved up to the second order in earlier papers (Buchert 1992, Buchert & Ehlers 1993) is evaluated up to the third order. On its basis a model for non-linear clustering applicable to the modeling of large-scale structure in the Universe for generic initial conditions is formulated. A truncated model is proposed which represents the “main body” of the perturbation sequence in the early non-linear regime by neglecting all gravitational sources which describe interaction of the perturbations. However, I also give the irrotational solutions generated by the interaction terms to the third order, which induce vorticity in Lagrangian space. The consequences and applicability of the solutions are put into perspective. In particular, the model presented enables the study of previrialization effects in gravitational clustering and the onset of non-dissipative gravitational turbulence within the cluster environment.
1. Introduction

For a long period of research in cosmology inhomogeneities in the Universe have been modeled on the basis of a perturbative approach exploiting the instability of the standard cosmologies of Friedman-Lemaître type against perturbations of the density and the velocity field. This approach is *Eulerian*, i.e., the perturbations are evaluated as a function of Eulerian coordinates (see, e.g., Peebles 1980). The limitations of this approach have been widely recognized, since it relies on the smallness of physical densities which is inappropriate for the modeling of the high density excesses observed in the Universe. Zel’dovich (1970, 1973) has realized this situation and has proposed an approximate extrapolation of the linear perturbation solution into the non-linear regime.

In recent papers the *Lagrangian* theory of gravitational instability of cosmologies of Friedman-Lemaître type for pressureless (“dust”) matter has been investigated and solved up to the second order in the deviations from homogeneity (Buchert 1992, Buchert & Ehlers 1993, henceforth abbreviated by B92 and BE93). This theory does not rely on the smallness of the density of the inhomogeneities, only the deviations of the particle trajectories from the homogeneous Hubble flow are treated perturbatively. This is possible, since the field of trajectories $\vec{f}(\vec{X},t)$ is the only dynamical variable in the Lagrangian picture. Interestingly, the widely applied “Zel’dovich approximation” for modeling the formation of large-scale structure in the Universe was found to be contained in a subclass of first order irrotational perturbation solutions in this theory (B92). The first order solutions have been analyzed in an earlier paper (Buchert 1989) in which they are shown to provide exact three-dimensional solutions in the case of *locally* one-dimensional motion, i.e., when two eigenvalues of the peculiar-velocity gradient vanish along trajectories of fluid elements. The general plane-symmetric case (*globally* one-dimensional motion) constitutes a subclass of these solutions.

It should be emphasized that solutions of the Euler-Poisson system, exact or in a perturbative sense, will depend *non-locally* on the initial conditions for the inhomogeneities. In general, they have to be constructed by solving elliptic boundary value problems, as will be discussed in full detail. Despite this, Zel’dovich’s approximation is local in this sense. It can be made rigorous that the first order perturbation solutions can be made local without loss of generality (B92). This will be recalled in APPENDIX B. The problem of uniqueness in
general perturbation solutions will be considered in a separate paper (Ehlers & Buchert 1993).

The success of Zel’dovich’s model as an approximation for irrotational self-gravitating “dust” flows in the early non-linear regime is commonly appreciated. Its range of applicability can be roughly limited to the epoch shortly after the first shell-crossing singularities in the flow develop, and provides an excellent approximation for the density field down to the non-linearity scale (i.e., where the r.m.s. deviations of the density from homogeneity exceed unity) in comparison with numerical N-body simulations (Coles et al. 1993). In contrast to other analytical models for the formation of large-scale structure, the Lagrangian perturbation theory offers a systematic and explicit way to extend the range of validity of Zel’dovich’s model for generic initial conditions (Buchert 1993). While the first order approximation chiefly covers the (up to the epoch of shell-crossing dominating) kinematical aspects of the structure formation process, the second order approximation firstly involves the tidal action of the gravitational field. The collapse process is significantly accelerated by this action. Also, tidal forces constitute the essence of so-called previrialization effects in gravitational clustering (Peebles 1990, BE93).

Zel’dovich’s model has shortcomings after the epoch when shell-crossing singularities develop, since shells of matter are predicted to cross freely, and the kinetic energy of particles at the caustic exceeds the gravitational potential energy. As a result no potential well is formed in contradiction with numerical simulations. The advantage of high-spatial resolution N-body simulations is obvious especially with respect to the modeling of multi-stream systems which develop after shell-crossing. Analytically, this modeling is not straightforward, since the particles are not treated as interacting bodies. Rather, the particles are viewed as tracers of a flow, which generally looses uniqueness as shells of matter overlap. However, while the Eulerian representation of any solution breaks down as soon as singularities in the density field (caustics) form, the Lagrangian representation allows to follow the trajectories $\vec{f}$ of fluid elements across caustics. The solution $\vec{f}$ remains regular, only the transformation from Eulerian to Lagrangian space is singular. Non-uniqueness of the flow is only realized in Eulerian space, where a patch of matter can originate from different Lagrangian ‘particles’, we say that the flow consists of several streams in such a region. Therefore, analytical Lagrangian approximations are in principle capable to model fully developed
non-linear situations including multiple shell-crossings.

The phenomenology of a self-gravitating multi-stream system is complex, the number of streams is systematically increasing in time. Numerical simulations demonstrate that a hierarchy of singularities forms (e.g. Doroshkevich et al. 1980), shells do not cross freely without limit as e.g. predicted by Zel’dovich’s model, rather the gravitational action of the firstly formed three-stream system (called ‘pancakes’ in the cosmological context) forces the inner fluid elements to recollapse and participate in a second shock structure and so on. This way a five-, seven-, nine-stream system etc. develops. A potential well is established leading to a self-trapped quasi-stationary structure around the firstly developed singularities. The morphogenesis of such a non-dissipative structure (e.g. a cluster of galaxies in the present context) is similar to that of a flower: the singular density peaks split up into shock structures which subsequently emerge from and move away from the center. Apparently, a cluster is composed of expanding shells, while permanently new shells are created in the center. A nested moving system of singularities is formed on smaller and smaller spatial scales. In real terms the situation is more complicated: in addition higher-order singularities (such as the ‘breaking’ of a pancake boundary) increase the number of streams as was originally discussed for Zel’dovich-Arnol’d pancakes by virtue of analyzing a numerical simulation in two spatial dimensions (Arnol’d et al. 1982). The resolution of higher-order singularities can also be appreciated in high-resolution plots of analytical mappings such as the “Zel’dovich approximation” (in 2D: Buchert 1989, in 3D: Buchert & Bartelmann 1991).

Analytical models have also been used to analyze self-gravitating multi-stream systems: Approximating a generic density peak locally by a triaxial ellipsoid, a model for the evolution of this peak has been formulated by Gurevich and Zybin (1988a,b) by performing a transformation of the known spherically symmetric solution. They show that the density distribution around the collapsed peak can be represented by a scaling law with a power of \( \approx -1.8 \) in this model. This law is found to hold on a wide range of spatial scales from the scale of clusters down to the scale of typical star distances. Also Moutarde et al. (1991) found a similar scaling behavior of the density field in the early non-linear regime of the collapse. The correspondence to the scaling behavior of observed galaxy concentrations as measured by the two-point correlation function is striking and suggests that multi-stream hierarchies in the density field of dark matter might be responsible
for the clustering properties of galaxies (Gurevich & Zybin 1988a, Berezinsky et al. 1992). However, this conclusion depends on the assumption that the dark matter dominates the matter density down to these scales (Gurevich & Zybin 1990). It is likely that the fraction of baryonic matter depends on scale and position which implies that the influence of hydrodynamic or radiative effects of the baryonic component might alter this picture. Also, it is not shown that a generic collapse occurs at the peaks of the initial density field. It is likely that, in general, a collapse occurs closer to the trajectory of the maximum of the largest eigenvalue function of the initial displacement tensor (Shandarin, priv. comm.). Gurevich and Zybin call this hierarchical structure “non-dissipative turbulence”. Its existence is related to the early conjecture by Mandelbrot (1976) that singularities in self-gravitating flows might form a fractal, although they rather display a multifractal scaling (Martínez 1991, Jones 1993). Indeed, a self-similarity of caustic patterns can be appreciated (see BE93) in accord with the existence of self-similar solutions (Filmore & Goldreich 1984, Bertshinger 1985). Moreover, in developed gravitational turbulence the density scaling law seems to be generic, i.e., it seems not to depend on the initial fluctuation spectrum according to Gurevich and Zybin’s model. However, it appears, at least with the amount of non-linearity one can resolve, that the impact of large-scale power will destroy the convergence to this local power law form, thus retaining the memory from the initial conditions, (compare the final power spectra for different initial conditions in Melott & Shandarin 1990).

The value of Lagrangian perturbation solutions has to be tested against the phenomenology of “non-dissipative turbulence”. Indeed it has been demonstrated that the second order solutions do describe the onset of such a hierarchy. They predict a second shell-crossing singularity within Zel’dovich-Arnol’d pancakes. A second bifurcation branch appears on the critical manifold of the flow in Lagrangian space (BE93). This prediction suggests that m-th order Lagrangian perturbations will transform pancakes into 2m+1-stream systems in the coarse of time. The stage until when the perturbation solutions are valid could be estimated by the time when the shell-crossing singularities of the corresponding order appear.

In the present paper, I evaluate the third order perturbation solutions. In this line the need for higher order solutions for the purpose of modeling large-scale structure should be questioned. Certainly, it will not be adequate to derive
higher orders unless we can expect to obtain an appropriate model for large-scale structure which remains sufficiently simple to handle with respect to applications, apart from the fact that the derivation of higher order perturbation solutions for generic initial conditions is cumbersome. As an argument to go to the third order I consider the structure of the equations to be solved: they are cubic in the basic dynamical variable (see B92, APPENDIX), so we can expect that a third order solution will cover the main effects of the perturbation sequence in the early non-linear regime. We first derive the longitudinal perturbations, i.e., for the case where the perturbations admit a potential in Lagrangian space. In contrast to opinions stated in the literature (Moutarde et al. 1991, Bouchet et al. 1992, Lachièze-Rey 1993), this restricts the generality of the solutions as will be discussed in detail. The restriction vanishes if interaction of perturbations (among the first and second order perturbations here) is neglected, an assumption which will define a class of third order models which we consider the “main body“ of the perturbation sequence. Since, in general, the interaction terms generate vorticity in Lagrangian space (even for irrotational flows considered throughout this paper), there will be a transverse part of the third order solution, which I shall derive.

As an example the approximation is evaluated on an Einstein-de Sitter background for initial conditions which correspond to Zel’dovich’s model, and the result is expressed in terms of the initial conditions for the peculiar-velocity potential, or the peculiar-gravitational potential, respectively. With this specification of the initial conditions (discussed in section 3) we have assumed a functional relationship between the peculiar-velocity potential and the peculiar-gravitational potential (here: proportionality). Relaxing this relationship will also induce vorticity in Lagrangian space already at the second order level (see BE93).
2. The perturbation equations in Lagrangian form

Let us briefly summarize the formal consequences of a Lagrangian description of self-gravitating flows in Newton’s theory. We introduce integral curves $\vec{x} = \vec{f}(\vec{X}, t)$ of the velocity field $\vec{v}(\vec{x}, t)$:

$$\frac{d\vec{f}}{dt} = \vec{v}(\vec{f}, t), \quad \vec{f}(\vec{X}, t_0) =: \vec{X}. \quad (1)$$

These curves are labelled by the Lagrangian coordinates $X_i$; $x_i$ are non-rotating Eulerian coordinates. We can express all Eulerian fields in terms of the field of trajectories $\vec{f}$ as was explained in Buchert (1989), B92 and BE93. We denote the determinant of the deformation tensor $(f_{i,k})$ by $J$, where the comma denotes partial differentiation with respect to the Lagrangian coordinates, a dot will denote Lagrangian time derivative $\frac{d}{dt} := \partial_t |_{\vec{x}} + \vec{v} \cdot \nabla_x = \partial_t |_{\vec{X}}$; comma and dot commute. Recall that mass conservation is guaranteed in the Lagrangian representation irrespective of any equations the trajectories $\vec{f}$ might obey. The Euler-Poisson system for “dust” matter can be cast into a set of four evolution equations for the single dynamical variable $\vec{f}$ (Buchert & Götz 1987, Buchert 1989):

$$\epsilon_{pq|j} \frac{\partial(f_{i|j}, f_p, f_q)}{\partial(X_1, X_2, X_3)} = 0, \quad i \neq j, \quad (2a, b, c)$$

$$\sum_{a,b,c} \frac{1}{2} \epsilon_{abc} \frac{\partial(f_{a|b}, f_{b|c}, f_{c|a})}{\partial(X_1, X_2, X_3)} - \Lambda J = -4\pi G \rho(\vec{X}) ; \quad \rho(\vec{X}) > 0. \quad (2d, e)$$

(indices run from 1 to 3, if not otherwise explicitly stated; henceforth, $\nabla_0$ denotes the nabla operator with respect to the Lagrangian frame which commutes with the dot).

We proceed to the evaluation of the perturbation equations. (The reader may consult earlier papers for more details.) We first make the following perturbation ansatz for longitudinal perturbations superposed on an isotropic homogeneous deformation:

$$\vec{f} = a(t)\vec{X} + \vec{p} ; \quad \vec{p} = \epsilon \nabla_0 \psi^{(1)} + \epsilon^2 \nabla_0 \psi^{(2)} + \epsilon^3 \nabla_0 \psi^{(3)}. \quad (3a)$$

The parameter $\epsilon$ is supposed to be small and dimensionless. It can be considered as the amplitude of the initial perturbation field. We formally split the initial density accordingly:

$$\rho = \rho_H + \epsilon \delta \rho^{(1)} + \epsilon^2 \delta \rho^{(2)} + \epsilon^3 \delta \rho^{(3)}. \quad (3b)$$
However, we can choose initial data at our convenience, so we put

\[
\delta \rho^{(1)} := \delta \rho =: \hat{\rho}_H \hat{\delta}, \quad \delta \rho^{(2)} := 0, \quad \delta \rho^{(3)} := 0,
\]

where \(\delta \rho\) denotes the total initial density perturbation, \(\hat{\delta}\) the initial density contrast. This formality is adequate, since the density needs not be perturbed in the Lagrangian framework.

In what follows, \(I(\psi_{i,k}) = tr(\psi_{i,k}) = \Delta_0 \psi\), \(II(\psi_{i,k}) = \frac{1}{2} [(tr(\psi_{i,k}))^2 - tr((\psi_{i,k})^2)]\) and \(III(\psi_{i,k}) = \det(\psi_{i,k})\) denote the three principal scalar invariants of the tensor \((\psi_{i,k})\). For the derivation of the perturbation solutions we shall first concentrate on the source equation (2d) and consider the conditions (2a,b,c) as constraint equations, which are checked after the solution is obtained; the constraint equations and the resulting constraints on initial conditions for the longitudinal third order approximation are given in APPENDIX A. We shall then construct the transverse part which removes the constraints obtained for the pure longitudinal solution.

Inserting the ansatz (3) into equation (2d), we obtain the following set of equations to be solved:

\[
\varepsilon^0 \left\{ -4\pi G \hat{\rho}_H = 3\ddot{a} a^2 - a^3 \Lambda \right\} = 0 \quad (4a)
\]

\[
\varepsilon^1 \left\{ -4\pi G \delta \rho^{(1)} = \left[ (2\ddot{a} a - a^2 \Lambda) + a^2 \frac{d^2}{dt^2} \right] \Delta_0 \psi^{(1)} \right\} = 0 \quad (4b)
\]

\[
\varepsilon^2 \left\{ -4\pi G \delta \rho^{(2)} = \left[ (2\ddot{a} a - a^2 \Lambda) + a^2 \frac{d^2}{dt^2} \right] \Delta_0 \psi^{(2)} \right\} = 0
\]

\[
+(\ddot{a} - a\Lambda)II(\psi_{i,k}) + a \sum_{a,b,c} \epsilon_{abc} \frac{\partial(\ddot{\psi}_{a}^{(1)}, \psi_{b}^{(1)}, X_c)}{\partial(X_1, X_2, X_3)} = 0 \quad (4c)
\]

\[
\varepsilon^3 \left\{ -4\pi G \delta \rho^{(3)} = \left[ (2\ddot{a} a - a^2 \Lambda) + a^2 \frac{d^2}{dt^2} \right] \Delta_0 \psi^{(3)} \right\} = 0
\]

\[
+ (\ddot{a} - a\Lambda) \sum_{a,b,c} \epsilon_{abc} \frac{\partial(\psi_{a}^{(2)}, \psi_{b}^{(1)}, X_c)}{\partial(X_1, X_2, X_3)} + a \sum_{a,b,c} \epsilon_{abc} \left( \frac{\partial(\ddot{\psi}_{a}^{(2)}, \psi_{b}^{(1)}, X_c)}{\partial(X_1, X_2, X_3)} + \frac{\partial(\ddot{\psi}_{a}^{(1)}, \psi_{b}^{(1)}, X_c)}{\partial(X_1, X_2, X_3)} \right)
\]

\[
+ \sum_{a,b,c} \frac{1}{2} \epsilon_{abc} \frac{\partial(\ddot{\psi}_{a}^{(1)}, \psi_{b}^{(1)}, \psi_{c}^{(1)})}{\partial(X_1, X_2, X_3)} - \Lambda III(\psi_{i,k}) \right\} = 0 \quad (4d)
\]

Each order (starting from \(\varepsilon^1\)) contains a term where a linear operator acts on the different potentials \(\psi^{(1)}, \psi^{(2)}, \psi^{(3)}\). Similarly, a quadratic and a cubic operator acts
on quadratic or cubic invariants formed by second derivatives of $\psi^{(1)}$. However, at the third order, there appears a quadratic term describing the interaction of linear and quadratic perturbations. Higher orders will subsequently add such interaction terms between the potentials at different or equal orders.

We now derive the solutions of the equations (4) in the case of a “flat” background. I present a simplified derivation for a special class of initial conditions which is relevant for the modeling of large-scale structure. Therefore, it is illustrative to derive all orders with the same restriction and procedure. The reader who is interested in applications of the solutions only may proceed directly to section 4.
3. Solution of the perturbation equations on an Einstein-de Sitter background

In what follows we make use of a restriction of the initial conditions to simplify the derivations. We require that, initially, the peculiar-velocity $\vec{u}(\vec{X}, t_0)$ be proportional to the peculiar-acceleration $\vec{w}(\vec{X}, t_0)$:

$$\vec{u}(\vec{X}, t_0) = \vec{w}(\vec{X}, t_0) t_0,$$

where we have defined the fields as usual (compare Peebles 1980, B92). This restriction has proved to be appropriate for the purpose of modeling large-scale structure since, for irrotational flows, the peculiar-velocity field tends to be parallel to the gravitational peculiar-field strength after some time. The reason for this tendency is related to the existence of growing and decaying perturbations, the growing part supports the tendency to parallelity. The assumption of irrotationality should be adequate down to the non-linearity scale. However, besides the possibility of non-linearly enhanced primordial vorticity (B92), shell-crossings generate vorticity on scales below the non-linearity scale (see, e.g., Doroshkevich 1973, Chernin 1993 and ref. therein). We should keep this in mind, since a third order approximation actually should give a proper description of these smaller scales. The treatment of vorticity is no longer academic, but might play a decisive role for the dynamics within clusters of galaxies. Also, decaying solutions should be considered with some care in the non-linear regime. They couple to the growing solution, the role of decaying and growing solutions in an expanding environment can be interchanged in a collapsing environment as was discussed by Gurevich and Zybin (1988b). We note that the restriction (5) is commonly used in the literature. The tendency of the flow expressed by (5) has been proved for general first order (Bildhauer & Buchert 1991) and a large class of second order irrotational flows (BE93). Note that in the case (5) we have to give one initial potential only, whereas the general initial value setting for irrotational flows would require two. Any restriction of this type simplifies the calculations enormously, but it implies the restriction to irrotational flows. (An alternative restriction of this type has been discussed in B92 and BE93).

We shall use in the following the initial peculiar-velocity potential $S$ defined as $\vec{u}(\vec{X}, t_0) =: \nabla_0 S(\vec{X})$. The initial peculiar-gravitational potential $\phi$, $\vec{w}(\vec{X}, t_0) =:$
\( -\nabla_0 \phi(\vec{X}) \) is related to it as \( S = -\phi t_0 \) (eq. (5)). Consequently, we seek solutions of the equations (4) of the form:

\[
\psi^{(1)} = q_z(t) \ S^{(1)}(\vec{X}) ,
\]

\[
\psi^{(2)} = q_{zz}(t) \ S^{(2)}(\vec{X}) ,
\]

\[
\psi^{(3)} = q_{zzz}(t) \ S^{(3)}(\vec{X}) ,
\]

where the potentials \( S^{(1)}, S^{(2)} \) and \( S^{(3)} \) have to be related to the initial condition \( S \). We shall see that this relation will require the solution of elliptic boundary value problems expressing non-locality of the solutions. Formally we require \( q_z(t_0) = 0, q_{zz}(t_0) = 0, q_{zzz}(t_0) = 0 \).

### 3.1. The zero order solution

One class of solutions of the Euler-Poisson system (2) is formed by the homogeneous and isotropic Friedman-Lemaître cosmologies:

\[
\vec{f}_H(\vec{X}, t) = a(t) \vec{X} .
\]

Inserting \( \vec{f}_H \) into the equations (2), we obtain for the function \( a(t) \) the zero order equation (4a). Its general solution is given by solutions of Friedman’s differential equation as an integral of (4a):

\[
\frac{\ddot{a}^2 + \text{const}}{a^2} = \frac{8\pi G \rho_H + \Lambda}{3} ,
\]

where \( \rho_H = \bar{\rho}_H a^{-3} \) is the background density. Henceforth we restrict all considerations to the Einstein-de Sitter case (\( \text{const} = 0, \Lambda = 0 \)). Then, the zero order solution reads:

\[
a(t) = \left( \frac{t}{t_0} \right)^\frac{2}{3} .
\]

For convenience, we shall express all time-dependent coefficients in terms of the solution \( a \). The constant \( -4\pi G \bar{\rho}_H \) will be written in terms of its value \( \frac{2}{3t_0^3} \) for the solution (7c).

### 3.2. The first order solution

By virtue of the ansatz (6a), equation (4b) simplifies to the following equation (in the sequel we put \( \Lambda = 0 \) and use (3c) and (7c)):

\[
\left( \frac{\ddot{q}_z + \frac{2}{a} \dot{a} q_z}{a} \right) \Delta_0 S^{(1)} = \frac{1}{a^2} \left( \frac{2}{3t_0^3} \right) \delta .
\]
Using Poisson’s equation for the initial potential $\phi$ and the relation $S = -\phi t_0$ we have:

$$\Delta_0 S = -\frac{2}{3t_0} \phi .$$  \hfill (8a)

Hence, solutions of (8) can be found as solutions of the linear ordinary differential equation:

$$\ddot{q}_z + 2\frac{\ddot{a}}{a} q_z = \frac{1}{a^2 t_0^2} \Delta_0 S = G^{(1)}(t)$$ \hfill (8b)

with:

$$\Delta_0 S^{(1)} = \Delta_0 S t_0 = I(S_{i,k}) t_0 .$$  \hfill (8c)

The solution to (8b) consists of two linearly independent solutions of the homogeneous part:

$$q_1^{(1)} = C_1^{(1)} a^2 , \quad q_2^{(1)} = C_2^{(1)} a^{-\frac{1}{2}} ,$$ \hfill (9a)

and a particular solution:

$$q_p^{(1)} = C^{(1)} \left( q_1^{(1)} \int^t q_2^{(1)} G^{(1)} dt - q_2^{(1)} \int^t q_1^{(1)} G^{(1)} dt \right) = C^{(1)} a .$$ \hfill (9b)

The coefficient $C^{(1)}$ is found by inserting $q_p^{(1)}$ into (8b); the coefficients $C_1^{(1)}$ and $C_2^{(1)}$ are found by the requirement that the coefficient functions of the peculiarity-velocity and -acceleration equal to 1 at $t = t_0$ for the solution $q^{(1)} = q_1^{(1)} + q_2^{(1)} + q_p^{(1)}$. Restricting the general solution $q^{(1)}$ according to (5) we find (B92):

$$q_z = \frac{3}{2}(a^2 - a) \ .$$ \hfill (10)

A discussion of the uniqueness of this solution related to the use of $St_0$ instead of $S^{(1)}$ is to be found in APPENDIX B.

3.3. The second order solution

Inserting the ansatz (6b) into the equation (4c) we find:

$$\left( \ddot{q}_{zz} + 2\frac{\ddot{a}}{a} q_{zz} \right) \Delta_0 S^{(2)} = - \left( \frac{\ddot{q}_z q_z}{a} + \frac{q_z^2 \ddot{a}}{2 a^2} \right) 2II(S_{i,k}) .$$ \hfill (11)

Solutions to (11) can be found by solving the linear ordinary differential equation (inserting the first order solution $q_z$):

$$\ddot{q}_{zz} + 2\frac{\ddot{a}}{a} q_{zz} = - \left( \frac{\ddot{q}_z q_z}{a} + \frac{q_z^2 \ddot{a}}{2 a^2} \right) =: G^{(2)}(t) \ ,$$ \hfill (11a)
with:
\[ \Delta_0 S^{(2)} = 2II(S^{(1)}_{i,k}) . \] (11b)

The solution to (11a) consists of two linearly independent solutions of the homogeneous part:
\[ q_1^{(2)} = C_1^{(2)} a^2 , \quad q_2^{(2)} = C_2^{(2)} a^{-\frac{1}{2}} , \] (12a)

and a particular solution:
\[ q_p^{(2)} = C^{(2)} \left( q_1^{(2)} \int q_2^{(2)} G^{(2)} dt - q_2^{(2)} \int q_1^{(2)} G^{(2)} dt \right) = C^{(2)} (-\frac{3}{4} + \frac{3}{4} a^{-2}) . \] (12b)

The coefficient \( C^{(2)} \) is found by inserting \( q_p^{(2)} \) into (11a); the coefficients \( C_1^{(2)} \) and \( C_2^{(2)} \) are found by the requirement that the coefficient functions of the peculiar-velocity and -acceleration vanish at \( t = t_0 \) for the solution \( q^{(2)} = q_1^{(2)} + q_2^{(2)} + q_p^{(2)} \).

We find for the restriction (5) the following result (compare BE93, section 5):
\[ q_{zz} = \left( \frac{3}{2} \right)^2 \left( -\frac{3}{14} a^3 + \frac{3}{5} a^2 - \frac{1}{2} a + \frac{4}{35} a^{-\frac{5}{2}} \right) . \] (13)

3.4. The third order solution - longitudinal part

Inserting the ansatz (6c) into the equation (4d) we find:
\[ \left( \ddot{q}_{zzz} + 2\dot{a} q_{zzz} \right) \Delta_0 S^{(3)} = - \left( \frac{\ddot{a}}{a^2} q_{zz} q_z + \frac{1}{a} (\dddot{q}_{zz} q_z + \ddot{q}_z q_{zz}) \right) \sum_{a,b,c} \epsilon_{abc} \frac{\partial(S^{(2)}_a ; S^{(1)}_b ; X_c)}{\partial(X_1, X_2, X_3)} \]
\[ - \left( \frac{1}{a^2} \dddot{q}_z q_z^2 \right) 3III(S^{(1)}_{i,k}) . \] (14)

To obtain solutions of (14) we use the linearity of Poisson’s equation and split the potential \( S^{(3)} \) into a part \( S^{(3a)} \) generating the cubic source term, and a part \( S^{(3b)} \) generating the quadratic source term of the interaction of first and second order perturbations. We then have to solve separately the linear ordinary differential equation (inserting the first order solution \( q_z \)):
\[ \ddot{q}_{zzz} + 2\dot{a} q_{zzz} = -\frac{1}{a^2} \dddot{q}_z q_z^2 =: G^{(3a)}(t) , \] (14a)

with:
\[ \Delta_0 S^{(3a)} = 3III(S^{(1)}_{i,k}) , \] (14b)
and the linear ordinary differential equation (inserting the first and second order solutions $q_z$ and $q_{zz}$):

$$\dddot{q}_{zzz} + 2 \frac{\ddot{a}}{a} q_{zzz} = -\frac{\ddot{a}}{a^2} q_{zzz} + \frac{1}{a} (\dddot{q}_{zz} + \dddot{q}_{z} q_z) =: G^{(3b)}(t),$$  \hspace{1cm} (14c)

with:

$$\Delta_0 S^{(3b)} = \sum_{a,b,c} \epsilon_{abc} \frac{\partial (S^{(2)}_a, S^{(1)}_b, X_c)}{\partial (X_1, X_2, X_3)}. \hspace{1cm} (14d)$$

A general solution to (14a) or (14c), respectively, consists of two linearly independent solutions of the homogeneous part:

$$q^{(3a,b)}_1 = C^{(3a,b)}_1 a^2, q^{(3a,b)}_2 = C^{(3a,b)}_2 a^{-\frac{3}{2}},$$  \hspace{1cm} (15a, b)

and a particular solution for each source term:

$$q^{(3a)}_p = C^{(3a)} \left( q^{(3a)}_1 \int^t q^{(3a)}_2 G^{(3a)} dt - q^{(3a)}_2 \int^t q^{(3a)}_1 G^{(3a)} dt \right)$$

$$= C^{(3a)} \frac{9}{4} \left( -\frac{2}{3} a + 1 - \frac{1}{3} a^{-2} \right),$$ \hspace{1cm} (16a)

$$q^{(3b)}_p = C^{(3b)} \left( q^{(3b)}_1 \int^t q^{(3b)}_2 G^{(3b)} dt - q^{(3b)}_2 \int^t q^{(3b)}_1 G^{(3b)} dt \right)$$

$$= C^{(3b)} \frac{9}{4} \left( \frac{5}{7} a - \frac{11}{10} + \frac{1}{2} a^{-2} - \frac{4}{35} a^{-\frac{3}{2}} \right).$$ \hspace{1cm} (16b)

The coefficients $C^{(3a,b)}$ are found by inserting $q^{(3a,b)}_p$ into (16a) or (16b), respectively; the coefficients $C^{(3a,b)}_1$ and $C^{(3a,b)}_2$ are found as in the second order case by the requirement that the coefficients of the initial peculiar-velocity and acceleration vanish for the general solutions. I find for the restriction (5) the following result:

$$q^{a}_{zzzz} = \left( \frac{3}{2} \right)^3 \left( -\frac{1}{9} a^4 + \frac{3}{7} a^3 - \frac{3}{5} a^2 + \frac{1}{3} a - \frac{16}{315} a^{-\frac{3}{2}} \right);$$ \hspace{1cm} (17a)

$$q^{b}_{zzzz} = \left( \frac{3}{2} \right)^3 \left( \frac{5}{42} a^4 - \frac{33}{70} a^3 + \frac{7}{10} a^2 - \frac{1}{2} a + \frac{4}{35} a^{\frac{3}{2}} + \frac{4}{105} a^{-\frac{3}{2}} \right).$$ \hspace{1cm} (17b)
3.5. The third order solution - transverse part

Inserting the longitudinal ansatz
\[ \vec{f}^L = \nabla_0 \left( a \frac{\vec{X}^2}{2} + q_z S^{(1)} + q_{zz} S^{(2)} + q_{zzz} S^{(3a)} + q_{zzzz} S^{(3b)} \right) \] (18)

into the equations (2a,b,c), we find restrictions on the initial conditions. These restrictions are derived in APPENDIX A. Note that, instead of the equations (2a,b,c) for the irrotationality of the gravitational field strength, we can use the corresponding equations for the irrotationality of the velocity in the case of irrotational flows (see the LEMMA proved in BE93), which we shall do.

The restrictions only arise at the third order level. This implies that the third order solution is not purely longitudinal, it is not fully covered by the ansatz (18). In the following we seek a transverse part of the third order solution by extending the ansatz (18):

\[ \vec{f} = \vec{f}^L + \vec{f}^T \quad ; \quad \vec{f}^T := q_{zzz} \vec{\Xi} \quad ; \quad \vec{\Xi} := -\nabla_0 \times \vec{S}^{(3c)} . \] (19a)

We have introduced the vector potential \( \vec{S}^{(3c)} \), on which we impose the following gauge condition (compare APPENDIX B):

\[ \nabla_0 \cdot \vec{S}^{(3c)} . \] (19b)

With the ansatz (19a) we generate no additional equations to be fulfilled in the source equation (2d). The integrability conditions for the velocity (BE93: equations (5d,e,f)) only yield an equation of the order \( \epsilon^3 \) as expected \((i, j, k = 1, 2, 3 \text{ cyclic ordering})\):

\[ \epsilon^3 \left\{ a (\dot{a} q_{zzz}^c - a \dot{q}_{zzz}^c) \left( \nabla_0 \times (-\nabla_0 \times \vec{S}^{(3c)}) \right) \right\}_k + a (\dot{q}_{zz} q_z - \dot{q}_z q_{zz}) \epsilon_{pqj} \frac{\partial (S_a^{(2)}, S_p^{(1)}, X_q)}{\partial (X_1, X_2, X_3)} = 0 \] . (20a, b, c)

Using the vector identity \( \nabla_0 \times (\nabla_0 \times \vec{S}^{(3c)}) = \nabla_0 (\nabla_0 \cdot \vec{S}^{(3c)}) - \Delta_0 \vec{S}^{(3c)} \) and (19b), we find solutions of the equations (20) by solving the linear ordinary differential equation (inserting the first and second order solutions \( q_z \) and \( q_{zz} \)):

\[ \dot{q}_{zzz}^c - \frac{\dot{a}}{a} q_{zzz}^c = -\frac{1}{a} (\dot{q}_{zz} q_z - \dot{q}_z q_{zz}) =: G^{(3c)}(t) , \] (21a)
with:

\[(\Delta_0 \vec{S}^{(3c)})_k = \epsilon_{pq[ij]} \frac{\partial (S^{(2)}_{ij}, S^{(1)}_p, X_q)}{\partial (X_1, X_2, X_3)} .\]  \hspace{1cm} (21b, c, d)

A general solution of (21a) is given by:

\[q^{c}_{zzz} = \frac{1}{M(t)} \left( \int_{t_0}^{t} G^{(3c)}(t)M(t)dt + C^{(3c)} \right) ,\]  \hspace{1cm} (22)

with the integrating factor \(M(t) = e^{-\int_{t_0}^{t} \dot{a} dt}\). The coefficient \(C^{(3c)}\) is found by the requirement \(q^{c}_{zzz}(t_0) = 0\). We finally obtain:

\[q^{c}_{zzz} = \left( \frac{3}{2} \right)^3 \left( \frac{1}{14} a^4 - \frac{3}{14} a^3 + \frac{1}{10} a^2 + \frac{1}{2} a - \frac{4}{7} a^{\frac{3}{2}} + \frac{4}{35} a^{-\frac{1}{2}} \right) .\]  \hspace{1cm} (23)
4. Result and Discussion

4.1. The solution

**THEOREM**

With a superposition ansatz for Lagrangian perturbations of an Einstein-de Sitter background we have obtained the following family of trajectories \( \vec{x} = \vec{f}(\vec{x}, a) \) as irrotational solution of the Euler-Poisson system up to the third order in the perturbations from homogeneity. The general set of initial conditions \((\phi(\vec{X}), S(\vec{X}))\) is restricted according to \( S = -\phi t_0 \). (The parameter \( \varepsilon \) is considered as the amplitude of the initial fluctuation field; \( a(t) = (t/t_0)^{2/3}, i, j, k = 1, 2, 3 \) with cyclic ordering):

\[
\vec{f} = a \vec{X} + q_z(a) \nabla_0 S^{(1)}(\vec{X}) + q_{zz}(a) \nabla_0 S^{(2)}(\vec{X})
+ q_{zzz}(a) \nabla_0 S^{(3a)}(\vec{X}) + q_{zzz}(a) \nabla_0 S^{(3b)}(\vec{X}) - q_{zzz}(a) \nabla_0 \vec{S}^{(3c)}(\vec{X}) ,
\]

with:

\[
q_z = \left( \frac{3}{2} \right) (a^2 - a) ,
\]

\[
q_{zz} = \left( \frac{3}{2} \right)^2 \left( -\frac{3}{14} a^3 + \frac{3}{5} a^2 - \frac{1}{2} a + \frac{4}{35} a^{-\frac{1}{2}} \right) ,
\]

\[
q_{zzz}^a = \left( \frac{3}{2} \right)^3 \left( -\frac{1}{9} a^4 + \frac{3}{7} a^3 - \frac{3}{5} a^2 + \frac{1}{3} a - \frac{16}{315} a^{-\frac{1}{2}} \right) ,
\]

\[
q_{zzz}^b = \left( \frac{3}{2} \right)^3 \left( \frac{5}{42} a^4 - \frac{33}{70} a^3 + \frac{7}{10} a^2 - \frac{1}{2} a + \frac{4}{35} a^{\frac{1}{2}} + \frac{4}{105} a^{-\frac{1}{2}} \right) ,
\]

\[
q_{zzz}^c = \left( \frac{3}{2} \right)^3 \left( \frac{1}{14} a^4 - \frac{3}{14} a^3 + \frac{1}{10} a^2 + \frac{1}{2} a - \frac{4}{7} a^{\frac{1}{2}} + \frac{4}{35} a^{-\frac{1}{2}} \right) ,
\]

and:

\[
\Delta_0 S^{(1)} = I(S_{i,k}) t_0 ,
\]

\[
\Delta_0 S^{(2)} = 2II(S_{i,k}) ,
\]

\[
\Delta_0 S^{(3a)} = 3III(S_{i,k}) ,
\]

\[
\Delta_0 S^{(3b)} = \sum_{a,b,c} \epsilon_{abc} \frac{\partial(S^{(2)}_a, S^{(1)}_b, X_c)}{\partial(X_1, X_2, X_3)} ,
\]

\[
(\Delta_0 \vec{S}^{(3c)})_k = \epsilon_{pql} \frac{\partial(S^{(2)}_r, S^{(1)}_p, X_q)}{\partial(X_1, X_2, X_3)} .
\]
REMARKS:
The potential $S^{(3b)}$ and the vector potential $\vec{S}^{(3c)}$ generate interaction among the first and second order perturbations. The general interaction term is not purely longitudinal. In order to satisfy the Euler-Poisson system with the longitudinal part only, we have to respect the following constraint (APPENDIX A):

$$\nabla_0 S^{(2)} = \mathcal{W}(\nabla_0 S^{(1)}) . \quad (24m)$$

The potential $S^{(3b)}$ generates the symmetric part, whereas the vector potential $\vec{S}^{(3c)}$ generates the anti-symmetric part of the interaction.

**Proof:** The proof follows by inserting the solution (24) into the perturbation equations (4), which has been done using the algebraic manipulation system REDUCE.

We now discuss some issues which are related to the practical use of the solution (24) as a model for the evolution of large-scale structure.

4.2. The construction of local forms

The usefulness of local approximations has been pointed out by Nusser et al. (1991) who apply the “Zel’dovich approximation” as a tool for locally reconstructing the density field from observed peculiar-velocity data. Contrary, solutions of Newtonian equations for the evolution of self-gravitating “dust” continua are non-local, since, e.g., the gravitational potential has to be a solution of the (elliptic) Poisson equation. This solution involves integrals over large space regions. As far as the first order Lagrangian perturbation solution is concerned, this is not a contradiction: without loss of generality, i.e., without change of physical quantities like the density contrast or the divergence of the peculiar-velocity, we can use the initial condition $S t_0$ instead of $S^{(1)}$ (APPENDIX B). With this reduction the iterative procedure of constructing the displacement vectors in (24) simplifies: the source terms are (except the interaction terms (24i-l)) completely expressible in terms of second derivatives of the initial potential $S$.

The situation is more delicate at higher order levels. We no longer are able to write second or higher order terms as a local approximation. Although it is computationally simple to obtain the displacement vectors in (24) by employing Poisson solvers (see 4.3), it is useful to know which classes of initial conditions admit the construction of local forms. With such closed form expressions the
study of special solutions is explicitly feasible without using a numerical Poisson solver.

According to COROLLARY 1 proved in (BE93), a local form can be obtained for second order displacements. It reads:

\[
\nabla_0 S^{(2)} = \nabla_0 S (\Delta_0 S) - (\nabla_0 S \cdot \nabla_0) \nabla_0 S \quad \nabla_0 S \times \Delta_0 \nabla_0 S = \vec{0} .
\]

(25a, b, c, d)

The local form (25a) is constructed such that its divergence agrees with the source term in (24g), its curl is, however, in general non-zero, it only vanishes if (25b-d) is satisfied. The latter equations express the fact that the form (25a) cannot be used in general, constraints have to be obeyed.

Similarly, we can ask for a local vector form whose divergence agrees with the third principal scalar invariant of \((S_{i,k})\), which is the source term in equation (24h). Indeed, the following expression has the required property:

**COROLLARY 1**

The vector \(\nabla_0 S^{(3a)}\) with the components

\[
(\nabla_0 S^{(3a)})_k = \sum_i (\nabla_0 S^{(1)})_{i,k} J^S_{i,k}
\]

(26a)

has the property:

\[
\Delta_0 S^{(3a)} = 3III(S^{(1)}_{i,k}) ,
\]

where \(J^S_{i,k}\) are the subdeterminants of the tensor \((S^{(1)}_{i,k})\). The following constraints have to be satisfied in order that \(\nabla_0 S^{(3a)}\) is curl-free:

\[
\sum_i (\nabla_0 S^{(1)})_{i,k} J^S_{i,[k,j]} = 0 , \quad k \neq j .
\]

(26b, c, d)

(The proof is done by explicit verification).

The source term in (24i) which describes the longitudinal part of the interaction of first and second order perturbations has a similar structure as the second order source term. We can construct a local form by analogy. The integral of the interaction source term in equation (24i) reads (BE93, COROLLARY 1):

\[
\nabla_0 S^{(3b)} = \\
\lambda_1 \left( \nabla_0 S^{(2)} (\Delta_0 S^{(1)}) - (\nabla_0 S^{(2)} \cdot \nabla_0) \nabla_0 S^{(1)} \right) + \lambda_2 \left( \nabla_0 S^{(1)} (\Delta_0 S^{(2)}) - (\nabla_0 S^{(1)} \cdot \nabla_0) \nabla_0 S^{(2)} \right) .
\]

(27a)
We have taken the linear combination of the two possible integrals as a general integral, where we have to assure $\lambda_1 + \lambda_2 = 1$. In order to satisfy the requirement that the vector \( (27a) \) be a solution of the Poisson equation \((24i)\), we have to assure that it is curl-free which implies \((\text{BE93, COROLLARY 1)}\):

$$
\lambda_1 \left( \nabla_0 S^{(2)} \times \Delta_0 \nabla_0 S^{(1)} \right) + \lambda_2 \left( \nabla_0 S^{(1)} \times \Delta_0 \nabla_0 S^{(2)} \right) = \vec{0} . \quad (27b, c, d)
$$

We finally give an integral expression for the source terms in \((24j,k,l)\) which describe the transverse part of the interaction of first and second order perturbations:

**COROLLARY 2**

The vector $\vec{\Xi}$:

$$
\vec{\Xi} := -\nabla_0 \times \vec{S}^{(3c)} = \\
\mu_1 \left( (\nabla_0 S^{(2)} \cdot \nabla_0) \nabla_0 S^{(1)} \right) + \mu_2 \left( -(\nabla_0 S^{(1)} \cdot \nabla_0) \nabla_0 S^{(2)} \right) \quad (28a, b, c)
$$

has the property:

$$
(\nabla_0 \times \vec{\Xi})_i = \epsilon_{pq|j} \frac{\partial(S^{(2)}_{,i}, S^{(1)}_{,p}, X_q)}{\partial(X_1, X_2, X_3)} , \ i \neq j .
$$

We have taken the linear combination of the two possible integrals as a general integral, where we have to assure $\mu_1 + \mu_2 = 1$. In order to satisfy the requirement that the vector components \((28a,b,c)\) be solutions of the Poisson equations \((24j,k,l)\), we have to assure that the displacement vector $\vec{\Xi}$ can be represented in terms of the vector potential $\vec{S}^{(3c)}$: $\vec{\Xi} = -\nabla_0 \times \vec{S}^{(3c)}$, i.e., the vector field $\vec{\Xi}$ has to be source-free:

$$
\mu_1 \nabla_0 \cdot \left( (\nabla_0 S^{(2)} \cdot \nabla_0) \nabla_0 S^{(1)} \right) - \mu_2 \nabla_0 \cdot \left( (\nabla_0 S^{(1)} \cdot \nabla_0) \nabla_0 S^{(2)} \right) = 0 . \quad (28d)
$$

(The proof is done by explicit verification).

In special cases it is possible to construct the four potentials $S^{(3b)}$ and $(\vec{S}^{(3c)})_i$ by fixing the parameters $\lambda_1$, $\lambda_2$, $\mu_1$, and $\mu_2$ suitably, thus fulfilling the constraints \((27b)\) and \((28d,e,f)\); in some cases the integration freedom (the curl of some vector in \((27a)\), and the gradient of some scalar in \((28a,b,c)\)) has to be used in order to fulfil the constraints, (see Buchert et al. 1993c).
4.3. Practical realization and comparison with numerical simulations

In order to realize the presented solution for generic initial conditions the following procedure is appropriate.

Let us start with a Gaussian random density contrast field $\delta(\vec{X})$ as initial condition. It is most convenient to express this field in terms of its discrete Fourier transform (see, e.g., Bertschinger 1992). Henceforth, we denote its Fourier components by $\hat{\delta}$, the Fourier sums may extend over $n$ wave vectors $\vec{K} = (K_1, K_2, K_3)$.

From the density perturbation we construct the initial peculiar-velocity potential in $K$–space using FFT (“Fast Fourier Transform”), here written for the “flat” background:

$$\hat{S}(\vec{K}) = -\frac{2}{3t_0} \Delta_0^{-1}(\vec{K}) \hat{\delta}(\vec{K}) .$$

From $\hat{S}(\vec{K})$ we are able to construct all displacement vectors which are generated by second derivatives of $S$ with respect to Lagrangian coordinates. This can be done by simply multiplying and summing the components $K_1, K_2, K_3$ of all $n$ wave vectors $\vec{K}$ in $K$–space. We first use the following property of Fourier transforms:

$$\nabla_0 \hat{S}(\vec{X})(\vec{K}) = i\vec{K} \hat{S}(\vec{K}) ;$$

$$(S_{i,k})(\vec{X})(\vec{K}) = -K_i K_k \hat{S}(\vec{K}) .$$

Transforming the gradient back to the $X$–space we obtain the displacement vector of the first order perturbation. Transforming all 6 different Fourier components of the symmetric tensor gradient $(\hat{S}_{i,k})$ back to the $X$–space we are able to construct the three principal scalar invariants of it:

$$I(S_{i,k}) = S_{1,1} + S_{2,2} + S_{3,3} ,$$

$$II(S_{i,k}) = S_{1,1}S_{2,2} + S_{1,1}S_{3,3} + S_{2,2}S_{3,3} - S_{1,2}S_{2,1} - S_{1,3}S_{3,1} - S_{2,3}S_{3,2}$$

$$III(S_{i,k}) = S_{3,3}S_{2,2}S_{1,1} - S_{3,3}S_{2,1}S_{1,2} - S_{3,2}S_{2,3}S_{1,1} + S_{3,2}S_{2,1}S_{1,3} + S_{3,1}S_{2,3}S_{1,2} - S_{3,1}S_{2,2}S_{1,3} .$$

From the invariants (31b) and (31c) we construct the potentials $S^{(2)}$ and $S^{(3a)}$ by using again FFT in analogy to (29). Note, that for the displacement vectors in (24) we need the gradients of these potentials which can be constructed again by multiplying with $i\vec{K}$ in $K$–space.
To construct the interaction terms, we have to use the components of $\nabla_0 S^{(2)}$ and mix them up with the components of $\nabla_0 S^{(1)}$. This is, of course, only possible after the construction of $S^{(2)}$: After building the mixed invariants, for the longitudinal interaction term:

$$II_{mix}^L = S_{1,1}^{(2)} S_{2,2} + S_{1,1}^{(2)} S_{3,3} + S_{2,2}^{(2)} S_{3,3} - S_{1,2}^{(2)} S_{2,1} - S_{1,3}^{(2)} S_{3,1} - S_{2,3}^{(2)} S_{3,2} + S_{1,1}^{(2)} S_{2,2} + S_{1,1}^{(2)} S_{3,3} + S_{2,2}^{(2)} S_{3,3} - S_{1,2}^{(2)} S_{2,1} - S_{1,3}^{(2)} S_{3,1} - S_{2,3}^{(2)} S_{3,2}$$

and for the transverse interaction terms:

$$(II_{mix}^T)_i = S_{i,j}^{(2)} S_{j,k} + S_{j,j}^{(2)} S_{k,k} + S_{i,j}^{(2)} S_{i,k} - S_{i,j}^{(2)} S_{i,k} - S_{j,j}^{(2)} S_{j,k} - S_{k,j}^{(2)} S_{k,k}$$

$$i,j,k = 1,2,3 \quad \text{cyclic ordering} \quad (31d,e,f,g)$$

we are able to construct the four generating functions of the interaction terms $S^{(3b)}$ and $\tilde{S}^{(3c)}$ via FFT, and from those functions the components of the displacement vectors.

The presented model is currently being compared with numerical simulations. Two of the comparisons (Buchert et al. 1993a,b) concern the cross-correlation of generic density fields as predicted by the Lagrangian perturbation solutions with N-body simulations. This is done for various different power spectra similar to the work by Coles et al. (1993). Another comparison (Buchert et al. 1993c) performs a study of special cluster models using a tree-code. The purpose of the latter work is to learn about the principal effects of the different orders at high resolution. In this study the local forms discussed in 4.2 can be used, the models employed are examples of third-order solutions which are fully expressible in closed form without using a numerical Poisson solver.

4.4. Discussion

In the sequel all statements are understood in terms of the restriction (5). Similar statements will hold for the general case.

The infinite sequence of a perturbation ansatz of the form:

$$\bar{f} = a \bar{X} + \sum_{\ell=1}^{\infty} \varepsilon^{\ell} \bar{p}^{(\ell)}$$

(32)
contains in its spatial part integrals of invariants which involve sums and products of first derivatives of the perturbation fields $p_\ell^i$. All these invariants are at most cubic in the products. This fact implies that the sequence (32) starts with those terms listed in (24) which grow as $q_z$, $q_{zz}$ and $q^a_{zzz}$, the whole rest of the sequence is made up of interaction terms resulting from products of perturbations which are not directly expressible in terms of the initial conditions. In view of this it is natural to retain those three terms as “main body” of the perturbation sequence for modeling the early non-linear regime, thus truncating all interaction terms. This way a truncated model is obtained which is the simplest third order model for the study of large-scale structure. In the sequel we list the advantages of the truncated model $\vec{f}^{\text{trunc}} = a\vec{X} + q_z \nabla_0 S^{(1)} + q_{zz} \nabla_0 S^{(2)} + q^a_{zzz} \nabla_0 S^{(3a)}$:

- The model respects all three scalar invariants of the initial displacement tensor $(S_{i,k})$.
- This (and only this) model maps the initial condition to the final stage directly without
  the need to construct the displacement vectors iteratively.
- The model is purely longitudinal.

However, the interaction terms (not contained in $\vec{f}^{\text{trunc}}$) grow at the rates $q^b_{zzz}$ and $q^c_{zzz}$ comparable with $q^a_{zzz}$. The comparison with numerical simulations will show, whether neglection of these terms is meaningful. Note, that a similar growth rate does not imply similar importance for the clustering process, because that depends on the detailed spatial structure of the source terms in (24i-l) compared to the source term in (24h). The other question, whether the local forms discussed in section 4.2 might be useful also for generic initial conditions will also be resumed in future work.

We finally emphasize that we can expect the Lagrangian perturbation ansatz to apply until $a(t)\sigma(t_0) = \mathcal{O}(1)$, where $\sigma(t_0) = \frac{3}{2}\varepsilon$ is the r.m.s. amplitude of the initial density contrast field. At this “accumulation point” all orders of the approximations result in displacements of the same order as the first order displacements. Obviously, this restriction is much less severe as the bound $|\delta| < 1$ in the Eulerian perturbation theory, since the density itself is not bounded in the Lagrangian framework.
Note: A REDUCE program of all equations in this paper and the solution can be obtained from the author via electronic mail: TOB at ibma.ipp-garching.mpg.de.

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APPENDIX A

In this APPENDIX the constraint equations resulting from the equations (2a,b,c) are listed. Since we restrict ourselves to irrotational flows, a LEMMA proved in BE93 applies, which enables us to reduce the integrability conditions (2a,b,c) for the irrotationality of the gravitational field strength to the conditions (BE93: 5d,e,f) for the irrotationality of the velocity. The components of the vorticity vector in Eulerian space \( \vec{\omega} = \frac{1}{2} \nabla \times \vec{v} \), \( \vec{v} = \vec{f} \), are expressed in Lagrangian space and are listed up to the second order for the ansatz (3) in (BE93, APPENDIX). The additional third order terms resulting from the same ansatz read explicitly (i,j,k = 1,2,3, cyclic ordering):

\[
\omega_i = \varepsilon^3 \left[ q_{zz} q_z - q_z q_{zzz} \right] S^{(2)}_{i,k} S^{(1)}_{i,j} + S^{(2)}_{j,k} S^{(1)}_{i,j} + S^{(2)}_{j,k} S^{(1)}_{i,j} - S^{(2)}_{j,k} S^{(1)}_{i,j} + S^{(2)}_{j,k} S^{(1)}_{i,j} - S^{(2)}_{j,k} S^{(1)}_{i,j} + S^{(2)}_{j,k} S^{(1)}_{i,j} - S^{(2)}_{j,k} S^{(1)}_{i,j} + S^{(2)}_{j,k} S^{(1)}_{i,j} - S^{(2)}_{j,k} S^{(1)}_{i,j} + S^{(2)}_{j,k} S^{(1)}_{i,j} - S^{(2)}_{j,k} S^{(1)}_{i,j} + S^{(2)}_{j,k} S^{(1)}_{i,j} - S^{(2)}_{j,k} S^{(1)}_{i,j} + S^{(2)}_{j,k} S^{(1)}_{i,j} - S^{(2)}_{j,k} S^{(1)}_{i,j} + S^{(2)}_{j,k} S^{(1)}_{i,j} - S^{(2)}_{j,k} S^{(1)}_{i,j} + S^{(2)}_{j,k} S^{(1)}_{i,j} - S^{(2)}_{j,k} S^{(1)}_{i,j} + S^{(2)}_{j,k} S^{(1)}_{i,j} - S^{(2)}_{j,k} S^{(1)}_{i,j} + S^{(2)}_{j,k} S^{(1)}_{i,j} - S^{(2)}_{j,k} S^{(1)}_{i,j} + S^{(2)}_{j,k} S^{(1)}_{i,j} - S^{(2)}_{j,k} S^{(1)}_{i,j} + S^{(2)}_{j,k} S^{(1)}_{i,j} - S^{(2)}_{j,k} S^{(1)}_{i,j} + S^{(2)}_{j,k} S^{(1)}_{i,j} - S^{(2)}_{j,k} S^{(1)}_{i,j} + S^{(2)}_{j,k} S^{(1)}_{i,j} - S^{(2)}_{j,k} S^{(1)}_{i,j} + S^{(2)}_{j,k} S^{(1)}_{i,j} - S^{(2)}_{j,k} S^{(1)}_{i,j} + S^{(2)}_{j,k} S^{(1)}_{i,j} - S^{(2)}_{j,k} S^{(1)}_{i,j} + S^{(2)}_{j,k} S^{(1)}_{i,j} - S^{(2)}_{j,k} S^{(1)}_{i,j} + S^{(2)}_{j,k} S^{(1)}_{i,j} - S^{(2)}_{j,k} S^{(1)}_{i,j} + S^{(2)}_{j,k} S^{(1)}_{i,j} - S^{(2)}_{j,k} S^{(1)}_{i,j} + S^{(2)}_{j,k} S^{(1)}_{i,j} - S^{(2)}_{j,k} S^{(1)}_{i,j} + S^{(2)}_{j,k} S^{(1)}_{i,j} - S^{(2)}_{j,k} S^{(1)}_{i,j} + S^{(2)}_{j,k} S^{(1)}_{i,j} - S^{(2)}_{j,k} S^{(1)}_{i,j} + S^{(2)}_{j,k} S^{(1)}_{i,j} - S^{(2)}_{j,k} S^{(1)}_{i,j} + S^{(2)}_{j,k} S^{(1)}_{i,j} - S^{(2)}_{j,k} S^{(1)}_{i,j} + S^{(2)}_{j,k} S^{(1)}_{i,j} - S^{(2)}_{j,k} S^{(1)}_{i,j} + S^{(2)}_{j,k} S^{(1)}_{i,j} - S^{(2)}_{j,k} S^{(1)}_{i,j} + S^{(2)}_{j,k} S^{(1)}_{i,j} - S^{(2)}_{j,k} S^{(1)}_{i,j} + S^{(2)}_{j,k} S^{(1)}_{i,j} - S^{(2)}_{j,k} S^{(1)}_{i,j} + S^{(2)}_{j,k} S^{(1)}_{i,j} - S^{(2)}_{j,k} S^{(1)}_{i,j} + S^{(2)}_{j,k} S^{(1)}_{i,j} - S^{(2)}_{j,k} S^{(1)}_{i,j} + S^{(2)}_{j,k} S^{(1)}_{i,j} - S^{(2)}_{j,k} S^{(1)}_{i,j} + S^{(2)}_{j,k} S^{(1)}_{i,j} - S^{(2)}_{j,k} S^{(1)}_{i,j} + S^{(2)}_{j,k} S^{(1)}_{i,j} - S^{(2)}_{j,k} S^{(1)}_{i,j} + S^{(2)}_{j,k} S^{(1)}_{i,j} - S^{(2)}_{j,k} S^{(1)}_{i,j} + S^{(2)}_{j,k} S^{(1)}_{i,j} - S^{(2)}_{j,k} S^{(1)}_{i,j} + S^{(2)}_{j,k} S^{(1)}_{i,j} - S^{(2)}_{j,k} S^{(1)}_{i,j} + S^{(2)}_{j,k} S^{(1)}_{i,j} - S^{(2)}_{j,k} S^{(1)}_{i,j} + S^{(2)}_{j,k} S^{(1)}_{i,j}.
\]

The second order terms do not imply restrictions in the case of solutions of the form (6), since a functional relationship between the peculiar-acceleration and the peculiar-velocity is assumed (see BE93). The third order terms depend on the peculiar-velocity is assumed (see BE93). The third order terms depend on the peculiar-acceleration and the peculiar-velocity are listed up to the second order for the ansatz (3) in (BE93, APPENDIX). We conclude that interaction of the perturbations add transversality to the third order solution which is not covered by the longitudinal ansatz. The Wronskian \( q_{zz} q_z - q_z q_{zzz} \) is non-zero, because \( q_z(t) \) and \( q_{zz}(t) \) are linearly independent solutions. Thus, in order to fulfill the constraint equations for the longitudinal ansatz, we have to assure a functional relationship of the form \( \nabla_0 S^{(2)} = \mathcal{W}(\nabla_0 S^{(1)}) \) with arbitrary \( \mathcal{W} \). Since \( \Delta_0 S^{(2)} = 2II(S_{i,k}) \), we obtain the following equation for the functional \( \mathcal{W} \) (a prime, here, denotes derivative with respect to the argument \( \nabla_0 S^{(1)} \); without loss of generality we can set \( S^{(1)} = St_0 \), see APPENDIX B):

\[
\mathcal{W}'I(S_{i,k}) - 2II(S_{i,k})t_0 = 0 .
\]

Neglecting interaction terms altogether, the third order terms are not constrained. The constraints are removed by the transverse part derived in section 3.5.
APPENDIX B

In this APPENDIX a short discussion is given of the uniqueness of the solutions obtained. In particular, I shall demonstrate this in the case of the first order solution, and for the transverse part of the third order solution.

For the realization of the first order solution we replace the perturbation potential $S^{(1)}$ by the initial peculiar-velocity potential $S_{t0}$. With this replacement the first order part of the solution (24) reduces to the “Zel’dovich approximation”. This can be done without loss of generality as will be shown below:

The solution to equation (24f) can be written as

$$S^{(1)} = S_{t0} + \psi$$

for any function $\psi$ which obeys the Laplace equation $\Delta_0 \psi = 0$. Thus, if we preserve the boundary conditions used for the initial condition $S$ (here assumed to be periodic), then the only solution of the Laplace equation is $\psi = \ell(t)$. Without loss of generality we can set $\ell(t) = 0$. According to the form of the solution (24), this argument holds for any time. q.e.d.

For the transverse part of the third order solution a vector potential $\vec{S}^{(3c)}$ has been introduced (eq. (19a)). On this a gauge condition has been imposed (eq. (19b)), which enables to write the transverse third order solution (eqs. (24j,k,l)) in terms of three Poisson equations. This is especially useful for technical reasons to realize the solution (4.3). In the following it is shown that the condition (19b) does not restrict the generality of the solution:

Firstly, note that the vector $\vec{\Xi} = -\nabla_0 \times \vec{S}^{(3c)}$ remains transverse, i.e., $\nabla_0 \cdot \vec{\Xi} = 0$, if we add the gradient of an arbitrary scalar function $\Omega$ to the vector potential $\vec{S}^{(3c)}$. The divergence of this vector potential is yet unspecified. If we impose the gauge condition (19b), then we have to choose the gauge function $\Omega$ in such a way that

$$\Delta_0 \Omega = -\nabla_0 \cdot \vec{S}^{(3c)}.$$

This is always possible according to a theorem by Brelot on the existence of solutions of Poisson equations (see Friedman 1963). Imposing the gauge condition (19b) is sufficient to solve the equations (24j,k,l), since (19b) implies $\nabla_0(\nabla_0 \cdot \vec{S}^{(3c)}) = \vec{0}$ required for the derivation of these equations. q.e.d.

A comprehensive discussion of the problem of uniqueness in general perturbation solutions will be given in a separate paper (Ehlers & Buchert 1993).
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