Applications of Waring’s formula to some identities of Chebyshev polynomials

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Abstract. Some identities of Chebyshev polynomials are deduced from Waring’s formula on symmetric functions. In particular, these formulae generalize some recent results of Grabner and Prodinger.

1 Introduction

Given a set of variables $X = \{x_1, x_2, \ldots\}$, the $k$th ($k \geq 0$) elementary symmetric polynomial $e_k(X)$ is defined by $e_0(X) = 1$, 

$$e_k(X) = \sum_{i_1 < \ldots < i_k} x_{i_1} \ldots x_{i_k}, \quad \text{for} \quad k \geq 1,$$

and the $k$th ($k \geq 0$) power sum symmetric polynomial $p_k(X)$ is defined by $p_0(X) = 1$, 

$$p_k(X) = \sum_i x_i^k, \quad \text{for} \quad k \geq 1.$$

Let $\lambda = 1^{m_1} 2^{m_2} \ldots$ be a partition of $n$, i.e., $m_1 + m_2 + \ldots + m_n n = n$, where $m_i \geq 0$ for $i = 1, 2, \ldots n$. Set $l(\lambda) = m_1 + m_2 + \ldots + m_n$. According to the fundamental theorem of symmetric polynomials, any symmetric polynomial can be written uniquely as a polynomial of elementary symmetric polynomials $e_i(X)$ ($i \geq 0$). In particular, for the power sum $p_k(x)$, the corresponding formula is usually attributed to Waring and reads as follows:

$$p_k(X) = \sum_\lambda (-1)^{k-l(\lambda)} \frac{k(l(\lambda) - 1)!}{\prod_i m_i!} e_1(X)^{m_1} e_2(X)^{m_2} \ldots, \quad (1)$$

where the sum is over all the partitions $\lambda = 1^{m_1} 2^{m_2} \ldots$ of $k$.

In a recent paper Grabner and Prodinger proved some identities about Chebyshev polynomials using generating functions, the aim of this paper is
to show that Waring’s formula provides a natural generalization of such kind of identities.

Let $U_n$ and $V_n$ be two sequences defined by the following recurrence relations:

$$U_n = pU_{n-1} - U_{n-2}, \quad U_0 = 0, U_1 = 1, \quad (2)$$
$$V_n = pV_{n-1} - V_{n-2}, \quad V_0 = 2, V_1 = p. \quad (3)$$

Hence $U_n$ and $V_n$ are rescaled versions of the first and second kind of Chebyshev polynomials $U_n(x)$ and $T_n(x)$, respectively:

$$U_n(x) = U_{n+1}(2x), \quad T_n(x) = \frac{1}{2}T_n(x).$$

**Theorem 1** For integers $m, n \geq 0$, let $W_n = aU_n + bV_n$ and $\Omega = a^2 + 4b^2 - b^2p^2$. Then the following identity holds

$$W_n^{2k} + W_{n+m}^{2k} = \sum_{r=0}^{k} \theta_{k,r}(m)\Omega^{k-r}W_n^rW_{n+m}^r, \quad (4)$$

where

$$\theta_{k,r}(m) = \sum_{0 \leq 2j \leq k} (-1)^j \frac{k(k-j-1)!}{j!(k-r)!(r-2j)!}V_m^{r-2j}U_m^{2k-2r}.$$  

Note that the identities of Grabner and Prodinger [3] correspond to the $m = 1$ and implicitly $m = 2$ cases of Theorem 1 (cf. Section 3).

**2 Proof of Theorem 1**

We first check the $k = 1$ case of (4):

$$W_n^2 + W_{n+m}^2 = V_nW_nW_{n+m} + U_m^2\Omega. \quad (5)$$

Set $\alpha = (p + \sqrt{p^2 - 4})/2$ and $\beta = (p - \sqrt{p^2 - 4})/2$ then it is easy to see that

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad V_n = \alpha^n + \beta^n,$$

it follows that

$$W_n = aU_n + bV_n = A\alpha^n + B\beta^n,$$

where $A = b + a/(\alpha - \beta)$ and $B = b - a/(\alpha - \beta)$. Therefore

$$V_nW_nW_{n+m} + U_m^2\Omega = (\alpha^n + \beta^n)(A\alpha^n + B\beta^n)(A\alpha^{n+m} + B\beta^{n+m})$$

$$+ \left(\frac{\alpha^n + \beta^n}{\alpha - \beta}\right)^2 (a^2 + 4b^2 - b^2p^2),$$
which is readily seen to be equal to $W_n^2 + W_{n+m}^2$.

Next we take the alphabet $X = \{W_n^2, W_{n+m}^2\}$, then the left-hand side of (1) is the power sum $p_k(X)$. On the other hand, since

$$e_1(X) = W_n^2 + W_{n+m}^2, \quad e_2(X) = W_n^2W_{n+m}^2, \quad e_i(X) = 0 \quad \text{if} \quad i \geq 3,$$

the summation at the right-hand side of (1) reduces to the partitions $\lambda = (1^{k-2j} 2^j)$, with $j \geq 0$. Now, using (5) Waring’s formula (1) infers that

$$W_n^{2k} + W_{n+m}^{2k} = \sum_{0 \leq 2j \leq k} (-1)^j \frac{k(k-j-1)!}{j!(k-2j)!} (V_n W_n W_{n+m} + U_m^2 \Omega)^{k-2j} (W_n^2 W_{n+m}^2)^j$$

$$= \sum_{0 \leq 2j \leq k} \sum_{i=0}^{k-2j} (-1)^j \frac{k(k-j-1)!}{j!(k-2j-i)!} V_m^{k-2j-i} U_m^2 \Omega^i (W_n W_{n+m})^{k-i}$$

Setting $k-i=r$ and exchanging the order of summations yields (4).

3 Some special cases

When $m=1$ or 2, as $U_1=1$, $V_1=p$ and $U_2=p$, $V_2=p^2-2$ the coefficient $\theta_{k,r}(r)$ of Theorem 1 is much simpler.

**Corollary 1** We have

$$\theta_{k,r}(1) = \sum_{0 \leq 2j \leq r} (-1)^j \frac{k(k-1-j)!}{(k-r)!j!(r+2j)!} p^{r-2j}, \quad (6)$$

$$\theta_{k,r}(2) = \sum_{0 \leq 2j \leq k} (-1)^j \frac{k(k-j-1)!}{j!(k-r)!j!(r-2j)!} (p^2-2)^{r-2j} p^{2k-2r}. \quad (7)$$

We notice that (6) is exactly the formula given by Grabner and Prodinger for $\theta_{k,r}(1)$, while for $\theta_{k,r}(2)$ they give a more involved formula than (7) as follows:

**Corollary 2** (Grabner and Prodinger) There holds

$$\theta_{k,r}(2) = \sum_{0 \leq \lambda \leq k} (-1)^\lambda p^{2k-2\lambda} \frac{k(k-\lfloor \frac{\lambda}{2} \rfloor - 1)!2^{\lfloor \frac{\lambda}{2} \rfloor}}{(k-r)!\lambda!(r-\lambda)!} \prod_{i=0}^{\lfloor \frac{\lambda}{2} \rfloor-1} (2k-2\lfloor \frac{\lambda}{2} \rfloor - 1-2i). \quad (8)$$

In order to identify (7) and (8), we need the following identity.
Lemma 2  We have
\[
\sum_{i=0}^{j/2} (-1)^i \frac{(k - i - 1)!2^{j - 2i}}{(j - 2i)!i!}
= \frac{(k - \lfloor j/2 \rfloor - 1)!}{j!} 2^{\lfloor j/2 \rfloor - 1} \prod_{i=0}^{\lfloor j/2 \rfloor - 1} (2k - 2\lfloor j/2 \rfloor - 1 - 2i). \tag{9}
\]

Proof: For \( n \geq 0 \) let \((a)_n = a(a + 1) \ldots (a + n - 1)\), then the Chu-Vandermonde formula [2, p.212] reads:
\[
2F_1(-n, a; c; 1) := \sum_{k \geq 0} \frac{(-n)_k (a)_k}{(c)_k k!} = \frac{(c - a)_n}{(c)_n}. \tag{10}
\]
Note that \(n! = (1)_n\), so using the simple transformation formulae:

\[
(a)_2n = \left(\frac{a}{2}\right)_n \left(\frac{a + 1}{2}\right)_n 2^{2n}, \quad (a)_{2n+1} = \left(\frac{a}{2}\right)_{n+1} \left(\frac{a + 1}{2}\right)_n 2^{2n+1},
\]

and
\[
(a)_{N-n} = \frac{(a)_N}{(a + N - n)_n} = (-1)^n \frac{(a)_N}{(-a - N + 1)_n},
\]
we can rewrite the left-hand side of identity (9) as follows:
\[
\begin{cases}
\frac{(k-1)!}{(\frac{1}{2})_{m(1)_{m}}} 2F_1(-m, -m + \frac{1}{2}; -k + 1; 1) & \text{if } j = 2m, \\
\frac{(k-1)!}{(\frac{1}{2})_{m+1}(1)_{m+1}} 2F_1(-m, -m - \frac{1}{2}; -k + 1; 1) & \text{if } j = 2m + 1,
\end{cases}
\]
which is clearly equal to the right-hand side of (9) in view of (10).

Now, expanding the right-hand side of (10) by binomial formula yields
\[
\sum_{0 \leq 2j \leq k} (-1)^j \frac{k(k - j - 1)!}{j!(k-r)!(r - 2j)!} \sum_{i=0}^{r-2j} \binom{r-2j}{i} p^{2i} (-2)^{r-2j-i} p^{2k-2r}.
\]
Writing \( \lambda = r - i \), so \( \lambda \leq r \leq k \), and exchanging the order of summations, the above quantity becomes
\[
\sum_{0 \leq \lambda \leq k} (-1)^\lambda p^{2k-2\lambda} \frac{k}{(k-r)!(r-\lambda)!} \sum_{0 \leq j \leq k/2} (-1)^j \frac{(k - j - 1)!2^{\lambda-2j}}{(\lambda - 2j)!j!},
\]
which yields (8) by applying Lemma 2.
References

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