TOPOLOGICAL OBJECTS AND CONFINEMENT ON THE LATTICE*

E.T. AKHMELOV, ITEP, B.Cheremushkinskaya 25, Moscow, 117259, Russia

M.N. CHERNODUB
ITEP, B.Cheremushkinskaya 25, Moscow, 117259, Russia

and

M.I. POLIKARPOV
ITEP, B.Cheremushkinskaya 25, Moscow, 117259, Russia
E-mail: polykarp@vxdesy.desy.de

ABSTRACT
First we discuss various topological objects (monopoles, “minopoles” and “hy-
brids”) which may be important for the confinement mechanism in various abelian projections. The second topic is the string between quark and anti-
quark. The standard quantum string with the Nambu-Goto action exists only in D=26. If we start from the field theory, in which the string excitations exist, and change the variables in the path integral to the string variables, then the Jacobian appears. This Jacobian generates the correction to the Nambu-Goto action. For this effective action the conformal anomaly cancels in D=4. Thus we get the quantum string theory in D=4.

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1. Introduction

Many numerical experiments confirm the monopole confinement mechanism in the $U(1)$ theory obtained by the abelian projection from the $SU(2)$ lattice gluodynamics. The well known examples are:

- The string tension $\sigma_{U(1)}$ calculated from the $U(1)$ Wilson loops (loops constructed only from the abelian gauge fields) coincides with the full $SU(2)$ string tension.
- The density of the monopoles seems to scale.
- The monopole currents satisfy the London equation for a superconductor.
- The $SU(2)$ string tension can be obtained, with good accuracy, from the contribution of the abelian monopole currents.

All these remarkable facts, however, have been obtained only for the so called maximal abelian (MaA) projection. Other abelian projections (such as the diagonalization of the plaquette matrix $U_{x,12}$) do not give evidence that the vacuum behaves as the dual superconductor. Below we give three examples.

First, it turns out that the fractal dimensionality of the monopole currents extracted from the lattice vacuum by means of the maximal abelian projection is strongly correlated with the string tension. If monopoles are extracted by means of other projections, this correlation is absent (cf. Fig.2 and Fig.4 of ref.12). Another example is the temperature dependence of the monopole condensate measured on the basis of the percolation properties of the clusters of monopole currents. For the maximal abelian projection the condensate is nonzero below the critical temperature $T_c$ and vanishes above it. For the projection which corresponds to the diagonalization of $U_{x,12}$, the condensate is nonzero at $T > T_c$, and it is not the order parameter for the phase transition. The last result has been obtained by the authors of ref.13, but has not been published. The space–time asymmetry of the monopole currents behaves as the order parameter for the deconfinement phase transition for the MaA projection, and is zero both below and above the critical temperature for the so called minimal Abelian projection. In Section 2 we discuss the dependence of the confinement

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*In the talk of A. Di Giacomo at this workshop (see also) it has been claimed that the value of the monopole condensate is the order parameter for the phase transition. This result is obtained for the abelian projection in which the Polyakov line is diagonalized; therefore there is a similarity between this gauge and the MaA projection.*
mechanism on the type of the abelian projection. In Section 3 we show what kind of quantum strings may exist between the quark and the antiquark.

2. Abelian Projection of the $SU(2)$ Gauge Theory

After the Abelian projection of the $SU(2)$ gluodynamics, the diagonal elements of the gauge field become the $U(1)$ gauge field, and the nondiagonal elements become charged matter fields; this is clear from the $U(1)$ gauge transformations: $A_{\mu}^{ii} \rightarrow A_{\mu}^{ii} + \partial_\mu \alpha$, $A_{\mu}^{\pm} \rightarrow A_{\mu}^{\pm} e^{\pm 2i\alpha}$. On the lattice, for the standard parametrization of the link matrix, $U_{l}^{11} = \cos \phi e^{i\theta}$, $U_{l}^{12} = \sin \phi e^{i\chi}$, the situation is similar. After the Abelian projection, $\theta$ becomes the compact abelian gauge field; and $\chi$ becomes the compact matter field. The $U(1)$ gauge transformations are:

\[ \theta \rightarrow \theta + \alpha_1 - \alpha_2, \]
\[ \chi \rightarrow \chi + \alpha_1 + \alpha_2. \]

2.1. Maximal and Minimal Abelian projections

The widely used MaA projection\cite{9}\cite{10} corresponds to the gauge transformation that makes the link matrices diagonal “as much as possible”. For the $SU(2)$ lattice gauge theory, the matrices of the gauge transformation $\Omega_x$ are defined by the following maximization condition:

\[ \max_{\{\Omega_x\}} R(U'), R(U') = \sum_{x,\mu} Tr(U''_{x\mu} \sigma_3 U''_{x\mu}^+ \sigma_3), \]

\[ U''_{x\mu} = \Omega^+_x U_{x\mu} \Omega_x. \]

In order to show that in discussing of the confinement mechanism one has to take into account the type of the Abelian projection, we consider the Minimal Abelian (MiA) projection\cite{14} defined as:

\[ \min_{\{\Omega_x\}} R(U'), \]

The action of the $SU(2)$ gluodynamics can be represented in the following form:

\[ S = \beta \ Tr U_P = \beta_1(\phi) \cos \theta_P + S^{\rm int}(\theta, \chi) + \beta_2(\phi) \cos \chi_P \]

For the MaA projection, $\beta_1$ is large, the first term in the sum dominates, and if we neglect fluctuations of the angle $\phi$, as well as the Faddeev-Popov determinant, the
\(SU(2)\) action in the MaA projection is well approximated by the \(U(1)\) action: \(S_P \approx \hat{\beta} \cos \theta_P, \hat{\beta} = \beta < \cos \phi >^4\). The fields in the MaA projection can be transformed into those in the MiA projection by the gauge transformation, and the roles of the fields \(\theta\) and \(\chi\) are interchanged \(\square\). For the MiA projection \(\beta_2\) is large, and the gluodynamics is approximately reduced to the theory of the vector matter field \(\chi\): \(S \approx \beta_2(\phi) \cos \chi_P\).

2.2. Monopoles and Minopoles

The monopoles extracted from the field \(\theta\) in the MaA projection turn, in the MiA projection, into certain topological defects constructed from the “matter” fields \(\chi\). We call these topological defects “minopoles”. Minopoles can be extracted from a given configuration of gauge fields similarly to monopoles: from the angles \(\chi\) we construct the \(U(1)\) invariant plaquette variables \(\chi_P = \chi_1 - \chi_2 + \chi_3 - \chi_4 \mod 2\pi\). From these plaquette variables we construct the variables attached to the elementary cubes \(*j = \frac{1}{2\pi} d\chi_P\); for \(*j \neq 0\) the link dual to the cube carries the minopole current. We use the notation \(\hat{d}\) (instead of \(d\)), since the gauge transformations of \(\chi\) given by (2) differ from the gauge transformations of \(\theta\) given by (1), and the construction of the plaquette variable from the link variables and the construction of the cube variable from the plaquette variables differ in an obvious way from the standard construction. In Fig.1(a) we illustrate the standard construction of the monopoles from the field \(\theta\), and in Fig.1(b) we show the construction of the minopoles from the field \(\chi\). The variables \(\theta\) are characterized by the direction, shown by arrows; the variables \(\chi\) are characterized by their sign, and the variables \(\chi\) which transforms as in eq. (2) are shown by solid lines; the variables \(\chi\) with the opposite sign are shown by the dashed lines.

Since monopoles, which exist in the MaA projection become minopoles in the MiA projection, then if in the MaA projection the confinement phenomenon is due to condensation of monopoles (constructed from the field \(\theta\)), then in the MiA projection the confinement is due (in some sense) to other topological objects (minopoles) constructed from the “matter” field \(\chi\). It should be stressed that monopoles still exist in the MiA projection; they can be extracted from the fields \(\theta\) in the usual way, but they are not related to the dynamics. From the point of view of the initial \(SU(2)\) gauge symmetry the fields \(\theta\) and \(\chi\) are equal; this fact explains the symmetry between the monopoles and the minopoles in MaA and MiA projections.

Now we make several simple remarks about various abelian projections from the point of view of the path integral. We start from the standard partition function for the \(SU(2)\) gluodynamics: \(Z = \int [dU_l] e^{-S(U_P)}\). After the abelian projection we have: \(Z = \int [d\theta][d\chi][d\phi] \text{Det}^{1/2}(\Delta) e^{-S_0(\theta,\chi,\phi)}\), where \(\Delta\) is the Faddeev–Popov operator. Integrating over the variables \(\phi\) we get:

\[
Z = \int [d\theta][d\chi] e^{-S_1(\theta,\chi)}; \quad (6)
\]
integration over the variables $\chi$ yields:

$$Z = \int [d\theta] e^{-S_2(\theta)} \tag{7}$$

and integration in (8) over the variables $\theta$ results in:

$$Z = \int [d\chi] e^{-S_3(\chi)} \tag{8}$$

In the introduction we give several examples which show that in the MaA the monopoles behave similarly to the Cooper pairs in a superconductor. This means that:

- In the MaA projection the action for the monopole fields (not the monopole currents) is close to the action of the Higgs boson in the Abelian Higgs model, the role of the Higgs boson being played by the monopole field.

- The quantum theory for this Abelian Higgs model is close to the (quasi)classical theory.

This seems to be related to the fact that $S_2(\theta)$ is sufficiently local and simple, and, as we have already mentioned, is close to the action of the compact QED\textsuperscript{b}. Due

\textsuperscript{b} $S_2$ can not be equal to the QED action, since, e.g. in QED the asymptotic freedom is absent.
to the symmetry between MiA and MaA projections the action $S_3(\mathbb{F})$ is sufficiently local and simple in the MiA projection.

2.3. Hybrids

MiA and MaA projections are in some sense opposite to each other; there are infinitely many abelian projections in between the MiA and the MaA projections. For these “intermediate” projections the action $S_1(\mathbb{E})$ may be simple. Now the fields $\theta$ and $\chi$ are important for the dynamics, and the topological objects constructed from both $\theta$ and $\chi$ may be also important. We call these objects “hybrids”. Two examples of hybrids are shown in Fig.2(a,b). As in Fig.1, the field $\theta$ is denoted by the line with an arrow, and the field $\chi$ is shown by the solid or dashed line depending on its sign.

The construction of hybrids is similar to the construction of monopoles and minopoles. Since the angles attached to the plaquettes are taken (mod $2\pi$), the sum over the phases of the cubes shown in Fig.2 always gives $2\pi \cdot Q$, $Q = 0, \pm 1, \pm 2$, for any $\theta$ and $\chi$. The charge $Q$, of the hybrid, is invariant under the $U(1)$ gauge transformations.

2.4. Extended Monopoles

We have widely used the phrases like “monopoles (minopoles) are important for

$S_2$ is a function of the $U(1)$ invariant loops, constructed from the field $\theta$; and $S_3$ is the same function of the loops, constructed from the field $\chi$. 

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Fig. 2. Hybrids constructed from the fields $\theta$ and $\chi$. 

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(a) (b)
the confinement”, but we have not specified the exact meaning of the word “important”. For a quantitative discussion we may use the following criteria. In the MaA projection the Creutz ratio of the Wilson loops constructed from the fields $\theta$ gives the string tension which is close to the full $SU(2)$ string tension $\sigma_{SU(2)}$. In the MiA projection, in order to get the same value of the string tension, we have to substitute the loops constructed from the field $\chi$ into the Creutz ratio. This follows from the exact symmetry $\theta \leftrightarrow \chi$. Similarly, if we calculate the string tension using monopole currents in the MaA projection, we have to use the minopole currents in the MiA projection to get the same result for $\sigma$. Unfortunately, the confinement scenario is unclear in terms of minopoles; moreover, minopoles may be, in some sense, lattice artifacts, which do not exist in the continuum limit.

Still, there exists a possibility that the vacuum is similar to the dual superconductor in any abelian projection. The idea is to use the extended monopoles $\mathbb{E}$ defined on the cubes of size $2^3, 3^3, \ldots$. A recent study $\mathbb{E}$ of the energy–entropy balance of the extended monopoles shows that extended monopoles are important for the dynamics of the temperature phase transition.

There are many open questions. For example: what is the action $S^{ext}$ of the extended monopoles; is it simple and/or local? What is the dependence of $S^{ext}$ on the type of the abelian projection? If some extended monopoles are important for the dynamics, what is their size; is it proportional to the correlation length in the gluodynamics? Is there any physical meaning in the extended minopoles and the extended hybrids?

3. What Kind of String may appear in the $D = 4$ Gluodynamics?

Here we briefly describe the results of our recent investigation $\mathbb{E}$. Numerical studies of the lattice gluodynamics clearly show the formation of a string between the quark and the antiquark (see, for example, the recent paper $\mathbb{E}$). The string is made of gluons, and is, therefore, the bosonic string. In the first approximation, the action is proportional to the area of the string world sheet:

$$S = \mu \cdot \text{Area} = \mu \int d^2 \sigma \sqrt{g}; \quad (9)$$

here the standard notations are used: the string world sheet $\tilde{x}(\sigma)$ is parametrized by $\sigma_a, a = 1, 2$; $g_{ab} = \partial_a \tilde{x}_\mu \partial_b \tilde{x}_\mu$ and $g = det|g_{ab}|$.

Attempts at numerical simulation of this string in four dimensions have led to the sophisticated world sheets similar to “branched polymers”. This is related to the well known difficulty $\mathbb{E}$ in the quantization of the bosonic string in four dimensions, which can be explained in the following way. For the Nambu-Goto action (9) we have the Virassoro algebra (algebra of the generators of the conformal transformations): $[L_n, L_m] = (n-m)L_{n+m} + \frac{D-26}{12}(m^3 - m)\delta_{n+m,0};$ and the last term in the right-hand side prevents quantization for $D \neq 26$. It occurs $\mathbb{E}$, that if we include the additional
term in the action:

$$ S \rightarrow S = \mu \int d\sigma \sqrt{\gamma} - \frac{\gamma}{96\pi} \int d\sigma (\partial_a \ln \sqrt{\gamma})^2, \quad (10) $$

the Virasoro algebra takes the form:

$$ [L_n, L_m] = (n - m) L_{n+m} + \frac{D - 26 + \gamma}{12} (m^3 - m) \delta_{n+m,0}. \quad (11) $$

If $\gamma = 22$, then for $D = 4$ the conformal anomaly is absent and the theory can be quantized. Below we show that this mechanism of cancellation of the conformal anomaly is natural if one starts from the field theory.

Consider a $4D$ theory in which strings exist, for example, the Abelian Higgs theory. Then it is possible to change the field variables to the string variables, and the Jacobian $J$ appears in the integral:

$$ Z = \int [d\phi][dA] e^{-S(\phi, A)} = \int [d\tilde{x}] e^{-S(\tilde{x})} J(\tilde{x}) \quad (12) $$

It occurs\[2\] that:

$$ J(\tilde{x}) = \exp \left\{ -\frac{11}{48\pi} \int d\sigma (\partial_a \ln \sqrt{\gamma})^2 + \ldots \right\}, \quad (13) $$

Comparison with eqs.\[10\], \[11\] shows that it is the Jacobian that gives the term in the string action which cancels the conformal anomaly in four dimensions! The Jacobian does not depend on the field theory from which we have started, and therefore the said mechanism is universal, and can be expected to work in gluodynamics and chromodynamics.

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