PIATETSKI–SHAPIRO PRIMES IN THE INTERSECTION OF MULTIPLE BEATTY SEQUENCES

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Abstract. Suppose that \( \alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R} \). Let \( \alpha_1, \alpha_2 > 1 \) be irrational and of finite type such that \( 1, \alpha_1^{-1}, \alpha_2^{-1} \) are linearly independent over \( \mathbb{Q} \). Let \( c \) be a real number in the range \( 1 < c < 12/11 \). In this paper, it is proved that there exist infinitely many primes in the intersection of Beatty sequences \( B_{\alpha_1, \beta_1} = \lfloor \alpha_1 n + \beta_1 \rfloor \), \( B_{\alpha_2, \beta_2} = \lfloor \alpha_2 n + \beta_2 \rfloor \) and the Piatetski–Shapiro sequence \( \mathcal{N}^{(c)} = \lfloor n^c \rfloor \). Moreover, we also give a sketch proof of Piatetski–Shapiro primes in the intersection of multiple Beatty sequences.

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1. Introduction

The Piatetski–Shapiro sequences are sequences of the form
\[
\mathcal{N}^{(c)} := (\lfloor n^c \rfloor)_{n=1}^{\infty} \quad (c > 1, \ c \not\in \mathbb{N}).
\]

Such sequences have been named in honor of Piatetski–Shapiro, who [10], in 1953, proved that \( \mathcal{N}^{(c)} \) contains infinitely many primes provided that \( c \in (1, \frac{12}{11}) \). More precisely, for such \( c \) he showed that the counting function
\[
\pi^{(c)}(x) := \#\{\text{prime } p \leq x : p \in \mathcal{N}^{(c)}\}
\]
satisfies the asymptotic property
\[
\pi^{(c)}(x) \sim \frac{x^{1/c}}{\log x} \quad \text{as } x \to \infty.
\]

The range for \( c \) of the above asymptotic formula in which it is known that \( \mathcal{N}^{(c)} \) contains infinitely many primes has been enlarged many times over the years and is currently known to hold for all \( c \in (1, \frac{2817}{2426}) \) thanks to Rivat and Sargos [11]. Rivat and Wu [12] also showed that there exist infinitely many Piatetski–Shapiro primes for \( c \in (1, \frac{243}{205}) \) by showing a lower bound of \( \pi^{(c)}(x) \) with the expected order of magnitude. The same result is expected to hold for all larger non–integer values of \( c \). We remark that if \( c \in (0, 1) \) then \( \mathcal{N}^{(c)} \) contains all natural numbers, and hence all primes, particularly. The investigation of Piatetski–Shapiro primes is an approximation of the well–known conjecture that there exist infinitely many primes of the form \( n^2 + 1 \).

For fixed real numbers \( \alpha \) and \( \beta \), the associated non–homogeneous Beatty sequence is the sequence of integers defined by
\[
B_{\alpha, \beta} := ([\alpha n + \beta])_{n=1}^{\infty},
\]

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where \( \lfloor t \rfloor \) denotes the integer part of any \( t \in \mathbb{R} \). Such sequences are also called generalized arithmetic progressions. If \( \alpha \) is irrational, it follows from a classical exponential sum estimate of Vinogradov \([18]\) that \( B_{\alpha, \beta} \) contains infinitely many prime numbers; in fact, one has the asymptotic estimate

\[
\#\{ \text{prime } p \leq x : p \in B_{\alpha, \beta} \} \sim \alpha^{-1} \pi(x) \quad \text{as } x \to \infty,
\]

where \( \pi(x) \) is the prime counting function. Moreover, Harman \([5]\) investigated the intersection of multiple Beatty sequences and showed the following result:

Let \( \xi \) be a positive integer and real numbers \( \alpha_1, \ldots, \alpha_\xi \) each exceeding 1 be given such that

\[
1, \alpha_1^{-1}, \alpha_2^{-1}, \ldots, \alpha_\xi^{-1} \text{ are linearly independent over } \mathbb{Q}.
\]

Then for any real numbers \( \beta_1, \ldots, \beta_\xi \), the intersection of \( B_{\alpha_1, \beta_1}, B_{\alpha_2, \beta_2}, \ldots, B_{\alpha_\xi, \beta_\xi} \) contains infinitely many primes. Indeed, the number of such primes up to \( x \) equals

\[
\frac{x}{\alpha_1 \cdots \alpha_\xi \log x} (1 + o(1)).
\]

It has been established in \([4]\) that for \( c \in (1, \frac{14}{13}) \), there exist infinitely many primes in the intersection of a Beatty sequence and a Piatetski–Shapiro sequence with an asymptotic formula. Motivated by Harman’s proof related to the intersection of multiple Beatty sequences, we show a corresponding result that there exist infinitely many primes in the intersection of multiple Beatty sequences and a Piatetski–Shapiro sequence. In this paper, we generalize and improve the previous theorem in \([4]\) and establish the following theorem.

**Theorem 1.1.** Suppose that \( \alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R} \). Let \( \alpha_1, \alpha_2 > 1 \) be irrational and of finite type such that

\[
1, \alpha_1^{-1}, \alpha_2^{-1} \text{ are linearly independent over } \mathbb{Q}.
\]

For \( c \in (1, \frac{12}{11}) \), there exist infinitely many primes in the intersection of Beatty sequences \( B_{\alpha_1, \beta_1}, B_{\alpha_2, \beta_2} \) and the Piatetski–Shapiro sequence \( \mathcal{N}^{(c)} \). Moreover, the counting function

\[
\pi_{\alpha_1, \beta_1, \alpha_2, \beta_2}^{(c)}(x) := \# \{ \text{prime } p \leq x : p \in B_{\alpha_1, \beta_1} \cap B_{\alpha_2, \beta_2} \cap \mathcal{N}^{(c)} \}
\]

satisfies

\[
\pi_{\alpha_1, \beta_1, \alpha_2, \beta_2}^{(c)}(x) = \frac{x^{1/c}}{\alpha_1 \alpha_2 \log x} + O \left( \frac{x^{1/c}}{\log^2 x} \right),
\]

where the implied constant depends only on \( \alpha_1, \alpha_2 \) and \( c \).

By a similar method, we can also give a stronger version of the theorem in \([4]\).

**Theorem 1.2.** Suppose that \( \alpha, \beta \in \mathbb{R} \). Let \( \alpha > 1 \) be irrational and of finite type. For \( c \in (1, \frac{12}{11}) \), there exist infinitely many primes in the intersection of the Beatty sequence \( B_{\alpha, \beta} \) and the Piatetski–Shapiro sequence \( \mathcal{N}^{(c)} \). Moreover, the counting function

\[
\pi_{\alpha, \beta}^{(c)}(x) := \# \{ \text{prime } p \leq x : p \in B_{\alpha, \beta} \cap \mathcal{N}^{(c)} \}
\]
satisfies
\[ \pi_{\alpha,\beta}^{(c)}(x) = \frac{x^{1/c}}{\alpha \log x} + O\left(\frac{x^{1/c}}{\log^2 x}\right), \]
where the implied constant depends only on \( \alpha \) and \( c \).

In the end, we give the corresponding result to Harman’s theorem in our case, which proves that there exist infinitely many primes in the intersection of a Piatetski–Shapiro sequence and multiple Beatty sequences.

**Theorem 1.3.** Suppose that \( \xi \) is a positive integer, and \( \alpha_1, \ldots, \alpha_\xi, \beta_1, \ldots, \beta_\xi \in \mathbb{R} \). Let \( \alpha_1, \ldots, \alpha_\xi > 1 \) be irrational and of finite type such that
\[ 1, \alpha_1^{-1}, \ldots, \alpha_\xi^{-1} \]
are linearly independent over \( \mathbb{Q} \).

For \( c \in (1, \frac{12}{11}) \), there are infinitely many primes in the intersection of Beatty sequences \( B_{\alpha_1,\beta_1}, \ldots, B_{\alpha_\xi,\beta_\xi} \) and the Piatetski–Shapiro sequence \( N^{(c)} \). Moreover, the counting function
\[ \pi_{\alpha_1,\beta_1; \ldots; \alpha_\xi,\beta_\xi}^{(c)}(x) := \# \{ \text{prime } p \leq x : p \in B_{\alpha_1,\beta_1} \cap \cdots \cap B_{\alpha_\xi,\beta_\xi} \cap N^{(c)} \} \]
satisfies
\[ \pi_{\alpha_1,\beta_1; \ldots; \alpha_\xi,\beta_\xi}^{(c)}(x) = \frac{x^{1/c}}{\alpha_1 \cdots \alpha_\xi \log x} + O\left(\frac{x^{1/c}}{\log^2 x}\right), \]
where the implied constant depends only on \( \alpha_1, \ldots, \alpha_\xi \) and \( c \).

### 2. Preliminaries

#### 2.1. Notation.

We denote by \([t]\) and \(\{t\}\) the integral part and the fractional part of \(t\), respectively. As is customary, we put
\[ e(t) := e^{2\pi it} \quad \text{and} \quad \{t\} := t - [t]. \]

Throughout the paper, we make considerable use of the sawtooth function defined by
\[ \psi(t) := t - [t] - \frac{1}{2} = \{t\} - \frac{1}{2}. \]

The notation \(\|t\|\) is used to denote the distance from the real number \(t\) to the nearest integer; that is,
\[ \|t\| := \min_{n \in \mathbb{Z}} |t - n|. \]

Let \(\mathbb{P}\) denote the set of primes in \(\mathbb{N}\). The letter \(p\) always denotes a prime. For a Beatty sequence \(\lceil an + \beta \rceil_{n=1}^\infty\), we denote \(\omega := \alpha^{-1}\). We represent \(\gamma := c^{-1}\) for the Piatetski–Shapiro sequence \(\lceil n^\gamma \rceil_{n=1}^\infty\). We use notation of the form \(m \sim M\) as an abbreviation for \(M < m \leq 2M\).

Throughout the paper, \(\varepsilon\) always denotes an arbitrarily small positive constant, which may not be the same at different occurrences; the implied constants in symbols \(O, \ll \text{ and } \gg\) may depend (where obvious) on the parameters \(\alpha_1, \ldots, \alpha_\xi, \beta_1, \ldots, \beta_\xi, c, \varepsilon\) but are absolute otherwise. For given functions \(F\) and \(G\), the notations \(F \ll G\), \(G \gg F\) and \(F = O(G)\) are all equivalent to the statement that the inequality \(|F| \leq C|G|\) holds with some constant \(C > 0\).
2.2. Type of an irrational number. For any irrational number $\alpha$, we define its type $\tau = \tau(\alpha)$ by the following definition

$$\tau := \sup \{ t \in \mathbb{R} : \liminf_{n \to \infty} n^t \|\alpha n\| = 0 \}.$$  

Using Dirichlet’s approximation theorem, one can see that $\tau \geq 1$ for every irrational number $\alpha$. Thanks to the work of Khinchin [8] and Roth [13, 14], it is known that $\tau = 1$ for almost all real numbers, in the sense of the Lebesgue measure, and for all irrational algebraic numbers, respectively. Moreover, if $\alpha$ is an irrational number of type $\tau < \infty$, then so are $\alpha - 1$ and $n\alpha - 1$ for all integer $n \geq 1$.

2.3. Technical lemmas. We need the following well–known approximation of Vaaler [17].

**Lemma 2.1.** For any $H \geq 1$, there exist numbers $a_h, b_h$ such that

$$\left| \psi(t) - \sum_{0 < |h| \leq H} a_h e(th) \right| \leq \sum_{|h| \leq H} b_h e(th), \quad a_h \ll \frac{1}{|h|}, \quad b_h \ll \frac{1}{H}.$$  

**Lemma 2.2.** For an arithmetic function $g$ and $N' \sim N$, we have

$$\sum_{N < p \leq N'} g(p) \ll \frac{1}{\log N} \max_{N < n \leq 2N} \left| \sum_{N < n \leq N_1} \Lambda(n) g(n) \right| + N^{1/2}.$$  

**Proof.** See the argument on page 48 of [3].

**Lemma 2.3.** Suppose that

$$\alpha = \frac{a}{q} + \frac{\theta}{q^2},$$  

with $(a, q) = 1, q \geq 1, |\theta| \leq 1$. Then there holds

$$\sum_{m \leq N} \Lambda(m) e(m\alpha) \ll (Nq^{-1/2} + N^{4/5} + N^{1/2}q^{1/2})(\log N)^4.$$  

**Proof.** See Chapter 25 of Davenport [2].

**Lemma 2.4.** Suppose that $a$ is a fixed irrational number of finite type $\tau < \infty$ and $h \geq 1, m$ are integers. Then we have

$$\sum_{m \leq M} \Lambda(m) e(ahm) \ll h^{1/2}M^{1-1/(2\tau)+\varepsilon} + M^{1-\varepsilon}.$$  

**Proof.** For any sufficiently small $\varepsilon > 0$, we set $q = \tau + \varepsilon$. Since $a$ is of type $\tau$, there exists some constant $c > 0$ such that

$$\|an\| > cn^{-\varepsilon}, \quad n \geq 1.$$  

(2.1)

For given $h$ with $0 < h \leq H$, let $b/d$ be the convergent in the continued fraction expansion of $ah$ which has the largest denominator $d$ not exceeding $M^{1-n}$ for a
sufficiently small positive number $\eta$. Then we derive that
\[
\left| ah - \frac{b}{d} \right| \leq \frac{1}{dM^{1-\eta}} \leq \frac{1}{d^2},
\] (2.2)
which combined with (2.1) yields
\[
M^{-1+\eta} \geq |ahd - b| \geq \|ahd\| > c(hd)^{-\epsilon}.
\]
Taking $C_0 := c^{1/\epsilon}$, we obtain
\[
d > C_0 h^{-1} M^{1/\epsilon - \eta/\epsilon}. \tag{2.3}
\]
Combining (2.2) and (2.3), applying Lemma 2.3 and the fact that $d \leq M^{1-\eta}$, we deduce that
\[
\sum_{m \leq M} \Lambda(m)e(ahm) \ll (Md^{-1/2} + M^{4/5} + M^{1/2}d^{1/2})(\log M)^4
\]
\[
\ll (h^{1/2}M^{1-1/(2\theta)} + h^{1/2} + M^{4/5} + M^{1-\eta/2})(\log M)^4
\]
\[
\ll h^{1/2}M^{1-1/(2\tau)+\epsilon} + M^{1-\epsilon}.
\]
This completes the proof of Lemma 2.4. \qed

**Lemma 2.5.** Suppose that $\xi$ is an integer, and $\omega_1, \omega_2, \ldots, \omega_\xi$ are irrational numbers of finite type such that
\[
1, \omega_1, \omega_2, \ldots, \omega_\xi
\]
are linearly independent over $\mathbb{Q}$. Then for any subset $\{i_1, i_2, \ldots, i_s\} \subset \{1, 2, \ldots, \xi\}$, the combination of $h_{i_1}\omega_{i_1} + \cdots + h_{i_s}\omega_{i_s}$ with $h_{i_1}, \ldots, h_{i_s} \in \mathbb{N}^s$ is of finite type.

**Proof.** For any $n \geq 1$, we have
\[
\|n(h_{i_1}\omega_{i_1} + \cdots + h_{i_s}\omega_{i_s})\| \leq \|nh_{i_1}\omega_{i_1}\| + \cdots + \|nh_{i_s}\omega_{i_s}\|.
\]
By the definition of the type of irrational number, we deduce that
\[
\tau(h_{i_1}\omega_{i_1} + \cdots + h_{i_s}\omega_{i_s}) \leq \tau(h_{i_1}\omega_{i_1}) + \cdots + \tau(h_{i_s}\omega_{i_s}),
\]
which implies the finite type of $h_{i_1}\omega_{i_1} + \cdots + h_{i_s}\omega_{i_s}$. \qed

The following lemma gives a characterization of the numbers in the Beatty sequence $B_{\alpha,\beta}$.

**Lemma 2.6.** A natural number $m$ has the form $\lfloor \alpha n + \beta \rfloor$ if and only if $\mathcal{X}_{\alpha,\beta}(m) = 1$, where $\mathcal{X}_{\alpha,\beta}(m) := [-\alpha^{-1}(m - \beta)] - [-\alpha^{-1}(m + 1 - \beta)]$.

**Proof.** Note that an integer $m$ has the form $m = \lfloor \alpha n + \beta \rfloor$ for some integer $n$ if and only if
\[
\frac{m - \beta}{\alpha} \leq n < \frac{m - \beta + 1}{\alpha}.
\]

Finally, we use the following lemma, which provides a characterization of the numbers that occur in the Piatetski–Shapiro sequence $\mathcal{N}^{(\sigma)}$. 

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**Lemma 2.7.** A natural number $m$ has the form $\lfloor n^\varepsilon \rfloor$ if and only if $X^{(c)}(m) = 1$, where $X^{(c)}(m) := \lfloor -m\gamma \rfloor - \lfloor -(m+1)\gamma \rfloor$. Moreover,
\[
X^{(c)}(m) = \gamma m^{\varepsilon - 1} + \psi(-(m+1)\gamma) - \psi(-m\gamma) + O(m^{\varepsilon - 2}).
\]

**Proof.** The proof of Lemma 2.7 is similar to that of Lemma 2.6, so we omit the details herein. \qed

**Lemma 2.8.** For $1 < c < \frac{2817}{2426}$, there holds
\[
\pi^{(c)}(x) = \sum_{p \leq x} X^{(c)}(p) = \frac{x^\gamma}{\log x} + O\left(\frac{x^\gamma}{\log^2 x}\right).
\] (2.4)

**Proof.** See Theorem 1 of Rivat and Sargos \cite{11}. \qed

2.4. **Upper bound estimate of exponential sums.** We begin with the decomposition of the von Mangoldt function by Heath–Brown.

**Lemma 2.9.** Let $z \geq 1$ and $k \geq 1$. Then, for any $n \leq 2z^k$, there holds
\[
\Lambda(n) = \sum_{j=1}^{k} (-1)^{j-1} \binom{k}{j} \sum_{n_1n_2\cdots n_j=n} \sum_{n_{j+1},\ldots,n_{2j} \leq z} (\log n_1)\mu(n_{j+1})\cdots\mu(n_{2j}).
\]

**Proof.** See the arguments on pp. 1366–1367 of Heath–Brown \cite{6}. \qed

**Lemma 2.10.** Suppose that
\[
L(H) = \sum_{i=1}^{m} A_i H^{a_i} + \sum_{j=1}^{n} B_j H^{-b_j},
\]
where $A_i$, $B_j$, $a_i$, and $b_j$ are positive. Assume further that $H_1 \leq H_2$. Then there exists some $\mathcal{H}$ with $H_1 \leq \mathcal{H} \leq H_2$ and
\[
L(\mathcal{H}) \ll \sum_{i=1}^{m} A_i H_1^{a_i} + \sum_{j=1}^{n} B_j H_2^{-b_j} + \sum_{i=1}^{m} \sum_{j=1}^{n} (A_i^{b_j} B_j^{a_i})^{1/(a_i+b_j)}.
\]
The implied constant depends only on $m$ and $n$.

**Proof.** See Lemma 3 of Srinivasan \cite{16}. \qed

For real numbers $m_1$ and $m_2$, the sum of the form
\[
\sum_{k \sim K} \sum_{\ell \sim L, KL > x} a_k b_\ell e(h(k\ell)^\gamma + m_1 k\ell + m_2)
\]
with $|a_k| \ll x^\varepsilon$, $|b_\ell| \ll x^\varepsilon$ for every fixed $\varepsilon > 0$ is usually called a “Type I” sum, denoted by $S_I(K,L)$, if $b_\ell = 1$ or $b_\ell = \log \ell$; otherwise it is called a “Type II” sum, denoted by $S_{II}(K,L)$.

**Lemma 2.11.** Suppose that $f(x) : [a,b] \to \mathbb{R}$ has continuous derivatives of arbitrary order on $[a,b]$, where $1 \leq a < b \leq 2a$. Suppose further that
\[
|f^{(j)}(x)| \ll \lambda_j, \quad j \geq 1, \quad x \in [a,b].
\]
Then we have
\[ \sum_{a<n\leq b} e(f(n)) \ll a\lambda_2^{1/2} + \lambda_2^{-1/2}, \quad (2.5) \]
and
\[ \sum_{a<n\leq b} e(f(n)) \ll a\lambda_3^{1/6} + \lambda_3^{-1/3}. \quad (2.6) \]

**Proof.** For (2.5), one can use Corollary 8.13 of Iwaniec and Kowalski [7], or Theorem 5 of Chapter 1 in Karatsuba [9]. For (2.6), one can use Corollary 4.2 of Sargos [15]. \(\square\)

**Lemma 2.12.** Suppose that \(|a_k| \ll 1, b_\ell = 1\) or \(\log \ell, KL \ll x\). Then if \(K \ll x^{1/2}\), there holds
\[ S_f(K, L) \ll |h|^{1/6}x^{7/6 + 1/3 + \epsilon} + |h|^{-1/3}x^{1/6 + 1/3 + \epsilon}. \]

**Proof.** Set \(f(\ell) = h(k\ell)\gamma + m_1k\ell + m_2\). It is easy to see that
\[ f'''(\ell) = \gamma(\gamma - 1)(\gamma - 2)hk^3\ell^{-3} \ll |h|K^\gamma L^{-3}. \]
If \(K \ll x^{1/2}\), then by (2.6) of Lemma 2.11, we deduce that
\[ x^{-\epsilon} \cdot S_f(K, L) \ll \sum_{k \sim K} \left| \sum_{\ell \sim L} e(f(\ell)) \right| \]
\[ \ll \sum_{k \sim K} \left( L \left( |h|K^\gamma L^{-3} \right)^{1/6} + (|h|K^\gamma L^{-3})^{-1/3} \right) \]
\[ \ll |h|^{1/6}x^{7/6 + 1/3}K^{1/2} + |h|^{-1/3}x^{1/6 + 1/3} \]
\[ \ll |h|^{1/6}x^{7/6 + 3/4} + |h|^{-1/3}x^{1/6 + 1/3}, \]
which completes the proof of Lemma 2.12. \(\square\)

**Lemma 2.13.** Suppose that \(|a_k| \ll 1, b_\ell | \ll 1\) with \(k \sim K, \ell \sim L\) and \(KL \ll x\). Then if \(x^{1/2} \ll K \ll x^{19/25}\), there holds
\[ S_{II}(K, L) \ll |h|^{1/4}x^{7/4 + 5/8} + |h|^{-1/4}x^{1/4 + 1/8} + x^{22/25} + |h|^{1/6}x^{7/6 + 3/4}. \]

**Proof.** Let \(Q\), which satisfies \(1 < Q < L\), be a parameter which will be chosen later. By the Weyl–van der Corput inequality (see Lemma 2.5 of Graham and Kolesnik [3]), we have
\[ \left| \sum_{k \sim K} \sum_{\ell \sim L} a_k b_\ell e(h(k\ell)\gamma + m_1k\ell + m_2) \right|^2 \]
\[ \ll K^2L^2Q^{-1} + KLQ^{-1} \sum_{\ell \sim L} \sum_{0 < |q| \leq Q} |\mathcal{G}(q; \ell)|, \quad (2.7) \]
where
\[ \mathcal{G}(q; \ell) = \sum_{k \in I(q; \ell)} e(g(k)) \]
with
\[ g(k) = hk^\gamma(\ell^\gamma - (\ell + q)^\gamma) - m_1kq. \]
It is easy to see that
\[ g''(k) = \gamma(\gamma - 1)hk^{\gamma - 2}(\ell' - (\ell + q)^\gamma) \times |h|K^{\gamma - 2}L^{\gamma - 1}|q|. \]
By (2.5) of Lemma 2.11, we have
\[ \mathcal{G}(q; \ell) \ll K\left(|h|K^{\gamma - 2}L^{\gamma - 1}|q|\right)^{1/2} + \left(|h|K^{\gamma - 2}L^{\gamma - 1}|q|\right)^{-1/2}. \quad (2.8) \]
Putting (2.8) into (2.7), we derive that
\[
\left| \sum_{k \sim K} \sum_{\ell \sim L} a_k b_{\ell} e(h(k\ell)^\gamma + m_1 k\ell + m_2) \right|^2 \\
\ll K^2L^2Q^{-1} + KLQ^{-1} \\
\times \sum_{\ell \sim L} \sum_{0 < |q| \leq Q \atop K \ll x} \left(|h|^{1/2}K^{\gamma/2}L^{\gamma/2 - 1/2}|q|^{1/2} + |h|^{-1/2}K^{1-\gamma/2}L^{1/2-\gamma/2}|q|^{-1/2}\right) \\
\ll K^2L^2Q^{-1} + KLQ^{-1}(|h|^{1/2}K^{\gamma/2}L^{\gamma/2+1/2}Q^{3/2} \\
+ |h|^{-1/2}K^{1-\gamma/2}L^{3/2-\gamma/2}Q^{1/2}) \\
\ll K^2L^2Q^{-1} + |h|^{1/2}K^{1+\gamma/2}L^{\gamma/2+3/2}Q^{1/2} + |h|^{-1/2}K^{2-\gamma/2}L^{5/2-\gamma/2}Q^{-1/2}.
\]

By noting that \(1 \leq Q \leq L\), it follows from Lemma 2.10 that there exists an optimal \(Q\) such that
\[
\left| \sum_{k \sim K} \sum_{\ell \sim L} a_k b_{\ell} e(h(k\ell)^\gamma + m_1 k\ell + m_2) \right|^2 \\
\ll |h|^{1/2}x^{\gamma/2+3/2}K^{-1/2} + Kx + |h|^{-1/2}x^{2-\gamma/2} + |h|^{1/3}x^{\gamma/3+5/3}K^{-1/3} + K^{-1/2}x^2,
\]
which implies
\[
\left| \sum_{k \sim K} \sum_{\ell \sim L} a_k b_{\ell} e(h(k\ell)^\gamma + m_1 k\ell + m_2) \right| \\
\ll |h|^{1/4}x^{\gamma/4+3/4}K^{-1/4} + |h|^{-1/4}x^{1-\gamma/4} + K^{1/2}x^{1/2} \\
+ |h|^{1/6}x^{\gamma/6+5/6}K^{-1/6} + K^{-1/4}x \\
\ll |h|^{1/4}x^{\gamma/4+5/8} + |h|^{-1/4}x^{1-\gamma/4} + x^{22/25} + |h|^{1/6}x^{\gamma/6+3/4},
\]
provided that \(x^{1/2} \ll K \ll x^{19/25}\), which completes the proof of Lemma 2.13.

**Lemma 2.14.** For real numbers \(m_1, m_2\) and \(x/2 < v \leq x\), we have
\[
\max_{x/2 < v \leq x} \left| \sum_{x/2 < n \leq v} \Lambda(n)e(hn^\gamma + m_1 n + m_2) \right| \\
\ll x^{\varepsilon}\left(|h|^{1/6}x^{\gamma/6+3/4} + |h|^{-1/3}x^{1-\gamma/3} + |h|^{1/4}x^{\gamma/4+5/8} + |h|^{-1/4}x^{1-\gamma/4} + x^{22/25}\right).
\]
Proof. By Heath–Brown’s identity, i.e. Lemma 2.9, with $k = 3$, one can see that the exponential sum
\[
\max_{x/2 < t \leq x} \left| \sum_{x/2 < n \leq t} \Lambda(n) e(hn^\gamma + m_1 n + m_2) \right|
\]
can be written as linear combination of $O(\log^6 x)$ sums, each of which is of the form
\[
T^* := \sum_{n_1 \sim N_1} \cdots \sum_{n_6 \sim N_6} (\log n_1) \mu(n_4) \mu(n_5) \mu(n_6)
\times e(h(n_1 n_2 \cdots n_6)^\gamma + (n_1 n_2 \cdots n_6)m_1 + m_2), \tag{2.9}
\]
where $N_1 N_2 \cdots N_6 \ll x$; $2N_i \leq (2x)^{1/3}$, $i = 4, 5, 6$ and some $n_i$ may only take value 1. Therefore, it is sufficient for us to give upper bound estimate for each $T^*$ defined as in (2.9). Next, we will consider three cases.

Case 1. If there exists an $N_j$ such that $N_j \geq x^{1/2}$, then we must have $j \leq 3$ for the fact that $N_i \ll x^{1/3}$ with $i = 4, 5, 6$. Let
\[
k = \prod_{1 \leq i \leq 6 \atop i \neq j} n_i, \quad \ell = n_j, \quad K = \prod_{1 \leq i \leq 6 \atop i \neq j} N_i, \quad L = N_j.
\]
In this case, we can see that $T^*$ is a sum of “Type I” satisfying $K \ll x^{1/2}$. By Lemma 2.12, we have
\[
x^{-\varepsilon} \cdot T^* \ll |h|^{1/6} x^{\gamma/6 + 3/4} + |h|^{-1/3} x^{1-\gamma/3}.
\]

Case 2. If there exists an $N_j$ such that $x^{6/25} \leq N_j < x^{1/2}$, then we take
\[
k = \prod_{1 \leq i \leq 6 \atop i \neq j} n_i, \quad \ell = n_j, \quad K = \prod_{1 \leq i \leq 6 \atop i \neq j} N_i, \quad L = N_j.
\]
Thus, $T^*$ is a sum of “Type II” satisfying $x^{1/2} \ll K \ll x^{19/25}$. By Lemma 2.13, we have
\[
x^{-\varepsilon} \cdot T^* \ll |h|^{1/4} x^{\gamma/4 + 5/8} + |h|^{-1/4} x^{1-\gamma/4} + x^{22/25} + |h|^{1/6} x^{\gamma/6 + 3/4}.
\]

Case 3. If $N_j < x^{6/25}$ ($j = 1, 2, 3, 4, 5, 6$), without loss of generality, we assume that $N_1 \geq N_2 \cdots \geq N_6$. Let $r$ denote the natural number $j$ such that
\[
N_1 N_2 \cdots N_{j-1} < x^{6/25}, \quad N_1 N_2 \cdots N_j \geq x^{6/25}.
\]
Since $N_1 < x^{6/25}$ and $N_6 < x^{6/25}$, then $2 \leq r \leq 5$. Thus, we have
\[
x^{6/25} \leq N_1 N_2 \cdots N_r = (N_1 \cdots N_{r-1}) \cdot N_r < x^{6/25} \cdot x^{25/25} < x^{1/2}.
\]
Let
\[
k = \prod_{i=r+1}^{6} n_i, \quad \ell = \prod_{i=1}^{r} n_i, \quad K = \prod_{i=r+1}^{6} N_i, \quad L = \prod_{i=1}^{r} N_i.
\]
At this time, $T^*$ is a sum of “Type II” satisfying $x^{1/2} \ll K \ll x^{19/25}$. By Lemma 2.13, we have
\[
x^{-\varepsilon} \cdot T^* \ll |h|^{1/4} x^{\gamma/4 + 5/8} + |h|^{-1/4} x^{1-\gamma/4} + x^{22/25} + |h|^{1/6} x^{\gamma/6 + 3/4}.
\]
Combining the above three cases, we derive that
\[ x^{-\varepsilon} \cdot \mathcal{T}^* \ll |h|^{1/6} x^{\gamma/6+3/4} + |h|^{-1/3} x^{1-\gamma/3} \]
\[ + |h|^{1/4} x^{\gamma/4+5/8} + |h|^{-1/4} x^{1-\gamma/4} + x^{22/25}, \]
which completes the proof of this lemma. \(\square\)

3. Proof of Theorem 1.1

3.1. Initial construction and the main term. We start by two Beatty sequences
\[ \mathcal{B}_{\alpha_1, \beta_1} := [\alpha_1 n + \beta_1] \quad \text{and} \quad \mathcal{B}_{\alpha_2, \beta_2} := [\alpha_2 n + \beta_2]. \]
We define that \( \omega_1 := \alpha_1^{-1} \) and \( \omega_2 := \alpha_2^{-1} \). By the definition of \( \pi_{\alpha_1, \beta_1; \alpha_2, \beta_2}^{(c)}(x) \), we obtain that
\[ \pi_{\alpha_1, \beta_1; \alpha_2, \beta_2}^{(c)}(x) = \sum_{p \leq x} \mathcal{X}_{\alpha_1, \beta_1}(p) \mathcal{X}_{\alpha_2, \beta_2}(p) \mathcal{X}^{(c)}(p), \quad (3.1) \]
where by Lemma 2.6
\[ \mathcal{X}_{\alpha_1, \beta_1}(p) := [-\omega_i(p - \beta_i)] - [-\omega_i(p + 1 - \beta_i)], \quad (i = 1, 2), \]
and by Lemma 2.7
\[ \mathcal{X}^{(c)}(p) := [-p^\gamma] - [(p + 1)^\gamma]. \]
Moreover, we see that
\[ \mathcal{X}_{\alpha_1, \beta_1}(p) = \omega_i + \psi(-\omega_i(p + 1 - \beta_i)) - \psi(-\omega_i(p - \beta_i)), \quad (i = 1, 2), \quad (3.2) \]
and
\[ \mathcal{X}^{(c)}(p) = \gamma p^{\gamma - 1} + O(p^{\gamma - 2}) + \psi(-(p + 1)^\gamma) - \psi(-p^\gamma). \quad (3.3) \]
Combining (3.1), (3.2) and (3.3), we obtain
\[ \pi_{\alpha_1, \beta_1; \alpha_2, \beta_2}^{(c)}(x) = S_1 + S_2 + S_3 + S_4 + S_5 + S_6 + S_7, \]
where
\[ S_1 := \sum_{p \leq x} \omega_1 \omega_2 \mathcal{X}^{(c)}(p), \]
\[ S_2 := \sum_{p \leq x} \omega_1 \left( \gamma p^{\gamma - 1} + O(p^{\gamma - 2}) \right) \]
\[ \times \left( \psi(-\omega_2(p + 1 - \beta_2)) - \psi(-\omega_2(p - \beta_2)) \right), \]
\[ S_3 := \sum_{p \leq x} \omega_2 \left( \gamma p^{\gamma - 1} + O(p^{\gamma - 2}) \right) \]
\[ \times \left( \psi(-\omega_1(p + 1 - \beta_1)) - \psi(-\omega_1(p - \beta_1)) \right), \]
\[ S_4 := \sum_{p \leq x} \left( \gamma p^{\gamma - 1} + O(p^{\gamma - 2}) \right) \left( \psi(-\omega_2(p + 1 - \beta_2)) - \psi(-\omega_2(p - \beta_2)) \right) \]
\[ \times \left( \psi(-\omega_1(p + 1 - \beta_1)) - \psi(-\omega_1(p - \beta_1)) \right), \]
\[ S_5 := \sum_{p \leq x} \omega_1 \left( -(p + 1)^\gamma - p^\gamma \right) \]
\[ \times \left( \psi(-\omega_2(p + 1 - \beta_2)) - \psi(-\omega_2(p - \beta_2)) \right), \]
\( S_6 := \sum_{p \leq x} \omega_2(\psi(-(p + 1)\gamma) - \psi(-p\gamma)) \times (\psi(-\omega_1(p + 1 - \beta_1)) - \psi(-\omega_1(p - \beta_1))) \)

\( S_7 := \sum_{p \leq x} (\psi(-(p + 1)\gamma) - \psi(-p\gamma))(\psi(-\omega_2(p + 1 - \beta_2)) - \psi(-\omega_2(p - \beta_2))) \times (\psi(-\omega_1(p + 1 - \beta_1)) - \psi(-\omega_1(p - \beta_1))) \).

By (2.4), we derive an asymptotic formula for \( S_1 \) with \( c \in (1, \frac{2817}{2426}) \), which is

\[ S_1 = \frac{x^\gamma}{\alpha_1 \alpha_2 \log x} + O\left(\frac{x^\gamma}{\log^2 x}\right). \]

Next, for \( i = 2, 3, 4, 5, 6, 7 \), we shall prove that

\[ S_i \ll x^\gamma \log^{-2} x. \]

Applying Vaaler’s approximation, i.e. Lemma 2.1, and taking

\[ H_1 = H_2 := x^\varepsilon \quad \text{and} \quad H_3 := x^{1-\gamma+\varepsilon} \]

with a sufficiently small positive number \( \varepsilon \), we have that

\[ \psi(-\omega_1(p + 1 - \beta_1)) - \psi(-\omega_1(p - \beta_1)) \]

\[ = \sum_{0 < |h_1| \leq H_1} a_{h_1} \left( e(\omega_1 h_1(p + 1 - \beta_1)) - e(\omega_1 h_1(p - \beta_1)) \right) + O\left(\sum_{|h_1| \leq H_1} b_{h_1} \left( e(\omega_1 h_1(p + 1 - \beta_1)) + e(\omega_1 h_1(p - \beta_1)) \right)\right), \quad (3.4) \]

and

\[ \psi(-\omega_2(p + 1 - \beta_2)) - \psi(-\omega_2(p - \beta_2)) \]

\[ = \sum_{0 < |h_2| \leq H_2} a_{h_2} \left( e(\omega_2 h_2(p + 1 - \beta_2)) - e(\omega_2 h_2(p - \beta_2)) \right) + O\left(\sum_{|h_2| \leq H_2} b_{h_2} \left( e(\omega_2 h_2(p + 1 - \beta_2)) + e(\omega_2 h_2(p - \beta_2)) \right)\right), \quad (3.5) \]

and

\[ \psi(-(p + 1)\gamma) - \psi(-p\gamma) \]

\[ = \sum_{0 < |h_3| \leq H_3} a_{h_3} \left( e(h_3(p + 1)\gamma) - e(h_3 p\gamma) \right) + O\left(\sum_{|h_3| \leq H_3} b_{h_3} \left( e(h_3(p + 1)\gamma) + e(h_3 p\gamma) \right)\right). \quad (3.6) \]

We mention that for \( j = 1, 2, 3 \), there holds

\[ a_{h_j} \ll |h_j|^{-1} \quad \text{and} \quad b_{h_j} \ll H_j^{-1}. \]
3.2. Upper bounds of $S_2$ and $S_3$. In order to prove that $S_2 \ll x^\gamma \log^{-2} x$, we write $S_2 = S_{21} + O(S_{22})$, where
\[
S_{21} := \sum_{p \leq x} \omega_1 \gamma p^{\gamma - 1} (\psi(-\omega_2(p + 1 - \beta_2)) - \psi(-\omega_2(p - \beta_2)))
\]
and
\[
S_{22} := \sum_{p \leq x} \omega_1 \gamma p^{\gamma - 2} (\psi(-\omega_2(p + 1 - \beta_2)) - \psi(-\omega_2(p - \beta_2))).
\]
By (3.5), we obtain that $S_{21} = S_{23} + O(S_{24})$, where
\[
S_{23} := \sum_{p \leq x} \omega_1 \gamma p^{\gamma - 1} \sum_{0 < |h_2| \leq H_2} a_{h_2} (e(\omega_2 h_2(p + 1 - \beta_2)) - e(\omega_2 h_2(p - \beta_2)))
\]
and
\[
S_{24} := \sum_{p \leq x} \omega_1 \gamma p^{\gamma - 1} \sum_{|h_2| \leq H_2} b_{h_2} (e(\omega_2 h_2(p + 1 - \beta_2)) + e(\omega_2 h_2(p - \beta_2))).
\]

3.2.1. Estimation of $S_{23}$. By Lemma 2.2 and a splitting argument, it suffices to prove that
\[
\sum_{x/2 < u \leq x} \sum_{0 < |h_2| \leq H_2} a_{h_2} n^{\gamma - 1} \Lambda(n) \times (e(\omega_2 h_2(n + 1 - \beta_2)) - e(\omega_2 h_2(n - \beta_2))) \ll x^{\gamma - \varepsilon}. \tag{3.7}
\]
Let
\[
\theta_{h_2} := e(\omega_2 h_2) - 1. \tag{3.8}
\]
It follows from partial summation and the trivial estimate $\theta_{h_2} \ll 1$ that the left–hand side of (3.7) is
\[
\ll x^{\gamma - 1} \max_{x/2 < u \leq x} \sum_{0 < |h_2| \leq H_2} h_2^{-1} \left| \sum_{x/2 < n \leq u} \Lambda(n)e(\omega_2 h_2 n) \right|.
\]
Hence it is sufficient to prove that
\[
\max_{x/2 < u \leq x} \sum_{0 < |h_2| \leq H_2} h_2^{-1} \left| \sum_{x/2 < n \leq u} \Lambda(n)e(\omega_2 h_2 n) \right| \ll x^{1 - \varepsilon}. \tag{3.9}
\]
By Lemma 2.4, the left–hand side of (3.9) is
\[
\ll \sum_{0 < |h_2| \leq H_2} h_2^{-1/2} x^{1 - 1/(2r) + \varepsilon} \ll x^{1 - \varepsilon}.
\]
\[
\ll \sum_{0 < |h_2| \leq H_2} h_2^{-1/2} x^{1 - 1/(2r) + \varepsilon} \ll x^{1 - 1/(2r) + \varepsilon} \ll x^{1 - \varepsilon} \ll x^{1 - \varepsilon}.
\]
3.2.2. Estimation of $S_{24}$ and conclusions. The contribution of $S_{24}$ from $h_2 = 0$ is
\[ \ll \sum_{p \leq x} p^{\gamma - 1} H_2^{-1} \ll x^{\gamma - \varepsilon}. \]
Hence we need to estimate the contribution from $h_2 \neq 0$ by $\ll x^{\gamma - \varepsilon}$. Then it is sufficient to prove that
\[ \sum_{x/2 < n \leq x} \sum_{0 < |h_2| \leq H_2} b_{h_2} n^{\gamma - 1} \Lambda(n) \times (e(\omega_2 h_2 (n + 1 - \beta_2)) + e(\omega_2 h_2 (n - \beta_2))) \ll x^{\gamma - \varepsilon}, \tag{3.10} \]
which can be proved by the same arguments that lead to (3.7). Hence we prove that $S_{21} \ll x^{\gamma - \varepsilon}$. Since $S_{22}$ can be bounded by the same method, the estimation of $S_2$ is done. Moreover, $S_3$ can be estimated by the same proof.

3.3. Upper bound of $S_4$. In order to prove that $S_4 \ll x^{\gamma \log^{-2} x}$, we write $S_4 = S_{41} + O(S_{42})$, where
\[ S_{41} := \sum_{p \leq x} \gamma p^{\gamma - 1} (\psi(-\omega_2(p + 1 - \beta_2)) - \psi(-\omega_2(p - \beta_2))) \times (\psi(-\omega_1(p + 1 - \beta_1)) - \psi(-\omega_1(p - \beta_1))), \]
and
\[ S_{42} := \sum_{p \leq x} p^{\gamma - 2} (\psi(-\omega_2(p + 1 - \beta_2)) - \psi(-\omega_2(p - \beta_2))) \times (\psi(-\omega_1(p + 1 - \beta_1)) - \psi(-\omega_1(p - \beta_1))). \]
By (3.4) and (3.5), we obtain that
\[ S_{41} = S_{43} + O(S_{44} + S_{45} + S_{46}), \]
where
\[ S_{43} := \sum_{p \leq x} \gamma p^{\gamma - 1} \sum_{0 < |h_1| \leq H_1} a_{h_1} (e(\omega_1 h_1 (p + 1 - \beta_1)) - e(\omega_1 h_1 (p - \beta_1))) \times \sum_{0 < |h_2| \leq H_2} a_{h_2} (e(\omega_2 h_2 (p + 1 - \beta_2)) - e(\omega_2 h_2 (p - \beta_2))), \]
\[ S_{44} := \sum_{p \leq x} \gamma p^{\gamma - 1} \sum_{0 < |h_1| \leq H_1} a_{h_1} (e(\omega_1 h_1 (p + 1 - \beta_1)) - e(\omega_1 h_1 (p - \beta_1))) \times \sum_{|h_2| \leq H_2} b_{h_2} (e(\omega_2 h_2 (p + 1 - \beta_2)) + e(\omega_2 h_2 (p - \beta_2))), \]
\[ S_{45} := \sum_{p \leq x} \gamma p^{\gamma - 1} \sum_{|h_1| \leq H_1} b_{h_1} (e(\omega_1 h_1 (p + 1 - \beta_1)) + e(\omega_1 h_1 (p - \beta_1))) \times \sum_{0 < |h_2| \leq H_2} a_{h_2} (e(\omega_2 h_2 (p + 1 - \beta_2)) - e(\omega_2 h_2 (p - \beta_2))), \]
\[ S_{46} := \sum_{p \leq x} \gamma p^{\gamma - 1} \sum_{|h_1| \leq H_1} b_{h_1} (e(\omega_1 h_1 (p + 1 - \beta_1)) + e(\omega_1 h_1 (p - \beta_1))) \]
\[ \times \sum_{|h_2| \leq H_2} b_{h_2} \left( e(\omega_2 h_2(p + 1 - \beta_2)) + e(\omega_2 h_2(p - \beta_2)) \right). \]

3.3.1. Estimation of \( S_{43} \). By Lemma 2.2 and a splitting argument, it is sufficient to prove that

\[ \sum_{x/2 < n \leq x} \sum_{0 < |h_1| \leq H_1} \sum_{0 < |h_2| \leq H_2} a_{h_1} a_{h_2} n^{-1} \Lambda(n) \times (e(\omega_1 h_1(n + 1 - \beta_1)) - e(\omega_1 h_1(n - \beta_1))) \times (e(\omega_2 h_2(n + 1 - \beta_2)) - e(\omega_2 h_2(n - \beta_2))) \ll x^{\gamma - \varepsilon}. \]  

(3.11)

Define

\[ \theta_{h_1} := e(\omega_1 h_1) - 1. \]  

(3.12)

Then by (3.8), (3.12) and the trivial estimate \( \theta_{h_1} \ll 1 \) and \( \theta_{h_2} \ll 1 \), we know that the left-hand side of (3.11) is

\[ \ll x^{\gamma - 1} \max_{x/2 < u \leq x} \left| \sum_{0 < |h_1| < H_1} |h_1|^{-1} \sum_{0 < |h_2| < H_2} |h_2|^{-1} \Lambda(n) e((\omega_1 h_1 + \omega_2 h_2) n) \right|, \]

which combined with Lemma 2.4, Lemma 2.5 and the fact that \( 1, \omega_1, \omega_2 \) are linearly independent over \( \mathbb{Q} \), yields the upper bound estimate of the left-hand side of (3.11)

\[ \ll x^{\gamma - 1} \sum_{0 < |h_1| < H_1} \sum_{0 < |h_2| < H_2} h_1^{-1} h_2^{-1} x^{1 - \varepsilon} \ll x^{\gamma - 1} x^{1 - \varepsilon} \ll x^{\gamma - \varepsilon}. \]

3.3.2. Estimations of \( S_{44} \) and \( S_{45} \). We work on \( S_{44} \) firstly. The contribution of \( S_{44} \) from \( h_2 = 0 \) is

\[ \ll \sum_{p \leq x} p^{\gamma - 1} H_2^{-1} \sum_{0 < |h_1| < H_1} a_{h_1} (e(\omega_1 h_1(p + 1 - \beta_1)) - e(\omega_1 h_1(p - \beta_1))) \]

\[ \ll H_2^{-1} S_{23} \ll x^{\gamma - \varepsilon}. \]

The contribution of \( S_{44} \) from \( h_2 \neq 0 \) is

\[ \sum_{p \leq x} \gamma p^{\gamma - 1} \sum_{0 < |h_1| < H_1} a_{h_1} (e(\omega_1 h_1(p + 1 - \beta_1)) - e(\omega_1 h_1(p - \beta_1))) \times \sum_{0 < |h_2| < H_2} b_{h_2} (e(\omega_2 h_2(p + 1 - \beta_2)) + e(\omega_2 h_2(p - \beta_2))). \]  

(3.13)

From Lemma 2.2, it is sufficient to show that, for \( x/2 < u \leq x \), there holds

\[ \sum_{x/2 < n \leq u} n^{-1} \Lambda(n) \sum_{0 < |h_1| \leq H_1} a_{h_1} (e(\omega_1 h_1(n + 1 - \beta_1)) - e(\omega_1 h_1(n - \beta_1))) \times \sum_{0 < |h_2| \leq H_2} b_{h_2} (e(\omega_2 h_2(n + 1 - \beta_2)) + e(\omega_2 h_2(n - \beta_2))) \ll x^{\gamma - \varepsilon}. \]  

(3.14)
By the same method of bounding $S_{43}$, the left-hand side of (3.14) is
\begin{align*}
&\ll x^{\gamma-1} \max_{x/2 < u \leq x} \sum_{0 < |h_1| \leq H_1} |h_1|^{-1} \sum_{0 < |h_2| \leq H_2} H_2^{-1} \sum_{x/2 < n \leq u} \Lambda(n) e((\omega_1 h_1 + \omega_2 h_2)n) \\
&\ll x^{\gamma-1} \cdot \left(x^{1-1/(2r)+\varepsilon} + x^{1-\varepsilon}\right) \ll x^{\gamma-\varepsilon}.
\end{align*}

Similarly, we can prove that $S_{45} \ll x^{\gamma-\varepsilon}$.

3.3.3. Estimation of $S_{46}$. The contribution from $h_1 = h_2 = 0$ is
\begin{align*}
\ll \sum_{p \leq x} p^{\gamma-1} H_1^{-1} H_2^{-1} \ll x^{\gamma-\varepsilon}.
\end{align*}

The contribution from $h_1 = 0$ and $h_2 \neq 0$ is
\begin{align*}
\ll \sum_{p \leq x} p^{\gamma-1} H_1^{-1} \sum_{|h_2| \leq H_2} b_{h_2} \left(e(\omega_2 h_2(p + 1 - \beta_2)) + e(\omega_2 h_2(p - \beta_2))\right) \\
\ll H_1^{-1} S_{24} \ll x^{\gamma-\varepsilon}.
\end{align*}

The contribution from $h_1 \neq 0$ and $h_2 = 0$ can be bounded by a similar method. In the end, the contribution from $h_1 \neq 0$ and $h_2 \neq 0$ is
\begin{align*}
&\sum_{p \leq x} \gamma p^{\gamma-1} \sum_{0 < |h_1| \leq H_1} b_{h_1} \left(e(\omega_1 h_1(p + 1 - \beta_1)) + e(\omega_1 h_1(p - \beta_1))\right) \\
&\times \sum_{0 < |h_2| \leq H_2} b_{h_2} \left(e(\omega_2 h_2(p + 1 - \beta_2)) + e(\omega_2 h_2(p - \beta_2))\right),
\end{align*}

which can be bounded by using the same method of (3.13).

3.4. Upper bounds of $S_5$ and $S_6$. We only give the details of the estimation of $S_5$, since the estimate of $S_6$ is exactly the same as that of $S_5$, and we omit it herein. By (3.5) and (3.6), it is easy to see that
\begin{align*}
S_5 = S_{51} + O(S_{52} + S_{53} + S_{54}),
\end{align*}

where
\begin{align*}
S_{51} &:= \sum_{p \leq x} \omega_1 \sum_{0 < |h_3| \leq H_3} a_{h_3} \left(e(h_3(p + 1)^\gamma) - e(h_3 p^\gamma)\right) \\
&\times \sum_{0 < |h_2| \leq H_2} b_{h_2} \left(e(\omega_2 h_2(p + 1 - \beta_2)) - e(\omega_2 h_2(p - \beta_2))\right), \\
S_{52} &:= \sum_{p \leq x} \omega_1 \sum_{0 < |h_3| \leq H_3} a_{h_3} \left(e(h_3(p + 1)^\gamma) - e(h_3 p^\gamma)\right) \\
&\times \sum_{|h_2| \leq H_2} b_{h_2} \left(e(\omega_2 h_2(p + 1 - \beta_2)) + e(\omega_2 h_2(p - \beta_2))\right), \\
S_{53} &:= \sum_{p \leq x} \omega_1 \sum_{|h_3| \leq H_3} b_{h_3} \left(e(h_3(p + 1)^\gamma) + e(h_3 p^\gamma)\right) \\
&\times \sum_{0 < |h_2| \leq H_2} a_{h_2} \left(e(\omega_2 h_2(p + 1 - \beta_2)) - e(\omega_2 h_2(p - \beta_2))\right),
\end{align*}

and
\[ S_{54} := \sum_{p \leq x} \sum_{|h_3| \leq H_3} \omega_1 b_{h_3} \left( e(h_3(p + 1)\gamma) + e(h_3p\gamma) \right) \times \sum_{|h_2| \leq H_2} b_{h_2} \left( e(\omega_2h_2(p + 1 - \beta_2)) + e(\omega_2h_2(p - \beta_2)) \right). \]

3.4.1. Estimation of \( S_{51} \). In order to show that \( S_{51} \ll x^{\gamma - \varepsilon} \), by Lemma 2.2 and a splitting argument, it suffices to prove that

\[
\sum_{x/2 < n \leq x} \Lambda(n) \sum_{0 < |h_3| \leq H_3} a_{h_3} \left( e(h_3(n + 1)\gamma) - e(h_3n\gamma) \right) \times \sum_{0 < |h_2| \leq H_2} a_{h_2} \left( e(\omega_2h_2(n + 1 - \beta_2)) - e(\omega_2h_2(n - \beta_2)) \right) \ll x^{\gamma - \varepsilon}. \tag{3.15}
\]

Define

\[ \phi_{h_3}(t) := e(h_3((t + 1)\gamma - t\gamma)) - 1. \]

Then we have

\[ \phi_{h_3}(t) \ll |h_3|t^{\gamma - 1} \quad \text{and} \quad \frac{\partial \phi_{h_3}(t)}{\partial t} \ll |h_3|t^{\gamma - 2}. \]

It follows from the above estimate, (3.8), Lemma 2.14 and partial summation that the left-hand side of (3.15) is

\[
\ll \sum_{0 < |h_3| \leq H_3} \frac{1}{|h_3|} \left| \sum_{x/2 < n \leq x} \Lambda(n) \phi_{h_3}(n)e(h_3n\gamma) \right|
\times \sum_{0 < |h_2| \leq H_2} a_{h_2} \left( e(\omega_2h_2(n + 1 - \beta_2)) - e(\omega_2h_2(n - \beta_2)) \right) \right|
\ll \sum_{0 < |h_3| \leq H_3} \frac{1}{|h_3|} \left| \phi_{h_3}(x) \right| \sum_{x/2 < n \leq x} \Lambda(n)e(h_3n\gamma)
\times \sum_{0 < |h_2| \leq H_2} a_{h_2} \left( e(\omega_2h_2(n + 1 - \beta_2)) - e(\omega_2h_2(n - \beta_2)) \right) \right|
+ \int_{x/2}^{x} \sum_{0 < |h_3| \leq H_3} \frac{1}{|h_3|} \left| \frac{\partial \phi_{h_3}(t)}{\partial t} \right| \sum_{x/2 < n \leq t} \Lambda(n)e(h_3n\gamma)
\times \sum_{0 < |h_2| \leq H_2} a_{h_2} \left( e(\omega_2h_2(n + 1 - \beta_2)) - e(\omega_2h_2(n - \beta_2)) \right) \, dt.
\]
\[ x^{-1} \cdot \max_{x/2 < t \leq x} \sum_{0 < |h_3| \leq H_3} \left| \sum_{x/2 < n \leq t} \Lambda(n)e(h_3n^\gamma) \right| \]
\[ \times \sum_{0 < |h_2| \leq H_2} a_{h_2} \left( e(\omega_2h_2(n + 1 - \beta_2)) - e(\omega_2h_2(n - \beta_2)) \right) \]
\[ = x^{-1} \cdot \max_{x/2 < t \leq x} \sum_{0 < |h_3| \leq H_3} \left| \sum_{0 < |h_2| \leq H_2} a_{h_2} \theta_{h_2} \right| \]
\[ \sum_{x/2 < n \leq t} \Lambda(n)e(h_3n^\gamma + \omega_2h_2n - \omega_2h_2\beta_2) \]
\[ \ll x^{-1} \cdot \max_{x/2 < t \leq x} \sum_{0 < |h_3| \leq H_3} \frac{1}{|h_2|} \left| \sum_{0 < |h_3| \leq H_3} \max_{x/2 < t \leq x} \left| \sum_{x/2 < n \leq t} \Lambda(n)e(h_3n^\gamma + \omega_2h_2n - \omega_2h_2\beta_2) \right| \right. \]
\[ \ll x^{-1+\varepsilon} \sum_{0 < |h_2| \leq H_2} \left( \frac{1}{|h_2|} \sum_{0 < |h_3| \leq H_3} \left( |h_3|^{1/6}x^{\gamma/6}3^{4/3} + |h_3|^{-1/3}x^{1-\gamma/3} + \right. \right. \]
\[ \left. \left. + |h_3|^{1/4}x^{\gamma/4}5^{1/3} + |h_3|^{-1/4}x^{1-\gamma/4} + x^{22/25} \right) \right) \]
\[ \ll x^{-1+\varepsilon} \left( H_3^{7/6}x^{\gamma/6}3^{4/3} + H_3^{2/3}x^{1-\gamma/3} + H_3^{5/4}x^{\gamma/4}5^{1/3} + H_3^{3/4}x^{1-\gamma/4} + H_3x^{22/25} \right) \]
\[ \ll x^{11/12+\varepsilon} + x^{2/3+\varepsilon} + x^{7/8+\varepsilon} + x^{3/4+\varepsilon} + x^{22/25+\varepsilon} \ll x^{\gamma-\varepsilon}, \]
provided that \( \gamma > \frac{11}{12} \).

3.4.2. Estimations of \( S_{52} \) and \( S_{53} \). We only give the proof of \( S_{52} \), since the bound of \( S_{53} \) can be obtained similarly. We mention that by assuming \( \gamma > \frac{11}{12} \), there holds
\[ \sum_{p \leq x} \sum_{0 < |h_3| \leq H_3} a_{h_3} \left( e(h_3(p + 1)^\gamma) - e(h_3p^\gamma) \right) \ll x^\gamma \log^{-2} x, \quad (3.16) \]
by a standard proof of the Piatetski-Shapiro counting function. A detailed proof can be found in [3, pp. 49–53]. By (3.16) the contribution of \( S_{52} \) from \( h_2 = 0 \) is
\[ \ll \sum_{p \leq x} H_2^{-1} \sum_{0 < |h_3| \leq H_3} a_{h_3} \left( e(h_3(p + 1)^\gamma) - e(h_3p^\gamma) \right) \]
\[ \ll H_2^{-1} x^\gamma \log^{-2} x \ll x^{\gamma-\varepsilon}. \]
The contribution of \( S_{52} \) from \( h_2 \neq 0 \) is
\[ \sum_{p \leq x} \sum_{0 < |h_3| \leq H_3} a_{h_3} \left( e(h_3(p + 1)^\gamma) - e(h_3p^\gamma) \right) \]
\[ \times \sum_{0 < |h_2| \leq H_2} b_{h_2} \left( e(\omega_2h_2(p + 1 - \beta_2)) + e(\omega_2h_2(p - \beta_2)) \right), \]
which can be bounded by the same method of (3.15).
3.4.3. Estimation of $S_{54}$. The contribution of $S_{54}$ from $h_2 = h_3 = 0$ is
\[\ll \sum_{p \leq x} H_3^{-1} H_2^{-1} \ll \frac{x}{\log x} \cdot x^{-(1-\gamma+\epsilon)} \cdot x^{-\epsilon} \ll x^{\gamma-\epsilon}.
\]
We mention that
\[\sum_{p \leq x} \sum_{0 < |h_3| \leq H_3} b_{h_3}(e(h_3(p + 1)\gamma) + e(h_3p\gamma)) \ll x^{\gamma} \log^2 x, \quad (3.17)
\]
by a standard method of exponent pair; see [3, pp. 48]. By (3.17) the contribution of $S_{54}$ from $h_2 = 0$ and $h_3 \neq 0$ is
\[\ll \sum_{p \leq x} H_2^{-1} \sum_{0 < |h_3| \leq H_3} b_{h_3}(e(h_3(p + 1)\gamma) + e(h_3p\gamma))
\]
\[\ll H_2^{-1} x^{\gamma} \log^2 x \ll x^{\gamma-\epsilon}.
\]
Similarly, the contribution of $S_{54}$ from $h_2 \neq 0$ and $h_3 = 0$ is
\[\ll \sum_{p \leq x} H_3^{-1} \sum_{|h_2| \leq H_2} b_{h_2}(e(\omega_2 h_2(p + 1 - \beta_2)) + e(\omega_2 h_2(p - \beta_2)))
\]
\[\ll H_3^{-1} \sum_{p \leq x} 1 \ll x^{\gamma-1-\epsilon} \cdot \frac{x}{\log x} \ll x^{\gamma-\epsilon}.
\]
The contribution of $S_{54}$ from $h_2 \neq 0$ and $h_3 \neq 0$ is
\[\ll \sum_{n \leq x} \Lambda(n) \sum_{0 < |h_3| \leq H_3} b_{h_3}(e(h_3(n + 1)\gamma) + e(h_3n\gamma))
\]
\[\times \sum_{0 < |h_2| \leq H_2} b_{h_2}(e(\omega_2 h_2(n + 1 - \beta_2)) + e(\omega_2 h_2(n - \beta_2))),
\]
which can be estimated by the method demonstrated on page 48 in [3] to derive the upper bound $x^{\gamma-\epsilon}$.

3.5. Upper bound of $S_7$. By (3.4), (3.5) and (3.6), we write
\[S_7 = S_{71} + O(S_{72} + S_{73} + S_{74} + S_{75} + S_{76} + S_{77} + S_{78}),
\]
where
\[S_{71} := \sum_{p \leq x} \sum_{0 < |h_1| \leq H_1} a_{h_1}(e(\omega_1 h_1(p + 1 - \beta_1)) - e(\omega_1 h_1(p - \beta_1)))
\]
\[\times \sum_{0 < |h_2| \leq H_2} a_{h_2}(e(\omega_2 h_2(p + 1 - \beta_2)) - e(\omega_2 h_2(p - \beta_2)))
\]
\[\times \sum_{0 < |h_3| \leq H_3} a_{h_3}(e(h_3(p + 1)\gamma) - e(h_3p\gamma)),
\]
\[S_{72} := \sum_{p \leq x} \sum_{0 < |h_1| \leq H_1} a_{h_1}(e(\omega_1 h_1(p + 1 - \beta_1)) - e(\omega_1 h_1(p - \beta_1)))
\]
\[\times \sum_{0 < |h_2| \leq H_2} a_{h_2}(e(\omega_2 h_2(p + 1 - \beta_2)) - e(\omega_2 h_2(p - \beta_2)))
\]
\[\times \sum_{|h_3| \leq H_3} b_{h_3}(e(h_3(p + 1)\gamma) + e(h_3p\gamma)),
\]
$S_{73} := \sum_{p \leq x} \sum_{0 < |h_1| \leq H_1} a_{h_1} \left( e(\omega_1 h_1 (p + 1 - \beta_1)) - e(\omega_1 h_1 (p - \beta_1)) \right)$
$$\times \sum_{|h_2| \leq H_2} b_{h_2} \left( e(\omega_2 h_2 (p + 1 - \beta_2)) + e(\omega_2 h_2 (p - \beta_2)) \right)$$
$$\times \sum_{0 < |h_3| \leq H_3} a_{h_3} \left( e(h_3 (p + 1)^\gamma) - e(h_3 p^\gamma) \right);$$

$S_{74} := \sum_{p \leq x} \sum_{0 < |h_1| \leq H_1} a_{h_1} \left( e(\omega_1 h_1 (p + 1 - \beta_1)) - e(\omega_1 h_1 (p - \beta_1)) \right)$
$$\times \sum_{|h_2| \leq H_2} b_{h_2} \left( e(\omega_2 h_2 (p + 1 - \beta_2)) + e(\omega_2 h_2 (p - \beta_2)) \right)$$
$$\times \sum_{|h_3| \leq H_3} b_{h_3} \left( e(h_3 (p + 1)^\gamma) + e(h_3 p^\gamma) \right);$$

$S_{75} := \sum_{p \leq x} \sum_{|h_1| \leq H_1} b_{h_1} \left( e(\omega_1 h_1 (p + 1 - \beta_1)) + e(\omega_1 h_1 (p - \beta_1)) \right)$
$$\times \sum_{0 < |h_2| \leq H_2} a_{h_2} \left( e(\omega_2 h_2 (p + 1 - \beta_2)) - e(\omega_2 h_2 (p - \beta_2)) \right)$$
$$\times \sum_{|h_3| \leq H_3} a_{h_3} \left( e(h_3 (p + 1)^\gamma) - e(h_3 p^\gamma) \right);$$

$S_{76} := \sum_{p \leq x} \sum_{|h_1| \leq H_1} b_{h_1} \left( e(\omega_1 h_1 (p + 1 - \beta_1)) + e(\omega_1 h_1 (p - \beta_1)) \right)$
$$\times \sum_{0 < |h_2| \leq H_2} a_{h_2} \left( e(\omega_2 h_2 (p + 1 - \beta_2)) - e(\omega_2 h_2 (p - \beta_2)) \right)$$
$$\times \sum_{|h_3| \leq H_3} b_{h_3} \left( e(h_3 (p + 1)^\gamma) + e(h_3 p^\gamma) \right);$$

$S_{77} := \sum_{p \leq x} \sum_{|h_1| \leq H_1} b_{h_1} \left( e(\omega_1 h_1 (p + 1 - \beta_1)) + e(\omega_1 h_1 (p - \beta_1)) \right)$
$$\times \sum_{|h_2| \leq H_2} b_{h_2} \left( e(\omega_2 h_2 (p + 1 - \beta_2)) + e(\omega_2 h_2 (p - \beta_2)) \right)$$
$$\times \sum_{0 < |h_3| \leq H_3} a_{h_3} \left( e(h_3 (p + 1)^\gamma) - e(h_3 p^\gamma) \right);$$

$S_{78} := \sum_{p \leq x} \sum_{|h_1| \leq H_1} b_{h_1} \left( e(\omega_1 h_1 (p + 1 - \beta_1)) + e(\omega_1 h_1 (p - \beta_1)) \right)$
$$\times \sum_{|h_2| \leq H_2} b_{h_2} \left( e(\omega_2 h_2 (p + 1 - \beta_2)) + e(\omega_2 h_2 (p - \beta_2)) \right)$$
$$\times \sum_{|h_3| \leq H_3} b_{h_3} \left( e(h_3 (p + 1)^\gamma) + e(h_3 p^\gamma) \right).$$

3.5.1. Estimation of $S_{71}$. In order to show that $S_{71} \ll x^{\gamma - \varepsilon}$, we apply the similar argument which derives the upper bound of $S_{51}$. Therefore, it is sufficient to
prove that
\[
\max_{x/2 < t \leq x} \sum_{0 < |h_1| \leq H_1} |h_1|^{-1} \sum_{0 < |h_2| \leq H_2} \sum_{0 < |h_3| \leq H_3} |T| \ll x^{1-\epsilon},
\] 
(3.18)
where
\[
T := \sum_{x/2 < n \leq t} \Lambda(n) \cdot \text{e} \left( h_3 n^\gamma + (\omega_1 h_1 + \omega_2 h_2) n - \omega_1 h_1 \beta_1 - \omega_2 h_2 \beta_2 \right),
\]
which can be bounded by Lemma 2.14. By a similar argument of the estimation of the left-hand side of (3.15), we derive (3.18) by assuming that \( \gamma > \frac{11}{12} \).

3.5.2. Estimation of \( S_{72} \). The contribution from \( h_3 = 0 \) is
\[
\sum_{p \leq x} H_3^{-1} \sum_{0 < |h_1| \leq H_1} a_{h_1} \left( \text{e}(\omega_1 h_1(p + 1 - \beta_1)) - \text{e}(\omega_1 h_1(p - \beta_1)) \right)
\times \sum_{0 < |h_2| \leq H_2} a_{h_2} \left( \text{e}(\omega_2 h_2(p + 1 - \beta_2)) - \text{e}(\omega_2 h_2(p - \beta_2)) \right)
\ll H_3^{-1} x^{1-\epsilon} \ll x^{\gamma-\epsilon},
\]
by the estimation of \( S_{33} \). The contribution from \( h_3 \neq 0 \) can be bounded by the same method of \( S_{71} \).

3.5.3. Estimations of \( S_{73} \) and \( S_{75} \). The contribution of \( S_{73} \) from \( h_2 = 0 \) is
\[
\sum_{p \leq x} H_2^{-1} H_3^{-1} \sum_{0 < |h_1| \leq H_1} a_{h_1} \left( \text{e}(\omega_1 h_1(p + 1 - \beta_1)) - \text{e}(\omega_1 h_1(p - \beta_1)) \right)
\times \sum_{0 < |h_3| \leq H_3} a_{h_3} \left( \text{e}(h_3(p + 1)\gamma) - \text{e}(h_3 p \gamma) \right),
\]
which can be bounded by the method of \( S_{51} \). The contribution of \( S_{73} \) from \( h_2 \neq 0 \) can be bounded by the similar arguments which deal with the upper bound of \( S_{51} \). The estimation of \( S_{75} \) is exactly the same as \( S_{72} \).

3.5.4. Estimations of \( S_{74} \) and \( S_{76} \). The contribution of \( S_{74} \) from \( h_2 = h_3 = 0 \) is
\[
\sum_{p \leq x} H_2^{-1} H_3^{-1} \sum_{0 < |h_1| \leq H_1} a_{h_1} \left( \text{e}(\omega_1 h_1(p + 1 - \beta_1)) - \text{e}(\omega_1 h_1(p - \beta_1)) \right)
\ll H_2^{-1} H_3^{-1} x^{1-\epsilon} \ll x^{\gamma-\epsilon},
\]
by the same method of \( S_{23} \). The contribution of \( S_{74} \) from \( h_2 = 0 \) and \( h_3 \neq 0 \) is
\[
\sum_{p \leq x} H_2^{-1} \sum_{0 < |h_1| \leq H_1} a_{h_1} \left( \text{e}(\omega_1 h_1(p + 1 - \beta_1)) - \text{e}(\omega_1 h_1(p - \beta_1)) \right)
\times \sum_{0 < |h_3| \leq H_3} b_{h_3} \left( \text{e}(h_3(p + 1)\gamma) + \text{e}(h_3 p \gamma) \right),
\]
which can be bounded by the similar method of \( S_{51} \). Similarly, the contribution of \( S_{74} \) from \( h_2 \neq 0 \) and \( h_3 = 0 \) is bounded by the same method of \( S_{44} \). The contribution of \( S_{74} \) from \( h_2 \neq 0 \) and \( h_3 \neq 0 \) can be bounded by the similar arguments which deal with the upper bound of \( S_{23} \). The estimation of \( S_{76} \) is almost the same as \( S_{74} \).
3.5.5. Estimation of $S_{77}$. The contribution from $h_1 = h_2 = 0$ can be bounded from the estimate (3.16). The contribution from $h_1 = 0$ and $h_2 \neq 0$ is bounded by the same method of $S_{43}$ and similar to the contribution from $h_1 \neq 0$ and $h_2 = 0$. The contribution from $h_1 \neq 0$ and $h_2 \neq 0$ can be bounded by following the similar argument of $S_{52}$.

3.5.6. Estimation of $S_{78}$. The contribution from $h_1 = h_2 = h_3 = 0$ is $\ll \sum_{p \leq x} H_1^{-1} H_2^{-1} H_3^{-1} \ll x^{\gamma - \varepsilon}$.

The contribution from $h_1 \neq 0$ and $h_2 = h_3 = 0$ is bounded by the same method of $S_{24}$ and similar to the contribution from $h_2 \neq 0$ and $h_1 = h_3 = 0$. The contribution from $h_1 \neq 0$, $h_2 \neq 0$ and $h_3 = 0$ is bounded by the same method of $S_{43}$. The contribution from $h_1 = h_2 = 0$ and $h_3 \neq 0$ can be bounded by following the argument on page 48 of [3]. The contribution from $h_1 \neq 0$, $h_2 = 0$ and $h_3 \neq 0$ is bounded by the same method of $S_{51}$ and similar to the contribution from $h_1 = 0$, $h_2 \neq 0$ and $h_3 \neq 0$. In the end, the contribution from $h_1 \neq 0$, $h_2 \neq 0$ and $h_3 \neq 0$ is bounded by the same method of $S_{71}$.

4. Proof of Theorem 2

For a Beatty sequence $B_{\alpha,\beta} := \lfloor \alpha n + \beta \rfloor$, recall that $\omega := \alpha^{-1}$. Similar to the construction of the proof of Theorem 1.1 and by the definition of $\pi^{(c)}_{\alpha,\beta}(x)$, we have that

$$
\pi^{(c)}_{\alpha,\beta}(x) = \sum_{p \leq x} \mathcal{X}_{\alpha,\beta}(p) \mathcal{X}^{(c)}(p) = S_1 + S_2 + S_3,
$$

where

$$
S_1 := \sum_{p \leq x} \omega \mathcal{X}^{(c)}(p);
$$

$$
S_2 := \sum_{p \leq x} (\gamma p^{\gamma - 1} + O(p^{\gamma - 2}))
\times \left( \psi(-\omega(p + 1 - \beta)) - \psi(-\omega(p - \beta)) \right),
$$

$$
S_3 := \sum_{p \leq x} \left( \psi(-(p + 1)^{\gamma}) - \psi(p^{\gamma}) \right)
\times \left( \psi(-\omega(p + 1 - \beta)) - \psi(-\omega(p - \beta)) \right).
$$

$S_1$ can be estimated by the same method of $S_1$ in the proof of Theorem 1.1, which is

$$
S_1 = \frac{x^{\gamma}}{\alpha \log x} + O \left( \frac{x}{\log^2 x} \right).
$$

$S_2$ can be bounded by the same method of $S_2$ in the proof of Theorem 1.1. By the assumption that $\alpha$ is of finite type, it gives that

$$
S_2 \ll x^{\gamma - \varepsilon}.
$$
We claim that
\[ S_3 \ll x^{\gamma-\varepsilon}, \]
provided that \( \gamma > \frac{11}{12} \).

### 5. Proof of Theorem 3

The method is similar to the proof of Theorem 1.1 with more technical summations since there are more characteristic functions of Beatty sequences. Therefore, we only give a sketch proof here. Let \( \omega_i := \alpha_i^{-1} \) for \( i \in \{1, \ldots, \xi\} \). By the similar construction, we have that
\[
\pi_{\alpha_1, \beta_1; \ldots; \alpha_\xi, \beta_\xi}(x) = \sum_{p \leq x} x^{\gamma} \prod_{i=1}^{\xi} \chi_{\alpha_i, \beta_i}(p),
\]
and
\[
\chi_{\alpha_i, \beta_i} = \omega_i + \psi(-\omega_i(p + 1 - \beta_i)) - \psi(-\omega_i(p - \beta_i)).
\]
We can break (5.1) into several sums by the Vaaler’s approximation, which is similar to the construction of the proof of Theorem 1.1. Since every sum with \( b_h \) can be bounded by separating the contribution of \( h = 0 \) and \( h \neq 0 \) and compared with the corresponding sum with \( a_h \), we only give a sketch of the estimations of the summations with \( a_h \).

The main term of (5.1) is
\[
\sum_{p \leq x} \omega_1 \cdots \omega_\xi \chi^{(c)}(p) = \frac{x^{\gamma}}{\alpha_1 \cdots \alpha_\xi \log x} + O \left( \frac{x^{\gamma}}{\log^2 x} \right),
\]
with \( c \in (1, \frac{2817}{2426}) \) by (2.4). Let \( \mathcal{L} \) be an arbitrary subset of \( \{1, \ldots, \xi\} \). Set
\[
H_1 = \cdots = H_\xi := x^{\varepsilon} \quad \text{and} \quad J := x^{1-\gamma+\varepsilon}.
\]
We claim that
\[
\pi_{\alpha_1, \beta_1; \ldots; \alpha_\xi, \beta_\xi}(x) = \frac{x^{\gamma}}{\alpha_1 \cdots \alpha_\xi \log x} + O \left( \frac{x^{\gamma}}{\log^2 x} + \mathcal{T}_1 + \mathcal{T}_2 \right),
\]
where
\[
\mathcal{T}_1 := \sum_{p \leq x} p^{\gamma-1} \prod_{i \in \mathcal{L}} \sum_{0 < |h_i| \leq H_i} a_{h_i} (e(\omega_i h_i(p + 1 - \beta_i)) - e(\omega_i h_i(p - \beta_i)))
\]
and
\[
\mathcal{T}_2 := \sum_{p \leq x} \left( \prod_{i \in \mathcal{L}} \sum_{0 < |h_i| \leq H_i} a_{h_i} (e(\omega_i h_i(p + 1 - \beta_i)) - e(\omega_i h_i(p - \beta_i))) \right) \times \sum_{0 < |j| \leq J} a_j (e(j(p + 1)^\gamma) - e(j p^\gamma)).
\]
We consider \( \mathcal{T}_1 \) firstly. By the same method of \( S_{33} \), in order to show that \( \mathcal{T}_1 \ll x^{\gamma-\varepsilon} \), it is sufficient to prove that
\[
\max_{x/2 < \xi \leq x} \prod_{i \in \mathcal{L}} \sum_{0 < |h_i| \leq H_i} |h_i|^{-1} |\mathcal{T}_3| \ll x^{1-\varepsilon}, \quad (5.2)
\]
where
\[ T_3 := \sum_{x/2 < n \leq u} \Lambda(n) e \left( \left( \sum_{i \in \mathcal{L}} \omega_i h_i \right)n \right). \]

By Lemma 2.4 and Lemma 2.5, the left-hand side of (5.2) is
\[ \ll \left( \prod_{i \in \mathcal{L}} \sum_{0 < |h_i| \leq H_i} |h_i|^{-1} \right) \left( x^{1 - 1/(2r) + \varepsilon} + x^{1 - \varepsilon} \right) \ll x^{1 - \varepsilon}, \]
provided that \( \alpha_i \) is of finite type with \( i \in \{1, 2, \ldots, \xi\} \).

Now we consider \( T_2 \). By the similar method of \( S_{71} \), it is sufficient to prove that
\[ \max_{x/2 < t \leq x} \prod_{i \in \mathcal{L}} \sum_{0 < |h_i| \leq H_i} |h_i|^{-1} \sum_{0 < j \leq J} |T_4| \ll x^{1 - \varepsilon}, \tag{5.3} \]
where
\[ T_4 := \sum_{x/2 < n \leq t} \Lambda(n) \cdot e \left( j n^\gamma + \sum_{i \in \mathcal{L}} (\omega_i h_i n - \omega_i h_i \beta_i) \right), \]
which can be bounded by Lemma 2.14. By a similar argument of the estimation of the left-hand side of (3.15), we derive (5.3) under the condition that \( \gamma > \frac{11}{12} \).

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